Nilpotent invariant motives I

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Abstract

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Introduction

The purpose of this article is to clarify the question what makes motives $A^1$-homotopy invariance. We give a guide for the structure of this article. In section 1, we recall the construction of the stable model category of nilpotent invariant motives $\text{Mot}_{\text{nilp}}^\text{dg}$ from [20] and define the nilpotent invariant motives associated with schemes and relative exact categories. For a noetherian scheme $X$, there are two kinds of motives associated with $X$ in the homotopy category $\text{Ho}(\text{Mot}_{\text{nilp}}^\text{dg})$, namely $M_{\text{nilp}}(X)$ and $M'_{\text{nilp}}(X)$. In general $M_{\text{nilp}}(X)$ is not isomorphic to $M'_{\text{nilp}}(X_{\text{red}})$. But there exists a canonical isomorphism $M'_{\text{nilp}}(X) \rightarrow M_{\text{nilp}}(X_{\text{red}})$ and if $X$ is regular noetherian separated, $M(X)$ is canonically isomorphic to $M'(X)$. In section 2, we will show absolute geometric presentation theorem 2.3. Roughly speaking this theorem says that for a regular local ring $A$ and for each non-negative integer $p \leq \dim A$, the topological weight $p$ part of $\text{Spec } A$ is isomorphic to the Adams weight $p$ part of $\text{Spec } A$ as nilpotent invariant motives. In section 3, we establish a concept of algebraic varieties over (locally) noetherian abelian categories. By utilizing this concept we clarify the question what makes motives $A^1$-homotopy invariance. We will show $A^1$-homotopy invariance of $M'$ which does not hold for $M$ in general.

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1 Stable model category of nilpotent invariant motives

In this section, we review the construction of the stable model category of nilpotent invariant motives $\mathcal{M}_{\text{ot}^{\text{nilp}}}$ from \cite{29} and define motives associated with coherent schemes and relative exact categories and dash motives associated with noetherian schemes.

We denote the category of small dg-categories over $\mathbb{Z}$ the ring of integers by $\text{dgCat}$ and let $\mathcal{M}_{\text{ot}^{\text{add}}}^{\text{dg}}$ and $\mathcal{M}_{\text{ot}^{\text{loc}}}^{\text{dg}}$ be symmetric monoidal stable model categories of additive noncommutative motives and localizing motives over $\mathbb{Z}$ respectively. (See \cite[§7]{3}.) In general we denote the homotopy category of a model category $\mathcal{M}$ by $\text{Ho}(\mathcal{M})$. There are functors from $\text{ExCat}$ the category of small exact categories to $\text{dgCat}$ which send sending a small exact category $E$ to its bounded dg-derived category $D_{\text{b}}^{\text{dg}}(E)$ (see \cite[§4.4]{23}). and the universal functors $U_{\text{add}}$: $\text{dgCat} \rightarrow \text{Ho}(\mathcal{M}_{\text{ot}^{\text{add}}}^{\text{dg}})$ and $U_{\text{loc}}$: $\text{dgCat} \rightarrow \text{Ho}(\mathcal{M}_{\text{ot}^{\text{loc}}}^{\text{dg}})$. We denote the compositions of these functors $\text{ExCat} \rightarrow \text{Ho}(\mathcal{M}_{\text{ot}^{\text{add}}}^{\text{dg}})$ and $\text{ExCat} \rightarrow \text{Ho}(\mathcal{M}_{\text{ot}^{\text{loc}}}^{\text{dg}})$ by $\mathcal{M}_{\text{add}}$ and $\mathcal{M}_{\text{loc}}$ respectively.

We start by recalling the construction of the stable model category of localizing non-commutative motives $\mathcal{M}_{\text{ot}^{\text{loc}}}^{\text{dg}}$ from \cite{41}. First notice that the category $\text{dgCat}$ carries a cofibrantly generated model structure whose weak equivalences are the derived Morita equivalences. \cite[Théorème 5.3]{40}. We fix on $\text{dgCat}$ a fibrant resolution functor $R$, a cofibrant resolution functor $Q$ and a left framing $\Gamma_{\ast}$ (see \cite[Definition 5.2.7 and Theorem 5.2.8]{19}). and we also fix a small full subcategory $\text{dgCat}_{f} \hookrightarrow \text{dgCat}$ such that it contains all finite dg cells and any objects in $\text{dgCat}_{f}$ are $(\mathbb{Z})$-flat and homotopically finitely presented (see \cite[Definition 2.1 (3)]{42}). and $\text{dgCat}_{f}$ is closed under the operations $Q$, $QR$ and $\Gamma_{\ast}$ and $\otimes$. The construction below does not depend upon a choice of $\text{dgCat}_{f}$ up to Dwyer-Kan equivalences. Let $\text{sdgCat}_{f}$ and $\text{sdgCat}_{f,\ast}$ be the category of simplicial presheaves and that of pointed simplicial preshaves on $\text{dgCat}_{f}$ respectively. We have the projective model structures on $\text{sdgCat}_{f}$ and $\text{sdgCat}_{f,\ast}$ where the weak equivalences and the fibrations are the termwise simplicial weak equivalences and termwise Kan fibrations respectively. (see \cite[Theorem 11.6.1]{18}.) We denote the class of derived Morita equivalences in $\text{dgCat}_{f}$ by $\Sigma$ and we also write $\Sigma_{\ast}$ for the image of $\Sigma$ by the composition of the Yoneda embedding $h$: $\text{dgCat}_{f} \rightarrow \text{sdgCat}_{f}$ and the canonical functor $(-)_{\ast}$: $\text{sdgCat}_{f} \rightarrow \text{sdgCat}_{f,\ast}$. Let $P$ be the canonical map $\emptyset \rightarrow h(\emptyset)$ in $\text{sdgCat}_{f}$ and we write $P_{\ast}$ for the image of $P$ by the functor $(-)_{\ast}$. We write $L_{\Sigma, \ast} \text{sdgCat}_{f,\ast}$ for the left Bosifield localization of $\text{sdgCat}_{f,\ast}$ by the set $\Sigma_{\ast} \cup \{P_{\ast}\}$. The Yoneda embedding functor induces a functor

$$R_{h}: \text{Ho}(\text{dgCat}) \rightarrow \text{Ho}(L_{\Sigma, \ast} \text{sdgCat}_{f,\ast})$$

which associates any dg category $\mathcal{A}$ to the pointed simplicial presheaves on $\text{dgCat}_{f}$:

$$R_{h}(\mathcal{A}): \mathcal{B} \mapsto \text{Hom}(\Gamma_{\ast}(Q \mathcal{B}), R(\mathcal{A}))_{\ast}.$$
Let $E$ be the class of morphisms in $L_{\Sigma, P \, \text{sdgCat}}$ of shape

$$\text{Cone}[\mathbb{R} h(A) \to \mathbb{R} h(B)] \to \mathbb{R} h(C)$$

associated to each exact sequence of dg categories $A \to B \to C$, with $B$ in $\text{dgCat}$ where Cone means homotopy cofiber. We write $\text{Mot}_{\text{uloc}}$ for the left Bousfield localization of $L_{\Sigma, P \, \text{sdgCat}}$ by $E$ and call it the model category of unstable localizing non-commutative motives. Finally we write $\text{Mot}_{\text{loc}}$ for the stable symmetric monoidal model category of $S^1 \otimes 1$-spectra on $\text{Mot}_{\text{uloc}}$ (see [20, §7].) and call it the model category of localizing non-commutative motives. Next we construct the stable model category $\text{Mot}_{\text{nilp}}$. First recall that we say that a non-empty full subcategory $Y$ of a Quillen exact category $X$ is a topologizing subcategory of $X$ if $Y$ is closed under finite direct sums and closed under admissible sub- and quotient objects. The naming of the term ‘topologizing’ comes from noncommutative geometry of abelian categories by Rosenberg. (See [25, Lecture 2 1.1].) We say that a full subcategory $Y$ of an exact category $X$ is a Serre subcategory if it is an extensional closed topologizing subcategory of $X$. For any full subcategory $Z$ of $X$, we write $\sqrt{Z}$ for intersection of all Serre subcategories which contain $Z$ and call it the Serre radical of $Z$ (in $X$). We say that an object $x$ in an exact category is noetherian if any ascending filtration of admissible subobjects of $x$ is stational. We say that an exact category $E$ is noetherian if it is essentially small and all objects in $E$ are noetherian.

1.1. Definition (Nilpotent immersion). Let $A$ be a noetherian abelian category and let $B$ a topologizing subcategory. We say that $B$ satisfies the dévissage condition (in $A$) or say that the inclusion $B \hookrightarrow A$ is a nilpotent immersion if one of the following equivalent conditions holds:

1. For any object $x$ in $A$, there exists a finite filtration of monomorphisms

$$x = x_0 \leftarrow x_1 \leftarrow x_2 \leftarrow \cdots \leftarrow x_n = 0$$

such that for every $i < n$, $x_i/x_{i+1}$ is isomorphic to an object in $B$.

2. We have the equality

$$A = \sqrt{B}.$$

(For the proof of the equivalence of the conditions above, see [16, 3.1], [10, 2.2].)

1.2. Definition (Nilpotent invariant motives). We write $N$ for the class of morphisms in $\text{Mot}_{\text{nilp}}$ of shape

$$\mathbb{R} h(C \otimes D^b_{\text{dg}}(B)) \to \mathbb{R} h(C \otimes D^b_{\text{dg}}(A))$$

associated with each noetherian abelian category $A$ and each nilpotent immersion $B \hookrightarrow A$ and each small dg-category $C$ in $\text{dgCat}$. We write $\text{Mot}_{\text{nilp}}$ the left Bousfield localization of $\text{Mot}_{\text{nilp}}$ by $N$ and call it the model category of unstable nilpotent invariant non-commutative motives. This category naturally becomes a symmetric monoidal model category by [3, Theorem 5.7.] Finally we write $\text{Mot}_{\text{nilp}}$ for the stable model category of symmetric $S^1 \otimes 1$-spectra on $\text{Mot}_{\text{nilp}}$ and call it the stable model
category of nilpotent invariant non-commutative motives. We denote the compositions of the following functors

\[ \text{dgCat} \to \text{Ho}(\text{dgCat}) \overset{\text{R}_{Dg}}{\to} \text{Ho}(L_{\Sigma, p} \text{sdgCat}^E) \to \text{Ho}(\text{Mot}^{\text{nilp}}_{Dg}) \overset{\Sigma}{\to} \text{Ho}(\text{Mot}^{\text{nilp}}_{Dg}) \]

by \( U_{\text{nilp}} \) and we write \( M_{\text{nilp}} \) for the compositions of the following functors

\[ \text{ExCat} \overset{D^{\text{nilp}}_{Dg}}{\to} \text{dgCat} \overset{U_{\text{nilp}}}{\to} \text{Ho}(\text{Mot}^{\text{nilp}}_{Dg}). \]

We recall the conventions of relative exact categories from [27]. A relative exact category \( E = (\mathcal{E}, w) \) is a pair of exact category \( \mathcal{E} \) and a class of morphisms \( w \) in \( \mathcal{E} \). We call \( \mathcal{E} \) and \( w \) the underlying exact category of \( E \) and the class of weak equivalences of \( E \) and denote it by \( \mathcal{E}_E \) and \( w_E \) respectively. We say that a relative exact category is small if its underlying exact category is small. A relative exact functor \( f : E \to E' \) is an exact functor \( f : \mathcal{E}_E \to \mathcal{E}_{E'} \) such that \( f(w_E) \subset w_{E'} \). We say that a relative exact category is extensional if its class of weak equivalences satisfies the extension axiom in [10]. We say that a relative exact category \( E \) is biWaldhausen if both \( w_E \) and \( w^{op}_E \) satisfy the gluing axiom in \( \mathcal{E}_E \) and \( \mathcal{E}_{E}^{op} \) respectively. For a relative exact category \((\mathcal{E}, w)\), let \( \mathcal{E}^w \) be the full subcategory of \( \mathcal{E} \) consisting of those object \( x \) such that the canonical morphism \( 0 \to x \) is in \( w \). If we assume \((\mathcal{E}, w)\) is either extensional or biWaldhausen, then \( \mathcal{E}^w \) is an exact category such that the exact functor \( \mathcal{E}^w \to \mathcal{E} \) is exact and reflects exactness. (See [27, Proposition 2.4].)

1.3. Definition (Motives associated with relative exact categories). Let \( E = (\mathcal{E}, w) \) be a small relative exact category which is either extensional or biWaldhausen. Then we define \( D^b_{Dg}(E) = D^b_{Dg}(\mathcal{E}, w) \) to be a dg-category by setting \( D^b_{Dg}(E) := D^b_{Dg}(\mathcal{E}) / D^b_{Dg}(\mathcal{E}^w) \) the Drinfeld quotient of \( D^b_{Dg}(\mathcal{E}) \) by \( D^b_{Dg}(\mathcal{E}^w) \) and we call it the bounded dg-derived category of \( E \). For \( \# \in \{\text{add, loc, nilp}\} \), we set \( M_\#(E) = M_\#(\mathcal{E}, w) := U_\#(D^b_{Dg}(E)) \) and call it additive (resp. localizing, nilpotent invariant) motives associated with \( E \). For a small exact category \( \mathcal{E} \), we write \( i_\mathcal{E} \) for the class of all isomorphisms in \( \mathcal{E} \). Then we have the canonical identification \( M_\#(i_\mathcal{E}) = M_\#(\mathcal{E}) \).

A scheme \( X \) is coherent if it is quasi-compact and quasi-separated. We say that a subset \( Y \) of a coherent scheme \( X \) is a Thomason-Ziegler closed subset if \( Y \) is a closed set with respect to Zariski topology of \( X \) and if \( X \setminus Y \) is a quasi-compact open subset of \( X \). The naming of the term ‘Thomason-Ziegler’ comes from the works [48] and [45].

1.4. Definition (Motives associated with schemes). Let \( X \) be a coherent scheme and \( Y \) be a Thomason-Ziegler closed subset of \( X \). We denote the dg-category of perfect complexes on \( X \) whose cohomological supports are in \( Y \) by \( \text{Perf}_X^Y \). In particular we write \( \text{Perf}_X \) for \( \text{Perf}_X^X \). We also write the same letters \( (\text{Perf}_X^Y, \text{qis}) \) for the Waldhausen category of perfect complexes on \( X \) whose cohomological supports are in \( Y \) with the class of quasi-isomorphisms. By virtue of [27, Theorem 0.1], the canonical dg-functor \( \text{Perf}_X^Y \to D^b_{Dg}(\text{Perf}_X^Y, \text{qis}) \) is a derived Morita equivalence. Thus it induces an isomorphism of motives \( U_\#(\text{Perf}_X^Y) \to M_\#(\text{Perf}_X^Y, \text{qis}) \) for \( \# \in \{\text{add, loc, nilp}\} \). We denote it by \( M_\#(X) \) and
call it the additive (resp. localizing, nilpotent invariant) motives associated with the pair \((X,Y)\). We denote the motives associated with the pair \((X,Y)\) by \(M_{\#}(X)\). For a non-negative integer \(p\), we write \(\text{Perf}_X^p\) for the dg-category of perfect complexes on \(X\) whose cohomological support has codimension \(\geq p\) and we set \(M_{\#}^{p,\text{top}}(X) := \mathcal{U}_{\#}(\text{Perf}_X^p)\) for \(\# \in \{\text{add}, \text{loc}, \text{nilp}\}\) and call it the topological weight \(p\) part of additive (resp. localizing, nilpotent invariant) motive associated with \(X\).

1.5. Remark (Fundamental properties of motives associated with schemes). Since the proofs of fundamental properties of algebraic \(K\)-theory in [43, 37] are based upon derived equivalences of the categories of perfect complexes on schemes, the proofs work still fine for motives. Namely by replacing \(K\) (or \(K^H\)) with \(M_{\#} (\# \in \{\text{loc}, \text{nilp}\}\), continuity ([43 3.20.2]), localization ([37 3.4.9]), excision ([43 7.1]), Mayer-Vietoris for Zariski open covers ([37 3.4.12]) and blow up formula for regular center ([37 3.5.4]) hold for \(M_{\#}\).

1.6. Definition (Dash motives associated with noetherian schemes). Let \(X\) be a noetherian scheme and \(Y\) be a closed subset of \(X\). We denote the abelian category of coherent sheaves on \(X\) whose supports are in \(Y\) by \(\text{Coh}_X^Y\). In particular we write \(\text{Coh}_X\) for \(\text{Coh}_X^X\). We set \(M_{\#}^Y(X) := \mathcal{U}_{\#}(\text{Coh}_X^Y)\) for \(\# \in \{\text{add}, \text{loc}, \text{nilp}\}\) and call it the additive (resp. localizing, nilpotent invariant) dash motive associated with the pair \((X,Y)\). We denote the dash motives associated with the pair \((X,X)\) by \(M_{\#}^X(X)\). Similarly for an non-negative integer \(p\), we denote the category of coherent sheaves on \(X\) whose support has codimension \(\geq p\) in \(X\) by \(\text{Coh}_X^{\text{nilp}}\). We set \(M_{\#}^{p,\text{top}}(X) := \mathcal{U}_{\#}(\text{Coh}_X^{\text{nilp}})\) for \(\# \in \{\text{add}, \text{loc}, \text{nilp}\}\) and call it topological weight \(p\) part of additive (resp. localizing, nilpotent invariant) dash motive associated with \(X\).

1.7. Remark (Fundamental properties of dash motives associated with noetherian schemes). As in Remark 1.5 since the proofs of fundamental properties of \(G\)-theory of noetherian schemes in [43] and [37] are based upon derived equivalences of the categories of pseudo-coherent complexes or bounded complexes of coherent modules on schemes, the proofs work still fine for dash motives. Namely by replacing \(G\) (or \(K\)) with \(M_{\#} (\# \in \{\text{loc}, \text{nilp}\}\), continuity ([43 3.20.2], 7.2]), localization ([37 3.3.2]), and excision ([37 3.19].] hold for \(M_{\#}\).

1.8. Remark (Nilpotent invariance). For a coherent scheme \(X\), in general \(M_{\text{nilp}}(X)\) is not isomorphic to \(M_{\text{nilp}}(X_{\text{red}})\). But for a noetherian scheme \(X\), since the inclusion functor \(\text{Coh}_{X_{\text{red}}} \hookrightarrow \text{Coh}_X\) is a nilpotent immersion, it induces an isomorphism \(M_{\text{nilp}}(X_{\text{red}}) \cong M_{\text{nilp}}(X)\). The naming of ‘nilpotent invariant’ comes from this fact.

Specific features of nilpotent invariant motives and relationship between motives and dash motives associated with schemes summed up with the following proposition.

1.9. Proposition. Let \(X\) be a regular noetherian separated scheme over \(\text{Spec} \, \mathbb{Z}\) and \(Y\) be a closed subscheme of \(X\) and \(p\) be a non-negative integer. Then

1. (Purity). The inclusion functor \(\text{Coh}_Y \hookrightarrow \text{Perf}_X^p\) induces an isomorphism of motives \(M_{\text{nilp}}^p(Y) \cong M_{\text{nilp}}^p(X)\). In particular if \(Y \hookrightarrow X\)

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is an regular closed embedding, then we have the canonical isomorphism $M_{\text{nilp}}(Y) \to M'_{\text{nilp}}(X)$.

(2) **Comparison of topological weight.** The inclusion functor $\text{Coh}^p_X \hookrightarrow \text{Perf}^p_X$ induces an isomorphism of motives $M'_{\text{nilp, top}}(X) \xrightarrow{\sim} M^p_{\text{nilp, top}}(X)$.

**Proof.** (1) In $\text{Mot}^\text{nilp,dg}$, there exists a commutative diagram of localization distinguished triangles

$$
\begin{array}{ccc}
M'_{\text{nilp}}(Y) & \longrightarrow & M'_{\text{nilp}}(X) \\
\downarrow & & \downarrow \\
M'_{\text{nilp}}(X) & \longrightarrow & M'_{\text{nilp}}(X) \setminus Y \\
\uparrow & & \cup \\
\Sigma M'_{\text{nilp}}(Y) & \longrightarrow & \Sigma M'_{\text{nilp}}(X) \\
\end{array}
$$

Then the morphisms $I$ and $\text{II}$ are isomorphism by (the proof of) [43, 3.21]. Thus by five lemma of distinguished triangles, the morphism $M'_{\text{nilp}}(Y) \to M'_{\text{nilp}}(X)$ is also an isomorphism.

(2) Consider the following commutative diagram

$$
\begin{array}{ccc}
M'^p_{\text{nilp, top}}(X) & \longrightarrow & M^p_{\text{nilp, top}}(X) \\
\downarrow & & \downarrow \\
\text{hocolim}_{\text{codim}_X Y \geq p} M'_{\text{nilp}}(Y) & \longrightarrow & \text{hocolim}_{\text{codim}_X Y \geq p} M^Y_{\text{nilp}}(X) \\
\end{array}
$$

in $\text{Ho}(\text{Mot}^\text{nilp,dg})$. By continuity, the vertical morphisms above are isomorphisms and by (1), the bottom morphism is an isomorphism. Thus we obtain the result.

In section 3, we will show $\mathbb{A}^1$-homotopy invariance and projective bundle formula for dash motives of noetherian schemes. (See Corollary 3.25)

1.10. **Remark (Relative version of nilpotent invariant motives).** For a commutative ring $B$ with 1, by replacing $\text{dgCat}$ with $\text{dgCat}_B$ the category of small dg-categories over $B$ in the construction $\text{Mot}^\text{nilp,dg}$, we can obtain the stable model category $\text{Mot}^\text{nilp,dg}_B$ of nilpotent invariant motives over $B$. Similar statements as in Proposition 1.9 are also true for schemes over $\text{Spec} B$.

## 2 Absolute geometric presentation theorem

Gersten’s conjecture [12] for Grothendieck groups is equivalent to the following generator conjecture. (See [24], [7], [8], [26] and [29].)

2.1. **Conjecture (Generator conjecture).** For any commutative regular local ring $R$ and any natural number $0 \leq p \leq \dim R$, the Grothendieck group $K_0(\text{Coh}^p_{\text{spec} R})$ is generated by cyclic modules $R/(f_1, \cdots, f_p)$ where the sequence $f_1, \cdots, f_p$ forms an $R$-regular sequence.
For historical background of generator conjecture, please see Introduction of [26]. Basically proofs of known cases are based upon structure theorems of $R$ (over base). For example, if $R$ is smooth over a field or a discrete valuation ring, we will utilize a version of noether normalization theorem. (See [33] and [13].) A version of noether normalization theorem sometimes called 'geometric presentation theorem'. (See for example [4].) So we would like to formulate an absolute version of such a presentation theorem.

On the other hands, by virtue of [13] and [43], we have the following proposition 2.2. Recall that we say that a scheme $X$ is divisorial if it is quasi-compact and if it has an ample family of line bundles. That is, there exists a family of line bundles $\{L_\lambda\}_{\lambda \in \Lambda}$ on $X$ indexed by a non-empty set $\Lambda$ which satisfies the following condition. For any $f \in \Gamma(X, L_\lambda^\otimes m)$, we set $X_f := \{x \in X : f(x) \neq 0\}$. Then the family $\{X_f\}$ is a basis of Zariski topology of $X$ where $m$ runs over all positive integer, $L_\lambda$ runs over the family of line bundles and $f$ runs over all global sections of all of $L_\lambda^\otimes m$. (See [21] 2.2.5., [43] 2.1.1. and [17] 2.12.]

### 2.2. Proposition

For a divisorial scheme $X$ and non-negative integer $p \leq \dim X$, there exists a family of Adams operations $\{\psi_k\}_{k \geq 0}$ indexed by the set of positive integers on the Grothendieck group $K_0(D(\text{Perf}^Y_X))$ of the triangulated category of perfect complexes on $X$ whose homological support has codimension $\geq p$. Moreover if $X$ is the spectrum $\text{Spec} R$ of a noetherian ring $R$ and if the sequence $f_1, \cdots, f_p$ is an $R$-regular sequence, then we have the equality

$$\psi_k([\text{Kos}(f_1, \cdots, f_p)]) = k^p[\text{Kos}(f_1, \cdots, f_p)]$$

for any positive integer $k > 0$ where $[\text{Kos}(f_1, \cdots, f_p)]$ is the class of $\text{Kos}(f_1, \cdots, f_p)$ the Koszul complex associated with the regular sequence $f_1, \cdots, f_p$ in $K_0(D(\text{Perf}^Y_{\text{Spec} R}))$.

**Proof.** In [13] 4.11], for a positive integer $k$, Gille and Soulé define an Adams operation of degree $k$ to be any collection of maps $\psi_k : K_0^Y(X) \to K_0^Y(X)$ where $Y$ runs over all closed subsets of $X$ and $K_0^Y(X)$ is the Grothendieck group of relative exact category $\text{Ch}^Y_{\geq 0, h}(P_X)$ the category of bounded complexes $x$ of vector bundles on $X$ such that $x_i = 0$ for $i < 0$ and whose homological support is in $Y$ with the class of all quasi-isomorphisms. By additivity theorem, the degree shift functor

$$[1] : \text{Ch}^Y_{\geq 0, h}(P_X) \to \text{Ch}^Y_{\geq 0, h}(P_X)$$

induces the multiplication by $-1$ on the Grothendieck group $K_0^Y(X)$. In particular $K_0^Y(X)$ is canonically isomorphic to the Grothendieck group of relative exact category of the strict perfect complexes (= bounded complexes of vector bundles) on $X$ whose homological support is in $Y$ with the class of all quasi-isomorphisms. Since $X$ is divisorial, this Grothendieck group is isomorphic to $K_0(\text{Perf}^Y_X, \text{qis})$ the Grothendieck group of relative exact category of perfect complexes on $X$ whose homological support is in $Y$ with the class of all quasi-isomorphisms. (See [33] 3.8.) Since the Adams operations on $K_0^Y(X)$ are compatible with the induced map.
$K^Z_0(X) \rightarrow K^Y_0(X)$ from the inclusion of closed subsets $Y \hookrightarrow Z$ \cite[3.4 A3], it induces the family of Adams operations on

$$K_0(D(\text{Perf}^p_X)) = \text{colim}_{\text{codim} Y \geq p} K^Y_0(X).$$

Finally the equation \cite{1} follows from \cite[4.11 A4'].

For a commutative regular ring $R$ and a non-negative integer $p \leq \dim R$, the inclusion functor $\text{Coh}^p_{\text{Spec} R} \hookrightarrow \text{Perf}^p_{\text{Spec} R}$ induces an isomorphism of the Grothendieck groups $K_0(\text{Coh}^p_{\text{Spec} R}) \rightarrow K_0(D(\text{Perf}^p_{\text{Spec} R}))$. (Compare Proposition \cite[21.3.0.7]{2} and see also \cite[Proposition 5.8]{20} and \cite[Theorem 3.3.]{17})] via this isomorphism, a class $[R/f_1, \cdots, f_p]$ of a cyclic module with an $R$-sequence $f_1, \cdots, f_p$ in $K_0(\text{Coh}^p_{\text{Spec} R})$ corresponds to the class $[\text{Kos}(f_1, \cdots, f_p)]$ of Koszul complex $\text{Kos}(f_1, \cdots, f_p)$ in $K_0(D(\text{Perf}^p_{\text{Spec} R}))$. Thus roughly saying, the generator conjecture says that if $X$ is the spectrum $\text{Spec} R$ of a regular local ring $R$, then $K_0(\text{Coh}^p_{\text{Spec} R})$ is generated by objects of Adams weight $p$. The purpose of this section is to provide a motivic version of the generator conjecture and in my viewpoint, it can be regarded as an absolute version of a structure theorem of $X$ and I would like to call it an absolute geometric presentation theorem \cite{22}.

We assume that $R$ is regular. Then since in particular $R$ is Cohen-Macaulay, the ordered set of all ideals of $R$ that contains an $R$-regular sequence of length $p$ with usual inclusion is directed. Thus $\text{Perf}^p_{\text{Spec} R}$ is the filtered limit $\text{colim}_I \text{Perf}^p_{\text{Spec} R}$ where $I$ runs through any ideal generated by any $R$-regular sequence of length $p$. Thus by continuity of motives, we obtain the following isomorphism

$$M^p_{\#}(\text{Perf}^p_{\text{Spec} R}) \cong \text{hocolim}_{\text{codim}_{\text{Perf}^p_{\text{Spec} R}} V(I) = p} M^V(I)(\text{Spec} R)$$

for $\# \in \{\text{loc}, \text{nilp}\}$. Thus we wish to show that for each ideal $I$ generated by $R$-regular sequence $f_1, \cdots, f_p$, $M^V(I)(\text{Spec} R)$ is spanned by objects of Adams weight $p$ in some sense.

To make the statement more precisely, we start by recalling the definitions of Koszul cubes from \cite{20}, \cite{21} and \cite{22}. Let $S$ be a finite set and $\mathcal{C}$ be a category. An $S$-cube in $\mathcal{C}$ is a contravariant functor from $\mathcal{P}(S)$ the power set of $S$ with the usual inclusion order to $\mathcal{C}$. A morphism of $S$-cubes is just a natural transformation. We denote the category of $S$-cubes by $\text{Cub}^S(\mathcal{C})$. For an $S$-cube $x$ and a subset $T$ and an element $t \in T$, we write $x_T$ for $x(T)$ and call it a vertex of $x$ (at $T$) and we denote $x(T \setminus \{ t \}) \hookrightarrow T$ by $d^T_T x = d^T_T$ and call it a (k-direction) boundary morphism of $x$.

Let $S$ be a non-empty finite set such that $\#S = n$ and $x$ an $S$-cube in an additive category $\mathcal{B}$. Let us fix a bijection $\alpha$ from $S$ to $[n]$ the set of all positive integers $1 \leq k \leq n$ and we will identify $S$ with the set $[n]$ via $\alpha$. We associate an $S$-cube $x$ with total complex $\text{Tot}_\alpha x = \text{Tot} x$ as follows. $\text{Tot} x$ is a chain complex in $\mathcal{B}$ concentrated in degrees $0, \ldots, n$ whose component at degree $k$ is given by $(\text{Tot} x)_k := \bigoplus_{T \in \mathcal{P}(S), \#T = k} x_T$
and whose boundary morphism $d^r_k: (\text{Tot } x)_k \to (\text{Tot } x)_{k-1}$ are defined
by $\sum_{t\neq j} \chi_T(t) d^r_j: x_T \to x_{T\setminus\{j\}}$ on its $x_T$ component to $x_{T\setminus\{j\}}$ component. Here $\chi_T$ is the characteristic function $\chi_T: S \to \{0,1\}$ of $T$.
Namely $\chi_T(s) = 1$ if $s$ is in $T$ and otherwise $\chi_T(s) = 0$.

Let $A$ be a commutative noetherian ring with 1. By an $A$-sequence we mean an $A$-regular sequence $f_1, \ldots, f_s$ such that any permutation of the $f_j$ is also an $A$-regular sequence. A Koszul cube (associated with an $A$-sequence $f_s = \{f_s\}_{s \in S}$) is an $S$-cube in the category of finitely generated projective $A$-modules such that for each subset $T$ of $S$ and each element $t$ in $T$, $d^r_t$ is an injection and $\text{Coker } d^r_t$ is annihilated by $f_j^m$ for some positive integer $m$. We denote the category of Koszul cubes associated with $f_s$ by $\text{Kos}_A^{f_s}$. Notice that if $S$ is the empty set, then $\text{Kos}_A^{f_s}$ is just the category of finitely generated projective modules $\text{P}_A$. A representative example is the following. Let $r$ be a non-negative integer and $n_s = \{n_s\}_{s \in S}$ a family of non-negative integers indexed by $S$ such that $r \geq n_s$ for each $s \in S$. We define $\text{Typ}_A(f_s; r, n_s)$ to be an $S$-cube of finitely generated free $A$-modules by setting for each element $s$ in $S$ and subsets $U \subset S$ and $V \subset S \setminus \{s\}$, $\text{Typ}_A(f_s; r, n_s)_V := A^{\otimes r}$ and $d^{\text{Typ}_A(f_s; r, n_s)}_{V\cup\{s\}} := \begin{pmatrix} f_s E_{n_s} & 0 \\ 0 & E_{r-n_s} \end{pmatrix}$ where $E_m$ is the $m \times m$ unit matrix.

We call $\text{Typ}_A(f_s; r, n_s)$ the typical cube of type $(r, n_s)$ associated with $f_s$. We denote the full subcategory of $\text{Kos}_A^{f_s}$ consisting of those object $\text{Typ}_A(f_s; r, n_s)$ for some pair $(r, n_s)$ by $\text{Kos}_A^{\text{typ}_s}$ and call it the category of typical Koszul cubes (associated with $f_s$). Roughly saying, $\text{Kos}_A^{\text{typ}_s}$ is just the category of objects of Adams weight $\# S$ supported in $V(f_s; A)$ where $f_s A$ is an ideal generated by the family $f_s$.

We say that a morphism $f: x \to y$ in $\text{Kos}_A^{f_s}$ is a total quasi-isomorphism if $\text{Tot } f: \text{Tot } x \to \text{Tot } y$ is a quasi-isomorphism. We denote the class of all total quasi-isomorphisms in $\text{Kos}_A^{f_s}$ and $\text{Kos}_A^{\text{typ}_s}$ respectively by the same letters $tq$. Then the pairs $(\text{Kos}_A^{\text{typ}_s}, tq)$ and $(\text{Kos}_A^{\text{typ}_s}, tq)$ are relative exact categories such that the inclusion functors $\text{Kos}_A^{\text{typ}_s} \hookrightarrow \text{Kos}_A^{\text{typ}_s} \hookrightarrow \text{Cub}^S(\text{P}_A)$ are exact and reflect exactness where the last category is the exact category of $S$-cubes in the exact category $\text{P}_A$ of finitely generated projective $A$-modules. We also denote the class of all isomorphisms in $\text{Kos}_A^{f_s}$ and $\text{Kos}_A^{\text{typ}_s}$ respectively by the same letter $i$. The absolute geometric presentation theorem is the following.

2.3. Theorem (Absolute geometric presentation theorem). Let $A$ be a commutative regular ring and $f_s = \{f_s\}_{s \in S}$ a regular sequence of $A$. Assume that the following two conditions:

1. $f_s$ is contained in the Jacobson radical of $A$.
2. For a subset $T$ of $S$, every finitely generated projective $A/ f_T A$-modules are free.

Then

1. The functor $\text{Tot}: (\text{Kos}_A^{f_s} , tq) \to (\text{Perf}_{\text{Spec } A}^{\text{qis}})$ induces an isomorphism of nilpotent invariant motives

$$M^\text{nil}(\text{Kos}_A^{f_s} , tq) \cong M^\text{nil}(\text{Spec } A).$$
The function \( \text{Tot} : (\text{Kos}^f_{A, \text{typ}}, i) \to (\text{Perf}^\mathfrak{q}_{\text{Spec } A}, q) \) induces a split epimorphism of nilpotent invariant motives

\[
M_{\text{nilp}}(\text{Kos}^f_{A, \text{typ}}) \twoheadrightarrow M^\mathfrak{q}_{\text{Spec } A}.
\]

If \( A \) is local, then assumptions in the statement automatically hold. If we replace \( (\text{Kos}^f_{A, \text{typ}}, w) \) with \( (\text{Kos}^f_{A, i}, w) \) for \( w = \mathfrak{f}q \) or \( w = i \), then the statement above is essentially proven in [27, Theorem 0.5, Corollary 9.6.].

We replace \( (\text{Kos}^f_{A, \text{typ}}, w) \) with \( (\text{Kos}^f_{A, i}, w) \) for \( w = \mathfrak{f}q \) or \( w = i \). To give a detailed proof, we need intermediate exact categories between \( \text{Kos}^f_{A, \text{typ}} \) and \( \text{Kos}^f_{A, i} \). We recall some notations from [25] and [28]. Fix an \( S \)-cube \( x \) in an abelian category \( A \). We say that \( x \) is monic if for any pair of subsets \( U \subseteq T \) in \( S \), \( x(U \subseteq V) \) is a monomorphism. For any element \( k \) in \( S \), we define \( H_k^0(x) \) to be an \( S \)-cube in \( A \) by setting \( H_k^0(x)_T := \text{Coker} d_{T \setminus \{k\}}^k \) for any \( T \in \mathcal{P}(S) \). We call \( H_k^0(x) \) the \( k \)-direction 0-th homology of \( x \).

When \( \#S = 1 \), we say that \( x \) is admissible if \( x \) is monic, namely if its unique boundary morphism is a monomorphism. For \( \#S > 1 \), we define the notion of an admissible cube inductively by saying that \( x \) is admissible if \( x \) is monic and if for every \( k \) in \( S \), \( H_k^0(x) \) is admissible. For an admissible \( S \)-cube \( x \) and a subset \( T = \{i_1, \ldots, i_k\} \subseteq S \), we set \( H_k^0(x) := H_{i_1}^0(H_{i_2}^0(\cdots (H_{i_k}^0(x)) \cdots)) \) and \( H_k^0(x) = x \). Notice that \( H_k^0(x) \) is an \( S \)-cube for any \( T \in \mathcal{P}(S) \) and we can show that the definition of \( H_k^0(x) \) does not depend upon an order \( i_1, \ldots, i_k \) up to the canonical isomorphism. Let \( \mathfrak{g} = \{F_T\}_{T \in \mathcal{P}(S)} \) be a family of full subcategories of \( A \) which are closed under isomorphisms. Namely for any subset \( T \) of \( S \) and for any object \( x \) in \( A \) such that it is isomorphic to an object in \( F_T \), \( x \) is also an object in \( F_T \). We define \( \mathfrak{g} \) is the full subcategory of \( \text{Cub}^S(A) \) consisting of those \( S \)-cubes \( x \) such that \( x \) is admissible and each vertex of \( H_k^0(x) \) is in \( F_T \) for any \( T \in \mathcal{P}(S) \).

Let \( R \) be a commutative noetherian ring with 1. We denote the category of finitely generated \( R \)-modules by \( \mathcal{M}_R \). Let the letter \( r \) be a natural number or \( \infty \) and \( I \) be an ideal of \( R \). Let \( \mathcal{M}_R^r(I) \) be the category of finitely generated \( R \)-modules \( M \) such that \( \text{Projdim}_R M \leq r \) and \( \text{Supp} M \subseteq V(I) \). We write \( \mathcal{M}_R^I \) for \( \mathcal{M}_R^{\infty} \). Since the category is closed under extensions in \( \mathcal{M}_R \), it can be considered to be an exact category in the natural way. Notice that if \( I \) is the zero ideal of \( R \), then \( \mathcal{M}_R^I(0) \) is just the category of finitely generated projective \( R \)-modules \( \mathcal{P}_R \). We call an \( R \)-module \( M \) in \( \mathcal{M}_R^I(r) \) is reduced (with respect to an ideal \( I \)) if \( IM = 0 \) and we write \( \mathcal{M}_R^I_{\text{red}}(r) \) for the full subcategory of reduced \( R \)-modules in \( \mathcal{M}_R^I(r) \). For \( R \)-sequence \( f_S = \{f_s\}_{s \in S} \) and a subset \( T \) of \( S \), we set \( f_T = \{f_t\}_{t \in T} \) and we denote the ideal in \( R \) spanned by \( f_T \) by \( f_T \). Then we have the following formula

\[
\text{Kos}^f_R = \bigtimes_{T \in \mathcal{P}(S)} \mathcal{M}_R^I_{\text{red}}(\#T).
\]

(See [26, 4.20].) Here by convention, we set \( f_0 R = (0) \) the zero ideal of \( R \) and \( \text{Kos}^f_R = \mathcal{P}_R \) the category of finitely generated projective \( R \)-modules.
Imitating the equality above, we define the following categories:

\[ \text{Kos}_{R, \text{red}}^S := \bigotimes_{T \in P(S)} M^{f_R}_{R, \text{red}} (#T), \]

\[ \text{Kos}_{R, \text{simp}}^S := \bigotimes_{T \in P(S)} P_{R/ f_R} R. \]

We call an \( S \)-cube in \( \text{Kos}_{R, \text{red}}^S \) (resp. \( \text{Kos}_{R, \text{simp}}^S \)) reduced (resp. simple) \( S \)-Koszul cubes associated with \( f_S \). We denote the class of all total quasi-isomorphisms and the class of all isomorphisms in \( \text{Kos}_{R, \text{red}}^S \) and \( \text{Kos}_{R, \text{simp}}^S \) by the same letters \( \text{tq} \) and \( \text{i} \) respectively.

**Proof of absolute geometric presentation theorem [27]** Under assumptions 1 and 2 in the statement, the inclusion functor \( \text{Kos}_{A, \text{typ}}^S \hookrightarrow \text{Kos}_{A, \text{simp}}^S \) is an equivalence of categories. (See [28, Proposition 1.2.15.].) Next we will show that the inclusion functor \( (\text{Kos}_{A, \text{simp}}^S, w) \to (\text{Kos}_{A, \text{red}}^S, w) \) induces an isomorphism of motives \( M_{\#} (\text{Kos}_{A, \text{simp}}^S, w) \to M_{\#} (\text{Kos}_{A, \text{red}}^S, w) \) for \( w = \text{tq} \) and \( w = \text{i} \) and \( \# \in \{\text{loc}, \text{nilp}\} \). For the case of \( w = \text{i} \), it follows from [28, Corollary 2.1.4.]. (Although the statement in the reference is written in terms of non-connective \( K \)-theory, the same proof works fine.) For the case of \( w = \text{tq} \), we consider the following commutative square of motives in \( \text{Mot}^S_{dg} \) for \( \# \in \{\text{loc}, \text{nilp}\} \)

\[
\begin{array}{ccc}
M_{\#}(\text{Kos}_{A, \text{simp}}^S, \text{tq}) & \longrightarrow & M_{\#}(\text{Kos}_{A, \text{red}}^S, \text{tq}) \\
\downarrow M_{\#}(W^S) & & \downarrow M_{\#}(W^S) \\
M_{\#}(P_{R/ f_R} R) & \longrightarrow & M_{\#}(M_{R, \text{red}}^S (#S))
\end{array}
\]

where the horizontal morphisms are induced from inclusion functors

\[ \text{Kos}_{A, \text{simp}}^S \hookrightarrow \text{Kos}_{A, \text{red}}^S \]

and \( P_{R/ f_R} R \hookrightarrow M_{R, \text{red}}^S (#S) \). The bottom horizontal morphism is isomorphism by [29, Lemma 1.2.5.]. To show that the vertical morphisms are isomorphisms by utilizing [24, Theorem 8.19], we need to check assumptions in Theorem 8.19 in [27]. They are follow from [26, Corollary 5.13.] for the right vertical morphism and [28, Lemma 2.1.3.] for the left vertical morphism. Finally we will prove that the inclusion functor \( (\text{Kos}_{A, \text{red}}^S, w) \to (\text{Kos}_{A}^S, w) \) induces an isomorphism \( M_{\text{nilp}}(\text{Kos}_{A, \text{red}}^S, w) \to M_{\text{nilp}}(\text{Kos}_{A}^S, w) \) of nilpotent invariant motives for \( w = \text{tq} \) or \( w = \text{i} \). The corresponding statement for \( K \)-theory is Corollary 6.3. in [26]. The proof is based upon derived equivalence and dévissage theorem for noetherian abelian categories. Thus the same proofs work fine for nilpotent invariant motives. We complete the proof. □

3 What makes motives \( \mathbb{A}^1 \)-homotopy invariance?

In this section, we consider what makes motives \( \mathbb{A}^1 \)-homotopy invariance. As in Introduction of [29], the main key idea of to prove homotopy prop-
erty of nilpotent invariant dash motives is, roughly speaking, that we recognize an affine space as to be a rudimental projective space

\[ \mathbb{A}^n = \mathbb{P}^n \setminus \{ [x_0 : \cdots : x_n] \in \mathbb{P}^n ; x_n = 0 \} = \mathbb{P}^n \setminus \mathbb{P}^{n-1}. \]  

(4)

This means that for a noetherian scheme \( X \), in the commutative diagram in \( \text{Ho}(\text{Mot}_{\text{nilp}}^\text{dg}) \) below

\[
\begin{array}{cccccc}
M'_{\text{nilp}}(\mathbb{P}_X^{n-1}) & \longrightarrow & M'_{\text{nilp}}(\mathbb{P}_X^n) & \longrightarrow & M'_{\text{nilp}}(X) & \longrightarrow & \Sigma M'_{\text{nilp}}(\mathbb{P}_X^{n-1}) \\
I & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \Sigma I \\
M'_{\text{nilp}}(\mathbb{P}_X^{n-1}) & \longrightarrow & M'_{\text{nilp}}(\mathbb{P}_X^n) & \longrightarrow & M'_{\text{nilp}}(\mathbb{A}_X^n) & \longrightarrow & \Sigma M'_{\text{nilp}}(\mathbb{P}_X^n),
\end{array}
\]

(5)

the bottom line is a localization distinguished triangle. Moreover by nilpotent invariance of dash motives, the vertical morphisms \( I \) and \( \Sigma I \) are isomorphisms. We will show that the top line above is also a distinguished triangle by a consequence of calculation of nilpotent invariant dash motives of projective spaces \[3.14\]. Thus by five lemma, we obtain homotopy invariance of dash motives. Therefore we would like to say that \( \mathbb{A}^1 \)-homotopy invariance is a degenerated version of projective bundle formula and nilpotent invariance makes dash motives of noetherian schemes \( \mathbb{A}^1 \)-homotopy invariance. To justify the argument above, the main task of this section is to calculate nilpotent invariant dash motives of projective spaces. In this calculation, nilpotent invariance of dash motives is again crucial and to make the points clarify, we establish an algebraic geometry over abelian categories which contains an algebraic geometry over \( \mathbb{F}_1 \) in some sense. (See Conventions \[3.2\].)

We will define for locally noetherian abelian category \( \mathcal{A} \) (For definition of locally noetherian abelian category, see after the next paragraph.) and a positive integer \( n \), we define the abelian category \( \mathbb{P}_\mathcal{A}^n \) which we call the \( n \)th projective space over \( \mathcal{A} \). The naming is justified by the fact that if \( \mathcal{A} \) is the category of quasi-coherent sheaves on a noetherian scheme \( X \), then \( \mathbb{P}_\mathcal{A}^n \) is equivalent to the category of quasi-coherent sheaves on \( \mathbb{P}_X^n \). (See Examples \[3.12\].)

To start by defining dash motives associated with locally noetherian abelian categories, first we recall the conventions and fundamental facts of abelian categories from \[9, 32, 39, 10 \] and \[13\]. Let \( \mathcal{A} \) be an abelian category and \( x \) be an object of \( \mathcal{A} \). For a family \( \{ x_i \rightarrow x \}_{i \in I} \) of subobjects of \( x \) indexed by a non-empty set \( I \), we denote the minimum object which contains all \( x_i \) in \( \text{P}(x) \) the partially ordered set of isomorphism classes of subobjects of \( x \) by \( \sum_{i \in I} x_i \). We say that \( x \) is \( \text{finitely generated} \) if for any family of subobjects \( \{ x_i \}_{i \in I} \) of \( x \) such that \( x \xrightarrow{\sim} \sum_{i \in I} x_i \), there exists a finite subset \( J \subset I \) such that \( x \xrightarrow{\sim} \sum_{i \in J} x_i \). We say that \( x \) is \( \text{finitely presented} \) if \( x \) is finitely generated and for any epimorphism \( a : y \twoheadrightarrow x \) with \( y \) finitely generated, \( \ker a \) is also finitely generated. We say that \( x \) is \( \text{coherent} \) if \( x \) is finitely presented and if every finitely generated subobject of \( x \) is finitely presented. We denote the full subcategory of \( \mathcal{A} \) spanned by coherent objects in \( \mathcal{A} \) by \( \text{Coh} \mathcal{A} \).
We say that an abelian category \(A\) is Grothendieck if \(A\) has a generator and has all small colimits and all small direct limits in \(A\) is exact. The first condition means that there exists an object \(u\) in a category \(A\) such that the corepresentable functor \(\text{Hom}(u, -)\) from \(A\) to the category of sets associated with \(u\) is faithful. We call such an object \(u\) a generator of \(A\).

The last condition means that for any filtered small category \(\mathcal{I}\), the colimit functor \(\text{colim}_\mathcal{I}: \text{Hom}(\mathcal{I}, A) \to A\) from the category of \(\mathcal{I}\)-diagrams in \(A\) to \(A\) is exact. We say that an abelian category \(A\) is locally noetherian if \(A\) is Grothendieck and if \(A\) has a family of generators consisting of noetherian objects. The last condition means that there exists a family of objects \(\{u_i\}_{i \in I}\) in \(A\) such that for each non-zero morphism \(a: x \to y\) in \(A\), there exists a morphism \(b: u_i \to x\) for some \(i \in I\) such that \(ab \neq 0\). We call such a family a family of generators of \(A\). If \(A\) is locally noetherian, then every finitely generated object in \(A\) is noetherian. (cf. [39, Chapter V §4].) In particular, every finitely presented (resp. coherent) object in \(A\) is also noetherian. Moreover \(\text{Coh}_A\) is a noetherian category.

For an essentially small abelian category \(A\), we denote the category of left exact functors from \(A^\text{op}\) to the category of abelian groups by \(\text{Lex}_A\). The category \(\text{Lex}_A\) is a Grothendieck category and there exists the Yoneda embedding functor \(y = y_A: A \to \text{Lex}_A\) which is exact and reflects exactness. (See [39, A.7.14, A.7.16].) If \(A\) is noetherian, then \(\text{Lex}_A\) is locally noetherian and the Yoneda functor induces an equivalence of categories \(A \to \text{Coh}_\text{Lex}_A\). If \(A\) is locally noetherian, then the inclusion functor \(\text{Coh}_A \hookrightarrow A\), induces an equivalence of categories \(\text{Lex}_\text{Coh}_A \to A\). (See [32, 5.8.8, 5.8.9].)

For example let \(X\) be a scheme. Then \(\text{Qcoh}_X\), the category of quasi-coherent \(\mathcal{O}_X\)-modules is Grothendieck category. (See for [2 3.14].) Moreover if we assume that \(X\) is noetherian, then \(\text{Qcoh}_X\) is a locally noetherian abelian category and the category \(\text{Coh}_\text{Qcoh}_X\) is just the category of coherent \(\mathcal{O}_X\)-modules \(\text{Coh}_X\) and \(\text{Lex}_\text{Coh}_X\) is equivalent to \(\text{Qcoh}_X\).

3.1. Definition (Dash motives of locally noetherian abelian categories). Let \(A\) be a locally noetherian abelian category. We define the additive (resp. localizing, nilpotent invariant) dash motive of \(A\) by setting 
\[ M_\#(A) := M_\#(\text{Coh}_A) \text{ for } \# \in \{\text{add, loc, nilp}\}. \]

3.2. Conventions (\(\mathbb{F}_1\)-algebra). In this article, the letters \(\mathbb{F}_1\) is just a symbol. By the term (graded) \(\mathbb{F}_1\)-algebra, we mean a graded commutative monoid.

For example, \(\mathbb{F}_1[t_1, \ldots, t_n]\) the \(n\)-variable polynomial ring over \(\mathbb{F}_1\) is just a commutative monoid \(\mathbb{N}^n\) with usual componentwise addition. If we regard it as an \(\mathbb{N}\)-grading algebra, then degree \(s\) part of \(\mathbb{F}_1[t_1, \ldots, t_n]\) is given by just the set
\[ \mathbb{F}_1[t_1, \ldots, t_n]_s := \left\{ (m_1, \ldots, m_n) \in \mathbb{N}^n; \sum_{i=1}^n m_i = s \right\}. \]

In this case, we denote an element \((i_1, \ldots, i_n)\) in \(\mathbb{N}^n\) by \(t_1^{i_1} \cdots t_n^{i_n}\).

For a commutative ring \(B\) with 1, by a \(B\)-category or a \(B\)-functor, we mean a category and a functor enriched by the category of \(B\)-modules respectively. Similarly by an \(\mathbb{F}_1\)-category, an \(\mathbb{F}_1\)-functor and \(\mathbb{F}_1\)-module, we mean a usual (locally small) category, a functor and a set respectively.
In the rest of this section, let $B$ be a commutative ring with 1 or the letters $\mathbb{F}_1$.

Our approach to define projective spaces over a locally noetherian abelian category is based upon the classical result of Serre and its globalization by Grothendieck and its non-commutative version by Artin-Zhang and the reconstruction result by Garkusha-Prest. To recall Serre’s theorem, we introduce the notion of stabilization of categories by an endofunctor. For a category $\mathcal{C}$ and a functor $T: \mathcal{C} \to \mathcal{C}$, we define $\text{Stab}_T \mathcal{C}$ to be a category by setting $\text{Ob Stab}_T \mathcal{C} := \text{Ob} \mathcal{C}$ and $\text{Hom}_{\text{Stab}_T \mathcal{C}}(x, y) := \lim_{m \to \infty} \text{Hom}_\mathcal{C}(T^m x, T^m y)$ for any pair of objects $x$ and $y$. We call $\text{Stab}_T \mathcal{C}$ the stabilization of $\mathcal{C}$ by the functor $T$.

If $\mathcal{C}$ is an abelian category and $T$ is an exact functor, then there exists another description of $\text{Stab}_T \mathcal{C}$. Namely let $\text{Nil}_T \mathcal{C}$ be a full subcategory of $\mathcal{C}$ consisting of those objects $x$ such that $T^m(x) = 0$ for sufficiently large positive integer $m$. Then $\text{Nil}_T \mathcal{C}$ is a Serre subcategory of $\mathcal{C}$ and for a certain case, there exists a canonical equivalence of categories $\text{Stab}_T \mathcal{C} \cong \mathcal{C} / \text{Nil}_T \mathcal{C}$.

For a $\mathbb{N}$-grading commutative ring $A$ with 1, we write $\text{GrCoh}_A$ for the category of finitely generated graded $A$-modules. For an integer $k$, there is Serre’s twist functor $(k): \text{GrCoh}_A \to \text{GrCoh}_A$. (For definition of Serre’s twist functor, see Definition 3.5. Here the definition of abstract Serre’s twist functor will be given.) Here is Serre’s theorem.

3.3. Theorem (Serre). (cf. [1, p.229 Theorem].) Let $k$ be a field and let $A$ be an $\mathbb{N}$-grading commutative ring of finite type over $k$ such that it generated by degree one elements. Then a functor

$$\text{Coh}_{\text{Proj}} A \to \text{Stab}_{(1)} \text{GrCoh}_k$$

which sends a coherent sheaf $\mathcal{F}$ on $\text{Proj} A$ to $\bigoplus_{n \geq 0} \Gamma(\text{Proj} A, \mathcal{F}(n))$ gives an equivalence of categories.

To imitate Serre’s theorem for more abstract situation, first we will define the notion of abstract graded categories associated with $\mathbb{N}$-grading $B$-algebras and (abelian) $B$-categories.

3.4. Notation (Categorified graded ring). Let $A$ be a $\mathbb{N}$-grading $B$-algebra. We regard $A$ as a category $A_{gr}$ in the following way. The class of objects is just the set of all non-negative integers $\mathbb{N}$. For any pair of non-negative integers $n$ and $m$, we set

$$\text{Hom}_{A_{gr}}(n, m) := \begin{cases} A_{m-n} & \text{if } m \geq n \\ \emptyset & \text{if } m < n. \end{cases}$$

The composition of morphisms is given by the multiplication of $A$.

Similarly, we define $\mathbb{F}_1_{gr}$ to be a category by setting $\text{Ob} \mathbb{F}_1_{gr} := \mathbb{N}$ the set of all non-negative integers and for any pair of non-negative integers $n$ and $m$,

$$\text{Hom}_{\mathbb{F}_1_{gr}}(n, m) := \begin{cases} \{\text{id}\} & \text{if } m = n \\ \emptyset & \text{if } m \neq n. \end{cases}$$
3.5. **Definition (Graded categories).** Let $A$ be an $\mathbb{N}$-grading commutative $B$-algebra and $C$ be a $B$-category. Then we define $\text{Gr}_A C$ to be a category by setting

$$\text{Gr}_A C := \text{Hom}_B(A_{\text{gr}}, C)$$

the category of $B$-functors from $A_{\text{gr}}$ to $C$ and natural transformations. Thus we regard an object $x$ in $\text{Gr}_A C$ as a system of a family of objects \(\{x_k\}_{k \geq 0}\) in $C$ indexed by non-negative integers and for each homogeneous element $f \in A_s$, a family of morphisms \(\{x_k \to x_{k+s}\}_{k \geq 0}\) in $C$ indexed by non-negative integers which we write the same letter $f$ such that this families of morphisms satisfy the equalities arose from the relations among elements in $A$.

We also define $\text{Gr}_{F_1} C$ to be a category by setting $\text{Gr}_{F_1} C := \text{Hom}_{F_1}(F_{1\text{gr}}, C)$. Namely $\text{Gr}_{F_1} C$ is just a countable products of copies of the category $C$.

Next we assume that $C$ admits a zero object. We fix the specific zero object $0$. Then for an integer $k$, we define Serre’s twist functor $(k): \text{Gr}_A C \to \text{Gr}_A C$ which sends an object $x$ in $\text{Gr}_A C$ to $x(k)$ in $\text{Gr}_A C$ which defined as follows.

$$x(k)_n := \begin{cases} x_{m+k} & \text{if } m \geq -k \\ 0 & \text{if } m < -k. \end{cases}$$

For a homogeneous element $f \in A_s$ and an object $x$ in $\text{Gr}_A C$, we define $x \to x(s)$ to be a morphism in $\text{Gr}_A C$ which we denote by the same letter $f: x \to x(s)$ by setting $f: x_m \to x(s)_m = x_{m+s}$ for each non-negative integer $m$.

For a finite non-empty family $f = \{f_k\}_{1 \leq k \leq r}$ of homogeneous elements in $A$ indexed by integers $1 \leq k \leq r$, we denote the full subcategory of $\text{Gr}_A C$ consisting of those objects $x$ such that there exists a sufficiently large integer $N$ such that the morphism $f_k^N: x \to x(\deg f_k^N)$ is the zero morphism for each $1 \leq k \leq r$ by $\text{Gr}_A^{V(0)} C$. Finally let $\text{Gr}_A^{V(0)} C$ be a full subcategory of $\text{Gr}_A^{V(0)} C$ consisting of those objects $x$ such that $f_k: x \to x(\deg f_k)$ is the zero morphism for $1 \leq k \leq r$.

3.6. **Examples (Graded category).**

1. An object $\text{Ob} \text{Gr}_F[t_1, \ldots, t_n] C \ni x: F[t_1, \ldots, t_n] \to C$ can be represented by the diagram

$$\begin{array}{ccc}
\cdot & \xrightarrow{i_3} & \cdot \\
\cdot & \xrightarrow{i_2} & \cdot \\
\cdot & \xrightarrow{i_3} & \cdot \\
\cdot & \xrightarrow{i_2} & \cdot \\
\cdot & \xrightarrow{i_3} & \cdot \\
\cdot & \xrightarrow{i_2} & \cdot \\
\end{array}$$

2. Let $A$ be $\mathbb{N}$-grading commutative $B$-algebra and $C$ be a commutative $B$-algebra and let $\text{Mod}_C$ be the category of $C$-modules. Then the category $\text{Gr}_A \text{Mod}_C$ is equivalent to the category $\text{GrMod}_{A \otimes_B C}$ the category of graded $A \otimes_B C$.

3. Let $(X, \mathcal{O}_X)$ be a ringed space and $n$ a positive integer and let $\text{Mod}_X$ be the category of $\mathcal{O}_X$-modules. Then the category $\text{Gr}_F[t_1, \ldots, t_n] \text{Mod}_X$ is equivalent to the category of graded $\mathcal{O}_X[t_1, \ldots, t_n]$-modules $\text{GrMod}_{\mathcal{O}_X[t_1, \ldots, t_n]}$.  

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3.7. Proposition (Fundamental properties of $\text{Gr}_A \mathcal{A}$). Let $\mathcal{A}$ be a category and $A$ an $\mathbb{N}$-grading commutative $B$-algebra. Then

1. We can calculate a (co)limit in $\text{Gr}_A \mathcal{A}$ by term-wise (co)limit in $\mathcal{A}$. In particular, if $\mathcal{A}$ is additive (resp. abelian, closed under small filtered colimits) then $\text{Gr}_A \mathcal{A}$ is also additive (resp. abelian, closed under small filtered colimits).

Moreover assume that $\mathcal{A}$ is abelian and let $\mathfrak{f} = \{ f_k \}_{1 \leq k \leq r}$ be a family of homogeneous elements of $A$. We write $\mathfrak{f} \mathcal{A}$ for the ideal of $A$ spanned by $\mathfrak{f}$. Then

2. $\text{Gr}^{V(\mathfrak{f})}_A \mathcal{A}$ is a Serre subcategory of $\text{Gr}_A \mathcal{A}$.

3. $\text{Gr}^{V(\mathfrak{f})}_{A,\text{red}} \mathcal{A}$ is a topologizing subcategory of $\text{Gr}^{V(\mathfrak{f})}_A \mathcal{A}$ and the inclusion functor $\text{Gr}^{V(\mathfrak{f})}_{A,\text{red}} \mathcal{A} \hookrightarrow \text{Gr}^{V(\mathfrak{f})}_A \mathcal{A}$ is a nilpotent immersion.

4. The canonical map $A \to A/\mathfrak{f} A$ induces an equivalence of categories $\text{Gr}_{A/\mathfrak{f} A} \mathcal{A} \sim \text{Gr}^{V(\mathfrak{f})}_{A,\text{red}} \mathcal{A}$.

5. The fully faithful embedding $\text{Gr}_{A/\mathfrak{f} A} \mathcal{A} \hookrightarrow \text{Gr}_A \mathcal{A}$ induced from the canonical map $A \to A/\mathfrak{f} A$ admits a left adjoint functor

$$- \otimes_A A/\mathfrak{f} A : \text{Gr}_A \mathcal{A} \to \text{Gr}_{A/\mathfrak{f} A} \mathcal{A}.$$

6. The canonical functor $\text{Gr}_A \mathcal{A} \to \text{Stab}_{(1)} \text{Gr}_A \mathcal{A}$ induces an isomorphism of categories

$$\text{Gr}_A \mathcal{A}/\text{Nil}_{(1)} \text{Gr}_A \mathcal{A} \sim \text{Stab}_{(1)} \text{Gr}_A \mathcal{A}.$$

In particular, $\text{Stab}_{(1)} \text{Gr}_A \mathcal{A}$ is also an abelian category.

Moreover assume that $A = B[t_1, \cdots, t_n]/(g_1, \cdots, g_t)$ where $B[t_1, \cdots, t_n]$ is the nth polynomial ring over $B$ and $g_i$ is a homogeneous polynomial in $B[t_1, \cdots, t_n]$ for $1 \leq i \leq t$. Then

7. The forgetful functor $U_A : \text{Gr}_A \mathcal{A} \to \mathcal{A}$ which sends an object $x$ in $\text{Gr}_A \mathcal{A}$ to $x_0$ admits a left adjoint functor $- \otimes_B A : \mathcal{A} \to \text{Gr}_A \mathcal{A}$.

8. If $x$ is a noetherian object, then $x \otimes_B A$ is also a noetherian object in $\text{Gr}_A \mathcal{A}$.

9. For any object $x$ in $\text{Gr}_A \mathcal{A}$ and for any non-negative integer $k \geq 0$, there exists a canonical morphism $x_k \otimes_B A(-k) \to x$.

10. If $\mathcal{A}$ admits a family of generators $\{ u_A \}_{A \in \Lambda}$ indexed by non-empty set $\Lambda$, then $\text{Gr}_A \mathcal{A}$ has a family of generators $\{ u_A \otimes_B A(-m) \}_{A \in \Lambda, m \geq 0}$. In particular if $A$ is Grothendieck (resp. locally noetherian), then $\text{Gr}_A \mathcal{A}$ is also.

Moreover assume that $A$ is a locally noetherian abelian category.

11. The inclusion functor $\text{Gr}_A \text{Coh} \mathcal{A} \to \text{Gr}_A \mathcal{A}$ induces an isomorphism of categories $\text{Coh} \text{Gr}_A \mathcal{A} \sim \text{Coh} \text{Gr}_A \mathcal{A}$. 
(12) Let $x$ be an object in $\text{Gr}_A \text{Coh}_A$. Then $x$ is a noetherian object in $\text{Gr}_A \text{Coh}_A$ if and only if there exists a positive integer $m > 0$ such that a canonical morphism $\bigoplus_{k=1}^m x_k \otimes_B A(-k) \to x$ induced from the morphisms in (9) is an epimorphism.

Proof. Assertions (1), (2), (3) and (4) are straightforward.

We define $f$ to be a subobject of $x$ by setting

$$f := \sum_{k=1}^r \text{Im}(f_k : x(-\deg f_k) \to x)$$

and we set $x \otimes_A A/ f := x/f$. Then we can show that the association which sends an object $x$ in $\text{Gr}_A \text{A}$ to an object $x \otimes_A A/ f$ gives a left adjoint functor of the fully faithful embedding $\text{Gr}_A/ f \to \text{Gr}_A A$. 

(6) We write $B$ and $C$ for $\text{Gr}_A A/ \text{Nil}_1 \text{Gr}_A A$ and $\text{Nil}_1 \text{Gr}_A A$ respectively. For any pair of objects $x$ and $y$ in $B$, we define $I_{(x,y)}$ to be a partially ordered direct set by

$$I_{(x,y)} := \{(x', y') \in \mathcal{P}(x) \times \mathcal{P}(y); x/x', y' \in \text{Ob} C\}$$

where $\mathcal{P}(z)$ is the partially ordered direct set of isomorphism classes of subobjects of $z$ and the ordering $(x', y') \leq (x'', y'')$ holds if and only if $x'' \subset x'$ and $y'' \subset y'$. Recall that the Hom set $\text{Hom}_{B}(x,y)$ is given by the formula

$$\text{Hom}_{B}(x,y) := \text{colim}_{(x', y') \in I_{(x,y)}} \text{Hom}_{\text{Gr}_A A}(x', y/y').$$

Let $(x', y')$ be an element in $I_{(x,y)}$. Then since $x/x'$ and $y'$ are (1)-nilpotent, there exists a non-negative integer $m$ such that $x/x'(m) \xrightarrow{\sim} 0$ and $y'(m) \xrightarrow{\sim} 0$. Then we have $(x', y') \leq ((x(m))(-m), y')$ and the equalities

$$\text{Hom}_{\text{Gr}_A A}((x(m))(-m), y/y') = \text{Hom}_{\text{Gr}_A A}((x(m))(-m), (y(m))(-m)) = \text{Hom}_{\text{Gr}_A A}(x(m), y(m)).$$

Therefore the canonical map $\text{Hom}_{B}(x,y) \to \text{Hom}_{\text{Stab}_{\text{Nil}_1} \text{Gr}_A A}(x,y)$ is an isomorphism.

(7) Since $U_A : \text{Gr}_A A \to A$ factors through $\text{Gr}_A A \hookrightarrow \text{Gr}_B [t_1, \cdots, t_n] A \xrightarrow{U_B [t_1, \cdots, t_n]} A$, we shall only prove the case where $A = B[t_1, \cdots, t_n]$ by (5). In this case, for simplicity, for an object $x$ in $A$, we denote $x \otimes_B B[t_1, \cdots, t_n]$ by $x[t_1, \cdots, t_n]$ which defined as follows. For each non-negative integer $s$, degree $s$ part of $x[t_1, \cdots, t_n]$ is given by the formula

$$x[t_1, \cdots, t_n] := \bigoplus_{\sum_{i=1}^n m_i = s} x t_1^{m_1} \cdots t_n^{m_n}$$
where $x t_1^{m_1} \cdots t_n^{m_n}$ is just a copy of $x$ and for each $1 \leq i \leq n$, $x [t_1, \cdots, t_n] \rightarrow x [t_1, \cdots, t_n]_{i+1}$ the multiplication by $t_i$ is induced from the identity morphisms $id_x: x t_1^{m_1} \cdots t_i^{m_i} \cdots t_n^{m_n} \rightarrow x t_1^{m_1} \cdots t_i^{m_i+1} \cdots t_n^{m_n}$.

Let $k$ be a non-negative integer and $x$ an object in $Gr_B[t_1, \cdots, t_n] A$. we define $x_k [t_1, \cdots, t_n] (-k) \rightarrow x$ to be a morphism by setting for any $m \geq k$ and any $i = (i_1, \cdots, i_n) \in \mathbb{N}^n$ such that $\sum_{j=1}^n t_j = m - k$,

$$t_1^{i_1} \cdots t_n^{i_n} : x_k t_1^{i_1} \cdots t_n^{i_n} \rightarrow x_m$$

on the $x_k t_1^{i_1} \cdots t_n^{i_n}$ component of $x_k [t_1, \cdots, t_n] (-k)_m$.

For any object $x$ in $A$ and any object $y$ in $Gr_B[t_1, \cdots, t_n] A$, we have a functorial isomorphism $\text{Hom}_A(x, y) \cong \text{Hom}_{Gr_B[t_1, \cdots, t_n] A}(x [t_1, \cdots, t_n], y)$ which sends $f$ to $(x[t_1, \cdots, t_n] f \otimes_B [t_1, \cdots, t_n] y_0[t_1, \cdots, t_n] \rightarrow y)$.

(8) If $A = B[t_1, \cdots, t_n]$, assertion is proven in [39 4.14 (1)]. (In the reference, the proof is written for $B = F_1$ with slightly different conventions. But a similar proof works fine for a general $B$.) In a general case, $x \otimes_B A$ is a quotient of a noetherian object $x[t_1, \cdots, t_n]$ by $g x [t_1, \cdots, t_n]$ where $g = \{g_i \}_{1 \leq i \leq \leq 2}$. Thus $x \otimes_B A$ is also a noetherian object.

(9) We regard $x$ as an object of $Gr_B[t_1, \cdots, t_n] A$. Then as in the proof of (7), there exists a canonical morphism $x_k [t_1, \cdots, t_n] (-k) \rightarrow x$. Since $x$ is in $Gr_A A$, it factors through $x_k [t_1, \cdots, t_n] (-k) \rightarrow x_k \otimes_B A(-k) \rightarrow x$ and the last morphism is the desired morphism.

(10) For any non-zero morphism $a: x \rightarrow y$ in $Gr_A A$, there exists a non-negative integer $k$ such that $a_k : x_k \rightarrow y_k$ is a non-zero morphism. Then there exists $\lambda \in \Lambda$ and $b : u_\lambda \rightarrow x_k$ such that $a_k b \neq 0$. Then the compositions $u_\lambda \otimes_B A(-k) \rightarrow x_k \otimes_B A(-k) \rightarrow x \rightarrow y$ is also a non-zero morphism. Therefore $\{u_\lambda \otimes_B A(-m)\}_{\lambda \in \Lambda, m \geq 0}$ is a family of generators of $Gr_A A$. The last assertion follows from (1) and (8).

(11) We need to prove that for any object $x$ in $\text{Coh} Gr_A A$, $x$ is in $\text{Coh} A$ for any $k \geq 0$. Obviously if $x$ is noetherian, then $x(k)$ is also noetherian. Therefore we shall assume that $k = 0$. Let $y_0 \rightarrow y_1 \rightarrow \cdots y_n \rightarrow \cdots$ be a sequence of subobjects in $x$, then we define subobject $\hat{y}_k$ of $x$ as follows:

$$(\hat{y}_k)_l := \begin{cases} y_k & \text{for } l = 0 \\ x_l & \text{for } l \geq 1. \end{cases}$$

Then since $x$ is noetherian, there exists a positive integer $m$ such that $\hat{y}_m = \hat{y}_{m+1} = \cdots$ and this means that $y_m = y_{m+1} = \cdots$. Hence $x_0$ is a noetherian object in $A$.

(12) Assume that $x$ is a noetherian object. For each integer $m \geq 0$, we set $y_i := \text{Im} \left( \bigoplus_{k=0}^m x_k \otimes_B A(-k) \rightarrow x \right)$. Then the ascending filtration $\{y_i\}_{i \geq 0}$ is stationar. Say $y_m = y_{m+1} = \cdots$. Since $y_\infty = x$, we obtain the equality $y_m = x$ and this means that $m$ is the desired integer.

Conversely if there exists an epimorphism $\bigoplus_{k=1}^{m} x_k \otimes_B A(-k) \rightarrow x$, then
By convention, we set $F$ filtration of $m$ exists a sufficiently large positive integer $x$ set deg $m$ is noetherian.

3.8. Theorem (Dash motive of $Gr_{\mathbb{F}_1[t_1, \ldots, t_n]} \mathcal{A}$). (cf. [30] 4.24.) Let $\mathcal{A}$ be a locally noetherian abelian category. Then for $\# \in \{\text{loc}, \text{nilp}\}$, we have the canonical isomorphism

$$M'_{\#} (Gr_{\mathbb{F}_1[t_1, \ldots, t_n]} \mathcal{A}) \rightarrow \bigoplus_{i \geq 0} M'_{\#} (\mathcal{A}) s^i$$

(6)

where $M'_{\#} (\mathcal{A}) s^i$ is a copy of $M'_{\#} (\mathcal{A})$ and we denote the identity morphism $\text{id}_{M'_{\#} (\mathcal{A})} : M'_{\#} (\mathcal{A}) s^i \rightarrow M'_{\#} (\mathcal{A}) s^{i+1}$ by $\times s^i$ (we write $\times s$ for $\times s^2$.) Then the isomorphism (6) makes the diagram below commutative.

$$
\begin{array}{ccc}
M'_{\#} (Gr_{\mathbb{F}_1[t_1, \ldots, t_n]} \mathcal{A}) & \sim & \bigoplus_{i \geq 0} M'_{\#} (\mathcal{A}) s^i \\
\downarrow (-1) & & \downarrow \times s \\
M'_{\#} (Gr_{\mathbb{F}_1[t_1, \ldots, t_n]} \mathcal{A}) & \sim & \bigoplus_{i \geq 0} M'_{\#} (\mathcal{A}) s^i.
\end{array}
$$

(7)

To give a proof of Theorem 3.8 we need to recall several concepts from [30] with slightly changing notations. Let $\mathcal{A}$ be an abelian category and let $x$ be an object in $Gr_{\mathbb{F}_1[t_1, \ldots, t_n]} \mathcal{A}$. For each non-negative integer $m \geq 0$, we define $F_m x$ to be a subobject of $x$ by setting

$$(F_m x)_k := \begin{cases} x_k & \text{if } k \leq m \\ \sum_{j \geq 1} \text{Im} t_1^{j_1} \cdots t_n^{j_n} & \text{if } k > m. \end{cases}$$

By convention, we set $F_{-1} x = 0$. We call a family $\{F_m x\}_{m \geq 0}$ a canonical filtration of $x$. Moreover if we assume that $x$ is noetherian, then there exists a sufficiently large positive integer $m$ such that $x = F_m x$ and we set deg $x := \min \{m \in \mathbb{N} : x = F_m x\}$ and call it degree of $x$.

We set $([n]) := \{k \in \mathbb{N} : 1 \leq k \leq n\}$. We define $\text{Kos}(x) : \mathcal{P}([n])^{op} \rightarrow Gr_{\mathbb{F}_1[t_1, \ldots, t_n]} \mathcal{A}$ to be an $(n)$-cube by sending a subset $T$ in $\mathcal{P}([n])$ to $x(-\# T)$, and an inclusion $T \setminus \{s\} \hookrightarrow T$ to $x(-\# T) \xrightarrow{t_s} x(-\# T + 1)$. We call $\text{Kos}(x)$ the Koszul cube associated with $x$. Moreover we define $T_i(x)$ to be the $i$th Koszul homology of $x$ by setting $T_i(x) := H_i(\text{Tot} \text{Kos}(x))$. We say that $x$ is $t$-regular if $T_i(x) = 0$ for any $i > 0$.

Proof of Theorem 3.8. By replacing $\mathcal{A}$ with $\text{Coh} \mathcal{A}$ and by Proposition 3.7 (3), we shall assume that $\mathcal{A}$ is noetherian. Let $\text{Coh} Gr_{\mathbb{F}_1[t_1, \ldots, t_n]} \mathcal{A}_{t \text{-reg}}$ be a full subcategory of $\text{Coh} Gr_{\mathbb{F}_1[t_1, \ldots, t_n]} \mathcal{A}$ consisting of $t$-regular objects. Then the inclusion functor $\text{Coh} Gr_{\mathbb{F}_1[t_1, \ldots, t_n]} \mathcal{A}_{t \text{-reg}} \hookrightarrow \text{Coh} Gr_{\mathbb{F}_1[t_1, \ldots, t_n]} \mathcal{A}$ induces an equivalence of triangulated categories

$$D^b \left( \text{Coh} Gr_{\mathbb{F}_1[t_1, \ldots, t_n]} \mathcal{A}_{t \text{-reg}} \right) \sim D^b \left( \text{Coh} Gr_{\mathbb{F}_1[t_1, \ldots, t_n]} \mathcal{A} \right)$$

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on bounded derived categories. (See [30] 4.24 (1).) Next for each non-negative integer \( m \), let \( \text{Coh} \mathcal{G}_F[t_1, \cdots, t_n] \mathcal{A}_{t-reg, deg \leq m} \) be the full subcategory of \( \text{Coh} \mathcal{G}_F[t_1, \cdots, t_n] \mathcal{A}_{t-reg} \) consisting of those objects of degree \( \leq m \). We define

\[
a: \mathcal{A}^{\times m+1} \to \text{Coh} \mathcal{G}_F[t_1, \cdots, t_n] \mathcal{A}_{t-reg, deg \leq m}
\]

and

\[
b: \text{Coh} \mathcal{G}_F[t_1, \cdots, t_n] \mathcal{A}_{t-reg, deg \leq m} \to \mathcal{A}^{\times m+1}
\]

to be the exact functors by sending an object \( (x_k)_{0 \leq k \leq m} \) to an object \( \bigoplus_{k=0}^m x_k[t_1, \cdots, t_n](-k) \) and an object \( x \) to an object \( (T_0(x))_{0 \leq k \leq m} \) respectively. We define \( F \mathcal{G}_F[t_1, \cdots, t_n] \mathcal{A}_{t-reg} \to \text{colim} \text{Coh} \mathcal{G}_F[t_1, \cdots, t_n] \mathcal{A}_{t-reg, deg \leq m} \), we have isomorphisms in \( \text{Ho} \text{Mot}_{dg}^\# \):}

\[
\begin{align*}
M'_\# \left( \mathcal{G}_F[t_1, \cdots, t_n] \mathcal{A} \right) & \xrightarrow{\sim} M'_\# \left( \mathcal{G}_F[t_1, \cdots, t_n] \mathcal{A}_{t-reg} \right) \\
& \xrightarrow{\sim} \text{colim}_{m \to \infty} M'_\# \left( \mathcal{G}_F[t_1, \cdots, t_n] \mathcal{A}_{t-reg, deg \leq m} \right) \\
& \xrightarrow{\sim} \text{colim}_{m \to \infty} \bigoplus_{i=0}^m M'_\#(\mathcal{A})s^i \\
& \xrightarrow{\sim} \bigoplus_{i=0}^\infty M'_\#(\mathcal{A})s^i.
\end{align*}
\]

Commutativity of the diagram follows from the definition of the functor \( a \) and the following commutative diagram.

\[
\begin{align*}
M'_\#(\mathcal{A}) \\
\bigoplus_{i=0}^\infty M'_\#(\mathcal{A})s^i \\
\end{align*}
\]

In the light of Theorem \[[3.3\] and Proposition \[[3.7\) (4), we would like to consider a quotient category of \( \mathcal{G}_A \mathcal{A} \) by \( \text{Nil}_{(1)} \mathcal{G}_A \mathcal{A} \). To examine quotient categories of abelian category, we recall the quotient theory of locally coherent abelian categories from [16]. We say that a Grothendieck category \( \mathcal{A} \) is locally coherent if every object of \( \mathcal{A} \) is a direct limit of coherent objects of \( \mathcal{A} \). The following lemma is fundamental.
3.9. Lemma. A locally noetherian category $\mathcal{A}$ is locally coherent.

Proof. Let $\{u_i\}_{i \in I}$ be a family of noetherian generators of $\mathcal{A}$ and let $x$ be an object of $\mathcal{A}$. For a non-zero morphism $\alpha : x \to y$, there exists $i_\alpha$ in $I$ and a morphism $\beta_\alpha : u_{i_\alpha} \to x$ such that $\alpha \beta_\alpha \neq 0$. Then the canonical morphism $\bigoplus_{\alpha} u_{i_\alpha} \to x$ induced from the family $\{\beta_\alpha : u_{i_\alpha} \to x\}_\alpha$ is epimorphism and we have the canonical isomorphism $x \cong \bigoplus_{\alpha} \operatorname{Im} \beta_\alpha$.

Here $\operatorname{Im} \beta_\alpha$ is noetherian, a fortiori, coherent for any $\alpha : x \to y$. Therefore $\mathcal{A}$ is locally coherent.

Let $\mathcal{A}$ be a Grothendieck category and $\mathcal{B}$ a full subcategory of $\mathcal{A}$. We say that $\mathcal{B}$ is a localizing subcategory (of $\mathcal{A}$) if $\mathcal{A}$ is closed under sub- and quotient objects and extensions and coproducts. (In [18, 2.1], we call a localizing subcategory a hereditary torsion subcategory.) We write $\sqrt{\mathcal{B}}$ for intersection of all localizing subcategories which contain $\mathcal{B}$ and call it the localizing radical of $\mathcal{B}$ (in $\mathcal{A}$). Assume that $\mathcal{A}$ is locally coherent and $\mathcal{B}$ is localizing. We say that $\mathcal{B}$ is of finite type if there exists a Serre subcategory $\mathcal{T}$ of $\operatorname{Coh} \mathcal{A}$ such that $\mathcal{B} = \sqrt{\mathcal{T}}$.

Let $\mathcal{A}$ be a locally coherent abelian category. There exists an inclusion preserving bijective correspondence between the class of Serre subcategories of $\operatorname{Coh} \mathcal{A}$ and the class of localizing subcategories of $\mathcal{A}$ of finite type. The correspondence given by sending a Serre subcategory $\mathcal{B}$ of $\operatorname{Coh} \mathcal{A}$ to $\sqrt{\mathcal{B}}$ and a localizing subcategory $\mathcal{T}$ of $\mathcal{A}$ of finite type to $\mathcal{T} \cap \operatorname{Coh} \mathcal{A}$ which are mutual inverse. (See [16 Theorem 2.8.].)

Let $\mathcal{A}$ be a locally coherent category and $\mathcal{B}$ a Serre subcategory of $\operatorname{Coh} \mathcal{A}$. The inclusion functor $\sqrt{\mathcal{B}} \hookrightarrow \mathcal{A}$ admits a right adjoint functor $t_{\sqrt{\mathcal{B}}} : \mathcal{A} \to \sqrt{\mathcal{B}}$ and we call it the torsion functor (associated to $\sqrt{\mathcal{B}}$). $t_{\sqrt{\mathcal{B}}}$ is a left exact functor and we write $t_{\sqrt{\mathcal{B}}}^1$ for the first derived functor of $t_{\sqrt{\mathcal{B}}}$. We say that an object $x$ of $\mathcal{A}$ is $\sqrt{\mathcal{B}}$-torsion free (resp. $\sqrt{\mathcal{B}}$-closed) if $t_{\sqrt{\mathcal{B}}}^1(x) = 0$. (resp. $t_{\sqrt{\mathcal{B}}}^1(x) = 0$.) We denote the full subcategory of $\mathcal{A}$ spanned by $\sqrt{\mathcal{B}}$-closed objects by $\mathcal{A}/\sqrt{\mathcal{B}}$ and call it the quotient category of $\mathcal{A}$ by $\sqrt{\mathcal{B}}$. $\mathcal{A}/\sqrt{\mathcal{B}}$ is a locally coherent category. The inclusion functor $j_{\sqrt{\mathcal{B}}} : \mathcal{A}/\sqrt{\mathcal{B}} \hookrightarrow \mathcal{A}$ admits an exact left adjoint functor $q_{\sqrt{\mathcal{B}}} : \mathcal{A} \to \mathcal{A}/\sqrt{\mathcal{B}}$ such that $q_{\sqrt{\mathcal{B}}} j_{\sqrt{\mathcal{B}}} = \operatorname{id}_{\mathcal{A}/\sqrt{\mathcal{B}}}$. The functor $q_{\sqrt{\mathcal{B}}}$ induces an exact functor $\operatorname{Coh} q_{\sqrt{\mathcal{B}}} : \operatorname{Coh} \mathcal{A} \to \operatorname{Coh} (\mathcal{A}/\sqrt{\mathcal{B}})$ and an equivalence of categories $(\operatorname{Coh} \mathcal{A})/\mathcal{B} \sim \operatorname{Coh} (\mathcal{A}/\sqrt{\mathcal{B}})$. Here the category $(\operatorname{Coh} \mathcal{A})/\mathcal{B}$ is the usual quotient abelian category of $\operatorname{Coh} \mathcal{A}$ by the Serre subcategory $\mathcal{B}$. (cf. [16 §2.2].)

3.10. Proposition. Let $\mathcal{A}$ be a locally noetherian category and $\mathcal{B}$ a Serre subcategory of $\operatorname{Coh} \mathcal{A}$. Then

(1) The localization functor $q_{\sqrt{\mathcal{B}}} : \mathcal{A} \to \mathcal{A}/\sqrt{\mathcal{B}}$ sends noetherian objects in $\mathcal{A}$ to noetherian objects in $\mathcal{A}/\sqrt{\mathcal{B}}$.

(2) The quotient category $\mathcal{A}/\sqrt{\mathcal{B}}$ is also locally noetherian.

Proof. For simplicity we write $q$ and $j$ for $q_{\sqrt{\mathcal{B}}}$ and $j_{\sqrt{\mathcal{B}}}$ respectively.
(1) Let \( x \) be a noetherian object in \( \mathcal{A} \). Then \( q(x) \) is coherent in \( \mathcal{A} / \sqrt{\mathcal{B}} \).
In particular \( jq(x) \) is finitely generated in \( \mathcal{A} \) by [16, 2.2]. Since \( \mathcal{A} \) is locally noetherian, \( jq(x) \) is \( \mathcal{A} \)-noetherian. In particular \( q(x) \) is noetherian in \( \mathcal{A} / \sqrt{\mathcal{B}} \).

(2) First notice that \( \mathcal{A} / \sqrt{\mathcal{B}} \) is Grothendieck. Let \( \{ u_{\lambda} \}_{\lambda \in \Lambda} \) be a family of noetherian generators of \( \mathcal{A} \). We will prove that \( \{ q(u_{\lambda}) \}_{\lambda \in \Lambda} \) is a family of generators of \( \mathcal{A} / \sqrt{\mathcal{B}} \). Then since \( q(u_{\lambda}) \) is noetherian in \( \mathcal{A} / \sqrt{\mathcal{B}} \) for any \( \lambda \) in \( \Lambda \), we complete the proof. Let \( \theta: \text{id}_{\Lambda} \rightarrow jq \) be an adjunction morphism and \( \beta: x \rightarrow y \) be a non-zero morphism in \( \mathcal{A} / \sqrt{\mathcal{B}} \). Then there exist an element \( \lambda \) in \( \Lambda \) and a morphism \( \alpha: u_{\lambda} \rightarrow j(x) \) such that \( j(\beta) \alpha \neq 0 \). Then by adjointness of \( q \), there exists a morphism \( \alpha': q(u_{\lambda}) \rightarrow x \) such that \( \alpha = \theta(u_{\lambda})j(\alpha') \). Obviously we have \( \beta \alpha' \neq 0 \). Hence we obtain the result. \( \square \)

Now we give a definition of projective varieties over a locally noetherian abelian categories.

3.11. Definition (Projective varieties). Let \( \mathcal{A} \) be a \( \mathbb{N} \)-grading \( B \)-algebra and let \( \mathcal{A} \) be a locally noetherian abelian \( B \)-category. Then we set \( \text{Proj}^A \mathcal{A} := \text{Gr}_A \mathcal{A} / \sqrt{\text{Nil}(1)} \text{Gr}_A \mathcal{A} \) and call it \( \text{proj} \overline{\mathcal{A}} \) over \( \mathcal{A} \).

Moreover let \( f = \{ f_k \}_{1 \leq k \leq r} \) be a family of homogeneous elements in \( \mathcal{A} \). Then we set \( \text{Proj}^A_{f} \mathcal{A} := \text{Gr}^V_A(1) \mathcal{A} / \sqrt{\text{Nil}(1)} \text{Gr}_A \mathcal{A} \).

By virtue of Proposition [3.10(2) and [16, §2], \( \text{Proj}^A \mathcal{A} \) and \( \text{Proj}^V_{f} \mathcal{A} \) are locally noetherian and we have equalities

\[
\text{Coh} \text{Proj}^A \mathcal{A} = \text{Coh} \text{Gr}_A \mathcal{A} / \text{Nil}(1) \text{Coh} \text{Gr}_A \mathcal{A} \quad \text{and}
\]

\[
\text{Coh} \text{Proj}^V_{f} \mathcal{A} = \text{Coh} \text{Gr}^V_A(1) \mathcal{A} / \text{Nil}(1) \text{Coh} \text{Gr}^V_A(1) \mathcal{A} .
\]

Thus \( \text{Proj} \mathcal{A} \) is a functor from the (large) category of locally noetherian abelian categories and cocontinuous functors to itself.

In particular, for a non-negative integer \( n \), we set \( \mathbb{P}_n := \text{Proj}^A \mathcal{F}_1[0, \ldots, n] \) and call it the \( n \)th projective space over \( \mathcal{A} \).

For \( \# \in \{ \text{add}, \text{loc}, \text{nilp} \} \), we write \( M^V_{\#}(\text{Proj}^A \mathcal{A}) \) for \( M(\text{Coh} \text{Proj}^V_{f} \mathcal{A}) \) a dash motive of \( \text{Proj}^V_{f} \mathcal{A} \).

3.12. Examples (Projective varieties).

(1) (cf. [31, p.49].) Let \( X \) be a noetherian scheme and \( n \) a positive integer. Then we have the canonical equivalence of categories \( \mathbb{P}_n^{\text{Coh}_{\text{coh}} X} \overset{\sim}{\rightarrow} \text{Qcoh}_{\text{coh}} X \).

(2) Let \( k \) be a field, \( A \) an \( \mathbb{N} \)-graded commutative ring of finite type over \( k \) such that \( A \) is generated by elements of degree one. Then by Serre’s theorem [3.3] we have the canonical equivalence of categories \( \text{Proj}^A_{\text{Mod}_{k}} \mathcal{A} \overset{\sim}{\rightarrow} \text{Qcoh} \text{Proj}^A \mathcal{A} \).

To study the localization sequence of dash motives associated with projective varieties over a locally noetherian abelian category and calculate the nilpotent invariant dash motives associated with projective spaces, we recall the notion of special filtering Serre subcategories. Let \( \mathcal{S} \) be a Serre subcategory of an abelian category \( \mathcal{A} \). We say that \( \mathcal{S} \) is (right) special filtering if for any monomorphism \( x \hookrightarrow y \) in \( \mathcal{A} \) with \( x \in \mathcal{S} \), there exists
a morphism \(y \to z\) with \(z\) in \(\mathcal{S}\) such that the composition \(x \to y \to z\) is a monomorphism. (cf. [35 Definition 1.5.]) If \(\mathcal{S}\) is special filtering, then the canonical sequence

\[
\mathcal{D}_\text{dg}^b(\mathcal{S}) \to \mathcal{D}_\text{dg}^b(\mathcal{A}) \to \mathcal{D}_\text{dg}^b(\mathcal{A}/\mathcal{S})
\]

is an exact sequence of dg-categories. (See [13 4.1] and [22 1.15].) Thus it induces a distinguished triangle

\[
M'_{\#}(\mathcal{S}) \to M'_{\#}(\mathcal{A}) \to M'_{\#}(\mathcal{A}/\mathcal{S}) \to \Sigma M'_{\#}(\mathcal{S})
\]

in \(\text{Ho}(\text{Mot}_M^\#)\) for \(\# \in \{\text{loc}, \text{nilp}\}\).

The following lemma is fundamental.

3.13. Lemma. Let \(A\) be an \(\mathbb{N}\)-grading \(B\)-algebra and \(\mathcal{J} = \{f_k\}_{1 \leq k \leq r}\) be a family of homogeneous elements in \(A\) and \(\mathcal{A}\) be a locally noetherian abelian category. Then \(\text{Nil}_{(1)}\mathbf{Coh}_{\text{Gr}_A}\mathcal{A}\) and \(\text{Nil}_{(1)}\mathbf{Coh}_{\text{Gr}_A}^{V(1)}\mathcal{A}\) are right special filtering in \(\text{Coh}_{\text{Gr}_A}\mathcal{A}\) and \(\text{Gr}_A^{V(1)}\mathcal{A}\) respectively. In particular we have the distinguished triangles

\[
M'_{\#}(\text{Nil}_{(1)}\text{Gr}_A\mathcal{A}) \to M'_{\#}(\text{Gr}_A\mathcal{A}) \to M'_{\#}(\text{Proj}_A\mathcal{A}) \to \Sigma M'_{\#}(\text{Nil}_{(1)}\text{Gr}_A\mathcal{A}) \quad \text{and}
\]

\[
M'_{\#}(\text{Nil}_{(1)}\text{Gr}_A^{V(1)}\mathcal{A}) \to M'_{\#}(\text{Gr}_A^{V(1)}\mathcal{A}) \to M'_{\#}(\text{Proj}_A^{V(1)}\mathcal{A}) \to \Sigma M'_{\#}(\text{Nil}_{(1)}\text{Gr}_A^{V(1)}\mathcal{A})
\]

in \(\text{Mot}_M^\#\) for \(\# \in \{\text{loc}, \text{nilp}\}\).

Proof. Let \(x \to y\) be a monomorphism in \(\text{Coh}_{\text{Gr}_A}\mathcal{A}\) with \(x\) in \(\text{Nil}_{(1)}\mathbf{Coh}_{\text{Gr}_A}\mathcal{A}\). Then there exists a non-negative integer \(k \geq 0\) such that \(x_m = 0\) for \(m \geq k\). Then we define \(z\) to be an object in \(\text{Nil}_{(1)}\mathbf{Coh}_{\text{Gr}_A}\mathcal{A}\) by setting \(z_l = y_l\) if \(l < k\) and \(z_l = 0\) if \(l \geq k\). There is a canonical epimorphism \(y \to z\) such that the composition \(x \to y \to z\) is a monomorphism. Thus \(\text{Nil}_{(1)}\mathbf{Coh}_{\text{Gr}_A}\mathcal{A}\) is right special filtering in \(\text{Coh}_{\text{Gr}_A}\mathcal{A}\). A proof for \(\text{Nil}_{(1)}\mathbf{Coh}_{\text{Gr}_A}^{V(1)}\mathcal{A}\) is similar. \(\square\)

3.14. Corollary (Nilpotent invariant dash motive of projective spaces). Let \(\mathcal{A}\) be a locally noetherian abelian category and let \(n\) be a non-negative integer. Then the exact functor \(\bigoplus_{i=0}^{n} \text{Coh}_\mathcal{A} \to \text{Coh}_\mathcal{P}_\mathcal{A}^n\) which sends an object \((x_i)_{0 \leq i \leq n}\) to \(\bigoplus_{i=0}^{n} x_i[t_1, \cdots, t_n](-i)\) induces an isomorphism of nilpotent invariant dash motives \(M'_{\text{nilp}}(\mathcal{P}_\mathcal{A}^n) \to \bigoplus_{i=0}^{n} M'_{\text{nilp}}(\mathcal{A})s^i\) where \(M'_{\text{nilp}}(\mathcal{A})s^i\) is just a copy of \(M'_{\text{nilp}}(\mathcal{A})\).

The proof is carried out in several lemmata. Recall the definition of admissibility of cubes from \(\S 2\) and the definition of Koszul cubes associated with graded objects from the paragraph before the proof of Theorem 3.8.

3.15. Lemma. Let \(\mathcal{A}\) be an abelian category and let \(n\) and \(k\) be non-negative integers. Then the cube \(\text{Kos}(x[t_1, \cdots, t_n](-k))\) is admissible.
Proof. First notice that $\text{Tot} \text{Kos}(x[t_1, \ldots, t_n](-k))$ is 0-spherical by Proposition 4.23 (1). We proceed by induction on $n$. For $n = 1$, assertion is trivial. For $n > 1$, notice that a face of $\text{Kos}(x[t_1, \ldots, t_{n-1}](-k-j))$ for $j = 0$ or $j = 1$. Therefore by the inductive hypothesis, it is admissible. Hence by Corollary 3.15, $\text{Kos}(x[t_1, \ldots, t_n](-k))$ is admissible. 

3.16. Lemma. Let $A$ be the $F_1[t_1, \ldots, t_n]$ the $n$th polynomial $\mathbb{N}$-grading ring over $F_1$. Then the functor $T_1(- \otimes_A A[t]) : \text{Coh Nil}_{(1)} A \rightarrow \text{Coh Gr}_{A[t]} A$ is exact and induced map on nilpotent invariant dash motives makes the diagram below commutative

$$
\begin{array}{c}
M_{\text{nilp}}(\text{Nil}_{(1)} \text{Gr}_A A) \\
\downarrow \\
\bigoplus_{i \geq 0} M_{\text{nilp}}(A)^{s^i} \\
\downarrow \\
\bigoplus_{i \geq 0} M_{\text{nilp}}(A)^{s^i}
\end{array} 
\xrightarrow{\text{id}} 
\begin{array}{c}
M_{\text{nilp}}(\text{Nil}_{(1)} \text{Gr}_A A(t)) \\
\downarrow \\
\bigoplus_{i \geq 0} M_{\text{nilp}}(A)^{s^i} \\
\downarrow \\
\bigoplus_{i \geq 0} M_{\text{nilp}}(A)^{s^i}
\end{array}
$$

where the vertical morphisms are isomorphisms in Theorem 3.8.

Proof. The functor $T_1(- \otimes_A A[t]) : \text{Coh Nil}_{(1)} A \rightarrow \text{Coh Gr}_{A[t]} A$ is exact by Lemma 3.15. For an object $x$ in $\text{Nil}_{(1)} A$, there exists an exact sequence $\begin{array}{c} x[t](-1) \rightarrow x[t] \rightarrow T_1(x[t]) \end{array}$ in $\text{Coh Gr}_{A[t]} A$. Therefore by additivity theorem and Theorem 3.8, it turns out that the square (8) is commutative.

In the next lemma, to make the statement simplify, we utilize the conventions of affine varieties over categories in Definition 3.18.

3.17. Lemma. Let $B$ be an additive category with countable coproduct and $x$ an object in $B$. We consider the object $x[t]$ in $\text{Spec}_B F_1[t]$ and we define $q_x : x[t] \rightarrow x[t]$, $\diamondsuit_x : x[t] \rightarrow x[t] = x$ and $\pi_x : x[t] \rightarrow \bigoplus_{i=0}^n x[t^i]$ to be morphisms in $\text{Spec}_B F_1[t]$ by setting $q_x := \begin{pmatrix} 0 & -\text{id} & -\text{id} & -\text{id} & \cdots \\ 0 & 0 & -\text{id} & -\text{id} & \cdots \\ 0 & 0 & 0 & -\text{id} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$, $\diamondsuit_x = (\text{id} \ 0 \ 0 \ \cdots)$ and $\pi_x := \begin{pmatrix} \diamondsuit_x q_x^n \\ \diamondsuit_x q_x^{n-1} \\ \vdots \\ \diamondsuit_x \end{pmatrix}$. Then the sequence

$$
x[t] \xrightarrow{(1-0)^{n+1}} x[t] \xrightarrow{\pi_x} \bigoplus_{i=0}^n x[t^i]
$$

where $x[t^i]$ is a just a copy of $x$ is a split exact sequence in $\text{Spec}_B F_1[t]$. 24
Proof. We set \( i_x : x \xrightarrow{id_x} x = xt^0 \hookrightarrow x[t] \) and \( j_x : \bigoplus_{i=0}^{n} xt^i \to x[t] \) by the formula
\[
j_x = ((1 - t)^n i_x, (1 - t)^{n-1} i_x, \ldots, i_x).
\]
Then we can check the equalities \( \pi_x (1 - t)^{n+1} = 0, q_x^{n+1} j_x = 0, q_x^{n+1} (1 - t)^{n+1} = \text{id}_x[t], \) \( \pi_x j_x = \text{id}_A x[t] \) and \( (1 - t)^{n+1} q_x^{n+1} j_x \) \( \pi_x = \text{id}_x[t]. \) This means that the sequence (0) is a split exact sequence.

Proof of Corollary 3.14. We define \( \textbf{Coh} \text{Gr}_{F_1} \mathcal{A} \to \text{Coh} \text{Gr}_{F_1[t_0, \ldots, t_n]} \mathcal{A} \) to be a functor by sending an object \((x_i)_{i \geq 0} \) to \( \bigoplus_{i \geq 0} T_0(x_i[t_0, \ldots, t_n][(-i)]) \).

Notice that for an object \((x_i)_{i \geq 0} \) in \( \text{Coh} \text{Gr}_{F_1} \mathcal{A} \), there exists an integer \( n > 0 \) such that \( x_i = 0 \) for \( i \geq n \). By Lemma 3.15 the functor \( T \) is exact and since \( \text{Coh} \text{Nil}_{(1)} \text{Gr}_{F_1[t_0, \ldots, t_n]} \mathcal{A} \) is contained in \( \text{Coh} \text{Gr}_{F_1[t_0, \ldots, t_n]} \mathcal{A} \) where \( t = \{t_i\}_{0 \leq i \leq n} \), it turns out that the inclusion \( \text{Coh} \text{Gr}_{F_1} \mathcal{A} \hookrightarrow \text{Coh} \text{Nil}_{(1)} \text{Gr}_{F_1[t_0, \ldots, t_n]} \mathcal{A} \) is a nilpotent immersion by Proposition 3.7(3) and (4). Thus it induces an isomorphism of nilpotent invariant dash motives \( M_{nlp}^{\text{Nil}_{(1)}} \text{Gr}_{F_1} \mathcal{A} \to M_{nlp}^{\text{Nil}_{(1)} \text{Gr}_{F_1[t_0, \ldots, t_n]} \mathcal{A}} \). We will show that this map makes the square I below commutative.

\[
\begin{array}{cccc}
\oplus_{i \geq 0} M_{nlp}(A)s^i & \bigoplus_{i \geq 0} M_{nlp}(A)s^i & \bigoplus_{i \geq 0} M_{nlp}(A)s^i & \bigoplus_{i \geq 0} M_{nlp}(A)s^i \\
\| & \| & \| & \|
\end{array}
\]

For an object \( x \) in \( \mathcal{A} \), Inspection shows that an equality
\[
T_0(x[t_0, \ldots, t_n](-k)) = T_0(T_0(x[t_0, \ldots, t_n](-k) @ F_1(t_0, \ldots, t_n)) @ F_1(t_0, \ldots, t_n)).
\]

Thus by induction on \( n \) and Lemma 3.16 it turns out that the square I is commutative.

Next since both the top and the bottom lines are distinguished triangles by Lemma 3.17 and Lemma 3.13 respectively, there exists a morphism \( \bigoplus_{i=0}^{n} M_{nlp}(A)s^i \to M_{nlp}(\mathcal{P}_A) \) which makes the square II and III commutative and by five lemmam this map is an isomorphism. Thus it
turns out that the bottom line is also a split exact sequence. Namely the map \( \partial \) in the diagram \( \square \) is trivial. Thus we learn that the map \( \bigoplus_{i=0}^n M_{n+1}(A) \rightarrow \bigoplus_{i=0}^n \text{Coh} \cdot A \rightarrow \mathbb{A} \cdot A \)
which sends an object \((x_i)_{0 \leq i \leq n} \rightarrow \bigoplus_{i=0}^n x_i[t_0, \cdots, t_n](-i)\) also makes the square \( \text{II} \) and \( \text{III} \) commutative and by five lemma again, this map is also an isomorphism. We complete the proof.

Similarly we define the notion of affine varieties over categories.

3.18. Definition (Affine varieties). Let \( A \) be a commutative \( B \)-algebra and let \( A \) be a noetherian abelian \( B \)-category. We regard \( A \) as a \( B \)-category by setting \( \text{Ob} \cdot A = \{ * \} \) and \( \text{Hom} \cdot A(*, *) := A \). Then we define \( \text{Spec} \cdot A \) to be a \( B \)-category by setting \( \text{Spec} \cdot A := \text{Hom}_B(A, A) \)
the category of \( B \)-functors from \( A \) to \( A \) and natural transformations and call it an affine scheme associated with \( A \) over \( A \). Namely an object in \( \text{Spec} \cdot A \) can be regarded as a pair of an object \( x \) in \( A \) and a family of endomorphisms \( \{ x_f : x \rightarrow x \}_{f \in A} \)
indexed by elements in \( A \) such that endomorphisms satisfy equalities arose from relations among elements in \( A \).

We sometimes abbreviate the morphism \( x_f : x \rightarrow x \rightarrow f \).

For a family of elements \( \mathcal{f} = \{ f_k \}_{1 \leq k \leq r} \) of \( A \), let \( \text{Spec}^{\mathcal{f}(t)} \cdot A \) be a full subcategory of \( \text{Spec} \cdot A \) consisting of those objects \( x \) such that there exists a positive integer \( N > 0 \) such that \( x_{f_k}^N = 0 \) for all \( 1 \leq k \leq r \) and we denote the full subcategory of \( \text{Spec}^{\mathcal{f}(t)} \cdot A \) consisting of those objects \( x \) such that \( x_{f_k} = 0 \) for all \( 1 \leq k \leq r \) by \( \text{Spec}^{\mathcal{f}(t)}(A) \).

By convention, we write \( \text{Spec}_A \cdot \mathbb{F}_1 \) for \( A \) and for any positive integer \( n \), we set \( \mathbb{A}^n_A := \text{Spec} \cdot A[t_1, \cdots, t_n] \)
and call it the \( n \)th affine space over \( A \).

By virtue of Proposition 3.20 (9) below, we can regard \( \text{Spec} \cdot A \) as a functor from (the large) category of locally noetherian abelian category and cocontinuous functors to it itself.

Let \( C \) be a \( N \)-grading commutative \( B \)-algebra. Then there exists a for-getting grading functor \( F_C : \text{Gr} \cdot C \rightarrow \text{Spec} \cdot C \)
which sends an object \( x \)
in \( \text{Gr} \cdot C \) to a pair \( \bigoplus_{n \geq 0} x_n, \{ F_A(x)_f \}_{f \in C} \). Here for a homogeneous element \( f \in C \) of degree \( s \), we set \( F_A(x)_f := \bigoplus_{k \geq 0} f : x_k \rightarrow x_{k+s} : \bigoplus_{n \geq 0} x_n \rightarrow \bigoplus_{n \geq 0} x_n \).

For a general element \( f \in C \), we denote it by the summation of homogeneous elements \( f = \sum_{k=1}^r f_k, f_k \in A_k \) and we set \( F_A(x)_f := \sum_{k=1}^r F_A(x)_{f_k} \).

Since \( \text{Gr}_A \cdot A \) is locally noetherian, the operation \( \bigoplus \) is exact and therefore \( F_A \) is a faithful and exact functor.

3.19. Examples (Affine varieties). Let \( \mathcal{A} \) be a locally noetherian abelian \( B \)-category.

(1) Let \( X \) be a noetherian scheme and \( n \) be a positive integer. Then we have the canonical equivalence of categories \( \mathbb{A}^n_{\text{Coh}_X} \rightarrow \text{Qcoh}_{\mathbb{A}^n_X} \).
2. Let $A$ and $C$ be a commutative $B$-algebras. Then we have the canonical equivalence of categories $\text{Spec}_{\text{Mod}_A} C \to \text{Mod}_{A \otimes_B C}$.

3.20. Proposition (Fundamental properties of affine varieties).

Let $A$ be a category and $A$ a commutative $B$-algebra. Then

(1) We can calculate a (co)limit in $\text{Spec}_A A$ by term-wise (co)limit in $A$. In particular, if $A$ is additive (resp. abelian, closed under small filtered colimits) then $\text{Spec}_A A$ is also additive (resp. abelian, closed under small filtered colimits).

Moreover assume that $A$ is abelian and let $\mathfrak{f} = \{f_k\}_{1 \leq k \leq r}$ be a family of homogeneous elements of $A$. We write $\mathfrak{f} A$ for the ideal of $A$ spanned by $\mathfrak{f}$. Then

(2) $\text{Spec}_A^{(f)} A$ is a Serre subcategory of $\text{Spec}_A A$.

(3) $\text{Spec}_{A,\text{red}}^{(f)} A$ is a topologizing subcategory of $\text{Spec}_A^{(f)} A$ and the inclusion functor $\text{Spec}_{A,\text{red}}^{(f)} A \hookrightarrow \text{Spec}_A^{(f)} A$ is a nilpotent immersion.

(4) The canonical map $A \to A/ \mathfrak{f} A$ induces an equivalence of categories $\text{Spec}_A A/ \mathfrak{f} A \cong \text{Spec}_{A,\text{red}}^{(f)} A$.

(5) The fully faithful embedding $\text{Spec}_A A/ \mathfrak{f} A \hookrightarrow \text{Spec}_A A$ induced from the canonical map $A \to A/ \mathfrak{f} A$ admits a left adjoint functor $- \otimes_A A/ \mathfrak{f} A$ to $\text{Spec}_A A$.

Moreover assume that $A = B[t_1, \ldots, t_n]/(g_1, \ldots, g_l)$ where $B[t_1, \ldots, t_n]$ is the $n$th polynomial ring over $B$ and $g_i$ is a polynomial in $B[t_1, \ldots, t_n]$ for $1 \leq i \leq l$. Then

(6) The forgetful functor $U_A : \text{Spec}_A A \to A$ which sends an object $(x, \{x_f\}_{f \in A})$ in $\text{Spec}_A A$ to $x$ admits a left adjoint functor $- \otimes_B A$ to $\text{Spec}_A A$.

(7) If $x$ in $A$ is a noetherian object, then $x \otimes_B A$ is also a noetherian object in $\text{Spec}_A A$.

(8) For any object $x$ in $A$, there exists a canonical epimorphism $U_A(x) \otimes_B A \to x$.

(9) If $A$ admits a family of generators $\{u_\lambda\}_{\lambda \in \Lambda}$ indexed by non-empty set $\Lambda$, then $\text{Spec}_A A$ has a family of generators $\{u_\lambda \otimes_B A\}_{\lambda \in \Lambda}$. In particular if $A$ is Grothendieck (resp. locally noetherian), then $\text{Spec}_A A$ is also.

Proof. (1), (2), (3) and (4) are straightforward.

(5) As in the proof of Proposition 3.17 (5), for an object $x$ in $\text{Spec}_A A$, we define $\mathfrak{f} x$ to be a subobject by setting $\mathfrak{f} x := \sum_{k=1}^r \text{Im}(f_k : x \to x)$ and we set $x \otimes_A A/ \mathfrak{f} A := x/ \mathfrak{f} x$. Then we can show that the association $x \mapsto x \otimes_A A/ \mathfrak{f} A$ gives a left adjoint functor of the fully faithful embedding $\text{Spec}_A A/ \mathfrak{f} A \hookrightarrow \text{Spec}_A A$.

(6) Since $U_A : \text{Spec}_A A \to A$ factors through

$$\text{Spec}_A A \hookrightarrow \text{Spec}_B B[t_1, \ldots, t_n] \to A,$$
by (5), we shall only prove the case where \( A = B[t_1, \cdots , t_n] \). In this case the functor \(- \otimes_B B[t_1, \cdots , t_n] \): \( \text{Spec}_A \to \text{Spec}_A \) is just a composition of \( A \to \otimes_B B[t_1, \cdots , t_n] \) \( \text{Gr}_{B[t_1, \cdots , t_n]} A \to \text{Spec}_A \).

(7) As in the proof of Proposition 3.18 (8), we can reduce to the case where \( A = B[t_1, \cdots , t_n] \) and in this case, assertion is proven in [34, 9.10 b].

(8) Let \( x \) be an object in \( \text{Spec}_A \). Then there exists the canonical projection

\[
U_A(x)[t_1, \cdots , t_n] = \bigoplus_{(t_1, \cdots , t_n) \in \mathbb{N}} xt_1^{t_1} \cdots t_n^{t_n} \Rightarrow xt_1^0 \cdots t_n^0 = x
\]

and this morphism factors through \( U_A(x)[t_1, \cdots , t_n] \to U_A(x) \otimes_B A \to x \).

The last morphism is a desired epimorphism.

(9) For a non-zero morphism \( x \to y \) in \( \text{Spec}_A \), since \( F_A(a) \) is a non-zero morphism, there exists a morphism \( b: u_A \to F_A(x) \) such that \( ab \neq 0 \).

Then the compositions \( u_A \otimes_B A \to \otimes As \to x \to y \) is a non-zero morphism.

\[\square\]

For an \( \mathbb{N} \)-grading commutative \( B \)-algebra \( A \) and a locally noetherian abelian \( B \)-category, recall the functor \( F_A: \text{Gr}_A A \to \text{Spec}_A A \) from Definition 3.18. The functor sends an object \( x \) in \( \text{Gr}_A A \) to a pair \( \left( \bigoplus_{n \geq 0} x_n, \{ F_A(x)_f \}_{f \in A} \right) \) in \( \text{Spec}_A A \).

**3.21. Proposition.** Let \( A \) be an \( \mathbb{N} \)-grading commutative \( B \)-algebra. Then

1. For an object \( x \) in \( \text{Gr}_A A \) and non-negative integer \( k \geq 0 \), \( F_A(x(-k)) = F_A(x) \).

2. For a family of homogeneous elements \( \mathfrak{f} = \{ f_k \}_{1 \leq k \leq r} \) in \( A \), we have the equality \( F_A(\mathfrak{f}) = F_A(\mathfrak{f})F_A(x) \) where \( F_A(\mathfrak{f}) := \{ F_A(f_k) \}_{1 \leq k \leq r} \).

3. Let \( y \) and \( z \) be a pair of subobjects of \( x \), then we have the equality \( F_A(y \cap z) = F_A(y) \cap F_A(z) \).

Moreover assume that there exists a finite family of homogeneous elements \( \{ g_i \}_{1 \leq i \leq r} \) of \( B[t_1, \cdots , t_n] \) such that \( A = B[t_1, \cdots , t_n]/(g_1, \cdots , g_r) \).

4. For an object \( x \) in \( A \), we have the equality \( F_A(x \otimes_B A) = x \otimes_B A \).

5. \( F_A \) induces the functor \( \text{Coh} \text{Gr}_A A \to \text{Coh} \text{Spec}_A A \). Namely \( F_A \) sends a coherent object \( x \) in \( \text{Gr}_A A \) to a coherent object \( F_A(x) \) in \( \text{Spec}_A A \).

**Proof.**

1. \( F_A(x(-k)) = \bigoplus_{n \geq k} x_{n-k} = \bigoplus_{n \geq 0} x_n = F_A(x) \).
let

A

3.22. Proposition. Let

(2)

noetherian object.

is a quotient of finite direct sum of noetherian objects,

(5) Assume that

x

Then

B

(4), we obtain the epimorphism

an epimorphism

(3)

F

Localization distinguished triangles).

(Purity).

(2)

F

\left( n \geq 0 \right) \rightarrow \bigoplus_{n \geq 0} x_n)

\sum_{k=1}^{r} \bigoplus_{n \geq 0} \text{Im}(f_k : x_{n-deg} f_k \rightarrow x_n)

\oplus \bigoplus_{k=1}^{r} \sum_{n \geq 0} \text{Im}(f_k : x_{n-deg} f_k \rightarrow x_n) = F_A(f) F_A(x).

(3) \ F_A(y \cap z) = \bigoplus_{n \geq 0} (y_n \cap z_n) = \left( \bigoplus_{n \geq 0} y_n \right) \cap \left( \bigoplus_{n \geq 0} z_n \right) = F_A(y) \cap F_A(z).

(4) By exactness of \ F_A \ and \ (2), \ we \ have \ the \ equalities

\begin{align*}
F_A(x \otimes_B A) &= F_A(x \otimes_B B[t_1, \ldots, t_n] \otimes_B [t_1, \ldots, t_n] A) \\
&= F_B[t_1, \ldots, t_n](x \otimes_B B[t_1, \ldots, t_n]) \otimes_B [t_1, \ldots, t_n] A \\
&= x \otimes_B B[t_1, \ldots, t_n] \otimes_B [t_1, \ldots, t_n] A = x \otimes_B A.
\end{align*}

(5) Assume that \ x \ is \ noetherian, \ then \ there \ exists \ an \ integer \ k > 0 \ and \ an \ epimorphism \ \bigoplus_{k=0}^{m} x_k \otimes_B A(-k) \rightarrow x. \ Then \ by \ exactness \ of \ F_A \ and \ (1), \ we \ obtain \ the \ epimorphism \ \bigoplus_{k=0}^{m} x_k \otimes_B A \rightarrow F_A(x). \ Thus \ since \ F_A(x) \ is \ a \ quotient \ of \ finite \ direct \ sum \ of \ noetherian \ objects, \ F_A(x) \ is \ also \ a \ noetherian \ object. \ □

3.22. Proposition. Let \ A \ be \ a \ locally \ noetherian \ abelian \ B-category \ and \ let \ A \ be \ a \ \mathbb{N}-grading \ commutative \ B-algebra \ \mathfrak{f} = \{f_i\}_{1 \leq i \leq s} \ a \ finite \ non-empty \ family \ of \ homogeneous \ elements \ in \ A \ and \ let \ C \ be \ a \ commutative \ B-algebra \ and \ \mathfrak{g} = \{g_i\}_{1 \leq i \leq t} \ a \ finite \ non-empty \ family \ of \ elements \ of \ C. \ Then

(1) (Localization distinguished triangles). For \ \# \in \{\text{loc, nilp}\},

there \ exists \ distinguished \ triangles \ of \ dash \ motives

\begin{align*}
M^V_{\text{nilp}}(\text{Spec}_A C) &\rightarrow M^V_{\text{nilp}}(\text{Spec}_A A) \rightarrow M^V_{\text{nilp}}(\text{Spec}_A C/\text{Spec}_A A) \rightarrow \Sigma M^V_{\text{nilp}}(\text{Spec}_A C), \\
M^V_{\text{nilp}}(\text{Proj}_A A) &\rightarrow M^V_{\text{nilp}}(\text{Proj}_A A) \rightarrow M^V_{\text{nilp}}(\text{Proj}_A A/\text{Proj}_A A) \rightarrow \Sigma M^V_{\text{nilp}}(\text{Proj}_A A) \rightarrow \Sigma M^V_{\text{nilp}}(\text{Proj}_A A)
\end{align*}

in \ Mot^\#_{\text{dg}}. \n
(2) (Purity). The canonical inclusions \ \text{Spec}_A C/\mathfrak{g} C \hookrightarrow \text{Spec}_A C \text{ and } \text{Proj}_A A/\mathfrak{f} A \hookrightarrow \text{Proj}_A A \text{ induce isomorphisms of nilpotent invariant dash motives } M^V_{\text{nilp}}(\text{Spec}_A C/\mathfrak{g} C) \cong M^V_{\text{nilp}}(\text{Spec}_A C) \text{ and } M^V_{\text{nilp}}(\text{Proj}_A A/\mathfrak{f} A) \cong M^V_{\text{nilp}}(\text{Proj}_A A). \n
29
Proof. (1) We will only give a proof for the projective case. A proof of
the affine case is similar. For simplicity, we set $\mathcal{B} = \text{Coh Gr}_A \mathcal{A}$ and
$\mathcal{C} = \text{Coh Gr}^{(i)}_A \mathcal{A}$. Then there exists a $3 \times 3$ commutative diagrams

\[
\begin{array}{ccc}
\text{Nil}(1) \mathcal{B} & \longrightarrow & \text{Nil}(1) \mathcal{C} \\
\downarrow & & \downarrow \\
\mathcal{B} & \longrightarrow & \mathcal{C} \\
\downarrow & & \downarrow \\
\text{Coh Proj}^{(i)}_A \mathcal{A} & \longrightarrow & \text{Coh Proj}_A \mathcal{A} \\
\downarrow & & \downarrow \\
\text{Coh} \left( \text{Proj}_A \mathcal{A} / \text{Proj}^{(i)}_A \mathcal{A} \right) & .
\end{array}
\]

What we need to prove is that $\mathcal{B}$ and $\text{Nil}(1) \mathcal{B}$ are special filtering in $\mathcal{C}$
and $\text{Nil}(1) \mathcal{C}$ respectively. Then assertion follows from $3 \times 3$-lemma for
distinguished triangles. (cf. [30 Exercise 10.2.6.].)

Let $x \rightarrow y$ be a monomorphism in $\mathcal{C}$ with $x \in \mathcal{B}$. Thus there exists
a non-negative integer $m \geq 0$ such that $f^m x = 0$. Then by Artin-Rees
lemma [3.22 below, we can show that there exists a non-negative integer
$n_0 \geq 0$ such that $f^n y \cap x = f^n - n_0 f^n y \cap x = 0$ for $n \geq n_0 + m$. Hence
for $n \geq n_0 + m$, the composition $x \mapsto y \rightarrow f^n y$ is a monomorphism
and $y / f^n y$ is in $\mathcal{B}$. Thus $\mathcal{B}$ is right special filtering in $\mathcal{C}$. A proof of that
$\text{Nil}(1) \mathcal{B}$ is right special filtering in $\text{Nil}(1) \mathcal{C}$ is similar.

(2) For affine case, it is a direct consequence of Proposition 3.20 (3) and
nilpotent invariance of $M_{\text{nilp}}$.

For projective case, we consider the commutative diagram of distin-
guished triangles

\[
\begin{array}{ccc}
M'_{\text{nilp}}(\text{Nil}(1) \text{ Gr}_{A/f} \mathcal{A}) & \longrightarrow & M'_{\text{nilp}}(\text{Gr}_{A/f} \mathcal{A}) \\
\downarrow & & \downarrow \\
M'_{\text{nilp}}(\text{Nil}(1) \text{ Gr}^{(i)} A) & \longrightarrow & M'_{\text{nilp}}(\text{Gr}^{(i)} A) \\
\downarrow & & \downarrow \\
M'_{\text{nilp}}(\text{Nil}(1) \text{ Gr}^{(i)} A) & \longrightarrow & M'_{\text{nilp}}(\text{Gr}^{(i)} A) \\
\downarrow & & \downarrow \\
M'_{\text{nilp}}(\text{Nil}(1) \text{ Gr}^{(i)} A) & \longrightarrow & M'_{\text{nilp}}(\text{Gr}^{(i)} A) \\
\downarrow & & \downarrow \\
M'_{\text{nilp}}(\text{Nil}(1) \text{ Gr}^{(i)} A) & \longrightarrow & M'_{\text{nilp}}(\text{Gr}^{(i)} A) \\
\end{array}
\]

As in Proposition 3.7 (3), we can also show that $\text{Nil}(1) \text{ Gr}_{A/f} \mathcal{A}$ is topon-
gizing subcategory of $\text{Nil}(1) \text{ Gr}^{(i)} A$ and the inclusion functor $\text{Nil}(1) \text{ Gr}_{A/f} \mathcal{A} \hookrightarrow
\text{Nil}(1) \text{ Gr}^{(i)} A$ is a nilpotent immersion. Thus by nilpotent invariance of
$M_{\text{nilp}}$ the morphisms I, II and III are isomorphisms and by five lemma,
the morphism III is also isomorphism.

\[\blacksquare\]

To state an abstract Artin-Rees lemma [3.23 below, we will introduce
some notations from [30 §2.5] with slight generaization. Let $A$ be a com-
mutative $B$-algebra of finite type over $B$ and let $\mathcal{A}$ be a locally noetherian
abelian $B$-category and $f = \{f_i\}_{1 \leq i \leq r}$ be a non-empty finite family of
elements in $A$. For an object $x$ in $\text{Spec}_A \mathcal{A}$, recall that we denote the sub-
object $\sum_{i=1}^r \text{Im}(f_i : x \rightarrow x)$ of $x$ by $f x$ and inductively we set $f^{n+1} x := f(f^n x)$
for a non-negative integer $n \geq 0$. 30
For a decreasing filtration $\mathfrak{f} = \{x_n\}_{n \geq 0}$ of $x$, $x = x_0 \leftarrow x_1 \leftarrow \cdots \leftarrow x_n \leftarrow \cdots$ is a $\mathfrak{f}$-filtration if $\mathfrak{f} x_n \subset x_{n+1}$ for any $n \geq 0$. A $\mathfrak{f}$-filtration $\mathfrak{f} = \{x_n\}_{n \geq 0}$ is stable if there exists an integer $n_0 \geq 0$ such that $\mathfrak{f} x_n = x_{n+1}$ for any $n \geq n_0$.

For a $\mathfrak{f}$-filtration $\mathfrak{f} = \{x_n\}_{n \geq 0}$ of $x$, We define $\text{Bl}_I \mathfrak{f}$ to be an object in $\text{Spec}_A$ by setting

$$\text{Bl}_I \mathfrak{f} := \left( \bigoplus_{n \geq 0} x_n, \left\{ \bigoplus_{n \geq 0} x_{nf} \right\}_{f \in A} \right).$$

We call $\text{Bl}_I \mathfrak{f}$ a blowing up object of $\mathfrak{f}$ along $I$. For each non-negative integer $n$ and each $1 \leq k \leq r$, the morphisms $f_{nk}^I : x_n \to x_{n+k}$ for $p > 1$ induce a morphism $\eta_{nk}^I : x_n \to \text{Bl}_I \mathfrak{f}$ in $\text{Spec}_A A$.

**3.23. Lemma.** Let $A$ be a commutative $B$-algebra of finite type over $B$ and $I = \{f_i\}_{1 \leq i \leq r}$ be a non-empty finite family of elements in $A$ and let $C$ be an $\mathbb{N}$-grading commutative $B$-algebra of finite type over $B$ and $g = \{g_i\}_{1 \leq i \leq s}$ be a non-empty finite family of homogeneous elements in $C$ and let $A$ be a locally noetherian abelian category. Then

(1) **(Characterization of stability).** For an object $x$ in $\text{Coh} \text{Spec}_A A$ and $\mathfrak{f}$-filtration $\mathfrak{f} = \{x_n\}_{n \geq 0}$ of $x$, the following conditions are equivalent:

(i) $\mathfrak{f}$ is stable.

(ii) There exists an integer $m \geq 0$ such that the canonical morphism

$$\left( \bigoplus_{1 \leq i \leq r} \bigoplus_{k=0}^m x_k \right) \to \mathfrak{f} \mathfrak{f}$$

induced by $\eta_{nk}^I$ ($0 \leq k \leq m$, $1 \leq i \leq r$), $\bigoplus_{1 \leq i \leq r} \bigoplus_{k=0}^m x_k \to \text{Bl}_I \mathfrak{f}$ is an epimorphism.

(iii) $\text{Bl}_I \mathfrak{f}$ is an object in $\text{Coh} \text{Spec}_A A$, namely a noetherian object in $\text{Spec}_A A$.

(2) **(Abstract Artin-Rees lemma).**

(i) Let $x$ be an object in $\text{Coh} \text{Spec}_A A$ and $y$ be a subobject of $x$. Then there exists a non-negative integer $n_0 \geq 0$ such that $f_{nk}^I x \cap y = f_{nk}^{n-n_0} (f_{nk}^{n_0} x \cap y)$ for any $n \geq n_0$.

(ii) Let $x$ be an object in $\text{Coh} \text{Gr}_C A$ and $y$ be a subobject of $x$. Then there exists a non-negative integer $n_0 \geq 0$ such that $g_{nk}^I x \cap y = g_{nk}^{n-n_0} (g_{nk}^{n_0} x \cap y)$ for any $n \geq n_0$.

Proof. (1) We assume that there exists an integer $m \geq 0$ such that $\mathfrak{f} x_n = x_{n+1}$ for any $n \geq m$. Then obviously the canonical morphism

$$\bigoplus_{1 \leq i \leq r} \bigoplus_{k=0}^m x_k \to \text{Bl}_I \mathfrak{f}$$

is an epimorphism.

Next assume the condition (ii). Since $\text{Bl}_I \mathfrak{f}$ is a quotient of finite direct sum of noetherian objects in $\text{Spec}_A A$, $\text{Bl}_I \mathfrak{f}$ is a noetherian object.

Finally we assume that $\text{Bl}_I \mathfrak{f}$ is noetherian. We set $z_m := \text{Im} \left( \bigoplus_{1 \leq i \leq r} \bigoplus_{k=0}^m x_k \to \text{Bl}_I \mathfrak{f} \right)$.

Then the sequence $z_0 \to z_1 \to z_2 \to \cdots$ is stational. Say $z_{n_0} = z_{n_0+1} =$
Thus we obtain the result by five lemma.

$\cdots$. Then for any $n \geq n_0$, we have

$$x_{n+1} \subset x_{n+1} \cap x_{n+1} = z_{n_0} \cap x_{n+1} \subset \sum_{k=1}^{r} \sum_{i=0}^{n_0} \text{Im}(f_k^{n+1-i} : x_i \to x_{n+1}) \subset f x_n.$$  

Hence $f$ is stable.

(2) (i) Consider the $f$-stable filtration $\mathfrak{f} = \{ f^n x \}_{n \geq 0}$ of $x$ and the induced $f$-filtration $\mathfrak{g} = \{(f^n x) \cap y \}_{n \geq 0}$ of $y$. Then $\text{Bl}_\mathfrak{f} \mathfrak{g}$ is a subobject of $\text{Bl}_\mathfrak{f} \mathfrak{f}$. Since $\text{Bl}_\mathfrak{f} \mathfrak{f}$ is noetherian by (1), $\text{Bl}_\mathfrak{f} \mathfrak{g}$ is also noetherian and by (1) again, we learn that $\mathfrak{g}$ is stable. Hence we obtain the result.

(ii) Let $F_C : \text{Coh} \text{Gr} \mathcal{A} \to \text{Coh} \text{Spec} \mathcal{C}$ be the forgetting grading functor. (Well-definedness follows from Proposition 3.21 (5).) By applying $F_C(\mathfrak{g})$, $F_C(x)$ and $F_C(y)$ to (2) (i) and Proposition 3.21 (2) and (3), we obtain the equality $F_C(\mathfrak{g}^n x \cap y) = F_C(\mathfrak{g}^n - \mathfrak{g}^n (\mathfrak{g}^n x \cap y))$. Since $F_C$ is faithful and exact, we obtain the desired equality.

\[\vspace{0.5em}\]

3.24. Corollary ($\mathcal{A}^1$-homotopy invariance). Let $\mathcal{A}$ be a locally noetherian category. Then the base change functor $- \otimes \mathbb{F}_1[t] : \mathcal{A} \to \mathcal{A}^1_{\mathcal{A}}$ is exact and induces an isomorphism $M'_{\text{nilp}}(\mathcal{A}) \sim M'_{\text{nilp}}(\mathcal{A}^1_{\mathcal{A}})$ of nilpotent invariant dash motives.

Proof. Since $\mathcal{A}$ is Grothendieck, the operation $\bigoplus$ is exact. Hence the functor $- \otimes \mathbb{F}_1[t] : \mathcal{A} \to \mathcal{A}^1_{\mathcal{A}}$ is exact. The functor $\text{Gr}_{\mathbb{F}_1[t_0]} \mathcal{A} \to \text{Spec} \mathcal{A} \mathbb{F}_1[t_0]$ which sends an object $x$ to $(\text{colim}_{n=0} x_n, \text{colim}_{n=0} t_0)$ induces an equivalence of categories $\text{Coh} \left( \mathbb{F}_1[t_0] \right) \sim \text{Coh} \mathcal{A}^1_{\mathcal{A}}$ by [20, Theorem 5.6]. Thus replacing $X$ with $\mathcal{A}$ and setting $n = 1$, we obtain the commutative diagram [5] where both the top and the bottom lines are distinguished triangles by Corollary 3.14 and Proposition 3.22 (1) respectively and the morphisms $\mathbf{I}$ and $\Sigma I$ are isomorphisms by Proposition 3.22 (2). Thus we obtain the result by five lemma.

\[\vspace{0.5em}\]

3.25. Corollary (Homotopy invariance and projective bundle formula for dash motives). Let $X$ be a noetherian scheme. Then

(1) Let $f : P \to X$ be a flat morphism of finite type whose fibers are affine spaces. Then $f$ induces an isomorphism of nilpotent invariant dash motives $M'_{\text{nilp}}(P) \sim M'_{\text{nilp}}(X)$. In particular if $X$ and $P$ are regular separated noetherian, then $f$ induces an isomorphism of nilpotent invariant dash motives $M_{\text{nilp}}(P) \sim M_{\text{nilp}}(X)$.

(2) Let $E$ be a vector bundle of rank $r$ on $X$ and let $\mathcal{P}(E)$ be associated projective bundle. Then we have a canonical isomorphism of nilpotent invariant dash motives $M'_{\text{nilp}}(\mathcal{P}(E)) \sim \bigoplus_{k=0}^r M'_{\text{nilp}}(X)$. 
Proof. (1) Let \( Z \) be a closed subset of \( X \), there exists a commutative diagram of distinguished triangles of nilpotent invariant motives

\[
\begin{array}{cccc}
M'_{\text{nilp}}(Z) & \rightarrow & M'_{\text{nilp}}(X) & \rightarrow M'_{\text{nilp}}(X \setminus Z) & \rightarrow \Sigma M'_{\text{nilp}}(Z) \\
\downarrow & & \downarrow & & \downarrow \\
M'_{\text{nilp}}(P_Z) & \rightarrow & M'_{\text{nilp}}(P) & \rightarrow M'_{\text{nilp}}(P_{X \setminus Z}) & \rightarrow \Sigma M'_{\text{nilp}}(P_Z).
\end{array}
\]

Then as in the proof of [33, §7 Proposition 4.1.], by utilizing noetherian induction, we can reduce the case where \( X \) is the spectrum \( \text{Spec} \ k \) of a field \( k \). Then assertion follows from Corollary 3.25.

(2) Similarly we can reduce to the case where \( X \) is the spectrum \( \text{Spec} \ k \) of a field \( k \). Then assertion follows from Corollary 3.14.

\[\square\]

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