On the classification of certain ternary codes of length 12

Makoto Araya and Masaaki Harada

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Abstract. Shimada and Zhang studied the existence of polarizations on some supersingular K3 surfaces by reducing the existence of the polarizations to that of ternary \([12,5]/C_{3^3}\) codes satisfying certain conditions. In this note, we give a classification of ternary \([12,5]/C_{3^3}\) codes satisfying the conditions. To do this, ternary \([10,5]/C_{3^3}\) codes are classified for minimum weights 3 and 4.

1. Introduction

A ternary \([n,k]\) code \(C\) is a \(k\)-dimensional vector subspace of \(F_3^n\), where \(F_3\) denotes the finite field of order 3. The weight \(\text{wt}(x)\) of a vector \(x\) is the number of non-zero components of \(x\). The minimum non-zero weight of all codewords in \(C\) is called the minimum weight of \(C\). A ternary \([n,k,d]\) code is a ternary \([n,k]\) code with minimum weight \(d\). Throughout this note, we denote the minimum weight of a code \(C\) by \(d(C)\).

Shimada and Zhang [9] studied the existence of polarizations on the supersingular K3 surfaces in characteristic 3 with Artin invariant 1 (see [9, Theorem 1.5] for the details). This was done by reducing the problem of the existence of the polarizations to a problem of the existence of ternary \([12,5]/C_{3^3}\) codes satisfying the following conditions:

\[
\text{wt}((x_1,x_2,\ldots,x_{10})) \equiv y_1y_2 \pmod{3},
\]

if \(c\) is not the zero vector, then \(\text{wt}((x_1,x_2,\ldots,x_{10})) \geq 3, \)

if \(\text{wt}((x_1,x_2,\ldots,x_{10})) = 3\), then \((y_1, y_2) \neq (0,0),\)

for any codeword \(c = (x_1,x_2,\ldots,x_{10},y_1,y_2) \in C\) (see [9, Claim 5.2]). Seven ternary \([12,5]/C_{3^3}\) codes satisfying the conditions (1)–(3) were found by Shimada and Zhang [9]. This motivates us to classify all such ternary \([12,5]/C_{3^3}\) codes.

For ternary \([12,5]/C_{3^3}\) codes satisfying the conditions (1)–(3), the following equivalence is considered in [9]. We say that two ternary \([12,5]/C_{3^3}\) codes
satisfying the conditions (1)–(3) are $SZ$-equivalent if one can be obtained from
the other by using the following:

$$(x_1, \ldots, x_{10}, y_1, y_2) \mapsto ((-1)^{a_1}x_{\sigma(1)}, \ldots, (-1)^{a_{10}}x_{\sigma(10)}, (-1)^{b}y_{\tau(1)}, (-1)^{b}y_{\tau(2)})$$

(4)

where $a_1, \ldots, a_{10}, b \in \{0, 1\}$ and $\sigma \in S_{10}, \tau \in S_2$ (see [9, Remark 5.3]). Here, $S_n$
denotes the symmetric group of degree $n$.

The main aim of this note is to give the following classification, which is
based on a computer calculation.

**Theorem 1.** Any ternary $[12, 5]$ code satisfying the conditions (1)–(3) is
$SZ$-equivalent to one of the seven codes given in [9, Remark 5.3].

To complete the above classification, ternary $[10, 5, d]$ codes are classified
for the cases $d = 3$ and 4.

2. Characterization of ternary $[12, 5]$ codes satisfying (1)–(3)

Let $C$ be a ternary $[n, k]$ code. The code obtained from $C$ by deleting
some coordinates $I$ in each codeword is called the punctured code of $C$ on $I$.
Throughout this note, we denote the punctured code of a ternary $[12, 5]$ code
$C$ on $\{11, 12\}$ by $\text{Pun}(C)$. Let $d_{\text{max}}(n, k)$ denote the largest minimum weight
among ternary $[n, k]$ codes. It is known that $d_{\text{max}}(10, 5) = 5$ and $d_{\text{max}}(12, 5) = 6$
(see [2], [5]).

**Lemma 1.** If $C$ is a ternary $[12, 5]$ code satisfying the condition (2), then
$\text{Pun}(C)$ is a ternary $[10, 5]$ code and $d(\text{Pun}(C)) \in \{3, 4, 5\}$.

**Proof.** Suppose that $\text{Pun}(C)$ has dimension at most 4. Then we may
assume without loss of generality that $C$ has generator matrix whose first row is
$(0, 0, \ldots, 0, y_1, y_2)$, where $(y_1, y_2) \neq (0, 0)$. This contradicts with the condition
(2). Hence, $\text{Pun}(C)$ is a ternary $[10, 5]$ code. Again, by the condition (2),
$\text{Pun}(C)$ has minimum weight at least 3. Since $d_{\text{max}}(10, 5) = 5$, the result
follows.

**Lemma 2.** Let $C$ be a ternary $[12, 5]$ code satisfying the conditions (1)–(3).
(i) $d(\text{Pun}(C)) \in \{4, 5\}$ if and only if $d(C) = 6$.
(ii) $d(\text{Pun}(C)) = 3$ if and only if $d(C) = 4$.

**Proof.** By Lemma 1, $\text{Pun}(C)$ is a ternary $[10, 5]$ code and $d(\text{Pun}(C)) \in$
$\{3, 4, 5\}$. It is trivial that $d(C) - d(\text{Pun}(C)) \in \{0, 1, 2\}$.

Suppose that $d(\text{Pun}(C)) \in \{4, 5\}$. Let $x = (x_1, \ldots, x_{10})$ be a codeword of
$\text{Pun}(C)$. If $\text{wt}(x) = 4$ (resp. 5), then any corresponding codeword $(x_1, \ldots, x_{10},$
$y_1, y_2)$ of $C$ has weight 6 (resp. 7), by the condition (1). Since $d_{\text{max}}(12, 5) = 6$,
we have that \( d(C) = 6 \). Conversely, if \( d(C) = 6 \), then it follows from \( d_{\text{max}}(10, 5) = 5 \) that \( d(\text{Pun}(C)) \in \{4, 5\} \).

Suppose that \( d(\text{Pun}(C)) = 3 \). Let \( x = (x_1, \ldots, x_{10}) \) be a codeword of \( \text{Pun}(C) \). If \( \text{wt}(x) = 3 \), then any corresponding codeword \( (x_1, \ldots, x_{10}, y_1, y_2) \) of \( C \) has weight 4, by the conditions (1) and (3). Hence, we have that \( d(C) = 4 \). Conversely, suppose that \( d(C) = 4 \). Then \( d(\text{Pun}(C)) \in \{2, 3, 4\} \).

By the condition (2), \( d(\text{Pun}(C)) \in \{3, 4\} \). From the statement (i), \( d(\text{Pun}(C)) = 3 \).

Recall that two ternary codes are equivalent if one can be obtained from the other by permuting the coordinates and (if necessary) changing the signs of certain coordinates. For ternary \([10, 5]\) codes, we consider this usual equivalence.

**Lemma 3.** Let \( C \) and \( C' \) be ternary \([12, 5]\) codes satisfying the conditions (1)–(3). Suppose that \( C \) and \( C' \) are SZ-equivalent. Then \( \text{Pun}(C) \) and \( \text{Pun}(C') \) are equivalent.

**Proof.** Suppose that \( C \) is obtained from \( C' \) by (4). Then \( \text{Pun}(C) \) can be obtained from \( \text{Pun}(C') \) by

\[
(x_1, \ldots, x_{10}) \mapsto ((-1)^{a_1} x_{a(1)}, \ldots, (-1)^{a_{10}} x_{a(10)}).
\]

By considering the inverse operation of puncturing, one can construct ternary \([12, 5]\) codes satisfying the conditions (1)–(3) as follows. Throughout this note, we denote the ternary code having generator matrix \( G \) by \( C(G) \). Suppose that \( C(G) \) is a ternary \([10, 5]\) code and \( d(C(G)) \in \{3, 4, 5\} \). Let \( g_i \) denote the \( i \)th row of \( G \). Consider the following generator matrix:

\[
G = \begin{pmatrix}
  a_1 & b_1 \\
  \vdots & \vdots \\
  a_5 & b_5
\end{pmatrix},
\]

where

\[
(a_i, b_i) = \begin{cases} 
  (0, 0), (0, 1), (0, 2), (1, 0), (2, 0) & \text{if } \text{wt}(g_i) \equiv 0 \pmod{3}, \\
  (1, 1), (2, 2) & \text{if } \text{wt}(g_i) \equiv 1 \pmod{3}, \\
  (1, 2), (2, 1) & \text{if } \text{wt}(g_i) \equiv 2 \pmod{3}.
\end{cases}
\]

We denote this generator matrix by \( G(a, b) \), where \( a = (a_1, \ldots, a_5) \) and \( b = (b_1, \ldots, b_5) \). The set of the codes \( C(G(a, b)) \) contains all ternary \([12, 5]\) codes \( C \) satisfying the conditions (1) and \( \text{Pun}(C(G(a, b))) = C(G) \). Hence, in this way, every ternary \([12, 5]\) code satisfying the conditions (1)–(3) can be obtained from some ternary \([10, 5]\) code. Here, by Lemma 2, its minimum weight is 3, 4 or 5. In addition, if \( C(G) \) and \( C(G') \) are equivalent \([10, 5]\)
codes, then the sets of all codes $C(G(a, b))$ satisfying the conditions (1)–(3) is obtained from the set of all codes $C(G'(a, b))$ satisfying the same conditions by considering (4) with $\beta = 0$ and $\tau$ is the identity permutation. Hence, it is sufficient to consider only inequivalent ternary $[10, 5, d]$ codes with $d \in \{3, 4, 5\}$ for the classification of ternary $[12, 5]$ codes satisfying the conditions (1)–(3). This is a reason why we consider the classification of ternary $[10, 5, d]$ codes with $d \in \{3, 4, 5\}$ in the next section.

3. Ternary $[10, 5, d]$ codes with $d \in \{3, 4, 5\}$

There is a unique ternary $[10, 5, 5]$ code, up to equivalence [6]. In this section, we give a classification of ternary $[10, 5, d]$ codes with $d \in \{3, 4\}$, which is based on a computer calculation.

We describe how ternary $[10, 5, 3]$ codes and $[10, 5, 4]$ codes were classified. Let $C$ be a ternary $[10, 5, 3]$ code (resp. $[10, 5, 4]$ code). We may assume without loss of generality that $C$ has generator matrix of the following form:

$$G = \begin{pmatrix} I_5 & A \end{pmatrix},$$

where $A$ is a $5 \times 5$ matrix over $F_3$ and $I_5$ denotes the identity matrix of order 5. Thus, we only need consider the set of $A$, rather than the set of generator matrices. The set of matrices $A$ was constructed, row by row, as follows, by a computer calculation. Let $r_i$ be the $i$th row of $A$. Then, we may assume without loss of generality that $r_1 = (0, 0, 0, 1, 1)$ (resp. $r_1 = (0, 0, 1, 1, 1)$), by permuting and (if necessary) changing the signs of the columns of $A$.

Let $e_1, \ldots, e_5$ denote the vectors $(1, 0, 0, 0, 0), \ldots, (0, 0, 0, 0, 1)$, respectively. We denote the ternary code generated by vectors $y_1, y_2, \ldots, y_s$ by $\langle y_1, y_2, \ldots, y_s \rangle$. For $x = (x_1, \ldots, x_5) \in F_3^5$, consider the following conditions:

- the first nonzero element of $x$ is 1,
- $\mathrm{wt}(x) \geq 2$ (resp. $\mathrm{wt}(x) \geq 3$),
- the ternary code $\langle (e_1, r_1), (e_2, x) \rangle$ has minimum weight 3 (resp. 4),
- $x_1 \leq x_2 \leq x_3 \leq 1$ and $x_4 \leq x_5$ (resp. $x_1 \leq x_2 \leq 1$ and $x_3 \leq x_4 \leq x_5$),

where we consider a natural order on the elements of $F_3 = \{0, 1, 2\}$ by $0 < 1 < 2$.

The determination of the minimum weights was done by a computer calculation for all codes in this note. Let $X_1$ be the set of vectors $x \in F_3^5$ satisfying the first three conditions. Let $X_2$ be the set of vectors $x \in X_1$ satisfying the fourth condition. Our computer calculation shows that $(\#X_1, \#X_2) = (115, 18)$ (resp. $(88, 14)$). Define a lexicographical order on $X_1$ induced by the above order of $F_3$, that is, $(a_1, \ldots, a_5) < (b_1, \ldots, b_5)$ if $a_1 < b_1$, or $a_1 = b_1, \ldots, a_k = b_k$ and $a_{k+1} < b_{k+1}$ for some $k \in \{1, 2, 3, 4\}$. The matrices $A$ were constructed, row by row, satisfying the following conditions:
• the ternary code \( \langle e_s, r_s \rangle \ | s = 1, 2, 3 \) has minimum weight 3 (resp. 4), where \( r_2 \in X_2, \ r_3 \in X_1 \),
• the ternary code \( \langle e_s, r_s \rangle \ | s = 1, 2, 3, 4 \) has minimum weight 3 (resp. 4), where \( r_2 \in X_2, \ r_3, r_4 \in X_1 \ (r_3 < r_4) \),
• the ternary code \( \langle e_s, r_s \rangle \ | s = 1, 2, 3, 4, 5 \) has minimum weight 3 (resp. 4), where \( r_2 \in X_2, \ r_3, r_4, r_5 \in X_1 \ (r_3 < r_4 < r_5) \).

It is obvious that the set of the matrices \( A \) which must be checked to achieve a complete classification, can be obtained in this way.

Then, by a computer calculation, we found 4328352 (resp. 650051) matrices \( A \). Our computer calculation shows the 4328352 ternary \([10, 5, 3]\) codes (resp. 650051 ternary \([10, 5, 4]\) codes) are divided into 527 (resp. 64) classes by comparing their Hamming weight enumerators. For each Hamming weight enumerator, to test equivalence of codes, we use the algorithm given in [7, Section 7.3.3] as follows. For a ternary \([n, k]\) code \( C \), define the digraph \( \Gamma(C) \) with vertex set

\[
(C - \{0\}) \cup (\{1, 2, \ldots, n\} \times (F_3 - \{0\}))
\]

and arc set

\[
\{(c, (j, c_j)) \ | \ c = (c_1, \ldots, c_n) \in C - \{0\}, c_j \neq 0, 1 \leq j \leq n \} \\
\cup \{(j, 1), (j, 2), ((j, 2), (j, 1)) \ | \ 1 \leq j \leq n \}.
\]

Then, two ternary \([n, k]\) codes \( C \) and \( C' \) are equivalent if and only if \( \Gamma(C) \) and \( \Gamma(C') \) are isomorphic. We use the package GRAPE [10] of GAP [4] for digraph isomorphism testing. After checking whether codes are equivalent or not by a computer calculation for each Hamming weight enumerator, we have the following:

**Proposition 1.** There are 135 ternary \([10, 5, 4]\) codes, up to equivalence. There are 1303 ternary \([10, 5, 3]\) codes, up to equivalence.

We denote the 135 ternary \([10, 5, 4]\) codes by \( C_{10, 4, i} \ (i = 1, 2, \ldots, 135) \), and we denote the 1303 ternary \([10, 5, 3]\) codes by \( C_{10, 3, i} \ (i = 1, 2, \ldots, 1303) \). Generator matrices of all codes can be obtained electronically from [1].

The unique ternary \([10, 5, 5]\) code \( C_{10, 5} \) is formally self-dual, that is, the Hamming weight enumerators of the code and its dual code are identical. In addition, the supports of the codewords of minimum weight in \( C_{10, 5} \) form a 3-design [3]. We verified by a computer calculation that 38 ternary \([10, 5, 4]\) codes and 242 ternary \([10, 5, 3]\) codes are formally self-dual. In addition, we verified by a computer calculation that the supports of the codewords of minimum weight in only the code \( C_{10, 4, 132} \) form a 2-design and the supports of the codewords of minimum weight in \( C_{10, 4, i} \) form a 1-design for only \( i = 6, 86, 87, 89, 132 \).
4. Ternary \([12,5]\) codes satisfying (1)–(3)

In this section, we give a classification of ternary \([12,5]\) codes satisfying the conditions (1)–(3), which is based on a computer calculation. This is obtained from the classification of ternary \([10,5,d]\) codes with \(d \in \{3,4,5\}\), by using the method given in Section 2.

4.1. From the \([10,5,5]\) code and the \([10,5,4]\) codes. As described in the previous section, there is a unique ternary \([10,5,5]\) code, up to equivalence \([6]\). It follows from \([3]\) that this code \(C_{10,5}\) has generator matrix \(G_{10,5} = (I_5 \ A)\), where \(A\) is the following circulant matrix:

\[
A = \begin{pmatrix}
12210 \\
01221 \\
10122 \\
21012 \\
22101
\end{pmatrix}.
\]

In order to construct all ternary \([12,5]\) codes \(C\) satisfying the conditions (1) and \(Pun(C) = C_{10,5}\), we consider generator matrices \(G_{10,5}(a,b)\) of the form (5). Since the weight of each row of \(G_{10,5}\) is 5, \((a_i,b_i) = (1,2)\) or \((2,1)\) for \(i = 1,2,3,4,5\). By (4), we may assume that \((a_1,b_1) = (1,2)\). Since the weight of the sum of the first row and the second row of \(G_{10,5}\) is 5, \((a_2,b_2)\) must be \((1,2)\). Similarly, we have that \((a_i,b_i) = (1,2)\) for \(i = 3,4,5\), since \(A\) is circulant. In addition, we verified by a computer calculation that this code satisfies the condition (1). Note that the code automatically satisfies the conditions (2) and (3). We denote the code by \(C_{12,1}\).

Now, consider the ternary \([10,5,4]\) codes \(C_{10,4,i}\) \((i = 1,2,\ldots,135)\). By considering generator matrices of the form (5), we found all ternary \([12,5]\) codes \(C\) satisfying the conditions (1) and \(Pun(C) = C_{10,4,i}\). This was done by a computer calculation. We denote by \(G_{10,4,i}\) the generator matrix \((I_5 \ A)\) of \(C_{10,4,i}\) for each \(i\). Since the weight of the first row of \(A\) is 3 (see Section 3), by (4), we may assume that \((a_1,b_1) = (1,1)\) in (5). Under this situation, we verified by a computer calculation that only the codes \(C_{10,4,60}\) and \(C_{10,4,132}\) give ternary \([12,5]\) codes satisfying the condition (1). Note that these codes automatically satisfy the conditions (2) and (3). In Table 1, we list the matrices \(A\) and \((a^T,b^T)\) in \(G_{10,4,i}(a,b)\) for \(i = 60,132\), where \(a^T\) denotes the transposed of a vector \(a\). It can be seen by hand that the two codes \(C(G_{10,4,60}(a,b))\) are SZ-equivalent. By Lemma 3, there are two ternary \([12,5]\) codes \(C\) satisfying the conditions (1)–(3) and the condition that \(Pun(C)\) is a ternary \([10,5,4]\) code. We denote the two codes by \(C_{12,2}\) and \(C_{12,3}\), respectively (note that take the first \((a^T,b^T)\) for \(i = 60\).
Lemma 2 shows that there are no other ternary \([12, 5, 6]\) codes satisfying the conditions (1)–(3). Hence, we have the following:

**Lemma 4.** Up to SZ-equivalence, there are three ternary \([12, 5, 6]\) codes satisfying the conditions (1)–(3).

### 4.2. From the \([10, 5, 3]\) codes.

By considering generator matrices of the form (5), we found all ternary \([12, 5]\) codes \(C\) satisfying the conditions (1) and \(\text{Pun}(C) = C_{10,3,i}\) \((i = 1, 2, \ldots, 1303)\). This was done by a computer calculation. We denote by \(G_{10,3,i}\) the generator matrix \((I_5 \ A)\) of \(C_{10,3,i}\) for each \(i\). Since the weight of the first row of \(A\) is 2 (see Section 3), by (4), we may assume that \((a_1, b_1) = (0, 1)\) in (5). Under this situation, we verified by a computer calculation that only the codes \(C_{10,3,i}\) give ternary \([12, 5]\) codes satisfying the condition (1) for

\[
i = 302, 639, 662, 666, 667, 756, 878, 957, 958, 987, 1210, 1215, 1241, 1245, 1263, 1285, 1297, 1298, 1299.
\]

In this case, there are codes satisfying the condition (1), but not (3). We verified by a computer calculation that only the codes \(C_{10,3,i}\) codes satisfying the conditions (1) and (3) for \(i = 302, 666, 987, 1245\). Note that these four codes automatically satisfy the condition (2). In Table 2, we list the matrices \(A\) and \((a^T, b^T)\) in \(G_{10,4,i}(a, b)\) for \(i = 302, 666, 987, 1245\). By Lemma 3, there are four ternary \([12, 5]\) codes satisfying the conditions (1)–(3) and the condition that \(\text{Pun}(C)\) is a ternary \([10, 5, 3]\) code. We denote the four codes by \(G_{12,i}\) \((i = 4, 5, 6, 7)\), respectively.

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**Table 1.** Generator matrices \(G_{10,4,i}(a, b)\) \((i = 60, 132)\)

| \(i\) | \(A\) | \((a^T, b^T)\) |
|-----|-----|-----|
| 60  | \[00111, 01011, 10101, 11001, 12210\] | \[11, 22, 22, 11, 12\] |
| 132 | \[00111, 01011, 10101, 11001, 11111\] | \[11, 22, 11, 00\] |
Lemma 2 shows that there are no other ternary $[12, 5, 4]$ codes satisfying the conditions (1)–(3). Hence, we have the following:

**Lemma 5.** Up to SZ-equivalence, there are four ternary $[12, 5, 4]$ codes satisfying the conditions (1)–(3).

Up to SZ-equivalence, seven ternary $[12, 5]$ codes satisfying the conditions (1)–(3) are known (see [9, Remark 5.3]). Lemmas 4 and 5 show that there are no other ternary $[12, 5]$ codes satisfying the conditions (1)–(3). Therefore, we have Theorem 1.

### 4.3. Some properties.

For the ternary $[12, 5]$ codes $C$ satisfying the conditions (1)–(3), instead of the Hamming weight enumerators, we consider the weight enumerators \( \sum_{(x_1, \ldots, x_{10}, y_1, y_2) \in C} x^{\text{wt}(x_1, \ldots, x_{10})} y^{n_1} z^{n_2}, \) where \( n_1 \) and \( n_2 \) are the numbers of 1’s and 2’s in \((y_1, y_2)\), respectively. We verified by a computer calculation that the codes $C_{12,i}$ \((i = 1, 2, \ldots, 7)\) have the following weight enumerators $W_i$:

\[
W_1 = 1 + 72x^5yz + 60x^6 + 90x^8yz + 20x^9,
\]

\[
W_2 = 1 + 9x^4z^2 + 9x^4y^2 + 18x^5yz + 24x^6 + 36x^6z + 36x^6y + 18x^7z^2 + 18x^7y^2 + 36x^8yz + 2x^9 + 18x^9z + 18x^9y,
\]

\[
W_3 = 1 + 15x^4z^2 + 15x^4y^2 + 60x^6 + 60x^7z^2 + 60x^7y^2 + 20x^9 + 6x^{10}z^2 + 6x^{10}y^2,
\]

\[
W_4 = 1 + 2x^3z + 2x^3y + 4x^4z^2 + 4x^4y^2 + 24x^5yz + 18x^6 + 38x^6z + 38x^6y + 22x^7z^2 + 22x^7y^2 + 30x^8yz + 8x^9 + 14x^9z + 14x^9y + x^{10}z^2 + x^{10}y^2,
\]

| $i$ | $A$ | $(a^T, b^T)$ | $i$ | $A$ | $(a^T, b^T)$ |
|-----|-----|-------------|-----|-----|-------------|
| 302 | \(\begin{pmatrix} 00011 \\ 01100 \\ 10101 \\ 11010 \\ 11221 \end{pmatrix}\) | \(\begin{pmatrix} 01 \\ 01 \\ 22 \\ 22 \\ 01 \end{pmatrix}\) | 987 | \(\begin{pmatrix} 00011 \\ 01010 \end{pmatrix}\) | \(\begin{pmatrix} 01 \\ 20 \end{pmatrix}\) |
| 666 | \(\begin{pmatrix} 00011 \\ 01100 \\ 10101 \\ 11010 \\ 12222 \end{pmatrix}\) | \(\begin{pmatrix} 01 \\ 01 \\ 22 \\ 22 \\ 20 \end{pmatrix}\) | 1245 | \(\begin{pmatrix} 00011 \\ 01010 \end{pmatrix}\) | \(\begin{pmatrix} 01 \\ 20 \end{pmatrix}\) |
$$W_5 = 1 + 3x^3z + 3x^3y + 3x^4z^2 + 3x^4y^2 + 18x^5yz + 24x^6 + 39x^6z + 39x^6y + 21x^7z^2 + 21x^7y^2 + 36x^8yz + 2x^9 + 12x^9z + 12x^9y + 3x^{10}z^2 + 3x^{10}y^2,$$

$$W_6 = 1 + 4x^3z + 4x^3y + 5x^4z^2 + 5x^4y^2 + 24x^5yz + 18x^6 + 34x^6z + 34x^6y + 20x^7z^2 + 20x^7y^2 + 30x^8yz + 8x^9 + 16x^9z + 16x^9y + 2x^{10}z^2 + 2x^{10}y^2,$$

$$W_7 = 1 + 6x^3z + 6x^3y + 9x^4z^2 + 9x^4y^2 + 36x^5yz + 24x^6 + 42x^6z + 42x^6y + 18x^7z^2 + 18x^7y^2 + 18x^8yz + 2x^9 + 6x^9z + 6x^9y,$$

respectively. These weight enumerators guarantee that the codes $C_{12,i}$ ($i = 1, 2, \ldots, 7$) satisfy the conditions (1)–(3). By putting $y = z = 1$, the above weight enumerators determine the Hamming weight enumerators of $Pun(C_{12,i})$ ($i = 1, 2, \ldots, 7$). This implies that $C_{12,1}$ is SZ-equivalent to $C_7$ in [9, Table 5.1]. In addition, by comparing generator matrices, it is easy to see that $C_{12,i}$ ($i = 2, 3, \ldots, 7$) are equal to $C_6$, $C_5$, $C_3$, $C_4$, $C_2$ and $C_1$ in [9, Table 5.1], respectively.

**Remark 1.** Shimada and Zhang [9] also considered the existence of ternary codes satisfying the condition that all codewords have weight divisible by three, in the proof of Theorem 1.4 (see [9, Claim 6.2]). We point out that a code satisfying the condition is self-orthogonal. There is a unique self-orthogonal ternary code, up to equivalence [8, Table 1].

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Makoto Araya
Department of Computer Science
Faculty of Informatics
Shizuoka University
Hamamatsu 432-8011, Japan
E-mail: araya@inf.shizuoka.ac.jp

Masaaki Harada
Research Center for Pure and Applied Mathematics
Graduate School of Information Sciences
Tohoku University
Sendai 980-8579, Japan
E-mail: mharada@m.tohoku.ac.jp