Equivariance of generalized Chern characters

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Abstract

In this note some generalization of the Chern character is discussed from the chromatic point of view. We construct a multiplicative $G_{n+1}$-equivariant natural transformation $\Theta$ from some height $n+1$ cohomology theory $E^*(-)$ to the height $n$ cohomology theory $K^*(-)\otimes_{\mathbb{F}} L$, where $K^*(-)$ is essentially the $n$th Morava $K$-theory. As a corollary, it is shown that the $G_n$-module $K^*(X)$ can be recovered from the $G_{n+1}$-module $E^*(X)$. We also construct a lift of $\Theta$ to a natural transformation between characteristic zero cohomology theories.

1 Introduction

In the stable homotopy category $\mathcal{S}$ of $p$-local spectra, there is a filtration of full subcategories $\mathcal{S}_n$, where the objects of $\mathcal{S}_n$ consist of $E(n)$-local spectra. The difference of each step of this filtration is equivalent to the $K(n)$-local category. So it can be considered that the stable homotopy category $\mathcal{S}$ is built up from $K(n)$-local category. In fact, the chromatic convergence theorem (cf. [15]) says that the tower $\cdots \rightarrow L_{n+1}X \rightarrow L_nX \rightarrow \cdots \rightarrow L_0X$ recovers a finite spectrum $X$, that is, $X$ is homotopy equivalent to the homotopy inverse limit of the tower. Furthermore, the chromatic splitting conjecture (cf. [5]) implies that the $p$-completion of a finite spectrum $X$ is a direct summand of the product $\prod_n L_{K(n)}X$. This means that it is not necessarily to reconstruct the tower but it is sufficient to know all $L_{K(n)}X$ to obtain some information of $X$.

The weak form of the chromatic splitting conjecture means that the canonical map $L_n(S^0)\wedge \mathbb{F}_p \rightarrow L_n L_{K(n+1)}S^0$ is a split monomorphism, where $S^0$ is the sphere spectrum and $(S^0)\wedge \mathbb{F}_p$ is its $p$-completion. In [14] Remark 3.1.(i)] Minami indicated that the weak form of the chromatic splitting conjecture implies that there is a natural map $\rho$ for a finite spectrum $X$ from the $K(n+1)$-localization $L_{K(n+1)}X$ to the $K(n)$-localization $L_{K(n)}X$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\eta_{K(n+1)} & X & \eta_{K(n)} \\
L_{K(n+1)}X & \rho & L_{K(n)}X,
\end{array}
$$

where $\eta_{K(n)}$ and $\eta_{K(n+1)}$ are the localization maps. In this note we would like to consider an algebraic analogue of this diagram.

Let $\mathbb{F}$ be an algebraic extension of the prime field $\mathbb{F}_p$ which contains $\mathbb{F}_{p^n}$ and $\mathbb{F}_{p^{n+1}}$. Let $E_n$ be the Morava $E$-theory with the coefficient ring $W[u_1, \ldots, u_{n-1}, u^{\pm 1}]$ where $W = W(\mathbb{F})$ is the ring

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of Witt vectors with coefficients in $F$. The group of multiplicative cohomology operations on $E_n$ is the extended Morava stabilizer group $G_n = \Gamma \ltimes S_n$, where $\Gamma = \text{Gal}(F/F_p)$ and $S_n$ is the $n$th Morava stabilizer group. Let $\mathcal{M}_n$ be the category of twisted $E_n$-modules. Then $E_n$-(co)homology gives a functor from $\mathcal{S}$ to $\mathcal{M}_n$, and $E_n$-(co)homology is considered to be an algebraic approximation of the localization map $\eta_{K(n)} : X \to L_{K(n)}X$. So an algebraic analogue of the problem to construct the diagram (1.1) is the following: Is there an algebraic functor $\mu : \mathcal{M}_{n+1} \to \mathcal{M}_n$ such that the following diagram commutes?

$$
\begin{array}{ccc}
E_{n+1} & \xrightarrow{\eta} & \mathcal{S} \\
\downarrow \mu & & \downarrow \mu \\
\mathcal{M}_{n+1} & \xrightarrow{\mu} & \mathcal{M}_n.
\end{array}
$$

We recall the classical Chern characters. The Chern character is a multiplicative natural transformation from $K$-theory to the rational cohomology:

$$
\text{ch} : K \to HQ[w^{\pm 1}] \cong \prod_{i \in \mathbb{Z}} \Sigma^{2i}HQ.
$$

The formal group law associated with $K$-theory is the multiplicative formal group law. So its height is one. On the other hand, the height of the formal group law associated with the rational cohomology is considered to be zero. So the Chern character is considered to be a transformation from a height 1 theory to a height zero theory. The map $\mu$ in (1.2) is a transformation from the height $n+1$ theory $E_{n+1}$ to the height $n$ theory $E_n$. So $\mu$ should be a generalization of Chern character in some sense. Such a generalized Chern character have been constructed and studied by Ando, Morava and Sadofsky [2].

There is a modulo $I_n$-version of the problem to construct the diagram (1.2), where $I_n$ is the invariant prime ideal $(p, v_1, \ldots, v_{n-1})$. Let $E^*(-)$ and $K^*(-)$ be the complex oriented cohomology theories with coefficient rings $E_n = F[u_n][w^{\pm 1}]$ and $K^*(-) = F[w^{\pm 1}]$, respectively, where $u^{-1}(p^{n+1}) = v_n + 1$ and $u_n u^{-1}(p^{n-1}) = v_n = w^{-(p^n-1)}$. We denote by $\mathcal{M}'_{n+1}$ (resp. $\mathcal{M}'_n$) the category of twisted $E_n$-modules (resp. $K_n$-modules). Then the modulo $I_n$-version of the problem to construct the diagram (1.2) is as follows: Is there an algebraic functor $\mu' : \mathcal{M}'_{n+1} \to \mathcal{M}'_n$ such that the following diagram commutes?

$$
\begin{array}{ccc}
E & \xrightarrow{\mu'} & \mathcal{S} \\
\downarrow \mu' & & \downarrow \mu' \\
\mathcal{M}'_{n+1} & \xrightarrow{\mu'} & \mathcal{M}'_n.
\end{array}
$$

In [16] we have studied the relationship between the formal group laws $F_{n+1}$ and $H_n$ associated with $E^*(-)$ and $K^*(-)$, respectively. There is a totally ramified Galois extension $L$ of infinite degree over the fraction field $F((u_n))$ of $E_0$, and there is an isomorphism between $F_{n+1}$ and $H_n$ over $L$. We have shown that the pro-finite group $G = \Gamma \ltimes (S_n \times S_{n+1})$ acts on $(F_{n+1}, L) \cong (H_n, L)$.

The following is the main theorem of this note.

**Theorem 1.1** (Theorem 1.1). Let $p$ be an odd prime. Then there is a $G_{n+1}$-equivariant multiplicative stable cohomology operation

$$
\Theta : E^*(-) \to K^*(-) \hat{\otimes} F L
$$

for the following reasons:
such that $\Theta$ induces a natural isomorphism of $G = \Gamma \ltimes (S_n \times S_{n+1})$-modules:

$$E^*(X) \otimes_{E_0} L \cong K^*(X) \otimes_{FL}$$

for all spectra $X$.

On the right hand side of the isomorphism in Theorem 1.1, the subgroup $S_{n+1}$ of $G$ acts on $L$ only and its invariant ring is the subfield $F$: $H^0(S_{n+1};L) = F$. This implies the following corollary.

**Corollary 1.2** (Corollary 4.3). There are natural isomorphisms of $G_n$-modules:

$$K^*(X) \cong H^0(S_{n+1}; E^*(X) \otimes_E L),$$

$$K_*(X) \cong H^0(S_{n+1}; E_*(X) \otimes_E L),$$

for all spectra $X$. If $X$ is a space, then these are also isomorphisms of graded commutative rings.

This corollary gives us an answer of the modulo $I_{n-1}$-version of the problem. We define $\mu'(M) = H^0(S_n; M \otimes_E L[u^{\pm 1}])$ for a twisted $E_n$-module $M$. Then we obtain a functor from the category of twisted $E_n$-modules to the category of twisted $K_n$-modules: $\mu' : M_n \to M_{n-1}$, which makes the triangle (1.3) commutative.

Furthermore, we can lift $\Theta$ to a natural transformation between characteristic $0$ cohomology theories. There is a complete discrete valuation ring $T$ of characteristic $0$ with uniformizer $p$ and residue fields $L$. We regard $T[w_i] = T[w_1, \ldots, w_{n-1}]$ as an $E_n = W[w_1, \ldots, w_{n-1}]$-algebra by obvious way. Also, we can regard $T[u_i] = T[u_1, \ldots, u_{n-1}]$ as an $E_{n+1} = W[u_1, \ldots, u_{n-1}]$-algebra.

**Theorem 1.3** (Theorem 5.11). There is a $G_{n+1}$-equivariant multiplicative stable operation

$$ch : E^*_{n+1}(-) \to E^*_{n}(T[w_i]),$$

such that this induces a natural isomorphism of $G$-modules:

$$E^*_{n+1}(X) \otimes_{E_{n+1}} T[u_i] \cong E^*_{n}(X) \otimes_{E_n} T[w_i]$$

for all spectra $X$.

The organization of this note is as follows: In §2 we review the Lubin-Tate’s deformation theory of formal group laws and the results of [16] on the degeneration of formal group laws. In §3 we study the relationship between the stable natural transformations of even-periodic complex oriented cohomology theories and the homomorphisms of their formal group laws. In §4 we construct a multiplicative $G_{n+1}$-equivariant natural transformation $\Theta$ from $E^*(-) \to K^*(-) \otimes L$ and prove the main theorem. In §5 we construct a lift of $\Theta$ to a natural transformation of characteristic zero cohomology theories.

## 2 Formal group laws

In this section we review the deformation theory of formal group laws. In the following of this note a formal group law means a one-dimensional commutative formal group law.
Let $R_1$ and $R_2$ be two (topological) commutative rings. Let $F_1$ (resp. $F_2$) be a formal group law over $R_1$ (resp. $R_2$). We understand that a homomorphism from $(F_1, R_1)$ to $(F_2, R_2)$ is a pair $(f, \alpha)$ of a (continuous) ring homomorphism $\alpha : R_2 \to R_1$ and a homomorphism $f : F_1 \to \alpha^*F_2$ in the usual sense, where $\alpha^*F_2$ is the formal group law obtained from $F_2$ by the base change induced by $\alpha$. We denote the set of all such pairs by

$$\text{FGL}((F_1, R_1), (F_2, R_2)).$$

If $R_1$ and $R_2$ are topological rings, then we denote the subset of $\text{FGL}((F_1, R_1), (F_2, R_2))$ consisting of $(f, \alpha)$ such that $\alpha$ is continuous by

$$\text{FGL}^c((F_1, R_1), (F_2, R_2)).$$

The composition of two homomorphisms $(f, \alpha) : (F_1, R_1) \to (F_2, R_2)$ and $(\beta, g) : (F_2, R_2) \to (F_3, R_3)$ is defined as $(\alpha^*g \circ f, \alpha \circ \beta) : (F_1, R_1) \to (F_3, R_3)$:

$$(f, \alpha) : F_1 \xrightarrow{f} \alpha^*F_2 \xrightarrow{\alpha^*g} \alpha^*(\beta^*F_3) = (\alpha \circ \beta)^*F_3.$$

A homomorphism $(f, \alpha) : (F_1, R_1) \to (F_2, R_2)$ is an isomorphism if there exists a homomorphism $(g, \beta) : (F_2, R_2) \to (F_1, R_1)$ such that $(f, \alpha) \circ (g, \beta) = (X, \text{id})$ and $(g, \beta) \circ (f, \alpha) = (X, \text{id})$. Then a homomorphism $(f, \alpha) : (F_1, R_1) \to (F_2, R_2)$ is an isomorphism if and only if $\alpha$ is a (topological) ring isomorphism and $f$ is an isomorphism in the usual sense.

There is a $p$-typical formal group law $H_n$ over the prime field $\mathbb{F}_p$ with $p$-series

$$[p]^{H_n}(X) = X^{p^n},$$

which is called the height $n$ Honda formal group law. Let $F$ be an algebraic extension of the finite field $\mathbb{F}_{p^n}$ with $p^n$ elements, and we suppose that $H_n$ is defined over $F$. The automorphism group $S_n$ of $H_n$ over $F$ in the usual sense is the $n$th Morava stabilizer group $S_n$, which is isomorphic to the unit group of the maximal order of the central division algebra over the $p$-adic number field $\mathbb{Q}_p$ with invariant $1/n$. We denote by $G_n$ the automorphism group of $H_n$ over $F$ in the above sense:

$$G_n = \text{Aut}(H_n, F).$$

Then the following lemma is well-known.

**Lemma 2.1.** The automorphism group $G_n$ is isomorphic to the semi-direct product $\Gamma \rtimes S_n$, where $\Gamma$ is the Galois group $\text{Gal}(F/\mathbb{F}_p)$.

We recall Lubin and Tate’s deformation theory of formal group laws [9]. Let $R$ be a complete Noetherian local ring with maximal ideal $I$ such that the residue field $k = R/I$ is of characteristic $p > 0$. Let $G$ be a formal group law over $k$ of height $n < \infty$. Let $A$ be a complete Noetherian local $R$-algebra with maximal ideal $m$. We denote by $\iota$ the canonical inclusion of residue fields $k \subset A/m$ induced by the $R$-algebra structure. A deformation of $G$ to $A$ is a formal group law $\tilde{G}$ over $A$ such that $\iota^*\tilde{G} = \pi^*\tilde{G}$ where $\pi : A \to A/m$ is the canonical projection. Let $\tilde{G}_1$ and $\tilde{G}_2$ be two deformations of $G$ to $A$. We define a $*$-isomorphism between $\tilde{G}_1$ and $\tilde{G}_2$ as an isomorphism $\tilde{u} : \tilde{G}_1 \to \tilde{G}_2$ over $A$ such that $\pi^*\tilde{u}$ is the identity map between $\pi^*\tilde{G}_1 = \iota^*\tilde{G}_1 = \iota^*\tilde{G}_2$. Then it is known that there is at most one $*$-isomorphism between $\tilde{G}_1$ and $\tilde{G}_2$. We denote by $\mathcal{C}(R)$ the category of
complete Noetherian local \( R \)-algebras with local \( R \)-algebra homomorphisms as morphisms. For an object \( A \) of \( \mathcal{C}(R) \), we let \( \text{DEF}(A) \) be the set of all \( * \)-isomorphism classes of the deformations of \( G \) to \( A \). Then \( \text{DEF} \) defines a functor from \( \mathcal{C}(R) \) to the category of sets. Let \( R[t_i] = R[t_1, \ldots, t_{n-1}] \) be a formal power series ring over \( R \) with \( n-1 \) indeterminates. Note that \( R[t_i] \) is an object of \( \mathcal{C}(R) \). Lubin and Tate constructed a formal group law \( F(t_i) = F(t_1, \ldots, t_{n-1}) \) over \( R[t_i] \) such that for every deformation \( \tilde{G} \) of \( G \) to \( A \), there is a unique local \( R \)-algebra homomorphism \( \alpha : R[t_i] \to A \) such that \( \alpha^* F(t_i) \) is \( * \)-isomorphic to \( G \). Hence the functor \( \text{DEF} \) is represented by \( R[t_i] \):

\[
\text{DEF}(A) \cong \text{Hom}_\mathcal{C}(R[t_i], A)
\]

and \( F(t_i) \) is a universal object.

**Lemma 2.2.** Let \( F \) and \( G \) be formal group laws of height \( n < \infty \) over a field \( k \) of characteristic \( p > 0 \) and \( (\tilde{f}, \tilde{\pi}) \) an isomorphism from \( (F, k) \) to \( (G, k) \). Let \( R \) be a complete Noetherian local ring with residue field \( k \) and \( \alpha \) a ring automorphism of \( R \) such that \( \alpha \) induces \( \tilde{\pi} \) on the residue field. Let \( F \) (resp. \( G \)) be a universal formal deformation of \( F \) (resp. \( G \)) over \( R[u_i] = R[u_1, \ldots, u_{n-1}] \) (resp. \( R[w_i] = R[w_1, \ldots, w_{n-1}] \)). Then there is a unique isomorphism \( (g, \beta) \) from \( (\tilde{F}, R[u_i]) \) to \( (\tilde{G}, R[w_i]) \) such that \( \beta \) induces \( (\tilde{f}, \tilde{\pi}) \) on the residue field and \( i \circ \alpha = \beta \circ j \), where \( i : R \to R[u_1] \) and \( j : R \to R[w_i] \) are canonical inclusions.

**Proof.** First, we show that there is such a homomorphism. Let \( f(X) \in R[X] \) be a lift of \( \tilde{f}(X) \in k[X] \) such that \( f(0) = 0 \). Set \( F'(X, Y) = f(\tilde{F}(f^{-1}(X), f^{-1}(Y))) \). Then \( (F', R[u_i]) \) is a deformation of \( F \) over \( R[u_i] \) with the \( R \)-algebra structure given by \( R \xrightarrow{\alpha} R \xrightarrow{u_i} R[u_i] \). Then \( (F', R[u_i]) \) is a deformation of \( G \). Since \( G \) is a universal deformation of \( G \), there exists a continuous \( R \)-algebra homomorphism \( \beta : R[w_i] \to R[u_i] \) and a \( * \)-isomorphism \( \tilde{u} : F' \to \beta^* \tilde{G} \). Then \( (g, \beta) = (\tilde{u} \circ f, \beta) : (\tilde{F}, R[u_i]) \to (\tilde{G}, R[w_i]) \) is a lift of \( (\tilde{f}, \tilde{\pi}) : (F, k) \to (G, k) \).

By the same way, we can construct a lift \( (h, \gamma) \) of \( (\tilde{f}, \tilde{\pi}) \). Then \( (h, \gamma) \circ (g, \beta) \) is a lift of \( (\tilde{g}, \tilde{\gamma}) : (F, k) \to (F, k) \). Note that \( \beta \circ \gamma : R[u_i] \to R[u_i] \) is a continuous \( R \)-algebra homomorphism. Since \( (F, R[u_i]) \) is a universal deformation, \( (h, \gamma) \circ (g, \beta) = (X, id) \) by the uniqueness. Similarly, we obtain that \( (g, \beta) \circ (h, \gamma) = (X, id) \). Hence we see that \( (g, \beta) \) is an isomorphism and a unique lift of \( (\tilde{f}, \tilde{\pi}) : (F, k) \to (G, k) \).

Let \( F \) be an algebraic extension of \( \mathbb{F}_p \) which contains \( \mathbb{F}_{p^n} \) and \( \mathbb{F}_{p^{n+1}} \). Let \( W = W(F) \) be the ring of Witt vectors with coefficients in \( F \). We define \( E_n \) to be a formal power series ring over \( W \) with \( (n-1) \) indeterminates:

\[
E_n = W[w_1, \ldots, w_{n-1}]
\]

The ring \( E_n \) is a complete Noetherian local ring with residue field \( F \). There is a \( p \)-typical formal group law \( \tilde{F}_n \) over \( E_n \) with the \( p \)-series:

\[
[p]\tilde{F}_n(X) = pX + \tilde{\nu}_n w_1 X^p + \tilde{\nu}_n w_2 X^{p^2} + \tilde{\nu}_n w_3 X^{p^3} + \cdots + \tilde{\nu}_n w_{n-1} X^{p^{n-1}} + \tilde{\nu}_n X^p.
\]

The formal group law \( \tilde{F}_n \) is a deformation of \( H_n \) to \( E_n \). The following lemmas are well-known.

**Lemma 2.3.** \((\tilde{F}_n, E_n)\) is a universal deformation of \((H_n, F)\).

**Lemma 2.4.** The automorphism group \( \text{Aut}^e(\tilde{F}_n, E_n) \) is isomorphic to \( G_n \).
As in $E_n$, we define $E_{n+1}$ to be a formal power series ring over $W$ with $n$ indeterminates:

$$E_{n+1} = W[u_1, \ldots, u_n],$$

and there is a universal deformation $(\tilde{F}_{n+1}, E_{n+1})$ of the height $(n+1)$ Honda group law $(H_{n+1}, F)$. Let $E = E_{n+1}/I_n = F[u_n]$, where $I_n = (p, u_1, \ldots, u_{n-1})$. Let $F_{n+1} = \pi^{*} \tilde{F}_{n+1}$, where $\pi$ is the quotient map $E_{n+1} \to E$. Then $F_{n+1}$ is a deformation of $H_{n+1}$ to $E$. The following lemma is easy.

**Lemma 2.5.** The automorphism group $\text{Aut}^F(F_{n+1}, E)$ is isomorphic to $G_{n+1}$.

If we suppose that $F_{n+1}$ is defined over the quotient field $M = F((u_n))$ of $E$, then its height is $n$. Since the formal group laws over a separably closed field is classified by their height, there is an isomorphism $\Phi$ between $F_{n+1}$ and $H_n$ over the separable closure $M^{sep}$ of $M$ (cf. [8, 4]). We fix such an isomorphism $\Phi$. Since $\Phi : F_{n+1} \to H_n$ is a homomorphism between $p$-typical formal group laws, $\Phi$ has a following form:

$$\Phi(X) = \sum_{i \geq 0} H_n \Phi_i X^p^i.$$ 

Let $L$ be the extension field of $M$ obtained by adjoining all the coefficients of the isomorphism $\Phi$. So $(\Phi, id_L)$ is an isomorphism from $(F_{n+1}, L)$ to $(H_n, L)$:

$$(\Phi, id_L) : (F_{n+1}, L) \xrightarrow{\cong} (H_n, L).$$

Note that $L$ is a totally ramified Galois extension of infinite degree over $M$ with Galois group isomorphic to $S_n$ [3, 10]. Set $G = \Gamma \ltimes (S_{n+1} \times S_n)$. Then $G_{n+1} = \Gamma \ltimes S_{n+1}$ and $G_n = \Gamma \ltimes S_n$ are subgroups of $G$. In [10] we have shown the following theorem.

**Theorem 2.6** (cf. [10, §2.4]). The pro-finite group $G$ acts on $(F_{n+1}, L) \cong (H_n, L)$. The action of the subgroup $G_{n+1}$ on $(F_{n+1}, L)$ is an extension of the action on $(F_{n+1}, E)$, and the action of the subgroup $G_n$ on $(H_n, L)$ is an extension of the action on $(H_n, F)$.

### 3 Stable operations of cohomology theories

In this section we recall and study the stable cohomology operations between Landweber exact cohomology theories over $P(n)$. The treatment is standard as in [10, 11, 17].

For a spectrum $h$, we denote by $h^{*}(-)$ (resp. $h_{*}(-)$) the associated generalized cohomology (resp. homology) theory. For spectra $h$ and $k$, we denote by $C(h, k)$ (resp. $H(h, k)$) the set of all degree 0 stable cohomology (resp. homology) operations from $h$ to $k$. Then $C(h, k)$ is naturally identified with the set of all degree 0 morphisms from $h$ to $k$ in the stable homotopy category. There is a natural surjection from $C(h, k)$ to $H(h, k)$ and the kernel consists of phantom maps (cf. [10] Chapter 4.3).

We say that a graded commutative ring $h_{*}$ is even-periodic if there is a unit $u \in h_2$ of degree 2 and $h_{odd} = 0$. Note that $h_{*} = h_{0}[u^{\pm 1}]$ if $h_{*}$ is even-periodic. We say that a ring spectrum $h$ is even-periodic if the coefficient ring $h_{*}$ is even-periodic.
Definition 3.1. Let $R$ be a commutative ring. A topological $R$-module $M$ is said to be *linearly topologized* if $M$ has a fundamental neighbourhood system at the zero consisting of the open submodules. A linearly topologized $R$-module $M$ is said to be *linearly compact* if it is Hausdorff and it has the finite intersection property with respect to the closed cosets $A$ linear topology ring $R$ is linearly compact if $R$ is linearly compact as an $R$-module. (cf. [6, Definition 2.3.13]).

Example 3.2. A linearly topologized compact Hausdorff (e.g. profinite) module is linearly compact. If $R$ is a complete Noetherian local ring, then a finitely generated $R$-module is linearly compact. In particular, a finite dimensional vector space over a field is linearly compact.

Lemma 3.3 (cf. [6, Corollary 2.3.15]). Let $\mathcal{I}$ be a filtered category. The inverse limit functor indexed by $\mathcal{I}$ is exact in the category of linearly compact modules and continuous homomorphisms.

For a spectrum $X$, we denote by $\Lambda(X)$ the category whose objects are maps $Z \xrightarrow{u} X$ such that $Z$ is finite, and whose morphisms are maps $Z \xrightarrow{v} Z'$ such that $u'v = u$. Then $\Lambda(X)$ is an essentially small filtered category.

Lemma 3.4. If $k$ is even-periodic, and $k_0$ is Noetherian and linearly compact, then there is no phantom maps to $k$.

**Proof.** For a finite spectrum $Z$, $k^0(Z)$ is a finitely generated module over $k^0$, and hence $k^0(Z)$ is linearly compact. By Lemma 3.3 $k^0(X) \cong \lim_{\mathcal{I}} k^0(Z)$, where the inverse limit is taken over $\Lambda(X)$. This means that there is no phantom maps to $k$.

Corollary 3.5. Suppose that a spectrum $k$ is even-periodic, and $k_0$ is Noetherian and linearly compact. Then the natural map $C(h, k) \to H(h, k)$ is an isomorphism.

**Proof.** Since $C(h, k) \to H(h, k)$ is surjective and the kernel consists of phantom maps, the corollary follows from Lemma 3.3.

Definition 3.6. We denote by $\text{Mult}(h, k)$ the set of all multiplicative stable cohomology operations from $h^*(-)$ to $k^*(-)$.

If $h^*(-)$ and $k^*(-)$ have their values in the category of linear compact modules, then we denote by $\text{Mult}^0(h, k)$ the subset of $\text{Mult}(h, k)$ consisting of $\theta$ such that $\theta : h^*(X) \to k^*(X)$ is continuous for all $X$.

If $h^*(-)$ is a complex oriented cohomology theory, then the orientation class $X_h \in h^2(\mathbb{C}P^\infty)$ gives a formal group law $F$ of degree $-2$. Furthermore, if $h$ is even-periodic, then a unit $u \in h_2$ gives a degree $0$ formal group law by $F_h(X, Y) = uF(u^{-1}X, u^{-1}Y)$. In the following of this section we suppose that $h^*(-)$ and $k^*(-)$ are complex orientable and even-periodic. Furthermore, we fix a unit $u \in h_2$ (resp. $v \in k_2$) and an orientation class $X_h \in h^2(\mathbb{C}P^\infty)$ (resp. $X_k \in k^2(\mathbb{C}P^\infty)$).

Then we obtain a degree 0 formal group law $F_h$ (resp. $F_k$) associated with $h$ (resp. $k$) as above. A multiplicative cohomology operation $\theta : h^*(-) \to k^*(-)$ gives a ring homomorphism $\alpha : h_0 \to k_0$ and an isomorphism $f : F_h \to \alpha_* F_h$ of formal group laws. Note that $f(X) = \theta(u)f(v^{-1}X)$, where $\bar{f}(X_k) = \theta(X_h)$. In particular, $f'(0) = \theta(u)v^{-1}$ is a unit of $k_0$. Hence we obtain a map from $\text{Mult}(h, k)$ to $\text{FGL}((F_k, k_0), (F_h, h_0))$:

$$\Xi : \text{Mult}(h, k) \to \text{FGL}((F_k, k_0), (F_h, h_0)).$$
If $h_0$ and $k_0$ are Noetherian and linearly compact, then $h^*(X)$ and $k^*(X)$ are linearly compact modules. Then we see that $\Xi$ induces
\[
\Xi^c : \text{Mult}^c(h, k) \to \text{FGL}^c((F_k, k_0), (F_h, h_0)).
\]

**Remark 3.7.** Let $h_0$ and $k_0$ be Noetherian and linearly compact. If $\theta \in \text{Mult}(h, k)$ induces a continuous ring homomorphism $h_0 \to k_0$, then $\theta \in \text{Mult}^c(h, k)$.

Let $p$ be a prime number and $BP$ the Brown-Peterson spectrum at $p$. There is a $BP$-module spectrum $P(n)$ with coefficient $P(n)_* = \mathbb{F}_p[v_n, v_{n+1}, \ldots]$. If $p$ is odd, then $P(n)$ is a commutative $BP$-algebra spectrum. As usual, we set $P(0) = BP$.

Let $R_*$ be a graded commutative ring over $\mathbb{Z}(p)$. We suppose that there is a $p$-typical formal group law $F$ of degree $-2$ over $R_*$. Since the associated formal group law to $BP$ is universal with respect to $p$-typical ones, there is a unique ring homomorphism $r : BP_* \to R_*$. Suppose that $r(v_i) = 0$ for $0 \leq i < n$. Then we obtain a ring homomorphism $\hat{r} : P(n)_* \to R_*$. The functor $R_* \otimes_{P(n)} P(n)_*(-)$ is a generalized homology theory and $R_* \otimes_{P(n)} P(n)_*^*(-)$ is a cohomology theory on the category of finite spectra if $v_n, v_{n+1}, \ldots$ is a regular sequence in $R_*$ by the exact functor theorem \[17\,13\]. We say that such a graded ring $R_*$ is Landweber exact over $P(n)_*$. For a spectrum $X$, we define
\[
R_* \otimes_{P(n)} P(n)_*^*(X) = \lim_{\rightarrow} \left( R_* \otimes_{P(n)} P(n)_*^*(Z) \right),
\]
where the inverse limit is taken over $\Lambda(X)$.

**Lemma 3.8.** Suppose that $R_*$ is Landweber exact over $P(n)_*$ and even-periodic. Furthermore, suppose that $R_0$ is Noetherian and linearly compact. Then the functor $R_* \otimes_{P(n)} P(n)_*^*(-)$ is a complex oriented multiplicative cohomology theory.

**Proof.** Set $R^*(-) = R_* \otimes_{P(n)} P(n)_*^*(-)$. It is easy to see that $R^*(-)$ takes coproducts to products. The exactness of $R^*(-)$ follows from Lemma \[3\,3\] (cf. \[3\] Proposition 2.3.16]). The natural transformation $P(n)_*^*(-) \to R^*(-)$ gives us an orientation of $R^*(-)$.

We suppose that $p$ is odd if $n > 0$. In \[17\] Würgler determined the structure of the co-operation ring $P(n)_*(P(n)_*)$, which is given as follows:
\[
P(n)_*(P(n)_*) = P(n)_*[t_1, t_2, \ldots] \otimes \Lambda(a_0, a_1, \ldots, a_{n-1}),
\]
where $|t_i| = 2(p^i - 1)$ and $|a_i| = 2p^i - 1$. In particular, $P(n)_*(P(n)_*)$ is free over $P(n)_*$. Let $k_*$ be an even-periodic $P(n)_*$-algebra. Hence we have a degree 0 formal group law $F_k$. Note that a $P(n)_*$-algebra homomorphism $\phi : P(n)_*(P(n)_*) \to k_*$ factors through $\overline{\phi} : P(n)_*(P(n)_*)/(a_0, \ldots, a_{n-1}) \cong P(n)_*[t_1, \ldots]$. It is known that there is a one-to-one correspondence between a $P(n)_*$-algebra homomorphism $\phi : P(n)_*[t_1, \ldots] \to k_*$ and the pair $(f, G)$, where $G$ is a $p$-typical formal group law of degree 0 such that the $p$-series $[p](X) \equiv 0 \mod (X^{p^n})$, and $f$ is an isomorphism from $F_k \to G$. Note that $G$ is the degree 0 formal group laws associated with $(\phi \circ \eta_R)_* F_P(n)_*$, where $\eta_R$ is the right unit of the Hopf algebroid $P(n)_*(P(n)_*)$, and $F_P(n)_*$ is the degree $-2$ formal group law associated with the complex oriented cohomology theory $P(n)_*^*(-)$.

Let $h_*$ and $k_*$ be graded rings which are Landweber exact over $P(n)_*$ and even-periodic. We suppose that $h_0$ and $k_0$ are Noetherian and linearly compact. Set $h^*(-) = h_* \otimes_{P(n)} P(n)_*^*(-)$ and $k^*(-) = k_* \otimes_{P(n)} P(n)_*^*(-)$. By Lemma \[3\,8\] $h^*(-)$ and $k^*(-)$ are generalized cohomology theories. We denote by $h$ and $k$ the representing ring spectra, respectively.
Proposition 3.9. We suppose that \( p \) is odd if \( n > 0 \). If \( h_{\ast} \) and \( k_{\ast} \) are Landweber exact over \( P(n)_{\ast} \) and even-periodic, then the map \( \Xi : \text{Mult}(h, k) \rightarrow \text{FGL}((F_{k}, k_{0}), (F_{h}, h_{0})) \) is a bijection. Furthermore, if \( h_{0} \) and \( k_{0} \) are Noetherian and linearly compact, then \( \Xi : \text{Mult}^{\ast}(h, k) \rightarrow \text{FGL}^{\ast}((F_{k}, k_{0}), (F_{h}, h_{0})) \) is also a bijection.

Proof. For \((f, \alpha) \in \text{FGL}(F_{k}, F_{h})\), we construct a multiplicative operation \( \theta : h_{\ast}(-) \rightarrow k_{\ast}(-) \). Since \( k_{\ast}(-) \cong k_{\ast} \otimes P(n)_{\ast}(-) \), we have \( k_{\ast}(P(n)) \cong k_{\ast} \otimes P(n)_{\ast}(P(n)) \), and hence \( k_{\ast}(P(n)) \) is free over \( k_{\ast} \). Then \( k^{0}(P(n)) \cong \text{Hom}_{k_{\ast}}(k_{\ast}(P(n)), k_{\ast}) \cong \text{Hom}_{P(n)_{\ast}}(P(n)_{\ast}(P(n)), k_{\ast}) \). It is easy to see that a multiplicative operation \( P(n)_{\ast}(-) \rightarrow k_{\ast}(-) \) corresponds to a \( P(n)_{\ast} \)-algebra homomorphism \( P(n)_{\ast}(P(n)) \rightarrow k_{\ast} \). Hence we obtain a \( P(n)_{\ast} \)-algebra homomorphism \( \phi : P(n)_{\ast}(P(n)) \rightarrow k_{\ast} \) such that \( \phi \circ \eta_{R} \) corresponds to \( F_{h} \). This gives a multiplicative operation \( \varphi : P(n)_{\ast}(-) \rightarrow k_{\ast}(-) \). Note that \( \varphi \) induces \( \phi \circ \eta_{R} \) on the coefficient rings, and \( F_{h} \) is the degree 0 formal group law associated with \( (\phi \circ \eta_{R})_{F(n)} \). By using the ring homomorphism \( \alpha : h_{\ast} \rightarrow k_{\ast} \), we may extend \( \varphi \) to a multiplicative operation \( \theta : h_{\ast}(-) = h_{\ast} \otimes P(n)_{\ast}(-) \rightarrow k_{\ast}(-) \). Then it is easy to check that this construction gives the inverse of \( \Xi \). If \( \alpha : h_{0} \rightarrow k_{0} \) is continuous, then \( \theta \in \text{Mult}^{\ast}(h, k) \) by Remark 3.7. \( \square \)

4 Multiplicative natural transformation \( \Theta \)

In this section we suppose that \( p \) is an odd prime. We construct a multiplicative natural transformation \( \Theta \) from \( E^{\ast}(-) \) to \( K^{\ast}(-) \otimes_{F} L \), which is equivariant under the action of \( G_{n+1} \). It is shown that \( \Theta \) induces an isomorphism of \( G_{n} \)-modules between \( E^{\ast}(X) \otimes_{E} L \) and \( K^{\ast}(X) \otimes_{F} L \). This implies that the \( G_{n} \)-module \( K^{\ast}(X) \) is naturally isomorphic to \( H^{0}(S_{n+1}; E^{\ast}(X) \otimes_{E} L) \) for all spectra \( X \). Hence we can recover the \( G_{n} \)-module structure of \( K^{\ast}(X) \) from the \( G_{n+1} \)-module structure of \( E^{\ast}(X) \).

Recall that \( F \) is an algebraic extension of \( \mathbb{F}_{p} \) which contains the finite fields \( \mathbb{F}_{p^{n}} \) and \( \mathbb{F}_{p^{n+1}} \). Set \( E_{\ast} = F[u_{n}]_{[u_{n}^{\pm 1}]} \), where the degree of \( u_{n} \) is 0 and the degree of \( u \) is \(-2 \). Abbreviate to \( E \) the degree 0 subring \( E_{0} = F[u_{n}] \). We consider that \( E_{\ast} \) is a \( P(n)_{\ast} \)-algebra by the ring homomorphism \( P(n)_{\ast} \rightarrow E_{\ast} \) given by \( v_{n} \mapsto u_{n} u^{-(p^{n}-1)} \), \( v_{n+1} \mapsto u^{-(p^{n+1}-1)} \), \( v_{i} \mapsto 0 \) (\( i > n+1 \)). Then \( E_{\ast} \) is an even-periodic Landweber exact \( P(n)_{\ast} \)-algebra, and \( E \) is complete Noetherian local ring. Hence, by Lemma 3.8, the functor \( E^{\ast}(-) = E_{\ast} \otimes_{P(n)_{\ast}} P(n)_{\ast}(-) \) is a generalized cohomology theory. We denote by \( E \) the representing ring spectrum. Then the degree 0 formal group law associated with \( E^{\ast}(-) \) is \( F_{n+1} \). By Proposition 3.9 there is a one-to-one correspondence between \( \text{Mult}^{\ast}(E, E) \) and \( \text{Aut}^{\ast}(F_{n+1}, E) \). By Lemma 2.5 the automorphism group of \( F_{n+1} \) over \( E \) is isomorphic to \( G_{n+1} \). In particular, \( G_{n+1} \) acts on the cohomology theory \( E^{\ast}(-) \) as multiplicative stable operations.

Let \( K_{\ast} = F[w^{\pm 1}] \), where \( |w| = -2 \). There is a ring homomorphism \( P(n)_{\ast} \rightarrow K_{\ast} \) given by \( v_{n} \mapsto w^{-(p^{n}-1)} \), \( v_{i} \mapsto 0 \) (\( i > n \)). By Lemma 3.8 \( K^{\ast}(-) = K_{\ast} \otimes_{P(n)_{\ast}} P(n)_{\ast}(-) \) is a complex oriented cohomology theory. We denote by \( K \) the representing ring spectrum. Then the associated degree 0 formal group law is \( H_{n} \). By definition, the automorphism group of \( (H_{n}, F) \) is \( G_{n} \). Hence the automorphism group of \( K^{\ast}(-) \) as multiplicative cohomology theory is \( G_{n} \) by Proposition 3.9.

The following is the main theorem of this note.

Theorem 4.1. There is a multiplicative stable cohomology operation

\[ \Theta : E^{\ast}(-) \rightarrow K^{\ast}(-) \otimes_{F} L \]
such that $\Theta$ is equivariant with respect to the action of $G_{n+1}$. Furthermore, $\Theta$ induces an isomorphism

$$E^*(X) \otimes E L \cong K^*(X) \otimes F L,$$

as $G$-modules for all $X$.

Proof. The even-periodic cohomology theory $K^*(-) \otimes_F L$ is obtained by the even periodic Landweber exact $P(n)_*$-algebra $L[u^{\pm 1}]$ given by $u_n \mapsto w^{-(p^n-1)}$, $v_i (i > n) \mapsto 0$. The associated degree 0 formal group law is the Honda group law $H_n$ of height $n$ over $L$. By Proposition 3.9 the automorphism group of $K^*(-) \otimes_F L$ is the Honda group law. By Theorem 2.6 $G$ acts on $K^*(-) \otimes_F L$ as multiplicative cohomology operations.

By Proposition 3.9 $\text{Mult}(E, K \otimes_F L) \cong \text{FGL}(H_n, L_{F}(F_{n+1}, E))$. We have the ring homomorphism $\alpha : F[v_n] = E \rightarrow M \leftarrow L = (K \otimes_F L)_0$, and the isomorphism $\Phi^{-1} : H_n \rightarrow \alpha_* F_{n+1}$ over $L$. Then $(\Phi^{-1}, \alpha) \in \text{FGL}(H_n, L)$. By 

By Proposition 3.9, the automorphism group of $K^*(-) \otimes_F L$ as a multiplicative cohomology theory is $\text{Aut}(H_n, L)$. By Theorem 2.6 $G$ acts on $K^*(-) \otimes_F L$ as multiplicative cohomology operations.

Let $X$ be a space, then these are also isomorphisms of graded commutative rings. Furthermore, by Theorem 2.6 and Proposition 3.9 $\Theta \otimes L : E^*(X) \otimes E L \cong K^*(X) \otimes F L$ is an isomorphism of $G$-modules, and $\Theta$ is equivariant under the action of $G_{n+1}$.

Lemma 4.2. The invariant ring of $L[u^{\pm 1}]$ under the action of $S_{n+1}$ is $K_*$. 

$$H^0(S_{n+1}; L[u^{\pm 1}]) = K_*.$$

Proof. Let $M^1_n = v_n^{-1} BP_r/(p, v_1, \ldots, v_{n-1}, v^\infty)$. By [12] Theorem 5.10, $\text{Ext}^0_{BP_r}(BP_r, M^1_n)$ is the direct sum of the finite torsion submodules and the $K(n)_*/k(n)_*$ generated by $1/v_n$, $j \geq 1$ as a $k(n)_*$-module. Then as in [10] §5.3 $H^0(S_{n+1}; R_* = F[v_n]$, where $v_n = u_n u^{-(p^n-1)}$. By [10] Lemma 5.9, $H^0(S_{n+1}; M_*)$ is the localization of $H^0(S_{n+1}; R_*)$ by inverting the invariant element $v_n$. Hence $H^0(S_{n+1}; M_*) = F[v_n^{\pm 1}]$. 

By [10] Lemma 3.7, $w = \Phi_0^{-1} u \in L$ is invariant under the action of $S_{n+1}$. Let $a$ be a degree 2n invariant element in $L[u^{\pm 1}]$. Then $b = aw^n$ is also invariant. Let $\phi(X) \in M[X]$ be the minimal polynomial of $b$. Then $\phi(b) = 0$. Since $b$ is invariant under the action of $S_{n+1}$, $\phi(b) = 0$ for all $g \in S_{n+1}$. Hence $\phi(X)$ is also the minimal polynomial of $b$. This implies that $\phi(X)$ is a polynomial over $H^0(S_{n+1}; M) = F$. Hence $b \in F \cap L = F$. This completes the proof.

Corollary 4.3. There are natural isomorphisms of $G_*$-modules:

$$K^*(X) \cong H^0(S_{n+1}; E^*(X) \otimes E L),$$

$$K_*(X) \cong H^0(S_{n+1}; E_*(X) \otimes E L),$$

for all spectra $X$. If $X$ is a space, then these are also isomorphisms of graded commutative rings.

Proof. We have the natural isomorphism of $G_*$-modules: $K^*(X) \otimes_F L \cong E^*(X) \otimes E L$. The action of the subgroup $S_{n+1} \subset G$ on the left hand side is obtained from the action on $L$ only. Hence $H^0(S_{n+1}; K^*(X) \otimes_F L) = K^*(X)$. This completes the proof of the cohomology case. The homology case is obtained by the similar way.
5 Lift to characteristic 0

In this section we lift Θ to the multiplicative natural transformation ch of the characteristic 0 cohomology theories. Then we prove that ch induces a natural isomorphism of cohomology theories with stable cohomology operations if the coefficients are sufficiently extended. Note that in this section we do not assume that p is an odd prime.

We recall that F is an algebraic extension of E, which contains the finite fields E_pⁿ and E_pⁿ⁺¹. We define graded rings E_n,* and E_{n+1,*} as follows:

\[ E_n,* = E_n[w^{±1}] = W[w_1, \ldots, w_{n-1}][w^{±1}], \]
\[ E_{n+1,*} = E_{n+1}[u^{±1}] = W[u_1, \ldots, u_n][u^{±1}], \]

where W = W(F) is the ring of Witt vectors with coefficients in F. The grading of E_n,* is given by |w_i| = 0 (1 ≤ i < n) and |w| = -2, and the grading of E_{n+1,*} is given by |u_i| = 0 (1 ≤ i ≤ n) and |u| = -2. Let \( r_n : BP_* \to E_{n,*} \) be the ring homomorphism given by \( r_n(v_i) = w_i w^{-(p_i-1)} \) (1 ≤ i < n), \( r_n(v_n) = w^{-(p_i-1)} \), \( r_n(v_i) = 0 \) (i > n), and let \( r_{n+1} : BP_* \to E_{n+1,*} \) be the ring homomorphism given by \( r_{n+1}(v_i) = u_i u^{-(p_i-1)} \) (1 ≤ i ≤ n), \( r_{n+1}(v_{n+1}) = u^{-(p^{n+1}-1)} \), \( r_{n+1}(v_i) = 0 \) (i > n + 1). These gives E_n,* and E_{n+1,*} even-periodic Landweber exact B*-algebra structures. Hence, by Lemma 3.3 \( E_n^*(-) = E_n,* \otimes_{BP_*} BP^*(-) \) and \( E_{n+1,*}^*(-) = E_{n+1,*} \otimes_{BP_*} BP^*(-) \) are generalized cohomology theories. Then there are associated degree 0 formal group laws \( \tilde{F}_n \) and \( \tilde{F}_{n+1} \) over \( E_n \) and \( E_{n+1} \), respectively. By Lemma 2.3 \( (\tilde{F}_n, E_n) \) and \( (\tilde{F}_{n+1}, E_{n+1}) \) are universal deformations of \( (H_n, F) \) and \( (H_{n+1}, F) \), respectively.

Let R = W[u_n]. We denote by S the p-adic completion of \( R[u^{-1}] : S = (W((u_n)))_p \). Then S is a complete discrete valuation ring with uniformizer p and residue field \( M = F((u_n)) \). In particular, S is a Henselian ring. We recall the following lemma on Henselian rings.

**Lemma 5.1** (cf. [13 Proposition I.4.4.]). Let A be a Henselian ring with residue field k. Then the functor \( B \mapsto B \otimes_A k \) induces an equivalence between the category of finite étale A-algebras and the category of finite étale k-algebras.

In [16], we have constructed a sequence of finite separable extensions of M:

\[ M = L_{-1} \to L_0 \to L_1 \to \cdots, \]

where \( L_i \) is obtained by adjoining the coefficients \( \Phi_0, \Phi_1, \ldots, \Phi_i \) of the isomorphism \( \Phi : F_{n+1} \xrightarrow{\cong} H_n \). By definition, \( L = \lim_{\to i} L_i = \cup_i L_i \) and we have shown that \( L_i \) is stable under the action of G for all \( i \). By Lemma 6.1 we obtain a sequence of finite étale S-algebras:

\[ S = S_{-1} \to S_0 \to S_1 \to \cdots. \]

We denote by \( S_\infty \) the direct limit \( \lim_{\to i} S_i \) and T the p-adic completion of \( S_\infty \).

**Lemma 5.2.** The ring T is a complete discrete valuation ring of characteristic 0 with uniformizer p and residue field \( L = \lim_{\to i} L_i \).
Proof. Since $L_i$ is a separable extension over $M$, we can take $a \in L_i$ such that $L_i = M(a)$. Let $f(X) \in M[X]$ be the minimal polynomial of $a$ and $\tilde{f}(X) \in S[X]$ a monic polynomial which is a lift of $f(X)$. Then $S_i \cong S[X]/(\tilde{f}(X))$. Then we see that $S_i$ is a complete discrete valuation ring with uniformizer $p$ and residue field $L_i$. This implies that $S_\infty$ is also a discrete valuation ring with uniformizer $p$ and residue field $L$. Then the lemma follows from the fact that $T$ is the $p$-adic completion of $S_\infty$. \qed

We abbreviate $T[u_1, \ldots, u_{n-1}]$ and $T[u_1, \ldots, u_{n-2}]$ by $T[u_1]$ and $T[u_i]$, respectively, etc. Then we obtain a sequence of finite étale $S[u_i]$-algebras:

$$S[u_i] = S_{-1}[u_i] \rightarrow S_0[u_i] \rightarrow S_1[u_i] \cdots,$$

and $T[u_i]$ is the $I_n$-adic completion of $\lim_{\overleftarrow{j}} S_j[u_i]$, where $I_n = (p, u_1, \ldots, u_{n-1})$.

The ring homomorphisms $B\nu \rightarrow E_n, \nu \rightarrow T[w_i][u^\pm 1]$ and $B\nu_n \rightarrow E_{n+1}, \nu_n \rightarrow T[u_i][u^\pm 1]$ satisfy the Landweber exact condition. Also $T[u_i][u^\pm 1]$ and $T[u_i][u^\pm 1]$ are even-periodic, and the degree 0 subring $T[u_i]$ and $T[u_i]$ are complete Noetherian local rings. By Lemma 3.8 the following two functors are generalized cohomology theories:

$$E_n^*(X) \hat{\otimes}_{E_n} T[u_i] := \lim_{\overleftarrow{\Lambda(X)}} (E_n^*(X) \otimes_{E_n} T[u_i]),$$

$$E_{n+1}^*(X) \hat{\otimes}_{E_{n+1}} T[u_i] := \lim_{\overleftarrow{\Lambda(X)}} (E_{n+1}^*(X) \otimes_{E_{n+1}} T[u_i]).$$

The degree 0 formal group laws associated with $E_n^*(-) \hat{\otimes} T[w_i]$ and $E_{n+1}^*(-) \hat{\otimes} T[u_i]$ are $(\tilde{F}_n, T[w_i])$ and $(\tilde{F}_{n+1}, T[u_i])$, respectively.

Lemma 5.3. The formal group laws $(\tilde{F}_{n+1}, T[w_i])$ and $(\tilde{F}_n, T[w_i])$ are universal deformations of $(F_{n+1}, L)$ and $(H_n, L)$ respectively, on the category of complete Noetherian local $T$-algebras.

Proof. From the fact that $(\tilde{F}_n, E_n)$ is a universal deformation of $(H_n, F)$, it is easy to see that $(\tilde{F}_{n+1}, T[w_i])$ is a universal deformation of $(H_n, L)$. From the form of the $p$-series of $\tilde{F}_{n+1}$ given by (2.1), we see that $(\tilde{F}_{n+1}, T[u_i])$ is a universal deformation of $(F_{n+1}, L)$. \qed

Corollary 5.4. The action of $G_{n+1}$ on $(\tilde{F}_{n+1}, E_{n+1})$ extends to an action on $(\tilde{F}_{n+1}, T[u_i])$ such that the induced action on $(F_{n+1}, L)$ coincides with the action of Theorem 2.4.

Proof. It is sufficient to show that the action of $G_{n+1}$ on $E_{n+1}$ extends to an action on $T[u_i]$. For $g \in G_{n+1}$, $u_i^g$ is a unit multiple of $u_i$ modulo $(p, u_1, \ldots, u_{n-1}, u_i^2)$. Hence the ring homomorphism $E_{n+1} \xrightarrow{g} E_{n+1} \xrightarrow{(E_{n+1}[u_i^{-1}])^g} S[u_i]$ extends to a ring homomorphism $E_{n+1}[u_i^{-1}] \rightarrow S[u_i]$. This induces a ring homomorphism $S[u_i] \rightarrow S[u_i]$ and defines an action of $G_{n+1}$ on $S[u_i]$. Since $S_j[u_i] \rightarrow S_{j+1}[u_i]$ is étale for $j \geq -1$ and $L_j$ is stable under the action of $G_{n+1}$ on $L$, the action on $S[u_i]$ extends to $S_j[u_i]$ uniquely and compatibly by Lemma 5.4. Hence we obtain an action on $\lim_{\overleftarrow{j}} (S_j[u_i])$ and its $I_n$-adic completion $T[u_i]$. \qed

We denote the action of $G_{n+1}$ on $(\tilde{F}_{n+1}, T[u_i])$ by $T(g) = (t(g), \nu(g)) : (\tilde{F}_{n+1}, T[u_i]) \rightarrow (\tilde{F}_{n+1}, T[u_i])$ for $g \in G_{n+1}$.
Corollary 5.5. The \((n+1)\)th extended Morava stabilizer group \(G_{n+1}\) acts on the cohomology theory \(E_{n+1}^*(-)\otimes T[u_i]\) as multiplicative cohomology operations.

Proof. This follows from Proposition 3.9. □

Recall that \(S_n\) and \(G_n\) are identified with the Galois groups \(\text{Gal}(L/M)\) and \(\text{Gal}(L/\mathbb{F}_p((u_n)))\), respectively, through the action of \(G_n\) on \(L\) (3.16).

Lemma 5.6. The action of \(G_n\) on \(L\) lifts to the action on \(T\).

Proof. Since \(L_i\) is stable under the action of \(G_n\) on \(L\) for all \(i \geq -1\), the action of \(G_n\) on \(L_i\) lifts to the action on \(S_i\) compatibly by Lemma 5.1. This induces an action on \(S_\infty\). Since \(T\) is the \(p\)-adic completion of \(S_\infty\), we obtain an action on \(T\) which is a lift of the action on \(L\). □

We denote this action of \(G_n\) on \(T\) by \(\tau(g) : T \to T\) for \(g \in G_n\). Since the actions of \(G_n\) on \(E_n\) and \(T\) are compatible on \(W\), the diagonal action defines an action of \(G_n\) on \(T[u_i] = T \otimes_W W[u_i]\). Then we obtain an extension of the action of \(G_n\) on \((\mathcal{F}_n, E_n)\) to \((\mathcal{F}_n, T[u_i])\). We denote this action of \(G_n\) on \((\mathcal{F}_n, T[u_i])\) by \(\Omega(g) = (s(g), \omega(g)) : (\mathcal{F}_n, T[u_i]) \to (\mathcal{F}_n, T[u_i])\) for \(g \in G_n\).

Corollary 5.7. The \(n\)th extended Morava stabilizer group \(G_n\) acts on \(E_{n+1}^*(-)\otimes T[u_i]\) as multiplicative cohomology operations.

Proof. This follows from Proposition 3.9. □

Lemma 5.8. There is a unique isomorphism \((\bar{\Phi}, \bar{\varphi}) : (\mathcal{F}_{n+1}, T[u_i]) \to (\mathcal{F}_n, T[u_i])\) such that \(\bar{\varphi}\) is a continuous \(T\)-algebra homomorphism and \(\bar{\Phi}\) induces \(\Phi\) on the residue fields.

Proof. Since there is an isomorphism \((\Phi, \text{id}_L) : (\mathcal{F}_{n+1}, L) \to (H_n, L)\), the lemma follows from Lemma 2.2. □

Lemma 5.9. For \(g \in G_n\) there is a commutative diagram:

\[
\begin{array}{ccc}
(\mathcal{F}_{n+1}, T[u_i]) & \xrightarrow{(X, \vartheta(g))} & (\mathcal{F}_{n+1}, T[u_i]) \\
(\bar{\Phi}, \bar{\varphi}) & \downarrow & (\bar{\Phi}, \bar{\varphi}) \\
(\mathcal{F}_n, T[u_i]) & \xrightarrow{\Omega(g)} & (\mathcal{F}_n, T[u_i]),
\end{array}
\]

where \(\vartheta(g) : T[u_i] \to T[u_i]\) is given by \(\vartheta(g)(t) = \tau(g)(t)\) for \(t \in T\) and \(\vartheta(g)(u_i) = u_i\) for \(1 \leq u_i < n\).

Proof. Note that \((\vartheta(g) \circ \bar{\varphi})|_T = \tau(g) = (\bar{\varphi} \circ \omega(g))|_T\). The diagram induced on the residue field is commutative by definition of the action of \(G_n\) on \((\mathcal{F}_{n+1}, L) \cong (H_n, L)\). Then the lemma follows from the universality of \((\mathcal{F}_n, T[u_i])\). □

Corollary 5.10. The pro-finite group \(\mathcal{G}\) acts on \((\mathcal{F}_{n+1}, T[u_i]) \cong (\mathcal{F}_n, T[u_i])\) such that the action of the subgroup \(G_{n+1}\) coincides with \(\bar{T}\), and the action of the subgroup \(G_n\) coincides with \(\Omega\).
Proof. We have the action \( \Upsilon \) of \( G_{n+1} \) on \( (\tilde{F}_{n+1}, T[u_i]) \) and the action \( \Omega \) of \( G_n \) on \( (\tilde{F}_n, T[w_i]) \).

The action of the subgroup \( \Gamma \) of \( G_{n+1} \) on \( (\tilde{F}_{n+1}, T[u_i]) \) coincides with the action on \( (\tilde{F}_n, T[w_i]) \) as the subgroup of \( G_n \) under the isomorphism \( (\Phi, \varphi) \). Hence it is sufficient to show that the following diagram commutes for \( g \in S_{n+1} \) and \( h \in S_n \):

\[
\begin{array}{ccc}
(\tilde{F}_{n+1}, T[u_i]) & \xrightarrow{\chi_{\theta(h)}} & (\tilde{F}_{n+1}, T[u_i]) \\
\Upsilon(g) \downarrow & & \downarrow \Upsilon(g) \\
(\tilde{F}_{n+1}, T[u_i]) & \xrightarrow{\chi_{\theta(h)}} & (\tilde{F}_{n+1}, T[u_i]).
\end{array}
\]

Note that the induced diagram on the residue field \( L \) commutes.

Since \( u_i^n \in E_{n+1} \subset T[u_i] \), \( (\theta(h) \circ v(g))(u_i^n) = u_i^n \) for \( \theta(h) \circ v(g) \mid_{S_i} = (v(g) \circ \theta(h)) \mid_{S_i} \). Hence \( (\theta(h) \circ v(g)) \mid_s = (v(g) \circ \theta(h)) \mid_s \) for all \( i \). Hence \( (\theta(h) \circ v(g)) \mid_{S_\infty} = (v(g) \circ \theta(h)) \mid_{S_\infty} \). Then the corollary follows from the universality of \( (\tilde{F}_{n+1}, T[u_i]) \).

\[\blacksquare\]

**Theorem 5.11.** There is a multiplicative stable cohomology operation

\[ ch : E^*_n(-) \rightarrow E^*_n(-) \otimes_{E_n} T[w_i] \]

such that \( ch \) is equivariant with respect to the action of \( G_{n+1} \). Furthermore, \( ch \) induces a natural isomorphism

\[ E^*_n(X) \otimes_{E_{n+1}} T[u_i] \cong E^*_n(X) \otimes_{E_n} T[w_i], \]

as \( G \)-modules for all spectra \( X \).

**Proof.** As in the proof of Theorem 4.1 this follows from Lemma 5.10 and Proposition 3.9 \[\blacksquare\]

**Remark 5.12.** As in Lemma 5.9 we can show that the following diagram commutes for \( g \in G_{n+1} \):

\[
\begin{array}{ccc}
(\tilde{F}_{n+1}, T[u_i]) & \xrightarrow{\chi_{\theta(h)}} & (\tilde{F}_{n+1}, T[u_i]) \\
(\tilde{F}_n, T[w_i]) & \xrightarrow{\chi_{\theta(h)}} & (\tilde{F}_n, T[w_i])
\end{array}
\]

where \( \mu(g) \) is given by \( \mu(g)(t) = \varphi^{-1}(v(g)(t)) \) for \( t \in T \) and \( \mu(g)(w_i) = w_i \) for \( 1 \leq i < n \). This implies that there is a \( G_n \)-equivariant natural homomorphism

\[ E^*_n(X) \rightarrow H^0(S_{n+1}; E^*_n(X) \otimes T[w_i]), \]

which is a homomorphism of graded commutative rings if \( X \) is a space.

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