Abstract

We investigate conjunctive normal form (CNF) encodings of a function represented with a decomposable negation normal form (DNNF). Several encodings of DNNFs and decision diagrams were considered by (Abío et al. 2016). The authors differentiate between encodings which implement consistency or domain consistency by unit propagation from encodings which are unit refutation complete or propagation complete. The difference is that in the former case we do not care about propagation strength of the encoding with respect to the auxiliary variables while in the latter case we treat all variables (the main and the auxiliary ones) in the same way. The currently known encodings of DNNF theories implement domain consistency. Building on these encodings we generalize the result of (Abío et al. 2016) on a propagation complete encoding of decision diagrams and present a propagation complete encoding of a DNNF and its generalization for variables with finite domains.

1 Introduction

Decomposable negation normal forms (DNNFs) were introduced in [13] where an approach of compiling a conjunctive normal form (CNF) into a DNNF was described. Since then DNNFs and their subclasses were extensively studied as a target language in knowledge compilation. DNNFs form the most succinct language within the knowledge compilation map [16] that allows efficient consistency checking, clause entailment and model enumeration queries. Specific restrictions are often put on DNNFs in order to get efficient answering of other queries as well. A prominent example is the language of deterministic DNNFs (d-DNNFs) which additionally allow model counting. For specific applications of DNNFs, see for example [7, 19, 29, 34, 40]. Ordered binary decision diagrams (OBDD) and their generalization in the form of multivalued decision diagrams (MDD) form subclasses of d-DNNFs which are also often used in both theory and applications.
If a constraint represented with an MDD, d-DNNF or a DNNF is a part of a larger problem, efficient propagators for these representations [23, 24, 25] can be used to maintain domain consistency of the constraint within a constraint programming solver. Recently, in [17] the authors have considered a general problem of compiling global constraints into multivalued decision diagrams (MDD) and d-DNNFs that can be used in this way. Besides detecting conflicts, a propagator can generate an explanation, which is a clause that can be added to the instance. This is the basis for lazy clause generation strategy in CP solvers which can increase their efficiency considerably and is analogous to the CDCL strategy in SAT solvers, see, for example [4, 22].

There are also approaches using an explicit CNF encoding of a constraint in a CP solver. A CNF encoding can be included into the instance at the beginning of the search or only if the constraint appears to be active in the search process as in lazy approaches [2], where a propagator that participates in many conflicts is replaced by a CNF decomposition during run-time. A solver can also resolve a conflict by extending the instance by clauses that contain auxiliary variables new in the instance by lazily expanding a corresponding part of a large known encoding of the constraint [3, 22]. Specific situations, where a CNF encoding of a DNNF is suitable for use in a SAT instance are described in [35, 38].

The running time of a clause learning SAT solver can be exponentially larger, if branching in the solver is restricted to make decisions only on the input variables of an encoding, see [30]. A similar effect was observed in [2], when a CP solver cannot make decisions on auxiliary variables of an encoding used for generating clauses by explaining conflicts in a propagator. This suggests that the solver can benefit from branching on auxiliary variables and then, it is natural to consider propagation strength on all variables of the encoding and not only on the main or input variables required by domain consistency. Here, by propagation, we mean unit propagation which is a standard tool in SAT solvers based on CDCL.

Let us recall several levels of propagation strength of a CNF encoding of a constraint \( f(x) \). For simplicity, we assume that the variables \( x \) have boolean domains although later we consider the direct encoding (see for example [1, 8]) to encode variables with arbitrary finite domains. We say that a CNF formula \( \varphi(x, y) \) is a CNF encoding of \( f(x) \) if \( f(x) \equiv (\exists y)\varphi(x, y) \). The variables \( y \) are called auxiliary variables and are used only for the purpose of the encoding. Following [1] we consider several properties of a CNF encoding \( \varphi(x, y) \) of a function \( f(x) \), which specify its propagation strength. In the description below, a partial assignment is represented by a consistent set of literals and implements means by unit propagation.

- **Encoding \( \varphi \) implements consistency** (called also a consistency checker in [8]) if for any partial assignment \( \alpha \) to the main variables \( x \), we have that if \( f(x) \land \alpha \) is inconsistent, then unit propagation derives a contradiction from \( \varphi(x, y) \land \alpha \).

- **Encoding \( \varphi \) implements domain consistency** (called also generalized arc consistency (GAC) or a propagator in [8]) if for every partial assignment \( \alpha \) to the main variables \( x \) and every literal \( l \) on a main variable, such that \( f(x) \land \alpha \models l \), unit propagation on \( \varphi(x, y) \land \alpha \) derives \( l \) or a contradiction.
• Encoding \( \varphi \) implements unit refutation completeness (called also unit refutation complete or URC encoding, [18]) if for any partial assignment \( \alpha \) to the variables \( x \cup y \), such that \( \varphi(x, y) \land \alpha \models \bot \), unit propagation derives a contradiction from \( \varphi \land \alpha \).

• Encoding \( \varphi \) implements propagation completeness (called also propagation complete or PC encoding, [10]) if for any partial assignment \( \alpha \) to the variables \( x \cup y \) and any literal \( l \) on these variables, such that \( \varphi(x, y) \land \alpha \models l \), unit propagation on \( \varphi \land \alpha \) derives \( l \) or a contradiction.

Clearly, a propagation complete encoding implements domain consistency and a unit refutation complete encoding implements consistency. By [27], the class of URC formulas coincides with the class SLUR introduced in [37]. Encodings which enforce GAC (domain consistency) by unit propagation were considered e.g. in [6, 8].

Encoding of a DNNF that implements consistency is relatively straightforward, since by decomposability propagating zeros from the inputs to the output is sufficient for testing consistency of partial assignments of the inputs. The basic idea how to obtain an encoding of a smooth DNNF of linear size implementing domain consistency (enforcing GAC by unit propagation), appeared first in [36] for a structure representing context free grammars. Although the authors do not mention it, the structure is a special case of a smooth DNNF and the construction of its encoding enforcing domain consistency by unit propagation can be generalized to an arbitrary smooth DNNF. This generalization is described in [31], although the necessary assumption of smoothness is not explicitly mentioned. This assumption is used in [23] and [1] where the description includes also the clauses that are needed, if some of the literals on input variables do not appear in the DNNF.

In [1], an encoding implementing propagation completeness for a function given by an MDD is presented. A decision diagram is a special case of a DNNF if we rewrite each decision node with a disjunction of two conjunctions in the standard way. The authors of [1] posed a question whether a propagation complete encoding of a sentential decision diagram [15] or some more general representation within the class of NNFs can be determined. In this paper, we present a polynomial time construction of such an encoding for a general DNNF and its generalization for variables with arbitrary finite domains which was also considered in [23] and which we call multivalued DNNF. For technical reasons, the construction is formulated for a smooth (multivalued) DNNF. Asymptotically, this is not a strong restriction, since a DNNF can be transformed into a smooth DNNF with a polynomial increase of size [14] and the same holds for a multivalued DNNF. The presented encoding was used for a construction of a propagation complete encoding for a related model more expressive than DNNF in [32].

For our construction, we need an encoding of the at-most-one constraint and the exactly-one constraint of linear size that can be used to replace a prime representation of these constraints inside a larger encoding while preserving the propagation strength of the whole encoding. We show that the usual 2-CNF encodings of the at-most-one constraint and the ladder encoding of the exactly-one constraint [29] satisfy this property. On the other hand, we demonstrate that the encoding of exactly-one constraint
obtained by combining an encoding of at-most-one constraint with the disjunction of all the variables is not suitable for this purpose.

In Section 2, we recall the necessary notions and terminology. Section 2.3 demonstrates an example of a smooth d-DNNF, for which the known encodings implementing domain consistency are not propagation complete. Section 3 introduces multivalued DNNF as a generalization of a DNNF for variables with finite domains, defines the notion of a separator cover used in our construction, and describes a URC and a PC encoding of a multivalued DNNF which is the main result. Sections 4 introduces the notions which we use to analyze our encodings. Section 5 describes a method how to obtain a separator cover. The proof of the main result is in Section 6. In Section 7, we prove the properties of encodings of the at-most-one and the exactly-one constraint used in our construction and estimate the size of the encoding of multivalued DNNF with encodings of at-most-one and exactly-one of linear size. Section 8 formulates some questions for further research.

2 Definitions

In this section, we recall the main notions used in our paper.

2.1 Propagation and Unit Refutation Complete Encodings

A formula in conjunctive normal form (CNF formula) is a conjunction of clauses. A clause is a disjunction of a set of literals and a literal is a variable \( x \) (positive literal) or its negation \( \neg x \) (negative literal). Given a set of variables \( x \), \( \text{lit}(x) \) denotes the set of literals on variables in \( x \).

A partial assignment \( \alpha \) of values to variables in \( x \) is a subset of \( \text{lit}(x) \) that does not contain a complementary pair of literals, so we have \( |\alpha \cap \text{lit}(x)| \leq 1 \) for each \( x \in x \). We identify a set of literals \( \alpha \) (in particular a partial assignment) with the conjunction of these literals if \( \alpha \) is used in a formula such as \( \varphi(x) \wedge \alpha \). A mapping \( a : x \rightarrow \{0,1\} \) or, equivalently, \( a \in \{0,1\}^x \) represents a full assignment of values to \( x \). Alternatively, a full assignment can be represented with the set of literals satisfied by the assignment. We use these representations interchangeably.

We consider encodings of boolean functions defined as follows.

**Definition 2.1 (Encoding).** Let \( f(x) \) be a boolean function on variables \( x = (x_1, \ldots, x_n) \). Let \( \varphi(x,y) \) be a CNF formula on \( n + m \) variables where \( y = (y_1, \ldots, y_m) \). We call \( \varphi \) a CNF encoding of \( f \) if for every \( a \in \{0,1\}^x \) we have

\[
f(a) \equiv \left( \exists b \in \{0,1\}^y \right) \varphi(a, b) .
\]

The variables in \( x \) and \( y \) are called main variables and auxiliary variables, respectively.

We are interested in encodings which are propagation complete or at least unit refutation complete. These notions rely on unit propagation which is a well known procedure in
SAT solving \[^9\]. For technical reasons, we represent unit propagation using the following two rules

\[
l \lor l_1 \lor \cdots \lor l_k, -l_1, \ldots, -l_k \vdash l
\]

\[
l, -l \vdash \bot.
\]

Derivation of a contradiction and derivation of the literals in non-contradictory cases yields the same result using these rules and using the unit propagation implemented in a SAT solver. In contradictory cases, the set of the literals derived together with a contradiction may be different. We say that a literal \(l\) or \(\bot\) can be derived from \(\varphi\) by unit propagation and denote this fact with \(\varphi \vdash l\) or \(\varphi \vdash \bot\), respectively, if the unit clause \(l\) or \(\bot\), respectively, is in \(\varphi\) or can be derived from \(\varphi\) by a series of applications of the rules (2) and (3).

We say that a set of literals, in particular, a partial assignment \(\alpha\) is closed under unit propagation in \(\varphi\), if for every literal \(l\) we have that \(\varphi \land \alpha \vdash l\) implies \(l \in \alpha\). The closure under unit propagation of a set of literals \(\alpha\) is the smallest superset of \(\alpha\) that is closed under unit propagation. If \(\varphi \land \alpha \not\vdash \bot\), then \(\alpha\) and its closure are partial assignments.

The notion of a propagation complete CNF formula was introduced in \[^10\] as a strengthening of a unit refutation complete CNF formula introduced in \[^18\]. These notions allow to distinguish different levels of propagation strength depending on the type of propagation (URC or PC) and the set of variables involved in the propagation.

**Definition 2.2.** Let \(\varphi(x,y)\) be a CNF encoding of a boolean function defined on a set of variables \(x\) and let \(v \subseteq x \cup y\) be non-empty.

- We say that the encoding \(\varphi\) is unit refutation complete (URC) on the variables \(v\), if the following implication holds for every partial assignment \(\alpha \subseteq \text{lit}(v)\)

\[
\varphi(x,y) \land \alpha \models \bot \implies \varphi(x,y) \land \alpha \vdash \bot
\]

(4)

- We say that the encoding \(\varphi\) is a propagation complete (PC) on the variables \(v\), if for every partial assignment \(\alpha \subseteq \text{lit}(v)\) and every \(l \in \text{lit}(v)\), such that

\[
\varphi(x,y) \land \alpha \models l
\]

(5)

we have

\[
\varphi(x,y) \land \alpha \vdash l
\]

or

\[
\varphi(x,y) \land \alpha \vdash \bot.
\]

(6)

- If an encoding is URC on the variables \(v = x\), we say that it implements consistency (by unit propagation).

- If an encoding is PC on the variables \(v = x\), we say that it implements domain consistency (by unit propagation).

- If an encoding is URC or PC on the variables \(v = x \cup y\), we say that it is URC or PC, respectively.
Given a set of literals \( A \), we use \( \text{AMO}(A) \) to denote the \textit{at-most-one} constraint which is satisfied if and only if at most one of the literals in \( A \) is satisfied. If \( A \) is specified within the specification of the constraint, we usually drop the curly brackets (e.g. we write \( \text{AMO}(x_1, \neg x_2, x_3) \) instead of \( \text{AMO}([x_1, \neg x_2, x_3]) \)). We use \( \text{amo}(A) \) to denote the CNF representation of \( \text{AMO}(A) \) which consists of all clauses \( \neg l_1 \lor \neg l_2, l_1, l_2 \in A, l_1 \neq l_2 \). Since \( \text{amo}(A) \) consists of all prime implicates of \( \text{AMO}(A) \), it is propagation complete.

Similarly, we use \( \text{EO}(A) \) to denote the \textit{exactly-one} constraint which is satisfied if and only if exactly one of the literals in \( A \) is satisfied. We use \( \text{eo}(A) \) to denote the CNF representation of \( \text{EO}(A) \) consisting of \( \text{amo}(A) \) together with the clause \( \bigvee l \in A l \). Since \( \text{eo}(A) \) consists of all prime implicates of \( \text{EO}(A) \), it is propagation complete.

### 2.2 Decomposable Negation Normal Form

The notion of a DNNF was introduced in \[13\] as a restricted NNF. A sentence in \textit{negation normal form} (NNF) \( D \) is a rooted DAG with vertices \( V \), root \( \rho \in V \), the set of edges \( E \), and the set of leaves \( L \subseteq V \). The inner vertices in \( V \) are labeled with \( \land \) or \( \lor \) and they represent connectives or gates in a monotone circuit. Each edge \((v, u)\) in \( D \) connects an inner vertex \( v \) labeled \( \land \) or \( \lor \) with one of its inputs \( u \). The edge is directed from \( v \) to \( u \), so the inputs of a vertex are its successors (or child vertices). The leaves are labeled with constants 0 or 1, or literals \( l \in \text{lit}(x) \) where \( x \) is a set of variables.

For a vertex \( v \in V \), let us denote \( \text{var}(v) \) the set of variables from \( x \) that appear in the leaves which can be reached from \( v \) by a directed path. More precisely, a variable \( x \in x \) belongs to \( \text{var}(v) \) if there is a directed path from \( v \) to a leaf vertex labeled with a literal from \( \text{lit}(x) \).

**Definition 2.3** \([14]\). We define the following structural restrictions of NNFs.

- We say that NNF \( D \) is \textit{decomposable} (DNNF), if for every vertex \( v = u_1 \land \cdots \land u_k \)
  we have that the sets of variables \( \text{var}(u_1), \ldots, \text{var}(u_k) \) are pairwise disjoint.

- We say that DNNF \( D \) is \textit{smooth} if for every vertex \( v = u_1 \lor \cdots \lor u_k \)
  we have \( \text{var}(v) = \text{var}(u_1) = \cdots = \text{var}(u_k) \).

Decomposability is a strong restriction. In particular, the satisfiability test is polynomial for a DNNF, while it is NP-complete for a general NNF. On the other hand, it is possible to transform a general DNNF into an equivalent smooth DNNF with a polynomial increase of size \[14\]. For example, a simple disjunction \( x_1 \lor x_2 \) is a DNNF that is not smooth. We can form an equivalent smooth DNNF by adding trivial subformulas to obtain \( x_1 \land (x_2 \lor \neg x_2) \lor (x_1 \lor \neg x_1) \land x_2 \).

### 2.3 Propagation Strength of Encodings of DNNFs and MDDs

The known encodings of DNNF implementing domain consistency are different from Tseitin encoding of a circuit in that the models of these encodings do not represent the exact computations. The models are assignments of values to gates which assign the value 1 to the output gate and to some of the gates satisfied in the computation.
The gates set to 1 are chosen so that the model is sufficient to certify that the output is 1, however, in general, not all of the gates satisfied in the computation are needed to certify the output. This modification appeared to be useful to construct encodings implementing domain consistency. Our PC encoding goes further in this direction in the sense that the models of the encoding are restricted to the minimal certificates of satisfiability of the circuit which is not the case of the previous encodings of DNNFs.

Let us point out that some modification of the set of models of a Tseitin encoding is not only useful, but necessary to obtain a URC encoding of a DNNF. It is a well-known fact that testing satisfiability of a conjunction of two DNNFs is NP-complete. See, for example, the proof of the fact that DNNFs do not satisfy bounded conjunction closure in [16]. If $F(x)$ and $G(x)$ are DNNFs, then $F \lor G$ is also a DNNF and its Tseitin encoding contains vertices $v_F$ and $v_G$ corresponding to the outputs of $F$ and $G$. If an encoding $\varphi$ of $F \lor G$ extends Tseitin encoding by additional clauses and is URC, then $\varphi \land \alpha \land v_F \land v_G$, where $\alpha$ is a partial assignment of the main variables $x$, should derive a contradiction by unit propagation if and only if $F(x) \land G(x) \land \alpha$ is unsatisfiable. Unless P is equal to NP, this cannot be achieved by additional clauses computable in polynomial time. On the other hand, the known encodings of DNNF avoid this problem and can be extended to a URC or even a PC encoding in polynomial time. The encodings contain the variables $v_F$ and $v_G$, however, they do not guarantee that both these variables are satisfied in every model representing a satisfying computation of $F(x) \lor G(x)$ for an assignment of the main variables $x$ which additionally satisfies $F(x) \land G(x)$.

Let us briefly describe an example of a DNNF for which the FullNNF encoding described in [1] is not PC. Assume $D$ is a smooth DNNF with vertices $V$, root $\rho \in V$, the set of edges $E$, and the set of leaves $L \subseteq V$. Let us assume that $D$ represents boolean function $f(x)$ and the leaves of $D$ are associated with the literals lit($x$). FullNNF encoding of $D$ is a CNF formula $\psi(x, v)$ where $v$ are auxiliary variables corresponding to the inner vertices of $D$. It consists of clauses [N1] to [N4] described in Table 1 and the unit clause $\rho$.

### Example 2.4.

Figure 1 presents a DNNF for a boolean function, for which FullNNF described in [1] is not a PC encoding, although it implements domain consistency. The boolean input variables are $x_1, x_2, x_3, x_4, x_5$ and the output is $\rho$.

Encoding FullNNF uses main variables $x_1, \ldots, x_5$ and auxiliary variables which represent the inner gates of the DNNF. Note that partial assignment $d_1 \land d_2$ is contradictory,

| group | clause | condition |
|-------|--------|-----------|
| N1    | $v \rightarrow v_1 \lor \cdots \lor v_k$ | $v = v_1 \lor \cdots \lor v_k$ |
| N2    | $v \rightarrow v_i$ | $v = v_1 \land \cdots \land v_k$, $i = 1, \ldots, k$ |
| N3    | $v \rightarrow p_1 \lor \cdots \lor p_k$ | $v$ has incoming edges from $p_1, \ldots, p_k$ |
| N4    | $\neg l \notin L$ | |

Table 1: Clauses of the FullNNF encoding.
Figure 1: An example DNNF. The root $\rho$ together with vertices $a_1$ and $a_2$ represents a decision vertex on variable $x_5$. Vertex labeled $d_1$ represents disjunctive normal form (DNF) $x_1x_2 \lor \overline{x_1}\overline{x_2}$ (where we use the usual compressed form of conjunctions of literals) which is equivalent to condition $x_1 = x_2$. Vertex $d_2$ represents DNF $\overline{x_1}x_2 \lor x_1\overline{x_2}$ which is equivalent to $x_1 \neq x_2$. Similarly, vertex $d_3$ represents $x_3x_4 \lor \overline{x_3}\overline{x_4}$ ($x_3 = x_4$) and vertex $d_4$ represents $x_3\overline{x_4} \lor \overline{x_3}x_4$ ($x_3 \neq x_4$).
since we cannot have simultaneously \( x_1 = x_2 \) and \( x_1 \neq x_2 \). However, unit propagation does not derive any literals except \( d_1 \), \( d_2 \) and \( \rho \), in particular it does not derive the contradiction. Let us look at the clauses in FullNNF which contain variables \( d_1 \) and \( d_2 \).

- Since \( d_1 \) and \( d_2 \) are \( \lor \) gates, they are in the following clauses of group \( N_1 \):
  \[
  d_1 \rightarrow e_1 \lor e_2 \\
  d_2 \rightarrow e_3 \lor e_4
  \]

- Since \( d_1 \) and \( d_2 \) are inputs to \( \land \) gates \( c_1 \) and \( c_2 \), they are in the following clauses of group \( N_2 \):
  \[
  c_1 \rightarrow d_1 \\
  c_2 \rightarrow d_2
  \]

- In addition, \( d_1 \) and \( d_2 \) are in the following clauses of group \( N_3 \):
  \[
  d_1 \rightarrow c_1 \lor c_4 \\
  e_1 \rightarrow d_1 \\
  e_2 \rightarrow d_1 \\
  d_2 \rightarrow c_2 \lor c_3 \\
  e_3 \rightarrow d_2 \\
  e_4 \rightarrow d_2
  \]

None of the clauses listed above becomes unit or empty after satisfying \( d_1 \) and \( d_2 \) and, hence, unit propagation cannot be applied to the formula. In particular, the contradiction is not derived which implies that FullNNF is not a URC encoding.

The vertices \( d_1 \) and \( d_2 \) in Example 2.4 are DNFs, however, they can be any mutually excluding DNNFs defined on the same set of variables. As pointed out before, testing satisfiability of a conjunction of two DNNFs is an NP-complete problem. This means that there is no known construction of an encoding of polynomial size that derives a contradiction from \( d_1 \land d_2 \), if and only if this assignment is contradictory with a partial assignment of the inputs. Instead, we use the fact that FullNNF encoding does not require that a variable representing a gate is assigned 1, if the gate evaluates to 1, and extend the encoding so that the partial assignment \( d_1 \land d_2 \) always leads to a contradiction. Our approach is a generalization of the idea used in the construction of CompletePath encoding of MDDs which we describe next.

CompletePath encoding introduced in [1] is a PC encoding of a multivalued decision diagram (MDD) generalizing an ordered binary decision diagram (OBDD) to arbitrary finite domains. Its construction uses the fact that an MDD is satisfied if and only if there is a path from the root to a leaf labeled 1 which consists only of activated edges. The encoding uses auxiliary variables for the edges and vertices of the MDD and consists of the following two parts.

- Clauses that describe local conditions implying that a model of the encoding represents the set of vertices and edges of the unique accepting path, if it exists.

- Exactly-one constraints implying that at every level of the MDD, exactly one vertex belongs to the model. These clauses are implied by the previous group of clauses, however, they are needed to achieve propagation completeness.
There are encodings of MDD whose models contain more vertices and edges than only those that belong to the unique accepting path, however, they are not propagation complete. Our PC encoding takes the idea of using exactly-one constraints to achieve propagation completeness of the encoding of the paths used in CompletePath and generalizes it to the case of DNNFs. The situation of DNNFs is different in that a minimal certificate of its satisfiability does not have a form of a path, but a tree, because we must make sure that every child of a satisfied \( \land \)-vertex is also satisfied. We call this tree a minimal satisfying subtree and introduce it in Section 4. The encoding of minimal satisfying subtrees that we describe later uses exactly-one constraints on specific subsets of the vertices of the DNNF. In particular, the vertices \( d_1 \) and \( d_2 \) in Figure 1 belong to one of such sets and, hence, the encoding implies \( \neg d_1 \lor \neg d_2 \).

3 Multivalued DNNFs and the Main Result

In this section, we shall describe a generalization of DNNFs to multivalued domains which was earlier considered in [23]. We shall call this generalization a multivalued DNNF (MDNNF) to distinguish it from the usual DNNF which is used to represent a boolean function. Later in this section, we formulate the main result of our paper which is the fact that we can construct a PC encoding for MDNNFs, the result holds for DNNFs on boolean variables as well.

Consider a set of variables \( x = (x_1, \ldots, x_n) \) where the domain of \( x_i \) denoted \( \text{dom}(x_i) \) is a non-empty finite set. A constraint \( f(x) \) is a mapping

\[
f : \text{dom}(x_1) \times \cdots \times \text{dom}(x_n) \to \{0, 1\},
\]

where 0 and 1 represent the truth values. In order to describe a tractable representation of such constraints, we introduce multivalued DNNF. The difference between a DNNF and a multivalued DNNF is the interpretation of the leaves, otherwise, they are the same.

Assume \( D \) is an acyclic directed graph as in a NNF with vertices \( V \), root \( \rho \in V \), the set of edges \( E \), and the set of leaves \( L \subseteq V \) in such that the inner vertices are labeled by \( \lor \) and \( \land \). We say that \( D \) is a multivalued NNF, if the leaves \( L \) are labeled with unary constraints \( x_i = a \) instead of the literals. We assume that each unary constraint \( x_i = a \) is used as a label of at most one leaf. Some of the unary constraints may be missing in \( D \), however, we assume that for each \( i = 1, \ldots, n \) at least one unary constraint on the variable \( x_i \) appears in \( D \).

If all the constraint variables \( x \) are boolean and we identify the unary constraint \( x_i = 1 \) with the literal \( x_i \) and the unary constraint \( x_i = 0 \) with the literal \( \neg x_i \), then NNF is a special case of a multivalued NNF.

For simplicity, we assume that no leaf of \( D \) is labeled with a constant unless \( D \) is a single vertex representing a constant function. This can be done without loss of generality, since one can always simplify \( D \) by propagating constant values in the leaves, if they are not the root. For the construction of our encodings, we assume that the root of \( D \) is not a constant.
We say that a multivalued NNF $D$ represents a constraint $f(x)$ if for every assignment $a$ of $x$ we have that $D$ evaluates to $f(a)$ if each leaf is evaluated according to the unary constraint in its label.

Following [23], decomposability and smoothness are required with respect to constraint variables $x_1, \ldots, x_n$. For a vertex $v \in V$ let us denote $\text{var}(v)$ the set of the variables from $x$ that appear in the leaves which can be reached from $v$ by a directed path. More precisely, a variable $x_i \in x$ belongs to $\text{var}(v)$ if there is a directed path from $v$ to a leaf labeled with a unary constraint $x_i = a$ for a value $a \in \text{dom}(x_i)$. In particular, by assumption we have that $\text{var}(\rho) = x$.

**Definition 3.1.** We define the following structural restrictions of multivalued NNFs.

- We say that multivalued NNF $D$ is **decomposable** (multivalued DNNF, MDNNF), if for every vertex $v = u_1 \land \cdots \land u_k$ the sets of variables $\text{var}(u_1), \ldots, \text{var}(u_k)$ are pairwise disjoint.

- We say that MDNNF $D$ is **smooth** if for every vertex $v = u_1 \lor \cdots \lor u_k$ we have $\text{var}(v) = \text{var}(u_1) = \cdots = \text{var}(u_k)$.

Requiring decomposability and smoothness is essential to our construction. Decomposability is a strong restriction. In particular, the satisfiability test is polynomial for MDNNF, while it is NP-complete for a general (multivalued) NNF. On the other hand, for a given non-smooth MDNNF, we can construct an equivalent smooth one in polynomial time by a simple generalization of the algorithm for DNNFs described in [14].

Smoothness is a property which simplifies interpretation of the models of a multivalued DNNF as sets of satisfied leaves. Let us also point out that when using an MDD to implement a constraint in CP in a way supporting explanations of the conflicts, it is suggested to use a complete MDD where each path tests all variables [24]. An MDD can be interpreted as a MDNNF with a special structure and then, the MDD is complete if and only if the corresponding MDNNF is smooth.

An important step of our construction is a construction of a cover by separators. A simple form of such a cover is used in order to achieve propagation completeness in CompletePath [1] encoding for MDDs where the cover by separators is formed by a partition of an acyclic graph into levels (layers). This is sufficient, since the computation of an MDD is represented by a single path. For MDNNF, the computation has more complex structure, however, the idea of a partition into levels generalized to a cover by separators can be used, if for each variable $x_i$, we consider the subgraph of $D$ induced on the scope of $x_i$ separately.

Assume an MDNNF $D$ and a variable $x_i$. The scope of $x_i$ in $D$ is the set $V_i$ of vertices $v$ satisfying $x_i \in \text{var}(v)$. Let $D_i$ be the subgraph of $D$ induced on $V_i$. Moreover, let us denote $L_i = L \cap V_i$ which is the set of the leaves of $D_i$. By assumptions on $D$, we have $1 \leq |L_i| \leq |\text{dom}(x_i)|$. Note that if $v \in V_i$ is labeled with $\land$, then by decomposability exactly one of the successors of $v$ belongs to $V_i$ as well. On the other hand, if $v \in V_i$ is labeled with $\lor$, then by smoothness all the successors of $v$ belong to $V_i$. If a vertex belongs to $V_i$, then all its predecessors in $V$ belong to $V_i$ as well.
Example 3.2. Consider DNNF $D$ from Figure 1. Then $D_5$ is the induced subgraph on vertices $V_5 = \{ \rho, a_1, a_2, x_5, \neg x_5 \}$ and $D_1$ is the induced subgraph on vertices $V_1 = \{ \rho, a_1, a_2, b_1, b_2, c_1, c_2, c_3, d_1, d_2, e_1, e_2, e_3, e_4, x_1, \neg x_1 \}$. Note also that $D_2$ differs from $D_1$ only by including the leaves $x_2$ and $\neg x_2$ instead of $x_1$ and $\neg x_1$.

Definition 3.3.

- A subset of vertices $S \subseteq V_i$ is called a separator in $D_i$, if every path in $D_i$ from the root to a leaf contains precisely one vertex from $S$.

- We say that $D$ is covered by separators, if for each $i = 1, \ldots, n$, there is a collection of separators $S_i$ in $D_i$, such that the union of $S \in S_i$ is $V_i$.

Not every MDNNF can be covered by separators. However, every MDNNF can be efficiently transformed into an equivalent one which admits a separator cover, see Section 5 for more detail.

The size of the encoding constructed using a separator cover depends on the total size of all separators. Without loss of generality, we can assume that this total size is polynomial by the following argument. For each $i \in \{1, \ldots, n\}$ and each vertex $v \in D_i$ choose a separator $S \in S_i$ containing $v$. The chosen separators form a subset of the original cover which is itself a separator cover. For each $i = 1, \ldots, n$, it consists of at most $|V_i|$ separators in $D_i$ each of size at most $|V_i|$ and, hence, the cover has polynomial total size. A better estimate will be formulated in Proposition 5.2.

A complete MDD or an MDD with no long edges as in [1] can be considered as a special case of a smooth strictly leveled MDNNF. An MDNNF is strictly leveled if for each $v \in V$, all paths from $\rho$ to $v$ have the same length. For such an MDNNF, there is a cover by separators in which for every $i \in \{1, \ldots, n\}$, every separator in $D_i$ consists of all vertices at a given level and the leaves above this level.

Example 3.4. Consider DNNF $D$ from Figure 1. Recall from Example 3.2 that $D_1$ is the induced subgraph on vertices $V_1 = \{ \rho, a_1, a_2, b_1, b_2, c_1, c_2, c_3, c_4, d_1, d_2, e_1, e_2, e_3, e_4, x_1, \neg x_1 \}$. Since $D$ is strictly leveled, we can form a separator cover of $D_1$ with separators being the levels of $D_1$. In particular $S_1$ is formed by the following separators

\[
\begin{align*}
S_\rho &= \{ \rho \} \\
S_b &= \{ b_1, b_2 \} \\
S_d &= \{ d_1, d_2 \} \\
S_x &= \{ x_1, \neg x_1 \}
\end{align*}
\]

However, this is not the only way of defining a separator cover, we can replace $S_d$ and $S_e$ with the sets

\[
\begin{align*}
S_1 &= \{ e_1, e_2, d_2 \} \\
S_2 &= \{ d_1, e_3, e_4 \}.
\end{align*}
\]
Our goal is to construct a CNF encoding of constraint (7) represented by an MDNNF. We encode the vector \( \mathbf{x} \) with the boolean variables representing the direct encoding (see for example [1, 8]) of the elements of finite domains. This encoding uses the domain variables \( \mathbf{J} \), where \( a \in \text{dom}(x_i) \), which are assumed to represent the truth value of the corresponding constraint \( x_i = a \). The vector of the domain variables related to \( x_i \) will be denoted \( \text{dvar}(x_i) \) and the vector of all the domain variables will be denoted \( \text{dvar}(\mathbf{x}) \). If we say that an encoding uses the direct encoding of variables, then we assume that it contains the direct encoding constraints which are the exactly-one constraint for the block of variables \( \text{dom}(x_i) \) for every \( i = 1, \ldots, n \). If an assignment of \( \text{dvar}(\mathbf{x}) \) satisfies the direct encoding constraints, we say that the assignment is direct encoding consistent (DE-consistent).

**Definition 3.5.** A CNF formula \( \varphi(\text{dvar}(\mathbf{x}), y) \) is an encoding of the constraint (7) using the direct encoding of the variables \( \mathbf{x} \), if it is an encoding of the boolean function \( f_{de}(\text{dvar}(\mathbf{x})) \) defined as follows

- every model of \( f_{de} \) is DE-consistent,
- a DE-consistent assignment of \( \text{dvar}(\mathbf{x}) \) is a model of \( f_{de} \) if and only if it represents a model of (7).

The encoding FullNNF [1] of a boolean function represented by a DNNF can be easily adapted to an encoding of a multivalued DNNF using the approach described in [23]. Recall that FullNNF consists of the unit clause \( \rho \) for the root of a DNNF and clauses of groups \( N1 \) to \( N4 \) described in Table 1. For an MDNNF \( D \), we use variables \( [x_i = a] \) to represent the unary constraints in the leaves and variables that represent the inner vertices (gates) of \( D \). The encoding consists of the direct encoding constraints for the variables \( \text{dvar}(\mathbf{x}) \), the clauses \( N1 \) to \( N3 \) from Table 1 and instead of clauses \( N4 \) we include unit clauses \( \neg[x_i = a] \) for the unary constraints \( x_i = a \) that are not associated with any leaf of \( D \). We denote the last group of clauses as \( N4' \) in Table 2. Since this is a straightforward modification of the original FullNNF encoding, we keep the name and refer to this modification as FullNNF encoding of a multivalued DNNF. Our encoding is based on this generalization of FullNNF, uses the same variables, and extends it by clauses which restrict its models.

We can now formulate our main result.

**Theorem 3.6.** A multivalued DNNF representing a constraint (7) can be converted in polynomial time into a URC and a PC encoding of the constraint using the direct encoding of the input variables \( \mathbf{x} \).

Every PC encoding is also URC, however, we include both a URC and a PC encoding in the theorem, since we shall construct them as different encodings. There is only a minor difference between the proof of URC and PC property for the basic form of the encodings, however, the constructions of the smaller variants of the encodings in Section 7 use quite different arguments. The proof is constructive and the construction of the encodings from an input MDNNF \( D^0 \) consists of the following three steps:
Table 2: Clauses used in the construction of our URC and PC encodings of a smooth MDNNF $D^c$ with a fixed collection $S_i$ of separators in each $D^c_i$.

(a) Construct a smooth MDNNF $D^s$ equivalent to $D^0$,

(b) extend $D^s$ to an MDNNF $D^c$ for which a separator cover can be obtained and construct such a cover,

(c) construct a URC and a PC encoding of $D^c$ using the chosen separator cover.

Step (a) can be carried out in polynomial time by a simple generalization of the algorithm for DNNFs described in [14]. Note that at most $n(s + d)$ additional vertices and edges are added during the construction, where $s$ is the number of the vertices of $D^0$ and $d$ is the maximum of $|\text{dom}(x_i)|$ over $i = 1, \ldots, n$. Step (b) can be carried out in polynomial time by Proposition 5.2.

In order to construct CNF encodings in step (c), denote by $V$ the set of the vertices of $D^c$. All the vertices in $V$ are considered as boolean variables and the vector of these variables will be denoted $v$. The list of clauses used in our encodings is presented in Table 2. All the groups of clauses in the table except $N4'$ are formulated using the variables as elements of $v$ for simplicity. However, the variables in the inner vertices and the variables in the leaves are treated differently in the encoding. The variables represented by inner vertices are used in the encoding themselves and the vector of these variables will be denoted $y \subseteq v$. The variables in the leaves are the variables $v \setminus y = \bigcup_{i=1}^{n} L_i$ and they are identified with the corresponding variables from $\text{dvar}(x_i)$. In particular, we have $L_i \subseteq \text{dvar}(x_i)$. Some of the variables from $\text{dvar}(x)$ may not appear as leaves of $D^c$ and these are forced to 0 by the clauses in group $N4'$.

As already mentioned, our encodings are based on a generalization of FullNNF encoding which is formed by clauses in groups $N1$, $N3$, $N4'$ in Table 2, the unit clause $\rho$, and the direct encoding constraints. Let us consider the following extensions of this encoding by further clauses from Table 2:

$\psi_c(\text{dvar}(x), y)$ consists of FullNNF of $D^c$ and the clauses in group $N5$,

$\psi_p(\text{dvar}(x), y)$ consists of FullNNF of $D^c$ and the clauses in group $N6$.
The use of the at-most-one and exactly-one constraints is motivated by CompletePath encoding \([1]\) as explained in Section 2.3. The effect of these constraints in an MDNNF is analyzed in Section 4 using the notion of a minimal satisfying subtree of \(D^c\). In Section 6, we prove that \(\psi_c\) is a URC encoding of \(D^c\), so it can verify consistency efficiently. Similarly, we prove that \(\psi_p\) is a PC encoding of \(D^c\), so it guarantees efficient propagation.

**Remark 3.7.** The FullNNF encoding contains the direct encoding constraints, since they are important to guarantee domain consistency on the direct encoding variables. The encodings \(\psi_c\) and \(\psi_p\) are extensions of FullNNF, so they contain direct encoding constraints as well. On the other hand, these constraints are redundant in both encodings in the sense that they are consequences of the remaining parts of the encodings and they are not needed for the proof of the claimed propagation strength. In particular, \(\psi_p\) is a PC encoding of \(f_{de}\) even without explicitly added direct encoding constraints, so these constraints do not increase propagation strength (they are absorbed by the other clauses in the encoding in the sense of \([10]\)). Moreover, if the leaves \(L_i\) of the DNNF for some \(1 \leq i \leq n\) contain all possible values of \(\text{dom}(x_i)\) and \(L_i\) is chosen as a separator, then the direct encoding constraints for \(\text{dvar}(x_i)\) are precisely the clauses \([N6]\) for this separator. The encoding \(\psi_c\) is a URC encoding of \(f_{de}\) even without explicitly added direct encoding constraints, however, these constraints can increase its propagation strength on the domain variables.

Let us look at an example how including the cardinality constraints on separators improves the strength of unit propagation.

**Example 3.8.** Consider DNNF \(D\) from Figure 1. In Example 2.4, we demonstrated that partial assignment \(d_1 \land d_2\) is contradictory and yet unit propagation on FullNNF with this partial assignment does not derive contradiction. Consider a cover which uses separators

\[
S_1 = \{e_1, e_2, d_2\} \\
S_2 = \{d_1, e_3, e_4\}
\]

introduced in Example 3.4. Let us show that \(\psi_c \land d_1 \land d_2 \vdash \bot\). Indeed, using \(\text{amo}(S_1)\) and \(d_2\), unit propagation derives \(\neg e_1\) and \(\neg e_2\). Using clause \(d_1 \rightarrow e_1 \lor e_2\) from group \([N1]\), unit propagation derives \(\neg d_1\) which together with \(d_1\) derives contradiction.

The exactly-one constraints in \(\psi_p\) offer stronger derivation properties. The encoding \(\psi_p\) contains FullNNF which semantically implies at-least-one condition on every separator. It follows that \(\psi_p \land \neg d_1 \vdash d_2\), however, unit propagation in FullNNF does not guarantee the corresponding derivations. Let us show that \(\psi_p \land \neg d_1 \vdash d_2\). Using clauses \(e_1 \rightarrow d_1\) and \(e_2 \rightarrow d_1\) from group \([N3]\), unit propagation derives \(\neg e_1\) and \(\neg e_2\). Then using eo\((S_1)\) which is part of \(\psi_p\), unit propagation derives \(d_2\). Note that the last step requires exactly-one constraints and the at-most-one constraint in \(\psi_c\) is not enough.

The encodings \(\psi_c\) and \(\psi_p\) contain prime representations of at-most-one and exactly-one constraints, respectively, for each separator. This is sufficient to prove that the time
complexity of the construction is polynomial, however, the encoding contains the above
-cardinality constraints used on large groups of variables and their prime representations
-have size quadratic in the number of the variables. The size of the encoding decreases, if
-linear size encodings of at-most-one and exactly-one constraints are used. In Section 7
-we present sufficient conditions under which this leads to a URC or a PC encoding and
in Section 7.4 we present estimates of the size of the resulting encodings.

A boolean variable $x_i$ has the binary domain $\text{dom}(x_i) = \{0, 1\}$, the variable $[x_i = 1]$ represents the positive literal $x_i$, and the variable $[x_i = 0]$ represents the negative literal $\neg x_i$. Let us demonstrate that the substitutions $[x_i = 1] \leftarrow x_i$ and $[x_i = 0] \leftarrow \neg x_i$ applied to the encodings from Theorem 3.6 preserve their propagation strength. Formally, the substitutions should be applied once the encoding is constructed, however, one can use the literals instead of the domain variables already during the construction, since the resulting encoding is the same.

**Corollary 3.9.** A DNNF representing a boolean function can be converted in polynomial
time into a URC and a PC encoding of the same function.

**Proof.** Assume, $f(x)$ is a boolean function. Let $\theta(\text{dvar}(x), y)$ be a URC or PC encoding of the boolean function $f_{\text{de}}(\text{dvar}(x))$ guaranteed by Theorem 3.6 or some of the formulas $\psi_c$ or $\psi_p$ with the direct encoding constraints removed. As argued in Remark 3.7, the clauses of direct encoding constraints can be removed without decreasing the required level of propagation strength.

Let us show that the formula $\theta''$ obtained from $\theta$ by the substitutions $[x_i = 1] \leftarrow x_i$ and $[x_i = 0] \leftarrow \neg x_i$ is an encoding of $f$ which is URC or PC, respectively. By definition of $f_{\text{de}}$, this formula is an encoding of $f$. In order to prove its propagation strength, it is more convenient to rename the variable $x_i$ to $[x_i = 1]$ in $\theta''$ and we denote $\theta'$ the formula after this renaming. Renaming a variable does not change the propagation strength and $\theta'$ can be obtained from $\theta$ by a simpler substitution, namely $[x_i = 0] \leftarrow \neg [x_i = 1]$ for all $i = 1, \ldots, n$.

Let $\Delta$ be the conjunction of the direct encoding constraints ($[x_i = 1] \lor [x_i = 0]$ \land
($\neg [x_i = 1] \lor \neg [x_i = 0]$) for $i = 1, \ldots, n$. Clearly, $\Delta \models \theta \equiv \theta'$ which implies that $\Delta \land \theta$ and $\Delta \land \theta'$ are equivalent.

Assume, $\theta$ is a PC formula. Since $\theta \equiv \theta \land \Delta$ according to Remark 3.7 and adding implicates to a PC formula does not decrease its propagation strength, we have that $\theta \land \Delta$ is PC as well. Let $\alpha \subseteq \text{lit}(\text{var}(\theta'))$ and $l \in \text{lit}(\text{var}(\theta'))$, where $\text{var}(\theta')$ is the set of variables of $\theta'$ and contains $[x_i = 1]$ as a replacement of $x_i$. Moreover, assume

$$\theta' \land \alpha \models l.$$

This implies

$$\Delta \land \theta' \land \alpha \models l$$

since $\Delta$ in this context is a definition of the variable $[x_i = 0]$ which is not used in $\theta'$, $\alpha$, and $l$. Since $\Delta \land \theta$ and $\Delta \land \theta'$ are equivalent, we obtain

$$\Delta \land \theta \land \alpha \models l$$

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and since $\theta \land \Delta$ is PC, we have

$$\Delta \land \theta \land \alpha \vdash l$$

or

$$\Delta \land \theta \land \alpha \vdash \bot.$$  

We can use $\Delta$ in unit propagation to derive $[x_i = 0]$ given $\neg[x_i = 1]$ and vice versa. Similarly, we can derive $\neg[x_i = 0]$ given $[x_i = 1]$ and vice versa. It follows that every unit propagation derivation in $\Delta \land \theta$ can be simulated in $\Delta \land \theta'$ and also vice versa. In particular, unit propagation derives the same set of literals from the formulas $\Delta \land \theta \land \alpha$ and $\Delta \land \theta' \land \alpha$. Hence, we have

$$\Delta \land \theta' \land \alpha \vdash l$$

or

$$\Delta \land \theta' \land \alpha \vdash \bot.$$  

It remains to show

$$\theta' \land \alpha \vdash l$$

or

$$\theta' \land \alpha \vdash \bot.$$  

A literal on the variable $[x_i = 0]$ for some $i$ can appear in a derivation from $\Delta \land \theta' \land \alpha$ only using some of the clauses $[x_i = 1] \lor [x_i = 0]$ and $\neg[x_i = 1] \lor \neg[x_i = 0]$ and both of these clauses are satisfied after this derivation step. It follows that the derived literal cannot be used to derive any new literal in a later step of the derivation. Hence, we can avoid the derivation of this literal and using the clauses of $\Delta$ in the derivation of $l$ or $\bot$.

If $\theta$ is a URC formula, we prove that $\theta'$ is a URC formula in a similar way, however, we consider only derivation of $\bot$.  

\[4\] Minimal Satisfying Subtrees and Separator Covers

The set of the variables of $\text{FullNNF}$ encoding of an MDNNF contains all $\text{dvar}(x)$ variables, however, the variables not used as leaves are forced to 0 by clauses $\text{N4}$. Hence, we can disregard them and identify models of $\text{FullNNF}$ with sets of vertices of an MDNNF corresponding to the variables with value 1 in the model.

The clauses of $\text{FullNNF}$ encoding guarantee that each of the models is a subset of the set of vertices satisfied in an accepting computation of an MDNNF. Moreover, the subgraph induced by this subset is sufficient as a certificate of the fact that the MDNNF is satisfied by an assignment of the input variables. This subgraph is not necessarily inclusion minimal. In particular, a $\lor$-vertex in the subgraph can have two or more successor vertices in the subgraph and one of them can be removed. In order to obtain propagation completeness, we extend $\text{FullNNF}$ by additional clauses which restrict the set of models of the encoding to inclusion minimal subgraphs that are certificates of satisfiability called a minimal satisfying subtree introduced below. Moreover, the resulting restrictions on the variables representing the vertices of a minimal satisfying subtree can be enforced by unit propagation.
In this section, \( D^s \) is an arbitrary smooth MDNNF and \( D^c \) is an arbitrary smooth MDNNF that is covered by separators. In both cases, the MDNNF has the set of vertices \( V \), root \( \rho \in V \), directed edges \( E \), and leaves \( L \subseteq V \) and represents constraint (7) on finite domain variables \( x = (x_1, \ldots, x_n) \). In particular, the leaves are associated with unary constraints of the form \( x_i = a \) for \( a \in \text{dom}(x_i) \). We denote \( D^s_i \) and \( D^c_i \) the subgraph of \( D^s \) and \( D^c \), respectively, induced on the scope of the variable \( x_i \).

**Definition 4.1.** A minimal satisfying subtree \( T \) of \( D^s \) is any subgraph of \( D^s \) which has the following properties:

1. \( T \) contains the root \( \rho \) of \( D^s \).
2. For every \( \wedge \)-vertex \( v \) in \( T \), all edges \( (v, u) \) in \( D^s \) are in \( T \).
3. For every \( \vee \)-vertex \( v \) in \( T \), exactly one of the edges \( (v, u) \) in \( D^s \) is in \( T \).
4. For every vertex \( v \) in \( T \), \( v \neq \rho \), there is an edge \( (u, v) \) in \( T \).

Condition 4 corresponds to clauses [N3] used in encodings of DNNFs that guarantee domain consistency. We could require exactly one incoming edge to \( v \), however, this is not necessary, since this is a consequence of the definition and the decomposability of \( D \), see Remark 4.3 below.

Let us note that a minimal satisfying subtree used as a certificate of satisfiability can be defined also as a minimal satisfied sub-DNNF, see e.g. [11]. The following lemma states a basic property of minimal satisfying subtrees that is important for our construction.

**Lemma 4.2.** If \( T \) is a minimal satisfying subtree of \( D^s \), then for each \( i = 1, \ldots, n \), \( T \cap D^s_i \) is a path from the root to a leaf.

**Proof.** Assume, \( T \) is a minimal satisfying subtree. Fix a variable \( x_i \). By assumption on \( D^s \), \( \rho \in D^s_i \). If \( v \in T \cap D^s_i \) and \( v \) is a \( \wedge \)-vertex, then by decomposability, exactly one of the successors of \( v \) in \( D^s \) is in \( D^s_i \) and by assumption on \( T \), the edge from \( v \) to this successor is in \( T \cap D^s_i \). If \( v \in T \cap D^s_i \) and \( v \) is a \( \vee \)-vertex, exactly one edge \( (v, u) \) is in \( T \), since \( T \) is a minimal satisfying subtree. By smoothness, \( u \in T \cap D^s_i \) and, since \( D^s_i \) is an induced subgraph, the edge \( (v, u) \) is in \( T \cap D^s_i \). It follows that there is exactly one maximal path \( P_i \) in \( T \cap D^s_i \) starting in the root. By Definition 4.1, the leaves of \( T \) are also leaves of \( D \) and thus the path ends in a leaf.

Every vertex \( v \in T \) is reachable from the root by a path in \( T \). Moreover, for every vertex \( u \) in this path, we have \( \text{var}(u) \supseteq \text{var}(v) \). It follows that every vertex \( v \in T \cap D^s_i \) is reachable from the root by a path in \( T \cap D^s_i \). Hence, \( T \cap D^s_i \) is equal to \( P_i \) and the proposition follows.

**Remark 4.3.** A minimal satisfying subtree is an out-arborescence (a rooted directed subtree) with root \( \rho \) and the leaves which are leaves of \( D \). This can be seen as follows. Lemma 4.2 implies that \( T \) is the union of paths \( T \cap D^s_i \) for \( i = 1, \ldots, n \). If paths \( P_i \) and \( P_j \) split at a vertex \( v \), then it is a \( \wedge \)-vertex and by decomposability, the parts of \( P_i \) and \( P_j \) after this vertex contain vertices with sets of variables which are disjoint subsets of \( \text{var}(v) \).
One can also verify that Lemma 4.2 can be reversed in the following sense. If $D^s$ is a smooth MDNNF and $T$ is an arbitrary subgraph of $D^s$ such that for every $i = 1, \ldots, n$ the intersection $T \cap D^s_i$ is a directed path from the root to a leaf in $L_i$, then $T$ is a minimal satisfying subtree of $D^s$. This can be considered as a basic idea behind the construction of our encodings. Later in Proposition 4.8 we formulate a similar argument in a form more suitable for the proof of the properties of the encodings.

The relationship between minimal satisfying subtrees and assignments of the variables $x$ is straightforward. By Lemma 4.2 every minimal satisfying subtree contains for every $i = 1, \ldots, n$ exactly one leaf containing the variable $x_i$. It follows that for every minimal satisfying subtree $T$ there is a unique assignment $a$ of $x$, such that $a_i \in \text{dom}(x_i)$ and the unary constraints in the leaves of $T$ are satisfied. More precisely, the leaves of $T$ are exactly the leaves of $D^s$ associated with the unary constraints satisfied by $a$. Moreover, we have the following.

**Proposition 4.4.** If $a$ is a total assignment of $x$, then $f(a) = 1$ if and only if there is a minimal satisfying subtree whose leaves are exactly the leaves of $D^s$ associated with the unary constraints satisfied by $a$.

**Proof.** Assume $f(a) = 1$ and consider the subgraph $G$ of $D^s$ induced by the vertices evaluating to 1 in the computation of $D^s$ for $a$. Then, a subset of $G$ satisfying the properties of the minimal satisfying subtree $T$ can be obtained by traversing $G$ top down. When a $\lor$-vertex $v$ is reached, an arbitrary edge $(v, u)$ in $G$ can be included into $T$.

For the opposite direction, assume a minimal satisfying subtree $T$ whose leaves are exactly the leaves of $D^s$ associated with the unary constraints satisfied by $a$. When evaluating $D^s$ on input $a$, all vertices of $T$ are evaluated to 1. Since $T$ contains the root, we have $f(a) = 1$. $\blacksquare$

It follows that the minimal satisfying subtrees can serve as certificates of acceptance by $D^s$ for assignments of the variables $x$. The models of the encodings which we construct correspond to the characteristic functions of the sets of vertices of minimal satisfying subtrees. Such an encoding will be called an *encoding of minimal satisfying subtrees*. It is sufficient to encode the set of vertices of a minimal satisfying subtree, since the assignment accepted by the tree depends only on its leaves.

A PC encoding of paths in an acyclic graph that can be partitioned into levels can be constructed using exactly-one constraints on the levels. This is used, for example, in CompletePath encoding for MDDs [1]. It appears that it is possible to generalize this approach to minimal satisfying subtrees in MDNNFs using Lemma 4.2 by which a minimal satisfying subtree is a union of paths, one for each variable $x_i$. In order to encode the condition that $T \cap D^s_i$ is a path we use a separator cover instead of a partition into levels. This allows to construct an encoding of an MDNNF without the requirement that it is strictly leveled.

**Remark 4.5.** In order to restrict the models of FullNNF encoding [1] to the minimal satisfying subtrees, a separator cover is not needed, if we do not require propagation.
completeness. The models of \textit{FullNNF} satisfy all the properties of a minimal satisfying subtree except that a $\lor$-vertex can have more than one successor. Let us discuss a smaller set of clauses that restricts the models to minimal satisfying subtrees, but is not propagation complete.

The conditions describing a minimal satisfying subtree in $D^s$ can be expressed in a straightforward way as a CNF formula whose variables are the vertices of $D^s$. To be exact, restricting the variables of the formula to the vertices and disregarding the edges of $D^s$ is correct only if $D^s$ does not contain a transitive edge. This is an edge $(v,u)$, such that $D^s$ contains also another path from $v$ to $u$. One can prove that a transitive edge is redundant in an MDNNF, so we can assume that $D^s$ does not contain a transitive edge.

The formula mentioned above contains \textit{FullNNF} and, additionally, the clauses representing the constraint $v \rightarrow \text{AMO}(u_1,\ldots,u_k)$ for every vertex $v = u_1 \lor \cdots \lor u_k$. The satisfying assignments of this formula correspond precisely to the characteristic functions of the sets of vertices of minimal satisfying subtrees.

One can show that in a smooth MDNNF a minimal satisfying subtree cannot contain two successors of the same $\lor$-vertex. It follows that we can strengthen the formula and use $\text{AMO}(u_1,\ldots,u_k)$ without the assumption $v$ for every vertex $v = u_1 \lor \cdots \lor u_k$. However, even this stronger formula is not suitable for our purposes, since it is not URC in general. For example, it is not URC for the DNNF from Example 2.4, since the formula is not URC for the DNNF from Example 2.4, since the formula

Instead of the above local approach to enforce minimal satisfying subtrees, we use a more global one using a separator cover. As explained at the beginning of this section, we can identify an assignment of values to the variables of \textit{FullNNF} encoding with a set of the vertices in $V$ satisfied by the assignment. We shall use the following property of \textit{FullNNF} to prove the main proposition of this section.

\textbf{Claim 4.6.} Let $M \subseteq V$ be a model of \textit{FullNNF} encoding of $D^s$ and let $T$ be the subgraph induced by $M$. For every $i \in \{1,\ldots,n\}$ and every $v_0 \in T \cap D^s_i$, there is a path in $T \cap D^s_i$ from the root to a leaf containing $v_0$.

\textit{Proof.} If $v \in T \cap D^s_i$ and $v \neq \rho$, then the clauses $\text{N3}$ guarantee that there is a vertex $u \in T$, such that $(u,v) \in T$. Moreover, $\text{var}(u) \supseteq \text{var}(v)$, so $u \in T \cap D^s_i$. By inductive use of this argument, we obtain a path in $T \cap D^s_i$ from the root to $v_0$.

If $v \in T \cap D^s_i$ and $v$ is a $\land$-vertex, then exactly one of its successors in $D^s$ is in $D^s_i$ and by clauses $\text{N2}$ this successor is in $T$. If $v \in T \cap D^s_i$ and $v$ is a $\lor$-vertex, then by clauses $\text{N1}$ at least one of its successors is in $T$. By smoothness, this successor is in $T \cap D^s_i$. By induction using these arguments, we obtain a path in $T \cap D^s_i$ from $v_0$ to a leaf. \hfill $\square$

Existence of a cover by separators has the following consequence for a smooth MDNNF. Note that it implies that the DNNF does not contain a transitive edge discussed in Remark 4.5.
Claim 4.7. If $u_1$ and $u_2$ are different successors of the same vertex in $D^c$, then there is no path from $u_1$ to $u_2$.

**Proof.** Let $v$ be the common predecessor of $u_1$ and $u_2$. If $v$ is a $\wedge$-vertex, the statement follows from decomposability. If $v$ is a $\vee$-vertex, then by smoothness, there is an index $i$, such that all the vertices $v, u_1$, and $u_2$ belong to $D^c_i$. Assume for a contradiction that there is a path from $u_1$ to $u_2$ in $D^c$. Clearly, this path belongs to $D^c_i$. Let $S$ be a separator in $D^c_i$ containing $u_1$. Let $P_1$ be a path from the root to a leaf in $D^c_i$ going through $v, u_1$, and $u_2$. Moreover, let $P_2$ be obtained from $P_1$ by skipping $u_1$ using the edge $(v, u_2)$. Since $u_1$ is the only vertex in $S \cap P_1$, the intersection $S \cap P_2$ is empty which contradicts the definition of a separator. It follows that there is no path from $u_1$ to $u_2$ in $D^c$.

We close this section by proving that the formulas $\psi_c$ and $\psi_p$ are encodings of the minimal satisfying subtrees.

**Proposition 4.8.** Assume, $D^c$ is smooth and covered by separators. For every $M \subseteq V$, the following are equivalent

(a) $M$ is a model of $\psi_c$,
(b) $M$ is a model of $\psi_p$,
(c) $M$ is the set of the vertices of a minimal satisfying subtree of $D^c$.

**Proof.** (a) $\Rightarrow$ (c) Assume, $M$ is a model of $\psi_c$ and let $T$ be the subgraph induced by $M$. Since $M$ is also a model of the FullNNF encoding of $D^c$, $T$ satisfies all the properties of a minimal satisfying subtree except that a $\vee$-vertex can have more than one successor in $T$. Let $v$ be any $\vee$-vertex of $T$ and let us prove using the assumption on the separators that at most one of its successors is in $T$.

Assume for a contradiction that two different successors $u_1, u_2$ of $v$ are in $M$. By smoothness, we have $v, u_1, u_2 \in D^c_i$ for some $i$. By Claim 4.6, there is a path $P$ in $T \cap D^c_i$ from the root to a leaf containing $u_1$. By assumption, there is a separator $S$ in $D^c_i$ containing $u_2$. The separator $S$ contains a vertex from $P \subseteq T$, however, this vertex is not $u_2$, since by Claim 4.7, $u_2$ is not in $P$. This implies $|M \cap S| \geq 2$ which is a contradiction, since $M$ is a model of $\psi_c$. It follows that every $\vee$-vertex of $T$ has at most one successor in $T$ and, hence, $T$ is a minimal satisfying subtree.

(c) $\Rightarrow$ (b) If $T$ is a minimal satisfying subtree, then the set $M$ of its vertices satisfies FullNNF encoding. Moreover, for every $i = 1, \ldots, n$, $P_i = T \cap D^c_i$ is a path by Lemma 4.2. If $S$ is a separator in $D^c_i$, we have

$$|M \cap S| = |P_i \cap S| = 1$$

by the properties of the separators. It follows that $M$ is a model of $\psi_p$.

(b) $\Rightarrow$ (a) is clear, since $\psi_c$ is a subset of $\psi_p$. 

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5 Covering a MDNNF with Separators

In this section, we describe step [b] of the proof of the main result. Namely, we show that for every smooth MDNNF \( D^s \) one can construct in polynomial time an equivalent smooth MDNNF \( D^c \) which can be covered by separators and moreover, a separator cover of \( D^c \) can be constructed in polynomial time as well.

In order to guarantee a cover by separators, we include auxiliary vertices into the MDNNF that subdivide some of its edges. The only purpose of these vertices is to create additional auxiliary variables in the encoding. We call them no-operation vertices, they represent the identity, and, formally, they are disjunctions with one argument. We describe a procedure to add suitable no-operation vertices controlled by a labeling \( \text{Level} : V \rightarrow \mathbb{N} \) which is edge consistent. By this we mean that \( \text{Level}(\rho) = 0 \) and for every edge \((v, u)\) in \( D^s \), we have \( \text{Level}(v) < \text{Level}(u) \).

If \( D^s \) is strictly leveled, then an edge consistent labeling \( \text{Level}_0 \) can be obtained, where for every vertex \( v \), \( \text{Level}_0(v) \) is the common length of the paths from \( \rho \) to \( v \). In this case, the separators in \( S \) can be obtained as the sets of all vertices \( v \) satisfying \( \text{Level}_0(v) = j \) and, additionally, the leaves \( v \) satisfying \( \text{Level}_0(v) < j \) for some \( j \geq 1 \). Hence, in this case, we get a cover by separators with no auxiliary vertices. For example, as already pointed out, if a complete MDD is considered as an MDNNF, it is a strictly leveled smooth MDNNF.

It is easy to construct an edge consistent labeling for an arbitrary MDNNF, however, it is unclear, how to choose a labeling that minimizes the number of new auxiliary vertices. For this reason, we leave the labeling unspecified, although we describe two simple examples later.

Assume a fixed labeling \( \text{Level} \). An extension \( D^c \) of \( D^s \) that can be covered by separators is obtained as follows. For each edge \((v, u)\) in \( D^s \), for which \( \text{Level}(v) + 2 \leq \text{Level}(u) \), include a new vertex \( u' \) representing no-operation, replace the edge \((v, u)\) by edges \((v, u')\) and \((u', u)\), and assume \( \text{var}(u') = \text{var}(u) \). The resulting MDNNF is \( D^c \) and it is clearly equivalent to \( D^s \). Let us denote \( D^c_i \) the subgraphs of \( D^c \) defined similarly as \( D^s_i \) in \( D^s \).

**Proposition 5.1.** If \( \text{Level} : V \rightarrow \mathbb{N} \) is an edge consistent labeling of the vertices of a smooth MDNNF \( D^s \) and \( D^c \) is the extension of \( D^s \) obtained using \( \text{Level} \) as above, then \( D^c \) is smooth, has size at most \(|V| + |E|\) and can be covered by separators.

**Proof.** Clearly, \( D^c \) has at most \(|V| + |E|\) vertices. If \( u' \) is a no-operation vertex subdividing \((v, u)\), we have \( \text{var}(u') = \text{var}(u) \). It follows that smoothness is preserved.

Construct separators for \( D^c \) as follows. For every \( i = 1, \ldots, n \), let \( d_i \) be the maximum of \( \text{Level}(v) \) for \( v \in D^c_i \). For every \( i = 1, \ldots, n \) and for each \( j \), \( 0 \leq j \leq d_i \), let \( S_{i,j} \) be the union of the following three sets: the set of the vertices \( v \in D^c_i \) satisfying \( \text{Level}(v) = j \), the set of the leaves \( v \in L_i \) satisfying \( \text{Level}(v) < j \), and the set of all no-operation vertices used to subdivide some of the edges \((v, u)\) in \( D^c_i \) satisfying \( \text{Level}(v) < j \leq \text{Level}(u) \). For each \( i = 1, \ldots, n \), let \( S_i \) be the collection of the sets \( S_{i,j} \) for \( 0 \leq j \leq d_i \).

Fix \( i \in \{1, \ldots, n\} \) and \( j \), \( 0 \leq j \leq d_i \). Every path from \( \rho \) to a leaf in \( D^c_i \) contains either a single vertex \( v \in D^c_i \) such that \( \text{Level}(v) = j \), a single leaf \( v \) satisfying \( \text{Level}(v) < j \), or a single pair of vertices \( v, u \in D^c_i \), such that \((v, u)\) is an edge of \( D^c_i \) and \( \text{Level}(v) < j <
Level(u). Each of these cases implies a vertex of the path that belongs to \( S_{i,j} \). Since the cases exclude each other, \( S_{i,j} \) is a separator in \( D^c_i \). Moreover, every vertex of \( D^c_i \) is contained in \( S_{i,j} \) for some \( j \).

Note that the construction used in the proof of Proposition 5.1 yields \( L_i \in S_i \), since \( S_{i,j} = L_i \) if \( j = d_i \). Let us estimate the size of the constructed separator cover and the complexity of the construction. By height of a MDNNF we mean the number of edges on a longest path from the root to a leaf.

By the width of an MDNNF corresponding to a chosen labeling Level, we mean the maximum over \( i = 1, \ldots, n \) of the maximum size of a separator in \( D^c_i \) constructed as in the proof of Proposition 5.1. We call this maximum the width, since it is also the size of a largest cut in any of the graphs \( D^c_i \) at a given level defined as the set of the vertices at this level, edges crossing this level, and leaves above this level.

**Proposition 5.2.** Assume \( D^a \) is a smooth MDNNF which represents a constraint on \( n \) variables \( x \) and has height \( h \). Let \( \text{Level} : V \rightarrow \mathbb{N} \) be an edge consistent labeling of the vertices such that \( \text{Level}(v) \leq h \) for every \( v \in V \). Let \( w \) be the width of \( D^a \) corresponding to the labeling \( \text{Level} \). Then there is a separator cover \( S = \bigcup_{i=1}^n S_i \) such that

- \( L_i \in S_i \) for every \( i = 1, \ldots, n \),
- the total size of all different separators is \( t = \sum_{S \in S} |S| \leq nwh + 1 \)
- the cover can be constructed in time \( O(t) \).

**Proof.** Let \( S = \bigcup_{i=1}^n S_i \) be the separator cover obtained by the construction in the proof of Proposition 5.1. One can verify that the first requirement is satisfied by inspecting the proof of Proposition 5.1. For each \( i = 1, \ldots, n \), the separator cover \( S_i \) consists of at most \( h \) nontrivial separators, each of which has size at most \( w \). One trivial separator consisting only of the root vertex \( \rho \) is part of \( S \) and thus \( t \leq nwh + 1 \). The complexity of the construction \( O(t) \) is clear from the construction in the proof of Proposition 5.1. \( \square \)

Each of the parameters \( w \) and \( h \) can be bounded from above by the total number of vertices and edges in the input MDNNF which implies that the construction can be done in time polynomial in the input size. However, the parameters \( w \) and \( h \) give a better insight into the size of the separator cover constructed in this way.

Let us present two examples of an edge consistent labeling bounded from above by the height of \( D^a \). A simple example is the function \( \text{Level}_1(v) \) defined for every vertex \( v \) as the maximum length of a path from the root to \( v \). Since \( D^a \) is acyclic, the values of \( \text{Level}_1(v) \) can be computed in linear time for every vertex \( v \). Another function that can be used is \( \text{Level}_2(v) \) that is equal to \( \text{Level}_1(v) \) for the leaves, however, for an inner vertex \( v \), it is defined as

\[
\text{Level}_2(v) = \min_{(v,u) \in D^a} \text{Level}_2(u) - 1.
\]

Note that \( \text{Level}_2(v) \) is the largest possible label of \( v \) if the labels of the successors of \( v \) are given and the labeling is edge consistent. The values of \( \text{Level}_2(v) \) can be computed in linear time by following the vertices in the reversed topological order of \( D^a \).
**Remark.** Using Level2(v), the levels assigned to the vertices of a path from the root to a leaf tend to concentrate on the values closer to the level of the leaf. This implies that the edges close to the leaves have higher chance to satisfy Level(v) + 1 = Level(u) for Level = Level2 than for Level = Level1. These edges then do not need to be subdivided by auxiliary vertices.

6 PC and URC Encodings of MDNNF

We are ready to show the main properties of encodings \(\psi_c\) and \(\psi_p\) introduced in Section 3 and used in step \(c\) of the proof of the main result. As in the previous sections, \(D^c\) is a smooth MDNNF that can be covered by separators. For each \(i = 1, \ldots, n\), consider a fixed collection of separators \(S_i\) in \(D^c_i\).

Models of our encodings exactly represent the set of minimal satisfying subtrees of \(D^c\). Since \(D^c\) is smooth, the leaves of a minimal satisfying subtree specify a full assignment of the variables in \(x\) by a vector of values from \(\text{dom}(x_1) \times \cdots \times \text{dom}(x_n)\). If a unary constraint \(x_i = a\) is not associated with any leaf of \(D^c\), then \(a\) cannot be assigned as a value of \(x_i\) by any minimal satisfying subtree of \(T\). The encoding thus forbids these assignments explicitly by clauses of group \([N4]\). This ensures that every variable gets exactly one value from its domain in every model of each of the encodings. Equivalently, every model of the encodings specifies a \(\text{DE}-\text{consistent} \) assignment of values in \(\text{dvar}(x)\).

**Proposition 6.1.** Each of the formulas \(\psi_c\) and \(\psi_p\) is an encoding of the constraint \(f(x)\) represented by \(D^c\).

**Proof.** By Proposition 4.8, \(\psi_c\) and \(\psi_p\) have the same set of models, in other words, they represent the same boolean function of variables \(\text{dvar}(x)\) and \(y\). It is thus enough to show that \(\psi_p\) is a CNF encoding of \(D^c\). By Definition 3.5, this means to show that \(\psi_p\) is a CNF encoding of the boolean function \(f_{de}(\text{dvar}(x))\). Assume, \(a\) is an assignment of \(\text{dvar}(x)\) and \(a'\) is an assignment of \(x\) encoded by \(a\). If \(a\) is a model of \(f_{de}\), then \(a'\) is accepted by \(D^c\) and by Proposition 4.3, there is a minimal satisfying subtree \(T\) which is consistent with \(a'\) in that the leaves of \(T\) are labeled with exactly the unary constraints satisfied by \(a'\). This means that the leaves of \(T\) are precisely the leaves of \(D^c\) satisfied by \(a\). By Proposition 4.8, the set \(M\) of the vertices of \(T\) is a model of \(\psi_p\). Since this model agrees with \(a\) on the variables \(\text{dvar}(x)\), \(\psi_p(a, y)\) is satisfiable.

Assume on the other hand a model \((a, b)\) of \(\psi_p\) where \(a\) assigns values to \(\text{dvar}(x)\) and \(b\) assigns values to \(y\). By Proposition 4.8, variables with value 1 in \((a, b)\) represent the set of vertices of a minimal satisfying subtree \(T\) of \(D^c\) and the leaves of \(T\) are consistent with \(a\). In particular \(a\) is \(\text{DE}-\text{consistent}\) and the existence of \(T\) certifies that \(D^c\) with input encoded by \(a\) evaluates to true. Assignment \(a\) is thus a model of \(f_{de}\). \(\square\)

In the rest of this section, we prove the claimed propagation strength of \(\psi_c\) and \(\psi_p\). Since the propagation is considered on all variables, it is a property of the formula and we may disregard the encoded constraint. Due to this, it is simpler not to distinguish
the main variables \( \text{dvar}(x) \) and the auxiliary variables \( y \), and we consider the encodings as formulas \( \psi_c(v) \) and \( \psi_p(v) \) whose variables are the vertices of \( D^c \). This is possible, since the only clauses which cannot be treated in this way are the clauses in group \( \mathbb{N}^4 \). These clauses are unit clauses on the \( \text{dvar}(x) \) variables not used in \( D^c \), so they do not share variables with the remaining parts of \( \psi_c \) and \( \psi_p \). It follows that we may ignore them for the proof of the propagation strength.

Let us prove that \( \psi_c \) and \( \psi_p \) are URC and complete for deriving negative literals.

**Lemma 6.2.** Let \( \psi \) be any of the formulas \( \psi_c \) and \( \psi_p \). Let \( \alpha \subseteq \text{lit}(V) \) be a partial assignment, such that \( \psi \land \alpha \not\vdash_1 \bot \). Then \( \psi \land \alpha \) is satisfiable. If, additionally, \( v_0 \in V \) is a vertex such that \( \psi \land \alpha \not\vdash_1 \neg v_0 \), then \( \psi \land \alpha \land v_0 \) is satisfiable.

**Proof.** It is sufficient to prove the second statement, since the first statement follows from the second one using \( v_0 = \rho \). Let \( V' = \{ v \in V \mid \psi \land \alpha \not\vdash_1 \neg v \} \) and let \( D' \) be the subgraph of \( D^c \) induced by the set \( V' \). Since unit clause \( \rho \) is contained in \( \psi \) and \( \psi \land \alpha \not\vdash_1 \bot \), we have \( \rho \in V' \). By assumption, \( v_0 \in V' \).

Let us show that there is a path \( P \) from the root \( \rho \) to \( v_0 \) in \( D' \) by constructing \( P \) backwards starting in \( v_0 \). At the beginning, \( P \) is initialized as \( \{v_0\} \) and then we repeat the following until we get to the root. Let \( v \) be the first vertex of \( P \) and assume it is not the root. Since \( v \in V' \), it has a predecessor \( u \) which is in \( V' \) and we prepend \( u \) to \( P \). The vertex \( u \) exists, since otherwise \( \neg v \) would be derived by unit propagation using the clause of group \( \mathbb{N}^3 \) corresponding to \( v \).

Let us construct a minimal satisfying subtree \( T \) of \( D' \) containing \( P \) by starting at the root \( \rho \) and successively extending \( T \) downwards. The set of the vertices of \( T \) is initialized as \( \{\rho\} \) and while there is a leaf \( v \) of \( T \) which is not a leaf of \( D^c \), we extend \( T \) as follows.

- Assume \( v = u_1 \lor \cdots \lor u_k \). Since \( v \in T \), we have that \( v \in V' \). If \( v \in P \) and \( v \neq v_0 \), then \( v \) has a successor \( u_j \) which belongs to \( P \) and we add \( u_j \) and \( (v,u_j) \) to \( T \). If \( v \notin P \) or \( v = v_0 \), then there is a successor \( u_j \) of \( v \) which belongs to \( V' \) and we add \( u_j \) and \( (v,u_j) \) to \( T \). The vertex \( u_j \) exists, since otherwise, the clause in group \( \mathbb{N}^4 \) corresponding to \( v \) derives \( \neg v \) by unit propagation.

- Assume \( v = u_1 \land \cdots \land u_k \). Since \( v \in T \), it belongs to \( V' \) and also all its successors \( u_j \) belong to \( V' \). Otherwise the clause \( v \rightarrow u_j \) in group \( \mathbb{N}^2 \) derives \( \neg v \) by unit propagation. We add all vertices \( u_1, \ldots, u_k \) and the corresponding edges to \( T \).

It follows from the construction that \( T \) is a minimal satisfying subtree of \( D^c \) in which all vertices belong to \( V' \). Moreover, by construction, \( T \) contains \( P \) and, hence, also \( v_0 \).

By Proposition \( 4.8 \) we have that the set of vertices in \( T \) specifies a model of \( \psi \), let us denote it \( \mathbf{a} \). Since \( v_0 \) is in \( T \) by construction, \( \mathbf{a}(v_0) = 1 \). It remains to show that \( \mathbf{a} \) is consistent with \( \alpha \).

If \( \neg v \in \alpha \) for some vertex \( v \in V \), then \( v \notin V' \) and thus \( v \) is not in \( T \). Assume a positive literal \( v \in \alpha \) for a vertex \( v \in V \). If \( v \) is the root, then it belongs to \( T \) by construction. Otherwise, consider an index \( i \in \{1, \ldots, n\} \), such that \( v \in D^c_i \). There is a separator \( S \in S_i \), such that \( v \in S \). Since the clauses in group \( \mathbb{N}^5 \) or group \( \mathbb{N}^6 \) are satisfied, every vertex \( u \in S \setminus \{v\} \) satisfies \( \psi \land \alpha \vdash_1 \neg u \) and thus \( u \notin V' \). It follows that \( T \) contains \( v \),
since by Lemma \[\text{Lemma 6.2}\] the intersection \(T \cap D_v^c\) is a path from the root to a leaf and, hence, has a nonempty intersection with \(S\). Thus \(a\) satisfies all literals from \(\alpha\). It follows that \(a\) is a model of \(\psi \land \alpha \land v_0\) and it is thus satisfiable. \(\square\)

The following is specific to \(\psi_p\) encoding.

**Lemma 6.3.** Encoding \(\psi_p\) is propagation complete.

**Proof.** Let \(\alpha \subseteq \text{lit}(V)\) be a partial assignment satisfying \(\psi_p \land \alpha \not\models 1\). Let \(l\) be a literal for which \(\psi_p \land \alpha \not\models l\). We shall show that \(\psi_p \land \alpha \land \neg l\) is satisfiable. Let \(v\) be the vertex in literal \(l\), so \(l = v\) or \(l = \neg v\). If \(l = \neg v\), then \(\psi_p \land \alpha \land v\) is satisfiable by application of Lemma \[\text{Lemma 6.2}\] with \(v_0 = v\). For the rest of the proof, assume \(l = v\).

Since \(\psi_p \land \alpha \not\models 1\), \(\psi_p \land \alpha\) is satisfiable by Lemma \[\text{Lemma 6.2}\]. In particular, \(\psi_p \land \alpha\) cannot simultaneously imply \(v\) and \(\neg v\). Thus if \(\psi_p \land \alpha \not\models \neg v\), then \(\psi_p \land \alpha \not\models v\) and \(\psi_p \land \alpha \land \neg v\) is satisfiable as required. For the rest of the proof, assume \(\psi_p \land \alpha \not\models \neg v\) in addition to the assumption \(\psi_p \land \alpha \not\models l\).

Let \(i \in \{1, \ldots, n\}\) be an index, such that \(v \in V_i\) and let \(S \in S_i\) be a separator containing \(v\). There must be another vertex \(v' \in S\) for which neither \(\psi_p \land \alpha \not\models l'\), nor \(\psi_p \land \alpha \not\models \neg l'\), since otherwise the clauses in group \(\text{N6}\) derive \(v\) or \(\neg v\). By Lemma \[\text{Lemma 6.2}\] applied with \(v_0 = v'\), we get that formula \(\psi_p \land \alpha \land v'\) is satisfiable. Since \(v\) and \(v'\) are both in the same separator \(S\), clauses \(\text{N6}\) imply that any model of \(\psi_p \land \alpha \land v'\) falsifies \(v\) and thus \(\psi_p \land \alpha \land \neg v\) is satisfiable.

In all cases we obtained that \(\psi_p \land \alpha \land \neg l\) is satisfiable as required. \(\square\)

Let us summarize the results of this section as follows.

**Theorem 6.4.** Formula \(\psi_c(\text{dvar}(x), y)\) is a URC encoding and formula \(\psi_p(\text{dvar}(x), y)\) is a PC encoding of constraint \((7)\)\text{.} Both of them can be constructed in polynomial time.

**Proof.** By Proposition \[\text{Proposition 6.1}\] the formulas \(\psi_c\) and \(\psi_p\) are encodings of constraint \((7)\). Encoding \(\psi_c\) is URC by Lemma \[\text{Lemma 6.2}\] and encoding \(\psi_p\) is PC by Lemma \[\text{Lemma 6.3}\].

The separator cover for \(D_v^c\) can be obtained in polynomial time by Proposition \[\text{Proposition 5.1}\]. The construction of both encodings can clearly be carried out in time which is polynomial in the size of these encodings. In order to estimate this size, we assume that the separator cover is constructed as in Proposition \[\text{Proposition 5.1}\] using a labeling Level bounded by the height of \(D_v^c\).

Denote \(d = \max_{i=1} \lvert \text{dom}(x_i) \rvert\) the maximum size of a domain, \(s\) the number of vertices of \(D_v^c\), \(e\) the number of edges of \(D_v^c\) and \(h\) the height of \(D_v^c\). Both encodings have at most \(nd + s\) variables, since \(\lvert \text{dvar}(x) \rvert \leq nd\) and \(\lvert y \rvert \leq s\).

The total number of clauses in groups \(\text{N1} to \text{N3}\) is \(O(e)\). The number of clauses in group \(\text{N4}\) is at most \(nd\) and the number of clauses in group \(\text{N5}\) is at most \(s^2\), because every variable is contained in at most \(s - 1\) negative binary clauses. Clauses of group \(\text{N6}\) include an additional clause for each separator \(S\) consisting of all its variables, the number of these clauses is thus bounded by \(\lvert S \rvert \leq nh + 1 \leq ns + 1\).

It follows that \(\psi_c\) and \(\psi_p\) consist of \(O(e + nd + s^2 + ns)\) clauses which is polynomial in the size of \(D_v^c\). The direct encoding constraints are not included in this estimate, since
they are not needed for the proof of the claimed properties of the encodings. On the other hand, they have polynomial size, so the size of the encoding together with these constraints is also polynomial.

For simplicity of proving the required propagation strength, the encodings used in Theorem 6.4 contain prime representations of the at-most-one or the exactly-one constraint on each separator. Due to this, the upper bound on the size of the encodings is quadratic in the number of the vertices of $D^c$. In Section 7.4 we prove that this is not necessary if we use suitably chosen encodings of linear size of the two cardinality constraints.

7 Embedding Basic Cardinality Constraints in an Encoding

CNF encodings of cardinality constraints frequently use auxiliary variables since this allows to reduce the size of the encoding. The main topic of this section is to investigate the consequences of using auxiliary variables in the at-most-one and the exactly-one constraints for the propagation strength of a larger encoding containing the constraint in question as its part. Sufficient conditions for using auxiliary variables for this purpose are presented in sections 7.2 and 7.3 and an estimate of the size of the resulting encodings is in Section 7.4.

7.1 Preliminary Considerations

Let us start with a more general question. Consider an encoding $\varphi(x, x', y) \land \theta(x)$ of a constraint on the variables $x \cup x'$ and let $\theta'(x, z)$ be an encoding of the subformula $\theta(x)$ using new auxiliary variables $z$. Then replacing $\theta$ by $\theta'$ yields an encoding $\varphi(x, x', y) \land \theta'(x, z)$ of the same constraint on the variables $x \cup x'$, because

$$\exists y \left[ \varphi(x, x', y) \land \theta(x) \right] \equiv \exists y (\exists z) [\varphi(x, x', y) \land \theta'(x, z)]. \quad (8)$$

A general question is what can be said about propagation strength of the right-hand side using assumptions on the propagation strength of the left-hand side and the properties of $\theta'(x, z)$.

There is a significant difference between replacing a subformula of an encoding if the required propagation strength is implementing domain consistency and if it is propagation completeness. In the case of domain consistency, the natural necessary conditions are also sufficient by Proposition 7.1. For technical reasons, the proposition is formulated in terms of propagation completeness on the main variables of the encoding which is equivalent to implementing domain consistency. On the other hand, we demonstrate by an example that a similar statement is not true for propagation completeness on all the variables of an encoding.

**Proposition 7.1.** Assume that $\theta'(x, z)$ is an encoding of $\theta(x)$ which is PC on variables $x$. Assume $\varphi(x, x', y) \land \theta(x)$ is an encoding of a constraint on the variables $x \cup x'$ that is PC on these variables. Then $\varphi(x, x', y) \land \theta'(x, z)$ is an encoding of the same constraint on the variables $x \cup x'$ that is also PC on these variables.
Proof. Using [5], we obtain that \( \varphi \land \theta \) and \( \varphi \land \theta' \) are encodings of the same constraint on the variables \( x \cup x' \). Let us show that \( \varphi(x, x', y) \land \theta'(x, z) \) is PC on variables \( x \cup x' \). For the rest of the proof, let \( \alpha \subseteq \text{lit}(x \cup x') \) be a partial assignment and \( l \in \text{lit}(x \cup x') \) be a literal such that \( \varphi(x, x', y) \land \theta'(x, z) \land \alpha \models l \). Our goal is to show that \( l \) or \( \bot \) can be derived by unit propagation.

By [5], we have \( \varphi(x, x', y) \land \theta(x) \land \alpha \models l \). Since \( \varphi \land \theta \) is PC on variables \( x \cup x' \), we have \( \varphi(x, x', y) \land \theta(x) \land \alpha \models l \). Any of these two unit propagation derivations can be simulated by unit propagation in \( \varphi \land \theta' \). This can be seen as follows. If a clause of \( \theta \) is used to derive a literal \( e \in \text{lit}(x) \) from assumptions \( \beta \subseteq \text{lit}(x) \), then \( \beta \models e \). This implies \( \theta' \land \beta \models e \) or \( \theta' \land \beta \models \bot \), because \( \theta' \) is PC on variables \( x \). In particular, using a clause of \( \theta \) in the original derivation can be replaced with a sequence of propagation steps in \( \theta' \) which either derive the same literal or a contradiction. We thus get that \( l \) or \( \bot \) can be derived from \( \varphi(x, x', y) \land \theta'(x, z) \land \alpha \) by unit propagation as required.

Let us now consider propagation completeness. The assumption that the encodings \( \varphi \land \theta \) and \( \theta' \) are PC is not sufficient to guarantee that \( \varphi \land \theta' \) is even URC. Let us demonstrate this by the following example, where \( \theta(x) \) represents the at-most-2 constraint. Consider the formula

\[
\varphi(x) = (x_1 \lor x_2 \lor x_3)(x_1 \lor x_2 \lor x_4)(x_1 \lor x_3 \lor x_4)(x_2 \lor x_3 \lor x_4)
\]

for the at-least-2 constraint on the variables \( (x_1, x_2, x_3, x_4) \) and the formula

\[
\theta(x) = (\neg x_1 \lor \neg x_2 \lor \neg x_3)(\neg x_1 \lor \neg x_2 \lor \neg x_4)(\neg x_1 \lor \neg x_3 \lor \neg x_4)(\neg x_2 \lor \neg x_3 \lor \neg x_4)
\]

for the at-most-2 constraint on these variables. Let \( \theta'(x, s) \) be the sequential encoding \( \text{LT}_{SEQ}^{4,2} \) from [39] simplified by eliminating pure literals \( \neg s_{1,2}, s_{3,1} \). This is an encoding of the constraint at-most-2 on the variables \( x = (x_1, x_2, x_3, x_4) \) with auxiliary variables \( s \). Namely, \( \theta'(x, s) \) is a Horn formula consisting of the clauses

\[
\begin{align*}
(\neg x_1 \lor s_{1,1}), & (\neg x_2 \lor \neg s_{1,1} \lor s_{2,2}), (\neg x_3 \lor \neg s_{2,2}), \\
(\neg s_{1,1} \lor s_{2,1}), & (\neg s_{2,2} \lor s_{3,2}), \\
(\neg x_2 \lor s_{2,1}), & (\neg x_3 \lor \neg s_{2,1} \lor s_{3,2}), (\neg x_4 \lor \neg s_{3,2}).
\end{align*}
\]

One can verify that \( \theta' \) is propagation complete. This follows by the results of [10] from the fact that all prime implicates of \( \theta' \) can be derived by a series of non-merge resolutions. The same argument can be used to show that \( \text{LT}_{SEQ}^{n,k} \) is PC for every \( n \geq 2 \) and \( n \geq k \geq 1 \).

The formula \( \varphi(x) \land \theta(x) \) consists of all the prime implicates of the exactly-2 constraint, so it is a propagation complete representation of this constraint. Using unit propagation and resolution, one can verify that the formula \( \varphi(x) \land \theta'(x, s) \land \neg s_{3,2} \land \neg x_4 \) implies \( \varphi(x) \land \neg x_1 \land \neg x_2 \land \neg x_3 \land \neg x_4 \land \neg x_4 \), so it is contradictory. On the other hand, unit propagation does not derive a contradiction. It follows that \( \varphi(x) \land \theta'(x, s) \) is an encoding of exactly-2 constraint on the variables \( x \) which is not unit refutation complete.

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In the next subsections, we consider the at-most-one and the exactly-one constraints as \( \theta \) and describe their encodings \( \theta' \) such that the replacement of \( \theta \) by \( \theta' \) in an arbitrary encoding preserves a specified propagation strength.

### 7.2 At-most-one Constraint

Recall that \( \text{AMO}(A) \) is the at-most-one constraint for a set of literals \( A \) and by \( \text{amo}(A) \) we denote the canonical representation of \( \text{AMO}(A) \) consisting of all the prime implicates of this function. For simplicity, let us assume that all the literals in \( A \) are positive and form a vector of the variables \( x \).

The number of clauses in \( \text{amo}(x) \) is quadratic, however, there are linear size encodings using auxiliary variables \([39, 12, 20, 21, 28, 33]\). Most of the encodings of the at-most-one constraint described in the literature are prime 2-CNFs, hence, they are propagation complete by \([5]\). We show that using these encodings in place of \( \text{amo}(x) \) inside another encoding preserves unit refutation completeness of the whole encoding.

The proof of the result uses the assumption that an encoding \( \text{amo}'(x, z) \) does not contain positive occurrences of the variables from \( x \). This is a natural assumption satisfied by any irredundant 2-CNF encoding of \( \text{AMO}(x) \), in particular, by most of the encodings suggested in the references above.

**Proposition 7.2.** Assume that \( \varphi(x, x', y) \land \text{amo}(x) \) is a URC encoding of a constraint \( f(x, x') \). Let \( \text{amo}'(x, z) \) be a PC encoding of \( \text{AMO}(x) \) with auxiliary variables \( z \) which does not contain positive literals on \( x \). Then \( \varphi(x, x', y) \land \text{amo}'(x, z) \) is a URC encoding of the constraint \( f(x, x') \).

**Proof.** Using \([5]\), we obtain that \( \varphi \land \text{amo}'(x, z) \) is an encoding of \( f(x, x') \) with auxiliary variables \( y \cup z \). Let us show that it is a URC encoding.

Let \( \alpha = \alpha_x \cup \alpha_{x', y} \cup \alpha_z \), where \( \alpha_x \subseteq \text{lit}(x) \), \( \alpha_{x', y} \subseteq \text{lit}(x' \cup y) \), \( \alpha_z \subseteq \text{lit}(z) \). Our goal is to show that if unit propagation does not derive a contradiction from the formula \( \varphi \land \text{amo}'(x, z) \land \alpha_x \land \alpha_{x', y} \land \alpha_z \), then this formula is satisfiable. So, assume

\[
\varphi \land \text{amo}'(x, z) \land \alpha_x \land \alpha_{x', y} \land \alpha_z \not\vdash \bot.
\]  

(9)

Without loss of generality, we can assume that \( \alpha \) is closed under unit propagation in \( \varphi \land \text{amo}'(x, z) \), since replacing \( \alpha \) with its closure does not affect satisfiability of the formula. Let us prove

\[
\varphi \land \text{amo}(x) \land \alpha_x \land \alpha_{x', y} \not\vdash \bot
\]

by contradiction. If \( \varphi \land \text{amo}(x) \land \alpha_x \land \alpha_{x', y} \vdash \bot \), then also \( \varphi \land \text{amo}'(x, z) \land \alpha_x \land \alpha_{x', y} \vdash \bot \), since \( \text{amo}'(x, z) \) is a PC encoding, but this contradicts \([5]\). Since \( \varphi \land \text{amo}(x) \) is a URC encoding, we get that \( \varphi \land \text{amo}(x) \land \alpha_x \land \alpha_{x', y} \) has a satisfying assignment \( \mathbf{a}_x \cup \mathbf{a}_{x', y} \), where \( \mathbf{a}_x \) and \( \mathbf{a}_{x', y} \) denote the parts of the satisfying assignment on the variables in \( x \) and \( x' \cup y \) respectively.

It remains to show that the formula \( \text{amo}'(x, z) \land \alpha_x \land \alpha_z \) is satisfiable by a suitable assignment of the variables \( z \). If \( \mathbf{a}_x \) contains only negative literals, then the formulas \( \text{amo}'(x, z) \land \mathbf{a}_x \land \alpha_z \) and \( \text{amo}'(x, z) \land \alpha_z \) are equisatisfiable, since each clause of \( \text{amo}' \)
containing a literal on a variable from \(x\) is satisfied by \(a_x\) by the assumption that the variables \(x\) occur only negatively. Equivalently, \(a_x\) is an autarky. Since amo’ is PC, amo’(x,z) \(\land\alpha_x\) is satisfiable by \([9]\) and the proof is finished in this case. If \(a_x\) contains a positive literal \(x\), we have \(\neg x \not\in \alpha_x\), since \(a_x\) is a satisfying assignment of a formula containing \(\alpha_x\). Since the partial assignment \(\alpha_x \land \alpha_z\) is closed under the unit propagation in amo’(x,z), we have that neither amo’(x,z) \(\land \alpha_x \land \alpha_z \not\models \neg x\), nor amo’(x,z) \(\land \alpha_x \land \alpha_z \not\models \bot\). Since amo’ is PC, we have that amo’(x,z) \(\land \alpha_x \land \alpha_z \not\models \neg x\). The formula amo’(x,z) \(\land \alpha_x \land \alpha_z \land x\) is thus satisfiable. Moreover, this formula has a satisfying assignment consistent with \(a_x\) since the literal \(x\) forces negative literals on all the other variables of AMO(x).

\[\square\]

7.3 Exactly-one Constraint

Recall that EO(A) is the exactly-one constraint on a set of literals \(A\) and by eo(A) we denote the canonical representation of EO(A) consisting of all the prime implicates of this function. For simplicity, let us assume that all the literals in \(A\) are positive and form a vector of the variables \(x = (x_1, \ldots, x_n)\).

The exactly-one constraint EO(x) can be represented as AMO(x) together with the clause \(x_1 \lor \cdots \lor x_n\). Let us demonstrate that the encoding obtained in this way is not in general propagation complete. For example, consider the encoding

\[
\varphi = (\neg x_1 \lor \neg x_2)(\neg x_1 \lor s_2)(\neg x_2 \lor s_2)(\neg s_2 \lor x_3)(\neg s_2 \lor \neg x_4)(\neg x_3 \lor \neg x_4)
\]

\[(x_1 \lor x_2 \lor x_3 \lor x_4).
\]

of EO(x_1, x_2, x_3, x_4) obtained, if AMO(x_1, x_2, x_3, x_4) is represented by the sequential encoding LT_{SEQ}^{1\text{st}} from \([33]\) (called AMO sequential counter encoding in \([33]\)) simplified by eliminating \(s_1\) and \(s_3\) by DP (Davis-Putnam) resolution. Assume the partial assignment \(\neg x_3 \land \neg x_4\). We have \(\varphi \land \neg x_3 \land \neg x_4 \models s_2\) and the unit propagation does not derive \(s_2\), so the encoding is not propagation complete. On the other hand, the encoding is URC, since it is obtained from a PC formula by adding a single clause.

Let us recall the ladder encoding \([26]\) of EO(x_1, \ldots, x_n) which uses auxiliary variables \(z_1, \ldots, z_{n-1}\). For simplicity, let \(z_0 = 1\) and \(z_n = 0\) be constants. The ladder encoding can then be obtained by an expansion of

\[
e_{n}(x_1, \ldots, x_n, z) = \bigwedge_{i=2}^{n} (z_i-1 \lor \neg z_i) \land \bigwedge_{i=1}^{n} (x_i \leftrightarrow z_{i-1} \land \neg z_i)
\]

(10)

into clauses. The resulting formula can be expressed as

\[
e_n(x_1, \ldots, x_n, z) = \bigwedge_{i=1}^{n} eo(\neg z_{i-1}, x_i, z_i)
\]

(11)

which implies that \(e_n\) is a conjunction of a sequence of \(n\) PC formulas \(eo(\neg z_{i-1}, x_i, z_i)\) where each two consecutive ones share a single variable. It follows that the ladder
Proposition 7.3. Assume that $i$ for every $l$ the following property: for every literal $f$ and $x$ have encoding satisfies its assumption, since it is propagation complete and using (13) we preserves propagation completeness, we prove a more general statement. The ladder Proof. Using (8), we obtain that $\varphi$ the constraint $f$ Proposition 7.1 with $x$ $\alpha$ $z$ where $z$ satisfies encoding is PC, see the proof of Proposition 5 in [10]. Moreover, the EO($x$) constraint satisfies

$$EO(x_1, \ldots, x_n) \iff (\exists z)[EO(x_1, \ldots, x_j, z) \land EO(\neg z, x_{j+1}, \ldots, x_n)] \tag{12}$$

where $z$ is an auxiliary variable and $1 \leq j \leq n - 1$. If we fix an index $i \in \{1, \ldots, n - 1\}$, we can use (12) to eliminate all auxiliary variables $z_j$, $j \neq i$ which gives us

$$\varepsilon_n(x, z) \models EO(x_1, \ldots, x_i, z_i) \land EO(\neg z_i, x_{i+1}, \ldots, x_n). \tag{13}$$

In order to show that replacing eo($x$) with the ladder encoding inside a larger encoding preserves propagation completeness, we prove a more general statement. The ladder encoding satisfies its assumption, since it is propagation complete and using (13) we have

$$\varepsilon_n(x, z) \models (z_i \iff \neg x_1 \land \cdots \land \neg x_i)$$

and

$$\varepsilon_n(x, z) \models (\neg z_i \iff \neg x_{i+1} \land \cdots \land \neg x_n)$$

for every $i = 1, \ldots, n - 1$.

**Proposition 7.3.** Assume that $\varphi(x, x', y) \land eo(x)$ is a PC encoding of a constraint $f(x, x')$. Let $eo'(x, z)$ be a PC encoding of $EO(x)$ with auxiliary variables $z$ which satisfies the following property: for every literal $l \in \text{lit}(z)$ there is a partial assignment $h(l) \subseteq \text{lit}(x)$, such that $eo'(x, z) \models (l \iff h(l))$. Then $\varphi(x, x', y) \land eo'(x, z)$ is a PC encoding of the constraint $f(x, x')$.

**Proof.** Using (8), we obtain that $\varphi \land eo'(x, z)$ is a CNF encoding of $f(x, x')$. Moreover, the assumption implies that the formula $\varphi(x, x', y) \land eo'(x, z)$ satisfies the assumption of Proposition 7.1 with $x'$ replaced with $x' \cup y$, $y$ replaced with an empty set of variables, $\theta(x)$ replaced with $eo(x)$, and $\theta'(x, z)$ replaced with $eo'(x, z)$. It follows that $\varphi(x, x', y) \land eo'(x, z)$ is PC on the variables $x \cup x' \cup y$.

In order to show that $\varphi \land eo'(x, z)$ is PC on all its variables, consider a partial assignment $\alpha = \alpha_x \cup \alpha_{x', y} \cup \alpha_z$ where $\alpha_x \subseteq \text{lit}(x)$, $\alpha_{x', y} \subseteq \text{lit}(x' \cup y)$, $\alpha_z \subseteq \text{lit}(z)$ and a literal $l \in \text{lit}(x \cup x' \cup y \cup z)$. Without loss of generality, we can assume that $\alpha$ is closed under unit propagation in $\varphi \land eo'(x, z)$, since replacing $\alpha$ with its closure does not affect validity of the conditions (5) and (6). In order to show that the implication from (5) to (6) holds also in this case, assume

$$\varphi \land eo'(x, z) \land \alpha_x \land \alpha_{x', y} \land \alpha_z \models l. \tag{14}$$

Our goal is to show that if

$$\varphi \land eo'(x, z) \land \alpha_x \land \alpha_{x', y} \land \alpha_z \not\models l$$

then

$$\varphi \land eo'(x, z) \land \alpha_x \land \alpha_{x', y} \land \alpha_z \not\models l. \tag{15}$$
Let $h(\alpha_z) \subseteq \text{lit}(x)$ denote the partial assignment consisting of $h(g)$ for all $g \in \alpha_z$. Since $eo'$ is PC, unit propagation on $eo'(x, z) \land \alpha_x$ derives all the literals in $h(\alpha_z)$ and unit propagation on $eo'(x, z) \land h(\alpha_z)$ derives all the literals in $\alpha_z$. Since $\alpha_x \cup \alpha_z$ is closed under unit propagation in $eo'$, we have $h(\alpha_z) \subseteq \alpha_x$. It follows that $eo'(x, z) \land \alpha_x \models \alpha_z$ and, consequently,

$$\varphi \land eo'(x, z) \land \alpha_x \land \alpha_x' \models l.$$  

If $l \in \text{lit}(x \cup x' \cup y)$, we get \[15\], since $\varphi \land eo'(x, z)$ is PC on the variables $x \cup x' \cup y$. If $l \in \text{lit}(z)$, then $eo'(x, z) \land l \models h(l)$ implies

$$\varphi \land eo'(x, z) \land \alpha_x \land \alpha_x' \models h(l)$$

and we obtain

$$\varphi \land eo'(x, z) \land \alpha_x \land \alpha_x' \models l \models g$$

for every literal $g \in h(l)$ again by propagation on the variables $x \cup x' \cup y$. Together with the fact that $eo'$ is a PC encoding, we obtain $eo'(x, z) \land h(l) \models l$ and, hence, $\[15\]$ as required.

Let us point out that the assumption of Proposition $7.3$ is satisfied also by an encoding $\varepsilon_n'$ which is slightly smaller than the ladder encoding. For $n \leq 4$, $\varepsilon_n' = eo(x_1, \ldots, x_n)$. If $n \geq 5$ is even, the encoding contains $n/2 - 2$ auxiliary variables and has the form

$$\varepsilon_n'(x_1, \ldots, x_n, z) = eo(x_1, x_2, x_3, z_1) \land$$

$$eo(\neg z_1, x_4, x_5, z_2) \land$$

$$\ldots$$

$$eo(\neg z_{j-1}, x_{2j}, x_{2j+1}, z_j) \land$$

$$\ldots$$

$$eo(\neg z_{n/2-3}, x_{n-4}, x_{n-3}, z_{n/2-2}) \land$$

$$eo(\neg z_{n/2-2}, x_{n-2}, x_{n-1}, x_n)$$

(16)

If $n \geq 5$ is odd, the number of auxiliary variables is $n/2 - 3/2$ and the encoding has the form $\[16\]$ except that the last two constraints are

$$eo(\neg z_{n/2-5/2}, x_{n-3}, x_{n-2}, z_{n/2-3/2}) \land eo(\neg z_{n/2-3/2}, x_{n-1}, x_n)$$

For both even and odd $n \geq 5$, the expansion $\[16\]$ consists of at most $\frac{1}{2} n$ constraints EO of 3 or 4 variables each of which consists of at most 7 clauses. It follows that $\varepsilon_n'$ has $\frac{7}{2} n + O(1)$ clauses and at most $\frac{1}{2} n$ auxiliary variables compared to $4n + O(1)$ clauses and $n - 1$ auxiliary variables of the ladder encoding.

### 7.4 Application to the Encodings of MDNNF

In this section, we present upper bounds on the size of URC and PC encodings of a smooth MDNNF, if the cardinality constraints on the separators are expressed using the linear size encodings instead of encodings of quadratic size with no auxiliary variables. Assume, $D^s$ is a smooth MDNNF and $D^c$ is a smooth MDNNF that is covered by
separators obtained from $D^s$ in step \([b]\) of the proof of the main result. Let $\psi_c$ and $\psi_p$ be the encodings obtained for $D^c$ in Section \([3]\)

Assume $S$ is a separator and denote $amo'(S, z_S)$ any irredundant 2-CNF encoding of $AMO(S)$ of linear size with auxiliary variables $z_S$, for example, the sequential counter encoding \([39, 33]\). Recall that every such encoding satisfies the assumption of Proposition \([7.2]\). Similarly, let $eo'(S, z_S)$ denote a linear size encoding of $EO(S)$ with auxiliary variables $z_S$ which satisfies the assumption of Proposition \([7.3]\) for example, the ladder encoding discussed in Section \([7.3]\).

Consider an encoding $\psi'_c(dvar(x), y, z)$ obtained from $\psi_c$ by using encoding $amo'(S, z_S)$ instead of $amo(S)$ for each separator $S$ in the cover. In addition to the main variables $dvar(x)$ and auxiliary variables $y$, the encoding $\psi'_c$ uses auxiliary variables $z$ which is the union of all sets $z_S$ for all separators.

Similarly, consider encoding $\psi'_p(dvar(x), y, z)$ which differs from $\psi_p$ by using encoding $eo'(S, z_S)$ instead of $eo(S)$ for each separator $S$ in the cover. In addition to the main variables $dvar(x)$ and auxiliary variables $y$, the encoding $\psi'_p$ uses auxiliary variables $z$ which is the union of all sets $z_S$ for all separators.

Encodings $\psi'_c$ and $\psi'_p$ are encodings of $D^c$. Moreover, encoding $\psi'_c$ is URC by Proposition \([7.2]\) and encoding $\psi'_p$ is PC by Proposition \([7.3]\). We assume the separator cover is constructed using the procedure from Proposition \([5.1]\). Hence, the main terms in the upper bound on the size of $\psi'_c$ and $\psi'_p$ are the sizes (vertices and edges together) of $D^c$ and the total size $t$ of the separators in $D^c$ estimated in Proposition \([5.2]\). By length of an encoding we mean the sum of the sizes of its clauses.

In the following estimate, we do not include direct encoding constraints to the size. One of the reasons is that they can be shared among several constraints in the instance, so they do not contribute to the size of each of them. On the other hand, if the encoding is used separately, then it is sufficient to consider these constraints after propagating the clauses \([N4]\). The estimate of the size of the separator cover from Proposition \([5.2]\) is valid for a cover which contains $L_i$ for each $i = 1, \ldots, n$ and the clauses $eo(L_i)$ are exactly the direct encoding constraints for $dvar(x_i)$ after propagating \([N4]\). As a consequence, we can assume that the direct encoding constraints can be replaced by the encodings $eo'(L_i, z_{L_i})$. The encoding $\psi'_p$ contains this encoding, so its size is included in the size estimate presented below. The encoding $\psi'_c$ contains only $amo'(L_i, z_{L_i})$, so replacing it with $eo'(L_i, z_{L_i})$ increases the size, however, both these encodings have size $\Theta(|L_i|)$, so the asymptotic estimate does not change also in this case.

**Theorem 7.4.** Denote $d = \max_{i=1} \left| \text{dom}(x_i) \right|$ the maximum size of a domain, $s$ the number of nodes of $D^s$, $e$ the number of edges of $D^s$ and $t$ the sum of the sizes of different separators constructed for $D^F$. Then $\psi'_c(dvar(x), y, z)$ and $\psi'_p(dvar(x), y, z)$ have $O(nd + t)$ variables, $O(nd + e + t)$ clauses, and length $O(nd + e + t)$.

**Proof.** Observe that the number of variables in $z$ is proportional to the total size of separators $t$. Since $|dvar(x)| \leq nd$ and $|y| \leq t$, we get that the number of variables in the encodings is bounded by $O(nd + t)$.

The total number of clauses in groups \([N1]\) to \([N3]\) is $O(e)$. The number of clauses in group \([N4]\) is at most $nd$ and the total number of clauses in the encodings of cardinality
constraints amo′(S,zS) and eo′(S,zS) respectively is proportional to the total size of separators t. Together, the number of clauses in the encodings is bounded by O(nd+e+t).

Clauses of group N3 have total length at most e and the same holds for the clauses of group N3. The rest of the clauses of both ψ′c and ψ′p have constant size and the length of both encodings is thus O(nd + e + t).

8 Conclusion and an Open Problem

We demonstrated a propagation complete encoding for a smooth DNNF, for which the previously known encodings implement only the domain consistency. In this context, it is natural to ask the following.

**Question 1.** Assume, ϕ(x,y) is an encoding of a boolean function f(x) with auxiliary variables y that implements domain consistency. Does this imply that there is an encoding ϕ′(x,z) of the same function of size polynomial in the size of ϕ with possibly a different set of auxiliary variables z that is unit refutation complete?

The results of Section 7 for the at-most-one and exactly-one constraints use quite specific properties of these constraints. So, one can expect a negative answer to the following question, however, a provable answer would be useful.

**Question 2.** Let θ(x) be the prime CNF representation of the constraint at-most-k, where k is a constant. Is there a PC encoding θ′(x,z) of θ(x) of linear size such that for every URC formula of the form ϕ(x,x′,y) ∧ θ(x) also the formula ϕ(x,x′,y) ∧ θ′(x,z) is URC?

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