Light-Cone Distribution Amplitudes
for Non-Relativistic Bound States

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Abstract. We calculate light-cone distribution amplitudes for non-relativistic bound states, including radiative corrections from relativistic gluon exchange to first order in the strong coupling constant. Our results apply to hard exclusive reactions with non-relativistic bound states in the QCD factorization approach like, for instance, $B_c \rightarrow \eta_c \ell \nu$ or $e^+e^- \rightarrow J/\psi \eta_c$. They also serve as a toy model for light-cone distribution amplitudes of light mesons or heavy $B$ and $D$ mesons.

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INTRODUCTION

Exclusive hadron reactions with large momentum transfer involve strong interaction dynamics at very different momentum scales. In cases where the hard-scattering process is dominated by light-like distances, the long-distance hadronic information is given in terms of so-called light-cone distribution amplitudes (LCDAs) which are defined from hadron-to-vacuum matrix elements of non-local operators with quark and gluon field operators separated along the light-cone. Representing universal hadronic properties, LCDAs can either be extracted from experimental data, or they have to be constrained by non-perturbative methods. The most extensively studied and probably best understood case is the leading-twist pion LCDA, for which experimental constraints from the $\pi - \gamma$ transition form factor [1], as well as estimates for the lowest moments from QCD sum rules [2, 3, 4] and lattice QCD [5] exist. On the other hand, our knowledge on LCDAs for heavy $B$ mesons [6, 7, 8], and even more so for heavy quarkonia [9, 10], had been relatively poor until recently.

The situation becomes somewhat simpler, if the hadron under consideration can be approximated as a non-relativistic bound state of two sufficiently heavy quarks. In this case we expect exclusive matrix elements – like transition form factors [11] and, in particular, the LCDAs – to be calculable perturbatively, since the quark masses provide an intrinsic physical infrared regulator. In these proceedings we report about results from [12], where we have calculated the LCDAs for non-relativistic meson bound states including relativistic QCD corrections to first order in the strong coupling constant at the non-relativistic matching scale which is set by the mass of the lighter quark in the hadron.
LIGHT-CONE DISTRIBUTION AMPLITUDES

The wave function for a non-relativistic (NR) bound state of a quark and an antiquark can be obtained from the solution of the Schrödinger equation with the QCD Coulomb potential. To first approximation it describes a quark with momentum $m_1 v_\mu$ and an antiquark with momentum $m_2 v_\mu$, where $v_\mu$ is the four-velocity of the meson. The spinor degrees of freedom for a non-relativistic pseudoscalar bound state are represented by the Dirac projector $\frac{1}{2}(1 + \gamma)\gamma$. The non-relativistic approximation can also serve as a toy model for bound states of light (relativistic) quarks. We will in the following refer to “heavy mesons” as "B" (where we mean the realistic example of a $B$ meson, or the toy model for a $B_q$ meson, with $m_1 \gg m_2$) and “light mesons” as "π" (where the realistic example is $\eta_c$, and the toy-model application would be the pion, with $m_1 \approx m_2$).

Definition of LCDAs for light pseudoscalar mesons

Following \cite{13,2} we define the 2-particle LCDAs of a light pseudoscalar meson via

\[
\langle \pi(P) | \bar{q}_1(y) \gamma_\mu \gamma_5 q_2(x) | 0 \rangle = -if_\pi \int_0^1 du \ e^{i(u \cdot p + \bar{u} \cdot p - x)} \left[ p_\mu \phi_\pi(u) + \frac{m_\pi^2}{2p \cdot z} z_\mu g_\pi(u) \right],
\]

\[
\langle \pi(P) | \bar{q}_1(y) i \gamma_5 q_2(x) | 0 \rangle = f_\pi \mu_\pi \int_0^1 du \ e^{i(u \cdot p + \bar{u} \cdot p - x)} \phi_p(u),
\]

\[
\langle \pi(P) | \bar{q}_1(y) \sigma_{\mu \nu} \gamma_5 q_2(x) | 0 \rangle = if_\pi \bar{\mu}_\pi (p_\mu z_\nu - p_\nu z_\mu) \int_0^1 du \ e^{i(u \cdot p + \bar{u} \cdot p - x)} \frac{2\phi_\sigma(u)}{2D - 2}
\]

with two light-like vectors $z_\mu = y_\mu - x_\mu$ and $p_\mu = P_\mu - m_\pi^2/(2P \cdot z) z_\mu$, with the usual gauge link factor $[y, x]$ (Wilson line) understood implicitly. Here $u = 1 - \bar{u}$ denotes the light-cone momentum fraction of the quark in the pion, with $\phi_\pi(u)$ being the twist-2 LCDA, while $\phi_p(u)$ and $\phi_\sigma(u)$ are of twist-3. For completeness, we have also quoted the twist-4 LCDA $g_\pi(u)$ which, like the 3-particle LCDAs, will not be considered further. All LCDAs are normalized to 1, such that the prefactors in (1) are defined in the local limit $x \to y$. In the definition of $\phi_\sigma(u)$, we have included a factor $3/(D - 1)$, such that the relation between $\mu_\pi$ and $\bar{\mu}_\pi$ from the equations of motion (eom),

\[
\bar{\mu}_\pi = \mu_\pi - (m_1 + m_2),
\]

is maintained in $D \neq 4$ dimensions. In the local limit the eom further imply

\[
\mu_\pi = \frac{m_\pi^2}{m_1 + m_2}, \quad \int_0^1 du u \phi_p(u) = \frac{1}{2} + \frac{m_1 - m_2}{2\mu_\pi}
\]

Notice that \cite{23} hold for the bare parameters and distribution amplitudes.

At tree level, and in leading order of the expansion in the non-relativistic velocities, the two quarks in the non-relativistic wave function simply share the momentum of the meson according to their masses, $p_\mu^q \simeq m_q/(m_1 + m_2) P_\mu$. For “light mesons” this implies

\[
\phi_\pi(u) \simeq \phi_p(u) \simeq g_\pi(u) \simeq \delta(u - u_0)
\]
where \( u_0 = m_1/(m_1 + m_2) \). Consequently, all positive and negative moments of the distribution amplitudes are simply given in terms of the corresponding power of \( u_0 \). Notice that \( \bar{\mu}_\pi \simeq 0 \) at tree-level, and the corresponding LCDA \( \phi_\sigma(u) \) can only be determined by considering the corresponding one-loop expressions (see [12]).

**Definition of LCDAs for heavy pseudoscalar mesons**

We define the 2-particle LCDAs of a heavy pseudoscalar \( B \) meson following [6, 14],

\[
\langle 0 | \bar{q}_{\beta}(z) h_\alpha^\pi(0) | B(v) \rangle = -i \hat{f}_B(\mu) M \frac{1 + \hat{\gamma}}{2} \left\{ 2 \tilde{\phi}_B^+ (t) + \tilde{\phi}_B^- (t) \right\} \gamma_5 \alpha \beta ,
\]

where \( v^\mu \) is the heavy meson’s velocity, \( t \equiv v \cdot z \) and \( z^2 = 0 \). Here \( \hat{f}_B \) is the (renormalization-scale dependent) decay constant in HQET. The Fourier-transformed expressions, which usually appear in factorization formulas, are given through

\[
\tilde{\phi}_B^\pm (t) = \int_0^\infty d\omega \, e^{-i\omega t} \phi_B^\pm (\omega) ,
\]

where \( \omega \) denotes the light-cone energy of the light quark in the \( B \) meson rest frame.

Including a finite spectator quark mass \( m \) and the effect of the 3-particle LCDAs \( \Psi_A, \Psi_V \) as defined in [15], the eom become

\[
\omega \phi_B^- (\omega) - m \phi_B^+ (\omega) + \frac{D-2}{2} \int_0^\omega d\eta \left[ \phi_B^+ (\eta) - \phi_B^- (\eta) \right] = (D-2) \int_0^\omega d\eta \int_0^\infty \frac{d\xi}{\xi} \frac{\partial}{\partial \xi} \left[ \Psi_A(\eta, \xi) - \Psi_V(\eta, \xi) \right] ,
\]

which is trivially fulfilled at tree-level, where \( \phi_B^+ (\omega) \simeq \phi_B^- (\omega) \simeq \delta (\omega - m) \), and \( \Psi_{V,A}(\eta, \xi) = \mathcal{O}(\alpha_s) \). We have shown in [12] that this relation also holds after including \( \alpha_s \) corrections to the NR limit. (A second relation, which has been presented in [15] and extended here to the case \( m \neq 0 \), is found to be not valid beyond tree-level.) Moreover, at tree level, the moments of the “heavy meson’s” LCDAs can be related to matrix elements of local operators in HQET [6].

**RELATIVISTIC CORRECTIONS AT ONE-LOOP**

The NR bound states are described by parton configurations with fixed momenta. Relativistic gluon exchange as in Fig. 1 leads to modifications: First, there is a correction from matching QCD (or, in the case of heavy mesons, the corresponding low-energy effective theory HQET) on the NR theory,

\[
\phi_M = \phi_M^{(0)} + \frac{\alpha_s C_F}{4\pi} \phi_M^{(1)} + \mathcal{O}(\alpha_s^2) .
\]

\[
\phi = \phi + \frac{\alpha_s C_F}{4\pi} \phi^{(1)} + \mathcal{O}(\alpha_s^2) .
\]
Secondly, there is the usual evolution under the change of the renormalization scale \[16, 7\]. In particular, the support region for the parton momenta is extended to \(0 \leq u \leq 1\) for light mesons and \(0 \leq \omega < \infty\) for heavy mesons, respectively.

**Light mesons**

We first consider the leading-order relativistic corrections to the local matrix elements. We will focus on the case of equal quark masses (results for \(m_1 \neq m_2\) can be found in \[12\]). Our result for the decay constant,

\[
f_\pi = f_{\pi}^{\text{NR}} \left[ 1 - 6 \frac{\alpha_s C_F}{4\pi} + \mathcal{O}(\alpha_s^2) \right],
\]

is in agreement with \[17\], and the result for \(\mu_\pi\) and \(\tilde{\mu}_\pi\) is consistent with the eom-constraint in \(2,3\), using \(m_\pi \simeq m_1^{\text{on}} + m_2^{\text{on}}\) in the on-shell scheme.

The remaining contributions to the NLO correction to the leading-twist LCDA contain an UV-divergent piece,

\[
\phi_\pi^{(1)}(u) \bigg|_{\text{div.}} = \frac{2}{\varepsilon} \int_0^1 dv V(u,v) \phi^{(0)}(v),
\]

which involves the well-known Efremov-Radyushkin-Brodsky-Lepage (ERBL) evolution kernel \[16\], and a finite term,

\[
\phi_\pi^{(1)}(u; \mu) = 4 \left\{ \left( \frac{\ln \frac{\mu^2}{m_\pi^2(1/2-u)^2} - 1}{m_\pi^2(1/2-u)^2} \right) \left[ \left( 1 + \frac{1}{1/2-u} \right) u \theta(1/2-u) + (u \leftrightarrow \bar{u}) \right] \right\}_+ + 4 \left\{ \frac{u(1-u)}{(1/2-u)^2} \right\}_{++}.
\]

(11)

Here the plus-distributions are defined as

\[
\int_0^1 du \left\{ \ldots \right\}_+ f(u) \equiv \int_0^1 du \left\{ \ldots \right\} \left( f(u) - f(1/2) \right),
\]

\[
\int_0^1 du \left\{ \ldots \right\}_{++} f(u) \equiv \int_0^1 du \left\{ \ldots \right\} \left( f(u) - f(1/2) - f'(1/2)(u - 1/2) \right).
\]

(12)
An independent calculation of the leading-twist LCDAs for the $\eta_c$ and $J/\psi$ meson has been presented in [9]. Our result is not in complete agreement with their findings. In particular, we find that the LCDA quoted in [9] is not normalized to unity as it should be.

At the non-relativistic scale, $\mu \simeq m$, the usual expansion of $\phi_\pi(u)$ into Gegenbauer polynomials (the eigenfunctions of the leading-order ERBL evolution equations), does not converge very well, i.e. the Gegenbauer coefficients $a_n$ drop off slower than $1/n$. A better characterization of the LCDAs at NLO is given in terms of the moments $\langle \xi^n \rangle_\pi \equiv \int_0^1 du (2u - 1)^n \phi_\pi(u)$, which are linear combinations of Gegenbauer coefficients of order $\leq n$. This corresponds to an expansion in terms of $\delta$-function and its derivatives,

$$\phi_\pi(u) = 2 \sum_n \langle \xi^n \rangle_\pi \frac{(-1)^n}{n!} \delta^{(n)}(2u - 1).$$

(14)

Results for the first few moments $\langle \xi^n \rangle_\pi$ are shown in Table 1. (Notice that the moments $\langle \xi^n \rangle_\pi$ also receive corrections from sub-leading terms in the non-relativistic expansion, see [10].)

Table 1 also contains the moments generated by ERBL evolution of the LO result (4) for two values of $\eta = \alpha_s(\mu)/\alpha_s(m)$ and in the asymptotic limit. To illustrate the change of the shape of $\phi_\pi(u)$ under evolution, we employ a parametrization which is obtained from a slight modification of the strategy developed in [18],

$$\phi_\pi(u) = \frac{3u\bar{u}}{\Gamma[a, -\ln t_c]} \int_0^{t_c} dt (-\ln t)^{\alpha - 1} \left( f(2u - 1, it^{1/b}) + f(2u - 1, -it^{1/b}) \right).$$

(15)

It involves three real parameters $a > 0, b > 0, t_c \leq 1$, which can be fitted to the first three moments $\langle \xi^{2,4,6} \rangle$, and the generating function for the Gegenbauer polynomials,

$$f(\xi, \theta) = \frac{1}{(1 - 2\xi \theta + \theta^2)^{3/2}} = \sum_{n=0}^{\infty} C_n^{3/2}(\xi) \theta^n.$$

(16)

Fig. 2(a) shows the evolution of the model LCDA as a function of $u$. For $\eta = 1/5$ the functional form still “remembers” the non-relativistic profile, while for $\eta = 1/25$ the LCDA gets close to the asymptotic form.

The twist-3 LCDAs for the 2-particle Fock states are obtained in the same way as the twist-2 one. Details can be found in [12]. In particular, all our results are in manifest agreement with the eom-constraint from (3).
Heavy mesons

The calculation of the LCDA for a heavy meson goes along the same lines as for the light-meson case. However, important differences arise because the heavier quark is to be treated in HQET which modifies the divergence structure of the loop integrals. As a consequence, the evolution equations for the LCDA of heavy mesons \[7\] differ from those of light mesons.

Let us focus on the distribution amplitude \( \phi_B^+ (\omega) \) which enters the QCD factorization formulas for exclusive heavy-to-light decays. In the local limit we derive the corrections from soft gluon exchange to the decay constant in HQET,

\[
\hat{f}_M (\mu) = f_M^{NR} \left[ 1 + \frac{\alpha_s C_F}{4\pi} \left( 3 \ln \frac{\mu}{m} - 4 \right) + \mathcal{O} (\alpha_s^2) \right]. \tag{17}
\]

Notice that the decay constant of a heavy meson exhibits the well-known scale dependence \[19\]. The remaining NLO corrections to the distribution amplitude \( \phi_B^+ (\omega) \) contain an UV-divergent piece, which involves the Lange-Neubert evolution kernel \( \gamma_+ (\omega, \omega', \mu) \) \[7\]. As has been shown by Lee and Neubert \[20\], the solution of the evolution equation can be written in closed form. The resulting evolution of \( \phi_B^+ (\omega, \mu) \) is illustrated in Fig. 2(b), starting from \( \delta (\omega - m) \) at the non-relativistic scale, for three different values of \( \eta = \alpha_s (\mu) / \alpha_s (m) \) and taking \( \alpha_s (m) = 1 \). From the double-logarithmic plot we can read off the asymptotic behaviour of the LCDA for \( \omega \to 0 \) and \( \omega \to \infty \). As argued on general grounds \[7\], \( \phi_B^+ (\omega) \) develops a linear behaviour for \( \omega \to 0 \), whereas for \( \omega \to \infty \) the evolution generates a radiative tail which tends to fall off slower than \( 1/\omega \) at higher scales.
The finite NLO correction to $\phi_B^+(\omega)$ reads

$$\frac{\phi_B^{(+,1)}(\omega; \mu)}{\omega} = 2 \left( \ln \left( \frac{\mu^2}{(\omega - m)^2} \right) - 1 \right) \left( \frac{\theta(m - \omega)}{m(m - \omega)} + \frac{\theta(\omega - m)}{\omega(\omega - m)} \right) +$$

$$+ 4 \left[ \frac{\theta(2m - \omega)}{(\omega - m)^2} \right]_{++} + \frac{4 \theta(2m - \omega)}{(\omega - m)^2} - \frac{\delta(\omega - m)}{m} \left( \frac{1}{2} \ln^2 \frac{\mu^2}{m^2} - \ln \frac{\mu^2}{m^2} + \frac{3\pi^2}{4} + 2 \right)$$

(18)

with an analogous definition of plus-distributions as in [12]. In contrast to the light-meson case, the normalization of the heavy meson distribution amplitude is ill-defined. Imposing a hard cutoff $\Lambda_{UV} \gg m$ and expanding to first order in $m/\Lambda_{UV}$, we derive

$$\int_0^{\Lambda_{UV}} d\omega \, \phi_B^+(\omega; \mu) \simeq 1 - \frac{\alpha_s C_F}{4\pi} \left[ \frac{1}{2} \ln^2 \frac{\mu^2}{\Lambda_{UV}^2} + \frac{\mu^2}{\Lambda_{UV}^2} + \frac{\pi^2}{12} \right] + \ldots$$

(19)

$$\int_0^{\Lambda_{UV}} d\omega \, \omega \phi_B^+(\omega; \mu) \simeq \frac{\alpha_s C_F}{4\pi} \left[ 2 \ln \frac{\mu^2}{\Lambda_{UV}^2} + 6 \right] \Lambda_{UV} + \ldots$$

(20)

The last two expressions provide model-independent properties of the distribution amplitude which agree with the general results in [20]. The two phenomenologically relevant moments in the factorization approach to heavy-to-light decays read

$$\lambda_B^{-1}(\mu) \equiv \int_0^{\infty} d\omega \, \phi_B^+(\omega; \mu) = \frac{1}{m} \left( 1 - \frac{\alpha_s C_F}{4\pi} \left[ \frac{1}{2} \ln^2 \frac{\mu^2}{m^2} - \ln \frac{\mu^2}{m^2} + \frac{3\pi^2}{4} - 2 \right] \right),$$

$$\sigma_B(\mu) \equiv \lambda_B(\mu) \int_0^{\infty} d\omega \, \frac{\phi_B^+(\omega; \mu)}{\omega} \ln \frac{\mu}{\omega} = \ln \frac{\mu}{m} + \frac{\alpha_s C_F}{4\pi} \left[ 8\zeta(3) \right].$$

(21)

A similar analysis can be performed for the LCDA $\phi_B^-(\omega)$, for details we refer to [12]. In particular, we can read off the anomalous dimension,

$$\gamma_{-}^{(1)}(\omega, \omega'; \mu) = \left( 4 \ln \frac{\mu}{\omega} - 2 \right) \delta(\omega - \omega') - 4 \frac{\theta(\omega' - \omega)}{\omega'}$$

$$- 4 \omega \frac{\theta(\omega' - \omega)}{\omega' (\omega' - \omega)} \right)_{++} - 4 \omega \frac{\theta(\omega - \omega')}{\omega (\omega - \omega')} \right),$$

(22)

which describes the evolution of $\phi_B^-(\omega, \mu)$ in the Wandzura-Wilczek approximation, where 3-particle LCDAs (and the light quark mass $m$) are neglected. Among others, $\gamma_{-}$ is needed to show the factorization of correlation functions in SCET sum rules for heavy-to-light form factors [21]. Another new result are the first positive moments of $\phi_B^-(\omega)$ as a function of the UV cutoff,

$$\int_0^{\Lambda_{UV}} d\omega \, \phi_B^-(\omega; \mu) \simeq 1 - \frac{\alpha_s C_F}{4\pi} \left[ \frac{1}{2} \ln^2 \frac{\mu^2}{\Lambda_{UV}^2} - \ln \frac{\mu^2}{\Lambda_{UV}^2} + \frac{\pi^2}{12} \right] + \ldots$$

(23)

$$\int_0^{\Lambda_{UV}} d\omega \, \omega \phi_B^-(\omega; \mu) \simeq \frac{\alpha_s C_F}{4\pi} \left[ 2 \ln \frac{\mu^2}{\Lambda_{UV}^2} + 2 \right] \Lambda_{UV} + \ldots$$

(24)

where the divergent pieces again are expected to be model-independent.
SUMMARY

Non-relativistic $q\bar{q}$ bound states have been used as a starting point to construct light-cone distribution amplitudes in QCD and in HQET. We considered relativistic gluon corrections at NLO in the strong coupling at the non-relativistic scale, as well as the leading logarithmic evolution towards higher scales needed in QCD factorization theorems. We also studied certain model-independent properties of light and heavy LCDAs, including new results for the LCDA $\phi^B(\omega, \mu)$ in HQET.

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