**Abstract.** A binary matrix $M$ has the Consecutive Ones Property (COP) if there exists a permutation of columns that arranges the ones consecutively in all the rows. Given a matrix, the $d$-COS-R problem is to determine if there exists a set of at most $d$ rows whose deletion results in a matrix with COP. We consider the parameterized complexity of this problem with respect to the number $d$ of rows to be deleted as the parameter. The closely related Interval Deletion problem has recently shown to be FPT \[22\]. In this work, we describe a recursive depth-bounded search tree algorithm in which the problems at the leaf-level are solved as instances of Interval Deletion. The running time of the algorithm is dominated by the running time of Interval Deletion, and therefore we show that $d$-COS-R is fixed-parameter tractable and has a run-time of $O^*(10^d)$.

1 Introduction

Testing COP for binary matrices is a classical algorithmic problem. COP testing has applications in physical mapping of DNA [5] and in recognizing interval graphs, planar graphs and Hamiltonian cubic graphs \[9,19\]. There are many linear-time algorithms known in the literature for COP testing \[9,14,21,20,7,4\]. There are many combinatorial properties of matrices with COP. They are known to be totally unimodular, and there are results connecting matrices with COP and Intersection Cardinality Preserving Interval assignments \[3,10\]. Further, the classical NP-hard problems, integer linear programming (ILP) and set cover, are polynomial-time solvable, when the associated binary matrix has COP \[12\]. In this paper our focus is on matrices that do not have COP, and we address the natural optimization problem to find a minimum set of rows whose deletion results in a submatrix with COP. The corresponding decision problem, referred to as $d$-COS-R, is known to be NP-complete \[15\] and is well-studied in the parameterized complexity framework \[13\]. A parameterized problem is said to be fixed-parameter tractable (FPT) with respect to $d$ as the parameter if there is an algorithm with run-time $O^*(f(d))$, where $f$ is a computable function depending only on $d$. For details on parameterized complexity, we refer the reader to \[17,18\]. In this paper, we consider the parameterized complexity of $d$-COS-R defined as follows:

| $d$-COS-R |
|---|
| **Instance:** $(M, d)$ - A binary matrix $M_{m \times n}$ and an integer $d \geq 1$. |
| **Parameter:** $d$ |
| **Question:** Does there exist a set of at most $d$ rows of $M$ whose deletion results in a matrix with COP? |

The problems of deleting a minimum number of rows or columns to transform a given matrix into a matrix with COP are called Min-COS-R and Min-COS-C, respectively. These two problems are known to be NP-hard even on very sparse matrices, containing only two 1-entries per row and at most three 1-entries per column \[8\]. These minimization and the corresponding maximization versions have been studied \[13\]. Min-COS-R and Min-COS-C are fixed-parameter tractable on matrices that have only two ones either per row or per column. In this work we focus only on the decision version of Min-COS-R which is the $d$-COS-R problem. On restricted classes of matrices, $d$-COS-R is known to be FPT \[13\]. These FPT algorithms are based on a refinement of the forbidden submatrix characterization of matrices with COP \[11\]. To the best of our knowledge, the parameterized complexity of $d$-COS-R on general binary matrices is still open. In this work, we show that $d$-COS-R admits an algorithm with run-time $O^*(10^d)$. Our result is obtained by a recursive branching algorithm in which the leaf instances are that of Interval-Deletion (defined below). Then, we employ the recent $O^*(10^d)$ algorithm for Interval Deletion \[22\] to solve $d$-COS-R. Thus, we answer the

\[1\] $O^*$ notation ignores polynomial terms.
natural open question on the parameterized complexity of $d$-COS-R by showing that it is FPT on all binary matrices. This is a significant advancement over the current knowledge on this problem, where current FPT results [13] are known only when there are bounds on the number of 1s in the rows or columns.

### Interval Deletion

**Instance:** $(G, d)$ - A graph $G$ and an integer $d \geq 1$.

**Parameter:** $d$

**Question:** Does $G$ have a set $V'$ of at most $d$ vertices such that $G \setminus V'$ is an interval graph?

**Our Approach:** A natural approach towards obtaining submatrices with COP is to identify the known classes of forbidden configurations [1], and to remove them by eliminating appropriate rows. While this is the broad approach in [13], we look at the well known fact that a graph is an interval graph if and only if its clique matrix (formally defined later) has COP [3][11]. We consider the question of how to convert a given 0-1 matrix into the clique matrix of some graph, and then attempt an interval deletion on that graph. From [11], a natural graph that can be associated with a binary matrix is a derived graph. Informally, the columns of the matrix correspond to cliques in the derived graph. However, a derived graph may have many other spurious cliques, and these cliques are the hinderances towards getting a clique matrix. Our first branching rule motivated by the Helly property, that must be satisfied by any set of intervals, ensures that these spurious cliques are localized to the derived graph associated with a pair of columns in the given matrix. We then design a second branching rule, based on induced 4-cycles, to ensure that the number of these spurious cliques is a polynomial in the input size, and they can be enumerated in polynomial time. Then with a third set of branching rules we eliminate these spurious cliques, the result being a matrix in which the maximal cliques of the derived graph are associated with some column of the matrix. We then consider an augmented matrix which becomes the clique matrix of a graph. We then show that Interval Deletion on this graph ensures that the augmented matrix has COP, which directly gives a submatrix with COP for the given matrix. All these branching rules, along with the recent FPT algorithm [22] for Interval Deletion are shown to solve the $d$-COS-R problem in FPT time.

## 2 COP, Intervals, and Clique-Matrices

In this section, we present the necessary structural results to describe our algorithm and the proofs of correctness. Some of the lemmas are cited from the appropriate papers, and some are proved by us. Graph theoretic definitions and notations are as per [2][11].

Throughout this paper we consider only binary matrices. For an $m \times n$ matrix $M$, let $R(M) = \{r_1, \ldots, r_m\}$ and $C(M) = \{c_1, \ldots, c_n\}$ denote the sets of rows and columns, respectively. The $(i, j)^{th}$ entry in $M$ is denoted as $M_{ij}$. For a subset $D \subseteq R(M)$ of rows, the submatrix induced on $D$ and $R(M) \setminus D$ are denoted by $M[D]$ and $M \setminus D$, respectively.

The derived graph associated with a 0-1 matrix $M$, defined in [11], is $G(M) = (V, E)$ is defined as $V = \{v_i \mid r_i \in R(M)\}$ and $E = \{\{v_i, v_j\} \mid \exists c_k \in C(M), M_{ik} = M_{jk} = 1\}$. In other words, $G(M)$ is obtained from $M$ by visualizing each column as a clique involving the vertices (corresponding to rows) which have a 1 in that column. For a column $c_k$ in $M$, the support of $c_k$, denoted by $supp(c_k)$, is defined as the set $\{r_i \in R(M) \mid M_{ik} = 1\}$. Also, for $c_k$, the set of vertices in $G(M)$ corresponding to the rows in $supp(c_k)$ is defined as $vert(c_k) = \{v_i \mid r_i \in supp(c_k)\}$.

### 2.1 Matrices with COP, Interval Assignments, and Interval Graphs

A graph is called an interval graph if its vertices can be assigned intervals such that two vertices are adjacent if and only if their corresponding intervals have nonempty intersection. Let $G$ be a graph on the vertex set $\{v_1, \cdots, v_n\}$ and let $\{Q_1, \cdots, Q_l\}$ be the set of maximal cliques in $G$. The clique matrix $M$ of $G$ is the matrix whose rows and columns correspond to the vertices and the maximal cliques, respectively, in $G$. The entry $M_{ij} = 1$ if the vertex $v_i$ is in the clique $Q_j$ and it is 0 otherwise. The following characterization relates COP and interval graphs.

**Theorem 1.** [3] A graph is an interval graph if and only if its clique matrix has COP.
Theorem 3. \([11]\) A graph \(G\) is an interval graph if and only if \(G\) has no induced cycle of length 4 and \(\overline{G}\) is a comparability graph.

Here we set up the framework to argue the correctness of our branching rules. An \(m \times n\) matrix \(M\) can be represented as a set system \((U, S(M))\) with \(S(M) = \{S_1, \ldots, S_m\}\) being a collection of subsets of \(U = \{1, \ldots, n\}\) where \(S_i = \{j \mid M_{ij} = 1\}\). A family of subsets is said to have the Helly property if every subfamily of it, formed by pairwise intersecting subsets, contains a common element \([10]\). An interval \(I_j\), denoted by \([i, k]\), is the ordered set of consecutive integers from \(i\) to \(k\). An interval assignment \(I\) to a set system \((U, S)\) is an assignment of an interval \(I_i\) to each \(S_i \in S\). An Intersection Cardinality Preserving Interval Assignment (ICPIA) to \(S\) is an interval assignment \(I\) that satisfies \(|S_i \cap S_j| = |I_i \cap I_j|\) for every pair \(S_i\) and \(S_j\) of elements in \(S\). A main property of the ICPIA, shown in \([3,16]\), is that for any collection of sets \(\{S_{i_1}, \ldots, S_{i_r}\}\),

\[
\bigcap_{j=1}^r S_{i_j} = \bigcap_{j=1}^r I_{i_j}.
\]

Theorem 3. \([3,16]\) A matrix \(M\) has COP if and only if \(S(M)\) has an ICPIA. Further, if \(I\) is an ICPIA for \(S(M)\), then for any collection of sets \(\{S_{i_1}, \ldots, S_{i_r}\} \subseteq S(M)\),

\[
\bigcap_{j=1}^r S_{i_j} = \bigcap_{j=1}^r I_{i_j}.
\]

We prove a key lemma that is necessary for the first rule in our branching algorithm.

Lemma 1. If \(M\) has COP then \(S(M)\) satisfies the Helly Property. Further, for every triple of pairwise intersecting sets in \(S(M)\), one of the sets is contained in the union of the other two.

Proof. Since \(M\) has COP, let \(M'\) be the column permuted matrix obtained from \(M\) which has consecutive ones in the rows. For each \(1 \leq i \leq m\), let \(I_i\) be the natural interval assigned to \(S_i\), obtained from \(M'\). Let \(I = \{I_1, \ldots, I_m\}\) be this interval assignment. From Theorem \([8]\), \(I\) is an ICPIA for \(S(M)\). Therefore, if there exists three sets \(S_1, S_2, S_3\) that violate the Helly property- we first observe that for each pair of them, say \(S_i\) and \(S_j\), \(|S_i \cap S_j| = |I_i \cap I_j| > 0\). Since intervals satisfy the Helly property, it follows that the 3 intervals have a common point. We now conclude that \(0 = |S_1 \cap S_2 \cap S_3| = |I_1 \cap I_2 \cap I_3| > 0\). The first equality comes from our hypothesis that the 3 sets violate Helly property, the second equality follows from Theorem \([3]\) and the third inequality follows from the fact that the 3 intervals share a common point, as intervals respect Helly Property. This is a contradiction to our premise that \(S_1, S_2, S_3\) violate the Helly Property, which is now shown to be false. To prove the second part of the lemma, let \(S_1, S_2, S_3\) be pairwise intersection sets. Then, we know that in the corresponding intervals \(I_1, I_2, I_3\), one of them is contained in the union of the other two, say \(I_3\) is contained in \(I_1 \cup I_2\). Since \(I\) is an ICPIA, it follows that \(S_3 \subseteq S_1 \cup S_2\). Hence the lemma. \(\square\)

2.2 Matrices with COP and Clique-Matrices of Derived Graphs

For \(M\), the \((n + m) \times n\) matrix \(\hat{M}\) is defined as \((I_M^{-1})\) where \(I\) is the \(n \times n\) identity matrix. The main reason for considering \(\hat{M}\) is that in \(G(\hat{M})\), each column corresponds to a maximal clique. This may not necessarily be the case in \(G(M)\). We first observe that \(M\) and \(\hat{M}\) behave the same with respect to COP, and the proof of this observation is very easy based on the fact that \(\hat{M}\) is obtained from \(M\) by padding an identity matrix.

Observation 4 \(\hat{M}\) has COP if and only if \(M\) has COP.

Corollary 1. Let \(D \subseteq R(M)\). Then, \(M \setminus D\) has COP if and only if \(\hat{M} \setminus D\) has COP.

Lemma 2. If \(M\) has COP, then \(G(M)\) is an interval graph. Further, for every maximal clique \(Q\) in \(G(M)\) there exists a column \(c_k\) in \(M\) such that \(\text{vert}(c_k) = Q\).

Proof. Consider the columns of \(M\) in the order of a permutation \(\sigma\) that results in COP. Now, for every vertex \(v_j\) in \(G(M)\) assign the interval \(I_j = [j, k]\) where \(j\) and \(k\) are the minimum and maximum column indices, respectively, with \(M_{ij} = M_{jk} = 1\). Consider two vertices \(v_a\) and \(v_b\) in \(G(M)\). Let \(I_a = [j_1, k_1]\) and \(I_b = [j_2, k_2]\) be the intervals assigned to \(v_a\) and \(v_b\) respectively. Now, by the definition of derived graphs, \(v_a\) and \(v_b\) are adjacent if and only if there is a column \(c_r\) \((\min\{j_1, j_2\} \leq r \leq \min\{k_1, k_2\})\) in \(M\) with \(M_{ar} = M_{br} = 1\). The existence of such a column \(c_r\) is well-defined if and only if \(I_a \cap I_b \neq \emptyset\). Therefore, \(v_a\)
and \( v_b \) are adjacent in \( G(M) \) if and only if \( I_a \cap I_b \neq \emptyset \). Thus, \( G(M) \) is an interval graph. We now prove the second part of the lemma. Let \( Q = \{ v_1, \ldots, v_q \} \) be a maximal clique in \( G(M) \). Consider the submatrix \( M' \) with \( R(M') = \{ r_i \in R(M) \mid v_i \in Q \} \). Recall that \( S(M') = \{ S_i \mid r_i \in R(M') \} \). Any two \( S_i, S_j \in S(M') \) have a non-empty intersection, and therefore, from Lemma 1, if follows that \( \bigcap_{i=1}^{q} S_i \neq \emptyset \). Let \( k \) be an element in \( \bigcap_{i=1}^{q} S_i \), then it follows that \( \text{vert}(c_k) = Q \). Note that \( \text{vert}(c_k) = Q \) because \( Q \) is a maximal clique. Hence the lemma.

**Corollary 2.** If \( M \) has COP, then \( G(\tilde{M}) \) is an interval graph, and \( \tilde{M} \) is the clique matrix of \( G(M) \).

**Proof.** From Observation 4, \( M \) has COP implies that \( \tilde{M} \) has COP, and from Lemma 2, it follows that \( G(\tilde{M}) \) is an interval graph, and that each maximal clique corresponds to a column in \( \tilde{M} \). Now in \( \tilde{M} \), each column has a distinguishing entry where there is a 1, and all other entries in that row are zero. This shows that each column corresponds to a maximal clique in \( G(\tilde{M}) \). Therefore, \( \tilde{M} \) is the clique matrix of \( G(\tilde{M}) \). \qed

### 3 \( d \)-COS-R via Interval Deletion

The basic idea in this algorithm is that we transform the given instance \((M, d)\) of \( d \)-COS-R to an instance \((M', d')\) where \( M' \) has the additional property that \( \tilde{M}' \) is the clique matrix of a graph \( G(M') \). Our recursive algorithm explores a recursion tree in which each leaf corresponds to an interval deletion problem.

**Algorithm COS-R**

**Input:** An instance \( I = (M_{m \times n}, d) \) where \( M \) is a binary matrix and \( d \geq 1 \).

**Output:** Return a set \( D \) of at most \( d \) rows (if one exists) such that \( M \setminus D \) has COP.

(Step 0) If \( M \) has COP and \( d \geq 0 \) then Return \( D \).

(Step 1) If \( d < 0 \) then Return ‘NO’ /* parameter budget exhausted */

(Step 2) (Branching Rule 1) If there exists three pairwise intersecting sets \( S_1, S_2, S_3 \in S(M) \) satisfying either of the following properties:

(H1) \( S_1 \cap S_2 \cap S_3 = \emptyset \).

(H2) None of \( S_1, S_2 \) and \( S_3 \) is contained in the union of the other two.

then branch into 3 instances \( I_i = (M_i, d_i) \) (where \( i \in \{1, 2, 3\} \))

Set \( D_i \leftarrow D \cup \{ r_i \} \) and \( M_i \leftarrow M \setminus \{ r_i \} \)

Update \( d_i \leftarrow d - 1 \) /* Parameter drops by 1 */

For some \( i \in \{1, 2, 3\} \), if \( \text{COS-R}(M_i, D_i, d_i) \) returns a solution \( D_i \), then Return \( D_i \), else Return ‘NO’

/* Invariant: See Lemma 4 and Corollary 3 */

(Step 3) (Branching Rule 2) If there exists two columns \( c_i \) and \( c_j \) in \( M \) such that \( G[\text{vert}(c_i) \cup \text{vert}(c_j)] \) has an induced cycle \( C = \{v_1, v_2, v_3, v_4\} \),

then branch into 4 instances \( I_i = (M_i, d_i) \) (where \( i \in \{1, 2, 3, 4\} \))

Set \( D_i \leftarrow D \cup \{ r_i \} \) and \( M_i \leftarrow M \setminus \{ r_i \} \)

Update \( d_i \leftarrow d - 1 \) /* Parameter drops by 1 */

For some \( i \in \{1, 2, 3, 4\} \), if \( \text{COS-R}(M_i, D_i, d_i) \) returns a solution \( D_i \), then Return \( D_i \), else Return ‘NO’

/* Invariant: See Lemma 4 */

(Step 4) (Branching Rule 3) If there is a maximal clique \( Q \) such that there does not exist a column \( c_l \) such that \( \text{vert}(c_l) = Q \) then, let \( Q' \) be a minimal subset of \( Q \) with the property that there is no column \( c_l \) such that \( Q \subseteq \text{vert}(c_l) \).

/* \( Q' \) is well-defined as it is a subset of \( Q \) and \( Q \) itself is in two columns */

Let \( v_1, v_2, v_3 \) be vertices in \( Q' \), and let the corresponding rows be \( r_1, r_2, r_3 \) respectively.

then branch into 3 instances \( I_i = (M_i, d_i) \) (where \( i \in \{1, 2, 3\} \))

Set \( D_i \leftarrow D \cup \{ r_i \} \) and \( M_i \leftarrow M \setminus \{ r_i \} \)

Update \( d_i \leftarrow d - 1 \) /* Parameter drops by 1 */

For some \( i \in \{1, 2, 3\} \), if \( \text{COS-R}(M_i, D_i, d_i) \) returns a solution \( D_i \), then Return \( D_i \), else Return ‘NO’

/* Invariant: See Lemma 4 */

(Step 5) (Interval Deletion) \( V' = \text{Interval-Deletion}(G(\tilde{M}), d) \).

(Step 6) If \( \text{Interval-Deletion} \) returns ‘NO’ then Return.

Otherwise, Return the set \( D = D \cup \{ r_i \in R(M) \mid v_i \in V' \} \).
At each leaf in the recursion tree, an interval deletion problem is solved. Each node in the recursion tree has at most 4 subproblems, and therefore, the tree has at most $4^d$ leaves, and then using the recent FPT algorithm for Interval Deletion [22], we get an overall running time of $O^*(10^d)$ for our algorithm. Recall that, for a matrix $M$, the derived graph is denoted by $G(M)$ and its system set is denoted by $S(M) = \{S_1, \ldots, S_m\}$. The recursive function COS-R is called initially with the input matrix $M$, the initial solution set $D = \emptyset$ and the parameter $d$ as inputs. It either returns a set $D$ of at most $d$ rows such that $M \setminus D$ has COP or returns ’NO’. COS-R makes a call to the function Interval-Deletion($G, d$) which either returns a set of vertices $X$ such that $|X| \leq d$, and $G \setminus X$ is an interval graph or returns ’NO’.

Correctness of the Algorithm: We prove the correctness of the algorithm by proving invariants that hold at the end of each branching rule.

Lemma 3. Let $M$ be a matrix for which branching rule 1 applies, and sets $S_1, S_2, S_3$ violate at least one of the two conditions checked in rule 1. Then, any solution $D$ of d-COS-R includes at least one of the corresponding rows $r_1, r_2, r_3$.

Proof. The proof follows from Lemma 1.

Branching Rule 1: To understand the effect of Branching Rule 1, consider this example of the matrices $M_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$, both do not have COP. In $M_1$ and $M_2$, the sets corresponding to the rows are pairwise intersecting. However, in $M_1$ the sets do not have a common element while in $M_2$, none of them is contained in the union of other two. The following lemma formalizes the crucial property satisfied by matrices for which branching rule 1 is not applicable.

Lemma 4. Let $M$ be a matrix on which branching rule 1 is not applicable. Then, for every maximal clique $Q$ in $G(M)$, there are at most two columns $c_i$ and $c_j$ such that $Q \subseteq \text{vert}(c_i) \cup \text{vert}(c_j)$.

Proof. Assume on the contrary that $Q$ is a maximal clique in $G(M)$ and $T$ is a minimum set of columns such that $Q \subseteq \bigcup_{c_i \in T} \text{vert}(c_i)$ with $|T| \geq 3$. Consider the submatrix $N$ with $R(N) = \{r_i \in R(M) \mid v_i \in Q\}$. Consider any 3 columns $c_1, c_2, c_3$ from $T$. Since $T$ is a minimum set of columns whose vertices contain $Q$ in $G(M)$, it follows that there are 3 vertices $v_1, v_2, v_3 \in Q$ such that the corresponding rows along with the columns $c_1, c_2, c_3$ form an identity submatrix which can be visualized as $\begin{pmatrix} \vdots & 1 & \vdots & \vdots & 1 \vdots \\ \vdots & 0 & \vdots & \vdots & 0 \vdots \\ \vdots & 0 & \vdots & \vdots & 0 \vdots \end{pmatrix}$. Thus each of the sets $S_1, S_2$ and $S_3$, corresponding to $r_1, r_2$, and $r_3$, has an element that is not present in the other two. Therefore, none of $S_1, S_2$ and $S_3$ is contained in the union of the other two, therefore branching rule 1 would have been applied. This is a contradiction to the hypothesis in the lemma that branching rule 1 is not applicable.

Corollary 3. Every maximal clique in $G(M)$ is a maximal clique in $G[\text{vert}(c_i) \cup \text{vert}(c_j)]$ for some pair of columns $c_i, c_j$ in $M$.

An example is shown in Figure 1. The maximal cliques $Q_1$ and $Q_2$ in $G(M)$ are such that $Q_1 = \{v_1, v_2, v_4, v_5, v_6\} = \text{vert}(c_1) \cup \text{vert}(c_5)$ and $Q_2 = \{v_2, v_3, v_4, v_5, v_6\} = \text{vert}(c_3) \cup \text{vert}(c_4)$. It is also clear from the figure that no five clique is present in a column.

![Fig. 1. Maximal cliques $Q_1$ and $Q_2$ in $G(M)$](image-url)
**Branching Rule 2:** Let $M$ be a matrix on which branching rule 1 is not applicable. Now for each maximal clique $Q$ in $G(M)$ for which there does not exist a column $c_k$ in $M$ such that $Q = \text{vert}(c_k)$, we need to branch according to Lemma 5. An important question here is that how do we check if there are such cliques. From Corollary 3, we know that the maximal cliques of $Q$ can be enumerated by enumerating the maximal cliques of $G[\text{vert}(c_i) \cup \text{vert}(c_j)]$ for each pair of columns $c_i$ and $c_j$ in $M$. However, there could be an exponential number of maximal cliques in $G[\text{vert}(c_i) \cup \text{vert}(c_j)]$. We handle this problem by checking if $G[\text{vert}(c_i) \cup \text{vert}(c_j)]$ is non-chordal. If it is chordal, then there are only a polynomial number of maximal cliques, and it is easy to enumerate each maximal clique, and branch as suggested in Lemma 2. If $G[\text{vert}(c_i) \cup \text{vert}(c_j)]$ is not chordal, there is a chordless cycle of length more than 3, and we show in the following lemma that such a chordless cycle can only be of length 4. However, from Lemma 2 and Theorem 2, it follows that any induced cycle of length 4 is forbidden in $G(M)$. Therefore, by our branching rule 2, we guarantee that $G[\text{vert}(c_i) \cup \text{vert}(c_j)]$ does not have an induced cycle of length 4. We now show in the following two lemmas that this guarantees that $G[\text{vert}(c_i) \cup \text{vert}(c_j)]$ is chordal.

**Lemma 5.** Let $c_p$ and $c_q$ be two columns in $M$ on which branching rule 1 is not applicable. Then, every induced cycle in $G' = G[\text{vert}(c_p) \cup \text{vert}(c_q)]$ is of length at most 4.

**Proof.** Any induced cycle in $G'$ can have at most two vertices from $\text{vert}(c_p)$ and $\text{vert}(c_q)$ each, as they both induce cliques in $G'$. Therefore, any induced cycle can be of length at most 4.

**Lemma 6.** Let $M$ be a matrix on which branching rule 1 is not applicable. For two columns $c_p$ and $c_q$ in $M$, let $C$ be an induced cycle of four vertices such that $C$ is in $G[\text{vert}(c_p) \cup \text{vert}(c_q)]$. Then, any solution $D$ of $d$-COS-R must include at least one of the four rows corresponding to the four vertices in $C$.

**Proof.** Let $D$ be a solution and let $M' = M \setminus D$. If $C$ is in $G(M')$, then it means that $G[M']$, an interval graph by Lemma 2, has an induced cycle of length 4, which is a contradiction to Theorem 2 which characterizes interval graphs as a subclass of graphs without induced cycles on 4 vertices. Hence the lemma.

**Lemma 7.** Let $M$ be a matrix on which branching rule 1 and branching rule 2 are not applicable. Then the following are true:

1. For each maximal clique $Q$ in $G(M)$, there exists at most two columns $c_p$ and $c_q$ in $M$ such that $Q \subseteq \text{vert}(c_p) \cup \text{vert}(c_q)$
2. For each pair of columns $c_p$ and $c_q$ in $M$, $G[\text{vert}(c_p) \cup \text{vert}(c_q)]$ is chordal.

**Proof.** If either of the two conditions are not true, then it would contradict the premise that the two branching rules are not applicable.

**Branching Rule 3:** After applying branching rule 2, $G[\text{vert}(c_p) \cup \text{vert}(c_q)]$ is chordal for each pair of columns $c_p, c_q$ in $M$. It is known that the maximal cliques of a chordal graph can be enumerated in linear time [11]. So, we enumerate the maximal cliques of $G[\text{vert}(c_i) \cup \text{vert}(c_j)]$ for each pair of columns $c_i$ and $c_j$ in $M$. From Corollary 3, this enumeration is guaranteed to list all the maximal cliques of $G(M)$. For each maximal clique $Q$ in this enumeration, if there is no column $c_k$ such that $\text{vert}(c_k) = Q$, then we identify two columns $c_p$ and $c_q$ such that $Q \subseteq \text{vert}(c_p) \cup \text{vert}(c_q)$, and apply branching rule 3. The following lemma proves that Branching Rule 3 is necessary.

**Lemma 8.** Let $M$ be a matrix on which branching rule 1 is not applicable. Let $Q$ be a maximal clique in $G(M)$ such that there is no column $c_i$ such that $\text{vert}(c_i) = Q$. Let $Q'$ be a minimal subset of $Q$ that has no column $c_i$ such that $Q' \subseteq \text{vert}(c_i)$. Let $v_1, v_2$, and $v_3$ be any three vertices in $Q'$, and let $r_1, r_2$, and $r_3$ respectively be the corresponding rows. Then, any solution $D$ of $d$-COS-R must include at least one of $r_1$, $r_2$, and $r_3$.

**Proof.** We prove this by contradiction. Suppose there exists a solution $D$ that contains none of $r_1$, $r_2$, and $r_3$. Let $M' = M \setminus D$ be the matrix with COP. Since there is no column $c_i$ in $M$ such that $Q' \subseteq \text{vert}(c_i)$ and $Q'$ is an inclusion minimal with this property, it follows that there exist distinct columns $c_1, c_2$, and $c_3$ such that $Q' \setminus \{v_i\} \subseteq \text{vert}(c_i)$ for each $i \in \{1, 2, 3\}$. Now, it follows that the rows $r_1$, $r_2$, and $r_3$ along with the columns $c_1, c_2, c_3$ form a submatrix of $M'$ which can be visualized as $\begin{pmatrix} \vdots & 0 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & 1 & \vdots & \vdots \end{pmatrix}$. This submatrix is forbidden for any matrix with COP [11]. This is a contradiction to the fact that $M'$ has COP. Therefore, our assumption is wrong, and hence the lemma is proved.
Let $M$ be a matrix for which branching rule 1, branching rule 2, and branching rule 3 are not applicable. Then, for each maximal clique $Q$ in $G(M)$, there exists a column $c_p$ such that $Q = \text{vert}(c_p)$. Further, $\tilde{M}$ is the clique matrix of $G(\tilde{M})$.

Proof. The proof of this lemma, too, is by contradiction. If $Q$ is a maximal clique such that there is no column $c_p$ such that $\text{vert}(c_p) = Q$, then since branching rule 1 and branching rule 2 are not applicable, by Lemma 7, it follows that there exist columns $c_p$ and $c_q$ such that $Q \subseteq \text{vert}(c_p) \cup \text{vert}(c_q)$. This implies that branching rule 3 is applicable for $M$, and this contradicts the premise of the lemma. Therefore, our assumption is wrong, and the first part of the lemma is proved. To prove the second part of the lemma—Any column $c_k$ whose vertices $\text{vert}(c_k)$ is not a maximal clique in $G(M)$ becomes a maximal clique in $G(\tilde{M})$. This is because $G(\tilde{M})$ can be viewed as a graph obtained from $G(M)$ by adding a new vertex for each clique $\text{vert}(c_k)$, $1 \leq k \leq n$, and making this vertex adjacent to all the vertices in $\text{vert}(c_k)$. Further, if $\text{vert}(c_k)$ is a maximal clique in $G(M)$, then in $G(\tilde{M})$, $\text{vert}(c_k)$ is a maximal clique with one additional vertex. This completes the proof that $\tilde{M}$ is the clique matrix of $G(\tilde{M})$. Hence the lemma.

Now we show that, solving $d$-COS-R on $\tilde{M}$ is equivalent to solving the Interval-Deletion problem on the graph $G(\tilde{M})$.

Theorem 5. Let $\tilde{M}$ be the clique matrix of $G(\tilde{M})$. Given $\tilde{M}$ and integer $d \geq 0$, there exists a set of rows $\mathcal{D}$ such that $|\mathcal{D}| \leq d$ and $\tilde{M} \setminus \mathcal{D}$ has COP if and only if $G(\tilde{M})$ has a set of vertices $V'$ such that $|V'| \leq d$ and $G(\tilde{M}) \setminus V'$ is an interval graph.

Proof. Let $\mathcal{D}$ be a set of rows in $\tilde{M}$, and let $V'$ be the corresponding vertices in $G(\tilde{M})$. From Lemma 2, it follows that $\tilde{M} \setminus \mathcal{D}$ has COP implies $G(\tilde{M} \setminus \mathcal{D})$ is an interval graph. Further, $G(\tilde{M} \setminus \mathcal{D})$ is basically the graph obtained by removing $V'$ from $G(\tilde{M})$. This completes the forward direction of the claim. In the reverse direction, let $V'$ be a minimal set of vertices such that $G(\tilde{M} \setminus \mathcal{D})$ is an interval graph. Due to the minimality of $V'$, observe that the vertices in $G(\tilde{M})$ which correspond to the rows of the identity matrix added to $M$ are not elements of $V'$. Let $\mathcal{D}$ be the set of rows in $\tilde{M}$ corresponding to $V'$. Note that the columns of $\tilde{M} \setminus \mathcal{D}$ are exactly the maximal cliques of $G(\tilde{M}) \setminus V'$. Therefore, $\tilde{M} \setminus \mathcal{D}$ is the clique matrix of $G(\tilde{M}) \setminus V'$. Since $G(\tilde{M}) \setminus V'$ is an interval graph, it follows from Theorem 1 that $\tilde{M} \setminus \mathcal{D}$ has COP. Hence the theorem is proved.

We now show that, the recursive function $d$-COS-R correctly decides whether a given matrix $M$ has a set of at most $d$-rows whose removal results in a matrix with COP.

Theorem 6. Given an instance $(M, d)$ of $d$-COS-R, the function call $\text{COS-R}(M, \emptyset, d)$ correctly decides in $O^*(10^d)$ time if there exists a set $\mathcal{D}$ of at most $d$-rows such that $M \setminus \mathcal{D}$ has COP.

Proof. Let $\mathcal{D}$ be a solution of size at most $d$. From the Lemma 3, Lemma 6, and Lemma 8 in each recursive subproblem, one of the rows to be added in the solution is an element of $\mathcal{D}$. Let $(M', d')$ be the instance of $d$-COS-R at a leaf node in the recursion, where this leaf node is one at which none of the first three branching rules apply, and each of recursive choices of rows to be added into the solution, in the computation starting at $\text{COS-R}(M, \emptyset, d)$ is selected from $\mathcal{D}$. Let $\mathcal{D}' \subseteq \mathcal{D}$ be the set of rows that have been added to the solution in recursive calls up to the leaf node at which $(M', d')$ is an instance of $d$-COS-R, and let $\mathcal{D}'' = \mathcal{D} \setminus \mathcal{D}'$. From Corollary 1, $M' \setminus \mathcal{D}''$ has COP if and only if $\tilde{M}' \setminus \mathcal{D}''$ has COP. Further, from Lemma 9, $\tilde{M}'$ is the clique matrix of $G(\tilde{M}')$ at the leaf node in the recursion tree. Therefore, it follows that $\tilde{M}' \setminus \mathcal{D}''$ has COP, and from Theorem 5, that $G(\tilde{M}') \setminus V'$ is an interval graph. Therefore, from [22], it follows that Interval-Deletion$(G(\tilde{M}'), d')$ will return a set of at most $d'$ vertices whose removal from $G(\tilde{M}')$ guarantees that the resulting graph is an interval graph. This proves that if there is a solution $\mathcal{D}$ to $(M, d)$, then $\text{COS-R}(M, \emptyset, d)$ will return a solution of size at most $d$. It is also clear that if there is no solution $\mathcal{D}$ of size at most $d$, the algorithm will not find one.

In each of the recursive subproblems generated by branching rules 1, 2, and 3, the parameter reduces by at least 1. Further, in each level of recursion, at most four recursive calls are made, in branching rules 1, 2, and 3. Therefore, in the recursion tree obtained by performing the 3 branching rules, there are at most $4^{d-d'}$ leaves at depth $d - d'$. At a leaf node, in which the problem is $(M', d')$, Interval-Deletion returns an answer in at most $O^*(10^d)$ time. Further, the checks made at each level of recursion takes only polynomial time. Therefore, this bounds the total running time of the algorithm by $O^*(10^d)$. Hence the theorem.
4 Concluding Remarks

Using our algorithm for \(d\)-COS-R, we observe that the Convex Bipartite Deletion problem is FPT. Let \(G = (V_1, V_2, E)\) be a bipartite graph with \(V_1 = \{x_1, \ldots, x_m\}\) and \(V_2 = \{y_1, \ldots, y_m\}\). Let \(M\) be the half adjacency matrix of \(G\). That is, \(M_{ij} = 1\) if and only if \(\{x_i, y_j\} \in E\). \(G\) is convex bipartite graph if and only if \(M\) has COP \([1,12]\). The Convex Bipartite Deletion problem is defined as follows.

| Convex Bipartite Deletion |
|----------------------------|
| **Input:** A bipartite graph \(G = (V_1, V_2, E)\), \(|V_1| = m\), \(|V_2| = n\) and \(d \geq 1\) |
| **Parameter:** \(d\) |
| **Question:** Does there exist a set \(D \subset V_1\) with \(|D| \leq d\) such that \(G[V_1 \setminus D, V_2]\) is a convex bipartite graph? |

This problem is known to be NP-complete from \([5]\). However, from Theorem \([6]\), the COS-R algorithm in Section 3 can be used to solve the problem in \(O^*(10^d)\) time. Here, the inputs to the algorithm are the half adjacency matrix \(M\) of \(G\) and the parameter \(d\). The algorithm returns a set \(D\) of at most \(d\) rows (if one exists) such that \(G[V_1 \setminus D, V_2]\) is convex bipartite where \(D\) is the subset of vertices of \(V_1\) corresponding to \(D\).

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