Nonexistence of global solutions for damped abstract wave equations with memory

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Abstract. We consider a class of abstract nonlinear wave equations with memory and linear dissipation. We give sufficient conditions in terms of the initial data to prove the nonexistence of global solutions. We improve recent results that have studied this problem for viscoelastic wave, Kirchhoff and Petrovsky equations with positive initial energy values.

1. Introduction

We shall study the following problem. For every initial data $u_0, u_1$, find a function $t \mapsto u(t), t \geq 0$, such that

\[ \begin{cases} Pu_{tt}(t) + Au(t) + \int_0^t g(t - \tau) Bu(\tau) \, d\tau + \delta Pu_t(t) = f(u(t)), & t > 0 \\ u(0) = u_0, \ u_t(0) = u_1, \end{cases} \]

where $u_t \equiv \frac{d}{dt} u$, $\delta > 0$ is a damping coefficient, $g(s) \geq 0$ for $s \geq 0$ is a relaxation function, and the following operators, defined on Banach spaces, are linear, continuous, positive and symmetric,

\[ P : H_P \to H'_P, \quad A : H_A \to H'_A, \quad B : H_B \to H'_B. \]

We assume that

\[ H_A \subset H_B \subset H_P \subset H, \]

are linear subspaces of a Hilbert space $H$ with inner product $(\cdot, \cdot)$, norm $\| \cdot \|$. The corresponding dual spaces $H', H'_P, H'_A, H'_B$, with the identification $H = H'$, satisfy

\[ H \subset H'_P \subset H'_B \subset H'_A. \]

We define the following bilinear forms through the operators $P, A, B$, and the corresponding duality pairs,

\[ \mathcal{P}(u, w) \equiv (Pu, w)_{H_P \times H'_P}, \ u, w \in H_P, \]

\[ \mathcal{A}(u, w) \equiv (Au, w)_{H_A \times H'_A}, \ u, w \in H_A, \quad \mathcal{B}(u, w) \equiv (Bu, w)_{H_B \times H'_B}, \ u, w \in H_B. \]
Then, we have corresponding norms
\[ \|u\|_{H_P}^2 = P(u, u), \quad u \in H_P, \quad \|u\|_{H_A}^2 = A(u, u), \quad u \in H_A, \quad \|u\|_{H_B}^2 = B(u, u), \quad u \in H_B. \]

To study the qualitative properties of the dynamics of problem (P) we define the phase space
\[ \mathcal{H} = H_A \times H_P, \]
with square norm
\[ \|(u, v)\|_{\mathcal{H}}^2 = \|v\|_{H_P}^2 + \|u\|_{H_A}^2. \]

Along the paper, the following hypotheses are assumed to be satisfied.

(i) There are constants \( c > 0, \hat{c} > 0 \), such that
\[ \|u\|_{H_A}^2 \geq \hat{c}\|u\|_{H_B}^2, \quad u \in H_A, \quad \|u\|_{H_B}^2 \geq c\|u\|_{H_P}^2, \quad u \in H_B \] (H0).

(ii) The nonlinear source term \( f : H_A \to H \), is a potential operator with potential \( F : H_A \to \mathbb{R} \), that is, \( f(u) = D_0F(u) \). We assume that \( f(0) = 0 \) and there exists a constant \( r > 2 \), such that
\[ (f(u), u) - rF(u) \geq 0, \quad u \in H_A, \] (H1).

(iii) The relaxation function \( g \in C^1(\mathbb{R}^+, \mathbb{R}^+) \), satisfies the following conditions
\[ g(0) > 0, \quad l = 1 - \int_0^{\infty} g(t) \, dt > 0, \quad \hat{g}(t) = \frac{d}{dt}g(t) \leq 0, \quad t \geq 0, \] (H2).

From the mechanics of continuous media, the modeling of the dynamics of deformable bodies with constitutive relations that include viscoelastic effects results in viscoelastic wave, Kirchhoff and Petrovsky equations like problem (P), see [1, 2]. One of the qualitative properties in the investigation of the dynamics of evolution equations is to characterize the set of initial data that produce global solutions, that is when the maximal time of existence is infinite. When this does not occur, sometimes is due to the blow-up, in a finite time, of the solution in the phase space, see [3]. This problem is the subject of this work.

The analysis will be done for weak solutions in the following sense.

**Definition 1.1.** For every initial data \((u_0, u_1) \in \mathcal{H} \), the map \((u_0, u_1) \mapsto (u(t), \dot{u}(t)) \in \mathcal{H} \), where \( \dot{u}(t) \equiv \frac{d}{dt}u(t) \), is a weak local solution of problem (P), if there exists some \( T > 0 \), such that \((u, \dot{u}) \in C([0, T]; \mathcal{H}) \), with
\[ u \in L^2([0, T]; H_B), \quad \dot{u} \in L^2([0, T]; H_P), \quad u(0) = u_0, \quad \dot{u}(0) = u_1, \]
and
\[
\frac{d}{dt} \mathcal{P}(\dot{u}(t), w) + \mathcal{A}(u(t), w) - \int_0^t g(t - \tau) \mathcal{B}(u(\tau), w) \, d\tau \\
+ \delta \mathcal{P}(\dot{u}(t), w) = (f(u(t)), w),
\]
a. e. in \((0, T)\), for every \(w \in H_A\).

Furthermore, the solution in this sense is unique and satisfies the following energy equation for \(T > t > t_0 \geq 0\),
\[
E(t) \equiv E(u(t), \dot{u}(t)) \equiv \frac{1}{2} \|\dot{u}(t)\|^2_{H_P} + J(u(t)),
\]
where
\[
J(u(t)) \equiv \frac{1}{2} (\|u(t)\|_{H_A}^2 + G(t)\|u(t)\|^2_{H_B} + (g \circ u(t)) - F(u(t)),
\]
then,
\[
E(t_0) \geq E(t) = \frac{1}{2} \|(u(t), \dot{u}(t))\|_{H_P}^2 - \frac{1}{2} G(t)\|u(t)\|^2_{H_B} + \frac{1}{2} (g \circ u(t)) - F(u(t)),
\]
where
\[
(g \circ u(t)) \equiv \int_0^t g(t - \tau)\|u(t) - u(\tau)\|^2_{H_B} \, d\tau, \quad G(t) \equiv \int_0^t g(\tau) \, d\tau.
\]
If the maximal time of existence \(T_{\text{MAX}} < \infty\) then
\[
\lim_{t \to T_{\text{MAX}}} \|(u(t), \dot{u}(t))\|_H = \infty,
\]
equivalently, if \(\hat{c} > 1 - \ell\),
\[
\lim_{t \to T_{\text{MAX}}} F(u(t)) = \infty,
\]
since, from (H0) and (H2),
\[
E(t) \geq \frac{1}{2} \left(1 - \frac{1 - t}{\ell}\right) \|(u(t), \dot{u}(t))\|_{H_P}^2 - F(u(t)).
\]

The analysis for \((P)\) is based in the study of the following differential inequality for a real function \(\psi(t) \geq 0\) that we investigated in [24],
\[
\psi(t) \frac{d^2}{dt^2} \psi(t) + \delta \psi(t) \frac{d}{dt} \psi(t) - (1 + \alpha) \left(\frac{d}{dt} \psi(t)\right)^2 - \beta \psi^2(t) + \gamma \psi(t) \geq 0, \quad t \geq 0, \quad (1.1)
\]
where \(\alpha, \beta, \gamma, \delta\) are strictly positive constants. To this end, we introduce the following functions,
\[
\phi(t) = \left(\frac{d}{dt} \psi(t) - \frac{\delta}{\alpha} \psi^\frac{1}{2}(t)\right)^2 + \frac{\beta}{\alpha} \psi(t), \quad \sigma_\nu(t) \equiv \frac{1 + 2\alpha}{2} \left(\phi(t) - \frac{\beta \nu}{\alpha} \psi(t)\right),
\]
\[
\mu_\lambda(t) \equiv \frac{1 + 2\alpha}{2} \left(\phi(t) - \frac{\beta}{\alpha(1 + 2\alpha)} \psi(t) \left(\frac{\lambda \beta \psi(t)}{\alpha \phi(t)}\right)^{2\alpha}\right),
\]
for $t \geq 0$, $\nu > 0$, and $\lambda \in (0, 1)$, and

$$\psi_0 \equiv \psi(0), \quad \phi_0 \equiv \phi(0) = \left(\frac{\psi_0'}{\psi_0^2} - \frac{\delta}{\alpha} \frac{\psi_0'}{\psi_0} \right)^2 + \frac{\beta}{\alpha} \psi_0, \quad \psi_0' \equiv \frac{d}{dt} \psi(0).$$

**Theorem 1.2.** [24] Consider any solution $\psi(t)$ of the differential inequality (1.1), such that

$$\psi_0' > \frac{\delta}{\alpha} \psi_0 > 0,$$

consequently $\phi_0 > \frac{\beta}{\alpha} \psi_0 > 0$. Then, there exists a nonempty interval

$$\mathcal{I} \equiv (a, b) \subset \left(0, \frac{1+2\alpha}{2} \phi_0\right),$$

with the following consequences:

(i) If $\gamma \in \mathcal{I}$, then $\psi(t)$ blows-up in finite time.

(ii) $a = \sigma_\nu(0)$ and $b = \mu_\lambda(0)$, moreover

$$a = \frac{\beta \psi_0}{(1 + 2\alpha)\nu^*} \frac{1}{2\alpha} < \frac{\beta \psi_0}{(1 + 2\alpha)^\frac{1}{2}};$$

$$b = \frac{\alpha \phi_0}{\lambda^*} > \frac{1+2\alpha}{2} \phi_0 - \left(\frac{1+2\alpha}{2\alpha} - \zeta(\lambda^*)\right) \beta \psi_0 > \frac{1+2\alpha}{2} \phi_0 - \frac{\beta}{2\alpha} \psi_0,$$

for some $\frac{2\alpha}{1+2\alpha} < \lambda^* < 1$ and $\nu^* > 1 + 2\alpha$, where $1 < \zeta(\lambda^*) < \frac{1+2\alpha}{2\alpha}$ is a function of $\lambda^*$.

(iii) For fixed $\psi_0$,

$$\psi_0' \mapsto t^*,$$

is strictly decreasing, and $\psi_0' \mapsto |\mathcal{I}|$, is strictly increasing.

(iv) We have the bounds

$$0 < \frac{1+2\alpha}{2} \phi_0 - |\mathcal{I}| < \left(\frac{1+2\alpha}{2\alpha} - \zeta(\lambda^*) + \frac{1}{((1+2\alpha)\nu^*)^{\frac{1}{2}}}\right) \beta \psi_0,$$

$$t^* \geq \left(\frac{\alpha \psi_0'}{\psi_0} - \delta\right)^{-1}.$$

(v) Furthermore, for fixed $\psi_0$, we have the limit values as $\psi_0' \to \infty$

$$a \to 0, \quad |b - \frac{1+2\alpha}{2} \phi_0| \to 0, \quad t^* \to 0,$$

$$\nu^* \to \infty, \quad \lambda^* \to \frac{2\alpha}{1+2\alpha}, \quad \zeta(\lambda^*) \to \frac{1+2\alpha}{2\alpha}.$$

**Corollary 1.3.** [24] Let the assumptions of Theorem 1.2 hold. For every number $\gamma > 0$, we can choose $\psi_0'$ large enough, so that $\gamma \in \mathcal{I}$, and then the corresponding solution $\psi(t)$ of (1.1) blows-up in finite time.
2. Main result
Here, we shall analyze the nonexistence of global solutions for the problem (P), for any positive size of the initial energy, applying Theorem 1.2 with

$$\psi(t) \equiv \|u(t)\|_{H_{P}}^2.$$  

To that end, the velocity is decomposed orthogonally as follows

$$\dot{u} = \frac{\mathcal{P}(\dot{u}, u)}{\|u\|_{H_{P}}^2} u + h, \quad \mathcal{P}(u, h) = 0,$$

$$\|\dot{u}\|_{H_{P}}^2 = \|h\|_{H_{P}}^2 + \frac{|\mathcal{P}(\dot{u}, u)|^2}{\|u\|_{H_{P}}^2} \geq \frac{|\mathcal{P}(\dot{u}, u)|^2}{\|u\|_{H_{P}}^2} \equiv Q(\dot{u}, u).$$

Theorem 2.1. Consider any solution of problem (P) in the sense of Definition 1.1. Assume that hypotheses \((H0) - (H2)\) hold. If

$$\hat{c} > 1 - l, \quad r > 1 + \frac{1}{\sqrt{1 - \frac{1}{r^2}}}$$

and

$$\mathcal{P}(u_0, u_1) > \frac{2\delta}{r - 2} \|u_0\|_{H_{P}}^2 > 0, \quad (2.1)$$

are satisfied, then the conclusions of Theorem 1.2 hold with

$$\alpha = \frac{r - 2}{4}, \quad \beta = c(r - 2) \left(\frac{(\hat{c} - (1 - l)(r - 1)^2 - \hat{c})}{(r - 1)^2 - 1}\right), \quad \gamma = 2rE_0.$$

In particular, \(\|u(t)\|_{H_{P}} \to \infty\) as \(t \to t^*\), that is blows-up in finite time and then the solution of (P) is not global.

Corollary 2.2. Assume that hypotheses of Theorem 2.1 are met. For every number \(K > 0\), we can choose initial data with \(\mathcal{P}(u_0, u_1)\) large enough, so that the corresponding solution with \(E_0 = K\) exists only up to a finite time.

Proof. (of Theorem 2.1.) We assume that the solution is global, then

$$\psi(t) \equiv \Psi(u(t))$$

is well defined for any \(t \geq 0\). We observe that

$$\frac{d}{dt} \psi(t) = 2(\mathcal{P}(u(t), \dot{u}(t)))$$

$$\frac{d^2}{dt^2} \psi(t) + \delta \frac{d}{dt} \psi(t) = 2 \left(\|\dot{u}(t)\|_{H_{P}}^2 - \|u(t)\|_{H_{A}}^2 + (f(u(t), u(t)))

+ 2 \int_0^t g(t - \tau) \mathcal{B}(u(\tau), u(t)) d\tau

= 2(\|\dot{u}(t)\|_{H_{P}}^2 - \|u(t)\|_{H_{A}}^2 + (f(u(t), u(t)))

+ 2G(t)\|u(t)\|_{H_{B}}^2 + 2 \int_0^t g(t - \tau) \mathcal{B}(u(\tau) - u(t), u(t)) d\tau.$$
By energy equation and hypothesis \((H1)\), we obtain the following
\[
2(||\dot{u}(t)||_{H^p}^2 - ||u(t)||_{H^A}^2 + (f(u(t), u(t)) + G(t)||u(t)||_{H^B}^2)
\]
\[
= 2(||\dot{u}(t)||_{H^p}^2 - ||u(t)||_{H^A}^2 + (f(u(t), u(t)) + G(t)||u(t)||_{H^B}^2 + rE(t) - rE(t))
\]
\[
\geq (r + 2)Q(u(t), \dot{u}(t)) + (r - 2)(\hat{c} - G(t))||u(t)||_{H^B}^2 + r(g \circ u)(t) - 2rE_0.
\]

Also, we get
\[
2\int_0^t g(t - \tau)B(u(\tau) - u(t), u(t)) \, d\tau
\]
\[
\leq 2\int_0^t g(t - \tau)||u(\tau) - u(t)||_{H^B}||u(t)||_{H^B} \, d\tau
\]
\[
\leq \int_0^t g(t - \tau)\left(\theta||u(\tau) - u(t)||_{H^B}^2 + \frac{1}{\theta}||u(t)||_{H^B}^2\right) \, d\tau
\]
\[
= \theta(g \circ u)(t) + \frac{1}{\theta}G(t)||u(t)||_{H^B}^2,
\]

where \(\theta > 0\) is a constant to be chosen later. Hence,
\[
\frac{d^2}{dt^2}\psi(t) + \delta \frac{d}{dt}\psi(t) - \left(\frac{r + 2}{4}\right) \left(\frac{d}{dt}\psi(t)\right)^2
\]
\[
\geq \left((r - 2)(\hat{c} - G(t)) - \frac{1}{\theta}G(t)\right)||u(t)||_{H^B}^2
\]
\[
+ (r - \theta)(g \circ u)(t) - 2rE_0.
\]

We take \(\theta = r\), and by \((H0), (H2)\), we get
\[
\frac{d^2}{dt^2}\psi(t) + \delta \frac{d}{dt}\psi(t) - \left(\frac{r + 2}{4}\right) \left(\frac{d}{dt}\psi(t)\right)^2
\]
\[
\geq \left(\hat{c}(r - 2) - (1 - l)\left(\frac{r - 1}{r}\right)^2\right)||u(t)||_{H^B}^2 - 2rE_0
\]
\[
\geq \frac{\hat{c}}{r} \left(\hat{c}(r - 1)^2 - 1\right) - (1 - l)(r - 1)^2 \psi(t) - 2rE_0,
\]

where \(\hat{c}(r - 1)^2 - 1 - (1 - l)(r - 1)^2 = (\hat{c} - (1 - l))(r - 1)^2 - \hat{c} > 0\) if and only if
\[
\hat{c} > 1 - l, \text{ and } r > 1 + \frac{1}{\sqrt{1 - \frac{1 - l}{\hat{c}}}}.
\]

Then, the following differential inequality for \(\psi(u(t))\) holds
\[
\psi(t) \frac{d^2}{dt^2}\psi(t) + \delta \psi(t) \frac{d}{dt}\psi(t) - \left(\frac{r + 2}{4}\right) \left(\frac{d}{dt}\psi(t)\right)^2
\]
\[
- c(r - 2) \left(\frac{(\hat{c} - (1 - l))(r - 1)^2 - \hat{c}}{(r - 1)^2 - 1}\right) \psi^2(t) + 2rE_0\psi(t) \geq 0.
\]

This equation is of the form \((1.1)\), and \((2.1)\) is \((1.2)\) with \(\alpha = \frac{r - 2}{4}\). Consequently, we can apply Theorem 1.2. □
Remark 2.1. We notice that the numbers $a$ and $b$ are closer one each other as the damping coefficient grows. Hence, the length of the blow-up interval $I$ decreases as $\delta$ increases. Therefore, as the damping coefficient grows, the set of initial energies where we can have global non existence becomes smaller. We also observe that the blow-up is obtained for a larger set of relaxation functions as long as $\hat{c}$ is larger. Moreover, the blow-up property is reached for a larger set of values of $r$, which is the parameter that measures the nonlinearity of the source term, as long as the quotient $0 < \frac{l-1}{r} < 1$ decreases. In particular, if $B = A$, then $\hat{c} = 1$ and then the set of relaxation functions is only restricted by hypothesis $(H2)$, and $r > 1 + \frac{1}{\sqrt{l}}$. Consequently, the set of values of $r$ to get blow-up is larger as long as $l$ is close to one, that is, almost without the memory term.

3. Discussion and applications

Several authors have studied the blow-up for solutions of the following problems, see [7]-[23]. Some of them have made the analysis for any positive initial energy. We next present and comment those results and our contribution.

3.1. Viscoelastic wave and Petrovsky equations

\begin{align*}
(WA) & \quad \left\{ \begin{array}{ll}
& u_{tt}(t) - \Delta u(t) + \int_0^t g(t - \tau) \Delta u(\tau) \, d\tau + \delta u_t(t) = f(u(t)), \quad x \in \Omega, \ t > 0, \\
& u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \\
& u(x, t) = 0, \quad x \in \partial \Omega, \ t > 0,
\end{array} \right.
\end{align*}

\begin{align*}
(PE) & \quad \left\{ \begin{array}{ll}
& u_{tt}(t) + \Delta^2 u(t) - \int_0^t g(t - \tau) \Delta^2 u(\tau) \, d\tau + \delta u_t(t) = f(u(t)), \quad x \in \Omega, \ t > 0, \\
& u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \\
& u(x, t) = 0 = \partial_n u(x, t), \quad x \in \partial \Omega, \ t > 0,
\end{array} \right.
\end{align*}

where $\Omega \subset \mathbb{R}^N$, is a bounded domain with smooth boundary and $\nu$ is the normal vector to $\partial \Omega$.

According to the abstract formulation, we identify the operators as follows. In $(WA)$, $A = B = -\Delta$, defined on $H_A = H_B = H_0^1(\Omega)$. In $(PE)$, $A = B = \Delta^2$, defined on $H_A = H_B = H_0^2(\Omega)$. In both problems, $P =$ identity operator and $H = H_P = L^2(\Omega)$. Then, $\hat{c} = 1$ and $r > 1 + \frac{1}{\sqrt{l}}$. Furthermore, the source term and the relaxation function satisfy $(H1)$ and $(H2)$.

Our main result implies that the solution of any of these two problems blows-up if $(2.1)$ holds for any positive value of the initial energy such that $2rE_0 \in I$. And, the size of the blow-up interval $I$ is as big as we wish if $(u_0, u_1) > 0$ is large enough.

Blow-up has been studied for the problem $(WA)$ with nonlinear damping in [7, 8]. Indeed, in [7] the initial energy was assumed to be negative, and in [8] the blow-up was showed by the potential well method if $E_0 < d$ and

$$I(u_0) \equiv \|\nabla u_0\|_2^2 - (f(u_0), u_0) < 0.$$ 

Here, $d > 0$ is the potential depth or the mountain pass level of the corresponding elliptic problem. Under similar conditions and for small positive positive energy, the blow-up was showed in [9] for the problem with $\delta = 0$ and in [12] for the equation with a strong linear damping.
The case with large positive values of the initial energy was analyzed in [9, 12, 16, 18, 22]. Certainly, under the assumptions

\[ \mathcal{P}(u_0, u_1) > 0, \quad I(u_0) < 0, \quad E_0 < C \| u_0 \|_{H_P}^2, \]

(3.1)

where \( C > 0 \), the blow-up was proved in [16]. See also [12] for a similar result. For the equation with \( \delta = 0 \), the same conclusion was proved in [9] under the conditions

\[ \mathcal{P}(u_0, u_1) > 0, \quad E_0 < C_1 \mathcal{P}(u_0, u_1) - C_2 \| u_0 \|_{H_P}^2, \]

(3.2)

for some constants \( C_j > 0, \ j = 1, 2 \). Moreover, for a nonlinear damping the blow-up was concluded in [22] if

\[ \mathcal{P}(u_0, u_1) > 0, \quad E_0 < C \mathcal{P}(u_0, u_1), \]

(3.3)

for some \( C > 0 \). The upper bound for \( E_0 \) given in Theorem 2.1 improves the previous ones. Furthermore, the additional condition \( I(u_0) < 0 \) is not required in our blow-up result. About this point, let us mention our previous work [25], where we studied the problem (P) with \( g(t) = 0 \), that is without memory, and we showed that the sign of \( I(u_0) \) has no relevance for the blow-up of solutions when the initial energy is positive and sufficiently large. On the other hand, by the potential well method, the negative sign of \( I(u_0) \) is completely essential for \( E_0 < d \).

If

\[ 0 < E_0 < 1 \frac{1}{8} \left( \frac{2 \mathcal{P}(u_0, u_1) - \delta \| u_0 \|_{H_P}^2}{\alpha \| u_0 \|_{H_P}^2} \right)^2, \]

holds, then the blow-up for (WA) is achieved in [18]. The lower bound of \( b \) in Theorem 2.1 is such that

\[ b > \alpha \phi_0 + \frac{1}{2} \left( \frac{\psi_0'}{\psi_0^2} - \frac{\delta \psi_0^2}{\alpha \psi_0} \right)^2 > 1 \frac{1}{2} \left( \frac{2 \mathcal{P}(u_0, u_1) - \delta \| u_0 \|_{H_P}^2}{\alpha \| u_0 \|_{H_P}^2} \right)^2. \]

Hence, the upper bound for \( E_0 \) in Theorem 2.1 is larger than the one in [18].

By the potential well method the blow-up of solutions for the problem (PE) with \( \delta = 0 \) and \( E_0 < d \) was showed in [15] and for any positive value of \( E_0 \) if the conditions given in (3.1) hold. See also [13] for the same problem and similar conditions and conclusions. In [21], under the assumption (3.3), the blow-up is showed for the problem (PE) with a nonlinear damping. The same result was proved in [23] if the following holds,

\[ \mathcal{P}(u_0, u_1) > 0, \quad E_0 < C \left( \mathcal{P}(u_0, u_1) + \| u_0 \|_{H_P}^2 \right), \]

(3.4)

where \( C > 0 \). As before, the upper bound for \( E_0 \) given in Theorem 2.1 improves the previous ones.

### 3.2. Dispersive wave and Petrovsky equations

The dispersive wave and Petrovsky equations

\[
(DW) \begin{cases}
  u_{tt}(t) - \Delta u_{tt}(t) - \Delta u(t) + \int_0^t g(t - \tau) \Delta u(\tau) \, d\tau \\
  + \delta u_t(t) - \delta \Delta u_t(t) = f(u(t)), \\
  u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \\
  u(x, t) = 0, \\
  x \in \Omega, \quad t > 0,
\end{cases}
\]

(\( x \in \Omega, \quad t > 0, \))
\[ \begin{aligned}
\text{(DP)} & \quad \left\{ \begin{array}{ll}
\partial_t u - \Delta u + \Delta^2 u - \int_0^t g(t - \tau) \Delta^2 u(\tau) \, d\tau + \delta u_t = f(u(t)), & x \in \Omega, \ t > 0, \\
u(0, x) = u_0(x), \ u_t(0, x) = u_1(x), & x \in \Omega, \\
u(x, t) = 0 & x \in \partial \Omega, \ t > 0,
\end{array} \right.
\end{aligned} \]

where \( \Omega \subset \mathbb{R}^N \), is a bounded domain with smooth boundary and \( \nu \) is the normal vector to \( \partial \Omega \).

In (DW) we have \( A = B = -\Delta \) defined on \( H_A = H_B = H_0^1(\Omega) \). In (DP) we have \( A = B = \Delta^2 \) defined on \( H_A = H_B = H_0^1(\Omega) \). In both problems \( P = I - \Delta \) defined on \( H_P = H_0^1(\Omega) \). Furthermore, \( H = L^2(\Omega), \ c = 1 \) and \( r > 1 + \frac{1}{\sqrt{1}} \). The relaxation function and the source term satisfy (H2) and (H1).

In [14] the problem (DW) was studied, where global and non-global solutions with \( E_0 < d \) are analyzed by means of the potential well method. Furthermore, arbitrary positive initial energy was considered to show the blow-up property if (3.1) holds. In [11] the blow-up property of the problem (DP) was analyzed for \( E_0 < d \) and \( I(u_0) < 0 \).

Applying Theorem 2.1 to these problems, we conclude that any solution that satisfies (2.1) blows-up if the initial energy is such that \( 2rE_0 \in I \), and Corollary 2.2 implies that if \( (u_0, u_1) > 0 \) is large enough, there exist initial data that imply blow-up. Hence, we conclude that Theorem (2.1) contributes to improve the understanding of the dynamics of these problems.

### 3.3. Kirchhoff equation

\[ \begin{aligned}
\text{(KI)} & \quad \left\{ \begin{array}{ll}
u_t(t) - \Delta u + \int_0^t g(t - \tau) \Delta u(\tau) \, d\tau + \delta u_t = g(u), & x \in \Omega, \ t > 0, \\
u(0, x) = u_0(x), \ u_t(0, x) = u_1(x), & x \in \Omega, \\
u(x, t) = 0 & x \in \partial \Omega, \ t > 0,
\end{array} \right.
\end{aligned} \]

where \( \Omega \subset \mathbb{R}^N \), is a bounded domain with smooth boundary and \( \phi(\|\nabla u\|_2^2) = 1 + \|\nabla u\|_2^{2p}, \quad p \geq 1 \).

The nonlinear source \( g \) is such that \( (g(u), u) - \rho G(u) \geq 0, \quad \rho > 2, \)

where \( G \) is the potential of \( g \). Then,

\[ f(u) = g(u) + \|\nabla u\|_2^{2p} \Delta u, \]

has the potential

\[ F(u) = G(u) - \frac{1}{2(p + 1)} \|\nabla u\|_2^{2(p + 1)}. \]

Hence, (H1) is satisfied if \( \rho \geq r \geq 2(p + 1) \).

That is, the nonlinearity of the source term \( g \) is stronger than the one of \( \phi \).

This problem has been studied in [10, 17, 19, 20] for initial data

\( (u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega), \)

In [10] the blow-up property the problem (KI) with nonlinear damping is analyzed for \( E_0 < d \) and \( I(u_0) < 0 \). For high initial energies the blow-up of solutions is proved in [17] and [19] under the conditions given in (3.1), and the same property is showed in [20] if (3.3) is satisfied. The remarks made in the previous examples regarding these conditions and our contribution also apply to this problem.
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4. References
[1] Renardy M, Hrusa W J and Nohel J A 1987 Mathematical problems in viscoelasticity (London: Pitman)
[2] Fabrizio M and Morro A 1992 Mathematical problems in linear viscoelasticity (Philadelphia: SIAM)
[3] Ball J M 1977 Remarks on blow up and nonexistence theorems for nonlinear evolutions equations Quart J Math Oxford 28 473-86.
[4] Korpusov M O, Ovchinnikov A V, Sveshnikov A G and Yushkov E V 2018 Blow-up in nonlinear equations of mathematical physics. Theory and methods (Berlin: De Gruyter)
[5] Levine H A 1973 Some nonexistence and instability theorems for solutions of formally parabolic equations of the form \( Pu_t = -Au + F(u) \) Arch. Rational Mech. Anal. 51 371-86
[6] Levine H A 1974 Instability and nonexistence of global solutions to nonlinear wave equations of the form \( Pu_{tt} = -Au + F(u) \) Trans. Amer. Math. Soc. 192 1-21
[7] Messaoudi S A 2003 Blow up and global existence in a nonlinear viscoelastic wave equation Math. Nachr. 260 58-66.
[8] Messaoudi S A 2006 Blow-up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation J. Math. Anal. Appl. 320 902-15
[9] Wu S-T 2006 Blow-up of solutions for an integro-differential equation with a nonlinear source Electron. J. Differential Equations 2006 45
[10] Wu S-T and Tsai L-Y 2010 Blow-up of positive initial energy solutions for an integro-differential equation with nonlinear damping Taiwanese J. Math. 14 2043-58
[11] Li G, Sun Y and Liu W 2013 On Asymptotic Behavior and Blow-Up of Solutions for a Nonlinear Viscoelastic Petrovsky Equation with Positive Initial Energy Journal of Function Spaces and Applications 2013 905867
[12] Liang F and Gao H 2012 Global existence and blow-up of solutions for a nonlinear wave equation with memory J. Inequal. Appl. 2012 33
[13] Tahamtani F and Shahrouzi M 2012 Existence and blow up of solutions to a Petrovsky equation with memory and nonlinear source term Bound. Value Probl. 2012 50
[14] Xu R, Yang Y and Liu Y 2013 Global well-posedness for strongly damped viscoelastic wave equation Appl. Anal. 92 138-57
[15] Yang Z and Fan G 2015 Blow-up for the Euler-Bernoulli viscoelastic equation with a nonlinear source Electron. J. Differential Equations 2015 306
[16] Wang Y 2009 A global nonexistence theorem for viscoelastic equations with arbitrarily positive initial energy Appl. Math. Lett. 22 1394-400
[17] Li G, Hong L and Liu W 2012 Global Nonexistence of Solutions for Viscoelastic Wave Equations of Kirchhoff Type with High Energy Journal of Function Spaces 2012 530861
[18] Kafini L. and Messaoudi S A 2013 A blow-up result in a nonlinear viscoelastic problem with arbitrary positive initial energy Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 20 657-65
[19] Jie L and Fei L 2015 Blow-up of solution for an integro-differential equation with arbitrary positive initial energy Bound. Value Probl. 2015 96
[20] Yang Z and Gong Z 2016 Blow-up solutions for viscoelastic equations of Kirchhoff type with arbitrary positive initial energy Electron. J. Differential Equations 2016 332
[21] Liu L, Sun F and Wu Y 2019 Blow-up of solutions for a nonlinear Petrovsky type equation with initial data at arbitrary high energy level Bound. Value Probl. 2019 15
[22] Sun F, Liu L and Wu Y 2019 Blow-up of solutions for a nonlinear viscoelastic wave equation with initial data at arbitrary energy level Appl. Anal. 98 2308-27
[23] Liu L, Sun F and Wu Y 2020 Finite time blow-up for a nonlinear viscoelastic Petrovsky equation with high initial energy SN Partial Differ. Equ. Appl. 1 31
[24] Esquivel-Avila J A 2021 A differential inequality and the blow-up of its solutions Appl. Math. E-Notes in press
[25] Esquivel-Avila J A 2020 Blow-up in damped abstract nonlinear equations Electron. Res. Arch. 28 347-67