Parity (XOR) Reasoning for the Index Calculus Attack

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Abstract
Models for cryptographic problems are often expressed as boolean polynomial systems, whose equivalent logical formulas can be treated using SAT solvers. Given the algebraic nature of the problem, the use of the logical XOR operator is common in SAT-based cryptanalysis. Recent works have focused on advanced techniques for handling parity (XOR) constraints, such as the Gaussian Elimination technique. First, we propose an original XOR-reasoning SAT solver, named WD-Sat, dedicated to a specific cryptographic problem. Secondly, we show that in some cases Gaussian Elimination on SAT instances does not work as well as Gaussian Elimination on algebraic systems. We demonstrate how this oversight is fixed in our solver, which is adapted to read instances in algebraic normal form (ANF). Finally, we propose a novel preprocessing technique based on the Minimal Vertex Cover Problem in graph theory. Our benchmarks use a model obtained from cryptographic instances for which a significant speedup is achieved using the findings in this paper. We further explain how our preprocessing technique can be used as an assessment of the security of a cryptographic system.

1 Introduction
Cryptanalysis is the study of methods to decrypt a ciphertext without any knowledge of the secret key. Academic research in cryptanalysis is focused on deciding whether a cryptosystem is secure enough to be used in the real world. In addition, a good understanding of the complexity of a cryptographic attack allows us to determine the secret key length, making sure that no cryptanalytic effort can find the key in a feasible amount of time. Recommendations for minimum key length requirements given by various academic and governmental organizations [3] are based on the complexity of known attacks.

In recent years, Boolean satisfiability (SAT) solvers have found use in cryptanalysis. Indeed, several cryptographic attacks can be reduced to the problem of solving a multivariate boolean polynomial system. In algebraic cryptanalysis, these systems are solved using Gröbner basis algorithms [11], exhaustive search [5] or hybrid methods [2]. These methods have been replaced by SAT solving techniques for attacks on various secret key cryptographic systems such as AES, Bivium, Trivium, Grain etc. [15; 17; 16; 22; 21]. Recent work has also focused on combining algebraic and SAT solving techniques [7]. In public key cryptography, SAT solvers have been considered for attacking binary elliptic curve cryptosystems using the index calculus attack [13]. We tackle the same question in this paper.

In this paper, we propose a built-from-scratch SAT solver dedicated to solving an important step of the index calculus attack. The solver, named WD-Sat, is adapted for XOR-reasoning and reads formulas in ANF form. In addition, we show certain limitations of the GE technique in XOR-enabled SAT solvers by pointing out a canceling property which is present in algebraic resolution methods but is overseen in current SAT-based GE implementations. We refer to this canceling property as the XG-ext method and we show how it is implemented in our solver. In implementations, the XG-ext method comes at a high computational cost, and is thus useful only for benchmarks where it reduces significantly the number of conflicts. Finally, we introduce a graph theory-based preprocessing technique, specifically designed for multivariate boolean polynomial systems that allows us to further accelerate the resolution of our benchmarks. This preprocessing technique is designed to allow rapid linearization of the underlying algebraic system and should be used coupled with the substitution technique implemented by the XG-ext method. In fact, when the XG-ext method is not applied, the positive outcome of the preprocessing technique cannot guaranteed. To confirm, we perform experiments using CryptoMiniSat [22] coupled with our preprocessing technique and show that this combination yields slower running times than CryptoMiniSat alone.

Experimental results in Section 6 show that the solver presented in this paper outperforms all existing solving approaches for the introduced problem. These approaches include Gröbner basis techniques [11] and state-of-the-art SAT solvers: MiniSat [10], Glucose [1] and CryptoMiniSat.
2 Background

Index Calculus

In cryptanalysis, the index calculus algorithm is a well-known method for attacking factoring and elliptic curve discrete logarithms, two computational problems which are at the heart of most used public-key cryptosystems. When performing this attack for elliptic curve discrete logarithms, a crucial step is the point decomposition phase. As proposed by Gaudry [14] and Diem [9] independently, a point on the elliptic curve can be decomposed into \( m \) other points by solving Semaev’s \((m + 1)\)th summation polynomial [20], that we denote by \( S_{m+1} \).

For an elliptic curve defined over the finite field \( \mathbb{F}_{2^n} \), the \( x \)-coordinate of a random point on the curve is an element in \( \mathbb{F}_{2^n} \). From an implementation point of view, this is represented as a \( n \)-bit vector. In index calculus attacks, the common approach is to decompose a random point given by a \( n \)-bit vector \( x \)-coordinate into \( m \) coordinates \( (m \approx \frac{n}{\log n}) \) [see for instance [12; 19]]. With this choice of parameters, the problem of decomposing a random point by solving the \((m + 1)\)th summation polynomial can be rewritten as a system of \( n \) boolean equations with \( ml \) variables.

A multivariate boolean polynomial system is a system of polynomials in more than one variable whose coefficients are in \( \mathbb{F}_2 \) (see for instance [18]). The following example shows a boolean polynomial system of three equations in variables \( \{x_1, x_2, x_3\} \):

\[
\begin{align*}
    x_1 + x_2 \cdot x_3 &= 0 \\
    x_1 \cdot x_2 + x_2 + x_3 &= 0 \\
    x_1 + x_1 \cdot x_2 \cdot x_3 + x_2 \cdot x_3 &= 0
\end{align*}
\]

In the literature, the modelisation process allowing to obtain a boolean polynomial system from a polynomial with coefficients in \( \mathbb{F}_2 \) (here the summation polynomial) is called a Weil Restriction [14] or Weil Descent [19]. The polynomial systems obtained in this way serve as our starting point for deriving SAT instances.

XOR-Enabled SAT Solvers

A boolean polynomial system can be rewritten as a conjunction of logical formulas in algebraic normal form (ANF) as follows: multiplication in \( \mathbb{F}_2 \) becomes the logical AND operation \( \land \) and addition in \( \mathbb{F}_2 \) becomes the logical XOR \( \oplus \). Consequently, solving a multivariate boolean polynomial system is equivalent to solving a conjunction of logical formulas in ANF form. To date, XOR-enabled SAT solvers are currently not adapted to tackle formulas in ANF. Consequently, every conjunction of two or more literals \( x_1 \land x_2 \land \ldots \land x_k \) has to be replaced by an additional and equivalent variable \( x' \) such that \( x' \iff x_1 \land x_2 \land \ldots \land x_k \). This then rewrites as \((x' \lor \neg x_1 \lor \neg x_2 \lor \ldots \lor \neg x_k) \land (\neg x' \lor x_1) \land (\neg x' \lor x_2) \land \ldots \land (\neg x' \lor x_k)\) in CNF. When we substitute all occurrences of conjunctions in a XOR clause by an additional variable, we obtain a formula in CNF-XOR form. This is the form used in CryptoMiniSat [22]. The CryptoMiniSat solver is an extension of MiniSat [10] specifically designed to work on cryptographic problems. Finally, one could of course consider generic solvers (i.e. MiniSat [10], Glucose [1]) for solving cryptographic problems, but this approach needs to further transform the ANF model in a CNF one. Describing a XOR-clause with \( k \) literals in CNF representation is a classical process which results in \( 2^{k-1} \) OR-clauses of \( k \) literals.

3 The WDSat solver

Our WDSat solver is based on the Davis-Putnam-Logemann-Loveland (DPLL) algorithm [8], which is a state-of-the-art complete SAT solving technique. The solver is designed to treat ANF formulae derived from the Weil Descent modelisation of cryptographic attacks, hence its name : WDSat.

WDSat implements three reasoning modules. These include the module for reasoning on the CNF part of the formula and the so-called XORSET and XORGAUSS (XG) modules designed for reasoning on XOR constraints. The CNF module is designed to perform classical unit propagation on OR-clauses. The XORSET module performs the operation equivalent to unit propagation, but adapted for XOR-clauses. Practically, this consists in checking the parity of the current interpretation and propagating the unassigned literal. Finally, the XG module is designed to perform GE on the XOR constraints dynamically. We also implement a XG extension, described in Section 4. The following is a detailed explanation of this module.

XOR clauses are normalized and represented as equivalence classes. Recall that a XOR-clause is said to be in normal form if it contains only positive literals and does not contain more than one occurrence of each literal. Since we consider that all variables in a clause belong to the same equivalence class (EC), we choose one literal from the EC to be the representative. A XOR-clause \((x_1 \oplus x_2 \oplus \ldots \oplus x_n) \iff \top\) rewrites as

\[
x_1 \iff (x_2 \oplus x_3 \oplus \ldots \oplus x_n \oplus \top)
\]

Finally, we replace all occurrences of a representative of a XOR clause with the right side of the equivalence. Applying this transformation, we obtain a simplified system having the following property: a representative of an EC will never be present in another EC.

Let \( R \) be the set of representatives and \( C \) be the set of clauses. \( R \) and \( C \) hold the right side and the left side of all equations of type (1) respectively. We denote by \( C_x \) the clause that is equivalent to \( x \). In other words, \( C_x \) is the EC that has \( x \) as representative. Finally, we denote by \( \text{var}(C_x) \) the set of literals (plus a \( \top/\bot \) constant) in the clause \( C_x \) and \( C[x_1/x_2] \) denotes \( \forall C \in C, C_1 \leftarrow C \land x_1 \land x_2 \), meaning that \( x_1 \) is replaced by \( x_2 \) in \( C_1 \).

Thus, assigning a literal \( x_1 \) to \( \top \) leads to computing one or the other rule of the Table 1 depending on whether \( x_1 \) belongs to \( R \) or not. In both cases, propagation occurs when \( \exists x_1 \neq x_1 \text{ s.t. } \text{var}(C_{x_1}) = \top/\bot \). Conflict occurs when one constraint leads to the propagation of \( x_1 \) to \( \top \) and another constraint leads to the propagation of \( x_1 \) to \( \bot \).

Table 1 presents inference rules for performing GE in the XG module of WDSat. Applying these rules allows us to maintain the property of the system which states that a representative of an EC will never be present in another EC. For
| Premises | Conclusions on $C$ | $R \leftarrow$ |
|----------|-------------------|-------------|
| $x_1, C$ | $C[x_1/\top]$ | N/A |
| $x_1 \not\in R$ | | |
| $x_1, C$ | $x_2 \leftarrow x_1 \oplus x_2 \oplus \top$ | $R \setminus \{x_1\}$ |
| $x_1 \in R$ | $R \cup \{x_2\}$ | |
| $x_2 \in \text{var}(C_{x_1})$ | $C[x_2/C_{x_2}]$ | |

Table 1: Gaussian elimination inference rules.

clarity of the notation, the first column of this table contains the premises, the second one contains the conclusion and the third one is an update on the set $R$ which has to be performed when the inference rule is used.

Let the number of variables in a SAT instance be $k$. At the implementation level, clauses are represented as $(k + 1)$-bit vectors: a bit for every variable and one for a $\top$, $\bot$ constant. Clauses are stored in an array indexed by the representatives. This representation allows us to perform Gaussian elimination only by XOR-ing bit-vectors and flipping the clause constant. For instance, let $k = 7$ and let us consider $x_2 \iff \top \oplus x_1 \oplus x_3 \oplus x_5$. Then we have that $\text{var}(C_{x_2}) = \{\top, x_1, x_3, x_5\}$ and the bit-vector in index 2 of our clauses array is 11010100, where the $\top$, $\bot$ constant takes the zero position. Assigning $x_1$ to $\top$ is equivalent to introducing the constraint $x_1 \oplus \top$. We apply the first rule, simply by XOR-ing this bit-vector with a mask of the form 11000000. The resulting vector is 00010100, which corresponds to $\text{var}(C_{x_2}) = \{\bot, x_3, x_5\}$. We implement a bit-vector structure which allows for a succinct representation: a $k$-bit vector is represented by a chain of $[k/64]$ integers.

The WDSat solver implements an ASSIGN function used to assign a truth value to a variable in a formula $F$, while synchronising all three modules in the following manner. First, the truth value is assigned in the CNF module and truth values of other variables are propagated. Next, the truth value of the initial variable, as well as the propagated ones are assigned in the XORSET module. If the XOR-adapted unit propagation discovers new truth values, they are assigned in the CNF module, going back to step one. We go back and forth with this process until the two modules are synchronized and there are no more propagations left. Finally, the list of all inferred literals is transferred in the XG module. If the XG module finds new XOR-implied literals, the list is sent to the CNF module and the process is restarted. If a conflict occurs in any of the reasoning modules, the ASSIGN function fails and a backtracking procedure is launched (see Algorithm 1).

We briefly detail the other functions used in the pseudo-code. There is one SET_IN function for each module which takes as input a list of literals and a propositional formula $F$ and sets all literals from the list to $\top$ in the corresponding modules. Specifically, the SET_IN function for the XG module (SET_IN_XG) function implements the rules in Table 1. Consequently, $F$ is simplified and truth values for other literals are inferred through the appropriate unit propagation technique. Finally, the function LAST_ASSIGNED returns the list of literals that were assigned during the last call to the respective SET_IN function.

**Algorithm 1** Function ASSIGN($F$, $x$) : Assigning a truth value to a literal $x$ in a formula $F$, simplifying $F$ and inferring truth values for other literals.

**Input:** Propositional formula $F$, Literal $x$  
**Output:** $\bot$ if a conflict is reached, $\top$ and a simplified $F$ otherwise

1. $\text{to_set} \leftarrow \{x\}$.
2. $\text{to_set_in}_XG \leftarrow \{x\}$.
3. **while** $\text{to_set} \neq \emptyset$ **do**
4. **while** $\text{to_set} \neq \emptyset$ **do**
5. **if** $\text{SET_IN}_\text{CNF}(\text{to_set}, F) \rightarrow \text{conflict}$ **then**
   **return** ($\bot$, $-$).
6. **end if**
7. $\text{to_set} \leftarrow \text{LAST_ASSIGNED}_\text{CNF}()$.
8. $\text{to_set}_XG \leftarrow \text{to_set}$.
9. **if** $\text{SET_IN}_\text{XORSET}(\text{to_set}, F) \rightarrow \text{conflict}$ **then**
   **return** ($\bot$, $-$).
10. **end if**
11. $\text{to_set}_XG \leftarrow \text{to_set} \cup \text{to_set}_XG$.
12. **end while**
13. **if** $\text{SET_IN}_XG(\text{to_set}_XG, F) \rightarrow \text{conflict}$ **then**
   **return** ($\bot$, $-$).
14. **end if**
15. $\text{to_set} \leftarrow \text{LAST_ASSIGNED}_XG()$.
16. **end while**
17. **return** ($\top$, $F$).

4 The XG-ext Method

In this section we show how we extend our XG module. First, we present the motivation for this work by giving an example of a case where GE in SAT solvers has certain limitations compared to Algebraic GE.

Secondly, we propose a solution to overcome these limitations and we implement it in our solver to develop the XORGAUSS-ext method (XG-ext in short). To introduce new rules for this method, we use the same notation as in Section 3.

Gaussian elimination on a boolean polynomial system consists in performing elementary operations on equations with the goal of reducing the number of equations as well as the number of terms in each equation. We cancel out terms by adding (XOR-ing) one equation to another. GE can be performed on instances in CNF-XOR form in the same way that it is performed on boolean polynomial systems presented in algebraic writing. However, we detected a case where a possible cancellation of terms is overseen due to the CNF-XOR form. Let us consider the following example:

$x_1 + x_2 x_3 + x_5 + x_6 + 1 = 0$
$x_3 + x_5 + x_6 = 0$

At some point during resolution, we try to assign the value of 1 to $x_2$. As the monomial $x_2 x_3$ will be equal to 1 only if both terms $x_2$ and $x_3$ are equal to 1, we get the following result:

$x_1 + x_3 + x_5 + x_6 + 1 = 0$
$x_3 + x_5 + x_6 = 0$
and when we XOR the second equation to the first we infer \(x_1 = 1\).

This example in CNF-XOR form writes as follows:
\[
\begin{align*}
x' \lor \neg x_2 \lor \neg x_3 \\
\neg x' \lor x_2 \\
\neg x' \lor x_3 \\
x_1 \oplus x' \oplus x_5 \oplus x_6 \oplus \bot \\
x_3 \oplus x_5 \oplus x_6.
\end{align*}
\]

There is an implicit \(\land\) at the end of each clause. When we assign \(x_2\) to \(\top\), as per unit propagation rules, we get the following result:
\[
\begin{align*}
x' \lor \neg x_3 \\
\neg x' \lor x_3 \\
x_1 \oplus x' \oplus x_5 \oplus x_6 \oplus \bot \\
x_3 \oplus x_5 \oplus x_6.
\end{align*}
\]

When we XOR the second clause to the first one we can not infer that \(x_1\) is \(\top\) at this point.

Note that \((x' \lor \neg x_3) \land (\neg x' \lor x_3)\) rewrites as \(x' \Leftrightarrow x_3\), but if the solver does not syntactically search for this type of occurrences regularly, \(x'\) will not be replaced by \(x_3\). Moreover, this type of search adds an additional computational cost to the resolution.

An omission of this sort can occur every time a variable is set to \(\top\). As a result, we define the following rule with the goal to improve the performance of XOR-enabled SAT solvers:
\[
\begin{align*}
x' & \quad x_1 \Leftrightarrow (x' \land x_2) \\
\hline
x_1 & \Leftrightarrow x_2 \quad .
\end{align*}
\]

This rule can be generalized for the resolution of higher-degree boolean polynomial systems:
\[
\begin{align*}
x' & \quad x_1 \Leftrightarrow (x' \land x_2 \land \ldots \land x_d) \\
\hline
x_1 & \Leftrightarrow (x_2 \land \ldots \land x_d) \quad .
\end{align*}
\]

Even though these rules are standard in boolean logic, they are presently not applied in XOR-enabled SAT solvers. Note that when a solver takes as input an instance in XOR-CNF form, the second premise is lost or has to be inferred by syntactic search. In order to have knowledge of the second premise, the solver needs to read the instance in ANF. To this purpose, we defined a new ANF input format for SAT solvers.

The following is a detailed explanation of how the rule in Equation (2) is applied in our implementation. Recall that the XG module has the following property: a representative of an EC will never be present in another EC. This property will be maintained in the XG-ext method as well. Using the conclusion in Equation (2), we derive in Table 2 six inference rules that allow us to perform the substitution of a variable \(x_1\) by a variable \(x_2\) while maintaining the unicity-of-representatives property. Applying one of the inference rules in Table 2 can result in conflict or it can propagate a newly discovered truth value. Note that \(\text{var}(C_{x_1} \oplus C_{x_2})\) is given by the symmetric difference \((\text{var}(C_{x_1}) \cup \text{var}(C_{x_2})) \setminus (\text{var}(C_{x_2}) \cap \text{var}(C_{x_1}))\).

| Premises | Conclusions on \(C\) | \(R\) \(\leftarrow\) |
|---------|----------------------|-----------------|
| \(C, x_1 \Leftrightarrow x_2\) | \(C[x_1/x_2]\) | \(N/A\) |
| \(x_1 \not\in R\), \(x_2 \not\in R\) | \(C[x_1/x_2]\) | \(N/A\) |
| \(C, x_1 \Leftrightarrow x_2\) | \(C_{x_2} \leftarrow C_{x_1}\) | \(R \setminus \{x_1\}\) |
| \(x_2 \not\in R\) | \(C[x_1/x_2]\) | \(R \cup \{x_2\}\) |
| \(x_1 \not\in R\) | \(C_{x_2} \leftarrow C_{x_1} \oplus x_2 \oplus x_3\) | \(R \setminus \{x_1\}\) |
| \(x_2 \in C_{x_1}\) | \(C[x_3/C_{x_3}]\) | \(R \cup \{x_3\}\) |
| \(x_3 \in C_{x_1}\) | \(C[x_1/x_2, x_3/C_{x_3}]\) | \(R \cup \{x_3\}\) |
| \(C, x_1 \Leftrightarrow x_2\) | \(C_{x_2} \leftarrow C_{x_1} \oplus C_{x_2} \oplus x_3\) | \(R \setminus \{x_1, x_2\}\) |
| \(x_1 \not\in R\) | \(C[x_3/C_{x_3}]\) | \(R \cup \{x_3\}\) |
| \(x_2 \not\in R\) | \(C_{x_2} \leftarrow C_{x_1} \oplus C_{x_2} \oplus x_3\) | \(R \setminus \{x_1, x_2\}\) |
| \(x_3 \not\in C_{x_1} + C_{x_2}\) | \(C[x_3/C_{x_3}]\) | \(R \cup \{x_3\}\) |

Table 2: Inference rules for the substitution of \(x_1\) by \(x_2\).

5 Our Preprocessing Technique

Let us reconsider the DPLL-based algorithm. It is well known that the number of conflicts is also influenced by the order in which the variables are assigned. Classical branching rules are based on Maximum number of Occurrences in the Minimum size clauses (MOMs) - type of heuristics.

In this work, we were interested in developing a criterion for defining the order of variables on XOR-CNF instances derived from boolean polynomial systems. Defining the order will serve as a preprocessing technique. We set the goal to choose branching variables that will lead as fast as possible to a linear polynomial system, which is solved using GE in polynomial time. In terms of SAT solving, the goal is to cancel out all clauses in the CNF part of the formula as a result of unit propagation. When only the XOR part of the CNF-XOR formula is left, the solver performs GE on the remaining XOR constraints in polynomial time. For simplicity, we only use the Boolean algebra terminology in this section.

However, the methods described are applicable to both SAT solving and algebraic techniques based on the process of recursively making assumptions on the truth values of variables in the system (as with the DPLL algorithm).

After setting this goal, choosing which variable to assign next according to the number of their occurrences in the system is no longer an optimal technique. Consider the following
Our preprocessing technique consists in (i) deriving a graph from a boolean polynomial system and (ii) finding the MVC of the resulting graph. During the solving process, variables corresponding to vertices in the MVC are assigned first. Even though the MVC problem is an NP-complete problem, its execution for graphs derived from cryptographic models always finishes in negligible running time due to the small number of variables.

When variables are assigned in the order defined by this preprocessing technique, the worst case time complexity of a DPLL-based algorithm drops from $O(2^k)$ to $O(2^{k'})$, where $k'$ is the number of vertices in the MVC set. Note that the MVC of a complete graph is equal to the number of its vertices. Consequently, when the corresponding graph of a boolean polynomial system is a complete graph, solving the system using this preprocessing technique is as hard as solving the system without it.

### 6 Experimental Results

To support our claims, we experimented with benchmarks derived from two variants of the index calculus attack on the discrete logarithm problem over binary elliptic curves. As explained in Section 2, a SAT solver can be used for solving Semaev’s summation polynomials in the point decomposition phase. Our model is derived from the boolean multivariate polynomial system given by the $m$th summation polynomial, $m \geq 3$. This model has previously been examined in [13].

We compare the WDSat solver presented in this paper to the following approaches: the best currently available implementation of Gröbner basis (F4 [11] in MAGMA [4]), solvers MiniSat [10], Glucose [1] and CryptoMiniSat with enabled GE\textsuperscript{1}. All tests were performed on a 2.40GHz Intel Xeon E5-2640 processor and are an average of 100 runs.

We conducted experiments using both the third and the fourth polynomials. Results on solving the third summation polynomial ($m = 2$) are shown in Table 4. The parameters used to obtain these benchmarks are $n = 41$ and $l = 20$. As a result, we obtained a boolean polynomial system of 41 equations in 40 variables (see Section 2). We show running-time and memory averages on satisfiable and unsatisfiable instances separately, since these values differ between the two cases.

As different variants of our solver can yield better results for different benchmarks, we compared all variants to decide on the optimal one. We also tested the solver with and without our preprocessing technique (denoted by mvc in tables). The results in Table 3 show that WDSat yields optimal results for these benchmarks when the XG-ext method is used coupled with the preprocessing technique. This outcome is not surprising when we examine the MVC obtained by the preprocessing technique. The number of variables in the system is $k = 40$, but the number of vertices in the MVC is 20.

\textsuperscript{1}Enabling GE in CryptoMiniSat yielded better performance for these benchmarks.
This means that when we use the optimization techniques described in this paper, the worst case time complexity of the examined models drops from $2^k$ to $2^{k^2}$. This is the case for every instance derived from the third summation polynomial.

By analyzing the average running time and the average number of conflicts in Table 4, we see that the chosen variant of the WDSat solver outperforms all other approaches for solving instances derived from the third summation polynomial. Running times for MiniSat and Glucose are not reported in the table as they both time out after 24 hours for these parameters.

Current versions of CryptoMiniSat do not allow to choose the order of the branching variables as its authors claim that this technique almost always results in slower running times. To verify this claim, we modified the source code of CryptoMiniSat in order to test our preprocessing technique coupled with this solver (see line CryptoMiniSat+mvc in Table 4). We set a timeout of 10 minutes and only 9 out of 100 unsatisfiable and 54 out of 100 satisfiable instances were solved. The MVC preprocessing technique is strongly linked to our XG-ext method. Indeed, when the XG-ext method is not used, one can not guarantee that when all variables from the MVC are assigned the system becomes linear. This is confirmed also by looking at the number of conflicts for the CryptoMiniSat+mvc approach, which is greater than $2^{\frac{k}{2}}$ even for benchmarks that were solved before the timeout. Recall that $\frac{k}{2}$ is the size of the MVC. On the other hand CryptoMiniSat without the preprocessing technique succeeds in solving these instances after less than $2^{\frac{k}{2}}$ conflicts. We conclude that the searching technique in CryptoMiniSat used to decide on the next branching variable is optimal for the CryptoMiniSat solver.

Experimental results in Table 5 are performed using benchmarks derived from the fourth summation polynomial. We obtain our model using a symmetrization technique proposed by Gaudry [14]. According to our parameters choice, the initial polynomial system contains 19 equations in 18 variables. Our experiments show that performing GE on these instances does not result in faster running times. On the contrary, running times are significantly slower when the XG module of the WDSat solver is enabled. Running times become even slower with the XG-ext method. We attribute this fallout to the particularly small improvement in the number of conflicts, compared to the significant computational cost of performing the GE technique. Indeed, the graph corresponding to the model for the fourth summation polynomial is complete and thus the size of the MVC is equivalent to the number of variables in the formula. This leads us to believe there is no optimal choice for the order of branching variables and the system generally does not become linear until the second-to-last branching. We conclude that for solving these instances WDSat without GE is the optimal variant, outperforming both the Gröbner basis method and current state-of-the-art solvers.

Our solver is dedicated to problems arising from a Weil descent, however we tested it on Trivium [6] benchmarks as they are extensively used in the SAT literature. Our experience is that CryptoMiniSat yields faster running times than all of the WDSat variants for Trivium benchmarks. WDSat does not implement any of the optimizations for Trivium such as dependent variable removal, sub-problem detection, etc. as there are no such occurrences in systems arising from a Weil descent.

### 7 Conclusion

We proposed a novel SAT solver dedicated to instances derived from the index calculus attack on binary elliptic curves. Our solver outperforms all existing resolution approaches for this problem.
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