On Randomized and Quantum Query Complexities

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Abstract. We study randomized and quantum query (a.k.a. decision tree) complexity for all total Boolean functions, with emphasis to de-randomization and dequantization (removing quantumness from algorithms). Firstly, we show that $D(f) = O(Q_1(f)^3)$ for any total function $f$, where $D(f)$ is the minimal number of queries made by a deterministic query algorithm and $Q_1(f)$ is the number of queries made by any quantum query algorithm (decision tree analog in quantum case) with one-sided constant error; both algorithms compute function $f$. Secondly, we show that for all total Boolean functions $f$ holds $R_0(f) = O(R_2(f)^2 \log N)$, where $R_0(f)$ and $R_2(f)$ are randomized zero-error (a.k.a Las Vegas) and two-sided (a.k.a. Monte Carlo) error query complexities.

1 Introduction

The Boolean query (a.k.a. decision tree) model is probably the simplest model of a non-uniform computation. In this model, an input is $N$ bit string $x_1, ..., x_N$ and we want to compute a Boolean function $f(x_1, ..., x_N)$. The decision tree complexity $D(f)$ is the minimal number of queries necessary to compute $f(x_1, ..., x_N)$, where each query asks the value of one variable. It is easy to see that to query all $x_i$’s is always enough to compute $f$, thus $D(f) \leq N$ for all $f$. Let us restrict to the case of total Boolean functions in the rest of the paper.

Randomness, as well as the laws of the quantum world, offer constructions of new models of computation. In this paper we study generalizations of the Boolean query model - randomized and quantum query algorithms. It is known that randomized and quantum query complexities are polynomially related with the deterministic query complexity. In this paper our goal is to tighten this result as much as possible.

Unfortunately, we know only some good randomized query algorithms [22][19] but surprisingly powerful the query model appeared in quantum case, since many quantum algorithms are stated in query model, i.e. Grover’s [11] algorithm for OR, Deutsch-Jozsa’s [10] algorithm for PARITY and Ambainis’s [4] element distinctness algorithm.

Like in uniform models, there are at least three variants how much we allow the randomness in the query model - zero-error (a.k.a. Las Vegas), one-sided
error and two-sided error (a.k.a. Monte Carlo); let $R_0$, $R_1$ and $R_2$ denote respective complexities (the optimal number of queries). Similar situation is in the quantum case also, so let $Q_0$, $Q_1$ and $Q_2$ denote the complexities in quantum case, respectively. It is interesting that there is meaningful to define exact complexity ($Q_E$) in the quantum model. Accurate definitions we will give in Section 3.

There are many ways how those complexities can be compared. Two of them are the most popular ones. The first, to compare them for some particular function (or some class of functions). Over the last years, a rich body of work has been investigated to show both upper and lower bounds of certain functions both for randomized complexity (i.e. [15,19,22]) and quantum complexity (i.e. [2,3,16,21,7,11]).

The second way, to which we focus in this paper, is to show relations between those models that hold for all functions. Other ways include studying the complexity of random functions and an average-case complexity.

The next section briefly survives known results and states our ones, as well as gives organization of the paper. Note: in the rest of the paper, unless otherwise specified, all results hold for every total Boolean function $f$.

## 2 Previous work and our results

### 2.1 The random case

Trivially, $R_2(f) \leq R_1(f) \leq R_0(f) \leq D(f)$. The first non-trivial result follows from an independent work of several authors [6,12,23] and states that $D(f) = O(R_0(f)^2)$. Nisan [17] generalized it to one-sided error case $D(f) = O(R_1(f)^2)$ and to two-sided error case $D(f) = O(R_2(f)^3)$. In this paper we show that $R_0(f) = O(R_2(f)^2 \log N)$, where $N$ is the length of input.

Much more progress has been made to study the complexity for certain classes of functions. For instance, it is known that $D(g) = O(R_0(g)^{1.96...})$ holds for every read-once formula $g$. Santha [20] showed that $R_0(g) = \Theta(R_2(g))$ holds for the class of read-once formulas $g$. For other classes of functions, like graph properties, monotone functions, random functions and symmetric functions, better results are known, too.

The best randomized algorithm is given by Snir [22]; he shows that for the recursive NAND function, $D(NAND) = \Omega(R_0(NAND)^{1.326...})$. For NAND function this algorithm is tight [19] and this gap is conjectured to be an optimal separation between deterministic and randomized complexities.

### 2.2 The quantum case

Trivially, $Q_2(f) \leq Q_1(f) \leq Q_0(f) \leq Q_E(f)$ and $Q_2(f) \leq R_2(f), Q_1(f) \leq R_1(f), Q_0(f) \leq R_0(f), Q_E(f) \leq D(f)$. Beals et al. [5] showed that $D(f) = O(Q_2(f)^6)$ and $D(f) = O(Q_E(f)^4)$. Buhrman et al. [2] improved the later to $D(f) =$
Aaronson [1] showed a relation between one-sided error randomized and quantum complexities, $R_0(f) = O(Q_1(f)^3 \log N)$ \(^2\). We give better result, $D(f) = O(Q_1(f)^3)$. Again, as well as in the random case, none of those relations are believed to be tight. Quantum algorithms usually are much more sophisticated than randomized ones. Countless papers have been written to find fast quantum algorithms as well as to characterize the power of quantum lower bound techniques. The best known quantum query algorithm is Grover’s algorithm [11] for OR function that gives $R_2(\text{OR}) = \Omega(Q_1(\text{OR})^2)$ and $Q_0(\text{OR}) = \Omega(Q_1(\text{OR})^2)$ \(^5\).

Buhrman et al. [7] showed that for any $\varepsilon > 0$ there is a function $g_\varepsilon$ such that $R_2(g_\varepsilon) = \Omega(Q_0(g_\varepsilon)^{2-\varepsilon})$ and $Q_E(g_\varepsilon) = \Omega(Q_0(g_\varepsilon)^{2-\varepsilon})$ \(^3\). The best known separation between $Q_E(f)$ and $D(f)$ is just by a factor 2 \(^10\). The result by van Dam [9] shows that $Q_2(f) \leq N^2 + \sqrt{N}$.

2.3 The organization of the paper

The rest of the paper is organized as follows. Section 3 gives definitions and some basic results we will use in proofs. Section 4 proves the relation between deterministic and quantum complexities. Section 5 proves the relation between randomized complexities. At the end, section 6 gives some immediate extensions of the results in this paper.

3 Preliminaries

We assume familiarity with classical and quantum query algorithms and basic complexity measures of them, so we will quickly breeze through definitions, notation and basic results. For more explicit statement one can look in superb (but somewhat outdated) survey by Buhrman and de Wolf [8]; mostly this section is based on the work done by Nisan [17], Beals et al. [5] and de Wolf [24].

We consider computing a Boolean function $f(x_1, ..., x_N): \{0,1\}^N \rightarrow \{0,1\}$ in the query model. In this model, the input bits can be accessed by queries to an oracle $X$ and the complexity of $f$ is the number of queries needed to compute $f$. The deterministic query complexity $D(f)$ is just a minimal number of queries necessary to compute function $f$.

A randomized query algorithm is just a probability distribution over deterministic query algorithms. We are interested in algorithms making minimal number of queries in the worst-case such that for all inputs it returns correct answer with probability at least $\rho \geq 4/5$ \(^4\).

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\(^1\) Explicitly they showed it only for $Q_0$ but it is easy to generalize to $Q_1$ case.

\(^2\) The same note as in the previous footnote.

\(^3\) Actually they showed only the first case but the second case is a trivial application of polynomials method.

\(^4\) $4/5$ can be replaced by arbitrary constant in $(1/2..1)$. 
A quantum computation with \( T \) queries is just a sequence of unitary transformations

\[
U_1 \rightarrow O \rightarrow \ldots \rightarrow U_{T-1} \rightarrow O \rightarrow U_T \rightarrow O.
\]

\( U_j \) can be arbitrary unitary transformation that do not depend on the input bits \( x_1, \ldots, x_N \). \( O \) are query transformations. To define \( O \), we represent basis states as \(|i, z\rangle\) where \( i \) consists of \( \lceil \log N \rceil \) bits describing an index of a variable to be queried, \( b \) is one bit and \( z \) consists of all other bits. Then, \( O \) maps \(|i, b, z\rangle\) to \((-1)^{bx_i}|i, b, z\rangle\) (i.e., we change phase depending on \( x_i \)). The computation starts with a state \(|0\rangle\). Then, we apply \( U_1, O, \ldots, O, U_T \) and measure the final state. The result of the computation is the bit \( b \) obtained by the measurement.

Now we can define the models depending on probability \( \rho \) such that for every \( x = (x_1, \ldots, x_N) \), the rightmost bit of \( U_T O_x \ldots O_x U_1 |0\rangle \) equals \( f(x_1, \ldots, x_N) \) with probability at least \( \rho = \rho(x_1, \ldots, x_N) \).

Both in randomized and quantum algorithms we are interested in those ones who compute functions asking as less queries as possible. With complexity of an algorithm we mean the number of queries it make. For such algorithms, if \( \rho = 1 \) then randomized query complexity is equal with deterministic complexity one but quantum complexity is denoted by \( Q_E(f) \). If \( \rho = 1 \) on all 0-instances or all 1-instances and always \( \rho \geq 4/5 \) then we call it one-sided error algorithm and \( R_1(f) \) \((Q_1(1))\) denote respective complexities. If \( \rho \geq 4/5 \) then we call the complexity two-sided and denote \( R_2(f) \) and \( Q_2(f) \), respectively. Zero-error case is special, because algorithms are allowed to output also "?" (meaning "I don’t know"). When it outputs 0 or 1 then it should be correct always but it can output ? with probability at most 1/5. Let \( R_0(f) \) and \( Q_0(f) \) denote the corresponding complexities.

It is well known fact that a Boolean function is unique represented by a multilinear polynomial. For example, \( OR \) function is represented by a polynomial \( 1 - (1 - x_1)(1 - x_2) \ldots (1 - x_N) \). Polynomials that approximate functions on every input will be interesting too.

Beals et al. showed the source lemma for polynomial method:

**Lemma 1** The probability to output correct answer for every quantum query algorithm making \( T \) queries is described by a multilinear polynomial with degree at most \( 2T \).

**Sketch of the proof.** The state of an algorithm running on input \( x \) can be described by

\[
\sum_{i, b, z} p_{i, b, z}(x) |i, b, z\rangle
\]

where \( p_k \) is an amplitude of basis state \( k \). At the beginning \( p_{i, b, z}(x) \) do not depends on input word \( x \). After a query, amplitudes changes. Since oracle is a linear operator, we can analyze each monomial separately. Oracle maps \( p_{i, b, z}(x) |i, b, z\rangle \) to \((-1)^{bx_i} p_{i, b, z}(x) |i, b, z\rangle \). Since \((-1)^y = (1 - 2y) \) then \((-1)^{bx_i} p_{i, b, z}(x) |i, b, z\rangle = (1 - 2bx_i) p_{i, b, z}(x) |i, b, z\rangle \). So the degree of amplitudes increase just by at most one.
Unitaries cannot increase the degree. Measurement can at most double the degree because the probability to observe a state to be 0 on $b$ is

$$\sum_{i,0,z} |p_{i,0,z}(x)|^2.$$ 

This allows us to lower bound the number of queries of quantum algorithm by finding lower bound of degree by polynomial representing function. More precisely, $Q_E(f) \geq \deg(f)/2$, $Q_0 \geq ndeg(f)/2$, $Q_1 \geq \min(ndeg(f), ndeg(1-f))/2$ and $Q_2 \geq \deg(f)/2$. There $\deg(f)$ denotes the degree of polynomial representing $f$, $ndeg(f)$ (degree of a nondeterministic polynomial) denotes the minimal degree of polynomial $p$ such that $p(x) = 0$ iff $f(x) = 0$, $\tilde{\deg}(f)$ denotes the minimal degree of polynomial approximating function $f$ on every input. It is easy to see that $\deg(f) \geq ndeg(f) \geq \tilde{\deg}(f)$.

The block sensitivity of $f$ on $x$ is the maximum number of disjoint $B_j \subseteq \{1, \ldots, n\}$ such that $f(x^{B_j}) \neq f(x)$, $x^{B_j}$ being $x$ with all $x_i$ for $i \in B_j$ changed to $1-x_i$. We denote it $bs_x(f)$. Let $bs(f) = \max bs_x(f)$. The sensitivity $s(f)$ is the same just all blocks are restricted to be with a size one. It is easy to see that $s(f) \leq bs(f)$.

It is known that

**Theorem 2** [IS] For any total Boolean function $f$,

$$bs(f) = O(\tilde{\deg}(f)^2).$$

**Theorem 3** [IS] For any Boolean function $f$,

$$R_2(f) \geq bs(f)/2.$$ 

**Proof.** Let $w$ be the input that achieves the block sensitivity, and let $B_1, B_2, ..., B_t$ be the disjoint sets s.t. $f$ is sensitive to $B_i$ on $w$. For each $1 \leq i \leq t$, any randomized algorithm running on $w$ must query some variable in $B_i$ with probability of at least $1/2$, since otherwise it cannot distinguish between $w$ and $w^{B_i}$. Thus the total number of queries has to be at least $t/2$. 

4 Deterministic vs. quantum one-sided error

To dequantize one-sided error algorithms we use polynomials method. Our result is improvement over that ones by Buhrman et al. [7] and Aaronson [1]. Here maxonomial of polynomial $p$ is a monomial with maximal degree.

The following generalization of a lemma attributed in [5] to Nisan and Smolensky was independently observed by Aaronson [1]. The key idea of it is that querying a maxonomial, we decrease the function’s block sensitivity on any input word by at least one.
Lemma 4 For any nondeterministic polynomial $p$ approximating function $f$, for every 0-instance $w \in \{0,1\}^N$ (s.t. $f(w) = 0$) and every maxonomial $M$ of $p$, there is a set $B$ of variables in $M$ such that $f(w^B) = 1$.

Proof. Obtain restricted polynomial $g$ from $p$ by setting all variables outside of $M$ according to $w$. Obtain word $w' \in \{0,1\}^{\vert M \vert}$ that assigns values from $w$ to variables in $M$. Since $g$ makes no errors on 0-instance, $g(w') = 0$. This $g$ contains monomial $M$ therefore it cannot be constant 0. Therefore there is some set $B$ of variables in $M$ that makes $g(w'^B) > 0$ and hence $f(w^B) = 1$. \hfill \Box

This we use in the following algorithm.

Lemma 5 For every total Boolean function $f$,

$$D(f) \leq (bs(f) + 1) \ast ndeg(f).$$

Proof. The deterministic query algorithm $\mathcal{A}$ is written in pseudo code, as a function of a complete description of a polynomial $q$ that nondeterministically represent the function $f$ \footnote{Remember, it means that for every input word $x$, $q(x) = 0$ if and only if $f(x) = 0$} (thus $deg(q) = ndeg(f)$) and a word $X \in \{0,1\}^N$ given by queries. $\mathcal{A}$ returns value of $f(X)$. A function $\text{sign} : \mathbb{R} \rightarrow \{0,1\}$ is defined as follows; if $p \neq 0$ let $\text{sign}(p) = 1$ otherwise let $\text{sign}(p) = 0$. The algorithm $\mathcal{A}$:

$\{0,1\}$ function Value of$\{$
  By value $q$ as polynomial,
  By queries $X \in \{0,1\}^N$;
  1 $p := q$;
  Repeat $bs(f) + 1$ times {
    3 If $p$ is constant then return $\text{sign}(p)$;
    4 Pick a maxonomial $M$ in $p$;
    5 Query $X$-values of $M$’s variables;
    6 Replace all queried variables in $p$
      by appropriate constants;
  };
  7 Return 1;
$\}$;

The nondeterministic ”pick a maxonomial” can easily be made deterministic by choosing the the first maxonomial in some fixed order.

It is easy to see that for every maxonomial $M$ holds $\vert M \vert = deg(p)$ and at every moment $deg(p) \leq deg(q)$, thus in every cycle $\mathcal{A}$ makes at most $deg(q)$ queries, hence the number of queries $\leq deg(q) \ast bs(f)$. If $\mathcal{A}$ returns the answer in 3rd line then it is right because $q$ represents $f$. If input word is 0-instance then by Lemma 4 querying each maxonomial decreases the function’s block sensitivity on $x$; after $bs(f)$ repetitions it should be a constant. Therefore algorithm can reach 7th line only on 1-instances. \hfill \Box
Theorem 6 For every total Boolean function \( f \),
\[
D(f) = O(Q_1(f)^3).
\]

Proof. Lemma 5 and Theorem 2 gives this relation whenever quantum algorithm happens to make error on 1-instances. However, if it makes error on 0-instances we could just dequantize the complementary function, and afterward just flip all the answers in the deterministic decision tree. \( \Box \)

5 Randomized zero-error vs. random two-sided error

Before this paper, the only nontrivial relation between \( R_0 \) and \( R_2 \) was
\[
R_0(f) = O(R_2(f) \deg(f) \log N)
\]
by Aaronson [1]. In this section we prove

Theorem 7 For every total Boolean function \( f \),
\[
R_0(f) = O(R_2(f)^2 \log N).
\]

Nisan introduced minimal sensitive blocks on input word \( X \) as sensitive blocks whom any strict subset is not sensitive on \( X \) and proved

Lemma 8 [17] For every word \( X \), for every minimal sensitive block \( B \) on \( X \),
\[
|B| \leq s(f) \leq bs(f).
\]

Proof. If we flip one of the B-variables in \( X^B \), then the function’s value must flip as well (otherwise \( B \) would not be minimal), so every B-variable is sensitive for \( f \) on \( X^B \).

We can easily get very rough estimate of the number of minimal sensitive blocks for \( f \) on word \( X \):

Lemma 9 For any total Boolean function \( f \) and word \( X \), the number of minimal sensitive blocks on word \( X \) is at most \( N^{bs(f)} \).

Proof. Since the previous lemma said that the size of any minimal sensitive block cannot be bigger than \( bs(f) \), then the maximal number of minimal sensitive blocks is less than the number of subsets of at most \( bs(f) \) variables, that is
\[
\sum_{k=1}^{bs(f)} \binom{N}{k} \leq N^{bs(f)}.
\]

The proof of the Theorem 7 just follows from the next lemma, by applying the Lemma 9.

\( \Box \)

\( 6 \) Notice, that in general a statement "\( Q_1(f) \geq \deg(f)/2 \)" is not true but in our case we have a relation that is true for all functions \( f \) therefore also for complementary functions.

\( 7 \) Special case of Nisan’s result - \( R_0(f) = O(R_2(f)^3) \) I call trivial.
Lemma 10 For every total Boolean function $f$

$$R_0(f) = O(R_2(f)bs(f) \log N).$$

Proof. The zero-error randomized algorithm running on word $X$ is as follows. Repeat two-sided error algorithm and take majority, until it gives estimation of expected error $\epsilon \leq \frac{1}{2N^{bs(f)}}$. We need $\Theta(bs(f) \log N)$ repetitions to get it (as usual, by Chenoff’s bounds). If at this moment the value of $f(X)$ is not determined for sure $^8$ then output ”?”, otherwise output the value. To finish the proof we have to show that the value of $f(X)$ is determined $^9$ with probability at least $1/2$.

Assume not; then there exists a block $B \subseteq [N]$ such that $f(X) \neq f(X^B)$, moreover, there should be such minimal block. On the other hand, by a simple adversary argument, every sensitive block $B$ of function $f$ on word $X$ should be queried with probability at least $\epsilon$. The expected number of blocks which are not touched is at most $\epsilon \cdot N^{bs(f)} \leq 1/2$ (by the Lemma 9). Therefore with probability at least $1/2$ there are no minimal sensitive blocks left, thus the value of $f(X)$ is determined.

$\square$

6 Extension of results

In the previous sections, to make picture simpler we compared just two complexities in each inequality. Actually, one could wish to see those results more precisely. Now we review all of them. All inequalities in the list hold for every total function $f$ up to constant factor:

* $D(f) \leq R_0(f)^2$ [6,12,23].
* $D(f) \leq R_1(f)R_2(f)$ [17].
* $D(f) \leq R_2(f)^3$ [17].
* $R_0(f) \leq R_2(f)^2 \log N$ [Theorem 7].
* $D(f) \leq Q(f)^6$ [3].
* $D(f) \leq Q_E(f)^2Q_2(f)^2$ [5].
* $D(f) \leq Q_1(f)^2Q_2(f)^2$ [7].
* $R_0(f) \leq Q_1(f)Q_2(f)^2 \log N$ [11].
* $D(f) \leq Q_1(f)Q_2(f)^2$ [Lemma 9].

Probably none of those inequalities are tight.

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$^8$ For those who know the notion of a ”certificate”. We output ”?” if we have not found a certificate.
$^9$ In other words, we have found a certificate.
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