Chernoff approximations of Feller semigroups in Riemannian manifolds

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Abstract

Chernoff approximations of Feller semigroups and the associated diffusion processes in Riemannian manifolds are studied. The manifolds are assumed to be of bounded geometry, thus including all compact manifolds and also a wide range of non-compact manifolds. Sufficient conditions are established for a class of second order elliptic operators to generate a Feller semigroup on a (generally non-compact) manifold of bounded geometry. A construction of Chernoff approximations is presented for these Feller semigroups in terms of shift operators. This provides approximations of solutions to initial value problems for parabolic equations with variable coefficients on the manifold. It also yields weak convergence of a sequence of random walks on the manifolds to the diffusion processes associated with the elliptic generator. For parallelizable manifolds this result is applied in particular to the representation of Brownian motion on the manifolds as limits of the corresponding random walks.

KEYWORDS

Chernoff product formula, diffusion processes, evolution equations, Feller semigroups, Feynman formula, Feynman–Kac formula, one-parameter operator semigroups

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1 INTRODUCTION

The relations between, on the one hand, the evolution equation and semigroup theory and, on the other hand, functional integration and the theory of stochastic processes is an extensively studied topic [22, 23, 28, 32, 34, 49] with a long history. Its roots can be traced back to the pioneering papers by Richard Feynman [22, 23], who proposed a heuristic representation of the solution to the Schrödinger equation in terms of limits of integrals over finite Cartesian powers of some spaces. Feynman’s ideas inspired Marc Kac [31], who rigorously proved a representation of the solution of the heat equation in terms of an integral on the space of continuous paths with respect to the Wiener measure. This formula, which is nowadays known as the celebrated “Feynman–Kac formula”, is the first and most famous example of the connections between parabolic equations associated with second order elliptic operators and stochastic processes. Remarkably, Feynman heuristically presented two mathematical constructions which are now associated with names of Trotter [58] and Chernoff [14], who rigorously proved them much later. Trotter and Chernoff formulas provide approximations of evolution (semigroups) that, in several cases, pave the way for the proof of representation formulas of Feynman–Kac type.

In the present paper, new Chernoff approximations are established for a particular class of Feller semigroups on a type of generally non-compact Riemannian manifolds. In addition, these formulas are also proved to have a nice
probabilistic interpretation on the said class of manifolds, since they allow the proof of the weak convergence of a sequence of random walks on the manifold to the diffusion process associated with the elliptic operator generating the said Feller semigroups.

1.1 Literature on the subject

From a general perspective, this work refers to the theory of some strongly continuous semigroups of linear operators \( (V(t))_{t \in \mathbb{R}^+} \) on the Banach space \( C_0(M) \) of continuous real-valued functions vanishing at \( \infty \) on a locally compact metric space \( M \). Such semigroups are called Feller semigroups. They are naturally associated with strong Markov stochastic processes \( (X^X(t))_{t \in \mathbb{R}^+} \) with values in the one-point compactification of \( M \) in such a way that the action of the operators \( V(t) \) on a function \( f \in C_0(M) \) can be represented in terms of the following formula

\[
(V(t)f)(x) = \mathbb{E}[f(X^X(t))], \quad x \in M, \quad t \in \mathbb{R}^+
\]

\( \mathbb{E} \) is the expected value. This paper considers the specific case where \( M \) is a smooth Riemannian manifold \( M \) and the generator of the Feller semigroup when restricted to the space \( C_0^\infty(M) \) of smooth functions with compact support is given by the second-order differential operator

\[
(L_0f)(x) = \frac{1}{2} \sum_{k=1}^{r} (A_k A_k f)(x) + A_0 f(x), \quad x \in M,
\]

where \( A_k, k = 0, \ldots, r \) are smooth vector fields. The stochastic processes associated with this particular kind of Feller semigroups are named Feller–Dynkin diffusions. They have continuous paths and can be constructed in terms of the (martingale) solution of stochastic differential equations of the form \([19, 27, 28, 60]\)

\[
dX(t) = \sum_{j=1}^{r} A_j(X(t)) dB^j(t) + A_0(X(t)) dt.
\]

This work in particular is devoted to the application of the Chernoff theorem (see Theorem 2.6) to the construction of an approximation formula for, on the one hand, the Feller semigroup and, on the other hand, the associated diffusion process and solutions to the evolution equation. This technique has been extensively implemented, for example, in the study of Chernoff approximations of Feller semigroups (and corresponding Feller processes) \([9, 10, 12, 13]\), in the construction of solutions to evolution equations \([4, 7, 8]\), and in the construction of the Wiener measure on compact manifolds \([3, 57]\) (see for overviews \([11, 52, 53]\)). Most of the results presented in literature are restricted to the case where either \( M = \mathbb{R}^d \) or \( M \) is compact. More general classes of \( C^k \) (with \( k = 1, 2, \ldots, \infty \) depending on the case) Riemannian manifolds were studied in \([29, 37, 42]\) (see also \([38]\) for an introductory overview of Brownian motion and diffusion processes on manifolds). In those papers, generally speaking, conditions are assumed about (a) the existence of a specific cover of open sets with both uniform metric properties and uniform bounds on the vector fields \( \{A_k\}_{k=0, \ldots, r} \) associated to the dynamical system \( (1.2) \) and (b) the validity of specific bounds on some curvatures. Under these conditions it is possible to prove the existence of Feller semigroups associated to the differential operator \( (1.1) \) as well as the non-explosive property of the associated process \( [37] \). In \([29, 42]\) similar conditions allow proving the convergence of geodesic random walks to the Brownian motion on the manifold. A recent remarkable book on semigroups on \( L^2(M) \) (instead of \( C_0(M) \)) for generally non-compact manifolds \( M \) and the special case of Schrödinger-like operators is \([25]\). There, heat kernels are extensively studied for Schrödinger-like operators on Hermitian bundles on generally non-compact base manifolds, extending many known results valid in \( \mathbb{R}^n \) to these geometric structures.

1.2 Results of this work

In contrast to the quoted literature, the present work focuses on continuous semigroups on \( C_0(M) \) with generators of the form \( (1.1) \) for the case of a generic smooth Riemannian manifolds \( (M, g) \) of bounded geometry, also requiring uniform
boundedness properties of the involved vector fields for general elliptic operators (1.1). Manifolds of bounded geometry are for instance \( \mathbb{R}^d \), compact manifolds, and a wide class of non-compact manifolds that are also relevant in applications, like Lie groups and homogeneous manifolds. The main results of this work follow.

(a) As the first result, in Section 3 we show that if the vector fields \( \{A_k\}_{k=0,\ldots,r} \) enjoy a property known as \( C^\infty \)-boundedness [51], then an extension of the differential operator \( L_0 \) in (1.1) is the generator of a Feller–Dynkin semigroup on \( C_0(M) \), and we provide a family of operator cores. This result paves the way for the proof of Theorem 3.15, the second result of this paper, where a Chernoff approximation formula (Equation (3.12)) for the Feller semigroup in terms of a family of rather simple shift operators is presented. The idea of using shift operators instead of integral operators on \( \mathbb{R}^d \) goes back to [46–48, 59] and is now applied to manifolds for the first time. We also extend the described results to more general operators \( L_0 + c \), where \( c \) is a bounded continuous scalar potential.

(b) The probabilistic interpretation of the approximation formulas (3.10) and (3.12) in the case of \( c = 0 \) is discussed in Section 4. There, as the third main result, we show that it allows us to construct the diffusion process associated to the Feller semigroup in terms of a weak limit of a sequence of random walks on \( M \). Several interesting convergence results for diffusion processes on manifolds can be found in literature, see, for example, [15, 29, 37, 39, 42]. It is worth mentioning the approximation schemes for the Wiener measure proposed in [1, 3], the proof of convergence of random walks to Brownian motion on sub-Riemannian manifolds [24] and the recent application of the notion of controlled rough path to Riemannian manifolds [18]. In contrast to the above mentioned results, in particular [29, 37, 42], where only geodesic paths are used in \( M \) so that the second order ODE are relevant, in this paper we provide three different approximation schemes associated to first order differential equations of curves in \( M \). These equations are the ones of integral lines of the aforementioned vector fields \( \{A_k\}_{k=0,\ldots,r} \). Indeed, the first approximation scheme involves a sequence of jump processes with random jumps along integral curves of the vector fields \( \{A_k\}_{k=0,\ldots,r} \). Notice that more than one vector field is necessary to change the direction of the random walk when dealing with vector fields in \( M \) instead of geodesics. The second approximation scheme is a sequence of random walks with continuous piecewise geodesic paths. Finally, the third approximation scheme involves a sequence of random walks with continuous paths where the single steps are integral curves of the vector fields \( \{A_k\}_{k=0,\ldots,r} \).

(c) These techniques are eventually applied in Section 5 to the Chernoff approximation of the specific case of the heat semigroup and the Brownian motion on parallelizable Riemannian manifolds. In this context we achieve the final results presented in this work. As noted above, besides the traditional approximation of Brownian motion in terms of the weak limit of a sequence of random walks with piecewise geodesic paths (Theorem 5.4), we provide a new approximation result in terms of the limit of random walks with paths along the integral curves of a family of parallelizing vector fields (Theorem 5.5).

1.3 Structure, notations, and conventions

The paper is organized as follows. Section 2 presents some basic definitions and results on Feller semigroups, Chernoff approximations and Riemannian geometry notions that are used throughout the paper. Section 3 presents the construction of the Feller semigroup and its Chernoff approximation. Section 4 is devoted to the probabilistic interpretation of the Chernoff approximation formula and to the construction of three different sequences of random walks on \( M \) converging weakly to the diffusion process associated to the Feller semigroup. Finally, Section 5 extends these results to the study of approximations of the heat semigroup and the Brownian motion on parallelizable manifolds of bounded geometry. The appendix contains the proofs of several technical propositions used in the main text.

From now on the notation \( A \subset B \) includes the case \( A = B \) and, referring to a universe set \( \mathcal{M} \), if \( A \subset \mathcal{M} \), then \( A^c := \mathcal{M} \setminus A \). Throughout the paper we adopt the definition \( \mathbb{R}^+ := [0, +\infty) \). If \( M \) is a smooth manifold the symbol \( C_0^\infty(M) \) denotes the complex space of smooth compactly supported complex-valued functions on \( M \).

An operator \( A \) is always understood as a linear operator and its domain, denoted by \( D(A) \), is always assumed to be a linear subspace. The symbol \( B \) denotes a Banach space over the field \( \mathbb{C} \) or \( \mathbb{R} \) and \( \mathcal{L}(B) \) denotes the set of all bounded linear operators in \( A : D(A) \to B \) with \( D(A) = B \).

If \( A : D(A) \to B \) and \( B : D(B) \to B \) are operators with \( D(A), D(B) \subset B \), then (i) the domain of \( A + B \) is defined as \( D(A + B) := D(A) \cap D(B) \), (ii) the domain of \( AB \) is defined as \( D(AB) := \{ x \in D(B) \mid Bx \in D(A) \} \), (iii) the domain of
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\( aA \), with \( a \in \mathbb{R} \) or \( \mathbb{C} \), is \( D(aA) := D(A) \) except for \( a = 0 \), where \( D(0A) = B \); finally, \( A \subset B \) means \( D(A) \subset D(B) \) and \( B|_{D(A)} = A \).

### 2 ANALYTIC AND GEOMETRIC PRELIMINARIES

We assume that the reader is familiar with the theory of \( C_0 \)-semigroups and we recall here just some basic definitions and results in order to fix the notation and the used terminology. We also recall some basic facts about the connection of the theory of \( C_0 \)-semigroups and the theory of random processes with particular emphasis on Feller semigroups and Feller processes. Generally speaking, we shall focus only on the notions and the results which are strictly necessary to state and prove the results in the work. Details appear in the classical monographs [5, 20, 21, 28, 34, 49] and references therein. Section 2.3 contains some basic notions about Chernoff-functions [14] which will be used in this work. In Sections 2.4 and 2.6 we shall remind the reader some basic notions of Riemannian geometry used throughout. Classical reference texts are [17, 33, 36, 41]. Section 2.5 introduces the basic notions and results on manifolds of bounded geometry. A recent review on the subject is [16].

#### 2.1 \( C_0 \)-semigroups and evolution equations

**Definition 2.1.** A mapping \( V : \mathbb{R}^+ \to \mathcal{L}(B) \), is called a \( C_0 \)-semigroup, or a strongly continuous one-parameter semigroup (of bounded operators) if it satisfies the following conditions,

1. \( V(0) = I \) the identity operator on \( B \),
2. \( V(t + s) = V(t)V(s) \) if \( t, s \in \mathbb{R}^+ \) (semigroup law),
3. \( \mathbb{R}^+ \ni t \mapsto V(t)x \) is continuous for every \( x \in B \), that is, \( V \) is continuous in the strong operator topology.

As is well known [20], if \( (V(t))_{t \geq 0} \) is a \( C_0 \)-semigroup in Banach space \( B \), then the set

\[
D(L) := \left\{ \varphi \in B \left| \exists \lim_{t \to +0} \frac{V(t)\varphi - \varphi}{t} \right\} \right.
\]

(2.1)

is a dense linear subspace of \( B \) invariant under the action of each \( V(t), t \geq 0 \). The operator \( L : D(L) \to B \)

\[
L\varphi = \lim_{t \to +0} \frac{V(t)\varphi - \varphi}{t}, \quad \varphi \in D(L)
\]

is called the (infinitesimal) generator of the \( C_0 \)-semigroup \( V \). The generator turns out to be a closed linear operator that defines \( V \) uniquely which, in turn, is denoted \( V(t) = e^{tL} \).

If \( L : D(L) \to B \) with \( D(L) \subset B \) is an operator, the problem of finding a function \( u : \mathbb{R}^+ \to B \) such that

\[
\begin{cases}
\frac{du}{dt}(t) = Lu(t); & t \geq 0, \\
u(0) = u_0, 
\end{cases}
\]

(2.2)

is called the abstract Cauchy problem (for the evolution equation) associated to \( L \). A function \( u : \mathbb{R}^+ \to B \) is called a classical solution to abstract Cauchy problem (2.2) if, for every \( t \geq 0 \), the function \( u \) has a continuous derivative (in the topology of \( B \) of \( u' : \mathbb{R}^+ \to B \), it holds \( u(t) \in D(L) \) for \( t \in \mathbb{R}^+ \), and (2.2) holds. The following fact can be found as Proposition 6.2 in [20], p. 145.

**Proposition 2.2.** Let the operator \( L : D(L) \to B \) be the generator of a strongly continuous semigroup \( (V(t))_{t \geq 0} \) in the Banach space \( B \). Then, for every \( u_0 \in D(L) \) there is a unique classical solution to abstract Cauchy problem (2.2), which is given by the formula \( u(t) = V(t)u_0 \).
2.2 Feller semigroups and random processes

**Feller semigroups** are of particular interest because of their strong interplay with the theory of evolution equations, on the one hand, and with probability theory, on the other hand; from the probabilistic point of view the so-called **Feller semigroups** [21, 34] are particularly important.

Let \( \mathcal{M} \) be a locally-compact metric space. With the symbol \( C(\mathcal{M}) \) we denote the space of continuous functions \( f : \mathcal{M} \to \mathbb{C} \). With \( C_0(\mathcal{M}) \) we shall denote the Banach space of continuous functions **vanishing at** \( \infty \), that is,

\[
C_0(\mathcal{M}) := \{ f \in C(\mathcal{M}) \mid \forall \varepsilon > 0 \exists K \subset \mathcal{M} \text{ compact} \mid |f(x)| < \varepsilon \forall x \in K \},
\]

endowed with the \( \| \cdot \|_\infty \)-norm. If \( \mathcal{M} \) is compact, it is natural to define \( C_0(\mathcal{M}) := C(\mathcal{M}) \).

A linear operator \( U : C_0(\mathcal{M}) \to C_0(\mathcal{M}) \) is said to be **positive** if \( (Uf)(x) \geq 0 \) for \( x \in \mathcal{M} \) whenever \( f \in C_0(\mathcal{M}) \) and \( f(x) \geq 0 \) if \( x \in \mathcal{M} \). \( U \) is said to be a **contraction** if \( \|Uf\| \leq \|f\| \) for \( f \in C_0(\mathcal{M}) \).

**Definition 2.3.** If \( \mathcal{M} \) is a locally-compact metric space, a strongly continuous semigroup made of positive contractions on \( C_0(\mathcal{M}) \) is called a **Feller semigroup**.

A crucial result is the following one (Theorem 2.2 Ch.4 in [21]):

**Theorem 2.4.** Let \( \mathcal{M} \) be a locally compact metric space and \( L_1 : D \to C_0(\mathcal{M}) \) an operator with domain \( D \subset C_0(\mathcal{M}) \) subspace. \( L_1 \) is closable and its closure \( L := \overline{L_1} \) is the generator of Feller semigroup if the following conditions are valid.

(a) \( D \) is dense in \( C_0(\mathcal{M}) \),
(b) \( L_1 \) satisfies the **positive maximum principle**:

\[
\text{for each } f \in D : \text{ if } \sup_{x \in \mathcal{M}} f(x) = f(x_0) \geq 0 \text{ for } x_0 \in \mathcal{M}, \text{ then } (L_1f)(x_0) \leq 0,
\]

(2.3)

(c) \( \text{ Ran}(L_1 - \lambda I) \) is dense in \( C_0(\mathcal{M}) \) for some \( \lambda > 0 \).

**Remark 2.5.**

(1) Given a closed operator \( L : D(L) \subset B \to B \) on a Banach space \( B \), a dense subspace \( D \subset D(L) \) is called a **core** for \( L \) if \( L|_{D} \) is closable and \( \overline{L|_{D}} = L \).

Theorem 2.4 in fact yields the existence of the semigroup as well as a core for its generator.

(2) In this paper, \( \mathcal{M} \) is a Riemannian manifold \( (\mathcal{M}, g) \). We will introduce and use three types of operators: \( L_0 \) is always a differential operator defined on the whole \( C^\infty(\mathcal{M}) \), \( L_1 \) is its restriction to a suitable subspace \( D_k \) satisfying the theorem above, \( L = \overline{L_1} \) is the generator of the Feller semigroup.

By the Riesz–Markov theorem, it is possible to associate to any Feller semigroup \( V \) a family \( (p_t(x))_{t \geq 0, x \in \mathcal{M}} \) of positive Borel measures on \( \mathcal{M} \) such that, for all \( t \geq 0 \),

\[
(V(t)f)(x) = \int_{\mathcal{M}} f(y)p_t(x, dy), \quad x \in \mathcal{M}
\]

and, for all \( f \in C_0(\mathcal{M}) \),

\[
\lim_{x_n \to x} \int_{\mathcal{M}} f(y)p_t(x_n, dy) = \int_{\mathcal{M}} f(y)p_t(x, dy).
\]

Moreover \( p_t(x, \mathcal{M}) \leq 1 \).

If all the measures of the family \( (p_t(x))_{t \geq 0, x \in \mathcal{M}} \) are probability measures, then the Feller semigroup is said **conservative**. In this case, from the semigroup law, the family of probability measures satisfies the Chapman–Kolmogorov
As a consequence, given an arbitrary probability measure \( \mu \) on the Borel \( \sigma \)-algebra \( \mathcal{B}(M) \) of \( M \), it is possible to construct a Markov process \( (X^\mu_t)_{t \geq 0} \) with values in \( M \) with finite dimensional distributions

\[
\mathbb{P}(X^\mu_{t_1} \in A_1, \ldots, X^\mu_{t_n} \in A_n) = \int 1_{A_1}(x_1) \cdots 1_{A_n}(x_n) p_{t_n-t_{n-1}}(x_{n-1}, dx_n) \cdots p_{t_1}(x_0, dx_1) d\mu(x_0),
\]

for \( 0 \leq t_1 \leq \cdots \leq t_n \) and \( A_1, \ldots, A_n \in \mathcal{B}(M) \). The existence of the process is guaranteed by the Kolmogorov existence theorem [5], the family of measures (2.5) being consistent due to the Chapman–Kolmogorov identity (2.4). In the general case, it is still possible to define the associated Markov process \( (X^\mu_t)_{t \geq 0} \) with values in the 1-point compactification \( M' := M \cup \partial \) of \( M \) and the process enjoys the strong Markov property [49]. If \( X_t = \partial \) \( \forall s \geq t \) whenever either \( X_{t-} = \partial \) or \( X_t = \partial \), then these processes are called **Feller–Dynkin (FD-)** processes. The random variable

\[
\xi := \inf \{ t \in \mathbb{R}^+ | X_t = \partial \}
\]

is called lifetime or explosion time of the process. In fact, if the Feller semigroup is conservative then \( \xi = +\infty \) almost surely, hence the FD-process can be thought as a stochastic process with values in \( M \) instead of \( M' \) and it is called conservative.

By Equation (2.5) the action of the semigroup admits the following probabilistic representation

\[
(V(t)f)(x) = \mathbb{E}[f(X^\mu_t)], \quad x \in M,
\]

where \( X^\mu_t \) is the aforementioned Markov process with initial distribution \( \mu = \delta_x \), the Dirac measure concentrated at \( x \in M \).

An important class of FD-processes are the diffusions, also called Feller–Dynkin diffusions [28, 49]. They are defined as FD-processes with continuous paths up to the explosion time. The generator \( L \) of the associated semigroup is a local operator with a domain that includes the set of smooth functions with compact support and \( L \) satisfies the maximum principle (2.3) there. If \( x \in M \) and \( (X^x_t) \) is the diffusion process starting at \( x \), then its law \( P^x \) is a probability measure on the metric space \( C(\mathbb{R}^+, M) \) of continuous paths on \( M \) or, more generally in the case of explosion, on \( C(\mathbb{R}^+, M') \). The family \( \{P^x\}_{x \in M} \) is called a system of diffusion measures.

In the case where the state space \( M \) of the Feller–Dynkin diffusion is \( \mathbb{R}^d \), it is well known (see, e.g., [34, 49]) that the restriction of \( L \) to \( C^\infty_c(\mathbb{R}^d) \) is a second-order elliptic operator of the form

\[
(L_0 f)(x) = \sum_{i,j} a^{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_j b^j(x) \frac{\partial f}{\partial x_j}(x) + c(x)f(x), \quad x \in \mathbb{R}^d, \quad f \in C^\infty_c(\mathbb{R}^d).
\]

where \( a^{ij}, b^j, c, i, j = 1, \ldots, d \), are real-valued continuous functions, \( c \leq 0 \) and the matrix of coefficients \( a^{ij}(x) \) is symmetric and non-negative definite. The corresponding semigroup \( V \) provides a classical solution of the Cauchy problem (in the above semigroup sense) for \( u_0 \in C^\infty_c(\mathbb{R}^d) \),

\[
\begin{cases}
  u'(t, x) = Lu(t, x) \text{ for } t > 0, x \in \mathbb{R}^d \\
  u(0, x) = u_0(x) \text{ for } x \in \mathbb{R}^d
\end{cases}
\]

Actually, by formula (2.6), the function \( u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R} \) admits the probabilistic representation formula \( u(t, x) = \mathbb{E}[u_0(X^x_t)] \).

Conversely, given globally Lipschitz maps \( \sigma^i_k : \mathbb{R}^d \to \mathbb{R} \) and \( b^i : \mathbb{R}^d \to \mathbb{R} \) and setting \( a^{ij} = \sum_k \sigma^i_k \sigma^j_k \), it is possible to prove that there exists a Feller semigroup whose generator restricted to \( C^\infty_c(\mathbb{R}^d) \) has the form (2.7) with \( c = 0 \). The associated diffusion process is constructed in terms of the so called martingale solution of the stochastic differential
equation
\[
\frac{dX^j_t}{dt} = \sum_{k=1}^{d} \sigma^j_k(X_t) dB^k_t + b^j(X_t) dt,
\]
(2.9)
where \((B_t)_{t \in \mathbb{R}^+}\), is a \(d\)-dimensional Brownian motion. For an extended discussion of this topic see, e.g., [28, 49].

2.3 Chernoff approximations for \(C_0\)-semigroups

Here we recall Chernoff’s theorem [6, 14, 20] which provides approximation method for \(C_0\)-semigroups on Banach spaces in terms of suitable operator valued functions.

**Theorem 2.6** (Chernoff theorem). Let \((e^{tL})_{t \geq 0}\) be a \(C_0\)-semigroup on a Banach space \(B\) with generator \(L : D(L) \to B\) and let \(S : \mathbb{R}^+ \to \mathcal{L}(B)\) be a map satisfying the following conditions:

1. There exists \(\omega \in \mathbb{R}\) such that \(\|S(t)\| \leq e^{\omega t}\) for all \(t \geq 0\);
2. The function \(S\) is continuous in the strong topology in \(\mathcal{L}(B)\);
3. \(S(0) = I\), that is, \(S(0)f = f\) for every \(f \in B\);
4. There exists a linear subspace \(D \subset D(L)\) that is a core for the operator \(L : D(L) \to B\) and such that \(\lim_{t \to 0} (S(t)f - f - tLf)/t = 0\) for each \(f \in D\).

Then the following holds:
\[
\lim_{n \to \infty} \sup_{t \in [0,T]} \left\| S(t/n)^n f - e^{tL} f \right\| = 0, \quad \text{for every } f \in B \text{ and every } T > 0,
\]
(2.10)
where \(S(t/n)^n\) is a composition of \(n\) copies of the linear bounded operator \(S(t/n)\).

**Remark 2.7.** Let \((e^{tL})_{t \geq 0}\) be a \(C_0\)-semigroup on a Banach space \(B\) with generator \(L : D(L) \to B\) and let \(S : \mathbb{R}^+ \to \mathcal{L}(B)\) be a map satisfying formula (2.10) then:

(a) \(S\) is called a Chernoff function for operator \(L\) or Chernoff-equivalent to \(C_0\)-semigroup \((e^{tL})_{t \geq 0}\) [54].
(b) The expression \(S(t/n)^n f\) is called a Chernoff approximation of \(e^{tL} f\).
(c) If \(u_0 \in D(L)\), the \(B\)-valued function
\[
U(t) := \lim_{n \to \infty} S(t/n)^n u_0 = e^{tL} u_0
\]
is the classical solution of the Cauchy problem (2.2) due to Proposition 2.2 and Theorem 2.6. Hence Chernoff approximation expressions become approximations of the solution with respect to norm in \(B\).

A definition of Chernoff equivalence and Chernoff function was suggested in 2002 [54] and developed in [45, 52, 53, 55–57]. New wording was proposed in [44, 46]. Every \(C_0\)-semigroup \(S(t) = e^{tL}\) is a Chernoff function for its generator \(L\); actually it is the only one having a semigroup composition property. There are also other statements known as Chernoff-type theorems and they produce different notions of Chernoff functions. Here we will not give an overview of this topic. We just fix one version of the Chernoff theorem, one definition of Chernoff function and work with it.

2.4 Structures on Riemannian manifolds

In this section we recall some general notions of Riemannian geometry. For more details we refer to [17, 30, 33, 36, 41]. Let \((M, g)\) be a smooth (i.e., \(C^\infty\)) Riemannian manifold, which we will always assume to be connected, Hausdorff, and
second countable. The **Riemannian distance** of \( p, q \in M \) is defined as

\[
d_{(M,g)}(p, q) = \inf_{\gamma \in C_{p,q}} L_g(\gamma). \tag{2.11}
\]

Above, \( C_{p,q} \) is the set of the smooth curves \( \gamma : [a, b] \to M \) with \( \gamma(a) = p \) and \( \gamma(b) = q \) \((a < b \text{ depend on } \gamma)\) and

\[
L_g(\gamma) := \int_a^b \| \dot{\gamma}(t) \|_g \, dt,
\]

where \( \dot{\gamma} \) is the tangent vector to \( \gamma \) and \( \| \dot{\gamma}(t) \|_g = \sqrt{g(\gamma(t))(\dot{\gamma}(t), \dot{\gamma}(t))} \) its standard \( g \)-norm (see below)—is the **length** of the curve \( \gamma \) computed with respect to \( g \). The Riemannian distance makes \( M \) a metric space whose metric topology coincides with the original topology of \( M \) as topological manifold.

If \( p \in M \) and \( U_p \subset T_p M \) is a sufficiently small open neighborhood of the origin 0 \( \in T_p M \), the **exponential map** at \( p \), denoted by \( \exp_p : U_p \to M \), is the map associating \( v \in U_p \) with \( \sigma(1, p, v) \), where \( [0,1] \ni s \mapsto \sigma(s, p, v) \in M \) is the restriction to \([0,1]\) of the maximal \( g \)-geodesic in \( M \) starting from \( p \), at \( s = 0 \), with initial tangent vector \( v \). It is known that if \( U_p \) is sufficiently small, \( \exp_p \) is a diffeomorphism from \( U_p \subset T_p M \) onto the open neighborhood \( V_p = \exp_p(U_p) \subset M \) of \( p \). Furthermore, such \( V_p \) can be chosen to be an open \( d_{(M,g)} \)-metric ball \( V_p = B_{(M,g)}(r, p) \) of sufficiently small radius \( r > 0 \) (in this case \( U_p \) will be the open ball in \( T_p M \) with radius \( r \)).

With the aforementioned choice of \( B_{(M,g)}(r, p) \), if \( N := \{e_1, \ldots, e_d\} \) is a \( g \)-orthonormal basis of \( T_p M \), we can construct a bijective map denoted by \( \exp_{p,N}^{-1} : B_{(M,g)}(r, 0) \subset \mathbb{R}^d \) as:

\[
\exp_{p,N}^{-1} : B_{(M,g)}(r, p) \ni q \mapsto (y^1(q), \ldots, y^d(q)) \in B_r(0) \subset \mathbb{R}^d \quad \text{where} \quad \sum_{j=1}^d y^j(q) e_j = \exp_p^{-1}(q).
\]

This map is smooth with its inverse and its image (i.e., the coordinate representation of the open neighborhood of the origin of \( T_p M \) previously denoted by \( U_p \)) is a standard ball \( B_r(0) \subset \mathbb{R}^d \) centered at the origin with the same radius \( r \) as \( B_{(M,g)}(r, p) \). The pair \((B_{(M,g)}(r, p), \exp_{p,N}^{-1})\) is called a (local) **normal Riemannian chart centered on** \( p \) and the coordinates \( y^1, \ldots, y^d \) Riemannian coordinates centered on \( p \).

It turns out that, referring to this coordinate patch,

(a) the components at \( y \in B_r(0) \) of the metric and its inverse respectively satisfy \( g_{ab}(0) = \delta_{ab} \) and \( g^{ab}(0) = \delta^{ab} \) for \( a, b = 1, \ldots, d \);
(b) the **Levi–Civita connection coefficients** (see Equation (2.16) below) \( \Gamma_{abc}^c(y) \) associated to metric satisfy \( \Gamma_{abc}^c(0) = 0 \) and it also holds \( \frac{\partial g_{ab}}{\partial y^c} |_0 = \frac{\partial g^{ab}}{\partial y^c} |_0 = 0 \) for \( a, b, c = 1, \ldots, d \);
(c) the \( \mathbb{R}^d \)-Euclidean norm in \( B_r(0) \) coincides with the distance from \( p \) in the following sense:

\[
\| y \| = d_{(M,g)} \left( \exp_p \left( \sum_{j=1}^d y^j(q) e_j \right), p \right);
\tag{2.12}
\]

(d) there is a unique geodesic segment \( \gamma \) joining \( p \) and \( q \in B_{(M,g)}(r, p) \) and completely included in \( B_{(M,g)}(r, p) \). In Riemannian coordinates centered on \( p \), it coincides with the \( \mathbb{R}^d \) segment joining the origin to \((y^1(q), \ldots, y^d(q))\). The length \( L_g(\gamma) \) is \( d_{(M,g)}(p, q) \).

\((M, g)\) is said to be **geodesically complete** if all geodesics are defined for all values of their affine parameter in \( \mathbb{R} \). This is equivalent to say that the exponential map \( \exp_x \), for every given \( x \in M \), is defined on the whole \( T_x M \) (even if this does not imply that it defines a diffeomorphism on the whole \( T_x M \)). The celebrated Hopf–Rinow theorem states that geodesical completeness is equivalent to the fact that \( M \) is complete as a metric space with respect to \( d_{(M,g)} \). In turn this is equivalent to the fact that closed bounded (with respect to the geodesical distance) subsets of \( M \) are compact. Finally for geodesically
complete manifolds, every pair \( p, q \in M \) admits a (not necessarily unique) geodesic joining them and the length of this geodesic segment coincides with \( d_{(M, g)}(p, q) \), since the said geodesic minimizes the length of the curves joining the points.

The **injectivity radius** at \( p \in M \), denoted by \( I_{(M, g)}(p) \in \mathbb{R}^+ \), is the supremum of the set of radii \( r \) of the open ball \( B^r_{(M, g)}(p) \subset M \) such that \( (B^r_{(M, g)}(p), \exp^{-1}_p N) \) is a normal Riemannian chart centered at \( p \) for an orthonormal basis \( N \) of \( T_p M \) (it does not depend on \( N \)). The **injectivity radius** of \((M, g)\) is

\[
I_{(M, g)} := \inf_{p \in M} I_{(M, g)}(p). \tag{2.13}
\]

**Remark 2.8.** Compact smooth Riemannian manifolds have always strictly positive injectivity radius, as the reader can easily prove. □

Strictly positivity of the injectivity radius has several important consequences, the following one in particular.

**Lemma 2.9.** If \((M, g)\) is a connected smooth manifold with strictly positive injectivity radius, then \((M, g)\) is geodesically complete and all closed bounded sets are compact.

**Proof.** See Appendix. □

### 2.5 Manifolds of bounded geometry

For future use, we introduce the definition of manifold \((M, g)\) of bounded geometry. This is a class of Riemannian manifolds where, in particular, the thesis of Lemma 2.9 is valid. See [16] for a recent extended review and [35, 51] for a summary of notions and results used in this paper. Roughly speaking (see Remark 2.11), bounded geometry means that, on the one hand, for every point \( p \in M \) on the manifold there is a geodesical ball \( B^r_{(M, g)}(p) \) covered by Riemannian coordinates centered on \( p \) of radius \( r > 0 \) independent of \( p \). On the other hand, there are uniform bounds on all derivatives of the component of the metric in the said Riemannian coordinates in \( B^r_{(M, g)}(p) \) independent of \( p \). Here is the formal definition.

**Definition 2.10.** A connected smooth Riemannian manifold \((M, g)\) is said of **bounded geometry** if \((M, g)\) has strictly positive injectivity radius and for some constants \( c_k < +\infty, k = 0, 1, ... \)

\[
\| \nabla^{(g)k} R \|_g \leq c_k, \quad k = 0, 1, ... . \tag{2.13}
\]

Above and henceforth, \( \nabla^{(g)k} \) indicates the covariant derivative of the Levi-Civita connection associated to \( g \), \( R \) indicates the Riemannian curvature tensor and \( \| \cdot \|_g \) denotes the natural point-wise norm associated to the metric \( g \) acting on smooth tensor fields of a given order \((1, 3 + k)\) concerning \( \nabla^{(g)k} \mathcal{R} \). For instance, if \( T \) is a smooth tensor field of order \((n, m)\), so that their components at \( q \in M \) in coordinates \( y^1, ..., y^d \) around \( q \) are \( T^{a_1 \cdots a_n b_1 \cdots b_m}(y(q)) \), we have

\[
\| T(q) \|_g^2 = \sum_{a_1, \ldots, a_n, b_1, \ldots, b_m, c_1, \ldots, c_r, d_1, \ldots, d_n} g_{a_1 c_1}(y(q)) \cdots g_{a_n c_n}(y(q)) g_{b_1 d_1}(y(q)) \cdots g_{b_m d_m}(y(q)) T^{a_1 \cdots a_n b_1 \cdots b_m}(y(q)) T^{c_1 \cdots c_r d_1 \cdots d_m}(y(q)). \tag{2.13}
\]

**Example 1.** From the definition above, the following manifolds in particular are of bounded geometry (Example 2.1 in [16, 51]):

(i) every smooth compact Riemannian manifold;
(ii) \( \mathbb{R}^m \) equipped with its natural metric;
(iii) every smooth Riemannian locally flat manifold with strictly positive injectivity radius;
(iv) some classical manifolds as the \( m \)-dimensional hyperbolic space (the unit ball \( B_1(0) \) in \( \mathbb{R}^m \) equipped with the Poincaré disk metric);
(v) Homogeneous manifolds with invariant metric;
(vi) covering manifolds of compact manifolds with a Riemannian metric which is lifted from the base manifold.

Another crucial feature of a smooth Riemannian manifolds of bounded geometry is the one that follows [16]. For every given \( r \in (0, I(M, g)) \), there is a sequence of finite constants \( C_k^{(r)} \in \mathbb{R}^+ \), \( k = 0, 1, 2, \ldots \) and a constant \( c^{(r)} > 0 \) such that

\[
\det[g_{ab}(y)] \geq c^{(r)}, \quad \text{if } y \in B_r(0) \quad \text{and} \quad \max_{|x| \leq k} \|\partial_x^2 g_{ab}(y)\|_{\infty}^{(B_r(0))} \leq C_k^{(r)}, \quad a, b = 1, \ldots, d \tag{2.14}
\]

where \( y^1, \ldots, y^n \) are the coordinates of every normal Riemannian chart with domain \( B_r(M, g)(p) \) centered at \( p \in M \) and \( g_{ab}(y) \) are the components of the metric in that local coordinate system. We stress that the constant \( C_k \) do not depend on \( p \) and all domains have the same geodesical radius \( r \).

From Equation (2.14) taking advantage of the Kramer rule to compute the element \( g^{ab}(y) \) of the inverse of the matrix of the coefficients \( g_{ab}(y) \), as well as recursively using the identity

\[
\frac{\partial g^{ab}}{\partial y^l} = - \sum_{c,d} g^{ac} g^{bd} \frac{\partial g_{cd}}{\partial y^l},
\]

it easily arises the existence of another sequence of finite constants \( H_k^{(r)} \in \mathbb{R}^+ \), \( k = 0, 1, 2, \ldots \) such that

\[
\max_{|x| \leq k} \|\partial_x^2 g^{ab}(y)\|_{\infty}^{(B_r(0))} \leq H_k^{(r)}, \quad a, b = 1, \ldots, d \tag{2.15}
\]

where, as above, \( y^1, \ldots, y^n \) are the coordinates of every normal Riemannian chart with domain \( B_r(M, g)(p) \) centered at \( p \in M \) of radius \( r \in (0, I(M, g)) \).

Finally, referring to Levi–Civita’s connection coefficients

\[
\Gamma_{bc}^a(y) := \frac{1}{2} \sum_d g^{ad}(y)(\partial_y^c g_{bd} + \partial_y^b g_{dc} - \partial_y^d g_{bc}),
\]

from the above pair of results, we obtain the existence of another sequence of finite constants \( J_k^{(r)} \in \mathbb{R}^+ \), \( k = 0, 1, 2, \ldots \) such that

\[
\max_{|x| \leq k} \|\partial_x^2 \Gamma_{bc}^a(y)\|_{\infty}^{(B_r(0))} \leq J_k^{(r)}, \quad a, b, c = 1, \ldots, d \tag{2.17}
\]

valid in every normal Riemannian chart around every \( p \in M \) as before defined on a metric ball of radius \( r \in (0, I(M, g)) \) with center \( p \).

Remark 2.11. We observe en passant that if \((M, g)\) has strictly positive injectivity radius and satisfies Equation (2.14) for a given \( r \in (0, I(M, g)) \)—so that it also satisfies Equations (2.15) and (2.17)—it is necessarily of bounded geometry, just in view of the polynomial expression in components of the Riemann tensor in terms of \( \Gamma_{bc}^a \) and their first derivatives.

2.6 Completeness of vector fields

Let \( M \) be a general smooth manifold. As a vector field \( A \) on \( M \) is a map \( A : M \to TM \), we use the notation \( A(p) \in T_p M \).

Assuming that \( A \) is smooth, let us consider the Cauchy problem

\[
\begin{cases}
\dot{\gamma}(s) = A(\gamma(s)) \\
\gamma(t_0) = x
\end{cases}
\] \tag{2.18}

A solution \( \gamma : (\alpha, \beta) \to M \) of Equation (2.18) is called maximal if it is not the proper restriction of any other solution of Equation (2.18). By the uniqueness of local solution of the Cauchy problem [41] there exists only one maximal solution \( \gamma \) of
Equation (2.18) and any other solution is one of its restrictions. \( \gamma \) is called the **maximal integral curve of \( A \) starting at \( x \).** A smooth vector field \( A \) on the smooth manifold \( M \) is said to be **complete** [41, p. 51] if each of its maximal integral curves is defined on the entire real line. We finally quote an elementary but crucial technical results whose proof is included for completeness in Appendix.

**Lemma 2.12.** Let \((M, g)\) be a connected geodesically complete Riemannian manifold. Let \( A \) be a smooth vector field such that

\[
\|A\|_g \leq +\infty. \tag{2.19}
\]

Then the maximal solutions of

\[
\frac{d}{dt} \gamma(t) = A(\gamma(t)) \tag{2.20}
\]

are complete.

**Remark 2.13.** The thesis of the lemma is automatically satisfied for a smooth field in the case of compact manifolds (for instance as consequence of Remark 2.8 and Lemma 2.9, but the result is elementary and valid also in absence of metric \( g \)). Yet, assuming that \( A \) is \( C^\infty \)-bounded (see Definition 3.7 below), the remaining hypotheses are true for manifolds of bounded geometry, as a consequence of Lemma 2.9. Hence the thesis of Lemma 2.12 is valid also in this case. \( \square \)

### 3 FELLER SEMIGROUPS AND CHERNOFF APPROXIMATIONS FOR DIFFUSIONS ON RIEMANNIAN MANIFOLDS

This section is devoted to the study of diffusions on Riemannian manifolds \((M, g)\) of bounded geometry. We consider second-order elliptic operators \( L_0 : C^\infty(M) \to C^\infty(M) \) of the form (3.1) proving that they admit an extension \( L : D(L) \subseteq C_0(M) \to C_0(M) \) that generates a Feller semigroup \((e^{tL})_{t \in \mathbb{R}^+}\) on \( C_0(M) \). We also provide a family of operator-cores for \( L \). This result is finally applied in Section 3.3 to the construction of Chernoff approximations of the semigroup \((e^{tL})_{t \in \mathbb{R}^+}\) in terms of a family of shift operators.

#### 3.1 Relevant operators and subspaces of \( C_0(M) \)

Let \((M, g)\) be a \( d \)-dimensional \( C^\infty \) connected Riemannian manifold which we also assume to be geodesically complete. Let \( \{A_k\}_{k=0,1, \ldots, r} \) be a family of \( C^\infty \) vector fields on \( M \). We start by considering the second order differential operator

\[
(L_0 f)(x) := \frac{1}{2} \sum_{k=1}^r A_k(A_k f)(x) + (A_0 f)(x), \quad x \in M, \quad f \in C^\infty(M) \tag{3.1}
\]

In every local coordinate neighborhood \( U \) containing \( x \), if \( \sigma_k^i(x) \) are the components of the vector \( A_k \), the operator \( L_0 \) can be represented by the differential operator

\[
(L_0 f)(x) = \frac{1}{2} \sum_{i,j} a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} f(x) + \sum_i b^i(x) \frac{\partial}{\partial x^i} f(x), \quad x \in U, \tag{3.2}
\]

with \( b^i(x) = \sigma_0^i(x) + \frac{1}{2} \sum_{j,k} \sigma_j^i(x) \frac{\partial}{\partial x^j} \sigma_k^i(x) \) and \( a^{ij}(x) = \sigma_k^i(x) \sigma_k^j(x) \) are the entries of a positive semidefinite matrix. \( (L_0 + c) : C^\infty(M) \to C(M) \) with \( L_0 \) taking the form (3.2) in every coordinate patch, and \( c \in C(M) \) used as a multiplicative operator, is said to be **elliptic** at \( x \in M \) if the matrix of coefficients \( a^{ij}(x) \) is positive semidefinite and non-singular in every local coordinate system of \( M \) around \( x \). It is easy to see that \( L_0 + c \) is elliptic if the matrices of coefficients \( a^{ij} \) are positive semidefinite and non-singular in every chart of an atlas of \( M \).
Remark 3.1. If \( A_k, k = 0, \ldots, r \), are smooth vector fields on the smooth manifold \( M \), then the 2nd order operator \( L_0 + c := \frac{1}{2} \sum_{i=1}^r A_i A_i + A_0 + c \) is elliptic at \( p \in M \) if and only if the vector fields \( A_k \), with \( k = 1, \ldots, r \), define a set of generators of \( T_p M \). (In particular, ellipticity requires \( r \ge d := \text{dim} M \) necessarily). In order to prove this fact, it is sufficient to notice that \( a^{ij}(p) = \sum_{k=1}^r \sigma_k(p) \sigma_k(p) \) is automatically positive semidefinite, hence ellipticity at \( p \) is equivalent to

\[
\sum_{k=1}^r \langle \sigma_k(p), \omega \rangle \sigma_k(p) = 0 \quad \text{iff} \quad \omega = 0 \quad \text{when} \quad \omega \in T_p^* M ,
\]

where \( \langle \cdot, \cdot \rangle \) is the standard pairing on \( T_p M \times T_p^* M \) and Equation (3.3) holds iff \( \{A_j(p)\}_{j=1, \ldots, d} \) generates \( T_p M \). □

\( L_0 + c \) is said uniformly elliptic (with respect to the metric \( g \)) if there is a constant \( C > 0 \) such that

\[
\sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j \ge C \sum_{i,j=1}^n g^{ij}(x) \xi_i \xi_j \quad \text{for every} \quad \xi_k \in \mathbb{R}, k = 1, \ldots, d, \quad \text{and every coordinate patch over} \quad M.
\]

It is easy to see that if the condition above is true for the local charts of an atlas of \( M \) and a given \( C > 0 \), then it is true for all local charts of \( M \) for the same \( C \).

Remark 3.2. It is elementary to prove that, if \( L_0 + c \) is elliptic and \( M \) is compact, then \( L_0 + c \) is uniformly elliptic. □

In general, the space \( C_c^\infty(M) \) is dense in \( C_0(M) \).

Proposition 3.3. If \( M \) is a smooth manifold, then \( C_c^\infty(M) \) is dense in \( C_0(M) \) in the norm \( \| \cdot \|_\infty \).

Proof. See Appendix. □

3.2 Generators of Feller semigroups on Riemannian manifolds

This section is devoted to the construction of generators of Feller semigroup on \( C_0(M) \) as well as to the description of their cores. In the following we shall always assume that \( (M, g) \) is a smooth manifold of bounded geometry. We start by giving the definition of some relevant subspaces of smooth functions.

Definition 3.4. Let \( (M, g) \) be a manifold of bounded geometry. A function \( f : M \to \mathbb{R} \) is said \( C^k \)-bounded if \( f \in C^k(M) \) and if for every \( r_0 \in (0, \text{I}(M, g)) \) and every multiindex \( \alpha \), with \( |\alpha| \le k \) there is a constant \( C_\alpha < \infty \) such that \( |\partial_\alpha^\gamma f(x)| \le C_\alpha < +\infty \) in every local Riemannian chart \( (B_{r_0}^{(M, g)}, \exp^{-1}_{p, N}) \) centered at every \( p \in M \).

A function \( f : M \to \mathbb{R} \) is said \( C_\infty \)-bounded if \( f \) is \( C^k \)-bounded for any \( k \ge 0 \).

The space of \( C^k \)-bounded functions on \( M \) is denoted with the symbol \( C^k_b(M) \) for \( k = 0, 1, \ldots, \infty \).

Remark 3.5. It is easy to prove [51] that \( f \in C^k_c(M) \) is \( C^k \)-bounded iff there exists a constant \( C < +\infty \) such that the covariant derivative \( \|\nabla^k f\|_\infty < C \).

Let us consider the operator \( L_0 (3.1) \) and define \( L_1 \) as its restriction to one of the linear subspaces \( D_k \subset C_0(M) \)

\[
D_k := \{ f \in C_0(M) \cap C^\infty(M) \cap C^k_b(M) \mid L_0 f \in C_0(M) \} \quad \text{for} \quad k = 0, 1, \ldots, \infty .
\]

Each \( D_k \) is non-trivial and dense in \( C_0(M) \) since \( C_c^\infty(M) \subset D_k \) and by Proposition 3.3. Actually, for every given \( k \), \( L_1 \) satisfies hypotheses (a) and (b) of Theorem 2.4, the latter can be trivially proved by direct inspection. If we are able to prove that also hypothesis (c) of Theorem 2.4 is fulfilled (there exists a \( \lambda > 0 \) such that \( \text{Ran}(L_1 - \lambda I) \) is dense in \( C_0(M) \)), then Theorem 2.4 proves that \( L := L_1 \) is the generator of a Feller semigroup \( (V(t))_{t \ge 0} \) on \( C_0(M) \).
Remark 3.6. In the case \( M = \mathbb{R}^d \) and the coefficients \( a^i, b^i \) of the differential operator (2.7) are bounded and globally Lipschitz (their smoothness is guaranteed by the assumptions that the vector fields \( A_k \) are smooth), probabilistic arguments [49] provide the existence of a Feller semigroup. The associated diffusion process is constructed in terms of the martingale solution of the stochastic PDE (2.9). In this case the representation formula (2.6) allows to prove that the generator restricted on the space \( C^\infty_\text{c}(\mathbb{R}^d) \) is actually given by the second order operator (2.7).

Analogous results can be obtained in the case where the manifold \( M \) is compact, extensively studied, for example, in [28]. If \( A_j, j = 0, \ldots, r \) are smooth vector fields, it is possible to construct a diffusion process \( X = (X(t)) \) solution of the stochastic PDE

\[
dX(t) = \sum_{j=1}^r A_j(X(t)) \circ dB^j(t) + A_0(X(t)) dt
\]

where \( \circ \) denotes the Stratonovich stochastic integral. The action of the Feller semigroup \( V(t) : C(M) \to C(M) \) given by \( V(t)f(x) = \mathbb{E}_x[f(X(t))] \) and the generator extends the operator (3.1) (see [27, 28] for details).

However, we stress that this technique does not directly provide a core for the generator.

This section presents some sufficient conditions for the validity of the hypothesis (c) in Riemannian manifolds different from \( \mathbb{R}^d \).

**Definition 3.7.** [51] Let \((M, g)\) a manifold of bounded geometry. A differential operator of order \( n, P : C^\infty(M) \to C^\infty(M)\), in local coordinates,

\[
(Pf)(x) = \sum_{|\alpha| \leq n} P_\alpha(x) \partial_\alpha^x f
\]

is said to be \( C^\infty \)-**bounded** if, for every \( r_0 \in (0, I(M, g)) \) and every pair of multiindices \( \alpha, \beta \) there is a constant \( C_{\alpha, \beta} \geq 0 \) such that \( |\partial_\beta^x P_\alpha(x)| \leq C_{\alpha, \beta} \) in every local Riemannian chart \((B_{r_0}(M, g), \exp^{-1}_p, N)\) centered at every \( p \in M \).

**Remark 3.8.**

1. It is possible to prove [51] that a \( C^\infty \)-bounded vector field \( A \) on \( M \) fulfills the following conditions

\[
\|\|\nabla(g)^k A\|\|_g \leq a_k, \quad k = 0, 1, \ldots
\]

for some constants \( a_k < +\infty, k = 0, 1, \ldots \).

2. It is possible to prove [51] that if a vector field \( A \) on \( M \) is \( C^\infty \)-**bounded**, then every differential operator given by the \( p \)-th power \( A^p \) is \( C^\infty \)-bounded. Obviously, linear combinations of \( C^\infty \)-bounded operators are \( C^\infty \)-bounded operators. Therefore the operator \( L_0 (3.1) \) is \( C^\infty \)-bounded if \((M, g)\) is of bounded geometry and the smooth vector fields \( A_j \) are \( C^\infty \)-bounded for \( j = 0, \ldots, r \).

3. Every \( C^\infty \) vector field on a compact Riemannian manifold is automatically \( C^\infty \)-bounded. Analogously, the operator \( L_0 (3.1) \) is \( C^\infty \)-bounded in the case the smooth Riemannian manifold \( M \) is compact and the fields \( \{A_j\}_{j=0, \ldots, r} \) are smooth.

From now on \( \nabla(g) \cdot A \) denotes the scalar field called **covariant divergence** of \( A \) completely defined in local coordinates around \( p \in M \) as

\[
\nabla(g) \cdot A := \sum_{j=1}^d (\nabla(g)^j A)^j = \sum_{j=1}^d \left( \partial_j A^i |_p + A^i \partial_j \log \sqrt{|g|} \right).
\]

Let us move on to state and prove the pivotal technical result of this section which we will use to prove that its closure \( L = \overline{L_1} \) generates a Feller semigroup. Everything relies upon the following technical result proved in the appendix and
based on fundamental achievements by Shubin (Theorem 2.2 in [51]), some of them already established in [35] where analytic semigroups in $L^p$-spaces are in particular studied in manifolds of bounded geometry.

**Proposition 3.9.** Let $(M, g)$ be a smooth Riemannian manifold of bounded geometry and consider a uniformly elliptic second order differential operator $L_0 : C^\infty(M) \to C^\infty(M)$ be of the form (3.1), where the $r \geq d$ real smooth vector fields $A_i$ are $C^\infty$-bounded and $A_0$ is defined as

$$A_0 := \frac{1}{2} \sum_{i=1}^{r} (\nabla^g \cdot A_i) A_i.$$  \hfill (3.5)

Then,

(i) $L := L_1$ with $L_1 := L_0|_{D_k}$ and $D_k$ defined in Equation (3.4) – is the generator of a Feller semigroup in $C_0(M)$ for every fixed $k = 0, 1, \ldots, \infty$.

(ii) Both the generator $L$ and the generated semigroup are independent of $k$.

**Proof.** (i) What we have to prove is nothing but that the three hypotheses of Theorem 2.4 are satisfied for $L_1 : D_k \to C_0(M)$. Condition (a) has been established in Proposition 3.3. Condition (b) immediately arises from the form of $L_0$ and the ellipticity property it satisfies. Regarding (c), the pivotal result appears in the following lemma proved in Appendix.

**Lemma 3.10.** With $(M, g)$ and $A_j$ $(j = 0, \ldots, r)$ and $L_0$ as in the hypothesis—in particular $A_0$ as in Equation (3.5)—for every $h \in C^\infty_c(M)$ and $\lambda > 0$ there exists $f \in C_0(M) \cap C^\infty_b(M)$ fulfilling

$$L_0 f - \lambda f = h.$$  \hfill (3.6)

**Proof.** See Appendix. 

Now observe that, due to Lemma 3.10, if $\lambda > 0$ and $h \in C^\infty_c(M)$, there is $f \in C_0(M) \cap C^\infty_b(M)$ (hence $f \in D_k$ for all $k = 0, 1, \ldots, \infty$) such that $L_0 f = \lambda f + h$. This fact can be rephrased to $(L_1 - \lambda I) f = h$. Since $C^\infty_c(M)$ is dense in $C_0(M)$ due to Proposition 3.3, we have proved that $\text{Ran}(L_1 - \lambda I)$ is dense in $C_0(M)$ for $\lambda > 0$, demonstrating that also the hypothesis (c) in Theorem 2.4 is satisfied. Let us finally prove (ii). This is consequence of the following general lemma.

**Lemma 3.11.** Let $M : D(M) \to B$ and $N : D(N) \to B$ be two closed densely defined operators in the Banach space $B$ which are generators of corresponding strongly continuous semigroups. If $M \subset N$, then $M = N$.

**Proof.** See Appendix. 

The proof ends observing that $L_0|_{D_{k+1}} \subset L_0|_{D_k}$ so that $L_0|_{D_{k+1}} \subset L_0|_{D_k}$ and both operators are generators of strongly-continuous semigroups on $C_0(M)$. The case $D_\infty$ is encompassed since, for example, $D_\infty \subset D_1$.  \hfill □

We can finally prove the main result of this section, by relaxing the requirement on the form of $A_0$.

**Theorem 3.12.** Let $(M, g)$ be a smooth Riemannian manifold of bounded geometry and consider a uniformly elliptic second order differential operator $L_0 : C^\infty(M) \to C^\infty(M)$ of the form (3.1), where $A_0$ and the $r \geq d$ vector fields $A_i$ are real, smooth, and $C^\infty$-bounded. Then,

(i) $L := L_1$ with $L_1 := L_0|_{D_k}$ and $D_k$ defined in Equation (3.4)—is the generator of a Feller semigroup in $C_0(M)$ for every fixed $k = 0, 1, \ldots, \infty$.

(ii) Both the generator $L$ and the generated semigroups are independent of $k$.

**Proof.** (ii) has the same proof as that of (ii) in Proposition 3.9. The proof of (i) is based on the following technical result.
Lemma 3.13. With \((M, g)\) and \(A_j \ (j = 1, \ldots, r)\) and \(L_0\) as in the hypothesis assume that

\[
A_0 := \frac{1}{2} \sum_{i=1}^{r} (\nabla^g \cdot A_i) A_i + B, \tag{3.7}
\]

for a real \(C^\infty\)-bounded vector field \(B\). If there exists \(c > 0\) independent of the used local chart around \(x \in M\) such that

\[
\sum_{a,b=1}^{d} B^a(x) B^b(x) \xi_a \xi_b \leq c \sum_{a,b=1}^{d} \sum_{i=1}^{r} A^a_i(x) A^b_i(x) \xi_a \xi_b \quad \text{for every } \xi_k \in \mathbb{R} \text{ and every } x \in M \tag{3.8}
\]

then \(L := \overline{L_1} - \text{with } L_1 := L_0\big|_{D_k}\) and \(D_k\) defined in Equation (3.4) is the generator of a Feller semigroup in \(C_0(M)\).

Proof. See Appendix □

In view of Lemma 3.13, to prove (i), it is sufficient to prove that Equation (3.8) is always satisfied however we choose the real smooth \(C^\infty\)-bounded vector field \(B\). If we think of the numbers \(\xi_k\) as the components of a form \(\xi \in T^*_x M\), dividing both sides for \(||\xi||_g \neq 0\), the inequality can be rephrased to

\[
\frac{|\langle B(x), \xi(x) \rangle|^2}{||\xi||_g^2} \leq c \sum_{i=1}^{r} \frac{|\langle A_i(x), \xi(x) \rangle|^2}{||\xi||_g^2},
\]

where \(\langle \cdot, \cdot \rangle\) is the standard pairing on \(T_x M \times T^*_x M\). The left-hand side above satisfies

\[
\frac{|\langle B(x), \xi(x) \rangle|^2}{||\xi||_g^2} \leq \frac{||B(x)||_g^2 ||\xi||_g^2}{||\xi||_g^2} \leq ||||B||_g||_\infty^2 < +\infty
\]

whereas the right-hand side fulfills

\[
\sum_{i=1}^{r} \frac{|\langle A_i(x), \xi(x) \rangle|^2}{||\xi||_g^2} \geq C \frac{||\xi||_g^2}{||\xi||_g^2} = C > 0
\]

just in view of the uniformly ellipticity condition. Choosing \(c := ||||B||_g||_\infty^2/C\), which is necessarily finite, Equation (3.8) is satisfied. □

To conclude, we prove that we can modify \(L_0\) by adding a zero-order term in a certain class of continuous functions preserving the results above.

Theorem 3.14. Let \((M, g)\) be a smooth Riemannian manifold of bounded geometry and consider a uniformly elliptic second order differential operator \(L_{0c} : C^\infty(M) \to C(M)\) of the form

\[
L_{0c} := L_0 + c, \tag{3.9}
\]

where \(L_0\) is the operator defined in Theorem 3.12 and \(c \in C^0(M)\) being bounded and continuous, defines a multiplicative operator \(c \in \mathcal{L}(C_0(M))\). Then,

(i) The operator \(L := \overline{L_{1c}} - \text{with } L_{1c} := L_{0c}\big|_{D_k}\) and \(D_k\) defined in Equation (3.4)—is the generator of a strongly continuous semigroup in \(C_0(M)\) for every fixed \(k = 0, 1, \ldots, \infty\).
(ii) Both the generator \(L\) and the generated semigroups are independent of \(k\).

If in addition \(c(x) \leq 0\) for all \(x \in M\), then the semigroup is Feller.

Proof. First step: Let us start by assuming that \(c(x) \leq 0\) for all \(x \in M\) and prove (i) and (ii) in this case.
(i) If \( c(x) \leq 0 \) for all \( x \in M \), then the multiplicative operator \(-c\) is accretive ([43] Definition on p. 240). In fact, if \( f \in C_0(M) \), let \( p \in M \) be such that \( |f(p)| = \sup_{x \in M} |f(x)| \). Let us construct a normalized functional \( \lambda \in C_0(M)' \) tangent to \( f \in C_0(M) \) as

\[
\lambda(h) := f(p)h(p), \quad h \in C_0(M),
\]

It holds trivially \( ||\lambda|| = ||f|| \) and \( \lambda(f) = ||f||^2 \), so that \( \lambda \) is normalized and tangent to \( f \), and also \( \lambda((-c)f) \geq 0 \) (notice that \( c \leq 0 \)), so that \(-c\) is accretive. At this juncture we can apply the lemma on p. 244 of [43] with \( a < 1/2, b := \sup_M |c|, A := L_0|D_k, and B := -c \in \mathcal{L}(C_0(M)) \). Since \( L_0|D_k \) generates a Feller semigroup which is a contraction semigroup by definition, we conclude from the above lemma that \( L_0 + c \) is the generator of a contraction semigroup. Since \( c \in \mathcal{L}(C_0(M)) \), we also have \( L_0|D_k + c = (L_0 + c)|D_k = L_{0c}|D_k \). Moreover the generated semigroup of contractions is made of positive operators. This fact immediately arises from the Trotter product formula

\[
\exp(-tA+B)f = \lim_{n \to +\infty} \left( \exp(-tA/n)e^{-tB/n} \right)^n f,
\]

that is, Theorem X.51 in [43], with \( A = -L_0|D_k \) and \( B = -c \), which is valid because \( A + B \) generates a contraction semigroup as established above. Now observe that \( \exp(-tA/n) \) is positive, since it is an element of a Feller semigroup, and \( \exp(-tB/n) \) is positive as well just because, by direct inspection, it is nothing but the multiplicative operator with a positive function \( e^{t\sigma(x)} \). Since the limit in the Trotter formula here is computed with respect to the norm \( || \cdot ||_\infty \), we find \( \exp(-tA+B)f \geq 0 \) if \( f \geq 0 \). According to Definition 2.3, the resulting semigroup is Feller.

(ii) The proof of this part is identical to that of (ii) in Theorem 3.12.

Second step: In the general case of \( c \in C_0^b(M) \) bounded and continuous, it is sufficient to write \( c(x) = \tilde{c}(x) + \sup_x c(x) \) with \( \tilde{c} = c - \sup_x c(x) \) and apply the first step to \( L_0 + \tilde{c} \), noting that the added constant \( \sup_x c(x) \) does not affect domains and closures. The resulting semigroup \( V_c(t) \) has the form \( V_c(t) = e^{t\sup_x c(x)}V_{\tilde{c}}(t) \), where \( V_{\tilde{c}}(t) \) is the Feller semigroup generated by \( L_0|D_k \).

\[ \square \]

3.3 Chernoff functions for the Feller semigroup

In this section we discuss how the semigroup \( V(t) \) generated by \( L \) can be obtained by a suitable Chernoff function \( S \) again constructed out of the vector fields \( A_j \).

In the following we shall assume that the smooth Riemannian manifold \((M, g)\) is of bounded geometry. In particular this implies that \((M, g)\) is geodesically complete (see Definition 2.10 and Lemma 2.9).

**Theorem 3.15.** Let \((M, g)\) be a smooth Riemannian manifold of bounded geometry and consider a uniformly elliptic second order differential operator \( L_0 : C^\infty(M) \to C^\infty(M) \) of the form (3.1), where \( A_0 \) and the \( r \geq d \) vector fields \( A_j \) are real, smooth and \( C^\infty \)-bounded. Let \( c \in C_0^b(M) \) and let \( L_0c := L_0 + c \) and \( L := L_1c \), with \( L_1c := L_0c|D_k \) and \( D_k \) defined in Equation (3.4) for \( k = 0, 1, \ldots, \infty \).

For any \( x \in M, t \geq 0 \) and \( f \in C_0(M) \) let us define

\[
(S(t)f)(x) = \frac{1}{4r} \sum_{j=1}^{r} \left( f(y_{x,A_j}(\sqrt{2rt})) + f(y_{x,-A_j}(\sqrt{2rt})) \right) + \frac{1}{2} f(y_{x,A_0}(2t)) + tc(x)f(x).
\]

where \( y_{x,A_j} : \mathbb{R}^+ \to M \) is the integral curve of the vector field \( A_j \) starting at time \( t = 0 \) at the point \( x \in M \), namely the solution of the initial value problem

\[
\begin{cases}
\frac{d}{dt} y_{x,A_j}(t) = A_j(y_{x,A_j}(t)), \\
y_{x,A_j}(0) = x.
\end{cases}
\]

Then the following holds.
1. For all $t \geq 0 S(t)(C_0(M)) \subset C_0(M)$.

2. If $(V(t))_{t \geq 0}$ is the strongly continuous semigroup on $C_0(M)$ generated by $L$ (according to Theorems 3.12 and 3.14) then for any $f \in C_0(M)$ and $T > 0$ the following holds

$$\lim_{n \to \infty} \sup_{t \in [0,T]} \| S(t/n)^n f - V(t)f \| = 0.$$  \hfill (3.12)

**Proof.** We remark that the right hand side of Equation (3.10) is well defined for all $t \geq 0$ since by Lemma (2.12) the maximal solution of the Cauchy problem (3.11) is defined for all $t \geq 0$, the manifold $(M, g)$ being geodesically complete by the assumption of bounded geometry. Let us first assume $c = 0$.

1. The continuity of the functions $x \mapsto f(x, A_0(2t))$ and $x \mapsto f(x, A_j(\sqrt{2r}t))$, $j = 1, \ldots, r$, follows from the continuity of the maps $x \mapsto \gamma(x, A_j(\tau))$ for all $j = 0, \ldots, r$ and $\tau \in \mathbb{R}_+$. Moreover, if $f \in C_0(M)$, then for any $x \in M$, $\tau \in \mathbb{R}_+$ and $k = 0, \ldots, r$, the map $x \mapsto f(x, A_j(\tau))$ belongs to $C_0(M)$ proving 1. in the thesis. Indeed, given $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset M$ such that $|f(y)| < \varepsilon$ for $y \in K_\varepsilon$. Set $\sup_{x \in M} \| A_j(x) \| : = c_j < \infty$ and consider the set $K_\varepsilon, \tau$ defined as the closure of the set of points $y \in M$ whose distance from $K_\varepsilon$ is less than $c_j \tau$:

$$K_{\varepsilon, \tau} : = \{ y \in M \mid d(y, K_\varepsilon) \leq c_j \tau \}. \hfill (3.13)$$

where $d(y, K_\varepsilon) := \inf_{x \in K_\varepsilon} d(y, x)$. Since $K_\varepsilon$ is compact, it is bounded, namely it is contained in some closed geodesical ball of finite radius $R$ centered on some $x_0 \in M$. Therefore, the closed set $K_{\varepsilon, \tau}$ is bounded as well since it is enclosed in a closed ball of radius $R + c_j \tau$ centered on $x_0$ and it is therefore compact by the Hopf–Rinow theorem because $(M, g)$ is complete. If $x \in K_{\varepsilon, \tau}$ then $\gamma(x, \tau) \in K_\varepsilon$, hence $|f(\gamma(x, \tau))| < \varepsilon$. Indeed if this were not true, that is, if $\gamma(x, \tau) \notin K_\varepsilon$, then:

$$d(x, K_\varepsilon) \leq d(x, \gamma(x, \tau)) \leq \int_0^\tau \| \dot{\gamma}(x, \tau) \| \, dt = \int_0^\tau \| A_j(\gamma(x, \tau)) \| \, ds < c_j \tau.$$  

2. (a) First of all we prove that if $f \in C_0(M)$ then $\sup_{x \in M} |(S(t)f)(x)| \leq \sup_{x \in M} |f(x)|$.

Indeed, for all $x \in M$ we use the fact that function $f$ is bounded and obtain

$$|(S(t)f)(x)| \leq \frac{1}{4r} \sum_{k=1}^r \left( \left| f(\gamma(x, A_j(\sqrt{2r}t))) \right| + \left| f(\gamma(x, A_k(\sqrt{2r}t))) \right| + \frac{1}{2} |f(\gamma(x, A_0(2t)))| \right) \leq \frac{1}{4r} \sum_{k=1}^r \left( 2 \sup_{z \in M} |f(z)| \right) + \frac{1}{2} \sup_{z \in M} |f(z)| = \sup_{z \in M} |f(z)|.$$  

(b) The mapping $\mathbb{R}_+ \ni t \mapsto S(tf)(x) \in C_0(M)$ is continuous.

It is sufficient to show that for any $k = 0, \ldots, r$ the map $\mathbb{R}_+ \ni \tau \mapsto S_\tau(f) \in C_0(M)$ given by $S_\tau(f)(x) := f(\gamma(x, A_j(\tau)))$ is continuous in the sup-norm.

Let $\tau_0 \in \mathbb{R}_+$ and fix $\varepsilon > 0$. Since $f \in C_0(M)$, there exists a compact set $K_\varepsilon$ such that $|f(y)| < \varepsilon/2$ for $y \in K_\varepsilon$. If $c_j := \sup_{x \in M} \| A_j(x) \|$ and considering the compact set $K_{\varepsilon, \tau}$ defined in Equation (3.13) with $\tau = \tau_0 + 1$, we have that if $t \in [0, \tau_0 + 1]$ then $\gamma(x, A_j(t)) \in K_\varepsilon$ for any $x \in K_{\varepsilon, \tau_0+1}$, hence

$$|f(\gamma(x, A_j(\tau))) - f(\gamma(x, A_j(\tau_0)))| < \varepsilon, \quad \forall x \in K_{\varepsilon, \tau_0+1}.$$  

If $x \in K_{\varepsilon, \tau_0+1}$, then for $t \in [0, \tau_0 + 1]$ we have $\gamma(x, A_j(t)) \in K_{\varepsilon, \tau_0+1}'$, where $K_{\varepsilon, \tau_0+1}'$ is the compact set defined as

$$K_{\varepsilon, \tau_0+1}' = \{ y \in M \mid d(y, K_{\varepsilon, \tau_0+1}) \leq c_j(\tau_0 + 1) \}.\hfill (3.14)$$

Since $f$ is continuous on $M$, it is uniformly continuous on the compact set $K_{\varepsilon, \tau_0+1}'$ and for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for $x, y \in K_{\varepsilon, \tau_0+1}'$ such that $|x - y| < \delta$. If $x \in K_{\varepsilon, \tau_0+1}$ and $|\tau - \tau_0| < \min\{1, \delta/c_j\}$,
then $\gamma_x(\tau), \gamma_x(\tau_0) \in K'_{\varepsilon, \tau_0+1}$ and $|\gamma_x(\tau) - \gamma_x(\tau_0)| < \delta$, hence:

$$|f(\gamma_{x,A}(\tau)) - f(\gamma_{x,A}(\tau_0))| < \varepsilon, \quad \forall x \in K'_{\varepsilon, \tau_0+1}.$$ 

(c) If $\varphi$ belongs to the core $D_k$ of $L$ with $k \geq 3$ we have

$$S(t)\varphi = \varphi + tL_1\varphi + o(t) \quad \text{as } \mathbb{R}^+ \ni t \to 0 \text{ in the uniform norm}$$

where $D_k$ is defined in Equation (3.4) and $L_1 := L_0|_{D_k}$ with $L_0$ defined in Equation (3.1).

For fixed $x \in M$ and $k \in \{1, \ldots, r\}$ let us consider the map $t \mapsto \varphi(\gamma_{x,A_j}(t))$ which is smooth by the stated assumptions on $\varphi \in D_k$ and $A_j$. By Taylor expansion we have for $t \downarrow 0$:

$$\varphi(\gamma_{x,A}(t)) = \varphi(\gamma_{x,A}(0)) + t \frac{d}{dt} \big|_{t=0} \varphi(\gamma_{x,A}(t)) + \frac{t^2}{2} \frac{d^2}{dt^2} \big|_{t=0} \varphi(\gamma_{x,A}(t)) + R_j(x, t)$$

(3.14)

$$= \varphi(x) + t(A_j \varphi)(x) + \frac{t^2}{2} (A_j A_j \varphi)(x) + R_j(x, t),$$

(3.15)

where

$$R_j(x, t) = \frac{t^3}{3!} (A_j A_j A_j \varphi)(u),$$

with $u = \gamma_{x,A_j}(\xi), \xi \in [0, t]$. Analogously for $j = 0$ we have:

$$\varphi(\gamma_{x,A_0}(t)) = \varphi(\gamma_{x,A_0}(0)) + t \frac{d}{dt} \big|_{t=0} \varphi(\gamma_{x,A_0}(t)) + R_0(x, t)$$

$$= \varphi(x) + t(A_0 \varphi)(x) + R_0(x, t),$$

with $R_0(x, t) = \frac{t^2}{2!} (A_0 A_0 \varphi)(u)$, with $u = \gamma_{x,A_0}(\xi), \xi \in [0, t]$. Hence

$$S(t)\varphi(x) = \frac{1}{4r} \sum_{j=1}^r \left( \varphi\left(\gamma_{x,A_j}(\sqrt{2rt})\right) + \varphi\left(\gamma_{x,-A_j}(\sqrt{2rt})\right) \right) + \frac{1}{2} \varphi(\gamma_{x,A_0}(2t))$$

$$= \varphi(x) + \frac{1}{4r} \sum_{j=1}^r 2rt (A_j \varphi)(x) + t(A_0 \varphi)(x) + t^{3/2} \tilde{R}(t, x)$$

$$= \varphi(x) + tL_1 \varphi(x) + t^{3/2} \tilde{R}(t, x)$$

where

$$\tilde{R}(t, x) = \sqrt{t} (A_0 A_0 \varphi)(u_0) + \frac{\sqrt{2r}}{12} \sum_{j=1}^r \left( (A_j A_j A_j \varphi)(u_j) + (A_j A_j A_j \varphi)(u'_j) \right),$$

for suitable $u_0, u_j, u'_j \in M, j = 1, \ldots, r$. The proof ends by proving that $\sup_{t \in [0,1], x \in M} |\tilde{R}(x, t)| < \infty$. This fact arises from the bounds

$$\|(A_0 A_0 \varphi)\|_\infty, \quad \|(A_j A_j A_j \varphi)\|_\infty, \quad j = 1, \ldots, r,$$

due to the very definition (3.4) of $D_k$ as well as on the assumption that the vector fields $\{A_j\}_{j=0}^r$ are $C^\infty$-bounded and $\varphi \in D_k$ with $k \geq 3$.

This concludes the proof of Equation (2) since the conditions (1)–(4) in Theorem 2.6 assuring the validity of Equation (2) are valid in view of the results above (Equation (3) is trivially true).
The case \( c \neq 0 \) has now an easy proof. Let \( S_0 \) denote the Chernoff function of \( L \) with \( c = 0 \) and let \( S \) denote the analog for the case \( c \neq 0 \). If \( f \in C_0(M) \) then \( S(t)f = S_0(t)f + tc f \in C_0(M) \) because \( S_0(t)f \in C_0(M) \), \( f \in C_0(M) \) and \( c \) is continuous and bounded. Hence (1) is true. Regarding Equation (2), the estimate \( \| S(t)f \| = \| S_0(t)f + tc f \| \leq \| S_0(t) \| \| f \| + t\|c\|\|f\| = (1 + t \sup_{x \in M} |c(x)|)\|f\| \leq e^{t\|c\|}\|f\| \) proves that condition (1) in Theorem 2.6 is valid. Requirement (2) is valid because \( S = S(t) \) is the sum of two continuous \( \mathcal{L}(C_0(M)) \)-valued functions of \( t \). Equation (3) is trivially true. Condition (4) is satisfied because if \( \varphi \in D_k \) with \( k \geq 3 \), exploiting condition (c) in Equation (2), and where \( L_1 \) is referred to the case \( c = 0 \),

\[
S(t)\varphi = S_0(t)\varphi + tc \varphi = \varphi + tL_1\varphi + o(t) + tc \varphi = \varphi + t(L_1 + c)\varphi + o(t) = \varphi + tL_1\varphi + o(t).
\]

Hence Theorem 2.6 implies that Equation (2) is valid.

**Theorem 3.16.** Under assumptions of Theorem 3.15, the following facts hold. 

(1) For the operator \( L \) defined in Theorem 3.14 and \( S(t) \) defined in Equations (3.10), we have that the classical solution \( u \) of the Cauchy problem

\[
\begin{aligned}
\frac{\partial}{\partial t} u(t,x) &= L u(t,x) \\
 u(0,x) &= u_0(x)
\end{aligned}
\]

is given for \( u_0 \in D(L) \) by

\[
u(t,x) = \lim_{n \to \infty} (S(t/n)^n u_0)(x). \tag{3.16}\]

(2) In the case \( A_0 = 0 \) and \( c = 0 \), then an alternative equivalent form for the operator \( S(t) : C_0(M) \to C_0(M), t \geq 0 \), is:

\[
(S(t)f)(x) = \frac{1}{2r} \sum_{j=1}^r \left( f\left( y_{x,A_j}(\sqrt{2rt}) \right) + f\left( y_{x,-A_j}(\sqrt{2rt}) \right) \right), \quad f \in C_0(M) \tag{3.17}
\]

**Proof.** Result (1) immediately arises from Equation (3.12), which is valid for all \( f \in C_0(M) \), for all \( x \in M \), and all \( t \geq 0 \). Equation (2) It can be proved with a proof strictly analogous to that of the corresponding statement in the Theorem 3.15.

\[ \square \]

4 | A PROBABILISTIC INTERPRETATION OF CHERNOFF CONSTRUCTION

The convergence result stated by Chernoff construction can be equivalently formulated (see [20] Th 5.2 Ch. III) in the following way for all \( t \geq 0 \) and \( f \in C_0(M) \):

\[
V(t) f = \lim_{n \to \infty} (S(1/n)^{nt} f). 
\]

Assuming that the function \( c = 0 \), the latter formula admits a probabilistic interpretation in terms of the limit of expectations with respect to a sequence of random walks on the manifold \( M \). Actually, in the following sections we shall set \( c = 0 \) and provide three different constructions.

4.1 | A jump process on \( M \)

Let \( \{X_n(t)\}_{n \geq 1} \) be a sequence of jump processes on \( M \) defined as

\[
\begin{cases}
X_n(0) = x, \\
X_n(t) := X_n([nt]/n) = Y_n([nt]) & t > 0,
\end{cases}
\tag{4.1}
\]
the jump chain \( \{Y_n(m)\}_{n \geq 1} \) is a Markov chain with transition probabilities given (for each Borel set \( B \subset M \)) by

\[
P(Y_n(m) \in B|Y_n(m-1) = y) = \frac{1}{4r} \sum_{j=1}^{r} \left( \delta_{y, A_j} \left( \sqrt{2r/n} \right) (B) + \delta_{y, -A_j} \left( \sqrt{2r/n} \right) (B) \right) + \frac{1}{2} \delta_{y, A_0} \left( \frac{2}{n} \right) (B), \quad B \in B(M). \tag{4.2}
\]

Actually \((X_n(t))_{t \geq 0}\) is a random walk on \( M \) with steps given by integral curves of the vector fields \( A_k, k = 0, \ldots, r \). Now Equation (3.16) can be written in the following form:

\[
u(t, x) = \lim_{n \to \infty} (S(1/n) \lfloor n t \rfloor) u_0)(x) = \lim_{n \to \infty} \mathbb{E}[u_0(X_n(t))] \tag{4.3}
\]

Actually, the sequence of jump processes \( \{X_n\} \) converges weakly to the diffusion process \((X(t))_{t \in \mathbb{R}^+}\) on \( M \) associated to the Feller semigroup \( V(t) \), as we are going to show.

Let \( D_M[0, +\infty) \) denote the space of càdlàg \( M \)-valued functions over the interval \([0, +\infty)\), that is, the functions which are right-continuous and admit left hand limits. It is possible to define a distance function (i.e., metric) on \( D_M[0, +\infty) \) under which it becomes a separable metric space. The topology induced by the metric is called Skorohod topology \([5, 21]\). In the following, with the symbol \( S_M \) we shall denote the corresponding Borel \( \sigma \)-algebra on \( D_M[0, +\infty) \). In fact \( S_M \) coincides with the \( \sigma \)-algebra generated by the projection maps \( \pi_t : D_M[0, +\infty) \to M \) defined as:

\[
S_M = \sigma(\pi_t, t \geq 0) \tag{4.4}
\]

where

\[
\pi_t(\gamma) := \gamma(t), \quad \gamma \in D_M[0, +\infty). \tag{4.5}
\]

As a consequence, a stochastic process \( X = (\Omega, \mathcal{F}, \mathbb{P}, (X(t)), \mathbb{P}) \) with trajectories in \( D_M[0, +\infty) \) can be looked at as a \( D_M[0, +\infty) \)-valued random variable, that is, as a map \( X : \Omega \to D_M[0, +\infty) \) defined as:

\[
X(\omega) := \gamma_\omega, \quad \gamma_\omega(t) := X(t)(\omega), \quad t \in [0, +\infty), \ \omega \in \Omega.
\]

The measurability of the map \( X \) from \((\Omega, \mathcal{F})\) to \((D_M[0, +\infty), S_M)\) follows from Equation (4.4). We shall denote with \( \mu_X \) the probability measure on \( S_M \) obtained as the pushforward of \( \mathbb{P} \) under \( X \), defined for any Borel set \( I \in S_M \) as \( \mu_X(I) = \mathbb{P}(X(\omega) \in I) \).

Considered the sequence of jump processes \( \{X_n\} \) defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) by Equation (4.1), let \( \mu_{X_n} \) be the corresponding distribution on \((D_M[0, +\infty), S_M)\). Further, let \( \mu_X \) be the analogous distribution corresponding to the Feller process \( X \).

**Theorem 4.1.** Under the assumptions of Theorem 3.15, the sequence of processes \( X_n \) converges weakly in \( D_M[0, +\infty) \) and its weak limit is the Feller process \( X \).

**Proof.** The proof is a direct application of Equation (4.3) and of Theorem 2.6 Ch 4 of [21], see also Theorem 19.25 in [32]. \( \square \)

### 4.2 A piecewise geodesic random walk

For any \( T > 0 \), let us consider the space \( C_M[0, T] \) of continuous functions \( \gamma : [0, T] \to M \) endowed with the topology of uniform convergence. The corresponding Borel \( \sigma \)-algebra \( B_M \) is generated by the coordinate projections \( \pi_t, t \in [0, T] \) defined as above (see Equation (4.5)) \([2]\).

The stochastic process \( X \) associated to the Feller semigroup \( V(t) \) is a diffusion process, hence it has continuous trajectories.
Let us consider the sequence of processes \( (Z_n)_n \) with sample paths in \( C_M[0,T] \), obtained by continuous interpolation of the paths of \( (X_n)_n \) by means of geodesic arcs. More precisely, the process \( (Z_n(t))_{t \geq 0} \) is defined as

\[
\begin{align*}
Z_n(0) & \equiv x, \\
Z_n(m/n) & \equiv X_n(m/n), \quad m \in \mathbb{N}, \\
Z_n(t) & = \gamma_{X_n(m/n),X_n((m+1)/n)}(t-m/n), \quad t \in [m/n, (m+1)/n],
\end{align*}
\]

(4.6)

where \( \gamma_{x,y}(t) \) denotes an arbitrary shortest geodesics in \( M \) such that \( \gamma_{x,y}(0) = x \) and \( \gamma_{x,y}(1/n) = y \).

Let us denote with \( \mu_n \), resp. \( \mu \), the Borel measure over the space \( C_M[0,T] \) induced by the process \( Z_n \), resp. \( X \). The following holds.

**Theorem 4.2.** Under the assumptions above, \( Z_n \) converges to \( X \) on \( C_M[0,T] \).

In other words, Theorem 4.2 states that the sequence of measures \( \{\mu_n\} \) converges weakly to \( \mu \). Before proving Theorem 4.2 we recall some preliminary results.

**Definition 4.3.** Let \((M,d)\) be a separable metric space. The **modulus of continuity** of a function \( \gamma : [0,T] \to M \) is defined for any \( \delta > 0 \) as:

\[
w(\gamma, \delta) := \sup\{d(\gamma(t), \gamma(s)), s, t \in [0,T], |t-s| < \delta\}.
\]

**Lemma 4.4.** Let \( \nu_n \) be a sequence of probability measures on \( D_M[0,T] \) converging weakly to a finite measure \( \nu \) which is concentrated on \( C_M[0,T] \). Then for any \( \varepsilon > 0 \)

\[
\lim_{\delta \to 0} \limsup_{n \to \infty} \nu_n\{\gamma \in D_M[0,T] : w(\gamma, \delta) > \varepsilon\} = 0.
\]

(4.7)

For a proof see [57].

**Proof of Theorem 4.2.** Let us consider the trajectories \( \gamma_\omega \) of the process \( Z_n \), defined as \( \gamma_\omega(t) := Z_n(t)(\omega) \). Fix \( \delta > 0 \) and take \( n \) sufficiently large in such a way that \( 1/n < \delta \). Consider \( s, t \in [0,T], s < t, |t-s| < \delta \). We will have \( s \in [m/n, (m+1)/n] \) and \( t \in [m'/n, (m'+1)/n] \), with \( m \leq m' \). We have:

\[
d(\gamma_\omega(s), \gamma_\omega(t)) \leq d(\gamma_\omega(s), \gamma_\omega((m+1)/n)) + d(\gamma_\omega((m+1)/n), \gamma_\omega(m'/n)) + d(\gamma_\omega(m'/n), \gamma_\omega(t))
\]

\[
\leq d(\gamma_\omega(m/n), \gamma_\omega((m+1)/n)) + d(\gamma_\omega((m+1)/n), \gamma_\omega(m'/n)) + d(\gamma_\omega(m'/n), \gamma_\omega((m'+1)/n))
\]

\[
\leq 3 \max\{d(\gamma_\omega(m/n), \gamma_\omega(m'/n)), |m/n - m'/n| < \delta\}
\]

We can then estimate the probability that the modulus of continuity of the trajectories of \( Z_n \) exceeds a given \( \varepsilon > 0 \) as

\[
\mu_n\{\gamma \in C_M[0,T] : w(\gamma, \delta) > \varepsilon\} \leq \mu_n\{\gamma \in C_M[0,T] : \max_m\{d(\gamma(m/n), \gamma(m+1/n))\} > \varepsilon/3\}
\]

\[
= \mu_{X_n}\{\gamma \in D_M[0,T] : w(\gamma, \delta) > \varepsilon/3\}
\]

By Theorem 4.1 and Lemma 4.4, we get for any \( \varepsilon > 0 \)

\[
\lim_{\delta \to 0} \limsup_{n \to \infty} \mu_n\{\gamma \in C_M[0,T] : w(\gamma, \delta) > \varepsilon\} = 0
\]

Since \( Z_n(0) = x \) for any \( n \), the sequence of probability measures \( \{\mu_n\} \) is tight [5] and the measure \( \mu \), that is, the law of \( X \) is the only possible limit point.
4.3 | A different interpolation scheme

Let us consider the sequence of processes \((Z_n)_n\) with sample paths in \(C_M[0, T]\), obtained by continuous interpolation of the paths of \((X_n)_n\) by means of integral curves of the vector fields \(A_k, k = 0, \ldots, r\). More precisely, introduced a sequence of i.i.d. discrete random variables \(\xi_j\) with distribution

\[
P(\xi_j = 0) = 1/2, \quad P(\xi_j = k) = \frac{1}{2r}, \quad k = 1, \ldots, 2r,
\]

and the map \(\tau : \{0, \ldots, 2r\} \times [0, 1] \rightarrow \mathbb{R}\) defined by

\[
\tau(k, t) = \begin{cases} 
2t & k = 0 \\
\sqrt{2rt} & k = 1, \ldots, 2r
\end{cases}
\]

the process \((\hat{Z}_n(t))_{t \in \mathbb{R}^+}\) can be defined as

\[
\begin{cases} 
\hat{Z}_n(0) \equiv x, \\
\hat{Z}_n(t) = y_{\hat{Z}_n(m/n),(-1)^{m/2}\tau(\xi_m, t-m/n)}(\tau(\xi_m, t-m/n)) \quad t \in [m/n, (m+1)n],
\end{cases}
\]

(4.8)

where for \(x \in M\) and a smooth vector field \(A\) on \(M\), \(\gamma_{x, A}\) denotes the maximal solution of the Cauchy problem (2.18). In particular the following holds:

\[
\hat{Z}_n(m/n) = X_n(m/n), \quad m \in \mathbb{N}. \quad \square
\]

Analogously to the case of geodesic interpolation studied in the previous section, it is possible to prove the weak convergence of \(\hat{Z}_n\) to \(X\) on \(C_M[0, T]\). Let \(\bar{\mu}_n\) (resp. \(\mu\)) be the Borel probability measure on \(C_M[0, T]\) induced by the process \(\hat{Z}_n\) (resp. \(X\)).

4.3.1 | A technical interlude

In this subsection we introduce some results that will be applied to the proof of Theorem 4.8.

In this section, if \(t = \sum_{i=1}^{d} t^i e_i\) and \(s = \sum_{i=1}^{d} s^i e_i\), where \((e_j)_{j=1,\ldots,d}\) is the standard orthonormal basis of \(\mathbb{R}^d\),

\[
\|t\| := \sqrt{\sum_{i=1}^{d} (t^i)^2} \quad \text{and} \quad \langle t, s \rangle := \sum_{i=1}^{d} t^i s^i
\]

respectively denote the standard Euclidean norm and the standard inner product in \(\mathbb{R}^d\). Furthermore, \(d_{\mathbb{R}^d}(p, q) := \|p - q\| \in [0, +\infty)\) denotes the usual Euclidean distance of \(p, q \in \mathbb{R}^d\).

Let us start by considering the case where \(M = \mathbb{R}^d\).

**Proposition 4.5.** Let \(A : \mathbb{R}^d \rightarrow \mathbb{R}^d\) be a smooth vector field such that, for some \(M_1, M_2 \in (0, +\infty)\),

1. \(\|A(x)\| \leq M_1\) if \(x \in \mathbb{R}^d\)
2. the components \(A^i : \mathbb{R}^d \rightarrow \mathbb{R}\) satisfy \(\|\nabla A^i(x)\| \leq M_2\) for all \(i = 1, \ldots, d\) if \(x \in \mathbb{R}^d\).

Consider the unique maximal and complete (for 1) smooth solution \(\gamma : \mathbb{R} \rightarrow \mathbb{R}^d\) of the Cauchy problem

\[
\begin{cases} 
\dot{\gamma}(t) = A(\gamma(t)) \\
\gamma(0) = \gamma_0
\end{cases}
\]

(4.9)

for every \(\gamma_0 \in \mathbb{R}^d\) and define \(d_{\gamma_0} : [0, +\infty) \rightarrow \mathbb{R}\) as

\[
d_{\gamma_0}(t) := d_{\mathbb{R}^d}(\gamma(0), \gamma(t)).
\]
Then there exists $T > 0$ independent of $\gamma_0$ such that the function $d_\gamma_0$ is non-decreasing in $[0, T]$. Even more, $d_\gamma_0$ is strictly increasing in $[0, T]$ if $A(\gamma_0) \neq 0$.

**Proof.** First of all let us remark that if $A(\gamma(0)) = 0$ then $d_\gamma_0(t) = d_{\mathbb{R}^d}(\gamma(0), \gamma(t)) = 0$ and the result holds trivially for any $T > 0$. Let us therefore restrict ourselves to the case $A(\gamma(0)) \neq 0$ where, by the local uniqueness of the solutions of the Cauchy problem (4.9), we have that $A(\gamma(t)) \neq 0$ for all $t \neq 0$. Let $f_{\gamma_0} : [0, +\infty) \to \mathbb{R}$ be the smooth map $f_{\gamma_0}(t) = d_\gamma_0(t)^2 = \|\gamma(t) - \gamma(0)\|^2$. To prove the thesis it is enough to demonstrate that

\[
\text{if } A(\gamma_0) \neq 0, \text{ then there exists } T > 0 \text{ independent of } \gamma_0 \text{ such that } f_{\gamma_0}(t) > 0 \text{ for all } t \in (0, T).
\] (4.10)

To prove Equation (4.10), we start by noticing that linearity and symmetry of the inner product in $\mathbb{R}^d$ and the trivial identity arizing from Equation (4.9)

\[
\gamma(t) - \gamma(u) = \int_u^t A(\gamma(s))\, ds
\] (4.11)

yield

\[
f_{\gamma_0}(t) = \left\langle \int_0^t A(\gamma(s))\, ds, \int_0^t A(\gamma(u))\, du \right\rangle = 2 \int_0^t \int_0^s \langle A(\gamma(s)), A(\gamma(u)) \rangle\, du\, ds.
\]

The derivative $f_{\gamma_0}'(t)$ appearing in Equation (4.10) therefore admits the explicit form

\[
f_{\gamma_0}'(t) = 2 \int_0^t \langle A(\gamma(t)), A(\gamma(u)) \rangle\, du.
\] (4.12)

The components $A^i(\gamma(u))$ $(i = 1, \ldots, d)$ of the vector field $A(\gamma(u))$ can be expanded as

\[
A^i(\gamma(u)) = A^i(\gamma(t)) + \langle \nabla A^i(\xi_{i,u,t}), \gamma(u) - \gamma(t) \rangle
\] (4.13)

where, according to the Lagrange for of the remainder of the $\mathbb{R}^d$ Taylor expansion,

\[
\xi_{i,u,t} = \gamma(t) + \delta_i(\gamma(u) - \gamma(t)) \quad \text{with } \delta_i \in [0, 1].
\] (4.14)

Plugging Equation (4.13) in the right-hand side of Equation (4.12), a trivial computation leads to

\[
f_{\gamma_0}'(t) = 2 \int_0^t \sum_{i=1}^d \left( |A^i(\gamma(t))|^2 + A^i(\gamma(t)) \langle \nabla A^i(\xi_{i,u,t}), \gamma(u) - \gamma(t) \rangle \right)\, du.
\] (4.15)

The proof the theorem ends proving that there exists $T > 0$ such that, if $0 \leq t \leq T$, then

\[
\sum_{i=1}^d |A^i(\gamma(t))\langle \nabla A^i(\xi_{i,u,t}), \gamma(u) - \gamma(t) \rangle| \quad \text{wish to prove } \sum_{i=1}^d |A^i(\gamma(t))|^2 = \|A(\gamma(t))\|^2.
\] (4.16)

Indeed, Equation (4.16) entails that the integrand in (4.15)—that is the one in Equation (4.12)—is strictly positive so that Equation (4.10) is valid because the integrand of Equation (4.12) is also $u$-continuous. To prove Equation (4.16), let us focus on its left-hand side. It is bounded by

\[
\sum_{i=1}^d |A^i(\gamma(t))\langle \nabla A^i(\xi_{i,u,t}), \gamma(u) - \gamma(t) \rangle| \leq \sum_{i=1}^d |A_i(\gamma(t))| \left| \langle \nabla A^i(\xi_{i,u,t}), \gamma(u) - \gamma(t) \rangle \right|
\]

\[
\leq \sum_{i=1}^d \|A(\gamma(t))\| \|\nabla A^i(\xi_{i,u,t})\| \|\gamma(u) - \gamma(t)\| \leq dM_2 \|A(\gamma(t))\| \|\gamma(u) - \gamma(t)\|.
\] (4.17)
The bound Equation (4.17) can be further improved estimating \(\|\gamma(u) - \gamma(t)\|\) with the following argument where we use the notation \(\gamma(t) = \sum_{i=1}^{d} \gamma_i(t) e_i\) and we exploit again Equations (4.11), (4.13), and (4.14).

\[
\gamma_i(u) - \gamma_i(t) = \int_{t}^{u} A_i(\gamma(s)) ds = \int_{t}^{u} A_i(\gamma(t)) ds + \int_{t}^{u} \langle \nabla A_i(\xi_{i,s}), \gamma(s) - \gamma(t) \rangle \ ds
\]

\[
= A_i(\gamma(t))(u-t) + \int_{t}^{u} \langle \nabla A_i(\xi_{i,s}), \gamma(s) - \gamma(t) \rangle \ ds.
\]

Since \(\|\nabla A_i(x)\| \leq M_2\) due to condition 2, we therefore have

\[
|\gamma_i(u) - \gamma_i(t)| \leq \|A(\gamma(t))\|(u-t) + \int_{t}^{u} M_2 \|\gamma(s) - \gamma(t)\| ds,
\]

so that

\[
\|\gamma(u) - \gamma(t)\| \leq \sqrt{d} \left( \|A(\gamma(t))\||(u-t) + \int_{u}^{t} M_2 \|\gamma(s) - \gamma(t)\| ds \right). \tag{4.18}
\]

Let us iterate this inequality for \(\|\gamma(u) - \gamma(t)\|\) finding an improved estimate in terms of \(\|A(\gamma(t))\|\) and \(t-u\), hence in terms of \(T\) because \(0 \leq u \leq t \leq T\). Let us start by applying inequality (4.18) to the term \(\|\gamma(s) - \gamma(t)\|\) on the integrand in the right-hand side of Equation (4.18):

\[
\|\gamma(u) - \gamma(t)\| \leq \sqrt{d} \|A(\gamma(t))\| \left( (t-u) + M_2 \sqrt{d} \int_{u}^{t} (t-s) ds_1 \right) + (M_2 \sqrt{d})^2 \int_{u}^{t} \int_{s_1}^{t} \|\gamma(s) - \gamma(t)\| ds_2 ds_1.
\]

Applying Equation (4.18) again, we obtain

\[
\|\gamma(u) - \gamma(t)\| \leq \sqrt{d} \|A(\gamma(t))\| \left( (t-u) + M_2 \sqrt{d} \int_{u}^{t} (t-s) ds_1 + (M_2 \sqrt{d})^2 \int_{u}^{t} \int_{s_1}^{t} \|\gamma(s) - \gamma(t)\| ds_2 ds_1 \right) + (M_2 \sqrt{d})^3 \int_{u}^{t} \int_{s_1}^{t} \int_{s_2}^{t} \|\gamma(s) - \gamma(t)\| ds_3 ds_2 ds_1.
\]

To state the general estimate, let us introduce the \(n\)-dimensional orthogonal simplex,

\[\Delta_n := \{(s_1, \ldots, s_n) \in [u, t]^n : u \leq s_1 \leq \ldots \leq s_n \leq t\}.
\]

The \(n\)-dimensional Lebesgue measure of \(\Delta_n\) equals \(|\Delta_n| = (t - u)^n / n!\) and a direct calculation shows that \(|\Delta_{n+1}| = \int_{\Delta_n} (t-s_n) ds_n \ldots ds_1\). With this notation, the last inequality reads

\[
\|\gamma(u) - \gamma(t)\| \leq \sqrt{d} \|A(\gamma(t))\| \left( |\Delta_1| + M_2 \sqrt{d} |\Delta_2| + (M_2 \sqrt{d})^2 |\Delta_3| \right) + (M_2 \sqrt{d})^3 \int_{\Delta_1} \|\gamma(s) - \gamma(t)\| ds_3 ds_2 ds_1.
\]

Applying Equation (4.18) to this inequality as many times as we need for each \(n \geq 1\), and recalling that \(M_2 \neq 0\), we have

\[
\|\gamma(u) - \gamma(t)\| \leq \sqrt{d} \|A(\gamma(t))\| \frac{1}{M_2 \sqrt{d}} \left( M_2 \sqrt{d} |\Delta_1| + (M_2 \sqrt{d})^2 |\Delta_2| + \cdots + (M_2 \sqrt{d})^n |\Delta_n| \right)
\]

\[
+ (M_2 \sqrt{d})^{n+1} \int_{\Delta_n} \|\gamma(s_n) - \gamma(t)\| ds_n \ldots ds_1,
\]

which, after exploiting \(|\Delta_n| = (t - u)^n / n!\), becomes

\[
\|\gamma(u) - \gamma(t)\| \leq \frac{\|A(\gamma(t))\|}{M_2} \sum_{n=1}^{\infty} \frac{(M_2 \sqrt{d} (t-u))^n}{n!} + (M_2 \sqrt{d})^{n+1} \int_{\Delta_n} \|\gamma(s_n) - \gamma(t)\| ds_n \ldots ds_1. \tag{4.19}
\]
An estimate of the remainder in this formula arises from the trivial bound

$$
\|\gamma(t) - \gamma(u)\| = \left\| \int_u^t A(\gamma(s)) \, ds \right\| \leq \int_u^t \|A(\gamma(s))\| \, ds \leq M_1(t - u) \quad \text{if } 0 \leq u \leq t
$$

which specializes to $$\|\gamma(s_n) - \gamma(t)\| \leq M_1(t - s_n)$$ in Equation (4.19), yielding

$$
\left| (M_2 \sqrt{d})^{n+1} \int_{\Delta_n} \|\gamma(s) - \gamma(t)\| \, ds_n \ldots ds_1 \right| \leq (M_2 \sqrt{d})^{n+1} \int_{\Delta_n} M_1(t - s_n) \, ds_n \ldots ds_1
$$

$$
= M_1(M_2 \sqrt{d})^{n+1} |\Delta_{n+1}| = \frac{M_1 \left( M_2 \sqrt{d}(t - u) \right)^{n+1}}{(n + 1)!} \rightarrow 0 \text{ as } n \rightarrow \infty.
$$

As a consequence, taking the limit as $$n \rightarrow \infty$$ in Equation (4.19), we finally obtain

$$
\|\gamma(u) - \gamma(t)\| \leq \frac{\|A(\gamma(t))\|}{M_2} \left( e^{M_2 \sqrt{d}(t - u)} - 1 \right).
$$

Combining Equation (4.21) with Equation (4.17) we find

$$
\sum_{i=1}^d \left| A_i(\gamma(t)) \langle \nabla A_i(\xi_{u,t}), \gamma(u) - \gamma(t) \rangle \right| \leq d M_2 \|A(\gamma(t))\| \|\gamma(u) - \gamma(t)\|
$$

$$
\leq d M_2 \|A(\gamma(t))\| \left( e^{M_2 \sqrt{d}(t - u)} - 1 \right) = d \|A(\gamma(t))\|^2 \left( e^{M_2 \sqrt{d}(t - u)} - 1 \right)
$$

$$
\leq d \|A(\gamma(t))\|^2 \left( e^{M_2 \sqrt{d}T} - 1 \right),
$$

where at the last step we used the fact that that $$\mathbb{R} \ni y \mapsto e^y$$ is monotonically increasing and that $$t - u \leq T$$ because $$0 \leq u \leq t \leq T$$. In summary, we have established that, for all $$T > 0$$, if $$0 \leq u \leq t \leq T$$, then

$$
\sum_{i=1}^d \left| A_i(\gamma(t)) \langle \nabla A_i(\xi_{u,t}), \gamma(u) - \gamma(t) \rangle \right| \leq d \|A(\gamma(t))\|^2 \left( e^{M_2 \sqrt{d}T} - 1 \right).
$$

This inequality is sufficient to prove Equation (4.16) concluding the proof, just choosing $$T > 0$$ such that

$$
d \|A(\gamma(t))\|^2 \left( e^{M_2 \sqrt{d}T} - 1 \right) < \|A(\gamma(t))\|^2 \quad \text{if } 0 \leq t \leq T.
$$

This is always feasible because, as observed at the beginning of the proof, $$A(\gamma(t)) \neq 0$$ if $$A(\gamma(0)) \neq 0$$ as we supposed in Equation (4.10). We can therefore divide both sides of Equation (4.23) by $$\|A(\gamma(t))\|^2$$ for $$\|A(\gamma(t))\| \neq 0$$, and the resulting inequality is solved as (taking the constraint $$T > 0$$ into account),

$$
0 < T < \frac{1}{M_2 \sqrt{d}} \ln \left( 1 + \frac{1}{d} \right)
$$

Notice that this $$T$$ can be chosen independent of $$\gamma_0 = \gamma(0)$$.

This result can be extended to Riemannian manifolds $$(M, g)$$ of bounded geometry. Indeed, in this case the following result allows to prove a bound for the Euclidean norm of the components of a vector field $$A$$ and of its covariant derivative $$\nabla A$$ in local normal charts in terms of their Riemannian norm $$\|A\|_g$$ and $$\|\nabla A\|_g$$.

**Proposition 4.6.** Let $$(M, g)$$ be a d-dimensional smooth Riemannian manifold of bounded geometry. If $$r_0 \in (0, I(M, g))$$ is sufficiently small, then there exist four constants $$k_1, k_2, k_3, k_4 \in [0, +\infty)$$ such that for every local normal Riemannian chart
centered at every \( p \in M(B_{r_0}^{(M,g)}(p), \exp_{p,N}^{-1}) \) with coordinates \( y^1, \ldots, y^n \) and every smooth vector field \( A \) on \( M \), the following uniform bounds hold:

(a) \( ||A(y(q))||^2 \leq k_1||A(q)||_g^2 \),

(b) \( ||\nabla A(y(q))||^2 \leq k_2||\nabla^{(g)}A(q)||_g^2 + k_3||A(q)||_g^2 + k_4||\nabla^{(g)}A(q)||_g||A(q)||_g \),

where \( A \in B_{r_0}^{(M,g)}(p) \) (i.e., \( y(q) \in B_{r_0}(0) \subset \mathbb{R}^d \)). Above \( \nabla \) denotes the standard gradient in \( \mathbb{R}^d \) and \( || \cdot || \) indicates the standard pointwise Euclidean norm of vectors and \( \mathbb{R}^d \)-(1,1) tensors referring to their components in Cartesian coordinates \( y^1, \ldots, y^d \):

\[
||A(y)||^2 = \sum_{a=1}^{d} |A^a(y)|^2 \quad \text{and} \quad ||T(y)||^2 := \sum_{a,b=1}^{d} |T^a_{,b}(y)|^2 ,
\]

whereas \( || \cdot ||_g \) denotes the previously defined natural point-wise norm associated to the metric \( g \) acting on vector fields and tensor fields of order (1,1) and \( \nabla^{(g)} \) is the Levi–Civita covariant derivative associated to the metric.

**Proof.** See Appendix \( \square \)

We are now in a position to state the final result which extends Proposition 4.5 to Riemannian manifolds of bounded geometry.

**Proposition 4.7.** Let \((M,g)\) be a smooth Riemannian manifold of bounded geometry (thus geodesically complete) and \( A \) a smooth vector field on \( M \) such that, for some \( c_1, c_2 \in (0, +\infty) \),

1. \( \sup_{x \in M} ||A(x)||_g \leq c_1 \),
2. \( \sup_{x \in M} ||\nabla^{(g)}A||_g \leq c_2 \).

Consider the unique maximal and complete (for 1) smooth solution \( \gamma : \mathbb{R} \to M \) of the Cauchy problem

\[
\begin{cases}
\dot{\gamma}(t) = A(\gamma(t)) \\
\gamma(0) = \gamma_0
\end{cases} \quad (4.25)
\]

for every \( \gamma_0 \in M \) and define \( d_{\gamma_0} : [0, +\infty) \to \mathbb{R} \) as

\[ d_{\gamma_0}(t) := d_{(M,g)}(\gamma(0), \gamma(t)). \]

Then, there exists \( T > 0 \) independent of \( \gamma_0 \) such that the function \( d_{\gamma_0} \) is non-decreasing in \([0, T]\). Even more, \( d_{\gamma_0} \) is strictly increasing in \([0, T]\) if \( A(\gamma_0) \neq 0 \).

**Proof.** First of all, exactly as for the case \( M = \mathbb{R}^d \), we remark that if \( A(\gamma(0)) = 0 \) then \( d_{\gamma_0}(t) = d_{(M,g)}(\gamma(0), \gamma(t)) = 0 \) and the result holds trivially for any \( T > 0 \). Let us therefore restrict ourselves to the case \( A(\gamma(0)) \neq 0 \) where, by the local uniqueness of the solutions of the Cauchy problem (4.25), we have that \( A(\gamma(t)) \neq 0 \) for all \( t \neq 0 \). Let \( f_{\gamma_0} : [0, +\infty) \to \mathbb{R} \) be the smooth map \( f_{\gamma_0}(t) = d_{\gamma_0}(t)^2 \). To prove the thesis it is enough to demonstrate that

\[
\text{if } A(\gamma_0) \neq 0, \text{ then there exists } T > 0 \text{ independent of } \gamma_0 \text{ such that } f'_{\gamma_0}(t) > 0 \text{ for all } t \in (0, T]. \quad (4.26)
\]

Statement (4.26) will be demonstrated by reducing to the analogous proof in \( \mathbb{R}^d \) here performed in a suitably Riemannian coordinate patch centered on \( \gamma(0) \). To this end it is fundamental to prove that the solution \( \gamma \) cannot exit from such a Riemannian coordinate domain. For a given \( \gamma(0) \in M \) take \( r \in (0, I_{(M,g)}) \) and consider the geodesical ball \( B^r_{(M,g)}(\gamma(0)) \).

We prove that there is \( T' > 0 \), independent of \( \gamma(0) \), such that \( \gamma(t) \in B^t_{(M,g)}(\gamma(0)) \) for \( t \in [0, T'] \). From the definition (2.11)
of \(d_{(M,g)}\) we have that
\[
d_{(M,g)}(\gamma(T'),\gamma(0)) \leq \int_0^{T'} \|\dot{\gamma}(t)\| \, dt = \int_0^{T'} \|A(\gamma(t))\| \, dt \leq \int_0^{T'} c_1 \, dt = T' c_1.
\]
We conclude that, defining \(T' := r/c_1\), we have that \(\gamma(t) \in B_{r}(\gamma(0))\) for \(t \in [0, T')\) as wanted. We henceforth restrict our attention to the ball \(B_{r}(\gamma(0))\), since the curve cannot exit it if \(t \in [0, T')\), looking for \(T \in (0, T')\) satisfying Equation (4.26). We can describe the curve \(\gamma\) in Riemannian coordinates \(y_1, \ldots, y^d\) centered on \(\gamma(0)\) inside the ball \(B_r(0) \subset \mathbb{R}^d\), taking advantage of the results already proved in \(\mathbb{R}^d\) in Proposition 4.5. Now, the crucial observation is that, due to Equation (2.12) and noticing that \(\gamma(0)\) coincides to the origin 0 of \(\mathbb{R}^d\) when describing it in Riemannian coordinates \(y_1, \ldots, y^d\), we have that
\[
d_{\gamma(0)}(t) = ||\gamma(t) - \gamma(0)||,
\]
where the norm is the Euclidean one in \(\mathbb{R}^d\) when describing the curve \(\gamma\) in coordinates \(\gamma(t) \equiv (y_1(t), \ldots, y^d(t))\). From now on the proof of Equation (4.26) is identical to that of Equation (4.10), using the fact that, in the said coordinate patch, conditions 1 and 2 in Proposition 4.5 are true for \(\mathbf{x} \in B_r(0)\) provided the initial \(r = r_0\) is chosen sufficiently small in such a way that Proposition 4.6 is valid (observe that this choice is independent of \(\gamma(0)\)). As a matter of fact, with the said \(r_0\), taking advantage of (a) and (b) in Proposition 4.6, we can choose
\[
M_1 \geq \sqrt{k_1 c_1} \quad \text{and} \quad M_2 \geq \sqrt{k_2 c_1^2 + k_3 c_2^2 + k_4 c_1 c_2}.
\]

With the proof of Proposition 4.5 and \(M_1, M_2\) as above (taking \(M_2 > 0\) as in the proof of Proposition 4.5), the wanted \(T\) is every \(T \in (0, T')\) which also satisfies Equation (4.24). It is clear from the procedure that \(T\) can be chosen independent of \(\gamma(0)\). \(\square\)

### 4.3.2 Weak convergence of the sequence \(Z_n\) to \(X\)

Coming back to the sequence \(\bar{Z}_n\) of random walks defined in Equation (4.8), the results of Proposition 4.7 allow to prove that for any \(T > 0\) the sequence of measures \(\bar{\mu}_n\) on \((C([0,T],M), B(C([0,T],M))\) induced by \(\bar{Z}_n\) converges weakly to the measure \(\mu\) induced by the diffusion process \(X\).

**Theorem 4.8.** Under the assumptions of Theorem 3.15, the sequence of measures \(\bar{\mu}_n\) on \((C([0,T],M), B(C([0,T],M))\) induced by the random walks \(\bar{Z}_n\) defined by Equation (4.8) converges weakly to the measure \(\mu\) on \((C([0,T],M), B(C([0,T],M))\) induced by the diffusion process \(X\) associated with the elliptic operator \(L\).

**Proof.** Since by assumptions \((M,g)\) is of bounded geometry and the vector fields \(\{A_k\}_{k=0,\ldots,r}\) are \(C^\infty\)-bounded, they satisfy the assumptions of Proposition 4.7. In particular there exists two constants \(c_1, c_2 \in \mathbb{R}^+\) such that for all \(k = 0, \ldots, r\)
\[
\sup_{x \in M} \|A_k(x)\| \leq c_1, \quad \sup_{x \in M} \|V^{(g)} A_k\| \leq c_2,
\]
and there exists a \(T > 0\) such that for all \(k = 0, \ldots, r\) and \(x \in M\) the functions \(d^k : \mathbb{R}^+ \to \mathbb{R}\) defined as \(d^k(t) := d(x, \gamma_x,t_A)(t)\) are non-decreasing for \(t \in [0, T]\), with \(\gamma_x,t_A\) denoting the maximal solution of the Cauchy problem (2.18).

The main argument is now completely similar to the one in the proof of Theorem 4.2. Let us consider the trajectories \(\gamma_{x,s}\) of the process \(Z_{x,s}\), defined as \(\gamma_{x,s}(t) := Z_{x,s}(t)(\omega)\). By Proposition 4.7 there exists \(T > 0\) such that for any \(x \in M\) we have \(d(x, \gamma_x,(t)) \leq d(x, \gamma_x(t'))\) for all \(0 \leq t \leq t' \leq T\), with \(\gamma_x : [0, +\infty) \to M\) is the maximal solution of the Cauchy problem (4.25). Fix \(\delta > 0\) and take \(n\) sufficiently large in such a way that \(\{1/n < \min(\delta, T)\}\) and. Consider \(s, t \in [0, T], s < t, |t-s| < \delta\). We will have \(s \in [m/n,(m+1)/n]\) and \(t \in [m'/n,(m'+1)/n]\), with \(m \leq m'\), hence:

\[
d(\gamma_{x,s}(t),\gamma_{x,t}(t)) \leq d(\gamma_{x,s}(s),\gamma_{x,s}((m+1)/n)) + d(\gamma_{x,s}((m+1)/n),\gamma_{x,s}(m'/n)) + d(\gamma_{x,s}(m'/n),\gamma_{x,t}(t)) \leq d(\gamma_{x,m/n},\gamma_{x,(m+1)/n}) + d(\gamma_{x,(m+1)/n},\gamma_{x,m'/n}) + d(\gamma_{x,m'/n},\gamma_{x,(m'+1)/n}) \leq 3 \max(d(\gamma_{x,m/n},\gamma_{x,m'/n}),|m/n-m'/n| < \delta)
\]
The probability that the modulus of continuity of the trajectories of $\tilde{Z}_n$ exceeds a given $\varepsilon > 0$ can be estimated by

$$\mu_n(\{\gamma \in C_M[0,T] : w(\gamma, \delta) > \varepsilon \}) \leq \mu_n(\{\gamma \in D_M[0,T] : \max_m(d(\gamma(m/n), \gamma(m+1/n))) > \varepsilon/3 \})$$

By Theorem 4.1 and Lemma 4.4, we get for any $\varepsilon > 0$

$$\lim_{\delta \downarrow 0} \limsup_n \mu_n(\{\gamma \in C_M[0,T] : w(\gamma, \delta) > \varepsilon \}) = 0$$

Since $\tilde{Z}_n(0) = x$ for any $n$, the sequence of probability measures $\{\mu_n\}$ is tight [5] and the measure $\mu$, that is, the law of $X$ is the only possible limit point.

5 | HEAT EQUATION AND BROWNIAN MOTION ON PARALLELIZABLE MANIFOLDS

The results of the previous sections can be also applied to the construction on the Brownian motion on $M$. Here we shall assume that the manifold $M$ is parallelizable, that is, that there exist smooth vector fields $\{e_k\}_{k=1,...,d}$ such that for any $x \in M$ the vectors $\{e_k\}_{k=1,...,d}$ provide a linear basis of $T_x M$. Examples of such manifolds are, for example, the spheres $S^1$, $S^3$, $S^7$ and Lie groups as well as orientable three-manifolds. Without loss of generality, we can take $\{e_k\}_{k=1,...,d}$ in such a way that for any $x \in M$ the vectors $\{e_k\}_{k=1,...,d}$ are orthonormal with respect to the metric tensor $g$. Further, given a local neighborhood $U$, the components $e^i_k$ of the vectors $e_k$ with respect to the local basis $\partial_i := \frac{\partial}{\partial x^i}$ satisfy the following equality:

$$\sum_{k=1}^{d} e^i_k(x) e^j_k(x) = g^{ij}(x)$$

Let us consider the Laplace–Beltrami operator $L_0 := \Delta_{LB}$ on $M$ defined in local coordinates on the smooth maps $u \in C^\infty(M)$ as:

$$\Delta_{LB} u = \sum_{i,j=1}^{d} g^{ij} V_i V_j(u) ,$$

or, more explicitly

$$(\Delta_{LB} u)(x) = \sum_{i,j=1}^{d} g^{ij}(x) \left( \frac{\partial^2 u}{\partial x^i \partial x^j}(x) - \sum_{k=1}^{d} \Gamma_{ij}^k \frac{\partial u}{\partial x^k}(x) \right).$$

Under suitable hypotheses, the results of previous sections can be applied to $\Delta_{LB}$ providing, on the one hand, the existence of an associated Feller semigroup—the heat semigroup—in $C_0(M)$ and, on the other hand, a Chernoff approximation in terms of translation operators of the form Equation (3.10) or Equation (5.1). From the probabilistic point of view, these results yield also an approximation for the Brownian motion on $M$, that is, the diffusion process associated to the heat semigroup, in terms of the weak limit of sequences of different types of random walks on $M$.

More precisely we have the following result.

**Theorem 5.1.** Let $(M, g)$ be a smooth Riemannian manifold of bounded geometry. Then the closure in $C_0(M)$ of $\Delta_{LB}|_{D_k}$, where $D_k$ is defined in Equation (3.4) with $L_0 := \frac{1}{2} \Delta_{LB}$, is the generator of a (unique) Feller semigroup on $C_0(M)$. Both the generator and the semigroup are independent of $k = 0, 1, \ldots$

**Proof.** Since $(M, g)$ is of bounded geometry $-\Delta_{LB}$ is $C^\infty$-bounded, furthermore $\Delta_{L_0}|_{C^\infty}$ is symmetric and $-\Delta_{LB}|_{C^\infty} \geq 0$. Finally $-\Delta_{LB}$ is automatically uniformly elliptic since the matrix defining its principal symbol is nothing but the metric $g$. Hence $\Delta_{LB}$ enjoys exactly the same properties as those of the operator $L_0$ we used in the proof of Lemma 3.10 and Proposition 3.9. The proof for $\Delta_{LB}$ is therefore identical. □
5.1 An approximation in terms of random walk with piecewise geodesic paths

Lemma 5.2. Let \((M, g)\) be a smooth parallelizable Riemannian manifold of bounded geometry.

For each \(x \in M\), \(t \geq 0\), \(f \in C_0(M)\) set

\[
(S(t)f)(x) = \frac{1}{2d} \sum_{k=1}^{d} \left( f\left( \gamma_{x, \sqrt{d} e_k} (\sqrt{t}) \right) + f\left( \gamma_{x, -\sqrt{d} e_k} (\sqrt{t}) \right) \right)
\]  

(5.1)

where \(\gamma_{x, v}\) denotes the geodesics starting at time 0 at the point \(x \in M\) with initial velocity \(v \in T_x M\). Further let \(L_0 : C^\infty(m) \to C^\infty(M)\) be the differential operator \(L_0 = \frac{1}{2} \Delta L_B\) and let \(L_1 := L_0|_D\), where \(D\) is given by Equation (3.4).

Then, with respect to the norm \(\|f\| = \sup_{x \in M} |f(x)|\), the following holds:

(I) for each \(t \geq 0\) and \(f \in C_0(M)\) we have \(S(t)f \in C_0(M)\) and \(\|S(t)f\| \leq \|f\|\).

(II) for each \(f \in D_k\), with \(k \geq 3\), we have \(\lim_{t \to +0} \|S(t)f - f - tL_1f\| / t = 0\).

(III) if \(t \to t_0\), \(t_n \geq 0\) and \(f \in C_0(M)\), then \(\lim_{t \to t_0} \|S(t)f - S(t_0)f\| = 0\) for each \(t_0 \geq 0\).

Proof. First of all we remark that under the stated assumptions the manifold is geodesically complete. Indeed, this follows from the bounded geometry assumption and Lemma 2.9.

The proof of (I) and (III) is completely analogous to the proof of points 2., 3a, and 3b. of Theorem 3.15. We can restrict ourselves to prove point (II).

For \(t \downarrow 0\), we have

\[
f(\gamma_{x, v}(t)) = f(x) + vf(x)t + \frac{1}{2} \frac{d}{ds}^2 f(\gamma_{x, v}(s))|_{s=0} t^2 + \frac{t^3}{3!} R(t, x),
\]

with \(R(t, x) = \frac{d}{ds}^3 f(\gamma_{x, v}(s))|_{s=u} \), \(u \in [0, t]\). In particular, by the geodesic equation

\[
\dddot{\gamma}_{x, v}^i(t) = -\Gamma^i_{jk} \ddot{\gamma}_{x, v}^j(t) \ddot{\gamma}_{x, v}^k(t),
\]

(5.2)

we obtain

\[
\frac{d^2}{dt^2} f(\gamma_{x, v}(t)) = \sum_{i,j} \partial^2_{ij} f(\gamma_{x, v}(t)) \ddot{\gamma}_{x, v}^i(t) \ddot{\gamma}_{x, v}^j(t) + \sum_i \partial_i f(\gamma_{x, v}(t)) \dddot{\gamma}_{x, v}^i(t),
\]

Analogously,

\[
\frac{d^3}{dt^3} f(\gamma_{x, v}(t)) = \left( (2\Gamma^i_{mj} \Gamma^m_{lk} - \partial_l \Gamma^i_{jk}) \dddot{\gamma}_{x, v}^i + \partial_{ljk} f + 3\Gamma^i_{kl} \partial_{ljk} f \right) \dddot{\gamma}_{x, v}^i(t) \dddot{\gamma}_{x, v}^j(t),
\]

(5.3)

(\text{where, for notational simplicity, we have used the convention on the sum over repeated indices}). Hence, by using the identity \(\sum_k e_k^i e_k^j = g(x)^{ij}\):

\[
S(t)f(x) = f(x) + \frac{1}{2} \sum_{k=1}^{d} \left( \sum_{i,j} \partial^2_{ij} f(x) e_k^i e_k^j - \sum_{k,i,j} \partial_k f(x) \Gamma^i_{jk} e_k^i e_k^j \right) t + t^3 / 2R(t, x),
\]

\[
= f(x) + L_1 f(x) + t^3 / 2R(t, x),
\]

with

\[
R(t, x) = \frac{1}{12d} \sum_{k=1}^{d} \left( \frac{d^3}{dt^3} f(\gamma_{x, \sqrt{d} e_k}(t))|_{t=u_k} + \frac{d^3}{dt^3} f(\gamma_{x, -\sqrt{d} e_k}(t))|_{t=u'_k} \right)
\]

with \(u_k, u'_k \in [0, \sqrt{d}]\), \(k = 1, \ldots, d\), and \(\frac{d^3}{dt^3} f(\gamma_{x, \sqrt{d} e_k}(t))\) is given by Equation (5.3).
Let us take an \(r_0 \in (0, \mathcal{I}(M, g))\) sufficiently small in such a way that the thesis of Proposition 4.6 holds and consider an atlas made of local normal Riemannian charts \((B_{r_0}^{(M, g)}(p), \exp^{-1}_p)\). By the assumption that \((M, g)\) is of bounded geometry, estimate Equation (2.17), the bound

\[
|\dot{\gamma}_{x, v}(t)| \leq \sqrt{\sum_{i=1}^{d} |\dot{\gamma}_{x, i}(t)|^2} \leq k_1 \|v\|_g
\]

resulting from statement (a) of Proposition 4.6 and by the geodesic Equation (5.2), and the condition \(f \in D_k\) with \(k \geq 3\), we obtain:

\[
\sup_{t \in [0, 1], x \in M} |R(t, x)| < \infty,
\]

which yields (II).

\[\square\]

**Corollary 5.3.** Under the assumptions of Lemma 5.2 the closure in \(C_0(M)\) of \(L_1\) is the generator of a Feller semigroup \(V\) and for any \(f \in C_0(M)\) and \(T > 0\):

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \|S(t/n)^n f - V(t)f\| = 0. \tag{5.4}
\]

The heat semigroup \(V\) provides a solution of the heat equation on \(M\)

\[
\begin{cases}
\frac{\partial}{\partial t} u(t, x) = \frac{1}{2} \Delta_L u(t, x) \\
u(0, x) = u_0(x)
\end{cases}
\tag{5.5}
\]

in the sense that if \(u_0 \in D(L)\) then \(u(t) := V(t)u_0 \in D(L)\) and \(\frac{d}{dt} u(t) = Lu(t)\) in the strong sense.

Analogously to the case of diffusion processes on manifolds, the approximation result stated in Corollary 5.3 admits a probabilistic interpretation. Indeed, we can still define a sequence of random walks on \(M\) with steps given by geodesic arcs according to the following construction.

For any \(n \in \mathbb{N}\), let \(X_n\) be a jump process defined as

\[
X_n(0) = x, \quad X_n(t) := X_n([nt]/n) = Y_n([nt]),
\]

where \(\{Y_n(m)\}_m\) is a Markov chain with transition probabilities

\[
P(Y_n(m) \in I | Y_n(m-1) = y) = \frac{1}{2d} \sum_{k=1}^{d} \left( \delta_{\gamma_{y, \sqrt{\mathcal{I}(y)}}(\sqrt{1/n})} I + \delta_{\gamma_{y, -\sqrt{\mathcal{I}(y)}}(\sqrt{1/n})} I \right), \quad I \in \mathcal{B}(M). \tag{5.6}
\]

Analogously, let \((Z_n)\) the sequence of processes with continuous paths obtained by \(X_n\) as geodesic interpolation, namely:

\[
Z_n(0) = x, \quad Z_n(m/n) = X_n(m/n), \quad Z_n(t) = \gamma_{X_n(m/n), X_n((m+1)/n)t-mn}, \quad t \in [m/n, (m+1)/n]
\]

where \(\gamma_{x,y}\) is the geodesic such that \(\gamma_{x,y}(0) = x\) and \(\gamma_{x,y}(1/n) = y\).

Denoted with \(X\) the diffusion process on \(M\) associated to the semigroup generated by the operator \(L = L_1\) we have the following result

**Theorem 5.4.** Under the assumption of Corollary 5.3, for any \(T > 0\), \(X_n\) converges weakly to \(X\) in \(D_M[0, T]\) and \(Z_n\) converges weakly to \(X\) in \(C_M[0, T]\)

The proof is completely similar to the proofs of Theorems 4.1 and 4.2.
An approximation in terms of random walk with steps along integral curves of the parallelizing vector fields

In the case where the parallelizing vector fields $e_1, \ldots, e_d$ of the manifold $(M, g)$ (simultaneously of bounded geometry and parallelizable) are $C^\infty$-bounded, we can view $\Delta_{LB}$ as a subcase of the operator $L_0$ discussed in Section 3 and recast all the discussion therein using the paths constructed out of the integral lines of the fields $e_k$ instead of the geodesics. In fact, since $\sum_{i=1}^d e_i^a(x)e_i^b(x) = g^{ab}(x)$ and using the fact that $\nabla(g) e_i = 0$, we can write

$$\Delta_{LB} = \sum_{a, b=1}^d g^{ij} \nabla_a^g \nabla_b^g = \sum_{a, b=1}^d \nabla_a^g g^{ab} \nabla_b^g = \sum_{i=1}^d \sum_{a, b=1}^d \nabla_a^g e_i^a \nabla_b^g e_i^b = \sum_{i=1}^d \sum_{a, b=1}^d \nabla_a^g (e_i^a e_i^b)$$

$$= \sum_{i=1}^d \sum_{a, b=1}^d e_i^a \nabla_a^g e_i^b + \sum_{i=1}^d (\nabla(g) \cdot e_i)e_i$$

In other words $\Delta_{LB}$ is the operator $L_0$ in Equation (3.1) generated by the vector fields $e_1, \ldots, e_d$, with a suitable choice for $e_0$ since, if $f \in C^\infty(M)$,

$$(\Delta_{LB}f)(x) = \sum_{i=1}^d e_i(f)(x) + (e_0 f)(x) \quad \text{where} \quad e_0 := \sum_{i=1}^d (\nabla(g) \cdot e_i)e_i.$$

In this case Theorem 4.8 holds yielding the Brownian motion on $M$, that is, the diffusion process associated with the Laplace–Beltrami operator $\Delta_{LB}$, as the weak limit of a sequence of random walks $\{\tilde{Z}_n\}$ of the form (4.8), with steps constructed out of integral curve of the vector fields $\{e_k\}_{k=1, \ldots, d}$. This result can be rephrased in following form.

**Theorem 5.5.** Let $(M, g)$ be a smooth parallelizable manifold of bounded geometry admitting a set of parallelizing vector fields $e_1, \ldots, e_d$ which are $C^\infty$-bounded. Then the Wiener measure $\mu$ on $(C([0, t], M), B(C([0, t], M)))$, that is, the law of the diffusion process associated to the Laplace–Beltrami operator $\Delta_{LB}$ is the weak limit of the sequence of probability measures $\tilde{\mu}_n$ on $(C([0, t], M), B(C([0, t], M)))$ induced by the random walks $\tilde{Z}_n$ defined by Equation (4.8) with $A_k = e_k$.

**DEDICATION**

Unfortunately, shortly after this article was accepted for publication, our dear colleague and co-author Professor Oleg Smolyanov died, and this article turned out to be one of the last for him. With the permission of the editorial board of the Mathematische Nachrichten journal, we place here a brief biographical note in memory of this remarkable mathematician. Oleg Georgievich Smolyanov (born February 8, 1938, Leningrad, died December 16, 2021, Moscow) was a Soviet and Russian mathematician whose entire life, from his student days to the last moments, was spent at the Lomonosov Moscow State University, at the Department of Function Theory and Functional Analysis of the Faculty of Mechanics and Mathematics. He obtained his Ph.D. in Physics and Mathematics in 1966 (with thesis titled “Measurable multilinear and power functionals in linear spaces with measure”) under supervision of two Professors: Georgi Shilov (who obtained his Ph.D. under supervision of Israel Gelfand) and Sergey Fomin (who obtained his Ph.D. under supervision of Andrey Kolmogorov). During his work, Smolyanov trained more than 40 Ph.D. students (at least eight of them later became Full Professors at universities around the world). Oleg Smolyanov is the author or co-author of more than 250 scientific papers, including six monographs. His main scientific results are connected with the development of functional analysis, in particular, infinite-dimensional analysis: the theory of topological vector spaces, the theory of differentiation and differential equations in infinite-dimensional spaces, functional integration and measure theory, as well as superanalysis and various problems of mathematical physics. In particular, it was he who, in 2000, drew attention to the possibilities provided by the Chernoff theorem, actively used them and familiarized us with them. Of course, this article would not have been written without his contribution.
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ENDNOTES

1As is well known, this notation is only formal in the general case even if in some situations it has a rigorous meaning in terms of norm-converging series if \( L \) is bounded respectively spectral functional calculus in Hilbert spaces when \( L \) is normal.

2Notice that in [43] semigroups are represented as \( e^{-tA} \) whereas for us they are represented as \( e^{tL} \); this explains the sign minus in front of the operators.

3In sense of Proposition 2.2.

4It is always possible to find such \( r_0 \) since the functions \( r \mapsto C_k(r) \) are monotone not-decreasing.

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A PROOF OF SOME TECHNICAL PROPOSITIONS

Proof of Lemma 2.9. Suppose there is a maximal geodesic $\gamma : I \ni t \to \gamma(t) \in M$, where $t$ is the a length parameter along $\gamma$ used as its affine parameter, such that $\sup I = \omega < +\infty$ (the case $-\infty < \inf I$ is analogous). Let $\{t_n\}_{n \in \mathbb{N}} \subset I$ be an increasing sequence such that $t_n \to \omega$ as $n \to +\infty$. Consider an element $t_n$. If there were an open ball $B_{t_n} \subset T_{\gamma(t_n)}M$ centered at the
Proof of Lemma 2.20. Let $\gamma : (a, b) \to M$ be a maximal solution of Equation (2.20) and let us assume ad absurdum that $b < +\infty$. Consider a $t_0 \in (a, b)$ and let $f : (t_0, b) \to \mathbb{R}$ be the continuous function defined as

$$f(t) := d(\gamma(t), \gamma(t_0)), \tag{A.1}$$

where $d := d_{(M,g)}$ is the above defined distance induced by the Riemannian metric. Since we have assumed that $b < \infty$, the function $f$ cannot be bounded on $[t_0, b)$. Indeed, $f$ were bounded, then there would exist an $R > 0$ such that $\gamma(t) \in B_R(\gamma(t_0))$ for all $t \in [t_0, b)$, where $B_R(\gamma(t_0))$ denotes the closed ball with radius $R$ and center $\gamma(t_0)$. On the other hand, under the stated assumptions on $M$, Hopf–Rinow theorem assures the compactness of the closed metric balls. By a classical result (see, e.g., Lemma 56, Ch. 1 in [41]), if there exists a compact set $K$ such that the maximal solution $\gamma : [t_0, b) \to M$ satisfies the condition $\gamma([t_0, b)) \subset K$, then $b = +\infty$. Hence, since $f$ cannot be bounded, there exists a monotonically increasing sequence $t_n \to b$ such that $d(\gamma(t_n), \gamma(t_0)) \to \infty$. Let $s : [t_0, b) \to \mathbb{R}$ be the curvilinear abscissa along the curve $\gamma$, namely:

$$s(t) = \int_{t_0}^{t} \sqrt{g(A(\gamma(u)), A(\gamma(u)))} \, du. \tag{A.1}$$

Clearly, for any $n \geq 1$, the following holds

$$\frac{d(\gamma(t_n), \gamma(t_0))}{t_n - t_0} \leq \frac{s(t_n) - s(t_0)}{t_n - t_0}, \tag{A.1}$$

the latter inequality, the boundedness of the sequence $\{t_n - t_0\}$ and the fact that $\{d(\gamma(t_n), \gamma(t_0))\}$ is unbounded and strictly positive gives

$$\lim_{n \to \infty} \frac{s(t_n) - s(t_0)}{t_n - t_0} = +\infty$$

On the other hand, by Lagrange's theorem applied to the (known to be differentiable) function $s : [t_0, b) \to \mathbb{R}$ defined in Equation (A.1), for any $n$ there exist a $u_n \in (t_0, t_n)$ such that

$$\sqrt{g(A(\gamma(u_n)), A(\gamma(u_n)))} = \frac{ds}{dt}(u_n) = \frac{s(t_n) - s(t_0)}{t_n - t_0}. \tag{A.2}$$

The left hand side of the above equality is bounded by the assumptions on $A$, while the right hand side is unbounded by the discussion above and we have obtained a contradiction. □

Proof of Proposition 3.3. Let us start with the following lemma.

Lemma A.1. Let $M$ be a smooth manifold and $f \in C_0(M)$. For every $\varepsilon > 0$ there is $\psi \in C^\infty(M) \cap C_0(M)$ such that $\|f - \psi\|_\infty < \varepsilon$.

Proof (There are different ways to prove this density result and this is just a possibility). It is sufficient to prove the thesis for real functions and, in turn, for $f \geq 0$. The general statement follows by decomposing $f = f_+ - f_-$ where $0 \leq f_\pm = \frac{1}{2}(f \pm \mathbb{1}) \in C_0(M)$. Let us therefore prove the thesis for $0 \leq f \in C_0(M)$.

If $p \in M$, there is a local chart $(U, \psi)$ such that $p \in U$. We can always restrict $U$ to a smaller open neighborhood $V$ of $p$, such that $\overline{V} \subset U$ is a compact set. Since there is such a local chart for every $p \in M$ and the topology of $M$ is second countable, we can extract a subcovering of $M$ made of charts $[V_j, \psi_j]_{j \in J}$ where $J$ is finite or countably infinite. Using paracompactness property of $M$, we can refine $\{V_j, \psi_j\}_{j \in J}$ to a locally finite covering (equipped with corresponding coordinate
maps $\psi_j$, the restrictions of the original ones) still indicated with the same symbol $\{V_j, \psi_j\}_{j \in J}$. Finally, we can define a partition of the unit $\{\chi_j\}_{j \in J}$ subordered to the covering $\{V_j\}_{j \in J}$. Therefore

(i) $\chi_j \in C^\infty_c(M)$,
(ii) $0 \leq \chi_j \leq 1$,
(iii) $\text{supp}(\chi_j) \subset V_j$,
(iv) $\sum_{j \in J} \chi_j(x) = 1$ where, due to locally finiteness property, for every $x \in M$ there is an open set containing $x$ whose intersection with the $V_j$ is not empty only for a finite number of indices $j \in J$, hence the sum is always finite.

To go on we assume that $J = \mathbb{N}$ (the case of $J$ finite is simpler). If $f \in C_0(M)$, the function $f|_{V_n} \geq 0$ represented in coordinates through the map $\psi_n$ turns out to be the restriction of a continuous function defined on a compact $\psi_n(V_n) \subset \mathbb{R}^n$. Using Stone–Weierstrass theorem we conclude that, for every $\varepsilon > 0$, there is a smooth function $p^{(n, \varepsilon)}$ defined on $V$ that, in coordinates is the restriction to $V$ of a polynomial defined in the compact set $\psi_n(V_n) \subset \mathbb{R}^n$, such that with obvious notation

$$\|f|_{V_n} - p^{(n, \varepsilon)}(V_n)\|_\infty < \varepsilon. \quad (A.2)$$

It is always possible to choose

$$0 \leq p^{(n, \varepsilon)} \leq f|_{V_n}. \quad (A.3)$$

In fact, for $\mu > 0$ define $g_\mu := f + \mu$. Using the same argument as above, there is a smooth function $q^{(n, \mu)}$ (in coordinates the restriction to the compact $\psi_n(V_n)$ of a polynomial) such that the inequality holds $\|q^{(n, \mu)} - g_\mu\|_\infty < \mu/3$, that is if $x \in V_n$

$$-\mu/3 \leq q^{(n, \mu)}(x) - f(x) - \mu < \mu/3$$

which implies

$$2\mu/3 < q^{(n, \mu)}(x) - f(x) < 4\mu/3$$

so that

$$0 < f(x) + 2\mu/3 < q^{(n, \mu)}(x) < f(x) + 4\mu/3$$

Defining $\varepsilon := 4\mu/3$ and $p^{(n, \varepsilon)} := q^{(n, \mu)}$ we have that Equations (A.2) and (A.3) are valid simultaneously. In view of the definition of the functions $\chi_n$, Equations (A.2) and (A.3) immediately imply

$$\|f \cdot \chi_n - p^{(n, \varepsilon)} \chi_n\|_\infty < \varepsilon. \quad (A.4)$$

and

$$0 \leq p^{(n, \varepsilon)} \cdot \chi_n \leq f \cdot \chi_n. \quad (A.5)$$

Notice that the functions $p^{(n, \varepsilon)} \cdot \chi_n$ and $f \cdot \chi_n$ are everywhere well defined on $M$ and belong to $C^\infty_c(M)$. To conclude the proof, for $\varepsilon > 0$ define

$$\psi := \sum_{n \in \mathbb{N}} \chi_n \cdot p^{(n, \varepsilon/2^{n+1})}$$

This function is well-defined and belongs to $C^\infty_c(M)$. Furthermore

$$0 \leq \psi = \sum_{n \in \mathbb{N}} \chi_n \cdot p^{(n, \varepsilon/2^{n+1})} \leq \sum_{n \in \mathbb{N}} \chi_n \cdot f = f$$

so that $\psi \in C^\infty_c(M) \cap C_0(M)$. Finally

$$\|f - \psi\|_\infty = \left\| \sum_{n \in \mathbb{N}} \chi_n \cdot p^{(n, \varepsilon/2^{n+1})} \cdot \chi_n \cdot f \right\|_\infty \leq \sum_{n \in \mathbb{N}} \|\chi_n \cdot p^{(n, \varepsilon/2^{n+1})} \cdot \chi_n \cdot f\|_\infty \leq \sum_{n \in \mathbb{N}} \varepsilon 2^{n+1} = \varepsilon.$$
In view of the lemma, in turn, it is sufficient to prove that \( C^\infty_c(M) \) is dense in \( C_0(M) \cap C^\infty(M) \). If \( f \in C_0(M) \cap C^\infty(M) \) and \( \varepsilon > 0 \), then there is a compact \( K \subset M \) such that \( |f(x)| < \varepsilon \) if \( x \notin K \). Let \( A \supset K \) be an open set whose closure is compact (It can be constructed as follows. Every \( p \in K \) admits an open neighborhood which is relatively compact—just work in a coordinate patch—due compactness, \( K \) is therefore covered by a finite class of those relatively-compact open sets. The union of those sets is the wanted \( A \).) Define \( B := M \setminus A \). Since \( K \) and \( B \) are disjoint closed sets \((K \text{ is closed because } M \text{ is Hausdorff by hypothesis)}\), from the smooth Urysohn lemma, there exists \( \chi \in C^\infty(M) \) such that \( |\chi(x)| \leq 1 \) for \( x \in M \) and \( K \subset \chi^{-1}(\{1\}) \), \( B \subset \chi^{-1}(\{0\}) \). Furthermore, from the construction, we see that \( \text{supp}(\chi) \subset A \cup \partial A = \overline{A} \). We conclude that \( \chi \in C^\infty_c(M) \). The function \( \psi := \chi \cdot f \) belongs to \( C^\infty_c(M) \) as well and furthermore

\[
\|f - \psi\|_\infty \leq \|f\|_K - \|\psi\|_\infty + \|f\|_{M \setminus K} - \|\psi\|_{M \setminus K} \leq \varepsilon,
\]

The proof is over since we have proved that if \( f \in C_0(M) \cap C^\infty(M) \) and \( \varepsilon > 0 \), then there exists \( \psi \in C^\infty_c(M) \) such that \( \|f - \psi\|_\infty < \varepsilon \). 

**Proof of Lemma 3.10.** Noticing that \( C^\infty_c(M) \) is dense in \( L^2(M, \mu_g) \), let us first establish that \( L_0|_{C^\infty_c(M)} \) is symmetric in \( L^2(M, \mu_g) \)—where from now on \( \mu_g \) is the volume form (a positive Borel measure) associated to the metric \( g \). Furthermore we also prove that \(-L_0|_{C^\infty_c(M)} \geq 0 \).

**Lemma A.2.** With the hypotheses of Lemma 3.10, Equation (3.5) in particular, \( L_0|_{C^\infty_c(M)} \) is symmetric and \(-\langle h, L_0 h \rangle \geq 0 \) if \( h \in C^\infty_c(M) \).

**Proof.** If \( A \) is a vector field viewed as differential operator, taking advantage of a partition of the unit, exploiting \( A f = \nabla_A^{(g)} f = \sum_k A^i \nabla^{(g)}_{A} f \) and the fact that \( \nabla^{(g)}_A \) is symmetric in \( L^2(M, \mu_g) \), one immediately sees that, if \( h, h' \in C^\infty_c(M) \),

\[
\langle h', Ah \rangle = -\langle Ah', h \rangle - \langle h', (\nabla^{(g)}_A) A h \rangle,
\]

where \( \nabla^{(g)}_A \) acts as multiplicative operator. Exploiting the fact that \( C^\infty_c(M) \) is invariant under the action of \( A_0 \) and \( A_i \) we find

\[
\langle L_0 h', h \rangle = \langle h', L_0 h \rangle - 2\langle h', A_0 h \rangle + \sum_{i=1}^r \langle h', (\nabla^{(g)}_A) A_i h \rangle - \langle h', (\nabla^{(g)}_A) A_i h \rangle
\]

\[
+ \frac{1}{2} \sum_{i=1}^r \langle h', (\nabla^{(g)}_A) (\nabla^{(g)}_A) A_i h \rangle = \langle h', L_0 h \rangle
\]

where we have used Equation (3.5) in the last passage. We have proved that \( L_0|_{C^\infty_c(M)} \) is symmetric because \( C^\infty_c(M) \) is dense and \( \langle L_0 h', h \rangle = \langle h', L_0 h \rangle \) for all \( h, h' \in C^\infty_c(M) \).

Regarding positivity, we have for \( h \in C^\infty_c(M) \),

\[
-\langle h, L_0 h \rangle = -\frac{1}{2} \sum_{i=1}^r \int_M \bar{h} A_i A_i h d\mu_g - \int_M \bar{h} A_0 h d\mu_g
\]

\[
= \frac{1}{2} \sum_{i=1}^r \langle A_i h, A_i h \rangle + \frac{1}{2} \sum_{i=1}^r \int_M (\bar{h} \nabla^{(g)}_A A_i h) d\mu_g - \int_M \bar{h} A_0 h d\mu_g = \frac{1}{2} \sum_{i=1}^r \langle A_i h, A_i h \rangle \geq 0
\]

where we have used again Equation (3.5) in the last passage. 

Let us pass to prove that there is a solution \( f \in C^\infty(M) \) of Equation (3.6) when \( h \in C^\infty_c(M) \). Since \( L_0|_{C^\infty_c(M)} \) is symmetric (“formally selfadjoint” in Shubin’s terminology), uniformly elliptic, and \( C^\infty \)-bounded, Corollary 4.2 in [51] implies that \( L_0|_{C^\infty_c(M)} \) is essentially selfadjoint in \( L^2(M, \mu_g) \) and we will denote by \( L' \) the unique selfadjoint extension of \( L_0|_{C^\infty_c(M)} \) (i.e.,
the closure of the latter in the Hilbert space $L^2(M, \mu_g)$). Let us focus on the equation for the unknown $f \in D(L')$

$$L' f - \lambda f = h,$$

(A.6)

when $h \in C_c^\infty(M) \subset L^2(M, \mu_g)$ and $\lambda > 0$ are given. By multiplying both sides with a test function $h' \in C_c^\infty(M)$ and integrating the result, using the fact that $L'$ is a selfadjoint extension of $L_0|_{C_c^\infty(M)}$, we find that an $f$ satisfying Equation (A.6), if any, must also satisfy Equation (3.6) (where $L_0$ appears instead of $L'$) in distributional sense, since $f \in D(L') \subset L^2(M, \mu_g) \subset D'(M)$. Elliptic regularity (Theorem 8.3.1 and Corollary 8.3.2 in [26]) applied to the elliptic operator $A = L_0 - \lambda I$ implies that, if $f$ exists, $f$ has to belong to $C^\infty(M)$ and also satisfies Equation (3.6) in classical sense. As a matter of fact, $f$ solving Equation (A.6) exists because every $\lambda > 0$ belongs to the resolvent set of $L'$. Indeed, $-L' \geq 0$ (that is true because $-L'$ is the Hilbert-space closure of $-L_0|_{C_c^\infty(M)}$ which is positive for the lemma above) entails $\sigma(-L') \subset [0, +\infty)$. A solution of Equation (A.6) (which also solves (3.6) and is smooth) therefore exists:

$$f = R_\lambda(L') h$$

(A.7)

where $R_\lambda(L') : L^2(M, \mu_g) \to L^2(M, \mu_g)$ is the resolvent operator of $L'$.

Let us move on to prove that $f \in C^0(M) \cap C_c^\infty(M)$ when $M$ is not compact (otherwise there is nothing to prove). We henceforth assume that $M$ is non-compact. We can say much more about $f$ in Equation (A.7). First of all we observe that the map $D(M) = C_c^\infty(M) \ni h \mapsto R_\lambda(L') = f \in L^2(M; \mu_g) \subset D'(M)$ is sequentially continuous with respect to the natural topologies of $C_c^\infty(M)$ and $D'(M)$ because $R_\lambda(L')$ is bounded in $L^2(M, \mu_g)$. Therefore we can apply Schwartz' kernel theorem [26] that establishes that there exists a distribution $G \in D'(M \times M)$ such that, for every pair $h, h' \in C_c^\infty(M)$,

$$\int_M h'(x)(R_\lambda(L') h)(x) \, d\mu_g(x) = \int_{M \times M} G(x, y) h'(x) h(y) \, d\mu_g(x) \otimes d\mu_g(y).$$

(A.8)

The integral on the left-hand side is a standard integral, the one on the right-hand side is just a formal expression accounting for the action of a distribution. However, Theorem 2.2 in [51] (in the case $p = 2$) proves that

(a) the distribution $G$ is smooth outside the diagonal, that is, $G \in C^\infty(M \times M \setminus \Delta)$, where $\Delta = \{(x, x) \mid x \in M\}$,

(b) there exists $\eta > 0$ such that for every $\delta > 0$ and every pair of multiindices $\alpha, \beta$, there exists $C_{\alpha, \beta, \delta} > 0$ with

$$|\partial^\alpha_x \partial^\beta_y G(x, y)| \leq C_{\alpha, \beta, \delta} e^{-\eta d_g(x, y)} \quad \text{if} \quad d_g(x, y) \geq \delta,$$

(A.9)

where $d_g$ is the geodesical distance on $(M, g)$ which is well defined since $M$ is connected and the derivatives $\partial_x$ and $\partial_y$ are computed in a pair of Riemannian charts (possibly the same). Let us take $x_0 \notin \text{supp}(h)$ and consider an open neighborhood $U$ of $x_0$ such that $\overline{U}$ is compact and $\overline{U} \cap \text{supp}(h) = \emptyset$. Since $U \times \text{supp}(h) \ni (x, y) \notin \Delta$, if $h' \in C_c^\infty(M)$ is supported in $U$ item (a) above permits us to interpret literally the integral on the right-hand side of Equation (A.8). Taking advantage of the Fubini theorem, we can rearrange Equation (A.8) to

$$\int_M h'(x) \left(f(x) - \int_M G(x, y) h(y) \, d\mu_g(y)\right) \, d\mu_g(x) = 0.$$

Since $C_c^\infty(U)$ is dense in $L^2(U, d\mu_g)$ and $x_0$ and $U$ as above are arbitrary, we can conclude that

$$f(x) = \int_M G(x, y) h(y) \, d\mu_g(y) \quad \text{almost everywhere if} \quad x \notin \text{supp}(h).$$

(A.10)

This result can be made even stronger observing that the function $\overline{U} \times \text{supp}(h) \ni (x, y) \mapsto G(x, y) h(y)$ is smooth due (a) and thus continuous and bounded. Hence, a direct use of dominated convergence theorem proves that

$$U \ni x \mapsto \int_M G(x, y) h(y) \, d\mu_g(y)$$

is continuous as well. Since the left-hand side of Equation (A.10) is also continuous, we have proved that

$$f(x) = \int_M G(x, y) h(y) \, d\mu_g(y) \quad \text{if} \quad x \in M \setminus \text{supp}(h).$$

(A.11)
Let us conclude the proof by establishing that \( f \) vanishes at infinity and \( \| A_k f \|_\infty < +\infty \) for \( k = 0, 1, \ldots, r \). Since \( \text{supp}(h) \) is compact and the open geodesical balls are a basis of the topology of \( M \), there is a finite covering \( \{ B_{r_n}(x_0) \}_{n=1}^{N} \) of \( \text{supp}(h) \) made of closed geodesical balls with finite radius. As a consequence there exist a sufficiently large closed ball \( B_{R}(x_0) \) including \( \text{supp}(f) \). It is sufficient to enlarge the radius \( r_0 \) of \( B_{r_0}(x_0) \) to \( R := D + P \), where \( D := \max\{ d_g(x_0, x_n) \mid n = 0, 1, \ldots, N \} \) and \( P = \max\{ r_n \mid n = 0, 1, \ldots, N \} \).

Notice that for every closed ball \( B_{R}(x_0) \), with arbitrary large \( R > 0 \), it must hold \( M \setminus B_{R}(x_0) \neq \emptyset \) necessarily, otherwise \( M \) would be compact due to Lemma 2.9 since \( M \) is of bounded geometry, and \( M \) is not compact by hypothesis. With \( \eta > 0 \) as in (b), choose \( \delta > 0 \) and define another closed ball \( B_{R'}(x_0) \) with \( R' > \delta + R \). If \( y \in B_{R}(x_0) \) and \( x \in M \setminus B_{R'}(x_0) \) we have \( d_g(x, y) \geq d_g(x, x_0) - R > R' - R > \delta + R - R > \delta \) so that we can use the inequality (A.9) with \( \alpha = \beta = 0 \), finding

\[
|f(x)| \leq \int_M |G(x, y)||h(y)||d\mu_g(y) \leq \text{vol}_g(B_{R}(x_0))C_{\delta}||h||_\infty e^{\eta R}e^{-\eta d_g(x, x_0)} \quad \text{if } x \in M \setminus B_{R'}(x_0)
\]  

where, for \( x \in M \setminus B_{R'}(x_0) \) and \( y \in B_{R}(x_0) \), we took advantage of

\[
R + d_g(x, y) \geq d_g(x, x_0)
\]

so that

\[
-\eta d_g(x, y) \leq -\eta d_g(x, x_0) + \eta R
\]

which implies Equation (A.12) through Equation (A.9). To conclude, with \( h, x_0, \eta, R, R', C_{\delta} \) fixed as above and if \( ||h||_\infty > 0 \) (otherwise there is nothing to prove since \( f = 0 \)), for every \( \varepsilon > 0 \) define

\[
R_\varepsilon := -\frac{1}{\eta} \log \left( \frac{\varepsilon}{\text{vol}_g(B_{R}(x_0))C_{\delta}||h||_\infty e^{\varepsilon R}} \right).
\]

For every \( \varepsilon > 0 \) (such small that \( R_\varepsilon > R' \)), consider the closed ball \( B_{R_\varepsilon}(x_0) \) which is compact in view of Lemma 2.9. Here, Equation (A.12) yields

\[
|f(x)| \leq \text{vol}_g(B_{R}(x_0))C_{\delta}||h||_\infty e^{R}e^{-R_\varepsilon} = \varepsilon \quad \text{if } x \in M \setminus B_{R_\varepsilon}(x_0).
\]

We have proved that \( f \in C_0(M) \). With a procedure similar to the we used to prove Equation (A.11) based on Lagrange theorem and dominated convergence theorem proves that in every Riemannian coordinate patch,

\[
\partial^2_x f(x) = \int_M \partial^2_x G(x, y)h(y)d\mu_g(y) \quad \text{if } x \in M \setminus \text{supp}(h).
\]

Every \( \partial^2_x f \) is necessarily bounded on a finite cover of Riemannian charts of a compact ball \( B_{R'} \) including \( \text{supp}(h) \). Outside \( B_{R'} \), a procedure similar to that followed to prove Equation (A.13) and relying on Equation (A.9) for \( \beta = 0 \) proves that there is a constant \( H_x < +\infty \) such that, in every local Riemannian coordinate patch on \( M \) and for \( i = 1, \ldots, d \),

\[
|\partial^i_x f(x)| < H_x.
\]

We have established that \( f \in C^\infty_b(M) \) concluding the proof.

**Proof of Lemma 3.11.** Let us consider \( u \in D(M) \) and the map \( u(t) := e^{t M} u \) for \( t \in [0, +\infty) \). Due to Proposition 2.2 (i.e., Proposition 6.2 in [20]) \( u(t) \in D(M) \) and this map is the unique classical solution of the Cauchy problem associated to \( M \) with initial datum \( u \). In particular it is continuously differentiable and satisfies \( \frac{du}{dt} = Mu(t) \). Since \( M \subset N \), it also satisfies \( \frac{du}{dt} = Nu(t) \) and thus, again for Proposition 2.2, it is also the unique solution of the Cauchy problem associated to \( N \) with initial datum \( u \). That is \( u(t) = e^{t N} u \). We have in particular found that, if \( u \in D(M) \), then \( e^{t N} u \in D(M) \) for \( t \in [0, +\infty) \), so that \( D(M) \) is invariant under the semigroup generated by \( N \). Proposition 6.2 in [20] implies that \( D(M) \) is a core for \( N \). Since \( M \subset N \) and both operators are closed, then \( M = N \). □

**Proof of Lemma 3.13.** Let us denote by \( L'' \) the Hilbert-space closure \( \|L_0|_{C^\infty_b(M)} \). We remark that \( L_0|_{C^\infty_b(M)} \) is closable since its adjoint has a dense domain, as one can easily prove by a integration-by-parts argument. We write \( L'' \) in place of \( L' \), to
stress that the differential operator $L_0$ whose $L''$ is the Hilbert space closure over the domain $C_c^\infty(M)$ now includes the perturbation $B$. The proof, except for a point, is identical to that of Proposition 3.9 using Proposition 4.1 in its Corollary 4.2 in [51], observing that elliptic regularity works also for $-L''$ since this property depends only on the second order part of $L_0$, and noticing that the properties of $G$ established in Theorem 2.2 of [51], Equation (A.9) in particular, are valid also if $L_0|_{C_c^\infty(M)}$ is not symmetric. The only new item to prove separately is that there is a $\lambda > 0$ in the resolvent set of $-L''$, which, differently from $-L'$, is no longer positive and selfadjoint due to the presence of the term $B$. With this result the proof of the thesis concludes. Let us prove the existence of such $\lambda > 0$ by establishing that $L''$ is the generator of a strongly continuous semigroup in $L^2(M, \mu_g)$ under the hypothesis (3.8): if this case, the standard spectral bound of generators of strongly continuous semigroups (Corollary 1.13 in [20]) implies that $\Re(\sigma(L''))$ has finite upper bound so that $\rho(L'') \cap (0, +\infty) \neq \emptyset$ and the requested $\lambda > 0$ exists. In the rest of the proof $-L'$ will denote again the positive selfadjoint operator used in the proof of Proposition 3.9, which is the Hilbert-space closure of $L_0|_{C_c^\infty(M)}$, where $A_0$ does not contain the perturbation $B$. As is known from Proposition 4.1 in [51], $D(L'') = D(L') = W^2_2(M)$ (see [51] for the definition of those Sobolev spaces on smooth Riemannian manifolds of bounded geometry). The operator $B|_{C_c^\infty(M)}$ is $L^2(M, \mu_g)$-closable since its adjoint has dense domain (it including $C_c^\infty(M)$) and the closure of $B|_{C_c^\infty(M)}$ has domain that evidently includes $W^2_2(M)$ because $C_c^\infty(M)$ is dense in $W^2_2(M) \supset W^2_2(M)$ [51]. We intend to prove that, defining $L'' + B|_{C_c^\infty(M)}$ on the domain $W^2_2(M)$ of the first addend, then $L' + B|_{C_c^\infty(M)}$ is (i) closed and (ii) it is the generator of a strongly continuous semigroup. Notice that, in this case $L' + B|_{C_c^\infty(M)} = L''$ since $L'' \subset L' + B|_{C_c^\infty(M)}$ by construction ($L''$ is the closure of $L_0|_{C_c^\infty}$ whereas the right-hand side is a closed extension of that) and the two sides of the inclusion have the same domain $W^2_2(M)$. Hence (i) and (ii) imply that $L''$ itself is the generator of a strongly continuous semigroup as wanted. To conclude the proof, we prove that (i) and (ii) are true if Equation (3.8) holds. Since $\sigma(L') \subset (-\infty, 0]$ and $L'$ is selfadjoint, $\{e^{tL'}\}_{t \in [0, +\infty)}$ is an analytic semigroup in $L^2(M, \mu_g)$. To prove (i) and (ii), according to Theorem X.54 in [43], it is sufficient to demonstrate that for every $a > 0$, there is a corresponding $b > 0$ such that (the norm is that of $L^2(M, \mu_g)$)

$$||B|_{C_c^\infty(M)}\psi|| \leq a||L'\psi|| + b||\psi|| \quad \text{for all } \psi \in W^2_2(M).$$

Observe that, since $C_c^\infty(M)$ is a core for $L'$ (it is essentially selfadjoint thereon) and $B|_{C_c^\infty(M)}$ is closed, the condition above is equivalent to

$$||B\psi|| \leq a||L'\psi|| + b||\psi|| \quad \text{for all } \psi \in C_c^\infty(M).$$

In turn, according to the remark on the condition (iii) on p. 162 of [43], the condition above is equivalent to the next statement: For every $a > 0$ there is $b > 0$ such that

$$||B\psi||^2 \leq a||L'\psi||^2 + b||\psi||^2 \quad \text{for all } \psi \in C_c^\infty(M) \quad (A.16)$$

(where these $a$, $b$ are generally different from those in the previous inequality). To conclude we prove that Equation (A.16) is consequence of Equation (3.8). From the latter, replacing $\xi_k$ with $V^k_h\psi$, if $\psi \in C_c^\infty(M)$, we have

$$\int_M (B\psi(x))(B\psi(x)) \, d\mu_g(x) \leq c \int_M \sum_{i=1}^r \sum_{a,b=1}^d (A^a_i V^g_a \psi(x))(A^b_i V^g_b \psi(x)) \, d\mu_g(x)$$

$$= -2c \int_M \bar{\psi}(x)(L'\psi)(x) \, d\mu_g(x).$$

Namely, if $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(M, \mu_g)$, standard results of spectral theory [40, 50] yield

$$||B\psi||^2 \leq 2c \langle \psi, -L'\psi \rangle = 2c \int_{\mathbb{R}^+} \lambda d\nu_{\psi}(\lambda)$$

where $\nu_{\psi}(E) = \langle \psi, P(-L')(E)\psi \rangle$, with $P(-L')$ being is the spectral measure of the selfadjoint positive operator $-L'$ and $E \subset \mathbb{R}$ any Borel set. Here observe that, since $c > 0$, for every $a > 0$ there is $b > 0$ such that

$$2c\lambda \leq a\lambda^2 + b \quad \text{for all } \lambda \geq 0.$$
It is in fact sufficient to use \( b = c^2 / a \). Therefore, again from standard results of spectral theory,

\[
||B\psi||^2 \leq 2c \int_{\mathbb{R}^+} \lambda dv_{\phi}(\lambda) \leq a \int_{\mathbb{R}^+} \lambda^2 dv_{\phi}(\lambda) + b \int_{\mathbb{R}^+} 1 dv_{\phi}(\lambda) = a ||L'\psi||^2 + b ||\psi||^2.
\]

In summary, for every \( a > 0 \), there is \( b > 0 \) such that Equation (A.16) holds

\[
||B\psi||^2 \leq a ||L'\psi||^2 + b ||\psi||^2 \quad \text{for all } \psi \in C^\infty_c(M),
\]

concluding the proof.

**Proof of Proposition 4.6.**

(a) Let us start with a given \( r \in (0, I(M,g)) \) and consider a Riemannian system of coordinates in the ball \( B_r^{(M,g)}(p) \). Expanding \( g_{ab}(y) \) around 0 up to the first order with the usual Taylor expansion, we have

\[
g_{ab}(y) = \delta_{ab} + 0 + R^{(2)}_{ab}(y)
\]

where, for some \( \xi \in B_r(0) \),

\[
R^{(2)}_{ab}(y) = \frac{1}{2!} \sum_{i,j} \frac{\partial^2 g_{ab}}{\partial y^i \partial y^j}(\xi) y^i y^j \quad y \in B_r(0), \quad i, j = 1, \ldots, d.
\]

Taking the second bound in Equation (2.14) into account for \( k = 2 \) and using \( |y^k| \leq r \) we have

\[
||A(y(q))||^2 - ||A(y(q))||^2_g \leq \sum_{a,b=1}^d A^a(y)g_{ab}(y)A^b(y) - A^a(y)\delta_{ab}A^b(y) = \sum_{a,b=1}^d A^a(y)A^b(y)R^{(2)}_{ab}(y)
\]

\[
\leq \sum_{a,b=1}^d |A^a(y)||A^b(y)| \frac{C_2^{(r)}}{2} d^2 r^2 \leq \frac{C_2^{(r)}}{2} \sum_{i,j=1}^d ||A(y)|| ||A(y)|| = \frac{C_2^{(r)}}{2} ||A(y)||^2.
\]

In particular

\[
||A(y(q))||^2 - ||A(y(q))||^2_g \leq \frac{C_2^{(r)}}{2} ||A(y)||^2
\]

namely, if \( ||y|| < r \), we have,

\[
\left( 1 - \frac{C_2^{(r)}}{2} d^2 r^2 \right) ||A(y)||^2 \leq ||A(y(q))||^2_g.
\]

Restricting \( r \) to \( r_0 > 0 \) such that \( r^4 (1 - d^4 r_0^2 C_2^{(r)}/2) > 0 \) and defining \( k_1 := (1 - d^4 r_0^2 C_2^{(r)}/2)^{-1} \), we conclude that (a) is valid for \( y \in B_{r_0}(0) \), that is, \( q \in B_{r_0}^{(M,g)}(p) \).

(b) Let us first show that, if \( r_0 > 0 \) is suitably small, then

\[
||T(y(q))||^2 \leq k_2 ||T(q)||^2_g, \quad \text{for all } q \in B_{r_0}^{(M,g)}(p)
\]

for some \( k_2 \geq 0 \) independent of \( T \) and \( p \), for every smooth tensor field \( T \) of order (1,1). The proof is strictly analogous to that of (a), observing that

\[
||T(y(q))||^2 - ||T(y(q))||^2_g = \sum_{a,b,i,j=1}^d T^a_i(y) (\delta_{ab}\delta_{ij} - g_{ab}(y)g_{ij}(y)) T^j_g(y)
\]

(A.17)
and
\[ g^{ab}(y)g_{ij}(y) = \delta^{ab}\delta_{ij} + R^{(2)ab}_{ij}(y) \]

where, for some \( \xi \in B_r(0) \),
\[ R^{(2)ab}_{ij}(y) = \frac{1}{2!} \sum_{i,j} \frac{\partial^2 g^{ab}}{\partial y^i \partial y^j} \bigg|_{\xi} y^i y^j \quad y \in B_r(0), \quad i, j = 1, \ldots, d. \]

Using in Equation (A.18) both the second bound in Equations (2.14) and (2.15) for \( k = 0, 1 \) as we did in the proof (a) we obtain Equation (A.17). To conclude the proof of (b), observe that, if \( y \in B_{r_0}(0) \),
\[ \partial_{y^a} A^i = (\nabla^{(g)} A)^i - \sum_{c=1}^d \Gamma_{ac}^i A^c \]

so that, using Equation (2.17) together with rough estimates \( |A'| \leq |A|, |\nabla^{(g)} A| \leq \|\nabla^{(g)} A\| \), we have
\[ \|\nabla A\|^2 \leq \|\nabla^{(g)} A\|^2 + 2d^3 J^{(r_0)} k_1 \|A\| \|\nabla^{(g)} A\| + d^4 (J^{(r_0)})^2 \|A\|^2. \]

Finally observe that (a) and Equation (A.17) respectively imply
\[ \|A\| \leq k_1 \|A\|_g \quad \text{and} \quad \|\nabla^{(g)} A\| \leq \sqrt{k_2} \|\nabla^{(g)} A\|_g \]

which, inserted in the previous inequality, yield
\[ \|\nabla A(y(q))\|^2 \leq k_2 \|\nabla^{(g)} A(q)\|^2 \|A(q)\|_g \|\nabla^{(g)} A(q)\|_g + d^4 (J^{(r_0)})^2 k_1^2 \|A(q)\|_g^2 \]

which must hold if \( q \in B_{r_0}^{(M,g)}(p) \). By construction, the constants, \( k_1, k_2, k_3 := d^4 (J^{(r_0)})^2 k_1^2 \), and \( k_4 := 2d^3 J^{(r_0)} k_1 \sqrt{k_2} \) do not depend on \( A \) and the estimate is valid for every \( p \in M \) provided \( q \in B_{r_0}^{(M,g)}(p) \). \( \square \)