Perturbative renormalization factors of $\Delta S = 1$ four-quark operators for domain-wall QCD

Sinya Aoki$^a$ and Yoshinobu Kuramashi$^b$ *

$^a$Institute of Physics, University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan
$^b$Department of Physics, Washington University, St. Louis, Missouri 63130, USA
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Abstract

Renormalization factors for $\Delta S = 1$ four-quark operators in the effective weak Hamiltonian are perturbatively evaluated in domain-wall QCD. The one-loop corrections of $\Delta S = 1$ four-quark operators consist of two types of diagrams: one is gluon exchange between quark lines and the other is penguin diagrams containing quark loops. Combining both contributions, our results allow a lattice calculation of the amplitude for $K \to \pi\pi$ decays with $O(g^2)$ corrections included.

11.15Ha, 11.30Rd, 12.38Bx, 12.38Ge

*On leave from Institute of Particle and Nuclear Studies, High Energy Accelerator Research Organization(KEK), Tsukuba, Ibaraki 305-0801, Japan
I. INTRODUCTION

$\Delta S = 1$ four-quark operators appear in the effective low-energy Hamiltonians for non-leptonic weak decays of kaons, which are relevant for the $\Delta I = 1/2$ rule and the $CP$-violation parameters $\epsilon'/\epsilon$. Although comparison of their experimental results with theoretical predictions based on the standard model is supposed to give a good testing ground for the model, it is not accomplished without any reliable non-perturbative estimates for the hadronic matrix elements of $\Delta S = 1$ four-quark operators.

Lattice QCD calculation allow us to evaluate the $\Delta S = 1$ hadronic matrix elements from first principles in QCD. While in the past decade the Wilson and the Kogut-Susskind quark actions are used to calculate the four-quark hadronic matrix elements, these quark actions have inherent defects: the explicit chiral symmetry breaking in the Wilson quark action, which causes the non-trivial operator mixing among different chiralities, and the mixture of space-time and flavor symmetries in the Kogut-Susskind quark action, which annoys us by demanding non-trivial matching between continuum and lattice operators. The domain-wall quark model [1,2] is a five-dimensional Wilson fermion with free boundaries imposed on the fifth direction of $N_s$ layers. This quark formulation is expected to have superior features in the large $N_s$ limit: no fine tuning for the chiral limit, $O(a^2)$ scaling violation and no mixing between four-quark operators with different chiralities. Perturbative calculations have shown that these expectations are fulfilled at the one-loop level in the limit $N_s \to \infty$ [3–7], and numerical studies support these features non-perturbatively [8–10]. These advantageous features over other quark formulations urge us to apply the domain-wall quark to a calculation of the $\Delta S = 1$ hadronic matrix elements.

Since the Wilson coefficients in the effective low-energy Hamiltonians for non-leptonic weak decays of kaons are calculated in some continuum renormalization scheme (e.g., $\overline{MS}$), the corresponding $\Delta S = 1$ hadronic matrix elements should be given in the same continuum scheme, which requires us to convert the matrix elements obtained by lattice simulations to those defined in continuum renormalization scheme. A necessary ingredient for this transformation is the renormalization factors matching the lattice operators to the continuum ones.

In this article we make a perturbative calculation of the renormalization factors for the $\Delta S = 1$ four-quark operators consisting of physical quark fields in the domain-wall QCD (DWQCD). The one-loop corrections of $\Delta S = 1$ four-quark operators contain the gluon exchange diagram and the penguin diagram, which leads to the following form for the relation between the lattice and continuum operators:

$$Q_{i}^{\text{cont}} = \sum_{j} Z_{ij}^{g} Q_{j}^{\text{latt}} + Z_{i}^{\text{pen}} Q_{\text{latt}}^{\text{pen}} + O(g^4),$$

where $Q_i$ denote a set of $\Delta S = 1$ four-quark operators. The gluon exchange contributions $Z_{ij}^{g}$ are already calculated in our previous paper [3], whose results are applicable to $\Delta S = 2$ four-quark operators. In this work we evaluate the additional contribution denoted by $Z_{i}^{\text{pen}} Q_{\text{latt}}^{\text{pen}}$ which originates from the penguin diagram in the $\Delta S = 1$ case. With the aid of the results in Ref. [6], we obtain complete expressions of the renormalization factors for the $\Delta S = 1$ four-quark operators.

This paper is organized as follows. In Sec. [4] we present the calculations of the penguin diagram in the continuum and on the lattice. Full expressions of the renormalization factors
for the $\Delta S = 1$ operators are given in Sec.III. Our conclusions are summarized in Sec.IV.

In Appendix we give the DWQCD action and the Feynman rule relevant for the present calculation to make this paper self-contained.

The physical quantities are expressed in lattice units and the lattice spacing $a$ is suppressed unless necessary. We take SU($N$) gauge group with the gauge coupling $g$, while $N = 3$ is specified in the numerical calculations. Hereafter, repeated indices are to be summed over unless otherwise indicated.

II. PENGUIN DIAGRAMS

A. $\Delta S=1$ four-quark operators

We consider the following $\Delta S=1$ four-quark operators,

$$Q_1 = (\bar{s}_a u_b) V_A (\bar{u}_b d_a) V_A,$$

$$Q_2 = (\bar{s}_a u_a) V_A (\bar{u}_b d_b) V_A,$$

$$Q_3 = (\bar{s}_a d_a) V_A \sum_q (\bar{q}_b q_a) V_A,$$

$$Q_4 = (\bar{s}_a d_b) V_A \sum_q (\bar{q}_b q_a) V_A,$$

$$Q_5 = (\bar{s}_a d_a) V_A \sum_q (\bar{q}_b q_a) V_A,$$

$$Q_6 = (\bar{s}_a d_b) V_A \sum_q (\bar{q}_b q_a) V_A,$$

$$Q_7 = \frac{3}{2} (\bar{s}_a d_a) V_A \sum_q e^q (\bar{q}_b q_a) V_A,$$

$$Q_8 = \frac{3}{2} (\bar{s}_a d_b) V_A \sum_q e^q (\bar{q}_b q_a) V_A,$$

$$Q_9 = \frac{3}{2} (\bar{s}_a d_a) V_A \sum_q e^q (\bar{q}_b q_a) V_A,$$

$$Q_{10} = \frac{3}{2} (\bar{s}_a d_b) V_A \sum_q e^q (\bar{q}_b q_a) V_A,$$

where $V \pm A$ indicates the chiral structures $\gamma_\mu (1 \pm \gamma_5)$. $a,b$ denote color indices and $e^q$ are quark charges: $2/3$ for the up-like quarks and $-1/3$ for the down-like quarks. The flavor sum is intended over those which are active below the renormalization scale of the operators. This set of operator basis closes under QCD and QED renormalization. We classify these 10 operators into the following four types:

$$Q_1 = (\bar{s}_a \Gamma^\mu_X u_b) (\bar{u}_b \Gamma^\mu_X d_a),$$

$$Q_2 = (\bar{s}_a \Gamma^\mu_X u_a) (\bar{u}_b \Gamma^\mu_X d_b),$$

$$Q_i^c = (\bar{s}_a \Gamma^\mu_X d_b) \sum_q \alpha_i^2 (\bar{q}_b \Gamma^\mu_Y q_a) \quad i = 4, 6, 8, 10,$$

$$Q_i^o = (\bar{s}_a \Gamma^\mu_X d_a) \sum_q \alpha_i^2 (\bar{q}_b \Gamma^\mu_Y q_b) \quad i = 3, 5, 7, 9,$$
where $\Gamma_X^\mu = \gamma_\mu (1 - \gamma_5)$ and $\Gamma_Y^\mu = \gamma_\mu (1 + \gamma_5)$. $\alpha_i^q$ are given by

$$\begin{align*}
\alpha_i^q &= 1 & i &= 3, 4, 5, 6, \\
\alpha_i^q &= \frac{3}{2} \epsilon^q & i &= 7, 8, 9, 10.
\end{align*}$$

B. Continuum calculation

It is instructive to first show the calculation of the penguin diagram in the continuum regularization schemes. For the present calculation we employ the Naive Dimensional Regularization (NDR) and the Dimensional Reduction (DRED) as the ultraviolet regularization scheme, in each of which the loop momenta of the Feynman integrals are defined in $D$ dimensions parameterized by $D = 4 - 2\epsilon$ ($\epsilon > 0$). The major difference between both regularization schemes are found in the definitions of the Dirac matrices: NDR scheme defines the Dirac matrices in $D$ dimension, while DRED scheme retains them in four dimensions.

We calculate the following Green function with massless quarks:

$$\langle Q_i \rangle_{ab;cd} \equiv \langle Q_i s_a(p_1) \bar{d}_b(p_2) q_c(p_3) \bar{q}_d(p_4) \rangle,$$  

where $a,b,c,d$ label color indices, while spinor indices are suppressed. Truncating the external quark propagators from $\langle Q_i \rangle$, where we multiply $i\not{p}_j$ ($j = 1, 2, 3, 4$) on $\langle Q_i \rangle$, we obtain the vertex functions, which is written in the following form up to the one-loop level

$$[\Lambda_i(p)]_{ab;cd} = \left[ \Lambda_i^{(0)}(p) + \Lambda_i^{(1-g)}(p) + \Lambda_i^{(1-p)}(p) \right]_{ab;cd},$$

where the momentum $p$ is defined by $p = p_1 - p_2 = p_4 - p_3$. The superscript $(i)$ refers to the $i$-th loop level. At the one-loop level $\Lambda_i^{(1-g)}$ and $\Lambda_i^{(1-p)}$ represent the gluon exchange diagram and the penguin diagram, respectively. In this paper, however, we do not consider $\Lambda_i^{(1-g)}$ because the gluon exchange contributions to the renormalization factors are already calculated in our previous paper [6].

The tree-level vertex functions $\Lambda_i^{(0)}$ are given by

$$\begin{align*}
[\Lambda_i^{(0)}]_{ab;cd} &= [1 \circ 1]_{ab;cd} \Gamma_X^\mu \otimes \Gamma_X^\nu, \\
[\Lambda_i^{(0)}]_{ab;cd} &= [1 \circ 1]_{ab;cd} \Gamma_Y^\mu \otimes \Gamma_Y^\nu, \\
[\Lambda_i^{(e(0))}]_{ab;cd} &= \alpha_i^{u} [1 \circ \tilde{1}]_{ab;cd} \Gamma_X^\mu \otimes \Gamma_Y^\nu, \\
[\Lambda_i^{(o(0))}]_{ab;cd} &= \alpha_i^{u} [1 \circ \tilde{1}]_{ab;cd} \Gamma_Y^\mu \otimes \Gamma_Y^\nu,
\end{align*}$$

where $\otimes$ represent the tensor structures in the Dirac spinor space and $\circ$ and $\tilde{\circ}$ act on the color space as

$$\begin{align*}
[1 \circ 1]_{ab;cd} &\equiv \delta_{ad} \delta_{eb}, \\
[1 \circ \tilde{1}]_{ab;cd} &\equiv \delta_{ab} \delta_{cd}.
\end{align*}$$
where $\bar{\gamma}$ eliminated by imposing on-shell conditions on external quark states. It should be noted that the terms proportional to integral of the quark loop, whose result is given by

$$T_i^{ab} = \bar{\gamma}_i \gamma_a \gamma_b \Gamma^{ab}_X \otimes \gamma_v,$$

where $1/(4 \pi)^2$ is the gauge propagator $\Gamma^{ab}_X$ as an evanescent $\delta^{ab}$. $L_{\alpha \beta}(p)$ denote the integral of the quark loop, whose result is

$$L_{\alpha \beta}(p) = \frac{1}{6(4 \pi)^2} \left[ (1 + \log(\mu^2/p^2) + \frac{5}{3} + 1) \frac{p^2}{2} \delta_{\alpha \beta} + \left( \frac{1}{\epsilon} + \log(\mu^2/p^2) + \frac{5}{3} \right) p_{\alpha} p_{\beta} \right],$$

where $1/\epsilon = 1/\epsilon - \gamma + \ln |4 \pi|$ and $\mu$ is the renormalization scale. We remark that the metric tensor $\delta_{\alpha \beta}$ is defined in $D$ dimension both for the NDR and DRED schemes. The products of the Dirac matrices are reduced as follows:

$$\begin{align*}
\text{tr} \left[ \Gamma^{\mu}_{\alpha} \gamma_{\alpha} \gamma_{\beta} \right] \delta_{\mu \nu} & = 4 \times \left\{ \begin{array}{ll}
(2 - D) \delta_{\mu \nu} & \text{NDR}, \\
2 \delta_{\mu \nu} - D \delta_{\mu \nu} & \text{DRED},
\end{array} \right. \\
\text{tr} \left[ \Gamma^{\mu}_{\gamma \alpha} \gamma_{\alpha} \gamma_{\beta} \right] p_{\alpha} p_{\beta} & = 4 \times \left\{ \begin{array}{ll}
2 p_{\mu} p_{\nu} - \delta_{\mu \nu} p^2 & \text{NDR}, \\
2 p_{\mu} p_{\nu} - \delta_{\mu \nu} p^2 & \text{DRED},
\end{array} \right. \\
\Gamma^{\nu}_{\alpha} \gamma_{\alpha} \gamma_{\beta} \Gamma^{\mu}_{\gamma} \delta_{\mu \nu} & = \delta_{XY} \times \left\{ \begin{array}{ll}
2(D - 2) \gamma_{\mu} (1 - \gamma_5) & \text{NDR}, \\
-4 [\gamma_{\mu} - D \gamma_{\mu}] (1 - \gamma_5) & \text{DRED},
\end{array} \right. \\
\Gamma^{\nu}_{\alpha} \gamma_{\alpha} \gamma_{\beta} \Gamma^{\mu}_{\gamma} p_{\alpha} p_{\beta} & = \delta_{XY} \times \left\{ \begin{array}{ll}
2(D - 2) \left[ p^2 \gamma_{\mu} - 2 p_{\mu} \gamma_{\mu} \right] (1 - \gamma_5) & \text{NDR}, \\
4 \left[ p^2 \gamma_{\mu} - 2 p_{\mu} \gamma_{\mu} \right] (1 - \gamma_5) & \text{DRED},
\end{array} \right.
\end{align*}$$

where $\gamma_{\mu} = \delta_{\mu \nu} \gamma_{\nu}$ in the DRED scheme, and we choose $E_{\mu} = \bar{\gamma}_{\mu} - \frac{D}{4} \gamma_{\mu}$ as an evanescent operator in the DRED scheme [1]. It should be noted that the terms proportional to $\gamma$ are eliminated by imposing on-shell conditions on external quark states.

After some algebra we obtain the following expressions for the vertex functions (26)–(29):

$$\begin{align*}
\left[ \Lambda^{(1-p)} \right]_{ab\phi} = - \frac{g^2}{12 \pi^2} \left[ T^A \otimes T^A \right]_{ab\phi} \gamma_{\mu} (1 - \gamma_5) \otimes \gamma_{\mu} L_{\alpha \beta}^{\gamma_{\mu}}
\end{align*}$$

where $\Lambda^{(1-p)}$ are generators of color SU($N$). $G_{AB}^{\mu \nu}$ is the gauge propagator given by

$$G_{\mu \nu}^{AB}(p) = \left\{ \begin{array}{ll}
\delta_{AB} \delta_{\mu \nu} \frac{1}{p^2} & \text{NDR}, \\
\delta_{AB} \delta_{\mu \nu} \frac{1}{p^2} & \text{DRED},
\end{array} \right.$$
\[
\left[ \Lambda_i^{(1-p)} \right]_{abcd} = -\frac{g^2}{12\pi^2} \sum_q \alpha_i^q \left[ T^A \otimes T^A \right]_{abcd} \gamma_\mu (1 - \gamma_5) \otimes \gamma_\mu \, L_i^{\text{cont}}
\]

\[
\left[ \Lambda_i^{0(1-p)} \right]_{abcd} = -\delta_{XY} \frac{g^2}{12\pi^2} (\alpha_i^s + \alpha_i^d) \left[ T^A \otimes T^A \right]_{abcd} \gamma_\mu (1 - \gamma_5) \otimes \gamma_\mu \, L_i^{\text{cont}}
\]

where

\[
L_i^{\text{cont}} = \frac{1}{\epsilon} + \log(\mu^2/p^2) + \frac{5}{3} + c_i
\]

with

\[
c_i = \begin{cases} 
-1 & Q_2 \text{ and } Q_i^u \text{ in NDR}, \\
0 & Q_i^c \text{ in NDR}, \\
\frac{1}{4} & Q_2, Q_i^c \text{ and } Q_i^d \text{ in DRED}.
\end{cases}
\]

The pole term \(1/\epsilon\) is subtracted from \(L_i^{\text{cont}}\) in the \(\overline{\text{MS}}\) scheme.

From the tree-level vertex functions \(\Lambda_i^{(0)}\) in eqs. (20)–(23) we find

\[
\left[ T^A \otimes T^A \right]_{abcd} \gamma_\mu (1 - \gamma_5) \otimes \gamma_\mu
\]

\[
= \frac{1}{4} \left[ \Lambda_4^{(0)} + \Lambda_6^{(0)} - \frac{1}{N} (\Lambda_3^{(0)} + \Lambda_5^{(0)}) \right]_{abcd}.
\]

Using this relation we finally obtain

\[
\Lambda_i^{(1-p)} = -\frac{g^2}{12\pi^2} \frac{1}{4} C(Q_i) \, L_i^{\text{cont}} \left[ \Lambda_4^{(0)} + \Lambda_6^{(0)} - \frac{1}{N} (\Lambda_3^{(0)} + \Lambda_5^{(0)}) \right]
\]

with \(C(Q_2) = 1, C(Q_3) = 2, C(Q_4) = C(Q_6) = f_q, C(Q_8) = f_u - f_d/2, C(Q_9) = -1, C(Q_{10}) = f_u - f_d/2\) and \(C(Q_i) = 0\) for other \(i\), where \(f_q, f_u\) and \(f_d\) denote the number of flavors, the number of charge 2/3 up-like quarks and the number of charge −1/3 down-like quarks, respectively.

### C. Lattice calculation

Let us turn to the calculation of the vertex functions on the lattice. In this subsection we use the same notations for quantities defined on the lattice as those for their counterparts in the continuum.

We consider the Green function of eq. (18) on the lattice. At the one-loop level the penguin diagram in Fig. [1] contributes to the vertex functions:

\[
\left[ \Lambda_1^{(1-p)} \right]_{abcd} = 0,
\]

\[
\left[ \Lambda_2^{(1-p)} \right]_{abcd} = -g^2 \left[ T^A \otimes T^B \right]_{abcd} (1 - w_0^3) G_{\mu\nu}^{AB}(p) \Gamma_X^\delta L_\mu(p) \Gamma_X^\delta \otimes \gamma_\nu,
\]

\[
\left[ \Lambda_3^{(1-p)} \right]_{abcd} = +g^2 \left[ T^A \otimes T^B \right]_{abcd} (1 - w_0^3) G_{\mu\nu}^{AB}(p) \sum_q \alpha_q^d \text{tr}(\Gamma_Y^\delta L_\mu(p)) \Gamma_X^\delta \otimes \gamma_\nu,
\]

\[
\left[ \Lambda_4^{(1-p)} \right]_{abcd} = -g^2 \left[ T^A \otimes T^B \right]_{abcd} (1 - w_0^3) G_{\mu\nu}^{AB}(p) \left[ \alpha_i^d \Gamma_X^\delta L_\mu(p) \Gamma_Y^\delta + \alpha_i^c \Gamma_Y^\delta L_\mu(p) \Gamma_X^\delta \right] \otimes \gamma_\nu,
\]

(37)
where \((1 - w_0^2)^3\) is the overlap factor to the four-dimensional quark fields which emerges through the truncation of the external quark legs by multiplying \(i \not{\!\!\!p}_i\) \((i = 1, \ldots, 4)\) on the Green function. \(L_\mu\) denotes the integral of the quark loop, which is expressed by

\[
L_\mu(p) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \left\{ P_R S_F(k)_{1s} + P_L S_F(k)_{N_\mu} \right\} (-i)v_\mu(k - p/2)_{st} \\
\times \left\{ S_F(k - p)_{t1} P_L + S_F(k - p)_{tN_\mu} P_R \right\} \\
= \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} I_\mu(k, p),
\]

with

\[
I_\mu(k, p) = -c_\mu(k - p/2) \gamma^\mu - c_\mu(k - p/2) \frac{1}{F' F(1 - e^{-\alpha + \alpha'})} \frac{1}{i \not{\!\!\!p} - \not{\!\!\!k}} \gamma^\mu \gamma^\nu + \frac{e^{-\alpha}}{F' (1 - e^{-\alpha + \alpha'})} \frac{e^{-\alpha'}}{F (1 - e^{-\alpha + \alpha'})} \frac{1}{i \not{\!\!\!p} - \not{\!\!\!k}} \gamma^\mu \gamma^\nu.
\]

where \(c_\mu(k) = \cos k_\mu, s_\mu(k) = \sin k_\mu, s'_\mu = s_\mu(k - p)\) \(\alpha = \alpha(k), \alpha' = \alpha(k - p), F = F(k)\) and \(F' = F(k - p)\). The expressions of \(v_\mu, S_F, \alpha\) and \(F\) are given in Appendix.

Since the function \(L_\mu(p)\) has an infrared divergence for \(p^2 \to 0\), we consider extracting its divergent part by employing an analytically integrable expression \(\tilde{I}_\mu(k, p)\) which has the same infrared behavior as \(I_\mu(k, p)\),

\[
L_\mu(p) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \tilde{I}_\mu(k, p) + \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \left\{ I_\mu(k, p) - \tilde{I}_\mu(k, p) \right\},
\]

where we choose

\[
\tilde{I}_\mu(k, p) = \theta (\Lambda^2 - k^2) (1 - w_0^2) \frac{1}{i \not{\!\!\!p} - \not{\!\!\!k}} \gamma^\mu \frac{1}{i \not{\!\!\!p} - \not{\!\!\!p}_i},
\]

with \(\Lambda (\leq \pi)\) a cut-off. Now the second term in the right hand side of eq.(51) is regular in terms of \(p\), we can make a Taylor expansion,

\[
L_\mu(p) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \tilde{I}_\mu(k, p) + \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \left\{ I_\mu(k, 0) - \tilde{I}_\mu(k, 0) \right\} \\
\quad + \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\partial}{\partial p_\rho} \left\{ I_\mu(k, p) - \tilde{I}_\mu(k, p) \right\} \bigg|_{p=0} \\
\quad + \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{1}{2} \frac{\partial^2}{\partial p_\rho \partial p_\lambda} \left\{ I_\mu(k, p) - \tilde{I}_\mu(k, p) \right\} \bigg|_{p=0},
\]

where \(\rho, \lambda = 1, 2, 3, 4\). Before evaluating each term of the right hand side, let us consider the transformation properties of \(L_\mu\) under various discrete symmetries, from which we can predict the types of terms to be allowed in \(L_\mu\). The same kind of discussion is found in the Wilson quark case \[12\].

Under the operation of charge conjugation matrix \(C = \gamma_4 \gamma_2\) we find
\[ CV_\mu(k)_{st} C^{-1} = -v_\mu^T (-k)_{N_s+1-t, N_s+1-s}, \]
\[ CS_F(k)_{st} C^{-1} = S_F^T (-k)_{N_s+1-t, N_s+1-s}, \]
\[ CL_\mu(p) C^{-1} = -L_\mu^T (p). \]

Parity transformation gives
\[ \gamma_4 v_4(k_4, \bar{k})_{st} \gamma_4 = v_4(k_4, -\bar{k})_{N_s+1-s, N_s+1-t}, \]
\[ \gamma_4 v_i(k_4, \bar{k})_{st} \gamma_4 = -v_i(k_4, -\bar{k})_{N_s+1-s, N_s+1-t}, \]
\[ \gamma_4 S_F(k_4, \bar{k})_{st} \gamma_4 = S_F(k_4, -\bar{k})_{N_s+1-s, N_s+1-t}, \]
\[ \gamma_4 L_4(p_4, \bar{p}) \gamma_4 = L_4(p_4, -\bar{p}), \]
\[ \gamma_4 L_i(p_4, \bar{p}) \gamma_4 = -L_i(p_4, -\bar{p}), \]

where \( i = 1, 2, 3 \). These discrete symmetries restrict the form of \( L_\mu \) as
\[ L_\mu(p) = \frac{1}{a^2} c_0 \gamma_\mu + \frac{1}{a} c_1 i \sigma_{\mu\nu} p_\nu + c_2 a \gamma_\mu p^2 + c_2 b \gamma_\mu \not{p}^2 + O(a), \]

with \( \sigma_{\mu\nu} = [\gamma_\mu, \gamma_\nu]/2 \), where the massless quark case is considered. We can eliminate further terms by the Ward-Takahashi identity:
\[ 2 \sin(p_\mu a/2) v_\mu(k - p/2)_{st} = [S_F(k)^{-1}]_{st} - [S_F(k - p)^{-1}]_{st}, \]
which leads to
\[ 2 \sin(p_\mu a/2) L_\mu(p) = i \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} [P_R \{ S_F(k)_{11} - S_F(k - p)_{11} \} P_L \]
\[ + P_R \{ S_F(k)_{1N_s} - S_F(k - p)_{1N_s} \} P_R \]
\[ + P_L \{ S_F(k)_{N_s1} - S_F(k - p)_{N_s1} \} P_L \]
\[ + P_L \{ S_F(k)_{N_sN_s} - S_F(k - p)_{N_sN_s} \} P_R ] \]
\[ = 0 \]

where the periodicity of \( S_F \) is used. Under the requirement that the terms of the left hand side must vanish order by order in terms of the lattice spacing, we find \( c_0 = c_2c = 0 \) and \( c_2a = -c_2b \) in eq.(61), with which the expression of \( L_\mu \) is simplified as
\[ L_\mu(p) = \frac{1}{a} c_1 i \sigma_{\mu\nu} p_\nu + c_2 a (\gamma_\mu p^2 - \not{p} \gamma_\mu \not{p}) + O(a). \]

Here we should note that
\[ \gamma_5 \sigma_{\mu\nu} \gamma_5 = 0, \]
\[ \text{tr} (\sigma_{\mu\nu} \gamma_5 (1 \pm \gamma_5)) = 0, \]
which means \( \sigma_{\mu\nu} p_\nu \) term in eq.(64) does not contribute to the wave functions (43)-(46).

Now we see that \( L_\mu(0) = 0 \) and linear term in \( p \) is irrelevant in the expansion (52), we focus on the remaining terms. Performing the integration of \( I_\mu(k, p) - I_\mu(k, 0) \) analytically, we obtain
\[ \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \left\{ \Delta \mu(k, p) - \Delta \mu(k, 0) \right\} = (1 - w_0^2) \frac{1}{16\pi^2} \left[ (p^2 \gamma_\mu + p_\mu \Phi) \frac{1}{3} \left( \log(\Lambda^2/p^2) + 5/6 \right) - \frac{p^2}{6} \gamma_\mu \right]. \] (67)

As for the last term in the expansion of eq. (52), some tedious algebra leads to the following expression for the integrand:

\[ p_\rho p_\lambda \frac{\partial^2}{\partial p_\rho \partial p_\lambda} \left\{ I_\mu(p) - \Delta I_\mu(p) \right\} |_{p=0} = p^2 \gamma_\mu \left( R^{\mu \rho \nu} - R^{\mu \nu} + R^{\mu \rho} \right) + p_\mu \Phi \left( 2R^c_{\mu \rho \nu} + R^d_{\mu \rho \nu} + R^d_{\mu \nu \rho} \right) + p_\mu \gamma_\nu \left( R^{\mu \rho \nu} + R^{\mu \rho \nu} - 2R^c_{\mu \rho \nu} + R^d_{\mu \rho} + R^d_{\mu \nu \rho} - R^d_{\mu \nu \rho} - R^d_{\mu \nu \rho} \right) + R^a_{\mu \rho \nu \rho} + R^b_{\mu \rho \nu} + R^c_{\mu \rho \nu}, \] (68)

where \( R^a_{\mu \rho \nu}, R^b_{\mu \rho \nu}, R^c_{\mu \rho \nu} \) and \( R^d_{\mu \rho \nu} \) are given by

\[ R^a_{\mu \rho} = -e^{-2\alpha} \frac{c_\mu}{4(e^\alpha - e^{-\alpha})} + \frac{s_\mu^2 f_{\mu}}{16} + \frac{c_\mu (10s_\mu^2 - s^2)}{4} - \frac{32(1 - w_0^2)\theta(\Lambda^2 - k^2)}{e^\alpha - e^{-\alpha}} \frac{k^4}{(k^2)^4}, \] (69)

\[ R^b_{\mu \rho \nu} = \frac{c_\mu}{(e^\alpha - e^{-\alpha})^3} \left[ \frac{2}{s_\nu^2 f_{\nu}^2(e^\alpha - e^{-\alpha}) - (h_\nu + s_\nu^2 g_{\nu \nu})(e^\alpha - e^{-\alpha})} \right] - \frac{c_\mu s_\nu^2}{F^2(1 - e^{-2\alpha})} + c_\mu (2s_\mu^2 - s^2)(h_\nu + s_\nu^2 g_{\nu \nu}) \frac{32(1 - w_0^2)\theta(\Lambda^2 - k^2)}{e^\alpha - e^{-\alpha}} \frac{k^4}{(k^2)^4}, \] (70)

\[ R^c_{\mu \rho \nu} = e^\alpha \frac{c_\nu s_\mu^2}{2F^2(e^\alpha - e^{-\alpha})} + s_\nu^2 F_{\nu} \frac{s_\mu^2}{F} + 2c_\mu s_\mu^2 G_{\mu \nu} \frac{2}{F} - \frac{c_\mu c_\nu}{2F(e^\alpha - e^{-\alpha})} + \frac{r e^{-2\alpha}}{2F(e^\alpha - e^{-\alpha})} \left[ s_\mu^2 s_\nu^2 G_{\mu \nu} - s_\mu^2 c_\nu F_{\nu} + c_\mu s_\nu^2 F_{\nu}/2 \right] \] \[ + \frac{r e^{-2\alpha}}{2F(e^\alpha - e^{-\alpha})} \left[ 2s_\mu^2 F_{\nu} + s_\nu^2 F_{\nu} \right] \left[ \frac{32(1 - w_0^2)\theta(\Lambda^2 - k^2)}{e^\alpha - e^{-\alpha}} \frac{k^4}{(k^2)^4}, \right. \] (71)

\[ R^d_{\mu \rho \nu} = -2c_\mu s_\mu^2 F_{\nu} \frac{c_\nu}{F} - 4(1 - w_0^2)\theta(\Lambda^2 - k^2) \frac{k^4}{(k^2)^4}. \] (72)
with
\[
\begin{align*}
    f_\mu &= \frac{-rW + c_\mu + r \cosh \alpha}{W \sinh \alpha}, \\
    h_\mu &= \frac{-c_\mu W + c_\mu^2 - s_\mu^2 + r c_\mu \cosh \alpha}{W \sinh \alpha}, \\
    g_{\mu\nu} &= g_{\nu\mu} = -\frac{f_\mu f_\nu}{\tan \alpha} + \frac{r^2}{W \sinh \alpha} + r \frac{f_\mu + f_\nu}{W}, \\
    F_\mu &= e^{2\alpha} \frac{-r + W f_\mu}{F^2(e^\alpha - e^{-\alpha})} - \frac{f_\mu}{F(e^\alpha - e^{-\alpha})^2}, \\
    H_\mu &= e^{2\alpha} \frac{r c_\mu - W h_\mu}{F(e^\alpha - e^{-\alpha})} + \frac{h_\mu}{F(e^\alpha - e^{-\alpha})^2}, \\
    G_{\mu\nu} &= G_{\nu\mu} = \frac{1}{F} \left[ \frac{g_{\mu\nu}}{(e^\alpha - e^{-\alpha})^2} - f_\mu f_\nu \frac{e^\alpha + e^{-\alpha}}{(e^\alpha - e^{-\alpha})^3} \right] + e^\alpha \frac{f_\mu(-r + W f_\nu) + f_\nu(-r + W f_\mu)}{F^2(e^\alpha - e^{-\alpha})^2} \\
    &\quad - \frac{e^\alpha}{F^3(e^\alpha - e^{-\alpha})} \left[ 2e^{2\alpha}(-r + W f_\mu)(-r + W f_\nu) + e^{\alpha} \{ -r(f_\mu + f_\nu) + W f_\mu f_\nu + W g_{\mu\nu} \} \right].
\end{align*}
\]  
(73)
(74)
(75)
(76)
(77)
(78)

We choose \( r = -1 \) in this calculation. Using the results of eqs. (64) and (68), \( L_\mu(p) \) is expressed as
\[
L_\mu(p) = \frac{(1 - w_0^2)}{16\pi^2} \left[ (p_\mu \beta - p^2 \gamma_\mu)^\frac{1}{3} \left( \log(\Lambda^2/p^2) + 5/6 \right) + A p^2 \gamma_\mu + B p_\mu \beta + \Delta R p^2_\mu \gamma_\mu + \delta R \gamma_\nu \gamma_\alpha p_\nu p_\alpha \gamma_\mu \right],
\]  
(79)
where
\[
\begin{align*}
    A &= -\frac{1}{6} + \frac{16\pi^2}{1 - w_0^2} \frac{1}{2} \int_{-\pi}^\pi \frac{d^4k}{(2\pi)^4} \left\{ R^b_{\mu\nu} - R^d_{\mu\nu\nu} \right\}, \\
    B &= \frac{16\pi^2}{1 - w_0^2} \frac{1}{2} \int_{-\pi}^\pi \frac{d^4k}{(2\pi)^4} \left\{ 2 R^c_{\mu\nu} + R^d_{\mu\nu\nu} + R^d_{\mu\nu\nu} \right\}, \\
    \Delta R &= \frac{16\pi^2}{1 - w_0^2} \frac{1}{2} \int_{-\pi}^\pi \frac{d^4k}{(2\pi)^4} \left\{ R^a_{\mu\nu} + R^b_{\mu\nu} - R^c_{\mu\nu} + 2(R^c_{\mu\nu} - R^c_{\mu\nu}) + R^d_{\mu\nu} + R^d_{\mu\nu\nu} - R^d_{\mu\nu\nu} \right\}, \quad \Delta R = \frac{16\pi^2}{1 - w_0^2} \frac{1}{2} \int_{-\pi}^\pi \frac{d^4k}{(2\pi)^4} \left\{ R^d_{\mu\nu\nu} - R^d_{\mu\nu\alpha} \right\} = 0,
\end{align*}
\]  
(80)
(81)
(82)
(83)

with \( \mu \neq \nu \neq \alpha \). We find that \( \delta R = 0 \) from the symmetry of the integrand. The integrals are estimated by a mode sum for a periodic box of a size \( L^4 \) with \( L = 64 \) after transforming the momentum variable through \( p_\mu = q_\mu - \sin q_\mu \). We choose \( \Lambda = \pi \) for the cut-off in the integrand \( \tilde{I}_\mu \). The numerical results for \( A \) are \( B \) are presented in Table I as a function of \( M \). We observe that the expected relation \( A = -B \) is well satisfied. We also checked that \( \Delta R \) is consistent with zero as expected. Finally we obtain
\[
L_\mu(p) = \frac{(1 - w_0^2)}{16\pi^2} \frac{L^{\text{latt}}}{3} \left[ p_\mu \beta - p^2 \gamma_\mu \right],
\]  
(84)
where
\[ L^{\text{latt}} = \log(\pi^2/p^2) + \frac{5}{6} + 3B. \]  

(85)

Substituting the above expression for \( L_\mu(p) \) in eqs. (13)–(14), we have

\[ [\Lambda_i^{(1-p)}]_{abcd} = 0, \]  

(86)

\[ [\Lambda_2^{(1-p)}]_{abcd} = -\frac{g^2}{12\pi^2} (1 - w_0^2)^4 L^{\text{latt}} T^A \tilde{T}^A \gamma_\mu (1 - \gamma_5) \otimes \gamma_\mu, \]  

(87)

\[ [\Lambda_i^{\epsilon(1-p)}]_{abcd} = -\frac{g^2}{12\pi^2} (1 - w_0^2)^4 \sum_q \alpha_q L^{\text{latt}} T^A \tilde{T}^A \gamma_\mu (1 - \gamma_5) \otimes \gamma_\mu, \]  

(88)

\[ [\Lambda_i^{\delta(1-p)}]_{abcd} = -\delta_{XY} \frac{g^2}{12\pi^2} (1 - w_0^2)^4 (\alpha_i^s + \alpha_i^d) L^{\text{latt}} T^A \tilde{T}^A \gamma_\mu (1 - \gamma_5) \otimes \gamma_\mu, \]  

(89)

where we used the on-shell conditions for the external quark states to eliminate the terms proportional to \( p / \). In the same way to derive eq.(42) in the continuum calculation, the lattice wave functions are written in the compact form as follows,

\[ \Lambda_i^{(1-p)} = -\frac{g^2}{12\pi^2} \frac{1}{4} C(Q_i) L^{\text{latt}} \left[ \Lambda_i^{(0)} + \Lambda_i^{(0)} - \frac{1}{N} (\Lambda_i^{(0)} + \Lambda_i^{(0)}) \right], \]  

(90)

where \( C(Q_i) \) are already obtained in the previous subsection.

Comparing the continuum vertex functions in eq. (42) and lattice ones in eq. (90), we find that the penguin diagram contributions to the renormalization factor in eq.(1) are written as

\[ Z_i^{\text{pen}} Q_i^{\text{latt}} = \frac{1}{(1 - w_0^2)^2} Z_i^{\text{pen}} [Q_4 + Q_6 - \frac{1}{N} (Q_3 + Q_5)]^{\text{latt}}, \]  

(91)

where the penguin operator and its coefficient are given by

\[ Q_i^{\text{pen}} = \left[ Q_4 + Q_6 - \frac{1}{N} (Q_3 + Q_5) \right]^\text{latt} \]  

(92)

\[ Z_i^{\text{pen}} = \frac{g^2}{12\pi^2} \frac{C(Q_i)}{4} (L^{\text{latt}} - L^{\text{cont}}) = \frac{g^2}{16\pi^2} \frac{C(Q_i)}{3} [-\log(\mu a)^2 + z_i^{\text{pen}}] \]  

(93)

with \( z_i^{\text{pen}} = \log(\pi^2) + 3B - \frac{5}{6} - c_i \). Numerical values of \( z_i^{\text{pen}} \) with the DRED scheme are presented in Table I as a function of \( M \).

III. FULL RENORMALIZATION FACTORS

In order to write down the complete expressions for the renormalization factors of the \( \Delta S = 1 \) operators \( Q_i \), we still need to know the contributions from the gluon exchange diagrams, which is denoted by \( Z_i^g Q_i^{\text{latt}} \) in eq. (11). Fortunately, they can be obtained from the combinations of the results in our previous paper [3], where we calculated the gluon exchange diagrams for the following four-quark operators,
\[ \mathcal{O}_\pm(q_1, q_2, q_3, q_4) = \frac{1}{8} \left[ (q_1 q_2)v_{-A}(q_3 q_4)v_{-A} \pm (\bar{q}_1 q_4)v_{-A}(\bar{q}_3 q_2)v_{-A} \right], \tag{94} \]
\[ \mathcal{O}_1(q_1, q_2, q_3, q_4) = \frac{1}{4} \left[ -C_F(q_1 q_2)v_{-A}(q_3 q_4)v_{+A} + (q_1 T^A q_2)v_{-A}(q_3 T^A q_4)v_{+A} \right], \tag{95} \]
\[ \mathcal{O}_2(q_1, q_2, q_3, q_4) = \frac{1}{4} \left[ \frac{1}{2N}(q_1 q_2)v_{-A}(q_3 q_4)v_{+A} + (q_1 T^A q_2)v_{-A}(q_3 T^A q_4)v_{+A} \right], \tag{96} \]

with \( C_F = (N^2 - 1)/(2N) \) the second Casimir of \( SU(N) \) group. The tensor structures in the color space are \( 1 \otimes 1 \) and \( T^A \otimes T^A \).

The operators \( Q_i \) are related to \( \mathcal{O}_{\pm,1,2} \) as

\[ Q_2 \pm Q_1 = 8 \mathcal{O}_\pm(s, u, u, d), \tag{97} \]
\[ Q_3 \pm Q_4 = 8 \sum_q \mathcal{O}_\pm(s, d, q, q), \tag{98} \]
\[ Q_9 \pm Q_{10} = 8 \sum_q \frac{3e^q}{2} \mathcal{O}_\pm(s, d, q, q), \tag{99} \]
\[ Q_6 = 8 \sum_q \mathcal{O}_2(s, d, q, q), \tag{100} \]
\[ Q_8 = 8 \sum_q \frac{3e^q}{2} \mathcal{O}_2(s, d, q, q), \tag{101} \]
\[ -NQ_5 + Q_6 = 8 \sum_q \mathcal{O}_1(s, d, q, q), \tag{102} \]
\[ -NQ_7 + Q_8 = 8 \sum_q \frac{3e^q}{2} \mathcal{O}_1(s, d, q, q). \tag{103} \]

From these relations the renormalized operators, which include the contributions of both the gluon exchange and penguin diagrams, are expressed in terms of the lattice operators as follows:

\[ Q_i^{cont} = \frac{1}{(1 - w^2_0)^2 Z_w^2} \left[ Z_j^g Q_j^{latt} + Z_i^{pen} Q_i^{latt} \right] \tag{104} \]

where

\[ Z_i^g = \begin{cases} 1 + \frac{g^2}{16\pi^2} \left[ \frac{3}{N} \log(\mu a)^2 + \frac{z_+ + z_-}{2} \right] & i = 1, 2, 3, 4, 9, 10, \\ 1 + \frac{g^2}{16\pi^2} \left[ -\frac{3}{N} \log(\mu a)^2 + z_1 - v_{21} \right] & i = 5, 7, \\ 1 + \frac{g^2}{16\pi^2} \left[ \frac{3(N^2 - 1)}{N} \log(\mu a)^2 + z_2 + v_{21} \right] & i = 6, 8, \\ \end{cases} \tag{105} \]

\[ Z_j^g = \begin{cases} = Z_{ji}^g = \frac{g^2}{16\pi^2} \left[ -3 \log(\mu a)^2 + \frac{z_+ - z_-}{2} \right] & (i, j) = (1, 2), (3, 4), (9, 10), \\ \frac{g^2}{16\pi^2} \left[ 3 \log(\mu a)^2 + \frac{z_2 - z_1 + v_{21} - v_{12}}{N} \right] & (i, j) = (5, 6), (7, 8), \\ -\frac{g^2}{16\pi^2} Nv_{21} & (i, j) = (6, 5), (8, 7), \\ 0 & \text{others} \end{cases} \tag{106} \]
with $z_{\pm,1,2}$, $v_{12}$, and $v_{21}$ presented in our previous paper [6]. The penguin operator $Q_{\text{latt}}^\text{pen}$ and its coefficient $Z_{\text{pen}}^i$ are given in eqs. (\ref{eq:Q_latt_pen}) and (\ref{eq:Z_pen}), respectively. For the sake of convenience, we list the differences between the NDR and DRED schemes in the finite part of the renormalization constant,

$$
\left(\frac{z_+ + z_-}{2}\right)_{\text{NDR}} = \left(\frac{z_+ + z_-}{2}\right)_{\text{DRED}} - \frac{N^2 - 6}{2N},
$$

(107)

$$
(z_1 - v_{21})_{\text{NDR}} = (z_1 - v_{21})_{\text{DRED}} - \frac{N^2 - 8}{2N},
$$

(108)

$$
(z_2 + v_{21})_{\text{NDR}} = (z_2 + v_{21})_{\text{DRED}} - \frac{N^2 - 4}{N},
$$

(109)

$$
\left(\frac{z_+ - z_-}{2}\right)_{\text{NDR}} = \left(\frac{z_+ - z_-}{2}\right)_{\text{DRED}} - \frac{5}{2},
$$

(110)

$$
\left(\frac{z_2 - z_1 + v_{21} - v_{12}}{N}\right)_{\text{NDR}} = \left(\frac{z_2 - z_1 + v_{21} - v_{12}}{N}\right)_{\text{DRED}} - \frac{7}{2},
$$

(111)

$$
-N(v_{21})_{\text{NDR}} = -N(v_{21})_{\text{DRED}} - 3,
$$

(112)

$$
(z_{\text{pen}}^i)_{\text{NDR}} = (z_{\text{pen}}^i)_{\text{DRED}} + \begin{cases} 
5 \\
4 \\
1 \\
i = 2, 3, 5, 7, 9, \\
i = 4, 6, 8, 10,
\end{cases}
$$

(113)

where $v_{12} = v_{21} = 0$ in the DRED scheme.

We are interested in the magnitude of the renormalization factors with the currently accessible coupling constant in numerical simulations. Let us estimate it taking $\beta = 6.0$ with $M = 1.8$ as a representative case. All the necessary ingredients in this analysis are given in our previous papers [5,6] and Table I of this report. With the use of the mean field improvement [13], we have

$$
(1 - w_0(\tilde{M})^2)Z_{\text{w}}^{\text{MF}}(\tilde{M}) = 0.9085,
$$

(114)

$$
\frac{z_+^{\text{MF}}(\tilde{M}) + z_-^{\text{MF}}(\tilde{M})}{2} = -12.848,
$$

(115)

$$
\frac{z_+^{\text{MF}}(\tilde{M}) - z_-^{\text{MF}}(\tilde{M})}{2} = 2.491,
$$

(116)

$$
z_1^{\text{MF}}(\tilde{M}) = -11.187,
$$

(117)

$$
z_2^{\text{MF}}(\tilde{M}) = -18.6,
$$

(118)

$$
z_2^{\text{MF}}(\tilde{M}) - z_1^{\text{MF}}(\tilde{M}) = -7.4,
$$

(119)

$$
z_{\text{pen}}^i(\tilde{M}) = 2.328,
$$

(120)

in the DRED scheme, where we employ

$$
g_{\text{MS}}^\beta(1/a) = \left[P_{\beta} - 0.13486\right]^{-1} = 2.1792,
$$

(121)

$$
\tilde{M} = M + 4(u - 1) = 1.3112,
$$

(122)

with $P = u^4 = 0.59374$ the plaquette value at $\beta = 6.0$ in the quenched approximation. Combining these results we obtain
\[
\left( \frac{Z_{ij}^g}{(1 - w_H^2)^2 Z_w^2} \right)^{\text{MF}} = \begin{cases} 
  u^2 \times 0.9968 & i = 1, 2, 3, 4, 9, 10, \\
  u^2 \times 1.0245 & i = 5, 7, \\
  u^2 \times 0.9006 & i = 6, 8,
\end{cases} 
\]

(123)

\[
\left( \frac{Z_{ij}^g}{(1 - w_H^2)^2 Z_w^2} \right)^{\text{MF}} = \begin{cases} 
  -0.0412 & (i, j) = (5, 6), (7, 8), \\
  0 & \text{others}
\end{cases} 
\]

(124)

\[
\left( \frac{Z_{i}^{\text{pen}}}{(1 - w_H^2)^2 Z_w^2} \right)^{\text{MF}} = C(Q_i) \times 0.0130,
\]

(125)

at \( \mu a = 1 \) in the DRED scheme, where we factor out the tadpole contributions. We find that the penguin diagram contributions to the renormalization factor are quite small.

IV. CONCLUSION

In this paper we have presented the full expressions for the renormalization factors of the \( \Delta S = 1 \) four-quark operators up to the one-loop level including the contributions of both the gluon exchange and penguin diagrams. Our calculation, however, does not include mixing with lower dimensional operators \( \bar{s}d \) and \( s \gamma_5 d \). The coefficients of these operators diverge as inverse powers of the lattice spacing, due to which it is practically inappropriate to subtract these lower dimensional operators by the perturbation theory. We are instead forced to use the non-perturbative methods such as those based on the chiral perturbation theory [14] and the Schrödinger functional scheme [15]. We leave it to future work.

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APPENDIX

In this appendix we explain the domain-wall fermion action and its Feynman rules relevant for the present calculation. The domain-wall fermion action is written as [11],

\[
S_{\text{DW}} = \sum_n \sum_{s=1}^{N_s} \left[ \frac{1}{2} \sum_{\mu} (\bar{\psi}(n)_s (-r + \gamma_\mu) U_{\mu}(n) \psi(n + \hat{\mu})_s + \bar{\psi}(n)_s (-r - \gamma_\mu) U^{\dagger}_{\mu}(n - \hat{\mu}) \psi(n - \hat{\mu})_s) \\
+ \frac{1}{2} \left( \bar{\psi}(n)_s (1 + \gamma_5) \psi(n)_{s+1} + \bar{\psi}(n)_s (1 - \gamma_5) \psi(n)_{s-1} \right) + (M - 1 + 4r) \bar{\psi}(n)_s \psi(n)_s \right] \\
+ m \sum_n \left( \bar{\psi}(n)_{N_s} P_R \psi(n)_1 + \bar{\psi}(n)_1 P_L \psi(n)_{N_s} \right),
\]

(126)

where \( U_{\mu} \) is the link variable of the SU(\( N \)) gauge group and the Wilson parameter is set to \( r = -1 \). Four dimensional space-time coordinate is labeled by \( n \) and \( s \) is an extra fifth
dimensional index which runs from 1 to \(N_s\). Since we impose no gauge interaction along the fifth dimension, it is also possible to consider that the index \(s\) labels the \(N_s\) “flavor” space. In our one-loop calculation we take \(N_s \to \infty\) to avoid complications arising from the finite \(N_s\) such as mixing among the four-quark operators with different chiralities. The parameter \(m\) denotes the physical quark mass and at \(m = 0\) one chiral zero mode is supposed to appear under the condition \(0 < M < 2\) for the Dirac “mass” \(M\). \(P_{R/L}\) are projection operators defined by \(P_{R/L} = (1 \pm \gamma_5)/2\). For the gauge part we employ a standard four dimensional Wilson plaquette action.

The quark fields on the four-dimensional space-time are given by the combinations of the fermion fields at the boundaries,

\[
q(n) = P_R \psi(n)_1 + P_L \psi(n)_{N_s},
\]

\[
\overline{q}(n) = \overline{\psi}(n)_{N_s} P_R + \overline{\psi}(n)_1 P_L.
\]  

We consider the composite operators constructed with these “physical” quark fields, and our renormalization procedure is based on the Green functions consisting of only these quark fields.

In order to obtain the Feynman rules we perform the weak coupling expansion of the quark and gauge actions. The fermion propagator with momentum \(p\) is obtained by inverting the domain-wall Dirac operator in eq.(126), which is expressed by \(S_F(p)_{st}\) as an \(N_s \times N_s\) matrix in the “flavor” space. In the present one-loop calculation, however, we do not need the whole matrix elements because we consider the Green functions consisting of the physical quark fields. The relevant fermion propagators are restricted to following three types:

\[
\bigg\langle q(-p) \overline{q}(p) \bigg\rangle = P_R S_F(p)_{11} P_L + P_L S_F(p)_{N_s N_s} P_R + P_L S_F(p)_{N_s 1} P_L + P_R S_F(p)_{1 N_s} P_R
\]

\[
= \frac{-i \gamma_\mu \sin p_\mu + (1 - W e^{-\alpha}) m}{- (1 - e^\alpha W) + m^2 (1 - W e^{-\alpha})},
\]

\[
\bigg\langle q(-p) \overline{\psi}_s(p) \bigg\rangle = P_R S_F(p)_{1s} + P_L S_F(p)_{N_s s}
\]

\[
= \frac{1}{F} \left( i \gamma_\mu \sin p_\mu - m \left(1 - W e^{-\alpha}\right) \right) \left( e^{-\alpha(N_s - s)} P_R + e^{-\alpha(s - 1)} P_L \right)
\]

\[
+ \frac{1}{F} \left[ m \left( i \gamma_\mu \sin p_\mu - m \left(1 - W e^{-\alpha}\right) \right) - F \right] e^{-\alpha} \left( e^{-\alpha(s - 1)} P_R + e^{-\alpha(N_s - s)} P_L \right),
\]

\[
\bigg\langle \overline{\psi}_s(-p) \overline{q}(p) \bigg\rangle = S_F(p)_{s1} P_L + S_F(p)_{sN_s} P_R
\]

\[
= \frac{1}{F} \left( e^{-\alpha(N_s - s)} P_L + e^{-\alpha(s - 1)} P_R \right) \left( i \gamma_\mu \sin p_\mu - m \left(1 - W e^{-\alpha}\right) \right)
\]

\[
+ \frac{1}{F} \left( e^{-\alpha(s - 1)} P_L + e^{-\alpha(N_s - s)} P_R \right) e^{-\alpha} \left[ m \left( i \gamma_\mu \sin p_\mu - m \left(1 - W e^{-\alpha}\right) \right) - F \right].
\]

with

\[
W = 1 - M - r \sum_\mu (1 - \cos p_\mu),
\]

\[
\cosh(\alpha) = \frac{1 + W^2 + \sum_\mu \sin^2 p_\mu}{2 |W|},
\]

\[
F = 1 - e^\alpha W - m^2 \left(1 - W e^{-\alpha}\right),
\]

15
where the argument $p$ in the factors $\alpha$ and $W$ is suppressed.

In the perturbative calculation of Green functions we assume that the external quark momenta and masses are much smaller than the lattice cut-off. In this case the external quark propagators can be expanded in terms of them. We have the following expressions as the leading term of the expansion:

$$\langle \bar{q}q \rangle(p) = \frac{1 - w_0^2}{ip + (1 - w_0^2)m},$$

$$\langle \bar{q}\bar{s} \rangle(p) = \langle \bar{q}q \rangle(p) \left(w_0^{s-1}P_L + w_0^{N-s}P_R\right),$$

$$\langle \bar{s}q \rangle(p) = \left(w_0^{s-1}P_R + w_0^{N-s}P_L\right) \langle \bar{q}q \rangle(p),$$

where $w_0 = 1 - M$. The form of $\langle \bar{q}q \rangle(p)$ tells us that $\sqrt{1 - w_0^2}$ is the overlap factor to the four-dimensional quark fields at the tree-level.

The one-gluon fermion vertex with outgoing and incoming fermion momenta $p$ and $q$, respectively, is given by

$$V_{1\mu}^A(q,p)_{st} = g T^A v_{\mu} \left(\frac{q_{\mu} + p_{\mu}}{2}\right)_{st} = g T^A \{i\gamma_{\mu} \cos \left(\frac{q_{\mu} + p_{\mu}}{2}\right) + r \sin \left(\frac{q_{\mu} + p_{\mu}}{2}\right)\} \delta_{st},$$

where $T^A$ ($A = 1, \ldots, N^2 - 1$) are generators of color SU($N$). We do not need two-gluon fermion vertex in the present calculation.

The gluon propagator for a gluon of momentum $p$ is written as

$$G^{AB}_{\mu\nu}(p) = \frac{\delta_{AB}}{4 \sin^2(p/2)} \left[\delta_{\mu\nu} - (1 - \alpha) \frac{4 \sin(p_{\mu}/2) \sin(p_{\nu}/2)}{4 \sin^2(p/2)}\right],$$

where $\sin^2(p/2) = \sum_{\mu} \sin^2(p_{\mu}/2)$ and we choose the Feynman gauge ($\alpha = 1$) in our calculation.
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TABLE I. Numerical values for $A$, $B$ and $z_i^{\text{pen}}$ (DRED) as a function of $M$. Note that $z_i^{\text{pen}}$ is independent of $i$ ($i = 1, \ldots, 10$) in the DRED scheme.

| $M$  | $A$        | $B$        | $z_i^{\text{pen}}$ (DRED) |
|------|------------|------------|-----------------------------|
| 0.05 | -1.5078(4) | 1.5077(3)  | 5.729(1)                    |
| 0.10 | -0.1130(4) | 0.1128(3)  | 1.544(1)                    |
| 0.15 | 0.2836(4)  | -0.2838(3) | 0.355(1)                    |
| 0.20 | 0.4456(4)  | -0.4458(3) | -0.131(1)                   |
| 0.25 | 0.5193(4)  | -0.5195(3) | -0.352(1)                   |
| 0.30 | 0.5513(4)  | -0.5515(3) | -0.448(1)                   |
| 0.35 | 0.5607(4)  | -0.5609(3) | -0.476(1)                   |
| 0.40 | 0.5565(4)  | -0.5567(3) | -0.464(1)                   |
| 0.45 | 0.5434(4)  | -0.5436(3) | -0.425(1)                   |
| 0.50 | 0.5242(4)  | -0.5244(3) | -0.367(1)                   |
| 0.55 | 0.5003(4)  | -0.5006(3) | -0.296(1)                   |
| 0.60 | 0.4728(4)  | -0.4729(3) | -0.213(1)                   |
| 0.65 | 0.4419(4)  | -0.4421(3) | -0.120(1)                   |
| 0.70 | 0.4081(4)  | -0.4082(3) | -0.019(1)                   |
| 0.75 | 0.3712(4)  | -0.3714(3) | 0.092(1)                    |
| 0.80 | 0.3314(4)  | -0.3316(4) | 0.211(1)                    |
| 0.85 | 0.2884(4)  | -0.2885(3) | 0.341(1)                    |
| 0.90 | 0.2417(4)  | -0.2419(3) | 0.480(1)                    |
| 0.95 | 0.1911(4)  | -0.1913(3) | 0.632(1)                    |
| 1.00 | 0.1360(4)  | -0.1362(4) | 0.797(1)                    |
| 1.05 | 0.0760(11) | -0.0754(4) | 0.980(1)                    |
| 1.10 | 0.00879(7) | -0.00944(65)| 1.178(2)                   |
| 1.15 | -0.0648(3) | 0.0647(4)  | 1.400(1)                    |
| 1.20 | -0.1476(3) | 0.1472(4)  | 1.648(1)                    |
| 1.25 | -0.2407(4) | 0.2405(3)  | 1.928(1)                    |
| 1.30 | -0.3470(4) | 0.3468(4)  | 2.246(1)                    |
| 1.35 | -0.4695(4) | 0.4693(4)  | 2.614(1)                    |
| 1.40 | -0.6125(4) | 0.6122(3)  | 3.043(1)                    |
| 1.45 | -0.7818(4) | 0.7816(4)  | 3.551(1)                    |
| 1.50 | -0.9859(4) | 0.9857(3)  | 4.163(1)                    |
| 1.55 | -1.2369(4) | 1.2367(3)  | 4.916(1)                    |
| 1.60 | -1.5531(4) | 1.5529(3)  | 5.865(1)                    |
| 1.65 | -1.9638(4) | 1.9636(3)  | 7.097(1)                    |
| 1.70 | -2.5179(4) | 2.5177(3)  | 8.759(1)                    |
| 1.75 | -3.3049(4) | 3.3047(4)  | 11.120(1)                   |
| 1.80 | -4.5053(4) | 4.5051(3)  | 14.721(1)                   |
| 1.85 | -6.5451(3) | 6.5449(3)  | 20.841(1)                   |
| 1.90 | -10.7173(1)| 10.717(3)  | 33.357(1)                   |
| 1.95 | -23.549(7) | 23.549(7)  | 71.853(22)                  |
FIG. 1. The penguin diagram. $s, d$ and $q$ label quark flavors. $p_1, p_2, p_3, p_4, p$ and $k$ are momenta. Solid circle denotes the four-quark operator.