Elements of Finite Order in the Riordan Group

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Abstract. We consider elements \((g(x), F(x))\) of finite order in the Riordan group over a field \(F\) of characteristic 0. We solve for all integers \(n \geq 2\) the two fundamental questions posed by L. Shapiro ([16], Q8., Q9.1) for the case \(n = 2\) (“involutions”). Given a formal power series \(F(x) = \omega x + f_2 x^2 + \cdots\), and an integer \(n \geq 2\), Theorem 1 states exactly which \(g(x)\) make \((g(x), F(x))\) a Riordan element of order \(n\). Theorem 2 classifies finite-order Riordan group elements up to conjugation. We then relate our work to papers [4], [5] of Cheon and Kim which motivated this paper. We supply a missing proof in [4] and we solve the Open question in Section 2 of [5]. Finally we show how this circle of ideas gives a new proof of C. Marshall’s theorem ([9], [10]), which finds the unique \(F(x)\), given bi-invertible \(g(x)\), such that \((g(x), F(x))\) is an involution.

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1 Introduction

The Riordan group over a field \(F\) is a group \(\mathcal{R} = \mathcal{R}(F)\) of infinite lower triangle matrices, introduced in [15] for combinatorial purposes. (See for example [1], [7], [8] and [17] for combinatorial applications.) Each element of \(\mathcal{R}\) is determined by a pair \((g(x), F(x))\) of generating functions and the group is isomorphic to the semi-direct product of two groups of formal power series. In this paper, we take this semi-direct product as the definition (see 3. immediately below) and we resolve some of the foundational algebraic problems concerning Riordan groups.

We fix a field \(F\) of characteristic zero and let \(F[[x]]\) denote the set of all formal power series \(g(x)\),

\[
g(x) = g_0 + g_1 x + g_2 x^2 + \cdots + g_n x^n + \cdots, \quad (g_n \in F).
\]

We consider the following three groups. (Technical background will be given in Section 2.)
1. \( F_0[[x]] = \{ g(x) \in \mathbb{F}[[x]] \mid g_0 \neq 0 \} \) is a group under multiplication.

2. \( F_1[[x]] = \{ G(x) \in \mathbb{F}[[x]] \mid G(x) = g_1 x + g_2 x^2 + \cdots, \text{with } g_1 \neq 0 \} \) is a group under composition.

3. \( F_0[[x]] \times F_1[[x]] \) becomes the Riordan group \( \mathcal{R}(\mathbb{F}) \) if we define the operation as
   \[
   (g(x), F(x)) (h(x), K(x)) = (g(x) \cdot h(F(x)), K(F(x)))
   \]

4. It is also useful to have specific notation for the subset \( F_+[[x]] \) of \( \mathbb{F}[[x]] \) given by:
   \( F_+[[x]] = \{ G(x) \in \mathbb{F}[[x]] \mid G(x) = g_k x^k + g_{k+1} x^{k+1} + \cdots, \text{with } k \geq 1 \text{ and } g_k \neq 0 \} \)

Notation:
- We denote series in \( F_0[[x]] \) with lower case letters \( g(x), h(x), \) etc.
- We denote series in \( F_+[[x]] \) with capital letters \( F(x), G(x), \) etc.
- A non-constant series \( g(x) \in F_0[[x]] \) may be written uniquely as \( g_0 + G(x) \) with \( 0 \neq g_0 \in \mathbb{F} \).

Definition 1.1.

The order of an element \( \gamma \in \Gamma \) (in this paper \( \Gamma \) will be one of the groups \( \mathbb{F}\setminus\{0\}, F_0[[x]], F_1[[x]], \mathcal{R}(\mathbb{F}) \)) is the least possible integer \( n \) such that \( \gamma^n = (\text{the identity element of } \Gamma) \). If no such integer exists, we say that \( \gamma \) has infinite order. If \( \gamma \) has order two, \( \gamma \) is called an involution in \( \Gamma \).

Remark 1.2.

If \( \mathbb{F} \) is a subfield of \( \mathbb{R} \) then it will be immediate from the definitions of the operations in our groups (Lemma 2.1) that the only non-trivial elements of finite order in these groups are involutions, since this is true in the multiplicative group \( \mathbb{R}\setminus\{0\} \). This fact is relevant when dealing with generating functions of integer (e.g., counting) sequences as is central in Combinatorics. Theorem 1 below gives a significant new result even when restricted to involutions. On the other hand, power series over the complex field \( \mathbb{F} = \mathbb{C} \) are central in mathematics, and both of our main theorems answer basic questions in this realm for all \( n \geq 2 \).
Remark 1.3.
While our Main Theorems below are logically independent of their work, we were much
influenced by the papers ([4], [5]) of Cheon and Kim. We discuss connections to their papers
in Section 5 and, in particular, we solve the Open question in Section 2 of [5]. In Section 6
we use an idea from [4] to give a new (but less elegant) proof of C. Marshall’s Theorem ([9],
[10]).

Main Theorems

Theorem 1.
Suppose that \( g(x) = g_0 + g_1 x + g_2 x^2 + \cdots \in \mathbb{F}_0[[x]] \), where \( g_0 \) has finite order (= \( \text{ord}(g_0) \)) in \( \mathbb{F} \setminus \{0\} \) and \( F(x) = \omega x + F_2 x^2 + \cdots \) is an element of finite (compositional) order in \( \mathbb{F}_1[[x]] \).
Then
\[
(g(x), F(x)) \text{ has finite order } n \text{ in the Riordan group } \iff
\]
1. \( n = \text{lcm}(\text{ord}(g_0), \text{ord}(F(x))) = \text{lcm}(\text{ord}(g_0), \text{ord}(\omega)) \)
and
2. there exists an element \( h(x) \in \mathbb{F}_0[[x]] \) such that \( g(x) = g_0 \cdot \frac{h(x)}{h(F(x))} \).

Indeed, if \( \frac{\hat{g}(x)}{g_0} = \frac{1}{g_0} \cdot g(x) \) and if we set \( b = \text{ord}(F(x)) \) then we may set
\[
h(x) = \left( \frac{\hat{g}^\frac{1}{b}(x)}{g_0} \right)^{b-1} \left( \frac{\hat{g}^\frac{1}{b}(F(x))}{g_0} \right)^{b-2} \left( \frac{\hat{g}^\frac{1}{b}(F^{(2)}(x))}{g_0} \right)^{b-3} \cdots \left( \frac{\hat{g}^\frac{1}{b}(F^{(b-2)}(x))}{g_0} \right)
\]

Remark 1.4. If \( m \in \mathbb{N} \) then \( \frac{\hat{g}(x)}{g_0} = \frac{1}{g_0} \cdot g(x) = (1 + b_1 x + \cdots) \) has a unique \( m^{\text{th}} \) root of the form \((1 + c_1 x + \cdots)\) by [11], Theorem 3. It is this \( m^{\text{th}} \) root which is denoted by \( \frac{\hat{g}^\frac{1}{m}}{g_0}(x) \). It is proved in [11] that all the usual laws of roots and exponents apply for these formal power series. In our proof of Theorem 1 we could replace \( \frac{\hat{g}^\frac{1}{b}(x)}{g_0} \) by \( \alpha \cdot \frac{\hat{g}^\frac{1}{b}(x)}{g_0} \) in the formula for \( h(x) \), where \( \alpha \) is any \( b^{\text{th}} \) root of unity.

Remark 1.5. If \((g(x), F(x))\) is an involution (i.e., \( n = 2 \)) then the proof gives \( h(x) = \left( \frac{1}{g_0} \cdot g(x) \right)^\frac{1}{2} \).

Theorem 2 (The Conjugacy Theorem).
A. Suppose that \( g(x) = g_0 + g_1 x + g_2 x^2 + \cdots \) and \( F(x) = \omega x + F_2 x^2 + \cdots \).

If \((g(x), F(x))\) has finite order in the Riordan group then \((g(x), F(x))\) is conjugate in \( R \) to \((g_0, \omega x)\).

Indeed if \( h(x) \in \mathbb{F}_0[[x]] \) with \( g(x) = g_0 \cdot \frac{h(x)}{h(F(x))} \), as in Theorem 1, if \( \text{ord}(F(x)) = b \) and \( \Sigma_F = \frac{1}{b} \sum_{j=1}^{b} \omega^j F(b-j)(x) \) then

\[
\left( h(x), \Sigma_F \right)^{-1} \left( g(x), F(x) \right) \left( h(x), \Sigma_F \right) = (g_0, \omega x)
\]

B. Suppose that \((g(x), F(x))\) and \((h(x), K(x))\) are Riordan elements of finite order with

\[
g(x) = \sum_i g_i x^i, \quad h(x) = \sum_i h_i x^i, \quad F(x) = \sum_j f_j x^j \quad \text{and} \quad K(x) = \sum_j k_j x^j.
\]

Then \((g(x), F(x)) \sim (h(x), K(x))\) (i.e., they are conjugate in \( R \))

\[\iff g_0 = h_0 \quad \text{and} \quad f_1 = k_1.\]

Remark 1.6. If \((g(x), F(x))\) is an involution and \((g(x), F(x)) \neq (-1, x)\) then the Conjugacy Theorem, Part A, states that

\[
\left( \left( \frac{1}{g_0} \cdot g(x) \right)^{\frac{1}{2}}, \frac{1}{2}(x - F(x)) \right)^{-1} \left( g(x), F(x) \right) \left( \left( \frac{1}{g_0} \cdot g(x) \right)^{\frac{1}{2}}, \frac{1}{2}(x - F(x)) \right) = (1, -x).
\]

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2 Basics of Formal Power Series and the Riordan Group

We summarize the initial facts about formal power series and the Riordan group with Lemmas 2.1 - 2.3. See [6] for proofs, references and historical background on formal power series, [1] and [15] for introduction to the Riordan group. In 2.4 - 2.7 we develop some basic facts about elements of finite order in \( R \) which we will need in the next section.

Our notational convention is to follow the pattern

\[
g(x) = \sum_{n=0}^{\infty} g_n x^n \in \mathbb{F}_0[[x]], \quad F(x) = \sum_{n=1}^{\infty} f_n x^n \in \mathbb{F}_+[[x]], \quad \text{etc.}
\]
Lemma 2.1.

- \( F_0[[x]] \) is an abelian group under multiplication:
  - \( g(x) \cdot h(x) \overset{\text{def}}{=} g_0 h_0 + (g_0 h_1 + g_1 h_0)x + \cdots + \left( \sum_{k=0}^{n} (g_k h_{n-k}) \right) x^n + \cdots \)
  - The identity element is \( g(x) = 1 \).
  - \( g(x)^{-1} \overset{\text{def}}{=} \frac{1}{g(x)} = \frac{1}{g_0} - \frac{g_1}{g_0^2} x + \cdots \) (found by solving inductively for coefficients).

- \( F_1[[x]] \) is a group under composition (= “Substitution”)
  - \((F \circ G)(x) = F(G(x)) \overset{\text{def}}{=} f_1 \cdot G(x) + f_2 \cdot G(x)^2 + f_3 G(x)^3 + \cdots \)
    \( = f_1 g_1 x + (f_1 g_2 + f_2 g_1^2) x^2 + \cdots \)
  - The identity is \( \text{id}(x) = x \).
  - The \( n \)th power of \( F \) is denoted by \( F^{(n)}(x) = F\left(F(\cdots F(x) \cdots)\right) \).
    \( F^{(n)}(x) = f_1^n x + \cdots \)
  - The inverse \( F^{-1}(x) \) of \( F(x) \) in \( F_1[[x]] \) is denoted by \( \overline{F}(x) = F^{-1}(x) \):
    \( \overline{F}(x) = \frac{1}{f_1} x - \frac{f_2}{f_1^2} x^2 + \cdots \) (solve \( F(G(x)) = x \) inductively for the coefficients \( g_j \)).

(For a general formula, see the Lagrange Inversion Formula, [19].)

- The set \( F_0[[x]] \times F_1[[x]] \) becomes a group – the Riordan group \( \mathcal{R} = \mathcal{R}(F) \) – if we define
  \( (g(x), F(x))(h(x), K(x)) = (g(x) \cdot h(F(x)), K(F(x))) \).

  - The identity element in \( \mathcal{R} \) is \((1, x)\)
  - Inverses are given by
    \( \left( g(x), F(x) \right)^{-1} = \left( \frac{1}{g(F(x))}, \overline{F}(x) \right) \).
  - If \( n \in \mathbb{N} \) then \( (g(x), F(x))^n = (g(x) \cdot g(F(x)) \cdots g(F^{(n-1)}(x)), F^{(n)}(x)) \)
    \( = (g_0^n + \cdots , f_1^n x + \cdots ) \).
Lemma 2.2.
If $F(x) = \omega x + f_2x^2 + f_3x^3 + \ldots$ has finite (compositional) order $n$ in $\mathbb{F}_1[[x]]$, then $\omega$ has order $n$ in $\mathbb{F} \setminus \{0\}$. (i.e., $\omega$ is a primitive $n$th root of unity.) □

The author first learned the following Conjugation Lemma from a much more general theorem (Theorem 8, page 19) in [2]. It appears explicitly in [5] and [14] and appears with different conjugator than $\Sigma_F$ in [3].

Lemma 2.3. (Conjugation of finite-order formal power series: $F(x) \sim \omega x$)
Suppose that $F(x) = \omega x + f_2x^2 + f_3x^3 + \cdots$ has finite order $n$ in $\mathbb{F}_1[[x]]$. Let

$$\Sigma_F = \frac{1}{n} \sum_{j=1}^{n} \omega^j F(n-j)(x).$$

Then $\Sigma_F$ conjugates $F(x)$ to $\omega x$ in $\mathbb{F}_1[[x]]$: $(\Sigma_F \circ F \circ \Sigma_F^{-1})(x) = \ell_\omega(x)$. □

The following simple examples of Riordan elements of order 6 in $\mathcal{R}(\mathbb{C})$ illustrate the next Proposition.

Example 2.4. $(a, bx)(c, dx) = (ac, (db)x), a, b, c, d \in \mathbb{F}$.
Thus all of the following elements of $\mathcal{R}(\mathbb{C})$ have order 6:

$$(1, e^{2\pi i / 6} z), \quad (e^{2\pi i / 3}, -z), \quad (e^{2\pi i / 3}, -z), \quad (e^{2\pi i / 3}, e^{2\pi i / 3} z)$$

The following Proposition is central to the study of Riordan elements of finite order $n \geq 2$ and to the proofs of our Main Theorems.

Proposition 2.5.
Let $g(x) = g_0 + g_1x + g_2x^2 + \cdots \in \mathbb{F}_0[[x]]$ and $F(x) \in \mathbb{F}_1[[x]]$. Suppose that

- $g_0$ has order $a$ in the multiplicative group $\mathbb{F} \setminus \{0\}$.
- $F(x)$ has (compositional) order $b$ in the group $\mathbb{F}_1[[x]]$
- $(g(x), F(x))$ has finite order $n$ in the Riordan group.

Then $n = \ellcm(a, b).$
Proof:

A straightforward inductive argument, using the definition of multiplication in \( R \) yields the fact that \((g(x), F(x))\) has finite order \( n \) iff \( n \) is the least positive integer such that

\[
\bullet \quad g(x) \cdot g(F(x)) \cdots g(F^{(n-1)}(x)) = 1 \quad \text{and} \quad F^{(n)}(x) = x.
\]

Since \( g(x) \cdot g(F(x)) \cdots g(F^{(n-1)}(x)) = (g_0^n + \cdots) \) we see that \( g_0^n = 1 \) so that \( a = \text{order}(g_0) \) divides \( n \). Also \( F^{(n)}(x) = x \) implies that \( b \) divides \( n \). Hence \( \ell cm(a, b) \) divides \( n \).

Let \( n = kb \). Then

\[
((1, x) = (g(x), F(x))^n = \left( g(x) \cdot g(F(x)) \cdots g(F^{(kb-1)}(x)), F^{(kb)}(x) \right) = \left( (g(x) \cdot g(F(x)) \cdots g(F^{(b-1)}(x)))^k, x \right) \quad \text{(because} F^{(j+b+r)}(x) = F^{(r)}(x)) \text{).}
\]

This implies that \( g(x) \cdot g(F(x)) \cdots g(F^{(b-1)}(x)) = (g_0^b + \cdots) \) is a constant (a \( k \)th root of unity in \( \mathbb{F} \)). Hence

\[
g(x) \cdot g(F(x)) \cdots g(F^{(b-1)}(x)) = g_0^b.
\]

Now let us set \( \ell cm = \ell cm(a, b) = qb \). Thus \( 1 = g_0^a = g_0^{\ell cm} = (g_0^b)^q \) and \( x = F^{(b)}(x) = F^{(\ell cm)}(x) \). Therefore

\[
(g(x), F(x))^{\ell cm} = \left( (g(x) \cdot g(F(x)) \cdots g(F^{(b-1)}(x)))^q, F^{\ell cm}(x) \right) = \left( (g_0^b)^q, x \right) = (1, x)
\]

Hence \( n = \text{order}(g(x), F(x)) \) divides \( \ell cm(a, b) \) and, as seen at the outset, \( \ell cm(a, b) \) divides \( n \). Thus \( n = \ell cm(a, b) \), as claimed. \( \square \)

Corollary 2.6.

Suppose that \( g(x) = 1 + g_2 x^2 + \cdots \) and that \((g(x), F(x))\) has finite order.

Then \( \text{order}(g(x), F(x)) = \text{order}(F(x)) \) \( \square \)

Corollary 2.7.

Suppose that \( g(x) = g_0 \cdot k(x) \) where \( g_0 \in \mathbb{F} \setminus \{0\} \) has finite order and \( k(x) \in \mathbb{F}_0[[x]] \). Let \( F(x) \in \mathbb{F}_1[[x]] \).
1. \((g(x), F(x))\) has finite order iff \((k(x), F(x))\) has finite order.

2. When these have finite order,
   
   \[ \text{ord}(g(x), F(x)) = \ell \text{cm}(\text{ord}(g_0k_0), \text{ord}(F(x))) \]
   \[ \text{ord}(k(x), F(x)) = \ell \text{cm}(\text{ord}(k_0), \text{ord}(F(x))) \]

Proof:

1. If \((g(x), F(x))\) has finite order \(n\) and if \(\ell = \ell \text{cm}(n, \text{ord}(g_0))\) then

   \[
   (1, x) = (g(x), F(x))^n = (g(x), F(x))^\ell \\
   = (g(x) \cdot g(F(x)) \cdots g(F^{(\ell-1)}(x)), F^{(\ell)}(x)) \\
   = (g_0^\ell \cdot k(x) \cdot k(F(x)) \cdots k(F^{(\ell-1)}(x)), F^{(\ell)}(x)) \\
   = (k(x) \cdot k(F(x)) \cdots k(F^{(\ell-1)}(x)), F^{(\ell)}(x))
   \]

   Thus \((k(x), F(x))\) has finite order dividing \(\ell\).

   Similarly, if \((k(x), F(x))\) has order \(m\), then \((g(x), F(x))^m = (g_0^m \cdot k(x) \cdot k(F(x)) \cdots (F^{(m-1)})(x), F^{(m)}(x))\).

   Thus \((g(x), F(x))^\ell \text{cm}(\text{ord}(g_0), m) = (1, x)\) and \((g(x), F(x))\) has finite order.

2. The evaluation of the orders of \((g(x), F(x))\) and \((k(x), F(x))\) follow from Proposition 2.5. \(\square\)

3. **Proof of Theorem 1:** \(g(x) = g_0 \cdot \frac{h(x)}{h(F(x))}\)

   \((\iff)\) Proof of Sufficiency in Theorem 1

   We are given that \(g(x) = g_0 + G(x) \in \mathbb{F}_0[[x]]\) and \(F(x) \in \mathbb{F}_1[[x]]\) with \(g_0\) of finite order and \(F(x)\) of finite order. We let \(n = \ell \text{cm}(\text{ord}(g_0), \text{ord}(F(x)))\). Also we are given \(h(x) \in \mathbb{F}_0[[x]]\).
with \( g(x) = g_0 \cdot \frac{h(x)}{h(F(x))} \). Then we have

\[
(g(x), F(x))^n = \left(g(x) \cdot g(F(x)) \cdots g(F^{(n-1)}(x)), F^{(n)}(x)\right)
\]

\[
= \left(\frac{g_0^n \cdot h(x)}{h(F(x))} \frac{h(F(x))}{h(F^{(2)}(x))} \cdots \frac{h(F^{(n-1)}(x))}{h(F^{(n)}(x))}, x\right)
\]

\[
= \left(1 \cdot \frac{h(x)}{h(F(x))} \frac{h(F(x))}{h(F^{(2)}(x))} \cdots \frac{h(F^{(n-1)}(x))}{h(F^{(n)}(x))}, x\right) = (1, x)
\]

Therefore \((g(x), F(x))\) has finite order. We apply Corollary 2.7, setting \(k(x) = \frac{h(x)}{h(F(x))} = 1 + K(x)\), and we see that \(\text{ord}(g(x), F(x)) = \ell cm(\text{ord}(g_0), \text{ord}(F(x))) = n\). \(\square\)

\(\implies\) Proof of Necessity in Theorem 1

We are given that \((g(x), F(x))\) has order \(n\). The fact that \(n = \ell cm(\text{ord}(g_0), \text{ord}(F(x)))\) is given by Proposition 2.5. It remains to prove that \(\exists h(x) \in \mathbb{F}_0[[x]]\) such that \(g(x) = g_0 \cdot \frac{h(x)}{h(F(x))}\).

If \(g(x) \in \mathbb{F}_0[[x]]\) we set \(\tilde{g}(x) = \frac{1}{g_0} \cdot g(x) = 1 + \frac{g_1}{g_0} x + \frac{g_2}{g_0} x^2 + \cdots\).

From Corollary 2.7, we know that \((\tilde{g}(x), F(x))\) is an element in the Riordan group of order equal to \(\ell cm(1, \text{ord}(F(x))) = \text{ord}(F(x))\). If we set \(b = \text{ord}(F(x))\) then we have

\[
1 = \tilde{g}(x)\tilde{g}(F(x)) \cdots \tilde{g}(F^{(b-1)}(x)) \quad \text{and} \quad \frac{1}{\tilde{g}(x)} = \tilde{g}(F(x)) \tilde{g}(F^{(2)}(x)) \cdots \tilde{g}(F^{(b-1)}(x)) \quad (**)
\]

Using Remark 1.4, the series \(\tilde{g}^\frac{1}{x}(x)\) is well-defined and for notational convenience we denote \(k(x) = \tilde{g}^\frac{1}{x}(x)\). Let us define

\[
h(x) \overset{\text{def}}{=} \left(\tilde{g}^\frac{1}{x}(x)\right)^{b-1} \left(\tilde{g}^\frac{1}{x}(F(x))\right)^{b-2} \left(\tilde{g}^\frac{1}{x}(F^{(2)}(x))\right)^{b-3} \cdots \left(\tilde{g}^\frac{1}{x}(F^{(b-2)}(x))\right)^1
\]

\[
= (k(x))^{b-1} \cdot (k(F(x)))^{b-2} \cdot (k(F^{(2)}(x)))^{b-3} \cdots (k(F^{(b-2)}(x))).
\]
Then

\[ g(x) = g_0 \cdot \hat{g}(x) = g_0 \left[ \frac{\hat{g}(x)^{b-1}}{(1/\hat{g}(x))} \right]^\dagger \]

\[ = g_0 \left[ \frac{\hat{g}(x)^{b-1}}{(\hat{g}(F(x)) \cdot \hat{g}(F'(x)) \cdots \hat{g}(F^{(b-1)}(x))} \right]^\dagger \text{ by equation (**)} \text{ above} \]

\[ = g_0 \cdot \frac{k(x)^{b-1}}{(k(F(x)) (k(F'(x))) (k(F''(x))) \cdots (k(F^{(b-1)}(x)))} \]

\[ = g_0 \cdot \frac{(k(x))^{b-1}}{(k(F(x)))^{b-1}} \cdot \frac{(k(F(x)))^{b-2}}{(k(F'(x)))^{b-2}} \cdot \frac{(k(F'(x)))^{b-3}}{(k(F''(x)))^{b-3}} \cdots \frac{(k(F^{(b-2)}(x)))}{(k(F^{(b-1)}(x)))} \]

\[ = g_0 \cdot \frac{h(x)}{h(F(x))} \]

This completes the proof of Theorem 1. \[ \square \]
Proof of Theorem 2 (The Conjugacy Theorem)

A. From Theorem 1, there exists \( h(x) \in F_0[[x]] \) such that \( g(x) = g_0 \cdot \frac{h(x)}{h(F(x))} \).

Let \( \Sigma_F = \sum_{j=1}^{b} \omega^j F(b-j)(x) \in F_1[[x]] \), as in Lemma 2.3. Denote \( \ell_\omega(x) = \omega x \). Then

\[
(g(x), F(x)) (h(x), \Sigma_F) = \left( g(x) \cdot h(F(x)), (\Sigma_F \circ F)(x) \right)
\]

\[
= \left( g_0 \cdot \frac{h(x)}{h(F(x))} \cdot h(F(x)), (\Sigma_F \circ F)(x) \right)
\]

\[
= \left( g_0 \cdot h(x), (\ell_\omega \circ \Sigma_F)(x) \right), \text{ by Lemma 2.3,}
\]

\[
= (h(x), \Sigma_F(x)) (g_0, \ell_\omega(x))
\]

\[
= (h(x), \Sigma_F(x)) (g_0, \omega x)
\]

Therefore \( (h(x), \Sigma_F)^{-1} (g(x), F(x)) (h(x), \Sigma_F) = (g_0, \omega x) \), as claimed. \( \square \)

B. If \( g_0 = h_0 \) and \( f_1 = k_1 \) then, from part A.,

\[
(g(x), F(x)) \sim (g_0, f_1) = (h_0, k_1) \sim (h(x), K(x)).
\]

Therefore \( (g(x), F(x)) \sim (h(x), K(x)). \)

Conversely suppose \( (g(x), F(x)) \sim (h(x), K(x)). \) We observe that if we conjugate \( (g(x), F(x)) \) by an element \( (a(x), B(x)) \), then the first coefficients of the two terms of the pair both remain unchanged:

\[
(a(x), B(x)) (g(x), F(x)) (a(x), B(x))^{-1}
\]

\[
= (a(x) \cdot g(B(x)), F(B(x))) \left( \frac{1}{(a(B(x)))}, \overline{B(x)} \right)
\]

\[
= \left( a(x) \cdot g(B(x)) \cdot \frac{1}{(a(B(F(x))))}, \overline{B(F(B))} \right)
\]

\[
= \left( (a_0 + \cdots)(g_0 + \cdots) \frac{1}{(a_0 + \cdots)}, \left( \frac{1}{b_1} \cdot f_1 \cdot b_1 \right) x + \cdots \right)
\]

\[
= \left( (g_0 + \cdots, f_1 \cdot x + \cdots) \right)
\]
Therefore $g_0 = h_0$ and $f_1 = k_1$. □

5 Connection of This Paper to the Papers of Cheon and Kim

We describe in this section how the results above evolved from our reading of the noted Cheon-Kim papers [4], [5]. We use the notation above in discussing these papers. Also, we deal only with formal power series, while these authors allow the series to be either formal power series or analytic functions.

A key concept in their papers is that of an anti-symmetric series $\Phi(x, z)$ of two variables and more generally a $k$-cyclic symmetric series $\varphi(z_1, \ldots, z_k)$.

5.1 Riordan involutions and anti-symmetric series

The key theorem in [4], relating to L. Shapiro’s question (Q9.1, [16]), is the following:

**Theorem.** ([4], Theorem 2.3 (Q9.1))

If $D = (g(x), F(x))$ is a Riordan involution then

$$g(x) = \pm \exp [\Phi(x, F(x))]$$

for some anti-symmetric function $\Phi(x, z)$. Conversely, if $F^{(2)}(x) = x$ and $g(x) = \pm \exp [\Phi(x, F(x))]$ for any antisymmetric function $\Phi(x, z)$ then $D = (g(x), F(x))$ is a Riordan involution.

**Comment on the proof of this Theorem:**

No proof is given in [4]. A reference is given to [12], where this result is stated, also without proof. The proof of the “Conversely ...” part of the theorem is straightforward. But the fact that such a $\Phi(x, z)$ exists, given that $(g(x), F(x))$ is a Riordan involution, has not appeared. The Main Results above developed from the author’s discovery of the following proof:

**Proof of Necessity in Cheon-Kim, Theorem 2.3:**

Let $g(x) = g_0 \cdot \hat{g}(x)$. (Necessarily $g_0 = \pm 1$ since $g(x) \cdot g(F(x)) = 1$.) Define

$$\Phi(x, z) = \frac{1}{2} \cdot \ln \left( \frac{\hat{g}(x)}{\hat{g}(z)} \right).$$
Clearly this is antisymmetric: $\Phi(x, z) = -\Phi(z, x)$. Moreover

$$g_0 \cdot \exp (\Phi(x, F(x))) = g_0 \cdot \exp \left( \frac{1}{2} \cdot \ln \left( \frac{\hat{g}(x)}{g(F(x))} \right) \right)$$

$$= g_0 \cdot \exp \left( \ln \left( \frac{\hat{g}(x)}{\hat{g}(F(x))} \right)^{\frac{1}{2}} \right)$$

$$= g_0 \cdot \exp \left( \ln \left( \frac{1}{g(x)} \right)^{\frac{1}{2}} \right)$$

$$= g_0 \cdot \exp (\ln(\hat{g}(x)))$$

$$= g_0 \cdot \hat{g}(x)$$

$$= g(x). \quad \square$$

**Remark 5.1.**

Within this proof we see from the second and the last equations that

$$g(x) = g_0 \cdot \frac{\hat{g}\frac{1}{2}(x)}{\hat{g}\frac{1}{2}(F(x))}$$

It was this observation – that we can let $h(x) = \hat{g}\frac{1}{2}(x)$ - which led to the author’s proof of necessity in Theorem 1 in the case $n = 2$. This observation and this proof in this special case led to the explorations which resulted in Theorems 1 and 2 above. As we shall see below, after we proved Theorem 2 above we saw that it gave the key to proving ”the Open question” in [5].

**Remark 5.2.** The ability to use the logarithm and square root (or more generally $n^{\text{th}}$ root) functions in these discussions is crucial. As noted in Remark 1.4, the $n^{\text{th}}$ root $\hat{g}\frac{1}{n}(x)$ is well-defined and the usual laws apply when $g(x) = 1 + G(x)$.

Similarly, $\ln\left(g(x)\right)$ is defined, and its laws justified, in [11] when $g(x) = 1 + G(x)$. We have

$$\ln\left(1 + G(x)\right) \overset{\text{def}}{=} G(x) - \frac{1}{2}(G(x))^2 + \cdots - \frac{1}{n}(G(x))^n + \cdots$$

In our definition of $\Phi(x, z)$, application of the logarithm is well-defined only because we may write $\hat{g}(x) = 1 + K(x)$ and we have

$$\frac{\hat{g}(x)}{\hat{g}(z)} = \frac{1 + K(x)}{1 + K(z)} = \left(1+K(x)\right)\cdot\left(1-K(z)+\left(K(z)\right)^2+\cdots(-1)^n(K(z))^n+\cdots\right) = 1+H(x, z).$$
Remark 5.3. Warning:

\[ \ln(g(x)) = \ln(1 + G(x)) \] is not an element of \( F_1[[x]] \) in the cases where \( g_1 \neq 0 \). Also, in general as in the proof above, one may not assume that \( \Phi(x, F(x)) \in F_1[[x]] \). So these may not, in general, be taken as the second coordinate of a Riordan element.

For example, there exist elements \((g(x), F(x))\) of every order \( n \geq 2 \) in the Bell subgroup of \( R \) for which \( \ln(\hat{g}(x)) \notin F_1[[x]] \). For we have the following theorem (See [3], Theorem 2.5.4 or [6]):

**Theorem.** If \( 2 \leq n \in \mathbb{N} \) and \( \omega \) is a primitive \( n \)'th root of unity in the field \( F \) then for every infinite sequence \( \{ a_k \}_{k \neq n,j+1}^\infty \in F \) there exists a unique sequence \( \{ a_{nj+1} \}_{j=1}^\infty \) such that the formal power series

\[ F(x) = \omega x + \sum_{k=2}^\infty a_k x^k \]

has order \( n \) in \( F_1[[x]] \). Moreover, \( a_{nj+1} = 0 \) if \( a_{2j} = 0 \) for all \( k \leq j \).

Thus for example, there is an involution \( F(x) = -x + a_{10}x^{10} + a_{11}x^{11} + \cdots \) with \( a_{10} \neq 0 \). Then

\( ((g(x), F(x))) = \left( \frac{F(x)}{x}, F(x) \right) \)

is an involution in \( R \), as is \( (\hat{g}(x), F(x)) \) where \( \hat{g}(x) = -g(x) = 1 - a_{10}x^9 - \cdots \). It follows that \( \ln(\hat{g}(x)) \) is defined in \( F_+[[x]] \) but it is not an element of \( F_1[[x]] \).

This Remark reduces the generality of the validity of Theorem 2.5 and of Corollary 2.6 of [4]. However, the idea in Corollary 2.6 of [4] of using the logarithm as second coordinate in a conjugating element will be very fruitful in empowering us to give a new proof of C. Marshall’s theorem in Section 6.

### 5.2 An Answer to an Open Question on \( k \)-Cyclic Symmetric Series

Generalizing the proof above, concerning antisymmetric series and involutions, we now answer the “Open question.” in [5] for Riordan elements of higher order.

**Definition 5.4.** A \( k \)-cyclic symmetric series is a formal series \( \varphi(x_1, x_2, \ldots, x_k) \) in \( k \) variables such that

\[ \varphi(x_1, x_2, \ldots, x_k) + \varphi(x_2, x_3, \ldots, x_k, x_1) + \cdots \varphi(x_k, x_1, \ldots, x_{k-1}) = 0 \]
**Open question.** ([5], Section 2)  
If \((g(x), F(x))\) is a Riordan element of order \(k\) with \(g_0 = 1\), is \(g(x)\) of the form  
\[g(x) = \exp \left( \varphi(x, F(x), \ldots, F^{(k-1)}(x)) \right)\]  
for some some \(k\)-cyclic series \(\varphi(x_1, x_2, \ldots, x_k)\)?

**Our Approach:**  
We know from Theorem 1, setting \(\hat{g}(x) = g(x)\) since we are given \(g_0 = 1\), that we may write  
\[g(x) = \frac{h(x)}{h(F(x))}.\]  
On the other hand, we may also write  
\[g(x) = \exp \left( \ln \left( g(x) \right) \right) = \exp \left( \ln \left( \frac{h(x)}{h(F(x))} \right) \right)\]  
so that:  
\[g(x) = \exp \left( \phi(X) \right) \implies \phi(X) = \ln \left( \frac{h(x)}{h(F(x))} \right)\]  

However, within the proof of necessity of Theorem 1 above, we see that  
\[\frac{h(x)}{h(F(x))} = \left[ \frac{g(x)^{k-1}}{g(F(x)) \cdot g(F^2(x)) \cdot \ldots \cdot g(F^{k-1}(x))} \right]^{1/k}\]  
This leads to the following  

**Theorem (Affirmative answer to the Open question).**  
If \((g(x), F(x))\) is a Riordan element of order \(k\) with \(g_0 = 1\), then \(g(x)\) is of the form  
\[g(x) = \exp \left( \varphi(x, F(x), \ldots, F^{(k-1)}(x)) \right)\]  
for some some \(k\)-cyclic series \(\varphi(x_1, x_2, \ldots, x_k)\).

**Proof:**  
Let  
\[\varphi(x_1, \ldots, x_k) = \frac{1}{k} \cdot \ln \left[ \frac{g(x_1)^{k-1}}{g(x_2) \cdot \ldots \cdot g(x_k)} \right].\]  
Then we have:  
1. \(\varphi(x_1, \ldots, x_k)\) is \(k\)-cyclic symmetric, by a straightforward calculation using the basic properties of logarithms.  
2. \(g(x) = \exp \left( \varphi(g(x), g(F(x)) \ldots, g(F^{(k-1)}(x))) \right)\), by the second line of the calculation done in the proof of necessity of Theorem 1 above.

Therefore the Open question is answered in the affirmative. \(\Box\)
6  A New Proof of C. Marshall’s Theorem

C. Marshall’s theorem, in contrast to Theorem 1, deals with the question of finding \( F(x) \) for given \( g(x) \), which will make \((g(x), F(x))\) into an involution. This theorem shows that there is a unique such \( F(x) \) if \( g(x) \) is bi-invertible. We shall show how the circle of ideas above can be used to give an alternate (though not as elegant) proof of her Theorem.

**Definition 6.1.**

An element \( g(x) \in \mathbb{F}_1[[x]] \) is bi-invertible if \( g(x) = g_0 + g_1 x + g_2 x^2 + \cdots = g_0 + G(x) \) where \( g_0 \neq 0 \) and \( g_1 \neq 0 \).

The term “bi-invertible” refers to the fact that \( g(x) \) is invertible in \( \mathbb{F}_0[[x]] \) and \( G(x) \) is invertible in \( \mathbb{F}_1[[x]] \).

**Theorem.** (C. Marshall ([9], [10]))

If \( g(x) \) is bi-invertible and \((g(x), F(x))\) is an involution in \( \mathcal{R} \) then

\[
F(x) = \overline{G} \left( \frac{-g_0 \cdot G(x)}{g(x)} \right)
\]

**Note:** For an involution, necessarily \( g_0 = \pm 1 \). We assume that \( F(x) = -x + \cdots \) (i.e., \( \omega = -1 \)) since the alternative, \((g(x), F(x)) = (-1, x)\), is not bi-invertible.

We shall prove this Theorem using the idea in Corollary 2.6 of [4] of conjugating by \( \left( \hat{g}(x)^{1/2}, \ln(\hat{g}(x)) \right) \).

Thus we replace \( \frac{1}{2} (x - F(x)) \), which (Remark 1.6) always works, by \( \ln(\hat{g}(x)) \). As pointed out in Remark 5.3, this may be done precisely when \( g(x) \) is bi-invertible; i.e., when \( \ln(\hat{g}(x)) \in \mathbb{F}_1[[x]] \).

**Proof of the Theorem:**

1. \((g(x), F(x)) = \left( \hat{g}(x)^{1/2}, \ln(\hat{g}(x)) \right) \left( g_0, -x \right) \left( \hat{g}(x)^{1/2}, \ln(\hat{g}(x)) \right)^{-1} \).

**Proof:** \((g(x), F(x)) \left( \hat{g}(x)^{1/2}, \ln(\hat{g}(x)) \right) = \left( g(x) \cdot \left( \hat{g}(F(x))^{1/2} \right), \ln(\hat{g}(F(x))) \right) \)

\[
= \left( g_0 \cdot \hat{g}(x) \cdot \frac{1}{\hat{g}(x)^{1/2}}, \ln \left( \frac{1}{\hat{g}(x)} \right) \right)
= \left( g_0 \cdot \hat{g}(x)^{1/2}, -\ln(\hat{g}(x)) \right)
= \left( \hat{g}(x)^{1/2}, \ln(\hat{g}(x)) \right) \left( g_0, -x \right) \quad \square
\]
2. We define $L(x) \in \mathbb{F}_1[[x]]$ by $L(x) = \ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$.

Using the fact that $g_0 = \pm 1$, define

$$
\hat{G}(x) = \frac{1}{g_0} \cdot G(x) = g_0 \cdot G(x) = (\ell_{g_0} \circ G)(x),
$$

where $\ell_{g_0}(x) \overset{\text{def}}{=} g_0 \cdot x$.

Notice that $\hat{g}(x) = 1 + \hat{G}(x)$ and that $\ln(\hat{g}(x)) = (L \circ \hat{G})(x) = (L \circ \ell_{g_0} \circ G)(x)$.

From 1. we have:

$$
F(x) = \ln(\hat{g})((-\text{id})(\ln(\hat{g}))(x)
= \ln(\hat{g})(-\ln(\hat{g}(x)))
= \ln(\hat{g})(\ln\left(\frac{1}{\hat{g}(x)}\right))
= (L \circ \ell_{g_0} \circ G)\left(L\left(\frac{1}{\hat{g}(x)}\right) - 1\right)
= (G \circ \ell_{g_0} \circ L)\left(L\left(\frac{1}{\hat{g}(x)}\right) - 1\right)
= G\left(g_0 \cdot \frac{1 - \hat{g}(x)}{\hat{g}(x)}\right)
= G\left(g_0 \cdot \frac{-\hat{G}(x)}{\hat{g}(x)}\right)
= G\left(-g_0 \cdot \frac{G(x)}{g(x)}\right).
$$

Remark 6.2. The uniqueness of $F(x)$ given $g(x)$, indicated by C. Marshall’s Theorem, does not hold if the order of $(g(x), F(x))$ is greater than two, for the following reason:

If $(g(x), F(x))$ has order $n$ and $\gcd(j, \text{ord}(F)) = 1$. Then $(g(x), F^{(j)}(x))$ also has order $n$.

Reason: $1 = g(x) \cdot g(F(x)) \cdots g(F^{(n-1)}(x))$

$$
= g(x) \cdot g(F^{(j)}(x)) \cdot g(F^{(2j)}(x)) \cdots g(F^{(n-1)j}(x)),
$$

as these are exactly the same product, written in different orders. □
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