Higher-order amplitude squeezing

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Abstract. A brief review of nonclassicality conditions in terms of moments of the creation and annihilation operators is given. By introducing $k$-th power amplitude squeezing, the notions of ordinary quadrature squeezing and amplitude-squared squeezing are generalized. Minimum uncertainty states are considered as a special class of $k$-th power amplitude squeezed states. These states can be characterized by a special, rather simple nonclassicality condition.

1. Introduction

Nonclassical effects have attracted substantial interest during the last decades. The definition of nonclassicality that is widely accepted in quantum optics is based on the existence of a well-behaved $P$-function. This means that a state is considered to have a classical counterpart if the $P$-function has the properties of a probability density [1, 2]. A quantum state is called a nonclassical one if $P$ fails to be interpreted as a probability.

New possibilities of measurement and characterization of quantum states render it possible to experimentally determine the complete information on elementary quantum states, for a review see [3]. Thus it became interesting to characterize nonclassical effects in terms of observable quantities that completely describe the quantum state, e.g. characteristic functions [4]. The attempt was made to formulate necessary and sufficient conditions for nonclassicality in terms of observable quantities. Such conditions have been expressed in different, equivalent forms. First, an infinite hierarchy of conditions has been formulated in terms of observable characteristic functions [5]. Second, the conditions have been reformulated in terms of normally ordered moments of two quadrature operators [6]. Third, the conditions have been expressed with normally ordered moments of annihilation and creation operators and the needed detection schemes for these moments have been studied [7].

In the present paper we will generalize the concept of amplitude-squared squeezing as introduced by Hillery [8], leading to the concept of higher-order amplitude squeezing. In Sec. 2 we give a brief review of the nonclassicality criteria that will turn out to be useful for this purpose. The definition of higher-order amplitude squeezing and the relation to observable nonclassicality criteria is given in Sec. 3. In Sec. 4 minimum uncertainty states showing higher-order amplitude squeezing are considered. A brief summary is given in Sec. 5.

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2. Nonclassical conditions

In the Glauber-Sudarshan $P$ representation [10, 11] any quantum state $\hat{\varrho}$ can be formally written as a "continuous mixture" of coherent states:

$$\hat{\varrho} = \int P(\alpha)|\alpha\rangle\langle\alpha| d^2\alpha. \quad (1)$$

the weight function $P(\alpha)$, the so-called $P$ function, is not necessarily a probability distribution on the complex plane. In general, it is even not an ordinary function of a complex argument. The $P$ function of a well defined quantum state can be a highly singular distribution. As it is widely accepted, a quantum state is called classical if its $P$ function has the properties of a probability distribution. In particular, this requires non-negativity and that $P$ is not more singular than a delta function. When the $P$ function fails to be a probability distribution on the complex plane the state under consideration is called nonclassical.

Since the $P$ function cannot be measured directly it is important to have conditions for nonclassicality in terms of measurable quantities. To derive such conditions we note that the quantum-mechanical mean value $\langle \hat{f} \hat{f} \rangle$ for states having a classical counterpart is nonnegative for any operator $\hat{f}$:

$$\langle \hat{f} \hat{f} \rangle = \int P(\alpha)|f(\alpha)|^2 d^2\alpha \geq 0, \quad (2)$$

provided that $P(\alpha) \geq 0$ according to the assumed classicality of the state under consideration. Due to this fact any negativity, $\langle \hat{f} \hat{f} \rangle < 0$, is a clear signature of nonclassicality. In fact, it is not necessary to test all operators $\hat{f}$ in this inequality, but those operators whose normally ordered form exists [6, 9]. Let us consider such an operator $\hat{f}$ and express it in the form $\hat{f} = \sum_{n,m=0}^{+\infty} c_{nm}\hat{a}^n\hat{a}^m$. The mean value $\langle \hat{f} \hat{f} \rangle$ can be expanded as

$$\langle \hat{f} \hat{f} \rangle = \sum_{n,m,k,l=0}^{+\infty} c_{kl} c_{nm} \langle \hat{a}^{n+k}\hat{a}^{m+l} \rangle. \quad (3)$$

This expression is a quadratic form for the coefficients $c_{nm}$. For any classical state this quadratic form must be nonnegative. According to the Sylvester criterion, for a classical state all the determinants $D_N$ must be nonnegative:

$$D_N = \begin{vmatrix}
1 & \langle \hat{a} \rangle & \langle \hat{a}^2 \rangle & \langle \hat{a}^3 \rangle & \langle \hat{a}^4 \rangle & \langle \hat{a}^5 \rangle & \ldots \\
\langle \hat{a}^2 \rangle & \langle \hat{a}^1 \hat{a} \rangle & \langle \hat{a}^2 \rangle & \langle \hat{a}^3 \rangle & \langle \hat{a}^4 \rangle & \langle \hat{a}^5 \rangle & \ldots \\
\langle \hat{a} \rangle & \langle \hat{a}^2 \rangle & \langle \hat{a}^4 \rangle & \langle \hat{a}^6 \rangle & \langle \hat{a}^8 \rangle & \langle \hat{a}^{10} \rangle & \ldots \\
\langle \hat{a}^3 \rangle & \langle \hat{a}^2 \hat{a} \rangle & \langle \hat{a}^4 \rangle & \langle \hat{a}^6 \rangle & \langle \hat{a}^8 \rangle & \langle \hat{a}^{10} \rangle & \ldots \\
\langle \hat{a}^4 \rangle & \langle \hat{a}^3 \hat{a} \rangle & \langle \hat{a}^5 \rangle & \langle \hat{a}^7 \rangle & \langle \hat{a}^9 \rangle & \langle \hat{a}^{11} \rangle & \ldots \\
\langle \hat{a}^5 \rangle & \langle \hat{a}^4 \hat{a} \rangle & \langle \hat{a}^6 \rangle & \langle \hat{a}^8 \rangle & \langle \hat{a}^{10} \rangle & \langle \hat{a}^{12} \rangle & \ldots \\
\langle \hat{a}^6 \rangle & \langle \hat{a}^5 \hat{a} \rangle & \langle \hat{a}^7 \rangle & \langle \hat{a}^9 \rangle & \langle \hat{a}^{11} \rangle & \langle \hat{a}^{13} \rangle & \ldots
\end{vmatrix} \geq 0. \quad (4)$$

This result can be formulated in the following equivalent way: a state is nonclassical if and only if at least one of the determinants $D_N$ is negative: $D_N < 0$, for more details see [7].

Note that for a classical state not only the determinants $D_N$ are nonnegative but all their subdeterminants which can be obtained by canceling rows and columns with the same indices. In some cases such subdeterminants can be more useful than the determinants $D_N$, i.e. it may be easier to find negativities of subdeterminants than those of the determinants $D_N$ themselves.

3. Higher-order amplitude squeezing

In the following we will deal with a special class of nonclassical states for which certain subdeterminants are especially useful. Consider the operators

$$\hat{F}_\varphi^{(k)} = \hat{a}^k e^{-i\varphi} + \hat{a}^k e^{i\varphi} \quad \text{and} \quad \hat{G}_\varphi^{(k)} = \hat{F}_\varphi^{(k)} e^{i\varphi/2} = -i(\hat{a}^k e^{-i\varphi} - \hat{a}^k e^{i\varphi}). \quad (5)$$
The operators $\hat{F}_\varphi^{(k)}$ and $\hat{G}_\varphi^{(k)}$ satisfy the uncertainty relation
\[
\Delta F_\varphi^{(k)} \Delta G_\varphi^{(k)} \geq \sum_{l=0}^{k-1} \binom{k}{l} k! \prod \langle \hat{a}^l \hat{a}^l \rangle.
\]
(6)

A state will be called $k$-th power amplitude squeezed if there exist values of $\varphi$ such that the variance of the operator $\hat{F}_\varphi^{(k)}$ becomes less than the limit set by the uncertainty relation (6),
\[
\min_\varphi (\Delta F_\varphi^{(k)})^2 < \sum_{l=0}^{k-1} \binom{k}{l} k! \prod \langle \hat{a}^l \hat{a}^l \rangle.
\]
(7)

This condition can be rewritten in the following, much simpler form:
\[
\min_\varphi \langle (\Delta \hat{F}_\varphi^{(k)})^2 \rangle : < 0,
\]
(8)

by using the normally ordered variance. Note that this particular nonclassical effect was considered in [12], for a review of a manifold of nonclassical states also see [13].

It is easily seen that for $k = 1$ this condition reduces to ordinary quadrature squeezing. For $k = 2$ we recover the condition for amplitude-squared squeezing [8]. From the condition (8) it is evident that a $k$-th power amplitude squeezed state is nonclassical for any $k$-value. This condition can be further simplified. One can easily verify the validity of the following expressions for the minimum and maximum of the variance $\langle (\Delta \hat{F}_\varphi^{(k)})^2 \rangle$ as a function of $\varphi$:
\[
\min_\varphi \langle (\Delta \hat{F}_\varphi^{(k)})^2 \rangle : = 2 \langle \Delta \hat{a}^k \Delta \hat{a}^k \rangle - 2 \langle (\Delta \hat{a}^k)^2 \rangle,
\]
\[
\max_\varphi \langle (\Delta \hat{F}_\varphi^{(k)})^2 \rangle : = 2 \langle \Delta \hat{a}^k \Delta \hat{a}^k \rangle + 2 \langle (\Delta \hat{a}^k)^2 \rangle.
\]
(9)

Let us consider the following determinant, which is a subdeterminant of a proper determinant $D_N$ defined in (4):
\[
\Delta_k = \left| \begin{array}{ccc}
1 & \langle \hat{a}^2 \rangle & \langle \hat{a}^2 \rangle \\
\langle \hat{a}^k \rangle & \langle \hat{a}^k \hat{a}^k \rangle & \langle \hat{a}^{2k} \rangle \\
\langle \hat{a}^{2k} \rangle & \langle \hat{a}^{2k} \hat{a}^k \rangle & \langle \hat{a}^{4k} \rangle \\
\end{array} \right|.
\]
(10)

By straightforward calculations it is easy to check that this subdeterminant is proportional to the product of the minimum and maximum variances,
\[
\Delta_k = \frac{1}{4} \min_\varphi \langle (\Delta \hat{F}_\varphi^{(k)})^2 \rangle \max_\varphi \langle (\Delta \hat{F}_\varphi^{(k)})^2 \rangle.
\]
(11)

Since the last term, $\max_\varphi \langle (\Delta \hat{F}_\varphi^{(k)})^2 \rangle$, is always nonnegative, the $k$-th power amplitude squeezing condition (7) is equivalent to $\Delta_k < 0$. Thus we arrive at the following conclusion: a state is $k$-th power amplitude squeezed if and only if the determinant (10) is negative. It is worth noting that the nonclassicality condition in this form is already optimized with respect to the phase-dependence of the $k$-th power effect.

4. Minimum uncertainty states

In the remainder of this paper we are interested in studying the minimum uncertainty states, the states for which the relation (6) becomes an equality. Pure quantum states of minimum uncertainty type can be found as solutions of the eigenvalue problem (for the case $k = 2$ of amplitude-squared squeezing cf. [14, 15])
\[
(\hat{F}_\varphi^{(k)} + i\lambda \hat{G}_\varphi^{(k)}) |\psi\rangle = \beta |\psi\rangle,
\]
(12)
Figure 1. Examples of $Q$ functions of $k$-th power amplitude squeezed states for the case of $k = 3$. 

where $\lambda$ is an arbitrary nonnegative real number and $\beta$ is arbitrary complex. The parameter $\lambda$ plays the role of a squeezing parameter, since from Eq. (12) it follows that $\Delta F_{\psi}^{(k)} = \lambda \Delta G_{\psi}^{(k)}$. Clearly, any minimum uncertainty state is $k$-th power amplitude squeezed, except for the case of $\lambda = 1$. Explicitly Eq. (12) can be written as

$$\left( (1 + \lambda) a^k e^{-i\varphi} + (1 - \lambda) a^\dagger k e^{i\varphi} \right) |\psi\rangle = \beta |\psi\rangle. \tag{13}$$

By the replacement $|\psi\rangle \rightarrow \hat{R}(-\varphi/k)|\psi\rangle$, where $\hat{R}(\varphi) = e^{i\varphi \hat{a}}$, Eq. (13) for a chosen phase $\varphi$ is reduced to the case of $\varphi = 0$:

$$\left( (1 + \lambda) a^k + (1 - \lambda) a^\dagger k \right) |\psi\rangle = \beta |\psi\rangle, \tag{14}$$

due to the relation $\hat{a} \hat{R}(\varphi) = e^{i\varphi} \hat{R}(\varphi) \hat{a}$. We see that there is a one-to-one correspondence between the solutions of the Eq. (13) for an arbitrary phase $\varphi$ and for the phase $\varphi = 0$, Eq. (14), by means of the operator $\hat{R}(\varphi)$. Therefore we may restrict our attention to the latter case.

A general analytical solution of Eq. (14) is unknown. Let us first consider the special situation for $\lambda = 1$. In this case Eq. (14) reduces to $\hat{a}^k |\psi\rangle = (\beta/2) |\psi\rangle$, whose solutions are combinations of coherent states $|\alpha \varepsilon_j^k\rangle$, $j = 0, \ldots, k - 1$. That is, $|\psi\rangle = \sum_{j=0}^{k-1} c_j |\alpha \varepsilon_j^k\rangle$, where $\alpha = \sqrt[2k]{\beta/2}$ and $\varepsilon_k$ is the primitive root of 1: $\varepsilon_k = e^{2\pi i/k}$. In general, Eq. (14) can be solved numerically.

We will show how to find the $Q$-function of the state $|\psi\rangle$, defined as $Q(\alpha) = |\langle \alpha |\psi\rangle|^2$. From the Eq. (14) one can easily get the equation for the scalar product $\langle \alpha |\psi\rangle$ of the state $|\psi\rangle$ with the coherent state $|\alpha\rangle$,

$$\left[ (1 + \lambda) \left( \frac{\alpha}{2} + \frac{\partial}{\partial \alpha^*} \right)^k + (1 - \lambda) \alpha^* k \right] \langle \alpha |\psi\rangle = \beta \langle \alpha |\psi\rangle. \tag{15}$$

We will look for a solution $\langle \alpha |\psi\rangle$ of the form $\langle \alpha |\psi\rangle = e^{-|\alpha|^2/2} g(\alpha^*)$. Then Eq. (15) reduces to

$$\frac{d^k g(\alpha^*)}{d \alpha^* k} = \left( -\frac{1 - \lambda}{1 + \lambda} \alpha^* k + \frac{\beta}{1 + \lambda} \right) g(\alpha^*). \tag{16}$$

The following simple trick allows one to solve this equation numerically on the whole complex plane. The complex parameter $\alpha^*$ is expressed in polar coordinates, $\alpha^* = re^{-i\theta}$. The
function \(g(\alpha^*)\) can be considered as a function of the radius \(r\) only for any fixed phase \(\theta\): 
\[ g(\alpha^*) = g(re^{-i\theta}) = g_\theta(r). \]
Now Eq. (16) can now be written in the form
\[
\frac{d^k g_\theta(r)}{dr^k} = \left( \frac{-1 - \lambda}{1 + \lambda} r^k e^{-2ik\theta} + \frac{\beta e^{-ik\theta}}{1 + \lambda} \right) g_\theta(r). \tag{17}
\]
The solutions \(g_\theta(r)\) for all phases \(\theta\) eventually form the solution \(g(\alpha^*)\) of Eq. (16). In Fig. 1 some illustrations of the \(Q\) function of minimum uncertainty states are given for the case of the third-order amplitude squeezing.

5. Summary
We have generalized the concept of amplitude-squared squeezing to higher-order amplitude squeezing. This more general nonclassical effect can be well characterized by a rather simple (observable) nonclassicality criterion formulated in terms of moments of annihilation and creation operators. Of particular interest are the minimum uncertainty states showing the general effect of higher-order amplitude squeezing. It is demonstrated that the \(Q\) functions of such states can be readily derived by numerical methods.

References
[1] Titulaer U M and Glauber R J 1965  *Phys. Rev.* **140** B676
[2] Mandel L 1986  *Phys. Scr.* **T12** 34
[3] Welsch D-G, Vogel W, Opatrny T 1999 *Homodyne Detection and Quantum State Reconstruction* in: *Progress in Optics*, Ed. Emil Wolf, Vol. XXXIX, Ch. II, pp. 63 - 211.
[4] Vogel W 2000  *Phys. Rev. Lett.* **84**, 1849
[5] Richter Th and Vogel W 2002  *Phys. Rev. Lett.* **89** 283601
[6] Shchukin E, Richter Th and Vogel W 2005  *Phys. Rev. A* **71** 011802(R)
[7] Shchukin V and Vogel W 2005  *Phys. Rev. A* **72** 043808
[8] Hillery M 1987  *Phys. Rev. A* **36** 3796
[9] Korbicz J K, Cirac J I, Wehr J and Lewenstein M 2005  *Phys. Rev. Lett.* **94** 153601
[10] Sudarshan E C G 1963  *Phys. Rev. Lett.* **10** 277
[11] Glauber R J 1963  *Phys. Rev.* **131** 2766
[12] Zhang Z M, Xu L, Chai J, Li F 1990  *Phys. Lett. A* **150** 27
[13] Dodonov V V 2002  *J. Opt. B* **4** R1
[14] Hillery M and Yu D 1994  *Quantum Opt.* **6** 37
[15] Bergou J A, Hillery M and Yu D 1991  *Phys. Rev. A* **43** 515