Exact Solutions to Supergravity

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Abstract

We find a general $p-1$-brane solution to supergravity coupled to a $p+1$-form field strength using the “standard ansatz” for the fields. In addition to the well-known elementary and solitonic $p-1$-brane solutions, which are the only ones preserving half of the supersymmetry (for the cases where the supersymmetry transformations are known), there are other possibilities, the most interesting of which is an elementary type I string solution with a non-dynamical string source and no conserved charge.
1 Introduction

Many people now believe that all string theories are different perturbative approximations of an, as yet, unknown theory. One consequence of this conjecture is that the strong coupling limit of a certain theory, with respect to a certain coupling constant, is the weak coupling limit of another theory. Candidates for these dual pairs have been found for (practically) all ten- and eleven-dimensional theories, and there are many interesting examples in lower dimensions. In such a strong-weak dual pair the fundamental states of one theory should occur as solitonic states of the dual one.

There is much evidence that in ten dimensions the $SO(32)$ heterotic string is dual to the type I string. Their low energy effective field theories are the same (after a field redefinition which, among other things, changes sign of the dilaton, thus also exchanging the coupling constant $g = \langle e^\phi \rangle$ with $1/g$), and some higher order terms have also been shown to agree after the same field redefinition. There is a “miraculous breakdown” of type I perturbation theory whenever the toroidally compactified heterotic string has an enhanced symmetry not present in the type I string. And finally, the type I string has the heterotic string as an exact classical solution. However, so far nobody has found the corresponding type I state in the effective theory for the heterotic string. This is not so very surprising, since the standard classical $p-1$-brane solutions which have been examined in this context, correspond to an infinite $p-1$-brane with constant energy per $p-1$-volume. For theories with only closed $p-1$-branes this is no problem. The solution can be interpreted as a $p-1$-brane winding around $p-1$ toroidally compactified coordinates, with very large compactification radii, but the type I string theory also contains open strings, so such a state obviously cannot be stable here. The best one can do without radically changing one’s ansatz is to look for a metastable configuration around such an unstable type I string explicitly put into the equations by hand.

In all, a careful study of exact solutions to low energy effective theories is certainly merited. In previous work on general $p-1$-brane solutions, one has always assumed supersymmetry (or rather, a generalization of the relations which tell us that half the supersymmetries remain unbroken for the cases where the fermionic theory and the supersymmetry transformations are known), see, for instance, [8, 9]. In this paper we will solve the supergravity equations of motion without prejudice. Given the “standard ansatz” for the fields, the complete exact solution turns out to exist in closed form. Putting this solution back into the equations of motion, however, gives us singularities, which we choose to put at the origin of the transverse coordinates. If we want the equations of motion to be consistent also at the origin, we have to choose the integration constants in a particular fashion. We can introduce a dynamical $p-1$-brane source at the origin in which case we find the (supersymmetric) elementary $p-1$-brane solution, or we can study the dual theory with no source, in which we find the
(supersymmetric) solitonic $p - 1$ brane. These well-known cases appear to be the only supersymmetric ones. As far as we know all the new solutions are configurations with a fixed source at the origin.

We will do the string case in great detail, keeping the metric general (all possible rescalings with $e^\phi$ included). We then find that one such fixed source solution exhibits the correct scaling behaviour to be the missing unstable type I solution. In the end of the paper we also sketch the case of generic dimensions, without trying to find any further possible interpretation.

2 The solution of ten-dimensional supergravity

Our starting point is the bosonic part of the combined string supergravity action in ten dimensions, here written in arbitrary metric

$$S = \frac{1}{\kappa^2} \int d^{10}x \sqrt{-g}e^{4(a + \frac{1}{3})\phi} \left[ R - 9a \left(a + \frac{2}{3}\right) \partial \phi^2 + \frac{3}{2}e^{-2(a + \frac{4}{3})\phi} H^2 \right]$$

$$-\frac{T_2}{2} \int d^2\xi \left[ \sqrt{-\gamma} \partial_i X^M \partial_j X^N e^{(a + \frac{4}{3})\phi} g_{MN} + 2\varepsilon^{ij} \partial_i X^M \partial_j X^N B_{MN} \right].$$

Here $a = -\frac{1}{3}$ gives us the usual (heterotic) string metric, $a = -\frac{1}{3}$ the Einstein metric, and $a = \frac{2}{3}$ the type I string metric. Choosing $a = 0$ gives us the five-brane metric, and if we also replace $H$ by its dual, $\tilde{H} = e^{2(a - \frac{4}{3})\phi} * H$ we obtain the action postulated to be the low energy effective action for five-brane theory.

For $b = a$ we have a heterotic string source, while $b = a - 2$ and, as it will turn out, no $B_{MN}$ term in the string part of the action will give us a type I source. Since we only consider solutions with the Yang-Mills fields identically zero, we can omit the $tr F^2$ term in the action just like we do with all the fermionic terms.

The variation of (2.1) with respect to $g_{MN}$, $B_{MN}$, $\phi$, $\gamma_{ij}$, and $X^M$, gives us

$$-\frac{1}{\kappa^2} \int d^2\xi \left[ \sqrt{-\gamma} \partial_i X^M \partial_j X^N e^{(a + \frac{4}{3})\phi} \delta_{10}^4(X - x) = 0. \right)$$

See Appendix for notation and conventions.
the antisymmetric tensor equation

\[ -3\partial_M \left( \sqrt{-g}e^{2(a+\frac{2}{3})\phi} H^{MNP} \right) - \kappa^2 T_2 \int d^2 \xi \varepsilon^{ij} \partial_i X^M \partial_j X^N \delta^{10}(X - x) = 0, \tag{2.3} \]

and the dilaton equation

\[
\sqrt{-g}e^{4(a+\frac{1}{3})\phi} \cdot \left[ 4 \left( a + \frac{1}{3} \right) R + 18a \left( a + \frac{2}{3} \right) \left( \Box \phi + 2 \left( a + \frac{1}{3} \right) \partial \phi^2 \right) + 3 \left( a - \frac{2}{3} \right) H^2 \right]
- \frac{\kappa^2 T_2}{2} \int d^2 \xi \sqrt{-\gamma} \gamma^{ij} \partial_i X^M \partial_j X^N e^{(b+\frac{4}{3})\phi} g_{MN} \left( b + \frac{4}{3} \right) \delta^{10}(X - x) = 0. \tag{2.4} \]

For the string degrees of freedom we obtain the equation for the string metric

\[
\frac{T_2}{2} \sqrt{-\gamma} \left[ \partial_i X^M \partial_j X^N e^{(b+\frac{4}{3})\phi} g_{MN} - \frac{1}{2} \gamma^{ij} \gamma_{kl} \partial_k X^M \partial_l X^N e^{(b+\frac{4}{3})\phi} g_{MN} \right] = 0, \tag{2.5} \]

and for the coordinate fields

\[
T_2 \left[ \partial_i \left( \sqrt{-\gamma} \gamma^{ij} \partial_j X^N e^{(b+\frac{4}{3})\phi} g_{MN} \right) + \frac{1}{2} \sqrt{-\gamma} \gamma^{ij} \partial_i X^N \partial_j X^P \partial_M \left( e^{(b+\frac{4}{3})\phi} g_{NP} \right) - 3 \varepsilon^{ij} \partial_i X^N \partial_j X^P H_{MNP} \right] = 0. \tag{2.6} \]

We now follow the standard procedure to find a string soliton. We split up the coordinates \( (M = 0, 1, \ldots 9) \)

\[ x^M = (x^\mu, y^m) \tag{2.7} \]

where \( \mu = 0, 1 \) and \( m = 2, \ldots 9 \), and make the ansatz

\[
ds^2 = e^{2A} \eta_{\mu\nu} dx^\mu dx^\nu - e^{2B} \delta_{mn} dy^m dy^n, \tag{2.8} \]

\[ B_{\mu\nu} = \gamma \varepsilon_{\mu\nu} \sqrt{g} e^C, \tag{2.9} \]

with \( g_2 = -\text{det}g_{\mu\nu} \). All other fields are put equal to zero, and the only coordinate dependence is on \( y = \sqrt{\delta_{mn} y^m y^n} \). We allow a numerical constant \( \gamma \) in the definition of \( B_{\mu\nu} \) to take care of the different normalization compared to \([7]\). The string coordinate \( X^M(\xi) \) is split up in the same way as the coordinates \((2.7)\), and we make the static gauge choice \( X^\mu = \xi^\mu \), and assume \( Y^m = \text{constant} \), put to zero for simplicity.

The \( \gamma_{ij} \) equation now immediately expresses \( \gamma_{ij} \) as a function of the metric and the string coordinates, inserting this, and all the expressions above into the remaining field equations we have

\[
\sqrt{-g}e^{4(a+\frac{4}{3})\phi} g^{\mu\nu} \left[ D^m \partial_m \left( A + 7B + 4 \left( a + \frac{1}{3} \right) \phi \right) + \partial A^2 - 21 \partial B^2 \\
+ 4 \left( a + \frac{1}{3} \right) \partial A \partial \phi + \left( 7a \left( a + \frac{2}{3} \right) + \frac{16}{9} \right) \partial \phi^2 + \gamma^2 e^{-4A-2(a+\frac{4}{3})\phi} \left( \partial e^{2A+C} \right)^2 \right] = \kappa^2 T_2 e^{2A+(b+\frac{4}{3})\phi} g^{\mu\nu} \delta^8(y), \tag{2.10} \]
\[ \sqrt{-g} e^{(a+\frac{4}{3})\phi} g^{mn} g^{\phi} \left[ D_m \partial_n (A + 3B + 2 \left( a + \frac{1}{3} \right) \phi) + \partial_m A \partial_n A + 3 \partial_m B \partial_n B \right. \]
\[- \left( a - \frac{2}{3} \right) \left( a + \frac{4}{3} \right) \partial_m \phi \partial_n \phi - \gamma^2 e^{-4A-2(a+\frac{4}{3})} \partial_m e^{2A+C} \partial_n e^{2A+C} \]
\[- \frac{g_{mn}}{2} \left[ D^p \partial_p (2A + 6B + 4 \left( a + \frac{1}{3} \right) \phi) + 3 \partial A^2 - 15 \partial B^2 + 8 \left( a + \frac{1}{3} \right) \partial A \partial \phi \right. \]
\[+ \left( 7a \left( a + \frac{2}{3} \right) + \frac{16}{9} \right) \partial \phi^2 - \gamma^2 e^{-4A-2(a+\frac{4}{3})} \left( \partial e^{2A+C} \right)^2 \right] = 0, \quad (2.11) \]
\[\gamma e^{2B} \varepsilon^{\mu \nu} \partial^m \left( e^{-2A+6B+2(a-\frac{2}{3})\phi} \partial_m e^{2A+C} \right) = -\kappa^2 T_2 \varepsilon^{\mu \nu} \delta^8 (y), \quad (2.12) \]
\[\sqrt{-g} e^{(a+\frac{4}{3})\phi} \left[ 4 \left( a + \frac{1}{3} \right) D^m \partial_m (2A + 7B) + 18a \left( a + \frac{2}{3} \right) D^m \partial_m \phi \right. \]
\[+ 12 \left( a + \frac{1}{3} \right) \left( \partial A^2 - 7 \partial B^2 \right) + 36a \left( a + \frac{2}{3} \right) \left( \partial A \partial \phi + \left( a + \frac{1}{3} \right) \partial \phi^2 \right) \]
\[- 2 \left( a - \frac{2}{3} \right) \gamma^2 e^{-4A-2(a+\frac{4}{3})} \left( \partial e^{2A+C} \right)^2 \right] \]
\[= \kappa^2 T_2 \left( b + \frac{4}{3} \right) e^{2A+(a+\frac{4}{3})\phi} \delta^8 (y), \quad (2.13) \]
\[\partial_m \left( e^{2A+(b+\frac{4}{3})\phi} \right) = 2 \gamma \partial_m e^{2A+C}. \quad (2.14) \]

Equations (2.10) and (2.12) only give one differential equation each, proportional to \( g^{\mu \nu} \) and \( \varepsilon^{\mu \nu} \), respectively, while (2.11) turns out to have two independent components. The easiest way to find them is to go to spherical coordinates, \( y, \theta_1, \theta_2 \ldots \theta_7 \). Then the \( yy \)-component will yield one equation, and the \( \theta_i \theta_j \)-components will yield one equation proportional to \( g_{\theta_i \theta_j} \).

The usual way to solve these equations is to require some unbroken supersymmetry, and therefore request that the variations of the spinor fields vanish for some nonzero supersymmetry parameter. If we do this we will find one unique solution (modulo some integration constants without physical significance). The solution found in this way also turns out to satisfy the field equations. This is very often the case since Einsteins equation, the dilaton equation and the Yang-Mills equation can be obtained as supersymmetry transformations of the fermionic equations of motion, which are already identically satisfied in our ansatz. But these equations, as well as the remaining ones, still need to be checked explicitly since the solution only keeps half of the supersymmetry.

We now want to show that the equations of motion have a more general solution. It turns out to be useful to make the field redefinitions
\[ X = 2A + 6B + 4 \left( a + \frac{1}{3} \right) \phi, \quad (2.15) \]
\[ Y = 2A + \left( a - \frac{8}{3} \right) \phi, \quad (2.16) \]
\[ Z = 2A + \left( a + \frac{4}{3} \right) \phi. \quad (2.17) \]

The equations of motion can then be written

\[
e^{X-2A} \left[ \frac{7}{6} (\nabla^2 X + \frac{1}{2} X'^2) - \frac{1}{6} (\nabla^2 Y + X'Y' - \frac{1}{2} Y'^2) \right. \\
- \frac{1}{2} Z'^2 \left] + \gamma^2 e^{-2Z} \partial \left( e^{2A+C} \right)^2 \right) = -\kappa^2 T_2 e^{-2A+(b-a)\phi} \delta^8(y), \quad (2.18) \]

\[
e^{X-2B} \left[ X'' + \frac{13X'}{y} + X'^2 \right] = 0, \quad (2.19) \]

\[
e^{X-2B} \left[ 7 \left( \frac{X''}{12} - \frac{1}{12} X'^2 \right) + \frac{13}{12} Y'^2 + \frac{13}{4} Z'^2 - 13\gamma^2 e^{-2Z} \left( \partial e^{2A+C} \right)^2 \right] = 0, \quad (2.20) \]

\[
\gamma \frac{1}{y'} \partial \left( y^7 e^{X-2Z} \partial e^{2A+C} \right) = \kappa^2 T_2 \delta^8(y), \quad (2.21) \]

\[
e^X \left[ \frac{14}{3} \left( a + \frac{1}{3} \right) \left( \nabla^2 X + \frac{1}{2} X'^2 \right) - \frac{1}{6} \left( a - \frac{8}{3} \right) \left( \nabla^2 Y + X'Y' \right) + \frac{1}{3} \left( a + \frac{1}{3} \right) Y'^2 \right. \\
- \frac{1}{2} \left( a + \frac{4}{3} \right) \left( \nabla^2 Z + X'Z' \right) + \left( a + \frac{1}{3} \right) Z'^2 \right. \\
- 2\gamma^2 \left( a - \frac{2}{3} \right) e^{-2Z} \left( \partial e^{2A+C} \right)^2 \left] = -\kappa^2 T_2 \left( b + \frac{4}{3} \right) e^{Z+(b-a)\phi} \delta^8(y), \quad (2.22) \right.

\[
\partial e^{Z+(b-a)\phi} = 2\gamma \partial e^{2A+C}. \quad (2.23) \]

All derivatives are now with respect to \( y \) only, with \( \nabla^2 F = \frac{1}{y'} \partial (y^7 F') \) and \( \partial F = F' = \frac{\partial}{\partial y} \).

Our strategy is now to solve all equations for \( y > 0 \), and only afterwards investigate the singularity structure at \( y = 0 \) to find a consistent choice for the values of some of the integration constants. We start by solving the supergravity equations alone. Equation \( 2.14 \) can be immediately solved

\[ e^X = e^{X_0} + \frac{K}{y^{12}}. \quad (2.24) \]

In the following, a subscript zero will always denote a constant which is equal to the value of the function in question at infinity, and \( K, L \ldots \) are integration constants. One of the conditions for a supersymmetric solution (see below) is \( X' = 0 \), so we immediately see that our nontrivial candidate to a generalization cannot preserve supersymmetry. Now \( 2.21 \) can be integrated once yielding

\[ e^{X-2Z} \partial e^{2A+C} = \frac{L}{y^8}, \quad (2.25) \]
and choosing $\gamma = \frac{1}{2}$ (cf (2.23) and Appendix) the remaining three equations are equivalent to

$$\nabla^2 Y + X'Y'' = 0,$$  \hspace{1cm} (2.26)

$$\nabla^2 Z + X'Z' - e^{2Z-2X} \frac{L^2}{y^{14}} = 0,$$  \hspace{1cm} (2.27)

$$7 \cdot 12 Ke^{X_0-2X} - \frac{1}{4} \frac{L^2 e^{2Z-2X}}{y^{14}} + \frac{1}{12} Y'^2 + \frac{1}{4} Z'^2 = 0.$$  \hspace{1cm} (2.28)

The first of these equations immediately gives

$$e^{X Y'} = \frac{M}{y^7}.$$  \hspace{1cm} (2.29)

This can be integrated once more, but the result depends on the sign of $K$, so we will put it off till this sign is known. Instead we first solve (2.28) for $Z'$

$$Z' = \left( L^2 e^{2Z} - \frac{M^2}{3} - 7 \cdot 48 Ke^{X_0} \right)^{1/2} = \pm \frac{e^{-X}}{y^7}.$$  \hspace{1cm} (2.30)

Equation (2.27) is then identically satisfied provided $Z' \neq 0$, and for $Z' = 0$ it gives us $L = 0$. We can also directly integrate, (2.30) and afterwards solve for $e^{-Z}$ in terms of $y$, for all the values of the integration constants for which the square root is real. Again we will not do so explicitly, since the functional form of the result depends on the signs of the constants. Combining (2.30) and (2.24) we get

$$\partial e^{2A+C} = \pm \frac{LZ' e^{2Z}}{\left( L^2 e^{2Z} - \frac{M^2}{3} - 7 \cdot 48 Ke^{X_0} \right)^{1/2}}.$$  \hspace{1cm} (2.31)

so

$$e^{2A+C} = \pm \frac{\left( L^2 e^{2Z} - \frac{M^2}{3} - 7 \cdot 48 Ke^{X_0} \right)^{1/2}}{L} + \text{constant}.$$  \hspace{1cm} (2.32)

This completes the solution of the supergravity equations, except for the case $L = 0$ which will be considered later. Depending on the values of $K$, $L$ and $M$, these equations have singularities at $y = 0$, and we must therefore add a source term. The string source we have so far neglected gives us equation (2.23) which is integrated

$$e^{2A+C} = e^{Z+(b-a)\phi} + \text{constant}.$$  \hspace{1cm} (2.33)

The precise value of the constant is not very interesting since it does not affect $H_{muv}$. We then have

$$e^{Z+\frac{b-a}{1}(Z-Y)} = \pm \frac{\left( L^2 e^{2Z} - \frac{M^2}{3} - 7 \cdot 48 Ke^{X_0} \right)^{1/2}}{L} + \text{constant}.$$  \hspace{1cm} (2.34)

This can be achieved in two different ways:  

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3We have not fully proven that there are no further solutions, but it seems unlikely.
The square root proportional to $e^Z$

$$(b - a)(Z - Y) = 0$$

$$M^2 + 7 \cdot 144Ke^{X_0} = 0$$

(2.35)

with the solutions

$$b = a$$

(2.36)

$$K = -\frac{M^2e^{-X_0}}{7 \cdot 144}$$

(2.37)

or

$$Z' = Y'$$

(2.38)

The square root constant

$$Z = \text{constant}$$

$$Z + \frac{(b - a)(Z - Y)}{4} = \text{constant}$$

Both $Z' = Y'$ and $Z' = 0$ have as a consequence $L = 0$. This case will be treated in Section 4.

### 3 The heterotic string solution

In this section, we will study the solution for $L \neq 0$ given in Section 2, and show that this is the one corresponding to the elementary string solution of [6]. That it is indeed a heterotic string was shown in [4, 5], where the zero modes are fully analyzed.

We insert (2.36) and (2.37) in our equations and find

$$Y' = Me^{-X_0} \frac{y^5}{y^{12} - \frac{M^2e^{-2X_0}}{7 \cdot 144}}$$

(3.1)

and

$$\partial e^{-Z} = -Le^{-X_0} \frac{y^5}{y^{12} - \frac{M^2e^{-2X_0}}{7 \cdot 144}}.$$  

(3.2)

This can now be integrated to

$$Y = \begin{cases} 
Y_0 + \frac{\sqrt{M}}{|M|} \log \left| \frac{y^6 - \frac{|M|e^{-X_0}}{y^{12}}}{y^{6} - \frac{|M|e^{-X_0}}{y^{12}}} \right| & M \neq 0 \\
Y_0 & M = 0 
\end{cases}$$

(3.3)
and
\[
e^{-Z} = \begin{cases} 
    e^{-Z_0} - \frac{\sqrt{7}f}{|M|} \log \left| \frac{y^6 - |M|e^{-X_0}}{\sqrt{7}12} \right| & M \neq 0 \\
    e^{-Z_0} + \frac{Le^{-X_0}}{6y^6} & M = 0
\end{cases}.
\]

(3.4)

A study of the singularities at \( y = 0 \) yields
\[
e^X Y' = M\Omega_7 f'
\]
(3.5)
\[
e^X \partial e^{-Z} = -L\Omega_7 f'
\]
(3.6)

where \( \nabla^2 f = \delta^8(y) \), and \( \Omega_7 \) is the volume of the seven-sphere. The nonvanishing parts of the equations of motion at \( y = 0 \) are then
\[
e^{-2A} \left[ -\frac{1}{6} M - \frac{1}{2} Le^Z \right] \Omega_7 \delta^8(y) = -\kappa^2 T_2 e^{Z-2A} \delta^8(y),
\]
(3.7)
\[
L\Omega_7 \delta^8(y) = 2\kappa^2 T_2 \delta^8(y),
\]
(3.8)
\[
\left[ -\frac{1}{6} \left( a - \frac{8}{3} \right) M - \frac{1}{2} \left( a + \frac{4}{3} \right) Le^Z \right] \Omega_7 \delta^8(y) = -\kappa^2 T_2 \left( a + \frac{4}{3} \right) e^Z \delta^8(y).
\]
(3.9)

So the constants must take the values
\[
L = \frac{2\kappa^2 T_2}{\Omega_7},
\]
(3.10)
\[
M = 0.
\]
(3.11)

Choosing \( M = 0 \) we get
\[
e^{Z(0)} = 0,
\]
(3.12)
\[
e^{Z(0)-2A(0)} = e^{\frac{1}{3}(a+\frac{4}{3})(Z(0)-Y(0))}.
\]
(3.13)

Here \( Y(0) = Y_0 \), so the factor multiplying the \( \delta \)-function in (3.7) is divergent for \( a + \frac{4}{3} < 0 \), and zero for \( a + \frac{4}{3} > 0 \). This is just a curiosity here, but will be important for the solitonic string solution.

The remaining integration constants are the values of the fields at infinity, \( X_0, Y_0 \) and \( Z_0 \), which can be rewritten in terms of \( A_0, B_0 \) and \( \phi_0 \). The first two of these have no physical meaning. They can be removed by constant rescaling of the coordinates, so we are left with only \( \phi_0 \), which is the vacuum expectation value of the dilaton field. This is exactly the standard elementary string solution of [6, 7], written for general \( a \), that is, with all possible rescalings of the metric explicitly given. In order to compare directly One should let \( a = -\frac{1}{3} \), and solve \( X \) and \( Y \) for \( A \) and \( B \) in terms of \( \phi \). We have a preserved Nöther charge
\[
e = \frac{6}{\sqrt{2\kappa}} \int e^{2(a+\frac{4}{3})\phi} * H = \sqrt{2\kappa} T_2.
\]
(3.14)

\(^4e^{Z(0)} \neq 0 \) for \( M \neq 0 \)
The factor 6 is put in to have the same definition as [7] using our normalization of $H$. The mass per unit string length is (assuming $g_{MN} \to e^{2A_0}g_{MN}$ for $y \to \infty$, and keeping $A_0 \neq 0$ as an extra check on the calculation.)

$$M_2 = -\frac{1}{2\kappa^2} e^{6A_0} \int d^8y e^{4\left(a + \frac{1}{3}\right)} \nabla^2 \left(e^{2(A - A_0)} + 7e^{2(B - A_0)}\right)$$

$$= 2 \left(a + \frac{5}{6}\right) T_2 e^{(a + \frac{4}{3})\phi_0} \quad (3.15)$$

Here we can see explicitly that $M_2$ scales with $g_{MN}$ as predicted in [8]. The fact that $M_2$ is negative in, for instance, the heterotic string metric, $a = -\frac{4}{3}$, should be no cause for worry as long as it can be interpreted as a (positive) energy density in the Einstein metric, $a = -\frac{1}{3}$.

The conditions that the solution preserve half the supersymmetry can be obtained just like in the papers quoted above. In our notation they are

$$3 \left(A' + \frac{a}{2}\phi'\right) = e^{-Z} \partial e^{2A+C} \quad (3.16)$$

$$6 \left(B' + \frac{a}{2}\phi'\right) = -e^{-Z} \partial e^{2A+C} \quad (3.17)$$

$$4\phi' = e^{-Z} \partial e^{2A+C} \quad (3.18)$$

which can be rewritten as

$$X' = 0 \quad (3.19)$$

$$Y' = 0 \quad (3.20)$$

$$\partial e^Z = \partial e^{2A+C} \quad (3.21)$$

These equations are all satisfied here as we know they should be.

The solution we have found can also be interpreted as a solitonic string solution of the dual version of ten-dimensional supergravity, nowadays usually interpreted as the five-brane theory [9]. We replace $H$ by $\tilde{H} = e^{2(a - \frac{4}{3})\phi} * H$. All supergravity equations remain unchanged, but the $H$-equation of motion is now interpreted as the Bianchi identity for $\tilde{H}$, and the Bianchi identity for $H$ becomes the equation of motion for the new field strength. This equation has no singularity at $y = 0$, and we have no source term. The constant $L$ is then no longer fixed by the equations at $y = 0$.

It gives us the conserved charge

$$\mu_2 = \frac{L\Omega_7}{\sqrt{2\kappa}} \quad (3.22)$$

which is now interpreted as a magnetic charge. However, the solution still has to satisfy (3.7) and (3.9) with zero on the rhs. For $M \neq 0$ the exponentials have finite values at $y = 0$, and we can only get the solution $L = M = 0$ in

5It still has to fulfill a Dirac quantization condition of the form $e_6\mu_2 = 2\pi n$, see [7].
contradiction with the assumption. We then have to choose $M = 0$. Equations 3.12 and 3.13 then tell us that we can have a non-zero $L$, and hence a non-trivial solution only for

$$a + \frac{4}{3} > 0. \quad (3.23)$$

This is the standard solitonic solution again satisfying the conditions for unbroken supersymmetry (3.19)-(3.21). In principle there is no reason to require (2.23) in this case. If we then study the singularity structure at $y = 0$ we find that we must still have $M = 0$, but that the other condition is generalized to

$$\left( L^2 e^{2Z(0)} - \frac{M^2}{9} - 7 \cdot 48K e^{X_0} \right)^{1/2} = 0. \quad \text{If we solve for } e^{-Z}, \text{ and require the solution to exist and be well-behaved on the full interval } 0 \leq y < \infty \text{ the condition can not be satisfied for } K \neq 0.$$

Notice that while the solitonic string solution exists in the type I metric, in the Einstein metric, and in the five-brane metric, there is no solution with $L \neq 0$ in the heterotic string metric, $a = -\frac{4}{3}$. This is probably an indication that the heterotic string is a fundamental state of “heterotic” supergravity, and not a solitonic one.

The supersymmetric solution to the supergravity equations has an extension to $\alpha' \neq 0$. If we use the framework of anomaly free supergravity, given explicitly in [13], where the Green-Schwarz anomaly cancellation term is supersymmetrized in a consistent fashion, we find an exact solution to the equations of motion reducing to the one above for $\alpha' = 0$ [10]. Assuming that there exists an action (probably containing infinitely many terms) for these equations of motion, we can ask ourselves what source terms we must have at $y = 0$, just like we did for the elementary string solution above. We will find that also the string action has to be modified with higher order terms, which might possibly be removable by an $\alpha'$-dependent field redefinition. In [3] it is argued that this elementary heterotic string is really a D-string in strongly coupled type I theory, since their world sheet structures are the same. This will then give us the exact conformal field theory construction corresponding to the full solution.

4 The type I string solution

The second possible way of finding a solution requires $L = 0$. In this case the lhs of equation (2.21) has no singularity at $y = 0$ and hence we cannot have a source at the rhs. We then have to redo the analysis putting $\partial e^{2A+C} = 0$ in (2.18), (2.20), (2.22) and (2.23), and removing the rhs of (2.21). In this case we will not find an explicit solution for $b$, but if we put $b = a - 2$ by hand, we find that the solution can be interpreted as an unstable type I configuration, which should be there if there is indeed a strong-weak coupling duality between ten dimensional heterotic and type I strings.
We first solve equations 2.18-2.22 for \( y > 0 \). Equation 2.19 gives the same solution for \( e^X \), and the remaining equations are now

\[
\nabla^2 Y + X'Y' = 0, \quad (4.1)
\]

\[
\nabla^2 Z + X'Z' = 0, \quad (4.2)
\]

\[
\partial e^{2A+C} = 0, \quad (4.3)
\]

\[
7 \cdot 12 \frac{Ke^{X_0-2X}}{y^{14}} + \frac{Y'^2}{12} + \frac{Z'^2}{4} = 0, \quad (4.4)
\]

with the solutions (for \( K \) different from zero)

\[
Y = Y_0 + \frac{Me^{-X_0/2}}{12\sqrt{-K}} \log \left| \frac{y^6 - \sqrt{-K}e^{-X_0/2}}{y^6 + \sqrt{-K}e^{-X_0/2}} \right|, \quad (4.5)
\]

\[
Z = Z_0 + \frac{Ne^{-X_0/2}}{12\sqrt{-K}} \log \left| \frac{y^6 - \sqrt{-K}e^{-X_0/2}}{y^6 + \sqrt{-K}e^{-X_0/2}} \right|, \quad (4.6)
\]

\[
K = -\frac{M^2 + 3N^2}{7 \cdot 144} e^{-X_0}. \quad (4.7)
\]

At \( y = 0 \) we now have

\[
\left[ -\frac{1}{6} M - \frac{1}{2} N \right] \Omega_7 \delta^8(y) = -\kappa^2 T_2 e^{Z-2A+(b-a)\phi} \delta^8(y), \quad (4.8)
\]

\[
\left[ -\frac{1}{6} \left( a - \frac{8}{3} \right) M - \frac{1}{2} \left( a + \frac{4}{3} \right) N \right] \Omega_7 \delta^8(y) = -\kappa^2 T_2 \left( b + \frac{4}{3} \right) e^{Z+(b-a)\phi} \delta^8(y). \quad (4.9)
\]

These equations have the solutions

\[
M = -\frac{3 \kappa^2 T_2}{2 \Omega_7} (b - a) e^{(Z+(b-a)\phi)(0)}, \quad (4.10)
\]

\[
N = \frac{1 \kappa^2 T_2}{2 \Omega_7} (b - a + 4) e^{(Z+(b-a)\phi)(0)}. \quad (4.11)
\]

This is in contradiction with (2.23), which, in this case, requires that \((b - a + 4)^2 + 3(b - a)^2 = 0\). However, equation 2.23 is the equation of motion for the string source. It can be interpreted as a no-force condition, stating that the graviton contribution to the force between two parallell source strings is cancelled by the “axion” contribution. In the present case we have no axion contribution (to the lowest order in \( \alpha' \) at least). Furthermore, this solution does not preserve supersymmetry, nor does it have a conserved Noether charge, so there is no reason to expect it to be stable. It is hence just the configuration around a source term corresponding to an infinitely long string put in by hand. There is no reason to expect that such an unstable test source should satisfy dynamical equations of
motion, or that there should be no force between two parallel unstable strings, so we just drop equation (2.23).

The reason we are still interested in this solution is that we are looking for just such a thing as the missing type I "soliton", [8]. The mass per unit string length for our solution is

$$M_2 = \frac{T_2}{2} \left[ 3 \left( a + \frac{1}{6} \right) (b - a) + \left( a + \frac{5}{6} \right) (b - a + 4) \right] e^{(b+\frac{4}{3})\phi_0}$$

(4.12)

If we choose $$b = a - 2$$, corresponding to a type I string source we obtain

$$M_2 = -2 \left( a - \frac{1}{2} \right) e^{(a-\frac{2}{3})\phi_0}.$$  

(4.13)

This is indeed the correct scaling behaviour for a type I soliton, since $$a = \frac{2}{3}$$ in the type I metric. The fact that there is no $$B_{MN}$$ term in the source is also consistent with the type I interpretation since $$B_{MN}$$ is here a Ramond-Ramond state corresponding to a sigma model term $$\gamma^{MNP} H_{MNP}$$ sandwiched between two spin fields, see [11], and such a term should vanish in our ansatz.

Can we also here reinterpret the solution as a "solitonic" type I string, that is a solution where we have exchanged the role of the equation of motion and the Bianchi identity for $$H$$, see Section 2, and with no sources at $$y = 0$$? We find

$$\tilde{H}_{\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6} \sim \sqrt{h} y^7 e^{X-Z} e^{2A+C}$$

(4.14)

with $$h_{\theta_i,\theta_j}$$ the metric on $$S^7$$ and $$h = \det h_{\theta_i,\theta_j}$$. The solution of the Bianchi identity now requires that $$\tilde{H}$$ be independent of $$y$$, and we are back at (2.25). With $$L = 0$$ we then have exactly the same equations as we already studied above, with the only difference that (4.9) and (4.10) have zero on the rhs. The only solution is then $$M = N = 0$$, so, as should have been expected, there is no interpretation as a solitonic string in this case.

As we already mentioned it has also been shown that the zero modes for the elementary string case indeed represent the degrees of freedom of a heterotic string [4, 5]. That analysis certainly does not work for the case of the non-supersymmetric solutions of the supergravity equations, since we have twice the wanted number of fermionic zero-modes.

Since, by our analysis, nonsupersymmetric, unstable solutions also exist around non-dynamical heterotic string sources put in by hand, it feels very tempting to believe that our solution to the supergravity equations should also contain an exact type I string solution (Even without invoking the duality arguments.) What we need is a non-vanishing field strength, providing a conserved charge, and a source term able to counteract the gravitational forces between two string sources. As I can see it there are two possibilities. The immediate one is that we have to employ a standard $$B_{MN}$$ term in the string action to higher orders in $$\alpha'$$, so that the good solution, which can then probably only be found if we solve the
equations to all orders in \( \alpha' \), vanishes, or becomes singular, for \( \alpha' \to 0 \). It might be difficult to find such a solution, which preserves some of the supersymmetry, however, since we must satisfy (the generalization of) (3.21). The second, perhaps more amusing possibility is that the Yang-Mills field will provide us with what we need. There are obvious difficulties with this possibility too. Solitons with Yang-Mills charge are 0-branes, not strings, and they also turn out to be non-supersymmetric [12]. If the 0-branes could somehow be interpreted as type I strings of zero length, one part of the problem would be solved, but we would still need to find the unbroken supersymmetry. (And show that the zero-modes of the solution provides us with the degrees of freedom for a type I string).

Our general solution also permits \( b \) to take any value, as long as the string is not dynamical. It is hard to think of any further physical interpretation of any of these solutions. The interpretation is probably just that one can find an exact supergravity configuration around many unusual sources put in by hand, while if you want an exact solution also to the string equations our equations forced us to put \( b = a \). We then expect that a stable type I solution should force us to choose \( b = a - 2 \).

\section{5 The general case}

Our general (non-supersymmetric?) solution actually exists for generic dimensions. We will briefly give this solution here without any attempts at interpretation.

We start with the action of a rank \( d \) antisymmetric tensor potential interacting with gravity and the dilaton in \( D \) space-time dimensions, and couple it to a \( p = d \) \( p - 1 \)-brane. The procedure is exactly analogous to the \( D = 10, d = 2 \) case we already did in great detail. Here we choose the Einstein metric so we use the same action (and equations of motion) as in [7] (except for the minor redefinitions of the fields, see Appendix). Splitting up the coordinates in \( D = d + (\tilde{d} + 2) \), and making the usual ansatz the equations corresponding to (2.18)-(2.23) are now

\begin{equation}
\begin{aligned}
e^{-2A} \left[ \frac{d+1}{d} (\nabla^2 X + \frac{1}{2} X'^2) - \frac{a^2(d+d)}{Ndd} (\nabla^2 Y + X'Y'' - \frac{1}{2} Y'^2) - \frac{2}{N} (\nabla^2 Z + X'Z' - \frac{1}{2} Z'^2) + \frac{1}{4} e^{-2Z} \partial \left( e^{2A+C} \right)^2 \right] = -\kappa^2 T_d e^{Z-2A+(b-a)\phi} \delta^{d+2}(y),
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
e^{-2B} \left[ \frac{X'' + (2\tilde{d} + 1)X'}{y} + X'^2 \right] = 0,
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
e^{-2B} \left[ (\tilde{d} + 1) \left( X'' - \frac{1}{2\tilde{d}} X'^2 \right) + (2\tilde{d} + 1) \left[ \frac{a^2(d+d)}{2Ndd} Y'^2 + \frac{1}{N} Z'^2 - \frac{1}{4} e^{-2Z} \left( \partial e^{2A+C} \right)^2 \right] \right] = 0,
\end{aligned}
\end{equation}

13
\[
\frac{1}{y^{d+1}} \partial \left( y^{d+1} e^{x-2z} \partial e^{2A+C} \right) = \kappa^2 T_d \delta^{d+2}(y), \tag{5.4}
\]

\[
e^x \left[ -\frac{2a}{N} (\nabla^2 Y + X'Y') + \frac{2a}{N} (\nabla^2 Z + X'Z') - \frac{a}{2} e^{-2z} \left( \partial e^{2A+C} \right)^2 \right]
= -\kappa^2 T_d e^Z (b-a) \phi \delta^{d+2}(y), \tag{5.5}
\]

\[
\partial e^{Z+(b-a)\phi} = \partial e^{2A+C}. \tag{5.6}
\]

Here
\[
X = dA + \tilde{d}B, \tag{5.7}
\]

\[
Y = dA - \frac{2\tilde{d}}{a(d+\tilde{d})} \phi, \tag{5.8}
\]

\[
Z = dA + a\phi \tag{5.9}
\]

and \( N = a^2 + \frac{2\tilde{d}}{d+\tilde{d}} \). The constants \( a \) and \( b \) here are really regarded as unknown, to be solved for. They have nothing to do with rescaling the metric since the equations are all written in the Einstein metric, but instead determine how the \( d \)-form potential couples to the dilaton.

We will now do the case where the source term does satisfy dynamical equations, to show it is exactly analogous to Section 2. Again we first study the supergravity part for \( y \neq 0 \) so we can put the rhs to zero in \((5.1),(5.4)\) and \((5.5)\). First we solve equation \((5.2)\) obtaining
\[
e^x = e^{x_0} + \frac{K}{y^{2\tilde{d}}}, \tag{5.10}
\]

We integrate \((5.4)\) once to get
\[
e^{x-2z} \partial e^{2A+C} = \frac{L}{y^{d+1}}. \tag{5.11}
\]

In exact analogy to the string case the remaining three equations then turn out to be equivalent to
\[
\nabla^2 (aY) + X'(aY') = 0, \tag{5.12}
\]

\[
\nabla^2 Z + X'Z' - \frac{N}{4} e^{2Z-2x} \frac{L^2}{y^{2(d+1)}} = 0, \tag{5.13}
\]

\[
2\tilde{d}(\tilde{d} + 1) \frac{K e^{X_0-2X}}{y^{2(d+1)}} - \frac{1}{4} \frac{L^2 e^{2Z-2x}}{y^{2(d+1)}} + \frac{(d+\tilde{d})}{2ddN} (aY')^2 + \frac{1}{N} Z'^2 = 0, \tag{5.14}
\]

where we must use \( aY \) instead of \( Y \) to have a field which is regular also for \( a = 0 \). The solution to equation \((5.12)\)
\[
e^x (aY') = \frac{M}{y^{d+1}}, \tag{5.15}
\]
can be inserted into (5.14) which is then solved for $Z'$. Just like in the string case this $Z'$ identically satisfies (5.13). We find

$$Z' = \left( e^{2Z} \frac{NL^2}{4} - \frac{(d+\bar{d})M^2}{2dd} - 2Nd(\bar{d} + 1)Ke^{X_0} \right)^{1/2} = \pm \frac{e^{-X}}{y^{d+1}}$$

(5.16)

Both (5.15) and (5.16) can be integrated, and $Y$ and $e^{-Z}$ expressed in terms of $Y$. Combining (5.16) with (5.11) we find

$$e^{2A+C} = \pm \frac{4}{NL} \left( e^{2Z} \frac{NL^2}{4} - \frac{(d + \bar{d})M^2}{2dd} - 2Nd(\bar{d} + 1)Ke^{X_0} \right)^{1/2} + \text{constant}.$$  

(5.17)

We will now only study the solution to this equation consistent with the full rhs. Integrating (5.6) gives

$$e^{2A+C} = e^{Z+(b-a)\phi} + \text{constant.}$$  

(5.18)

We must then choose

$$b = a,$$  

(5.19)

$$K = -\frac{(d + \bar{d})M^2e^{-X_0}}{4Nd\bar{d}(\bar{d} + 1)}$$  

(5.20)

$$N = 4$$  

(5.21)

Then we have

$$e^{-Z} = \begin{cases} 
  e^{Z_0} - \sqrt{\frac{d(d+1)}{\bar{d}+d}} \frac{2L}{|M|} \log \left| y^d + \sqrt{\frac{d+\bar{d}}{d(d+1)}} \frac{|M|e^{-X_0}}{4d} \right| & M \neq 0 \\
  e^{-Z_0} + \frac{Le^{-X_0}}{dy^d} & M = 0
\end{cases}.$$  

(5.22)

If we now continue and analyze the equations at $y = 0$ we will again find that we have to put in a $p - 1$-brane source term at the origin getting

$$L = \frac{2\kappa^2 T_d}{\Omega_{d+1}},$$  

(5.23)

$$K = M = 0,$$  

(5.24)

which is the elementary solution found in [7]. Alternatively we can again examine the dual solution and again find the result of [7].

Also in the general case we have to study $L = 0$ separately. The analysis of this case closely follows Section 4, only yielding unstable solutions. We will not show these solutions explicitly here since the only interesting interpretation we know of is the type I string solution already considered.
6 Conclusions

We have found a general solution to the supergravity equations with the standard ansatz for all fields. A specific choice of integration constants gives us the known supersymmetric elementary and solitonic $p-1$-branes. These are the only solutions with a dynamical $p-1$-brane source, and with no source term. We also show that there is a string solution with a fixed source giving us a probably unstable type I configuration.

Making other choices for the integration constants, will give us other solutions, were the singularities at $y = 0$ can be accounted for with a source term with an arbitrary relative constant between the $g_{MN}$-term and the $B_{MN}$-term, as well as an arbitrary $b$. There is still a possibility that some other such choice will give us a solution with a reasonable physical interpretation.

It would now be very interesting to see if we could somehow relax some of the assumptions here to find a stable, or at least less unstable type I solution. What we really would like is of course a solution where the zero-modes match the degrees of freedom of the type I string. Only then can we feel really convinced that what we have found is indeed the type I state in heterotic string theory. We discussed some tentative ideas about how to achieve this at the end of Section 4, but, in view of the discussion in [8], it might, unfortunately, turn out to be impossible to find such a state at all.

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Appendix

We use the conventions of [13] for the fields, and of [7] for the coupling constants with, for instance

\[
\{\gamma_M, \gamma_N\} = 2g_{MN},
\]

\[
g_{MN} = (+, -, - \ldots),
\]

(A.1)

and for the string case, Section 2–Section 4, our action and equations of motion are the ones from these papers (with \(\gamma_1 \sim \alpha' = 0\), and \(R = R(\omega)\), and with \(g_{MN}\) rescaled to \(e^{\alpha}g_{MN}\). These conventions are the ones which came out naturally in the superspace derivation of [13], and will later make it easier to find solutions consistent to all orders in \(\alpha'\). In order to compare to, for instance, [7] we put \(a = \frac{1}{3}\) to obtain the Einstein metric, and find

\[
R = \frac{1}{2} \tilde{R}
\]

(A.2)

\[
g_{MN} = -\tilde{g}_{MN}
\]

(A.3)

\[
B_{MN} = \frac{1}{2} \tilde{B}_{MN}
\]

(A.4)

\[
H_{MNP} = \frac{1}{6} \tilde{H}_{MNP}
\]

(A.5)

\[
\phi = \frac{1}{2} \tilde{\phi}
\]

(A.6)

\[
2A + C = \tilde{C}
\]

(A.7)

where the fields with a tilde are the ones of [7]. In the generic case we use the Einstein metric only, so here we are much closer to this report, only using the generalizations of (A.2)-(A.7), as our fields.

Finally, we give some explicit expressions using the ansatz (2.7)-(2.9).

\[
R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} [D^m \partial_m A + 2 \partial^m A \partial_m A]
\]

(A.8)

\[
R_{mn} = D_m \partial_n A + \partial_m A \partial_n A + 3D_m \partial_n B + \frac{1}{2} g_{mn} D^p \partial_p B + 3 \partial_m B \partial_n B - 3 g_{mn} \partial^p B \partial_p B
\]

(A.9)

where

\[
D_m \partial_n F = \partial_m \partial_n F - \Gamma^p_{mn} \partial_p F = \partial_m \partial_n F - \partial_m B \partial_n F - \partial_n B \partial_m F + g_{mn} \partial_p B \partial_p F
\]

(A.10)

\[
D_{\mu} \partial_{\nu} F = -\Gamma^m_{\mu\nu} \partial_m F = g_{\mu\nu} \partial^m A \partial_m F
\]

(A.11)

The generalizations to generic dimension can easily be found from the expressions given in [7].
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