Optimal Robustness Results for Some Bayesian Procedures and the Relationship to Prior-Data Conflict

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Abstract

The robustness to the prior of Bayesian inference procedures based on a measure of statistical evidence are considered. These inferences are shown to have optimal properties with respect to robustness. Furthermore, a connection between robustness and prior-data conflict is established. In particular, the inferences are shown to be effectively robust when the choice of prior does not lead to prior-data conflict. When there is prior-data conflict, however, robustness may fail to hold.

1 Introduction

Robustness to the choice of the prior is an issue of considerable importance in a Bayesian statistical analysis. If an inference is very sensitive to the choice of the prior, then this could be viewed as either a negative for the inference method being used or for the choice of prior. In this paper it is shown that certain inferences are in a sense optimally robust to the choice of the prior. Furthermore, when the sensitivity of the inferences to the prior is measured quantitatively, it is shown that there is an intimate connection between the effective robustness of the inferences and whether or not there is prior-data conflict. So by choice of the inferential methodology and the avoidance of prior-data conflict, robustness of the inferences to the choice of prior is achieved.

The basic ingredients for a statistical analysis are taken here to be the data $x$, a statistical model $\{f_\theta : \theta \in \Theta\}$, where each $f_\theta$ is a probability density with respect to volume measure $\mu$ on the sample space $\mathcal{X}$, and a proper prior density $\pi$ with respect to volume measure $\nu$ on $\Theta$. Note that volume measure on a discrete set is taken to be counting measure. Furthermore, suppose that interest is in making inferences about the quantity $\psi = \Psi(\theta)$ where $\Psi : \Theta \to \Psi$ is onto and we don’t distinguish between the function and its range to save notation.

Let $\pi_{\psi}(\cdot \mid x)$ and $\pi_{\psi}$ denote the posterior and prior densities of $\psi$ where these are both taken with respect to support measure $\nu_{\psi}$ on $\Psi$. It follows that,
under smoothness assumptions, \( \pi_\psi(x) = \int_{\Psi^{-1}\{\psi\}} \pi(\theta)J_\psi(\theta) \nu_{\Psi^{-1}\{\psi\}}(d\theta) \) where 
\( J_\psi(\theta) = (\det(d\Psi(\theta) \circ d\Psi(\theta)^t))^{-1/2} \), 
\( d\Psi \) is the differential of \( \Psi \) and \( \nu_{\Psi^{-1}\{\psi\}} \) is volume measure on \( \Psi^{-1}\{\psi\} \). Also, 
\( \pi_\psi(x \mid |x) = \int_{\Psi^{-1}\{\psi\}} \pi(\theta \mid |x)J_\psi(\theta) \nu_{\Psi^{-1}\{\psi\}}(d\theta) \)
where \( \pi(\theta \mid |x) = \pi(\theta)f_\theta(x) / m(x) \), with \( m(x) = \int_\theta \pi(\theta)f_\theta(x) \nu(d\theta) \), is the posterior density of \( \theta \) with respect to \( \nu \). Note that \( m \) is the prior predictive density of the data with respect to \( \mu \). The conditional prior of \( \theta \) given \( \psi = \Psi(\theta) \) has density 
\( \pi(\theta \mid |\psi) = \pi(\theta)J_\psi(\theta) / \pi_\psi(\psi) \) with respect to \( \nu_{\Psi^{-1}\{\psi\}} \) on the set 
\( \Psi^{-1}\{\psi\} \). The conditional prior predictive density of \( x \) is then given by 
\( m(x \mid |\psi) = \int_{\Psi^{-1}\{\psi\}} \pi(\theta \mid |\psi)f_\theta(x) \nu_{\Psi^{-1}\{\psi\}}(d\theta) \). A simple argument, see Baskurt and Evans (2013), gives the Savage-Dickey ratio result that 
\[
\frac{\pi_\psi(\psi \mid |x)}{\pi_\psi(\psi)} = \frac{m(x \mid |\psi)}{m(x)},
\]
which has some use in the developments here.

Robustness to the prior has been considered by many authors and there are a number of different approaches. Many discussions are concerned with determining the range of values that some characteristic of interest takes when the prior is allowed to vary over some class. Berger (1990, 1994) contain broad reviews of work on this topic and Rios Insua and Ruggeri (2000) is a collection of papers by key contributors. Dey and Birmiwal (1994) considers global robustness measures based upon measures of distance from the posterior distribution.

The approach taken here is to study robustness to the prior for relative belief inferences for \( \psi \) rather than all possible inferences. Relative belief inferences are based on the relative belief ratio defined
\[
RB_\psi(\psi \mid |x) = \lim_{\delta \to 0} \frac{\Pi_\Psi(N_\delta(\psi) \mid |x)}{\Pi_\Psi(N_\delta(\psi))},
\]
whenever this limit exists for a sequence of neighborhoods \( N_\delta(\psi) \) of \( \psi \) converging nicely to \( \psi \) (see Rudin (1974) for the definition of 'converging nicely'). Under mild regularity conditions the limit exists and is given by \( RB_\psi(\psi \mid |x) = \pi_\psi(\psi \mid |x) / \pi_\psi(\psi) \). Since \( RB_\psi(\psi \mid |x) \) measures the change in belief that \( \psi \) is the true value it is a measure of evidence. Here \( RB_\psi(\psi \mid |x) > 1 \) means that there is evidence in favor of \( \psi \) being the true value, as belief in \( \psi \) has increased after seeing the data, and \( RB_\psi(\psi \mid |x) < 1 \) means that there is evidence against \( \psi \) being the true value, as belief in \( \psi \) has decreased after seeing the data. Section 2 provides some more details concerning relative belief inferences for both estimation and hypothesis assessment but also see Baskurt and Evans (2013). Results in Section 3 establish that these inferences have optimal robustness properties when the marginal prior for \( \psi \) is allowed to vary over all possibilities in the class of \( \epsilon \)-contaminated priors. This generalizes results found in Wasserman (1989), Ruggeri and Wasserman (1993) and de la Horra and Fernandez (1994). Furthermore, an ambiguity concerning the interpretation of the results is resolved. As such this provides further justifications for these inferences.

While inferences may be optimally robust, this does not imply that they are in fact robust. In Section 4 quantitative measures of the sensitivity of relative
belief inferences to both the marginal prior of $\psi$ and the conditional prior for $\theta$ given $\Psi(\theta) = \psi$ are derived. In Section 5 it is shown that these inferences are indeed robust when the base prior $\pi$ does not suffer from prior-data conflict. This adds weight to arguments concerning the importance of checking for prior-data conflict before reporting inferences, as prior-data conflict can imply sensitivity of the inferences to the choice of the prior. Prior-data conflict is interpreted as the true value lying in the tails of the prior and consistent methods have been developed for assessing this in Evans and Moshonov (2006) and Evans and Jang (2011a). Methodology for modifying a prior when prior-data conflict is encountered, through the selection of a prior weakly informative with respect to the base prior, is developed in Evans and Jang (2011b).

2 Relative Belief Inferences

When $RB_\psi(\psi \mid x) > 1$ this is the factor by which prior belief in the truth of $\psi$ has increased after seeing the data. Clearly the bigger $RB_\psi(\psi \mid x)$ the more evidence there is in favor of $\psi$ while, when $RB_\psi(\psi \mid x) < 1$, the smaller $RB_\psi(\psi \mid x)$ is the more evidence there is against $\psi$. This leads to a total preference ordering on $\Psi$, namely, $\psi_1$ is not preferred to $\psi_2$ whenever $RB_\psi(\psi_1 \mid x) \leq RB_\psi(\psi_2 \mid x)$ since there is at least as much evidence for $\psi_2$ as there is for $\psi_1$. This in turn leads to unambiguous solutions to inference problems.

The best estimate of $\psi$ is the value for which the evidence is greatest, namely,

$$\psi(x) = \arg \sup RB_\psi(\psi \mid x).$$

Associated with this estimate is a $\gamma$-relative belief credible region $C_{\Psi,\gamma}(x) = \{\psi : RB_\psi(\psi \mid x) \geq c_{\Psi,\gamma}(x)\}$ where $c_{\Psi,\gamma}(x) = \inf\{k : \Pi_\Psi(RB_\psi(\psi \mid x) \leq k \mid x) \geq 1 - \gamma\}$. Notice that $\psi(x) \in C_{\Psi,\gamma}(x)$ for every $\gamma \in [0,1]$ and so, for selected $\gamma$, the size of $C_{\Psi,\gamma}(x)$ can be taken as a measure of the accuracy of the estimate $\psi(x)$. The interpretation of $RB_\psi(\psi \mid x)$ as the evidence for $\psi$, forces the use of the sets $C_{\Psi,\gamma}(x)$ for our credible regions. For if $\psi_1$ is in such a region and $RB_\psi(\psi_2 \mid x) \geq RB_\psi(\psi_1 \mid x)$, then $\psi_2$ must be in the region as well as there is at least as much evidence for $\psi_2$ as for $\psi_1$. Optimal properties for relative belief credible regions, in the class of all credible regions, have been established in Evans, Guttmann and Swartz (2006) and Evans and Shakhare (2008) and optimal properties for $\psi(x)$ are established in Evans and Jang (2011c).

For the assessment of the hypothesis $H_0 : \Psi(\theta) = \psi_0$, the evidence is given by $RB_\psi(\psi_0 \mid x)$. One problem that both the relative belief ratio and the Bayes factor share as measures of evidence, is that it is not clear how they should be calibrated. Certainly the bigger $RB_\psi(\psi_0 \mid x)$ is than 1, the more evidence we have in favor of $\psi_0$ while the smaller $RB_\psi(\psi_0 \mid x)$ is than 1, the more evidence we have against $\psi_0$. But what exactly does a value of $RB_\psi(\psi_0 \mid x) = 20$ mean? It would appear to be strong evidence in favor of $\psi_0$ because beliefs have increased by a factor of 20 after seeing the data. But what if other values of $\psi$ had even larger increases? For example, the discussion in Baskurt and Evans (2013) of the Jeffreys-Lindley paradox makes it clear that the value of a relative belief
ratio or a Bayes factor cannot always be interpreted as an indication of the strength of the evidence.

The value $RB_\psi(\psi_0 \mid x)$ can be calibrated by comparing it to the other possible values $RB_\psi(\cdot \mid x)$ through its posterior distribution. For example, one possible measure of the strength is

$$\Pi_\psi(RB_\psi(\psi \mid x) \leq RB_\psi(\psi_0 \mid x) \mid x)$$

which is the posterior probability that the true value of $\psi$ has a relative belief ratio no greater than that of the hypothesized value $\psi_0$. While (3) may look like a p-value, it has a very different interpretation. For when $RB_\psi(\psi_0 \mid x) < 1$, so there is evidence against $\psi_0$, then a small value for (3) indicates a large posterior probability that the true value has a relative belief ratio greater than $RB_\psi(\psi_0 \mid x)$ and so there is strong evidence against $\psi_0$. If $RB_\psi(\psi_0 \mid x) > 1$, so there is evidence in favor of $\psi_0$, then a large value for (3) indicates a small posterior probability that the true value has a relative belief ratio greater than $RB_\psi(\psi_0 \mid x)$ and so there is strong evidence in favor of $\psi_0$. Notice that, in the set $\{\psi : RB_\psi(\psi \mid x) \leq RB_\psi(\psi_0 \mid x)\}$, the “best” estimate of the true value is given by $\psi_0$ simply because the evidence for this value is the largest in this set.

Various results have been established in Baskurt and Evans (2013) supporting both $RB_\psi(\psi_0 \mid x)$, as the measure of the evidence for $H_0$, and (3), as a measure of the strength of that evidence. For example, the following simple inequalities are useful in assessing the strength of the evidence, namely, $\Pi_\psi(RB_\psi(\psi \mid x) = RB_\psi(\psi_0 \mid x) \mid x) \leq \Pi_\psi(RB_\psi(\psi \mid x) \leq RB_\psi(\psi_0 \mid x) \mid x) \leq RB_\psi(\psi_0 \mid x)$. So if $RB_\psi(\psi_0 \mid x) > 1$ and $\Pi_\psi(RB_\psi(\psi_0 \mid x) \mid x)$ is large, there is strong evidence in favor of $\psi_0$ while, if $RB_\psi(\psi_0 \mid x) < 1$ is very small, then there is immediately strong evidence against $\psi_0$. Also, in situations where there are only a few possible values of $\psi$, then $\Pi_\psi(RB_\psi(\psi \mid x) = RB_\psi(\psi_0 \mid x) \mid x)$ can be a more appropriate measure of strength.

When interest is in making inferences about $\psi = \Psi(\theta)$, it is reasonable to ask how sensitive the belief relief approach is to the ingredients given by the prior. This entails examining how dependent $\psi(x), C_\Psi(\cdot), RB_\psi(\psi_0 \mid x)$ and $\Pi_\psi(RB_\psi(\psi \mid x) \leq RB_\psi(\psi_0 \mid x) \mid x)$ are to changes in the prior, as these four objects represent the essential relative belief inferences.

The full prior $\pi$ for $\theta$ can always be factored as $\pi(\theta) = \pi_\Psi(\psi)\pi(\theta \mid \psi)$. In contrast to other discussions of robustness with respect to the prior, the sensitivity of the inferences to $\pi_\Psi$ and the sensitivity of the inferences to $\pi(\cdot \mid \psi)$ are considered separately, as this leads to more information concerning where the lack of robustness arises when this occurs.

3 Optimal Robustness With Respect to the Marginal Prior

The result (1) implies that $RB_\psi(\psi \mid x) = m(x \mid \psi)/m(x)$. From this it is immediate that $\psi(x) = \arg\sup_\psi RB_\psi(\psi \mid x) = \arg\sup_\psi m(x \mid \psi)$ and so the relative
belief estimate is optimally robust to $\pi_\psi$ as the estimate has no dependence on the marginal prior. Furthermore, $C_{\psi, \gamma}(x)$ is of the form $\{\psi : m(x | \psi) \geq k\}$ for some $k$ and so the form of relative belief regions for $\psi$ is optimally robust to $\pi_\psi$. The specific region chosen for the assessment of the accuracy of $\psi(x)$ depends on the posterior and so is not independent of $\pi_\psi$. It is now proved that $C_{\psi, \gamma}(x)$ has an optimal robustness property among all credible regions for $\psi$.

Consider $\epsilon$-contaminated priors for $\theta$ of the form

$$\Pi_\epsilon = \Pi(\cdot | \psi) \times [(1 - \epsilon)\Pi_\psi + \epsilon Q],$$

where $Q$ is a probability measure on $\Psi$ and $\Pi$ is the base prior as described in the Introduction. Note that the conditional prior of $\theta$ given $\Psi(\theta) = \psi$ is fixed and independent of $\epsilon$.

To assess the robustness of the posterior content of a set $A \subset \Psi$ it makes sense to look at $\delta(A) = \Pi_\psi^{upper} (A | x) - \Pi_\psi^{lower} (A | x)$ where $\Pi_\psi^{upper} (A | x) = \sup_{\psi \in A} \Pi_\psi (A | x)$ and $\Pi_\psi^{lower} (A | x) = \inf_{\psi \in A} \Pi_\psi (A | x)$. For this let $\epsilon^* = \epsilon/(1-\epsilon)$ and $r(A) = \sup_{\psi \in A} R\psi(\psi | x) = \sup_{\psi \in A} m(x | \psi)/m(x)$, so $r(\Psi) = R\psi_{LRSE}(x) | x)$ and always one and only one of $r(A), r(A^c)$ equals $r(\Psi)$.

The following result is needed and a proof is provided in the Appendix.

**Lemma 1** (Huber (1973)) Let $Q$ denote a probability measure on $\Psi$. For prior measure $\Pi_\psi = (1-\epsilon)\Pi_\psi + \epsilon Q$ on $\Psi$ and $A \subset \Psi$, (i) $\Pi_\psi^{upper} (A | x) = (\Pi_\psi (A | x) + \epsilon^* r(A))/(1 + \epsilon^* r(A))$, (ii) $\Pi_\psi^{lower} (A | x) = \Pi_\psi (A | x)/(1 + \epsilon^* r(A^c))$, (iii)

$$\delta(A) = \frac{\Pi_\psi (A | x) \epsilon^*(r(A^c) - r(A))}{(1 + \epsilon^* r(A))(1 + \epsilon^* r(A^c))} + \frac{\epsilon^* r(A)}{(1 + \epsilon^* r(A))}$$

and (iv) $\delta(A^c) = \delta(A)$.

Let $\gamma^*(x) = \Pi_\psi (C_{\psi, \gamma}(x) | x)$ be the exact posterior content of the $\gamma$-relative belief region. The following result generalizes results found in Wasserman (1989) and de la Horra and Fernandez (1994) who considered robustness to the prior of credible regions for the full parameter $\theta$. In particular, this result applies to arbitrary parameters $\psi = \Psi(\theta)$ and does not require continuity.

**Proposition 2** The following hold,

(i) among all sets $A \subset \Psi$ satisfying $\Pi_\psi (A | x) \leq \gamma^*(x)$ and $r(A) = r(\Psi)$, the set $C_{\psi, \gamma}(x)$ minimizes $\delta(A)$,

(ii) among all sets $A \subset \Psi$ satisfying $\Pi_\psi (A | x) \geq \gamma^*(x)$ and $r(A^c) = r(\Psi)$, the set $C^c_{\psi, 1-\gamma^*(x)}(x)$ minimizes $\delta(A)$,

(iii) when $\gamma^*(x) = \gamma \geq 1/2$ then, among all sets $A \subset \Psi$ satisfying $\Pi_\psi (A | x) = \gamma$, the set $C_{\psi, \gamma}(x)$ minimizes $\delta(A)$.

**Proof.** (i) For any set $A$ with $r(A) = r(\Psi)$ then $r(A^c) - r(A) = r(A^c) - r(\Psi) \leq 0$. 

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Therefore,
\[
\delta(A) = \frac{\Pi_{\Psi}(A \mid x) e^r(r(A^c) - r(\Psi))}{(1 + e^r(\Psi))(1 + e^r(A^c))} + \frac{e^r(\Psi)}{1 + e^r(\Psi)}
\geq \frac{\Pi_{\Psi}(C_{\Psi,\gamma}(x) \mid x) e^r(r(A^c) - r(\Psi))}{(1 + e^r(\Psi))(1 + e^r(A^c))} + \frac{e^r(\Psi)}{1 + e^r(\Psi)}.
\]

Now
\[
\frac{r(A^c) - r(\Psi)}{1 + e^r(A^c)} \quad (5)
\]
is increasing in \(r(A^c)\), so we need to show that \(A = C_{\Psi,\gamma}(x)\) minimizes \(r(A^c)\) among all \(A\) satisfying \(\Pi_{\Psi}(A \mid x) \leq \Pi_{\Psi}(C_{\Psi,\gamma}(x) \mid x)\) and \(r(A) = r(\Psi)\). Suppose that \(r(A^c) < r(C_{\Psi,\gamma}(x))\) and let \(B = \{\psi : R_B\psi(x) > r(A^c)\}\). Note that \(r(C_{\Psi,\gamma}(x)) \leq \inf_{\psi \in C_{\Psi,\gamma}(x)} R_B\psi(x)\) and so \(C_{\Psi,\gamma}(x) \subset B\), which implies \(\Pi_{\Psi}(B \mid x) > \Pi_{\Psi}(C_{\Psi,\gamma}(x) \mid x)\) with the strictness of the inequality following from the definition of \(C_{\Psi,\gamma}(x)\). But also \(B \subset A\) which contradicts \(\Pi_{\Psi}(A \mid x) \leq \Pi_{\Psi}(C_{\Psi,\gamma}(x) \mid x)\) and so we must have \(r(A^c) \geq r(C_{\Psi,\gamma}(x))\). This establishes that (5) is minimized by \(A = C_{\Psi,\gamma}(x)\).

(ii) Now consider all the sets \(A\) with \(r(A^c) = r(\Psi)\). Since \(\delta(A) = \delta(A^c)\), it is equivalent to minimize \(\delta(A^c)\) among all sets \(A^c\) satisfying \(\Pi_{\Psi}(A^c \mid x) \leq \Pi_{\Psi}(C_{\Psi,\gamma}(x) \mid x) = 1 - \gamma^*(x)\) and \(r(A^c) = r(\Psi)\). By part (i) this is minimized by taking \(A^c = C_{\Psi,\gamma}(x)\) and the result is proved.

(iii) The solutions to the optimization problems in parts (i) and (ii), namely, \(C_{\Psi,\gamma}(x)\) and \(C_{\Psi,1-\gamma}(x)\) respectively, both have posterior content equal to \(\gamma\). As such one of these sets is the solution to the optimization problem stated in (iii). We have that
\[
\delta(C_{\Psi,\gamma}(x)) - \delta(C_{\Psi,1-\gamma}(x)) = \delta(C_{\Psi,\gamma}(x)) - \delta(C_{\Psi,1-\gamma}(x))
\]
\[
= \frac{\gamma e^r(r(C_{\Psi,\gamma}(x)) - r(\Psi))}{(1 + e^r(\Psi))(1 + e^r(C_{\Psi,\gamma}(x)))} - \frac{\gamma e^r(r(C_{\Psi,1-\gamma}(x)) - r(\Psi))}{(1 + e^r(\Psi))(1 + e^r(C_{\Psi,1-\gamma}(x)))}
\]
\[
= \frac{\gamma e^r}{(1 + e^r(\Psi))} \left\{ \frac{r(C_{\Psi,\gamma}(x)) - r(\Psi)}{1 + e^r(C_{\Psi,\gamma}(x))} - \frac{r(C_{\Psi,1-\gamma}(x)) - r(\Psi)}{1 + e^r(C_{\Psi,1-\gamma}(x))} \right\}.
\]

The result follows from this because \(C_{\Psi,\gamma}(x) \subset C_{\Psi,1-\gamma}(x)\), so \(r(C_{\Psi,\gamma}(x)) \leq r(C_{\Psi,1-\gamma}(x))\), and \(\delta\) is increasing in \(r(A^c)\). 

It is interesting to consider the statistical meaning of the separate parts of Proposition 2 as the statements create a degree of ambiguity. If a system of credible regions is being used, say \(B_{\Psi,\gamma}(x)\), then it makes sense to require that these sets are monotonically increasing in \(\gamma\) and the smallest set \(\lim_{\gamma \to 0} B_{\Psi,\gamma}(x)\) contains a single point which is taken as the estimate of \(\psi\). The size of \(B_{\Psi,\gamma}(x)\), for some specific \(\gamma\), can then be taken as an assessment of the accuracy of the estimate where size is measured in some application dependent way. The relative belief regions satisfy this and the estimate, under the assumption of a unique maximizer of \(RB\psi(x)\), is \(\psi(x)\). So effectively (i) is saying that \(C_{\Psi,\gamma}(x)\) is the most robust system of credible regions with respect to posterior content. Note
that we have to exclude sets $A$ with $\Pi_\Psi(A \mid x) > \gamma^*(x)$ because, for example, the set $A = \Psi$ is always optimally robust with respect to content but does not provide a meaningful assessment of the accuracy of the estimate. Given that $\psi(x)$ and the form of $C_{\Psi,\gamma}(x)$ are optimally robust, this further supports the claim that relative belief estimation is optimally robust to the choice of the marginal prior. Note that the sets in (ii) do not satisfy the stated criteria for being a system of credible regions.

Part (iii) indicates that, when there are many sets with posterior content exactly equal to $\gamma$, and this is typically true in the continuous case, then $C_{\Psi,\gamma}(x)$ is optimally robust among these sets with respect to content. It makes sense to require $\gamma^*(x) \geq 1/2$ for any credible region as, if $\gamma^*(x) < 1/2$, then there is more belief that the true value is in $C_{\Psi,\gamma}(x)$ than in $C_{\Psi,\gamma}(x)$.

Applying Lemma 1 gives

$$\delta(C_{\Psi,\gamma}(x)) = \frac{\epsilon^* RB_\Psi(\psi(x) \pi x)}{(1 + \epsilon^* RB_\Psi(\psi(x) \mid x))} \times \left\{ 1 - \frac{\Pi_\Psi(C_{\Psi,\gamma}(x) \mid x)}{RB_\Psi(\psi(x) \mid x)} \frac{1 - \inf_{\psi \in C_{\Psi,\gamma}(x)} RB_\Psi(\psi \mid x)}{(1 + \epsilon^* \inf_{\psi \in C_{\Psi,\gamma}(x)} RB_\Psi(\psi \mid x))} \right\}$$

and this can be close to 1 when $RB_\Psi(\psi(x) \mid x)$ is large. So, while $C_{\Psi,\gamma}(x)$ possesses an optimal robustness property with respect to posterior content, this does not imply that the posterior content is necessarily robust. This depends on other aspects of the particular problem which will be discussed.

## 4 Measuring Robustness Quantitatively

To measure the robustness of an inference to the prior $\pi$, when using the $\epsilon$-contaminated class, it is natural to look at Gâteaux derivatives of the relevant quantity at $\pi$ in various directions $Q$. The derivative is a measure of the sensitivity of the inference to small changes in the prior and so is local in nature. When the derivative is large for some $Q$, the inference is highly sensitive to the prior chosen and naturally this is viewed negatively. In this section this behavior of relative belief inferences is analyzed separately for $\epsilon$-contaminated classes for the marginal $\pi_\Psi$ and the conditional $\pi(\cdot \mid \psi)$.

### 4.1 Sensitivity to the Marginal Prior

Consider the family of priors given by (6) but now restricted to those $Q$ that are also absolutely continuous with respect to $\nu_\Psi$ on $\Psi$ and let $q$ denote the density of $Q$. The posterior of $\psi$ based on the contaminated prior is $\Pi_\epsilon,\Psi(\cdot \mid x) = (1 - \epsilon_x)\Pi_\Psi(\cdot \mid x) + \epsilon_x Q(\cdot \mid x)$ where $\epsilon_x = em_Q(x) / [(1 - \epsilon)m(x) + em_Q(x)], m_Q(x) = \int_\Psi m(x \mid \psi) Q(d\psi)$ and $Q(A \mid x) = \int_A (m(x \mid \psi)/m_Q(x)) Q(d\psi)$. The relative belief ratio for $\psi$ based on a general $\Pi_\epsilon$ equals $RB_{\epsilon,\Psi}(\psi \mid x) = (1 - \epsilon_x)RB_\Psi(\psi \mid x) + \epsilon_x RB_{Q,\Psi}(\psi \mid x)$ and here, using (6), $RB_{Q,\Psi}(\psi \mid x) = m(x \mid \psi)/m_Q(x)$ so

$$RB_{\epsilon,\Psi}(\psi \mid x) = \frac{RB_\Psi(\psi \mid x)}{1 - \epsilon(1 - m_Q(x)/m(x))}.$$  \hspace{1cm} (6)
The following result gives the Gâteaux derivative of the relative belief ratio.

**Proposition 3** The Gâteaux derivative of $RB_\psi(\cdot | x)$ at $\psi$ in the direction $Q$ equals

$$RB_\psi(\psi | x) \{ 1 - m_Q(x) / m(x) \} .$$

**Proof.** From (3),

$$\lim_{\epsilon \to 0} \frac{RB_\psi(\psi | x) - RB_\psi(\psi | x)}{\epsilon} = RB_\psi(\psi | x) \lim_{\epsilon \to 0} \left\{ \frac{(1 - m_Q(x) / m(x))}{1 - \epsilon(1 - m_Q(x) / m(x))} \right\} .$$

The value of (7) can be large simply because $RB_\psi(\psi | x)$ is large, so it makes more sense to look at the relative change as given by $1 - m_Q(x) / m(x)$.

The Gâteaux derivative of the strength of the evidence is now computed.

**Proposition 4** The Gâteaux derivative of $\Pi_\psi(RB_\psi(\psi | x) \leq RB_\psi(\psi_0 | x) | x)$ at $\psi_0$ in the direction $Q$ is

$$\frac{m_Q(x)}{m(x)} \left\{ \frac{Q(RB_\psi(\psi | x) \leq RB_\psi(\psi_0 | x) | x) - \Pi_\psi(RB_\psi(\psi | x) \leq RB_\psi(\psi_0 | x) | x)}{\Pi_\psi(RB_\psi(\psi_0 | x) \leq RB_\psi(\psi_0 | x) | x)} \right\} .$$

**Proof.** The strength based on $\Pi_\psi$ satisfies $\Pi_\psi(RB_\psi(\psi | x) \leq RB_\psi(\psi_0 | x) | x)$

$$= (1 - \epsilon_x) \Pi_\psi(RB_\psi(\psi | x) \leq RB_\psi(\psi_0 | x) | x) + \epsilon_x Q(RB_\psi(\psi | x) \leq RB_\psi(\psi_0 | x) | x).$$

So, using (6),

$$\Pi_\psi(RB_\psi(\psi | x) \leq RB_\psi(\psi_0 | x) | x) = \Pi_\psi(m(x | \psi) \leq m(x | \psi_0) | x) + \epsilon_x \{ Q(m(x | \psi) \leq m(x | \psi_0) | x) - \Pi_\psi(m(x | \psi) \leq m(x | \psi_0) | x) \} .$$

This implies that

$$\lim_{\epsilon \to 0} \frac{\Pi_\psi(RB_\psi(\psi_0 | x) \leq RB_\psi(\psi_0 | x) | x) - \Pi_\psi(RB_\psi(\psi | x) \leq RB_\psi(\psi_0 | x) | x)}{\epsilon}$$

$$= \frac{m_Q(x)}{m(x)} \left\{ Q(RB_\psi(\psi | x) \leq RB_\psi(\psi_0 | x) | x) - \Pi_\psi(m(x | \psi) \leq m(x | \psi_0) | x) \right\} .$$

So the strength is robust to choice of the marginal prior $\pi_\psi$ whenever $m_Q(x)/m(x)$ is small.

For both the measure of evidence $RB_\psi(\psi_0 | x)$ and its strength, the ratio $m_Q(x)/m(x)$ plays a key role in determining the robustness. The implications
Consider now priors for \( p \)-value in Evans and Zou (2001). This leads to the following result.

**Proof.** Since \( \pi \) is \( \epsilon \), we see that for small \( \epsilon \),

\[
\lim_{\epsilon \to 0} \frac{\pi_{\epsilon, \Psi}(\psi_0 \mid x) - \pi_{\Psi}(\psi_0 \mid x)}{\pi_{\Psi}(\psi_0 \mid x)} = \frac{m_Q(x)}{m(x)} \left( 1 - \frac{q(\psi_0 \mid x) - \pi_{\Psi}(\psi_0 \mid x)}{\pi_{\Psi}(\psi_0 \mid x)} \right) \epsilon
\]

and the relative change in \( \pi_{\Psi}(\psi_0 \mid x) \) is dependent on the ratio of the posteriors as well as \( m_Q(x) \). So if \( \pi_{\Psi}(\psi_0 \mid x) \) is small relative to \( q(\psi_0 \mid x) \) we will get a big relative change and this suggests that MAP inferences are much less robust than relative belief inferences. A similar result is obtained for the Bayesian \( p \)-value in Evans and Zou (2001).

### Proposition 5

The Gâteaux derivative of the posterior density of \( \psi \) in the direction \( Q \) at \( \psi_0 \) is given by \( \{m_Q(x)/m(x)\} \{q(\psi_0 \mid x) - \pi_{\Psi}(\psi_0 \mid x)\} \).

**Proof.** Since \( \pi_{\epsilon, \Psi}(\psi \mid x) = (1 - \epsilon)\pi_{\Psi}(\psi \mid x) + \epsilon q(\psi \mid x) \) it follows that

\[
\lim_{\epsilon \to 0} \frac{\pi_{\epsilon, \Psi}(\psi_0 \mid x) - \pi_{\Psi}(\psi_0 \mid x)}{\epsilon} = \frac{m_Q(x)}{m(x)} \left( 1 - \frac{q(\psi_0 \mid x) - \pi_{\Psi}(\psi_0 \mid x)}{\pi_{\Psi}(\psi_0 \mid x)} \right).
\]

Note that MAP-based inferences implicitly use \( \pi_{\Psi}(\psi_0 \mid x) \) as a measure of the evidence that \( \psi_0 \) is the true value. Comparing this with the relative belief ratio we see that for small \( \epsilon \),

\[
\frac{\pi_{\epsilon, \Psi}(\psi_0 \mid x) - \pi_{\Psi}(\psi_0 \mid x)}{\pi_{\Psi}(\psi_0 \mid x)} \approx \frac{m_Q(x)}{m(x)} \left( 1 - \frac{q(\psi_0 \mid x) - \pi_{\Psi}(\psi_0 \mid x)}{\pi_{\Psi}(\psi_0 \mid x)} \right) \epsilon
\]

of this are discussed in Section 5. Note that \( \sup_Q m_Q(x)/m(x) = RB(\psi(x) \mid x) \) gives the worst case behavior of this ratio.

It is of interest to contrast these results with those for the commonly used MAP inferences which are based on the posterior density \( \pi_{\Psi}(\cdot \mid x) \).

**Proposition 6**

The Gâteaux derivative of \( RB_{\Psi}(\cdot \mid x) \) at \( \psi_0 \) in the direction \( Q \) is \( \{m_Q(x)/m(x)\} \{RB_{Q, \Psi}(\psi_0 \mid x) - RB_{\Psi}(\psi_0 \mid x)\} \).

**Proof.** Clearly,

\[
\lim_{\epsilon \to 0} \frac{RB_{\epsilon, \Psi}(\psi_0 \mid x) - RB_{\Psi}(\psi_0 \mid x)}{\epsilon} = \frac{m_Q(x)}{m(x)} \left( RB_{Q, \Psi}(\psi_0 \mid x) - RB_{\Psi}(\psi_0 \mid x) \right).
\]

The implications of this result for robustness are discussed in Section 5.

Now consider the robustness of the strength of the evidence.
Proposition 7 If $RB_\psi(\cdot \mid x)$ has a discrete distribution with support containing no limit points, the Gâteaux derivative of $\Pi_\psi(RB_\psi(\psi \mid x) \leq RB_\psi(\psi_0 \mid x) \mid x)$ at $\psi_0$ in the direction $Q$ equals $0$. When $RB_\psi(\cdot \mid x)$ has a continuous distribution under $\Pi_\psi(\cdot \mid x)$ with density $g(\cdot \mid x)$, the Gâteaux derivative of $\Pi_\psi(RB_\psi(\psi \mid x) \leq RB_\psi(\psi_0 \mid x) \mid x)$ at $\psi_0$ in the direction $Q$ equals

$$\{m_Q(x)/m(x)\}RB_{Q,\psi}(\psi_0 \mid x)g(RB_\psi(\psi_0 \mid x) \mid x).$$

Proof. Since

$$\Pi_\psi(RB_{\epsilon,\psi}(\psi \mid x) \leq RB_{\epsilon,\psi}(\psi_0 \mid x) \mid x)$$

$$= \Pi_\psi \left( (1 - \epsilon_x)RB_\psi(\psi \mid x) + \epsilon_x RB_{Q,\psi}(\psi \mid x) \leq (1 - \epsilon_x)RB_\psi(\psi_0 \mid x) + \epsilon_x RB_{Q,\psi}(\psi_0 \mid x) \mid x \right),$$

then, for all $\epsilon > 0$ such that $\epsilon_x \leq 1$,

$$\Pi_\psi(RB_{\epsilon,\psi}(\psi \mid x) \leq RB_{\epsilon,\psi}(\psi_0 \mid x) \mid x)$$

$$\leq \Pi_\psi \left( RB_\psi(\psi \mid x) \leq RB_\psi(\psi_0 \mid x) + \frac{\epsilon_x}{1 - \epsilon_x} RB_{Q,\psi}(\psi_0 \mid x) \mid x \right)$$

and for all $\epsilon < 0$,

$$\Pi_\psi(RB_{\epsilon,\psi}(\psi \mid x) \leq RB_{\epsilon,\psi}(\psi_0 \mid x) \mid x)$$

$$\geq \Pi_\psi \left( RB_\psi(\psi \mid x) \leq RB_\psi(\psi_0 \mid x) + \frac{\epsilon_x}{1 - \epsilon_x} RB_{Q,\psi}(\psi_0 \mid x) \mid x \right).$$

When $RB_\psi(\cdot \mid x)$ has a discrete distribution with support containing no limit points, then the lower and upper bounds equal $\Pi_\psi(RB_\psi(\psi \mid x) \leq RB_\psi(\psi_0 \mid x) \mid x)$ for all $\epsilon$ small enough and the result follows. When $RB_\psi(\cdot \mid x)$ has a continuous distribution with density $g(\cdot \mid x)$, then

$$\lim_{\epsilon \to 0} \frac{\Pi_\psi(RB_{\epsilon,\psi}(\psi_0 \mid x) \leq RB_{\epsilon,\psi}(\psi_0 \mid x) \mid x) - \Pi_\psi(RB_\psi(\psi \mid x) \leq RB_\psi(\psi_0 \mid x) \mid x)}{\epsilon}$$

$$= \{m_Q(x)/m(x)\}RB_{Q,\psi}(\psi_0 \mid x)g(RB_\psi(\psi_0 \mid x) \mid x).$$

From this it is seen that in the discrete case the strength is insensitive to local changes in the prior.

Consider the continuous case. When there is strong evidence either for or against $\psi_0$, then $RB_\psi(\psi_0 \mid x)$ will be in the right or left tail correspondingly of the posterior distribution of $RB_\psi(\cdot \mid x)$ and so $g(RB_\psi(\psi_0 \mid x) \mid x)$ will tend to be small. As such the strength will be robust to small changes in the prior provided $m_Q(x)/m(x)$ is not large. When there is not strong evidence however, then $g(RB_\psi(\psi_0 \mid x) \mid x)$ could be large and, if $m_Q(x)/m(x)$ is not small, then the strength is not robust. This underscores a recommendation in Baskurt and Evans (2013) that in the continuous case the parameter be discretized when assessing the evidence and its strength. For this, when $\psi$ is real-valued, let
\[ \delta > 0 \] be the difference between two \( \psi \) values that is deemed to be of practical importance. The prior and posterior distributions of \( \psi \) discretized to the intervals \( [\psi_0 + (2i - 1)\delta/2, \psi_0 + (2i + 1)\delta/2] \) for \( i \in \mathbb{Z} \) are then used to assess the hypothesis corresponds to the interval \( [\psi_0 - \delta/2, \psi_0 + \delta/2] \). By Proposition 7 the strength is then insensitive to small changes in the prior.

It is perhaps not surprising that the robustness behavior of the relative belief ratio and its strength is more complicated when considering the effect of the conditional prior than with the marginal prior. The optimality results concerning robustness to the marginal prior underscore this.

5 Robustness and Prior-Data Conflict

The existence of a prior-data conflict means that the data support certain values of \( \psi = \Psi(\theta) \) being the true value but the prior places little or no mass there. While various measures can be used to determine whether or not such a conflict has occurred, a logical approach is based on the factorization of the joint probability measure for \( (\theta, x) \) given by \( \Pi \times P_\theta = \Pi(\cdot \mid T) \times M_T \times P(\cdot \mid T) \), where \( T \) is a minimal sufficient statistic, \( \Pi(\cdot \mid T) \) is the posterior probability measure for \( \theta \), \( M_T \) is the prior predictive probability measure of \( T \) and \( P(\cdot \mid T) \) is the conditional probability measure of the data given \( T \). The measure \( P(\cdot \mid T) \) is then available for computing probabilities relevant to checking the model \( \{ f_\theta : \theta \in \Theta \} \), the measure \( M_T \) is available for computing probabilities relevant to checking the prior and \( \Pi(\cdot \mid T) \) is the relevant probability measure for computing probabilities for \( \theta \). A statistical analysis then proceeds by checking the model, perhaps via a tail probability based on a discrepancy statistic, and then proceeding to check the prior if the data does not contradict the model. If both the model and prior are not contradicted by the data, then we can proceed to inference about \( \theta \). The logic behind this sequence lies in part with the fact that it makes no sense to check a prior if the model fails. Furthermore, separating the check of the prior from that of the model provides more information in the event of a conflict arising, as it is then possible to identify where the failure lies, namely, with the model or with the prior.

In Evans and Moshonov (2006) this factorization was adhered to and the tail probability

\[ M_T(m_T(t) \leq m_T(T(x))) \] (8)

was advocated for checking the prior where \( m_T \) is the density of \( M_T \) with respect to some support measure. So if \( \mathcal{S} \) is small, then the observed value \( T(x) \) of the minimal sufficient statistic lies in the tails of \( M_T \) and there is an indication of a prior-data conflict. In Evans and Jang (2011a) the validity of this approach was firmly established by the proof that \( \mathcal{S} \) converges to \( \Pi(\pi(\theta) \leq \pi(\theta_{true})) \) under i.i.d. sampling and some additional weak conditions. Furthermore, it was shown how to modify \( \mathcal{S} \) so as to achieve invariance under choice of the minimal sufficient statistic. Also, Evans and Moshonov (2006) argued that \( \mathcal{S} \) should be replaced by \( M_T(m_T(t) \leq m_T(T(x)) \mid U(T(x))) \) for any maximal ancillary \( U(T) \) as the variation in \( T \) due to to \( U(T) \) has nothing to do with \( \theta \) and so reflects
nothing about the prior. The tail probability is a check on the full prior and Evans and Moshonov (2006) also developed methods for checking factors of the prior so a failure in the prior could be isolated to a particular aspect.

First, however, consider the case when $\Psi(\theta) = \theta$ and interest is in the robustness of inferences to the whole prior. From the results in Section 11 it is seen that the ratio $m_Q(x)/m(x) = m_{Q,T}(T(x))/m_T(T(x))$, where $m_{Q,T}(T(x)) = \int_{\Theta} f_{\theta,T}(T(x)) Q(d\theta)$, plays a key role in determining the local sensitivity in the direction given by $Q$, of the inferences for given observed data $x$. This depends on $Q$ and the worst case is given by

$$
\sup_{Q} m_{Q,T}(T(x))/m_T(T(x)) = \sup_{Q} \frac{\int_{\Theta} f_{\theta,T}(T(x)) Q(d\theta)}{m_T(T(x))} = RB(\theta(x) \mid x) \tag{9}
$$

and note that $\theta(x)$ is the MLE in this case as well as the relative belief estimate. Notice that when $S$ is small, so there is an indication of a prior-data conflict existing, then $m_T(T(x))$ is relatively small when compared to other values of $m_T(t)$ which are not influenced by the data. This implies that the prior is having a big influence relative to the data and so a lack of robustness can be expected.

This phenomenon is well-illustrated in the following examples where ancillaries play no role because of Basu’s theorem.

**Example 1. Location normal model.**

Suppose that $x = (x_1, \ldots, x_n)$ is a sample from the $N(\mu, 1)$ distribution with $\mu \sim N(\mu_0, \sigma_0^2)$. Then $M_T$ is given by $T(x) = \bar{x} \sim N(\mu_0, 1/n + \sigma_0^2)$. When $Q$ is the $N(\mu_1, \sigma_1^2)$ distribution, then $M_{Q,T}$ is given by $\bar{x} \sim N(\mu_1, 1/n + \sigma_1^2)$. This implies that

$$
m_{Q,T}(T(x))/m_T(T(x)) = \sqrt{\frac{1/n + \sigma_0^2}{1/n + \sigma_1^2}} \exp \left\{ -\frac{1}{2} \left[ (1/n + \sigma_0^2)^{-1} (\bar{x} - \mu_0)^2 - (1/n + \sigma_1^2)^{-1} (\bar{x} - \mu_1)^2 \right] \right\} \tag{10}
$$

and, as a function of $(\mu_1, \sigma_1^2)$ this is maximized when $\mu_1 = \bar{x}, \sigma_1^2 = 0$. Notice that this supremum converges to $\infty$ as $\bar{x} \to \pm \infty$ and such values correspond to prior-data conflict with respect to the $N(\mu_0, \sigma_0^2)$ prior.

Now consider a numerical example. A sample of size $n = 20$ was generated from the $N(0, 1)$ distribution obtaining $\bar{x} = 0.2591$. When the base prior is $N(0.5, 1)$ then $S$ equals 0.8141 and accordingly there is no indication of any prior-data conflict. Also, $\sup_Q (m_Q(x)/m(x)) = 4.7109$ which seems modest as it describes the worst case robustness behavior. In Table 1 some values of $m_Q(x)/m(x)$ are recorded when $Q$ is a $N(\mu_1, \sigma_1^2)$ distribution for various values of $\mu_1$ and $\sigma_1^2$ as these might be expected to be realistic directions in which to perturb the base prior. In all cases the value of $m_Q(x)/m(x)$ is quite modest and the maximum value of $S$ is 1.0534. Overall it can be concluded here that the analysis is robust to local perturbations of the prior.

Now consider an example where there is prior-data conflict. In this case a sample of $n = 20$ is generated from a $N(4, 1)$ distribution obtaining $\bar{x} = 4.0867$ and the same base prior is used. The value of $S$ is 0.0005 and so there is a strong indication of prior-data conflict. Furthermore, $\sup_Q (m_Q(x)/m(x)) =$
Table 1: The ratio $m_Q(x)/m(x)$ in Example 1 when there is no conflict.

| $\mu_1$ | $\sigma_T^2$ | $m_Q(x)/m(x)$ | $\mu_1$ | $\sigma_T^2$ | $m_Q(x)/m(x)$ |
|---------|--------------|----------------|---------|--------------|----------------|
| -3.0    | 1            | 0.0065         | 0.5     | 0.5          | 1.3474         |
| -2.0    | 1            | 0.0905         | 0.5     | 1.0          | 1.0000         |
| -1.0    | 1            | 0.4832         | 0.5     | 2.0          | 0.7254         |
| 1.0     | 1            | 0.7917         | 0.5     | 3.0          | 0.5975         |
| 2.0     | 1            | 0.2428         | 0.5     | 50.0         | 0.1488         |
| 3.0     | 1            | 0.0287         | 0.5     | 100.0        | 0.1053         |

Table 2: The ratio $m_Q(x)/m(x)$ in Example 1 when there is conflict.

| $\mu_1$ | $\sigma_T^2$ | $m_Q(x)/m(x)$ | $\mu_1$ | $\sigma_T^2$ | $m_Q(x)/m(x)$ |
|---------|--------------|----------------|---------|--------------|----------------|
| -3.0    | 1            | $1.88 \times 10^{-8}$ | 0.5     | 0.5          | 0.0053         |
| -2.0    | 1            | $9.97 \times 10^{-8}$ | 0.5     | 1.0          | 1.0000         |
| -1.0    | 1            | $2.00 \times 10^{-9}$ | 0.5     | 2.0          | 14.2070        |
| 1.0     | 1            | $4.90 \times 10^{-1}$ | 0.5     | 3.0          | 32.5842        |
| 2.0     | 1            | $5.75 \times 10^{1}$  | 0.5     | 50.0         | 58.2823        |
| 3.0     | 1            | $2.61 \times 10^{2}$  | 0.5     | 100.0        | 43.9565        |

2096.85 which certainly indicates a lack of robustness. In Table 2 some values of $m_Q(x)/m(x)$ are recorded when $Q$ is a $N(\mu_1, \sigma_T^2)$ distribution for various values of $\mu_1$ and $\sigma_T^2$. It is seen that the value of $m_Q(x)/m(x)$ can be relatively large and the maximum value of $Q$ is 468.86. So it can be concluded that the analysis based on the model, prior and observed data, will not be robust to local perturbations of the prior when there is prior-data conflict. □

**Example 2. Bernoulli model.**

Suppose that $x = (x_1, \ldots, x_n)$ is a sample from a Bernoulli($\theta$) and the prior is $\theta \sim \text{beta}(\alpha_0, \beta_0)$ for some choice of $(\alpha_0, \beta_0)$. A minimal sufficient statistic is $T(x) = \sum_{i=1}^{n} x_i \sim \text{Binomial}(n, \theta)$ and then

$$m_T(t) = \binom{n}{t} \frac{\Gamma(\alpha_0 + \beta_0) \Gamma(t + \alpha_0) \Gamma(n-t + \beta_0)}{\Gamma(\alpha_0) \Gamma(\beta_0) \Gamma(n + \alpha_0 + \beta_0)}.$$

Also,

$$\sup_Q (m_Q(x)/m(x)) = \frac{\Gamma(\alpha_0) \Gamma(\beta_0)}{\Gamma(\alpha_0 + \beta_0)} \frac{\Gamma(n + \alpha_0 + \beta_0)}{\Gamma(t + \alpha_0) \Gamma(n-t + \beta_0)} \bar{x}^{n\beta_0} (1 - \bar{x})^{n(1-\alpha_0)}.$$

To illustrate the relationship between prior-data conflict and robustness, consider a numerical example. Suppose that $\alpha_0 = 5$ and $\beta_0 = 20$. Generating a sample of size $n = 20$ from the Bernoulli(0.25) gave the value $n\bar{x} = 3$. In this case equals 0.7100 and there is no indication of any prior-data conflict. Also, $\sup_Q (m_Q(x)/m(x)) = 1.4211$ which indicates that the inferences will be generally robust to small deviations. If $m_Q(x)/m(x)$ is computed for various $Q$, then...
was due to where they were to be used before (11), then it would not be possible to assess if a failure quite reasonable in value as indeed it is bounded above by 1.4211.

A sample of $n = 20$ was also generated from a Bernoulli(0.9) with the same prior being used. In this case $n\bar{x} = 17$ and (3) equals $6.2 \times 10^{-6}$, so there is a strong indication of prior-data conflict. Also, $\sup_Q \{m_Q(x)/m(x)\} = 46396.43$ which indicates that the inferences will be generally not be robust to small deviations. Table 3 provides some values of $m_Q(x)/m(x)$ for $Q$ given by a beta$(\alpha_1, \beta_1)$ for various choices of $(\alpha_1, \beta_1)$ and there are several large values. ■

Now consider the case when $\theta = (\theta_1, \theta_2) \in \Theta_1 \times \Theta_2$ so the prior factors as $\pi(\theta) = \pi_2(\theta_2 | \theta_1) \pi_1(\theta_1)$. Presumably the conditional prior $\pi_2(\cdot | \theta_1)$ and the marginal prior $\pi_1$ are elicited and the goal is inference about some $\psi = \Psi(\theta)$. It is then preferable to check the prior by checking each individual component for prior-data conflict as this leads to more information about where a conflict exists when it does.

In general, it is not clear how to check the individual components but in certain contexts a particular structure holds that allows for this. Suppose that all ancillaries are independent of the minimal sufficient statistic and so can be ignored. The more general situation is covered in Evans and Moshonov (2006).

As discussed in Evans and Moshonov (2006), suppose there is a statistic $V(T)$ such that the marginal distribution of $V(T)$ is dependent only on $\theta_1$. Such a statistic is referred to as being ancillary for $\theta_2$ given $\theta_1$. Naturally we want $V(T)$ to be a maximal ancillary for $\theta_2$ given $\theta_1$. An appropriate tail probability for checking $\pi_1$ is then given by

$$M_{V(T)}(m_{V(T)}(v) \leq m_{V(T)}(V(T(x)))),$$

as $M_{V(T)}$ does not depend on $\pi_2(\cdot | \theta_1)$. A natural order is to check $\pi_1$ first and then check $\pi_2(\cdot | \theta_1)$ for prior-data conflict, whenever no prior-data conflict is found for $\pi_1$. The appropriate tail probability for checking $\pi_2(\cdot | \theta_1)$ is given by

$$M_T(m_T(t | V(T(x))) \leq m_T(T(x) | V(T(x))) | V(T(x))).$$

Note that this is assessing whether or not $\pi_2(\cdot | \theta_1)$ is a suitable prior for $\theta_2$ among those $\theta_1$ values deemed to be suitable according to the prior $\pi_1$. If (12) were to be used before (11), then it would not be possible to assess if a failure was due to where $\pi_1$ was placing the bulk of its mass or was caused by where

| $\alpha_1$ | $\beta_1$ | $m_Q(x)/m(x)$ | $\alpha_1$ | $\beta_1$ | $m_Q(x)/m(x)$ |
|-----------|-----------|----------------|-----------|-----------|----------------|
| 20        | 5         | 32647.89       | 5         | 1         | 21523.28       |
| 15        | 5         | 25729.50       | 5         | 25        | 0.12           |
| 10        | 5         | 15010.95       | 5         | 22        | 0.41           |
| 5         | 5         | 3996.37        | 5         | 20        | 1.00           |
| 1         | 5         | 125.87         | 5         | 16        | 6.77           |

Table 3: The ratio $m_Q(x)/m(x)$ in Example 2 when there is conflict.

where $Q$ is a beta$(\alpha_1, \beta_1)$, then in all cases it is readily seen that this ratio is quite reasonable in value as indeed it is bounded above by 1.4211.
the conditional priors were placing their mass. Notice that

\[
\frac{m_{Q,T}(T(x))}{m_T(T(x))} = \frac{m_{Q,T}(T(x) | V(T(x)))}{m_T(T(x) | V(T(x)))} \frac{m_{Q,V(T)}(V(T(x)))}{m_{V(T)}(V(T(x)))},
\]

so prior-data conflict with either \(\pi_1\) or \(\pi_2(\cdot | \theta_1)\) could lead to large values of the ratio on the left for certain choices of \(Q\). When only the conditional prior of \(\theta_2\) given \(\theta_1\) is perturbed, then \(m_{Q,V(T)}(V(T(x))) = m_{V(T)}(V(T(x)))\).

Letting \(f_{\theta_1,V}\) denote the density of \(V\), then

\[
\frac{m_{Q,V(T)}(V(T(x)))}{m_{V(T)}(V(T(x)))} = \int_{\theta_1} \frac{f_{\theta_1,V}(V(T(x)))}{f_{\theta_1,V}(V(T(x)))} Q_1(d\theta_1) \leq RB_1(\theta_1(V(T(x))) | V(T(x)))
\]

where \(RB_1(\cdot | V(T(x)))\) gives the relative belief ratios for \(\theta_1\) based on having observed \(V(T(x))\). The right-hand side gives the worst-case behavior of the second factor in (13).

Now consider the robustness of relative belief inferences for a general \(\psi = \Psi(\theta)\). The following result generalizes Propositions 3 and 6 as we consider a general perturbation to the prior, namely, \(\Pi_l = (1-\epsilon)\Pi + \epsilon Q\) and the proof is the same as that of Proposition 6.

**Proposition 8** The Gâteaux derivative of \(RB_\Psi(\cdot | x)\) at \(\psi\) in the direction \(Q\) is \(\{m_Q(x)/m(x)\}(RB_{Q,\Psi}(\psi | x) - RB_\Psi(\psi | x))\).

The factor \(RB_{Q,\Psi}(\psi | x) - RB_\Psi(\psi | x)\) can be big simply because we choose a prior \(Q\) that is very different than \(\Pi\). For example, \(RB_\Psi(\psi | x)\) may be big (small) because there is considerable evidence in favor of (against) \(\psi\) being the true value and we can choose a prior \(Q\) that doesn’t (does) place mass near \(\psi\). As such, it makes sense to standardize the derivative by dividing by this factor and this leaves the robustness determined again by \(m_Q(x)/m(x)\).

Suppose now that \(Q\) and \(\Pi\) have the same marginal for \(\zeta = \Xi(\theta)\). Then,

\[
m_Q(x) = m(x) \int_Z \int_{\mathbb{R}^{n(x)}} RB(\theta | x) Q(d\theta | \psi) \Pi_{\Xi}(dk) \leq m(x) \int_Z RB(\theta_\zeta(x) | x) \Pi_{\Xi}(dk)\]

where \(\theta_\zeta(x) = \arg \sup \{RB(\theta | x) : \Xi(\theta) = \zeta\}\). Therefore,

\[
\frac{m_Q(x)}{m(x)} \leq \int_Z RB(\theta_\zeta(x), x) \Pi_{\Xi}(dk)
\]

and the right-hand side gives the worst-case behavior of the first factor in (13) when \(\Xi(\theta) = \theta_1\) which is related to prior-data conflict with the prior on \(\theta_2\).

The following is a standard example where priors are specified hierarchically.

**Example 3** Location-scale normal model.

Suppose that \(x = (x_1, \ldots, x_n)\) is a sample from the \(N(\mu, \sigma^2)\) distribution with \(\mu | \sigma^2 \sim N(\mu_0, \sigma^2)\), \(\sigma^2 \sim \text{gamma}(\alpha_0, \beta_0)\). Then \(T(x) = (\bar{x}, ||x - \bar{x}1||^2)\) is a minimal sufficient statistic for the model. Note that the prior is chosen...
by eliciting values for $\mu_0, \sigma_0^2, \alpha_0, \beta_0$ and so there is interest in how sensitive inferences are to perturbations in each component separately. The posterior distribution of $(\mu, \sigma^2)$ is given by $\mu \mid \sigma^2, T(x) \sim N(\mu_x, (n + 1/\tau^2_0)^{-1} \sigma^2)$, $\sigma^{-2} \mid T(x) \sim$ gamma$_{rate} \{\alpha_0 + n/2, \beta(\bar{x}, s^2)\}$ where $\mu_x = (n+1/\tau^2_0)^{-1}(n\bar{x} + \mu_0/\tau^2_0)$ and $\beta(\bar{x}, s^2)$

Consider first inferences for $\psi = \Psi(\theta) = \sigma^2$ and note that $V(T(x)) = ||x - \bar{x}||^2$ is ancillary given $\psi$ and its distribution depends on $\psi$. Therefore, the prior on $\sigma^2$ is checked first using the prior predictive for $V(T(x))$. An easy calculation gives that the prior of $\sigma^2 = V(T(x))/(n-1)$ is $(\beta_0/\alpha_0)F(n-1, 2\alpha_0)$ and this specifies $\{1\}$. While the results of Section 4.1 apply here, consider the behavior of the relative belief ratio $RB_1(\sigma^2 \mid V(T(x)))$ which is based on only observing $V(T(x))$ rather than $T(x)$. By Proposition 3 this has Gâteaux derivative depending on $m_{Q \mid V(T)}(V(T(x)))m_{V(T)}(V(T(x)))$. Notice, however, that relative belief ratios accumulate evidence in a simple way. For any statistic $V(T(x))$, then \[RB_\psi(\psi \mid T(x)) = \frac{\pi_\psi(\psi \mid T(x))}{\pi_\psi(\psi)} = \frac{\pi_\psi(\psi \mid V(T(x)))}{\pi_\psi(\psi)} \frac{\pi_\psi(\psi \mid T(x))}{\pi_\psi(\psi)} \]

where the first factor gives the evidence obtained after observing $V(T(x))$ and the second factor gives the evidence obtained after observing $T(x)$ having already observed $V(T(x))$. So $RB_1(\sigma^2 \mid T(x)) = R(y_1(\sigma^2 \mid V(T(x))))[RB_1(\sigma^2 \mid T(x)) / \sigma^2 \mid V(T(x))]$ with the same interpretation for the factors. As such, a lack of robustness of $RB_1(\sigma^2 \mid V(T(x)))$, which can be connected to prior-data conflict through $\{1\}$, implies a lack of robustness for $RB_1(\sigma^2 \mid x)$.

When no prior-data conflict is obtained for the prior on $\sigma^2$, then it makes sense to look for prior-data conflict with the prior on $\mu$ which is typically the parameter of primary interest. Now consider perturbations to the prior on $\mu$ and the relationship to prior-data conflict with this prior. The conditional distribution of $T(x)$ given $V(T(x))$ is given by the conditional prior predictive of $x$ given $s^2$ which is distributed as $\mu_0 + \hat{\sigma} \sigma_{n+2\sigma_0-1}$ where $\hat{\sigma}^2 = (n\tau^2_0 + 1) / (n\sigma^2_0 + (n-1)s^2 + 1)$ / $n\tau^2_0(n+2\sigma_0-1)$ specifying $\{1\}$. Furthermore, for $\{1\}$, $\Xi(\theta) = \sigma^2$ and $\theta_{\sigma^2}(x) = (\bar{x}, \sigma^2)$ with \[RB((\bar{x}, \sigma^2) \mid x) = \frac{(n\tau^2_0 + 1)^{1/2}}{\beta_0^{\alpha_0}} \frac{1}{\Gamma(\alpha_0 + n/2)} (\beta(\bar{x}, s^2))^{\alpha_0+\frac{1}{2}} \exp \left\{-\frac{(n-1)s^2}{2\sigma^2} \right\} \]

and so \[\int_0^\infty RB((\bar{x}, \sigma^2) \mid x) \Pi_1(da^{-2}) = (n\tau^2_0 + 1)^{1/2} \frac{\beta(\bar{x}, s^2)}{\beta_0 + (n-1)s^2/2} \alpha_0+\frac{1}{2}. \]

Now consider a number of numerical examples where the base prior is always specified by $\mu_0 = 0, \tau^2_0 = 1, \alpha_0 = 5$ and $\beta_0 = 5$. The behavior of the two factors in $\{1\}$ is examined when there is no prior-data conflict and when there is.
Table 4: The ratio \(m_{Q,V(T)}(V(T(x)))/m_{V(T)}(V(T(x)))\) in Example 3 when there is no conflict with the prior on \(\sigma^2\).

| \(\alpha_1\) | \(\beta_1\) | \(\frac{m_{Q,V(T)}(V(T(x)))}{m_{V(T)}(V(T(x)))}\) | \(\alpha_1\) | \(\beta_1\) | \(\frac{m_{Q,V(T)}(V(T(x)))}{m_{V(T)}(V(T(x)))}\) |
|---|---|---|---|---|---|
| 5  | 1  | 0.05 | 1  | 5  | 0.07 |
| 5  | 2  | 0.38 | 2  | 5  | 0.25 |
| 5  | 4  | 0.99 | 4  | 5  | 0.81 |
| 5  | 10 | 0.34 | 10 | 5  | 0.53 |

Table 5: The ratio \(m_{Q,V(T)}(V(T(x)))/m_{V(T)}(V(T(x)))\) in Example 3 when there is conflict with the prior on \(\sigma^2\).

| \(\alpha_1\) | \(\beta_1\) | \(\frac{m_{Q,V(T)}(V(T(x)))}{m_{V(T)}(V(T(x)))}\) | \(\alpha_1\) | \(\beta_1\) | \(\frac{m_{Q,V(T)}(V(T(x)))}{m_{V(T)}(V(T(x)))}\) |
|---|---|---|---|---|---|
| 5  | 1  | 0.00 | 1  | 5  | 5517.42 |
| 5  | 2  | 0.01 | 2  | 5  | 1245.26 |
| 5  | 4  | 2.34 | 4  | 5  | 13.78 |
| 5  | 10 | 23.51 | 10 | 5  | 0.00 |

A sample of size \(n = 20\) was generated from the \(N(0, 1)\) distribution obtaining \(\bar{x} = -0.1066, s^2 = 0.9087\). So there should be no prior-data conflict with the prior on \(\sigma^2\). Indeed, \(RB_1\) equals 0.7626 so there is no indication of any problems with the prior on \(\sigma^2\). Values of \(m_{Q,V(T)}(V(T(x)))/m_{V(T)}(V(T(x)))\) are recorded in Table 4 when the marginal prior on \(\sigma^2\) is perturbed by a gamma rate \((\alpha_1, \beta_1)\) distribution for various values of \(\alpha_1\) and \(\beta_1\). In all cases, the ratio is small and indicates robustness to local perturbations of the prior on \(\sigma^2\). Note that the worst case behavior, over all possible directions, is given by the maximized relative belief ratio for \(\sigma^2\) based on \(V(T(x))\) which occurs at \(\sigma^2 = s^2\) and equals

\[
RB_1(s^2 \mid V(T(x))) = \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_0 + (n - 1)/2)} \theta_0^{-\alpha_0} e^{-\frac{n-1}{2}(s^2) - \frac{(n-1)s^2}{2} + \alpha_0 + \beta_0}.
\]

In this case \(RB_1(s^2 \mid V(T(x))) = 1.7479\).

Next a sample of size \(n = 20\) from the \(N(0, 25)\) was generated obtaining \(\bar{x} = 0.0950, s^2 = 23.9503\). So there is clearly prior-data conflict with the prior on \(\sigma^2\). This is reflected in the value of \(RB_1\) which equals \(0.64 \times 10^{-5}\). Table 5 shows that there is a serious lack of robustness. The worst case behavior is given by \(RB_1(s^2 \mid V(T(x))) = 40484.68\).

It is also relevant to consider what happens concerning the robustness of inferences about \(\sigma^2\) when there is prior-data conflict with the prior on \(\mu\) but not with the prior on \(\sigma^2\). A sample of \(n = 20\) was generated from the \(N(10, 1)\) distribution obtaining \(\bar{x} = 9.7041, s^2 = 1.0082\), so there is clearly prior-data conflict with the prior on \(\mu\) but not with the prior on \(\sigma^2\). The value of \(RB_1\) equals 0.6406 which gives no reason to doubt the relevance of the prior on \(\sigma^2\).
Table 6: The ratio $m_{Q,V(T)}(V(T(x))) / m_{V(T)}(V(T(x)))$ in Example 3 when there is conflict with the prior on $\mu$ but not with the prior on $\sigma^2$.

| $\alpha_1$ | $\beta_1$ | $m_{Q,V(T)}(V(T(x))) / m_{V(T)}(V(T(x)))$ | $\alpha_1$ | $\beta_1$ | $m_{Q,V(T)}(V(T(x))) / m_{V(T)}(V(T(x)))$ |
|------------|------------|---------------------------------------------|------------|------------|---------------------------------------------|
| 5          | 1          | 0.03                                        | 1          | 5          | 0.09                                        |
| 5          | 2          | 0.29                                        | 2          | 5          | 0.31                                        |
| 5          | 4          | 0.92                                        | 4          | 5          | 0.86                                        |
| 5          | 10         | 0.44                                        | 10         | 5          | 0.38                                        |

Table 7: The ratio $m_{Q,T}(V(T(x))) / m_{T}(V(T(x)))$ in Example 3 when there is no conflict with the prior on $\sigma^2$ or with the prior on $\mu$.

| $\mu_1$ | $\tau_1^2$ | $m_{Q,T}(V(T(x))) / m_{T}(V(T(x)))$ | $\mu_1$ | $\tau_1^2$ | $m_{Q,T}(V(T(x))) / m_{T}(V(T(x)))$ |
|---------|------------|--------------------------------------|---------|------------|--------------------------------------|
| $-2$    | 1          | 0.17                                 | 0       | 2          | 0.51                                 |
| $-1$    | 1          | 0.66                                 | 0       | 3          | 0.34                                 |
| 1       | 1          | 0.54                                 | 0       | 4          | 0.26                                 |
| 2       | 1          | 0.12                                 | 0       | 5          | 0.21                                 |

Table 6 shows that $m_{Q,V(T)}(V(T(x))) / m_{V(T)}((T(x)))$ is small and indicates robustness to local perturbations of the prior on $\sigma^2$. The worst case behavior is given by $RB_1(s^2 \mid V(T(x))) = 1.7218$. This reinforces the claim that the tail probabilities (11) and (12) are measuring different aspects of the data conflicting with the prior.

Now consider perturbations to the prior on $\mu$ with the prior on $\sigma^2$ fixed. A sample of $n = 20$ was generated from a $N(0,1)$ obtaining $\bar{x} = -0.1066, s^2 = 0.9087$ so there is clearly no prior-data conflict with either component. This is reflected in the value of (12) which equals 0.9150. Table 7 shows that the first factor $m_{Q,T}(V(T(x))) / m_{T}(V(T(x)))$ in (12) is small when the conditional prior on $\mu$ is perturbed by $N(\mu_1, \tau_1^2)$ priors and thus demonstrates robustness to perturbations in these directions. The worst case behavior is given by $\int_0^\infty RB((\bar{x}, \sigma^2) \mid x) \Pi_1(d\sigma^{-2}) = 4.6099$ which is comparatively small.

Table 8 gives some values of $m_{Q,T}(V(T(x))) / m_{T}(V(T(x)))$ when a sample of $n = 20$ was generated from a $N(0,25)$, obtaining $\bar{x} = 0.0950, s^2 = 23.9593$. So in this case there is prior-data conflict with the prior on $\sigma^2$ but not with the prior on $\mu$. The value of (12) equals 0.9150 which gives no indication of prior-data conflict with the prior on $\mu$. The tabulated values also indicate no serious robustness concerns as does $\int_0^\infty RB((\bar{x}, \sigma^2) \mid x) \Pi_1(d\sigma^{-2}) = 4.5838$. This also reinforces the claim that the tail probabilities (11) and (12) are measuring different aspects of the data conflicting with the prior.

Table 9 gives some values of $m_{Q,T}(V(T(x))) / m_{T}(V(T(x)))$ when a sample of $n = 20$ was generated from a $N(10,1)$ obtaining $\bar{x} = 9.7941, s^2 = 1.0082$. So in this case there is prior-data conflict with the prior on $\mu$ but not with the prior on $\sigma^2$. The value of (12) equals $0.1691 \times 10^{-9}$ which gives a clear indication of prior-data conflict with the prior on $\mu$. In this case the tab-
\[\mu_1 \quad \tau_1 \quad \frac{m_{Q,T}(T(x) | V(T(x)))}{m_{T}(T(x) | V(T(x)))} \quad \mu_1 \quad \tau_2 \quad \frac{m_{Q,T}(T(x) | V(T(x)))}{m_{T}(T(x) | V(T(x)))}\]

| $\mu_1$ | $\tau_1$ | $\frac{m_{Q,T}(T(x) | V(T(x)))}{m_{T}(T(x) | V(T(x)))}$ | $\mu_1$ | $\tau_2$ | $\frac{m_{Q,T}(T(x) | V(T(x)))}{m_{T}(T(x) | V(T(x)))}$ |
|---|---|---|---|---|---|
| -2 | 1 | 0.87 | 0 | 2 | 0.51 |
| -1 | 1 | 0.96 | 0 | 3 | 0.34 |
| 1 | 1 | 0.98 | 0 | 4 | 0.26 |
| 2 | 1 | 0.90 | 0 | 5 | 0.21 |

Table 8: The ratio $m_{Q,T}(T(x) | V(T(x))) / m_{T}(T(x) | V(T(x)))$ in Example 3 when there is conflict with the prior on $\sigma^2$ but not with the prior on $\mu$.

\[\mu_1 \quad \tau_1 \quad \frac{m_{Q,T}(T(x) | V(T(x)))}{m_{T}(T(x) | V(T(x)))} \quad \mu_1 \quad \tau_2 \quad \frac{m_{Q,T}(T(x) | V(T(x)))}{m_{T}(T(x) | V(T(x)))}\]

| $\mu_1$ | $\tau_1$ | $\frac{m_{Q,T}(T(x) | V(T(x)))}{m_{T}(T(x) | V(T(x)))}$ | $\mu_1$ | $\tau_2$ | $\frac{m_{Q,T}(T(x) | V(T(x)))}{m_{T}(T(x) | V(T(x)))}$ |
|---|---|---|---|---|---|
| -2 | 1 | 0.01 | 0 | 2 | 117.584 |
| -1 | 1 | 0.10 | 0 | 3 | 5,611,980 |
| 1 | 1 | 10.83 | 0 | 4 | 26,012,609 |
| 2 | 1 | 132.09 | 0 | 5 | 55,478,630 |

Table 9: The ratio $m_{Q,T}(T(x) | V(T(x))) / m_{T}(T(x) | V(T(x)))$ in Example 3 when there is no conflict with the prior on $\sigma^2$ but there is with the prior on $\mu$.

ulated values indicate a clear lack of robustness with respect to the prior on $\mu$. Also, $\int_0^\infty RB(\bar{x}, \sigma^2 | x) \Pi_1(d\sigma^{-2}) = 8.046,933,962$ indicates that the worst case behavior with respect to robustness is terrible.

### 6 Conclusions

Several optimal robustness results have been derived here for relative belief inferences. These and other results suggest a natural preference for these inferences over other Bayesian inferences for estimation and hypothesis assessment. Even though relative belief inferences may be the most robust to choice of prior, this does not guarantee that they are robust in practice. The issue of practical robustness in a given problem is seen to be connected with whether or not there is prior-data conflict. With no prior-data conflict the inferences are robust to small changes in the prior, at least in the sense measured here. This adds support to the point-of-view that checking for prior-data conflict is an essential aspect of good statistical practice.

It is interesting that the worst case behavior of the measure of sensitivity is associated with the maximized value of a relative belief ratio. The actual maximum value attained is meaningless, however, as there is no way to calibrate this as opposed to calibrating the relative belief ratio at a fixed value via the strength. The relative belief estimate is consistent, however, and the relative belief ratio at this value will, at least in the continuous case, converge to infinity. So large values would seem to be associated with high evidence in favor. What has been shown here is that large values can be associated with prior-data conflict and a lack of robustness rather than providing high evidence. When prior-data conflict is encountered the prior can be modified, following Evans.
and Jang (2011b), to avoid this. While objections can be raised to taking such a step, it seems necessary if we want to report a valid characterization of the evidence obtained.

In Baskurt and Evans (2013) the relationship between relative belief ratios and Bayes factors is examined. Both serve as measures of evidence but the relative belief ratio is a simpler, more direct measure and it has many nice mathematical properties. It is the case too that often relative belief ratios and Bayes factors agree. For example, in the case of continuous priors, when the Bayes factor at a point is defined as a limit, then these quantities are the same. As such, it is reasonable to expect that the results derived here about relative belief inferences will apply equally well to inferences based on Bayes factors.

7 References

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Appendix

Proof of Lemma 1

Note first that
\[ Q(A | x) = \int_A \frac{m(x | \psi)}{m_q(x)} Q(d\psi). \]  
(15)

Therefore, using (15),
\[
\Pi_{\psi} (A | x) = \frac{(1 - \epsilon)m\langle A | x \rangle + \epsilon m_q(x) Q(A | x)}{(1 - \epsilon)m(x) + \epsilon m_q(x)} 
\]
\[
= \frac{\Pi_{\psi} (A | x) + \epsilon \frac{\epsilon}{(1 - \epsilon)m(x)} \int_A m(x | \psi) Q(d\psi)}{1 + \frac{\epsilon}{(1 - \epsilon)m(x)} \left\{ \int_A m(x | \psi) Q(d\psi) + \int_{A^c} m(x | \psi) Q(d\psi) \right\}} 
\]
\[
\leq \frac{\Pi_{\psi} (A | x) + \epsilon \frac{\epsilon}{(1 - \epsilon)m(x)} \int_A m(x | \psi) Q(d\psi)}{1 + \frac{\epsilon}{(1 - \epsilon)m(x)} \sup_{\psi \in A^c} m(x | \psi)} 
\]

and the last inequality is an equality when \( Q(A^c) = 0 \). Result (i) then follows since \((p+\gamma)/(1+\gamma) = 1 - (1-p)/(1+y)\) is increasing in \( y \geq 0 \) when \( 1 - p > 0 \) and clearly \( \sup_{\psi} \int_A m(x | \psi) Q(d\psi) = \lim_{\delta \downarrow 0} \int_A m(x | \psi) Q_\delta(d\psi) = \sup_{\psi \in A} m(x | \psi) \) where \( Q_\delta \) places all of its mass on the set \( \{ \psi : m(x | \psi) \geq \sup_{\psi \in A} m(x | \psi) - \delta \} \cap A \).

For result (ii) we have that
\[
\Pi_{\psi} (A | x) \geq \frac{\Pi_{\psi} (A | x) + \epsilon \frac{\epsilon}{(1 - \epsilon)m(x)} \int_A m(x | \psi) Q(d\psi)}{1 + \frac{\epsilon}{(1 - \epsilon)m(x)} \sup_{\psi \in A^c} m(x | \psi)} 
\]
\[
\geq \frac{\Pi_{\psi} (A | x)}{1 + \frac{\epsilon}{(1 - \epsilon)m(x)} \sup_{\psi \in A^c} m(x | \psi)} 
\]

where the first inequality is obvious and the second follows since \((p+y)/(1+y+b) = 1 - (1-p+b)/(1+y+b)\) is increasing in \( y \geq 0 \) when \( 1 - p + b > 0 \) and so the minimum is attained at \( y = 0 \). The inequalities are equalities whenever \( Q(A) = 0 \). We then argue as in (i).
For (iii) a direct calculation yields

\[ \delta(A) = \Pi_{\Psi}^{upper}(A \mid x) - \Pi_{\Psi}^{lower}(A \mid x) \]

\[ = \frac{\Pi_{\Psi}(A \mid x)(1 + \epsilon^*r(A^c)) + \epsilon^*r(A)(1 + \epsilon^*r(A^c)) - \Pi_{\Psi}(A \mid x)(1 + \epsilon^*r(A))}{(1 + \epsilon^*r(A))(1 + \epsilon^*r(A^c))} \]

\[ = \frac{\Pi_{\Psi}(A \mid x)\epsilon^*(r(A^c) - r(A)) + \epsilon^*r(A)}{(1 + \epsilon^*r(A))(1 + \epsilon^*r(A^c))} \]

Result (iv) follows from \( \delta(A^c) = \sup_{Q}(1 - \Pi_{\Psi}(A \mid x)) - \inf_{Q}(1 - \Pi_{\Psi}(A \mid x)) = \sup_{Q}(-\Pi_{\Psi}(A \mid x)) - \inf_{Q}(-\Pi_{\Psi}(A \mid x)) = \delta(A) \).