Detect black holes using photons for coupling model of electromagnetic and gravitational fields

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Abstract: For a model of the electromagnetic field coupled to Weyl tensor, Maxwell equations are modified and photons at low frequencies no longer propagate along light cone. If we detect a black hole using these photons, some difficulties appear because we can not determine the position of event horizon which is defined by null surface. To overcome these difficulties, the simplest way may be an effective description by introducing an effective spacetime in which the photons propagate along the light cone. Then, we find, comparing the results with those of the original spacetime, that the event horizon and temperature do not change, but the area of the event horizon and Bekenstein-Hawking entropy become different. We show that the total entropy for this system, which is still the same as that of original spacetime, consists of two parts, one is the Bekenstein-Hawking entropy and the other is the entropy arising from the coupling of electromagnetic field and Weyl tensor. We also present the effective descriptions for the Smarr relation and first law of thermodynamics.

Keywords: modified theory, faster than light, entropy, Smarr relation, thermodynamical law.

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1. Introduction

The studies of coupling models of the electromagnetic and gravitational fields have attracted much attention in many fields recently [1, 2, 3, 4, 5, 6, 7, 8] due to the facts: (a) the interaction between the electromagnetic and gravitational fields could be appeared naturally in quantum electrodynamics in curved spacetime [1], and (b) the Maxwell and Einstein equations modified by the coupling terms in the Lagrangian will lead to emergence of new properties of the gravitational and electromagnetic fields. Inspired by a well motivated non-minimal coupling between gravity and electromagnetism, Balakin and Zimdahl [6] demonstrated that the richer structure of the corresponding theory gives rise to novel features of the cosmological dynamics. Bamba and Nojiri [7] reviewed cosmology in non-minimal Maxwell theory in which the electromagnetic field couples to a function of the scalar curvature of the spacetime and explored the forms of the non-minimal gravitational coupling which generate the finite-time future singularities and the general conditions for this
coupling in order that the finite-time future singularities cannot appear. Balakin and Muharlyamov [8] showed that the resonance interactions between particles and transversal waves in plasma can take place due to the curvature coupling effect in a coupling model of gravity with electromagnetic fields. Drummond and Hathrell [1] noted that the characteristics of propagation of the light in the coupling models are altered by the tidal gravitational forces on the photons, so that in some cases photons travel at speeds greater than unity. Daniels, Cai, Shore, and Hollowood et al. [9, 10, 11, 12, 13, 14, 15, 16] proved that the interactions between the electromagnetic field and the spacetime curvature lead to a dependence of the photon velocity on the motion and polarization directions. In these coupling models the electromagnetic field is described by the modified theory rather than the usual free Maxwell theory. Therefore, the characteristics of propagation of the light are altered and the laws of geometric optics, in general, are invalid in the original spacetime. We [17] showed that, by introducing an effective spacetime, the wave vector can be cast into null and then obeys the null geodesic equation, the polarization vector is perpendicular to the rays, and the number of photons is conserved. We also found that the focusing theorem of light rays in the effective spacetime can also be written as the usual form.

Noting that photons at low frequencies no longer propagate along the light cone in the coupling models of the electromagnetic and gravitational fields, if we detect a black hole by using these photons, we can not determine the event horizon of a black hole by means of the null surface, and we can not calculate quantum entropy of black hole using statistical mechanics because we do not have any knowledge of the statistical mechanics if the speed of the light is greater than unity. We also do not know whether the thermodynamical laws still valid in these models. It should be noted that all the difficulties arise from the fact that photons no longer propagate along the light cone in these coupling models. Therefore, to overcome these difficulties, the simplest way may be to introduce an effective spacetime in which the photons propagate along the light cone. In this manuscript we systematically present an effective description for the thermodynamics in a coupling model of electromagnetic and gravitational fields, and some interesting results are obtained.

The plan of the paper is as follows. In the next section we introduce the difficulties arise in coupling model of electromagnetic and gravitational fields. In section III we will study the temperature and entropy of black hole described by the effective spacetime. In section IV we will obtain the effective Smarr formula and first law of thermodynamics. We present our conclusions and give some discussions in the last section.
2. Difficulties for physics of black hole in coupling model of electromagnetic and gravitational fields

The electromagnetic theory with the Weyl correction has been studied extensively [18, 19, 20, 21, 22, 23]. The action for a model of the electromagnetic field coupled to the Weyl tensor can be taken as

\[ S = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} - \frac{1}{4} \left( F_{\mu\nu} F^{\mu\nu} - 4\alpha C_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \right) \right], \tag{2.1} \]

where \( F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \) is the usual electromagnetic tensor with a vector potential \( A_{\mu} \), \( \alpha \) is the coupling constant, and \( C_{\mu\nu\rho\sigma} \) is the Weyl tensor of the spacetime. Varying the action (2.1) with respect to vector potential \( A_{\mu} \), we get

\[ \nabla_\mu (F^{\mu\nu} - 4\alpha C^{\mu\nu\rho\sigma} F_{\rho\sigma}) = 0. \tag{2.2} \]

Under the geometric optics assumption in which the wavelength \( \lambda \) of photon is much smaller than the minimum \( L \) of the radius of curvature of a wavefront \( \mathcal{L} \) and the radius of curvature of the spacetime \( \mathcal{R} \), the electromagnetic field strength can be written as

\[ F_{\mu\nu} = f_{\mu\nu} e^{i\Theta(t,r,\theta,\phi)}, \tag{2.3} \]

where \( f_{\mu\nu} \) is a slowly varying amplitude, and \( \Theta(t, r, \theta, \phi) \) is a rapidly varying phase and the wave vector can be defined as \( k_\mu = \partial_\mu \Theta(t, r, \theta, \phi) \). According to the Bianchi identity, we note that \( f_{\mu\nu} = k_\mu a_\nu - k_\nu a_\mu \) in which \( a_\mu \) is the polarization vector satisfying the condition that \( k_\mu a^\mu = 0 \). Substituting Eq. (2.3) into Eq. (2.2), we get the equation of motion of photon coupling to Weyl tensor

\[ k_\mu k^\mu a^\nu + 8\alpha C^{\mu\nu\rho\sigma} k_\sigma k_\mu a_\rho = 0. \tag{2.4} \]

Eq. (2.4) shows that the wave vector for this model is not null vector. That is to say, the interactions between the electromagnetic field and the spacetime curvature lead to a dependence of the photon velocity on the motion and polarization directions, so that in some cases photons travel at speeds greater or less than unity.

If we use this signal to detect the black hole, the fact that photons no longer propagate along the light cone will make the physics of black hole more complicated for the following reasons: 1) We can not get the position of the event horizon of a black hole by means of the null surface because the photons no longer propagate along the light cone. 2) We do not have any knowledge of the statistical mechanics if the speed of the light is greater than unity but the usual method for calculating quantum entropy of black hole is based on the statistical mechanics. 3) We do not
know whether the thermodynamical laws are valid for the case that the speed of the light is greater than unity.

It should be pointed out, in this coupling model, that all the difficulties arise from the fact that photons no longer propagate along the light cone. Therefore, to overcome these difficulties, the simplest way may be to introduce an effective spacetime in which photons propagate along the light cone. Therefore, we will find the effective spacetime in next section.

3. Effective spacetime for the coupling model

In a general four-dimensional static and spherically symmetric spacetime

\[ ds^2 = g_{00}dt^2 + g_{11}dr^2 + g_{22}(d\theta^2 + \sin^2 \theta d\phi^2), \]  

(3.1)

where \( g_{00}, g_{11} \) and \( g_{22} \) are functions of the polar coordinate \( r \) only, by means of \( g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu \) where \( \eta_{ab} \) is the Minkowski metric with the signature \((-,+,+,+),\) for metric (3.1) we get the vierbeins

\[ e^a_\mu = \text{diag}(\sqrt{-g_{00}}, \sqrt{g_{11}}, \sqrt{g_{22}}, \sqrt{g_{33}}). \]  

(3.2)

Taking advantage of the antisymmetric combination of vierbeins \[ U_{\mu\nu} = e^a_\mu e^b_\nu - e^a_\nu e^b_\mu, \]  

(3.3)

the Weyl tensor can be expressed as

\[ C_{\mu\nu\rho\sigma} = \mathcal{A} \left( 2U^{01}_\mu U^{01}_\rho - U^{02}_\mu U^{02}_\rho - U^{03}_\mu U^{03}_\rho + U^{12}_\mu U^{12}_\rho + U^{13}_\mu U^{13}_\rho - 2U^{23}_\mu U^{23}_\rho \right), \]  

(3.4)

with

\[
\mathcal{A} = -\frac{1}{12(g_{00}g_{11})^2g_{22}} \left\{ \left[ g_{00}g_{11}g''_{00} - \frac{1}{2}(g_{00}g_{11})'g''_{00} \right] g_{22} - g_{00}g_{11}g''_{22} \right. \\
\left. + \frac{1}{2} \left( g_{00}g''_{11} - g_{00}g_{11}g''_{00} \right) g_{22} - 2(g_{00}g_{11})^2 \right\},
\]  

(3.5)

where the prime represents the derivative with respect to \( r. \) To simplify the equation of motion for the coupled photon propagation, we introduce three linear combinations of momentum components \[ l_\nu = k^\mu U^{01}_{\mu\nu}, \quad n_\nu = k^\mu U^{02}_{\mu\nu}, \quad m_\nu = k^\mu U^{23}_{\mu\nu}, \]  

(3.6)
written as

\[ p_\nu = k^\mu U_{\mu}^{12} = \frac{1}{e_0^2 k^0} \left( e_1^1 k^1 n_\nu - e_2^2 k^2 l_\nu \right), \]
\[ r_\nu = k^\mu U_{\mu}^{03} = \frac{1}{e_2^2 k^2} \left( e_0^0 k^0 m_\nu + e_3^3 k^3 l_\nu \right), \]
\[ q_\nu = k^\mu U_{\mu}^{13} = \frac{e_1^1 k^1}{e_2^2 k^2} m_\nu + \frac{e_1^1 e_3^3 k^1 k^3}{e_0^0 e_2^2 k^2} n_\nu - \frac{e_3^3 k^3}{e_0^0 k^0} l_\nu, \quad (3.7) \]

where the vectors \( l_\nu, n_\nu, m_\nu \) are independent and orthogonal to the wave vector \( k_\nu \). Contracting Eq. (2.4) with \( l_\nu, n_\nu, m_\nu \) respectively, using the relation (3.7) and introducing three independent polarisation components \((a \cdot l), (a \cdot n), \) and \((a \cdot m),\) we find that the equation of motion of the photon coupling with the Weyl tensor can be written as

\[ \begin{pmatrix} K_{11} & 0 & 0 \\ K_{21} & K_{22} & K_{23} \\ 0 & 0 & K_{33} \end{pmatrix} \begin{pmatrix} a \cdot l \\ a \cdot n \\ a \cdot m \end{pmatrix} = 0, \quad (3.8) \]

with

\[ K_{11} = (1 + 16 \alpha \mathcal{A}) (g^{00} k_0 k_0 + g^{11} k_1 k_1) + (1 - 8 \alpha \mathcal{A}) (g^{22} k_2 k_2 + g^{33} k_3 k_3), \]
\[ K_{22} = (1 - 8 \alpha \mathcal{A}) (g^{00} k_0 k_0 + g^{11} k_1 k_1 + g^{22} k_2 k_2 + g^{33} k_3 k_3), \]
\[ K_{21} = 24 \alpha \mathcal{A} \sqrt{g^{11} g^{22} k_1 k_2}, \quad K_{23} = 8 \alpha \mathcal{A} \sqrt{-g^{00} g^{33} k_0 k_3}, \]
\[ K_{33} = (1 - 8 \alpha \mathcal{A}) (g^{00} k_0 k_0 + g^{11} k_1 k_1) + (1 + 16 \alpha \mathcal{A}) (g^{22} k_2 k_2 + g^{33} k_3 k_3). \quad (3.9) \]

The condition of Eq. (3.8) with the non-zero solution is \( K_{11} K_{22} K_{33} = 0. \) The first root \( K_{11} = 0 \) leads to the modified light cone

\[ (1 + 16 \alpha \mathcal{A}) (g^{00} k_0 k_0 + g^{11} k_1 k_1) + (1 - 8 \alpha \mathcal{A}) (g^{22} k_2 k_2 + g^{33} k_3 k_3) = 0, \quad (3.10) \]

which corresponds to the case where the polarisation vector \( a_\mu \) is proportional to \( l_\mu, \) i.e., the photon with the polarization along \( l_\mu. \) The second root \( K_{22} = 0 \) corresponds to an unphysical polarisation and should be neglected. The third root is \( K_{33} = 0, \) i.e.,

\[ (1 - 8 \alpha \mathcal{A}) (g^{00} k_0 k_0 + g^{11} k_1 k_1) + (1 + 16 \alpha \mathcal{A}) (g^{22} k_2 k_2 + g^{33} k_3 k_3) = 0, \quad (3.11) \]

which means that the vector \( a_\mu = \lambda m_\mu, \) i.e., the photon with the polarization along \( m_\mu. \)

Eqs. (3.10) and (3.11) show that the light cone condition depends on not only the
coupling between the photon and the Weyl tensor, but also on the polarizations. We know that the light cone condition is not modified for the radially directed photons (i.e., \( k_2 = k_3 = 0 \)) but is modified for the orbital photons (i.e., \( k_1 = k_2 = 0 \)), and the velocities of the photons for the two polarizations are different, i.e., the phenomenon of gravitational birefringence. It is interesting to note that if photons travel at speeds greater than unity for the case that the photon travels with the polarization along \( l_\mu \), then photons will travel at speeds less than unity for the case that the photon travels with the polarization along \( m_\mu \), and vice versa.

Moreover, the light cone conditions (3.10) and (3.11) imply that the coupled photons propagation dose not along the null geodesic in the original metric (3.1). However, it is interesting to note that we can cast the light cone conditions (3.10) and (3.11) into null form

\[
\tilde{k}^\mu \tilde{k}_\mu = 0,
\]

by defining \( \tilde{k}^\mu = G^{\mu\nu} k_\nu \), \( \tilde{k}_\mu = k_\mu \) with effective contravariant metric \( G^{\mu\nu} \)

\[
G^{00} = g^{00}, \\
G^{11} = g^{11}, \\
G^{22} = \left( \frac{1 - 8\alpha A}{1 + 16\alpha A} \right)^n g^{22}, \\
G^{33} = \left( \frac{1 - 8\alpha A}{1 + 16\alpha A} \right)^n g^{33},
\]

where \( n = 1 \) for the case (3.10), and \( n = -1 \) for the case (3.11). It should be noted that, in physics, the metric (3.13) with \( n = 1 \) corresponds to the case where the photon travels with the polarization along direction \( l_\mu \), and the metric with \( n = -1 \) to the case that the photon travels with the polarization along direction \( m_\mu \).

4. Effective temperature and entropy of black hole

We now study the event horizon, the area of event horizon, the temperature and entropy of black hole in the effective spacetime.

4.1 Effective event horizon of black hole

Since photons propagate along the light cone in the effective spacetime, we can get the position of the event horizon of the black hole by means of the null surface. For the static spherically symmetric spacetime described by effective metric (3.13), by using usual way that the event horizon should be a null surface \( f(r) \) with normal vector \( n_\mu = \frac{\partial f(\nu)}{\partial x^\nu} \), we know that the event horizon for the black hole can be obtained
from the equation

\[ G^{11} = g^{11} = 0, \quad (4.1) \]

which shows that the event horizon in the effective spacetime is the same as that of the original spacetime. Although the event horizon does not change, the area of the event horizon is given by

\[ \tilde{A}_H = \int r_H^2 d\theta d\varphi = 4\pi \left[ \left( \frac{1 + 16\alpha A}{1 - 8\alpha A} \right)^n \frac{G_{22} G_{33}}{g_{22}} \right] r_H, \quad (4.2) \]

which is different from that of the original spacetime. It is interesting to note that if the area of the event horizon for case \( n = 1 \) is greater than that of the original spacetime, then the area for case \( n = -1 \) will be less than that of the original spacetime.

### 4.2 Effective temperature of black hole

A semi-classical method of modeling Hawking radiation as a tunnelling effect has been developed and has attracted a lot of interest \[24, 25, 26, 27, 28, 29, 30, 31\]. Tunnelling provides not only a useful verification of thermodynamic properties of black holes but also an alternate conceptual means for understanding the underlying physical process of black hole radiation.

Noting that the metric (3.13) take the same form as Eq. (2.1) in Ref. \[32\] in which the Hawking radiation was studied by Hamilton-Jacobi ansatz method, we know that the Hawking temperature of the effective spacetime is described by

\[ \tilde{T} = \frac{1}{4\pi} \left( \frac{1}{\sqrt{g_{00} g_{11}}} \frac{dG_{00}}{dr} \right)_{r_H} = \frac{1}{4\pi} \left( \frac{1}{\sqrt{g_{00} g_{11}}} \frac{dg_{00}}{dr} \right)_{r_H} = T, \quad (4.3) \]

which shows that the Hawking temperature \( \tilde{T} \) of the effective spacetime is the same with the temperature \( T \) of the original spacetime.

### 4.3 Effective entropy of black hole

The statistical-mechanical entropy of black hole can be derived from the canonical formulation \[33\]. The corresponding free energy, \( F \), can be defined in term of the one-particle spectrum. One of the ways to calculate \( F \) is “brick wall” model (BWM) proposed by 't Hooft \[34\] which has been used extensively \[35, 36, 37, 38, 40\]. In this model, in order to eliminate divergence which appears due to the infinite growth of the density of states close to the horizon, 't Hooft introduces a “brick wall” cutoff. We now use this approach to find the entropy of the effective spacetime.
For the effective black hole, taking the phase as
\[ \Theta(t, r, \theta, \phi) = e^{\frac{-iEt + im\phi + iW(r, \theta)}} \]
and using the equation of null wave vector \( \tilde{k}_\mu \tilde{k}^\mu = 0 \), we get
\[ \tilde{k}_r^2 = -\frac{G_{rr}}{G_{tt}} \left[ E^2 + G_{tt} \left( \frac{m^2}{G_{\phi\phi}} + \frac{\tilde{k}_\theta^2}{G_{\theta\theta}} \right) \right]. \]

Therefore, the number of modes with \( E, m \) and \( \tilde{k}_\theta \) in phase space is shown by
\[ n(E, m, \tilde{k}_\theta) = \frac{1}{\pi} \int d\theta \int_{r_{H+h}}^r \tilde{k}_r dr = \frac{1}{\pi} \int d\theta \int_{r_{H+h}}^r \sqrt{-\frac{G_{rr}}{G_{tt}} \left[ E^2 + G_{tt} \left( \frac{m^2}{G_{\phi\phi}} + \frac{\tilde{k}_\theta^2}{G_{\theta\theta}} \right) \right]} dr. \]

The free energy is given by
\[ \beta F = \int dm \int d\tilde{k}_\theta \int dn(E, m, \tilde{k}_\theta) \ln \left[ 1 - e^{-\beta E} \right] = -\beta \int dm \int d\tilde{k}_\theta \int n(E, m, \tilde{k}_\theta) e^{\beta E - 1} dE. \]

Substituting Eq. (4.6) into Eq. (4.7) and then taking the integration over \( E \), we can work out the free energy
\[ F = \frac{1}{192\pi} \frac{\beta^3}{\beta^4 \epsilon^2} \int d\theta \left\{ \left( \frac{1 + 16\alpha A}{1 - 8\alpha A} \right)^n \sqrt{g_{\theta\theta} g_{\phi\phi}} \delta^r g_{rr} \right\}_{r_H} + \frac{1}{1440} \frac{\beta^3}{\beta^4} \int d\theta \left\{ \left( \frac{1 + 16\alpha A}{1 - 8\alpha A} \right)^n \sqrt{g_{\theta\theta} g_{\phi\phi}} \right\}_{r_H} \times \left[ \frac{\partial^2 g_{rr}}{\partial r^2} + 2 \frac{\partial g_{rr}}{\partial r} \ln |g_{tt} g_{rr}| + \frac{2\pi}{\beta_H} \sqrt{-g_{tt} g_{rr}} \left( \frac{1}{g_{\theta\theta} \partial r} - \frac{1}{1 + 16\alpha A} \right) \right] \ln \Lambda \right. \frac{1}{\epsilon}, \]

where \( \delta^2 = \frac{2\epsilon^2}{15} \) and \( \Lambda^2 = \frac{L^2}{h} \) ( \( \delta \approx 2 \sqrt{h/(\partial g^{rr}/\partial r)}_{r_H} \) is the proper distance from the horizon to \( \Sigma_h \), \( \epsilon \) is the ultraviolet cutoff, and \( \Lambda \) is the infrared cutoff). Using the relation between the entropy and the free energy, \( S = \beta^2 \frac{\partial F}{\partial \beta} \), we find that the leading term of the entropy is given by
\[ S = \frac{\tilde{A}_H}{48\pi \epsilon^2}, \]

where \( \tilde{A}_H \) is area of the event horizon defined by (4.2). Therefore, renormalizing the entropy by using the standard approach shown in Refs. [12, 13, 14], we find that the
Bekenstein-Hawking entropy of the effective black hole can be expressed as

\[ \tilde{S}_H = \frac{\tilde{A}_H}{4} = \pi \left[ \left( \frac{1 + 16 \alpha A}{1 - 8 \alpha A} \right)^n g_{22} \right] \]

which is different from that of the original black hole. It is interesting to note that if the Bekenstein-Hawking entropy for case \( n = 1 \) is greater than that of the original black hole, the Bekenstein-Hawking entropy for case \( n = -1 \) will be less than that of the original black hole.

5. Effective Smarr formula and the first law of thermodynamics

Since the Time-like Killing vector for the metric (3.13) is given by \( \tilde{\xi}(t) = (1, 0, 0, 0) \), we can take the mass as

\[ \tilde{M} = \frac{1}{4\pi} \int \tilde{R}_\mu^\nu \tilde{\xi}(t)^\nu d\tilde{\Sigma}_\mu + \frac{1}{4\pi} \int_{\partial S_B} \tilde{\xi}(t)^\mu d\tilde{\Sigma}_{\mu\nu}, \]

\[ = \frac{1}{4\pi} \int \tilde{R}_\mu^\nu \tilde{\xi}(t)^\nu d\tilde{\Sigma}_\mu + 2\tilde{T} \tilde{S}, \]

where \( \partial S_B \) is the boundary of the black hole. Because the value of the integral in Eq. (5.1) will be different for different black hole, we now show the Smarr formula and first law of thermodynamics for the effective Schwarzschild and Reissner-Nordström black holes, respectively.

5.1 Effective Smarr formula and the first law of thermodynamics of Schwarzschild black hole

For the effective Schwarzschild black hole

\[ d\tilde{S}^2 = - \left( 1 - \frac{r_+}{r} \right) dt^2 + \left( 1 - \frac{r_+}{r} \right)^{-1} dr^2 + r^2 \left( \frac{r^3 - 16 \alpha r_+}{r^3 + 8 \alpha r_+} \right)^n (d\theta^2 + \sin^2 \theta d\varphi^2) \]

where \( r_+ = 2M \) is the radius of event horizon, since the integral term in Eq. (5.1) can be expressed as

\[ \frac{1}{4\pi} \int \tilde{R}_\mu^\nu \tilde{\xi}(t)^\nu d\tilde{\Sigma}_\mu = 2\tilde{T} \tilde{S}_C = 2T \tilde{S}_C, \]

we can obtain the Smarr relation and first law of the thermodynamics

\[ M = 2T(\tilde{S}_H + \tilde{S}_C), \]

\[ dM = T(d\tilde{S}_H + d\tilde{S}_C), \]
where $\tilde{S}_H$ is the Bekenstein-Hawking entropy and the entropy $\tilde{S}_C$ can be considered as a contribution arising from the coupling of electromagnetic field and the Weyl tensor, which are given by

$$\tilde{S}_H = \begin{cases} \pi r_+^2 \left( \frac{r_+^2 - 16\alpha}{r_+^2 + 8\alpha} \right), & (\text{for } n = 1), \\ \pi r_+^2 \left( \frac{r_+^2 + 8\alpha}{r_+^2 - 16\alpha} \right), & (\text{for } n = -1), \end{cases}$$ (5.6)

$$\tilde{S}_C = \begin{cases} \frac{24\pi \alpha r_+^2}{r_+^2 + 8\alpha}, & (\text{for } n = 1), \\ -\frac{24\pi \alpha r_+^2}{r_+^2 - 16\alpha}, & (\text{for } n = -1). \end{cases}$$ (5.7)

The Smarr relation and first law of the thermodynamics show that the entropy of the system consists of two parts, $\tilde{S}_H$ and $\tilde{S}_C$, and although $\tilde{S}_H$ and $\tilde{S}_C$ are different for the cases of $n = 1$ and $n = -1$, the total entropy is the same, i.e., $\tilde{S}_H + \tilde{S}_C = \pi r_+^2$ for the both cases.

### 5.2 Effective Smarr formula and the first law of thermodynamics of Reissner-Nordström black hole

The effective Reissner-Nordström black hole is described by

$$dS^2 = -\frac{(r - r_+)(r - r_-)}{r^2} dt^2 + \frac{r^2}{(r - r_+)(r - r_-)} dr^2 + r^2 \left[ \frac{r^4 + 32\alpha r_+ r_- - 16\alpha (r_+ + r_-)}{r^4 - 16\alpha r_+ r_- + 8\alpha (r_+ + r_-)} \right]^n (d\theta^2 + \sin^2 \theta d\phi^2),$$ (5.8)

where $r_+ = M + \sqrt{M^2 - Q^2}$ and $r_- = M - \sqrt{M^2 - Q^2}$. Noting that

$$\frac{1}{4\pi} \int \tilde{R}^{\mu\nu} \xi_0(t)^\nu d\tilde{S}_\mu = 2\tilde{T} \tilde{S}_C + \tilde{\Phi}_Q Q = 2T \tilde{S}_C + \tilde{\Phi}_Q Q,$$ (5.9)

from Eq. (5.1) we can obtain the following Smarr relation and the first law of the thermodynamics

$$M = 2T(\tilde{S}_H + \tilde{S}_C) + \tilde{\Phi}_Q Q,$$ (5.10)

$$dM = T(d\tilde{S}_H + d\tilde{S}_C) + \tilde{\Phi}_Q dQ,$$ (5.11)

where $Q$ is the electric charge and $\tilde{\Phi}_Q$ is the corresponding potential of the charge, $\tilde{S}_H$ is the Bekenstein entropy and the entropy $\tilde{S}_C$ can be considered as a contribution arising from the coupling of electromagnetic field and the Weyl tensor, which are
given by
\[ \Phi = \frac{Q}{r_+} \quad (5.12) \]

\[ \tilde{S}_H = \begin{cases} 
\pi r_+^2 \left[ \frac{r_+^3}{r_+^3 + 8\alpha (r_+ - r_-)} \right], & (\text{for } n = 1), \\
\pi r_+^2 \left[ \frac{r_+^3 + 8\alpha (r_+ - r_-)}{r_+^3 - 16\alpha (r_+ - r_-)} \right], & (\text{for } n = -1). 
\end{cases} \quad (5.13) \]

\[ \tilde{S}_C = \begin{cases} 
\frac{24\pi \alpha r_+^2}{r_+^3 + 8\alpha (r_+ - r_-)} (r_+ - r_-) & (\text{for } n = 1), \\
-\frac{24\pi \alpha r_+^2}{r_+^3 - 16\alpha (r_+ - r_-)} (r_+ - r_-) & (\text{for } n = -1). 
\end{cases} \quad (5.14) \]

Although \( \tilde{S}_H \) and \( \tilde{S}_C \) for the case of \( n = 1 \) are different from those for the case of \( n = -1 \), the total entropy is also the same for the both cases, i.e., \( \tilde{S}_H + \tilde{S}_C = \pi r_+^2 \).

From above discussions we can obtain two interesting results: 1) The entropy for this spacetime consists of two parts, one is the Bekenstein-Hawking entropy, the other is the entropy arising from the coupling of electromagnetic field and the Weyl tensor. 2) Although the Bekenstein-Hawking entropy \( \tilde{S}_H \) and the entropy \( \tilde{S}_C \) arising from the coupling of electromagnetic field and the Weyl tensor are different for the cases of \( n = 1 \) and \( n = -1 \), the total entropy is the same, i.e., \( \tilde{S}_H + \tilde{S}_C = \pi r_+^2 \) for the both cases. That is to say, if the entropy \( \tilde{S}_C \) arising from the coupling of electromagnetic field and the Weyl tensor increases, the Bekenstein-Hawking entropy \( \tilde{S}_H \) will decrease, and vice versa.

### 6. Conclusions and discussions

For the model of the electromagnetic field coupled to the Weyl tensor of gravitational fields, the photons at low frequencies no longer propagate along the light cone in the original spacetime. That is to say, the interactions between the electromagnetic field and the spacetime curvature lead to a dependence of the photon velocity on the motion and polarization directions, so that in some cases photons travel at speeds greater or less than unity. The photons which travel at speeds greater or less than unity arise some difficulties for the physics of black hole because we can not get the position of the event horizon by means of the null surface, and we can not calculate quantum entropy of the black hole by the statistical mechanics method because we do not have any knowledge of the statistical mechanics. It is interesting to note that all these difficulties can be overcome by introducing an effective spacetime in which photons propagate along the light cone.

Using this effective descriptions, we found, comparing with the results of original spacetime, that the event horizon and Hawking temperature do not change, but the area of the event horizon and Bekenstein-Hawking entropy become different. We noted that the total entropy for this system, which is still the same as that of
original spacetime, consists of two parts: one is the Bekenstein-Hawking entropy $\tilde{S}_H$, the other is the entropy $\tilde{S}_C$ arising from the coupling of electromagnetic field and the Weyl tensor. Fortunately, we also obtained effective descriptions for the Smarr relation and first law of thermodynamics.

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