Approximate Random Matrix Models for Generalized Fading MIMO Channels

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Abstract

Approximate random matrix models for $\kappa - \mu$ and $\eta - \mu$ faded multiple input multiple output (MIMO) communication channels are derived in terms of a complex Wishart matrix by keeping the degree of freedom constrained to the number of transmitting antennas and by matching matrix variate moments. The utility of the result is demonstrated in a) computing the average capacity/rate of $\kappa - \mu/\eta - \mu$ MIMO systems b) computing Symbol Error Rate (SER) for optimum combining with Rayleigh faded users and an arbitrary number of $\kappa - \mu$ and $\eta - \mu$ faded interferers. These approximate expressions are compared with Monte-Carlo simulations and a close match is observed.

Index Terms

Random matrices, Wishart matrices, Generalized fading, $\kappa - \mu$, $\eta - \mu$, MIMO, capacity, optimum combining

I. INTRODUCTION

The need for high data rates has been one of the driving factors for the evolution of the wireless systems from Single Input Single Output (SISO) systems to Multiple Input Multiple Output (MIMO) systems. MIMO systems are being used increasingly in modern wireless standards and it is imperative to study the channel capacity and other Quality of Service (QoS) metrics of such systems.

The capacity of wireless channels depends on channel fading statistics and also on whether the statistics are known at the receiver and the transmitter. To capture the fading statistics, channel
gain is characterized by a single random variable in SISO systems. But for MIMO systems, the channel is in the form of a matrix, hence the characterization of random matrices plays an indispensable role in studying MIMO channel metrics. The MIMO system is mathematically modeled by an $N_R \times N_T$ channel gain matrix $H$, where $N_R$ is the number of receive antennas and $N_T$ is the number of transmit antennas. Various performance metrics such as capacity, rate, etc. require the eigenvalue statistics of the Gram matrix $HH^H$ (or $H^HH$). When the elements of $H$ are i.i.d. circular symmetric complex Gaussian with zero or non-zero mean, i.e., when Rayleigh or Rician faded MIMO channels are considered, the Gram $HH^H$ can be characterized by Wishart matrices - central or non-central respectively [1]. These random matrix models have been used widely for deriving capacity expressions in the case of Rayleigh faded MIMO channels [2]–[4] and also Rician faded MIMO channels [5], [6].

Recently, there has been significant focus on generalized fading models namely $\kappa - \mu$ and $\eta - \mu$ models introduced in [7]. These distributions model the small scale variations in the fading channel in the line of sight and non-line of sight conditions respectively. Further, these generalized fading distributions include Rayleigh, Rician, Nakagami, One-sided Gaussian distributions as special cases. These generalized fading distributions have been widely used in capacity and outage probability analysis for SISO and MISO systems. Average channel capacity of single branch $\kappa - \mu$ and $\eta - \mu$ faded receivers is studied in [8]. The outage, coverage probability and rate of these generalized fading channels are analyzed in [9]–[13] and references therein. Secrecy capacity analysis is carried out in [14] and effective throughput in MISO systems is determined in [15], [16]. Outage probability of MRC in $\kappa - \mu$ fading channels in the presence of co-channel interference is studied in [17]. Analysis of decode and forward relay system for generalized fading models is performed in [18]–[20].

The capacity of MIMO systems for these generalized fading channels has been less analyzed for want of a random matrix model that characterizes the channel matrix. Nevertheless, some random matrix models have been developed for Nakagami and Rician-shadowed fading channels. A random matrix model has been developed for Nakagami-q fading in [21] and the pdf of eigen values of the Gram $HH^H$ is obtained in terms of a Pfaffian. In [22], the ergodic capacity of MIMO correlated Nakagami-m fading channel has been derived using the concept of a copula. But the work presents an analysis only for $2 \times 2$ MIMO channel and determining the capacity of MIMO channels with a larger number of receive and transmit antennas using this method is cumbersome. Recently a MIMO capacity upper bound was derived for the $\kappa - \mu$ and $\eta - \mu$...
fading channels in [23]. In [24], a MIMO model has been developed for Rician-shadowed fading as a unification model for MIMO-Rayleigh and MIMO-Rician fading models.

Given the complicated pdf structure of complex $\kappa - \mu$ and $\eta - \mu$ fading distributions [25], [26], it is challenging to develop the matrix distribution and the eigenvalue statistics for $HH^H$, even when the elements of $H$ are assumed to be i.i.d. $\kappa - \mu$ or $\eta - \mu$ random variables. Hence, in this paper, we develop an approximate matrix model for $HH^H$ (or $H^H H$) in terms of a Wishart distribution, which is a very well-studied matrix distribution [1]. Approximating any matrix distributions by central Wishart by means of Taylor expansion is studied in [27], but the approximation requires the knowledge of not only one or more cumulants and moments of the random matrix that is to be approximated but also the derivatives of central Wishart matrix. Also, the approximation of non-central Wishart matrix by a central Wishart by means of Laguerre polynomial expansion is given in [28] and by means of the moment generating functions in [29]. In this paper, we propose a Wishart distributed approximation of $HH^H$, such that the approximation has its first moment matched with the original matrix distribution of $HH^H$ and the degree of freedom is constrained to be the number of columns of the matrix $H$. This method requires only the knowledge of the expectation of $HH^H$ with respect to the original distribution and this can be found out for both the $\kappa - \mu$ and $\eta - \mu$ case. We also show that our method is equivalent to minimizing the K-L divergence between the actual MIMO matrix and the Wishart distributed approximation\(^1\).

The proposed approximation is discussed in Section II. In Section III, the utility of the approximation is shown in two applications. In one application, the proposed approximation is used to determine the capacity of MIMO systems with i.i.d. $\kappa - \mu$ or $\eta - \mu$ channel gains. Further, the approximation is also used to determine the asymptotic capacity of these MIMO systems. To the best of our knowledge, ours is the first work to derive even an approximate capacity expression in the presence of $\kappa - \mu/\eta - \mu$ MIMO channels. Another application of the approximation is in determining the Symbol Error Rate (SER) of an optimum combining (OC) receiver [30] for Rayleigh faded user and $\kappa - \mu$ or $\eta - \mu$ faded interferers. The SINR expression for OC involves a covariance matrix formed by interferer channel gains. Using the derived Wishart approximation, closed form expressions for SER of OC systems for $\kappa - \mu$ or $\eta - \mu$ faded

\(^1\)Since the $\kappa - \mu/\eta - \mu$ fading distributions are fairly complicated, computing and matching the higher moments is fairly difficult.
interferers are obtained. To the best of our knowledge, no prior work has given SER expressions for a receiver diversity system employing OC under the case of Rayleigh faded user and \( \kappa - \mu \) or \( \eta - \mu \) faded interferers. In Section IV, the derived capacity and SER approximations are compared with Monte-Carlo simulations and a close match is found between the theoretical results and simulation results. While we have only shown the utility of the approximation in two applications namely capacity computation and SER computation in OC system, the approximation can be used in any application which deals with random \( \kappa - \mu / \eta - \mu \) matrix models.

Basic notation: \( \mathbb{E}_x(.) \) denotes expectation with respect to distribution \( x \). \( |X| \) and \( \det(X) \) denote determinant of a matrix \( X \). \( \etr(X) \) denotes an exponential raised to trace of the matrix \( X \).

II. PROPOSED APPROXIMATION

Let \( H \) be an \( n_1 \times n_2 \) random matrix with independent and identically distributed elements. and \( X = HH^H \) be an \( n_1 \times n_1 \) random matrix. The exact matrix distribution of \( X \) denoted by \( p(X) \) is not known. Hence, we propose to approximate the density \( p(X) \) by an \( n_1 \times n_1 \) Wishart matrix whose distribution is \( q(X) = \mathcal{CW}_{n_1}(n_2, \Sigma) \) with \( n_2 \) degrees of freedom and covariance matrix \( \Sigma \), such that \( E_p[X] = E_q[X] \), i.e., their first moments are matched.

\[
E_q(X) = n_2 \Sigma = E_p[X]. \tag{1}
\]

We now show that this is also equivalent to minimizing the the K-L divergence between \( p(X) \) and a Wishart distribution. Let \( q(X) \) be that Wishart distribution which minimizes the K-L divergence between \( p(X) \) and all the Complex Wishart distributions \( \mathcal{CW}_{n_1}(n, \Sigma) \), i.e.,

\[
q(X) = \arg \min \text{KL}(p(X)||q(X)) = \arg \max q(X) \int p(X)[\ln(q(X)) - \ln(p(X))]dX
\]

\[
= \arg \max q(X) \int p(X)\ln(q(X))dX. \tag{2}
\]

Note that we assume an unknown degrees of freedom as \( n \) in this case, while in the Wishart approximation we had constrained the degree of freedom to \( n_2 \). The density of an \( n_1 \times n_1 \) complex Wishart matrix \( X \sim \mathcal{CW}_{n_1}(n, \Sigma) \) is given by [31],

\[
q(X) = \frac{1}{C\Gamma_{n_1}(n)(\det(S))^n} \etr(-X^{-1})\etr(X)^{n-n_1},
\]

\( ^2 \)The pdf of entries of \( H \) are known. But finding the matrix variate pdf of \( X = HH^H \) is fairly complicated, since we require the joint pdf of all entries of \( X \). On the other hand, the diagonal elements of \( X \) are well characterized.
where $C\Gamma(.)$ is the complex multivariate gamma function \[31\]. Substituting the density in \[2\], we obtain

$$q(X) = \arg\max_{q(X)} \int p(X) \left[ -\ln(C\Gamma_{n_1}(n)) - n\ln|\Sigma| + Tr(-\Sigma^{-1}X) + (n - n_1)\ln|X| \right] dX$$

$$= \arg\max_{q(X)} \left[ -\ln(C\Gamma_{n_1}(n)) - n\ln|\Sigma| + Tr(-\Sigma^{-1}E_p[X]) + (n - n_1)E_p[\ln|X|] \right].$$

Denoting $Z = E_p[X]$ and $Y = E_p[\ln|X|]$, we get

$$q(X) = \arg\max_{q(X)} \left[ -\ln(C\Gamma_{n_1}(n)) - n\ln|\Sigma| + Tr(-\Sigma^{-1}Z) + (n - n_1)Y \right]. \quad (3)$$

To obtain the minimizing distribution, we can differentiate the above equation with respect to two variables namely, $\Sigma$ and $n$. Differentiating equation \(3\) w.r.t. $\Sigma$, we obtain

$$\frac{dq(X)}{d\Sigma} = -n\Sigma^{-1} + \Sigma^{-1}Z^T\Sigma^{-1}.$$

When the above equation is equated to zero, we obtain

$$\Sigma = \frac{1}{n}Z^T = \frac{1}{n}E_p[X]. \quad (4)$$

Note that we obtain the same $\Sigma$ when we equate the expectations of the matrix with respect to distributions $p(X)$ and $q(X)$, i.e., $E_p[X] = E_q[X]$ and by fixing the degrees of freedom to be $n_2$. But the question arises as to whether the degree of freedom of the distribution obtained from minimizing K-L divergence is indeed $n_2$, which is nothing but the number of columns of the matrix $H$. In order to answer this question, we will continue the K-L divergence minimization by differentiating \(3\) w.r.t. $n$. Now differentiating equation \(3\) w.r.t. $n$, we obtain

$$\frac{dq(X)}{dn} = -\ln|\Sigma| + Y - \sum_{i=1}^{n_1} \psi(n - i + 1),$$

where $\psi(.)$ is the digamma function \[32\]. Equating the derivative to zero, we get

$$-\ln|\Sigma| + Y - \sum_{i=1}^{n_1} \psi(n - i + 1) = 0.$$

By substituting $\Sigma = \frac{1}{n}Z^T$ from \(4\), we obtain,

$$\ln(n) - \ln|Z| + Y - \sum_{i=1}^{n_1} \psi(n - i + 1) = 0. \quad (5)$$

Matching the expectations $E_p[\ln|X|] = E_q[\ln|X|]$ also leads to \(5\). Hence minimizing the K-L divergence has reduced to a simple case of matching expectations $E_p[X]$ and $E_p[\ln|X|]$ with $E_q[X]$ and $E_q[\ln|X|]$ respectively.
Solving (5) for \( n \) requires the knowledge of \( Y = E_p[ln|X|] \). Since finding the random matrix variate distribution of \( X \) is mathematically intractable, finding the expectation of the log-determinant analytically is not possible. We now argue that approximating \( n \) by \( n_2 \) is a plausible solution to the problem.

a) While \( Y = E_p[ln|X|] \) cannot be theoretically computed, it can be empirically computed by using simulated \( \kappa - \mu/\eta - \mu \) matrix elements and then solving for \( n \) in (5). We observe that in all our simulations, this leads to \( n \) being a real number which is very close to \( n_2 \).

b) Note, \( n_2 \) denotes the number of transmitter antennas or the number of interferers in the MIMO channel matrix. Hence, it makes sense to retain the same number \( n_2 \), even in approximation, given that there is no correlation in the transmitter side and all the elements of the matrix are i.i.d.

c) For any Wishart distributed matrix \( A = BB^H \sim CW_{n_1}(n_2, \Sigma) \), the degrees of freedom also denote the number of columns of complex Gaussian \( B \). In fact, when a non-central Wishart matrix was approximated by a Wishart matrix in [29], the approach of keeping \( n = n_2 \) was followed.

Based on the above reasoning it can be argued that, \( n \approx n_2 \). Hence, the minimizer complex Wishart distribution, \( q(X) \) given by \( CW_{n_1}(n_2, \frac{1}{n_2}Z^T) \), where \( Z = E_p[X] \), is the closest to the actual unknown distribution among all central Wishart distributions in terms of K-L divergence. We will now apply the approximation procedure to i.i.d. \( \kappa - \mu \) and \( \eta - \mu \) fading MIMO channels and theoretically determine the covariance matrix \( \Sigma = \frac{1}{n_2}(E_p[X])^T = \frac{1}{n_2}Z^T \) of the corresponding approximate Wishart matrix.

Once the first moment is matched by the above procedure, we now quantify how close the second moment of the two matrices are to one another. For this, we determine NMSE (Normalized Mean Square Error) between the second moments of the two matrices. First, let \( \hat{X} = \sum [XX^H] - \sum [X'X'^H] \), where \( X' \) is the Wishart approximated matrix and \( \sum [\cdot] \) denotes the empirical average of a matrix. Also, \( X = HH^H \) where \( H \) is a matrix with i.i.d. \( \kappa - \mu \) elements. In other words, \( \hat{X} \) is used to measure the discrepancy between the second moments of the actual matrix and Wishart distributed approximation. Second, we determine the absolute value of all the elements of the matrix \( \hat{X} \), i.e., \( \hat{X}(i,j) = |\hat{X}(i,j)| \). We now determine NMSE, i.e., the ratio of average of elements of \( \hat{X} \) to the average of elements of \( \sum [XX^H] \) as

\[
NMSE = \frac{1\hat{X}1^T}{1\sum [XX^H]1^T}.
\]
Table I: NMSE for $n_1 = 2$

| $n_2$ | $\kappa = 2$ and $\mu = 3$ | $\kappa = 6$ and $\mu = 3$ | $\kappa = 2$ and $\mu = 6$ | $\kappa = 6$ and $\mu = 6$ |
|-------|----------------------------|----------------------------|----------------------------|----------------------------|
| 2     | 0.4247                     | 0.4633                     | 0.4631                     | 0.4769                     |
| 4     | 0.2182                     | 0.2328                     | 0.2362                     | 0.2417                     |
| 6     | 0.1511                     | 0.1563                     | 0.1574                     | 0.1631                     |
| 8     | 0.1115                     | 0.1198                     | 0.1195                     | 0.1205                     |

Table I shows the NMSE for various values of degree of freedom $n_2$, $\kappa$ and $\mu$. We can see that as the degree of freedom $n_2$ increases, NMSE decreases, i.e., the approximation becomes tighter, whereas, if $\kappa$ or $\mu$ increases, NMSE increases.

A. $\kappa - \mu$ model

In $\kappa - \mu$ fading model, the signal is divided into different clusters of waves. The number of clusters is $\mu$ and in each of the clusters, there is a deterministic LOS component with arbitrary power and scattered waves with identical powers. Note, $\kappa$ is the ratio between the total power of the dominant components and the total power of the scattered waves. Suppose the elements $h_{ij} = x_{ij} + jy_{ij}$ of $H$ are i.i.d. $\kappa - \mu$ random variables, where $x_{ij}$ and $y_{ij}$ are the real and imaginary components respectively, then the joint distribution is given by [25],

$$f_{xy}(x_{ij}, y_{ij}) = \frac{|x_{ij}y_{ij}|^{\mu/2}}{4\sigma^4|p|q|^{\mu/2-1}} e^{-\left(\frac{(x_{ij} - p)^2 + (y_{ij} - q)^2}{2\sigma^2}\right)} \text{sech}\left(\frac{px_{ij}}{\sigma^2}\right) \text{sech}\left(\frac{qy_{ij}}{\sigma^2}\right) I_{\frac{\mu}{2}-1}\left(\frac{|px_{ij}|}{\sigma^2}\right) I_{\frac{\mu}{2}-1}\left(\frac{|qy_{ij}|}{\sigma^2}\right).$$ (6)

Here $p^2 = \sum_{i=1}^{\mu} p_i^2$ and $q^2 = \sum_{i=1}^{\mu} q_i^2$, where $p_i$ and $q_i$ are the LOS components of in-phase and quadrature components respectively of multipath waves of each cluster. $\kappa = \frac{p^2 + q^2}{2\mu\sigma^2}$, where $\sigma^2$ is the power of the scattered waves. We are interested in the distribution of $HH^H$. However, an exact characterization of the matrix variate distribution is intractable, because it involves finding the joint pdf of entries of $HH^H$. Hence, we now approximate $HH^H$ by the Wishart matrix with $n_2$ degrees of freedom and $\Sigma = \frac{1}{n_2} \mathbb{E}_p[HH^H]$, where the expectation is with respect to the $\kappa - \mu$ pdf. The diagonal elements of $Z = \mathbb{E}_p[HH^H]$, i.e., $z_{ii}$ are nothing but the mean of $n_2$ $\kappa - \mu$
The envelope square variables, i.e., $z_{ii} = \mathbb{E} \left[ \sum_{j=1}^{n_2} [x_{ij}^2 + y_{ij}^2] \right]$. Hence, $z_{ii} = 2\sigma^2 n_2 (1 + \kappa) \mu$ from (7), where $\kappa = \frac{p^2 + q^2}{2\mu \sigma^2}$. The off diagonal elements of $z_{ij}$ are given by

$$z_{ij} = \mathbb{E} \left[ \sum_{k=1}^{n_2} [(x_{ik} + j y_{jk})(x_{kj} - j y_{kj})] \right] = \sum_{k=1}^{n_2} \mathbb{E}[x_{ik}x_{kj} + y_{ik}y_{kj} - jx_{ik}y_{kj} + jx_{kj}y_{ik}].$$

Since $x_{ik}$ and $y_{ik}$ are i.i.d., we obtain $\forall i, j,$

$$z_{ij} = \sum_{k=1}^{n_2} (\mathbb{E}[x_{ik}])^2 + (\mathbb{E}[y_{kj}])^2. \quad (7)$$

$\mathbb{E}[x_{ik}]$ and $\mathbb{E}[y_{ik}]$ are given by,

$$\mathbb{E}[x_{ik}] = \int_{-\infty}^{\infty} x \frac{|x|^{\mu/2}}{2 \sigma^2 |p|^{\mu/2} - 1} \exp\left(-\frac{(x - p)^2}{2 \sigma^2}\right) \text{sech}\left(\frac{px}{\sigma^2}\right) I_{\frac{\mu}{2} - 1}(\frac{|px|}{\sigma^2}) dx \quad (8)$$

$$\mathbb{E}[y_{ik}] = \int_{-\infty}^{\infty} y \frac{|y|^{\mu/2}}{2 \sigma^2 |q|^{\mu/2} - 1} \exp\left(-\frac{(y - q)^2}{2 \sigma^2}\right) \text{sech}\left(\frac{qy}{\sigma^2}\right) I_{\frac{\mu}{2} - 1}(\frac{|qy|}{\sigma^2}) dy. \quad (9)$$

$\forall i, j$. The closed form expressions for the above integrals seem mathematically intractable. However an approximation for the integral is derived in Appendix A and given by (39) and (40). Since all the $\kappa - \mu$ elements of the matrix $H$ are i.i.d., the mean of all the off-diagonal elements are equal. Substituting the results from (39) and (40) in (7), we obtain $\forall i, j$ and $i \neq j$,

$$z_{ij} \approx n_2 \left[ 2p e^{-\frac{p^2}{2\sigma^2}} \left(\frac{4\sigma^2}{4\sigma^2 + 2p^2}\right)^{\frac{\mu}{2} + 1} \left(\frac{2\sigma^2}{2\sigma^2 + 4\sigma^2}\right)^{\frac{\mu}{2} + 1} \Psi_1\left(\frac{\mu}{2} + 1, 1, 3/2, \frac{2p^2}{2\sigma^2 + 4\sigma^2}, \frac{2p^2}{4\sigma^2 + 2p^2}\right)^2 \right. \right.$$

$$+ \left. \left[ 2q e^{-\frac{q^2}{2\sigma^2}} \left(\frac{4\sigma^2}{4\sigma^2 + 2q^2}\right)^{\frac{\mu}{2} + 1} \left(\frac{2q^2}{2q^2 + 4\sigma^2}\right)^{\frac{\mu}{2} + 1} \Psi_1\left(\frac{\mu}{2} + 1, 1, 3/2, \frac{2q^2}{2q^2 + 4\sigma^2}, \frac{2q^2}{4\sigma^2 + 2q^2}\right)^2 \right] \right]. \quad (10)$$

Since $\Sigma = \frac{Z}{n_2}$, we have

$$\Sigma_{ii} = 2\sigma^2 (1 + \kappa) \mu \text{ and } \Sigma_{ij} = z_{ij}/n_2, \ i \neq j \quad (11)$$

Now we look at a special case of $\kappa - \mu$ distribution namely the Rician distribution with $\mu = 1$.

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3These closed form approximations are also compared with both numerical integration evaluation of (8) and (9) and Monte-Carlo simulation and an excellent match is observed with both.
**Rician:** Suppose the elements $h_{i,j} = x_{ij} + jy_{ij} \sim CN(p_{ij} + jq_{ij}, 2\sigma^2)$ of $H$ are independent Rician distributed random variables, where $x_{ij}$ and $y_{ij}$ are the real and imaginary components respectively, then the joint distribution is given by [25],

$$f_{xy}(x_{ij}, y_{ij}) = \frac{1}{2\pi\sigma^2}e^{\exp\left(-\frac{(x_{ij} - p_{ij})^2 + (y_{ij} - q_{ij})^2}{2\sigma^2}\right)}$$

with identical Rice factor $\kappa = \frac{p_{ij}^2 + q_{ij}^2}{2\sigma^2}$. The diagonal elements $z_{ii}$ of $n_2\Sigma$ are nothing but mean of $n_2$ sum of Rician envelope square variables, i.e., $z_{ii} = \mathbb{E}[\sum_{j=1}^{n_2}[x_{ij}^2 + y_{ij}^2]] = 2\sigma^2n_2(1 + \kappa)$ and the off-diagonal elements $z_{ij} = \sum_{k=1}^{n_2}(p_{ik} + jq_{ik})(p_{kj} - jq_{kj})$.

Since the elements of $H$ have a Rician envelope, $X$ is a non-central Wishart matrix. In this case, we are simply approximating a non-central Wishart matrix by a central Wishart matrix. This has been well studied in [29], which obtains the same approximation as us, but by deriving the moment generating function (mgf) of the non-central Wishart matrix and retaining the same degree of freedom. If the columns of the matrix $H$ with dimension $n_1 \times n_2$ are distributed as complex Gaussian $CN(m, \Sigma')$, then $X$ is a non-central Wishart matrix distributed as $CW_{n_1}(n_2, \Sigma', MM^H)$, where $M = [m_1, m_2, ..., m_s]$. From [29], a non-central Wishart matrix $CW_{n_1}(n_2, \Sigma', MM^H)$ can be approximated by a Wishart matrix $CW_{n_1}(n_2, \Sigma)$, where $\Sigma = \Sigma' + \frac{1}{n_2}MM^H$. It is also known that the approximation becomes tighter as the degree of freedom $n_2$ increases. In our case also, it can be shown that $\Sigma = \Sigma' + \frac{1}{n_2}MM^H$.

If the variables are identically distributed, i.e., $p_{ij}$s are equal to $p$ and $q_{ij}$s are equal to $q$, then the off-diagonal elements are $n_2(p^2 + q^2) = 2\sigma^2n_2\kappa$ and it can be seen from a numerical evaluation that it is approximately equal to (10) evaluated at $\mu = 1$.

### B. $\eta - \mu$ model

The $\eta - \mu$ is a fading distribution that represents small scale fading effects in non-line of sight condition. The elements $h_{ij}$ of $H$ are independent and identical $\eta - \mu$ distributed random variables with density [26],

$$f_{xy}(x_{ij}, y_{ij}) = \frac{\mu^{2\mu}|x_{ij}y_{ij}|^{2\mu-1}}{\Omega_X^\mu\Omega_Y^\mu\Gamma^2(\mu)}\exp\left(-\mu \left(\frac{x_{ij}^2}{\Omega_X} + \frac{y_{ij}^2}{\Omega_Y}\right)\right)$$ (12)

where $\Omega$ is the power parameter given by $\Omega = 2\sigma^2\mu$, $\sigma^2$ is the power of the Gaussian variable in each cluster, $\mu$ is the number of clusters. Note, $\Omega_X = (1 - \eta)\Omega/2$, $\Omega_Y = (1 + \eta)\Omega/2$, $-1 \leq \eta \leq 1$ and the diagonal elements $z_{ii}$ are means of sums of $\eta - \mu$ envelope square
variables. i.e., $z_{ii} = \mathbb{E}[\sum_{j=1}^{n_2} (x_{ij}^2 + y_{ij}^2)]$. Hence by (7), $z_{ii} = n_2(\Omega_X + \Omega_Y)$. The off-diagonal elements are given by,

$$z_{ij} = \mathbb{E}\left[\sum_{k=1}^{n_2} ((x_{ik} + jy_{ik})(x_{kj} - jy_{kj}))\right]$$

Since $x_{ik}$ and $y_{ik}$ are i.i.d., we obtain $\forall i, j$ and $i \neq j$,

$$z_{ij} = \sum_{k=1}^{n_2} (\mathbb{E}[x_{ik}^2] + \mathbb{E}[y_{kj}^2]) = 0$$

The off-diagonal elements are zero, because distributions $f_x(x_{ij})$ and $f_y(y_{ij})$ are odd functions. Therefore, $\Sigma = (\Omega_X + \Omega_Y)I_{n_1}$. It is interesting to note that, the approximation doesn’t depend on $\eta$.

C. Nakagami-m model

The elements $h_{ij}$ of $H$ are independent and identically distributed Nakagami-m variables with density,

$$f_{xy}(x_{ij}, y_{ij}) = \frac{m^m|x_{ij}|^{m-1}|y_{ij}|^{m-1}}{\Gamma^2(m/2)} \exp\left(-\frac{m}{\Omega} (x_{ij}^2 + y_{ij}^2)\right)$$

We can obtain a Nakagami random variable by substituting $\eta = 0$ i.e., $\Omega_X = \Omega_Y = \Omega/2$ and $m = 2\mu$ in $\eta - \mu$ random variable given by (12). It can also be obtained by substituting $\kappa = 0$ i.e., $p = q = 0$ and $\Omega = 2\mu\sigma^2$ in $\kappa - \mu$ random variable given by (6). Hence, both the approaches yield the same result, i.e., $\Sigma = \Omega I_{n_1}$. It is interesting to note that, existing work [22] has analyzed MIMO ergodic capacity of correlated Nakagami-m fading channels and derived a joint pdf of eigen-values of $HH^H$ using copula. But the analysis is performed only for $2 \times 2$ channel matrix and it becomes fairly difficult even for a $3 \times 3$ channel matrix.

III. APPLICATIONS OF THE APPROXIMATION

In this section, to demonstrate the utility of our work, we apply the above approximation in two very different applications namely, finding MIMO channel capacity for $\kappa - \mu/\eta - \mu$ faded channel coefficients and finding SER expressions for optimum combining with $\kappa - \mu/\eta - \mu$ faded interferers. Finding ergodic MIMO channel capacity involves finding the expectation of log determinant of Gram matrix $HH^H$, where entries of $H$ are $\kappa - \mu/\eta - \mu$ faded. On the other hand, finding SER expressions for optimum combining involves determining the SINR given by $\eta = c^H R^{-1} c$, where $R = HH^H$ is the Gram matrix formed by $\kappa - \mu/\eta - \mu$ faded interferers and $c$ is the user signal.
A. MIMO channel Capacity

We consider an $N_R \times N_T$ MIMO channel matrix $H$, where $N_R$ denotes the number of receive antennas and $N_T$ denotes the number of transmit antennas. Let $x$ be the $N_T \times 1$ transmitted vector and $n$ be the $N_R \times 1$ zero mean i.i.d. complex Gaussian noise vector. The $N_R \times 1$ received vector $y$ is given by, $y = Hx + n$. Assuming that the transmitter has no channel state information (CSI), the capacity of the MIMO channels when the transmitter has no CSI, is given by \[3\],

$$C' = \log_2 \det(I + \frac{\rho}{N_T} HH^H),$$ \hspace{1cm} (13)

where $\rho$ is the average signal to noise ratio (SNR) per receiving antenna. Since $HH^H$ and $H^HH$ have the same non-zero eigenvalue statistics, from \[3\],

$$C' = \sum_{i=1}^{n_1} \log_2(1 + \frac{\rho}{N_T} \lambda_i),$$

where $n_1 = \min(N_R, N_T)$ and $\lambda_1, \ldots, \lambda_{n_1}$ are the non-zero eigenvalues of $R$, which is given by,

$$R = \begin{cases} HH^H & \text{if } N_R \leq N_T \\ H^HH & \text{if } N_R > N_T. \end{cases}$$

Hence, the mean value of $C'$ is given by \[2\],

$$C = \mathbb{E}_\Lambda \left[ \sum_{i=1}^{n_1} \log_2(1 + \frac{\rho}{N_T} \lambda_i) \right].$$

We do not know the exact eigenvalue distribution of $R$, when $H$ comprises i.i.d. $\kappa - \mu$ or $\eta - \mu$ variables. Hence, we apply the Wishart approximation developed in the last section and then determine $C$. We approximate $R$ by a Wishart matrix with a degree of freedom $N_T$ and $N_R \times N_R$ covariance matrix $\Sigma$, given by \[11\] for $\kappa - \mu$ interferers and $(\Omega_X + \Omega_Y)I_{N_R}$ for $\eta - \mu$ interferers.

1) $\kappa - \mu$: The approximate expression for $C$ is derived in Appendix. B. For $N_T \geq N_R$, by substituting $n_2 = N_T$ and $n_1 = N_R$ in \[44\], we can get the average capacity approximation as,

$$C \approx (-1)^{\frac{1}{2}N_R(N_R-1)} \frac{1}{\prod_{j=1}^{N_R} (N_T - j)!} (\det \Sigma)^{N_T} (w_2 - w_1)^{N_R-1} \prod_{j=1}^{N_R-2} j! \sum_{k=1}^{N_R-1} |N^k|, $$ \hspace{1cm} (14)

where $N^k$ is given in \[45\] and $w_1 = 2\sigma^2 (1 + \kappa) \mu - y$ and $w_2 = 2\sigma^2 (1 + \kappa) \mu + (N_R - 1) y$ are the eigenvalues of $\Sigma^{-1}$ with multiplicity $N_R - 1$ and 1 respectively.

In case $N_T < N_R$, we approximate $H^HH$ instead of $HH^H$ since both have the same non-zero eigenvalues. We therefore approximate $H^HH$ by a central Wishart $W \sim \mathcal{CW}_{N_R}(N_R, \Sigma)$. For
$N_T \leq N_R$, by substituting $n_2 = N_R$ and $n_1 = N_T$ in (44), we can get the capacity approximation as,

$$C \approx (-1)^{\frac{1}{2}N_T(N_T-1)} \frac{1}{\prod_{j=1}^{N_T} (N_R - j)!} \frac{1}{\prod_{i=1}^{N_R} (N_T - i)!} \sum_{k=1}^{N_T-1} |N^k|,$$

(15)

where $N^k$ is given in (45) and $w_1 = 2\sigma^2(1 + \kappa)\mu - y$ and $w_2 = 2\sigma^2(1 + \kappa)\mu + (N_T - 1)y$ are the eigenvalues of $\Sigma$ with multiplicity $N_T - 1$ and 1 respectively.

2) $\eta - \mu$: If $R = HH^H$ with $H$ having i.i.d. $\eta - \mu$ elements, we use the Wishart approximation of $R$ and follow a procedure similar to that used for $\kappa - \mu$. For $N_T \geq N_R$, by substituting $n_2 = N_T$ and $n_1 = N_R$ in (46), we can obtain the capacity approximation as,

$$C' \approx \frac{((\Omega_X + \Omega_Y))^{-N_RN_T}}{\prod_{i=1}^{N_R} (N_T - i)!} \frac{\sum_{k=1}^{N_R} |N^k|}{\prod_{i=1}^{N_T} (N_T - i)!},$$

(16)

where $N^k$ is given by (47). Similarly for $N_T \leq N_R$, by substituting $n_2 = N_R$ and $n_1 = N_T$ in (46), we can obtain the capacity approximation as,

$$C' \approx \frac{((\Omega_X + \Omega_Y))^{-N_TN_T}}{\prod_{i=1}^{N_R} (N_T - i)!} \frac{\sum_{k=1}^{N_T} |N^k|}{\prod_{i=1}^{N_T} (N_T - i)!},$$

(17)

where $N^k$ is given by (47). Since, $\Omega_X = (1 - \eta)\Omega/2$ and $\Omega_Y = (1 + \eta)\Omega/2$, the approximate capacity expressions depends only on the power parameter $\Omega$ and not on the $\eta$ parameter. In [5] and [2], exact capacity expressions are derived for Rayleigh faded MIMO channels. The results from these expressions match our $\eta/\mu$ capacity expressions for Rayleigh faded MIMO channels, i.e., for $\eta = 0$ and $\mu = 1$. Also, an upper bound for the ergodic capacity of $\kappa - \mu$ and $\eta - \mu$ faded MIMO channels is derived in [23]. However, the upper bound requires computation of the mean of each entry of $H$ given by $E[h_{ij}]$, for which a numerical computation is done in [23]. Hence, we can apply our mean approximation in [23] to evaluate the upper bound. The upper bound is plotted in Section IV and compared with our theoretical approximation.

3) Asymptotics: Since we have approximated $\eta - \mu$ faded MIMO channels by a complex Wishart matrix, a lot of existing properties and results of complex Wishart matrix can be exploited to get interesting results for these channels. One such application is in determining the asymptotic capacity of generalized fading channels, especially $\eta - \mu$ faded channel. The asymptotic capacity of Rayleigh faded channels is studied in detail in [33]. We now use their analysis to study $\eta - \mu$ asymptotics. For $N_T = N_R = N$, and $\eta - \mu$ fading, the capacity is given by

$$C = E_{\Omega} \sum_{i=1}^{N} \log_2\left(1 + \frac{\rho(\Omega_X + \Omega_Y)}{N} \lambda_i\right),$$
where $\lambda_i$ are the eigen values of $H \sim \mathcal{C}\mathcal{N}_N(0, I_N)$. Using [33], we obtain the asymptotic capacity as

$$\lim_{N \to \infty} E_{\Lambda} \frac{1}{N} \sum_{i=1}^{N} \log_2(1 + \frac{\rho(\Omega_X + \Omega_Y)}{N} \lambda_i) = \int_{0}^{\infty} \log_2(1 + \rho(\Omega_X + \Omega_Y)\lambda)g(\lambda)d\lambda$$

where

$$g(\lambda) = \begin{cases} \frac{1}{\pi} \sqrt{\lambda - \frac{1}{4}} & 0 \leq \lambda \leq 4 \\ 0 & \text{o.w.} \end{cases}$$

Solving the above integral, we obtain the asymptotic capacity as

$$\lim_{N \to \infty} E_{\Lambda} \frac{1}{N} \sum_{i=1}^{N} \log_2(1 + \frac{\rho}{N} \lambda_i) = \frac{\rho(\Omega_X + \Omega_Y)}{\ln 2} 3F_2(1, 1, 3/2; 2, 3; -4\rho(\Omega_X + \Omega_Y))$$

where $_3F_2(\cdot)$ is a Hypergeometric function. With a first order approximation of the logarithm at low SNR as in [33],

$$\lim_{N \to \infty} \frac{1}{N} C \approx \int_{0}^{\infty} \rho(\Omega_X + \Omega_Y)\lambda g(\lambda)d\lambda = \rho(\Omega_X + \Omega_Y).$$

Similarly at high SNR, we obtain from [33],

$$\lim_{N \to \infty} \frac{1}{N} C \approx \log_2(\rho(\Omega_X + \Omega_Y)/e).$$

Since $(\Omega_X + \Omega_Y) = \Omega = 2\mu\sigma^2$, the capacity grows as a linear function of $\mu$ and SNR $\rho$, at low SNR and capacity grows as a logarithmic function of $\mu$ and SNR $\rho$, at high SNR.

For $\kappa - \mu$ random variables, the capacity is given by

$$C = E[\log_2 \text{det}(I + \frac{\rho}{N} HH^H)],$$

where $H \sim \mathcal{C}\mathcal{N}_N(0, \Sigma)$ and $w_1 = 2\sigma^2(1 + \kappa)\mu - y$ and $w_2 = 2\sigma^2(1 + \kappa)\mu + (N - 1)y$ are the eigenvalues of $\Sigma$ with multiplicity $N - 1$ and 1. Unlike $\eta - \mu$ random variable, it is difficult to obtain the asymptotic capacity like in (20) for all SNR values, due to the presence of correlation matrix $\Sigma$. Hence, we will derive approximate asymptotic capacity only at high SNR. At high SNR, the capacity is given by

$$C \approx E[\log_2 \text{det}(\frac{\rho}{N} HH^H)] = E[\log_2 \text{det}(\frac{\rho}{N} \Sigma^{1/2} H' H^H \Sigma^{1/2})] = E[\log_2 \text{det}(\frac{\rho}{N} H' H^H)] + \log_2 \text{det}(\Sigma)$$

$$= E_{\Lambda} \sum_{i=1}^{N} \log_2(\frac{\rho}{N} \lambda_i) + \log_2(w_1^{N-1} w_2).$$
where $H' \sim C \mathcal{N}_N(0, I)$. Therefore, the asymptotic capacity is given by,

$$
\lim_{N \to \infty} \frac{1}{N} C \approx \lim_{N \to \infty} E_N \frac{1}{N} \sum_{i=1}^{N} \log_2(\frac{\rho}{N} \lambda_i) + \lim_{N \to \infty} \frac{N - 1}{N} \log_2(w_1) + \lim_{N \to \infty} \frac{1}{N} \log_2(w_2)
$$

$$
= \int_0^4 \log_2(\rho \lambda) g(\lambda) + \log_2(w_1) + 0 = \log_2(\rho) + \log_2(w_1).
$$

(22)

From the above equation it is clear that, at high SNR, capacity grows as a logarithmic function of SNR $\rho$.

**B. SER Optimum combining**

One other application where our approximation can be used is in determining SER expressions for OC with $\kappa - \mu / \eta - \mu$ interferers. Though there exist some results that compute bounds for the capacity of $\kappa - \mu / \eta - \mu$ faded MIMO channels, there exists no such prior literature for OC, to the best of our knowledge, where the interferers are $\kappa - \mu / \eta - \mu$ faded. Let $c$ denote the $N_R \times 1$ channel from the desired transmitter to the user, $c_i$ denote the $N_R \times 1$ channel from the $i^{th}$ interferer to the user, $x$ denotes the desired user symbol belonging to unit energy QAM constellation and $x_i$ denote the $i^{th}$ interferer symbol also belonging to a unit energy QAM constellation. The $N_R \times 1$ received vector is given by,

$$
y = cx + \sqrt{E_I} \sum_{i=1}^{N_I} c_i x_i + n,
$$

(23)

where $E_I$ is the mean interferers power, $n$ is the $N_R \times 1$ additive white complex Gaussian noise vector with power $\sigma^2$ per dimension, i.e., $n \sim C \mathcal{N}(0, \sigma^2 I_{N_R})$ and $N_I$ denotes the number of interferers. The user channel is modeled as i.i.d. Rayleigh i.e., $c \sim C \mathcal{N}(0, I_{N_R})$. The interferer channels are modeled as equal power i.i.d. $\kappa - \mu$ or $\eta - \mu$. The $N_R \times N_R$ covariance matrix of the interference term is,

$$
R = \mathbb{E}[(\sum_{i=1}^{N_I} c_i x_i) (\sum_{i=1}^{N_I} c_i x_i)^H].
$$

(24)

In order to derive the expression for SER, we first consider the expression for SINR of OC given by [34],

$$
\eta = \frac{1}{E_I} c^H (R + \frac{\sigma^2}{E_I} I)^{-1} c.
$$

(25)

Let $\lambda_1, \lambda_2, ..., \lambda_{N_R}$ denote the eigenvalues of $R$. Then, $R = U \Lambda U^H$ by eigen-value decomposition, where $U$ is the matrix composed of orthonormal eigen vectors, corresponding to the
eigenvalues of \( R \) and \( \Lambda \) is the diagonal matrix of eigenvalues. The received SINR is now given by,

\[
\eta = \frac{1}{E_I} c^H (R + \sigma^2 I)^{-1} c = \frac{1}{E_I} \tilde{c}^H (\Lambda + \sigma^2 E_I I)^{-1} \tilde{c},
\]

(26)

where \( \tilde{c} = U^H c \). Defining \( p_k = |\tilde{c}_k|^2 \),

\[
\eta = \sum_{k=1}^{N_R} \frac{p_k}{E_I} \lambda_k + \frac{\sigma^2}{E_I}.
\]

(27)

Since \( \tilde{c} \) is spherically invariant, it will have the same distribution as \( c \). Since \( c_k \) are i.i.d. complex Gaussian with zero means, \( p_k \) are i.i.d. exponential random variables with unit means. Using the standard assumption that the contribution of the interference and the noise at the output of optimal combiner, for a fixed \( \eta \), can be well-approximated to be Gaussian, as in [35] and [36] and references therein, the probability of symbol error for an M-ary square QAM constellation is given by [37],

\[
P_e = k_1 Q(\sqrt{k_2 \eta}) - k_3 Q(\sqrt{k_2 \eta})^2,
\]

(28)

where \( k_1 = 4(1 - \frac{1}{\sqrt(M)}) \), \( k_2 = \frac{3}{M-1} \), \( k_3 = \frac{k_2}{4} \) and the Q-function is given by \( Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-u^2/2} du \). The assumption is valid even when the number of interferers \( N_I \) is small [36] and such a system model assumption is made in a number of papers [34], [38], [39] to derive the SER expression. Using the popular approximation \( Q(x) \approx \frac{1}{12} e^{-\frac{x^2}{2}} + \frac{1}{4} e^{-\frac{3x^2}{4}} \) from [40], one can write \( P_e \) as,

\[
P_e = \sum_{l=1}^{5} a_l e^{-b_l \eta},
\]

(29)

where \( a_1 = \frac{k_1}{12} \), \( a_2 = \frac{k_2}{4} \), \( a_3 = -\frac{k_3}{144} \), \( a_4 = -\frac{k_3}{16} \), \( a_5 = -\frac{k_3}{24} \), \( b_1 = \frac{k_2}{2} \), \( b_2 = \frac{2k_3}{3} \), \( b_3 = k_2 \), \( b_4 = \frac{4k_3}{3} \) and \( b_5 = \frac{2k_2}{6} \). The exponential approximation of the Q-function is shown to be tight in [40] and a similar approximation is used in [41], [42]. The average SER obtained by averaging \( P_e \) over all channel realizations is derived as follows:

\[
SER = \mathbb{E}_\eta[P_e] = \mathbb{E}_\eta[\sum_{l=1}^{5} a_l e^{-b_l \eta}] = \sum_{l=1}^{5} a_l \mathbb{E}_\eta[e^{-b_l \eta}].
\]

(30)

Substituting for \( \eta \) from (27) in the above equation and also rewriting the expectation over \( \eta \) using the fact that \( \Lambda \) and \( \mathbf{p} = [p_1 p_2 ... p_{N_R}] \) are independent, we get,

\[
SER = \sum_{l=1}^{5} a_l \mathbb{E}_\Lambda \left[ \mathbb{E}_\mathbf{p} \left[ e^{-b_l \sum_{k=1}^{N_R} \frac{p_k}{E_I} \lambda_k + \frac{\sigma^2}{E_I}} \right] \right].
\]

(31)
Each of the $p_k$ is an independent exponential random variable and the m.g.f. of an exponential random variable $X$ with mean $\omega$ is given by $\mathbb{E}[e^{tX}] = \frac{\omega}{\omega-t}$ for $t < \omega$. Hence, we can write (31) as,

$$SER = \sum_{l=1}^{5} a_l \mathbb{E}_X \mathbb{E}_P \left[ \prod_{k=1}^{N_R} \left( e^{-b_k \frac{p_k}{E_I}} \right) \right] = \sum_{l=1}^{5} a_l \mathbb{E}_\Lambda \left[ \prod_{k=1}^{N_R} \frac{\lambda_k + \frac{\sigma^2}{E_I} + \frac{b_k}{E_I}}{\lambda_k + \frac{\sigma^2}{E_I} + \frac{b_k}{E_I}} \right].$$ (32)

The problem reduces to determining an expectation $\mathbb{E}_\Lambda \left[ \prod_{k=1}^{n_1} \frac{\lambda_k + \frac{\sigma^2}{E_I} + \frac{b_k}{E_I}}{\lambda_k + \frac{\sigma^2}{E_I} + \frac{b_k}{E_I}} \right]$, where $l = 1, \ldots, 5$ and $\lambda_1, \lambda_2, \ldots, \lambda_{n_1}$ denote the eigenvalues of $R$. Let $J(l, n_1) = \mathbb{E}_\Lambda \left[ \prod_{k=1}^{n_1} \frac{\lambda_k + \frac{\sigma^2}{E_I} + \frac{b_k}{E_I}}{\lambda_k + \frac{\sigma^2}{E_I} + \frac{b_k}{E_I}} \right]$. For this case also, we use the Wishart approximation of $R$, i.e., we approximate $R$ to a Wishart matrix with degree of freedom $N_I$ and $N_R \times N_R$ covariance matrix $\Sigma$.

1) $\kappa - \mu$: The approximate expression for $J(l, n_1)$ is given in (49) in Appendix C. For $N_I \geq N_R$, we can get the SER approximation directly by substituting $n_2 = N_I$ and $n_1 = N_R$ in the approximation for $J(l, n_1)$ given in (49) in Appendix C as,

$$SER \approx \sum_{l=1}^{5} a_l (-1)^{\frac{1}{2}N_R(N_R-1)} \frac{1}{\prod_{j=1}^{N_R}(N_I-j)!} |\Sigma|^{N_I(N_R-1)} \prod_{j=1}^{N_R-2} j!$$

where $M$ matrix is given in (50) and $w_1 = 2\sigma^2$ and $w_2 = N_R^2 \sigma^2 (1 + \kappa) \mu - (N_R - 1) 2\sigma^2$ are the eigenvalues of $\Sigma^{-1}$ with multiplicity $N_R - 1$ and 1 respectively. For $N_I \leq N_R$, the number of non-zero eigenvalues of $\Sigma$ is only $N_I$. Hence,

$$SER = \sum_{l=1}^{5} a_l \left( \frac{\sigma^2}{E_I} + \frac{b_l}{E_I} \right)^{N_R-N_I} \mathbb{E}_\Lambda \left[ \prod_{k=1}^{N_I} \frac{\lambda_k + \frac{\sigma^2}{E_I} + \frac{b_k}{E_I}}{\lambda_k + \frac{\sigma^2}{E_I} + \frac{b_k}{E_I}} \right].$$ (33)

Hence, for $N_I \leq N_R$, we apply the same logic that was applied in the capacity calculations for $N_T \leq N_R$. We thus obtain the SER approximation from $J(l, n_1)$ in (49), but with $n_2 = N_R$ and $n_1 = N_I$,

$$SER \approx \sum_{l=1}^{5} a_l \left( 1 + \frac{b_l}{\sigma^2} \right)^{N_I-N_R} (-1)^{\frac{1}{2}N_I(N_I-1)} \frac{1}{\prod_{j=1}^{N_I}(N_R-j)!} (w_1-w_2)^{N_I-1} \prod_{j=1}^{N_I-2} j!$$

(34)

where $M$ matrix is given in (50) and $w_1 \approx 2\sigma^2$ and $w_2 \approx N_T^2 \sigma^2 (1 + \kappa) \mu - (N_T - 1) 2\sigma^2$ are the eigenvalues of $\Sigma^{-1}$ with multiplicity $N_T - 1$ and 1 respectively.
2) $\eta - \mu$: For $N_I \geq N_R$, we can get the SER approximation directly by substituting $n_2 = N_I$ and $n_1 = N_R$ in (51) in Appendix C as,

$$SER \approx \sum_{i=1}^{5} a_i \left( \frac{\Gamma(N_I-N_R+i+j-1)(\Omega_X+\Omega_Y)^{N_I-N_R+i+j-2}}{(\Omega_X+\Omega_Y-\frac{b_l}{\sigma^2}E_l e^{\frac{\sigma^2}{\Omega_X+\Omega_Y}E_N}-\frac{b_l}{\sigma^2}E_l e^{\frac{\sigma^2}{\Omega_X+\Omega_Y}E_N})} \right) \times det \{ \Gamma(N_I-N_R+i+j-1)(\Omega_X+\Omega_Y)^{N_I-N_R+i+j-2} \}
$$

For $N_I \leq N_R$, the number of non-zero eigenvalues of $\mathbf{HH}^H$ is $N_I$ and $N_R-N_I$ zero eigenvalues. Hence,

$$SER = \sum_{i=1}^{5} a_i \left( 1 + \frac{b_l}{\sigma^2} \right)^{N_I-N_R} \left( \frac{\Gamma(N_I-N_R+i+j-1)(\Omega_X+\Omega_Y)^{N_I-N_R+i+j-2}}{(\Omega_X+\Omega_Y-\frac{b_l}{\sigma^2}E_l e^{\frac{\sigma^2}{\Omega_X+\Omega_Y}E_N}-\frac{b_l}{\sigma^2}E_l e^{\frac{\sigma^2}{\Omega_X+\Omega_Y}E_N})} \right) \times det \{ \Gamma(N_R-N_I+i+j-1)(\Omega_X+\Omega_Y)^{N_R-N_I+i+j-2} \}
$$

For $N_R \leq N_I$, we obtain the SER approximation by using (51) in Appendix C but with $n_2 = N_R$ and $n_1 = N_I$. Hence,

$$SER \approx \sum_{i=1}^{5} a_i \left( \frac{b_l}{\sigma^2} \right)^{N_I-N_R} \left( \frac{\Gamma(N_I-N_R+i+j-1)(\Omega_X+\Omega_Y)^{N_I-N_R+i+j-2}}{(\Omega_X+\Omega_Y-\frac{b_l}{\sigma^2}E_l e^{\frac{\sigma^2}{\Omega_X+\Omega_Y}E_N}-\frac{b_l}{\sigma^2}E_l e^{\frac{\sigma^2}{\Omega_X+\Omega_Y}E_N})} \right) \times det \{ \Gamma(N_R-N_I+i+j-1)(\Omega_X+\Omega_Y)^{N_R-N_I+i+j-2} \}
$$

Since $\Omega_X = (1-\eta)\Omega/2$ and $\Omega_Y = (1+\eta)\Omega/2$, similar to the capacity case, the approximate SER expressions depends only on the power parameter $\Omega$ and not on the $\eta$ parameter.

IV. NUMERICAL RESULTS AND SIMULATIONS

A. Capacity

The derived capacity expressions are verified using Monte-Carlo simulations for both $N_R \geq N_T$ and $N_R \leq N_T$. For each Monte-Carlo simulation, the $N_R \times N_R$ random matrix $\mathbf{HH}^H$ is generated such that $\mathbf{H}$ has i.i.d. $\kappa - \mu$ or $\eta - \mu$ complex variables following the distribution that is given in [25], [26]. For a given SNR $\rho$ and $N_T$, capacity is evaluated using (13). This procedure is repeated over many realizations of $\mathbf{HH}^H$ and the mean is taken to obtain the average capacity. The approximate average capacity value is obtained by using the expressions (14) and
A close match is found between the theoretical and simulation results for all the cases as can be seen from the Fig. 1 - Fig. 2.

It can be observed from Fig. 1 (a), that capacity increases with $N_T$ for a fixed $N_R$, but...
saturates for large values of $N_T$. For any further increase in capacity one has to increase either $N_R$ or the SNR. From Fig. 1 (b) and Fig. 2 (a), it can be seen that the average capacity, increases with increase in the number of clusters $\mu$ of $\kappa - \mu$ or $\eta - \mu$ distribution. But the increase is diminished as $\mu$ increases. Similarly, the asymptotic capacity increases with increase in the number of clusters $\mu$ of $\eta - \mu$ distribution, as seen in Fig. 2 (b). Also, the average capacity increases with $\kappa$, as observed in Fig. 1 (b). The capacity upper bound from [23] is plotted in Fig. 1 (a). Similarly, the existing results for Rayleigh faded MIMO channels from [5] are plotted.
in Fig. 2 and a close match with our $\eta - \mu$ results are observed for $\eta = 0$ and $\mu = 1$.

**B. Optimum combining**

The derived SER expressions are verified using Monte-Carlo simulations for both $N_R \geq N_I$ and $N_R \leq N_I$. For each Monte-Carlo simulation, the random matrix $R = HH^H$ is generated, where $H$ has i.i.d. $\kappa - \mu$ or $\eta - \mu$ complex variables following the distribution that is given in [25], [26]. $R$ is decomposed into its eigen-values $\lambda_1, \lambda_2, ..., \lambda_{NR}$ and exponential random variables with unit mean, $p_k$ for $k = 1, ..., N_R$, are generated for the user channel. For a given noise value $\sigma^2$, SINR $\eta$ is evaluated using (27) and is substituted in (28), to obtain the exact probability of error over one iteration. This procedure is repeated over many realizations of $R$ and the exponential random variables $p_k$ and the average of all these values is taken to get the final SER. Instead of using (28) to compute the probability of error, one can use the approximation given in (29) and average over many realizations of $C$ and $p_k$ to get the final SER.

The approximate SER value is obtained by using the expressions (33) and (34) for $N_R \geq N_I$ and $N_R \leq N_I$ respectively for the $\kappa - \mu$ case. Similarly, the approximate SER value is obtained by using the expressions (35) and (36) for $N_R \geq N_I$ and $N_R \leq N_I$ respectively for the $\eta - \mu$ case. This procedure is repeated for various values of $\kappa$ or $\eta$, $\mu$, $N_R$, $N_I$ and $E_I$. A close match is found between the theoretical and simulation results for all the cases as can be seen from Fig. 3 and Fig. 4.

We can observe from Fig. 3(a) that SER increases with increase in $\kappa$ or $\mu$. As we keep $\kappa$ constant and increase $\mu$, the increase in SER diminishes as $\mu$ becomes larger. The same can be said for an increase in $\kappa$ with $\mu$ kept constant. Even for the case of $\eta - \mu$, we can observe from Fig. 4(a) that, the SER increases as there is an increase in either $E_I$ or $\mu$. As $\mu$ increases, the increase in SER also diminishes, as seen from the plots for $\mu = 1, 5$ and 9, for $E_I = -10$dB.

**V. Conclusions**

Approximate random matrix models have been derived for $HH^H$ when the elements of $H$ are i.i.d $\kappa - \mu$ or $\eta - \mu$ random variables. The approximation is terms of a complex Wishart matrix having the same first moment as the original matrix distribution with the degree of freedom being constrained to the number of columns of $H$. The utility of our result is shown by a) deriving approximate capacity expressions for $\kappa - \mu$ or $\eta - \mu$ MIMO models b) deriving approximate expressions for the SER of an optimum combining system with Rayleigh faded users and $\kappa - \mu$.
or \( \eta - \mu \) faded interferers. For both these applications, extensive Monte-Carlo simulations have been performed and an excellent match with the approximate expressions has been observed.

**APPENDIX A**

**APPROXIMATE MEAN OF COMPLEX \( \kappa - \mu \) RANDOM VARIABLES**

The expectations to be approximated are,

\[
\mathbb{E}[x_{ik}] = \int_{-\infty}^{\infty} x \frac{|x|^\mu/2}{2\sigma^2|p|^{\mu/2-1}} \exp\left(-\frac{(x-p)^2}{2\sigma^2}\right) \text{sech}\left(\frac{px}{\sigma^2}\right) I_{\frac{\mu}{2}-1}\left(\frac{|px|}{\sigma^2}\right) dx
\]

\[
\mathbb{E}[y_{ik}] = \int_{-\infty}^{\infty} y \frac{|y|^\mu/2}{2\sigma^2|q|^{\mu/2-1}} \exp\left(-\frac{(y-q)^2}{2\sigma^2}\right) \text{sech}\left(\frac{qy}{\sigma^2}\right) I_{\frac{\mu}{2}-1}\left(\frac{|qy|}{\sigma^2}\right) dy.
\]

The expectation \( \mathbb{E}[x_{ik}] \) is rewritten, using the trigonometric identity \( \text{tanh}(z) = 1 - e^{-2 \text{sech}(z)} \), as,

\[
\mathbb{E}[x_{ik}] = 2 \int_{0}^{\infty} \frac{x^{\mu/2+1}}{2\sigma^2|p|^{\mu/2-1}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \exp\left(-\frac{p^2}{2\sigma^2}\right) \text{erf}\left(\frac{\sqrt{\pi} px}{2\sigma^2}\right) I_{\frac{\mu}{2}-1}\left(\frac{px}{\sigma^2}\right) dx.
\]

The above integral cannot be solved to obtain a solution in closed form. Alternatively, we can approximate like in [43], \( \text{tanh}(\frac{px}{\sigma^2}) \) by \( \text{erf}\left(\frac{\sqrt{\pi} px}{2\sigma^2}\right) \) to obtain,

\[
\mathbb{E}[x_{ik}] \approx 2 \int_{0}^{\infty} \frac{x^{\mu/2+1}}{2\sigma^2|p|^{\mu/2-1}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \exp\left(-\frac{p^2}{2\sigma^2}\right) \text{erf}\left(\frac{\sqrt{\pi} px}{2\sigma^2}\right) I_{\frac{\mu}{2}-1}\left(\frac{px}{\sigma^2}\right) dx.
\]

Using the identity \( I_v(z) = \frac{1}{\Gamma(v+1)} (\frac{z}{2})^v F_1(v+1, \frac{3}{2} \frac{z}{2}) \) from [44], we get,

\[
\mathbb{E}[x_{ik}] \approx 2 \int_{0}^{\infty} \frac{x^{\mu/2+1}}{2\sigma^2|p|^{\mu/2-1}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \exp\left(-\frac{p^2}{2\sigma^2}\right) \text{erf}\left(\frac{\sqrt{\pi} px}{2\sigma^2}\right) \frac{1}{\Gamma(\frac{\mu}{2})} \left(\frac{px}{2\sigma^2}\right)^{\frac{\mu}{2}-1} F_1\left(\frac{\mu}{2}, \frac{p^2 x^2}{4\sigma^4}\right) dx.
\]

Expanding the hypergeometric series and interchanging the integration and summation, we obtain,

\[
\mathbb{E}[x_{ik}] \approx 2 \int_{0}^{\infty} \frac{x^{\frac{\mu}{2}+1}}{2\sigma^2|p|^{\mu/2-1}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \exp\left(-\frac{p^2}{2\sigma^2}\right) \text{erf}\left(\frac{\sqrt{\pi} px}{2\sigma^2}\right) \left(\frac{px}{2\sigma^2}\right)^{\mu/2-1} \\
\times \sum_{n=0}^{\infty} \frac{1}{(\frac{\mu}{2})_n n! \left(\frac{p^2 x^2}{4\sigma^4}\right)^n} dx
\]

\[
= 2 \sum_{n=0}^{\infty} \frac{p^{2n}}{(\frac{\mu}{2})_n n!} e^{-\frac{x^2}{2\sigma^2}} \frac{1}{\Gamma(\frac{\mu}{2}) (2\sigma^2)^{\frac{\mu}{2}+2n}} \int_{0}^{\infty} x^{\mu+2n} \exp\left(-\frac{x^2}{2\sigma^2}\right) \text{erf}\left(\frac{\sqrt{\pi} px}{2\sigma^2}\right) dx
\]

Now using the integration identity \( \int_{0}^{\infty} \text{erf}(ax) e^{-b^2x^2} x^{p} dx = \frac{\sqrt{\pi}}{\sqrt{a}} b^{-p-2} \Gamma(\frac{p}{2} + 1) 2 F_1\left(\frac{1}{2}, \frac{p+1}{2}, \frac{3}{2}, -\frac{a^2}{b^2}\right) \) for \( b^2 > 0 \) and \( p > -2 \) from [45], we obtain,

\[
\mathbb{E}[x_{ik}] = 2 p e^{-\frac{x^2}{2\sigma^2}} \sum_{n=0}^{\infty} \frac{p^{2n}}{(\frac{\mu}{2})_n n! \Gamma(\mu/2)} \Gamma(\mu/2 + n + 1) 2 F_1\left(\frac{1}{2}, \frac{\mu}{2} + n + 1, \frac{3}{2}, -\frac{2p^2 \pi}{\sigma^2}\right).
\]
Using the transformation $2F_1(a, b, c, z) = (1 - z)^{-b}2F_1(c - a, b, c, \frac{z}{z-1})$ for the Gauss Hypergeometric function from [46], we obtain,

$$
\mathbb{E}[x_{ik}] \approx 2pe^{-\frac{x^2}{2\sigma^2}} \sum_{n=0}^{\infty} \left( \frac{x^2}{2\sigma^2} \right)^n \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma\left(\frac{n}{2}\right)} (1 + \frac{2p^2}{\sigma^2})^{-\frac{n}{2} - 1} \frac{2p^2}{\sigma^2} \Gamma\left(\frac{\mu}{2} + n + 1, \frac{3}{2}, \frac{2p^2}{2p^2 + 4\sigma^2}\right)
$$

$\Rightarrow 2pe^{-\frac{x^2}{2\sigma^2}} \left( \frac{4\sigma^2}{4\sigma^2 + 2p^2}\right)^{\mu/2 + 1} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(\frac{n}{2} + 1)_{n+k}(1)_k}{(\frac{\mu}{2})_n } \frac{2p^2}{(4\sigma^2 + 2p^2\pi)}^k \frac{2p^2}{2p^2 + 4\sigma^2}.\]

Expanding the $_2F_1$ as series

$$
\mathbb{E}[x_{ik}] = 2pe^{-\frac{x^2}{2\sigma^2}} \left( \frac{4\sigma^2}{4\sigma^2 + 2p^2}\right)^{\mu/2 + 1} \frac{\Gamma\left(\frac{\mu}{2} + 1\right)}{\Gamma\left(\frac{\mu}{2}\right)} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(\frac{n}{2} + 1)_{n+k}(1)_k}{(\frac{\mu}{2})_n } \frac{2p^2}{(4\sigma^2 + 2p^2\pi)}^k \frac{2p^2}{2p^2 + 4\sigma^2}.\]

Rewriting the above using confluent Appell function $\Psi_1[47],$

$$
\mathbb{E}[x_{ik}] \approx 2pe^{-\frac{x^2}{2\sigma^2}} \left( \frac{4\sigma^2}{4\sigma^2 + 2p^2}\right)^{\mu/2 + 1} \frac{\Gamma\left(\frac{\mu}{2} + 1\right)}{\Gamma\left(\frac{\mu}{2}\right)} \Psi_1(\mu/2 + 1, 1, 3/2, \mu/2, \frac{2p^2}{2p^2 + 4\sigma^2}, \frac{2p^2}{4\sigma^2 + 2p^2\pi}). \tag{39}
$$

Similarly,

$$
\mathbb{E}[y_{ik}] \approx 2qe^{-\frac{y^2}{2\sigma^2}} \left( \frac{4\sigma^2}{4\sigma^2 + 2q^2}\right)^{\mu/2 + 1} \frac{\Gamma\left(\frac{\mu}{2} + 1\right)}{\Gamma\left(\frac{\mu}{2}\right)} \Psi_1(\mu/2 + 1, 1, 3/2, \mu/2, \frac{2q^2}{2q^2 + 4\sigma^2}, \frac{2q^2}{4\sigma^2 + 2q^2\pi}). \tag{40}
$$

We have compared (39) and (40) with numerical evaluation of the expectation integrals and also empirical average of simulated $\kappa - \mu$ variables for a wide range of parameters. In all cases, an excellent match has been observed.

**APPENDIX B**

**CAPACITY FOR $\kappa - \mu$ AND $\eta - \mu$**

We have to determine an approximation for $C = E \Lambda \left[ \sum_{i=1}^{n_1} \log_2(1 + \frac{\phi}{N\eta\lambda_i}) \right]$, where $\lambda_k$ for $k = 1, .., n_1$ are eigenvalues of a $n_1 \times n_1$ random matrix $R = HH^H$, where $H$ have i.i.d. $\kappa - \mu$ or $\eta - \mu$ elements.
A. $\kappa - \mu$

We approximate the matrix $R$ by a $n_1 \times n_1$ central Wishart matrix $W \sim \mathcal{W}_{n_1}(n_2, \Sigma)$, such that $n_1 \leq n_2$ and $\Sigma$ as in (11). The eigenvalue distribution of the unordered eigenvalues of $W$ is given by,

$$f(\Lambda) = (-1)^{\frac{1}{2}n_1(n_1-1)} \frac{1}{n_1!} \frac{\det(\{e^{-\lambda_i w_j}\})}{\Delta(\Lambda)} \frac{\Delta(\Lambda)}{\Delta(\Sigma^{-1})} \prod_{j=1}^{n_1} \frac{\lambda_j^{n_2-n_1}}{(n_2 - j)!}$$  \hspace{1cm} (41)

where $w_1 > w_2 > \ldots > w_{n_1}$ are the eigenvalues of $\Sigma^{-1}$ and $\lambda_1, \ldots, \lambda_{n_1}$ are the eigenvalues of $W$. But if some eigenvalues of $\Sigma^{-1}$ are not distinct, then the above distribution cannot be used because $\det(\{e^{-\lambda_i w_j}\}) = \Delta(\Sigma^{-1}) = 0$ leading to an indeterminate form. Hence, we apply the following theorem from [31], to modify the distribution and account for non-distinct eigenvalues.

**Theorem 1.** Let $f_1, \ldots, f_N$ be a family of infinitely differentiable functions and let $x_1, \ldots, x_N \in \mathcal{R}$. Denote

$$R(x_1, \ldots, x_N) \triangleq \frac{\det(\{f_i(x_j)\})}{\prod_{i<j}(x_j - x_i)}.$$

Then, for $N_1, \ldots, N_p$ such that $N_1 + \ldots + N_p = N$ and for $y_1, \ldots, y_p \in R$ distinct,

$$\lim_{\substack{x_1,\ldots,x_{N_1} \to y_1 \ldots \ldots x_{N-N_p+1},x_{N-N_p+2} \to y_p}} R(x_1, \ldots, x_N)$$

$$= \frac{\det[f_i(y_1), f'_i(y_1), \ldots, f_i^{(N_1-1)}(y_1), \ldots, f_i(y_p), f'_i(y_p), \ldots, f_i^{(N_p-1)}(y_p)]}{\prod_{1 \leq i < j \leq p} (y_j - y_i)^{N_i N_j} \prod_{i=1}^{p} N_i^{N_i-1} j!}.$$

In our case, $\Sigma^{-1}$ has two eigenvalues $w_1$ and $w_2$ with multiplicity $n_1 - 1$ and 1 respectively. Hence, applying the above theorem to (41), we obtain, the eigenvalue distribution as,

$$f(\Lambda) = (-1)^{\frac{1}{2}n_1(n_1-1)} \frac{1}{n_1!} \frac{\det(\{e^{-\lambda_i w_1} (-\lambda_i) e^{-\lambda_i w_1} \cdots (-\lambda_i)^{n_1-2} e^{-\lambda_i w_1} e^{-\lambda_i w_2}\})}{(w_2 - w_1)^{n_1-1} \prod_{j=1}^{n_1-2} j!} \frac{\Delta(\Lambda)}{\Delta(\Sigma^{-1})} \prod_{j=1}^{n_1} \frac{\lambda_j^{n_2-n_1}}{(n_2 - j)!}.$$  \hspace{1cm} (42)

Hence

$$C \approx (-1)^{\frac{1}{2}n_1(n_1-1)} \frac{1}{n_1! \prod_{j=1}^{n_1} (n_2 - j)!} \frac{1}{\Delta(\Pi^2(w_2 - w_1)^{n_1-1} \prod_{j=1}^{n_1-2} j!} \int_0^{\infty} \sum_{i=1}^{n_1} \log(1 + \frac{\rho}{N_T} \lambda_i) \lambda_i^{n_1-1} \Delta(\Lambda) \det(\{e^{-\lambda_i w_1} (-\lambda_i) e^{-\lambda_i w_1} \cdots (-\lambda_i)^{n_1-2} e^{-\lambda_i w_1} e^{-\lambda_i w_2}\}) d\Lambda.$$
From Theorem 3 in Appendix of [3], it can be observed that, for two arbitrary \( n_1 \times n_1 \) matrices \( \Phi(y) \) and \( \Psi(y) \) with \( ij \)th elements \( \phi_i(y_j) \) and \( \Psi_i(y_j) \), and two arbitrary functions \( \xi(.) \) and \( \xi'(.) \), where \( y = [y_1 \ y_2 \ ... \ y_{n_1}]^T \), the following identity holds:

\[
\int \cdots \int_{d \geq y \geq c} \phi_i(y) \Psi_j(y) \xi(y) U_{k,l}(\xi(y)) dy = \sum_{k=1}^{N} \det \left( \left\{ \int_{c}^{d} \phi_i(y) \Psi_j(y) \xi(y) U_{k,l}(\xi(y)) dy \right\}_{1 \leq i, j \leq n_1} \right),
\]

(43)

where, \( U_{i,j}(x) = x \), if \( k = j \) and \( U_{i,k}(x) = 1 \), if \( k \neq j \). Applying the above identity, we obtain,

\[
C \approx (-1)^{\frac{1}{2}n_1(n-1)} \frac{1}{n_1! \prod_{j=1}^{n_1} n_2 - j! |\sum_{j=2}^{n_1} n_2(w_2 - w_1)|} \prod_{j=1}^{n_1} \sum_{k=1}^{n_1} det(N^k).
\]

(44)

where

\[
\begin{align*}
N^k_{i,j}(n_1, n_2) &= \begin{cases}
\int_{0}^{\infty} \lambda^{n_2-n_1} \lambda^{i-j} \lambda^{n_1-j-1} e^{-\lambda w_1} \lambda^{i-1} d\lambda; & 1 \leq i \leq n_1, 1 \leq j \leq n_1 - 1, j \neq k, \\
\int_{0}^{\infty} \log_2(1 + \frac{\rho}{N_T}) \lambda^{n_2-n_1} \lambda^{i-j} \lambda^{n_1-j-1} e^{-\lambda w_1} \lambda^{i-1} d\lambda; & 1 \leq i \leq n_1, 1 \leq j \leq n_1 - 1, j = k,
\end{cases}
\end{align*}
\]

First writing the logarithm in terms of Meijer-G function using the identity \( \ln(1 + x) = G_{1,2}^{1,1}(1,0 | x) \) and solving the integrals using identities \( \int_{0}^{\infty} x^{n-1} e^{-\mu x} dx = \Gamma(n) \mu^{-n} \) and \( \int_{0}^{\infty} x^{-\rho} e^{-\beta x} G_{p,q}^{m,n}(a_1, ..., a_p | b_1, ..., b_q | ax) dx = \beta^{-\rho} G_{p+1,q}^{m+n+1}(\rho a_1, ..., a_p | b_1, ..., b_q | ax) \) from [49], we obtain,

\[
\begin{align*}
N^k_{i,j}(n_1, n_2) &= \begin{cases}
(-1)^{j-1} \Gamma(n_2 - n_1 + i + j - 1) w_1^{n_2-n_1-i-j+1}; & 1 \leq i \leq n_1, 1 \leq j \leq n_1 - 1, j \neq k, \\
(-1)^{j-1} \frac{1}{\lambda^2} G_{3,2}^{1,3}(1-n_2+n_1-i-j+1,1,1 | \rho N_T w_1) w_1^{n_2-n_1-i-j+1}; & 1 \leq i \leq n_1, 1 \leq j \leq n_1 - 1, j = k,
\end{cases}
\end{align*}
\]

(45)
B. \( \eta - \mu \)

We approximate the matrix \( \mathbf{R} \) by a \( n_1 \times n_1 \) central Wishart matrix \( \mathbf{W} \sim \mathcal{C}\mathcal{W}_{n_1}(n_2, c \mathbf{I}_{n_1}) \) such that \( n_1 \leq n_2 \). The eigenvalue distribution of the unordered eigenvalues of \( \mathbf{W} \) is given by,

\[
f(\lambda) = \frac{(c)^{-n_1 n_2}}{\prod_{i=1}^{n_1} (n_2 - i)! \prod_{i=1}^{n_1} (n_1 - i)!} \prod_{i=1}^{n_1} \lambda_i^{n_2 - n_1} |V(\lambda)|^2 \exp\left(-\frac{1}{c} \sum_{i=1}^{n_1} \lambda_i\right),
\]

where \( \lambda_1, ..., \lambda_{n_1} \) are the eigenvalues of \( \mathbf{W} \) and the term \( V(\lambda) \) denotes the Vandermonde matrix formed by the eigenvalues. For our case, \( c = (\Omega_X + \Omega_Y) \). Hence,

\[
C \approx \int_0^\infty \left[ \sum_{i=1}^{n_1} \log_2\left(1 + \frac{\rho}{N_T} \lambda_i\right) \right] \frac{(\Omega_X + \Omega_Y)^{-n_1 n_2}}{\prod_{i=1}^{n_1} (n_2 - i)! \prod_{i=1}^{n_1} (n_1 - i)!} \prod_{i=1}^{n_1} \lambda_i^{n_2 - n_1} |V(\lambda)|^2 e^{-\sum_{i=1}^{n_1} \frac{\lambda_i}{(\Omega_X + \Omega_Y)}} d\lambda
\]

\[
\approx \frac{(\Omega_X + \Omega_Y)^{-n_1 n_2}}{\prod_{i=1}^{n_1} (n_2 - i)! \prod_{i=1}^{n_1} (n_1 - i)!} \int_0^\infty \left[ \sum_{i=1}^{n_1} \log_2\left(1 + \frac{\rho}{N_T} \lambda_i\right) \right] \lambda_i^{n_2 - n_1} |V(\lambda)|^2 e^{-\frac{\lambda_i}{(\Omega_X + \Omega_Y)}} d\lambda.
\]

By applying Theorem 3 in Appendix of [3], we obtain,

\[
C \approx \frac{(\Omega_X + \Omega_Y)^{-n_1 n_2}}{\prod_{i=1}^{n_1} (n_2 - i)! \prod_{i=1}^{n_1} (n_1 - i)!} \sum_{k=1}^{n_1} |N^k|,
\]

where

\[
N_{i,j}^k(n_1, n_2) = \begin{cases} \int_0^\infty \lambda_i^{n_2 - n_1} \lambda_j^{i+j-2} e^{-\frac{\lambda}{\Omega_X + \Omega_Y}} d\lambda, & 1 \leq i, j \leq n_1, j \neq k, \\ \int_0^\infty \log_2\left(1 + \frac{\rho}{N_T} \lambda\right) \lambda_i^{n_2 - n_1} \lambda_j^{i+j-2} e^{-\frac{\lambda}{\Omega_X + \Omega_Y}} d\lambda, & 1 \leq i, j \leq n_1, j = k. \end{cases}
\]

Solving similar to the \( \kappa - \mu \) case, we obtain,

\[
N_{i,j}^k(n_1, n_2) = \begin{cases} \Gamma(n_2 - n_1 + i + j - 1)(\Omega_X + \Omega_Y)^{n_2 - n_1 + i + j - 1}, & 1 \leq i, j \leq n_1, j \neq k, \\ \frac{1}{n_2} C_{3,2}^{1,3} \left(1 - n_2 + n_1 - i - j + 1, 1, 1, 1\right) \frac{\rho \Omega_X + \Omega_Y}{N_T} \left(\Omega_X + \Omega_Y\right)^{n_2 - n_1 + i + j - 1}, & 1 \leq i, j \leq n_1, j = k. \end{cases}
\]

APPENDIX C

OPTIMUM COMBINING FOR \( \kappa - \mu \)

We have to determine an approximation for \( J(l) = \mathbb{E}_\Lambda \left[ \prod_{k=1}^{n_1} \frac{\lambda_k + \frac{\rho \sigma^2}{N_T}}{\lambda_k + \frac{\rho \sigma^2}{N_T} + \frac{1}{N_T}} \right] \), where \( \lambda_k \) for \( k = 1, ..., n_1 \) are eigenvalues of a \( n_1 \times n_1 \) random matrix \( \mathbf{R} = \mathbf{H}\mathbf{H}^H \), where \( \mathbf{H} \) have i.i.d. \( \kappa - \mu \) or \( \eta - \mu \) elements.
\[ J(l) \approx \int_0^\infty \prod_{k=1}^{n_1} \frac{\lambda_k + \frac{\sigma^2}{E_l}}{\lambda_k + \frac{\sigma^2}{E_l} + \frac{b_j}{E_l}} \frac{(-1)^{\frac{1}{2}(n_1-1)}}{n_1!} \times \frac{\det \left( \{ e^{-\lambda_i w_1} (-\lambda_i) e^{-\lambda_i w_1} \ldots (-\lambda_i)^{n_2-1} e^{-\lambda_i w_1} \} \right)}{(w_2 - w_1)^{n_1-1} \prod_{j=1}^{n_2-2} j!} \frac{\Delta(\Lambda)}{\Sigma|\Sigma|^{n_2}} \prod_{j=1}^{n_1} \frac{\lambda_j^{n_2-n_1}}{(n_2-j)!} \ d\Lambda \]

\[ \approx (-1)^{\frac{1}{2}(n_1-1)} \frac{1}{n_1! \prod_{j=1}^{n_1} (n_2-j)!} \frac{1}{\Sigma|\Sigma|^{n_2}} \int_0^\infty \prod_{k=1}^{n_1} \frac{\lambda_k + \frac{\sigma^2}{E_l}}{\lambda_k + \frac{\sigma^2}{E_l} + \frac{b_j}{E_l}} \times \lambda_k^{n_2-n_1} \Delta(\Lambda) \det \left( \{ e^{-\lambda_i w_1} (-\lambda_i) e^{-\lambda_i w_1} \ldots (-\lambda_i)^{n_2-1} e^{-\lambda_i w_1} \} \right) d\Lambda. \]

From Theorem 2 in Appendix of [3], it can be observed that, for two arbitrary \( n_1 \times n_1 \) matrices \( \rho(y) \) and \( \Psi(y) \) with \( ij^{th} \) elements \( \rho_i(y_j) \) and \( \Psi_i(y_j) \), and an arbitrary function \( \rho(.) \), where \( y = [y_1 y_2 \ldots y_{n_1}]^T \), the following identity holds:

\[ \int \cdots \int_{d \geq w \geq c} |\Xi(y)||\Psi(y)| \prod_{k=1}^{n_1} \rho(y_k) dy_1 \cdots dy_{n_1} = n_1! \det \left( \left\{ \int_c^d \xi_i(y)\Psi_j(y)\rho(y) dy \right\}_{1 \leq i, j \leq n_1} \right). \]

(48)

Using the above relation to simplify the expectation, we obtain,

\[ J(l) \approx (-1)^{\frac{1}{2}(n_1-1)} \frac{1}{n_1! \prod_{j=1}^{n_1} (n_2-j)!} \frac{1}{\Sigma|\Sigma|^{n_2}} \int_0^\infty \prod_{k=1}^{n_1} \frac{\lambda_k + \frac{\sigma^2}{E_l}}{\lambda_k + \frac{\sigma^2}{E_l} + \frac{b_j}{E_l}} \times \lambda_k^{n_2-n_1} \det(\{ e^{-\lambda_i w_1} (-\lambda_i) e^{-\lambda_i w_1} \ldots (-\lambda_i)^{n_2-1} e^{-\lambda_i w_1} \}) \ d\Lambda. \]

(49)

where \( M \) is given by,

\[ M_{ij}(n_1, n_2) = \begin{cases} \int_0^\infty \frac{\lambda^{n_2-n_1} \lambda^{i-1} e^{-\lambda w_1} d\lambda}{\lambda^{n_1} (n_2-j)! \prod_{j=1}^{n_1} (n_2-j)!} & 1 \leq i \leq n_1, 1 \leq j \leq n_1 - 1, \\ \int_0^\infty \frac{\lambda^{n_2-n_1} \lambda^{i-1} e^{-\lambda w_2} d\lambda}{\lambda^{n_1} (n_2-j)! \prod_{j=1}^{n_1} (n_2-j)!} & 1 \leq i \leq n_1, j = n_1, \end{cases} \]

which can be simplified using identities \( \int_0^\infty x^{v-1} e^{-\mu x} dx = \Gamma(v) \mu^{-v} \) from [49] as,

\[ M_{ij}(n_1, n_2) = \begin{cases} (-1)^{j-i} w_1^{-i+1-n_1} \left( 1 - \frac{b_j}{E_l} \frac{\sigma^2}{E_l} w_1 w_2 \right)^{n_1-i+j-1} \left( 1 - \frac{b_j}{E_l} \frac{\sigma^2}{E_l} w_1 \right)^{n_1-i+j} \\ \times \Gamma(n_2-n_1+i+j-1) & 1 \leq i \leq n_1, 1 \leq j \leq n_1 - 1, \\ w_2^{-i+n_1-n_2} \left( 1 - \frac{b_j}{E_l} \frac{\sigma^2}{E_l} w_2 \right)^{n_1-i+j} \Gamma(n_2-n_1+i) & 1 \leq i \leq n_1, j = n_1. \end{cases} \]

(50)

where \( E(.) \) is the exponential integral function [44].
B. $\eta - \mu$

We approximate the matrix $R$ by a $n_1 \times n_1$ central Wishart matrix $W \sim CW_{n_1}(n_2, cI_{n_1})$ such that $n_1 \leq n_2$. The eigenvalue distribution of the unordered eigenvalues of $W$ is given by,

$$f(\Lambda) = \prod_{i=1}^{n_1} \left( \frac{(c)^{-n_1n_2}}{(n_2-i)!} \prod_{i=1}^{n_1} \frac{\lambda_i^{n_2-n_1}}{(n_1-i)!} \lambda_i^{n_2-n_1} |V(\Lambda)|^2 \exp\left(-\frac{1}{c} \sum_{i=1}^{n_1} \lambda_i\right) \right)$$

where $\lambda_1, ..., \lambda_{n_1}$ are the eigenvalues of $W$ and the term $V(\Lambda)$ denotes the Vandermonde matrix formed by the eigenvalues. For our case, $c = (\Omega_X + \Omega_Y)$. Hence,

$$J(l, n_1) \approx \int_0^\infty \prod_{i=1}^{n_1} \lambda_i^{n_2-n_1} |V(\Lambda)|^2 \exp\left(-\frac{1}{(\Omega_X + \Omega_Y)} \sum_{i=1}^{n_1} \lambda_i\right) \lambda_i^{n_2-n_1} |V(\Lambda)|^2 e^{-\frac{\lambda}{\Omega_X+\Omega_Y}} d\Lambda.$$

Using Theorem 2 in Appendix of [3], we obtain for $1 \leq i, j \leq n_1$

$$J(l, n_1) \approx \frac{\left(\Omega_X + \Omega_Y\right)^{n_2-n_1} \Gamma\left(n_2-n_1+1\right)}{\prod_{i=1}^{n_1} \left(\Omega_X + \Omega_Y\right)^{n_2-n_1+i+j-2}}.$$

This can be further simplified using identities $\int_0^\infty x^{v-1} e^{-\mu x} dx = \Gamma(v)\mu^{-v}$ from [49] to obtain,

$$J(l, n_1) \approx \frac{\left(\Omega_X + \Omega_Y\right)^{n_2-n_1} \Gamma\left(n_2-n_1+1\right)}{\prod_{i=1}^{n_1} \left(\Omega_X + \Omega_Y\right)^{n_2-n_1+i+j-2}}.$$
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