On quasi successful couplings of Markov processes

Michael Blank, Sergey Pirogov

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Abstract

The notion of a successful coupling of Markov processes, based on the idea that both components of the coupled system “intersect” in finite time with probability one, is extended to cover situations when the coupling is unnecessarily Markovian and its components are only converging (in a certain sense) to each other with time. Under these assumptions the unique ergodicity of the original Markov process is proven. A price for this generalization is the weak convergence to the unique invariant measure instead of the strong one. Applying these ideas to infinite interacting particle systems we consider even more involved situations when the unique ergodicity can be proven only for a restriction of the original system to a certain class of initial distributions (e.g. translational invariant ones). Questions about the existence of invariant measures with a given particle density are discussed as well.

1 Introduction

Let \((X, \mathcal{B})\) be a measurable space equipped with a metric \(\rho(\cdot, \cdot)\) and assume that the resulting metric space is locally compact. A Markov chain \(\xi^t\) acting on this phase space is defined by a family of transition probabilities \(P^t(x, A)\) to jump from a point \(x \in X\) to a measurable set \(A \in \mathcal{B}\) during the time \(t \geq 0\) and an initial distribution of the random variable \(\xi^0\) representing the initial state of the Markov chain. The Markov chain \(\xi^t\) generates two semigroups of operators \(P^t \phi(x) := \int \phi(y) P^t(x, dy)\) acting on bounded measurable functions and \(P^t_* \mu(\phi) := \mu(P^t \phi)\) acting on measures.

One of the first questions in the analysis of Markov chains is the existence/uniqueness of their invariant (stationary) measures (solutions to the equation \(P^t \mu = \mu \ \forall t\)) and convergence to them for various initial distributions. We shall describe a novel approach to study these questions applicable to a reasonably broad class of Markov chains.

To a large extent various coupling results about Markov chains are based on the so called coupling inequality applied as follows: let \(\xi^t\) and \(\hat{\xi}^t\) be two Markov chains with the same transition probabilities and initial conditions \(\xi^0 = x, \ \hat{\xi}^0 = y\) defined on a common probability space \((\Omega, \mathcal{F}, \mathbb{P}_{x,y})\) where the distribution \(\mathbb{P}_{x,y}\) depends on \(x, y\) measurably. Denote by \(\tau^0\) a random variable, called intersection time, equal to the first moment of time such that \(\rho(\xi^t, \hat{\xi}^t) = 0\ \forall t \geq \tau^0\). In other words \(\tau^0\) is the moment of intersection of realizations of two Markov processes started from the points \(x, y \in X\). Using this notation the coupling inequality can be written as follows:

\[
\sup_{A \in \mathcal{B}} |\mathbb{P}_{x,y}(\xi^t \in A) - \mathbb{P}_{x,y}(\hat{\xi}^t \in A)| \leq \mathbb{P}(\tau^0 > t).
\] (1.1)
Assuming now that there exists a probability invariant measure \( \mu \) (called also a stationary distribution) of the Markov chain \( \xi_t^\epsilon \) and that the process \( \xi_t^\epsilon \) is started in the stationary distribution \( \mu \), we have that the probability that \( \xi_t^\epsilon \in A \) is equal to \( \mu(A) \) for any set \( A \in \mathcal{B} \) and \( t \geq 0 \).

Thus if the intersection time is finite with probability one the inequality (1.1) proves the convergence in total variation of the law of \( \xi_t^\epsilon \) to \( \mu \) (see Section 2). We refer the reader for excellent reviews of various results based on the coupling inequality to [11, 12, 14].

Recall that a coupling is an arrangement of a pair of processes on a common probability space to facilitate their direct comparison, namely the coupling of Markov chains \( \xi_t^\epsilon \) and \( \xi_t^\prime \) is a pairs process \((\xi_t^\epsilon, \xi_t^\prime)\) defined on the direct product space \( X \times X \) and satisfying the assumptions

\[
\mathbb{P}_{x,y}(\xi_t^\epsilon, \xi_t^\prime) \in A \times X) = \mathbb{P}_x(\xi_t^\epsilon \in A),
\]

\[
\mathbb{P}_{x,y}(\xi_t^\epsilon, \xi_t^\prime) \in X \times A) = \mathbb{P}_y(\xi_t^\epsilon \in A)
\]

for each \( t \geq 0, \ A \in \mathcal{B}. \) Extending this definition for arbitrary initial distributions \( \mu, \nu \) instead of initial points \( x, y \) one gets the coupling in this case as well.

One says that a coupling is successful if the components of the coupled processes coincide starting from the finite moment of the intersection time. Under certain assumptions on the Markov chain \( \xi_t^\epsilon \) it is believed that the existence of the successful coupling is equivalent to the convergence of distributions \( \mathbb{P}(\xi_t^\epsilon \in \cdot) \) and \( \mathbb{P}(\xi_t^\prime \in \cdot) \) (see, e.g. [12]). Of course, such equivalence cannot hold for arbitrary Markov chains. Let \( \xi_t^\epsilon \) is described by a deterministic difference relation \( x \to x/2 \). This relation defines a one-dimensional Markov chain with the unique invariant measure \( \mu \) – the delta-measure at the origin. Moreover, any probability measure converges weakly under the action of this Markov chain to \( \mu \). On the other hand, two realizations of this Markov chain started at nonzero points \( x, y \) will never intersect in finite time and thus no successful coupling is possible.\(^1\) Nevertheless any two realizations are becoming arbitrary close to each other, which leads to a generalization of the notion of the successful coupling.

Fix a certain coupling of the Markov chains \( \xi_t^\epsilon \) and \( \xi_t^\prime \) and for a given \( \epsilon > 0 \) denote by \( \tau^\epsilon \) a random variable, called quasi intersection time, equal to the first moment of time such that \( g(\xi_t^\epsilon, \xi_t^\prime) \leq \epsilon \ \forall t \geq \tau^\epsilon. \)

We shall say that the coupling is metrically quasi successful if

\[
\mathbb{P}_{x,y}(\tau^\epsilon < \infty) = 1 \quad \forall x, y \in X, \ \epsilon > 0;
\]

and topologically quasi successful if

\[
\mathbb{P}_{x,y}(1_A(\xi_t^\epsilon) \neq 1_A(\xi_t^\prime)) \xrightarrow{t \to \infty} 0 \quad \forall x, y \in X
\]

and any open set \( A \in \mathcal{B} \). Here \( 1_A(\cdot) \) stands for the indicator function of the set \( A \in \mathcal{B} \).

These definitions represent quite natural generalizations of the usual successful coupling. To the best of our knowledge the only published result in this direction is the discussion of the so called shift-coupling with close enough intervals of time in [14], see also some particular results about Markovian couplings for TASEP in [4] [11].

Both definitions of the quasi successful coupling look very similar, but neither of them implies another one. Indeed, in the above example of the coupling of two copies of the processes defined by the map \( x \to x/2 \) the coupling is clearly metrically quasi successful. To show that it is not topologically quasi successful set \( A := \mathbb{R} \setminus \{0\}, x = 0, y = 1. \) Then \( 1_A(\xi_t^\epsilon) \equiv 0 \) while \( 1_A(\xi_t^\prime) = 1 \ \forall t \geq 0, \) which implies \( \mathbb{P}_{x,y}(1_A(\xi_t^\epsilon) \neq 1_A(\xi_t^\prime)) = 1. \) On the other hand, it is obvious

\(^1\)One might argue that in this toy model all “states” of the phase space except for the origin are “unessential” and this is the main reason of such behavior. This is not the case and we shall return to this question in the analysis of infinite particle systems in Section 5.
that the property (1.3) cannot imply (1.2) since arbitrary large deviations occurring with small probabilities are allowed in (1.3) but not in (1.2).

The coupling we consider needs not to be Markovian, however in the latter case the arguments might be significantly simplified (see Section 4). Another important question also discussed in Section 4 is whether it suffices to have two coupled Markov chains become equal (close enough) at a single moment of time, or whether it is necessary to have them remain equal for all future times as it is assumed in the definition of the (quasi) intersection time.

The situation is becoming even more involved when we control the convergence between the components of a coupled system only up to a semimetric rather than a metric. This is typical for Markov chains described infinite interacting particle systems (see Section 5 and [11] for a general review). In such cases there is no hope to study all invariant measures but as we shall show, using the ideas developed in the analysis of quasi successful couplings, one can prove the uniqueness of an invariant measure for a system restricted to a reasonably large class of initial distributions (e.g. translational invariant ones).

It is worth note that various models of continuous time infinite interacting particle systems are very thoroughly studied (see, e.g. [11] and further references therein). Methods used in these references are very model specific and are based on rather different arguments (monotonicity, positivity, duality, etc.) instead of a direct application of a version of a successful coupling. Moreover, there are only a few mathematical results [7, 5, 8, 15, 2, 3, 1] about discrete time infinite interacting particle systems. The main reason here is that in the continuous time case with probability one only one “interaction” between particles can happen at a given time, while in the discrete time setting an arbitrary (even infinite) number of “interactions” may happen simultaneously.

One of the main purposes of the present paper is to fill this gap and to develop some direct tools to study the latter situation. Therefore we shall discuss mainly the discrete time case, explaining (where it is necessary) how to change the arguments to prove the corresponding continuous time versions of our arguments. The construction of couplings satisfying our assumptions for specific particle systems is not trivial and will be discussed in a separate paper.

Throughout this paper we shall not discuss the existence of invariant measures for Markov processes under consideration except for a special case of invariant measures with a given particle density discussed in Section 4. In general to prove the existence one might need to assume that these processes satisfy the Feller property (i.e. corresponding Markov operators leave the space of bounded continuous functions invariant) or some its generalizations and the compactness of the phase space. The situation is rather different if one wants to study the existence of an invariant measure supported by a non compact subset of the phase space as in the case considered in Section 6.

We follow the convention: the upper index is always reserved for time, while the lower index is used for space variables. The only exception is the upper index of the shift map $\sigma^\ell$ which means the spatial translation by the vector $\ell$ and, in fact, can be interpreted as a time variable having in mind that we apply $|\ell|$ times the translation by the length one in the direction $\ell$.

2 Functional-analytic approach

For a Markov chain $\xi^\ell$ on a metric space $(X, \varrho)$ denote by

$$
P_x(\xi^\ell \in A) := P^\ell(x, A), \quad E_x(\phi(\xi^\ell)) := P^\ell \phi(x)$$

\[2\text{Compare this to a rather extensive list of publications related to the continuous time case in [11]. Note also that the last three references are dedicated to a pure deterministic setting.}\]
the corresponding probability and mathematical expectation for the trajectory started from the point \( x \in X \). Recall that a measure \( \mu \) is invariant (stationary) for the Markov chain \( \xi_t \) if
\[
\mu(\mathbb{E}_x(\phi(\xi_t))) = \mu(\phi)
\]
for any continuous function \( \phi : X \to \mathbb{R} \), where \( \mu(\phi) := \int_X \phi \, d\mu \).

We start with a well known folklore conditional result about the uniqueness of an invariant measure of a Markov process assuming that this measure does exist.

**Theorem 2.1** Assume that there exists a probability invariant measure \( \mu \) for a Markov chain \( \xi_t \) and let
\[
\mathbb{P}_{x,y}(\tau^0 < \infty) = 1 \quad \forall x, y \in X.
\]
Then \( \mu \) is the only probability invariant measure of this Markov chain and \( P_t^\pi \nu \) converges in the total variation metric to \( \mu \) as \( t \to \infty \) for any probability measure \( \nu \).

**Proof.** Consider a coupled system \( (\xi_t, \tilde{\xi}_t) \) whose trajectories are assumed to coincide after their first intersection. Integrating the inequality (1.1) with respect the measures \( \mu \) and \( \nu \) we have
\[
|P_t^\pi \nu(A) - \mu(A)| \leq \int \mathbb{P}_{x,y}(\tau^0 > t) \, d\nu(x) \, d\mu(y)
\]
which vanishes as \( t \to \infty \) due to the assumption (2.1). \( \Box \)

In a more general setting different realizations of a Markov chain may not intersect but only converge to each other in time. It turns out that the previous result can be generalized to this setting as well (at least partially).

**Theorem 2.2** Assume that there exists a probability invariant measure \( \mu \) for a Markov chain \( \xi_t \) and let
\[
\mathbb{P}_{x,y}(\tau^\varepsilon < \infty) = 1 \quad \forall x, y \in X, \varepsilon > 0.
\]
Then \( \mu \) is the only probability invariant measure of this Markov chain and \( P_t^\pi \nu \overset{t \to \infty}{\to} \mu \) weakly for any probability measure \( \nu \).

**Proof.** Recall that a function \( \phi : X \to \mathbb{R} \) is called Lipschitz continuous if there exists a finite constant \( \text{Lip}(\phi) \) such that
\[
|\phi(x) - \phi(y)| \leq \text{Lip}(\phi) \rho(x, y)
\]
for any \( x, y \in X \).

Define \( |\phi|_L := |\phi|_\infty + \text{Lip}(\phi) \) and consider only functions \( \phi \) with bounded \( | \cdot |_L \)-norm. We have:
\[
|\mathbb{E}_x(\phi(\xi_t)) - \mathbb{E}_y(\phi(\tilde{\xi}_t))| = |\mathbb{E}_{x,y}(\phi(\xi_t) - \mathbb{E}_x,\phi(\xi_t))| \leq \mathbb{E}_{x,y}|\phi(\xi_t) - \phi(\tilde{\xi}_t)|
\leq 2|\phi|_\infty \mathbb{P}_{x,y}(\tau^\varepsilon \geq t) + \varepsilon \text{Lip}(\phi) \mathbb{P}_{x,y}(\tau^\varepsilon < t) \overset{t \to \infty}{\to} 0.
\]

Now since the left hand side does not depend on \( \varepsilon > 0 \) we deduce that \( |\mathbb{E}_x(\phi(\xi_t)) - \mathbb{E}_y(\phi(\tilde{\xi}_t))| \overset{t \to \infty}{\to} 0 \).

On the other hand, for any measure \( \nu \)
\[
\mu(\phi) - P_t^\pi \nu(\phi) = (\mu \times \nu)(\mathbb{E}_x(\phi(\xi_t)) - \mathbb{E}_y(\phi(\tilde{\xi}_t))).
\]

Observe that on a locally compact space the Lipschitz continuous functions are dense in the set of all continuous functions being constant at infinity (by Stone-Weierstrass Theorem). So to distinguish between any two probability measures on such space it is enough to consider
only integrals of Lipschitz continuous functions. Hence the invariant measure is unique and
\( P^t \nu \xrightarrow{t \to \infty} \mu \) weakly.

A close look at the proof shows that in the relation \( P^t \nu \xrightarrow{t \to \infty} \mu \) one needs to control the integrands only up to the sets of zero measure. Therefore the uniformity of the assumption \( P^t \nu \xrightarrow{t \to \infty} \mu \) on the choice of pairs of initial conditions \((x,y)\) may be significantly relaxed:

**Corollary 2.1** Let \( \mu, \nu \) be probability invariant measures for a Markov chain \( \xi^t \) and let

\[
P_{x,y}(\tau^\varepsilon < \infty) = 1 \quad \forall \varepsilon > 0
\]

for \( \mu \times \nu \) almost all pairs of points \( x, y \in X \). Then \( \mu = \nu \).

### 3 Probability techniques

In this section we shall study the topological quasi successful couplings.

**Theorem 3.1** Let \( \xi^t \) be a Markov chain with an invariant measure \( \mu \) and let \( \xi^t \) be another version of the same Markov chain. Assume that there exists a topologically quasi successful coupling for the Markov chains \( \xi^t, \xi^t \), i.e. the relation \( \xi^t, \xi^t \) holds. Then \( \mu \) is the only probability invariant measure of this Markov chain and \( P^t \nu \xrightarrow{t \to \infty} \mu \) weakly for any probability measure \( \nu \).

**Proof.** For an open set \( A \in \mathcal{B} \) and a pair of points \( x, y \in X \) we introduce the notation

\[
\Phi^t(x, y, A) := |P_{x,y}(\xi^t \in A) - P_{x,y}(\xi^t \notin A)|.
\]

Then

\[
P_{x,y}(\xi^t \in A) = P_{x,y}(\xi^t \in A, \xi^t \notin A) + P_{x,y}(\xi^t \notin A, \xi^t \in A)
\]

and

\[
P_{x,y}(\xi^t \notin A) = P_{x,y}(\xi^t \in A, \xi^t \notin A) + P_{x,y}(\xi^t \notin A, \xi^t \notin A).
\]

Hence,

\[
\Phi^t(x, y, A) = |P_{x,y}(\xi^t \in A, \xi^t \notin A) - P_{x,y}(\xi^t \notin A, \xi^t \notin A)|
\]

\[
\leq P_{x,y}(1_A(\xi^t) \neq 1_A(\xi^t)) \xrightarrow{t \to \infty} 0,
\]

by the assumption \( [3.3] \).

On the other hand, since

\[
P_{x,y}(\xi^t \in A) = P_x(\xi^t \in A) = P^t(x, A)
\]

and

\[
P_{x,y}(\xi^t \notin A) = P_y(\xi^t \in A) = P^t(y, A)
\]

from the relation \( [3.1] \) we get

\[-\Phi^t(x, y, A) \leq P^t(x, A) - P^t(y, A) \leq \Phi^t(x, y, A).
\]

Integrating the last inequality with respect to the probability invariant measure \( d\mu(x) \) and using that

\[
\int P^t(x, A) \, d\mu(x) = P^t_\mu(A) = \mu(A)
\]
we get
\[ -\int \Phi^t(x, y, A) \, d\mu(x) \leq \mu(A) - P^t(y, A) \leq \int \Phi^t(x, y, A) \, d\mu(x). \]

Thus
\[ |\mu(A) - P^t(y, A)| \leq \int \Phi^t(x, y, A) \, d\mu(x). \]

Therefore \( P^t(y, A) \xrightarrow{t \to \infty} \mu(A) \). Any bounded continuous function \( \phi \) can be approximated uniformly by a finite combination of indicator functions. Therefore we have
\[ P^t_* \nu(\phi) = \nu(P^t \phi) = \nu(\int \phi(z) P^t(\cdot, dz)) \xrightarrow{t \to \infty} \nu(\phi) \nu(1) = \mu(\phi) \]
which implies the weak convergence of \( P^t_* \nu \).

A close look at the above proof shows that if the relation (1.3) holds for a given set \( A \in \mathcal{B} \) and any \( x, y \in X \) one has
\[ P^t_* \nu(A) \xrightarrow{t \to \infty} \mu(A) \]
for any probability measure \( \nu \). Thus at least on the set \( A \) the limit measure coincides with \( \mu \).

The following simple generalization of this result is useful for the analysis of interacting particle systems. Let \( S : X \to X \) be a measurable bijection such that \( P^t(x, A) = P^t(Sx, SA) \) for any \( x \in X, A \in \mathcal{B}, t \geq 0 \).

**Corollary 3.1** Assume that all conditions of Theorem 3.1 hold except that instead of the topologically quasi successful coupling condition one assumes that there exists a coupling such that
\[ \mathbb{P}_{x,y}(1_A(\xi^t) \neq 1_A(S\xi^t)) \xrightarrow{t \to \infty} 0 \quad \forall \mathcal{B}, x, y \in X, \]
(3.2)

Then the claim of Theorem 3.1 remains valid.

The proof of this result follows exactly the same lines as above except that one uses \( S\xi^t \) instead of \( \xi^t \) and later \( Sy \) instead of \( y \) since
\[ \mathbb{P}_y(S\xi^t \in A) = \mathbb{P}_y(\xi^t \in S^{-1}A) = P^t(y, S^{-1}A) = P^t(Sy, A). \]

Similarly to Corollary 2.1 the result of Theorem 3.1 can be generalized for the case when the relation (1.3) holds only for almost all pairs of initial points \( x, y \in X \).

## 4 Markovian couplings

A coupling is called **Markovian** if the pairs process \( (\xi^t, \hat{\xi}^t) \) is a Markov chain on the direct product space \( X \times X \). Under this assumption the proofs of our results about the uniqueness of the invariant measure might be significantly simplified by means of the following idea borrowed from [11, 4]. Observe that if the coupling is Markovian and the Feller Markov chain \( \xi^t \) admits two probabilistic invariant measures \( \mu \neq \nu \) then the Markov pairs process \( (\xi^t, \hat{\xi}^t) \) admits an invariant measure having marginals equal to \( \mu \) and \( \nu \) correspondingly. Assume now that any version of the quasi successful coupling takes place. Then under the coupling the processes \( \xi^t \) and \( \hat{\xi}^t \) eventually behave the same, which contradicts to the assumption that \( \mu \neq \nu \).

Let \( (\xi^t, \hat{\xi}^t) \) be a Markovian coupling of two copies of the Markov chain \( \xi^t \). Denote by \( \mathcal{M}^t \) the distribution of the pair \( \xi^t, \hat{\xi}^t \) at time \( t \geq 0 \) and by \( \pi_* \mathcal{M}^t \) and \( \hat{\pi}_* \mathcal{M}^t \) the projections of the
measure $\mathcal{M}^t$ to the components of the coupled system. Then from a measure-theoretic point of view both our quasi successful coupling assumptions mean the the projections $\pi_\ast \mathcal{M}^t$ and $\hat{\pi}_\ast \mathcal{M}^t$ are becoming close to each other in a certain sense. On the other hand, the convergence of the projections $\pi_\ast \mathcal{M}^t$ and $\hat{\pi}_\ast \mathcal{M}^t$ immediately implies that under the action of the original Markov operator $P_t^x$ any two probability measures converge to each other.

The Markovian assumption is very natural and, in fact, is satisfied in all main coupling constructions. Using the idea proposed in [13] under this assumption one can improve significantly the result of Theorem 2.1.

**Theorem 4.1**

Let $(\xi^t, \hat{\xi}^t)$ be a Markovian coupling of two copies of a Markov chain on the probability space $(\Omega, \mathbb{P}_{x,y})$ and let $\tau$ be a random variable such that $\xi^\tau = \hat{\xi}^\tau$ and $\mathbb{P}_{x,y}(\tau < \infty) = 1$. Then if there exists a probability invariant measure $\mu$ of the Markov chain $\xi^t$ then $P_t^x \nu$ converges in the total variation metric to $\mu$ as $t \to \infty$ for any probability measure $\nu$.

**Proof.** Define a new process:

$$\tilde{\xi}^t := \begin{cases} 
\hat{\xi}^t & \text{if } t \leq \tau \\
\xi^t & \text{otherwise.}
\end{cases}$$

In [13] it has been shown that if the coupling is Markovian then the process $\tilde{\xi}^t$ is Markov with the same transition probabilities $P^t(\cdot, \cdot)$. Thus the coupling $(\xi^t, \tilde{\xi}^t)$ is successful according to our definition and the result follows from Theorem 2.1. \hfill \square

One is tempted to use a similar argument to simplify the assumptions about the quasi successful couplings. Unfortunately this does not work even in the deterministic setting. Consider a binary map $T x := 2x \pmod{1}$ defined on the unit interval $X := [0, 1]$. Then the direct product of these maps is a coupled process with the Markovian coupling. Moreover, for Lebesgue almost any initial point $x \in X$ and any $y \in X$ the trajectories $\xi^t := T^t x$ and $\hat{\xi}^t := T^t y$ are becoming arbitrary close infinitely many times. On the other hand, the dynamical system $(T, X)$ has infinitely many ergodic probability invariant measures, in particular, the Lebesgue measure and the delta-measure at zero, which contradicts to Corollary 2.1.

5 Applications to particle systems

By discrete space infinite particle systems one means the translational invariant dynamics of infinite configurations of particles on an integer lattice. We refer the reader to [11] for a general discussion of systems of this sort.

To apply the coupling technique to discrete space particle dynamics one needs to construct a coupling between different realizations of the dynamics. In distinction to the finite dimensional systems there is no much hope to achieve the usual successful coupling since this would mean that an infinite number of pairs of particles in the components of the coupled system need to share the same positions simultaneously. It seems more probable to get a version of a quasi successful coupling but even this setting turns out to be restrictive. The only known way to overcome this difficulty is to use a kind of a consecutive coupling based on pairing. By a pairing of two particles belonging to two different configurations we shall mean that up to a certain moment of time, which we call the pairing time, these particles behave independently of each other and after that time the rules applied to these two particles in both processes are assumed to be the same if this does not contradict to the definition of the process. If the latter event happen the pair needs to be broken.

The most known construction, which we shall call an equal pairing is based on a very simple idea that two particles belonging to different configurations are becoming paired when they are
located simultaneously at the same lattice site. This construction has been successfully applied e.g. in [11] for a continuous time totally asymmetric exclusion process (TASEP). However even for a discrete time version of the TASEP (not speaking about more general exclusion type processes) the equal coupling does not work well since in the discrete time setting several particles may move simultaneously (which cannot happen in the continuous time case). Another also consecutive coupling making use of a non equal pairing has been proposed to solve this problem for the discrete time TASEP by L. Gray [6], see also our forthcoming paper for the analysis of more general consecutive coupling constructions. Roughly speaking instead of pairing of particles sharing the same positions L. Gray proposed to pair particles located close enough. The main difference to the equal pairing is that one proves that eventually with time only pairs of particles with the same distances between their members survive almost surely.

Even in the case of the equal pairing one cannot apply directly our results about quasi successful couplings since their conditions are still too strong. Typically one has to deal with one of the following scenarios:

1. the upper density of the set of discrepancies between particle configurations (possibly calculated up to a finite spatial shift) vanishes with time;

2. for any finite segment \( I \in \mathbb{Z}^d \) the probability that particle configurations do not coincide on this segment (possibly calculated after a finite spatial shift) vanishes with time.

Without spatial shifts these scenarios can be considered as direct generalizations of the quasi successful coupling and the weak quasi successful coupling respectively. Much weaker versions with spatial shifts are necessary to take into account situations when non equal pairs of particles survive in a coupled system. Note that despite both scenarios look very similar none of them is a consequence of another one. Indeed the presence of a finite number of fixed discrepancies does not change their density while it changes completely the situation described by the second scenario.

Specific models of particle systems and specific pairing for them leading to above conditions will be studied in a separate paper and here we restrict ourselves to ergodic consequences of these scenarios.

5.1 Functional-analytic setting

Let \( X := \mathcal{A}^\mathbb{Z}^d, \ d \geq 1 \) be a space of be-infinite sequences with elements from a finite alphabet \( \mathcal{A} \) endowed with the standard metric \( g(\cdot, \cdot) \) based on cylinders, namely

\[
g(x, y) := 2^{-\kappa(x, y)},
\]

where \( x, y \in X, \kappa(x, y) \) is defined as the largest positive integer \( n \) such that \( x_\ell = y_\ell \) for all \( \ell \in \mathbb{Z}^d \) such that \(|\ell| \leq n\), and \(|\ell| := \max_i \ell_i\).

It is straightforward to check that the metric space \((X, g)\) is compact. Recall that two probability measures \( \mu, \nu \) are equal if \( \mu(\phi) = \nu(\phi) \) for any continuous function \( \phi : X \to \mathbb{R} \) and different otherwise. In this sense measures can be distinguished by means of continuous functions.

To be consistent with the previous results we fix also the standard sigma-algebra of Borel sets \( \mathcal{B} \) on \( X \) based on cylindric sets and denote by \( \mathcal{M} \) the set of all translational invariant probability measures \( m \) on \( X \). The triple \((X, \mathcal{B}, g)\) forms a compact measurable metric space on which we consider our Markov chains.

Since Lipschitz continuous functions are dense in the set of all continuous functions it is enough to use only Lipschitz continuous test functions to distinguish between any two distinct measures.
We introduce a semimetric \( \tilde{\rho}(x, y) \) equal to the upper density of the set of discrepancies between the configurations \( x, y \in X \) as follows:

\[
\tilde{\rho}(x, y) := \limsup_{n \to \infty} \frac{\# \{ k \in \mathbb{Z}^d : x_k \neq y_k, |k| \leq n \}}{(2n+1)^d}
\]

and say that a function \( \phi : X \to \mathbb{R} \) is \( \tilde{\rho} \)-Lipschitz continuous if there exists a finite constant \( \tilde{\text{Lip}}(\phi) \) such that the inequality

\[
|\phi(x) - \phi(y)| \leq \tilde{\text{Lip}}(\phi) \tilde{\rho}(x, y)
\]

holds almost everywhere (a.e.) with respect to any translationally invariant measure.

Using this notation the first scenario mentioned above can be reduced to the question up to which extent measures can be distinguished by means of \( \tilde{\rho} \)-Lipschitz test functions.

Clearly there are measures which cannot be distinguished this way. Therefore we restrict this question to translationally invariant measures. Moreover, we shall restrict ourselves to some subset \( X_r \) of the set of configurations invariant with respect to dynamics and to spatial shifts (e.g. the set of configurations of a given particle density). Denote by \( \sigma^\ell \xi_t \) the spatial shift of the configuration \( \xi_t \) by the integer vector \( \ell \in \mathbb{Z}^d \), i.e. \( (\sigma^\ell \xi_t)_i := \xi_t_i + \ell \).

**Theorem 5.1** Let there exist a coupling of two identical particle systems \( \xi^0, \hat{\xi}^0 \) such that for any two initial configurations \( \xi^0, \hat{\xi}^0 \in X_r \) we have

\[
\tilde{\rho}(\xi^t, \sigma^\ell \hat{\xi}^t) \xrightarrow{t \to \infty} 0
\]

for some \( \ell \in \mathbb{Z}^d, |\ell| < L \) with probability one. Then the cardinality of the set of translational (both in space and time) invariant probability measures of the Markov chain \( \xi^t \) supported by \( X_r \) cannot exceed one.

**Proof.** We start with the stronger version of the assumption without the spatial shifts namely we assume that:

\[
\tilde{\rho}(\xi^t, \hat{\xi}^t) \xrightarrow{t \to \infty} 0
\]

with probability one.

Let \( \mu, \nu \) be two different translational, both in space and time, invariant probability measures of the particle system. Then there exists a continuous function \( \phi \) which can distinguish between them, i.e. \( \mu(\phi) \neq \nu(\phi) \). On the other hand, any continuous function can be approximated by indicators of finite cylinders. Thus there exists a finite cylinder \( \langle I \rangle \) with the base \( I \subset \mathbb{Z}^d \) for which \( \mu(\langle I \rangle) \neq \nu(\langle I \rangle) \). Recall that a cylinder with the base \( I \) is the set of configurations \( x \in X \) with the given subset of coordinates \( x_i \) for \( i \in I \), i.e. \( \langle I \rangle := \{ x \in X : x_i = a_i, i \in I \} \).

Consider a sequence of functions

\[
\psi_n := \frac{1}{(2n+1)^d} \sum_{|\ell| \leq n} 1_{\langle I \rangle} \circ \sigma^\ell,
\]

where \( 1_A(\cdot) \) is the indicator function of the set \( A \subset X \).

Observe that the function \( \psi_0 := 1_{\langle I \rangle} \) distinguishes between the measures \( \mu, \nu \). Therefore due to the translation invariance of these measures each of the functions \( \psi_n(\cdot) \) again can distinguish between the measures \( \mu, \nu \). Moreover,

\[
\mu(\psi_n) = \frac{1}{(2n+1)^d} \sum_{|\ell| \leq n} 1_{\langle I \rangle} \circ \sigma^\ell = \frac{1}{(2n+1)^d} \sum_{|\ell| \leq n} \mu(1_{\langle I \rangle}) = \mu(\psi_0)
\]
for any $n \in \mathbb{Z}_+$ and thus
\[ |\mu(\psi_n) - \nu(\psi_n)| = |\mu(\psi_0) - \nu(\psi_0)| = |\mu(1_{I}) - \nu(1_{I})| > 0.\]

The limit function $\psi(x) := \lim_{n \to \infty} \psi_n(x)$ exists a.e. (with respect to any translational invariant measure) by the Birkhoff Ergodic Theorem. Due to the estimates above the limit function again distinguishes between the measures $\mu, \nu$.

Let us show that this limit function $\psi$ is $\tilde{\rho}$-Lipschitz continuous. For $x \in X$ and $n \in \mathbb{Z}_+$ denote
\[ J_n(x) := \{ \ell \in \mathbb{Z}^d : |\ell| \leq n, \ \sigma^\ell(x) \in \langle I \rangle \}. \]

Then $\psi_n(x) = (2n + 1)^{-d}|J_n(x)|$. A single site may contribute to the value $|J_n(x)|$ at most a certain finite constant $C$ which depends only on the cylinder $\langle I \rangle$ but neither on $n \in \mathbb{Z}_+$ nor on the configuration $x \in X$.

For a pair of configurations $x, y \in X$ and $n \in \mathbb{Z}_+$ denote
\[ D_n(x, y) := \{ k \in \mathbb{Z}^d : x_k \neq y_k, |k| \leq n \}. \]

Then
\[ |\psi_n(x) - \psi_n(y)| \leq C(2n + 1)^{-d} |D_{n+|I|}(x, y)|. \]

On the other hand,
\[ \limsup_{n \to \infty} (2n + 1)^{-d} |D_n(x, y)| = \tilde{\rho}_n(x, y). \]

Therefore passing to the limit as $n \to \infty$ we get that the limit function $\psi(\cdot)$ is $\tilde{\rho}$-Lipschitz continuous with the constant $\text{Lip}(\psi) := C$.

Applying now the argument used in the first part\textsuperscript{3} of the proof of Theorem 2.2 we get the result.

Using similar arguments one considers also the case when the components of the coupled process converge to each other in terms of the semimetric $\tilde{\rho}$ only after a finite spatial shift. \qed

5.2 Probabilistic setting

Assume now that there exist a coupling of two identical particle systems such that for any given finite subset $I \in \mathbb{Z}$ the restriction of the components of the coupled system to $I$ coincide at moment $t \geq 0$ with probability going to one with time, i.e.
\[ \mathbb{P}_{x, y}(\xi^t_I = \xi^t_I) \xrightarrow{t \to \infty} 1 \]

for any two initial configurations $\xi^0, \tilde{\xi}^0$ from a certain dynamically invariant subset $X_r \subseteq X$. Similar to the deterministic setting we say that a set $Y \subseteq X$ is dynamically invariant if $\xi^0 \in Y$ implies that $\xi^t \in Y$ a.s. for any $t \geq 0$.

Thus the probability that the configurations $\xi^t, \tilde{\xi}^t$ coincide on any finite cylinder goes to one with time. Hence the relation (8.1) is satisfied for any cylinder set. Using now that linear combinations of indicator functions of the cylinder sets are dense in the space of bounded continuous functions on $X$ we may apply the claim of Theorem 8.1.

In the case when the restriction of the configurations to a finite segment $I$ coincide only after a finite spatial shift $\sigma^\ell$ follows from the same argument if we consider only (spatially) translational invariant distributions and use Corollary 8.1 with $S := \sigma^\ell$ instead of Theorem 8.1.

Therefore we get the following result. Denote by $\mathcal{M}_r$ the set of translational invariant probability measures supported by $X_r$.

\textsuperscript{3}Since we deal here only with Lipschitz continuous functions we do not need to apply the last part of the proof of Theorem 2.2 which deals with the approximations of continuous functions by the Lipschitz continuous ones.
Theorem 5.2 Let \( \xi^t \) be a Markov chain with an invariant measure \( \mu \in \mathcal{M}_r \) and let \( \check{\xi}^t \) be another version of the same Markov chain. Assume that there exists a coupling under which

\[
P_{x,y}(\xi^t_{i,j} = \check{\xi}^t_{i,j}) \xrightarrow{t \to \infty} 1
\]

for any pair of configurations \( \xi^0 = x, \check{\xi}^0 = y \in X_r \), any finite segment \( I \subset \mathbb{Z}^d \), and some \( \ell \in \mathbb{Z}^d \), \( |\ell| \leq L < \infty \). Then \( P^t \nu \) converges weakly to \( \mu \) as \( t \to \infty \) for any \( \nu \in \mathcal{M}_r \).

Corollary 5.1 Under all the assumption of the previous Theorem except for \( \mu \in \mathcal{M}_r \) the sequence \( P^t \nu \) converges weakly to some invariant measure which does not depend on \( \nu \).

6 Invariant measures with a given particle density

As we already mentioned normally the proof of the existence of invariant measures can be reduced to the checking of some very general topological and metrical assumptions. The situation changes drastically if one wants to study invariant measures having some additional properties, e.g., supported by some dynamically invariant subsets. The problem here is that even if the phase space of the system is compact the dynamically invariant subset may not satisfy this property.

To be more specific consider a class of locally interacting particle systems on the lattice \( \mathbb{Z}^d \) represented by Markov chains with the phase space \( X := \mathcal{A}^{\mathbb{Z}^d} \), \( \mathcal{A} := \{0, 1, \ldots, |A| - 1\} \), \( |A| < \infty \). For a configuration \( x \in X \) and a lattice site \( i \in \mathbb{Z}^d \) we associate the value \( x_i \in \mathcal{A} \) to the number of particles located at \( i \). The dynamics is defined as follows. At each site of the lattice there is an alarm-clock and at time \( t > 0 \) we consider only those lattice sites at which the alarm rings. For each particle located at such site \( i \in \mathbb{Z}^d \) we calculate its velocity \( v_i \), such that \( |v_i| \leq V < \infty \), using a (random or deterministic) procedure which does not depend neither on time nor on the configuration \( x \). Here \( V \) plays the role of the largest allowed velocity and its boundedness defines the locality of interactions. Then one checks a certain admissibility condition related to the possibility to move a particle from the site \( i \) to the site \( i + v_i \). We assume that the admissibility condition is again local and depends only on the present positions of the particles in the \( 2V \)-lattice neighborhood of the site \( i \). Only if this condition is satisfied the particle is moved to a new position. Then for all sites to where the particles were moved we restart the alarm-clocks (again using a certain random or deterministic procedure).

Depending on the way how one restarts the alarm-clocks both continuous and discrete time particle systems can be considered. In what follows we restrict ourselves to a (more interesting from our point of view and much less studied) discrete time case, assuming that the alarm-clocks start with the same setting and after each restart we add one to the time. Therefore all particles are trying to move simultaneously.

There is an important property that holds for all systems we consider here: particle number conservation. For a configuration \( x \in X := \mathcal{A}^{\mathbb{Z}^d} \) and a finite subset \( I \subset \mathbb{Z}^d \) denote by \( \rho(x, I) \) the number of particles from the configuration \( x \) located in \( I \) divided by the total number of sites in \( I \), which we denote by \( |I| \). Clearly \( 0 \leq \rho(x, I) \leq |A| - 1 \). Choosing a sequence of lattice cubes \( I_n := \{ \ell \in \mathbb{Z}^d : |\ell| \leq n \} \) we consider the limit \( \lim_{n \to \infty} \rho(x, I_n) \). If this limit exists we call it the particle density of the configuration \( x \in X \) and denote by \( \rho(x) \).

According to the above informal description the particle systems under consideration are Markov chains on \( X \). We assume the they satisfy the Feller condition, which together with the compactness of the phase space implies immediately the existence of an invariant measure. Moreover, due to the particle number conservation it seems that for each \( r \in [0, |A| - 1] \) there should be an invariant measure \( \mu_r \) having a support on the set of configurations of density \( r \).
The following simple deterministic example shows that this is absolutely not the case. For $d = 1$ define a map $T : X \to X$ as follows:

$$(Tx)_i := \begin{cases} x_i & \text{if } i = 0 \\ 0 & \text{if } |i| = 1 \\ x_{i-1} & \text{if } i > 1 \\ x_{i+1} & \text{otherwise.} \end{cases}$$

In words, we preserve the central site, shift all other sites to the left and to the right, and set the sites $\pm 1$ to zero.

Clearly all above assumptions are satisfied in this example and the particle density is conserved. On the other hand, for any initial configuration $x \in X$ the sequence of configurations $T^t x$ converges in the standard metric to the configuration of all zeros, having, of course, zero particle density. Thus this system has only one invariant measure supported by the configuration of all zeros.

One might think that the result above is a consequence of the absence of the conservation of vacancies. Let us describe a simple probabilistic generalization of this example conserving both particles and vacancies and having exactly two invariant measures supported by the configuration of all zeros or all ones. Choose $p \in (0, 1)$. For each particle in a configuration $x \in X$ located at a nonnegative site and having a vacancy immediately to the right we exchange the positions of these particle/vacancy with the probability $p$. If the particle is located at a negative site we do the same but for the vacancy immediately to the left of it. This rule defines a discrete time Markov chain having the desired property.

In what follows we discuss a coupling argument\textsuperscript{4} which gives sufficient conditions for the existence of invariant measures supported by configurations of a given density.

Let $\xi^t$ be a Markov chain on $X := \mathcal{A}^{\mathbb{Z}^d}$ with translational invariant transition probabilities, i.e. $P(x, A) \equiv P(\sigma^\ell x, \sigma^\ell A)$ for any $x \in X$, $A \in \mathcal{B}$, $\ell \in \mathbb{Z}^d$, which describes the dynamics of a locally interacting particle system. We shall say that a particle density $\rho(\mu)$ of a translational invariant probability measure $\mu$ is the mathematical expectation (with respect to this measure) of the number of particles located at the origin, i.e. $\rho(\mu) := \mu(x_0)$.

The following simple result explains the reason why we use the same notation for the the particle densities of configurations and measures.

\textbf{Lemma 6.1} For each $r \in [0, |A| - 1]$ there exists a translational both in space and time invariant measure $\mu_r$ such that $\rho(\mu_r) = r$.

\textbf{Proof.} Choose any translational invariant measure $\mu$ and consider a sequence of measures

$$\mu^n := \frac{1}{n} \sum_{t=0}^{n-1} P^t_{\mu} \mu.$$ 

Due to the compactness of $X$ there exists a subsequence $n_k \to \infty$ such that $\mu^{n_k}$ converges weakly to a limit measure $\nu$. Now using the Feller property we see that the measure $\nu$ is invariant. The translation invariance of $\nu$ follows from the construction and since the particle density of each measure $\mu^n$ is the same as for the initial measure $\mu$, we get $\rho(\nu) = \rho(\mu)$. Consider a distribution $\pi := \{\pi_i\}_{i \in A}$ such that $\sum_{i} i \pi_i = r$. To finish the proof it remains to notice that the Bernoulli

\textsuperscript{4}The idea of this construction has been proposed by L. Gray \cite{gray} for the case of the one-dimensional totally asymmetric exclusion process. It is worth note that the situation we consider differs significantly from the one studied by L. Gray in that particles may have velocities greater than one and thus long range interactions should be taken into account.
Proof. Fix a spatially ergodic distribution sequence of measures \( \mu_r \) with the alphabet \( \mathcal{A} \) and the distribution of symbols \( \pi \) has the particle density which is exactly equal to \( r \).

\[ \square \]

**Corollary 6.2** For each \( r \in [0, |A| - 1] \) the measure \( \mu_r \) with properties described above can be constructed as a limit point of the sequence of ergodic averages of the measures \( P^n_s \mu_r, B \) where \( \mu_r, B \) is a Bernoulli measure with the particle density \( r \).

Unfortunately a translational invariant measure with the particle density \( r \) is not necessarily supported by configurations of density \( r \).

We shall say that a measure \( \mu \) is spatially ergodic if the dynamical system \( (\sigma, X, \mu) \) is ergodic. We refer the reader to [10] for general definitions related to ergodic theory of dynamical systems.

Clearly each spatially ergodic measure is translational invariant and by the Birkhoff theorem it is supported by the configurations of the same particle density.

**Lemma 6.3** Let the initial distribution \( \mu \) of the process \( \xi^t \) be spatially ergodic. Then almost surely for any \( t \geq 0 \) the density \( \rho(\xi^t) \) for the configuration \( \xi^t \) is well defined, does not depend on \( t \), and is equal to the particle density \( \rho(\mu) \) of the initial distribution.

**Proof.** For \( t = 0 \) the existence of the limit in the definition of the particle density for the configuration \( \xi^0 \) and its coincidence with \( \rho(\mu) \) follows from the Birkhoff ergodic theorem due to the spatial ergodicity of the initial distribution \( \mu \).

Choose \( n \in \mathbb{Z}_+ \) large enough and consider the lattice cube \( I_n \). We can estimate the difference between the number of particles located in \( I_n \) at moments \( t \) and \( t + 1 \) as follows. Only particles belonging to the set \( I_n \setminus I_{n-V} \) are able to leave the the cube \( I_n \) during one time step. Similarly only particles located at the set \( I_{n+V} \setminus I_n \) are able to enter the cube \( I_n \) during this time. Therefore using that \( 0 \leq \rho(\xi^t, I) \leq |A| - 1 \) we get

\[
|\rho(\xi^{t+1}, I_n) - \rho(\xi^t, I_n)| \\
\leq \frac{|I_n \setminus I_{n-V}|}{|I_n|} \rho(\xi^t, I_n \setminus I_{n-V}) + \frac{|I_{n+V} \setminus I_n|}{|I_n|} \rho(\xi^t, I_{n+V} \setminus I_n) \\
\leq 2|A| \max \left( \frac{(2n + 1)^d - (2n + 1 - 2V)^d}{(2n + 1)^d}, \frac{(2n + 1 + 2V)^d - (2n + 1)^d}{(2n + 1)^d} \right) \\
= 2|A| \max \left( 1 - \left( 1 - \frac{2V}{2n + 1} \right)^d, \left( 1 + \frac{2V}{2n + 1} \right)^d - 1 \right) \to 0 \quad \text{as} \quad n \to \infty.
\]

This proves both the existence and the conservation of the particle density for the configurations \( \xi^t \). \( \square \)

**Lemma 6.4** Let there exist a coupling of two copies \( \xi^t, \tilde{\xi}^t \) of our Markov chain such that for all initial configurations with the same particle density the conditions of either Theorem \( \ref{Theorem 5.1} \) or Theorem \( \ref{Theorem 6.2} \) hold. Then for each spatially ergodic initial distribution \( \mu \) all limit points of the sequence of measures \( \{ P^n_{s^t} \mu \}_{n \in \mathbb{Z}_+} \) coincide and the limit measure has the same particle density.

**Proof.** Fix a spatially ergodic distribution \( \mu \) on \( X \). Due to the compactness of \( X \) there exists a sequence of integers \( n_k \to \infty \) such that the sequence of measures \( P^n_{s^t} \mu \) converges to a certain limit point, call it \( \tilde{\mu} \). Passing to the limit as \( n_k \to \infty \) and using that \( \rho(P^n_{s^t} \mu) \) does not depend on \( n_k \in \mathbb{Z}_+ \) we get that the limit distribution \( \tilde{\mu} \) also has the same particle density. The spatial ergodicity of \( \mu \) implies the spatial ergodicity of \( P^n_{s^t} \mu \) for each \( n \in \mathbb{Z}_+ \).
Coupling\textsuperscript{5} the process with the initial distribution $\mu$ and the one with $P_\ast \mu$ we see that according to our assumptions these two processes eventually behave the same. Using the same arguments as in the proofs of Theorems 2.2 and 3.1 we get $\lim_{n \to \infty} P_n \mu \sim \mu$. Thus using the semigroup property $P_{n+1} = P_n P_\ast$ we have:

$$\tilde{\mu} = \lim_{k \to \infty} P_n \mu = P_\ast \left( \lim_{k \to \infty} P_n \mu \right) = P_\ast \tilde{\mu},$$

which proves the invariance of the measure $\tilde{\mu}$, and in turn implies that all limit points of the sequence of measures $P_n \mu$ coincide and do not depend on $\mu$.

Finally let us discuss briefly a prototype coupling between two copies $\xi_t, \xi_t'$ of the Markov chain we consider here. To each particle we attach an additional parameter $s \in \{0, 1\}$ called state and if a particle is paired we set its state to 1 and to 0 otherwise. We introduce also a parameter $L > 0$ which will be used in pairing.

At $t = 0$ all particles in the configurations $\xi_0$ and $\xi_0'$ are at state 0 (unpaired). At time $t > 0$ first we enumerate all particles in the configurations $\xi_t, \xi'_t$ with respect to their distance to the origin. If several particles have the same distance we enumerate them at random (similarly later when choosing the closest particles we shall take the one with the smallest index if there are more than one of them). Then we start the procedure with the particle $\eta$ from the first configuration having the smallest index.

If the particle $\eta$ is at state 1 (i.e. paired) we check its distance $\ell$ to another member of the pair $\eta'$, and if $\ell > L$ we change the states of both these particles to 0 (i.e. they become unpaired). Otherwise we check the distance $\ell'$ to the closest particle $\eta''$ with state 0 from another process. If $\ell' < \ell$ we swap the pairing setting the state of the particle $\eta'$ to 0 and the state of the particle $\eta''$ to 1 pairing it with $\eta$.

If the particle $\eta$ is at state 0 (may be after the application of the previous step of the procedure) we check the distance $\ell$ to the closest particle in the second process $\eta'$ and if $\ell > L$ we stop the consideration of the particle $\eta$. Otherwise if the state of $\eta'$ is 0 we pair these particles setting their types to 1. If the state of $\eta'$ is 1 we compare $\ell$ to the distance to another member of the paired particle $\eta'$ and proceed exactly as in the case above.

Then we continue the procedure with the next particle from the process $\xi^\prime_t$ until all the particles will be taken into consideration. Since only a finite number of particles may belong to the $L$-neighborhood of a given particle the procedure is well defined.

During the time when two particles are paired all choices of their velocities in the coupled process are assumed to be identical. Therefore the particles from the same pair move synchronously until either the admissibility condition breaks down for only one of the particles (which basically means that its movement is blocked by another particle) or an unpaired particle comes close enough to one of the members of the pair (see Fig. 1).

This construction clearly defines a Markovian coupling between two copies of the Markov chain $\xi_t$. Of course, without details of the procedures defining velocities and their admissibility one cannot check the validity of the conditions of Theorems 5.1 and 5.2. Sufficient conditions for this will be a subject of a forthcoming paper.

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\textsuperscript{5}Observe that to make our coupling between two copies of the process with certain initial distributions we need those distributions to be spatially ergodic and to apply any kind of a quasi successful property they need to be of the same particle density.
Figure 1: Pairing of particles. Black circles corresponds to the particles from the first component of the process and open circles with primed numbers to the second component. The paired particles are connected by straight lines. At time $t$ the particles 1 and $1'$ are paired, while at time $t + 1$ the particle 1 becomes unpaired while the particle $1'$ becomes paired with the particle 2. The unpaired initially particles 3 and $2'$ become paired at time $t + 1$. The velocities at time $t$ are shown by vectors.

References

[1] Belitsky V., Ferrari P.A. Invariant Measures and Convergence for Cellular Automaton 184 and Related Processes. J. Stat Phys 118:3-4(2005), 589-623.

[2] Blank M. Ergodic properties of a simple deterministic traffic flow model. J. Stat. Phys., 111:3-4(2003), 903-930.

[3] Blank M. Hysteresis phenomenon in deterministic traffic flows. J. Stat. Phys. 120: 3-4(2005), 627-658.

[4] Ekhaus M., Gray L. Convergence to equilibrium and a strong law for the motion of restricted interfaces. Unpublished manuscript.

[5] Evans M. R., Rajewsky N., Speer E. R. Exact solution of a cellular automaton for traffic. J. Stat. Phys. 95(1999), 45-98.

[6] Gray L. (private communication).

[7] Gray L., Griffeath D. The ergodic theory of traffic jams. J. Stat. Phys., 105:3/4 (2001), 413-452.

[8] Griffeath D. An ergodic theorem for a class of spin systems. Annales de l’institut Henri Poincar (B) Probabilits et Statistiques, 13:2 (1977), 141-157.

[9] Harris T.E. Contact interactions on a lattice. Ann. Probab., 2 (1974), 969-988.

[10] Kornfeld E.P., Sinai Ya.G., Fomin S.V. Ergodic theory. M.: Nauka, 1980. (Springer Verlag, 1982)

[11] Liggett T.M. Interacting particle systems. Springer-Verlag, NY, 1985.

[12] Nummelin E. General irreducible Markov chains and non-negative operators. Cambridge Univ. Press, 1984.
[13] Rosenthal J.S. Faithful couplings of Markov chains: now equals forever. Adv. Appl. Math. 18(1997), 372-381.

[14] Thorisson H. Coupling, Stationarity, and Regeneration. Springer-Verlag, NY, 2000.

[15] Stochastic Cellular Systems: ergodicity, memory, morphogenesis. Ed. by R. Dobrushin, V, Kryukov and A. Toom. Nonlinear Science: theory and applications. Manchester University Press, 1990.