Why clothes don’t fall apart: tension transmission in staple yarns

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The problem of how staple yarns transmit tension is addressed within abstract models in which the Amontons-Coulomb friction laws yield a linear programming (LP) problem for the tensions in the fiber elements. We find there is a percolation transition such that above the percolation threshold the transmitted tension is in principle unbounded. We determine that the mean slack in the LP constraints is a suitable order parameter to characterize this supercritical state. We argue the mechanism is generic, and in practical terms corresponds to a switch from a ductile to a brittle failure mode accompanied by a significant increase in mechanical strength.

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In his celebrated Dialogues Concerning Two New Sciences, Galileo identified a fascinating puzzle in the mechanics of ropes [1]. His fictitious discussant Salviati asks: “How are fibers, each not more than two or three cubits in length, so tightly bound together in the case of a rope one hundred cubits long that great force is required to break it?” Galileo’s answer to this is to assert that “the very act of twisting causes the threads to bind one another in such a way that when the rope is stretched . . . the fibers break rather than separate from each other.” From a modern perspective, we would say the mechanical integrity of ropes derives from frictional contacts between fibers, and Galileo’s rope problem but one exemplar of a host of related frictional phenomena in fiber assemblies, of which perhaps the canonical case is the ‘staple’ yarn [2–4]. Spun from fibers only 2–3 cm long [5], such a yarn is nevertheless patently capable of transmitting tension over indefinite distances. Accompanying these seemingly innocuous puzzles is an even more existential question: why don’t clothes fall apart? After all, like Galileo’s rope and the staple yarn, woven fabrics and sewn garments are only held together by friction.

A typical yarn (Fig. 1) is ∼100 fibers in cross section, and there are likely several frictional contacts per pitch length (∼100µm), per fiber, hence we estimate > 50 contacts per fiber, and an overall contact density 10^3–10^4 cm^-1. Clearly, the problem of tension transmission in such a structure is a problem in statistical physics. Here we introduce and explore a class of abstract yarn models which isolate the key frictional ingredients of such a problem. Our analysis supports the idea that given sufficient friction and contact points, a random fiber assembly can in principle transmit an indefinitely large tension, by means of a collective friction locking mechanism that resembles a percolation transition.

The underlying premise is that normal forces acting between pairs of fibers facilitate tension transfer between fibers. The Amontons-Coulomb friction laws [6] then imply there is an upper bound on the tension ∆T that can be transferred before slip occurs. Away from the fiber ends, the fibers are in a tension-dominated regime even under modest loads [7], hence this tension transfer ‘cap’ can be expressed as |∆T| ≤ λTm, where Tm is the mean tension in the notionally over-wrapped fiber [8] and λ is what we term a tension transfer coefficient. In the spirit of the approach we shall take the transfer coefficients from a random distribution to reflect the quenched disorder rather than attempting to solve the ‘inner’ elastic problem [9] for each pair of fibers. The key insight is that if ⟨λ⟩ is large enough, this mechanism ‘bootstraps’ a percolation transition for tension transmission.

We accommodate the remnant yet singular effect of bending stiffness in a lower bound to the tension in the

FIG. 1. Gütermann cotton sewing thread. The composite 3-ply structure prevents untwisting under load. One ply (yarn) has been artificially tinted to emphasize the structure. Note the halo of stray fiber ends. Main image: SEM (Hitachi S-3400N); inset: flatbed scanner (Canon LiDE 220).
fiber ends, which we estimate as $T^* \lesssim 1$ mN \[10\]. This is illustrated in Fig. 3a, where the tension in each section of fiber between frictional contacts is shown building from zero at the free end. What $T^*$ means in practice is that we expect the percolation transition to correspond to a switch from a ‘ductile’ failure mode where the yarn fails by fiber slippage, at around $T^*$ per fiber, cf. \[11\], to a ‘brittle’ failure mode where the failure mechanism is fiber breakage, at $T^* \approx 20–130$ mN per fiber \[12\]. As we shall argue, twisting fibers together (à la Galileo) pushes the assembly over the percolation threshold, resulting in perhaps a hundred-fold increase in the tensile strength. Note that the scale separation between $T^*$ and $T_\parallel$ means there is a significant loading regime, of practical relevance, where tension can only be carried by the percolation mechanism identified in the present work.

Given the transfer coefficients, the problem of computing the set of tensions $T_i$ in the fiber elements translates into a system of linear inequalities which can be solved by techniques imported from linear programming (LP). From this perspective, the question of whether the yarn transmits an arbitrarily large tension becomes a linear satisfiability problem. In this form it is fairly easy to show that $T^*$ is ‘irrelevant’, in the language of renormalisation group theory \[13\], and as such we can carry out all our calculations setting $T^* = 0$ \[14\]. Our approach shares elements with Bayman’s ‘theory of hitches’ \[15\], although in our model a yarn is more akin to a random continuous splice, comprised of many short fibers, rather than single-rope hitches.

To explain the above we introduce a ‘toy’ model of an actual splice, shown in Fig. 2. Suppose that the tensions in the various elements are as in Fig. 2b, and the transfer coefficients are $\lambda_1$ and $\lambda_2$. Then,

$$|T_1 - T_0| \leq \frac{1}{2}\lambda_1(T_0 + T_1), \quad T_0 = T_1 + T_2, \quad (1a)$$

$$|T_2 - T_3| \leq \frac{1}{2}\lambda_2(T_2 + T_3), \quad T_1 + T_2 = T_3, \quad (1b)$$

where the inequalities are the tension transfer caps, and the equalities are force balance constraints. As mentioned, we simplify by assuming tension-free fiber ends, and in this particular case make a judicious choice for the over-wrapping direction (otherwise, the splice would unravel). We define the LP objective function $z = \sum T_i$, and determine the percolation threshold by requiring $z > 0$. This, together with Eqs. \[1\] and the constraints $T_i \geq 0$, specifies the LP problem.

This case can be solved by hand. Defining $x = T_1/T_0$ and $1 - x = T_2/T_0$, with $0 \leq x \leq 1$, the caps yield $x \geq (1 - \frac{1}{2}\lambda_1)/(1 + \frac{1}{2}\lambda_1)$ and $x \leq \lambda_2/(1 + \frac{1}{2}\lambda_2)$. A solution thus requires $(1 - \frac{1}{2}\lambda_1)/(1 + \frac{1}{2}\lambda_1) \leq \lambda_2/(1 + \frac{1}{2}\lambda_2)$, or $\lambda_1 + \lambda_2 + \frac{3}{2}\lambda_1\lambda_2 \geq 2$. If this inequality is satisfied, one is in a ‘locked’ state where there are unbounded solutions with $z \to \infty$. Intuitively (Fig. 3b), such solutions exist in the high-friction region. Determining the value of $x$ (i.e. the individual tensions) in the supercritical locked state is complex; it may depend on the history of loading, forces beyond static friction, or the frictional contacts may adapt to the load, altering the transfer coefficients.
The centerpiece of the theory is the bi-dimensional function $\langle \psi \rangle$, which in this case is an order parameter. In these terms $\langle \lambda \rangle$ is the mean tension transferred undershoots the friction limit [18]. The system-wide mean slack $\langle S \rangle$ is an order parameter. In the supercritical state there is generally not a unique set of tensions (cf. selection of $x$ in the splice toy model), rather there is a feasible solution space, which in this case is an open polytope: a convex, high-dimensional cone in the positive hyper-quadrant of the space of fiber element tensions. To compute $\langle S \rangle$, we select a random edge of the solution cone, and average over such edges. The results (Fig. 3) support the notion of a second-order phase transition in the limit of long fibers [19], although there are significant finite-size effects.

To understand the nature of the percolation transition we now develop a mean field theory for the tension $T(s)$ in a fiber as a continuous function of fractional arc length $s$. We assume $N \gg 1$ and correspondingly the tension transfer coefficient $\lambda \ll 1$. The centerpiece of the theory is the bi-dimensional function $\psi(s,s')$ which gives the actual tension transfer between fibers in contact at $s$ and $s'$, i.e., as $\Delta T = \psi \lambda T_m$ with $|\psi| \leq 1$. In these terms $T(s)$ satisfies the integro-differential equation

$$
\frac{dT}{ds} = \Lambda \int_0^1 ds' \left[ \psi(s,s') T(s') - \psi(s',s) T(s) \right],
$$

where $\Lambda = \frac{1}{2} N \lambda$, noting that on average there are $\frac{1}{2} N$ frictional contacts of each type per fiber. We additionally require $T(0) = T(1) = 0$. The mean slack is given by
Load bearing is enhanced by transferring tension to the fiber with longer to go, so for \( s < s' \) we transfer from \( s' \) to \( s \), and vice versa, and at criticality we must maximize this opportunity. Thus as an ansatz we choose \( \psi(s, s') = \text{sgn}(s' - s) \) (and concomitantly, \( \langle S \rangle = 0 \)). Eq. (2) becomes \( dT/ds = \Lambda_c \int_0^s ds' \text{sgn}(s' - s) [T(s) + T(s')] \).

The resulting Sturm-Liouville-like problem can be solved, with normalized solution \( T(s) = 2x_0^2[1 - 2xF(x)] \) where \( x = x_0(2x - 1), F(x) = \int_0^x dy \exp(y^2 - x^2) \) (Dawson’s integral [20]), and \( x_0 \approx 0.924 \) solves \( 2x_0F(x_0) = 1 \). The critical value \( \Lambda_c = 4x_0^2 \approx 3.416. \) For the tension profiles, Fig. 4 precise agreement with the numerical results is observed; we speculate the theory becomes exact in the limit of long fibers. The critical value yields \( N\Lambda_c \approx 6.83, \) in good agreement with Fig. 3, for long fibers.

Turning now to the supercritical behaviour, states with slack are under-determined by the friction constraints alone. The challenge is to determine the fractional tension transfer \( \psi(s, s') \) in Eq. (2) within the friction constraint that \( |\psi| \leq 1 \). The results are dependent on the choice of physics in the supercritical state. Here we sketch the main results [21]. For example, maximizing the slack selects \( \psi(s, s') = 0 \) in a diagonal band \( |s' - s| < w \) whilst retaining the critical form \( \psi(s, s') = \text{sgn}(s' - s) \) outside this. Treating the band as a perturbation \( (w \ll 1) \) recovers the critical Sturm-Liouville-like problem with \( \Lambda_c \) replaced by \( \Lambda/(1 + \Lambda_w^2) \).

As a result the latter expression must match \( \Lambda_c \), leading to \( 1/\Lambda + w^2 = 1/\Lambda_c \). For this form of \( \psi \) we readily find

\[
N\langle S \rangle = 2w\Lambda\langle T \rangle = 2\langle T \rangle \sqrt{\Lambda/\Lambda_c}(\Lambda - \Lambda_c). \tag{3}
\]

To test this we used linear programming to solve the max-slack condition for a single, long fiber transferring tension to itself. The result (Fig. 5) shows good agreement with the theoretical curve in Eq. (3) over six decades [22].

A more physical model is to presume that as we load the sample, contact points displace affinely where they can within a ‘core’ region \( |s - \frac{1}{2}| < w \) and otherwise slide under locally critical conditions (a ‘stretch-and-slip’ model [4b]). This means that all contacts associated with the ‘tails’ \( i.e., |s - \frac{1}{2}| \geq w \) are at their sliding condition with \( \psi(s, s') = \text{sgn}(s' - s) \), including their contacts with points \( s' \) in the core. For points \( s \) in the core we have affine deformation so anticipate uniform strain leading to uniform tension. This turns out to be exactly compatible with Eq. (2) choosing \( \psi(s, s') = 0 \) when both \( s \) and \( s' \) are in the core, and noting that the contributions to \( dT/\text{ds} \) in the core from left and right tails cancel each other out. This leads to a second curve shown in Fig. 4.

Yet another possible scenario is to postulate that all supercritical states within the allowed solution cone are equally likely (a ‘max-entropy’ model), akin to the microcanonical ensemble in statistical mechanics, or the Edwards’ conjecture in granular packings [23]. A numerical investigation of this case is also shown in Fig. 5.

These possibilities lead to different values for the critical exponent in \( \langle S \rangle \sim (\Lambda - \Lambda_c)^\beta \), ranging from \( \beta = 1/2 \) for the max-slack model, Eq. (3), to \( \beta \approx 0.75 \pm 0.05 \) for the max-entropy case (fitting to a power law). The near-critical behavior of the stretch-and-slip model is \( w \propto (\Lambda - \Lambda_c)^{1/3} \) and \( \langle S \rangle \propto w^2 \), leading to the intermediate value \( \beta = 2/3 \).

To summarize, we propose a generic percolation transition as the explanation for how staple yarns, woven fabrics, sewn garments (and Galileo’s rope) transmit tension over arbitrary distances. Our assertion is supported by the appearance of a transition in abstract models, where the friction laws are recast as a linear satisfiability problem. This transition appears to be second-order, although the critical exponents are dependent on physics beyond simple static friction. The abstract model may be generalised and applied in various ways. For example one can investigate fiber blends, with applications to optimising the properties of functionalized sewing threads. In another direction, failure could be modeled by iteratively breaking the most highly loaded fiber elements (Fig. 4 inset), cf. elastic fiber bundle models [24]. More generally, the LP approach to Amontons-Coulomb friction problems may have applications in stress transmission in granular media such as sand piles and grain silos.
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APPENDIX: SUPPLEMENTAL MATERIAL

We provide a short proof that the inhomogeneous LP problem has unbounded solutions if and only if the homogeneous LP problem has unbounded solutions, thus the percolation threshold can be computed with $T^* = 0$.

Suppose that the homogeneous LP problem is specified by $T_i \geq 0$, subject to tension transfer limits at mechanical contacts $|\Delta T| \leq \Delta T_{\alpha}$, and $T_i = 0$ at the fiber ends, with the objective function $z = \sum T_i$. The inhomogeneous LP problem is the same, but with $T_i \leq T^*$ at the fiber ends.

First note that every solution to the homogeneous LP problem is also a solution to the inhomogeneous LP problem, since setting $T_i = 0$ in the fiber ends satisfies $T_i \leq T^*$. Therefore if the homogeneous LP problem has unbounded solutions, so does the inhomogeneous problem. To prove the converse, let an unbounded class of solutions of the inhomogeneous LP problem be

$$T_i = T_i^{(0)} + \alpha R_i$$

where $\alpha > 0$. The ‘ray’ $R_i$ represents a direction in $T_i$-space in which unbounded solutions exist. Certainly, some of the $R_i$ will vanish, but not all of them, since we know that if $z$ is unbounded at least one of the $T_i$ should be unbounded. Therefore solutions of this kind exists. Also, if the lower bound to $\alpha$ is not zero, it can be absorbed into the definition of $T_i^{(0)}$. Finally, we must have $R_i \geq 0$ since if any $R_i$ was negative we could violate the constraint $T_i \geq 0$ by making $\alpha$ large enough.

By substitution, and dividing through by $\alpha > 0$, we find the $R_i$ satisfy

$$|\Delta R + \alpha^{-1} \Delta T^{(0)}| \leq \lambda (R_m + \alpha^{-1} T_m^{(0)}),$$

$$R_i + \alpha^{-1} T_i^{(0)} \leq \alpha^{-1} T^* \quad \text{(fibre ends)},$$

where the notation is hopefully obvious. Letting $\alpha \to \infty$ in this gives

$$|\Delta R| \leq \lambda R_m, \quad R_i = 0 \quad \text{(fibre ends)}.$$

Thus the ray direction $R_i$ solves the homogeneous LP problem. This means that given unbounded class of solutions to the inhomogeneous LP problem, one can construct a solution to the homogeneous LP problem. Then, since any positive multiple $\beta R_i (\beta > 0)$ also solves Eqs. (6), one has in fact constructed a class of unbounded solutions to the homogeneous problem. This proves the equivalence.

[1] G. Galilei, Discorsi e Dimostrazioni Matematiche Intorno a Due Nuove Scienze (Elzevir, Leiden, 1638); our quote is from the translation by H. Crew and A. de Salvio, Dialogues Concerning Two New Sciences (Macmillan, New York, 1914).

[2] J. W. S. Hearle, P. Grosberg, and S. Backer, Structural mechanics of fibers, yarns, and fabrics, Vol. 1 (Wiley-Interscience, New York, 1969).

[3] N. Pan, J.-H. He, and J. Yu, Textile Res. J. 77, 205 (2007).

[4] N. Pan, Textile Res. J. 62, 749 (1992); 63, 504 (1993).

[5] W. E. Morton and J. W. S. Hearle, Physical Properties of Textile Fibres (Textile Institute, Manchester, 1993).

[6] J. Gao et al., J. Phys. Chem. B 108, 3410 (2004).

[7] Tension-dominated means $T_i \gg T^*$ where $T^*$ is defined below; in this regime (yarn load $\geq$ few g wgt.), bending stiffness can be safely ignored away from the fiber ends.

[8] The results are insensitive to the choice of which fiber is notionally overwrapped since this subtlety is erased by randomly shuffling the pinning directions. Also, we expect adjacent frictional contacts are elastically decoupled in the tension-dominated regime, justifying the choice to use independent and identically distributed tension transfer coefficients.

[9] I. M. Stuart, Brit. J. Appl. Phys. 12, 559 (1961).

[10] The estimate derives from $T^* = \mu N$ where the friction coefficient $\mu = O(1)$ for these purposes. The beam equation then gives the normal force $N \approx B/L^2$, where the bending stiffness $B \approx 3-35 \mu N \text{mm}^2$ and the fiber ‘crimp’ length scale (or the distance between frictional contacts) we estimate as $L \gtrsim 100 \mu m$. The bending stiffness in turn derives from the reported specific flexural rigidity of cotton, $0.53 \text{mNmm}^2/\text{tex}^2$, in combination with the reported range of cotton fiber linear densities 0.7–2.3 denier, where 1 denier = 1 g/9000 m = 0.11 tex.

[11] H. Alarcón et al., Phys. Rev. Lett. 116, 015502 (2016). Note that the giant amplification of their equivalent $T^*$ is peculiar to their geometry of interleaved clamped sheets; we do not expect it to apply to fibers in a staple yarn.

[12] This estimate derives from the reported cotton tensile strength 0.2–0.5 N/tex, multiplied by the above linear fiber density, and is in accord with the yarn tensile strength $\sim 1 \text{kgwgt}$.

[13] M. E. Fisher, Rev. Mod. Phys. 70, 653 (1998).

[14] To see this, note that the percolation threshold corresponds to the appearance of unbounded LP solution(s).
By multiplicative rescaling, we can renormalise $T^*$ to zero in the problem, without changing the nature of the transition. See Appendix ‘Supplemental Material’.

[15] B. F. Bayman, Am. J. Phys. 45, 185 (1977); see also J. Walker, Sci. Am. 249, 120 (1983); J. H. Maddocks and J. B. Keller, SIAM J. Appl. Math. 47, 1185 (1987).

[16] Fig. 2a is redrawn from the entry on knots in the Encyclopædia Britannica, 11th edn (New York, 1911).

[17] That is to say, $\kappa = (d/4)/[(d/4)^2 + (3\lambda/2\pi)^2]$.

[18] This definition of slack is closely related to, but not exactly the same as, the LP notion of slack.

[19] For another slant on the problem we also monitored the number of iterations of the LP simplex solver required to decide satisfiability as a measure of computational complexity. We find this exhibits a peak at the critical point, reminiscent of the behavior of the heat capacity at a conventional second-order phase transition.

[20] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions (Dover, New York, 1965).

[21] P. B. Warren, R. C. Ball, and R. E. Goldstein, in preparation (2017).

[22] The success in extrapolating the perturbation calculation is explained by the fact that Eq. 3 asymptotes to $N\langle S \rangle / \Lambda \langle T \rangle = 2\Lambda^{-1/2} = 1/x_0 \approx 1.082$; this is almost the correct large $\Lambda$ behaviour, $N\langle S \rangle / \Lambda \langle T \rangle = 1$.

[23] S. F. Edwards and R. B. S. Oakeshott, Phys. A 157, 1080 (1989); S. Martiniani et al., Nat. Phys. 13, 848 (2017).

[24] S. Pradhan, A. Hansen, and B. K. Chakrabarti, Rev. Mod. Phys. 82, 499 (2010).