We will consider the type II superstring four-point amplitude for massless incoming states of momenta \( k_i \) \((i = 1, \ldots, 4)\). The 10D maximal supersymmetry implies that information on the four external states is encoded in a kinematic prefactor \( R^4 \) \([13]\), such that the supergravity tree-level amplitude is \( \sim R^4/(s_{12}s_{13}s_{14}) \). We define the Mandelstam variables as \( s_{ij} = 2k_i \cdot k_j \). Our working assumption will be that, up to three loops \([14]\), the \( g \)-loop superstring amplitude \( A^{(g)}_S \) takes the form

\[
\frac{A^{(g)}_S}{R^4} = \int_{M_{g,4}} \left| \prod_{1 \leq J \leq 4} d\Omega_{1J} \right|^2 \int d\ell \left| \mathcal{Y}^{(g)}_S \right|^2 \prod_{i < j} \left| E(z_i, z_j) \right|^{\omega_{ij}} \times \exp \frac{\alpha'}{2} \left( i \pi \Omega_{1J} \ell^I \cdot \ell^J + 2\pi i \sum_j \ell^I \cdot k_j \int_{z_0}^{z_j} \omega_I \right)^2.
\]

(1)

The integration denoted by \( M_{g,4} \) is over a genus-\( g \) fundamental domain parametrised by the period matrix \( \Omega_{1J} \) \((I, J = 1, \ldots, g)\) and over four marked points \( z_i \). We use a ‘chiral splitting’ representation \([15, 16]\), made possible by the introduction of the loop momenta \( \ell^I \), with \( d\ell \) denoting \( \prod_I d^{10} \ell^I \). The appearance of the prime form \( E(z_i, z_j) \) and the exponential (involving the holomorphic Abelian differentials \( \omega_I \) whose cycles define the period matrix) constitute the chiral×anti-chiral loop-level Koba-Nielsen factors. The interesting object is \( \mathcal{Y}^{(g)}_S \). We make no distinction between type IIA and type IIB apart from the details of \( R^4 \), since at four points there is no contribution from odd spin structures at least up to three loops \([17]\).

We will exploit the analogy between the formula (1) for the superstring and the following expected formula for supergravity:

\[
\frac{A^{(g)}_A}{R^4} = \int d\ell \int_{M_{g,4}} \prod_{1 \leq J \leq 4} d\Omega_{1J} \left( \mathcal{Y}^{(g)}_A \right)^2 \prod_{i=1}^4 \delta(E_i) \prod_{l \leq J} \delta(u^J). \]

(2)

This type of formula for a scattering amplitude was discovered at tree level by Cachazo, He and Yuan \([18, 19]\),...
generalising a previous formula from twistor string theory [20, 21]. The loop-level extension [22–28] was derived from the type II ambitwistor string [29], which is a worldsheet model of type II supergravity. The 10D loop integration in (2) is UV divergent, so the expression is formal only, and we understand it as defining a loop integrand. The genus-\(g\) moduli-space integration is fully localised on a set of critical points, determined by the genus-\(g\) scattering equations: \(E_i = 0\) and \(u_{IJ} = 0\) [30]. An extensive discussion of the loop-level version of this formalism was presented in [28]: the brief discussion below will be sufficient for our purposes. There is a clear analogy between (1) and (2). Our proposal, under conditions to be discussed, is to identify the ‘chiral half-integrands’,

\[
Y^{(g)}_S = Y^{(g)}_A,
\]

which is known to be possible for \(g \leq 2\). Notice that (1) is a simplified expression where \(Y^{(g)}_S\) is independent of \(\alpha'\). The idea is that we can import an ambitwistor string—i.e. supergravity—result into the superstring.

The only known procedure to evaluate (2) reflects the fact that the ambitwistor string is a field theory in disguise: the genus-\(g\) formula can be localised on a maximal non-separating degeneration, i.e. a Riemann sphere with \(g\) nodes, as in FIG. 1. This follows from a residue argument in moduli space at one [25, 26] and two [27, 28] loops, and our three-loop results provide evidence that it holds at higher order. The formula on the nodal sphere is

\[
\frac{A^{(g)}_A}{\mathcal{R}^4} = \int \frac{d\ell}{\ell^2} \int_{\mathcal{M}_{0,4+2g}} c^{(g)}(\mathcal{J}^{(g)} Y^{(g)})^2 \prod_{A=1}^{4+2g} \delta(E_A).
\]

Here, \(\mathcal{M}_{0,4+2g}\) is the moduli space of the Riemann sphere with \(4+2g\) marked points, corresponding to 4 external particles and \(2g\) ‘loop marked points’, one pair per node as in FIG. 1. The factors \(c^{(g)}\) and \(\mathcal{J}^{(g)}\) arise from the degeneration of \(\mathcal{M}_{g,4}\) to \(\mathcal{M}_{0,4+2g}\) [28]. We will give an example momentarily. The object \(Y^{(g)}\) in this expression is the limit of \(Y^{(g)}_A\) in the maximal non-separating degeneration. Finally, the delta functions impose the loop-level scattering equations on the nodal sphere, \(E_A = 0\), on whose finite set of solutions the moduli-space integral fully localises; in fact, this integral can be understood as a multi-dimensional residue integral.

Let us be more concrete. The degeneration to the \(g\)-nodal sphere is achieved in a limit involving the diagonal components of the period matrix: \(q_{IJ} = e^{2\pi i \Omega_{IJ}} \to 0\). In this limit, the holomorphic Abelian differentials whose periods define the period matrix acquire simple poles at the corresponding node: with \(\sigma \in \mathbb{CP}^1\),

\[
\omega_I = \frac{\omega_{I+} - \omega_{I-}}{2\pi i}, \quad \omega_{I+} - \omega_{I-}(\sigma) = \frac{(\sigma_{I+} - \sigma_{I-}) d\sigma}{(\sigma - \sigma_{I+})(\sigma - \sigma_{I-})},
\]

where the \(\sigma_{I\pm}\) are the marked points for node \(I\). Together with the marked points \(\sigma_i\) associated to the four external particles, we have the total of \(4 + 2g\) marked points parametrising \(\mathcal{M}_{0,4+2g}\) up to \(SL(2,\mathbb{C})\). For \(g \geq 2\), the off-diagonal components of the period matrix are expressed in this limit in terms of cross-ratios of the nodal marked points,

\[
q_{IJ} = e^{2\pi i \Omega_{IJ}} = \frac{\sigma_{I+} - \sigma_{J-}}{\sigma_{I+} - \sigma_{J+}} + \frac{\sigma_{I-} - \sigma_{J+}}{\sigma_{I-} - \sigma_{J-}},
\]

where we denote \(\sigma_{AB} = \sigma_A - \sigma_B\). This change of integration variables leads to the \((\mathcal{J}^{(g)})^2\) appearing in (4). One \(\mathcal{J}^{(g)}\) arises from the moduli-space measure,

\[
\prod_{I<J} \frac{dq_{IJ}}{q_{IJ}} = \frac{\mathcal{J}^{(g)}}{\text{vol } SL(2,\mathbb{C})}, \quad \mathcal{J}^{(g)} = \mathcal{J}^{(g)} \prod_{I \pm} d\sigma_{I\pm},
\]

while the other arises from rewriting higher-genus scattering equations as nodal sphere ones. Finally, the scattering equations on the nodal sphere are equivalent to the vanishing of a meromorphic quadratic differential \(\mathcal{P}^{(g)}\) with only simple poles, and can be read off from the residues of this differential at the \(4 + 2g\) marked points,

\[
E_A = \text{Res}_{\sigma_i} \mathcal{P}^{(g)}.
\]

The ingredients of (4) can be illustrated with the two-loop example. We have \(c^{(2)} = 1/(1 - q_{12})\) [31] and

\[
\mathcal{P}^{(2)} = P^2 - (\ell_1^I \omega_{I+} - \ell_1^J \omega_{J+})^2 + (\ell_1^I + \ell_2^J)^2 \omega_{1+} \omega_{2+} - \omega_{2+} \omega_{2-} - ,
\]

where

\[
P_{\mu}(\sigma) = \ell_1^I \omega_{I+}(\sigma) + \sum \frac{k_{i\mu}}{\sigma - \sigma_i} d\sigma.
\]

Effectively, \(\mathcal{P}^{(g)}\) encodes all the potential loop-integrand propagators in an expression like (4), while \(c^{(g)}\) projects out certain unphysical propagators. These details are not important for this paper, where we are concerned with \(\mathcal{J}^{(g)}\) and especially \(\mathcal{Y}^{(g)}\). At two loops, we have

\[
J^{(2)} = \frac{1}{\sigma_{1+} \sigma_{1+} - \sigma_{1-} \sigma_{1-}},
\]

and

\[
\mathcal{Y}^{(2)} = \frac{1}{3} \left( (s_{14} - s_{13}) \Delta^{(2)}_{12} \Delta^{(2)}_{34} + \text{cyc}(234) \right).
\]
where we used the determinant
\[
\Delta_{i_1, \ldots, i_g}^{(g)} = \varepsilon^{I_1 \ldots I_g} \omega_{I_1}(\sigma_{i_1}) \cdots \omega_{I_g}(\sigma_{i_g})
\]
(13)
defined for any \( g \). The expression (12) is built from the differentials \( \omega_I \), which naturally lift from the nodal sphere to become the holomorphic Abelian differentials on the genus-2 surface. Indeed, the genus-2 expression is also valid as \( \mathcal{Y}_A^{(2)} \) in (2) and, crucially for us, as \( \mathcal{Y}_S^{(2)} \) in (1). The object \( \Delta^{(g)} \) is a modular form of weight \(-1\) at any genus, which at genus 2 gives \( \mathcal{Y}_S^{(2)} \) the appropriate weight such that the moduli-space integral is well defined. At three loops, the answer is not as simple as (12): \( \Delta^{(3)} \) still arises [10], but additional ingredients are needed, as discussed e.g. in [32], and as we will see here.

**\( \mathcal{Y}_S^{(g)} \) FROM BCJ NUMERATORS**

Let us present and test our strategy. The steps are to:

(i) take a supergravity loop integrand written in a BCJ double-copy representation,

(ii) translate that integrand into the ambitwistor string moduli-space integrand localised on the nodal Riemann sphere, i.e. obtain \( \mathcal{Y}^{(g)} \),

(iii) uplift that formula to a higher-genus modular form valid as \( \mathcal{Y}_S^{(g)} \) such that \( \mathcal{Y}_S^{(g)} \to \mathcal{Y}^{(g)} \) as \( q_{11} \to 0 \).

With our current understanding, step (iii) relies on an educated guess, as we will exemplify.

Starting with step (i), a BCJ representation is one in which the loop integrand is written in terms of trivalent diagrams, whose numerators are the square of analogous numerators in non-planar SYM obeying the BCJ colour-kinematics duality [12, 33] [34]. See [35] for a review of this remarkable construction, which was motivated by the KLT relations of string theory [36]. Indeed, there is a large body of work relating this construction to aspects of string theory, e.g. [37–53]. Step (ii) is based on the connection to the scattering equations story, for which we use the following relation based on a differential form with logarithmic singularities [54]

\[
(2\pi i)^4 J^{(g)} \mathcal{Y}^{(g)} = \sum_{\rho \in S_{2g+2}} N^{(g)}(1^+, \rho, 1^-) \prod_{\Lambda=1}^{4+2g} d\sigma_{\Lambda},
\]
(14)
where \( (ABC \ldots D) = \sigma_{AB} \sigma_{BC} \ldots \sigma_{DA} \) is a Parke-Taylor numerator. The BCJ numerators \( N^{(g)} \), which depend on a particle ordering, are SYM numerators whose square gives the supergravity numerators; this square effectively translates into the square of \( J^{(g)} \mathcal{Y}^{(g)} \) in (4). Notice, however, that we have extracted the overall factor \( R^4 \) in (4), whose ‘square root’ is therefore not included in the SYM numerators. The correspondence between the numerators \( N^{(g)} \) and trivalent diagrams is best understood in an explicit example, to be discussed below. Before that, let us make two comments. The first is that two marked points singled out in (14) were chosen to be \( \sigma_{1 \pm} \), but the sum is independent of that choice. The second, for the reader familiar with the scattering equations formalism including the developments [55–58], is that equalities like (14) often hold only when the marked points satisfy the scattering equations (e.g. for CHY Pfaffians). Here, on the other hand, we propose that (14) defines \( \mathcal{Y}^{(g)} \) such that it may be uplifted to the superstring, as happens up to two loops.

Let us test the strategy at two loops, for which the BCJ representation of the four-point supergravity loop integrand is long known [59] [60]. The two-loop BCJ numerators can be compactly written as

\[
N^{(2)}(1^+, \rho_1, 2^+, \rho_2, 2^-, \rho_3, 1^-) = \begin{cases} 
  s_{ij} & \{ i, j \} \\
  0 & \text{otherwise}
\end{cases}
\]
(15)
They correspond to half-ladder diagrams with loop momenta \( \pm \ell_1 \) at the ends; see FIG. 2. A standard two-loop diagram is then obtained by gluing the nodal legs, i.e. \( I^+ \) with \( I^- \). Taking the result (15) from the literature, it is possible to obtain \( \mathcal{Y}^{(2)} \) via (14). Then, it is both natural and easy to rewrite \( \mathcal{Y}^{(2)} \) in the form (12), which as explained earlier can be uplifted to genus 2, matching the superstring result \( \mathcal{Y}_S^{(2)} \). This achieves step (iii).

**THREE LOOPS**

We now apply our strategy to the much more intricate three-loop case. From the general form of a three-loop field theory integrand, namely the inclusion of the relevant diagram topologies, we can determine \( c^{(3)} \) and \( \mathfrak{P}^{(3)} \). However, they do not appear in (14), so they are not important for the goal of this paper [61]. The important quantities are \( J^{(3)} \) and \( \mathcal{Y}^{(3)} \). The Jacobian is straightforwardly obtained from (7) and can be written as

\[
J^{(3)} = J_{\text{hyp}} \prod_{i < j} \frac{\sigma_{i+1-j} \cdot \sigma_{j+1-i} \cdot \sigma_{i-j} \cdot \sigma_{j-i} \cdot \sigma_{j-i-j+}}{\sigma_{i+1-j} \cdot \sigma_{j+1-i}},
\]
(16)
where in the factor

\[
J_{\text{hyp}} = \sigma_{1+2} \cdot \sigma_{2+3} \cdot \sigma_{3+1} - \sigma_{1+3} \cdot \sigma_{3+2} \cdot \sigma_{2+1},
\]
(17)
the subscript refers to hyperelliptic, as we will explain.

We can now determine $\gamma^{(3)}$ using (14). The right-hand side is obtained from the known BCJ representation of the three-loop supergravity integrand, a landmark application of the double copy [12] [62]. The BCJ numerators, listed in table I of [12], are not as simple as at two loops and depend linearly on the loop momenta, e.g. [63]

$$N(1^+, 1, 2, 2^+, 3, 3^+, 2^-, 4, 3^-, 1^-) = \frac{1}{3} s_{12} (s_{12} - s_{14}) + \frac{2}{3} \ell_1 \cdot (k_2 (s_{13} - s_{14}) + k_3 (s_{13} - s_{12}) + k_4 (s_{12} - s_{14})) .$$

Via (14), this property implies

$$2\pi i \gamma^{(3)}_S = \gamma_0 + 2\pi i \ell \cdot \gamma^{(3)}_I ,$$

where the factors were chosen for later convenience. We write our results already in uplifted form, i.e. for $\gamma^{(3)}_S$ (which we claim is $\gamma^{(3)}_{\omega}$) instead of its degeneration $\gamma^{(3)}$. To determine $\gamma^{(3)}_S$, we construct a well-motivated ansatz with the required modular weight of $-1$, and fix the coefficients of that ansatz by matching numerically the degeneration limit to (14). This requires expanding in the degeneration parameters the Jacobi theta functions which define various objects, a straightforward if computationally heavy procedure.

The second term in (18) is the easiest: we can write

$$\gamma^u_I = \frac{2}{3} (a^I \omega I(z_1) \Delta^{(3)}_{123} + \text{cyc}(234)) ,$$

with $a^I = k^I \cdot (k_3 - k_4) \cdot k_1 + \text{cyc}(234)$ . All the ingredients have been introduced previously.

The object $\gamma_0$ is more involved. It is convenient to extricate the kinematic dependence by writing

$$\gamma_0 = s_{13} s_{14} Y_{12,34} + \text{cyc}(234) ,$$

where $Y_{12,34}$ is independent of the $s_{ij}$ and is symmetric when exchanging: $z_1 \leftrightarrow z_2$, $z_3 \leftrightarrow z_4$. Let us first state the result and then discuss it:

$$Y_{12,34} = \frac{1}{3} D_{12,34} - \frac{1}{15} \Psi_9 \left( S^{(a)}_{12,34} - \frac{1}{8} S^{(b)}_{12,34} \right) ,$$

where

$$D_{12,34} = \omega_{3,4}(z_1) \Delta^{(3)}_{234} + \omega_{3,4}(z_2) \Delta^{(3)}_{134} + \omega_{1,2}(z_3) \Delta^{(3)}_{412} + \omega_{1,2}(z_4) \Delta^{(3)}_{312} ,$$

$$S^{(a)}_{12,34} = \sum \delta \Xi_8[\delta] \left( S_\delta(z_1, z_2) S_\delta(z_2, z_3) S_\delta(z_3, z_4) S_\delta(z_4, z_1) + S_\delta(z_2, z_1) S_\delta(z_1, z_3) S_\delta(z_3, z_4) S_\delta(z_4, z_2) \right) ,$$

$$S^{(b)}_{12,34} = \sum \delta \Xi_8[\delta] S_\delta(z_1, z_2)^2 S_\delta(z_3, z_4)^2 .$$

Starting with the expression (22), the object $\omega_{i,j}(z_k)$ is the normalised Abelian differential of the third kind, whose degeneration limit is

$$\omega_{i,j}(\sigma) = \frac{\sigma_{ij}}{(\sigma - \sigma_i)(\sigma - \sigma_j)} d\sigma .$$

A consistency check is that the contribution (22), including the kinematic coefficient, is completely fixed by (19). This follows from the condition of ‘homology invariance’: distinct choices of homology cycles of the Riemann surface with respect to the marked points $z_j$ obey monodromy relations dictated by the chiral splitting procedure [16], and this connects the two contributions [64].

The contributions (23) and (24) are more elaborate, but the structure is familiar from the RNS formalism [4, 15, 65–70]. The sums are over the 36 even spin structures at genus 3, labelled by $\delta$, and the objects $S_\delta(z_i, z_j)$ are the Szegö kernels arising from the OPEs of worldsheet fermions. The ‘chiral measure’ $\Xi_8[\delta]/\Psi_9$ is the crucial ingredient. Here, $\Psi_9 = \sqrt{-\prod} \theta[\delta](0)$ is a modular form of weight 9 (note our non-standard definition for the sign), defined in terms of the even Jacobi theta functions. The general properties of the chiral measure were described in [7, 8] and the precise definition of $\Xi_8[\delta]$ was given in [9]. It is a sophisticated definition, so we will not repeat it here; we found ref. [71] very helpful. The RNS derivation of this measure remains obscure; see appendix C of [72].

In the degeneration limit $q_{11} \to 0$, $\Psi_9$ vanishes with leading behaviour $\Psi_9 = (\prod q_{I{\bar I}}^{2}) \psi_9 + \ldots$,

$$\psi_9 = 2^{14} J_{\text{hyp}} \prod_{I < J} \frac{\left( \prod \sigma_{I+J}\right)^3}{\sigma_{I+J-}\sigma_{I+J-}\sigma_{I+J-}} ,$$

where $J_{\text{hyp}}$ is given in (17). It is opportune to note that only a codimension-1c subset of genus-3 Riemann surfaces are hyperelliptic (whereas for $g \leq 2$ all surfaces are), and these are precisely identified by the vanishing of $\Psi_9$ [73]. The condition $J_{\text{hyp}} = 0$ identifies hyperelliptic surfaces in the degeneration limit. The factors of $J_{\text{hyp}}$ in $J^{(3)}$ and in $1/\Psi_9$ cancel, such that $J^{(3)} \psi^{(3)}$ does not vanish in the hyperelliptic sector.

The sums (23) and (24), which are modular forms of weight 8, vanish in the degeneration limit in a manner analogous to $\Psi_9$, so that the ratio appearing in (21) yields a finite result on the nodal sphere [74]. As consistency checks on our implementation of the chiral measure, we verified to order $O(q_{11}^4)$ the following identities (respectively, from [9, 75, 76]):

$$\sum \delta \Xi_8[\delta] = 0 , \quad \sum \delta \Xi_8[\delta] S_\delta(z_1, z_2)^2 = 0 ,$$

$$\sum \delta \Xi_8[\delta] S_\delta(z_1, z_2) S_\delta(z_2, z_3) S_\delta(z_3, z_1) = C \Psi_9 \Delta^{(3)}_{123} ,$$

where we determined the previously unknown coefficient $C = 15 (2\pi)^3$. We could not find simplified expressions...
for (23) and (24); they are not proportional to \( \Psi \), i.e. not proportional to \( J_{\psi\psi} \) in the degeneration limit.

Comparing our result to the pure spinor computation of \([10]\), the latter was restricted to part of the correlator and was not manifestly modular invariant, but appears to be consistent at least with (19). The main goal of \([10]\), for which the partial computation was sufficient, was to match a prediction from S-duality \([77]\) for the low-energy amplitude, where the overall normalisation is important. We neglected the normalisation here, and leave this aspect and a proper comparison to \([10]\) for future work. Due to manifest supersymmetry, the splitting of spin structures does not arise in the pure spinor approach \([5, 6, 78–81]\), so this approach may be helpful in simplifying the sums seen above.

**DISCUSSION**

We have constructed a conjectured expression for the three-loop four-point amplitude of massless states in the type II superstring. The crucial ingredient is the chiral half-integrand (18). As at two loops \([4, 82]\), this object can also in principle be imported into the Heterotic superstring, paired with a bosonic counterpart.

In place of a first-principles worldsheet calculation, we wrote down an ansatz inspired by insights from the RNS and pure spinor formalisms, and then constrained that ansatz using supergravity data mined with modern amplitudes techniques. Our focus was on briefly delineating a strategy, with very concrete results. Additional technical details will be presented elsewhere. We hope that our conjecture can guide rigorous derivations using established worldsheet methods. Alternatively, in the spirit of the amplitudes programme, perhaps the proof can follow from a set of basic constraints, such as unitarity.

Natural future directions are: the study of the moduli-space integration in the low-energy limit, building on \([10, 83–86]\), which is newly motivated by beautiful advances in the non-perturbative amplitudes bootstrap \([87]\); and the consideration of higher-point \([40, 42, 44, 82, 86, 88–95]\) or higher-loop \([7, 9, 96–102]\) amplitudes. We expect our strategy to prove useful, not least because there are BCJ numerators for \(N = 8\) supergravity up to five loops \([103–105]\), although the five-loop case required a generalisation of this representation. Also at this loop order, the relation between supermoduli space and ordinary moduli space becomes more intricate \([106]\), calling into question the structure of our starting point (1). The interplay between field theory and string theory amplitudes continues to present us with many challenges and fruitful surprises.

**Note** As this work was concluded, it came to our knowledge that the authors of \([82]\) have independently constructed the contribution to the half-integrand that is linear in the loop momenta, equation (19).

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This follows from supersymmetry and is clear if we consider factorisable NS-NS external states $\varepsilon^\mu_k = \varepsilon^\mu_i \varepsilon^\nu_j$. The supersymmetric pre-factor is $R^4(\varepsilon, \tilde{\varepsilon}) = F^4(\varepsilon) F^4(\tilde{\varepsilon})$, where $F^4$ is the pre-factor for the open superstring and includes products $\varepsilon_i \cdot \varepsilon_j$. At three loops and four points, a 10D Levi-Civita tensor arising from an odd spin structure’s zero mode may just about be saturated: $\varepsilon_{10}(k_1, k_2, k_3, \varepsilon_2, \varepsilon_3, \varepsilon_4, \ell_1, \ell_2, \ell_3)$, but it would never give rise to any $\varepsilon_i \cdot \varepsilon_j$. Moreover, the contraction of the two Levi-Civita tensors (with $\varepsilon_i$ and with $\ell_i$) over three indices after loop integration, required for a potentially non-vanishing contribution, yields products $\varepsilon_i \cdot \ell_j$, inconsistent with $F^4(\varepsilon) F^4(\tilde{\varepsilon})$. This discussion is consistent with the results of [10, 107].

It should be noted that the supergravity loop integrand in (4), once the moduli-space integral is performed, is not written in terms of Feynman-like propagators, as in the original BCJ representation [12]. It is instead written in an alternative representation, which was discovered in the ambitwistor string; see [25, 26, 108–117] for discussions. The supergravity BCJ numerators relevant in our two- and three-loop problems are valid in both representations.

[18] F. Cachazo, S. He, and E. Y. Yuan, Scattering of Massless Particles in Arbitrary Dimensions, Phys.Rev.Lett. 113, 171601 (2014), arXiv:1307.2199 [hep-th].

[19] F. Cachazo, S. He, and E. Y. Yuan, Scattering of Massless Particles: Scalars, Gluons and Gravitons, JHEP 1407, 033, arXiv:1309.0885 [hep-th].

[20] E. Witten, Perturbative gauge theory as a string theory in twistor space, Comm.Math.Phys. 252, 189 (2004), arXiv:hep-th/0312171 [hep-th].

[21] R. Roiban, M. Spradlin, and A. Volovich, On the tree level S matrix of Yang-Mills theory, Phys.Rev. D70, 026009 (2004), arXiv:hep-th/0403190 [hep-th].

[22] T. Adamo, E. Casali, and D. Skinner, Ambitwistor strings and the scattering equations at one loop, JHEP 1404, 104, arXiv:1312.3828 [hep-th].

[23] E. Casali and P. Tourkine, Infrared behaviour of the one-loop scattering equations and supergravity integrands, JHEP 1504, 013, arXiv:1412.3787 [hep-th].

[24] T. Adamo and E. Casali, Scattering equations, supergravity integrands, and pure spinors, JHEP 1505, 120, arXiv:1502.06826 [hep-th].

[25] Y. Geyer, M. Mason, R. Monteiro, and P. Tourkine, Loop Integrands for Scattering Amplitudes from the Riemann Sphere, Phys. Rev. Lett. 115, 121603 (2015), arXiv:1507.00321 [hep-th].

[26] Y. Geyer, M. Mason, R. Monteiro, and P. Tourkine, One-loop amplitudes on the Riemann sphere, JHEP 03, 114, arXiv:1511.06315 [hep-th].

[27] Y. Geyer, M. Mason, R. Monteiro, and P. Tourkine, Two-Loop Scattering Amplitudes from the Riemann Sphere, Phys. Rev. D94, 125029 (2016), arXiv:1607.08887 [hep-th].

[28] Y. Geyer and R. Monteiro, Two-Loop Scattering Amplitudes from Ambitwistor Strings: from Genus Two to the Nodal Riemann Sphere, JHEP 11, 008, arXiv:1805.05344 [hep-th].

[29] L. Mason and D. Skinner, Ambitwistor strings and the scattering equations, JHEP 1407, 048, arXiv:1311.2564 [hep-th].

[30] For the delta functions, we use the definition $\delta(z) = \frac{1}{\pi} \frac{1}{\pi iz}$, standard in this context.

[31] $c^{(2)}$ is associated to the genus-2 fundamental domain constraint $|q_{12}| < 1$. In particular, $c^{(2)}$ arises from relaxing this constraint when degenerating to the sphere $\mathbb{CP}^1$. It is instead written in an alternative representation, which was discovered in the ambitwistor string; see [25, 26, 108–117] for discussions. The supergravity BCJ numerators relevant in our two- and three-loop problems are valid in both representations.

[32] P. Tourkine, Tropical Amplitudes, Annales Henri Poincare 18, 2199 (2017), arXiv:1309.3551 [hep-th].
The tree-level ($\mathcal{F}^{(0)} = 1$) version of this relation was revealed in string theory in [39, 118] and in the scattering equations (CHY) formalism in [19]; see also [119, 120].

Our higher-loop formula is motivated by its verification at two loops in [115]. The higher-multiplicity extension is trivial.

In summary, if a marked point $z_i$ is shifted by a ‘bicycle’, (i) the loop momentum associated to that cycle is shifted by $k_i$ and (ii) the Abelian differential of the third kind has non-trivial monodromy. These two effects combine precisely to achieve homology invariance. See the very clear discussion for the two-loop five-point amplitude in [82], where the objects $g_{i j}^{(2)}$ relate to $\omega_{ A , j} (z_1)$ as $\omega_{ A , j} (z_1) = ( g_{i,3} - g_{i,4} ) \omega_ A (z_1) .

In the classical reference [121].

They will be discussed elsewhere.

The result in [12] applies to $4D \mathcal{N} = 8$ supergravity, but we will assume that it ‘oxidates’ trivially to 10D type II supergravity for similar reasons as in the two loop case, given the absence of contributions from odd spin structures.

Our convention for the external momenta is that they are incoming, whereas the convention in [12] was that they are outgoing. This affects the sign of the term linear in the loop momenta.

In summary, if a marked point $z_i$ is shifted by a ‘bicycle’, (i) the loop momentum associated to that cycle is shifted by $k_i$ and (ii) the Abelian differential of the third kind has non-trivial monodromy. These two effects combine precisely to achieve homology invariance. See the very clear discussion for the two-loop five-point amplitude in [82], where the objects $g_{i j}^{(2)}$ relate to $\omega_{ A , j} (z_1)$ as $\omega_{ A , j} (z_1) = ( g_{i,3} - g_{i,4} ) \omega_ A (z_1) .

E. D’Hoker and D. H. Phong, Lectures on two loop superstrings, Superstring Theory, Proceedings, International Conference, Hangzhou, P.R. China, August 12-15, 2002, Conf. Proc. C0208124, 85 (2002), [arXiv:hep-th/0211111 [hep-th]].

E. D’Hoker and D. H. Phong, Two loop superstrings. 1. Main formulas, Phys. Lett. B529, 241 (2002), arXiv:hep-th/0110247 [hep-th].

E. D’Hoker and D. H. Phong, Two loop superstrings. 2. The Chiral measure on moduli space, Nucl. Phys. B636, 3 (2002), arXiv:hep-th/0110283 [hep-th].

E. D’Hoker and D. H. Phong, Two loop superstrings. 3. Slice independence and absence of ambiguities, Nucl. Phys. B636, 61 (2002), arXiv:hep-th/0111016 [hep-th].

E. D’Hoker and D. H. Phong, Two loop superstrings 4: The Cosmological constant and modular forms, Nucl. Phys. B639, 129 (2002), arXiv:hep-th/0111040 [hep-th].

E. D’Hoker and D. H. Phong, Two-loop superstrings. V. Gauge slice independence of the N-point function, Nucl. Phys. B715, 91 (2005), arXiv:hep-th/0501196 [hep-th].

S. Tsujimura, On Siegel modular forms of degree three, Amer. J. Math. 108, 755 (1986).

E. Witten, The Super Period Matrix With Ramond Punctures, J. Geom. Phys. 92, 210 (2015), arXiv:1501.02490 [hep-th].
II. Low energy expansion and S-duality, JHEP 02, 139, arXiv:2008.08687 [hep-th].

[87] A. Guerrieri, J. Penedones, and P. Vieira, Where is String Theory?, (2021), arXiv:2102.02847 [hep-th].

[88] A. Tsuchiya, More on One Loop Massless Amplitudes of Superstring Theories, Phys. Rev. D39, 1626 (1989).

[89] M. B. Green, C. R. Mafra, and O. Schlotterer, Multiparticle one-loop amplitudes and S-duality in closed superstring theory, JHEP 10, 188, arXiv:1307.3534 [hep-th].

[90] C. R. Mafra and O. Schlotterer, One-loop superstring six-point amplitudes and anomalies in pure spinor superspace, JHEP 04, 148, arXiv:1603.04790 [hep-th].

[91] C. R. Mafra and O. Schlotterer, Towards the n-point one-loop superstring amplitude. Part I. Pure spinors and superfield kinematics, JHEP 08, 090, arXiv:1812.10969 [hep-th].

[92] C. R. Mafra and O. Schlotterer, Towards the n-point one-loop superstring amplitude. Part II. Worldsheet functions and their duality to kinematics, JHEP 08, 091, arXiv:1812.10970 [hep-th].

[93] C. R. Mafra and O. Schlotterer, Towards the n-point one-loop superstring amplitude. Part III. One-loop correlators and their double-copy structure, JHEP 08, 092, arXiv:1812.10971 [hep-th].

[94] S. Grushkov, Superstring scattering amplitudes in higher genus, Commun. Math. Phys. 287, 749 (2009), arXiv:0803.3469 [hep-th].

[95] S. L. Cacciatori, F. Dalla Piazza, and B. van Geemen, Genus four superstring measures, Lett. Math. Phys. 85, 185 (2008), arXiv:0804.0457 [hep-th].

[96] R. Salvati-Manni, Remarks on Superstring amplitudes in higher genus, Nucl. Phys. B 801, 163 (2008), arXiv:0804.0512 [hep-th].

[97] A. Morozov, NSR Superstring Measures Revisited, JHEP 05, 086, arXiv:0804.3167 [hep-th].

[98] S. Grushkov and R. Salvati Manni, The superstring cosmological constant and the Schottky form in genus 5, Am. J. Math. 133, 1007 (2011), arXiv:0809.1391 [math.AG].

[99] M. Matone and R. Volpato, Getting superstring amplitudes by degenerating Riemann surfaces, Nucl. Phys. B 839, 21 (2010), arXiv:1003.3452 [hep-th].

[100] M. Matone and R. Volpato, Higher genus superstring amplitudes from the geometry of moduli space, Nucl. Phys. B 732, 321 (2006), arXiv:hep-th/0506231.

[101] Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson, and R. Roiban, Simplifying Multiloop Integrands and Ultraviolet Divergences of Gauge Theory and Gravity Amplitudes, Phys. Rev. D 85, 105014 (2012), arXiv:1201.5366 [hep-th].

[102] Z. Bern, J. J. Carrasco, W.-M. Chen, H. Johansson, and R. Roiban, Gravity Amplitudes as Generalized Double Copies of Gauge-Theory Amplitudes, Phys. Rev. Lett. 118, 181602 (2017), arXiv:1701.02519 [hep-th].