SHARP EIGENVALUE ASYMPTOTICS FOR FOURTH ORDER OPERATORS ON THE CIRCLE

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Abstract. We determine high energy asymptotics of eigenvalues of fourth order operator on the circle.

1. Introduction and main results

We consider the operator $H$ acting on the circle $2(\mathbb{R}/\mathbb{Z})$ and given by

$$H = \partial^4 + 2\partial p \partial + q.$$  \hfill (1.1)

i.e., we consider operator $H$ on the interval $[0, 2]$ with the 2-periodic boundary conditions. We assume that $p, q$ are the real 1-periodic functions, which satisfy the conditions:

$$p, q \in L^1(\mathbb{T}), \quad \mathbb{T} = \mathbb{R}/\mathbb{Z}, \quad \int_0^1 q(t)dt = 0.$$  \hfill (1.2)

It is well known that the operator $H$ is self-adjoint and its spectrum consists of eigenvalues of multiplicity $\leq 4$. Really, eigenvalues of multiplicity 4 exist, see Remarks 1.2.

Let $\lambda^+_0, \lambda^+_n, n \in \mathbb{N}$, be the eigenvalues of the operator $H$ labeled by $\lambda^+_0 \leq \lambda^-_1 \leq \lambda^+_1 \leq \lambda^-_2 \leq \ldots$, counted with multiplicities. These eigenvalues satisfy the asymptotics

$$\lambda^+_n = (\pi n)^4 + 2(\pi n)^2( - p_0 \pm |\hat{p}_n| + O(n^{-\frac{3}{2}}))$$  \hfill (1.3)

as $n \to +\infty$, see [BK1], where

$$f_0 = \int_0^1 f(t)dt, \quad \hat{f}_n = \int_0^1 f(t)e^{-i2\pi nt}dt \quad \forall \ n \in \mathbb{Z}.$$

There are no coefficients $\hat{q}_n$ in the leading term of asymptotics (1.3), since $p, q$ satisfy (1.2). Our goal is to determine asymptotics of eigenvalues, when there are the coefficients $\hat{q}_n$ in the leading term. In this case we need an additional condition $p'' \in L^2(\mathbb{T})$.

1.1. Second order operators. Firstly we recall the well-known asymptotics of eigenvalues for the second order operator $-\partial^2 + q$ on the circle. The spectral properties of Schrödinger operators $-\partial^2 + q$ on $\mathbb{T}$ with the periodic $q$ are well understood, see, e.g., the books of Levitan–Sargsyan [LeS], Magnus–Winkler [MW], Marchenko [M], Titchmarsh [T], and references therein. Consider the second order operator $-\partial^2 + q$ on the interval $[0, 2]$ with the 2-periodic boundary conditions. If $q \in C^\infty(\mathbb{T})$, then the eigenvalues $\alpha^+_n, \alpha^-_n, n \in \mathbb{N}$, of this operator
have multiplicity 1 or 2 and satisfy the inequalities \( \alpha_0^+ < \alpha_1^- \leq \alpha_1^+ < \alpha_2^- \leq \ldots \) and the asymptotics
\[
\alpha_n^\pm = (\pi n)^2 + \frac{c_0}{(\pi n)^2} + \frac{c_1}{(\pi n)^4} + \ldots \tag{1.4}
\]
Here the numbers \( c_j \) are expressed in terms of \( q \) and the derivatives \( q^{(s)}, s \leq j \), see [M, Th 1.5.2]. The full asymptotic expansion for the case \( q, q^{(m)} \in L^2(T) \), where \( m \in \mathbb{N} \), was determined by Marchenko–Ostrovski [MO]. The case, when the potential \( q \) is a distribution, \( q = u', u \in L^2(T) \), was considered by Korotyaev [K2].

1.2. Fourth order operators. The standard applications of the fourth order differential equations are bending vibrations of thin beams and plates described by the Euler-Bernoulli equation, see [Gl]. Moreover, these equations arise in many physical models: hydrodynamic stability (Orr–Sommerfeld equation, [Li]), kinetic of liquid phase (Cahn–Hilliard equation, [CH]), elastic buckling [HW], thin films [BGW], see also the book [PeT] and references therein.

Many papers are devoted to the study of the spectral problems for the fourth order operators, see papers Caudill–Perry–Schueller [CPS], McLaughlin [McL], Hoppe–Laptev–Östensson [HLO], Mikhailets–Molyboga [MMo], Papanicolaou [P1], [P2], Badanin–Korotyaev [BK1] and references therein.

Caudill, Perry and Schueller [CPS] described the so-called iso-spectral potentials for the operator \( \partial^4 + 2\partial p \partial + q \) with the boundary condition \( f(0) = f'(0) = f(1) = f'(1) = 0 \).

McLaughlin [McL] studied the inverse spectral problems by the spectrum and the normalization constants for the operator \( \partial^4 + \partial p \partial + q \) on the interval \([0, 1]\) with the boundary conditions \( f(0) = f'(0) = f(1) = f'(1) = 0 \).

Hoppe, Laptev and Östensson [HLO] considered the inverse spectral problem for the operator \( \partial^4 + \partial p \partial + q \) on the line in the case of rapidly decaying \( p, q \) at infinity.

Mikhailets and Molyboga [MMo] considered the operator \( \partial^4 + q \) on the circle, where the function \( q \) is a distribution. They determined asymptotics of eigenvalues for this operator.

In the series of papers Papanicolaou [P1], [P3] and jointly with Kravvaritis [PK1], [PK2] the Euler-Bernoulli operator \( \frac{1}{2}(af'')'' \) on the line with the periodic positive \( a, b \) is considered. The unitary Liouville’s type transformation reduces this operator to the operator \( \partial^4 + \partial p \partial + q \) with some special periodic \( p, q \). The spectral properties of the Euler-Bernoulli operator are simpler, than for the general operator \( \partial^4 + \partial p \partial + q \). The spectrum lies on \( \mathbb{R}_+ \) and its structure is similar to the structure of the spectrum of the Hill operator: the spectrum has multiplicity 2, consists of non-degenerated intervals separated by gaps, the endpoints of gaps are eigenvalues of the 2-periodic problem. The necessary and sufficient conditions for the Euler-Bernoulli operator to be a perfect square of the operator \( -af'' \) are found in [P3].

Badanin and Korotyaev [BK1] considered the operator \( \partial^4 + \partial p \partial + q \) with periodic \( p, q \) on the line. The spectrum of this operator consists of intervals, separated by gaps, has multiplicity 2 or 4. Authors determine asymptotics of the gaps. This asymptotics shows that the spectrum has multiplicity 2 at high energy and for the generic coefficients there exists an infinite number of gaps.

Spectral asymptotics for the higher order operators are much less investigated. Numerous results about the regular and singular boundary value problems for these operators are expounded in the book of Naimark [Na]. High energy asymptotics of the spectral gaps
for even order operator with periodic coefficients is determined by Badanin and Korotyaev [BK2]. The scattering theory for higher order operators is a subject of the book of Beals, Deift, Tomei [BDT].

1.3. Main results. In order to determine asymptotics including the term \( \hat{q}_n \) we need an additional condition \( p'' \in L^1(\mathbb{T}) \), and finally we have also consider the case \( p''' \), \( q' \in L^1(\mathbb{T}) \).

**Theorem 1.1.** i) If \( p, p'', q \in L^1(\mathbb{T}) \), then the eigenvalues of \( H \) satisfy

\[
\lambda_n^\pm = (\pi n)^4 - (\pi n)^2 p_0 - \frac{||p||^2 - p_0^2}{2} \pm |\hat{V}_n| + o\left(\frac{1}{n^2}\right) \quad \text{as} \quad n \to +\infty, 
\]

where

\[
V = q - \frac{p''}{2}, \quad ||p||^2 = \int_0^1 |p(t)|^2 dt.
\]

ii) Let, in addition, \( p''' \), \( q' \in L^1(\mathbb{T}) \). Then the eigenvalues of \( H \) satisfy

\[
\lambda_n^\pm = (\pi n)^4 - (\pi n)^2 p_0 - \frac{||p||^2 - p_0^2}{2} \pm |\hat{V}_n| + o\left(\frac{1}{n^3}\right) \quad \text{as} \quad n \to +\infty. 
\]

**Remark 1.2.** i) Consider the operator \((-\partial^2 - 10\pi^2)^2\). This operator has a 1-periodic eigenvalue \( 36\pi^4 \) of multiplicity 4. The corresponding eigenfunctions have the form \( e^{\pm 2\pi it}, e^{\pm 4\pi it} \).

ii) Erovenko [E] considered the operator (1.1) and determined the following asymptotics for its eigenvalues

\[
\lambda_n^+ - \lambda_n^- = 2\left(|\hat{q}_n|^2 + |\hat{p}_n''|^2\right)\left(1 + o(1)\right) \quad \text{as} \quad n \to +\infty. 
\]

Unfortunately, this asymptotics is not correct, see more in Section 4.

iii) We use asymptotics (1.6) in order to obtain the trace formula for the operator \( H \). [BK3].

iv) There is an open problem: to determine the full asymptotic expansion in the case \( p, q \in C^\infty(\mathbb{T}) \).

1.4. Describe briefly our proofs. We rewrite the initial differential equation (2.1) into the matrix form (2.4). The 4\times4 matrix solution \( M(t, \lambda) \) of equation (2.4) satisfying the condition \( M(0, \lambda) = I_4 \) is the fundamental matrix. The matrix \( M(1, \lambda) \) is the monodromy matrix. Our main tool is an analysis of this matrix. Such analysis for the Schrödinger operator with the periodic matrix potential was made by Chelkak, Korotyaev [CK]. But in our case we meet additional difficulties. For the second order operators (even with the matrix coefficients) all entries of the monodromy matrix are bounded for \( \lambda \to +\infty \) (in the unperturbed case they have the form \( \cos \sqrt{\lambda}, \sin \sqrt{\lambda} \)). For the fourth order differential operator some entries of the monodromy matrix are bounded as \( \lambda \to +\infty \) and all other entries are unbounded (in the unperturbed case they have the form \( \cos \lambda^{1/4}, \sin \lambda^{1/4}, \cosh \lambda^{1/4}, \sinh \lambda^{1/4} \)).

Asymptotics of eigenvalues for the higher order operators in the case of smooth coefficients was determined by Birkhoff (see [Na, Ch. II.4]). Here we consistently apply the matrix form of Birkhoff’s method (the similar method was used in [BK2]). The calculations in the matrix form are simpler than in the original scalar form. Note that the matrix approach was used before by Chelkak, Korotyaev [CK] for the second order operator with the periodic matrix potential. However, in the case [CK] the situation is simpler, since the monodromy matrix
is bounded at $\lambda \to +\infty$. Roughly speaking the determining of asymptotics for higher order operators has the difficulties of analysis of the systems plus additional difficulties from the increasing solutions.

In order to overcome these problems we need some modifications of the matrix differential equation, see Lemma 2.1 Following to Fedoruk [Fe, Ch. V.1.3], Korotyaev [K1], we modify equation (2.2) to the quasi-diagonal form, where the matrix coefficient of the equation is the sum of the diagonal matrix and the small perturbation as $|\lambda| \to \infty$, see (2.13). After that, following to Birkhoff (see [Na, Ch. II.4]), we rewrite the matrix differential equation into the form of an integral equation, and show that this equation is uniquely solvable for all large $|\lambda|$. Finally we obtain the special representation of the monodromy matrix in the form of the multiplication of the simple diagonal matrix and the matrix $F$ bounded for all large $|\lambda|$, see (3.15). Iterations of the integral equation allow us to determine the asymptotics of the matrix $F$, and then asymptotics (1.5), (1.6).

The plan of the paper is as follows. In Section 2 we transform the differential equation for the fundamental matrix to the quasi-diagonal form. In Section 3 we reduce the matrix differential equation to the equivalent integral equation. Iterations of this integral equation give us the asymptotics of the fundamental matrix. In Section 4 we analyze the characteristic determinant of the monodromy matrix and determine the asymptotics of eigenvalues of $H$.

2. Fundamental matrix

Consider the equation

$$f^{(4)} + 2(p f')' + q f = \lambda f$$

(2.1)

on $\mathbb{R}$, where $\lambda \in \mathbb{C}$. Rewrite equation (2.1) in the matrix form

$$f' - \Lambda f = -(2pJ + qJ_1)f,$$

(2.2)

where the vector $f(t)$ and the matrices $\Lambda(\lambda), J, J_1$ are given by

$$f = \begin{pmatrix} f \\ f' \\ f'' \\ f''' + 2pf \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda & 0 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Define the $4 \times 4$ - matrix valued solution $M(t, \lambda)$ of equation (2.2):

$$M' - \Lambda M = -(2pJ + qJ_1)M, \quad M(0, \lambda) = I_4,$$

(2.3)

where $I_4$ is the identity $4 \times 4$ - matrix. $M(t, \lambda)$ is called the fundamental matrix of equation (2.1). Each function $M(t, \cdot), t \in \mathbb{R}$, is entire and real on $\mathbb{R}$. The matrix valued function $M(1, \lambda)$ is called the monodromy matrix. The spectrum $\sigma(H)$ of the operator $H$ satisfies the identity

$$\sigma(H) = \{\lambda_0^+, \lambda_0^-, n \geq 1\} = \{\lambda \in \mathbb{R} : \det(M(1, \lambda) \pm I_4) = 0\}.$$

Introduce the unitary $4 \times 4$ - matrices $\omega$ and $U$ given by

$$\omega = \text{diag}(\omega_1, \omega_2, \omega_3, \omega_4) = \text{diag}(i, 1, -1, -i),$$

(2.5)
\[ U = \frac{1}{2} (\omega_k^{j-1})_{j,k=1}^4 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ i & 1 & -1 & -i \\ -1 & 1 & 1 & -1 \\ -i & 1 & -1 & i \end{pmatrix}. \]

Define the 4 \times 4 matrices

\[ A = U^{-1} J U = \frac{1}{4} \begin{pmatrix} -i & -1 & 1 & i \\ i & 1 & -1 & -i \\ i & 1 & 1 & -i \\ -i & 1 & 1 & i \end{pmatrix}, \quad A_1 = -U^{-1} J U = \frac{1}{4} \begin{pmatrix} -i & -i & -i & -i \\ -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ i & i & i & i \end{pmatrix}, \quad (2.6) \]

\[ B = \frac{1}{8} \begin{pmatrix} 0 & 1 + i & 1 - i & 1 \\ -1 + i & 0 & -1 - i & 1 \\ -1 - i & -1 & 0 & 1 + i \\ 1 & 1 - i & 1 + i & 0 \end{pmatrix}, \quad B_1 = \frac{1}{16} \begin{pmatrix} 0 & -2 & -2 & -1 \\ 2i & 0 & i & 2i \\ -2i & -i & 0 & -i \\ 1 & 2 & 2 & 0 \end{pmatrix}, \quad (2.7) \]

\[ B_2 = \frac{1}{32} \begin{pmatrix} i & i & i & 0 \\ 1 & 1 & 0 & 1 \\ -1 & 0 & -1 & -1 \\ 0 & -i & -i & -i \end{pmatrix}. \quad (2.8) \]

Introduce the variable \( z = \lambda^2, \lambda \in \mathbb{C}, \) by the condition

\[ \arg z \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right] \quad \text{as} \quad \arg \lambda \in [0, 2\pi). \]

If \( \arg \lambda \in [0, \pi), \) then

\[ z \in S = \left\{ z \in \mathbb{C} : \arg z \in \left[0, \frac{\pi}{4}\right) \right\}, \]

and we have the following estimates:

\[ \text{Re}(i \omega_1 z) \leq \text{Re}(i \omega_2 z) \leq \text{Re}(i \omega_3 z) \leq \text{Re}(i \omega_4 z) \quad \forall \quad z \in S. \quad (2.9) \]

Introduce the 4 \times 4 - matrix valued functions

\[ Z = \begin{pmatrix} 1, iz, (iz)^2, (iz)^3 \end{pmatrix}, \quad (2.10) \]

\[ W(t, z) = \mathds{1}_4 - \frac{2p(t)}{z^2} B - \frac{2p'(t)}{z^3} B_1, \quad (2.11) \]

\((t, z) \in \mathbb{R} \times S.\) In the following Lemma we modify equation \((2.4)\) into the quasi-diagonal form \((2.13).\)

**Lemma 2.1.** Let

\[ z \in S_r = \{ z \in S : |z| > r \} \]

for some \( r > 0 \) large enough. Then the matrix valued function \( \Phi(t, z), t \in \mathbb{R}, \) given by the identity

\[ \Phi(t, z) = \left( Z(z) W(t, z) \right)^{-1} M(t, z^4) Z(z) W(0, z), \quad (2.12) \]

satisfies the equation on the real line

\[ \Phi' - iz \xi \Phi = Q \Phi, \quad \Phi(0, z) = \mathds{1}_4, \quad (2.13) \]
where

\[
\xi(t, z) = \omega + \frac{p(t)}{2z^2}\omega^* = \text{diag}(\xi_j(t, z))_{j=1}^4,
\]

\[
\xi_j(t, z) = \omega_j + \frac{\omega_j p(t)}{2z^2} \quad \forall \quad j \in \mathbb{N}_4 = \{1, 2, 3, 4\},
\]

\[
Q(t, z) = \frac{Q_0(t)}{z^3} + \frac{4p(t)p'(t)Q_1}{z^4} + \frac{Q_2(t, z)}{z^5},
\]

\[
Q_0(t) = iq(t)A_1 + 2p''(t)B_1 + i4p^2(t)B_2,
\]

\[
Q_1 = -iAB_1 + B(B - i\omega B_1) + iB_1(A - \omega B),
\]

the matrix valued function \(Q_2(t, z)\), given by (2.24), is uniformly bounded on \(\mathbb{R} \times S_r\), and each function \(Q_2(t, \cdot)\), \(t \in \mathbb{R}\), is analytic in \(S_r\).

**Proof.** Multiplying equation (2.21) by the matrix \(ZU\) from the right and by the matrix \((ZU)^{-1}\) from the left and using the identities

\[
(ZU)^{-1}AZU = \omega,
\]

\[-(ZU)^{-1}(2pJ + qJ_1)ZU = i\frac{U^{-1}(2pJ - \frac{q}{z^2}J_1)U}{z} + \frac{i2p}{z}A + \frac{iq}{z^3}A_1,
\]

we obtain the equation

\[
\Phi' - iz\omega \Phi = \left(\frac{i2p}{z}A + \frac{iq}{z^3}A_1\right)\Phi,
\]

(2.18)

with respect to the matrix valued function \(\Phi = (ZU)^{-1}MZU\).

Identity (2.12) gives

\[
\tilde{\Phi}(t, z) = W(t, z)\Phi(t, z)W^{-1}(0, z).
\]

Substituting this identity into (2.18), we obtain

\[
W\Phi' = \left(i\omega W - W' + \frac{i2p}{z}AW + \frac{iq}{z^3}A_1W\right)\Phi, \quad W = W(t, z).
\]

Substituting (2.11) into the last identity we obtain

\[
W\Phi' = \left(i\omega + \frac{i2p}{z}(A - \omega B) + \frac{2p'}{z^2}(B - i\omega B_1) + \frac{Q_0}{z^3} + \frac{\tilde{A}}{z^4}\right)\Phi,
\]

(2.19)

where

\[
Q_0 = i4p^2[B, A] + i4p^2B[B, \omega] + iqA_1 + 2p''B_1, \quad [A, B] = AB - BA,
\]

\[
\tilde{A} = -i4pp'AB_1 - i\frac{2pq}{z}A_1B - i\frac{2p'q}{z^2}A_1B_1,
\]

(2.20)

(2.21)

Introduce the matrix valued function \(\tilde{B}(t, z), (t, z) \in \mathbb{R} \times S_r\), by the identity

\[
W^{-1}(t, z) = \mathbb{I} + \frac{2p(t)}{z^2}B + \frac{2p'(t)}{z^3}B_1 + \frac{\tilde{B}(t, z)}{z^4}.
\]

(2.22)

Then \(\tilde{B}(t, z)\) is uniformly bounded on \(\mathbb{R} \times S_r\) and \(B(t, \cdot)\) is analytic in \(S_r\) for all \(t \in \mathbb{R}\).

Substituting (2.22) into (2.19) we obtain

\[
\Phi' = \left(i\omega + \frac{i2p}{z}([B, \omega] + A) + \frac{2p'}{z^2}(B + i[B_1, \omega]) + \frac{Q_0}{z^3} + \frac{4pp'Q_1}{z^4} + \frac{Q_2}{z^5}\right)\Phi,
\]

(2.23)
where the constant matrix $Q_1$ is given by (2.17),
\[
Q_2 = -i2qA_1 \left( pB + \frac{p'}{z}B_1 \right) + 2pB \left( Q_0 + \frac{\tilde{A}}{z} \right) + 2p'B_1 \left( 2p'(B - i\omega B_1) + \frac{Q_0}{z} + \frac{\tilde{A}}{z^2} \right) + \tilde{B} \left( 2ip(A - \omega B) + \frac{2p'}{z}(B - i\omega B_1) + \frac{Q_0}{z^2} + \frac{\tilde{A}}{z^3} \right). \tag{2.24}
\]
This identity shows that $Q_2(t, z)$ is uniformly bounded on $\mathbb{R} \times S_r$ and each function $Q_2(t, \cdot), t \in \mathbb{R}$, is analytic in $S_r$.

The matrices $B, B_1, B_2$ satisfy the identities
\[
[B, \omega] + A = \frac{\omega^*}{4}, \quad B + i[B_1, \omega] = 0, \quad B_2 = \frac{B\omega^*}{4} - AB.
\]
Substituting these identities into (2.23) and (2.20) we obtain (2.13) and (2.16). □

3. ASYMPTOTICS OF THE MONODROMY MATRIX

Let $r > 0$ be large enough and let $z \in S_r$. In equation (2.13) we introduce the new unknown $4 \times 4$ matrix valued function $G(t, z), t \in [0, 1]$, given by the identity
\[
\Phi(t, z) = G(t, z)e^{iz\int_0^t \xi(t, s)ds}G^{-1}(0, z). \tag{3.1}
\]
Substituting (3.1) into (2.13) we obtain the matrix differential equation on the interval $[0, 1]$
\[
G' + iz(G\xi - \xi G) = QG. \tag{3.2}
\]
If the matrix valued function $G$ satisfies equation (3.2), then the matrix valued function $\Phi$, given by (3.1), satisfies (2.13) on the interval $[0, 1]$.

We rewrite equation (3.2) in the form of the matrix integral equation on the interval $[0, 1]$
\[
G = I + KG, \tag{3.3}
\]
where
\[
(KG)_{\ell j}(t, z) = \int_0^1 K_{\ell j}(t, s, z)(QG)_{\ell j}(s, z)ds \quad \forall \quad \ell, j \in \mathbb{N}_4, \tag{3.4}
\]
\[
K_{\ell j}(t, s, z) = \begin{cases} e^{iz\int_0^s (\xi(t, u, z) - \xi(s, u, z))du} \chi(t - s), & \ell < j, \\ -e^{iz\int_0^t (\xi(t, u, z) - \xi(s, u, z))du} \chi(s - t), & \ell \geq j \end{cases}, \quad \chi(s) = \begin{cases} 1, & s \geq 0 \\ 0, & s < 0 \end{cases}. \tag{3.5}
\]

In the following Lemma we will show that equation (3.3) for $z \in S_r$ has the unique solution in the class of $4 \times 4$ matrix valued functions from $L^\infty(0, 1)$ with the norm given by
\[
\|A\|_\infty^2 = \sup_{t \in [0, 1]} |A(t)|^2, \quad |A|^2 = \max_{h \in \mathbb{C}^4} \sum_{|h_j|^2 = 1} \sum_{j=1}^4 |(Ah_j)|^2.
\]

**Lemma 3.1.** Let $z \in S_r$, for some $r > 0$ large enough. Then equation (3.3) has the unique solution $G(\cdot, z) \in L^\infty(0, 1)$. Moreover,
\[
G(t, z) = \sum_{n=0}^\infty G_n(t, z) \quad \forall \quad (t, z) \in [0, 1] \times S_r, \tag{3.6}
\]
where
\[
G_0 = I_4, \quad G_n = KG_{n-1} = K^n I_4 \quad \forall \quad n \in \mathbb{N}. \tag{3.7}
\]
The series (3.6) converges absolutely and uniformly on any compact set in \([0, 1] \times S_r\). Each matrix valued function \(G(t, \cdot), t \in [0, 1]\), is analytic in \(S_r\) and it satisfies the asymptotics
\[
G(t, z) = \mathbb{1}_4 + O(z^{-3}), \quad G(t, z) = \mathbb{1}_4 + G_1(t, z) + O(z^{-6}) \tag{3.8}
\]
as \(|z| \to \infty, z \in S_r\), uniformly on \(t \in [0, 1]\).

**Proof.** Identity (2.14) gives
\[
\xi(t, z) - \xi_j(t, z) = \omega_\ell - \omega_j + \frac{p(t)}{2z^2}(\omega_\ell - \omega_j) \quad \forall \; \ell, j \in N_4.
\]
Then estimates (2.9) yield
\[
\text{Re}iz(\xi(t, z) - \xi_j(t, z)) = \text{Re}iz(\omega_\ell - \omega_j) + \frac{p(t)}{2} \text{Re} \frac{i(\omega_\ell - \omega_j)}{z} \leq \frac{|p(t)|}{|z|}
\]
for all \(t \in [0, 1], \ell, j \in N_4 : \ell \leq j\). Substituting this estimate into (3.5) we obtain
\[
|K_{ij}(t, s, z)| \leq e^{\frac{|t|}{|z|}} \leq 2 \tag{3.9}
\]
for all \(t, s \in [0, 1], \ell, j \in N_4, z \in S_r\).

Let the \(4 \times 4\) - matrix valued function \(A\) belongs to the class \(L^\infty(0, 1)\). Substituting estimates (3.9) into (3.4), we obtain
\[
\|(KA)(\cdot, z)\|_\infty = \sup_{t \in [0, 1]} \|(KA)(t, z)\| \leq 4 \sup_{t \in [0, 1]} \max_{(i, j) \in N_4^2} |(KA)_{ij}(t, z)|
\]
\[
\leq 8 \max_{(i, j) \in N_4^2} \int_0^1 |(QA)_{ij}(s, z)|ds \leq 8 \int_0^1 |Q(s, z)||A(s)||ds \leq 8\|A\|_\infty \int_0^1 |Q(s, z)|ds.
\]
Identity (2.15) gives
\[
\int_0^1 |Q(s, z)|ds \leq \frac{C}{|z|^3} \quad \forall \; (t, z) \in [0, 1] \times S_r
\]
for some \(C > 0\). Therefore,
\[
\|(KA)(\cdot, z)\|_\infty \leq \frac{8C}{|z|^3}\|A\|_\infty \quad \forall \; z \in S_r. \tag{3.10}
\]

Iterations in (3.3) give (3.6). Estimate (3.10) yields
\[
\|G_n(\cdot, z)\|_\infty = \|(K^n\mathbb{1}_4)(\cdot, z)\|_\infty \leq \left(\frac{8C}{|z|^3}\right)^n \quad \forall \; (n, z) \in \mathbb{N} \times S_r. \tag{3.11}
\]

These estimates show that the formal series (3.6) converges absolutely and uniformly on any compact in \(S_r\). Therefore, it gives the unique solution of equation (3.3). Each term of this series is analytic with respect to \(z\) in \(S_r\). Therefore, the function \(G\) is analytic also. Substituting estimates (3.11) into series (3.6) we obtain asymptotics (3.8). ■

Identity (3.7) yields
\[
\Phi(1, z) = G(0, z)F(z)e^{izv(z)}G^{-1}(0, z) \quad \forall \; z \in S_r, \tag{3.12}
\]
where the \(4 \times 4\) - matrix valued functions \(F(z), v(z)\) are given by
\[
F(z) = G^{-1}(0, z)G(1, z), \quad v(z) = \int_0^1 \xi(t, z)dt. \tag{3.13}
\]
Proof. i) Asymptotics (3.8) and identity (3.13) give (3.16) and the asymptotics we obtain (3.17).

\[
\int_1^\infty \left( \omega + \frac{p(t)}{2z^2} \right) dt = \text{diag}(v_j(z))_{j=1}^4, \quad v_j(z) = \omega_j + \frac{p_0}{2z^2}\omega_j. \tag{3.14}
\]

Lemma 3.1 shows that \( F(z) \) is analytic and uniformly bounded on \( S_r \).

Identities (2.5), (2.14) yield

\[
G(z) = \int_1^\infty \left( \omega + \frac{p(t)}{2z^2} \right) dt = \text{diag}(v_j(z))_{j=1}^4, \quad v_j(z) = \omega_j + \frac{p_0}{2z^2}\omega_j.
\]

where

\[
M(1, z^4) = U(z)F(z)e^{izn(z)}U^{-1}(z) \quad \forall \quad z \in S_r, \tag{3.15}
\]

In the following Lemma we determine asymptotics of the matrix valued function \( F \).

Lemma 3.2. i) The function \( F(z) \) satisfies the asymptotics

\[
F_{jk}(z) = O(z^{-3}) \quad \forall \quad j, k \in \mathbb{N}_4 : j \neq k, \tag{3.16}
\]

\[
F_{jj}(z) = 1 + \frac{i\omega_j\|p\|^2}{8z^3} + \frac{O(1)}{z^5} \quad \forall \quad j \in \mathbb{N}_4 \tag{3.17}
\]

as \(|z| \to \infty, z \in S, \)

\[
F_{23}(z) = -\frac{i\hat{V}_n}{4(\pi n)^3} + \frac{o(1)}{n^4}, \quad F_{32}(z) = \frac{i\hat{V}_n}{4(\pi n)^3} + \frac{o(1)}{n^4} \tag{3.18}
\]

as \( z = (\lambda_n^\pm)^\frac{1}{4}, n \to +\infty. \)

ii) Let \( p, p'' \), \( q, q' \in L^1(\mathbb{T}) \). Then

\[
F_{23}(z) = -\frac{i\hat{V}_n}{4(\pi n)^3} + \frac{o(1)}{n^5}, \quad F_{32}(z) = \frac{i\hat{V}_n}{4(\pi n)^3} + \frac{o(1)}{n^5} \tag{3.19}
\]

as \( z = (\lambda_n^\pm)^\frac{1}{4}, n \to +\infty. \)

Proof. i) Asymptotics (3.8) and identity (3.13) give (3.16) and the asymptotics

\[
F(z) = \mathbb{I}_4 + G(1, z) - G(0, z) + O(z^{-6}) \quad \text{as} \quad |z| \to \infty, \quad z \in S, \tag{3.20}
\]

where \( G = G_1 = K\mathbb{I}_4 \). Identity (3.4) implies

\[
G_{\ell j}(1, z) = \begin{cases} f_1 e^{iz} f_1(\xi_{\ell}(t,z) - \xi_{j}(t,z))dt Q_{\ell j}(s, z)ds, & \ell < j, \\ 0, & \ell \geq j, \end{cases} \tag{3.21}
\]

\[
G_{\ell j}(0, z) = \begin{cases} \int_0^1 e^{-iz} f_1(\xi_{\ell}(t,z) - \xi_{j}(t,z))dt Q_{\ell j}(s, z)ds, & \ell < j, \\ 0, & \ell \geq j \end{cases}
\]

Let \( j \in \mathbb{N}_4 \). Substituting (2.15) into (3.21) and using (2.16), (2.17), the identities

\[
\int_0^1 q(t)dt = \int_0^1 p''(t)dt = \int_0^1 p(t)p'(t)dt = 0 \quad \text{and} \quad (2.5), \]

we obtain

\[
G_{\ell j}(0, z) = -\int_0^1 \left( \frac{4i}{z^3} \right) f_1(p^2(t)(B_2)_{\ell j})dt + \frac{O(1)}{z^5} = -\frac{i\omega_j\|p\|^2}{8z^3} + \frac{O(1)}{z^5}
\]

as \(|z| \to \infty, z \in S. \) Substituting this asymptotics and the identity \( G_{\ell j}(1, z) = 0 \) into (3.20) we obtain (3.17).
Let $z = (\lambda^\pm_n)^i$ and let $n \to +\infty$. Then asymptotics (1.3) gives $z = \pi n - \frac{p_0 + o(1)}{2\pi n}$. Substituting (2.14), (2.15) into (3.21) and integrating by parts we obtain

\[
\mathcal{G}_{23}(1, z) = \int_0^1 e^{2iz(1-s)} e^{\frac{i}{2} \int_0^s p(t)dt} Q_{\ell_j}(s, z) ds = \frac{G_1^+}{z^3} + \frac{G_2^+}{z^4} + O(1) n^5, \tag{3.22}
\]

where

\[
G_1^+ = \int_0^1 e^{2iz(1-s)} (Q_0)_{23}(s) ds, \quad G_2^+ = i \int_0^1 e^{2iz(1-s)} (Q_0)_{23}(s) \int_s^1 p(t) dt ds.
\]

Similarly,

\[
\mathcal{G}_{32}(0, z) = - \int_0^1 e^{2izs} e^{\frac{i}{2} \int_0^s p(u) du} Q_{32}(s, z) ds = \frac{G_1^-}{z^3} + \frac{G_2^-}{z^4} + O(1) n^5, \tag{3.24}
\]

where

\[
G_1^- = - \int_0^1 e^{2isz} (Q_0)_{32}(s) ds, \quad G_2^- = i \int_0^1 e^{2isz} (Q_0)_{32}(s) \int_0^s p(t) dt ds.
\]

Substituting these identities into (3.23), (3.25) we have

\[
\mathcal{G}_1^+ = - \frac{i}{4} \int_0^1 e^{2iz(1-s)} V(s) ds = - \frac{i}{4} \int_0^1 e^{-2i\pi ns - (1-s) \frac{p_0 + o(1)}{2\pi n}} V(s) ds
\]

\[
= - \frac{i}{4} \int_0^1 e^{-2i\pi ns} \left(1 - (1-s) \frac{p_0 + o(1)}{2\pi n}\right) V(s) ds + O(1) n^2, \tag{3.26}
\]

\[
\mathcal{G}_1^- = \frac{i}{4} \int_0^1 e^{2isz} V(s) ds = \frac{i}{4} \int_0^1 e^{2isz} \left(1 - s \frac{p_0 + o(1)}{2\pi n}\right) V(s) ds + O(1) n^2, \tag{3.27}
\]

which yields

\[
\mathcal{G}_1^+ = - \frac{i\hat{V}_n}{4} + o(1) n^3, \quad \mathcal{G}_1^- = \frac{i\hat{V}_n}{4} + o(1) n^3.
\]

Similarly, $G_2^\pm = o(1)$, which yields

\[
\mathcal{G}_{23}(1, z) = - \frac{i\hat{V}_n}{4(\pi n)^3} + o(1) \frac{n^4}{n^4}, \quad \mathcal{G}_{32}(0, z) = \frac{i\hat{V}_n}{4(\pi n)^3} + o(1) \frac{n^4}{n^4}.
\]

Substituting these asymptotics into (3.20) we obtain (3.18).

(1) Let $q, q', p, p'' \in L^1(\mathbb{T})$ and let $z = (\lambda^\pm_n)^i, n \to +\infty$. The integration by parts in (3.26), (3.27) gives

\[
\mathcal{G}_1^+ = - \frac{i\hat{V}_n}{4} + O(1) \frac{n}{n^2}, \quad \mathcal{G}_1^- = \frac{i\hat{V}_n}{4} + O(1) \frac{n}{n^2}.
\]

Similarly, $G_2^\pm = O(n^{-1})$, which yields

\[
\mathcal{G}_{23}(1, z) = - \frac{i\hat{V}_n}{4(\pi n)^3} + O(1) \frac{n^4}{n^5}, \quad \mathcal{G}_{32}(0, z) = \frac{i\hat{V}_n}{4(\pi n)^3} + O(1) \frac{n^4}{n^5}.
\]
Substituting these asymptotics into (3.20) we obtain (3.19). ■

4. Eigenvalue asymptotics

In this Section we will prove our main results. Introduce the function

$$D(\tau, \lambda) = \text{det}(M(1, \lambda) - \tau I_4), \quad (\tau, \lambda) \in \mathbb{C}^2.$$  

The eigenvalues $\lambda_n^\pm$ satisfy the identity $D((-1)^n, \lambda_n^\pm) = 0$ for all $n \in \mathbb{N}$ large enough, see [BK1].

**Proof of Theorem 1.1.** i) Identity (3.15) gives

$$D(\tau, z^\pm) = \text{det} \begin{pmatrix} F_{11}(z)e^{iv_1(z)z} - \tau & F_{12}(z)e^{iv_2(z)z} & F_{13}(z)e^{iv_3(z)z} & F_{14}(z)e^{iv_4(z)z} \\ F_{21}(z)e^{iv_1(z)z} & F_{22}(z)e^{iv_2(z)z} - \tau & F_{23}(z)e^{iv_3(z)z} & F_{24}(z)e^{iv_4(z)z} \\ F_{31}(z)e^{iv_1(z)z} & F_{32}(z)e^{iv_2(z)z} & F_{33}(z)e^{iv_3(z)z} - \tau & F_{34}(z)e^{iv_4(z)z} \\ F_{41}(z)e^{iv_1(z)z} & F_{42}(z)e^{iv_2(z)z} & F_{43}(z)e^{iv_3(z)z} & F_{44}(z)e^{iv_4(z)z} - \tau \end{pmatrix}$$  

(4.1)

for all $(\tau, z) \in \mathbb{C} \times S_r$ for some $r > 0$ large enough, where $v(z)$ is given by (3.14).

Let $\tau = (-1)^n, \lambda = \lambda_n^\pm$ and let $n \to +\infty$. Then asymptotics (1.3) gives

$$z = \lambda_n^\pm = \pi n - \frac{p_0 + o(1)}{2\pi n}, \quad (4.2)$$

Identities (2.5) imply $v_1(z) = -v_4(z), v_2(z) = -v_3(z)$. Moreover, (4.2) gives

$$e^{-iv_1(z)z} = e^{iv_4(z)z} = e^{\pi n}(1 + o(1)), \quad v_2(z)z = z + \frac{p_0}{2z} = \pi n + \delta_n, \quad \delta_n = o(n^{-1}), \quad (4.3)$$

and then

$$e^{iv_2(z)z} = (-1)^n e^{i\delta_n}. \quad (4.4)$$

Substituting asymptotics (3.16), (4.3) into identity (4.1) we obtain

$$D(\tau, z^4) = e^{iv_4(z)z} \text{det} \begin{pmatrix} O(e^{-\pi n}) - \tau & O(n^{-3}) & O(n^{-3}) & O(n^{-3}) \\ O(e^{-\pi n}) & F_{22}(z) - \tau e^{-iv_2(z)z} & F_{23}(z) & O(n^{-3}) \\ O(e^{-\pi n}) & F_{32}(z) & F_{33}(z) - \tau e^{iv_2(z)z} & O(n^{-3}) \\ O(e^{-\pi n}) & O(n^{-3}) & O(n^{-3}) & 1 + O(n^{-3}) \end{pmatrix}. \quad (4.5)$$

Substituting (3.17), (4.4) into the last asymptotics we obtain

$$D(\tau, z^4) = -\tau e^{iv_2(z)z} d_n, \quad (4.5)$$

where

$$d_n = \text{det} \begin{pmatrix} 1 - e^{-i\delta_n} + \frac{\|p\|^2}{8\pi^3} + O(n^{-5}) & F_{23}(z) & O(n^{-3}) & O(n^{-3}) \\ F_{32}(z) & 1 - e^{-i\delta_n} - \frac{\|p\|^2}{8\pi^3} + O(n^{-5}) & O(n^{-3}) & 1 + O(n^{-3}) \end{pmatrix} + O(e^{-\pi n}). \quad (4.6)$$

Asymptotics $\delta_n = o(n^{-1})$ implies

$$d_n = \left(2 \sin \frac{\delta_n}{2} + \frac{\|p\|^2}{8\pi^3} \cos \frac{\delta_n}{2}\right)^2 + \frac{O(1)}{n^6}. \quad (4.7)$$

Identity $D((-1)^n, z^4) = 0$ gives $d_n = 0$. Asymptotics (4.7) implies $\delta_n = O(n^{-3})$.  


Then we have the asymptotics
\[ e^{i\delta_n} + \frac{i\|p\|^2}{8z^3} = e^{i\tilde{\delta}_n} + \frac{O(1)}{n^6}, \quad \text{where} \quad \tilde{\delta}_n = \delta_n + \frac{\|p\|^2}{8z^3}. \]

Asymptotics (4.6) yields
\[ d_n = \left( 4 \sin^2 \frac{\tilde{\delta}_n}{2} + \frac{O(\delta_n)}{n^5} + \frac{O(1)}{n^{10}} \right) \left( 1 + \frac{O(1)}{n^3} \right) - F_{23}(z)F_{32}(z) + \frac{O(\tilde{\delta}_n)}{n^6} + o(1), \tag{4.8} \]
where we used (3.18) and \( \hat{V}_n = o(1) \).

Substituting asymptotics (3.18) into (4.8) and using \( \tilde{\delta}_n = O(n^{-3}) \) we obtain
\[ d_n = 4 \sin^2 \frac{\tilde{\delta}_n}{2} - \frac{\|\hat{V}_n\|^2}{16(\pi n)^3} = \frac{O(1)}{n^7}. \tag{4.9} \]

Identity (4.3) gives
\[ z + \frac{p_0}{2z} = \pi n - \frac{\|p\|^2}{8z^3} \pm \frac{|\hat{V}_n|}{4(\pi n)^3} + \frac{o(1)}{n^7}. \tag{4.10} \]

Let \( \varepsilon \in \mathbb{C} \) and let \([-i\varepsilon, i\varepsilon]\) be the segment of the line. Define the single valued analytic function \( f(w) = (w^2 + \varepsilon^2)^{\frac{1}{2}} - w \) in the domain \( \mathbb{C} \setminus [-i\varepsilon, i\varepsilon] \) by the condition \( \lim_{|w| \to \infty} f(w) = 0 \).

The maximal value of the function \(|f(w)|\) on the segment \([-i\varepsilon, i\varepsilon]\) is equal to \(|\varepsilon|\). Using the Maximum Principle we obtain the estimate
\[ |f(w)| = |(w^2 + \varepsilon^2)^{\frac{1}{2}} - w| \leq |\varepsilon| \quad \forall \ w \in \mathbb{C}. \tag{4.11} \]

Substituting \( w = \frac{|\hat{V}_n|}{4(\pi n)^3} \) and \( \varepsilon^2 = o(n^{-7}) \) into inequality (4.11) we obtain
\[ \left( \frac{|\hat{V}_n|}{4(\pi n)^3} \right)^2 + o(n^{-7}) \geq \frac{|\hat{V}_n|}{16(\pi n)^3} = \frac{o(1)}{n^7}. \]

Asymptotics (4.9) yields
\[ 2 \sin \frac{\tilde{\delta}_n}{2} = \pm \frac{|\hat{V}_n|}{4(\pi n)^3} + \frac{o(1)}{n^7}. \]

This asymptotics implies
\[ \delta_n = \frac{-\|p\|^2}{8z^3} \pm \frac{|\hat{V}_n|}{4(\pi n)^3} + \frac{o(1)}{n^7}, \]
then (4.3) yields
\[ v_2 z = \pi n - \frac{\|p\|^2}{8z^3} \pm \frac{|\hat{V}_n|}{4(\pi n)^3} + \frac{o(1)}{n^7}. \]

Identity (4.3) gives
\[ z + \frac{p_0}{2z} = \pi n - \frac{\|p\|^2}{8z^3} \pm \frac{|\hat{V}_n|}{4(\pi n)^3} + \frac{o(1)}{n^7}. \]

Then
\[ z = \pi n - \frac{p_0}{2\pi n} \frac{\|p\|^2}{8(\pi n)^3} \pm \frac{|\hat{V}_n|}{4(\pi n)^3} + \frac{o(1)}{n^7}, \]
which yields (1.5).
ii) Let \( p, p''', q, q' \in L^1(\mathbb{T}) \) and let \( n \to +\infty \). Substituting (3.19) into (4.8) and using the asymptotics \( \hat{V}_n = o(n^{-1}) \) we obtain
\[
d_n = 4 \sin^2 \frac{\delta_n}{2} - \frac{|\hat{V}_n|^2}{16(\pi n)^6} + o(1) \frac{n^9}{n^9}.
\]
Repeating the previous arguments we obtain
\[
\delta_n = - \|p\|^2 + \frac{|\hat{V}_n|^2}{4(\pi n)^3} + o(1) \frac{n^9}{n^9},
\]
which yields (1.6).

The perfect square case. Consider the operator \( h^2 \), where the operator \( h = -\partial^2 - p \) acts on the functions, satisfying the 2-periodic conditions \( f(0) = f(2), f'(0) = f'(2) \). The operator \( h^2 - \|p\|^2 \) is equal to the operator \( H \) with \( q = p'' + p^2 - \|p\|^2 \). Let \( p, p'' \in L^1(0,1) \) and let \( \alpha_0^+ < \alpha_1^- \leq \alpha_1^+ < ... \) be eigenvalues of the operator \( h \). It is well known (see [M, Th 1.5.2]), that
\[
\alpha_n^\pm = \left( \pi n \right)^2 - p_0^2 + \frac{|p''|^2}{(2\pi n)^2} \pm \frac{|\hat{p}''|^2}{(2\pi n)^2} + o(1) \frac{n^{1/2}}{n^{1/2}} \quad \text{as} \quad n \to +\infty.
\]
Therefore, the eigenvalues \( \lambda_n^\pm \) satisfy
\[
\lambda_n^\pm = \left( \pi n \right)^4 - 2p_0(\pi n)^2 - \frac{|p|^2}{2} + o(1) \frac{n^{1/2}}{n^{1/2}} \quad \text{as} \quad n \to +\infty.
\]
This asymptotics shows that asymptotics (1.7) is not correct.

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