Hypermatrix factors for string and membrane junctions

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Abstract

The adjoint representations of the Lie algebras of the classical groups $SU(n)$, $SO(n)$ and $Sp(n)$ are, respectively, tensor, antisymmetric and symmetric products of two vector spaces, and hence are matrix representations. We consider the analogous products of three vector spaces and study when they appear as summands in Lie algebra decompositions. The $\mathbb{Z}_3$-grading of the exceptional Lie algebras provides such summands and provides representations of classical groups on hypermatrices. The main natural application is a formal study of 3-junctions of strings and membranes. Generalizations are also considered.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Classical Lie groups admit representations on vector spaces as well as on second powers of the vector spaces, underlying the Lie algebra. The first is the fundamental representation and the second is the adjoint representation. The next step is representations on triple product of vector spaces, which is the context of this paper. We consider $\mathbb{Z}_3$-graded decomposition of Lie exceptional Lie algebras $g$ along the lines of [56, 57].

The degree 0 piece is a classical Lie algebra which acts on degree 1 and degree $−1$ pieces via the module structure inherited from the Lie bracket on $g$. We study this action from the viewpoint of representations on hypermatrices, which are higher-dimensional generalizations of matrices. The latter are two-dimensional arrays of numbers while the former are $n$-dimensional such arrays, $n \geq 0$. These can take values in $\mathbb{C}$ or $\mathbb{R}$ or even finite fields. On these,
we will use scalar invariants, the corresponding trace and hyperdeterminant [28] generalizing the usual trace and determinant of matrices. Generally, the situations we encounter are summarized as follows:

\[
\text{[Exceptional Lie algebra]} = \{\text{‘dual’ cubic hypermatrix}\} \oplus \text{[classical Lie algebra]}
\]

\[
\oplus \text{[cubic hypermatrix]}.
\]

All exceptional algebras appear, most notably \(E_8\) and \(E_6\), corresponding to summands the tensor power \(\otimes^3 V\) and the exterior power \(\wedge^3 V\) in the graded Lie algebra decomposition, as well as their subgroups such as \(D_4\), which corresponds to a summand the symmetric power \(S^3 V\).

Hyperdeterminants have appeared in applications to string theory, starting in [23] (see [12] for a review). We consider other applications, where not only hyperdeterminants but hypermatrices also appear, in the following context. One of the original motivations for string theory was to describe mesons. A meson is formed of a quark \(q\) and an antiquark \(\bar{q}\) i.e. \(q\bar{q}\).

The modern viewpoint (see [45] for details) is that the endpoints of the strings carry \(U(1)\) degrees of freedom and can end on D-branes. The gauge group arising from \(n\) coincident D-branes becomes non-Abelian \(U(n)\). \(U(1)\) corresponds to degrees of freedom for the center of mass and \(SU(n)\) for the relative degrees of freedom. There are also models that extend the above description to baryons (see e.g. [36]). A baryon is formed of a triplet of quarks, i.e. \(qqq\). The modern incarnation of this is string junctions or prongs [50].

Since the prongs of a three-pronged string are mutually non-local, they cannot all end on D-branes in general. The exception is the D3-brane, on which any \((p, q)\) string can end [9]. Here \((p, q)\) denotes an \(SL(2, \mathbb{Z})\) doublet with \(p\) and \(q\) coprime integers. Note that by S-duality one can have \((p, q)\) strings and \((p, q)\) D3-branes. Thus D3-branes are allowed boundaries for three-pronged open strings. Since one needs at least three D3-branes to support a three-pronged string, the states should arise for gauge groups at least as large as \(SU(3)\) [9].

In gauge theory, the number of degrees of freedom corresponding to \(U(n)\) is \(n^2\), which is the dimension of the adjoint representation. This appears for theory of the open string ending on \(n\) coincident D-branes. On the other hand, the membrane in M-theory can end on the fivebrane [52]. This M-brane configuration is T-dual to the abovementioned picture of having strings end on D3-branes. One can consider open membranes ending on multiple fivebranes in analogy to open strings ending on multiple D-branes. The triple string junction arises from M-theory by starting with a pant configuration of membrane and wrapping each of the membrane prongs on different cycles of the compactified two-dimensional torus [2, 50]. The resulting field theory is not well understood. The number of degrees of freedom in this case scales as the cube \(n^3\) of the number of M5-branes [6, 31, 38]. This suggests that a description might fall outside the scope of finite dimensional semi-simple Lie groups and algebras as none of those have a dimension growing as fast as \(n^2\) (where \(n\) would be the dimension of the Cartan sub-algebra) [7, 8, 38].

Configurations with multiple membranes are also allowed. The membrane fields do not have to be in the Lie algebra of \(N \times N\) matrices \(\text{Mat}_N(\mathbb{C})\) and the membrane fivebrane interaction seems to be out of the realm of matrix theory at the moment [9]. It is shown in [20] that the Lie 3-algebras proposed in [4] to model multiple membranes can be encoded in an ‘ordinary’ Lie algebra together with some representation. In fact, the relation found in [20] between classes of metric 3-algebras and unitary representations of Lie algebras is much more general than for just the 3-Lie algebras, which only encode maximally supersymmetric M2-brane theories. For example, it applies to the ABJM theory in [1], where the 3-algebra corresponds to a so-called anti-Jordan triple system rather than a Lie 3-algebra. The precise
relationship was clarified in [19] for all the M2-brane theories which are at least half-BPS. Hence, Lie 3-algebras do not seem to be absolutely indispensable for models of multiple membranes, e.g. [1]. Therefore, in this paper we propose to keep working with Lie algebras, but to view them from a different angle as above.

The representations we consider are not fundamental. Other representations, which are direct sums, were considered in [49] to implement an exceptional symmetry, namely that of the real Lie algebra of type $G_2$. The complex case, $G_2(\mathbb{C})$, cannot be seen within the Lie 3-algebra formalism since in this case the vector space $V = \mathbb{C}^3$ has dimension 3, and hence cannot support (complex) Lie 3-algebras. However, the complex case can be implemented in the current context. The implementation of the above proposal further leads to exceptional algebras of type $E_6$ and $E_7$, as well as $F_4$.

As a natural byproduct of our formalism, we show that the symmetry of the gauge fields resulting from the dimensional reduction of 11-dimensional supergravity to three dimensions is that of the exceptional Lie algebra $e_8$. This is obtained in section 5.1 using a $\mathbb{Z}_3$-graded model for $e_8$, and thus proves an assertion in [17].

To make the paper as self-contained as possible, we have kept enough expository parts both on elementary—but perhaps not widely known—discussions of nonlinear algebra and representation theory, as well as on the applications to strings and branes in physics.

2. Tensor product decompositions and Lie algebras

Reminder on Lie algebra representations. We start by reviewing some basic notions which we will use in this paper.

(1) A vector space $W$ is called a representation of a Lie algebra $\mathfrak{g}$, or a $\mathfrak{g}$-module, if there is a Lie algebra homomorphism $\mathfrak{g} \to gl(W)$.

(2) When $W = \mathfrak{g}$, the map $\text{ad}_\mathfrak{g} : \mathfrak{g} \to gl(\mathfrak{g})$ defined by $\text{ad}_X(Y) = [X, Y]$, the Lie bracket of $X, Y \in \mathfrak{g}$, is the adjoint representation of $\mathfrak{g}$.

(3) Let $V$ be a $\mathfrak{g}$-module. Then $W^*$ is the dual (or contragredient) representation given by $(X \cdot f)(v) = -f(X \cdot v)$ for $X \in \mathfrak{g}$, $f \in W^*$, $v \in W$.

(4) The dual of the adjoint representation is the coadjoint representation of $\mathfrak{g}$ and $\mathfrak{g}^*$ is a map $\text{ad}^* : \mathfrak{g} \to gl(\mathfrak{g}^*)$ defined by $\text{ad}_X^* a(Y) = a(-\text{ad}_X Y) = -a([X, Y])$, for $a \in \mathfrak{g}^*$, $X, Y \in \mathfrak{g}$.

2.1. The case of an open string

The open string Chan–Paton [44] factors lead to matrix Lie groups as follows (see [29]).

(1) Assign a vector space $V$ to each of the two point boundaries of the open string.

(2) Form the tensor product $V \otimes V$ in the case of unoriented string and $V \otimes \overline{V}$, where $\overline{V}$ is the complex conjugate, in the case of oriented strings. The former tensor product is a special case of the latter when $\overline{V} = V$.

(3) Explicitly, the states for the two-ended open string are represented by matrices $\lambda^i_j$, where $i$ is an index for the states of a ‘quark’ and $j$ is an index for the states of the corresponding ‘antiquark’.

(4) Require that the adjoint representation $\text{ad} \mathfrak{g}$ be (inside) $V \otimes \overline{V}$ so that the spectrum of the string contains a vector gauge field.

(5) The set of anti-Hermitian operators is required to form an algebra, as well as the set of linear combinations of Hermitian and anti-Hermitian operators. This means that $\mathfrak{g} = \mathfrak{g}_s \oplus \mathfrak{g}_a$, with $\mathfrak{g}_a$ required to be a Lie algebra. By a theorem of Wedderburn, the
algebra \( \mathfrak{g} \) corresponds to the group \( GL(n, \mathbb{C}) \), whose anti-Hermitian part corresponds to the group \( U(n) \). Taking a real form first, the anti-Hermitian part gives two cases: the orthogonal group \( SO(n) \) and the symplectic group \( Sp(n) \). Thus the following cases are realized:

\[
\begin{align*}
U(n) &: \quad \text{ad}_g \cong V \otimes \overline{V}, \\
SO(n) &: \quad \text{ad}_g \cong \wedge^2 V, \\
Sp(n) &: \quad \text{ad}_g \cong S^2 V.
\end{align*}
\] (2.1)

For any finite-dimensional vector space \( V \) there is a decomposition of \( V \otimes V \), under the action of \( GL(V) \), into a direct sum of irreducible \( GL(V) \)-modules

\[
V \otimes V = \wedge^3 V \oplus S^3 V.
\] (2.2)

This means that the above three cases correspond, respectively, to the left-hand side, to the first summand and to the second summand in (2.2).

**Remark 2.1.** Note that in the complex case the amplitude is invariant under \( GL(n, \mathbb{C}) \), while insisting on the norm of the states to be invariant requires \( U(n) \) (see [45]). Similarly for the real and quaternionic cases.

### 2.2. The case of a junction

We would like to carry out the corresponding process for the 3-junction. We proceed as follows.

1. We assign a vector space \( V_i, i = 1, 2, 3 \), to each of the three vertices.
2. We form the tensor product \( V_1 \otimes V_2 \otimes V_3 \). Then we identify this with a representation of some Lie (or Kac–Moody) group. If this is not possible then identify a summand of this triple tensor product with a representation of a group. If we require to have a field in string theory or in M-theory to be included in the spectrum, then for the latter an obvious choice would be a 3-form corresponding to the \( C \)-field. But we will not insist on this.
3. The action of \( GL(V) \) breaks \( V \otimes V \otimes V \) into a direct sum of four \( GL(V) \)-modules

\[
V \otimes V \otimes V = \wedge^3 V \oplus S^3 V \oplus (S_{(2,1)} V)^{\otimes 2},
\] (2.3)

where \( S_{(2,1)} V \) is defined as (see e.g. [26])

\[
S_{(2,1)} V = \ker(\wedge^2 V \otimes V \longrightarrow \wedge^3 V).
\] (2.4)

Elements of \( \wedge^2 V \otimes V \) are of the form \((v_1 \wedge v_3) \otimes v_2\), and are embedded in \( \wedge^3 V \) as \( v_1 \otimes v_2 \otimes v_3 - v_3 \otimes v_2 \otimes v_1 \).
4. We thus ask for the summands in the triple tensor product (2.3) to be representation spaces for Lie groups or Lie algebras—as they cannot be Lie algebras by themselves—so as to give a ‘higher analog’ of the adjoint representation.
5. We could also ask for a summand in the graded decomposition of the Lie algebra \( \mathfrak{g} \) (cf section 2.3) to be identified with a summand in \( V \otimes V \otimes V \).
6. The states for the 3-junction are represented by higher matrices \( \lambda_{ijk} \), where each of the indices represents a state of a quark. The study of this \( \lambda_{ijk} \) is the main subject of this paper.

A representation ‘with three indices’ mentioned in the introduction should correspond to a product of three vector representations, each corresponding to a vector space \( V_i, i = 1, 2, 3 \). There are three possibilities.

1. Tensor product: \( V_1 \otimes V_2 \otimes V_3 \).
(2) Symmetric power: $S^3 V$, where $V$ is isomorphic to each of the $V_i$.
(3) Antisymmetric power: $\wedge^3 V$, where again $V$ is isomorphic to each of the $V_i$.

In the desired cases, the grading naturally provides an action of the general (or special) linear group on $\wedge^3 V \otimes^3 V$ or $S^3 V$. The dimensions all grow $\sim n^3$

\begin{align}
\dim S^3(V) &= \frac{1}{6} n(n+1)(n+2), \\
\dim \otimes^3(V) &= n^3, \\
\dim \wedge^3(V) &= \frac{1}{6} n(n-1)(n-2).
\end{align}

The question now is what replaces the list (2.1) in the degree 3 case? We will answer this in section 2.3. It turns out that they correspond not to classical Lie groups but to exceptional Lie groups!

The factors $\lambda_{ijk}$ a priori admit no symmetry, i.e. belong to $V \otimes V \otimes V$. If we require antisymmetry upon exchange of the first two indices $\lambda_{ijk} = -\lambda_{jik}$, then $\lambda \in \wedge^2 V \otimes V$. Using the decomposition

$$\wedge^2 V \otimes V = \wedge^3 V \oplus S_{(2,1)}(V),$$

this gives two types for $\lambda$:

(1) $\lambda_{ijk} \in \wedge^3 V$ totally skew-symmetric,
(2) $\lambda_{ijk} \in S_{(2,1)}(V)$ which is such that
$$\lambda_{ijk} + \lambda_{kij} + \lambda_{jki} = 0.$$

Furthermore,

(3) if $\lambda_{ijk} = \lambda_{ji k} = \lambda_{ki j} = \lambda_{kj i} = \lambda_{jki} = \lambda_{kj i}$, then $\lambda \in S^3 V$.

Remarks 2.2.

(1) The above procedure can be performed on the dual vector space $V^*$ leading to factors $\wedge^3 V^*, \otimes^3 V^*$ and $S^3 V^*$, with corresponding factors $\lambda^{*ijk}$. The forms related to the dual vector space are contravariant while those related to the initial vector space are covariant. The duality between $V$ (and its powers) and $V^*$ (and its powers) is occurring as a duality on the brane.

(2) An alternative considered in [10] is the fuzzy 3-sphere algebra $A_n(S^3)$, which reduces to the classical algebra of functions on the 3-sphere in the large $N$ limit. This algebra is not closed under multiplication and so a projection is needed after multiplication. This leads to a nonassociative algebra. The number of degrees of freedom is given by $D = \frac{1}{6} (n+1)(n+2)(2n+3)$ so that in the large $n \sim \sqrt{N}$ limit this scales as $D \sim N^{3/2}$.

2.3. Graded Lie algebras

(1) A Lie algebra $g$ is called the direct sum of two Lie subalgebras $g = g_1 \oplus g_2$ if the underlying vector spaces obey the direct sum with
$$g_1 \cap g_2 = \emptyset; \quad [g_1, g_2] = 0.\quad (2.10)$$
So both $g_1$ and $g_2$ are ideals of the direct sum.

(2) A Lie algebra $g$ is called a semidirect sum of two Lie subalgebras $g = g_1 \oplus_s g_2$ if we replace the second condition in (2.10) by $[g_1, g_2] \subset g_1$, so that $g_1$ is an ideal but $g_2$ is not.

---

2 The closedness in our context is considered at the end of section 2.3.
If \( g \) is a Lie algebra, then the tensor product space \( \mathbb{C} \otimes g \) is a complex vector space since we can define
\[
\tau(\mu \otimes x) = (\tau \mu) \otimes x, \quad \forall \tau, \mu \in \mathbb{C} \text{ and all } x \in g.
\]
This can be regarded as a complex Lie algebra \( g_{\mathbb{C}} \), the complexification of \( g \), if we set for the Lie bracket
\[
[\tau \otimes x, \mu \otimes y] = (\tau \mu) \otimes [x, y],
\]
as then this would still satisfy antisymmetry and the Jacobi identity.

A graded Lie algebra is an ordinary Lie algebra \( g \), together with a gradation of vector spaces
\[
g = \bigoplus_{i \in \mathbb{Z} \text{ or } \mathbb{Z}_m} g_i,
\]
such that the Lie bracket respects this gradation:
\[
[g_i, g_j] \subseteq g_{i+j}.
\]

- A \( \mathbb{Z}_2 \)-grading \( g = g_0 \oplus g_1 \) corresponds to coset spaces.
- A \( \mathbb{Z}_3 \)-grading is of the form \( g = g_{-1} \oplus g_0 \oplus g_1 \).
- A \( \mathbb{Z} \)-grading is of the form \( g = g_{-d} \oplus \cdots \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus \cdots \oplus g_d \), where \( d = \max\{p \mid g_p \neq 0\} \) is the depth of the grading.

The grading via the Weyl group. Vinberg [57] extended the concept of the Weyl group \( W \) to semisimple complex Lie algebras which are graded modulo any \( m \). \( W \) is generated by complex reflections, i.e. linear transformations that can be described in some basis by a matrix of the form
\[
\begin{pmatrix}
\omega & & \\
1 & & \\
& \ddots & \\
& & 1
\end{pmatrix},
\]
where \( \omega \) is a root of unity. If \( g \) is \( \mathbb{Z}_m \)-graded for finite \( m \), then the linear transformation \( d\theta \) defined by \( d\theta(x) = \omega^k x \), for \( x \in g_k \), gives the gradation so that \( g_k \) are the eigenspaces of \( d\theta \) as follows [57]. For any \( \tau \in \mathbb{C} \), set
\[
g(\tau) = \{ x \in g \mid d\theta(x) = \tau x \},
\]
so that \( g = \bigoplus_{\tau} g(\tau) \), and
\[
[g(\tau), g(\omega)] \subseteq g(\tau \omega).
\]
The eigenvalues of the operator \( d\theta \) can be assumed, without loss of generality, to be of the form \( \omega^k \), with \( k \in \mathbb{Z} \). Setting \( g(\omega^k) = g_k \) gives a \( \mathbb{Z} \)-grading of \( g \) if \( \theta \) has an infinite order, and a \( \mathbb{Z}_m \)-grading if \( \theta \) has a finite order \( m \).

Tensor representations of Lie algebras. Let \( W \) be a \( g \)-module and let \( T, S \) and \( \wedge \) denote tensor, symmetric and antisymmetric powers, respectively. Then

1. \( T(W) = \bigoplus_{i=0}^{\infty} T^i(W) \) (or \( T^i(W) \)) is the tensor product representation of \( g \);
2. \( S(W) = \bigoplus_{i=0}^{\infty} S^i(W) \) (or \( S^i(W) \)) is the symmetric product representation of \( g \);
3. \( \wedge(W) = \bigoplus_{i=0}^{\infty} \wedge^i(W) \) (or \( \wedge^i(W) \)) is the antisymmetric product representation of \( g \).
As also mentioned in the introduction, there are no Lie groups or algebras whose dimension grows like the cube of their rank. Therefore one cannot find a representation of dimension \( n^3 \) to make up a whole of a Lie algebra. However, the next best thing one could hope for is to find inside a Lie algebra a representation that grows like \( n^3 \). Thus we seek those Lie algebras \( \mathfrak{g} \) which admit a decomposition of the form

\[
\mathfrak{g} \supset \otimes^3 V, \quad \text{or} \quad (2.18)
\]
\[
\mathfrak{g} \supset \wedge^3 V, \quad \text{or} \quad (2.19)
\]
\[
\mathfrak{g} \supset S^3 V. \quad (2.20)
\]

Similar requirements can be made for the dual spaces \( \otimes^3 V^*, \wedge^3 V^* \) and \( S^3 V^* \). It turns out that the above decompositions are realized for the Lie algebras \( \mathfrak{e}_6 \) and \( \mathfrak{e}_8 \) defined in [56]. From [56] we have

**Proposition 1.** Consider the decomposition \( \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \), where \( \mathfrak{g}_0 \) is of type \( \mathfrak{sl} \) or \( \mathfrak{gl} \), and \( \mathfrak{g}_{-1} \) and \( \mathfrak{g}_1 \) are third tensor, symmetric or antisymmetric powers of some vector space \( V \) or the tensor product of three vector spaces \( V_1, V_2, V_3 \). The only possibilities are

1. \( \mathfrak{d}_4 = S^3 V^* \oplus \mathfrak{sl}(V) \oplus S^3 V, \dim(V) = 3; \)
2. \( \mathfrak{e}_6 = (V_1^* \otimes V_2^* \otimes V_3^*) \oplus (\mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2) \oplus \mathfrak{sl}(V_3)) \oplus (V_1 \otimes V_2 \otimes V_3), \dim(V) = 3; \)
3. \( \mathfrak{e}_8 = \wedge^3 V^* \oplus \mathfrak{sl}(V) \oplus \wedge^3 V, \dim(V) = 9. \)

**Remarks 2.3.**

1. In proposition 1, we think of \( V \) as \( \mathbb{C}^3 \) in (1) and as \( \mathbb{C}^9 \) in (3), while we think of \( V_i, i = 1, 2, 3 \), as \( \mathbb{C}^3 \) in (2).
2. While in the open string case \( \mathfrak{g}_6 \) was a Lie algebra, in the 3-junction case \( \otimes^3, \wedge^3 V \) and \( S^3 V \) are not algebras, but only modules. However, in one model they close in the \( \mathfrak{g} \)-summand \( \wedge^3 V^* \) (see equation 4.7) and in another they close in the \( \mathfrak{g} \)-summand \( \wedge^6 V \) (cf equation (5.6)).

### 2.4. Representations of the corresponding groups

Let \( G \) be a connected reductive algebraic group over \( \mathbb{C} \) and let \( \mathfrak{g} \) be its Lie algebra. Let \( \theta \) be a semisimple automorphism of \( G \). This is the ‘antiderivative’ of \( d\theta \), the automorphism of the algebras considered in the previous section.

Let \( G_0 \) be the identity component of the group \( G^\theta \) of elements invariant under \( \theta \). The two coincide if \( G \) is simply connected and semisimple. Let \( \hat{G}_0 \) be the simply connected group locally isomorphic to \( G_0 \). The adjoint representation of \( G \) induces a linear representation of \( G_0 \) in each of the subspaces \( \mathfrak{g}(\tau) \). The algebra of invariant polynomials \( \mathbb{C}[\mathfrak{g}_1]^{G_0} \) is finitely generated and free [56].

We seek \( G_0 \)-invariant rank 3 tensors. For \( m = 3 \) there are the following cases corresponding to the ones in proposition 1.

**Proposition 2.** 3-junctions (with no physical constraints) may admit the following group symmetries.

1. \( G = D_4, G_0 = SL(3), \) and the elements of \( \mathfrak{g}_1 \) are symmetric forms of degree \( 3 \) in three variables.
2. \( G = E_6, \hat{G}_0 = SL(3) \times SL(3) \times SL(3), \) and \( \mathfrak{g}_1 \) can be interpreted as \( \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^3 \).
G = E_8, G_0 = SL(9), and the space g_1 can be interpreted as the third exterior power \wedge^3 C^9 of C^9.

Lie groups and Lie algebras have natural representations, e.g. the adjoint, on matrices. Given the above decompositions containing cube powers, corresponding to tensor representations, it is natural to ask what are the corresponding objects replacing matrices. The answer is hypermatrices. What replaces linear algebra is multi-linear algebra.

3. Tensors and hypermatrices

3.1. Hypermatrices and hyperdeterminants

A three-dimensional hypermatrix is a 3-way array of complex numbers \( A = [a_{j_1,j_2,j_3}]_{j_1,j_2,j_3=1}^{n_1,n_2,n_3} \), where \( a_{j_1,j_2,j_3} \in \mathbb{C} \) is the \((j_1,j_2,j_3)\)-entry of the array, and the notation \([\cdot]_{j_1,j_2,j_3=1}^{n_1,n_2,n_3} \) means that the indices \( j_i \) run as \( 1 \leq j_i \leq n_i \), for \( i = 1, 2, 3 \). This array is denoted as \( C_{n_1 \times n_2 \times n_3} \), which is a complex vector space of dimension \( n_1 \times n_2 \times n_3 \).

In general, hypermatrices are higher-dimensional arrays generalizing matrices, which are viewed as two-dimensional arrays of numbers. The latter admits scalar invariants which include the determinant, and likewise the former admits the hyperdeterminant. For a \( k \)-dimensional hypermatrix \( A = (A_{i_1,...,i_k})_{1\leq i_1,...,i_k \leq n} \) of order \( n \), the hyperdeterminant of \( A \) is

\[
\text{Det}_k(A) = \frac{1}{n!} \sum_{\sigma_1,...,\sigma_k \in \Sigma_n} \text{sign}(\sigma_1) \cdots \text{sign}(\sigma_k) \prod_{i=1}^{n} A_{\sigma_1(i) \cdots \sigma_k(i)}. \tag{3.1}
\]

(1) When \( k = 2 \) this expression for the hyperdeterminant coincides with that of the determinant

\[
\text{Det}_2(A) = \det(A) = \frac{1}{n!} \sum_{\sigma_1,\sigma_2 \in \Sigma_n} \text{sign}(\sigma_1) \text{sign}(\sigma_2) \prod_{i=1}^{n} A_{\sigma_1(i)\sigma_2(i)}. \tag{3.2}
\]

(2) Expression (3.1) is the zero polynomial when the dimension \( k \) of the hypermatrix is odd. This will be used later in section 5.4, where an extension to the odd-dimensional case is considered.

In the \( n \)-dimensional case, rows and columns are replaced by slices which come in \( n \) types. For example, for \( n = 3 \) we have vertical, horizontal and lateral slices. Row and column operations are replaced by slab (or slice) operations and hence it is natural to check for behavior of hypermatrices under those, that is to check analogs for hypermatrices of Gaussian elimination for matrices. The following are essentially known since [15] (see [51] for a more recent reference).

**Properties of a hyperdeterminant under hypermatrix operations.**

(a) Interchanging two parallel slices leaves the hyperdeterminant invariant up to sign (which may equal 1).

(b) The hyperdeterminant is a homogeneous polynomial in the entries of each slice. The degree of homogeneity is the same for parallel slices.

(c) The hyperdeterminant does not change if we add to some slice a scalar multiple of a parallel slice.

(d) The hyperdeterminant of a matrix having two parallel slices proportional to each other is equal to 0. In particular, \( \text{Det}(A) = 0 \) if \( A \) has a zero slice.
3.2. Equivalence of tensors and hypermatrices

A 3-array can be formed out of 3 vectors as follows. The Segre product of three vectors, \( u \in \mathbb{C}^{n_1}, v \in \mathbb{C}^{n_2} \text{ and } w \in \mathbb{C}^{n_3} \), is defined as
\[
\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} := [u_{j_1} v_{j_2} w_{j_3}]_{j_1,j_2,j_3=1}^{n_1,n_2,n_3}.
\] (3.3)

Next, for arrays themselves we have that the outer product of two 3-arrays \( A \) and \( B \) is a 6-array \( C = A \otimes B \) with entries
\[
c_{i_1,i_2,j_1,j_2,j_3} := a_{i_1,i_2,j_1} b_{j_1,j_2,j_3}.
\] (3.4)

The relation of a hypermatrix to a tensor. A tensor is an element in the tensor product of vector spaces. The Segre map
\[
\varphi : \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{C}^{n_3} \rightarrow \mathbb{C}^{n_1 \times n_2 \times n_3}, \quad (u, v, w) \mapsto u \otimes v \otimes w
\] (3.5)
is multilinear with kernel the decomposable tensors, i.e. those that are of the form \( A = e_{i_1} \otimes e_{j_2} \otimes e_{k_3} \). By the universal property of the tensor product there exists a linear map \( \theta : \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3} \xrightarrow{\sim} \mathbb{C}^{n_1 \times n_2 \times n_3} \).

Since the spaces have the same dimension, \( \theta \) is an isomorphism of vector spaces. Consider the canonical basis of \( \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3} \)
\[
\{ e_{j_1}^{(1)} \otimes e_{j_2}^{(2)} \otimes e_{j_3}^{(3)} | 1 \leq j_1 \leq n_1, 1 \leq j_2 \leq n_2, 1 \leq j_3 \leq n_3 \}.
\] (3.7)

where \( \{ e_{i_1}^{(1)}, \ldots, e_{i_3}^{(3)} \} \) denotes the canonical basis in \( \mathbb{C}^{n_i}, i = 1, 2, 3 \). Then \( \theta \) may be described as [16]
\[
\theta \left( \sum_{j_1,j_2,j_3} a_{j_1,j_2,j_3} e_{j_1}^{(1)} \otimes e_{j_2}^{(2)} \otimes e_{j_3}^{(3)} \right) = [a_{j_1,j_2,j_3}]_{j_1,j_2,j_3=1}^{n_1,n_2,n_3}.
\] (3.8)

Thus, we have

**Proposition 3.** An order 3 tensor in \( \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3} \) is the same as a three-dimensional hypermatrix in \( \mathbb{C}^{n_1 \times n_2 \times n_3} \) in the above basis. Similarly for the real case.

3.3. Relation to matrices

**Change of basis.** Let \( A = [a_{ijk}] \in \mathbb{C}^{n_1 \times n_2 \times n_3} \) and let \( L, M \) and \( N \) be three \( n_1 \times n_1, n_2 \times n_2 \text{ and } n_3 \times n_3 \) nonsingular matrices, respectively. This means that \( L = [l_{ij}] \in GL(n_1, \mathbb{C}), M = [m_{ij}] \in GL(n_2, \mathbb{C}) \text{ and } N = [n_{ij}] \in GL(n_3, \mathbb{C}) \). The result of the transformation of the multilinear map \( (L, M, N) \) on \( A \) is a tensor \( A' = (L, M, N) \cdot A = [a'_{pq}] \in \mathbb{C}^{n_1 \times n_2 \times n_3} \), the multilinear transform of \( A \), defined by
\[
a'_{pqr} = \sum_{i,j,k} l_{pi} m_{qj} n_{rk} a_{ijk}.
\] (3.9)

**Multilinear matrix multiplication.** The following properties hold (see [21]).

1. For \( A, B \in \mathbb{C}^{n_1 \times n_2 \times n_3}, L_i \in GL(n_1, \mathbb{C}), \text{ and } \alpha, \beta \in \mathbb{C}, \)
\[
(L_1, L_2, L_3) \cdot (\alpha A + \beta B) = \alpha (L_1, L_2, L_3) \cdot A + \beta (L_1, L_2, L_3) \cdot B.
\]
(2) For \( L_i \in \mathbb{C}^{n_i \times n_i}, M_i \in \mathbb{C}^{n_i \times n_i}, i = 1, \ldots, k, \)
\[
(M_1, M_2, M_3) \cdot [(L_1, L_2, L_3) \cdot A] = (M_1 L_1, M_2 L_2, M_3 L_3) \cdot A.
\]

(3) For any \( M_i, N_i \in \mathbb{C}^{n_i \times n_i}, \alpha, \beta \in \mathbb{C}, \)
\[
(\alpha M_1 + \beta N_1, L_2, L_3) \cdot A = \alpha (M_1, L_2, L_3) \cdot A + \beta (N_1, L_2, L_3) \cdot A,
\]
and similarly for the other two slots.

Before going to their applications, we work with cubic hypermatrices of general ‘size’ \( n \) for which we have the following result:

**Proposition 4.** Let \( A \in \mathbb{R}^{n \times n \times n} \), let \( A' \) be obtained from \( A \) by permuting the three factors in the tensor product and let \( (L, M, N) \in GL_n(\mathbb{R})^3 \). Then \( \Delta(A') = \Delta(A) \) and
\[
\Delta((L, M, N) \cdot A) = \det(L)^a \det(M)^b \det(N)^c \Delta(A).
\]

This is a generalization of proposition 5.6 in [21], and the proof is similar. Here \( \Delta \) is the discriminant defined right after the proof of proposition 5 and \( G^{ \times k } \) is the product \( G \times \cdots \times G \).

**Remark 3.1.** The hyperdeterminant as defined in expression (3.1) is not the same as the discriminant of a tensor. The notion used in corollary 1.5 of [28] is the discriminant \( \Delta(A) \). In fact, both \( \det \) and \( \Delta \) are invariant under matrix operations. More precisely, they are invariant under the action of \( SL_n^{ \times k } \). However, the polynomial \( \Delta \) is in general much more complicated than \( \det \). For example, if one considers \( 2 \times 2 \times 2 \) hypermatrices, then \( \det \) is a polynomial of degree 2 while \( \Delta \) is a polynomial of degree 24.

**Symmetric tensors and hypermatrices.** A three-dimensional cubic hypermatrix \( A = [a_{ijk}] \in \mathbb{C}^{n \times n \times n} \) is symmetric if \( a_{i_1 i_2 i_3} = a_{i_2 i_1 i_3} = a_{i_1 i_3 i_2} = a_{i_3 i_2 i_1}, \) with \( i_1, i_2, i_3 \in \{1, \ldots, n\} \) for all permutations \( \sigma \) of the symmetric group \( \Sigma_3 \). Explicitly this is
\[
a_{ijk} = a_{ikj} = a_{jki} = a_{kji} = a_{jik}, \quad \forall i, j, k \in \{1, \ldots, n\}.
\]

An order 3 tensor \( A \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n \) is symmetric if \( \sigma(A) = A \) for all permutations \( \sigma \in \Sigma_3 \), where the group action is given by
\[
\sigma(x_{i_1} \otimes x_{i_2} \otimes x_{i_3}) = x_{i_{\sigma(1)}} \otimes x_{i_{\sigma(2)}} \otimes x_{i_{\sigma(3)}},
\]

Given a basis \( \{e_1, \ldots, e_n\} \) of \( \mathbb{C}^n \), a basis of the set \( S^3(\mathbb{C}^n) \) of symmetric 3-tensors in \( \mathbb{C}^n \) is given by
\[
\left\{ \frac{1}{3} \sum_{\sigma \in \Sigma_3} e_{i_{\sigma(1)}} \otimes e_{i_{\sigma(2)}} \otimes e_{i_{\sigma(3)}}, 1 \leq i_1 \leq i_2 \leq i_3 \leq n \right\},
\]
whose (complex) dimension is
\[
\dim_{\mathbb{C}} S^3(\mathbb{C}^n) = \binom{n+2}{3} = \frac{1}{6}n(n+1)(n+2).
\]

This corresponds to the number of partitions of 3 into a sum of \( n \) nonnegative integers, so that for \( n = 1, 2, 3 \) this is 1, 4 and 10, respectively.

There is a bijective correspondence between symmetric tensors and homogeneous polynomials of degree 3 in \( n \) variables:
\[
S^3(\mathbb{C}^n) \cong \mathbb{C}[x_1, \ldots, x_n].
\]
For \( n = 1 \) this is \( \mathbb{C}[x]_3 \) which is of the form \( x^3 \). For \( n = 2 \) this is \( \mathbb{C}[x, y]_3 \) which is formed of the four monomials \( x^3, x^2y, xy^2, y^3 \). For \( n = 3 \) this is formed of the ten monomials \( y^3z, xz^2, y^2z, xy^2, x^2y, xyz, x^3, y^3, z^3 \).

**Direct sum.** The direct sum of two order 3 tensors/hypermatrices \( A \in \mathbb{C}^{l_1 \times m_1 \times n_1} \) and \( B \in \mathbb{C}^{l_2 \times m_2 \times n_2} \) is a ‘block tensor’ with \( A \) in the \((1, 1, 1)\)-block and \( B \) in the \((2, 2, 2)\)-block:

\[
A \oplus B = \begin{bmatrix} A & 0 \\ 0 & 0 & B \end{bmatrix} \in \mathbb{C}^{(l_1 + l_2) \times (m_1 + m_2) \times (n_1 + n_2)}.
\]  

(3.15)

In terms of vector spaces, if \( U_l, V_i, W_j \) are vector spaces such that \( W_i \equiv U_l \oplus V_i \) for \( i = 1, \ldots, k \), then the tensors \( A \in U_1 \otimes U_2 \otimes U_3 \), \( B \in V_1 \otimes V_2 \otimes V_3 \) have direct sum \( A \oplus B \in W_1 \otimes W_2 \otimes W_3 \).

**Tensor rank.** Such a notion goes back as far as [33]. A tensor has a tensor rank \( r \) if it can be written as a sum of \( r \) decomposable tensors, but no fewer.

\[
\text{rank}_{\otimes}(A) := \min \left\{ r \mid A = \sum_{i=1}^{r} u_i \otimes v_i \otimes \cdots \otimes z_i \right\}.
\]  

(3.16)

A nonzero decomposable tensor has tensor rank 1.

We have the following general result on tensor rank.

**Proposition 5.** Let \( A \in \mathbb{C}^{n_1 \times \cdots \times n_k} \). Then

\[
\text{rank}(A) \leq \frac{\prod_{i=1}^{k} n_i}{\max\{n_i\}}.
\]

**Proof.** \( k = 2 \) is obvious. We proceed by induction on \( k - 1 \). Define \( l \) by \( n_l = \min\{n_i\} \).

Without loss of generality, \( l = 1 \). Each slice \( A_t = \{a_{(l-1)t}, \ldots, k\} \) for \( t = 1, \ldots, n_1 \) is a \((k - 1)\)-dimensional hypermatrix. By our induction hypothesis, each of these slices has rank at most

\[
\frac{\prod_{i=2}^{k} n_i}{\max\{n_i\}}.
\]  

(3.17)

Write

\[
A_t = \sum_{m=1}^{r} [v_1 \otimes \cdots \otimes v_{(k-1)m}].
\]  

(3.18)

and express \( A = \bar{e}_1 \otimes A_1 + \cdots + \bar{e}_{n_l} \otimes A_{n_l} \). Expanding the sum, the result follows. \( \square \)

The discriminant can be defined for homogeneous forms in \( k + 1 \) variables of degree \( d \) as follows [28]. The discriminant is an irreducible polynomial \( \Delta(f) \) in the coefficients of a form \( f = f(x_0, x_1, \ldots, x_k) \) which vanishes if and only if all the partial derivatives \( \partial f/\partial x_0, \partial f/\partial x_1, \ldots, \partial f/\partial x_k \) have a common zero in \( \mathbb{C}^{k+1} - \{0\} \). Note that \( \Delta(f) \) depends on the degree \( d \). The requirement that the polynomial \( \Delta(f) \) be irreducible over \( \mathbb{Z} \), i.e. it has relatively prime integer coefficients, makes it defined uniquely up to a sign. The importance of the discriminant is that it vanishes whenever \( f \) has multiple roots. This is familiar from the low degree cases, namely the quadratic and cubic polynomials. For a tensor \( A \), the discriminant \( \Delta(A) \) is the hyperdeterminant. We have the following generalization of proposition 5.9 in [21].

---

3 The referee informed us that he thinks that this might have appeared before, perhaps in [33].
Proposition 6. Let $A \in \mathbb{R}^{n \times n \times n}$. If $\Delta(A) > 0$ implies that $\det \left( \sum_{i=1}^{n} \lambda_i A_i \right)$ has $n$ distinct real sets of roots, then $\text{rank}(A) \leq n(n-1)$.

Proof. By hypothesis, we have $n$ distinct real sets of roots for $\det \left( \sum_{i=1}^{n} \lambda_i A_i \right)$, for $i = 1, \ldots, n, \lambda_1, \ldots, \lambda_n$. Then we can transform $[A_1] \cdots [A_n]$ by slab operations into $[B_1] \cdots [B_n] = B$, where $B_i = \sum_{j=1}^{n} \lambda_j A_j$. By construction, $\det(B_i) = 0$, so $B_i$ is of non-maximal rank, such that $B_i = \sum_{j=1}^{n-1} f_i \otimes g_{ij}$. Taking the tensor product $e_i \otimes B_i$ and summing over all $i$ gives an expression of $n(n-1)$ rank-$1$ hypermatrices for $B$. Since the rank is invariant under Gaussian processes, $\text{rank}(A) = \text{rank}(B) \leq n(n-1)$.

3.4. The action of the general linear group on wedge products

The natural action of $GL(V)$ on $V$ extends canonically to the exterior powers of $V$. For completeness we review this briefly, following [55]. The elements of $\wedge^n(V)$ are called $m$-vectors of polyvectors of degree $m$. Polyvectors which can be written in the form $u_1, \wedge \cdots \wedge u_m$, for some vectors $u_1, \ldots, u_m$ are called decomposable polyvectors. On decomposable polyvectors, multiplication is defined by the formula

$$\sum_{\text{perm}(i_1, \ldots, i_m)} u_1 \wedge \cdots \wedge u_{i_1} \wedge \cdots \wedge u_{i_m}.$$ 

In particular, the degree of the product equals the sum of the degrees of its factors.

Elements of $\wedge^m(V^*)$ are exterior (differential) $m$-forms. The exterior power $\wedge^m(V^*)$ can be identified with $\wedge^m(V)^*$ by means of the canonical pairing. On the decomposable polyvectors and decomposable exterior forms, this pairing is given by

$$\sum_{\text{perm}(i_1, \ldots, i_m)} u_1 \wedge \cdots \wedge u_{i_1} \wedge \cdots \wedge u_{i_m} \mapsto \det(v_1(u_{i_1})), \quad v_1, \ldots, v_m \in V^*,$$ 

the image of which on polyvectors and exterior forms of distinct degrees equals zero. Choosing a basis $e_1, \ldots, e_m \in V$, we can identify the automorphism group $GL(V)$ of the module $V$ with $GL(n, \mathbb{R})$. For every $m$, the group $GL(n, \mathbb{R})$ acts naturally on $\wedge^m(V)$. The action of $g \in GL(n, \mathbb{R})$ on decomposable $m$-vectors is given by

$$\wedge^m(g)(v_1 \wedge \cdots \wedge v_m) = g v_1 \wedge \cdots \wedge g v_m, \quad \forall v_1, \ldots, v_m \in V.$$ 

The Binet–Cauchy theorem asserts that the map $\wedge^m : GL(n, \mathbb{R}) \rightarrow GL(C^m_n, \mathbb{R})$, $g \mapsto \wedge^m(g)$, where $C^m_n$ are the binomial coefficients, is in fact a homomorphism

$$\wedge^m(hg) = \wedge^m(h) \wedge^m(g).$$ 

Thus the map $g \mapsto \wedge^m(g)$ is a degree-$C^m_n$ representation of the group $GL(n, \mathbb{R})$ [55]. It is called the $m$-vector representation of the $m$th fundamental representation.

As in section 2.4, consider the following subgroup of the group $G$ corresponding to the Lie algebra $\mathfrak{g}$:

$$G^0 = \{ g \in G | \theta(g) = g \}. \quad (3.23)$$

Let $G_0 \subset G^0$ be the group corresponding to the subalgebra $\mathfrak{g}_0$. From the property $[\mathfrak{g}_0, \mathfrak{g}_k] \subset \mathfrak{g}_k$, it follows that the adjoint representation of the group $G$ induces, by restriction, a linear representation $\rho_k$ of $G_0$ in $\mathfrak{g}_k$ (for any $k$) [57].

3.5. Ranks and orbits of 3-vectors: admissible dimensions for $V$

A generic 3-tensor is an element of $\otimes^3(C^0_n)$ with open $GL(n, \mathbb{C})$ orbit. Similarly for symmetric and antisymmetric powers. The isotropy group of a tensor consists of all group elements leaving the tensor invariant

$$G_T := \{ A \in GL(n, \mathbb{C}) | T = AT \}. \quad (3.24)$$
The dimension of this space is $n^2 - \dim(\otimes^3(\mathbb{C}^n))$. Similarly for the antisymmetric and symmetric powers, in which cases the tensor power $\otimes^3$ is replaced by either $\wedge^3$ or $S^3$, respectively.

Consider the orbits of the group $\wedge^m(GL(n, \mathbb{R})$ acting on $\wedge^m(V)$. For bivectors the situation is very simple as every bivector is equivalent to one of the bivectors $e_1 \wedge e_2 + \cdots + e_{2r-1} \wedge e_{2r}$, $1 \leq 2r \leq n$, under the action of $\wedge^m(GL(n, \mathbb{R})$. We are interested mainly in the cases when $\mathcal{R}$ is $\mathbb{R}$ or $\mathbb{C}$.

The rank of an orbit of an $m$-vector in an $n$-dimensional vector space can take only the values (see e.g. [25])

$$0, m, m + 2, \ldots, n,$$

so that for trivectors the only possible ranks are 0, 3, 5, \ldots, $n$.

The complex case: The orbits of a given rank are known and are described as follows.

- **Rank 0**: only 0 is possible.
- **Rank 3**: any such trivector is equivalent to $e_1 \wedge e_2 \wedge e_3$.
- **Rank 5**: any such trivector is equivalent to $e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_4 \wedge e_5$.
- **Rank 6**: there are two orbits of complex trivectors of rank 6, with representatives (Reichel 1907)

$$e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6, \quad e_1 \wedge e_2 \wedge e_4 + e_1 \wedge e_3 \wedge e_5 + e_2 \wedge e_3 \wedge e_6.$$

- **Rank 7**: the complex trivectors of rank 7 have 5 orbits.
- **Rank 8**: the complex trivectors of rank 8 have 13 orbits.
- **Rank 9**: in this case two new interesting features occur [58]. First, there are infinitely many orbits. Second, here close connections to Lie algebras start to become apparent. This uses the exceptional embedding $A_8 \subseteq E_8$ and the graded Lie algebra decomposition

$$\mathfrak{e}_8 = \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \wedge^3 V^* \oplus \mathfrak{sl}(V) \oplus \wedge^3 V,$$

where $V = \mathbb{C}^9$. There is a nontrivial homomorphism of $SL(9, \mathbb{C})$ into the adjoint group of $\mathfrak{g}$ whose kernel is the central subgroup of order 3. This action of $SL(9, \mathbb{C})$ on $\mathfrak{g}$ preserves the grading. Restricting the action to $\mathfrak{g}_1$ gives the desired action of $SL(9, \mathbb{C})$ on $\wedge^3(\mathbb{C}^9)$. The general method of Vinberg [56, 57] can be applied to classify the orbits of $SL(9, \mathbb{C})$ in $\mathfrak{g}_1$.

The real case. $\wedge^3(\mathbb{R}^n)$ is a real subspace of the complexification $\wedge^3(\mathbb{C}^n)$. For $x \in \wedge^3(\mathbb{R}^n)$ the real orbit $GL(n, \mathbb{R}) \cdot x$ is contained in the complex orbit $GL(n, \mathbb{C}) \cdot x$. This orbit is called a **real form** of the complex orbit containing it. Every complex orbit has only finitely many real forms [11]. The problem of classifying the orbits of $GL(n, \mathbb{R})$ in $\wedge^3(\mathbb{R}^n)$ thus reduces to the problem of classifying the real forms of the orbits of $GL(n, \mathbb{C})$ in $\wedge^3(\mathbb{C}^n)$. The orbits of a given rank are known and are described as follows.

- **Rank 0**: the classification is trivial.
- **Rank 6**: the classification is obtained by Gurevich in the 1930s, and then by [14] and [47].
- **Rank 7**: given in [59] and [47].
- **Rank 8**: all real forms of the 23 orbits of $GL(8, \mathbb{C})$ in $\wedge^3(\mathbb{C}^8)$ are enumerated in [22].

From the above classic results, we state the following.

**Proposition 7.** In representing 3-junctions by finite-dimensional exceptional Lie algebras (or groups) according to the graded Lie algebra decomposition, the highest dimension for the corresponding vector space $V$ is 9.
Interpreting the orbits. The classification of the orbits of the action of the general linear group on \( g_1 \), which generically is either a symmetric, tensor or antisymmetric power of some vector space, involves considering a three-dimensional subalgebra \([22, 58]\). The elements \( h \in g_0, x \in g_1 \) and \( y \in g_{-1} \) form a graded \( sl_2 \)-triple \((x, h, y)\) with

\[
[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h \neq 0.
\]

The vector space will decompose as \( V = V_1 \oplus \cdots \oplus V_k \), with \( k \) depending on the algebra and its grading. The centralizer \( Z(h) \) will be of the form \( SL_{m_1} \times \cdots \times SL_{m_k} \), with \( m_1 + \cdots + m_k = \text{dim}(V) \), where \( SL_{m_i} = SL(V_i) \), \( 1 \leq i \leq k \).

We interpret \( Z(h) \) as the breaking of the original symmetry \( SL(V) \) into the corresponding product pieces. What this means dynamically is that we are moving apart the stack of branes into a final \( m_k \) sets of thinner stacks. For example, start with a decomposition \( V = V_1 \oplus V_2 \) so that \( SL(V) \) breaks into \( SL(V_1) \times SL(V_2) \). This decomposition means, for the ends of the junction, that the original stack of branes corresponding to \( V \) will break into two separate stacks, formed (for the case of D-branes) of \( \text{dim}(V_1) \) and \( \text{dim}(V_2) \) overlapping branes, respectively.

The tensor product \( V \otimes^3 \) will involve pieces \( V \otimes^3 = V_1 \otimes^3 \otimes V_2 \), corresponding to a junction joining \( \text{dim}(V_1) \) stacks together, and \( V \otimes^3 \) corresponding to a junction joining the other stacks, i.e. the one with \( \text{dim}(V_2) \) components. In addition, there are ‘mixed junctions’, i.e. ones which connect \( i \) stacks of first type to \((3 - i)\) stacks of the second type, with \( i = 1, 2 \). The extension to the general case is straightforward. Thus we see that one can have more junction configurations, starting with basic ones which correspond to \( g_0 \).

Proposition 8. The \( SL(V) \)-action on \( g_1 \) leads, via breaking of symmetry, to admissible junction configurations according to the corresponding orbits.

3.6. The traces and invariants

We have seen that single strings can be represented simply as matrices \( \lambda^i \). When joining multiple strings together the indices which correspond to adjoined ends are contracted:

\[
(\lambda^{(1)})^{i_1 j_1} \cdot (\lambda^{(2)})^{j_2 i_2} \cdot \cdots \cdot (\lambda^{(m)})^{i_m j_m} = \text{tr}(\lambda^{(1)} \lambda^{(2)} \cdots \lambda^{(m)}).
\]

Similarly, as we saw in section 2.2, we use a cubic hypermatrix \( \lambda_{ijk} \) to represent three-pronged junctions. An analogous expression for trace is then

\[
\sum_{i,j,k=1}^n \lambda^{(1)}_{ijk} \lambda^{(2)}_{jki} \lambda^{(3)}_{kij} = \text{tr}(\lambda^{(1)} \lambda^{(2)} \lambda^{(3)}).
\]

We set up the indices cyclically so as to satisfy the conservation of the charge condition for each endpoint.

Note that if we model the vertex by \( \Lambda^3 V \), our generalized expression for trace is trivial, for by renaming the indices and applying antisymmetry we get

\[
\sum_{i,j,k=1}^n \lambda^{(1)}_{ijk} \lambda^{(2)}_{jki} \lambda^{(3)}_{kij} = \sum_{i,j,k=1}^n \lambda^{(1)}_{ijk} \lambda^{(2)}_{jki} \lambda^{(3)}_{kij}
\]

\[
= \sum_{i,j,k=1}^n (-\lambda^{(1)}_{ijk})(-\lambda^{(2)}_{jki})(-\lambda^{(3)}_{kij}),
\]

whence it is zero.

We would like to consider the question of admissible traces more systematically, using invariant theory, as follows. Any \( \lambda \in g_1 \) admits a Jordan decomposition \( \lambda^{ss} + \lambda^{\text{nil}} \) into a
Proposition 9. String and membrane 3-junctions provide (and hence can be described by) representations of \( \mathfrak{g}_\text{sl}(9) \). Hence can be described by representations of the Cartan subspaces \( \mathfrak{g}_\text{sl}(9) \) consisting of commuting semisimple elements

\[
\mathfrak{c} = \{ \lambda_1^{\text{ss}}, \lambda_2^{\text{ss}} \in \mathfrak{g}_1 \mid [\lambda_1^{\text{ss}}, \lambda_2^{\text{ss}}] = 0 \}.
\]  

(3.31)

Thus a basis for \( \mathcal{G} = \mathcal{D}_3 \) will be composed of tensor products of the vectors \( e_1, e_2, e_3 \) that form a basis for \( \mathbb{C}^3 \), while that of \( \mathcal{G} = \mathcal{E}_8 \) will be wedge products of the vectors \( e_1, \ldots, e_9 \) that form a basis for \( \mathbb{C}^9 \).

The Weyl group is the group \( W \) of linear transformations of \( \mathfrak{c} \) generated by elements of \( \mathcal{G}_0 \), the adjoint action of which leaves \( \mathfrak{c} \) invariant.

The algebra \( \mathbb{C}[\mathfrak{g}_1]^W \) of \( \mathcal{G}_0 \)-invariant polynomials in \( \mathfrak{g}_1 \) is free and is isomorphic to the algebra \( \mathbb{C}[\mathfrak{c}]^W \) of Weyl-invariant polynomials in the Cartan subspace [56]. We will consider specific examples in the next section.

4. The main examples

In case the vector spaces \( V \) are of dimensions less than or equal to 9, we have

Proposition 9. String and membrane 3-junctions provide (and hence can be described by) representations of \( \mathfrak{g}_\text{sl}(9, \mathbb{C}) \) (and hence of its subgroups by restriction) on \( \lambda^3 V \). (Similarly for the compact subgroups \( SU(9) \) and their subgroups when requiring that norms of states be preserved.)

Given proposition 1 we see that \( \mathfrak{sl}(V) \) arises as the \( \mathfrak{g}_0 \) factor in the graded decomposition of \( \mathfrak{g}_3, \mathfrak{g}_2 \) and \( \mathfrak{g}_1 \). Thus, it is natural to consider these Lie algebras. We summarize the main result of the examples in the following four sections, i.e. sections 4.1–4.4, as

Theorem 4.1. String and membrane 3-junctions allow for \( \mathfrak{g}_3, \mathfrak{g}_2, \mathfrak{g}_1 \), or \( \mathfrak{g}_0 \) symmetries, depending on whether we take hypermatrix factors for the junctions to be symmetric, antisymmetric, symmetric on two indices, or of no symmetries, respectively.

4.1. Representations of the Lie algebra \( \mathfrak{g}_0 \) on \( \wedge^3 V \): the \( E_8 \) example

Let \( V \) be a vector space and \( V^* \) the dual vector space to \( V \). Consider \( \wedge^3 V \), the third exterior power of \( V \). This can be identified with \( V^{\otimes 3} = V \otimes V \otimes V \), the space of third tensor power of \( V \), so that for any \( v_1, v_2, v_3 \in V \),

\[
v_1 \wedge v_2 \wedge v_3 = \sum_{\text{perm}} \text{sgn}(i_1, i_2, i_3) v_{i_1} \otimes v_{i_2} \otimes v_{i_3}.
\]  

(4.1)

Form the third exterior power \( \wedge^3 V^* \) of \( V^* \). There is a duality between \( \wedge^3 V \) and \( \wedge^3 V^* \) (using Einstein’s summation convention henceforth):

\[
\langle \lambda, \lambda^* \rangle = \frac{1}{3!} \lambda^{i_1 i_2 i_3} \lambda^*_{i_1 i_2 i_3}, \quad \lambda \in \wedge^3 V, \quad \lambda^* \in \wedge^3 V^*.
\]  

(4.2)

Similarly, if \( \epsilon \) is a nonzero element of the space \( \wedge^0 V \), then \( \epsilon^* \) will denote the element of the space \( \wedge^0 V^* \) that satisfies \( \langle \epsilon, \epsilon^* \rangle = 1 \). Let \( L(V) = V \otimes V^* \) be the space of linear transformations of \( V \) and

\[
L_0(V) = \{ S \in L(V) \mid \text{Tr}(S) = 0 \}.
\]  

(4.3)

These form the algebras \( \mathfrak{gl}(V) \) and \( \mathfrak{sl}(V) \), respectively.
An important question to ask is whether the factor Remark 4.2.
the contraction of two three-junctions, and the commutator of a junction and its dual.

Since \( \wedge \)

\[ 16 \]

\[ \epsilon_p \]

where \( e \)

writing equations explicitly we will use component notation. For \( \epsilon_6 \), with \( X, Y \in \mathfrak{g}_0 \), \( \lambda, \lambda_1, \lambda_2 \in \mathfrak{g}_1 \) and \( \lambda^*, \lambda^{(1)}, \lambda^{(2)} \in \mathfrak{g}_{-1} \), the commutation relations, which result from breaking the original Lie bracket on \( \mathfrak{g} \) into components corresponding to the grading, are \[ [X, Y]_\mathfrak{g} = X_s^j Y^j_s - Y_s^j X^j_s \quad \subset \mathfrak{g}_0 \] \[ [X, \lambda^1]_{jk} = X^i_{jk} \lambda^1_{ij} + X^i_{jk} \lambda^1_{ij} \quad \subset \mathfrak{g}_1 \] \[ [X, \lambda^*]_{jk} = -\lambda^*_s_{jk} X^i_s - \lambda^*_i_{jk} X^s_i \quad \subset \mathfrak{g}_{-1} \] \[ [\lambda_1, \lambda_2]_{jk} = \frac{1}{(3!)^2} \epsilon^{pqrrijpqrjkl} \lambda^{(1)}_{pqijl} \lambda^{(2)}_{pqijl} \quad \subset \mathfrak{g}_1 \] \[ [\lambda, \lambda^*]_{jk} = \frac{1}{2} \lambda^{pqr} \lambda^*_{pqj} - \frac{1}{18} \lambda^{pqr} \lambda^*_{pqj} \quad \subset \mathfrak{g}_0. \] where \( \epsilon^{pqrrijpqrjkl} \) and \( \epsilon^{*_{pqripgrijl}} \) are the components of \( \epsilon \) and \( \epsilon^* \), respectively. Alternatively, if \( \lambda = x \wedge y \wedge z \) and \( \lambda^* = f \wedge g \wedge h \), the last commutator can be written as

\[ [x \wedge y \wedge z, f \wedge g \wedge h] = \begin{vmatrix}
  f(x) & f(y) & f(z) & f \\
  g(x) & g(y) & g(z) & g \\
  h(x) & h(y) & h(z) & h \\
  x & y & z & 1
\end{vmatrix}. \]

Figures 1, 2 and 3 illustrate, respectively, the action of the algebra on string configurations, the contraction of two three-junctions, and the commutator of a junction and its dual.

**Remark 4.2.** An important question to ask is whether the factor \( \mathfrak{g}_1 = \wedge^3 V \) forms an algebra by itself. The answer is no, as hinted earlier. However, while this is not the case we see from the commutation relation (4.7) that two factors in \( \wedge^3 V \) close into an element of \( \wedge^3 V^* \), which is a degree 3 element but for the dual vector space. Thus, this process does produce the desired form provided that we also introduce the operation of dualization for the vector spaces. Similarly, for starting with the dual vector space the commutation relation (4.8) of two dual degree 3 forms gives a degree 3 form of the original vector space. We will see another model for the multiplication of two 3-forms in section 5.3.

**Which traces can occur?** As mentioned in section 3.6 we will consider traces using invariant theory. The free generators of the invariant algebra of the action of \( G_0 = SL(V) \) on \( \mathfrak{g}_1 = \wedge^3 V \), \( \dim(V) = 9 \), have degrees [56] 12, 18, 24, 30. They can be constructed as follows [24, 37]. Consider the linear transformation

\[ L_{\otimes}(\lambda) : \text{V} \otimes \text{V} \otimes \text{V} \longrightarrow \text{V} \otimes \text{V} \otimes \text{V}. \]

Since \( \wedge^3 V \subset \text{V} \otimes \text{V} \otimes \text{V} \) (cf equation (2.3)), then the restriction of the above linear transformation to \( \wedge^3 V \) is

\[ L_{\wedge}(\lambda) : \wedge^3 V \longrightarrow \wedge^3 V, \]

which is the cube of the action of \( \lambda \in \wedge^3 V \) on \( \wedge^3 V^* \), given in (4.9). Starting with \( \lambda \in \wedge^3 V \), the tensor defining the linear transformation (4.10) is of type (3, 3) and given by

\[ \begin{align*}
(C(\lambda))^{imn}_{pqr} &= \epsilon_{ijklpqrisjkt} \lambda^{ijkl}(\lambda^{mkn} \lambda^{imp} + \lambda^{mkn} \lambda^{imp} + \lambda^{mkn} \lambda^{imp}).
\end{align*} \]

This tensor is skew-symmetric in both superscripts and subscripts.
One can take for the generators $P_1, \ldots, P_r$ of the algebra of invariants of the adjoint representation of the algebra $\mathfrak{e}_8$ to be the trace of the $k$th power of the action of an element of $\mathfrak{e}_8$ on $\mathfrak{e}_8$. In our case $k = 3$ and the trace of $L_\otimes$ coincides with that of $L_\wedge$. 

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By general results of [37], the restriction of the algebra $\mathbb{C}[\varepsilon_8]^E_6$ on $\wedge^3 V$ coincides with $\mathbb{C}[\wedge^3 V]_{SL(9)}$, and the degrees $n$ of the free generators are 12, 18, 24 and 30. The explicit form of the generators is given in [24, 37] as
\[
f_{3n}(\lambda) = \text{tr} L_{0}(\lambda)^n, \quad n = 4, 6, 8, 10.
\] (4.13)

Because of the isomorphism $\mathbb{C}[\mathfrak{g}_1]^{G_2} \cong \mathbb{C}[\mathfrak{g}]^W$, we can also look at the invariants using Weyl invariance instead. For $\varepsilon_8$, every semisimple trivector is equivalent to the linear combination
\[
\lambda^{ss} = \eta_1 \lambda_1^{ss} + \eta_2 \lambda_2^{ss} + \eta_3 \lambda_3^{ss} + \eta_4 \lambda_4^{ss},
\] (4.14)
of the trivectors
\[
\begin{align*}
\lambda_1^{ss} &= e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6 + e_7 \wedge e_8 \wedge e_9, \\
\lambda_2^{ss} &= e_1 \wedge e_4 \wedge e_7 + e_2 \wedge e_5 \wedge e_8 + e_3 \wedge e_6 \wedge e_9, \\
\lambda_3^{ss} &= e_1 \wedge e_5 \wedge e_9 + e_2 \wedge e_6 \wedge e_7 + e_3 \wedge e_4 \wedge e_8, \\
\lambda_4^{ss} &= e_1 \wedge e_6 \wedge e_8 + e_2 \wedge e_4 \wedge e_9 + e_3 \wedge e_5 \wedge e_7,
\end{align*}
\] where $\{e_1, \ldots, e_9\}$ is a basis for $\mathbb{C}^9$ and the coefficients $\eta_i, i = 1, 2, 3, 4$, are determined up to a linear transformation by the Weyl group $W(\varepsilon_8)$ associated with the $Z_3$-grading of $\varepsilon_8$ (see [58]). We also know that this group is generated by complex reflections with a parameter $\omega = e^{2\pi i/3}$, and the same result follows.

**Proposition 10.** The $SL(9)$-invariant configurations of junctions correspond to the admissible traces (4.13).

### 4.2. Representations of the Lie algebra $\mathfrak{g}_0$ on $\wedge^3 V$: the $E_6$ example

We can embed $\varepsilon_6$ in $\varepsilon_8$ and compute the associated Lie commutators. Recall that $\varepsilon_8 \supset \wedge^3 V$, where $V$ is a nine-dimensional vector space, and that $\varepsilon_6 \supset V_1 \otimes V_2 \otimes V_3$, where $V_i, i = 1, 2, 3$, is a three-dimensional vector space, and similarly for the duals.

Let $v_i \in V_i$ and $f_j \in V_j^*$, $i, j = 1, 2, 3$. Denote by $\overline{v}_i \in V$ ($\overline{f}_j \in V^*$) the extension of each vector (dual) to nine dimensions by zero entries. (That is, $V_1 \oplus V_2 \oplus V_3 \subset V_\ast$.) Then $V_1 \otimes V_2 \otimes V_3 \subset \wedge^3 V$ by taking $v_1 \otimes v_2 \otimes v_3 = \overline{v}_1 \wedge \overline{v}_2 \wedge \overline{v}_3 \in \wedge^3 V$, and
\[
[\overline{v}_1 \wedge \overline{v}_2 \wedge \overline{v}_3, \overline{f}_1 \wedge \overline{f}_2 \wedge \overline{f}_3] = -
\begin{vmatrix}
 f_1(v_1) & 0 & 0 & \overline{f}_1 \\
 0 & g_2(v_2) & 0 & \overline{f}_2 \\
 0 & 0 & h_3(v_3) & \overline{f}_3 \\
 v_1 & v_2 & v_3 & \frac{1}{3} I
\end{vmatrix},
\] (4.15)
from which we obtain
\[
[v_1 \otimes v_2 \otimes v_3, f_1 \otimes f_2 \otimes f_3]
\]
\[
= f_1(v_1) f_2(v_2) f_3(v_3)
\]
\[
\begin{vmatrix}
 \frac{1}{f_1(v_1)} v_1 \otimes f_1 - \frac{1}{3} I \\
 \frac{1}{f_2(v_2)} v_2 \otimes f_2 - \frac{1}{3} I \\
 \frac{1}{f_3(v_3)} v_3 \otimes f_3 - \frac{1}{3} I
\end{vmatrix} \in \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2) \oplus \mathfrak{sl}(V_3).
\] (4.16)

Given an order $r$ hypermatrix $A_{i_1 \cdots i_r}, 1 \leq i_j \leq n_j$, the hyperdeterminant of $A$ is invariant under $SL(n_1) \times \cdots \times SL(n_r)$ transformations. In fact more generally it is relatively

\[\text{Note that this group is the huge Witting complex reflection group of order 155 520. Hence, the fact that the normal form is determined up the action of the Witting group is not a trivial remark!}\]
invariant under the action of $GL(n_1) \times \cdots \times GL(n_r)$ by [28] (proposition 1.4, chapter 14). This means that the hyperdeterminant of an order 3 hypermatrix $A_{i_1i_2i_3}$ is invariant under $SL(n_1) \times SL(n_2) \times SL(n_3)$, or $sl(n_1) \oplus sl(n_2) \oplus sl(n_3)$ at the Lie algebra level. Thus,

**Proposition 11.** For the $E_6$ model, $g_0$ is the algebra leaving invariant the hypermatrix factor.

Here a result similar to that of proposition 10 also holds. However, to get the invariants explicitly requires calculations that are outside the scope of this paper (we plan to get back to this in the future). Semisimple and nilpotent elements, as well as the invariants are obtained in [41]. Note that the $\mathbb{Z}_3$-grading of $e_6$ and the computation of the normal forms have been investigated in the context of quantum information in [13].

### 4.3. Representations of the Lie algebra $g_0$ on $S^3V$: the $D_4$ example

Similarly, we have $d_4 \subset e_6$ by taking

$$w_1w_2w_3 = \sum_{\sigma \in S_3} \phi_1(w_{\sigma(i_1)}) \wedge \phi_2(w_{\sigma(i_2)}) \wedge \phi_3(w_{\sigma(i_3)}) \in S^3W,$$

(4.17)

for isomorphisms $\phi_i : W \rightarrow V_i$, $i = 1, 2, 3$. That is, $S^3W = V_1 \otimes V_2 \otimes V_3$ for $W = V_1 = V_2 = V_3$. The same method applies to the dual.

Now we wish to express the Lie bracket (4.16) for this algebra. Denoting by $a'_{ji}$ the vector $\phi_i(w_j)$ and by $f'_{ji}$ the vector $(\phi^*)^{-1}(u^i)$ we have

$$[w_1w_2w_3, u^1u^2u^3] = \sum_{(j_1,j_2,j_3) \in S_3} f'_{j_1}(a'_{j_1}) f'_{j_2}(a'_{j_2}) f'_{j_3}(a'_{j_3}) \otimes f'_{j_3} - \frac{1}{3} I.$$

(4.18)

We may express the action $[g_0, g_1] \rightarrow g_1$ in terms of the matrix $M \in sl(W)$ and $\lambda \in S^3W$. Given $\lambda = u \otimes v \otimes w$, we have the transformed $\lambda' = Mu \otimes Mv \otimes Mw$, or using the notation from before

$$\lambda' = (M, M, M) \cdot \lambda.$$

(4.19)

The action is similarly defined for $g_{-1}$.

A sufficient condition for $\lambda$ to be left invariant is that $u$, $v$ and $w$ are eigenvectors of $M$ with eigenvalue 1, or

$$\det(M - I) = 0.$$

(4.20)

From proposition 4 we have that the transformation formula for the hyperdeterminant

$$\Delta(\lambda') = \det(M)^9 \Delta(\lambda),$$

(4.21)

so for invariance of $\lambda$ we must have $\det(M)^9 = 1$. Therefore, we get

**Proposition 12.** A state in a junction in the $d_4$ model is invariant if $\det(M)^9 = 1$.

This can happen, for example, for $M = I e^{2\pi i/9}$, i.e. a ninth root of unity.

**Remark 4.3.** Again, a result similar to that of proposition 10 also holds here. However, as we noted right after proposition 11, we leave the explicit computation of the invariants for a future treatment.
4.4. The non-simply laced Lie algebras: types $F_4$ and $G_2$

The non-simply laced exceptional groups do not include a third (antisymmetric, symmetric or tensor) power in their graded decomposition. However, there is a 3-tensor symmetric on two indices in the case of $F_4$, and an extra 3-form is involved in the case of $G_2$.

**Representations of the Lie algebra $g_0$ on $S^2 V \otimes V$: the $F_4$ example.** The Lie algebra $f_4$ of the Lie group $F_4$ admits the $\mathbb{Z}_3$-graded decomposition

$$f_4 = (S^2 V_1^* \otimes V_2^*) \oplus (\mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)) \oplus (S^2 V_1 \otimes V_2), \quad \dim V_i = 3.$$  \hfill (4.22)

We see that the factor $S^2 V_1 \otimes V_2$ is the part of $V_1 \otimes V_2 \otimes V_3$ where two vector spaces $V_1$ and $V_2$ are identified. The Lie algebra $\mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)$ can be embedded in $\mathfrak{sl}(V)$ so that any element $(X, Y)$ in the former corresponds to the block-diagonal matrix with blocks $X, X, Y$. This allows for computation of the commutators in the algebra. Semisimple and nilpotent elements, as well as invariants can be found (also for $E_6$ and $D_4$) in \cite{3}.

**Remark 4.4.** Note that what appears here as a summand is a symmetric analog of the degree 3 element that is antisymmetric on the first two indices which appears in the $GL(n)$-decomposition of $V^\otimes 3$, for instance in expression (2.4).

**Invariant 3-forms and the $G_2$ case.** $G_2$ does not admit a cubic factor in its graded Lie algebra decomposition. The dimension of the Lie algebra $g_2 = \text{Lie}(G_2)$ is too small to admit such a factor, but it admits the decomposition

$$g_2(\mathbb{C}) = \text{Lie}(G_2) = V^* \oplus \mathfrak{sl}(V) \oplus V, \quad \dim_{\mathbb{C}} V = 3.$$  \hfill (4.23)

By duality, the factors $g_{-1}$ and $g_1$ can alternatively be taken to be $\wedge^2 V^*$ and $\wedge^2 V$, respectively. The real version admits a similar decomposition but with a real vector space dimension 4. This is used in \cite{49} to give a superalgebra structure on $g_2(\mathbb{R})$ and to make connection to symmetries of multiple membranes and Lie 3-algebras.

We see from (4.23) that in this case a 3-form, for instance, would have to be an additional piece of data, i.e. $\lambda \notin g_1$. Consider invertible complex linear transformations $S$ on a 3-form $\lambda$ on a complex seven-dimensional vector space $V$ such that

$$\lambda(M \cdot, M \cdot, M \cdot) = \lambda(\cdot, \cdot, \cdot).$$  \hfill (4.24)

The group of such $M$ is $G_2 \times \mathbb{Z}_3$ \cite{32}.

Given the above, the transformation (4.19) and proposition 4, we therefore have

**Proposition 13.** The states of a 3-junction, represented by a 3-form on a complex seven-dimensional vector space $V$, are invariant under the algebra $g_2$ or the group $G_2 \times \mathbb{Z}_3$.

In the non-simply laces case, a result similar to that of proposition 10 also holds.

5. Further applications and extensions

5.1. Symmetry of dimensionally-reduced supergravity

In \cite{17} it was shown that the underlying algebras for all the $D$-dimensional maximal supergravities that come from 11 dimensions are deformations of $G \oplus G^*$, where $G$ itself is the semi-direct sum of the Borel subalgebra of the superalgebra $\mathfrak{sl}(11 - D|1)$ and a rank-3 tensor representation, and $G^*$ is the coadjoint representation of $G$. The fields coming from
the 3-form potential in $D = 11$ form a linear graded antisymmetric 3-tensor representation of $\mathfrak{sl}(n|1)$. The algebra $\mathcal{G}$ for $D$-dimensional supergravity can be denoted by

$$\mathcal{G} = \mathfrak{sl}_+(n|1) \oplus \mathbb{Z}_3 (\wedge V)^3,$$

where $V$ is the appropriate fundamental representation, and $\mathfrak{sl}_+(n|1)$ is the Borel subalgebra of the superalgebra $\mathfrak{sl}(n|1)$.

In the special case of a reduction to $D = 3$ dimensions, the obvious $\mathfrak{gl}(n, \mathbb{R})$ symmetry from the dimensional reduction on an $n$-torus can be enlarged to the bosonic algebra $\mathfrak{sl}(n+1, \mathbb{R})$ rather than the superalgebra $\mathfrak{sl}_+(n|1)$. In the case of the doubled system of equations for maximal supergravity in $D = 11 - n$ dimensions, the global part of the gauge field preserving symmetry is $\mathfrak{e}_8^*$, the Borel subalgebra of the algebra $\mathfrak{e}_8$. In [17] it was expected that the doubled formalism should be invariant under the full global $\mathfrak{e}_n$ algebra. Here we provide a proof of that for the case $n = 8$.

This is actually straightforward in our setting. For $n = 8$, the enhanced algebra from the 8-torus will be $\mathfrak{sll}(V)$, with $\dim(V) = 9$ (rather than $\dim(V) = 8$). The algebras $\mathcal{G}$ and $\mathcal{G}^*$ are then

$$\mathcal{G} = \mathfrak{sll}(V) \oplus (\wedge V)^3, \quad \mathcal{G}^* = \mathfrak{sll}(V^*) \oplus (\wedge V^*)^3.$$

Now forming the semidirect sum gives

$$\mathcal{G} \oplus \mathcal{G}^* = (\wedge V^*) \oplus \mathfrak{sll}(V) \oplus (\wedge V)^3.$$

But this is exactly the $\mathbb{Z}_3$-graded model of $\mathfrak{e}_8$ (see proposition 1). Thus we immediately have the following.

**Theorem 5.1.** In the doubled formalism, the symmetry of gauge fields resulting from the dimensional reduction of 11-dimensional supergravity on the 8-torus is $\mathfrak{e}_8$.

Note that at the level of Lie algebras, we take $\mathfrak{e}_8$ to be a real form of the corresponding complex Lie algebra.

### 5.2. Valuedness of the fields

Our discussion suggests that the fields in the adjoint representation of $\mathfrak{sl}$, $\mathfrak{so}$ and $\mathfrak{sp}$, respectively, would be replaced by fields in the corresponding degree 3 antisymmetric, tensor, and symmetric powers:

$$A^{[ij]} e_i \wedge e_j \rightarrow A^{[ijk]} e_i \wedge e_j \wedge e_k,$$

$$A^{ij} e_i \otimes e_j \rightarrow A^{ijk} e_i \otimes e_j \otimes e_k,$$

$$A^{(ij)} e_i \circ e_j \rightarrow A^{(ijk)} e_i \circ e_j \circ e_k.$$

What possible combinations of the wedge $\wedge$, tensor $\otimes$ and symmetric $\circ$ products can occur, i.e. which indices $i, j, k$ are admissible? This of course depends on the Lie algebra $g$. In general, there is a Jordan decomposition of such degree 3 tensors into a semisimple part and a nilpotent part and the admissible tensors are known (see the discussion in sections 3.6 and 4.1).

### 5.3. Higher m-vectors

Here we provide alternatives to the models presented in sections 4.1 and 4.2. We have seen from equation (4.7) that $\wedge^3 V$ does not close on itself but rather on the dual $\wedge^3 V^*$ (cf the
remarks at the end of section 4.1). Here we describe a model in which the closure is on $\wedge^6 V$, i.e. for $\lambda_1, \lambda_2 \in \wedge^1 V$ we have the commutator
\[ [\lambda_1, \lambda_2] = \lambda_1 \wedge \lambda_2. \tag{5.5} \]
Thus we seek a graded Lie algebra decomposition which includes $\wedge^6 V$ as a summand. Then we would have the following extra cases (see [42]):

1. $e_6 = \wedge^6 V^* \oplus \wedge^3 V^* \oplus \text{gl}(V) \oplus \wedge^3 V \oplus \wedge^6 V$, $\text{dim}(V) = 6$.
2. $e_7 = \wedge^6 V^* \oplus \wedge^3 V^* \oplus \text{gl}(V) \oplus \wedge^3 V \oplus \wedge^6 V \oplus (V \otimes \wedge^8 V)$, $\text{dim}(V) = 7$.
3. $e_8 = (V^* \otimes \wedge^8 V^*) \oplus \wedge^6 V^* \oplus \wedge^3 V^* \oplus \text{gl}(V) \oplus \wedge^3 V \oplus \wedge^6 V \oplus (V \otimes \wedge^8 V)$, $\text{dim}(V) = 8$.

In addition to (5.5) there are other brackets corresponding to each pair of summands in the above decompositions of $e_i$, $i = 6, 7, 8$. Most relevant for us is the bracket of $\lambda \in \wedge^3 V^*$ and $C \in \wedge^6 V$:
\[ [\lambda, C] = \frac{1}{6} \lambda^{ijl} C_{ijkl}. \tag{5.6} \]
The bracket between $\lambda \in \wedge^3 V$ and $C^* \in \wedge^6 V^*$ is obtained from (5.6) by simply raising all lower indices and lowering all upper indices. The complete brackets are given for instance in [42].

**Remark 5.2.**

1. Because of the identity $[g_0, g_i] \subset g_i$ we get representations of the algebra $g_0 = \text{gl}(V)$ on $\wedge^6 V$. As in section 2.4 we also get representations of the corresponding general linear groups on $\wedge^6 V$.
2. One can in principle consider the action of $GL(V)$ which breaks $V^0G$ into a direct sum of $GL(V)$-modules which include
\[ V^0G \supset \wedge^6 V \oplus S^6 V. \tag{5.7} \]
This is a special cases of the more general action of $GL(V) \times \Sigma_n$ on $V^{0n}$ leading to the canonical isomorphism $V^{0n} \cong \bigoplus S_\rho(V) \otimes V_\rho$, where the sum is over all partitions $\rho$ of $n$ into at most $\text{dim}(V)$ parts, and $S_\rho(V)$ is the (image of the) Schur functor, i.e. the image $S_\rho(V) = \text{Im}(c_\rho : V^{0n} \rightarrow V^{0n})$ of the Young symmetrizer $c_\rho \in \mathbb{C} \Sigma_\rho$ (see [26]). Thus we could have posed the question as that of seeking graded Lie algebra decompositions that include (one of) the summands $\otimes^6 V$ or $\wedge^6 V$ or $S^6 V$. The question in the antisymmetric cases is provided by the above three cases of exceptional Lie algebras of type $E$.
3. What does (5.5) mean in terms of states and configurations? It represents a composite of two 3-junctions that are not joined or do not intersect.
4. The bracket (5.6) represents the contraction between a dual 3-junction state and a composite of two 3-junction states, giving rise to a single 3-junction state. This is a degree 3 analog of the contraction of a degree 2 tensor by a metric.
5. The degree 6 factor suggests the field coupling to the fivebrane. This forms part of the discussion in the next section.

### 5.4. Generalized Born–Infeld for membranes and fivebranes?

**D-branes.** The dynamics of D-p-branes, with $d = p + 1$ spacetime dimensions, is described in part by the Born–Infeld action of nonlinear electrodynamics. This can be seen from the sigma model approach [39] or using path integrals [54]. The action is given by
\[ S_d = \int d^d x e^{-\phi} \sqrt{\det(g_{mn} + F_{mn})}, \tag{5.8} \]
where \( F_{mn} = F_{mn} - B_{mn} \) is the difference (or, alternatively, sum) of the components of the curvature 2-form \( F_2 \) of the \( U(1) \) bundle and the B-field \( B_2 \).

The membrane. The fields on an open membrane include a 3-form field strength \( F_3 \), whose potential is a 2-form \( A_2 \) on the boundary. The 3-form can be combined with the pullback of the background C-field \( C_3 \) to form the shifted field
\[
H_3 = F_3 - C_3. \tag{5.9}
\]
This is a higher degree analog of the gauge invariant combination \( F_2 - B_2 \) for the open string, where \( F_2 \) is the curvature of the \( U(1) \) bundle and \( B_2 \) is the connection on a gerbe.

The fivebrane. The topological part of the worldvolume action involve combinations of expression (5.9) as well as \([53, 61]\)
\[
H_6 = F_6 + C_6 + \text{composite}. \tag{5.10}
\]
There exist proposed extensions that involve metric-dependent terms. One is the PST action which has an auxiliary scalar \( a \) and a dual field \( H_2 = \ast_6(da \wedge H_3) \) in six dimensions. The gauge-invariant action involves \([43]\)
\[
S_{PST} \supset \int \left( -d^6 \sigma \sqrt{\det(g + H_2)} + \left( C_6 + \frac{1}{2} F_3 \wedge C_3 \right) \right). \tag{5.11}
\]
The dimensional reduction reproduces the action of the D4-brane via the identifications \( F_3 \rightarrow (F_3, F_2), C_3 \rightarrow (C_3, B_2) \) and \( C_6 \rightarrow C_5 \).

Higher ‘gerby’ Born–Infeld. The boundaries of the membrane—which can end on M5-branes—are strings and hence do not carry gauge but gerbe degrees of freedom. Gerbes model higher form electrodynamics so that it is natural to ask for a nonlinear version of such a higher form. Thus, we propose a higher generalization of Born–Infeld action to accommodate degree 3 and degree 6 field strengths corresponding to the membrane and the fivebrane, respectively. As recalled in section 3.1, what replaces the determinant \( \det \) is naturally the \textit{hyperdeterminant} \( \text{Det} \). Furthermore, there is no obvious metric part in this case (unless we consider the idea of the dual of the graviton; see \([18, 60]\)). Thus, a generalization of the action (5.8) and (5.11), without the gravity part, would involve a scalar built out of the fields (5.9) and (5.10) for the case of the membrane and fivebrane, respectively.

We consider the antisymmetric tensor fields \( H_3 \) and \( H_6 \) as hypermatrices of the following form.

- For membrane: \( H_3 = (H_{ijk})_{1 \leq i, j, k \leq 3} \).
- For fivebrane:
  - \( H_3 = (H_{ijk})_{1 \leq i, j, k \leq 6} \).
  - \( H_6 = (H_{i_1 \ldots i_6})_{1 \leq i_1 \leq 6, k = 1, \ldots, 6} \).

The desired action will involve a square root of a generalization of the determinant. In the case of an antisymmetric matrix, this has an interesting description in terms of a Pfaffian, which is the ‘square root’ of the determinant of an antisymmetric matrix. In fact, a Pfaffian can be described in several ways, all of which turn out to be equivalent.

The analog of an antisymmetric matrix can be defined as follows. A \( k \)-dimensional alternating tensor \( A \) of order \( n \) can be defined as a function \( A \) on the product set \( \{1, \ldots, n\}^k \) such that
\[
A(i_1, \ldots, i_k) = \text{sign}(\sigma) A(i_{\sigma(1)}, \ldots, i_{\sigma(k)}) \tag{5.12}
\]
for any permutation \( \sigma \in \Sigma_k \) and \( 1 \leq i_1, \ldots, i_k \leq n \).
The higher degree analog of the Pfaffian will be the hyper-Pfaffian, which plays the analogous role for the hyperdeterminant of an alternating tensor as the Pfaffian plays for the determinant of an antisymmetric tensor. Like the Pfaffian, there are several ways of defining the hyper-Pfaffian. However, in contrast to the Pfaffian, those definitions are not all equivalent (for a discussion on this see the first section in [46]). Some definitions of the Pfaffian are, like the hyperdeterminant (see section 3.1), the zero polynomial for the case when \( k \) is odd. This will not be useful for us because we are seeking an expression involving \( H_3 \), i.e. for \( k = 3 \). Luckily, there is a definition that works for both even as well as odd \( k \) [40] and which is the one we will follow.

Let \( \Sigma_{k,m,k} \subseteq \Sigma_{km,k} \) be the set of permutations \( \sigma \) such that
\[
\sigma(ki + j) < \sigma(ki + j + 1) \quad \text{and} \quad \sigma(ki + 1) < \sigma(k(i + 1) = 1)
\]
for all \( 0 \leq i < m \)
\[
\text{and} \quad 1 \leq j < k.
\]

Then for a \( k \)-dimensional alternating tensor \( A \) of order \( km \), the hyper-Pfaffian of \( A \) is defined to be [40]
\[
Pf_k^*(A) = \sum_{\sigma \in \Sigma_{k,m,k}} \text{sign}(\sigma) \prod_{i=0}^{m-1} A(\sigma(ki + 1), \ldots, \sigma(ki + k)).
\]
(5.13)

For \( k = 2 \), this reproduces the formula for the Pfaffian as follows (see [46]). Define \( S_{2n} \subseteq \Sigma_{2n} \) to be the set of all \( \sigma \in \Sigma_{2n} \) such that \( \sigma(i) < \sigma(i + 1) \) and \( \sigma(i) < \sigma(i + 2) \) for all odd \( i \). Then for a \( 2n \times 2n \) antisymmetric matrix \( A \) the Pfaffian is
\[
Pf(A) = \sum_{\sigma \in S_{2n}} \text{sign}(\sigma) \prod_{i=0}^{n-1} A(\sigma(2i + 1), \sigma(2i + 2)),
\]
(5.14)
which indeed coincides with (5.13) for \( k = 2 \).

We note from (5.13) that the order of the antisymmetric tensor should be a nontrivial multiple of its dimension. This means that the case for \( H_3 \) on the membrane worldvolume cannot be described by expression (5.13), whereas both \( H_3 \) and \( H_6 \) on the fivebrane worldvolume do. The proposed action for the fivebrane would then contain
\[
S_{M5} \supset \int \text{Pf}(H_6),
\]
(5.15)
where \( H_6 \) has expression (5.10). For the membrane, while we cannot write a similar expression using the same definition for the hyper-Pfaffian, we expect something analogous to occur once a convenient definition for the hyper-Pfaffian is obtained which can be adapted for the case when the dimension of the tensor is equal to its rank.

**Proposal 1.** \( M5 \)-branes (and \( M2 \)-branes) can be described (in part) by the generalized Born–Infeld action (5.15) (and a similar action for the \( M2 \)-brane).

**Correspondence with the string/D-brane case.** The determinant is part of the formula for the hyperdeterminant. We consider \( F \) to be sitting inside \( H \) as a slab, so that we get a matrix if we start with a hypermatrix all of whose slabs in one direction are the same, i.e. if the hypermatrix is a stack of identical slabs. By slab operations, this is equivalent to a hypermatrix with all zero entries except in one full slab. For a visualization see figure 4.

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5 There is an earlier definition of the hyper-Pfaffian in [5], but that definition matches only for the even case.
5.5. Final remarks

(1) **Relation to Kac–Moody Algebras.** We discussed a duality-symmetric model of the $E$-series in section 5.3. In fact $E_6$ is duality-symmetric in a different setting. This is one main aspect of the $E_6$, $E_{10}$ and $E_{11}$ models aiming to describe 10- and 11-dimensional supergravity and M-theory—see [60]. There is a correspondence [35] between a $\mathbb{Z}_m$-graded Lie algebra $g$ with its ‘covering’ infinite-dimensional $\mathbb{Z}$-graded Lie algebra

$$\hat{g} = \sum_{k \in \mathbb{Z}} \hat{g}_k \subset \mathbb{C}[t, t^{-1}] \otimes g,$$

where $\hat{g}_k$ denotes the grading subspace of $g$ whose index is the residue class $k$ modulo $m$. So obviously any $g_k \subset g$ will also be a summand in $\hat{g}$.

The algebra $g$ is obtained, as an algebra over $\mathbb{C}$, from $\hat{g}$ by factoring the ideal $(u - 1)\hat{g}$, where the multiplication $u$ is defined by the formula $ux = t^m x$, $x \in \hat{g}$, which make $\hat{g}$ a finite-dimensional $\mathbb{C}[u, u^{-1}]$ algebra.

The models we have seen in this paper use finite-dimensional—and in fact relatively low-dimensional—vector spaces. On the other hand, we would be interested in the large $N$ limit, which thus cannot be immediately seen in such models. It might be possible that embedding in a Kac–Moody algebra might allow for this possibility, but we will not discuss this further in the current paper.

(2) **Relation to 3-algebras.** In the main part of this paper we focused on keeping Lie algebras in the discussion. The Lie bracket on a Lie algebra $g$ is defined as a map $[, ] : \wedge^2 g \rightarrow g$. There has been very interesting recent activity (starting with [4] and [30]) on modeling multiple M-branes using Lie 3-algebras with bracket $[, , ] : \wedge^3 g \rightarrow g$.

In [20] a Lie-algebraic origin of certain metric 3-algebras is provided. In particular, it is proved that certain metric 3-algebras correspond to pairs $(g, V)$ consisting of a metric Lie algebra $g \leq \mathfrak{so}(V)$, $[g, g] \subset g$, and a real faithful orthogonal representation $V$. In this paper we kept working with Lie algebras (justified by [20]) and used third powers of $V$ instead of $V$ itself. Thus, we have taken a different path from the ones in the above cited works. Hence the work in this paper will not directly connect to Lie 3-algebras but could be seen as complimentary. Further work might require higher algebras as in [48].

We have presented models that capture some aspects of the description of 3-junctions which introduces hypermatrices and their hyperdeterminants. This made natural and novel connection to exceptional Lie algebras. However, as we discussed throughout the paper, there are many unanswered questions. Our treatment has been mostly formal, and further physical arguments would be needed to tell how the physics of D-branes would...
favor a model over the other. Furthermore, a more refined mathematical discussion might be needed. We hope to address such matters in a future project to at a more final answer. The full answer is likely to go beyond usual (non)linear algebra.

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References

[1] Aharony O, Bergman O, Jafferis D L and Maldacena J 2008 \( N = 6 \) superconformal Chern–Simons-matter theories, M2-branes and their gravity duals J. High Energy Phys. JHEP10(2008)091 (arXiv:0806.1218 [hep-th])
[2] Aharony O, Sonnenschein J and Yankielowicz S 1996 Interactions of strings and D-branes from M-theory Nucl. Phys. B 474 309 (arXiv:hep-th/9603009)
[3] Artamkin D I and Nurmiev A G 2002 Orbits and invariants of third-order cubic matrices with symmetric layers Math. Notes 72 447–53
[4] Bagger J and Lambert N 2007 Modeling multiple M2's Phys. Rev. D 75 045020 (arXiv:hep-th/0611108)
[5] Barvinok A I 1995 New algorithms for linear \( k \)-matroid intersection and matroid \( k \)-parity problems Math. Programming 69 449–70
[6] Bastianelli F, Frolov S and Tseytlin A A 2000 Conformal anomaly of (2, 0) tensor multiplet in six dimensions and AdS/CFT correspondence J. High Energy Phys. JHEP02(2000)013 (arXiv:hep-th/0001041)
[7] Bekkaert X, Henneaux M and Sevrin A 1999 Deformations of chiral two-forms in six dimensions Phys. Lett. B 468 228
[8] Bekkaert X, Henneaux M and Sevrin A 2001 Chiral forms and their deformations Commun. Math. Phys. 224 683–703 (arXiv:hep-th/0004049)
[9] Berman D S 2008 M-theory branes and their interactions Phys. Rep. 456 89–126 (arXiv:0710.1707 [hep-th])
[10] Berman D S and Copland N B 2006 A note on the M2-M5 brane system and fuzzy spheres Phys. Lett. B 639 553 (arXiv:hep-th/0605086)
[11] Borel A and Harish-Chandra 1962 Arithmetic subgroups of algebraic groups Ann. Math. 75 485–535
[12] Borsten L, Dahanayake D, Duff M J, Ebrahim H and Rubens W 2009 Black holes, qubits and octonions Phys. Rep. 471 113–219 (arXiv:0809.4685 [hep-th])
[13] Briand E, Luque J-G, Thibon J-Y and Verstraete F 2004 The moduli space of three-qutrit states J. Math. Phys. 45 4855–67
[14] Capdevielle B 1972 Classification des formes trilinéaires alternées en dimension 6 Enseignement Math. 18 225–43
[15] Cayley A 1846 Mémoire sur les hyperdeterminants J. Reine Agnew. Math. 30 1–37
[16] Comon P, Golub G, Lim L-H and Mourrain B 2008 Symmetric tensors and symmetric tensor rank SIAM J. Matrix Anal. Appl. 30 1254–79
[17] Cremmer E, Julia B, Lu H and Pope C N 1998 Dualization of dualities II Nucl. Phys. B 535 242–92 (arXiv:hep-th/9806106)
[18] Curtright T 1985 Generalized gauge fields Phys. Lett. B 165 304–8
[19] de Medeiros P, de, Figueroa-O’Farrill J and Méndez-Escobar E 2009 Superpotentials for superconformal Chern–Simons theories from representation theory J. Phys. A: Math. Theor. 42 485204 (arXiv:0908.2125 [hep-th])
[20] de Medeiros P, Figueroa-O’Farrill J, Méndez-Escobar E and Ritter P 2009 On the Lie-algebraic origin of metric 3-algebras Commun. Math. Phys. 290 871–902 (arXiv:0809.1086 [hep-th])
[21] de Silva V and Lim L-H 2008 Tensor rank and the ill-posedness of the best low-rank approximation problem SIAM J. Matrix Anal. Appl. 30 1084–127
[22] Djokovic D Z 1983 Classification of trivectors of an eight-dimensional real vector space Linear Multilinear Algebra 13 3–9
[23] Duff M J 2006 Hidden symmetries of the Nambu–Goto action Phys. Lett. B 641 335–7 (arXiv:hep-th/0602160)
[24] Egorov G V 1981 Invariants of 3-vectors of 9-dimensional space Problems in Group Theory and Homology Algebra (Yaroslavl: Yaroslavl. Gos. Univ.) pp 123–31
[25] Fehér L M, Némethi A and Rimányi R 2005 Degeneracy of 2-forms and 3-forms Can. Math. Bull. 48 547–60
[26] Fulton W and Harris J 1991 Representation Theory: A First Course (New York: Springer)
[27] Gabdul M R and Zwiebach B 1998 Exceptional groups from open strings Nucl. Phys. B 518 151–72 (arXiv:hep-th/9709013)
[28] Gelfand I M, Kapranov M M and Zelevinsky A V 1994 Discriminants, Resultants and Multidimensional Determinants (Basel: Birkhäuser)
[29] Green M, Schwarz J and Witten E 1986 Superstring Theory vol 1 (Cambridge: Cambridge University Press)
[30] Gustavsson A 2009 Algebraic structures on parallel M2-branes Nucl. Phys. B 811 66–76 (arXiv:0709.1260 [hep-th])
[31] Henningson M and Skenderis K 1998 The holographic Weyl anomaly J. High Energy Phys. JHEP07(1998)023 (arXiv:hep-th/9806087)
[32] Herz C 1983 Alternating 3-forms and exceptional simple Lie groups of type $G_2$ Can. J. Math. 35 776–806
[33] Hitchcock F L 1927 The expression of a tensor or a polyadic as a sum of products J. Math. Phys. 6 164–89
[34] Johansen A 1997 A comment on BPS states in $\mathcal{N}=1$ super-symmetric $\gamma$-theory in 8 dimensions Phys. Lett. B 395 36 (arXiv:hep-th/9608186)
[35] Kac V G 1990 Infinite Dimensional Lie Algebras 3rd edn (Cambridge: Cambridge University Press)
[36] Kalashnikova Yu S and Nefediev A V 1996 String junction as a baryonic constituent Linear Multilinear Algebra 13 3–9
[37] Katanova A A 1992 Explicit form of certain multivector invariants Adv. Sov. Math. 8 87–93
[38] Klebanov I R and Tseytlin A A 1996 Entropy of near-extremal black $p$-branes Nucl. Phys. B 475 164 (arXiv:hep-th/9604089)
[39] Leigh R 1989 Dirac–Born–Infeld action from Dirichlet sigma model Mod. Phys. Lett. A 4 2767
[40] Luque J-G and Thibon J-Y 2002 Pfaffian and Hafnian identities in shuffle algebras Adv. Appl. Math. 29 620–46
[41] Mavlyutov A G 2000 Orbits and invariants of third-order matrices Sh. Math. 191 717–24
[42] Nkurumila B and Winberg E B (ed) 1994 Lie Groups and Lie Algebra III: Structure of Lie Groups and Lie Algebras (Berlin: Springer)
[43] Pasti P, Sorokin D P and Tonin M 1997 Covariant action for a $\mathcal{N}=1$ super-symmetric $\gamma$-theory in 8 dimensions Phys. Lett. B 398 41 (arXiv:hep-th/9701057)
[44] Paton J E and Chan H M 1969 Generalized Veneziano model with isospin Nucl. Phys. B 10 519
[45] Polchinski J 1998 String Theory vol 2 (Cambridge: Cambridge University Press)
[46] Redelmeier D 2006 Hyperpfaffians in algebraic combinatorics Sb. Math. 191 717–24
[47] Revy Ph 1988 Formes trilinéaires alternées en dimension 6 et $\gamma$ Bull. Sci. Math. 112 357–68
[48] Satō H, Schreiber U and Stasheff J 2009 $L_{\infty}$-connections and applications to string- and Chern–Simons $n$-transport Recent Developments in QFT ed B Fauser et al (Basel: Birkhäuser) 303–424 (arXiv:0801.3480 [math.DG])
[49] Schwab J H 1997 Lectures on superstring and M-theory dualities Nucl. Phys. B 55 1 (arXiv:hep-th/9607201)
[50] Sokolov N P 1972 Introduction to the Theory of Multidimensional Matrices (Kiev: Naukova Dumka)
[51] Strominger A 1996 Open $p$-branes Phys. Lett. B 383 44–7 (arXiv:hep-th/9512059)
[52] Townsend P K 1996 D-branes from M-branes Phys. Lett. B 373 68–75 (arXiv:hep-th/9512062)
[53] Tseytlin A A 1996 Self-duality of Born–Infeld action and Dirichlet 3-brane of type IIB superstring theory Nucl. Phys. B 469 51–67 (arXiv:hep-th/9602064)
[54] Vasil’ev N A and Perelom E 2007 Polyvector representations of $GL_n$ J. Math. Sci. (NY) 145 4737–50
[55] Vinberg E B 1975 On the linear groups that are associated to periodic automorphisms of semisimple Lie groups Sov. Math. Dokl. 16 406–9
[56] Vinberg E B 1976 The Weyl group of a graded Lie algebra Math. USSR Izv. 10 463–95
[57] Vinberg E B and Elasvili A G 1978 A classification of the three-vectors of nine-dimensional space Trudy Sem. Vektor. Tenzor. Anal. 18 197–233
[58] Witten R 1981 Real trivectors of rank seven Linear Multilinear Algebra 10 183–204
[59] West P 2001 E 11 and M theory Class. Quantum Grav. 18 4443–60
[60] Witten E 1997 Five-brane effective action In M-theory J. Geom. Phys. 22 103–33 (arXiv:hep-th/9610234)