A SEARCH FOR EXACT SUPERSTRING VACUA

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ABSTRACT

We investigate $2d$ sigma-models with a $2 + N$ dimensional Minkowski signature target space metric and Killing symmetry, specifically supersymmetrized, and see under which conditions they might lead to corresponding exact string vacua. It appears that the issue relies heavily on the properties of the vector $M_\mu$, a reparametrization term, which needs to possess a definite form for the Weyl invariance to be satisfied. We give, in the $n = 1$ supersymmetric case, two non-renormalization theorems from which we can relate the $u$ component of $M_\mu$ to the $\beta^G_{uu}$ function. We work out this $(u, u)$ component of the $\beta^G$ function and find a non-vanishing contribution at four loops. Therefore, it turns out that at order $\alpha'^4$, there are in general non-vanishing contributions to $M_u$ that prevent us from deducing superstring vacua in closed form.
1 Introduction

To find exact string vacua is one of the most interesting problems of string theory. In fact, the symmetries of the vacuum are the symmetries of the World [1], the constants of the vacuum are the constants of the World [2] and therefore to know the string vacua is indeed the first crucial step towards a TOE.

In the past few months, attention has been brought to $2d$ finite sigma-models with a $2 + N$ dimensional Minkowski signature target space metric with a covariantly constant null Killing vector. They can be considered as describing string tree-level backgrounds consisting in plane gravitational wave-type, supplemented by a dilaton background. Such kinds of solution of Einstein equations have been discussed long ago by Brinkmann [3].

These models have been extensively studied [4] in the bosonic case and show several important features such as:

(1) the UV-finiteness (on shell) of these models;

(2) that, given a non-conformal $\sigma$-model with Euclidean $N$ dimensional target space (the so-called transverse space), there exists a conformal invariant Minkowskian $\sigma$-model in $2 + N$ dimensions;

(3) that, because of the Killing symmetry, the $2 + N$ dimensional metric does not depend on one of the two extra coordinates $u$ and $v$;

(4) the fact that the $2 + N$ dimensional metric is expressed in terms of the running coupling of the transverse theory.

Later, Tseytlin [5] and the present authors [6] discussed independently a specific supersymmetric extension of this class of models, which allows one to know exactly the transverse beta function, thanks to a non-renormalization theorem [7], and therefore to be able to give the line element in closed form.

But, having in mind exact string vacua identification, we need also to know an appropriate dilaton field such that the Weyl invariance conditions are satisfied. Only then will the resulting models correspond to string vacua. That such a dilaton field exists has been proved order by order in a perturbation expansion in $\alpha'$. However, this does not allow in general a formulation in closed form. The crucial point is a piece of information on $W_u$, the $u$-component of $W_\mu$, a vector that enters linearly in the reparametrization vector $M_\mu$ (cf. section 2); $W_\mu$ is a covariant vector originating through the mixing under renormalization of dimension 2 composite operators [8]. While earlier discussions on this point had been based on conjectures (see [9], [10]), none of them seems to be satisfactory, being either incomplete [5], [10] or only necessary but not sufficient [3] (see also [11]).

It is the aim of this paper to go deeper into this question. In order to get concrete information, we have been looking at the first non-vanishing contribution to $W_u$, if it exists at all.
In the bosonic case, it is well known \[4\] that \(W_u\) starts to be non-vanishing at \(\alpha'^3\) and behaves like \(u^{-3}\) (\(u\) is a light-cone coordinate). In the \(n = 1\) supersymmetric case under examination in the present paper, \(W_u\) has been computed to order \(\alpha'^4\) in two essentially different ways: first directly; and secondly via the non-renormalization theorem, which binds \(\partial_u W_u \equiv \dot{W}_u\) and the \(\beta^G_{uu}\) component of \(\beta^G_{\mu\nu}\), the beta function of \(G_{\mu\nu}\). The two coinciding results show that \(W_u\), with a \(u^{-4}\) behaviour, starts to be non-vanishing at order \(\alpha'^4\) for general \(N\).

This work is organized as follows: in section 2 we recall the general features of the \(n = 1\) supersymmetric model, which we shall investigate and the two non-renormalization theorems uncovered in this model, in addition to the well-known theorems on the homogeneous Kähler transverse space (hence \(n = 2\) supersymmetric). In section 3, we give the results for the \(\beta^G_{uu}\) component as derived from two different sources and for the direct computation of the \(W_u\) component of \(W_\mu\). In section 4, we give and discuss these results for a simple homogeneous manifold as transverse space. Finally, in section 5, we present our conclusions.

## 2 The supersymmetric model and the non-renormalization theorems

In the class of finite 2d \(\sigma\)-models introduced in Ref. \[4\], the \(N\)-dimensional transverse space was supposed to be a Euclidean symmetric space. Order by order it was shown that there exists a dilaton which, together with the \(2 + N\) dimensional metric background, solves the Weyl invariance conditions. The metric of the \(2 + N\) dimensional space turns out to give the line element

\[
ds^2 = G_{\mu\nu}dx^\mu dx^\nu = -2 du dv + f(u)\gamma_{ij}dx^i dx^j
\]

\[
\mu, \nu = 0, \ldots, N+1; \quad i, j = 1, \ldots, N
\]

With a specific choice of \(f(u)\), the model is shown to be UV-finite; \(f(u)\) is bound to satisfy a first-order RG equation

\[
p\dot{f} = \beta(f); \quad \dot{f} = \partial_u f; \quad p = \text{constant}
\]

with \(\beta(f)\) defined by the transverse \(\beta^G_{ij}\)

\[
\beta^G_{ij} = \beta(f)\gamma_{ij}.
\]

In order to complete the proof of finiteness of the model on a flat 2d background, the beta function of the \(2 + N\sigma\)-model with target space metric \(G_{\mu\nu}\) has to vanish up to a reparametrization term \[8\]:

\[
\beta^G_{\mu\nu} + 2D_{(\mu}M_{\nu)} = 0
\]

\[^1\text{For instance, the three-loop contribution to } \beta^G_{uu} \text{ due to the term } D_u R_{\alpha\beta\gamma\delta} D_u R^{\alpha\beta\gamma\delta} \text{ reads } -\frac{\alpha'^3}{16} \int f^2 f^{-4} R_{ijkl} R^{ijkl}; \quad i, j, k, \ell \text{ are transverse indices, and } f = bu.\]
$M_\mu$ is not arbitrary and to establish that the $\sigma$-model based on (1) is Weyl-invariant, one needs to show that a dilaton field $\phi$ exists such that $M_\nu$ in (4) can be represented by

$$M_\mu = \alpha' \partial_\mu \phi + \frac{1}{2} W_\mu$$

(5)

the origin of $W_\mu$ having been specified in the Introduction.

The $n = 1$ supersymmetric extension ([5], [6]) of the model with bosonic action [4] has been done as schematically indicated below.

One replaces the bosonic action

$$I_b = (4\pi \alpha')^{-1} \int d^2 z \sqrt{g} [G_{\mu\nu}(x) \partial_\alpha x^\mu \partial^\alpha x^\nu + \alpha' R^{(2)}(x)]$$

by the following superfield action

$$I = I_G + I_\phi$$

(6)

with

$$I_G = (4\pi \alpha')^{-1} \int d^2 z d^2 \theta G_{\mu\nu}(X) DX^\mu DX^\nu$$

(7)

$$I_\phi = (4\pi)^{-1} \int d^2 z d^2 \theta R^{(2)} E^{-1} \phi(X)$$

(8)

$D = \partial/\partial \bar{\theta} + \bar{\theta} \gamma^a \partial_a$ and $E^{-1}$ is the determinant of the $n = 1$ super-vielbein.

It is then obvious how to specialize the general metric $G_{\mu\nu}$ to a null Killing vector metric as in (1), in terms of the real superfields $U$, $V$ and $X^i$.

$$I = (4\pi \alpha')^{-1} \int d^2 z d^2 \theta [-2 D U \bar{D} V + g_{ij}(U, X) \cdot DX^i \bar{D} X^j]$$

(9)

The generalization of the bosonic case studied in [4] to the supersymmetric case is then straightforward and the finiteness condition for symmetric spaces (2) will also be determined by the beta-function of the transverse part of (9). If this transverse space is chosen to be Kählerian, the $N$-dimensional part is $n = 2$ supersymmetric. The choice in [5] and [6] was even more restrictive, assuming the transverse space to be a symmetric homogeneous Kähler manifold. This was dictated by the fact that known examples of these manifolds exist, for which the beta-function reduces to its one-loop expression and is therefore exactly known [12], [7]. In this case, $\beta(f)$ reduces to a constant $a$, depending on the manifold chosen and its symmetries (remember that $f$ is the inverse of the generic transverse $\sigma$-model coupling $\lambda$); $f(u)$ becomes equal to

$$f(u) = bu$$

(10)

with $b = \alpha p^{-1}$, and the transverse metric $g_{ij}(u, x)$ is $bu^2 \gamma_{ij}$. Evidently the $n = 2$ supersymmetry is not shared by the full $2 + N$ model with Minkowski signature studied here and has only $n = 1$ by construction [cf. eqs. (7), (9)]. However, use can be made of the result [13] that in the $n = 2$

\footnote{With metric tensor given by $g_{ij}(u, x) = f(u)\gamma_{ij}(x)$.}
case, the dilaton coupling does not get renormalized, so that some quantities appearing in the Weyl anomaly coefficients of the transverse part do vanish in the minimal subtraction scheme we use throughout (see [3] for details). Therefore the 2 + N model with \( n = 1 \) supersymmetry and with homogeneous symmetrical Kähler transverse subspace has a simplified structure, as compared with the generic \( n = 1 \) \( \sigma \)-models. However, the \( u \) components of the various key quantities such as \( W_u, \beta^G_{uu} \) and the “interaction” part \( \Phi(u) \) of the dilaton field \( \phi \), expressed as

\[
\phi(u, v) = pv + qu + \Phi(u)
\]

for symmetric transverse spaces, do not seem to benefit from the special properties of the transverse part.

Nevertheless, one can formulate two non-renormalization theorems concerning the three quantities \( \Phi(u), W_u \) and \( \beta^G_{uu} \):

1. \( 4 \dot{\Phi} + W_u = 0 \) beyond one loop,
2. \( \dot{W}_u + 2\beta^G_{uu} = 0 \) beyond one loop,

(at the one-loop level, however, \( 4 \dot{\Phi} + W_u = N/2u \) and \( \dot{W}_u + 2\beta^G_{uu} = -N/2u^2 \)).

As already emphasized in the Introduction, the issue of whether \( W_u \) is identically zero or starts being non-vanishing at some high order is crucial. For, if zero, then the dilaton field can be integrated to a closed form. If not, this closed form will only be the starting value of a perturbation expansion in \( \alpha' \) and the associated string vacuum, though existing, can only be expressed perturbatively. No string vacuum can be given in closed form if \( W_u \neq 0 \), according to the present methods.

In a first attempt to clarify the situation, we computed both \( W_u \) and \( \beta^G_{uu} \) in the \( n = 1 \) supersymmetric case. The results are shortly presented in the next section.

### 3 \( W_u \) and \( \beta^G_{uu} \) at four loops

We have explicitly worked out at four-loops the quantity \( W_u \) and the \((u, u)\) component of the \( n = 1 \) supersymmetric beta-function in the simplest non-trivial case, when the subspace \((N\text{-dimensional transverse space})\) is locally symmetric.

The direct calculation of \( W_u \) comes out with the result

\[
W_u = -\frac{\zeta(3)}{3(4\pi)^4} \cdot \dot{f} f^{-4} \cdot T_1 ;
\]

with \( T_1 = (R_a^{[bc]} d + R_a^{bc} d \cdot R^d e f \cdot R_{bc}^{e f} \).

Although we can derive the \( \beta^G_{uu} \) from (12) and the non-renormalization theorem, we evaluated it, as a cross-check, from two different sources:
(1) the direct calculation of the beta-function in \( n = 1 \) supersymmetric non-linear generic \( \sigma \) models, with the result

\[
\beta_{uu}^G = -\frac{\zeta(3)}{3(4\pi)^4} \hat{f}^2 f^{-5} \cdot T_2 ;
\]

with \( T_2 = R_{a b}^{c d} \cdot R_{e f}^{d a} \cdot R_{b (e f)}^{c} + \frac{3}{4} R_{b c d a} \cdot R_{d e f}^{l} \cdot R_{b c f}^{e} \cdot R_{ijkl} \) means \( R_{ijkl}(\gamma) \);

(b) the simple derivatives of a scalar built out from the sum of two different contractions of the product of three Riemann tensors, confirming eq. (13).

Accessoryly, it can be verified that

\[
\ddot{\Phi} = \frac{1}{2} \beta_{uu}^G = -\frac{1}{4} \dot{W}_u, \text{ at four loops}
\]

so that

\[
\beta_{uu}^G + \dot{W}_u + 2 \ddot{\Phi} = \bar{\beta}_{uu}^G = 0 ,
\]

which is the Weyl invariance condition for the \((u, u)\) component of the gravitational \( \bar{\beta}_{\mu\nu}^G \) Weyl anomaly coefficient. This shows, if necessary, that the dilaton field \( \phi \), eq. (11), will receive contributions from higher order, which are due to \( \sigma \)-model interactions.

4 Concrete example

We want to apply here the general formulas we have found in the previous sections and verify accessorily that the non-renormalization theorems are satisfied. There are general Riemannian manifolds called spaces of constant curvature, i.e. whose curvature is independent both of the surface direction and the position (for details, see e.g. Ref. [3b], section 18). These manifolds have a particularly simple curvature tensor, said maximally symmetric, which reads

\[
R_{hijk} = \frac{R}{N(N-1)} (\gamma_{hj} \gamma_{ik} - \gamma_{hk} \gamma_{ij}) ,
\]

\( R \) being the constant curvature and \( N \) the dimension of space (transverse in our case).

Of course these manifolds, for instance the \( N \)-dimensional sphere \( S^N \) embedded in a Euclidean \( R_{N+1} \) space, are Riemannian but not Kählerian in general. Therefore they are not suitable for our transverse space, which we assumed to be homogeneous Kähler. However, it happens that for \( N = 2 \) this Riemannian space is also homogeneous Kähler, due to the accidental isomorphism between \( S^2 = SO(3)/SO(2) \) and \( CP^1 \). So, for \( N = 2 \), this model is homogeneous Kähler and possesses all the properties wanted for our \( N \)-dimensional transverse space. Also the metric (16) can be used and it is straightforward to establish that (12) takes the value

\[
W_u = \frac{\zeta(3)}{6(4\pi)^4} \hat{f} f^{-4} \cdot R^3
\]

\(^3\)The covariant derivatives \( D_u \) differ from ordinary \( \partial_u \), the connections \( \Gamma_{ju}^{i} \) being non-vanishing for \( i, j \) transverse.
and (13)
\[ \beta_{uu}^G = \frac{\zeta(3)}{3(4\pi)^4} J f^2 f^{-5} \cdot R^3 \]  

Equations (17) and (18) obviously verify (14) and the non-renormalization theorem of section 2.

One notes also that the generic four-loop term contributes, for \( n = 1 \) supersymmetry, a quartic expression in terms of the curvature tensor, which is the only one to survive in locally symmetric spaces for general \( N \). However, using the metric (16), it gives a contribution proportional to \( (N - 2) \) and therefore vanishes in our example for which \( N = 2 \). This exemplifies the consistency of the approach: the quartic term has a pure Riemannian origin, but must be absent in a Kähler geometry, as it does in our example.

5 Conclusions

In spite of the well-known fact that, even in the simplest models with \( n = 1 \) supersymmetry, the four-loop beta-function is non-vanishing, witnessing the tree-level string theory graviton scattering modification to Einstein action [10], the model introduced by Tseytlin [4] and conveniently supersymmetrized [3], [4] is so specific that the hope was not unreasonable to see it escaping the four-loop contributions to the \((u, u)\) component of the beta-function. As a matter of fact, and as a posteriori justification of our attempt, the genuine Riemannian \( \alpha'^4 \) contribution to \( \beta_{\mu\nu} \), proportional to \( R^4 \), produces zero contribution to \( \beta_{uu}^G \) in the present model [1], while it is fully contributing to the generic \( n = 1 \) supersymmetric \( \sigma \)-model (see for instance [11]). This is an example of its specificity.

Summarizing, it turns out that only an \( n = 4 \) supersymmetry allows a perfect knowledge of the backgrounds in closed form. With a hyper-Kähler transverse space, the beta-function of this transverse part is identically zero, in all renormalization schemes (RS) and \( f \) is constant. The metric is trivially simple. Moreover, if non-renormalization theorems might exist too, they will not necessarily be the same as those of section 2. But if \( W_u \) starts getting non-zero contributions at some high order, even if it is at the four-loop level, there will necessarily be a scheme in which it vanishes identically and for which the beta function is still identically zero, since this property is RG-invariant. This is in contrast with the situation in the present paper. We can indeed find a scheme in which \( W_u \) is zero at all orders. However, we will lose the exact knowledge of the metric due to \( \beta(f) = a \), the “vanishing of all loop contributions but the first” being not an RG-invariant property, but specific to particular schemes. We plan to come back on these general questions in a forthcoming publication [12].

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4As it does also of course to \( \beta_{ij}^G \) or \( \beta_{iu}^G \) since the \( N \)-space has been assumed (homogeneous) Kählerian.
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