SHARP \((H_p, L_p)\) AND \((H_p, \text{weak} - L_p)\) TYPE INEQUALITIES OF WEIGHTED MAXIMAL OPERATORS OF \(T\) MEANS WITH RESPECT TO VILENKIN SYSTEMS

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Abstract. We discuss \((H_p, L_p)\) and \((H_p, \text{weak} - L_p)\) type inequalities of weighted maximal operators of \(T\) means with respect to the Vilenkin systems with monotone coefficients, considered in [50] and prove that these results are the best possible in a special sense. As applications, both some well-known and new results are pointed out.

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1. Introduction

The definitions and notations used in this introduction can be found in our next Section.

It is well-known that Vilenkin systems do not form bases in the space \(L_1\). Moreover, there is a function in the Hardy space \(H_p\), such that the partial sums of \(f\) are not bounded in \(L_p\)-norm, for \(0 < p \leq 1\). Approximation properties of Vilenkin-Fourier series with respect to one- and two-dimensional cases can be found in Persson, Tephnadze and Wall [30], Simon [37], Blahota [3] and Gát [8], Tephnadze [44, 45, 46], Tutberidze [47] (see also [18]). In the one-dimensional case the weak \((1,1)\)-type inequality for the maximal operator of Fejér means

\[
\sigma^* f := \sup_{n \in \mathbb{N}} |\sigma_n f|
\]

can be found in Schipp [34] for Walsh series and in Pál, Simon [28] for bounded Vilenkin series. Fujji [15] and Simon [36] verified that \(\sigma^*\) is bounded from \(H_1\) to \(L_1\). Weisz [53] generalized this result and proved boundedness of \(\sigma^*\) from the martingale space \(H_p\) to the space \(L_p\), for \(p > 1/2\). Simon [35] gave a counterexample, which shows that boundedness does not hold for \(0 < p < 1/2\). A counterexample for \(p = 1/2\) was given by Goginava [10] (see also Tephnadze [38]). Moreover, Weisz [55] proved that the maximal operator of the Fejér means \(\sigma^*\) is bounded from the Hardy space \(H_{1/2}\) to the space \(\text{weak} - L_{1/2}\). In [39] and [40] the following result was proved:

**Theorem T1:** Let \(0 < p \leq 1/2\). Then the weighted maximal operator of Fejér means \(\tilde{\sigma}^*_p\), defined by

\[
\tilde{\sigma}^*_p f := \sup_{n \in \mathbb{N}} \frac{|\sigma_n f|}{(n + 1)^{1/p - 2} \log^{2[1/2+p]}(n + 1)}
\]

is bounded from the martingale Hardy space \(H_p\) to the Lebesgue space \(L_p\).

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Moreover, the rate of the weights \( \left\{ 1/(n + 1)^{1/p - 2} \log^{2[p+1/2]}(n + 1) \right\}_{n=1}^{\infty} \) in \( n \)-th Fejér mean was given exactly.

In [43] (see also [2] and [16]) it was proved that the maximal operator of Riesz means
\[
R^* f := \sup_{n \in \mathbb{N}} |R_n f|
\]
is bounded from the Hardy space \( H_{1/2} \) to the space weak \(-L_{1/2}\) and is not bounded from \( H_p \) to the space \( L_p \), for \( 0 < p \leq 1/2 \). There was also proved that Riesz summability has better properties than Fejér means. In particular, the following weighted maximal operators
\[
\log n |R_n f| \quad \frac{(n + 1)^{1/p - 2} \log^{2[1/2+p]}(n + 1)}{(n + 1)}
\]
are bounded from \( H_p \) to the space \( L_p \), for \( 0 < p \leq 1/2 \) and the rate of weights are sharp.

Similar results with respect to Walsh-Kaczmarz systems were obtained in [11] for \( p = 1/2 \) and in [41] for \( 0 < p < 1/2 \). Approximation properties of Fejér means with respect to Vilenkin and Kaczmarz systems can be found in Tephnadze [42], Tutberidze [49], Persson, Tephnadze and Tutberidze [32], Blahota and Tephnadze [6] and Persson, Tephnadze, Tutberidze and Wall [33], Gogolashvili, Nagy and Tephnadze [12] and Persson, Tephnadze and Wall [31].

Móricz and Siddiqi [20] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of \( L_p \) function in norm. In the two-dimensional case approximation properties of Nörlund means were considered by Nagy [21, 22, 23] (see also [24, 25, 26, 27]). In [29] (see also [7, 17] and [31]) it was proved that the maximal operators of Nörlund means \( t^* \) defined by
\[
t^* f := \sup_{n \in \mathbb{N}} |t_n f|
\]
either with non-decreasing coefficients, or non-increasing coefficients, satisfying the condition
\[
(1) \quad \frac{1}{Q_n} = O \left( \frac{1}{n} \right), \quad \text{as} \quad n \to \infty
\]
are bounded from the Hardy space \( H_{1/2} \) to the space weak \(-L_{1/2}\) and are not bounded from the Hardy space \( H_p \) to the space \( L_p \), when \( 0 < p \leq 1/2 \).

In [48] was proved that the maximal operators \( T^* \) of \( T \) means defined by
\[
T^* f := \sup_{n \in \mathbb{N}} |T_n f|
\]
either with non-increasing coefficients, or non-decreasing sequence satisfying condition
\[
(2) \quad \frac{q_{n-1}}{Q_n} = O \left( \frac{1}{n} \right), \quad \text{as} \quad n \to \infty,
\]
are bounded from the Hardy space \( H_{1/2} \) to the space weak \(-L_{1/2}\). Moreover, there exists a martingale and such \( T \) means for which boundedness from the Hardy space \( H_p \) to the space \( L_p \) does not hold when \( 0 < p \leq 1/2 \).

In [50] (see also [13, 14]) it was proved that if \( T \) is either with non-increasing coefficients, or non-decreasing sequence satisfying condition (2) that the weighted maximal operator of \( T \) means \( \tilde{T}_p^* \) defined by
\[
\tilde{T}_p^* f := \sup_{n \in \mathbb{N},} \frac{|T_n f|}{(n + 1)^{1/p - 2} \log^{2[1/2+p]}(n + 1)}
\]
is bounded from the martingale Hardy space $H_p$ to the Lebesgue space $L_p$.

Some general means related to $T$ means was investigated by Blahota and Nagy [4] (see also [3]).

In this paper we discuss $(H_p, L_p)$ and $(H_p, \text{weak } - L_p)$ type inequalities of weighted maximal operators of $T$ means with respect to the Vilenkin systems with monotone coefficients, considered in [50] and prove that the rate of the weights in (3) are the best possible in a special sense. As applications, both some well-known and new results are pointed out.

This paper is organized as follows: In order not to disturb our discussions later on some definitions and notations are presented in Section 2. The main results with their proof and some of consequences can be found in Section 3.

2. Definitions and Notation

Denote by $\mathbb{N}_+$ the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \ldots)$ be a sequence of the positive integers not less than 2. Denote by

$$Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$$

the additive group of integers modulo $m_k$.

Define the group $G_m$ as the complete direct product of the groups $Z_{m_i}$ with the product of the discrete topologies of $Z_{m_i}$'s.

The direct product $\mu$ of the measures $\mu_k(\{j\}) := 1/m_k$ $(j \in Z_{m_k})$ is the Haar measure on $G_m$ with $\mu(G_m) = 1$.

In this paper we discuss bounded Vilenkin groups, i.e. the case when $\sup_n m_n < \infty$.

The elements of $G_m$ are represented by sequences

$$x := (x_0, x_1, \ldots, x_j, \ldots), \quad (x_j \in Z_{m_j})$$

Set $e_n := (0, \ldots, 0, 1, 0, \ldots) \in G_m$, the $n$-th coordinate of which is 1 and the rest are zeros $(n \in \mathbb{N})$. It is easy to give a basis for the neighborhoods of $G_m$:

$$I_0(x) := G_m, \quad I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}$$

where $x \in G_m, n \in \mathbb{N}$.

If we define the so-called generalized number system based on $m$ in the following way:

$$M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in \mathbb{N})$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N}_+)$ and only a finite number of $n_j$'s differ from zero.

We introduce on $G_m$ an orthonormal system which is called the Vilenkin system. At first, we define the complex-valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions, by

$$r_k(x) := \exp \left(2\pi i x_k / m_k \right), \quad (i^2 = -1, x \in G_m, \ k \in \mathbb{N})$$

Next, we define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on $G_m$ by:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_{nk}^n(x), \quad (n \in \mathbb{N})$$

Specifically, we call this system the Walsh-Paley system when $m \equiv 2$. 
The norms (or quasi-norms) of the spaces $L_p(G_m)$ and weak $- L_p(G_m)$ ($0 < p < \infty$) are respectively defined by

$$
\|f\|_p := \int_{G_m} |f|^p d\mu, \quad \|f\|_{\text{weak}-L_p} := \sup_{\lambda > 0} \lambda^p \mu(f > \lambda) < +\infty.
$$

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see [51]).

Now, we introduce analogues of the usual definitions in Fourier-analysis. If $f \in L_1(G_m)$ we can define Fourier coefficients, partial sums and Dirichlet kernels with respect to the Vilenkin system in the usual manner:

$$
\hat{f}(n) := \int_{G_m} f \overline{\psi_n} d\mu, \quad S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k, \quad D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}_+).
$$

Let $\{q_k : k \geq 0\}$ be a sequence of non-negative numbers. The $n$-th $T$ means $T_n$ for a Fourier series of $f$ are defined by

$$
T_n f := \frac{1}{Q_n} \sum_{k=0}^{n-1} q_k S_k f, \quad \text{where} \quad Q_n := \sum_{k=0}^{n-1} q_k.
$$

We always assume that $\{q_k : k \geq 0\}$ is a sequence of non-negative numbers and $q_0 > 0$. Then the summability method (4) generated by $\{q_k : k \geq 0\}$ is regular if and only if $\lim_{n \to \infty} Q_n = \infty$.

Let $\{q_k : k \geq 0\}$ be a sequence of nonnegative numbers. The $n$-th Nörlund mean $t_n$ for a Fourier series of $f$ is defined by

$$
t_n f = \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f, \quad \text{where} \quad Q_n := \sum_{k=0}^{n-1} q_k.
$$

If $q_k \equiv 1$ in (4) and (5) we respectively define the Fejér means $\sigma_n$ and Kernels $K_n$ as follows:

$$
\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f, \quad K_n := \frac{1}{n} \sum_{k=1}^{n} D_k.
$$

The well-known example of Nörlund summability is the so-called $(C, \alpha)$ mean (Cesàro means) for $0 < \alpha < 1$, which are defined by

$$
\sigma^\alpha_n f := \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_k^{\alpha-1} S_k f,
$$

where

$$
A^\alpha_0 := 0, \quad A^\alpha_n := \frac{(\alpha + 1) \ldots (\alpha + n)}{n!}.
$$

We also consider the "inverse" $(C, \alpha)$ means, which is an example of $T$ means:

$$
U_n^\alpha f := \frac{1}{A_n^\alpha} \sum_{k=0}^{n-1} A_k^{\alpha-1} S_k f, \quad 0 < \alpha < 1.
$$
Let $V_n^\alpha$ denote the $T$ mean, where $\{q_0 = 0, q_k = k^{\alpha-1} : k \in \mathbb{N}_+\}$, that is

$$V_n^\alpha f := \frac{1}{Q^n} \sum_{k=1}^{n-1} k^{\alpha-1} S_k f, \quad 0 < \alpha < 1.$$ 

The $n$-th Riesz logarithmic mean $R_n$ and the Nörlund logarithmic mean $L_n$ are defined by

$$R_n f := \frac{1}{l_n} \sum_{k=1}^{n-1} S_k f, \quad \text{and} \quad L_n f := \frac{1}{l_n} \sum_{k=1}^{n-1} S_k f, \quad n - k,$$

respectively, where $l_n := \sum_{k=1}^{n-1} 1/k$.

If $\{q_k : k \in \mathbb{N}\}$ is monotone and bounded sequence, then we get the class $B_n$ of $T$ means with non-decreasing coefficients:

$$B_n f := \frac{1}{Q^n} \sum_{k=1}^{n-1} q_k S_k f.$$

The $\sigma$-algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by $\mathcal{F}_n (n \in \mathbb{N})$. Denote by $f = (f^{(n)}, n \in \mathbb{N})$ a martingale with respect to $\mathcal{F}_n (n \in \mathbb{N})$. (for details see e.g. [52]). The maximal function of a martingale $f$ is defined by $f^* := \sup_{n \in \mathbb{N}} |f^{(n)}|$. For $0 < p \leq \infty$ the Hardy martingale spaces $H_p$ consist of all martingales $f$ for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f = (f^{(n)}, n \in \mathbb{N})$ is a martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\hat{f}(i) := \lim_{k \to \infty} \int_{G_m} f^{(k)} \overline{\psi_i} d\mu.$$

A bounded measurable function $a$ is called a $p$-atom, if there exists an interval $I$, such that

$$\int_I a d\mu = 0, \quad \|a\|_\infty \leq \mu(I)^{-1/p}, \quad \text{supp}(a) \subset I.$$

We need the following auxiliary Lemmas:

**Proposition 1** (see e.g. [54]). A martingale $f = (f^{(n)}, n \in \mathbb{N})$ is in $H_p (0 < p \leq 1)$ if and only if there exists a sequence $(\mu_k, k \in \mathbb{N})$ of $p$-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that, for every $n \in \mathbb{N},$

$$\sum_{k=0}^{\infty} \mu_k S_M_a f = f^{(n)}, \quad \text{a.e.}, \quad \text{where} \quad \sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover,

$$\|f\|_{H_p} \sim \inf \left( \sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decompositions of $f$ of the form (6).
Our first main result reads:

**Theorem 1.** a) Let sequence \( \{q_k : k \geq 0\} \) is nondecreasing, satisfying condition

\[
\frac{q_0}{Q_{M_{2n_k}+2}} \geq \frac{c}{M_{2n_k}}, \quad \text{for some constant } c \text{ and } n \in \mathbb{N}.
\]

or let sequence \( \{q_k : k \geq 0\} \) is nonincreasing, satisfying condition

\[
\frac{q_{M_{2n_k}+1}}{Q_{M_{2n_k}+2}} \geq \frac{c}{M_{2n_k}}, \quad \text{for some constant } c \text{ and } n \in \mathbb{N}.
\]

Then for any increasing function \( \varphi : \mathbb{N}_+ \to [1, \infty) \) satisfying the conditions

\[
\lim_{n \to \infty} \varphi(n) = \infty
\]

and

\[
\lim_{n \to \infty} \frac{\log^2 (n + 1)}{\varphi (n + 1)} = +\infty.
\]

Then there exists a martingale \( f \in H_{1/2} \), such that

\[
\left\| \sup_{n \in \mathbb{N}} \left| T_n f \right| \varphi (n) \right\|_{1/2} = \infty.
\]

b) Let \( 0 < p < 1/2 \) and sequence \( \{q_k : k \geq 0\} \) is nondecreasing, or let sequence \( q_k \) is nonincreasing, satisfying condition \( \text{n.4} \). Then for any increasing function \( \varphi : \mathbb{N}_+ \to [1, \infty) \) satisfying the condition

\[
\lim_{n \to \infty} \frac{(n + 1)^{1/p-2}}{\varphi (n + 1)} = +\infty,
\]

there exists a martingale \( f \in H_p \), such that

\[
\left\| \sup_{n \in \mathbb{N}} \left| T_n f \right| \varphi (n) \right\|_{\text{weak-}L_p} = \infty.
\]

**Proof.** According to condition \( \text{(9)} \) in case a) we conclude that there exists an increasing sequence \( \{n_k : k \in \mathbb{N}\} \) of positive integers such that

\[
\lim_{k \to \infty} \frac{\log^2 (M_{2n_k+1})}{\varphi (M_{2n_k+1})} = +\infty.
\]

According to condition \( \text{(10)} \) we conclude that there exists an increasing sequence \( \{n_k : k \in \mathbb{N}\} \) of positive integers such that (Here we use same indexes \( n_k \), but it could be different)

\[
\lim_{k \to \infty} \frac{(M_{2n_k}+2)^{1/p-2}}{\varphi (M_{2n_k}+2)} = +\infty, \quad \text{for } 0 < p < 1/2.
\]

Let

\[
f_{n_k}(x) := D_{M_{2n_k+1}}(x) - D_{M_{2n_k}}(x).
\]

It is evident that
\( \hat{f}_{n_k}(i) = \begin{cases} 
1, & \text{if } i = M_{2n_k}, \ldots, M_{2n_k+1} - 1, \\
0, & \text{otherwise.} 
\end{cases} \)

and

\[ S_i f_{n_k}(x) = \begin{cases} 
D_i(x) - D_{2n_k}(x), & i = 2n_k + 1, \ldots, 2n_k+1 - 1, \\
f_{n_k}(x), & i \geq 2n_k+1, \\
0, & \text{otherwise.} 
\end{cases} \]  

(11)

Since

\[ f_{n_k}^*(x) = \sup_{n \in \mathbb{N}} |S_M(f_{n_k};x)| = |f_{n_k}(x)|, \]

we get

\[ \|f_{n_k}\|_p = \|f_{n_k}^*\|_p = \|D_{2n_k}\|_p = M_{2n_k}^{1 - 1/p}. \]

First, for case a) we consider \( p = 1/2 \). By using (11) and equality (see 1)

\[ D_n(x) = D_{n_1}(x) + r_{[n]}(x) D_{n - n_1}(x) \]

for \( 1 \leq s \leq n_k \) we get that

\[
\frac{|T_{M_{2n_k} + M_{2s}} f_{n_k}|}{\varphi(M_{2n_{k}} + M_{2s})} = \begin{cases} 
\frac{1}{\varphi(M_{2n_k} + M_{2s}) Q_{M_{2n_k} + M_{2s}}} \sum_{j=0}^{M_{2n_k} + M_{2s} - 1} q_j S_j f_{n_k} \\
\frac{1}{\varphi(M_{2n_k} + M_{2s}) Q_{M_{2n_k} + M_{2s}}} \sum_{j=M_{2n_k}}^{M_{2n_k} + M_{2s} - 1} q_j S_j f_{n_k} \\
\frac{1}{\varphi(M_{2n_k} + M_{2s}) Q_{M_{2n_k} + M_{2s}}} \sum_{j=M_{2n_k}}^{M_{2n_k} + M_{2s} - 1} q_j (D_j - D_{M_{2n_k}}) \\
\frac{1}{\varphi(M_{2n_k} + M_{2s}) Q_{M_{2n_k} + M_{2s}}} \sum_{j=M_{2n_k}}^{M_{2n_k} - 1} q_j + M_{2n_k} (D_{j + M_{2n_k} - D_{M_{2n_k}}}) \\
\frac{1}{\varphi(M_{2n_k} + M_{2s}) Q_{M_{2n_k} + M_{2s}}} \sum_{j=0}^{M_{2n_k} - 1} q_j + M_{2n_k} D_j 
\end{cases} \]

Let \( x \in I_{2s} \setminus I_{2s+1} \). Then

\[
\frac{|T_{M_{2n_k} + M_{2s}} f_{n_k}(x)|}{\varphi(M_{2n_k} + M_{2s})} = \begin{cases} 
\frac{1}{\varphi(M_{2n_k} + M_{2s}) Q_{M_{2n_k} + M_{2s}}} \sum_{j=0}^{M_{2n_k} - 1} q_j + M_{2n_k} D_j 
\end{cases} \]

Let sequence \( \{q_k : k \geq 0\} \) is nondecreasing. Then according to condition (7) we find that

\[
\frac{|T_{M_{2n_k} + M_{2s}} f_{n_k}(x)|}{\varphi(M_{2n_k} + M_{2s})} \geq \frac{1}{\varphi(M_{2n_{k+1}}) Q_{M_{2n_{k+1}} + M_{2s}}} \sum_{j=0}^{M_{2s} - 1} j \geq \frac{c M_{2s}^2}{M_{2n_k} \varphi(M_{2n_{k+1}})}. 
\]
Let sequence \( \{ q_k : k \geq 0 \} \) is nonincreasing. Since \( \varphi : \mathbb{N} \to [1, \infty) \) is increasing sequence, by using condition (8) we get that

\[
\left| \frac{T_{M_{2n_k} + M_{2s}} f_{n_k}(x)}{\varphi(M_{2n_k} + M_{2s})} \right| \geq \frac{1}{\varphi(M_{2n_k} + M_{2s})} \frac{q_{M_{2n_k} + M_{2s} - 1}}{Q_{M_{2n_k} + M_{2s}}} \sum_{j=0}^{M_{2s} - 1} j \geq \frac{cM_{2s}^2}{M_{2n_k} \varphi(M_{2n_k + 1})}.
\]

Hence,

\[
\frac{\left( \sup_{n \in \mathbb{N}} \frac{|T_n f_{n_k}|}{\varphi(n)} \right)^{1/2}}{\|f_{n_k}\|_{H^{1/2}}} \geq \frac{c n_k}{(M_{2n_k} \varphi(M_{2n_k + 1}))^{1/2}} \geq \frac{c n_k}{(M_{2n_k} \varphi(M_{2n_k + 1}))^{1/2}}.
\]

From (12) we get that

\[
\left( \frac{\left( \sup_{n \in \mathbb{N}} \frac{|T_n f_{n_k}|}{\varphi(n)} \right)^{1/2}}{\|f_{n_k}\|_{H^{1/2}}} \right)^2 \geq \frac{cn_k^2 M_{2n_k}}{(M_{2n_k} \varphi(M_{2n_k + 1}))^{1/2}} \geq \frac{cn_k^2}{\varphi(M_{2n_k + 1})} \geq \frac{c}{\varphi(M_{2n_k + 1})} \rightarrow \infty, \text{ as } k \to \infty.
\]

This complete proof of part a).

Next, we consider case \( 0 < p < 1/2 \). In the view of identities (11) of Fourier coefficients we find that

\[
\left| \frac{T_{M_{2n_k} + 2} f_{n_k}}{\varphi(M_{2n_k} + 2)} \right| = \frac{1}{\varphi(M_{2n_k} + 2)} \frac{1}{Q_{M_{2n_k} + 2}} \sum_{j=0}^{M_{2n_k} + 1} j \psi_j S_j f_{n_k} = \frac{1}{\varphi(M_{2n_k} + 2)} \frac{1}{Q_{M_{2n_k} + 2}} q_{M_{2n_k} + 1} \left( D_{M_{2n_k} + 1} - D_{M_{2n_k}} \right) = \frac{1}{\varphi(M_{2n_k} + 2)} \frac{1}{Q_{M_{2n_k} + 2}} q_{M_{2n_k} + 1} \psi_{M_{2n_k}}
\]

Let sequence \( \{ q_k : k \geq 0 \} \) is nondecreasing. Then

\[
\left| \frac{T_{M_{2n_k} + 2} f(x)}{\varphi(M_{2n_k} + 2)} \right| \geq \frac{1}{\varphi(M_{2n_k} + 2)} \frac{q_{M_{2n_k} + 1}}{q_{M_{2n_k} + 1} (M_{2n_k} + 2)} \geq \frac{c}{M_{2n_k} \varphi(M_{2n_k} + 2)}.
\]

Let sequence \( \{ q_k : k \geq 0 \} \) is nonincreasing. Then, according condition (8) we find that

\[
\left| \frac{T_{M_{2n_k} + 2} f(x)}{\varphi(M_{2n_k} + 2)} \right| = \frac{1}{\varphi(M_{2n_k} + 2)} \frac{q_{M_{2n_k} + 1}}{Q_{M_{2n_k} + 2}} \geq \frac{c}{M_{2n_k} \varphi(M_{2n_k} + 2)}
\]
Hence,
\[
\mu \left\{ x \in G_m : \frac{T_{M_{2n_k}^2} f(x)}{\varphi(M_{2n_k} + 2)} \geq \frac{c}{M_{2n_k} \varphi(M_{2n_k} + 2)} \right\} = |G_m| = 1.
\]

Then from (12) we get that
\[
\frac{c}{M_{2n_k} \varphi(M_{2n_k} + 2)} \left\{ \mu \left\{ x \in G_m : \frac{T_{M_{2n_k}^2} f(x)}{\varphi(M_{2n_k} + 2)} \geq \frac{c}{M_{2n_k} \varphi(M_{2n_k} + 2)} \right\} \right\}^{1/p} \geq \frac{\| f_{n_k} \|_{H_p}}{c M_{2n_k}^{1/p - 2} \varphi(M_{2n_k} + 2)} \geq \frac{c (M_{2n_k} + 2)^{1/p - 2}}{\varphi(M_{2n_k} + 2)} \to \infty, \quad \text{as} \quad k \to \infty.
\]

The proof is complete. □

As application we get well-known result for the weighted maximal operator of Fejér means which was considered in Tephnadze [39, 40]:

**Corollary 1.** Let \( \varphi : \mathbb{N}_+ \to [1, \infty) \) be any increasing function satisfying the conditions
\[
\lim_{n \to \infty} \varphi(n) = \infty
\]
and
\[
\lim_{n \to \infty} \frac{(n + 1)^{1/p - 2} \log^{2[1/2+p]} (n + 1)}{\varphi(n)} = +\infty.
\]
Then
\[
\left\| \sup_{n \in \mathbb{N}} \frac{|\sigma_n f|}{\varphi(n)} \right\|_{H_{1/2}}^{1/2} = \infty
\]
and
\[
\left\| \sup_{n \in \mathbb{N}} \frac{|\sigma_n f|}{\varphi(n)} \right\|_{weak-L_p} = \infty.
\]

We also present some new results on \( T \) means with respect to Vilenkin systems which follows Theorem 1:

**Corollary 2.** Theorem 1 holds true for \( U_{n_k}^\alpha f, V_n^\alpha f \) and \( B_n f \) means with respect to Vilenkin systems.

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