THE ALPERIN-MCKAY CONJECTURE FOR METACYCLIC, MINIMAL NON-ABELIAN DEFECT GROUPS

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Abstract. We prove the Alperin-McKay Conjecture for all $p$-blocks of finite groups with metacyclic, minimal non-abelian defect groups. These are precisely the metacyclic groups whose derived subgroup have order $p$. In the special case $p = 3$, we also verify Alperin’s Weight Conjecture for these defect groups. Moreover, in case $p = 5$ we do the same for the non-abelian defect groups $C_{25} \rtimes C_5^n$. The proofs do not rely on the classification of the finite simple groups.

1. Introduction

Let $B$ be a $p$-block of a finite group $G$ with respect to an algebraically closed field of characteristic $p$. Suppose that $B$ has a metacyclic defect group $D$. We are interested in the number $k(B)$ (respectively $k_i(B)$) of irreducible characters of $B$ (of height $i \geq 0$), and the number $l(B)$ of irreducible Brauer characters of $B$. If $p = 2$, these invariants are well understood and the major conjectures are known to be true by the work of several authors (see [4,9,11,31,35,37]). Thus we will focus on the case $p > 2$ in the present work. Here at least Brauer’s $k(B)$-Conjecture, Olsson’s Conjecture and Brauer’s Height Zero Conjecture are satisfied for $B$ (see [14,38,43]). By a result of Stancu [40], $B$ is a controlled block. Moreover, if $D$ is a non-split extension of two cyclic groups, it is known that $B$ is nilpotent (see [7]). Then a result by Puig [33] describes the source algebra of $B$ in full detail. Thus we may assume in the following that $D$ is a split extension of two cyclic groups. A famous theorem by Dade [6] handles the case where $D$ itself is cyclic by making use of Brauer trees. The general situation is much harder – even the case $D \cong C_3 \times C_3$ is still open (see [24,26,42]). Now consider the subcase where $D$ is non-abelian. Then a work by An [11] shows that $G$ is not a quasisimple group. On the other hand, the algebra structure of $B$ in the $p$-solvable case can be obtained from Külshammer [27]. If $B$ has maximal defect (i.e. $D \in \text{Syl}_p(G)$), the block invariants of $B$ were determined in [15]. If $B$ is the principal block, Horimoto and Watanabe [20] constructed a perfect isometry between $B$ and its Brauer correspondent in $N_G(D)$.

Let us suppose further that $D$ is a split extension of a cyclic group and a group of order $p$ (i.e. $D$ is the unique non-abelian group with a cyclic subgroup of index $p$). Here the difference $k(B) - l(B)$ is known from [16]. Moreover, under additional assumptions on $G$, Holloway, Koshitani and Kunugi [19] obtained the block invariants precisely. In the special case where $D$ has order $p^3$, incomplete information is...
given by Hendren [17]. Finally, one has full information in case \(|D| = 27\) by work of the present author [38, Theorem 4.5].

In the present work we consider the following class of non-abelian split metacyclic groups

\[(1.1) \quad D = \langle x, y \mid x^{p^m} = y^{p^n} = 1, \quad yxy^{-1} = x^{1+p^{m-1}} \rangle \cong C_{p^m} \rtimes C_{p^n} \]

where \(m \geq 2\) and \(n \geq 1\). By a result of Rédei (see [21, Aufgabe III.7.22]) these are precisely the metacyclic, minimal non-abelian groups. A result by Knoche (see [21, Aufgabe III.7.24]) implies further that these are exactly the metacyclic groups with derived subgroup of order \(p\). In particular the family includes the non-abelian group with a cyclic subgroup of index \(p\) mentioned above. The main theorem of the present paper states that \(k_0(B)\) is locally determined. In particular the Alperin-McKay Conjecture holds for \(B\). Recall that the Alperin-McKay Conjecture asserts that \(k_0(B) = k_0(b)\) where \(b\) is the Brauer correspondent of \(B\) in \(N_G(D)\). This improves some of the results mentioned above. We also prove that every irreducible character of \(B\) has height 0 or 1. This is in accordance with the situation in \(\text{Irr}(D)\).

In the second part of the paper we investigate the special case \(p = 3\). Here we are able to determine \(k(B), k_1(B)\) and \(l(B)\). This gives an example of Alperin’s Weight Conjecture and the Ordinary Weight Conjecture. Finally, we determine the block invariants for \(p = 5\) and \(D \cong C_{25} \rtimes C_{5^n}\) where \(n \geq 1\).

As a new ingredient (compared to [38]) we make use of the focal subgroup of \(B\).

2. The Alperin-McKay Conjecture

Let \(p\) be an odd prime, and let \(B\) be a \(p\)-block with split metacyclic, non-abelian defect group \(D\). Then \(D\) has a presentation of the form

\[D = \langle x, y \mid x^{p^m} = y^{p^n} = 1, \quad yxy^{-1} = x^{1+p^{m-1}} \rangle\]

where \(0 < l < m\) and \(m - l \leq n\). Elementary properties of \(D\) are stated in the following lemma.

**Lemma 2.1.**

(i) \(D' = \langle x^{p^l} \rangle \cong C_{p^{m-l}}\).

(ii) \(Z(D) = \langle x^{p^{m-l}} \rangle \times \langle y^{p^{m-l}} \rangle \cong C_{p^l} \rtimes C_{p^{n-m+l}}\).

**Proof.** Omitted. \(\square\)

We fix a Sylow subgroup \((D, b_D)\) of \(B\). Then the conjugation of subpairs \((Q, b_Q) \leq (D, b_D)\) forms a saturated fusion system \(\mathcal{F}\) on \(D\) (see [2 Proposition IV.3.14]). Here \(Q \leq D\) and \(b_Q\) is a uniquely determined block of \(C_G(Q)\). We also have subsections \((u, b_u)\) where \(u \in D\) and \(b_u := b_{(u)}\). By Proposition 5.4 in [10], \(\mathcal{F}\) is controlled. Moreover by Theorem 2.5 in [14] we may assume that the inertial group of \(B\) has the form \(N_G(D, b_D)/C_G(D) = \text{Aut}_\mathcal{F}(D) = \langle \text{Inn}(D), a \rangle\) where \(a \in \text{Aut}(D)\) such that \(a(x) \in \langle x \rangle\) and \(a(y) = y\). By a slight abuse of notation we often write \(\text{Out}_\mathcal{F}(D) = \langle a \rangle\). In particular the inertial index \(e(B) := |\text{Out}_\mathcal{F}(D)|\) is a divisor of \(p - 1\). Let

\[\text{foc}(B) := \langle f(a)a^{-1} : a \in Q \leq D, \ f \in \text{Aut}\mathcal{F}(Q) \rangle\]

be the focal subgroup of \(B\) (or of \(\mathcal{F}\)). Then it is easy to see that \(\text{foc}(B) \subseteq \langle x \rangle\). In case \(e(B) = 1\), \(B\) is nilpotent and \(\text{foc}(B) = D'\). Otherwise \(\text{foc}(B) = \langle x \rangle\).

For the convenience of the reader we collect some estimates on the block invariants of \(B\).
Proposition 2.2. Let $B$ be as above. Then
\[
\left(\frac{p^i + p^{i-1} - p^{2l-m-1} - 1}{e(B)} + e(B)\right) p^n \leq k(B) \leq \left(\frac{p^{i-1} - 1}{e(B)} + e(B)\right) (p^{n+m-i-2} + p^n - p^{n-2}),
\]
\[
2p^n \leq k_0(B) \leq \left(\frac{p^{i-1} - 1}{e(B)} + e(B)\right) p^n,
\]
\[
\sum_{i=0}^{\infty} p^{2i} k_i(B) \leq \left(\frac{p^{i-1} - 1}{e(B)} + e(B)\right) p^{n+m-l},
\]
\[
l(B) \geq e(B) | p - 1,
\]
\[
p^n | k_0(B), \quad p^{n-m+i} | k_i(B) \quad \text{for } i \geq 1,
\]
\[
k_i(B) = 0 \quad \text{for } i > 2(m-l).
\]

Proof. Most of the inequalities are contained in Proposition 2.1 to Corollary 2.5 in [38]. By Theorem 1 in [36] we have $p^n | [D : \text{foc}(B)] | k_0(B)$. In particular $p^n \leq k_0(B)$. In case $k_0(B) = p^n$ it follows from [23] that $B$ is nilpotent. However then we would have $k_0(B) = |D : D'| = p^{n+l} > p^n$. Therefore $2p^n \leq k_0(B)$. Theorem 2 in [36] implies $p^{n-m+i} | [Z(D) : Z(D) \cap \text{foc}(B)] | k_i(B)$ for $i \geq 1$. □

Now we consider the special case where $m = l + 1$. As mentioned in the introduction these are precisely the metacyclic, minimal non-abelian groups. We prove the main theorem of this section.

Theorem 2.3. Let $B$ be a $p$-block of a finite group with metacyclic, minimal non-abelian defect groups for an odd prime $p$. Then
\[
k_0(B) = \left(\frac{p^{m-1} - 1}{e(B)} + e(B)\right) p^n
\]
with the notation from (1.1). In particular the Alperin-McKay Conjecture holds for $B$.

Proof. By Proposition 2.2 we have
\[
p^n | k_0(B) \leq \left(\frac{p^{m-1} - 1}{e(B)} + e(B)\right) p^n.
\]
Thus, by way of contradiction we may assume that
\[
k_0(B) \leq \left(\frac{p^{m-1} - 1}{e(B)} + e(B) - 1\right) p^n.
\]
We also have
\[
k(B) \geq \left(\frac{p^{m-1} + p^{m-2} - p^{m-3} - 1}{e(B)} + e(B)\right) p^n
\]
from Proposition 2.2. Hence the sum $\sum_{i=0}^{\infty} p^{2i} k_i(B)$ will be small if $k_0(B)$ is large and $k_1(B) = k(B) - k_0(B)$. This implies the following contradiction:

$$\left(\frac{p^n - 1}{e(B)} + p^2 + e(B) - 1\right) p^{n} = \left(\frac{p^{n-1} - 1}{e(B)} + e(B) - 1\right) p^n$$

$$+ \left(\frac{p^{n-2} - p^{n-3}}{e(B)} + 1\right) p^{n+2}$$

$$\leq \sum_{i=0}^{\infty} p^{2i} k_i(B) \leq \left(\frac{p^n - p}{e(B)} + pe(B)\right) p^n$$

$$< \left(\frac{p^n - 1}{e(B)} + p^2\right) p^n.$$ 

Since the Brauer correspondent of $B$ in $N_G(D)$ has the same fusion system, the Alperin-McKay Conjecture follows. \hfill \Box

Isaacs and Navarro [22] Conjecture D] proposed a refinement of the Alperin-McKay Conjecture by invoking Galois automorphisms. We show (as an improvement of Theorem 4.3 in [38]) that this conjecture holds in the special case $|D| = p^3$ of Theorem 2.3. We will denote the subset of $\operatorname{Irr}(B)$ of characters of height 0 by $\operatorname{Irr}_0(B)$. 

**Corollary 2.4.** Let $B$ be a $p$-block of a finite group $G$ with non-abelian, metacyclic defect group of order $p^3$. Then Conjecture D in [22] holds for $B$. 

**Proof.** Let $D$ be a defect group of $B$. For $k \in \mathbb{N}$, let $\mathbb{Q}_k$ be the cyclotomic field of degree $k$. Let $|G|_{p'}$ be the $p'$-part of the order of $G$. It is well known that the Galois group $G := \text{Gal}(\mathbb{Q}_k|\mathbb{Q}_k)$ acts canonically on $\operatorname{Irr}(B)$. Let $\gamma \in G$ be a $p$-element. Then it suffices to show that $\gamma$ acts trivially on $\operatorname{Irr}_0(B)$. By Lemma IV.6.10 in [12] it is enough to prove that $\gamma$ acts trivially on the $\mathcal{F}$-conjugacy classes of subsections of $B$ via $\gamma(u, b_u) := (u^\overline{\gamma}, b_u)$ where $u \in D$ and $\overline{\gamma} \in \mathbb{Z}$. Since $\gamma$ is a $p$-element, this action is certainly trivial unless $|\langle u \rangle| = p^2$. Here however, the action of $\gamma$ on $\langle u \rangle$ is just the $D$-conjugation. The result follows. \hfill \Box

In the situation of Corollary 2.3 one can say a bit more: By Proposition 3.3 in [38], $\operatorname{Irr}(B)$ splits into the following families of $p$-conjugate characters:

- $(p-1)/e(B) + e(B)$ orbits of length $p-1$,
- two orbits of length $(p-1)/e(B)$,
- at least $e(B)$ $p$-rational characters.

Without loss of generality, let $e(B) > 1$. By Theorem 4.1 in [38] we have $k_1(B) \leq (p-1)/e(B) + e(B) - 1$. Moreover, Proposition 4.1 of the same paper implies $k_1(B) < p-1$. In particular, all orbits of length $p-1$ of $p$-conjugate characters must lie in $\operatorname{Irr}_0(B)$. In case $e(B) = p-1$ the remaining $(p-1)/e(B) + e(B)$ characters in $\operatorname{Irr}_0(B)$ must be $p$-rational. Now let $e(B) < \sqrt{p-1}$. Then it is easy to see that $\operatorname{Irr}_0(B)$ contains just one orbit of length $(p-1)/e(B)$ of $p$-conjugate characters. Unfortunately, it is not clear if this also holds for $e(B) \geq \sqrt{p-1}$.

Next we improve the bound coming from Proposition 2.2 on the heights of characters.
**Proposition 2.5.** Let $B$ be a $p$-block of a finite group with metacyclic, minimal non-abelian defect groups. Then $k_1(B) = k(B) - k_0(B)$. In particular, $B$ satisfies the following conjectures.

- Eaton’s Conjecture [8],
- Eaton-Moretó Conjecture [10],
- Robinson’s Conjecture [28], Conjecture 4.14.7,
- Malle-Navarro Conjecture [29].

**Proof.** By Theorem 2 in [37] we may assume $p > 2$ as before. By way of contradiction suppose that $k_i(B) > 0$ for some $i \geq 2$. Since

$$k(B) \geq \left( \frac{p^{m-1} + p^{m-2} - p^{m-3} - 1}{e(B)} + e(B) \right) p^n,$$

we have $k(B) - k_0(B) \geq (p^{m-1} - p^{m-2})p^{n-1}/e(B)$ by Theorem 2.3. By Proposition 2.2, $k_1(B)$ and $k_i(B)$ are divisible by $p^{n-1}$. This shows

$$\left( \frac{p^{n-1} - 1}{e(B)} + e(B) \right) p^n + \left( \frac{p^{m-1} - 1}{e(B)} + e(B) \right) p^{n+1}$$

$$+ p^{n+3} = \sum_{i=0}^{\infty} p^{2i} k_i(B) \leq \left( \frac{p^{m-1} - 1}{e(B)} + e(B) \right) p^{n+1}.$$

Hence, we derive the following contradiction:

$$p^{n+3} - p^{n+1} \leq \left( \frac{1 - p}{e(B)} + e(B)(p - 1) \right) p^n \leq p^{n+2}.$$

This shows $k_1(B) = k(B) - k_0(B)$. Now Eaton’s Conjecture is equivalent to Brauer’s $k(B)$-Conjecture and Olsson’s Conjecture. Both are known to hold for all metacyclic defect groups. Also, the Eaton-Moretó Conjecture and Robinson’s Conjecture are trivially satisfied for $B$. The Malle-Navarro Conjecture asserts that $k(B)/k_0(B) \leq k(D') = p$ and $k(B)/l(B) \leq k(D)$. By Theorem 2.3 and Proposition 2.2, the first inequality reduces to $p^{n-1} + p^n - p^{n-2} \leq p^{n+1}$ which is true. For the second inequality we observe that every conjugacy class of $D$ has at most $p$ elements, since $|D : Z(D)| = p^2$. Hence, $k(D) = |Z(D)| + \frac{|D| - |Z(D)|}{p} = p^{n+m-1} + p^{n+m-2} - p^{n+m-3}$. Now Proposition 2.2 gives

$$\frac{k(B)}{l(B)} \leq \frac{p^{m-1} - 1}{e(B)} + e(B) \left( p^{n-1} + p^n - p^{n-2} \right)$$

$$\leq p^{n+m-1} + p^{n+m-2} - p^{n+m-3} = k(D).$$

We use the opportunity to present a result for $p = 3$ and a different class of metacyclic defect groups (where $l = 1$ with the notation above).

**Theorem 2.6.** Let $B$ be a 3-block of a finite group $G$ with defect group $D = \langle x, y \mid x^m = y^n = 1, yxy^{-1} = x^4 \rangle$ where $2 \leq m \leq n + 1$. Then $k_0(B) = 3^{n+1}$. In particular, the Alperin-McKay Conjecture holds for $B$.

**Proof.** We may assume that $B$ is non-nilpotent. By Proposition 2.2 we have $k_0(B) \in \{2 \cdot 3^n, 3^{n+1}\}$. By way of contradiction, suppose that $k_0(B) = 2 \cdot 3^n$. Let $P \in \text{Syl}_p(G)$. Since $D/\text{Z}(B)$ acts freely on $\text{Irr}_0(B)$, there are 3 $n$ characters of degree
A generalization of the argument in the proof shows that in the situation of Proposition 2.2, \( k_0(B) = 2p^n \) can only occur if \( p \equiv 1 \pmod{4} \).

3. LOWER DEFECT GROUPS

In the following we use the theory of lower defect groups in order to estimate \( l(B) \). We cite a few results from the literature. Let \( B \) be a \( p \)-block of a finite group \( G \) with defect group \( D \) and Cartan matrix \( C \). We denote the multiplicity of an integer \( a \) as elementary divisor of \( C \) by \( m(a) \). Then \( m(a) = 0 \) unless \( a \) is a \( p \)-power. It is well known that \( m(|D|) = 1 \). Brauer [3] expressed \( m(p^n) \) \((n \geq 0)\) in terms of 1-multiplicities of lower defect groups (see also Corollary V.10.12 in [12]):

\[
(3.1) \quad m(p^n) = \sum_{R \in \mathcal{R}} m_B^{(1)}(R)
\]

where \( \mathcal{R} \) is a set of representatives for the \( G \)-conjugacy classes of subgroups \( R \leq D \) of order \( p^n \). Later (3.1) was refined by Broué and Olsson by invoking the fusion system \( \mathcal{F} \) of \( B \).

**Proposition 3.1** (Broué-Olsson [5]). For \( n \geq 0 \) we have

\[
m(p^n) = \sum_{R \in \mathcal{R}} m_B^{(1)}(R, b_R)
\]

where \( \mathcal{R} \) is a set of representatives for the \( \mathcal{F} \)-conjugacy classes of subgroups \( R \leq D \) of order \( p^n \).

**Proof.** This is (2S) of [5]. \( \square \)

In the present paper we do not need the precise (and complicated) definition of the non-negative numbers \( m_B^{(1)}(R) \) and \( m_B^{(1)}(R, b_R) \). We say that \( R \) is a lower defect group for \( B \) if \( m_B^{(1)}(R, b_R) > 0 \). In particular, \( m_B^{(1)}(D, b_D) = m_B^{(1)}(D) = m(|D|) = 1 \). A crucial property of lower defect groups is that their multiplicities can usually be determined locally. In the next lemma, \( b_R^{N_G(R,b_R)} \) denotes the (unique) Brauer correspondent of \( b_R \) in \( N_G(R, b_R) \).

**Lemma 3.2.** For \( R \leq D \) and \( b_R := b_R^{N_G(R,b_R)} \) we have \( m_B^{(1)}(R, b_R) = m_B^{(1)}(R) \). If \( R \) is fully \( \mathcal{F} \)-normalized, then \( B_R \) has defect group \( N_D(R) \) and fusion system \( N_{\mathcal{F}}(R) \).

**Proof.** The first claim follows from (2Q) in [5]. For the second claim we refer to Theorem IV.3.19 in [2]. \( \square \)

Another important reduction is given by the following lemma.

**Lemma 3.3.** For \( R \leq D \) we have \( \sum_{Q \in \mathcal{R}} m_{B_R}^{(1)}(Q) \leq l(b_R) \) where \( \mathcal{R} \) is a set of representatives for the \( N_G(R, b_R) \)-conjugacy classes of subgroups \( Q \) such that \( R \leq Q \leq N_D(R) \).
Proof. This is implied by Theorem 5.11 in [32] and the remark following it. Notice that in Theorem 5.11 it should read $B \in \text{Bl}(G)$ instead of $B \in \text{Bl}(Q)$. □

In the local situation for $B_R$ the next lemma is also useful.

Lemma 3.4. If $O_p(Z(G)) \not\subseteq R$, then $m_B^{(1)}(R) = 0$.

Proof. See Corollary 3.7 in [32]. □

Now we apply these results.

Lemma 3.5. Let $B$ be a $p$-block of a finite group with metacyclic, minimal non-abelian defect group $D$ for an odd prime $p$. Then every lower defect group of $B$ is $D$-conjugate either to $\langle y \rangle$, $\langle y^p \rangle$, or to $D$.

Proof. Let $R < D$ be a lower defect group of $B$. Then $m(|R|) > 0$ by Proposition 3.1 Corollary 5 in [36] shows that $p^{n-1} \mid |R|$. Since $F$ is controlled, the subgroup $R$ is fully $F$-centralized and fully $F$-normalized. The fusion system of $b_R$ (on $C_D(R)$) is given by $C_F(R)$ (see Theorem IV.3.19 in [2]). Suppose for the moment that $C_F(R)$ is trivial. Then $b_R$ is nilpotent and $l(b_R) = 1$. Let $B_R := b_R^{N_G(R,b_R)}$. Then $B_R$ has defect group $N_D(R)$ and $m^{(1)}_{b_R}(N_D(R)) = 1$. Hence, Lemmas 3.2 and 3.3 imply $m^{(1)}_B(R,b_R) = m^{(1)}_{b_R}(R) = 0$. This contradiction shows that $C_F(R)$ is non-trivial. In particular $R$ is centralized by a non-trivial $p'$-automorphism $\beta \in \text{Aut}_F(D)$. By the Schur-Zassenhaus Theorem, $\beta$ is $\text{Inn}(D)$-conjugate to a power of $\alpha$. Thus, $R$ is $D$-conjugate to a subgroup of $\langle y \rangle$. The result follows. □

Proposition 3.6. Let $B$ be a $p$-block of a finite group with metacyclic, minimal non-abelian defect groups for an odd prime $p$. Then $e(B) \leq l(B) \leq 2e(B) - 1$.

Proof. Let

$$D = \langle x, y \mid x^{p^n} = y^{p^n} = 1, yxy^{-1} = x^{1+p^{m-1}} \rangle$$

be a defect group of $B$. We argue by induction on $n$. Let $n = 1$. By Proposition 2.2 we have $e(B) \leq l(B)$ and

$$k(B) \leq \left( \frac{p^{m-1} - 1}{e(B)} + e(B) \right) (1 + p - p^{-1}).$$

Moreover, Theorem 3.2 in [38] gives

$$k(B) - l(B) = \frac{p^m + p^{m-1} - p^{m-2} - p}{e(B)} = e(B)(p - 1).$$

Hence,

$$l(B) = k(B) - (k(B) - l(B)) \leq \left( \frac{p^{m-1} - 1}{e(B)} + e(B) \right) (1 + p - p^{-1})$$

$$- \frac{p^m + p^{m-1} - p^{m-2} - p}{e(B)} = e(B)(p - 1)$$

$$= 2e(B) - \frac{1}{p} \left( e(B) - \frac{1}{e(B)} \right) - \frac{1}{e(B)},$$

and the claim follows in this case.

Now suppose $n \geq 2$. We determine the multiplicities of the lower defect groups by using Lemma 3.5. As usual $m(|D|) = 1$. Consider the subpair $(\langle y \rangle, b_y)$. By
Lemmas 3.1 and 3.2 we have $m(p^n) = m_B^{(1)}((y), b_y) = m_B^{(1)}((y))$ where $B_y := b_y N_G(y, b_y)$. Since $N_D(y) = C_D(y)$, it follows easily that $N_G((y), b_y) = C_G(y)$ and $B_y = b_y$. By Theorem IV.3.19 in [2] the block $b_y$ has defect group $C_D(y)$ and fusion system $C_F((y))$. In particular $e(b_y) = e(B)$. It is well known that $b_y$ dominates a block $\overline{b_y}$ of $C_D(y)/y$ with cyclic defect group $C_D(y)/\langle y \rangle$ and $e(\overline{b_y}) = e(b_y) = e(B)$ (see [30, Theorem 5.8.11]). By Dade’s Theorem [6] on blocks with cyclic defect groups we obtain $l(b_y) = e(B)$. Moreover, the Cartan matrix of $\overline{b_y}$ has elementary divisors $1$ and $|C_D(y)/\langle y \rangle|$ where $1$ occurs with multiplicity $e(B) - 1$ (this follows for example from [13]). Therefore, the Cartan matrix of $b_y$ has elementary divisors $p^n$ and $|C_D(y)|$ where $p^n$ occurs with multiplicity $e(B) - 1$. Since $\langle y \rangle \subseteq Z(C_G(y))$, Lemma 3.3 implies $m(p^n) = m_{b_y}^{(1)}((y)) = e(B) - 1$.

Now consider $(\langle u \rangle, b_u)$ where $u := y^p \in Z(D)$. Here $b_u$ has defect group $D$. By the first part of the proof (the case $n = 1$) we obtain $l(b_u) = l(b_u) \leq 2e(B) - 1$. As above we have $m(p^{n-1}) = m_B^{(1)}((u), b_u) = m_{b_u}^{(1)}((u))$. Since $p^n$ occurs as elementary divisor of the Cartan matrix of $b_u$ with multiplicity $e(B) - 1$ (see above), it follows that $m(p^{n-1}) = m_{b_u}^{(1)}((u)) \leq e(B) - 1$. Now $l(B)$ is the sum over the multiplicities of elementary divisors of the Cartan matrix of $B$ which is at most $m(|D|) + m(\langle y \rangle) + m((u)) \leq 1 + e(B) - 1 + e(B) - 1 = 2e(B) - 1$.

The next proposition gives a reduction method.

**Proposition 3.7.** Let $p > 2$, $m \geq 2$ and $e \mid p - 1$ be fixed. Suppose that $l(B) = e$ holds for every block $B$ with defect group

$$D = \langle x, y \mid x^{p^n} = y^p = 1, yxy^{-1} = x^{1+p^{n-1}} \rangle$$

and $e(B) = e$. Then every block $B$ with $e(B) = e$ and defect group

$$D = \langle x, y \mid x^{p^n} = y^n = 1, yxy^{-1} = x^{1+p^{n-1}} \rangle$$

where $n \geq 1$ satisfies the following:

$$k_0(B) = \frac{p^{n-1} - 1}{e(B)} + e(B) p^n, \quad k_1(B) = \frac{p^{m-1} - p^{m-2}}{e(B)} p^{n-1},$$

$$k(B) = \frac{p^m + p^{m-1} - p^{m-2} - p + e(B)p}{e(B)} p^{n-1}, \quad l(B) = e(B).$$

**Proof.** We use induction on $n$. In case $n = 1$ the result follows from Theorem 3.2 in [38], Theorem 2.3 and Proposition 2.5.

Now let $n \geq 2$. Let $R$ be a set of representatives for the $\mathcal{F}$-conjugacy classes of elements of $D$. We are going to use Theorem 5.9.4 in [30]. For $1 \neq u \in R$, $b_u$ has metacyclic defect group $C_D(u)$ and fusion system $C_F((u))$. If $C_F((u))$ is non-trivial, $\alpha \in \text{Aut}_F(D)$ centralizes a $D$-conjugate of $u$. Hence, we may assume that $u \in \langle y \rangle$ in this case. If $\langle u \rangle = \langle y \rangle$, then $b_u$ dominates a block $\overline{b_u}$ of $C_G(u)/\langle u \rangle$ with cyclic defect group $C_D(u)/\langle u \rangle$. Hence, $l(b_u) = l(\overline{b_u}) = e(B)$. Now suppose that $\langle u \rangle < \langle y \rangle$. Then by induction we obtain $l(b_u) = l(\overline{b_u}) = e(B)$. Finally assume that $C_F((u))$ is trivial. Then $b_u$ is nilpotent and $l(b_u) = 1$. It remains to determine $R$. The powers of $y$ are pairwise non-conjugate in $\mathcal{F}$. As in the proof of Proposition 2.5 $D$ has precisely $p^{n+3} - (p^2 + p - 1)$ conjugacy classes. Let $C$ be one of these classes which do not intersect $\langle y \rangle$. Assume $\alpha(C) = C$ for some $i \in \mathbb{Z}$ such that $\alpha^i \neq 1$.

Then there are elements $u \in C$ and $w \in D$ such that $\alpha^i(u) = wuw^{-1}$. Hence
\[
\gamma := w^{-1} \alpha^i \in N_G(D, b_D) \cap C_G(u). \text{ Since } \gamma \text{ is not a } p\text{-element, we conclude that } u \text{ is conjugate to a power of } y \text{ which was excluded. This shows that no non-trivial power of } \alpha \text{ can fix } C \text{ as a set. Thus, all these conjugacy classes split in }
\]
\[
\frac{p^2 + p - p^{2m-1} - 1}{e(B)} p^{n+m-3}
\]
orbits of length \(e(B)\) under the action of \(\text{Out}_F(D)\). Now Theorem 5.9.4 in [30] implies
\[
k(B) - l(B) = \left(\frac{p^{m-1} + p^{m-2} - p^{m-3} - 1}{e(B)} + e(B)\right) p^n - e(B).
\]
By Proposition 3.6 it follows that
\[
(3.2) \quad k(B) \leq \left(\frac{p^m + p^{m-1} - p^{m-2} - p}{e(B)} + e(B)p\right) p^{n-1} + e(B) - 1.
\]
By Proposition 2.2 the left-hand side of (3.2) is divisible by \(p^{n-1}\). Since \(e(B) - 1 < p^{n-1}\), we obtain the exact value of \(k(B)\). It follows that \(l(B) = e(B)\). Finally, Theorem 2.3 and Proposition 2.5 give \(k_i(B)\). \(\Box\)

For \(p = 3\), Proposition 3.6 implies \(l(B) \leq 3\). Here we are able to determine all block invariants.

**Theorem 3.8.** Let \(B\) be a non-nilpotent 3-block of a finite group with metacyclic, minimal non-abelian defect groups. Then
\[
k_0(B) = \frac{3^{m-2} + 1}{2} 3^{n+1}, \quad k_1(B) = 3^{m+n-3}, \quad k(B) = \frac{11 \cdot 3^{m-2} + 9}{2} 3^{n-1}, \quad l(B) = e(B) = 2
\]
with the notation from (1.1).

**Proof.** By Proposition 3.7 it suffices to settle the case \(n = 1\). Here the claim holds for \(m \leq 3\) by Theorem 3.7 in [38]. We will extend the proof of this result in order to handle the remaining \(m \geq 4\). Since \(B\) is non-nilpotent, we have \(e(B) = 2\). By Theorem 2.3 we know \(k_0(B) = (3^m+9)/2\). By way of contradiction, we may assume that \(l(B) = 3\) and \(k_1(B) = 3^{m-2} + 1\) (see Theorem 3.4 in [38]).

We consider the generalized decomposition numbers \(d^z_{\chi, \varphi_z}\), where \(z := x^3 \in Z(D)\) and \(\varphi_z\) is the unique irreducible Brauer character of \(b_z\). Let \(d^z := (d^z_{\chi, \varphi_z}) \chi \in \text{Irr}(B)\). By the orthogonality relations we have \((d^z, d^z) = 3^{m+1}\). As in [18] Section 4] we can write
\[
d^z = \sum_{i=0}^{2 \cdot 3^{m-2} - 1} a_i \zeta_{3^m}^i
\]
for integral vectors \(a_i\) and a primitive \(3^{m-1}\)-th root of unity \(\zeta_{3^m} \in \mathbb{C}\). Since \(z\) is \(F\)-conjugate to \(z^{-1}\), the vector \(d^z\) is real. Hence, the vectors \(a_i\) are linearly dependent. More precisely, it turns out that the vectors \(a_i\) are spanned by \(\{a_j : j \in J\}\) for a subset \(J \subseteq \{0, \ldots, 2 \cdot 3^{m-2} - 1\}\) such that \(0 \in J\) and \(|J| = 3^{m-2}\).
Let \( q \) be the quadratic form corresponding to the Dynkin diagram of type \( A_{3m-2} \). We set \( a(\chi) := (a_j(\chi) : j \in J) \) for \( \chi \in \text{Irr}(B) \). Since the subsection \((z,b_x)\) gives equality in Theorem 4.10 in [18], we have

\[
k_0(B) + 9k_1(B) = \sum_{\chi \in \text{Irr}(B)} q(a(\chi))
\]

for a suitable ordering of \( J \). This implies \( q(a(\chi)) = 3^{2h(\chi)} \) for \( \chi \in \text{Irr}(B) \) where \( h(\chi) \) is the height of \( \chi \). Moreover, if \( a_0(\chi) \neq 0 \), then \( a_0(\chi) = \pm 3^{h(\chi)} \) by Lemma 3.6 in [38]. By Lemma 4.7 in [18] we have \((d^x,k) = 0\) for characters \( \chi \in \text{Irr}(B) \) of height 1.

In the next step we determine the number \( \beta \) of 3-rational characters of height 1. Since \((a_0,a_0) = 27\), we have \( \beta < 4 \). On the other hand, the Galois group \( \mathcal{G} \) of \( \mathbb{Q}((\zeta_{3^m-1}) \cap \mathbb{Q} \) acts on \( d^x \) and the length of every non-trivial orbit is divisible by 3 (because \( \mathcal{G} \) is a 3-group). This implies \( \beta = 1 \), since \( k_1(B) = 3^{m-2} + 1 \).

In order to derive a contradiction, we repeat the argument with the subsection \((x,b_y)\). Again we get equality in Theorem 4.10 in [18], but this time for \( k_0(B) \) instead of \( k_0(B) + 9k_1(B) \). Hence, \( d^x(\chi) = 0 \) for characters \( \chi \in \text{Irr}(B) \) of height 1.

Again we can write \( d^x = \sum_{i=0}^{2 \cdot 3^{m-1} - 1} \overline{\alpha}_i \zeta_{3^m}^i \) where \( \overline{\alpha}_i \) are integral vectors. Lemma 4.7 in [18] implies \( (\overline{\alpha}_0,\overline{\alpha}_0) = 9 \). Using Lemma 3.6 in [38] we also have \( \overline{\alpha}_0(\chi) \in \{0,1\} \) by Proposition 3.3 in [38]. We have precisely three 3-rational characters \( \chi_1, \chi_2, \chi_3 \in \text{Irr}(B) \) of height 0 (note that altogether we have four 3-rational characters). Then \( a_0(\chi_i) = \pm \overline{\alpha}_0(\chi_i) = \pm 1 \) for \( i = 1,2,3 \). By [38] Section 1] we have \( \lambda \ast \chi_i \in \text{Irr}(B) \) and \( (\lambda \ast \chi_i)(u) = \chi_i(u) \) for \( \lambda \in \text{Irr}(D/\text{oc}(B)) \cong C_3 \) and \( u \in \{x,z\} \). Since this \( \text{Irr}(B) \) is free, we have nine characters \( \psi \in \text{Irr}(B) \) such that \( a_0(\psi) = \pm \overline{\alpha}_0(\psi) = \pm 1 \). In particular \( (a_0,\overline{\alpha}_0) \equiv 1 \) (mod 2). By the orthogonality relations we have \( (d^x,d^x) = 0 \) for all \( j \in \mathbb{Z} \) such that \( 3 \nmid j \). Using Galois theory we get the final contradiction \( 0 = (d^x,\overline{\alpha}_0) = (a_0,\overline{\alpha}_0) \equiv 1 \) (mod 2).

In the smallest case \( D \cong C_9 \times C_3 \) of Theorem 3.8 even more information on \( B \) was given in Theorem 4.5 in [38].

**Corollary 3.9.** Alperin’s Weight Conjecture and the Ordinary Weight Conjecture are satisfied for every 3-block with metacyclic, minimal non-abelian defect groups.

**Proof.** Let \( D \) be a defect group of \( B \). Since \( B \) is controlled, Alperin’s Weight Conjecture asserts that \( l(B) = l(B_D) \) where \( B_D \) is a Brauer correspondent of \( B \) in \( N_G(D) \). Since both numbers equal \( e(B) \), the conjecture holds.

Now we prove the Ordinary Weight Conjecture in the form of [2] Conjecture IV.5.49. Since \( \text{Out}_F(D) \) is cyclic, all 2-cocycles appearing in this version are trivial. Therefore the conjecture asserts that \( k_1(B) \) only depends on \( F \) and thus on \( e(B) \). Since the conjecture is known to hold for the principal block of the solvable group \( G = D \rtimes C_{e(B)} \), the claim follows.

We remark that Alperin’s Weight Conjecture is also true for the abelian defect groups \( D \cong C_{3^n} \times C_{3^m} \) where \( n \neq m \) (see [34],[41]).

We observe another consequence for arbitrary defect groups.

**Corollary 3.10.** Let \( B \) be a 3-block of a finite group with defect group \( D \). Suppose that \( D/(z) \) is metacyclic, minimal non-abelian for some \( z \in \mathbb{Z}(D) \). Then Brauer’s \( k(B)\)-Conjecture holds for \( B \), i.e. \( k(B) \leq |D| \).
Proof. Let \((z, b_z)\) be a major subsection of \(B\). Then \(b_z\) dominates a block \(b_z\) of \(C_G(z)/\langle z \rangle\) with metacyclic, minimal non-abelian defect group \(D/\langle z \rangle\). Hence, Theorem 3.8 implies \(l(b_z) = l(b_z) \leq 2\). Now the claim follows from Theorem 2.1 in [39]. □

In the situation of Theorem 3.8 it is straight-forward to distribute \(\text{Irr}(B)\) into families of 3-conjugate and 3-rational characters (cf. Proposition 3.3 in [38]). However, it is not so easy to see which of these families lie in \(\text{Irr}_0(B)\).

Now we turn to \(p = 5\).

**Theorem 3.11.** Let \(B\) be a 5-block of a finite group with non-abelian defect group \(C_{25} \rtimes C_5^n\) where \(n \geq 1\). Then

\[
\begin{align*}
k_0(B) &= \left( \frac{4}{e(B)} + e(B) \right) 5^n, & k_1(B) &= \frac{4}{e(B)} 5^{n-1}, \\
k(B) &= \left( \frac{24}{e(B)} + 5e(B) \right) 5^{n-1}, & l(B) &= e(B).
\end{align*}
\]

**Proof.** By Proposition 3.7 it suffices to settle the case \(n = 1\). Moreover by Theorem 4.4 in [38] we may assume that \(e(B) = 4\). Then by Theorem 2.3 above and Proposition 4.2 in [38] we have \(k_0(B) = 25, 1 \leq k_1(B) \leq 3, 26 \leq k(B) \leq 28\) and \(4 \leq l(B) \leq 6\). We consider the generalized decomposition numbers \(d_{\chi \varphi_z}^z\) where \(z := x^5 \in \mathbb{Z}(D)\) and \(\varphi_z\) is the unique irreducible Brauer character of \(b_z\). Since all non-trivial powers of \(z\) are \(F\)-conjugate, the numbers \(d_{\chi \varphi_z}^z\) are integral. Also, these numbers are non-zero, because \((z, b_z)\) is a major subsection. Moreover, \(d_{\chi \varphi_z}^z \equiv 0 \pmod{p}\) for characters \(\chi \in \text{Irr}(B)\) of height 1 (see Theorem V.9.4 in [12]). Let \(d_z := (d_{\chi \varphi_z}^z : \chi \in \text{Irr}(B))\). By the orthogonality relations we have \((d_z, d_z) = 125\).

Assume by way of contradiction that \(k_1(B) > 1\). Then it is easy to see that \(d_{\chi \varphi_z}^z = \pm 5\) for characters \(\chi \in \text{Irr}(B)\) of height 1. By [38, Section 1], the numbers \(d_{\chi \varphi_z}^z (\chi \in \text{Irr}_0(B))\) split in five orbits of length 5 each. Let \(\alpha\) (respectively \(\beta, \gamma\)) be the number of orbits of entries \(\pm 1\) (respectively \(\pm 2, \pm 3\)) in \(d_z^z\). Then the orthogonality relations read

\[
\alpha + 4\beta + 9\gamma + 5k_1(B) = 25.
\]

Since \(\alpha + \beta + \gamma = 5\), we obtain

\[
3\beta + 8\gamma = 20 - 5k_1(B) \in \{5, 10\}.
\]

However, this equation cannot hold for any choice of \(\alpha, \beta, \gamma\). Therefore we have proved that \(k_1(B) = 1\). Now Theorem 4.1 in [38] implies \(l(B) = 4\). □

**Corollary 3.12.** Alperin’s Weight Conjecture and the Ordinary Weight Conjecture are satisfied for every 5-block with non-abelian defect group \(C_{25} \rtimes C_5^n\).

**Proof.** See Corollary 3.9. □

Unfortunately, the proof of Theorem 3.11 does not work for \(p = 7\) and \(e(B) = 6\) (even by invoking the other generalized decomposition numbers). However, we have the following partial result.
Proposition 3.13. Let \( p \in \{7, 11, 13, 17, 23, 29\} \) and let \( B \) be a \( p \)-block of a finite group with defect group \( \mathbb{C}_p^2 \times \mathbb{C}_p^n \) where \( n \geq 1 \). If \( e(B) = 2 \), then
\[
\begin{align*}
  k_0(B) &= \frac{p^2 + 4p - 1}{2} p^{n-1}, \\
  k_1(B) &= \frac{p^2 + 3}{2} p^n, \\
  k(B) &= \frac{p^2 + 4p - 1}{2} p^{n-1}, \\
  l(B) &= 2.
\end{align*}
\]

Proof. We follow the proof of Theorem 4.4 in [38] in order to handle the case \( n = 1 \).

In fact the first part of the proof of Theorem 4.4 in [38] applies to any prime \( p \geq 7 \). Hence, we know that the generalized decomposition numbers \( d_z^\chi \phi \) for \( z := x^p \) and \( \chi \in \text{Irr}_0(B) \) are integral. Moreover,
\[
\sum_{\chi \in \text{Irr}_0(B)} a_0(\chi)^2 = p^2.
\]

The action of \( D/foc(B) \) on \( \text{Irr}_0(B) \) shows that the values \( a(\chi) \) distribute in \( (p+3)/2 \) parts of \( p \) equal numbers each. Therefore, (4.1) in [38] becomes
\[
\sum_{i=2}^{\infty} r_i (i^2 - 1) = \frac{p - 3}{2}
\]
for some \( r_i \geq 0 \). This gives a contradiction. \qed

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References

[1] Jianbei An, Controlled blocks of the finite quasisimple groups for odd primes, Adv. Math. 227 (2011), no. 3, 1165–1194, DOI 10.1016/j.aim.2011.03.003. MR2799604 (2012f:20023)

[2] Michael Aschbacher, Radha Kessar, and Bob Oliver, Fusion systems in algebra and topology, London Mathematical Society Lecture Note Series, vol. 391, Cambridge University Press, Cambridge, 2011. MR2848834 (2012m:20015)

[3] Richard Brauer, Defect groups in the theory of representations of finite groups, Illinois J. Math. 13 (1969), 53–73. MR0246979 (40 #248)

[4] Richard Brauer, On 2-blocks with dihedral defect groups, Symposia Mathematica, Vol. XIII (Convegno di Gruppi e loro Rappresentazioni, INDAM, Rome, 1972), Academic Press, London, 1974, pp. 367–393. MR0354838 (50 #7315)

[5] Michel Broué and Jørn B. Olsson, Subpair multiplicities in finite groups, J. Reine Angew. Math. 371 (1986), 125–143. MR0859323 (87j:20026)

[6] E. C. Dade, Blocks with cyclic defect groups, Ann. of Math. (2) 84 (1966), 20–48. MR0200355 (34 #251)

[7] Jill Dietz, Stable splittings of classifying spaces of metacyclic \( p \)-groups, \( p \) odd, J. Pure Appl. Algebra 90 (1993), no. 2, 115–136, DOI 10.1016/0022-4049(93)90125-D. MR1250764 (95f:55014)

[8] Charles W. Eaton, Generalisations of conjectures of Brauer and Olsson, Arch. Math. (Basel) 81 (2003), no. 6, 621–626, DOI 10.1007/s00013-003-0832-y. MR2029237 (2004i:20012)

[9] Charles W. Eaton, Radha Kessar, Burkhard Külshammer, and Benjamin Sambale, 2-blocks with abelian defect groups, Adv. Math. 254 (2014), 706–735, DOI 10.1016/j.aim.2013.12.024. MR3161112

[10] Charles W. Eaton and Alexander Moretó, Extending Brauer’s height zero conjecture to blocks with nonabelian defect groups, Int. Math. Res. Not. IMRN 20 (2014), 5581–5601. MR3271182

[11] Karin Erdmann, Blocks of tame representation type and related algebras, Lecture Notes in Mathematics, vol. 1428, Springer-Verlag, Berlin, 1990. MR1084107 (91c:20016)
[12] Walter Feit, *The representation theory of finite groups*, North-Holland Mathematical Library, vol. 25, North-Holland Publishing Co., Amsterdam-New York, 1982. MR661045 [83g:20001]

[13] Mitsuo Fujii, *On determinants of Cartan matrices of p-blocks*, Proc. Japan Acad. Ser. A Math. Sci. 56 (1980), no. 8, 401–403. MR596014 [81m:20001]

[14] Sheng Gao, *On Brauer’s k(B)-problem for blocks with metacyclic defect groups of odd order*, Arch. Math. (Basel) 96 (2011), no. 6, 507–512, DOI 10.1007/s00013-011-0265-y. MR2821468 (2012h:20020)

[15] Sheng Gao, *Blocks of full defect with nonabelian metacyclic defect groups*, Arch. Math. (Basel) 98 (2012), no. 1, 1–12, DOI 10.1007/s00013-011-0337-z. MR2885527

[16] Sheng Gao and Ji-wen Zeng, *On the number of ordinary irreducible characters in a p-block with a minimal nonabelian defect group*, Comm. Algebra 39 (2011), no. 9, 3278–3297, DOI 10.1080/00927872.2010.501775. MR2845573 (2012h:20021)

[17] Stuart Hendren, *Extra special defect groups of order p^3 and exponent p^2*, J. Algebra 291 (2005), no. 2, 457–491, DOI 10.1016/j.jalgebra.2005.05.002. MR2163548 (2006e:20015)

[18] László Héthelyi, Burkhard Külshammer, and Benjamin Sambale, *A note on Olsson’s conjecture*, J. Algebra 398 (2014), 364–385, DOI 10.1016/j.jalgebra.2012.08.010. MR3123771

[19] Miles Holloway, Shigeo Koshitani, and Naoko Kunugi, *Blocks with nonabelian defect groups which have cyclic subgroups of index p*, Arch. Math. (Basel) 94 (2010), no. 2, 101–116, DOI 10.1007/s00013-009-0075-7. MR2592757 (2011c:20013)

[20] H. Horimoto and A. Watanabe, *On a perfect isometry between principal p-blocks of finite groups with cyclic p-hyperfocal subgroups*, preprint.

[21] B. Huppert, *Endliche Gruppen. I* (German), Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin-New York, 1967. MR0224703 (37 #302)

[22] I. M. Isaacs and Gabriel Navarro, *New refinements of the McKay conjecture for arbitrary finite groups*, Ann. of Math. (2) 156 (2002), no. 1, 333–344, DOI 10.2307/3597192. MR1935849 (2003h:20018)

[23] R. Kessar, M. Linckelmann and G. Navarro, *A characterisation of nilpotent blocks*, arXiv:1402.5871v1.

[24] Masao Kiyota, *On 3-blocks with an elementary abelian defect group of order 9*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 31 (1984), no. 1, 33–58. MR743518 (85k:20036)

[25] Shigeo Koshitani and Naoko Kunugi, *Broué’s conjecture holds for principal 3-blocks with elementary abelian defect group of order 9*, J. Algebra 248 (2002), no. 2, 575–604, DOI 10.1006/jabr.2001.9048. MR1882112 (2002i:20021)

[26] Shigeo Koshitani and Hyoue Miyachi, *Donovan conjecture and Loewy length for principal 3-blocks of finite groups with elementary abelian Sylow 3-subgroup of order 9*, Comm. Algebra 29 (2001), no. 10, 4509–4522, DOI 10.1081/AGB-100106771. MR1854148 (2002f:20009)

[27] Burkhard Külshammer, *On p-blocks of p-solvable groups*, Comm. Algebra 9 (1981), no. 17, 1763–1785, DOI 10.1080/00927878108822682. MR631888 (83c:20017)

[28] Klaus Lux and Herbert Pahlings, *Representations of groups*, Cambridge Studies in Advanced Mathematics, vol. 124, Cambridge University Press, Cambridge, 2010. A computational approach. MR2680716 (2011j:20016)

[29] Gunter Malle and Gabriel Navarro, *Inequalities for some blocks of finite groups*, Arch. Math. (Basel) 87 (2006), no. 5, 390–399, DOI 10.1007/s00013-006-1769-8. MR2269211 (2007f:20073)

[30] Hiroshi Nagao and Yukio Tsushima, *Representations of finite groups*, Academic Press, Inc., Boston, MA, 1989. Translated from the Japanese. MR998775 (90h:20008)

[31] Jørn Børling Olsson, *On 2-blocks with quaternion and quasidihedral defect groups*, J. Algebra 36 (1975), no. 2, 212–241. MR0376841 (51 #13016)

[32] Jørn B. Olsson, *Lower defect groups*, Comm. Algebra 8 (1980), no. 3, 261–288, DOI 10.1080/00927878008822458. MR558114 (81g:20024)

[33] Luis Puig, *Nilpotent blocks and their source algebras*, Invent. Math. 93 (1988), no. 1, 77–116, DOI 10.1007/BF01393688. MR943924 (89e:20023)

[34] Luis Puig and Yoko Usami, *Perfect isometries for blocks with abelian defect groups and Klein four inertial quotients*, J. Algebra 160 (1993), no. 1, 192–225, DOI 10.1006/jabr.1993.1184. MR1237084 (94g:20017)

[35] Geoffrey R. Robinson, *Large character heights, Qd(p), and the ordinary weight conjecture*, J. Algebra 319 (2008), no. 2, 657–679, DOI 10.1016/j.jalgebra.2006.05.038. MR2381801 (2008j:20020)
Geoffrey R. Robinson, *On the focal defect group of a block, characters of height zero, and lower defect group multiplicities*, J. Algebra 320 (2008), no. 6, 2624–2628, DOI 10.1016/j.jalgebra.2008.04.032. MR2441776 (2009f:20010)

Benjamin Sambale, *Fusion systems on metacyclic 2-groups*, Osaka J. Math. 49 (2012), no. 2, 325–329. MR2945751

Benjamin Sambale, *Brauer’s height zero conjecture for metacyclic defect groups*, Pacific J. Math. 262 (2013), no. 2, 481–507, DOI 10.2140/pjm.2013.262.481. MR3069071

Benjamin Sambale, *Further evidence for conjectures in block theory*, Algebra Number Theory 7 (2013), no. 9, 2241–2273, DOI 10.2140/ant.2013.7.2241. MR3152013

Radu Stancu, *Control of fusion in fusion systems*, J. Algebra Appl. 5 (2006), no. 6, 817–837, DOI 10.1142/S0219498806002034. MR2286725 (2007j:20025)

Yoko Usami, *On p-blocks with abelian defect groups and inertial index 2 or 3. I*, J. Algebra 119 (1988), no. 1, 123–146, DOI 10.1016/0021-8693(88)90079-8. MR0971349 (89i:20024)

A. Watanabe, *Appendix on blocks with elementary abelian defect group of order 9*, in: Representation Theory of Finite Groups and Algebras, and Related Topics (Kyoto, 2008), 9–17, Kyoto University Research Institute for Mathematical Sciences, Kyoto, 2010.

Sheng Yang, *On Olsson’s conjecture for blocks with metacyclic defect groups of odd order*, Arch. Math. (Basel) 96 (2011), no. 5, 401–408, DOI 10.1007/s00013-011-0251-4. MR2805343 (2012d:20020)

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