On Ricci-Tensor of Randers Metrics

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Abstract

In this paper, we study Randers metrics and find a condition on Ricci tensor of these metrics to be Berwaldian. This generalize Shen’s Theorem which says: every R-flat complete Randers metric is locally Minkowskian. Then we find a necessary and sufficient condition on Ricci tensor under which a Randers metric of scalar flag curvature is of zero flag curvature.

Keywords: Randers metric, Berwald metric, Ricci-Tensor.

1 Introduction.

For a Finsler metric $F = F(x, y)$, its geodesics are characterized by the system of differential equations $\dddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients. A Finsler metric $F$ is called a Berwald metric if

$$G^i = \frac{1}{2} \Gamma^i_{jk}(x)y^j y^k$$

are quadratic in $y \in T_xM$ for any $x \in M$. It is proved that on a Berwald space, the parallel translation along any geodesic preserves the Minkowski functionals [4]. Thus Berwald spaces can be viewed as Finsler spaces modeled on a single Minkowski space.

In order to find explicit examples of Berwald metrics, we consider Randers metrics. By definition a Randers metric is a scalar function on $TM$ defined by $F = \alpha + \beta$ where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on $M$. The Randers metrics were introduced by G. Randers in the context of general relativity [9]. For a Randers metric $F = \alpha + \beta$, it is proved that if $\beta$ is parallel with respect to $\alpha$, then $F$ is a Berwald metric [3].

In [12], Shen prove that every regular $(\alpha, \beta)$-metric with vanishing Landsberg curvature is a Berwald metric. Therefore it is interesting to find another Finslerian curvature with the same property. For this work, we study the Riemannian curvature and its averaging quantity, i.e, the Ricci curvature of Randers metrics.

A Randers metric and its Ricci curvature, are related by their histories in physics. The well-known Ricci curvature was introduced by Ricci which used to formulate the Einstein’s of gravitation. Nine years later, Ricci’s work was used to formulate the Einstein’s of gravitation. Recently Robles investigated

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the Ricci curvature of Randers metrics and obtained the necessary and sufficient conditions for a Randers metric to be Einstein. This is one of our motivations to investigation on the Ricci tensor of Randers metrics.

In this paper, we study the Randers metric and its Ricci tensor. We find a condition on Ricci tensor of Randers metric\[ F = \alpha + \beta, \] such that the mean Landsberg tensor of \( F \) satisfies in a ODE (see Lemma 1). By this ODE, we get a condition for complete Randers metric to be Berwald metric (Theorem 3). For this reason, let us define

\[
\mathfrak{R}_{ij} := R_{ij} - \frac{1}{n+1} \left( I^k R_{ijk} + I^k R_{kji} + F^{-1} \ell_i I^k R_{k0j} + I_i R_{0j} \right),
\]

where \( R_{ijkl} \) is the Riemannian curvature of the Cartan connection, \( R_{ij} := R^r_{ijr} \) is the Ricci tensor, \( R_{nk} := R_{ijk} \theta^j, R^r_{ijr} := g^{rm} R_{irmjr}, R_{0k} := R_{jk} \theta^j, \ell_i := F_y^i \), \( I_i := g^{kl} C_{kli} \) is the mean Cartan tensor, \( C_{kli} \) is the Cartan tensor, \( g^{ij} \) is the inverse of the fundamental tensor \( g_{ij} \) and \( I^k = g^{k\ell} I_\ell \) \cite{14}. By definition, \( \mathfrak{R}_{ij} \) is not a symmetric tensor. But in a Riemannian space, \( \mathfrak{R}_{ij} \) is equal to \( R_{ij} \) which is a symmetric tensor. It is interesting to consider non-Riemannian Finsler spaces with vanishing tensor \( \mathfrak{R}_{ij} \).

According to \cite{12}, every Randers metric with vanishing Landsberg curvature is a Berwald metric. Is there any other interesting Finslerian quantity which has the same property for Randers metrics? We will show that the \( \mathfrak{R} \)-tensor is another candidate. A Finsler metric \( F \) is called \( \mathfrak{R} \)-flat if \( \mathfrak{R}_{ij} = 0 \). We prove the following.

**Theorem 1.** Every complete \( \mathfrak{R} \)-flat Randers metric \( F = \alpha + \beta \) on a manifold \( M \) is Berwaldian.

A Finsler metric \( F \) is called \( R \)-flat, if \( R_{ijkl} = 0 \). Then every \( R \)-flat metric is \( \mathfrak{R} \)-flat. In \cite{13}, it is proved that every \( R \)-flat Berwald metric is locally Minkowskian. Therefore by Theorem 1 we get the following corollary.

**Corollary 1.** Every complete \( R \)-flat Randers metric \( F = \alpha + \beta \) on a manifold \( M \) is locally Minkowskian.

The Corollary was proved by Prof. Z. Shen as Theorem 1.2 in \cite{11}. Then Theorem 1 can be regarded as a generalization of Theorem 1.2 in \cite{11}.

As we mentioned, \( \mathfrak{R} \)-tensor is not symmetric. Therefore, it is a natural problem, that we find some conditions under which the \( \mathfrak{R} \)-tensor became symmetric. In section 3, we study Randers metric of scalar flag curvature \( K \) with symmetric \( \mathfrak{R} \)-tensor. To our surprise that for a non-Riemannian Randers metric of scalar flag curvature \( K \), the \( \mathfrak{R} \)-tensor is symmetric if and only if \( K = 0 \).

**Theorem 2.** Let \( F = \alpha + \beta \) be a non-Riemannian Randers metric of scalar flag curvature \( K \). Then \( K = 0 \) if and only if \( \mathfrak{R} \)-tensor is symmetric.

There are many connections in Finsler geometry \cite{15}. Throughout this paper, we use the Cartan connection on Finsler manifolds. The \( h \)- and \( v \)- covariant derivatives of a Finsler tensor field are denoted by “|” and “\ Virasoro,” respectively.
2 Preliminaries

Let $M$ be a n-dimensional $C^\infty$ manifold. Denote by $T_xM$ the tangent space at $x \in M$, by $TM = \bigcup_{x \in M} T_xM$ the tangent bundle and by $TM_0 := TM \setminus \{0\}$ the slit tangent bundle of $M$.

Let $x \in M$ and $F_x = F|_{T_xM}$. To measure the non-Euclidean feature of $F_x$, define $C_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by $C_y(u, v, w) := C_{ijk}(y)u^iv^jw^k$ where

$$C_{ijk}(y) := \frac{1}{4}\frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k}(y).$$

The family $C := \{C_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that $C=0$ if and only if $F$ is Riemannian [13]. Define mean Cartan torsion $I_y$ by $I_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$ and $g^{jk} := (g_{jk})^{-1}$. By Deicke’s Theorem, $I=0$ if and only if $F$ is Riemannian [13].

For $y \in T_xM_0$, define the Matsumoto torsion $M_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $M_y(u, v, w) := M_{ijk}(y)u^iv^jw^k$ where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\{I_ih_{jk} + I_jh_{ik} + I_kh_{ij}\}.$$

A Finsler metric $F$ is said to be $C$-reducible if $M_y = 0$. This quantity is introduced by Matsumoto [9]. Matsumoto proves that every Randers metric satisfies that $M_y = 0$ [5].

The horizontal covariant derivatives of $C$ and $I$ along geodesics give rise to the Landsberg curvature $L_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ and mean Landsberg curvature $J_y : T_xM \to \mathbb{R}$ defined by $L_y(u, v, w) := L_{ijk}(y)u^iv^jw^k$ and $J_y(u) := J_i(y)u^i$ where

$$L_{ijk} := C_{ijk|s}y^s \quad \text{and} \quad J_i := I_{i|s}y^s.$$

The families $L := \{L_y\}_{y \in TM_0}$ and $J := \{J_y\}_{y \in TM_0}$ are called the Landsberg curvature and mean Landsberg curvature. A Finsler metric is called Landsberg metric and weakly Landsberg metric if $L=0$ and $J = 0$, respectively [13].

Given a Finsler manifold $(M, F)$, then a global vector field $G$ is induced by $F$ on $TM_0$, which in a standard coordinate $(x^i, y^i)$ for $TM_0$ is given by

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where $G^i(x, y) := \frac{1}{2}g^{ij}(x, y)[F^2]_{x^jx^k}y^k - [F]_{x^i}^2$ are called the spray coefficients of $G$. The vector field $G$ is called the associated spray to $(M, F)$. The projection of an integral curve of $G$ to $M$ is a geodesic of $(M, F)$. Using the spray coefficients of $G$, one can define

$$B_{ijkl}(y) := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}(y).$$
For a vector $y \in T_x M_0$, define the quantity $B_y : T_x M \otimes T_x M \otimes T_x M \rightarrow T_x M$ by $B_y(u, v, w) := B^i_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i}|_x$. The quantity $B$ is called the Berwald curvature. A Finsler metric is called a Berwald metric if $B=0$ [13]. Every Berwald metric is a Landsberg metric [13].

The Riemann curvature $R_y = R^i_{jkl} (dx^k \otimes \frac{\partial}{\partial x^j})|_y : T_x M \rightarrow T_x M$ is a family of linear maps on tangent spaces, defined by

$$R^i_{jkl} = \frac{\partial G^i}{\partial x^k} - v^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^i \frac{\partial G^j}{\partial y^p} \frac{\partial G^l}{\partial y^p} - \frac{\partial G^j}{\partial y^p} \frac{\partial G^l}{\partial y^q}.$$

For a flag $F = \text{span}\{y, u\} \subset T_x M$ with flagpole $y$, the flag curvature $K = K(P, y)$ is defined by

$$K(P, y) := \frac{g_y(u, R_y(u))}{g_y(y, y)g_y(u, u) - g_y(y, u)^2},$$

where $g_y = g_{ij}(x, y) dx^i \otimes dx^j$. We say that a Finsler metric $F$ is of scalar curvature if for any $y \in T_x M$, the flag curvature $K = K(x, y)$ is a scalar function on $TM$. If $K = \text{constant}$, then $F$ is said to be of constant flag curvature.

Let us consider the pull-back tangent bundle $\pi^* TM$ over $TM$ defined by $\pi^* TM = \{(u, v) \in TM \times TM | x(u) = v(x)\}$. Take a local coordinate system $(x^i)$ in $M$, the local natural frame $(\frac{\partial}{\partial x^i})$ of $T_x M$ determines a local natural frame $\partial|_v$ for $\pi^* TM$ the fibers of $\pi^* TM$, where $\partial|_v = (v, \frac{\partial}{\partial x^i})|_x$, and $v = y^i \frac{\partial}{\partial x^i}|_x \in TM$. The fiber $\pi^*_v TM$ is isomorphic to $T_{\pi(v)} M$ where $\pi(v) = x$. There is a canonical section $\ell$ of $\pi^* TM$ defined by $\ell_v = (v, v)/F(v)$.

Now let $\nabla$ be the Cartan connection on $\pi^* TM$ and $\{e_i\}_{i=1}^n$ be a local orthonormal frame for $\pi^* TM$ such that $e_n = \ell$. Let $\{\omega^i\}_{i=1}^n$ be its dual co-frame field. Put $\nabla e_i = \omega^j \otimes e_j$ and $\Omega e_i = 2\Omega^j_i \otimes e_j$, where $\{\Omega^j_i\}$ and $\{\omega^i\}$ are called respectively, the curvature forms and connection forms of $\nabla$ with respect to $\{e_i\}$. Put $\omega_i := \omega^j_i + d(log F)\delta^i_j$. Then $\{\omega^i, \omega^{n+i}\}_{i=1}^n$ is a local basis for $T^* (TM)$. Since $\{\Omega^j_i\}$ are 2-forms on $TM$, they can be expanded as

$$\Omega^j_i = \frac{1}{2} R^j_{i kl} \omega^k \wedge \omega^l + P^j_{i kl} \omega^k \wedge \omega^{n+l} + \frac{1}{2} Q^j_{i kl} \omega^{n+k} \wedge \omega^{n+l}.$$

Let $\{e_i, \bar{e}_i\}_{i=1}^n$ be the local basis for $T(TM)$, which is dual to $\{\omega^i, \omega^{n+i}\}_{i=1}^n$. The objects $R$, $P$ and $Q$ are called, respectively, the hh-, hv- and vv-curvature tensors of the Cartan connection with the components $R(\bar{e}_k, \bar{e}_l)e_i = R^j_{i kl} e_j$, $P(\bar{e}_k, \bar{e}_l)e_i = P^j_{i kl} e_j$ and $Q(\bar{e}_k, \bar{e}_l)e_i = Q^j_{i kl} e_j$ [13].

### 3 Proof of Theorem [1]

In this section, we will prove a generalized version of Theorem [1]. Indeed we study complete Randers metric $F = \alpha + \beta$ with assumption $\mathfrak{K}_{0j} = \mathfrak{K}_{i0} = 0$ where $\mathfrak{K}_{0j} = \mathfrak{K}_{ij} y^j$ and $\mathfrak{K}_{i0} = \mathfrak{K}_{ij} y^j$. More precisely, we prove the following.
Theorem 3. Let $F = \alpha + \beta$ be a complete Randers metric on a manifold $M$. Suppose that $\mathcal{R}_{0i} = \mathcal{R}_{i0} = 0$. Then $F$ is a Berwald metric.

To prove Theorem 3 we are going to establish a relation between the mean Landsberg curvature of a Randers metric and its Ricci tensor in the case that $\mathcal{R}_{0j} = \mathcal{R}_{i0} = 0$. In this case, we find that the mean Landsberg tensor $\mathbf{J}$ of Rander metric satisfies a special equation along geodesics:

Lemma 1. Let $F = \alpha + \beta$ be a Randers metric on a $n$-dimensional manifold $M$. Suppose that the Ricci tensor of $F$ satisfy $\mathcal{R}_{0i} = \mathcal{R}_{i0} = 0$. Then for any linearly parallel vector fields $u = u(t)$ and $v = v(t)$ along a geodesic $c(t)$, we have

$$\frac{d}{dt}[\mathbf{J}_c(u, v)] = 0. \quad (3.1)$$

Proof. We are specially concerned with the Cartan connection and the $h$- and $v$-covariant derivatives are denoted by “$|$” and “$\|$”, respectively. The following Bianchi identity of Cartan connection is hold

$$R^h_{i,j,k} + Q^h_{kr} R^r_{i,j} + S_{(ij)} \{ R^h_{i,r} C^r_{jk} + P^h_{l,i} R^r_{jk} + P^h_{l,j} k_{j|k} \} = 0, \quad (3.2)$$

where $S_{(ij)}$ means interchange of indices $i$ and $j$ (for more details see the formula (17.15) page 113 in [6]). We take a contraction for $h$ and $j$ in (3.2) and get the following

$$R_{ik,i} = P^s_{i,r} L^r_{ki} - R_{ir} C^r_{ki} - P^s_{i,r} k_{i|k} + P^{r}_{l,s|k} - R^r_{l,k} C^r_{si} - P^m_{l,m} L^r_{si} + Q^m_{l,m} R^r_{mk}, \quad (3.3)$$

(see (2.5) in [14]). Contracting (3.3) by $y^l$ yields

$$R_{0k,i} = R_{ik} - R_0 \cdot C^r_{ki} - J_{i|k} + L^r_{ki} J_r + L^r_{k|r} - R^s_{kr} C^r_{si} - L^s_{kr} L^r_{si}. \quad (3.4)$$

For more details see (2.5)’ in [14]. Since $F$ is a Randers metric, then $F$ is $C$-reducible

$$C_{ijk} = \frac{1}{n+1} \{ I_i h_{jk} + I_j h_{ik} + I_k h_{ij} \}. \quad (3.5)$$

Taking a horizontal derivative of (3.3), using $h_{ij|k} = 0$ and $C_{ijk}|y^s = L_{ijk}$ we get

$$L_{ijk} = \frac{1}{n+1} \{ J_i h_{jk} + J_j h_{ik} + J_k h_{ij} \}. \quad (3.6)$$

From (3.3) we derive

$$L_{k|i|r} = \frac{1}{n+1} \{ J_{r|k} h_{ki} + J_{i|k} + J_{k|i} - F^{-1}(J_{i|s} y^s t^k + J_{k|s} y^s t^i) \}. \quad (3.7)$$

Putting (3.3), (3.6) and (3.7) in (3.4), after long computations, we have

$$R_{0k,i} = \frac{S_{(ik)}}{n+1} \{ J_{i|k} - F^{-1} J_{i|k} + \frac{(n-3)}{2} J_{k|i} - \frac{1}{n+1} (R_{0k} - F^{-1} R_{00} \ell_k) J_{i} \}
+ \frac{1}{n+1} (J_{r|k} + (n-1) J_r J_{r}- \frac{1}{n+1} m R_{0m}) h_{ik} + \mathcal{R}_{ik}, \quad (3.8)$$

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where
\[ \mathfrak{R}_{ik} := R_{ik} - \frac{1}{n+1} \left( I^m R_{ikm} + I^m R_{mki} + F^{-1} \ell_i I^m R_{m0k} + I_i R_{0k} \right). \] (3.9)

Multiplying (3.8) with \( y^k \) yields
\[ R_{00,i} = R_{i0} + R_{0i} - \frac{1}{n+1} (2I^m R_{m0i} + R_{00} I_i) - J_{i|s} y^s. \] (3.10)

Contracting (3.9) by \( y^i \) and \( y^k \), we get respectively
\[ \mathfrak{R}_{0i} = R_{0i}, \] (3.11)
\[ \mathfrak{R}_{i0} = R_{i0} - \frac{1}{n+1} (2I^m R_{m0i} + I_i R_{00}). \] (3.12)

By assumption we have \( \mathfrak{R}_{i0} = \mathfrak{R}_{0i} = 0 \). From (3.11) and (3.12) it follows that
\[ R_{0i} = 0, \] (3.13)
\[ R_{i0} = \frac{2}{n+1} I^m R_{m0i}. \] (3.14)

By considering (3.13) and (3.14), the equation (3.10) reduces to (3.1).

**Lemma 2.** Let \((M, F)\) be Finsler manifold. Suppose that the mean Landsberg curvature of \( F \) satisfy equation (3.1). Then for any geodesic \( c(t) \) and any parallel vector field \( V(t) \) along \( c \), the following function
\[ I(t) := I_c(V(t)) \] (3.15)
must be in the following forms
\[ I(t) = t J(0) + I(0). \] (3.16)

Proof. By assumption we have
\[ J_{i|s} y^s = I_{i|s} y^s y^s = 0. \] (3.17)

Let
\[ J(t) := J_c(V(t)) \] (3.18)
From our definition of \( J_y \), we get
\[ J(t) = I'(t). \] (3.19)

By (3.17) we conclude
\[ I''(t) = 0, \] (3.20)
which implies that \( I'(t) = I'(0) \). By (3.19), we get (3.16). \( \square \)
Remark 1. Let \((M,F)\) be a Finsler space and \(c : [a,b] \to M\) be a geodesic. For a parallel vector field \(V(t)\) along the geodesic \(c\),
\[
g_c(V(t), V(t)) = \text{constant}. \tag{3.21}
\]

Proof of Theorem \([3]\) To prove Theorem \([3]\) take an arbitrary unit vector \(y \in T_xM\) and an arbitrary vector \(v \in T_xM\). Let \(c(t)\) be the geodesic with \(\dot{c}(0) = y\) and \(V(t)\) the parallel vector field along \(c\) with \(V(0) = v\). Define \(I(t)\) and \(J(t)\) as in (3.15) and (3.18), respectively. Then by Lemma \([2]\) we have
\[
I(t) = tJ(0) + I(0). \tag{3.22}
\]
The norm of mean Cartan torsion at a point \(x \in M\) is defined by
\[
\|I\|_x := \sup_{0 \neq y \in T_xM} \sqrt{I_i(x,y)g^{ij}(x,y)I_j(x,y)}. \tag{3.23}
\]
It is known that if \(F = \alpha + \beta\) is a Randers metric, then
\[
\|I\|_x \leq \frac{n+1}{2} \sqrt{1 - \sqrt{1 - \|\beta\|^2}} < \frac{n+1}{\sqrt{2}}. \tag{3.24}
\]
See \([11]\) or \([13]\) for a proof. So \(I_y\) is bounded, i.e., there is a constant \(N < \infty\) such that
\[
\|I\|_x := \sup_{y \in T_xM} \sup_{v \in T_xM} \frac{I_y(v)}{|g_y(v, v)|^2} \leq N \tag{3.25}
\]
By Remark \([1]\) \(Q := g_c(V(t), V(t)) = \text{constant}\) is a positive constant. Thus
\[
|I(t)| \leq NQ^2 < \infty,
\]
and \(I(t)\) is a bounded function on \([0, \infty)\). Using \(|I| < \infty\) and letting \(t \to +\infty\) or \(t \to -\infty\), we conclude that
\[
J_y(v) = J(0) = 0. \tag{3.26}
\]
Therefore \(J = 0\) and \(F\) is a weakly Landsberg metric. From (3.6), we conclude that \(F\) is a Landsberg metric. It is known that \(F = \alpha + \beta\) is a Landsberg metric if and only if is a Berwald metric \([5]\). Then \(F\) is a Berwald metric.

Corollary 2. Let \((M,F)\) be a compact negatively curved Finsler manifold of dimension \(n \geq 3\). Suppose that \(F\) is of non-zero scalar flag curvature \(K\). If \(\mathcal{R}_{0i} = \mathcal{R}_{i0} = 0\), then \(F\) is a Riemannian metric.

Proof. In \([7]\), Mo-Shen prove that every closed \(n\)-dimensional Finsler manifold of negative scalar flag curvature is a Randers metric \((n \geq 3)\). By Theorem \([3]\) \(F\) is a Berwald metric. Numata showed that every Landsberg metric of non-zero scalar flag curvature is Riemannian \([8]\). This completes the proof.
Corollary 3. Let \((M, F)\) be a complete Randers manifold with vanishing \(\mathcal{R}\)-tensor. Suppose that \(F\) is of non-zero scalar flag curvature. Then \(F\) is a Riemannian metric.

Proof. By Theorem 1, \(F\) is a Berwald metric. According to Akbar-Zadeh’s theorem, every complete Berwald metric of non-zero scalar flag curvature with bounded Cartan tensor is Riemannian [1]. This completes the proof. \(\square\)

4 Proof of Theorem 2

Proof of Theorem 2: Let \(F = \alpha + \beta\) be of scalar flag curvature \(K\). In this case, it is known that the Ricci tensor \(R_{ij}\) is symmetric and we have

\[ R_{ijk} = \frac{F^2}{3}(K_{ij}h_k - K_{ik}h_j) + K(y_jh_k - y_kh_j), \quad (4.1) \]

where \(K_{ij} = \frac{\partial K}{\partial y_i} [1]\). Suppose that \(F\) be a non-Riemannian Randers metric of scalar flag curvature \(K\). Then \(\mathfrak{R}_{ij}\) is written in the following form

\[ \mathfrak{R}_{ij} = R_{ij} + \frac{F^2}{3(n+1)}(Kh_{ij} + K_iI_j + K_jI_i) - \frac{I_i}{3}(F^2K_{ij} + 3Kg_{ij}). \quad (4.2) \]

Let \(\mathfrak{R}_{ij}\) is symmetric. Recall that \(K_{ij}y^i = 0\). Since \(I_i \neq 0\), then we have \(K = 0\). Conversely, it is easy to see that if \(K = 0\) then \(\mathfrak{R}_{ij}\) is symmetric. \(\square\)

Corollary 4. Let \((M, F)\) be a non-Riemannian Randers manifold. Suppose that \(F\) is of scalar flag curvature \(K\). Then \(K = 0\) if and only if \(\mathfrak{R}_{i0} = \mathfrak{R}_{0i}\).

Proof. By (4.2) we have

\[ \mathfrak{R}_{i0} = R_{i0} - KF^2I_i, \quad \mathfrak{R}_{0i} = R_{0i}. \quad (4.3) \]

For a Finsler metrics of scalar flag curvature, the Ricci tensor is symmetric. Then by (4.3) we have \(KF^2I_i = 0\). Since \(I_i \neq 0\), then \(K = 0\). \(\square\)

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