HYPERBOLICITY IN PRESENCE OF A LARGE LOCAL SYSTEM

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Abstract. We prove that the projective complex algebraic varieties admitting a large complex local system satisfy a strong version of the Green-Griffiths-Lang conjecture.

1. Introduction

Let $X$ be a (non-necessarily smooth nor irreducible, but reduced) proper complex algebraic variety. Following Lang, we define three ‘special’ subsets of $X$ that measure three different kind of hyperbolic behaviour:

- The subset $\text{Sp}(X) \subset X$ is defined as the union of the (positive-dimensional) integral closed subvarieties of $X$ that are not of general type.
- The subset $\text{Sp}_{ab}(X) \subset X$ is defined as the union of all images of non-constant rational maps $A \to X$ from an abelian variety $A$.
- The subset $\text{Sp}_{h}(X) \subset X$ is defined as the union of all entire curves in $X$, i.e. the images of non-constant holomorphic maps $\mathbb{C} \to X$.

It is not clear from their definition whether these special subsets are Zariski-closed in $X$. The inclusions $\text{Sp}_{ab}(X) \subset \text{Sp}(X)$ and $\text{Sp}_{ab}(X) \subset \text{Sp}_{h}(X)$ always hold, see Proposition 2.1.

The following conjecture is a strong version of conjectures of Green-Griffiths [GG80] and Lang [Lan86].

Conjecture A. Let $X$ be a projective complex algebraic variety. Then

1. $\text{Sp}(X) = \text{Sp}_{ab}(X) = \text{Sp}_{h}(X)$,
2. $\text{Sp}(X)$ is a closed algebraic subvariety of $X$,
3. $\text{Sp}(X) \neq X$ if and only if $X$ is of general type.

The main result in this paper is the following (see [Zuo96, Yam10, CCE15, JR20] for other hyperbolicity results in presence of a local system).

Theorem A. Let $X$ be a projective complex algebraic variety. Assume that for every integral closed subvariety $Z \to X$ there exists a complex local system on $X$ whose pull-back to the normalization of $Z$ is not isotrivial. Then $X$ satisfies Conjecture A.

In particular, Theorem A shows that Conjecture A holds if $X$ admits a large complex local system, i.e. a complex local system $\mathcal{L}$ such that for every integral closed subvariety $Z \to X$ the pull-back of $\mathcal{L}$ to the normalization of $Z$ is not isotrivial. Equivalently, the Galois étale cover $X^\mathcal{L} \to X$ associated to the kernel of

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1To be consistent with the conjecture, one says that a non-necessarily irreducible projective variety is of general type if at least one of its irreducible component is of general type.

2A local system on an algebraic variety $X$ is called isotrivial if it becomes trivial on a finite étale cover of $X$.

3At least when $X$ is normal, the assumptions of Theorem A are in fact equivalent to the existence of a large complex local system on $X$, see Proposition 6.3.
the monodromy representation of $\mathcal{L}$ does not have any positive-dimensional compact complex subspaces, cf. Proposition [3.4]. This holds for example when the complex space $X^\mathbb{C}$ is Stein, and turns out to be equivalent at least when $X$ is normal [Eys04, EKPR12]. Examples include:

(1) Projective complex algebraic varieties admitting a finite morphism to an abelian variety. In that case, Theorem [A] follows from works of Bloch [Blo26], Ueno [Uen75], Ochiai [Och77], Kawamata [Kaw80] and Yamanoi [Yam15a].

(2) Projective complex algebraic varieties admitting a (graded-polarizable) variation of $\mathbb{Z}$-mixed Hodge structure with a finite period map. When in addition the Hodge structures are pure, then it follows from works of Griffiths and Schmid [GS69] that $Sp(X) = Sp_{ab}(X) = Sp_h(X) = \emptyset$.

In a nutshell, the proof of Theorem [A] consists in reducing the general case to these two special cases by using general structure results from non-abelian Hodge theory.

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Conventions. A complex algebraic variety is a separated reduced finite type $\mathbb{C}$-scheme. One often makes no distinction between a complex algebraic variety and the associated complex analytic space. A fibration between two normal complex algebraic varieties is a proper surjective morphism $X \to Y$ with connected fibers.

2. Generalities on special sets

We gather some easy properties of special sets for future reference.

Proposition 2.1. Let $X$ be a proper complex algebraic variety. Then

$$Sp_{ab}(X) \subset Sp(X) \text{ and } Sp_{ab}(X) \subset Sp_h(X).$$

Proof. The inclusion $Sp_{ab}(X) \subset Sp(X)$ follows from the fact that the image of a non-constant rational map $A \dashrightarrow X$ from an abelian variety $A$ is not of general type (this is an easy consequence of a special case of Iitaka conjecture proved by Viehweg, cf. [Vie83, Corollary IV]). On the other hand, for every rational map $Y \dashrightarrow X$ with $Y$ smooth projective, there exists a sequence of blow-ups $Y' \to Y$ along smooth subvarieties such that the composite rational map $Y' \to X$ is defined everywhere. Note that the exceptional locus of $Y' \to Y$ is covered by rational curves. The inclusion $Sp_{ab}(X) \subset Sp_h(X)$ follows, since every abelian variety is covered by entire curves. □

Proposition 2.2. Let $f : X \to Y$ be a finite étale cover between projective complex algebraic varieties. Then, for any $* \in \{\emptyset, ab, h\}$,

$$Sp_*(X) = f^{-1}(Sp_*(Y)).$$

Proposition 2.3. Let $f : X \to Y$ be a birational morphism between irreducible projective complex algebraic varieties. Then, for any $* \in \{\emptyset, ab, h\}$, $Sp_*(X)$ is Zariski-dense in $X$ if, and only if, $Sp_*(Y)$ is Zariski-dense in $Y$.

Proof. Let $Z \subset Y$ be a strict closed subvariety such that $f$ is an isomorphism over $Y \setminus Z$. Then $f(\text{Sp}_*(X)) \setminus Z = \text{Sp}_*(Y) \setminus Z$, and the result follows. □
Definition 2.4. Let $X \to Y$ be a proper morphism between projective complex algebraic varieties. For any $* \in \{\emptyset, ab, h\}$, we let

$$\text{Sp}_*(X/Y) := \bigcup_{y \in Y} \text{Sp}_*(X_y).$$

Proposition 2.5. Let $f : X \to Y$ be a proper morphism between projective complex algebraic varieties. Then, for any $* \in \{\emptyset, ab, h\}$,

$$\text{Sp}_*(X) \subset f^{-1}(\text{Sp}_*(Y)) \cup \text{Sp}_*(X/Y).$$

Lemma 2.6. Consider a proper complex algebraic variety $X$ and a finite collection of surjective morphisms $X \to S_i$, $i \in I$, where the $S_i$'s are proper complex algebraic varieties. If the induced morphism $p_i : X \to \prod_{i \in I} S_i$ is finite, then:

1. If $\text{Sp}_b(S_i) \neq S_i$ for every $i \in I$, then $\text{Sp}_b(X) \neq X$.
2. If $\text{Sp}_b(ab(S_i)) \neq S_i$ for every $i \in I$, then $\text{Sp}_b(ab(X)) \neq X$.

Proof. Let $\C \to X$ be a non-constant holomorphic map. Since the morphism $f : X \to \prod_{i \in I} S_i$ is finite, at least one of the map $\C \to S_i$ obtained by composing $\C \to X$ with one of the map $p_i : X \to S_i$ is non-constant. Therefore $\text{Sp}_b(X) \subset \bigcup_{i \in I} h_{-1}(\text{Sp}_b(S_i))$. Since the $p_i$'s are surjective, it follows that $\text{Sp}_b(X) \neq X$. The proof of the second item is similar. \qed

3. Generalities on large local systems

3.1. Monodromy groups. Let $k$ be a field. Let $\mathcal{L}$ be a $k$-local system on a connected complex space $X$. For any $x \in X$, one has the corresponding monodromy representation $\pi_1(X, x) \to \text{GL}(\mathcal{L}_x)$. By definition, the monodromy group of $\mathcal{L}$ is the image of the monodromy representation, and the algebraic monodromy group of $\mathcal{L}$ is the Zariski-closure of the image of the monodromy representation. Different points in $X$ yield isomorphic groups, respectively $k$-algebraic groups.

3.2. Equivalent definitions.

Definition 3.1. A local system $\mathcal{L}$ on a projective complex algebraic variety $X$ is called large if for every integral closed subvariety $Z \hookrightarrow X$, the pull-back of $\mathcal{L}$ to the normalization of $Z$ is not isotrivial.

If $f : X \to Y$ is a dominant morphism between two irreducible normal complex algebraic varieties, the image of the induced morphism of groups $f_* : \pi_1(X) \to \pi_1(Y)$ has finite index in $\pi_1(Y)$ [Cam91]. Therefore, a local system $\mathcal{L}$ on a projective complex algebraic variety $X$ is large if, and only if, for any non-constant morphism $f : Y \to X$ from an irreducible normal projective complex algebraic variety $Y$ the local system $f^{-1}\mathcal{L}$ is not isotrivial. With this observation, the following results are immediate.

Proposition 3.2. Let $X$ be a projective complex algebraic variety and $\mathcal{L}$ be a large local system on $X$. Let $f : Y \to X$ be a finite morphism from another projective complex algebraic variety $Y$. Then the pull-back local system $f^{-1}\mathcal{L}$ on $Y$ is large.

Proposition 3.3. Let $f : Y \to X$ be a finite étale morphism between projective complex algebraic varieties. Let $\mathcal{L}$ be a complex local system on $X$. Then $\mathcal{L}$ is large if, and only if, the pull-back local system $f^{-1}\mathcal{L}$ on $Y$ is large.

The following result is useful to prove that a local system is large.

Proposition 3.4 (compare with [Kol93 Proposition 2.12]). Consider a local system $\mathcal{L}$ on a connected projective complex algebraic variety $X$, and denote by $X^\mathcal{L} \to X$ the associated connected covering space. Then the local system $\mathcal{L}$ is large if, and
only if, the complex analytic space $X^L$ does not contain any positive dimensional compact complex subspaces.

Proof. This is essentially the same proof as in \[Kol93, Proposition 2.12\]. Let $Y \subset X^L$ be an irreducible compact complex subspace with normalization $\overline{Y}$. The induced holomorphic map $Y \to X$ has discrete (hence finite) fibres. Let $Z \subset X$ be the image of $Y$ and $\overline{Z}$ denote its normalization. Then the monodromy group of $L|_{\overline{Z}}$ is isomorphic to the Galois group of $\overline{Y}/\overline{Z}$, in particular it is finite. This shows that (up to deck transformations of $X^L \to X$) there is a one-to-one correspondence between irreducible compact complex subspaces $Z \subset X$ such that the pull-back of $L$ to the normalization $\overline{Z}$ is isotrivial. $\square$

3.3. Shafarevitch morphisms. We will use in several places the existence of the Shafarevitch morphism associated to a complex local system.

**Theorem 3.5.** Let $L$ be a complex local system on a projective normal complex algebraic variety. Then there exists a surjective morphism with connected fibres $\sh_L^X : X \to \Sh_X^L$ onto a projective normal algebraic variety, unique up to unique isomorphism, such that the following property holds: for any connected normal projective complex algebraic variety $Z$ and any morphism $f : Z \to X$, the composite map $\sh_L^X \circ f : Z \to \Sh_X^L$ is constant if, and only if, the local system $f^*L$ is isotrivial. Moreover, when the monodromy group of $L$ is torsion-free, there exists a complex local system $M$ on $\Sh_X^L$ such that $L = (\sh_L^X)^*M$. In particular, the local system $M$ is large.

The map $\sh_L^X : X \to \Sh_X^L$ is called the Shafarevitch morphism associated to $L$. The theorem above is proved in \[Eys04\] when $X$ is smooth projective and the complex local system $L$ is semisimple, and more generally in \[EKPR12\] only assuming that $X$ is smooth projective. See also \[CCE15\] for the case where $X$ is a compact Kähler manifold. The existence of the Shafarevitch morphism when $X$ is only normal is proved by applying the following result to a desingularization of $X$.

**Proposition 3.6.** Let $X' \to X$ be a surjective morphism with connected fibres between normal projective complex algebraic varieties. Let $L$ be a complex local system on $X$ and $\sh_{X'}^L : X' \to \Sh_{X'}^L$ be the Shafarevitch morphism associated to $L' := \nu^{-1}L$. Then there is a (unique) factorization

\[
\begin{array}{ccc}
X' & \longrightarrow & \Sh_{X'}^L \\
\downarrow & & \downarrow \\
X & \longrightarrow & \Sh_X^L
\end{array}
\]

and the induced morphism $X \to \Sh_{X'}^L$ is the Shafarevitch morphism associated to $L$.

Proof. Let $F$ be an irreducible component of the normalization of a fiber of $\nu$. Since the induced morphism $F \to X$ is constant, the restriction of $L$ to $F$ is trivial, therefore $F$ is mapped to a point by the composition $F \to X' \to \Sh_{X'}^L$. Since the fibers of $\nu$ are connected and $X$ is normal, this shows that $\sh_{X'}^L : X' \to \Sh_{X'}^L$ factorizes through a map $X \to \Sh_X^L$. The easy verification that this is the Shafarevitch morphism associated to $L$ is left to the reader. $\square$
3.3.1. The abelian case. We recall a construction of the Shafarevitch morphism when the monodromy group is abelian. Let $X$ be a smooth projective complex algebraic variety. Let $\mathcal{L}$ be a complex local system on $X$ with abelian monodromy, and let $\pi_1(X) \to \Gamma$ denote its monodromy representation. Since by assumption $\Gamma$ is abelian, the monodromy representation factorizes through the abelianization $\pi_1(X) \to H_1(X, \mathbb{Z})$ of $\pi_1(X)$. Let $\text{Alb}(X) \to A_{\mathcal{L}}$ be the quotient of $\text{Alb}(X)$ by the biggest abelian subvariety $T$ of $\text{Alb}(X)$ such that $\pi_1(T) \subset \ker(H_1(X, \mathbb{Z}) \to \Gamma)$. Then the Shafarevitch morphism associated to $\mathcal{L}$ coincides with the Stein factorization of the composition of the Albanese morphism $X \to \text{Alb}(X)$ with $\text{Alb}(X) \to A_{\mathcal{L}}$.

3.3.2. The solvable case. We recall a construction of the Shafarevitch morphism when the monodromy group is solvable, cf. [CCE15, Théorème 4.4].

**Proposition 3.7.** Let $\mathcal{L}$ be a complex local system on a normal projective complex algebraic variety $X$. Let $\pi_1(X) \to \Gamma \subset \text{Gl}(n, \mathbb{C})$ be its monodromy representation. Assume that its monodromy group $\Gamma$ is solvable. If moreover the derived group $[\Gamma, \Gamma]$ is nilpotent, then the Shafarevitch morphism associated to $\mathcal{L}$ is equal to the Shafarevitch morphism associated to the representation $\pi_1(X) \to \Gamma^{ab}$. In particular, $X$ admits a finite morphism to an abelian variety.

**Proof.** Thanks to Proposition 3.6, one can assume that $X$ is smooth. Then the Shafarevitch morphism of $\pi_1(X) \to \Gamma^{ab}$ is the Stein factorization of a morphism from $X$ to an abelian variety $T$. Let $Z$ be a smooth projective complex algebraic variety and $Z \to X$ a morphism. Assume that the composition $f : Z \to X \to T$ is constant. Then, up to replace $Z$ by a finite étale cover, one can assume that the induced homomorphism $\pi_1(Z) \to \Gamma^{ab}$ is trivial, so that $f_\ast \pi_1(Z) \subset [\Gamma, \Gamma]$. Thanks to Lemma 3.8 below, $f_\ast (\pi_1(Z)/C^k \pi_1(Z)) = f_\ast \pi_1(Z)/C^k f_\ast \pi_1(Z)$ has finite image for every positive integer $k$. Since $f_\ast \pi_1(Z) \subset [\Gamma, \Gamma]$ and $[\Gamma, \Gamma]$ is nilpotent by assumption, it follows that $C^k f_\ast \pi_1(Z) = \{0\}$ for $k \gg 1$. Therefore $f_\ast \pi_1(Z)$ is finite. 

**Lemma 3.8.** Let $f : X \to Y$ be a morphism between two smooth projective complex algebraic varieties. If the induced $\mathbb{Q}$-linear map $f_\ast : H_1(X, \mathbb{Q}) \to H_1(Y, \mathbb{Q})$ is zero, then $f_\ast (\pi_1(X)/C^k \pi_1(X))$ has finite image in $\pi_1(Y)/C^k \pi_1(Y)$ for every positive integer $k$. Here, for every group $G$, $\{C^k G\}_{k \geq 0}$ denote the descending central series of $G$, defined by $C^0 G = G$ and $C^{k+1} = [G, C^k G]$ for every integer $k \geq 0$.

In Proposition 3.6, the condition that $[\Gamma, \Gamma]$ is nilpotent is true up to replacing $X$ by a finite étale cover defined from a finite index subgroup of $\Gamma$ thanks to the following observation.

**Proposition 3.9.** If $\Gamma$ is a subgroup of $\text{Gl}(n, \mathbb{C})$ whose Zariski-closure is connected and solvable, then $[\Gamma, \Gamma]$ is nilpotent.

**Proof.** Let $G$ denote the Zariski-closure of $\Gamma$ in $\text{Gl}(n, \mathbb{C})$. Since $G$ is a connected solvable algebraic group, its derived group $[G, G]$ is a connected nilpotent algebraic group. It follows that $[\Gamma, \Gamma] \subset [G, G]$ is nilpotent. 

3.3.3. The case of variations of Hodge structure with discrete monodromy. We recall the construction of the Shafarevitch morphism when the complex local system underlies a variation of Hodge structure with discrete monodromy.

**Proposition 3.10.** Let $X$ be a connected normal projective complex algebraic variety and $\mathcal{L}$ a large complex local system on $X$. Assume that $\mathcal{L}$ underlies a polarized complex variation of pure Hodge structure $(\mathcal{L}, F^\bullet, h)$. Assume moreover that the monodromy group $\Gamma$ of $\mathcal{L}$ is discrete, so that the associated period map induces a holomorphic map $X \to \Gamma \setminus \mathcal{D}$. Then the Shafarevitch morphism associated to $\mathcal{L}$ coincide with the Stein factorization of the proper holomorphic map $X \to \Gamma \setminus \mathcal{D}$. 

Proposition 3.14. Let \( X \) be a normal irreducible projective complex algebraic variety supporting a large complex local system \( L \), and \( \mathcal{F} \) a large complex local system with torsion-free monodromy and a semisimple algebraic monodromy group. Assume that \( L \) underlies a polarized complex variation of pure Hodge structure with discrete monodromy. Then
\[
\text{Sp}(X) = \text{Sp}_{ab}(X) = \text{Sp}_h(X) = \emptyset.
\]

**Proof.** Thanks to \([GS69, Corollary 9.4]\), every horizontal holomorphic map \( C \to D \) is constant. This implies that \( \text{Sp}_{ab}(X) = \emptyset \). A fortiori, \( \text{Sp}_{ab}(X) = \emptyset \) thanks to Lemma 2.1. Finally, \( \text{Sp}(X) = \emptyset \) is a reformulation of the fact that any smooth projective complex variety that admits a polarized complex variation of pure Hodge structure with discrete monodromy and a generically finite period map is of general type, see e.g. \([CCE15, Proposition 3.5]\). \( \square \)

### 3.4. Three canonical decompositions.

**Proposition 3.13.** Let \( X \) be a normal irreducible projective complex algebraic variety supporting a large complex local system \( L \). Up to replacing \( X \) with a finite étale cover, there exists a surjective morphism with connected fibers \( f : X \to Y \) onto a normal irreducible projective complex algebraic variety \( Y \) such that:

- \( Y \) admits a large complex local system with torsion-free monodromy and a semisimple algebraic monodromy group;
- the monodromy of the restriction of \( L \) to the normalization of any fiber of \( f \) is solvable.

**Proof.** Consider the monodromy representation \( \pi_1(X) \to G(\mathbb{C}) \) of \( L \), where \( G \) is the algebraic monodromy group of \( L \). Up to replacing \( X \) with a finite étale cover, one can assume that \( G \) is connected. Let \( N \) be the (solvable) radical of \( G \), so that \( N \) is a (connected) solvable complex algebraic group. Let \( H \) be the quotient of \( G \) by \( N \), so that \( H \) is a connected semisimple complex algebraic group. The induced representation \( \pi_1(X) \to H(\mathbb{C}) \) has a Zariski-dense image, and replacing \( X \) with a finite étale cover, one can assume that it has torsion-free image thanks to Selberg Lemma.

Let \( f : X \to Y \) denote the Shafarevich morphism associated to \( \pi_1(X) \to H(\mathbb{C}) \). In particular, \( Y \) is a normal irreducible projective complex algebraic variety and \( f \) is surjective with connected fibres. Since the representation \( \pi_1(X) \to H(\mathbb{C}) \) has torsion-free image, it factorizes through the homomorphism \( \pi_1(X) \to \pi_1(Y) \), cf. Theorem 3.3. The induced homomorphism \( \pi_1(Y) \to H(\mathbb{C}) \) corresponds to a large complex local system with a semisimple algebraic monodromy group.

Let \( F \) be the normalization of an irreducible component of a fiber of \( f \). Since the induced morphism \( F \to Y \) is constant, the induced homomorphism \( \pi_1(F) \to \pi_1(X) \to H(\mathbb{C}) \) has finite image. But the image of \( \pi_1(F) \to H(\mathbb{C}) \) is torsion-free, hence the image of \( \pi_1(F) \to H(\mathbb{C}) \) is in fact trivial. Therefore it is contained in \( N(\mathbb{C}) \), from what it follows that it is solvable. \( \square \)

**Proposition 3.14.** Let \( X \) be a normal irreducible projective complex algebraic variety supporting a large complex local system \( L \). Up to replacing \( X \) with a finite
étale cover, there exists a surjective morphism with connected fibers \( f : X \to Y \) onto a normal irreducible projective complex algebraic variety \( Y \) such that:

- \( Y \) admits a large complex local system with torsion-free monodromy and a reductive algebraic monodromy group;
- the monodromy of the restriction of \( \mathcal{L} \) to the normalization of any fiber of \( f \) is unipotent;
- the normalization of every fiber of \( f \) admits a finite morphism to an abelian variety.

Proof. The two first assertions are proved exactly as in the proof of Proposition 3.13, replacing the solvable radical \( N \) of \( G \) with the unipotent radical \( U \) of \( G \). For the last assertion, let \( F \) be the normalization of an irreducible component of a fiber of \( f \). Since the monodromy of the restriction of \( \mathcal{L} \) to \( F \) is unipotent, one can apply Proposition 3.6 and Proposition 3.7 to infer that \( F \) admits a finite morphism to an abelian variety. The statement follows by taking a product over the irreducible components. \( \square \)

Proposition 3.15. Let \( X \) be a normal irreducible projective complex algebraic variety supporting a large complex local system \( \mathcal{L} \). Assume that the algebraic monodromy group \( G \) is reductive (equivalently, the complex local system \( \mathcal{L} \) is semisimple). Up to replacing \( X \) with a finite étale cover, there exists a surjective morphism with connected fibers \( f : X \to Y \) onto a normal irreducible projective complex algebraic variety \( Y \) and a morphism to an abelian variety \( g : X \to A \) such that:

- \( Y \) admits a large complex local system with torsion-free monodromy and a semisimple algebraic monodromy group;
- the monodromy of the restriction of \( \mathcal{L} \) to the normalization of any fiber of \( f \) is commutative;
- the induced morphism \((f, g) : X \to Y \times A\) is finite.

Proof. Consider the monodromy representation \( \rho : \pi_1(X) \to G(\mathbb{C}) \) of \( \mathcal{L} \), where \( G \) is the algebraic monodromy group of \( \mathcal{L} \). Up to replacing \( X \) with a finite étale cover, one can assume that \( G \) is connected. Then \( T := G/[G, G] \) is a torus and \( H := G/Z(G) \) is a connected semisimple algebraic group. Let \( \rho_T : \pi_1(X) \to T(\mathbb{C}) \) and \( \rho_H : \pi_1(X) \to H(\mathbb{C}) \) denote the induced representations. Since \( \rho \) has Zariski-dense image, both \( \rho_T \) and \( \rho_H \) have Zariski-dense image. Since \( \rho \) is large and the canonical morphism \( G \to T \times H \) is an isogeny, the morphism \( \mathrm{sh}^{\mathrm{ct}}_X \times \mathrm{sh}^{\mathrm{un}}_X : X \to \mathrm{Sh}^{\mathrm{ct}}_X \times \mathrm{Sh}^{\mathrm{un}}_X \) is finite. Up to replacing \( X \) with a finite étale cover, one can assume that the image of \( \rho_H \) is torsion-free, so that \( \rho_H \) factorizes through \( \mathrm{Sh}^{\mathrm{un}}_X \). Therefore, \( \mathrm{Sh}^{\mathrm{un}}_X \) admits a large complex local system with a semisimple algebraic monodromy group. On the other hand, recalling the construction of the Shafarevich morphism in the abelian case (cf. Proposition 3.6 and section 3.3.1), it follows that \( \mathrm{Sh}^{\mathrm{ct}}_X \) admits a finite morphism to an abelian variety \( A_{\text{pr}} \).

This proves the first and the third assertions if one lets \( f : X \to Y \) be the morphism \( \mathrm{sh}^{\mathrm{un}}_X : X \to \mathrm{Sh}^{\mathrm{un}}_X \) and \( g : X \to A \) be the composition of \( \mathrm{sh}^{\mathrm{ct}}_X : X \to \mathrm{Sh}^{\mathrm{ct}}_X \) with \( \mathrm{Sh}^{\mathrm{ct}}_X \to A_{\text{pr}} \).

Let \( F \) be the normalization of an irreducible component of a fiber of \( f \). Since the induced morphism \( F \to Y \) is constant, the induced homomorphism \( \pi_1(F) \to \pi_1(X) \to H(\mathbb{C}) \) has finite image. But the image of \( \pi_1(X) \to H(\mathbb{C}) \) is torsion-free, hence the image of \( \pi_1(F) \to H(\mathbb{C}) \) is in fact trivial. Therefore it is contained in \( T(\mathbb{C}) \), from what it follows that it is commutative. \( \square \)

3.5. Algebraic varieties with a large complex local system. The following statement collect some known results on algebraic varieties supporting a large complex local system.
Theorem 3.16. Let $X$ be a connected normal projective complex algebraic variety with a large complex local system $\mathcal{L}$.

1. Assume that the monodromy group of $\mathcal{L}$ is solvable (equivalently the algebraic monodromy group of $\mathcal{L}$ is solvable). Then, up to a finite étale cover, $X$ is isomorphic to the product of an abelian variety by a variety of general type.

2. Assume that $X$ is Brody-special$^4$. Then, up to a finite étale cover, $X$ is isomorphic to an abelian variety.

3. Assume that the algebraic monodromy group of $\mathcal{L}$ is semisimple. Then $X$ is of general type.

4. Assume that $X$ is weakly-special$^5$. Then, up to a finite étale cover, $X$ is isomorphic to an abelian variety.

Proof. For the first item, up to replacing $X$ by a finite étale cover, one can assume that the derived group of the monodromy group is nilpotent, cf. Proposition 3.9. It follows from Proposition 3.7 that $A$ admits a finite morphism to an abelian variety. Therefore, thanks to a result of Kawamata [Kaw81, Theorem 13], after passing to another finite étale cover, $X$ is biholomorphic to a product $B \times X'$ of an abelian variety $B$ and a projective variety of general type $X'$ whose dimension is equal to the Kodaira dimension $\kappa(X)$ of $X$. The second item is due to Yamanoi, see [Yam10] and [Yam15b, Theorem 2.17]. The third item is due to Zuo [Zuo96], see also [CCE15, Théorème 6.3] for an alternative proof. For the last item, it follows from Proposition 3.13 that, up to replacing $X$ with a finite étale cover, there exists a surjective morphism with connected fibers $f : X \to Y$ onto a normal irreducible projective complex algebraic variety $Y$ such that:

- $Y$ admits a large complex local system with a semisimple algebraic monodromy group, and
- the monodromy of the restriction of $\mathcal{L}$ to the normalization of any fiber of $f$ is solvable.

Thanks to the third item, $Y$ is of general type. Since $X$ is weakly-special, it follows that $Y$ is a point and that $\mathcal{L}$ has solvable monodromy. But then the result follows from the first item. \qed

4. The special subsets coincide

Theorem 4.1. Let $X$ be a projective complex algebraic variety supporting a large complex local system $\mathcal{L}$. If $X$ is not of general type, then $\text{Sp}(X) = \text{Sp}_{ab}(X) = \text{Sp}_{h}(X) = X$.

Proof. Thanks to Lemma 2.1, it is sufficient to prove that $\text{Sp}_{ab}(X) = X$. It is harmless to assume that $X$ is irreducible. Note also that one can freely replace $X$ with any projective complex algebraic variety $X'$ not of general type and such that there exists a finite surjective morphism $X' \to X$. Indeed, the pull-back to $X'$ of the local system $\mathcal{L}$ is still large thanks to Proposition 3.2 whereas $\text{Sp}_{ab}(X') = X'$ implies $\text{Sp}_{ab}(X) = X$. In particular, one can assume that $X$ is normal. We will also freely replace $X$ by any finite étale cover.

Up to replacing $X$ with a finite étale cover, one can assume that there exists a fibration $f : X \to Y$ as in Proposition 3.13. The normal projective variety $Y$ admits a large complex local system with a semisimple algebraic group, hence it is

$^4$A proper complex algebraic variety $X$ is called Brody-special if there exists a Zariski-dense entire curve $\mathbb{C} \to X$.

$^5$Following Campana [Cam04], a proper complex algebraic variety $X$ is called weakly special if it does not admit a finite étale cover $X'$ with a rational dominant map $X' \to Y$ to a positive dimensional variety of general type $Y$. 

of general type thanks to Theorem 3.10. Since by assumption $X$ is not of general type, the (geometric) generic fibre of $f$ is a positive-dimensional variety which is not of general type.

Let $C$ be an irreducible component of a fibre of $f$. Thanks to Theorem 4.2 below, $C$ is not of general type. On the other hand, by definition of $f$, the restriction of $L$ to the normalization of $C$ is a large complex local system with solvable monodromy. Therefore, thanks to Theorem 3.16, the normalization of $C$ is, up to a finite étale cover, a product of a positive dimensional abelian variety by a variety of general type. This proves that the fibers of $f$ are covered by images of abelian varieties by finite maps, hence a fortiori $Sp_{ab}(X) = X$. □

Theorem 4.2 (Nakayama, [Nak04, Theorem VI.4.3]). Let $X \to S$ be a projective surjective morphism with connected fibres from a normal complex analytic variety onto a smooth curve and $0 \in S$. Let $X_0 = \cup_{i \in I} \Gamma_i$ the decomposition into irreducible components. If there is at least one irreducible component $\Gamma_j$ which is of general type (i.e. its desingularisation is such), then for $s \in S$ general the fiber $X_s$ is of general type.

Corollary 4.3. Let $X$ be a projective complex algebraic variety supporting a large complex local system $L$. Then:

$$Sp(X) = Sp_{ab}(X) = Sp_h(X).$$

Proof. It follows from Theorem 4.1 that $Sp(X) \subset Sp_{ab}(X)$, hence the equality thanks to Lemma 2.1. Moreover, given an entire curve $C \to X$, the normalization $Z$ of the Zariski-closure of its image in $X$ is connected and Brody-special. Since the induced morphism $Z \to X$ is finite, the pull-back of $L$ to $Z$ is still a large complex local system (cf. Proposition 3.2). Therefore a finite étale cover of $Z$ is isomorphic to an abelian variety thanks to Theorem 3.16. This proves the inclusion $Sp_h(X) \subset Sp_{ab}(X)$, hence the equality thanks to Lemma 2.1. □

5. A non-Archimedean detour

5.1. Katzarkov-Zuo reductions. The following result is due to Eyssidieux [Eys04, Proposition 1.4.7], based on former works of Katzarkov and Zuo [Kat97, Zuo96, Zuo99].

Theorem 5.1. Let $k$ be a non-Archimedean local field\footnote{A local field is assumed to be locally compact by definition. Therefore, a non-Archimedean local field is either a finite extension of $\mathbb{Q}_p$ for some prime $p$ or a field of formal Laurent series $\mathbb{F}_q((T))$ over a finite field.} and $G$ be a reductive algebraic group over $k$. Let $X$ be a connected normal compact Kähler analytic space and $\rho : \pi_1(X) \to G(k)$ a representation with Zariski-dense image. Then there exists a surjective holomorphic map with connected fibres $\sigma_X^\rho : X \to S_X^\rho$ onto a connected normal compact Kähler analytic space such that the following property holds: for any connected normal compact complex analytic space $Z$ and any holomorphic map $f : Z \to X$, the composition $\sigma_X^\rho \circ f : Z \to S_X^\rho$ is constant if, and only if, the representation $f^*\rho$ has bounded image.

Observe that a fibration $\sigma_X^\rho : X \to S_X^\rho$ with this property is unique, up to unique isomorphism. It is called the Katzarkov-Zuo reduction of $(X, \rho)$. Its existence is proved in [Eys04] for a smooth $X$. However, one can argue as in Proposition 3.6 to prove its existence more generally when $X$ is normal.

The following result is a key ingredient in the proof of Theorem A.
Theorem 5.2. Let $k$ be a non-Archimedean local field and $G$ be a reductive algebraic group over $k$. Let $X$ be a connected normal compact Kähler analytic space and $\rho : \pi_1(X) \to G(k)$ a representation with Zariski-dense image. Assume moreover that $G/k$ is absolutely simple and that $\rho$ is a large representation with unbounded image. Then:

1. the Katzarkov-Zuo reduction $\sigma_X^\rho : X \to S_X^\rho$ is birational.
2. $\overline{Sp}_{ab}(X) \neq X$.

The first assertion is a result of Zuo [Zuo96]. It is a key ingredient in both Zuo’s paper [Zuo96] and Yamanoi’s paper [Yam10]. We give an alternative and arguably simpler argument below. The rest of this section is devoted to the proof of this theorem.

5.2. A construction of the Katzarkov-Zuo reduction. We give an alternative construction of the Katzarkov-Zuo reduction based on a construction of Klingler [Kli03].

Let $k$ be a non-Archimedean local field and $G$ be a reductive algebraic group over $k$. Let $\mathcal{B}(G, k)$ be the Bruhat-Tits building of $G/k$, $W$ the Weyl group of $G/k$ and $W^{aff}$ the affine Weyl group of $G/k$. Let $\mathcal{A}$ be the affine real vector space on which the apartments of $\mathcal{B}(G, k)$ are modelled, and let $\mathcal{A}_C$ be its complexification. The group $W^{aff}$ acts on $\mathcal{A}$ by affine reflections.

Let $X$ be a Riemannian manifold and $f : X \to \mathcal{B}(G, k)$ be a continuous map. A point $x \in X$ is called regular for $f$ if there is an apartment in $\mathcal{B}(G, k)$ that contains a sufficiently small neighborhood of $x$ [GS92, p.225]. Otherwise $x$ is called singular. The map $f$ is called harmonic if for every point $x \in X$, there exists a small ball $B$ centered at $x$ on which $f$ minimizes the energy relatively to $f|_{\partial B}$ [GS92, p.232].

Assume now that $X$ is a connected compact Kähler manifold and let $\rho : \pi_1(X) \to G(k)$ be a representation with Zariski-dense image. Thanks to [GS92, Theorem 7.8, Lemma 8.1] there exists a Lipschitz harmonic $\rho$-equivariant pluriharmonic map $f : \tilde{X} \to \mathcal{B}(G, k)$ from the universal covering of $X$ (with finite energy since $X$ is compact). The subset $R(\tilde{X}, \rho) \subset \tilde{X}$ of regular points for $f$ is a $\pi_1(X)$-invariant open subset of $\tilde{X}$, and one denotes by $R(X, \rho)$ its image in $X$. The Hausdorff codimension of its complementary $S(X, \rho) \subset X$ is at least 2 [GS92 Theorem 6.4]. Moreover $f$ is pluriharmonic, i.e. $\partial \overline{\partial} f = 0$ on $R(\tilde{X}, \rho)$ [GS92 Theorem 7.3], and for any holomorphic map $g : Y \to X$ from a connected compact Kähler manifold $Y$ with universal covering $\tilde{Y}$, the representation $g^* \rho : \pi_1(Y) \to G(k)$ is reductive (i.e. the Zariski-closure of its image is a reductive group) and the composition $f \circ \tilde{g} : \tilde{Y} \to \mathcal{B}(G, k)$ is $g^* \rho$-equivariant and pluriharmonic [Eys04, Corollaire 1.3.8].

Klingler explains in [Kli03 section 2.2.2] the construction of a complex local system $F(X, \rho)$ with finite monodromy on $R(X, \rho)$ that corresponds intuitively to the pull-back by $f$ of the complexified tangent bundle of the building $\mathcal{B}(G, k)$. We briefly recall the construction and refer to loc. cit. for the details.

Let $x \in R(\tilde{X}, \rho)$, so that there exists an isometric embedding $i : A \subset \mathcal{B}(G, k)$ and a neighborhood $B$ of $x$ in $\tilde{X}$ such that the map $f|_B : B \to \mathcal{B}(G, k)$ factorizes through a pluriharmonic map $h : B \to A$. The map $h$ is well-defined up to the action of $W^{aff}$ on $A$. Since $h$ is pluriharmonic, by taking the $(1,0)$-part of the complexification of its differential, one obtains a $\mathbb{C}$-linear map $A^{\omega}_C \to \Omega^1_B$, well-defined up to the action of $W$ on $A^{\omega}_C$. Globalizing, the map $f$ defines a real local system...
with monodromy representation \( \pi_1(R(X, \rho)) \to W^{aff} \subset \text{Aut}(A) \). By composing with the homomorphism \( W^{aff} \to W \), one obtains a real local system \( F_\mathbb{R}(X, \rho) \) corresponding to the monodromy representation \( \pi_1(R(X, \rho)) \to W \subset \text{Aut}(A) \), and the derivative of \( f \) yields a real one-form \( \mu_X^{\mathbb{R}} \) with values in \( F_\mathbb{R}(X, \rho) \). We denote by \( F(X, \rho) \) the complex local system associated to \( F_\mathbb{R}(X, \rho) \). Since \( h \) is pluriharmonic, the complexification of \( \mu_X^{\mathbb{R}} \) is a holomorphic one-form \( \mu_X \) with values in \( F(X, \rho) \).

Since \( S(X, \rho) \) is a closed analytic subset of \( X \) [Eys04] Proposition 1.3.3] distinct from \( X \) and since the monodromy of \( F(X, \rho) \) is finite, there exists a normal ramified Galois covering \( p : Z \to X \) with Galois group \( \Lambda \subset W \) such that the complex local system \( p^*F(X, \rho) \) on \( p^{-1}(R(X, \rho)) \) is trivial, hence extends to \( Z \). Since \( f \) is Lipschitz, the holomorphic one-form \( \mu_Z = p^*\mu_X \) on \( p^{-1}(R(X, \rho)) \) is bounded, hence it extends as a holomorphic one-form \( \mu_Z \in H^0(Z, \Omega^1_Z \otimes \mathcal{A}) \). Moreover, \( \mu_Z \) is the zero one-form if, and only if, the map \( f \) is constant.

Let \( Z \to \text{Alb}(Z) \) denote the Albanese morphism of \( Z \) (we refer to [Eys11] Section 3.4.2] for the definitions of holomorphic forms and Albanese morphism for normal Kähler spaces). The map \( Z \to \text{Alb}(Z) \) is an initial object in the category of holomorphic maps \( Z \to T \) where \( T \) is a compact complex torus. In particular, \( \text{Alb}(Z) \) is equipped with a \( \Lambda \)-action such that \( Z \to \text{Alb}(Z) \) is \( \Lambda \)-equivariant. Moreover, the induced \( \mathbb{C} \)-linear \( \Lambda \)-equivariant map \( H^0(\text{Alb}(Z), \Omega^1) \to H^0(Z, \Omega^1) \) is an isomorphism.

There is a the largest subtorus \( B \) of \( \text{Alb}(Z) \) such that the composition of \( \mu_Z : \mathcal{A}^\vee \to H^0(Z, \Omega^1_Z) \) with \( H^0(\text{Alb}(Z), \Omega^1) \to H^0(B, \Omega^1) \) is zero. Since \( B \) is preserved by the action of \( \Lambda \), we get a commutative diagram:

\[
\begin{array}{ccc}
Z & \to & \text{Alb}(Z)/B \\
\downarrow & & \downarrow \\
X & \to & (\text{Alb}(Z)/B)/\Lambda
\end{array}
\]

It is easy to check that the Stein factorization of the holomorphic map \( X \to (\text{Alb}(Z)/B)/\Lambda \) is the Katzarkov-Zuo reduction of \( (X, \rho) \).

5.3. A lemma.

**Lemma 5.3.** Let \( k \) be a non-Archimedean local field. Let \( G \) be an absolutely simple \( k \)-algebraic group. Let \( \Gamma \subset G(k) \) be a Zariski-dense and unbounded subgroup. Let \( \Delta \subset \Gamma \) be a normal subgroup. If \( \Delta \) is bounded in \( G(k) \), then \( \Delta \) is finite.

**Proof.** We briefly recall a class of compactifications of Bruhat-Tits buildings introduced in [Ber90] and generalized in [RTW10].

Up to replacing \( k \) with a finite extension, one can assume that \( G \) is \( k \)-isotropic. (One can even assume for simplicity that \( G/k \) is split.) Let \( \mathcal{B}(G, k) \) be the Euclidean Bruhat-Tits building associated to \( G/k \) equipped with its natural \( G(k) \)-action by isometries. For any \( G(k) \)-conjugacy class \( t \) of parabolic subgroups of \( G \), there exists a continuous, \( G(k) \)-equivariant map \( \theta_t : \mathcal{B}(G, k) \to \text{Par}_t(G)^{an} \) which is a homeomorphism onto its image. Here \( \text{Par}_t(G) \) denotes the connected component of type \( t \) in the proper \( k \)-algebraic variety \( \text{Par}(G) \) of all parabolic subgroups in \( G \) (on which \( G \) acts by conjugation). The superscript \(^{an}\) means that we pass from the \( k \)-variety \( \text{Par}_t(G) \) to the Berkovich \( k \)-analytic space associated with it. Note that the space \( \text{Par}(G)^{an} \) is compact since \( \text{Par}(G) \) is projective. Let \( \mathcal{B}_k(G, k) \) denote the
closure of the image of $\theta_t$.

Two parabolic subgroups $P$ and $Q$ of $G$ are called osculatory if their intersection $P \cap Q$ is also a parabolic subgroup of $G$. Moreover, each parabolic subgroup $P \in \text{Par}(G)$ defines a closed osculatory subvariety $\text{Osc}_t(P)$ of $\text{Par}_t(G)$, namely the one consisting of all parabolics of type $t$ whose intersection with $P$ is a parabolic subgroup. Then $P$ is $t$-relevant if it is maximal among all parabolic $k$-subgroups defining the same osculatory subvariety. It is readily seen that each parabolic subgroup is contained in a unique $t$-relevant one.

The space $\overline{\mathcal{F}}_t(G, k)$ admits a stratification in disjoint locally closed subspaces that are indexed by the $t$-relevant parabolic $k$-subgroups in $G$. If $x \in \overline{\mathcal{F}}_t(G, k)$ belongs to the stratum indexed by the $t$-relevant parabolic $k$-subgroup $Q$, then the subgroup of $G(k)$ that fixes $x$ is contained in $Q(k)$, cf. [RTW10, Theorem 4.11].

In particular, a subgroup of $G(k)$ that fixes a point in the boundary $\partial \overline{\mathcal{F}}_t(G, k) = \overline{\mathcal{F}}_t(G, k) \setminus \mathcal{F}_t(G, k)$ cannot be Zariski-dense in $G$.

Let us finish the proof of the lemma. Since $\Delta$ is normal in $\Gamma$, its Zariski-closure $\overline{\Delta}^{\text{Zar}}$ is normal in $\overline{\Gamma}^{\text{Zar}} = G$. Since $G$ is simple, it follows that $\overline{\Delta}^{\text{Zar}}$ is either equal to $G$ or is a finite group.

Let $\mathcal{F} \subset \overline{\mathcal{F}}_t(G, k)$ be the subset of points that are fixed by the induced action of $\Delta$. It is compact, since the action of $G(k)$ on $\overline{\mathcal{F}}_t(G, k)$ is continuous and $\overline{\mathcal{F}}_t(G, k)$ is compact. Moreover, $\mathcal{F}$ is non-empty since $\Delta$ is bounded by assumption. Since $\Delta$ is a normal subgroup of $\Gamma$, the action of $\Gamma$ on $\overline{\mathcal{F}}_t(G, k)$ stabilizes $\mathcal{F}$. If $\mathcal{F}$ was contained in $\mathcal{F}_t(G, k)$, then $\Gamma$ would fix the barycenter of $\mathcal{F}$. This would be in contradiction with the assumption that $\Gamma$ is unbounded. Therefore, $\mathcal{F}$ meets the boundary $\partial \overline{\mathcal{F}}_t(G, k)$. By the preceding discussion, it follows that $\overline{\Delta}^{\text{Zar}}$ cannot be equal to $G$, so that it is necessary a finite group. This proves that $\Delta$ is finite. □

5.4. Proof of Theorem 5.2. Let us now turn to the proof of Theorem 5.2. We keep the notations introduced in section 5.2. We can assume without loss of generality that $X$ is smooth. Let $X_s$ be a general fiber of the Katzarkov-Zuo reduction $\sigma_X^k : X \to S_X^k$, so that there is an exact sequence of groups:

$$
\pi_1(X_s) \to \pi_1(X) \to \pi_1(S_X^k) \to 1.
$$

Let $\Gamma$ be the image of $\rho : \pi_1(X) \to G(k)$ and $\Delta$ be the image of the composition $\pi_1(X_s) \to \pi_1(X) \to G(k)$. The assumption of Lemma 5.3 are fulfilled, hence $\Delta$ is finite. Since the representation $\rho$ is large, this is only possible if $X_s$ has dimension zero. This proves that $\sigma_X^k$ is birational.

Let $Z \to X$ denote the Galois covering associated to the pair $(X, \rho)$, whose construction is recalled in section 5.2. By construction, $Z$ is a connected normal compact Kähler space.

Let us prove that $Z$ is of general type, using an argument of Zuo [Zuo96]. Since the induced map $Z \to X$ is finite surjective, the image of the homomorphism $\pi_1(Z) \to \pi_1(X)$ has finite index in $\pi_1(X)$, hence the image of the induced representation $\rho_Z : \pi_1(Z) \to G(k)$ is also Zariski-dense and unbounded. Moreover, $\rho_Z$ is also a large representation, hence it follows from the preceding paragraph that the Katzarkov-Zuo reduction of $(Z, \rho_Z)$ is birational. But by construction, the image of the Katzarkov-Zuo reduction of $(Z, \rho_Z)$ admits a finite morphism to a compact complex torus. Therefore, assuming by contradiction that $Z$ is not of
general type, there would exist a finite étale cover $Z' \to Z$ and a non-trivial fibration $f : Z' \to Y$ whose general fibre is a compact complex torus [Kaw81, Theorem 23]. Consider the induced representation $\rho_{Z'} : \pi_1(Z') \to G(k)$. It is again large since $Z' \to Z$ is finite. Let $F$ be a general fibre of $f$. There is an exact sequence of groups $\pi_1(F) \to \pi_1(Z') \to \pi_1(Y)$. Since $\pi_1(F)$ is normal in $\pi_1(Z')$ and the image of $\pi_1(Z')$ is Zariski-dense in $G$, the Zariski-closure $H$ of the image of $\pi_1(F)$ in $G$ is a normal algebraic subgroup of $G$. But $\pi_1(F)$ is abelian, hence $H$ is abelian too. Since $G$ is absolutely simple, $H$ is necessarily a finite subgroup of $G$. This is a contradiction, since $\rho_{Z'}$ is large.

Let $a : A \dashrightarrow X$ be a non-constant rational map from an abelian variety. Since $X$ possesses a large local system, it has no rational curves. It follows that $a$ is defined everywhere. Since the fundamental group of $A$ is abelian, the Zariski-closure $H$ of the image of $\rho_{A} : \pi_1(A) \to G(k)$ is a torus. Therefore, the Bruhat-Tits building $\mathcal{B}(H, k)$ is an Euclidean space. On the other hand, if $\tilde{A}$ is a universal covering of $A$, the composition $\tilde{A} \to X \to \mathcal{B}(G, k)$ is $\rho_{A}$-equivariant and pluriharmonic [Kys04, Corollaire 1.3.8]. By unicity (up to translation) of the harmonic map associated to a reductive representation, it follows that the composition $\tilde{A} \to X \to \mathcal{B}(G, k)$ takes values in a totally geodesic embedded copy of $\mathcal{B}(H, k)$ in $\mathcal{B}(G, k)$. Since the Weyl group of $H/k$ is trivial, it follows that the complex local system $F(A, \rho_{A})$ introduced in section 5.2 is trivial.

Assume that the image of $a$ is not contained in $S(X, \rho)$. By construction, the pullback to $R(A, \rho_{A})$ along $a$ of the complex local system $F(X, \rho)$ is equal to $F(A, \rho_{A})$. Since the local system $F(A, \rho_{A})$ is trivial, there is a rational map $A \dashrightarrow Z$ lifting $a : A \to X$. It follows that $\overline{\text{Sp}_{\text{ab}}}(X)$ is contained in the union of $S(X, \rho)$ and the image of $\overline{\text{Sp}_{\text{ab}}}(Z)$. But $Z$ is a projective variety of general type with a generically finite map to an abelian variety. It follows from a result of Yamanoi [Yam15a, Corollary 1] that $\overline{\text{Sp}_{\text{ab}}}(Z) \neq Z$, so that $\overline{\text{Sp}_{\text{ab}}}(X) \neq X$ by the preceding discussion.

6. Proof of Theorem A

6.1. A first reduction.

Definition 6.1. Let $X$ be a projective complex algebraic variety. A collection of complex local systems $\{\mathcal{L}_i\}_{i \in \mathcal{I}}$ on $X$ is large if for every integral closed subvariety $Z \hookrightarrow X$ the pullback to the normalization of $Z$ of one of the $\mathcal{L}_i$’s is not isotrivial. Equivalently, for any non-constant morphism $f : Y \to X$ from an irreducible normal projective complex algebraic variety $Y$, there exists $i \in \mathcal{I}$ such that the local system $f^{-1}\mathcal{L}_i$ is not isotrivial.

In particular, a complex local system $\mathcal{L}$ is large when the collection $\{\mathcal{L}\}$ is large.

Proposition 6.2. Let $X$ be a projective complex algebraic variety and $\{\mathcal{L}_i\}_{i \in \mathcal{I}}$ a large collection of complex local systems on $X$. Let $f : Y \to X$ be a finite morphism from another projective complex algebraic variety $Y$. Then the collection $\{f^{-1}\mathcal{L}_i\}_{i \in \mathcal{I}}$ of complex local systems on $Y$ is large.

Proposition 6.3. Let $X$ be a projective normal complex algebraic variety. If $X$ admits a large collection of complex local systems, then $X$ admits a large complex local system.

Proof. For any complex local system $\mathcal{L}$, we have the associated Shafarevitch morphism $X \to \text{Sh}^\mathcal{L}$. If $\{\mathcal{L}_i\}_{i \in \mathcal{I}}$ is a large collection of complex local systems on $X$, then the morphism $X \to \prod_{i \in \mathcal{I}} \text{Sh}_{\mathcal{L}}^i$ does not contract any subvariety, therefore it is finite. By Noetherianity, there exists finitely many local systems $\mathcal{L}_{i_1}, \cdots, \mathcal{L}_{i_N}$ such
that the morphism $X \to \prod_{k=1}^{N} \text{Sh}^{L_{ik}}_{X}$ is finite. It follows that the complex local system $\bigoplus_{k=1}^{N} L_{ik}$ is large. \hfill \Box

In view of Proposition 6.2 and Proposition 6.3, the proofs of Theorem 4.1 and Corollary 4.3 immediately generalize as follows:

**Theorem 6.4.** Let $X$ be a projective complex algebraic variety supporting a large collection of complex local systems. Then $\text{Sp}(X) = \text{Sp}_{\text{ab}}(X) = \text{Sp}_{h}(X)$, and $\text{Sp}(X) = \text{Sp}_{\text{ab}}(X) = \text{Sp}_{h}(X) = X$ if $X$ is not of general type.

With the preceding results at hand, we first reduce Theorem A to the following result.

**Theorem 6.5.** Let $X$ be a projective complex algebraic variety supporting a large collection of complex local systems. If $X$ is of general type, then $\text{Sp}_{\text{ab}}(X)$ is not Zariski-dense in $X$.

**Proof of Theorem A assuming Theorem 6.5.** Recalling Theorem 6.4, we need only to check that $\text{Sp}(X) = \text{Sp}_{\text{ab}}(X) = \text{Sp}_{h}(X)$ is Zariski-closed in $X$. There is nothing to prove when $\dim X = 0$, so that one can assume that $\dim X > 0$. By Noetherian induction, let us assume that the result holds for all strict subvarieties of $X$. If $X$ is not of general type, then the result follows from Theorem 6.4. Otherwise, if $X$ is of general type, then $S := \text{Sp}_{\text{ab}}(X)$ is not Zariski-dense in $X$ thanks to Theorem 6.5. Denoting by $\bar{S}$ its Zariski-closure in $X$, observe that $\text{Sp}_{\text{ab}}(\bar{S}) = \text{Sp}_{\text{ab}}(X) = S$. Therefore, $\text{Sp}_{\text{ab}}(\bar{S})$ is Zariski-dense in $\bar{S}$. Since $\bar{S}$ is a strict subvariety of $X$, it follows by Noetherian induction that $\bar{S}$ is not of general type. Thanks to Theorem 6.4, $\bar{S}$ is covered by images of abelian varieties, so that $S = \bar{S}$. \hfill \Box

### 6.2. Proof of Theorem 6.5

Let $X$ be a projective complex algebraic variety supporting a large collection of complex local systems. Assume that $X$ is of general type. Our goal in this section is to prove that $\text{Sp}_{\text{ab}}(X)$ is not Zariski-dense in $X$.

Let $X = \bigcup_{i} X_{i}$ be the decomposition of $X$ in its irreducible components. Since $\text{Sp}_{\text{ab}}(X_{i}) \subset X_{i}$ for every $i$, it is sufficient to prove that $\text{Sp}_{\text{ab}}(X_{i})$ is not Zariski-dense in $X_{i}$ for at least one of the $X_{i}$’s which is of general type. Therefore one can assume from now on that $X$ is irreducible.

Let $\nu : \tilde{X} \to X$ denote the normalization of $X$ and $Z \subset X$ the non-normal locus of $X$. Then $\nu(\text{Sp}_{\text{ab}}(\tilde{X})) \subset \text{Sp}_{\text{ab}}(X)$ since $\nu$ is finite, and $\text{Sp}_{\text{ab}}(X) \subset \nu(\text{Sp}_{\text{ab}}(\tilde{X})) \cup Z$ since $\nu$ is an isomorphism onto its image outside $\nu^{-1}(Z)$. Therefore, $\text{Sp}_{\text{ab}}(X)$ is Zariski-dense in $X$ if, and only if, $\text{Sp}_{\text{ab}}(\tilde{X})$ is Zariski-dense in $\tilde{X}$. In view of Proposition 6.2 and Proposition 6.3, $\tilde{X}$ admits a large complex local system. As a consequence, one can assume from now on that $X$ is normal irreducible projective complex algebraic variety with a large complex local system. We denote by $\rho : \pi_{1}(X) \to G(\mathbb{C})$ the corresponding monodromy representation, with $G$ its algebraic monodromy group.

Note also that it is harmless to replace $X$ with a finite étale cover, cf. Proposition 2.2 and Proposition 3.3. In particular, one can assume that the algebraic monodromy group $G$ is a connected algebraic group.

#### 6.2.1. The semisimple case

Let us first finish the proof under the additional assumption that the algebraic monodromy group $G$ is semisimple. Thanks to [Eys04], there exist finitely many representations $\rho_{i}$ such that:
(1) Every $\rho_i$ is of the form $\pi_i(X) \to G(k)$, with $k$ a non-Archimedean local field and $G$ an absolutely simple $k$-algebraic group, and $\rho_i$ has Zariski-dense and unbounded image;

(2) If $X \to S_X^{ab}$ denote the Katzarkov-Zuo reduction of $\rho_i$, then the restriction of $L$ to the normalization of any fiber of $X \to \prod_i S_X^{ab}$ underlies a polarized variation of pure Hodge structures with discrete monodromy.

Let $f : X \to Y$ denote the Stein factorization of the morphism $X \to \prod_i S_X^{ab}$. Note that for every $i$ the induced morphism $Y \to S_X^{ab}$ is surjective, since its precomposition $X \to S_X^{ab}$ with the canonical morphism $X \to Y$ is surjective. Thanks to Theorem 6.6 and Proposition 6.2, $\text{Sp}_{ab}(S_X^{ab}) \neq S_X^{ab}$ for every $i \in I$. Using Proposition 2.6, it follows that $\text{Sp}_{ab}(Y) \neq Y$.

Therefore, in view of Proposition 2.3 it is sufficient to prove that $\text{Sp}_{ab}(X/Y)$ is not Zariski-dense in $X$. Since $X$ is normal, there exists a Zariski-dense open $Y^o$ of $Y$ over which the (geometric) fibers of $f$ are normal. Moreover, by construction, any such fiber $X_y$ admits a large complex local system that underlies a polarized variation of pure Hodge structures with discrete monodromy. It follows from Proposition 3.12 that $\text{Sp}_{ab}(X_y) = \emptyset$ for any $y \in Y^o$. This proves that $\text{Sp}_{ab}(X/Y)$ is contained in $f^{-1}(Y\setminus Y^o)$, hence it is not Zariski-dense in $X$. This concludes the proof.

6.2.2. The reductive case. Assume that the algebraic monodromy group $G$ is reductive. Thanks to Proposition 3.15 replacing $X$ with a finite étale cover, we can assume that there exists a surjective morphism with connected fibers $f : X \to Y$ onto a normal irreducible projective complex algebraic variety $Y$ such that:

- $Y$ admits a large complex local system with torsion-free monodromy and a semisimple algebraic monodromy group;
- for $y$ in a Zariski-dense open subset of $Y$, the fiber $X_y$ admits a finite morphism to an abelian variety.

Let $X' \to X$ be a projective desingularization of $X$. Thanks to Proposition 2.3 it is equivalent to prove that $\text{Sp}_{ab}(X')$ is not Zariski-dense in $X'$. The composition $X' \to X \to Y$ is still a fibration, and there exists a Zariski-dense open subset $Y^o$ of $Y$ such that $X'_y$ is a desingularization of $X_y$ for every $y \in Y^o$. In particular, $X'_y$ has maximal Albanese dimension (since $X'_y$ admits a finite morphism to an abelian variety) and is of general type (since $X'$ is of general type by assumption).

The set $\text{Sp}_{ab}(Y)$ is not Zariski-dense in $Y$ thanks to the preceding section. Therefore, thanks to Proposition 2.3 it is sufficient to prove that $\text{Sp}_{ab}(X'/Y)$ is not Zariski-dense in $X'$. Since one can freely shrink $Y$ to make the morphism $X' \to Y$ smooth, this is a consequence of the following result.

**Theorem 6.6 (cf. [Brs16]).** Let $f : X \to Y$ be a smooth projective surjective morphism with connected fibers between smooth complex algebraic varieties. Assume that the fibers of $f$ over a Zariski-dense open subset of $Y$ are of general type and of maximal Albanese dimension. Then $\text{Sp}_{ab}(X/Y)$ is not Zariski-dense in $X$.

6.2.3. The general case. For the general case, we can argue exactly as in the reductive case, making use Proposition 3.14 instead of Proposition 3.15.

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