Kirchberg algebras with the same homotopy groups of their automorphism groups

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Abstract

We determine when two unital UCT Kirchberg algebras with finitely generated K-groups have the same homotopy groups of their automorphism groups and reveal a kind of duality between two algebras given by the Spanier–Whitehead duality for \( KK \)-theory.

1 Introduction

Bundles of C*-algebras naturally appear in several areas, including Kasparov’s KK-theory for continuous fields of C*-algebras \[15,5\], the classification of group actions via topological invariants introduced in M. Izumi and H. Matui’s work \[10\], the Dixmier–Douady and the Dadarlat–Pennig theory \[9,8\]. If we restrict ourselves to considering locally trivial bundles over \( X \) whose fiber is a fixed C*-algebra \( A \), they are classified by the homotopy set \([X, \text{BAut}(A)]\) consisting of homotopy classes of continuous maps from \( X \) to the classifying space of the automorphism group \( \text{Aut}(A) \). If two spaces \( Y_1, Y_2 \) are (weakly) homotopy equivalent, their homotopy sets \([X, Y_i]\) are naturally identified. Thus, the homotopy groups \( \pi_n(\text{BAut}(A)) \cong \pi_{n-1}(\text{Aut}(A)) \) are also important objects.

The homotopy set \([X, \text{BAut}(A)]\) gives a cohomology group in the Dadarlat–Pennig theory, and it also gives the topological invariant of group actions in \[10\]. Thus, from the viewpoint of C*-algebra theory, it is an interesting problem to investigate \([X, \text{Aut}(A)]\) and \([X, \text{BAut}(A)]\) using the K-theory of \( X \) and \( A \), and the Kirchberg algebras \( A \) including the Cuntz algebras \( \mathcal{O}_n, \mathcal{O}_\infty \) are appropriate targets to which we can apply powerful K-theoretic tools (see Section 2.6).

The research on the bundles of \( \mathcal{O}_n \) were initiated by M. Dadarlat, and, in \[6\ Sec. 2\], he introduced a crucial idea of focusing on the Puppe sequences appearing in the construction of bundles of \( \mathcal{O}_n \) as the Cuntz–Pimsner algebras. In our previous work, the above idea and the Dadarlat–Pennig theory lead us to \[21\ Thm. 3.8\] which gives a natural bijection

\[ [X, \text{BAut}(\mathcal{O}_n)] \rightarrow [X, \text{BAut}(\mathcal{M}_{n-1}(\mathcal{O}_\infty))]. \]
This bijection implies that two different unital Kirchberg algebras \( O_n \) and \( M_{n-1}(O_\infty) \) have the same homotopy groups of their automorphism groups:

\[ \pi_i(\text{Aut}(O_n)) \cong \pi_i(\text{Aut}(M_{n-1}(O_\infty))). \]

Thus, it is natural to ask the following question.

**Question 1.1.**

1. When two unital Kirchberg algebras \((A, B)\) have the same homotopy groups of their automorphism groups?

2. Can we construct a natural bijection \([X, B\text{Aut}(A)] \to [X, B\text{Aut}(B)]\) for the pair \((A, B)\) sharing the same homotopy groups of their automorphism groups?

In this paper, we answer the first question in the case that K-groups of Kirchberg algebras are finitely generated.

Let \( A \) and \( B \) be unital UCT Kirchberg algebras with finitely generated K-groups. In [7], M. Dadarlat computes the groups \([X, \text{Aut}(A)]\) using the mapping cone \( C_u A \) of the unital map \( u_A : C \to A \), and obtains the group isomorphisms

\[ \pi_i(\text{Aut}(A)) \cong KK^{i+1}(C_{u_A}, A), \quad i \geq 1, \]

for every unital Kirchberg algebra \( A \) (see [7, Cor. 5.10.]). One can rewrite the above formula using the Spanier–Whitehead duality for \( KK\)-theory established in [13, 11, 12] by D. S. Kahn, J. Kaminker, I. Putnam, C. Schoch et.

For a separable nuclear UCT C*-algebra \( A \) with finitely generated K-groups, one can find another separable nuclear UCT C*-algebra \( D(A) \) uniquely up to \( KK\)-equivalence, which is called the Spanier–Whitehead K-dual of \( A \). The algebra \( D(A) \) is characterized by \( KK^i(A, C) \cong KK^i(C, D(A)) \), and this is a duality between the covariant and contravariant inputs of the \( KK\)-theory \( KK(-, -) \).

The K-duality shows

\[ \pi_i(\text{Aut}(A)) = K^{i+1}(C_{u_A} \otimes A), \quad i \geq 1, \]

and the condition \( \pi_i(\text{Aut}(A)) \cong \pi_i(\text{Aut}(B)), i \geq 1 \) is equivalent to

\[ D(C_{u_A}) \otimes A \sim_K D(C_{u_B}) \otimes B. \]

This \( KK\)-equivalence places restrictions on the choices of \((A, B)\), and we obtain the following main result.

**Theorem 1.2** (Thm. 3.1). Let \( A \) and \( B \) be two unital UCT Kirchberg algebras with finitely generated K-groups. We have the following:

1. For every \( A \), there uniquely exists \( B \) satisfying

\[ D(C_{u_A}) \sim_{KK} B, \quad D(C_{u_B}) \sim_{KK} A, \]

and such \( B \) is non-isomorphic to \( A \).
2. For $A \not\sim B$, one has $\pi_i(\text{Aut}(A)) \cong \pi_i(\text{Aut}(B))$, $i \geq 1$ if and only if the pair $(A, B)$ satisfies the above $KK$-equivalence relations.

3. For $(A, B)$ in 2 and a compact metrizable space $X$, there exists a natural anti-isomorphism of groups

$$[X, \text{Aut}(A)] \to [X, \text{Aut}(B)].$$

For the Puppe sequence $C_uA \to \mathbb{C} \xrightarrow{u_A} A$, the K-duality gives another sequence $D(C_uA) \leftarrow D(\mathbb{C}) \leftarrow D(A)$ which is the origin of $B \xleftarrow{u_B} \mathbb{C} \leftarrow C_uB$. This construction can be generalized for $C(X)$-algebras’ setting if every $C(X)$-algebra admits the K-duality (see Proposition 4.4).

**Question 1.3.** Does the Spanier–Whitehead K-duality hold for appropriate $C(X)$-algebras via the Kasparov’s parametrized $KK$-groups $KK_X(\cdot, \cdot)$?

This question is at present far from being solved. However, the Dadarlat–Pennig theory solves this problem for locally trivial continuous $C(X)$-algebras whose fibers are $KK$-equivalent to $\mathbb{C}$. For a $C(X)$-algebra $A$ with fiber $O_\infty \otimes \mathbb{K}$, the Dadarlat–Pennig theory provides another $C(X)$-algebra $A^{-1}$ with the same fiber satisfying $A \otimes C(X) \cong A^{-1} \cong C(X) \otimes O_\infty \otimes \mathbb{K}$, which implies $KK_X(A, C(X)) \cong KK_X(C(X), A^{-1})$ (see [19] Sec. 3.2.), and $A$ and $A^{-1}$ are K-dual. Using this result, we also generalize [21], and obtain new examples of the pairs of unital Kirchberg algebras $(A, B)$ that admit a natural bijection $[X, B\text{Aut}(A)] \to [X, B\text{Aut}(B)]$ (see Corollary 4.5).

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**2 Preliminaries**

**2.1 Notation**

Let $A$ be a unital C*-algebra with the unit $1_A$ and map $u_A : \mathbb{C} \ni \lambda \mapsto \lambda 1_A \in A$. Let $SA$ denote the suspension $C_0(0, 1) \otimes A$, and we write $S^iA := S^i \otimes A$ for short. Let $\mathbb{K}$ denote the algebra of compact operators on the separable infinite dimensional Hilbert space, and we denote by $M_n$ the $n$ by $n$ matrix algebra. For a *-homomorphism $\varphi : A \to B$, we denote by $C_\varphi$ the mapping cone $\{(f, a) \in$
(C₀(0, 1) ⊗ B) ⊕ A \mid \varphi(a) = f(1)), and the following sequence is called the Puppe sequence (see [11 Sec. 19])

\[ SA \xrightarrow{id_\mathbb{S} \otimes \varphi} SB \to C_\varphi \to A \not\to B. \]

We denote by \( KK(\varphi) \in KK(A, B) \) the Kasparov module represented by the \(*\)-homomorphism, and the Kasparov product of two elements \( KK(\varphi) \in KK(A, B) \), \( KK(\psi) \in KK(B, C) \) is denoted by \( KK(\varphi) \otimes KK(\psi) = KK(\psi \circ \varphi) \) (see [11 Sec. 18]). We write \( I_A := KK(id_A) \in KK(A, A) \).

In this paper, we assume that the space \( X \) is compact metrizable and of finite covering dimension, for example, a finite CW complex. Let \( C(X) \) be the \( C^*\)-algebra of continuous functions on \( X \), and let \( C_0(X, Y) \) be the ideal of functions vanishing on the closed subset \( Y \subset X \). For a unital \( C(X)\)-algebra \( \mathcal{A} \), we denote by \( u_\mathcal{A} : C(X) \to \mathcal{A} \) the unital \(*\)-homomorphism which defines the \( C(X)\)-linear structure of the \( C^*\)-algebra. Since \( C_0(X, Y) \mathcal{A} = \{ u_\mathcal{A}(f) a \in \mathcal{A} \mid f \in C_0(X, Y), a \in \mathcal{A} \} \) is a closed ideal of \( \mathcal{A} \) by Cohen’s factorization (see [11 Th. 4.6.4]), the algebra \( \mathcal{A}(Y) := \mathcal{A}/C_0(X, Y) \mathcal{A} \) is a \( C(Y)\)-algebra with the quotient map \( \pi_Y : \mathcal{A} \to \mathcal{A}(Y) \).

Let \([X, Y]_1\) be the set of homotopy classes of continuous maps from \( X \) to \( Y \), and let \([X, Y]_0\) be the homotopy set of the maps between the pointed spaces concerning the base point preserving homotopy. The \( i \)-th homotopy group of a space \( Y \) is denoted by \( \pi_i(Y) := [S^i, Y]_0 \), where \( S^i \) denotes the \( i \)-dimensional sphere. For a unital \( C^*\)-algebra \( \mathcal{A} \), we denote by \( \text{Aut}(\mathcal{A}) \) (resp. \( \text{End}(\mathcal{A}) \)) the set of automorphisms (resp. unital endomorphisms) equipped with the point-norm topology, and the homotopy sets \([X, \text{Aut}(\mathcal{A})]\) and \([X, \text{End}(\mathcal{A})]\) have the natural semi-group structures.

### 2.2 Bundles of \( C^*\)-algebras and Dadarlat–Pennig theory

A \( C(X)\)-algebra such that the map

\[ \chi \ni x \mapsto \| \pi_x(a) \|_{\mathcal{A}_x} \in \mathbb{R}, \quad a \in \mathcal{A} \]

is continuous is called continuous \( C(X)\)-algebra. The continuous \( C(X)\)-algebra \( \mathcal{A} \) is called locally trivial if there exists a closed neighborhood \( Y \) for every \( x \in \chi \) and a \( C(Y)\)-linear isomorphism \( \mathcal{A}(Y) \cong C(Y) \otimes \mathcal{A} \) for a \( C^*\)-algebra \( \mathcal{A} \). We always assume that \( \mathcal{A} \) is separable nuclear, and one can take the tensor product \( \mathcal{A} \otimes_{C(X)} \mathcal{B} \) for two nuclear continuous \( C(X)\)-algebras (see [2]). One can identify locally trivial continuous \( C(X)\)-algebras with fiber \( \mathcal{A} \) and locally trivial principal \( \text{Aut}(\mathcal{A})\)-bundles by the following elementary fact.
Proposition 2.1. Let $X$ be a compact metrizable space, and let $A$ be a $C^*$-algebra. For every locally trivial continuous $C(X)$-algebra $A$ with fiber $A$, there exists a principal $\text{Aut}(A)$ bundle $P \to X$ such that the section algebra of the associated bundle, denoted by $\Gamma(X, P \times_{\text{Aut}(A)} A)$, is $C(X)$-linearly isomorphic to $A$. Using the above correspondence, the set of isomorphism classes of principal $\text{Aut}(A)$-bundles over $X$ is identified with the set of $C(X)$-linear isomorphism classes of locally trivial continuous $C(X)$-algebras with fiber $A$.

Recall that the isomorphism class of a principal $\text{Aut}(A)$-bundle $P \to X$ is determined by the homotopy class of the classifying map $f_P$ making the following diagram commute:

$$
\begin{array}{ccc}
P & \longrightarrow & \text{EAut}(A) \\
\downarrow & & \downarrow \\
X & \longrightarrow & \text{BAut}(A).
\end{array}
$$

Therefore, the set of isomorphism classes of locally trivial continuous $C(X)$-algebras with fiber $A$ is equal to $[X, \text{BAut}(A)]$. Thus, we denote by $[A] \in [X, \text{BAut}(A)]$ the isomorphism class of a locally trivial continuous $C(X)$-algebra $A$ with fiber $A$.

In the case of $A$ is a stabilized strongly self-absorbing $C^*$-algebra introduced in [22], M. Dadarlat and U. Pennig reveal a surprising structure of $[X, \text{BAut}(A)]$.

Theorem 2.2 ([8, Thm. 3.8, Cor. 4.5]). For every strongly self-absorbing $C^*$-algebra $D$, the group $\text{Aut}(D \otimes \mathbb{K})$ is an infinite loop space providing a generalized cohomology $E_D^1$. In particular, the homotopy set

$$
E_D^1(X) = [X, \text{BAut}(D \otimes \mathbb{K})]
$$

has a commutative group structure defined by the tensor product of locally trivial continuous $C(X)$-algebras with fiber $D \otimes \mathbb{K}$.

The infinite Cuntz algebra $O_\infty$ is a typical strongly self-absorbing Kirchberg algebra with the following K-groups:

$$(K_0(O_\infty), [1_{O_\infty}], K_1(O_\infty)) \cong (\mathbb{Z}, 1, 0).$$

For every locally trivial continuous $C(X)$-algebra $A$ with fiber $O_\infty \otimes \mathbb{K}$, there exists another one $A^{-1}$, and we have a $C(X)\otimes O_\infty \otimes \mathbb{K} \cong A \otimes_{C(X)} A^{-1}$ (i.e., $[A] + [A^{-1}] = 0 \in E_{O_\infty}^1(X)$).

2.3 $KK_X$-groups

We review some of the basic facts on $KK_X$-groups. All $C^*$-algebras in this paper are ungraded. As in the usual $KK$-theory, one has the natural maps

$$\otimes \mathcal{I}_C : KK_X(A, B) \ni KK(\phi) \mapsto KK(\phi \otimes \text{id}_C) \in KK_X(A \otimes_{C(X)} C, B \otimes_{C(X)} C),$$

where $C$ is a $C^*$-algebra with the trivial $C(X)$-action.
$I_C \otimes - : KK_X(A, B) \ni KK(\phi) \mapsto KK(id_C \otimes \phi) \in KK_X(C \otimes_{C(X)} A, C \otimes_{C(X)} B)$, and the Kasparov product is denoted by

$$- \hat{\otimes} - : KK_X(A, B) \times KK_X(B, C) \to KK_X(A, C).$$

For the isomorphisms $\sigma : A \otimes_{C(X)} C \to C \otimes_{C(X)} A$ and $\theta : B \otimes_{C(X)} C \to C \otimes_{C(X)} B$ exchanging two tensor factors, one has $\sigma \hat{\otimes} (I_C \otimes -) \hat{\otimes} \theta^{-1} = - \otimes I_C$. In particular, the following equations hold for $a \in KK_X(A, C(X))$ and $b \in KK_X(C(X), B)$:

$$\sigma \hat{\otimes} (I_C \otimes a) = a \otimes I_C \in KK_X(A \otimes_{C(X)} C, C), \quad (I_C \otimes b) \hat{\otimes} \theta^{-1} = b \otimes I_C \in KK_X(C, B \otimes_{C(X)} C).$$

In this paper, the Kasparov products are computed categorically, and the following commutativity is most important. For $\alpha \in KK_X(A, B)$, $\gamma \in KK_X(C, D)$, one has

$$(\alpha \otimes I_C) \hat{\otimes} (I_B \otimes \gamma) = (I_A \otimes \gamma) \hat{\otimes} (\alpha \otimes I_D) \in KK_X(A \otimes_{C(X)} C, B \otimes_{C(X)} D)$$

by [15 Thm. 2.14. 8], Prop. 2.21.] that is quite obvious if $\alpha, \gamma$ are given by the homomorphisms.

For the Puppe sequence

$$SA \xrightarrow{I_S \otimes \varphi} SB \xrightarrow{\nu} C_\varphi \xrightarrow{\varepsilon} A \xrightarrow{\varphi} B,$$

one has the following exact sequences (see [17 Sec. 2])

$$KK_X(C, SA) \to KK_X(C, SB) \to KK_X(C, C_\varphi) \to KK_X(C, A) \to KK_X(C, B),$$

$$KK_X(B, C) \to KK_X(A, C) \to KK_X(C_\varphi, C) \to KK_X(SB, C) \to KK_X(SA, C).$$

The definition of the $KK_X$-equivalence is the same as in the usual $KK$-theory and we denote by $\sim_{KK_X}$ the equivalence relation. We denote by $KK_X(A, B)^{-1}$ the set of $KK_X$-equivalences. Since $KK_X$ is contravariant with respect to $X$ (see [15 Prop. 2.20.]), one has the evaluation map

$$KK_X(A, B) \ni \sigma \mapsto \sigma_x \in KK(A_x, B_x), \quad x \in X,$$

and M. Dadarlat characterizes the $KK_X$-equivalence using this map.

**Theorem 2.3** ([5 Thm. 1.1, Thm. 2.7]). Let $A, B$ be separable nuclear continuous $C(X)$-algebras, where $X$ is a compact metrizable space of finite covering dimension. Then, $\sigma \in KK_X(A, B)$ is a $KK_X$-equivalence if and only if $\sigma_x \in KK(A_x, B_x)^{-1}$ for every $x \in X$.

If the fibers $A_x, B_x$ are Kirchberg algebras, there is a $C(X)$-linear isomorphisms

$$\phi : A \otimes K \to B \otimes K$$

with $\sigma = KK(\phi)$. 

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For a given separable nuclear C*-algebra \( A \), there is a unital Kirchberg algebra \( \text{KK} \)-equivalent to \( A \). By [5, Proof of Thm. 2.5, Proof of Lem. 2.2.], one can easily check a similar statement for locally trivial continuous \( C(X) \)-algebras.

**Theorem 2.4** ([5 Thm. 2.5.]). Let \( X \) be a compact metrizable space of finite covering dimension. Let \( A \) be a locally trivial continuous \( C(X) \)-algebra with fiber a separable nuclear C*-algebra \( A \). Then, there exists a unital locally trivial continuous \( C(X) \)-algebra \( A^\sharp \) satisfying \( A \sim_{\text{KK}} A^\sharp \), and the fiber \( A^\sharp_x(\sim_{\text{KK}} A) \) is a unital Kirchberg algebra.

### 2.4 Spanier–Whitehead K-duality

We recall the Spanier–Whitehead K-duality formulated in [12].

**Definition 2.5** ([12 Def. 2.1.]). Two \( C(X) \)-algebras \( A \) and \( B \) are said to have duality classes if there exist

\[
\mu \in KK_X(C(X), A \otimes C(X) B), \quad \nu \in KK_X(B \otimes C(X) A, C(X))
\]

satisfying

\[
(\mu \otimes I_A) \hat{\otimes} (I_A \otimes \nu) = I_A,
\]

\[
(I_B \otimes \mu) \hat{\otimes} (\nu \otimes I_B) = I_B.
\]

Then, \( A \) and \( B \) are said to be Spanier–Whitehead K-dual with duality classes \((\mu, \nu)\).

**Proposition 2.6** ([12 Thm. 2.6.]). The above definition is symmetric for \( A \) and \( B \). More precisely, using the isomorphism

\[
\sigma : A \otimes C(X) B \ni a \otimes b \mapsto b \otimes a \in B \otimes C(X) A,
\]

\( B \) and \( A \) are also K-dual with duality classes \((\mu \hat{\otimes} \sigma, \sigma \hat{\otimes} \nu)\).

**Proof.** Using the isomorphisms \( \theta_{B,BA} : B \otimes C(X) (B \otimes C(X) A) \to (B \otimes C(X) A) \otimes C(X) \)

\( B \) and \( \theta_{B,AB} : B \otimes C(X) (A \otimes C(X) B) \to (A \otimes C(X) B) \otimes C(X) B \) exchanging tensor factors, one can obtain the following equations

\[
(\mu \hat{\otimes} \sigma) \otimes I_B = (I_B \otimes (\mu \hat{\otimes} \sigma)) \hat{\otimes} \theta_{B,BA},
\]

\[
I_B \otimes (\sigma \hat{\otimes} \nu) = \theta_{B,AB} \hat{\otimes} ((\sigma \hat{\otimes} \nu) \otimes I_B).
\]

It is straightforward to check

\[
((\mu \hat{\otimes} \sigma) \otimes I_B) \hat{\otimes} (I_B \otimes (\sigma \hat{\otimes} \nu)) = (I_B \otimes \mu) \hat{\otimes} (\nu \otimes I_B) = I_B,
\]

and the other equation for the duality classes is proved similarly. \( \square \)
The following results which are obtained by a similar argument as in [12, Proof of Thm. 2.2] are known for the specialists, but we write their proofs for the convenience of readers.

**Lemma 2.7.** Let $A$ and $B$ be $C(X)$-algebras with two elements

$$\mu \in KK_X(C(X), A \otimes_{C(X)} B), \quad \nu \in KK_X(B \otimes_{C(X)} A, C(X)).$$

1. If the above elements satisfy

$$(\mu \otimes I_A) \hat{\otimes} (I_A \otimes \nu) = \alpha \in KK_X(A, A)^{-1},$$

$$(I_B \otimes \mu) \hat{\otimes} (\nu \otimes I_B) = \beta \in KK_X(B, B)^{-1},$$

then $A$ and $B$ are K-dual with duality classes $(\mu \hat{\otimes} (\alpha^{-1} \otimes I_B), \nu)$.

2. Assume that the both $(\mu, \nu)$ and $(\mu', \nu')$ provide duality classes for $A$ and $B$. Then, they give invertible elements

$$\alpha := (\mu \otimes I_A) \hat{\otimes} (I_A \otimes \nu'), \quad \alpha' := (\mu' \otimes I_A) \hat{\otimes} (I_A \otimes \nu) \in KK_X(A, A)^{-1},$$

$$\beta := (I_B \otimes \mu) \hat{\otimes} (\nu' \otimes I_B), \quad \beta' := (I_B \otimes \mu') \hat{\otimes} (\nu \otimes I_B) \in KK_X(B, B)^{-1}$$

satisfying

$$\mu = \mu' \hat{\otimes} (\alpha \otimes I_B), \quad \nu = (I_B \otimes \alpha') \hat{\otimes} \nu',$$

$$\mu = \mu' \hat{\otimes} (I_A \otimes \beta), \quad \nu = (\beta' \otimes I_A) \hat{\otimes} \nu'.$$

3. Let $A$ and $B_i$ be K-dual with duality classes $(\mu_i, \nu_i)$ for $i = 1, 2$. Then, we have $\gamma := (I_{B_2} \otimes \mu_1) \hat{\otimes} (\nu_2 \otimes I_{B_1}) \in KK_X(B_2, B_1)^{-1}$ (i.e., $B_1 \sim_{KK_X} B_2$) with $\gamma^{-1} = (I_{B_1} \otimes \mu_2) \hat{\otimes} (\nu_1 \otimes I_{B_2})$. In particular, the K-dual is uniquely determined up to KK$_X$-equivalence.

4. Let $A$ and $B$ be $C(X)$-algebras with duality classes $(\mu, \nu)$, and let $\varphi : A \cong \tilde{A}$ and $\beta : B \cong \tilde{B}$ be KK$_X$-equivalences. Then, the both of $(\mu \hat{\otimes} (\alpha \otimes I_B), (I_B \otimes \alpha^{-1}) \hat{\otimes} \nu)$ and $(\mu \hat{\otimes} (I_A \otimes \beta), (\beta^{-1} \otimes I_A) \hat{\otimes} \nu)$ are duality classes.

**Proof.** First, we prove 1. Let $\mu \hat{\otimes} (\alpha^{-1} \otimes I_B)$ denote by $\mu_\alpha$, then one has

$$I_A = (\mu \otimes I_A) \hat{\otimes} (I_A \otimes \nu) \hat{\otimes} \alpha^{-1},$$

$$= (\mu \otimes I_A) \hat{\otimes} (I_A \otimes \nu) \hat{\otimes} (\alpha^{-1} \otimes I_{C(X)}),$$

$$= (\mu \otimes I_A) \hat{\otimes} (\alpha^{-1} \otimes I_{B \otimes_{C(X)} A}) \hat{\otimes} (I_A \otimes \nu),$$

$$= (\mu_\alpha \otimes I_A) \hat{\otimes} (I_A \otimes \nu),$$

and direct computation yields

$$\beta \hat{\otimes} (I_B \otimes \mu_\alpha) \hat{\otimes} (\nu \otimes I_B) = (\beta \otimes I_{C(X)}) \hat{\otimes} (I_B \otimes \mu_\alpha) \hat{\otimes} (\nu \otimes I_B),$$

$$= (I_B \otimes \mu_\alpha) \hat{\otimes} (\beta \otimes I_{A \otimes_{C(X)} B}) \hat{\otimes} (\nu \otimes I_B),$$

$$= (I_B \otimes \mu) \hat{\otimes} (I_B \otimes \alpha^{-1} \otimes I_B) \hat{\otimes} (((\beta \otimes I_A) \hat{\otimes} \nu) \otimes I_B).$$

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We also have
\[(\beta \otimes I_A) \otimes I_B = (I_B \otimes I_C) \otimes (\nu \otimes \mu \otimes I_A) \otimes (I_A \otimes I_B) \otimes (I_B \otimes I_C) \otimes (I_C \otimes I_D) \otimes (I_D \otimes I_E) \otimes (I_E \otimes I_F) \]
and this yields
\[\beta \otimes \mu \otimes \nu = \beta \]
which proves 1.

The proof of 2 is done by a similar computation. So we only show
\[\alpha^{-1} = \alpha',\]
and it is enough to check
\[\mu \hat{\otimes} \nu,\]
the direct computation yields
\[\alpha \hat{\otimes} \alpha' = \mu \otimes I_A \otimes I_B \otimes I_C \otimes I_D \otimes I_E \otimes I_F \]
\[= \mu \otimes I_A \otimes I_B \otimes I_C \otimes I_D \otimes I_E \otimes I_F \]
\[= I_A.\]

Statement 3 is shown by the same computation, and 4 follows by definition. □

From the above lemma and its proof, one can easily show the following.

**Corollary 2.8** ([12, Cor. 2.3]). Let \(A\) and \(B\) be Spanier–Whitehead K-dual with the duality classes \((\mu, \nu)\). Then, the map
\[\mu \hat{\otimes} (\nu \otimes I_B) : KK_X(A, C(X)) \to KK_X(C(X), B)\]
is an isomorphism with the inverse map \((- \otimes I_A) \hat{\otimes} \nu).

**Corollary 2.9.** Let \(A\) be a locally trivial continuous \(C(X)\)-algebra with fiber \(K \otimes O_\infty\), and let \(A^{-1}\) be its inverse in \(E^0_{\infty}(X)\) (i.e., \(A \otimes C(X), A^{-1} \cong C(X) \otimes K \otimes O_\infty\)). Then, \(A\) and \(A^{-1}\) are Spanier–Whitehead K-dual.

We will discuss the Spanier–Whitehead K-duality for \(C(X)\)-algebras in Section 4, and we restrict ourselves to the case \(X = pt\) in the remainder of this section and Section 3. Not all \(C(X)\)-algebras admit the Spanier–Whitehead K-duality, but we have the following fact in the case \(X = pt\).

**Theorem 2.10** ([12, Thm. 3.1, 6.2], [13, Sec. 4A]). Let \(A\) be an arbitrary separable nuclear UCT C*-algebra. One has a separable nuclear UCT C*-algebra \(D(A)\) such that \(A\) and \(D(A)\) are Spanier–Whitehead K-dual if and only if \(A\) has finitely generated K-groups.
Lemma 2.12 ([12] Thm. 2.2, Cor. 2.3]). Let $A$ (resp. $B$ and $C$) be $C^*$-algebras with the duality classes

$$\mu_A \in KK(\mathbb{C}, A \otimes D(A)), \quad \nu_A \in KK(D(A) \otimes A, \mathbb{C}).$$

The map $d^{X}_{\mu_A,\nu_B} : KK(A, B \otimes C(X)) \to KK(D(B), C(X) \otimes D(A))$ defined by

$$d^{X}_{\mu_A,\nu_B}(x) := (I_{D(B)} \otimes \mu_A) \hat{\otimes} (I_{D(B)} \otimes x \otimes I_{D(A)}) \hat{\otimes} (\nu_B \otimes I_{C(X) \otimes D(A)})$$

is an isomorphism satisfying

$$d^{X \times Y}_{\mu_A,\nu_C}(y \hat{\otimes} (x \otimes I_{C(Y)})) = d^{X}_{\mu_B,\nu_C}(x) \hat{\otimes} (I_{C(X)} \otimes \mu_B \hat{\otimes} \nu_D(y))$$

for $y \in KK(A, B \otimes C(Y))$ and $x \in KK(B, C \otimes C(X))$.

In particular, the map

$$d_{\mu_A,\nu_A} := d^{\text{pt}}_{\mu_A,\nu_A} : KK(A, A) \to KK(D(A), D(A))$$

is an anti-isomorphism of the rings.

Proof. By a similar computation as in Lemma [2,7] the inverse map of $d^{X}_{\mu_A,\nu_B}$ is given by

$$KK(D(B), C(X) \otimes D(A)) \ni z \mapsto (\mu_B \otimes I_A) \hat{\otimes} (I_B \otimes z \otimes I_A) \hat{\otimes} (I_{B \otimes C(X)} \otimes \nu_A) \in KK(A, B \otimes C(X)).$$

We show the equation

$$d^{X \times Y}_{\mu_A,\nu_C}(y \hat{\otimes} (x \otimes I_{C(Y)})) = d^{X}_{\mu_B,\nu_C}(x) \hat{\otimes} (I_{C(X)} \otimes \mu_B \hat{\otimes} \nu_D(y)).$$

By $(\mu_B \otimes I_B) \hat{\otimes} (I_B \otimes \nu_B) = I_B$ and the commutativity of $\hat{\otimes}$, we have

$$y \hat{\otimes} (x \otimes I_{C(Y)}) = (\mu_B \otimes I_A) \hat{\otimes} (I_B \otimes D(B) \otimes (x \otimes I_D(B) \otimes I_{C(Y)})) \hat{\otimes} (I_{C(X)} \otimes \nu_B \otimes I_{C(Y)})$$

$$= (\mu_B \otimes I_A) \hat{\otimes} (x \otimes I_D(B) \otimes A) \hat{\otimes} (I_{C(X)} \otimes D(B) \otimes y) \hat{\otimes} (I_{C(X)} \otimes \nu_B \otimes I_{C(Y)}).$$

We write $Z := I_{D(C)} \otimes ((\mu_B \otimes I_A) \hat{\otimes} (x \otimes I_{D(B) \otimes A})) \otimes I_{D(A)}$, and direct computation yields

$$(I_{D(C)} \otimes \mu_A) \hat{\otimes} Z = (I_{D(C)} \otimes \mu_A) \hat{\otimes} (I_{D(C)} \otimes \mu_B \otimes I_A \otimes I_{D(A)}) \hat{\otimes} (I_{D(C)} \otimes x \otimes I_{D(B) \otimes A \otimes D(A)})$$

$$= (I_{D(C)} \otimes \mu_B) \hat{\otimes} (I_{D(C)} \otimes B \otimes I_{D(B) \otimes A} \otimes \mu_A) \hat{\otimes} (I_{D(C)} \otimes x \otimes I_{D(B) \otimes A \otimes D(A)})$$

$$= (I_{D(C)} \otimes \mu_B) \hat{\otimes} (I_{D(C)} \otimes x \otimes I_{D(B)}) \hat{\otimes} (I_{D(C)} \otimes C \otimes I_{C(X)} \otimes D(B) \otimes \mu_A).$$
Similarly, one can check

\[(I_D(C) \otimes CC(X) \otimes D(B) \otimes y \otimes I_D(A)) \cong (I_D(C) \otimes CC(X) \otimes I_C(Y) \otimes D(A)) \cong \nu_C \otimes I_C(X \otimes Y) \otimes D(A) \]

and the commutativity shows

\[(I_D(C) \otimes CC(X) \otimes D(B) \otimes A_0 \otimes D(A)) \cong (I_D(C) \otimes (I_D(B) \otimes y \otimes I_D(A)) \cong \nu_B \otimes I_C(Y) \otimes D(A)) =: W,
\]

Finally, we get

\[d_{\mu_A,\nu_C}^X \otimes y(x \otimes I_C(Y)) = (I_D(C) \otimes \mu_A) \otimes Z \otimes Y \]

\[= d_{\mu_B,\nu_C}^X (x) \otimes (I_C(X) \otimes d_{\mu_A,\nu_B}^Y (y))\]

\[\square\]

The map \(d_{\mu_A,\nu_B}^X\) is the composition of two maps

\[KK(A, B \otimes C(X)) \ni x \mapsto (I_D(B) \otimes x) \otimes (\nu_B \otimes I_C(X)) \in KK(D(B) \otimes A, C(X)),\]

\[KK(D(B) \otimes A, C(X)) \ni y \mapsto (I_D(B) \otimes \mu_A) \otimes (y \otimes I_D(A)) \in KK(D(B), C(X) \otimes D(A)).\]

So \(d_{\mu_A,\nu_B}^X\) is natural with respect to \(X\).

**Lemma 2.13.** Let \(A, B\) and \(C\) be \(C^*\)-algebras as in Lemma 2.12. Assume

\[KK(C, S^i) \xrightarrow{h \otimes} KK(B, S^i) \xrightarrow{f \otimes} KK(A, S^i) \quad i = 0, 1,\]

are exact for \(f \in KK(A, B), \ h \in KK(B, C)\). Then, the following sequences are exact

\[K_i(D(C)) \xrightarrow{d_{\mu_C,\nu_S}^i(h)} K_i(D(B)) \xrightarrow{d_{\mu_A,\nu_B}^i(f)} K_i(D(A)) \quad i = 0, 1.\]

**Proof.** We identify \(K_i(-)\) with \(KK(S^i, -)\) and choose duality classes

\[\mu_S, \in KK(C, S^i \otimes S^i), \ \nu_S, \in KK(S^i \otimes S^i, C)\]

By Lemma 2.12 every element of \(KK(S^i, D(C))\) is of the form \(d_{\mu_C,\nu_S}^i(c), \ c \in KK(C, S^i)\), and one has

\[d_{\mu_C,\nu_S}^i(c) \otimes d_{\mu_B,\nu_C}^i(h) \otimes d_{\mu_A,\nu_B}^i(f) = d_{\mu_A,\nu_S}^i(f \otimes h \otimes c) = 0\]

by the assumption of \(f\) and \(h\).

Suppose that \(b \in KK(B, S^i)\) satisfies

\[0 = d_{\mu_B,\nu_S}^i(b) \otimes d_{\mu_A,\nu_B}^i(f) = d_{\mu_A,\nu_S}^i(f \otimes b).
\]

Since the map \(d_{\mu_A,\nu_S}^i\) is injective, one has \(f \otimes b = 0\). By assumption, there is \(c \in KK(C, S^i)\) satisfying \(b = h \otimes c\), which implies

\[d_{\mu_B,\nu_S}^i(b) = d_{\mu_C,\nu_S}^i(c) \otimes d_{\mu_B,\nu_C}(h).
\]

This proves the statement.\[\square\]

11
Recall the mapping cone sequence (see [1, Thm. 19.4.3.])

\[
\begin{array}{ccccccccc}
SSA & \overset{I_S \otimes \iota_A}{\longrightarrow} & SCu_A & \overset{I_S \otimes e_A}{\longrightarrow} & SC & \overset{e_A}{\longrightarrow} & C & \overset{\mu_S}{\longrightarrow} & \mu_S \otimes I_A \\
& & & & & & & & \downarrow \\
& & & & & & & & SSA.
\end{array}
\]

Applying Lemma 2.13 to

\[
S \rightarrow SA \rightarrow C_{u_A} \overset{e_A}{\longrightarrow} \mathbb{C} \overset{u_A}{\longrightarrow} A \overset{\mu_S}{\longrightarrow} \mu_S \otimes I_A \\
\downarrow \downarrow \\
SS \overset{I_S \otimes u_A}{\longrightarrow} SSA.
\]

one has an exact sequence

\[
0 \leftarrow K_1(D(A)) \leftarrow K_0(D(C_{u_A})) \leftarrow \mathbb{Z} \leftarrow K_0(D(A)) \leftarrow K_1(D(C_{u_A})) \leftarrow 0,
\]

and there is a unital UCT Kirchberg algebra \(B \sim_{KK} D(C_{u_A})\) making the following diagram commute (see [18, Sec.4.3.])

\[
\begin{array}{cccccc}
K_0(D(C_{u_A})) & \longrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow \\
K_0(B) & \overset{(u_B)^*}{\longrightarrow} & K_0(\mathbb{C}).
\end{array}
\]

Then, the equations \(K_i(C_{u_B}) \cong K_i(D(A))\) automatically hold. Since the rank of the free part of \(K_1(A)\) is equal to that of \(K_1(D(A))\), one can observe \(A \not\sim_{KK} B\) by comparing ranks of the free parts of their K-groups. Furthermore, if \([1_A]_0 \in K_0(A)\) is a torsion element, \([1_B]_0 \in K_0(B)\) is not. Conversely, if \([1_A]_0 \in K_0(A)\) is not a torsion element, then \([1_B]_0 \in K_0(B)\) is a torsion element. So one has the following corollary.

**Corollary 2.14.** For every unital UCT Kirchberg algebra \(A\) with finitely generated K-groups, there exists a unital UCT Kirchberg algebra \(B \not\sim_{KK} D(C_{u_A})\) satisfying

\[
D(C_{u_A}) \sim_{KK} B, \quad D(C_{u_B}) \sim_{KK} A.
\]

### 2.5 \(K_0(A)\) and \(K_1(C_{u_A})\)

In this section, we investigate a relationship between \(K_0(A)\) and \(K_1(C_{u_A})\).

Every finitely generated Abel group \(G\) admits the following presentation

\[
G = \mathbb{Z}^F \oplus \bigoplus_{p:\text{prime}} G(p),
\]

where we write \(G(p) = \mathbb{Z}_{p^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{k_p}}, \ k_i > 0\) for finitely many \(p\) and otherwise \(G(p) = 0\). We write \(I(G(p)) := \{k_1, \cdots, k_p\}\) if \(G(p) \neq 0\). For the above \(p\)-group \(G(p)\), we define \(L(G(p))\) by

\[
L(G(p)) := t_p = |I(G(p))| \quad \text{for } G(p) \neq 0, \quad L(0) := 0.
\]
Proposition 2.15. For a unital C*-algebra \( A \), the following hold.

In particular, if \( (G, \mathcal{G}) \) be two elements satisfying

\[
I(Z_p \oplus Z_p) = \{1, 1\}, \quad I(Z_p \oplus Z_p \oplus Z_{p^2}) = I(Z_p \oplus Z_{p^2} \oplus Z_p) = \{1, 1, 2\},
\]

\[
L(Z_p \oplus Z_p \oplus Z_{p^2}) = 3, \quad I(Z_p \oplus Z_p) \cap I(Z_p \oplus Z_p \oplus Z_{p^2}) = \{1, 1\}.
\]

We note the precise meaning of the intersection \( I(G(p)) \cap I(H(p)) \). It is the collection of numbers identified with the subset of both \( I(G(p)) \) and \( I(H(p)) \) such that the complements \( I(G(p)) \setminus I(G(p)) \cap I(H(p)) \) and \( I(H(p)) \setminus I(G(p)) \cap I(H(p)) \) do not share the same number. Since every \( I(G(p)) \) is a finite set, the intersection is well-defined and uniquely determined by \( I(G(p)) \) and \( I(H(p)) \). We also write \( I(G(p)) \setminus I(H(p)) := I(G(p)) \cap I(H(p)) \).

Let \( (G, \mathcal{G}) \) be a pair of finite Abel \( p \)-groups (i.e., \( G = G(p) \), \( \mathcal{G} = G(p) \)). The pair \( (G, \mathcal{G}) \) is said to satisfy \((*)\), if either \( G = \mathcal{G} \), or one can take the strictly increasing sequence by alternating all elements of \( I(G) \setminus I(\mathcal{G}) \) and \( I(\mathcal{G}) \setminus I(G) \).

More precisely, a pair \( (G, \mathcal{G}) \) with \( G \neq \mathcal{G} \) satisfies \((*)\) if and only if one has

\[
G = N \oplus Z_{p^{k_1}} \oplus \cdots \oplus Z_{p^{k_s}}, \quad \mathcal{G} = N \oplus Z_{p^{\tilde{k}_1}} \oplus \cdots \oplus Z_{p^{\tilde{k}_s}}
\]

for some finite Abel \( p \)-group \( N \) and \( k_1, \tilde{k}_j \) that satisfy either \( 0 \leq k_1 < \tilde{k}_1 < \cdots < k_s < \tilde{k}_s \), or \( 0 \leq \tilde{k}_1 < k_1 < \cdots < \tilde{k}_s < k_s \) for \( 1 \leq s \).

For a presentation \( N = Z_{p^{k_1}} \oplus \cdots \oplus Z_{p^{k_s}} \), we have

\[
I(G) \setminus I(\mathcal{G}) = \{k_i \mid 0 < k_i\}, \quad I(\mathcal{G}) \setminus I(G) = \{\tilde{k}_j \mid 0 < \tilde{k}_j\},
\]

\[
I(G) \cap I(\mathcal{G}) = \{n_1, \cdots, n_t\}.
\]

In particular, if \( (G, \mathcal{G}) \) satisfies \((*)\), then so does \( (\mathcal{G}, G) \), and one has \( |L(G) - L(\mathcal{G})| \leq 1 \). For a pair \((G, \mathcal{G}) \neq (0, 0)\) with \((*)\), we write \( m(G, \mathcal{G}) := \max \{k \mid k \in I(G) \cup I(\mathcal{G})\} \).

A pair \((G, \mathcal{G})\) with \((*)\) is said to satisfy \((**)\) if either \( G = \mathcal{G} \) or one has

\[
\max \{k \mid k \in (I(\mathcal{G}) \setminus I(G)) \cup \{0\}\} > \max \{k \mid k \in (I(G) \setminus I(\mathcal{G})) \cup \{0\}\}.
\]

**Proposition 2.15.** For a unital C*-algebra \( A \) with finitely generated \( K \)-groups, the following hold.

1. If \( [1_A]_0 \in K_0(A) \) is a torsion element, the pair \((K_1(C_{uA})(p), K_0(A)(p))\) satisfies \((**)\) for every prime \( p \).

2. If \( [1_A]_0 \in K_0(A) \) is not a torsion element, the pair \((K_0(A)(p), K_1(C_{uA})(p))\) satisfies \((**)\) for every prime \( p \).

**Proposition 2.16.** Let \( G \) be a finitely generated Abel group, and let \( g_1, g_2 \in G \) be two elements satisfying \( G/\langle g_1 \rangle \cong G/\langle g_2 \rangle \). Then, there exists an isomorphism \( G \ni g_1 \mapsto g_2 \in G \).
The above propositions follow from elementary arguments on finitely generated Abel groups, and we prove them in Section 5. We note that Proposition 2.16 does not hold for non-finitely generated groups such as $G = \bigoplus_{\mathbb{N}} \mathbb{Z}$.

**Corollary 2.17.** Let $B_1, B_2$ be unital $C^*$-algebras with finitely generated $K$-groups satisfying

$$K_0(B_1) \cong K_0(B_2), \quad K_1(C_{uB_1}) \cong K_1(C_{uB_2}).$$

Then, there exists an isomorphism $K_0(B_1) \ni [1_{B_1}]_0 \mapsto [1_{B_2}]_0 \in K_0(B_2)$. In particular, the choice of the unital UCT Kirchberg algebra $B$ in Corollary 2.14 is unique.

### 2.6 Homotopy set $[X, \text{Aut}(A)]$

We recall M. Dadarlat’s result [7, Thm. 4.6] which asserts

$$[X, \text{End}(A)] = KK(C_{uA}, SA \otimes C(X))$$

for every unital Kirchberg algebra $A$. Every continuous map $\alpha : X \to \text{End}(A)$ is identified with a $*$-homomorphism $\alpha : A \to A \otimes C(X)$ sending $a \in A$ to the function $X \ni x \mapsto \alpha_x(a) \in A$, which is also identified with the $C(X)$-linear map

$$\tilde{\alpha} : A \otimes C(X) \ni f \otimes a \mapsto f \alpha(a) \in A \otimes C(X).$$

Let $l$ denote the constant map $l_x = \text{id}_A$, and we write $C\alpha : C_{uA} \ni a(t) \mapsto \alpha_x(a(t)) \in C_{uA} \otimes C(X)$. Since $a(1) \in C1_A$, one has $C\alpha(a(t)) - Cl(a(t)) \in SA \otimes C(X)$, and the Cuntz picture of KK-theory gives an element $\langle C\alpha, Cl \rangle \in KK(C_{uA}, SA \otimes C(X))$ (see [7, Sec. 3], [1, Sec. 17.6.]).

Let $\iota_A$ denote the inclusion $SA \hookrightarrow C_{uA}$, and let $\beta : X \to \text{End}(A)$ be another map. For the composition $(\alpha \circ \beta)_x = \alpha_x \circ \beta_x$, the direct computation yields

$$\langle C(\alpha \circ \beta), Cl \rangle = \langle C(\alpha \circ \beta), C\alpha \rangle + \langle C\alpha, Cl \rangle$$

$$= \langle (C\beta, Cl) \hat{\otimes} (I_S \otimes KK(\hat{\alpha})) \rangle + \langle C\alpha, Cl \rangle$$

$$= \langle C\beta, Cl \rangle \hat{\otimes} (I_S \otimes (KK(\hat{\alpha}) - KK(\hat{\beta}))) + \langle C\alpha, Cl \rangle + \langle C\beta, Cl \rangle$$

$$= \langle C\beta, Cl \rangle \hat{\otimes} (\hat{K}(\iota_A) \hat{\otimes} (C\alpha, Cl) \otimes I_{C(X)}) \hat{\otimes} (I_{SA} \otimes \Delta_X)$$

$$+ \langle C\alpha, Cl \rangle + \langle C\beta, Cl \rangle,$$

where $\Delta_X : C(X) \otimes C(X) \ni f(x,y) \mapsto f(x,x) \in C(X)$ is the diagonal map. For $x,y \in KK(C_{uA}, SA \otimes C(X))$, we define a multiplication $\circ_A$ by

$$x \circ_A y := x + y \hat{\otimes} (\hat{K}(\iota_A) \hat{\otimes} x) \otimes I_{C(X)} \hat{\otimes} (I_{SA} \otimes \Delta_X).$$

Then, one has the following theorem.
Theorem 2.18 ([7 Thm. 4.6, Prop. 5.8]). Let $X$ be a compact metrizable space, and let $A$ be a unital Kirchberg algebra.

1. The following map is an isomorphism of semigroups:

$$[X, \text{End}(A)] \ni [\alpha] \mapsto (C\alpha, Cl) \in (KK(CuA, SA \otimes C(X)), \circ_A).$$

2. If $A$ satisfies the UCT and has finitely generated $K$-groups, by the inclusion $\text{Aut}(A) \hookrightarrow \text{End}(A)$, the homotopy set $[X, \text{Aut}(A)]$ is identified with the set of invertible elements of $[X, \text{End}(A)]$.

The invertible element means the element of $[X, \text{End}(A)]$ which admits both left and right inverse, and this is an algebraic condition.

3 The main result

In this section, we prove our main theorem.

Theorem 3.1. Let $A$ and $B$ be two unital UCT Kirchberg algebras with finitely generated $K$-groups. We have the following:

1. For every $A$, there uniquely exists $B$ satisfying

$$D(CuA) \sim_{KK} B, \quad D(CuB) \sim_{KK} A,$$

and such $B$ is non-isomorphic to $A$.

2. For $A \not\cong B$, one has $\pi_i(\text{Aut}(A)) \cong \pi_i(\text{Aut}(B))$, $i \geq 1$ if and only if the pair $(A, B)$ satisfies the above $KK$-equivalence relations.

3. For $(A, B)$ in 2. and a compact metrizable space $X$, there exists a natural anti-isomorphism of groups

$$[X, \text{Aut}(A)] \rightarrow [X, \text{Aut}(B)].$$

Statement 1 immediately follows from Lemma 2.14 and Corollary 2.17.

Remark 3.2. One can not drop the assumption that the $K$-groups are finitely generated. First of all, as mentioned in Theorem 2.10, the K-duals do not exist for $C^*$-algebras whose $K$-groups are not finitely generated. Statement 2 asserts that, for a given $A$, there is only one non-trivial $B$ sharing the same homotopy groups of automorphism groups in the category of $C^*$-algebras with finitely generated $K$-groups. However, there are many different $B$ whose homotopy groups of the automorphism groups are equal to that of $A$ if we allow $C^*$-algebras with non-finitely generated $K$-groups. For example, $A = O_{n+1}$ and $B = M_n(O_\infty) \otimes M_p\infty$ with $\text{GCD}(n, p) = 1$ give such pairs.
3.1 Proof of the statement 2.

To prove the statement 2, we need the following proposition (see Section 2.5 for notation).

**Proposition 3.3.** Let \((G, \tilde{G})\) and \((H, \tilde{H})\) be finite Abel p-groups satisfying (*). Let \(f \geq 1, F \geq 0\) be integers with \(f - F \in 2\mathbb{Z}\). Then the following equation holds if and only if \((G, \tilde{G}) = (H, \tilde{H})\):

\[
G^f \oplus \tilde{G}^{(F+1)} \oplus (G \otimes \tilde{G})^2 \cong H^f \oplus \tilde{H}^{(F+1)} \oplus (H \otimes \tilde{H})^2.
\]

**Proof.** We prove the statement by the induction over \(h = L(G) + L(\tilde{G}) \geq 0\). In the case of \(h = 0\), we have \(0 = H^f \oplus \tilde{H}^{(F+1)} \oplus (H \otimes \tilde{H})^2\) that implies \((G, \tilde{G}) = (0, 0) = (H, \tilde{H})\). So we will prove the statement for a pair \((G, \tilde{G})\) satisfying (*), \(L(G) + L(\tilde{G}) = h + 1, h \geq 0\) under the assumption that the statement holds for every pair \((G', \tilde{G}')\) satisfying (*), \(L(G') + L(\tilde{G}') \leq h\).

One has the following two cases:

1) \(m(G, \tilde{G}) \in I(G) \cap I(\tilde{G})\),

2) \(m(G, \tilde{G}) \notin I(G) \cap I(\tilde{G})\).

First, we discuss case I). Let \((H, \tilde{H})\) satisfy (*) and the equation

\[
G^f \oplus \tilde{G}^{(F+1)} \oplus (G \otimes \tilde{G})^2 \cong H^f \oplus \tilde{H}^{(F+1)} \oplus (H \otimes \tilde{H})^2.
\]

Since \(f \geq 1\) and \(F + 1 \geq 1\), one has \(m(G, \tilde{G}) = m(H, \tilde{H}) = m\). The left hand side has at least \(f + (F + 1) + 2\) copies of \(\mathbb{Z}_{p^m}\) as direct summands. If \(m \notin I(H) \cap I(\tilde{H})\), one has \(m > k\) for every \(k \in I(H) \cap I(\tilde{H})\) and it follows that \(H \otimes \tilde{H}\) does not contain \(\mathbb{Z}_{p^m}\) as a direct summand by the definition of (*). It also follows that \(H^f \oplus \tilde{H}^{(F+1)}\) has at most \(\max\{f, F + 1\}\) copies of \(\mathbb{Z}_{p^m}\) as direct summands that implies \(f + (F + 1) + 2 \leq \max\{f, F + 1\}\). This is a contradiction and we have \(m \in I(H) \cap I(\tilde{H})\). Thus, one can write

\[
(G, \tilde{G}) = (\mathbb{Z}_{p^m} \oplus G', \mathbb{Z}_{p^m} \oplus \tilde{G}'), \quad (H, \tilde{H}) = (\mathbb{Z}_{p^m} \oplus H', \mathbb{Z}_{p^m} \oplus \tilde{H}')
\]

by the pairs \((G', \tilde{G}')\), \((H', \tilde{H}')\) satisfying (*), \(L(G') + L(\tilde{G}') = h - 1 \leq h\), and the following equation holds

\[
G'^{(f+2)} \oplus \tilde{G}'^{(F+2+1)} \oplus (G' \otimes \tilde{G}')^2 \cong H'^{(f+2)} \oplus \tilde{H}'^{(F+2+1)} \oplus (H' \otimes \tilde{H}')^2.
\]

The assumption of the induction shows \((G', \tilde{G}') = (H', \tilde{H}')\) and the statement is proved for \(L(G) + L(\tilde{G}) = h + 1\) in case I).

Next, we discuss case II). As in case I), we have \(m(G, \tilde{G}) = m(H, \tilde{H}) = m\) for a pair \((H, \tilde{H})\) satisfying (*) and

\[
G^f \oplus \tilde{G}^{(F+1)} \oplus (G \otimes \tilde{G})^2 \cong H^f \oplus \tilde{H}^{(F+1)} \oplus (H \otimes \tilde{H})^2.
\]
The same argument as in case I) shows that \( m \notin I(H) \cap I(\tilde{H}) \) and both \( G \cong \tilde{G} \) and \( H \otimes \tilde{H} \) contain no copies of \( \mathbb{Z}_{p^n} \) as direct summands.

If \( m \in I(G) \setminus I(\tilde{G}) \), the left hand side of the above equation contains exactly \( f \) copies of \( \mathbb{Z}_{p^n} \) as direct summands. If \( m \notin I(H) \setminus I(\tilde{H}) \), it means \( m \in I(\tilde{H}) \setminus I(H) \) and the right hand side contains exactly \( F + 1 \) copies of \( \mathbb{Z}_{p^n} \). Since \( f - F = 1 \) contradicts to the assumption, one has \( m \in I(H) \setminus I(\tilde{H}) \). One can write \( (G, \tilde{G}) = (\mathbb{Z}_{p^n} \oplus G', \tilde{G}), (H, \tilde{H}) = (\mathbb{Z}_{p^n} \oplus H', \tilde{H}) \) and it is easy to check that \( (G', \tilde{G}), (H', \tilde{H}) \) also satisfy \((*)\) and

\[
G^f \oplus \tilde{G}^{(F+2+1)} \oplus (G' \otimes \tilde{G})^2 \cong H^f \oplus \tilde{H}^{(F+2+1)} \oplus (H' \otimes \tilde{H})^2.
\]

Since \( L(G') + L(\tilde{G}) = h \), the assumption of the induction yields \( (G', \tilde{G}) = (H', \tilde{H}) \) which implies \((G, \tilde{G}) = (H, \tilde{H})\).

If \( m \in I(\tilde{G}) \setminus I(G) \) the same argument shows \( (G, \tilde{G}) = (H, \tilde{H}) \), and we have proven the statement for \( L(G) + L(\tilde{G}) = h + 1 \) in case II). This completes the induction.

**Corollary 3.4.** Let \((G, \tilde{G})\) and \((H, \tilde{H})\) be two pairs of finite Abel \( p \)-groups satisfying \((**)\). Let \( f \geq 0, F \geq 0 \) be integers with \( f - F \in 2\mathbb{Z} \). Then, the following equation holds if and only if \((G, \tilde{G}) = (H, \tilde{H})\):

\[
G^f \oplus \tilde{G}^{(F+1)} \oplus (G' \otimes \tilde{G})^2 \cong H^f \oplus \tilde{H}^{(F+1)} \oplus (H' \otimes \tilde{H})^2.
\]

**Proof.** Since \((G, \tilde{G})\) (resp. \((H, \tilde{H})\)) satisfies \((**)\), one has \( m(G, \tilde{G}) \in I(\tilde{G})\) (resp. \( m(H, \tilde{H}) \in I(\tilde{H})\)), and \( F + 1 \geq 1 \) implies \( m(G, \tilde{G}) = m(H, \tilde{H}) = m \).

We first consider the case \( m \in I(G, p) \cap I(\tilde{G}, p) \). The left hand side of the above equation has at least \( f + (F + 1) + 2 \) copies of \( \mathbb{Z}_{p^n} \) as direct summands. If \( m \notin I(H, p) \cap I(\tilde{H}, p) \), the right hand side has exactly \( F + 1 \) copies of \( \mathbb{Z}_{p^n} \) as direct summands that implies \( f + (F + 1) + 2 \leq F + 1 \). This is a contradiction, and we have \( m \in I(H, p) \cap I(\tilde{H}, p) \). Thus, there exist \((G', \tilde{G}')\) and \((H', \tilde{H}')\) satisfying \((*)\) and

\[
(G, \tilde{G}) = (\mathbb{Z}_{p^n} \oplus G', \mathbb{Z}_{p^n} \oplus \tilde{G}'), \quad (H, \tilde{H}) = (\mathbb{Z}_{p^n} \oplus H', \mathbb{Z}_{p^n} \oplus \tilde{H}'),
\]

\[
G^{f+2} \oplus \tilde{G}^{(F+2+1)} \oplus (G' \otimes \tilde{G}')^2 \cong H^{f+2} \oplus \tilde{H}^{(F+2+1)} \oplus (H' \otimes \tilde{H}')^2,
\]

and Proposition 3.3 shows \((G', \tilde{G}') = (H', \tilde{H}')\).

Next, we discuss the case \( m \notin I(G, p) \cap I(\tilde{G}, p) \). The same argument as in the previous case shows \( m \notin I(H, p) \cap I(\tilde{H}, p) \), and we write

\[
(G, \tilde{G}) = (G, \mathbb{Z}_{p^n} \oplus \tilde{G}'), \quad (H, \tilde{H}) = (H, \mathbb{Z}_{p^n} \oplus \tilde{H}'),
\]

where \((G, \tilde{G}')\) and \((H, \tilde{H}')\) satisfy \((*)\) and

\[
G^{f+2} \oplus \tilde{G}^{(F+1)} \oplus (G \otimes \tilde{G}')^2 \cong H^{f+2} \oplus \tilde{H}^{(F+1)} \oplus (H \otimes \tilde{H}')^2.
\]

Proposition 3.3 shows \((G, \tilde{G}') = (H, \tilde{H}')\), and this proves the statement. \(\square\)
Following the notation in Section \[2.5\] we write
\[
K_i(A) = \mathbb{Z}^{F'_i} \oplus \bigoplus_p K_i(A)(p), \quad K_i(C_{u_A}) = \mathbb{Z}^{F'_i} \oplus \bigoplus_p K_i(C_{u_A})(p).
\]
Note that Remark \[2.11\] implies
\[
K_i(D(A)) = \mathbb{Z}^{F'_i} \oplus \bigoplus_p K_{i+1}(A)(p), \quad K_i(D(C_{u_A})) = \mathbb{Z}^{F'_i} \oplus \bigoplus_p K_{i+1}(C_{u_A})(p).
\]
Using the mapping cone sequence \( SA \to C_{u_A} \to \mathbb{C} \to A \), one can easily check
\( K_0(C_{u_A})(p) \cong K_1(A)(p), f_0^A \geq F_1^A, F_0^A \geq f_1^A \) and \( F_1^A - f_0^A + 1 - F_0^A + f_1^A = 0 \). The Künneth theorem yields
\[
K_0(A \otimes D(C_{u_A})) = \mathbb{Z}^{F'_0 f_0^A + F_1^A f_i^A} \oplus \bigoplus_p \left( A_p^{F'_1} \oplus \tilde{A}_p^{F'_1} \oplus (A_p \otimes \tilde{A}_p) \oplus K_0(A)(p) f_0^A - F_0^A + K_1(C_{u_A})(p) F_0^A - f_1^A \right),
\]
\[
K_1(A \otimes D(C_{u_A})) = \mathbb{Z}^{F'_1 f_0^A + F_0^A f_i^A} \oplus \bigoplus_p \left( A_p^{F'_1} \oplus \tilde{A}_p^{F'_1} \oplus (A_p \otimes \tilde{A}_p) \oplus K_1(A)(p) (F_0^A - f_1^A + (F_0^A - f_1^A)) \right)
\]
\[
= \mathbb{Z}^{F'_1 f_0^A + F_0^A f_i^A} \oplus \bigoplus_p \left( A_p^{F'_1} \oplus \tilde{A}_p^{F'_1} \oplus (A_p \otimes \tilde{A}_p) \oplus K_1(A)(p) \right),
\]
where we write
\( A_p := K_1(A)(p) \oplus K_0(A)(p), \quad \tilde{A}_p := K_1(A)(p) \oplus K_1(C_{u_A})(p). \)

If \([1_A]_0 \in K_0(A)\) is a torsion element, one has \((f_0^A - F_1^A, F_0^A - f_1^A) = (1, 0)\). Conversely, one has \((f_0^A - F_1^A, F_0^A - f_1^A) = (0, 1)\) if \([1_A]_0 \in K_0(A)\) is not a torsion. One can easily check the following lemma.

**Lemma 3.5.** Let \( n, m, s, t \) be non-negative integers satisfying
\[
2mn + m = 2st + s, \quad n^2 + n + m^2 = t^2 + t + s^2.
\]
Then, we have \( m = n = s \).

*Proof of Theorem \[3.1\] 2.* Let \( A \) and \( B \) be unital UCT Kirchberg algebras with finitely generated K-groups satisfying
\[
\pi_i(\text{Aut}(A)) = K_{i+1}(A \otimes D(C_{u_A})) \cong K_{i+1}(B \otimes D(C_{u_B})) = \pi_i(\text{Aut}(B)), \quad i \geq 1,
\]
and we use the following notation
\[
A_p := K_1(A)(p) \oplus K_0(A)(p), \quad \tilde{A}_p := K_1(A)(p) \oplus K_1(C_{u_A})(p),
\]
\[
B_p := K_1(B)(p) \oplus K_0(B)(p), \quad \tilde{B}_p := K_1(B)(p) \oplus K_1(C_{u_B})(p).
\]
We consider the following three cases:
1. \([A]_0\) is a torsion element and \([B]_0\) is not.
2. Both \([A]_0\) and \([B]_0\) are torsion elements.
3. Both \([A]_0\) and \([B]_0\) are non-torsion elements.

First, we discuss case 1 and prove that \(A\) and \(B\) satisfy

\[ K_i(B) = K_i(D(Cu_A)), \quad K_i(A) = K_i(D(Cu_B)). \]

One has \(f_0^A = F_1^A + 1, f_1^A = F_0^A\) and \(F_1^B = f_0^B, F_0^B = f_1^B + 1\), and the comparison of ranks of the free parts yields

\[ 2F_0^A F_1^A + F_0^A = 2f_0^B f_1^B + f_0^B, \quad F_1^A + F_0^A + F_0^A = f_1^B + f_1^B + f_0^B. \]

Lemma 3.5 shows \(F_i^A = f_i^B, f_i^A = F_i^B\), and we have

\[ \tilde{A}_p^F \oplus A_p^{F+1} \oplus (\tilde{A}_p \otimes A_p)^2 \cong B_p^F \oplus \tilde{B}_p^{F+1} \oplus (B_p \otimes \tilde{B}_p)^2 \]

for \(F = F_1^A + f_1^A = f_1^B + F_B\). Since \((\tilde{A}_p, A_p)\) and \((B_p, \tilde{B}_p)\) satisfy \((**)\) by Proposition 2.15 Corollary 3.4 shows

\[ (\tilde{A}_p, A_p) = (B_p, \tilde{B}_p). \]

Thus, the assumption \(K_i(A \otimes D(Cu_A)) = K_i(B \otimes D(Cu_B))\) shows

\[ K_0(A)(p) = K_1(Cu_B)(p), \quad K_1(A)(p) = K_1(B)(p), \]

and we have

\[ K_0(A) = Z^{F_0^A} \oplus \bigoplus_p K_0(A)(p) \]
\[ = Z^{f_0^B} \oplus \bigoplus_p K_0(Cu_B)(p) \]
\[ = K_0(D(Cu_B)), \]
\[ K_1(A) = Z^{F_1^A} \oplus \bigoplus_p K_1(A)(p) \]
\[ = Z^{f_1^B} \oplus \bigoplus_p K_1(B)(p) \]
\[ = Z^{f_1^B} \oplus \bigoplus_p K_0(Cu_B)(p) \]
\[ = K_1(D(Cu_B)). \]

Similar argument shows \(K_i(B) \cong K_i(D(Cu_A)), \quad i = 0, 1\).
Next, we show $A = B$ in case 2. In case 2, one has $f_0^A = F_0^A + 1$, $f_1^A = F_0^A$ and $f_0^B = F_1^B + 1$, $f_1^B = F_0^B$. Comparing ranks of free parts of the homotopy groups, one has
\[2F_0^A F_1^A + F_0^A = 2F_0^B F_1^B + F_0^B, \quad F_1^A + F_1^A + F_0^A = F_1^B + F_1^B + F_0^B,\]
and Lemma 3.5 implies $F_i^A = F_i^B$, $f_i^A = f_i^B$. Thus, we obtain
\[
\tilde{A}_p^F \oplus A_p^{F+1} \oplus (\tilde{A}_p \otimes A_p)^2 \cong \tilde{B}_p^F \oplus B_p^{F+1} \oplus (\tilde{B}_p \otimes B_p)^2
\]
for $F := F_1^A + f_1^A = F_1^B + f_1^B$. Since $(\tilde{A}_p, A_p)$ and $(\tilde{B}_p, B_p)$ satisfy $(\ast \ast)$ by Proposition 2.15, Corollary 3.4 shows $(\tilde{A}_p, A_p) = (\tilde{B}_p, B_p)$, which implies $A \sim_{KK} B$, $C_{u_A} \sim_{KK} C_{u_B}$ by a similar argument as in case 1, and Corollary 2.17 gives $A \cong B$.

The same argument also shows $A = B$ in case 3, and this completes the proof.

### 3.2 Proof of the statement 3.

In this section, $A$ and $B$ are unital separable UCT Kirchberg algebras satisfying
\[D(C_{u_A}) \sim_{KK} B, \quad D(SA) \sim_{KK} SD(A) \sim_{KK} SC_{u_B},\]
and we denote their duality classes by
\[
\mu_{CA} \in KK(C, C_{u_A} \otimes B), \quad \nu_{CA} \in KK(B \otimes C_{u_A}, \mathbb{C}),
\]
\[
\mu_{SA} \in KK(C, SA \otimes SC_{u_B}), \quad \nu_{SA} \in KK(SC_{u_B} \otimes SA, \mathbb{C}).
\]
Let $\gamma \in KK(C_{u_B}, SC_{u_B} \otimes S)^{-1}$ be a KK-equivalence, and we also denote by $\iota_B$ the natural map $B \otimes S \to C_{u_B}$. Then, as in Section 2.6, the semi-group $(KK(C_{u_B}, C(X) \otimes B \otimes S), \iota_B \circ)$ with the multiplication
\[x_B \circ y := x + y \hat{\otimes} (I_C(X) \otimes (KK(\iota_B) \hat{\otimes} x)) \hat{\otimes} (\Delta_X \otimes I_B \hat{\otimes} S)\]
is isomorphic to $[X, End(B)]$.

We may assume that $[1_A]_0$ is a torsion element and $[1_B]_0$ is not, and we will prove $(KK(C_{u_A}, SA \otimes C(X), \circ_A)$ and $(KK(C_{u_B}, C(X) \otimes B \otimes S), \circ_B)$ are anti-isomorphic. Combining Theorem 2.18 and Lemma 2.12, one can expect that the map
\[d^{X}_{\mu_{CA}, \nu_{SA}} : KK(C_{u_A}, SA \otimes C(X)) \to KK(SC_{u_B}, C(X) \otimes B)\]
provides the anti-isomorphism which proves the statement 3. To do this end, we need the following lemmas.
Lemma 3.6. There exists $\alpha \in KK(B \otimes S, B \otimes S)^{-1}$ satisfying
\[
(d_{\mu_C,A,e_C}(KK(e_A)) \otimes I_S) \otimes \alpha = KK(u_B) \otimes I_S \in KK(S, B \otimes S)
\]
for the map $e_A : C_{u_A} \to \mathbb{C}$.

Proof. The isomorphism $d_{\mu_C,A,e_C}$ and the Puppe sequence
\[
KK(\mathbb{C}, \mathbb{C}) \xrightarrow{\hat{e}_A} KK(C_{u_A}, \mathbb{C}) \to KK(SA, \mathbb{C}) \to KK(S, \mathbb{C})
\]
show
\[
KK(\mathbb{C}, B)/(d_{\mu_C,A,e_C}(KK(e_A))) \cong KK(C_{u_A}, \mathbb{C})/(KK(e_A)) \cong KK(SA, \mathbb{C}),
\]
and the right hand side is isomorphic to
\[
KK(\mathbb{C}, SC_{u_B}) = KK(\mathbb{C}, B)/(KK(u_B)).
\]
Thus, Proposition 2.16 gives a $KK$-equivalence $\alpha \in KK(B \otimes S, B \otimes S)^{-1}$ satisfying
\[
(d_{\mu_C,A,e_C}(KK(e_A)) \otimes I_S) \otimes \alpha = KK(u_B) \otimes I_S.
\]

Lemma 3.7. There exists $\beta \in KK(C_{u_B}, C_{u_B})^{-1}$ satisfying
\[
KK(\iota_B) = \alpha^{-1} \otimes (d_{\mu_{SA},e_C}(KK(\iota_A)) \otimes I_S) \otimes \gamma^{-1} \otimes e^{-1} \in KK(B \otimes S, C_{u_B}).
\]

Proof. By the Puppe sequence $KK(SA, SA) \xrightarrow{-\hat{\otimes} \alpha} KK(SA, C_{u_A}) \xrightarrow{-\hat{\otimes} e_A} KK(SA, \mathbb{C})$, the kernel of the map $-\hat{\otimes} KK(e_A)$ is $KK(SA, SA) \otimes KK(\iota_A) \subset KK(SA, C_{u_A})$, Lemma 2.12 implies that the kernel of the map
\[
d_{\mu_{CA},e_C}(KK(e_A)) \otimes - : KK(B, SC_{u_B}) \to KK(\mathbb{C}, SC_{u_B})
\]
is $d_{\mu_{SA},e_C}(KK(\iota_A)) \otimes KK(SC_{u_B}, SC_{u_B})$. Thus, the kernel of the map
\[
(d_{\mu_{CA},e_C}(KK(e_A)) \otimes I_S) \otimes - : KK(B \otimes S, C_{u_B}) \to KK(S, C_{u_B})
\]
is equal to $(d_{\mu_{SA},e_C}(KK(\iota_A)) \otimes I_S) \otimes \gamma^{-1} \otimes KK(C_{u_B}, C_{u_B})$. By Lemma 3.6, one can identify the kernel of the map
\[
(KK(u_B) \otimes I_S) \otimes - : KK(B \otimes S, C_{u_B}) \to KK(S, C_{u_B})
\]
with
\[
\alpha^{-1} \otimes (d_{\mu_{SA},e_C}(KK(\iota_A)) \otimes I_S) \otimes \gamma^{-1} \otimes KK(C_{u_B}, C_{u_B}),
\]
and this implies
\[
KK(\iota_B) \otimes KK(C_{u_B}, C_{u_B}) = \alpha^{-1} \otimes (d_{\mu_{SA},e_C}(KK(\iota_A)) \otimes I_S) \otimes \gamma^{-1} \otimes KK(C_{u_B}, C_{u_B}).
\]
There exist two elements $Z_1, Z_2 \in KK(C_{uB}, C_{uB})$ satisfying
\[
KK(t_B) \otimes Z_1 = \alpha^{-1} \hat{\otimes} (d_{\mu_{SA}, \nu_{CA}}(KK(t_A)) \otimes I_S) \hat{\otimes} \gamma^{-1},
\]
\[
\alpha^{-1} \hat{\otimes} (d_{\mu_{SA}, \nu_{CA}}(KK(t_A)) \otimes I_S) \hat{\otimes} \gamma^{-1} \hat{\otimes} Z_2 = KK(t_B),
\]
and we have $KK(t_B) \otimes Z_1 \otimes Z_2 = KK(t_B)$. Since $[1_B]_0$ is not a torsion element, the Puppe sequence shows that $- \hat{\otimes} KK(t_B)$ induces the following surjections
\[
K_0(B \otimes S) \to K_0(C_{uB}), \quad K_1(B \otimes S) \to K_1(C_{uB}).
\]
Since $K_i(C_{uB})$ are finitely generated, one can easily check that $Z_i$ induce automorphisms of $K_i(C_{uB})$, and the UCT proves $Z_i \in KK(C_{uB}, C_{uB})^{-1}$.

One has a natural isomorphism of KK-groups
\[
D_{A,B}^X : KK(C_{uA}, SA \otimes C(X)) \to KK(C_{uB}, C(X) \otimes B \otimes S)
\]
defined by
\[
D_{A,B}^X(x) := \beta \hat{\otimes} \gamma \hat{\otimes} (d_{\mu_{CA}, \nu_{SA}}^X(x) \otimes I_S) \hat{\otimes} (I_{C(X)} \otimes \alpha).
\]
Now the following theorem proves the statement 3.

**Theorem 3.8.** For any $x, y \in KK(C_{uA}, SA \otimes C(X))$, we have
\[
D_{A,B}^X(x \circ_A y) = D_{A,B}^X(y) \circ D_{A,B}^X(x).
\]

**Proof.** We write
\[
t := y \hat{\otimes} ((KK(t_A) \hat{\otimes} x) \otimes I_{C(X)}) \in KK(C_{uA}, SA \otimes C(X \times X))
\]
and direct computation yields
\[
d_{\mu_{CA}, \nu_{SA}}^X(t \otimes (I_{SA} \otimes \Delta_X))
\]
\[
= (I_{SC_{uB}} \otimes (\mu_{CA} \hat{\otimes} ((t \hat{\otimes} (I_{SA} \otimes \Delta_X)) \otimes I_B))) \hat{\otimes} (\nu_{SA} \otimes I_{C(X) \otimes B})
\]
\[
= (I_{SC_{uB}} \otimes (\mu_{CA} \hat{\otimes} (t \otimes I_B))) \hat{\otimes} (I_{SC_{uB}} \otimes \Delta_X \otimes I_B) \hat{\otimes} (\nu_{SA} \otimes I_{C(X) \otimes B})
\]
\[
= (I_{SC_{uB}} \otimes (\mu_{CA} \hat{\otimes} (t \otimes I_B))) \hat{\otimes} (\nu_{SA} \otimes I_{C(X \times X) \otimes B}) \hat{\otimes} (\Delta_X \otimes I_B)
\]
\[
= d_{\mu_{CA}, \nu_{SA}}^X(t) \hat{\otimes} (\Delta_X \otimes I_B).
\]
By Lemma 2.12 one has
\[
d_{\mu_{CA}, \nu_{SA}}^X(t) = d_{\mu_{CA}, \nu_{SA}}^X(KK(t_A) \hat{\otimes} x) \hat{\otimes} (I_{C(X)} \otimes d_{\mu_{CA}, \nu_{SA}}^X(y))
\]
\[
= d_{\mu_{CA}, \nu_{SA}}^X(x) \hat{\otimes} (I_{C(X)} \otimes (d_{\mu_{CA}, \nu_{CA}}^X(KK(t_A)) \hat{\otimes} d_{\mu_{CA}, \nu_{SA}}^X(y))).
\]
By Lemma 3.7 it is straightforward to check
\[
D_{A,B}^X(x) \hat{\otimes} (I_{C(X)} \otimes (KK(t_B) \hat{\otimes} D_{A,B}^X(y))) \hat{\otimes} (\Delta_X \otimes I_{B \otimes S})
\]
\[
= D_{A,B}^X(t \hat{\otimes} (I_{SA} \otimes \Delta_X)),
\]
and this proves the statement. \qed
Proof of the statement 3. We have natural isomorphisms of the semi-groups

\[ [X, \text{End}(A)] \rightarrow (KK(C_{uA}, S A \otimes C(X)), \circ_A), \]

\[ (KK(C_{uB}, C(X) \otimes B \otimes S), B \circ) \rightarrow [X, \text{End}(B)]. \]

Thus, Theorem \ref{thm:3.8} gives a natural anti-isomorphism \([X, \text{End}(A)] \rightarrow [X, \text{End}(B)]\), and this induces a natural anti-isomorphism of groups \([X, \text{Aut}(A)] \rightarrow [X, \text{Aut}(B)]\) by Theorem \ref{thm:2.18}.

4 Examples and Problems

In this section, we give new examples of the pairs of unital Kirchberg algebras \((A, B)\) with natural bijections \([X, \text{BAut}(A)] \rightarrow [X, \text{BAut}(B)]\) using the Spanier–Whitehead K-duality. The key ingredient is the Dadarlat–Pennig theory which proves the duality for locally trivial continuous \(C(X)\)-algebras with fiber \(KK\)-equivalent to \(C\) (see Theorem \ref{thm:2.4} Corollary \ref{cor:2.9}).

A \(C^*\)-algebra \(C\) with finitely generated \(K\)-groups is said to satisfy (d), if \(C\) has the following property:

For every locally trivial continuous \(C(X)\)-algebra \(C\) with fiber \(KK\)-equivalent to \(C\), there exists another (locally trivial) continuous \(C(X)\)-algebra \(D\) with fiber \(D(C)\), and two \(C(X)\)-algebras are Spanier–Whitehead K-dual.

Lemma 4.1. Two algebras \(C\) and \(S\) satisfy (d).

Proof. By Theorem \ref{thm:2.4} it is enough to consider \(C(X)\)-algebras whose fibers are stable Kirchberg algebras \(KK\)-equivalent to \(C\) or \(S\). The stable Kirchberg algebra \(KK\)-equivalent to \(C\) is \(O_\infty \otimes K\) and Corollary \ref{cor:2.9} shows the statement. Let \(P_\infty^s\) denote the stable Kirchberg algebra \(KK\)-equivalent to \(S\). By [5, Thm. 2.7], every locally trivial continuous \(C(X)\)-algebra \(P\) with fiber \(P_\infty^s\) is isomorphic to \(P \otimes P_\infty^s \otimes \mathbb{C}\) and \(P \otimes P_\infty^s\) is a locally trivial continuous \(C(X)\)-algebra with fiber \(O_\infty \otimes K\). By Corollary \ref{cor:2.9} and Lemma \ref{lem:2.7} \(P\) and \((P \otimes P_\infty^s)^{-1} \otimes P_\infty^s\) are Spanier–Whitehead K-dual.

In the rest of this section, \(A\) and \(B\) are unital UCT Kirchberg algebras satisfying \(D(C_{uA}) \sim_{KK} B\), \(D(C_{uB}) \sim_{KK} A\), and we assume that \(C_{uA}\) satisfies (d). Let \(A\) be an arbitrary locally trivial continuous \(C(X)\)-algebra with fiber \(A\). In the Puppe sequence

\[ C_{uA} \xrightarrow{e_A} C(X) \xrightarrow{u_A} A, \]

the mapping cone \(C_{uA}\) is a locally trivial continuous \(C(X)\)-algebra with fiber \(C_{uA}\). By Theorem \ref{thm:2.4} and the assumption of \(C_{uA}\), there exists a locally trivial continuous \(C(X)\)-algebra \(B^s\) with fiber \(B \otimes \mathbb{K}\) such that \(C_{uA}\) and \(B^s\) are K-dual with duality classes

\[ \mu \in KK_X(C(X), C_{uA} \otimes_C C(X) B^s), \quad \nu \in KK_X(B^s \otimes_C C(X) C_{uA}, C(X)). \]
Since $KK_X(C(X), B^s) = K_0(B^s)$ (see [5] Proof of Cor. 2.8.), one obtains a properly infinite full projection $p \in B^s$ with $[p]_0 = \mu \hat{\otimes} (e_{A} \otimes I_{B}) \in K_0(B^s) = KK_X(C(X), B^s)$.

**Lemma 4.2.** For every $x \in X$, we have $\pi_x(p)B_x^s \pi_x(p) \cong B$. Thus, $\mathcal{B} := pB^s p$ is a locally trivial continuous $C(X)$-algebra with fiber $B$.

**Proof.** Since $B^s$ is locally trivial, the local triviality of $\mathcal{B}$ immediately follows. So we prove $\pi_x(p)B_x^s \pi_x(p) \cong B$. By the evaluation map, one has

$$[\pi_x(p)]_0 = \mu \hat{\otimes} (e_{uA_x} \otimes I_{B^s}) \in KK(C, B^s_x).$$

Corollary [2,8] shows

$$KK(C, B^s_x)/([\pi_x(p)]_0) \cong KK(C_{uA_x}, C)/KK(e_{A_x}),$$

and the right hand side is isomorphic to

$$KK(SA_x, C) \cong KK(SA, C) \cong K_1(C_{uB})$$

by the Puppe sequence

$$KK(C, C) \xrightarrow{e_{uA_x} \hat{\otimes} -} KK(C_{uA_x}, C) \xrightarrow{e_{A_x} \hat{\otimes} -} KK(SA_x, C) \rightarrow KK(S, C).$$

Since $B^s_x \cong B \otimes K$, we have

$$K_0(B \otimes K)/([\pi_x(p)]_0) \cong K_1(C_{uB}) \cong K_0(B \otimes K)/([1_B]_0),$$

and Proposition [2,16] proves the statement. \hfill \Box

**Proposition 4.3.** Let $A_i$, $i = 1, 2$ be locally trivial continuous $C(X)$-algebras with $C(X)$-linear isomorphism $\varphi : A_1 \rightarrow A_2$, and let $B_i$, $i = 1, 2$ be the $C(X)$-algebras obtained from $A_i$ as in Lemma [4,2]. Then, $B_1$ and $B_2$ are $C(X)$-linearly isomorphic.

**Proof.** Applying [17] Appendix A. to the following diagram

$$\begin{array}{ccc}
SA_1 & \xrightarrow{e_{A_1}} & C(X) & \xrightarrow{u_{A_1}} & A_1 \\
\downarrow_{I_{S} \otimes \varphi} & & & & \\
SA_2 & \xrightarrow{e_{A_2}} & C(X) & \xrightarrow{u_{A_2}} & A_2,
\end{array}$$

one has a $KK_X$-equivalence $\gamma : C_{uA_1} \rightarrow C_{uA_2}$ making the diagram commutes. We take the $C(X)$-algebras $B^s_i$ and duality classes $\mu_i \in KK_X(C(X), C_{uA_i} \otimes C(X) B^s_i)$
for \( i = 1, 2 \). By Lemma 2.7, one has a \( KK_X \)-equivalence \( B_1^s \cong B_2^s \). Let \( p_i \) denote the projection defined by \([p_i]_0 = \mu_i \otimes (e_{A_i} \otimes I_{B_i^s})\). Direct computation yields
\[
[p_1]_0 \otimes \alpha = \mu_1 \otimes (e_{A_1} \otimes I_{B_1^s}) \otimes \alpha \\
= \mu_1 \otimes (\gamma \otimes I_{B_1^s}) \otimes (e_{A_2} \otimes I_{B_1^s}) \otimes (I_{C(X)} \otimes \alpha) \\
= \mu_1 \otimes (\gamma \otimes I_{B_1^s}) \otimes (I_{C_u A_2} \otimes \alpha) \otimes (e_{A_2} \otimes I_{B_2^s}).
\]

Lemma 2.7 shows \( \mu_1 \otimes (\gamma \otimes I_{B_1^s}) \otimes (I_{C_u A_2} \otimes \alpha) \) also provides duality classes and one has
\[
\mu_1 \otimes (\gamma \otimes I_{B_1^s}) \otimes (I_{C_u A_2} \otimes \alpha) = \mu_2 \otimes (I_{C_u A_2} \otimes \alpha')
\]
for some \( \alpha' \in KK_X(B_2^s, B_2^s)^{-1} \). Finally, we have \([p_1]_0 \otimes \alpha = [p_2]_0 \otimes \alpha' \in K_0(B_2^s)\), and this shows
\[
B_1 = p_1 B_i^s p_1 \cong p_2 B_2^s p_2 = B_2.
\]

By Proposition 4.3, we obtain a well-defined natural map
\[
D_{A,B} : [X, B, \text{Aut}(A)] \ni [A] \mapsto [B] \in [X, B, \text{Aut}(B)]
\]
for a unital Kirchberg algebra \( A \) with \( C_u A \) satisfying (d), and we prove the following proposition in the same spirit as [6, Sec. 2].

**Proposition 4.4.** The map \( D_{A,B} \) is injective. If \( D(C_u A) \) also satisfies (d), the map \( D_{A,B} \) is surjective.

**Proof.** First, we prove the injectivity. Take two Puppe sequences \( C_u A_i \xrightarrow{e_{A_i}} C(X) \xrightarrow{u_{A_i}} A_i, i = 1, 2 \) and \( C(X) \)-algebras \( B_i^s \) with duality classes
\[
\mu_i \in KK_X(C(X), C_u A_i \otimes_{C(X)} B_i^s), \quad \nu_i \in KK_X(B_i^s \otimes_{C(X)} C_u A_i, C(X)).
\]
The unital \( C(X) \)-algebra \( B_i \) with \([B_i] = D_{A,B}([A_i])\) is obtained as a corner of \( B_i^s \) and the following diagram commutes
\[
\begin{array}{c}
C(X) \xrightarrow{\mu_i \otimes (e_{A_i} \otimes I_{B_i^s})} B_i^s \\
\downarrow \quad \downarrow \\
C(X) \xrightarrow{u_{B_i}} B_i,
\end{array}
\]
where the right vertical map is the inclusion providing a \( KK_X \)-equivalence. We show \( A_1 \cong A_2 \) under the assumption that there is a \( C(X) \)-linear isomorphism
By assumption, one has a $KK_X$-equivalence $\alpha$ making the following diagram commute

$$
\begin{array}{ccc}
C(X) & \xrightarrow{\mu_1 \hat{\otimes} (e_{A_1} \otimes I_{B_1})} & B_1 \\
\downarrow & & \downarrow \alpha \\
C(X) & \xrightarrow{\mu_2 \hat{\otimes} (e_{A_2} \otimes I_{B_2})} & B_2
\end{array}
$$

and Lemma 2.7 also gives $\beta \in KK_X(C_{uA_1}, C_{uA_2})^{-1}$.

Let $f_i$ denote the element $(\mu_i \hat{\otimes} (e_{A_i} \otimes I_{B_i})) \otimes I_{C_{uA_i}}$, and direct computation yields

$$f_1 \hat{\otimes} (\alpha \otimes I_{C_{uA_1}}) \hat{\otimes} (I_{B_2} \otimes \beta) = ((\mu_2 \hat{\otimes} (e_{A_2} \otimes I_{B_2})) \otimes I_{C_{uA_1}}) \hat{\otimes} (I_{B_2} \otimes \beta)$$

$$= \beta \hat{\otimes} f_2.$$  

The element $\nu'_1 := (\alpha \otimes I_{C_{uA_1}}) \hat{\otimes} (I_{B_2} \otimes \beta) \hat{\otimes} \nu_2$ provides duality classes for $C_{uA_1}$ and $B_1$ and $\nu_1 = (I_{B_1} \otimes \gamma) \hat{\otimes} \nu'_1$.

Now one has the following commutative diagram

$$
\begin{array}{ccc}
C_{uA_1} & \xrightarrow{f_1} & B_1^s \otimes C(X) \\
\downarrow \gamma & & \downarrow I \otimes \gamma \\
C_{uA_1} & \xrightarrow{f_1} & B_1^s \otimes C(X) \\
\downarrow \beta & & \downarrow (\alpha \otimes I) \hat{\otimes} (I \otimes \beta) \\
C_{uA_2} & \xrightarrow{f_2} & B_2^s \otimes C(X)
\end{array}
$$

Since $f_i \hat{\otimes} \nu_i = e_{A_i}$, we can apply [17, Appendix A.] for

$$
\begin{array}{ccc}
SC(X) & \xrightarrow{I_{S \otimes uA_1}} & SA_1 \\
\downarrow & & \downarrow e_{A_1} \\
SC(X) & \xrightarrow{I_{S \otimes uA_2}} & SA_2 \\
\downarrow \gamma \hat{\otimes} \beta & & \downarrow e_{A_2} \\
SC(X) & \xrightarrow{I_{S \otimes uA_1}} & SA_1 & \xrightarrow{I_{S \otimes uA_2}} & SC(X)
\end{array}
$$

and obtain the isomorphism $A_1 \cong A_2$.

Next, we show the surjectivity under the assumption that $D(C_{uA})(\sim KK_B)$ satisfies (d). Fix an arbitrary locally trivial continuous $C(X)$-algebra $B$ with fiber $B$, and we construct $[A] \in [X, BAut(A)]$ with $[B] = D_{A,B}([A])$. There exists
a locally trivial continuous $C(X)$-algebra $\mathcal{D}$ such that $\mathcal{D}$ and $\mathcal{B}$ are Spanier–Whitehead K-dual with duality classes

$$\mu \in KK_X(C(X), \mathcal{D} \otimes C(X) \mathcal{B}), \quad \nu \in KK_X(\mathcal{B} \otimes C(X) \mathcal{D}, C(X)).$$

Using the Cuntz picture, one can replace $(u_B \otimes I_D) \hat{\otimes} \nu \in KK_X(\mathcal{D}, C(X))$ by a $C(X)$-linear $\ast$-homomorphism as in [17, Appendix A]. There exist another $C(X)$-algebra $q_X\mathcal{D}$ with $f \in KK_X(\mathcal{D}, q_X\mathcal{D})^{-1}$ and a $C(X)$-linear $\ast$-homomorphism $d : q_X\mathcal{D} \to C(X) \otimes K$ making the following diagram commute:

$$\begin{array}{ccc}
D & \xrightarrow{(u_B \otimes I_D) \hat{\otimes} \nu} & C(X) \\
\downarrow f & & \downarrow \\
q_X\mathcal{D} & \xrightarrow{d} & C(X) \otimes K.
\end{array}$$

The map $d$ gives the Puppe sequence $SC(X) \otimes K \xrightarrow{\iota_x} C_d \to q_X\mathcal{D} \to C(X) \otimes K$, and we show that the fiber $C_{dx}$ is $KK$-equivalent to $SA$. Since $KK(d_x) \in KK((q_X\mathcal{D})_x, K)$ is identified with $(u_B \otimes I_D_x) \hat{\otimes} \nu_x \in KK(D_x, \mathcal{C})$, the isomorphism $(- \otimes I_{D_x}) \hat{\otimes} \nu_x : KK(\mathcal{C}, B_x) \to KK(D_x, \mathcal{C})$ shows

$$KK((q_X\mathcal{D})_x, K)/\langle KK(d_x) \rangle \cong KK(C_{dx})/\langle KK(u_{B_x}) \rangle \cong K_1(C_{u_B}) = KK(SA, C).$$

Thus, we have isomorphisms

$$KK((q_X\mathcal{D})_x, K)/\langle KK(d_x) \rangle = KK(SA, C) \cong KK(C_{u_A}, C)/\langle KK(e_A) \rangle.$$

Combining Lemma 2.7 and Proposition 2.16, a similar argument as in the proof of Lemma 4.2 gives a $KK$-equivalence $h \in KK(C_{u_A}, (q_X\mathcal{D})_x)^{-1}$ with the following commutative diagram:

$$\begin{array}{ccc}
SK & \xrightarrow{\iota_x} & C_{dx} \\
\downarrow & & \downarrow \\
S & \xrightarrow{I_S \otimes u_A} & SA \xrightarrow{e_A} C_{u_A},
\end{array}$$

and one has $\sigma \in KK(SA, C_{dx})^{-1}$ making the above square commute. Thanks to Theorem 2.4 one has a diagram:

$$\begin{array}{ccc}
SC(X) & \xrightarrow{I_S \otimes [p]_0} & S(SC_d)^0 \otimes K \\
\downarrow & & \downarrow \\
S^2 SC(X) \otimes K & \xrightarrow{I_{S^2} \otimes \iota} & S^2 C_d \\
\downarrow & & \downarrow \\
SC(X) \otimes K & \xrightarrow{\iota} & C_d,
\end{array}$$

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where all vertical maps are $KK_X$-equivalences and the broken arrow determines a properly infinite full projection $p \in (SC_d)^{\sharp} \otimes K$ satisfying
\[
\pi_x(p)((SC_d)^{\sharp} \otimes K)p \cong A.
\]
Thus, we obtain a locally trivial continuous $C(X)$-algebras $A := p((SC_d)^{\sharp} \otimes K)p$ with fiber $A$ and the following commutative diagram

\[
\begin{array}{ccccccccc}
S^2A & \xrightarrow{I_S \otimes e_A} & SC_{uA} & \xrightarrow{I_S \otimes u_A} & SC(X) & \xrightarrow{I_S \otimes u_D} & SA \\
SC_d & \xrightarrow{S(q_X D)} & SC(X) \otimes K & \xrightarrow{\iota} & C_d \\
SD & \xrightarrow{IS \otimes (u_B \otimes I_D) \otimes \nu} & SC(X)
\end{array}
\]

where every vertical arrow is a $KK_X$-equivalence and the broken arrow is given by [17, Appendix A.]. Finally, we obtain a $KK_X$-equivalence $\gamma \in KK_X(C_{uA}, D)$ with $\gamma \otimes (u_B \otimes I_D) \otimes \nu = e_A$. It is straightforward to check $\mu \otimes (\gamma^{-1} \otimes I_B) \otimes (e_A \otimes I_B) = u_B$, and $\mu \otimes (\gamma^{-1} \otimes I_B) \in KK_X(C(X), C_{uA} \otimes C(X) B)$ provides duality classes for $C_{uA}$ and $B$. Since the definition of $D_{A,B}$ is independent of the choice of the duality classes, this implies $[B] = D_{A,B}([A])$.

**Corollary 4.5.** Let $B$ be either the Cuntz standard form of $O_{\infty}$, $M_n(O_{\infty})$, $n \geq 1$, or the unital Kirchberg algebra $KK$-equivalent to $S$. Then, the map $D_{A,B}$ is bijective.

**Remark 4.6.** This map $D_{A,B}$ allows us to obtain the maps constructed in [20, 21] without using Brown’s representability theorem nor the homotopy theory for the automorphism groups of the Cuntz–Toeplitz algebras.

It seems to be not so bad idea that the map $D_{A,B}$ should exist and be bijective for other $B$, and we propose the following question.

**Question 4.7.** Does every separable UCT $C^*$-algebra with finitely generated $K$-groups satisfy (d)?

**Remark 4.8.** By the path connectedness of $End(O_n)$, one can show that every locally trivial continuous $C(X)$-algebra with fiber $O_n$ satisfies the $K$-duality. However, this says nothing for the continuous $C(X)$-algebras whose fiber is the standard form of the Cuntz algebras, and the problem of whether $O_n$ satisfies (d) is not solved yet.
5 Appendix

We use the same notation and terminology as in Section 2.5.

Lemma 5.1. Let $G = G(p) \neq 0$ be a finite Abel $p$-group with $L(G) = t$. For $1 \leq l \leq \max\{k \in I(G)\}$ and an element $g \in G$ of order $p^l$, there exists $t \geq s \geq 1$ and we have an isomorphism

$$G \ni g \mapsto (0, \cdots, 0, p^{k_1-r_1}, \cdots, p^{k_s-r_s}) \in (\mathbb{Z}_{p^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{n_{l-s}}}) \oplus (\mathbb{Z}_{p^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{k_s}})$$

with $r_s = l, 0 < r_s$, where the set $\{n_1, \cdots, n_{l-s}, k_1, \cdots, k_s\}$ is equal to $I(G)$. For $s \geq 2$, the above $k_i, r_i$ satisfy

$$k_i < k_{i+1}, \quad r_i < r_{i+1}, \quad 0 \leq k_1 - r_1, \quad k_i - r_i < k_{i+1} - r_{i+1}.$$

We note that the set $\{n_1, \cdots, n_{l-s}\}$ is empty if $t = s$.

Proof. Every element $x \in \mathbb{Z}_{p^k}$ of order $p^r$ ($r \leq k$) is of the form $x = a p^{k-r}$ with $\text{GCD}(a, p) = 1, a \in \mathbb{Z}$. One has $a^{-1} \in \mathbb{Z}_{p^k}$ and the multiplication by $a^{-1}$ is an automorphism of $\mathbb{Z}_{p^k}$ that sends $x$ to $p^{k-r}$. Thus we have an isomorphism

$$G \ni g \mapsto (x_1, \cdots, x_t) \in \mathbb{Z}_{p^k} \oplus \cdots \oplus \mathbb{Z}_{p^k},$$

where every $x_i$ is of the form $x_i = p^{k_i-R_i}, 0 \leq R_i \leq l$.

We may assume $k_1' < k_2' < \cdots < k_t'$, and write $i_1 := \min\{i \mid R_i = l\}$. We will consider the following two cases :

$I)$: there exists $i > i_1$ with $x_i = p^{k_i'-R_i} \neq 0 \in \mathbb{Z}_{p^{k_i'}}$,

$II)$: there exists $i < i_1$ with $x_i = p^{k_i'-R_i} \neq 0 \in \mathbb{Z}_{p^{k_i'}}$.

If there exists $i > i_1$ satisfying $I)$, we have

$$k_i' \geq k_{i_1}', \quad R_i \leq R_{i_1} = l, \quad (k_i' - R_i) - (k_{i_1}' - R_{i_1}) + k_{i_1}' \geq k_i'.$$

Thus, the following isomorphism

$$\mathbb{Z}_{p^{k_i'}} \oplus \mathbb{Z}_{p^{k_i'}} \ni (x, y) \mapsto (x, xp^{(k_i'-R_i)-(k_{i_1}'-R_{i_1})} + y) \in \mathbb{Z}_{p^{k_i'}} \oplus \mathbb{Z}_{p^{k_i'}}$$

is well-defined and this sends $(x_{i_1}, 0)$ to $(x_{i_1}, x_i)$.

If there exists $i < i_1$ satisfying $II)$, we have

$$(k_i' - R_i) - (k_{i_1}' - R_{i_1}) + k_{i_1}' \geq k_i',$$

and the following isomorphism is well-defined

$$\mathbb{Z}_{p^{k_i'}} \oplus \mathbb{Z}_{p^{k_i'}} \ni (y, x) \mapsto (xp^{(k_i'-R_i)-(k_{i_1}'-R_{i_1})} + y, x) \in \mathbb{Z}_{p^{k_i'}} \oplus \mathbb{Z}_{p^{k_i'}}.$$

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and this sends \((0, x_i)\) to \((x_i, x_i)\).

Now one can obtain an isomorphism

\[
\mathbb{Z}_{p^{j_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{j_s}} \ni (x_1, \cdots, x_t) \mapsto (y_1, \cdots, y_t) \in \mathbb{Z}_{p^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{k_t}}
\]

with \(y_i = p^{k_i - R_i}, R_i \leq l\) such that there uniquely exists \(i_1\) satisfying

\[
R_{i_1} = l, \quad y_j = 0 \text{ for } i_1 < j, \quad R_j < R_{i_1} \text{ for } j < i_1, \quad k_j' - R_j < k_{i_1}' - R_{i_1} \text{ for } y_j \neq 0.
\]

Since \(k_j' - R_j < k_{i_1}' - R_{i_1}\) and \(R_j < R_{i_1}\) imply \(k_j' < k_{i_1}'\), we can prove the statement by applying the same argument for \((y_1, \cdots, y_{(i_1-1)}) \in \mathbb{Z}_{p^{k_1'}} \oplus \cdots \oplus \mathbb{Z}_{p^{k_{i_1-1}'}}\) inductively.

**Corollary 5.2.** Let \(g \in G = G(p), I(G) = \{n_1, \cdots, n_{l-s}, k_1, \cdots, k_s\}\) and \(r_i\) be as in Lemma 5.1. Then the quotient group \(G/\langle g \rangle\) is isomorphic to

\[
(\mathbb{Z}_p n_1 \oplus \cdots \oplus \mathbb{Z}_p n_{l-s}) \oplus (\bigoplus_{i=2}^s \mathbb{Z}_{p^{k_i-r_i+r_{i-1}}} (s \geq 2),
\]

\[
(\mathbb{Z}_p n_1 \oplus \cdots \oplus \mathbb{Z}_p n_{l-1}) \oplus \mathbb{Z}_{p^{k_{l-1}-1}} (s = 1).
\]

In particular, we have \(I(G) \cap I(G/\langle g \rangle) = \{n_1, \cdots, n_{l-s}\}\), and \((G/\langle g \rangle, G)\) satisfies (**).

**Proof.** It is easy to verify the case of \(s = 1\), and we prove the case of \(s \geq 2\). By Lemma 5.1 we have

\[
G/\langle g \rangle \cong (\mathbb{Z}_p n_1 \oplus \cdots \oplus \mathbb{Z}_p n_{l-s}) \oplus ((\mathbb{Z}_p k_1 \oplus \cdots \oplus \mathbb{Z}_p k_s)/((p^{k_1-r_1}, \cdots, p^{k_s-r_s}))).
\]

We write \(h_i := (y_i^1, \cdots, y_i^s) \in \mathbb{Z}_{p^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{k_s}}\) with

\[
y_j^i = 0 \quad (j < i), \quad y_i^i = 1, \quad y_j^i = p^{(k_j'-r_j)-(k_i-r_i)} \quad (j > i).
\]

Note that \(\{h_i\}_{i=1}^s\) generate \(\mathbb{Z}_{p^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{k_s}}\) and they satisfy the following equations

\[
p^{k_1-r_1}h_1 = (p^{k_1-r_1}, \cdots, p^{k_s-r_s}),
\]

\[
p^{k_1-r_1+r_{i-1}}h_i = p^{r_{i-1}}(p^{k_1-r_1}, \cdots, p^{k_s-r_s})
\]

which provide the surjection

\[
\mathbb{Z}_{p^{k_1-r_1}} \oplus \bigoplus_{i=2}^s \mathbb{Z}_{p^{k_i-r_i+r_{i-1}}} \rightarrow (\mathbb{Z}_{p^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{k_s}})/((p^{k_1-r_1}, \cdots, p^{k_s-r_s})).
\]

Since \(|(\mathbb{Z}_{p^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{k_s}})/((p^{k_1-r_1}, \cdots, p^{k_s-r_s}))| = p^{k_1+r_1+\cdots+k_s/r_{s-1}}\), the above map is bijective.
By Lemma 5.1, one has $k_{i-1} < k_i - r_i + r_{i-1} < k_i$ and this proves

$$I(G/\langle g \rangle) \cap I(G) \subset \{n_1, \ldots, n_{t-s}, k_1 - r_1, k_2 - r_2 + r_1, \ldots, k_s - r_s + r_{s-1}\} \cap \{n_1, \ldots, n_{t-s}, k_1, \ldots, k_s\} = \{n_1, \ldots, n_{t-s}\}.$$ 

Since $\{n_1, \ldots, n_{t-s}\} \subset I(G/\langle g \rangle) \cap I(G)$, we have $\{n_1, \ldots, n_{t-s}\} = I(G/\langle g \rangle) \cap I(G)$. \hfill \Box

**Corollary 5.3.** Let $g_m \in G = G(p), m = 1, 2$ be elements of the finite Abel $p$-group satisfying $G/\langle g_1 \rangle \cong G/\langle g_2 \rangle$. Then, there exists an automorphism of $G$ that sends $g_1$ to $g_2$.

**Proof.** If $g_1 = 0$, the statement is trivial. So we may assume that $g_1$ and $g_2$ have the same order $p^l, l \geq 1$. By assumption, one has

$$I(G) \cap I(G/\langle g_1 \rangle) = I(G) \cap I(G/\langle g_2 \rangle) = \{n_1, \ldots, n_{t-s}\},$$

where we write $I(G) = \{n_1, \ldots, n_{t-s}, k_1, \ldots, k_s\}$. There exist isomorphisms

$$G \ni g_m \mapsto (0, \ldots, 0, p^{k_1-r_1^m}, \ldots, p^{k_s-r_s^m}) \in (\mathbb{Z}_{p^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{n_{t-s}}}) \oplus (\mathbb{Z}_{p^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{k_s}})$$

by Lemma 5.1 and Corollary 5.2 and $k_i, r_i^m$ satisfy

$$k_i < k_{i+1}, \quad 0 < r_i^m < r_{i+1}^m, \quad 0 \leq k_i - r_i^m < k_{i+1} - r_{i+1}^m, \quad r_s^m = l.$$ 

The assumption $G/\langle g_1 \rangle \cong G/\langle g_2 \rangle$ and Corollary 5.2 show $r_i^1 = r_i^2$, and this proves the statement. \hfill \Box

For a finite Abel $p$-group $G = G(p)$ and $g \in G$, $l \geq 0$, the map

$$G \ni x \mapsto [(x, 0)] \in (G \oplus \mathbb{Z})/\langle (g, p^l) \rangle$$

is injective and gives an exact sequence

$$0 \to G \to (G \oplus \mathbb{Z})/\langle (g, p^l) \rangle \to \mathbb{Z}_{p^l} \to 0.$$ 

Thus, one has $|(G \oplus \mathbb{Z})/\langle (g, p^l) \rangle| = p^l |G|$, and we will see that $(G, (G \oplus \mathbb{Z})/\langle (g, p^l) \rangle)$ satisfies (**). If $l = 0$, one can easily find an isomorphism $G \oplus \mathbb{Z} \ni (g, 1) \mapsto (0, 1) \in G \oplus \mathbb{Z}$.

**Lemma 5.4.** For $l > 0$ and $g \neq 0$, we assume that there are no automorphisms of $G \oplus \mathbb{Z}$ which send $(g, p^l)$ to $(0, p^l)$. Then, for some $s \geq 1$, there exists an isomorphism

$$G \oplus \mathbb{Z} \to (\mathbb{Z}_{p^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{n_{t-s}}}) \oplus (\mathbb{Z}_{p^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{k_s}}) \oplus \mathbb{Z}$$

which sends $(g, p^l)$ to $(0, \ldots, 0, p^{k_1-r_1}, \ldots, p^{k_s-r_s}, p^l)$ with $k_i - r_i < l, 0 < r_i$, and we have

$$k_i < k_{i+1}, \quad 0 \leq k_i - r_i < k_{i+1} - r_{i+1}, \quad 0 < r_i < r_{i+1} \text{ for } s \geq 2.$$ 

Here, we write $I(G) = \{n_1, \ldots, n_{t-s}, k_1, \ldots, k_s\}$ as in Lemma 5.1.
Proof. Since $g \neq 0$, the same argument as in Lemma 5.1 gives an isomorphism

$$G \oplus \mathbb{Z} \to (\mathbb{Z}_{p^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{n_{t-s}}}) \oplus (\mathbb{Z}_{p^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{k_s}}) \oplus \mathbb{Z}$$

which sends $(g, p^j)$ to $(0, \cdots, 0, p^{k_1-r_1}, \cdots, p^{k_s-r_s}, p^j)$ with $0 < r_i$, and we have

$$k_i < k_{i+1}, \quad 0 \leq k_i - r_i < k_{i+1} - r_{i+1}, \quad 0 < r_i < r_{i+1}$$

If $k_s - r_s \geq l$, the following isomorphism

$$\mathbb{Z}_{p^{k_s}} \oplus \mathbb{Z} \ni (y, x) \mapsto (xp^{k_s-r_s}-l + y, x) \in \mathbb{Z}_{p^{k_s}} \oplus \mathbb{Z}$$

sends $(0, p^j)$ to $(p^{k_s-r_s}, p^j)$. Therefore we may assume $k_i - r_i < k_s - r_s < l$, and this proves the statement. \[ \square \]

Corollary 5.5. Let $(g, p^j), k_i, r_i$ and $n_1, \cdots, n_{t-s}$ be as in Lemma 5.4. Then the quotient group $(G \oplus \mathbb{Z})/\langle (g, p^j) \rangle$ is isomorphic to

$$(\mathbb{Z}_{p^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{n_{t-s}}}) \oplus (\mathbb{Z}_{p^{k_1-r_1}} \oplus (\bigoplus_{i=2}^{s} \mathbb{Z}_{p^{k_i-r_i+r_{i-1}}}) \oplus \mathbb{Z}_{p^{l+r_s}}) \quad (s \geq 2),$$

$$(\mathbb{Z}_{p^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{n_{t-s}}}) \oplus (\mathbb{Z}_{p^{k_1-r_1}} \oplus \mathbb{Z}_{p^{l+r_1}}) \quad (s = 1).$$

If $s = 1$, one has $0 \leq k_1 - r_1 < k_1 < l + r_1$. If $s \geq 2$, one has

$$0 \leq k_1 - r_1 < k_1 < k_2 - r_2 + r_1, \quad k_i - r_i + r_{i-1} < k_i < k_{i+1} - r_{i+1} + r_i, \quad k_s - r_s + r_{s-1} < k_s < l + r_s.$$

In particular, we have

$$I(G) \cap I((G \oplus \mathbb{Z})/\langle (g, p^j) \rangle) = \{n_1, \cdots, n_{t-s}\} \neq I(G),$$

and $(G, (G \oplus \mathbb{Z})/\langle (g, p^j) \rangle)$ satisfies (**).

Proof. We consider the case of $s \geq 2$, and the proof for $s = 1$ is the same. We write

$$(g, p^j) = (0, \cdots, 0, p^{k_1-r_1}, \cdots, p^{k_s-r_s}, p^j) \in (\mathbb{Z}_{p^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{n_{t-s}}}) \oplus (\mathbb{Z}_{p^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{k_s}}) \oplus \mathbb{Z},$$

and it is enough to show

$$(\mathbb{Z}_{p^{k_1-r_1}} \oplus (\bigoplus_{i=2}^{s} \mathbb{Z}_{p^{k_i-r_i+r_{i-1}}}) \oplus \mathbb{Z}_{p^{l+r_s}}) \cong (\mathbb{Z}_{p^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{k_s}}) \oplus \langle (p^{k_1-r_1}, \cdots, p^{k_s-r_s}, p^j) \rangle.$$
with
\[ h'_j = 0 \ (j < i), \quad h'_i = 1, \quad h''_j = p^{(k_j - r_j) - (k_i - r_i)} \ (j > i). \]

We have
\[ p^{k_1 - r_1} u_1 = (p^{k_1 - r_1}, \ldots, p^{k_s - r_s}, p^l), \quad p^{l + r_s} u_{s+1} = p^{r_s} (p^{k_1 - r_1}, \ldots, p^{k_s - r_s}, p^l) \]

and
\[ p^{k_i - r_i + r_i - 1} u_i = p^{r_i - 1} (p^{k_1 - r_1}, \ldots, p^{k_s - r_s}, p^l) \]

for \( 2 \leq i \leq s \). Since \( \{u_j\}_{j=1}^{s+1} \) generate \( \mathbb{Z}_{p^1} \oplus \cdots \oplus \mathbb{Z}_{p^k} \oplus \mathbb{Z} \), one has a surjection
\[ (\mathbb{Z}_{p^1} \oplus \cdots \oplus \mathbb{Z}_{p^k} \oplus \mathbb{Z}) \to (\mathbb{Z}_{p^1} \oplus \cdots \oplus \mathbb{Z}_{p^k} \oplus \mathbb{Z})/\langle (p^{k_1 - r_1}, \ldots, p^{k_s - r_s}, p^l) \rangle, \]

which is bijective by the equation
\[
\begin{align*}
|\langle (\mathbb{Z}_{p^1} \oplus \cdots \oplus \mathbb{Z}_{p^k} \oplus \mathbb{Z})/\langle (p^{k_1 - r_1}, \ldots, p^{k_s - r_s}, p^l) \rangle| &= p^l |\mathbb{Z}_{p^1} \oplus \cdots \oplus \mathbb{Z}_{p^k} | \\
&= p^{k_1 + \cdots + k_s + l}.
\end{align*}
\]

If \((g, p^l)\) is sent to \((0, p^l)\), the quotient group \((G \oplus \mathbb{Z})/\langle (g, p^l) \rangle\) is equal to \(G \oplus \mathbb{Z}_{p^l}\), and the pair \((G, (G \oplus \mathbb{Z})/\langle (g, p^l) \rangle)\) satisfies (**)．

**Corollary 5.6.** Let \(G = G(p)\) be a finite Abelian \(p\)-group. For \((g_m, p^l) \in G \oplus \mathbb{Z}, (m = 1, 2, l \geq 0)\), there exists an automorphism \(\Theta : G \oplus \mathbb{Z} \to G \oplus \mathbb{Z}\) with \(\Theta((g_1, p^l)) = (g_2, p^l)\) if and only if we have \((G \oplus \mathbb{Z})/\langle (g_1, p^l) \rangle \cong (G \oplus \mathbb{Z})/\langle (g_2, p^l) \rangle\).

**Proof.** We may assume \(l > 0\). If there exists an automorphism of \(G \oplus \mathbb{Z}\) which sends \((g_1, p^l)\) to \((0, p^l)\), the quotient group \((G \oplus \mathbb{Z})/\langle (g_1, p^l) \rangle\) is isomorphic to \(G \oplus \mathbb{Z}_{p^l}\), and we have
\[
\begin{align*}
I(G) &= I(G) \cap I((G \oplus \mathbb{Z})/\langle (g_1, p^l) \rangle) \\
&= I(G) \cap I((G \oplus \mathbb{Z})/\langle (g_2, p^l) \rangle)
\end{align*}
\]

and Corollary 5.5 shows that \((g_2, p^l)\) must be sent to \((0, p^l)\) by an automorphism. So we may assume that there are no automorphisms that send \((g_m, p^l)\) to \((0, p^l)\). Since
\[
I(G) \cap I((G \oplus \mathbb{Z})/\langle (g_1, p^l) \rangle) = I(G) \cap I((G \oplus \mathbb{Z})/\langle (g_2, p^l) \rangle),
\]
Corollary 5.5 and Lemma 5.4 give an isomorphism which sends \((g_m, p^l) \in G \oplus \mathbb{Z}\) to the following element:
\[
(0, \ldots, 0, p^{k_1 - r_1}, \ldots, p^{k_s - r_s}, p^l) \in (\mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_{k-1}}) \oplus (\mathbb{Z}_{p_{k-1}} \oplus \cdots \oplus \mathbb{Z}_{p_k}) \oplus \mathbb{Z},
\]

and Corollary 5.5 proves \(r_i^1 = r_i^2\) for \(1 \leq i \leq s\).
Lemma 5.7. Let $G = \bigoplus_p G(p)$ be a finite Abel group with $g = (g_p)_p \in G$, and let $n = \prod_p p^{n_p}$ be the prime decomposition of $n \in \mathbb{Z}_{\geq 1}$. Then, the following hold:

1. $(G/\langle g \rangle)(p) \cong G(p)/\langle g_p \rangle$,
2. $((G \oplus \mathbb{Z})/\langle (g, n) \rangle)(p) \cong (G(p) \oplus \mathbb{Z})/\langle (g_p, p^{n_p}) \rangle$.

Proof. We write $n = \prod_p p^{n_p} = p^{n_p}r_p$, and let $R_p$ denote $\prod_{q \neq p} \text{ord}(g_q)$. One has $\langle g \rangle = \bigoplus_p \langle g_p \rangle \subset G$ by $\text{GCD}(R_p, p) = 1$, and this proves the statement 1.

Next, we check the statement 2. For a finite set $F := \{p \mid G(p) \neq 0, \text{ or } n_p \neq 0\}$, one has $\text{GCD}\{R_p r_p \mid p \in F\} = 1$, which implies the surjectivity of the map

$$\bigoplus_{p \in F} (G(p) \oplus \mathbb{Z}) \ni ((h_p, x_p))_p \mapsto ((R_p h_p)_p, \sum_p R_p r_p x_p) \in \bigoplus_{p \in F} G(p) \oplus \mathbb{Z}.$$ 

The above map induces a well-defined surjection

$$\bigoplus_{p \in F} ((G(p) \oplus \mathbb{Z})/\langle (g_p, p^{n_p}) \rangle) \twoheadrightarrow (\bigoplus_{p \in F} G(p) \oplus \mathbb{Z})/\langle (g_p)_p, n \rangle.$$ 

Since $\prod_{p \in F} p^{n_p}|G(p)| = n|G|$, the above map is injective, and this proves the statement 2.

We prove Proposition 2.15 and 2.16 using the above preliminaries.

Proof of Proposition 2.15. First, we discuss the case that $[1_A]_0 \in K_0(A)$ is a torsion element. We write $K_0(A) = (\bigoplus_p K_0(A)(p)) \oplus F \ni [1_A]_0 = ((g_p)_p, 0, \cdots, 0)$, and it is easy to check that $K_1(C_{u_A})(p) = K_0(A)(p)/\langle g_p \rangle$. Thus, Corollary 5.2 proves that $(K_1(C_{u_A})(p), K_0(A)(p))$ satisfies $(\ast \ast)\hspace{1cm}(\ast \ast)$.

Next, we consider the case that $[1_A]_0 \in K_0(A)$ is not a torsion element. Then, we may assume

$$[1_A]_0 = ((g_p)_p, n, 0, \cdots, 0) \in K_0(A) = (\bigoplus_p K_0(A)(p)) \oplus \mathbb{Z}^{1+F}$$

for some $n \geq 1$. For the prime decomposition $n = \prod_p p^{n_p}$, Lemma 5.7 proves $K_1(C_{u_A})(p) = (K_0(A)(p) \oplus \mathbb{Z})/\langle (g_p, p^{n_p}) \rangle$, and Corollary 5.5 shows that $(K_0(A)(p), K_1(C_{u_A})(p))$ satisfies $(\ast \ast)$.

Proof of Proposition 2.16. Let $G$ be a finitely generated Abel group, and let $g_1, g_2 \in G$ satisfy $G/\langle g_1 \rangle \cong G/\langle g_2 \rangle$. Then, $g_1$ is a torsion element (resp. a non-torsion element) if and only if so is $g_2$ by the comparison of ranks of $G/\langle g_i \rangle$.

By comparison of the torsion parts of $G/\langle g_i \rangle$, we may assume that $G = H \oplus \mathbb{Z}$ for a finite Abel group $H$ and $g_i = (h_i, n) \in H \oplus \mathbb{Z}$ for some $n \in \mathbb{Z}_{\geq 0}$. We write $h_i = (h^i_p)_p \in H = \bigoplus_p H(p), n = \prod_p p^{n_p}$, and Lemma 5.7 shows

$$(G/\langle g_i \rangle)(p) \cong H(p)/\langle h^i_p \rangle \hspace{1cm} (n = 0),$$

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\( (G/\langle g_i \rangle)(p) \cong \frac{H(p) \oplus \mathbb{Z}}{\langle (h^1_p, p^n) \rangle} \quad (n \neq 0). \)

We write \( r_p = \prod_{q \neq p} q^{n_q} = n/p^n, \) and denote by \( r_p^{-1} \) the inverse map of \( H(p) \ni x \mapsto r_p x \in H(p). \) Now one has

\[
H(p)/\langle h^1_p \rangle \cong H(p)/\langle h^2_p \rangle \quad (n = 0),
\]

\[
(H(p) \oplus \mathbb{Z})/\langle (r_p^{-1}(h^1_p), p^n) \rangle \cong (H(p) \oplus \mathbb{Z})/\langle (r_p^{-1}(h^2_p), p^n) \rangle \quad (n \neq 0),
\]

and Corollary 5.3 proves the statement for the case \( n = 0. \) Applying Corollary 5.6 to the case \( n \neq 0, \) one has an automorphism \( \theta \) of \( H(p) \oplus \mathbb{Z} \) with

\[
\theta((r_p^{-1}(h^1_p), p^n)) = (r_p^{-1}(h^2_p), p^n).
\]

Since the map \( (\bigoplus_{q \neq p} \text{id}_{H(q)}) \otimes \theta \) sends \( ((h^1_q, h^1_p, n) \in (\bigoplus_p H(p)) \oplus \mathbb{Z} \) to \( ((h^1_q, h^2_p, n), \) one can obtain desired isomorphism \( H \oplus \mathbb{Z} \ni ((h^1_p, n) \mapsto ((h^2_p), n) \in H \oplus \mathbb{Z} \) by applying the same argument for every \( q. \)

\[ \square \]

References

[1] B. Blackadar, K-theory for operator algebras, 2nd ed., Math. Sci. Inst. Publ., Vol. 5, Cambridge University Press, Cambridge, 1998.

[2] E. Blanchard, Tensor products of \( C(X) \)-algebras over \( C(X), \) Recent advances in operator algebras, No. 232, (1995), 81–91.

[3] E. Blanchard and E. Kirchberg, Non-simple purely infinite \( C^* \)-algebras : the Hausdorff case. J. Funct. Anal., 207: 461–513, 2004.

[4] N. P. Brown and N. Ozawa, \( C^* \)-algebras and finite dimensional approximations, \) vol. 88, Amer. Math. Soc., 2008.

[5] M. Dadarlat, Fiberwise KK-equivalence of continuous fields of \( C^* \)-algebras, J. K-theory 3 (2009), no. 2, 205–219.

[6] M. Dadarlat, The \( C^* \)-algebra of a vector bundle, J. Reine Angew. Math. 670 (2012), 121–143.

[7] M. Dadarlat, The homotopy groups of the automorphism groups of Kirchberg algebras, J. Noncommut. Geom. 1 (2007), 113–139.

[8] M. Dadarlat, and U. Pennig, A Dixmier–Douady theory for strongly self-absorbing \( C^* \)-algebras, J. Reine Angew. Math. 718 (2016), 153–181.

[9] J. Dixmier and A. Douady, Champs continus d’espaces hilbertiens et de \( C^* \)-algebres. Bull. Soc. Math. France, 91: 227–284, 1963.

[10] M. Izumi, and H. Matui, Poly-\( \mathbb{Z} \) group actions on Kirchberg algebras II, Invent. Math. 224 (2021), 699-766.
[11] D. S. Kahn, J. Kaminker and C. Schochet, Generalized homology theories on compact metric spaces, Michigan Mathematical Journal 24.2 (1977), 203-224.

[12] J. Kaminker and C. Schochet, Spanier–Whitehead K-duality for C*-algebras, Journal of Topology and Analysis, Vol. 11, No. 01, 21–52 (2019).

[13] J. Kaminker and I. Putnam, K-theoretic duality for shifts of finite type, Comm. Math. Phys. 187 (1997), 509–522.

[14] J. Kaminker, I. Putnam and M.F. Whittaker, K-theoretic duality for hyperbolic dynamical systems, J. Rein Angew. Math., DOI 10.1515/crelle-2014-0126.

[15] G. G. Kasparov, Equivariant KK-theory and the Novikov conjecture, Invent. Math., 91 (1) : 147–201, 1988.

[16] E. Kirchberg, Das nicht-kommutative Michael-auswahlprinzip und die klassifikation nicht-einfacher algebra, In C*-algebras, 92–141, Berlin, 2000. Springer.

[17] R. Meyer and R. Nest, The Baum–Connes conjecture via localisation of categories, Topology 45 (2006), n.0. 2, 209–259.

[18] M. Rørdam and E. Størmer, Classification of nuclear C*-algebras. Entropy in operator algebras, Encyclopaedia of Mathematical Sciences Volume 126, Springer-Verlag Berlin Heidelberg 2002.

[19] U. Pennig, A noncommutative model for higher twisted K-theory, Journal of Topology 9 (1), 27–50.

[20] T. Sogabe, A topological invariant for continuous fields of Cuntz algebras. Math. Ann. 380, 91–117 (2021). https://doi.org/10.1007/s00208-020-02101-6

[21] T. Sogabe, A topological invariant for continuous fields of Cuntz algebras II, Proc. Amer. Math. Soc. 150 (2022), 1059-1070.

[22] A. S. Toms and W. Winter, Strongly self-absorbing C*-algebras, Trans. Amer. Math. Soc. 359 (2007), no. 8, 3999–4029.