Three Euler’s Sieves and a Fast Prime Generator
(Functional Pearl)

Ivano Salvo†, Agnese Pacifico‡

† Departement of Computer Science, Sapienza University of Rome
‡ Departement of Mathematics, Sapienza University of Rome

Abstract

The Euler’s Sieve refines the Sieve of Eratosthenes to compute prime numbers, by crossing off each non prime number just once. Euler’s Sieve is considered hard to be faithfully and efficiently coded as a purely functional stream based program. We propose three Haskell programs implementing the Euler’s Sieve, all based on the idea of generating just once each composite to be crossed off. Their faithfulness with respect to the Euler’s Sieve is up to costly stream unions imposed by the sequential nature of streams. Our programs outperform classical naïve stream based prime generators such as trial division, but they are asymptotically worse than the O’Neill ‘faithful’ Sieve of Eratosthenes. To circumvent the bottleneck of union of streams, we integrate our techniques inside the O’Neill program, thus obtaining a fast prime generator based on the Euler’s Sieve and priority queues.

1 Introduction

The generation of the stream of prime numbers is a classical and well-studied problem, that has been deeply investigated for a long time (see for example [Pri87]). The first algorithm, and probably the best known, is the Sieve of Eratosthenes: in this algorithm, after discovering a new prime \( p \), all multiples of \( p \), \( \{ p \cdot k \mid k \in \mathbb{N}_{\geq 2} \} \), are crossed off as non-primes. Since composites less than \( p^2 \) have at least a prime factor \( p' < p \), this process can be quicken, by starting from \( p^2 \), because smaller composites have been already crossed off as multiples of some \( p' < p \). The complexity of this algorithm is \( O(n \ln \ln n) \) to generate all primes less than \( n \).

Nevertheless, a lot of useless work is carried out in the execution of the Sieve of Eratosthenes: as an example, crossing off multiples of 3, we cross off again even numbers such as 12, 18, 24, . . . that have already been crossed off as multiples of 2. This suggests a refinement of this algorithm, the Euler’s Sieve, in which during the \( k^{th} \) iteration, only numbers that are multiple of \( p_k \), but not multiple of \( p_1, \ldots, p_{k-1} \) are crossed off. The asymptotic speed-up of the Euler’s Sieve with respect to the Sieve of Eratosthenes is \( \ln \ln n \), that is (on average) the number of distinct prime factors of \( n \). Therefore its complexity is \( O(n) \).

The natural implementation of the Sieve of Eratosthenes requires direct access to prime candidates to be crossed off, and hence array based imperative programs reflects the original procedure more than stream based purely functional programs. As a matter of fact, despite several elegant and concise Haskell programs that lazily generates the stream of primes, a ‘faithful’ (from the point
of view of both efficiency and the computations actually performed) functional implementation of the Sieve of Eratosthenes is far from trivial [O’N09].

The so-called stream-based Sieve of Eratosthenes (see Fig. 1) is still on the official home page of Haskell [Hhp18] as a paradigmatic example of the conciseness and the level of abstraction that one can achieve in Haskell. Since its first appearance [Tur75], however, its performances appeared not to be worthy of its cleanliness, as they are rather poor compared to other simple algorithms that lazily generate the stream of primes, such as trial division (e.g., see [Hwp18]).

\[
\text{primes} = \text{filterPrime [2..]} \text{ where}
\]
\[
\text{filterPrime (p:xs)} =
\]
\[
p : \text{filterPrime [x | x <- xs, x ‘mod’ p /= 0]}
\]

Figure 1: The ‘unfaithful’ Sieve of Eratosthenes [Tur75].

Even in imperative languages, to make the Euler’s Sieve efficient requires some ingenuity and it is less obvious than for the Sieve of Eratosthenes (e.g., see [Sor90]). The traditional Haskell program implementing the Euler’s Sieve (see Fig. 2) performs even worse than the unfaithful Sieve of Eratosthenes. The problem is that the prime \( p_k \), before being recognised as prime, as the head of the not yet crossed off numbers, must survive to \( k - 1 \) stream ‘complementation’ (via the \texttt{minus} function). In turn, each stream complementation generates a new stream and this generation is a never-ending process, causing very soon also a memory explosion problem.

\[
\text{minus xs@(x:txs) ys@(y:tys)}
\]
\[
| x < y \quad = x : \text{minus txs ys}
\]
\[
| x > y \quad = \text{minus xs txs}
\]
\[
| \text{otherwise} = \text{minus txs tys}
\]

\[
\text{primes} = \text{eulerSieve [2..] where}
\]
\[
\text{eulerSieve cs@(p:tcs)} = p : \text{eulerSieve (tcs ‘minus’ map (p*) cs)}
\]

Figure 2: The classical Euler’s Sieve in Haskell [Hwp18].

The Euler’s Sieve is considered hard to be coded in a stream based fashion or even impossible in principle (e.g., see [Hwp18], Section 5.3) because, differently from multiples, numbers to be crossed off depend on all \( p_1, \ldots, p_k \) and they appear not to be efficiently computable from the stream of primes under construction.

1.1 The Euler’s Sieve, Formally

Euler’s Sieve can be formally defined by specifying the set of numbers to be crossed off and the set of those that survive as prime candidates at each iteration of the algorithm. We start by giving some notation and definitions.

Let \( A \cdot B = \{ a \cdot b \mid a \in A, b \in B \} \) be the set of all products of numbers in \( A \) and \( B \). We write \( a \cdot B \) for \( \{ a \} \cdot B \). We use the notation \( p \mid n \) (resp. \( p \nmid n \)) to mean that \( p \) is (resp. is not) a factor of \( n \). We denote with \( \mathcal{P} \) the sequence \([2, 3, 5, 7, \ldots, p_k, \ldots]\) of prime numbers that, when convenient, we regard as a set. We denote with \( C(P) \) the set of composites of a set of primes \( P \subseteq \mathcal{P} \). Given
a sequence $A$, we will denote with $A|_k$ the prefix of its first $k$ elements and with $A|^{k}$ the suffix starting at its $k$ element. Accordingly, we stipulate that $p_1$ is 2 and we denote with $P|_k$ the first $k$ primes, and with $P|^{k}$ the suffix of primes starting in $p_k$.

For a natural number $k > 0$, $E_k$ is the set of natural numbers crossed off at the $k^{th}$ iteration of the Euler’s Sieve (‘E’ stands for ‘erased’), that is $E_k = \{ n \in \mathbb{N}_{\geq 2} \mid \forall p \in P|_{k-1} p \nmid n \land p_k \nmid n \}$. For $k \geq 0$, $S_k$ is the set of naturals that are still candidate to be prime after the $k^{th}$ iteration of the Euler’s Sieve (‘S’ stands for ‘survived’), that is $S_k = \{ n \in \mathbb{N}_{\geq 2} \mid \forall p \in P|_{k} p \nmid n \}$. In view of our recursive programs, it is useful to give an inductive definition of these sequences of sets, as follows:

$$E_0 = \emptyset \quad S_0 = \mathbb{N}_{\geq 2} \quad E_{k+1} = p_{k+1} \cdot S_k|^{k+1} \quad S_{k+1} = S_k \setminus E_{k+1} \quad (1)$$

Clearly, we have that $S_k \supset S_{k'}$ for $k < k'$, and $E_k \cap E_{k'} = \emptyset$ for $k \neq k'$. Observe also that the property $S_k \cup \bigcup_{i=0}^{k} E_i = \mathbb{N}_{\geq 2}$ is invariant for all $k$. Essentially, the Euler’s Sieve in Fig. 2 computes the stream of prime numbers accordingly to the fact that $P = \bigcap_{k=0}^{\infty} S_k$, using our mutual inductive definition of the sequences of sets $E_k$ and $S_k$. The set of all composite numbers, $C(P)$ can be characterised as $\bigcup_{k=0}^{\infty} E_k$.

1.2 Our Contributions

Inspired mainly by a stream based Sieve of Eratosthenes attributed to Richard Bird in [O’N09] (see Fig. 3), we present three Haskell programs implementing the Euler’s Sieve. All these programs inherit from the Bird program a change of perspective: we look at primes as the ground of composites and therefore our goal becomes to generate each composite just once.

Our first solution is based on a new solution to a generalisation of the Hamming problem [BW88]. Our second solution makes use of wheels as a tool to generate composites to be crossed off, rather than to generate primes, as they have been usually used in prime generators [Run97]. Our third solution essentially applies Equation (1), making efficient the idea behind the Euler’s Sieve of Fig. 2.

The faithfulness of these programs with respect to the original Euler’s Sieve is up to the bottleneck of costly union operations required because of the sequential nature of streams. To overcome such problem, we have finally integrated our second and third solutions into the priority-queue based ‘faithful’ Sieve of Eratosthenes in [O’N09]. In particular, we regard our priority-queue wheel based Euler’s Sieve as our fast prime generator, as it is fast enough and resistant to performance degradation due to memory (de)allocation when it computes millions of primes.

2 Figure and Ground: Primes vs Composites

As in Escher lithographs and woodcuts, prime and composite numbers form a figure-ground picture [Hol79]. In the Epilogue of the Melissa O’Neill paper [O’N09] about faithful sieves, it is reported a brilliant purely stream based program attributed to Richard Bird (see Fig. 3) essentially based on the following recursive equation:

$$P = \mathbb{N}_{\geq 2} \setminus C(P) \quad (2)$$

To make the computation productive, we extract the first prime, 2, and we characterise $C(P)$ as union of multiples of primes according to the Sieve of Eratosthenes, thus obtaining:

$$P = \{2\} \cup \mathbb{N}_{\geq 3} \setminus \bigcup_{k=1}^{\infty} p_k \cdot \mathbb{N}_{\geq p_k} \quad (3)$$

3
Roughly speaking, Bird’s sieve simply computes the list of all composite numbers as the union of multiples of all numbers in the list of primes that, in turn, is under construction as the complement of composites numbers (‘union’ here means merge of ordered lists, possibly avoiding duplicates). As in the classical Sieve of Eratosthenes, in this program each composite number is crossed off once for each of its distinct prime factors. To be precise, here ‘crossed off’ means generated in the list of composites. For example, 120 will be generated 3 times as a multiple of 2, 3, and 5.

```haskell
union xs@(x:txs) ys@(y:tys)
| x == y = x:union txs tys
| x < y = x:union txs ys
| x > y = y:union xs tys

primes = 2:(\[3..\] 'minus' composites) where
  composites = foldr unionP [] [multiples p | p <- primes]
  multiples n = map (n*) \[n..\]
  unionP (x:xs) ys = x:union xs ys
```

Figure 3: Stream based Sieve of Eratosthenes by R. Bird [O’N09]

We observe that Bird’s program uses a smart trick to make productive the computation of the union of a stream of streams, by using a union function (we use the name `unionP`, where ‘P’ stands for ‘productive’) that always chooses the first element of the first stream and then proceeds as an usual `union`. In this particular case, we know that this is correct, because primes are a ordered list and each list of multiples of \(p\) starts at \(p^2\) and hence for \(i < j\), we have \(p_i^2 < p_j^2\).

Motivated by improving Bird’s sieve by generating each composite number just once, in this functional pearl, we essentially look for a purely stream-based Haskell program implementing the Euler’s Sieve.

### 3 Primes as Background of Generalised Hamming Numbers

The problem of generating composites is tightly related to a generalised Hamming problem. Given a set of generators \(P\) (usually, but not necessarily, primes), the problem consists of generating in increasing order the smallest set \(H(P)\) such that \(1 \in H(P)\) and for all \(p \in P\) and \(h \in H(P)\), \(p \cdot h \in H(P)\). Using our notations, \(H(P)\) is the smallest set containing 1 and satisfying the equation \(P \cdot H(P) = H(P)\).

In [BW88], it is presented a solution to the classic version of the Hamming problem with \(P = \{2, 3, 5\}\) that can be easily generalised to generate Hamming numbers starting from any list of generators of arbitrary length (Exercise 7.6.5 in [BW88]). Some solutions can be found in [SP18].

Unfortunately, such classical solutions do not serve to our main purpose, that is generate each composite number just once, because, for example computing \(H(\{2, 3, 5\})\) they generate the number 30 six times (all permutations of factors of 30).

#### 3.1 The Hamming Problem Revisited

In the Afterwords of [SP18], we presented a solution to the Hamming problem that generates once each composite. We can do much better, however, thinking the set of \(H(P)\) as the smallest set
satisfying the following equation \((p \in P)\):

\[
H(P) = p \cdot H(P) \cup H(P \setminus \{p\})
\]  

(4)

Observe that, if we are interested in \(H'(P) = H(P) \setminus \{1\}\), from Equation (4) we have \(H'(P) = \{p\} \cup p \cdot H'(P) \cup H'(P \setminus \{p\}) = \emptyset\), because all numbers in \(p \cdot H'(P)\) have \(p\) as a prime factor, whereas numbers in \(H'(P \setminus \{p\})\) do not. This implies that we generate just once each composite as desired. This equation leads to a small (and highly circular) Haskell program, that always outperforms the classical solution to the Hamming problem (to be precise, function \texttt{hamming} in Fig. 4 never generates 1).

```haskell
import Data.List (union)

dUnion :: [a] -> [a] -> [a]
dUnion xs@(x:txs) ys@(y:tys)
    | x < y = x:dUnion txs ys
    | otherwise = y:dUnion xs tys
dUnion xs [] = xs

hamming :: [Int] -> [Int]
hamming [] = []
hamming (x:xs) = hmsg where
    hmsg = x:map (x*) hmsg 'dUnion' hamming xs
```

Figure 4: A solution to the Hamming problem that generates each number exactly once.

Thanks to the fact that we generate disjoint streams of composites, we can also slightly optimise the \texttt{union} function: the function \texttt{dUnion} (‘d’ stands for ‘disjoint’), assuming as precondition that its parameters are disjoint ordered lists, avoids to consider the case \(x=y\) that never occurs: this small optimisation has a significant impact on running time of the function \texttt{hamming}.

### 3.2 Euler’s Sieve from Hamming Numbers

Function \texttt{hamming} correctly computes also Hamming numbers of an infinite list of generators, such as the stream of primes. Moreover, its recursive calls are tightly related to the iterations of the Euler’s Sieve. As a matter of fact, \(H(P) = \mathbb{N}\) and \(2 \cdot H(P)\) corresponds to all even numbers, that are in turn \(\{2\} \cup E_1\), that is numbers crossed off in the first iteration of the Euler’s Sieve (in this particular case, this also corresponds to the set of numbers crossed off by the first iteration of the Sieve of Eratosthenes). Similarly, the set \(3 \cdot H(P|1)\) is \(\{3\} \cup E_2\), that is all multiples of 3 that are not multiples of 2, again 3 plus the set of numbers crossed off in the second iteration of the Euler’s Sieve, and so on.

Stemming from this solution of the Hamming problem, we can write an efficient lazy generator of prime numbers that ideally implements the Euler’s Sieve in Haskell (see Fig. 5). Since we are interested in computing the set of composites \(C(P) = H(P) \setminus P\), we need just to avoid to insert generators (i.e. prime numbers) in the resulting list of composites. This makes the code just a bit more involved, because the suffix \(P|^{k+1}\) of prime numbers is needed to compute \(C(P|^{k})\), but they are not present in the list of composites \(C(P|^{k+1})\) that we get from the recursive call. Therefore, they must be reinserted before computing \(C(P|^{k})\).

We observe that in Fig. 5 we use the function \texttt{sMinus} (instead of the standard \texttt{minus}, ‘s’ stands for ‘subset’). Function \texttt{sMinus} assumes as a precondition that input lists are ordered and the set of elements of the second list is contained in the first one: under this precondition, we can avoid to
check the case \( x > y \), that never happens. Similarly to \( d\text{Union} \), this small optimisation has a significant impact on the running time of this program.

Unfortunately, each composite number is generated just once, but several comparisons in nested calls of the \( d\text{Union} \) function are needed for a number before joining the list of composites. This is, of course, the main reason why this program cannot achieve the expected speed-up.

4 Reinventing Wheels

Wheels are a typical tool to generate primes (e.g., see [Pri82]). In this paper, we consider the sequence of wheels \( w_0, w_1, \ldots \) such that by ‘rolling’ the wheel \( w_k \) starting from \( p_{k+1} \), we efficiently generate all numbers that are not multiples of \( p_1, \ldots, p_k \), that is in turn \( S_k \mid k+1 \).

We define \( w_0 \) as the sequence \([1]\), that starting in 2 generates \( N_{\geq 2} = S_0 \). Let \( \Pi_k \) be the product \( p_1 p_2 \ldots p_k \) of the first \( k \) primes. Let \([q_1, \ldots, q_m]\) be the ordered sequence of numbers in the interval \([p_{k+1} \ldots p_{k+1} + \Pi_k]\) such that \( p_i \nmid q_j \) for all \( i \in [1 .. k] \) and \( j \in [1 .. m] \). The wheel \( w_k \) of circumference \( \Pi_k \) starting in \( p_{k+1} \) is the sequence \([q_1 - p_{k+1}, q_2 - q_1, \ldots, \Pi_k - q_m]\).

Wheels of arbitrary size are used in [Run97] to generate all prime numbers following the Wheel Sieve in [Pri82]. Unfortunately, the resulting Haskell programs are not so performant. More usually, a fixed size pre-computed wheel is used to dramatically improve the running time of a prime generator, even without changing its asymptotic complexity (see Section 6).

Again, by changing our point of view, we can use wheels to generate composites to be crossed off, rather than primes as in [Run97]. Since rolling the wheel \( w_k \) starting from \( p_{k+1} \), we get the sequence \( S_k \mid k+1 \), we can use \( w_k \) to generate composites to be crossed off after finding a new prime \( p_{k+1} \). \( S_k \mid k+1 \) \( = E_{k+1} \). Therefore, we can replace multiples of a prime \( p_k \) in the Bird’s Sieve of Fig. 3 with \( E_k \) just by rolling the wheel \( w_{k-1} \) (see Fig. 6).

Even though this program performs quite well and it does not suffer from huge memory allocation of that one in Fig. 5, we present it mainly because it shows very clearly the idea of using wheels
nextWheel [] _ _ = [1]
nextWheel (w:ws) p np = nWAux (rep p (w:ws)) np p where
  nWAux [] _ _ = []
  nWAux [w] _ _ = [w]
  nWAux (w:ws) s p =
    | mod (w+s) p == 0 = nWAux ((w+head ws):(tail ws)) s p
    | otherwise = w:nWAux ws (w+s) p
  rep 0 _ = []
  rep n xs = xs ++ rep (n-1) xs

nextWheel1 ws@(w:_):p = nextWheel ws p (p+w)
circ w = w ++ circ w
spin (w:ws) n = n:spin ws (n+w)

Figure 7: Computing wheels, incrementally.

to implement the Euler’s Sieve. However, it contains a couple of evident inefficiencies: 1. function
wP is called for each prime number p, and it recomputes at each invocation the prefix of primes up to p; 2. at each invocation, function wheel computes the wheel w_k from scratch, taking as input primes p_1, . . . , p_k and therefore it has to perform a trial division on the finite interval of naturals [p_k+1 . . . Π_k + p_k+1] (that becomes huge also for relatively small k).

Both these two computations can be quickened by saving information on parameters of the function composites. In particular, since we need all wheels w_1, w_2, . . . , w_k, . . ., we add a wheel as a parameter in order to compute the wheel w_{k+1} from the wheel w_k as in [Run97].

The key observation is that the wheel w_{k+1} consists of p_{k+1} copies of w_k and merging intervals when we hit a multiple of p_{k+1}. Instead of just picking the stream of wheels defined in [Run97], in our programs based on explicit recursion, we find convenient to use function nextWheel as in Fig. 7. Usually, we find convenient use function nextWheel1 that needs just one prime as parameter. Its correctness depends on the fact that having p_k and w_k, p_{k+1} is always p_k + (head w_k). Finally, function circ in Fig. 7 defines the repetitive stream w‘ generated by the wheel w, and function spin rolls a wheel (usually made repetitive by circ) starting from a given number.

composites (p:ps) w =
  map (p*) (spin (circ w) p) ‘dUnionP’ composites ps w’ where
  w’ = nextWheel1 w p
  dUnionP (x:xs) ys = x : dUnion xs ys

primes = 2:([3..] ‘sMinus’ (composites primes [1]))

Figure 8: W: The wheel based Euler’s Sieve.

As an example, taking as input the wheel w_2 = [2, 4] that avoids to generate multiples of 2 and 3 starting from 5, and the primes 5 and 7, function nextWheel w2 5 7 returns the wheel w_3 = [4, 2, 4, 2, 4, 6, 2, 6] that avoids to generate multiples of 2, 3, and 5 starting from 7 as follows: 1. first it makes 5 copies of the ‘shifted’ wheel w’_2 = [4, 2], that is [4, 2, 4, 2, 4, 2, 4, 2, 4, 2]. Shifting is needed because we will roll this wheel starting from 7 and not from 5; 2. then it generates the corre-
sponding sequence of numbers starting in 7, that is \([11, 13, 17, 19, 23, 25, 29, 31, 35, 37]\); and 3. finally it ‘merges’ delta’s that corresponds to multiples of 5, thus obtaining \(w_3 = [4, 2, 4, 2, 4, 2 + 4, 2, 4 + 2]\).

Having the wheel machinery and the already mentioned functions \(s\text{Minus}\) and \(d\text{Union}\), the result is the very small program \(W\) of Fig. 5 that again follows the figure-ground idea of prime-composite numbers formalised in Equation (2) \((d\text{Union}P\text{P}k\text{P} is for \(d\text{Union}\) the analogous of \(\text{unionP}\) for \(\text{union}\))

At each invocation of the function \(\text{composite} (p:ps) w\), if \(p\) is the prime \(p_k\) then \(w\) is the wheel \(w_{k-1}\) and hence \(\text{map} (p*) (\text{spin} (\text{circ} w) p)\) is \(p_k \cdot S_{k-1} = E_k\).

5 Back to the Origins

Are wheels really necessary? Probably, the bad behaviour of the classical Haskell program implementing the Euler’s Sieve in Fig. 2 prevented us to start with a characterisation of composites to be crossed off along the lines of that program.

As we have seen in Section 1.1, sets \(E_k\) and \(S_k\) can be defined by mutual induction, without the need of additional machinery such as wheels. Again starting from Equation (2), the idea is to compute the set of primes as the ground of composites as:

\[
P = \mathbb{N}_{\geq 2} \setminus \bigcup_{k=1}^{\infty} E_k
\]  

By contrast, as already observed, the program of Fig. 2 is essentially based on the equation \(P = \bigcap_{k=0}^{\infty} S_k = \bigcap_{k=1}^{\infty} (\mathbb{N}_{\geq 2} \setminus E_k)\) that is set-theoretically equivalent, but it leads to a much more expensive computational process due to a deep nesting of stream complementations via the \(\text{minus}\) function.

As usual, the corresponding Haskell program \(\text{ES}\) in Fig. 9 just rewrites the recursive Equations (5), extracting the first prime to make the lazy computation productive. At each invocation of \(\text{composites} (p:ps) ss@(s:tss)\), if \(p\) is the prime \(p_k\), \(ss\) is the iterator generating \(S_{k-1} = E_k\) and therefore \(es = \text{map} (p*) ss\) generates \(E_k\), following the mutual induction schema defined in Equation (1).

6 Pit Stop: Mounting Wheels on Sieves

As noted in Section 4 a fixed size pre-computed wheel can improve the running time of a prime generator, even without changing its asymptotic complexity. A common choice is the wheel \(w_4\) that, starting from 11, generates all numbers that are not multiples of 2, 3, 5, and 7: this wheel avoids to check about 77% of numbers for large \(n\).

This optimisation can be easily integrated in our programs, just modifying the definition of \(\text{primes}\). For all programs, mounting the wheel \(w_k\) prunes the set of candidate primes to be sieved,
from $N \geq 2$ to $S_k[pk+1]$. However, the impact on performance varies a lot among programs, due to the different meaning that mounting a pre-computed wheel has in the generation of composites.

In the Hamming Sieve of Fig. 5, mounting the wheel $w_k$, requires to compute $C(P|^k)$ instead of $C(P)$: here, this is not only useful to significantly speed up its computation, but it is necessary for program correctness, to satisfy preconditions of function $sMinus$. This holds for all our programs.

Mounting the wheel $w_k$ on the sieve $W$ of Fig. 8 means just avoiding its first $k$ recursive calls, i.e. starting the computation from the wheel $w_k$ rather than from the wheel $w_0 = [1]$. Since the computation of the first 4 small wheels is quite efficient, this optimisation has a limited impact on its performance.

Mounting the wheel $w_k$ on the sieve $ES$ of Fig. 9 means starting the computation from $S_k[pk+1]$; this is quite relevant, because in that program $S_4[11]$ is obtained by the first 4 stream complementations via $sMinus$, that are the most expensive.

In Fig. 10, we give the new definitions of streams $primesH4$ (for the Hamming based sieve $H$), $primesW4$ (for the wheel based sieve $W$), and $primesES4$ (for the sieve $ES$).

\begin{verbatim}
primesH4 = 2:3:5:7:11:ts4 'sMinus' composites (drop 4 primesH4)
primesW4 = 2:3:5:7:11:ts4 'sMinus' composites (drop 4 primesW4) w4
primesES4 = 2:3:5:7:11:ts4 'sMinus' composites (drop 4 primesES4) s4
\end{verbatim}

Figure 10: Mounting Wheels on our sieves.

Of course, wheels can be mounted on other programs, such as trial division and the Bird’s Sieve of Fig. 3. As already observed in [O’N09], the trial division program does not gain so much from being equipped with the wheel $w_4$. The reason is that in such program, the wheel $w_4$ just prunes the stream of candidates primes to $S_4[11]$, but the erased numbers are those that trial division quickly recognises as non primes, as they are multiples of the first 4 primes.

In the Epilogue of [O’N09], the author hints to the fact that it is nontrivial to modify the Bird’s sieve to support the wheel optimisation: this is true if we look at the elegant program of Fig. 3 that makes use of list comprehension, but mounting a pre-computed wheel is almost trivial if we rewrite that program by following the same structure as all our sieves, based on explicit recursion. Adding suitable parameters to the function $composites$, along the same lines of equipping our sieves with the wheel optimisation, we mount the $w_4$ wheel to the Bird’s sieve as in Fig. 11.

We observe that in this case, this optimisation is really significant, because also multiples of any prime $p$ are computed as $p \cdot S_4[p]$, rather than as $p \cdot N \geq p$, as in the original algorithm of Eratosthenes.

In other words, the program of Fig. 11 is not, strictly speaking, a genuine Sieve of Eratosthenes, but rather it implicitly encompasses in its computation the fourth iteration of the Euler’s Sieve.

\begin{verbatim}
primes = 2:3:5:7:11:s4 'sMinus' composites (drop 4 primes) s4
  where
  composites (p:ps) ss@(s:tss) =
    multiples p ss 'unionP' composites ps tss
  multiples n ss = map (n*) (n:ss)
\end{verbatim}

Figure 11: Bird’s Sieve equipped with the wheel $w_4$. 
The same holds for the O’Neill ‘faithful’ Sieve of Eratosthenes, and this explains why this program is so efficient when equipped with the pre-computed wheel $w_4$, even though without $w_4$ it is clearly outperformed even by our stream based Euler’s sieves, even without the $w_4$ optimisation (see Section 8 for details).

7 Haskeller shall not Live by Streams Alone

As already discussed, the need of merging streams is the main bottleneck of our programs. As a matter of fact, extracting $n$ numbers in an ordered list from $m$ ordered lists is not linear in $n$ when $m$ is not constant, but rather $O(n \cdot m)$. This problem looks impossible to be circumvented when generating composites as the union of a stream of ordered streams as we do in our programs.

The main virtue of the O’Neill Sieve [O’N09] is to circumvent this problem by using a priority queue to store/extract ‘efficiently’ composites to be crossed off. In that program, composites are stored in a priority queue as pairs $(k, v)$, where the key $k$ is the next composite to be extracted from the iterator that generates all multiples of a certain prime, and the value $v$ is such iterator.

Both wheels (in the sieve $W$ of Fig. 8) and the streams generating the set $E_{k+1}$ (in the sieve $ES$ of Fig. 9) are indeed iterators to generate composites, with the advantage, with respect to multiples, that two generators generate disjoint streams of composites. Therefore, in the priority queue based O’Neill Sieve, we can easily replace iterators generating multiples with iterators generated by wheels or those generating sets $E_k$ leading to two Euler’s sieve programs without the bottleneck of stream union.

In Fig. 12 we show the program that integrates the computation of the $ES$ sieve of Fig. 9 into the O’Neill Sieve. We have just made the O’Neill code more compact and modified insertion into the table of composites: we do not just insert multiples of the tail of the stream to be sieved, but we insert a stream $es$ that generates the set $E_{k+1}$. We have used variables $es$ and $ss$ with the same meaning as in the program of Fig. 9 that is, at each invocation of the function $sieve’$, if the head $c$ of the stream of prime candidates $cs$ is the prime $p_k$, then $es$ is $E_k$ and $ss$ is $S_{k|k+1}$. As before, $E_{k+1}$ is computed from $S_k$ that in turn is generated by the parameter $ss$.

Along the same lines, we can use wheels for the same purpose. The resulting program is in Fig. 13 (in this case we present the version equipped with the wheel $w4$). We incrementally compute wheels

```
sieve (c:cs) = c:sieve’ cs ss (insertPQ (c*c) (tail es) emptyPQ)
where es = map (c*) (c:cs)
ss = cs ‘sMinus’ es
sieve’ cs@(c:tcs) ss tbl
  | c < n = c : sieve’ tcs ss’ tbl'
  | otherwise = sieve’ tcs ss tbl''
where (n, m:ms) = minKeyValuePQ tbl
  es = map (c*) ss
  ss’ = tail (ss ‘sMinus’ es)
  tbl’ = insertPQ (c*c) (tail es) tbl
  tbl’’ = deleteMinAndInsertPQ m ms tbl
primes = sieve [2..]
```

Figure 12: EPQ: the priority-queue based version of ES.
as in the program of Fig. 8 in such a way that at each invocation \texttt{sieve' cs@(c:cs) w tbl}, if \(c\) is the prime \(p_k\), then \(w\) is the wheel \(w_k\). Even though this program essentially consider the same sequence of composites as the one in Fig. 12, it wastes less memory thanks to the circular representation of wheels (see function \texttt{circ}). As we will see in Section 8 this allows this program to compute efficiently the stream of primes for very large \(n\), even though it is slightly less efficient than the program of Fig. 12 for small values of \(n\).

```
sieve (c:cs) w = c:sieve' cs (nextWheel w c) (insertPQ (c*c) (circ (map (c*) w)) emptyPQ)
  where sieve' cs@(c:cs) w tbl
    | c < n = c : sieve' tcs w' tbl'
    | otherwise = sieve' tcs w tbl''
      where (n, m:ms) = minKeyValuePQ tbl
                 w' = nextWheel1 w c
                 tbl' = insertPQ (c*c) (circ (map (c*) w)) tbl
                 tbl'' = deleteMinAndInsertPQ (n+m) ms tbl
primes = 2:3:5:7:sieve s4 w4
```

Figure 13: \texttt{WPQ}: the priority-queue based version of \texttt{W}.

8 The Operation Was Successful, but the Patient Died

In this section, we present an experimental evaluation of our programs, by comparing their running time to trial division [Hwp18], the stream based Sieve of Eratosthenes by Richard Bird of Fig. 3, and the Melissa O’Neill faithful Sieve of Eratosthenes in [O’N09]. The ‘unfaithful’ Sieve of Eratosthenes in Fig. 1 and the ‘naïve’ Euler’s Sieve in Fig. 2 do not fit in our results as their running time are huge compared to those of above mentioned programs. As an example, they compute \(p_{214}\) in \(\sim 2'\) and in \(\sim 2'30''\) respectively (under the \texttt{GHCi} interpreter), and they reveal a (more than) quadratic experimental complexity (see also [Hwp18]). As we report in our experiments, \(p_{216}\) is computed in few seconds by all other programs we listed above (see Table 1).

8.1 Experimental Details

Experimental results are reported in Tables 1–4. Tables 1 and 2 report the running time of stream based programs, whereas those of priority-queue based programs are in Tables 3 and 4. Tables 1 and 3 report the running time obtained running programs under the interpreter, whereas Tables 2 and 4 report those of compiled programs.

In all tables, we call \texttt{TD} the standard Trial Division algorithm [Hwp18], \texttt{BS} the Bird’s Sieve in Fig. 3 and \texttt{O’N} the faithful Sieve of Eratosthenes in [O’N09]. Our programs are referred to as in pictures, that is \texttt{H} is the Hamming sieve of Fig. 5, \texttt{W} is the wheel based sieve of Fig. 8, \texttt{ES} is the sieve of Fig. 9, and \texttt{EPQ} and \texttt{WPQ} are sieves of Fig. 12 and 13 based on a priority queue. The superscript \(^4\) denotes the program in which we mount the pre-computed wheel \(w_4\).

We show the running time computed by the \texttt{GHCi} interpreter by using the option \texttt{:set +s} and the running time of compiled programs using the \texttt{time} Unix command (\texttt{sys} plus \texttt{user} time). All experiments have been performed on an Intel i7 quad core, 2.5 GHz, 16Gb of RAM, under MacOSX.
In all tables, $n$ means 'compute the prime $p_n$' and an asterisk * means that more than 10% of the running time has been spent in system calls, that here means essentially that the program has allocated a huge amount of memory and sometimes, the program has used virtual memory.

| $n$ | TD | BS | BS$^4$ | H | W | ES | H$^4$ | W$^4$ | ES$^4$ |
|-----|----|----|--------|---|---|----|------|------|------|
| 2$^{16}$ | 6$^8$ | 12$^4$ | 2$^5$ | 2$^5$ | 2$^4$ | 1$^5$ | 1$^4$ | 1$^1$ |
| 2$^{17}$ | 18$^5$ | 34$^6$ | 6$^6$ | 7$^2$ | 6$^0$ | 5$^8$ | 3$^6$ | 3$^9$ | 3$^0$ |
| 2$^{18}$ | 47$^7$ | 1$^{31}$ | 1$^{95}$ | 15$^5$ | 14$^1$ | 13$^5$ | 8$^5$ | 10$^4$ | 7$^1$ |
| 2$^{19}$ | 2$^{05}$ | 6$^{31}$ | 53$^8$ | 37$^7$ | 34$^8$ | 30$^3$ | 24$^2$ | 24$^4$ | 19$^8$ |
| 2$^{20}$ | 5$^{34}$ | 11$^{08}$ | 2$^{24}$ | 1$^{42}$ | 1$^{29}$ | 1$^{16}$ | 1$^{09}$ | 1$^{05}$ | 52$^1$ |

Table 1: Running time: stream programs under the GHCi interpreter and the :set $+$s option.

### 8.2 The Moral of our Experiments

The Hamming Sieve H is the slowest of our programs, but still much faster than the Bird’s Sieve (even than BS$^4$) and Trial Division when we run all these programs under the GHCi interpreter. Strangely, H gains much less than all other programs from compilation: in this case it is even slower than Trial Division and memory becomes quite early a big trouble for it.

Our stream-based programs ES and W dramatically outperforms Trial Division, the Bird’s Sieve, and even the Melissa O’Neill faithful Sieve of Eratosthenes without pre-computed wheels (even though O’N experimentally exhibits a better asymptotical complexity than ES$^4$ and W$^4$). Remarkably, for small $n$, ES$^4$ is also faster than priority-queue based programs (for $n \leq 2^{18}$ running under the interpreter, and for $n \leq 2^{21}$ in the compiled arena).

Running time of W is better than that of ES when compiled, but once the pre-computed wheel $w_4$ is mounted on both programs, ES$^4$ is slightly faster than W$^4$, both in interpreted and compiled version. This probably depends on the overhead of (lazily) computing huge wheels, whose savings in terms of composites is not so important once the wheel $w_4$ has been mounted on. When memory becomes a critical resource, W appears to be more parsimonious than ES and W$^4$ can solve efficiently problem instances in which ES$^4$ severely slows down because of memory allocation.

Nevertheless, experimental results are a bit disappointing from our point of view. Our best programs, EPQ$^4$ and WPQ$^4$, are definitively the fastest when we execute all programs inside the GHCi interpreter (Table 3), but they fail to be convincingly faster than O’N$^4$, when programs are compiled (Table 4). Similar to ES, EPQ$^4$ severely slows down computing primes beyond $p_{2^{24}}$ because

| $n$ | TD | BS | BS$^4$ | H | W | ES | H$^4$ | W$^4$ | ES$^4$ |
|-----|----|----|--------|---|---|----|------|------|------|
| 2$^{19}$ | 8$^9$ | 10$^9$ | 3$^0$ | 12$^1$ | 2$^7$ | 4$^2$ | 6$^3$ | 2$^1$ | 1$^7$ |
| 2$^{20}$ | 2$^{26}$ | 28$^1$ | 8$^3$ | 25$^5$ | 6$^7$ | 10$^7$ | 21$^1$ | 5$^0$ | 4$^3$ |
| 2$^{21}$ | 58$^7$ | 1$^{22}$ | 25$^2$ | 1$^{21}$ | 17$^0$ | 24$^5$ | 49$^0$ | 14$^1$ | 11$^8$ |
| 2$^{22}$ | 2$^{44}$ | 4$^{06}$ | 1$^{19}$ | *9$^{27}$ | 46$^5$ | 1$^{00}$ | *6$^{13}$ | 37$^4$ | 33$^8$ |
| 2$^{23}$ | 7$^{03}$ | 11$^{49}$ | 4$^{13}$ | – | 2$^{16}$ | *3$^{33}$ | – | 1$^{50}$ | 1$^{36}$ |
| 2$^{24}$ | 19$^{13}$ | – | 14$^{13}$ | – | 6$^{33}$ | – | – | 5$^{43}$ | *5$^{51}$ |

Table 2: Running time: compiled stream programs using time Unix function (usr+sys).
of memory allocation. By contrast, WPQ and WPQ⁴ are still quite efficient in the computation of the first 2²⁵ primes, when also O’N⁴ goes through a significant performance degradation (Table 4). WPQ⁴ succeeds in computing primes beyond 2²⁵, when all other programs are killed by the operating system, due to excessive memory requirements.

As we observed in Section 6, O’N⁴ encompasses the fourth iteration of the Euler’s Sieve, and this explains the huge speed-up with respect to O’N. By contrast, for our WPQ and EPQ, mounting the wheel w₄ means just saving their first 4 recursive calls, and this is not as important for them as it is for O’N. In particular, for wheel based sieves (W and WPQ) mounting the wheel w₄ just prunes the stream of candidate primes to be examined, as they efficiently compute the first 4 small wheels. Indeed, WPQ is slower but competitive with all prime generators mounting the wheel w₄ (see Table 4) and we can consider WPQ⁴ our fast prime generator as it is fast enough for small n and the most resistant to performance degradation for large n.

| n   | O’N | WPQ | EPQ  | O’N⁴ | WPQ⁴ | EPQ⁴ |
|-----|-----|-----|------|------|------|------|
| 2¹⁸ | 58⁵ | 19³ | 20⁷ | 11¹ | 8⁶ | 8⁰ |
| 2¹⁹ | 2⁰⁸⁹ | 41⁸ | 4⁴⁸ | 2⁴⁸ | 18⁹ | 1⁷⁶ |
| 2²⁰ | 4’2¹⁵ | 1’2⁹⁴ | 1’3⁵¹ | 5²⁵ | 3⁹⁷ | 3⁸⁵ |
| 2²¹ | 9’0⁸⁶ | 2’4⁵⁸ | 3’0¹⁶ | 1’5⁵⁴ | 1’2⁷⁸ | 1’²⁴⁶ |
| 2²² | – | 5’5⁵⁷ | 7’⁰⁴⁷ | 4’1⁴⁴ | 2’⁵⁷⁹ | 2’⁵³³ |

Table 3: Running time: Priority queue programs under the GHCi interpreter.

| n   | O’N | WPQ | EPQ  | O’N⁴ | WPQ⁴ | EPQ⁴ |
|-----|-----|-----|------|------|------|------|
| 2²¹ | 3⁴⁸ | 1⁵⁰ | 2³¹ | 1¹⁸ | 1²³ | 1³² |
| 2²² | 1’1¹⁵ | 3¹⁸ | 5¹ | 2⁵² | 2⁷³ | 2⁸² |
| 2²³ | *2⁴⁹⁶ | 1’1⁵¹ | *2²⁵⁴ | 5⁶³ | 1’⁰³³ | 1⁰⁰⁷ |
| 3. 2²² | *5’3⁴⁰ | 1’5⁷⁸ | *5¹⁵¹ | 1’²⁹⁹ | 1’³⁸⁵ | 1³¹⁶ |
| 2²⁴ | – | 2’3³³ | – | 2¹¹³ | 2¹⁰⁹ | 2⁰⁸² |
| 7. 2²² | – | 5⁵⁵⁹ | – | *5’⁴³⁷ | 5’⁰⁷⁹ | *⁰’²²⁹ |
| 2²⁵ | – | *8⁴⁵⁹ | – | *8’2³⁷ | *7’³⁴² | – |

Table 4: Running time: compiled priority queue programs.

9 Conclusion and Future Work

We have presented three stream based Haskell implementation of the Euler’s Sieve. The resulting programs are pretty efficient with respect to other stream based prime sieves. Their faithfulness with respect to the original algorithm is up to the overhead of union operation over streams: even though we succeed in generating each composite to be crossed off just once, a composite number will be compared several times before joining the list of composites. To overcome such problem, we have integrated ideas behind two of our Euler’s Sieves in the priority queue structure of the O’Neill’s ‘faithful’ Sieve of Eratosthenes. Our sieve of Fig. 13 based on wheels and a priority queue is our fast prime generator as it results both fast enough on small instances and the most robust to performance degradation because of memory allocation on large instances.

Several interesting questions still remains open. First of all, it would be interesting to investigate why the Hamming Sieve is so ‘resistant’ to compiler optimisations. Moreover, it would be interesting...
to look for some advanced data structure that could improve its performances, since it appears not trivial use priority queues to speed-up its computation.

An intriguing question is about using wheels to cross off composites in an array based imperative implementation of the Euler’s Sieve. Of course, the lazy computation of wheels is crucial as already observed by [Run97], and this could be hard to code properly in an eager imperative language, but the advantage would be to avoid additional data structures such as a double linked list in [Sor90].

Finally, it would be interesting to carefully look for optimisations to make our faster sieves competitive with the current prime generator in the Data.Number Haskell library.

References

[BW88] Richard Bird and Philip Wadler. *Introduction to Functional Programming*. Prentice Hall, 1988.

[Hhp18] Haskell Home Page. [https://wiki.haskell.org/](https://wiki.haskell.org/) 2018. Consulted October 2018.

[Hof79] Douglas R. Hofstadter. *Gödel, Escher, Bach: an Eternal Golden Braid*. Basic Books, 1979.

[Hwp18] Prime numbers (haskell wiki). [https://wiki.haskell.org/Prime_numbers](https://wiki.haskell.org/Prime_numbers) 2018. Consulted October 2018.

[O’N09] Melissa E. O’Neill. The genuine sieve of Erathosthenes. *Journal of Functional Programming*, 19(1):95–106, 2009.

[Pri82] Paul Pritchard. Explaining the wheel sieve. *Acta Informatica*, pages 477–485, 1982.

[Pri87] Paul Pritchard. Linear prime-number sieves: a family tree. *Science of Computer Programming*, 9:17–35, 1987.

[Run97] Colin Runciman. Lazy wheel sieves and spirals of primes. *Journal of Functional Programming*, 7(2):219–225, 1997.

[Sor90] Jonathan Sorenson. An introduction to prime number sieves. University of Wisconsin, Computer Science Technical Report 909, 1990.

[SP18] Ivano Salvo and Agnese Pacifico. Computing integer sequences: Filtering vs generation. Available at [https://arxiv.org/abs/1807.11792](https://arxiv.org/abs/1807.11792), 2018.

[Tur75] David A. Turner. SASL language manual, 1975. Tech. rept. CS/75/1. Department of Computational Science, University of St. Andrews.