d-Orthogonal Charlier Polynomials and the Weyl Algebra

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Abstract. It is shown that d-orthogonal Charlier polynomials arise as matrix elements of non unitary automorphisms of the Weyl algebra. The structural formulas that these polynomials obey are derived from this algebraic setting.

1. Charlier polynomials
The (normalized) Charlier polynomials \( p_n(k) \) are functions of a discrete variable \( k \) that satisfy the 3-term recurrence relation [1]

\[
p_{n+1}(k) = (k - n - a)p_n(k) - na p_{n-1}(k)
\]  

and obey the orthogonality relation

\[
\sum_{k=0}^{\infty} p_m(k)p_n(k)e^{-a\frac{a^k}{k!}} = a^n n! \delta_{m,n},
\]  

with \( a \) a real parameter.

2. Weyl algebra
The Weyl algebra is generated by 3 elements \( a, a^* \) and \( I \) that satisfy the commutation relations

\[
[a, a^*] = 1, \quad [a, I] = [a^*, I] = 0.
\]  

It possesses a canonical irreducible representation defined as follows on basis states \( |n\rangle \), \( n = 0, 1, 2, \ldots \)

\[
a|n\rangle = \sqrt{n}|n - 1\rangle, \quad a^*|n\rangle = \sqrt{n+1}|n + 1\rangle, \quad I |n\rangle = |n\rangle,
\]  

\[
N = a^*a, \quad N|n\rangle = n|n\rangle, \quad a|0\rangle = 0.
\]
3. Charlier polynomials and Weyl algebra

We recall how Charlier polynomials arise in the context of the Weyl algebra [2]. This algebra admits the following oscillator realization:

\[ a = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + x \right), \quad a^* = \frac{1}{\sqrt{2}} \left( -\frac{d}{dx} + x \right), \]

\[ N = \frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 - \frac{1}{2}. \]  

(6)

Now effect the shift \( x \rightarrow x - \lambda \). It induces the following transformations:

\[ \tilde{a}(\lambda) = a - \frac{\lambda}{\sqrt{2}}, \quad \tilde{a}^*(\lambda) = a^* - \frac{\lambda}{\sqrt{2}}, \]

\[ \tilde{N}(\lambda) = N - \lambda x + \frac{\lambda^2}{2}. \]  

(7)

Let \( \tilde{N}|\tilde{n}\rangle = n|n\rangle \) and set

\[ |k\rangle = \sum_n \langle n|k\rangle|n\rangle. \]  

(8)

From

\[ N|k\rangle = \sum_n \langle n|k\rangle \left[ \tilde{N} + \frac{\lambda}{\sqrt{2}} (\tilde{a} + \tilde{a}^*) + \frac{\lambda^2}{2} \right]|n\rangle \]  

it is found that the overlaps \( \langle n|k\rangle \) satisfy the 3-term recurrence relation

\[ k\langle n|k\rangle = \left( n + \frac{\lambda^2}{2} \right)\langle n|k\rangle + \frac{\lambda}{\sqrt{2}} \sqrt{n+1} \langle n+1|k\rangle + \frac{\lambda}{\sqrt{2}} \sqrt{n} \langle n-1|k\rangle \]  

(9)

and hence are given in terms of Charlier polynomials

\[ \langle n|k\rangle = \left( \frac{\sqrt{2}}{\lambda} \right)^n \frac{1}{\sqrt{n!}} \nu(k)p_n(k) \]  

(11)

where \( \nu(k) \) is a normalization factor.

Let us remark that translations obviously provide an automorphism of the Weyl algebra and that various properties of the Charlier polynomials can be derived from the above context (see [2]).

4. d-orthogonal polynomials

Monic d-orthogonal polynomials [3, 4] of degree \( n \) are defined through recurrence relations of order \( d+1 \) that are of the form

\[ P_{n+1}(x) = xP_n(x) - \sum_{s=0}^d a_{n,n-s}P_{n-s}(x) \]  

(12)

with initial conditions

\[ P_k(x) = 0, \quad k < 0; \quad P_0(x) = 1. \]  

(13)

For \( d = 1 \), one recovers the standard orthogonal polynomials. When \( d > 1 \), they satisfy vector orthogonality relations. Namely there exist \( d \) linear functionals \( \sigma_k, k = 0, 1, \ldots, d-1 \) such that

\[ \langle \sigma_k, P_mP_n \rangle = 0, \quad m > dn + k \]

\[ \langle \sigma_k, P_mP_{dn+k} \rangle \neq 0, \quad n \geq 0. \]  

(14)

These have applications in various areas, in particular in Padé approximation problems and in random matrix theory.
5. A nonlinear automorphism of the Weyl algebra

We shall show that such polynomials also arise as matrix elements of certain automorphisms of the Weyl algebra [5].

Take this automorphism to be

\[ S = e^{\beta a^*} e^{Q(a)}, \quad S^{-1} = e^{-Q(a)} e^{-\beta a^*} \]

(15)

where \( Q \) is a polynomial of degree \( N > 1 \) and \( \beta \) a parameter.

We have

\[
SaS^{-1} = a - \beta, \quad S^{-1}aS = a + \beta, \\
Sa^*S^{-1} = a^* + Q'(a - \beta), \quad S^{-1}a^*S = a^* - Q'(a).
\]

(16)

6. d-Charlier polynomials

We shall calculate the matrix element

\[ \psi_{n,k} = \langle k | S | n \rangle \]

(17)

by obtaining for it a recurrence relation. From

\[
\langle k|a^*aS|n\rangle = \langle k|SS^{-1}a^*aS|n\rangle = \langle k|S(a^* - Q'(a))(a + \beta)|n\rangle
\]

(18)

and using the fact that \( Q'(x)(x + \beta) \) is a polynomial of degree \( N \) if this is so for \( Q \), we find

\[
k\psi_{n,k} = (n - \xi_0)\psi_{n,k} + \beta\sqrt{n + 1}\psi_{n+1,k} - \sum_{i=1}^{N} \xi_i \sqrt{n(n-1)\ldots(N-i+1)}\psi_{n-i,k}
\]

(19)

where the coefficients \( \xi_i \) are defined by the expansion

\[ Q'(x)(x + \beta) = \sum_{i=1}^{N} \xi_i x^i. \]

(20)

Formula (19) gives \( \psi_{n,k} \) recursively from \( \psi_{0,k} \) and we can write

\[ \psi_{n,k} = P_n(k)\psi_{0,k} \]

(21)

where \( P_n(k) \) is a polynomial that satisfy

\[
\beta\sqrt{n + 1}P_{n+1}(k) = (k - n + \xi_0)P_n(k) + \sum_{i=1}^{N} \xi_i \sqrt{n(n-1)\ldots(n-i+1)}P_{n-i}(k),
\]

\[ P_0(k) = 1, \quad P_n(k) = 0, \quad n < 0 \]

(22)

that is a recurrence relation of order \( N + 1 \).

The matrix element \( \psi_{0,k} \) is straightforwardly found to be

\[ \psi_{0,k} = e^{Q(0)} \frac{\beta^k}{\sqrt{k!}}. \]

(23)
To make the polynomials monic, set
\[ \hat{P}_n(k) = \beta^n \sqrt{n!} P_n(k) \] (24)
so as to find the defining \( N + 1 \) recurrence relation
\[ \hat{P}_{n+1}(k) = (k - n + \xi_0) \hat{P}_n(k) + \sum_{i=1}^{N} \beta^i \xi_i n(n-1) \ldots (n-i+1) \hat{P}_{n-i}(k) \] (25)
for the \( d \)-Charlier polynomials [6].

For \( N = 1 \), we recover the relation for the ordinary Charlier polynomials (1) up to the shift \( k \rightarrow k + \beta \xi_1 - \xi_0 \).

7. Properties
The characterization of the \( d \)-Charlier polynomials is easily performed using the algebraic tools now at our disposal.

- Difference equation

Start in this case from the “dual” recurrence relation in the variable \( k \) that the following equation entail
\[ \langle k|Sa^*a|n \rangle = \langle k|Sa^*aS^{-1}S|n \rangle = \langle k(a^* + Q'(a - \beta))(a - \beta)S|n \rangle \] (26)
to find that
\[ (n - k) \hat{P}_n(k) + k \hat{P}_n(k - 1) = \sum_{i=0}^{N} \eta_i \beta^i \hat{P}_n(k + 1). \] (27)
Here the parameters \( \eta_i \) are defined through the expansion
\[ (x - \beta)Q'(x - \beta) = \sum_{i=1}^{N} \eta_i x^i. \] (28)

Using the difference operators
\[ \Delta f(k) = f(k + 1) - f(x), \quad \nabla f(k) = f(k) - f(k - 1) \] (29)
and the identity
\[ F(x + m) = \sum_{i=0}^{m} \binom{m}{i} \Delta^i F(x). \] (30)

Eq. (27) becomes
\[ n \hat{P}_n(k) = k \nabla \hat{P}_n(k) + \sum_{i=0}^{N} \eta_i \beta^i \sum_{\ell=0}^{i} \binom{i}{\ell} \Delta^\ell \hat{P}_n(k). \] (31)

Recalling (28), we have
\[ yQ'(y) = \sum_{i=0}^{N} \eta_i (y + \beta)^i = \sum_{i=0}^{N} \eta_i \sum_{\ell=0}^{i} \binom{i}{\ell} y^\ell \beta^{i-\ell} \]
\[ = \sum_{i=0}^{N} \eta_i \beta^i \sum_{\ell=0}^{i} \binom{i}{\ell} (y/\beta)^\ell \equiv \sum_{j=0}^{N} \mu_j (y/\beta)^j \] (32)
thus defining new parameters $\mu_i$.

As a result the d-Charlier polynomials $\hat{P}_n$ are observed to satisfy a finite difference eigenvalue equation

$$H \hat{P}_n = n \hat{P}_n$$

(33)

with

$$H = k\nabla + \sum_{i=0}^{N} \mu_i \Delta^i.$$  

(34)

- Ladder operators
  The relation
  $$\langle k | S a | n \rangle = \langle k | (a - \beta) S | n \rangle$$

(35)

is easily seen to give

$$\Delta \hat{P}_n(k) = n \hat{P}_{n-1}(k)$$

(36)

thereby showing that the polynomials $\hat{P}_n(k)$ satisfy the difference Appell property.

Similarly

$$\langle k | S a^\ast | n \rangle = \langle k | (a^\ast + Q'(a - \beta)) S | n \rangle$$

(37)

yields the Rodrigues’ formula

$$\hat{P}_n(k) = R^n \{ 1 \}$$

(38)

with

$$R = kT_+ + \sum_{i=0}^{N-1} \nu_i \beta^{i+1} T_+^i$$

(39)

where $Q'(x - \beta) = \sum_{i=0}^{N-1} \nu_i x^i$ and $T_\pm F(k) = F(k \pm 1)$.

- Generating function
  Consider the generating function

$$F(z; k) = e^{-Q(0)} \beta^{-k} \sqrt{k!} \sum_{n=0}^{\infty} \frac{\psi_n, k z^n}{\sqrt{n!}}.$$  

(40)

Recalling that

$$\psi_{n, k} = e^{Q(0)} \beta^{k-n} \sqrt{n! k!} \hat{P}_n(k)$$

(41)

it yields

$$F(z, k) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{z}{\beta} \right)^n \hat{P}_n(k).$$

(42)

The oscillator coherent states enjoy the following properties:

$$|z\rangle = e^{za^\ast} |0\rangle, \quad (n|z\rangle = \frac{z^n}{\sqrt{n!}}$$

(43)

$$a|z\rangle = z|z\rangle, \quad F(a)|z\rangle = F(z)|z\rangle.$$

In terms of these coherent states, $F(z, k)$ can be written

$$F(z, k) = e^{-Q(0)} \beta^{-k} \sqrt{k!} \langle k | S | z \rangle.$$  

(44)
The matrix element \( \langle k | S | z \rangle \) is easily computed using (43):

\[
\langle k | e^{\beta a^* e^Q(a)} | z \rangle = e^{Q(z)} \langle k | z + \beta \rangle = e^{\varphi(x)} (z + \beta)^k \sqrt{k!}
\]

(45)

to find the following generating function for the d-Charlier polynomials:

\[
e^{Q(z) - Q(0)} \left( 1 + \frac{z}{\beta} \right)^k = \sum_{n=0}^{\infty} \left( \frac{z}{\beta} \right)^n \frac{\hat{P}_n(k)}{n!}.
\]

(46)

**Characterization**

It can be proved that the \( N + 1 \) order recurrence relation, the matrix elements of \( S \), the Appell difference property, the eigenvectors of \( H \) can all be derived from one another and represent equivalent characterizations of d-orthogonal Charlier polynomials.

**8. Explicit expression in a special case**

For the special case where the polynomial \( Q(x) \) is a monomial, i.e.

\[
Q(x) = \sigma x^N
\]

(47)

\[
F(z; k) = \left( 1 + \frac{z}{\beta} \right)^k \exp(\sigma z^N) = \sum_{n=0}^{\infty} \left( \frac{z}{\beta} \right)^n \frac{\hat{P}_n(k)}{n!}.
\]

(48)

Expanding \( F \) and performing some transformations give the explicit expression for \( \hat{P}_n(k) \):

\[
\hat{P}_n(k) = \frac{(-1)^q \beta^N j \sigma^j (-k)^q}{q!j!} \sum_{N=1}^{N+1} F_N \left( \begin{array}{c} -j, \{a_m\} \\ \{b_m\} \end{array} \right) \frac{(-1)^{N+1}}{\sigma \beta^n}
\]

(49)

\[
j = 0, 1, \ldots, \quad q = 0, 1, \ldots, N - 1 \quad a_m = (q - k + m)/N, \quad b_m = (q + m + 1)/N
\]

\[
m = 0, \ldots, N - 1, \quad (-k)_q = (-k) (-k + 1) \ldots (-k + q + 1).
\]

For \( N = 1 \), the standard expression

\[
\hat{P}_n(k) = \frac{\sigma^n \beta^n}{n!} 2F_0 \left( \begin{array}{c} -n, -k \\ \frac{1}{\sigma \beta} \end{array} \right)
\]

(50)

for the ordinary Charlier polynomial is recovered.

**9. Dual functions and orthogonality**

Much more can be obtained from this algebraic framework. Most interesting is probably what it allows to say about the orthogonality properties of the d-orthogonal polynomials that the matrix elements \( \psi_{n,k} = \langle k | S | n \rangle \) entail.

Assume \( S \) invertible and define

\[
\varphi_{n,k} = \langle n | S^{-1} | k \rangle.
\]

(51)

It can be shown, proceeding as above, that the matrix elements are similarly expressible in terms of d-Charlier polynomials, albeit with different parameters. A generating function for \( \varphi_{n,k} \) can also be obtained.
As a result, the identity

\[
\sum_{k=0}^{\infty} \varphi_{m,k} \psi_{n,k} = \sum_{k=0}^{\infty} \langle m|S^{-1}|k \rangle \langle k|S|n \rangle = \delta_{m,n}
\]  

(52)

is tantamount to a biorthogonality relation for the d-Charlier polynomials.

Moreover, it is possible to relate this biorthogonality between \( \varphi_{m,k} \) and \( \psi_{n,k} \) to the vector orthogonality of the underlying d-orthogonal polynomials. In fact, from the biorthogonality relation and the knowledge of the matrix elements \( \varphi_{m,k} \) and \( \psi_{n,k} \), it is possible to construct explicit realizations of the linear functionals expressing the vector orthogonality. (See ref. [5]).

References

[1] Koekoek R and Swarttouw R F 1994 The Askey-scheme of hypergeometric orthogonal polynomials and its \( q \)-analogue Tech. Rep. 94-05 Delft University of Technology
[2] Floreanini R, LeTourneux J and Vinet L 1993 *Ann. Phys.* 226 331
[3] van Iseghem J 1987 *J. Comput. Appl. Math.* 19 141
[4] Maroni P 1989 *Ann. Fac. Sci. Toulouse Math.* 10 105
[5] Vinet L and Zhedanov A 2009 *J. Math. Phys.* 50 33511
[6] Cheikh Y B and Zaghouani A 2003 *J. Comput. Appl. Math.* 156 253