EQUIVARIANT VECTOR BUNDLES ON DRINFELD’S HALFSPACE OVER A FINITE FIELD

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ABSTRACT. Let \( \mathcal{X} \subset \mathbb{P}^d_k \) be Drinfeld’s halfspace over a finite field \( k \) and let \( \mathcal{E} \) be a homogeneous vector bundle on \( \mathbb{P}^d_k \). The paper deals with two different descriptions of the space of global sections \( H^0(\mathcal{X}, \mathcal{E}) \) as \( \text{GL}_{d+1}(k) \)-representation. This is an infinite dimensional modular \( G \)-representation. Here we follow the ideas of [O2, OS] treating the \( p \)-adic case. As a replacement for the universal enveloping algebra we consider both the crystalline universal enveloping algebra and the ring of differential operators on the flag variety with respect to \( \mathcal{E} \).

INTRODUCTION

Let \( k \) be a finite field and denote by \( \mathcal{X} \) Drinfeld’s halfspace of dimension \( d \geq 1 \) over \( k \). This is the complement of all \( k \)-rational hyperplanes in projective space \( \mathbb{P}^d_k \), i.e.,

\[
\mathcal{X} = \mathbb{P}^d_k \setminus \bigcup_{H \subseteq k^{d+1}} \mathbb{P}(H).
\]

This object is equipped with an action of \( G = \text{GL}_{d+1}(k) \) and can be viewed as a Deligne-Lusztig variety, as well as a period domain over a finite field [OR]. In particular we get for every homogeneous vector bundle \( \mathcal{E} \) on \( \mathbb{P}^d_k \) an induced action of \( G \) on the space of global sections \( H^0(\mathcal{X}, \mathcal{E}) \) which is an infinite-dimensional modular \( G \)-representation.

In [O2] we considered the same problem for the Drinfeld halfspace over a \( p \)-adic field \( K \). We constructed for every homogeneous vector bundle \( \mathcal{E} \) a filtration by closed \( \text{GL}_{d+1}(K) \)-subspaces and determined the graded pieces in terms of locally analytic \( G \)-representations in the sense of Schneider and Teitelbaum [ST1]. The definition of the filtration above involves the geometry of \( \mathcal{X} \) being the complement of an hyperplane arrangement. In the \( p \)-adic case \( H^0(\mathcal{X}, \mathcal{E}) \) is a ”bigger” object, it is a reflexive \( K \)-Fréchet space with a continuous \( G \)-action. Its strong dual is a locally analytic \( G \)-representation. The interest here for studying those objects lies in the connection to the \( p \)-adic Langland correspondence.
In his thesis [Ku] Kuschkowitz adapts the strategy of the $p$-adic case to the situation considered here.

**Theorem (Kuschkowitz):** Let $\mathcal{E}$ be a homogeneous vector bundle on $\mathbb{P}^d_k$. There is a filtration

\[ \mathcal{E}(\mathcal{X})^0 \supset \mathcal{E}(\mathcal{X})^1 \supset \cdots \supset \mathcal{E}(\mathcal{X})^{d-1} \supset \mathcal{E}(\mathcal{X})^d = H^0(\mathbb{P}^d, \mathcal{E}) \]

on $\mathcal{E}(\mathcal{X})^0 = H^0(\mathcal{X}, \mathcal{E})$ such that for $j = 0, \ldots, d - 1$, there is an extension of $G$-representations

\[ 0 \to \text{Ind}_{P_{(j+1,d-j)}}(\hat{H}^{d-j}_{\mathbb{P}^d_k}(\mathbb{P}^d, \mathcal{E}) \otimes \text{St}_{d+1-j}) \to \mathcal{E}(\mathcal{X})^j/\mathcal{E}(\mathcal{X})^{j+1} \to \mathcal{E}_{P_{(j+1,1,\ldots,1)}}(\mathbb{P}^d_k, \mathcal{E}) \to 0. \]

Here the module $\mathcal{E}_{P_{(j+1,1,\ldots,1)}}$ is a generalized Steinberg representation corresponding to the decomposition $(j, 1, 1, \ldots, 1)$ of $d + 1$. Further $P_{(j+1, d-j)} = P_{(j, d+1-j)} \subset G$ is the (lower) standard-parabolic subgroup attached to the decomposition $(j, d+1-j)$ of $d+1$ and $\text{St}_{d+1-j}$ is the Steinberg representation of $\text{GL}_{d+1-j}(k)$. Here the action of the parabolic is induced by the composite

\[ P_{(j, d+1-j)} \to L_{(j, d+1-j)} = \text{GL}_j(k) \times \text{GL}_{d+1-j}(k) \to \text{GL}_{d+1-j}(k). \]

Finally we have the reduced local cohomology

\[ \hat{H}^{d-j}_{\mathbb{P}^d_k}(\mathbb{P}^d_k, \mathcal{E}) := \ker \left( H^{d-j}_{\mathbb{P}^d_k}(\mathbb{P}^d_k, \mathcal{E}) \to H^{d-j}(\mathbb{P}^d_k, \mathcal{E}) \right) \]

which is a $P_{(j+1,d-j)}$-module.

In the $p$-adic setting the substitute of the LHS of this extension has the structure of an admissible module over the locally analytic distribution algebra. Here we were able to give a description of the dual representation in terms of a series of functors

\[ \mathcal{F}^G_P : \mathcal{O}^p_{\text{alg}} \times \text{Rep}^{\infty, \text{adm}}_K(P) \to \text{Rep}^\ell_K(G) \]

where $\mathcal{O}^p_{\text{alg}}$ consist of the algebraic objects of type $p$ in the category $\mathcal{O}$, $\text{Rep}^{\infty, \text{adm}}_K(P)$ is the category of smooth admissible $P$-representations and $\text{Rep}^\ell_K(G)$ denotes the category of locally analytic $G$-representations.

In positive characteristic Lie algebra methods do not behave so nice. E.g. the local cohomology groups are not finitely generated over the universal enveloping algebra of the Lie algebra of $\text{GL}_{d+1}$ so that the same machinery does not apply. Our goal in this paper is to concentrate on the latter aspect and to present two candidates for a substitution in this situation. The first approach considers the crystalline universal enveloping algebra $\hat{U}(\mathfrak{g})$ (or Kostant form) which coincides with the distribution algebra of $G$, cf. [Ja]. The action of $\mathfrak{g}$ extends to one of $\hat{U}(\mathfrak{g})$, so that $H^0(\mathcal{X}, \mathcal{E})$ becomes
a module over the smash product \( k[G] \# \mathcal{U}(g) \). We define a positive characteristic version of \( \mathcal{F}_p^G \) and prove analogously properties of them as in the \( p \)-adic case, e.g. we give an irreducibility criterion, cf. \([OS]\).

The second approach uses instead of \( \mathcal{U}(g) \) the ring of distributions \( D^\mathcal{E} \) on the flag variety with respect to \( \mathcal{E} \). The important point is that the natural map \( \mathcal{U}(g) \rightarrow D^\mathcal{E} \) is in contrast to the field of complex numbers not surjective as shown by Smith \([Sm]\). We will show that the above local cohomology modules are finitely generated leading to a category \( \mathcal{O}_{D^\mathcal{E}} \) where we can define similar our functors \( \mathcal{F}_p^G \).

Notation: We let \( p \) be a prime number, \( q = p^n \) some power and let \( k = \mathbb{F}_q \) the corresponding field with \( q \) elements. We fix an algebraic closure \( \mathbb{F} := \overline{\mathbb{F}_q} \) and denote by \( \mathbb{P}^d_F \) the projective space of dimension \( d \) over \( \mathbb{F} \). If \( Y \subset \mathbb{P}^d_F \) is a closed algebraic \( \mathbb{F} \)-subvariety and \( \mathcal{F} \) is a sheaf on \( \mathbb{P}^d_F \) we write \( H^*_Y(\mathbb{P}^d_F, \mathcal{F}) \) for the corresponding local cohomology. We consider the algebraic action \( G \times \mathbb{P}^d_F \rightarrow \mathbb{P}^d_F \) of \( G \) on \( \mathbb{P}^d_F \) given by

\[
g \cdot [q_0 : \cdots : q_d] := m(g, [q_0 : \cdots : q_d]) := [q_0 : \cdots : q_d]g^{-1}.\]

We use bold letters \( \mathbf{H} \) to denote algebraic group schemes over \( \mathbb{F}_q \), whereas we use normal letters \( H \) for their \( \mathbb{F}_q \)-valued points. We denote by \( \mathbf{H}_F := \mathbf{H} \times_{\mathbb{F}_q} \mathbb{F} \) their base change to \( \mathbb{F} \). We use Gothic letters \( \mathfrak{h} \) for their Lie algebras (over \( \mathbb{F} \)). The corresponding enveloping algebras are denoted as usual by \( U(\mathfrak{h}) \).

We denote by \( G_Z \) a split reductive algebraic group over \( \mathbb{Z} \). We fix a Borel subgroup \( B_Z \subset G_Z \) and let \( U_Z \) be its unipotent radical and \( U_{Z}^{-} \) the opposite radical. Let \( T_Z \subset G_Z \) be a fixed split torus and denote the root system by \( \Phi \) and its subset of simple roots by \( \Delta \).

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1. The theorem of Kuschkowitz

In this section we recall shortly the strategy for proving the theorem of Kuschkowitz. Here we consider for \( G \) the general linear group \( \text{GL}_{d+1} \) and for \( B \subset G \) the Borel
which is consequently a product is defined via the adjoint action of $P$. For these functions, the action of $g$ be the standard generator of the weight space $E$. Write $\mu \in X^*(T)$ be a character of the torus $\mathbf{T}$. Write $\mu$ in the shape $\mu = \sum_{i=0}^d m_i \epsilon_i$ with $\sum_{i=0}^d m_i = 0$. Define $\Xi_\mu \in \mathcal{O}(\mathcal{X})$ by

$$\Xi_\mu(q_0, \ldots, q_d) = q_0^{m_0} \cdots q_d^{m_d}.$$  

For these functions, the action of $g$ is given by

$$L_{(i,j)} : \Xi_\mu = m_j \cdot \Xi_{\mu + \alpha_{i,j}}$$

and

$$t \cdot \Xi_\mu = (\sum_i m_i t_i) \cdot \Xi_\mu, \quad t \in \mathfrak{t}.$$  

Fix an integer $0 \leq j \leq d - 1$. Let

$$\mathbb{P}_F^j = V(X_{j+1}, \ldots, X_d) \subset \mathbb{P}_F^d$$

be the closed subvariety defined by the vanishing of the coordinates $X_{j+1}, \ldots, X_d$. The algebraic local cohomology modules $H^i_{\mathbb{P}_F^j}(\mathbb{P}_F^d, \mathcal{E})$, $i \in \mathbb{N}$, sit in a long exact sequence

$$\cdots \rightarrow H^{i-1}(\mathbb{P}_F^d \setminus \mathbb{P}_F^j, \mathcal{F}) \rightarrow H^i_{\mathbb{P}_F^j}(\mathbb{P}_F^d, \mathcal{F}) \rightarrow H^i(\mathbb{P}_F^d, \mathcal{F}) \rightarrow H^i(\mathbb{P}_F^d \setminus \mathbb{P}_F^j, \mathcal{F}) \rightarrow \cdots$$

which is equivariant for the induced action of $P_{(j+1,d-j)} \ltimes U(\mathfrak{g})$. Here the semi-direct product is defined via the adjoint action of $P_{(j+1,d-j)}$ on $\mathfrak{g}$. We set

$$\tilde{H}^{d-j}_{\mathbb{P}_F^j}(\mathbb{P}_F^d, \mathcal{E}) := \ker \left( H^{d-j}_{\mathbb{P}_F^j}(\mathbb{P}_F^d, \mathcal{E}) \rightarrow H^{d-j}(\mathbb{P}_F^d, \mathcal{E}) \right)$$

which is consequently a $P_{(j+1,d-j)} \ltimes U(\mathfrak{g})$-module.
Consider the exact sequence of $\mathbb{F}$-vector spaces with $G$-action
\[ 0 \rightarrow H^0(\mathbb{P}^d_{\mathbb{F}}, \mathcal{E}) \rightarrow H^0(\mathcal{X}, \mathcal{E}) \rightarrow H^1(\mathbb{P}^d_{\mathbb{F}}, \mathcal{E}) \rightarrow H^1(\mathcal{Y}, \mathbb{P}^d_{\mathbb{F}}, \mathcal{E}) \rightarrow 0. \]
Note that the higher cohomology groups $H^i(\mathcal{X}, \mathcal{E})$, $i > 0$, vanish since $\mathcal{X}$ is an affine space. The $G$-representations $H^0(\mathbb{P}^d_{\mathbb{F}}, \mathcal{F})$, $H^1(\mathbb{P}^d_{\mathbb{F}}, \mathcal{F})$ are finite-dimensional algebraic.

Let $i : \mathcal{Y} \hookrightarrow (\mathbb{P}^d_{\mathbb{F}})$ denote the closed embedding and let $\mathcal{Z}$ be constant sheaf on $\mathcal{Y}$. Then by [SGA2, Proposition 2.3 bis.], we conclude that
\[ \text{Ext}^*(i_*(\mathcal{Z}), \mathcal{E}) = H^*_\mathcal{Y}(\mathbb{P}^d_{\mathbb{F}}, \mathcal{E}). \]

The idea is now to plug in a resolution of the sheaf $\mathcal{Z}$ on the boundary and works as follows.

Let $\{e_0, \ldots, e_d\}$ be the standard basis of $V = \mathbb{P}^{d+1}$. For any $\alpha_i \in \Delta$, put
\[ V_i = \bigoplus_{j=0}^{i} \mathbb{F} \cdot e_j \quad \text{and} \quad Y_i = \mathbb{P}(V_i) \]

For any subset $I \subset \Delta$ with $\Delta \setminus I = \{\alpha_{i_1} < \ldots < \alpha_{i_r}\}$, set $Y_I = \mathbb{P}(V_{i_1})$ and consider it as a closed subvariety of $\mathbb{P}^d_{\mathbb{F}}$. Furthermore, let $P_I$ be the lower parabolic subgroup of $G$, such that $I$ coincides with the simple roots appearing in the Levi factor of $P_I$. Hence the group $P_I$ stabilizes $Y_I$. We obtain
\[ \mathcal{Y} = \bigcup_{g \in G} g \cdot Y_{\Delta \setminus \{\alpha_{d-1}\}}. \]

Consider the natural closed embeddings
\[ \Phi_{g,I} : gY_I \longrightarrow \mathcal{Y} \]
and put
\[ \mathcal{Z}_{g,I} := (\Phi_{g,I})_*(\Phi_{g,I}^* \mathcal{Z}). \]

We obtain the following complex of sheaves on $\mathcal{Y}$:
\[ 0 \rightarrow \mathcal{Z} \rightarrow \bigoplus_{I \subset \Delta, |\Delta \setminus I| = 1} g \in G/P_I \mathcal{Z}_{g,I} \rightarrow \bigoplus_{I \subset \Delta, |\Delta \setminus I| = 2} g \in G/P_I \mathcal{Z}_{g,I} \rightarrow \cdots \rightarrow \bigoplus_{I \subset \Delta, |\Delta \setminus I| = d-1} g \in G/P_I \mathcal{Z}_{g,I} \rightarrow \cdots \]
\[ \cdots \rightarrow \bigoplus_{I \subset \Delta, |\Delta \setminus I| = d-1} g \in G/P_I \mathcal{Z}_{g,I} \rightarrow \bigoplus_{g \in G/P_0} \mathcal{Z}_{g,0} \rightarrow 0. \]

which is acyclic by [O1].
In a next step one considers the spectral sequence which is induced by this complex applied to \( \text{Ext}^* \left( i_*(-), \mathcal{E} \right) \). Here one uses that for all \( I \subset \Delta \), there is an isomorphism
\[
\text{Ext}^* \left( i_* \left( \bigoplus_{g \in G/P_I} \mathbb{Z}_{g,I} \right), \mathcal{E} \right) = \bigoplus_{g \in G/P_I} H^*_g \left( \mathbb{P}^d_{g\gamma_I}, \mathcal{E} \right).
\]

By evaluating the spectral sequence Kuschkowitz arrives in [Ku] at the theorem mentioned in the introduction.

2. First approach

In this section we replace \( U(\mathfrak{g}) \) by its crystalline version and transform the results of [OS] to this setting.

Let \( G_{\mathbb{Z}} \) be a split reductive algebraic group over \( \mathbb{Z} \) and let \( \mathfrak{g}_{\mathbb{C}} \) be the Lie algebra of \( G_{\mathbb{Z}}(\mathbb{C}) \). On the other hand let \( D(G_{\mathbb{F}}) \) be the distribution algebra of \( G_{\mathbb{F}} = G_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{F} \). We identify \( D(G_{\mathbb{F}}) \) with the universal crystalline enveloping algebra (Kostant form) \( \dot{\mathcal{U}}(\mathfrak{g}) \). Thus \( \dot{\mathcal{U}}(\mathfrak{g}) = \dot{\mathcal{U}}(\mathfrak{g})_{\mathbb{Z}} \otimes \mathbb{F} \) where \( \dot{\mathcal{U}}(\mathfrak{g})_{\mathbb{Z}} \) is the \( \mathbb{Z} \)-subalgebra of \( U(\mathfrak{g}_{\mathbb{C}}) \) generated by the expressions
\[
x_{\alpha}^{[n]} := x_{\alpha}^n/n!, \quad y_{\alpha}^{[n]} := y_{\alpha}^n/n!, \quad \alpha \in \Phi^+, \quad n \in \mathbb{N}
\]
and \( \left( h_{\alpha} \right)/n \), \( \alpha \in \Delta, \quad n \in \mathbb{N} \),

where \( x_{\alpha} \in \mathfrak{g}_{\alpha}, y_{\alpha} \in \mathfrak{g}_{-\alpha} \) are generators and \( h_{\alpha} = [x_{\alpha}, y_{\alpha}] \) for all \( \alpha \in \Delta \). We have a PBW-decomposition
\[
\dot{\mathcal{U}}(\mathfrak{g}) = \dot{\mathcal{U}}(\mathfrak{u}) \otimes_{\mathbb{F}} \dot{\mathcal{U}}(\mathfrak{t}) \otimes_{\mathbb{F}} \dot{\mathcal{U}}(\mathfrak{u}^-)
\]
where the crystalline enveloping algebras for \( \mathfrak{u}, \mathfrak{u}^- \) and \( \mathfrak{t} \) are defined analogously.

We mimic the definition of the category \( \mathcal{O} \) in the sense of BGG.

**Definition 2.1.** Let \( \hat{\mathcal{O}} \) be the full subcategory of all \( \dot{\mathcal{U}}(\mathfrak{g}) \)-modules such that

i) \( M \) is finitely generated as \( \dot{\mathcal{U}}(\mathfrak{g}) \)-module

ii) \( \dot{\mathcal{U}}(\mathfrak{t}) \) acts semisimple with finite-dimensional weight spaces.

iii) \( \dot{\mathcal{U}}(\mathfrak{u}) \) acts locally finite-dimensional, i.e., for all \( m \in M \) we have \( \dim \dot{\mathcal{U}}(\mathfrak{u}) \cdot m < \infty \).

**Remark 2.2.** In [Hab] Def. 3.2 Haboush calls \( \dot{\mathcal{U}}(\mathfrak{g}) \)-modules satisfying properties i) and ii) admissible. The category \( \hat{\mathcal{O}} \) has been also recently considered by Andersen [An] and Fiebig [Fi] (even more generally for weight modules) discussing among others the structure of these objects.
Similarly, for a parabolic subgroup $P \subset G$ with Levi decomposition $P = L_P \cdot U_P$ (induced by one over $\mathbb{Z}$), we let $\hat{O}^P$ be the full subcategory of $\hat{O}$ consisting of objects which are direct sums of finite-dimensional $\hat{U}(l_P)$-modules. We let $\hat{O}_{\text{alg}}$ be the full subcategory of $\hat{O}$ such that the action of $\hat{U}(t)$ is induced on the weight spaces by algebraic characters $X^*(T_F)$ of $T_F$. Finally we set $\hat{O}^P_{\text{alg}} := \hat{O}_{\text{alg}} \cap \hat{O}^P$.

As in the classical case there is for every object $M \in \hat{O}^P_{\text{alg}}$ some finite-dimensional algebraic $P$-representation $W \subset M$ which generates $M$ as a $\hat{U}(g)$-module, i.e., there is a surjective homomorphism $\hat{U}(g) \otimes_{\hat{U}(P)} W \to M$. Again there is a PBW-decomposition $\hat{U}(g) = \hat{U}(u_P) \otimes_F \hat{U}(l_P) \otimes_F \hat{U}(u_P^{-1})$ such that the latter homomorphism can be seen as a map $\hat{U}(u_P) \otimes_F W \to M$.

According to [Hab] there is the notion of maximal vectors, highest weights, highest weight module etc. and we may define Verma modules, cf. Def. 3.1 in loc.cit. In fact let $\lambda$ be a one-dimensional $\hat{U}(t)$-module. Then we consider it as usual via the trivial $\hat{U}(u)$-action as a one-dimensional $\hat{U}(b)$-module $F_\lambda$. Then

$$M(\lambda) = \hat{U}(g) \otimes_{\hat{U}(b)} F_\lambda$$

is the attached Verma module of weight $\lambda$. As in the classical case Theorem of [Hu, 1.2] holds true for our highest weight modules. In particular it has a unique maximal proper submodule and therefore a unique simple quotient $L(\lambda)$, cf. [Hab, Prop. 4.4], [An, Thm 2.3], [Fi, Prop. 2.3].

**Proposition 2.3.** The simple modules in $\hat{O}_{\text{alg}}$ are exactly of the shape $L(\lambda)$ for $\lambda \in X^*(T_F)$.

**Proof.** We need to show that every simple object in $\hat{O}_{\text{alg}}$ is of this form. But by [Hab, Thm 4.9 i)] simple admissible highest weight modules are of the form $L(\lambda)$ for a one-dimensional $\hat{U}(t)$-module $\lambda$. The algebraic condition forces $\lambda$ to be an algebraic character $\lambda \in X^*(T_F)$. \qed

We also consider the full subcategory $M^d_{\hat{U}(g)}$ for all $\hat{U}(g)$-modules which satisfy condition ii) in the definition of $\hat{O}$. For any such object $M$ we define a dual object $M'$ (the graded dual) following the classical concept: consider the weight space decomposition $M = \bigoplus \lambda M_\lambda$ where $\lambda$ is as above a one-dimensional $\hat{U}(t)$-module. Then the

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1 Meaning that we restrict an algebraic $P$-representation to the its rational points $P$. 

underlying vector space of \( M' \) is the direct sum \( \bigoplus_{\lambda} \text{Hom}(M_{\lambda}, K) \). The \( \hat{\mathcal{U}}(\mathfrak{g}) \)-structure on it is given by the natural one. Clearly one has \((M')' = M\).

We consider the natural action of \( u_{\mathfrak{p}} \) on \( \mathcal{O}(\mathfrak{U}_{\mathfrak{p}, \mathfrak{f}}) \). This extends to a non-degenerate pairing

\[
(\hat{\mathcal{U}}(\mathfrak{u}_{\mathfrak{p}}) \otimes \mathcal{O}(\mathfrak{U}_{\mathfrak{p}, \mathfrak{f}})) \rightarrow \mathbb{F}
\]

such that \( \mathcal{O}(\mathfrak{U}_{\mathfrak{p}, \mathfrak{f}}) \) identifies with the graded dual of \( \hat{\mathcal{U}}(\mathfrak{u}_{\mathfrak{p}}) \). Moreover we pull back via this identification the action of \( \mathfrak{p} \) on \( (\hat{\mathcal{U}}(\mathfrak{g}) \otimes_{\hat{\mathcal{U}}(\mathfrak{p})} 1)' \) to \( \mathcal{O}(\mathfrak{U}_{\mathfrak{p}, \mathfrak{f}}) \). By construction we obtain the following statement.

**Lemma 2.4.** There is an isomorphism of \( \mathfrak{p} \ltimes \hat{\mathcal{U}}(\mathfrak{g}) \)-modules \( \mathcal{O}(\mathfrak{U}_{\mathfrak{p}, \mathfrak{f}}) \cong (\hat{\mathcal{U}}(\mathfrak{g}) \otimes_{\hat{\mathcal{U}}(\mathfrak{p})} 1)' \).

\( \square \)

The pairing (2.1) extends for every algebraic \( \mathfrak{p} \)-representation \( W \) to a pairing

\[
(\hat{\mathcal{U}}(\mathfrak{u}_{\mathfrak{p}}) \otimes W') \otimes (\mathcal{O}(\mathfrak{U}_{\mathfrak{p}, \mathfrak{f}}) \otimes W) \rightarrow \mathbb{F}
\]

such that \( \mathcal{O}(\mathfrak{U}_{\mathfrak{p}, \mathfrak{f}}) \otimes W \) becomes isomorphic to \( \hat{\mathcal{U}}(\mathfrak{u}_{\mathfrak{p}})' \otimes W' \) as \( \mathfrak{p} \ltimes \hat{\mathcal{U}}(\mathfrak{g}) \)-modules.

Let \( \mathbb{F}[G, \mathfrak{g}] := \mathbb{F}[G] \# \hat{\mathcal{U}}(\mathfrak{g}) \) be the smash product of \( \hat{\mathcal{U}}(\mathfrak{g}) \) and the group algebra \( \mathbb{F}[G] \) of \( G \). Recall that this \( \mathbb{F} \)-algebra has as underlying vector space the tensor product \( \mathbb{F}[G] \otimes \hat{\mathcal{U}}(\mathfrak{g}) \) and where the multiplication is induced by \((g_1 \otimes z_1) \cdot (g_2 \otimes z_2) = g_1 g_2 \otimes \text{Ad}(g_2)(z_1) z_2\) for elements \( g_i \in G, z_i \in \hat{\mathcal{U}}(\mathfrak{g}), i = 1, 2\).

**Definition 2.5.** i) We denote by \( \text{Mod}_{\mathbb{F}[G, \mathfrak{g}]}^d \) be the full subcategory of all \( \mathbb{F}[G, \mathfrak{g}] \)-modules for which the action of \( \hat{\mathcal{U}}(\mathfrak{t}) \) is diagonalisable into finite-dimensional weight spaces.

ii) We denote by \( \text{Mod}_{\mathbb{F}[G, \mathfrak{g}]}^{fg,d} \) be the full subcategory of \( \text{Mod}_{\mathbb{F}[G, \mathfrak{g}]}^d \) which are finitely generated.

For an object \( \mathcal{M} \) of \( \text{Mod}_{\mathbb{F}[G, \mathfrak{g}]}^d \), we define the dual \( \mathcal{M}' \) as the graded dual of the underlying \( \hat{\mathcal{U}}(\mathfrak{g}) \)-module together with the contragradient action of \( G \).

Let \( M \) be an object of \( \hat{\mathcal{O}}_{\text{alg}}^{\mathfrak{p}} \). Then there is a surjection

\[
p : \hat{\mathcal{U}}(\mathfrak{u}_{\mathfrak{p}}) \otimes W \rightarrow M
\]

\(^2\)Without the composition with the Cartan involution.
for some finite-dimensional algebraic $P$-module $W$. Let $\mathfrak{d} := \ker(p)$ be its kernel. Then set

$$\mathcal{F}_{P}^G(M) := \text{Ind}_{G}^{P}((\mathcal{O}(U_{P,F}) \otimes W)^{\mathfrak{d}})$$

where $(\mathcal{O}(U_{P,F}) \otimes W)^{\mathfrak{d}}$ is the orthogonal complement of $\mathfrak{d}$ with respect to the pairing (2.2). The latter submodule can be interpreted as the graded dual of $M$. In particular we get

$$\mathcal{F}_{P}^G(M)' = \text{Ind}_{P}^{G}(M).$$

**Lemma 2.6.** Let $M$ be an object of $\mathcal{O}_{\text{alg}}$. Then $\mathcal{F}_{P}^G(M)$ is an object of the category $\text{Mod}_{\mathbb{F}[G,\mathfrak{g}]}^{d}$. Its dual $\mathcal{F}_{P}^G(M)'$ is an object of the category $\text{Mod}_{\mathbb{F}[G,\mathfrak{g}]}^{d,f,g,d}$.

**Proof.** It suffices to show the second assertion. As $G/P$ is a finite set, we need only to show that $\mathcal{F}_{P}^G(M)'$ has a decomposition into finite-dimensional weight spaces. Let $M = \bigoplus_{\lambda} M^{\lambda}$. We write $\mathcal{F}_{P}^G(M) = \bigoplus_{g \in G/P} \delta_{g} \star M$ where $\delta_{g} \star M$ is the $\hat{U}(\mathfrak{g})$-module with the same underlying vector space but where the Lie algebra action is twisted by $Ad(g)$. We consider the Bruhat decomposition $G/P = \bigcup_{w \in W} U_{B,w} P/P$ where $U_{B,w} = U \cap wUw^{-1}$ and take the obvious representatives for $G/P$. Thus we have

$$\mathcal{F}_{P}^G(M)' = \bigoplus_{w \in W_{P}} \bigoplus_{u \in U_{B,w}} \delta_{uw} \star M.$$ 

In the case of $\delta_{w}, w \in W$, the grading of $\delta_{w} \star M$ is given by $\bigoplus_{\lambda} M^{w\lambda}$. In the case of $\delta_{u}, u \in U_{B,w}$ the grading is given by $\bigoplus_{\lambda} u \cdot M^{\lambda}$ (Note that we have an action of $U$ on $M$). In general we consider the mixture of these cases. \qed

Let $V$ be additionally a finite-dimensional $P$-module. Then we set

$$\mathcal{F}_{P}^G(M, V) := \text{Ind}_{P}^{G}((\mathcal{O}(U_{P,F}) \otimes W)^{\mathfrak{d}} \otimes V).$$

This is an object of $\text{Mod}_{\mathbb{F}[G,\mathfrak{g}]}^{d}$ by a slight generalization of the above lemma. In this way we get a bi-functor

$$\mathcal{F}_{P}^G : \mathcal{O}_{\text{alg}} \times \text{Rep}(P) \to \text{Mod}_{\mathbb{F}[G,\mathfrak{g}]}^{d}.$$ 

By the following statement the dual $\mathcal{F}_{P}^G(M, V)'$ is an object of $\text{Mod}_{\mathbb{F}[G,\mathfrak{g}]}^{d,f,g,d}$.

**Lemma 2.7.** The dual of $\mathcal{F}_{P}^G(M, V)$ is given by

$$\mathcal{F}_{P}^G(M, V)' = \mathbb{F}[G,\mathfrak{g}] \otimes_{\mathbb{F}[P,\mathfrak{g}]} (M \otimes V').$$

**Proof.** We have $\mathcal{F}_{P}^G(M, V)' = \text{Ind}_{P}^{G}(M' \otimes V)' = \text{Ind}_{P}^{G}((M')' \otimes V') = \text{Ind}_{P}^{G}(M \otimes V').$ \qed
Proposition 2.8. The functor $F^G_P$ is exact in both arguments.

Proof. We start to prove that the functor is exact in the first argument. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence in the category $\mathcal{O}^p_{\text{alg}}$. Then the sequence $0 \to \text{Ind}_P^G M_1 \to \text{Ind}_P^G M_2 \to \text{Ind}_P^G M_3 \to 0$ is also exact. But the graded dual of this sequence is exactly $0 \to F^G_P(M_3) \to F^G_P(M_2) \to F^G_P(M_1) \to 0$.

As for exactness in the second argument let $0 \to V_1 \to V_2 \to V_3 \to 0$ be an exact sequence of $P$-representations. As $F^G_P(M, V) := \text{Ind}_P^G((\mathcal{O}(U_{P,W}) \otimes W') \otimes V_i)$ and $\text{Ind}_P^G$ is an exact functor we see easily the claim. $\square$

Now let $Q \supset P$ be a parabolic subgroup and let $M \in \dot{\mathcal{O}}^q_{\text{alg}}$. Then we may consider it also as an object of $\dot{\mathcal{O}}^q_{\text{alg}}$.

Proposition 2.9. If $Q \supset P$ is a parabolic subgroup, $M$ an object of $\dot{\mathcal{O}}^q_{\text{alg}}$ and $V$ a finite-dimensional $P$-module, then

$$F^G_P(M, V) = F^G_Q(M, \text{Ind}_P^Q(V)).$$

Proof. We have

$$F^G_P(M, V) = \text{Ind}_P^G(M \otimes V) = \text{Ind}_Q^G(\text{Ind}_P^Q(M \otimes V)) = \text{Ind}_Q^G(M' \otimes \text{Ind}_P^Q(V)) = F^G_Q(M, \text{Ind}_P^Q(V))$$

by the projection formula. Hence we deduce the claim. $\square$

As in [OS] a parabolic Lie algebra $p$ is called maximal for an object $M \in \dot{\mathcal{O}}^q_{\text{alg}}$ if there does not exist a parabolic Lie algebra $q \supset p$ with $M \in \dot{\mathcal{O}}^q_{\text{alg}}$.

Theorem 2.10. Let $p > 3$. Let $M$ be an simple object of $\dot{\mathcal{O}}^q_{\text{alg}}$ such that $p$ is maximal for $M$. Then $F^G_P(M)$ is a simple $\mathbb{F}[G, \mathfrak{g}]$-module.

Proof. The proof follows the idea of loc.cit. and is even simpler. We start with the observation that by duality $F^G_P(M, V)$ is simple as $\mathbb{F}[G, \mathfrak{g}]$-module iff $F^G_P(M, V)'$ is simple as $\mathbb{F}[G, \mathfrak{g}]$-module. We consider again the Bruhat decomposition $G/P = \bigcup_{w \in W_P} U^-_B, wB / B$ and the induced decomposition

$$F^G_P(M)' = \bigoplus_{w \in W_P} \bigoplus_{u \in U^-_B, w} \delta_{uw} \star M.$$
We denote (with respect to \( \delta_{uw} \ast M \)) for elements \( z \in \hat{\mathcal{U}}(g) \) and \( m \in M \) the action of \( z \) on \( m \) by \( z \cdot_{uw} m \). Now each summand \( \delta_{uw} \ast M \) is simple since \( M \) is simple. Thus it suffices to show that the summands are pairwise non isomorphic as \( \hat{\mathcal{U}}(g) \)-modules. Suppose that there is an isomorphism \( \beta \). Then there is a positive root \( \lambda \). Consider a non-zero element \( y \) element \( y \in g_{-\beta} \), and let \( v^+ \in M \) be a weight vector of weight \( \lambda \). Then we have for \( n \in \mathbb{N} \), the following identity

\[
y^{[n]} \cdot_{w} v^+ = \text{Ad}(w^{-1})(y^{[n]}) \cdot v^+ = 0
\]
as \( \text{Ad}(w^{-1})(y^{[n]}) \in g_{-w^{-1} \beta} \) annihilates \( v^+ \). We have \( \phi(v^+) = v \) for some nonzero \( v \in M \). And therefore

\[
0 = \phi(y^{[n]} \cdot_{w} v^+) = y^{[n]} \cdot \phi(v^+) = y^{[n]} \cdot v.
\]

But \( y \) is an element of \( u_{G/P} \), hence we get a contradiction by Proposition 2.13 since \( n \) was arbitrary.

**Theorem 2.11.** Let \( p > 3 \). Let \( M \) be an simple object of \( \hat{\mathcal{O}}_{\text{alg}}^p \) such that \( p \) is maximal for \( M \) and let \( V \) be an irreducible \( P \)-representation. Then \( \mathcal{F}_G^P(M, V) \) and its dual \( \mathcal{F}_G^P(M, V)' \) are simple as \( \hat{\mathbb{F}}[G, g] \)-module.

**Proof.** Again by duality it is enough to check the assertion for \( \mathcal{F}_G^P(M, V)' \). So let \( U \subset \mathcal{F}_G^P(M, V)' \) be a non-zero \( G \)-invariant subspace. Recall that \( \mathcal{F}_G^P(M)' = \bigoplus_{\gamma \in G/P} \delta_\gamma \ast L(\lambda) \) so that

\[
\mathcal{F}_G^P(M, V) = \bigoplus_{\gamma \in G/P} \delta_\gamma \ast L(\lambda) \otimes V^\gamma.
\]

Considered as \( \hat{\mathcal{U}}(g) \)-module \( \mathcal{F}_G^P(M, V) \) is isomorphic to \( \bigoplus_{\gamma \in G/P} \delta_\gamma \ast L(\lambda)' \otimes V \). Hence by the simplicity of \( M \) and since the summands \( \delta_\gamma \ast L(\lambda)' \) are pairwise not isomorphic the \( \hat{\mathcal{U}}(g) \)-module \( U \) is equal to

\[
\bigoplus_{\gamma \in G/P} \delta_\gamma \ast L(\lambda)' \otimes \mathcal{F} V_\gamma,
\]
with subspaces, $V_{\gamma, \gamma}$, of $V$. Here $\delta_1 \ast L(\lambda)' \otimes V_1 = L(\lambda)' \otimes V_1$ is a $\mathbb{F}[P, \mathfrak{g}]$-submodule of $L(\lambda)' \otimes V$. Since $V$ is irreducible the latter object is irreducible, as well. Hence $V_1 = V$. But since $G$ permutes the summands of $U$ we see that $U = \mathcal{F}_p^G(M, V)'$. \hspace{1cm} \Box

In the following statement we merely consider elements in a root space by the very definition of $\hat{\mathcal{U}}(\mathfrak{g})$.

**Lemma 2.12.** Let $p > 3$. Let $x \in \mathfrak{g}_{\gamma}$ some element for $\gamma \in \Phi$. Let $M$ be a $\hat{\mathcal{U}}(\mathfrak{g})$-module and $v \in M$.

(i) If $x$ acts locally finitely on $v$ (i.e., the $K$-vector space generated by $(x^i.v)_{i \geq 0}$ is finite-dimensional), then $x$ acts locally finitely on $\hat{\mathcal{U}}(\mathfrak{g}).v$.

(ii) If $x.v = 0$ and $[x, [x, y]] = 0$ for some $y \in \mathfrak{g}_\beta$, where $\beta \in \Phi$ then $x^n y^n.v = [x, y]^n.v$.

**Proof.** (i) The idea is to apply Lemma 8.1 of loc.cit. which gives in characteristic 0 the formula

$$x^k \cdot z_1 z_2 \ldots z_n = \sum_{i_1 + \ldots + i_n+1 = k} \frac{k!}{i_1! \ldots i_n!} [x^{(i_1)}, z_1] \cdot \ldots \cdot [x^{(i_n)}, z_n] x^{j_{n+1}}.$$ 

Here the expression $[x^{(i)}, z]$ means $ad(x)^i(z)$. We may rewrite this as

$$x^{[k]} \cdot z_1 z_2 \ldots z_n = \sum_{i_1 + \ldots + i_n+1 = k} \frac{1}{i_1! \ldots i_n!} [x^{(i_1)}, z_1] \cdot \ldots \cdot [x^{(i_n)}, z_n] x^{[j_{n+1}]}.$$ 

Indeed we consider the PBW-decomposition $\hat{\mathcal{U}}(\mathfrak{g}) = \hat{\mathcal{U}}(\mathfrak{u}) \otimes \hat{\mathcal{U}}(t) \otimes \hat{\mathcal{U}}(\mathfrak{u})$ and assume that the elements $z_i$ lie without loss of generality in one of these factors. For any element $z$ in some root space it follows from [Hu 0.2] that $[x^{(k)}, z] = 0$ for all $k \geq 4$. Since we avoid the situation $p = 2, 3$ we my divide my the denominators 2! and 3!.

Now in contrast to loc.cit. we have again to consider $z_i$ as elements of $\hat{\mathcal{U}}(\mathfrak{g})$ instead of elements in $\mathfrak{g}$. Let $d_i$ be the order of the differential $z_i$. Then $[x^{(i_1)}, z_1] \ldots [x^{(i_n)}, z_n]$ is an differential of order less than $4(d_1 + \ldots + d_n)$. In particular we can conclude as in loc.cit. that the term lies in a finite dimensional vector space which gives now easily the claim.

(ii) In characteristic 0 we have the formula $x^n y^n.v = n! \cdot [x, y]^n.v$, cf. [OS Lemma 8.2 ii)]. We only have to divide two times by $n!$. \hspace{1cm} \Box

\(^3\)Note that this definition is stronger than the one in characteristic 0.
Proposition 2.13. Let $p > 3$. Let $p = p_I$ for some $I \subset \Delta$. Suppose $M \in \mathfrak{O}^p$ is a highest weight module with highest weight $\lambda$ and

$$I = \{ \alpha \in \Delta \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \}.$$ 

Then no non-zero element of a root space of $u_p^-$ acts locally finitely on $M$.

Proof. The proof is in principal the same as in the case of characteristic 0 [OS, Cor. 8.2]. However we have to modify some technical ingredients of the necessary lemmas due the different characteristic.

Let $y \in (u_p^-)_\gamma$ for some root $\gamma$. Let $v^+$ be a weight vector with weight $\lambda$. Write $\gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha$ (with non-negative integers $c_\alpha$). We show by induction on the height $ht(\gamma)$ of $\gamma$ (Recall that $ht(\gamma) = \sum_{\alpha \in \Delta} c_\alpha$) that $y_\gamma$ can not act locally finite. For this it suffices by weight reasons to show that $y_\gamma^{[n]}v^+ \neq 0$ for infinitely many positive integers $n$.

If $ht(\gamma) = 1$, then $\gamma$ is an element of $\Delta \setminus I$. Rescaling $y_\gamma$ we can choose $x_\gamma \in g_\gamma$ such that $[x_\gamma, y_\gamma] = h_\gamma$ and $[h_\gamma, x_\gamma] = 2x_\gamma$ and $[h_\gamma, y_\gamma] = -2y_\gamma$. Then by [Hab, 5.2] we get

$$x_\gamma^{[n]}y_\gamma^{[n]}v^+ = \binom{\lambda(h_\gamma)}{n}.v^+ = \frac{1}{n!} \prod_{i=0}^{n-1}((\lambda, \gamma^\vee) - i).v^+. \quad (2.3)$$

As $I = \{ \alpha \in \Delta \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \}$, it follows that $\langle \lambda, \gamma^\vee \rangle \notin \mathbb{Z}_{\geq 0}$ and the term on the right of $2.3$ does not vanish for infinitely many $n$. In particular, $y_\gamma^{[n]}v^+ \neq 0$ for infinitely many $n \geq 0$.

Now suppose $ht(\gamma) > 1$. Then we can write $\gamma = \alpha + \beta$ with $\alpha \in \Delta$ and $\beta \in \Phi^+$. Clearly, not both $\alpha$ and $\beta$ can be contained in $\Phi_I$. We distinguish two cases.

(a) Let $\beta - \alpha \notin \Phi$. Then we get for $\alpha \notin I$ by Lemma 2.12

$$x_\beta^{[n]}y_\gamma^{[n]}v^+ = [x_\beta, y_\gamma]^{[n]}v^+$$

where $x_\beta$ is a non-zero element of $g_\beta$. We conclude by induction that $[x_\beta, y_\gamma]^{[n]}v^+ \neq 0$ for infinitely many $n \geq 0$.

For $\alpha \in I$ we have by Lemma 2.12

$$x_\alpha^{[n]}y_\gamma^{[n]}v^+ = [x_\alpha, y_\gamma]^{[n]}v^+.$$ 

where $x_\alpha$ be a non-zero element of $g_\alpha$. Again we conclude by induction the claim. And thus $y_\gamma^{[n]}v^+ \neq 0$ for infinitely many $n \geq 0$. 

(b) Let $\beta - \alpha$ is in $\Phi$. Then we have $\gamma - k\alpha \in \Phi^+$ for $0 \leq k \leq k_0$ (with $k_0 \leq 3$, cf. [Hu, 0.2]), and $\gamma - k\alpha \notin \Phi \cup \{0\}$ for $k > k_0$. This implies $[x_x^{(i)} \cdot y_\gamma] = 0$ for $i > k_0$. By Lemma 2.12 we conclude as in loc.cit.

$x^{[nk_0]}_\alpha y^n_\gamma v^+ = \sum_{i_1 + \ldots + i_n = nk_0} \frac{1}{i_1! \ldots i_n!} [x_x^{(i_1)} \cdot y_\gamma] \cdot \ldots \cdot [x_x^{(i_n)} \cdot y_\gamma] v^+

which can be rewritten as (the corresponding term vanishes if there is one $i_j > k_0$)

$$\frac{1}{(k_0)!} [x_x^{(k_0)} \cdot y_\gamma]^n v^+. $$

Thus we get

$$x^{[nk_0]}_\alpha y^n_\gamma v^+ = \frac{1}{(k_0)!} [x_x^{(k_0)} \cdot y_\gamma]^n v^+.$$

If $\gamma - k_0\alpha$ is not in $\Phi_I$ we are done by induction. Otherwise we necessarily have $\alpha \notin I$. In this case, if we choose some $x_\beta \in g_\beta \setminus \{0\}$ and deduce as in loc.cit that

$$x^{[n]}_\beta y_\gamma^n v^+ = [x_\beta \cdot y_\gamma]^n v^+.$$ 

As we are now in the case of height one, we can thus conclude again.

\[\square\]

**Remark 2.14.** Unfortunately objects in the category $\hat{\mathcal{O}}$ do not have finite length in general. This holds in particular for the local cohomology modules $H^d_{P_1}(\mathbb{P}^d, \mathcal{O})$ as discussed in [Ku]. However in loc.cit. it was pointed out that one can consider composition series of countable length in the sense of Birkhoff [Bi]. In this way one can use similar to the $p$-adic case [OS] the functors $\mathcal{F}_{G}$ for a description of the composition factors of the terms $\text{Ind}^G_{P_{j+1,d-j}}(\tilde{H}^{d-j}_{P_j}(\mathbb{P}^n, \mathcal{E}) \otimes St_{d+1-j})$ appearing in the Theorem of Kuschkowitz.

3. Second approach

This section is inspired by the theory of $\mathcal{D}$-modules. Here we carry out the theory presented in the previous section for the rings of differential operators on the flag variety $X := B_F \setminus G_F$.

Let $D^{(\mathbb{P}^d)}_{\mathbb{P}^d}$ be the space of global sections of the $\mathcal{D}$-module sheaf $D^{(\mathbb{P}^d)}_F$ on the projective variety $\mathbb{P}^d_F$. For a homogeneous vector bundle $\mathcal{E}$ on $\mathbb{P}^d_F$, set

$$D^{\mathcal{E}}_{\mathbb{P}^d} = \mathcal{E}(\mathbb{P}^d) \otimes D^{(\mathbb{P}^d)}_{\mathbb{P}^d} \otimes \mathcal{E}^*(\mathbb{P}^d).$$
Then $D^\mathcal{E}_{F_p}$ acts naturally on $\mathcal{E}(X)$ and the filtration appearing in Kuschkowitz’s theorem. Instead we consider (which become clear later) the space of global sections $D = D_X(X)$ of the differential operators on $X$ and
\[ D^\mathcal{E} = \mathcal{E}(X) \otimes D \otimes \mathcal{E}(X) \]
for any homogeneous vector bundle $\mathcal{E}$ on $B \setminus G$. There is an action of $D^\mathcal{E}$ on all the above objects as well. We consider further the Beilinson-Bernstein homomorphism
\[ \pi^\mathcal{E} : \mathcal{U}(g) \to D^\mathcal{E} \]
which is not surjective (for $\mathcal{E} = \mathcal{O}_X$) in positive characteristic as shown by Smith in [Sm].

Consider the covering $X = \bigcup_{w \in W} B \setminus B U^{-} w$ by translates of the big open cell $B \setminus B U^{-}$. Let $D^1 = D(B \setminus B U^{-})$. Thus $D^1$ is the crystalline Weyl algebra
\[ D^1 = \mathbb{F}[T_{\alpha} | \alpha \in \Phi^-] \langle y_{\alpha}^{[n]} | \alpha \in \Phi^-, n \in \mathbb{N} \rangle. \]

By the sheaf property we see that $D$ coincides with the set
\[ \{ \Theta \in D^1 | \Theta(\mathcal{O}(B \setminus B U^{-} w)) \subset \mathcal{O}(B \setminus B U^{-}) \forall w \}. \]

For any prime power $q = p^n$ we let $D_q^1$ be the differential operators which are $\mathbb{F}[T_{\alpha}^q | \alpha \in \Phi^-]$-linear. Then we have $D = \bigcup_n D_{p^n}$. The next statement is a generalization of [Sm, lemma 3.1]. We set for $\alpha > 0$, $T_{\alpha} := T_{-\alpha}^{-1}$.

**Lemma 3.1.** Let $\Theta \in D_q^1$. Then $\Theta \in D$ iff

i) $\Theta(1) \in \mathbb{F}$

and

ii) $\Theta(\prod_{\alpha \in \Phi^-} T_{\alpha}^{i_{\alpha}}) \in V := \bigoplus_{0 \leq i \leq q} \prod_{\alpha \in \Phi^-} T_{\alpha}^{i_{\alpha}}$ for all tuples $(i_{\alpha})_{\alpha}$ with $0 \leq i_{\alpha} \leq q - 1$.

**Proof.** $\Rightarrow$: The first item follows from the sheaf property (3.1) since $\mathcal{O}(B \setminus G) = \mathbb{F}$. Now let $\Theta \in D \cap D_q^1$. Let $w_0 \in W$ be the longest element and $f = \prod_{\alpha < 0} T_{\alpha}^{i_{\alpha}}$ as above. Then $g = f : \prod_{\alpha > 0} T_{\alpha}^q \in \mathcal{O}(B \setminus B U^{-} w_0)$. But then
\[ \Theta(f) = \prod_{\alpha < 0} T_{\alpha}^q \Theta(g) \in \prod_{\alpha} T_{\alpha}^q \mathcal{O}(B \setminus B U^{-} w_0) \cap \mathcal{O}(B \setminus B U^{-}) \subset V. \]
⇐: We show that \( \Theta(O(B \setminus BU \omega)) \subset O(B \setminus BU \omega) \) \( \forall \omega \in W \). We consider the element \( f = \prod_{\beta \in w(\Phi^-)} T_{\beta}^{i_{\beta}} \in O(B \setminus BU \omega) \). Write
\[
f = \prod_{\beta \in w(\Phi^-)} T_{\beta}^{i_{\beta}} \prod_{\beta \in w(\Phi^-)} T_{\beta}^{i_{\beta}} = \prod_{\beta \in w(\Phi^-)} T_{\beta}^{i_{\beta}} \prod_{\beta \in w(\Phi^-)} T_{-\beta}^{-i_{\beta}}.\]

For each \( \beta > 0 \) let \( m_{\beta} \) be the integer with \( m_{\beta} q < i_{\beta} \leq (m_{\beta} + 1)q \). On the other hand, for each \( \beta < 0 \) let \( m_{\beta} \) be the integer with \( m_{\beta} q \leq i_{\beta} < (m_{\beta} + 1)q \). Then \( \prod_{\beta \in w(\Phi^-)} T_{\beta}^{i_{\beta}} = \prod_{\beta \in w(\Phi^-)} T_{\beta}^{m_{\beta} q} T_{\beta}^{i_{\beta} - m_{\beta} q} \). Putting this together we get by assumption (ii)
\[
\Theta(\prod_{\beta \in w(\Phi^-)} T_{\beta}^{m_{\beta} q} T_{\beta}^{i_{\beta} - m_{\beta} q}) \in V.
\]

Thus \( \Theta(f) \in \prod_{\beta \in w(\Phi^-)} T_{\beta}^{(m_{\beta} q + 1) - i_{\beta}} \prod_{\beta \in w(\Phi^-)} T_{\beta}^{i_{\beta} - m_{\beta} q} \forall \omega \subset O(B \setminus BU \omega) \).

We fix the same setup as in the previous section. I.e. \( P \subset G \) is a parabolic subgroup, \( U_P \) its unipotent radical and \( U^-_{P} \) its opposite unipotent radical. Moreover we have fixed as before lifts \( P \subset G \) etc. inside \( G \). We consider the following subalgebras of \( D \) in terms of generators:
\[
D(P) = \langle T_{\alpha}^{m} \cdot y_{\alpha}^{[n]} | m \leq n \text{ for } y_{\alpha} \in p \cap b, m \geq n \text{ for } L_{-\alpha} \in u \rangle.
\]
\[
D(U) = \langle (T_{\alpha})^{m} \cdot y_{\alpha}^{[n]} | m > n, L_{-\alpha} \in u \rangle.
\]
\[
D(U^-_{P}) = \langle (T_{\alpha})^{m} \cdot y_{\alpha}^{[n]} | m < n, y_{\alpha} \in u \rangle.
\]
\[
D(L_{P}) = \langle (T_{\alpha})^{m} \cdot y_{\alpha}^{[n]} | m \leq n \text{ for } y_{\alpha} \in l_{P} \cap b, m > n \text{ for } L_{-\alpha} \in l_{P} \cap u \rangle.
\]
\[
D(T) = \langle (T_{\alpha})^{m} \cdot y_{\alpha}^{[n]} | m = n, \alpha \in \Delta \rangle.
\]

**Remark 3.2.** i) Note that \( D(T) \) is for \( p \neq 2 \) nothing else but \( \pi^{O_{x}}(\hat{U}(t)) \) as \( T_{\alpha} y_{\alpha} = \pi(2h_{\alpha}) \) for all \( \alpha \in \Delta \). Hence if \( \lambda \in X^{*}(T) \), it induces a \( D(T) \)-module structure on \( \mathbb{F} \) which we denote by \( \mathbb{F}_{\lambda} \).

ii) By Lemma 3.1 one checks that \( D(U_{P}) = \pi^{O_{x}}(\hat{U}(u_{P})) \) since \( T_{\alpha}^{2} y_{\alpha} = \pi(L_{-\alpha}) \forall \alpha \in \Phi^- \).

**Lemma 3.3.** There is for all \( n \in \mathbb{N} \) and \( \alpha \in \Delta \) the identity \( (T_{\alpha} y_{\alpha})^{n} = T_{\alpha}^{n} y_{\alpha}^{[n]} \).

**Proof.** This is left to the reader. \( \square \)
We set $D^\mathcal{E}(P) = \mathcal{E}(X) \otimes D(P) \otimes \mathcal{E}^*(X)$ etc. Then there is a product decomposition $D^\mathcal{E} = D^\mathcal{E}(P) D^\mathcal{E}(U_B \setminus P)$ (an almost PBW-decomposition).

Again we mimic the definition of the category $\mathcal{O}$ in the sense of BGG. Let $\mathcal{O}^P_{D^\mathcal{E}}$ be the category of $D^\mathcal{E}$-modules such that

i) $M$ is finitely generated as a $D^\mathcal{E}$-module

ii) As a $D^\mathcal{E}(L_p)$-module it is a direct sum of finite-dimensional modules.

iii) $D^\mathcal{E}(U_P)$ acts locally finite-dimensional, i.e. for all $m \in M$ the subspace $D^\mathcal{E}(U_P) \cdot v$ is finite-dimensional.

**Remark 3.4.** For $\mathcal{E} = \mathcal{O}_X$ this category corresponds in analogy to the classical case to the principal block.

We define the algebraic part of $\mathcal{O}^P_{D^\mathcal{E}}$ as usual, i.e. we denote by $\mathcal{O}^P_{D^\mathcal{E}, \text{alg}}$ the full subcategory of $\mathcal{O}^P_{D^\mathcal{E}}$ consisting of objects such that the action of $\hat{\mathfrak{u}}(\mathfrak{t})$ on the weight spaces is given by algebraic characters $\lambda \in X^*(T)$. Note that axioms ii) and iii) induce together with the map $\pi^\mathcal{E} : \hat{\mathfrak{u}}(\mathfrak{g}) \to D^\mathcal{E}$ an algebraic $P$-module structure on any object in $\mathcal{O}^P_{D^\mathcal{E}, \text{alg}}$.

As in the classical case we see that the axioms imply the existence of a finite-dimensional $D^\mathcal{E}(P)$-module $N$ which generates $M$ as a $D^\mathcal{E}$-module. Further there are similar definitions. E.g. a vector in an $D^\mathcal{E}$-module $M \in \mathcal{O}_{D^\mathcal{E}}$ is called a maximal vector of weight $\lambda \in t^*$ if $v \in M_\lambda$ and $D^\mathcal{E}(U_P) \cdot v = 0$. A $D^\mathcal{E}$-module $M$ is called a highest weight module of weight $\lambda$ if there is a maximal vector $v \in M_\lambda$ such that $M = D^\mathcal{E} \cdot v$. By the very definition such a module satisfies $M = D^\mathcal{E}(U_B \setminus P) \cdot v$. For a one-dimensional $\hat{\mathfrak{u}}(\mathfrak{t})$-module $\lambda$ we consider it as usual via the trivial $D^\mathcal{E}(U_B)$-action as a one-dimensional $D^\mathcal{E}(B)$-module $\mathbb{F}_\lambda$ and set $M(\lambda) = D^\mathcal{E} \otimes D^\mathcal{E}(B) \mathbb{F}_\lambda$. More generally we may define for every finite-dimensional $D^\mathcal{E}(P)$-module $W$ the generalized Verma module $M(W) = D^\mathcal{E} \otimes_{D(P)} W$. Note that we have surjections $D^\mathcal{E}(U_B) \otimes \mathbb{F}_\lambda \to M(\lambda)$ and $D^\mathcal{E}(U_B \setminus P) \otimes \mathbb{F} W \to M(W)$. We see by the above surjections that [Hu, Thm. 1.3] holds true in our category, i.e. if $M(\lambda) \neq 0$ then it has a unique simple quotient $L(\lambda)$. Moreover these modules form a complete list of simple modules in the "union" of our categories $\mathcal{O}_{D^\mathcal{E}}$. 
Consider the local cohomology module $\tilde{H}_{P_j}^{d-j}(\mathbb{P}^d, \mathcal{O})$. For $d-j \geq 2$ this coincides with the vector space of polynomials
\[
\bigoplus_{n_0, \ldots, n_j \geq 0} \mathbb{F} \cdot X_0^{n_0} \cdots X_j^{n_j} X_{j+1}^{n_{j+1}} \cdots X_d^{n_d}
\]
cf. [O2]. In general there is some finite-dimensional $P_{(j+1,d-j)}$-module $V$ such that $\tilde{H}_{P_j}^{d-j}(\mathbb{P}^d, \mathcal{E})$ is a quotient of $\bigoplus_{n_0, \ldots, n_j \geq 0} \mathbb{F} \cdot X_0^{n_0} \cdots X_j^{n_j} X_{j+1}^{n_{j+1}} \cdots X_d^{n_d} \otimes V$.

**Proposition 3.5.** Let $E$ be a homogeneous vector bundle on $\mathbb{P}^d$. Then $\tilde{H}_{P_j}^{d-j}(\mathbb{P}^d, \mathcal{E})$ is an object of $\mathcal{O}_{D^E}^{P_{(j+1,d-j)}}$.

**Proof.** The non-trivial aspect is to show that $\tilde{H}_{P_j}^{d-j}(\mathbb{P}^d, \mathcal{E})$ is finitely generated. We will show this for $E = \mathcal{O}$. We claim that
\[
\bigoplus_{n_0, \ldots, n_j \geq 0} \mathbb{F} \cdot X_0^{n_0} \cdots X_j^{n_j} X_{j+1}^{n_{j+1}} \cdots X_d^{n_d}
\]
is as in characteristic 0 a generating system of $H_{P_j}^{d-j}(\mathbb{P}^d, \mathcal{O})$. Indeed, as in the latter case we can apply successively the differential operators $L_\alpha \in \mathfrak{u}_{P_{(j+1,d-j)}}$ to obtain all expressions $X_0^{n_0} \cdots X_j^{n_j} X_{j+1}^{n_{j+1}} \cdots X_d^{n_d}$ such that $|n_i| \leq p$ for all $i \geq j + 1$. In order to obtain those where $n_i = -(p+1)$ for some $i \geq j + 1$ we can apply $y_{(-,j+1)}^{[p]}$ to get the desired denominators. However, we do not get all possible nominators. But in our algebra $D$ we have in contrast to $\mathcal{U}(\mathfrak{g})$ the differential operator $T_{(a,b)}^{r-1} L_{(a,b)}^{[p]}$ with $j+1 \leq a < b \leq d$ at our disposal. Applying these operators we can realize all nominators. For $|n_i| > p + 1$ in particular for $|n_i| = rp + 1, r \geq 2$ we use the same method as above etc.

**Proposition 3.6.** The object $\tilde{H}_{P_j}^i(\mathbb{P}^d, \mathcal{O})$ is a simple module isomorphic to $L(s_i \cdots s_1, 0)$.

**Proof.** In characteristic 0 we gave a proof in [OS, Prop. 7.5]. Here we can argue with the differential operators at our disposal in the same way. Note that for general $\lambda \in X^*(T)$ the simple module $L(\lambda)$ is an avatar of the characteristic 0 version.

We let
\[
A^E_G := \mathbb{F}[G] \# D^E
\]
be the smash product of the group algebra $\mathbb{F}[G]$ and $D^E$. 

Let $M$ be an object of $\mathcal{O}^P_{D,\text{alg}}$ and let $V$ be a finite-dimensional $P$-module. Then we set
\[ \mathcal{F}_G^P(M, V) := F[G] \otimes_{F[P]} (M \otimes V). \]
Note that $\mathcal{F}_P^G(M, V) = \text{Ind}_P^G(M \otimes V)$. This is an $A_G$-module. In this way we get a bi-functor
\[ \mathcal{F}_P^G : \mathcal{O}^P_{D,\text{alg}} \times \text{Rep}(P) \to \text{Mod}_{A_G}. \]

The proof of the next statement is the same as in Propositions 2.8 and 2.9.

**Proposition 3.7.** a) The bi-functor $\mathcal{F}_P^G$ is exact in both arguments.
b) If $Q \supset P$ is a parabolic subgroup, $M$ an object of $\mathcal{O}^Q_{D,\text{alg}}$, then
\[ \mathcal{F}_P^G(M, V) = \mathcal{F}_Q^G(M, \text{Ind}_P^Q(V)), \]
where $\text{Ind}_P^Q(V)$ denotes the corresponding induced representation. \bbox

**Theorem 3.8.** Let $M$ be an simple object of $\mathcal{O}^P_{D,\text{alg}}$ such that $P$ is maximal for $M$ and let $V$ be a simple $P$-representation. Then $\mathcal{F}_P^G(M, V)$ is simple as $A_G$-module.

**Proof.** The proof follows the strategy of Theorems 2.10 and 2.11. Note that Proposition 2.13 does also holds true for our objects $L(\lambda)$ as avatars of their character zero versions. \bbox

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