Truncated Rayleigh Pareto Distribution

Reyah Zeadan Khalaf  Kareema Abad Al-Kadim
College of Education of Pure Sciences/ university of Babylon
Kareema.kadim@yahoo.com zeaden123@gmail.com

Abstract. In this paper, we introduced a new distribution which is the truncated Rayleigh Pareto distribution of a variable, and some useful functions, and some mathematical and statistical properties such as density function, the cumulative distribution function, limit function, reliability function, hazard function, stress - strength reliability, engineering arithmetic mean, mode and median, harmonic arithmetic mean, the rth moment about the mean and the origin, coefficients of variation, of kurtosis and Skewedness, order statistic, and some estimation methods.

Keywords: Truncated Rayleigh Pareto distribution, density function, cumulative function, Reliability function, hazard function, reverse hazard function, cumulative hazard function, mode, median and quantile, moment generating function, mea and variance, moments, order statistics, estimation methods.
1. Introduction

In practical and scientific life we need to invent or discover a new distribution or develop a new discovery, so the importance of lifelong distribution is used in many areas of existent life like biostatistics, survival function analysis or reliability, and in this field we will look at the amputated Rayleigh Pareto distribution the new. The aim of the research is to find the new (TRPD) distribution by amputating the period for the distribution (RPD). Using methods such as those used by Wen-lian Hung and ching-yichen (2004), Approximate MLE of the scale parameter of the truncated Rayleigh distribution under the first – censored data[1], David R-clark, FCAS (2013), A note on the upper- truncated Pareto distribution[2], Taylor and francis (2014), Parameter Estimation for the Truncated Pareto Distribution[3], Mathias Reachke (2012), Inference for the truncated exponential distribution[4], and Ahmed Yassin Taqi M.CS. Thesis (2014), Estimation of Parameters and Reliability Function for truncated Logistic Distribution[5]. In the second section, we examined the description of the amputated Rayleigh Pareto distribution and the study of its mathematical and statistical properties, and the methods of estimating and comparing them.

2. Truncated Rayleigh – Pareto distribution.

This is the distribution in which the data is amputated and at the point \( x_\cdot \), where \( x_\cdot \) is a constant value. This means that the random drawn values are located between \( 0 < x < x_\cdot \) the pdf for Rayleigh – Pareto distribution is.[6],[7],[8]

\[
f_{RP}(x; q, w, \alpha) = \frac{\alpha}{2q^2w} \left( \frac{x}{w} \right)^{\alpha-1} e^{-\frac{1}{2q^2w^\alpha} x^\alpha}, \quad 0 < x \leq x_\cdot \quad \& \quad q, w, \alpha > 0
\]  

(1)

2.1 The PDF and CDF of TRPD

The p.d.f of the (TRPD) can be given .

\[
g(x; q, w, \alpha) = \frac{f(xq,w,\alpha)}{F(b)-F(a)} = \frac{\alpha}{2q^2w} \left( \frac{x}{w} \right)^{\alpha-1} e^{-\frac{1}{2q^2w^\alpha} x^\alpha} \frac{1}{1- e^{-\frac{1}{2q^2w^\alpha} (x_\cdot)^\alpha}}, \quad 0 < x \leq x_\cdot \quad \& \quad q, w, \alpha > 0
\]  

(2)

Such as \[
\int_0^{x_\cdot} \frac{\alpha}{2q^2w} \left( \frac{x}{w} \right)^{\alpha-1} e^{-\frac{1}{2q^2w^\alpha} x^\alpha} \frac{1}{1- e^{-\frac{1}{2q^2w^\alpha} (x_\cdot)^\alpha}} \, dx = 1
\]

The plot of p.d.f. and p.d.f. for the(TRPD) as follows.
Figure 1: The plot to p.d.f. for (TRPD), the parameter $\alpha=2.2$ $q=1.1,1.4,1.9,2.3$ $w=0.5,0.9,1.3,1.6$

Figure 2: The plot to p.d.f. for (TRPD), the parameter $q=1.1$ $c=0.9,1.1,1.6,1.7$ $w=1.5,1.1,0.7,0.4$
Figure 3: The plot to p.d.f. for (TRPD), the parameter $w=0.8$ $c=2.9,2.6,1.7,1.1$ ; $q=2.3,1.7,1.3,0.8$

And the c.d.f of the (TRPD)is given by

$$G(x; q, w, \alpha) = \int_0^x g(x; q, w, \alpha) \, dx = \int_0^x \frac{\alpha}{2q^2w} \left( \frac{x}{w} \right)^{q-1} e^{-\frac{1}{2q^2w}x} \, dx$$

$$= \frac{1 - e^{-\frac{1}{2q^2w}x^q}}{1 - e^{-\frac{1}{2q^2w}x^q}}, \quad 0 < x \leq x_c \text{ & } q, w, \alpha > 0 \quad (3)$$
**Figure 4:** The plot to c.d.f. for (TRPD), the parameter $\alpha=2.7; q=1.6,1.9,2.3,2.7; w=0.7,1.1,1.6,2.2$.

**Figure 5:** The plot to c.d.f. for (TRPD), the parameter $q=3.4; c=2.2,1.9,1.6,0.9; w=1.4,0.9,0.8,0.6$.

**Figure 6:** The plot to c.d.f. for (TRPD), the parameter $w=3.6; c=1.1,0.9,0.6,0.2; q=1.2,1.1,0.8,0.5$.
2.2. Limit of p.d.f and c.d.f

The limit of this distribution is given by the form a:

\[
\lim_{x \to 0} g(x; q, w, \alpha) = 0 \\
\lim_{x \to 0} \frac{\alpha}{2q^2 w} \left( \frac{x}{w} \right)^{\alpha-1} e^{-\frac{1}{2q^2 w} x^2} = \frac{\alpha}{2q^2 [1 - e^{-\frac{1}{2q^2 w} x^2}]} \lim_{x \to 0} \left( \frac{x}{w} \right)^{\alpha-1} e^{-\frac{1}{2q^2 w} x^2} = 0
\]

Also

\[
\lim_{x \to 0} \frac{1}{2q^2 w} (\frac{x}{w})^{\alpha-1} e^{-\frac{1}{2q^2 w} x^2} = \frac{1}{2q^2 [1 - e^{-\frac{1}{2q^2 w} x^2}]} > 0 \\
\lim_{x \to 0} \frac{e^{-\frac{1}{2q^2 w} x^2}}{1 - e^{-\frac{1}{2q^2 w} x^2}} = 0
\]

Because \(1 - e^{-\frac{1}{2q^2 w} x^2} > 0\), and \(\lim_{x \to 0} \frac{\alpha}{2q^2 w} (\frac{x}{w})^{\alpha-1} e^{-\frac{1}{2q^2 w} x^2} > 0\)

Also the c.d.f. of this distribution is

\[
\lim_{x \to 0} G(x; q, w, \alpha) = \lim_{x \to 0} \frac{1 - e^{-\frac{1}{2q^2 w} x^2}}{1 - e^{-\frac{1}{2q^2 w} x^2}} = 0
\]

Also,

\[
\lim_{x \to x_0} G(x; q, w, \alpha) = \lim_{x \to x_0} \frac{1 - e^{-\frac{1}{2q^2 w} x^2}}{1 - e^{-\frac{1}{2q^2 w} x^2}} = 1
\]

i.e \(0 \leq G(x; q, w, \alpha) \leq 1\)

3. Some Reliability Functions. [9]

In this section, we introduce some reliability functions for the TRPD.

3.1. Reliability Function

The function of survival or reliability to TRPD are.

\[
R(x) = 1 - G(x) = 1 - \frac{1 - e^{-\frac{1}{2q^2 w} x^2}}{1 - e^{-\frac{1}{2q^2 w} x^2}} = \frac{e^{-\frac{1}{2q^2 w} x^2} - e^{-\frac{1}{2q^2 w} x^2}}{1 - e^{-\frac{1}{2q^2 w} x^2}}
\]

Such as
\[
\lim_{x \to 0} R(x) = \lim_{x \to 0} \frac{e^{\frac{-1}{2q^2w^\alpha}} - e^{\frac{-1}{2q^2w^\alpha}}}{1 - e^{\frac{-1}{2q^2w^\alpha}}} = 1
\]

\[
\lim_{x \to x_0} R(x) = \lim_{x \to x_0} \frac{e^{\frac{-1}{2q^2w^\alpha}} - e^{\frac{-1}{2q^2w^\alpha}}}{1 - e^{\frac{-1}{2q^2w^\alpha}}} = 0
\]

### 3.2. Hazard Function

TRPD’s hazard function:

\[
h(x) = \frac{g(x; q, w, \alpha)}{R(x)}
\]

\[
h(x) = \frac{\frac{\alpha}{2q^2w} \left(\frac{x}{w}\right)^{\alpha-1} e^{\frac{-1}{2q^2w^\alpha}}}{1 - e^{\frac{-1}{2q^2w^\alpha}}} = \frac{\frac{\alpha}{2q^2w} \left(\frac{x}{w}\right)^{\alpha-1} e^{\frac{-1}{2q^2w^\alpha}}}{1 - e^{\frac{-1}{2q^2w^\alpha}}}
\]

### 3.3. Reverse Hazard Function

TRPD’s Reverse Hazard function:

\[
r(x) = \frac{g(x; q, w, \alpha)}{G(x; q, w, \alpha)} = \frac{\frac{\alpha}{2q^2w} \left(\frac{x}{w}\right)^{\alpha-1} e^{\frac{-1}{2q^2w^\alpha}}}{1 - e^{\frac{-1}{2q^2w^\alpha}}} = \frac{\frac{\alpha}{2q^2w} \left(\frac{x}{w}\right)^{\alpha-1} e^{\frac{-1}{2q^2w^\alpha}}}{1 - e^{\frac{-1}{2q^2w^\alpha}}}
\]

Such as \(x, \alpha, q, w > 0\)

### 3.4. The Cumulative Hazard Function

TRPD’s cumulative hazard function:

\[
H(x) = -\ln R(x) = -\ln \frac{e^{\frac{-1}{2q^2w^\alpha}} - e^{\frac{-1}{2q^2w^\alpha}}}{1 - e^{\frac{-1}{2q^2w^\alpha}}}
\]

Such as \(x, \alpha, q, w > 0\)
3.5. Stress Strength Reliability.[10]

Let Y represent the pressure applied to a specific device and X represents the strength to maintain the pressure, since the reliability of the stress force is indicated by \( R = P(Y < X) \), if X and Y are assumed to be random.

\[
R = P(Y < X) = \int_0^X g_y(x)G_x(x)\,dx
\]

By using series expansion of \( e^{-\frac{1}{2q^2w^\alpha}x^\alpha} \) then.

\[
e^{-\frac{1}{2q^2w^\alpha}x^\alpha} = \sum_{j=0}^{\infty} \left( -\frac{1}{2q^2w^\alpha} \right)^j \frac{(x^\alpha)^j}{j!}
\]

Then

\[
P(Y < X) = \frac{1}{1 - e^{-\frac{1}{2q^2w^\alpha}x^\alpha}} - \sum_{j=0}^{\infty} \frac{(-2q^2w^\alpha)^{2j+1}}{(2j+1)\alpha} \int_0^X x^{(2j+1)\alpha-1} \,dx
\]

the stress-strength reliability is

\[
P(Y < X) = \frac{1}{1 - e^{-\frac{1}{2q^2w^\alpha}x^\alpha}} - \sum_{j=0}^{\infty} \frac{\alpha(-1)^{2j}(x)^{(2j+1)\alpha}}{(2j+1)\alpha(2q^2w^\alpha)^{2j+1}}
\]

4. Some properties of the TRPD:

Proposition(1)

4.1 mode: The mode of the TRPD is

\[
x_{\text{mode}} = w\left( \frac{\alpha-1}{\alpha} \right) \frac{1}{\sqrt{2q}}
\]

Proof:

\[
x_{\text{mode}} = \frac{\partial \ln g(x_{\text{mode}}, q, w, \alpha)}{\partial x} = 0
\]
\[
\frac{\partial \ln g(x_{\text{mode}}, q, w, \alpha)}{\partial x} = \frac{\partial}{\partial x} \left[ \ln \frac{\alpha}{2q^2w} \frac{x^{\alpha-1} e^{-\frac{1}{2q^2w} x^\alpha}}{1 - e^{-\frac{1}{2q^2w} x^\alpha}} \right] = 0 \Rightarrow \\
\frac{\partial}{\partial x} [\ln \alpha - \ln 2q^2 - \ln w + (\alpha - 1) \ln x - (\alpha - 1) \ln w - \frac{1}{2q^2w} x^\alpha - \ln(1 - e^{-\frac{1}{2q^2w} x^\alpha})] = 0 \Rightarrow \\
(\alpha - 1) - \frac{\alpha}{2q^2w} (x_{\text{mode}})^{\alpha-1} = 0 \Rightarrow (\alpha - 1) = \frac{\alpha}{2q^2w} (x_{\text{mode}})^{\alpha-1} \\
x_{\text{mode}} = w \left( \frac{\alpha - 1}{\alpha} \right)^{\frac{1}{\alpha}} \\
(16)
\]

**Proposition(2)**

4.2 Median and Quintile:

The quintile \(x_q\) and median of the TRPD is

\[
x_q = w \left[ -2q^2 \ln \left(1 - Q \left[1 - e^{-\frac{1}{2q^2w} x^\alpha} \right] \right) \right]^{\frac{1}{\alpha}} \\
(17)
\]

\[
x_{\text{median}} = w \left[ -2q^2 \ln \left( \frac{1}{2} \left[1 + e^{-\frac{1}{2q^2w} x^\alpha} \right] \right) \right]^{\frac{1}{\alpha}} \\
(18)
\]

**Proof:**

\[
G(x_q; q, w, \alpha) = Q \left( x_q \leq Q \right) = Q \\
(19)
\]

From equation (2) we obtain.

\[
1 - e^{-\frac{1}{2q^2w} x_q^\alpha} = Q \Rightarrow 1 - e^{-\frac{1}{2q^2w} x_q^\alpha} = Q \left[1 - e^{-\frac{1}{2q^2w} x_q^\alpha} \right] \Rightarrow \\
e^{-\frac{1}{2q^2w} x_q^\alpha} = 1 - Q \left[1 - e^{-\frac{1}{2q^2w} x_q^\alpha} \right] \text{ Take Ln to the two parties}
\]

\[
-\frac{1}{2q^2w} x_q^\alpha = \ln \left(1 - Q \left[1 - e^{-\frac{1}{2q^2w} x_q^\alpha} \right] \right)
\]

Then

\[
x_q = w \left[ -2q^2 \ln \left(1 - Q \left[1 - e^{-\frac{1}{2q^2w} x_q^\alpha} \right] \right) \right]^{\frac{1}{\alpha}} \\
(20)
\]

To find the median of (TRPD) we set \(Q = \frac{1}{2}\) in equation (20).
\[ x_{\text{median}} = w[-2q^2 \ln \left( 1 - \frac{1}{2} \left[ 1 - e^{-\frac{1}{2q^2 w^\alpha}} \right] \right) \Rightarrow \]

\[ x_{\text{median}} = w[-2q^2 \ln \frac{1}{2} \left[ 1 + e^{-\frac{1}{2q^2 w^\alpha}} \right] \frac{1}{\alpha}] \]

4.3 Moment Generating Function:

Proposition (3)

The moment generating function of TRPD as follows:

\[ M_x(t) = \sum_{j=0}^{\infty} \sum_{s=0}^{\infty} \frac{t^s \alpha (-1)^j (x_e)^{s+j+1} \alpha}{j! s! (2q^2)^{j+1}w^{(j+1)}} \left[ 1 - e^{-\frac{1}{2q^2 w^\alpha}} \right] \frac{x_e}{s + (j + 1)\alpha} \]

(22)

Proof:

\[ M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} \frac{\alpha}{2q^2} \frac{x_e^{\alpha-1} e^{-\frac{1}{2q^2 w^\alpha}}}{1 - e^{-\frac{1}{2q^2 w^\alpha}}} \]

By using series expansion of \( e^{tx} \) then.

\[ e^{tx} = \sum_{s=0}^{\infty} \frac{t^s x^s}{s!} \]

So

\[ = \frac{\alpha}{2q^2 w^\alpha} \left[ 1 - e^{-\frac{1}{2q^2 w^\alpha}} \right] \sum_{s=0}^{\infty} \frac{t^s x_e}{s!} \frac{x_e^{s+\alpha-1} e^{-\frac{1}{2q^2 w^\alpha}}}{s + (j + 1)\alpha} \]

By using series expansion of \( e^{-\frac{1}{2q^2 w^\alpha}} \) then.

\[ e^{-\frac{1}{2q^2 w^\alpha}} = \sum_{j=0}^{\infty} \left( -\frac{1}{2q^2 w^\alpha} \right)^j \left( \frac{x_e}{w} \right)^j \]

Then

\[ M_x(t) = \sum_{j=0}^{\infty} \sum_{s=0}^{\infty} \frac{t^s \alpha (-1)^j (x_e)^{s+j+1} \alpha}{j! s! (2q^2)^{j+1}w^{(j+1)}} \left[ 1 - e^{-\frac{1}{2q^2 w^\alpha}} \right] \frac{x_e}{s + (j + 1)\alpha} \]

(23)
4.4. Mean and variance:

Proposition (4)

The mean and variance of TRPD

\[
E(X) = \sum_{j=0}^{\infty} \frac{\alpha(-1)^j (x_j^{(j+1)\alpha+1})}{j! (2q^2)^{j+1}(w^\alpha)^j} \left[ 1 - e^{-\frac{1}{2q^2}} x_j^{\alpha} \right] (j + 1)\alpha + 1
\]

(24)

\[
Var = E(X^2) - [E(X)]^2 = \sum_{j=0}^{\infty} \frac{\alpha(-1)^j (x_j^{(j+1)\alpha+2})}{j! (2q^2)^{j+1}(w^\alpha)^j} \left[ 1 - e^{-\frac{1}{2q^2}} x_j^{\alpha} \right] (j + 1)\alpha + 1
\]

- \sum_{j=0}^{\infty} \frac{\alpha(-1)^j (x_j^{(j+1)\alpha+1})}{j! (2q^2)^{j+1}(w^\alpha)^j} \left[ 1 - e^{-\frac{1}{2q^2}} x_j^{\alpha} \right] (j + 1)\alpha + 1

(25)

Proof:

\[
E(X) = \int_0^x \frac{x}{2q^2 w^\alpha} e^{-\frac{1}{2q^2} x^{\alpha}} \ dx = \frac{\alpha}{2q^2 w^\alpha \left[ 1 - e^{-\frac{1}{2q^2} x^{\alpha}} \right]} \int_0^x x^{\alpha} e^{-\frac{1}{2q^2} x^{\alpha}} \ dx
\]

Then

\[
e^{-\frac{1}{2q^2} x^{\alpha}} = \sum_{j=0}^{\infty} \frac{(-1)^j \left( x^{(j+1)\alpha} \right)}{j!}
\]

Then

\[
E(X) = \frac{\alpha}{2q^2 w^\alpha \left[ 1 - e^{-\frac{1}{2q^2} x^{\alpha}} \right]} \sum_{j=0}^{\infty} \frac{(-1)^j}{j! w^{j\alpha} (2q^2)^j} \int_0^x x^{(j+1)\alpha} \ dx
\]

We can find \( \int_0^x x^{(j+1)\alpha} \ dx \) as the follows.

\[
\int_0^x x^{(j+1)\alpha} \ dx = \left[ \frac{x^{(j+1)\alpha+1}}{(j+1)\alpha + 1} \right]_0^x = \frac{x^{(j+1)\alpha+1}}{(j+1)\alpha + 1}
\]

Then

\[
E(X) = \sum_{j=0}^{\infty} \frac{\alpha(-1)^j x_j^{(j+1)\alpha+1}}{j! (2q^2)^{j+1} w^\alpha (j + 1)\alpha + 1}
\]

(26)
The variance of TRPD as follows.

\[
E(X^2) = \int_{0}^{x} x^2 \frac{\alpha}{2q^2w} \left( \frac{x}{w} \right)^{a-1} e^{-\frac{1}{2q^2w}x^a} \partial x = \sum_{j=0}^{\infty} \frac{\alpha(-1)^j x_j^{(j+1)a+2}}{j! \left( 2q^2 \right)^{j+1} w^{(j+1)a} \left[ 1 - e^{-\frac{1}{2q^2w}x^a} \right] (j+1)a + 2}
\]

Then

\[
Var = \left[ \sum_{j=0}^{\infty} \frac{\alpha(-1)^j \left( x_j \right)^{(j+1)a+1}}{j! \left( 2q^2 \right)^{j+1} w^{(j+1)a} \left[ 1 - e^{-\frac{1}{2q^2w}x^a} \right] (j+1)a + 1} \right]^2
\]

(27)

4.5 Moments

Proposition (5)

The \( r \)-th moment about the origin and mean of the TRPD.

\[
E(X^r) = \sum_{j=0}^{\infty} \frac{\alpha(-1)^j \left( x_j \right)^{(j+1)a+r}}{j! \left( 2q^2 \right)^{j+1} w^{(j+1)a} \left[ 1 - e^{-\frac{1}{2q^2w}x^a} \right] (j+1)a + r}
\]

(28)

\[
r = 1, 2, 3
\]

\[
E(X - \mu)^r = \sum_{l=0}^{\infty} C_l^r (\mu)^l \sum_{j=0}^{\infty} \frac{\alpha(-1)^j \left( x_j \right)^{(j+1)a+1}}{j! \left( 2q^2 \right)^{j+1} w^{(j+1)a} \left[ 1 - e^{-\frac{1}{2q^2w}x^a} \right] (j+1)a + 1}
\]

(29)

Proof: The \( r \)-th about the origin is:

\[
E(X^r) = \int_{0}^{x} x^r g(x; q, w, \alpha) \partial x = \int_{0}^{x} x^r \frac{\alpha}{2q^2w} \left( \frac{x}{w} \right)^{a-1} e^{-\frac{1}{2q^2w}x^a} \partial x
\]

\[
= \frac{\alpha}{2q^2w^a \left[ 1 - e^{-\frac{1}{2q^2w}x^a} \right]} \int_{0}^{x} x^{a+r-1} e^{-\frac{1}{2q^2w}x^a} \partial x
\]

\[
= \frac{\alpha}{2q^2w^a \left[ 1 - e^{-\frac{1}{2q^2w}x^a} \right]} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \left( \frac{x}{w} \right)^{a+j} \int_{0}^{x} x^{j+1} \alpha+r-1 \partial x
\]
Then
\[
E(X') = \sum_{j=0}^{\infty} \frac{\alpha(-1)^j (x_c)^{(j+1)\alpha + r}}{j! (2q^2)^{j+1} w^{(j+1)\alpha} \left[ 1 - e^{-\frac{1}{2q^2 w^{\frac{1}{\alpha}}}} \right] [(j + 1)\alpha + r]}
\]

\[E(X - \mu)^r = \int_0^x (X - \mu)^r \frac{\alpha}{2q^2 w} \frac{x^{\alpha-1} e^{-\frac{x}{2q^2 w^{\frac{1}{\alpha}}}}}{1 - e^{-\frac{1}{2q^2 w^{\frac{1}{\alpha}}}}} \, dx
\]

We have
\[
(X - \mu)^r = \sum_{i=0}^{r} C_i^r (x)^{r-i} (-\mu)^i
\]

Thus the \(r\)th moment about mean is given by:
\[
E(X - \mu)^r = \sum_{i=0}^{r} \sum_{j=0}^{\infty} C_i^r (-\mu)^i \frac{\alpha(-1)^j (x_c)^{(j+1)\alpha + r-i}}{j! (2q^2)^{j+1} w^{(j+1)\alpha} \left[ 1 - e^{-\frac{1}{2q^2 w^{\frac{1}{\alpha}}}} \right] [(j + 1)\alpha + r - i]}
\]

5. Order Statistics

Suppose that \(x_1, x_2, \ldots, x_n\) denoted a random sample of size \(n\) from a TRPD with \(G(x; q, w, \alpha)\) and \(g(x; q, w, \alpha)\) in the equation (2) and (3). Let \(Y_{k,1}, Y_{k,2}, \ldots, Y_{k,n}\) express the corresponding order statistics; then.

The p.d.f of \(X_{k,n}\) is given by:
\[
g_{k,n}(x, q, w, \alpha) = \frac{n!}{(k-1)!(n-k)!} g(x; q, w, \alpha) G(x; q, w, \alpha)^{k-1} \left[ 1 - g(x; q, w, \alpha) \right]^{n-k}
\]

From the p.d.f of TRPD we have the p.d.f of the \(r\)-th order statistics
\[
g_{k,n}(x, q, w, \alpha) = \frac{n!}{(k-1)!(n-k)!} \left[ \frac{a}{2q^2 w} \frac{x^{\alpha-1} e^{-\frac{1}{2q^2 w^{\frac{1}{\alpha}}}}}{1 - e^{-\frac{1}{2q^2 w^{\frac{1}{\alpha}}}}} \right] \left[ 1 - e^{-\frac{1}{2q^2 w^{\frac{1}{\alpha}}}} \right]^{k-1} \left[ 1 - e^{-\frac{1}{2q^2 w^{\frac{1}{\alpha}}}} \right]^{n-k}
\]

\[
= \frac{n!}{(k-1)!(n-k)!} \left[ \frac{a}{2q^2 w} \frac{x^{\alpha-1} e^{-\frac{1}{2q^2 w^{\frac{1}{\alpha}}}}}{1 - e^{-\frac{1}{2q^2 w^{\frac{1}{\alpha}}}}} \right] \left[ 1 - e^{-\frac{1}{2q^2 w^{\frac{1}{\alpha}}}} \right]^{k-1} \left[ 1 - e^{-\frac{1}{2q^2 w^{\frac{1}{\alpha}}}} \right]^{n-k}
\]

Then the p.d.f of the maximum, minimum and the median are explain as follows.
When $k=1$, we have the p.d.f of minimum

$$g_{1,n}(x, q, w, \alpha) = \frac{n \left[ e^{-\frac{1}{2q^2 w^\alpha}} - 1 \right]}{1 - e^{-\frac{1}{2q^2 w^\alpha}}} \left[ e^{-\frac{1}{2q^2 w^\alpha}} - e^{-\frac{1}{2q^2 w^\alpha}} \right]^{n-1}$$

When $k = n$, we have the p.d.f of maximum

$$g_{n,n}(x, q, w, \alpha) = \frac{n \left[ e^{-\frac{1}{2q^2 w^\alpha}} - 1 \right]}{1 - e^{-\frac{1}{2q^2 w^\alpha}}} \left[ e^{-\frac{1}{2q^2 w^\alpha}} - e^{-\frac{1}{2q^2 w^\alpha}} \right]^{n-1}$$

3. When $k= m+1$, we have the p.d.f of the median

$$g_{m+1,n}(x, q, w, \alpha) = \frac{m!}{m!(n-m-1)!} \left[ \frac{e^{-\frac{1}{2q^2 w^\alpha}} - 1}{1 - e^{-\frac{1}{2q^2 w^\alpha}}} \right]^m \left[ e^{-\frac{1}{2q^2 w^\alpha}} - 1 \right]^{n-m-1}$$

6. The Coefficients of Variation, Kurtosis and Skewedness.

Proposition (6)

The coefficients of variation, skewness and kurtosis of the TRPD are respectively as:

$$CS = \frac{E(X - \mu)^3}{\sigma^3}$$

Let $CS = \frac{A}{B}$

By equation (28) we put $r=3$ then

$$A = E(X - \mu)^3 = \sum_{i=0}^{3} \sum_{j=0}^{\infty} C_1^i (-\mu)^j \frac{\alpha(-1)^j (x)^{(j+1)\alpha+3-i}}{j! (2q^2)^{j+1} w^{\alpha(j+1)} [1 - e^{-\frac{1}{2q^2 w^\alpha}}]} (j+1)\alpha + 3 - i$$

$$B = \sigma^3 = [E(X - \mu)^2]^\frac{3}{2} = \sum_{i=0}^{2} \sum_{j=0}^{\infty} C_1^i (-\mu)^j \frac{\alpha(-1)^j (x)^{(j+1)\alpha+2-i}}{j! (2q^2)^{j+1} w^{\alpha(j+1)} [1 - e^{-\frac{1}{2q^2 w^\alpha}}]} (j+1)\alpha + 2 - i$$

$$CK = \frac{E(X - \mu)^4}{\sigma^4}$$

Let $CK = \frac{C}{D}$
Where
\[ C = E(X - \mu)^4 \]

By equation (28) we put r=4 then
\[
C = \sum_{l=0}^{4} \sum_{j=0}^{\infty} C_l^3 (-\mu)_l \frac{\alpha(-1)^j (x_j)^{(j+1)\alpha+4-l}}{j! (2q^2)^{j+1}(w^\alpha(j+1)) \left[1 - e^{-\frac{1}{2q^2}(\frac{x_j}{w})^\alpha}\right] (j + 1)\alpha + 4 - i}
\]

\[ D = \sigma^4 = [E(X - \mu)^2]^2 \]

Then
\[
D = \sum_{l=0}^{2} \sum_{j=0}^{\infty} C_l^2 (-\mu)_l \frac{\alpha(-1)^j (x_j)^{(j+1)\alpha+2-l}}{j! (2q^2)^{j+1}(w^\alpha(j+1)) \left[1 - e^{-\frac{1}{2q^2}(\frac{x_j}{w})^\alpha}\right] (j + 1)\alpha + 2 - i} \]

\[ CV = \frac{\sigma}{\mu} = \frac{E}{F} \]

Let \( \sigma = E \)

\[
E = \sum_{l=0}^{2} \sum_{j=0}^{\infty} C_l^2 (-\mu)_l \frac{(-1)^j (x_j)^{(j+1)\alpha+2-l}}{j! (2q^2)^{j+1}(w^\alpha(j+1)) \left[1 - e^{-\frac{1}{2q^2}(\frac{x_j}{w})^\alpha}\right] (j + 1)\alpha + 2 - i}
\]

\[ F = E(X) = \sum_{j=0}^{\infty} \frac{\alpha(-1)^j x_j^{(j+1)\alpha+1}}{j! (2q^2)^{j+1}w^\alpha(j+1) \left[1 - e^{-\frac{1}{2q^2}(\frac{x_j}{w})^\alpha}\right] (j + 1)\alpha + 1} \]

Proposition (7)

The harmonic mean is given by:
\[
H = E \left( \frac{1}{X} \right) = \sum_{j=0}^{\infty} \frac{\alpha(-1)^j x_j^{(j+1)\alpha-1}}{j! (2q^2)^{j+1}w^\alpha(j+1) \left[1 - e^{-\frac{1}{2q^2}(\frac{x_j}{w})^\alpha}\right] (j + 1)\alpha - 1}
\]

Proof:

\[ E \left( \frac{1}{X} \right) = \int_0^X \frac{1}{X} g(x; q, w, \alpha) dx \]

\[
= \int_0^x \frac{\alpha}{2q^2w^\alpha \left[1 - e^{-\frac{1}{2q^2}(\frac{x}{w})^\alpha}\right]} \int_0^x X^{\alpha-2} e^{-\frac{1}{2q^2}(\frac{x}{w})^\alpha} dx
\]

\[
= \frac{\alpha}{2q^2w^\alpha \left[1 - e^{-\frac{1}{2q^2}(\frac{x}{w})^\alpha}\right]} \int_0^x X^{\alpha-2} e^{-\frac{1}{2q^2}(\frac{x}{w})^\alpha} dx
\]

\[ E \left( \frac{1}{X} \right) = \frac{\alpha}{2q^2w^\alpha \left[1 - e^{-\frac{1}{2q^2}(\frac{x}{w})^\alpha}\right]} \int_0^x X^{\alpha-2} e^{-\frac{1}{2q^2}(\frac{x}{w})^\alpha} dx \]
Thus

\[
E \left( \frac{1}{X} \right) = \frac{\alpha}{2q^2w^\alpha[1 - e^{-\frac{x}{2q^2w^\alpha}}]} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \left( \frac{x}{w} \right)^j \int_0^x x^{(j+1)\alpha - 2} \, dx
\]

We can find \( \int_0^x x^{(j+1)\alpha - 2} \, dx \) as follows

\[
\int_0^x x^{(j+1)\alpha - 2} \, dx = \frac{x^{(j+1)\alpha - 1}}{\alpha(j + 1) - 1}
\]

Thus

\[
E \left( \frac{1}{X} \right) = \sum_{j=0}^{\infty} \frac{\alpha(-1)^j}{j! (2q^2)^{j+1} w^{(j+1)\alpha}} \left[ 1 - e^{-\frac{x}{2q^2w^\alpha}} \right] (j + 1)\alpha - 1
\]

(40)

**Proposition (8)**

The geometric mean is by the form:

\[
G = E(\sqrt{X}) = \sum_{j=0}^{\infty} \frac{\alpha(-1)^j}{j! (2q^2)^{j+1} w^{(j+1)\alpha}} \left[ 1 - e^{-\frac{x}{2q^2w^\alpha}} \right] (j + 1)\alpha + \frac{1}{2}
\]

(41)

**Proof:**

\[
G = E(\sqrt{X}) = \int_0^x \sqrt{X} g(x; q, w, \alpha) \, dx
\]

\[
= \int_0^x \frac{\alpha}{\sqrt{X} 2q^2w^{\alpha} \left[ 1 - e^{-\frac{x}{2q^2w^\alpha}} \right]} \left[ 1 - e^{-\frac{x}{2q^2w^\alpha}} \right] \, dx
\]

Thus

\[
E(\sqrt{X}) = \frac{\alpha}{2q^2w^\alpha[1 - e^{-\frac{x}{2q^2w^\alpha}}]} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \left( \frac{x}{w} \right)^j \int_0^x x^{(j+1)\alpha - 1/2} \, dx
\]

Then

\[
G = E(\sqrt{X}) = \sum_{j=0}^{\infty} \frac{\alpha(-1)^j}{j! (2q^2)^{j+1} w^{(j+1)\alpha}} \left[ 1 - e^{-\frac{x}{2q^2w^\alpha}} \right] (j + 1)\alpha + \frac{1}{2}
\]

(42)
7. Estimation Methods

We shall discuss some methods to estimate the unknown parameters of TRPD.

7.1. Maximum Likelihood Estimation.

Estimating the parameters for TRPD using the maximum likelihood estimation method for p.d.f are:

\[
L = \prod_{i=1}^{n} g(X_i, \theta) = \frac{\alpha^n}{(2q^2)^nw^n} \left( \prod_{i=1}^{n} \frac{x_i}{w} \right)^{\alpha-1} e^{-\frac{1}{2q^2}\sum_{i=1}^{n} \left( \frac{x_i}{w} \right)^2} = 1 - e^{-\frac{1}{2q^2}\frac{x_i}{w}^2}
\]

We take the log of both sides to the likelihood function.

\[
\ell = \ln L = \ln \left( \frac{\alpha^n}{(2q^2)^nw^n} \left( \prod_{i=1}^{n} \frac{x_i}{w} \right)^{\alpha-1} e^{-\frac{1}{2q^2}\sum_{i=1}^{n} \left( \frac{x_i}{w} \right)^2} \right) = n\ln \alpha - n\ln(2q^2) - n\ln w + (\alpha - 1) \sum_{i=1}^{n} \ln x_i - n(\alpha - 1)\ln w - \frac{1}{2q^2} \sum_{i=1}^{n} \frac{x_i^2}{w^\alpha} - \ln \left( 1 - e^{-\frac{1}{2q^2}\frac{x_i}{w}^2} \right)
\]

\[
\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln x_i - n\ln w - \frac{1}{2q^2} \sum_{i=1}^{n} \left( \frac{x_i}{w} \right)^2 \ln \left( \frac{x_i}{w} \right) - \frac{1}{2q^2} \sum_{i=1}^{n} \frac{x_i^2}{w^\alpha} e^{-\frac{1}{2q^2}\frac{x_i}{w}^2} \ln \left( \frac{x_i}{w} \right) e^{-\frac{1}{2q^2}\frac{x_i}{w}^2} \frac{1}{1 - e^{-\frac{1}{2q^2}\frac{x_i}{w}^2}}
\]

\[
\frac{\partial \ell}{\partial q} = -\frac{2n}{q} + \frac{1}{q^3} \sum_{i=1}^{n} \frac{x_i^2}{w^\alpha} - \frac{1}{q^3} \left( \frac{x_i}{w} \right)^2 e^{-\frac{1}{2q^2}\frac{x_i}{w}^2} \frac{1}{1 - e^{-\frac{1}{2q^2}\frac{x_i}{w}^2}}
\]

\[
\frac{\partial \ell}{\partial w} = -\frac{n}{w} - \frac{n(\alpha - 1)}{w} + \frac{\alpha}{2q^2} \frac{\sum_{i=1}^{n} x_i^2}{w^{\alpha+1}} - \frac{\alpha}{2q^2} \frac{(\sum_{i=1}^{n} x_i)}{w^{\alpha+1}} e^{-\frac{1}{2q^2}\frac{(\sum_{i=1}^{n} x_i)}{w}}
\]

Now equaling each these equations to zero.

\[
\frac{n}{\alpha} + \sum_{i=1}^{n} \ln x_i - n\ln w - \frac{1}{2q^2} \sum_{i=1}^{n} \left( \frac{x_i}{w} \right)^2 \ln \left( \frac{x_i}{w} \right) - \frac{1}{2q^2} \frac{\sum_{i=1}^{n} x_i^2}{w^\alpha} e^{-\frac{1}{2q^2}\frac{(\sum_{i=1}^{n} x_i)}{w}} \frac{1}{1 - e^{-\frac{1}{2q^2}\frac{x_i}{w}^2}} = 0
\]
\[-2n \frac{q}{\hat{q}} + \frac{1}{\hat{q}^2} \sum_{i=1}^{n} \frac{x_i^{\hat{q}} (x_i^{\hat{q}} - 1)}{\hat{q}^2 \hat{w}^{\hat{q}}} = 0 \quad (48)\]

\[-n \frac{\hat{w}}{\hat{w}} - \frac{n(\hat{a} - 1)}{\hat{w}} + \frac{\hat{a}}{2q^2} \sum_{i=1}^{n} x_i^{\hat{q}} - \frac{\hat{a}}{2q^2 \hat{w}^{\hat{q} + 1}} \left[ 1 - e^{-\frac{1}{2q^2 \hat{x}^{\hat{q}}}} \right] = 0 \quad (49)\]

The solution of these equation are that the MLEs that can be found of the parameters p,b,α by numerical method.

7.2. The Least Square Method(LS):[13]

The method defined is. Let Y₁, Y₂, ..., Yₙ is a random sample of size n from a distribution function G(.) and Y(1), Y(2), ..., Y(n) indicates the order statistics of the observed sample. It is well know that

\[E(G(Y(i))) = \frac{i}{n+1}\]

Obtain the estimators by minimizing

\[\sum_{i=1}^{n} (G(Y(i)) - \frac{i}{n+1})^2\]

So

\[\sum_{i=1}^{n} \left( \frac{1 - e^{-\frac{1}{2q^2 \hat{x}^{\hat{q}}}}} {1 - e^{-\frac{1}{2q^2 \hat{x}^{\hat{q}}}}} \right) - \frac{i}{n+1} = 0 \quad (50)\]

The \(\hat{q}_{\text{LS}}\) can be found by derive the equation (50) with respect to q, it is given.

\[\sum_{i=1}^{n} 2 \left( \frac{1 - e^{-\frac{1}{2q^2 \hat{x}^{\hat{q}}}}}{1 - e^{-\frac{1}{2q^2 \hat{x}^{\hat{q}}}}} - \frac{i}{n+1} \right) \cdot \frac{(-1)^2 e^{-\frac{1}{2q^2 \hat{x}^{\hat{q}}}} + (\frac{1}{q^2 \hat{x}^{\hat{q}}})^2}{(1 - e^{-\frac{1}{2q^2 \hat{x}^{\hat{q}}}})^2} \times \left(1 - e^{-\frac{1}{2q^2 \hat{x}^{\hat{q}}}}\right)^2 = 0\]

Then

\[\sum_{i=1}^{n} \left( \frac{1 - e^{-\frac{1}{2q^2 \hat{x}^{\hat{q}}}}}{1 - e^{-\frac{1}{2q^2 \hat{x}^{\hat{q}}}}} - \frac{i}{n+1} \right) \times \frac{(-1)^2 e^{-\frac{1}{2q^2 \hat{x}^{\hat{q}}}} + (\frac{1}{q^2 \hat{x}^{\hat{q}}})^2}{(1 - e^{-\frac{1}{2q^2 \hat{x}^{\hat{q}}}})^2} \times \left(1 - e^{-\frac{1}{2q^2 \hat{x}^{\hat{q}}}}\right)^2 = 0 \quad (51)\]
The \( \hat{\alpha}_{LES} \) can be found by derive the equation (50) with respect to \( \alpha \), it is given.

\[
\sum_{l=1}^{n} 2 \left[ \frac{1}{1 - e^{\frac{-1}{2q^2w^\alpha}}^{x_{(l)}}} \right] \times \\
\left[ \left( 1 - e^{\frac{-1}{2q^2w^\alpha}}^{x_{(l)}} \right) \left( \frac{1}{2q^2w^\alpha e} \right) \right] = 0
\]

Then

\[
\sum_{l=1}^{n} 2 \left[ \frac{1}{1 - e^{\frac{-1}{2q^2w^\alpha}}^{x_{(l)}}} \right] \times \\
\left[ \left( 1 - e^{\frac{-1}{2q^2w^\alpha}}^{x_{(l)}} \right) \left( \frac{1}{2q^2w^\alpha e} \right) \right] = 0 \tag{52}
\]

The \( \hat{\alpha}_{LES} \) can be found by derive the equation (50) with respect to \( \alpha \), it is given.

\[
\sum_{l=1}^{n} 2 \left[ \frac{1}{1 - e^{\frac{-1}{2q^2w^\alpha}}^{x_{(l)}}} \right] \times \\
\left[ \left( 1 - e^{\frac{-1}{2q^2w^\alpha}}^{x_{(l)}} \right) \left( \frac{1}{2q^2w^\alpha e} \right) \right] = 0
\]

Then

\[
\sum_{l=1}^{n} 2 \left[ \frac{1}{1 - e^{\frac{-1}{2q^2w^\alpha}}^{x_{(l)}}} \right] \times \\
\left[ \left( 1 - e^{\frac{-1}{2q^2w^\alpha}}^{x_{(l)}} \right) \left( \frac{1}{2q^2w^\alpha e} \right) \right] = 0 \tag{53}
\]

The solution to these equations from (51) – (53) is by numerical method.

7.3. Percentile Estimation Method (PEM):

This method was used effectively for generalized exponential distribution and Weibull distribution. Where it was first discovered by Kao (1958, 1959). [13]

Since \( G(x; q, w, \alpha) = \frac{1 - e^{-\frac{x^{\alpha}}{2q^2w}}}{1 - e^{-\frac{x^{\alpha}}{2q^2w}}} \), then for

\[
\ln[G(x; q, w, \alpha)] = \ln \left[ \frac{1 - e^{-\frac{x^{\alpha}}{2q^2w}}}{1 - e^{-\frac{x^{\alpha}}{2q^2w}}} \right] \tag{54}
\]

Suppose \( X_{(j)} \) is the \( j \)-th order statistic, i.e \( X_{(1)} < X_{(2)} < \cdots < X_{(n)} \). If \( P \) indicates some estimate of \( G(x_{(j)}; q, w, \alpha) \), then the estimate of \( p, b \) and \( \alpha \) can be obtained by minimizing .
\[
\sum_{j=0}^{n} \left[ \ln(P_j) - \ln \left( \frac{1 - e^{-\frac{1}{2q^2\omega^a}}} {1 - e^{-\frac{1}{2q^2\omega^a}}} \right) \right]^2
\]

We have

\[
\sum_{j=0}^{n} \left[ \ln(P_j) - \ln \left( 1 - e^{-\frac{1}{2q^2\omega^a}} \right) + \left( 1 - e^{-\frac{1}{2q^2\omega^a}} \right) \right]^2
\]

(55)

We mainly consider \( P_j = \frac{j}{n+1} \)

The \( \tilde{q}_{PCE} \) can be found by deriving the equation (55) with respect to \( q \), it is given.

\[
\sum_{j=1}^{n} \left[ 2 \ln \left( P_j \right) - \ln \left( 1 - e^{-\frac{1}{2q^2\omega^a}} \right) + \ln \left( 1 - e^{-\frac{1}{2q^2\omega^a}} \right) \right] \left[ \frac{\frac{1}{q^3} \left( \frac{X^a}{\omega} \right) e^{-\frac{1}{2q^2\omega^a}}} {1 - e^{-\frac{1}{2q^2\omega^a}}} \right] = 0
\]

Then

\[
\sum_{j=1}^{n} \left[ \ln \left( P_j \right) - \ln \left( 1 - e^{-\frac{1}{2q^2\omega^a}} \right) + \ln \left( 1 - e^{-\frac{1}{2q^2\omega^a}} \right) \right] \left[ \frac{\frac{1}{q^3} \left( \frac{X^a}{\omega} \right) e^{-\frac{1}{2q^2\omega^a}}} {1 - e^{-\frac{1}{2q^2\omega^a}}} \right] = 0
\]

(56)

The \( \tilde{w}_{PCE} \) can be found by deriving the equation (55) with respect to \( w \), it is given

\[
\sum_{j=0}^{n} \left[ 2 \ln(P_j) - \ln \left( 1 - e^{-\frac{1}{2q^2\omega^a}} \right) + \ln \left( 1 - e^{-\frac{1}{2q^2\omega^a}} \right) \right] \left[ \frac{\frac{\alpha}{2q^2} \left( \frac{X^a}{\omega^{a+1}} \right) e^{-\frac{1}{2q^2\omega^a}}} {1 - e^{-\frac{1}{2q^2\omega^a}}} \right] = 0
\]
Then
\[
\sum_{j=0}^{n} [\ln(P_j) - \ln(1 - e^{-\frac{1}{2q^2}x_j^\alpha}) + \ln(1 - e^{-\frac{1}{2q^2}x_j^\alpha})] [\left(\frac{\alpha}{2q^2} x_j^\alpha e^{-\frac{1}{2q^2}x_j^\alpha}}{1 - e^{-\frac{1}{2q^2}x_j^\alpha}}\right) + \left(\frac{\alpha}{2q^2} x_j^\alpha e^{-\frac{1}{2q^2}x_j^\alpha}}{1 - e^{-\frac{1}{2q^2}x_j^\alpha}}\right) = 0
\]

(57)

The \( \hat{a}_{PCE} \) can be found by derive the equation (55) with respect to \( b \), it is given.
\[
\sum_{j=0}^{n} 2[\ln(P_j) - \ln(1 - e^{-\frac{1}{2q^2}x_j^\hat{a}})] [\left(\frac{\alpha}{2q^2} \frac{x_j^\hat{a}}{\hat{w}^{\hat{a}+1}} e^{-\frac{1}{2q^2}x_j^\hat{a}}}{1 - e^{-\frac{1}{2q^2}x_j^\hat{a}}}\right) + \left(\frac{\alpha}{2q^2} \frac{x_j^\hat{a}}{\hat{w}^{\hat{a}+1}} e^{-\frac{1}{2q^2}x_j^\hat{a}}}{1 - e^{-\frac{1}{2q^2}x_j^\hat{a}}}\right) = 0
\]

Then
\[
\sum_{j=0}^{n} [\ln(P_j) - \ln(1 - e^{-\frac{1}{2q^2}x_j^\hat{a}})] [\left(\frac{-1}{2q^2} \frac{x_j^\hat{a}}{\hat{w}} \ln(\frac{x_j^\hat{a}}{\hat{w}}) e^{-\frac{1}{2q^2}x_j^\hat{a}}}{1 - e^{-\frac{1}{2q^2}x_j^\hat{a}}}\right) + \left(\frac{-1}{2q^2} \frac{x_j^\hat{a}}{\hat{w}} \ln(\frac{x_j^\hat{a}}{\hat{w}}) e^{-\frac{1}{2q^2}x_j^\hat{a}}}{1 - e^{-\frac{1}{2q^2}x_j^\hat{a}}}\right) = 0
\]

(58)

The solution to these equations from (56) – (58) is by numerical method.

7.4. Moments Method (W):[14]

Let \( X_1, X_2, ..., X_3 \) be a random sample from \( X \sim TRPD(x; q, w, \alpha) \) This method can be found through .

\[
E(X^r) = \frac{1}{n} \sum_{i=1}^{n} X_i^r
\]

(59)

Where \( E(X^r) \) is the r-th moment about origin .

If \( r = 1 \) then the equation (59) becomes as follows
\[ E(X) = \sum_{j=0}^{\infty} \frac{\alpha (-1)^j x_{(j+1)\alpha+1}}{j! (2q^2)^{j+1} w^{(j+1)} [1 - e^{-\frac{1}{2q^2 \bar{x}^2 \alpha}}] (j + 1) \alpha + 1} = \sum_{i=1}^{n} \frac{1}{n} X_i = \bar{X} \]  

(60)

If \( r = 2 \) then the equation (59) becomes as follows

\[ E(X^2) = \sum_{j=0}^{\infty} \frac{\alpha (-1)^j x_{(j+1)\alpha+2}}{j! (2q^2)^{j+1} w^{(j+1)} [1 - e^{-\frac{1}{2q^2 \bar{x}^2 \alpha}}] (j + 1) \alpha + 2} = \sum_{i=1}^{n} \frac{1}{n} X_i^2 \]  

(61)

If \( r = 3 \) then the equation (59) becomes as follows

\[ E(X^3) = \sum_{j=0}^{\infty} \frac{\alpha (-1)^j x_{(j+1)\alpha+3}}{j! (2q^2)^{j+1} w^{(j+1)} [1 - e^{-\frac{1}{2q^2 \bar{x}^2 \alpha}}] (j + 1) \alpha + 3} = \sum_{i=1}^{n} \frac{1}{n} X_i^3 \]  

(62)

To find the estimates of the parameters \( p, b, \alpha \) we solve these equation from (60) – (62) by a numerical path such as Newton Raphson Method.

**Application**

These are dataset agree to remissful (in months) of a random sample of (128) for sick bladder cancer [9].

\[ \{0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 1.20, 2.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69\]  

We have installed the truncated Rayleigh Pareto distribution on the dataset using (MLE), and compare the proposal with the Rayleigh Lomax distribution (ROD) and Lomax distribution. The model selection is carried out using the AIC, the BIC, The CAIC, and the HQIC.

\[ \text{AIC} = -2 \hat{\ell} + 2q, \]  

(63)

\[ \text{BIC} = -2 \hat{\ell} + qlog(n), \]  

(64)

\[ \text{HQIC} = -2 \hat{\ell} + 2qlog(log(n)), \]  

(65)

\[ \text{CAIC} = -2 \hat{\ell} + \frac{2q}{n-q-1} \]  

(67)

Whereas \( \hat{\ell} \) represent the log-likelihood function evaluate at the maximum likelihood estimates, \( n \) the sample size, and \( q \) the number of parameters.

The MLEs of the pattern parameters for the data are given in Table (1), and the numerical values of the model selection statistics \( \hat{\ell} \), AIC,BIC,CAIC and HQIC are listed in Table (2).
From Table (2) we see that the (TRPD) model gives the smallest values for the criteria AIC, BIC, CAIC and HQIC so it represents the data set better than the other chosen models.

**Table 1. Parameters Estimates for the Data**

| Model                  | Parameters Estimates |
|------------------------|----------------------|
| TRPD(x;α,q,w)          | \( \hat{\alpha} = 0.823 \)  \( \hat{q} = 2.658 \)  \( \hat{w} = 1.802 \) |
| R-LD(x;β,θ,b)          | \( \hat{\beta} = 1.107 \)  \( \hat{\theta} = 4.454 \)  \( \hat{b} = 5.798 \) |
| LD(x;α,λ)              | \( \hat{\alpha} = 2.889 \)  \( \hat{\lambda} = 19.614 \) |
| R-PD(x;p,b,α)          | \( \hat{\alpha} = 1.048 \)  \( \hat{b} = 1.262 \)  \( \hat{p} = 3.164 \) |

**Table 2. the values of Statistics \( \hat{\ell} \), AIC, BIC, CAIC and HQIC for the Data set.**

| Model                  | \( \hat{\ell} \) | AIC       | BIC       | CAIC      | HQIC      |
|------------------------|------------------|-----------|-----------|-----------|-----------|
| TRPD(x;α,q,w)          | -391.8296        | 789.6593  | 803.0674  | 789.8528  | 793.1357  |
| R-LD(x;β,θ,b)          | -474.3170        | 954.6339  | 968.0421  | 954.8275  | 958.1103  |
| LD(x;α,λ)              | -420.8238        | 847.6477  | 861.0558  | 847.8412  | 851.1240  |
| R-PD(x;p,b,α)          | -414.0869        | 834.1738  | 847.5819  | 834.3673  | 837.6502  |
References

[1]. Wen-lian Hung and ching-yichen. (2004). “Approximate MLE of the scale parameter of the truncated Rayleigh distribution under the first – censored data” Department of Statistics Tamkang University Tamsui, Taipei Taiwan, 25137 R.O.C.
Department of Mathematics Education National Hsinchu Teachers College Hsin-Chu Taiwan R.O.C.

[2]. David R-clark, FCAS. (2013). “A note on the upper-truncated Pareto distribution”, Presented at the. 2013 Enterprise Risk Management SymposiumApril 22-24, 2013.

[3]. Taylor and francis. (2014). “Parameter Estimation for the Truncated Pareto Distribution”, Informa Ltd Registered in England and Wales Registered Number: 1072954
Registered office: Mortimer House, 37-41 Mortimer Street, London W1T 3JH, UK

[4]. Mathias Reachke. (2012). “Inference for the truncated exponential distribution”, Published online: 5 February 2011.

[5]. AL_ Yasseri, Ahmad Yassin Taqi. (2014). “Estimation of parameter and Reliability function for truncated Logistic distribution”, M.CS. Thesis. AL-Mustansiriya University.

[6]. Dibagay, Dler Mustafa. (2007). “Numerical Estimation of the Parameters of the Gamma and Truncated Exponential Distribution”, M.Sc. Thesis, University of Mosul.

[7]. Hormuz. Amir Hanna. (1990). ”Mathematical Statistics”, Directorate of Dar Al-Kutub for Printing and Publishing, University of Mosul.
[8] Blumenthal & Goel. (1988). “Estimation With Truncated Data” OHIO State Univresity For columus VOL.4

[9]. AL_ Sultani ,Bnadheer Dhea’a Mohmmad. (2017). “ New Distribution on Proposed Classes” M.cs. Thesis. University of Babylon.

[10]. Hassan, Habe Ali. (2015). “Issues related with uniform distribution”, M.CS. Thesis. AL-Mustansiriya University.
[11] AL_Ghalib, Sabah Muhammad Khudair. (2019). “On some Special oge distributions”. M.CS. Thesis. University of Pabylon.

[12] AL-Sultani, B.D. (2017). “New Distributions on proposed classes”, M.Sc. Thesis, University of Babylon.

[13] Hanaa H. Abu – Zinadah. (2014). “Six Method of Estimations for the Shape Parameter of Exponentiated Gompertz Distribution” Department of Statistics, sciences Faculty for Girls King Abaulaziz University, P.O. BOX 3269 Jededah 21436, Saudi Arabia.