Quantum line operators from Lax pairs

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Abstract: Motivated by the realisation of Yang-Baxter equation of 2d Integrable models in the 4d gauge theory of Costello-Witten-Yamazaki (CWY), we study the embedding of integrable 2d Toda field models inside this construction. This is done by using the Lax formulation of 2d integrable systems and by thinking of the standard Lax pair $L_\pm$ in terms of components of CWY gauge connection propagating along particular directions in the gauge bundle. We also use results of the CWY theory to build quantum line operators for 2d Toda models and compute the one loop contribution of two intersecting lines exchanging one gluon. Other features like local symmetries and comments on extension of our method to other 2d integrable models are also discussed.

Keywords: 2d integrable models, Liouville equation, Costello-Witten-Yamazaki theory, line operators, R-matrix and solution of Yang-Baxter equation.
1 Introduction

Yang-Baxter equation (YBE) generally expressed $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ is a basic hypermatrix relation of 2d quantum integrable models that has been subject to many studies [1]-[3] and has been revealed to be important for several issues; for example in the formulation of Hopf algebras and quantum groups [4]-[8], and in dealing with knots of 3d Chern-Simons
gauge theory [9–12]; as well as with relationships concerning integrable lattice models from the view of topological quantum field theories [13, 14]; see also [15] for a Gauge/YBE correspondence linking SU(N) quiver gauge theories with the partition function of 2d integrable spin models. This non linear equation in R-matrix has initially appeared in two different contexts of integrable models as a sufficient condition for exact solvability. It appeared first in the factorisation property of many body scattering amplitudes of relativistic QFT [16, 17]; and second for the transfer matrix of statistical models to commute for different values of the spectral parameters [18, 19].

Recently a formal topological four-dimensional gauge theory has been constructed in [20] to deal with the fundamentals of the Yang-Baxter equation in terms of a non abelian gauge potential $A_\mu$ and of gauge invariant quantum line operators $W_{\epsilon_i}(K_i)$ as basic quantities that are behind the derivation of the solutions of the matrix R and also behind the study of its quantum properties. Based on previous results of [21, 22] and nicely motivated in [23], this construction uses the power of the quantum field theory (QFT) method to study quantum properties of intersecting multi-line configurations $K_1,...,K_n$ supporting non local observables $W_{\epsilon_1}(K_1),...,W_{\epsilon_n}(K_n)$. It has allowed to rederive known results on 2d integrable systems in a nice manner; and has permitted moreover to obtain new involved ones like the RTT presentation for Yangians $\mathcal{Y}(G_c)$ [24]. For other applications, see also [25] dealing with unification of integrability in supersymmetric models and for the six dimensional origin of topological invariant constructions. The formal gauge field theory of [20] to which we refer hereafter to as the CWY theory — CWY for Costello-Witten-Yamazaki — is a topological gauge theory with 1-form gauge connection $A_\mu = A_\mu dX^\mu$ described by a partial gauge potential $A_\mu = (A_x, A_y, A_\zeta)$ living on 4d manifolds $M_4$ that factorises as the product of two Riemann surfaces $\Sigma$ and $C$. In the CWY approach, the well known three kinds of quasi-classical solutions of the Yang-Baxter equation (rational, trigonometric, elliptic) and their underlying quantum group symmetries [26] have been derived from specific aspects of the 4d space $M_4 = \Sigma \times C$, hosting the CWY gauge theory, and from properties the quantum line operators like framing anomaly, fusion of lines due to scaling symmetry as well as line operator product expansions [24].

In this paper, we contribute to this matter by studying the embedding of a family of 2d integrable QFT models inside the CWY theory and use this approach to get more inside on properties of their quantum integrability. Concretely, we consider conformal Toda field theory in 2d, which constitute a class of 2d integrable QFT models based on finite dimensional Lie algebras, and study its embedding into the 4d CWY theory. The leading 2d QFT model in this finite Toda QFT$_2$ family is given by the well known integrable Liouville theory which is associated with sl(2) algebra and which we take it here as an illustrating example. By focussing on this leading sl(2) based model, we show that the insertion of the Liouville
field into CWY can be done by using the Lax formalism allowing to linearise the Liouville equation by help of a pair of operators \( L_\pm \). Here, the Lax pair \((L_+, L_-)\) is thought of as given by a particular gauge field configuration that solves the field equation of motion of the CWY vector potential. Using this approach, we develop a method to build quantum line operators \( W(K_i) \) for Liouville theory that define non local observables of the theory and which give the bridge between Liouville field and the Yang-Baxter equation of 2d integrable models. As an application of the construction, we calculate as well the amplitude at one loop order of two intersecting lines exchanging one gluon.

The organisation of this paper is as follows: In section 2, we review some useful aspects on the 4d CWY theory. In section 3, we recall the classical Liouville equation and give a list of some of its remarkable properties which are relevant to the present study. In section 4, we develop the 2d Lax formalism for Liouville field and study links with the CWY connection. In section 5, we study the embedding of the Liouville equation into the CYW modeling and build the associated quantum lines. In section 6, we compute the amplitude of two intersecting lines exchanging one gluon. Section 7 is devoted to the conclusion and to comments.

2 CYW theory: an overview

In this section, we give a brief review of those tools of the Costello-Witten-Yamazaki gauge theory that are useful for dealing with the modeling of the solutions of the Yang-Baxter equation and for the study of interacting line operators. Some of these tools will be rephrased so that they can be used in next sections when considering the application of the CWY theory to approach finite Toda QFT\(_2\)'s; in particular the integrable 2d Liouville theory and the building of its quantum line operators.

2.1 Formal field action

A manner to introduce the 4d CWY theory modeling the solutions of Yang-Baxter (YB) equation \( R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \) in terms of partial gauge fields \( A_\mu (X) \) is to start from the explicit expression of the field action \( S_{cyw}[A] \) and the gauge invariant observables \( W_{\xi_i} [K_{\zeta_i}] \) of the theory. Then, use the path integral method and Feynman diagram rules to approach the quasi-classical solutions of the R-matrices \( R_{ij} (\zeta_{ij}) \) by using crossing quantum line operators \( W_{\xi_i} [K_{\zeta_i}] \) exchanging gauge particles as in figure 2.

The action \( S_{cyw} \) is a formal 4d functional living on a 4 space \( \mathbb{M}_4 \) given by the cross product of two Riemann surfaces; \( \mathbb{M}_4 = \Sigma \times \mathbb{C} \). It reads in terms of differential forms as

\[
S_{cyw}[A, \omega] = \frac{1}{2\pi} \int_{\Sigma \times \mathbb{C}} L_4(A, \omega)
\]  

(2.1)
where the 4-form Lagrangian $L_4$ has some special features of which the two following ones: (i) it is holomorphic in the partial gauge connection $A$ valued in a complexified finite dimensional Lie algebra $\mathcal{G}_c$; i.e: no adjoint conjugate $A^\dagger \equiv \overline{A}$, and (ii) it is given by the exterior product on $\Sigma \times C$,

$$L_4 = \omega_1 \wedge \Omega_3$$

with 3-form $\Omega_3$ of Chern-Simons (CS) type

$$\Omega_3 = A \wedge dA + \frac{2}{3} A \wedge A \wedge A$$

and $\omega_1 = \omega_\zeta d\zeta$ a holomorphic 1-form living on the complex curve $C$ with no zeros; but may have poles. The two real surfaces $\Sigma$ and $C$ play an important role in the CWY construction; the $\Sigma$ hosts the real curves $K_\zeta$ supporting the line operators $W_\theta [K_\zeta]$; and the $C$ gives the coordinate $\zeta$ of the curve $K_\zeta$ inside $\mathbb{M}_4$ and then the spectral parameter of the R-matrices. By denoting the four local coordinates of $\mathbb{M}_4 = \Sigma \times C$ like

$$X^M = (x, y; \bar{\zeta}, \zeta)$$

with $(X_1, X_2) = (x, y)$ for the Riemann surface $\Sigma$ and $\zeta = X_3 + iX_4$ for the complex line $C$, we have

$$\Omega_3 = \Omega_{[\mu\nu\sigma]} dX^\mu \wedge dX^\nu \wedge dX^\sigma$$

with

$$\Omega_{[\mu\nu\sigma]} = A_{[\mu} \partial_{\nu} A_{\sigma]} + \frac{2}{3} A_{[\mu} A_{\nu} A_{\sigma]}$$

and where $A_\mu = t_\alpha A_\alpha^\mu$ with $t_\alpha$ standing for a basis of generators of $\mathcal{G}_c$. For the holomorphic 1-form, we have the three following possibilities depending on the nature of the complex curve $C$,

| complex curve $C$ | holomorphic 1-form $\omega_1$ |
|-------------------|--------------------------|
| $\mathbb{C}$      | $d\zeta$                 |
| $\mathbb{C}^\times$ | $d (\log \zeta)$         |
| $\mathbb{C} / (\mathbb{Z} + \tau \mathbb{Z})$ | $d\zeta$                 |

Notice that for convenience, we use below three kinds of space indices to refer to the space coordinates and to the fields living on $\Sigma \times C$. The capital latin index $M$ to refer to full space vectors like $X^M = (x, y; \bar{\zeta}, \zeta)$ or equivalently $(X^+, X^-; \bar{\zeta}, \zeta)$ with $X^\pm = x \pm iy$. The Greek index $X^\mu$ to refer to the subspace $(X^+, X^-; \bar{\zeta})$ where spread the 3-form $\Omega_3$ directions. Finally the tiny latin index like in $X^m$ to refer to $(X^+, X^-)$ and in general for vectors on $\Sigma$. So, we have the convention notation

$$M = \mu; \zeta , \quad \mu = m; \bar{\zeta} , \quad m = +, -$$
The field equation of the \textit{partial} 1-form gauge potential \( A = A_\mu dX^\mu \) is obtained by the functional variation of \( S_{cyw} \); it is given, in differential form language, by

\[
\frac{\delta S_{cyw}}{\delta A} = 0 \quad \Rightarrow \quad \omega_1 \wedge F = 0 \tag{2.8}
\]

with non-vanishing holomorphic 1-form, \( \omega_1 \neq 0 \); and where the 2-form \( F = F_{\mu\nu} dX^\mu \wedge dX^\nu \) is nothing but the gauge curvature of the partial vector potential \( A \) filling the \((x, y; \bar{\zeta})\) space directions and reading as follows

\[
F = dA + A \wedge A \tag{2.9}
\]

The gauge field equation of motion (2.8) is naturally solved as \( F = 0 \) requiring \( F_{\mu\nu} = 0 \) and which reads in terms of the \( A_\mu \) components of the partial gauge connection as follows

\[
\begin{align*}
\partial_+ A_- - \partial_- A_+ + [A_+, A_-] &= 0 \\
\partial_\bar{\zeta} A_+ - \partial_+ A_\bar{\zeta} + [A_\bar{\zeta}, A_+] &= 0 \\
\partial_\bar{\zeta} A_- - \partial_- A_\bar{\zeta} + [A_\bar{\zeta}, A_-] &= 0
\end{align*} \tag{2.10}
\]

The \( A_\mu \) defines a flat gauge bundle on \( \Sigma \) which varies holomorphically as we move on \( C \). In section 5, we will study a particular solution of these component field equations; before that, notice the three useful features. First, eqs (2.1) and (2.8) have rich local symmetries allowing to perform several operations like for instance doing gauge transformations or also moving in a safe way line operators from the left to the right in a system with apparently crossing lines configurations as schematized by figure (1). In addition to the complexified gauge symmetry with Lie algebra \( \mathcal{G}_e \) allowing freedom in changing the vector potential as \( A'_\mu = g^{-1} A_\mu g + g^{-1} \partial_\mu g \), the action \( S_{cyw} \) and the field equation are also invariant under \( \text{diff}(\Sigma) \), the group of diffeomorphisms of \( \Sigma \) given by general coordinates transformations \( X'^m = f^m (X^+, X^-) \); and are invariant as well under \( \text{Hol}(C) \), the group of holomorphic
transformations on $C$ with local coordinate $\zeta$. Second, from $S_{cyw}[A, \omega]$, one can determine the free gauge propagators $G_{\mu\nu}^{ab}(X - X') = \langle A_a^\mu(X) A_b^\nu(X') \rangle$ represented by the wavy red line in figure 2. One can also learn from the interacting part of the action the structure of the 3-vertex $\Gamma_{\mu\nu\sigma}^{abc}$ of the tri-vector fields coupling $\langle A_a^\mu, A_b^\nu, A_c^\sigma \rangle$. Using p-form language by killing the $\mu$- space indices with the help of the differentials $dX^\mu$, the free propagators $G_{\mu\nu}$ get mapped to 2-form propagators

$$P_{ab} = \delta_{ab} P$$

with $P = G_{\mu\nu} dX^\mu \wedge dX^\nu$ and where

$$G_{\mu\nu} = \frac{1}{4\pi} \varepsilon_{\mu\nu\sigma\tau} \frac{\partial}{\partial X^\tau} \left( \frac{1}{(x - x')^2 + (y - y')^2 + |\zeta - \zeta'|^2} \right)$$

(2.12)

Similarly, the vertex of the coupling of the three gauge fields carries only adjoint representation group indices and reads in terms of the $f_{abc}$ structure constant of $G_c$ as follows

$$\Gamma_{abc} = \frac{i}{2\pi} f_{abc} d\zeta$$

(2.13)

The third feature we want to comment here concerns observables $O$ in the CWY theory and their quantum properties. In this regards, it is interesting to notice that the on shell vanishing of the gauge curvature $F$ teaches us that the CWY construction behaves somehow like the 3d topological CS gauge theory in the sense that there are no local observables $O(F)$ in the CWY theory that can be built by using gauge invariant polynomials in the curvature $F$; they vanish identically due to gauge invariance and the field equation of motion. However, the construction of non trivial observables in the CWY theory is still possible; it needs considering other gauge invariants that are non local quantities as described in what follows for the example $M_4 = \mathbb{R}^2 \times \mathbb{C}$.

### 2.2 Observables

Despite the on shell vanishing of the 2-form gauge curvature $F = 0$, we can still construct observables $O$ in the 4d gauge theory of Costello-Witten-Yamazaki; they are given by non local invariant operators, some of them will be described in a moment. The vacuum expectation values (VEV) of these observable are given by the usual path integral formulation

$$\langle O \rangle = \frac{\int [DA] O \exp \left( \frac{i}{\hbar} S_{cyw} \right)}{\int [DA] \exp \left( \frac{i}{\hbar} S_{cyw} \right)}$$

(2.14)

with $S_{cyw} = \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{C}} d\zeta \wedge \Omega_3$; and where $\hbar$ is a parameter scaling as length with powers capturing the quantum loop corrections. A particular class of these gauge invariant quantities
Figure 2. An example of Feynman diagram with three interacting line operators $W_{\phi_1}(K_1), W_{\phi_2}(K_2)$ and $W_{\phi_3}(K_3)$ with spectral parameters $\zeta_1, \zeta_2$ and $\zeta_3$. Two line operators are taken as parallel in the topological plane with equations $y = 0$ and $y = \epsilon$. The third is given by $x = 0$. The three lines interact through gluons with bulk interaction located at $(x, y, \bar{\zeta}, \zeta)$.

is given by the line operators

$$W_{\phi} [ \varphi(K_\zeta)] = Tr_{\phi} \left[ P \exp \left( \oint_{K_\zeta} A \right) \right]$$  \hspace{1cm} (2.15)$$

with $P$ referring to the path ordering and $\phi$ to some representation of the finite dimensional $G_c$. The presence of the trace is to ensure invariance under gauge transformations. Like for Wilson line operators of the 3d Chern-Simons gauge theory, these gauge invariant quantities of the CWY theory are based as well on the holonomy term of the 1-form gauge potential $A$; but along particular loops $K_\zeta$ in the 4d space

$$\varphi(K_\zeta) = \oint_{K_\zeta} A$$  \hspace{1cm} (2.16)$$

Indeed, the real curves $K_\zeta$ involved in the building of the $W_{\phi} [ \varphi(K_\zeta)]$ operators are very special in the sense that they should belong to the surface $\Sigma$ part of $M_4$; but also live at some point $\zeta$ in $C$. These $K_\zeta$'s are then described by real algebraic equations relating the $x,y$ variables like $y = f_\zeta (x)$ where the complex $\zeta \in C$ plays the role of a spectral parameter. Therefore, the above holonomy should be treated as

$$\varphi(K_\zeta) = \oint_{K_\zeta} t_a A^a_n dX^n$$  \hspace{1cm} (2.17)$$
with $X^m = (x, y)$ and $t_a$ the generators of $G_c$. By using $X^m = (X^+, X^-)$, we also have $\mathcal{A} = \mathcal{A}_+ dX^+ + \mathcal{A}_- dX^-$ on the curve $K_\zeta$. Notice that the gauge components $\mathcal{A}_m^a$ in above (2.17) have a hidden structure due to the presence of the spectral parameter $\zeta$. Because of the coordinate dependence $\mathcal{A}_m^a = \mathcal{A}_{m+n}^a (x, y, \zeta, \bar{\zeta})$, one can define generalised Wilson lines $W_\hat{\phi} (K_\zeta)$ extending (2.15). This is done by substituting the $\mathcal{A}_\pm = t_a \mathcal{A}_m^a$ in (2.17) by a holomorphic expansion in the spectral parameter $\zeta$ like

$$\hat{A}_\pm (x, y, \zeta) := \sum_{k=0}^{\infty} t_{a,k} \hat{A}_\pm^{a(k)} (x, y)$$

with

$$t_{a,n} = t_a \otimes \zeta^n$$

and where the $\hat{A}_\pm^{a(n)} (x, y)$ modes in eq(2.18) may be imagined as given by the following modes in the $\zeta$- expansion where $\bar{\zeta}$ has been omitted,

$$\hat{A}_\pm^{a(n)} (x, y) := \frac{1}{n!} \frac{\partial^n}{\partial \zeta^n} A_\pm^a (x, y, \zeta, \bar{\zeta}) \bigg|_{\zeta = \bar{\zeta} = 0}$$

Notice that the induced operators $t_{a,n} = t_a \otimes \zeta^n$ generate an infinite dimensional Lie algebra $\mathcal{G}[[\zeta]]$ containing the finite dimensional $\mathcal{G}_c$ as the subalgebra of zero modes. In terms of the $\hat{A} = \hat{A} (x, y, \zeta)$, the generalised Wilson line operators read therefore as follows

$$W_\hat{\phi} (K_\zeta) = T \hat{\phi} \left[ P \exp \left( \int_{K_\zeta} \hat{A} \right) \right]$$

where now $\hat{\phi}$ stands for a representation of $\mathcal{G}[[\zeta]]$. Notice that the presence of the $T \hat{\phi}$ in defining above operator is somehow undesirable as it kills the effect of $\mathcal{G}[[\zeta]]$ and makes the generalisation meaningless; by following [20], this difficulty may be overcome by dropping the trace in defining $W_\hat{\phi} (K_\zeta)$; that is restricting (2.21) to

$$W_\hat{\phi} (K_\zeta) \sim P e^{\hat{\phi} (K_\zeta)} \quad , \quad \hat{\phi} (K_\zeta) = \int_{K_\zeta} \hat{A}$$

Though apparently not invariant under gauge transformations $g = g (X^+, X^-)$ since the $e^{\hat{\phi} (K_\zeta)}$ holonomy varies like $g^{-1} e^{\hat{\phi} (K_\zeta)} g$; however this holonomy can be made gauge invariant if considering the limit of lines $K_\zeta$ spreading to infinity in $\Sigma$ with the property $g \to I$ when $|X^\pm| \to \infty$. This restriction can be also justified by the infrared-free limit of the CWY theory where the holonomy term behaves as a gauge invariant quantity.

With this brief review of CWY formalism and the rephrasing of some of its tools, we come now to address the question on how to embed known 2d integrable QFTs in the Costello-Witten-Yamazaki theory. In what follows, we shall focus on the explicit solutions of eqs(2.10) by
first considering the restriction of these relations to the subspace \( \Sigma \); so the three relations reduces to the first one of (2.10) and will be interpreted in terms of 2d Lax equations of 2d integrable systems. After that, we turn to study the solution of (2.10) for the full \( \Sigma \times \mathcal{C} \) with \( \mathcal{G}_c \) a complex finite dimensional Lie algebra.

3 Liouville equation and special aspects

We begin by introducing briefly the classical Liouville equation in 2d space by considering first both Lorentzian \( \mathbb{R}^{1,1} \) and euclidian \( \mathbb{R}^2 \) signature; but focusing later on \( \mathbb{R}^2 \). We also use this description to fix some convention notations. After that, we make three comments on this 2d field equation which are helpful when studying the embedding of Liouville field into CWY theory. Some aspects on finite 2d Toda theory will be also commented.

3.1 2d field action

In real 2d space-time space \( \mathbb{R}^{1,1} \) with 1+1 signature and local coordinates \( \rho^\alpha = (\sigma, \tau) \), the classical Liouville equation is an integrable 2d field equation of the form

\[
\frac{\partial^2 \phi}{\partial \tau^2} - \frac{\partial^2 \phi}{\partial \sigma^2} + \tilde{\kappa} e^{2\phi} = 0
\]  

with \( \phi = \phi(\sigma, \tau) \) a real 2d field and \( \tilde{\kappa} \) a real constant parameter scaling as \((\text{length})^{-2}\).

This 2d field equation, which can be also presented like \( \frac{\partial^2 \phi}{\partial \rho^+ \partial \rho^-} + \kappa e^{2\phi} = 0 \) with light cone coordinates \( \rho^\pm = \tau \pm \sigma \) and \( \frac{\partial}{\partial \rho^\pm} = \frac{1}{2}(\frac{\partial}{\partial \tau} \pm \frac{\partial}{\partial \sigma}) \), can be derived from an action principle \( \delta S_L[\phi] = 0 \) with

\[
S_L[\phi] \sim \int_{\Sigma} \left( \frac{\partial \phi}{\partial \rho^-} \frac{\partial \phi}{\partial \rho^+} - \frac{\kappa}{2} e^{2\phi} \right) 
\]

where the real surface \( \Sigma \) is given here by \( \mathbb{R}^{1,1} \). This action describes a non linear dynamics of the real 2d scalar field \( \phi = \phi(\rho^+, \rho^-) \) to which we refer below to as the classical Liouville field; the integral measure in (3.2) is given by \( d\tau d\sigma = \frac{1}{2} d\rho^- \wedge d\rho^+ \) with \( \varepsilon_{-+} = \varepsilon^{+-} = 1 \). The scalar potential of the Liouville theory namely

\[
V(\phi) = \frac{\kappa}{2} e^{2\phi}
\]

has two special aspects: First, the factor 2 in the argument of \( e^{2\phi} \) may be imagined in terms of the Cartan matrix \( C_{11} = 2 \) of \( \text{sl}(2) \), it indicates how the Liouville theory can be extended to 2d Toda theories\(^1\) based on finite dimensional Lie algebra \([27–29]\). There, the 2d Toda fields

\(^1\)We refer to this class of 2d field models as finite Toda QFT\(_2\); this family is sometimes designated as conformal Toda QFT\(_2\). Notice that there exists also another class of Toda QFT\(_2\) based on affine Lie algebras and known as affine Toda theories \([30–32]\).
\{\phi_1, \ldots, \phi_r\} extending the Liouville \(\phi\) are given by \(r\) real scalars that can be also presented like
\[ \vec{\phi} = \vec{\alpha}_1 \phi_1 + \vec{\alpha}_2 \phi_2 + \ldots + \vec{\alpha}_r \phi_r \] (3.4)
with \(\vec{\alpha}_1, \ldots, \vec{\alpha}_r\) standing for the simple roots of \(G_c\). In this extension, the field action reads as follows [33–35],
\[ S_{Toda} [\phi_1, \ldots, \phi_r] \sim \int_{\Sigma} \left[ C_{ij} \frac{\partial \phi_i}{\partial \rho^-} \frac{\partial \phi_j}{\partial \rho^+} - \sum_{i=1}^{r} \kappa_i e^{C_{ij} \phi_j} \right] \] (3.5)
with \(C_{ij} = \frac{2}{\vec{\alpha}_i \cdot \vec{\alpha}_i} \vec{\alpha}_i \cdot \vec{\alpha}_j\) the Cartan matrix of \(G_c\) and a scalar potential as follows
\[ V_{Toda} (\phi_1, \ldots, \phi_r) = \sum_{i=1}^{r} \kappa_i \exp \left( \sum_{j=1}^{r} C_{ij} \phi_j \right) \] (3.6)
For the case of \(sl(r+1)\), the \(C_{ij}\) is a symmetric \(r \times r\) matrix reading as.
\[ C_{ij} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & \ddots & 0 \\ 0 & \ddots & \ddots & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}_{r \times r} = \vec{\alpha}_i \cdot \vec{\alpha}_j \] (3.7)
Second, the scalar potential \(V(\phi)\) is highly non linear (non polynomial) and its minimum takes place at \(\phi \to -\infty\) making the quantisation of the 2d Liouville field \(\phi\) difficult to do by using the standard canonical QFT manner [36–38].
A similar equation to (3.1) and related expressions can be also written down for the 2d euclidian plane \(\mathbb{R}^2\) parameterised \(X^m = (x, y)\); and the same comments given above are still valid. By using \(\mathbb{R}^2 \sim \mathbb{C}\), we can also use the complex coordinate \(\xi = x + iy\) and its conjugate \(\bar{\xi} = x - iy\) to deal with
\[ \frac{\partial^2 \phi}{\partial \xi \partial \bar{\xi}} + \kappa e^{2\phi} = 0 \] (3.8)
Here, the real Liouville field \(\phi = \phi (\xi, \bar{\xi})\) is a function of the complex variable \(\xi\) and its conjugate that may be formally denoted like \(\xi \equiv \xi^+\) and \(\bar{\xi} \equiv \xi^-\) where now \(\pm\) stand for \(U(1) \simeq SO(2)\) charges. As the two local coordinates \(\rho^\pm\) and \(\xi^\pm\) may be related by a Wick rotation of the time direction (say \(y = i\sigma\)), the general analysis of the two flat 2d geometries is quite similar; so one may treat both of the Lorentzian and euclidian equations collectively like
\[ \frac{\partial^2 \phi}{\partial X^+ \partial X^-} + \kappa e^{2\phi} = 0 \] (3.9)
where \(X^\pm\) designate either \(\rho^\pm\) or \(\xi^\pm\) and parameterise a real surface \(\Sigma\) as in eq(3.2). In what follows, we shall focus on the euclidian 2d geometry with \(X^\pm = x \pm iy\), we sometimes use
also \((\xi, \bar{\xi})\) to designate \((X^+, X^-)\) in order to simplify the notations. To fix ideas, notice also that the 2d space \(\Sigma\) may viewed as the analogous one in the 4d space \(\mathcal{M}_4 = \Sigma \times \mathcal{C}\) used in the CWY theory.

### 3.2 Special properties of eq(3.9)

First, recall that the properties of 2d Liouville theory are very well known and have been extensively studied in the mathematical physics literature from several views \[39]-[45]; so we will target in what follows only on those useful aspects directly relevant for our present construction; three of these aspects concern particularly: (i) the infinite dimensional conformal symmetry of eq(3.9), (ii) the factorisation of the modulus \(\kappa\) of the Liouville theory as the product of two terms; and (iii) the classical solvability of (3.1-3.9).

1) **Conformal symmetry of (3.9)**

Under the holomorphic coordinate change \(\xi \rightarrow \xi' = f(\xi)\), the Liouville eq(3.9) remains invariant provided the scalar potential \(e^{2\phi}\) transforms in same manner like \(\frac{\partial^2 \phi}{\partial \xi' \partial \bar{\xi}'}\), that is

\[
\frac{\partial^2 \phi'}{\partial \xi' \partial \bar{\xi}'} = \left| \frac{\partial f}{\partial \xi} \right|^{-2} \frac{\partial^2 \phi}{\partial \xi \partial \bar{\xi}} \tag{3.10}
\]

and

\[
e^{2\phi'} = \left| \frac{\partial f}{\partial \xi} \right|^{-2} e^{2\phi} \tag{3.11}
\]

Holomorphy of \(f(\xi)\) and the symmetry of eq(3.9) lead therefore to the following relationship between \(\phi\) and \(\phi'\),

\[
2\phi' (\xi', \bar{\xi'}) = 2\phi (\xi, \bar{\xi}) - \ln \left| \frac{\partial f}{\partial \xi} \right|^2 \tag{3.12}
\]

defining the conformal transformation of the Liouville field \(\phi\). Quite similar relations can be also written down for 2d Toda fields.

2) **Factoring the coupling \(\kappa\) in eq (3.9)**

From now on, we will think on the real parameter \(\kappa\) of the Liouville theory as given by the product of two non zero real numbers \(\alpha\) and \(\beta\) as follows

\[
\kappa = \alpha \times \beta > 0 \quad , \quad \alpha \beta \neq 0 \tag{3.13}
\]

This factorisation is important in figuring out novel properties on the integrability of the equation as well as its embedding into the CWY theory. For example, the two \(\alpha\) and \(\beta\) parameters will be used in building a general form of the Lax pair \(L_m = (L_+, L_-)\) underlying the linearization of (3.9). In terms of this pair of field operators, the Liouville equation can be brought to the form \[46]-[51],

\[
\partial_+ L_- - \partial_- L_+ + [L_+, L_-] = 0 \tag{3.14}
\]
where the relations between the Lax pair \((L_+, L_-)\) and the Liouville field \(\phi\) will be given later on.

3) Classical integrability

Before formulating the solvability of the Liouville equation as in eq(3.14), recall that a particular solution of the Liouville equation (3.9) can be explicitly written down. Up to the conformal transformation (3.12) that leaves the Liouville field action invariant, it not difficult to check that

\[
\phi = \ln \left( \frac{1}{1 + \kappa \bar{\xi} \xi} \right) , \quad e^{2\phi} = \frac{1}{(1 + \kappa \bar{\xi} \xi)^2}
\]  

(3.15)
is an exact solution of (3.9). From this expression, we have

\[
\partial_\xi \phi = -\frac{\kappa \bar{\xi}}{1 + \kappa \bar{\xi} \xi} , \quad \partial_\bar{\xi} \partial_\xi \phi = -\kappa \left( 1 + \kappa \bar{\xi} \xi \right)^{-2}
\]  

(3.16)

Notice that for the limit of small \(\kappa \xi \bar{\xi}\), say near the origin \(\xi \to 0\), the Liouville field \(\phi \sim -\kappa \xi \bar{\xi} \to 0\); so the linear term \(\frac{\partial^2 \phi}{\partial \xi \partial \bar{\xi}}\) behaves as \(-\kappa \left(1 - 2\kappa \xi \bar{\xi}\right) \to -\kappa\) in the same manner as the opposite of \(\kappa e^{2\phi}\) which behaves like \(\kappa \left(1 - 2\kappa \xi \bar{\xi}\right) \to \kappa\). For the large limit \(\kappa \xi \bar{\xi} \gg 1\), say near \(|\xi| \to \infty\), the Liouville field \(\phi \sim \ln \frac{1}{\kappa \xi \bar{\xi}}\) goes to \(-\infty\). In this case, the linear term \(\frac{\partial^2 \phi}{\partial \xi \partial \bar{\xi}}\) behaves as \(\frac{\kappa}{(\kappa \xi \bar{\xi})^2}\) and goes to 0 in the same way as \(\kappa e^{2\phi}\) which behaves as \(\frac{\kappa}{(\kappa \xi \bar{\xi})^2}\) and then tends to zero as well. A quite general form of the solution of the Liouville equation is given by \(\phi = \frac{1}{2} \ln \left( \frac{|\partial_\xi f|^2}{(1 + \kappa |f|^2)^2} \right)\) with \(f = f(\xi)\).

4 More on Lax formulation

Here, we study some useful properties of the Lax pair \((L_+, L_-)\) appearing in eq(3.14). First, we describe the link between the Lax equation (3.14) and the gauge field eqs(2.10) in the CWY theory. Then, we derive the relation between the 2d Liouville field \(\phi(X^\pm)\) and the \(L_\pm\) Lax pair. We also study the set \(H \times Hol(\Sigma)\) of local symmetries of the Lax pair; this set is contained into \(SL(2) \times Diff(\Sigma)\) and is obtained by solving constraint relations on some components of the CWY gauge connection imposed by the embedding of Liouville equation. The construction given here below for Liouville model applies as well to the full set of the finite 2d Toda QFT’s.

4.1 Lax pair as a particular CWY gauge configuration

The Lax pair \((L_+, L_-)\), which satisfy the Lax equation eq(3.14) linearising the Liouville equation, is very suggestive. Comparing (3.14) with the first relation of eqs(2.10) in the CWY theory namely

\[
\mathcal{F}_{[+ -]} = \partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0
\]  

(4.1)
one may think of eq(3.9) and then of eq(3.14) as following from the 4d eq(4.1) by imposing constraints on some components of \( A_\pm \) along the sl(2) fiber directions. In this view, the \( L_\pm \) operators can be imagined as describing a particular non abelian gauge configuration solving the field equation of the CWY gauge field namely

\[
\mathcal{F}_{\mu \nu} = \mathcal{F}_{\mu \nu} \left( X^\pm, \zeta, \bar{\zeta} \right) = 0
\]  

This antisymmetric \( \mathcal{F}_{\mu \nu} \) tensor splits in the \( X^\pm \) and \( \zeta \) directions as follows

\[
\mathcal{F}_{\mu \nu} = \begin{pmatrix} F_{mn} & F_{m\bar{\zeta}} \\ F_{\bar{n} \bar{\zeta}} & 0 \end{pmatrix}
\]  

with \( F_{mn} = \varepsilon_{mn}F_{[+]} \) as in (4.1) and \( F_{m\bar{\zeta}} = (F_{+\bar{\zeta}}, F_{-\bar{\zeta}}) \). It splits as well with respect to the sl(2) fiber directions like \( \mathcal{F}_{\mu \nu} = t_a F_{\mu \nu}^a \). By expanding the partial 1-form gauge connection \( A = A \left( X^\pm, \zeta, \bar{\zeta} \right) \) along the \( dX^\pm \) and \( d\bar{\zeta} \) dimensions like

\[
A = A_+ dX^+ + A_- dX^- + A_\zeta d\zeta
\]  

and setting the differential

\[
d\bar{\zeta} = 0
\]  

by demanding to the variable \( \bar{\zeta} \) to sit at some fixed constant value — for example by setting \( \bar{\zeta} = 0 \) —, the above expansion reduces to 2d space gauge connection \( A = A_+ dX^+ + A_- dX^- \) with \( A_\pm = A_\pm \left( X^\pm \right) \). So, one can imagine the Lax pair \( L_\pm = L_\pm \left( X^\pm \right) \) of the 2d Liouville theory as contained in \( A_\pm \left( X^\pm \right) \) as follows

\[
L_+ \subset A_+ \left( X^\pm \right) \\
L_- \subset A_- \left( X^\pm \right)
\]  

Notice that the \( A_\pm \) are reductions of the CWY vector potential components \( A_\pm = A_\pm \left( X^\pm, \zeta, \bar{\zeta} \right) \) down to 2d as they are given by

\[
A_+ \left( X^\pm \right) = A_+ \left( X^\pm, \zeta, \bar{\zeta} \right) \bigg|_{\zeta=\bar{\zeta}=0}
A_- \left( X^\pm \right) = A_- \left( X^\pm, \zeta, \bar{\zeta} \right) \bigg|_{\zeta=\bar{\zeta}=0}
\]  

So they are zero modes on the complex curve \( C \) of the 4d space \( \mathbb{M}_4 = \Sigma \times C \), and they satisfy the vanishing 2d space curvature condition

\[
\partial_\zeta A_- - \partial_- A_+ + [A_+, A_-] = 0
\]  

that follows from eq(4.1) by fixing \( \bar{\zeta} \) to a constant. The inclusion \( \subset \) symbol in eqs(4.6) means that \( L_\pm \) are given by pieces of the Lie algebra expansion of the non abelian vector potential
\[ A_\pm = \sum_a t_a A_a^\pm. \]  

The \( t_a \)'s are the generators of the \( \mathcal{G}_c \) Lie algebra satisfying the commutation relation

\[ [t_a, t_b] = f_{abc} t_c \quad (4.9) \]

Put differently, the 2-dimensional \( L_\pm \) Lax operators of Liouville equation can be recovered from the \( A_\pm \) components of CWY gauge connection by taking \( \mathcal{G}_c = sl(2) \) and imposing constraints on some of the direction of propagation of the gauge potential \( A_\mu \) in gauge fiber bundle. These constraints will be derived in what follows; but after deriving the explicit expression of \( L_\pm \) in terms of the Liouville field.

### 4.2 From Liouville field to Lax pair \( L_\pm \)

First, we construct the relationship between the Liouville field \( \phi \) and the Lax pair \((L_+, L_-)\) by using two manners (top-down and bottom-up) to get more insight into the construction:

(i) top-down: this is a short and somehow heuristic manner using specific features to build easily \( L_\pm (\phi) \); it applies to 2d models like the finite Toda QFTs considered in this study.

(ii) bottom-up: this a systematic manner based on general arguments and which may be used for generic cases. Then, we give some properties on the link \( L_\pm = L_\pm (\phi) \) as well as the interpretation of \( L_+, L_- \) as particular components of a non-abelian gauge potential \( A_\pm = t_a A_a^\pm \).

#### 4.2.1 Relationship between \( \phi \) and \( L_\pm \)

To begin, notice that the link between the Liouville field \( \phi \) and the Lax pair defines the transformations \( L_\pm = L_\pm (\phi) \) one has to do in order to linearise the Liouville equation. The use of a pair \( L_+ \) and \( L_- \) of variables at the place of the unique field variable \( \phi \) is the price to pay for linearization. The relationship between \( \phi \) and \( L_\pm \) have the following dependence

\[ L_+ = L_+ (\phi, \partial \phi ; t_a) \quad , \quad L_- = L_- (\phi, \partial \phi ; t_a) \quad (4.10) \]

where \( \partial \phi \) stands for the 2d gradient of the Liouville field and \( t_a \) for the \( sl(2) \) generators to be taken in the Cartan basis; i.e: \( t_a \equiv (h, E^-, E^+) \).

1) Heuristic derivation of \( L_\pm \)

By thinking of the Liouville equation eq(3.9) as a matrix relation valued in the diagonal \( h \)-direction of \( sl(2) \) like

\[ \left( \frac{\partial^2 \phi}{\partial X^+ \partial X^-} + \kappa e^{2\phi} \right) h = 0 \quad (4.11) \]

with coupling \( \kappa = \alpha \beta \), and equating this matrix with the Lax equation \( \partial_+ L_- - \partial_- L_+ + [L_+, L_-] = 0 \) of eq(3.14), we can derive the explicit expressions of the \( L_\pm \) Lax pair in terms of the constants \( \alpha, \beta \); the Liouville field \( \phi = \phi (X^+, X^-) \) and its gradient \( \partial_\pm \phi \) with \( \partial_\pm = \frac{1}{2} (\frac{\partial}{\partial x} \mp i \frac{\partial}{\partial y}) \). Using the following commutation relations

\[ [h, E^\pm] = \pm 2 E^\pm \quad , \quad [E^+, E^-] = h \quad (4.12) \]
and thinking of the non linear term $\kappa e^{2\phi} h$ in the Liouville field as intimately related with the commutator $[L_+, L_-]$ especially with $[\alpha E^+, \beta e^{2\phi} E^-]$; i.e:

$$\kappa e^{2\phi} h = [\alpha E^+, \beta e^{2\phi} E^-]$$

(4.13)

one can easily check that the following expressions give a realisation of the Lax operators in terms of $\phi$ and $h, E^+, E^-$,

$$L_+ = (\partial_+ \phi) h - \alpha E^+$$

$$L_- = \beta e^{2\phi} E^-$$

(4.14)

Putting these quantities back into $L_x = L_+ + L_-$ and $L_y = \frac{1}{i}(L_+ - L_-)$, we obtain the following complex quantities

$$L_x = \left( \frac{\partial \phi}{\partial x} - i \epsilon_{xy} \frac{\partial \phi}{\partial y} \right) h - \alpha E^+ + \beta e^{2\phi} E^-$$

$$L_y = \left( \frac{\partial \phi}{\partial y} + i \epsilon_{yx} \frac{\partial \phi}{\partial x} \right) h + i \alpha E^+ + i \beta e^{2\phi} E^-$$

(4.15)

where we have used $\epsilon_{xy} = -\epsilon_{yx} = 1$ and $\epsilon_{xx} = \epsilon_{yy} = 0$. Substituting eqs(4.14) back into the Lax equation, we rediscover (4.11).

2) Rigourous derivation of $L_\pm$

A quite rigourous manner to get the expressions in (4.14) compared to the previous heuristic one is to proceed as follows: (i) start from eq(4.8) describing the condition of vanishing curvature $F^+_- = \epsilon^+_- F = 0$ of a generic non abelian sl(2) vector potential $A_m = (A_+, A_-)$,

$$F = \partial_+ A_- - \partial_- A_+ + [A_+, A_-]$$

(4.16)

and (ii) look for a particular solution that fit with the Liouville equation. In this manner of doing one has to impose constraints on some components of the vector potential $A_m^a$; this may be achieved in two steps as follows:

- **step 1**: project the vanishing curvature matrix condition $F = 0$ along the three directions of sl(2) like $Tr (t^a F) = F^a = 0$ with

$$F^a = \partial_+ A_-^a - \partial_- A_+^a + f^{abc} A^b_+ A^c_-$$

(4.17)

The resulting three scalar conditions are nicely formulated by using the Cartan basis of sl(2) as follows:

$$Tr \left( \frac{h}{2} F \right) = F^0 = 0$$

$$Tr (E^+ F) = F^+ = 0$$

$$Tr (E^- F) = F^- = 0$$

(4.18)
with neutral \( F^0 \) component given by
\[
F^0 = \partial_+ A_0^- - \partial_- A_0^+ + A_+^- A_+^+ - A_+^+ A_+^-
\] (4.19)
and two charged \( F^\pm \) ones like
\[
F^+ = \partial_+ A_+^+ - \partial_- A_+^- + 2 (A_+^+ A_0^- - A_0^+ A_+^-) \\
F^- = \partial_+ A_-^- - \partial_- A_-^+ + 2 (A_0^- A_-^- - A_-^- A_0^-)
\] (4.20)

\* step 2: solve the two charged equations \( F^\pm = 0 \) in terms of a real scalar field \( \phi \); and put the obtained solution back into (4.19). However, the solving of (4.20) should be such that one ends with the Liouville equation; this requires imposing appropriate constraints on some of the components appearing the following expansion
\[
A_m = A_m^0 h + A_m^- E^+ + A_m^+ E^- \] (4.21)
The determination of the appropriate constraints on the \( A_m^0, \pm \)'s can be motivated from the structure of the Liouville equation which indicates that we should have an \( F^0 \) equation containing two terms like for instance
\[
F^0 = \partial_- A_+^- - A_+^- A_+^+ = 0 \] (4.22)
The first \( \partial_- A_+^- \) term in above \( F^0 \) is needed to generate the laplacian \( \partial_- \partial_+ \phi \); and the second one namely \( A_+^- A_+^+ \) is needed to recover the contribution coming from the scalar potential \( \kappa e^{2\phi} \). By comparing (4.22) with (4.19), we end with the following constraint relations we have to impose
\[
A_+^- = 0 \quad A_0^- = 0 \quad A_-^- = 0 \] (4.23)
Putting these constraints back into (4.20), we end with reduced charged \( F^\pm \) curvatures that we have to solve in terms of the Liouville field \( \phi \) and other parameters,
\[
F^- = \frac{\partial A_-^-}{\partial X^-} = 0 \\
F^+ = \left( \frac{\partial}{\partial X^+} - 2A_0^+ \right) A_+^+ = 0
\] (4.24)
The first relation of (4.24) namely \( \frac{\partial A_-^-}{\partial X^-} = 0 \) is solved by \( A_+^- = A_+^- (X^+) \); but because of conformal symmetry (3.12), it can be thought of as just given by the constant \( A_+^- = -\alpha \). The second relation can be also solved exactly as follows
\[
A_0^+ = \frac{\partial \phi}{\partial X^+}, \quad A_+^+ = \beta e^{2\phi}
\] (4.25)
By putting these solutions back into (4.22), we get \( \frac{\partial^2 \phi}{\partial X^- \partial X^+} + \alpha \beta e^{2\phi} = 0 \) coinciding exactly with the Liouville equation.
4.2.2 More on the link $L_{\pm}/A_{\pm}$

The realisation of the Lax operators $L_{\pm}$ in terms of $\phi$ and the sl(2) generators given by eqs(4.14) has some particular features that are interesting for the study of the embedding of the Liouville equation into the CWY theory with SL(2) gauge symmetry. Here, we want to describe five of these remarkable properties; they are as listed here below.

i) complexified Lie algebra sl(2):

The Lax operators $L_+$ and $L_-$ linearising Liouville equation are non hermitian operators since from (4.14) we learn

$$(L_+) ^\dagger \neq L_- \quad , \quad (L_-) ^\dagger \neq L_+ \quad (4.26)$$

Similarly, we have from (4.15),

$$(L_x) ^\dagger \neq L_x \quad , \quad (L_y) ^\dagger \neq L_y \quad (4.27)$$

The non hermiticity property of the Lax formalism may be also exhibited by using the $2 \times 2$ matrix representation of the generators $(h, E^\pm)$; we have

$$L_+ = \begin{pmatrix} \partial_+ \phi & -\alpha \\ 0 & -\partial_+ \phi \end{pmatrix}, \quad L_- = \begin{pmatrix} 0 & 0 \\ \beta e^{2\phi} & 0 \end{pmatrix} \quad (4.28)$$

The complex behavior of the $L_x$ and $L_y$ operators fits well with the formal action of the CWY theory requiring a complexified gauge symmetry $G_c$ which in the Liouville model is given by $sl(2)$.

ii) constrained vector potential $B_m$:

By comparing eq(4.14) with the expansion of a generic vector potential $A_\pm$ along the sl(2) directions namely

$$A_+ = A_0^+ h + A_+ E^+ + A_+^E^- \quad , \quad A_- = A_0^- h + A_- E^+ + A_-^E^- \quad (4.29)$$

we end with the relationships

$$A_0^+ = \partial_+ \phi \quad , \quad A_+ = -\alpha \quad , \quad A_-^E^- = \beta e^{2\phi} \quad (4.30)$$

together with the following constraints

$$A_+^E^- = 0 \quad , \quad A_0^- = 0 \quad , \quad A_-^- = 0 \quad (4.31)$$

which are precisely the ones obtained by the rigourous manner in deriving $L_{\pm} = L_{\pm}(\phi)$. These constraint relations play an important role in our construction; they show that
the $A_+^+$ component of the left chirality $A_+^+$ of vector potential $A_\pm$ should not propagate in the $E^-$ direction of the $sl(2)$ gauge bundle; and similarly the $A_0^-, A_-^- A_-$ components of the right chirality $A_-$ which should not spread in the $h$ and $E^+$ directions; this feature is illustrated on the figure 3. This means that the gauge potential associated with the

![Figure 3](image)

**Figure 3.** A schematic representation of the propagation directions in the SL(2) gauge bundle of the non abelian vector potential $A_\pm = A_\pm (X)$ inducing the Liouville equation. The direction of $A_+ = A_+ (X)$ is shown on the left panel; it has no component on $E^-$. The direction of $A_- = A_- (X)$ is shown on the right panel; it has one non zero component along the $E^-$ direction. The $(h, E^+, E^-)$ is the basis of $sl(2)$ at each $X = (X^+, X^-)$; and the projection of $A_m = A_m (X)$ along a given direction axis $t$ is given by $Tr [t A_m]$. Liouville theory is not a generic vector potential; but a constrained $B_m$ vector potential given by the following restriction

$$B_m = \eta_{m-} \left( A_0^+ h + A_-^+ E^+ \right) + \eta_{m+} A_-^+ E^- \tag{4.32}$$

In section 5, we will give more details on the properties of this non abelian vector potential; see for instance eq(5.36) and eq(5.45).

**iii) behavior of $L_\pm$ in the limit $\kappa X^+ X \to \infty$:**

By using eq(3.15) giving an exact solution of the Liouville equation, up to the conformal transformation (3.12) namely $\tilde{\phi} = \phi - \frac{1}{\kappa} \ln \left| \frac{\partial X^+}{\partial X^+} \right|^2$ with $\tilde{X}^+ = \tilde{X}^+ (X^+)$, we can express the Lax pair as

$$L_\pm = \frac{\kappa X^-}{1 + \kappa X^+ X^-} h - \alpha E^+$$

$$L_- = \frac{\beta}{(1 + \kappa X^+ X^-)^2} E^- \tag{4.33}$$
and then the corresponding non zero components of the non abelian vector potential like

\[
A_0^+ = -\frac{\kappa X^-}{1 + \kappa X^+ X^-} \\
A_- = -\frac{\kappa}{\beta} \\
A_0^- = \beta \left( 1 + \kappa X^+ X^- \right) \frac{1}{(1 + \kappa X^+ X^-)^2}
\]

(4.34)

These relations teach us that in large limit

\[
\kappa X^+ X^- = \kappa (x^2 + y^2) >> 1
\]

(4.35)

the non abelian vector potential \( A_m = A_m^0 h + A_m^- E^+ + A_m^+ E^- \) in Liouville theory tends towards the constant \(-\frac{2}{\beta} E^+ \eta_m^-\) with metric components \( \eta_{++} = \eta_{--} = 1 \) and \( \eta_{+-} = \eta_{-+} = 0 \).

**iv) reduced gauge symmetry \( H \subset SL(2) \):**

Because of the constraints (4.23) on components of the non abelian gauge connection, the usual SL(2) gauge transformation of the vector potential namely

\[
A_m \rightarrow A'_m = g^{-1} A_m g + g^{-1} \partial_m g
\]

(4.36)

gets reduced down to a subgroup of SL(2) that preserve (4.23). Indeed for the constraint eqs(4.23) to be invariant, the gauge transformation

\[
g = e^{\vartheta} \quad , \quad g = g \left( X^+, X^- \right) \quad , \quad \vartheta = \vartheta \left( X^+, X^- \right)
\]

(4.37)

with \( \vartheta = \vartheta^0 h + \vartheta^- E^+ + \vartheta^+ E^- \) should be in a subset \( H \) of \( SL(2) \) whose elements satisfy the following conditions

\[
\begin{align*}
Tr \left( E^+ g^{-1} \partial_+ g \right) &= 0 \\
Tr \left( E^- g^{-1} \partial_- g \right) &= 0 \\
Tr \left( h g^{-1} \partial_- g \right) &= 0
\end{align*}
\]

(4.38)

Clearly, this subset \( H \) contains the abelian subgroup of \( SL(2) \) whose elements \( g = \exp \left( \vartheta^0 h \right) \) with local parameter \( \vartheta^0 \) a holomorphic function in the complex variable \( X^+, \)

\[
\frac{\partial \vartheta^0}{\partial X^-} = 0 \quad \Rightarrow \quad \vartheta^0 = \vartheta^0 \left( X^+ \right)
\]

(4.39)

This is because the two first relations in (4.38) are trivially solved due to \( Tr \left( E^+ h \right) = 0 = Tr \left( E^- h \right) \); while the third one requires \( \frac{\partial \vartheta^0}{\partial X^-} = 0 \) since \( Tr \left( h h \right) \neq 0 \).
Notice also that as far as chiral transformations are concerned, we distinguish two subgroups $H_\(\oplus\)$ and $H_\(\ominus\)$ depending on whether $\partial_- g = 0$ or $\partial_+ g = 0$.

* Subgroup $H_\(\oplus\)$: It corresponds to the case where $g = g\(X^\oplus\)$ satisfying $\frac{\partial g}{\partial X^\ominus} = 0$. In this situation, the eqs\(4.38\) reduce to the first relation

$$Tr\(E^+ g^{-1} \partial_+ g\) = 0 \tag{4.40}$$

since the two $Tr\(E^- g^{-1} \partial_- g\)$ and $Tr\(h g^{-1} \partial_- g\)$ vanish identically due to $\frac{\partial g}{\partial X^\ominus} = 0$ and $\frac{\partial g}{\partial X^\ominus} = 0$. The above relation \(4.40\) is solved by those non abelian transformations

$$g = \exp\(\vartheta^0 h + \vartheta^- E^+\) \tag{4.41}$$

with analytic parameters

$$\vartheta^0 = \vartheta^0\(X^\oplus\) \quad , \quad \vartheta^- = \vartheta^-\(X^\oplus\) \tag{4.42}$$

The restriction to eq\(4.41\) with no dependence into $\vartheta^+$ parameter follows from the fact that $g^{-1} \partial_m g$ is an element of the Lie algebra; and the identity $Tr\[E^+ (ah + bE^+)\] = 0$ which holds for arbitrary numbers $a$ and $b$ but not for $Tr\[E^+ (ah + bE^+ + cE^-)\]$ which does not vanish for $c \neq 0$. The $H_\(\oplus\)$ contains the the diagonal $\exp\[h \vartheta^0\(X^\oplus\)\]$.

* Subgroup $H_\(\ominus\)$: It corresponds to $g = g\(X^\ominus\)$ satisfying $\frac{\partial g}{\partial X^\oplus} = 0$. In this case, eq\(4.38\) reduce to its two last relations seen that the first one vanishes identically

$$Tr\(E^- g^{-1} \partial_- g\) = 0 \tag{4.43}$$

$$Tr\(h g^{-1} \partial_- g\) = 0$$

Because of the properties $Tr\(E^- E^\ominus\) \neq 0$ and $Tr\(hh\) \neq 0$, it results that the solution of these constraints is given by

$$g\(X^\ominus\) = \exp\[E^- \vartheta^+\(X^\ominus\)\] \tag{4.44}$$

with $\frac{\partial g}{\partial X^\ominus} = 0$. So, the $H_\(\ominus\)$ does not contain the diagonal $\exp\[h \vartheta^0\(X^\ominus\)\]$.

v) holomorphic diffeomorphisms:

Because of the constraints \(4.23\), the invariance under $Diff\(\Sigma\)$ gets reduced to the invariance under holomorphic transformations

$$X^+ \rightarrow \tilde{X}^+ = f\(X^+\) \quad , \quad X^- \rightarrow \tilde{X}^- = \bar{f}\(X^-\) \tag{4.45}$$

This holomorphic feature can be derived by using the language of differential forms on $\Sigma$; the gauge connection $A\(X\) = A_+ dX^+ + A_- dX^-$, in coordinate frame $X^\pm$,
transforms in $\tilde{X}^\pm$-coordinate frame into $\tilde{A}(\tilde{X}) = \tilde{A}_+ d\tilde{X}^+ + \tilde{A}_- d\tilde{X}^-$. Solving the identity $A(X) = \tilde{A}(\tilde{X})$ and substituting back into (4.23), we obtain

$$
\begin{align*}
A^0_0 &= \frac{\partial \tilde{X}^+}{\partial \tilde{X}^-} \tilde{A}^0_0 + \frac{\partial \tilde{X}^-}{\partial \tilde{X}^-} \tilde{A}^0_- = 0 \\
A^-_0 &= \frac{\partial \tilde{X}^+}{\partial \tilde{X}^-} \tilde{A}^-_0 + \frac{\partial \tilde{X}^-}{\partial \tilde{X}^-} \tilde{A}^-_- = 0 \\
A^+_0 &= \frac{\partial \tilde{X}^+}{\partial \tilde{X}^+} \tilde{A}^+_0 + \frac{\partial \tilde{X}^-}{\partial \tilde{X}^+} \tilde{A}^+_-_ = 0
\end{align*}
(4.46)
$$

Demanding invariance under diffeomorphism of the constraint eqs (4.23); that is $\tilde{A}_0^0 = A^-_0 = A^+_0 = 0$, we end with

$$
\begin{align*}
\frac{\partial \tilde{X}^+}{\partial \tilde{X}^-} \tilde{A}^0_+ &= 0 \\
\frac{\partial \tilde{X}^+}{\partial \tilde{X}^-} \tilde{A}^-_+ &= 0 \\
\frac{\partial \tilde{X}^-}{\partial \tilde{X}^-} \tilde{A}^+_+ &= 0
\end{align*}
(4.47)
$$

requiring the conditions

$$
\frac{\partial \tilde{X}^+}{\partial \tilde{X}^-} = 0, \quad \frac{\partial \tilde{X}^-}{\partial \tilde{X}^-} = 0
(4.48)
$$

and then holomorphic coordinates transformations. This feature indicates that the properties of the real surface $\Sigma$ have to be approached in terms of properties of a complex curve.

5 From Liouville to CWY

In section 4, we have shown that the Lax pair $(L_+, L_-)$ given by eq (4.14), linearising the 2d Liouville equation, is in fact a particular solution of the equations of motion (2.21) describing the dynamics of the CWY gauge connection in 4d; see the figure 3. In this section, we complete this analysis by studying in subsection 5.1 the generalisation of these Lax operators to the 4d space $\Sigma \times C$ as well as some of their properties. With this link between the standard integrable 2d Lax formalism and the formal CWY gauge theory, we dispose of a manner to construct observables in Toda QFT$_2$’s by using quantum Wilson line operators and their generalisations. In subsection 5.2, we build these quantum line operators for the example of Liouville theory and show how they are characterised by a rank four tensor $\Gamma^{\nu b}_{\mu a}$ where $\mu, \nu$ are space indices and $a, b$ Lie algebra ones.
5.1 4d extended Lax equations

Using results from [20] on Costello-Witten-Yamazaki 4d topological gauge theory, whose useful aspects to the present analysis were presented in section 2, and focusing on the case of a 4d space $M_4$ given by $\mathbb{R}^2 \times \mathbb{C}$ parameterised by the local coordinates

$$X^M = (X^+, X^-; \zeta, \bar{\zeta})$$ (5.1)

with $X^\pm = x \pm iy$ for $\Sigma = \mathbb{R}^2 \sim \mathbb{C}$ and $\zeta = X_3 + iX_4$ for the complex line $\mathcal{L} = \mathbb{C}$, we can write down an extension of the 2d Lax equation on $\mathbb{R}^2$ to the 4 space $\mathbb{R}^2 \times \mathbb{C} \simeq \mathbb{C} \times \mathbb{C}$. In this 4d extension, the usual 2d Lax equation

$$\partial_+ L_- - \partial_- L_+ + [L_+, L_-] = 0$$ (5.2)

expressed in terms of the 2d Lax pair $(L_+, L_-)$ living on $\mathbb{R}^2$ with

$$L_+ = L_+ (X^+, X^-; t_a), \quad L_- = L_- (X^+, X^-; t_a)$$ (5.3)

gets promoted to three 4d extended equations

$$\partial_+ \mathcal{L}_- - \partial_- \mathcal{L}_+ + [\mathcal{L}_+, \mathcal{L}_-] = 0$$

$$\partial_\zeta \mathcal{L}_+ - \partial_+ \mathcal{L}_\zeta + [\mathcal{L}_\zeta, \mathcal{L}_+] = 0$$ (5.4)

$$\partial_\zeta \mathcal{L}_- - \partial_- \mathcal{L}_\zeta + [\mathcal{L}_\zeta, \mathcal{L}_-] = 0$$

involving three pairs of generalised Lax pairs $(\mathcal{L}_+, \mathcal{L}_-)$, $(\mathcal{L}_\zeta, \mathcal{L}_+)$ and $(\mathcal{L}_\bar{\zeta}, \mathcal{L}_-)$ with coordinate dependence as

$$\mathcal{L}_+ = \mathcal{L}_+ (X^+, X^-; \zeta, \bar{\zeta}; t_a)$$

$$\mathcal{L}_- = \mathcal{L}_- (X^+, X^-; \zeta, \bar{\zeta}; t_a)$$

$$\mathcal{L}_\zeta = \mathcal{L}_\zeta (X^+, X^-; \zeta, \bar{\zeta}; t_a)$$ (5.5)

Roughly speaking, these three extended Lax pairs $(\mathcal{L}_+, \mathcal{L}_-)$, $(\mathcal{L}_\zeta, \mathcal{L}_+)$ and $(\mathcal{L}_\bar{\zeta}, \mathcal{L}_-)$ could be interpreted as obeying Lax-type equation in the 2d subspaces $(X^+, X^-), (X^+, \zeta)$ and $(X^-, \bar{\zeta})$. A field realisation of the $\mathcal{L}_\pm$ and $\mathcal{L}_\zeta$ operators extending the 2d realisation (4.14) can be obtained by thinking of the 4d $\mathcal{L}_\pm$ as related to the 2d $L_\pm$ like

$$L_+ = \mathcal{L}_+|_{\zeta=\bar{\zeta}=0}, \quad L_- = \mathcal{L}_-|_{\zeta=\bar{\zeta}=0}$$ (5.6)

This feature suggests that $\mathcal{L}_\pm$ and $\mathcal{L}_\zeta$ can be realised in terms of the following field system

$$\Phi = \Phi (X^+, X^-; \zeta, \bar{\zeta})$$

$$\Psi = \Psi^+ (X^-, \zeta, \bar{\zeta})$$

$$N = N^+ (X^+, \zeta, \bar{\zeta})$$

$$\Gamma = \Gamma^\zeta (\zeta, \bar{\zeta})$$ (5.7)
extending the 2d system \( \{ \phi, \alpha, \beta \} \) used in the 2d Lax construction (4.14) with \( \phi = \phi (X^+, X^-) \) but \( \alpha \) and \( \beta \) constant parameters; \( \frac{\partial \phi}{\partial X^\pm} = \frac{\partial \phi}{\partial X^\mp} = 0 \). General arguments indicate that \( \mathcal{L}_+ \) and \( \mathcal{L}_\zeta \) are realised like

\[
\mathcal{L}_+ = \frac{\partial \Phi}{\partial X^+} h - NE^+ \\
\mathcal{L}_- = e^{2\Phi} \Psi E^- \\
\mathcal{L}_\zeta = -\frac{\partial \log N}{2\partial \zeta} h + \Gamma e^{2\Phi} E^-
\]

The charges carried by the fields in (5.7) are as before and are of two types: space and Lie algebraic; they should be understood like for instance \( \mathcal{L}_+ = \frac{\partial \Phi}{\partial X^+} h - N_2 E^+ \), \( \mathcal{L}_- = e^{2\Phi} \Psi E^- \) and so on; for convenience the charges of \( N_2 \), \( \Psi \), and \( \Gamma \) have been omitted in (5.8). Moreover, the dependence of the fields \( \{ \Phi, N, \Psi, \Gamma \} \) into the variables \( X^+, X^- \) and \( \zeta, \bar{\zeta} \) as specified in (5.7) is motivated by the fact that we want to reproduce the Liouville equation; for instance when computing \( \partial_- \mathcal{L}_+ \) we need \( \frac{\partial N}{\partial X^-} = 0 \) requiring that \( N \) should not depend on \( X^- \). By substituting (5.8) back into eqs(5.4) and using the commutation relations of \( \text{sl}(2) \), we obtain the three following equations

\[
\left( \frac{\partial^2 \Phi}{\partial X^+ \partial X_-} + N \Psi e^{2\Phi} \right) h = 0 \tag{5.9}
\]

\[
\left( \frac{\partial^2 \Phi}{\partial \zeta \partial X^+} + N \Gamma e^{2\Phi} \right) h = 0 \tag{5.10}
\]

\[
\left( \frac{\partial \Psi}{\partial \zeta} + \Psi \frac{\partial N}{2\partial \zeta} + 2\Psi \frac{\partial \Phi}{\partial \zeta} - 2\Gamma \frac{\partial \Phi}{\partial X^-} \right) E^+ = 0 \tag{5.11}
\]

Two of these relations point in the diagonal \( h \)-direction of \( \text{sl}(2) \), and look like generalisations of the standard Liouville equation; the third relation points in \( E^+ \)-direction and behaves as a constraint relation between the fields (5.7).

5.1.1 More on eqs (5.9-5.11)

The above relations (5.9-5.11) obey some special features capturing data on the 2d Liouville equation among which the two following ones; other properties like exact solution will be given in next sub-subsection:

1. **Symmetry under holomorphic change on \( C \).**

Eqs(5.9-5.11) are invariant under the holomorphic local change

\[
\Psi \to e^{f(\zeta)} \times \Psi \\
N \to e^{-k(\zeta)} \times N \\
\Gamma \to e^{f(\zeta)} \times \Gamma \tag{5.12}
\]

\[
\Phi \to \Phi + \frac{1}{2} k(\zeta) - \frac{1}{2} f(\zeta)
\]
where \( f(\zeta) = f \) and \( k(\zeta) = k \) are arbitrary holomorphic functions in the complex coordinate \( \zeta \) of the base space \( \mathcal{C} \). Under these transformations, we also have

\[
\begin{align*}
\frac{\partial \Psi}{\partial \zeta} & \to e^f \frac{\partial \Psi}{\partial \zeta} \\
\frac{\partial N}{\partial \zeta} & \to e^{-k} \frac{\partial N}{\partial \zeta} \\
\frac{\partial \Gamma}{\partial \zeta} & \to e^f \frac{\partial \Gamma}{\partial \zeta} \\
\frac{\partial \Phi}{\partial \zeta} & \to \frac{\partial \Phi}{\partial \zeta}
\end{align*}
\]

(5.13)

Observe moreover that if choosing \( f = k \), then the 4d scalar field \( \Phi \) becomes invariant; and by thinking of \( N \) and \( \Psi \) as real quantities like

\[
\begin{align*}
\Psi & = \beta e^{f(\zeta) + \bar{f}(\bar{\zeta})} \\
N & = \frac{\alpha}{e^{k(\zeta) + \bar{k}(\bar{\zeta})}}
\end{align*}
\]

(5.14)

where \( \alpha \) and \( \beta \) are as above, then the product

\[
N \Psi = \alpha \beta \times \frac{e^f}{e^k} \times \frac{e^f}{e^k}
\]

(5.15)

behaves as

\[
N \Psi = \alpha \beta \times \left| \frac{e^f}{e^k} \right|^2
\]

(5.16)

and reduced further to \( N \Psi = \alpha \beta \) if we take \( f = k \).

(2) From (5.9-5.11) to Liouville equation.

The formal similarity between the two relations (5.9) and (5.10) is striking an suggestive since both of them describe an extension of the 2d Liouville equation. However, these two relations can be brought to one equation namely

\[
\frac{\partial^2 \Phi}{\partial X^- \partial X^+} + \alpha \beta e^{2\Phi} = 0
\]

(5.17)

if we think of the third (5.11) like

\[
\left( \frac{\partial \Psi}{\partial \zeta} + \frac{\Psi}{N} \frac{\partial N}{\partial \zeta} \right) + 2 \left( \Psi \frac{\partial \Phi}{\partial \zeta} - \Gamma \frac{\partial \Phi}{\partial X^-} \right) = 0
\]

(5.18)

and cast it as follows

\[
\begin{align*}
\frac{\partial \Psi}{\partial \zeta} + \frac{\Psi}{N} \frac{\partial N}{\partial \zeta} &= 0 \\
\Psi \frac{\partial \Phi}{\partial \zeta} - \Gamma \frac{\partial \Phi}{\partial X^-} &= 0
\end{align*}
\]

(5.19)
or equivalently
\[ \frac{\partial}{\partial \zeta} \log (N \Psi) = 0, \quad \frac{\partial \Phi}{\partial \zeta} = \frac{\Gamma}{\Psi} \frac{\partial \Phi}{\partial x} \] (5.20)

The first relation shows that \( \log (N \Psi) \) is independent of \( \bar{\zeta} \) and \( \zeta \); and then \( N \times \Psi \) is a constant precisely given by the Liouville coupling constant \( \kappa = \alpha \times \beta \) as one sees from (5.16). The other relation in (5.19) namely \( \frac{\partial \Phi}{\partial \bar{\zeta}} = \frac{\Gamma \Psi}{\partial \Phi/\partial x} = \Gamma \Psi \) \( \frac{\partial \Phi}{\partial x} \) is also remarkable in the sense it relates \( \frac{\partial \Phi}{\partial \bar{\zeta}} \) to the gradient \( \frac{\partial \Phi}{\partial x} \). By substituting it back into eq(5.10), we obtain
\[ \Gamma \frac{\partial^2 \Phi}{\Psi \partial x^{-} \partial x^{+}} + N \Gamma e^{2\Phi} = 0 \] (5.21)

Multiplying with \( \frac{\Psi}{\Gamma} \) and using \( N \Psi = \kappa \), we obtain \( \frac{\partial^2 \Phi}{\partial x^{-} \partial x^{+}} + \kappa e^{2\Phi} \) which is exactly with the generalised Liouville equation (5.17) with \( \Phi_{\zeta} = \Phi (X^{+}, X^{-}; \bar{\zeta}, \zeta) \).

### 5.1.2 Exact solution of (5.9-5.11)

To work out the exact solution of eqs(5.9-5.11), we use results from 2d Liouville theory and its conformal symmetry. To that purpose, we shall solve each of the eqs (5.9-5.11) separately by starting from (5.9), then solving (5.10) and finally solving (5.11). First, notice that (5.9) looks like a generalised Liouville equation in the plane \( (X^{+}, X^{-}) \); but with coupling given by \( K_{+-} = N \Psi \),
\[ \frac{\partial^2 \Phi}{\partial x^{-} \partial x^{+}} + K_{+-} e^{2\Phi} = 0 \] (5.22)

Comparing this equation with (3.9), one can easily wonder an exact solution of (5.22) in terms of the variables \( X^{\pm} \) and the coupling \( K_{+-} \). The solution has a similar structure as eq(3.15) and reads, up to a conformal transformation, as follows
\[ \Phi = \ln \left( \frac{1}{1 + K_{+-} X^{+} X^{-}} \right) \] (5.23)
with \( K_{+-} \) having no dependence on the \( X^{\pm} \) variables; i.e:
\[ K_{+-} = K (\zeta, \bar{\zeta}) \] (5.24)

Viewed from the 2d Liouville side, this solution corresponds to a fibration of (3.15) on the complex curve \( C \). Notice also the following expression to be encountered later on
\[ X^{-} \frac{\partial \Phi}{\partial X^{-}} = -\frac{K_{+-} X^{+} X^{-}}{1 + K_{+-} X^{+} X^{-}} \] (5.25)

Regarding the second relation (5.10), it looks as well like a Liouville equation; but in the plane \( (X^{+}, \bar{\zeta}) \) and a different coupling given by \( K_{+\bar{\zeta}} = N \Gamma \),
\[ \frac{\partial^2 \Phi}{\partial \zeta \partial X^{+}} + K_{+\bar{\zeta}} e^{2\Phi} = 0 \] (5.26)
An exact solution of this equation is derived easily by using the same trick as above; it reads
up to a conformal transformation like
\[ \Phi = \ln \left( \frac{1}{1 + K_+ \zeta X^+ \bar{\zeta}} \right) \] (5.27)
with \( K_+ \zeta \) having no dependence on the variables \( X^+ \) and \( \bar{\zeta} \); i.e:
\[ K_+ \zeta = \tilde{K} (X^-, \zeta) \] (5.28)

Notice as well the following expression
\[ \bar{\zeta} \frac{\partial \Phi}{\partial \bar{\zeta}} = - \frac{K_+ \zeta X^+ \bar{\zeta}}{1 + K_+ \zeta X^+ X^-} \] (5.29)

By equating the two solutions (5.23) and (5.27) as they concern the same field \( \Phi \), we end
with the identification \( K_+ \zeta X^+ \bar{\zeta} = K_{+-} X^+ X^- \) leading to
\[ K_+ \zeta = K_{+-} X^- \] (5.30)
and implying in turns
\[ \frac{K_+ \zeta}{K_{+-}} = \frac{\Gamma}{\Psi} = \frac{X^-}{\bar{\zeta}} \] (5.31)

Notice that the equality \( K_+ \zeta X^+ \bar{\zeta} = K_{+-} X^+ X^- \) shows that we also have
\[ \bar{\zeta} \frac{\partial \Phi}{\partial \bar{\zeta}} = X^- \frac{\partial \Phi}{\partial X^-} \] (5.32)

To determine the solution of the third relation (5.11), it is interesting to put it into a
convenient form. By dividing (5.11) by \( \Psi \), we get
\[ \frac{\partial \log \Psi}{\partial \zeta} + \frac{\partial \log N}{\partial \zeta} + 2 \frac{\partial \Phi}{\partial \zeta} - 2 \frac{\Gamma}{\Psi} \frac{\partial \Phi}{\partial X^-} = 0 \] (5.33)

Then by substituting \( \frac{\Gamma}{\Psi} = \frac{X^-}{\bar{\zeta}} \), we can put it into the form
\[ \bar{\zeta} \frac{\partial \log (N \Psi)}{\partial \zeta} + 2 \bar{\zeta} \frac{\partial \Phi}{\partial \zeta} - 2 X^- \frac{\partial \Phi}{\partial X^-} = 0 \] (5.34)

But because of the identity (5.32), we end with
\[ \bar{\zeta} \frac{\partial \log (N \Psi)}{\partial \zeta} = 0 \] (5.35)
showing that \( (N \Psi) \) is independent from \( \bar{\zeta} \).
5.2 Quantum line operators

In this subsection, we build line operators associated with the pair \((L_+, L_-)\) extending the usual Lax pair \((L_+, L_-)\) linearising the 2d Liouville equation. The structure of these operators are given by eqs\((2.15)\) and \((2.21)\); but with the partial gauge connection \(A = A_\mu dX^\mu\) replaced by a constrained gauge connection \(B = B_\mu dX^\mu\) related to \(A\) like

\[
B = dX^\mu \Gamma^\rho_\mu i_{\partial_\rho} (A)
\]

(5.36)

where \(i_V\) indicates contraction with a vector field \(V\), and where the operator \(\Gamma^\rho_\mu = t_b \Gamma^\rho_\mu b\) will be determined below. To get the explicit of \(\Gamma^\rho_\mu i_{\partial_\rho} (A)\), we first need to determine the analogue of the constraint eqs\((4.23)\) to the 4d space \(M_4 = \Sigma \times \mathcal{C}\).

5.2.1 Deriving the constraint equations

To begin, notice that the realisation \((5.8)\) of the \(L_+, L^-\) operators in terms of the fields \((5.7)\) can be rigorously derived by solving the vanishing condition of the CWY curvature

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]
\]

(5.37)

by using the same method as the one used in sub-subsection 4.2.1 for the 2d Liouville theory: see the analysis between eq\((4.16)\) and eq\((4.25)\). This solution is obtained by imposing constraints on some components of the non abelian vector potential \(A^a_\mu\) as done for \((4.23)\). Recall that \(A = A_\mu dX^\mu\) is a partial gauge connection valued in \(\mathfrak{sl}(2)\) with 3d subspace index as \(\mu = +, -, \bar{\zeta}\) and metric \(\eta_{\mu\nu}\) like

\[
\eta_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad \eta^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}
\]

(5.38)

Each one of the \(A_\mu\) components expands along the \(\mathfrak{sl}(2)\) generators as \(t_a A^a_\mu\) with \(t_1, t_2, t_3\) standing for the generators of \(\mathfrak{sl}(2)\). To exhibit the constraint eqs on the \(A^a_\mu\) that lead to \((5.8)\), we use the Cartan basis \((h, E^\pm)\) of \(\mathfrak{sl}(2)\) and expand the non abelian vector potential as follows

\[
\begin{align*}
A_+ &= A^0_+ h + A^-_+ E^+ + A^+_+ E^- \\
A_- &= A^0_- h + A^-_- E^+ + A^+_+ E^- \\
A_{\bar{\zeta}} &= A^0_{\bar{\zeta}} h + A^-_{\bar{\zeta}} E^+ + A^+_{\bar{\zeta}} E^- 
\end{align*}
\]

(5.39)

Generally speaking, the above equations teach us that in the \(\mathfrak{sl}(2)\) case, the non abelian vector potential has nine components \(A^a_\mu\) since \(\mu = \pm, \bar{\zeta}\) and \(a = 0, \pm\). By comparing the
above $A_{0,\pm}$ expansions with those of the $L_{0,\pm}$’s given by eqs (5.8) that we rewrite like,

\[
\begin{align*}
L_+ &= \frac{\partial \Phi}{\partial X^+} h - NE^+ + 0E^- \\
L_- &= 0 h + 0E^+ + e^{2\Phi}E^- \\
\zeta &= -\frac{h}{2N} \frac{\partial N}{\partial \zeta} + 0 E^+ + \Gamma e^{2\Phi}E^- 
\end{align*}
\]

we deduce the constraint relations that we have to impose some of the $A_{0,\pm}$’s in order to embed the Liouville equation into the CWY theory. These constraints are given by

\[
A_+ = 0 , \quad A_0^0 = 0 , \quad A_- = 0 , \quad A^- = 0 (5.41)
\]

leaving then five non zero components filling particular directions in the gauge bundle. The above constraints extend (4.23); their gauge invariance requires reducing down the volume of the SL(2) set of gauge transformations on $M_4 = \Sigma \times C$. This is because the generic SL(2) change

\[
A'_\mu = g^{-1}A_\mu g + g^{-1}\partial_\mu g
\]

does not preserve (5.41). The above constraints require also reducing down the volume of the Diff($\Sigma$) set of general coordinate transformations $\tilde{X}^\pm = f^\pm (X^+, X^-)$. For the gauge symmetry, we have in addition to (4.38), the extra condition along the complex $C$ curve dimension

\[
Tr \left( E^- g^{-1} \frac{\partial}{\partial \zeta} g \right) = 0 , \quad g = e^\vartheta , \quad \vartheta = \vartheta^0 h + \vartheta^- E^+ + \vartheta^+ E^- (5.43)
\]

with $\vartheta$ a priori a generic function of the coordinates $(X^+, X^-, \zeta, \bar{\zeta})$. The conditions (4.38) and (5.43) on the local matrix transformations $g = e^\vartheta$ can be solved as before with analytic gauge parameters $\vartheta^0 = \vartheta^0(X^+, \zeta)$ and $\vartheta^- = \vartheta^- (X^+, \zeta)$. The same thing holds for the set of general coordinate transformations which gets reduced to invariance under holomorphic transformations $\tilde{X}^+ = f^\pm (X^+)$ and $\tilde{X}^- = f^\pm (X^-)$.

### 5.2.2 Building line operators

To build line operators associated with the embedding of Liouville equation in the CWY theory, we use the eqs (2.15) and (2.21) and impose the constraints in the holonomy like

\[
\varphi (K_\zeta) \bigg|_{Eq(5.41)} = \left( \oint_{K_\zeta} \mathcal{A} \right)_{Eq(5.41)} (5.44)
\]

Substituting the constraint eqs (5.41) back into (5.39), one gets a restriction on the allowed directions of the vector potentials $A_\mu^a$ in the $SL(2)$ gauge bundle. Thinking of these restricted
directions in terms of projections in $sl(2)$, we can define the constrained vector potentials like $B^b_{\mu}$ related to the generic $A^a_{\rho}$ as follows
\[ B^b_{\mu} = \Gamma^b_{\mu a} A^a_{\rho} \] (5.45)
The tensor $\Gamma^b_{\mu a}$ which links the projected $B^b_{\mu}$ to the generic $A^a_{\rho}$ can be presented in different, but equivalent, manners depending on the index we want to exhibit; that is the space indices $\mu, \rho$; or the $sl(2)$ Lie algebra ones $a, b$. For instance, by multiplying both sides of (5.45) by $t_b$ generator, we can put the above relation into the matrix form $B^b_{\mu} = \Gamma^b_{\mu a} A^a_{\rho}$ where now we have $\Gamma^b_{\mu a} = t_b \Gamma^b_{\mu a}$. Moreover, using the Killing form normalised like $\text{Tr}(t_a t_b) = \delta_{ab}$, we can express the above relation in terms of $B^b_{\mu} = t_b B^b_{\mu}$ and $A^a_{\rho} = t_a A^a_{\rho}$ as follows
\[ B^b_{\mu} = \Gamma^b_{\mu a} (A^a_{\rho}) \] (5.46)
where $\Gamma^b_{\mu a}$ refer to some representation of $sl(2)$ and the curve $K_\zeta$ lies in the topological plane $\Sigma$ and is defined as in the CWY theory of section 2. Notice that instead of the usual generic partial gauge $A = dX^\mu A^a_{\mu}$ we have now the constrained gauge connection $B = B_b dX^b$ which is related to the generic $A$ through (5.45). Denoting by $\psi_{K_\zeta} = t_b \psi_{K_\zeta}^b$ the holonomy of $B$ along the curve $K_\zeta$, we then have
\[ \psi_{K_\zeta}^b = \int_{K_\zeta} \Gamma^b_{\mu a} A^a_{\mu} dX^\mu \] (5.49)
Moreover, because of the fact that $K_\zeta$ belongs to the topological plane $\Sigma$ taken here as $\mathbb{R}^2$, it is clear that the contribution to the holonomy is given by
\[ \psi_{K_\zeta}^b = \int_{K_\zeta} A^a_{\mu} (\Gamma^b_{\mu a} dX^+ + \Gamma^b_{\mu a} dX^-) \] (5.50)
Notice also that, in addition to the coordinate variables \((x, y)\) of \(\mathbb{R}^2\), the \(B^\mu_\alpha\) depends as well on the spectral parameter \(\zeta\). Using the same trick as done in (2.18) by keeping only the holomorphic variable \(\zeta\), we obtain a new operator \(\hat{B}_\pm (x, y, \zeta)\) defined as

\[
\hat{B}_\pm (x, y, \zeta) := \sum_{k=0}^{\infty} t_{a,n} \hat{B}_\pm^{a(n)} (x, y)
\]

with extended generators \(t_{a,n} = t_a \otimes \zeta^n\). By using these quantities, we can construct generalised like operators for Liouville theory like

\[
\hat{W}_{\hat{\theta}_{sl_2}} [K_{\zeta}] = Tr_{\hat{\theta}_{sl_2}} \left( P \exp \left( \int_{K_{\zeta}} \hat{B}_+ dX^+ + \hat{B}_- dX^- \right) \right) \]

where now \(\hat{\theta}_{sl_2}\) is a representation of the infinite dimensional algebras \(sl_2[[\zeta]]\) induced by the fibration of \(sl(2)\) on the holomorphic complex line \(\mathbb{C}\). Similar comments that have been done for the derivation of eq(2.22) applies as well here for the above \(\hat{W}_{\hat{\theta}_{sl_2}} [K_{\zeta}]\).

6 One loop quantum effect

In this section, we calculate the expression of the amplitude of two intersecting lines \(K_{\zeta_1}\) and \(K_{\zeta_2}\) supporting two quantum Wilson operators \(W_{\theta_1} [K_{\zeta_1}]\) and \(W_{\theta_2} [K_{\zeta_2}]\) as schematised in the figure 4. Then, we compare the obtained result with a similar amplitude calculated in [20] for generic vector potentials. The two quantum line operators of the figure 4 are

![Figure 4](image-url) - One gluon exchange between two Wilson lines operators \(W_{\theta_1} (K_1)\) and \(W_{\theta_2} (K_2)\) supported by \(K_1\) and \(K_2\) with spectral parameters \(\zeta_1\) and \(\zeta_2\). These line operators are respectively chosen as given by the x- and y- axes in the topological plane \(\mathbb{R}^2\).
loop Feynman diagram. The two Wilson operators are respectively characterised by the spectral parameters $\zeta_1$ and $\zeta_2$; and carry $\varrho_1$ and $\varrho_2$ representations of $\text{sl}(2)$. The amplitude $\mathcal{I}_1 = \mathcal{I}_1 [\varrho_1, \zeta_1; \varrho_2, \zeta_2]$ of this one-loop Feynman diagram is proportional to $\hbar$ and, because of translation invariance in the 4d space $\mathcal{M}_4 = \mathbb{C} \times \mathbb{C}$, is a function of $\zeta_1 - \zeta_2$; so the $\mathcal{I}_1$ has the form

$$\mathcal{I}_1 = \hbar F_{\varrho_1, \varrho_2} (\zeta_1 - \zeta_2) \quad (6.1)$$

where $F_{\varrho_1, \varrho_2} (\zeta_1 - \zeta_2)$ is obtained by computing the contribution of the diagram by using the Feynman rules given in section 2. The calculation of $\mathcal{I}_1$ follows the same manner as done in [20] for a generic non abelian gauge potential $A_{\mu}$. The main difference is that now the two quantum lines $W_{\varrho_1} [K_{\zeta_1}]$ and $W_{\varrho_2} [K_{\zeta_2}]$ are built out of the vector potential $B_{\mu}$ which is related to the CWY gauge field $A_{\mu}$ like in (5.45). In what follows, we repeat this computation for the one-loop diagram 4 built out of the vector potential $B_{\mu}$; and show that the quantum contribution has the form

$$\mathcal{I}_1 = \hbar \tilde{c}_{\varrho_1, \varrho_2} \frac{\zeta_1 - \zeta_2}{\zeta_1 - \zeta_2} \quad (6.2)$$

with coefficient $\tilde{c}_{\varrho_1, \varrho_2}$ given by

$$\tilde{c}_{\varrho_1, \varrho_2} = \sum_{a,b,c} \Gamma_+^{a_1} \times (t_{a_1, \varrho_1} \otimes t_{a_2, \varrho_2}) \times \Gamma_-^{a_2} \quad (6.3)$$

and where $\Gamma_+^{a_1}$ and $\Gamma_-^{a_2}$ are as in eqs(5.47). To perform the explicit calculation of $\mathcal{I}_1$, we must know the expression of the propagators $\langle B_{\mu}^a (X) B_{\nu}^b (X') \rangle$; they are obtained form the propagators $G_{\rho\sigma}^{cd} (X - X') = \langle A_{\rho}^{a} (X) A_{\sigma}^{d} (X') \rangle$ of the CWY gauge field by using the relation $B_{\mu} = \Gamma_{\mu a} A_{a}^{\rho}$. So, we have the following relation

$$\langle B_{\mu}^a (X) B_{\nu}^b (X') \rangle = \Gamma_{\mu a}^\rho \times \Gamma_{\nu b}^\sigma \times G_{\rho\sigma}^{cd} (X - X') \quad (6.4)$$

involving the two factor product $\Gamma_{\mu a}^\rho \times \Gamma_{\nu b}^\sigma$. Recall that in the CWY theory, the free propagators $G_{\rho\sigma}^{cd} (X - X')$ are given by (2.12) that we re-express like

$$G_{\rho\sigma}^{cd} (X_1 - X_2) = \delta^{cd} G_{\rho\sigma} (X_1 - X_2) \quad (6.5)$$

with

$$G_{\rho\sigma} (X_1 - X_2) = \frac{1}{2\pi} \varepsilon_{\rho\sigma\tau\lambda} \eta^{\tau\lambda} \frac{\partial R_{12}}{\partial X_1^{\lambda}} \quad (6.6)$$

and

$$R_{12} = \frac{1}{(X_1^+ - X_2^+) (X_1^- - X_2^-) + |\zeta_1 - \zeta_2|^2} \quad (6.7)$$

By substituting in (6.4), we get

$$\langle B_{\mu}^a (X) B_{\nu}^b (X') \rangle = \frac{1}{2} \Gamma_{\mu a}^\rho \times G_{\rho\sigma} \times \Gamma_{\nu c}^{\sigma b} \quad (6.8)$$
where we have used eq (2.12) namely $G_{\rho \sigma} = \frac{1}{4\epsilon} \epsilon_{\rho \sigma \tau} \partial R^\tau \partial X^\rho$; and where summation on the repeated index $c$ is understood. Using these $B_{\rho \sigma}$ propagators and following the method of [20, 24] by choosing the first line operator as supported on the axis $X^+ = x$ at $\zeta = \zeta_1$ and the second line operator as supported on the axis $X^- = y$ at $\zeta = \zeta_2$, we can determine the explicit value of then quantum contribution of the two line operators $K_{\zeta_1}$ and $K_{\zeta_2}$ exchanging one gluon. The calculations are quite similar to the ones done in [20, 24] by using the generic $A_{\mu}$'s; the novelty here is given by the fact that now we have a restriction coming from the factors $\Gamma_{\mu \nu \xi}^{ab}$ of (5.45). By setting $X = X_1 - X_2$, the 2-form propagator $Q^{ab}$ associated with $\langle B^a_{\mu} (X_1) B^b_{\nu} (X_2) \rangle$ is given by

$$Q^{ab} = \frac{1}{2} \langle B^a_{\mu} (X_1) B^b_{\nu} (X_2) \rangle dX^\mu \wedge dX^\nu \tag{6.9}$$

and, by using (6.8), it reads explicitly like

$$Q^{ab} = \frac{1}{2} (\Gamma_{\mu \nu \xi}^{a b} \times G_{\rho \sigma \xi} \times \Gamma_{\nu \xi \rho}^{c a}) dX^\mu \wedge dX^\nu \tag{6.10}$$

This expression can be simplified by noticing from eq (5.47) that the components of the tensor $\Gamma_{\mu \nu \xi}^{a b}$ are proportional to Kronecker $\eta_{\rho}^{\nu}$, it results therefore the following relation between $Q^{ab} = Q^{ab} (X - X')$ and $P = P (X - X')$,

$$Q^{ab} = \Gamma_{+ c}^{+ a} \times P \times \Gamma_{- c}^{- b} \tag{6.11}$$

with $P = P_+ dX^+ \wedge dX^-$ that reads explicitly as

$$P = \frac{1}{2\pi} \frac{2 (\bar{\zeta}_1 - \bar{\zeta}_2)}{(x_1 - x_2)^2 + (y_1 - y_2)^2 + |\zeta_1 - \zeta_2|^2} dx \wedge dy \tag{6.12}$$

Comparing the above $Q^{ab}$ quantity with the analogous $P^{ab} = \delta^{ab} P$ given by (2.11), used in [20] for generic vector potentials $A_{\mu}$, we end with the following expression for the one-loop contribution

$$I_1 = \hbar (t_{a, \rho} \otimes t_{b, \sigma}) \int_{x,y} Q^{ab} (x - x', y - y'; \zeta_1 - \zeta_2, \bar{\zeta}_1 - \bar{\zeta}_2) \tag{6.13}$$

By substituting $Q^{ab}$ by its expression (6.11), we also have

$$I_1 = \hbar \tilde{c}_{e, \rho} \int_{x,y} P (x - x', y - y'; \zeta_1 - \zeta_2, \bar{\zeta}_1 - \bar{\zeta}_2) \tag{6.14}$$

with $c_{e, \rho}$ reading as

$$\tilde{c}_{e, \rho} = \sum_{a,b,c} \Gamma_{+ c}^{+ a} \times (t_{a, \rho} \otimes t_{b, \sigma}) \times \Gamma_{- c}^{- b} \tag{6.15}$$

and the two integrations over $P$ given by $\frac{1}{\zeta_1 - \zeta_2}$ in agreement with the scaling dimension and the invariance of the 1-form $\omega_1 = d\zeta$ in the field action $S_{cay}$ under global translation $\zeta' = \zeta + cte$. The construction we have done above for the sl(2) case of the Liouville equation extends straightforwardly to finite dimensional Lie algebras of 2d Toda QFT.
7 Conclusion

By using the Lax formalism of 2d integrable models, we have studied in this paper the embedding of finite Toda QFT\textsubscript{2}'s into the Costello-Witten-Yamazaki theory by focussing on the first term in this family of 2d integrable models namely the Liouville model. After reviewing briefly some useful aspects of Liouville theory and properties of the Lax method of 2d integrable systems, we have shown how the two Lax operators $L_{\pm}$ of Liouville equation can be derived from the 4d CWY gauge connection $A = t_a A^a_{\mu} dx^\mu$ with SL(2) gauge symmetry. This has been done by imposing appropriate contraints on some $A^a_{\mu}$ components of the non abelian gauge potential $A_{\mu} = t_a A^a_{\mu}$ and interpreted in terms of turning off propagation of $A^a_{\mu}$ in some directions of the gauge bundle. By using results from [20, 24] regarding the observables into CWY theory, we have constructed quantum line operators that are associated with the Lax pair describing the Liouville field equation; this construction extends straightforwardly to the family of 2d Toda QFT\textsubscript{2}'s based on finite Lie algebras $G_c$. As an illustration of quantum excitations of these lines, we have also computed the one loop contribution of two interesting line operators exchanging one gauge boson and shown how the quasi-classical solution of the R-matrix gets modified compared to the result of [20]. We suspect that the method developed in this study may be applied to a large class of 2d integrable systems especially to those systems having a Lax pair formulation; the ones considered in this paper concern the class of 2d Toda QFT\textsubscript{2} based on finite dimensional Lie algebra. It would be interesting to extend this construction to other families like for instance the affine 2d Toda QFT\textsubscript{2}'s containing the sinh-Gordon model. From the analysis given in this study, we expect that these kinds of integrable systems correspond as well to particular orientations of the non abelian gauge potential $A_{\mu} = t_a A^a_{\mu}$ in the gauge bundle; these directions are given by the $\Gamma^{ab}_{\mu}$ of eq(5.45) and so this tensor may be also interpreted as an object that characterise the 2d integrable models. We end this conclusion by noticing that it would be interesting to approach the constraint eqs(5.41) by using Lagrange multipliers and look how CWY theory and its observables may be modified. Progress in this direction will be reported elsewhere.

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