Non-minimal Einstein–Maxwell theory: the Fresnel equation and the Petrov classification of a trace-free susceptibility tensor

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Abstract
We construct a classification of dispersion relations for electromagnetic waves non-minimally coupled to space-time curvature, based on analysis of the susceptibility tensor which appears in the non-minimal Einstein–Maxwell theory. We classify solutions to the Fresnel equation for the model with a trace-free non-minimal susceptibility tensor according to the Petrov scheme. For all Petrov types we discuss specific features of the dispersion relations, and plot the corresponding wave surfaces.

Keywords: Petrov type, dispersion relation, wave surface, non-minimal coupling

(Some figures may appear in colour only in the online journal)

1. Introduction

The Einstein theory of gravitation includes the postulate that in vacuum, test photons move along null geodesic lines of the corresponding space-time, i.e. the space-time geometry predetermines photon trajectories. In order to obtain detailed information about photon behavior in vacuum in a space-time with given symmetry, the classification of metrics with respect to Lie groups [1, 2] can be used.

The Einstein–Maxwell theory deals with more sophisticated situations, when the propagating electromagnetic waves are coupled to a material medium [3–6], or to quasi-media of various types (see, e.g. [7–10]). In this case, the geometric classification of space-times is not enough, and one needs the additional classification of the electromagnetically active media.
What are the objects and tools of the corresponding classifications? To answer this question, we would like to recall two well-known facts.

First, when we are interested in the analysis of space-times from the symmetry point of view, we study the solutions of the equations containing the Lie derivative of the metric along the vector field $\xi$, e.g. $\mathcal{L}_\xi g_{ik} = 0$ (for the group of isometries), or $\mathcal{L}_\xi g_{ik} = 2\phi g_{ik}$ (for the group of conformal motions), etc [1, 11]. In other words, the object of classification is the metric, while a vector field $\xi$ (the Killing vector, conformal Killing vector, etc) is the tool (key element) of classification.

Second, when one deals with the algebraic Petrov classification of Einstein space-times [2, 12], the object of classification is the Weyl tensor, $W_{ikmn}$—the trace-free constituent of the Riemann tensor $R_{ikmn}$ [1]—and the tool of classification is the set of eigenbivectors with corresponding eigenvalues. When the space-time is not Einsteinian, but is conformally flat, i.e. $W_{ikmn} = 0$, an additional classification of the Ricci tensor $R_{ik}$ [1, 13, 14] becomes necessary.

The natural question arises: what is the object for media classification in the linear electrodynamics, and what is the appropriate tool? In fact, the answer follows from the works of Tam [15, 16], in which the idea was proposed to use the so-called linear response tensor (or the constitutive tensor in the alternative terminology) $C_{ikmn}$, which links the excitation tensor, $H^k$ with the Maxwell tensor $F_{mn}$, through the linear relationship $H^k = C_{ikmn}F_{mn}$. This tensor $C_{ikmn}$ depends on the properties of the medium only, contains the information about the dielectric permittivity, magnetic permeability, magneto-electric cross effects [4–6], and thus can be taken as the intrinsic characteristic of the electromagnetically active medium. We follow this approach below, and consider the tensor $C_{ikmn}$ as the object for classification; as for the corresponding tool, the situation in the general case depends on the approach used.

Historically, the first attempt to classify the linear response tensor $C_{ikmn}$ is connected with the formalism of optical metrics (see, e.g. [17–19] and [3, 20]). According to the idea of Gordon, one can introduce the metric of some effective space-time, in which the photons—coupled, in fact, to the medium in the real space-time—‘propagate’ along the null geodesic lines attributed to this effective space-time. It was proved, in particular, that in terms of the optical metric, $g^{ik} = \eta^{-2} \left[ \delta^{ik} + (n^2 - 1)U^iU^k \right]$, found for a spatially isotropic medium with refraction index $n$ and magnetic permeability $\mu$, moving with macroscopic velocity $U$, the linear response tensor $C_{ikmn}^{(\text{opt})} = \frac{\mu}{\eta^2} \left[ g^{im}g^{kn} - g^{in}g^{km} \right]$ has the same form as that for the vacuum: $C_{ikmn}^{(\text{vac})} = \frac{1}{2} \left( g^{im}g^{kn} - g^{in}g^{km} \right)$ (see, e.g. [18]). Based on this idea, in [21] the tensor of linear response was decomposed using two associated metrics $g^{(a)}_i$ and $g^{(b)}_i$ for the case of a uniaxial medium. The decomposition of the same type, but for the colored linear response tensor $C_{(a)(b)}^{ikmn}$ with respect to color (effective) metrics $g^{(a)}_i$ and $g^{(a)}_i$, was made in [22, 23] in the framework of the $SU(N)$ symmetric Einstein–Yang–Mills theory ((a) is the group index). As the result, it was shown that the classification of the linear response tensor with respect to effective (associated, color, optical) metrics is very useful for spatially isotropic and uniaxial media, but this method is not effective when the medium has biaxial symmetry [20, 21]. In other words, the effective metrics can be used as a tool for the classification of the linear response tensor, but the classification is not complete.

Generally, the linear response tensor is skew-symmetric with respect to permutation of indices inside the pairs $[ik]$ and $[mn]$; the symmetry of the pairs is not obligatory. When $C_{ikmn} \neq C_{mnik}$ the decomposition of this constitutive tensor with respect to irreducible tensor elements shows that in addition to two standard symmetric permittivity tensors, $\varepsilon^{ii}$, $(\mu^{-1})_{pq}$ and one non-symmetric cross-effect (pseudo-)tensor $\xi^{im}$ [5], the so-called skewonic and axionic parts appear (see, e.g. [6], [24–30] and [31–37] for details and references).
The so-called premetric axiomatic scheme in the electrodynamics elaborated by Hehl and colleagues (see, e.g. [38–41]) can also be considered as a realization of the idea of the representation of the constitutive tensor $C^{ikmn}$ with respect to metric (but now there is no basic metric, and the metric, which the authors extract from the constitutive tensor, is not, generally speaking, an analog of Gordon’s optical metric).

The non-minimal Einstein–Maxwell theory (see, e.g. [42–47] for details and references) can also be described in terms of (quasi-)vacuum electrodynamics. Indeed, the coupling of photons to the space-time curvature produces variations of the phase and group velocities of the electromagnetic waves propagating in the non-minimal vacuum (see, e.g. [7, 8, 10, 43, 45]). Non-minimal coupling induces the effect of birefringence [48], the Cherenkov effect [49], optical activity [50], i.e. in the framework of the non-minimal Einstein–Maxwell theory the physical vacuum behaves as a specific medium: the quasi-medium. The main purpose of this paper is to classify the non-minimally extended linear response tensor $C^{ikmn}$ and related Fresnel equations (the objects of classification) in terms of the Petrov classification. The non-minimal Einstein–Maxwell theory is the unique theory for which the Petrov scheme can be applied for both classifications: of the space-time and of the non-minimal electromagnetically active quasi-medium.

The paper is organized as follows. In section 2.1, we recall the key elements of the Einstein–Maxwell theory. In section 2.2, we describe the geometrical optics approach, the Fresnel equations for a general linear response tensor $C^{ikmn}$ and their algebraic structure. In section 3, we discuss the essential details of the non-minimal theory and the properties of the non-minimal susceptibility tensor. In section 4, we analyze the structure of the Fresnel equation for all classes appearing in the Petrov classification of the space-times, and plot the corresponding wave surfaces in section 5. Results and outlook are formulated in section 6.

2. Einstein–Maxwell theory and the problem of electromagnetic wave propagation

2.1. Basic formalism of the medium electrodynamics

The Einstein–Maxwell theory deals with the action functional

$$S_{(EM)} = \int d^4x \sqrt{-g} \left( \frac{R}{\kappa} + \frac{1}{2} C^{ikmn} F_{ik} F_{mn} \right),$$

where $R$ is the Ricci scalar. The quantity $F_{ik} \equiv \nabla_i A_k - \nabla_k A_i$ is the Maxwell tensor, defined on the base of the electromagnetic field potential four-vector $A_i$. The information concerning specific features of interactions in the electromagnetically active medium (or quasi-medium) is encoded in the linear response tensor $C^{ikmn}$ (which can also be called as a constitutive tensor). Due to the structure of the second term in (1), this tensor possesses evident symmetry of indices

$$C^{ikmn} = -C^{ikmn} = C^{mnik} = -C^{ikmn}. \quad (2)$$

Variation of the action functional (1) with respect to potential $A_i$ yields the electrodynamic equation

$$\nabla_k \left( C^{ikmn} F_{mn} \right) = 0. \quad (3)$$

The equation $\nabla_k F^{ik} = 0$ is the consequence of the Maxwell tensor definition. Hereafter, the asterisk denotes the dualization procedure, e.g. $F^{ik} \equiv \frac{1}{2} \epsilon^{ikmn} F_{mn}$ for the second-rank tensor,
where $\epsilon_{ikmn}$ is the completely skew-symmetric Levi-Civita tensor. If we deal with the fourth-rank tensors, the dualization can be of one of two types: left dualization, $C^*_{ikmn} \equiv \frac{1}{2} \epsilon_{ikls} C_{lsmn}$; or right dualization, $C^*_m{}^{ikn} \equiv \frac{1}{2} \epsilon_{nmpl} C_{lkpl}$. When the left-dual tensor and the right-dual tensor coincide, we will use the centered asterisk, $C^*_{ikmn}$. The double-dual tensor, for which both types of dualization are applied simultaneously, we will denote as $C^{**}_{ikmn}$, i.e. $C^{**}_{ikmn} \equiv \frac{1}{4} \epsilon_{ikls} \epsilon_{mnpl} C_{lspl}$.

The tensor $C_{ikmn}$ contains information about dielectric and magnetic permeabilities, as well as about the magneto-electric coefficients [4–6]. Using the medium velocity four-vector $U_i$, normalized such that $U_i U_i = 1$, one can decompose $C_{ikmn}$ uniquely as follows

$$
C_{ikmn} = \frac{1}{2} \left( \epsilon_{im} U_k U_n - \epsilon_{kn} U_i U_m - \epsilon_{in} U_k U_m - \epsilon_{km} U_i U_n \right)
$$

$$
- \frac{1}{2} (\mu^{-1})_i U_i \epsilon_{klpq} U_p U_q - \frac{1}{2} \left[ \epsilon_{ik} (U^m U^p \nu^l U^n - U^m U^p \nu^l U^n) + \epsilon_{mn} (U^i U^p \nu^k U^l - U^i U^p \nu^k U^l) \right].
$$

Here, $\epsilon_{im}$ is the dielectric permittivity tensor, $\mu^{-1}$ is the magnetic permeability tensor and $\nu_{lm}$ is the magneto-electric coefficients pseudo-tensor. These quantities are defined through

$$
\epsilon_{im} = 2 C_{ikmn} U^k U^m,
$$

$$
(\mu^{-1})_i = - 2 C^*_{ikmn} U^k U^m,
$$

$$
\nu_{lm} = 2^* C_{kmln} U^k U^l.
$$

They are space-like, i.e. they are orthogonal to $U^i$ with respect to each of their indices. The symmetry conditions (2) for the tensor $C_{ikmn}$ yield

$$
\epsilon_{im} = \epsilon_{mi}, \mu^{-1}_i = \mu^{-1}_s, \nu_{lm} = \nu_{ml}
$$

and indicate that our model has no skewons. The tensor $\nu_{lm}$ is generally non-symmetric. If, in addition, the linear response tensor satisfies the relation

$$
C_{ikmn} + C_{inkm} + C_{imkn} = 0,
$$

the model is free from axions, and the trace of the magneto-electric coefficients tensor has to vanish $\nu_{mm} = 0$. The last condition is also known as the Post constraint (see [34, 51]).

In vacuum, the linear response tensor has the simplest form

$$
C_{ikmn}^{(\text{vac})} = \frac{1}{2} \left( g^{ik} g^{mn} - g^{in} g^{km} \right).
$$

The difference $\chi_{ikmn} \equiv C_{ikmn} - \frac{1}{2} \left( g^{ik} g^{mn} - g^{in} g^{km} \right)$ is called the susceptibility tensor.

### 2.2. Fresnel equation

When short-wavelength electromagnetic radiation propagates in curved space-time, the approximation called geometrical optics is the appropriate tool for analysis. In this approach, the potential four-vector and the field strength tensor can be represented, respectively, as

$$
A_m = a_m e^{i \Theta}, \quad F_{mn} = i (K_m a_n - K_n a_m) e^{i \Theta},
$$

where $\Theta$ is the phase, $a_m$ is a slowly varying amplitude, and $K_m$ is a wave four-vector defined as the gradient of the phase, $K_m = \nabla_m \Theta$. In the leading-order approximation, the Maxwell equations can be reduced to the system of algebraic equations
This set of linear equations with respect to $a_i$ evidently admits the pure gauge solution $a_i \sim K_i$.

In order to find non-trivial solutions to equation (12), we have to require that the rank of the matrix $\mathfrak{A}_{ij} \equiv C_{pq} K^p K^q$ is less than three, i.e. all third-order subdeterminants of $\mathfrak{A}_{ij}$ have to vanish:

$$\Delta^{ij} \equiv \frac{1}{3!} \epsilon^{ipmn} \epsilon^{jqrs} \mathfrak{A}_{pq} \mathfrak{A}_{mr} \mathfrak{A}_{ns} = 0.$$  

(13)

After some routine calculations, we can rearrange the expression for $\Delta^{ij}$ in the form

$$\Delta^{ij} = \frac{1}{8} K^i K^j \cdot G^{pqrs} K^p K^q K^r K^s,$$  

(14)

$$G^{pqrs} \equiv -\frac{4}{3} C_{ipmq} C_{krns}^* C_{ikmn}.$$  

(15)

Thus, equation (12) admits non-trivial solutions when the four components of $K^p$ satisfy the Fresnel equation, which is usually called the dispersion relation:

$$T[K] = G^{pqrs} K^p K^q K^r K^s = 0.$$  

(16)

The tensor $G^{pqrs}$ in the form (15) is known as the Kummer tensor (see, e.g. [52–54]), while its totally symmetric part $G^{(pqrs)}$ is usually called the Tamm–Rubilar tensor (see, e.g. [16, 41, 55, 56]). It is worth noting that, first, the factor $-4/3$ in (15) is chosen merely for the sake of convenience: in the vacuum case, the Tamm–Rubilar tensor and the dispersion relation take the simplest form with a unit factor $G^{(pqrs)}(\text{vac}) = g^{pq} g^{rs}$, $T[\text{vac}] K^p K^q K^r K^s = (K^p K^p)^2 = 0$; second, the Fresnel equation (16) actually does not change, if the tensor $C_{ikmn}$ transforms as follows

$$C_{ikmn} \rightarrow \Omega \cdot C_{ikmn}, \quad \Omega \neq 0,$$  

(18)

because the Tamm–Rubilar tensor and equation (16) are homogeneous with respect to the tensor $C_{ikmn}$ components.

On the other hand, the Fresnel equation (16) is a quartic homogeneous equation in the wave vector components, and it gives $K^p$ up to a factor. Therefore, equation (16) defines a quartic surface in a three-dimensional projective space $\mathbb{R}P^3$. This surface may possess, in principle, isolated singular points (maximal number of such points is sixteen [52]) and/or singularities located on a line. The positions of these singularities are determined by the condition

$$\frac{\partial T[K]}{\partial K_s} = 4 G^{(pqrs)} K^p K^q K^r K_s = 0.$$  

(19)

To clarify the physical significance of equation (19), we should recall that the system (12) with the wave vector $K^p$ satisfying the dispersion relation (16) gives only one non-trivial polarization vector $a^i$, when $\text{rank}(\mathfrak{A}_{ij}) = 2$. If $\text{rank}(\mathfrak{A}_{ij}) = 1$, we have two non-trivial polarizations for the fixed wave vector and, as a corollary, we have no birefringence phenomenon along this direction in our (quasi-)medium. Due to differential consequence of the identity (14)

$$\frac{1}{2!} \epsilon^{ipmn} \epsilon^{jqrs} \mathfrak{A}_{pq} \mathfrak{A}_{mr} \mathfrak{A}_{ns} \frac{\partial \mathfrak{A}_{ij}}{\partial K_s} = \frac{1}{8} K^i K^j \frac{\partial T[K]}{\partial K_s} + \frac{1}{8} T[K] (g^{kl} K^l + g^{ks} K^s),$$

(20)
we can conclude that the necessary condition for existence of such directions is equation (19).

3. Non-minimal Einstein–Maxwell model

3.1. Non-minimal susceptibility tensor

When one speaks about the three-parameter non-minimal Einstein–Maxwell model, one deals with the specific action functional

$$ S_{(NMEM)} = \int d^4x \sqrt{-g} \left( R - \frac{1}{2} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} \mathcal{R}^{ikmn} F_{ik} F_{mn} \right), $$

where tensor $\mathcal{R}^{ikmn}$ is defined as follows:

$$ \mathcal{R}^{ikmn} = \frac{q_1}{2} R (g^{im} g^{kn} - g^{in} g^{km}) + \frac{q_2}{2} (R^{im} g^{kn} - R^{km} g^{im} + R^{kn} g^{im} - R^{im} g^{kn}) + q_3 R^{ikmn}. $$

It is composed of the Ricci scalar $R$, the Ricci tensor $R_{ik}$, and the Riemann curvature tensor $R^{ikmn}$ (see, e.g. [47] for references and terminology); the phenomenological parameters $q_1$, $q_2$, $q_3$ describe the non-minimal coupling of electromagnetic and gravitational fields. Obviously, the action of the non-minimal Einstein–Maxwell model looks like (1), where the linear response tensor takes the form

$$ C^{ikmn} = \frac{1}{2} (g^{im} g^{kn} - g^{in} g^{km}) + \mathcal{R}^{ikmn}. $$

The latter means that the tensor $\mathcal{R}^{ikmn}$ plays the role of the non-minimal susceptibility tensor.

Let us discuss some properties of $\mathcal{R}^{ikmn}$ and $C^{ikmn}$. Firstly, they inherit symmetries of indices from the curvature tensor,

$$ \mathcal{R}^{ikmn} = -\mathcal{R}^{ikmn} = \mathcal{R}^{mnik} = -\mathcal{R}^{ikmn}, $$

$$ C^{ikmn} = -C^{ikmn} = C^{mnik} = -C^{ikmn}, $$

and the Bianchi-type identity

$$ \mathcal{R}_{ikmn} + \mathcal{R}_{inkm} + \mathcal{R}_{imnk} = 0, $$

$$ C_{ikmn} + C_{inkm} + C_{imnk} = 0. $$

Thus, the gravitational field non-minimally coupled with curvature can be considered as a quasi-medium without ‘skewons’ and ‘axions’.

Secondly, the non-minimal susceptibility tensor $\mathcal{R}^{ikmn}$ can be rewritten in terms of irreducible parts of the curvature tensor (see [11])

$$ \mathcal{R}^{ikmn} = \zeta_1 G^{ikmn} + \zeta_2 E^{ikmn} + \zeta_3 W^{ikmn}, $$

where $W^{ikmn}$ is the traceless Weyl tensor,

$$ W^{ikmn} = R^{ikmn} - \frac{1}{2} \left( R^{im} g^{kn} + R^{kn} g^{im} - R^{km} g^{im} - R^{im} g^{km} \right) + \frac{R}{6} \left( g^{im} g^{kn} - g^{in} g^{km} \right), $$

and

$$ E^{ikmn} = \frac{1}{2} \left( S^{im} g^{kn} + S^{kn} g^{im} - S^{ik} g^{mn} - S^{mn} g^{ik} \right), \quad S^{mn} = R^{mn} - \frac{1}{4} R g^{mn}, \quad S^m_m = 0, $$

Can be expressed as

$$ S^{mn} = R^{mn} - \frac{1}{4} R g^{mn}, \quad S^m_m = 0, $$

The latter means that the tensor $\mathcal{R}^{ikmn}$ plays the role of the non-minimal susceptibility tensor.
The parameters $\zeta_1$, $\zeta_2$, and $\zeta_3$ are connected with $q_1$, $q_2$, and $q_3$ by the linear relations

$$\zeta_1 = 6q_1 + 3q_2 + q_3, \quad \zeta_2 = q_2 + q_3, \quad \zeta_3 = q_3.$$  

(31)

It is worth noting that the tensor $E^{ikmn}$ does not change after the double duality procedure, while $W^{ikmn}$ and $G^{ikmn}$ change their sign

$$W^{ikmn} = -W_{ikmn}, \quad G^{ikmn} = -G_{ikmn}, \quad E^{ikmn} = E_{ikmn}.$$  

(32)

### 3.2. Trace-free susceptibility tensor model

In this paper, we focus on the model with a special type of the non-minimal susceptibility tensor $R_{ikmn}$: we will consider the case when

$$R^{**}_{ikmn} = -R_{ikmn}$$  

(33)

and, as a consequence,

$$C^{**}_{ikmn} = -C_{ikmn}.$$  

(34)

This relation means that the decomposition (27) does not contain the tensor $E_{ikmn}$, or, in other words, we have to require $\zeta_2 E_{ikmn} = 0$. This is possible in two cases.

(i) The first case arises when $\zeta_2 = q_2 + q_3 = 0$; in this case the tensor $E_{ikmn}$ and thus the Ricci tensor $R_{im}$ can be arbitrary, and the second term in (27) is provided to be vanishing due to the choice of the phenomenological parameters $q_2$ and $q_3$.

(ii) In the second case, we assume that $E_{ikmn} = 0$; this means that the traceless part of the Ricci tensor, $S_{im}$, vanishes. This variant corresponds to all Einstein space-times, $R_{im} = -\Lambda g_{im}$, $R = -4\Lambda$ (Schwarzschild, Kerr, de Sitter space-times, etc). Thus, when we consider light propagation on a certain Einstein space-time background in the framework of the non-minimal theory, we deal with our model.

Below, in the analysis of the Fresnel equation, we do not attract the reader’s attention to the question: ‘Which version do we use, the first or the second one?’; for both cases, the algebraic structures of the susceptibility tensor coincide, and the analysis is identical.

The linear response tensor $C^{ikmn}$ can be written as follows:

$$C^{ikmn} = \frac{1}{2} \left[ 1 + \frac{6q_1 + 3q_2 + q_3}{6} R \right] \left( g^{im} g^{kn} - g^{in} g^{km} \right) + q_3 W^{ikmn}.$$  

(35)

Since this tensor satisfies the Bianchi identity (9) and the duality constraint (34) $C^{**}_{ikmn} = -C_{ikmn}$, the effective tensors of dielectric permittivity and magnetic impermeability coincide with each other

$$\varepsilon_{im} = \mu^{-1}_{im}.$$  

(36)

The magneto-electric coefficients tensor in this case becomes symmetric with respect to its indices, and traceless:

$$\nu_{im} = \nu_{mi}, \quad \nu^m_m = 0.$$  

(37)
The first term in (35) describes an isotropic part of the tensor $C^{ikmn}$ related to the traces of $\varepsilon_{im}$ and $\mu_{im}^{-1}$

$$\varepsilon_{m}^m = \mu_{m}^{-1} = 3\varepsilon, \quad \varepsilon \equiv 3 \left[ 1 + \frac{6q_1 + 3q_2 + q_3}{6} P \right].$$  \hspace{1cm} (38)

When $\varepsilon \neq 0$, the expression (35) can be rearranged as follows

$$C^{ikmn} = \varepsilon \left[ \frac{1}{2} (g^{im} s^{kn} - g^{im} s^{km}) + \frac{q_3}{\varepsilon} W^{ikmn} \right],$$  \hspace{1cm} (39)

and, due to the homogeneity of the Fresnel equation (16), we may drop the factor in front of the brackets in equation (39). As a result, the effective susceptibility tensor $\chi_{ikmn}$ appears to be proportional to the Weyl tensor, $\chi_{ikmn} = (q_3/\varepsilon) W_{ikmn}$, and therefore our model can be indicated as the trace-free susceptibility tensor model.

Lastly, it is worth mentioning that for the model considered here, the linear response tensor has eleven independent components—six components of the dielectricity tensor $\varepsilon_{im}$ and five components of the magneto-electric tensor $\nu_{im}$. On the other hand, this number can be calculated via another route—the Weyl tensor has ten independent components, the eleventh one being the factor in front of $\frac{1}{2} (g^{im} s^{kn} - g^{im} s^{km})$ in (35).

The Kummer tensor $G^{pqrs}$, in accordance with (34), can be rewritten as

$$G^{pqrs} = \frac{4}{3} C^{pqmn} C^{klns} C^{ikmn}.$$  \hspace{1cm} (40)

Substituting here the representation for the linear response tensor $C^{ikmn}$ (35), we obtain

$$G^{pqrs} = \varepsilon^3 g^{pq} g^{rs} - 2\varepsilon q_1^2 W^{ipmj} W_{jlm} + \frac{4}{3} q_3 W^{ipmq} W^{klrs} W_{klnm} + \text{non-sym. terms},$$  \hspace{1cm} (41)

and, after symmetrization with respect to indices

$$G^{(pqrs)} = \left[ \varepsilon^3 - \frac{1}{8} \varepsilon q_1^2 W^{ikmn} W^{ikmn} + \frac{1}{24} q_3^2 W^{ijkl} W^{ijkl} W^{iplm} W^{iplm} \right] g^{[p} g^{q]} + 4 \varepsilon q_3^2 B^{pqrs}$$

$$+ \frac{4}{3} q_3 \left( B^{pqlm} W_{jlm}^{r s} + B^{pqlm} W_{jlm}^{r q} + B^{pqlm} W_{jlm}^{r q} \right),$$  \hspace{1cm} (42)

where $B^{pqrs}$ is the totally symmetric traceless tensor of the second order with respect to $W^{ikmn}$, also known as the Bel–Robinson tensor [57],

$$B^{pqrs} = \frac{1}{4} \left( W^{plkm} W_{klm}^{r s} + W^{plkm} W_{klm}^{r s} \right).$$  \hspace{1cm} (43)

Thus, if we need a classification of the dispersion relations for the trace-free susceptibility tensor model, we should evidently apply the classification of the Weyl tensor.

3.3. Our further strategy

In the framework of the trace-free susceptibility tensor model, the non-minimal susceptibility tensor $R^{ikmn}$, and thus the linear response tensor $C^{ikmn}$ and the Tamm–Rubilar tensor $G^{(pqrs)}$ can be represented in terms of eleven quantities: ten components of the Weyl tensor, and one real scalar $\varepsilon$ related to the isotropic part of the tensor $C^{ikmn}$ (see (38)). Then using the classification of the Weyl tensor given by Petrov [2], one can classify the Tamm–Rubilar
tensor and therefore describe all the types of electromagnetic waves coupled to curvature for our model.

From the geometrical point of view, this scheme is firmly associated with the type classification of quartic surfaces in the three-dimensional projective space \( \mathbb{RP}^3 \), which are defined by the Fresnel equations (16) with (42). The main purpose of such classification is to find and investigate singularities of these surfaces, their number and properties. In order to illustrate each surface type, we will depict a set of corresponding wave surfaces.

In this paper, we make just the first step towards a complete classification of the dispersion relations. To achieve this, we have to consider, in principle, a combined classification of the Weyl and Ricci tensors—but this is an idea for the next few years.

4. Petrov classification of the model with trace-free non-minimal susceptibility tensor

4.1. The tools for analysis: the Newman–Penrose formalism and Petrov classification

It is well-known that the Weyl tensor can be classified according to the Petrov scheme [1, 2], thus providing the corresponding classification of the trace-free susceptibility tensor.

We follow the standard Newman–Penrose formalism, but in order to avoid differences, we fix the definitions given in the book [58]. The signature of the metric is \( \{ +−−− \} \), and the null tetrads

\[
e^p_{(1)} = l^p, \quad e^p_{(2)} = n^p, \quad e^p_{(3)} = m^p, \quad e^p_{(4)} = \bar{m}^p,
\]

where \( l^p \) and \( n^p \) are real vectors, and \( m^p \) and \( \bar{m}^p \) compose the complex-conjugate pair of vectors, satisfy the conditions

\[
l^p l^p = n^p n^p = m^p m^p = \bar{m}^p \bar{m}^p = 0, \quad l^p m^p = l^p \bar{m}^p = n^p m^p = n^p \bar{m}^p = 0,
\]

\[
l^p n^p = 1, \quad m^p \bar{m}^p = -1.
\]

The corresponding tetrad components of a vector \( F_p \) are denoted as

\[
F_{(1)} = F_p l^p, \quad F_{(2)} = F_p n^p, \quad F_{(3)} = F_p m^p, \quad F_{(4)} = F_p \bar{m}^p.
\]

When the vector \( F_p \) are real one, we obtain that \( F_{(4)} = \bar{F}_{(3)} \).

In these terms, the space-time metric \( g^{ik} \) and the basic self-dual—\( U^{ik}, V^{ik}, M^{ik}, \) and anti-self-dual bivectors, \( U^{ik}, \bar{V}^{ik}, \bar{M}^{ik}, \) can be written as follows

\[
g^{pq} = l^p l^q + n^p n^q - m^p m^q - \bar{m}^p \bar{m}^q,
\]

\[
U^{ik} = -2m^i m^k, \quad V^{ik} = 2l^i m^k, \quad M^{ik} = 2m^i \bar{m}^k - 2l^i n^k,
\]

\[
U^{ik} = -2n^i n^k, \quad \bar{V}^{ik} = 2l^i \bar{m}^k, \quad \bar{M}^{ik} = -2m^i \bar{m}^k - 2l^i n^k.
\]

Using these, we can reconstruct the tensor \( q_3 W^{ikmn} \):

\[
q_3 W^{ikmn} = -\Psi_0 U^{ik} U^{mn} - \Psi_1 (U^{ik} M^{mn} + M^{ik} U^{mn}) - \Psi_2 (V^{ik} U^{mn} + U^{ik} V^{mn} + M^{ik} M^{mn}) - \Psi_3 (V^{ik} M^{mn} + M^{ik} U^{mn}) - \Psi_4 V^{ik} V^{mn} - \Psi_5 U^{ik} \bar{V}^{mn} - \Psi_6 \bar{U}^{ik} U^{mn} - \Psi_7 \bar{U}^{ik} \bar{V}^{mn} + \bar{M}^{ik} M^{mn} - \Psi_8 \bar{V}^{ik} \bar{V}^{mn} - \Psi_9 \bar{V}^{ik} V^{mn}.
\]
Five complex scalars \( \Psi_0, \Psi_1, \Psi_2, \Psi_3, \) and \( \Psi_4 \) are of the form
\[
\Psi_0 = -q_3 W_{pqrs} l^p n^q m^r m', \quad \Psi_1 = -q_3 W_{pqrs} l^p n^q m', \quad \Psi_2 = -q_3 W_{pqrs} l^p n^q m', \\
\Psi_3 = -q_3 W_{pqrs} l^p n^q m', \quad \Psi_4 = -q_3 W_{pqrs} n^p m^q n',
\]
and the bar above the scalar symbols denotes complex conjugation.

According to the Petrov classification the Weyl-type tensor belongs to one of six types: I, II, III, D, N, or O. The type I is said to be algebraically general; the other types are known as algebraically special. In the case of the type O, all scalars are equal to zero, \( \Psi_0 = \ldots = \Psi_4 = 0 \).

For the remaining types, there exist one or more scalars, which can be converted into zero by the appropriate admissible turn of the tetrad vectors. The simplest set of the scalars corresponding to each type takes the form (see [1])

(a) \( \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0 \) for the type O;
(b) \( \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0, \Psi_4 = -2 \) for the type N;
(c) \( \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \Psi_2 \neq 0 \) for the type D;
(d) \( \Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0, \Psi_3 = -i \) for the type III;
(e) \( \Psi_0 = \Psi_1 = \Psi_3 = 0, \Psi_0 \neq 0, \Psi_4 = -2 \) for the type II;
(f) \( \Psi_1 = \Psi_3 = 0, \Psi_0 = \Psi_4 \neq 0, \Psi_2 \neq 0 \) for the type I.

The type O corresponds to a conformally flat space-time, for which the Weyl tensor—and therefore the susceptibility tensor \( \mathcal{R}_{pqrs} \) vanishes. The Tamm–Rubilar tensor \( G^{(pqrs)} \) and the Fresnel equation take the simplest form
\[
G^{(pqrs)} = \epsilon^3 g^{(pqg^{rs})},
\]
and if \( \epsilon \neq 0 \), the dispersion relation (53) for the type O actually does not differ from the vacuum case. Thus, this case can be indicated as trivial. When \( \epsilon = 0 \), we deal with a degenerate case, for which the Tamm–Rubilar tensor is identically equal to zero, \( G^{(pqrs)} = 0 \).

### 4.2. General properties of the Fresnel equation

Before we begin to investigate other types of the Weyl tensor, we would like to formulate some general propositions about the Fresnel equation (16).

Let us consider the case, when the wave vector \( K_p \) in (16) is null. For the sake of simplicity we assume that \( K_p = l_p \). Direct calculation yields
\[
G^{pqrs} l_p l_q l_r l_s = -16 \text{Re} \left[ \Psi_0^2 \left( \Psi_2 + \frac{\epsilon}{4} \right) - \Psi_1^2 \Psi_0 \right].
\]

Obviously, this expression is equal to zero, if \( \Psi_0 = 0 \). Thus, when the vector \( l_p \) defines a principal null direction of the Weyl tensor (see [1]), it satisfies the Fresnel equation. Therefore, for this direction, the phase velocity of light propagation is equal to the vacuum speed of light, and the refractive index is equal to one.

Let us proceed to the case in which the principal null direction \( l_p \) defines the position of a singularity, i.e. when the vector \( K_p = l_p \) is a solution to (16):
\[
F^s \equiv G^{(pqrs)} l_p l_q l_r = 0.
\]
Calculation of every tetrad component for \( F^s \), provided \( \Psi_0 = 0 \), yields...


\begin{align}
F_{(1)} &= 0, \quad F_{(2)} = -24|\Psi_1|^2 \left( \text{Re} \Psi_2 - \frac{\varepsilon}{2} \right), \quad F_{(3)} = -24\Psi_1 \bar{\Psi}_1, \quad F_{(4)} = -24\bar{\Psi}_1 \Psi_1. \tag{56}
\end{align}

From the obtained expressions we can conclude that the principal null direction \( l_p \) defines a singularity of the surface (19), if and only if \( \Psi_0 = \Psi_1 = 0 \), i.e. if the Weyl tensor is of an algebraically special type. For this case, the equation for the amplitude vector (12) gives

\[ a_1 = c_1I_1 + c_2M_1 + c_3M_2, \tag{57} \]

where \( c_1, c_2 \) and \( c_3 \) are arbitrary constants. The first term corresponds to the pure gauge solution \( a_1 \sim K_p \), while other terms define two independent polarizations for one wave vector \( K_s = l_p \). Thus, for algebraically special types of the Weyl tensor, the principal null direction \( l_p \) determines the direction along which light propagates without refraction or birefringence.

### 4.3. Type N

For the type \( N \), when \( \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0 \), and \( \Psi_4 = -2 \), the Fresnel equation (16) gives

\[ T[K] = 4\varepsilon^3 \left( K_{(1)} - |K_{(3)}|^2 \right)^2 - 16\varepsilon K_{(1)}^4 = 0. \tag{58} \]

In a bizarre case—when the trace of the effective non-minimal quasi-medium dielectric tensor \( \varepsilon = 1 + (6q_1 + 3q_2 + q_3)R/6 = 0 \)—the Tamm–Rubilar tensor vanishes, and the Fresnel equation is trivial. When \( \varepsilon \neq 0 \), this equation can be rearranged as follows:

\[ T[K] = \left( 2K_{(1)}K_{(2)} - 2|K_{(3)}|^2 + \frac{4}{\varepsilon}K_{(1)}^3 \right) \left( 2K_{(1)}K_{(2)} - 2|K_{(3)}|^2 - \frac{4}{\varepsilon}K_{(1)}^3 \right) = 0; \tag{59} \]

therefore, the quartic surface defined by (58) splits into two quadrics. The surface possesses only one singularity, at \( K_{(1)} = K_{(3)} = K_{(4)} = 0 \)—i.e. at \( K_p = l_p \), where the two sheets of the quartic have a common point.

Each multiplier in (59) can be rewritten in the form \( g^{pq}K_pK_q \), where the tensor \( g^{pq} \) is said to be an optical metric tensor. Solutions to the Fresnel equation, in this case, have to satisfy to a relation of the following type \( g^{pq}K_pK_q = 0 \). This means that any solution \( K_p \) is a null vector for an appropriate optical metric tensor. For the type \( N \), we have two types of the optical metrics, for instance, \( A \) and \( B \),

\[ g^{pq}_A = g^{pq} + \frac{4}{\varepsilon}l^pl^q = l^pl^q + l^pl^p - m^pl^q - m^q l^p - \frac{4}{\varepsilon}l^pl^q, \]

\[ g^{pq}_B = g^{pq} - \frac{4}{\varepsilon}l^pl^q = l^pl^q + l^pl^p - m^pl^q - m^q l^p - \frac{4}{\varepsilon}l^pl^q, \tag{60} \]

which relate to the corresponding types of different polarizations \( a_i^A \) and \( a_i^B \). These formulas generalize the result obtained in [59] for gravitational pp-waves. The birefringence is absent for the wave vector \( K_p \) associated with the principal null direction \( l_p \). For other directions of the wave vector, one can derive the following polarization vector expressions as solutions to equation (12)

\[ a_i^A = i \left[ (m_p - m_p)I_1 - (m_s - m_s)l_p \right] K^p = i \left( V_{sp} - \bar{V}_{sp} \right) K^p. \tag{61} \]
\[ a^p = [(m_p + \bar{m}_p)l_p - (m_p + \bar{m}_p)\hat{p}] K^p = (V_{sp} + \bar{V}_{sp}) K^p, \]  

or

\[ a^A_{(1)} = 0, \ a^A_{(2)} = i(K_{(3)} - K_{(4)}), \ a^A_{(3)} = -iK_{(1)}, \ a^A_{(4)} = iK_{(1)}, \]

\[ a^B_{(1)} = 0, \ a^B_{(2)} = K_{(3)} + K_{(4)}, \ a^B_{(3)} = K_{(1)}, \ a^B_{(4)} = K_{(1)}. \]

Both these vectors are real, non-null and orthogonal to each other and to the wave vector:

\[ g^{pq}a^pa^q_B = g^{pq}a^pa^q_B = -2(K_p l)^2 \neq 0, \ g^{pq}a^pa^q_B = 0, \ a^p_B K^p = a^p_B K^p = 0. \]

### 4.4. Type III

For the type III, when \( \Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0, \) and \( \Psi_3 = -i, \) the dispersion relation (16) gives

\[ 4\varepsilon^3 (K_{(1)} K_{(2)} - |K_{(3)}|^2)^2 - 16\varepsilon K_{(1)}^2 (K_{(1)} K_{(2)} + 3|K_{(3)}|^2) + 32 i K_{(1)}^4 (K_{(1)} - K_{(4)}) = 0. \]  

(66)

In the case \( \varepsilon = 0, \) this equation splits into

\[ K_{(1)} = 0, \ K_{(3)} = K_{(4)}, \]

(67)

and describes two intersecting linear surfaces. When \( \varepsilon \neq 0, \) we deal with a qualitatively different situation, because the surface defined by (66) does not split, e.g. into quadrics, but is an essentially quartic one. This surface possesses two sheets, which intersect at two singular points in \( \mathbb{R}P^3: \) the first corresponds to the principal null direction, \( K_p = l_p, \) and the second is located at

\[ K_p = n_p + \frac{9}{4\varepsilon^2} l_p - \frac{i}{2\varepsilon} (m_p - \bar{m}_p). \]

(68)

Along these directions, the birefringence phenomenon is absent, and the polarization vector for the latter case being orthogonal to \( K_p \) takes the form

\[ a_p = C_1 \left[ \frac{1}{2} l_p - \frac{2\varepsilon^2}{3} n_p + i\varepsilon (m_p - \bar{m}_p) \right] + C_2 (m_p + \bar{m}_p). \]

(69)

### 4.5. Type D

When we deal with the type D, i.e. if \( \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \) and \( \Psi_2 \neq 0, \) like for the type N, we have a bizarre case, namely, \( \text{Re} \Psi_2 = \varepsilon/2, \) for which the Fresnel equation is trivial. If \( \text{Re} \Psi_2 \neq \varepsilon/2 \) the dispersion relation (16) reduces to the form

\[ (\varepsilon + 4 \text{Re} \Psi_2) (K_{(1)} K_{(2)} - |K_{(3)}|^2)^2 - \frac{36 |\Psi_2|^2}{\varepsilon - 2 \text{Re} \Psi_2} K_{(1)} K_{(2)} |K_{(3)}|^2 = 0. \]

(70)

In the case \( \text{Re} \Psi_2 = -\varepsilon/4, \) the first term vanishes and equation (70) gives three linear equations, \( K_{(1)} = 0, K_{(2)} = 0 \) and \( K_{(3)} = K_{(4)} = 0, \) describing three intersecting surfaces. Lastly, if \( \text{Re} \Psi_2 \neq \varepsilon/2 \) and \( \text{Re} \Psi_2 \neq -\varepsilon/4, \) we have the most interesting case, for which the fourth-order equation (70) splits into two second-order equations,
\[ K_{(1)}K_{(2)} - M|K_{(3)}|^2 = 0, \] (71)

\[ K_{(1)}K_{(2)} - \frac{1}{M}|K_{(3)}|^2 = 0, \] (72)

where the factor \( M \) is determined as follows:

\[ M = \frac{|\varepsilon + 2\Psi_2 - \bar{\Psi}_2| + 3|\Psi_2|}{|\varepsilon + 2\Psi_2 - \Psi_2| - 3|\Psi_2|}. \] (73)

These formulas generalize the result obtained by Drummond and Hathrell [43] for the Schwarzschild space-time.

Like in the case of the type N, for the type D we can represent two optical metrics:

\[ g_A^{pq} = l^p n^q + l^q n^p - M(m^p \bar{m}^q + m^q \bar{m}^p), \]
\[ g_B^{pq} = l^p n^q + l^q n^p - M^{-1}(m^p \bar{m}^q + m^q \bar{m}^p). \] (74)

The quartic surface described by (70) has two singularities, at \( K_p = l_p \) and \( K_p = n_p \), which are common points of the quadric sheets (71) and (72) of the main surface. Here, the vector \( n^p \) is another principal null direction of the type D Weyl tensor. When \( K_i \neq l_i \) and \( K_i \neq n_i \), there exist two different polarizations related to the corresponding optical metric

\[ a_A^p = K_p [S(l^p n_i - n^p l_i) + m^p \bar{m}_i - \bar{m}^p m_i], \] (75)
\[ a_B^p = K_p [l^p n_i - n^p l_i + S(m^p \bar{m}_i - \bar{m}^p m_i)], \] (76)

where the factor \( S \) takes the form

\[ S = \frac{i}{\text{Im}\Psi_2} \frac{|\varepsilon + 2\Psi_2 - \bar{\Psi}_2| \text{Re}\Psi_2 - (\varepsilon + \text{Re}\Psi_2)|\Psi_2|}{|\varepsilon + 2\Psi_2 - \Psi_2| + 3|\Psi_2|}. \] (77)

When the imaginary part of \( \Psi_2 \) tends to zero, \( S \) is finite, and vanishes if \( \text{Re}\Psi_2 \neq -\varepsilon \).

### 4.6. Type II

For the last algebraically special type II, when \( \Psi_0 = \Psi_1 = \Psi_3 = 0, \Psi_2 \neq 0, \) and \( \Psi_4 = -2 \), the Fresnel equation takes the form

\[ T[K] = 4(\varepsilon - 2 \text{Re}\Psi_2)^2 (\varepsilon + 4 \text{Re}\Psi_2) \left( K_{(1)}K_{(2)} - |K_{(3)}|^2 \right)^2 - 144(\varepsilon - 2 \text{Re}\Psi_2)|\Psi_2|^2 K_{(1)}K_{(2)} |K_{(3)}|^2 - 16(\varepsilon + 4 \text{Re}\Psi_2)K_{(1)}^4 + 48(\varepsilon + 2\Psi_2 - \bar{\Psi}_2)|\Psi_2|K_{(1)}^2K_{(2)}^2 + 48(\varepsilon - \Psi_2 + 2\bar{\Psi}_2)|\bar{\Psi}_2|K_{(1)}K_{(2)}K_{(3)} = 0. \] (78)

Obviously, this equation reduces to (58) for the type N, when we put \( \Psi_2 = 0 \).

For the first specific case, \( \varepsilon = 2 \text{Re}\Psi_2 \), we obtain

\[ -48K_{(1)}^2 \left[ 2\text{Re}\Psi_2 K_{(2)}^2 - 3|\Psi_2|^2(K_{(3)}^2 + K_{(4)}^2) \right] = 0. \] (79)

This equation describes two intersecting surfaces, of first and second orders, respectively. For the second specific case, \( \varepsilon = -4 \text{Re}\Psi_2 \), the relation (78) yields
Here, the surface \( T[K] = 0 \) splits into two parts—one of which, \( K(1) = 0 \), is of first order, the other of third order. For other cases, when \( \varepsilon \neq 2 \) Re \( \Psi_2 \) and \( \varepsilon \neq -4 \) Re \( \Psi_2 \), the surface \( T[K] = 0 \) does not split into lower order surfaces. However, it possesses three singularities: the first one is located at \( K(1) = K(3) = K(4) = 0 \) and the second and the third singularities are defined by the relations

\[
K(1) = (\varepsilon - 2 \text{ Re } \Psi_2) \text{ sgn}(\varepsilon + 4 \text{ Re } \Psi_2), \quad K(2) = \frac{|\varepsilon - \Psi_2 + 2\Psi_2|^2 + 9\Psi_2^2}{3|\Psi_2||\varepsilon - \Psi_2 + 2\Psi_2|}, \quad K(3)^2 = (K(4))^2 = \frac{(\varepsilon - 2 \text{ Re } \Psi_2)^2(\varepsilon + 4 \text{ Re } \Psi_2)}{3\Psi_2(\varepsilon - \Psi_2 + 2\Psi_2)}.
\]

These two points differ from each other by the sign of \( K(3) \), and therefore \( K(4) \).

### 4.7 Type I

Now, let us proceed to discussion of the algebraically general type I. For this case, we can define the set of the Weyl scalars as follows (see [1]):

\[
\Psi_1 = \Psi_3 = 0, \quad \Psi_0 = \Psi_4 = \frac{\lambda_2 - \lambda_1}{2}, \quad \Psi_2 = -\frac{\lambda_3}{2},
\]

where the three invariants \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) satisfy the relations

\[
\lambda_1 + \lambda_2 + \lambda_3 = 0,
\]

\[
\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = -\frac{q_1^2}{16} \left( W_{ikmn}W^{ikmn} - iW_{ikmn}W^{ikmn} \right) = -I,
\]

\[
\lambda_1\lambda_2\lambda_3 = -\frac{q_1^3}{48} \left( W_{ikmn}W^{mpqr}W_{pq} - iW_{ikmn}W^{mpqr}W_{pq} \right) = 2J,
\]

i.e. these invariants are the roots of the cubic equation \( \lambda^3 - I\lambda - 2J = 0 \). All these quantities have to be different, because if any pair of them coincides (or, \( I^2 = 27J^2 \)) the type I transforms to the type D.

The Fresnel equation can be written as follows:

\[
\beta_3 \left[ K(1)^4 + K(2)^4 + K(3)^4 + K(4)^4 \right] - 2(\beta_3 + 2\alpha) \left[ K(1)^2K(2)^2 + |K(3)|^4 \right] + 2(\beta_1 - \beta_2 + i\gamma) \left[ K(2)^2K(3)^2 + K(1)^2K(4)^2 \right] + 8(\beta_1 + \beta_2 + \alpha)K(1)K(2)|K(3)|^2 = 0,
\]

where the quantities \( \alpha, \beta_1, \beta_2, \beta_3, \) and \( \gamma \) are related to the invariants \( \lambda_1, \lambda_2, \lambda_3 \),

\[
\alpha = (\varepsilon - 2 \text{ Re } \lambda_1)(\varepsilon - 2 \text{ Re } \lambda_2)(\varepsilon - 2 \text{ Re } \lambda_3),
\]

\[
\beta_1 = (\varepsilon - 2 \text{ Re } \lambda_1)|\lambda_2 - \lambda_3|^2,
\]

\[
\beta_2 = (\varepsilon - 2 \text{ Re } \lambda_2)|\lambda_3 - \lambda_1|^2,
\]

\[
\beta_3 = (\varepsilon - 2 \text{ Re } \lambda_3)|\lambda_1 - \lambda_2|^2.
\]
\[ \beta_3 = (\epsilon - 2 \Re \lambda_3)|\lambda_1 - \lambda_2|^2. \]  

\[ \gamma = -\frac{2}{3} \text{Im}\left\{ (\lambda_1 - \lambda_2)[\epsilon + (\lambda_2 - \lambda_3)](\lambda_3 - \lambda_1) \\
+ \epsilon + (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1) + (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)[\epsilon + (\lambda_3 - \lambda_1)] \right\}. \]  

Firstly: let us consider the case in which none of the invariants \( \lambda_i \) has real part equal to \( \epsilon / 2 \). For this case, \( \alpha \neq 0 \) and every parameter \( \beta_i, i = 1,2,3 \), is non-vanishing. We can calculate number of singularities for the quartic surface defined by (86). Let us assume this surface possesses at least one singularity at the point \( \mathcal{M}_1(\hat{K}_1, \hat{K}_2, \hat{K}_3, \hat{K}_4) \). Then equations (19) yield

\[ \beta_3 \hat{K}_1^4 - (\beta_3 + 2\alpha)K_1\hat{K}_1 \hat{K}_2^2 + (\beta_1 - \beta_2 + i\gamma)K_1^2 \hat{K}_2^2 + (\beta_1 - \beta_2 - i\gamma)K_1 \hat{K}_2 \hat{K}_3 \hat{K}_4 = 0, \]  

\[ \beta_3 \hat{K}_2^4 - (\beta_3 + 2\alpha)K_2\hat{K}_2 \hat{K}_1^2 + (\beta_1 - \beta_2 + i\gamma)K_2^2 \hat{K}_1^2 + (\beta_1 - \beta_2 - i\gamma)K_2 \hat{K}_1 \hat{K}_3 \hat{K}_4 = 0, \]  

\[ \beta_3 \hat{K}_3^4 - (\beta_3 + 2\alpha)K_3\hat{K}_3 \hat{K}_4^2 + (\beta_1 - \beta_2 + i\gamma)K_3^2 \hat{K}_4^2 + (\beta_1 - \beta_2 - i\gamma)K_3 \hat{K}_4 \hat{K}_1 \hat{K}_2 = 0, \]  

\[ \beta_3 \hat{K}_4^4 - (\beta_3 + 2\alpha)K_4\hat{K}_4 \hat{K}_1^2 + (\beta_1 - \beta_2 + i\gamma)K_4^2 \hat{K}_1^2 + (\beta_1 - \beta_2 - i\gamma)K_4 \hat{K}_1 \hat{K}_3 \hat{K}_2 = 0. \]  

From these equations, we obtain that the parameters \( \beta_1, \beta_2, \beta_3, \) and \( \gamma \) are determined by \( \alpha \) and the coordinates of the point \( \mathcal{M}_1 \):

\[ \beta_1 = \frac{\alpha(K_1^2 + K_2^2 - K_3^2 - K_4^2) - K_1 \hat{K}_2 \hat{K}_3 - K_2 \hat{K}_1 \hat{K}_4 + 2K_3 \hat{K}_2 \hat{K}_1 \hat{K}_4}{2[\hat{K}_1 \hat{K}_2 (\hat{K}_3^2 + \hat{K}_4^2) + \hat{K}_3 \hat{K}_4 (\hat{K}_1^2 + \hat{K}_2^2)]}, \]  

\[ \beta_2 = \frac{\alpha(K_1^2 + K_2^2 - K_3^2 - K_4^2) - K_2 \hat{K}_1 \hat{K}_3 - K_3 \hat{K}_1 \hat{K}_4 + 2K_4 \hat{K}_1 \hat{K}_2 \hat{K}_3}{2[\hat{K}_3 \hat{K}_4 (\hat{K}_1^2 + \hat{K}_2^2) + \hat{K}_1 \hat{K}_2 (\hat{K}_3^2 + \hat{K}_4^2)]}, \]  

\[ \beta_3 = \frac{4\alpha(K_1^2 + K_2^2 - K_3^2 - K_4^2)}{(K_1^2 - K_2^2)^2 - (K_3^2 - K_4^2)^2}, \]  

\[ \gamma = -\frac{2}{3} \text{Im}\left\{ (\lambda_1 - \lambda_2)[\epsilon + (\lambda_2 - \lambda_3)](\lambda_3 - \lambda_1) \\
+ \epsilon + (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1) + (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)[\epsilon + (\lambda_3 - \lambda_1)] \right\}. \]
\[
\gamma = \frac{i\alpha(\hat{K}_1^2 - \hat{K}_2^2)(\hat{K}_3^2 - \hat{K}_4^2) + (\hat{K}_1^2 - \hat{K}_3^2)(\hat{K}_2^2 - \hat{K}_4^2)}{(\hat{K}_1^2 - \hat{K}_2^2)^2 - (\hat{K}_3^2 - \hat{K}_4^2)^2} \left[ \hat{K}_1^2(\hat{K}_2(\hat{K}_1^2 + \hat{K}_2^2) + (\hat{K}_3^2 + \hat{K}_4^2)) \right].
\]

Excluding from equations (96)–(99) the coordinates \(\hat{K}_1, \ldots, \hat{K}_4\), we find this singular point \(\mathcal{M}_1\) exists if

\[
\alpha(\beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2) - 2\alpha(\beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1) - 4\beta_1\beta_2\beta_3 = 0,
\]

and the parameters \(\alpha, \beta_1, \beta_2, \beta_3, \) and \(\gamma\) defined by (87)–(91) satisfy this constraint identically.

Due to the symmetry of equation (95), in addition to the point \(\mathcal{M}_1\), there exist fifteen more solutions:

\[
\begin{align*}
\mathcal{M}_2(\hat{K}_2, \hat{K}_1, \hat{K}_4, \hat{K}_3), & \quad \mathcal{M}_3(\hat{K}_1, \hat{K}_2, -\hat{K}_3, -\hat{K}_4), & \quad \mathcal{M}_4(\hat{K}_2, \hat{K}_1, -\hat{K}_4, -\hat{K}_3), \\
\mathcal{M}_5(\hat{K}_1, -\hat{K}_2, \hat{K}_3, \hat{K}_4), & \quad \mathcal{M}_6(-\hat{K}_2, \hat{K}_1, -\hat{K}_4, \hat{K}_3), & \quad \mathcal{M}_7(\hat{K}_1, -\hat{K}_2, -\hat{K}_3, \hat{K}_4), \\
\mathcal{M}_8(-\hat{K}_2, \hat{K}_1, \hat{K}_4, -\hat{K}_3), & \quad \mathcal{M}_9(-\hat{K}_2, \hat{K}_1, -\hat{K}_4, \hat{K}_3), & \quad \mathcal{M}_{10}(\hat{K}_4, \hat{K}_3, \hat{K}_2, \hat{K}_1), \\
\mathcal{M}_{11}(\hat{K}_3, \hat{K}_4, -\hat{K}_2, \hat{K}_1), & \quad \mathcal{M}_{12}(\hat{K}_4, \hat{K}_3, \hat{K}_2, \hat{K}_1), & \quad \mathcal{M}_{13}(\hat{K}_3, -\hat{K}_4, \hat{K}_2, \hat{K}_1), \\
\mathcal{M}_{14}(\hat{K}_3, \hat{K}_4, -\hat{K}_2, \hat{K}_1), & \quad \mathcal{M}_{15}(\hat{K}_3, -\hat{K}_4, -\hat{K}_2, \hat{K}_1), & \quad \mathcal{M}_{16}(\hat{K}_4, \hat{K}_3, -\hat{K}_2, \hat{K}_1).
\end{align*}
\]

Since a quartic surface may possess only 16 isolated singular points, other points do not exist. It is necessary to note that only four points, \(\mathcal{M}_1, \ldots, \mathcal{M}_4\), have well-defined coordinates. For these, the first pair of coordinates is real, while the second pair is complex conjugated. For the remaining points, \(\mathcal{M}_5, \ldots, \mathcal{M}_{16}\), this rule is violated. Thus, the quartic surface (86) has four isolated singular points.

With respect to the arrangement of these points, one can divide the type I into two subtypes. For the first subtype, the singularities are located on a plane as for a biaxial crystal, i.e. the determinant of the coordinates of \(\mathcal{M}_1, \ldots, \mathcal{M}_4\) vanishes,

\[
\Delta = \begin{vmatrix}
\hat{K}_1 & \hat{K}_2 & \hat{K}_3 & \hat{K}_4 \\
\hat{K}_2 & \hat{K}_1 & \hat{K}_4 & \hat{K}_3 \\
\hat{K}_1 & \hat{K}_2 & -\hat{K}_3 & -\hat{K}_4 \\
\hat{K}_2 & \hat{K}_1 & -\hat{K}_4 & -\hat{K}_3
\end{vmatrix} = 4(\hat{K}_1^2 - \hat{K}_2^2)(\hat{K}_3^2 - \hat{K}_4^2).
\]

This subtype can be indicated as the Fresnelian subtype, \(I_F\). The second type, for which \(\Delta \neq 0\), should be accordingly called ‘non-Fresnelian’, and it can be denoted as \(I_{NF}\).

The subtype \(I_F\) corresponds evidently to the case \(\gamma = 0\). It is realized, for instance, when each invariant \(\lambda_i\) is real. When \(\gamma \neq 0\) and \(\varepsilon \neq 2q_3\text{Re } \lambda_i\), we deal with the non-Fresnelian subtype.

Secondly; if, however, only one of the invariants \(\lambda_i\) satisfies the condition \(\varepsilon = 2q_3\text{Re } \lambda_i\), two parameters in (86), \(\alpha\) and \(\beta_i\) (for instance, \(\beta_3\)) vanish. The Fresnel equation for this case takes the form

\[
(\beta_1 - \beta_2 + i\gamma) \left[ K_{13}^2 + K_{14}^2 \right] + (\beta_1 - \beta_2 - i\gamma) \left[ K_{23}^2 + K_{24}^2 \right] + 4(\beta_1 + \beta_2)K_{16}K_{25}K_{34} = 0.
\]
The surface that is defined by (103) is essentially quartic, and has two singular lines: \(K_{(1)} = K_{(2)} = 0\) and \(K_{(3)} = K_{(4)} = 0\).

Thirdly, if two invariants satisfy the condition \(\varepsilon = 2\Re \lambda\), one more parameter, e.g. \(\beta_2\), is also equal to zero, and the Fresnel equation can be written as follows:

\[
(\beta_1 + i\gamma) \left[ K_{(2)} K_{(3)} + K_{(1)} K_{(4)} \right]^2 + (\beta_1 - i\gamma) \left[ K_{(2)} K_{(4)} + K_{(1)} K_{(3)} \right]^2 = 0.
\]

This surface has four singular lines: \(K_{(1)} = K_{(2)} = 0\), \(K_{(3)} = K_{(4)} = 0\), and in addition \(K_{(1)} = K_{(2)}\), \(K_{(3)} = -K_{(2)}\), \(K_{(3)} = K_{(4)}\).

5. Wave surfaces

The wave surfaces describing the propagation of light inside a (quasi-)medium play a considerable role in the investigation of refraction properties, and they are an apt illustration of the dispersion relations. In order to proceed to discussion of wave surfaces, it is necessary to define new non-null tetrads. One of these, \(e_i^{(s)}\), has to be time-like, normalized by the condition \(g_{ij} e_i^{(s)} e_j^{(s)} = 1\). The remaining vectors \(e_i^{(s)}\) and \(e_j^{(s)}\) have to be space-like and satisfy the conditions

\[
g_{ij} e_i^{(s)} e_j^{(s)} = g_{ij} e_i^{(s)} e_j^{(s)} = g_{ij} e_i^{(s)} e_j^{(s)} = -1.
\]

The simplest way is to choose these tetrads in the following form

\[
e_i^{(t)} = \frac{e^{-\phi Ke_i} + e^{\phi Ke_i}}{\sqrt{2}}, \quad e_i^{(s)} = \frac{e^{\psi Ke_i} + e^{-\psi Ke_i}}{\sqrt{2}}, \quad e_i^{(u)} = \frac{e^{\psi Ke_i} + e^{-\psi Ke_i}}{\sqrt{2}}, \quad e_i^{(c)} = \frac{e^{\psi Ke_i} + e^{-\psi Ke_i}}{\sqrt{2}},
\]

where \(\phi, \psi\) are arbitrary real constants (\(\phi\) is related to the speed of a frame), and their values can be fixed later. Relations between the corresponding tetrad components of the wave vector \(K\) can be accordingly written as

\[
\Omega \equiv K_{(i)} = \frac{e^{\psi Ke_{(i)}} + e^{-\psi Ke_{(i)}}}{\sqrt{2}}, \quad K_{(i)} = \frac{e^\theta K_{(i)} - e^{-\theta K_{(i)}}}{\sqrt{2}},
\]

\[
K_{(i)} = \frac{e^{i\psi K_{(i)}} + e^{-i\psi K_{(i)}}}{\sqrt{2}}, \quad K_{(i)} = \frac{e^{i\psi K_{(i)}} + e^{-i\psi K_{(i)}}}{\sqrt{2}},
\]

where the quantity \(\Omega\) may be identified as a frequency.

A wave surface determines a space distribution of the refraction indices; hence, we will apply the following coordinates

\[
n_x = \frac{K_{(1)}}{\Omega}, \quad n_y = \frac{K_{(2)}}{\Omega}, \quad n_z = \frac{K_{(3)}}{\Omega}, \quad n_z = \frac{K_{(4)}}{\Omega}.
\]

If \(K = l\), then \(K_{(i)} = \Omega\), \(K_{(i)} = 0\) or \(n_x = 1\), \(n_y = n_z = 0\). In this case, light propagates along the positive direction of the axis \(Ox\), and its phase velocity is equal to the speed of
light. When $K_i = n_i$, we have $n_x = -1, n_y = n_z = 0$, and light propagates along the negative direction of this axis with the same velocity.

Generally speaking, such a surface is quartic. However, for the case of the type $O$, the wave surface (of course, when $\varepsilon \neq 0$) is the unit sphere

$$n_x^2 + n_y^2 + n_z^2 = 1.$$  \hfill (110)

We will consider the remaining types of wave surface below.

5.1. Type $N$

For this type, the Fresnel equation (59) in the non-trivial case $\varepsilon \neq 0$ can be rewritten as follows:

$$\left( 1 - (n_x^2 + n_y^2 + n_z^2) + \frac{2e^{2\phi}}{\varepsilon} (1 - n_i)^2 \right) \left( 1 - (n_x^2 + n_y^2 + n_z^2) - \frac{2e^{2\phi}}{\varepsilon} (1 - n_i)^2 \right) = 0.$$  \hfill (111)

If we fix the value of the fitting parameter by the condition $\exp(2\phi) < |\varepsilon/2|$, we find the wave surface for the type $N$ splits into two ellipsoids of revolution,

$$\frac{\langle n_x - 1 + (1 - 2e^{2\phi}/\varepsilon)^{-1} \rangle}{(1 - 2e^{2\phi}/\varepsilon)^{-2}} + \frac{n_y^2 + n_z^2}{(1 - 2e^{2\phi}/\varepsilon)^{-1}} = 1,$$

$$\frac{\langle n_x - 1 + (1 + 2e^{2\phi}/\varepsilon)^{-1} \rangle}{(1 + 2e^{2\phi}/\varepsilon)^{-2}} + \frac{n_y^2 + n_z^2}{(1 + 2e^{2\phi}/\varepsilon)^{-1}} = 1,$$  \hfill (112)

which touch each other at the point $n_x = 1, n_y = n_z = 0$ (see figure 1(a)). If we put $\exp(2\phi) = |\varepsilon/2|$, the wave surface consists of an ellipsoid and a paraboloid of revolution (see figure 1(b)). When $\exp(2\phi) > |\varepsilon/2|$, it splits into an ellipsoid and a two-sheet hyperboloid (see figure 1(c)).

5.2. Type $D$

For the type $D$, as was mentioned above, there exist two specific cases, $\varepsilon - 2 \Re \Psi_2 = 0$ and $\varepsilon + 4 \Re \Psi_2 = 0$. The first of these is trivial, while for the second case we obtain that the wave surface consists of two planes $n_x = \pm 1$ and the straight line $n_y = n_z = 0$.

When $\varepsilon \neq 2 \Re \Psi_2$ and $\varepsilon \neq -4 \Re \Psi_2$, the wave surface splits into two parts,

$$n_x^2 + M(n_y^2 + n_z^2) = 1, \quad n_y^2 + \frac{n_z^2 + n_x^2}{M} = 1,$$  \hfill (113)

which touch each other at two points: $n_x = \pm 1, n_y = n_z = 0$. If $(\varepsilon - 2 \Re \Psi_2)(\varepsilon + 4 \Re \Psi_2) > 0$, the quantity $M$ defined by (73) is positive, and both parts of the wave surface are ellipsoids of revolution (see figure 2(a)). If $(\varepsilon - 2 \Re \Psi_2)(\varepsilon + 4 \Re \Psi_2) < 0$, we deal with two hyperboloids (see figure 2(b)). It is worth noting that in the case $\Psi_2 = -\varepsilon$, we obtain $M = -1$, and these hyperboloids coincide.

5.3. Type $III$

For this type, the Fresnel equation (66) in the non-trivial case $\varepsilon \neq 0$ can be rewritten as follows:
\[(n_x^2 + n_y^2 + n_z^2 - 1)^2 - 4e^{2\phi}/\epsilon^2 (1 - n_z^2) [1 - n_x^2 + 3(n_y^2 + n_z^2)] + \frac{16e^{3\phi}}{\epsilon^3} (1 - n_z^2) [n_x \sin \psi - n_z \cos \psi] = 0.\]  

The surface defined by (114) is essentially quartic: unlike the previous cases, it does not split into two quadrics, e.g. ellipsoids. This wave surface (see figure 3) possesses two singular points. The first, at \(n_x = 1, n_y = n_z = 0\), is standard for the algebraically special types. The second is located at the point

\[
n_x = \frac{9e^{2\phi} = 4\epsilon^2}{9e^{2\phi} + 4\epsilon^2}, \quad n_y = \frac{4e^{\phi}\epsilon \sin \psi}{9e^{2\phi} + 4\epsilon^2}, \quad n_z = -\frac{4e^{\phi}\epsilon \cos \psi}{9e^{2\phi} + 4\epsilon^2}.\]  

Finally, the surface has mirror symmetry with respect to a plane determined by the origin and two singular points. It appears to be finite for the case \(e^{\phi}/|\epsilon| < 1/2\) (see figure 3(a)), or infinite for the remaining values (see, e.g. figure 3(b)).

When \(\epsilon = 0\), the wave surface transforms to the set of two planes, \(n_x = 1\) and \(n_y \sin \psi = n_z \cos \psi\).
In this case, the wave surface equation (116) reduces to
\[
(\varepsilon - 2 \text{Re} \Psi_2)^2 (\varepsilon + 4 \text{Re} \Psi_2) \left[ n_x^2 + n_y^2 + n_z^2 - 1 \right]^2 - 36(\varepsilon - 2 \text{Re} \Psi_2)|\Psi_2|^2 (1 - n_x^2)(n_y^2 + n_z^2) \\
-4 e^{i\phi}(\varepsilon + 4 \text{Re} \Psi_2)(1 - n_x)^4 + 24 e^{i\phi}(1 - n_y)^2 \text{Re} \left\{ e^{-2i\phi}\Psi_2(\varepsilon - \Psi_2 + 2 \Psi_2)(n_y + in_z)^2 \right\} = 0.
\]  
(116)

5.4. Type II

For the type II, the wave surface equation in terms of refraction indices takes the form
\[
(\varepsilon - 2 \text{Re} \Psi_2)^2 (\varepsilon + 4 \text{Re} \Psi_2) \left[ n_x^2 + n_y^2 + n_z^2 - 1 \right]^2 - 36(\varepsilon - 2 \text{Re} \Psi_2)|\Psi_2|^2 (1 - n_x^2)(n_y^2 + n_z^2) \\
-4 e^{i\phi}(\varepsilon + 4 \text{Re} \Psi_2)(1 - n_x)^4 + 24 e^{i\phi}(1 - n_y)^2 \text{Re} \left\{ e^{-2i\phi}\Psi_2(\varepsilon - \Psi_2 + 2 \Psi_2)(n_y + in_z)^2 \right\} = 0.
\]  
(116)

This equation describes an essentially quartic surface as well as for the type III. It possesses mirror symmetry with respect to the planes \( n_y = 0 \) and \( n_z = 0 \) and three singular points: one of these is located at \( n_x = 1, n_y = n_z = 0 \); the position of the two remaining points is determined by (81) with (117). This surface appears to be finite (see figure 4) only when
\[
\begin{cases}
(\varepsilon - 2 \text{Re} \Psi_2)^2 > 4 e^{i\phi}, \\
e^{2i\phi} > \frac{(\varepsilon - 2 \text{Re} \Psi_2)^2 (\varepsilon + 4 \text{Re} \Psi_2) |\Psi_2|^2 - 9 |\Psi_2|^4 - |\Psi_2 + 2 \Psi_2|^2}{12 |\Psi_2|^2 |\varepsilon - \Psi_2 + 2 \Psi_2|^2}.
\end{cases}
\]  
(119)

otherwise, it will be infinite (see possible examples in figure 5).

For the first specific case, \( \varepsilon = 2 \text{Re} \Psi_2 \neq 0 \), the equation (118) can be transformed to the form

\[
Figure 3. Cross-section \( n_y = 0 \) of the wave surfaces for the type III. On panel (a), an example for the case \( e^{i\phi}/|\phi| < 1/2 \) is depicted, while panel (b) corresponds to the case \( e^{i\phi}/|\phi| > 1/2 \). All these surfaces possess mirror symmetry with respect to the plane \( n_y = 0 \).
Figure 4. An example of a finite wave surface for the type II ($\varepsilon = 6, \Psi_2 = 0.1 + 1.9i$). Panels (a) and (b) correspond to two orthogonal cross-sections of this surface, $n_y = 0$ and $n_z = 0$. This surface has mirror symmetry with respect to these planes.

Figure 5. Examples of infinite wave surfaces for the type II. The panels present the cross-section $n_y = 0$ of such surfaces for $\varepsilon = 6$ and various values of $\Psi_2$. (a) $\Psi_2 = 3.4$, (b) $\Psi_2 = 2.5$, (c) $\Psi_2 = 2$, and (d) $\Psi_2 = -2.5$. 
\[(1 - n_i)^2 \left[ e^{i\phi} \text{Re} \Psi_2 \right] (1 - n_i)^2 + 3 |\Psi_2|^2 (n_i^2 - n_i^2) \right] = 0. \quad (120)\]

It describes a surface consisted of two parts: a plane \( n_i = 1 \) and a cone \( e^{i\phi} \text{Re} \Psi_2 \right] (1 - n_i)^2 + 3 |\Psi_2|^2 (n_i^2 - n_i^2) = 0 \), whose vertex is located at the point \( n_i = 1 \), \( n_i = 0 \).

For the second specific case, \( \varepsilon = -4 \text{Re} \Psi_2 \neq 0 \), the wave surface equation \( (116) \) can be rewritten as follows:

\[3 \text{Re} \Psi_2 |\Psi_2|^2 (1 - n_i)^2 (n_i^2 + n_i^2) - e^{2i\phi} (1 - n_i)^2 \text{Re} \left\{ e^{-2i\phi} \Psi_2 (n_i + i n_i)^2 \right\} = 0. \quad (121)\]

As was done for the non-trivial case, we can choose the value of the parameter \( \psi \) to simplify this expression. We assume that \( \Psi_2 = |\Psi_2| e^{i\phi} \). This makes it possible to reduce \( (121) \) to the form

\[(1 - n_i) \left[ 3 \text{Re} \Psi_2 (1 + n_i)(n_i^2 + n_i^2) - e^{2i\phi} (1 - n_i)(n_i^2 - n_i^2) \right] = 0. \quad (122)\]

The surface defined by this equation splits into two parts: a plane \( n_i = 1 \) and a cubic surface

\[3 \text{Re} \Psi_2 e^{-2i\phi} (1 + n_i)(n_i^2 + n_i^2) - (1 - n_i)(n_i^2 - n_i^2) = 0. \quad (123)\]

When \( \varepsilon = 0 \) and \( \text{Re} \Psi_2 = 0 \), the wave surface for both previous cases degenerates to a set of planes: \( n_i = 1 \), \( n_i = \pm n_i \).

5.5. Type I

For this algebraically general type, the wave surface equation takes the sufficiently complicated form

\[
\begin{align*}
\alpha (n_i^2 + n_i^2 + n_i^2 - 1)^2 &+ 2(n_i^2 n_i^2 - n_i^2) \left[ \beta_1 (1 - \cos 2\phi \cos 2\psi) + \beta_2 (1 + \cosh 2\phi \cos 2\psi) + \gamma \sinh 2\phi \sin 2\psi \right] \\
+ 2(n_i^2 n_i^2 - n_i^2) \left[ \beta_1 (1 + \cos 2\phi \cos 2\psi) + \beta_2 (1 - \cosh 2\phi \cos 2\psi) - \gamma \sinh 2\phi \sin 2\psi \right] \\
+ 4\beta_3 (n_i^2 n_i^2 - n_i^2 \cos 4\phi) + \beta_i \left[ - (n_i^2 - 1)^2 \sinh^2 2\phi + (n_i^2 + n_i^2)^2 \right] \\
\sin^2 2\phi + 2n_i (n_i^2 + 1) \sinh 4\phi - 2n_i n_i (n_i^2 - n_i^2) \sin 4\psi \\
+ 4n_i (n_i^2 - n_i^2) \left[ (\beta_1 - \beta_2) \sin 2\phi \cos 2\psi - \gamma \cos 2\phi \sin 2\psi \right] \\
- 4n_i (n_i^2 - n_i^2) \left[ (\beta_1 - \beta_2) \cosh 2\phi \sin 2\phi + \gamma \sinh 2\phi \cos 2\psi \right] \\
+ 8 \left[ \gamma \cos 2\phi \cos 2\psi + (\beta_1 - \beta_2) \sin 2\phi \sin 2\psi \right] n_i n_i n_i n_i = 0, \quad (124)
\end{align*}
\]

where \( \alpha, \beta_1, \beta_2, \beta_3, \) and \( \gamma \) are defined by \( (87)-(91) \). In order to simplify it, we will assume that \( \phi = \psi = 0 \). The wave surface equation then reduces to

\[
\alpha (n_i^2 + n_i^2 + n_i^2 - 1)^2 + 4\beta_1 (n_i^2 n_i^2 - n_i^2) + 4\beta_2 (n_i^2 n_i^2 - n_i^2) + 4\beta_3 (n_i^2 n_i^2 - n_i^2) + 8 \gamma n_i n_i n_i n_i = 0. \quad (125)
\]

As was demonstrated in section 4.7, if \( \varepsilon \neq 2 \text{Re} \lambda_i \) \((i = 1, 2, 3)\) this equation defines a quartic surface with four singular points. When \( \gamma = 0 \), all these points belong to one plane, like a biaxial quasi-crystal (the Fresnelian subtype \( I_F \)). If \( \gamma \neq 0 \), these points do not lie on a plane (the non-Fresnelian subtype \( I_{NF} \)). Examples of wave surfaces for both subtypes are presented in figures 6 and 7. This surface will be infinite when at least one quantity \( \beta_i / \alpha \) \((i = 1, 2, 3)\) satisfies the inequality \( \beta_i / \alpha < -1 \) (see, e.g. figure 8); otherwise, it will be finite.

As an example for the non-Fresnelian subtype \( I_{NF} \), we can consider the model with \( \varepsilon = 1 \), \( \lambda_1 = -i\nu, \lambda_2 = i\nu, \) and \( \lambda_3 = -\lambda_1 - \lambda_2 = 0. \) From the physical point of view, this means that
Figure 6. An example of a finite wave surface for the Fresnelian subtype \( I_F \) (\( \varepsilon = 1, \lambda_1 = 0.1, \lambda_2 = 0.15 \)). Panels (a) and (b) correspond to two orthogonal cross-sections of this surface: \( n_y = 0 \) and \( n_z = 0 \). This surface has mirror symmetry with respect to these planes, and all four singular points lie on one plane: \( n_y = 0 \). This subtype corresponds to the case of a biaxial quasi-medium.

Figure 7. An example of a finite wave surface for the non-Fresnelian subtype \( I_{NF} \) (\( \varepsilon = 1, \lambda_1 = 0.5i, \lambda_2 = -0.5i \)). Panels (a) and (b) correspond to cross-sections with inclination to the plane \( n_z = 0 \) being equal to \( \pi/4 \) and \( -\pi/4 \) respectively. Panel (c) describes the cross-section \( n_y = 0 \).

Figure 8. An example of an infinite wave surface the type \( I \) (\( \varepsilon = 1, \lambda_1 = 0.6 + 0.2i, \lambda_2 = 0.2 \)). Panels (a) and (b) correspond to cross-sections with inclination to the plane \( n_y = 0 \) being equal to 0.43 and \(-0.43 \) respectively. Panel (c) describes the cross-section \( n_z = 0 \).
the dielectric permittivity tensor $\varepsilon_{m}^{i}$ and the magnetic permeability tensor $\mu_{m}^{i}$ are equal to the Kroneker delta tensor, but there exists one non-zero component of the magneto-electric coefficients tensor. In this case, we obtain

$$\alpha = 1, \quad \beta_{1} = \beta_{2} = \nu^{2}, \quad \beta_{3} = 4\nu^{2}, \quad \gamma = -4\nu^{3} \neq 0.$$ (126)

The wave surface equation takes the form

$$(n_{x}^{2} + n_{y}^{2} + n_{z}^{2} - 1)^{2} + 16\nu^{2}(n_{y}^{2}n_{z}^{2} - n_{x}^{2}) + 4\nu^{2}(n_{x}^{2} - 1)(n_{z}^{2} + n_{z}^{2}) - 32\nu^{3}n_{x}n_{y}n_{z} = 0.$$ (127)

It has four singular points, which are located at

$$M_{1}(x_{0}, y_{0}, z_{0}), \quad M_{2}(x_{0}, -y_{0}, -z_{0}), \quad M_{3}(-x_{0}, y_{0}, -z_{0}), \quad M_{4}(-x_{0}, -y_{0}, z_{0}),$$

where

$$x_{0} = \frac{\nu}{1 + 2\nu^{2} + \sqrt{1 + 5\nu^{2} + 4\nu^{4}}}, \quad y_{0} = z_{0} = \frac{1}{1 + 2\nu^{2} + \sqrt{1 + 5\nu^{2} + 4\nu^{4}}}.$$  

For the specific case when $\alpha = 0$ and, say, $\beta_{3} = 0$, the wave surface will always be infinite, because it contains the straight line $n_{x} = n_{y} = 0$. Any cross-section through this line may contain, in addition, not more than two perpendicular straight lines.

### 6. Conclusion

In this paper, we have constructed a classification of dispersion relations for the non-minimal Einstein–Maxwell model with trace-free susceptibility tensor, i.e. when the linear response tensor $C^{ikmn}$ takes the form

$$C^{ikmn} = \frac{\varepsilon}{2} \left( g^{im} g^{kn} - g^{in} g^{km} \right) + q_{3} W^{ikmn}, \quad W^{ikmn} g_{im} = 0.$$  

Since the tensor $q_{3} W^{ikmn}$ has the same algebraic properties as the Weyl tensor, our classification is based on the Petrov type distribution scheme.

On the other hand, the dispersion relation can be considered as a homogeneous equation of the fourth order with respect to the wave vector components $K_{mn}$ which determines a quartic surface in the three-dimensional real projective space $\mathbb{R}P^{3}$. Therefore, we could classify the dispersion relations according to the number of singular points for such surfaces. From the physical point of view, the singularities relate to absence of birefringence phenomenon along these directions. Apart from a few specific, bizarre cases (see details in the corresponding subsections of sect. 4), the first and second classification schemes appear to be interrelated:

| Petrov type | Number of singular points |
|-------------|---------------------------|
| O           | 0                         |
| N           | 1                         |
| III, D      | 2                         |
| II          | 3                         |
| I           | 4                         |

As was shown in section 4, only for the types N and D does the dispersion relation split into two second-order equations, and thus only these types admit application of the optical
(or effective) metric approach to describe trajectories of photons. Expressions for the optical metrics in these two cases are presented in (60) and (74).

In section 5, for each Petrov type we have presented equations and plots of corresponding wave surfaces. As was demonstrated, these wave surfaces can be either finite or infinite depending on relationships between the frame rate parameter \( \phi \), the invariant scalars \( \Psi_i \) and the trace parameter \( \varepsilon \). Furthermore, the Petrov type I should be divided into two subtypes, depending on the distribution of singular points of the wave surface:

(i) the Fresnelian subtype \( I_F \) — singular points lie on one plane, this subtype corresponds to a biaxial quasi-medium. For instance, this subtype arises when the magneto-electric coefficients tensor \( \nu^i_m \) vanishes;

(ii) the non-Fresnelian subtype \( I_{NF} \) — singular points do not lie on one plane; for instance, this subtype arises when the dielectric permittivity tensor \( \varepsilon^i_m \), the magnetic permeability tensor \( (\mu^{-1})^i_m \), and the magneto-electric coefficients tensor \( \nu^i_m \) take the form

\[
\varepsilon^i_m = (\mu^{-1})^i_m = \delta^i_m, \quad \nu^i_m = \nu \left( \delta^i_1 \delta^1_m - \delta^i_2 \delta^2_m \right).
\]

Thus, in the framework of the non-minimal Einstein–Maxwell model with trace-free susceptibility tensor, we can divide the set of dispersion relations into seven basic sorts (\( I_F, I_{NF}, II, III, D, N \) and \( O \)), according to the number and position of their singular points.

The Petrov classification scheme appears to be very productive and revealing, in arranging dispersion relations and wave surface types. However, it does not involve a lot of interesting cases [10, 26, 29, 61–63], e.g. the case for which the wave surface has 16 real singular points [63] (in contrast with the type I, where the surface possesses only four real singular points). Therefore, we are going to broaden the scope of our classification, and try to apply this approach to models with non-vanishing tensor \( S_{mn} = R_{mn} - \frac{1}{4}Rg_{mn} \)—and, in principle, non-vanishing skewon part of the linear response tensor \( C_{iklm} \). We believe that this detailed classification will help us to find new types of dispersion relations and new types of wave surfaces, and to study other geometrical optics effects: distortion, caustics etc.

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