Abstract
We show that, in string theory, as a result of the $W_\infty$-symmetries that preserve quantum coherence in the full string theory by coupling different mass levels, transitions between initial- and final-state density matrices for the effective light-particle theory involve non-Hamiltonian terms $\delta H$ in their time evolution, and are described by a $S$ matrix that is not factorizable as a product of field-theoretical $S$ and $S^\dagger$ matrices. We exhibit non-trivial string contributions to $\delta H$ and the $S$ matrix associated with topological fluctuations related to the coset model that describes an s-wave black hole. These include monopole-antimonopole configurations on the world-sheet that correspond to black hole creation and annihilation, and instantons that represent back-reaction via quantum jumps between black holes of different mass, both of which make the string supercritical. The resulting Liouville mode is interpreted as the time variable, and the arrow of time is associated with black hole decay. Since conformal invariance is broken in the non-critical string theory, monopole and antimonopole, or instanton and anti-instanton, are not separable, and the valley trajectories between them contribute to the $S$ matrix. Ultraviolet divergences in the valley actions contribute non-Hamiltonian terms $\delta H$ to the time evolution of the density matrix, causing it to become mixed and suppressing off-diagonal interference terms in the configuration-space representation of the final-state density matrix. This can be understood as a consequence of the fact that the arrow of time causes world-sheets to contract in target space.
1 Introduction and Summary

The two greatest revolutions in physics in the first half of the twentieth century were those leading to quantum mechanics and general relativity. The greatest piece of unfinished business for physics in the second half of the century is their reconciliation in a consistent and complete quantum theory of gravity. Such a synthesis is likely to require that either or both of quantum mechanics and general relativity be modified in an essential way. It has been argued that any consistent quantum theory of gravity must indeed modify traditional quantum field theory [1] and quantum mechanics [2] in such a way as to accommodate the passage from pure to mixed states that seems to be unavoidable when information is lost across an event horizon [3], for example that surrounding a black hole, whether macroscopic or microscopic.

The original argument was presented in the context of conventional point-like field theory [1]. However, there are other reasons, such as the appearance of uncontrollable infinities in perturbation theory, to think that any point-like quantum theory is doomed. Indeed, the only candidate we have for a consistent quantum theory of gravity is string theory, in which point-like particles are replaced by extended objects. Thus general relativity is modified in an essential way by this candidate for a complete quantum theory of gravity. The next question within the string framework is whether conventional quantum mechanics is also modified in an essential way [4].

It has been suggested [1] that the conventional $S$-matrix description of scattering in quantum field theory would need to be abandoned in quantum gravity, that one should work in a density matrix formalism, and consider the superscattering $S$ matrix that relates the outgoing density matrix to the incoming one,

$$\rho_{\text{out}} = S \rho_{\text{in}}$$

(1)

In conventional quantum field theory, the $S$ matrix is factorizable as the product of an $S$ and an $S^\dagger$ matrix: $S = S S^\dagger$, but some model calculations [5] indicated that the $S$ matrix would not factorize in a true quantum theory of gravity. This argument has been questioned in the literature [6]: in this paper we demonstrate the correctness of the conclusion in the context of string theory.

Some time ago, two of us (J.E. and D.V.N.), together with J.S. Hagelin and M. Srednicki, pointed out [2] that if asymptotic scattering was described by a non-factorizable $S$ matrix as in equation (1), there should be a corresponding modification of the quantum Liouville equation which describes the continuous evolution of the density matrix when averaged over time scales long compared with the Planck time:

$$\partial_t \rho \equiv \dot{\rho} = i[\rho, H] + \delta H \rho$$

(2)

This form of equation is characteristic of an open quantum-mechanical system, and expresses the intuition that observable particles are coupled to degrees of freedom
that are associated with microscopic space-time singularities and unobservable in conventional laboratory experiments. Subsequently, J.E. and D.V.N. together with S. Mohanty showed \[7\] that the modification \(2\) of the quantum Liouville equation caused off-diagonal interference terms in the density matrix to collapse over time, and that this collapse could be very rapid for large objects\[8\] such as Schroedinger’s cat.

During the last couple of years we have been investigating\[9, 10, 11, 12, 13, 14\] whether modifications of quantum mechanics of the forms \(1\) or \(2\) appear in string theory, using the s-wave stringy black hole as a theoretical laboratory\[1\]. We have found that quantum mechanics is maintained in a fixed black hole background, because of an infinite-dimensional \(W\)-algebra which inter-relates string states with different masses\[9\]. Its Cartan subalgebra provides an infinite set of conserved, non-local gauge charges that label all the black hole states and enable the information apparently lost across the event horizon to be encoded and retained\[9\]. Thus there is a well-defined \(S\) matrix for the scattering of light particles off a black hole, black hole decay is a quantum-mechanical effect of higher genera\[11\], not a thermodynamic effect, and we have proposed gedanken experiments capable in principle of measuring all the \(W\) quantum numbers of the black hole\[12\]. However, this does not mean that quantum mechanics applies unmodified to conventional laboratory experiments, in which these quantum numbers are not measurable at the microscopic level.

It is necessary \[4\] to extend the discussion of fixed space-time backgrounds to include quantum fluctuations, namely space-time foam, which includes microscopic black holes that are described by a statistical gas of monopoles and vortices on the world-sheet\[13\]. A laboratory experiment conducted with light particles does not measure the massive degrees of freedom associated with these microscopic black holes. However, light and heavy states are coupled via the \(W\)-algebra\[9\], which expresses the back-reaction of particles on the background metric. Laboratory measurements therefore truncate string theory in a manner incompatible with the coherence-preserving \(W\) symmetry. Unlike the full string theory, which is finite, the truncated theory has infinities that require renormalization. We interpret \[4, 16\] the renormalization scale as the time variable, and renormalization group flow in the sense of Zamolodchikov \[17\] provides an arrow of time. Non-zero renormalization coefficients associated with space-time background fluctuations contribute to the \(\delta H\) term in the modified Liouville equation \(2\):

\[
\delta H = \beta^i G_{ij} \frac{\partial \rho}{\partial p_i}
\]

where \(G_{ij} = \langle V_i V_j \rangle\) denotes the Zamolodchikov metric in coupling constant space. This extra term causes a monotonic increase in entropy:

\[
\dot{S} = \beta^i G_{ij} \beta^j S
\]

As we discuss elsewhere\[15\], we do not consider a two-dimensional dilaton gravity field theory to be an adequate model for studying black hole physics in string theory.
and the asymptotic suppression of off-diagonal elements in the density matrix \([7, 4]\). These results indicate strongly that the scattering of light particles is described in string theory by a non-factorizing \(S\) matrix, and the present paper exhibits explicit non-trivial contributions to the \(S\) matrix.

We base our discussion on the Wess-Zumino coset model \([18]\) that characterizes a spherically-symmetric black hole in string theory. This is an illustration of the structures that appear in space-time foam in string theory. We use it as a convenient tool because it is exactly solvable. We discuss two types of topological fluctuations on the world-sheet: monopole-antimonopole pairs \([13, 19]\) corresponding to the quantum creation and annihilation of a black hole \([13]\), and instantons \([20]\) in the Wess-Zumino coset model. As we show in section 2, these instantons describe quantum jumps between black holes of different masses. Both monopoles and instantons increase the central charge of the string, making it locally supercritical with \(c > 26\). This would be a general feature of fluctuations in the space-time foam. We show that instantons rescale the metric in the same way as one of the marginal operators associated with massive string modes. Thus instantons represent the back-reaction expressed by the \(W\)-induced coupling of light particles to massive string states.

As we discuss in section 3, the transition between initial- and final-state density matrices is given by an absorptive part of a world-sheet correlation function with an even number of external tachyon fields, reminiscent of the Mueller \([21]\) description of inclusive reactions in hadronic physics. We argue that, as in the analysis of non-perturbative electroweak cross-sections, there are identifiable non-factorizable contributions to the correlation function, and hence the \(S\) matrix, from monopole-antimonopole and instanton-anti-instanton configurations. Furthermore, the dominant transitory space-time may correspond to a valley trajectory \([22, 23]\).

It is generally believed \([24]\) that the motion along a valley trajectory is equivalent to the scattering of solitons in one higher dimension. In section 4 we use this equivalence to identify valley trajectories in the monopole-antimonopole and instanton-anti-instanton systems. In the case of the monopole-antimonopole valley, breaking of conformal symmetry at the quantum level plays a key rôle. In the case of the instanton-anti-instanton valley, the energy integral must first be made finite by introducing an infrared cut-off.

As we discuss in section 5, the ultraviolet renormalization scale can be identified with the Liouville field, which can in turn be identified with target time in a supercritical string theory \([25]\), which is our case. The arrow of time corresponds to black hole decay.

We discuss first in section 6 string contributions to the conventional \(S\) matrix within our renormalization group approach, associated with topologically-trivial
space-time backgrounds. Then we exhibit the valley contributions to the $\mathcal{S}$ matrix, and show that they are renormalization scale-, and hence time-, dependent, and therefore make contributions to $\delta H$. By analogy with conventional models of quantum friction \cite{26, 27}, these contributions suppress off-diagonal density matrix elements. This can be understood intuitively from the fact that the size of the string world-sheet in target space shrinks as time increases. This means that final-state fields that are located at arbitrary points on the world-sheet are asymptotically coincident in target space. Thus off-diagonal terms in the final-state density matrix in configuration space, which represent interferences between objects at different points in space, vanish asymptotically at large times \cite{7, 4}. This partial “collapse of the wave function” is more rapid for systems containing larger numbers of particles \cite{8, 4}. The final state is described probabilistically: Schroedinger’s cat is either dead or alive, but not in a superposition of the two states.

Finally, in section 7 we relate our work to previous ideas \cite{28, 29} about wave function collapse and the possible rôle of quantum gravity.

## 2 Stringy Black Holes and Instantons

In string theory, spherically-symmetric black holes, which are equivalent for our purposes to black holes with a two-dimensional target space, are described by a gauged SL(2,R)/U(1) Wess-Zumino coset model \cite{18}. The black hole exists in both Minkowski and Euclidean versions, obtained by gauging different subgroups of SL(2,R). As usual in instanton calculations, it is more convenient to work with the Euclidean version, and make an analytic continuation at the end of the calculation. The Euclidean stringy black hole is obtained by gauging the axial subgroup of SL(2,R): after eliminating the U(1) gauge field one arrives at the action

$$S = \frac{k}{4\pi} \int d^2z [(\partial_\mu r)^2 + \tan hr^2(\partial_\mu \theta)^2 + \ldots]$$

(5)

where $r$ and $\theta$ are radial and angular coordinates respectively, and the $\ldots$ denote dilaton terms that arise from the path-integral measure of the gauge-field. It is convenient to rewrite (5) using the complex coordinate

$$w = \sinh re^{-i\theta}$$

$$\bar{w} = \sinh re^{i\theta}$$

(6)

in terms of which the action (5) becomes

$$S = \frac{k}{4\pi} \int d^2z \frac{1}{1 + |w|^2} \partial_\mu \bar{w} \partial^\mu w + \ldots$$

(7)
whence we see that the target space-time line element is
\[ ds^2 = \frac{dw d\overline{w}}{1 + w \overline{w}} = dr^2 + \tanh^2 r d\theta^2 \] (8)

**World-Sheet Vortices/Monopoles**

As discussed in ref. [13], the Euclidean black hole (5, 6, 7, 8) can be written as a world-sheet vortex-antivortex pair, which is a solution \( X_v \) of Green function equations on a spherical world-sheet of the type
\[ \partial_z \partial_{\overline{z}} X_v = i\pi \frac{q_v}{2} \left[ \delta(z - z_1) - \delta(z - z_2) \right] \] (9)
where \( z_1 \) is the location of the vortex and \( z_2 \) the location of the anti-vortex (the net vorticity is always zero on a compact world-sheet). If the vortex is located at the origin and the antivortex at infinity (corresponding to the South and North Poles in a stereographic projection), the corresponding \( \sigma \)-model coordinate solution of (9) is
\[ X_v = q_v \text{Im} ln z \] (10)

It is clear that for \( X_v \) to be angle-valued it must have period 2\( \pi \), and hence the vortex charge \( q_v \) must be an integer.

To see the interpretation of (10) as a Euclidean black hole [13], we rewrite the vortex configuration \( X_v \equiv \theta \) as
\[ e^{2i\theta} = \frac{z}{\overline{z}} \] (11)
and complexify the phase by introducing a real part \( r \), defined by the following embedding of the world-sheet in a two-dimensional target space \((r, \theta)\):
\[ z = (e^r - e^{-r})e^{i\theta} \] (12)
The induced target-space metric (8) is inferred from the world-sheet arc-length, which is \( dl = \frac{dz}{1 + z \overline{z}} \), after the stereographic projection, and the infinitesimal Euclidean displacement \( d\overline{z} \) induced by a corresponding shift in the \( r \)-coordinate.

The representation of the Euclidean black hole as a coset \( SL(2,R)/U(1) \) Wess-Zumino model means that the world-sheet vortices (8) induce gauge defects of the underlying world-sheet gauge theory. Eliminating the non-propagating gauge field \( A_z \) of the gauged Wess-Zumino model via its equations of motion yields
\[ A_z = -\frac{u \partial_z (a - b) - (a - b) \partial_z u}{(a + b)^2} \] (13)
where \( u = \sinh r \sin \theta \), \( a = \cosh r + \sinh r \cos \theta \) and \( b = \cosh r - \sinh r \sin \theta \). In the neighbourhood of the singularity \( r \to \epsilon \), \( u \to \epsilon \sin \theta \), and \( a - b \to 2\epsilon \cos \theta \), so that the gauge potential (13) becomes that of a singular gauge transformation
\[ A_z \to \epsilon^2 \partial_z \theta \] (14)
which has the interpretation of a \( U(1) \) gauge monopole [13].
In addition to vortex configurations, one can have monopole-antimonopole (or better, “spike-anti-spike”) pairs on the world-sheet. These are solutions $X_m$ of Green function equations of the following type [19]:

$$\partial_z \overline{\partial}_z X_m = -\frac{q_m \pi}{2} \left[ \delta(z - z_1) - \delta(z - z_2) \right]$$

which corresponds to a “spike” at $z_1$ and an “antispike” at $z_2$. Once again, the compactness of the closed string world-sheet imposes zero net “spikiness”, but in this case $X_m$ is non-compact and hence aperiodic, so the spike charge $q_m$ is not quantized a priori. After stereographic projection of the South Pole on the sphere onto the origin and of the North Pole onto the point at infinity in the complex plane,

$$X_m = q_m \text{Relnz} = q_m \ln|z|$$

The world-sheet monopoles are related to Minkowski black holes [13], and the corresponding charges are proportional to the black hole mass.

Both vortex and spike configurations can be described in terms of sine-Gordon deformations of the Lagrangian for the field $X$. Viewing these defects as thermal excitations on the world sheet, corresponding to a ‘pseudo-temperature’ $\beta^{-1}$, and assuming a statistical population, the corresponding effective action reads [13]

$$Z = \int D\tilde{X} \exp(-\beta S_{\text{eff}}(\tilde{X}))$$

where $\tilde{X} \equiv \beta^{\frac{1}{2}} X$, and

$$\beta S_{\text{eff}} = \int d^2z [2\partial \tilde{X} \overline{\partial} \tilde{X} + \frac{1}{4\pi} [\delta_v \omega^{\frac{\omega}{2}} - 2(2\sqrt{|g(z)|})^{1-\frac{\omega}{2}} \cos(\sqrt{2\pi \alpha} [\tilde{X}(z) + \tilde{X}(\bar{z})])] + (\delta_v, \alpha, \tilde{X}(z) + \tilde{X}(\bar{z})) \to (\delta_m, \alpha', \tilde{X}(z) - \tilde{X}(\bar{z}))]$$

Above we have made a stereographic projection of the sphere of radius $R$ onto the complex plane, inducing an effective metric $g(z)$; $\omega$ is an angular ultraviolet cut-off on the world-sheet, which accompanies the normal ordering of the cosine (deformation) terms. For future use, we note that the projection of the cut-off on the sphere of radius $R$ is $2\omega R$. Its stereographic projection on the plane through a point $z$, as in figure 1, yields $2\omega R(1 + |z|^2/4R^2)$. In the Liouville theory framework, that we follow in this paper, we can keep the radius of the sphere fixed and let $\omega$ vary, which thus incorporates the effects of the Liouville mode. Equivalently, one may assume a spherical contraction of the world sheet (as the theory approaches close to the ultraviolet fixed point), keeping $\omega$ fixed. The quantities $\delta_v, m$ in eq. (18) are the fugacities for vortices and spikes respectively, and

$$\alpha \equiv 2\pi \beta q_v^2 \quad \alpha' \equiv 2\pi \beta q_m^2$$

(19)
are related to the conformal dimensions $\Delta_{v,m}$ of the vortex and spike creation operators respectively, namely
\begin{align*}
\alpha &= 4\Delta_v \\
\alpha' &= 4\Delta_m \\
\Delta_m &= \frac{\pi \beta}{2} q_m^2 = \frac{(eq_v)^2}{16\Delta_v}
\end{align*}
where we took into account the vortex-induced quantization condition for monopoles [19] $2\pi \beta q_m = e : e = 1, 2, \ldots$. The corresponding deformations are irrelevant - and the pertinent topological defects are thus bound in pairs - if the conformal dimensions are larger than one.

For future use we mention that these considerations have also been extended to Liouville theory [19]. If $\phi$ is a Liouville mode, the corresponding action on a curved world-sheet with metric $g_{ab}$ reads
\begin{equation}
S_L = \left(\frac{25 - c}{96\pi}\right) \int d^2x \sqrt{g}(g^{ab}\partial_a\phi \partial_b\phi + \phi R^{(2)})
\end{equation}
where $c$ is the central charge of the matter theory. For $c \geq 25$ the requirement of the boundedness of the action [21] implies a Wick rotated, angle-valued Liouville field $\phi$, and the theory admits both monopoles and vortices. The rôle of the pseudo-temperature is now played by the quantity [19]
\begin{equation}
\beta_L = \frac{3}{\pi(c - 25)}
\end{equation}
An interesting consequence of (22), which will be of use to us later, is that for $e = 1$ the monopole operator is irrelevant in the region where $c \geq 49$. In this region the vortex operator is relevant. In fact, in general for $c \geq 25$ there is no region where both operators are irrelevant, and thus the system appears strongly coupled. We also note for future reference that, in accordance with the Zamolodchikov c-theorem [17], the part of the central charge pertaining to the conformal field theory subsystem that possesses the topological defects increases when the latter appear bound in dipole-like pairs as irrelevant deformations.

**Instantons**

These appear [20] in the $SL(2,R)/U(1)$ Wess-Zumino coset model as follows. A topological charge is defined in this model by analogy with compact $\sigma$ models:
\begin{equation}
Q = \frac{1}{\pi} \int d^2z \frac{1}{1 + |w|^2} \overline{\partial w \partial w - h.c.}
\end{equation}
It is easy to verify that there are topological classes labelled by an integer $n$, and that the action is minimized in each topological class by holomorphic functions
\begin{equation}
w(z) = b \frac{(z - c_1)\ldots(z - c_n)}{(z - z_1)\ldots(z - z_n)}
\end{equation}
where the parameters $b, c_{1,2,\ldots,n}, z_{1,2,\ldots,n}$ are complex, and $n$ is the winding number of the map (24). The topological charge (23) is proportional to the winding number $n$, but with a logarithmically-divergent coefficient:

$$Q = -2n \ln(a) + \text{const} \quad (25)$$

where $a$ is the ultraviolet cutoff (in configuration space) to be discussed in more detail later. One should stress at this point that the black hole metric $g(|w|) = (1+|w|^2)^{-1}$ is the limiting case allowing for the existence of instanton solutions, as it leads to logarithmic, and not to power, dependence of the topological charge $Q$ on the cutoff. The coefficient of the leading logarithm in $Q$ is regularization-independent and hence the quantity $Q$, although divergent, still allows for a definition of a winding number $n$ and, therefore, of topological classes for the configuration space of the model.

We concentrate here on the simplest $n = 1$ instanton which can be written as

$$w(z) = \frac{\rho}{z - z_0} \quad (26)$$

where $\rho$ is the instanton size parameter. The contribution of this holomorphic instanton and its antiholomorphic anti-instanton partner to the free energy of the model is given in the dilute gas approximation by

$$F_{\text{dilute-gas}} \simeq -\frac{d}{2\pi} \int d^2 z_0 \frac{d^2 \rho}{|\rho|^2} e^{-S_0} \quad (27)$$

where $d$ is an appropriate constant and $S_0$ is the action of the instanton (26), which is easily evaluated to be

$$S_0 = \frac{k}{2} \ln(a^{-2} |\rho|^2 + 1) \quad (28)$$

Inserting this result into the formula (27) for the free energy, we find a decrease in the free energy:

$$F_{\text{dilute-gas}} \simeq -d \int d^2 z_0 \frac{d|\rho|}{|\rho|^2} (a^{-2} |\rho|^2 + 1)^{-\frac{5}{2}} \equiv -d' a^{-2} V^{(2)} \quad (29)$$

where $V^{(2)}$ is the (two-dimensional) world-sheet volume. The constant $d'$ is the instanton density. We shall discuss it in more detail later on. The effect of the instanton (26) can be reproduced by the vertex operator

$$V_I \propto \int d^2 z \frac{d^2 \rho}{|\rho|^4} e^{-S_0} e^{k[\rho\partial\bar{\rho} + h.c.]} \quad (30)$$

which can be combined with the corresponding anti-instanton vertex to give

$$V_{1\bar{I}} \propto -\frac{d}{2\pi} \int d^2 z \frac{d^2 \rho}{|\rho|^4} e^{-S_0} (e^{\left(\frac{k[\rho\partial\bar{\rho} + h.c. + \ldots]}{f(|w|)}\right)} + e^{\left(\frac{k[\rho\partial\bar{w} + h.c. + \ldots]}{f(|\bar{w}|)}\right)}) \quad (31)$$

\(^{2}\text{To make contact with the case of spherical world sheets, discussed above, we remind the reader that this cut-off is } 2\omega R, \text{ in the notation above.}\)
where $f(|w|)$ is an unknown function that is regular at $w = 0$. It can be expanded as $f(|w|) = 1 + a_1|w|^2 + a_2|w|^4 + \ldots$. In the dilute-gas approximation it is not possible to compute this function. It is conceivable that $f(|w|)$ is the inverse metric $(1 + |w|^2)^{-1}$. In this case the exact effects of instantons would be correctly captured by the large-$k$ expansion, as we discuss immediately below.

At leading order in large $k$ we need only retain the first two terms in a power series expansion of $f(|w|)$ about the origin, and the effect of the instanton vertex

$$V_{\mathcal{I}} = -d'[a^{-2}V^{(2)} + \frac{k^2}{2} \int d^2z \frac{\partial \bar{w} \partial^\mu w}{f^2(|w|)}]$$

(32)

is simply to add a vacuum energy term, which we discuss later, and to renormalize the kinetic energy term of the coset model (7):

$$k \to k - 2\pi k^2 d'$$

(33)

In the above expression $d'$ denotes the instanton density which is given via (29) as

$$d' \equiv d \int \frac{d|\rho|}{|\rho|^3} \frac{a^2}{[(\rho/a)^2 + 1]^{\frac{\gamma}{2}}}$$

(34)

The integral (34) has a power divergence for small-size instantons ($\rho \to 0$), and one is obliged to cut such contributions off at a scale $a$. Such a procedure results in an effective renormalization of the Wess-Zumino level parameter $k$. Indeed, consider the right-hand-side of the expression (34) integrated over small-size instantons only, i.e. $d' = d \int_0^1 d\hat{\rho} \frac{1}{\hat{\rho}^3(1+\hat{\rho}^2)^{\frac{\gamma}{2}}}$, and concentrate in the region $\hat{\rho} \equiv \frac{a}{\Lambda} << 1$. In this region we can parametrize $\hat{\rho} \equiv \frac{a}{\Lambda} \gamma$, where $\Lambda$ is the world-sheet infrared cut-off in configuration space, and $\gamma$ a small positive number. Converting the integration over $\hat{\rho}$ into an integral over $\gamma$ around a small value $\gamma_0$, of radius $\frac{1}{|\ln (a/\Lambda)|}$ (which is the region corresponding to the leading ultraviolet divergences) we get, in the limit $a \to 0$

$$d' \approx \frac{a}{\Lambda} |\gamma - 2\gamma_0|$$

(35)

It can be shown that if one considers matter deformations in the theory, as we shall do later on, the relevant order of magnitude of $\gamma_0$ is, to leading approximation in the deformation couplings, of the order of the anomalous dimension of the matter deformation. The induced divergences can be absorbed, as a result of (33) in a ‘renormalization’ of $k$, which, thus, becomes scale-dependent. It should be stressed that the above computation is only indicative, in order to demonstrate the scale dependence of $d'$ due to regularization. The anomalous dimension coefficient $\gamma_0$ is assumed small enough so as to guarantee the validity of a perturbative $\epsilon$-expansion when discussing contributions to the light-matter $S$ matrix, as we shall see in section 6. There we will extrapolate the results to finite $\gamma_0$, analogously to the $\epsilon$-expansion in local field theory, whose qualitative features are assumed valid in the region of finite $\epsilon$. 
Performing the above regularization procedure, and rescaling \( \rho \rightarrow \rho/a \), we arrive easily, after a change of integration variable \( |\rho/a| \rightarrow \tau = \frac{1}{|\rho/a|^2} \), at

\[
d'_{\text{reg}} = \frac{d}{2} \int_0^1 d\tau (\tau\frac{k}{2} (1 + \tau)^{-\frac{k}{2}} = \frac{d}{2(\frac{k}{2} + 1)} 2^{-\frac{k}{2}} F(1, \frac{k}{2}, \frac{k}{2} + 2; \frac{1}{2})
\] (36)

The series expansion of the hypergeometric function yields terms of the form \[ \]

\[
F(1, \frac{k}{2}, \frac{k}{2} + 2; \frac{1}{2}) = \frac{1}{2} \Gamma(\frac{k}{2} + 1) + \{ \frac{1}{2} (\frac{k}{2} + 1) \frac{k}{2} \times \\
\times \sum_{n=0}^{\infty} \frac{(2n)(\frac{k}{2} + 1)_n}{n!(n + 1)!} (2)^{-n}[\ln(1/2) - \psi(n + 1) + \psi(\frac{k}{2} + n + 1)]\}
\] (37)

Hence in the large-\( k \) limit \( d'(k) \ll 1 \) and the convergence of the expression (33) is guaranteed.

Since the level parameter \( k \) is inversely related to the central charge \( c \)

\[
c = \frac{2(k + 1)}{k - 2} = 2 + \frac{6}{k - 2}
\] (38)

the effect of the decrease (33) in \( k \) is to increase the central charge. If we assume that the stringy black hole had a space-time interpretation before adding instantons, so that its central charge was 26, either because \( k = 9/4 \) or because additional matter fields were present, the string model will be noncritical after including instanton effects. Indeed, it will be supercritical with \( c > 26 \), reflecting the fact that the instanton vertex is an irrelevant operator. As already commented in section 1, the fact that both monopoles and instantons make the string locally supercritical is presumably a general feature of incorporating foamy backgrounds in string theory.

As is apparent from the representation (5) of the stringy black hole action and the form (11) of the instanton vertex, the shift (33) in the level parameter \( k \) renormalizes the target space metric. This rescaling is correlated with the renormalization of the black hole mass:

\[
M_{bh} = \sqrt{\frac{2}{k - 2}} e^{const}
\] (39)

The effect of the instanton, at least as seen in the large \( k \) limit (1) is therefore to increase the black hole mass in a quantum jump which may represent back-reaction or infall. Its effect is opposite to that of higher genera (loops) (11), which increase \( k \), and hence decrease \( c \), corresponding to black hole decay. Changes in the stringy black hole background contribution to the total central charge can always be compensated by a change in the contribution of other, matter, fields. This reflects the fact that operators which appear irrelevant within a specific conformal model may become marginal in the context of models with a larger value of the central charge.

\[3\] We note once again that this large-\( k \) effect would be exact if the function \( f(|w|) \) were the inverse target-space metric.
In what follows, we would like to make a connection of these results with the \( SL(2, R) \) current algebra deformation approach of ref. \[31\]. We shall be interested in exactly marginal deformations of conformal string backgrounds. Such deformations are \((1,1)\) operators which however retain their conformal dimension in the deformed theory, thereby generating families of conformally invariant backgrounds. Such operators in the Wess-Zumino coset model describing the stringy black hole have been studied in \[31\]. They consist in general of vertices for both massless (tachyon) and massive states. Since the latter are solitons in the stringy background, these exactly marginal operators represent the back-reaction of matter fields on the metric. One particular example of such an exactly marginal operator is \[31\]

\[ L^2_0 \propto \psi^{++} + \psi^{--} + \psi^{-+} + \psi^{+-} + \ldots \]  

(40)

where \( \psi^{\pm\pm} \equiv (\mathcal{T}^\pm)^N (J^\pm)^j_{m-N} \), are composite fields constructed out of the \( SL(2, R) \) currents \( J \) and the Wess-Zumino field \( g \), and represent \textit{massive} discrete string states at level \( N \). The \( \ldots \) in eq. (40) imply summation over all such states. We see immediately \[31\] that this operator also rescales the target space metric. In the context of the critical \( k = 9/4 \) string theory, such global rescalings of the metric define families of conformally invariant black-hole backgrounds, which may be thought of as corresponding to constant shifts in the dilaton field \[15\]. From a target-space point of view, however, such global rescalings of the metric field amount to a change in the black hole mass \( \Delta \) which might also be attributed to a change in \( k \) analogously to \( \Delta \).

To understand this interpretation better, we give another example of an exactly marginal deformation of the Wess-Zumino stringy black hole. It assumes the form:

\[ L^1_0 \mathcal{T}_0 \propto \Phi_{-c}^{c-} \Psi^{++} + i(\psi^{++} - \psi^{--}) + \ldots \]  

(41)

where the massive string modes \( i(\psi^{++} - \psi^{--}) + \ldots \) are given in terms of \( SL(2, R) \) currents as in \[10\] \[31\]. The operator \( \Phi_{-c}^{c-} \) generates the light string matter. The level-one massive modes indicated explicitly in (41) represent back-reaction on the space-time metric, as can be checked explicitly in the case \( k \rightarrow \infty \). The deformation \( \Phi_{-c}^{c-} \) alone is relevant. Thus, the \textit{combined} deformation consisting of (40) and the \textit{massive} mode (irrelevant) parts of (41) is an \textit{irrelevant} deformation of the Wess-Zumino theory, which represents back-reaction effects of matter on the space-time black-hole geometry. Such effects are analogous to those induced by instantons \[32\]. Although at this stage this analogy cannot be made rigorous, however it follows that the instanton can represent the effects of massive string modes\footnote{These are related to each other and to the massless excitations by a particular quantum version of the \( W_{1+\infty} \) algebra, with the level parameter \( k \) interpreted as a quantum deformation parameter.}.
There is a formal way of demonstrating this, to lowest order in perturbation theory, by looking at the behaviour of the model with instantons under renormalization group flow. We first observe that, due to (28), the dominant contributions to the path-integral in the dilute gas approximation come from instantons with sizes of the order of the cut-off. This is expected because the instanton perturbations (31) are irrelevant operators in a renormalization group sense. To see this, we first note that on dimensional grounds one expects the instanton contributions to be of order

$$\left(\frac{\rho}{a}\right)^{2\lambda}$$

where \(\lambda\) is the first coefficient of the \(\beta\)-function. In our case the instanton \(\beta\) function can be estimated by looking at the contribution of the deformation (31) to the trace of the stress-tensor of the model. The result is \(\beta = -\frac{k}{2}\) (43)

which is negative for \(k > 0\), hence the irrelevance of the respective deformation. The result (43),(42) is compatible with (28). This result implies that the dominant contributions in the path-integral are instantons of the size of the cut-off

$$\rho \simeq a$$

(44)

To discuss the role of contributions close to the instanton center, it is convenient to keep \(\rho\) fixed and finite in a correlation function involving \(N\) tachyon operators \(T_i \equiv T(z_i)\), while we send \(|z_i - z_0| \to 0\) (\(i=1,2,...N\)). One may represent such a process as a sum of zero and non-zero instanton sectors appropriately normalized. From the general analysis in the zero-instanton sector \(\beta\) one may represent the tachyon deformations in the \(\sigma\)-model as

$$\Phi^{c-c}_{\frac{1}{2}b,0,0}(r) = \frac{1}{\cosh r} F\left(\frac{1}{2}, \frac{1}{2}; 1, \tanh^2 r\right)$$

(45)

where \(r\) is a target spatial coordinate defined in (3).

In the region of weak matter deformations, \(r\) is large and one may use the following expansion of the hypergeometric function \(\beta\)

$$F\left(\frac{1}{2}, \frac{1}{2}; 1, \tanh^2 r\right) \simeq \frac{1}{\Gamma^2\left(\frac{1}{2}\right)} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{(n!)^2} \left[2\psi(n + 1) - 2\psi(n + \frac{1}{2}) + \ln(1 + |w|^2)(\sqrt{1 + |w|^2})^{-n}\right]$$

(46)

The leading behaviour for \(r\) large comes from the terms with \(n = 0\). In this limit the tachyon function yields the usual result of the tachyon background \(\beta\)

$$T(r) = \text{sech}(r) \ln(\cosh(r))$$

(47)
which after an appropriate change of variable becomes identical to the result of the perturbative $\beta$-function approach to $c = 1$ string theory \[18, 33\]. In this black hole string theory, however, the tachyon operator \[15\] alone is no longer a marginal deformation in the presence of a black hole, but a relevant deformation, since the massive string mode effects in the truly marginal operator \[11\] are not vanishing. This non-vanishing of the $\beta$-functions of the light string modes is the key ingredient to define target time as an evolution parameter in a renormalization-group sense \[4\]. If we denote generically the ‘bare’ couplings of the tachyon deformations by $T^i$ then, to leading order, the renormalized couplings are of the form

$$T_R^i \propto |\frac{a}{\Lambda}|^\lambda T^i$$

where $\Lambda$ is the infrared cut-off introduced previously for dimensional reasons\[5\].

To demonstrate things clearly we consider first the one-point tachyon correlator in the one-instanton background. This means that we evaluate $< V_T V_{\text{inst}} > T^i$, where $V_T$ is the vertex operator for the tachyon given in \[14\], and the instanton vertex is given by \[31\]. In the coset model \[7\] the following relation is valid to leading order in $k$ \[20\]:

$$\partial_z < w(z_1)w(z_2) > \propto \frac{1}{k} \frac{1}{z_1 - z_2}$$

which implies that when evaluating correlation functions of $w$ in the one-instanton sector for large $k$ one effectively replaces $w$ by its classical expression \[26\], and its complex conjugate for anti-instantons. Taking into account the expansion \[46\] of the hypergeometric function in \(45\), we find for the one-point function in the short-distance region of the $z$-integration terms proportional to

$$T_R(\frac{|a|}{\Lambda})^{-\lambda} \int_{z \to z_0} d^2zd^2z_0d^2\rho \frac{1}{|\rho|^4} \left( \frac{|\rho|^2}{|a|^2} + 1 \right)^{-\frac{1}{2}} \frac{|z - z_0|}{|\rho|^2} 2ln|\rho| |z - z_0|$$

Recalling that the dominant contributions to this integral come from instantons with sizes of order $a$, and cutting off the instanton size integration at $a$, we observe that the tachyon deformation \[17\] is marginal in the one-instanton sector, in the sense that the integrand \[50\] has a smooth short distance behaviour. This implies a scale-independent $T$ in this sector, because the anomalous dimension induced by the instantons cancels that of the ordinary renormalization of $T$ in the zero-instanton sector. This is exactly what should happen in the dilute gas approximation, given the argued rôle of the instantons as representing massive string mode effects. The combined effect of the (exact) instanton and light matter deformations is expected

\[5\] Here we follow Wilson’s renormalization approach, where the cut-off is never removed and the variable scale is the ratio of infrared to ultraviolet cut-offs. It is this approach that was used in \[34\] to prove the gradient flow of the renormalization group $\beta$-functions used in the equivalence of the $\sigma$-model conformal conditions with the $S$-matrix approach to string theory \[33\]. In the Wilson approach, elimination of the irrelevant operators leads eventually to the perturbative Gell-Mann-Low $\beta$-functions, that do not have an explicit cut-off dependence.
to be marginal, and the above analysis checks this conjecture in the single instanton sector for large $k$.

3 Topologically Non-Trivial Contributions to Transitions between Density Matrices

In string theory, all physical quantities can be related to the calculation of correlation functions on the world-sheet. In our case, these are to be evaluated in a black hole background in target space, including the effects of world-sheet monopoles or instantons. In the dilute gas approximation, correlation functions for light matter (tachyon) fields may be expanded in powers of the density of topological structures:

$$<A|B >_{\text{inst}} = <A|0 > + <A|B >_{1} <0|0 > + <A|B >_{0} <0|0 >_{0} + \ldots$$

where $< \ldots >_{1(0)}$ denotes correlators evaluated in the one(zero)-monopole or-instanton sector. From the discussion in the previous section, we see that the zero-monopole or -instanton contributions correspond to neglecting the back-reaction of matter fields on the metric. These constitute the approximation made up to now in the string treatment of black hole physics. There is a well-defined $S$ matrix for the scattering of light particles off a black hole in this approximation, whose only non-zero elements are those with just one incoming or outgoing tachyon [36, 12]:

$$G_{++--}, G_{+-+-} \neq 0$$

whilst all other Green functions vanish.

We wish to calculate the first non-trivial corrections to this result, corresponding to the first non-trivial terms in the expansion (51). Rather than calculate this correction for all Green functions, we adapt a convenient approach used recently [23] to estimate non-perturbative effects in electroweak theory, which exploits the optical theorem.

We are interested in the transition between a generic initial-state density matrix $\rho^{A}_{B}^{\text{in}}$ and final-state density matrix $\rho^{C}_{D}^{\text{out}}$. Here the index $A$ labels a generic bra vector $<A|_{\text{in}}$, and similarly for the other indices: $|B >_{\text{in}}, <C|_{\text{out}}$ and $|D >_{\text{out}}$. These states may each contain many matter fields at different target-space locations. If not stated explicitly, we will consider the indices $A$, etc., as referring to the target-space locations. In string theory, these are coordinates given by the values of field variables on the world-sheet: $X(z_{A}) = A$, etc. We note that the field for a single light particle (`tachyon') may involve an infinite number of world-sheet operators at
the same world-sheet point. For example, far from the centre of the string black hole, in the weak field region, one may Fourier expand a tachyon field:

\[ T(X) = \int d^Dk e^{ik \cdot X} T(k) \]  

(53)

where \( T(k) \) is a ‘tachyon’ polarization tensor in momentum space. In the stringy black hole, there is a one-to-one mapping between points in target space and on the world-sheet: \( A \leftrightarrow z_A \), etc.. In general, transition matrix elements are calculated from matter (tachyon) correlation functions on the world-sheet. In the simplest case of single-particle initial and final states, the relevant object is a four-point correlation function. The quantity required is a specific absorptive part of this correlation function, analogous to the discontinuity that describes [21] an inclusive cross-section in conventional hadronic physics:

\[
\sum_{X_{\text{out}}} \left< A\right| D, X_{\text{out}} \left| X, C \right| B\left< X\right| T\left(\phi(z_A)\phi(z_D)\right)\left| X\right>_{\text{out}} = \\
\sum_{X_{\text{out}}} \left< 0\right| T\left(\phi(z_A)\phi(z_D)\right)\left| 0\right>_{\text{in}}
\]

(54)

where the symbols \( T \) and \( \overline{T} \) denote time-ordered (anti-ordered) products, and equation (54) has the diagrammatic representation shown in figure 2.

Note that we have used the optical theorem on the world-sheet to replace the sum over unseen states \( X \) by unity. This application of the optical theorem is justified by the fact that conventional quantum mechanics and quantum field theory remain valid on the world-sheet: it is only their elevation to space-time that we challenge. We recall that, in such a hadronic inclusive reaction at high energies, the final-state particle distribution is described probabilistically, with all interferences vanishing after summation over the unseen parts \( X \) of the final states. We will argue later that something similar happens when one calculates the analogous absorptive part of the correlation function on the world-sheet.

Since the quantity of interest (54) is an expectation value in a definite vacuum state, all topologically non-trivial contributions must have equal numbers of monopoles and anti-monopoles or instantons and anti-instantons. If the string theory were critical, the monopole anti-monopole or instanton-anti-instanton configurations would be trivial, since they could be moved around arbitrarily by conformal transformations. However, as discussed in the previous section, monopoles or instantons make the stringy black hole non-critical, so that the action depends on the relative locations. The question then arises: what is the dominant contribution to the functional integral in the topologically non-trivial sector? It is presumably given by a saddle point in the action associated with a valley trajectory [22, 23], as in figure 3. Thus, to evaluate the absorptive part of a correlation function in the conventional
path-integral formalism, we first continue analytically to Euclidean space, and then make a saddle-point approximation using a valley configuration. The path integral is thereby converted into an integral over the collective coordinates that describe the valley. The leading semi-classical behaviour of an $S$ matrix element in the single defect-antidefect valley approximation takes the form

$$S \propto \text{Abs} \int D\phi^c \exp(-S^{(1)}_v(\phi^c)) F_{\text{kin}}$$

(55)

where the integral is over the collective coordinates $\phi^c$ of the valley, whose action is described by $S^{(1)}_v(\phi^c)$. The function $F_{\text{kin}}$ depends on kinematic factors, in particular the total centre-of-mass energy $E$ in the case of a forward four-point Green function, and on collective coordinates, in particular the separation parameter $\Delta R$ of the valley. It is a generic feature of valley configurations that in the four-point case, for large $E\Delta R >> 1$,

$$F_{\text{kin}} \simeq \exp(E\Delta R)$$

(56)

allowing for a saddle-point approximation to the integral (55). To evaluate the relevant absorptive part of (55) it is necessary to continue back to Minkowski space [23]. We assume that, as usual, the quantitative features of this semi-classical approximation are not affected by quantum fluctuations.

4 Valley Trajectories in the String Black Hole

We demonstrate in this section the existence in the $SL(2,R)/U(1)$ coset Wess-Zumino model of suitable monopole-antimonopole and instanton-anti-instanton valley trajectories, following the general method outlined in ref. [37] and applied to the two-dimensional $O(3)$ $\sigma$-model. Since there are many similarities, but also some important formal differences, we first review briefly the case of the $O(3)$ $\sigma$-model, and then move on to the $SL(2,R)/U(1)$ case. The reader who is not interested in the details of the valley trajectories, but is content to accept their existence and eager to study their physical consequences, is advised to go to section 5.

$O(3)$ $\sigma$-model valley

The ”target space-time” metric in the $O(3)$ model is given in complex notation by $g(w) = 1/(1 + |w|^2)^2$, and the action is

$$S_\sigma = \int d^2z g(|w|)[\partial w \bar{\partial} \bar{w} + \text{h.c.}]$$

(57)

This model has well-known instanton solutions:

$$w(z) = b\frac{(z - c_1)\ldots(z - c_n)}{(z - z_1)\ldots(z - z_N)}$$

(58)
In order for the separation between an instanton and an anti-instanton to be defined, one must break conformal invariance, specifically by adding to the action a term of the form

\[ S_1 = \frac{2m^2}{g^2} \int d^2z \left| w \right|^2 \frac{1}{1 + \left| w \right|^2} \]  

(59)

When evaluated in the one (anti-)instanton sector, this term yields

\[ S_I(I) = \frac{2\pi}{g^2} m^2 \left( \rho^2_I(I) \right) \]  

(60)

and so serves as an infrared cutoff for large-size instantons in the partition function of the theory. For our purposes, the importance of this term is that it allows for static finite-energy solutions that resemble the sphalerons of four-dimensional gauge theories. Such solutions describe the passage between topologically-inequivalent sectors without tunnelling at centre-of-mass energies comparable to the sphaleron mass, \( E_s = \frac{2\pi m^2}{g^2} \), i.e., the height of the barrier between different vacua. The sphaleron is the highest-energy point on the valley trajectory which we now discuss.

The construction \[37, 24\] of the valley trajectory is simplified by observing that if one restricts one’s attention to radially-symmetric configurations of the field \( w = f(r)e^{-in\theta} \ (n \in Z^+) \), one can map the action (57) onto that of the sine-Gordon model by setting \( r = e^{y/n}, \ f(r) = tan(\psi/4) \):

\[ S_0 = \frac{n\pi}{2g^2} \int dy \left[ \frac{1}{2} (\psi')^2 + (1 - \cos \psi) \right] : \ (\ldots)' = \frac{\partial}{\partial y} (\ldots) \]  

(61)

This transformation maps the instanton solution of the \( O(3) \) \( \sigma \)-model into the kink solution of the sine-Gordon model:

\[ \psi_K(y; y_0) = 4tan^{-1}exp(y - y_0) \]  

(62)

The sine-Gordon model has a kink-antikink valley with homotopic valley parameter \( \mu \) that obeys the equation

\[ \frac{\partial S_0}{\partial \psi} \big|_{\psi = \psi_v} = W_\psi(y, \mu) \frac{\partial \psi_v}{\partial \mu} \]  

(63)

where \( W_\psi(y, \mu) \) is a weight function, with the following properties \[23\]: it is positive definite and decays rapidly at large \( y \). Thus one recovers at large \( y \) and/or \( \mu \) the asymptotic solution

\[ \psi_v(y; \mu) \rightarrow \psi_K(y, -y_0) + \psi_K(y; y_0) \]  

(64)

Note that the weight function \( W_\psi(y, \mu) \) is not fixed uniquely by the above properties, and the valley is an approximate classical solution that interpolates smoothly between a soliton-antisoliton pair at large separation and the trivial vacuum.
A general method for constructing valleys in two-dimensional models has been suggested in ref. [24]. It identifies the valley parameter \( \mu \) with time, and considers the valley as a kink-antikink scattering problem. The reliability of this identification was conjectured in ref. [24] to be a general property of theories whose instantons become solitons in one dimension higher. Following ref. [24] let \( \psi_* \) denote a soliton-antisoliton scattering solution, obeying by construction the time-dependent equation

\[
- \frac{\partial^2 \psi_*}{\partial \mu^2} = \frac{\delta S_0[\psi_*]}{\delta \psi_*}
\]

We see immediately that if we choose as weight function

\[
W_\psi(y, \mu) = -\frac{\partial^2 \psi_*}{\partial \mu^2} \frac{\partial \psi_*}{\partial \mu}
\]

equation (63) becomes identical to equation (65). In the case of the \( O(3) \) \( \sigma \)-model, the weight function (66) is positive definite and decays exponentially as \( |y| \) tends to infinity. The boundary condition (64) is automatically satisfied by the scattering solution \( \psi_* \), which is therefore an exact valley solution of the \( O(3) \) theory. The introduction of the conformal symmetry breaking term (59) complicates the problem, but we follow ref. [23] in assuming that it does not change significantly the valley configuration, but only has the effect of introducing an infrared cutoff in the valley action when either the instanton or the anti-instanton becomes large. In the case of \( O(3) \) \( \sigma \)-model for \( 2\pi m << g^2 \) the effect of this term on the valley path-integral amounts [37] to introducing a harmless pre-factor \( e^{-\delta_1} \) of order one.

**Reduction of the SL(2, R)/U(1) coset model**

The above procedure can be applied to the \( SL(2, R)/U(1) \) model [18] of interest to us, but with certain differences which we shall mention as we proceed. We start from the action (7), concentrate on radially-symmetric configurations \( w = f(r)e^{-i\theta} \), and restrict our attention to the one-instanton sector. Setting \( y = \ln r \), the action (7) becomes

\[
S_{\text{red}} = \int_{-\infty}^{+\infty} dy \left[ \frac{1}{1 + f^2} [(f')^2 + f^2] \right]
\]

and redefining \( \phi = c + \sinh^{-1} f \) we get

\[
S_{\text{red}} = \int_{-\infty}^{+\infty} dy [(\phi')^2 + 1 - \frac{1}{\cosh^2 \phi}] \]

The corresponding equations of motion remain unchanged if the constant is subtracted from (68). Then, the effective action of the model becomes

\[
S_{\text{red}} = \int_{-\infty}^{+\infty} dy [(\phi')^2 - \frac{1}{\cosh^2 \phi}] \]
There are static solitons which are extrema of the action (69) of the form
\[ \phi(y, y_0) = \pm \sinh^{-1}(\sqrt{2}(y - y_0)) \] (70)
where the prefactors + (−) correspond to kink (antikink) solutions respectively, which are sketched in figure 4. It is easy to check that these solitons have finite energy, like the sphalerons:
\[ E = \int_{-\infty}^{+\infty} dy [(\phi')^2 + |V(\phi)|^2] = \frac{3\pi}{\sqrt{2}} \] (71)

It is interesting to notice that these solutions, when expressed in terms of \( \sigma \)-model variables, correspond to monopoles on the world-sheet, with unit charge [19], centred at the origin of the stereographically-projected plane, as becomes evident from the discussion in section 2. In the string case, monopoles have been interpreted [13] as the singularities of black holes in target space, with the charge \( q_m \) playing the rôle of the black hole mass in units of the Planck mass, whilst the corresponding antimonopoles required by the zero net “spikiness” of a compact world-sheet represent their horizons. Within this interpretation the kink-antikink solutions should be viewed as microscopic (Planck mass) black holes in the space-time foam.

Having identified the black holes with kink solutions in the reduced model (69), a natural question concerns the rôle of two-dimensional world-sheet instantons, which, as we have argued, describe massive string mode effects, and induce transitions between black holes of different mass. To answer this question it is necessary to consider time-dependent solutions of (69) by solving the equation
\[ S(\mu, y) = \frac{\partial^2 \phi}{\partial \mu^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{1}{2} \frac{\partial |V(\phi)|}{\partial \phi} = 0 \] (72)
where \( V(\phi) = -1/\cosh^2\phi \), \( \mu \) is a Minkowskian time variable, and \( y \) is a space variable. The full time-dependent equation for \( F = \sinh\phi(\mu, y) \) reads
\[ \frac{F}{1 + F^2}[-(\frac{\partial F}{\partial \mu})^2 + (\frac{\partial F}{\partial y})^2 + 1] = -\frac{\partial^2 F}{\partial \mu^2} + \frac{\partial^2 F}{\partial y^2} \] (73)
whose solutions can be split into two branches:

\[ -(\frac{\partial F}{\partial \mu})^2 + \frac{\partial^2 F}{\partial y^2} = 0 \]
\[ -(\frac{\partial F}{\partial \mu})^2 + (\frac{\partial F}{\partial y})^2 + 1 = 0 \] (74)
and:

\[ -(\frac{\partial F}{\partial \mu})^2 + \frac{\partial^2 F}{\partial y^2} = F \]
\[ -(\frac{\partial F}{\partial \mu})^2 + (\frac{\partial F}{\partial y})^2 = F^2 \] (75)
The branch of solutions (74) admits Lorentz-boosted soliton solutions

$$\phi(y, \mu) = \pm \sinh^{-1}\left[ \frac{y - y_0}{\sqrt{1 - u^2}} + \frac{u\mu}{\sqrt{1 - u^2}} \right]$$

(76)

where $u$ is the velocity of the soliton smaller than unity.

The other branch (75) resembles a massive Klein-Gordon equation and the corresponding condition for vanishing of the action. Solutions of this branch assume the form

$$F(\mu, y) = e^{\pm \frac{y_0}{\sqrt{1 - u^2}} \pm \frac{y - y_0}{\sqrt{1 - u^2}}}$$

(77)

Upon passing into the $\sigma$-model formalism, such solutions describe instantons, centered at the origin of the world-sheet, with scale sizes determined appropriately by $\mu$ and $y_0$. Notice that the static ($\mu$-independent) configurations (77), are not finite-energy solutions of the action (69), but have a logarithmic infrared singularity analogous to that of the topological charge (25).

Monopole-Antimonopole Valley in the reduced $SL(2, R)/U(1)$ model

We are now ready to demonstrate the existence of monopole-antimonopole and instanton-anti-instanton valleys in the $SL(2, R)/U(1)$ model. To this end, following the conjecture of ref. [24], we shall consider scattering solutions of the time-dependent equation (72), finding approximate solutions to the scattering of a kink and an antikink. To understand physically what is going on, we recall that a valley is a field configuration that interpolates smoothly between the trivial vacuum and a far separated soliton-antisoliton pair. In the prescription of [24], which we follow here, the ‘time’ $\mu$ in equation (72) corresponds to the separation of the soliton-anti-soliton, and thus to the valley separation parameter. In a conformally invariant theory, the concept of a finite-separation valley is meaningless. Thus conformal symmetry breaking terms must be introduced. The physics of the valley configuration is supposed to be independent of the particular form of the symmetry breaking term.

In the case of the $SL(2, R)/U(1)$ model there are two kinds of such symmetry breaking terms. One corresponds to classical symmetry-breaking: light-string-matter background deformations of the $\sigma$-model, which are by themselves relevant perturbations, causing a flow of the system towards a non trivial fixed point, and thus leading to the interpretation of the renormalization scale parameter as target time [4], as discussed in the next section. The form of such deformations is found by $W_\infty$-symmetry considerations in the target space of the $SL(2, R)/U(1)$ model; in fact one constructs marginal deformations out of $W_\infty$ currents, which assume the form (11):

$$L_0^1 T_0^1 \propto \Phi^{-c}_{\frac{1}{2}, 0, 0} + i(\psi^{++} - \psi^{-}) + \ldots$$

(78)

where the massive string modes $i(\psi^{++} - \psi^{-}) + \ldots$ are given in terms of $SL(2, R)$ currents [31]. The operator $\Phi^{-c}_{\frac{1}{2}, 0, 0}(r)$ is given in (13), and generates the light string
matter. As discussed in sec. 2, the deformation $\Phi^{c-c}_{c,0,0} (r)$ alone is relevant, leading to a breaking of the conformal invariance of the model. The truncation of the marginal deformation (78) to its relevant light-matter part has physical significance, reflecting the fact that local scattering experiments cannot observe the delocalized (solitonic) massive string modes.

When correlation functions of such light-matter deformations are evaluated in the presence of monopole-antimonopole configurations, the corresponding valleys in the $\sigma$-model theory correspond to saddle-point configurations in the path integrals for these correlators, leading in turn to non-factorizable contributions to the $S$-matrix for the light string modes. However, in such computations, the deformations appear as scattering terms in a theory whose action is conformally invariant. One needs additional conformal symmetry breaking terms to give meaning to finite-separation valleys. These are provided by dilaton terms, which constitute the second kind of conformal symmetry breaking terms. They appear at the quantum level and assume the form

$$S_{\text{dil}} = - \int d^2z \frac{1}{8\pi} \sqrt{h} R^{(2)} \ln(1 + |w|^2) + \ldots$$  \hspace{1cm} (79)$$

This term does not have any coefficient $k$, unlike the Wess-Zumino action term (7). Redefinition of the fields $r$ by a factor $\sqrt{k}$, so as to ensure a canonical kinetic term, leads to a rescaling of the conformal symmetry breaking term by a factor $\frac{1}{\sqrt{k}}$, reflecting its quantum nature[6]. For relatively large $r$, such terms become

$$S_{\text{dil}} = - \frac{1}{8\pi \sqrt{k}} \int d^2 z \sqrt{h} R^{(2)} r(z, \bar{z})$$  \hspace{1cm} (80)$$

Concentrating all the curvature at a single point $z^*$ we have, for surfaces with the topology of a sphere (Euler characteristic $\chi = 2$), $R^{(2)} = 8\pi \delta^{(2)}(z - z^*)$, and thus

$$S_{\text{dil}} = - \frac{1}{\sqrt{k}} r(z^*)$$  \hspace{1cm} (81)$$

Hence, in the large $k$ limit, the dilaton term becomes important for target space points at a distance $r >> 1$, far from the black-hole singularity at $r = 0$. For finite $k$ (as is the case of critical strings $c = 26$), the dilaton term becomes important in a region localized around the singularity $r = 0$.

The scattering solution may be found by first noticing that the function

$$\bar{\phi}(y, \mu) = \sinh^{-1}(\mu/cosh(y))$$  \hspace{1cm} (82)$$

\[\text{We remind the reader that in the stringy Wess-Zumino models (7), the level parameter } k \text{ is inversely proportional to the string Regge slope } \alpha' \text{. The later is a quantum-effect book-keeping parameter.}\]
is an exact solution of the equation (72), with a potential modified by a dilaton source-term (79). We can demonstrate this analytically by computing the left-hand-side $S(\mu, y)$ of eq. (72) for the function (82). The result is

$$S(\mu, y) = \frac{y}{\cosh^2\mu + \cosh^2y} = \frac{1}{\cosh^2y} \tanh[\phi(\mu, y)] =$$

$$= \frac{1}{\cosh^2y} \times \left| \frac{\delta}{\delta \phi(\mu, y)} \ln(\cosh(\phi(\mu, y))) \right|_{\phi = \bar{\phi}} \tag{83}$$

Thus, the function (82) is an exact solution of a σ-model whose potential energy has been modified by a conformal-symmetry breaking term of the form

$$-2 \int d\mu dy \frac{1}{\cosh^2y} \ln[\cosh(\phi(\mu, y))] \tag{84}$$

From figure 5, it is clear that for large $\mu$ the valley solution (82) yields a far-separated pair of a boosted kink and an antikink approaching each other with an ultra-relativistic velocity $u \simeq 1$.

When expressed in terms of the original σ-model variables this relation yields:

$$-2 \int d\mu |z|d|z| \frac{1}{(|z|^2 + 1)^2} \ln(1 + |w(z, \bar{z}; \mu)|^2) \tag{85}$$

One should now remember that we are effectively working on a world-sheer which has the topology of a sphere, and is projected stereographically onto the complex plane. Without loss of generality we can fix the geometry to be that of a sphere of radius $1/2$, and, hence, of curvature 8. In that case there is an induced $O(3)$ invariant measure on the complex plane defined by

$$d^2z \sqrt{|g(z, \bar{z})|} = \frac{1}{2(1 + |z|^2)^2} d^2z \tag{86}$$

thereby implying that (up to an overall normalization) the conformal symmetry breaking term (84) coincides with the the usual σ-model dilaton term (79) after a homotopic extension in $\mu$. The normalization factor in this case determines the value of $k$ corresponding to the solution (82), and a simple calculation yields $k = \frac{1}{4}$. To understand this value, one should note that (79) is only correct in the large $k$ approximation. If instead one assumes that the theory (4) together with (79) represents an exact conformal field theory for all $k$, then the corresponding central charge can be computed from the asymptotically linear dilaton term (79) far away from the singularity. The corresponding central charge is

$$c = 2 + 6/k \tag{87}$$

for large $k$, which agrees with the exact answer (88) within an error $1/k^2$. For $k = 1/4$ eq. (87) yields $c = 26$, i.e. the target-space interpretation remains. However, there
are, of course, corrections in higher orders in $\frac{1}{k}$ in the expression for the dilaton and graviton which we do not discuss here. We assume that the qualitative features of the valley remain the same for the exact black hole solution $k = 9/4$. As we shall see later, the results of importance for us are, in any case, independent of the details of the valley configuration.

To check whether (82) is a sensible valley configuration, we should verify that the corresponding weight function (66) is positive definite (for $\mu \geq 0$), and decays fast enough for large $y$. From (82),(66) we then have

$$W(\mu, y) = \frac{\mu \text{sech}^2(y)}{1 + \mu^2 \text{sech}^2(y)}$$

which is indeed positive definite and decays exponentially with large $y$. We also note as a curiosity that the valley function (88) coincides formally with the corresponding function for the $O(3)\sigma$-model [24]. It can easily be checked that, for highly separated monopole-antimonopole pairs, the solution (82), when substituted back to the Wess-Zumino action, yields the characteristic logarithmic infinities of an isolated monopole and an anti-monopole [19]. Also, for zero separation $\mu \to 0$, it induces the collapse into the trivial vacuum, since the corresponding action vanishes. When mapped back to the original $\sigma$-model variables, upon making the replacement $\mu \to v - \frac{1}{v}$, with $v$ the conventional $\sigma$-model separation parameter, the valley configuration yields a concentric valley,

$$w(z, \bar{z}) = \frac{(v - 1/v)\bar{z}}{1 + |z|^2}$$

The latter may then be mapped to the desired valley by applying a conformal transformation in the world-sheet, involving an appropriate inversion followed by a translation, in a similar spirit to the case of $O(3)\sigma$-model [24]. The function (89) is shown in figure 6.

**Instanton-Anti-instanton Valley in the reduced $SL(2, R)/U(1)$ model**

The method of the previous subsection for the construction of the $\sigma$-model valley using the soliton-anti-soliton scattering solution of the homotopically-extended ('time-dependent') reduced model (69), cannot be applied directly to the case of valley configurations for instanton-anti-instanton pairs in the reduced model of the form (77). The latter are delocalized classical solutions of (63), that do not have finite energy. This complicates technically the construction of a classical 'scattering' solution, because the definition of an asymptotic (freely-propagating) state in the infinite future needs elaboration.

However, this does not imply that a valley configuration cannot be constructed. Recalling that the latter is a smooth interpolating function between an infinitely-separated pair of topological defects and the trivial vacuum, we observe that all one
needs for the construction of a valley in the reduced-model formalism is the first half of a ‘scattering-like’ process, i.e., the part that extends from the infinite past, where the defects are infinitely separated, till the time their centres come close to each other. This last stage corresponds to the collapse of the valley into the trivial vacuum. Thus, as a trial valley for our case, we consider, following [23, 37], a simple superposition of a instanton and an anti-instanton (77),

\[
g(y, \mu) = \sinh^{-1}[\exp(y + \mu)] - \sinh^{-1}[\exp(y - \mu)]
\]

We observe that (90) can be derived from a monopole-antimonopole pair

\[
\sinh^{-1}(y + \mu) - \sinh^{-1}(y - \mu)
\]

by a conformal transformation in the \((y, \mu)\) plane :

\[
\begin{align*}
\mu + y & \rightarrow e^{y+\mu} \\
y - \mu & \rightarrow e^{y-\mu}
\end{align*}
\]

To be more accurate, recall that the scattering solution (82) represents to a very good approximation a monopole-antimonopole pair approaching each other at an ultra-relativistic velocity \(u \simeq 1\). This implies that the pertinent conformal transformation that connects (90) to the monopole-antimonopole valley solution (82) is

\[
\begin{align*}
\mu + y & \rightarrow e^{\sqrt{1-u^2}(y+\mu)} \\
y - \mu & \rightarrow e^{\sqrt{1-u^2}(y-\mu)}
\end{align*}
\]

It should be stressed that the transformation (93) is only \textit{approximate}. Due to the velocity \(u\), the actual transformation that connects the instanton valley with the monopole-antimonopole valley is \textit{not conformal}, since it involves Lorentz-boosted solutions. However, for \(u \simeq 1\) such non-conformal effects may be ignored, and hence one is effectively working with light-cone coordinates \(y \pm \mu\), which can be complexified in the Euclidean formalism to yield the usual \(z, \bar{z}\) variables of the complex \((y, \mu)\) plane (caution: this is not the original complex plane of the Wess-Zumino model).

One may express the time-dependent action that reproduces (72) in light-cone coordinates, and then apply the transformation (93). The kinetic term is conformally invariant, whilst the potential term changes as follows:

\[
(1 - u^2) \int \frac{dz'd\bar{z}'}{|z'|^2 \cosh^2 \phi(z', \bar{z}')} \frac{1}{\cosh^2 \phi(z', \bar{z}')}
\]

where we took into account the fact that the field \(\phi\) is a scalar under coordinate transformations. Due to the \((1 - u^2)\) factor, the potential term becomes important only for small \(|z| \simeq O(1 - u^2)\), where it has a similar form to the original potential term, yielding (72). This is the region where the original potential term is dominant.
for intermediate and/or large separations $y$. Thus, we can argue that the instanton valley may be derived, to a good approximation, by applying the transformation (93) to the exact monopole-antimonopole valley. Taking into account the above remarks on the approximate invariance of the valley action under conformal transformations, we observe that the weight function for the instanton valley is induced by the conformal transform of the monopole weight function (88), thereby maintaining all the necessary properties. We therefore consider the following representation of the instanton valley, expressed in terms of the original $\sigma$-model variables

$$w(z, \bar{z})_{\text{inst}} \simeq \frac{1}{\cosh(\mu \sqrt{1-u^2})} \sinh(\frac{\mu}{\sqrt{1-u^2}})$$

(95)

The function (95) is represented in figure 7. That this function corresponds to a valley is guaranteed by the general property that a conformal transformation of a valley is also a valley configuration [37, 24]. As in the monopole-antimonopole valley case, the conformal symmetry breaking term in the action that is necessary to give a meaning to the finite-separation valleys is provided by the dilaton term (29).

**Contributions to Correlation Functions**

Having constructed approximate valley configurations in the $SL(2, R)/U(1)$ model, we now discuss the physical consequences for the correlation functions of the matter operators (45) $\Phi_{\text{inst}}^{-1,0}(r)$. From a world-sheet point of view, such correlators are nothing but complicated algebraic functions of $\sigma$-model fields $r(z, \bar{z})$. The above-described valley configurations are, then, viewed as dominant configurations in a path-integral evaluation of the forward scattering amplitude on the world-sheet, which by means of the optical theorem is related to the total cross section for the fields $r(z, \bar{z})$. For completeness, we mention that the conformal symmetry breaking terms in this picture allow for non-vanishing inelastic scattering amplitudes on the world-sheet.

Although from a world-sheet point of view one can define perfect quantum-mechanical scattering in the presence of valley configurations, and apply the optical theorem as a consequence of world-sheet unitarity, this is not the case in target space. As a result of the valleys, there are obstructions to the interpretation of the tachyon correlators as generating functionals for target-space scattering amplitudes. In the case of black-hole $SL(2, R)/U(1)$ string models, the spherically-symmetric four-dimensional target space is obtained [13] as a topologically non-trivial homomorphism of the world-sheet onto a two-dimensional target subspace. Under such a mapping, the non-separable contributions of the valleys to the world-sheet scattering matrix result - as we shall discuss in sec. 6 - in non-analytic contributions to the target-space $S$-matrix, thereby leading to the $S$-formalism. This result has been anticipated in [4] using a pure $\beta$-function approach. Here we rederive it using
an alternative - and physically appealing - point of view exploiting the topologically non-trivial structure of the world-sheet of a string propagating in target-space black-hole backgrounds.

5 Renormalization-Group Scale as Target Time

We review in this section our interpretation of the renormalization group scale in the subsystem of the light string modes as target time. Most of the material is contained in ref. [4], and here we give only the general idea and technical details that are needed for the development of the material presented in this article.

We use the concept of the local (on the world-sheet) renormalization group equation, which was originally introduced as a formal tool for analyzing $\sigma$-models [38]. It was used to prove the “off-shell” corollary of the c-theorem [39], according to which perturbative $\beta$-functions are gradient flows of the string effective action in target space, but no physical significance was attached to the local dependence of the cutoff. However, we identify the local cutoff as the Liouville mode, whose kinetic term has a temporal signature for supercritical strings ($c > 26$) [25]. This was the first paper where cosmological time was introduced into string theory by exploiting the flow of the subsystem pertaining to the extra compact dimensions under the renormalization group. Subsequently, this formalism was interpreted in terms of Liouville theory [40], but there was not any serious attempt, at the time, to associate the target time with a local renormalization group scale on the world-sheet in the usual $\sigma$-model sense, and exploit the consequences of such a formalism. As we have already pointed out, the inclusion of topological fluctuations (such as monopole-antimonopole or instanton-anti-instanton pairs) in a critical string theory (such as the $SL(2,R)/U(1)$ coset model with $k = 9/4$) makes it supercritical, necessitating the introduction of such a time-like Liouville field, whose interpretation as target time we now explain.

Consider a general fixed-point world-sheet theory, such as the critical black hole model [4]. Upon deforming the theory by light matter background fields, one gets formally for the $\sigma$-model action

$$S_{\text{def}} = \int d^2x (L_0 + \Phi(x) R^{(2)}(x) - \Sigma(x) + g^i(x) V_i)$$  \hspace{1cm} (96)$$

where the renormalized couplings $g^i(x)$ depend on the world-sheet coordinates $x$ through the local renormalization group scale $-lna(z, \bar{z})$ to be identified as the Liouville mode $\phi(x)$. The corresponding normal-ordered vertex operators $V_i(x)$ are also dressed by the Liouville field cut-off, so as to be $(1,1)$ operators at the fixed point. Simple power counting arguments show that the extra ‘dilaton’ and ‘tachyon’

\footnote{The fact that this classical background configuration depends only on target space is already suggestive of a time interpretation of the renormalization group cutoff.}
counterterms $\Phi(x)$, $\Sigma(x)$ in (96) are necessary when renormalizing in curved space-time, although they vanish in any global renormalization group scheme. The explicit form of the ‘tachyon’ counterterm in, for example, dimensional regularization is

$$\Sigma_{\text{Bare}} = a'[\Sigma + L_\Sigma(g^i)]$$

(97)

where $a$ is the ultraviolet cut-off introduced in section 2, $\epsilon$ is an anomalous dimension, and

$$L_\Sigma = \frac{1}{2} G_{ij} \partial_\alpha g^i \partial^\alpha g^j$$

(98)

and, as already mentioned, the $g^i$ depend on the local renormalization group scale (Liouville mode) $\phi(x)$. It can be shown [39] that $G_{ij}$ is related to divergences of the two-point function $<V_i V_j>$, but it differs from the Zamolodchikov metric, so that the positivity for unitary theories is not immediate: however, this does not concern us here. The $\beta$-functions corresponding to $\Sigma(x)$, $g^i$ are defined by

$$\hat{\beta}_\Sigma \equiv -d\Sigma/d\ln(a) = \epsilon \Sigma + \beta_\Sigma(g)$$

$$\hat{\beta}_i \equiv -dg^i/d\ln(a) = \epsilon^i + \beta^i(g)$$

(99)

where

$$\beta_\Sigma = \frac{1}{2} \chi_{ij} \partial_\alpha g^i \partial^\alpha g^j = \frac{1}{2} \beta^i \chi_{ij} \beta^j \partial_\alpha \phi \partial_\beta \phi$$

(100)

The renormalizability of the model (96) now specifies $\chi_{ij}$ in terms of $G_{ij}$. Imposing the invariance of (96) under the renormalization group operator

$$(\epsilon - \hat{\beta}^i \partial_i - \hat{\beta}^\lambda \partial_\lambda) S = 0 \quad : \quad \lambda \equiv (\Phi, \Sigma)$$

(101)

we find

$$\chi_{ij} = (\epsilon - \hat{\beta}^k \partial_k) G_{ij} - (\partial_i \hat{\beta}^k G_{kj} - (i \leftrightarrow j))$$

(102)

In the limit $\epsilon \to 0$, the ‘tachyon’ $\Sigma(x)$ contributes the following terms to the $\sigma$-model action:

$$- G^{(1)}_{ij} \beta^j (g^i + \beta^i \phi) \partial_\alpha \phi \partial^\alpha \phi \equiv G_{00} \partial_\alpha X^0 \partial^\alpha X^0$$

(103)

where $G^{(1)}_{ij}$ is the residue of the simple $\epsilon$-pole. We now see that $G_{00}$ may be viewed as a perturbation in the temporal component of the background metric in the two-dimensional black-hole string model, and denote the Liouville mode $\phi(x)$ henceforward as the target time $X^0(x)$. Notice that in this picture the Liouville field acquires non-trivial corrections to its kinetic term as compared to eq. (21). We see clearly that the temporal (cutoff) dependence of the metric tensor is a back reaction due to matter emission. Indeed, if the matter couplings $g^i$ were marginal, then the $\beta^i$ would vanish, as would the corrections to the background metric.

---

8In covariant cut-off approaches this should be replaced by the leading logarithmic divergences, but the qualitative results remain the same.
At this stage it is useful to compare the above picture to that of ref. [25]. Contrary to that reference, in our approach the target time (local renormalization group scale) appears already in the fixed-point action (5). At first sight this seems strange from a renormalization group point of view. Indeed, even in the case of a local scale \( \phi(z, \bar{z}) \), the corresponding renormalization group equation for the effective action \( Z \) reads [38]:

\[
\frac{\partial}{\partial \phi(z, \bar{z})} + \beta^i(g_i(\phi)) \frac{\partial}{\partial g^i(\phi)} Z = 0
\]

In our interpretation of target time as a renormalization group scale, this equation should be considered as a constraint on the physical states. In the Liouville theory framework, which we follow here, the constraint (104) is incorporated automatically as a result of the conformal dressing of the various \( \sigma \)-model operators [41]. In the conformal Wess-Zumino theory, the constraint (104) is satisfied as a result of the world-sheet equations of motion of the \( \theta \) field, which in our language is the Liouville mode/time \( X^0 \). This is the crucial formal difference of our approach from that of ref. [25, 41]. The Liouville field at the fixed point must satisfy the equation of motion of a free \( \sigma \)-model field. Under this condition, any apparent time-dependence in the action (5) disappears at the level of the background fields, which are static, and hence their local renormalization group \( \beta \)-functions vanish. It should be stressed at this point, that these \( \beta \)-functions are not the \( \beta \)-functions leading to the black-hole solution of [18]. From that point of view the Liouville mode is the spatial coordinate, because in that picture the string is subcritical with \( c = 1 \) [18].

The arrow of target time can now be found by analyzing the correlation functions \( A_N = \langle V_{i_1} \ldots V_{i_N} \rangle \) of matter vertex operators \( V_i \). We use two representations of the same physics. One describes the Euclidean black hole without matter deformations as a subcritical \( c = 1 \) string model on a flat target-space-time with a modified cosmological constant term [14]

\[
2\pi \nu \beta \bar{\beta} \exp\left(-\frac{2}{\alpha_+} \zeta\right)
\]

(105)

Here, \( \beta \) and \( \bar{\beta} \) are chiral ghost fields and \( \zeta \) is a free field entering the representation of the \( SL(2, R)/U(1) \) current algebra, \( \nu \) is the black-hole mass \( M_{bh} \), and \( \alpha_+ = 2k - 4 \) is related to the level parameter \( k \) by the requirement that the conformal dimension of the modified area term [105] be unity. Because the matter field describing target time is compact in this picture, it represents a Euclidean black hole. The second representation, applicable to a Minkowski black hole, is to use marginal deformations of the coset model that respect the \( W_\infty \) coherence-preserving symmetry. As mentioned earlier, truncating the theory to the light propagating modes corresponds to including the instantons that represent massive modes as discussed in section 2, which make the conformal theory supercritical with \( c > 26 \), resulting in a time-like

\( ^9 \) The precise functional relation between \( \theta \) and \( X^0 \) [12, 13] is not directly relevant for our purpose here.
local cutoff. This latter representation is the more relevant for our purpose here, but it is instructive to compare the two formalisms.

In the $c = 1$ Euclidean picture vertex operators for the simplest $(1,1) \ SL(2,R)$ primary states take the form

$$V_{jm}^T \propto \gamma^j - m \exp \left( \frac{2j}{\alpha_+} \zeta \right) \exp \left( im \sqrt{2/k} X \right)$$

(106)

where the boson $X$ is compact with radius $\sqrt{k}$ and represents Euclidean time, as mentioned above, $\gamma$ is another chiral boson in the free-field realization of the $SL(2,R)$ current algebra, and $(j,m)$ is the $SL(2,R)$ isospin and its third component, which are subject to the on-shell condition

$$-j(j+1) - \frac{m^2}{k} = 0$$

(107)

The $W_\infty$ symmetry algebra tells us that the deformation (106) is exactly marginal only in the infinite-mass limit for the heavy string modes. The integration over the zero mode of $\zeta$ in the path integral for a correlation function $A_N$ may be performed by inserting the formal identity

$$\int dA \delta \left( \int d^2z |\beta|^2 e^{-\frac{A}{\alpha_+} \zeta} - A \right) = 1$$

(108)

resulting in the representation

$$A_N = \nu^s \Gamma(-s) \times \langle V_{j_1m_1} \cdots V_{j_Nm_N} | \int d^2z |\beta|^2 e^{-\frac{2}{\alpha_+} \zeta(z)} \rangle_{\nu=0}$$

(109)

where

$$\Gamma(X) = \int_0^\infty dAA^{-s-1}e^{-A}$$

(110)

and $\nu$ is the coupling constant of the modified area term (105) which is related to the black-hole mass. The symbol $\langle \ldots \rangle_{\nu=0}$ denotes the integration over $X$ and the world-sheet dependent non-zero modes of $\zeta$, in the limit of vanishing mass for the black hole.

Because of the $\Gamma(-s)$ factor in (109), the unregularized correlation functions $A_N$ are ill-defined whenever a discrete massive string state is excited. In this case, $s$ is a positive integer $n^+$ and one can regularize the $\Gamma(-n^+)$ pole by analytic continuation, replacing the integral representation (110) of $\Gamma(-s)$ by

$$I(s)_{reg} \equiv \int_{C} A^{-s-1}exp(-A)dA$$

(111)
where the contour $C$ is shown in figure 8. This introduces imaginary parts in the correlation functions:

$$I(n^+) = \lim_{\epsilon \to 0} [\exp(-2\pi i \epsilon) - 1] \Gamma(-n^+ - \epsilon) = (-1)^{n^+} \frac{2\pi i}{(n^+)!}$$  \hspace{1cm} (112)

which can be interpreted as instabilities reflecting the renormalization group flow of the theory. The parameter $A$ appearing in (111) is naturally interpreted as the effective area of the world-sheet as measured in target space. This follows from the way in which (108) is related to the free field $\zeta$ and the ghost fields $\beta$ and $\bar{\beta}$, which is the same as in the cosmological constant term (103) in the world-sheet action. However, although the $\phi$ field of the subcritical $c = 1$ model is analogous to the Liouville mode discussed earlier, it is space-like rather than time-like, and we need to extend the above discussion to the supercritical Minkowski black hole representation.

In this case, as already discussed, a time-like Liouville field can be introduced to restore criticality in the sense that the total central charge $c = 26$, and one finds a ‘tachyon’ area-like Liouville term similar to (105), with $\nu$ having the traditional interpretation as a cosmological constant on the world-sheet. The standard analysis [41] of Liouville theory correlation functions then applies, leading again to expressions of the form (109), where the parameter $A$ in the representation of the $\Gamma$-function regularized à la (111) now admits a world-sheet area interpretation. The imaginary parts (112) of the correlation functions can now be interpreted directly as instabilities of the string vacuum. In the supercritical string, deformed by light matter and truncated to its light subsystem, this is just the usual flow of the latter from one fixed point towards another with lower central charge $c$, in accordance with Zamolodchikov’s $c$-theorem [17] for any unitary theory.

It is evident from the form of the contour integral (111) illustrated in figure 8 that there are two phases in the renormalization group flow. It starts from an infrared fixed point with large world-sheet area, corresponding to a large negative value of the Liouville field, passes through an ultraviolet fixed point with small world-sheet area, corresponding to a large positive value of the Liouville field, and returns to the infrared fixed point. In this picture, it is natural to regard the ultraviolet fixed point as the end-point of the actual time flow in target space-time. This interpretation is suggested by the identification made in the previous paragraph of the renormalization group flow with the decay of a metastable vacuum. Our interpretation is reinforced by the analogy with the divergences in the path integral representation of ‘bounce’ solutions in ordinary field theory [46]. For the convenience of the reader we give here the formula for the decay probability per unit volume $\Gamma/V$ of a false vacuum state $\phi_f$ in the conventional field theory case

$$\frac{\Gamma}{V} \propto \frac{B^2}{\hbar} \exp\left(-\frac{B}{\hbar} - S^{(1)}(\phi_B) + S^{(1)}(\phi_f) + \ldots\right) \times$$

\footnote{This is an explicit demonstration of a generic conjecture made in ref. [45].}
\[
\begin{align*}
\left\{ \frac{\text{det}[-\nabla^2 + U''(\phi_B)]}{\text{det}[-\nabla^2 + U''(\phi_f)]} \right\}^{\frac{1}{2}}(1 + O[\hbar])
\end{align*}
\]

where the prime in the determinant denotes omission of zero-modes, \( \nabla^2 \equiv \partial_\mu \partial^\mu \), \( U(\phi) \) is the field-theory potential energy, \( B = S_R(\phi_B) \), \( S_R \) is the renormalized effective action, and \( S^{(1)} \) denotes first-order renormalization counterterms with the \( \ldots \) denoting higher-loop corrections. The bounce \( \phi_B \) is a stationary point of \( S_R(\phi) \), i.e.

\[\partial S_R/\partial \phi_B = 0\]

In our Liouville-theory analogue, upon the identification of the renormalization group flow with the decay of metastable vacua in target space, the condition (114) is nothing other than the renormalization group equation (104) of the \( \sigma \) model action, which follows from the renormalizability of the model in two dimensions [47].

The ultraviolet divergence arising in the integration over small areas in (111) leads to the above flow of target time. As is shown in ref. [14] this flow is characterized by energy conservation [14] and a monotonic increase in entropy [1, 46], which is particularly rapid at early times when the volume of the Universe expands exponentially. Thus this framework provides an alternative to conventional field-theoretical inflation that does not rely on an inflaton field to generate the initial entropy of the Universe.

The flow towards the ultraviolet fixed point implies that the apparent size of the string world-sheet, as measured in target space, becomes smaller as time increases. We argue in the next section that this suppresses off-diagonal interference terms in the configuration-space representation of the density matrix for observable light states.

6 Contributions to the \( S \) Matrix

We now demonstrate that the valley configurations discussed in section 4 make non-trivial contributions to the \( S \) matrix, i.e. contributions that cannot be factorized as a product of \( S \) and \( S^\dagger \) matrix elements. We will see explicitly that these contributions diverge logarithmically in the renormalization scale parameter, i.e. in the time variable, as discussed in section 5. Since the string world-sheet shrinks in target space as time progresses, these contributions have the effect of suppressing off-diagonal elements in the configuration-space representation of the density matrix, removing the quantum-mechanical interference between systems in different locations.

String Derivation of the \( S \) matrix
Before deriving the contributions of the monopole-antimonopole and instanton-anti-instanton valleys to the $S$ matrix, we first discuss light particle (‘tachyon’) scattering in the absence of topological fluctuations in the space-time background, showing how the $S$-matrix of conventional quantum field theory and the Hamiltonian evolution of conventional quantum mechanics emerge within our treatment of target time as a renormalization scale parameter.

We consider the model (5) perturbed by a ‘tachyon’ operator $T(X^\mu): X^\mu = (r, \theta)$ in a region of target space far from the black hole singularity. Since the target space is almost flat in this region, we can Fourier transform the ‘tachyon’ perturbation as in equation (53). The relevant $\sigma$-model action is

$$S = \frac{1}{4\pi\alpha'} \int d^2z \partial^\mu X^\mu \partial_\alpha X^\nu \eta_{\mu\nu} + \int d^2z \frac{1}{\alpha^2} T(k) e^{ik^\mu X_\mu}$$ (115)

where we have exhibited explicitly an ultraviolet world-sheet cut-off $\alpha$ which appears for dimensional reasons. We can split the field $X^\mu$ into a zero-mode part $x^\mu$ and a quantum part $\xi^\mu$:

$$X(z)^\mu = x^\mu + \xi^\mu (z)$$ (116)

and the consequent integration over $X^\mu$ in the path-integral yields energy-momentum conservation for the ‘tachyon’ perturbations\footnote{At this point one should note that in subcritical string theory ($c < 1$) the “energies” $k^0$ are purely imaginary (Liouville energies), and so the zero-mode field integration does not imply energy conservation. However, in such a case one can still define an energy-conserving string theory by restricting oneself in the residues of the poles of the respective amplitudes\footnote{[48]}. In our case, we are effectively working with a supercritical string ($c > 26$), where the Liouville mode has a temporal signature in its kinetic term, and hence energy-momentum conservation is a consequence of the zero-mode field integration.}. Expanding (115) in powers of $T(k)$, one obtains

$$Z \propto \int D\xi e^{-\int d^2z \partial^\mu \xi^\nu \partial_\alpha \xi^\sigma \eta_{\mu\nu}} \sum_n \frac{1}{n!} \int d^2z_1 \ldots d^2z_n \left( \frac{1}{\alpha^2 n} \right) \int \ldots$$

$$\ldots \int d^Dk_1 \ldots d^Dk_n \delta^{(D)} \left( \sum_i k_i \right) e^{ik^\mu \xi(z_1)} \ldots e^{ik^\mu \xi(z_n)} T(k_1) \ldots T(k_n)$$ (117)

This expression depends on the world-sheet cut-off, because of ultraviolet infinities. We can exhibit this by examining the first power of $T(k)$ in the expansion of (117). Denoting $<\ldots> \equiv \int D\xi e^{-\int \partial^\mu \xi^\nu \partial_\alpha \xi^\mu \xi^\nu} (\ldots)$ and ignoring the zero mode, we obtain

$$Z^{(1)} \propto \int d^Dk \frac{1}{\alpha^2} T(k) \int d^2z \left< e^{ik^\mu \xi(z)} \right> = \int d^2z \frac{1}{\alpha^2} \int k T(k) e^{-\frac{1}{2}k^\mu \alpha \xi(z)} \left< \xi^\mu(z) \xi^\nu(z') \right> = \int d^2z \frac{1}{\alpha^2} \int_k T(k) a \frac{k^2}{2}$$ (118)

where we have cut the flat-space propagator as follows:

$$\left< \xi^\mu(z) \xi^\nu(0) \right> = \eta^{\mu\nu} \log|z + a|^2$$ (119)
It is evident that the explicit $a$-dependence in (118) can be absorbed in a renormalized ‘tachyon’ polarization ‘tensor’ \[49\]

\[ T_R(k) = a \frac{k^2}{2} T(k) \]  

at the lowest order in perturbation theory. In this way we recover the leading anomalous dimension term $\lambda$ in the perturbative renormalization group approach to string theory. The generic structure of the latter, in a theory with a scale $t = -\ln a$, and renormalized couplings $g^i(t)$ is \[49\],

\[ \beta^i = \frac{dg^i}{dt} = \lambda^i g^i + a^j_{jk} g^j g^k + \gamma^{ijkl}_{ijkl} g^j g^k g^l g^i \]  

where the couplings $g(t)$ are related to the bare ones $g(0)$ by:

\[ g^i(t) = e^{\lambda^i t} g^i(0) + e^{(\lambda_j + \lambda_k) t} \left[ \frac{a^j_{jk}}{\lambda_j + \lambda_k - \lambda_i} g^j(0) g^k(0) + b^i_{ijkl}(t) g^j(0) g^k(0) g^l(0) + \ldots \right] \] \[122\]

with

\[ b^i_{ijkl}(0) g^j(0) g^k(0) g^l(0) = e^{\lambda^i t} \left( \frac{2a^i_{jm} a^m_{kl}}{\lambda_j + \lambda_m + \lambda_i} - \gamma^{ijkl}_{ijkl} \right) \frac{1}{\lambda_j + \lambda_k + \lambda_l - \lambda_i} + \right. \]

\[ \left. \frac{2a^i_{jm} a^m_{kl}}{\lambda_k + \lambda_l - \lambda_m + \gamma^{ijkl}_{ijkl}} e^{(\lambda_j + \lambda_k + \lambda_l) t} \frac{1}{\lambda_j + \lambda_k + \lambda_l - \lambda_i} \right] \] \[123\]

We recall that the lower indices in the coefficients $a^i_{jk}, b^i_{ijkl}, \ldots$ of the $\beta$-function \[121\] are trivially symmetrized, whilst symmetrizing the upper and lower indices requires the Zamolodchikov ‘metric’ in coupling constant space \[17\],

\[ G_{ij} = 2|z|^{4-\lambda_i-\lambda_j} < [V_i(z)] [V_i(0)] > g \] \[124\]

where the $[V_i]$ are normal-ordered perturbations corresponding to the couplings $g^i$, and the vev is taken with respect to the perturbed theory.

Due to the composite operators existing in \[124\] there are extra infinities in the limit $z \to 0$ that cannot be taken care by the normal ordering of the operators $V_i$ \[50\]. To see this, consider the metric \[124\] expressed in terms of bare (in a renormalization group sense) quantities. For the sake of formal convenience, we shall work in the (abstract) Wilson scheme where the exact $\beta$-function is quadratic in the couplings\[\overline{\beta}\]

\[ \beta^i_{\text{Wilson}} = \lambda^i g^i + A^i_{jk} g^j g^k \] \[125\]

\[12\] The $\epsilon$-poles in the metric \[124\] can also be seen in the practical dimensional regularization scheme as discussed in ref. \[50\], to which we refer the interested reader for details. Here we choose the Wilson approach, which is a coupling constant expansion, since this is more relevant for our purposes.
To connect bare and renormalized quantities in this scheme we use the concept of scaling fields \( G^i \) defined by
\[
dG^i / dt = \lambda^i G^i
\] (126)
which are related to the \( g^i \) via:
\[
g^i = G^i + \frac{1}{2} B^i_{jk} G^j G^k + O(G^3)
\] (127)
with
\[
B^i_{jk} = A^i_{jk}(\lambda_j + \lambda_k - \lambda_i)^{-1}
\] (128)
For our purposes, an important relation is
\[
\int d^2 z [V^i] = \int d^2 z (V^B B^i_{jk} g^j V^B_j + O(g^2))
\] (129)
where the superscript ‘B’ denotes bare quantities. Standard scaling arguments can be used to express the two-point function of two bare vertex operators as
\[
< V^B_i(r) V^B_j(0) >_0 = G_{ij} |z|^{\lambda_i + \lambda_j - 4}
\] (130)
where the subscript ‘0’ in the vev denotes path-integral average with respect to the fixed point action. It can be shown that the following relation is true:
\[
G_{im} A^m_{jk} = -2\pi C_{ijk}
\] (131)
where the \( C_{ijk} \) are the operator product expansion structure constants, which are totally symmetric in their indices. These are universal in a renormalization group sense. The covariant \( \beta \)-function
\[
\beta_i \equiv G_{ij} \beta^j
\] (132)
has, therefore, coefficients that are totally symmetric in their indices, and the renormalization group can be represented as a gradient flow. The corresponding flow function is the effective action in target space, which generates ‘tachyon’ scattering amplitudes.

In conventional treatments the metric (124) is defined at a fixed point on the complex plane, \(|z| = 1\). In our Liouville formalism, this no longer makes sense, given that the stereographic projection \( \Delta R(z_1, z_2) \) on the complex plane of the distance between two points on the world-sheet surface changes as the area of the

\footnote{For the case of two-dimensional \( \sigma \)-models this has also been proven rigorously in dimensional regularization, using composite operator renormalization techniques by means of a local renormalization-group scale on the world-sheet.}
surface shrinks along the direction of the target time flow (which is opposite to the renormalization group flow). Indeed $\Delta R(z_1, z_2)$ is given by

$$\Delta R(z_1, z_2) = \frac{\pi}{2} \frac{|z_1 - z_2|}{(1 + |z_1|^2/4R^2)^{\frac{3}{2}}(1 + |z_2|^2/4R^2)^{\frac{3}{2}}}$$ (133)

Here $R$ is the radius of the world-sheet, which is assumed to have spherical topology. It is related to the Liouville mode $\phi(z_1, z_2)$ as follows: $\int_{\Sigma} \sqrt{\hat{g}} e^{-2\phi} = 4\pi R^2$, where $\hat{g}$ is a fiducial metric on the world-sheet, which is assumed to be fixed [41]. Close to the ultraviolet fixed point $\phi \to \infty$ and hence $R, \Delta R$ in (133) vanish. In that case the metric (124) can be shown [34] to exhibit poles in the anomalous dimensions of almost-marginal operators\(^{14}\). To see this, we recall from ref. [34] the expression for $\partial_i G_{jk}|_{g^i=0,|z|}$:

$$\partial_i G_{jk}|_{g^i=0,|z|} = \frac{4\pi}{\epsilon} C_{ijk}(1 - |z|^{2+\epsilon})$$ (134)

This expression, when evaluated in standard treatments at the point $|z| = 1$, yields a vanishing coefficient for the first-order term in a coupling-constant power expansion for the metric. This corresponds to a scheme choice, and is the analogue of the diffeomorphism invariance in metric theories, which allows the choice of a frame where the connexion vanishes. In our Liouville framework, the distance $|z|$ is replaced essentially by the stereographic projection (133), which is vanishing close to the ultraviolet fixed point, thereby leading to an explicit example of a simple $\epsilon$-pole in the 'metric' in coupling constant space. It should be understood that these extra infinities correspond to extra divergences when the arguments of the vertex operators in (124) come close to each other. In physical expressions one should take the finite parts of the pertinent expressions involving $G_{ij}$. This is always done in the context of this work, and will not be explicitly stated.

We now discuss the relevance of the string conformal invariance conditions to the construction of the scattering amplitudes in target space for the light modes. To this end, we make a perturbative expansion of the solution $g^i = g_0^i + g_1^i + g_2^i + \ldots$ to the conformal invariance condition $\beta^i = 0$. Vanishing of the $\beta$-function at leading order yields the mass-shell condition $\lambda_i g_0^i = 0$. Vanishing at first order yields

$$g_1 = -\frac{1}{\lambda_i} a^i_{jk} g_0^j g_0^k$$ (135)

which reproduces the three-tachyon amplitude, whilst the next-order vanishing condition reproduces the four-point amplitude,

$$g_2 = \frac{1}{\lambda_i} \left( \frac{2a^i_{jm} a^m_{kl}}{\lambda_m} - \gamma^i_{jkl} \right) g_0^j g_0^k$$

\(^{14}\)Such small anomalous dimensions $\lambda_i \simeq \epsilon << 1$ serve as a regulator for the theory, like an $\epsilon$ expansion. The same of course is true in dimensional regularization [41] where the rôle of the anomalous dimension is played by the $\epsilon = d - 2$, where $d$ is the dimensionality of the world-sheet. We do not have a precise estimate of $\epsilon$ in our case, but note that the $\epsilon$-expansion give successful qualitative results even when $\epsilon$ is not small.
\[
\frac{1}{\lambda_i} \left( \frac{2a^i_{jm} a^m_{kl}}{\lambda_m} + D^i_{jkl} - \frac{2a^i_{jm} a^m_{kl}}{\lambda_j + \lambda_m - \lambda_i} \right) g^j_0 g^k_0 g^l_0
\]  

(136)

where \( D^i_{jkl} \) is the connected part. This is the only part of the amplitude that remains on-shell, since the leading order conditions imply \( \lambda_i = 0 \) [49].

In our approach described in section 5, the ultraviolet cut-off \( \alpha \) introduced above is interpreted as the Liouville mode, which is in turn identified with target time. Thus the cut-off dependences of couplings, etc, above should be regarded as equivalent to conformal dressings with the Liouville field, which lead in turn to dependence on target time. Renormalizability of the theory is reflected in the independence of any physical quantity such as the density matrix \( \rho \) from the global cut-off scale, i.e. the zero-mode of the Liouville field, i.e. the target time:

\[
\frac{d}{dt} \rho(g^i, p^i, t) = 0
\]

(137)

The total derivative (137) may be split into an explicit partial derivative term and an implicit term that may be represented by the commutator with the Hamiltonian, so that (137) becomes

\[
\partial_t \rho = i[\rho, H]
\]

(138)

In the case of an exactly marginal perturbation, the corresponding \( \beta \)-function vanishes identically implying that the partial time derivative \( \partial \rho / \partial t = 0 \) also, and equation (138) becomes trivial\(^{15} \). This is the case of the Wheeler-De-Witt equation in cosmology [10]:

\[
\rho(g^i, p^i, t), \mathcal{H} = 0
\]

(139)

where \( \mathcal{H} \) is the Hamiltonian of the entire Universe. However, as we have already discussed at length, in our case the ‘tachyon’ perturbation (53) is not exactly marginal, and as a result equation (138) is non-trivial. Recalling the usual field-theoretical relationships between the Hamiltonian, the \( T \)-matrix \( T = \int H \) and the \( S \)-matrix \( S = 1 + iT \) we see that (138), corresponds to the conventional factorizable form

\[
\rho_{\text{out}} = \$ \rho_{\text{in}} \quad : \quad \$ = SS^\dagger
\]

(140)

of the \( \$ \) matrix.

The asymptotic analogue of the Wheeler-De-Witt equation (139) is the statement that the \( T \) matrix of the full theory vanishes. This is precisely what happens in the \( SL(2, R)/U(1) \) coset model close to the singularity where the manifest target-space \( W \) symmetry [4] is enhanced, and contains a twisted \( N = 2 \) world-sheet supersymmetry and hence double bosonic \( W \) symmetry [53]. This enhanced symmetry is understood from the observation that the model is in fact a topological string

\(^{15} \) These vanishing conditions should be understood in the weak sense, i.e. as vevs of the corresponding quantum mechanical operators between physical states.
theory \[18, 24, 23, 53, 56\]. The fact that the \(T\) matrix of the full theory vanishes for symmetry reasons does not mean that the theory is itself trivial. Correlation functions for the light modes are non-trivial \[4, 55\], but this does not mean that scattering is described by an \(S\) matrix, as we now demonstrate.

**Monopole-Antimonopole Valley Contributions to the \(S\) Matrix**

The first non-factorizable contribution to the \(S\) matrix that we present is that associated with the \(SL(2, R)/U(1)\) monopole-antimonopole valley. In the two-dimensional \(O(3)\) \(\sigma\)-model discussed in section 4, the collective coordinates are well-known and the action in the dilute gas approximation for instantons has the approximate form

\[
S^{(1)}_{O(3)}(\Delta R, \rho_I, \bar{\rho}_I) \propto \frac{8\pi}{g^2} \left[ 1 - \frac{2\rho\bar{\rho}}{(\Delta R)^2} + O\left(\frac{\rho^2\bar{\rho}^2}{(\Delta R)^4}\right) \right] \ldots
\] (141)

for instanton sizes \(\rho_I\) and \(\bar{\rho}_I\) much smaller than the separation \(\Delta R\). Note that the action is finite in the infrared limit of large separations \(\Delta R \to \infty\). In the case of the monopole-antimonopole valley \([89]\) in the \(SL(2, R)/U(1)\) model, the corresponding collective coordinates are the monopole charges \(q_m\) and \(\bar{q}_m\) and their relative separation \(\Delta R\), which can be projected on the complex plane by means of the stereographic projection \([133]\). We work in a phase where the monopoles and antimonopoles are confined in dipole pairs with \(q_m = -\bar{q}_m = q\). The classical action of such a well-separated monopole-antimonopole pair takes the following form in stereographic coordinates:

\[
S_m = 8\pi q^2 \ln(2) \sqrt{2e^\gamma} + 2\pi q^2 \ln\frac{2R}{\omega} + 2\pi q^2 \ln\left[ \frac{\left| z_1 - z_2 \right|}{(4R^2 + \left| z_1 \right|^2)^{3/2}} \right] - \frac{4}{(4R^2 + \left| z_2 \right|^2)^{1/2}}
\] (142)

where \(\gamma\) is Euler’s constant, and \(\omega\) is the angular cut-off on the world-sheet introduced in section 2, whose radius \(R\) is related in our approach to the renormalization group scale, i.e. the Liouville field, i.e. target time. Expression \([143]\) substituted into the generic formula \([55]\) gives our first non-trivial contributions to the \(S\) matrix elements. The action \([142]\) contains first a finite part related to the monopole charge \(q\) (over which we must integrate), that does not exhibit any particular suppression factor, then a logarithmically-divergent self-energy term, and finally a dipole interaction energy. In the limit of finite separations \(0 < |z_1 - z_2| < \infty\) and very small world-sheets \(R = O(a) \to 0\), the action \([142]\) yields

\[
S_m = 2\pi q^2 \ln^{\frac{a}{\omega}} + \text{finite parts}
\] (143)

Recalling the discussion in section 2, about the effective variation of either \(a\) or \(\omega\), but not both, under the renormalization group evolution, we observe that, since \(-\ln \omega\) corresponds to target time in our approach, equation \([143]\) exhibits an additional time-dependence in the absorptive part of the forward scattering amplitude, beyond
that included in the conventional quantum Liouville equation (138) obtained in the zero-defect sector, as discussed in the previous subsection. It therefore contributes to the non-Hamiltonian term \( \delta H \) in the equation (2) \[2\]. It should be remarked that the correct time dependence implied by (143) should take into account quantum corrections to dipole interactions, which are not exhibited in the classical expression (143). We shall do that later on, by following the approach of [19] to represent these effects as \( \sigma \)-model sine-Gordon deformations.

This extra cut-off (time) dependence can be understood from a world-sheet point of view. The dipole phase of monopole-antimonopole configurations can be represented in the \( \sigma \)-model framework as a sine-Gordon deformation [19] of the string action (5). For monopoles and antimonopoles to be confined into pairs, as in the valley case, this deformation must be irrelevant, breaking the conformal invariance of the \( \sigma \)-model action. This introduces extra renormalization scale dependence into ‘tachyon’ correlation functions, causing the \( \beta \)-functions for propagating light-particle operators to be non-zero. However, as we saw in the first part of this section, target-space scattering amplitudes are defined by the vanishing of the respective \( \beta \)-functions, and so cannot be defined in the presence of monopole-antimonopole pairs.

The presence of \( \delta H \neq 0 \) (143) induces a monotonic increase in entropy (4), whose origin can be understood intuitively as follows. Monopoles correspond to black holes in Minkowski space, so the monopole-antimonopole valley configuration corresponds to a quantum fluctuation producing a microscopic black hole in the space-time foam. Traditional quantum gravity arguments suggested that information be ‘lost’ across the corresponding microscopic event horizon. In our string approach, this information is passed by the \( W \)-symmetry to unmeasured (by local scattering experiments) massive modes (78), and it is the truncation of the effective theory to light modes that induces the renormalization scale (time) dependence (143).

**Instanton-Anti-Instanton Valley Contribution to the \$ S \$ Matrix**

The instanton-anti-instanton valley in the \( SL(2, R)/U(1) \) model corresponds more closely to the \( O(3) \) valley [37, 24]. As in that case, one introduces as collective coordinates the sizes \( \rho_I \) and \( \overline{\rho_I} \) of the instanton and anti-instanton, and their relative separation \( \Delta R \). The dilute gas approximation of section 2 corresponds to the limit \( \Delta R \gg |\rho_I, \overline{\rho_I}| \) in which the valley action is approximated by a separated instanton and anti-instanton pair with dipole interactions [20]. The form of the latter can be inferred from that of (31) of the vertex operator describing an instanton-anti-instanton pair, but its detailed form is not important for our purposes. In contrast to the \( O(3) \) \( \sigma \)-model case [141], however, the action of an isolated instanton or anti-instanton depends on the instanton size and the ultraviolet cutoff, as we discussed in section 2 (see equation (28)). Thus in the large-separation limit of the dilute-gas
approximation the single instanton-anti-instanton valley action is

\[ S^{(1)} = k \ln(1 + |\rho|^2/a^2) + O(\rho \bar{\rho}/(\Delta R)^2) \] (144)

Expression (144) substituted into the generic formula (55) gives our second set of non-trivial contributions to the $S$ matrix elements. As in the monopole case, the expression (144) has extra renormalization scale (target time) dependence beyond that induced in the quantum Liouville equation (138), associated in this case with the scale dependence of $k$ discussed in section 2 (c.f. (35)):  

\[ k \propto e^{-2\gamma_0 \ln|a/\Lambda|} \] (145)

where $\gamma_0$ is the anomalous dimension of the tachyon $\beta$-function. It therefore contributes to the non-Hamiltonian term $\delta H$ in the equation (2).

As in the monopole case, this extra cut-off (time) dependence can be understood from the world-sheet point of view as being due to the irrelevant nature (as renormalization group operators) of the deformations (31) that describe instanton excitations. We recall that instantons induce transitions between black holes with different values of $k$ (33) and hence different masses (39). Moreover, the resulting scattering operator may have extra non-analytic behaviour for some values of $k$. Indeed, upon performing the collective coordinate integral (55) over a region of the size parameter $\rho$ in the action (144) where instanton effects are dominant, i.e. $\rho = O(a)$, we obtain

\[ \sigma_{\text{total}} \propto \int d(\Delta R) \frac{1}{k - 1} e^{E \Delta R - O(a^2/(\Delta R)^2)} \] (146)

The region of $k \approx 1$, which is easily reached by instanton transitions that can be interpreted as renormalization group flow, induces non-analytic behaviour in the world-sheet correlation functions.

The origin of the increase in entropy (4) associated with this contribution to $\delta H$ can also be understood intuitively. Instantons correspond to quantum jumps in the black hole mass, interpreted in section 2 as back reaction in the presence of light matter ‘tachyon’ perturbations. Thus they increase the ‘area’ of the microscopic event horizon associated with a microscopic black hole in the space-time foam, entailing the loss of information. As in the monopole case, this is passed by the $W$-symmetry to unmeasured (by local scattering experiments) massive modes (78), whose truncation induces the renormalization scale (time) dependence (144), (145).

*Induced Von Neumann Collapse of the Density Matrix*
We now address the key question: what is the general form of the final state density matrix $\rho_{\text{out}}$? In our treatment of time as a renormalization group scale represented by a Liouville field, the answer to this question lies at the ultraviolet fixed point of the renormalization group flow. We consider the $S$ matrix which relates $in$ and $out$ states in target configuration space:

$$\hat{\rho}_{\text{out}}(x,x') = \int dy dy' S(x,x';y,y') \rho_{\text{in}}(y,y')$$ (147)

In the two-dimensional string approach we have taken, light particles are represented by the massless ‘tachyon’ mode, which is to be regarded as a collective representation of the $s$-wave matter fields in four dimensions. As discussed in section 3, the $S$ matrix in (147) is related to a world-sheet tachyon correlation function. The target space coordinates $(x,x',y,y')$ are to be understood as values of the zero-mode part of the decomposition (116) of the $\sigma$-model variables $(X,X',Y,Y')$. This decomposition results in a modification of the corresponding vertex operators that represent various string backgrounds:

$$\int d^2 z g^I(X(\sigma)) V_I(X(\sigma)) \rightarrow \int d^2 z \int d^D x g^I(x) \delta^{(4)}(x - X(\sigma)) V_I(X(\sigma)) \equiv \int d^2 z g^I V_I$$ (148)

where the summation over the index $i$ now includes integration over the target-space coordinates $x$. This notation should be understood in the following discussion.

In our definition of target time, the latter is a ‘non-relativistic’ evolution parameter of the system of light string modes\footnote{We recall that Lorentz invariance can be regarded as an \textit{a posteriori} concept in critical strings, which merits re-evaluation when one deals with supercritical strings, as in our treatment of topological defects on the world-sheet.}. The evolution of the density matrix is given in the $\sigma$-model formalism (148) by

$$\partial_t \rho = i[\rho, H] + i\beta^I G_{jk}\{g^k, \rho\}$$ (149)

We pointed out\footnote{\textit{We recall that Lorentz invariance can be regarded as an \textit{a posteriori} concept in critical strings, which merits re-evaluation when one deals with supercritical strings, as in our treatment of topological defects on the world-sheet.}} that this equation resembles closely the Drude model of quantum friction discussed in ref. [26, 27], with the massive string modes playing the rôles of ‘environmental oscillators’.

The analysis of ref. [27] led to the following expression for the reduced density matrix of the observable states:

$$\rho(g_I, g_F, t)/\rho_S(g_I, g_F, t) \simeq e^{\eta \int_0^t dr \int_{r-\epsilon}^{r+\epsilon} dr' \frac{(g(r)-g(r'))^2}{(r-r')^2}} \simeq e^{-\eta \int_0^t dr \int_{r-\epsilon}^{r+\epsilon} dr' G_{ij}(S_0)^{\beta J}} \simeq e^{-D(t(g_I-g_F)^2 + ...}$$ (150)
quantum mechanics. The constant $D$ is proportional to the sum of the squares of the effective anomalous dimensions of the renormalised couplings $g^i$. As in ref. [26, 27] we interpret (150) as representing the overlap, after an elapsed time interval $t$, between quantum systems localized at different values $g = g_I, g_F$ of the coordinates. Equation (150) exhibits a simple quadratic dependence on the $\beta$-functions, which is supported by the identification [35] in string theory of the target-space effective action with the Zamolodchikov $c$-function $C\{\{g\}\}$. The latter evolves according to

$$\partial_t C\{\{g\}\} = \beta^i G_{ij} \beta^j \quad (151)$$

justifying in string theory the simple Drude model formula (150) for the evolution of density matrix elements. This is because in the Drude model of ref. [26] the reduced density matrix at time $t$ is given in a path-integral formalism in terms of the ‘effective action’

$$\rho(t) \propto \exp[-\int_0^t d\tau F(g^i(\tau))] \quad (152)$$

where $F(g_t, g'_t)$ is

$$F(g, g') = \exp(-i(V(g) - V(g'))) \quad (154)$$

The integrand in (153) can then be expressed in terms of the effective action $S_{eff} = S_0(g) - V(g)$ of the reduced subsystem, including the interaction with the environment

$$\rho(g, g'_t) = \int \exp(iS_{eff}(g) - iS_{eff}(g'))\rho(g_0, g'_0)Dg(\tau) \ldots dg'_0 \quad (155)$$

The important difference of the density matrix of Feynman and Vernon (153) from the reduced density matrix (150) lies in the fact that the former describes the overlap between two states at different locations at a time $t$, while the latter is actually a transition amplitude from a state located at $g_I$ at time 0 to the same state located...
at \( g_F \) at time \( t \). However, when considering the renormalization group evolution as a real time evolution of the string light-mode-subsystem, one may view the two different locations \( g \) and \( g' \) appearing in the density matrix (153), as obtained from one another by a coordinate transformation in coupling constant space. This in turn may be interpreted as a renormalization scale (time) evolution, by solving the equation

\[
\Delta t = \int_g^{g'} \frac{\beta^i G_{ij} dg^j}{\partial C(g)/\partial \tau}
\]

where \( C(g) \) is the Zamolodchikov \( C \) function. In this case \( \rho(g_t, g'_t) \) becomes

\[
\rho(g_t, g'_t) = \int \exp\left(-i \int_0^t d\tau \beta^i G_{ij} \beta^j \right) \rho(g_0, g'_0) \frac{Dg(\tau)}{...dg'_0}
\]

which has a similar structure to the Drude model integrand (150). We then interpret this result as representing the overlap between quantum systems localized at different values of the coordinates \( g^i \). In the case of the Wess-Zumino string theory studied here, the coordinates are the ‘tachyon’ fields \( T(X) \). Wave-function renormalization effects\(^\text{17}\) in the \( \sigma \)-model induce a ‘time’ dependence on the target-space coordinates \( X \rightarrow X + \Delta X(g^i(t), t) \) in the form of target-space diffeomorphisms [57]. The shifts \( \Delta X \) are related to space-time diffeomorphisms, thereby allowing the re-interpretation of the points \( X \) and \( X + \Delta X \) as representing a renormalization scheme change, so the above formalism applies. Thus, we can express the differences between \( \sigma \)-model couplings at two neighbouring points in the \( g^i \) coordinate space by a simple Taylor expansion

\[
|g^i(t) - g'^i(t)|^2 \rightarrow (\nabla_X T(X))^2|X(t) - X'(t)|^2 + \ldots
\]

We, therefore, expect the exponent in the suppression factors (153) \( \rho_{out}(x, x') \) to vanish as \( x \rightarrow x' \).

Before presenting the valley contributions to the Drude coefficient \( D \), we first make some general comments on its structure in string theory. Recall the general decomposition (99) \( \hat{\beta}^i = \epsilon g^i + \beta^i \) of the \( \beta \)-functions, where \( \beta^i = O(g^2) \) and \( \epsilon \neq 0 \) in the presence of a cut-off. Recall also from the discussion following (124) that the Zamolodchikov metric, has a pole contribution

\[
G_{ij} = \frac{1}{\epsilon} G_{ij}^{(1)} + \text{regular}
\]

in the presence of a cut-off. Thus there is a contribution to the exponent \( K \) of the Drude model (150) from the \( \epsilon \)-term in \( \hat{\beta}^i \) and the pole term in \( G_{ij} \):

\[
K = \int_0^t \hat{\beta}^i G_{ij} \hat{\beta}^j d\tau \equiv 2 \int_0^t g^i \hat{G}_{ij}^{(1)} \beta^j d\tau + \epsilon \int_0^t g^i \hat{G}_{ij}^{(1)} g^j d\tau
\]

\(^\text{17}\)It should be remarked that such effects cannot be seen in the first and second orders of the \( \alpha' \)-expansion of perturbative \( \sigma \)-models, but appear at third and higher orders [57]. In our case we are effectively dealing with a (formal) expansion in powers of the couplings, and hence all orders in \( \alpha' \) are taken into account. Therefore such wave-function renormalization effects are unambiguously present.
Now, using the relation \( \beta^i = dg^i/dt - \epsilon g^i \) we find

\[
K \ni 2 \int_0^t \int_0^t g^{i(1)}_{ij} \frac{dg^i}{d\tau} d\tau - \epsilon \int_0^t g^{i(1)}_{ij} g^j d\tau =
\]

\[
= [g^i(t)]^2 - [g^i(0)]^2 - \epsilon \int_0^t (g^i(\tau))^2 d\tau
\]

(161)

Above we work in a scheme where \( G_\infty \) is constant. We assume by convention that \( g^i(0) = 0 \), and that \( g^i(t) \) remains small and \( O(\epsilon) \). We extract from (161) the time dependence of \( K \), assuming that \( g^i(t) \) approaches a fixed point which is of \( O(\epsilon) \) as \( t \to \infty \). Then

\[
\dot{K} = -\epsilon [g^i(t)]^2 + O(\epsilon^3)
\]

(162)

and

\[
K \simeq \epsilon [g^i(t)]^2 t
\]

(163)

for long times \( t \gg \frac{1}{\epsilon} \). Notice that the suppression factor in the exponent is linear in both time and the anomalous dimension coefficient \( \epsilon \). This is a very important feature of the string space-time foam, that opens the possibility of observing these non-quantum mechanical effects at next-generation experiments, for instance through induced CPT violation in the light-state subsystem (e.g. neutral kaons).

We now show explicitly how the monopole-antimonopole and instanton-anti-instanton valleys contribute to the exponential suppression factor in (150). We recall from equations (142),(143) that the monopole-antimonopole valley contributes to the $S$ matrix elements a factor

\[
S \simeq \int \exp(-S_m + \ldots)
\]

(164)

where \( S_m \) is the classical action for a monopole-antimonopole interacting dipole pair (142),(143),

\[
S_m = 2\pi q^2 \ln \frac{a}{\omega} + \text{finite parts}
\]

(165)

and the integration is over all possible separations. The dots in (164) indicate dipole interactions at finite separations. It has been shown in [19] that world-sheet dipoles can be described by an effective action (18) which contains sine-Gordon type deformations of a free \( \sigma \)-model action. Integration over the valley parameters can be converted in this case into integration over the relevant positions of the monopole and the antimonopole. In addition, multi-valley configurations [24] can be taken into account in a straightforward manner by applying the formalism of [19] for the description of a dipole gas of monopoles. In the dilute gas approximation one can represent the result by restricting oneself to the first power in an expansion over the sine-Gordon deformation. Close to the ultraviolet fixed-point (asymptotic

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\[18\text{In the Wilson scheme (134) this is at least linear in the renormalized couplings, in which case there are cubic (and higher order) contributions in } K.\]
(future), one obtains the following result for the leading monopole-antimonopole valley contribution to the $S$ matrix elements:

$$S \simeq e^{2(\Delta_m - 1) \ln \omega + \ldots} \equiv e^{-2(\Delta_m - 1) t + \ldots}$$ \hspace{1cm} (166)$$

where $\Delta_m$ is the conformal dimension \(^{(20)}\) for the sine-Gordon deformation for monopoles, and is proportional to the black-hole mass. The latter is assumed constant in the dilute gas approximation for monopoles. Notice that the quantum corrections coming from dipole-interactions in \(^{(18)}\) result in an extra factor of $-1$ in the coefficient of the leading logarithmically divergent terms of \(^{(166)}\), as compared to the classical result \(^{(165)}\). The dots $\ldots$ in \(^{(166)}\) represent the finite (renormalized) valley contributions that can be computed by evaluating averages of the sine-Gordon vertex operator \(^{(19)}\). Their explicit form is irrelevant for our purposes. Notice that the relation \(^{(166)}\) describes the leading contributions close to the ultraviolet fixed point where $\omega \to 0$, and that $\Delta_m - 1$ has the interpretation of an anomalous dimension \(^{(19)}\). Therefore the monopole-antimonopole valley suppression factor has exactly the form of the factor $K$ in the string version of the Drude model \(^{(163)}\). Because we are in the dipole phase, $\Delta_m > 1$ and hence the contributions \(^{(166)}\) to the $S$ matrix vanish asymptotically at the ultraviolet fixed point. The irrelevance of the respective coupling constants (in this case the fugacities of the monopoles) was crucial to the effect. We stress the fact that the above computation exhibits (in a generic manner) the effects - on the light string matter - of the space-time foam that are associated with the creation of black holes in target space. The representation of the latter by topologically non-trivial objects (monopoles) on the world sheet converted this complicated problem into a simple quantum mechanical computation on the world-sheet.

We now discuss the instanton valleys that represent back-reaction effects of the matter on the foam, i.e. transitions among black hole states. In the case of the instanton-anti-instanton valley, we recall that it contributes to the light matter $S$ matrix a factor

$$S \simeq \frac{1}{k - 1} \times \text{(constant)}$$ \hspace{1cm} (167)$$

where $k$ is renormalization-scale- (time-) dependent according to \(^{(35)}\):

$$k \simeq (\frac{a}{\Lambda})^{-2\gamma_0} \simeq \exp(-2\gamma_0 \ln(\frac{a}{\Lambda}))$$ \hspace{1cm} (168)$$

so that

$$S \simeq \exp(2\gamma_0 \ln(\frac{a}{\Lambda})) \simeq \exp(-2\gamma_0 t + \ldots)$$ \hspace{1cm} (169)$$

The coefficient of the exponential suppression can be interpreted as the anomalous dimension of the tachyon matter operator, as is clear from the discussion in section 2. Notice that it is the same anomalous dimension of the matter deformations, though with the opposite sign, that appears in the leading instanton contributions to the $S$ matrix of the light matter. This is consistent with the back-reaction interpretation
of the instanton effects. In the absence of any matter deformations, such effects go away. As in the monopole case, this suppression factor causes all off-diagonal entries in the final state density matrix to vanish asymptotically. Again, this result is a consequence of the irrelevance of the corresponding deformations (31) of the string black hole $\sigma$-model.

The time-dependences of both the expressions (166,169) mean that monopole-antimonopole and instanton-anti-instanton valleys both contribute to $\delta H$ (2,3), as in the Drude model. The time-dependent expressions (166, 169) were both derived on the assumption that the relevant defect and anti-defect were far separated, and cannot be applied directly to estimate $\rho_{\text{out}}(x, x')$ for $(x - x') \to 0$. However this can be inferred from the analogous Drude model result (158). In such a case, according to our previous discussion, the exponent in the suppression factors (166,169) vanishes as $x \to x'$.

Thus, both monopole-antimonopole (166) and instanton-anti-instanton (169) valleys yield the following result for the density-matrix elements for light string out-states:

$$\rho_{\text{out}}(x, x') = \hat{\rho}(x)\delta^{(D)}(x - x')$$

(170)

As already mentioned, we believe that such contributions to the $\mathcal{S}$ matrix and to the collapse (171) are generic for string contributions to the space-time foam. Such a collapse of off-diagonal elements is known as Von Neumann collapse, and is inherent to any transition from pure to mixed states. It had been suggested previously on the basis of one particular approach to the wormhole calculus [8]. The above analysis puts the conclusion (170) on a secure basis. It can be understood intuitively as follows. We recall from section 5 that, in our interpretation of time as a renormalization scale, the string theory evolves towards an ultraviolet fixed point as the time $t \to \infty$. This means that the apparent physical size of the world-sheet, as measured in target space, contracts as time progresses. This implies that interferences between string configurations located at different values of the zero-mode variable $x$ vanish asymptotically, in agreement with the explicit calculations presented above.

We have argued previously [7, 4] that the rate of suppression of off-diagonal entries in the density matrix is larger for systems containing many light particles (‘tachyons’). In particular, we found in the Drude analogue model that $D \propto N$, where $N$ is the number of particles in the in- (or out-) states. The previous arguments apply in particular to the specific contributions calculated above. Moreover, in these specific cases we can give additional intuitive arguments. It is well-known from other examples that instanton and hence valley amplitudes are larger for systems containing many particles [58, 23]. Moreover, in the shrinking world-sheet picture discussed in the previous paragraph, it is clear that the overlap of multi-particle systems in different locations will decrease as a product of single-particle overlap factors. Thus quantum-mechanical interferences vanish more rapidly for
macroscopic bodies: Schrödinger’s cat is either alive or dead, but not a superposition of the two!

7 Discussion

The original intuition behind the postulation of the $\mathcal{S}$ matrix and the modified quantum Liouville equation was that pure states should evolve into mixed states within a density matrix formalism. It was shown in [2] that the generic effect of such a modification of quantum mechanics would be to suppress off-diagonal entries in the density matrix. In a single two-state system, the general form of $\delta H$ consistent with the conservation of probability and energy is

$$-\delta H = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} : \alpha, \gamma > 0, \alpha \gamma \geq \beta^2$$

leading to the following asymptotic form for the density matrix:

$$\rho(t) \simeq \frac{1}{2} \begin{pmatrix} 1 & exp[-(\alpha + \gamma)t/2]exp[-i\Delta Et] \\ exp[-(\alpha + \gamma)t/2]exp[i\Delta Et] & 1 \end{pmatrix}$$

(172)

Note that the off-diagonal entries in $\rho(t)$ decay exponentially, so that the state becomes completely mixed. It was suggested in ref. [7] that the collapse of off-diagonal density matrix elements should be a general feature of this type of modification of quantum mechanics, in particular for the configuration space representation $\rho(x, x', t)$ of the density matrix:

$$\rho_{\text{out}}(x, x') = \rho_{\text{in}}(x', x)[1 - D t(x - x')^2 + \ldots]$$

(173)

and a possible way to motivate such a collapse on the basis of one version of the wormhole calculus was exhibited.

We have previously demonstrated [4] that just such a collapse of off-diagonal entries in the density matrix occurs in the string modification of quantum mechanics. We showed that the $W_\infty$-induced couplings of light particles to unobserved massive solitonic string modes induced quantum gravitational friction leading to the collapse of the reduced density matrix describing the light particle system. The formalism and conclusion resembled that of the Feynman and Vernon description of an open quantum-mechanical system coupled to simple harmonic oscillators representing the environment [26, 27].

However, in our approach the collapse is an intrinsic and inevitable manifestation of quantum gravity that occurs due to the nature of space-time foam and independently of any environmental conditions. In this paper we demonstrate this collapse phenomenon from the $\mathcal{S}$-matrix point of view, using topological structures on the world-sheet (monopoles, instantons) to represent the effects of massive string modes.
For clarity and the convenience of the reader, we summarize the relationship of our results to general discussions \[28, 29\] of the “collapse of the wave function”. We distinguish two aspects of this collapse: one is the suppression of off-diagonal density matrix elements:

$$\rho_{ij} \rightarrow \text{diag}(p_1, p_2, \ldots p_N) : \sum_{i=1}^{N} p_i = 1$$ (174)

generally called Von Neumann collapse, and the other is the collapse of entities along the diagonal:

$$\text{diag}(p_1, p_2, \ldots p_N) \rightarrow \text{diag}(0, 0, \ldots 0, 1, 0, \ldots 0)$$ (175)

generally called Dirac-Heisenberg collapse. When do these occur in conventional quantum mechanics? In the conventional framework an isolated quantum mechanical system develops in a unitary manner between measurements, given by the normal Hamiltonian evolution embodied in the Schrödinger equation which becomes the $S$ matrix asymptotically. This is called the $U$ process in ref. \[29\]. A measurement involves a reduction of the wave function to an eigenvector of the operator measured, and is called the $R$ process in ref. \[29\]. Both the Von Neumann (174) and Dirac-Heisenberg (175) collapses are supposed in conventional quantum mechanics to occur during the measurement or $R$ process. The decoherence of different eigenstates, i.e. the Von Neumann collapse (174), during a measurement has indeed been demonstrated (see, e.g., \[28\] for a review), but the Dirac-Heisenberg collapse (175) remains a supplementary postulate in conventional quantum mechanics.

This conventional picture has several unsatisfactory features. First is the question whether quantum-mechanical coherence is retained throughout the $U$ process. Is it reasonable for an isolated and possibly macroscopic system to exhibit quantum-mechanical interference indefinitely? how long can Schrödinger’s cat remain in a superposition of alive and dead states? Put another way: do large objects behave classically while they are not observed? A second question is: whence the mysterious mechanism of state reduction during the $R$ process?

In the modification of quantum mechanics and quantum field theory that we derive from string, Von Neumann collapse occurs during the $U$ process, which is therefore not unitary. In our framework, entropy increases continuously throughout the $U$ process as the state becomes more mixed. This collapse occurs more rapidly for more macroscopic systems: Schrödinger’s cat is not in a superposition of alive and dead states, even before we observe it. Just such a gradual collapse between measurements has been posited \[29\] on the basis of general arguments related to quantum gravity, but without any concrete mechanism, still less a derivation. Subsequent to our quantum-gravitational \[2\] approach, a stochastic mechanism for suppressing off-diagonal density matrix elements (Von Neumann collapse) during the $U$ process was postulated phenomenologically in ref. \[59\], but without any motivation from quantum gravity and as a supplementary postulate. We believe that string theory...
also has important implications for the measurement or $R$ process, to which we shall return in a forthcoming paper [61]. For the moment, we just note that Dirac-Heisenberg collapse (173) involves a reduction of the entropy of the system under consideration, which can only occur if it is coupled to an external system (observer) which increases its entropy sufficiently for the total entropy of the observer and the measured system to increase overall. Thus Dirac-Heisenberg collapse can only occur during the measurement or $R$ process, and a string theory of measurement is the next item on our agenda [61].

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Figure Captions

Figure 1. Stereographic projection of a two-sphere on the complex plane. The projection of the ultraviolet cut-off of the sphere onto the plane is indicated.

Figure 2. A diagrammatic representation of eq. (54), that represents a specific absorptive part of a world-sheet correlation function.

Figure 3. Valley trajectory: the quantity plotted represents the corresponding energy density.

Figure 4. Kink solution of the Euler-Lagrange equations obtained from the reduced $SL(2,R)/U(1)$ model (67).

Figure 5. Comparison of the scattering solution $\phi(y, \mu) = sinh^{-1}[\mu/cosh y]$ at large $y$ with the sum of an undistorted kink and an antikink at infinity, $G(y, \mu) = sinh^{-1}(\frac{y+\mu}{\sqrt{1-u^2}}) - sinh^{-1}(\frac{y-\mu}{\sqrt{1-u^2}})$, approaching each other with an ultra-relativistic velocity $u \simeq 1$.

Figure 6. Plot of the monopole-anti-monopole valley function (89) in the $z$-plane, for $\mu \equiv v - \frac{1}{v} = 1$.

Figure 7. As in figure 6, but for the instanton-anti-instanton valley (95), for $\mu/\sqrt{1-u^2} = 1$, obtained from (89) by a conformal transformation. For convenience we plot the function (95) on the rescaled $z\sqrt{1-u^2}$-plane.

Figure 8. Contour of integration in the analytically-continued (regularized) version (111) of $\Gamma(-s)$ for $s \in Z^+$.