THE BERRY-ESSÉEN UPPER BOUNDS OF VASICEK MODEL ESTIMATORS

YONG CHEN AND YUMIN CHENG*

Abstract. The Berry-Esséen upper bounds of moment estimators and least squares estimators of the mean and drift coefficients in Vasicek models driven by general Gaussian processes are studied. When studying the parameter estimation problem of Ornstein-Uhlenbeck (OU) process driven by fractional Brownian motion, the commonly used methods are mainly given by Kim and Park, they show the upper bound of Kolmogorov distance between the distribution of the ratio of two double Wiener-Itô stochastic integrals and the Normal distribution. The main innovation in this paper is extending the above ratio process, that is to say, the numerator and denominator respectively contain triple Wiener-Itô stochastic integrals at most. As far as we know, the upper bounds between the distribution of above estimators and the Normal distribution are novel.

1. Introduction

Vasicek model is a type of 1-dimensional stochastic processes, it is used in various fields, such as economy, finance, environment. It was originally used to describe short-term interest rate fluctuations influenced by single market factors. Proposed by O. Vasicek [1], it is the first stochastic process model to describe the “mean reversion” characteristic of short-term interest rates. In the financial field, it can also be used as a random investment model in Wu et al.[2] and Han et al.[3].

Definition 1. Consider the Vasicek model driven by general Gaussian process, it satisfies the following Stochastic Differential Equation (SDE):

\[ dV_t = k(\mu - V_t) \, dt + \sigma \, dG_t, \quad t \in [0,T], \]

where \( k, T > 0 \), \( V_0 = 0 \) and \( G = \{G_t\}_{t \geq 0} \) is a general one-dimensional centered Gaussian process that satisfies Hypothesis 1.

This paper mainly focuses on the convergence rate of estimators of coefficient \( k, \mu \). Without loss of generality, we assume \( \sigma = 1 \), then Vasicek model can be represent by the following form:

\[ V_t = \mu(1 - e^{-kt}) + \int_0^t e^{-k(t-s)} \, dG_s. \]

When the coefficients in the drift function is unknown, an important problem is to estimate the drift coefficients based on the observation. Based on the Brownian motion, Fergusson and Platen [4] present the maximum likelihood estimators of coefficients in Vasicek model. When the Vasicek model driven by the fractional Brownian motion, Xiao and Yu [5] consider the least squares estimators and their asymptotic behaviors. When \( k > 0 \), Hu and Nualart [6] study the moment estimation problem.

2020 Mathematics Subject Classification. 60H07, 60G15, 60G22.

Key words and phrases. Vasicek model, Malliavin calculus, Central limit theorem, Berry-Esséen upper bounds.
Since the Gaussian process $G_t$ mainly determines the trajectory properties of Vasicek model. Therefore, following the assumptions in Chen and Zhou [7], we make the following Hypothesis about $G_t$.

**Hypothesis 1** ([7] Hypothesis 1.1). Let $\beta \in (\frac{1}{2}, 1)$ and $t \neq s \in [0, \infty)$, Covariance function $R(t, s) = \mathbb{E}[G_tG_s]$ of Gaussian process $G_t$ satisfies the following condition:

$$\frac{\partial^2}{\partial t \partial s} R(t, s) = C_\beta |t - s|^{2\beta - 2} + \Psi(t, s),$$

(2)

where

$$|\Psi(t, s)| \leq C_\beta' |ts|^{\beta - 1},$$

$\beta, C_\beta > 0, C_\beta' \geq 0$ are constants independent with $T$. Besides, $R(0, t) = 0$ for any $t \geq 0$.

**Remark.** The covariance functions of Gaussian processes such as fractional Brownian motion, subfractional Brownian motion and double fractional Brownian motion satisfy the above Hypothesis [7, Examples 1.5-1.8].

Assuming that there is only one trajectory $(V_t, t \geq 0)$, we can construct the least squares estimators (LSE) and the moment estimators (ME) (See [5, 8, 9, 10] for more details).

**Proposition 1** ([11] (4) and (5)). The estimator of $\mu$ is the continuous-time sample mean:

$$\hat{\mu} = \frac{1}{T} \int_0^T V_t \, dt.$$  

(3)

The second moment estimator of $k$ is given by

$$\hat{k} = \left[ \frac{1}{T} \int_0^T V_t^2 \, dt - \left( \frac{1}{T} \int_0^T V_t \, dt \right)^2 \right]^{-\frac{1}{2\beta}}.$$  

(4)

Following from Xiao and Yu [5], we present the LSE in Vasicek model.

**Proposition 2** ([11] (7) and (8)). The LSE is motivated by the argument of minimize a quadratic function $L(k, \mu)$ of $k$ and $\mu$:

$$L(k, \mu) = \int_0^T \left( V_t - k(\mu - V_t) \right)^2 \, dt,$$

Solving the equation, we can obtain the LSE of $k$ and $\mu$, denoted by $\hat{k}_{LS}$ and $\hat{\mu}_{LS}$ respectively.

$$\hat{k}_{LS} = \frac{V_T \int_0^T V_t \, dt - T \int_0^T V_t \, dV_t}{T \int_0^T V_t^2 \, dt - (\int_0^T V_t \, dt)^2},$$  

(5)

$$\hat{\mu}_{LS} = \frac{V_T \int_0^T V_t^2 \, dt - \int_0^T V_t \, dV_t \int_0^T V_t \, dt}{V_T \int_0^T V_t \, dt - T \int_0^T V_t \, dV_t},$$  

(6)

where the integral $\int_0^T V_t \, dV_t$ is an Itô-Skorohod integral.

Pei et al.[11] prove the following consistencies and central limit theorems (CLT) of estimators.
Theorem 3 ([11], Theorem 1.2). When Hypothesis 1 is satisfied, both ME and LSE of µ, k are strongly consistent, that is
\[
\lim_{T \to \infty} \hat{\mu} = \mu, \quad \lim_{T \to \infty} \hat{k} = k, \quad \text{a.s.;}
\]
\[
\lim_{T \to \infty} \hat{\mu}_{LS} = \mu, \quad \lim_{T \to \infty} \hat{k}_{LS} = k, \quad \text{a.s.}
\]

Theorem 4 ([11], Theorem 1.3). Assume Hypothesis 1 is satisfied. When \(G_t\) is self-similar and \(\mathbb{E}[G_t^2] = 1\), \(\hat{\mu}\) and \(\hat{\mu}_{LS}\) are asymptotically normal as \(T \to \infty\), that is,
\[
T^{1-\beta}(\hat{\mu} - \mu) \xrightarrow{\text{law}} \mathcal{N}(0, 1/k^2), \quad \sqrt{T}(\hat{\mu}_{LS} - \mu) \xrightarrow{\text{law}} \mathcal{N}(0, 1/k^2).
\]
When \(\beta \in (1/2, 3/4)\),
\[
\sqrt{T}(\hat{k} - k) \xrightarrow{\text{law}} \mathcal{N}(0, k\sigma_{\beta}^2/4\beta^2),
\]
where
\[
\sigma_{\beta}^2 = (4\beta - 1) \left[ 1 + \frac{\Gamma(3 - 4\beta)\Gamma(4\beta - 1)}{\Gamma(2\beta)\Gamma(2 - 2\beta)} \right]. (7)
\]
Similarly, \(\sqrt{T}(\hat{k}_{LS} - k)\) is also asymptotically normal as \(T \to \infty\):
\[
\sqrt{T}(\hat{k}_{LS} - k) \xrightarrow{\text{law}} \mathcal{N}(0, k\sigma_{\beta}^2).
\]

We now present the main Theorems for the whole paper, and their details are given in the following sections.

Theorem 5. Let \(Z\) be a standard Normal random variable, and \(\sigma_{\beta}^2\) be the constant defined by (7). Assume \(\beta \in (1/2, 3/4)\) and Hypothesis 1 is satisfied. When \(T\) is large enough, there exists a constant \(C_{\beta,V}\) such that
\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left( \sqrt{\frac{4\beta^2T}{k\sigma_{\beta}^2}}(\hat{k} - k) \leq z \right) - \mathbb{P}(Z \leq z) \right| \leq \frac{C_{\beta,V}}{T^m}, (8)
\]
\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left( \sqrt{\frac{T}{k\sigma_{\beta}^2}}(\hat{k}_{LS} - k) \right) - \mathbb{P}(Z \leq z) \right| \leq \frac{C_{\beta,V}}{T^{3/4 - \beta}}, (9)
\]
where \(m = \min\{1/3, (3 - 4\beta)/2\}\).

Next, we show the convergence speed of mean coefficient estimators \(\hat{\mu}\) and \(\hat{\mu}_{LS}\).

Theorem 6. Assume \(\beta \in (1/2, 1)\), and \(G_t\) is a self-similar Gaussian process satisfying Hypothesis 1 and \(\mathbb{E}[G_t^3] = 1\). Then there exists a constant \(C_{\beta,V}\) such that
\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left( \frac{k}{T^{\beta - 1}}(\hat{\mu} - \mu) \leq z \right) - \mathbb{P}(Z \leq z) \right| \leq \frac{C_{\beta,V}}{T^{\beta/2}}, (10)
\]
\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left( \frac{k}{T^{\beta - 1}}(\hat{\mu}_{LS} - \mu) \right) - \mathbb{P}(Z \leq z) \right| \leq \frac{C_{\beta,V}}{T^{(1-\beta)/2}}. (11)
\]

2. Preliminary

In this section, we recall some basic facts about Malliavin calculus with respect to Gaussian process. The reader is referred to [12, 13, 14] for a more detailed explanation. Let \(G = \{G_t, t \in [0, T]\}\) be a continuous centered Gaussian process with \(G_0 = 0\) and covariance function
\[
\mathbb{E}(G_tG_s) = R(s, t), \quad s, t \in [0, T], (12)
\]
defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \(\mathcal{F}\) is generated by the Gaussian family \(G\). Denote \(\mathcal{E}\) as the the space of all real valued step functions on \([0, T]\). The Hilbert space \(\mathcal{H}\) is defined as the closure of \(\mathcal{E}\) endowed with the inner product:

\[
\langle 1_{[a,b)}, 1_{[c,d)} \rangle_{\mathcal{H}} = \mathbb{E}((G_b - G_a)(G_d - G_c)).
\]  

(13)

We denote \(G = \{G(h), h \in \mathcal{H}\}\) as the isonormal Gaussian process on the probability space, indexed by the elements in \(\mathcal{H}\), which satisfies the following isometry relationship:

\[
\mathbb{E}(G(h)) = 0, \quad \mathbb{E}[G(g)G(h)] = \langle g, h \rangle_{\mathcal{H}}, \quad \forall \ g, h \in \mathcal{H}.
\]  

(14)

The following Proposition shows the inner products representation of the Hilbert space \(\mathcal{H}\) [15].

**Proposition 7** ([7] Proposition 2.1). Denote \(\mathcal{V}_{[0,T]}\) as the set of bounded variation functions on \([0, T]\). Then \(\mathcal{V}_{[0,T]}\) is dense in \(\mathcal{H}\) and

\[
\langle f, g \rangle_{\mathcal{H}} = \int_{[0,T]^2} R(t, s) v_f(dt)v_g(ds), \quad \forall \ f, g \in \mathcal{V}_{[0,T]},
\]

where \(v_g\) is the Lebesgue-Stieljes signed measure associated with \(g^0\) defined as

\[
g^0 = \begin{cases} 
  g(x), & \text{if } x \in [0, T]; \\
  0, & \text{otherwise}.
\end{cases}
\]

When the covariance function \(R(t, s)\) satisfies Hypothesis 1,

\[
\langle f, g \rangle_{\mathcal{H}} = \int_{[0,T]^2} f(t)g(s) \frac{\partial^2 R(t, s)}{\partial t \partial s} \, dt \, ds, \quad \forall \ f, g \in \mathcal{V}_{[0,T]}.
\]  

(15)

Furthermore, the norm \(\|\cdot\|_{\mathcal{H}}\) of the elements in \(\mathcal{H}\) can be induced naturally:

\[
\|\phi\|_{\mathcal{H}}^2 = \int_{[0,T]^2} \phi(r_1)\phi(r_2) \frac{\partial^2 R(r_1, r_2)}{\partial r_1 \partial r_2} \, dr_1 \, dr_2, \quad \forall \ \phi \in \mathcal{H}.
\]

**Remark** ([7] Notation 1). Let \(C_\beta\) and \(C'_\beta\) be the constants given in Hypothesis 1. For any \(\phi(r) \in \mathcal{V}_{[0,T]}\), we define two norms as

\[
\|\phi\|_{\mathcal{H}_1}^2 = C_\beta \int_{[0,T]^2} \phi(r_1)\phi(r_2) |r_1 - r_2|^{2\beta-2} \, dr_1 \, dr_2,
\]

\[
\|\phi\|_{\mathcal{H}_2}^2 = C'_\beta \int_{[0,T]^2} |\phi(r_1)\phi(r_2)| (r_1 r_2)^{\beta-1} \, dr_1 \, dr_2.
\]

For any \(\varphi(r, s)\) in \([0, T]^2\), define an operator \(K\) from \(\mathcal{V}_{[0,T]}^\otimes 2\) to \(\mathcal{V}_{[0,T]}\) to be

\[
(K\varphi)(r) = \int_0^T |\varphi(r, u)| u^{\beta-1} \, du.
\]  

(16)

**Proposition 8** ([7] Proposition 3.2). Suppose that Hypothesis 1 holds, then for any \(\phi(r) \in \mathcal{V}_{[0,T]}\),

\[
\|\phi\|_{\mathcal{H}_1}^2 - \|\phi\|_{\mathcal{H}_1}^2 \leq \|\phi\|_{\mathcal{H}_2}^2,
\]  

(17)

and for any \(\varphi, \psi \in (\mathcal{V}_{[0,T]}^\otimes 2)\),

\[
\|\phi\|_{\mathcal{H}_1}^2 - \|\phi\|_{\mathcal{H}_1}^2 \leq 2C'_\beta \|K\phi\|_{\mathcal{H}_1}^2,
\]

\[
\|\phi\|_{\mathcal{H}_2}^2 - \|\phi\|_{\mathcal{H}_2}^2 \leq \|\phi\|_{\mathcal{H}_2}^2 + 2C'_\beta \|K\phi\|_{\mathcal{H}_1}^2,
\]

\[
\|\phi\|_{\mathcal{H}_1}^2 - \|\phi\|_{\mathcal{H}_1}^2 \leq \|\phi\|_{\mathcal{H}_2}^2 + 2C'_\beta \|K\phi\|_{\mathcal{H}_1}^2.
\]
Let $\mathcal{H}^\otimes p$ and $\mathcal{H}^\otimes q$ be the $p$-th tensor product and the $p$-th symmetric tensor product of $\mathcal{H}$. For every $p \geq 1$, denote $\mathcal{H}_p$ as the $p$-th Wiener chaos of $G$. It is defined as the closed linear subspace of $L^2(\Omega)$ generated by $\{H_p(G(h)) : h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$, where $H_p$ is the $p$-th Hermite polynomial. Let $h \in \mathcal{H}$ such that $\|h\|_{\mathcal{H}} = 1$, then for every $p \geq 1$ and $h \in \mathcal{H}$,

$$I_p(h^\otimes p) = H_p(G(h)),$$

where $I_p(\cdot)$ is the $p$-th Wiener-Itô stochastic integral.

Denote $\{e_k, k \geq 1\}$ as a complete orthonormal system in $\mathcal{H}$. The $q$-th contraction between $f \in \mathcal{H}^{\otimes m}$ and $g \in \mathcal{H}^{\otimes n}$ is an element in $\mathcal{H}^{\otimes (m+n-2q)}$: $f \otimes_q g = \sum_{i_1, \ldots, i_q = 1}^{\infty} \langle f, e_{i_1}, \ldots, e_{i_q} \rangle_{\mathcal{H}^{\otimes q}} \otimes \langle g, e_{i_1}, \ldots, e_{i_q} \rangle_{\mathcal{H}^{\otimes q}},$ $q = 1, \ldots, m \wedge n.$

The following proposition shows the product formula for the multiple integrals.

**Proposition 9** ([12] Theorem 2.7.10). Let $f \in \mathcal{H}^\otimes p$ and $g \in \mathcal{H}^\otimes q$ be two symmetric function. Then

$$I_p(f)I_q(g) = \sum_{r=0}^{p \wedge q} r^! \binom{p}{r}^! \binom{q}{r}^! I_{p+q-2r}(f \otimes_r g),$$

where $f \otimes_r g$ is the symmetrization of $f \otimes_q g$.

We then introduce the derivative operator and the divergence operator. For these details, see sections 2.3-2.5 of [12]. Let $\mathcal{T}$ be the class of smooth random variables of the form:

$$F = f(G(\psi_1), G(\psi_2), \ldots, G(\psi_n)),$$

where $n \geq 1$, $f \in C^\infty_c(\mathbb{R}^n)$ which partial derivatives have at most polynomial growth, and for $i = 1, \ldots, n$, $\psi_i \in \mathcal{H}$. Then, the Malliavin derivative of $F$ (with respect to $G$) is the element of $L^2(\Omega, \mathcal{H})$ defined by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(G(\psi_1), \ldots, G(\psi_n)) \psi_i.$$

Given $q \in [1, \infty)$ and integer $p \geq 1$, let $D_{p,q}$ denote the closure of $\mathcal{T}$ with respect to the norm

$$\|F\|_{D_{p,q}} = \left[\mathbb{E}(|F|^q) + \sum_{k=1}^{p} \mathbb{E}(\|D^k F\|^q_{\mathcal{H}^{\otimes k}})\right]^{1/q}.$$

Denote $\delta$ (the divergence operator) as the adjoint of $D$. The domain of $\delta$ is composed of those elements:

$$|\mathbb{E}([D^p F, u]_{\mathcal{H}^{\otimes p}})| \leq C[\mathbb{E}(|F|^2)]^{1/2}, \quad \forall F \in D^{p,2},$$

and is denoted by $\text{Dom}(\delta)$. If $u \in \text{Dom}(\delta)$, then $\delta(u)$ is the unique element of $L^2(\Omega)$ characterized by the duality formula:

$$\mathbb{E}[F \delta(u)] = \mathbb{E}(\langle DF, u \rangle_{\mathcal{H}}), \quad \forall F \in D^{1,2}.$$

We now introduce the infinitesimal generator $L$ of the Ornstein-Uhlenbeck semigroup. Let $F \in L^2(\Omega)$ be a square integrable random variable. Denote $\mathcal{J}_n :$
provides the Berry-Esséen upper bound on the sum of two random variables. Assuming that for every \( z \), \( A \) is a zero-mean process, and \( G_T \in \mathbb{D}^{1,2} \) satisfies \( G_T > 0 \) a.s. For simplicity, we define the following four functions:

\[
\begin{align*}
\Psi_1(T) &= \frac{1}{(\mathbb{E}G_T)^2} \sqrt{\mathbb{E} \left[ \left( (\mathbb{E}G_T)^2 - (\langle DF_T, -DL^{-1}F_T \rangle) \right)^2 \right]}, \\
\Psi_2(T) &= \frac{1}{(\mathbb{E}G_T)^2} \sqrt{\mathbb{E} \left[ \langle DF_T, -DL^{-1}(G_T - \mathbb{E}G_T) \rangle \right]^2],} \\
\Psi_3(T) &= \frac{1}{(\mathbb{E}G_T)^2} \sqrt{\mathbb{E} \left[ \langle DG_T, -DL^{-1}F_T \rangle \right]^2]}, \\
\Psi_4(T) &= \frac{1}{(\mathbb{E}G_T)^2} \sqrt{\mathbb{E} \left[ \langle DG_T, -DL^{-1}(G_T - \mathbb{E}G_T) \rangle \right]^2]}.
\end{align*}
\]

**Theorem 11** ([17] Theorem 2 and Corollary 1). Let \( Z \) be a standard Normal variable. Assuming that for every \( z \in \mathbb{R} \), \( F_T + zG_T \) has an absolutely continuous law with respect to Lebesgue measure and \( \Psi_i(T) \to 0 \), \( i = 1, \ldots, 4 \), as \( T \to \infty \). Then, there exists a constant \( c \) such that for \( T \) large enough,

\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left( \frac{F_T}{G_T} \leq z \right) - \mathbb{P}(Z \leq z) \right| \leq c \cdot \max_{i=1,\ldots,4} \Psi_i(T).
\]

3. **Berry-Esséen upper bounds of moment estimators**

In this section, we will prove the Berry-Esséen upper bounds of Vasicek model moment estimators \( \hat{\mu} \) and \( k \). For the convenience of the following discussion, we first define \( A(z) \):

\[
A(z) := \mathbb{P} \left( \frac{k}{T^{-\beta}}(\hat{\mu} - \mu) \leq z \right) - \mathbb{P}(Z \leq z),
\]

where \( Z \sim \mathcal{N}(0, 1) \) is a standard Normal variable. Next, we introduce the CLT of \( \hat{\mu} \).
Theorem 12 ([11] Proposition 4.19). Assume $\beta \in (1/2, 1)$, and $G_t$ is a self-similar Gaussian process satisfying Hypothesis 1 and $\mathbb{E}[G_1^2] = 1$. Then $T^{1-\beta}(\hat{\mu} - \mu)$ is asymptotically normal as $T \to \infty$:

$$T^{1-\beta}(\hat{\mu} - \mu) = \frac{\mu e^{-kT} - 1}{T^\beta} + \frac{1}{k} \frac{G_T - Z_T}{T^\beta} \xrightarrow{law} \mathcal{N}(0, 1/k^2),$$  \hspace{1cm} (21)

where

$$Z_T = I_1(e^{-k(T-s)}I_{[0,T]}(s))$$

is stochastic integral with respect to $G_t$.

Following from the above Theorem, we can obtain the expanded form of (20):

$$A(z) = \mathbb{P}\left(kT^{1-\beta}(\hat{\mu} - \mu) \leq z\right) - \mathbb{P}(Z \leq z)$$

$$= \mathbb{P}\left(\frac{G_T - Z_T + \mu(e^{-kT} - 1)}{T^\beta} \leq z\right) - \mathbb{P}(Z \leq z).$$

Then, we can prove the convergence speed of $\hat{\mu}$.

Proof of formula (10). Let $a = T^{-\beta/2}$, According to Lemma 10, we have

$$\sup_{z \in \mathbb{R}} |A(z)| \leq \sup_{z \in \mathbb{R}} \left| \mathbb{P}\left(\frac{G_T}{T^\beta} \leq z\right) - \mathbb{P}(Z \leq z) \right|$$

$$+ \mathbb{P}\left(\left|\frac{-Z_T + \mu(e^{-kT} - 1)}{T^\beta}\right| > T^{-\frac{\beta}{2}}\right) + \frac{T^{-\frac{\beta}{2}}}{\sqrt{2\pi}}.$$

Since $G_T$ is self-similar, $(G_T/T^\beta)$ is standard Normal variable,

$$\sup_{z \in \mathbb{R}} \left| \mathbb{P}\left(\frac{G_T}{T^\beta} \leq z\right) - \mathbb{P}(Z \leq z) \right| = 0.$$

Following from Chebyshev inequality, we can obtain

$$\mathbb{P}\left(\left|\frac{-Z_T + \mu(e^{-kT} - 1)}{T^\beta}\right| > T^{-\frac{\beta}{2}}\right) = \mathbb{P}\left(|Z_T - \mu(e^{-kT} - 1)| > T^{\frac{\beta}{2}}\right) \leq \frac{C_1}{T^\beta},$$

where

$$C_1 = \mathbb{E}(|Z_T - \mu(e^{-kT} - 1)|^2).$$

The Proposition 3.10 of [11] ensures that $C_1$ is bounded. Combining the above results, we have

$$\sup_{z \in \mathbb{R}} |A(z)| \leq \frac{1}{\sqrt{2\pi T^{\beta/2}}} + \frac{C_1}{T^\beta}.$$  \hspace{1cm} (22)

When $T$ is sufficiently large, there exists the constant $C_{\beta,V}$ such that the formula (10) holds. \hfill \Box

Similarly, we review the central limit theorem of $\hat{k}$.

Theorem 13 ([11] Proposition 4.18). Assume $\beta \in (1/2, 3/4)$ and $G_t$ is a Gaussian process satisfying Hypothesis 1. Then $\sqrt{T}(\hat{k} - k)$ is asymptotically normal as $T \to \infty$:

$$\sqrt{T}(\hat{k} - k) = \sqrt{T}\left(\frac{1}{C_\beta \Gamma(2\beta - 1)} \left[\frac{1}{T} \int_0^T V_t^2 dt - \frac{1}{T} (\int_0^T V_t dt)^2\right]^{-\frac{1}{2\beta}} - k\right) \xrightarrow{law} \mathcal{N}(0, k\sigma^2/4\beta^2).$$

The following Lemma shows the upper bound of the expectation of $(\int_0^T e^{-ks} dG_s)^2$. 

Lemma 14. Let $M_T$ be the process defined by
\[ M_T = I_1(e^{-ku}1_{[0,T]}(u)) = \int_0^T e^{-ks}dG_s. \]
When $\beta \in (1/2, 1)$, there exists constant $C$ independent of $T$ such that
\[ \mathbb{E}(M_T^2) \leq C. \] (23)

Proof. According to (14) and (17), we can obtain
\[ \mathbb{E}[|M_T|^2] = \|m_T\|^2_{\mathcal{H}} \leq \|m_T\|^2_{\mathcal{H}_1} + \|m_T\|^2_{\mathcal{H}_2}, \]
where $m_T(u) = e^{-ku}1_{[0,T]}(u)$.

It is easy to see that
\[ \|m_T\|^2_{\mathcal{H}_1} = 2C_{\beta} \int_0^T \int_0^u e^{-k(u+v)}|u-v|^{2\beta-2}du dv \leq C \int_0^T \int_0^u e^{-k(2x+y)}x^{2\beta-2}dx dy \leq C', \]
where $C'$ is a constant. Also, we have
\[ \|m_T\|^2_{\mathcal{H}_2} = C_{\beta} \int_0^T e^{-ku}u^{\beta-1}du \int_0^T e^{-kv}v^{\beta-1}dv \leq C''. \]
Combining the above two formulas, we obtain (23). □

Denote $B(z)$ as
\[ B(z) := \mathbb{P}\left( \sqrt{\frac{4\beta^2T}{k\sigma^2_\beta}}(\hat{k} - k) \leq z \right) - \mathbb{P}(Z \leq z). \]

Then we can obtain the Berry-Esseen upper bound of $\text{ME} \hat{k}$.

Proof of formula (8). According to [11] Proposition 4.18, we have
\[ B(z) = \mathbb{P}\left( \sqrt{\frac{4\beta^2T}{k\sigma^2_\beta}}(\hat{k} - k) \leq z \right) - \mathbb{P}(Z \leq z) \]
\[ = \mathbb{P}\left( \hat{k} - k \leq \sqrt{\frac{k\sigma^2_\beta}{4\beta^2T}} \right) - \mathbb{P}(Z \leq z) \]
\[ = \mathbb{P}\left( \frac{1}{T} \int_0^T V_t^2 dt - \left( \frac{1}{T} \int_0^T V_t dt \right)^2 \geq \left( \sqrt{\frac{k\sigma^2_\beta}{4\beta^2T}} + k \right)^{-2\beta} \right) - \mathbb{P}(Z \leq z) \]
\[ = \mathbb{P}\left( \frac{1}{T} \left( \int_0^T V_t^2 dt - \left( \frac{1}{T} \int_0^T V_t dt \right)^2 \right) - \alpha \geq 0 \right) \]
\[ \geq C_{\beta} \Gamma(2\beta - 1) \left[ \left( \sqrt{\frac{k\sigma^2_\beta}{4\beta^2T}} + k \right)^{-2\beta} - k^{-2\beta} \right] - \mathbb{P}(Z \leq z) \]
\[ = \mathbb{P}\left( \frac{1}{T} \left( \int_0^T V_t^2 dt - \left( \frac{1}{T} \int_0^T V_t dt \right)^2 \right) - \alpha \geq \alpha \left( 1 + \frac{z\sigma_\beta}{2\beta \sqrt{kT}} \right)^{-2\beta} - 1 \right) \]
\[ - \mathbb{P}(Z \leq z). \]
We denote \( \Phi(z) \) as the tail probability \( 1 - \P(Z \leq z) \) and
\[
\nu = \sqrt{\frac{kT}{\sigma^2}} \left[ \left(1 + \frac{z\sigma_{\beta}}{2\beta\sqrt{kT}}\right)^{-2\beta} - 1 \right].
\]

Then we can obtain
\[
|B(z)| = \left| \P \left( \sqrt{\frac{kT}{\alpha^2\sigma^2}} \left[ \frac{1}{T} \int_0^T V_t^2 \, dt - \left( \frac{1}{T} \int_0^T V_t \, dt \right)^2 \right] \geq \nu \right) \right| - \P(Z \leq z)
\]
\[
\leq \left| \P \left( \sqrt{\frac{kT}{\alpha^2\sigma^2}} \left[ \frac{1}{T} \int_0^T V_t^2 \, dt - \left( \frac{1}{T} \int_0^T V_t \, dt \right)^2 \right] \geq \nu \right) \right| - \Phi(\nu)
\]
\[
+ |\Phi(\nu) - \P(Z \leq z)|
\]
\[
= |D(\nu)| + |\Phi(\nu) - \P(Z \leq z)|.
\]

Denote \( D(\nu) \) as
\[
D(\nu) := \left| \P \left( \sqrt{\frac{kT}{\alpha^2\sigma^2}} \left[ \frac{1}{T} \left( \int_0^T V_t^2 \, dt - \left( \frac{1}{T} \int_0^T V_t \, dt \right)^2 \right) \right] \geq \nu \right) \right| - \Phi(\nu),
\]
where \( \nu \in \mathbb{R}, \; \alpha := C_{\beta} \Gamma(2\beta - 1)k^{-2\beta} \). The Lemma 5.4 of [7] ensures that
\[
|\Phi(\nu) - \P(Z \leq z)| \leq \frac{C}{\sqrt{T}}.
\]
Combining with Lemma 15, we obtain the desired result. \( \square \)

The following Lemma provides the upper bound of \( D(\nu) \).

**Lemma 15.** When \( T \) is large enough, there exists constant \( C'_{\beta,V} \) such that
\[
\sup_{\nu \in \mathbb{R}} |D(\nu)| \leq \frac{C'_{\beta,V}}{T^m},
\]
where \( m = \min \{1/3, (3 - 4\beta)/2\} \).

**Proof.** Since the Normal distribution is symmetric, we have
\[
D(\nu) = \left| \P \left( \sqrt{\frac{kT}{\alpha^2\sigma^2}} \left[ \frac{1}{T} \left( \int_0^T V_t^2 \, dt - \left( \frac{1}{T} \int_0^T V_t \, dt \right)^2 \right) \right] \geq \nu \right) \right| - \Phi(\nu)
\]
\[
= \left| \P \left( \sqrt{\frac{kT}{\alpha^2\sigma^2}} \left[ \frac{1}{T} \left( \int_0^T V_t^2 \, dt - \left( \frac{1}{T} \int_0^T V_t \, dt \right)^2 \right) \right] \leq \nu \right) \right| - \Phi(\nu).
\]

Consider the following processes:
\[
X_t = \int_0^t e^{-k(t-s)} \, dG_s, \quad I_5 = \frac{1}{T} \left( \int_0^T X_t^2 \, dt \right),
\]
\[
F_T = \int_0^T \int_0^t e^{-k(t-s)} \, dG_s \, dt,
\]
where \( X_t \) is an OU process driven by \( G_t \). According to [11] formula (63), we can obtain
\[
\frac{1}{T} \int_0^T V_t^2 \, dt - \left( \frac{1}{T} \int_0^T V_t \, dt \right)^2 - \alpha = \frac{1}{T} \left( \int_0^T X_t^2 \, dt \right) - \alpha + K_T + I_4,
\]
where
\[ K_T = \frac{\mu^2}{2k} \cdot \frac{1 - e^{-2kT}}{T} - \frac{\mu^2(e^{-kT} - 1)^2}{k^2T^2}, \]
\[ I_4 = \frac{\mu}{k} \left[ \frac{Z_T}{T} e^{-kT} - \frac{M_T}{T} + 2(1 - e^{-kT}) \frac{F_T}{T^2} \right] - \frac{F_T^2}{T^2}. \]

Let \( a = T^{-1/3} \). Lemma 10 ensures that
\[
\sup_{\nu \in \mathbb{R}} \left| D(\nu) \right| \leq \sup_{\nu \in \mathbb{R}} \left| \mathbb{P} \left( \sqrt{\frac{kT}{\alpha^2 \sigma^2_{\beta}}^2} (I_5 - \alpha + K_T) \leq \nu \right) - \mathbb{P}(Z \leq \nu) \right| + \mathbb{P} \left( \left| \sqrt{\frac{kT}{\alpha^2 \sigma^2_{\beta}}^2} I_4 \right| > T^{-1/3} \right) + \frac{T^{-1/3}}{\sqrt{2\pi}} .
\] (28)

According to [7] Theorem 1.4, we have
\[
\sup_{\nu \in \mathbb{R}} \left| \mathbb{P} \left( \sqrt{\frac{kT}{\alpha^2 \sigma^2_{\beta}}^2} (I_5 - \alpha + K_T) \leq \nu \right) - \mathbb{P}(Z \leq \nu) \right| \leq \frac{C_{k,\beta}}{T^{3/4}}, \tag{29}
\]
where \( C_{k,\beta} \) independent of \( T \) is a constant. Denote \( K_1 = 4 \sqrt{\frac{k}{\alpha^2 \sigma^2_{\beta}}} \). We then consider the second term of right side of (28).
\[
\mathbb{P} \left( \left| \sqrt{\frac{kT}{\alpha^2 \sigma^2_{\beta}}^2} I_4 \right| > T^{-1/3} \right) \leq \mathbb{P} \left( \left| K_1 \frac{\mu \cdot e^{-kT} \cdot Z_T}{k} \right| > T^{1/6} \right)
+ \mathbb{P} \left( \left| K_1 \frac{\mu \cdot M_T}{k} \right| > T^{1/6} \right)
+ \mathbb{P} \left( \left| K_1 \frac{\mu \cdot (2 - 2e^{-kT}) \cdot F_T}{k} \right| > T^{7/6} \right)
+ \mathbb{P} \left( \left| K_1 \cdot F_T^2 \right| > T^{7/6} \right).
\]

Combining the Chebyshev inequality, Lemma 14 and the Proposition 3.10 of [11], we can obtain
\[
\mathbb{P} \left( \left| K_1 \frac{\mu \cdot e^{-kT} \cdot Z_T}{k} \right| > T^{1/6} \right) \leq \frac{C'_1 \mathbb{E}(Z_T^2)}{T^{1/3}} \leq \frac{C_1}{T^{1/3}} ;
\]
\[
\mathbb{P} \left( \left| K_1 \frac{\mu \cdot M_T}{k} \right| > T^{1/6} \right) \leq \frac{C'_2 \mathbb{E}(M_T^2)}{T^{1/3}} \leq \frac{C_2}{T^{1/3}} ;
\]
\[
\mathbb{P} \left( \left| K_1 \frac{\mu \cdot (2 - 2e^{-kT}) \cdot F_T}{k} \right| > T^{13/6} \right) \leq \frac{C'_3 \mathbb{E}(F_T^2)}{T^{13/6}} \leq \frac{C_3}{T^{13/6-2/3}} ;
\]
\[
\mathbb{P} \left( \left| K_1 \cdot F_T^2 \right| > T^{13/6} \right) \leq \frac{C'_4 \mathbb{E}(F_T^2)}{T^{13/6}} \leq \frac{C_4}{T^{13/6-2/3}} .
\]

Then we have
\[
\mathbb{P} \left( \left| \sqrt{\frac{kT}{\alpha^2 \sigma^2_{\beta}}^2} I_4 \right| > T^{-1/3} \right) + \frac{T^{-1/3}}{\sqrt{2\pi}} \leq \frac{C'_{k,\beta}}{T^{1/3}}, \tag{30}
\]
where \( C'_{k,\beta} \) is a constant. Combining formulas (29) and (30), we obtain the desired result. \( \square \)
4. Berry-Éssèen upper bounds of least squares estimators

For the convenience of following proof, we introduce some variables:
\[
\begin{align*}
    a_T &= 1 - e^{-kT}, \\
    b_T &= \frac{1}{T} \int_0^T \|e^{-k(t-\cdot)}\mathbb{1}_{[0,t]}(\cdot)\|^2_B \, dt \to C_\beta \Gamma(2\beta - 1)k^{-2\beta}, \\
    c_T &= \int_0^T \mu^2(1-e^{-kt})^2 \, dt = \mu^2(T + \frac{2}{k}(e^{-kT} - 1) + \frac{1}{2k}(1-e^{-2kT})), \\
    d_T &= \int_0^T (1-e^{-kt}) \, dt = T + \frac{1}{k}(e^{-kT} - 1).
\end{align*}
\]

The Proposition 3.10 of [11] and Proposition 9 ensure that
\[
\begin{align*}
    e_T &= \|l_T \otimes_1 l_T\|^2_B \leq CT^{2\beta}, \\
    q_T &= \|l_T \otimes_1 k_T\|^2_B \leq \|k_T \otimes_1 \frac{1}{k}\|^2_B \leq CT^{2\beta},
\end{align*}
\]
where \(C\) is a constant independent of \(T\). Also, we show \(l_T\) and other functions:
\[
\begin{align*}
    f_T(t,s) &= e^{-k|t-s|} \mathbb{1}_{[0,T]}(t,s), \\
    h_T(t,s) &= e^{-k(T-t)-k(T-s)} \mathbb{1}_{[0,T]}(t,s), \\
    g_T(t,s) &= \frac{1}{2kT}f_T - h_T, \\
    k_T(s) &= e^{-k(T-s)} \mathbb{1}_{[0,T]}(s), \\
    l_T(s) &= \frac{1}{k}(1-e^{-k(T-s)}) \mathbb{1}_{[0,T]}(s), \\
    m_T(s) &= e^{-ks} \mathbb{1}_{[0,T]}(s), \\
    n_T(s) &= \frac{1}{2k}(e^{-k(2T-s)} - 1) \mathbb{1}_{[0,T]}(s).
\end{align*}
\]
Furthermore, we denote \(I_1(f_T)\) as \(I_1(f_T(t,\cdot)\mathbb{1}_{[0,T]}(\cdot))\).

We now extend the Corollary 1 of [17].

**Theorem 16.** Let \(F_T/H_T\) be a zero-mean ratio process that contains at most triple Wiener-Itô stochastic integrals, where \(F_T = I_1(f_{1,T}) + I_2(f_{2,T}) + I_3(f_{3,T})\), \(H_T = E_T + I_1(h_{1,T}) + I_2(h_{2,T}) + I_3(h_{3,T})\) and \(E_T\) is a positive function of \(T\) converging to a constant \(\alpha\). Suppose that \(\Psi_i(T) \to 0\), \(i = 1, \ldots, 4\), as \(T \to \infty\). Then, there exists constant \(C\) such that when \(T\) is large enough,
\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P}\left( \frac{F_T}{H_T} \leq z \right) - \mathbb{P}(Z \leq z) \right| \leq C \cdot \max_{m,n=1,2,3} \left( \left\| p_{m,T} \otimes_1 q_{n,T} \right\|_{B^{(m+n-2)}} \right), \tag{31}
\]
\[
\langle p_{m,T}, q_{m,T} \rangle_{B^{2m}} \left( \sqrt{E_T^2 + \sum_{j=1}^m j! \left\| f_j \right\|_{B^2}^2} \right).
\]

**Proof.** We first consider \(\Psi_4(T) = \frac{1}{(EH_T)^2} \sqrt{\mathbb{E}[(DH_T, -DL^{-1}(H_T - EH_T))_B^2]}\). It is easy to see that \((EH_T)^2 \to E_T^2 \to \alpha^2\) a.s.. Then we deal with \(\sqrt{\mathbb{E}[(DH_T, -DL^{-1}(H_T - EH_T))_B^2]}\).

Denote \(\langle DH_T, -DL^{-1}(H_T - EH_T) \rangle_B = \psi_4\), we have
\[
\begin{align*}
    \psi_4 &= \langle h_{1,T} + 2I_1(h_{2,T}) + 3I_2(h_{3,T}), h_{1,T} + I_1(h_{2,T}) + I_2(h_{3,T}) \rangle_B \\
    &= \left\| h_{1,T} \right\|_B^2 + 3h_{1,T}I_1(h_{2,T}) + 4h_{1,T}I_2(h_{3,T}) \\
    &+ 2 \left\| h_{2,T} \right\|_{B^{2}}^2 + 2I_2(h_{2,T} \otimes_1 h_{3,T}) \\
    &+ 5I_2(h_{2,T} \otimes_1 h_{3,T}) + 10I_1(h_{2,T} \otimes_2 h_{3,T}) \\
    &+ 6 \left\| h_{3,T} \right\|_{B^{2}}^2 + 3I_4(h_{3,T} \otimes_1 h_{3,T}) + 12I_2(h_{3,T} \otimes_2 h_{3,T}). \tag{32}
\end{align*}
\]
Following from the orthogonality property of multiple integrals, we can obtain
\[
\Psi_4(T) \leq C_1 \cdot \max_{m,n=1,2,3, \atop i=1,\ldots,(m\wedge n)} \left( \|h_{m,T} \hat{\otimes}_i h_{n,T}\|_{\mathcal{B}^{(m+n-2i)}} \cdot \|h_{m,T}\|_{\mathcal{B}^{m}}^2 \right). \tag{33}
\]
where \( C_1 \) is a constant independent of \( T \).

We next consider \( \Psi_2(T) \) and \( \Psi_3(T) \). Denote \((DF_T, -DL^{-1}(H_T - \mathbb{E}H_T))_{\mathcal{B}} = \psi_2\), we have
\[
\psi_2 = \langle f_{1,T} + 2I_1(f_{2,T}) + 3I_2(f_{3,T}), h_1, T + I_1(h_2, T) + I_2(h_3, T) \rangle_{\mathcal{B}}
\]
\[
= f_{1,T}h_1, T + f_{1,T}I_1(h_2, T) + f_{1,T}I_2(h_3, T)
\]
\[
+ 2h_1,T I_1(f_{2,T}) + 3h_1,T I_2(f_{3,T})
\]
\[
+ 2\langle f_{2,T}, h_2, T \rangle_{\mathcal{B}} + 2I_2(f_{2,T} \otimes_1 h_{2, T})
\]
\[
+ 2I_3(f_{2,T} \otimes_1 h_{3, T}) + 4I_1(f_{2,T} \otimes_2 h_{3, T})
\]
\[
+ 3I_3(f_{3,T} \otimes_1 h_{2, T}) + 6I_1(f_{3,T} \otimes_2 h_{2, T})
\]
\[
+ 6\langle f_{3,T}, h_3, T \rangle_{\mathcal{B}} + 3I_4(f_{3,T} \otimes_1 h_{3, T}) + 12I_2(f_{3,T} \otimes_2 h_{3, T}).
\]

Similarly, there exists a constant \( C_2 \) such that
\[
\Psi_2(T) \leq C_2 \cdot \max_{m,n=1,2,3, \atop i=1,\ldots,(m\wedge n)} \left( \|p_{m,T} \hat{\otimes}_i q_{n,T}\|_{\mathcal{B}^{(m+n-2i)}} \cdot \langle f_{m,T}, h_{m,T} \rangle_{\mathcal{B}^{m}} \right). \tag{34}
\]

Following from the above result, we can obtain the bound of \( \Psi_3(T) \).

We then deal with \( \Psi_1(T) = \frac{1}{(\mathbb{E}H_T)^2} \sqrt{\mathbb{E}[((\mathbb{E}H_T)^2 - (DF_T, -DL^{-1}F_T)_{\mathcal{B}})]}^2 \). According to the orthogonality and (32), we have
\[
\Psi_1(T) \leq C_3 \cdot \max_{m,n=1,2,3, \atop i=1,\ldots,(m\wedge n)} \left( \|f_{m,T} \hat{\otimes}_i f_{n,T}\|_{\mathcal{B}^{(m+n-2i)}} \cdot \langle E_T^2 - \sum_{i=1}^m j! \|f_{j,T}\|_{\mathcal{B}^{j}}^2 \rangle \right). \tag{35}
\]
Combining formulas (33), (34) and (35), we obtain the desired result. \( \square \)

We now prove the convergence speed of \( \hat{k}_{LS} \). First, we review the CLT of \( \hat{k}_{LS} \).

**Theorem 17** ([11] Propositions 4.20). When \( \beta \in (1/2, 3/4) \) and Hypothesis 1 holds, \( \hat{k}_{LS} \) satisfies the following central limit theorem:
\[
\sqrt{T}(\hat{k}_{LS} - k) = \frac{V_T}{\sqrt{T}} \hat{\mu} + k\sqrt{T} (\hat{\mu} - \mu) - \frac{1}{\sqrt{T}} \int_0^T V_t \ dG_t \ \overset{law}{\longrightarrow} \mathcal{N}(0, k\sigma^2). \tag{36}
\]

We then transform the above \( \sqrt{T}(\hat{k}_{LS} - k) \) as multiple Wiener-Itô integrals.

**Proposition 18.** Let \( X_t = \int_0^t e^{-k(t-s)} \ dG_s \) be a OU process driven by \( G_t \). \( \sqrt{T}(\hat{k}_{LS} - k) \) can be rewritten as:
\[
\sqrt{T}(\hat{k}_{LS} - k) = \frac{I_0 + I_1(f_{1,T}) + I_2(f_{2,T})}{J_0 + I_1(h_{1,T}) + J_2(h_{2,T})}. \tag{37}
\]
where

\[ I_0 = \frac{1}{T^{3/2}} \left( \mu^2 a_T \cdot d_T + q_T + \frac{\mu^2}{k} \sigma_T^2 + k \varepsilon_T \right) + \frac{\mu^2}{T^{1/2}}(e^{-kT} - 1), \quad (38a) \]

\[ f_{1,T} = \frac{\mu}{T^{3/2}} (d_T k_T - a_T l_T) + \frac{\mu}{\sqrt{T}} (m_T - k_T), \quad (38b) \]

\[ f_{2,T} = \frac{1}{T^{3/2}} l_T \otimes k_T + \frac{k}{T^{3/2}} l_T \otimes l_T - \frac{1}{2T} f_T, \quad (38c) \]

\[ J_0 = \frac{c_T}{T} + b_T - \frac{1}{T^2} (\mu^2 d_T^2 + e_T), \quad (38d) \]

\[ h_{1,T} = \frac{2\mu}{T} (l_T + n_T) - \frac{2}{T^2} d_T l_T, \quad (38e) \]

\[ h_{2,T} = g_T - \frac{1}{T^2} l_T \otimes l_T. \quad (38f) \]

**Proof.** We first deal with the numerator of (36):

\[ \frac{V_T}{\sqrt{T}} \hat{\mu} + k \sqrt{T} \hat{\mu} (\hat{\mu} - \mu) - \frac{1}{\sqrt{T}} \int_0^T V_t dG_t. \quad (39) \]

According to the definition of above functions, \( V_T = \mu a_T + I_1(k_T) \). Combining with Proposition 9, we can obtain

\[
\frac{V_T}{\sqrt{T}} \hat{\mu} = \frac{\mu a_T + I_1(k_T) \mu d_T + I_1(l_T)}{T^{1/2}}
= \frac{1}{T^{3/2}} \left( \mu^2 a_T \cdot d_T + \mu a_T I_1(l_T) + \mu d_T I_1(k_T) + I_1(l_T) I_1(k_T) \right)
\]

\[
= \frac{1}{T^{3/2}} \left( \mu^2 a_T \cdot d_T + \mu a_T I_1(l_T) + \mu d_T I_1(k_T) \right)
+ I_2(l_T \otimes k_T) + \| l_T \otimes 1_k T \|_2^2 \bigg). \]

Then the second term of (39). Let \( \text{Item}_2 := k \sqrt{T} \hat{\mu}(\hat{\mu} - \mu) \), we have

\[
\text{Item}_2 = k \sqrt{T} (\hat{\mu} - \mu + \mu)(\hat{\mu} - \mu)
= k \sqrt{T} (\hat{\mu} - \mu) + k \sqrt{T} \mu (\hat{\mu} - \mu)
= \frac{k}{T^{3/2}} \left( \mu^2 \frac{e^{-kT} - 1}{k^2} + \frac{2\mu}{k} (e^{-kT} - 1) I_1(l_T) + I_2(l_T \otimes l_T) + e_T \right)
+ \frac{k\mu}{T^{3/2}} \left( \frac{\mu}{k} (e^{-kT} - 1) I_1(l_T) + I_2(l_T \otimes l_T) + e_T \right)
\]

\[
= \frac{k}{T^{3/2}} \left( \mu^2 \frac{e^{-kT} - 1}{k^2} + \frac{2\mu}{k} (e^{-kT} - 1) I_1(l_T) + I_2(l_T \otimes l_T) + e_T \right)
+ \frac{\mu^2}{T^{3/2}} (e^{-kT} - 1) + \frac{\mu}{T^{3/2}} \left( G_T - I_1(k_T) \right). \]

Besides, we can obtain

\[- \frac{1}{\sqrt{T}} \int_0^T V_t dG_t = - \frac{1}{\sqrt{T}} \left( \int_0^T \mu(1 - e^{-kt}) dG_t + \frac{1}{2} I_2(f_T) \right)
= - \frac{1}{\sqrt{T}} \left( \mu(G_T - I_1(m_T)) + \frac{1}{2} I_2(f_T) \right). \]
Next, we consider the denominator. For \( \frac{1}{T} \int_0^T V_t^2 \, dt \), we have

\[
\frac{1}{T} \int_0^T V_t^2 \, dt = \frac{1}{T} \int_0^T (\mu(1-e^{-kt}) + X_t)^2 \, dt
\]

\[
= \frac{1}{T} \left( \int_0^T \mu^2(1-e^{-kt})^2 \, dt + 2 \int_0^t \mu(1-e^{-kt})X_t \, dt + \int_0^T X_t^2 \, dt \right)
\]

\[
= \frac{1}{T} \left( c_T + 2\mu \int_0^T \int_0^t (1-e^{-kt})e^{-k(t-s)} \, dG_s \, dt \right) + I_2(g_T) + b_T
\]

\[
= \frac{c_T}{T} + \frac{2\mu}{T}(I_1(l_T) + I_1(n_T)) + I_2(g_T) + b_T.
\]

Since \( \hat{\mu}^2 = \left( \frac{1}{T} \int_0^T V_t \, dt \right)^2 \), we can obtain

\[
\left( \frac{1}{T} \int_0^T V_t \, dt \right)^2 = \frac{1}{T^2} \left( \mu d_T + I_1(l_T) \right)^2
\]

\[
= \frac{1}{T^2} \left( \mu^2 d_T^2 + 2\mu d_T I_1(l_T) + I_1^2(l_T) \right)
\]

\[
= \frac{1}{T^2} \left( \mu^2 d_T^2 + 2\mu d_T I_1(l_T) + I_2(l_T \otimes l_T) + e_T \right).
\]

Combining the above formulas, we obtain the desired result. \( \square \)

For simplicity, let \( F_T := (I_1(f_1,T) + I_2(f_2,T))/(\sqrt{k\sigma_\beta}), \) \( H_T := J_0 + I_1(h_{1,T}) + I_2(h_{2,T}) \). We show the convergence speed of zero-mean part.

**Lemma 19.** Let \( Z \sim \mathcal{N}(0,1) \) be a standard Normal variable. Assume \( \beta \in (1/2, 3/4) \) and Hypothesis 1 holds. When \( T \) is large enough, there exists constant \( C_{\beta,V} \) such that

\[
\sup_{z \in \mathbb{R}} \left| \Pr \left( \frac{F_T}{H_T} \leq z \right) - \Pr(Z \leq z) \right| \leq \frac{C'_{\beta,V}}{T^\gamma},
\]

where \( \gamma = \min \{1/2, 3 - 4\beta\} \).

**Proof.** According to [11] formulas (9) and (47),

\[
\mathbb{E}H_T = \mathbb{E}J_0 \rightarrow \alpha = C_3 \Gamma(2\beta - 1)k^{-2\beta} \text{ a.s.}
\]

Combining with [12] Lemma 5.2.4, we can obtain

\[
\max\{\|l_T\|_{\mathcal{B}_2}^2, \|n_T\|_{\mathcal{B}_2}^2\} \leq C^{(1)} T^{2\beta},
\]

(41)

where \( C^{(1)} \) is a constant, and \( \|h_{1,T}\|_{\mathcal{B}_2}^2 \leq C^{(2)} / T^{2-2\beta} \).

Following from [7] Theorem 1.4 and (41), we have

\[
\|g_T\|_{\mathcal{B}_2} \leq \frac{C^{(3)}}{\sqrt{T}}, \quad \|g_T \otimes 1 g_T\|_{\mathcal{B}_2} \leq \frac{C^{(3)}}{T}, \quad \left\| \frac{2}{T^2} l_T \otimes l_T \right\|_{\mathcal{B}_2}^2 \leq \frac{C^{(3)}}{T^{4-4\beta}},
\]

where \( C^{(3)} \) is a constant independent of \( T \). Minkowski inequality ensures that \( \|h_{2,T}\|_{\mathcal{B}_2}^2 \leq C^{(4)} / T \). Similarly, the Proposition 3.10 of [11] induces that

\[
J_0^2 - \sum_{j=1}^m \frac{j! \|f_j,T\|_{\mathcal{B}_2}^2}{k\sigma_\beta^2} \leq C^{(5)} / T^{3-4\beta},
\]

(42)

and \( \|f_1,T\|_{\mathcal{B}}^2 \leq C^{(5)} / T \). The formula (5.13) of [7] ensures that

\[
\left\| f_{2,T} \otimes 1 h_{2,T} \right\|_{\mathcal{B}_2} \leq C^{(5)} / T^{1/2}.
\]

(42)
Combining the above results, Theorem 16 and Cauchy-Schwarz inequality, we obtain the Lemma.

The following Lemma shows the bound of non-zero mean part.

**Lemma 20.** Assume \( \beta \in (1/2, 3/4) \) and Hypothesis 1 holds. When \( T \) is large enough, there exists constant \( C_1 \) such that

\[
P\left(|L_1| > \frac{C_1}{T^{(3/4-\beta)}}\right) = 0,
\]

where \( L_1 = I_0/(\sqrt{k} \sigma_\beta H_T) \).

**Proof.** The Proposition 3.14 and Corollary 3.15 of [11] ensure that when \( T \) is large enough,

\[
\frac{1}{T} \int_0^T V_t^2 \, dt - \left( \frac{1}{T} \int_0^T V_t \, dt \right)^2 \to C_\beta \Gamma(2\beta - 1)k^{-2\beta} = \alpha \quad \text{a.s.} \quad (43)
\]

Then we can obtain

\[
|L_1| = \left| \frac{1}{\sqrt{k} \sigma_\beta} I_0 \right| \leq C' |I_0|,
\]

where \( C' \) is a constant. Furthermore, According to (38a), there exists \( C'' \) such that

\[
|I_0| \leq \frac{1}{T^{3/2}} \left( \mu^2 a_T \cdot d_T + q_T + \frac{\mu^2}{k} a_T^2 + k e_T \right) + \frac{\mu^2}{T^{1/2}} a_T \leq \frac{C''}{T^{3/2 - 2\beta}}.
\]

Combining the above two formulas, we have

\[
|L_1| \leq \frac{C(6)}{T^{3/2 - 2\beta}} \quad \text{a.s.},
\]

where \( C(6) = 2C' \cdot C'' \). Then we obtain the desired result. \( \square \)

We now prove the formula (9).

**Proof of formula (9).** According to Lemma 10,

\[
\sup_{z \in \mathbb{R}} \left| P \left( \sqrt{\frac{T}{k} \sigma_\beta^2} (\hat{k}_{LS} - k) \leq z \right) - P(Z \leq z) \right| \leq \sup_{z \in \mathbb{R}} \left| P \left( \frac{F_T}{H_T} \leq z \right) - P(Z \leq z) \right| \\
+ P \left( |L_1| \leq \frac{C_1}{T^{3/4 - \beta}} \right) + \frac{1}{\sqrt{2\pi}} \frac{C_1}{T^{3/4 - \beta}}.
\]

Combining Lemmas 19 and 20, we obtain the desired result. \( \square \)

Pei et al.[11] show the CLT of least squares estimator \( \hat{\mu}_{LS} \) of mean coefficient \( \mu \).

**Theorem 21** ([11] Propositions 4.21). Assume \( \beta \in (1/2, 1) \) and \( G_t \) is a self-similar Gaussian process satisfying Hypothesis 1 and \( \mathbb{E}[G_t^2] = 1 \). \( T^{1-\beta}(\hat{\mu}_{LS} - \mu) \) is asymptotically normal as \( T \to \infty \):

\[
T^{1-\beta}(\hat{\mu}_{LS} - \mu) = \frac{V_T}{\sqrt{T}} \left[ \frac{1}{T} \int_0^T V_t^2 \, dt - \mu \hat{\mu} \right] - \frac{1}{T} \int_0^T V_t \, dV_t \cdot T^{1-\beta} [\hat{\mu} - \mu] \xrightarrow{\text{law}} \mathcal{N}(0, 1/k^2).
\]

We also transform \( T^{1-\beta}(\hat{\mu}_{LS} - \mu) \) as the following multiple integrals.
Proposition 22. $T^{1-\beta}(\hat{\mu}_L - \mu)$ can be represented by as:

$$T^{1-\beta}(\hat{\mu}_L - \mu) = \frac{I^*_0 + I_1(f^*_1T) + I_2(f^*_2T) + I_3(f^*_3T)}{J^*_0 + I_1(h^*_1T) + I_2(h^*_1T)},$$

where

$$I^*_0 = \frac{1}{T^{1+\beta}} \left( 2\mu(\|kt\|_2 l_T^2) + \|kt\|_2 n_T^2 + \|kT\|_2 \|nT\|_2^2 + \mu \|mt\|_2 l_T^2 \right),$$

$$f^*_1T = \frac{1}{T^{1+\beta}} \left( (cT - \mu^2 dt - \frac{\mu^2 a_T}{k} I_{0,T}) + 2\mu a_T l_T + f_T \|l_T\|_2 \right) - \frac{1}{T^{1+\beta} m_T} \left( b_T I_{0,T} + 2g_T \|l_T\|_2 \right),$$

$$f^*_2T = \frac{1}{T^{1+\beta}} \left( 2\mu (kt + kT \|nT\|_2) + 2k\mu (nT \|l_T\|_2) + \mu mT \|l_T\|_2 - \frac{\mu}{2k} a_T f_T \right),$$

$$f^*_3T = \frac{1}{T^{1+\beta}} \left( gT \|kT\|_2 + k T \|l_T\|_2 \right) + \frac{1}{2T^{1+\beta} f_T} \left( kT - \mu^2 dt \right) + kT,$$

$$h^*_1T = \frac{1}{T} \left( \mu a_T l_T + \mu d_T kT \right) + \frac{1}{T} \left( \mu kT + 2k\mu mT + \mu mT \right),$$

$$h^*_2T = \frac{1}{T^2} l_T \|kT\|_2 - \frac{1}{2T} f_T + kT.$$}

Proof. We first consider the denominator.

$$\frac{V_T}{T} \cdot \hat{\mu} - \frac{1}{T} \int_0^T V_t \, dV_t.$$ According to Proposition 18,

$$\frac{V_T}{T} \cdot \hat{\mu} = \frac{\mu a_T + I_1(kt) \mu d_T + I_1(lT)}{T}$$

$$= \frac{1}{T^2} \left( \mu^2 a_T \cdot d_T + \mu a_T I_1(lT) + \mu d_T I_1(kT) + I_1(lT) I_1(kT) \right)$$

$$= \frac{1}{T^2} \left( \mu^2 a_T \cdot d_T + \mu a_T I_1(lT) + \mu d_T I_1(kT) + I_2(lT \otimes kT) + \|lT \otimes kT\|_2^2 \right).$$

Then, we deal with $-\frac{1}{T} \int_0^T V_t \, dV_t$:

$$-\frac{1}{T} \int_0^T V_t \, dV_t = -\frac{1}{T} \left( \int_0^T k\mu V_t \, dt - \int_0^T kV_t^2 \, dt + \int_0^T V_t \, dG_t \right)$$

$$= -\frac{1}{T} \left( \int_0^T k\mu V_t \, dt + \frac{k}{T} \int_0^T V_t^2 \, dt - \frac{1}{T} \int_0^T V_t \, dG_t \right)$$

$$= -\frac{1}{T} \left( k\mu^2 d_T + k\mu I_1(lT) + \frac{k}{T} \left( \mu kT \right) + \mu I_1(lT) + I_1(nT) \right)$$

$$- kI_2(g_T) + kT$$

$$= \frac{1}{T} \left( k\mu I_1(lT) + 2k\mu I_1(nT) + \mu I_1(kT) + I_1(mT) + \frac{1}{2} I_2(f_T) \right)$$

$$+ kI_2(g_T) + kT.$$ We next consider the numerator:

$$\frac{V_T}{T^{\beta}} \left[ \frac{1}{T} \int_0^T V_t^2 \, dt - \mu \hat{\mu} \right] - \frac{1}{T} \int_0^T V_t \, dV_t \cdot T^{1-\beta} [\hat{\mu} - \mu].$$
Since the Proposition 18, we have
\[
\frac{V_T}{T^\beta} \frac{1}{T} \int_0^T V_t^2 dt = \frac{1}{T^\beta} (\mu a_T + I_1(k_T)) \left( \frac{c_T}{T} + \frac{2\mu}{T} (I_1(l_T) + I_1(n_T)) + I_2(g_T) + b_T \right)
\]
\[
= \frac{1}{T^{1+\beta}} \left( \mu a_T \cdot c_T + 2\mu^2 a_T (I_1(l_T) + I_1(n_T)) + c_T I_1(k_T) \right.
\]
\[
+ 2\mu (I_2(k_T \otimes l_T) + I_2(k_T \otimes n_T))
\]
\[
+ 2\mu (\|k_T \otimes_1 l_T\|^2 + \|k_T \otimes_1 n_T\|^2)
\]
\[
+ \frac{1}{T^\beta} \left( \mu a_T I_2(g_T) + \mu a_T b_T + I_3(k_T \otimes g_T) \right.
\]
\[
+ \left. 2I_1(k_T \otimes g_T) + b_T I_1(k_T) \right).
\]

Similarly, we can obtain
\[
- \frac{V_T}{T^\beta} \cdot \mu \hat{\mu} = -\mu \frac{\mu a_T + I_1(k_T)}{T^\beta} - \mu \frac{d_T + I_1(l_T)}{T}
\]
\[
= -\frac{\mu}{T^{1+\beta}} \left( \mu^2 a_T \cdot d_T + \mu a_T I_1(l_T) + \mu d_T I_1(k_T) \right)
\]
\[
+ \frac{\mu}{T^{1+\beta}} \left( \mu^2 a_T \cdot d_T + \mu a_T I_1(l_T) + \mu d_T I_1(k_T) \right)
\]
\[
+ I_2(l_T \otimes k_T) + \|l_T \otimes_1 k_T\|^2_B \right).
\]

Let Item_3 = \(-\frac{1}{T} \int_0^T V_t dV_t \cdot T^{1-\beta} (\hat{\mu} - \mu)\), we have
\[
Item_3 = \frac{\left( I_1(l_T) - \frac{\mu a_T}{k_T^\beta} \right)}{T^\beta} \frac{1}{T} \int_0^T -V_t dV_t
\]
\[
= \frac{1}{T^{1+\beta}} \left( k_T^\beta I_1(l_T) - k_T^\beta I_1(l_T) + 2k_T^\beta I_2(n_T \otimes l_T) + 2k_T^\beta \|n_T \otimes l_T\|^2_B \right.
\]
\[
+ \mu I_2(k_T \otimes l_T) + \mu \|k_T \otimes l_T\|^2_B + \mu I_2(m_T \otimes l_T) + \mu \|m_T \otimes l_T\|^2_B \right)
\]
\[
+ \frac{k}{T^\beta} \left( b_T I_1(l_T) + I_3(g_T \otimes l_T) + 2I_1(g_T \otimes l_T) \right)
\]
\[
+ \frac{1}{2T^{1+\beta}} \left( I_3(f_T \otimes l_T) + 2I_1(f_T \otimes l_T) \right)
\]
\[
+ \frac{1}{T^{1+\beta}} \left( \mu a_T \cdot d_T - \mu a_T \cdot c_T - 2\mu^2 a_T I_1(n_T) \right)
\]
\[
- \frac{1}{k T^{1+\beta}} \left( \mu^2 a_T I_1(k_T) + \mu^2 a_T I_1(m_T) + \frac{1}{2} \mu a_T I_2(f_T) \right)
\]
\[
- \frac{1}{T^\beta} (\mu a_T \cdot b_T + \mu a_T I_2(g_T)).
\]

Combining the above formulas, we obtain the Proposition. \(\square\)

Denote \(F_T^* := I_0^* + I_1(f_1^* T) + I_2(f_2^* T) + I_3(f_3^* T)\) and \(H_T^* = \frac{1}{k} (J_0^* + I_1(h_1^* T) + I_2(h_2^* T))\). We now prove the upper bounds of zero-mean part.

**Lemma 23.** Let \(Z \sim \mathcal{N}(0, 1)\) be a standard Normal variable. Assume \(\beta \in (1/2, 1)\) and \(G_t\) is a self-similar Gaussian process satisfying Hypothesis 1 and \(\mathbb{E}[G_1^2] = 1\). When \(T\) is large enough, there exists a constant \(C_{\beta, V}'\) such that
\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left( \frac{F_T^*}{H_T^*} \leq z \right) - \mathbb{P}(Z \leq z) \right| \leq \frac{C_{\beta, V}'}{T^{2-2\beta}}, \quad (46)
\]
Proof. According to Lemma 19, (45f) and (45g), we have
\[ \|h_{1,T}\|^2 \leq \frac{K_1}{T^{2-2\beta}}, \quad \|h_{2,T}\|^2 \leq \frac{K_1}{T}, \]
where \( K_1 \) is a constant independent of \( T \). It is easy to see that Lemma 19 and equation (45e) implies \( \mathbb{E} H_T \rightarrow \alpha \) a.s. Furthermore, when \( T \) is large enough, there exists constant \( K_2 \) such that
\[ |(\mathbb{E} H_T^2 - \alpha^2)| \leq \frac{K_2}{T^{2-2\beta}}. \] (47)
Since \( G_t \) is self-similar, Lemma 19 implies \( \|f_{1,T}\|^2 \leq \frac{K_3}{T^2} \). Also, we can obtain \( \|f_{2,T}\|^2 \leq \frac{K_3}{T^2} \). Combining above three formulas, we have
\[ J_{0,2}^* - \sum_{j=1}^{m} j \|f_{j,T}\|^2 \leq \frac{K_4}{T^2}, \]
Following from Theorem 16 and Cauchy-Schwarz inequality, we obtain the result. \( \square \)

We next consider the non-zero mean part.

Lemma 24. Assume \( \beta \in (1/2, 1) \) and \( G_t \) is a self-similar Gaussian process satisfying Hypothesis 1 and \( \mathbb{E}[G_T^2] = 1 \). Let \( M_1 = I_0^* / H_T \). When \( T \) is large enough, there exists a constant \( C_1 \) independent of \( T \) such that
\[ \mathbb{P}
\bigg( |M_1| > \frac{C_1}{T^{1/2-\beta/2}} \bigg) = 0. \]
Proof. Following from Lemma 23, we have \( |M_1| \leq C' |I_0^*| \), where \( C' \) is a constant independent of \( T \). Furthermore, Equation (45a) implies that there exists \( C'' \) such that
\[ |I_0^*| \leq \frac{C''}{T^{1-\beta}}, \quad |M_1| \leq \frac{C_1}{T^{1-\beta}} \] a.s.,
where \( C_1 = 2C' \cdot C'' \). Combining the above formulas, we obtain the desired result. \( \square \)

We now prove the formula (11).

Proof of formula (11). Following from Lemma 10, we have
\[ \sup_{z \in \mathbb{R}} \left| \mathbb{P}
\left( \frac{k}{T^{\frac{1}{2}-1}} (\mu_{LS} - \mu) \leq z \right) - \mathbb{P}(Z \leq z) \right| \leq \sup_{z \in \mathbb{R}} \left| \mathbb{P}
\left( \frac{F_T}{H_T} \leq z \right) - \mathbb{P}(Z \leq z) \right| + \mathbb{P}
\left( |M_1| > \frac{C_1}{T^{1/2-\beta/2}} \right) + \frac{1}{\sqrt{2\pi}} \frac{C_1}{T^{1/2-\beta/2}}. \]
Combining Lemmas 23 and 24, we obtain the formula (11). \( \square \)

ACKNOWLEDGMENTS

We gratefully acknowledge the very valuable suggestions by referees. Y. Chen is supported by National Natural Science Foundation of China (NSFC) with grant No.11961033.
Declarations

• Availability of data and materials
This manuscript has no associated data.

References

[1] Oldrich Vasicek. An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5(2):177–188, 1977.
[2] Sang Wu, YingHui Dong, WenXin Lv, and GuoJing Wang. Optimal asset allocation for participating contracts with mortality risk under minimum guarantee. *Communications in Statistics - Theory and Methods*, 49(14):3481–3497, 2020.
[3] YueCai Han and Nan Li. Calibrating fractional vasicek model. *Communications in Statistics - Theory and Methods*, pages 1–15, 2021.
[4] Kevin Fergusson and Eckhard Platen. Application of maximum likelihood estimation to stochastic short rate models. *Annals of Financial Economics*, 10(02):1–26, 2015. doi: 10.1142/S2010495215500098. URL http://www.worldscientific.com/doi/10.1142/S2010495215500098.
[5] WeiLin Xiao and Jun Yu. Asymptotic theory for rough fractional vasicek models. *Economics Letters*, 177:26–29, 2019. ISSN 0165-1765. doi: https://doi.org/10.1016/j.econlet.2019.01.020. URL https://www.sciencedirect.com/science/article/pii/S0165176519300266.
[6] YaoZhong Hu and David Nualart. Parameter estimation for fractional ornstein–uhlenbeck processes. *Statistics & Probability Letters*, 80(11-12):1030–1038, 2010.
[7] Yong Chen and HongJuan Zhou. Parameter estimation for an ornstein-uhlenbeck process driven by a general gaussian noise. *Acta Mathematica Scientia*, 41(2):573–595, 2021.
[8] Qian Yu. Statistical inference for vasicek–type model driven by self-similar gaussian processes. *Communications in Statistics - Theory and Methods*, pages 1–14, 2018. doi: 10.1080/03610926.2018.1543774. URL http://www.tandfonline.com/doi/full/10.1080/03610926.2018.1543774.
[9] WeiLin Xiao, XiLi Zhang, and Ying Zuo. Least squares estimation for the drift parameters in the sub-fractional vasicek processes. *Journal of Statistical Planning and Inference*, 197:141–155, 2018. ISSN 0378-3758. doi: https://doi.org/10.1016/j.jspi.2018.01.003. URL https://www.sciencedirect.com/science/article/pii/S0378375818300053.
[10] ChunHao Cai, QingHua Wang, and WeiLin Xiao. Mixed sub-fractional brownian motion and drift estimation of related ornstein–uhlenbeck process. *Communications in Mathematics and Statistics*, pages 1–27, 2022.
[11] Yong Chen, Ying Li, and XingZhi Pei. Parameter estimation for vasicek model driven by a general gaussian noise. *Communications in Statistics - Theory and Methods*, pages 1–17, 2021.
[12] Ivan Nourdin and Giovanni Peccati. *Normal Approximations with Malliavin Calculus: From Stein’s Method to Universality*, volume 192. Cambridge University Press, Cambridge, 2012. doi: 10.1017/CBO9781139084659.
[13] G. Peccati. Gaussian approximations of multiple integrals. *Electronic Communications in Probability*, 12(23):350–364, 2007. doi: 10.1214/ECP.v12-1322. URL http://www.ams.org/mathscinet-getitem?mr=2350573.
[14] David Nualart. *The Malliavin Calculus and Related Topics*. Springer Science & Business Media, Berlin, Heidelberg, 2006. URL http://link.springer.com/978-1-4757-2437-0.

[15] Maria Jolis. On the wiener integral with respect to the fractional brownian motion on an interval. *Journal of Mathematical Analysis and Applications*, 330(2):1115–1127, 2007. doi: 10.1016/j.jmaa.2006.07.100. URL http://www.sciencedirect.com/science/article/pii/S0022247X06007864.

[16] Myron N Chang and PV Rao. Berry-esseen bound for the kaplan-meier estimator. *Communications in Statistics - Theory and Methods*, 18(12):4647–4664, 1989.

[17] Yoon Tae Kim and Hyun Suk Park. Optimal berry–esseen bound for statistical estimations and its application to spde. *Journal of Multivariate Analysis*, 155:284–304, 2017.

[18] Hermine Biermé, Aline Bonami, Ivan Nourdin, and Giovanni Peccati. Optimal berry-esseen rates on the wiener space: the barrier of third and fourth cumulants. *ALEA : Latin American Journal of Probability and Mathematical Statistics*, 9(2):473–500, 2012. URL https://hal.archives-ouvertes.fr/hal-00620384. 29 pages.

[19] Alexandre Brouste and Marina Kleptsyna. Asymptotic properties of mle for partially observed fractional diffusion system. *Statistical Inference for Stochastic Processes*, 13(1):1–13, 2010.

[20] Yong Chen, NengHui Kuang, and Ying Li. Berry–esséen bound for the parameter estimation of fractional ornstein–uhlenbeck processes. *Stochastics and Dynamics*, 20(04):1–11, 2020.

[21] Stanislav Lohvinenko, Kostiantyn Ralchenko, and Olga Zhuchenko. Asymptotic properties of parameter estimators in fractional vasicek model. *Lithuanian Journal of Statistics*, 55(1):102–111, 2016.

[22] Ivan Nourdin and Giovanni Peccati. Stein’s method and exact berry–esseen asymptotics for functionals of gaussian fields. *The Annals of Probability*, 37(6):2231–2261, 2009.

[23] Ivan Nourdin and Giovanni Peccati. The optimal fourth moment theorem. *Proceedings of the American Mathematical Society*, 143(7):3123–3133, 2015.

School of Mathematics and Statistics, Jiangxi Normal University, Nanchang, 330022, Jiangxi, China

Email address: zhishi@pku.org.cn

School of Mathematics and Statistics, Jiangxi Normal University, Nanchang, 330022, Jiangxi, China

Email address: chengym@jxnu.edu.cn