The complexity of Boolean surjective general-valued CSPs

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Abstract

Valued constraint satisfaction problems (VCSPs) are discrete optimisation problems with a \( \mathbb{Q} \)-valued objective function given as a sum of fixed-arity functions, where \( \mathbb{Q} = \mathbb{Q} \cup \{ \infty \} \) is the set of extended rationals. In Boolean surjective VCSPs variables take on labels from \( D = \{ 0, 1 \} \) and an optimal assignment is required to use both labels from \( D \). A classic example is the global min-cut problem in graphs. Building on the work of Uppman, we establish a dichotomy theorem and thus give a complete complexity classification of Boolean surjective VCSPs. The newly discovered tractable case has an interesting structure related to projections of downsets and upsets. Our work generalises the dichotomy for \( \{ 0, \infty \} \)-valued constraint languages (corresponding to CSPs) obtained by Creignou and Hébrard, and the dichotomy for \( \{ 0, 1 \} \)-valued constraint languages (corresponding to Min-CSPs) obtained by Uppman.

1 Introduction

The \((s, t)\)-Min-Cut problem asks, given a digraph \( G = (V, E) \) with source \( s \in V \), sink \( t \in V \), and edge weights \( w : E \to \mathbb{Q}_{>0} \), for a subset \( C \subseteq V \) with \( s \in C \) and \( t \notin C \) minimising \( w(C) = \sum_{(u,v)\in E,u\in C,v\notin C} w(u,v) \) [24]. This fundamental problem is an example of a Boolean valued constraint satisfaction problem (VCSP).

Let \( D \) be an arbitrary finite set called the domain. A valued constraint language, or just a language, \( \Gamma \) is a set of weighted relations; each weighted relation \( \gamma \in \Gamma \) is a function \( \gamma : D^{\ar(\gamma)} \to \mathbb{Q} \), where \( \ar(\gamma) \in \mathbb{N} \) is the arity of \( \gamma \) and \( \mathbb{Q} = \mathbb{Q} \cup \{ \infty \} \) is the set of extended rationals. If \( |D| = 2 \) then \( \Gamma \) is called a Boolean language. An instance \( I = (V, D, \phi_I) \) of the VCSP on domain \( D \) is given by a finite set of \( n \) variables \( V = \{ x_1, \ldots, x_n \} \) and an objective function \( \phi_I : D^n \to \mathbb{Q} \) expressed as a weighted sum of valued constraints over \( V \), i.e. \( \phi_I(x_1, \ldots, x_n) = \sum_{i=1}^n w_i \cdot \gamma_i(x_i) \), where \( \gamma_i \) is a weighted relation, \( w_i \in \mathbb{Q}_{\geq 0} \) is the weight and \( x_i \in V^{\ar(\gamma_i)} \) the scope of the \( i \)th valued constraint. (We note that we allow zero weights and for \( w_i = 0 \) we define \( w_i \cdot \infty = \infty \).) Given an instance \( I \), the goal is to find an assignment \( s : V \to D \) of domain labels to the variables that \( \text{minimises} \ \phi_I \). Given a language \( \Gamma \), we denote by \( \text{VCSP}(\Gamma) \) the class of all instances \( I \) that use only weighted relations from \( \Gamma \) in their objective function. Valued CSPs are also called general-valued CSPs to emphasise that (decision) CSPs are a special case of valued CSPs.

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To continue with the \((s,t)\)-Min-Cut example, let \(D = \{0, 1\}\). We will use the following three weighted relations: \(\gamma : D^2 \to \mathbb{Q}\) is defined by \(\gamma(x, y) = 1\) if \(x = 0\) and \(y = 1\), and \(\gamma(x, y) = 0\) otherwise; \(\rho_d : D \to \mathbb{Q}\), for \(d \in D\), is defined by \(\rho_d(x) = 0\) if \(x = d\) and \(\rho_d(x) = \infty\) if \(x \neq d\). Now, given an \((s,t)\)-Min-Cut instance \(G = (V, E)\), \(s, t \in V = \{x_1, \ldots, x_n\}\), and \(w : E \to \mathbb{Q}_{\geq 0}\) as before, the problem of finding an optimal \((s,t)\)-Min-Cut in \(G\) is equivalent to solving the following instance of \(\text{VCSP}(\Gamma_{\text{cut}})\), where \(\Gamma_{\text{cut}} = \{\gamma, \rho_0, \rho_1\}: I = (V, D, \phi_I)\) and 

\[
\phi_I(x_1, \ldots, x_n) = \sum_{(u,v) \in E} w(u,v) \cdot \gamma(x_u, x_v) + \rho_0(s) + \rho_1(t).
\]

It is well known that the \((s,t)\)-Min-Cut problem is solvable in polynomial time. Since every instance \(I\) of \(\text{VCSP}(\Gamma_{\text{cut}})\) can be reduced to an instance of the \((s,t)\)-Min-Cut problem, \(I\) is solvable in polynomial time. Thus, \(\Gamma_{\text{cut}}\) is an example of a tractable constraint language. When \(\text{VCSP}(\Gamma)\) is NP-hard, we call \(\Gamma\) an intractable language.

It is natural to ask about the complexity of \(\text{VCSP}(\Gamma)\) in terms of \(\Gamma\). For Boolean valued constraint languages, we have a complete answer: Cohen et al. [8] showed that every Boolean valued constraint language is either tractable or intractable, thus obtaining what is known as a dichotomy theorem. In fact, [8] identified eight different types of tractable valued constraint languages; one of these types corresponds to submodularity [24] and includes \(\Gamma_{\text{cut}}\). The dichotomy theorem from [8] is an extension of Schaefer’s celebrated result, which gave a dichotomy for \(\{0, \infty\}\)-valued constraint languages [23], and the work of Creignou [9], who gave a dichotomy theorem for \(\{0, 1\}\)-valued constraint languages.

The (global) Min-Cut problem asks, given a graph \(G = (V, E)\) and edge weights \(w : E \to \mathbb{Q}_{\geq 0}\), for a subset \(C \subseteq V\) with \(\emptyset \subseteq C \subseteq V\) minimising \(w(C) = \sum_{(u,v) \in E} w(u,v)\) [24]. This fundamental problem is an example of a Boolean surjective \(\text{VCSP}\). Given a \(\text{VCSP}\) instance \(I = (V, D, \phi_I)\), in the surjective setting the goal is to find a surjective assignment \(s : V \to D\) minimising \(\phi_I\); here \(s\) is called surjective if for every \(d \in D\) there is \(x \in V\) such that \(s(x) = d\). For Boolean \(\text{VCSPs}\) with \(D = \{0, 1\}\), this simply means that the all-zero and all-one assignments are not allowed. Given a language \(\Gamma\), we denote by \(\text{VCSP}_s(\Gamma)\) the class of all instances \(I\) of the surjective \(\text{VCSP}\) that use only weighted relations from \(\Gamma\) in their objective function.

Let \(D = \{0, 1\}\) and define \(\gamma : D \to \mathbb{Q}\) by \(\gamma(x, y) = 0\) if \(x = y\) and \(\gamma(x, y) = 1\) if \(x \neq y\). \(\text{VCSP}_s(\{\gamma\})\) captures Min-Cut as every instance of \(\text{VCSP}_s(\{\gamma\})\) is a Min-Cut instance and vice versa. Since Min-Cut is solvable in polynomial time (say, by a reduction to the \((s,t)\)-Min-Cut problem but other algorithms exist [25]), \(\{\gamma\}\) is an example of a surjectively tractable, or \(s\)-tractable for short, valued constraint language. As before, if \(\text{VCSP}_s(\Gamma)\) is NP-hard we call \(\Gamma\) a surjectively intractable, or \(s\)-intractable for short, language.

**Surjective \(\text{VCSPs}\)**

What can we say about the complexity of \(\text{VCSP}_s(\Gamma)\) for arbitrary \(\Gamma\)? In particular, is every \(\Gamma\) \(s\)-tractable or \(s\)-intractable? What is the mathematical structure of \(s\)-tractable languages?

First, observe that for a language \(\Gamma\) defined on \(D\), we have \(\text{VCSP}(\Gamma) \leq_p \text{VCSP}_s(\Gamma)\). Indeed, given an instance \(I\) of \(\text{VCSP}(\Gamma)\), construct a new instance \(I'\) of \(\text{VCSP}_s(\Gamma)\) by adding \(|D|\) extra variables. Then, any solution to \(I\) can be extended to a surjective solution to \(I'\) of the same value and conversely, any (surjective) solution to \(I'\) induces a solution to \(I\) of the same value.

Second, observe that for a language \(\Gamma\) defined on \(D\), we have \(\text{VCSP}_s(\Gamma) \leq_p \text{VCSP}(\Gamma \cup C_D)\), where \(C_D\) is the set of constants on \(D\); that is, \(C_D = \{\rho_d \mid d \in D\}\), where \(\rho_d\) is defined by \(\rho_d(x) = 0\) if \(x = d\) and \(\rho_d(x) = \infty\) if \(x \neq d\). Indeed, given an instance of \(\text{VCSP}_s(\Gamma)\), constants can be used to go through all \(O(|D|^{|D|})\) ways to assign all the labels from \(D\) to a \(|D|\)-subset
of the $n$ variables, each resulting in an instance of VCSP($\Gamma \cup C_D$). Consequently, a tractable language $\Gamma$ defined on $D$ with $C_D \subseteq \Gamma$ is also an $s$-tractable language.

In this paper we deal with Boolean valued constraint languages defined on $D = \{0, 1\}$. By the two observations above, the only Boolean valued constraint languages for which tractability could be different from $s$-tractability are tractable languages that do not include constants.

For Boolean $\{0, \infty\}$-valued languages, Schaefer’s dichotomy [23] gives six tractable cases, four of which include constants. Creignou and Hébrard showed that the remaining two cases (0-valid and 1-valid) are $s$-intractable, thus obtaining a dichotomy in the surjective setting [10].

For Boolean $\{0, 1\}$-valued languages, Creignou’s dichotomy [9] gives three tractable cases, one of which includes constants. Uppman showed that the remaining two cases (0-valid and 1-valid) are $s$-tractable if they are almost-min-min or almost-max-max, respectively, and $s$-intractable otherwise, thus obtaining a dichotomy in the surjective setting [26].

**Contributions**

As our main contribution we classify all Boolean valued constraint languages (i.e. $\mathbb{Q}$-valued languages) as $s$-tractable or $s$-intractable. Our result extends the classifications from [10] and [26]. Six of the eight tractable cases identified for Boolean valued constraint languages [8] include constants and thus are also $s$-tractable. The remaining two cases (0-optimal and 1-optimal) are $s$-tractable if they satisfy a certain condition. This condition, defined formally in Definition 8, says that both the feasibility and optimality relations of every weighted relation in the language have to be a projection of a downset (in the 0-optimal case), or a projection of an upset (in the 1-optimal case). This shows that, surprisingly, $s$-tractability of valued constraint languages (that are not covered by the tractable languages with constants) does not depend on the rational-values in the weighted relations. It is only the structure of the underlying feasibility and optimality relations that matters. (However, the running time of our algorithm depends on these values.) Identifying this condition and establishing that it captures the precise borderline of $s$-tractability is our main contribution.

The hardness part of our result is proved in the same spirit as for $\{0, \infty\}$-valued and $\{0, 1\}$-valued languages by carefully analysing the types of weighted relations that can be obtained in gadgets in the surjective setting, and relying on the explicit dichotomy for Boolean VCSPs [8].

While 0-optimal and 1-optimal languages are trivially tractable for VCSPs, the algorithm for surjective VCSPs over the newly identified languages is nontrivial. The $s$-tractability part of our result is established by a reduction from $\mathbb{Q}$-valued VCSPs to the *Generalised Min-Cut* problem (defined in Section 4), in which we require to find in polynomial time all $\alpha$-optimal solutions, where $\alpha$ is a constant depending on the (finite) valued constraint language. The algorithm for the Generalised Min-Cut problem is essentially the same as in [26]. We show that the algorithm works in the more general setting with one part of the objective function being a $(\mathbb{Q}_{\geq 0} \cup \{\infty\})$-valued superadditive set function given by an oracle; see Section 4 for the details. By providing a tighter analysis we are able to improve the bound on the running time from roughly $O(n^{3\alpha})$ to $O(n^{20\alpha})$, thus answering one of the open problems from [26]. We also show that the dependence of the running time on the language is unavoidable unless P = NP (cf. Example 41).
Related work

Recent years have seen some remarkable progress on the computational complexity of CSPs and VCSPs parametrised by the (valued) constraint language, see [1] for a survey. We highlight the resolution of the “bounded width conjecture” [2] and the result that a dichotomy for CSPs, conjectured in [11], implies a dichotomy for VCSPs [20, 19]. All this work is for arbitrary (and thus not necessarily Boolean) finite domains and relies on the so-called algebraic approach initiated in [6] and nicely described in a recent survey [3]. One of the important aspects of the algebraic approach is the assumption that constants are present in (valued) constraint languages. (This is without loss of generality with respect to polynomial-time solvability.) It is the lack of constants in the surjective setting that makes it difficult, if not impossible, to employ the algebraic approach in this setting. See the work of Chen [7] for an initial attempt.

For a binary (unweighted) relation $\gamma$, VCSP$_s(\{\gamma\})$ has been studied under the name of surjective $\gamma$-Colouring [4, 21]. We remark that our notion of surjectivity is global. For the $\gamma$-Colouring problem, a local version of surjectivity has also been studied [13, 12].

2 Preliminaries

We denote by $\leq_p$ the standard polynomial-time Turing reduction. If $A \leq_p B$ and $B \leq_p A$ we write $A \equiv_p B$.

2.1 Weighted relations and VCSPs

We use the notation $[n] = \{1, \ldots, n\}$. For any tuple $x \in D^n$, we refer to its $i$th element as $x_i$. For $x, y \in D^n$, we define $x \leq y$ if and only if $x_i \leq y_i$ for all $i \in [r]$.

We define relations as a special case of weighted relations with range \{0, \infty\}, where value 0 is assigned to tuples that are elements of the relation in the conventional sense. We will use both views interchangeably. Relations are also called unweighted or crisp.

If $s \in [r]^n$ is a tuple of coordinates then for any $x \in D^n$ we denote its projection to $s$ by $\text{Pr}_s(x) = (x_{s_1}, \ldots, x_{s_n}) \in D^n$. For any relation $\rho$, we define $\text{Pr}_s(\rho) = \{\text{Pr}_s(x) \mid x \in \rho\}$.

We denote by $\rho_\equiv$ the binary equality relation $\{(x, x) \mid x \in D\}$. Recall from Section 1 that we denote, for any $d \in D$, by $\rho_d$ the unary relation $\{(d)\}$, i.e. $\rho_d(x) = 0$ if $x = d$ and $\rho_d(x) = \infty$ if $x \neq d$.

**Definition 1.** For a weighted relation $\gamma : D^n \rightarrow \overline{\mathbb{Q}}$, we denote by

- $\text{Feas}(\gamma) = \{x \in D^n \mid \gamma(x) < \infty\}$ the underlying feasibility relation; and by
- $\text{Opt}(\gamma) = \{x \in \text{Feas}(\gamma) \mid \gamma(x) \leq \gamma(y) \text{ for every } y \in D^n\}$ the relation of optimal tuples.

We define $\text{Feas}(\Gamma) = \{\text{Feas}(\gamma) \mid \gamma \in \Gamma\}$ and $\text{Opt}(\Gamma) = \{\text{Opt}(\gamma) \mid \gamma \in \Gamma\}$.

An assignment $s : V \rightarrow D$ for a VCSP instance $I = (V, D, \phi_I)$ with $V = \{x_1, \ldots, x_n\}$ is called feasible if $\phi_I(s(x_1), \ldots, s(x_n)) < \infty$; $s$ is called optimal if $\phi_I(s) \leq \phi_I(s')$ for every assignment $s'$.

Recall from Section 1 that any set of weighted relation $\Gamma$ is called a valued constraint language. $\Gamma$ is called $s$-tractable if for any finite subset $\Gamma' \subseteq \Gamma$ any instance of VCSP$_s(\Gamma')$ can be solved in polynomial time. $\Gamma$ is called $s$-intractable if VCSP($\Gamma'$) is NP-hard for some finite $\Gamma' \subseteq \Gamma$. 

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Definition 2. Let $\gamma : D^r \to \mathbb{Q}$ be a weighted relation.

1. For $c \in \mathbb{Q}_{\geq 0}$, the scaling of $\gamma$ by $c$ is the weighted relation $c \cdot \gamma$. If $c = 0$ then $c \cdot \gamma = \text{Feas}(\gamma)$, where we define $0 \cdot \infty = \infty$.

2. For a permutation $\pi$ on $[r]$, we define $\gamma_{\pi}(x_1, \ldots, x_r) = \gamma(x_{\pi(1)}, \ldots, x_{\pi(r)})$ and call $\gamma_{\pi}$ a permutation of $\gamma$.

3. The identification of $\gamma$ is the weighted relation $\gamma' : D^{r-1} \to \mathbb{Q}$ defined by $\gamma'(x_1, \ldots, x_{r-1}) = \gamma(x_1, x_1, x_2, \ldots, x_{r-1})$.

4. A $d$-pinning of $\gamma$ at coordinate $i \in [r]$ is the weighted relation $\gamma'' : D^{r-1} \to \mathbb{Q}$ defined by $\gamma''(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r) = \gamma(x_1, \ldots, x_{i-1}, d, x_{i+1}, \ldots, x_r)$.

The following notion of closure will be important in our proofs.

Definition 3. A valued constraint language $\Gamma$ is closed if it is closed under scaling, permutation, identification, addition, adding rational constants, the $\text{Feas}(\cdot)$ operator, the $\text{Opt}(\cdot)$ operator, and $d$-pinning (if $\rho_d \in \Gamma$) for $d \in D$.

For a valued constraint language $\Gamma$, we define $\Gamma^*$ to be the smallest closed language containing $\Gamma$.

Note that since $\Gamma^*$ is closed under permutations it allows to identify any coordinates, not necessarily the first two.

Lemma 4. Let $\Gamma$ be a valued constraint language. Then, $\text{VCSP}_s(\Gamma^*) \leq_p \text{VCSP}_s(\Gamma)$.

Proof. Most reductions are fairly standard and easy to prove [8]. Scaling, including scaling by 0, can be achieved by adjusting the weight of affected constraints. If $\gamma(x) = \gamma_1(x_1) + \gamma_2(x_2)$ then $\gamma(x)$ is replaced by two constraints $\gamma_1(x_1)$, $\gamma_2(x_2)$. Replacing $\gamma(x) + c$, for some $c \in \mathbb{Q}$, by $\gamma(x)$ changes the value of all feasible assignments by the same amount $c$. If $\gamma_{\pi}$ is a permutation of $\gamma$ then $\gamma_{\pi}(x)$ is replaced by $\gamma(\pi(x))$. If $\gamma'$ is the identification of $\gamma$ then $\gamma'(x_1, \ldots, x_r)$ is replaced by $\gamma(x_1, x_1, x_2, \ldots, x_r)$.

We now show that if $\phi \in \Gamma$ then $\text{VCSP}_s(\Gamma \cup \{\text{Opt}(\phi)\}) \leq_p \text{VCSP}_s(\Gamma)$. Relation $\text{Opt}(\gamma)$ can be simulated by adjusting the weight of the corresponding valued constraint by a sufficiently large constant. Let $w$ be an upper bound on the maximum value of a feasible assignment of $I$. By adding rational constants to weighted relations, we may assume that no weighted relation of $I$ assigns a negative value and that the smallest value assigned by $\gamma$ is 0. Let $\delta$ equal the second smallest (finite) value assigned by $\gamma$. We replace $\text{Opt}(\gamma)(x)$ with $(w/\delta + 1) \cdot \gamma(x)$, so that any assignment of $I$ that would incur an infinite value from $\text{Opt}(\gamma)(x)$ has now objective value exceeding $w$.

Finally, we show that if $\rho_d \in \Gamma$ then $\text{VCSP}_s(\Gamma \cup \{\gamma\}) \leq_p \text{VCSP}_s(\Gamma)$, where $\gamma$ is a $d$-pinning of $\phi \in \Gamma$. Without loss of generality let $\gamma$ be the $d$-pinning of $\phi \in \Gamma$ at the first coordinate with $\text{ar}(\gamma) = r$; i.e. $\gamma(x_1, \ldots, x_r) = \phi(d, x_1, \ldots, x_r)$. Let $I = (V, D, \phi_I)$ be an instance of $\text{VCSP}_s(\Gamma \cup \{\gamma\})$ with $V = \{x_1, \ldots, x_n\}$. For every $1 \leq i \leq n$, we construct instance $I_i = (V, D, \phi_{I_i})$ of $\text{VCSP}_s(\Gamma)$ as follows. If $\phi_I = \sum_{j=1}^n w_j \gamma_j(x_j)$ then $\phi_{I_i} = \sum_{j=1}^n w_j \gamma'_j(x_j) + \rho_d(x_i)$, where $\gamma'_j(x_j) = \gamma_j(x_j)$ if $\gamma_j \neq \gamma$ and $\gamma'_j(x_j) = \phi(x_i, x_j)$ if $\gamma_j = \gamma$. Intuitively, variable $x_i$ is forced to take label $d$ in the instance $I_i$. Now an optimal assignment to $I$ is the best (i.e. of smallest value) of optimal assignments to the instances $I_1, \ldots, I_n$. \qed
2.2 Polymorphisms and multimorphisms

We apply a k-ary operation $h : D^k \to D$ to $k$ $r$-tuples componentwise; i.e. $h(x^1, \ldots, x^k) = (h(x^1_1, x^2_1, \ldots, x^k_1), h(x^1_2, x^2_2, \ldots, x^k_2), \ldots, h(x^1_r, x^2_r, \ldots, x^k_r))$.

The following notion is at the heart of the algebraic approach to decision CSPs [6].

Definition 5. Let $\gamma$ be a weighted relation on $D$. A k-ary operation $h : D^k \to D$ is a polymorphism of $\gamma$ (and $\gamma$ is invariant under or admits $h$) if, for every $x^1, \ldots, x^k \in \text{Feas}(\gamma)$, we have $h(x^1, \ldots, x^k) \in \text{Feas}(\gamma)$. We say that $h$ is a polymorphism of a language $\Gamma$ if it is a polymorphism of every $\gamma \in \Gamma$.

The following notion, which involves a collection of $k$ k-ary polymorphisms, plays an important role in the complexity classification of Boolean valued constraint languages [8], as we will see in Theorem 7 in Section 2.3.

Definition 6. Let $\gamma$ be a weighted relation on $D$. A list $\langle h_1, \ldots, h_k \rangle$ of k-ary polymorphisms of $\gamma$ is a k-ary multimorphism of $\gamma$ (and $\gamma$ admits $\langle h_1, \ldots, h_k \rangle$) if, for every $x^1, \ldots, x^k \in \text{Feas}(\gamma)$, we have

$$\sum_{i=1}^{k} \gamma(h_i(x^1, \ldots, x^k)) \leq \sum_{i=1}^{k} \gamma(x^i).$$

$\langle h_1, \ldots, h_k \rangle$ is a multimorphism of a language $\Gamma$ if it is a multimorphism of every $\gamma \in \Gamma$.

It is known that all operators in Definition 3 preserve multimorphisms [8, 15]. In particular, for a crisp relation all polymorphisms are preserved.

2.3 Boolean VCSPs

For the rest of the paper let $D = \{0, 1\}$. We define some important operations on $D$. For any $a \in D$, $c_a$ is the constant unary operation such that $c_a(x) = a$ for all $x \in D$. Operation $\neg$ is the unary negation, i.e. $\neg(0) = 1$ and $\neg(1) = 0$. Binary operation $\min$ (max) returns the smaller (larger) of its two arguments with respect to the order $0 < 1$. Ternary operation $\text{Mn}$ (for minority) is the unique ternary operation on $D$ satisfying $\text{Mn}(x, x, y) = \text{Mn}(x, y, x) = \text{Mn}(y, x, x) = y$ for all $x, y \in D$. Ternary operation $\text{Mj}$ (for majority) is the unique ternary operation on $D$ satisfying $\text{Mj}(x, x, y) = \text{Mj}(x, y, x) = \text{Mj}(y, x, x) = x$ for all $x, y \in D$.

Theorem 7 ([8]). Let $\Gamma$ be a Boolean valued constraint language. Then, $\Gamma$ is tractable if it admits any the following eight multimorphisms $\langle c_0 \rangle$, $\langle c_1 \rangle$, $\langle \min, \min \rangle$, $\langle \max, \max \rangle$, $\langle \min, \max \rangle$, $\langle \min, \max \rangle$, $\langle \text{Mn}, \text{Mn}, \text{Mn} \rangle$, $\langle \text{Mj}, \text{Mj}, \text{Mj} \rangle$, $\langle \text{Mj}, \text{Mj}, \text{Mn} \rangle$. Otherwise, $\Gamma$ is intractable.

We note that Theorem 7 is a generalisation of Schaefer’s classification of $\{0, \infty\}$-valued constraint languages [23] and Creignou’s classification of $\{0, 1\}$-valued constraint languages [9].

2.4 Boolean surjective VCSPs

The following definition is new in this paper and crucial for our main result.

Definition 8. A relation $\rho$ is a downset (upset) if for any $x, y$ such that $x \geq y$ ($x \leq y$) and $x \in \rho$ it holds $y \in \rho$. We will refer to relations that are a projection of a downset (upset) as PDS (PUS). A weighted relation $\gamma$ is called a PDS (PUS) weighted relation if both $\text{Feas}(\gamma)$ and $\text{Opt}(\gamma)$ are PDS (PUS). A language $\Gamma$ is called PDS (PUS) if every weighted relation from $\Gamma$ is PDS (PUS).
Example 9. Relation $\rho = \{(0,0),(0,1),(1,0)\}$ is a downset and hence also a PDS, while $\rho' = \{(0,0,0),(0,1,1),(1,0,0)\}$ is a PDS (as $\rho' = \Pr_{(1,2,2)}(\rho)$) but not a downset. Relation $\rho_\pi$ is a PDS relation and also a PDS weighted relation.

Observation 10. Any PDS relation can be written as a sum of a downset and binary equality relations. More formally, if $\rho : D^r \to \{0, \infty\}$ is a PDS then we can write

$$\rho(x_1, \ldots, x_r) = \rho'(x_{\pi(1)}, \ldots, x_{\pi(r')}) + \sum_{j=r'+1}^r \rho_\pi(x_{\pi(j)}, x_{i_j}),$$

where $\rho' : D' \to \{0, \infty\}$ is a downset, $\pi$ is a permutation of $[r]$, and $i_j \in \{\pi(1), \ldots, \pi(r')\}$ for every $r' + 1 \leq j \leq r$.

The following result is our main contribution.

Theorem 11 (Main). Let $\Gamma$ be a Boolean valued constraint language. Then, $\Gamma$ is s-tractable if it admits any of the following six multimorphisms $\langle \min, \min \rangle$, $\langle \max, \max \rangle$, $\langle \min, \max \rangle$, $\langle Mn, Mn, Mn \rangle$, $\langle Mj, Mj, Mj \rangle$, or $\Gamma$ is PDS, or $\Gamma$ is PUS. Otherwise, $\Gamma$ is s-intractable.

Theorem 11 generalises the following two previously established results.

Theorem 12 ([10]). Let $\Gamma$ be a Boolean $\{0, \infty\}$-valued constraint language. Then, $\Gamma$ is s-tractable if it admits any of the following four polymorphisms $\min$, $\max$, $Mn$, $Mj$. Otherwise, $\Gamma$ is s-intractable.

Theorem 13 ([26]). Let $\Gamma$ be a Boolean $\{0, 1\}$-valued constraint language. Then, $\Gamma$ is s-tractable if it admits the $\langle \min, \max \rangle$ multimorphism, or $\Gamma$ is PDS, or $\Gamma$ is PUS. Otherwise, $\Gamma$ is s-intractable.

Theorem 13 is stated differently in [26] as the definition of PDS (PUS) languages is introduced in this paper. In fact, it was not a priori clear what the right condition for tractability should be for $\mathbb{Q}$-valued constraint languages. Note that for $\{0, 1\}$-valued languages, the condition $\text{Feas}(\Gamma)$ being PDS or PUS is vacuously true. Thus, a $\{0, 1\}$-valued language $\Gamma$ is PDS (PUS) if $\text{Opt}(\Gamma)$ is PDS (PUS) and this is equivalent to $\Gamma$ being almost-min-min (almost-max-max) [26].

Recall that $\neg$ is the unary negation operation. For a weighted relation $\gamma$, we define $\neg(\gamma)$ to be the weighted relation $\neg(\gamma)(x) = \gamma(\neg(x))$. For a language $\Gamma$, we define $\neg(\Gamma) = \{\neg(\gamma) \mid \gamma \in \Gamma\}$.

The following observation follows from Definition 8.

Observation 14. A valued constraint language $\Gamma$ is PDS if and only if $\neg(\Gamma)$ is PUS.

Lemma 15. Let $\Gamma$ be a Boolean valued constraint language. Then, $\text{VCSP}_s(\Gamma) \equiv_p \text{VCSP}_s(\neg(\Gamma))$.

Proof. Since $\neg(\neg(\Gamma)) = \Gamma$, it suffices to show $\text{VCSP}_s(\Gamma) \leq_p \text{VCSP}_s(\neg(\Gamma))$. For an instance $I = (V, D, \phi_I)$ of $\text{VCSP}_s(\Gamma)$ with $V = \{x_1, \ldots, x_n\}$, construct the instance $I' = (V, D, \phi_{I'})$ of $\text{VCSP}_s(\neg(\Gamma))$ by replacing every valued constraint $\gamma(x)$ in $\phi_I$ by $\neg(\gamma)(x)$. By construction, for every assignment $s : V \to D$ we have $\phi_I(s(x_1), \ldots, s(x_n)) = \phi_{I'}(\neg(s(x_1)), \ldots, \neg(s(x_n)))$.

Thus $s'$ is an optimal assignment to $I'$ if and only if $\neg(s')$ is an optimal assignment to $I$.

\begin{flushright} \qed \end{flushright}
The following reductions were shown in Section 1.

**Lemma 16.**
\[
\text{VCSP}(\Gamma) \leq_p \text{VCSP}_s(\Gamma)
\]
\[
\text{VCSP}_s(\Gamma) \leq_p \text{VCSP}(\Gamma \cup \{\rho_0, \rho_1\})
\]

**Proof of Theorem 11.** The s-tractability of languages admitting any of the six multimorphisms in the statement of the theorem follows from Theorem 7 via (2). The s-tractability of PDS languages follows from Theorem 40, proved in Section 4. The s-tractability of PUS languages is then a simple corollary of Theorem 40, Observation 14, and Lemma 15. Finally, the s-intractability of the remaining languages follows from Theorem 25, proved in Section 3. □

## 3 Hardness proofs

Recall that $D = \{0, 1\}$. We denote by $\text{Soft}(\rho)$ the soft variant of relation $\rho$, i.e., $\text{Soft}(\rho)(x) = 0$ if $x \in \rho$ and $\text{Soft}(\rho)(x) = 1$ otherwise. Let us define weighted relations $\gamma_0 = \text{Soft}(\rho_0)$, $\gamma_1 = \text{Soft}(\rho_1)$, and $\gamma_\infty = \text{Soft}(\rho_\infty)$.

We define the binary relation $\rho_{\leq} = \{(0, 0), (0, 1), (1, 1)\}$. For $r \in \{3, 4\}$, we define the $r$-ary relation $A_r = \{x \mid \sum_{i=1}^{r} x_i \equiv 0 \pmod{2}\}$. For any $r \geq 1$, we denote by $0^r$ ($1^r$) the zero (one) $r$-tuple. By $\oplus$ we denote the addition modulo 2 operation on $\{0, 1\}$ and its extension to tuples, to which it is applied componentwise.

The structure of our hardness proofs essentially follows [26] but we need to deal with $\mathbb{Q}$-valued constraint languages as opposed to the special case of $\{0, 1\}$-valued languages.

**Lemma 17.** Let $\rho$ be a relation invariant under $c_0$ but not invariant under $c_1$. Then $\rho_0 \in \{\rho\}^*$.

**Proof.** Let $\rho'$ be a smallest-arity relation in $\{\rho\}^*$ that is not invariant under $c_1$, and denote its arity by $r$. It holds $0^r \in \rho'$ and $1^r \not\in \rho'$. If $r \geq 2$, we would obtain an $(r-1)$-ary relation that is not invariant under $c_1$ by taking an identification of $\rho'$. Therefore, $r = 1$ and $\rho' = \rho_0$. □

**Lemma 18.** Let $\gamma$ be a non-crisp weighted relation that admits multimorphism $\langle c_0 \rangle$. Then either $\gamma_0$, $\gamma_\infty$, or $\rho_{\leq}$ belongs to $\{\gamma\}^*$.

**Proof.** Let $\gamma'$ be a smallest-arity non-crisp weighted relation in $\{\gamma\}^*$, and denote its arity by $r$. There exists an $r$-tuple $u$ such that $\gamma'(0^r) < \gamma'(u) < \infty$. If for any distinct coordinates $i, j$ it holds $u_i = u_j$, we would obtain an $(r-1)$-ary non-crisp weighted relation from $\gamma'$ by identifying coordinates $i, j$. Therefore, $r \leq 2$. If $r = 1$, it holds $\gamma'(0) < \gamma'(1) < \infty$, and hence we can obtain $\gamma_0$ from $\gamma'$ by adding a constant and scaling.

Assume now $r = 2$. It may not hold $\gamma'(0, 0) < \gamma'(1, 1) < \infty$, because we would obtain a unary non-crisp weighted relation by taking an identification of $\gamma'$, and hence either $\gamma'(1, 1) = \infty$ or $\gamma'(1, 1) = \gamma'(0, 0)$. Without loss of generality (by permutation of coordinates), assume that $\gamma'(0, 0) < \gamma'(0, 1) < \infty$. If $\gamma'(1, 1) = \infty$, relation $\text{Feas}(\gamma')$ is not invariant under $c_1$, and hence we have $\rho_0 \in \{\gamma\}^*$ by Lemma 17. But then we would obtain a unary non-crisp weighted relation by $0$-pinning $\gamma'$ at the first coordinate, which contradicts the minimality of $\gamma'$. Therefore, it holds $\gamma'(1, 1) = \gamma'(0, 0)$.

If $\gamma'(1, 0) = \infty$, we have $\text{Feas}(\gamma') = \rho_{\leq}$. Otherwise, we take $\gamma'' \in \{\gamma\}^*$ defined as $\gamma''(x, y) = \gamma'(x, y) + \gamma'(y, x)$. It holds $\gamma''(0, 0) = \gamma''(1, 1) < \gamma''(0, 1) = \gamma''(1, 0) < \infty$. We obtain $\gamma_\infty$ from $\gamma''$ by adding a constant and scaling. □
Lemma 19. Let \( \rho \) be a relation invariant under \( c_0 \) and \( c_1 \) but not invariant under \( \neg \). Then \( \rho \in \{ \rho \}^* \).

Proof. Let \( \rho' \) be a smallest-arity relation in \( \{ \rho \}^* \) that is not invariant under \( \neg \), and denote its arity by \( r \). It holds \( 0', 1^r \in \rho' \). There exists an \( r \)-tuple \( u \in \rho' \) such that \( \neg(u) \notin \rho' \). For any distinct coordinates \( i, j \) where \( u_i = u_j \), we can obtain an \((r - 1)\)-ary relation that is not invariant under \( \neg \) by identifying coordinates \( i, j \) of \( \rho' \). Therefore, no such \( i, j \) exist, which implies \( r = 2 \). Without loss of generality (by permutation of coordinates), \( (0, 1) \in \rho' \), and hence \( \rho' = \rho \).

Lemma 20. Let \( \rho \) be a relation invariant under \( c_0 \) and max but not invariant under min. Then \( \rho \in \{ \rho \}^* \).

Proof. Let \( \rho' \) be a smallest-arity relation in \( \{ \rho \}^* \) that is not invariant under min, and denote its arity by \( r \). There exist \( r \)-tuples \( u, v \in \rho' \) such that \( \min(u, v) \notin \rho' \). Suppose that \( \rho' \) is not invariant under \( c_1 \), and hence we have \( \rho_0 \in \{ \rho \}^* \) by Lemma 17. For any coordinate \( i \) such that \( u_i = v_i = 0 \), we obtain an \((r - 1)\)-ary relation that is not invariant under min by 0-pinning \( \rho' \) at coordinate \( i \), which contradicts the minimality of \( \rho' \). Therefore, there is no such coordinate, and hence \( \max(u, v) = 1' \in \rho' \). But then \( \rho' \) is invariant under \( c_1 \), which is a contradiction.

As \( \min(x, y) = \neg(\max(\neg(x), \neg(y))) \), relation \( \rho' \) is not invariant under \( \neg \). By Lemma 19, we have \( \rho \in \{ \rho \}^* \).

Lemma 21. Let \( \rho \) be a relation invariant under \( c_0 \) and min but not a PDS. Then \( \rho \in \{ \rho \}^* \).

Proof. Let \( \rho' \) be a smallest-arity relation in \( \{ \rho \}^* \) that is not a PDS, and denote its arity by \( r \). Suppose there are distinct coordinates \( i, j \) such that \( x_i = x_j \) for all \( x \in \rho' \). Denote by \( \rho'' \) the \((r - 1)\)-ary relation obtained from \( \rho' \) by identifying coordinates \( i, j \). As \( \rho' \) is a projection of \( \rho'' \), \( \rho'' \) is not a PDS, which contradicts the minimality of \( \rho' \). Therefore, for any distinct coordinates \( i, j \), there exists a \( z^{(i,j)} \in \rho' \) with \( z^{(i,j)}_i \neq z^{(i,j)}_j \). By Observation 10, a relation with this property is a PDS if and only if it is a downset.

Suppose that \( \rho' \) is not invariant under \( c_1 \), and hence we have \( \rho_0 \in \{ \rho \}^* \) by Lemma 17. As \( \rho' \) is not a downset, there exist \( r \)-tuples \( u, v \) such that \( u \geq v, u \in \rho' \), and \( v \notin \rho' \). Let \( K \) be the set of coordinates \( k \) where \( u_k = 1 \); it holds \( K \subseteq [r] \) because \( 1^r \notin \rho' \). Denote by \( \rho'' \) the \(|K|\)-ary relation obtained by 0-pinning \( \rho' \) at coordinates \([r] \setminus K \). Relation \( \rho'' \) is not a downset. Moreover, for any distinct coordinates \( i, j \in K \), the \( r \)-tuple \( w = \min(z^{(i,j)}, u) \in \rho' \) satisfies the property that \( w_i \neq w_j \) and \( w_k = 0 \) for all \( k \in [r] \setminus K \). Therefore, relation \( \rho'' \) is not a PDS, which contradicts the minimality of \( \rho' \).

Relation \( \rho' \) is thus invariant under \( c_1 \). Suppose that it is also invariant under \( \neg \). Therefore, we may assume that \( z^{(i,j)}_i = 0, z^{(i,j)}_j = 1 \). For \( i \in [r] \), let \( e^{(i)} \) denote the \( r \)-tuple with \( e^{(i)}_i = 1 \) and \( e^{(i)}_j = 0 \) for all \( j \neq i \). It holds \( e^{(i)} = \min(z^{(i,j)}, \ldots, z^{(r,i)}) \), and hence \( e^{(i)} \in \rho' \). As \( \max(x, y) = \neg(\min(-x, -y)) \), relation \( \rho' \) is invariant under max. Any \( r \)-tuple \( x \) can be expressed as the max of \( e^{(i)} \) for \( i \) such that \( x_i = 1 \), and therefore \( x \in \rho' \). But then \( \rho' \) contains all \( r \)-tuples, which is a contradiction.

Relation \( \rho' \) is thus not invariant under \( \neg \). By Lemma 19, we have \( \rho \in \{ \rho \}^* \).

Lemma 22. Let \( \rho \) be a relation invariant under \( c_0 \) and \( M_n \) but not invariant under \( M_{M_j} \). Then either \( A_3 \) or \( A_4 \) belongs to \( \{ \rho \}^* \).
Proof. Let $\rho'$ be a smallest-arity relation in $\{\rho\}^*$ that is not invariant under $M_j$, and denote its arity by $r$. As $0^r \in \rho'$ and $M_n(x,y,0^r) = x \oplus y$, relation $\rho'$ is closed under the $\oplus$ operation. Let $u,v,w \in \rho'$ be $r$-tuples such that $M_j(u,v,w) \not\in \rho'$. Let $s = u \oplus v \oplus w$, $u' = u \oplus s$, $v' = v \oplus s$, $w' = w \oplus s$; it holds $s,u',v',w' \in \rho'$. Because $M_j(u',v',w') = M_j(u,v,w) \oplus s$, it holds $M_j(u',v',w') \not\in \rho'$.

Moreover, $u' \oplus v' \oplus w' = 0^r$, and hence $(u'_i,v'_i,w'_i) \in \{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\}$ at any coordinate $i$. For any distinct $i,j$ where $u'_i = u'_j$, $v'_i = v'_j$, and $w'_i = w'_j$, we can obtain an $(r-1)$-ary relation that is not invariant under $M_j$ by identifying coordinates $i,j$ of $\rho'$. Therefore, no such $i,j$ exist, which implies $r \leq 4$. All unary and binary relations are invariant under $M_j$, and hence $r \geq 3$.

If $r = 3$, we have without loss of generality (by permutation of coordinates) $u' = (0,1,1)$, $v' = (1,0,1)$, $w' = (1,1,0)$, and $M_j(u',v',w') = (1,1,1) \not\in \rho'$. Then $(1,0,0) \not\in \rho'$ as $u' \oplus (1,0,0) = (1,1,1)$: similarly $(0,1,0) \not\in \rho'$ and $(0,0,1) \not\in \rho'$. Therefore, $\rho' = A_3$.

If $r = 4$, we have without loss of generality (again, by permutation of coordinates) $u' = (0,0,1,1)$, $v' = (0,1,0,1)$, $w' = (0,1,1,0)$, and $M_j(u',v',w') = (0,1,1,1) \not\in \rho'$. Suppose that $\rho'$ is not invariant under $c_1$. By Lemma 17, we have $\rho_0 \in \{\rho\}^*$, and hence we get a contradiction with the minimality of $\rho'$ by taking its 0-pinning at the first coordinate. Therefore, $(1,1,1,1) \in \rho'$. As $\neg(x) = x \oplus 1^r$, relation $\rho'$ is invariant under $\neg$, and hence $A_4 \subseteq \rho'$. For any $x \not\in A_4$, it holds $x \oplus (0,1,1,1) \in A_4 \subseteq \rho'$, and therefore $x \not\in \rho'$, which gives us $\rho' = A_4$.

Lemma 23. If $\rho_0 \in \Gamma$ then $VCSP_\rho(\Gamma \cup \{\rho_0, \rho_1\}) \leq_p VCSP_\rho(\Gamma)$.

Proof. Let $I$ be an instance of $VCSP_\rho(\Gamma \cup \{\rho_0, \rho_1\})$ with $n$ variables. For each pair of variables $y_0,y_1$, we construct an instance of $VCSP_\rho(\Gamma)$ assuming that $y_0$ and $y_1$ are assigned labels 0 and 1 respectively in an optimal solution to $I$. Solving these $O(n^2)$ instances and taking the best assignment among them then solves $I$.

We impose constraints $\rho_0(y_0,x)$ and $\rho_0(x,y_1)$ for all variables $x$ to ensure that $y_0 = 0$ and $y_1 = 1$ in any surjective assignment. Then we replace each constraint of the form $\rho_0(x)$ with $\rho_0(x,y_0)$, each constraint of the form $\rho_1(x)$ with $\rho_0(y_1,x)$, and keep the remaining constraints from $I$. The obtained instance preserves the objective value of all assignments satisfying $y_0 = 0, y_1 = 1$.

Lemma 24. Languages $\{A_3, \gamma_0\}$ and $\{A_4, \gamma_\ast\}$ are both $s$-intractable.

Proof. First we show a reduction from the optimisation variant of the Minimum Distance problem, which is NP-hard [27], to $VCSP_\rho(\{A_3, \gamma_0\})$. A problem instance is given as an $m \times n$ matrix $H$ over the field $D = \{0,1\}$, and the objective is to find a non-zero vector $x \in D^n$ satisfying $H \cdot x = 0^n$ with the minimum weight (i.e. $\sum_{i=1}^n x_i$).

Note that $\rho_0 = \text{Opt}(\gamma_0)$, and therefore we may use relation $\rho_0$ as well. We construct a $VCSP_\rho$ instance $I$ as follows: Let $x_1,\ldots,x_n$ be variables corresponding to the elements of the sought vector $x$. The requirement $H \cdot x = 0^n$ can be seen as a system of $m$ linear equations, each in the form $\bigoplus_{i=1}^k x_{a_i} = 0$ for a set $\{a_1,\ldots,a_k\} \subseteq [n]$. We encode such an equation by introducing new variables $y_0,\ldots,y_k$ and imposing constraints $\rho_0(y_0), A_3(y_{i-1},x_{a_i},y_i)$ for all $i \in [k]$, and $\rho_0(y_k)$. These ensure that each variable $y_j$ is assigned the value of the prefix sum $\bigoplus_{i=1}^j x_{a_i}$, and that the total sum equals 0. Finally, we encode the objective function of the Minimum Distance problem by imposing constraints $\gamma_0(x_1),\ldots,\gamma_0(x_n)$.
Observe that any non-zero vector \( \mathbf{x} \in \mathbb{D}^n \) satisfying \( H \cdot \mathbf{x} = 0^m \) corresponds to a surjective feasible assignment of \( I \) and vice versa, while the weight of \( \mathbf{x} \) equals the objective value of the assignment. Therefore, solving \( I \) gives an optimal solution to the Minimum Distance problem.

Finally, we show that \( \{A_4, \gamma_\neg\} \) is s-intractable by a reduction from \( \text{VCSP}_4(\{A_3, \gamma_0\}) \) to \( \text{VCSP}_4(\{A_4, \gamma_\neg\}) \). Given an instance \( I \), we construct an instance \( I' \) by introducing a new variable \( w \) and replacing each constraint of the form \( A_3(x, y, z) \) with \( A_4(x, y, z, w) \) and each constraint of the form \( \gamma_0(x) \) with \( \gamma_\neg(x, w) \). Because language \( \{A_4, \gamma_\neg\} \) admits multimorphism \( (\neg) \), any assignment to \( I' \) has the same objective value as its negation, and hence we may without loss of generality assume \( w = 0 \). It holds \( A_3(x, y, z) = A_4(x, y, z, 0) \) \( \gamma_0(x) = \gamma_\neg(x, 0) \), and therefore the objective value of any assignment to \( I \) is preserved by the corresponding assignment to \( I' \). As for the disallowed (non-surjective) assignments to \( I \): The all-zero assignment corresponds to a disallowed assignment to \( I' \); the all-one assignment corresponds to a surjective assignment to \( I' \), but it maximises the objective function.

**Theorem 25.** Let \( \Gamma \) be a Boolean valued constraint language that does not admit any of the following six multimorphisms \( (\min, \min) \), \( (\max, \max) \), \( (\min, \max) \), \( (\text{Mn, Mn, Mn}) \), \( (\text{Mj, Mj, Mj}) \), \( (\text{Mj, Mj, Mn}) \). If \( \Gamma \) is neither a PDS language nor a PUS language then it is s-intractable.

**Proof.** Suppose that \( \Gamma \) is not s-intractable. Because of the reduction (1), \( \Gamma \) is not intractable and hence, by Theorem 7, it admits at least one of the multimorphisms \( (\rho_0) \) and \( (\rho_1) \). Suppose that \( \Gamma \) admits \( (\rho_0) \); our goal is to prove that it is a PDS language. If \( \Gamma \) admits \( (\rho_1) \) then we consider \( (\neg) \), which admits \( (\rho_0) \). Thus proving that \( (\neg) \) is PDS gives, by Observation 14, that \( \Gamma \) is PUS.

If \( \rho_\leq \in \Gamma^* \) then, by Lemma 23, the language \( \Gamma^* \cup \{\rho_0, \rho_1\} \) is not s-intractable. Since \( \{\rho_0, \rho_1\} \) is not invariant under \( (\rho_0) \) and \( (\rho_1) \), \( \Gamma^* \cup \{\rho_0, \rho_1\} \) is intractable by Theorem 7. However, this contradicts the reduction (1). Therefore, we may assume that \( \rho_\leq \notin \Gamma^* \).

By the classification of \( \{0, \infty\} \)-valued languages given in Theorem 12, set \( \Phi = \text{Feas}(\Gamma) \cup \text{Opt}(\Gamma) \) is invariant under \( \min, \max, \text{Mn, Mn, Mn} \), \( \text{Mj, Mj, Mj} \). Let us consider these cases; we will show they all imply that \( \Phi \) is in fact invariant under \( \min \).

If \( \Phi \) is invariant under \( \max \), then by Lemma 20 it is also invariant under \( \min \). If \( \Phi \) is invariant under \( \text{Mj} \), then it is also invariant under \( \min(\mathbb{x}, \mathbb{y}) = \text{Mj}(\mathbb{0}, \mathbb{x}, \mathbb{y}) \).

If \( \Phi \) is invariant under \( \text{Mn} \), then \( \Gamma \) is not crisp, otherwise it would admit multimorphism \( (\text{Mn, Mn, Mn}) \). By Lemma 18, we have \( \gamma_0 \in \Gamma^* \) or \( \gamma_\neg \in \Gamma^* \). Suppose that \( \Phi \) is not invariant under \( \text{Mj} \) (otherwise we are done); by Lemma 22 we have \( A_3 \in \Gamma^* \) or \( A_4 \in \Gamma^* \). If \( A_3 \in \Gamma^* \) then we have \( \rho_0 \in \Gamma^* \) as \( \rho_0(x) = A_3(x, x, x) \), and hence \( \gamma_0 \) can be obtained from \( \gamma_\neg \) by pinning. If \( \gamma_0 \in \Gamma^* \) then also \( \rho_0 = \text{Opt}(\gamma_0) \in \Gamma^* \), and hence \( A_3 \) can be obtained from \( A_4 \) by pinning. Therefore, it holds \( \{A_3, \gamma_0\} \subseteq \Gamma^* \) or \( \{A_4, \gamma_\neg\} \subseteq \Gamma^* \), and \( \Gamma^* \) is s-intractable by Lemma 24.

All relations in \( \Phi \) are therefore invariant under \( \min \) and, by Lemma 21, they are PDS. \( \square \)

### 4 Tractability of PDS languages

Let \( V \) be a finite set. A **set function** on \( V \) is a function \( f : 2^V \rightarrow \mathbb{Q}_{\geq 0} \cup \{\infty\} \) with \( f(\emptyset) = 0 \).

**Definition 26.** A set function \( f : 2^V \rightarrow \mathbb{Q}_{\geq 0} \cup \{\infty\} \) is **increasing** if \( f(X) \leq f(Y) \) for all \( X \subseteq Y \subseteq V \); it is **superadditive** if \( f(X) + f(Y) \leq f(X \cup Y) \) for all disjoint \( X, Y \subseteq V \); it is **posimodular** if \( f(X) + f(Y) \geq f(X \setminus Y) + f(Y \setminus X) \) for all \( X, Y \subseteq V \); and finally it is **submodular** if \( f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y) \) for all \( X, Y \subseteq V \).
It is known and easy to show that any superadditive set function is also increasing.

**Example 27.** Let \( U \) be a finite set and \( T \subseteq U \) a non-empty subset. We define a set function \( f \) on \( U \) by \( f(X) = 1 \) if \( T \subseteq X \) and \( f(X) = 0 \) otherwise. Intuitively, this corresponds to a soft NAND constraint if we interpret \( T \) as its scope and \( X \) as the set of variables assigned \( \text{true} \). The set function \( f \) is superadditive, and hence also increasing.

We now formally define the Min-Cut problem introduced in Section 1.

**Definition 28.** An instance of the Min-Cut (MC) problem is given by a graph \( G = (V, E) \) with edge weights \( w : E \to \mathbb{Q}_{>0} \). The goal is to minimise the objective function \( g \), which is a set function on \( V \) defined by \( g(X) = \sum_{\{u,v\} \in E, |\{u,v\} \cap X| = 1} w(u,v) \).

Note that \( g \) from Definition 28 is posimodular.

Any set \( X \) such that \( \emptyset \subseteq X \subseteq V \) is called a solution of the MC problem. Note that a cut \( (X, V \setminus X) \) corresponds to two solutions, namely \( X \) and \( V \setminus X \). Any solution that minimises the objective function \( g \) is called optimal, and any optimal solution with no proper subset being an optimal solution is called minimal. Any two different minimal optimal solutions \( X, Y \) are disjoint, as \( X \setminus Y \) and \( Y \setminus X \) are also optimal solutions (by the posimodularity of \( g \)).

We will need the following lemma relating the number of optimal and minimal optimal solutions to the MC problem. Note that this bound is tight for (unweighted) paths and cycles with at most one path attached to each vertex.

**Lemma 29.** For any instance of the MC problem on a connected graph with \( n \) vertices and \( p \) minimal optimal solutions, there are at most \( p(p-1) + 2(n-p) \) optimal solutions.

We will prove the lemma by induction on \( n \), closely following the proof that establishes the cactus representation of minimum cuts in [14]. We note that the cactus representation could be applied directly to obtain a weaker bound of \( p(p-1) + O(n) \) but we do not know how to achieve the exact bound using it.

**Proof.** Let us denote the number of optimal solutions by \( s \). The lemma trivially holds for \( n = 1 \) as there are no solutions at all. For \( n = 2 \), both solutions are minimal optimal and the bound holds. Assume \( n \geq 3 \). A solution \( X \) is called a star if \( |X| = 1 \) or \( |X| = n-1 \), otherwise it is called proper. First we consider the case where every optimal solution is a star. Let us denote the minimum cuts by \( (\{v_1\}, V \setminus \{v_1\}), \ldots, (\{v_h\}, V \setminus \{v_h\}) \). If \( h = 1 \), then we have \( s = p = 2 \) and the bound holds. Otherwise there are \( 2h \) optimal solutions but only \( h \) of them are minimal \( (\{v_1\}, \ldots, \{v_h\}) \), and hence

\[
 p(p-1) + 2(n-p) = 2h + (h-1) \cdot (h-2) - 2 + 2(n-h) \geq 2h = s
\]

as it holds \( n \geq h \) and \( n \geq 3 \).

From now on we assume that there is a proper optimal solution, and hence \( n \geq 4 \). We say that solutions \( X, Y \) cross if none of \( X \setminus Y, Y \setminus X, X \cap Y, V \setminus (X \cup Y) \) is empty. Note that only proper solutions might cross. If every proper optimal solution is crossed by some optimal solution, then the graph is a cycle with edges of equal weight [14, Lemma 7.1.3]. In that case, there are \( n(n-1) \) optimal solutions (all sets of contiguous vertices except for \( \emptyset \) and \( V \)) and \( n \) minimal optimal solutions (all singletons), and therefore the bound holds.

Finally, assume that there is a proper optimal solution that is not crossed by any optimal solution, and denote the corresponding minimum cut by \((V_1, V_2)\). For any optimal solution
Lemma 31. There is a polynomial-time algorithm that, given an instance of the GMC problem, either finds a solution $X$ with $f(X) + g(X) = 0$, or determines that $\lambda = \infty$, or determines that $0 < \lambda < \infty$.

Proof. A solution $X$ with $f(X) + g(X) = 0$ satisfies $f(X) = g(X) = 0$, and hence it does not cut any edge. Because set function $f$ is increasing, we may assume that $X$ is a single connected component. The algorithm simply tries each connected component as a solution, which takes a linear number of queries to the oracle for $f$.

The case of $\lambda = \infty$ occurs only if $f(X) = \infty$ for all solutions $X$. Because $f$ is increasing, it is sufficient to check all solutions of size 1.

In view of Lemma 31, we can assume that $0 < \lambda < \infty$. Our goal is to show that, for a given $\alpha \geq 1$, all $\alpha$-optimal solutions to a GMC instance can be found in polynomial time. This will be proved in Theorem 36. But first we need the following four lemmas exhibiting certain structural properties of optimal solutions to GMC instances.
Lemma 32. For any instance $J$ of the GMC problem on a graph $G = (V, E)$ and any non-empty set $V' \subseteq V$, there is an instance $J'$ on the induced subgraph $G[V']$ that preserves the objective value of all solutions $X \not\subseteq V'$. In particular, any $\alpha$-optimal solution $X$ of $J$ such that $X \not\subseteq V'$ is $\alpha$-optimal for $J'$ as well.

Proof. Edges with exactly one endpoint in $V'$ need to be taken into account separately because they do not appear in the induced subgraph. We accomplish that by defining the new set function $f'$ as $f'(X) = f(X) + \sum_{(u,v) \in E, u \in V, v \not\in V'} w(u,v)$ for all $X \subseteq V'$. By construction, $f'$ is superadditive, and the objective value $f'(X) + g'(X)$ for any $X \not\subseteq V'$ equals $f(X) + g(X)$.

Note that the minimum objective value for $J'$ is greater than or equal to the minimum objective value for $J$. Therefore, any solution $X \not\subseteq V'$ that is $\alpha$-optimal for $J$ is also $\alpha$-optimal for $J'$.

Lemma 33. Let $X$ be an optimal solution to an instance of the GMC problem over vertices $V$ with $\lambda < \infty$, and $Y$ a minimal optimal solution to the underlying MC problem. Then $X \subseteq Y$, $X \not\subseteq V \setminus Y$, or $X$ is an optimal solution to the underlying MC problem.

Proof. Assume that $X \not\subseteq Y$ and $X \not\subseteq V \setminus Y$. If $Y \subseteq X$, we have $f(Y) \leq f(X)$ as $f$ is increasing, and hence $f(Y) + g(Y) \leq f(X) + g(X) < \infty$. Therefore, $Y$ is optimal for the GMC problem and $X$ is optimal for the MC problem. In the rest, we assume that $Y \not\subseteq X$.

By the posimodularity of $g$ we have $g(X) + g(Y) \geq g(X \setminus Y) + g(Y \setminus X)$. Because $Y \setminus X$ is a proper non-empty subset of $Y$, it holds $g(Y \setminus X) > g(Y)$, and hence $g(X) > g(X \setminus Y)$. But then $f(X) + g(X) > f(X \setminus Y) + g(X \setminus Y)$ as $\infty > f(X) \geq f(X \setminus Y)$. Set $X \setminus Y$ is non-empty, and therefore contradicts the optimality of $X$.

Lemma 34. For any instance of the GMC problem on $n$ vertices with $0 < \lambda < \infty$, the number of optimal solutions is at most $n(n-1)$. There is an algorithm that finds all of them in polynomial time.

Note that the bound of $n(n-1)$ optimal solutions precisely matches the known upper bound of $\binom{n}{2}$ for the number of minimum cuts [18]; the bound is tight for cycles.

Proof. Let $t(n)$ denote the maximum number of optimal solutions for such instances on $n$ vertices. We prove the bound by induction on $n$. If $n = 1$, there are no solutions and hence $t(1) = 0$. For $n \geq 2$, let $Y_1, \ldots, Y_p$ be the minimal optimal solutions to the underlying MC problem. As there exists at least one minimum cut and the minimal optimal solutions are all disjoint, it holds $2 \leq p \leq n$.

Suppose that the minimal optimal solutions cover all vertices, i.e. $\bigcup Y_i = V$. By Lemma 33, any optimal solution to the GMC problem is either a proper subset of some $Y_i$ or an optimal solution to the underlying MC problem. Restricting solutions to a proper subset of $Y_i$ is by Lemma 32 equivalent to considering a GMC problem instance on vertices $Y_i$, and hence the number of such optimal solutions is bounded by $t(|Y_i|) \leq |Y_i| \cdot (|Y_i| - 1)$. If the graph is connected, then by Lemma 29 there are at most $p(p-1) + 2(n-p)$ optimal solutions to the
underlying MC problem. Adding these upper bounds we get
\[ p(p - 1) + 2(n - p) + \sum_{i=1}^{p} |Y_i| \cdot (|Y_i| - 1) \]
\[ \leq p(p - 1) + 2(n - p) + (p - 1) \cdot 0 + (n - p + 1) \cdot (n - p) \]
\[ = n(n - 1) - 2(p - 2) \cdot (n - p) \]
\[ \leq n(n - 1). \]

If the graph is disconnected, then sets \( Y_1, \ldots, Y_p \) are precisely its connected components and the optimal solutions to the underlying MC problem are precisely unions of connected components (with the exception of \( \emptyset \) and \( V \)). We have \( 0 < \lambda \leq f(Y_i) + g(Y_i) = f(Y_i) \). Because \( f \) is superadditive, it holds \( f(Y_{i_1} \cup \cdots \cup Y_{i_k}) \geq f(Y_{i_1}) + \cdots + f(Y_{i_k}) \geq k\lambda \) for any distinct \( i_1, \ldots, i_k \), and hence no union of two or more connected components can be an optimal solution to the GMC problem. This gives us an upper bound of \( p \leq p(p - 1) + 2(n - p) \), and the rest follows as in the previous case.

Finally, suppose that \( \bigcup Y_i \neq V \), and hence the graph is connected. Let \( Z = V \setminus \bigcup Y_i \). By Lemma 33, any optimal solution to the GMC problem is a proper subset of some \( Y_i \), a proper subset of \( Z \), set \( Z \) itself, or an optimal solution to the underlying MC problem. Similarly as before, we get an upper bound of

\[ p(p - 1) + 2(n - p) + \sum_{i=1}^{p} |Y_i| \cdot (|Y_i| - 1) + |Z| \cdot (|Z| - 1) + 1 \]
\[ \leq p(p - 1) + 2(n - p) + p \cdot 1 \cdot 0 + (n - p) \cdot (n - p - 1) + 1 \]
\[ = n(n - 1) - 2(p - 1) \cdot (n - p) + 1 \]
\[ \leq n(n - 1). \]

Using a procedure generating all minimum cuts [28], it is straightforward to turn the above proof into a recursive algorithm that finds all optimal solutions in polynomial time.

**Lemma 35.** Let \( \alpha, \beta \geq 1 \). Let \( X \) be an \( \alpha \)-optimal solution to an instance of the GMC problem over vertices \( V \) with \( 0 < \lambda < \infty \), and \( Y \) an optimal solution to the underlying MC problem. If \( g(Y) < \lambda/\beta \), then
\[ (f(X \setminus Y) + g(X \setminus Y)) + (f(X \cap Y) + g(X \cap Y)) < \left( \alpha + \frac{2}{\beta} \right) \lambda; \]

otherwise, \( X \) is an \( \alpha \beta \)-optimal solution to the underlying MC problem.

**Proof.** If \( g(Y) \geq \lambda/\beta \), it holds \( g(X) \leq f(X) + g(X) \leq \alpha \lambda \leq \alpha \beta \cdot g(Y) \), and hence \( X \) is an \( \alpha \beta \)-optimal solution to the underlying MC problem. In the rest we assume that \( g(Y) < \lambda/\beta \).

Because \( g \) is posimodular, we have
\[ g(X) + g(Y) \geq g(X \setminus Y) + g(Y \setminus X) \]
\[ g(Y) + g(Y \setminus X) \geq g(X \cap Y) + g(\emptyset), \]
and hence
\[ g(X) + 2g(Y) \geq g(X \setminus Y) + g(X \cap Y). \]

By superadditivity of \( f \), it holds \( f(X) \geq f(X \setminus Y) + f(X \cap Y) \). The claim then follows from the fact that \( f(X) + g(X) + 2g(Y) < \left( \alpha + 2/\beta \right) \lambda \). \( \square \)
Finally, we can now prove that \( \alpha \)-optimal solutions to the GMC problem can be found in polynomial time.

**Theorem 36.** For any instance of the GMC problem on \( n \) vertices with \( 0 < \lambda < \infty \) and \( \alpha \in \mathbb{Z}_{\geq 1} \), the number of \( \alpha \)-optimal solutions is at most \( n^{20\alpha - 15} \). There is an algorithm that finds all of them in polynomial time.

Note that for a cycle on \( n \) vertices, the number of \( \alpha \)-optimal solutions to the MC problem is \( \Theta(n^{2\alpha}) \), and thus the exponent in our bound is asymptotically tight in \( \alpha \).

**Proof.** Let \( \beta \in \mathbb{Z}_{\geq 3} \) be a parameter and define \( \ell(x) = \frac{2(\beta+1)}{\beta-2} \cdot (\beta x - 3) \). We will prove, for any \( \alpha \geq 1 + 1/\beta \) such that \( \alpha \beta \) is an integer, that the number of \( \alpha \)-optimal solutions is at most \( n^{\ell(\alpha)} \). Choosing \( \beta = 4 \) then together with Lemma 34 gives the claimed bound.

Function \( \ell \) was chosen as the slowest growing function satisfying the following properties required in this proof: It holds \( \ell(x) + \ell(y) \leq \ell(x + y - 3/\beta) \) for any \( x, y \); and \( \ell(x) \geq 2\beta x \) for any \( x \geq 1 + 1/\beta \).

We prove the bound by induction on \( n + \alpha \). As it trivially holds for \( n \leq 2 \), we will assume \( n \geq 3 \) in the rest of the proof. Let \( Y \) be an optimal solution to the underlying MC problem with \( k = |Y| \leq n/2 \). If \( g(Y) \geq \lambda/\beta \), then, by Lemma 35, any \( \alpha \)-optimal solution to the GMC problem is an \( \alpha \beta \)-optimal solution to the underlying MC problem. Because \( g(Y) \geq \lambda/\beta > 0 \), the graph is connected, and hence there are at most \( 2^{2\alpha \beta} (\frac{n}{2\alpha \beta}) \leq n^{2\alpha \beta} \leq n^{\ell(\alpha)} \) such solutions by [18]. (In detail, [18, Theorem 6.2] shows that the number of \( \alpha \beta \)-optimal cuts in an \( n \)-vertex graph is \( 2^{2\alpha \beta-1} \cdot (\frac{n}{\alpha \beta}) \), and every cut corresponds to two solutions.)

From now on we assume that \( g(Y) < \lambda/\beta \), and hence inequality (15) holds. We have \((k/n)^{\ell(\alpha)} \leq (k/n)^{\ell(1+1/\beta)} = (k/n)^{2(\beta+1)} \leq (k/n)^{2(\beta+1)} = (k/n)^{2} \leq (k/n)^{2} \leq (k/n)^{2} \leq (1/n)^{2} \leq (1/n)^{2} \leq (1/n)^{5} \leq (1/n)^{5} \leq 1/(3^{5}n) < 1/128n \).

Consider any \( \alpha \)-optimal solution to the GMC problem \( X \).

If \( X \subseteq Y \), then, by Lemma 32, \( X \) is an \( \alpha \)-optimal solution to an instance on vertices \( Y \). By the induction hypothesis, there are at most \( k^{\ell(\alpha)} \leq (k/128n) \cdot n^{\ell(\alpha)} \) such solutions.

Similarly, if \( X \not\subseteq Y \), then \( X \) is an \( \alpha \)-optimal solution to an instance on vertices \( V \setminus Y \), and there are at most \((n - k)^{\ell(\alpha)} = (1 - k)^{\ell(\alpha)} \cdot n^{\ell(\alpha)} \leq (1 - k) \cdot n^{\ell(\alpha)} \) such solutions.

If \( Y \setminus X \) then, since \( Y = X \cap Y \) is a solution, \( f(X \cap Y) + g(X \cap Y) \geq \lambda \), which together with (15) implies that \( X \setminus Y \) is an \((\alpha - 1 + 2/\beta)\)-optimal solution on vertices \( V \setminus Y \). Similarly, if \( V \setminus Y \subseteq X \), then \( X \cap Y \) is an \((\alpha - 1 + 2/\beta)\)-optimal solution on vertices \( Y \). In either case, we bound the number of solutions depending on the value of \( \alpha \): For \( \alpha < 2 - 2/\beta \), there are trivially none; for \( \alpha = 2 - 2/\beta \), Lemma 34 gives a bound of \( n(n - 1) \leq n^{\ell(\alpha)-2\beta} \); and for \( \alpha > 2 - 2/\beta \) we get an upper bound of \( n^{\ell(\alpha)-2\beta} \leq n^{\ell(\alpha)-2\beta} \) by the induction hypothesis. The number of solutions is thus at least \( n^{\ell(\alpha)-2\beta} \leq (1/128n) \cdot n^{\ell(\alpha)} \) for any \( \alpha \).

Finally, we consider \( X \) such that \( \emptyset \not\subseteq X \setminus Y \subseteq V \setminus Y \) and \( \emptyset \not\subseteq X \cap Y \subseteq Y \), i.e. \( X \setminus Y \) and \( X \cap Y \) are solutions on vertices \( V \setminus Y \) and \( Y \) respectively. Let \( i \) be the integer for which

\[
\left(1 + \frac{i}{\beta}\right) \lambda \leq f(X \cap Y) + g(X \cap Y) < \left(1 + \frac{i+1}{\beta}\right) \lambda. \tag{19}
\]

Then, by (15), it holds \( f(X \setminus Y) + g(X \setminus Y) < (\alpha - 1 - (i - 2)/\beta) \lambda \). Therefore, \( X \cap Y \) is a \((1 + (i + 1)/\beta)\)-optimal solution on vertices \( Y \) and \( X \setminus Y \) is an \((\alpha - 1 - (i - 2)/\beta)\)-optimal solution on vertices \( V \setminus Y \). Because \( 0 \leq i \leq (\alpha - 2)/\beta + 1 \), we can bound the number of such
solutions by the induction hypothesis as at most

\[ k^{\ell \left(1 + \frac{1}{n} \right)} \cdot (n - k)^{\ell \left(\alpha - 1 - \frac{1}{n} \right)} \leq \left( \frac{k}{n} \right)^{\ell \left(1 + \frac{1}{n} \right)} \cdot n^{\ell \left(\alpha - 1 - \frac{1}{n} \right)} \leq \left( \frac{k}{n} \right)^{2(\beta + 1)} \cdot \frac{1}{2^\beta} \cdot n^{\ell(\alpha)}, \]

which is at most \(2 \cdot (k/128n) \cdot n^{\ell(\alpha)}\) in total for all \(i\).

By adding up the bounds we get that the number of \(\alpha\)-optimal solutions is at most \(n^{\ell(\alpha)}\). A polynomial-time algorithm that finds the \(\alpha\)-optimal solutions follows from the above proof using a procedure generating all \(\alpha\beta\)-optimal cuts \[28\].

**Remark 37.** For our reduction from the VCSPs over PDS languages, we need to find all \(\alpha\)-optimal solutions to the GMC problem. However, if one is only interested in a single optimal solution, the presented algorithm can be easily adapted to a more general problem.

Let \(f, g\) be set functions on \(V\) given by an oracle such that \(f : 2^V \to \mathbb{Q}_{\geq 0} \cup \{\infty\}\) is increasing and \(g : 2^V \to \mathbb{Q}_{\geq 0}\) satisfies the posimodularity and submodularity inequalities for intersecting pairs of sets (i.e. sets \(X, Y\) such that neither of \(X \cap Y, X \setminus Y, Y \setminus X\) is empty). The objective is to minimise the sum of \(f\) and \(g\).

The case when the optimum value \(\lambda = \infty\) can be recognised by checking all solutions of size 1. Assuming \(\lambda < \infty\), note that the proof of Lemma 33 works even for this more general problem. Let \(Y\) be a minimal optimal solution to \(g\). It follows that there is an optimal solution \(X\) to \(f + g\) such that \(X \subseteq Y, X \subseteq V \setminus Y,\) or \(X\) is itself a minimal optimal solution to \(g\) (as \(f\) is increasing). We can find all minimal optimal solutions to \(g\) in polynomial time [22, Theorem 10.11]. Restricting \(f, g\) to a subset of \(V\) preserves the required properties, and hence we can recursively solve the problem on \(Y\) and \(V \setminus Y\). Therefore, an optimal solution to \(f + g\) can be found in polynomial time.

We finish this section with establishing the \(s\)-tractability of PDS valued constraint languages. This is achieved in Theorem 40 via Lemma 39.

**Definition 38.** Let \(\gamma\) be an \(r\)-ary weighted relation on domain \(D = \{0, 1\}\). We will associate any \(r\)-tuple \(x \in D^r\) with the set \(X = \{i \in [r] \mid x_i = 1\}\) and use them interchangeably.

Let \(J\) be an instance of the GMC problem on vertices \([r]\) with \(J(X)\) denoting the objective value for any \(x \in D^r\) (including the all-zero and all-one tuples, which correspond to non-solutions \(\emptyset\) and \([r]\)). For any \(\alpha \geq 1\), we say that \(J\) \(\alpha\)-approximates \(\gamma\) if \(J(X) \leq \gamma(x) \leq \alpha \cdot J(X)\) for all \(x \in D^r\).

**Lemma 39.** Let \(\gamma\) be a weighted relation such that \(\text{Feas}(\gamma)\) is a downset, \(\text{Opt}(\gamma)\) is a PDS, and \(\gamma(x) = 0\) for \(x \in \text{Opt}(\gamma)\). There is a constant \(\alpha\) and an instance of the GMC problem that \(\alpha\)-approximates \(\gamma\).

**Proof.** We define a set function \(f_{\text{Feas}}\) on \([r]\) as \(f_{\text{Feas}}(X) = 0\) if \(x \in \text{Feas}(\gamma)\) and \(f_{\text{Feas}}(X) = \infty\) otherwise. Because \(\text{Feas}(\gamma)\) is a downset, \(f_{\text{Feas}}\) is superadditive.

By Observation 10, we can write \(\text{Opt}(\gamma)\) as a sum of a downset \(\rho\) on coordinates \(A \subseteq [r]\) and equalities \(x_i = x_j\) for \((i, j) \in E\) with \(|A| + |E| = r\). Let \(x|_A\) denote the projection of an \(r\)-tuple \(x\) to coordinates \(A\) in the same order as in \(\rho\). We define a set function \(f_{\text{Opt}}\) on \([r]\) as \(f_{\text{Opt}}(X) = 0\) if \(x|_A \in \rho\) and \(f_{\text{Opt}}(X) = |X \cap A|\) otherwise. Because \(\rho\) is a downset, \(f_{\text{Opt}}\) is superadditive.
We define a GMC instance $J'$ on vertices $[r]$, unit-weight edges $E$, and the superadditive set function $f_{\text{Feas}} + f_{\text{Opt}}$. By the construction, it holds

$$J'(X) = \infty \iff f_{\text{Feas}}(X) = \infty \iff x \notin \text{Feas}(\gamma) \iff \gamma(x) = \infty$$

and

$$J'(X) = 0 \iff f_{\text{Feas}}(X) = f_{\text{Opt}}(X) = 0 \land |\{i, j\} \cap X| \neq 1 \text{ for all } (i, j) \in E \\
\iff x \in \text{Feas}(\gamma) \land x|_A \in \rho \land x_i = x_j \text{ for all } (i, j) \in E \\
\iff x \in \text{Opt}(\gamma) \iff \gamma(x) = 0.$$  

Moreover, for any $X$ such that $0 < J'(X) < \infty$ it holds $1 \leq J'(X) \leq r$.

If $\gamma$ is crisp then the instance $J'$ $1$-approximates $\gamma$; otherwise let $\delta_{\min}, \delta_{\max}$ denote the minimum and maximum of $\{\gamma(x) \mid 0 < \gamma(x) < \infty\}$. We scale the weights of the edges and the superadditive function of $J'$ by a factor of $\delta_{\min}/r$ to obtain an instance $J$ such that $J(X) \leq \gamma(x)$ for all $X$. Setting $\alpha = r \cdot \delta_{\max}/\delta_{\min}$ then gives $\gamma(x) \leq \alpha \cdot J(X)$ for all $X$. \qed

**Theorem 40.** Let $\Gamma$ be a Boolean valued constraint language. If $\Gamma$ is PDS, then it is tractable.

**Proof.** Let $\Gamma' \subseteq \Gamma$ be a finite language. The feasibility relation $\text{Feas}(\gamma)$ for any $\gamma \in \Gamma'$ is a PDS and hence, by Observation 10, a sum of a downset and binary equality relations. A crisp equality constraint $\rho=(x, y)$ in an instance can be omitted after identifying the variables $x$ and $y$. Therefore, we will assume that $\text{Feas}(\gamma)$ is a downset. Moreover, we will assume that the minimum value assigned by $\gamma$ is 0, as changing values $\gamma(x)$ by the same constant for all $x \in D^{\min(\gamma)}$ affects all assignments equally.

By Lemma 39, for any $\gamma \in \Gamma'$, there is a constant $\alpha_{\gamma}$ and a GMC instance $J_\gamma$ that $\alpha_{\gamma}$-approximates $\gamma$. Let $\alpha$ be the smallest integer such that $\alpha \geq \alpha_{\gamma}$ for all $\gamma \in \Gamma'$.

Given a VCSP$_{\text{a}}(\Gamma')$ instance $I$ with an objective function $\phi_I(x_1, \ldots, x_n) = \sum_{i=1}^q w_i \cdot \gamma_i(x^i)$, we construct a GMC instance $J$ that $\alpha$-approximates $\phi_I$. For $i \in [q]$, we relabel the vertices of $J_{\gamma_i}$ to match the variables in the scope $x^i$ of the $i$th constraint (i.e. vertex $j$ is relabelled to $x_j^i$) and identify vertices in case of repeated variables. We also scale both the weights of the edges of $J_{\gamma_i}$ and the superadditive function by $w_i$. The instance $J$ is obtained by adding up the GMC instances $J_{\gamma_i}$ for all $i \in [q]$.

Let $x \in D^n$ denote a surjective assignment minimising $\phi_I$ and $y \in D^n$ an optimal solution to $J$ with $J(Y) = \lambda$. Because $J \alpha$-approximates $\phi_I$, it holds

$$\lambda \leq J(X) \leq \phi_I(x) \leq \phi_I(y) \leq \alpha \cdot J(Y) = \alpha \lambda,$$

and hence $x$ is an $\alpha$-optimal solution to $J$. By Lemma 31, we can determine whether $\lambda = 0$, in which case any optimal solution to $J$ is also optimal for $\phi_I$; and whether $\lambda = \infty$. If $0 < \lambda < \infty$, we find all $\alpha$-optimal solutions by Theorem 36. \qed

**5 Conclusions**

While the complexity of (valued) constraint languages is, as in this paper, studied mostly for finite languages, in all known cases the same result also holds for languages of infinite size. We now show that this is **not** the case for Boolean surjective VCSPs.
Example 41. We give an example of an infinite Boolean valued constraint language \( \Gamma \) that is a PDS language but \( \text{VCSP}_s(\Gamma) \) is NP-hard.

Let \( D = \{0, 1\} \). For any \( w \in \mathbb{Z}_{\geq 1} \), we define \( \gamma_w : D^3 \to \mathbb{Q} \) by \( \gamma(0, 0, 0) = 0 \), \( \gamma(\cdot, \cdot, 0) = w \), \( \gamma(x, y, 1) = 2 \) if \( x = y \) and \( \gamma(x, y, 1) = 1 \) if \( x \neq y \). Note that \( \text{Feas}(\gamma_w) = D^3 \) and \( \text{Opt}(\gamma_w) = \{(0, 0, 0)\} \) are PDS relations. Let \( \Gamma = \{\gamma_w \mid w \in \mathbb{Z}_{\geq 1}\} \).

Given an instance \( G = (V, E) \) of the Max-Cut problem with \( V = \{x_1, \ldots, x_n\} \) and no isolated vertices, we choose a value \( w > 2|E| \) and construct a \( \text{VCSP}_s(\Gamma) \) instance \( I = (V \cup \{z\}, D, \phi_I) \) with \( \phi_I(x_1, \ldots, x_n, z) = \sum_{(i,j) \in E, i < j} \gamma_w(x_i, x_j, z) \). Cuts in \( G \) are in one-to-one correspondence with assignments to \( I \) satisfying \( z = 1 \). In particular, a cut of value \( k \) corresponds to an assignment to \( I \) of value \( k + 2(|E| - k) = 2|E| - k \). Moreover, any surjective assignment that assigns label 0 to variable \( z \) is of value at least \( w > 2|E| \geq 2|E| - k \). Thus, solving \( I \) amounts to solving Max-Cut in \( G \).

An obvious open problem is to consider surjective VCSPs on a three-element domain. A complexity classification is known for \( \{0, \infty\} \)-valued languages [5] and \( \mathbb{Q} \)-valued languages [16] (the latter generalises the \( \{0, 1\} \)-valued case obtained in [17]). In fact [19] implies a dichotomy for \( \mathbb{Q} \)-valued languages on a three-element domain. However, all these results depend on the notion of core and the presence of constants in the language, and thus it is unclear how to use them to obtain a complexity classification in the surjective setting. Moreover, one special case of the CSP on a three-element domain is the 3-No-Rainbow-Colouring problem [4], whose complexity status is open.

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