ON C*-EXTREME MAPS AND *-HOMOMORPHISMS OF A
COMMUTATIVE C*-ALGEBRA

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Abstract. The generalized state space of a commutative C*-algebra, denoted $S_{\mathcal{H}}(C(X))$, is the set of positive unital maps from $C(X)$ to the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. C*-convexity is one of several non-commutative analogs of convexity which have been discussed in this context. In this paper we show that a C*-extreme point of $S_{\mathcal{H}}(C(X))$ satisfies a certain spectral condition on the operators in the range of the associated positive operator-valued measure. This result enables us to show that C*-extreme maps from $C(X)$ into $K^+$, the algebra generated by the compact and scalar operators, are multiplicative. This generalizes a result of D. Farenick and P. Morenz. We then determine the structure of these maps.

Several non-commutative analogs of convexity have appeared in the literature including CP-convexity [4] and matrix convexity [2], as well as C*-convexity [3], [6], which is the topic of this paper. In [6], Hopenwasser, Moore, and Paulsen characterized operators which are C*-extreme in the unit ball of $\mathcal{B}(\mathcal{H})$ and obtained results about other C*-convex sets and their extreme points. In [3], Farenick and Morenz extend the idea of C*-convexity to the space of completely positive maps on a C*-algebra. They show that C*-extreme maps with their range in $K^+$ are also extreme (in the classical sense) and obtain a characterization of C*-extreme maps on a commutative C*-algebra which have their range in $M_n$, the C*-algebra of $n \times n$ complex matrices. Subsequently, Zhou [9] gave two necessary and sufficient conditions for a completely positive map to be C*-extreme, and described the structure of C*-extreme maps with range in $M_n$. The main results presented here are Theorem 5 which gives a necessary condition for a map $\phi : C(X) \to \mathcal{B}(\mathcal{H})$ to be C*-extreme, and Theorem 10 which then shows that a positive unital map $\phi : C(X) \to K^+$ is C*-extreme if and only if it is multiplicative. We then determine the structure of such maps.

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Throughout, let $X$ be a compact Hausdorff space, $C(X)$ the C*-algebra of continuous functions on $X$, and $\mathcal{H}$ a Hilbert space.

Definition 1. The generalized state space of $C(X)$ is

$$S_{\mathcal{H}}(C(X)) = \{ \phi : C(X) \to \mathcal{B}(\mathcal{H}) | \phi \text{ is positive and } \phi(1) = I \}.$$
Note that in the case of a non-commutative $C^*$-algebra $\mathfrak{A}$, the generalized state space $S_{\mathfrak{H}}(A)$ is the set of completely positive unital maps. However, for a commutative $C^*$-algebra, every positive map is also completely positive [8, Theorem 4]. If $\mathcal{H} = \mathbb{C}$, the generalized state space $S_{\mathbb{C}}(\mathfrak{A})$ coincides with the classical state space of $\mathfrak{A}$.

**Definition 2.** We say that $\phi, \psi \in S_{\mathcal{H}}(C(X))$ are unitarily equivalent, and write $\phi \sim \psi$, if there is a unitary $u \in B(\mathcal{H})$ such that $\phi(f) = u^* \psi(f) u$ for every $f \in C(X)$.

**Definition 3.** If $\phi, \psi_1, \ldots, \psi_n \in S_{\mathcal{H}}(C(X))$ and $t_1, \ldots, t_n \in B(\mathcal{H})$ are invertible with $t_1^* t_1 + \ldots + t_n^* t_n = I$, then we say

$$\phi(f) = t_1^* \psi_1(f) t_1 + \ldots + t_n^* \psi_n(f) t_n$$

for every $f \in C(X)$, is a proper $C^*$-convex combination. We call a map $\phi \in S_{\mathcal{H}}(C(X))$ $C^*$-extreme if, whenever $\phi$ is written as a proper $C^*$-convex combination of $\psi_1, \ldots, \psi_n$, then $\psi_j \sim \phi$ for each $j = 1, \ldots, n$.

We begin with a discussion of $B(\mathcal{H})$-valued measures, which closely follows the development given in Paulsen [7]. These operator valued measures play a key role in the proof of Theorem 5, below. Given a bounded linear map $\phi : C(X) \to B(\mathcal{H})$ and vectors $x, y \in \mathcal{H}$, the bounded linear functional

$$f \mapsto \langle \phi(f)x, y \rangle$$

corresponds to a unique regular Borel measure $\mu_{x,y}$ on $X$ such that

$$\int_X f d\mu_{x,y} := \langle \phi(f)x, y \rangle$$

for any $f \in C(X)$. Denote the $\sigma$-algebra of Borel sets of $X$ by $\mathcal{S}$. For a set $B \in \mathcal{S}$, the sesquilinear form

$$(x, y) \mapsto \mu_{x,y}(B)$$

then determines an operator $\mu(B)$. Thus we obtain an operator-valued measure $\mu : \mathcal{S} \to B(\mathcal{H})$ which is:

1. weakly countably additive, i.e., if $\{B_i\}_{i=1}^\infty \subseteq \mathcal{S}$ are pairwise disjoint, and $B = \bigcup_{i=1}^\infty B_i$ then

$$\langle \mu(B)x, y \rangle = \sum_{i=1}^\infty \langle \mu(B_i)x, y \rangle$$

for every $x, y \in \mathcal{H}$.

2. bounded, i.e., $\|\mu\| := \sup\{\|\mu(B)\| : B \in \mathcal{S}\} < \infty$.

3. regular, i.e., for each pair of vectors $x$ and $y$ in $\mathcal{H}$, the complex measure $\mu_{x,y}$ is regular.

Furthermore, this process works in reverse: given a regular bounded operator-valued measure $\mu : \mathcal{S} \to B(\mathcal{H})$, define Borel measures

$$\mu_{x,y}(B) := \langle \mu(B)x, y \rangle$$

for each $x, y \in \mathcal{H}$. Then the operator $\phi(f)$ is uniquely defined by the equations

$$\langle \phi(f)x, y \rangle := \int_X f d\mu_{x,y};$$

the map $\phi : C(X) \to B(\mathcal{H})$ is then seen to be bounded and linear. This construction shows that each operator valued-measure gives rise to a unique bounded linear map, and vice-versa. The following proposition summarizes properties shared by
operator valued-measures and their associated linear maps. We will be most concerned with parts (2) and (4) of Proposition 4; part (4) is, of course, the Spectral Theorem.

**Proposition 4.** [Proposition 4.5] Given an operator valued measure $\mu$ and its associated linear map $\phi$,

1. $\phi$ is self-adjoint if and only if $\mu$ is self-adjoint,
2. $\phi$ is positive if and only if $\mu$ is positive,
3. $\phi$ is a homomorphism if and only if $\mu(B_1 \cap B_2) = \mu(B_1)\mu(B_2)$ for all $B_1, B_2 \in S$,
4. $\phi$ is a *-homomorphism if and only if $\mu$ is spectral (i.e., projection-valued).

We note the following important features of positive operator valued measures, and their relationship to the associated positive maps.

1. Let $\mathfrak{g}(X) = \{f : X \to \mathbb{C} \mid f$ is a bounded Borel measurable function\}. If $\phi : C(X) \to B(\mathcal{H})$ is a positive map, we may use the corresponding positive operator-valued measure to extend $\phi$ to a map $\tilde{\phi} : \mathfrak{g}(X) \to B(\mathcal{H})$ by defining

$$\tilde{\phi}(f) = \int_X f d\mu_\phi,$$

for every $f \in \mathfrak{g}(X)$. The measure $\mu_\phi$ may then be viewed as the restriction of $\tilde{\phi}$ to the characteristic functions of Borel sets. For simplicity, we will simply write $\tilde{\phi}$, rather than $\phi$, and use the notations $\mu_\phi(F)$ and $\phi(\chi_F)$ interchangeably.

2. A positive unital map $\phi \in S_H(C(X))$ is $C^*$-extreme if and only if the associated operator-valued measure $\mu_\phi$ is $C^*$-extreme. (Here, a positive operator-valued measure $\mu_\phi$ is called $C^*$-extreme if, whenever $\mu_\phi$ is written

$$\mu_\phi = t_1^*\mu_1 t_1 + \ldots + t_n^*\mu_n t_n,$$

where $\sum_{j=1}^n t_j^* t_j = I$ and each $\mu_j$ is a positive operator-valued measure, then $\mu_j \sim \mu_\phi$ for each $j = 1, \ldots, n$.)

3. Finally, if $\phi : C(X) \to B(\mathcal{H})$ is a positive bounded linear map, and $\mu_\phi$ the associated operator-valued measure, then for each Borel set $F \subseteq X$, $\mu_\phi(F) \in \text{wot-cl } \phi(C(X))$, the weak operator topology closure of $\phi(C(X))$. The proof of this fact requires some care, because while $\phi(C(X))$ is an operator space, it is not generally an algebra.

**Proof.** Let $G \subseteq X$ be an open set. Then a basic WOT-open set in $B(\mathcal{H})$ centered at $\phi(\chi_G)$ has the form:

$$\mathcal{O} = \{T \in B(\mathcal{H}) : \|((T - \phi(\chi_G))x_i, y_i)\| < \varepsilon \text{ for } i = 1, \ldots, n\},$$

where $x_i, y_i \in \mathcal{H}$ and $\varepsilon > 0$. We wish to show that for any such open set there is a function $f \in C(X)$ with $\phi(f) \in \mathcal{O}$. For each $j$, we can write

$$\mu_{x_j, y_j} = \mu_{j,1} - \mu_{j,2} + i(\mu_{j,3} - \mu_{j,4}),$$

where each $\mu_{j,k}$ is a positive measure. Since each of these measures is regular, we may choose compact sets $K_{j,k} \subseteq G$ for $j = 1, \ldots, n$ and $k = 1, \ldots, 4$ such that

$$\mu_{j,k}(G \setminus K_{j,k}) < \frac{\varepsilon}{4}.$$

Then, setting

$$K = \bigcup_{j=1}^n \bigcup_{k=1}^4 K_{j,k}$$
we have, for \(j = 1, \ldots, n\),
\[
|\mu_{x_j,y_j}(G \setminus K)| \leq |\mu_{j,1}(G \setminus K)| + \cdots + |\mu_{j,n}(G \setminus K)| < \varepsilon.
\]
Urysohn’s Lemma now guarantees the existence of a continuous function \(f : X \to [0, 1]\) with \(f|_K = 1\) and \(f|_{G^C} = 0\). Hence, for each \(j = 1, \ldots, n\),
\[
|\langle \phi(f) - \mu_\phi(K) \rangle_{x_j,y_j}| = \left| \int_X (f - \chi_K) d\mu_{j,1} - \int_X (f - \chi_K) d\mu_{j,2} + i \left( \int_X (f - \chi_K) d\mu_{j,1} - \int_X (f - \chi_K) d\mu_{j,4} \right) \right|
\leq \int_X \chi_{G^C} d\mu_{j,1} + \cdots + \int_X \chi_{G^C} d\mu_{j,4}
< \varepsilon.
\]
Therefore \(\phi(f) \in \mathcal{O}\), as required; hence \(\phi(f) \in \text{wot-cl } \phi(C(X))\).

Now let
\[
\mathcal{F} = \{ F \subseteq X : \text{ F is a Borel set and } \phi(\chi_F) \subseteq \text{wot-cl } \phi(C(X)) \}.
\]
We will prove that \(\mathcal{F}\) is a \(\sigma\)-algebra containing the Borel sets, and hence that \(\mathcal{F} = \mathcal{S}\).

Our discussion above shows that \(\mathcal{F}\) contains every open set of \(X\). Suppose that \(\{B_i\}\) is a countable family of sets in \(\mathcal{F}\) and set \(B = \bigcup_{i=1}^\infty B_i\). Assume without loss of generality that \(\{B_i\}\) are a disjoint family. Then, since \(\mu_\phi\) is weakly countably additive,
\[
\langle \mu_\phi(B) x, y \rangle = \sum_{i=1}^\infty \langle \mu_\phi(B_i) x, y \rangle
\]
for any \(x, y \in \mathcal{H}\). That is
\[
\mu_\phi(B) = \text{wot-lim } \mu_\phi \left( \bigcup_{i=1}^N B_i \right);
\]
It follows that \(B \in \mathcal{F}\). Furthermore, if \(F \in \mathcal{F}\), then
\[
\phi(\chi_{F^C}) = \phi(1 - \chi_F) = I - \phi(\chi_F),
\]
so that \(F^C \in \mathcal{F}\) also. Therefore \(\mathcal{F}\) is the \(\sigma\)-algebra of Borel sets of \(X\). \(\Box\)

Thus, if the range of \(\phi\) is contained in a \(C^*\)-subalgebra \(\mathcal{A}\) of \(\mathcal{B}(\mathcal{H})\), then the range of \(\mu_\phi\) is contained in the weak operator topology closure of \(\mathcal{A}\), i.e., \(\mathcal{A}''\).

We can now prove the following theorem, which gives a necessary condition for a positive map \(\phi\) on a commutative \(C^*\)-algebra (or equivalently its associated positive operator-valued measure) to be \(C^*\)-extreme.

**Theorem 5.** Let \(X\) be a compact Hausdorff space, and \(\phi : C(X) \to \mathcal{B}(\mathcal{H})\) a unital, positive map. Denote by \(\mu_\phi\) the unique positive operator-valued measure associated to \(\phi\). If \(\phi\) is \(C^*\)-extreme, then for every Borel set \(F \subseteq X\), either

1. \(\mu_\phi(F)\) is a projection, in which case \(\mu_\phi(F) \in \phi(C(X))'\), or
2. \(\sigma(\mu_\phi(F)) = [0, 1]\).

Moreover, if (2) occurs and \(\mu_\phi(F)\) has an eigenvalue in \((0, 1)\), then the point spectrum of \(\mu_\phi(F)\) must contain \((0, 1)\).
Proof. Suppose there is a Borel set \( F \subseteq X \) so that \( \mu_\phi(F) \) is not a projection and \( \sigma(\mu_\phi(F)) \neq [0, 1] \). We will show that \( \phi \) is not \( C^* \)-extreme by constructing a proper \( C^* \)-convex combination

\[
t^*_1 \psi_1 t_1 + t^*_2 \psi_2 t_2 = \phi
\]

in which \( \psi_1 \) and \( \psi_2 \) are not unitarily equivalent to \( \phi \). Choose \( x \in (0, 1) \setminus \sigma(\mu_\phi(F)) \) and let \( (a, b) \) be the largest open subinterval of \( (0, 1) \) which contains \( x \) but does not intersect \( \sigma(\mu_\phi(F)) \). To be precise, let

\[
(a, b) = \bigcup \{ (\alpha, \beta) \subseteq (0, 1) : x \in (\alpha, \beta), (\alpha, \beta) \cap \sigma(\mu_\phi(F)) = \emptyset \}
\]

Note that this choice of the interval \( (a, b) \) insures that at least one of the pair \( \{a, b\} \) is in \( \sigma(\mu_\phi(F)) \). In particular, if \( a > 0 \) then \( a \in \sigma(\mu_\phi(F)) \) and if \( b < 1 \) then \( b \in \sigma(\mu_\phi(F)) \). Choose \( s_1 \in \left( \frac{1}{2}, \frac{1}{2} \right) \) with \( s_1 > \frac{1}{2} \left( \frac{a - b}{b - ab} \right) \), and set \( s_2 = 1 - s_1 \). For \( k = 1, 2 \), define

\[
Q_k = \frac{1}{2} \mu_\phi(F) + s_k \mu_\phi(F^C) = s_k I + \left( \frac{1}{2} - s_k \right) \mu_\phi(F).
\]

Note that \( 0 \not\in \sigma(Q_k) = s_k + \left( \frac{1}{2} - s_k \right) \sigma(\mu_\phi(F)) \), so that both \( Q_k \)'s are invertible. Now define new positive operator-valued measures \( \mu_1 \) and \( \mu_2 \) by

\[
\mu_k(B) = Q^{-\frac{1}{2}}_k \left( \frac{1}{2} \mu_\phi(B \cap F) + s_k \mu_\phi(B \cap F^C) \right) Q_\phi^{-\frac{1}{2}},
\]

where \( B \) is any Borel set of \( X \). Observe that each of the \( \mu_k \)'s is a positive operator-valued measure with \( \mu_k(X) = I \). Next, define \( t_k = Q^{-\frac{1}{2}}_k \), for \( k = 1, 2 \). Then, for any Borel set \( B \) of \( X \),

\[
t^*_1 \mu_1(B) t_1 + t^*_2 \mu_2(B) t_2 = \frac{1}{2} \mu_\phi(B \cap F) + s_1 \mu_\phi(B \cap F^C)
\]

\[
+ \frac{1}{2} \mu_\phi(B \cap F) + s_2 \mu_\phi(B \cap F^C)
\]

\[
= \mu_\phi(B).
\]

Each \( t_k \) is invertible and

\[
t^*_1 t_1 + t^*_2 t_2 = Q_1 + Q_2 = \mu_\phi(F) + \mu_\phi(F^C) = I.
\]

Thus \( t^*_1 \mu_1 t_1 + t^*_2 \mu_2 t_2 \) is a proper \( C^* \)-convex combination of \( \mu_1 \) and \( \mu_2 \).

It is still necessary to show that \( \phi \) is not unitarily equivalent to at least one of \( \mu_1 \) or \( \mu_2 \). For \( k = 1, 2 \), set \( g_k(t) = [s_k + (s_k - \frac{1}{2}) t^{-\frac{1}{2}}]^{-1} \). As each \( g_k \) is continuous on \([0, 1]\), and \( Q_k^{-\frac{1}{2}} = g_k(\mu_\phi(F)) \), \( Q_k^{-\frac{1}{2}} \) commutes with \( \mu_\phi(F) \). Thus, for \( k = 1, 2 \), we have

\[
\mu_k(F) = Q_k^{-\frac{1}{2}} \left( \frac{1}{2} \mu_\phi(F) \right) Q_k^{-\frac{1}{2}}
\]

\[
= \frac{1}{2} \mu_\phi(F) \left( s_k I + \left( \frac{1}{2} - s_k \right) \mu_\phi(F) \right)^{-1}.
\]

Let \( f_k(t) = \frac{1}{2} t (s_k + (s_k - \frac{1}{2}) t^{-\frac{1}{2}}) \)^{-1}. Observe that each \( f_k \) is continuous on \([0, 1]\), and that \( \mu_k(F) = f_k(\mu_\phi(F)) \). Therefore, by the spectral mapping theorem, \( \sigma(\mu_k(F)) = f_k(\sigma(\mu_\phi(F))) \). It is easy to check that for \( t \in (0, 1) \), \( t < f_k(1) < 1 \),
while $0 < f_2(t) < t$, and that both $f_k$’s are strictly increasing. In addition, since $s_1 > \frac{1}{2} \left( \frac{a-b}{b-a} \right)$, if $a > 0$,
\[ a < f_1(a) = \frac{1}{2} a \left( \frac{1}{s_1 + (\frac{1}{2} - s_1)a} < \frac{a}{\frac{a(1-b)}{b(1-a)}(1-a) + a} = b \leq f_1(b). \]

Consider the following two cases:

**Case (i):** $a \neq 0$. In this case $a \in \sigma(\mu_\phi(F))$. Thus $f_1(a) \in \sigma(\mu_1(F))$, but since $f_1(a) \in (0, b)$, $f_1(a) \notin \sigma(\mu_\phi(F))$. This shows that $\sigma(\mu_\phi(F)) \neq \sigma(\mu_1(F))$; therefore $\mu_\phi$ and $\mu_1$ are not unitarily equivalent.

**Case (ii):** $a = 0$. In this case, $b < 1$ and $b \in \sigma(\mu_\phi(F))$. As $a = 0 < f_2(b) < b$, we have $f_2(b) \in \sigma(\mu_2(F)) \setminus \sigma(\mu_\phi(F))$. In this case, $\mu_2$ is not unitarily equivalent to $\mu_\phi$.

Let $\psi_k$ be the positive map determined by $\mu_k$. Then $\phi = t_1^* \psi_1 t_1 + t_2^* \psi_2 t_2$; this is a proper $C^*$-convex combination of $\psi_1$ and $\psi_2$, where $\phi$ is not unitarily equivalent to at least one of the maps $\psi_k$. Therefore, $\phi$ is not $C^*$-extreme.

Now suppose that $\sigma(\mu_\phi(F)) = [0, 1]$ and that $\sigma_{pt}(\mu_\phi(F))$ intersects $(0, 1)$, but does not contain $(0, 1)$. It is not difficult to convince oneself that it is possible to choose $a, b \in (0, 1)$ satisfying both

(i) $a < b < \frac{2b}{a+1}$, and

(ii) exactly one of the pair $\{a, b\}$ is an eigenvalue.

Set $s_1 = \frac{1}{2} \left( \frac{a-b}{b-a} \right)$ and define positive operator-valued measures $\mu_1$ and $\mu_2$ as above. As in the previous computation, $\mu_1(F) = f_1(\mu_\phi(F))$. As a result of our choice of $s_1$, $f_1(a) = b$. Application of the Spectral Mapping Theorem then shows that either

\[ b \in \sigma_{pt}(\mu_1(F)) \setminus \sigma_{pt}(\mu_\phi(F)), \text{ or} \]

\[ b \in \sigma_{pt}(\mu_\phi(F)) \setminus \sigma_{pt}(\mu_1(F)). \]

Since the point spectrum is also a unitary invariant, and $\mu_\phi = t_1^* \mu_1 t_1 + t_2^* \mu_2 t_2$, this shows that $\phi$ is not $C^*$-extreme.

Finally, we wish to show that any projection in the range of $\mu_\phi$ must commute with $\phi(C(X))$. Suppose that $\mu_\phi(F)$ is a projection and choose $f \in C(X)$ with $0 \leq f \leq 1$. Write

\[ f = \chi_f f + (1 - \chi_f) f. \]

Then $\phi(\chi_f f) \leq \mu_\phi(F)$, so these operators commute. Similarly,

\[ \phi((1 - \chi_f) f) \leq \mu_\phi(X \setminus F) = I - \mu_\phi(F), \]

so that $\phi((1 - \chi_f) f)$ also commutes with $\mu_\phi(F)$. Therefore $\phi(f)$ commutes with $\mu_\phi(F)$. If $f$ is an arbitrary continuous function, we can express $f$ as a linear combination of positive functions with ranges in $[0, 1]$. Thus $f$ will commute with $\mu_\phi(F)$. \[ \square \]

In their paper of 1997 [3], Farenick and Morenz show that a positive map from a commutative $C^*$-algebra into a matrix algebra $M_n$ is $C^*$-extreme if and only if it is a $\ast$-homomorphism. In view of the spectral condition given by Theorem [5], a shorter proof is possible.
Corollary 6. [3, Proposition 2.2] Let $X$ be a compact Hausdorff space and $\phi : C(X) \to M_n$ a positive map. Then $\phi$ is $C^*$-extreme if and only if it is a $*$-homomorphism.

Proof. It is already known that if $\phi$ is a representation (i.e., $*$-homomorphism), then $\phi$ is $C^*$-extreme [3 Proposition 1.2]. On the other hand, if $\phi$ is not a representation, then the associated positive operator-valued measure $\mu_\phi$ is not a spectral measure. In this case, there is a Borel set $F \subset X$ for which $\mu_\phi(F)$ is not a projection. As $\mu_\phi(F)$ is an $n \times n$ matrix, $\sigma(\mu_\phi(F))$ consists of at most $n$ isolated points. We may therefore apply the theorem to conclude that $\phi$ is not $C^*$-extreme. \hfill $\Box$

Note that in the proof of Theorem [5, Proposition 2.2] $Q_k$, $Q_k^\perp$ and $t_k = Q_k^\perp$ are elements of the $C^*$-algebra generated by $\mu_\phi(F)$. As noted in the remark preceding Theorem 5, the range of $\mu_\phi$ is contained in the WOT-closure of the range of $\phi$. Thus we have the following corollary:

Corollary 7. Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, $\phi : C(X) \to M$ a unital positive map, and $\mu_\phi$ the positive operator-valued measure associated to $\phi$. If $\phi$ fails to meet the spectral condition described in Theorem [5], then $\phi$ can be written as a proper $C^*$-convex combination

$$\phi = t_1^* \psi_1 t_1 + t_2^* \psi_2 t_2,$$

where each $t_k \in M$, each $\psi_k : C(X) \to M$, and, for at least one choice of $k$, $\psi_k$ is not unitarily equivalent to $\phi$ in $\mathcal{B}(\mathcal{H})$.

We now consider an example of a $C^*$-extreme map which is not multiplicative. The positive map $\phi$ defined below was considered by Arveson [1, p. 164] as an example of an extreme point in the generalized state space. Farenick and Morenz [3, Example 2] subsequently showed that $\phi$ is also a $C^*$-extreme point, although not a homomorphism. Consider the Hilbert spaces $L^2(\mathbb{T}, m)$, where $m$ is normalized Lebesgue measure on $\mathbb{T}$, and $H^2$, the classical Hardy space. Let $P$ be the projection of $L^2(\mathbb{T}, m)$ onto $H^2$. For a function $f \in L^2(\mathbb{T}, m)$ denote by $M_f$ multiplication by $f$ and by $T_f = PM_f P$ the Toeplitz operator for $f$.

Example 8. [1, 3] Consider the representation $\pi : C(\mathbb{T}) \to \mathcal{B}(L^2(\mathbb{T}, m))$ given by $\pi(f) = M_f$. The spectral measure associated to $\pi$ is given by $\mu_\pi(B) = M_{\chi_B}$, where $B \subseteq X$ is a Borel set. Define a unital positive map

$$\phi : C(\mathbb{T}) \to \mathcal{B}(H^2)$$

by

$$\phi(f) = PM_f P.$$  

Since $\mu_\pi(B) = M_{\chi_B}$, we have $\mu_\phi(B) = PM_{\chi_B} P = T_{\chi_B}$, a Toeplitz operator. Thus $\sigma(\mu_\phi(B)) = \sigma(T_{\chi_B})$. Since $\chi_B$ is a real-valued $L^\infty$ function, $\sigma(T_{\chi_B})$ is the closed convex hull of the essential range of $\chi_B$ [5, p. 868]. Therefore, if $\mu_\phi(B) \notin \{0, I\}$, then $\sigma(\mu_\phi(B)) = [0, 1]$. Thus, for any Borel set $B \subseteq X$, either $\mu_\phi(B) = [0, 1]$ or $\mu_\phi(B)$ is a trivial projection; that is, $\phi$ satisfies the conditions of the theorem.

Now let us consider the case of a unital positive map $\phi$ on a commutative $C^*$-algebra $C(X)$ whose range is in $\mathcal{K}^+$, the $C^*$-algebra generated by the compact operators and the identity operator. In [3, Proposition 1.1] Farenick and Morenz show that if such a map $\phi$ is $C^*$-extreme, then $\phi$ is also extreme. It is possible, however, to say more. Theorem 5 requires the operators in the range of the positive
operator-valued measure $\mu_\phi$ either to be projections, or to have spectrum equal to $[0,1]$. In contrast, the spectrum of a positive operator $K + \alpha I \in \mathcal{K}^+$ must be a sequence of positive numbers with a single limit point at $\alpha$. This dichotomy suggests that Theorem 5 may give additional information about these maps. In fact, both the result of Theorem 5 (the spectral condition on the operators in the range of $\mu_\phi$) and the technique used in its proof, will be used below. The result is Theorem 10, which shows that such maps must be multiplicative, and gives their structure.

In the succeeding lemma and theorem, let $q$ be the usual quotient map $q : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, and set $\tau = q \circ \phi$. Then $\tau$ is a positive linear functional, so there is a unique positive real-valued Borel measure $\mu_\tau$ on $X$ so that

$$\tau(f) = \int_X f d\mu_\tau \text{ for every } f \in C(X).$$

For any function $f \in C(X)$, write

$$\phi(f) = K_f + \tau(f)I,$$

where $K_f \in \mathcal{K}$ is a compact operator.

**Lemma 9.** Let $\phi : C(X) \to \mathcal{K}^+$ be unital, positive, and $C^*$-extreme. Then the map $\tau$ is multiplicative.

**Proof.** As in the proof of Theorem 5 we will prove the contrapositive. Assume that $\tau$ is not multiplicative; then the support of $\mu_\tau$ must contain at least two distinct points, which we will call $s_1$ and $s_2$. Let $N_1$ be a neighborhood of $s_1$ which does not contain $s_2$. By Urysohn’s Lemma, there exists a continuous function $f : X \to [0,1]$ such that $f(s_1) = 1$ and $f\big|_{N_1^c} = 0$.

Choose $\alpha$ and $\beta$ in $(0,1)$ with $\alpha > \beta$ and let

$$Q_1 = \alpha\phi(f) + \beta\phi(1-f) = (\alpha - \beta)\phi(f) + \beta I,$$

and

$$Q_2 = (1 - \alpha)\phi(f) + (1 - \beta)\phi(1-f) = (\beta - \alpha)\phi(f) + (1 - \beta)I.$$

Note that since $0 \leq f \leq 1$, the spectrum of $\phi(f)$ is contained in the closed unit interval. Thus,

$$\sigma(Q_1) \subseteq [\beta, \alpha], \text{ and}$$

$$\sigma(Q_2) \subseteq [1 - \alpha, 1 - \beta].$$

So both $Q_j$’s are invertible positive operators. Define maps $\psi_1$ and $\psi_2$ by

$$\psi_1(g) = Q_1^{-\frac{1}{2}}[\alpha\phi(fg) + \beta\phi((1-f)g)]Q_1^{-\frac{1}{2}}, \text{ and}$$

$$\psi_2(g) = Q_2^{-\frac{1}{2}}[(1 - \alpha)\phi(fg) + (1 - \beta)\phi((1-f)g)]Q_2^{-\frac{1}{2}}.$$

Both $\psi_j$’s are positive, unital maps with ranges in $\mathcal{K}^+$. Setting $t_j = Q_j^{-\frac{1}{2}}$, we have

$$t_1^*\psi_1(g)t_1 + t_2^*\psi_2(g)t_2 = \alpha\phi(fg) + \beta\phi(g - fg) + (1 - \alpha)\phi(fg) + (1 - \beta)\phi(g - fg) = \phi(fg) + \phi(g - fg) = \phi(g), \text{ for every } g \in C(X).$$

Since $t_1^*t_1 + t_2^*t_2 = I$, the above expression gives $\phi$ as a proper $C^*$-convex combination of $\psi_1$ and $\psi_2$.

We now wish to show that $\psi_1$ and $\psi_2$ are not unitarily equivalent. To this end, let $N_2$ be a neighborhood of $s_2$ with $N_1 \cap N_2 = \emptyset$. Then we may choose a continuous
function \( h : X \to [0, 1] \) with \( h|_{N_2^C} = 0 \) (i.e., \( \text{supp} h \subseteq N_2 \)) and \( h(s_2) = 1 \); thus \( fh = 0 \) and \( (1 - f)h = h \).

Since \( h \in C(X), \phi(h) = K_h + \tau(h)I \in K^+ \). Note that \( \tau(h) > 0 \), since \( h > 0 \) on some neighborhood of \( s_2 \), and that the essential spectrum of \( \phi(h) \) is \( \{\tau(h)\} \). Now compute

\[
\psi_1(h) = Q_1^{1/2} (\alpha\phi(fh) + \beta\phi((1 - f)h)) Q_1^{-1/2} \\
= \beta Q_1^{1/2} \phi(h) Q_1^{-1/2} \\
= \beta Q_1^{1/2} K_h Q_1^{-1/2} + \beta \tau(h) Q_1^{-1}
\]

The first term in this sum is compact, while the second term can be written

\[
\beta \tau(h) Q_1^{-1} = \beta \tau(h)[(\alpha - \beta)K_f + ((\alpha - \beta)\tau(f) + \beta)I]^{-1},
\]

where \( \phi(f) = K_f + \tau(f)I \). Thus

\[
(q \circ \psi_1)(h) = \frac{\beta \tau(h)}{(\alpha - \beta)\tau(f) + \beta} I + K.
\]

Similar computations yield

\[
\psi_2(h) = (1 - \beta)Q_2^{1/2} K_h Q_2^{-1/2} + (1 - \beta)\tau(h) Q_2^{-1}, \quad \text{and} \\
(q \circ \psi_2)(h) = \frac{(1 - \beta)\tau(h)}{(\beta - \alpha)\tau(f) + (1 - \beta)} I + K.
\]

So the essential spectra of \( \psi_1(h) \) and \( \psi_2(h) \) are

\[
\left\{ \frac{\beta \tau(h)}{(\alpha - \beta)\tau(f) + \beta} \right\} \quad \text{and} \quad \left\{ \frac{(1 - \beta)\tau(h)}{((\beta - \alpha)\tau(f) + (1 - \beta)} \right\},
\]

respectively. However, if these are equal, then

\[
\beta(\beta - \alpha)\tau(f) + \beta(1 - \beta) = (1 - \beta)(\alpha - \beta)\tau(f) + \beta(1 - \beta),
\]

so that

\[
\beta = \beta - 1,
\]

which is clearly impossible. This shows that the essential spectra of \( \psi_1(h) \) and \( \psi_2(h) \) are distinct, so that \( \psi_1(h) \) and \( \psi_2(h) \) are not unitarily equivalent. Thus

\[
\phi = t_1^* \psi_1 t_1 + t_2^* \psi_2 t_2
\]

expresses \( \phi \) as a proper \( C^* \)-convex combination of positive unital maps \( \psi_1 \) and \( \psi_2 \) which are not both unitarily equivalent to \( \phi \), demonstrating that \( \phi \) is not \( C^* \)-extreme. This proves the lemma.

\[\square\]

We can now prove the following:

**Theorem 10.** Let \( \phi : C(X) \to K^+ \) be unital and positive. Then \( \phi \) is \( C^* \)-extreme if and only if \( \phi \) is a homomorphism.

**Proof.** If \( \phi \) is multiplicative, then \( \phi \) is \( C^* \)-extreme [Proposition 1.2]. Conversely, if \( \phi \) is \( C^* \)-extreme, Lemma [3] shows that the map \( \tau = q \circ \phi \) is multiplicative, so \( \tau \) is a point evaluation \( \tau(f) = f(s_0) \) for some point \( s_0 \in X \).

Let \( N \) be any neighborhood of \( s_0 \). Then there exists a continuous function \( g_N : X \to [0, 1] \) with \( g_N(s_0) = 0 \) and \( g_N|_{N C} = 1 \).
In this case \( \tau(g_N) = 0 \), so

\[
\phi(g_N) = Kg_N \in \mathcal{K}.
\]

Note that \( \chi_{NC} \leq g_N \), so that \( \phi(\chi_{NC}) \leq \phi(g_N) \). Since \( \mathcal{K} \) is hereditary, it follows that \( \phi(\chi_{NC}) \) is a projection. By Theorem 5, either \( \phi(\chi_{NC}) \) is a projection or \( \sigma(\phi(\chi_{NC})) = [0, 1] \). As a compact operator cannot have the unit interval as its spectrum, \( \phi(\chi_{NC}) \) must be a projection of finite rank. Thus \( \phi(\chi_{NC}) \) is also a projection.

Let \( B \) be any Borel set of \( X \) which does not contain \( s_0 \). Set

\[
\Lambda := \{ K \subseteq B : K \text{ closed} \},
\]

and partially order \( \Lambda \) by inclusion. Then \( \mu_\phi(K) \) is an increasing net of projections. Thus the SOT-lim \( \bigcup_{K \in \Lambda} \text{ran } \mu_\phi(K) \) exists, and is a projection, namely the projection onto \( \bigcup_{K \in \Lambda} \text{ran } \mu_\phi(K) \). Since the measures \( \mu_{x,x} \) are regular for any choice of \( x \in \mathcal{H} \), we have

\[
\mu_{x,x}(B) = \sup_{K \in \Lambda} \mu_{x,x}(K)
\]

or, equivalently,

\[
\langle \mu_\phi(B)x, x \rangle = \sup_{K \in \Lambda} \langle \mu_\phi(K)x, x \rangle = \langle Qx, x \rangle.
\]

As this holds for any \( x \in \mathcal{H} \),

\[
Q = \mu_\phi(B).
\]

If \( B \) is a Borel set in \( X \) which does contain \( s_0 \), then the preceding argument shows that \( \mu_\phi(B^c) \) is a projection. Thus \( \mu_\phi(B) \) is also a projection. Hence \( \mu_\phi \) is a projection valued measure, and \( \phi \) is a homomorphism.

\[ \square \]

**Remark 11.** When \( \phi : C(X) \to \mathcal{K}^+ \), as in Theorem 10, we can obtain more information regarding the support of \( \mu_\phi \). We have shown above that for any closed set \( K \) with \( s_0 \notin K \), \( \mu_\phi(K) \) is a finite rank projection, say of rank \( n \). If \( s_1, s_2 \) are distinct points of \( K \cap \text{supp } \mu_\phi \), let \( N_1 \subseteq K \) be a neighborhood of \( s_1 \) which does not contain \( s_2 \). Then \( K \setminus N_1 \) is closed and \( s_0 \notin K \setminus N_1 \), so \( \mu_\phi(K \setminus N_1) \) is a projection of finite rank and

\[
0 < \text{rank } \mu_\phi(K \setminus N_1) < \text{rank } \mu_\phi(K) = n.
\]

Since

\[
\mu_\phi(K) = \mu_\phi(K \setminus N_1) + \mu_\phi(N_1),
\]

it follows that \( \mu_\phi(N_1) \) is also a projection with

\[
0 < \text{rank } \mu_\phi(N_1) < n.
\]

Clearly this process can be iterated at most \( n \) times; we conclude that any closed set \( K \notin s_0 \) contains at most finitely many points of \( \text{supp } \mu_\phi \). Consequently, \( \text{supp } \mu_\phi \setminus \{ s_0 \} \) is a discrete set with at most one accumulation point at \( s_0 \).
If \( \mathcal{H} \) is a separable Hilbert space, then it is clear from the proof of Theorem 10 and the preceding remark that the support of \( \mu_\phi \) must be at most countable with a single limit point at \( s_0 \). In this case, \( \phi \) must have the form

\[
\phi(f) = \sum_{s \in \text{supp}(\mu_\phi)} f(s)P_s,
\]

where \( P_s = \mu_\phi(\{s\}) \) is a finite rank projection for each \( s \neq s_0 \). The rank of \( P_{s_0} \), on the other hand, may be finite or infinite. The following example, in which we consider the case of a nonseparable Hilbert space, illustrates the structure of unital positive maps \( \phi : C(X) \to K^+ \).

**Example 12.** Let \( \mathcal{H} \) be a nonseparable Hilbert space with dimension at least as great as the cardinality of \( \mathbb{R} \), and let \( X = \mathbb{R} \cup \{\omega\} \) be the one point compactification of \( (\mathbb{R}, d) \), the reals equipped with the discrete topology. Choose an orthonormal set \( \{e_s\}_{s \in \mathbb{R}} \) in \( \mathcal{H} \) indexed by the reals, and write \( P_s \) for the projection onto the span of \( e_s \). Then, for any function \( f \in C(X) \), the set

\[
S(f) := \{s \in X : f(s) \neq f(\omega)\}
\]

is at most countable, and

\[
\lim_{n \to \infty} f(s_n) = f(\omega),
\]

where \( \{s_n\} \) is any enumeration of \( S(f) \). Define a positive map \( \phi \) on \( C(X) \) by

\[
\phi(f) = \sum_{s \in S(f)} [f(s) - f(\omega)]P_s + f(\omega)I.
\]

Then for each \( s \in \mathbb{R} \), the function \( \delta_s = \chi_{\{s\}} \) is continuous and \( \phi(\delta_s) = \mu_\phi(\{s\}) = P_s \).

As in the proof of Theorem 10 if \( G \) is any neighborhood of \( \omega \), then \( C^G \) is a closed set not containing \( \omega \), and \( \phi(\chi_G) \) is a projection. In this case the descending net \( \phi(\chi_G) \) of projections converges to the projection \( \phi(\chi_{\{\omega\}}) = 0 \). Thus \( \mu_\phi \) is a projection valued measure.

Note that we could define similar maps \( \phi_1 \) and \( \phi_2 \) by

\[
\phi_1(f) = \sum_{s \in S(f)} [f(s) - f(\omega)]P_{1/s} + f(\omega)I,
\]

and

\[
\phi_2(f) = \sum_{s \in S(f)} [f(s) - f(\omega)]P_{\arctan s} + f(\omega)I.
\]

For these two maps, we have \( \phi_1(\chi_{\{\omega\}}) = P_0 \), while \( \phi_2(\chi_{\{\omega\}}) \) is the projection onto the closed span \( \{\text{ran} P_s : s \in (-\infty, \pi/2] \cup [\pi/2, \infty)\} \). Thus, the measure of \( \{\omega\} \) may be a projection of either finite or infinite rank.

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