Tolerant Distribution Testing in the Conditional Sampling Model

Shyam Narayanan*

July 21, 2020

Abstract

Recently, there has been significant work studying distribution testing under the Conditional Sampling model. In this model, a query involves specifying a subset $S$ of the domain, and the output received is a sample drawn from the distribution conditioned on being in $S$. In this paper, we primarily study the tolerant versions of the classic uniformity and identity testing problems, providing improved query complexity bounds in the conditional sampling model.

In this paper, we prove that tolerant uniformity testing in the conditional sampling model can be solved using $\tilde{O}(\epsilon^{-2})$ queries, which is known to be optimal and improves upon the $\tilde{O}(\epsilon^{-20})$-query algorithm of [CRS15]. Our bound even holds under the Pair Conditional Sampling model, a restricted version of the conditional sampling model where every queried subset $S$ either must have exactly 2 elements, or must be the entire domain of the distribution. We also prove that tolerant identity testing in the conditional sampling model can be solved in $\tilde{O}(\epsilon^{-4})$ queries, which is the first known bound independent of the support size of the distribution for this problem. Finally, we study (non-tolerant) identity testing under the pair conditional sampling model, and provide a tight bound of $\tilde{\Theta}(\sqrt{\log n \cdot \epsilon^{-2}})$ for the query complexity, improving upon both the known upper and lower bounds in [CRS15].

*Massachusetts Institute of Technology, Cambridge, MA, 02139. Email: shyamsn@mit.edu. Research supported by an MIT Akamai Fellowship and an NSF Graduate Fellowship.
1 Introduction

1.1 Distribution Testing in the Sampling Model

Distribution testing is a fundamental problem in statistics where the goal is to learn properties of a distribution $D$ over an $N$-element set $[N]$, given an oracle that can draw independent samples from $D$. Given an arbitrary number of samples, it is quite simple to learn the distribution fully, but we wish to learn properties of $D$ using a sublinear number of samples, and preferably as few samples as possible. This implies that we cannot fully learn the distribution of $D$. Instead, as in many problems in the field of property testing, the algorithm should make a sublinear number of samples, and output ACCEPT with high probability if the distribution $D$ has the desired property but should output REJECT with high probability if $D$ has total variation distance at least $\varepsilon$ from every distribution with the desired property. However, if $D$ neither has the desired property nor has total variation distance at least $\varepsilon$ from every distribution with the desired property, the algorithm is allowed to output either ACCEPT or REJECT. This freedom is what makes it possible for an algorithm to make a sublinear number of queries to determine properties of the distribution.

The study of distribution testing in the framework of property testing began nearly two decades ago with [BFR+00], which initiated a long line of study in this area (e.g., [GR00, BFF+01, BKR04, Pan08, ADJ+11, BFRV11, Val11, VV11, ILR12, DDS+13, VV13, CDVV14, Wag15, ADK15, BV15, DKN15, DK16, Can16, CDGR16, WY16, BCG17, BC17, VV17, DW18, DKPP18, WY19], or the surveys [Rub12, Can17, BW18, Kam18]). Problems which have been studied in the distribution testing framework have included uniformity testing, or testing whether an unknown distribution $D$ is uniform; identity testing, or testing whether $D$ is identical to a known distribution $D^*$; equivalence testing, or testing whether $D_1$ and $D_2$ are identical where we have sample access to both $D_1$ and $D_2$; and monotonicity testing. Problems such as estimating the entropy of $D$ and the support size of $D$ have also been well-studied. A final important class of studied problems are the tolerant versions of uniformity, identity, and equivalence testing, which mean trying to approximate the total variation distance between $D$ and the Uniform distribution, $D$ and $D^*$, and $D_1$ and $D_2$, respectively. Tolerant testing provides much stronger functionality than basic hypothesis testing, as it provides meaningful guarantees even if the underlying distribution is known not to satisfy the hypothesis exactly.

While there exist sublinear algorithms for all of the problems listed above, the optimal algorithms are often only slightly sublinear, so they are not significantly more efficient than naive algorithms. For instance, estimating the entropy, the support size, and the distance from the uniform distribution or a fixed distribution $D^*$ all have optimal sample complexities of $\Theta(N/\log N)$. Even for the simple problem of uniformity testing, the optimal algorithm requires $\Theta(\sqrt{N})$ samples. This motivated different models where one is allowed additional information besides simply sampling from the distribution. Some models involve sampling from distributions with additional structure. For instance, the distributions may be promised to be monotone [RS09], to be $k$-modal [DDS+13], to be a low-degree Bayesian network [CDKS17, DP17, ABDK18], or to have some other property [DK18, GLP18, BBC+19, DKP19]. In other settings, one is allowed additional types of queries, such as to either the Probability Mass Function (PMF) or the Cumulative Distribution Function (CDF) of the distribution [BDKR05, GMV06, RS09, CR14]. A related model is the probability-revealing samples model, where one is allowed samples $(x, p(x))$ where $x \sim D$ and $p(x) = \Pr_{y \sim D}(y = x)$ [OS18]. In this paper, however, we primarily study the Conditional Sampling Model, which allows for sampling conditioned on some query set $S \subset [N]$. 
1.2 The Conditional Sampling Model and Motivation

In the conditional sampling model, our goal is again to test properties of a distribution \( \mathcal{D} \) supported over \([N]\), but now we are given a stronger sampling oracle. This time, for each query, we are allowed to choose a subset \( S \subseteq [N] \), and the oracle draws from the distribution \( \mathcal{D} \) conditioned on \( S \). Formally, we have the following.

**Definition 1.1.** \([\text{CRS15}]\) Fix a distribution \( \mathcal{D} \) over \([N]\). A Cond oracle for \( \mathcal{D} \) is defined as follows. The oracle is given as input a query set \( S \subseteq [N] \) and outputs an element \( i \in S \), where the probability that element \( i \) is returned is \( D(i)/D(S) \), where \( D(i) = \mathbb{P}_{x \sim \mathcal{D}}(x = i) \) and \( D(S) = \mathbb{P}_{x \sim \mathcal{D}}(x \in S) \). Moreover, each output of the oracle is independent of all previous calls.

We note that the above definition only makes sense if \( D(S) > 0 \). One way to fix this is to make sure that all probabilities are nonzero by slightly modifying any \( i \) such that \( D(i) = 0 \) to be barely positive. In our case, this will not even matter, because every time that we call \( \text{Cond}(S) \), we will have previously sampled at least one element in \( S \), so we know that \( D(S) > 0 \).

The simplest motivation for the conditional sampling model is that for many traditional distribution testing problems, the standard sampling model cannot provide a strongly sublinear-query algorithm. However, as we will see, for most standard distribution testing problems, only a poly-logarithmic or often even a constant number of queries to the conditional sampling oracle is required (assuming a fixed error parameter \( \varepsilon \)). This makes the conditional sampling model very powerful even though at a first glance it may not look significantly more powerful than the regular sampling oracle. Moreover, for many problems, the full extent of the COND sampling power is not even needed. As we will discuss later, some problems can be solved using an oracle which only samples from either all of \([N]\) or pairs of elements in \([N]\) (PAIRCOND queries); or from other restricted versions of COND.

At the same time, various forms of conditional sampling are supported in multiple applied scenarios. For instance, the BlinkDB database system \([\text{APM+13}]\) uses stratified random sampling to provide approximate answers to SQL aggregation queries over large volumes of data. Their system enables generating random samples of the data satisfying user-specified predicates, which are then used to approximate the aggregates of interests (counts, sums, etc). In particular, PAIRCOND queries considered here correspond to simple disjunctive predicates, where an attribute can have one of two fixed values.

Another appealing aspect of the conditional sampling model, as noted in \([\text{CRS15}]\), is that unlike in the standard sampling model, we are now able to deal with adaptive queries, since we can choose at each step which set \( S \) to sample from. This leads to a richer class of algorithms than in the standard sampling model, where the only queries allowed are samples from the full data set \( \mathcal{D} \). Therefore, the conditional sampling model leads to a much broader range of potential algorithms.

1.3 Prior Work in the Conditional Sampling Model

The conditional sampling model was initially studied in \([\text{CRS15}, \text{CFGM16}]\). The work of Chakraborty et. al. \([\text{CFGM16}]\) proved that uniformity testing with error \( \varepsilon \) could be done in \( \text{poly}(\varepsilon^{-1}) \) queries and that identity testing with error \( \varepsilon \) could be done in \( \text{poly}(\varepsilon^{-1}, \log^* (N)) \) queries. They also proved that computing entropy could be done in \( \text{poly}(\varepsilon^{-1}, \log N) \) queries, by providing an algorithm for testing any label-invariant property of a distribution, i.e., a property which was invariant under a
permutation of the elements of $[N]$. The work of Canonne et. al. [CRS15] gave nearly tight bounds for uniformity testing, by providing an $O(\varepsilon^{-2})$-query algorithm and a nearly matching lower bound of $\Omega(\varepsilon^{-2})$. [CRS15] also provided an $O(\varepsilon^{-4})$-query algorithm for the identity testing problem, which was later improved to a nearly optimal $\tilde{O}(\varepsilon^{-2})$-query algorithm by Falahatgar et. al. [FJO+15]. [CRS15] also provided an $O(\varepsilon^{-4}\log^5 N)$-query algorithm for equivalence testing, which was also improved by [FJO+15] to $\tilde{O}(\varepsilon^{-5}\cdot \log N)$-queries. The best known lower bound for equivalence testing, however, is $\Omega(\sqrt{\log \log N})$, done by [ACK18], [CRS15] also provided an $\tilde{O}(\varepsilon^{-20})$-query algorithm for tolerant uniformity testing. Canonne [Can15] also studied monotonicity testing in the conditional sampling model, providing an $O(\varepsilon^{-22})$-query algorithm for testing monotonicity. For an in-depth summary of results in the standard sampling model, conditional sampling model, and other related models, we point the interested reader towards Canonne’s survey paper [Can17].

We note that many variants of the conditional sampling model have been studied, most of which are more restrictive versions of the standard conditional sampling model. For instance, [CFGM16, ACK18, KT19] also looked at testing in the nonadaptive conditional sampling model, where queries are not allowed to depend on the previous outputs of the oracle. Another variant is the subcube conditioning problem, where $N = 2^n$ and we treat $[N]$ as the set of vertices of an $n$-dimensional cube, but we are only allowed to conditionally sample from subcubes of the $n$-dimensional cube [BCG17, BC18, CCK+19, CJLW20]. Two more variants are the PAIRCOND and INTCOND sampling models. In the PAIRCOND model, all samples must either be sampled from the entire set $[N]$ or from the oracle $\text{PCond}(x, y)$, which samples from the conditional distribution of $\{x, y\} \subseteq [N]$. In other words, if we ever conditionally sample from a set $S \subseteq [N]$, either $S = [N]$ or $|S| = 2$. In the INTCOND model, all conditional samples must be conditionally sampled on some interval $S = [a, b] = \{a, a+1, \ldots, b\}$, i.e., we can only sample in intervals. The paper [CRS15] investigated many problems in the PAIRCOND model, and both papers [CRS15, CFGM16] investigated some problems in the INTCOND model. In this work, we will provide new upper bounds in the COND model, as well as new upper and lower bounds in the PAIRCOND model.

1.4 Our Results

For a fixed $D$, the $\text{Samp}$ oracle simply draws a random element $i$ from the distribution $D$. Recall the $\text{Cond}$ oracle from Definition 1.1, and finally, define the $\text{PCond}$ oracle as follows. $\text{PCond}$ takes as input $x, y \in [N]$, and returns $x$ with probability $\frac{D(x)}{D(x)+D(y)}$ and returns $y$ with probability $\frac{D(y)}{D(x)+D(y)}$. Equivalently, the outputs of $\text{PCond}(x, y)$ and $\text{Cond}(\{x, y\})$ have the same distributions.

We now describe the main results of this paper. See Table 1 for a summary of our main results as well as previous results (both upper and lower bounds).

**Tolerant Uniformity Testing:** The first main result we prove is a nearly optimal query complexity algorithm for tolerant uniformity testing, improving on the $\tilde{O}(\varepsilon^{-20})$-query algorithm of [CRS15]. Like the result in [CRS15], we only need the weaker PAIRCOND model.

**Theorem 1.1.** Let $\mathcal{U}$ be the uniform distribution over $[N]$. Given any distribution $D$ and access to $\text{Samp}$ and $\text{PCond}$, there is an algorithm $\text{TolerantUnif}$ that uses $\tilde{O}(\varepsilon^{-2})$ queries and determines the total variation distance $d_{TV}(D, \mathcal{U})$ up to an additive error of $O(\varepsilon)$ with probability at least $2/3$.

The above theorem is known to be nearly optimal in both the COND and PAIRCOND models, since even the standard uniformity testing problem, i.e., distinguishing between $D = \mathcal{U}$ and $d_{TV}(D, \mathcal{U}) > \varepsilon$, requires at least $\Omega(\varepsilon^{-2})$ queries, even in the COND model.
| Problem | COND | PAIRCOND | SAMP |
|---------|------|----------|------|
| Uniformity Testing Is $\mathcal{D}$ uniform? | $\Theta(\varepsilon^{-2})$ [CRS15] | $\tilde{O}(\varepsilon^{-2})$ [CRS15] | $\Theta\left(\frac{n}{\varepsilon^2}\right)$ [Pan08] |
| Identity Testing Does $\mathcal{D} = \mathcal{D}^*$? | $O(\varepsilon^{-2})$ [FJO+15] | $\tilde{O}\left(\frac{\log^2 N}{\varepsilon^2}\right)$ [CRS15] | $\Theta\left(\frac{n}{\varepsilon^2}\right)$ [VV17] |
| Tolerant Uniformity What is $d_{TV}(\mathcal{D}, \mathcal{U})$? | $\tilde{O}(\varepsilon^{-20})$ [CRS15] | $\tilde{O}(\varepsilon^{-20})$ [CRS15] | $\tilde{O}\left(\frac{1}{\varepsilon^2} \cdot \frac{n}{\log n}\right)$ [VV11] |
| Tolerant Identity What is $d_{TV}(\mathcal{D}, \mathcal{D}^*)$? | $\tilde{O}(\varepsilon^{-4})$ | $\tilde{O}(\varepsilon^{-2})$ | $\tilde{O}\left(\frac{1}{\varepsilon^2} \cdot \frac{n}{\log n}\right)$ [VV11] |

Table 1: List of query complexity bounds in the COND, PAIRCOND, and standard sampling (SAMP) models. Our new results are in bold. We note that the $\tilde{O}(\varepsilon^{-2})$ upper bound for Tolerant Uniformity in the COND model is directly implied by our $\tilde{O}(\varepsilon^{-2})$ upper bound for Tolerant Uniformity in the PAIRCOND model, since PAIRCOND is a more restrictive model.

**Tolerant Identity Testing:** Our next main result is an algorithm for tolerant identity testing in the conditional sampling model, which has never been directly addressed in the previous literature. While our bounds are not optimal, we provide an algorithm with query complexity that does not depend on the support size $N$ at all and grows only polynomially in $\varepsilon^{-1}$, though we require the full power of the COND model.

**Theorem 1.2.** Let $\mathcal{D}^*$ be some fixed distribution over $[N]$. Given any distribution $\mathcal{D}$ and access to COND, there is an algorithm TOLerantID that uses $\tilde{O}(\varepsilon^{-4})$ queries and determines the total variation distance $d_{TV}(\mathcal{D}, \mathcal{D}^*)$ up to an additive error of $O(\varepsilon)$.

A natural open question is whether the bound of $\tilde{O}(\varepsilon^{-4})$ can be improved to $\tilde{O}(\varepsilon^{-2})$.

**Identity Testing in PAIRCOND:** Our final result is a nearly optimal query complexity algorithm for identity testing in the PAIRCOND model, as well as a nearly matching lower bound.

**Theorem 1.3.** Let $\mathcal{D}^*$ be some fixed distribution over $[N]$. Given any distribution $\mathcal{D}$ and access to SAMP and PCOND, there is an algorithm PCONId that, if $\mathcal{D} = \mathcal{D}^*$, outputs ACCEPT with probability at least $2/3$, and if $d_{TV}(\mathcal{D}, \mathcal{D}^*) \geq \varepsilon$, outputs REJECT with probability at least $2/3$. Moreover, PCONId uses $\tilde{O}(\sqrt{\log N} \cdot \varepsilon^{-2})$ queries.

**Theorem 1.4.** There exists a distribution $\mathcal{D}^*$ with the following property. If any algorithm that, given access to SAMP and PCOND, outputs ACCEPT with probability at least $2/3$ if $\mathcal{D} = \mathcal{D}^*$ and outputs REJECT with probability at least $2/3$ if $d_{TV}(\mathcal{D}, \mathcal{D}^*) \geq \varepsilon$, then the algorithm must make at least $\Omega\left(\sqrt{\frac{\log N}{\log(\varepsilon^{-2})}} \cdot \varepsilon^{-2}\right)$ queries.

While there exists a previously known $\tilde{O}(\varepsilon^{-2})$-query algorithm in the COND model [FJO+15], their algorithm requires the full power of the COND model. Our upper and lower bounds improve upon the results of [CRS15], which provides an $O(\log^4 N \cdot \varepsilon^{-4})$-query algorithm and a $\Omega\left(\sqrt{\frac{\log N}{\log \log N}}\right)$-query lower bound. Importantly, our upper and lower bounds are now tight, up to a poly($\log \varepsilon^{-1}, \log \log N$) multiplicative factor.
1.5 Outline

We briefly outline the rest of the paper. In Section 2, we go over some definitions and preliminary results. In Section 3, we outline the ideas of our main results. In Section 4, we prove Theorem 1.1. In Section 5, we prove Theorem 1.2. In Section 6, we prove Theorem 1.3. We defer the proof of Theorem 1.4 to Appendix B as the proof is very similar to [CRS15, Theorem 8]. We also include pseudocode for the algorithms (divided into subroutines corresponding to lemmas) in Appendix A.

2 Preliminaries

First, for any integer $N \geq 1$, we use $[N]$ to denote the set $\{1, 2, \ldots, N\}$, and for integers $b \geq a \geq 1$, we use $[a:b]$ to denote the set $\{a, a+1, \ldots, b\}$. In this paper, we will usually be working with an unknown distribution over the set $[N]$, unless specified otherwise. For a distribution $D$ over $[N]$, we write $x \sim D$ to mean that $x$ was drawn from the distribution $D$. For any element $i \in [N]$, we let $D(i) := \Pr_{x \sim D}(x = i)$, and for any subset $S \subseteq [N]$, we let $D(S) := \Pr_{x \sim D}(x \in S) = \sum_{i \in S} D(i)$. Likewise, we define $D^*(i) := \Pr_{x \sim D^*}(x = i)$ for a distribution $D^*$, and so on. We also let $U$ denote the uniform distribution over $[N]$.

Recall that the Total Variation Distance $d_{TV}$ between two distributions $D_1$ and $D_2$ is defined as

$$d_{TV}(D_1, D_2) := \frac{1}{2} \|D_1 - D_2\|_1 = \frac{1}{2} \sum_{i=1}^{N} |D_1(i) - D_2(i)|.$$

Regarding definitions, we finally recall that in the COND model, we are allowed queries $\text{COND}(S)$, which draws from $x \sim D$ conditional on $x \in S$, and in the PAIRCOND model, we are allowed samples $\text{SAMP}$, which draws from $x \sim D$ and queries $\text{PCOND}(x, y) = \text{COND}(\{x, y\})$, where $x, y$ are distinct elements in $[N]$. If dealing with a distribution $Q \neq D$, we will write $\text{COND}(S)_Q$ to denote sampling from the distribution $Q$, conditioned on being in $S$.

We note a simple result about total variation distance, which is straightforward to verify.

**Proposition 2.1.** [CR14, Can17] For distributions $D, D^*$ on $[N]$, we have that

$$d_{TV}(D, D^*) = \sum_{i=1}^{N} \max(0, D(i) - D^*(i)) = \sum_{i=1}^{N} \max(0, D^*(i) - D(i)).$$

We also have that

$$1 - d_{TV}(D, D^*) = \sum_{i=1}^{N} \min(D(i), D^*(i)).$$

We next note the following proposition, proven as a part of [CRS15, Theorem 14].

**Proposition 2.2.** [CRS15] Suppose that $\Pr_{x \sim D} \left[D(x) \geq \frac{1}{\kappa N}\right] \geq 1 - \kappa$. Then, $d_{TV}(D, U) \geq 1 - 2\kappa$.

We will also be using the Chernoff bound numerous times. We state it formally here.

**Theorem 2.1.** [DP02] Let $X_1, X_2, \ldots, X_n$ be independent random variables bounded between 0 and some value $A$. Let $X = \sum_{i=1}^{n} X_i$ and let $\mu = \mathbb{E}[X]$. Then, for any $\delta \leq 1$,

$$\Pr(X \not\in [(1-\delta)\mu, (1+\delta)\mu]) \leq 2 \exp\left(-\frac{\delta^2 \mu}{3A}\right).$$
Finally, we note a simple primitive algorithm that will be very useful for us. The algorithm explains how to get a good approximation to the ratio of $D(x)$ and $D(y)$ using $\text{PCOND}$.

**Proposition 2.3.** For any $\gamma < 1$ and elements $x, y \in [N]$, there is an algorithm $\text{PAIRCOMP}(x, y, \gamma)$ that uses $O(\gamma^{-2} \log \varepsilon^{-1})$ calls to $\text{PCOND}$ and returns $\alpha$ such that

1. if $\frac{D(x)}{D(y)} \leq 20$, then with probability at least $1 - \varepsilon^{10}$, $\left| \alpha - \frac{D(x)}{D(y)} \right| \leq \gamma$.
2. if $\frac{D(x)}{D(y)} \geq \frac{1}{20}$, then with probability at least $1 - \varepsilon^{10}$, $\left| \frac{1}{\alpha} - \frac{D(y)}{D(x)} \right| \leq \gamma$.

**Proof.** The algorithm simply runs $k = O(\gamma^{-2} \log \varepsilon^{-1})$ samples of $\text{PCOND}(x, y)$ and computes $\alpha$ as the ratio of the number of times $x$ is returned to the number of times $y$ is returned. The analysis of parts 1 and 2 are standard applications of the Chernoff bound. 

3  Proof Overview

In this section, we provide the general proof outlines for Theorems 1.1, 1.2, and 1.3.

3.1  Overview of Theorem 1.1

Canonne et. al. [CRSL5] provided an $\tilde{O}(\varepsilon^{-20})$-query algorithm for estimating distance to uniformity. Their algorithm can be broken into two steps. The first step is to find a pair $(x, \hat{D}(x))$ for $x \in [N]$ such that $\hat{D}(x) = (1 \pm \varepsilon) \cdot D(x)$ and $D(x) \in \left[ \frac{\varepsilon}{\sqrt{N}}, \frac{\varepsilon^{-1}}{\sqrt{N}} \right]$ - they use $\tilde{O}(\varepsilon^{-20})$ queries to achieve this. The second step is to use $x$ and the estimate for $D(x)$ to estimate $D(y)$ for randomly sampled $y$, using $\text{PCOND}$. However, they need to sample up to $O(\varepsilon^{-2})$ elements $y_1, \ldots, y_{O(1/\varepsilon^2)}$ from the distribution, and for each sample $y_i$, they obtain a $1 \pm \varepsilon$ multiplicative estimate for the ratio $\frac{D(x)}{D(y_i)}$, so that they can estimate $D(y_i)$ properly. By ignoring $y$ such that $\frac{D(x)}{D(y)}$ is too small or too large, they show that only $O(\varepsilon^{-6})$ queries are needed for the second step in the worst case.

We therefore need improved algorithms for both steps. To do this, we first deal with the case that all elements $i \in [N]$ are promised to satisfy $D(i) \in \left[ \frac{1}{4N}, \frac{2}{3N} \right]$. For the first step, given some $x$, we first create an unbiased estimator for $\frac{D(y)}{D(x)}$ using an average of $O(1)$ queries to $\text{PCOND}$. The idea is a “Geometric Distribution” trick. Normally, if we call $\text{PCOND}(x, y)$, we get $y$ with probability $\frac{D(y)}{D(x)}$. But by calling the variable until we see $x$, the number of times we see $y$ is distributed as $\text{Geom} \left( \frac{D(y)}{D(x)} \right)$, which has mean $\frac{D(y)}{D(x)}$. If $y$ were sampled uniformly, then this is actually an unbiased estimator for $\frac{1}{N \cdot D(x)}$, since the expected value of $D(y)$ is $\frac{1}{N}$ if $y$ is uniformly sampled. Thus, by sampling $\tilde{O}(\varepsilon^{-2})$ random $y$ and using an expected $O(1)$ samples for each to get an unbiased estimator of $\frac{D(y)}{D(x)}$, we can estimate $D(x)$ up to a $1 \pm \varepsilon$ multiplicative factor.

For the second step, we want to estimate $\sum |D(i) - \frac{1}{N}|$. The natural intuition is to pick $\varepsilon^{-2}$ samples $i$ and estimate $D(i)$ for each $i$, but each $i$ would need $\varepsilon^{-2}$ samples. To improve this from $\varepsilon^{-4}$ to $\tilde{O}(\varepsilon^{-2})$, we again try another geometric trick to give an unbiased estimator of $\frac{D(i)}{D(x)}$, where $D(x)$ is already approximately known. However, we need some way of distinguishing between $i$ with $D(i) > \frac{1}{N}$ and $i$ with $D(i) < \frac{1}{N}$. The rough idea to fix this is to say that if $D(i) \approx \frac{1}{N} \cdot (1 \pm \delta)$, then we can estimate $D(i)$ up to a $\frac{\delta}{2}$ error using $O(\delta^{-2})$ queries to $\text{PCOND}(x, i)$. For each $\delta$, we
estimate the fraction of $i$ with $D(i) \in \left[ \frac{1+j}{N}, \frac{1+j+\delta}{N} \right]$ (or $D(i) \in \left[ \frac{1-\delta}{N}, \frac{1-\delta}{N} \right]$), and also try to estimate the expected value of $N \cdot D(i)$ up to an $\varepsilon$ error conditioned on this range. While we need $\delta^{-2}$ samples to estimate each $D(i)$, we only need to sample $\tilde{O}(\varepsilon/\delta)^{-2}$ such $i$ rather than $\varepsilon^{-2}$ such $i$, since the range of $N \cdot D(i)$ has width $\delta$. Therefore, we still use $\tilde{O}(\varepsilon^{-2})$ samples. We look at $\delta = \pm 2^{-j}$ for $j = 1, 2, \ldots, \log \varepsilon^{-1}$, and in total we will only use $\tilde{O}(\varepsilon^{-2})$ samples.

One issue, however, is that we are not guaranteed that $D(i) \in \left[ \frac{1}{2N}, \frac{2}{3} \right]$ for all $i$. To fix this, we create an oracle which essentially only accepts elements $i$ with $D(i)$ close to $\frac{1}{N}$. To do this, the rough idea is to first find an element $x$ with $D(x) \in \left[ \frac{1}{2N}, \frac{2}{3} \right]$. We do this by sampling $R = \tilde{O}(\varepsilon^{-1})$ points $x_1, \ldots, x_R$, and for each $x_i$, we determine whether a random element $y$ drawn from the uniform distribution or a random element $z$ drawn from $D$ is more likely to be close in probability to $x_i$. For $D(x_i) \approx \frac{1}{N}$, the probability that $D(y) \approx D(x_i)$ and the probability that $D(z) \approx D(x_i)$ should be approximately the same, so for each $x_i$, we sample $y_1, \ldots, y_R \leftarrow U$ and $z_1, \ldots, z_R \leftarrow D$ to compare $x_i$ to. The total number of comparisons will thus be $\tilde{O}(R^2) = \tilde{O}(\varepsilon^{-2})$. Once we have found such an $x$, the oracle simply accepts $i$ iff $D(i)$ is within a constant factor of $D(x)$, which can be verified by $P_{\text{COND}}$. While there may be borderline elements which are accepted with some probability, they will not end up being particularly difficult to deal with. Now, we already have an algorithm to deal with elements that are accepted. However, we also need to determine $\sum |D(i) - \frac{1}{N}|$ where the sum is over $i$ rejected by the oracle. To deal with these elements, we split these $i$’s into $i$ with $D(i)$ too small and $i$ with $D(i)$ too large. First, we want $\sum \left( \frac{1}{N} - D(i) \right)$ over the values $i$ with $D(i)$ too small, which equals the probability of a uniformly chosen element $i$ having $D(i)$ too small minus the probability of an element $i \leftarrow D$ having $D(i)$ too small. We determine the probability of a uniformly random element $i$ having $D(i)$ too small by sampling $O(\varepsilon^{-2})$ random $i$ from the uniform distribution, and determining what fraction of sampled $i$’s have $D(i)$ too small by using the oracle on each sampled $i$ (the oracle will reject such $i$’s, and also say that $D(i)$ is too small). We determine the probability that $D(i)$ too small for $i \leftarrow D$ with the same process, except we sample each $i$ from $D$ instead of the uniform distribution. For $i$ with $D(i)$ too large, we make a symmetric argument.

### 3.2 Overview of Theorem 1.2

For identity testing in the $\text{PAIRCOND}$ model, there must be some dependence on $N$. The rough reason for why is that if we are trying to determine if $D = D^*$, some values $D^*(i)$ can be much bigger than other values $D^*(j)$, so there are groups of elements which can never be compared to each other. Thus, one of the crucial ideas for identity testing in the COND model [CRS15, FJO+15] is to, rather than compare $D(i)$ with $D(j)$ when $D^*(i)$ and $D^*(j)$ may be vastly different, compare $D(i)$ with $D(S)$ for some subset $S$ with $D^*(S) \approx D^*(i)$, which is doable in the COND model.

As done in [CRS15, FJO+15], we assume WLOG that $D^*(1) \leq D^*(2) \leq \cdots \leq D^*(N)$. Since $D^*$ is known, we can permute the elements accordingly. Our goal will be to determine

$$d_{\text{TV}}(D, D^*) = 1 - \sum_{i=1}^{N} \min(D(i), D^*(i)) = 1 - \mathbb{E}_{i \sim D^*} \left[ \min \left( 1, \frac{D(i)}{D^*(i)} \right) \right].$$

Now, let’s sample some $i \leftarrow D^*$. We have that since $D^*(1), \ldots, D^*(i-1) \leq D^*(i)$, we can partition $[i-1]$ into sets $S_1, \ldots, S_{k-1}$ so that $0.5 \cdot D^*(i) \leq D^*(S_i) \leq D^*(i)$, unless $D^*(i-1) \leq 0.5 \cdot D^*(i)$. Ignoring the latter case, assuming $D([i]) \approx D^*([i])$, we can determine $\frac{D(i)}{D^*(i)}$ by looking at the set $\{S_1, \ldots, S_{k-1}, S_k\}$ (where $S_k = \{i\}$) and determining both $D([i])$ and $\frac{D(i)}{D^*(i)}$. To compute $\frac{D(i)}{D^*(i)}$, we
will show how to compute \( \frac{D(S)}{D(i)} \) and \( \frac{D(i)}{D(S)} \) for some \( j \leq k \) that we will choose, using ideas somewhat similar to our ideas for tolerant uniformity testing. However, a problem with our methods may arise if \( \frac{D(i)}{D(S)} \) is close to 1, but \( \frac{D(i)}{D(D(i))} \) is much smaller than 1 and \( \frac{D(i)}{D(S)} \) is much larger than \( \frac{1}{2} \). In this case, we may not be able to determine \( \frac{D(i)}{D(S)} \), even with a poly(\( \varepsilon^{-1} \)) number of queries. However, if \( \frac{D(i)}{D(S)} \) is very small, this means that \( \sum_{j=1}^{i} \min(D^*(j), D(j)) \leq \sum_{j=1}^{i} D(j) = D([i]) \) will be so small that we can approximate it as 0. As a result, we can remove the subset \([i]\) and look at \( \sum_{j=i+1}^{N} \min(D^*(j), D(j)) \). In most situations, however, we will be able to approximate \( \frac{D(i)}{D(S)} \), and therefore by sampling many \( i \leftarrow D^* \), we can get a good estimate for \( \mathbb{E}_{i \leftarrow D^*} \min \left( 1, \frac{D(i)}{D(S)} \right) \).

One additional problem is that when we remove the subset \([i]\), we may be removing only a small fraction of the mass under \( D^* \), so if we keep doing this, the query complexity could get very large. As a result, we sample \( i \) from \( D^* \) conditioned on \( i \) being in the “right” half of the distribution under \( D^* \), i.e., we sample \( i \) conditioned on \( i \geq L \) for some \( L \) with \( D^*([L]) \approx \frac{1}{2} \). We will be able to estimate \( \mathbb{E}_{i \leftarrow D^*, |i| \geq L} \min(\frac{D(i)}{D(S)}, 1) = \frac{1}{D^*([L:N])} \sum_{i=L}^{N} \min(D(i), D^*(i)) \), unless \( \sum_{j=1}^{i} \min(D(j), D^*(j)) \leq O(\varepsilon) \). Therefore, we can remove either \([L:N]\) or \([z]\) for some \( z \geq L \), so we remove at least half of the weight under \( D^* \). We may need to solve this recursively, but since half of the mass is removed at each step, only \( \log \varepsilon^{-1} \) steps of the recursion are necessary.

In the case where we have removed some of the elements, and for instance just have to approximate \( \sum_{i \in S} \min(D(i), D^*(i)) \) for some \( S \subset [N] \), we condition on the data coming from \( S \), but this may cause the values \( D^*(i) \) to scale differently from \( D(i) \). As a result, we will actually attempt to solve a slightly more general question of estimating \( \mathbb{E}_{i \leftarrow D^*} \min(c_1 P(i), c_2 P^*(i)) \), for some constants \( c \leq c_2 \leq 1 \), where \( P, P^* \) are the conditional distributions of \( D, D^* \) conditioned on \( i \in S \). This will not end up being much more difficult than the original problem of estimating \( \sum \min(D(i), D^*(i)) \).

### 3.3 Overview of Theorem 1.3

A crucial observation motivating this proof, also noted in [FJO+15, Theorem 6], is that if \( D \neq D^* \), then if we select \( i \) from \( D \) and \( j \) from \( U \), we should expect \( \frac{D(i)}{D(j)} \) to be larger than \( \frac{D^*(i)}{D^*(j)} \). Intuitively, this is true since drawing \( i \leftarrow D \) is biased in favor of elements with high values of \( D(i) \). We formalize this by proving that if all expected probabilities \( D^*(i) \) are in the range \([\frac{1}{N}, \frac{2}{N}] \), and if we draw \( i \leftarrow D \) and \( j \leftarrow U \), then \( \mathbb{E} \left| \frac{D(i)}{D(i)+D(j)} - \frac{D^*(i)}{D^*(i)+D^*(j)} \right| \geq \Omega(d_{TV}(D, D^*)) \). It is well known that this implies if \( d_{TV}(D, D^*) \geq \varepsilon \), we can find some \( \alpha, \beta \) with \( \alpha \beta \geq \hat{\Omega}(\varepsilon) \) and \( \mathbb{P} \left( \left| \frac{D(i)}{D(i)+D(j)} - \frac{D^*(i)}{D^*(i)+D^*(j)} \right| \geq \alpha \right) \geq \beta \).

But we can determine \( \left| \frac{D(i)}{D(i)+D(j)} - \frac{D^*(i)}{D^*(i)+D^*(j)} \right| \) up to an \( \frac{\varepsilon}{\bar{D}} \) error using \( \hat{O}(\alpha^{-2}) \) queries, so we can then sample \( \hat{O}(\beta^{-1}) \) samples \( i \leftarrow D, j \leftarrow U \) and determine \( \left| \frac{D(i)}{D(i)+D(j)} - \frac{D^*(i)}{D^*(i)+D^*(j)} \right| \) for each of them, with a total of \( O(\alpha^{-2} \beta^{-1}) = \hat{O}(\varepsilon^{-2}) \) queries in PAIRCOND.

However, to generalize to arbitrary distributions \( D \) and \( D^* \) using only the power of PAIRCOND, we need a different approach from [FJO+15]. Notably, [FJO+15] was able to utilize the COND model by comparing \( D(i) \) with \( D(S) \) for some set \( S \) with \( D^*(S) \approx D^*(i) \), as noted in the overview of Theorem 1.2 but in the PAIRCOND model, we cannot do this. Instead, we partition \([N]\) into sets \( S_1, S_2, \ldots, S_{O(\log n)} \) where elements \( i \) in \( S_k \) have \( D^*(i) \approx 2^{-k} \). We similarly can show that if we draw \( i \) from \( D \), and then draw \( j \) uniformly from the set \( S_k \) containing \( i \), then either
\[ \mathbb{E} \left| \frac{D(i)}{D(i) + D(j)} - \frac{D^*(i)}{D^*(i) + D^*(j)} \right| = \Omega(\varepsilon), \]

or the distribution \( S \) over \( [O(\log n)] \) where we draw \( k \leftarrow S \) with probability \( D(S_k) \) differs from the distribution \( S^* \) where we draw \( k \leftarrow S \) with probability \( D^*(S_k) \) by Total Variation Distance \( \Omega(\varepsilon) \). We verify the first possibility using the argument in the previous paragraph, with \( \tilde{O}(\varepsilon^{-2}) \) queries. We verify the second possibility using sampling from \( S \), which can be simulated by sampling from \( D \), which is doable with \( O\left(\frac{\log n}{\varepsilon^2}\right) \) queries by \[ \text{VW17}, \] since the support size of \( S \) is \( O(\log n) \).

### 4 An \( \tilde{O}(\varepsilon^{-2}) \)-query algorithm for Tolerant Uniformity Testing

In this section, we present an algorithm that makes \( \tilde{O}(\varepsilon^{-2}) \) queries to \textsc{Samp} and \textsc{Pcond} and determines \( d_{TV}(D, \mathcal{U}) \) up to an \( O(\varepsilon) \) additive error. This algorithm is known to be optimal even in the stronger \textsc{Cond} model and even for standard uniformity testing.

The algorithm can be broadly divided into three steps. The first step of the proof, done in subsection 4.1, determines the distance from uniformity when all probabilities are known to be close to \( \frac{1}{N} \). To get this to work for general distributions, we will assume an oracle which discards elements with probabilities too far away from \( \frac{1}{N} \). In subsection 4.2, we show how to generate the oracle, which roughly works by first finding a single element \( x \) and an approximation \( \tilde{D}(x) \approx D(x) \), which will not be a \( 1 + \varepsilon \) approximation but will be an \( O(1) \) approximation. We then combine the two steps together and finish the proof in subsection 4.3.

#### 4.1 An Algorithm with Access to an Oracle

In this subsection, our goal is to determine \( \sum_{i=1}^{N} |D(i) - \frac{1}{N}| \) but only among the \( D(i) \)'s which are within a constant factor of \( \frac{1}{N} \). To make this usable for the general algorithm, we assume we have an oracle that roughly accepts no elements that are not close to \( \frac{1}{N} \) but accepts elements that are very close to \( \frac{1}{N} \) with some probability. In the case where \( D(i) \) is within a constant factor of \( \frac{1}{N} \) for all \( i \), this subsection immediately implies an algorithm for estimating Total Variation Distance from uniform.

**Lemma 4.1.** Let \( s : [N] \rightarrow [0, 1] \) be an (unknown) function so that \( \sum_{i=1}^{N} s(i) \cdot D(i) = \gamma_2 \), and \( s(i) = 0 \) whenever \( D(i) \not\in \left[\frac{1}{2N}, \frac{5}{N}\right] \), where \( \gamma_1, \gamma_2 \) are also unknown. Then, given an oracle \( O \) which accepts an element \( i \) with probability \( s(i) \), as well as an element \( x \) with probability between \( \frac{1}{2N} \) and \( \frac{5}{N} \), there is an algorithm \textsc{SingleElement} that can determine \( D(x) \) up to a \( 1 \pm c\varepsilon \) multiplicative factor with probability at least 0.98 for some small constant \( c \), using an expected \( O(\varepsilon^{-2}) \) queries to \textsc{Pcond} and \( O \).

**Proof.** First, let \( K = O(\varepsilon^{-2}) \) and sample elements \( y_1, \ldots, y_K \leftarrow \mathcal{U} \) and \( z_1, \ldots, z_K \leftarrow \mathcal{D} \). Note that for \( y \leftarrow \mathcal{U} \), the probability of \( O(y) \) accepting is \( \sum_{i=1}^{N} \frac{s(i)}{N} = \gamma_1 \), and for \( z \leftarrow \mathcal{D} \), the probability of \( O(z) \) accepting is \( \sum_{i=1}^{N} s(i) \cdot D(i) = \gamma_2 \). Thus, by checking if the oracle \( O \) accepts the \( y_i \)'s and \( z_i \)'s for each \( 1 \leq i \leq K \), we can determine \( \hat{\gamma}_1 \approx \gamma_1 \) and \( \hat{\gamma}_2 \approx \gamma_2 \), which are accurate up to a \( c_1 \cdot \varepsilon \) additive error with probability at least 0.99 for some small constant \( c_1 \). Also, recall that \( \gamma_1 \geq 10\varepsilon \), and as \( s(i) \) is nonzero only when \( D(i) \in \left[\frac{1}{2N}, \frac{5}{N}\right] \), we have that \( \gamma_2 \in \left[\frac{\varepsilon}{\gamma_1}, 5\gamma_1\right] \), so \( \gamma_2 \geq 2\varepsilon \). Therefore, \( \tilde{\gamma}_1 \approx \gamma_1 \) and \( \tilde{\gamma}_2 \approx \gamma_2 \) up to a multiplicative factor of 2 as well.

Now, we will attempt to determine \( D(x) \). To do so, for each \( y_i \) accepted by \( O \), we run \textsc{Pcond}(\( x, y_i \)), which outputs \( y_i \) with probability \( \frac{D(y_i)}{D(y_i) + D(x)} \). This probability will be in the range \([1/11, 10/11]\), as
$D(y_i) \in \left[\frac{1}{10N}, \frac{5}{N}\right]$ and $D(x) \in \left[\frac{1}{10N}, \frac{2}{N}\right]$. Keep running $P_{\text{cond}}(x, y_i)$ until it outputs $x$ and consider the number of times $y_i$ is returned. This is some random variable $Y$, which, conditioned on $y_i$, is a Geometric Random variable with mean $\frac{\sum_{j=1}^{\infty} \left(\frac{D(y_i)}{D(y_i) + D(x)}\right)^j}{D(y_i) / D(x)}$ and variance $O(1)$. Moreover, in expectation we only call $P_{\text{cond}}(O(1))$ times for each $i$.

Next, let $A$ be the indicator random variable of $O$ accepting $y$. In this case, we output the random variable $Y$ computed above, and we know that $\mathbb{E}[Y|y, A = 1] = \frac{D(y)}{D(x)}$ and $\text{Var}(Y|y, A = 1) = O(1)$ for all $y$. Otherwise, if $A = 0$, we output $Y = 0$. Therefore, $Y$ has mean $\frac{1}{D(x)} \cdot \sum_{y=1}^{N} \frac{1}{N}s(y) \cdot D(y) = \frac{\gamma_3}{N \cdot D(x)}$ (since $y$ is chosen from $U$) and variance

$$\text{Var}(Y) = \mathbb{E}[\text{Var}(Y|y, A)] + \mathbb{E}[\text{Var}(Y|y, A)] = O(1).$$

The first equality in the above equation follows from the Law of Total Variance. The second inequality is true since if $A = 0$, then $\mathbb{E}[Y|y, A = 0] = \text{Var}(Y|y, A = 0) = 0$, and if $A = 1$, then as $\frac{D(y)}{D(y) + D(x)} \in \left[\frac{1}{11}, \frac{10}{11}\right]$ for all $y$ accepted by $O$, $\mathbb{E}[Y|y, A = 1], \text{Var}(Y|y, A = 1) = O(1)$. By averaging over $K$ samples $Y_1, \ldots, Y_K$, where each $Y_i$ depends on $y_i \sim D$, we get a random variable with mean $\frac{\gamma_3}{N \cdot D(x)}$ and standard deviation $O(\varepsilon)$, so we get some number $\gamma_3$ which equals $\frac{\gamma_2}{\gamma_2} \pm c_2 \varepsilon$ with probability at least 0.98 for some small constant $c_2$.

Noting that $N \cdot D(x) \in [1/2, 2]$ and $\gamma_2 \geq 2\varepsilon$, we have that with probability at least 0.98,

$$\frac{\gamma_2}{\gamma_3} = \frac{\gamma_2 \pm c_1 \varepsilon}{N \cdot D(x)} \pm c_2 \varepsilon = N \cdot D(x) \cdot \left(1 \pm \frac{c}{5} \cdot \frac{\varepsilon}{\gamma_2}\right).$$

Finally, $\gamma_2 \geq \frac{1}{5} \cdot \gamma_1$, so we thus get a $1 \pm c_2 \varepsilon$ multiplicative approximation to $D(x)$. \hfill $\square$

**Lemma 4.2.** Let $z$ be in $[N]$ such that we know $D(z) \in \left[\frac{1}{5N}, \frac{5}{N}\right]$. Suppose we also know some element $x$ with $D(x) \in \left[\frac{1}{10N}, \frac{2}{N}\right]$, along with an estimate $\tilde{D}(x)$ such that $\frac{\tilde{D}(x)}{D(x)} \in [1 - \beta, 1 + \beta]$ for some known $\varepsilon \leq \beta \leq \frac{1}{1000}$. Then, there is an algorithm $Z\text{ESTIMATE}$ that gives an estimate $\tilde{D}(z)$ of $D(z)$ with the following properties:

1. If $|N \cdot D(z) - 1| \geq 160 \cdot \beta$, then with probability at least $1 - \varepsilon^9$, $\frac{N \cdot \tilde{D}(z) - 1}{N \cdot D(z) - 1} \in \left[\frac{1}{3}, 2\right]$.
2. If $|N \cdot D(z) - 1| \leq 160 \cdot \beta$, then with probability at least $1 - \varepsilon^9$, $|N \cdot \tilde{D}(z) - 1| \leq 320 \cdot \beta$.
3. The algorithm uses $O\left(\min\left((N \cdot D(z) - 1)^{-2}, \beta^{-2}\right) \cdot \log \varepsilon^{-1}\right)$ queries to $P_{\text{cond}}$.

**Remark.** By setting $c = \frac{1}{100}$ in Lemma 4.1 we have that $\beta \leq \frac{1}{1000}$ is an acceptable assumption.

**Proof.** Fix some integer $i \geq 1$ with $2^{-i} \geq 40 \cdot \beta$, and suppose we run $P_{\text{PAIR}}(z, x, 2^{-i}/20)$ to get $\alpha$, which is a $2^{-i}/20$ additive approximation to $\frac{D(z)}{D(x)}$. Moreover, we know $\tilde{D}(x)$ and we know that $\frac{\tilde{D}(x)}{D(x)} \in [1 - \beta, 1 + \beta]$, so

$$\alpha \cdot \tilde{D}(x) = \left(\frac{D(z)}{D(x)} \pm \frac{2^{-i}}{20}\right) \cdot D(x) \cdot \frac{\tilde{D}(x)}{D(x)} = \left(\frac{D(z)}{20} \cdot D(x) \right) \cdot \left(1 \pm \beta\right).$$

Now, noting that $\frac{2^{-i}}{20} \cdot D(x) \leq 2^{-i} \cdot \frac{1}{10N}$ and that $\beta \leq 0.025 \cdot 2^{-i}$, we have that $|\alpha \cdot \tilde{D}(x) - D(z)| \leq 2^{-i} \cdot \frac{1.025}{10N} + 0.025 \cdot 2^{-i} \cdot \frac{3}{N} \leq 2^{-i} \cdot \frac{1.025}{N} + 0.025 \cdot \frac{5}{N} \leq 2^{-i} \cdot \frac{11}{N}$. For $D(z) < \frac{2}{N}$, we can improve this to $2^{-i} \cdot \frac{1.025}{10N} + 0.025 \cdot \frac{2}{N} \leq 2^{-i} \cdot \frac{11}{N}$. 

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Therefore, our algorithm will work as follows. First, we set \( i = 1 \). For each \( i \), we run \( \text{PAIRCOMP}(z, x, 2^{-i}/20) \) and multiply it by \( \tilde{D}(x) \) to get an approximation to \( D(z) \). If our approximation is not in the range 
\[
\left[ \frac{1}{N} \cdot (1 - 2^{-i}), \frac{1}{N} \cdot (1 + 2^{-i}) \right]
\]
we output the approximation as our estimate \( \tilde{D}(z) \). Otherwise, we increment \( i \), until we reach \( 2^{-i} < 40 \beta \). In this case, we simply output \( \tilde{D}(z) = \frac{1}{N} \).

Note that for \( D(z) > \frac{1}{N} \), with probability at least \( 1 - \epsilon^{10} \), we will just output \( \tilde{D}(z) \) on the iteration \( i = 1 \) and that \( |\tilde{D}(z) - D(z)| \leq \frac{1}{2N}, \) which means that in fact \( \frac{N \cdot \tilde{D}(z) - 1}{N \cdot D(z) - 1} \in \left[ \frac{1}{16}, \frac{5}{16} \right] \). The total number of calls to \( \text{Pcond} \) in this case is just \( O(\log \epsilon^{-1}) \). Otherwise, if \( 80 \beta \leq 2^{-(j+1)} \leq |N \cdot D(z) - 1| \leq 2^{-j} \), with probability at least \( 1 - \epsilon^{9} \), for all \( i < j - 1 \) we will not output an estimate \( \tilde{D}(z) \) and we will output an estimate \( \tilde{D}(z) \) at least by the time of reaching \( i = j + 2 \). Note that \( 40 \beta \leq 2^{-j+2} \). The estimate will be off by at most \( \frac{2^{-(j+1)}}{6N} \) but \( |N \cdot D(z) - 1| \geq 2^{-(j+1)} = \frac{2^{-(j-1)}}{4} \), so \( \frac{N \cdot \tilde{D}(z) - 1}{N \cdot D(z) - 1} \in \left[ \frac{1}{16}, \frac{5}{16} \right] \). The number of queries in this case is \( O(2^{2j} \cdot \log \epsilon^{-1}) \). Otherwise, we must have that \( |N \cdot D(z) - 1| \leq 160 \beta \), so we will either never output an estimate until the end (so \( \tilde{D}(z) = \frac{1}{N} \)), or we output \( \tilde{D}(z) \) at some step \( i \), so \( 2^{-(i+1)} < |N \cdot D(z) - 1| \). In this case, \( |\tilde{D}(z) - D(z)| \leq \frac{2}{6N} \leq \frac{|N \cdot D(z) - 1|}{3N} \), so \( |N \cdot \tilde{D}(z) - 1| \leq 2 \cdot |N \cdot D(z) - 1| \leq 320 \beta \). The number of queries in this case is \( O(\beta^{-2} \log \epsilon^{-1}) \).

**Lemma 4.3.** Let \( x, s, O, \gamma_1, \gamma_2 \) be as in Lemma 4.1. Then, using an expected \( O(\epsilon^{-2} \log^{2} \epsilon^{-1}) \) queries to \( \text{Pcond} \) and \( O \), there is an algorithm \( \text{ESTIMATECLOSETERMS} \) that can determine \( \sum_{i=1}^{N} s(i) \cdot |D(i) - \frac{1}{N}| \) up to an additive \( O(\epsilon) \) error with probability at least 0.97.

**Proof.** Let \( \tilde{D}(z) \) be our estimate of \( D(z) \) based on Lemma 4.1 where we know that \( D(x) \in \left[ \frac{1}{2N}, \frac{2}{N} \right] \) and \( \frac{D(z)}{D(x)} = 1 + O(\epsilon/\gamma_1) \) (with probability at least 0.98). For any \( z \sim \mathcal{U} \) with \( D(z) \in \left[ \frac{1}{5N}, \frac{5}{N} \right] \), let \( \tilde{D}(z) \) be our guess for \( D(z) \) based on Algorithm 4.2 with \( \beta = \frac{1}{100 \gamma_1} \). Note that \( \tilde{D}(z) \) is a random variable even if we fix \( z \). However, for any \( z \) with \( |N \cdot D(z) - 1| > \frac{\epsilon}{\gamma_1} \), with \( 1 - \epsilon^{9} \) probability, \n\[
\frac{N \cdot \tilde{D}(z) - 1}{N \cdot D(z) - 1} \in \left[ \frac{3}{16}, \frac{7}{16} \right].
\]

Now, we choose \( T \) so that \( 2^{-T} = \Theta(\epsilon/\gamma_1) \). While we don’t know \( \gamma_1 \), we know \( \tilde{\gamma}_1 = \Theta(\gamma_1) \), so we can choose \( T \). For each \( 0 \leq t \leq T - 1 \), let \( \mathbb{I}_{+1}(z) \) be the indicator event that \( O \) accepts \( z \) and \( 2^{-(t+1)} < N \cdot \tilde{D}(z) - 1 \leq 2^{-t} \). (For \( t = 1 \), we also let \( \mathbb{I}_{+1}(z) \) indicate when \( \tilde{D}(z) > \frac{2}{N} \), i.e., \( N \cdot \tilde{D}(z) - 1 > 1 \) also means \( \mathbb{I}_{+1}(z) = 1 \).) Likewise, let \( \mathbb{I}_{-1}(z) \) be the indicator event that \( O \) accepts \( z \) and \( 2^{-(t+1)} < 1 - N \cdot \tilde{D}(z) < 2^{-t} \) (we also let \( \mathbb{I}_{-1}(z) \) indicate when \( \tilde{D}(z) < \frac{1}{2N} \), and let \( \mathbb{I}_{T}(z) \) be the indicator event that \( O \) accepts \( z \) and \( |N \cdot \tilde{D}(z) - 1| \leq 2^{-T} \). Next, let \( q_{+1}(z) = \mathbb{P}(\mathbb{I}_{+1}(z)), q_{-1}(z) = \mathbb{P}(\mathbb{I}_{-1}(z)), \) and \( q_{T}(z) = \mathbb{P}(\mathbb{I}_{T}(z)) \). Also, let \( q_{t+1} = \mathbb{P}(\mathbb{I}_{+1+t}(z)), q_{t-1} = \mathbb{P}(\mathbb{I}_{-1+t}(z)), \) and \( q_{T} = \mathbb{P}(\mathbb{I}_{T+t}(z)) \), where the probability is now also over \( z \leftarrow \mathcal{U} \). Finally, let \( g_{+1} = \mathbb{E}[\mathbb{I}_{+1+t}(z) \cdot (N - D(z) - 1)], g_{-1} = \mathbb{E}[\mathbb{I}_{-1+t}(z) \cdot (N - D(z) - 1)], \) and \( g_{T} = \mathbb{E}[\mathbb{I}_{T+t}(z) \cdot (N - D(z) - 1)] \), where the expectations again are also over \( z \leftarrow \mathcal{U} \).

Now, fix some \( 0 \leq t \leq T - 1 \) and consider \( \delta = 2^{-(t+1)} \). For a sample \( z \leftarrow \mathcal{U} \), we create a random variable \( Z_{+t} \) as follows. First, we determine if \( O \) accepts \( z \) and, if so, we use the algorithm of Lemma 4.2 with \( \beta = O(\frac{1}{\gamma_1}) \) (recall that we know \( \tilde{\gamma}_1 = \Theta(\gamma_1) \)) to sample the indicator variable \( \mathbb{I}_{+1+t}(z) \) and determine if \( \mathbb{I}_{+1+t}(z) = 1 \). If either \( O \) rejects \( z \) or \( \mathbb{I}_{+1+t}(z) = 0 \), we set \( Z_{+t} = 0 \). Determining \( \mathbb{I}_{+1+t}(z) \) normally needs \( O((\epsilon/\gamma_1)^{-2} \cdot \log \epsilon^{-1}) \) queries to \( \text{Pcond} \), but can be improved to \( O(\delta^{-2} \log \epsilon^{-1}) \) queries by stopping the algorithm at \( t' = t + O(1) \). Else, if \( O \) accepts \( z \) and \( \mathbb{I}_{+1+t}(z) = 1 \), we saw in Lemma 4.1 that using an expected \( O(1) \) queries to \( \text{Pcond} \), we could create a random variable \( Y \) with expectation \( \frac{D(z)}{D(x)} \) and variance \( O(1) \) (since \( \frac{D(z)}{D(x)} = \Theta(1) \)). Thus, by averaging \( O \left( \frac{1}{\sqrt{T}} \right) \) copies of
the random variable, we get a random variable $\tilde{Y}$, which when conditioned on $z$, has expectation $D(z)$ and variance $O(\delta^2)$. We will finally return $Z_{+,t} = N \cdot \tilde{D}(x) \cdot \tilde{Y} - 1$ in this case. In total, we use an expected $O(\delta^{-2} \log \epsilon^{-1})$ queries to $P_{\text{COND}}$ and $O$ to generate $Z_{+,t}$.

Conditioned on $z$, we have that
\[
E[Z_{+,t}|z] = \mathbb{P}(I_{+,t}(z) = 1) \cdot E[Z_{+,t}|z, I_{+,t}(z) = 1]
\]
\[
\quad = q_{+,t}(z) \cdot \left( N \cdot \tilde{D}(x) \cdot \frac{D(z)}{D(x)} - 1 \right)
\]
\[
\quad = q_{+,t}(z) \cdot \left( (N \cdot D(z) - 1) \cdot \frac{\tilde{D}(x)}{D(x)} + \frac{\tilde{D}(x) - D(x)}{D(x)} \right).
\]

Taking the expectation over $z$, we have that
\[
E[Z_{+,t}] = \frac{\tilde{D}(x)}{D(x)} \cdot E[q_{+,t}(z) \cdot (N \cdot D(z) - 1)] + \frac{\tilde{D}(x) - D(x)}{D(x) \cdot q_{+,t}} \cdot E[q_{+,t}(z)]
\]
\[
\quad = \frac{\tilde{D}(x)}{D(x)} \cdot g_{+,t} + \frac{\tilde{D}(x) - D(x)}{D(x)} \cdot q_{+,t}
\]
\[
\quad = g_{+,t} + \frac{\tilde{D}(x) - D(x)}{D(x)} \cdot (g_{+,t} + q_{+,t}).
\]

Next, note that since $\text{Var}(\tilde{Y}|z, I_{+,t} = 1) = O(\delta^2)$, and since $N \cdot \tilde{D}(x) = O(1)$, $\text{Var}(Z_{+,t}|z, I_{+,t} = 1) = O(\delta^2)$. We also have $\text{Var}(Z_{+,t}|z, I_{+,t} = 0) = 0$. We also have that $E[Z_{+,t}|z, I_{+,t} = 1] = N \cdot D(z) \cdot \frac{\tilde{D}(x)}{D(x)} - 1 = O(\delta)$ for all $z$ that can allow $I_{+,t} = 1$, since $\delta > \frac{\epsilon}{\gamma_1}$ and $E[Z_{+,t}|z, I_{+,t} = 0] = 0$. Therefore, by the Law of Total Variance,
\[
\text{Var}(Z_{+,t}) = E[\text{Var}(Z_{+,t}|z, I_{+,t} = 1)] + \text{Var}(E[Z_{+,t}|z, I_{+,t} = 0]) = O(\delta^2).
\]

Therefore, by using $O(\delta^{-2} \log \epsilon^{-1})$ queries to $P_{\text{COND}}$ and $O$, we can generate a random variable $Z_{+,t}$ with mean $g_{+,t} + O \left( \frac{\epsilon}{\gamma_1} \cdot (g_{+,t} + q_{+,t}) \right)$ and variance $O(\delta^2)$ for $\delta = 2^{-(t+1)}$. By the same argument, we can also generate a random variable $Z_{-,t}$ with mean $g_{-,t} + O \left( \frac{\epsilon}{\gamma_1} \cdot (g_{-,t} + q_{-,t}) \right)$ and variance $O(\delta^2)$. Finally, we can also generate $Z_T$ with mean $g_T + O \left( \frac{\epsilon}{\gamma_1} \cdot (g_T + q_T) \right)$ and variance $O(\delta^2)$ for $\delta = 2^{-T}$. By generating $O((\delta/\epsilon)^2 \cdot \log \epsilon^{-1})$ repetitions of $Z_{+,t}$ and averaging them, we can get a random variable $W_{+,t}$ with the same mean but variance $O((\epsilon^2/(\log \epsilon^{-1}))$, using $O(\epsilon^{-2} \log^2 \epsilon^{-1})$ queries to $P_{\text{COND}}$ and $O$. The same is true for $W_{-,t}$ and $W_T$.

Our final estimate will be
\[
W := W_T + \sum_{t=1}^{T-1} (W_{+,t} + W_{-,t}),
\]
where $W_T, W_{+,t}, W_{-,t}$ are all determined using independent samples. We have that
\[
E[W] = g_T + O \left( \frac{\epsilon}{\gamma_1} \right) \cdot (g_T + q_T) + \sum_{t=1}^{T-1} \left( g_{+,t} + g_{-,t} + O \left( \frac{\epsilon}{\gamma_1} \right) \cdot (g_{+,t} + g_{-,t} + q_{+,t} + q_{-,t}) \right)
\]
\[
= \left( 1 + O \left( \frac{\epsilon}{\gamma_1} \right) \right) \cdot \left( g_T + \sum_{t=1}^{T-1} (g_{+,t} + g_{-,t}) \right) + O \left( \frac{\epsilon}{\gamma_1} \right) \cdot \left( q_T + \sum_{t=1}^{T-1} (q_{+,t} + q_{-,t}) \right).
\]
Now, note that if \( z \) is accepted by \( O \) and we draw a sample \( \tilde{D}(z) \), then exactly one \( \mathbb{I}_{+,t}, \mathbb{I}_{-,t}, \mathbb{I}_T \) be a 1, and all others be 0. Therefore,

\[
g_T + \sum_{t=1}^{T-1} (g_{+,t} + g_{-,t}) = \mathbb{E}_{z \sim \mathcal{U}} (\mathbb{I}(O \text{ accepts } z) \cdot |N \cdot D(z) - 1|) = \sum_{z=1}^{N} s(z) \cdot \left| D(z) - \frac{1}{N} \right| = O(\gamma_1),
\]

since \( s(z) > 0 \) implies that \( D(z) = O(1/N) \). Also,

\[
g_T + \sum_{t=1}^{T-1} (q_{+,t} + q_{-,t}) = \mathbb{E}_{z \sim \mathcal{U}} (\mathbb{I}(O \text{ accepts } z)) = \mathbb{P}_{z \sim \mathcal{U}}(O \text{ accepts } z) = \sum_{z=1}^{N} s(z) \cdot \frac{1}{N} = \gamma_1.
\]

Therefore,

\[
\mathbb{E}[W] = \left(1 + O\left(\frac{\varepsilon}{\gamma_1}\right)\right) \cdot \left(\sum_{z=1}^{N} s(z) \cdot \left| D(z) - \frac{1}{N} \right|\right) + O\left(\frac{\varepsilon}{\gamma_1}\right) = \left(\sum_{z=1}^{N} s(z) \cdot \left| D(z) - \frac{1}{N} \right|\right) + O(\varepsilon).
\]

Moreover, since generating the \( W_{+,t} \)'s, \( W_{-,t} \)'s, and \( W_T \) use independent queries, we have

\[
\text{Var}(W) = \text{Var}(W_T) + \sum_{t=1}^{T-1} \text{Var}(W_{+,t}) + \sum_{t=1}^{T-1} \text{Var}(W_{-,t}) = O(\log \varepsilon^{-1}) \cdot O\left(\frac{\varepsilon}{\log \varepsilon^{-1}}\right) = O(\varepsilon^2).
\]

Therefore, with probability at least 0.97, we have that \( W = O(\varepsilon) + \sum_{z=1}^{N} s(z) \cdot \left| D(z) - \frac{1}{N} \right| \).

\[\square\]

### 4.2 Creating the Oracle

In this section, we create an oracle \( O' \) which is a slightly modified version of the oracle \( O \) used in Subsection 4.1. In some cases for \( D \), we will not be able to create such an oracle, but we show in the next subsection that in both cases, regardless of whether we have found an oracle or not, we can still determine \( d_{TV}(D, \mathcal{U}) \) up to additive error \( O(\varepsilon) \).

We first prove the following lemma, which we note is very similar in idea to Proposition 2.2.

**Lemma 4.4.** Suppose that \( d_{TV}(D, \mathcal{U}) \leq 1 - 3\varepsilon \). Then, there exists an integer \( t \) such that \(-\log_{1.01} \varepsilon^{-1} \leq t \leq \log_{1.01} \varepsilon^{-1} - 1 \) and that

\[
\mathbb{P}_{x \sim D} \left( D(x) \in \left[ \frac{1.01^t}{N}, \frac{1.01^{t+1}}{N} \right] \right) \geq \frac{\varepsilon}{210 \log \varepsilon^{-1}}, \quad \mathbb{P}_{x \sim \mathcal{U}} \left( D(x) \in \left[ \frac{1.01^t}{N}, \frac{1.01^{t+1}}{N} \right] \right) \geq \frac{\varepsilon}{210 \log \varepsilon^{-1}}.
\]

**Proof.** Assume WLOG that \( \varepsilon^{-1} \) is an integer power of 1.01. For each integer \( t \) such that \(-\log_{1.01} \varepsilon^{-1} \leq t \leq \log_{1.01} \varepsilon^{-1} - 1 \), define \( p_t = \mathbb{P}_{x \sim D} \left( D(x) \in \left[ \frac{1.01^t}{N}, \frac{1.01^{t+1}}{N} \right] \right) \) and \( q_t = \mathbb{P}_{x \sim \mathcal{U}} \left( D(x) \in \left[ \frac{1.01^t}{N}, \frac{1.01^{t+1}}{N} \right] \right) \).

Also let \( p_- = \mathbb{P}_{x \sim D}(D(x) \leq \varepsilon/N) \), \( q_- = \mathbb{P}_{x \sim \mathcal{U}}(D(x) \leq \varepsilon/N) \), \( p_+ = \mathbb{P}_{x \sim D}(D(x) \geq \varepsilon^{-1}/N) \), and \( q_+ = \mathbb{P}_{x \sim \mathcal{U}}(D(x) \geq \varepsilon^{-1}/N) \).

Now, note that by Proposition 2.1

\[
1 - d_{TV}(D, \mathcal{U}) = \sum_{i=1}^{n} \min \left( D(i), \frac{1}{n} \right) \leq \min(p_-, q_-) + \min(p_+, q_+) + \sum_{t=-\log_{1.01} \varepsilon^{-1}}^{\log_{1.01} \varepsilon^{-1} - 1} \min(p_t, q_t).
\]
However, note that $p_- \leq \varepsilon$, since there are at most $N$ elements $x$ with $D(x) \leq \frac{1}{N}$. Likewise, $q_+ \leq \varepsilon$, since there are at most $\varepsilon \cdot N$ elements $x$ with $D(x) \geq \frac{\varepsilon}{N}$. Therefore, if $d_{TV}(D,U) \leq 1 - 3\varepsilon$, we have that $1 - d_{TV}(D,U) \geq 3\varepsilon$, so there must exist some $t$ in the desired range such that $\min(p_t, q_t) \geq \frac{\varepsilon}{2 \log_2 1.01 \varepsilon} \geq \frac{\varepsilon}{200 \log \varepsilon}$.

Now, for any $x \in [N]$, recall that we can use PairComp to compare $x$ and some other element $w$. Using the notation of Proposition 23, if we set $\gamma = 0.01$, PairComp$(w, x, \gamma)$ returns $\alpha$ such that if $\frac{D(w)}{D(x)} \in [0.99, 1.01]$, then $\alpha \in [0.98, 1.02]$ with probability $1 - \varepsilon^{10}$, but if $\frac{D(w)}{D(x)} \not\in [0.97, 1.03]$, then $\alpha \not\in [0.98, 1.02]$ with probability $1 - \varepsilon^{10}$. To make use of this observation, we first define the following.

**Definition 4.1.** For any element $w \in [n]$, let $d(x) = \mathbb{P}_{w \sim D}(\text{PairComp}(w, x, 0.01)) \in [0.98, 1.02]$, and let $u(x) = \mathbb{P}_{w \sim U}(\text{PairComp}(w, x, 0.01)) \in [0.98, 1.02]$.

Next, we will find some element $x$ as well as a constant-factor approximation $\hat{D}(x)$ of $D(x)$.

**Lemma 4.5.** There exists an algorithm ConstantApprox using $O(\varepsilon^{-2} \log^5 \varepsilon^{-1})$ queries to PCond and Samp that returns a set $S$ with elements of the form $(x_r, \hat{D}(x_r))$ with the following guarantees.

1. For all $(x, \hat{D}(x)) \in S$, with probability at least $1 - \varepsilon^6$, we have that $\frac{\hat{D}(x)}{D(x)} \in [0.9, 1.1]$ and $
\hat{D}(x) \in \left[\frac{0.9\varepsilon}{N}, \frac{1.1\varepsilon}{N}\right]$. 

2. If $d_{TV}(D,U) \leq 1 - 3\varepsilon$, then with probability at least $1 - \varepsilon^6$, $S$ is nonempty.

3. With probability at least $1 - \varepsilon^6$, at least one of the following is true:

   (a) Some $(x, \hat{D}(x)) \in S$ satisfies $\hat{D}(x) \in \left[\frac{5}{3N}, \frac{9}{5N}\right]$.

   (b) If some $(x, \hat{D}(x)) \in S$ satisfies $\hat{D}(x) \geq \frac{1}{N}$, the $x$ with the smallest such $\hat{D}(x)$ satisfies $
\mathbb{P}_{w \sim U}(\frac{1}{N} \leq D(w) \leq 0.8 \cdot D(x)) \leq 2\varepsilon$. Likewise, if some $(x, \hat{D}(x)) \in S$ satisfies $\hat{D}(x) \leq \frac{1}{N}$, then the $x$ with the largest such $\hat{D}(x)$ satisfies $\mathbb{P}_{w \sim D}(\frac{1}{N} \geq D(w) \geq 1.25 \cdot D(x)) \leq 2\varepsilon$.

**Proof.** First, choose $R = O(\varepsilon^{-1} \log^2 \varepsilon^{-1})$ and sample elements $x_1, \ldots, x_R \leftarrow D$. Also, sample elements $w_1, \ldots, w_R \leftarrow D$ and $y_1, \ldots, y_R \leftarrow U$.

For each $1 \leq r \leq R$ and each $1 \leq i \leq R$, we run PairComp$(w_i, x_r, 0.01)$ and PairComp$(y_i, x_r, 0.01)$. Let $\hat{d}(x_r) = \frac{1}{R} \cdot \#\{i \in [R] : \text{PairComp}(w_i, x_r, 0.01) \in [0.98, 1.02]\}$, and let $\hat{u}(x_r) = \frac{1}{R} \cdot \#\{i \in [R] : \text{PairComp}(y_i, x_r, 0.01) \in [0.98, 1.02]\}$. We note that $\hat{d}(x_r)$ is distributed as $\frac{1}{R} \cdot \text{Bin}(R, d(x_r))$ and $\hat{u}(x_r)$ is distributed as $\frac{1}{R} \cdot \text{Bin}(R, u(x_r))$. Therefore, by a basic application of the Chernoff bound, if $d(x_r), u(x_r) \geq \frac{\varepsilon}{300 \log \varepsilon^{-1}}$, then assuming $R \geq C \varepsilon^{-1} \log^2 \varepsilon^{-1}$ for a sufficiently large constant $C$, we have that $\hat{d}(x_r) \in [0.99 \cdot d(x_r), 1.01 \cdot d(x_r)]$ and $\hat{u}(x_r) \in [0.99 \cdot u(x_r), 1.01 \cdot u(x_r)]$ with probability at least $1 - \varepsilon^8$. Moreover, if $\hat{d}(x_r) \leq \frac{\varepsilon}{300 \log \varepsilon^{-1}}$, we have that $\hat{d}(x_r) \leq \frac{\varepsilon}{250 \log \varepsilon^{-1}}$, and if $u(r) \leq \frac{\varepsilon}{300 \log \varepsilon^{-1}}$, we have that $\hat{u}(x_r) \leq \frac{\varepsilon}{250 \log \varepsilon^{-1}}$.

Now, the algorithm proceeds as follows. For each $r$, we check whether both $\hat{d}(x_r) > \frac{\varepsilon}{250 \log \varepsilon^{-1}}$ and $\hat{u}(x_r) > \frac{\varepsilon}{250 \log \varepsilon^{-1}}$. In this case, we let $\hat{D}(x_r) = \frac{\hat{d}(x_r)}{\hat{u}(x_r)} \cdot \frac{1}{N}$. We will ignore any $\hat{D}(x_r) \not\in \left[\frac{0.9\varepsilon}{N}, \frac{1.1\varepsilon}{N}\right]$. With probability at least $1 - \varepsilon^6$, for any $x_r$ with either $d(x_r)$ or $u(x_r)$ less than $\frac{\varepsilon}{300 \log \varepsilon^{-1}}$, we will output either $\hat{d}(x_r) \leq \frac{\varepsilon}{250 \log \varepsilon^{-1}}$ or $\hat{u}(x_r) \leq \frac{\varepsilon}{250 \log \varepsilon^{-1}}$. Therefore, for any $x_r$
with both \( \tilde{d}(x_r), \tilde{u}(x_r) > \frac{\varepsilon}{300 \log \varepsilon} \), we must have that \( d(x_r), u(x_r) \geq \frac{\varepsilon}{300 \log \varepsilon} \), which means that \( \tilde{d}(x_r) \) and \( \tilde{u}(x_r) \) are accurate up to a multiplicative error of 0.01 by a simple application of Chernoff.

Therefore, for all \( r \) such that we output some \( \hat{D}(x_r) \), we have that \( \hat{D}(x_r) \in \left[ 0.99 \frac{1}{N} \cdot \frac{d(x_r)}{u(x_r)}, 1.01 \frac{1}{N} \cdot \frac{d(x_r)}{u(x_r)} \right] \).

Next, we will look at the ratio \( \frac{d(x_r)}{u(x_r)} \). For each \( w \), let \( p(w, x_r) \) be the probability that \( \text{PARCOMP}(w, x_r, 0.01) \in [0.98, 1.02] \). Then, \( d(x_r) = \sum_w D(w) \cdot p(w, x_r) \) and \( u(x_r) = \sum_w \frac{1}{N} \cdot p(w, x_r) \). However, recalling that \( p(w, x) \leq \varepsilon^{10} \) whenever \( D(x) \notin [0.97D(w), 1.03D(w)] \), we have that

\[
d(x_r) = O(\varepsilon^{10}) + \sum_{w:D(w)\in[0.97D(x_r),1.03D(x_r)]} D(w) \cdot p(w, x_r)
\]

and

\[
u(x_r) = O(\varepsilon^{10}) + \sum_{w:D(w)\in[0.97D(x_r),1.03D(x_r)]} \frac{1}{N} \cdot p(w, x_r).
\]

However, note that

\[
\sum_{w:D(w)\in[0.97D(x_r),1.03D(x_r)]} D(w) \cdot p(w, x_r) \in [0.97, 1.03] \cdot N \cdot D(x_r) \cdot \sum_{w:D(w)\in[0.97D(x_r),1.03D(x_r)]} \frac{1}{N} \cdot p(w, x_r)
\]

due to our restriction of \( w \in [0.97D(x_r), 1.03D(x_r)] \). Therefore, if \( d(x_r), u(x_r) \geq \frac{\varepsilon}{300 \log \varepsilon} \), the \( O(\varepsilon^{10}) \) additive errors are negligible, and we have that \( d(x_r) \in [0.96, 1.04] \cdot N \cdot D(x_r) \cdot u(x_r) \). Thus, whenever we output \( \hat{D}(x_r) \), we have that

\[
\hat{D}(x_r) \in [0.96, 1.04] \cdot N \cdot D(x_r) \cdot \left[ 0.99 \frac{1}{N} \cdot \frac{1}{0.99} \cdot \frac{1}{0.99} \right] \subseteq [0.9 \cdot D(x_r), 1.1 \cdot D(x_r)].
\]

Thus, we have proven that with probability at least \( 1 - \varepsilon^6 \), all returned \( \hat{D}(x_r) \)'s are accurate. Next, we prove the second condition, i.e., if \( d_{TV}(D, \mathcal{U}) \leq 1 - 3\varepsilon \), then at least one \( r \) will result in a \( \hat{D}(x_r) \) being returned. To see why, if \( d_{TV}(D, \mathcal{U}) \leq 1 - 3\varepsilon \), then by Lemma [4.3] there exists an integer \( t \) such that \( [1.01^t, 1.01^{t+1}] \subset [\varepsilon, \varepsilon^{-1}] \) and

\[
P_{x \sim D} \left( D(x) \in \left[ \frac{1.01^t}{N}, \frac{1.01^{t+1}}{N} \right] \right) \geq \frac{\varepsilon}{210 \log \varepsilon^{-1}}, \quad P_{x \sim \mathcal{U}} \left( D(x) \in \left[ \frac{1.01^t}{N}, \frac{1.01^{t+1}}{N} \right] \right) \geq \frac{\varepsilon}{210 \log \varepsilon^{-1}}.
\]

Therefore, with probability at least \( 1 - \varepsilon^7 \), some \( x_r \) with \( D(x_r) \in [1.01^t, 1.01^{t+1}] \) will be sampled. It is clear that \( d(x_r), u(x_r) \geq \frac{\varepsilon}{210 \log \varepsilon^{-1}} \), and we have seen that for all such \( x_r \), we will output \( \hat{D}(x_r) \in [0.9D(x_r), 1.1D(x_r)] \), with failure probability at most \( \varepsilon^6 \). Moreover, \( \hat{D}(x_r) \in [0.9 \cdot \frac{1}{N}, 1.1 \cdot \frac{1}{N}] \), since \( [1.01^t, 1.01^{t+1}] \subset [\varepsilon, \varepsilon^{-1}] \).

Finally, we verify the third condition. Suppose no \( x_r \) has a returned \( \hat{D}(x_r) \in \left[ \frac{5}{2N}, \frac{9}{5N} \right] \), but that some \( x_r \) has \( \hat{D}(x_r) > \frac{9}{5N} \). Choose such an \( x_r \) that minimizes \( \hat{D}(x_r) \). Now, let \( t_1 \) be the smallest nonnegative integer \( t \leq \log_{1.01} \varepsilon^{-1} - 1 \) such that \( P_{w \sim \mathcal{U}} \left( D(w) \in \left[ \frac{1.01^t}{N}, \frac{1.01^{t+1}}{N} \right] \right) \geq \frac{\varepsilon}{210 \log \varepsilon^{-1}} \). Then, since \( \frac{1.01^t}{N} \geq \frac{1}{N} \), we will also have that \( P_{w \sim D} \left( D(w) \in \left[ \frac{1.01^t}{N}, \frac{1.01^{t+1}}{N} \right] \right) \geq \frac{\varepsilon}{210 \log \varepsilon^{-1}} \). Thus, some \( x \) with \( D(x) \in \left[ \frac{1.01^t}{N}, \frac{1.01^{t+1}}{N} \right] \) and some output \( \hat{D}(x) \) of \( D(x) \) will be returned, with probability at least \( 1 - \varepsilon^6 \). Therefore, since \( \hat{D}(x) \geq \frac{9}{5N} \), we have that

\[
0.9 \cdot D(x_r) \leq \hat{D}(x_r) \leq D(x) \leq 1.1 \cdot D(x) \leq 1.1 \cdot 1.01 \cdot 1.01^t.
\]

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This implies that \(0.8 \cdot D(x_r) \leq 1.01 t_1\), so \(P \left( \frac{1}{N} \leq D(w) \leq 0.8 \cdot D(x_r) \right) \leq t_1 \cdot \frac{\varepsilon}{9N \log \varepsilon} \leq \varepsilon.

Now, if no such \(t_1\) exists, then \(P_{w \sim \mathcal{U}} \left( \frac{1}{N} \leq D(w) \leq \frac{\varepsilon - 1}{N} \right) \leq \varepsilon\) and \(P_{w \sim \mathcal{U}} \left( D(w) \geq \frac{\varepsilon - 1}{N} \right) \leq \varepsilon\), so we even have that \(P_{w \sim \mathcal{U}} \left( \frac{1}{N} \leq D(w) \right) \leq 2 \varepsilon\). Finally, we note that the proof for the other direction, i.e., if \(D(x) \leq \frac{1}{N}\), is identical.

We are now ready to prove the main lemma of this subsection. Informally, we show that as long as we found some \(D(x)\) that is very close to \(\frac{1}{N}\), we also find an oracle \(\mathcal{O}'\) which essentially separates between elements with probabilities less than \(\frac{1}{N}\) and probabilities greater than \(\frac{1}{N}\), but allows for elements with probabilities close to \(\frac{1}{N}\) to be “unknown.”

**Lemma 4.6.** Suppose that Lemma 4.5 outputs some \((x, \hat{D}(x))\) such that \(\hat{D}(x) \in \left[\frac{5}{9N}, \frac{9}{5N}\right]\) and \(D(x) \in [0.9 \cdot D(x), 1.1 \cdot D(x)]\). Then, there exists an algorithm ORACLE using \(O(\varepsilon^{-2} \log^3 \varepsilon^{-1})\) additional queries to PCOND and SAMP that generates a randomized oracle \(\mathcal{O}'\) which takes as input an element \(z \in [N]\) and outputs either 0, 1, or \(-1\). Moreover, the oracle satisfies the following four properties for all \(z \in [N]\).

1. If \(D(z) > \frac{5}{9N}\), then \(\mathcal{O}'(z) = 1\) with probability at least \(1 - O(\varepsilon^6)\).
2. If \(D(z) < \frac{1}{5N}\), then \(\mathcal{O}'(z) = -1\) with probability at least \(1 - O(\varepsilon^6)\).
3. If \(\frac{1}{5N} \leq D(z) \leq \frac{1}{N}\), then \(\mathcal{O}'(z)\) is either 0 or \(-1\) with probability at least \(1 - O(\varepsilon^6)\).
4. If \(\frac{1}{N} \leq D(z) \leq \frac{5}{9N}\), then \(\mathcal{O}'(z)\) is either 0 or 1 with probability at least \(1 - O(\varepsilon^6)\).

Finally, calling the oracle \(\mathcal{O}'\) requires \(O(\log \varepsilon^{-1})\) calls to PCOND.

**Proof.** By our assumption about what was returned by Lemma 4.5, we have that \(D(x) \in \left[\frac{\varepsilon - 1}{2N}, \frac{2 - 1}{N}\right]\).

The oracle works as follows. For any \(z\), the oracle runs PairComp\((z, x, 0.01)\) and returns some \(\alpha\). If \(\alpha \in [0.45, 2.2]\), then we know that \(\frac{D(z)}{D(x)} \in [0.4, 2.5]\), so \(D(z) \in \left[\frac{1}{5N}, \frac{5}{9N}\right]\), so \(\mathcal{O}'\) returns 0. If \(\alpha < 0.45\), then we know that \(\frac{D(z)}{D(x)} < 0.5\), so \(D(z) < \frac{1}{N}\), so \(\mathcal{O}'\) returns \(-1\). Finally, if \(\alpha > 2.2\), then we know that \(\frac{D(z)}{D(x)} > 2\), so \(D(z) > \frac{1}{N}\), so \(\mathcal{O}'\) returns 1. Finally, note that calling the oracle just requires calling PairComp\((z, x, 0.01)\), which needs \(O(\log \varepsilon^{-1})\) calls to PCOND.

### 4.3 Finishing the Algorithm

In this section, we show how to combine subsections 4.1 and 4.2 to prove Theorem 1.1.

**Lemma 4.7.** Suppose Lemma 4.5 finds some \((x, \hat{D}(x))\) such that \(\hat{D}(x) \in \left[\frac{5}{9N}, \frac{9}{5N}\right]\) and \(D(x) \in [0.9 \cdot D(x), 1.1 \cdot D(x)]\). Then, there is an algorithm GIVENGOODELT that uses \(O(\varepsilon^{-2} \log^3 \varepsilon^{-1})\) additional queries to PCOND and SAMP and with probability at least 0.9 returns \(d_{TV}(\mathcal{D}, \mathcal{U})\) with error \(O(\varepsilon)\).

**Proof.** First, we use Lemma 4.6 to get the oracle \(\mathcal{O}'\), and convert this to the oracle \(\mathcal{O}\) of Subsection 4.1 by having the oracle \(\mathcal{O}\) accept an input \(i\) if and only if \(\mathcal{O}'\) returns 0 on input \(i\). Note that \(\mathcal{O}\) satisfies the requirements except that the probability that \(\mathcal{O}\) accepts an element \(i\) with \(D(i) \notin \left[\frac{5}{9N}, \frac{9}{5N}\right]\) is at most \(e^6\) rather than just never accepting. This, however, is fine, since we will only make at most \(O(\varepsilon^{-2})\) calls to \(\mathcal{O}\). Importantly, note that if we sum \(s(i) \cdot |D(i) - \frac{1}{N}|\) over the \(i\)
such that $D(i) \not\in \left[\frac{1}{N}, \frac{3}{N}\right]$, where $s(i) = \mathbb{P}(O'(i) = 0)$, we get $O(\varepsilon^6)$, since $\sum_i D(i), \sum_i \frac{1}{N} \leq 1$ and $s(i) \leq \varepsilon^6$ for all such $i$. Therefore, using $O(\varepsilon^{-2}\log^2 \varepsilon^{-1})$ queries to PCOND and our oracle $O$ which we have created, we can determine $\sum_{i=1}^N s(i) \cdot |D(i) - \frac{1}{N}|$ up to an $O(\varepsilon)$ additive error using Lemma 4.3, where $s(i)$ is the probability of $O'$ returning 0 on $i$. For Lemma 4.3 to work, we will need $\gamma_1 = \sum_{i=1}^N s(i) \cdot \frac{1}{N}$ to be at least 10ε, but this can be checked easily, and if $\gamma_1 = O(\varepsilon)$, then $\sum_{i=1}^N s(i) \cdot |D(i) - \frac{1}{N}| = O(\varepsilon)$ so we can estimate it as 0.

Now, let $r(i)$ be the probability that $O'$ returns −1 on $i$ and $t(i)$ be the probability that $O'$ returns 1 on $i$. Note that $\mathbb{P}_{y \sim \mathcal{U}}(O'(y) = -1) = \sum_{i=1}^N r(i) \cdot \frac{1}{N}$ and $\mathbb{P}_{z \sim \mathcal{D}}(O'(z) = -1) = \sum_{i=1}^N r(i) \cdot D(i)$. Likewise, $\mathbb{P}_{y \sim \mathcal{U}}(O'(y) = 1) = \sum_{i=1}^N t(i) \cdot \frac{1}{N}$ and $\mathbb{P}_{z \sim \mathcal{D}}(O'(z) = 1) = \sum_{i=1}^N t(i) \cdot D(i)$.

Now, we note that since $t(i) = O(\varepsilon^6)$ for all $i$ such that $D(i) \leq \frac{1}{N}$ and $r(i) = O(\varepsilon^6)$ for all $i$ such that $D(i) \geq \frac{1}{N}$, we have that $\sum_{i=1}^N r(i) \cdot \left(\frac{1}{N} - D(i)\right) = O(\varepsilon^6) + \sum_{i=1}^N r(i) \cdot |\frac{1}{N} - D(i)|$. Likewise, we have that $\sum_{i=1}^N t(i) \cdot (D(i) - \frac{1}{N}) = O(\varepsilon^6) + \sum_{i=1}^N t(i) \cdot |D(i) - \frac{1}{N}|$. Recall that we know $\sum_{i=1}^N s(i) \cdot |D(i) - \frac{1}{N}|$ up to an $O(\varepsilon)$ additive factor, and that we can compute $\mathbb{P}_{y \sim \mathcal{U}}(O'(y) = -1), \mathbb{P}_{z \sim \mathcal{D}}(O'(z) = -1), \mathbb{P}_{y \sim \mathcal{U}}(O'(y) = 1), \mathbb{P}_{z \sim \mathcal{D}}(O'(z) = 1)$ each up to an $O(\varepsilon)$ error using $O(\varepsilon^{-2})$ queries to $O'$ with probability at least 0.9, where we either sample from $\mathcal{D}$ or $\mathcal{U}$. Therefore, we compute:

$$\sum_{i=1}^N s(i) \cdot |D(i) - \frac{1}{N}| + \sum_{i=1}^N r(i) \cdot |D(i) - \frac{1}{N}| + \sum_{i=1}^N t(i) \cdot |D(i) - \frac{1}{N}| = \sum_{i=1}^N D(i) - \frac{1}{N} = 2 \cdot d_{\text{TV}}(\mathcal{D}, \mathcal{U})$$

up to an $O(\varepsilon)$ factor, where we used the fact that $r(i) + s(i) + t(i) = 1$ for all $i$.

In total, we used $O(\varepsilon^{-2}\log^2 \varepsilon^{-1})$ queries to PCOND, SAMP, and $O'$. But since each call to $O'$ uses $O(\log \varepsilon^{-1})$ calls to PCOND, the final query complexity of $O(\varepsilon^{-2}\log^3 \varepsilon^{-1})$.

---

**Proof of Theorem 4.1** We assume that the output of Lemma 4.5 satisfies the guarantees, ignoring the $O(\varepsilon^6)$ failure probability.

First, suppose that in Lemma 4.5 we return $S = \{\}$. Then, we can say that $d_{\text{TV}}(\mathcal{D}, \mathcal{U}) = 1$ and we are off by at most 3ε.

Next, suppose that in Lemma 4.5 we return some $(x, \hat{D}(x)) \in S$ with $\hat{D}(x) \in \left[\frac{5}{9N}, \frac{9}{9N}\right]$. Then, we can use Lemma 4.7 to finish the proof.

Otherwise, we have that in Lemma 4.5 at least some pair $(x, \hat{D}(x)) \in S$ was found with $\hat{D}(x) \in \left[0.9\varepsilon, \frac{1.1\varepsilon - 1}{N}\right]$, but every such $(x_r, \hat{D}(x_r)) \in S$ satisfies $\hat{D}(x_r) \not\in \left[\frac{5}{9N}, \frac{9}{9N}\right]$. We will show first how to estimate $\mathbb{P}_{y \sim \mathcal{U}}(D(y) \geq \frac{1}{N})$ and then how to estimate $\mathbb{P}_{z \sim \mathcal{D}}(D(z) < \frac{1}{N})$. We combine these together to get the final estimate.

Suppose there exists some $x_r$ returned by Lemma 4.5 such that $\hat{D}(x_r) \geq \frac{1}{N}$. In that case, choose $x_r$ such that $\hat{D}(x_r) \leq \hat{D}(x_r)$ for all $x_r$ returned by Lemma 4.5 with $\hat{D}(x_r) \geq \frac{1}{N}$. With probability at least $1 - \varepsilon^6$, all $\hat{D}(x_r)$’s are accurate up to a 1 ± 0.1 factor, so $\hat{D}(x_r) \geq \frac{3}{2N}$ since $\hat{D}(x_r) \geq \frac{9}{5N}$. We also know that

$$\mathbb{P}_{y \sim \mathcal{U}}\left(\frac{1}{N} \leq D(y) < 0.8D(x_r)\right) \leq 2\varepsilon$$

by Lemma 4.5. Therefore, the probability over $y \sim \mathcal{U}$ that $\text{PAIRCOMP}(y, x_r, 0.01)$ is at least 0.78 equals $\mathbb{P}_{y \sim \mathcal{U}}(y \geq \frac{1}{N})$, up to a 3ε additive error. This is true because

$$\mathbb{P}_{y \sim \mathcal{U}}(\text{PAIRCOMP}(y, x_r, 0.01) \geq 0.78) \geq \mathbb{P}_{y \sim \mathcal{U}}\left(\frac{D(y)}{D(x_r)} \geq 0.79\right) - \varepsilon \geq \mathbb{P}_{y \sim \mathcal{U}}\left(D(y) \geq \frac{1}{N}\right) - 3\varepsilon,$$
but
\[ P_{y \sim U}(\text{PairComp}(y, x, 0.01) \geq 0.78) \leq P_{y \sim U} \left( \frac{D(y)}{D(x_r)} \geq 0.77 \right) + \varepsilon \leq P_{y \sim U} \left( D(y) \geq \frac{1}{N} \right) + \varepsilon. \]

We can estimate \( P_{y \sim U}(\text{PairComp}(y, x, 0.01) \geq 0.78) \) up to a \( 2\varepsilon \) error, using \( O(\varepsilon^{-2}) \) samples of \( y_i \sim U \) and computing \( \text{PairComp}(y_i, x_r, 0.01) \) for each of them. Now, if no such \( x_r \) with \( D(x_r) \geq \frac{1}{N} \) exists, then saw in the proof of Lemma 4.5 that even
\[ P_{y \sim U} \left( \frac{1}{N} \leq D(y) \right) \leq 2\varepsilon. \]

Thus, we estimate \( P_{x \sim U}(D(x) \geq \frac{1}{N}) \) as 0, which is correct up to a \( 2\varepsilon \) additive error.

Similarly, suppose there exists some \( x_s \) such that \( \tilde{D}(x_s) < \frac{1}{N} \). In that case, choose \( x_s \) such that \( \tilde{D}(x_s) \geq \tilde{D}(x_{s'}) \) for all \( x_{s'} \) returned by Lemma 4.5 with \( \tilde{D}(x_{s'}) < \frac{1}{N} \). Also, with probability at least \( 1 - \varepsilon^6 \), all \( \tilde{D}(x_r) \)'s are accurate up to a \( 1 \pm 0.1 \) factor, so \( D(x_s) \leq \frac{27}{N} \) since \( \tilde{D}(x_s) \leq \frac{5}{N} \). We also know that
\[ P_{z \sim D} \left( \frac{1}{N} > D(z) > 1.25 D(x_r) \right) < 2\varepsilon \]
by Lemma 4.5. Therefore, the probability over \( z \sim D \) that \( \text{PairComp}(z, x_s, 0.01) \) is at most \( 1.27 \) equals \( P_{z \sim D} (z < \frac{1}{N}) \), up to a \( 3\varepsilon \) additive error. This is true because
\[ P_{z \sim D}(\text{PairComp}(z, x_s, 0.01) \leq 1.27) \geq P_{z \sim D} \left( \frac{D(z)}{D(x_s)} \leq 1.26 \right) - \varepsilon \geq P_{z \sim D} \left( D(z) < \frac{1}{N} \right) - 3\varepsilon, \]

but
\[ P_{z \sim D}(\text{PairComp}(z, x_s, 0.01) \leq 1.27) \leq P_{z \sim D} \left( \frac{D(z)}{D(x_s)} \leq 1.28 \right) + \varepsilon \leq P_{z \sim D} \left( D(z) < \frac{1}{N} \right) + \varepsilon. \]

We can estimate \( P_{z \sim D}(\text{PairComp}(z, x_s, 0.01) \leq 1.27) \) up to a \( 2\varepsilon \) error, using \( O(\varepsilon^{-2}) \) samples of \( z_i \sim D \) and computing \( \text{PairComp}(z_i, x_s, 0.01) \) for each of them. Now, if no \( x_s \) exists, then we know that
\[ P_{z \sim D} \left( D(z) < \frac{1}{N} \right) \leq 2\varepsilon. \]

by the same proof as in of Lemma 4.5. Namely, for any distribution \( D \), \( P_{z \sim D} \left( D(z) < \frac{\varepsilon N}{210 \log \epsilon^{-t}} \right) \leq \varepsilon \), and if \( P_{z \sim D} \left( \frac{\varepsilon N}{210 \log \epsilon^{-t}} > D(z) \geq \frac{\varepsilon N}{210 \log \epsilon^{-t}} \right) \geq \varepsilon \), then some \( 0 \leq t < \log_{1.01} \varepsilon^{-1} \) satisfies \( P_{z \sim D} \left( D(z) \in \left[ \frac{1.01^{-(t+1)} - 1.01^{-t}}{N}, \frac{1.01^{-(t+1)} + 1.01^{-t}}{N} \right] \right) \geq \frac{\varepsilon}{210 \log \epsilon^{-t}} \). Thus, Lemma 4.5 would find some \( (x, \tilde{D}(x)) \) with \( D(x) \) in the range \( \left[ \frac{1.01^{-(t+1)} - 1.01^{-t}}{N}, \frac{1.01^{-(t+1)} + 1.01^{-t}}{N} \right] \subset \left[ \frac{\varepsilon N}{210 \log \epsilon^{-t}}, \frac{\varepsilon}{210 \log \epsilon^{-t}} \right]. \)

To finish, note that
\[
P_{x \sim U}(D(x) \geq \frac{1}{N}) + P_{x \sim D}(D(x) < \frac{1}{N}) = \sum_{x: D(x) \geq \frac{1}{N}} \frac{1}{N} + \sum_{x: D(x) < \frac{1}{N}} D(x) = \sum_{x=1}^{N} \min \left( D(x), \frac{1}{N} \right),
\]
which equals \( 1 - d_{TV}(\mathcal{D}, \mathcal{U}) \) by Proposition 2.1. Therefore, we can estimate \( d_{TV}(\mathcal{D}, \mathcal{U}) \) up to an \( O(\epsilon) \) additive error, which concludes all cases. \( \square \)
5 An $\tilde{O}(\varepsilon^{-4})$-query algorithm for Tolerant Identity Testing

In this section, we present an algorithm that, given a known distribution $D^*$ over $[N]$, makes $\tilde{O}(\varepsilon^{-4})$ queries to $\text{COND}$ with distribution $D$ and determines $d_{TV}(D, D^*)$ up to an $O(\varepsilon)$ additive error. The dependence on the support size $N$ is optimal (i.e., no dependence), though it is possible that the dependence on $\varepsilon$ can be improved to $\tilde{O}(\varepsilon^{-2})$.

First, assume that $D^*$ is ordered so that $D^*(1) \leq D^*(2) \leq \cdots \leq D^*(N)$. We are allowed to permute the elements of both $D$ and $D^*$ with the same permutation, since this will not affect $d_{TV}(D, D^*)$ and we can still make the same $\text{COND}$ queries (after the same permutation is applied).

We will consider a slightly more general distribution problem. Suppose we have two distributions $\mathcal{P}$ and $\mathcal{P}^*$ over $[M]$, where $\mathcal{P}^*$ is known and $P^*(1) \leq P^*(2) \leq \cdots \leq P^*(M)$, and we are given $\text{COND}$ access to $\mathcal{P}$. Our goal is to determine $\sum_{i=1}^{M} \min(c_1 P(i), c_2 P^*(i))$ up to an additive $O(\varepsilon)$ error, where $c_1, c_2 \leq 1$ are known constants. Since $\sum_{i=1}^{M} P(i) = \sum_{i=1}^{M} P^*(i) = 1$, if either $c_1 = O(\varepsilon)$ or $c_2 = O(\varepsilon)$, then $\sum_{i=1}^{M} \min(c_1 P(i), c_2 P^*(i)) = O(\varepsilon)$ so we can just output 0 as our estimate. For the case where $c_1, c_2 \gg \varepsilon$, we give an inductive approach. Namely, we show how to find some set $S \subseteq [M]$ such that either $P(S) \geq \frac{1}{2}$ or $P^*(S) \geq \frac{1}{2}$ and estimate $\sum_{i \in S} \min(c_1 P(i), c_2 P^*(i))$ up to an $O(\varepsilon)$ additive error. To estimate $\sum_{i \notin S} \min(c_1 P(i), c_2 P^*(i))$, we can modify the distributions $P, P^*$ to be conditioned on $i \notin S$. Both $c_1$ and $c_2$ will either increase or stay the same, and either $c_1$ or $c_2$ will multiply by a factor of at most $\frac{2}{3}$, since either $P(S) \geq \frac{1}{2}$ or $P^*(S) \geq \frac{1}{2}$. Therefore, we only need to repeat this process $O(\log \varepsilon^{-1})$ times, until either $c_1$ or $c_2$ is $O(\varepsilon)$. Our final error will be $O(\varepsilon \cdot \text{poly log } \varepsilon^{-1})$, but we can fix this by replacing $\varepsilon$ with $\varepsilon' = \frac{\varepsilon}{\text{poly log } \varepsilon^{-1}}$.

**Theorem 5.1.** Suppose $\mathcal{P}, \mathcal{P}^*$ are distributions over $[M]$, where $\mathcal{P}^*$ is known, $P^*(1) \leq P^*(2) \leq \cdots \leq P^*(M)$, and we have $\text{COND}$ access to $\mathcal{P}$. Also, let $1 \geq c_1, c_2 \geq \varepsilon$ be known constants. Then, there is an algorithm $\text{PARTIALDERTING}$ that uses $O(\varepsilon^{-4} \log^6 \varepsilon^{-1})$ queries to $\text{COND}_{\mathcal{P}}$, such that with probability at least $1 - \varepsilon$, the algorithm finds a set $S$ such that $P^*(S) \geq \frac{1}{2}$ as well as an estimate of $\sum_{i \in S} \min(c_1 P(i), c_2 P^*(i))$ which is accurate to an additive $O(\varepsilon \log \varepsilon^{-1})$ error.

To begin the proof, we first let $L$ be the smallest integer such that $P^*(\lfloor L \rfloor) = \mathbb{P}_{x \sim \mathcal{P}^*}(x \leq L) \geq \frac{1}{2}$. Then, $P^*(\lceil L \rceil) \geq \frac{1}{2}$ and $P^*(\lfloor L : M \rfloor) \geq \frac{1}{2}$. We attempt to make $S = \lfloor L : M \rfloor$ or $S = \lceil z \rceil$ for some $z \geq L$. Note that for any set $S \subseteq [M]$, 

$$\sum_{x \in S} \min(c_1 P(x), c_2 P^*(x)) = \sum_{x \in S} P^*(x) \cdot \min \left( c_1 \frac{P(x)}{P^*(x)}, c_2 \right) = \mathbb{E}_{x \sim \mathcal{P}^*} \left( \mathbb{I}(x \in S) \cdot \min \left( c_1 \frac{P(x)}{P^*(x)}, c_2 \right) \right),$$

where $\mathbb{I}(x \in S)$ is the indicator variable of $x \in S$. Our rough goal will therefore be to provide estimates of $\frac{P(x)}{P^*(x)}$ for $x$ drawn from $\mathcal{P}^*$ with $x \in S$.

First, suppose that $P^*(L) \geq \frac{1}{3}$. Then, we can let $S = \{ L \}$. To estimate $\sum_{x \in S} \min(c_1 P(x), c_2 P^*(x)) = \min(c_1 P(L), c_2 P^*(L))$, we just need to estimate $P(L)$ up to a $\varepsilon$ additive error, since $c_1, c_2 \leq 1$ and we already know $P^*(z)$. This, however, can be done with $O(\varepsilon^{-2} \log \varepsilon^{-1})$ samples to $\mathcal{P}$ with failure probability $1 - \varepsilon^{10}$ by a simple Chernoff bound argument. Otherwise, $P^*(\lceil z - 1 \rceil) > \frac{1}{2} P^*(z)$, since if $z > L$ then $P^*(\lceil z - 1 \rceil) \geq \frac{1}{2}$ and for $z = L$, $P^*(z) < \frac{1}{3} \leq \frac{2}{3} \cdot P^*(\lceil z \rceil)$. Therefore, we can partition $[z - 1]$ into sets $S_1, \ldots, S_{k-1}$ such that $\frac{1}{2} P^*(z) \leq P^*(S_i) \leq P^*(z)$ for all $1 \leq i \leq k - 1$, using a simple greedy procedure [FJO+15]. We finally let $S_k := \{ z \}$. Thus, $S_1, \ldots, S_k$ partition $[z]$ so that $\frac{1}{2k} \leq \frac{P^*(S_i)}{P([z])} \leq \frac{2}{k}$ for all $1 \leq i \leq k$. 

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To estimate $\frac{P(z)}{P^*(z)}$, write

$$\frac{P(z)}{P^*(z)} = \frac{P([z])}{P^*[z]} \cdot \frac{P(S_j)}{P^*(S_j)} \cdot \frac{P(z)}{P(S_j)}$$

for some $1 \leq j \leq k$ to be chosen later. Now, let $Q, Q^*$ be distributions over $[k]$ so that $Q(i) = \frac{P(S_i)}{P^*(S_i)}$ and $Q^*(i) = \frac{P^*(S_i)}{P^*[z]}$. Then, recalling that $S_k = \{z\}$ and that $P^*(\{z\}) = Q^*(\{k\}) = 1$, we have that

$$\frac{P(z)}{P^*(z)} = \frac{1}{P^*(z)} \cdot P([z]) \cdot Q(j) \cdot \frac{Q(k)}{Q(j)}$$

We will assume $\textsf{COND}$ access to both $P$ and $Q$. We note that $\textsf{COND}$ access to $Q$ can easily be simulated by $\textsf{COND}$ access to $P$, since we just condition on a subset $T \subset [z]$ which is the union of some $S_i$’s and return which $S_i$ the $\textsf{COND}(T)_P$ query outputs an element in. We will show how to output a $1 \pm \delta$ multiplicative approximation to each of $P([z])$, $Q_j$ for some $j$, and $\frac{Q(k)}{Q(j)}$ using a small number of queries. Moreover, our estimates will also be nearly unbiased. These algorithms will not work under a few extreme cases, but we will deal with these cases accordingly.

**Lemma 5.1.** Fix $\delta \geq \varepsilon$. Then, there is an algorithm $\textsf{Est1}$ that uses $O(\varepsilon^{-1} \cdot \log \varepsilon^{-1} \cdot \delta^{-2})$ queries to $\textsf{COND}_P$, such that conditioned on some event $E_1$ with $\Pr(E_1) \geq 1 - \varepsilon^6$, the following happens:

1. If $P([z]) \geq \frac{\varepsilon}{2}$, the algorithm outputs a random variable $\tilde{P}([z])$ such that $\frac{\tilde{P}([z])}{P([z])} \in [1 - \delta, 1 + \delta]$ whenever $E_1$ is true, and $\EE[\tilde{P}([z])|E_1] \in [1 - \varepsilon, 1 + \varepsilon] \cdot P([z])$.

2. If $P([z]) < \frac{\varepsilon}{2}$, the algorithm outputs $\tilde{P}([z]) \leq \varepsilon$ whenever $E_1$ is true.

**Proof.** Our algorithm will be quite straightforward. Namely, we sample $R = O(\varepsilon^{-1} \cdot \log \varepsilon^{-1} \cdot \delta^{-2})$ samples $x_1, \ldots, x_R$ from $P$ and let $\tilde{P}([z])$ denote the fraction of the $x_i$’s such that $x_i \leq z$. We let $E_1$ be the event $\frac{\tilde{P}([z])}{P([z])} \in [1 - \delta, 1 + \delta]$ in the case $P([z]) \geq \frac{\varepsilon}{2}$ and the event $\tilde{P}([z]) \leq \varepsilon$ in the case $P([z]) < \frac{\varepsilon}{2}$. Since each $x_i$ has a $P([z])$ probability of being in $[z]$, the estimate $\tilde{P}([z])$ has distribution $\frac{1}{R} \cdot \text{Bin}(R, P([z]))$, so $\EE[\tilde{P}([z])] = P([z])$. Moreover, if $P([z]) \geq \frac{\varepsilon}{2}$, we have that $\frac{\tilde{P}([z])}{P([z])} \in [1 - \delta, 1 + \delta]$ with probability at least $1 - \varepsilon^6$ by the Chernoff bound. Even if we condition on $E_1$, the expectation of $\tilde{P}([z])$ changes by at most $\varepsilon^6$ which is at most $\varepsilon \cdot P([z])$. Likewise, if $P([z]) \leq \frac{\varepsilon}{2}$, then $\tilde{P}([z]) \leq \varepsilon$ with probability at least $1 - \varepsilon^6$, by the Chernoff bound. 

**Lemma 5.2.** Fix $\delta \geq \varepsilon$. Then, there is an algorithm $\textsf{Est2}$ that uses $O(\varepsilon^{-2} \log^5 \varepsilon^{-1} \cdot \delta^{-2})$ queries to $\textsf{COND}_P$, such that conditioned on some event $E_2$ with $\Pr(E_2) \geq 1 - O(\varepsilon^6)$, the following happens:

1. Suppose that $d_{TV}(Q, \mathcal{U}) < 1 - 3\varepsilon$, where $\mathcal{U}$ is the uniform distribution over $[k]$. Then, the algorithm outputs some pair $(j, \tilde{Q}(j))$ with $j \in [k]$ such that $Q(j) \in \left[\frac{2}{3}, \varepsilon, \frac{3}{2}, \frac{\varepsilon}{k}\right]$ and $\frac{\tilde{Q}(j)}{Q(j)} \in [1 - \delta, 1 + \delta]$ whenever $E_2$ is true. Moreover, conditioning on any fixed $j$ being returned, $\frac{E[\tilde{Q}(j)|E_2]}{Q(j)} \in [1 - \varepsilon, 1 + \varepsilon]$.

2. If $d_{TV}(Q, \mathcal{U}) \geq 1 - 3\varepsilon$, then conditioned on $E_2$, the algorithm either outputs NULL or a pair $(j, \tilde{Q}(j))$ with $Q(j) \in \left[\frac{2}{3}, \varepsilon, \frac{3}{2}, \frac{\varepsilon}{k}\right]$, $\frac{\tilde{Q}(j)}{Q(j)} \in [1 - \delta, 1 + \delta]$, and conditioned on any fixed $j$ being returned, $\frac{E[\tilde{Q}(j)|E_2]}{Q(j)} \in [1 - \varepsilon, 1 + \varepsilon]$. 

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Proof. First, by running the procedure of Lemma 4.5 with PCONDQ and SAMPO access, we find some \((j, \tilde{Q}(j))\) with \(\tilde{Q}(j) \in \left[\frac{0.9\varepsilon}{k}, \frac{1.1\varepsilon}{k}\right]\) and \(\frac{Q(j)}{Q(j)} \in [0.9, 1.1]\) if \(d_{TV}(Q, U) \leq 1 - 3\varepsilon\). If we don’t find such a pair (i.e., Lemma 4.5 returns \(S = \{\}\)) then we just return NULL.

Else, let \(j\) be the first \(j\) returned by Lemma 4.5. Recall that in Lemma 4.5 we proved that \(d(j, u(j)) \geq \frac{\varepsilon}{300 \log \varepsilon}\), where \(d(j, u(j))\) are defined in Definition 1.1 (we assume the guarantees of Lemma 4.5 are met). To approximate \(Q(j)\), we modify the approach of Lemma 4.5. We choose \(R = O(\varepsilon^{-2} \log^3 \varepsilon^{-1} \cdot \delta^{-2})\) and sample \(x_1, \ldots, x_R \leftarrow Q\) and \(y_1, \ldots, y_R \leftarrow U\). Also, we define \(\tilde{u}(j) = \frac{1}{R} \cdot \sum\{i \in [R] : \text{PAIRCOMP}(y_i, j, 0.01) \in [0.98, 1.02]\}\), so \(\tilde{u}(j)\) has distribution \(\frac{1}{R} \cdot \text{Bin}(R, u(j))\).

Next, for each \(i\), we create a random variable \(X_i\) and run \(\text{PAIRCOMP}(x_i, j, 0.01)\). If the returned value is between 0.98 and 1.02, then we keep calling \(\text{COND}\{\{x_i, j\}\}_Q\) until \(x_i\) is returned, and define \(X_i\) to be the number of times we see \(j\) returned before the first time \(x_i\) is returned. Otherwise, \(X_i = 0\). Finally, we let \(\tilde{q}(j)\) be the average of \(\text{min}(X_1, C \log \varepsilon^{-1}), \ldots, \text{min}(X_R, C \log \varepsilon^{-1})\) for some sufficiently large constant \(C\).

In our implementation, for each \(1 \leq i \leq R\), we will stop calling \(\text{COND}\{\{x_i, j\}\}_Q\) once we have already made \(C \log \varepsilon^{-1}\) calls to \(\text{COND}\{\{x_i, j\}\}_Q\).

If we define \(p(x, j)\) to be the probability that \(\text{PAIRCOMP}(x, j, 0.01) \in [0.98, 1.02]\), then

\[
\mathbb{E}[X_i] = \sum_{x=1}^{k} Q(x) \cdot p(x, j) \cdot \frac{Q(j)}{Q(x)} = Q(j) \cdot \sum_{x=1}^{k} p(x, j) = k \cdot Q(j) \cdot \frac{1}{k} = k \cdot Q(j) \cdot u(j),
\]

To see why, recall that to create the variable \(X_i\), we first sample \(x_i \leftarrow Q\), then \(X_i\) is only nonzero if \(\text{PAIRCOMP}(x_i, j, 0.01) \in [0.98, 1.02]\), in which case \(X_i\) is a Geometric random variable with parameter \(p = \frac{Q(x_i)}{Q(x_i) + Q(j)}\) and thus has mean \(\frac{Q(j)}{Q(x)}\). The last equality is true since \(u(j)\) is just the probability that \(\text{PAIRCOMP}(x, j, 0.01) \in [0.98, 1.02]\), where \(x\) is now uniformly distributed. Therefore, \(\mathbb{E}[\text{min}(X_i, C \log \varepsilon^{-1})] \leq k \cdot Q(j) \cdot u(j)\). However, for any \(x\) with \(\frac{Q(x)}{Q(j)} \in [0.97, 1.03]\), if we choose \(C\) large enough, then \(\mathbb{E}\left[\text{min}\left(\text{Geom}\left(\frac{Q(x)}{Q(x) + Q(j)}\right), C \log \varepsilon^{-1}\right)\right] = (1 + O(\varepsilon^6)) \cdot \frac{Q(j)}{Q(x)}\). Thus,

\[
\mathbb{E}[\text{min}(X_i, C \log \varepsilon^{-1})] \geq (1 - O(\varepsilon^6)) \cdot \sum_{x:Q(x) \in [0.97, 1.03]:Q(j)} Q(x) \cdot p(x, j) \cdot \frac{Q(j)}{Q(x)}
\]

\[
= (1 - O(\varepsilon^6)) \cdot k \cdot Q(j) \cdot \left[ u(j) - \sum_{x:Q(x) \notin [0.97, 1.03]:Q(j)} p(x, j) \right]
\]

\[
\geq (1 - O(\varepsilon^6)) \cdot k \cdot Q(j) \cdot [u(j) - O(\varepsilon^{10})],
\]

since for \(x\) with \(\frac{Q(x)}{Q(j)} \notin [0.97, 1.03]\), the probability of \(\text{PAIRCOMP}(x, j, 0.01) \in [0.98, 1.02]\) is at most \(\varepsilon^{10}\). But since \(u(j) \geq \frac{\varepsilon}{300 \log \varepsilon}\) and \(k \cdot Q(j) \geq \frac{3}{2} \varepsilon\), we have that \(\mathbb{E}[\text{max}(X_i, C \log \varepsilon^{-1})] = k \cdot Q(j) \cdot u(j) \cdot (1 \pm \frac{\varepsilon}{10}) \geq \frac{3}{500 \log \varepsilon}\). But since \(\text{max}(X_i, C \log \varepsilon^{-1})\) is bounded by \(C \log \varepsilon^{-1}\), the Chernoff bound tells us that the average of \(O(\varepsilon^{-2} \log^3 \varepsilon^{-1} \cdot \delta^{-2})\) samples \(X_i\) will be within a \(1 \pm \frac{\varepsilon}{10}\) multiplicative factor of \(\mathbb{E}[X_i]\) with probability at least \(e^{-6}\). Thus, \(\tilde{q}(j) = k \cdot Q(j) \cdot u(j) \cdot (1 \pm \frac{\varepsilon}{10})\), and with probability at least \(1 - e^{-6}\), \(\tilde{q}(j) = k \cdot Q(j) \cdot u(j) \cdot (1 \pm \frac{\varepsilon}{4})\).

We output \((j, \frac{\tilde{d}(j)}{k \cdot \tilde{u}(j)})\), unless no \((j, \tilde{Q}(j))\) was found by Lemma 4.5 in which case we return NULL. We know that since \(u(j) \geq \frac{\varepsilon}{300 \log \varepsilon}\) and \(\tilde{u}(j) \sim \frac{1}{R} \cdot \text{Bin}(R, u(j))\), with probability \(1 - e^{-6}\), \(\tilde{u}(j) \in (1 \pm \frac{\varepsilon}{4}) \cdot u(j)\). Therefore,

\[
\mathbb{E}[\tilde{u}(j)] = \frac{Q(j)}{1 + \frac{e^6}{4}} = Q(j) \cdot (1 \pm \frac{\varepsilon}{4})
\]

with probability at least 22
$1-3e^6$. We let $E_2$ be the event that the claims in Lemma 4.5 are satisfied, as well as that $	ilde{d}(j), \tilde{u}(j)$ are accurate up to a $1 \pm \frac{3}{2}$ and $1 \pm \frac{\delta \sqrt{e}}{2}$ multiplicative approximation, respectively, if Lemma 4.5 doesn’t return NULL.

To compute $E[\tilde{Q}(j)|E_2]$ for a fixed $j$, first note that if $\tilde{u}(j) = (1 + \gamma)u(j)$ for $\gamma \in [-0.5, 0.5]$, then $\frac{1}{u(j)} = \frac{1}{u(j)} \cdot (1 - \gamma + O(\gamma^2))$. Since $\tilde{u}(j) \in (1 \pm \frac{\delta \sqrt{e}}{4}) \cdot u(j)$ assuming $E_2$, $\frac{1}{u(j)} = \frac{1}{u(j)} \cdot \left(2 - \frac{\tilde{u}(j)}{u(j)} + O(\epsilon)\right)$. As $\tilde{u}(j)$ has distribution $\frac{1}{R} \cdot Bin(R, u(j))$, and conditioning on an event with $O(\epsilon^6)$ failure probability will not change $E[\tilde{u}(j)]$ by more than $O(\epsilon^6)$, we have $E[\tilde{u}(j)|E_2] = u(j) \cdot (1 \pm \epsilon)$. Therefore, $E \left[\frac{1}{u(j)}|E_2\right] = \frac{1}{u(j)} \cdot \left(2 - \frac{E[\tilde{u}(j)|E_2]}{u(j)} + O(\epsilon)\right) = \frac{1 + O(\epsilon)}{u(j)}$. Also, since $\tilde{q}(j)$ is bounded by $O(\log \epsilon^{-1})$, $E[\tilde{q}(j)] \geq \Omega(\frac{\epsilon^2}{\log \epsilon^{-1}})$, and $E_2$ occurs with probability $1 - O(\epsilon^6)$, conditioning on $E_2$ marginally affects the expectation of $\tilde{q}(j)$, and we will still have that $E[\tilde{q}(j)|E_2] = k \cdot Q(j) \cdot u(j) \cdot (1 \pm O(\epsilon))$. Thus, conditioned on $E_2$ (and $j$ being returned for some fixed $j$), the expected value of $\frac{\tilde{q}(j)}{k\cdot u(j)}$ is $E[\tilde{q}(j)|E_2] \cdot E \left[\frac{1}{u(j)}|E_2\right] \cdot \frac{1}{k} = Q(j) \cdot (1 \pm O(\epsilon))$. We can split the expectation into a product because for any fixed $j$, $\tilde{q}(j)$ and $\tilde{u}(j)$ are independent conditioned on $E_2$ and $j$, as we used disjoint samples to compute $\tilde{q}(j)$ and $\tilde{u}(j)$ once we found $j$.

\textbf{Lemma 5.3.} Fix $\delta \geq \epsilon$ and let $j$ be as returned in Lemma 5.2. Then, there is an algorithm $Est_3$ that uses an expected $O(\epsilon^{-2} \log^2 \epsilon^{-1} \cdot \delta^{-2})$ queries to $Cond_Q$, such that conditioned on some event $E_3$ with $\mathbb{P}(E_3) \geq 1 - O(\epsilon^6)$, the following happens:

1. If $\frac{Q(k)}{Q(j)} \in [0.12, 10 \epsilon^{-2}]$, then the algorithm finds an estimator $Y$ of $\frac{Q(k)}{Q(j)}$ such that $Y \in \left[(1 - \delta) \cdot \frac{Q(k)}{Q(j)}, (1 + \delta) \cdot \frac{Q(k)}{Q(j)}\right]$. Moreover, $E[Y|E_3] \in \left[(1 - O(\epsilon)) \cdot \frac{Q(k)}{Q(j)}, (1 + O(\epsilon)) \cdot \frac{Q(k)}{Q(j)}\right]$.

2. If $\frac{Q(k)}{Q(j)} \leq 0.12 \epsilon^2$, then $Y \leq 0.2 \epsilon^{-2}$, and if $\frac{Q(k)}{Q(j)} \geq 0 \epsilon^{-2}$, then $Y \geq 5 \epsilon^{-2}$.

\textbf{Proof.} First, we sample $R = O(\epsilon^{-2} \log \epsilon^{-1})$ queries of $Cond(\{j, k\})$. If $\frac{Q(k)}{Q(j)} \in [0.052, 20 \epsilon^{-2}]$, a simple application of the Chernoff bound tells us that with at least $1 - \epsilon^6$ probability, the sample ratio of the number of times $k$ is returned to the number of times $j$ is returned will be correct up to a factor of $1 \pm 0.1$. Likewise, with $1 - \epsilon^6$ probability, if $\frac{Q(k)}{Q(j)} < 0.052 \epsilon^2$, the sample ratio will be at most $0.062^2$, and if $\frac{Q(k)}{Q(j)} > 20 \epsilon^{-2}$, the sample ratio will be at least $18 \epsilon^{-2}$.

Let $E_3'$ be the event that the above sample ratio is sufficiently accurate. We know that $\mathbb{P}(E_3') \geq 1 - O(\epsilon^6)$. Now, assuming $E_3'$, if our estimate is not in the range $[0.062^2, 18 \epsilon^{-2}]$, we output the sample ratio as our estimate $Y$, and we know that the true ratio $\frac{Q(k)}{Q(j)}$ is not in the range $[0.12^2, 10 \epsilon^{-2}]$. Otherwise, we know that our sample ratio, which we will call $\alpha$, is accurate up to a factor of $1 \pm 0.1$, and that $\frac{Q(k)}{Q(j)} \in [0.052 \epsilon^2, 20 \epsilon^{-2}]$.

Now, consider the following algorithm. If $\alpha \geq 1$, we create random variables $X_1, \ldots, X_T$ where $T = O(\log^2 \epsilon^{-1} \cdot \delta^{-2})$. For each $1 \leq t \leq T$, we create $X_t$ by sampling from $Cond(\{j, k\})$ until $k$ is returned, and letting $X_t$ be the number of times we saw $j$ returned before $k$ was returned. We know this is a Geometric random variable with parameter $p = \frac{Q(j)}{Q(j) + Q(k)}$ and thus has mean $\frac{Q(j)}{Q(j) + Q(k)}$. We will truncate this random variable at $O(\alpha \cdot \log \epsilon^{-1})$, i.e., we will really let $X_t = \min \left(C \alpha \cdot \log \epsilon^{-1}, Geo(\frac{Q(j)}{Q(j) + Q(k)})\right)$ for some large constant $C$. That way, since $\alpha$ is within a $1 \pm 0.1$ factor of $\frac{Q(k)}{Q(j)}$, we will still have $E[X_t] = \frac{Q(k)}{Q(j)} \cdot (1 \pm \epsilon)$, conditioned on $E_3'$. However, the
Chernoff bound tells us that \( Y := \frac{1}{T} (X_1 + X_2 + \cdots + X_T) \in [1 - \frac{\delta}{2}, 1 + \frac{\delta}{2}] \cdot E[X_i] \) with probability at least \( 1 - \varepsilon^6 \), conditioned on \( E_3 \). Moreover, the number of calls to \( \text{COND}_Q \) in expectation is \( O(\alpha \cdot T) = O(\varepsilon^{-2} \log^2 \varepsilon^{-1} \cdot \delta^{-2}) \).

Likewise, if \( \alpha < 1 \), we create random variables \( X_1, \ldots, X_T \) where \( T = O(\alpha^{-1} \cdot \log \varepsilon^{-1} \cdot \delta^{-2}) \). For each \( 1 \leq t \leq T \), we create \( X_t \) by sampling from \( \text{COND} \{j, k\} \) until \( k \) is returned, and letting \( X_t \) be the number of times we saw \( k \) returned before \( j \) was returned, but we also truncate this random variable at \( O(\log \varepsilon^{-1}) \). Thus, \( X_t = \min \left( C \cdot \log \varepsilon^{-1}, \text{Geom} \left( \frac{Q(j)}{Q(U) + Q(k)} \right) \right) \) for some large constant \( C \). Since \( \alpha \leq 1 \), this means \( \frac{Q(k)}{Q(U)} \leq 1.2 \), so we still have \( E[X_t] = \frac{Q(k)}{Q(U)} \cdot (1 \pm \varepsilon) \), conditioned on \( E_3' \). However, the Chernoff bound tells us that \( Y := \frac{1}{T} (X_1 + X_2 + \cdots + X_T) \in [1 - \frac{\delta}{2}, 1 + \frac{\delta}{2}] \cdot E[X_i] \) with probability at least \( 1 - \varepsilon^6 \), conditioned on \( E_3' \). Moreover, the expected number of calls to \( \text{COND}_Q \) is \( O(T) = O(\varepsilon^{-2} \log^2 \varepsilon^{-1} \cdot \delta^{-2}) \).

Finally, let \( E_3 \) be the event that \( E_3' \) is true and that if our initial estimate \( \alpha \) is in the range \( [0.06 \varepsilon^2, 18 \varepsilon^{-2}] \), then \( Y \in [1 - \frac{\delta}{2}, 1 + \frac{\delta}{2}] \cdot E[X_i | E_3'] \). Clearly, \( \mathbb{P}(E_3) \geq 1 - O(\varepsilon^6) \). If we condition on \( E_3 \) instead of \( E_3' \) we trivially achieve all the guarantees by our previous analysis, except possibly the bound on \( \mathbb{E}[Y | E_3] \). However, note that \( Y \) is uniformly bounded by \( O(\varepsilon^{-2} \log \varepsilon^{-1}) \) assuming \( E_3' \) and \( \mathbb{E}[Y | E_3'] = \Omega(\varepsilon^2) \), so conditioning on \( E_3 \) instead of \( E_3' \), where \( \mathbb{P}(E_3 | E_3') \geq 1 - O(\varepsilon^6) \), can only change the expectation of \( Y \) by a \( 1 \pm \varepsilon \) multiplicative error. Thus, \( \mathbb{E}[Y | E_3] = (1 \pm O(\varepsilon)) \cdot \frac{Q(k)}{Q(U)} \).

Next, we show how to combine Lemmas 5.1, 5.2 and 5.3 to get a good estimator for \( c_1 \cdot \frac{P(z)}{P(z)} \).

**Lemma 5.4.** Suppose \( z \geq L \) and \( \varepsilon \leq \delta \leq \frac{1}{10} \) are fixed. Then, there is an algorithm \( \text{EST} \) using \( O(\varepsilon^{-2} \log^5 \varepsilon^{-1} \cdot \delta^2) \) queries to \( \text{COND}_P \) and \( \text{COND}_Q \), such that conditioned on some event \( E_4 := E_4(z, \delta) \) with \( \mathbb{P}(E_4) \geq 1 - O(\varepsilon^6) \), the following happens.

1. If \( \sum_{i \leq z} \min(c_1 P(i), c_2 P^*(i)) > 6 \varepsilon \), then assuming \( E_4 \):

   - (a) If \( c_1 \cdot \frac{P(z)}{P(z)} < \varepsilon \), we return an estimator \( X := X(z, \delta) \) such that \( X \leq 2 \varepsilon \).
   - (b) If \( c_1 \cdot \frac{P(z)}{P(z)} > \frac{5}{3} \), we return an estimator \( X \) such that \( X \geq \frac{3}{\varepsilon} \).
   - (c) If \( \varepsilon \leq c_1 \cdot \frac{P(z)}{P(z)} \leq \frac{5}{3} \), we return an estimator \( X \) such that \( X \in [1 - 4 \delta, 1 + 4 \delta] \cdot \frac{P(z)}{P(z)} \) and \( \mathbb{E}[X | E_4] \in [1 - 4 \varepsilon, 1 + 4 \varepsilon] \cdot \frac{P(z)}{P(z)} \).

2. If \( \sum_{i \leq z} \min(c_1 P(i), c_2 P^*(i)) \leq 6 \varepsilon \), then assuming \( E_4 \), we either will return \([z, 0]\) or return some \( X \) with the same guarantees as above.

**Proof.** We use Lemma 5.1 to find an estimate \( \hat{P}([z]) \), then Lemma 5.2 to find some pair \( (j, \hat{Q}(j)) \), and then Lemma 5.3 to find \( Y \) (assuming Lemma 5.2 did not return NULL). We let \( E_4 \) indicate that \( E_1 \) from Lemma 5.1, \( E_2 \) from Lemma 5.2 and \( E_3 \) from Lemma 5.3 (assuming Lemma 5.2 didn’t return NULL) are all true. Clearly, \( \mathbb{P}(E_4) \geq 1 - O(\varepsilon^6) \).

Now, suppose Lemma 5.1 returns \( \hat{P}([z]) \leq \frac{\varepsilon}{c_1} \). Then, we know that \( P([z]) \leq 2 \frac{\varepsilon}{c_1} \), which means that

\[
\sum_{i \leq z} \min(c_1 P(i), c_2 P^*(i)) \leq \sum_{i \leq z} c_1 P(i) \leq c_1 \cdot 2 \frac{\varepsilon}{c_1} = 2 \varepsilon.
\]

Therefore, we can output \( S = [z] \) and our estimate as 0, i.e., we output \(([z], 0)\).
Otherwise, suppose that Lemma 5.2 returns NULL. Then, we have that $d_{TV}(Q, U) \geq 1 - 3\varepsilon$, so by Proposition 2.1, $\sum_{j=1}^{k} \min \left( \frac{1}{k}, Q(j) \right) \leq 3\varepsilon$. Therefore,

$$\sum_{i \leq \varepsilon} \min(c_1 P(i), c_2 P^*(i)) \leq \sum_{i \leq \varepsilon} \min (P(i), P^*(i)) \leq \sum_{j \leq k} \min \left( Q(j), Q^*(j) \right) \leq \sum_{j \leq k} \min \left( Q(j) \frac{2}{k} \right) \leq 6\varepsilon,$$

where we used the facts that $c_1, c_2 \leq 1$, $\sum_{i \in S_j} P(i) = \frac{Q(j)}{P([z]])} \leq Q(j)$, $\sum_{i \in S_j} P^*(i) = \frac{Q^*(j)}{P^*([z]])} \leq Q^*(j)$, and $Q^*(j) \leq \frac{2}{k}$ for all $1 \leq j \leq k$, due to how we chose the sets $S_1, \ldots, S_{k-1}, S_k$. Therefore, we can again output $S = [z]$ and our estimate as 0, i.e., we output $([z], 0)$.

Otherwise, suppose that Lemma 5.2 returns NULL. Then, we have that $\sum_{j \leq k} \min \left( Q(j) \frac{2}{k} \right) \leq 6\varepsilon$. Finally, if $\frac{Q(k)}{Q(j)} < 0.1\varepsilon^2$, then

$$c_1 \frac{P(z)}{P^*(z)} = c_1 \cdot \frac{1}{P^*(z)} \cdot P([z]) \cdot Q(j) \cdot \frac{Q(k)}{Q(j)} \leq c_1 \cdot 4k \cdot 1 \cdot \frac{3}{2} \varepsilon^{-1} \cdot 0.1\varepsilon^2 \leq 0.6\varepsilon,$$

and conditioned on $E_4$,

$$X = c_1 \cdot \frac{1}{P^*(z)} \cdot \tilde{P}([z]) \cdot \tilde{Q}(j) \cdot Y \leq c_1 \cdot 4k \cdot 1 \cdot (1 + \delta) \cdot \frac{3}{2} \varepsilon^{-1} \cdot 0.2\varepsilon^2 \leq 2\varepsilon,$$

for $\delta \leq \frac{1}{10}$. Likewise, if $\frac{Q(k)}{Q(j)} > 10\varepsilon^2$, then

$$c_1 \frac{P(z)}{P^*(z)} = c_1 \cdot \frac{1}{P^*(z)} \cdot P([z]) \cdot Q(j) \cdot \frac{Q(k)}{Q(j)} \geq c_1 \cdot \frac{k}{2c_1} \cdot \frac{2}{3} \frac{\varepsilon}{k} \cdot 10\varepsilon^{-2} \geq \frac{5}{3},$$

and conditioned on $E_4$,

$$X = c_1 \cdot \frac{1}{P^*(z)} \cdot \tilde{P}([z]) \cdot \tilde{Q}(j) \cdot Y \geq c_1 \cdot \frac{k}{2c_1} \cdot \frac{\varepsilon}{c_1} \cdot (1 - \delta) \cdot \frac{2}{3} \frac{\varepsilon}{k} \cdot 5\varepsilon^{-2} \geq 1.5,$$

for $\delta \leq \frac{1}{10}$.

Finally, if $\frac{Q(k)}{Q(j)} \in [0.1\varepsilon^2, 10\varepsilon^{-2}]$, then conditioned on $E_4$, $\tilde{P}([z]) \in [1 - \delta, 1 + \delta] \cdot P([z])$, $\tilde{Q}(j) \in [1 - \delta, 1 + \delta] \cdot Q(j)$, and $Y \in [1 - \delta, 1 + \delta] \cdot \frac{Q(k)}{Q(j)}$. Moreover, conditioned on $E_4, E_2, E_3$, we have that $\mathbb{E}[\tilde{P}(z)] \in [1 - \varepsilon, 1 + \varepsilon] \cdot P([z])$, $\mathbb{E}[\tilde{Q}(j)] \in [1 - \varepsilon, 1 + \varepsilon] \cdot Q(j)$, and $\mathbb{E}[Y] \in [1 - \varepsilon, 1 + \varepsilon] \cdot \frac{Q(k)}{Q(j)}$. Due to using independent samples, this means that $X \in [1 - 4\delta, 1 + 4\delta] \cdot c_1 \cdot \frac{P(i)}{P^*(i)}$, and $\mathbb{E}[X_{z, \delta}|E_1, E_2, E_3] \in [1 - 4\varepsilon, 1 + 4\varepsilon] \cdot c_1 \cdot \frac{P(i)}{P^*(i)}$.

Overall, assuming we have not already returned $([z], 0)$ we have that, assuming $E_4$, if $c_1 \cdot \frac{P(z)}{P^*(z)} \geq \frac{5}{3}$, we will output an estimate $X \geq \frac{2}{3}$, if $c_1 \cdot \frac{P(z)}{P^*(z)} \leq \varepsilon$, then $X \leq 2\varepsilon$, and otherwise, $X \in [1 - 4\delta, 1 + 4\delta] \cdot c_1 \cdot \frac{P(z)}{P^*(z)}$ and $\mathbb{E}[X|E_4] \in [1 - 4\varepsilon, 1 + 4\varepsilon] \cdot c_1 \cdot \frac{P(z)}{P^*(z)}$. □
We are now ready to finish the proof of Theorem 5.1. The rest of the proof will be similar to that of Lemmas 4.2 and 4.3.

**Proof of Theorem 5.1.** Fix \(L \leq z \leq M\), where \(L\) is the smallest integer such that \(\mathbb{P}_{x \sim P^\ast}(x \leq L) \geq \frac{1}{4}\), and \(T\) such that \(2^{-T} = c\varepsilon\) for some small constant \(c\). We show a method for estimating \(c_1 \cdot \frac{P(z)}{P^\ast(z)}\) and create indicator random variables \(I_{+,t}(z), I_{-,t}(z)\) to go along with this for each \(1 \leq t \leq T - 1\), as well as \(I_T(z)\). Recall that Lemma 5.4 conditioned on some event \(E_4 =: E_4(z, \delta)\) occurring with probability \(1 - O(\varepsilon^6)\), either outputs an estimate \(X(z, \delta)\) that is a \(1 \pm \delta\) multiplicative approximation of \(c_1 \cdot \frac{P(z)}{P^\ast(z)}\), or returns \(([z], 0)\), meaning that \(\sum_{i \leq \delta} \min(c_1 P(i), c_2 P^\ast(i)) \leq 6\varepsilon\). We set \(\delta_i = \frac{1}{20} \cdot 2^{-i}\), beginning with \(i = 1\). At any step, we compute \(X(z, \delta_i)\) using Lemma 5.4 and check if \(X(z, \delta_i) \in [(1 - 2^{-i})c_2, (1 + 2^{-i})c_2]\). If so, we increment \(i\). We repeat this process of incrementing \(i\) (i.e., dividing \(\delta\) by 2) and running Lemma 5.4 until one of the following three things occur: either \(X(z, \delta_i) \notin [(1 - 2^{-i})c_2, (1 + 2^{-i})c_2]\), or \(([z], 0)\) is returned instead of an estimate \(X\), or \(i \geq T\). If \(([z], 0)\) is ever returned, we will simply output \(S = [z]\) and \(\sum_{x \in S} \min(c_1 P(x), c_2 P^\ast(x)) = 0\). Else, for some \(1 \leq t \leq T - 1\), if \(t\) is the first value such that \(X(z, \delta_i) \notin [(1 - 2^{-t})c_2, (1 + 2^{-t})c_2]\), then \(X(z, \delta_t) \leq (1 - 2^{-t})c_2\), then we set \(I_{-,t}(z) := 1\) and \(X(z, \delta_t) \geq (1 + 2^{-t})c_2\), then we set \(I_{+,t}(z) := 1\). However, if we reach \(i = T\), we just set \(I_T(z) = 1\). All variables not set to 1 will be 0.

For the rest of the proof, we implicitly condition on the events \(E_4 = E_4(z; \delta)\) being true for every call to the algorithm of Lemma 5.4. We will only make \(O(\varepsilon^2)\) calls to the algorithm 5.4 and as \(\mathbb{P}(E_4) \geq 1 - \varepsilon^6\), the event we are conditioning on will happen with probability \(1 - O(\varepsilon^3)\).

Now, when running the above procedure on some element \(z\), we will always have that exactly one of the indicator variables is 1 (unless \([z], 0\) is returned). If \(c_1 \cdot \frac{P(z)}{P^\ast(z)} \geq c_2\), the nonzero indicator is either \(I_{+,t}(z)\) for some \(t\) or \(I_T(z)\). Next, if \(c_1 \cdot \frac{P(z)}{P^\ast(z)} = c_2(1 - \gamma)\) for some \(\gamma \geq \varepsilon\), then exactly one value \(I_{-,t}\) will be nonzero for some \(t = \log_2 \gamma^{-1} + O(1)\). Finally, if \(c_1(1 - \varepsilon) \leq c_1 \cdot \frac{P(z)}{P^\ast(z)} \leq c_2\), either \(I_T(z)\) or some \(I_{-,t}(z)\) will be nonzero: in the latter case, \(t = \log_2 \varepsilon^{-1} + O(1)\). Define \(q_{+,t}(z) := \mathbb{P}(I_{+,t}(z))\), \(q_{-,t}(z) := \mathbb{P}(I_{-,t}(z))\), and \(q_T(z) := \mathbb{P}(I_T(z))\), where we implicitly condition on \(E_4\) being true for all calls to the algorithm of Lemma 5.4 and that \(([z], 0)\) is never returned by the algorithm of Lemma 5.4. Therefore, if \(c_1 \cdot \frac{P(z)}{P^\ast(z)} \leq c_2\), then \(\sum_{t=1}^{T-1} q_{+,t}(z) = 0\); and if \(c_1 \cdot \frac{P(z)}{P^\ast(z)} \geq c_2\), then \(\sum_{t=1}^{T-1} q_{-,t}(z) = 0\). This means that for any \(L \leq z \leq M\),

\[
\min \left( c_1 \cdot \frac{P(z)}{P^\ast(z)}, c_2 \right) = \sum_{t=1}^{T-1} q_{+,t}(z) \cdot c_2 + q_T(z) \cdot \min \left( c_1 \cdot \frac{P(z)}{P^\ast(z)}, c_2 \right) + \sum_{t=1}^{T-1} q_{-,t}(z) \cdot \left( c_2 - c_1 \cdot \frac{P(z)}{P^\ast(z)} \right) = c_2 + O(\varepsilon) - \sum_{t=1}^{T-1} q_{-,t}(z) \cdot \left( c_2 - c_1 \cdot \frac{P(z)}{P^\ast(z)} \right),
\]

where we used the fact that \(\sum_{t=1}^{T-1} q_{+,t}(z) + q_T(z) + \sum_{t=1}^{T-1} q_{-,t}(z)\), and if \(q_T(z) > 0\) then \(c_1 \cdot \frac{P(z)}{P^\ast(z)} = c_2(1 \pm O(\varepsilon))\).

Since we want to compute

\[
\sum_{z=L}^{M} \min(c_1 P(z), c_2 P^\ast(z)) = \mathbb{E}_{z \sim P^\ast} \left[ \mathbb{I}(z \geq L) \min \left( c_1 \cdot \frac{P(z)}{P^\ast(z)}, c_2 \right) \right],
\]

26
where \(|z| \geq L\) is the indicator function of \(z \geq L\), it suffices to approximate

\[
\mathbb{E}_{z \sim P^*} \left[ \mathbb{I}(z \geq L) \cdot q_{-t}(z) \cdot \left( c_2 - c_1 \cdot \frac{P(z)}{P^*(z)} \right) \right] = \mathbb{E}_{z \sim P^*} \left[ \mathbb{I}(z \geq L) \cdot \mathbb{I}_{-t}(z) \cdot \left( c_2 - c_1 \cdot \frac{P(z)}{P^*(z)} \right) \right]
\]

for each \(1 \leq t \leq T - 1\), where the final expectation is over \(z\) drawn from \(P^*\) and the randomness in the algorithm determining \(\mathbb{I}_{-t}(z)\), and is implicitly conditioned on all events \(E_4\) being true and \((z, 0)\) never being returned for all calls to Lemma 5.4. To approximate this quantity, we set \(\delta := \delta_t = \frac{1}{20} \cdot 2^{-t}\) and we sample \(S = O((\delta/\varepsilon)^2 \log \varepsilon^{-1})\) samples \(z_1, z_2, \ldots, z_S \leftarrow P^*\). For each sample \(z_s\), we determine \(\mathbb{I}_{-t}(z_s)\), which can be done using \(O(\varepsilon^{-2} \log^5 \varepsilon^{-1} \cdot \delta^{-2})\) calls to \(\text{COND}_P\) (since we run Lemma 5.4 for \(\delta_1, \delta_2, \ldots, \delta_t+O(1)\) and then we can stop). If \(z_s \leq L\) or \(\mathbb{I}_{-t}(z_s) = 0\), then we can just set some variable \(Z_s = 0\). Otherwise, we again run Lemma 5.4 on \(z_s\) but with fresh randomness and with \(\delta = \delta_t\), which we do to get an estimate \(X_s\) such that \(X_s \in [1 - 4\delta, 1 + 4\delta] \cdot c_1 \cdot P(z_s)/P^*(z_s)\), and we set \(Z_s = c_2 - X_s\). Importantly, since we are using fresh randomness, if \(z_s \geq L\), then \(\mathbb{E} \left[ c_2 - c_1 \cdot \frac{P(z)}{P^*(z)} \right] \leq 1 \) and \(\mathbb{E} [X_s|z_s, \mathbb{I}_{-t}(z_s) = 1] \in [1 - 4\varepsilon, 1 + 4\varepsilon] \cdot c_1 \cdot P(z_s)/P^*(z_s)\), unless \(P(z_s)/P^*(z_s) < \varepsilon\), in which case we have \(X = c_1 \cdot P(z_s)/P^*(z_s) \pm O(\varepsilon)\) uniformly, since \(c_1 \cdot P(z_s)/P^*(z_s) \leq \varepsilon\) and \(X \leq 2\varepsilon\) always by Condition 1a) of Lemma 5.4. Therefore, \(\mathbb{E} [Z_s|z_s, \mathbb{I}_{-t}(z_s) = 1] = c_2 - c_1 \cdot P(z_s)/P^*(z_s) + O(\varepsilon)\). Moreover, since \(c_1 \cdot P(z_s)/P^*(z_s) = c_2 \cdot (1 - O(\delta))\), we always have that \(Z_s = O(\delta)\) (even if \(z_s < L\) or \(\mathbb{I}_{-t}(z_s) = 0\)). Therefore,

\[
\mathbb{E}_{z \sim P^*} \left[ \mathbb{I}(z \geq L) \cdot \mathbb{I}_{-t}(z) \cdot \left( c_2 - c_1 \cdot \frac{P(z)}{P^*(z)} \right) \right] = \mathbb{E}[Z_s] + O(\varepsilon)
\]

and \(Z_s\) is uniformly bounded in magnitude by \(O(\delta)\). Therefore, by sampling \(S = O((\delta/\varepsilon)^2 \log \varepsilon^{-1})\) samples of \(z_1, \ldots, z_S\) and computing \(Z_s\) for each \(1 \leq s \leq S\), with probability at least \(1 - \varepsilon^2\), we will have that the average of the \(Z_s\)'s is within \(O(\varepsilon)\) of \(\mathbb{E}_{z \sim P^*} \left[ \mathbb{I}(z \geq L) \cdot \mathbb{I}_{-t}(z) \cdot \left( c_2 - c_1 \cdot \frac{P(z)}{P^*(z)} \right) \right]\). (The exception is if Lemma 5.4 ever returns \(([z_s], 0)\) for some sampled \(z_s\), but by our assumption that \(E_4(z_s, \delta)\) is always true, in this case we can instead return \(([z_s], 0)\).) Therefore, the overall error will be \(O(\varepsilon \log \varepsilon^{-1})\), since we need to compute this for all \(1 \leq t \leq T - 1 = O(\log \varepsilon^{-1})\). Moreover, each \(t\) will require \(O((\delta/\varepsilon)^2 \cdot \varepsilon^{-2} \log^5 \varepsilon^{-1} \cdot \delta^{-2}) = O(\varepsilon^{-4} \log^5 \varepsilon^{-1})\) queries to \(\text{COND}_P\), so the total number of queries to \(\text{COND}_P\) is \(O(\varepsilon^{-4} \log^6 \varepsilon^{-1})\).

**Proof of Theorem 7.2.** It suffices to show that we can estimate \(d_{TV}(\mathcal{D}, \mathcal{D}^*)\) up to an \(O(\varepsilon^{-1} \log^2 \varepsilon^{-1})\) additive error using \(\hat{O}(\varepsilon^{-4})\) queries, since we can then replace \(\varepsilon\) with \(\varepsilon' = \frac{\varepsilon}{\log \varepsilon^{-1}}\).

Let \(T_0 \supset T_1 \supset T_2 \supset \cdots \supset T_r\) be subsets of \([N]\) with \(T_0 = [N]\) - we will decide the remaining sets later. Also, let \(P_k\) be the distribution \(\mathcal{D}\) conditioned on being in \(T_k\), and let \(P_k^*\) be the distribution \(\mathcal{D}^*\) conditioned on being in \(T_k\). Also, let \(c_{1,k} = D(T_k)\) and \(c_{2,k} = D^*(T_k)\). Importantly, note that \(P_0 = \mathcal{D}, P_0^* = \mathcal{D}^*\), and for any \(i \in T_k\), \(c_{1,k}P_k(i) = D(i)\) and \(c_{2,k}P_k^*(i) = D^*(i)\).

Now, our algorithm proceeds as follows. Suppose we have determined \(T_k\). Then, we can compute \(c_{2,k}\) as \(\sum_{i \in T_k} D^*(i)\), and we can estimate \(c_{1,k}\) up to an additive \(\varepsilon\) error using \(\hat{O}(\varepsilon^{-2})\) samples from the distribution \(\mathcal{D}\) and determining what fraction of the samples are in \(T_k\).

If either our estimate \(\hat{c}_{1,k}\) (for \(c_{1,k}\)) or \(c_{2,k}\) is less than \(2\varepsilon\), then either \(c_{1,k}\) or \(c_{2,k}\) is at most \(3\varepsilon\), so

\[
\sum_{i \in T_k} \min(D(i), D^*(i)) \leq 3\varepsilon.
\]
In this case, set $S = T_k$. Else, $c_{1,k}, \tilde{c}_{1,k}, c_{2,k} \geq \varepsilon$, so we can find a set $S \subset T_k$ such that $D^*(S) \geq \frac{D^*(T_k)}{3}$ and determine

$$\sum_{i \in S} \min(\tilde{c}_{1,k}P_k(i), c_{2,k}P_k^*(i))$$

up to an additive $O(\varepsilon \log \varepsilon^{-1})$ error by Theorem 5.1 since we can conditionally sample subsets in $T_k$. But noting that $\sum_{i \in S} P_k(i), \sum_{i \in S} P_k^*(i) \leq 1$ and that $|\tilde{c}_{1,k} - c_{1,k}| \leq \varepsilon$, we therefore have that our estimate is an $O(\varepsilon \log \varepsilon^{-1})$ additive error approximation to

$$\sum_{i \in S} \min(c_{1,k}P_k(i), c_{2,k}P_k^*(i)) = \sum_{i \in S} \min(D(i), D^*(i)).$$

We now let $T_{k+1} = T_k \setminus S$ for each $k$. At each stage, we can determine $\sum_{i \in T_k \setminus T_k} \min(D(i), D^*(i))$ up to an additive $O(\varepsilon \log \varepsilon^{-1})$ error. Moreover, $D^*(T_{k+1}) \leq \frac{2}{3}D^*(T_k)$, so this process will only continue $O(\log \varepsilon^{-1})$ times until we reach some $r$ such that either $D(T_r) \leq \varepsilon$ or $D^*(T_r) \leq O(\varepsilon)$, in which case we can just estimate $\sum_{i \in T_r} \min(D(i), D^*(i))$ as 0. Therefore, by adding our estimates for $\sum_{i \in T_k \setminus T_{k+1}} \min(\tilde{c}_{1,k}P_k(i), \tilde{c}_{2,k}P_k^*(i))$ for all $0 \leq k \leq r$ and $T_{r+1} = \emptyset$, we get an $O(\varepsilon \log^2 \varepsilon^{-1})$ additive error for $\sum_{i=1}^N \min(D(i), D^*(i))$, which equals $1 - d_{TV}(D, D^*)$ by Proposition 2.1.

6. Identity Testing in the PAIRCOND model

In this section, we prove Theorem 1.3. In other words, we show that in the PAIRCOND model, we can test whether an unknown distribution $\mathcal{D}$ equals $\mathcal{D}^*$ or is $\varepsilon$-far in Total Variation Distance from $\mathcal{D}^*$, using $\tilde{O}\left(\frac{\sqrt{\log n}}{\varepsilon^2}\right)$ samples.

We first note the following simple proposition.

**Proposition 6.1.** Suppose $X$ is a positive random variable bounded by 1, such that $\mathbb{E}X \geq \varepsilon$. Then, there exist $\alpha = 2^{-a}, \beta = 2^{-b}$ with $a, b$ nonnegative integers such that $\alpha \geq \Theta(\varepsilon), \alpha: \beta \geq \Theta(\varepsilon/\log \varepsilon^{-1})$, and $\mathbb{P}(X \geq \alpha) \geq \beta$.

We next prove the following lemma.

**Lemma 6.1.** Suppose that $\mathcal{P}$ is an unknown probability distribution over $[m]$ with probabilities $P(1), \ldots, P(m)$, and suppose $\mathcal{P}^*$ is a known probability distribution with probabilities $P^*(1), \ldots, P^*(m)$, with $P^*(i) \in \left[\frac{1}{2m}, \frac{2}{m}\right]$. Finally, suppose that $d_{TV}(\mathcal{P}, \mathcal{P}^*) \geq \varepsilon$. Then, we have that if $i$ is drawn from $\mathcal{P}$ and $j$ is drawn uniformly from $\mathcal{U}$, the uniform distribution over $[m]$, then

$$\mathbb{E}_{i \sim \mathcal{P}, j \sim \mathcal{U}} \left| \frac{P(i)}{P(i) + P(j)} - \frac{P^*(i)}{P^*(i) + P^*(j)} \right| \geq \frac{\varepsilon}{16}.$$
Proof. First, let \( r_i = \frac{P(i)}{P^*(i)} \). Then, note that

\[
\frac{P(i)}{P(i) + P(j)} - \frac{P^*(i)}{P^*(i) + P^*(j)} = \frac{P(i)}{P(i) + P(j)} \cdot \frac{P(i) - P^*(i)(P(i) + P(j))}{P(i) + P(j) + P^*(i) + P^*(j)}
\]

Since \( \frac{P^*(i)P^*(j)}{P^*(i) + P^*(j)} \geq 1 \),

Therefore,

\[
\mathbb{E}_{i \sim \mathcal{P}, j \sim \mathcal{D}} \left| \frac{P(i)}{P(i) + P(j)} - \frac{P^*(i)}{P^*(i) + P^*(j)} \right| = \sum_{i,j=1}^{m} P(i) \cdot \frac{1}{m} \left| \frac{P(i)}{P(i) + P(j)} - \frac{P^*(i)}{P^*(i) + P^*(j)} \right|
\]

\[
\geq \frac{1}{4m^2} \sum_{i,j=1}^{m} \frac{P(i)}{P(i) + P(j)} \cdot |r_i - r_j|
\]

\[
= \frac{1}{4m^2} \sum_{1 \leq i < j \leq m} |r_i - r_j|
\]

\[
= \frac{1}{8m^2} \sum_{i,j=1}^{m} |r_i - r_j|
\]

Next, since \( \sum P(i) = \sum P^*(i) = 1 \) but \( \sum |P(i) - P^*(i)| \geq 2\varepsilon \) since \( d_{\text{TV}}(\mathcal{P}, \mathcal{P}^*) \geq \varepsilon \), we have that \( \sum_{i : r_i \geq 1} (r_i - 1) P^*(i) = \sum_{i : r_i \leq 1} (1 - r_i) P^*(i) \geq \varepsilon \). But since \( P^*(i) \leq \frac{2}{m} \) for all \( i \), this means that \( \sum_{i : r_i \geq 1} (r_i - 1) \geq m \cdot \frac{\varepsilon}{2} \) and \( \sum_{i : r_i \leq 1} (1 - r_i) \geq m \cdot \frac{\varepsilon}{2} \). Thus, for any \( r_i \), if \( r_i \geq 1 \) then

\[
\sum_{j=1}^{m} |r_i - r_j| \geq \sum_{r_j \leq 1} (r_i - r_j) \geq \sum_{r_j \leq 1} (1 - r_j) \geq \frac{\varepsilon}{2} \cdot m,
\]

and if \( r_i \leq 1 \) then

\[
\sum_{j=1}^{m} |r_i - r_j| \geq \sum_{r_j \geq 1} (r_j - r_i) \geq \sum_{r_j \geq 1} (r_j - 1) \geq \frac{\varepsilon}{2} \cdot m.
\]

Therefore,

\[
\sum_{i,j=1}^{m} |r_i - r_j| \geq m \cdot \frac{\varepsilon}{2} \cdot m \geq \frac{\varepsilon}{2} \cdot m^2,
\]

so we get the final bound of \( \frac{1}{8m^2} \cdot \frac{\varepsilon}{2} m^2 = \frac{\varepsilon}{10} \). \( \square \)

Recall we are trying to determine if \( \mathcal{D} = \mathcal{D}^* \) or \( d_{\text{TV}}(\mathcal{D}, \mathcal{D}^*) \geq \varepsilon \). The algorithm proceeds as follows. First, we split \( [N] \) into sets \( S_1, S_2, \ldots, S_{\log(10N/\varepsilon)} \) where \( i \in S_k \) if and only if \( 2^{-k} < D_i^* \leq 2 \cdot 2^{-k} \). Let \( K = \log \frac{10N}{\varepsilon} \), and define \( S_{K+1} := [n] \setminus \left( \bigcup_{i=1}^{K} S_i \right) \), so that \( S_1, \ldots, S_{K+1} \) partition \( [N] \).
We define $S$ as the distribution over $[K+1]$ with $P_{x \sim S}(x = k) := D(S_k) = P_{x \sim D}(x \in S_k)$. Likewise, we define $S^*$ as the distribution over $[K+1]$ with $P_{x \sim S^*}(x = k) := D^*(S_k) = P_{x \sim D^*}(x \in S_k)$. Also, for all $1 \leq k \leq K + 1$, we define $P_k$ as the conditional distribution of $x \sim D$ conditioned on $x \in S_k$, and $P_k^*$ as the conditional distribution of $x \sim D^*$ conditioned on $x \in S_k$. We finally define $s_k = P_{x \sim S}(x = k)$, $s_k^* = P_{x \sim S^*}(x = k)$, $P_k(i) = P_{x \sim P}(x = i)$, and $P_k^*(i) = P_{x \sim P^*}(x = i)$. (Note: we use $s_k$ and $s_k^*$ instead of $S_k$ and $S_k^*$ to avoid confusion with the sets $S_k$.)

First, we wish to compare the distributions $S$ and $S^*$. Since $S^*$ is a known distribution, we can test whether $S = S^*$ or $d_{TV}(S, S^*) \geq \frac{\varepsilon}{10}$ using $O\left(\frac{\log(n/\varepsilon)}{\varepsilon^2}\right)$ samples [], since the support size of $S$ and $S^*$ are $O(\log(n/\varepsilon))$. However, since a sample from $S$ can be simulated by a single call to SAMP for $D$, we only need $O\left(\frac{\log(n/\varepsilon)}{\varepsilon^2}\right)$ calls to SAMP. If the test tells us that $d_{TV}(S, S^*) \geq \frac{\varepsilon}{10}$ then we know $S \neq S^*$, so $D \neq D^*$ and we can output NO. Otherwise, we know that $d_{TV}(S, S^*) \leq \frac{\varepsilon}{10}$.

Assuming we haven’t yet output NO, we let $D'$ be the distribution where we draw $k$ according to $S$, and then draw $i$ according to $P_k^*$. It is simple to see that $d_{TV}(D^*, D') = d_{TV}(S^*, S) \leq \frac{\varepsilon}{10}$.

Now, suppose $X$ is a random variable constructed as follows. First, draw $k \sim S$, and draw $i \sim P_k$ and $j$ uniformly from $S_k$. Then, $X$ is defined as $\left|\frac{D(i)}{D(i) + D(j)} - \frac{D^*(i)}{D^*(i) + D^*(j)}\right|$. Note that $0 \leq X \leq 1$. Moreover,

$$
\mathbb{E}X = \mathbb{E}_{k \sim S} \mathbb{E}_{i \sim P_k} \mathbb{E}_{j \sim \text{Uniform}[S_k]} \left|\frac{D(i)}{D(i) + D(j)} - \frac{D^*(i)}{D^*(i) + D^*(j)}\right|.
$$

We next note the following proposition.

**Proposition 6.2.** We have that $\mathbb{E}X \geq \frac{1}{16} \cdot (d_{TV}(D', D) - \frac{3\varepsilon}{10})$.

**Proof.** First, we condition on $k$ for $1 \leq k \leq K$. In this case, note that for any $i, j \in S_k$, $\frac{D(i)}{D(i) + D(j)} = \frac{P_k(i)}{P_k(i) + P_k(j)}$ and $\frac{D^*(i)}{D^*(i) + D^*(j)} = \frac{P_k^*(i)}{P_k^*(i) + P_k^*(j)}$. Moreover, since $2^{-k} \leq D^*(i) \leq 2 \cdot 2^{-k}$ for all $i \in S_k$, we have that $P_k^*(i) \in \left[\frac{1}{2|S_k|}, \frac{2}{|S_k|}\right]$ for all $i \in S_k$. Therefore, by Lemma 6.1, we have that

$$
\mathbb{E}_{i \sim P_k} \mathbb{E}_{j \sim \text{Uniform}[S_k]} \left|\frac{D(i)}{D(i) + D(j)} - \frac{D^*(i)}{D^*(i) + D^*(j)}\right| \geq \frac{1}{16} \cdot d_{TV}(P_k^*, P_k)
$$

for some constant $c > 0$. Taking the expected value over $k$, we get that

$$
\mathbb{E}X \geq \sum_{k=1}^{K+1} \frac{1}{16} \cdot s_k \cdot d_{TV}(P_k^*, P_k) \geq \frac{1}{16} \cdot \left(\sum_{k=1}^{K+1} s_k \cdot d_{TV}(P_k^*, P_k)\right) - \frac{1}{16} \cdot s_{K+1},
$$

since $d_{TV}$ is always in the range $[0, 1]$. Now, note that $\sum_{k=1}^{K+1} s_k \cdot d_{TV}(P_k^*, P_k) = d_{TV}(D', D)$ by definition of $D'$. Also, $s_{K+1} \leq \frac{\varepsilon}{10}$, since $|S_{K+1}| \leq n$ and each element $i \in S_{K+1}$ satisfies $P^*(i) \leq \frac{\varepsilon}{10}$. But since $d_{TV}(S, S^*) \leq \frac{\varepsilon}{10}$, this means that $s_{K+1} \leq \frac{\varepsilon}{10} + 2 \cdot \frac{\varepsilon}{10} \leq \frac{3\varepsilon}{10}$. Thus, we have that $\mathbb{E}X \geq \frac{1}{16} \cdot (d_{TV}(D', D) - \frac{3\varepsilon}{10})$, as desired. \qed

Therefore, if $d_{TV}(D, D^*) \geq \varepsilon$, we have that $d_{TV}(D', D) \geq \frac{9\varepsilon}{10}$ by the triangle inequality, so $\mathbb{E}X \geq \frac{\varepsilon}{30}$ by Proposition 6.2. Therefore, by Proposition 6.1 there exist constants $\alpha = 2^{-a}, \beta = 2^{-b}$
for \(a, b\) nonnegative integers, such that \(\alpha = \Omega(\varepsilon), \alpha \cdot \beta = \Omega\left(\frac{\varepsilon}{\log 1/\varepsilon}\right)\) and \(\mathbb{P}(X \geq \alpha) \geq \beta\). However, if \(D = D'\), then \(X\) is uniformly 0.

Now, for each pair \((\alpha, \beta)\), let \(R = O(\beta^{-1} \cdot \log \varepsilon^{-1})\), and we choose \(R\) samples \(i_1, i_2, \ldots, i_R\) from the distribution \(D\). For each \(i_r\) for \(1 \leq r \leq R\), we let \(k_r\) denote the index of the set \(S_{k_r}\) that contains \(i_r\) and draw \(j_r\) uniformly from \(S_{k_r}\). This is indeed equivalent to drawing \(k_r \sim S, i_r \sim P_{k_r},\) and \(j_r \sim Unif[S_{k_r}]\). Therefore, if \(d_{TV}(D, D') \geq \varepsilon\), with at least \(9/10\) probability, some \(\alpha, \beta, i_r, j_r\) satisfies

\[
\left| \frac{D(i_r)}{D(i_r) + D(j_r)} - \frac{D^*(i_r)}{D^*(i_r) + D^*(j_r)} \right| \geq \alpha.
\]

However, if \(D = D^*\), then for all \(\alpha, \beta, i_r, j_r\), we have \(\frac{D(i_r)}{D(i_r) + D(j_r)} - \frac{D^*(i_r)}{D^*(i_r) + D^*(j_r)} = 0\).

Now, for each \((\alpha, \beta)\) and \(1 \leq r \leq R\), we use \(O(\alpha^{-2} \log \varepsilon^{-1})\) Pairwise Conditional Samples from \(D_{i_r, j_r}\) to determine \(\frac{D(i_r)}{D(i_r) + D(j_r)}\) up to an \(\frac{\alpha}{3}\) additive factor. If \(d_{TV}(D, D^*) \geq \varepsilon\), then there will be some \((\alpha, \beta)\) and some \(r \leq R\) such that our estimate for \(\frac{D(i_r)}{D(i_r) + D(j_r)}\) differs from \(\frac{D^*(i_r)}{D^*(i_r) + D^*(j_r)}\) by at least \(\frac{\alpha}{3}\) with probability at least \(1 - \varepsilon^{10}\) by the Chernoff bound. However, if \(D = D^*\), then for all \((\alpha, \beta)\) and all \(r \leq R\), our estimate for \(\frac{D(i_r)}{D(i_r) + D(j_r)}\) differs from \(\frac{D^*(i_r)}{D^*(i_r) + D^*(j_r)}\) by at most \(\frac{\alpha}{3}\) with probability at least \(1 - \varepsilon^{10}\) by the Chernoff bound. Therefore, the total number of samples from the total distribution and from Pair conditional samples is at most

\[
\sum_{(\alpha, \beta)} O(\alpha^{-2} \log \varepsilon^{-1} \cdot \beta^{-1} \cdot \log \varepsilon^{-1}) = O(\log^2 \varepsilon^{-1}) \cdot \sum_{(\alpha, \beta)} \alpha^{-2} \beta^{-1}.
\]

But since we only need to look at \(\alpha = \Omega(\varepsilon), \alpha \cdot \beta = \Omega(\varepsilon / \log \varepsilon^{-1})\), and \(\alpha, \beta\) as negative powers of 2, the sum on the right hand side is \(O(\varepsilon^{-2} \cdot \log \varepsilon^{-1})\), so the total number of calls to \text{SAMP} and \text{PCOND} in this step is at most \(O(\varepsilon^{-2} \cdot \log^3 \varepsilon^{-1})\). Adding this to the initial \(O\left(\frac{\sqrt{\log(n/\varepsilon)}}{\varepsilon^2}\right)\) calls to \text{SAMP}, we have made a total of \(\tilde{O}\left(\frac{\sqrt{\log n}}{\varepsilon^2}\right)\) queries.

**Acknowledgments**

I would like to thank Piotr Indyk for many helpful discussions, reading and editing (several) drafts of this paper, and pointing me to useful references. I would also like to thank Ted Pyne for reading and editing a draft of this paper. I would also like to thank Clément Canonne for answering some questions about the state of the art in the conditional sampling model. Finally, I would like to thank Ronitt Rubinfeld, Talya Eden, Sandeep Silwal, and Tal Wagner for helpful discussions.

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A Pseudocode

A.1 Algorithms in Section 2

In this subsection, we write the pseudocode for Proposition 2.3.

Algorithm 1 Proposition 2.3: Compares the probabilities of two given elements $x, y$.

\begin{algorithm}
1: \textbf{procedure} PairComp($x, y, \gamma$) \hfill \triangleright \text{Will estimate } P(x)/P(y) \text{ with } O(\gamma^{-2}) \text{ repetitions}
2: \hspace{1em} c = 0
3: \hspace{1em} k = O(\gamma^{-2} \log \epsilon^{-1})
4: \hspace{1em} \textbf{for } i = 1 \text{ to } k \text{ do}
5: \hspace{2em} z = \text{Pcond}(x, y)
6: \hspace{2em} \textbf{if } z = x \text{ then}
7: \hspace{3em} c \leftarrow c + 1
8: \hspace{1em} \textbf{return } \frac{c}{k-c} \hfill \triangleright \text{If } k - c = 0 \text{ then return } \infty
\end{algorithm}
A.2 Algorithms for Section 4

In this subsection, we write the pseudocode for all algorithms in Section 4 leading to the final algorithm for Theorem 1.1. We assume Samp and Pcond access to $D$ and that we already know the size $N$. Recall that $D$ is a distribution over $[N]$ and we are trying to determine $d_{TV}(D, U)$ where $U$ is uniform over $[N]$.

**Algorithm 2** Lemma 4.1 Determines the probability of a given element $x$, based on the oracle.

1: procedure SINGLEELEMENT($\varepsilon, x, \hat{D}(x)$) $\triangleright$ The oracle $O$ returns ACCEPT on $z$ if and only if $\text{ORACLE}(\varepsilon, x, \hat{D}(x), z) = 0$. The $\hat{D}(x)$ estimate comes from Lemma 4.5, and will be a weaker estimate than $\hat{\hat{D}}(x)$. See Algorithm 4.6 for the $\text{ORACLE}$ procedure.

2: $K = O(\varepsilon^{-2})$
3: for $k = 1$ to $K$ do
4: $y_k \leftarrow U$
5: $z_k \leftarrow D$
6: $\hat{\gamma}_1, \hat{\gamma}_2, \gamma_3 = 0$
7: for $k = 1$ to $K$ do
8: if $\text{ORACLE}(\varepsilon, x, \hat{D}(x), y_k) = 0$ then
9: $\hat{\gamma}_1 \leftarrow \hat{\gamma}_1 + \frac{1}{K}$
10: while Pcond($x, y_k$) = $y_k$ do
11: $\gamma_3 \leftarrow \gamma_3 + \frac{1}{K}$
12: if $\text{ORACLE}(\varepsilon, x, \hat{D}(x), z_k) = 0$ then
13: $\hat{\gamma}_2 \leftarrow \hat{\gamma}_2 + \frac{1}{K}$
14: Return $\hat{\gamma}_1, \frac{\hat{\gamma}_2}{\gamma_3}$ $\triangleright \frac{\hat{\gamma}_2}{\gamma_3}$ is our improved estimate $\hat{D}(x)$.

**Algorithm 3** Lemma 4.2 Estimates the probability of an element $z$ based on distance to $\frac{1}{N}$.

1: procedure ZESTIMATE($\beta, x, \hat{D}(x), z$)
2: Initialize $\hat{D}(z)$
3: for $i = 1$ to $\log_2 \beta^{-1} + O(1)$ do
4: $\alpha = \text{PAIRCOMP}(z, x, 2^{-i}/20)$
5: $\hat{D}(z) = \alpha \cdot \hat{D}(x)$
6: if $\frac{1}{N} \cdot (1 - 2^{-i}) > \hat{D}(z)$ or $\frac{1}{N} \cdot (1 + 2^{-i}) < \hat{D}(z)$ then
7: Return $\hat{D}(z)$
8: Return $\hat{D}(z)$
Algorithm 4 Lemma\[4.3\] Estimates average distance to $\frac{1}{N}$ among elements accepted by the oracle.

1: procedure \textsc{EstimateCloseTerms}($\varepsilon, x, \hat{D}(x)$)
2: \hspace{1em} $\tilde{\gamma}_1, \tilde{D}(x) = \textsc{SingleElement}($\varepsilon, x, \hat{D}(x)$)
3: \hspace{1em} $T = \log_2(\tilde{\gamma}_1/\varepsilon) + O(1)$
4: \hspace{1em} for $t = 1$ to $T - 1$ do
5: \hspace{2em} $W_{+,t} = 0$
6: \hspace{2em} $\delta = 2^{-(t+1)}$
7: \hspace{2em} $C = O((\delta/\varepsilon)^2 \cdot \log \varepsilon^{-1})$
8: \hspace{1em} for $i = 1$ to $C$ do
9: \hspace{2em} $z \sim U$
10: \hspace{2em} if $\textsc{Oracle}(\varepsilon, x, \hat{D}(x), z) = 0$ and $\textsc{ZEstimate}(c \cdot \delta, x, \tilde{D}(x), z) \in \left[\frac{1+\delta}{N}, \frac{1+2\delta}{N}\right]$ then ▷
11: \hspace{2em} $C' = O(\delta^{-2})$
12: \hspace{2em} $Y = 0$
13: \hspace{2em} for $j = 1$ to $C'$ do
14: \hspace{3em} while $\textsc{PCond}(z, x) = z$ do
15: \hspace{4em} $Y \leftarrow Y + \frac{\delta}{C}$
16: \hspace{4em} $W_{+,t} \leftarrow W_{+,t} + \frac{N \cdot \tilde{D}(x) \cdot Y - 1}{C}$
17: Similarly create $W_{-,1}, \ldots, W_{-,T-1}, W_T$.
18: Return $\sum_{t=1}^{T-1} W_{+,t} + \sum_{t=1}^{T-1} W_{-,t} + W_T$

Algorithm 5 Lemma\[4.5\] Determines the probability of a set of elements up to a $1 \pm 0.1$ factor.

1: procedure \textsc{ConstantApprox}($\varepsilon$)
2: \hspace{1em} $R = O(\varepsilon^{-1} \log^2 \varepsilon^{-1})$
3: \hspace{1em} for $r = 1$ to $R$ do
4: \hspace{2em} $x_r \leftarrow D$
5: \hspace{2em} $w_r \leftarrow D$
6: \hspace{2em} $y_r \leftarrow U$
7: \hspace{1em} for $r = 1$ to $R$ do
8: \hspace{2em} $\tilde{d}(x_r) = 0$
9: \hspace{2em} $\tilde{u}(x_r) = 0$
10: \hspace{2em} $\tilde{D}(x_r) = \text{NULL}$
11: \hspace{1em} for $i = 1$ to $R$ do
12: \hspace{2em} if $0.98 < \textsc{PairComp}(x_r, w_i, 0.01) < 1.02$ then
13: \hspace{3em} $\tilde{d}(x_r) \leftarrow \tilde{d}(x_r) + \frac{\varepsilon}{R}$
14: \hspace{2em} if $0.98 < \textsc{PairComp}(x_r, y_i, 0.01) < 1.02$ then
15: \hspace{3em} $\tilde{u}(x_r) \leftarrow \tilde{U}(x_r) + \frac{\varepsilon}{R}$
16: \hspace{2em} if $\tilde{d}(x_r) > \frac{\varepsilon}{250 \log \varepsilon^{-1}}$ and $\tilde{u}(x_r) > \frac{\varepsilon}{250 \log \varepsilon^{-1}}$ then
17: \hspace{3em} $\tilde{D}(x_r) = \frac{1}{N} \cdot \frac{\tilde{d}(x_r)}{\tilde{u}(x_r)}$
18: Return $\{ (x_r, \tilde{D}(x_r)) : \tilde{D}(x_r) \neq \text{NULL}, \tilde{D}(x_r) \in \left[\frac{0.98 \varepsilon}{N}, \frac{1.02 \varepsilon^{-1}}{N}\right] \}$
Algorithm 6 Lemma 4.6: Generates the oracle used in Subsection 4.1

1: procedure Oracle(\(\varepsilon, x, \hat{D}(x), z\)) \(\triangleright\) Oracle(\(\varepsilon, x, \hat{D}(x), z\)) is the same procedure as \(O'(z)\). We assume that \(\hat{D}(x) \in \left[\frac{5}{9N}, \frac{9}{5N}\right]\).
2: \(\alpha = \text{PairComp}(z, x, 0.01)\)
3: if \(\alpha < 0.45\) then
4: Return \(-1\)
5: else if \(\alpha \leq 2.2\) then
6: Return \(0\)
7: else
8: Return \(1\)


Algorithm 7 Lemma 4.7: Solves tolerant uniformity in COND assuming we found an element \(x\) with \(\hat{D}(x) \approx \frac{1}{N}\).

1: procedure GivenGoodElt(\(\varepsilon, x, \hat{D}(x)\)) \(\triangleright\) We assume that \(\hat{D}(x) \in \left[\frac{5}{9N}, \frac{9}{5N}\right]\)
2: \(K = O(\varepsilon^{-2})\)
3: \(a, b, c, d, e = 0\)
4: \(\tilde{\gamma}_1 = 0\) \(\triangleright\) Must verify that \(\gamma_1 \geq 10\varepsilon\) to use \text{SingleElement} properly
5: for \(i = 1\) to \(K\) do
6: \(z \leftarrow \mathcal{D}\)
7: if Oracle(\(\varepsilon, x, \hat{D}(x), z\)) = \(-1\) then
8: \(a \leftarrow a + \frac{1}{K}\)
9: else if Oracle(\(\varepsilon, x, \hat{D}(x), z\)) = \(-1\) then
10: \(c \leftarrow c + \frac{1}{K}\)
11: else
12: \(\tilde{\gamma}_1 \leftarrow \tilde{\gamma}_1 + \frac{1}{K}\)
13: \(y \leftarrow \mathcal{U}\)
14: if Oracle(\(\varepsilon, x, \hat{D}(x), y\)) = \(-1\) then
15: \(b \leftarrow b + \frac{1}{K}\)
16: else if Oracle(\(\varepsilon, x, \hat{D}(x), y\)) = \(-1\) then
17: \(d \leftarrow d + \frac{1}{K}\)
18: if \(\tilde{\gamma}_1 \geq 11 \cdot \varepsilon\) then
19: \(e = \text{EstimateCloseTerms}(\varepsilon, x, \hat{D}(x)) \triangleright\) If not, we know this value will be \(O(\varepsilon)\) so we can just have \(e = 0\)
20: Return \(0.5 \cdot (e + a - b - c + d)\)
Algorithm 8 Theorem 1.1: Main algorithm for tolerant uniformity testing in COND/PAIRCOND.

1: procedure TolerantUnif(ε) 
2:     S = ConstantApprox(ε) 
3:     if S = {} then 
4:         Return 1 
5:     else if ∃(x, ˆD(x)) ∈ S such that ˆD(x) ∈ [5/9N, 9/5N] then 
6:         Return GivenGoodElt(ε, x, ˆD(x)) 
7:     else 
8:         a, b = 0 
9:         K = O(ε⁻²) 
10:        if ∃(x, ˆD(x)) ∈ S such that ˆD(x) ≥ 1/N then 
11:           for i = 1 to K do 
12:               y ← U 
13:                   if PairComp(y, x, 0.01) ≥ 0.78 then 
14:                       a ← a + 1/R 
15:                 if ∃(x, ˆD(x)) ∈ S such that ˆD(x) ≤ 1/N then 
16:                   for i = 1 to K do 
17:                       z ← D 
18:                        if PairComp(z, x, 0.01) ≤ 1.27 then 
19:                           b ← b + 1/R 
20:      Return (1 − a − b) 

A.3 Algorithms for Section 5

In this subsection, we write the pseudocode for all algorithms in Section 5 leading to the final algorithm for Theorem 1.2. We assume COND access to D and that we already know the size N. Recall that D is a distribution over [N] and we are trying to determine d_{TV}(D, D^*) where D^* is a known distribution over [N].

Algorithm 9 Lemma 5.1: Estimates P([z]) for given z.

1: procedure Est1(ε, δ, z, P) ⋄ P is a distribution over [M] that we assume we can conditionally sample from. 
2:     R = O(ε⁻¹ · log ε⁻¹ · δ⁻²) 
3:     ˆP([z]) = 0 
4:     for i = 1 to R do 
5:         x_i ← P 
6:         if x_i ≤ z then 
7:             ˆP([z]) ← ˆP([z]) + 1/R. 
8:      Return ˆP([z])
**Algorithm 10** Lemma 5.2: Finds some $j \leq z$ along with an estimate $\hat{Q}(j)$ of $Q(j)$.

1: procedure Est2($\varepsilon, \delta, z, Q$) \texttt{⊲} $Q$ is a distribution over $[k]$ that we assume we can conditionally sample from.

2: Run Algorithm ConstantApprox with $\text{Pcond}_Q$ and $\text{Samp}_Q$ access: outputs a set $S$.

3: if $S = \emptyset$ then

4: Return NULL

5: Else $S$ is nonempty: let $j$ such that $(j, \hat{Q}(j))$ is the first pair returned in $S$.

6: $R = O(\varepsilon^{-2} \cdot \log^3 \varepsilon^{-1} \cdot \delta^{-2})$

7: $\hat{u}(j), \hat{q}(j) = 0$

8: for $i = 1$ to $R$ do

9: $y_i \leftarrow U$

10: if $\text{PairComp}(y_i, j, 0.01)_Q \in [0.98, 1.02]$ then

11: $\hat{u}(j) \leftarrow \hat{u}(j) + \frac{1}{R}$.

12: $x_i \leftarrow Q$

13: $X_i = 0$

14: if $\text{PairComp}(x_i, j, 0.01)_Q \in [0.98, 1.02]$ then

15: $\text{ctr} = 0$ \texttt{⊲} Counter to make sure we don’t call COND more than $C \log \varepsilon^{-1}$ times

16: while $\text{COND}([x_i, j])_Q = j$ and $\text{ctr} \leq O(\log \varepsilon^{-1})$ do

17: $X_i \leftarrow X_i + 1$

18: $\text{ctr} \leftarrow \text{ctr} + 1$

19: $\hat{q}(j) \leftarrow \frac{\hat{q}(j) + X_i}{R}$ \texttt{⊲} $\hat{q}_i$ is the average of the $X_i$'s

20: Return $(j, \frac{\hat{d}(j)}{k \cdot \hat{u}(j)})$
Algorithm 11 Lemma 5.3: Estimates $Q(k)/Q(j)$.

1: procedure $\text{Est3}(\varepsilon, \delta, j, Q)$ \triangleright $Q$ is a distribution over $[k]$ that we assume we can conditionally sample from.
2: $R = O(\varepsilon^{-2} \log \varepsilon^{-1})$
3: $ctr = 0$
4: for $i = 1$ to $R$ do
5:  if $\text{COND}({j, k}) = k$ then
6:     $ctr \leftarrow ctr + 1$
7: $\alpha = \frac{ctr}{R-ctr}$
8: if $\alpha < 0.06\varepsilon^2$ or $\alpha > 18\varepsilon^{-2}$ then
9:     Return $\alpha$
else
10:   \triangleright We will combine the $\alpha \geq 1$ and $\alpha < 1$ case here
11:   $Y = 0$
12:   $T = O(\max(\alpha^{-1}, 1) \cdot \log^2 \varepsilon^{-1} \cdot \delta^{-2})$
13:   for $t = 1$ to $T$ do
14:     $X_t = 0$
15:     while $\text{COND}({j, k}) = k$ do
16:         $X_t \leftarrow X_t + 1$
17:         $X_t = \min(X_t, O(\max(\alpha, 1) \cdot \log \varepsilon^{-1}))$
18:     $Y \leftarrow Y + \frac{X_t}{T}$ \triangleright $Y$ is the average of the $X_t$'s
19: Return $Y$

Algorithm 12 Lemma 5.4: Estimates either $P(z)/P^*(z)$ or tells us that $\sum_{i \leq z} \min(c_1 P(i), c_2 P^*(i))$ is small. Assumed that $P, P^*$ have support size $M$ with $P^*(1) \leq P^*(2) \leq \cdots \leq P^*(M)$.

1: procedure $\text{Est}(\varepsilon, \delta, z, P, P^*, c_1, c_2)$
2: $\tilde{P}([z]) \leftarrow \text{Est1}(\varepsilon, \delta, z, P)$
3: if $P(z) \leq \frac{\varepsilon}{c_1}$ then
4:     Return $([z], 0)$
5: else
6: \triangleright Can be done with a simple greedy procedure
7:   Partition $[z]$ into sets $S_1, S_2, \ldots, S_k$ such that $S_k = \{z\}$ and $\max P^*(S_i) \leq 2 \min P^*(S_i)$.
8:   Let $Q$ be the distribution over $[k]$ with $Q(i) = P(S_i)/P([z])$ \triangleright Easy to conditionally sample from $Q$ if we can conditionally sample from $P$
9:   $(j, \tilde{Q}(j)) \leftarrow \text{Est2}(\varepsilon, \delta, z, Q)$
10: if $(j, \tilde{Q}(j)) = \text{NULL}$ then \triangleright i.e., if Lemma 5.2 returned NULL
11:     Return $([z], 0)$
12: $Y \leftarrow \text{Est3}(\varepsilon, \delta, j, Q)$
13: Return $\frac{P(z)}{\tilde{P}(z)} \cdot P([z]) \cdot \tilde{Q}(j) \cdot Y$. 

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Algorithm 13 Theorem 5.1 Returns \((S, \beta)\), where \(S\) is a subset with \(P^*(S) \geq \frac{1}{3}\), and \(\beta\) is an estimate for \(\sum_{x \in S} \min(c_1 D(x), c_2 D^*(x)) = O(\varepsilon^{-1} \log \varepsilon^{-1})\). Assumed that \(\mathcal{P}, \mathcal{P}^*\) have support size \(M\) with \(P^*(1) \leq P^*(2) \leq \cdots \leq P^*(M)\).

1: procedure PartialDetermining\((\varepsilon, \mathcal{P}, \mathcal{P}^*, c_1, c_2)\)
2: Let \(L\) be the smallest integer such that \(P^*(L) \geq \frac{1}{3}\) \(\triangleright\) Deal with this edge case first
3: if \(P^*(L) \geq \frac{1}{3}\) then
4: Compute an estimate \(\tilde{P}(L) = P(L) \pm O(\varepsilon)\) using \(\tilde{O}(\varepsilon^{-2})\) samples to \(\mathcal{P}\).
5: Return \((\{L\}, \min(c_1 \tilde{P}(L), c_2 P^*(L)))\).
6: \(T = \log_2 \varepsilon^{-1} + O(1)\)
7: \(Y_1, \ldots, Y_{T-1} = 0\) \(\triangleright\) \(Y_t\) will be the average of the \(Z_t\) estimates
8: for \(t = 1\) to \(T - 1\) do
9: \(\delta = 2^{-t}/20\)
10: \(S = O((\delta/\varepsilon)^2 \cdot \log \varepsilon^{-1})\)
11: for \(s = 1\) to \(S\) do
12: \(Z_s = 0\)
13: if \(z_s \geq L\) then
14: \(I_{-t} = 0\) \(\triangleright\) Lines 15 to 24 are solely for determining \(I_{-t}(z)\)
15: for \(i = 1\) to \(t\) do
16: \(\delta' = 2^{-i}/20\)
17: \(X = \text{Est}(\varepsilon, \delta', z_s, \mathcal{P}, \mathcal{P}^*, c_1, c_2)\)
18: if \(X = ([z_s], 0)\) then
19: Return \(([z_s], 0)\) \(\triangleright\) Whenever Algorithm Est returns \(([z], 0)\), this algorithm can also automatically return \(([z], 0)\).
20: if \(i = t\) and \(X < c_2(1 - 2^{-i})\) then
21: \(I_{-t} = 1\)
22: else if \(X \notin [c_2(1 - 2^{-i}), c_2(1 + 2^{-i})]\) then
23: break (out of the for loop starting at line 15)
24: if \(I_{-t} = 1\) then
25: \(X_s = \text{Est}(\varepsilon, \delta, z_s, \mathcal{P}, \mathcal{P}^*, c_1, c_2)\)
26: if \(X_s = ([z_s], 0)\) then
27: Return \(([z_s], 0)\)
28: \(Z_s = c_2 - X_s\) \(\triangleright\) In all other cases, we set \(Z_s = 0\)
29: end if
30: end if
31: end if
32: \(Y_t \leftarrow Y_t + \frac{Z_s}{S}\)
33: Return \(([L:M], Y_1 + \cdots + Y_{t-1})\).
Theorem 1.2: Main algorithm for tolerant identity testing in COND

1: procedure TolerantId(ε)
2: ε′ = O(ε / log^2 ε^{-1})
3: k = 0
4: T_0 = [N].
5: ˜c_{1,0}, c_{2,0} = 1.
6: γ = 0 \quad \triangleright \gamma \text{ will be our approximation to } 1 - d_{TV}(D, D^*)
7: \textbf{while } ˜c_{1,k}, c_{2,k} \geq 2ε' \textbf{ do}
8: \quad \mathcal{P}_k, \mathcal{P}_k^* \text{ are the distributions of } D, D^* \text{ conditioned on } T_k. \quad \triangleright \text{ We can conditionally sample from } \mathcal{P}_k \text{ assuming we can conditionally sample from } D. \text{ We treat } \mathcal{P}_k, \mathcal{P}_k^* \text{ as distributions over } [M_k], \text{ where } M_k = |T_k|.
9: \quad (S, \beta) \leftarrow \text{PartialDetermining}(ε, \mathcal{P}, \mathcal{P}^*, ˜c_{1,k}, c_{2,k})
10: \quad γ \leftarrow γ + β
11: \quad T_{k+1} = T_k \setminus S
12: \quad c_{2,k+1} = D^*(T_{k+1})
13: \quad \text{Compute } ˜c_{1,k+1} = c_{1,k+1} \pm ε' \text{ using } \tilde{O}(ε'^{-2}) \text{ samples}
14: \quad k \leftarrow k + 1
15: \textbf{Return } 1 - γ
A.4 Algorithm for Section 6

In this subsection, we write the pseudocode for the algorithm of Section 6. The full procedure for Theorem 1.3 is done by Algorithm 15. We assume SAMP and PCOND access to D and that we already know the size $N$. Recall that $D$ is a distribution over $[N]$ and we are trying to determine $d_{TV}(D, D^*)$ where $D^*$ is a known distribution over $[N]$.

**Algorithm 15** Theorem 1.3: Full algorithm for identity testing in PAIRCOND

1: procedure $PcondId(\varepsilon)$
2: \[ K = \log \frac{10N}{\varepsilon} \]
3: for $k = 1$ to $K$ do
4: \[ S_k = \{ i : 2^{-k} < D^*_i \leq 2 \cdot 2^{-k} \} \]
5: \[ S_{K+1} = [n] \setminus \left( \bigcup_{i=1}^{K} S_i \right) \]
6: $S$ is the distribution over $[K+1]$ where we sample $k \leftarrow S$ if we sample $i \leftarrow D$ and $i \in S_k$.
7: $S^*$ is the distribution over $[K+1]$ where $\mathbb{P}_{x \sim S^*}(x = k) = \mathbb{P}_{x \sim D^*}(x \in S_k)$.
8: Use [VV17] to determine if $S = S^*$. Outputs ACCEPT with at least 9/10 probability if $S = S^*$ and REJECT with at least 9/10 probability if $d_{TV}(S, S^*) \geq \varepsilon$.
9: if [VV17] outputs REJECT then
10: Return REJECT
11: for $a, b \geq 0$, $2^{-a} = \Omega(\varepsilon), 2^{-(a+b)} = \Omega(\varepsilon / \log \varepsilon^{-1})$ do
12: \[ \alpha = 2^{-a}, \beta = 2^{-b} \]
13: \[ R = O(\beta^{-1} \log \varepsilon^{-1}) \]
14: for $r = 1$ to $R$ do
15: \[ i_r \leftarrow D \]
16: \[ k_r := \text{set index such that } S_{k_r} \text{ contains } i_r \]
17: \[ j_r \leftarrow \text{Unif}[S_{k_r}] \]
18: Let $c$ be the approximation to $\frac{D(i_r)}{D(i_r) + D^*(j_r)}$ formed by making $O(\alpha^{-2} \log \varepsilon^{-1})$ calls to $Pcond(i_r, j_r)$.
19: if $|c - \frac{D^*(i_r)}{D^*(i_r) + D^*(j_r)}| \geq 2\alpha$ then
20: Return REJECT
21: Return ACCEPT
B A Near-Tight Lower Bound for Identity Testing in PAIRCOND

In this section, we prove Theorem B.1. As we noted in the introduction, the proof is very similar to [CRS15 Theorem 8], and to maintain consistency, we will adopt a similar proof structure.

B.1 The Distribution \( \mathcal{D}^* \) and Proof Intuition

Let \( K = \Theta \left( \varepsilon^{-2} \log N \right) \) and let \( R = \Theta \left( \frac{\log N}{\log K} \right) = \Theta \left( \frac{\log N}{\log(\varepsilon^{-1} \log N)} \right) \), so that \( N = K + K^2 + K^3 + \cdots + K^{2R} \). Now, for each \( 1 \leq r \leq 2R \), let \( B_r \) be the interval of integers starting from \( 1 + \sum_{i=1}^{r-1} K^i \) and ending with \( \sum_{i=1}^{r} K^i \), so that \( |B_r| = K^r \), and the \( B_r \)'s partition \([N]\). We now define \( \mathcal{D}^* \) as follows. For \( 1 \leq i \leq N \), if \( i \in B_r \), then \( D^*(i) = \frac{1}{2RK^r} \). This way, each bucket has equal probability, and for each fixed bucket, the elements all have the same probability.

We will show that it is difficult to distinguish between this distribution and a distribution randomly selected from \( \mathcal{P} \), where \( \mathcal{P} \) is a collection of distributions each having total variation distance at least \( \frac{\varepsilon}{2} \) from \( \mathcal{D}^* \). We select a distribution \( \mathcal{D} \leftarrow \mathcal{P} \) based on a random string \( s \in \{0,1\}^R \). For each \( 1 \leq r \leq R \), if \( s = 0 \), then for all \( i \in B_{2r-1} \), we choose \( D(i) = \frac{1+\varepsilon}{2RK^r} \) and for all \( i \in B_{2r} \), we choose \( D(i) = \frac{1-\varepsilon}{2RK^r} \). This way, it is clear that for all strings \( s \), \( \sum_{i=1}^{N} D(i) = 1 \), so \( \mathcal{D} \) is in fact a distribution, and that \( d_{TV}(\mathcal{D}, \mathcal{D}^*) = \frac{\varepsilon}{2} \) for all \( \mathcal{D} \).

For intuition as to why it is difficult to distinguish between \( \mathcal{D}^* \) and \( \mathcal{D} \leftarrow \mathcal{P} \), first note that intuitively, PCOND is useless. This is because if we ever call PCOND\((x, y) \) and \( x \) and \( y \) are not in the same bucket, we will almost always call the one in the smaller bucket, but if \( x \) and \( y \) are in the same bucket, PCOND\((x, y) \) is equivalent to choosing a random element in \( \{x, y\} \) regardless of whether our distribution is \( \mathcal{D}^* \) or \( \mathcal{D} \). Thus, the only useful information we get is from SAMP. However, the only real information we get from SAMP is which bucket the sampled element is in. This is because beyond that, we are just sampling a uniformly random element in the bucket regardless of whether our distribution is \( \mathcal{D}^* \) or \( \mathcal{D} \). However, we have \( 2R \) buckets, and it is known that in the sampling model, at least \( \sqrt{R} \cdot \varepsilon^{-2} \) samples are needed to test uniformity [Pan08]. And indeed, \( \mathcal{D}^* \) which is uniform on the buckets and \( \mathcal{D} \), when restricted to the buckets, has half of its elements with probability \( \frac{1+\varepsilon}{2RK} \) and half of its elements with probability \( \frac{1-\varepsilon}{2RK} \) and thus has total variation distance \( \frac{\varepsilon}{2} \) from uniform. This suggests that we need \( \Omega \left( \sqrt{\frac{\log N}{\log(\varepsilon^{-1} \log N)} \cdot \varepsilon^{-2}} \right) \) queries.

B.2 Preliminaries

First, we need the following well-known lemma, known as the Data Processing Inequality for Total Variation Distance.

**Lemma B.1.** Let \( \mathcal{D}, \mathcal{D}' \) be two distributions over some probability space \( \Omega \). Let \( F \) be a randomized function over \( \Omega \), which can be thought of as a distribution over functions \( f \) on \( \Omega \). In other words, \( F(\mathcal{D}) \) is the distribution of \( f(x) \) where \( x \leftarrow \mathcal{D} \) and \( f \leftarrow F \) (and likewise for \( F(\mathcal{D}') \)). Then, we have that

\[
d_{TV}(F(\mathcal{D}), F(\mathcal{D}')) \leq d_{TV}(\mathcal{D}, \mathcal{D}').
\]

We will also need the following result, which is used to prove a lower bound for uniformity testing in the sampling model.

**Theorem B.1.** [Pan08, rephrased] Let \( m \geq 1 \) be a positive integer, and let \( \mathcal{U} \) be the uniform distribution over \([2m]\). Next, draw random \( s_1, \ldots, s_m \leftarrow \{0,1\} \) and let \( Q_s \) be a distribution over
samples drawn from \( Q \).

In this subsection, we prove the following variant of Theorem 1.4:

**B.3 The Proof**

Let \( \text{Tr}(\cdot) \)

\[
\text{Proof.}
\]

We shall use the same setup used in the proof of [CRS15, Theorem 9]. Namely, we first fix

\[
\text{let } \text{Tr}(\cdot)\text{.}
\]

depend on any previous input or output. This implication was also used in [CRS15].

\[
\text{distribution } \text{Tr}(\cdot)\text{.}
\]

where \( A \) (and likewise for \( P \))

\[
\text{by making } q \text{ calls to } \text{SAMP and } \text{Pcond access to the distribution } \mathcal{D}\text{ (and likewise for } \mathcal{D}^*).\]

Remark. To see why this implies Theorem 1.4, note that if an algorithm can use

\[
\text{to see why this implies Theorem 1.4, note that if an algorithm can use } q \text{ queries to } \text{SAMP and } \text{Pcond, it can make all the SAMP queries first, since they are nonadaptive queries and do not depend on any previous input or output. This implication was also used in [CRS15].}
\]

**Proof.** We shall use the same setup used in the proof of [CRS15, Theorem 9]. Namely, we first fix such an algorithm \( A \), and define a transcript for \( A \) to be a pair \((Y, Z)\), such that \( Y = (s_1, \ldots, s_q) \in [N]^q \) and \( Z = ((x_1, y_1), p_1), \ldots, ((x_q, y_q), p_q)\), where each \( x_i, y_i \in [N] \) and each \( p_i \) is either \( x_i \) or \( y_i \). The \( Y \) represents the \( q \) samples drawn from SAMP, and \( Z \) represents the queries \( \{x_i, y_i\} \) and the output \( p_i = \text{Pcond}(x_i, y_i) \).

Let \( \text{Tr}(\mathcal{D}^*) \) denote the distribution of transcripts generated by running \( A \) on distribution \( \mathcal{D}^* \), and let \( \text{Tr}(\mathcal{P}) \) denote the distribution of transcripts generated by first sampling \( \mathcal{D} \leftarrow \mathcal{P} \) and then running \( A \) on distribution \( \mathcal{D} \). Our goal will be to show that the total variation distance between the transcript distribution \( \text{Tr}(\mathcal{D}^*) \) and the transcript distribution \( \text{Tr}(\mathcal{P}) \) is less than \( \frac{1}{3} \) for \( q = o(\sqrt{R} \cdot \epsilon^{-2}) \).

To do this, we consider the following modified algorithm \( A^{(k)} \) for \( 0 \leq k \leq q \).

1. \( A^{(k)} \) simulates \( A \) by making \( q \) calls to \( \text{SAMP} \) and then simulates the first \( k \) calls to \( \text{Pcond} \).
2. For each \( k' > k \), for the \( (k') \text{th} \) call to \( \text{Pcond} \), \( A^{(k)} \) generates \((x_{k'}, y_{k'})\) as \( A \) would given the output \( Y \) and the output \((\{x_1, y_1\}, p_1), \ldots, (\{x_{k'}-1, y_{k'}-1\}, p_{k'-1})\) that it has already seen. However, instead of calling \( \text{Pcond} \), \( A^{(k)} \) does the following:
   (a) If \( x_{k'} \) and \( y_{k'} \) belong to the same block \( B \), then \( p_{k'} \) is chosen uniformly from \( \{x_{k'}, y_{k'}\} \).
   (b) If \( x_{k'} \) and \( y_{k'} \) belong to different blocks, then \( p_{k'} \) will just be the smaller of \( x_{k'} \) and \( y_{k'} \) (since the smaller element is in the smaller block).

We now define \( \text{Tr}^{(k)}(\mathcal{D}^*) \) as the distribution of transcripts generated by running \( A^{(k)} \) on \( \mathcal{D}^* \) and \( \text{Tr}^{(0)}(\mathcal{P}) \) as the distribution of transcripts generated sampling \( \mathcal{D} \leftarrow \mathcal{P} \) and running \( A^{(k)} \) on \( \mathcal{D} \). Note that \( \text{Tr}^{(q)}(\mathcal{D}^*) \) is just \( \text{Tr}(\mathcal{D}^*) \) and \( \text{Tr}^{(q)}(\mathcal{P}) \) is just \( \text{Tr}(\mathcal{P}) \). By an immediate application of the Triangle Inequality, the proof follows from the following two lemmas.

**Lemma B.2.** \( d_{TV}\left(\text{Tr}^{(0)}(\mathcal{D}^*), \text{Tr}^{(0)}(\mathcal{P})\right) = o(1) \).
Lemma B.3. For all $0 \leq k \leq q - 1$, if $q = o(\sqrt{R} \cdot \varepsilon^{-2})$ we have that $d_{TV}(T^{(k)}(D^*), T^{(k+1)}(D^*)) \leq \frac{1}{20q}$ and $d_{TV}(T^{(k)}(P), T^{(k+1)}(P)) \leq \frac{1}{20q}$.

The theorem follows since by the triangle inequality, $d_{TV}(Tr(D^*), Tr(P)) \leq o(1) + 2q \cdot \frac{1}{20q} = o(1) + \frac{1}{10} < \frac{1}{3}$, so the algorithm cannot even statistically distinguish between $D^*$ and $D \leftarrow P$ using $q$ queries with advantage at least $\frac{1}{3}$.

Thus, we just need to prove Lemmas B.2 and B.3. In fact, the proof of Lemma B.3 doesn’t need to be changed from the corresponding proof in Canonne et al. ([CRS15, Lemma 16]) at all. So, we just prove Lemma B.2 and give an outline of the proof of Lemma B.3.

Proof of Lemma B.2. Note that when $A^{(0)}$ runs on $D^*$ or $D$, it does not ever call from the PCOND oracle, and only calls upon $\text{SAMP}$ $q$ times, followed by generating $\{x_k, y_k\}$, $p_k$ for all $1 \leq k \leq q$, which is done only using the return values of the previous calls to $\text{SAMP}$ and the randomness which can be generated by the algorithm $A^{(0)}$ itself. Therefore, if $d_{TV}(Tr^{(0)}(D^*), Tr^{(0)}(P)) = \Omega(1)$, we would have a way of distinguishing between $D^*$ and $D \leftarrow P$ with $\Omega(1)$ advantage using only $q$ queries.

This, however, would give us a way of distinguishing between a uniform distribution $U$ over $[2R]$ and $Q_s$ for $s \leftarrow \{0, 1\}^R$, which contradicts Theorem B.1. To see why, suppose we were trying to distinguish between $U$ and $Q_s$. Then, we simply choose $N = K + K^2 + \cdots + K^{2R}$. For $1 \leq k \leq q$, we sample $i$ from either $U$ or $Q_s$, and output a random element in bucket $B_i \subset [N]$. This will have the distribution $D^*$ if the original distribution were $U$, and would have the distribution $D \leftarrow P$ with $s$ being the randomness drawn. Therefore, if we could distinguish between $D^*$ and $D \leftarrow P$, we could also distinguish between $U$ and $Q_s$. This proves the lemma.

Proof Sketch of Lemma B.3. For simplicity, let’s see how to show $d_{TV}(Tr^{(k)}(D^*), Tr^{(k+1)}(D^*)) \leq \frac{1}{20q}$. Note that for both $Tr^{(k)}(D^*)$ and $Tr^{(k+1)}(D^*)$, the algorithms $A^{(k)}$ and $A^{(k+1)}$ operate identically until the point of calling $\text{PCOND}(x_{k+1}, y_{k+1})$. They still generate $x_{k+1}, y_{k+1}$ in the same way, which means that the transcripts until this point have the same distribution.

Now, if $x_{k+1}, y_{k+1}$ are in the same bucket $B_\ell$, the output $p_{k+1}$ will be a random element in $\{x_{k+1}, y_{k+1}\}$ for both $A^{(k+1)}$ and $A^{(k)}$, since $A^{(k)}$ just chooses a random element, and $A^{(k+1)}$ actually calls $\text{PCOND}(x_{k+1}, y_{k+1})$, which will give a random element. However, if $x_{k+1}$ is in bucket $\ell$ and $y_{k+1}$ is in bucket $\ell'$ for $\ell < \ell'$, the transcript of $A^{(k)}$ will always choose $p_{k+1} = x_{k+1}$ and the transcript of $A^{(k+1)}$ will choose $p_{k+1} = y_{k+1}$ with probability at least $1 - O\left(\frac{1}{K}\right)$ and will choose $y_{k+1}$ otherwise. Thus, only the rare event that the transcript chooses $p_{k+1} = y_{k+1}$ will cause the transcripts to deviate in distribution, so up to this point, the total variation distance is at most $O\left(\frac{1}{K}\right)$.

Now, the rest of the protocol is the same for both $A^{(k)}$ and $A^{(k+1)}$ - namely, the rest of the transcript is just a randomized function of the current transcript. Thus, we can use Lemma B.1 to say that the overall total variation distance is at most $O\left(\frac{1}{K}\right) \leq \frac{1}{20q}$, since $K = \Theta(\varepsilon^{-2} \log N)$ and $q = o(\sqrt{R} \cdot \varepsilon^{-2}) = o(K)$.