OPERATORS RELATED TO SUBORDINATION FOR FREE MULTIPLICATIVE CONVOLUTIONS

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ABSTRACT. It has been shown by Voiculescu and Biane that the analytic subordination property holds for free additive and multiplicative convolutions. In this paper, we present an operatorial approach to subordination for free multiplicative convolutions. This study is based on the concepts of ‘freeness with subordination’, or ‘s-free independence’, and ‘orthogonal independence’, introduced recently in the context of free additive convolutions. In particular, we introduce and study the associated multiplicative convolutions and construct related operators, called ‘subordination operators’ and ‘subordination branches’. Using orthogonal independence, we derive decompositions of subordination branches and related decompositions of s-free and free multiplicative convolutions. The operatorial methods lead to several new types of graph products, called ‘loop products’, associated with different notions of independence (monotone, boolean, orthogonal, s-free). We also prove that the enumeration of rooted ‘alternating double return walks’ on the loop products of graphs and on the free product of graphs gives the moments of the corresponding multiplicative convolutions.

1. Introduction

Multiplication of free random variables $X_1$ and $X_2$ with distributions $\mu_1$ and $\mu_2$, respectively, leads to the multiplicative convolution $\mu_1 \boxtimes \mu_2$, introduced by Voiculescu [23] in the $C^*$-algebra framework, which gives the distribution of the product of $X_1$ and $X_2$ (the general case of measures with unbounded support was studied by Bercovici and Voiculescu [7]).

Let $\mathcal{M}_{\mathbb{R}_+}$ denote the set of probability measures on $\mathbb{R}_+ = [0, \infty)$. If $\mu \in \mathcal{M}_{\mathbb{R}_+}$, we can define

\begin{equation}
\psi_\mu(z) = \int_{\mathbb{R}_+} \frac{zt}{1-zt} d\mu(t), \quad z \in \mathbb{C}\setminus\mathbb{R}_+,
\end{equation}

which, in the case when $\mu$ has finite moments of all orders, becomes the moment generating function $\psi_\mu(z) = \sum_{n=1}^{\infty} \mu(X^n)z^n$, where $\mu(X^n)$ are the moments of the unique functional $\mu : C[X] \to \mathbb{C}$ defined by $\mu$. In order to study $\mu_1 \boxtimes \mu_2$, Voiculescu introduced the S-transform of $\mu$ defined by $S_\mu(z) = (1+z)\psi_\mu^{-1}(z)/z$, where $\psi_\mu^{-1}(z)$ denotes the inverse of $\psi_\mu(z)$ with respect to composition. The key multiplicative formula for the S-transforms is given by $S_{\mu_1 \boxtimes \mu_2}(z) = S_{\mu_1}(z)S_{\mu_2}(z)$.

2000 Mathematics Subject Classification: Primary 46L54, 46L53; Secondary 05C50

Key words and phrases: free probability, free random variable, free multiplicative convolution, s-free multiplicative convolution, orthogonal multiplicative convolution, s-free independence, subordination, free product of graphs

This work is partially supported by MNiSW research grant No 1 P03A 013 30
In our approach, a central role is played by the transform related to $\psi_\mu(z)$, namely
\begin{equation}
\eta_\mu(z) = \frac{\psi_\mu(z)}{1 + \psi_\mu(z)}
\end{equation}
where $z \in \mathbb{C}\setminus\mathbb{R}_+$. Using this transform, Biane [8] has proved the subordination property for free multiplicative convolutions of probability measures on $\mathbb{R}_+$ (and also for probability measures on the unit circle $\mathbb{T}$). Earlier, the subordination property for free additive convolutions [22] was discovered by Voiculescu [24] for compactly supported measures on $\mathbb{R}$, generalized by Biane [8] to arbitrary measures on $\mathbb{R}$ (see also [17] for a related approach).

For instance, subordination for free multiplicative convolutions of probability measures on $\mathbb{R}_+$ says that for given $\mu_1, \mu_2 \in \mathcal{M}_{\mathbb{R}_+}\setminus\{\delta_0\}$, there exist analytic self-maps $\eta_1, \eta_2$ of $\mathbb{C}\setminus\mathbb{R}_+$, such that
\begin{equation}
\eta_{\mu_1 \boxtimes \mu_2}(z) = \eta_{\mu_1}(\eta_1(z)) = \eta_{\mu_2}(\eta_2(z)).
\end{equation}
One says that $\eta_{\mu_1 \boxtimes \mu_2}$ is subordinate to both $\eta_{\mu_1}$ and $\eta_{\mu_2}$, with $\eta_1$ and $\eta_2$ being the so-called subordination functions. These functions play a key role in the analytical study of free convolutions [3,4,10].

The functions $\eta_1$ and $\eta_2$ are unique and can be viewed as $\eta$-transforms of certain probability measures on $\mathbb{R}_+$ which are not concentrated at zero. This defines a binary operation $\boxtimes$ on $\mathcal{M}_{\mathbb{R}_+}\setminus\{\delta_0\}$, namely
\begin{equation}
\eta_1(z) = \eta_{\mu_2 \boxtimes \mu_1}(z), \quad \text{and} \quad \eta_2(z) = \eta_{\mu_1 \boxtimes \mu_2}(z).
\end{equation}
The associated convolution $\mu_1 \boxtimes \mu_2$, introduced in this paper, is called the $s$-free multiplicative convolution and it plays the role of the multiplicative analog of the $s$-free additive convolution $\mu_1 \boxplus \mu_2$ studied in [16].

The subordination formulas (1.3) are related to the so-called monotone multiplicative convolution of probability measures on $\mathbb{R}_+$, introduced and studied by Bercovici [6]. This convolution can be defined by the equation
\begin{equation}
\eta_{\mu_1 \circ \mu_2}(z) = \eta_{\mu_1}(\eta_{\mu_2}(z))
\end{equation}
for $z \in \mathbb{C}\setminus\mathbb{R}_+$, where $\mu_1, \mu_2 \in \mathcal{M}_{\mathbb{R}_+}$. Using this convolution and the $s$-free multiplicative convolution, we can write subordination equations (1.3) in the convolution form
\begin{equation}
\mu_1 \boxtimes \mu_2 = \mu_1 \circ (\mu_2 \boxtimes \mu_1) = \mu_2 \circ (\mu_1 \boxtimes \mu_2)
\end{equation}
for $\mu_1, \mu_2 \in \mathcal{M}_{\mathbb{R}_+}\setminus\{\delta_0\}$.

Further, we show that one can decompose both $s$-free and free multiplicative convolutions of compactly supported measures on $\mathbb{R}_+$ which are not concentrated at zero in terms of simpler convolutions. For that purpose we introduce another convolution of $\mu_1, \mu_2 \in \mathcal{M}_{\mathbb{R}_+}\setminus\{\delta_0\}$ called the orthogonal multiplicative convolution, denoted $\mu_1 \angle \mu_2$. It plays the role of a multiplicative analog of the orthogonal additive convolution on $\mathbb{R}$, introduced and studied in [16]. Using transforms, one can define this convolution by
\begin{equation}
\eta_{\mu_1 \angle \mu_2}(z) = \frac{z \eta_{\mu_1}(\eta_{\mu_2}(z))}{\eta_{\mu_2}(z)}
\end{equation}
for $z \in \mathbb{C}\setminus \mathbb{R}_+$. Using this convolution, we obtain a decomposition of the s-free multiplicative convolution of $\mu_1 \boxtimes \mu_2$ in terms of an infinite sequence of alternating $\mu_1, \mu_2$ if these are compactly supported. In view of (1.6), this leads to a decomposition of the free multiplicative convolution.

In a similar way one can treat probability measures on the unit circle $\mathbb{T}$, denoted $\mathcal{M}_\mathbb{T}$. In that case, $\eta_\mu$ is the integral over $\mathbb{T}$ and $z$ lies inside the open unit disc $\mathbb{D}$. One defines subordination functions for $\mu_1, \mu_2$ in $\mathbb{M}_\mathbb{T}$, where $\mathbb{M}_\mathbb{T} = \mathcal{M}_\mathbb{T} \setminus \{\mu : \mu(X) = 0\}$.

Let us also observe that a number of results hold for distributions $\mu_1, \mu_2$, where $\mu_1$ is the integral over $\mathbb{T}$ and $z$ lies inside the open unit disc $\mathbb{D}$. One defines subordination functions for $\mu_1, \mu_2$ in $\mathbb{M}_\mathbb{T}$, where $\mathbb{M}_\mathbb{T} = \mathcal{M}_\mathbb{T} \setminus \{\mu : \mu(X) = 0\}$.

One of the main points of this paper is that our study of the relations between subordination functions and their decompositions uses operatorial techniques. Namely, we construct bounded operators on Hilbert spaces, which correspond to all compactly supported convolutions which appear in the subordination equations and in the decompositions of s-free convolutions. Our approach is based on the decomposition of the free product of Hilbert spaces $\mathcal{H}$, $\xi$ as (two different) orthogonal direct sums

\begin{equation}
\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}^{(n-1)}(\iota) \oplus \mathcal{H}^{(n)}(\bar{\iota})
\end{equation}

for each $\iota \in I = \{1, 2\}$, where $\mathcal{T} = 2$ and $\overline{\mathcal{T}} = 1$, with $\mathcal{H}^{(n)}(\iota)$ denoting the subspace spanned by alternating tensor products of order $n$ which do not end with a vector from $\mathcal{H}_0^{\iota} := \mathcal{H}_\iota \oplus \mathbb{C}\xi$, following the original notation of Voiculescu [21] (see also [25]), and we set $\mathcal{H}^{(0)}(\iota) = \mathbb{C}\xi$ for each $\iota \in I$. By $P_\iota(n)$ we denote the canonical projection from $\mathcal{H}$ onto $\mathcal{H}^{(n-1)}(\iota) \oplus \mathcal{H}^{(n)}(\bar{\iota})$ and we set $P_\xi$ to be the projection onto $\mathbb{C}\xi$. Finally, we define the so-called vacuum state $\varphi(\cdot) = \langle \xi, \xi \rangle$ on $\mathcal{B}(\mathcal{H})$.

According to the above decomposition, we can represent bounded free random variables in the form of ‘orthogonal series’

\begin{equation}
X_\iota = \sum_{j=1}^{\infty} X_\iota(j),
\end{equation}

where $\iota \in I$ and $X_\iota(j) = P_\iota(j)X_\iota P_\iota(j)$. These series play a crucial role in our study of subordination for both free additive convolutions [16] and free multiplicative convolutions, studied in this paper. Let us remark that the ‘orthogonal series’ were introduced in a more general context of ‘monotone closed *-algebras’ of operators ‘affiliated’ with unital *-algebras [14]. In that approach, we used the ‘monotone tensor product’ $\boxtimes$ (reminding the von Neumann tensor product) to represent free random variables.

Let $\mu_1, \mu_2$ be $\varphi$–distributions of bounded random variables $X_1$ and $X_2$, respectively, which are free with respect to $\varphi$. In the case of free additive convolutions, crucial was the decomposition of their sum in terms of ‘additive subordination branches’, namely $X_1 + X_2 = B_1 + B_2$, with

\begin{equation}
B_\iota = S^{\text{odd}}_\iota + S^{\text{even}}_\iota,
\end{equation}

(1.10)
where
\[
S^\text{odd}_i = \sum_{j \text{ odd}} X_i(j) \quad \text{and} \quad S^\text{even}_i = \sum_{j \text{ even}} X_i(j),
\]
for each \( i \in I \). Although the subordination branch \( B_i \) is a bounded operator on the ‘free Fock space’ \( \mathcal{H} \) for each \( i \in I \), it acts non-trivially only on its subspace, the ‘s-free Fock space’ \( \mathcal{K}_i \), defined as
\[
\mathcal{K}_i = \bigoplus_{n \text{ odd}} \mathcal{H}^{(n-1)}(i) \bigoplus \mathcal{H}^{(n)}(7),
\]
and therefore, it is appropriate to view \( B_i \) as an element of \( B(D)_K \).

For operatorial subordination in the multiplicative case, we need to modify the operators \( S^\text{odd}_i, S^\text{even}_i \) and take
\[
R^\text{odd}_i = S^\text{odd}_i \quad \text{and} \quad R^\text{even}_i = P + S^\text{even}_i.
\]
Moreover, in order to include all bounded positive random variables, we shall need the operation \( \boxtimes \) to be defined for all compactly supported \( \mu_1, \mu_2 \in \mathcal{M}_{+} \). For that purpose we set
\[
\mu \boxtimes \delta_0 = \delta_{\mu}(X) \quad \text{and} \quad \delta_0 \boxtimes \mu = \delta_0
\]
for compactly supported \( \mu \in \mathcal{M}_{+} \), which turns out natural in our operatorial setting.

Using these notations, and choosing, for simplicity, an existence-type formulation, we can summarize our subordination result for positive operators as follows.

**Subordination for Positive Operators.** If \( X_1 \) and \( X_2 \) are positive, then there exists a decomposition
\[
\sqrt{X_1 X_2} = \sqrt{Y} \sqrt{\mathcal{F}_0},
\]
such that the \( \varphi \)-distributions of \( y \) and \( Y \) are \( \mu_1 \) and \( \mu_2 \boxtimes \mu_1 \), respectively, and the pair \( (y-1, Y-1) \) is monotone independent w.r.t. \( \varphi \). Moreover, the \( \varphi \)-distribution of \( Y \) agrees with that of \( \sqrt{R_2 \boxtimes R_1} \), and the pair \( (R_2^\text{odd} - 1, R_1^\text{even} - 1) \) is s-free independent w.r.t. \( \varphi, \psi \), where \( \psi \) is the state associated with any unit vector \( \zeta \in \mathcal{H}_2^0 \).

Moreover, we further decompose the ‘positive subordination branches’
\[
\sqrt{R_2^\text{odd} R_1^\text{even}} \quad \text{using the notion of ‘orthogonal independence’}. \quad \text{These decompositions correspond to decompositions of s-free convolutions in terms of orthogonal convolutions. In a similar way we obtain operatorial subordination results for unitary operators, or even more generally, for bounded operators.}

The main examples of operators related to subordination for the free additive convolution, or, more generally, to various notions of independence \( I \), are the adjacency matrices of subgraphs of the corresponding \( I \)-product graphs \( G_1 I G_2 \) (in our study, \( I \) stands for orthogonal, comb, star, s-free or free). In fact, to each independence \( I \), we can associate an additive and a multiplicative convolution,
\[
\mu_1 +_I \mu_2 \quad \text{and} \quad \mu_1 \times_I \mu_2
\]
respectively. Recall that the additive $\mathcal{I}$–convolution of spectral distributions of rooted graphs corresponds to the addition of ‘monochromatic’ $\mathcal{I}$–independent adjacency matrices $S_i$, $i \in I$, and that, in turn, is related to the enumeration of rooted (i.e. root-to-root) walks on $G_1 \mathcal{I} G_2$ (for details, see [2] and its references).

In order to find a ‘universal’ multiplicative analog of this theorem, one needs to introduce a new concept of a product of $G_1$ and $G_2$ for each $\mathcal{I}$–independence, which we call the $\mathcal{I}$–loop product and denote $G_1 \mathcal{I} G_2$. Roughly speaking, $G_1 \mathcal{I} G_2$ is obtained by adding colored loops to $G_1 \mathcal{I} G_2$ in a suitable way. We assume that the product graph has a ‘natural coloring’, by which we mean that each copy of $G_i$ is colored with color $\iota$, where $\iota \in I$. The procedure of adding loops is equivalent to replacing each $S_i$ by its ‘unitization’ $R_i$ obtained from $S_i$ by adding some projection, in such a way that makes the pair $(R_1 - 1, R_2 - 1)$ $\mathcal{I}$–independent.

Quite naturally, in order to formulate our result, we shall use the formal power series $\eta_Z$ corresponding to the variable $Z = R_2 R_1$, namely

$$\eta_Z(z) = \sum_{n=1}^{\infty} N_Z(n) z^n,$$

and interpret $N_Z(n)$ as the ‘first return moments’ in the state $\varphi_e$ associated with the root $e$. These moments are related to the enumeration of walks of the same class, which we find to be rooted alternating $d$-walks (‘$d$-walk’ is our abbreviation of ‘double return walks originating with color 1’) counted on different products. Finally, in view of the relation to independence $\mathcal{I}$ just mentioned, these moments agree with the $\eta$–moments of the corresponding multiplicative convolutions $\mu_1 \times_\mathcal{I} \mu_2$. It is not hard to see that this result can be generalized to the framework of random walks [26].

**Multiplication theorem.** Let $G_1 \mathcal{I} G_2$ be naturally colored and let $A(G_1 \mathcal{I} G_2) = R_1 + R_2$ be the decomposition of its adjacency matrix induced by the coloring. Then

$$N_Z(n) = N_{\mu_1 \times_\mathcal{I} \mu_2}(n) = |D_{2n}(e)|$$

where $Z = R_2 R_1$ and $D_{2n}(e)$ denotes the set of rooted alternating $d$-walks on $G_1 \mathcal{I} G_2$ of length $2n$, where $n \in \mathbb{N}$. Moreover, the pair $(R_1 - 1, R_2 - 1)$ is $\mathcal{I}$–independent.

The paper is organized as follows. In Section 2, we introduce basic notions, including the s-free multiplicative convolution. In Section 3, we introduce and study the concepts of comb– and star loop products of graphs and find relations between rooted alternating $d$-walks on these graphs and the monotone and boolean multiplicative convolutions, respectively. In Section 4, we introduce and study the orthogonal multiplicative convolution. In Section 5 we define ans study the corresponding notion of the orthogonal loop product of rooted graphs. In Section 6 we show, by means of analytical methods, that the definition of the orthogonal multiplicative convolution can be extended to arbitrary measures on $\mathbb{R}_+$ and $\mathbb{T}$. The main operatorial results of the paper are contained in Sections 7 and 8, where we introduce and study operators on the free and s-free Fock spaces which are related to subordination for multiplicative free convolutions and their decompositions. Finally, in Section 9 we find a relation between free and s-free
multative convolutions and the enumeration of rooted alternating d-walks on the
free product of graphs and on the s-free loop product of graphs, respectively.

Throughout the whole paper we understand that \( I = \{ 1,2 \} \) and we adopt the notation \( \overline{1} = 2 \) and \( \overline{2} = 1 \).

2. Preliminaries

By a non-commutative probability space we understand a pair \((\mathcal{A}, \varphi)\), where \( \mathcal{A} \) is a
unital algebra over \( \mathbb{C} \) and \( \varphi \) is a linear functional \( \varphi : \mathcal{A} \rightarrow \mathbb{C} \) such that \( \varphi(1) = 1 \). If \( \mathcal{A} \) is
a unital *-algebra and \( \varphi \) is positive (called a state), then \((\mathcal{A}, \varphi)\) is called a *-probability space. If, in addition, \( \mathcal{A} \) is a \( C^* \)-algebra, then \((\mathcal{A}, \varphi)\) is called a \( C^* \)-probability space.

By the Gelfand-Naimark-Segal theorem, a \( C^* \)-algebra, then \( \mathcal{A} \) is a \( C^* \)-probability space can always be realized
as a subalgebra of bounded operators on a Hilbert space \( \mathcal{H} \) with a distinguished unit
vector \( \xi \), for which \( \varphi(a) = \langle a\xi, \xi \rangle \) for \( a \in \mathcal{A} \).

By a random variable we will understand any element \( a \) of the considered algebra \( \mathcal{A} \). If \( \mathcal{A} \) is equipped with an involution, then a random variable \( a \) will be called self-adjoint
if \( a = a^* \). The \( \varphi \)-distribution of a random variable \( a \) is the functional \( \mu_a : \mathbb{C}[X] \rightarrow \mathbb{C} \) given by \( \mu_a(1) = 1 \), \( \mu_a(X^n) = \varphi(a^n) \). In particular, if \((\mathcal{A}, \varphi)\) is a \( C^* \)-probability space, then the distribution \( \mu_a \) of a self-adjoint random variable \( a \in \mathcal{A} \) extends to a compactly
supported probability measure \( \mu \) on the real line. In that case we will often use the
same notation \( \mu \) for both the distribution of \( a \) and the associated compactly supported
probability measure.

Various notions of ‘independence’ \( \mathcal{I} \) lead to several types of convolutions of distributions (probability measures). If we have two random variables, \( X_1 \in \mathcal{A}_1 \) and \( X_2 \in \mathcal{A}_2 \) with \( \varphi \)-distributions \( \mu_1 \) and \( \mu_2 \), respectively, , where \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are \( \mathcal{I} \)-independent sub-

algebras of a noncommutative probability space \((\mathcal{A}, \varphi)\), then the additive convolution of \( \mu \) and \( \nu \) associated with \( \mathcal{I} \)-independence is the \( \varphi \)-distribution of the sum \( X_1 + X_2 \). In turn, the multiplicative convolution of \( \mu \) and \( \nu \) is the \( \varphi \)-distribution of \( X_2X_1 \). In this paper we are interested in the free and s-free multiplicative convolutions associated
with free and s-free independence, respectively. In order to decompose them, we
shall use the monotone multiplicative convolution introduced by Bercovici [6] and we
will introduce the orthogonal multiplicative convolution associated with the notion of
‘orthogonal independence’ [16].

**Definition 2.1.** Let \( \mu_1, \mu_2 \in \mathcal{M}_{\mathbb{R}_+} \setminus \{ \delta_0 \} \) and let \( \eta_1 \) and \( \eta_2 \) be the associated subordina-
tion functions. The unique probability measures \( \sigma_1, \sigma_2 \in \mathcal{M}_{\mathbb{R}_+} \setminus \{ \delta_0 \} \) such that \( \eta_1 = \eta_{\sigma_1} \)
and \( \eta_2 = \eta_{\sigma_2} \) will be called the s-free multiplicative convolution of \( \mu_1, \mu_2 \) and \( \mu_2, \mu_1 \), respectively. We set \( \sigma_1 = \mu_1 \boxplus \mu_2 \) and \( \sigma_2 = \mu_2 \boxplus \mu_1 \). In a similar way we define the s-free multiplicative convolution of \( \mu_1, \mu_2 \in \mathcal{M}_* \).

The s-free multiplicative convolution defines a binary operation \( \boxplus \) on both \( \mathcal{M}_{\mathbb{R}_+} \setminus \{ \delta_0 \} \) and \( \mathcal{M}_* \). It can be seen that it is neither commutative nor associative. We will show
later that it is related to the concept of ‘freeness with subordination’, or ‘s-free indepen-
dence’, introduced in [16] (a relation between freeness and monotone independence was
also studied in [15]). Moreover, it allows us to rewrite (1.3) in terms of convolutions.
Proposition 2.1. If \( \mu_1, \mu_2 \in \mathcal{M}_{\mathbb{R}_+}\setminus \{\delta_0\} \), then
\[
\mu_1 \boxplus \mu_2 = \mu_1 \bigcirc (\mu_2 \boxtimes \mu_1) = \mu_2 \bigcirc (\mu_1 \boxtimes \mu_2)
\]
is the decomposition of the free multiplicative convolution corresponding to the subordination equations (1.3). A similar result holds for \( \mu_1, \mu_2 \in \mathcal{M}_* \).

Proof. This fact is an immediate consequence of (1.3), in view of (1.5). \( \blacksquare \)

By a rooted set we understand a pair \( (X, e) \), where \( X \) is a countable set and \( e \) is a distinguished element of \( X \) called root. By a rooted graph we understand a pair \( (\mathcal{G}, e) \), where \( \mathcal{G} = (V, E) \) is a non-oriented graph with the set of vertices \( V = V(\mathcal{G}) \), and the set of edges \( E = E(\mathcal{G}) \subseteq \{(x, x') : x, x' \in V\} \) and \( e \in V \) is a distinguished vertex called the root. We identify \( (x, x') \) with \( (x', x) \) since we consider non-oriented graphs, but when speaking of walks we find it convenient to say that \( (x, x') \) begins with \( x \) and terminates with \( x' \). Another reason for using this notation for edges is that of main interest to us are graphs which have loops, i.e. edges of the form \( (x, x) \), where \( x \in V \). The notion of a rooted graph can be easily generalized to allow multiple edges. Formally, we then obtain a rooted multigraph, but we will still use the term ‘rooted graph’, or simply ‘graph’ since all graphs will be considered to have a root. Moreover, we will very often omit the root in our notation and denote by \( \mathcal{G} \) the rooted graph \( (\mathcal{G}, e) \) if no confusion arises. Thus, in a graph \( \mathcal{G} \) we denote by \( n(x, x') \) the number of edges connecting \( x \) and \( x' \) (it may be zero).

For (rooted) graphs we will also use the notation
\[
V^0 = V\setminus \{e\}.
\]
Two vertices \( x, x' \in V \) are called adjacent if \( x \) and \( x' \) are connected by an edge, which we denote \( x \sim x' \). The degree of \( x \in V \) is defined by \( \kappa(x) = \sum_{x' \sim x} n(x, x') \). A graph is called locally finite if \( \kappa(x) < \infty \) for every \( x \in V \). It is called uniformly locally finite if \( \sup \{\kappa(x) : x \in V\} < \infty \). All graphs considered in this paper will be connected and uniformly locally finite.

For \( x \in V \), let \( \delta(x) \) be the indicator function of the one-element set \( \{x\} \). Then \( \{\delta(x), x \in V\} \) is an orthonormal basis of the Hilbert space \( l^2(V) \) of square integrable functions on the set \( V \), with the usual inner product. The adjacency matrix \( A = A(\mathcal{G}) \) of a graph \( \mathcal{G} \) is a matrix defined by \( A(\mathcal{G})_{x, x'} = n(x, x') \) We identify \( A \) with the densely defined symmetric operator on \( l^2(V) \) defined by
\[
A(\mathcal{G})\delta(x) = \sum_{x \sim x'} n(x, x')\delta(x')
\]
for \( x \in V \). Notice that the sum on the right-hand-side is finite since our graph is assumed to be locally finite. It is known that \( A(\mathcal{G}) \) is bounded iff \( \mathcal{G} \) is uniformly locally finite.

By the spectral distribution of \( A(\mathcal{G}) \) in a state \( \psi \) on \( l_2(V) \) we understand the measure \( \mu \) for which
\[
\psi(A^n) = \int_{\mathbb{R}} x^n \mu(dx), \quad n \in \mathbb{N} \cup \{0\},
\]
and by the spectral distribution of the rooted graph \((G, e)\) we understand the spectral distribution of \(A(G)\) in the state \(\varphi_e(.) = \langle \delta(e), \delta(e) \rangle\).

A walk from \(v_0\) to \(v_n\) on a graph \(G\) is an alternating sequence of vertices and edges of the form
\[
(2.5)\quad w = (v_0, \beta_1, v_1, \beta_2, v_2, \ldots, v_{n-1}, \beta_n, v_n),
\]
with vertices \(v_0, v_1, \ldots, v_n \in V(G)\) and edges \(\beta_1, \beta_2, \ldots, \beta_n \in E(G)\), such that \(\beta_i\) is an edge connecting \(v_{i-1}\) and \(v_i\). The length of \(w\) is given by \(l(w) = n\). We allow \(v_{i-1} = v_i\), in particular this happens if \(\beta_i\) is a loop.

A subwalk of \(w\) is a subsequence of \(w\) of the form
\[
(2.6)\quad w' = (v_i, \beta_i, v_{i+1}, \beta_{i+1}, \ldots, \beta_j, v_j),
\]
where \(0 \leq i < j \leq n\). In the case when \(G\) does not have multiple edges, we can identify \(w\) with the sequence \((v_0, v_1, \ldots, v_n)\) with the understanding that \(v_{i-1} \sim v_i\) (in this case there is no confusion which edge connecting \(v_{i-1}\) and \(v_i\) is chosen) for \(1 \leq i \leq n\).

The set of walks from \(v\) to \(v\) will be denoted \(W(v, w)\) and we set \(W(v, v) = W(v)\). A walk from \(v = v_0\) to \(v = v_n\), in which \(v_i \neq v\) for all \(1 \leq i \leq n - 1\) will be called an \(f\)-walk. Note that we do not require all vertices \(v_0, v_1, \ldots, v_{n-1}\) in an \(f\)-walk to be distinct. The set of \(f\)-walks from \(v\) to \(v\) will be denoted \(F(v)\). Subsets of \(W(v)\) and \(F(v)\) consisting of walks and \(f\)-walks of length \(n\), respectively, will be denoted \(W_n(v)\) and \(F_n(v)\), respectively.

A walk \(w\) can also be represented as a sequence of subwalks; in particular, an alternating sequence of subwalks and subwalks of the form
\[
(2.7)\quad w = (w_0, \beta_1, w_1, \beta_2, w_2, \ldots, w_{r-1}, \beta_r, w_r),
\]
where \(w_0\) is a walk from \(v_0\) to some \(v_{i(1)}\), \(\beta_1\) is an edge connecting \(v_{i(1)}\) with \(v_{i(1)+1}\), \(w_1\) is a walk from \(v_{i(1)+1}\) to some \(v_{i(2)}\), \(\beta_2\) is an edge connecting \(v_{i(2)}\) with \(v_{i(2)+1}\), and so on, finally, \(w_r\) is a walk from \(v_{i(r)+1}\) to \(v_{i(r+1)} = v_n\). We shall also use similar representations of walks which begin or end with an edge. Representing a walk in terms of edges and subwalks is very convenient when the subwalks \(w_1, w_2, \ldots, w_r\) are of special type (for instance, are \(f\)-walks).

A graph \(G\) whose edges are colored with colors from the set \(I\) will be called an \(I\)-edge-colored graph. In our case, we will always assume that \(I = \{1, 2\}\), so we can also use the term ‘2-edge-colored graph’. A walk in such a graph will be called alternating if the colors of its edges alternate.

**Definition 2.2.** Let \(G\) be \(I\)-colored. A walk \(w \in W(v)\) on \(G\) will be called an alternating double return walk, or simply an alternating \(d\)-walk, if it is alternating, begins with an edge of color 1, and can be represented as a pair \((u_1, u_2)\) of subwalks, such that \(u_1, u_2 \in F(v)\). The set of alternating \(d\)-walks from \(v\) to \(v\) will be denoted \(D(v)\), and we set \(D_m(v) = W_m(v) \cap D(v)\).

Edge-coloring by a two-element set is natural in the case of many products of rooted graphs. In particular, the products of rooted graphs of type \(G_1 \square G_2\) have the property that every edge of the product graph belongs either to a copy of \(G_1\), or to a copy of \(G_2\). Thus we can color the edges of all copies of \(G\), with color \(i\), where \(i \in I\), in which case
we will say that $G_1I_2$ is naturally colored. In turn, we will say that the product $G_1I_2G_2$ is naturally colored if its coloring is inherited from $G_1I_2G_2$. Let us also remark that our condition that an alternating walk should begin with an edge of color 1 (and thus end with an edge of color 2) is caused by the fact that we want to identify the given walk $w$ and its ‘reverse’ obtained by reversing the order in (2.12).

We end this Section with elementary formulas for the ‘first return moments’ of a random variable $X$, by which we understand the coefficients of the formal power series $\eta_X$ associated with the distribution of $X$. If $X$ is a random variable in a non-commutative probability space $(\mathcal{A}, \varphi)$, then the ‘moment generating function’ and the ‘first return moment generating function’, respectively, associated with the $\varphi$-distribution $\mu$ of $X$, are given by formal power series

$$\psi_X(z) = \sum_{n=1}^{\infty} \mu(X^n)(n)z^n, \quad \eta_X(z) = \frac{\psi_X(z)}{1 + \psi_X(z)} = \sum_{n=1}^{\infty} N_X(n)z^n,$$

where the numbers $N_X(n)$, $n \in \mathbb{N}$, will sometimes be called ‘first return moments of $X$’.

Below we give a convenient algebraic formula for these ‘first return moments’. For that purpose, let us take the extension of $(\mathcal{A}, \varphi)$ to a larger noncommutative probability space, namely, the free product with identified units $A \ast \mathbb{C}[P]$, where $P^2 = P$, with the state given by the linear extension of

$$\varphi(P) = 1, \quad \varphi(w_1Pw_2) = \varphi(w_1)\varphi(w_2)$$

for any $w_1, w_2 \in A \ast \mathbb{C}[P]$, where, slightly abusing notation, we also denote the new state by $\varphi$ [13]. In particular, in the $C^*$-algebra context, $P$ can be interpreted as the projection onto $\mathbb{C}\xi$, where $\xi$ is the cyclic unit vector of the GNS triple associated with $(\mathcal{A}, \varphi)$.

**Proposition 2.2.** Let $X$ be a random variable in a noncommutative probability space $(\mathcal{A}, \varphi)$. Then

$$N_X(n) = \varphi(X(P^\perp X)^{n-1})$$

for $n \in \mathbb{N}$, where $P^\perp = 1 - P$.

*Proof.* This is a straightforward consequence of (1.2). ■

Finally, let us prove an elementary fact about a relation between the ‘first return moments’ of products of ‘monochromatic’ adjacency matrices in certain $I$-colored graphs and the cardinalities of the sets $D_{2n}(e)$.

**Proposition 2.3.** Let $(\mathcal{G}, e)$ be an $I$-colored uniformly locally finite graph, and let $A(\mathcal{G}) = A_1 + A_2$ be the decomposition of $A(\mathcal{G})$ induced by the coloring. If the set of rooted alternating $f$-walks of even lengths is empty, then

$$N_Z(n) = |D_{2n}(e)|,$$

where $Z = A_2A_1$ and the numbers $N_Z(n)$ correspond to the state $\varphi_e(.) = \langle \cdot, \delta(e), \delta(e) \rangle$.

*Proof.* If we take $Z = A_2A_1$ in Proposition 2.2, then $N_Z(n)$ is equal to the number of rooted alternating walks of length $2n$, which begin with an edge of color 1 and such that intermediate returns to the root can occur only after odd numbers of steps. If there
are no rooted alternating f-walks of even length, then the first return to the root must occur after an odd number of steps and thus the second return to the root must occur after an even number of steps. Therefore, if we consider rooted alternating walks which have an even number of steps and such that intermediate returns occur only after an odd number of steps, these are d-walks. This proves our assertion.

Let us note that all graph products considered in this paper satisfy the assumptions of Proposition 2.3, and therefore, walk-counting is always reduced to rooted alternating d-walks.

3. COMB AND STAR LOOP PRODUCTS OF GRAPHS

We begin with recalling the notions of the additive [18] and multiplicative [6] monotone convolutions and show that the \( \eta \)-moments of the latter are related to the enumeration of alternating d-walks on a new version of the comb product of graphs called the ‘comb loop product’.

**Definition 3.1.** Two subalgebras \( A_1, A_2 \) of a unital algebra \( A \) are monotone independent with respect to a normalized linear functional \( \varphi \) on \( A \) if

\[
\varphi(w_1a_1b) = \varphi(b)\varphi(w_1a_1) \quad \text{and} \quad \varphi(ba_2w_2) = \varphi(b)\varphi(a_2w_2),
\]

\[
\varphi(w_1a_1ba_2w_2) = \varphi(b)\varphi(w_1a_1a_2w_2),
\]

whenever \( a_1, a_2 \in A_1, b \in A_2 \) and \( w_1, w_2 \) are arbitrary elements of the unital algebra \( \text{alg}(A_1, A_2) \) generated by \( A_1 \) and \( A_2 \). In particular, we will say that the pair \((a, b)\) of elements of \( A \) is monotone independent w.r.t. \( \varphi \) if the algebras generated by these elements are monotone independent.

Suppose \( \varphi \)-distributions of random variables \( a_1 \) and \( a_2 \) are \( \mu_1 \) and \( \mu_2 \), respectively. If the pair \( (a_1, a_2) \) is monotone independent w.r.t. \( \varphi \), then the \( \varphi \)-distribution of the sum \( a_1 + a_2 \) is the monotone additive convolution \( \mu_1 \rhd \mu_2 \). In turn, if the pair \( (a_1 - 1, a_2 - 1) \) is monotone independent w.r.t. \( \varphi \), then the \( \varphi \)-distribution of the product \( a_2a_1 \) is the monotone multiplicative convolution \( \mu_1 \odot \mu_2 \). The corresponding formal power series satisfy (1.5). If \( \mu_1, \mu_2 \in \mathcal{M}_{\mathbb{R}_+} \) or \( \mu_1, \mu_2 \in \mathcal{M}_T \), then (1.5) is understood in terms of analytic self-maps of \( \mathbb{C} \setminus \mathbb{R}_+ \) or \( \mathbb{T} \), respectively. For details, see [19] (additive case) and [6] (multiplicative case).

It is known that the monotone independence can be associated with the comb product of graphs [1]. Let us recall the definition of the comb product and follow up with the corresponding loop product.

**Definition 3.2.** The comb product of rooted graphs \((G_1, e_1)\) and \((G_2, e_2)\) is the rooted graph \((G_1 \rhd G_2, e)\) obtained by attaching a copy of \( G_2 \) by its root \( e_2 \) to each vertex of \( G_1 \), where we denote by \( e \) the vertex obtained by identifying \( e_1 \) and \( e_2 \). If no confusion arises, we denote the comb product by \( G_1 \rhd G_2 \). If we identify its set of vertices with \( V_1 \times V_2 \), then its root is identified with \( e_1 \times e_2 \).
Definition 3.3. Suppose the edges of the comb product \((G_1 \triangleright\triangleright G_2, e)\) are naturally colored. The \textit{comb loop product} of rooted graphs \((G_1, e_1)\) and \((G_2, e_2)\) is the rooted graph \((G_1 \triangleright\triangleright G_2, e)\) obtained from \((G_1 \triangleright\triangleright G_2, e)\) by attaching a loop of color 1 to each vertex but the root of each copy of \(G_2\).

Let us justify the above definition. We follow the observation made by Bercovici [6] that in order to introduce a multiplicative convolution associated with monotone independence, one needs to ‘unitize’ the usual monotone independent variables. It has been shown in [5] that one of the possible choices is to take variables of the form

\begin{equation}
R_1 = a_1 \otimes P_2 + 1_1 \otimes P_2^\perp \quad \text{and} \quad R_2 = 1_1 \otimes a_2,
\end{equation}

where \(a_i, 1_i \in B(H_i),\) with \((H_i, \xi_i),\) being Hilbert spaces with distinguished unit vectors and identity operators \(1_i,\) and \(i \in I.\) Then the moments of the product \(R_2R_1\) in the state associated with the vector \(\xi_1 \otimes \xi_2\) agree with the moments of \(\mu_1 \oslash \mu_2.\)

If \(a_i\) is taken to be the adjacency matrix of a (uniformly locally finite) graph \(G_i,\) \(i \in I,\) then the term \(1_1 \otimes P_2^\perp\) corresponds to gluing a loop of color 1 to each vertex but the root of each copy of \(G_2.\) Thus, \(R_1\) is obtained from the usual comb product adjacency matrix \(S_1\) by adding a projection \(L_1,\) whereas \(R_2 = S_2,\) where

\begin{equation}
S_1 = a_1 \otimes P_2, \quad L_1 = 1_1 \otimes P_2^\perp
\end{equation}

and \(R_1, R_2\) are the adjacency matrices of ‘monochromatic’, 1 and 2-colored subgraphs, respectively. This realization can be generalized to arbitrary random variables in non-commutative probability spaces, using the extensions of functionals given by (2.9).

Remark 3.1. It is not difficult to show that the comb loop product of graphs \(G_1\) and \(G_2\) is isomorphic to the usual comb product of larger graphs \(\tilde{G}_1\) and \(\tilde{G}_2.\) However, in order to keep the same coloring of the product graph (which is needed for counting alternating walks) one has to use two colors for \(\tilde{G}_2.\) Namely, \(\tilde{G}_2\) is obtained from \(G_2\) by gluing a loop of color 1 to each vertex but the root of \(G_2.\) Then we indeed have \(G_1 \triangleright\triangleright G_2 = \tilde{G}_1 \triangleright\triangleright \tilde{G}_2,\) but usefulness of this relation seems to be limited (roughly speaking, we have a simpler product but a more complicated coloring).

Example 3.1. Let us consider the example of the comb loop product of graphs given in Figure 1. We choose both \(G_1\) and \(G_2\) to have a loop at the root (to distinguish the loops, we draw loops of color 1 smaller than loops of color 2, which is helpful in the enumeration of alternating walks). Then \(G_1 \triangleright\triangleright G_2\) has loops of two types: loops whose origin can be traced back to graphs \(G_1\) and \(G_2\) (these are all loops which are at the glueing points), or are added to the usual comb product in the process of forming the comb loop product. Denote by \((v, v)\) the loop from \(v\) to \(v\) of color \(i.\) The following walk is a rooted alternating d-walk:

\((\beta_1, w_1, \beta_2, w_2, \beta_3, w_3, \beta_4, w_4) \in D_8(e),\)

where the sequence of edges of color 1,

\((\beta_1, \beta_2, \beta_3, \beta_4) = ((e, x), (x, x'), (x', x), (x, e))\)

forms a rooted f-walk of color 1, which is interlaced with the sequence \((w_1, w_2, w_3, w_4)\) of loops of color 2: \(w_1 = w_3 = (x, x) \in F_1(x)\) and \(w_2 = (x', x') \in F_1(x')\) and \(w_4 = (e, e) \in F_1(e)\) and \(w_4 = (e, e) \in F_1(e).\)
The above combinatorial formula allows us to find a correspondence between these numbers \( N \) where the pair \((D_4, w)\) occurs as the end of \( G \) associated loop product \( G \) moments and rooted alternating f-walks on \( G \)s. For any of the loops, \( w_i, 1 \leq i \leq 4 \), is replaced by an arbitrary alternating f-walk from the corresponding vertex to itself which begins and ends with an edge of color 2, we still get a rooted alternating d-walk. The simplest examples of a rooted alternating d-walk are of course given by: \((\beta_0, w_0) \in D_2(e)\), where \( \beta_0 = (e, e)_1, w_0 = (e, e)_2 \) and \((\beta_1, w_1, \beta_4, w_4) \in D_4(e)\).

**Theorem 3.1.** The Multiplication Theorem holds for monotone independence, the associated loop product \( G_1 \vDash \vDash G_2 \), and the multiplicative convolution \( \mu_1 \odot \mu_2 \).

**Proof.** From Definition 3.3 and (3.1) it follows that

\[
Z = A(G_1 \vDash \vDash G_2) = R_1 + R_2,
\]

where the pair \((R_1 - 1, R_2 - 1)\) is monotone independent w.r.t. \( \varphi_e \), the state associated with \( e \) and \( R_i, i \in I \), is the adjacency matrix of the \( i \)-colored subgraph. Moreover, by Proposition 2.2, \( N_2(n) \) is the number of alternating d-walks on \( G_1 \vDash \vDash G_2 \) of length \( 2n \). This proves the first equation in (1.13) for monotone independence. Now, from equation (1.5), we obtain the combinatorial formula

\[
N_{\mu_1 \odot \mu_2}(n) = \sum_{r=1}^{n} N_{\mu_1}(r) \sum_{k_1+k_2+\cdots+k_r=n} N_{\mu_2}(k_1)N_{\mu_2}(k_2)\cdots N_{\mu_2}(k_r),
\]

where it is tacitly assumed that the indices \( k_1, k_2, \ldots, k_r \) are positive integers and numbers \( N_{\mu}(n) \) are coefficients of \( \eta_{\mu} \) treated as formal power series

\[
\eta_{\mu}(z) = \sum_{n=1}^{\infty} N_{\mu}(n)z^n.
\]

The above combinatorial formula allows us to find a correspondence between these \( \eta \)-moments and rooted alternating d-walks on \( G_1 \vDash \vDash G_2 \). Let us observe that \( N_{\mu_1}(r) \) is the number of rooted f-walks of length \( r \) on \( G_1 \) and \( N_{\mu_2}(k) \) is the number of rooted f-walks of length \( k \) on \( G_2 \). On the other hand, recall that in the comb loop product \( G_1 \vDash \vDash G_2 \) there is only one copy of \( G_1 \) (with \( c_1 \) identified with the root \( e \) of the product graph), with a copy of \( G_2 \) attached by its root to every vertex \( x \) of \( G_1 \) (the vertex \( x \) becomes then the only common vertex of this copy of \( G_2 \) and the original copy of \( G_1 \)). In addition, loops of color 1 are glued to all vertices but the roots of these copies of \( G_2 \). Therefore, each rooted alternating d-walk on \( G_1 \vDash \vDash G_2 \) consists of a sequence of edges of \( G_1 \) which forms an f-walk of color 1, namely \( c = (v_0, v_1, v_2, \ldots, v_{r-1}, v_r) \in F(e) \), and alternating f-walks \( c_i \in F(v_i), i = 1, \ldots, r \), with the first edge of color 2, attached to every vertex

\[
\begin{align*}
F_1(e). & \quad \text{Note that the first return to } e \text{ occurs at the end of } \beta_4, \text{ whereas the second return occurs as the end of } w_4. \text{ When any of the loops, } w_i, 1 \leq i \leq 4, \text{ is replaced by an arbitrary alternating f-walk from the corresponding vertex to itself which begins and ends with an edge of color 2, we still get a rooted alternating d-walk. The simplest examples of a rooted alternating d-walk are of course given by: } (\beta_0, w_0) \in D_2(e), \text{ where } \beta_0 = (e, e)_1, w_0 = (e, e)_2 \text{ and } (\beta_1, w_1, \beta_4, w_4) \in D_4(e). \\
\end{align*}
\]
of \( c \). Note that the number of alternating \( f \)-walks from \( v_i \) to \( v_i \) with the first edge of color 2 is equal to the number of walks from \( e_2 \) to \( e_2 \) in graph \( G_2 \) – the only difference is that if the latter has length \( k_i \), the former has length \( 2k_i - 1 \) since each edge must be followed by a loop of color 2. Thus, the contribution from each product of the form

\[
N_{\mu_1}(r)N_{\mu_2}(k_1)N_{\mu_2}(k_2) \ldots N_{\mu_2}(k_r)
\]

to the RHS of the formula for \( N_{\mu_1 \otimes \mu_2}(n) \) is equal to the number of all such \( f \)-walks \( c \) of color 1 which have \( r \) edges, interlaced with \( r \) alternating \( f \)-walks of lengths \( 2k_1 - 1, 2k_2 - 1, \ldots, 2k_r - 1 \). The summation over \( k_1 + k_2 + \ldots + k_r = n \) indicates that all alternating \( f \)-walks involved must have \( 2n - r \) edges together. The summation over \( 1 \leq r \leq n \) gives exactly the cardinality of \( D_{2n}(c) \), which finishes the proof.

Let us observe here that (3.1) can be generalized to arbitrary random variables in noncommutative probability spaces with distributions \( \mu_1, \mu_2 \in \Sigma \) if one treats \( P_1 \) and \( P_2 \) as idempotents and uses extensions of states \( \varphi_1, \varphi_2 \) given by (2.9). Then, the combinatorial formula in the proof of Theorem 3.1 remains valid for \( \mu_1, \mu_2 \in \Sigma \).

**Example 3.2.** If \( \mu_1, \mu_2 \in \Sigma \), the lowest order moments of \( N_{\mu_1 \otimes \mu_2} \) are given by

\[
\begin{align*}
N_{\mu_1 \otimes \mu_2}(1) &= N_{\mu_1}(1)N_{\mu_2}(1), \\
N_{\mu_1 \otimes \mu_2}(2) &= N_{\mu_1}(2)N_{\mu_2}^2(1) + N_{\mu_1}(1)N_{\mu_2}(2), \\
N_{\mu_1 \otimes \mu_2}(3) &= N_{\mu_1}(3)N_{\mu_2}^3(1) + 2N_{\mu_1}(2)N_{\mu_2}(2)N_{\mu_2}(1) + N_{\mu_1}(1)N_{\mu_2}(3), \\
N_{\mu_1 \otimes \mu_2}(4) &= N_{\mu_1}(4)N_{\mu_2}^4(1) + 3N_{\mu_1}(3)N_{\mu_2}(2)N_{\mu_2}(1) + N_{\mu_1}(2)N_{\mu_2}^2(2) \\
&\quad + 2N_{\mu_1}(2)N_{\mu_2}(3)N_{\mu_2}(1) + N_{\mu_1}(1)N_{\mu_2}(4).
\end{align*}
\]

Let us apply these formulas to the graph \( G_1 \bowtie \bowtie G_2 \) in Fig.1. Let \( \mu_1 \) and \( \mu_2 \) be the spectral distributions associated with \( G_1 \) and \( G_2 \), respectively. By counting \( f \)-walks on \( G_1 \) and \( G_2 \), we easily get \( N_{\mu_1}(1) = N_{\mu_1}(2) = N_{\mu_1}(4) = 1, N_{\mu_1}(3) = 0 \) and \( N_{\mu_2}(1) = N_{\mu_2}(2) = 1, N_{\mu_2}(3) = 0, N_{\mu_2}(4) = 2 \). In turn, using the above formulas and Theorem 3.1, we get \( D_2(e) = 1, D_4(e) = D_6(e) = 2 \) and \( D_8(e) = 4 \), which can also be verified directly by counting comb walks on \( G_1 \bowtie \bowtie G_2 \).

For completeness, let us also briefly discuss the case of the star product of graphs \( G_1 \ast G_2 \) and find its relation to the boolean multiplicative convolution.

**Definition 3.4.** The *star product* of rooted graphs \((G_1, e_1)\) and \((G_2, e_2)\) is the rooted graph \((G_1 \ast G_2, e)\) obtained by gluing \( G_1 \) and \( G_2 \) at their roots and taking this common root to be the root of the product. The *star loop product* \((G_1 \star G_2, e)\) is obtained from naturally colored \((G_1 \ast G_2, e)\) by adding loops of color \( \iota \) to every vertex but the root of \( G_\iota \), where \( \iota \in I \).

Recall here the definition of the boolean multiplicative convolution of distributions \( \mu_1, \mu_2 \in \Sigma [5] \). Treating \( \eta_\mu(z) \) as a formal power series, we define

\[
\rho_\mu(z) = \frac{\eta_\mu(z)}{z},
\]

(3.3)
Figure 2. An example of the star loop product

which allows us to define the boolean multiplicative convolution of \( \mu_1, \mu_2 \in \Sigma \) by the formula

\[
\rho_{\mu_1 \boxtimes \mu_2}(z) = \rho_{\mu_1}(z) \rho_{\mu_2}(z).
\]

This formula defines \( \boxtimes \) as a binary operation on \( \mathcal{M}_T \) \([11]\). However, if \( \mu_1, \mu_2 \in \mathcal{M}_{\mathbb{R}_+} \), then the RHS does not always give the \( \rho \)-transform of a probability measure on \( \mathbb{R}_+ \) \([5]\).

Nevertheless, the well-known relation \([8,24]\)

\[
\rho_{\mu_1 \boxtimes \mu_2}(z) = \rho_1(z) \rho_2(z)
\]

where \( \rho_\iota(z) = \eta_\iota(z)/\iota, \iota \in I \), can still be written in terms of convolutions as

\[
\mu_1 \boxtimes \mu_2 = (\mu_1 \boxtimes \mu_2) \boxtimes (\mu_2 \boxtimes \mu_1).
\]

which was our motivation to use the symbol \( \boxtimes \) for the s-free convolution (an analogous formula holds for \( \mu_1, \mu_2 \in \mathcal{M}_a \)).

If \( a_\iota \) is the adjacency matrix of a graph \( G_\iota \), where \( \iota \in I \), then, using a similar reasoning as in the case of the comb loop product, we introduce operators of the form

\[
R_1 = a_1 \otimes P_2 + 1 \otimes P_2^1,
\]

\[
R_2 = P_1 \otimes a_2 + P_1^1 \otimes 1_2,
\]

which can be viewed as the adjacency matrices of the 1- and 2-colored subgraphs of the product graph \( (G_1 \star_\iota G_2, e) \). Moreover, these are exactly the operators which give the moments of the boolean multiplicative convolution of spectral distributions of \( G_1 \) and \( G_2 \) since the pair \( (R_1 - 1, R_2 - 1) \) is boolean independent w.r.t. \( \varphi_\iota \). An example of a star loop product is given in Fig.2.

Let us also note that formulas (3.3)-(3.4) can be used if \( a_1, a_2 \) are random variables in arbitrary noncommutative probability spaces – this is obtained if one uses extended states \([2.9]\). This gives a realization of the boolean multiplicative convolution of arbitrary distributions \( \mu_1, \mu_2 \in \Sigma \).

Proposition 3.1. The Multiplication Theorem holds for boolean independence, the associated loop product \( G_1 \star_\iota G_2 \), and the multiplicative convolution \( \mu_1 \boxtimes \mu_2 \).

Proof. The proof is similar to that of Theorem 3.1 and is based on the easy combinatorial formula

\[
N_{\mu_1 \boxtimes \mu_2}(n) = \sum_{j+k=n+1} N_{\mu_1}(j)N_{\mu_2}(k),
\]

where the sum runs over positive indices \( j, k \), obtained from (1.2) and (3.4). Details are left to the reader. \( \blacksquare \)
Example 3.3. If \( \mu_1, \mu_2 \in \Sigma \), the lowest order ‘first return moments’ \( N_{\mu_1 \mu_2}(n) \) are given by

\[
N_{\mu_1 \mu_2}(1) = N_{\mu_1}(1)N_{\mu_2}(1),
\]
\[
N_{\mu_1 \mu_2}(2) = N_{\mu_1}(1)N_{\mu_2}(2) + N_{\mu_1}(2)N_{\mu_2}(1),
\]
\[
N_{\mu_1 \mu_2}(3) = N_{\mu_1}(1)N_{\mu_2}(3) + N_{\mu_1}(2)N_{\mu_2}(2) + N_{\mu_1}(3)N_{\mu_2}(1),
\]
\[
N_{\mu_1 \mu_2}(4) = N_{\mu_1}(1)N_{\mu_2}(4) + N_{\mu_1}(2)N_{\mu_2}(3) + N_{\mu_1}(3)N_{\mu_2}(2) + N_{\mu_1}(4)N_{\mu_2}(1).
\]

These formulas can be used to count rooted alternating d-walks on the product graph given in Fig. 2. Using Proposition 3.2 and the values of \( N_{\mu_1}(j) \) and \( N_{\mu_2}(k) \), given in Example 3.1, we get \( D_2(e) = D_4(e) = D_6(e) = 1 \) and \( D_8(e) = 3 \), which can be also verified by direct computations.

4. Orthogonal multiplicative convolution

For given positive bounded random variables \( X_1 \) and \( X_2 \) with distributions \( \mu_1 \) and \( \mu_2 \), we want to find a positive random variable \( Z \) on some Hilbert space whose distribution is given by the multiplicative analog of the orthogonal convolution, denoted \( \mu_1 \perp \mu_2 \).

Let us first recall the notion of orthogonal independence [16].

\textbf{Definition 4.1.} Let \( (\mathcal{A}, \varphi, \psi) \) be a unital algebra with a pair of linear normalized functionals and let \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) be non-unital subalgebras of \( \mathcal{A} \). We say that \( \mathcal{A}_2 \) is orthogonal to \( \mathcal{A}_1 \) with respect to \( (\varphi, \psi) \) if

\begin{enumerate}[(i)]
  \item \( \varphi(bw_2) = \varphi(w_1b) = 0 \),
  \item \( \varphi(w_1a_1ba_2w_2) = \psi(b) (\varphi(w_1a_1a_2w_2) - \varphi(w_1a_1)\varphi(a_2w_2)) \),
\end{enumerate}

for any \( a_1, a_2 \in \mathcal{A}_1 \), \( b \in \mathcal{A}_2 \), and any elements \( w_1, w_2 \) of the unital algebra \( \text{alg}(\mathcal{A}_1, \mathcal{A}_2) \) generated by \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \). We say that the pair \( (a, b) \) of elements of \( \mathcal{A} \) is orthogonal with respect to \( (\varphi, \psi) \) if the algebra generated by \( b \in \mathcal{A} \) is orthogonal to the algebra generated by \( a \in \mathcal{A} \).

\textbf{Definition 4.2.} Let \( (\mathcal{A}, \varphi, \psi) \) be a unital algebra with a pair of normalized linear functionals and let \( \mathcal{A}_1, \mathcal{A}_2 \) be a pair of (in general, non-unital) subalgebras of \( \mathcal{A} \), such that \( \mathcal{A}_2 \) is orthogonal to \( \mathcal{A}_1 \) w.r.t. \( (\varphi, \psi) \). Let \( a_1, a_2 \in \mathcal{A} \) be random variables with \( \varphi \)-distribution \( \mu_1 \) and \( \psi \)-distribution \( \mu_2 \), respectively. The orthogonal multiplicative convolution of \( \mu_1 \) and \( \mu_2 \), denoted \( \mu_1 \perp \mu_2 \), is the distribution of \( a_2a_1 \), where \( (a_1 - 1, a_2 - 1) \) is orthogonal w.r.t. \( (\varphi, \psi) \).

In order to find a Hilbert space realization of the orthogonal multiplicative convolution of compactly supported probability measures, recall the definition of the orthogonal product of two Hilbert spaces with distinguished unit vectors \( (\mathcal{H}_1, \xi_1) \) and \( (\mathcal{H}_2, \xi_2) \). Thus, the orthogonal product of \( (\mathcal{H}_1, \xi_1) \) and \( (\mathcal{H}_2, \xi_2) \) is the pair \( (\mathcal{H}, \xi) \), where

\begin{equation}
(4.1) \quad \mathcal{H} = \mathbb{C}\xi \oplus \mathcal{H}_1^0 \oplus (\mathcal{H}_2^0 \otimes \mathcal{H}_1^0),
\end{equation}

with \( \mathcal{H}_i^0 = \mathcal{H}_i \ominus \mathbb{C}\xi_i \) and where \( \xi \) is a unit vector. We denote it by \( (\mathcal{H}_1, \xi_1) \leftarrow (\mathcal{H}_2, \xi_2) \), or simply \( \mathcal{H}_1 \leftarrow \mathcal{H}_2 \), if no confusion arises. It has been shown in [16] that \( \mathcal{H} \) gives a
Hilbert space realization of orthogonal random variables. For that purpose we defined an isometry \( U : \mathcal{H} \to \mathcal{H}_1 \otimes \mathcal{H}_2 \) given by

\[
U(\xi) = \xi_1 \otimes \xi_2, \quad U(h_1) = h_1 \otimes \xi_2, \quad U(h_2 \otimes h_1) = h_1 \otimes h_2
\]

for any \( h_1 \in \mathcal{H}_1^0 \) and \( h_2 \in \mathcal{H}_2^0 \). Using \( U \), we define faithful non-unital *-homomorphisms \( \tau_i : \mathcal{B}(\mathcal{H}_i) \to \mathcal{B}(\mathcal{H}) \) by

\[
\tau_1(a_1) = U^*(a_1 \otimes P_2)U, \quad \tau_2(a_2) = U^*(P_1^\perp \otimes a_2)U,
\]

where \( P_1, P_2 \) are the projections onto \( \mathbb{C} \xi_1 \) and \( \mathbb{C} \xi_2 \), respectively. Then the pair \((\tau_1(a_1), \tau_2(a_2))\) is orthogonal w.r.t. \((\varphi, \psi)\), where \( \varphi \) and \( \psi \) are states on \( \mathcal{B}(\mathcal{H}) \) associated with the unit vector \( \xi \) and any unit vector \( \zeta \in \mathcal{H}_1^0 \), respectively. For details, see [16].

For simplicity, we will identify vectors from \( \mathcal{H} \) with their images in \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) under the isometry \( U \) as well as bounded operators \( a, b \) with \( \tau_1(a_1), \tau_2(a_2) \), respectively, and we will carry out computations in \( \mathcal{H}_1 \otimes \mathcal{H}_2 \).

**Proposition 4.1.** Let \( a_1 \in B(\mathcal{H}_1) \) have \( \varphi_1 \)-distribution \( \mu_1 \) and \( a_2 \in B(\mathcal{H}_2) \) \( \varphi_2 \)-distribution \( \mu_2 \). Then there exist a Hilbert space \( \mathcal{H} \) with unit vectors \( \xi \) and \( \zeta \), and random variables \( R_1, R_2 \in B(\mathcal{H}) \) such that \( R_1 \) has \( \varphi \)-distribution \( \mu_1 \), \( R_2 \) has \( \psi \)-distribution \( \mu_2 \) and \( R_2 R_1 \) has \( \varphi \)-distribution \( \mu_1 \wedge \mu_2 \), where \( \varphi \) and \( \psi \) are states associated with \( \xi \) and \( \zeta \), respectively.

**Proof.** Let \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) with the canonical unit vector \( \xi = \xi_1 \otimes \xi_2 \) and let \( \zeta = \zeta_1 \otimes \zeta_2 \), where \( \zeta_1 \) is an arbitrary unit vector from \( \mathcal{H}_1^0 \). Define

\[
R_1 = a_1 \otimes P_2 + 1_1 \otimes P_2^\perp,
R_2 = P_1^\perp \otimes a_2 + P_1 \otimes 1_2,
\]

where \( P_1^\perp = 1_1 - P_1 \) and \( P_1 \) is the canonical projection onto \( \mathbb{C} \xi_1 \) and \( 1_i \) denotes the identity in \( B(\mathcal{H}_i) \), \( i = 1, 2 \). Therefore, denoting \( 1 = 1_1 \otimes 1_2 \), we get

\[
R_1 - 1 = (a_1 - 1_1) \otimes P_2,
R_2 - 1 = P_1^\perp \otimes (a_2 - 1_2),
\]

which, due to the appropriate tensor product form, implies that the pair \((R_1 - 1, R_2 - 1)\) is orthogonal w.r.t. \((\varphi, \psi)\), where \( \varphi \) is the state associated with \( \xi \) and \( \psi \) is the state associated with \( \zeta \). Moreover, elementary calculations show that the \( \varphi \)-distribution of \( R_1 \) is \( \mu_1 \) and that the \( \psi \)-distribution of \( R_2 \) is \( \mu_2 \), which completes the proof.

**Corollary 4.1.** Let \( \mu_1, \mu_2 \) be compactly supported probability measures on \( \mathbb{R}_+ \). Then there exist bounded positive random variables \( R_1, R_2 \) on some Hilbert space \( \mathcal{H} \) and unit vectors \( \xi, \zeta \in \mathcal{H} \), such that \( R_1 \) has \( \varphi \)-distribution \( \mu_1 \), \( R_2 \) has \( \psi \)-distribution \( \mu_2 \) and \( \sqrt{R_2 R_1} \sqrt{R_2} \) has \( \varphi \)-distribution \( \mu_1 \wedge \mu_2 \).

**Proof.** There exist positive operators \( x_1 \in B(L^2(\mathbb{R}_+, \mu_1)) \) and \( x_2 \in B(L^2(\mathbb{R}_+, \mu_2)) \) (standard multiplication operators) which have distributions \( \mu_1 \) and \( \mu_2 \), respectively, with respect to the states \( \varphi_1 \) and \( \varphi_2 \) associated with the constant functions equal to
one, \( h_1 \) and \( h_2 \), respectively. We can then use the framework of Proposition 4.1 to get self-adjoint bounded random variables on \( \mathcal{H} = L^2(\mathbb{R}, \mu_1) \otimes L^2(\mathbb{R}, \mu_2) \) of the form

\[
    R'_1 = (x_1 - 1) \otimes P_2, \\
    R'_2 = P_1^\perp \otimes (x_2 - 1),
\]

which are orthogonal w.r.t. \((\varphi, \psi)\), where \( \varphi \) is associated with vector \( \xi = h_1 \otimes h_2 \) and \( \psi \) is associated with \( \zeta = f \otimes h_2 \), where \( f \) is any function such that \( \int f(x)d\mu_1(x) = 0 \) and \( \int f^2(x)d\mu_1(x) = 1 \). Now, let \( R_1 = R'_1 + 1 \) and \( R_2 = R'_2 + 1 \). It follows from the proof of Proposition 4.1 that \( R_1 \) and \( R_2 \) have distributions \( \mu_1 \) and \( \mu_2 \), respectively. Moreover, both are positive operators and thus, in particular, \( \sqrt{R_1} \) exists. In fact,

\[
    \sqrt{R_2} = P_1^\perp \sqrt{a_2} + P_1 \otimes 1_2,
\]

and \( \sqrt{R_2}R_1\sqrt{R_2} \) is a bounded positive operator on \( \mathcal{H} \). Moreover, \( \sqrt{R_2}R_1\sqrt{R_2} \) has the same \( \varphi \)-distribution as \( R_1R_2 \), which is obtained from the following calculation:

\[
    \varphi((\sqrt{R_2}R_1\sqrt{R_2})^n) = \varphi(\sqrt{R_2}(R_1R_2)^{n-1}R_1\sqrt{R_2}) = (\varphi_1 \otimes \varphi_2)((\sqrt{P_1} \otimes 1_2)(R_1R_2)^{n-1}R_1(P_1 \otimes 1_2)) = (\varphi_1 \otimes \varphi_2)((R_1R_2)^{n-1}R_1(P_1 \otimes 1_2)) = \varphi((R_1R_2)^n).
\]

By using the adjoints, we can interchange the order of \( R_1 \) and \( R_2 \) under the symbol \( \varphi \). Therefore, the probability distribution of \( \sqrt{R_2}R_1\sqrt{R_2} \) extends to the compactly supported measure \( \mu_1 \preceq \mu_2 \) on \( \mathbb{R}_+ \).

We will now compute the \( \varphi \)-distribution of the product \( Z := R_2R_1 \), where \( R_1 \) and \( R_2 \) are given by Proposition 4.1, and derive a formula for \( \eta_Z \). In this context, let us observe that one can generalize Proposition 4.1 to arbitrary random variables in non-commutative probability spaces since formulas for \( R_1 \) and \( R_2 \) remain valid provided one takes extensions of states given by (2.9) and uses Proposition 2.2. Therefore, further developments in this Section will hold for arbitrary random variables in noncommutative probability spaces. This corresponds to taking distributions \( \mu \in \Sigma \) and the corresponding formal power series \( \eta_{\mu} \) and \( \rho_{\mu} \).

In order to compute the \( \varphi \)-distribution of \( Z \), let us decompose \( Z \) as

\[
    Z = Z_1 + Z_2 + Z_3 + Z_4,
\]

where

\[
    Z_1 = P_1^\perp a_1 \otimes a_2 P_2, \quad Z_2 = P_1^\perp \otimes a_2 P_2^\perp, \quad Z_3 = P_1 a_1 \otimes P_2, \quad Z_4 = P_1 \otimes P_2^\perp.
\]

In order to compute the ‘first-return moments’ \( N_Z(n) \) in the state \( \varphi \), we need to compute the mixed \( \varphi \)-moments of \( Z_i \)’s. However, it turns out that there are only two types of them which give a non-vanishing contribution. We compute them in Propositions 4.3-4.4.

**Proposition 4.2.** For \( r \) odd and \( j_1, j_2, \ldots, j_r \geq 1 \), we have

\[
    \varphi(Z_3Z_1^{j_1}Z_2^{j_2}Z_1^{j_3} \ldots Z_1^{j_r}) = N_{a_1}(m)N_{a_2}(k_1) \ldots N_{a_2}(k_{m-1}),
\]
where $m = 1 + \sum_{k \text{ odd}} j_k$ and all $k_i$'s are equal to 1, except

$$k_{j_1+1} = j_2 + 1, \quad k_{j_1+j_3+1} = j_4 + 1, \ldots, k_{j_1+j_3+\ldots+j_r+1} = j_{r-1} + 1,$$

and where the ‘first return moments’ $N_{a_i}(s)$ refer to the state $\varphi_i$, $i \in I$.

Proof. We have

$$\varphi(Z_1 Z_2^j Z_3^j \ldots Z_r^j) = \varphi_1(P_1 a_1 P_1^j a_1 P_1^{j_1} a_1 \ldots P_1^{j_r} a_1)$$

$$\varphi_2(P_2 a_2 P_2^j a_2 P_2^{j_2} a_2 \ldots P_2^{j_r} a_2)$$

$$= \varphi_1(a_1 P_1^{j_1+j_3+\ldots+j_r})$$

$$\varphi_2((a_2 P_2^{j_2}) a_2 P_2^{j_2} a_2 \ldots a_2)$$

$$= N_{a_1}(1 + j_1 + j_3 + \ldots + j_r)(N_{a_2}(1)^{(j_2-1)+\ldots+(j_r-1)}$$

$$N_{a_2}(j_2+1)N_{a_2}(j_4+1)\ldots N_{a_2}(j_{r-1}+1),$$

using the following properties:

$$P_1 a_1 P_1 = \varphi_1(1)P_1 = N_{a_1}(1)P_1, \quad P_2 a_2 P_2 = \varphi_2(P_2)P_2 = N_{a_2}(1)P_2.$$

Setting $k_1 = k_2 = \ldots = k_{j_1} = 1$, $k_{j_1+1} = j_2 + 1, k_{j_1+j_3+1} = j_4 + 1$, $k_{j_1+j_3+j_4+1} = j_6 + 1, \ldots, k_{j_1+j_3+\ldots+j_r+1} = j_{r-1} + 1$, we get the desired result.

Proposition 4.3. For $r$ even and $j_1, j_2, \ldots, j_r \geq 1$, we have

$$\varphi(Z_1^j Z_2^j Z_3^j \ldots Z_r^j) = N_{a_1}(m)N_{a_2}(k_1)N_{a_2}(k_{m-1}),$$

where $m = 1 + \sum_{k \text{ even}} j_k$ and all $k_i$'s are equal to 1, except

$$k_{j_2+1} = j_3 + 1, \quad k_{j_2+j_4+1} = j_5 + 1, \ldots, k_{j_2+j_4+\ldots+j_r+1} = j_{r-1} + 1,$$

and where the ‘first return moments’ $N_{a_i}(s)$ refer to the state $\varphi_i$, $i \in I$.

Proof. Using similar arguments as in the proof of Proposition 4.3, we get

$$\varphi(Z_1^j Z_2^j Z_3^j \ldots Z_r^j) = \varphi_1(P_1 a_1 P_1^j a_1 P_1^{j_1} a_1 \ldots P_1^{j_r} a_1)$$

$$\varphi_2(P_2 a_2 P_2^j a_2 P_2^{j_2} a_2 \ldots P_2^{j_r} a_2)$$

$$= \varphi_1((a_1 P_1^{j_1}) a_1 P_1^{j_1+j_3+\ldots+j_r})$$

$$\varphi_2((a_2 P_2^{j_2}) a_2 P_2^{j_2} a_2 \ldots a_2)$$

$$= N_{a_1}(1 + j_2 + j_4 + \ldots + j_r)(N_{a_2}(1)^{(j_2-1)+\ldots+(j_r-1)}$$

$$N_{a_2}(j_1+1)N_{a_2}(j_3+1)\ldots N_{a_2}(j_{r-1}+1).$$

Setting $k_1 = j_1 + 1, k_2 = \ldots = k_{j_2} = 1, k_{j_2+1} = j_3 + 1, k_{j_2+j_4+1} = j_5 + 1, \ldots, k_{j_2+j_4+\ldots+j_r+1} = j_{r-1} + 1$, we get the desired result.

Theorem 4.1. If the $\varphi$-distribution of $a_2$ is not concentrated at zero, then the formal power series corresponding to random variables $a_1, a_2$ and $Z$ satisfy the relation

$$\eta_Z(z) = \frac{z \eta_{a_2}(\eta_{a_2}(z))}{\eta_{a_2}(z)}. $$
If the distribution of $a_2$ is concentrated at zero, then $\eta_Z(z) = z\mu_1(X)$, where $\mu_1$ is the $\varphi$–distribution of $a_1$.

Proof. Suppose the $\varphi$–distribution of $a_2$ is not the Dirac delta. Then, writing the RHS of equation (4.4) as $D(z) = \sum_{m=1}^{\infty} d_n z^n$, we need to show that $d_n = \eta_Z(n)$ for $n \in \mathbb{N}$. We clearly have

$$N_Z(1) = \varphi(Z_3) = \varphi_1(a_1) = N_{a_1}(1) = d_1.$$ 

Now, for $n > 1$, it holds that

$$d_n = \sum_{m=1}^{n} N_{a_1}(m) \sum_{k_1 + k_2 + \ldots + k_{m-1} = n-1} N_{a_1}(k_1)N_{a_2}(k_2) \ldots N_{a_2}(k_{m-1}),$$

where all $k_i$’s are assumed to be positive integers, Using Proposition 2.2 and the explicit form of $Z_i$’s, we get

$$N_Z(n) = \varphi(Z(P\perp Z)^{n-1}) = \varphi(Z_3(Z_1 + Z_2)^{n-1}) = \varphi(Z_3(Z_1 + Z_2)^{n-2}Z_1) = \sum_{r=1}^{n-2} \sum_{i_1 \neq i_2 \neq \ldots \neq i_r \neq j_1 + j_2 + \ldots + j_r = n-2} \varphi(Z_3Z_{i_1}^{j_1}Z_{i_2}^{j_2} \ldots Z_{i_r}^{j_r}Z_1),$$

where $j_1, j_2, \ldots, j_r$ are positive integers. Note that on the RHS of the formula for $d_n$ we have the sum of products

$$N_{a_1}(m)N_{a_2}(k_1)N_{a_2}(k_2) \ldots N_{a_2}(k_{m-1}),$$

where $1 \leq m \leq n$ and $(k_1, k_2, \ldots, k_{m-1})$ is a sequence of positive integers which add up to $n - 1$. Now, every product obtained in Propositions 4.3-4.4 is of exactly this form, with indices $(k_1, k_2, \ldots, k_{m-1})$ satisfying the conditions just mentioned. Moreover, all these products are different from each other and exhaust all possible values of $(m, k_1, \ldots, k_{m-1})$. In fact, there are two disjoint cases distinguished: Proposition 4.3 covers the case $k_1 = 1$ and Proposition 4.4 – the case $k_1 \neq 1$. In the first case, even $j_i$’s determine the subsequence of $(k_1, k_2, \ldots, k_{m-1})$ consisting of numbers which are greater than 1 and odd $j_i$’s determine their positions in the sequence by determining all $k_i$’s which are equal to 1. In the second case, odd $j_i$’s determine the subsequence of $(k_1, k_2, \ldots, k_{m-1})$ consisting of numbers which are greater than 1 and even $j_i$’s determine their positions in the sequence by determining all $k_i$’s which are equal to 1. Therefore, both cases give together every tuple $(m, k_1, k_2, \ldots, k_{m-1})$ under consideration exactly once. This proves (4.4) in the case when the distribution of $a_2$ is not concentrated at zero. Finally, if the $\varphi$–distribution of $a_2$ is $\delta_0$, then $\varphi_2(a_2^n) = \delta_{a_2,0}$, which easily gives $N_Z(n) = N_{a_1}(1)\delta_{n,1}$, where $\delta_{i,j}$ is the Kronecker symbol, and since $N_{a_1}(1) = \varphi_1(a_1) = \mu_1(X)$, we have $\eta_Z(z) = z\mu_1(X)$. This completes the proof.

This shows that Eq.(1.7) holds for the orthogonal multiplicative convolution of distributions $\mu_1, \mu_2 \in \Sigma$, provided $\mu_2 \neq \delta_0$, where the functions $\eta$ are treated as formal
power series. In turn, explicit computations of distributions $\mu_1 \triangleleft \mu_2$ based on Definition 4.2 give

\begin{equation}
\delta_0 \triangleleft \mu = \delta_0 \quad \text{and} \quad \mu \triangleleft \delta_0 = \delta_{\mu(X)}
\end{equation}

for any $\mu \in \Sigma$. Let us also remark that Theorem 4.5 gives another formula for formal power series corresponding to $\varphi$–distributions, namely

\begin{equation}
\rho_{\mu_1 \triangleleft \mu_2}(z) = \rho_{\mu_1}(\eta_{\mu_2}(z)),
\end{equation}

where $\rho_{\mu_1}$ is given by (3.3), which turns out slightly more useful in computations involving $\mu_1 \triangleleft \mu_2$ than (4.4) (see Section 5).

**Corollary 4.2.** If $\mu_1$ and $\mu_2$ are $\varphi$–distributions of certain random variables, then the $\varphi$–moment of $\mu_1 \triangleleft \mu_2$ of order $n \in \mathbb{N}$ depends on $\varphi$–moments of $\mu_1$ of orders $1 \leq k \leq n$ and $\varphi$–moments of $\mu_2$ of orders $1 \leq k \leq n-1$.

**Proof.** This fact is a consequence of Proposition 4.1 and Theorem 4.5. In fact, the combinatorial formula for $d_n$ in the proof of Theorem 4.5 gives the explicit formula for $N_{\mu \triangleleft \nu}(n)$ in terms of $N_\mu(m)$ and $N_\nu(k)$ for $1 \leq m \leq n$ and $1 \leq k \leq n-1$. Thus, in view of (1.2), a similar property holds for the $\varphi$–moments of $\mu \triangleleft \nu$. $\blacksquare$

**Example 4.1.** If $\mu_1, \mu_2 \in \Sigma$, the lowest order first return moments $N_{\mu_1 \triangleleft \mu_2}(n)$ are given by

\[
\begin{align*}
N_{\mu_1 \triangleleft \mu_2}(1) &= N_{\mu_1}(1), \\
N_{\mu_1 \triangleleft \mu_2}(2) &= N_{\mu_1}(2)N_{\mu_2}(1), \\
N_{\mu_1 \triangleleft \mu_2}(3) &= N_{\mu_1}(3)N_{\mu_2}^2(1) + N_{\mu_1}(2)N_{\mu_2}(2), \\
N_{\mu_1 \triangleleft \mu_2}(4) &= N_{\mu_1}(4)N_{\mu_2}(1) + 2N_{\mu_1}(3)N_{\mu_2}(2)N_{\mu_2}(1) + N_{\mu_1}(2)N_{\mu_2}(3).
\end{align*}
\]

When we return to the $\varphi$–moments $\mu_1 \triangleleft \mu_2(X^n)$, using (1.2), we can express them in terms of ‘universal polynomials’ $Q_n(\mu_1(X), \ldots, \mu_1(X^n), \mu_2(X), \ldots, \mu_2(X^{n-1}))$, but one should note that they are not homogenous as in the case of free convolutions [25].

5. **Convolutions of measures on $\mathbb{R}_+$ and $\mathbb{T}$**.

In this section we show that the orthogonal multiplicative convolution can be defined for arbitrary probability measures on $\mathbb{R}_+$ which are not concentrated at zero, as well as for arbitrary probability measures on the unit circle $\mathbb{T}$.

For the following result, we refer the reader to the works of Belinschi and Bercovici [3,4].

**Theorem 5.1.** There is a bijection between $\mathcal{M}_{\mathbb{R}_+}$ and the class of analytic self-maps $\eta$ of $\mathbb{C} \setminus \mathbb{R}_+$, such that

1. $\eta(\overline{z}) = \overline{\eta(z)}$ for all $z \in \mathbb{C} \setminus \mathbb{R}_+$.
2. $\lim_{x \to -0, -z \to 0} \eta(z) = 0$.
3. $\arg(\eta(z)) \in [\arg(z), \pi]$ for all $z \in \mathbb{C}^+$.

Moreover, the map $\eta$ corresponding to $\mu \in \mathcal{M}_{\mathbb{R}_+}$ is given by the transform $\eta_\mu$. 


We shall use this result to prove that formula (4.4), which was shown to hold for formal power series corresponding to distributions of random variables, can also be used to define $\mu_1 \mu_2$ for $\mu_1, \mu_2 \in \mathcal{M}_{\mathbb{R}^+}$ and $\mu_2 \neq \delta_0$, if formal power series are replaced by analytic functions on $\mathbb{C} \setminus \mathbb{R}^+$.

**Proposition 5.1.** If $\mu_1, \mu_2 \in \mathcal{M}_{\mathbb{R}^+}$ and $\mu_2$ is not concentrated at zero, then there exists a unique probability measure $\mu \in \mathcal{M}_{\mathbb{R}^+}$, such that

$$
\eta_\mu(z) = \frac{z \eta_{\mu_1}(\eta_{\mu_2}(z))}{\eta_{\mu_2}(z)}
$$

for $z \in \mathbb{C} \setminus \mathbb{R}^+$. The measure $\mu$ will be defined to be the orthogonal multiplicative convolution of $\mu_1$ and $\mu_2$, denoted $\mu_1 \mu_2$.

**Proof.** For $z \in \mathbb{C} \setminus \mathbb{R}^+$, let us define

$$
\eta(z) = \frac{z \eta_{\mu_1}(\eta_{\mu_2}(z))}{\eta_{\mu_2}(z)}.
$$

Since $\mu_1, \mu_2 \in \mathcal{M}_{\mathbb{R}^+}$, transforms $\eta_{\mu_1}$ and $\eta_{\mu_2}$ are analytic self-maps of $\mathbb{C} \setminus \mathbb{R}^+$ which satisfy the conditions of Theorem 5.1. This implies that $\eta(z)$ is an analytic self-map of $\mathbb{C} \setminus \mathbb{R}^+$ since $\mu_2 \neq \delta_0$. Condition (1) of Theorem 5.1 holds for $\eta_{\mu_1}$ and $\eta_{\mu_2}$ and therefore it holds for $\eta$. Then, it is known that

$$
0 \leq \lim_{z \to 0^-} \eta_{\mu_2}(z) \leq \infty,
$$

and this limit is different from zero (it equals zero only if $\mu_2 = \delta_0$). Thus, we obtain

$$
\lim_{z \to 0^-} \eta(z) = \lim_{z \to 0^-} \eta_{\mu_2}(z) \lim_{z \to 0^-} \eta_{\mu_1}(\eta_{\mu_2}(z)) = c \cdot \lim_{z \to 0^-} \eta_{\mu_1}(\eta_{\mu_2}(z)) = 0
$$

for some $0 \leq c < \infty$, where we used condition 2 of Theorem 5.1 for the function $\eta_{\mu_1}$ (if $z \to 0^-$, $\psi_{\mu_2}(z) \to 0^-$ and thus $\eta_{\mu_2}(z) \to 0^-$ and therefore $\eta_{\mu_1}(\eta_{\mu_2}(z)) \to 0$ as $z \to 0^-$). This proves that condition (2) of Theorem 5.1 holds for the map $\eta$.

Next, for any $z \in \mathbb{C}^+$, we have $\psi_{\mu_2}(z) \in \mathbb{C}^+$, using the integral representation of $\psi_{\mu_2}(z)$ given by (1.1). This implies that $\eta_{\mu_2}(z) \in \mathbb{C}^+$, since

$$
\eta_{\mu_2}(z) = \frac{\psi_{\mu_2}(z)}{1 + \psi_{\mu_2}(z)} = \frac{\psi_{\mu_2}(z) + |\psi_{\mu_2}(z)|^2}{1 + |\psi_{\mu_2}(z)|^2},
$$

and therefore

$$
\Im \eta_{\mu_2}(z) = \frac{\Im \psi_{\mu_2}(z)}{1 + |\psi_{\mu_2}(z)|^2}.
$$

Hence, applying Theorem 5.1 to the function $\eta_{\mu_1}$, we get

$$
\arg \eta_{\mu_1}(\eta_{\mu_2}(z)) \geq \arg \eta_{\mu_2}(z),
$$

and thus

$$
\arg \eta(z) = \arg z + \arg \eta_{\mu_1}(\eta_{\mu_2}(z)) - \arg \eta_{\mu_2}(z) \geq \arg z.
$$
Similarly,

\[ \arg \eta(z) = \arg \eta_{\mu_1}(\eta_{\mu_2}(z)) + \arg z - \arg \eta_{\mu_2}(z) \]
\[ = \arg \eta_{\mu_1}(\eta_{\mu_2}(z)) - (\arg \eta_{\mu_2}(z) - \arg z) \]
\[ < \pi \]

since we have

\[ \arg \eta_{\mu_2}(z) \geq \arg z \quad \text{and} \quad \arg \eta_{\mu_1}(\eta_{\mu_2}(z)) < \pi, \]

which follows from \( \eta_{\mu_2}(z) \in \mathbb{C}^+ \). This proves that \( \eta \) satisfies condition (3) of Theorem 5.1, which completes the proof. 

For computations, it is convenient to use (4.6) and represent the transforms involved as continued fractions. The continued fraction representation of \( \eta_{\mu} \) for compactly supported \( \mu \in \mathcal{M}_{\mathbb{R}^+} \) is obtained from that of the Cauchy transform \( G_\mu \), or its reciprocal \( F_\mu \), by using the formula

\[ \eta_{\mu}(z) = 1 - z F_\mu \left( \frac{1}{z} \right). \]

If \( \mu \in \mathcal{M}_{\mathbb{R}^+} \) is associated with sequences of Jacobi parameters \( (\alpha, \omega) \), where \( \alpha = (\alpha_n)_{n \geq 0} \) and \( \omega = (\omega_n)_{n \geq 0} \), which we write \( J(\mu) = (\alpha, \omega) \), then, using the continued fraction representation of \( G_\mu \), we obtain

\[ \eta_{\mu}(z) = \alpha_0 z + \frac{\omega_0 z^2}{1 - \alpha_1 z - \frac{\omega_1 z^2}{1 - \alpha_2 z - \frac{\omega_2 z^2}{1 - \alpha_3 z - \frac{\omega_3 z^2}{\ldots} \}}}, \]

and a related continued fraction for \( \rho_{\mu} \). Conversely, this formula, together with the first formula of (1.7), enables us to compute the Jacobi sequences corresponding to the orthogonal multiplicative convolution. In the case of compactly supported measures, the Jacobi sequences uniquely determine the corresponding measure.

**Example 5.1.** Let \( \delta_a \) be the Dirac measure with \( a > 0 \) and let \( \mu \in \mathcal{M}_{\mathbb{R}^+} \) be compactly supported, with the associated Jacobi sequences \( J(\mu) = (\alpha, \omega) \). Then \( \rho_{\mu}(z) = a \) and therefore, \( \rho_{\delta_a \mu}(z) = \rho_{\delta_a}(\eta_{\mu}(z)) = a \), which gives \( \delta_a \mu = \delta_a \). In turn, \( \rho_{\mu \delta_a}(z) = \rho_{\mu}(az) \).

Let us denote the corresponding transformation of compactly supported measures by \( S_a \). In terms of reciprocal Cauchy transforms, it can be defined by a convex linear combination

\[ F_{S_a \mu}(z) = \frac{1}{a} F_{D_a \mu}(z) + (1 - \frac{1}{a})z, \]

where \( D_\lambda \mu \) is the dilation of measure \( \mu \) by \( \lambda \), defined by \( D_\lambda \mu(E) = \mu(\lambda^{-1} E) \). In terms of the transforms \( \eta \), we have \( \eta_{S_a \mu}(z) = \eta_{D_a \mu}(z)/a \), using (5.2). We also have \( \eta_{D_a \mu}(z) = \eta_a(az) \). Note that \( S_a \mu = (T_a \circ D_a) \mu \), where \( T_{-1} \) is the so-called \( t \)-transformation of measures [9]. If \( \mu \) is compactly supported, we can write \( \eta_{S_a \mu} \) in the form of a continued
and therefore, in view of (4.6), we obtain

\[ \text{Proposition 5.2.} \]

If \( \mu \) is given by the transform \( \eta \) on the unit circle \( T \) and \( \eta(z) = \frac{\omega_0 a z^2}{1 - \alpha_1 a z} \), then there exists a unique probability measure \( \mu \) such that \( J(\mu \angle \delta_1) = (\alpha, \omega) \), with \( \alpha = (p, 1 - pq, 0, 0, \ldots) \) and \( \omega = ((p - p^2)q, 0, 0, \ldots) \). The corresponding measure has two atoms (explicit dependence on \( p \) and \( q \) is rather complicated and is omitted).

Example 5.2. If \( \mu_1 = (1 - p)\delta_0 + p\delta_1 \) and \( \mu_2 = (1 - q)\delta_0 + q\delta_1 \), then the corresponding reciprocal Cauchy transforms are

\[
F_{\mu_1}(z) = \frac{z - 1 + p}{z(z - 1)}, \quad F_{\mu_2}(z) = \frac{z - 1 + q}{z(z - 1)},
\]

which gives

\[
\rho_{\mu_1} = p + \frac{(p - p^2)z}{1 - (1 - p)z}, \quad \eta_{\mu_2} = qz + \frac{(q - q^2)z^2}{1 - (1 - q)z},
\]

and therefore, in view of (4.6), we obtain

\[
\rho_{\mu_1 \angle \mu_2}(z) = p + \frac{(p - p^2)qz}{1 + (pq - 1)z}.
\]

Using (5.3), we obtain \( J(\mu_1 \angle \mu_2) = (\alpha, \omega) \), with \( \alpha = (p, 1 - pq, 0, 0, \ldots) \) and \( \omega = ((p - p^2)q, 0, 0, \ldots) \). The corresponding measure has two atoms (explicit dependence on \( p \) and \( q \) is rather complicated and is omitted).

A result analogous to Theorem 5.1 also holds for the set \( \mathcal{M}_T \) of probability measures on the unit circle \( T = \{ z \in \mathbb{C} : |z| = 1 \} \). Here, the class of self-maps \( \eta \) of the open unit disc \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) is used, which is again related to the transforms \( \eta_{\mu} \). For details, we refer the reader to the works of Belinschi and Bercovici [3,4].

Theorem 5.2. There is a bijection between \( \mathcal{M}_T \) and the class of analytic self-maps \( \eta \) of \( D \) such that \( \eta(0) = 0 \) and \( |\eta(z)| \leq |z| \) for all \( z \in D \). Moreover, the map \( \eta \) corresponding to \( \mu \) is given by the transform \( \eta_{\mu} \).

Proposition 5.2. If \( \mu_1, \mu_2 \in \mathcal{M}_T \), then there exists a unique probability measure \( \mu \in \mathcal{M}_T \) such that

\[
\eta_{\mu}(z) = \frac{z \eta_{\mu_1}(\eta_{\mu_2}(z))}{\eta_{\mu_2}(z)}
\]

for \( z \in D \), where we understand that \( \eta_{\mu_1}(0)/0 = \eta_{\mu_1}'(0) \). This measure \( \mu \) will be defined to be the orthogonal multiplicative convolution \( \mu_1 \angle \mu_2 \).

Proof. First of all, observe that \( |\eta(z)| \leq |z| \) since \( \eta_{\mu_1} \) has the same property by Theorem 5.3. Moreover, \( \eta \) is a quotient of analytic functions on \( D \), and if \( \eta_{\mu_2}(z_0) = 0 \), then

\[
\eta(z_0) = \lim_{z \to z_0} \frac{z \eta_{\mu_1}(\eta_{\mu_2}(z))}{\eta_{\mu_2}(z)} = z_0 \eta_{\mu_1}'(0),
\]

fraction

\[
\eta_{S_{\mu}}(z) = \alpha_0 z + \frac{\omega_0 a z^2}{1 - \alpha_1 a z} - \frac{\omega_1 a z^2}{1 - \alpha_2 a z} - \frac{\omega_2 a z^2}{1 - \alpha_3 a z} - \frac{\omega_3 a z^2}{1 - \alpha_4 a z} - \cdots
\]

In particular, \( \mu \angle \delta_1 = \mu \), thus \( \delta_1 \) is the right unit w.r.t. the operation \( \angle \) on \( \mathcal{M}_R \), and \( \delta_1 \angle \mu = \delta_1 \).
thus \( \eta \) is a well-defined analytic self-map of \( \mathbb{D} \). Finally, \( \eta(0) = 0 \cdot \eta_{\mu}^I(0) = 0 \). Thus, \( \eta \) satisfies the conditions of Theorem 5.3 and therefore \( \eta = \eta_{\mu} \) for some \( \mu \in \mathcal{M}_T \), which completes the proof.

**Example 5.3.** For any \( \mu \in \mathcal{M}_T \) and \( a \in \mathbb{T} \), we get \( \delta_a \mu = \delta_a \) and \( \mu \Delta \delta_a = S_a \mu \), where \( S_a \mu \) is defined by the same equation \( \eta S_a \mu(z) = \eta D_a \mu(z)/a \) as in the case of measures on \( \mathbb{R}_+ \) (on \( \mathcal{M}_T \), the operation \( D_a \) should be interpreted as a rotation).

### 6. Orthogonal Loop Product of Graphs

In this section we establish a relation between the orthogonal multiplicative convolution of compactly supported probability measures and a new type of product of graphs called the ‘orthogonal loop product’.

As in the monotone case, we begin with recalling the definition of the orthogonal product of graphs [2,16]. Then we will modify this product in an appropriate manner to find a relation with the orthogonal multiplicative convolution.

**Definition 6.1.** The orthogonal product of two rooted graphs \((G_1,e_1)\) and \((G_2,e_2)\) is the rooted graph \((G_1 \leftarrow G_2, e)\) obtained by attaching a copy of \(G_2\) by its root \(e_2\) to each vertex of \(G_1\) but the root \(e_1\), where \( e \) is taken to be equal to \( e_1 \). If its set of vertices is identified with \( V_1 \leftarrow V_2 := (V_1^0 \times V_2) \cup \{e_1 \times e_2\} \), then \( e \) is identified with \( e_1 \times e_2 \).

It is worth noting that the orthogonal product of graphs resembles their comb product. The difference is that in the comb product the second graph is glued by its root to all vertices of the first graph, whereas in the orthogonal product the second graph is glued to all vertices but the root of the first graph.

**Definition 6.2.** Suppose the edges of the orthogonal product \((G_1 \leftarrow G_2, e)\) are \( I \)-colored. The orthogonal loop product of rooted graphs \((G_1,e_1)\) and \((G_2,e_2)\) is the rooted graph \((G_1 \leftarrow \ell G_2, e)\) obtained from \((G_1 \leftarrow G_2, e)\) by attaching a loop of color 1 to all vertices but the root of each copy of \(G_2\), and a loop of color 2 to the root of \(G_1\).

The justification of the above definition is similar to that in Remark 3.1. It is enough to deduce the gluing rules from the expressions for \( R_1 \) and \( R_2 \), given in the proof of Proposition 4.1, interpreting \( a_\ell \) as the adjacency matrix of graph \( G_\ell \), \( \ell \in I \). Also, there is an easy analog of Remark 3.1. Finally, it follows from [2] that the additive analog of the Multiplication Theorem holds for orthogonal independence. An example of the orthogonal loop product of graphs is given in Fig.3 (cf. Figs.1-2).
Example 6.1. In the orthogonal loop product of graphs given in Fig. 3, we have

$$ (\beta_1, w_1, \beta_2, w_2, \beta_3, w_3, \beta_4, w_4) \in D_8(e), $$

where the sequence of edges of color 1 is the same as in Example 3.1, namely

$$ (\beta_1, \beta_2, \beta_3, \beta_4) = ((e, x), (x, x'), (x', x), (x, e)), $$

and the (alternating) f-walks attached to the vertices $x, x', x, e$ (in that order) are: $w_1 = w_3 = (x, x)_2$, $w_2 = (x', x')_2$ and $w_4 = (e, e)$ Note that this d-walk has length 8. Again, as in the case of d-walks on the comb loop product, if any of the loops at $x$ or $x'$ is replaced by an alternating f-walk, we shall still get a rooted alternating d-walk. Of course, the simplest rooted alternating d-walk is given by $(\beta_1, w_1, \beta_4, w_4) \in D_4(e)$ and $(\beta_0, w_4) \in D_2(e)$, where $\beta_0 = (e, e)_1$.

Theorem 6.1. The Multiplication Theorem holds for orthogonal independence, the associated loop product $G_1 \rightarrow_\ell G_2$, and the multiplicative convolution $\mu_1 \triangleleft \mu_2$.

Proof. The proof is similar to that of Theorem 3.1, but the combinatorics is based on a different formula for formal power series, namely

$$ \eta_{\mu_1 \triangleleft \mu_2}(z) = \frac{z \eta_{\mu_1}(\eta_{\mu_2}(z))}{\eta_{\mu_2}(z)}, $$

which leads to the combinatorial formula

$$ N_{\mu_1 \triangleleft \mu_2}(n) = \sum_{r=1}^{n} N_{\mu_1}(r) \sum_{k_1 + k_2 + \ldots + k_{r-1} = n-1} N_{\mu_2}(k_1) N_{\mu_2}(k_2) \ldots N_{\mu_2}(k_{r-1}), $$

where it is assumed that $k_1, k_2, \ldots, k_{r-1}$ are positive integers. Recall that in the orthogonal product $G_1 \rightarrow_\ell G_2$ there is one copy of $G_1$ (with $e_1$ identified with the root $e$ of the product graph) with a copy of $G_2$ attached by its root to every vertex $x$ of $G_1$ but the root $e_1$. Therefore, each d-walk $w \in D_{2n}(e)$ is a sequence of $r$ edges of $G_1$, which themselves must form an f-walk $c = (e, v_1, v_2, \ldots, v_{r-1}, e)$ of color 1, interlaced with alternating f-walks $w_i \in F(v_i)$, $1 \leq i \leq r - 1$ and a loop of color 2, $w_r = (e, e)_2$. Note that this is the only way to produce a rooted alternating d-walk since the only edge of color 2 incident on $e$ is the loop and therefore, in order to get an alternating walk, the first f-walk which begins and ends with an edge of color 1 must be followed by the loop at $e$ of color 2 to make a double return to $e$. The contribution from each product of type

$$ N_{\mu_1}(r) N_{\mu_2}(k_1) N_{\mu_2}(k_2) \ldots N_{\mu_2}(k_{r-1}) $$

to the RHS of the above formula is equal to the number of all such f-walks $c$ of color 1 which have $r$ edges and are interlaced with $r - 1$ f-walks of lengths $2k_1 - 1, 2k_2 - 1, \ldots, 2k_{r-1} - 1$ attached to all vertices of $c$ but one (we choose this vertex to be the root since we want the considered walk to be an f-walk). The summation over all $1 \leq r \leq n$ and $2k_1 + 2k_2 + \ldots + 2k_{r-1} - r + 1 = 2n - 1$ indicates that the first f-walk $w_1$ in the d-walk $w = (u_1, u_2)$ is of length $2n - 1$, and $u_2$ is the loop of length 1. The summation over $1 \leq r \leq n$ gives exactly the cardinality of $D_{2n}(e)$, which finishes the proof. \[\blacksquare\]
Example 6.2. Let us apply the formulas of Example 4.1 to the enumeration of rooted alternating d-walks on the graph $G_1 \rightarrow_\ell G_2$ in Fig. 3 (we keep the notation of Example 3.2 for spectral distributions $\mu_1$ and $\mu_2$ and thus we get the same values of the $N_{\mu_1}(k)$'s and the $N_{\mu_2}(j)$'s). Using Example 4.1 and Theorem 6.1, we get $D_2(e) = D_4(e) = D_6(e) = D_8(e) = 1$, which can be verified directly by counting rooted alternating d-walks on $G_1 \rightarrow_\ell G_2$.

7. Subordination operators

In analogy to the additive case [16], where we introduced and studied operators related to the subordination property for the free additive convolution, we shall now present an analogous approach to the subordination property for the free multiplicative convolution. We will mainly refer to sets $\mathcal{M}_{\mathbb{R}}$ and $\mathcal{M}_{\mathbb{C}}$, which correspond to positive and unitary operators, but the operatorial subordination will also hold for all bounded operators.

Let $(\mathcal{H}_i, \xi_i)$, where $i \in I$, be Hilbert spaces with distinguished unit vectors. Then their Hilbert space free product $(\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2)$ is $(\mathcal{H}, \xi)$ where

$$H = \mathbb{C} \xi \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{i_1 \neq i_2 \neq \ldots \neq i_n} \mathcal{H}_{i_1}^0 \otimes \mathcal{H}_{i_2}^0 \otimes \ldots \otimes \mathcal{H}_{i_n}^0,$$

with $\mathcal{H}_i^0 = \mathcal{H}_i \ominus \mathbb{C} \xi_i$ and $\xi$ denoting a unit vector (canonical scalar product is used). For any $h \in \mathcal{H}_i$, denote by $h^0$ the orthogonal projection of $h$ onto $\mathcal{H}_i^0$. Moreover, let

$$\mathcal{H}^{(n)}(i) = \bigoplus_{i_1 \neq i_2 \neq \ldots \neq i_n} \mathcal{H}_{i_1}^0 \otimes \mathcal{H}_{i_2}^0 \otimes \ldots \otimes \mathcal{H}_{i_n}^0,$$

$$\mathcal{K}^{(n)}(i) = \bigoplus_{i_1 \neq i_2 \neq \ldots \neq i_n} \mathcal{H}_{i_1}^0 \otimes \mathcal{H}_{i_2}^0 \otimes \ldots \otimes \mathcal{H}_{i_n}^0,$$

for $i \in I$ and $n \in \mathbb{N}$, and, for convenience, we set $\mathcal{H}^{(0)}(i) = \mathcal{K}^{(0)}(i) = \mathbb{C} \xi$ with the canonical projection $P_\xi : \mathcal{H} \rightarrow \mathbb{C} \xi$.

Since our index set $I$ consists of two elements, the above notation gives identifications

$$\mathcal{H}^{(n)}(i) = \begin{cases} \mathcal{K}^{(n)}(i) & n \text{ odd,} \\ \mathcal{K}^{(n)}(\bar{i}) & n \text{ even.} \end{cases}$$

Nevertheless, it is convenient to use both sequences, (7.2) and (7.3), as well as direct sums

$$\mathcal{H}(i) = \bigoplus_{n=1}^{\infty} \mathcal{H}^{(n)}(i) \quad \text{and} \quad \mathcal{K}(i) = \bigoplus_{n=1}^{\infty} \mathcal{K}^{(n)}(i),$$

where $i \in I$.

As in our previous work, we decompose the free product of Hilbert spaces as (two different) orthogonal direct sums

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}^{(n-1)}(\bar{i}) \oplus \mathcal{H}^{(n)}(i),$$
where \( \iota \in I \), and denote by
\[
(7.7) \quad P_\iota(n) : \mathcal{H} \to \mathcal{H}^{(n-1)}(\iota) \oplus \mathcal{H}^{(n)}(\bar{\iota})
\]
the associated canonical projections. Finally, we define the so-called vacuum state \( \varphi(\cdot) = \langle \xi, \xi \rangle \) on \( \mathcal{B}(\mathcal{H}) \).

Of particular interest will be \( s \)-free products of Hilbert spaces, \((\mathcal{H}_\iota, \xi_\iota)\) and \((\mathcal{H}_{\bar{\iota}}, \xi_{\bar{\iota}})\), denoted \((\mathcal{H}_\iota \oplus \mathcal{H}_{\bar{\iota}}, \xi)\) and defined [16] as the pair \((\mathcal{K}_\iota, \xi)\) where
\[
(7.8) \quad \mathcal{K}_\iota = \mathbb{C}\xi \oplus \mathcal{K}(\bar{\iota}),
\]
and \( \iota \in I \). The direct sum decompositions
\[
(7.9) \quad \mathcal{K}_\iota = \bigoplus_{n \text{ odd}} \mathcal{H}^{(n-1)}(\iota) \oplus \mathcal{H}^{(n)}(\bar{\iota})
\]
hold for each \( \iota \in I \).

Let \( x \in B(\mathcal{H}_1) \) and \( y \in B(\mathcal{H}_2) \) be fixed random variables. The corresponding free random variables \( X_1 := \lambda(x) \) and \( X_2 := \lambda(y) \) are elements of \( B(\mathcal{H}) \), where \( \lambda \) is the free product representation on \( \mathcal{H} \), and can be decomposed according to (7.6) as
\[
(7.10) \quad X_\iota = \sum_{j=1}^{\infty} X_\iota(j),
\]
where \( X_\iota(j) = P_\iota(j)X_\iota P_\iota(j), j \in \mathbb{N} \), can be viewed as replicas of \( x \) and \( y \), respectively, where \( \iota \in I \). Using (7.6) and (7.9), we also have the decompositions
\[
(7.11) \quad 1 = \sum_{j=1}^{\infty} P_\iota(j) \quad \text{and} \quad 1_{\mathcal{K}_\iota} = \sum_{j \text{ odd}} P_\iota(j)
\]
of the units in \( B(\mathcal{H}) \) and \( B(\mathcal{K}_\iota) \), where \( \iota \in I \), respectively. Here, and also in the sequel, we denote by the same symbols, \( X_\iota(j) \) and \( P_\iota(j) \), the corresponding operators on \( \mathcal{K}_\iota \).

We shall use representations of free random variables as ‘orthogonal series’ with the unit singled out, namely
\[
(7.12) \quad X_\iota = 1 + \sum_{j=1}^{\infty} x_\iota(j)
\]
where \( x_\iota(j) = X_\iota(j) - P_\iota(j) \) and \( \iota \in I, j \in \mathbb{N} \). If no confusion arises, we will distinguish the first term in the above series by a special notation and write \( x_\iota = x_\iota(1) \). Let us remark that the above form is suitable for the study of multiplicative convolutions, where units play a special role (roughly speaking, they have to be subtracted when it comes to proving some kind of independence).

We begin with the decomposition of the product \( X_{\bar{\iota}}X_\iota \) which corresponds to Eq.(1.3) for \( \mu_1, \mu_2 \in \mathcal{M}_{\mathbb{R}+,} \), and therefore involves positive operators.

**Theorem 7.1.** Let \( X_\iota, \iota \in I \), be positive random variables which are free with respect to \( \varphi \), and denote \( t_\iota = 1 + x_\iota \), \( T_{\bar{\iota}} = \sqrt{X_\iota} - x_\iota X_{\bar{\iota}} \sqrt{X_\iota} - x_\iota \). Then \( X_{\bar{\iota}}X_\iota \) has the same \( \varphi \)-distribution as \( \sqrt{t_\iota}T_{\bar{\iota}}\sqrt{t_\iota} \) and the pair \((t_\iota - 1, T_{\bar{\iota}} - 1)\) is monotone independent with respect to \( \varphi \).
Proof. Note that $T_\tau$ introduced above is well-defined since

$$X_i - x_i = P_i(1) + \sum_{j=2}^{\infty} X_i(j)$$

is positive as an orthogonal direct sum of positive random variables. Using a similar argument, we get positivity of

$$t_i = X_i(1) + \sum_{j=2}^{\infty} P_i(j).$$

Computations of square roots give

$$\sqrt{t_i} = \sqrt{X_i(1)} + \sum_{j=2}^{\infty} P_i(j) \quad \text{and} \quad \sqrt{X_i - x_i} = P_i(1) + \sum_{j=2}^{\infty} \sqrt{X_i(j)},$$

where $\sqrt{X_i(j)} = P_i(j)\sqrt{z}P_i(j)$ for $j \in \mathbb{N}$, with $z = x$ or $z = y$ if $t = 1$ or $t = 2$, respectively, which leads to

$$\sqrt{t_i} \sqrt{X_i - x_i} = \sqrt{X_i}.$$

Now, we use the fact that $X_\tau X_i$ has the same $\varphi$-distribution as $\sqrt{X_i} X_\tau \sqrt{X_i}$, which completes the proof of the first part of the theorem.

We will show now that the pair $(t_i - 1, T_\tau - 1)$ is monotone independent w.r.t. $\varphi$. For simplicity, we denote $x = t_i - 1$ and $y = T_\tau - 1$.

Case 1. If $y^n$ is in the middle of the moment, we compute $\varphi(w_1 xy^n x w_2)$, where $n \in \mathbb{N}$ and $w_1, w_2 \in \text{alg}(x, y)$. Note that the range of $x$ is $\mathbb{C} \oplus \mathcal{H}_t^0$, therefore, we only need to find the action of $y^n$ onto $\xi$ and $h \in \mathcal{H}_t^0$. Using the explicit form of $\sqrt{X_i - x_i}$, we get

$$y^n \xi = (1 + \sum_{k=2}^{\infty} r_k)(1 + \sum_{j=1}^{\infty} z_j)(1 + \sum_{k=2}^{\infty} r_k) - 1)^n \xi$$

$$= \varphi(y^n) \xi \mod \mathcal{K}(\iota),$$

where $r_k = \sqrt{X_i(k)} - P_i(k)$ and $z_j = X_\tau(j) - P_\tau(j)$ and therefore $r_k \xi = 0$ for any $k \geq 2$. Similarly, we get

$$y^n h = \langle y^n h, h \rangle h \mod \mathcal{K}(\tau) \ominus \mathcal{K}^{(1)}(\tau)$$

for any $h \in \mathcal{H}_t^0$ of norm $\| h \| = 1$. Note that $h$ plays the role of a cyclic vector for the restriction of the free product representation $\pi_1 \ast \pi_2$ to the unital algebra generated by $y$. Thus $\langle y^n h, h \rangle = \varphi(y^n) h$, which gives

$$\varphi(w_1 xy^n x w_2) = \varphi(y^n) \varphi(w_1 x^2 w_2),$$

i.e. the required condition for monotone independence.

Case 2. If $y^n$ is at the end of the moment, we get

$$\varphi(w_1 xy^n) = \varphi(y^n) \varphi(w_1 x),$$

using the relation for $y^n \xi$ obtained above and the fact that $\mathcal{K}(\iota) \subset \text{Ker}x$, which finishes the proof.

Corollary 7.1. If, in addition, $X_i$ has $\varphi$-distribution $\mu_i \neq \delta_0$ for $i \in I$, then the $\varphi$-distribution of $T_i$ is given by the $s$-free multiplicative convolution $\mu_i \boxtimes \mu_\tau$. \hfill $\blacksquare$
Proof. First, note that the variable \( t_i \) has \( \varphi \)-distribution \( \mu_i \). In view of (1.5) and Theorem 7.1, we have \( \eta_{\mu_1 \boxtimes \mu_2} = \eta_{\mu_1} \circ \eta_{\mu_2} = \eta_{\mu_2} \circ \eta_{\mu_1}, \) where \( \sigma_1 \) and \( \sigma_2 \) are the \( \varphi \)-distributions of \( T_1 \) and \( T_2 \), respectively. From the definition of the s-free multiplicative convolution, it follows that \( \sigma_1 = \mu_1 \boxtimes \mu_2 \) and \( \sigma_2 = \mu_2 \boxtimes \mu_1 \), which proves our assertion. \[ \blacksquare \]

Remark 7.1. In analogy to the terminology used in analytic subordination \[8,23\], we can say that \( X_\tau X_i \) is ‘subordinate to \( t_i \) and \( t_\tau \) with \( T_\tau \) and \( T_i \), respectively, being the corresponding ‘subordination operators’. Let us remark, however, that operators \( T_i, i \in I \), are not good candidates for ‘subordination branches’ of the product of free random variables since they are not suitable for further decomposition. In that sense, they are not multiplicative analogues of the ‘additive subordination branches’ \[16\], or the branches of the free product of graphs introduced by Quenell \[20\].

In the next section we will introduce and study such analogues, whose decompositions will correspond to decompositions of s-free multiplicative convolutions.

A similar operatorial subordination result can be established for bounded operators which includes unitary operators (the latter are related to \( \mathcal{M}_\tau \)).

Theorem 7.2. Let \( X_i, i \in I \), be bounded random variables which are free with respect to \( \varphi \). Then, for each \( i \in I \), the operator \( X_\tau X_i \) has the same \( \varphi \)-distribution as \( Q_\tau q_i \), where \( q_i = 1 + x_i \) and \( Q_\tau = X_\tau (X_i - x_i) \). Moreover, the pair \((q_i - 1, Q_\tau - 1)\) is monotone independent with respect to \( \varphi \). Finally, if \( X_i, i \in I \), are unitary, then the operators \( q_i, Q_i, i \in I \), are unitary.

Proof. The proof of the statements concerning bounded operators is similar to that of Theorem 7.1. Therefore, we shall just prove unitarity of \( Q_i \) and \( q_i \). Writing operators \( q_i \) and \( X_i - x_i \) in the form of orthogonal series

\[
q_i = X_i(1) + \sum_{j=2}^{\infty} P_i(j), \quad X_i - x_i = P_i(1) + \sum_{j=2}^{\infty} X_i(j),
\]

we obtain

\[
q_i q_i^* = X_i(1)X_i^*(1) + \sum_{j=2}^{\infty} P_i(j) = 1
\]

and

\[
(X_i - x_i)(X_i - x_i)^* = P_i(1) + \sum_{j=2}^{\infty} X_i(j)X_i^*(j) = 1.
\]

Similarly, \( q_i^* q_i = 1 \) and \( (X_i - x_i)^*(X_i - x_i) = 1 \). This proves unitarity of \( q_i \) and \( X_i - x_i \), from which we obtain unitarity of \( Q_i \). The remaining part of the proof is similar to that of Theorem 7.1. \[ \blacksquare \]

Corollary 7.2. Let \( \mu_i \) and \( \sigma_i \) be the \( \varphi \)-distributions of \( X_i \) and \( Q_i \), respectively, for \( i \in I \). Then the \( \varphi \)-distribution of \( X_\tau X_i \) is given by \( \mu_i \otimes_{\sigma_\tau} \sigma_i \) for each \( i \in I \). In particular, if the operators \( X_i \) are unitary, then \( \sigma_i = \mu_i \boxtimes \mu_\tau \) for each \( i \in I \).

Proof. The first statement follows from Theorem 7.3 and the definition of the monotone multiplicative convolution of distributions. The proof of the second statement is
similar to that of Corollary 7.2 (the s-free convolution of measures from $\mathcal{M}_u$ is used).

8. Subordination branches

In order to introduce operator-valued ‘multiplicative subordination branches’ associated with the product of bounded free random variables, let us introduce random variables

\begin{equation}
R_\iota(m) = 1 + \sum_{j \in J(m)} x_\iota(j)
\end{equation}

where the notation $J(m) = \{m, m+2, m+4, \ldots\}$ is used for index sets and, by abuse of notation, 1 denotes $1_{K_\iota}$ if $m$ is odd, or $1_{K_\iota}$ if $m$ is even. Note that operators $R_\iota(m)$ are elements of $B(K_\iota)$ or $B(K_\iota)$, depending on whether we have $m$ odd or even, respectively.

Therefore, on each of the ‘s-free Fock spaces’, $K_\iota$, we get an ‘interlaced’ sequence

\begin{equation}
R_\iota(1), R_\xi(2), R_\iota(3), R_\xi(4), \ldots \in B(K_\iota).
\end{equation}

Each of these two sequences is used in the definition of one sequence of operator-valued ‘multiplicative subordination branches’.

**Proposition 8.1.** If $X_1, X_2$ are positive (unitary), then operators $R_\iota(m)$, where $\iota \in I$ and $m \in \mathbb{N}$, are positive (unitary).

**Proof.** This fact can be easily checked by decomposing the units according to (7.11). We choose to show unitarity of

\begin{align*}
R_\iota^{\text{odd}} &= \sum_{j \text{ odd}} X_\iota(j) \quad \text{and} \quad R_\iota^{\text{even}} = P_\xi + \sum_{j \text{ even}} X_\iota(j)
\end{align*}

for unitary $X_\iota$. Since $X_\iota(j)X_\iota^*(j) = P_\iota(j)X_\iota^*P_\iota(j) = P_\iota(j)$ for each $\iota \in I$ and $j \in \mathbb{N}$, we get

\begin{align*}
R_\iota^{\text{odd}}(R_\iota^{\text{odd}})^* &= \sum_{j \text{ odd}} X_\iota(j)X_\iota^*(j) = \sum_{j \text{ odd}} P_\iota(j) = 1, \\
R_\iota^{\text{even}}(R_\iota^{\text{even}})^* &= P_\xi + \sum_{j \text{ even}} X_\iota(j)X_\iota^*(j) = P_\xi + \sum_{j \text{ even}} P_\iota(j) = 1.
\end{align*}

Unitarity of $R_\iota(m)$ for arbitrary $m$ can be proved in similar way. Positivity of $R_\iota(m)$ follows easily from the appropriate decomposition of the unit. \hfill \blacksquare

**Definition 8.1.** Let $X_\iota$, $\iota \in I$, be random variables which are free w.r.t. $\varphi$. Random variables

\begin{equation}
U_\iota(m) = R_\iota(m)R_\xi(m + 1),
\end{equation}

where $\iota \in I$ and $m \in \mathbb{N}$, will be called multiplicative bounded subordination branches. If $X_\iota$, $\iota \in I$ are unitary, then they will be called multiplicative unitary subordination branches.
In addition to branches $U_\iota(m)$, we shall need operators obtained from $U_\iota(m)$ by ‘order reversal’. Namely, let
\begin{equation}
V_\iota(m) = R_\iota(m + 1)R_\iota(m)
\end{equation}
where $\iota \in I$ and $m \in \mathbb{N}$. Again, we use a simpler notation for $m = 1$, namely $V_\iota = V_\iota(m)$.

**Definition 8.2.** Let $X_\iota$, $\iota \in I$ be positive random variables which are free w.r.t. $\varphi$. Random variables
\begin{equation}
Y_\iota(m) = \sqrt{R_\iota(m + 1)R_\iota(m)}\sqrt{R_\iota(m + 1)},
\end{equation}
where $\iota \in I$ and $m \in \mathbb{N}$, will be called multiplicative positive subordination branches.

For simplicity, we will also use the terms: ‘bounded branches’, ‘unitary branches’ and ‘positive branches’. As in the case of operators $R_\iota(m)$, where $m \in \mathbb{N}$ and $\iota \in I$, we get two sequences of alternating (bounded, unitary) branches
\begin{equation}
U_\iota(1), U_\iota(2), U_\iota(3), U_\iota(4), \ldots \in B(K_\iota)
\end{equation}
for each $\iota \in I$, and similar sequences of positive branches. Recall that existence of ‘interlaced branches’ was also observed for the additive (self-adjoint) subordination branches [16].

Of particular interest are branches of 1st order, and it is therefore of advantage to use a special notation for the first two operators in each ‘interlaced’ sequence given by (8.1). For $\iota \in I$, we shall use
\begin{equation}
R_\iota^{\text{odd}} = 1 + \sum_{j \text{ odd}} x_\iota(j) \quad \text{and} \quad R_\iota^{\text{even}} = 1 + \sum_{j \text{ even}} x_\iota(j)
\end{equation}
where again, the unit 1 has to be interpreted as $1_{K_\iota}$ and $1_{K_\iota'}$, respectively. Note that although we have two different units here, the operators which appear in the same ‘interlaced’ sequence contain the same unit.

**Remark 8.1.** Let us recall that in the additive case [15, Theorem 8.4], the branches $B_1$ and $B_2$ of the sum $X_1 + X_2$ can be decomposed as
\begin{equation}
B_1 = S_1^{\text{odd}} + S_2^{\text{even}},
\end{equation}
where $S_1^{\text{odd}} = \sum_{j \text{ odd}} X_1(j)$, $S_2^{\text{even}} = \sum_{j \text{ even}} X_2(j)$, and that the pair $(S_1^{\text{odd}}, S_2^{\text{even}})$ is s-free w.r.t. the pair of states $(\varphi, \psi)$, where $\psi$ is associated with any unit vector $\zeta \in H_0^\iota$. Note that we have $R_1^{\text{odd}} = S_1^{\text{odd}}$ and $R_2^{\text{even}} = P_\zeta + S_2^{\text{even}}$, but we prefer to have a new notation to maintain a uniform style for ‘subtracting units’.

This, in turn, leads to a special notation for branches of 1st order, $U_\iota = U_\iota(1)$, $Y_\iota = Y_\iota(1)$, $\iota \in I$. For instance,
\begin{align*}
Y_1 &= Y_1(1) = \sqrt{R_2^{\text{even}}}R_1^{\text{odd}}\sqrt{R_2^{\text{even}}}, \\
Y_2 &= Y_2(1) = \sqrt{R_1^{\text{even}}}R_2^{\text{odd}}\sqrt{R_1^{\text{even}}}.
\end{align*}
Many computations can be reduced to branches of 1st order since their $\varphi$-distributions agree with the distributions of branches of higher orders with respect to suitably chosen states.

Let us examine the $\varphi$-distributions of branches of first order. In fact, we will show that $Y_\iota$ and $U_\iota$ have the same $\varphi$-distributions as $T_\iota$ for given $\iota \in I$. 


Lemma 8.1. The variables $Q_i$, $U_i$ and $V_i$ have the same $\varphi$-distributions for any given $i \in I$. If $X_i$, $i \in I$, are positive, then the variables $T_i, Y_i, U_i$ and $V_i$ have the same $\varphi$-distributions for any given $i \in I$.

Proof. We shall give the proof in the more difficult case of positive operators (the proof for the case of bounded operators is similar). For $n \geq 1$, we have

$$\varphi(Y_i^n) = \left(\sqrt{R_{X_i}^{\text{odd}}} (R_{X_i}^{\text{odd}} R_{X_i}^{\text{even}})^{n-1} \sqrt{R_{X_i}^{\text{even}}} \right)$$

$$= \varphi \left( (R_{X_i}^{\text{odd}} R_{X_i}^{\text{even}})^{n-1} R_{X_i}^{\text{odd}} \right)$$

$$= \varphi \left( U_i^n \right)$$

since $\sqrt{R_{X_i}^{\text{even}}} \xi = \xi$ (the only summand in the direct sum decomposition of this square root which gives a non-zero contribution is $1_{\mathbb{P}}(1)$). This shows that $Y_i$ and $U_i$ have the same $\varphi$-distributions.

Let us now compare the moments of $T_i$ with those of $Y_i$. We have

$$\varphi(T_i^n) = \varphi \left( \sqrt{X_i - x_i} (X_i(X_i - x_i))^{n-1} \sqrt{X_i - x_i} \right)$$

$$= \varphi \left( (X_i(X_i - x_i))^{n-1} \right)$$

since $\sqrt{X_i - x_i} \xi = \xi$. To fix attention, suppose that $i = 2$ and denote, for convenience, $z_k = x_1(k)$ and $y_k = x_2(k)$. Now, let us observe that in the finite set of variables of type $z_k$ and $y_j$, where $k \geq 2$ and $j \geq 1$, which are used when calculating $X_2(X_1 - x_1)^{n-1} X_2 \xi$ there are no $z_k$’s for odd $k$’s, or $y_j$’s for even $j$’s. This is because the only variable which may give a non-zero contribution when acting on $\xi$ is $y_1$, then only $y_1$ and $z_2$ may follow, of which the first operator can be followed by $y_1$ or $z_2$, whereas the second — by $z_2, y_1$ or $y_3$, etc. This leads to

$$\varphi(P(X_2, X_1 - x_1)) = \varphi(P(R_2^{\text{odd}}, R_1^{\text{even}}))$$

for any polynomial $P$ in two noncommuting variables. This shows that $T_2$ and $Y_2$ have the same $\varphi$-distributions.

Finally, we will show that $U_2$ and $V_2$ have the same $\varphi$-distributions. Using the notation used in the previous paragraph, we have

$$\varphi((U_2 - 1)^n) = \varphi( \left( \sum_{k \text{ even}} z_k + \sum_{j \text{ odd}} y_j + \sum_{|j-k| < 1} y_j z_k \right)^n)$$

$$\varphi((V_2 - 1)^n) = \varphi( \left( \sum_{k \text{ even}} z_k + \sum_{j \text{ odd}} y_j + \sum_{|j-k| < 1} z_k y_j \right)^n)$$

for any natural $n$, and therefore, we need to show that the right-hand-sides of the above equations are equal to each other. Note that the products of variables $z_k$ and $y_j$ in words $w$ which appear in mixed moments $\varphi(w)$ giving non-zero contributions to the above moments satisfy the following three conditions:

(i) whenever $z_k$ stands next to $z_m$ or $y_k$ next to $y_m$, it holds that $k = m$,
(ii) whenever $y_j$ stands next to $z_k$, it holds that $|j - k| = 1$, 
(iii) the word \( w \) begins and ends with the letter \( y_1 \).
Let us denote by \( W \) the set of words in letters
\[
L = \{z_k, y_j : k \text{ even, } j \text{ odd}\}
\]
subject to conditions (i)-(iii). In turn, let \( W' \) and \( W'' \), respectively, denote the sets of all words in letters
\[
L' = \{z_k, y_j, z_k y_{k-1}, z_k y_{k+1} : k \text{ even, } j \text{ odd}\},
\]
\[
L'' = \{z_k, y_j, y_{k-1} z_k, y_{k+1} z_k : k \text{ even, } j \text{ odd}\}
\]
subject to conditions (i) and (iii), where products of type \( z_k y_j \) and \( y_j z_k \) are treated as letters. When they are used to form words, they are denoted by \( [z_k y_j] \) and \( [y_j z_k] \), respectively. Note that \( W \subset W' \) and \( W \subset W'' \). More importantly, there is a bijection
\[
\tau : W' \rightarrow W''
\]
defined by ‘shifting brackets to the left’. Namely, \( \tau \) is uniquely defined by \( \tau (w_1) = w_1 \) and the recursion
\[
\tau(w_1 y_m z_k^p[z_k y_j] w_2) = w_1[y_m z_k] z_k^p \tau(y_j w_2)
\]
for any \( w_1 \in W, w_2 \in W', p \geq 0 \) and \( |j-k| = 1, |k-m| = 1 \), i.e. we assume that \( [z_k y_j] \) is the first such pair (counting from the left). In this recursion, we shift the brackets to the left from this pair to the closest possible pair \( y_m, z_k \) lying to its left (in view of (iii), such a pair must exist). It is not hard to see that its inverse is given by \( \tau^{-1}(u_2) = u_2 \) and the recursion
\[
\tau^{-1}(w'_1[y_j z_k] z_k^p[y_m] w'_2) = \tau^{-1}(w'_1 y_j) z_k^p[z_k y_m] w'_2
\]
where \( w'_1 \in W'', w'_2 \in W, p \geq 0 \) and \( |j-k| = 1, |k-m| = 1 \). Again, note that for every such pair \( [y_j z_k] \) there must exist \( y_m \) standing to its right in view of (iii). Clearly, for every \( w \in W' \) it holds that \( \varphi(\tau(w)) = \varphi(w) \). This completes the proof.

We are ready to prove a theorem, which will lead to a relation between the distributions of two consecutive branches. For that purpose, introduce operators
\[
(8.9) \quad Z_\iota(m) = 1 + x_\iota(m)
\]
with the same convention concerning the units as before (1 = \( 1_{K_\iota} \) if \( m \) is odd and \( 1 = 1_{K_\iota} \) if \( m \) is even), as well as operators obtained from the subordination branches by subtracting the (appropriate) unit:
\[
(8.10) \quad y_\iota(m) = Y_\iota(m) - 1
\]
for any \( m \in \mathbb{N} \) and \( \iota \in I \).

**Theorem 8.1.** For given \( m \in \mathbb{N} \) and \( \iota \in I \), let \( \varphi, \psi \) be states associated with any unit vectors \( \zeta \in \mathcal{H}_{\iota}^{(m-1)}(i) \) and \( \zeta' \in \mathcal{H}_{\iota}^{(m)}(\tau) \). Then, operators \( U_\iota(m) \) and \( Z_\iota(m)U_{\tau}(m+1) \) have the same \( \varphi \)-distributions. Similarly, if \( X_\iota, \iota \in I \), are positive, then operators \( Y_\iota(m) \) and \( Z_\iota(m)Y_{\tau}(m+1) \) have the same \( \varphi \)-distributions. Moreover, the pairs \( (x_\iota(m), y_\iota(m+1)) \) and \( (x_\iota(m), y_{\tau}(m+1)) \) are orthogonal with respect to \( (\varphi, \psi) \) for each \( \iota \in I \) and \( m \in \mathbb{N} \).
Proof. We shall prove the more difficult case of positive random variables. Without
loss of generality, let \( \nu = 1 \) and, for notational simplicity, consider the case \( m = 1 \) (the
proof of the general case is similar). Denote
\[
Z_1 = Z_1(1), \quad R_1 = R_1(1), \quad R_3 = R_1(3), \quad R_2 = R_2(2), \quad Y_2 = Y_2(2).
\]
In the first part of the theorem we need to prove that \( Y_1 \) has the same \( \varphi \)-distribution
as \( Z_1Y_2 \). Using orthogonal decompositions and 'square root calculus', we obtain
\[
Z_1 = X_1(1) + \sum_{j \in J(3)} P_1(j), \quad \text{and} \quad \sqrt{R_3} = P_1(1) + \sum_{j \in J(3)} \sqrt{X_1(j)}
\]
which gives
\[
\sqrt{R_3}Z_1\sqrt{R_3} = \sum_{j \text{ odd}} X_1(j) = R_1,
\]
and therefore
\[
Y_1 = \sqrt{R_2}R_1\sqrt{R_2} = \sqrt{R_2}\sqrt{R_3}Z_1\sqrt{R_3}\sqrt{R_2}.
\]
Therefore,
\[
\varphi(Y_1^n) = \varphi\left(\left(\sqrt{R_2}\sqrt{R_3}Z_1\sqrt{R_3}\sqrt{R_2}\right)^n\right)
\]
\[
= \varphi\left(\sqrt{R_2}\sqrt{R_3}(Z_1\sqrt{R_3}R_2\sqrt{R_3})^{n-1}Z_1\sqrt{R_3}\sqrt{R_2}\right)
\]
\[
= \varphi\left(\sqrt{R_2}\sqrt{R_3}(Z_1Y_2)^{n-1}Z_1\right)
\]
\[
= \varphi\left((Z_1Y_2)^n\right)
\]
since \( Y_2\xi = \xi \) and also \( \sqrt{R_3}\sqrt{R_2}\xi = \xi \). Therefore, the \( \varphi \)-distribution of \( Y_1 \) agrees with
the \( \varphi \)-distribution of \( Z_1Y_2 \).

In the proof of orthogonality, denote, as before, \( x_1 = x_1(1) \), and \( y_2 = Y_2 - 1_{K_1} \). Using
orthogonal decompositions and 'square root calculus', we can write
\[
Y_2 = \sqrt{R_3}R_2\sqrt{R_3}
\]
\[
= (P_1(1) + \sum_{j \in J(3)} \sqrt{X_1(j)}) (P_\xi + \sum_{k \in J(2)} X_2(k)) (P_1(1) + \sum_{j \in J(3)} \sqrt{X_1(j)})
\]
and therefore \( Y_2\xi = \xi \), which implies that \( y_2\xi = 0 \). Therefore, we get \( \varphi(wy_2) = 0 \) and
therefore, by taking the adjoints, \( \varphi(y_2w) = 0 \) for any \( w \in \text{alg}(x_1, y_2) \). This gives the
first orthogonality condition. Now, let us prove the second orthogonality condition, i.e.
\[
\varphi(w_1x_1y_2^kx_1w_2) = \psi(y_2^k)(\varphi(w_1x_1^2w_2) - \varphi(w_1x_1)\varphi(x_1w_2))
\]
for any \( k \in \mathbb{N} \) and \( w_1, w_2 \in \text{alg}(x_1, y_2) \). Note that the 'lowest order term' in the
expression for \( y_2 \), which turns out to be of special importance when \( x_1 \) stands next to
\( y_2 \), is of the form
\[
P_1(1)X_2(2)P_1(1) = P_1(1)P_2(2)X_2P_2(2)P_1(1).
\]
Recall that \( P_2(2) \) is the projection onto \( \mathcal{H}_2^0 \oplus (\mathcal{H}_2 \otimes \mathcal{H}_1^0) \) and \( P_1(1) \) is the projection
onto \( \mathbb{C}^2 \oplus \mathcal{H}_1^0 \), thus \( P_1(1)P_2(2) = P_{\mathcal{H}_2^0} \), and therefore we get
\[
x_1y_2^mx_1 = x_1P_{\mathcal{H}_2^0}y_2^mx_1.
\]
for $m \geq 1$. This gives
\[ \varphi(w_1 x_1 y_2^m x_1 w_2) = \varphi(w_1 x_1 P_{\mathcal{H}_1^0} x_1 w_2) \psi(y_2^m), \]
since, for any $\zeta \in \mathcal{H}_1^0$ of norm one, we have
\[ y_2^m \zeta = \psi(y_2^m) \zeta \mod \mathcal{K}(2) \otimes \mathcal{K}(1)(2), \]
where $\psi$ is the state associated with $\zeta$. Note that $\psi$ gives the same moments of the variable $y_2$ irrespective which $\zeta$ is taken and they agree with the corresponding moments in the state $\varphi_2$. Finally, since $P_{\mathcal{H}_1^0} = P_{\mathcal{C} \otimes \mathcal{H}_1^0} - P_{\mathcal{C} \otimes 1}$, we have
\[ \varphi(w_1 x_1 P_{\mathcal{H}_1^0} x_1 w_2) = \varphi(w_1 x_1^2 w_2) - \varphi(w_1 x_1) \varphi(x_1 w_2), \]
which completes the proof of the second orthogonality condition. ■

**Corollary 8.1.** Let $\mu_i$ and $\sigma_i$ be $\varphi$-distributions of (bounded operators) $X_i$ and $U_i$, respectively, for each $i \in I$, and let $\psi_{m-1}$ be the state associated with any unit vector $\zeta \in \mathcal{H}^{(m-1)}(\mu)$, where $m \in \mathbb{N}$. Then

1. the $\psi_{m-1}$-distribution of $U_i(m)$ agrees with $\sigma_i$ for every $m \geq 2$ and $i \in I$,
2. it holds that $\sigma_i = \mu_i \angle \sigma_2$ for each $i \in I$,
3. if $X_1, X_2$ are positive and $\mu_1, \mu_2 \neq \delta_0$, then $\sigma_i$ agrees with the s-free convolution $\mu_i \boxplus \mu_2$ of probability measures on $\mathbb{R}_+$ for each $i \in I$,
4. if $X_1, X_2$ are unitary, then $\sigma_i$ agrees with the s-free convolution $\mu_i \boxplus \mu_2$ of probability measures on $\mathbb{T}$ for each $i \in I$.

**Proof.** Assertion (1) follows from the definition of operators $U_i(m)$. Then, (2) is a consequence of (1) and Theorem 8.3. Finally, it follows from Theorems 7.3 and 8.3 that the $\varphi$-distributions of $U_i$ satisfy the subordination equations (2.1). Uniqueness of the subordination functions for the cases considered in (3) and (4) give the assertions. ■

Corollary 8.1, together with Corollary 4.6, naturally lead to the definition of a sequence of iterations of the s-free multiplicative convolution. Thus, for given $\mu_1, \mu_2 \in \Sigma$ let
\[ (8.11) \quad \mu_1 \angle_1 \mu_2 = \mu_1 \angle \mu_2, \quad \mu_1 \angle n \mu_2 = \mu_1 \angle (\mu_2 \angle_{n-1} \mu_1), \]
for $n \geq 2$ (of course, this sequence can also be defined by for any measures, for which the operation $\angle$ has been defined).

**Corollary 8.2.** If $\mu_1, \mu_2 \in \mathcal{M}_{\mathbb{R}_+} \setminus \{\delta_0\}$ are compactly supported, then
\[ (8.12) \quad w - \lim_{n \to \delta} (\mu_1 \angle_{n} \mu_2) = \mu_1 \boxplus \mu_2 \]
\[ (8.13) \quad w - \lim_{n \to \delta} (\mu_1 \circ (\mu_2 \angle_{n} \mu_1)) = \mu_1 \boxminus \mu_2, \]
and the corresponding $\eta$-transforms converge uniformly on the compact subsets of $\mathbb{C} \setminus \mathbb{R}_+$ to the $\eta$-transforms of the limit measures. An analogous result holds for $\mu_1, \mu_2 \in \mathcal{M}_{\mathbb{R}}$.

**Proof.** In view of Corollary 4.6, for fixed $m \in \mathbb{N}$, $(\mu_1 \angle_{n} \mu_2)(m) = (\mu_1 \angle_{n} \mu_2)(m)$ for all $n \geq m$, from which we obtain convergence of moments as $\rightarrow \infty$ for any distributions $\mu_1, \mu_2 \in \Sigma$. Since the measures are compactly supported, this implies weak convergence
of the corresponding measures. Therefore, the $\eta$-transforms converge uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}_+$. Moreover, by Corollary 8.1, the limit distributions agree with the distributions of the positive subordination branch $Y_1$, which is $\mu_1 \boxtimes \mu_2$. The arguments for the weak convergence of the second sequence are similar. An analogous proof holds for $\mu_1, \mu_2 \in \mathcal{M}_\ast$. ■

**Remark 8.2.** Informally, the weak limits of Corollary 8.2 can be written in the following form:

$$
\mu_1 \boxtimes \mu_2 = \mu_1 \triangleleft (\mu_2 \triangleleft (\mu_1 \triangleleft (\mu_2 \triangleleft (\ldots)))),
$$
$$
\mu_1 \blackboxtimes \mu_2 = \mu_1 \blacktriangleleft (\mu_2 \blacktriangleleft (\mu_1 \blacktriangleleft (\mu_2 \blacktriangleleft (\ldots)))),
$$
whereas their transforms in the ‘continued composition form’:

$$
\rho_{\mu_1 \boxtimes \mu_2}(z) = \rho_{\mu_1}(z\rho_{\mu_2}(z\rho_{\mu_1}(z\rho_{\mu_2}(\ldots)))),
$$
$$
\eta_{\mu_1 \blackboxtimes \mu_2}(z) = \eta_{\mu_1}(z\rho_{\mu_2}(z\rho_{\mu_1}(z\rho_{\mu_2}(\ldots)))),
$$
where the right-hand sides are understood as the uniform limits on compact subsets of $\mathbb{C}\setminus\mathbb{R}_+$ or $\mathbb{D}$. Actually, these formulas can be used to compute some examples, including simple examples of $\mu_1 \blackboxtimes \mu_2$ (without using S-transforms).

**Remark 8.3.** By Corollary 8.2, one can define the s-free and free convolutions of any compactly supported measures $\mu_1, \mu_2$ on $\mathbb{C}$ as the weak limits of the form (8.12)-(8.13). This allows us to use operatorial subordination results of Sections 7-8, including those which concern distributions of products of free bounded random variables as well as distributions of bounded branches and express them in terms of s-free convolutions, denoted with the same symbol $\boxtimes$ and understood as weak limits of type (8.12)-(8.13).

Using the weak limits, we also get

$$
\delta_0 \boxtimes \mu = \delta_0 \quad \text{and} \quad \mu \boxtimes \delta_0 = \delta_{\mu(X)}
$$
and

$$
\mu \blackboxtimes \delta_0 = \delta_0 = \delta_0 \blackboxtimes \mu,
$$
for any compactly supported probability measure on $\mathbb{C}$, where we used the well-known relations $\mu \blacktriangleleft \delta_0 = \delta_0 = \delta_0 \blacktriangleleft \mu$. 

**Example 8.1.** Using Corollary 8.2 and the results of Example 5.1, we obtain $\mu \boxtimes \delta_a = \mu \triangleleft \delta_a = S_a(\mu)$ and $\delta_a \boxtimes \mu = \delta_a \triangleleft \mu = \delta_a$ for compactly supported $\mu \in \mathcal{M}_{\mathbb{R}_+}$ and $a > 0$. This gives

$$
\delta_a \blackboxtimes \mu = \delta_a \blacktriangleleft S_a\mu = D_a\mu,
$$
in view of (1.5), since the corresponding $\eta$-transform is of the form

$$
\eta_{\delta_a}(\eta_{S_a\mu}(z)) = \eta_{D_a\mu}(z),
$$
where we used $\eta_{\delta_a}(z) = az$ and $\eta_{S_a\mu}(z) = \eta_{D_a\mu}(z)/a$. On the other hand,

$$
\mu \blackboxtimes \delta_a = \mu \blacktriangleleft \delta_a = D_a\mu,
$$
since $\eta_{\mu \blackboxtimes \delta_a}(z) = \eta_{\mu}(az) = \eta_{D_a\mu}(z)$. In particular, we have $\delta_a \boxtimes \delta_b = \delta_a$ for $a, b > 0$. 

Example 8.2. Using the results of Example 5.3, we obtain
\[ \mu \boxtimes \delta_a = S_a \mu \quad \text{and} \quad \delta_a \boxtimes \mu = \delta_a , \]
for compactly supported \( \mu \in \mathcal{M}_\tau \) and \( a \in \mathbb{T} \), which gives
\[ \mu \boxtimes \delta_a = D_a \mu = \delta_a \boxtimes \mu , \]
by the same arguments as those used in Example 8.1.

Let us finally show that the multiplicative subordination branches are related to the notion of s-free independence, as it was the case of their additive counterparts. For that purpose, let us recall this concept [16].

Definition 8.3. Let \( (\mathcal{A}, \varphi, \psi) \) be a unital algebra with a pair of linear normalized functionals. Let \( \mathcal{A}_1 \) be a unital subalgebra of \( \mathcal{A} \) and let \( \mathcal{A}_2 \) be a non-unital subalgebra with an ‘internal’ unit \( 1_2 \), i.e. \( 1_2 b = b = b 1_2 \) for every \( b \in \mathcal{A}_2 \). We say that the pair \( (\mathcal{A}_1, \mathcal{A}_2) \) is free with subordination, or simply s-free, with respect to \( (\varphi, \psi) \) if \( \psi(1_2) = 1 \) and it holds that

1. \( \varphi(a_1 a_2 \ldots a_n) = 0 \) whenever \( a_j \in \mathcal{A}_1^0 \) and \( i_1 \neq i_2 \neq \ldots \neq i_n \)
2. \( \varphi(w_1 w_2) = \varphi(w_1) \varphi(w_2) \) for any \( w_1, w_2 \in \text{alg}(\mathcal{A}_1, \mathcal{A}_2) \),

where \( \mathcal{A}_1^0 = \mathcal{A}_1 \cap \ker \varphi \) and \( \mathcal{A}_2^0 = \mathcal{A}_2 \cap \ker \psi \). We say that the pair \( (a, b) \) of random variables from \( \mathcal{A} \) is s-free with respect to \( (\varphi, \psi) \) if there exists \( 1_2 \in \mathcal{A} \) such that \( (\mathcal{A}_1, \mathcal{A}_2) \), where \( \mathcal{A}_1 := \text{alg}(1, a) \) and \( \mathcal{A}_2 := \text{alg}(1_2, b) \), is s-free w.r.t. \( (\varphi, \psi) \).

Proposition 8.2. The pair \( (R_{1_2}^{\text{odd}} - 1, R_{1_2}^{\text{even}} - 1) \) is s-free with respect to the pair \( (\varphi, \psi) \), where \( \psi \) is the state associated with any unit vector \( \zeta \in \mathcal{H}_1^0 \), and the projection associated with the second variable is \( 1_2 = 1 - P_{\zeta} \).

Proof. For simplicity, choose \( \iota = 1 \) and denote \( r_1 = R_{1_2}^{\text{odd}} - 1, r_2 = R_{1_2}^{\text{even}} - 1 \). We have
\[ r_1 = \sum_{j \text{ odd}} x_1(j) \quad \text{and} \quad r_2 = \sum_{j \text{ even}} x_2(j) . \]
Clearly, the definition of \( 1_2 \) immediately gives \( \psi(1_2) = 1 \) as well as condition (ii) of s-freeness. Let us show that condition (i) also holds. Let \( \mathcal{A}_1 = \text{alg}(r_1, 1) \) and \( \mathcal{A}_2 = \text{alg}(r_2, 1_2) \). If in (i) we take \( a_n \in \mathcal{A}_2 \cap \ker \psi \), then \( \lambda(a_n) \zeta = 0 \), where \( \lambda \) is the free product representation, and hence (i) holds. If \( a_n, a_{n-2}, \ldots \in \mathcal{A}_1 \cap \ker \varphi \) and \( a_{n-1}, a_{n-3}, \ldots \in \mathcal{A}_2 \cap \ker \psi \), then the GNS representation space for such a mixed moment is \( K_1 \) (none of \( a_{n-1}, a_{n-3}, \ldots \) ever gets to act on \( \zeta \)). Therefore, in the computation of this kind of moment, one can replace \( r_1, r_2 \) and \( 1_2 \) by \( X_1, X_2 \) and \( 1 \), respectively. But then (i) follows from the usual freeness condition.

9. Free Product and s-Free Loop Product of Graphs

In this section we prove the Multiplication Theorem for s-free independence and free independence.

Let us recall the definition of the s-free product of rooted graphs [2,16].
Definition 9.1. By the s-free product of rooted graphs \((G_1, e_1)\) and \((G_2, e_2)\), denoted \((G_1 \oplus G_1, e_1)\), we understand the inductive limit of the sequence \((B_1(m), e_1)_{m \in \mathbb{N}}\) of rooted graphs, where \(B_1(1) = G_1 \rightharpoonup G_2\) and \(B_1(m)\) is obtained from \(B_1(m - 1)\) by attaching by its root a copy of \(G_1\) (if \(m\) is even), or a copy of \(G_2\) (if \(m\) is odd) to every vertex of the difference \(B_1(m - 1) \setminus B_1(m - 2)\).

Remark 9.1. It can be seen that \((B_1(m), e_1)_{m \in \mathbb{N}}\) is a sequence of growing graphs with the root kept to be \(e_1\), and therefore, the inductive limit exists. In a similar way we define \((G_2 \oplus G_1, e_2)\). It is easy to see that

\[
B_1(m) = G_1 \rightharpoonup G_2(m - 1) \quad \text{and} \quad B_2(m) = G_2 \rightharpoonup G_1(m - 1),
\]

which can also serve an inductive definition of the considered sequences. These formulas are important since they correspond to the decomposition of the s-free additive convolution of the form

\[
\mu_1 \boxplus \mu_2 = \mu_1 \rightharpoonup (\mu_2 \rightharpoonup (\mu_1 \rightharpoonup (\mu_2 \rightharpoonup \ldots))),
\]

where \(\mu_1\) is the spectral distribution of \(G_1\) [2].

Using the decomposition (9.2), we proved the Addition Theorem for s-free independence in [2]. Our first goal in this Section is to prove the corresponding Multiplication Theorem. For that purpose, we introduce a new notion of graph product, called the ‘s-free loop product’. We follow it up with a proposition which justifies the definition.

Definition 9.2. Suppose that the s-free product of graphs \((G_1 \oplus G_2, e_1)\) is naturally colored. The s-free loop product of \((G_1, e_1)\) and \((G_2, e_2)\) is the graph \((G_1 \oplus e G_2, e_2)\) obtained from \((G_1 \oplus G_2, e_1)\) by attaching a loop of color 2 to the root \(e_1\).

Example 9.1. Consider the s-free loop product of graphs given in Fig.4, which can be viewed as a ‘binary tree with loops’ (the product graph is of course infinite, only our picture is truncated at distance 5 from the root). Note that graph \(G_1\) has a loop at the root, and so do copies of \(G_1\) in the product graph, whereas \(G_2\) does not have any loops. We produce \(G_1 \oplus G_2\) in the inductive way (Definition 9.1 is used). Now, in order to obtain the s-free loop product, we attach one additional loop of color 2 to the root (we draw this loop larger than loops of color 1). On the level of the adjacency matrices, this additional loop corresponds to the projection \(P_2\) as the proof of Proposition 9.1 shows. Note that the main purpose of attaching this extra loop is to fit the s-free multiplicative convolution into the general scheme of the Multiplication Theorem, where counting...
rooted alternating d-walks leads to the moments of multiplicative convolutions. Direct computations give, for instance, $D_2(e_1) = 1$, $D_4(e_1) = 0$, $D_6(e_1) = 4$, $D_8(e_1) = 0$.

**Proposition 9.1.** Let $A_i$ be the adjacency matrix of $\mathcal{G}_i$, where $i \in I$. The adjacency matrix of the s-free loop product of $\mathcal{G}_1$ and $\mathcal{G}_2$ takes the form

$$A(\mathcal{G}_1 \oplus \mathcal{G}_2) = R_1 + R_2$$

(9.3)

where

$$R_1 = \sum_{j \text{ odd}} A_1(j) \quad \text{and} \quad R_2 = P_\xi + \sum_{k \text{ even}} A_2(k).$$

Moreover, the $\varphi_e$-distribution of $R_1$ is $\mu_1$, the $\psi$-distribution of $R_2$ is $\mu_2$ and the pair $(R_1 - 1, R_2 - 1)$ is s-free independent w.r.t. $(\varphi_e, \psi)$, where $\psi$ is the state associated with any unit vector $\zeta \in \mathcal{H}_1^0 = l_2(V_1^0)$.

**Proof.** It follows from [2,16] that the adjacency matrix of the s-free product of graphs is of the form

$$A(\mathcal{G}_1 \odot \mathcal{G}_2) = S_1 + S_2,$$

where

$$S_1 = \sum_{j \text{ odd}} A_1(j) \quad \text{and} \quad S_2 = \sum_{k \text{ even}} A_2(k).$$

By Definition 9.1, the adjacency matrix of the corresponding s-free loop product is obtained by adding the projection $P_0$ to the adjacency matrix of the subgraph of color 2, namely $S_2$, which corresponds to attaching the loop of color 2 to the root. This gives

$$A(\mathcal{G}_1 \oplus \mathcal{G}_2) = R_1 + R_2,$$

where

$$R_1 = S_1 \quad \text{and} \quad R_2 = P_\xi + S_2.$$ 

In view of Proposition 8.2, the pair $(R_1 - 1, R_2 - 1)$ is s-free independent w.r.t. the pair $(\varphi_e, \psi)$, where $\psi$ is the state associated with any unit vector $\zeta \in \mathcal{H}_1^0$ (here, 1 denotes the unit on $l_2(V)$, where $V$ is the set of vertices of $\mathcal{G}_1 \odot \mathcal{G}_2$, which corresponds to $K_1$). Moreover, it is easy to check that the $\varphi_e$-distribution of $R_1$ is $\mu_1$ and the $\psi$-distribution of $R_2$ is $\mu_2$. This completes the proof. 

Thus, the first part of the Multiplication Theorem for s-free independence is proved. We now need to find a connection between the ‘first return moments’ of $R_2R_1$ and cardinalities of the sets $D_{2n}$. In that context, note that in order that the set $D$ of all rooted alternating d-walks on $\mathcal{G}_1 \oplus \mathcal{G}_2$ be non-empty, either $\mathcal{G}_1$ or $\mathcal{G}_2$ must have a loop at $e_1$ or $e_2$, respectively. In fact, otherwise, starting an alternating walk from the root of $\mathcal{G}_1 \oplus \mathcal{G}_2$, one cannot return to it since the $m$-th edge of that walk must be of different color than the $m-1$-th edge and thus it must belong to $\mathcal{E}_j(m) \setminus \mathcal{E}_j(m-1)$, and therefore its distance from the root (understood as the distance of the vertex closer to the root) equals $m - 1$ and thus tends to $\infty$ as $m \to \infty$.

**Theorem 9.1.** The Multiplication Theorem holds for s-free independence, the associated loop product $\mathcal{G}_1 \oplus \mathcal{G}_2$, and the multiplicative convolution $\mu_1 \boxtimes \mu_2$. 

Proof. For notational simplicity, denote \( \sigma_1 = \mu_1 \boxtimes \mu_2 \), \( \sigma_2 = \mu_2 \boxtimes \mu_1 \) and
\[
\mathcal{B}_i = \mathcal{G}_i \oplus \mathcal{G}_r, \quad \mathcal{B}^\ell_i = \mathcal{G}_i \otimes \mathcal{G}_r
\]
From Proposition 9.1 and the results of Section 8, it follows that
\[
N_{R_2R_1}(n) = N_{\sigma_1}(n)
\]
for \( n \in \mathbb{N} \). Now, we need to prove that
\[
N_{\sigma_1}(n) = |D_{2n}(e_1)|
\]
for \( k \in \mathbb{N} \). Note first that any f-walk \( w \in F(e_1) \) on \( \mathcal{B}_1 \) must begin and terminate with an edge of color 1 since no edge of color 2 is incident on \( e_1 \). Since \( \mathcal{B}^\ell_1 \) is obtained from \( \mathcal{B}_1 \) by attaching a loop of color 2 to the root \( e_1 \), the set of rooted alternating d-walks on \( \mathcal{B}^\ell_1 \) of length 2\( n \) is in 1-1 correspondence with the set of rooted alternating f-walks on \( \mathcal{B}_1 \) of length 2\( n - 1 \). In fact, each rooted alternating d-walk of length 2\( n \) on \( \mathcal{B}^\ell_1 \) is of the form \( w = (u_1, u_2) \), where \( u_1 \) is a rooted alternating f-walk of length 2\( n - 1 \) and \( u_2 \) is the rooted loop of color 2. Therefore,
\[
|D_{2n}(e_1)| = |F_{2n-1}(e_1)|,
\]
where \( D_{2n}(e_1) \) refers to the loop product and \( F_{2n-1}(e_1) \) to the usual product. This fact will be used in the induction proof given below. If \( n = 1 \), we use this correspondence to get \( N_{\sigma_1}(1) = N_{\mu_1}(1) = |F_1(e_1)| = |D_2(e_1)| \). Suppose now that (9.6) holds if \( n \) is replaced by \( 1 \leq k \leq n - 1 \), where \( \sigma_1 \) is understood to be the s-free convolution of any \( \sigma' \) and \( \sigma'' \) (thus, in particular, \( \mu_2 \) and \( \mu_1 \)). Using the proof of Theorem 6.1, we have
\[
N_{\sigma_1}(n) = \sum_{r=1}^{n} N_{\mu_1}(r) \sum_{k_1 + k_2 + \ldots + k_{r-1} = n-1} N_{\sigma_2}(k_1) N_{\sigma_2}(k_2) \ldots N_{\sigma_2}(k_{r-1}).
\]
In view of (9.7), it is enough to show that the expression on the RHS is equal to \(|F_{2n-1}(e_1)|\). By the inductive assumption, we replace in the above formula each \( N_{\sigma_2}(k_i) \) by \(|F_{2k_i}(e_1)|\), and then, using (9.7), by \(|F_{2k_i-1}(e_1)|\). Therefore, we need to justify the above formula, viewing \( N_{\sigma_1}(n) \) as the number of rooted alternating f-walks on \( \mathcal{B}_1 \) and \( N_{\sigma_2}(k_i) \), \( 1 \leq i \leq r-1 \), as numbers of rooted alternating f-walks on \( \mathcal{B}_2 \). Since
\[
\mathcal{B}_1 = \mathcal{G}_1 \vdash \mathcal{B}_2
\]
we can observe that each rooted alternating f-walk of length 2\( n - 1 \) on \( \mathcal{B}_1 \) consists of a rooted f-walk \( c = (v_0, v_1, \ldots, v_r) \in F_r(e_1) \) of color 1, for some \( 1 \leq r \leq n \), with \( r-1 \) alternating f-walks \( w_i \in F(v_i) \) on copies of \( \mathcal{B}_2 \) attached to vertices \( v_i \), where \( 1 \leq i \leq r-1 \). The latter must begin and terminate with edges of color 2 since no edge of color 1 is incident on the roots of these copies. Thus, the expression given on the RHS of the above formula gives the numbers of all alternating rooted f-walks of total length
\[
r + (2k_1 - 1) + (2k_2 - 1) + \ldots + (2k_{r-1} - 1) = r + 2(n - 1) - (r - 1) = 2n - 1
\]
summed over \( 1 \leq r \leq n \), which completes the proof. \( \blacksquare \)

Example 9.2. The lowest order first return moments \( N_{\mu_1 \boxtimes \mu_2}(n) \) are given by
\[
N_{\mu_1 \boxtimes \mu_2}(1) = N_{\mu_1}(1), \quad N_{\mu_1 \boxtimes \mu_2}(2) = N_{\mu_1}(2)N_{\mu_2}(1),
\]
\[
N_{\mu_1 \boxtimes \mu_2}(3) = N_{\mu_1}(3)N_{\mu_2}^2(1) + N_{\mu_1}(2)N_{\mu_1}(1)N_{\mu_2}(2), \\
N_{\mu_1 \boxtimes \mu_2}(4) = N_{\mu_1}(4)N_{\mu_2}^3(1) + 2N_{\mu_1}(3)N_{\mu_1}(1)N_{\mu_2}(2)N_{\mu_2}(1) + N_{\mu_1}(2)N_{\mu_1}^2(1)N_{\mu_2}(3) + N_{\mu_1}(2)N_{\mu_2}^2(2)N_{\mu_2}(1)
\]

where we used the results of Example 6.2. The enumeration of d-walks given in Example 9.1 can be easily verified by substituting to the above formulas only the non-vanishing moments: \(N_{\mu_1}(1) = 1, \ N_{\mu_1}(2) = 2, \ N_{\mu_2}(2) = 2\). We have \(N_{\sigma_1}(1) = 1 = D_2(e_1)\), \(N_{\sigma_1}(2) = 0 = D_4(e_1)\), \(N_{\sigma_1}(3) = 4 = D_6(e_1)\) and \(N_{\sigma_1}(4) = 0 = D_8(e_1)\), where \(\sigma_1 = \mu_1 \boxtimes \mu_2\).

Finally, we will prove the Multiplication Theorem for free independence. In this case, one does not need to introduce a new type of graph product since no ‘unitization’ of adjacency matrices is necessary to make them freely independent (basically, this is because units are identified in the case of the free product of algebras). Nevertheless, if neither of the graphs, \(G_1\) or \(G_2\), have loops at their roots, then the set of rooted alternating d-walks on \(G_1 \ast G_2\) is empty.

**Theorem 9.2.** The Multiplication Theorem holds for free independence, the associated product graph \(G_1 \ast G_2\), and the multiplicative convolution \(\mu_1 \boxtimes \mu_2\).

**Proof.** First of all, we know [2] that

\[A(G_1 \ast G_2) = S_1 + S_2\]

where \(S_i = \sum_{k=1}^n A_i(k), \ i \in I\). Moreover, the pairs \((S_1, S_2)\) and \((S_1 - 1, S_2 - 1)\) are free w.r.t. \(\varphi\). Also, it is clear that the moments of \(S_2S_1\) agree with the moments of \(\mu_1 \boxtimes \mu_2\). It remains to be shown that the latter coincide with the corresponding numbers of rooted alternating d-walks. Using Proposition 2.1, we have \(\mu_1 \boxtimes \mu_2 = \mu_1 \odot (\mu_2 \boxtimes \mu_1)\) and therefore, we can employ the combinatorial formula used in the proof of Theorem 3.1 to get

\[N_{\sigma_1}(n) = \sum_{r=1}^n N_{\mu_1}(r) \sum_{k_1+k_2+...+k_r=n} N_{\sigma_2}(k_1)N_{\sigma_2}(k_2)\ldots N_{\sigma_2}(k_r)\]

Now, we can decompose the free product of graphs as

\[G_1 \ast G_2 = G_1 \rhd B_2\]

with the root \(e\) of \(G_1 \ast G_2\) obtained by identifying the root \(e_1\) of \(G_1\) with the root \(e_2\) of \(B_2\) (see [2,16,20]). In view of Proposition 9.1,

\[N_{\sigma_2}(k_i) = |F_{2k_i-1}(e_2)|,\]

where \(e_2\) is the root in \(B_2\). Of course, \(N_{\mu_1}(r) = |F_r(e_1)|\), where \(e_1\) is treated as the root of \(G_1\). Therefore, the RHS of the above combinatorial formula gives the number of rooted walks \(w\) on \(G_1 \ast G_2\) consisting of an f-walk \(c = (v_0, v_1, \ldots, v_r) \in F_r(e)\), interlaced with rooted alternating f-walks \(w_1, w_2, \ldots, w_r\) on copies of \(B_2\) attached to vertices \(v_1, v_2, \ldots, v_r\), respectively. In other words, we have

\[w = (\beta_1, w_1, \beta_2, w_2, \ldots, \beta_r, w_r),\]
and therefore, it consists of two rooted alternating f-walks, namely

\[ u_1 = (\beta_1, w_1, \beta_2, w_2, \ldots, \beta_r) \quad \text{and} \quad u_2 = w_r, \]

where \( u_1 \) begins and terminates with an edge of color 1, and \( u_2 \) begins and terminates with an edge of color 2. Therefore, \( w \) is a rooted alternating d-walk of length 2\( n \) which is ‘subordinate’ to an f-walk \( c \) of length \( r \). The summation over \( 1 \leq r \leq n \) gives all rooted d-walks of length 2\( n \), which completes the proof.

In view of this result, the free product of graphs is ‘complete’ in our ‘category’ of product graphs since it is naturally related to the rooted alternating d-walks and no additional loops are necessary to fit it into the scheme of the Multiplication Theorem.

**Example 9.3.** The first return moments \( N_{\mu_1 \boxtimes \mu_2}(n) = N_{\mu_2 \boxtimes \mu_1}(n) \) of lowest orders can be computed using (2.1) and the results of Example 3.2, with the \( N_{\sigma_1}(n) \), where \( \sigma_1 = \mu_1 \boxtimes \mu_2 \), taken from Example 9.2. For the free product in Fig.5, we obtain \( D_2(e) = 0, D_4(e) = 2, D_6(e) = 0 \) and \( D_8(e) = 16 \).

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