The Representations of the Automorphism Groups and the Frobenius Invariants of K3 Surfaces

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Abstract. For a complex algebraic K3 surface, it is known that the representations of the automorphism group on the transcendental cycles is finite and is isomorphic to the representation on the two-forms. In this paper, we prove similar results for a K3 surface defined over a field of odd characteristic. Also, we prove that the height and the Artin invariant of a K3 surface equipped with a nonsymplectic automorphism of some high order are determined by a congruence class of the base characteristic.

1. Introduction

When \( X \) is an algebraic complex K3 surface, the second integral singular cohomology \( H^2(X, \mathbb{Z}) \) is a free Abelian group of rank 22 equipped with a lattice structure isomorphic to \( U^3 \oplus E_8^2 \). Here \( U \) is the hyperbolic plane, and \( E_8 \) is the unique unimodular, even, and negative definite lattice of rank 8. The cycle map gives a primitive embedding of the Neron–Severi group of \( X \) into the second cohomology \( \text{NS}(X) \hookrightarrow H^2(X, \mathbb{Z}) \). The rank of \( \text{NS}(X) \) is called the Picard number of \( X \) and is denoted by \( \rho(X) \). The orthogonal complement of this embedding is called the transcendental lattice of \( X \) and is denoted by \( T(X) \). The rank of the transcendental lattice is \( 22 - \rho(X) \). Cohomology \( H^2(X, \mathbb{Z}) \) is an overlattice of \( \text{NS}(X) \oplus T(X) \), and

\[
|H^2(X, \mathbb{Z})/(\text{NS}(X) \oplus T(X))| = |d(\text{NS}(X))|.
\]

The one-dimensional complex space of global holomorphic two-forms of \( X \), \( H^0(X, \Omega^2_{X/\mathbb{C}}) \) is a direct factor of \( H^2(X, \mathbb{Z}) \otimes \mathbb{C} = H^2(X, \mathbb{C}) \), and by the Lefschetz (1, 1) theorem,

\[
\text{NS}(X) = H^0(X, \Omega^2_{X/\mathbb{C}})^\perp \cap H^2(X, \mathbb{Z})
\]

in \( H^2(X, \mathbb{C}) \). In particular, \( H^0(X, \Omega^2_{X/\mathbb{C}}) \) is a direct factor of \( T(X) \otimes \mathbb{C} \). The automorphism group of \( X \), \( \text{Aut}(X) \), has natural actions on \( T(X) \) and on \( H^0(X, \Omega^2_{X/\mathbb{C}}) \). Let us denote the actions of \( \text{Aut}(X) \) on the transcendental lattice and the two-forms by

\[
\chi_X : \text{Aut}(X) \to O(T(X)) \quad \text{and} \quad \rho_X : \text{Aut}(X) \to \text{Gl}(H^0(X, \Omega^2_{X/\mathbb{C}})).
\]
We say that an automorphism of \( X, \alpha : X \to X \) is symplectic if \( \rho_X(\alpha) = 1 \). If \( \alpha \) is of finite order greater than 1 and the order of \( \alpha \) is equal to the order of \( \rho_X(\alpha) \), we say that \( \alpha \) is purely nonsymplectic. Since \( H^0(X, \Omega^2_{X/\mathbb{C}}) \) is a direct factor of \( T(X) \otimes \mathbb{C} \), there is a canonical surjection \( p_X : \text{Im} \chi_X \to \text{Im} \rho_X \). It is known that \( p_X \) is an isomorphism and \( \text{Im} \chi_X \) and \( \text{Im} \rho_X \) are finite cyclic groups [21]. The proof of this result is based on the Lefschetz \((1, 1)\) theorem and the Torelli theorem for K3 surfaces. If the order of \( \text{Im} \rho_X \) is \( N \), then there is an automorphism \( \alpha \in \text{Aut} X \) such that \( \xi_N = \rho_X(\alpha) \) is a primitive \( N \)th root of unity. Then \( T(X) \) has a free \( \mathbb{Z}[\xi_N] \)-module structure in a natural way [18], and \( 22 - \rho(X) \) is a multiple of \( \phi(N) \). Here \( \phi \) is the Euler \( \phi \)-function.

Assume that \( k \) is an algebraically closed field of odd characteristic \( p \) and \( W \) is the ring of Witt vectors of \( k \). Let \( X \) be a K3 surface defined over \( k \). The second crystalline cohomology \( H^2_{\text{cris}}(X/W) \) and the second étale cohomology \( H^2_{\text{ét}}(X, \mathbb{Z}_l) \) are unimodular lattices of rank 22 over \( W \) and \( \mathbb{Z}_l \), respectively. Here \( l \) is a prime number different from \( p \). The cycle maps to the crystalline cohomology and étale cohomology give an embedding of \( W \)-modules

\[
\text{NS}(X) \otimes W \hookrightarrow H^2_{\text{cris}}(X/W)
\]

and an embedding of \( \mathbb{Z}_l \)-modules

\[
\text{NS}(X) \otimes \mathbb{Z}_l \hookrightarrow H^2_{\text{ét}}(X, \mathbb{Z}_l).
\]

The Newton polygon of \( H^2_{\text{cris}}(X/W) \) is determined by the height of the formal Brauer group of \( X \) (see Section 2). This height is a positive integer between 1 and 10 or \( \infty \). If the height of \( X \) is \( \infty \), then we say that \( X \) is supersingular. We again denote the representation of the automorphism group of \( X \) on the global two-forms by

\[
\rho_X : \text{Aut}(X) \to \text{Gl}(H^0(X, \Omega^2_{X/k})).
\]

When \( X \) is a supersingular K3 surface over \( k \), \( \rho(X) \) is 22 [20; 5; 19], and the discriminant group of the Neron–Severi group is \( (\mathbb{Z}/p)^{2\sigma} \) for a positive integer \( \sigma \) between 1 and 10. We say that \( \sigma \) is the Artin-invariant of \( X \). By the Frobenius invariant of a K3 surface in positive characteristic we mean the height and the Artin-invariant. If the Artin-invariant of \( X \) is \( \sigma \), then \( H^2_{\text{cris}}(X/W)/(\text{NS}(X) \otimes W) \) is a \( \sigma \)-dimensional \( k \)-space, and there is a canonical projection

\[
H^2_{\text{cris}}(X/W)/(\text{NS}(X) \otimes W) \to H^2(X, \mathcal{O}_X).
\]

Moreover, \( H^2_{\text{cris}}(X/W)/(\text{NS}(X) \otimes W) \) is an invariant isotropic subspace of the discriminant group \( (\text{NS}(X)^*/\text{NS}(X)) \otimes k \). Let

\[
v_X : \text{Aut}(X) \to O(\text{NS}(X)^*/\text{NS}(X))
\]

be the representations on the discriminant group of the Neron–Severi group. We prove that there is a canonical isomorphism \( \text{Im} v_X \to \text{Im} \rho_X \) and \( \text{Im} v_X \simeq \text{Im} \rho_X \) is a finite cyclic group (Prop. 3.1).

When \( X \) is a K3 surface of finite height \( h \) over \( k \), \( \rho(X) \) is at most \( 22 - 2h \) [3]. For a K3 surface of finite height \( X \), we call the orthogonal complements of the
cycle maps
\[ NS(X) \otimes \mathbb{Z}_l \hookrightarrow H^2_{\text{et}}(X, \mathbb{Z}_l) \quad \text{and} \quad NS(X) \otimes W \hookrightarrow H^2_{\text{cris}}(X/W) \]
the \( l \)-adic transcendental lattice of \( X \) and the crystalline transcendental lattice of \( X \), respectively. We denote those lattices by \( T_l(X) \) and \( T_{\text{cris}}(X) \). The representation of \( \text{Aut}(X) \) on \( T_l(X) \) and \( T_{\text{cris}}(X) \) are denoted by \( \chi_{l,X} : \text{Aut}(X) \to O(T_l(X)) \) and \( \chi_{\text{cris},X} : \text{Aut}(X) \to O(T_{\text{cris}}(X)) \).

We will see that \( \ker \chi_{l,X} \) is equal to \( \ker \chi_{X,\text{cris}} \) and for any automorphism \( \alpha \in \text{Aut}(X) \), the characteristic polynomial of \( \chi_{l,X}(\alpha) \) is equal to the characteristic polynomial of \( \chi_{\text{cris},X}(\alpha) \) (Prop. 3.6). We will also construct canonical projections \( p_{\text{cris},X} : \text{Im} \chi_{\text{cris},X} \to \text{Im} \rho_X \) and \( p_{l,X} : \text{Im} \chi_{l,X} \to \text{Im} \rho_X \) satisfying \( \rho_X = p_{\text{cris},X} \circ \chi_{\text{cris},X} = p_{l,X} \circ \chi_{l,X} \). Also, using a Neron–Severi group preserving lifting of \( X \), we prove that \( \text{Im} \chi_{\text{cris},X}, \text{Im} \chi_{l,X}, \) and \( \text{Im} \rho_X \) are finite (Props. 3.5 and 3.6). It follows that, for any \( \alpha \in \text{Aut}(X) \), all the eigenvalues of \( \chi_{l,X}(\alpha) \) are roots of unity. In addition to that, if the order of \( \text{Im} \chi_{l,X}(\alpha) \) is not divisible by \( p \), the order of \( \rho_X(\alpha) \) is \( n \), every primitive \( n \)-th root of unity occurs as an eigenvalue of \( \chi_{l,X}(\alpha) \) (Prop. 3.7). This generalizes Proposition 2.1 in [13].

When \( \alpha \) is an automorphism of a K3 surface \( X \) over \( k \), under certain conditions, some parts of eigenvalues of \( \alpha^*|H^2_{\text{et}}(X, \mathbb{Q}_l) \) are decided by the Frobenius invariant of \( X \) and \( \rho_X(\alpha) \). More precisely, we have the following result.

**Theorem 3.9.** Let \( k \) be an algebraically closed field of odd characteristic \( p \). Assume that \( X \) is a K3 surface over \( k \) and \( \alpha \) is an automorphism of \( X \). We assume either of the following:

1. \( X \) is of finite height \( h \), and the order of \( \chi_{l,X}(\alpha) \) is prime to \( p \)
2. \( X \) is supersingular of Artin-invariant \( \sigma \), and the order of \( \alpha \) is finite and prime to \( p \).

Suppose that \( \rho_X(\alpha)(u) = \zeta \cdot u \) for a generator \( u \) of \( H^0(X, \Omega^2_X/k) \) and that \( \xi \) is the Teichmüller lift of \( \zeta \) in \( W \).

Then in case (1), \( \xi \pm p^0, \xi \pm p^{-1}, \ldots, \xi \pm p^{1-h} \) appear as eigenvalues of \( \chi_{l,X}(\alpha) \) and in case (2), \( \xi \pm p^0, \xi \pm p^{-1}, \ldots, \xi \pm p^{1-\sigma} \) appear as eigenvalues of \( \alpha^*|H^2_{\text{et}}(X, \mathbb{Q}_l) \).

Based on Theorem 3.9 and the Tate conjecture for K3 surfaces of finite height [23; 19], we can prove the followings.

**Theorem 3.10.** Let \( k \) be an algebraic closure of a finite field of odd characteristic \( p \), and \( X \) be a K3 surface of finite height \( h \) over \( k \). If the order of \( \text{Im} \chi_{l,X} \) is not divisible by \( p \), the projection \( p_{l,X} : \text{Im} \chi_{l,X} \to \text{Im} \rho_X \) is bijective.

**Corollary 3.11.** Let \( k \) be an algebraic closure of a finite field of odd characteristic \( p \), and \( X \) be a K3 surface of finite height over \( k \). If \( N \) is the order of \( \text{Im} \rho_X \), then the rank of \( T_l(X) = 22 - \rho(X) \) is divisible by \( \phi(N) \).
We can apply these results to study the relation of Frobenius invariant and non-symplectic automorphisms for K3 surfaces. We prove the followings.

**Corollary 4.3.** Let \( k \) be an algebraically closed field of odd characteristic \( p \), and \( X \) be a K3 surface over \( k \). Let \( \alpha \) be an automorphism of \( X \). We assume that the order of \( \rho_X(\alpha) \) is \( N (> 2) \) and that the rank of the Neron–Severi group of \( X \) is at least \( 22 - \phi(N) \). If \( p^m \equiv -1 \text{ modulo } N \) for some \( m \), then \( X \) is supersingular. If \( p^m \not\equiv -1 \text{ modulo } N \) for any \( m \) and the order of \( p \) in \( (\mathbb{Z}/N\mathbb{Z})^* \) is \( n \), then the height of \( X \) is \( n \).

**Corollary 4.4.** Assume that \( X \) is a K3 surface over \( k \) and \( \alpha \) is an automorphism of \( X \) such that the order of \( \rho_X(\alpha) \) is \( N (> 2) \). We assume that \( \alpha \) is of finite order prime to \( p \) and that a primitive \( N \)th root of unity appears only one time in the eigenvalues of \( \alpha^*|H^2_{\text{et}}(X, \mathbb{Q}_l) \). If the order of \( p \) in \( (\mathbb{Z}/N\mathbb{Z})^* \) is \( 2n \) and \( p^n \equiv -1 \text{ modulo } N \), then \( X \) is supersingular of Artin-invariant \( n \).

If \( X \) is a complex algebraic K3 surface with \( N = |\text{Im} \rho_X| \) and the rank of \( T(X) \) is equal to \( \phi(N) \), then \( X \) has a model defined over a number field [26]. By the last two corollaries we deduce that for almost all places, the Frobenius invariant of the reduction of the model of \( X \) over the number field is determined by the congruence class of the residue characteristic modulo \( N \) (Theorem 4.7). This generalizes the results on the Delsarte K3 surfaces in [28; 30; 7].

### 2. Crystalline Cohomology of K3 Surfaces

In this section, we review some facts on the Neron–Severi group and the crystalline cohomology of K3 surfaces over a field of odd characteristic. Assume that \( k \) is an algebraically closed field of characteristic \( p > 2 \). Let \( W \) be the ring of Witt vectors of \( k \), and \( K \) be the fraction field of \( W \). Assume that \( X \) is a K3 surface over \( k \). Let \( \hat{Br}_X \) be the formal Brauer group of \( X \). \( \hat{Br}_X \) is a smooth formal group of dimension 1 over \( k \) [2]. A smooth formal group of dimension 1 over an algebraically closed field of positive characteristic is classified by its height. The height \( h \) of \( \hat{Br}_X \) is a positive integer \((1 \leq h \leq 10) \) or \( \infty \). When \( h = \infty \), we say \( X \) is supersingular. The Dieudonné module of \( \hat{Br}_X \) is

\[
\mathbb{D}(\hat{Br}_X) = W[F, V]/(FV = p, F = V^{h-1})
\]

if \( h \) is finite or

\[
\mathbb{D}(\hat{Br}_X) = k[[V]]
\]

if \( h = \infty \). Here \( F \) is a Frobenius linear operator, and \( V \) is a Frobenius inverse linear operator.

The crystalline cohomologies \( H^i_{\text{cris}}(X/W) \) are finite free \( W \)-modules of ranks 1, 0, 22, 0, 1 for \( i = 0, 1, 2, 3, 4 \), respectively, equipped with Frobenius linear operators

\[
F : H^i_{\text{cris}}(X/W) \to H^i_{\text{cris}}(X/W).
\]
If the height \( h \) is finite, then the Frobenius slopes of \( H^2_{\text{cris}}(X/W) \) are \( 1 - 1/h, 1, 1 + 1/h \) of lengths \( h, 22 - 2h, h \), respectively. If \( X \) is supersingular, then the only Frobenius slope of \( H^2_{\text{cris}}(X/W) \) is \( 1 \) of length 22.

The crystalline cohomology \( H^i_{\text{cris}}(X/W) \) can be realized as the hypercohomology of the DeRham–Witt complex [9],

\[
0 \to W\mathcal{O}_X \to W\Omega^1_{X/k} \to W\Omega^2_{X/k} \to 0.
\]

The naive filtration of the DeRham–Witt complex gives the slope spectral sequence

\[
H^i(X, W\Omega^j_{X/k}) \Rightarrow H^{i+j}_{\text{cris}}(X/W).
\]

The \( E_1 \)-level page of the slope spectral sequence of \( X \) is

\[
\begin{align*}
H^2(X, W\mathcal{O}_X) &\to H^2(X, W\Omega^1_X) \\
W &\to H^1(X, W\Omega^1_X) \\
W &\to 0 \\
0 &\to H^0(X, W\Omega^2_X).
\end{align*}
\]

Here \( H^2(X, W\mathcal{O}_X) \) is isomorphic to \( \mathbb{D}(\widehat{Br}_X) \) [3]. By an exact sequence of sheaves on \( X \),

\[
0 \to W\mathcal{O}_X \to W\mathcal{O}_X \to \mathcal{O}_X \to 0,
\]

we have an isomorphism \( H^2(W, W\mathcal{O}_X)/V \cong H^2(X, \mathcal{O}_X) \).

If \( X \) is of finite height \( h \), then \( H^2(X, W\mathcal{O}_X)/V = 0 \), and the slope spectral sequence degenerates at \( E_1 \)-level. Moreover, \( H^2_{\text{cris}}(X/W) \) has an \( F \)-crystal decomposition [9], II.7.2, [12], Theorem 1.6.1,

\[
H^2(X/W) = H^2_{\text{cris}}(X/W)_{[1-1/h]} \oplus H^2_{\text{cris}}(X/W)_{[1]} \oplus H^2_{\text{cris}}(X/W)_{[1+1/h]}.
\]

Here

\[
H^2_{\text{cris}}(X/W)_{[1-1/h]} = H^2(X, W\mathcal{O}_X) = \mathbb{D}(\widehat{Br}_X)
\]

and

\[
H^2_{\text{cris}}(X/W)_{[1+1/h]} = \text{Hom}(H^2(X, W\mathcal{O}_X), H^4(X/W)).
\]

Note that \( H^4(X/W) \) is a free \( W \)-module of rank 1 equipped with a Frobenius linear operator of slope 2. For the cup product pairing on \( H^2_{\text{cris}}(X/W) \), \( H^2_{\text{cris}}(X/W)_{[1-1/h]} \) and \( H^2_{\text{cris}}(X/W)_{[1+1/h]} \) are isotropic and dual to each other. On the other hand, \( H^2_{\text{cris}}(X/W)_{[1]} \) is unimodular. The discriminant of the \( \mathbb{Z}_p \)-lattice \( H^2_{\text{cris}}(X/W)_{[1]}^{F=p} \) is \( (-1)^{h+1} \). When \( X \) is of finite height \( h \), the Frobenius morphism and the lattice structure of \( H^2_{\text{cris}}(X/W) \) are completely determined by \( h \) ([25], p. 363). Because there exists a canonical embedding ([9], Proposition II.5.12),

\[
\text{NS}(X) \otimes \mathbb{Z}_p \hookrightarrow H^1(X, W\Omega^1_X)^{F=p},
\]

the Picard number \( \rho(X) \) of \( X \) is not greater than the length of slope 1 part of \( H^2_{\text{cris}}(X/W) \). It follows that \( \rho(X) \leq 22 - 2h \) if \( h \) is finite.

In odd characteristic, it is known that \( X \) is supersingular if and only if the Picard number of \( X \) is 22 [20; 5; 19]. Assume that \( X \) is a supersingular K3 surface. The discriminant of \( \text{NS}(X) \) is \(-p^{2\sigma}\) for an integer \( \sigma \) between 1 and 10, and \( \sigma \) is called the Artin-invariant of \( X \). The discriminant group of \( \text{NS}(X) \) is isomorphic
to \((\mathbb{Z}/p)^{2\sigma}\). Moreover, \(\text{NS}(X)\) is determined by the base characteristic \(p\) and \(\sigma\) [27].

For a supersingular K3 surface \(X\), \(H^0(X, \omega_X^2) = 0\), and the slope spectral sequence degenerates at \(E_2\)-level ([9], Corollaire II.3.13). The only nontrivial map in the \(E_1\)-page of the slope spectral sequence is
\[
d : H^2(X, \omega_X) \to H^2(X, \omega_X^1).
\]
Here \(d\) is surjective, and
\[
\ker d = H^2_{\text{cris}}(X/W)/F^1 H^2_{\text{cris}}(X/W),
\]
where \(F^1 H^2_{\text{cris}}(X/W)\) is the filtration given by the slope spectral sequence. We can identify \(F^1 H^2_{\text{cris}}(X/W)\) with the image of the cycle map ([9], II.7.2)
\[
\text{NS}(X) \otimes W \hookrightarrow H^2_{\text{cris}}(X/W).
\]
Since \(H^2_{\text{cris}}(X/W)\) is a unimodular \(W\)-lattice and the cycle map preserves the paring, we have a chain
\[
\text{NS}(X) \otimes W \subset H^2_{\text{cris}}(X/W) \subset (\text{NS}(X)^*) \otimes W.
\]
Moreover, \(\ker d = H^2_{\text{cris}}(X/W)/\text{NS}(X) \otimes W\) is a \(\sigma\)-dimensional isotropic \(k\)-subspace of the discriminant group \((\text{NS}(X)^* \otimes W)/(\text{NS}(X) \otimes W) = (\text{NS}(X)^*/\text{NS}(X)) \otimes k\). It is also known that
\[
\ker dV^i : H^2(X, \omega_X) \to H^2(X, \omega_X^1)
\]
is a \((\sigma - i)\)-dimensional \(k\)-space for \(i \leq \sigma\) and \(\ker dV^{i+1} \subseteq \ker dV^i\) ([22], Thm. 0.1). When \(x\) is a nonzero element of \(\ker dV^{\sigma-1}\),
\[
x, Vx, \ldots, V^{\sigma-1}x
\]
generate \(\ker V^i d\) over \(k\), and \(x\) is a \(V\)-adic topological generator of \(H^2(X, \omega_X)\). The composition
\[
\ker dV^{\sigma-1} \hookrightarrow H^2(X, \omega_X) \to H^2(X, \omega_X)
\]
is an isomorphism.

3. Representations of the Automorphism Groups on the Two-Forms and Transcendental Cycles

Let \(k\) be an algebraically closed field of characteristic \(p > 2\). Let \(W\) be the ring of Witt vectors of \(k\), and \(K\) be the fraction field of \(W\). Let \(X\) be a K3 surface over \(k\). Let
\[
\rho_X : \text{Aut}(X) \to GL(H^0(X, \omega_X^2_{X/k})) \quad \text{and} \quad \lambda_X : \text{Aut}(X) \to GL(H^2(X, \omega_X))
\]
be the representation of \(\text{Aut}(X)\) on \(H^0(X, \omega_X^2_{X/k})\) and \(H^2(X, \omega_X)\). By the Serre duality, for any \(\alpha \in \text{Aut} X\), \(\rho_X(\alpha)^{-1} = \lambda_X(\alpha)\) and \(\ker \rho_X = \ker \lambda_X\).

Assume that \(X\) is supersingular. Let
\[
\nu_X : \text{Aut}(X) \to O((\text{NS}(X)^*/\text{NS}(X)) \otimes k)
\]
be the representation of Aut(X) on the discriminant group \((NS(X)^*/NS(X)) \otimes k\). Because \(\nu_X\) factors through the action of Aut(X) on \((NS(X)^*/NS(X))\), a finite-dimensional space over a finite field \(\mathbb{Z}/p\), \(\text{Im} \nu_X\) is finite. Since \(\ker d : H^2(X, W\mathcal{O}_X) \rightarrow H^2(X, W\Omega_X^1)\) is an invariant subspace of \((NS(X)^*/NS(X)) \otimes k\) for the action of Aut X and there is a projection \(\ker d \rightarrow H^2(X, \mathcal{O}_X)\), we have a surjective map \(q_X : \text{Im} \nu_X \twoheadrightarrow \text{Im} \lambda\) such that \(q_X \circ \nu_X = \lambda\). Then a surjective map \(p_X = q_X^{-1} : \text{Im} \nu_X \twoheadrightarrow \text{Im} \rho\) satisfies \(p_X \circ \nu_X = \rho\).

**Proposition 3.1.** Let \(X\) be a supersingular K3 surface in odd characteristic. Then \(p_X : \text{Im} \nu_X \to \text{Im} \rho\) is an isomorphism.

**Proof.** Let \(\sigma\) be the Artin-invariant of \(X\), and \(x\) be a nonzero element of \(\ker d V^{\sigma-1}\). Then, \(x_i = V^i x, i = 1, \ldots, \sigma - 1\), is a basis of \(\ker d\). Let \(y_i\) be the dual basis for \(x_i\) of the dual isotropic subspace of \(\ker d\) in \((NS(X)^*/NS(X)) \otimes k\). Then any automorphism \(\alpha \in \text{Aut}(X)\) preserves all the lines \(k \cdot x_i\) and \(k \cdot y_i\). In other words, all \(x_i\) and \(y_i\) are eigenvectors of \(\nu_X(\alpha)\). Since \(\alpha^*(V^i x) = V^i \alpha^*(x)\) and \(y_i\) is dual to \(x_i\), \(\nu_X(\alpha)\) is decided by the eigenvalue at \(x = x_0\). But, for any \(\alpha \in \text{Aut}(X)\), \(\rho(\alpha)\) is the inverse of the eigenvalue of \(\nu_X(\alpha)\) for an eigenvector \(x_0\), so \(p_X\) is injective. \(\square\)

**Remark 3.2.** In [16], a supersingular K3 surface is defined to be generic if the order of \(\text{Im} \nu_X\) is 1 or 2. Also, it is proved that there exists a generic supersingular K3 surface of Artin-invariant \(\sigma \geq 2\) in odd characteristic ([16], Thm. 1.7). By the last proposition, a supersingular K3 surface in odd characteristic is generic if and only if the order of \(\text{Im} \rho\) is 1 or 2.

For the order of the \(\text{Im} \rho\), the following is known.

**Proposition 3.3** ([22], Thm. 2.1). The cardinality of \(\text{Im} \rho\) divides \(p^\sigma + 1\).

**Remark 3.4.** If \(X\) is a supersingular K3 surface of Artin-invariant 1, then we have shown, using the crystalline Torelli theorem, that \(\text{Im} \rho \simeq \text{Im} \nu\) is a cyclic group of rank \(p + 1\) ([11], Thm. 3.3). By this result, if \(\phi(p + 1) > 20\), then \(X\) has an automorphism that cannot be lifted to characteristic 0.

Now we assume that \(X\) is a K3 surface of finite height over \(k\). There is a smooth lifting of \(X\) over \(W, \mathcal{X}/W\), with the generic fiber \(\mathcal{X}_k = \mathcal{X} \otimes K\) such that the reduction map

\[
\text{NS}(\mathcal{X}_k) \rightarrow \text{NS}(X)
\]

is an isomorphism [23; 17; 10]. We say that a lifting of \(X\) satisfying this condition is a Neron–Severi group preserving lifting of \(X\). When \(\mathcal{X}\) is a Neron–Severi group preserving lifting and \(\tilde{K}\) is an algebraic closure of \(K\), the specialization map

\[
\text{Aut}(\mathcal{X}_k \otimes \tilde{K}) \rightarrow \text{Aut}(X)
\]

is an injection of finite index ([17], Thm. 6.1).
Proposition 3.5. Let $X$ be a K3 surface of finite height over $k$. Then $\text{Im} \rho_X$ is finite.

Proof. Let $\mathcal{X}/W$ be a Neron–Severi group preserving lifting of $X$, and $\mathcal{X}_K/K$ be the generic fiber of $\mathcal{X}/W$. Since $K$ is of characteristic 0, the image of $\text{Aut}(\mathcal{X}_K) \rightarrow GL(H^0(\mathcal{X}_K, \Omega^2_{\mathcal{X}_K/K}))$ is finite. Therefore, the image of $\text{Aut}(\mathcal{X}_K) \hookrightarrow \text{Aut}(X) \rightarrow GL(H^0(X, \Omega^2_{X/k}))$ is also finite. Since $\text{Aut}(\mathcal{X}_K)$ is of finite index in $\text{Aut}(X)$, the image of $\rho_X$ is finite. □

Let $X$ be a K3 surface of finite height over $k$. Let $T_l(X)$ be the orthogonal complement of the cycle map $\text{NS}(X) \otimes \mathbb{Z}_l \hookrightarrow H^2_{\text{et}}(X, \mathbb{Z}_l)$ for $l \neq p$, and $T_{\text{cris}}(X)$ be the orthogonal complement of the cycle map $\text{NS}(X) \otimes W \hookrightarrow H^2_{\text{cris}}(X/W)$. We say $T_l(X)$ and $T_{\text{cris}}(X)$ are the $l$-adic transcendental lattice of $X$ and the crystalline transcendental lattice of $X$, respectively. When $\rho(X)$ is the Picard number of $X$, the ranks of $T_l(X)$ and $T_{\text{cris}}(X)$ are $22 - \rho(X)$. Note that $H^2(X, W\mathcal{O}_X) \oplus H^2_{\text{cris}}(X/W)\{1+1/h\}$ is a direct factor of $T_{\text{cris}}(X)$ and $H^2(X, W\mathcal{O}_X)/V \simeq H^2(X, \mathcal{O}_X)$. Hence, there is a canonical projection $T_{\text{cris}}(X) \rightarrow H^2(X, \mathcal{O}_X)$. Let

$$\chi_{l, X} : \text{Aut}(X) \rightarrow O(T_l(X)) \quad \text{and} \quad \chi_{\text{cris}, X} : \text{Aut}(X) \rightarrow O(T_{\text{cris}}(X))$$

be the canonical representations.

Proposition 3.6. Let $X$ be a K3 surface of finite height over $k$. The images of $\chi_{l, X}$ and $\chi_{\text{cris}, X}$ are finite, and there is an isomorphism $\psi_l : \text{Im} \chi_{l, X} \rightarrow \text{Im} \chi_{\text{cris}, X}$ such that $\psi_l \circ \chi_{l, X} = \chi_{\text{cris}, X}$.

Proof. Let $\mathcal{X}/W$ be a Neron–Severi group preserving lifting of $X$ with the generic fiber $\mathcal{X}_K = \mathcal{X} \otimes K$. $\text{Aut}(\mathcal{X}_K)$ is a subgroup of $\text{Aut}(X)$ of finite index. If we identify $H^2_{\text{et}}(\mathcal{X}_K, \mathbb{Z}_l)$ with $H^2_{\text{et}}(X, \mathbb{Z}_l)$, $T_l(X)$ is equal to the orthogonal complement of the cycle map $\text{NS}(\mathcal{X}_K) \otimes \mathbb{Z}_l \hookrightarrow H^2_{\text{et}}(\mathcal{X}_K, \mathbb{Z}_l)$. Because $\tilde{K}$ is of characteristic 0, the action of $\text{Aut}(\mathcal{X}_K)$ on $T_l(X)$ has a finite image. Therefore, $\text{Im} \chi_{l, X}$ is finite. In a similar way, there is a canonical isomorphism

$$H^2_{\text{dr}}(\mathcal{X}_K/\tilde{K}) \simeq H^2_{\text{cris}}(X/W) \otimes \tilde{K},$$

which is compatible with the action of $\text{Aut}(\mathcal{X}_K)$ on both sides ([4, Cor. 2.5]). Also, this isomorphism is compatible with two cycle maps (loc. cit., Cor. 3.7),

$$\text{NS}(\mathcal{X}_K) \rightarrow H^2_{\text{dr}}(\mathcal{X}_K/\tilde{K}) \quad \text{and} \quad \text{NS}(X) \rightarrow H^2_{\text{cris}}(X/W) \otimes \tilde{K}.$$
It follows that the action of $\text{Aut}(X^\text{cris}_K)$ on $T^\text{cris}(X)$ has a finite image, and so does the action of $\text{Aut}(X)$ on $T^\text{cris}(X)$. When $\alpha$ is an automorphism of $X$, the characteristic polynomials of $\alpha^*|H^2_{\text{cris}}(X/K)$ and $\alpha^*|H^2_{\text{et}}(X, \mathbb{Q}_l)$ are equal to each other and have integer coefficients ([8], 3.7.3). Note that the characteristic polynomial of $\alpha^*|H^2_{\text{et}}(X, \mathbb{Q}_l)$ is the product of the characteristic polynomial of $\alpha^*|\text{NS}(X)$ and the characteristic polynomial of $\chi_{l,X}(\alpha)$. Also, the characteristic polynomial of $\alpha^*|H^2_{\text{cris}}(X/K)$ is the product of the characteristic polynomial of $\alpha^*|\text{NS}(X)$ and the characteristic polynomial of $\chi_{\text{cris},X}(\alpha)$. Because $\text{NS}(X)$ is an integral lattice, the characteristic polynomial of $\alpha^*|\text{NS}(X)$ is also integral, and the characteristic polynomials of $\chi_{l,X}(\alpha)$ and $\chi_{\text{cris},X}(\alpha)$ are equal to each other and integral. Since $\chi_{l,X}(\alpha)$ and $\chi_{\text{cris},X}(\alpha)$ are of finite orders, they are semisimple, and all their eigenvalues are roots of unity. It follows that $\chi_{l,X}(\alpha) = \text{id}$ if and only if $\chi_{\text{cris},X}(\alpha) = \text{id}$. Therefore, $\ker \chi_{l,X} = \ker \chi_{\text{cris},X}$, and there exists a compatible isomorphism $\psi_l : \text{Im} \chi_{l,X} \to \text{Im} \chi_{\text{cris},X}$. □

Using the projection $T^\text{cris}(X) \to H^2(X, \mathcal{O}_X)$ and the Serre duality, we have a canonical projection $p_{\text{cris},X} : \text{Im} \chi_{\text{cris},X} \to \text{Im} \rho_X$ such that $p_{\text{cris},X} \circ \chi_{\text{cris},X} = \rho_X$. Composing with $\psi_l$, we have a canonical projection $p_{l,X} = p_{\text{cris},X} \circ \psi_l : \text{Im} \chi_{l,X} \to \text{Im} \rho_X$.

**Proposition 3.7.** Let $X$ be a $K$3 surface of finite height, and $\alpha$ be an automorphism of $X$. If the order of $\chi_{l,X}(\alpha)$ is prime to $p$ and the order of $\rho_X(\alpha)$ is $n$, then all the primitive $n$th roots of unity appear as eigenvalues of $\chi_{\text{cris},X}(\alpha)$.

**Proof.** Let $\zeta = \rho_X(\alpha) \in k^*$, and let $\xi$ be the Teichmüller lifting of $\zeta$ in $W$. Since there is a projection $T^\text{cris}(X)/p \to H^2(X, \mathcal{O}_X)$, $\xi^{-1}$ is an eigenvalue of $\alpha^*|(T^\text{cris}(X)/p)$, and $\xi^{-1}$ is an eigenvalue of $\chi_{\text{cris},X}(\alpha)$. Because the characteristic polynomial of $\chi_{\text{cris},X}(\alpha)$ is integral and $\xi^{-1}$ is a primitive $n$th root of unity, the $n$th cyclotomic polynomial divides the characteristic polynomial of $\chi_{\text{cris},X}(\alpha)$. Therefore, every primitive $n$th root of unity is an eigenvalue of $\chi_{\text{cris},X}(\alpha)$. □

**Remark 3.8.** Because the rank of the transcendental lattice is not greater than 21 and the degree of the $n$th cyclotomic polynomial is $\phi(n)$, if an $n$th root of unity appears as an eigenvalue of $\chi_{\text{cris},X}(\alpha)$, then $\phi(n) \leq 21$. Here $\phi$ is the Euler $\phi$-function. In particular, if $p \geq 23$, then a $p$th root of unity can not appear as an eigenvalue of $\chi_{\text{cris},X}(\alpha)$.

**Theorem 3.9.** Let $k$ be an algebraically closed field of odd characteristic $p$. Let $X$ be a $K$3 surface over $k$, and $\alpha$ be an automorphism of $X$. We assume either of the following:

1. $X$ is of finite height $h$, and the order of $\chi_{l,X}(\alpha)$ is prime to $p$.

2. $X$ is supersingular of Artin-invariant $\sigma$, and the order of $\alpha$ is finite and prime to $p$. 
Suppose \( \zeta = \rho_X(\alpha) \) and \( \xi \) is the Teichmüller lift of \( \zeta \) in \( W \).

Then in case (1), \( \xi^{\pm p^0}, \xi^{\pm p^{-1}}, \ldots, \xi^{\pm p^{1-h}} \) appear as eigenvalues of \( \chi_{\text{cris}, X}(\alpha) \), and in case (2), \( \xi^{\pm p^0}, \xi^{\pm p^{-1}}, \ldots, \xi^{\pm p^{1-\sigma}} \) appear as eigenvalues of \( \alpha^*|H^2_{\text{cris}}(X/W) \).

**Proof.** First case: \( X \) is of finite height. Assume that \( X \) is of finite height \( h \) and the order of \( \chi_l(\alpha) \) is prime to \( p \). Let us identify \( H^2(X, W\mathcal{O}_X) \) with \( W[[F, V]/(FV - p, F - V^{h-1})] \). Let \( f: W \to W \) be the Frobenius morphism. We assume that

\[
\alpha^*(1) = a_01 + a_1V + \cdots + a_{h-1}V^{h-1}.
\]

Here \( 1 \in W[F, V]/(FV - p, F - V^{h-1}) \) is a \( V \)-adic topological generator, and \( a_i \in W \). Note that \( a_0 \) is a unit of \( W \). Then

\[
\alpha^*(V^i) = V^i\alpha^*(1) = f^{-i}(a_0)V^i + f^{-i}(a_1)V^{i+1} + \cdots + f^{-i}(a_{h-1})V^{h+i-1}
\]

for \( i \leq h - 1 \). But \( H^2(X, W\mathcal{O}_X)/V = H^2(X, \mathcal{O}_X) \), so \( a_0 \equiv \zeta^{-1} \) modulo \( p \). The matrix of \( \alpha^*|H^2(X, W\mathcal{O}_X)/p) \) with respect to a basis \( 1 + (p), V + (p), \ldots, V^{h-1} + (p) \) is

\[
\begin{pmatrix}
\zeta^{-1} & \cdots & \cdots & \cdots \\
0 & \zeta^{-p-1} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \zeta^{-p^{1-h}}
\end{pmatrix},
\]

and the characteristic polynomial of \( \alpha^*|H^2(X, W\mathcal{O}_X)/p) \) is

\[
\prod_{i=0}^{h-1}(T - \zeta^{-p^{-i}}).
\]

Since \( \alpha^*|H^2(X, W\mathcal{O}_X) \) is of finite order prime to \( p \), the characteristic polynomial of \( \alpha^*|H^2(X, W\mathcal{O}_X) \) is

\[
\prod_{i=0}^{h-1}(T - \zeta^{-p^{-i}}).
\]

Because \( H^2_{\text{cris}}(X/W)[1+1/h] \) is dual to \( H^2(X, W\mathcal{O}_X) \), the characteristic polynomial of \( \alpha^*|H^2_{\text{cris}}(X/W)[1+1/h] \) is

\[
\prod_{i=0}^{h-1}(T - \zeta^{p^{-i}}).
\]

Hence, the claim follows.

Second case: \( X \) is supersingular. Assume that \( X \) is supersingular of Artin-invariant \( \sigma \) and \( \alpha \) is of finite order prime to \( p \). Fix \( x_0 \), a nonzero element of

\[
\ker dV^{\sigma-1}: H^2(X, W\mathcal{O}_X) \to H^2(X, W\Omega_X^1).
\]
Let $x_i = V^i X$ for $i = 0, 1, \ldots, \sigma - 1$, and let $y_i$ be the dual basis of $x_i$ in $(\text{NS}(X^*)/\text{NS}(X)) \otimes k$. Then $\alpha^* x_i = \zeta^{-p^{-i}} x_i$ and $\alpha^* y_i = \zeta^{p^{-i}} y_i$. Because there is an embedding

$$(\text{NS}(X^*)/\text{NS}(X)) \otimes k \simeq (p\text{NS}(X^*)/p\text{NS}(X)) \otimes k \subseteq (\text{NS}(X) \otimes W)/p,$$

$\xi \pm p^0, \xi \pm p^{-1}, \ldots, \xi \pm p^{-\sigma}$ occur as eigenvalues of $\alpha^*|\text{NS}(X) \otimes W$ and so as eigenvalues of $\alpha^*|\text{NS}(X)$. Since $H^{2\text{cris}}_\ell(X, W) \otimes K = \text{NS}(X) \otimes K$, the claim follows. □

When $X$ is a complex algebraic K3 surface, the projection $p_X : \text{Im} \chi_X \to \text{Im} \rho_X$ is an isomorphism, and the action of $\text{Aut}(X)$ on the transcendental lattice $T(X)$ is determined by the action on $H^0(X, \Omega^2_{X/C})$, [21]. Moreover, if $N$ is the order of $\text{Im} \rho_X$ and $\xi_N$ is a primitive $N$th root of unity, then by the Lefschetz (1, 1) theorem $T(X)$ is a torsion-free $\mathbb{Z}[\xi_N]$-module. Because $\phi(N) < 22$, $\mathbb{Z}[\xi_N]$ is a P.I.D. [18], so $T(X)$ is a free $\mathbb{Z}[\xi_N]$-module. It follows that the rank of $T(X)$ is a multiple of $\phi(N)$. We can ask if the same result holds for a K3 surface of finite height in odd characteristic.

**Theorem 3.10.** Let $k$ be an algebraic closure of a finite field of odd characteristic $p$, and $X$ be a K3 surface of finite height $h$ over $k$. If the order of $\text{Im} \chi_{l,X}$ is not divisible by $p$, the projection $p_{l,X} : \text{Im} \chi_{l,X} \to \text{Im} \rho_X$ is bijective.

**Proof.** Clearly, $p_{l,X}$ is surjective. Suppose $X$ is defined over $\overline{\mathbb{F}}_q$ for $q = p^m$. The $m$-iterative relative Frobenius morphism of $X/k$ is an endomorphism of $X$ over $k$. We denote this morphism by $F : X \to X$. The induced morphism $F^*|H^2_{\text{cris}}(X, \mathbb{Q}_l)$ is equal to the Galois action of the geometric Frobenius element in $\text{Gal}(k/\overline{\mathbb{F}}_q)$ on $H^2_{\text{cris}}(X, \mathbb{Q}_l)$. Let $V_l(X) = T_l(X) \otimes \mathbb{Q}_l$. Then

$$H^2_{\text{cris}}(X, \mathbb{Q}_l) = V_l(X) \oplus (\text{NS}(X) \otimes \mathbb{Q}_l).$$

Let $\varphi(T)$ be the characteristic polynomial of $F^*|V_l(X)$. It is a polynomial over $\mathbb{Q}$ and equal to the characteristic polynomial of $F^*|T_{\text{cris}}(X)$ ([8], 3.7.3). Let $s_1, s_2, \ldots, s_r$ be the roots of $\varphi(T)$. After replacing $\overline{\mathbb{F}}_q$ by a suitable finite extension, we may assume that if $s_i/s_j$ is a root of unity, then $s_i = s_j$. Let $\alpha$ be an automorphism of $X$. We may assume that $\alpha$ is defined over $\overline{\mathbb{F}}_q$ after replacing the base field $\overline{\mathbb{F}}_q$ by a finite extension. In this case, $F \circ \alpha = \alpha \circ F$. Since $F^*$ and $\alpha^*$ are semisimple on $V_l(X)$ [6], there exists a basis of $V_l(X)$ consisting of common eigenvectors for $F^*$ and $\alpha^*$. We assume that $t_1, \ldots, t_r$ are eigenvalues of $\chi_{l,X}(\alpha)$ and $s_1 t_1, \ldots, s_r t_r$ are eigenvalues of $F^* \circ \alpha^*|V_l(X)$. Let $\psi(T) \in \mathbb{Q}[T]$ be the characteristic polynomial of $F^* \circ \alpha^*|T_l(V)$. Let us fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{K}}$. There is a unique $q$-adic order $\text{ord}_q(\cdot)$ on $\overline{\mathbb{Q}}$ associated to this embedding.

Because the height of $X$ is $h$, exactly $h$ roots of $\varphi(T)$ have order $1 - 1/h$ for the $q$-adic order $\text{ord}_q(\cdot)$. Assume that $\text{ord}_q(s_i) = 1 - 1/h$ for $i = 1, \ldots, h$. Then $s_1, \ldots, s_h$ are roots of characteristic polynomial of $F^*|H^2(X, W\mathcal{O}_X)$. We assume that $\rho_X(\alpha) = 1$. By the proof of Theorem 3.9, $\alpha^*|H^2(X, W\mathcal{O}_X) = id$. Because the characteristic polynomial of $(F \circ \alpha^*)|V_l(X)$ is equal to the characteristic polynomial of $(F \circ \alpha^*)|T_{\text{cris}}(X)$, if $\text{ord}_q(s_i) < 1$, then $t_i = 1$. Now assume
that \( t_i \neq 1 \) for some \( i > h \). Because the Tate conjecture is valid for K3 surfaces \([23; 19]\), \( s_i \) is conjugate to \( s_j \) over \( \mathbb{Q} \) for some \( j \leq h \). Suppose \( \tau(s_i) = s_j \) for some \( \tau \in \text{Gal}(\mathbb{Q}/\mathbb{Q}) \). Then

\[
\tau(s_i t_i) = s_j \tau(t_i) = s_k t_k = s_k
\]

for some \( k \leq h \). But it is impossible since \( \tau(t_i) \neq 1 \) is a root of unity. Therefore, \( \chi_{l, X}(\alpha) = id \) and \( p_{l, X} : \text{Im} \chi_{l, X} \rightarrow \text{Im} \rho_X \) is injective.

**Corollary 3.11.** Let \( k \) be an algebraic closure of a finite field of odd characteristic \( p \), and \( X \) be a K3 surface of finite height over \( k \). If \( N \) is the order of \( \text{Im} \rho_X \), then the rank of \( T_l(X) \), \( 22 - \rho(X) \) is divisible by \( \phi(N) \).

**Proof.** We choose \( \alpha \in \text{Aut}(X) \) such that the order of \( \rho_X(\alpha) \) is \( N \). Assume that the order of \( \chi_{l, X}(\alpha) \) is \( p^m n \) for some nonnegative integers \( m \) and \( n \) where \( p \) does not divide \( n \). Since the order of \( \rho_X(\alpha^{p^m}) \) is still \( N \), replacing \( \alpha \) by \( \alpha^{p^m} \), we may assume that the order of \( \chi_{l, X}(\alpha) \) is not divisible by \( p \). Let \( t_i \) be an eigenvalue of \( \chi_{l, X}(\alpha) \). By the proof of Theorem 3.10, \( t_i \) is a primitive \( N \)th root of unity, and \( n \) is equal to \( N \). It follows that the characteristic polynomial of \( \chi_{l, X}(\alpha) \) is a power of the \( N \)th cyclotomic polynomial over \( \mathbb{Q} \) and the rank of \( T_l(X) \) is a multiple of \( \phi(N) \).

**4. Nonsymplectic Automorphism of Some High Order and Frobenius Invariant**

**Proposition 4.1.** Let \( k \) be an algebraic closure of a finite field of odd characteristic \( p \), and \( X \) be a K3 surface over \( k \). Let \( \alpha \) be an automorphism of \( X \). We assume that the order of \( \rho_X(\alpha) \) is \( N \ (> 2) \) and that the rank of the Neron–Severi group of \( X \) is at least \( 22 - \phi(N) \). If \( p^m \equiv -1 \) modulo \( N \) for some \( m \), then \( X \) is supersingular. If \( p^m \neq -1 \) modulo \( N \) for any \( m \) and the order of \( p \) in \( (\mathbb{Z}/N\mathbb{Z})^* \) is \( n \), then the height of \( X \) is \( n \).

**Proof.** Assume that \( p^m \neq -1 \) modulo \( N \) for any \( m \). Then by Proposition 3.3, \( X \) is of finite height.

We assume that \( X \) is of finite height. Then, by the assumption and Corollary 3.11, the rank of \( T_l(X) \) is \( \phi(N) \), and the order of \( \chi_{l, X}(\alpha) \) is equal to the order of \( \rho(\alpha) \). Every eigenvalue of \( \chi_{l, X}(\alpha) \) is a primitive \( N \)th root of unity, and the characteristic polynomial of \( \chi_{l, X}(\alpha) \) is the \( N \)th cyclotomic polynomial over \( \mathbb{Q} \). We denote the \( N \)th cyclotomic polynomial over \( \mathbb{Q} \) by \( \Phi_N(T) \). Let \( \zeta = \rho_X(\alpha) \), and let \( \xi \) be the Teichmüller lift of \( \zeta \) in \( W \). Let \( V_l(X) = T_l(X) \otimes \mathbb{Q}_l \). Since every primitive \( N \)th root of unity appears once as an eigenvalue of \( \chi_{l, X}(\alpha) \), \( V_l(X) \) is a rank 1 free module over \( \mathbb{Q}_l[T]/\Phi_N(T) \) by the action of \( \alpha^* \). Note that

\[
\mathbb{Q}_l[T]/\Phi_N(T) = (\mathbb{Q}[T]/\Phi_N(T)) \otimes \mathbb{Q}_l \simeq \bigoplus \mathbb{Q}_l(\zeta^{ak})
\]

for suitable primitive \( N \)th roots of unity \( \zeta^{ak} \). Suppose that \( X \) and \( \alpha \) are defined over \( \mathbb{F}_q \) for \( q = p^r \) and \( F : X \to X \) is the \( r \)-iterative relative Frobenius morphism of \( X/k \). Let \( \varphi(T) \in \mathbb{Q}[T] \) be the characteristic polynomial of \( F^*|V_l(X) \). Since
$F^* \circ \alpha^* = \alpha^* \circ F^*$, $F^*|_{V_l(X)}$ is a $\mathbb{Q}_l[T]/\Phi_N(T)$-module endomorphism, so it is the multiplication by a unit element of $\mathbb{Q}_l[T]/\Phi_N(T)$. Hence, all the roots of $\varphi(T)$ are contained in $\mathbb{Q}_l(\xi)$ for any $l \neq p$. It follows that, by the Chebotarev density theorem, all the roots of $\varphi(T)$ are contained in $\mathbb{Q}(\xi)$. Let $n$ be the order of $p$ in $(\mathbb{Z}/N\mathbb{Z})^*$. Then $\xi^{p^{-n}} = \xi$. Since a primitive $N$th root of unity appears only once in the eigenvalues of $\alpha^*|_{T_l(X)}$, by Theorem 3.9, the height of $X$ is at most $n$. We fix a $q$-adic order $\text{ord}_q(\cdot)$ on $\hat{\mathbb{Q}}$. For a root of $\varphi(T)$, $s \in \mathbb{Q}(\xi)$, by the Tate conjecture, there is $\tau \in \text{Gal}(\mathbb{Q}/\mathbb{Q})$ such that $\text{ord}_q(\tau(s)) < 1$. Since $\deg \varphi(T)$ is $\phi(N)$ and the number of primes of $\mathbb{Q}(\xi)$ dividing $p$ is $\phi(N)/n$, the number of roots of $\varphi(T)$ whose $\text{ord}_q(\cdot)$ orders are less than 1 is at least $\phi(N)/(\phi(N)/n) = n$. Therefore, the height of $X$ is at least $n$, so the height of $X$ is $n$. Now suppose $n = 2m$ and $p^m \equiv -1 \pmod{N}$. Then the height of $X$ is $2m$. But among $\xi^{\pm 1}, \xi^{\pm p^{-1}}, \ldots, \xi^{\pm p^{-2m+1}}$, $\xi$ appears twice as an eigenvalue of $\chi_{l,X}(\alpha)$. It contradicts to the assumption, and $X$ is supersingular.

**Remark 4.2.** In the statement of the theorem, the assumption that the rank of the Neron–Severi group is at least $22 - \phi(N)$ is satisfied if $\phi(N) > 10$ by Corollary 3.11.

**Corollary 4.3.** Let $k$ be an algebraically closed field of odd characteristic $p$, and $X$ be a K3 surface over $k$. Let $\alpha$ be an automorphism of $X$. We assume that the order of $\rho_X(\alpha)$ is $N (> 2)$ and that the rank of the Neron–Severi group of $X$ is at least $22 - \phi(N)$.

1. If $p^m \equiv -1 \pmod{N}$ for some $m$, then $X$ is supersingular.
2. If $p^m \not\equiv -1 \pmod{N}$ for any $m$ and the order of $p$ in $(\mathbb{Z}/N\mathbb{Z})^*$ is $n$, then the height of $X$ is $n$.

**Proof.** There exists an integral model $\mathcal{X}/R$ of $X/k$, where $R$ is a Noetherian domain of finite type over $\overline{\mathbb{F}}_p$ equipped with an embedding $R \hookrightarrow k$ such that a geometric generic fiber $k \otimes_R \mathcal{X}$ is isomorphic to $X/k$. After shrinking the base $\text{Spec} R$, we may assume that $\text{NS}(X)$ and $\alpha$ extend to $\mathcal{X}/R$. But the locus of degeneration of the Frobenius invariant is closed ([2], Sect. 8), so we may assume that every geometric fiber of $\mathcal{X}/R$ has the same Frobenius invariant as the generic fiber. We choose a closed fiber $X_0$ of $\mathcal{X}/R$, which is a K3 surface defined over a finite field. By the assumption, the rank of the Neron–Severi group of $X_0 \otimes \overline{\mathbb{F}}_p$ is at least $22 - \phi(N)$. Then the claim follows by Proposition 4.1.

**Corollary 4.4.** Let $k$ be an algebraically closed field of odd characteristic $p$. Assume that $X$ is a K3 surface over $k$ and $\alpha$ is an automorphism of $X$ such that the order of $\rho_X(\alpha)$ is $N (> 2)$. We assume that $\alpha$ is of finite order prime to $p$ and that a primitive $N$th root of unity appears only once in the eigenvalues of $\alpha^*|_{H^2_{\text{et}}(X, \mathbb{Q}_l)}$. If the order of $p$ in $(\mathbb{Z}/N\mathbb{Z})^*$ is $2n$ and $p^n \equiv -1 \pmod{N}$, then $X$ is supersingular of Artin-invariant $n$.

**Proof.** By the proof of Corollary 4.3, $X$ is supersingular. Since $n$ is the least number satisfying $p^n \equiv -1 \pmod{N}$, the Artin-invariant of $X$ is at least $n$ by
Proposition 3.3. On the other hand, by Theorem 3.9, the Artin-invariant of $X$ cannot be greater than $n$, so it is equal to $n$. □

Because a supersingular K3 surface of Artin-invariant 1 is unique up to isomorphism, we obtain the following.

**Corollary 4.5.** Let $k$ is an algebraically closed field of odd characteristic $p$. If $10 < \phi(N) < 22$, $N \neq 60$, and $p \equiv -1$ modulo $N$, then there exists a unique K3 surface over $k$ up to isomorphism that has a purely nonsymplectic automorphism of order $N$.

**Proof.** The existence can be checked in Section 3 of [13]. □

**Remark 4.6.** Over $\mathbb{C}$, a K3 surface equipped with a purely nonsymplectic automorphism of some high order is unique [18; 24; 1; 29]. Also, there is a unique K3 surface with an automorphism of order 60 in characteristic $\neq 2$, and there is a unique K3 surface with an automorphism of order 66 in characteristic $\neq 2$ [14; 15].

Assume that $X$ is a complex algebraic K3 surface such that the order of $\text{Im} \chi_X$ is $N$ ($>2$) and the rank of the transcendental lattice of $X$ is $\phi(N)$. By [26], Corollary 3.9.4, $X$ corresponds to a CM point in a moduli Shimura variety and is defined over a number field. We assume that $X$, $NS(X)$, and $\text{Aut}(X)$ are defined over a number field $F$, and we fix a smooth projective integral model $X_R$ of $X$ over a ring $R$, where $\text{Spec} R$ is an affine open set of the affine scheme of the ring of integers of $F$, $\text{Spec} \sigma_F$. For each place $v \in \text{Spec} R$, let $p_v$ be the residue characteristic of $v$. We may assume that $p_v \nmid \text{Nd}(NS(X))$ and $p_v$ is unramified in $F$ for any $v \in \text{Spec} R$. We denote the reduction of $X_R$ over an algebraic closure of the residue field $k(v)$ by $X_v$.

**Theorem 4.7.** If $p_v^m \neq -1$ modulo $N$ for all $m \in \mathbb{Z}$, then $X_v$ is of finite height, and the height of $X_v$ is the order of $p_v$ in $(\mathbb{Z}/N)^*$. If the order of $p_v$ in $(\mathbb{Z}/N)^*$ is $2m$ and $p_v^m \equiv -1$ modulo $N$, then $X_v$ is supersingular of Artin-invariant $m$.

**Proof.** There is an embedding $NS(X) \hookrightarrow NS(X_v)$, so the rank of $NS(X_v)$ is at least $22 - \phi(N)$. By Corollary 4.3, $p_v^m \neq -1$ modulo $N$ for any $m \in \mathbb{Z}$ if and only if $X_v$ is of finite height, and in this case, the height is equal to the order of $p_v$ in $(\mathbb{Z}/n\mathbb{Z})^*$.

Now assume that $X_v$ is supersingular and $2m$ is the order of $p_v$ in $(\mathbb{Z}/N)^*$. We fix an automorphism $\alpha \in \text{Aut}(X)$ such that $\xi = \rho_X(\alpha)$ is a primitive $N$th root of unity. Note that we do not assume that $\alpha$ is of finite order. Let $T_{NS}(X)$ be the orthogonal complement of the embedding $NS(X) \otimes W \hookrightarrow NS(X_v) \otimes W$.

Here $W$ is the ring of Witt vectors of the algebraic closure of $k(v)$. Because $NS(X_v) \otimes K$ is canonically isomorphic to $H^2_{dR}(X_R/R) \otimes K$, $\alpha^* | T_{NS}(X)$ is of finite
Table 1

| Congruence class of $p_\nu$ modulo 36 | Frobenius invariant of $X_\nu$ |
|-------------------------------------|-------------------------------|
| 1                                  | ordinary                      |
| 17                                 | height 2                      |
| 13, 25                             | height 3                      |
| 5, 7, 19, 29, 31                   | height 6                      |
| 35                                 | supersingular of Artin-invariant 1 |
| 11, 23                             | supersingular of Artin invariant 3 |

order, and every $N$th root of unity appears once as an eigenvalue of $\alpha^*|T_{NS}(X)$. Since $p$ does not divide $d(NS(X))$, $NS(X) \otimes W$ is unimodular. Because there is a unimodular sublattice of $NS(X_\nu) \otimes W$ of rank $22 - \phi(N)$, the Artin-invariant of $X_\nu$ is at most $\phi(N)/2$. If $\sigma$ is the Artin-invariant of $X_\nu$, then $N$ divides $p^\sigma + 1$, so $p^\sigma \equiv -1$ modulo $N$, and $\sigma$ is an odd multiple of $n$. We have an inclusion

$$NS(X_\nu)^*/NS(X_\nu) \simeq T_{NS}(X)^*/T_{NS}(X) \subseteq T_{NS}(X)/pT_{NS}(X),$$

which is compatible with the actions of $Aut(X)$. All the eigenvalues of $\alpha^*(T_{NS}(X)/pT_{NS}(X))$ are distinct. But if $\sigma$ is greater than $n$, then $\rho_{X_\nu}(\alpha)$ appears more than once in the eigenvalues of $\alpha^*(NS(X_\nu)^*/NS(X_\nu))$ by Theorem 3.9. This contradicts the assumption. Therefore, the Artin-invariant of $X_\nu$ is $n$.

\[\square\]

Example 4.8 (cf. [28; 30]). Let $X$ be a K3 surface defined over a number field $F$ such that the order of $Im \rho_X$ is 36. The rank of the transcendental lattice of $X$, $T(X)$ is $12 = \phi(36)$. For example, an elliptic K3 surface $X_{36}/\mathbb{Q}$ defined by the equation

$$y^2 = x^3 + t^5(t^6 - 1)$$

has a purely nonsymplectic automorphism of order 36, $(t, x, y) \mapsto (\xi^{30}t, \xi^2x, \xi^3y)$, where $\xi$ is a primitive 36th root of unity. Although it is quite believable, we do not know whether $X_{36}$ is a unique complex K3 surface satisfying this condition. For almost all places $\nu$ of $F$, $X$ has a good reduction $X_\nu$. The Frobenius invariant of $X_\nu$ is in Table 1.

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