Abstract

We prove that the dimension of the space of primitive Vassiliev invariants of degree $n$ grows - as $n$ tends to infinity - faster than $e^{c\sqrt{n}}$ for any $c < \pi \sqrt{2/3}$.

The proof relies on the use of the weight systems coming from the Lie algebra $\mathfrak{gl}(N)$. In fact, we show that our bound is - up to a multiplication with a rational function in $n$ - the best possible that one can get with $\mathfrak{gl}(N)$-weight systems.

1 Introduction

The space $\mathcal{V}$ of Vassiliev knot invariants is still mysterious. Although we have a perfect combinatorial description of it (see e.g. [BL93, Kon93, BN95]), even the asymptotic behavior of the dimension of Vassiliev invariants in degree $n$ is unknown.

Vassiliev invariants form an algebra which is isomorphic to a free polynomial algebra. A basis for this algebra is given by a basis of the primitive Vassiliev invariants. Therefore all information about Vassiliev invariants is included in the primitive ones. Soon after the discovery of Vassiliev invariants, it became clear that there is at least one primitive Vassiliev invariant in each degree [CDL94]. From that it follows that the dimension of the space of (not necessarily primitive) Vassiliev invariants of order $n$ grows - as $n$ tends to infinity - faster than $e^{c\sqrt{n}}$ for any $c < \pi \sqrt{2/3}$ [Kon93].

Other subspaces of the primitive space can be obtained in the following way:

For an arbitrary subspace of Vassiliev invariants one can take the subalgebra generated by this subspace and look at the intersection in degree $n$ with the primitive space. One can see [CDV97] that this construction gives for the HOMFLY-Vassiliev invariants [BL93] a contribution to the primitive space of dimension $\left\lfloor \frac{n}{2} \right\rfloor$ in degree $n$. The dimension of the subspace of the primitive space coming in this way from the (unframed) colored Jones polynomial [MM95] in degree $n$ is again $\left\lfloor \frac{n}{2} \right\rfloor$ [CDV97].

The best known lower bound for the dimension of the primitive space was recently given in a nice paper by Chmutov and Duzhin [CD]. They proved that the primitive space in degree $n$ is at least $n^{\log n}$ dimensional as $n$ tends to infinity.

The first aim of this paper is to prove the so called Kontsevich-Bar-Natan conjecture (for the history of it see [CD]) which states that the dimension of the space of primitive Vassiliev invariants grows - as $n$ tends to infinity - faster than $e^{c\sqrt{n}}$ for any $c < \pi \sqrt{2/3}$ [Kon93].
invariants of degree \( n \) grows - as \( n \) tends to infinity - faster than \( e^{c\sqrt{n}} \) for any \( c < \pi \sqrt{2/3} \). Therefore we will get a much better lower bound than \( n^{\log n} \).

For showing this we will make use of the universal weight system coming from the Lie algebra \( \mathfrak{gl}(N) \), which is related to the cablings of the HOMFLY-Vassiliev invariants.

As a by-product of the proof we will get that the vector space \( C_a H(n) \) of all Vassiliev invariants that come (in the usual way) from the HOMFLY-Vassiliev invariants in degree \( n \) and all of their cablings, connected and disconnected, behaves like

\[
g_1(n)e^{c\sqrt{n}} \leq \dim C_a H(n) \leq g_2(n)e^{c\sqrt{n}}
\]

for two rational functions \( g_1 \) and \( g_2 \) in \( n \) and \( c = \pi \sqrt{2/3} \).

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This paper covers a talk given at the "Knot theory week", Bonn, July 1997, well-organized by C.-F. Bödigheimer and his knot theory group.

In a fax to Sergei Chmutov [Kon97], Maxim Kontsevich independently gave a proof for a weaker lower bound estimate. His bound is - roughly speaking - the square root of ours. He uses the same techniques that we use.

## 2 Preliminaries

Vassiliev invariants form a filtered vector space \( \mathcal{V} = \bigcup \mathcal{V}_n \) with \( \mathcal{V}_n \) the space of Vassiliev invariants of order at most \( n \). Kontsevich (see [Kon93, BN95]) gave a combinatorial description of \( \mathcal{V}_n/\mathcal{V}_{n-1} \):

The algebra \( \mathcal{A} \) is the algebra generated by all chord-diagrams modulo the four-term-relation, with the connected sum as a multiplication. It is graded by the number of chords in a diagram and the graded \( n \)-part of it is denoted by \( \mathcal{A}_n \). By the result of Kontsevich we know that \( \mathcal{V}_n/\mathcal{V}_{n-1} \) is isomorphic to the subspace of the dual \( \mathcal{A}_n^* \) of \( \mathcal{A}_n \) that is generated by all functionals vanishing on chord diagrams with an isolated chord. Elements of \( \mathcal{A}_n^* \) are called weight systems.

It turns out that with a suitable coproduct \( \mathcal{A} \) becomes an associative, commutative, coassociative and cocommutative Hopf-algebra. By the classical structure theory of these algebras we know that \( \mathcal{A} \) is isomorphic to the polynomial algebra over its primitive space \( \mathcal{P}(\mathcal{A}) \). The space \( \mathcal{P}(\mathcal{A}) \) corresponds to the subspace of \( \mathcal{V} \) generated by invariants \( v \) that are additive for connected sums of knots: \( v(K_1 \# K_2) = v(K_1) + v(K_2) \).

We have another description of \( \mathcal{A} \): The algebra \( \mathcal{B} \) is the algebra of all (finite) diagrams (graphs) having only trivalent and univalent vertices, each trivalent vertex equipped with one of the two cyclic orientations. It turns out that with a suitable coproduct \( \mathcal{A} \) becomes an associative, commutative, coassociative and cocommutative Hopf-algebra. By the classical structure theory of these algebras we know that \( \mathcal{A} \) is isomorphic to the polynomial algebra over its primitive space \( \mathcal{P}(\mathcal{A}) \). The space \( \mathcal{P}(\mathcal{A}) \) corresponds to the subspace of \( \mathcal{V} \) generated by invariants \( v \) that are additive for connected sums of knots: \( v(K_1 \# K_2) = v(K_1) + v(K_2) \).

We have another description of \( \mathcal{A} \): The algebra \( \mathcal{B} \) is the algebra of all (finite) diagrams (graphs) having only trivalent and univalent vertices, each trivalent vertex equipped with one of the two cyclic orientations. Furthermore the following two types of relations hold:

(i) The IHX-relation: \( \begin{array}{c} \hline \end{array} = \begin{array}{c} \hline \end{array} \times \)

(ii) The antisymmetry relation: if in a diagram \( D \) the orientation at one trivalent vertex is changed then \( D \) changes to \( -D \).

As usual, in all pictures of diagrams (or subdiagrams) in \( \mathcal{B} \), it is assumed that the three edges meeting at one trivalent vertex are oriented counterclockwise.
The gradation in $\mathcal{B}$ is given by half of the number of vertices in a diagram. The $n$-graded part is denoted by $\mathcal{B}_n$. As vector spaces $\mathcal{A}_n$ and $\mathcal{B}_n$ are isomorphic and we have two different products in $\mathcal{B}$: the natural product induced by the disjoint union of diagrams and the product coming from the product in $\mathcal{A}$.

Since the IHX-relation and the antisymmetry relation are homogenous in the number of univalent vertices we get a splitting $\mathcal{B} = \bigoplus \mathcal{B}^{(u)}$ where $\mathcal{B}^{(u)}$ is the subspace generated by all diagrams in $\mathcal{B}$ with $u$ univalent vertices. By $\mathcal{B}^{(u)}_n$ we denote the graded $n$-part of $\mathcal{B}^{(u)}$. Of special interest is the subspace $\mathcal{B}^c$ (resp. $\mathcal{B}_n^c$ or $\mathcal{B}^{(u)}_n$) spanned by all diagrams in $\mathcal{B}$ (resp. $\mathcal{B}_n$ or $\mathcal{B}^{(u)}_n$) that are connected.

It is well-known that the primitive space in $\mathcal{A}$ is isomorphic to $\mathcal{B}^c$ and therefore the aim of this paper is to give a lower bound for the dimension of $\mathcal{B}_n^c$.

We choose the following notation for diagrams in $\mathcal{B}$: By the antisymmetry relation we know that two univalent vertices in a nontrivial diagram cannot be adjacent to the same trivalent vertex. Hence we think of a connected diagram $\Gamma \in \mathcal{B}$ as a cubic graph, i.e. all vertices are trivalent, $G(\Gamma)$ with some edges (“legs”) - with a free end - attached to the edges of $G(\Gamma)$.

In pictures the legs will be given as a number that is posed according to the cyclic ordering at the trivalent vertices of the legs, e.g.:

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### 3 Weight systems coming from Lie algebras

Let $\mathcal{L}$ be a finite dimensional Lie algebra equipped with a metric $t$, i.e. an $ad$-invariant, non-degenerated, symmetric bilinear form. There is a well-known and often described way (e.g. [Kon93], [BN95], [CD], [Vog] and [Vai94]) to use $\mathcal{L}$ for the construction of weight systems.

A diagram $\Gamma \in \mathcal{A}$ will be mapped to an element $\mathcal{W}(\mathcal{L}, t)(\Gamma)$ in the center $Z(U(\mathcal{L}))$ of the universal enveloping algebra of $\mathcal{L}$, which is weighted by the word length of the elements. $\mathcal{W}(\mathcal{L}, t)(\Gamma)$ is of weight less or equal to the number of vertices of $\Gamma$ lying on the oriented circle. This map yields a map $\mathcal{B} \to Z(U(\mathcal{L}))$ (also denoted by $\mathcal{W}(\mathcal{L}, t)$) by the isomorphism $\mathcal{B} \to \mathcal{A}$.

Now let $\mathcal{L}$ be the Lie algebra $gl(N)$, let $e_{ij}$ be the standard generators and let $t$ be the trace of the product of matrices.

The elements

$$c_j := \sum_{i_1, \ldots, i_j} e_{i_1 i_2} e_{i_2 i_3} \cdots e_{i_j i_1}, \quad 1 \leq j \leq N,$$

in $U(gl(N))$ are called generalized Casimir elements. It is well known (e.g. [Zel73]) that $Z(U(gl(N)))$ is a free commutative polynomial algebra in the $c_j$, $j = 1, \ldots, N$. We regard $N$ as a variable and set $c_0 := N$.

As in [CD] we will make only use of the part $\mathcal{W}_{\mathfrak{gl}(N)}$ of $\mathcal{W}(\mathfrak{gl}(N))$ with highest weight; that means for a diagram $\Gamma \in \mathcal{B}$ with $u$ univalent vertices that $\mathcal{W}_{\mathfrak{gl}(N)}(\Gamma)$ is the part of $\mathcal{W}_{\mathfrak{gl}(N)}(\Gamma)$ with weight $u$. Here the weight of $c_j$ is $j$. For a diagram $\Gamma \in \mathcal{B}^{(u)}$ and an arbitrary $N \geq u$ we have a nice combinatorial description of $\mathcal{W}_{\mathfrak{gl}(N)}$ (see [CD] and compare with [BN93]):

**Definition 3.1** A $\mathcal{B}$-state $s$ of a diagram $\Gamma \in \mathcal{B}$ is a map from the internal (i.e. trivalent) vertices of $\Gamma$ to $\{-1, 1\}$. The number $|s|$ is the number of $-1$’s in $s$. 


It is convenient to distinguish between two types of internal vertices: Proper internal vertices that are not adjacent to univalent vertices and non-proper internal vertices that are. We will denote the part of \( s \) corresponding to proper internal (resp. non-proper internal) vertices by \( s_p \) (resp. \( s_{np} \)).

Let \( F(\Gamma, s) \) be the orientable surface with some missing points on the boundary that we get by the construction:

- each edge of \( \Gamma \) will be thickened:

- each trivalent vertex will be resolved according to the value of \( s \) at it:

- each univalent vertex is responsible for a missing point in the boundary.

Now, for a diagram \( \Gamma \in \mathcal{B}_n^{c,(u)} \) and a \( \mathcal{B} \)-state \( s = (s_p, s_{np}) \) we map via a function \( \Omega \) the orientable surface \( F(\Gamma, s) \), to a monomial in the polynomial ring \( \mathbb{C}[c_0, c_1, \ldots, c_N] \), \( N \geq u \), in the generalized Casimir elements: If \( F(\Gamma, s) \) has boundary components \( K_1, \ldots, K_j \) and the number of missing points on it are \( r_1, \ldots, r_j \) then \( \Omega(F(\Gamma, s)) \) will be the monomial \( c_1^{r_1} \cdot \cdots \cdot c_j^{r_j} \).

**Proposition 3.2 (see [CD] and also [BN95])** Let \( \Gamma \) be a connected diagram in \( \mathcal{B}_n^{c,(u)} \) - i.e. \( \Gamma \) has \((2n-u)\) trivalent vertices - and let \( N \geq u \). Then:

\[
\mathcal{W}_{gl}(N)(\Gamma) = \sum_{s_p \in \{\pm 1\}^{2n-2u}} \sum_{s_{np} \in \{\pm 1\}^u} (-1)^{|s|} \Omega(F(\Gamma, s)).
\]

The following is quite easy to see:

**Lemma 3.3** Let \( \Gamma \) be a diagram in \( \mathcal{B} \) and let \( s \) be a \( \mathcal{B} \)-state of \( \Gamma \). The conjugate \( \overline{s} \) of \( s \) is the state that we get by multiplying each component of \( s \) by \(-1\).

We have:

\[
F(\Gamma, s) = F(\Gamma, \overline{s}).
\]

We give an example for the computation of \( \mathcal{W}_{gl}(N) \):

**Example 3.4** For \( u \) even let \( \Gamma \in \mathcal{B}_u^{c,(u)} \) be the diagram:

Then

\[
\mathcal{W}_{gl}(N)(\Gamma) = \sum_{j=0}^{u} (-1)^j \binom{u}{j} c_j c_{u-j}
\]
Since in general the whole polynomial $\overline{W}_{g(N)}$ is arduous to handle, with great success in CD the following part of it is used instead:

**Definition 3.5** For a diagram $\Gamma \in \mathcal{B}_n^{c,(u)}$, $u \leq N$, the polynomial $CD(\Gamma)$ is the highest degree homogeneous part of $\overline{W}_{g(N)}$, that means now the degree of each $c_j$ is one.

### 3.1 Evaluations for a special type of diagrams

**Lemma 3.6** Let $\Gamma$ be a diagram in $\mathcal{B}_n^{c,(u)}$, $u$ even, $u \leq n$, so that the underlying cubic graph of $\Gamma$ is planar and 3-connected. Furthermore we assume that $\Gamma$ is embedded in the 2-sphere. Then the polynomial $CD(\Gamma)$ is

$$CD(\Gamma) = 2 \sum_{snp \in \{\pm 1\}^{2n-2u}} (-1)^{|s|}\Omega \circ F(\Gamma, s).$$

**Proof** (Compare with [BN97].) For a state $s$ let $F(\Gamma, s)$ be the closed orientable surface, obtained by gluing disks into the boundary components of $F(\Gamma, s)$. The Euler characteristic of $F(\Gamma, s)$ is

$$\chi(F(\Gamma, s)) = 2 - 2g(F(\Gamma, s))$$

where $g$ is the genus.

On the other hand we know that

$$\chi(F(\Gamma, s)) = \#\partial(F(\Gamma, s)) = -\#e(G(\Gamma)) + \#v(G(\Gamma))$$

where $\#\partial(F(\Gamma, s))$ is the number of boundary components and $\#e(G(\Gamma))$ (resp. $\#v(G(\Gamma))$) is the number of edges (resp. vertices) of the underlying cubic graph of $\Gamma$.

For $s = (s_p, s_{np})$ the genus $g(F(\Gamma, s))$ only depends on $s_p$. Furthermore it is easy to see that each $s_p$ that lead to a 2-sphere $F(\Gamma, s)$ induces an embedding of the underlying cubic graph $G(\Gamma)$ into the 2-sphere and vice versa. Hence, the number of $s_p$ such that $g(F(\Gamma, s)) = 0$ is equal to the number of embeddings of the underlying cubic graph $G(\Gamma)$ into the oriented 2-sphere.

By the classical result of Whitney we know that a 3-connected planar graph has only one embedding into the 2-sphere up to homeomorphisms. Therefore the number of $s_p$ so that $F(\Gamma, s_p, s_{np})$ is the 2-sphere is two, i.e. corresponds to an embedding and to its mirror image. Because the diagram is already embedded this means $s_p = (1, \ldots, 1)$ or $s_p = (-1, \ldots, -1)$. Using Lemma 3.3 we get the desired formula. $\square$

**Example 3.7** For $u = a_1 + \ldots + a_k + 2b$ even and $n = k + u$ let $\Gamma(a_1, \ldots, a_k, b)$ be the following 'Pont Neuf' diagram in $\mathcal{B}_n^{c,(u)}$:
This diagram is 3-connected and we have:

\[
CD(\Gamma(a_1, \ldots, a_k, b)) = 2 \sum_{j_1, \ldots, j_k, l} (-1)^{j_1+\cdots+j_k+l} \binom{a_1}{j_1} \cdots \binom{a_k}{j_k} (\frac{2b}{l}) c_{j_1} \cdots c_{j_k} c_{u-j_1-\cdots-j_k-l}.
\]

4 Asymptotic behavior of partition numbers

Let \( p(n) \) be the number of partitions of \( n = a_1 + \cdots + a_k \) into numbers \( 1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \) and \( p_2(n) \) be the number of partitions so that \( a_1 \geq 2 \), that means \( p_2(n) = p(n) - p(n-1) \).

A theorem of Hardy and Ramanujan (see for example [Har59]), which is as beautiful as it is famous, gives us an asymptotic formula for \( p(n) \). The asymptotic for \( p_2(n) \) follows by a straightforward computation and should be well-known:

**Theorem 4.1 (Hardy and Ramanujan)**

\[
p(n) \approx \frac{1}{4n \sqrt{3}} e^{\frac{\pi}{\sqrt{2}} \sqrt{\frac{2\sqrt{3}}{n}}}
\]

\[
p_2(n) \approx \frac{\pi \sqrt{\frac{2}{3}}}{8n \sqrt{3\sqrt{n}}} e^{\frac{\pi}{\sqrt{2}} \sqrt{\frac{2\sqrt{3}}{n}}} = \frac{\pi \sqrt{\frac{2}{3}}}{24n} e^{\frac{\pi}{\sqrt{2}} \sqrt{\frac{2\sqrt{3}}{n}}}
\]

In the course of this text we will need the following lemma:

**Lemma 4.2** Let \( adm_2(n) \) be the number of partitions of \( n = a_1 + \cdots + a_k \), so that \( 2 \leq a_1 \leq \cdots \leq a_k \) and \( n - a_k \) is even. Then

\[
\frac{1}{2} p_2(n) \leq adm_2(n) \leq p_2(n).
\]

**Proof (S. Chmutov, O.D.)** Call a partition \((a_1, \ldots, a_k)\) with \( a_1 \geq 2 \) admissible if \( n - a_k \) is even. It is non-admissible if \( n - a_k \) is odd.

Now let a partition \((a_1, \ldots, a_k)\) be non-admissible. Therefore one of \( a_1, \ldots, a_{k-1} \) must be odd and hence there is an \( l < k \) so that \( a_l \geq 3 \) and either \( l = 1 \) or \( a_{l-1} = 2 \).

With this choice for \( l \) the map

\[
(a_1, \ldots, a_l, \ldots, a_k) \mapsto (a_1, \ldots, a_l - 1, \ldots, a_k + 1)
\]

extends to an injective map from the set of non-admissible partitions of cardinality \( p_2(n) - adm_2(n) \) to the set of admissible ones. Hence

\[
p_2(n) - adm_2(n) \leq adm_2(n)
\]

\(\square\)

5 A lower bound

**Theorem 5.1** For fixed \( u \) and \( k \) with \( u \) even let \( S_{k,u} \) be the set of all \((a_1, \ldots, a_k, b)\) so that

(i) \( 0 \leq a_1 \leq \cdots \leq a_k \leq b \)
(ii) \( u := a_1 + \ldots + a_k + 2b. \)

Let \( \Gamma(a_1, \ldots, a_k, b) \in B_{u+k}^c \) as in Example 3.7. Then the polynomials \( CD(\Gamma(a_1, \ldots, a_k, b)) \) are linearly independent on \( S_{k,u} \).

**Proof** In \( S_{k,u} \) we have an ordering by the lexicographical ordering on \( (a_1, \ldots, a_k) \).

Let \( (a_1, \ldots, a_k, b) \) be an element of \( S_{k,u} \). Then in \( CD(\Gamma(a_1, \ldots, a_k, b)) \) the monomial \( c_{a_1} \cdots c_{a_k} c_b^2 \) has a nontrivial coefficient and it does not occur in any \( CD(\Gamma(\tilde{a}_1, \ldots, \tilde{a}_k, \tilde{b})) \) for \( (\tilde{a}_1, \ldots, \tilde{a}_k, \tilde{b}) \) in \( S_{k,u} \) less than \( (a_1, \ldots, a_k, b) \). \( \square \)

**Theorem 5.2** The dimension of the subspace \( B_n^c \) generated by connected graphs in \( B_n \) is greater than or equal to the number of partitions

\[
p_1 + \ldots + p_r = n + 2
\]

\[
2 \leq p_1 \leq \ldots \leq p_r
\]

\[
n - p_r \quad \text{even}.
\]

**Proof** For all \( u_1 \neq u_2 \) it holds \( B_{n(u_1)}^c \cap B_{n(u_2)}^c = \{0\} \). Hence, by Theorem 5.1 we know that the dimension of \( B_n^c \) is greater than or equal to the number of \( (a_1, \ldots, a_k, b) \) with

\[
0 \leq a_1 \leq \ldots \leq a_k \leq b
\]

\[
a_1 + \ldots + a_k + b + b = u, \quad u \text{ even}
\]

\[
k + u = n.
\]

By increasing each \( a_j, j = 1, \ldots, k \) and \( b \) by 1 we see that the dimension of \( B_n^c \) is greater than or equal to the number of \( (a_1, \ldots, a_k, b) \) with

\[
1 \leq a_1 \leq \ldots \leq a_k \leq b
\]

\[
a_1 + \ldots + a_k + b + b = n + 2
\]

\[
n - k \quad \text{even}.
\]

Now we will look at the Young diagram (also called Ferrers diagram) of a partition. To each partition corresponds its conjugate partition defined by a reflection of the diagram (see Figure 5). By looking at the conjugates of the partitions in (4) we get the set defined by (3). Hence, the claim follows. \( \square \)

Combining Theorem 5.2 with the Hardy-Ramanujan formula 4.1 and Lemma 4.2 we get

**MAIN THEOREM 5.3** The dimension of the primitive space in the space of Vassiliev invariants grows in degree \( n \) faster than \( e^{c\sqrt{n}} \) for any \( c < \pi \sqrt{2/3} \), as \( n \) tends to infinity.
6 The dimensions of $B_{u+k}^{c,(u)}$ for low $k$.

For $k = 0, 1$ and $2$ the dimensions of $B_{u+k}^{c,(u)}$ are known. For $k \leq 5$ and $u$ odd we have shown in [Das97] that the spaces are trivial. So for the rest of this section we assume that $u$ is even.

It is easy to see that $B_{u}^{c,(u)}$ is one-dimensional. Furthermore it is proved in [Das97] (see also [Das98]) that $B_{u+1}^{c,(u)}$ is $\lfloor \frac{u}{6} \rfloor + 1$ dimensional.

For $k = 2$ we know [Das98] that

$$\dim B_{u+2}^{c,(u)} = \left\lfloor \frac{u^2 + 12u}{48} \right\rfloor + 1.$$ 

We will look at these dimension formulas from the settings given in this paper:

(i) $k = 1$: We have seen in section 5 that the dimension of $B_{u+1}^{c,(u)}$ is greater than or equal to the number of partitions

$$u = a_1 + 2b, \quad 0 \leq a_1 \leq b, \quad a_1 \text{ even.} \quad (5)$$

With $r_1 := a_1/2$ and $r_2 := (b - a_1)$ we see that the number of partitions fulfilling (5) is equal to the number of partitions $u = 6r_1 + 2r_2$, $r_1, r_2 \geq 0$.

Hence, the generating function for the lower bound is

$$\frac{1}{(1 - x^2)(1 - x^6)} = \sum_{u \text{ even}} \left( \left\lfloor \frac{u}{6} \right\rfloor + 1 \right) x^u.$$ 

Therefore, the lower bound gives the exact dimensions.

(ii) $k = 2$: The dimension of $B_{u+2}^{c,(u)}$ is greater than or equal to the number of partitions

$$u = a_1 + a_2 + 2b, \quad 0 \leq a_1 \leq a_2 \leq b. \quad \text{Moreover } a_2 - a_1 \text{ must be even.}$$

With $r_1 := a_1$, $r_2 := \frac{a_2 - a_1}{2}$ and $r_3 := b - a_2$ we see that the number of these partitions is equal to the number of partitions $u = 4r_1 + 6r_2 + 2r_3$, $r_1, r_2, r_3 \geq 0$.

The generating function is

$$\frac{1}{(1 - x^2)(1 - x^4)(1 - x^6)} = \sum_{u \text{ even}} \left( \left\lfloor \frac{u^2 + 12u}{48} \right\rfloor + 1 \right) x^u,$$

where one can see the equality as in [Das98].

Hence, still for $k = 2$ the estimate is sharp.

(iii) $k = 3$: The dimension of $B_{u+3}^{c,(u)}$ is greater than or equal to the number of partitions

$$u = a_1 + a_2 + a_3 + 2b, \quad 0 \leq a_1 \leq a_2 \leq a_3 \leq b. \quad (6)$$

Because $u$ is even we see that $a_1 + a_3 - a_2$ must be even. The set of partitions in (6) divides into two subsets:
(a) \(a_1\) even: In this case we set
\[
r_1 := \frac{a_1}{2}, \quad r_2 := a_2 - a_1, \quad r_3 := \frac{a_3 - a_2}{2}, \quad r_4 := b - a_3
\]
and hence we look at the number of partitions
\[
u = 10 r_1 + 4r_2 + 6r_3 + 2r_4 \quad r_1, r_2, r_3, r_4 \geq 0.
\]
A generating function for this number is
\[
\frac{1}{(1 - x^2)(1 - x^4)(1 - x^6)(1 - x^{10})}.
\]

(b) \(a_1\) odd: We set \(\tilde{a}_1 := a_1 - 1, \tilde{a}_2 := a_2 - 1, \tilde{a}_3 := a_3 - 2, \tilde{b} := b - 2\). This yields a partition of \(u - 8 = \tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3 + 2\tilde{b}\) satisfying that \(\tilde{a}_1\) is even. Hence we have a generating function for their numbers:
\[
\frac{x^8}{(1 - x^2)(1 - x^4)(1 - x^6)(1 - x^{10})}.
\]

We only have to add these two generating functions and we get a generating function for the lower bound of the dimension of \(\mathcal{B}^{c,(u)}_{u+3}\) coming from our construction:
\[
\frac{1 + x^8}{(1 - x^2)(1 - x^4)(1 - x^6)(1 - x^{10})}.
\]

Remark Dror Bar-Natan \cite{BN96} has computed - with the help of weight systems coming from \(so(N)\) - the dimensions of \(\mathcal{B}^{c,(u)}_{u+3}, u = 2, 4, 6, 8\) and gave lower bounds for \(u = 10, 12, 14\). With a different approach Jan Kneissler verified these dimensions \cite{Kne97}, showed that the lower bounds are actually the dimensions and in addition gave the dimension for \(u = 16\):
\[
2, 3, 5, 8, 10, 15, 19, 24.
\]

Our estimate only gives \(\dim \mathcal{B}^{c,(2)}_{5} \geq 1\) and is therefore not sharp even for the most simple case. (In fact, the reason for this is that \(CD(\Gamma(a_1, a_2, a_3, b))\) is invariant under any permutation of \(a_1, a_2\) and \(a_3\), while in general the diagrams \(\Gamma(a_1, a_2, a_3, b)\) are not.)

However, the estimates (8) are the first terms of the generating function in the

Conjecture The dimension of the space \(\mathcal{B}^{c,(u)}_{u+3}\) is given by the generating function:
\[
\frac{1 + x^2 + x^8 - x^{10}}{(1 - x^2)(1 - x^4)(1 - x^6)(1 - x^{10})}.
\]
7 Upper bound for the dimension in $Z(U(\mathfrak{g}l(N)))$

We have proved that the image of the map

$$\mathcal{W}_{\mathfrak{g}l(N)} : \mathcal{B}_n \to Z(U(\mathfrak{g}l(N)))$$

is greater than $g_1(n) e^{c\sqrt{n}}$ for some rational function $g_1(n)$ if $N > n$ and $c = \pi \sqrt{2/3}$.

Now let $\mathcal{B}_n^r$ (resp. $\mathcal{B}_n^{r,(u)}$) be the subspace of $\mathcal{B}_n$ (resp. $\mathcal{B}_n^{u}$) that is generated by diagrams with at least one trivalent vertex in each connected component. Let $\Gamma$ be a diagram in $\mathcal{B}_n^{r,(u)}$. Specifically this means that $u \leq n$.

$\mathcal{W}_{\mathfrak{g}l(N)}(\Gamma)$, $N > u$, is a polynomial in the generalized Casimir elements $c_1, \ldots, c_u$ and $c_0 := N$.

A monomial $c_{j_1} \cdots c_{j_r}$ in $\mathcal{W}_{\mathfrak{g}l(N)}(\Gamma)$ fulfills $\sum_{k=1}^r j_k \leq u$. Furthermore as in the proof of Lemma 3.6 we know that the homogenous degree of the polynomial $\mathcal{W}_{\mathfrak{g}l(N)}(\Gamma)$ is less or equal to $n$.

Therefore, $\dim \mathcal{W}_{\mathfrak{g}l(N)}(\mathcal{B}_n^{u})$ is less or equal to the number of partitions

$$0 \leq j_1 \leq \ldots \leq j_r$$

$$j_1 + \ldots + j_r \leq n$$

$$r \leq n.$$

A rough estimate gives:

**Lemma 7.1**

$$\dim \mathcal{W}_{\mathfrak{g}l(N)}(\mathcal{B}_n^{r}) \leq n^2 p(n),$$

where $p(n)$ is the partition number.

**Remark** In fact, we only used the space $\mathcal{B}_n^r$ instead of $\mathcal{B}_n$ itself to avoid some messy details. Let $l$ be the element in $\mathcal{B}$ with two univalent vertices and no trivalent vertex. $l^k$ is the disjoint union of $k$ copies of $l$.

So $\mathcal{B}_n = \bigoplus_{k=0}^{n} l^k \mathcal{B}_n^{r-k}$. Looking carefully at the isomorphism of $\mathcal{B}$ and $\mathcal{A}$ one can see that

$$\dim \mathcal{W}_{\mathfrak{g}l(N)}(\mathcal{B}_n) \leq \sum_{k=0}^{n} \dim \mathcal{W}_{\mathfrak{g}l(N)}(\mathcal{B}_n^{r-k}) \leq n^3 p(n).$$

Recently there was some interest in operations on the space of Vassiliev invariants that where induced by doing cabling operations (see e.g. [KSA97], [MR97] or [CDV97]). One can interpret the facts given in this section as (see [CDV97] for details):

**Corollary 7.2** Let $C_n H(n)$ be the space of all Vassiliev invariants coming from the HOMFLY-Vassiliev invariants and all of their cablings, connected and disconnected. Then there are rational functions $g_1(n)$ and $g_2(n)$ so that

$$g_1(n)e^{c\sqrt{n}} \leq \dim C_n H(n) \leq g_2(n)e^{c\sqrt{n}}.$$
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