High-contrast random composites: homogenisation framework and new spectral phenomena

Mikhail Cherdantsev¹, Kirill Cherednichenko², and Igor Velčić³

¹School of Mathematics, Cardiff University, Senghennydd Road, Cardiff, CF24 4AG, United Kingdom
²Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, United Kingdom
³Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, 10000 Zagreb, Croatia

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Abstract

We study the homogenisation problem for elliptic high-contrast operators \( A_\varepsilon \) whose coefficients degenerate as \( \varepsilon \to 0 \) on a set of randomly distributed inclusions. In our earlier paper [Stochastic homogenisation of high-contrast media. Applicable Analysis (2019)] we proved the Hausdorff convergence of the spectra of \( A_\varepsilon \) to the spectrum of a two-scale limit homogenised operator \( A_{\text{hom}} \) in the bounded domain setting and provided a partial analysis of \( A_{\text{hom}} \) and its spectrum. In the present work we offer a comprehensive study of their properties. Our main focus, however, is on the spectra of \( A_\varepsilon \) in the whole space setting, when their structure is significantly different to the case of a bounded domain. We show that for the whole space the limit of \( \text{Sp}(A_\varepsilon) \) is, in general, strictly larger than \( \text{Sp}(A_{\text{hom}}) \) and illustrate how this effect is connected with the stochastic nature of the operators in question. Under an additional assumption of finite range correlation of the random inclusions, we are able to characterise the limit \( \lim_{\varepsilon \to 0} \text{Sp}(A_\varepsilon) \) via a stochastic non-local analogue of Zhikov’s \( \beta \)-function. Furthermore, we introduce the notion of a statistically relevant (limiting) spectrum and develop a qualitative and quantitative tool for describing the part \( \text{Sp}(A_\varepsilon) \) that converges to \( \text{Sp}(A_{\text{hom}}) \).

Keywords: High contrast · Random media · Stochastic homogenisation · Spectrum

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1 Introduction

A recent drive towards the development of mathematical tools for understanding the behaviour of realistic inhomogeneous media has led to a renewal of interest in stochastic homogenisation and a similarly explosive activity in the analysis of composite media, in particular, of those with contrasting
material components. In those fields the new step change is characterised by moving from qualitative analysis to quantitative results: from the general multiscale frameworks of De Giorgi and Spagnolo [DS73], Bakhvalov [Bak75], Murat–Tartar [T77], MT98, Allaire [A92], Kamotski–Smyshlyaev KS19 (in the periodic setting), Yurinskii Yu78, Papanicolaou–Varadhan [PV79], Kozlov [K79], Zhikov–Piatnitski ZP06 (in the stochastic setting), Zhikov [Zi00, Zi04] (in the high-contrast periodic setting) to error bounds in appropriate topologies Griso [G04], Zhikov–Pastukhova ZP05, Birman–Suslina BS06, Kenig–Lin–Shen KLS12 (periodic), Gloria–Otto GO12, Armstrong–Smart AS16, Armstrong–Kuusi–Mourrat AKM19 (stochastic), Cherednichenko–Cooper CCT16, Cherednichenko–Ershova–Kiselev CEK20 (high-contrast periodic).

The high-contrast setting occupies a special place in homogenisation theory, for it enables resonant phenomena leading to micro-to-macro scale interactions, thereby bringing about new, often not naturally occurring, material properties. While the periodic high-contrast setting has now been analysed in good detail, its stochastic counterpart, which can be argued to be even more relevant to applications in material science, has not yet been addressed (apart from our tentative study [CCV18] in the bounded domain setting). The goal of the present paper is to remedy this by providing a general framework and a comprehensive toolbox for multiscale analysis of random high-contrast media, which lays a foundation for the related research avenue.

We study the problem of homogenisation of operators of the form $A^\varepsilon = -\nabla \cdot A^\varepsilon \nabla$ with high-contrast random (stochastic) coefficients, represented by the matrix $A^\varepsilon$, which models a two-component material with randomly distributed and randomly shaped “soft” inclusions, whose typical size and spacing are both of order $\varepsilon \ll 1$, embedded in a “stiff” component. It is assumed that the ellipticity constants of $A^\varepsilon$ is of order 1 in the stiff component and of order $\varepsilon^2$ in the inclusions.

Our interest in high-contrast homogenisation problems is motivated by the band-gap structure of the spectra of the associated operators. Composites exhibiting spectral gaps are widely used for manipulating acoustic and electromagnetic waves. From a mathematically rigorous perspective, these were first analysed in [Zh00, Zh04] in the periodic setting. It was shown that the spectra of $A^\varepsilon$ converge in the sense of Hausdorff to the spectrum of a limit homogenised operator $A^{\text{hom}}$. The latter has a two-scale structure that captures the macro- and microscopic behaviour of the operator $A^\varepsilon$ for small values of $\varepsilon$. The spectrum of $A^{\text{hom}}$ has a band-gap spectrum (in the whole-space setting), quantified with respect to the spectral parameter $\lambda$ by a function $\beta(\lambda)$, which in turn is explicitly determined by the microscopic part of $A^{\text{hom}}$. In the bounded domain case the analysis of the problem and the results are very similar, in particular, the discrete part of the spectrum of $A^{\text{hom}}$ “populates” the bands (accumulating at their right ends) corresponding to the whole-space case.

In [CCV18] we considered the high-contrast stochastic homogenisation problem in a bounded domain. To a certain extent, our findings as well as the basic techniques (modulo replacing the reference periodicity cell with the probability space, and the standard two-scale convergence with its stochastic counterpart) were similar to those of [Zh00] in the periodic case. Namely, we have shown that the homogenised operator has a similar two-scale structure with the “macroscopic” component $-\nabla \cdot A^{\text{hom}} \nabla$ acting in the physical space, and the “microscopic” one $-\Delta_O$ acting in the probability space on a prototype “inclusion” $O$. We have proved an appropriate (i.e. stochastic two-scale) version of the resolvent convergence of $A^\varepsilon$ to $A^{\text{hom}}$, and the Hausdorff convergence of their spectra. However, due to the technical challenges of the stochastic setting our understanding of the homogenised operator $A^{\text{hom}}$ was limited. In particular, we have been able to describe its spectrum only for a range of explicit examples. Notice that the stochastic two-scale resolvent convergence of
the operators (implying \( \lim_{\varepsilon \to 0} \text{Sp}(A^\varepsilon) \supset \text{Sp}(A^{\text{hom}}) \)) proven in [CCV18] is valid both in a bounded domain and the whole space settings without any changes to the proof.

Unlike the periodic high-contrast operators, whose spectra are described by similar multiscale arguments for a bounded domain and for the whole space (leading to closely related features in the two settings, albeit resulting in spectra of different types from the operator-theoretic perspective), in the stochastic case the situation is fundamentally different. In the present paper we show that two settings, albeit resulting in spectra of different types from the operator-theoretic perspective,

arguments for a bounded domain and for the whole space (leading to closely related features in the domain and the whole space settings without any changes to the proof.

\[ \lim_{\varepsilon \to 0} \text{Sp}(A^\varepsilon) \]

where \( \beta \) is an element of the probability space, which is a “global” analogue of \( \beta(\lambda) \), and a set \( \mathcal{G} \), characterised by \( \beta_\infty(\lambda, \omega) \) and the spectrum of \( -\Delta_{\mathcal{O}} \). While \( \beta(\lambda) \) is deterministic by definition, \( \beta_\infty(\lambda, \omega) \) is defined via realisations of a probabilistic quantity in the physical space. We prove that \( \beta_\infty(\lambda, \omega) \) is, in fact, deterministic, i.e. \( \beta_\infty(\lambda, \omega) = \beta_\infty(\lambda) \), for a.e. \( \omega \) (Section 4.2). An intuitive explanation of the difference between the two functions is as follows: the values of \( \beta(\lambda) \) are determined via the ergodic limit (in other words from the global average distribution of inclusions), whereas the values of \( \beta_\infty(\lambda) \) are determined, loosely speaking, by the areas with the least dense distribution of the inclusions (non-typical areas). The first of the two main results of the section (Theorem 4.2) is that the Hausdorff limit of \( \text{Sp}(A^\varepsilon) \) is a subset of \( \mathcal{G} \). We do not know whether \( \mathcal{G} \) is the actual limit in general (in fact, under ergodicity assumption alone, it is not clear if the limit of the spectra is deterministic at all). However, under an additional assumption of finite range correlation of the distribution of inclusions in space, we are able to claim this (Theorem 4.5), which...
is the second main result of the section. The key observation for the proof of this statement is that under the finite correlation assumption, the existence of arbitrary large cubes with almost periodic copies of any sample of space is guaranteed due to the law of large numbers, leading to recovering the values of $\beta_\infty(\lambda)$ for $\lambda \in \mathcal{G}$ on such cubes and relatively explicit construction of approximate eigenfunctions of $A^\varepsilon$. We give the proof of Theorem 4.2 in Section 4.3 in Section 4.4 we provide auxiliary statements on existence of cubes with almost periodic arrangement of inclusions, in Section 4.4 we prove Theorem 4.5. In the last subsection we consider a range of example in order to illustrate the results of the section.

In Section 5 we explore the connection between the spectrum of $A^{\text{hom}}$ and the limiting behaviour of the spectra of $A^\varepsilon$. We introduce a notion of statistically relevant limiting spectrum of $A^\varepsilon$, denoted by $\text{SR-lim} \text{Sp}(A^\varepsilon)$ - a subset of $\lim_{\varepsilon \to 0} \text{Sp}(A^\varepsilon)$ whose points are characterised by existence of families of approximate eigenfunctions of $A^\varepsilon$ satisfying certain bounds. The essence of the definition is that one takes only those $\lambda$ that have approximate eigenfunctions (of $A^\varepsilon$) significant part of whose energy stays inside (large) fixed neighbourhoods of the origin as $\varepsilon \to 0$, and disregards those $\lambda$ that have only (approximate) eigenfunctions whose energy mainly concentrated in the areas of space with non-typical (in the same sense as above) distribution of inclusions (since for small $\varepsilon$ non-typical areas large enough to support approximate eigenfunctions are located far from the origin). The main result of this section is that $\text{SR-lim} \text{Sp}(A^\varepsilon)$ coincides with $\text{Sp}(A^{\text{hom}})$, the proof is given in Sections 5.1 and 5.2. A natural question that arises is “can one somehow characterise the statistically relevant subset of $\text{Sp}(A^\varepsilon)$”? In Section 5.3 we address this question by introducing “approximate statistically relevant spectrum” $\text{Sp}_\varepsilon(A^\varepsilon) \subset \text{Sp}(A^\varepsilon)$. The idea behind the definition is very similar to the one used for $\text{SR-lim} \text{Sp}(A^\varepsilon)$. We show that $\text{Sp}_\varepsilon(A^\varepsilon)$ converges in the sense of Hausdorff to $\text{Sp}(A^{\text{hom}})$. It is important to mention that $\text{Sp}_\varepsilon(A^\varepsilon)$ is not defined in a unique way. By changing the bounds on the approximate eigenfunctions in the definition (fundamentally determined by the stochastic homogenisation correctors) one changes the set $\text{Sp}_\varepsilon(A^\varepsilon)$. We do not know whether the exact (rather than approximate) decomposition of $\text{Sp}(A^\varepsilon)$ into statistically relevant and irrelevant parts is possible — the intuitive concept behind our definition is difficult to quantify precisely, so this question remains open.

A number of technical preliminaries, constructions and auxiliary statements that we use in the paper, are presented in the appendix. Some of these statements are of interest in their own right. In particular, we would like to highlight the following results: a higher regularity of the periodic homogenisation corrector for perforated domains, see Theorem 4.16 whose proof is presented in Appendix C; the extension property for potential vector fields in the probability space, Proposition D.2, and its corollary Lemma D.3 important for the stochastic homogenisation corrector for randomly perforated domains; construction of an ergodic random marking function for inclusion, Proposition D.8.

2 Preliminaries and problem setting

The setting of the problem is almost identical to [CCV18] with the only exception that we simplify our main assumption and deal mostly with the whole space situation. We have placed a number of definitions and technical statements, such as Sobolev spaces in the probability space, the notion of stochastic two-scale convergence, and measurability properties of various mappings, which we will refer to or often use tacitly throughout the text, in Appendices A and B.
2.1 Probability framework

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space. We assume that the \(\sigma\)-algebra \(\mathcal{F}\) is countably generated, which implies that the spaces \(L^p(\Omega)\), \(p \in [1, \infty)\), are separable.

**Definition 2.1.** A family \((T_x)_{x \in \mathbb{R}^d}\) of measurable bijective mappings \(T_x : \Omega \rightarrow \Omega\) on a probability space \((\Omega, \mathcal{F}, P)\) is called a dynamical system on \(\Omega\) with respect to \(P\) if:

a. \(T_x \circ T_y = T_{x+y} \ \forall x, y \in \mathbb{R}^d\);

b. \(P(T_x F) = P(F) \ \forall x \in \mathbb{R}^d, F \in \mathcal{F}\);

c. \(\mathcal{T} : \mathbb{R}^d \times \Omega \rightarrow \Omega, \ (x, \omega) \rightarrow T_x(\omega)\) is measurable (for the standard \(\sigma\)-algebra on the product space, where on \(\mathbb{R}^d\) we take the Borel \(\sigma\)-algebra).

**Definition 2.2.** A dynamical system is called ergodic if one of the following equivalent conditions is fulfilled:

a. \(f\) measurable, \(f(\omega) = f(T_x \omega) \ \forall x \in \mathbb{R}^d, \ \text{a.e.} \ \omega \in \Omega \implies f(\omega)\) is constant \(P\)-a.e. \(\omega \in \Omega\).

b. \(P((T_x B \cup B)\setminus(T_x B \cap B)) = 0 \ \forall x \in \mathbb{R}^d \implies P(B) \in \{0, 1\}\).

Henceforth we assume that the dynamical system \(T_x\) is ergodic.

**Remark 2.3.** Note that if the symmetric difference between \(T_x B\) and \(B\) is a null set, then for the condition \(b\) in the above definition the implication \(P(B) \in \{0, 1\}\) has to hold. It can be shown (e.g., [CFSS82]) that ergodicity is also equivalent to the following (a priori weaker) implication:

\[ T_x B = B \ \forall x \in \mathbb{R}^d \implies P(B) \in \{0, 1\}. \]

For \(f \in L^p(\Omega)\), we write \(f(x, \omega) := f(T_x \omega)\), defining the realisation \(f \in L^p_{\text{loc}}(\mathbb{R}^d, L^p(\Omega))\). There is a natural unitary action on \(L^2(\Omega)\) associated with \(T_x\):

\[ U(x)f = f \circ T_x, \quad f \in L^2(\Omega), \]

which is a strongly continuous group. It is often necessary that a set of full measure be invariant in the sense that together with the point \(\omega\) it contains the whole "trajectory" \(\{T_x \omega, x \in \mathbb{R}^m\}\). This requirement can always be met on the basis of the following simple lemma, see [ZKO94, Lemma 7.1].

**Lemma 2.4.** Let \(\Omega_0\) be a set of full measure in \(\Omega\). Then there exists a set of full measure \(\Omega_1\) such that \(\Omega_1 \subseteq \Omega_0\), and for a given \(\omega \in \Omega_1\) we have \(T_x \omega \in \Omega_0\) for almost all \(x \in \mathbb{R}^m\).

The differentiation operators in \(L^2(\Omega)\) (as the infinitesimal generators \(D_j\) of the groups (147)) and spaces \(C^\infty(\Omega), W^{k,2}(\Omega), W^{2,2}(\Omega), C^\infty_0(\mathcal{O})\) and \(W^{1,2}_0(\mathcal{O})\) are introduced in a standard way. We also use the spaces of potential and solenoidal vector fields \(L^2_{\text{pot}}(\Omega)\) and \(L^2_{\text{sol}}(\Omega)\) and their zero-mean subspaces \(\mathcal{V}^2_{\text{pot}}\) and \(\mathcal{V}^2_{\text{sol}}\), see [ZKO94]. We provide the relevant definitions in Appendix A.

The following theorem is the key tool in setting the basis for our analysis.
Theorem 2.5 ("Ergodic Theorem"). Let $(\Omega, \mathcal{F}, P)$ be a probability space with an ergodic dynamical system $(T_x)_{x \in \mathbb{R}^d}$ on $\Omega$. Suppose $f \in L^1(\Omega)$ and $S \subset \mathbb{R}^d$ is a bounded open set. Then for $P$-a.e. $\omega \in \Omega$ one has
\[
\lim_{\varepsilon \to 0} \int_S f(T_{x/\varepsilon} \omega) dx = |S| \int_\Omega f(\omega) dP(\omega).
\] (1)
Furthermore, for all $f \in L^p(\Omega)$, $1 \leq p \leq \infty$, and a.e. $\omega \in \Omega$, the function $f(x, \omega) = f(T_x \omega)$ satisfies $f(\cdot, \omega) \in L^{p}_{\text{loc}}(\mathbb{R}^d)$. For $p < \infty$ one has $f(\cdot/\varepsilon, \omega) \rightarrow f(T_{\cdot/\varepsilon} \omega)$ weakly in $L^{p}_{\text{loc}}(\mathbb{R}^d)$ as $\varepsilon \to 0$.

The elements $\omega$ such that (1) holds for all $f \in L^1(\Omega)$ and bounded open $S \subset \mathbb{R}^d$ are usually referred to as "typical", and the corresponding sets $(T_x \omega)_{x \in \mathbb{R}^d}$ are called "typical trajectories". Note that the separability of $L^1(\Omega)$ (which is guaranteed if $\mathcal{F}$ is countably generated) implies that almost every $\omega \in \Omega$ is typical. In general, statements that only require a simple convergence argument, such as passing to the limit in the resolvent problem, are valid for all such $\omega$. However, throughout the paper we have a number of results, notably Theorem 4.5 and Proposition 4.9, where one does not invoke the ergodic theorem but uses the law of large numbers instead for the analysis of non-typical areas (as discussed in Introduction). Such results hold on a full-measure subset of $\Omega$ that in general is different from the one provided by the ergodic theorem. Thus in what follows all results can be viewed as valid on a set of full measure, which we henceforth always assume $\omega$ to be an element of.

2.2 The main assumption and its implications

We define the inclusions in the following way. Let a set $O \subseteq \Omega$ be such that $0 < P(O) < 1$ and for each $\omega \in \Omega$ consider its realisation
\[
O_\omega := \{ x \in \mathbb{R}^d : T_x \omega \in O \}.
\]

As we have already mentioned we have simplified the main assumptions used in our previous paper [CCC17]. It turns out that it is sufficient to require the uniform minimal smoothness of the inclusions together with their "extension domains" and their boundedness in order to guarantee the validity of the main results therein, as well as the results of the present paper. In particular, Assumption 2.7 implies the key extension property, Theorem 2.8 (simply postulated in our previous paper [CCC17]), and the density result, Lemma A.1.

We first introduce the definition of a minimally smooth set, see [St70].

Definition 2.6. An open set $P \subset \mathbb{R}^d$ is said to be minimally smooth with constants $(\rho, N, \gamma)$ if we may cover $\partial P$ by a countable sequence of open sets $\{U_i\}_i$ such that

- Each $x \in \mathbb{R}^d$ is contained in at most $N$ of the open sets $U_i$.
- For any $x \in \partial P$ the ball $B_\rho(x)$ is contained in at least one $U_i$.
- For any $i$ the portion of the boundary $\partial P$ inside $U_i$ agrees (in some Cartesian system of coordinates) with the graph of a Lipschitz function whose Lipschitz semi-norm is at most $\gamma$.

Our main assumption is as follows.
Assumption 2.7. For a.e. $\omega \in \Omega$ the set $\mathbb{R}^d \setminus \overline{\Omega}_\omega$ is connected and

$$O_\omega := \bigcup_{k=1}^{\infty} O^k_\omega,$$

where:

1) For every $k \in \mathbb{N}$ the set $O^k_\omega$ is open, connected, and there exists an open bounded set $B^k_\omega \supset \overline{O}^k_\omega$ such that $B^k_\omega \cap O_\omega = O^k_\omega$ and the set $B^k_\omega \setminus \overline{O}^k_\omega$ is minimally smooth with constants $(\rho, N, \gamma)$;

2) For every $k \in \mathbb{N}$ one has $\text{diam } O^k_\omega < 1/2$. Appropriately translated, the domains $O^k_\omega$ and $B^k_\omega$ fit into the unit cube $\square := [-1/2, 1/2]^d$. More precisely, let the vector $D^k \in \mathbb{R}^d$ be defined by

$$(D^k)_j := \inf \{x_j : x \in O^k_\omega \}, j = 1, \ldots, d,$$

and denote $d_{1/4} := (1/4, \ldots, 1/4)^\top$, then $O^k_\omega - D^k - d_{1/4} \subset B^k_\omega - D^k - d_{1/4} \subset \square$.

The specific bounds on the size of the inclusions and their “extension sets” $B^k_\omega$ are imposed for the sake of convenience and can be obtained from more general boundedness assumptions by a simple scaling argument. The minimal smoothness assumption implies, in particular, that $\text{dist}(O^k_\omega, \mathbb{R}^d \setminus B^k_\omega) \geq \rho$ and the distance between inclusions is at least $\rho$. Moreover, it is not difficult to see that the uniform minimal smoothness and boundedness of the inclusions imply that their surface area (perimeter if $d = 2$) is also uniformly bounded:

$$|\partial O^k_\omega| \leq C, \quad k \in \mathbb{N}, \text{ for a.e. } \omega \in \Omega. \quad (3)$$

Theorem 2.8. There exist bounded linear extension operators $E_k : W^{1,p}(B^k_\omega \setminus O^k_\omega) \to W^{1,p}(B^k_\omega)$, $p \geq 1$, such that the extension $\tilde{u} := E_k u$, $u \in W^{1,p}(B^k_\omega \setminus O^k_\omega)$, satisfies the bounds

$$\|\nabla \tilde{u}\|_{L^p(B^k_\omega)} \leq C_{\text{ext}} \|\nabla u\|_{L^p(B^k_\omega \setminus O^k_\omega)}, \quad (4)$$

$$\|\tilde{u}\|_{L^p(B^k_\omega)} \leq C_{\text{ext}} \left(\|u\|_{L^p(B^k_\omega \setminus O^k_\omega)} + \|\nabla u\|_{L^p(B^k_\omega \setminus O^k_\omega)}\right), \quad (5)$$

where $C_{\text{ext}}$ depends only on $\rho, N, \gamma, p$, and is independent of $\omega$ and $k$. Additionally, in case $p = 2$ the extension can be chosen to be harmonic in $O^k_\omega$,

$$\Delta \tilde{u} = 0 \text{ on } O^k_\omega.$$

Proof. Inequalities $(4)$ and $(5)$ are direct corollaries of a classical result due to Calderón and Stein on the existence of uniformly bounded extension operators [St70, Chapter IV, Theorem 5] and a uniform Poincaré inequality [GK15, Proposition 3.2] for minimally smooth domains. We make three comments. First, though it is not stated explicitly in [St70, Chapter IV, Theorem 5], the norm of the extension operator depends only on the constants of the minimal smoothness assumption, which can be seen from the proof. Second, the proof of the uniform Poincaré inequality in [GK15, Proposition 3.2] works for any $p \geq 1$ without amendments. Third, the authors of [GK15] tacitly assume that a fixed radius neighbourhood of a minimally smooth set is also minimally smooth with the same constants, which is wrong in general: it is not difficult to construct a counterexample when such a neighbourhood is not even Lipschitz. However, assuming from the outset that the
extension set is minimally smooth solves the problem. Confer also [BC07], where a very similar approach to the uniform Poincaré inequality is used. (And again the authors make an insufficient, in our opinion, assumption of a uniform cone property only from inside, whereas the uniform cone property should be imposed on both sides of the boundary.)

Now let $p = 2$ and fix $O_k^{\omega}$. We look for the harmonic extension $\tilde{u}$ in the form $\tilde{u} = E_k u + \hat{u}$, where $\hat{u} \in W^{1,2}_0(O_k^{\omega})$ (here $E_k$ denotes the extension operator from [St70]). If such $\hat{u}$ exists, it has to satisfy the equation

$$
\int_{O_k^{\omega}} \nabla \hat{u} \cdot \nabla \varphi = -\int_{O_k^{\omega}} \nabla (E_k u) \cdot \nabla \varphi \quad \forall \varphi \in W^{1,2}_0(O_k^{\omega}).
$$

By the uniform extension property have

$$
\left| \int_{O_k^{\omega}} \nabla (E_k u) \cdot \nabla \varphi \right| \leq C_{\text{ext}} \| \nabla u \|_{L^2(B_k^{\omega}\setminus O_k^{\omega})} \| \nabla \varphi \|_{L^2(O_k^{\omega})},
$$

i.e. the right-hand side is a bounded linear functional on $W^{1,2}_0(O_k^{\omega})$ equipped with the norm $\| \nabla \varphi \|_{L^2(O_k^{\omega})}$. Therefore, by the Riesz representation theorem the solution $\hat{u}$ of (6) exists and satisfies the bound

$$
\| \nabla \hat{u} \|_{L^2(O_k^{\omega})} \leq C_{\text{ext}} \| \nabla u \|_{L^2(B_k^{\omega}\setminus O_k^{\omega})}.
$$

Then (4) and (5) follow easily.

**Remark 2.9.** The above theorem can be reformulated in an obvious way for any family of domains (i.e. not related to our stochastic setting) satisfying the same minimal smoothness condition.

### 2.3 Problem setting and an overview of the existing results

For $\varepsilon > 0$ we define

$$
S^\varepsilon_0(\omega) := \varepsilon O_\omega = \bigcup_k \varepsilon O_k^{\omega}, \quad S^\varepsilon_1(\omega) := \mathbb{R}^d \setminus S^\varepsilon_0(\omega).
$$

The corresponding set indicator functions are denoted by $\chi^\varepsilon_0(\omega)$ and $\chi^\varepsilon_1(\omega)$ respectively. We consider the self-adjoint operator $A^\varepsilon(\cdot, \omega)$ in $L^2(S)$ (where $S \subset \mathbb{R}^d$ denotes either a bounded domain with Lipschitz boundary or the whole space $\mathbb{R}^d$) generated by the bilinear form

$$
\int_{\mathbb{R}^d} A^\varepsilon(\cdot, \omega) \nabla u \cdot \nabla v, \quad u, v \in W^{1,2}_0(S),
$$

where

$$
A^\varepsilon(\cdot, \omega) = \chi^\varepsilon_1(\omega) A_1 + \varepsilon^2 \chi^\varepsilon_0(\omega) I,
$$

and $A_1$ is a symmetric positive-definite matrix. Note that $W^{1,2}_0(\mathbb{R}^d) = W^{1,2}(\mathbb{R}^d)$ when $S = \mathbb{R}^d$. In what follows we assume that $\omega$ is fixed and will drop in from the notation of the operator, writing simply $A^\varepsilon$.

We recall the definition of the limit homogenised operator $A^{\text{hom}}$ given in [CCV18]. The matrix of homogenised coefficients $A_1^{\text{hom}}$ arising from the stiff component is defined by

$$
A_1^{\text{hom}} \xi : \xi := \inf_{p \in V_2^{\text{pot}}} \int_{\Omega \setminus \mathcal{O}} A_1(\xi + p) \cdot (\xi + p), \quad \xi \in \mathbb{R}^d.
$$
It is well known, see [ZKO94, Chapter 8], that in order for the matrix $A_{1}^{\text{hom}}$ to be positive definite it suffices to have a certain kind of extension property. It is not difficult to see that a slight adaptation of the argument in [ZKO94, Lemma 8.8] ensures that $A_{1}^{\text{hom}}$ is indeed positive definite under Assumption 2.7.

The variational problem (8) has a unique solution $p_{\xi}$ in $X$, where $X$ denotes the completion in $L^{2}(\Omega \setminus \mathcal{O})$ of $V_{2}^{\text{pot}}$. This solution satisfies the following equation

$$\int_{\Omega \setminus \mathcal{O}} A_{1}(\xi + p_{\xi}) \cdot \varphi = 0, \quad \forall \varphi \in V_{2}^{\text{pot}}. \quad (9)$$

In Appendix D we prove that under our assumptions on the regularity of the inclusions the space $X$ can be viewed merely as the restriction of $V_{2}^{\text{pot}}$ to $\Omega \setminus \mathcal{O}$ (see Lemma D.3). We define the space

$$H := L^{2}(S) + L^{2}(S \times \mathcal{O}),$$

which is naturally embedded in $L^{2}(S \times \Omega)$, and its dense (by Lemma A.1) subspace

$$V := W_{0}^{1,2}(S) + L^{2}(S, W_{0}^{1,2}(\mathcal{O})).$$

For $f \in H$, we consider the following resolvent problem: find $u_{0} + u_{1} \in V$ such that

$$\int_{S} A_{1}^{\text{hom}} \nabla u_{0} \cdot \nabla \varphi_{0} + \int_{S \times \Omega} \nabla_{\omega} u_{1} \cdot \nabla_{\omega} \varphi_{1} - \lambda \int_{S \times \Omega} (u_{0} + u_{1})(\varphi_{0} + \varphi_{1}) = \int_{S \times \Omega} f(\varphi_{0} + \varphi_{1}) \quad \forall \varphi_{0} + \varphi_{1} \in V. \quad (10)$$

This problem gives rise to a positive definite operator $A_{1}^{\text{hom}}$ in $H$, see [CCV18] for details. Notice that one can take $f \in L^{2}(S \times \Omega)$ in (10): in this case the solution coincides with the solution for the right-hand side $Pf$, where $P$ is the orthogonal projection onto $H$. Furthermore, the identity (10) can be written as a coupled system

$$\int_{S} A_{1}^{\text{hom}} \nabla u_{0} \cdot \nabla \varphi_{0} - \lambda \int_{S} (u_{0} + (u_{1})) \varphi_{0} = \int_{S} f \varphi_{0} \quad \forall \varphi_{0} \in W_{0}^{1,2}(S), \quad (11)$$

$$\int_{\mathcal{O}} \nabla_{\omega} u_{1}(x, \cdot) \cdot \nabla_{\omega} \varphi_{1} - \lambda \int_{\mathcal{O}} (u_{0}(x) + u_{1}(x, \cdot)) \varphi_{1} = \int_{\mathcal{O}} f(x, \cdot) \varphi_{1} \quad \forall \varphi_{1} \in W_{0}^{1,2}(\mathcal{O}). \quad (12)$$

The following statement of weak (strong) stochastic two-scale resolvent convergence, which was proved in [CCV18] in the case of bounded $S$, remains valid for $S = \mathbb{R}^{d}$ with the proof being exactly the same.

**Theorem 2.10.** Under Assumption 2.7 let $\lambda > 0$ and suppose that $f^{\varepsilon}$ is a bounded sequence in $L^{2}(S)$ such that $f^{\varepsilon} \overset{2}{\rightharpoonup} (\overset{2}{\rightharpoonup}) f \in L^{2}(S \times \Omega)$. Consider the sequence of solutions $u^{\varepsilon}$ to the resolvent problem

$$A^{\varepsilon} u^{\varepsilon} + \lambda u^{\varepsilon} = f^{\varepsilon}.$$

Then for a.e. $\omega \in \Omega$ one has $u^{\varepsilon} \overset{2}{\rightharpoonup} (\overset{2}{\rightharpoonup}) u_{0} + u_{1} \in V$, where $u_{0} + u_{1}$ is the solution to (10).
Denote by \(-\Delta_O\) the positive definite self-adjoint operator generated by the bilinear form
\[
\int_O \nabla \omega \cdot \nabla \omega, \quad u, v \in W^{1,2}_0(O).
\] (13)
The following results we obtained in [CCV18]:

- \(\text{Sp}(\Delta_O) = \bigcup_{k \in \mathbb{N}} \text{Sp}(\Delta_{O_k})\) for a.e. \(\omega \in \Omega\), \((14)\)
  where \(-\Delta_{O_k}\) is the Dirichlet Laplace operator on \(O_k\) for each \(\omega, k\).

- In the case of bounded \(S\) and under the condition that \(\text{Sp}(\Delta_O) \subseteq \text{Sp}(A^{\text{hom}})\), the strong stochastic two-scale convergence of eigenfunctions of \(A^\epsilon\) holds, i.e. if
  \[
  A^\epsilon u^\epsilon = \lambda^\epsilon u^\epsilon, \quad \int_{S} |u^\epsilon|^2 = 1,
  \]
  and if \(\lambda^\epsilon \to \lambda\), then \(u^\epsilon \overset{2}{\to} u\), where \(A^{\text{hom}} u = \lambda u\).

- In the case of bounded \(S\), for two explicit examples of random distribution of inclusions, it was shown that \(\text{Sp}(\Delta_O) \subseteq \text{Sp}(A^{\text{hom}})\), the Zhikov function \(\beta(\lambda)\) was calculated, and a complete characterisation of \(\text{Sp}(A^{\text{hom}})\) was given.

Remark 2.11. From \((14)\) and property 2 of Assumption 2.7 it follows, via the Poincaré inequality and the min-max theorem, that \(\inf \text{Sp}(\Delta_O) > 0\).

Remark 2.12. The identity \((14)\) was proven in [CCV18] under an additional assumption requiring that the normalised eigenfunctions of the operators \(-\Delta_{O_k}, \omega \in \Omega, k \in \mathbb{N}\), were bounded in \(L^\infty\)-norm uniformly in \(\omega\) and \(k\) in any bounded spectral interval. It is not difficult to see that this property holds true under Assumption 2.7: the proof of this fact follows closely the standard argument of De Giorgi–Nash–Moser regularity theory, see e.g. [BF02, Chapter 2.3], cf. a similar result in [CCV21, Lemma 4.3] for more detail. Another observation we make is that in order to prove \((14)\) one does not necessarily need a uniform bound of \(L^\infty\)-norms of the eigenfunctions of \(-\Delta_{O_k}\). Instead, it is sufficient to note that the random variable \(\psi_{a,b}\) utilised in the proof of Theorem 5.1 in [CCV18] belongs to \(L^2(\Omega)\), which follows by a standard argument (cf. Proposition 3.6) via integrating its realisation in the physical space first, then integrating over \(\Omega\) and using the Fubini theorem.

Remark 2.13. Another and perhaps more common approach to describing random media consists in taking \(\Omega\) to be a subset of locally bounded Borel measures on \(\mathbb{R}^d\), which additionally can be chosen so that it is a separable metric space (then appropriate \(\sigma\)-algebra is the Borel \(\sigma\)-algebra). Thus, a random material consisting of inclusions and a matrix material can be seen as a random measure. The dynamical system is then defined using the stationarity property of the random measure, however, the ergodicity assumption does not follow from this construction and has to be made additionally. In fact, the ergodicity is not a necessary assumption for the ergodic theorem (in that case the expectation on the right-hand side of \((1)\) has to be replaced by the conditional expectation). The ergodicity property in the classical (non-high-contrast) setting guarantees that the coefficients of the limit equation are deterministic. In the case when ergodicity assumption
is dropped, the coefficients of the limit equation are measurable with respect to the $\sigma$-algebra of the invariant sets of the dynamical system (under the assumption of ergodicity this $\sigma$-algebra consists of sets of measure zero and one).

Point processes such as the Poisson process and the random parking model, used in the present work in the construction of examples of a random medium, are usually looked at as random measures in probability theory. The benefit of this approach is that one can use Palm’s theory (see [H26, Section 2.7, Section 2.8]). This is especially useful in the analysis of random structures (see [ZP06]). In the present work we do not use the Palm’s theory and we find the chosen framework sufficient for our analysis.

3 Homogenised operator and its spectrum

In this section we study the operator $A_{\text{hom}}$ and its spectrum. In particular, our analysis allows us to improve a number of results from [CCV18].

We begin with two general statements about the spectrum of $A_{\text{hom}}$, Proposition 3.1 and Theorem 3.3 which were proved in [CCV18] only for certain examples.

**Proposition 3.1.** For the spectra of $A_{\text{hom}}$, defined by (11)–(12), and $-\Delta_O$, defined by (13), one has the inclusion $\text{Sp}(-\Delta_O) \subset \text{Sp}(A_{\text{hom}})$.

**Proof.** Suppose that $\lambda \in \mathbb{R}$ is in the resolvent set of $A_{\text{hom}}$, so that (10) has a solution $u = u_0 + u_1$ for any $f \in H$. Taking first a non-trivial $f \in L^2(S)$ (we recall that $S$ is either bounded or $S = \mathbb{R}^d$), for the corresponding solution we have

$$-\Delta_O u_1 - \lambda u_1 = (\lambda u_0 + f)1_O,$$

where $1_O$ is the indicator function of the set $O$. We observe that for two arbitrary functions $w \in L^2(S, O)$ and $h \in L^2(S)$ one has $\int_S wh \in L^2(O)$. Then, multiplying the above identity by $(\lambda u_0 + f)\|\lambda u_0 + f\|^{-2}_{L^2(S)}$ and integrating over $S$, we conclude that the function

$$\phi := \int_S \frac{u_1(\lambda u_0 + f)}{\|\lambda u_0 + f\|_{L^2(S)}^2} \in L^2(O)$$

solves the equation

$$-\Delta_O \phi - \lambda \phi = 1.$$

Next, we take $f = g\psi$ with arbitrary $g \in L^2(S)$ and $\psi \in L^2(O)$. Then for the corresponding solution of (10), which we denote by $\tilde{u}_0 + \tilde{u}_1$, we have

$$-\Delta_O \tilde{u}_1 - \lambda \tilde{u}_1 = \lambda \tilde{u}_0 1_O + g\psi.$$

The difference between $\tilde{u}_1$ and $\hat{u}_1 := \lambda \tilde{u}_0 \phi$ solves

$$-\Delta_O(\hat{u}_1 - \hat{u}_1) - \lambda(\hat{u}_1 - \hat{u}_1) = g\psi.$$

Multiplying the last equation by $\|g\|^{-2}_{L^2(S)}$ and integrating over $S$, we see that

$$\hat{u}_1 := \int_S \frac{(\hat{u}_1 - \hat{u}_1)g}{\|g\|_{L^2(S)}^2}$$

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is a solution of

\[ -\Delta_{\Omega} \hat{u}_1 - \lambda \hat{u}_1 = \psi. \]

We have shown that \(-\Delta_{\Omega} - \lambda I\) acts onto, therefore, by the bounded inverse theorem one concludes that \((-\Delta_{\Omega} - \lambda I)^{-1}\) is bounded.

Now we are in a position to define Zhikov’s \(\beta\)-function. For \(\lambda \notin \text{Sp}(-\Delta_{\Omega})\) we set

\[ \beta(\lambda) := \lambda + \lambda^2 \int_{\Omega} (-\Delta_{\Omega} - \lambda I)^{-1} 1_{\Omega} dP(\omega) = \lambda + \lambda^2 \langle b \rangle, \tag{15} \]

where \(b = b(\omega, \lambda)\) is the solution to the problem

\[ -\Delta_{\Omega} b = \lambda b + 1. \tag{16} \]

**Remark 3.2.** If our probability space is such that the set of inclusions is periodic, then the above formula gives exactly the Zhikov’s \(\beta\)-function of the periodic setting.

The next theorem provides a characterisation of the spectrum of \(\mathcal{A}_{\text{hom}}\).

**Theorem 3.3.**

a. The spectrum of the homogenised operator is fully characterised by \(\beta(\lambda)\) and the spectra of the “microscopic” \(-\Delta_{\Omega}\) and the “macroscopic” \(-\nabla \cdot A_{\text{hom}}^{\text{hom}} \nabla\) operators as follows:

\[ \text{Sp}(\mathcal{A}_{\text{hom}}) = \text{Sp}(-\Delta_{\Omega}) \cup \{ \lambda : \beta(\lambda) \in \text{Sp}(-\nabla \cdot A_{\text{hom}}^{\text{hom}} \nabla) \}. \tag{17} \]

b. If \(S\) is bounded, then for a.e. \(\omega \in \Omega\) we have convergence of the spectra \(\text{Sp}(\mathcal{A}^\varepsilon) \to \text{Sp}(\mathcal{A}_{\text{hom}})\) in the sense of Hausdorff.

**Proof.** Suppose that \(\lambda \notin \text{Sp}(-\Delta_{\Omega})\). In order to solve (11)–(12) we first solve (12) for \(u_1\) in terms of an arbitrary fixed \(u_0 \in W^{1,2}_0(S)\):

\[ u_1(x, \cdot) = \lambda u_0(x) (-\Delta_{\Omega} - \lambda I)^{-1} 1_{\Omega} + (-\Delta_{\Omega} - \lambda I)^{-1} f(x, \cdot). \]

We then substitute the obtained expression into (11),

\[ -\nabla \cdot A_{\text{hom}}^{\text{hom}} \nabla u_0 - \beta(\lambda) u_0 = \langle \lambda (-\Delta_{\Omega} - \lambda I)^{-1} f(x, \cdot) + f(x, \cdot) \rangle. \tag{18} \]

Taking \(f(x, \omega) = g(x) \psi(\omega)\) with arbitrary \(g \in L^2(S)\) and \(\psi \in \text{dom}(-\Delta_{\Omega})\), we see that the solvability of (18) is equivalent to \(\beta(\lambda) \notin \text{Sp}(-\nabla \cdot A_{\text{hom}}^{\text{hom}} \nabla)\), and (17) follows.

The second part of the statement is proved in [CCV18]: the resolvent converges implies that the spectrum of the limit operator is contained in the limit of \(\text{Sp}(\mathcal{A}^\varepsilon)\) as \(\varepsilon \to 0\), while strong two-scale convergence of eigenfunctions of \(\varepsilon\) problem implies that any limiting point of \(\text{Sp}(\mathcal{A}^\varepsilon)\) is contained in the spectrum of the limiting operator.

**Remark 3.4.** In the case \(S = \mathbb{R}^d\) we can write

\[ \text{Sp}(\mathcal{A}_{\text{hom}}) = \text{Sp}(-\Delta_{\Omega}) \cup \{ \lambda : \beta(\lambda) \geq 0 \}. \]

**Remark 3.5.** While in the periodic case \(\beta(\lambda)\) blows up to \(\pm \infty\) as \(\lambda\) approaches \(\text{Sp}(-\Delta_{\Omega})\), this is not the case in the stochastic setting in general. It is not difficult to construct an example where \(\beta(\lambda)\) does not blow up. For instance, taking the example of randomly scaled inclusions in Section 4.6.4 below, and choosing the probability density of \(\omega_0\) on the scaling interval \([r_1, r_2]\) so that it is positive on \((r_1, r_2)\) and converges to zero sufficiently quickly near the ends, one gets that \(\beta(\lambda)\) converges to finite values as \(\lambda\) approaches \(\text{Sp}(-\Delta_{\Omega})\), cf. also formula (96).
3.1 Properties of Zhikov’s $\beta$-function

We begin by showing that the solution to (16) can be reconstructed from its physical realisations.

For the subsequent construction we need to use some notation provided in Appendix B. Namely, by $P_\omega$ we denote the set obtained by shifting the inclusion containing the origin in a specific way defined in (149). For the Dirichlet Laplacian operator $-\Delta_{P_\omega}$ we denote the set of its eigenvalues arranged in the increasing order, $\Lambda_s = \Lambda_s(\omega)$, $s \in \mathbb{N}$, and by $\{\Psi^p_s\}_{p=1,\ldots,N_s}$ the corresponding system of the orthonormalised eigenfunctions, $\Psi^p_s = \Psi^p_s(\cdot, \omega)$, where $N_s$ is the multiplicity of $\Lambda_s$. See Lemmata B.9 and B.11 for the measurability properties.

For $\omega \in \mathcal{O}$ and $\lambda \notin \text{Sp}(-\Delta_\mathcal{O})$ we define $\hat{b}(\omega, \lambda; \cdot) \in W^{1,2}_0(P_\omega)$ as the solution to the problem

$$(-\Delta_{P_\omega} - \lambda)\hat{b} = 1. \quad (19)$$

We assume that $\hat{b}$ is extended by zero outside $P_\omega$. By applying the spectral decomposition, we can write it in terms of the eigenfunctions of $-\Delta_{P_\omega}$,

$$\hat{b}(\omega, \lambda; x) = \sum_{s=1}^{\infty} \sum_{p=1}^{N_s(\omega)} \left( \frac{\int_{\mathbb{R}^d} \Psi^p_s(\cdot, \omega) \Psi^p_s(x, \omega)}{\Lambda_s - \lambda} \right), \quad x \in \mathbb{R}^d. \quad (20)$$

The results of Appendix B imply that for a fixed $\lambda$ the mapping $\omega \mapsto \hat{b}(\omega, \lambda; \cdot)$, taking values in $L^2(\mathbb{R}^d)$, is measurable.

For any locally integrable function $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ we denote by $f_{\text{reg}}$ its regularisation, defined at $x \in \mathbb{R}^d$ by

$$f_{\text{reg}}(x) := \begin{cases} \lim_{r \to 0} |B_r(x)|^{-1} \int_{B_r(x)} f(y)dy, & \text{if the limit exists,} \\ 0, & \text{otherwise.} \end{cases}$$

The Lebesgue differentiation theorem states that $f_{\text{reg}} = f$ a.e.

Consider the following function,

$$\hat{b}(\omega, \lambda) := \begin{cases} \hat{b}_{\text{reg}}(\omega, \lambda; D(\omega)), \quad \omega \in \mathcal{O}, \\ 0, \quad \text{otherwise.} \end{cases} \quad (21)$$

We make a few observations. First, by using the regularised representative of $\hat{b}$ we ensure that the right-hand side of (21) is measurable. Second, for fixed $\omega$ and $k$ one has that $\hat{b}_{\text{reg}}(T_x \omega, \lambda; \cdot) = \hat{b}_{\text{reg}}(T_y \omega, \lambda; \cdot)$ for all $x, y \in \mathcal{O}_\omega^k$, therefore, fixing some $y \in \mathcal{O}_\omega^k$ we have

$$\hat{b}(T_x \omega, \lambda) = \hat{b}_{\text{reg}}(T_x \omega, \lambda; D(T_x \omega)) = \hat{b}_{\text{reg}}(T_y \omega, \lambda; D(T_y \omega) + x - y) \quad \forall x \in \mathcal{O}_\omega^k. \quad (22)$$

Third, for the realisations we have $\hat{b}(T_x \omega, \lambda) \in W^{1,2}_{\text{loc}}(\mathbb{R}^d)$. Lastly, for a.e. $\omega \in \mathcal{O}$ one has $x = D(\omega)$ is a Lebesgue point of $\hat{b}_{\text{reg}}(\omega, \lambda; \cdot)$, i.e.

$$\hat{b}_{\text{reg}}(\omega, \lambda; D(\omega)) = \lim_{r \to 0} |B_r(D(\omega))|^{-1} \int_{B_r(D(\omega))} \hat{b}_{\text{reg}}(\omega, \lambda; y)dy.$$

Indeed, denote by $N$ the subset of $\mathcal{O}$ where it is not true. By construction we have

$$\int_{B_R(0)} 1_{\{T_x \omega \in N\}} dx = 0 \quad \text{a.e. } \omega \in \Omega,$$
for a positive $R$. Applying Fubini Theorem, we have
\[
0 = \int_{\Omega} \int_{B_{R}(0)} 1_{\{T_x \omega \in N\}} \text{d}x \text{d}P = \int_{B_{R}(0)} \int_{\Omega} 1_{\{T_x \omega \in N\}} \text{d}P \text{d}x = |B_{R}| |N|.
\]

**Proposition 3.6.** For $\lambda \notin \text{Sp}(\Delta_{\mathcal{O}})$ the function $\tilde{b}(\cdot, \lambda)$ defined by (21) is the solution to (16), i.e. $\tilde{b}$ coincides with $b$.

**Proof.** First we show that $\tilde{b}(\cdot, \lambda) \in L^{2}(\mathcal{O})$. Since $b(\omega, \lambda, \cdot)$ is the solution to (19), taking into account (14) and that by the assumption $|\mathcal{O}_{\omega_{k_{0}}}^{\Delta}| \leq 2^{-d} < 1$, we have
\[
\|b(\omega, \lambda, \cdot)\|_{L^{2}(\mathbb{R}^{d})} < \frac{1}{\text{dist}^{2}(\lambda, \text{Sp}(\Delta_{\mathcal{O}}))}, \quad \omega \in \mathcal{O}.
\] (23)

It follows from Assumption 2.7 that the ball $B_{\rho}$ is contained in $B_{\delta}^{k_{0}}$, in particular, it has no intersections with inclusions $\mathcal{O}_{\omega}^{k}$, $k \neq k_{0}$. Invoking (22) and (23), it follows that
\[
\int_{B_{\rho}} \tilde{b}^{2}(T_{x} \omega, \lambda) \text{d}x < \frac{1}{\text{dist}^{2}(\lambda, \text{Sp}(\Delta_{\mathcal{O}}))}.
\]

Integrating over $\Omega$ and using Fubini Theorem, we obtain
\[
|B_{\rho}| \|b(\cdot, \lambda)\|_{L^{2}(\mathcal{O})}^{2} < \frac{1}{\text{dist}^{2}(\lambda, \text{Sp}(\Delta_{\mathcal{O}}))}.
\]

Next we argue that $\tilde{b}(\cdot, \lambda) \in W^{1,2}(\mathcal{O})$. To this end we define the function $g : \Omega \rightarrow \mathbb{R}^{d}$ as follows:
\[
g(\omega, \lambda) := \begin{cases} (\nabla \tilde{b})_{\text{reg}}(\omega, \lambda; D(\omega)), & \omega \in \mathcal{O}, \\ 0, & \omega \notin \mathcal{O}. \end{cases}
\]

For a fixed $\omega \in \mathcal{O}$ we clearly have $(\nabla \tilde{b})_{\text{reg}}(\omega, \lambda; x) = \nabla \tilde{b}_{\text{reg}}(\omega, \lambda; x) = \nabla \tilde{b}(\omega, \lambda; x)$ for a.e. $x \in \mathbb{R}^{d}$. By observing that
\[
\|\nabla \tilde{b}(\omega, \lambda, \cdot)\|_{L^{2}(\mathbb{R}^{d})} \leq \lambda \|\tilde{b}(\omega, \lambda, \cdot)\|_{L^{2}(\mathbb{R}^{d})}^{2} + \|\tilde{b}(\omega, \lambda, \cdot)\|_{L^{2}(\mathbb{R}^{d})}^{2}
\]
(which follows from (19)), it is not difficult to show that $g \in L^{2}(\mathcal{O}; \mathbb{R}^{d})$, similarly to the claim that $\tilde{b}(\cdot, \lambda) \in L^{2}(\mathcal{O})$. It remains to show that $\nabla_{\omega} \tilde{b} = g$, which is equivalent to
\[
\int_{\Omega} g_{j} v = -\int_{\Omega} \tilde{b} \mathcal{D}_{j} v \quad \forall v \in W^{1,2}(\Omega), \quad j = 1, \ldots, d.
\] (24)

Denote by $\tilde{B}_{R}(\omega)$ the union of inclusions contained in the ball $B_{R}$,
\[
\tilde{B}_{R}(\omega) := \bigcup_{\mathcal{O}_{\omega}^{k} \subset B_{R}} \mathcal{O}_{\omega}^{k}.
\]

We need the following simple assertion.
Lemma 3.7. Suppose \( f \in L^1(\Omega) \), let \( U \subset \mathbb{R}^d \) be an open bounded star-shaped set, and for a fixed \( \delta > 0 \) let \( V_R \) be a sequence of measurable sets satisfying \( (R - \delta)U \subset V_R \subset RU \). Then one has almost surely

\[
\int_{\Omega} f = \lim_{R \to \infty} \frac{1}{|RU|} \int_{V_R} f(T_x\omega)dx.
\]

The proof is straightforward and follows from the ergodic theorem and the observation that

\[
\lim_{R \to \infty} \frac{1}{|RU|} \int_{RU \setminus (R-\delta)U} |f(T_x\omega)| \, dx = 0.
\]

Applying the above lemma, we get

\[
\int_{\Omega} g_j v = \lim_{R \to \infty} \frac{1}{|BR|} \int_{BR(\omega)} g_j(T_x\omega)v(T_x\omega)dx, \quad j = 1, \ldots, d,
\]

\[
\int_{\Omega} \tilde{b} D_j v = \lim_{R \to \infty} \frac{1}{|BR|} \int_{BR(\omega)} \tilde{b}(T_x\omega, \lambda)(D_j v)(T_x\omega)dx, \quad j = 1, \ldots, d.
\]  \hspace{1cm} \text{(25)}

Similarly to (22), for a fixed \( O_\omega^k \subset B_R \) and a fixed \( y \in O_\omega^k \) we have

\[
g(T_x\omega) = (\nabla \tilde{b})_{reg}(T_x\omega, \lambda; D(T_x\omega)) = \nabla_x h_{reg}(T_y\omega, \lambda; D(T_y\omega) + x - y) \, \text{a.e. } x \in O_\omega^k.
\]

Integrating by parts in each of the inclusions, we obtain

\[
\int_{BR(\omega)} g_j(T_x\omega)v(T_x\omega)dx = -\int_{BR(\omega)} \tilde{b}(T_x\omega, \lambda)D_j v(T_x\omega), \quad j = 1, \ldots, d,
\]

which in conjunction with (25) implies (24).

Writing (19) in the weak form,

\[
\int_{P_\omega} \nabla \tilde{b}(\omega, \lambda; x) \cdot \nabla v(T_x\omega)dx - \lambda \int_{P_\omega} \tilde{b}(\omega, \lambda; x)v(T_x\omega)dx = \int_{P_\omega} v(T_x\omega)dx \quad \forall v \in W^{1,2}_0(\Omega),
\]

and arguing in a similar way to the above via Lemma 3.7 one easily arrives at

\[
\int_{O} \nabla \omega \tilde{b}(\cdot, \lambda) \cdot \nabla v - \lambda \int_{O} \tilde{b}(\cdot, \lambda)v = \int_{O} v \quad \forall v \in W^{1,2}_0(O),
\]

which concludes the proof. \( \square \)

Remark 3.8. Applying Lemma 3.7 to the definition (15) we obtain the following representation:

\[
\beta(\lambda) = \lim_{R \to \infty} \left( \lambda + \lambda^2 \frac{1}{|BR|} \int_{BR(\omega)} b(T_x\omega, \lambda) dx \right). \hspace{1cm} \text{(26)}
\]

(In fact, one can replace \( \hat{B}_R(\omega) \) with \( B_R \) here.) We emphasise the importance of the above formula: in order to calculate \( \beta(\lambda) \) one does not need to know \( b(\cdot, \lambda) \) on \( \Omega \), but only its realisation \( b(T_x\omega, \lambda) \) for a single \( \omega \), which, as a consequence of Proposition 3.6, can be obtained from (22) (with \( \hat{b} \) replaced by \( b \)) by solving the resolvent problem (19) on every inclusion.
Proposition 3.9. Zhikov’s \(\beta\)-function is differentiable and its derivative is uniformly positive:

\[ \beta'(\lambda) \geq 1 - P(O). \]

Proof. For \(\omega \in O\) we have (cf. (20))

\[
\int_{\mathbb{R}^d} \tilde{b}(\omega, \lambda, \cdot) = \sum_{s=1}^{\infty} \sum_{p=1}^{N_s(\omega)} \left( \int_{\mathbb{R}^d} \Psi_s^p \right)^2, \tag{27}
\]

Observing that

\[
\sum_{s=1}^{\infty} \sum_{p=1}^{N_s(\omega)} \left( \int_{\mathbb{R}^d} \Psi_s^p \right)^2 = |P_\omega|, \tag{28}
\]

a direct calculation yields

\[
\frac{\partial}{\partial \lambda} \left( \lambda^2 \int_{\mathbb{R}^d} \tilde{b}(\omega, \lambda, \cdot) \right) = -\sum_{s=1}^{\infty} \sum_{p=1}^{N_s(\omega)} \left( \int_{\mathbb{R}^d} \Psi_s^p \right)^2 + \sum_{s=1}^{\infty} \sum_{p=1}^{N_s(\omega)} \frac{\Lambda_s^2 \left( \int_{\mathbb{R}^d} \Psi_s^p \right)^2}{(A_s - \lambda)^2} \geq -|P_\omega|, \tag{29}
\]

By construction, one has

\[
\int_{\mathcal{O}_\omega^k} b(T_x \omega, \lambda) dx = \int_{\mathbb{R}^d} \tilde{b}(T_y \omega, \lambda, \cdot) \quad \forall \text{ fixed } y \in \mathcal{O}_\omega^k.
\]

Then is not difficult to see that on any closed interval \([\lambda_1, \lambda_2] \subset \text{Dom}(\beta)\) the right-hand side of (26) and its derivative

\[
\frac{\partial}{\partial \lambda} \left( \lambda + \lambda^2 \frac{1}{|B_R|} \int_{B_R(\omega)} b(T_x \omega, \lambda) dx \right)
\]

converge uniformly (cf. (28) and (29) and Lemma 3.7). It follows that \(\beta(\lambda)\) is differentiable. Moreover, from (26) and (29) we have

\[
\beta'(\lambda) = 1 - \lim_{R \to \infty} \frac{1}{|B_R|} \int_{B_R(\omega)} 1_{\mathcal{O}_\omega} = 1 - P(O).
\]

Remark 3.10. It was shown in [CCV18] that

\[
\text{Sp}(-\Delta_O) = \bigcup_{x \in \mathbb{R}^d} \bigcup_{s \in \mathbb{N}} \{ \Lambda_s(T_x \omega) \}
\]

almost surely. In particular, this implies that the set on the right-hand side is deterministic almost surely. In fact, it is possible to prove the latter using a different simpler argument based on ergodicity, see the discussion in Section 4.
3.2 Resolution of identity for $-\Delta \Omega$

For a fixed $\omega \in \Omega$ we can think of the family of Dirichlet Laplace operators on each individual inclusion as the realisation of $-\Delta \Omega$ in the physical space. Moreover, we can characterise the resolution of identity for $-\Delta \Omega$ by using the spectral projectors of these Dirichlet Laplace operators on the inclusions and then pulling it back to the probability space.

For $0 \leq t_1 \leq t_2$ we define the mappings $\tilde{E}_{(t_1,t_2]}$ and $E_{(t_1,t_2]}$ as follows. For all $\varphi \in L^2(\Omega)$ let

$$
(\tilde{E}_{(t_1,t_2]} \varphi)(\omega, x) := \sum_{s=1}^{\infty} \sum_{p=1}^{N_s(\omega)} 1\{t_1 < \Lambda_s(\omega) \leq t_2\} \left( \int_{\mathbb{R}^d} \Psi_p^s(\cdot, \omega) \Phi(T_s x, \omega) \right) \Phi_p^s(x, \omega), \quad \omega \in \Omega,
$$

and

$$
(E_{(t_1,t_2]} \varphi)(\omega) = (\tilde{E}_{(t_1,t_2]} \varphi)_{\text{reg}}(\omega, D(\omega)).
$$

The mapping $E_{[t_1,t_2]}$ is defined analogously. In what follows we extend functions from $L^2(\Omega)$ by zero in $\Omega \setminus \mathcal{O}$ without mentioning it explicitly.

**Proposition 3.11.** $E_{[0,t]}$ is the resolution of identity for the operator $-\Delta \Omega$.

**Proof.** Arguing as in the proof of Proposition 3.6 we conclude that there exists $C > 0$, independent of $t_1$ and $t_2$, such that

$$
\|E_{(t_1,t_2]} \varphi\|_{L^2(\Omega)} \leq C \|\varphi\|_{L^2(\Omega)} \quad \forall \varphi \in L^2(\Omega),
$$

and that for a.e. $\omega \in \Omega$ we have

$$
(E_{(t_1,t_2]}\varphi_1, \varphi_2) = \lim_{R \to \infty} \frac{1}{|B_R|} \int_{B_R(\omega)} (E_{(t_1,t_2]} \varphi_1) (T_x \omega) (T_x \varphi_2) dx \quad \forall \varphi_1, \varphi_2 \in L^2(\Omega). \quad (31)
$$

In order prove that $E_{[0,t]}$ is the resolution of identity, we next verify the following properties:

(a) $E_{[0,t]}$ is an orthogonal projection;
(b) $E_{[0,t_1]} \leq E_{[0,t_2]}$ if $t_1 \leq t_2$;
(c) $E_{[0,t]}$ is right continuous in the strong topology;
(d) $E_{[0,t]} \to 0$ if $t \to 0$ and $E_{[0,t]} \to I$ if $t \to +\infty$ in the strong topology.

By using (31) and a representation of $(E_{(t_1,t_2]} \varphi_1)(T_x \omega)$ analogous to (22), it is easy to see that

$$
(E_{(t_1,t_2]} \varphi_1, \varphi_2) = (\varphi_1, E_{(t_1,t_2]} \varphi_2),
$$

$$
(E_{(t_1,t_2]} E_{(t_1,t_2]} \varphi_1, \varphi_2) = (E_{(t_1,t_2]} \varphi_1, E_{(t_1,t_2]} \varphi_2) = (E_{(t_1,t_2]} \varphi_1, \varphi_2).
$$

This implies (a). In the same way one can see that for $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4$ one has

$$
E_{(t_1,t_2]} E_{(t_3,t_4]} = 0, \text{ and } E_{(t_1,t_3]} = E_{(t_1,t_2]} + E_{(t_2,t_3]},
$$

so (b) holds.

Finally, we prove (c). For a fixed a.e. $\omega \in \Omega$ and $\varepsilon > 0$ small enough we have

$$
\int_{B_\rho} \left| E_{[0,t+\varepsilon]} \varphi(T_x \omega) - E_{[0,t]} \varphi(T_x \omega) \right|^2 dx = 0.
$$
only by the coefficients $m$

Comparing the right-hand sides of (32) and (33) on realisations, we see that on each inclusion $O$ for passing to realisations, and using the ergodic theorem), one can show that $Y \subset \phi$ they are linear combinations of the projections of Arguing in a similar way as in the proof of Proposition 3.6 (recall that $B\hat{\phi}$ we have that $|E_{[0,t]}|\phi(T_\omega)\phi(T_\omega)|^2 dx \leq \int_\Omega |\phi^2(T_\omega)| dx$.

Integrating the right-hand side of the last inequality over $\Omega$ we get

$$\int_\Omega \int_\Omega |\phi^2(T_\omega)| dx d\omega = \|\phi\|_{L^2(\Omega)}^2.$$ 

Thus by the Lebesgue theorem on dominated convergence we conclude that

$$\int_\Omega \int_\Omega |E_{[0,t]}|\phi(T_\omega)\phi(T_\omega)|^2 dx d\omega = |B\rho|\|E_{[0,t]}|\phi|_{L^2(\Omega)}^2 \to 0,$n and \(\phi\)

and (c) is proven. In an analogous way we prove (d), cf. also Remark 2.11.

No we prove that $E_{[0,t]}$ is the resolution of identity associated with the operator $-\Delta O$, i.e.

$$-\Delta O = T := \int_0^{+\infty} s dE_s, \quad \text{Dom}(T) = \left\{ \phi \in L^2(O) : \int_0^{+\infty} s^2 dE_s \phi < \infty \right\}.$$ 

Consider the space

$$\mathcal{Y} := \bigcup_{t>0} \text{Im} E_{[0,t]}.$$

By property (d) we have that $\mathcal{Y}$ is dense in $L^2(O)$. Let $\phi \in \mathcal{Y}$, then $\phi = E_{[0,t]}|\phi|$ for some $t$ and

$$T \phi = \int_0^t s dE_s \phi = \lim_{n \to \infty} \sum_{m=1}^n m \frac{t}{n} E_{[(m-1)\frac{t}{n},m\frac{t}{n}]} \phi.$$ 

(32)

Arguing in a similar way as in the proof of Proposition 3.6 (i.e. using the definition of $E_{[0,t]}|\phi|$, passing to realisations, and using the ergodic theorem), one can show that $\mathcal{Y} \subset \text{Dom}(-\Delta O)$, and for $\phi$ as above one has

$$-\Delta O \phi(\omega) = \sum_{s=1}^\infty \Lambda_s(\omega) \chi_{\{0 \leq \Lambda_s(\omega) \leq t\}} \left( \tilde{E}_{[\Lambda_s(\omega),\Lambda_s(\omega)]} \phi \right)_{\text{reg}}(\omega, D(\omega)).$$ 

(33)

Comparing the right-hand sides of (32) and (33) on realisations, we see that on each inclusion $O_k^k$ they are linear combinations of the projections of $\phi(T_\omega)$ on the eigenspaces of $-\Delta O_k$ and differ only by the coefficients $m \frac{t}{n}$ and $\Lambda_s(T_\omega)\phi$ respectively. Thus we have

$$\frac{1}{|B_R|} \int_{B_R(\omega)} \sum_{m=1}^n m \frac{t}{n} \left( E_{[(m-1)\frac{t}{n},m\frac{t}{n}]} \phi \right) (T_\omega)$$

$$- \sum_{s=1}^\infty \Lambda_s(T_\omega) \chi_{\{0 \leq \Lambda_s(T_\omega) \leq t\}} \left( \tilde{E}_{[\Lambda_s(T_\omega),\Lambda_s(T_\omega)]} \phi \right)_{\text{reg}}(T_\omega, D(T_\omega)) \left| \frac{t^2}{n^2} \frac{1}{|B_R|} \int_{B_R(\omega)} \left| (E_{[0,t]} \phi) (T_\omega) \right|^2 dx \right| dx \leq \frac{t^2}{n^2} \frac{1}{|B_R|} \int_{B_R(\omega)} \left| (E_{[0,t]} \phi) (T_\omega) \right|^2 dx.$$
Passing to the limit as $R \to \infty$ via Lemma 3.7 we get

$$
\left\| \sum_{m=1}^{n} \frac{t}{n} E\left((m-1)\frac{l}{n},m\frac{l}{n}\right) \varphi - \sum_{s=1}^{\infty} \Lambda_s 1_{\{0 \leq \Lambda_s \leq t\}} E[\Lambda_s, \Lambda_s] \varphi \right\|_{L^2(\mathcal{O})} \leq \frac{t}{n} \| E[0,t] \varphi \|_{L^2(\mathcal{O})}.
$$

Now passing to the limit as $n \to \infty$ we conclude that $T \varphi = -\Delta_\mathcal{O} \varphi$, i.e. the (symmetric) restrictions of the operators $T$ and $-\Delta_\mathcal{O}$ to $\mathcal{Y}$ coincide.

To conclude the proof it remains to show that either of the restrictions is essentially self-adjoint. To this end it suffices to prove that the image of $T|_{\mathcal{Y}} \pm i$ is dense in $L^2(\mathcal{O})$. But this is obvious by observing that for any $\varphi \in L^2(\mathcal{O})$ one has

$$
E[0,t] \varphi = \lim_{n \to \infty} \sum_{m=1}^{n} \frac{1}{m} \int_{(m-1)\frac{l}{n}}^{m\frac{l}{n}} (s \pm i) dE_s \varphi,
$$

where the limit is understood in the $L^2(\mathcal{O})$ sense.

\[\square\]

**Remark 3.12.** Similarly to Remark 3.8 one can recover the resolution of identity from realisations for a single (typical) $\omega$ via formulae (30)–(31).

### 3.3 Point spectrum of the limit operator on $\mathbb{R}^d$

In this section we first provide a general characterisation of the point spectrum of $A_{\text{hom}}$, arguing that it is determined completely by the point spectrum of $-\Delta_\mathcal{O}$ (in complete analogy with the periodic setting [Zh04]). Combining it with the above construction of the resolution of identity, we then illustrate it by considering two classes of examples.

**Proposition 3.13.** The point spectrum of $A_{\text{hom}}$ consists of those eigenvalues of $-\Delta_\mathcal{O}$ whose eigenfunctions have zero mean:

$$
\text{Sp}_p(A_{\text{hom}}) = \{ \lambda \in \text{Sp}_p(-\Delta_\mathcal{O}) : \exists \psi \in W_0^{1,2}(\mathcal{O}) \text{ such that } -\Delta_\mathcal{O} \psi = \lambda \psi, \langle \psi \rangle = 0 \}.
$$

**Proof.** Consider the eigenvalue problem for $A_{\text{hom}}$ (cf. (11)–(12)):

$$
-\nabla A_{\text{hom}}^1 \nabla u_0 = \lambda (u_0 + \langle u_1 \rangle),
$$

$$
-\Delta_\mathcal{O} u_1(x, \cdot) = \lambda (u_0(x) + u_1(x, \cdot)),
$$

where $u_0 + u_1 \in V$. Suppose that $\lambda \in \text{Sp}_p(-\Delta_\mathcal{O})$ and the corresponding eigenfunction $\psi$ has zero mean. Then the pair $u_0 = 0, u_1 = v \psi$, where $v \in L^2(\mathbb{R}^d)$, clearly satisfies (34). Hence $\lambda \in \text{Sp}_p(A_{\text{hom}})$.

Now suppose $\lambda \in \text{Sp}_p(A_{\text{hom}})$, i.e. (34) holds. We prove that $u_0 = 0$. Indeed, assuming the contrary, we infer from the second equation in (34) that the problem

$$
-\Delta_\mathcal{O} v = \lambda v + 1
$$

is solvable. Notice that all its solutions have the form $v = v_0 + \psi$, where $v_0$ is a fixed solution and $\psi \in E_{[\lambda, \lambda]}(L^2(\mathcal{O}))$. Moreover, all elements of $E_{[\lambda, \lambda]}(L^2(\mathcal{O}))$ have zero mean by the solvability
condition for (35). It is easy to see that the solution of the second equation in (34) has the form
\[ u_1 = \lambda u_0 v_0 + \varphi, \]
where \( \varphi \in L^2(\mathbb{R}^d; E_{[\lambda,\lambda]}(L^2(\mathcal{O}))) \). Substituting this into the first equation we, arrive at
\[ -\nabla A^{\text{hom}}_1 \nabla u_0 = (\lambda + \lambda^2 v_0) u_0. \]
But this contradicts to the fact that the spectrum of \(-\nabla A^{\text{hom}}_1 \nabla \) in \( \mathbb{R}^d \) has no eigenvalues, therefore \( u_0 = 0 \), as required. Then necessarily \( \langle u_1 \rangle = 0 \), and
\[ u_1(x, \cdot) \] is an eigenfunction of \(-\Delta_{\mathcal{O}}\) for a.e. \( x \in \mathbb{R}^d \).

### 3.3.1 An example with a finite number of shapes of inclusions

Assume that for a.e. \( \omega \) the inclusions \( \mathcal{O}^k_{\omega} \) are (translated and rotated) copies of a finite number of sets (shapes). By (14) the spectrum of \(-\Delta_{\mathcal{O}}\) is the union of the spectra of the Dirichlet Laplacian on each of these sets. In particular, \( \text{Sp}(-\Delta_{\mathcal{O}}) \) is discrete. It is not difficult to see (cf. the construction of the resolution of identity) that the eigenspaces of \(-\Delta_{\mathcal{O}}\) consist of those functions whose realisations are appropriate eigenfunctions on each inclusion. Thus, the point spectrum of \( A^{\text{hom}}_\omega \) consists of the union of the eigenvalues of each shape whose all eigenfunctions have zero mean.

### 3.3.2 An example with continuum family of scaled copies of one shape

Assume that for a.e. \( \omega \) the inclusions \( \mathcal{O}^k_{\omega} \) are scaled (translated and rotated) copies of an open set \( \check{\mathcal{O}} \subset \mathbb{R}^d \) (shape), where the scaling parameter \( r \) takes values in some interval \( [r_1, r_2] \subset (0, +\infty) \). We will write \( \mathcal{O}^k_{\omega} \sim r\check{\mathcal{O}} \) if the inclusion \( \mathcal{O}^k_{\omega} \) is a translation and rotation of \( r\check{\mathcal{O}} \). Let \( \{\nu_j\}_{j=1}^\infty \) be the sequence of eigenvalues for the Dirichlet Laplacian on \( \check{\mathcal{O}} \). Then the spectrum of the Dirichlet Laplacian on \( \mathcal{O}^k_{\omega} \sim r\check{\mathcal{O}} \) is \( \{r^{-2}\nu_j\}_{j=1}^\infty \). Clearly, for any \( t \in \mathbb{R} \) there are at most finitely many values of the scaling parameter satisfying \( t = r^{-2}\nu_j \) for some \( j \).

Consider the situation when the probability measure is absolutely continuous with respect to \( r \). We can make this assumption precise using the ergodicity rather than talking about an abstract probability space. Namely, we assume that for a.e. \( \omega \) and all \( r \in [r_1, r_2] \) one has
\[ \lim_{R \to \infty} \frac{1}{|B_R|} \int_{B_R(\omega)} \sum_{k: \mathcal{O}^k_{\omega} \sim r\check{\mathcal{O}}} 1_{\mathcal{O}^k_{\omega}} = 0, \]
equivalently, that the quantity
\[ \lim_{R \to \infty} \frac{1}{|B_R|} \int_{B_R(\omega)} \sum_{k: \mathcal{O}^k_{\omega} \sim r\check{\mathcal{O}}, r \in [r-\delta, r+\delta]} 1_{\mathcal{O}^k_{\omega}} \]
vanishes as \( \delta \to 0 \). It is not difficult to see, cf. (31), that in this case the resolution of identity \( E_{[0,t]}, t \geq 0 \), has no jumps, i.e. \( \text{Sp}_p(-\Delta_{\mathcal{O}}) = \emptyset \).

### 4 Convergence of the spectra

As we have already discussed in the introduction, the spectrum of \( A^\varepsilon \) and its limit behaviour in the whole space setting is very different from the bounded domain situation. Many ideas we used in [CCV18] were an adaptation of well-known techniques from periodic high-contrast homogenisation. The analysis of the whole space setting, however, calls for methods that go beyond those used in...
the bounded domain case. The rest of the paper is devoted wholly to this problem, and in what follows we will always assume that \( S = \mathbb{R}^d \).

In this section we analyse the limit of the spectra of \( \mathcal{A}^\varepsilon \). In order to understand the basic difference between the bounded domain and the whole space settings, one may think of the following simple example: the set of inclusions \( \Omega_\omega \) is generated by putting equal size balls with probability \( \frac{1}{2} \) at the nodes of a periodic lattice. In this example the spectrum of \( \mathcal{A}_{\text{hom}} \) has a band-gap structure very similar (but with the left ends of the bands moved slightly to the left) to the corresponding periodic case. Yet, \( \text{Sp}(\mathcal{A}^\varepsilon) = \mathbb{R}^+ \). Indeed, due to the law of large numbers, there exists a sequence of balls \( B_R(x_R), R \in \mathbb{N} \), that is completely void of inclusions from \( \varepsilon \Omega_\omega \). Then, for an arbitrary \( \lambda \geq 0 \), by taking \( e^{ikx} \) with \( |k|^2 = \lambda \), multiplying it by suitable cut-off functions and normalizing, one constructs an explicit, supported on \( B_R(x_R), R \in \mathbb{N} \), Weyl sequence for the operator \( \mathcal{A}^\varepsilon \).

In more advanced (and more realistic as far as applications concerned) examples, such as the random parking model, see Section 4.6.6 below, the spectrum of \( \text{Sp}(\mathcal{A}^\varepsilon) \) may exhibit gaps which persist in the limit. However, the general idea is the same: the additional part of the spectrum, not accounted for by \( \text{Sp}(\mathcal{A}_{\text{hom}}) \) in the limit, is present due to the arbitrary large areas of space with non-typical distribution of inclusions.

In order to account for this part of the spectrum in the limit, one, somewhat surprisingly, may use a very close analogue of the \( \beta \)-function as given by the formula (26). Intuitively, one can think of a "local \( \beta \)-function", i.e. a local average of the expression \( \lambda + \lambda^2 b(T_x, \lambda) \), whose "gaps" (intervals where it is negative) get bigger when the local volume fraction of the inclusions is relatively large, and, conversely, shrink when the local volume fraction of the inclusions is small (or completely disappear if the volume fraction tends to zero). In order to make this more precise one needs to look at the behaviour of the distribution of inclusions in large randomly located sets.

### 4.1 Main results

For \( \lambda \notin \text{Sp}(-\Delta_\mathcal{O}) \) we define:

\[
\beta_\infty(\lambda, \omega) := \liminf_{M \to \infty} \sup_{x \in \mathbb{R}^d} \ell(x, M, \lambda, \omega),
\]

where

\[
\ell(x, M, \lambda, \omega) := \lambda + \lambda^2 \frac{1}{M^d} \int_{\square^M_x} b(T_y, \lambda) \, dy,
\]

and \( \square^M_x \) denotes the cube of edge length \( M \) centred at \( x \),

\[
\square^M_x := x + M \square.
\]

We will use the notation \( \square^M \) for the cube of edge length \( M \) centred at the origin.

Notice that \( \ell(x, M, \lambda, \cdot) \) is measurable, which implies the measurability of \( \beta_\infty(\lambda, \cdot) \).

**Remark 4.1.** One may interpret the term \( \lambda^2 M^{-d} \int_{\square^M} b(T_y, \lambda) \, dy \) as the local averaged resonant (or anti-resonant, if negative) contribution from the inclusions, which, from physical point of view play a role of micro-resonators, to the term \( \ell(x, M, \lambda, \omega) \). The latter in turn may be interpreted as a "local spectral average" – the precursor for the spectral parameter \( \beta_\infty(\lambda, \omega) \).

By Proposition 4.9 the function \( \beta_\infty \) is deterministic almost surely, that is

\[
\beta_\infty(\lambda, \omega) = \beta_\infty(\lambda) \text{ for a.e. } \omega.
\]
The set
\[ \mathcal{G} := \text{Sp}(-\Delta_{\mathcal{O}}) \cup \{ \lambda : \beta_{\infty}(\lambda) \geq 0 \} \]
is an “upper bound” for the limit of the spectra of \( A^\varepsilon \), as stated next.

**Theorem 4.2.** One has
\[ \lim_{\varepsilon \to 0} \text{Sp}(A^\varepsilon) \subset \mathcal{G}, \]
where the limit is understood in the sense of Hausdorff.

We provide the proof of the theorem in Section 4.3.

In order to prove that \( \mathcal{G} \) is actually the limit of \( \text{Sp}(A^\varepsilon) \) we make an additional finite range correlation assumption, which we describe next. For a compact set \( K \subset \mathbb{R}^d \) we denote by \( \mathcal{H}_K \) the set-valued mapping
\[ \mathcal{H}_K(\omega) := \overline{\mathcal{O}_\omega} \cap K. \]
By \( \mathcal{F}_{H,K} \) we denote the \( \sigma \)-algebra on the set of all compact subsets of \( K \) generated by the Hausdorff metric \( d_H \). The proof of the following lemma is completely analogous to that of Lemma B.4.

**Lemma 4.3.** The set-valued mapping \( \mathcal{H}_K \) is measurable with respect to the \( \sigma \)-algebra \( \mathcal{F}_{H,K} \).

For a compact set \( K \subset \mathbb{R}^d \) we define the \( \sigma \)-algebra \( \mathcal{F}_K \) by
\[ \mathcal{F}_K := \{ \mathcal{H}_K^{-1}(\mathcal{F}_{H,K}) : \mathcal{F}_{H,K} \in \mathcal{F}_{H,K} \}. \]

**Assumption 4.4.** There exists \( \kappa \in \mathbb{R}^+ \) such that for every two compact sets \( K_1, K_2 \subset \mathbb{R}^d \) satisfying \( \text{dist}(K_1, K_2) > \kappa \) the \( \sigma \)-algebras \( \mathcal{F}_{K_1} \) and \( \mathcal{F}_{K_2} \) are independent.

**Theorem 4.5.** Under Assumption 4.4 one has
\[ \lim_{\varepsilon \to 0} \text{Sp}(A^\varepsilon) \supset \mathcal{G} \]
in the sense of Hausdorff (and hence, in view of Theorem 4.2, the two sets are equal).

The proof of the theorem is given in Section 4.5.

Assumption 4.4 is only needed for the proof of Theorem 4.5. It is not required for any other result in this paper. One can probably relax Assumption 4.4 by allowing for sufficiently quickly decaying (e.g. exponentially) rather than finite range correlation. However, we do not know whether any version of Assumption 4.4 can be discarded altogether. Finite range correlation assumption guarantees the almost sure existence of arbitrarily large cubes with almost periodic arrangement of inclusions. This allows an explicit construction of approximate eigenfunctions starting from the generalised eigenfunctions of an operator with constant coefficients, which is the macroscopic component of the homogenisation limit for the corresponding periodic operator. In the absence of such (almost) periodic structures one needs to analyse “arbitrary” sequences of operators \( \tilde{A}^\varepsilon \), each capturing a specific point of the limit spectrum of \( A^\varepsilon \). These \( \tilde{A}^\varepsilon \) are obtained from \( A^\varepsilon \) by shifting to the origin the areas of non-typical distribution of inclusions on which relevant quasimodes are supported (see the proof of Theorem 4.2 below). The limits of these sequences, understood in an appropriate sense, correspond neither to stochastic nor to periodic homogenisation, and therefore have unpredictable spectral properties in general.
4.2 Properties of $\beta_\infty(\lambda, \omega)$

The main purpose of this subsection is to assert the deterministic nature of $\beta_\infty(\lambda, \omega)$, as well as its continuity and monotonicity. We begin with some obvious observations.

**Remark 4.6.** (i) The use of cubes in the definition of $\ell(x, M, \lambda, \omega)$ is not essential. In fact, one can replace cubes with the scaled and translated versions of any sufficiently regular bounded open set — the resulting function $\beta_\infty(\lambda, \omega)$ will be exactly the same (cf. also (26)). The reason we use cubes is that it will be convenient for our constructions in the proofs of both theorems.

(ii) Clearly, one has

$$\beta(\lambda) = \lim_{M \to \infty} \sup_{x \in \mathbb{R}^d} \left( \lambda + \lambda^2 \frac{1}{M^d} \int_{\Box^M} b(T_y \omega, \lambda) dy \right) = \beta_\infty(\lambda, \omega).$$

We first prove two auxiliary results which will be used throughout the remaining part of Section 4. The proof of the following lemma is straightforward and requires only part 2 of Assumption 2.7 and the bound (23).

**Lemma 4.7.** Let $\kappa > 0$ be fixed, and denote (cf. (27))

$$\ell_\kappa(x, M, \lambda, \omega) := \lambda + \lambda^2 \frac{1}{M^d} \int_{S_{x,\kappa}^M} b(T_y \omega, \lambda) dy,$$

where $S_{x,\kappa}^M$ is a measurable set satisfying $\Box^M - \kappa \subset S_{x,\kappa}^M \subset \Box^M$ (clearly, the value of $\ell_\kappa(x, M, \lambda, \omega)$ depends on the choice of $S_{x,\kappa}^M$, but we do not reflect it in the notation). Denote also

$$\tilde{\ell}(x, M, \lambda, \omega) := \lambda + \lambda^2 \frac{1}{M^d} \int_{\Box^M \setminus \partial_x^M(\omega)} b(T_y \omega, \lambda) dy,$$

where $\Box^M(\omega) := \bigcup_{O_{x,\kappa}^M} \Box_x^M$ (i.e. the inclusions touching the boundary of the cube are removed). Then

$$\beta_\infty(\lambda, \omega) = \liminf_{M \to \infty} \sup_{x \in \mathbb{R}^d} \ell_\kappa(x, M, \lambda, \omega) = \liminf_{M \to \infty} \sup_{x \in \mathbb{R}^d} \tilde{\ell}(x, M, \lambda, \omega).$$

Note that $\tilde{\ell}(x, M, \lambda, \omega)$ is equal to $\ell_\kappa(x, M, \lambda, \omega)$ for an appropriate choice of $S_{x,\kappa}^M$.

**Lemma 4.8.** Let $(M_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ be an arbitrary sequence diverging to infinity. Then almost surely one has

$$\beta_\infty(\lambda, \omega) = \lim_{n \to \infty} \sup_{x \in \mathbb{R}^d} \ell(x, M_n, \lambda, \omega).$$

**Proof.** For a fixed $\omega \in \Omega$ let a sequence $C_n \to \infty$ be such that

$$\beta_\infty(\lambda, \omega) = \lim_{n \to \infty} \sup_{x \in \mathbb{R}^d} \ell(x, C_n, \lambda, \omega).$$
and suppose that there exists a subsequence of $M_n$ (still indexed by $n$) such that

$$
\lim_{n \to \infty} \sup_{x \in \mathbb{R}^d} \ell(x, M_n, \lambda, \omega) > \beta_\infty(\lambda, \omega).
$$

We take a sequence $x_n$ such that

$$
\lim_{n \to \infty} \sup_{x \in \mathbb{R}^d} \ell(x, M_n, \lambda, \omega) = \lim_{n \to \infty} \ell(x_n, M_n, \lambda, \omega).
$$

Thus there exists $n_0 \in \mathbb{N}$ and $\delta > 0$ such that for every $n \geq n_0$, we have that

$$
\ell(x_n, M_n, \lambda, \omega) > \beta_\infty(\lambda, \omega) + \delta,
$$

$$
\left| \sup_{x \in \mathbb{R}^d} \ell(x, C_n, \lambda, \omega) - \beta_\infty(\lambda, \omega) \right| < \frac{\delta}{2}.
$$

Next, we cover each cube $\Box_{x_n}^{M_n}$ with cubes of edge length $C_n$. More precisely, for $N_n := \lfloor M_n/C_n \rfloor C_n$ ($\lfloor . \rfloor$ denotes the integer part) the cube $\Box_{x_n}^{N_n} \subset \Box_{x_n}^{M_n}$ is a union of $\lfloor M_n/C_n \rfloor^d$ disjoint cubes of edge length $C_n$. On the one hand, observing that $N_n/M_n \to 1$ as $n \to \infty$ and applying Lemma 4.7 with $\kappa = C_n$ we have

$$
\lim_{n \to \infty} \ell(x_n, N_n, \lambda, \omega) = \lim_{n \to \infty} \ell(x_n, M_n, \lambda, \omega) \geq \beta_\infty(\lambda, \omega) + \delta. \tag{38}
$$

On the other hand, since

$$
\ell(x_n, N_n, \lambda, \omega) = \frac{1}{\lfloor M_n/C_n \rfloor^d} \sum_{i=1}^{\lfloor M_n/C_n \rfloor^d} \ell(x'_i, C_n, \lambda, \omega),
$$

where $x'_i$ denote the centres of the cubes in the partition described above, we conclude that

$$
lim_{n \to \infty} \ell(x_n, N_n, \lambda, \omega) \leq \beta_\infty(\lambda, \omega) + \frac{\delta}{2},
$$

which is a contradiction with (38).

Finally, we present the main result of this subsection.

**Proposition 4.9.** The function

$$(\lambda, \omega) \mapsto \beta_\infty(\lambda, \omega),$$

is deterministic almost surely, continuous on $\mathbb{R}^+ \setminus \text{Sp}(-\Delta_C)$ and strictly increasing on every interval contained in this set.

**Proof.** By Lemma 4.7 and Lemma 4.8 we have

$$
\beta_\infty(\lambda, \omega) = \lim_{M \to \infty} \sup_{x \in \mathbb{Q}^d} \ell(x, M, \lambda, \omega).
$$

The measurability of $\beta_\infty(\lambda, \cdot)$ now follows directly from the measurability of $b(\cdot, \lambda)$. 

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It is easy to see from (27), (28) and (29) that \( \lambda \mapsto \tilde{\ell}(x, M, \lambda) \) has locally bounded (uniformly in \( x, M, \omega \)) and uniformly positive derivative almost surely,

\[
\frac{\partial}{\partial \lambda} \tilde{\ell}(x, M, \lambda, \omega) \geq 1 - \frac{\sum_{O_k \subset \square M} |O_k|}{M^d} \geq C > 0.
\]

Hence it is locally Lipschitz (uniformly in \( x, M, \omega \)) and increasing on every open interval from its domain. More specifically, for \( \lambda_1 < \lambda_2 \) contained in the same open interval from \( \mathbb{R}^+ \setminus \text{Sp}(-\Delta_\infty) \) one has

\[
\tilde{\ell}(x, M, \lambda_2, \omega) - \tilde{\ell}(x, M, \lambda_1, \omega) \geq C(\lambda_2 - \lambda_1). \tag{39}
\]

The bound (39) implies a similar property for \( \beta_\infty \). Indeed, let \( x_n \) be a sequence such that \( \beta_\infty(\lambda_1, \omega) = \lim_{n \to \infty} \tilde{\ell}(x_n, n, \lambda_1, \omega) \). Passing to the limit along \( x_n \) in the above inequality, we arrive at

\[
\beta_\infty(\lambda_2, \omega) - \beta_\infty(\lambda_1, \omega) \geq \limsup_{n \to \infty} \tilde{\ell}(x_n, n, \lambda_2, \omega) - \liminf_{n \to \infty} \tilde{\ell}(x_n, n, \lambda_1, \omega) \geq C(\lambda_2 - \lambda_1).
\]

Utilising a similar argument one can easily see that \( \beta_\infty(\lambda, \omega) \) is locally Lipschitz.

Finally, for a fixed \( \lambda \) the function \( \beta_\infty(\lambda, \cdot) \) is translation invariant and is thus constant almost surely. To conclude that the function \( \beta_\infty(\lambda, \omega) \) is deterministic almost surely, i.e. \( \beta_\infty(\lambda, \omega) = \beta_\infty(\lambda) \), it is enough to take the set of probability one such that for every \( \lambda \in \mathcal{Q} \), \( \lambda \notin \text{Sp}(-\Delta_\infty) \), \( \beta(\lambda, \omega) \) is deterministic and use the almost sure continuity of \( \lambda \mapsto \beta_\infty(\lambda, \omega) \).

**Remark 4.10.** Notice that in general \( \beta_\infty(\lambda) \) is not necessarily differentiable, as can be seen from examples provided at the end of this section.

### 4.3 Proof of Theorem 4.2

Let \( \lambda \) be a limit point of the spectra of \( \mathcal{A}_\varepsilon \), i.e. \( \lambda = \lim_{\varepsilon \to 0} \lambda_\varepsilon \), \( \lambda_\varepsilon \in \text{Sp}(\mathcal{A}_\varepsilon) \). Without loss of generality we may assume that

\[
\lambda \notin \text{Sp}(-\Delta_\infty). \tag{40}
\]

Since \( \lambda_\varepsilon \in \text{Sp} \mathcal{A}_\varepsilon \), there exists a sequence \( u_\varepsilon \in \text{Dom}(\mathcal{A}_\varepsilon) \), \( \|u_\varepsilon\|_{L^2(\mathbb{R}^d)} = 1 \), such that

\[
(\mathcal{A}_\varepsilon - \lambda_\varepsilon)u_\varepsilon = : f_\varepsilon, \quad \text{with} \quad \|f_\varepsilon\|_{L^2(\mathbb{R}^d)} = : \delta_\varepsilon \to 0
\]

(such a sequence can be extracted by the diagonalisation procedure from Weyl sequences corresponding to \( \lambda_\varepsilon \) for each \( \varepsilon \)). Multiplying the above equation by \( u_\varepsilon \) and integrating by parts, we obtain the estimate

\[
\|\varepsilon \nabla u_\varepsilon\|_{L^2(S_0^\varepsilon)} + \|\nabla u_\varepsilon\|_{L^2(S_0^\varepsilon)} \leq C.\tag{41}
\]

By Theorem 2.8 the harmonic extension \( \tilde{u}_\varepsilon \in W^{1,2}(\mathbb{R}^d) \) of \( u_\varepsilon|_{S_0^\varepsilon} \) satisfies the bounds (4), (5) with \( \mathcal{O}_0^\varepsilon \) and \( B_0^\varepsilon \) replaces by their scaled counterparts. We also need to ensure that \( \|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^d)} \) does not vanish the limit as \( \varepsilon \to 0 \). Denoting

\[
v_\varepsilon := u_\varepsilon - \tilde{u}_\varepsilon, \quad v_\varepsilon \in W^{1,2}_0(S_0^\varepsilon),
\]

we have (since \( \tilde{u}_\varepsilon \) is harmonic in the inclusions)

\[
- (\varepsilon^2 \Delta + \lambda^\varepsilon) v_\varepsilon = \lambda^\varepsilon u_\varepsilon^\varepsilon + f_\varepsilon, \quad x \in S_0^\varepsilon, \tag{42}
\]

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or, after the rescaling,
\[-(\Delta_y + \lambda^\varepsilon)v^\varepsilon(ey) = \lambda^\varepsilon\tilde{u}^\varepsilon (ey) + f^\varepsilon (ey), \quad y \in \mathcal{O}_\omega.\]

Since \(\lambda \notin \text{Sp}(-\Delta_\Omega)\), for sufficiently small \(\varepsilon\) the resolvent is uniformly bounded, and hence
\[
\|v^\varepsilon\|_{L^2(\varepsilon\mathcal{O}_\Omega^k)} \leq \|\lambda^\varepsilon\tilde{u}^\varepsilon + f^\varepsilon\|_{L^2(\varepsilon\mathcal{O}_\Omega^k)} \quad \forall \mathcal{O}_\Omega^k \subset \mathcal{O}_\omega. 
\]

A direct implication of this bound is the following statement: for an open set \(U \subset \mathbb{R}^d\) denote
\[
U^\varepsilon := U \bigcup \left( \bigcup \varepsilon\mathcal{O}_\Omega^k \right),
\]
then
\[
\|u^\varepsilon\|_{L^2(U)} \leq C \left( \|\tilde{u}^\varepsilon\|_{L^2(U^\varepsilon)} + \|f^\varepsilon\|_{L^2(U^\varepsilon)} \right),
\]
where the constant \(C\) does not depend on \(U\) or \(\varepsilon\). Furthermore, it is not difficult to see, by using \(v^\varepsilon\) as a test function in (42), and taking into account (43), that
\[
\varepsilon \|
abla v^\varepsilon\|_{L^2(\varepsilon\mathcal{O}_\Omega^k)} \leq C \left( \|	ilde{u}^\varepsilon\|_{L^2(\varepsilon\mathcal{O}_\Omega^k)} + \|f^\varepsilon\|_{L^2(\varepsilon\mathcal{O}_\Omega^k)} \right) \quad \forall \mathcal{O}_\Omega^k \subset \mathcal{O}_\omega.
\]

The sequence \(u^\varepsilon\) converges weakly to zero in \(L^2(\mathbb{R}^d)\) (otherwise its weak stochastic two-scale limit would be an eigenfunction of \(\mathcal{A}^{\text{hom}}\), which is impossible). At the same time, in order to retrieve the information about the relation of \(\lambda\) to the set \(\mathcal{G}\), we need a non-zero limit object associated with the sequence \(u^\varepsilon\). In what follows we devise a compactification procedure for \(u^\varepsilon\) which allows to obtain such an object.

We fix an arbitrary \(L > 0\) and cover \(\mathbb{R}^d\) by the cubes \(\square^L_\xi, \xi \in LZ^d\). A key ingredient of the compactification is the following obvious assertion.

**Lemma 4.11.** Let (possibly finite) sequences \(a_\xi, b_\xi, c_\xi, d_\xi, \xi \in \Pi\), where \(\Pi\) is a countable or finite set of indices, be such that
\[
\sum_\xi a_\xi = \sum_\xi b_\xi = \sum_\xi c_\xi = \sum_\xi d_\xi = 1.
\]

Then there exists \(\xi \in \Pi\) such that \(b_\xi + c_\xi + d_\xi \leq 3a_\xi\).

For each \(\varepsilon\) we apply the above lemma to the sequences
\[
\|u^\varepsilon\|^2_{L^2(\square^L_\xi)}, \quad \frac{\|(A^\varepsilon - \lambda^\varepsilon)u^\varepsilon\|^2_{L^2(\square^L_\xi)}}{3d(\delta^\varepsilon)^2}, \quad \frac{1}{3d}\|u^\varepsilon\|^2_{L^2(\square^L_\xi)}, \quad \frac{\|\chi^\varepsilon_1 \nabla u^\varepsilon\|^2_{L^2(\square^L_\xi)}}{3d\|\chi^\varepsilon\|_{L^2(\mathbb{R}^d)}}, 
\]
\(\xi \in LZ^d, \|u^\varepsilon\|_{L^2(\square^L_\xi)} \neq 0\) (we only consider those cubes \(\square^L_\xi\) where \(u^\varepsilon\) does not vanish identically). Taking into account (41), we infer that for each \(\varepsilon\) there exists \(\xi^\varepsilon\) such that
\[
\|(A^\varepsilon - \lambda^\varepsilon)u^\varepsilon\|_{L^2(\square^L_{\xi^\varepsilon})} \leq C\delta^\varepsilon\|u^\varepsilon\|_{L^2(\square^L_{\xi^\varepsilon})},
\]
\[
\|u^\varepsilon\|_{L^2(\square^L_{\xi^\varepsilon})} + \|\chi^\varepsilon_1 \nabla u^\varepsilon\|_{L^2(\square^L_{\xi^\varepsilon})} \leq C\|u^\varepsilon\|_{L^2(\square^L_{\xi^\varepsilon})}.
\]
Now we shift the cubes $\Box_{\xi}^L$ to the origin and re-normalise the sequence $u^\varepsilon$. Namely, we define

$$w^\varepsilon_L(x) := \frac{u^\varepsilon(x + \xi^\varepsilon)}{\|u^\varepsilon\|_{L^2(\Box_{\xi}^L)}}.$$  

We denote by $\hat{A}^\varepsilon$ the operator obtained from $A^\varepsilon$ by shifting its coefficients: $\hat{A}^\varepsilon(x, \omega) := A^\varepsilon(x + \xi^\varepsilon, \omega)$. Analogously, $\hat{S}^\varepsilon_0$, $\hat{S}^\varepsilon_1$, and $\hat{\chi}^\varepsilon_0$, $\hat{\chi}^\varepsilon_1$, denote appropriately shifted set of inclusions, its complement, and the corresponding characteristic functions. Note that for the harmonic extension of $w^\varepsilon_L|_{\hat{S}_1}$ one has

$$\tilde{w}^\varepsilon_L(x) = \frac{\tilde{u}^\varepsilon(x + \xi)}{\|\tilde{u}^\varepsilon\|_{L^2(\Box_{\xi}^L)}}.$$  

From (46) we immediately have

$$\|f^\varepsilon_L\|_{L^2(\Box_{0}^{L/2})} \leq C\delta^\varepsilon,$$

where $f^\varepsilon_L := (\hat{A}^\varepsilon - \chi^\varepsilon)w^\varepsilon_L = \frac{f^\varepsilon(\cdot + \xi^\varepsilon)}{\|u^\varepsilon\|_{L^2(\Box_{\xi}^L)}}$, (47)

and

$$\|w^\varepsilon_L\|_{L^2(\Box_{0}^{L/2})} + \|\chi^\varepsilon_1\nabla w^\varepsilon_L\|_{L^2(\Box_{0}^{L/2})} \leq C.$$  

Then utilising Theorem 2.8, the bounds (44), (45) rewritten for $w^\varepsilon_L$, and (47) we arrive at

$$\|\tilde{w}^\varepsilon_L\|_{L^2(\Box_{0}^{L/2})} + \|\tilde{\nabla}\tilde{w}^\varepsilon_L\|_{L^2(\Box_{0}^{L/2})} \leq C,$

$$0 < C \leq \|\tilde{w}^\varepsilon_L\|_{L^2(\Box_{0}^{L/2})} \text{ for small enough } \varepsilon,$$

(48)

$$\varepsilon\|\nabla w^\varepsilon_L\|_{L^2(\Box_{0}^{L/2})} \leq C.$$

From the first two bounds in (48) we have that, up to a subsequence,

$$\tilde{w}^\varepsilon_L \to w^0_L \neq 0 \text{ weakly in } W^{1,2}(\Box_{0}^{L/2}) \text{ and strongly in } L^2(\Box_{0}^{L/2}).$$

(49)

Let $\eta \in C^\infty_c(\Box)$ be a cut-off function satisfying $0 \leq \eta \leq 1$, $\eta|_{\Box_{0}^{1/2}} = 1$, and for each $L > 0$ denote

$$\eta_L(\cdot) := \eta(\cdot/L),$$

(50)

so that $\eta_{2L} \in C^\infty_{c}(\Box_{0}^{2L})$, $\eta_{2L}|_{\Box_{0}^1} = 1$, $|\nabla \eta_{2L}| \leq C/L$ for all $L > 0$, with the constant $C$ independent of $L$. Using $\tilde{w}^\varepsilon_L \eta_{2L}$ as a test function in (47) and integrating by parts, we get

$$I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon := \int_{\Box_{2L}} \chi^\varepsilon_1\eta_{2L}A_1\nabla w^\varepsilon_L \cdot \nabla w^\varepsilon_L + \int_{\Box_{2L}} \chi^\varepsilon_1 w^\varepsilon_L A_1\nabla w^\varepsilon_L \cdot \nabla \eta_{2L}$$

$$+ \int_{\Box_{2L}} \varepsilon^2 \chi^\varepsilon_0\nabla w^\varepsilon_L \cdot \nabla (\tilde{w}^\varepsilon_L \eta_{2L}) = \int_{\Box_{2L}} \chi^\varepsilon_1 w^\varepsilon_L \tilde{w}^\varepsilon_L \eta_{2L} + \int_{\Box_{2L}} f^\varepsilon_L \tilde{w}^\varepsilon_L \eta_{2L}.\tag{51}$$

We estimate all the terms but one via (47) and (48) as follows:

$$\lim_{\varepsilon \to 0} I_1^\varepsilon \geq 0, \quad |I_2^\varepsilon| \leq \frac{C}{L}, \quad \lim_{\varepsilon \to 0} I_3^\varepsilon = 0, \quad \lim_{\varepsilon \to 0} \int_{\Box_{2L}} f^\varepsilon_L \tilde{w}^\varepsilon_L \eta_{2L} = 0.\tag{52}$$

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It remains to analyse the behaviour of the first term on the right-hand side of (51). Since $\tilde{w}_L^\varepsilon$ is harmonic in the inclusions, we see that $z^\varepsilon := w_L^\varepsilon - \tilde{w}_L^\varepsilon \in W^{1,2}_0(\hat{S}_0^\varepsilon)$ satisfies the equation
\[-\varepsilon^2 \Delta z^\varepsilon - \lambda \varepsilon z^\varepsilon = \lambda \varepsilon \tilde{w}_L^\varepsilon + f_L^\varepsilon.\]

We use the family of local averaging operators $P^\varepsilon$ on $L^2(\mathbb{R}^d)$ defined in Lemma [E.1] taking for $X_k^\varepsilon$ the inclusions $\varepsilon \mathcal{O}_k^\varepsilon - \xi \subset \hat{S}_0^\varepsilon$ and considering the decomposition $z^\varepsilon = \hat{z}^\varepsilon + \tilde{z}^\varepsilon$, where $\hat{z}^\varepsilon, \tilde{z}^\varepsilon \in W^{1,2}_0(\hat{S}_0^\varepsilon)$ satisfy the following equations:
\[-\varepsilon^2 \Delta \hat{z}^\varepsilon - \lambda \hat{z}^\varepsilon = \lambda P^\varepsilon w_L^0 \quad \text{in } \hat{S}_0^\varepsilon,\]
\[-\varepsilon^2 \Delta \tilde{z}^\varepsilon - \lambda \varepsilon \tilde{z}^\varepsilon = (\lambda - \varepsilon) \hat{z}^\varepsilon + \lambda \varepsilon \tilde{w}_L^0 - \lambda \varepsilon P^\varepsilon w_L^0 + f_L^\varepsilon \quad \text{in } \hat{S}_0^\varepsilon.\] (53)

First, it is easy to see that
\[\hat{z}^\varepsilon(x) = \lambda(P^\varepsilon w_L^0)(x) b(T_{x/\varepsilon^2+\xi \varepsilon \omega}, \lambda).\]

Furthermore, it follows from Lemma [E.1], the bound (23), and a rescaling argument that for sufficiently small $\varepsilon$ one has
\[\|\hat{z}^\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C \frac{\lambda \|w_L^0\|_{L^2(\mathbb{R}^d)}}{\text{dist}(\lambda, \text{Sp}(-\Delta))} \leq C.\] (54)

From the bound (54), (49), Lemma [E.1] and the bound (47) we infer that the right-hand side of (53) vanishes in the limit as $\varepsilon \to 0$. Then, using again the above rescaling argument, we infer that
\[\|\hat{z}^\varepsilon\|_{L^2(\mathbb{R}^d)} \to 0.\] (55)

Next, denote
\[g_\lambda^\varepsilon(x) := b(T_{x/\varepsilon^2+\xi \varepsilon \omega}, \lambda).\]

It follows from (23) that $g_\lambda^\varepsilon$ is bounded in $L^2(\mathbb{R}^d)$, and thus converges weakly in $L^2(\mathbb{R}^d)$, as $\varepsilon \to 0$, up to a subsequence to some $g_\lambda$. Integrating $\lambda + \lambda^2 g_\lambda^\varepsilon$ over an arbitrary fixed cube contained in $\mathbb{R}^d$, passing to the limit, and using the definition of $\beta_\infty(\lambda)$, it is not difficult to see that
\[\lambda + \lambda^2 g_\lambda \leq \beta_\infty(\lambda) \text{ for a.e. } x \in \mathbb{R}^d.\]

Using Lemma [E.1] and the convergence (55), we obtain
\[z^\varepsilon \rightharpoonup \lambda g_\lambda w_L^0, \text{ weakly in } L^2(\mathbb{R}^d).\]

Finally, the strong convergence of $\tilde{w}_L^\varepsilon$, the relations (52), and a passage to the limit in (51) as $\varepsilon \to 0$ yield
\[-\frac{C}{L} \leq \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \lambda (w_L^\varepsilon + z^\varepsilon) \tilde{w}_L^\varepsilon \eta_{2L} = \int_{\mathbb{R}^d} (\lambda + \lambda^2 g_\lambda)(w_L^0)^2 \eta_{2L} \leq \beta_\infty(\lambda) \int_{\mathbb{R}^d} (w_L^0)^2 \eta_{2L} \leq C \beta_\infty(\lambda),\]

where in the last inequality we used the estimate (48). Since $L$ is arbitrary, noting that all constants in the bounds obtained in the proof are independent of $L$, we conclude that $\beta_\infty(\lambda) \geq 0$.  

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4.4 Existence of almost periodic cubes

In this subsection we present preparatory constructions and results necessary for the proof of Theorem 4.5.

By $P_\#$ we denote the push forward of the probability measure given by the map $\mathcal{H}_K$ on the $\sigma$-algebra $\mathcal{F}_{H,K}$ (see Section 4.1). We use the following notation for a closed Hausdorff ball of radius $r$ around a fixed compact set $K'$:

$$B_{H,K}(K',r) := \{ U \subset K : U \text{ is compact, } d_H(K', U) \leq r \}.$$ 

Next lemma is the key assertion that implies the existence of “almost periodic” arrangements of inclusions.

**Lemma 4.12.** Let $K \subset \mathbb{R}^d$ be a compact set. There exists a subset $\Omega_K \subset \Omega$ of probability one such that for every $\omega \in \Omega_K$ and every $r > 0$ one has

$$P_\#(B_{H,K}(\mathcal{H}_K(\omega),r)) > 0. \tag{56}$$

**Proof.** It is sufficient to prove that the set where the inequality (56) does not hold for any $r > 0$ is a set of probability zero. Recall that the Hausdorff topology on the compact subsets of $K$ is compact and thus separable. Consider a countable family $\{K_m\}_{m \in \mathbb{N}}$ of compact subsets of $K$ that is dense in this topology and let

$$r_m := \sup \{ r > 0 : P_\#(B_{H,K}(K_m,r)) = 0 \}. \tag{57}$$

Additionally, we set $r_m = -\infty$ if $K$ is empty. Notice that by the continuity of probability measure we also have $P_\#(B_{H,K}(\mathcal{H}_K(\omega),r)) = 0$. Let $\omega$ be such that there exists $r > 0$ with $P_\#(B_{H,K}(\mathcal{H}_K(\omega),r)) = 0$. By the density of the family $\{K_m\}_m$ we have that there exists $K_i$ and $r' > 0$ satisfying $\mathcal{H}(\omega) \in B_{H,K}(K_i,r') \subset B_{H,K}(\mathcal{H}_K(\omega),r)$. Clearly, we have $P_\#(B_{H,K}(K_i,r')) = 0$. From (57) and the latter we infer the following representation,

$$\{ \omega \in \Omega : \exists r > 0 \text{ such that } P_\#(B_{H,K}(\mathcal{H}_K(\omega),r)) = 0 \} = \bigcup_{m \in \mathbb{N}, r_m > 0} \{ \omega \in \Omega : \mathcal{H}_K(\omega) \in B_{H,K}(K_m,r_m) \}.$$ 

The right-hand side of the last equality is clearly a set of probability zero.

**Corollary 4.13.** There exists a subset $\Omega_1 \subset \Omega$ of probability one such that every $\omega \in \Omega_1$ satisfies the following property: for every $n \in \mathbb{N}$, $q \in \mathbb{Q}^d$, and $r > 0$ one has

$$P_\# \left( B_{H,\square_q^d}(\mathcal{H}_{\square_q^d}(\omega),r) \right) = P_\# \left( B_{H,\square_n^d}(\mathcal{H}_{\square_n^d}(\omega) - q,r) \right) > 0.$$

**Proof.** The equality follows from a straightforward translation argument. The existence of $\Omega_1$ follows from Lemma 4.12 and a simple observation that the intersection of a countable family of sets of probability one is a set of probability one.

The following two theorems contain the main result of the present subsection. Though the notation is somewhat involved, in simple words their meaning is the following: for a fixed $\lambda$ there is an arbitrarily large cube that can be divided into smaller sub-cubes with almost periodic arrangement of inclusions (apart from a fixed size boundary layer), and on each of the sub-cubes the corresponding quantity $\ell(\ldots, \omega, \lambda)$ approximates $\beta_\infty(\lambda)$. 

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Theorem 4.14. Under Assumptions \[4.4\] there exists a set \( \Omega_1 \subset \Omega \) of probability one such that for any \( \omega \in \Omega_1, N, M \in \mathbb{N}, x \in \mathbb{R}^d, \delta > 0, \) and the cube \( \square^M \) there exist \( N^d \) cubes \( \square^M_{x_i}, i = 1, \ldots, N^d, \) such that the cubes \( \square^M_{x_i} \) (discarding the boundary) are mutually disjoint, their union is a cube \( \bigcup_i \square^M_{x_i} = \square^N_{x^*(M+\kappa)}, \) and the following estimate holds:

\[
d_H(\mathcal{H}^M(T_x\omega), \mathcal{H}^M(T_{x^*}\omega)) < \delta.
\]

Proof. It is direct consequence of Corollary \[4.13\] Assumption \[4.4\] and the law of large numbers that for any \( N, M, n \in \mathbb{N}, x \in \mathbb{Q}^d, \) for almost every \( \omega \in \Omega \) there exist \( N^d \) cubes \( \square^M_{x_i}, i = 1, \ldots, N^d, \) such that \( \bigcup_i \square^M_{x_i} = \square^N_{x^*(M+\kappa)} \) for a suitable \( x^* \in \mathbb{R}^d, \) and

\[
d_H(\mathcal{H}^M(T_x\omega), \mathcal{H}^M(T_{x^*}\omega)) < \frac{1}{n}.
\]

We denote the corresponding set of probability one by \( \Omega_{N,M,n,x} \) and set

\[
\Omega_1 := \bigcap_{N,M,n \in \mathbb{N}, x \in \mathbb{Q}^d} \Omega_{N,M,n,x}.
\]

Clearly, \( \Omega_1 \) has probability one and the statement of the theorem holds for any \( x \in \mathbb{Q}^d, \) but then, by the density argument it holds for any \( x \in \mathbb{R}^d. \)

Theorem 4.15. Under Assumption \[4.4\] there exists a set \( \Omega_1 \subset \Omega \) of probability one such that for any \( \lambda \in \mathbb{R} \setminus \text{Sp}(-\Delta_\Omega), \) \( \omega \in \Omega_1, \delta > 0, N, M_0 \in \mathbb{N} \) there exist \( N^d \) cubes \( \square^M_{x_i}, i = 1, \ldots, N^d, \) \( N \geq M > M_0, \) as described in Theorem \[4.14\], such that

\[
\max_i |\ell(x_i, M + \kappa, \lambda, \omega) - \bar{\beta}_\infty(\lambda)| < \delta.
\]

Proof. Let us fix \( \lambda \in \mathbb{Q} \setminus \text{Sp}(-\Delta_\Omega). \) By Lemma \[4.8\] Proposition \[4.9\] and Lemma \[4.7\] for a.e. \( \omega \) for all sufficiently large \( M \in \mathbb{N} \) there exist cubes \( \square^M_{x_M} \) such that

\[
|\bar{\beta}_\infty(\lambda) - \bar{\ell}(x_M, M, \lambda, \omega)| < \frac{\delta}{3}.
\]

Similarly to Lemma \[4.7\] it is not difficult to show via part 2 of Assumption \[2.7\] and the bound \[23\] that for all sufficiently large \( M \) and any \( x \)

\[
|\ell(x, M + \kappa, \lambda, \omega) - \bar{\ell}(x, M, \lambda, \omega)| < \frac{\delta}{3}.
\]

By Theorem \[4.14\] there exists a sequence of cubes \( \square^N_{x^n_M}, n \in \mathbb{N}, \) such that for each \( n \) the cube \( \square^N_{x^n_M} \) is a union of mutually disjoint cubes \( \square^{M+\kappa}_{x^n_i}, i = 1, \ldots, N^d, \) and

\[
\lim_{n \to \infty} \max_i d_H(\mathcal{H}^M(T_{x^n_M}\omega), \mathcal{H}^M(T_{x^n}\omega)) = 0.
\]

It is clear, by direct inspection, that the difference between \( \bar{\ell}(x_M, M, \lambda, \omega) \) and \( \hat{\ell}(x^n_M, M, \lambda, \omega) \) is controlled (uniformly in \( M \)) by the maximal difference between the values of \( \int_{\mathbb{R}^d} b(T_x, \lambda, \cdot) \) corresponding to the “matching” inclusions in \( \square^M_{x_M} \) and \( \square^{M+\kappa}_{x^n_i}. \) Hence, by Lemma \[E.3\] for sufficiently large (fixed) \( n \) one has

\[
|\hat{\ell}(x^n_M, M, \lambda, \omega) - \bar{\ell}(x_M, M, \lambda, \omega)| < \frac{\delta}{3}.
\]
Combining inequalities (60), (61) and (62) we obtain that (59) holds (with \( x_i := x_i^n \)) for each \( \lambda \in Q \setminus \text{Sp}(-\Delta_\sigma) \) on a set of probability one, which, in general, depends on \( \lambda \). We define \( \Omega_1 \) as the intersection of these sets for all \( \lambda \in Q \setminus \text{Sp}(-\Delta_\sigma) \). Then by continuity of \( \beta_\infty(\lambda) \) we conclude that the assertion holds for all \( \lambda \in R \setminus \text{Sp}(-\Delta_\sigma) \).

\[ \square \]

### 4.5 Proof of Theorem 4.5

Let \( \lambda \) be such that \( \beta_\infty(\lambda) \geq 0 \). In what follows the parameters \( L \) and \( M \) are assumed sufficiently large, and \( \varepsilon \) and \( \delta \) are sufficiently small. The constants that appear in the subsequent bounds may depend on \( \lambda \) but are independent of \( \varepsilon, \delta, L \) and \( M \). We will emphasise this through the notation in two key bounds below. We fix \( \delta > 0 \) and \( L > 0 \) and choose \( M \) so that (61) holds and (60) is satisfied for some cube \( \square_{x_M}^M \) (by Lemmata 4.7 and 4.8). Let \( N = N(\varepsilon) \) be the smallest integer such that \( \varepsilon N(M + \kappa) \geq L \). By Theorems 4.14 and 4.15 there exist \( N^d \) cubes \( \square_{x_i}^{M + \kappa}, i = 1, \ldots, N^d \), as described in Theorem 4.14, satisfying (58) and (59). In particular, their union is \( \square_{x_i}^{N(M + \kappa)} \) and we have \( \bigcap_{x_i}^{N(M + \kappa)} \subseteq \bigcap_{x_i}^{M + \kappa} \). Clearly, the choice of the cubes (i.e. their centres \( x_i, x^* \)) depends on \( L, M, \delta \) and \( \varepsilon \), which we omit in the notation for brevity.

Denote by \( \square_{x_M}^{M + \kappa,1} \) the set obtained from \( \square_{x_M}^{M + \kappa} \) by removing all the sets \( O^*_\omega \) whose closures are contained in \( \square_{x_M}^{M + \kappa} \):

\[
\square_{x_M}^{M + \kappa,1} := \square_{x_M}^{M + \kappa} \setminus \bigcup_{\square_{x_M}^{\kappa,1} \subseteq \square_{x_M}^{M + \kappa}} O^*_\omega.
\]

Let \( \hat{N}_j \in W_{\text{per}}^{1,2}(\square_{x_M}^{M + \kappa,1}) \), \( j = 1, \ldots, d \), be solutions to the periodic corrector problem for the perforated cube \( \square_{x_M}^{M + \kappa,1} \):

\[
\int_{\square_{x_M}^{M + \kappa,1}} A_1(e_j + \nabla \hat{N}_j) \cdot \nabla \varphi = 0 \quad \forall \varphi \in W_{\text{per}}^{1,2}(\square_{x_M}^{M + \kappa,1}).
\]

We assume that each \( \hat{N}_j \) is extended inside the inclusions in \( \square_{x_M}^{M + \kappa} \) according to Theorem 2.8 has zero mean over \( \square_{x_M}^{M + \kappa} \), and extended by periodicity to the whole of \( R^d \). Observe the following estimates with the constant depending only on \( A_1 \) and the extension constant \( C_{\text{ext}} \):

\[
\begin{align*}
\| \nabla \hat{N}_j \|_{L^2(\square_{x_M}^{M + \kappa})} & \leq C M^{d/2}, \\
\| \hat{N}_j \|_{L^2(\square_{x_M}^{M + \kappa})} & \leq C M^{d/2 + 1}.
\end{align*}
\]

(64)

The first easily follows from the identity

\[
\int_{\square_{x_M}^{M + \kappa,1}} A_1 \nabla \hat{N}_j \cdot \nabla \hat{N}_j = - \int_{\square_{x_M}^{M + \kappa,1}} A_1 e_j \cdot \nabla \hat{N}_j,
\]

and the second from the Poincaré inequality (recall that \( M \) is sufficiently large, in particular \( M \geq \kappa \)).

An essential component of the construction is the higher (than \( L^2 \)) regularity of the correctors \( \hat{N}_j \). The proof closely follows the argument of [BF02] and is based on the use of special versions of two well-known results: the Poincaré-Sobolev inequality and the reverse Hölder’s inequality. In particular, the uniform scalable version of Poincaré-Sobolev inequality for perforated domains is valid under the minimal smoothness assumption.
Theorem 4.16 (Higher regularity of the periodic corrector). Under Assumption 2.7 there exists $p > 2$ and $C > 0$ such that for a.e. $\omega$ one has
\[
\left( \int_{\Box_{x_M}^{M+\kappa}} |\nabla \hat{N}_j|^p \right)^{1/p} \leq C + C \left( \int_{\Box_{x_M}^{M+\kappa}} |\nabla \hat{N}_j|^2 \right)^{1/2},
\]
uniformly in $M$.

We provide the proof of the theorem in Appendix C. As a corollary of the theorem and the bound (64), we have that
\[
\|\nabla \hat{N}_j\|_{L^p(\Box_{x_M}^{M+\kappa})} \leq CM^{d/p}.
\]

We denote by $\hat{A}_{1}^{\text{hom}}$ the matrix of homogenised coefficients associated with $\hat{N}_j$, $\hat{A}_{1}^{\text{hom}} \xi = \frac{1}{\Box_{x_M}^{M+\kappa}} \int_{\Box_{x_M}^{M+\kappa,1}} A_1(\xi + \xi_j \nabla \hat{N}_j) \forall \xi \in \mathbb{R}^d$.

Take some $k$ such that $\hat{A}_{1}^{\text{hom}} k \cdot k = \beta_\infty(\lambda)$. Then $u(x) := e^{ik \cdot x}$ satisfies
\[-\nabla \cdot \hat{A}_{1}^{\text{hom}} \nabla u = \beta_\infty(\lambda) u.\]

We define $N_j \in W^{1,2}_{\text{per}}(\Box_{x_M}^{M+\kappa})$, extended periodically to $\mathbb{R}^d$, as $N_j(x) := \hat{N}_j(x - x_1 + x_M)$. Denote by $\hat{\chi}_1^\varepsilon$ the characteristic function of the set $\Box_{x_M}^{M+\kappa,1} - x_M + x_1$ extended periodically to $\mathbb{R}^d$.

Recall the cut-off function $\eta$ defined in Section 4.3. We redefine $\eta_L$ as follows:
\[\eta_L(\cdot) := \eta(\cdot/L + \varepsilon x^\ast).\]

Multiplying $u$ by $\eta_L$ and normalising the resulting expression,
\[u_L := \frac{\eta_L u}{\|\eta_L u\|_{L^2(\mathbb{R}^d)}},\]
we obtain a standard Weyl sequence for the operator $-\nabla \cdot \hat{A}_{1}^{\text{hom}} \nabla$ (that is if we let $L \to \infty$):
\[\|( - \nabla \cdot \hat{A}_{1}^{\text{hom}} \nabla - \beta_\infty(\lambda)) u_L\|_{L^2(\mathbb{R}^d)} \leq \frac{C}{L}.\]

Notice that
\[\|u_L\|_{L^\infty(\Box_{x_M}^L)} + \|\nabla u_L\|_{L^\infty(\Box_{x_M}^L)} + \|\nabla^2 u_L\|_{L^\infty(\Box_{x_M}^L)} \leq \frac{C}{L^{d/2}}.\]

Let $b$ be the solution to (16) and denote by $b^\varepsilon$ its $\varepsilon$-realisation, $b^\varepsilon(x) := b(T_{x/\varepsilon} \omega)$. We define
\[u_L^\varepsilon := (1 + \lambda b^\varepsilon) u_L.\]

It is not difficult to see that the norm of $u_L^\varepsilon$ is bounded from below uniformly in $\varepsilon$ and $L$,
\[\|u_L^\varepsilon\|_{L^2(\mathbb{R}^d)} \geq C > 0.\]

We denote by $\tilde{u}_L^\varepsilon$ the solution to the resolvent problem
\[(A^\varepsilon + 1) \tilde{u}_L^\varepsilon = (\lambda + 1) u_L^\varepsilon.\]
Then
\[(A^\varepsilon - \lambda)\hat{\tilde{\omega}}_L = (\lambda + 1)(u_L^\varepsilon - \hat{\tilde{\omega}}_L^\varepsilon).\]

Our goal is to show that \((A^\varepsilon - \lambda)\hat{\tilde{\omega}}_L^\varepsilon\) can be made arbitrary small by choosing appropriately the parameters introduced at the beginning of this subsection. To this end we will estimate the difference between \(\hat{\tilde{\omega}}_L^\varepsilon\) and a further approximation \(u_{LC}^\varepsilon\) which incorporates the homogenisation corrector term using the sesquilinear form associated with the operator \(A^\varepsilon\). We define
\[u_{LC}^\varepsilon := u_L^\varepsilon + \varepsilon \partial_j u_L N_j (\cdot / \varepsilon).\]

**Remark 4.17.** The technical details in what follows built up to the bounds \([92] - [94]\), which represent the essence of the argument for the remainder of the proof. The reader may therefore wish to inspect these bounds before proceeding. The general scheme for this argument is adopted from [KS18] and is useful in settings where one needs to establish proximity to a spectrum. We will resort to it once again in the proof of Theorem \([5,2]\) below.

We estimate the corrector term via \([64]\) and \([69]\),
\[
\| u_{LC}^\varepsilon - u_L^\varepsilon \|_{L^2(\mathbb{R}^d)} = \varepsilon \| \partial_j u_L N_j \|_{L^2(\mathbb{R}^d)} \leq C \varepsilon M. \tag{73}
\]
In what follows we will often use the bounds \([64]\) and \([69]\) without mentioning.

Denote by \(a^\varepsilon\) the sesquilinear form associated with the operator \(A^\varepsilon + 1\),
\[a^\varepsilon(u, v) := \int_{\mathbb{R}^d} (A^\varepsilon \nabla u \cdot \nabla v + u \overline{v}).\]
We substitute \(u_{LC}^\varepsilon\) into the form with an arbitrary test function \(v \in W^{1,2}(\mathbb{R}^d)\) and consider separately its behaviour on the set of inclusions and its complement,
\[
a^\varepsilon(u_{LC}^\varepsilon, v) = \int_{S_1^e} A_1 \nabla u_{LC}^\varepsilon \cdot \nabla v + \int_{S_0^e} \varepsilon^2 \nabla u_{LC}^\varepsilon \cdot \nabla v + \int_{\mathbb{R}^d} u_{LC}^\varepsilon \nabla v. \tag{74}
\]
To proceed with the argument we need to use the following decomposition:
\[v = \overline{v}^\varepsilon + v_0^\varepsilon, \tag{75}\]
where \(\overline{v}^\varepsilon \in W^{1,2}(\mathbb{R}^d)\) is the extension of the restriction \(v|_{S_1^e}\) into the set of inclusions \(S_0^e\) by Theorem \([2,8]\) and \(v_0^\varepsilon = v - \overline{v}^\varepsilon \in W^{1,2}(S_0^e)\). The next bounds are straightforward consequence of Theorem \([2,8]\) and the Poincaré inequality:
\[
\| \nabla \overline{v}^\varepsilon \|_{L^2(\mathbb{R}^d)} \leq C \| \nabla v \|_{L^2(S_1^e)}, \|
\| \nabla v_0^\varepsilon \|_{L^2(\mathbb{R}^d)} \leq C \| \nabla v \|_{L^2(\mathbb{R}^d)}, \|
\| v_0^\varepsilon \|_{L^2(\mathbb{R}^d)} \leq C \varepsilon \| \nabla v_0^\varepsilon \|_{L^2(\mathbb{R}^d)}, \|
\| \overline{v}^\varepsilon \|_{L^2(\mathbb{R}^d)} + \| \nabla \overline{v}^\varepsilon \|_{L^2(\mathbb{R}^d)} + \| \varepsilon \nabla v \|_{L^2(\mathbb{R}^d)} \leq C \sqrt{a^\varepsilon(v, v)}. \tag{77}
\]
We begin by analysing the first term on the right-hand side of \([74]\).
\[
\int_{S_1^e} A_1 \nabla u_{LC}^\varepsilon \cdot \nabla \overline{v} = \int_{\mathbb{R}^d} \left( A_1^{\text{hom}} \nabla u_L + [\chi_1 A_1(e_j + \nabla N_j) - \tilde{A}_1^{\text{hom}} e_j] \partial_j u_L + \varepsilon \chi_1 N_j \nabla \partial_j u_L \right) \cdot \nabla \overline{v}^\varepsilon
\]
We can estimate the last term on the right-hand side of the latter as follows,

$$
\left| \varepsilon \int_{\mathbb{R}^d} \chi_1^\varepsilon N_j \nabla \partial_j u_L \cdot \nabla \hat{v}^\varepsilon \right| \leq C \varepsilon M \| \nabla \hat{v}^\varepsilon \|_{L^2(\mathbb{R}^d)}.
$$

Next we estimate the term containing

$$
\chi_1^\varepsilon A_1(e_j + \nabla N_j) - \hat{A}_1^\text{hom} e_j = (\chi_1^\varepsilon - \hat{\chi}_1^\varepsilon) A_1(e_j + \nabla N_j) + \hat{\chi}_1^\varepsilon A_1(e_j + \nabla N_j) - \hat{A}_1^\text{hom} e_j. \quad (78)
$$

In order to estimate the difference between $\chi_1^\varepsilon$ and $\hat{\chi}_1^\varepsilon$ inside the cube $\Box_{L_\varepsilon}^{L_\varepsilon}$, we make the following two observations: 1) by (3) the total surface area of the inclusions contained in a cube is bounded by its volume times a constant, hence the difference $\chi_1^\varepsilon$ and $\hat{\chi}_1^\varepsilon$ in every cube $\Box_{L_\varepsilon}^{L_\varepsilon}$ can be estimated via (58); 2) the volume the “boundary layers” $\Box_{L_\varepsilon}^{L_\varepsilon} \setminus \Box_{L_\varepsilon}^{M_\varepsilon}$, relative to the volume of $\Box_{L_\varepsilon}^{L_\varepsilon}$, can be made arbitrary small by choosing $M$ sufficiently large. Therefore, for sufficiently large $M$ we have

$$
\int_{\Box_{L_\varepsilon}^{L_\varepsilon}} |\chi_1^\varepsilon - \hat{\chi}_1^\varepsilon| \leq C \delta L^d. \quad (79)
$$

From this we infer

$$
\left| \int_{\mathbb{R}^d} (\chi_1^\varepsilon - \hat{\chi}_1^\varepsilon) A_1 e_j \nabla \partial_j u_L \cdot \nabla \hat{v}^\varepsilon \right| \leq C \delta^{1/2} \| \nabla \hat{v}^\varepsilon \|_{L^2(\mathbb{R}^d)}. \quad (80)
$$

For the next estimate we invoke Theorem 4.16. Applying Hölder’s inequality twice and taking into account (66) and (79), we get

$$
\left| \int_{\mathbb{R}^d} (\chi_1^\varepsilon - \hat{\chi}_1^\varepsilon) A_1 \nabla N_j \partial_j u_L \cdot \nabla \hat{v}^\varepsilon \right|
\leq \frac{C}{L^d/2} \| \chi_1^\varepsilon - \hat{\chi}_1^\varepsilon \|_{L^2 p/(p-2)(\Box_{\varepsilon L_\varepsilon}^{\varepsilon L_\varepsilon})} \sum_j \| \nabla N_j \|_{L^{p/(p-2)}(\Box_{\varepsilon L_\varepsilon}^{\varepsilon L_\varepsilon})} \| \nabla \hat{v}^\varepsilon \|_{L^2(\mathbb{R}^d)} 
\leq C \delta^{(p-2)/2p} \| \nabla \hat{v}^\varepsilon \|_{L^2(\mathbb{R}^d)}. \quad (81)
$$

Now we deal with the last two terms on the right-hand side of (78). The vector field

$$
g_j(y) := \hat{\chi}_1^\varepsilon(y) A_1(e_j + \nabla N_j(y)) - \hat{A}_1^\text{hom} e_j \quad (82)
$$

is solenoidal, i.e. $\int_{\Box_{L_1}^{L+\varepsilon}} g_j \cdot \nabla \varphi = 0 \ \forall \varphi \in W_{\text{per}}^{1,2}(\Box_{L_1}^{L+\varepsilon})$, and has zero mean, by the definition of the homogenised matrix $\hat{A}_1^\text{hom}$. It is well known, see an explicit construction via Fourier series in [ZKO94, Section 1.1], that such field can be represented as the “divergence” of a skew-symmetric zero-mean field $G^j$, $G^j_{ik} = -G^j_{ki}$, $G^j_{ik} \in W_{\text{per}}^{1,2}(\Box_{L_1}^{L+\varepsilon})$:

$$
g_j = \nabla \cdot G^j, \text{ i.e. } (g_j)_k = \partial_k G^j_{ik}. \quad (83)
$$

Moreover, it is not difficult to see (by estimating the coefficients of the said Fourier series) that

$$
\| G^j \|_{L^2(\Box_{L_1}^{L+\varepsilon})} \leq C \| g_j \|_{L^2(\Box_{L_1}^{L+\varepsilon})} \leq C M^{d/2} \quad (84)
$$
(the second inequality here is obtained with the help of (64)). Applying (83) (note that \(g_j(\cdot/\varepsilon) = \varepsilon \nabla \cdot G^j(\cdot/\varepsilon)\)) and integrating by parts we get

\[
\int_{\mathbb{R}^d} \partial_j u_L g_j \cdot \nabla \bar{\varepsilon} = \int_{\mathbb{R}^d} \varepsilon \partial_j u_L (\nabla \cdot G^j) \cdot \nabla \bar{\varepsilon} = \int_{\mathbb{R}^d} \varepsilon [\partial_j u_L G^j : \nabla^2 \bar{\varepsilon} - (G^j \nabla \partial_j u_L) \cdot \nabla \bar{\varepsilon}],
\]

(85)

where the colon denotes the Frobenius inner product. The first term in the last integral is identically zero due to the anti-symmetry of \(G^j\). Therefore, combining (78), (80), (81), (84) and the last identity, we arrive at

\[
\left| \int_{\mathbb{R}^d} [\hat{\chi}_1 A_1 (e_j + \nabla N_j) - \hat{A}_1^{\text{hom}} e_j ] \partial_j u_L \cdot \nabla \bar{\varepsilon} \right| \leq C \left( \delta^{(p-2)/2p + \varepsilon} \right) \| \nabla \bar{\varepsilon} \|_{L^2(\mathbb{R}^d)}.
\]

(86)

Now we address the second integral in (74). Integrating by parts one of the terms we get

\[
\varepsilon^2 \int_{S_0^e} \nabla u_L \varepsilon \cdot \nabla \bar{\varepsilon} = \varepsilon^2 \int_{S_0^e} \left( \lambda u_L \nabla b^\varepsilon \cdot (\nabla \bar{v}_0 + \bar{\varepsilon}) + [\nabla u_L (1 + \lambda b^\varepsilon) + \varepsilon \nabla (\partial_j u_L N_j)] \cdot \nabla \bar{v} \right)
\]

\[
= \varepsilon^2 \int_{S_0^e} \left( - \lambda u_L \Delta b^\varepsilon \bar{v}_0 - \lambda \nabla u_L \cdot \nabla b^\varepsilon \bar{v}_0 + \lambda u_L \nabla b^\varepsilon \cdot \nabla \bar{\varepsilon} \right)
\]

\[
+ [\nabla u_L (1 + \lambda b^\varepsilon) + \varepsilon \nabla (\partial_j u_L N_j)] \cdot \nabla \bar{v}.
\]

(87)

Since \(b\) is the solution to (16), its \(\varepsilon\)-realisation satisfies

\[-\varepsilon^2 \Delta b^\varepsilon = \lambda b^\varepsilon + 1,
\]

thus we have

\[- \varepsilon^2 \int_{S_0^e} \lambda u_L \Delta b^\varepsilon \bar{v}_0 = \int_{S_0^e} \lambda u_L (\lambda b^\varepsilon + 1) \bar{v}_0.
\]

(88)

The bound for the rest of the terms via (64), (69), (70), (77) and Lemma E.2 is straightforward:

\[
\left| \varepsilon^2 \int_{S_0^e} \left( - \lambda \nabla u_L \cdot \nabla b^\varepsilon \bar{v}_0 + \lambda u_L \nabla b^\varepsilon \cdot \nabla \bar{\varepsilon} + [\nabla u_L (1 + \lambda b^\varepsilon) + \varepsilon \nabla (\partial_j u_L N_j)] \cdot \nabla \bar{v} \right) \right|
\]

\[
\leq C (\varepsilon + \varepsilon^2 M) \sqrt{a^\varepsilon(v, v)}.
\]

Combining (74)–(78), (80), (81), (86)–(88) and the last bound we arrive at

\[
a^\varepsilon(u_{LC}^\varepsilon, v) = \int_{\mathbb{R}^d} \hat{A}_1^{\text{hom}} \nabla u_L \cdot \nabla \bar{\varepsilon} + \int_{S_0^e} \lambda u_L (1 + \lambda b^\varepsilon) \bar{v}_0 + \int_{\mathbb{R}^d} \nabla u_{LC} \tilde{v} + \mathcal{R}^\varepsilon,
\]

(89)

where the remainder \(\mathcal{R}^\varepsilon\) satisfies

\[
|\mathcal{R}^\varepsilon| \leq C(\lambda) \sqrt{a^\varepsilon(v, v)} \left( \varepsilon M + \delta^{(p-2)/2p} \right).
\]

(90)
Replacing \( v_0^{\varepsilon} \) with \( v - \overline{v}^{\varepsilon} \) in (\ref{eq:89}) and recalling the definition of \( u_L^{\varepsilon} \) we can rewrite it as follows
\[
a^{\varepsilon}(u_{LC}^{\varepsilon}, v) = \int_{\mathbb{R}^d} \left( (-\nabla \cdot A_1^{\hom} \nabla - \beta_\infty(\lambda)) u_L \bar{v}^{\varepsilon} + (\beta_\infty(\lambda) - (\lambda + \lambda^2 b^\varepsilon)) u_L \bar{v}^{\varepsilon} \right. \\
\left. \quad + ((\lambda + 1) u_L^{\varepsilon} + \varepsilon \partial_j u_L N_j) \bar{v} \right) + R^{\varepsilon}.
\]

It remains to estimate the second term under the integral sign.

Consider the piece-wise averaging operator \( M_{M+k}^{\varepsilon} : L^2(\varepsilon \Box_x^{N(M+k)}) \rightarrow L^2(\varepsilon \Box_x^{N(M+k)}) \) defined by
\[
 M_{M+k}^{\varepsilon} f(x) = \int_{\varepsilon \Box_x^{M+k}} f, \quad x \in \varepsilon \Box_x^{(M+k)}.
\]
The proof of the following lemma is a straightforward application of the Poincaré inequality.

**Lemma 4.18.** Let \( f \in W^{1,2}(\varepsilon \Box_x^{N(M+k)}) \), then
\[
\| f - M_{M+k}^{\varepsilon} f \|_{L^2(\varepsilon \Box_x^{N(M+k)})} \leq C\varepsilon M \| \nabla f \|_{L^2(\varepsilon \Box_x^{N(M+k)})}.
\]

Now notice that by (\ref{eq:59}) on every cube \( \varepsilon \Box_x^{M+k} \) we have
\[
\left| \int_{\varepsilon \Box_x^{M+k}} (\beta_\infty(\lambda) - (\lambda + \lambda^2 b^\varepsilon)) M_{M+k}^{\varepsilon} \left( u_L \bar{v}^{\varepsilon} \right) \right| = |\varepsilon \Box_x^{M+k}| |\beta_\infty(\lambda) - \ell(x, M + k, \lambda, \omega)| \leq |\varepsilon \Box_x^{M+k}| \delta.
\]
Therefore
\[
\left| \int_{\mathbb{R}^d} (\beta_\infty(\lambda) - (\lambda + \lambda^2 b^\varepsilon)) M_{M+k}^{\varepsilon} \left( u_L \bar{v}^{\varepsilon} \right) \right| \\
\leq \delta \sum_{i=1}^{N^d} \left| \varepsilon \Box_x^{M+k} \right| \left( M_{M+k}^{\varepsilon} \left( u_L \bar{v}^{\varepsilon} \right) \right)(x_i) = \delta \int_{\mathbb{R}^d} |u_L \bar{v}^{\varepsilon}| \leq C \delta \| \bar{v}^{\varepsilon} \|_{L^2(\mathbb{R}^d)}.
\]

Applying Lemma 4.18 to the product \( u_L \bar{v}^{\varepsilon} \) and observing that \( \| \lambda + \lambda^2 b^\varepsilon \|_{L^2(\mathbb{R}^d)} \leq CL^{d/2} \), cf. (\ref{eq:23}), we get
\[
\left| \int_{\mathbb{R}^d} (\beta_\infty(\lambda) - (\lambda + \lambda^2 b^\varepsilon)) \left( u_L \bar{v}^{\varepsilon} - M_{M+k}^{\varepsilon} \left( u_L \bar{v}^{\varepsilon} \right) \right) \right| \leq C\varepsilon M \left( \| \bar{v}^{\varepsilon} \|_{L^2(\mathbb{R}^d)} + \| \nabla \bar{v}^{\varepsilon} \|_{L^2(\mathbb{R}^d)} \right). \quad (\text{91})
\]

Combining (\ref{eq:68}), (\ref{eq:72}), (\ref{eq:73}), (\ref{eq:77}), (\ref{eq:90})—(\ref{eq:91}) we arrive at
\[
|a^{\varepsilon}(u_{LC}^{\varepsilon} - \bar{u}_L^{\varepsilon}, v)| \leq \widehat{R}(\varepsilon, L, \lambda) \sqrt{a^{\varepsilon}(v, v)}, \quad (\text{92})
\]
where
\[
\widehat{R}(\varepsilon, L, \lambda) := C(\lambda) \left( \varepsilon M + \delta^{(p-2)/2p} + L^{-1} \right). \quad (\text{93})
\]
Substituting \( v = u_{LC}^{\varepsilon} - \bar{u}_L^{\varepsilon} \) in (\ref{eq:92}) yields
\[
\| u_{LC}^{\varepsilon} - \bar{u}_L^{\varepsilon} \|_{L^2(\mathbb{R}^d)}^2 \leq a^{\varepsilon}(u_{LC}^{\varepsilon} - \bar{u}_L^{\varepsilon}, u_{LC}^{\varepsilon} - \bar{u}_L^{\varepsilon}) \leq \widehat{R}(\varepsilon, L, \lambda)^2,
\]

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and since \( u^\varepsilon - u^\varepsilon_L = \varepsilon \partial_j u^\varepsilon N_j \) we have the same estimate (cf. (73)) for the difference between \( u^\varepsilon_L \) and \( \tilde{u}^\varepsilon_L \) (possibly with a different constant in (93)),

\[
\| u^\varepsilon_L - \tilde{u}^\varepsilon_L \|_{L^2(\mathbb{R}^d)} \leq \tilde{\mathcal{R}}(\varepsilon, L, \lambda).
\]

The latter immediately implies (cf. also (71)) that

\[
\frac{\|(A^\varepsilon - \lambda) \tilde{u}^\varepsilon_L \|_{L^2(\mathbb{R}^d)}}{\| \tilde{u}^\varepsilon_L \|_{L^2(\mathbb{R}^d)}} = \frac{\|(\lambda + 1)(u^\varepsilon_L - \tilde{u}^\varepsilon_L) \|_{L^2(\mathbb{R}^d)}}{\| u^\varepsilon_L \|_{L^2(\mathbb{R}^d)}} \leq |\lambda + 1| \tilde{\mathcal{R}}(\varepsilon, L, \lambda)
\]

(again, possibly with a different constant in (93)). It follows that

\[
\limsup_{\varepsilon \to 0} \text{dist}(\text{Sp}(A^\varepsilon), \lambda) \leq C \left( \delta^{(p-2)/2p} + L^{-1} \right),
\]

and, since \( \delta \) and \( L \) are arbitrary, we conclude that

\[
\lim_{\varepsilon \to 0} \text{dist}(\text{Sp}(A^\varepsilon), \lambda) = 0.
\]

**Remark 4.19.** In the present work we assume that the coefficients are constant in the stiff component and are a multiple of the identity matrix in the inclusions in order to simplify the exposition of the main ideas. The coefficients can be described in the following way. Consider a sequence of random variables of the form

\[
A^\varepsilon(\omega) = A_1 1_{\Omega \setminus \mathcal{O}} + \varepsilon I_1 \Omega,
\]

where \( A_1 \) is a positive definite matrix, then the coefficients defined by (7) are simply the \( \varepsilon \)-realisation of \( A^\varepsilon(\omega) \):

\[
A^\varepsilon(x, \omega) = A^\varepsilon(T_{x/\varepsilon} \omega).
\]

(Notice that the operator \( A^\varepsilon \) may be formally considered as the \( \varepsilon \)-realization of the operator \( -\nabla \omega \cdot A^\varepsilon(\omega) \nabla \omega \).) In a similar way we could analyse a more general problem when the coefficients in the constitutive parts of the composite are not assumed to be constant, namely, by setting

\[
A^\varepsilon(\omega) = A_1(\omega) 1_{\Omega \setminus \mathcal{O}} + \varepsilon^2 A_0(\omega) 1_{\mathcal{O}},
\]

where \( A_0, A_1 \in L^\infty(\Omega, \mathbb{R}^{d \times d}) \) are uniformly coercive. (One can look at an interesting simpler case when one does not change the geometry of the inclusions, but only the coefficients on the soft component.) We believe that our analysis can be easily adapted to this more general setting. It will manifest, in particular, in the need to analyse operator \( -\nabla \omega \cdot A^\varepsilon(\omega) \nabla \omega \) with the domain being a suitable subspace of \( W^{1,2}_0(\mathcal{O}) \) instead of the operator \( -\Delta \mathcal{O} \). In this case one will get completely analogous results, in particular, the spectrum of \( A^\varepsilon_{\text{hom}} \) will be characterised by a suitable version of the \( \beta \)-function, and the limit set \( \mathcal{G} \) by a suitable version of \( \beta_\infty(\lambda) \) (cf. (15) and (36)). Moreover, the notions introduced and the results presented in Section 5 can also be easily adapted to this more general setting.

### 4.6 Examples

In general, constructing explicitly a probability space that would provide a “truly random” distribution of inclusions is a challenging task. We consider two general setups: inclusions randomly placed at the nodes of a periodic lattice and the random parking model. We begin with the former.
4.6.1 Probability space setup

Let \((\tilde{\omega}_j)_{j \in \mathbb{Z}^d}\) be a sequence of independent and identically distributed random vectors taking values in \(N_0^l \times [r_1, r_2]\), where \(0 < r_1 \leq r_2 \leq 1\) and \((\bar{\Omega}, \bar{F}, \bar{P})\) is the canonical probability space obtained by the Kolmogorov construction. Let \(Y_k \subset \square := [0, 1)^d, k \in N_0^l\), be open, connected sets satisfying minimal smoothness assumption and not touching the boundary of \(\square\). These sets model the shapes of the inclusions, namely, for every \(j \in \mathbb{Z}^d\) (which determines the location of the inclusion) the first components of \(\tilde{\omega}_j = (k_j, r_j)\) determines the shape of the inclusion and the second component – its size. We also set \(Y_0 = \emptyset\). On \(\bar{\Omega}\) there is a natural shift \(\bar{T}_z(\tilde{\omega}_j) = (\tilde{\omega}_{j-z})\), which is ergodic.

We next state the discrete analogue of Lemma 2.4.

**Lemma 4.20.** Assume that \(\tilde{\Omega}_0 \subseteq \tilde{\Omega}\) is a set of full measure. Then there exists a subset \(\tilde{\Omega}_1 \subseteq \tilde{\Omega}_0\) of full measure such that for each \(\tilde{\omega} \in \tilde{\Omega}_1\), \(z \in \mathbb{Z}^d\) we have \(\bar{T}_z \tilde{\omega} \in \tilde{\Omega}_0\).

We treat \(\square\) as a probability space with Lebesgue measure \(dy\) and the standard algebra \(\mathcal{L}\) of Lebesgue measurable sets, and define

\[
\Omega = \tilde{\Omega} \times \square, \quad \mathcal{F} = \bar{\mathcal{F}} \times \mathcal{L}, \quad \mathcal{P} = \bar{\mathcal{P}} \times dy.
\]

On \(\Omega\) we introduce a dynamical system \(T_x(\tilde{\omega}, y) = (\bar{T}_{[x+y]} \tilde{\omega}, x + y - [x+y])\), and define

\[
\mathcal{O} := \{ (\tilde{\omega}, y) : y \in r_0 Y_{k_0} \}.
\]

It is easy to see that \(\mathcal{O}\) is measurable. For a fixed \(\omega = (\tilde{\omega}, y)\) the realisation \(\mathcal{O}_\omega\) consists of the inclusions \(r_j Y_{k_j} + j - y, j \in \mathbb{Z}^d\).

Next we consider three special cases of this general setup.

4.6.2 One shape randomly placed at a periodic lattice nodes

In this example we set \(l = 1, r_1 = r_2 = 1\). The second component of \(\tilde{\omega}\) is redundant in this example, so we disregard it in the notation. The value 0 or 1 of \(\tilde{\omega}_z, z \in \mathbb{Z}^d\), corresponds to the absence or the presence of the inclusion at the lattice node \(z\) respectively. We have

\[
\mathcal{O} = \{ \omega = (\tilde{\omega}, y) : \tilde{\omega}_0 = 1, y \in Y_1 \} \subseteq \Omega,
\]

so \(\text{Sp}(-\Delta_{\mathcal{O}}) = \text{Sp}(-\Delta_{Y_1})\). For a given \(\omega = (\tilde{\omega}, y) \in \Omega\) the realisation \(\mathcal{O}_\omega = \{ x : T_x \omega \in \mathcal{O} \}\) is the union of the sets \(Y_1 + z - y\) for all \(z \in \mathbb{Z}^d\) such that \(\tilde{\omega}_z = 1\). By the law of large numbers for a.e. \(\omega\) and there exist arbitrary large cubes both contain no inclusions and containing an inclusion at every lattice node. Thus we have that

\[
\beta_\infty(\lambda) \geq \max\{ \lambda, \beta_{1\text{-per}, Y_1}(\lambda) \},
\]

where

\[
\beta_{t\text{-per}, Y_1}(\lambda) = \lambda + \chi^2 \sum_{j=1}^{\infty} \frac{\langle \varphi_j \rangle^2}{\nu_j - \lambda}
\]

is Zhikov’s \(\beta\)-function corresponding to the \(t\)-periodic distribution of inclusions. Here \(\nu_j\) and \(\varphi_j\) denote the eigenvalues and orthonormalised eigenfunctions (extended by zero outside \(Y_1\)) of \(-\Delta_{Y_1}\), and \(\langle f \rangle := \int_{\mathbb{R}^d} f\). (In fact, it is not difficult to see that the equality holds in (95).) It follows from an observation that on arbitrary converging sequence one has that \(\lim_{M \to \infty} \ell(x_M, M, \lambda, \omega)\) is a convex combination of \(\lambda\) and \(\beta_{1\text{-per}, Y_1}(\lambda)\). Thus we have \(\mathcal{G} = \mathbb{R}_0^+\).
Remark 4.21. It was shown in [CCV18] that
\[ \beta(\lambda) := \lambda + \lambda^2 \sum_{j=1}^{\infty} \frac{\langle \varphi_j \rangle^2}{\nu_j - \lambda}. \]

4.6.3 Finite number of shapes at the lattice nodes

Now we take finite \( l > 1, r_1 = r_2 = 1 \) (again second component of \( \tilde{\omega} \) is redundant in this example) and assume that \( P\{\tilde{\omega}_0 = 0\} = 0 \), which is equivalent to saying that \((\tilde{\omega}_j)_{j \in \mathbb{Z}^d}\) take values in \( \mathbb{N}^d \).

By the law of large numbers for each \( k \in \mathbb{N}^l \) there exist arbitrary large cubes containing only the shapes \( Y_k \) at the lattice nodes. We have that
\[ \text{Sp}(\nabla) = \bigcup_{k=1}^{l} \text{Sp}(\nabla Y_k), \beta_\infty(\lambda) = \max_{k=1,\ldots,l} \beta_{1\text{-per},Y_k}(\lambda). \]

In particular,
\[ \mathcal{G} = \bigcup_{k=1}^{l} \text{Sp}(\nabla Y_k) \cup \bigcup_{k=1}^{l} \{\lambda \geq 0 : \beta_{1\text{-per},Y_k}(\lambda) \geq 0\}. \]

Remark 4.22. It was shown in [CCV18] that
\[ \beta(\lambda) := \lambda + \lambda^2 \sum_{k=1}^{l} P\{\tilde{\omega}_0 = k\} \sum_{j=1}^{\infty} \frac{\langle \varphi_j \rangle^2}{\nu_j^r - \lambda}. \]

Here \( \nu_j^r \) and \( \varphi_j^r \) denote the eigenvalues and orthonormalised eigenfunctions of \( -\Delta_{Y_k} \).

4.6.4 Randomly scaled inclusions

Here we take \( l = 1 \) and \( 0 < r_1 < r_2 \leq 1 \) and assume that \( P\{\tilde{\omega}_0 = 0\} = 0 \). So the first component of \( \tilde{\omega}_0 \) is redundant and we drop it from the notation. We denote by \( S \subset [r_1, r_2] \) the support of the random variable \( \tilde{\omega}_0 \).

Thus for every value of the scaling parameter \( r \in S \) and every \( \delta > 0 \) the set \( \{\tilde{\omega}_0 \in [r - \delta, r + \delta]\} \) has positive probability. It follows that for any \( \delta > 0 \) there exist arbitrary large cubes containing only inclusions whose scaling parameter belongs to the interval \([r - \delta, r + \delta]\).

Thus
\[ \text{Sp}(\nabla) = \bigcup_{r \in S} \bigcup_{j \in \mathbb{N}} \{r^{-2} \nu_j\}, \beta_\infty(\lambda) = \max_{r \in S} \beta_{1\text{-per},rY_1}(\lambda), \]

where \( \nu_j \) are the eigenvalues of \( -\Delta_{Y_1} \). So
\[ \mathcal{G} = \bigcup_{r \in S} \bigcup_{j \in \mathbb{N}} \{r^{-2} \nu_j\} \cup \bigcup_{r \in S} \{\lambda > 0 : \beta_{1\text{-per},rY_1}(\lambda) \geq 0\}. \]

Remark 4.23. It was shown in [CCV18] that
\[ \beta(\lambda) := \lambda + \lambda^2 \sum_{j=1}^{\infty} \frac{(\varphi_{j,\tilde{\omega}_0})^2}{\nu_j \tilde{\omega}_0 - \lambda}. \] (96)

Here \( \nu_{j,r} \) and \( \varphi_{j,r} \) are the eigenvalues and orthonormalised eigenfunctions of \( -\Delta_{rY_1} \). They can be obtained from those of \( -\Delta_{Y_1} \) by scaling, in particular, \( \nu_{j,r} = r^{-2} \nu_j \).
4.6.5 Modifications of the above examples

It is not difficult to advise an explicit modification of the above setting in which the inclusions are randomly rotated and shifted within the lattice cells (still respecting the minimal distance between them in order for the extension property to hold). As can be seen from the definition of $\beta_\infty(\lambda)$ and formula (26) for $\beta(\lambda)$, this additional degree of freedom does not affect these functions (but only $\Lambda_1^{\text{hom}}$), hence both the limit set $G$ and $\text{Sp}(\Lambda^{\text{hom}})$ (in the whole space setting) remain the same.

Another possible modification is to assume that random variables $(\tilde{\omega}_j)_{j \in \mathbb{Z}^d}$ have finite correlation distance. In this case, in order to recover the above formulae for $\beta_\infty(\lambda)$ arbitrary large cubes containing only the inclusions of a specific type (as in the above examples) should have positive probability. If this is not the case, then only the general formula (36) is available.

4.6.6 Random parking model

In this example we consider the random parking model described in [P01] (see also [GP13]). The intuitive description of the model is the following: copies of a set $V$ arrive sequentially at random without overlapping until jamming occurs.

We quickly recall the graphical construction of the random parking measure (RPM) in $\mathbb{R}^d$. Let $\mathcal{P}$ be a homogeneous Poisson process of unit intensity in $\mathbb{R}^d \times \mathbb{R}^+$ and let $V \subset \mathbb{R}^d$ be the bounded set that contains origin (in our case this will be a unit cube). An oriented graph is a special kind of directed graph in which there is no pair of vertices $\{x, y\}$ are included as directed edges. We shall say that $x$ is a parent of $y$ and $y$ is an offspring of $x$ if there is an oriented edge from $x$ to $y$. By a root of an oriented graph we mean a vertex with no parent. The graphical construction goes as follows. Make the points of the Poisson process $\mathcal{P}$ on $\mathbb{R}^d \times \mathbb{R}^+$ into the vertices of an infinite oriented graph, denoted by $\Gamma$, by putting in an oriented edge $(X, T) \rightarrow (X', T')$ whenever $(X' + V) \cap (X + V) \neq \emptyset$ and $T < T'$. For completeness we also put an edge $(X, T) \rightarrow (X', T')$ whenever $(X' + V) \cap (X + V) \neq \emptyset$, $T = T'$, and $X$ precedes $X'$ in the lexicographical order - although in practice the probability that $\mathcal{P}$ generates such an edge is zero. It can be useful to think of the oriented graph as representing the spread of an "epidemic" through space over time; each time an individual is "born" at a Poisson point in space-time, it becomes (and stays) infected if there is an earlier infected point nearby in space (in the sense that the translates of $V$ centred at the two points overlap). This graph determines which items have to be accepted.

For $(X, T) \in \mathcal{P}$, let $C(X, T)$ (the "cluster at $(X, T)$") be the (random) set of ancestors of $(X, T)$, that is, the set of $(Y, U) \in \mathcal{P}$ such that there is an oriented path in $\Gamma$ from $(Y, U)$ to $(X, T)$. As shown in [P01] Corollary 3.1], the "cluster" $C(X, T)$ is finite for $(X, T) \in \mathcal{P}$ with probability 1. It represents the set of all items that can potentially affect the acceptance status of the incoming particle represented by the Poisson point $(X, T)$. The set of accepted items may be reconstructed from the graph $\Gamma$, as follows. Recursively define subsets $F_i$, $G_i$, $H_i$ of $\Gamma$ as follows. Let $F_1$ be the set of roots of the oriented graph $\Gamma$, and let $G_1$ be the set of offspring of roots. Set $H_1 = F_1 \cup G_1$. For the next step, remove the set $H_1$ from the vertex set, and define $F_2$ and $G_2$ in the same way; so $F_2$ is the set of roots of the restriction of $\Gamma$ to vertices in $\mathcal{P}\backslash H_1$, and $G_2$ is the set of vertices in $\mathcal{P}\backslash H_1$ which are offspring of $F_2$. Set $H_2 = F_2 \cup G_2$, remove the set $H_2$ from $\mathcal{P}\backslash H_1$, and repeat the process to obtain $F_3, G_3, H_3$. Continuing ad infinitum gives us subsets $F_i, G_i, \mathcal{P}$ defined for $i = 1, 2, 3, \ldots$. These sets are disjoint by construction. As proved in [P01] Lemma 3.2], the sets $F_1, G_1, F_2, G_2 \ldots$ form a partition of $\mathcal{P}$ with probability 1.
Definition 4.24. The random parking measure in $\mathbb{R}^d$ is given by the counting measure $N(A)$, $A \subset \mathbb{R}^d$, generated by the projection of the union $\cup_{i=1}^{\infty} F_i$ on $\mathbb{R}^d$.

It is proven in e.g. [P01, GP13] that there exists probability space which supports RPM and that RPM is an ergodic process with respect to translations. Furthermore, there exists $\kappa = \kappa(d, V)$ such that

$$\lim_{M \to \infty} \frac{N(\square^M)}{|\square^M|} = \kappa$$

almost surely. (97)

In our example we set $V = \square$, thus we have a collection of randomly parked non-overlapping cubes $\square^1_{X_j}, j \in \mathbb{N}, \cup_j X_j = \cup_i F_i$, such that no more unit cubes can be fitted without overlapping. At each cube we place an inclusion $X_j + Y_1$, where $Y_1 \subset \square$ is a reference inclusion observing a positive distance from the boundary of the cube. We have

$$\text{Sp}(-\Delta_\sigma) = \text{Sp}(-\Delta_{Y_1}).$$

From (26) and (97) we easily get that

$$\beta(\lambda) := \lambda + \lambda^2 \kappa \sum_{j=1}^\infty \frac{\langle \varphi_j \rangle^2}{\nu_j - \lambda}.$$ (98)

In order to derive a formula for $\beta_\infty$ we need to look at the areas with the smallest and the greatest density of the distribution of inclusions (hence the choice of the set $V$, for which this can be done explicitly).

For an arbitrary (small) $\delta > 0$ consider the periodic lattice $(2 - 4\delta)\mathbb{Z}^d$, denoting its points by $\xi_i, i \in \mathbb{N}$, and consider the balls of radius $\delta$ centred at $\xi_i, i \in \mathbb{N}$. For an arbitrary $T > 0$ there is a positive probability of that $P$ has exactly one point $(X_i, T_i)$ in each of the sets $B_\delta(\xi_i) \times (0, T) \subset \square^{M+4} \times (0, T)$ and no other points in the set $\square^{M+4} \times (0, T)$. Then, by the above construction, the random parking measure has exactly one point $X_i$ inside each ball $B_\delta(\xi_i)$ with $\xi_i \in \square^M$ and no other points inside $\square^M$, as no more unit cubes could fit between the cubes $\square^1_{X_j}$. By the law of large numbers, with probability one there exist arbitrary large cubes $\square^M$ containing only almost periodically distributed inclusions $Y_1 + X_i, X_i \in B_\delta(\xi_i), \xi_i \in (2 - 4\delta)\mathbb{Z}^d$. Since $M$ and $\delta$ are arbitrary, we conclude that

$$\beta_\infty(\lambda) \geq \beta_{2\text{-per}, Y_1}(\lambda).$$ (98)

Making a completely analogous construction now with the periodic lattice $(1 + 4\delta)\mathbb{Z}^d, \delta > 0$, we can see that

$$\beta_\infty(\lambda) \geq \beta_{1\text{-per}, Y_1}(\lambda).$$ (99)

On the other hand, since 1-periodic and 2-periodic distribution of inclusions provide the greatest and the smallest possible density of the distribution of inclusions in this example, one can easily conclude that equalities are attained in (98) and (99) when $\int_{Y_1} b(\cdot, \lambda) < 0$ and $\int_{Y_1} b(\cdot, \lambda) \geq 0$, respectively.

Thus we conclude that

$$\beta_\infty(\lambda) = \max\{\beta_{1\text{-per}, Y_1}(\lambda), \beta_{2\text{-per}, Y_1}(\lambda)\}.$$

Remark 4.25. As in the previous examples, one can elaborate the random parking model example further by allowing inclusions of different shapes, sizes, randomly rotated, through the use of the marked point processes framework.
Remark 4.26. The random parking model is a more realistic (from the applications point of view) model of random distribution of inclusions. On one hand, it does not allow arbitrarily large areas where inclusions do not appear (more precisely, there exists a radius \( r \) such that any ball \( B_r(x), x \in \mathbb{R}^d \), contains at least one inclusion). On the other hand, the inclusions are not too close to each other, thus satisfying Assumption 2.7. If one only uses the Poisson point process in the construction, then one ends up with arbitrarily large areas free of inclusions and at the same time having arbitrary many overlapping inclusions.

The problem of overlapping inclusions can also be dealt with by using a Matérn hardcore process (see e.g. [H20]) which is constructed from a given point process by mutually erasing all points with the distance to the nearest neighbour smaller than a given constant. It can be shown that if the original process is stationary (ergodic), the resulting hardcore process is stationary (ergodic) respectively. If one starts from a Poisson process and makes Poison-Matérn process, this gives the point process which can be further used to construct random inclusions satisfying Assumption 2.7. However, the limit spectrum in this case would coincide with \( \mathbb{R}_0^+ \) due to the existence of arbitrarily large areas free of inclusions (in this case \( \beta_\infty(\lambda) \geq \lambda \) for all \( \lambda \geq 0 \)).

5 Statistically relevant (limiting) spectrum

In this section we address the problem of the spectrum of \( A^\varepsilon \) in the gaps of \( \text{Sp}(A^\text{hom}) \). As we have seen in the previous section, the limit set \( \mathcal{G} \) of \( \text{Sp}(A^\varepsilon) \) is larger that \( \text{Sp}(A^\text{hom}) \) in general. On the other hand, in the bounded domain setting we have the Hausdorff convergence of the spectra, i.e. the gaps of \( \text{Sp}(A^\text{hom}) \) are free from the spectrum of \( A^\varepsilon \) in the limit. These facts indicate that the elements of the Weyl sequences for \( A^\varepsilon \) corresponding to \( \lambda \notin \text{Sp}(A^\text{hom}) \) are supported further and further away from the origin (as \( \varepsilon \to 0 \)) and could not be contained in a bounded domain, however large it is. Another observation that can be drawn from Section 4 is that the elements of such Weyl sequences are concentrated in the areas of space with non-typical distribution of inclusions. Both observations may be interpreted in the following way: in the applications, where one deals with finite distances (finite size of a material) and manufactured composites with relatively uniform distribution of inclusions, the afore mentioned part of the spectrum may not be relevant. On the other hand, this remark should be taken with a certain level of caution: since in the applications \( \varepsilon \) is finite, one can not be sure that this part of the spectrum is not present at least until one has some quantitative results.

In this lies our motivation to refer to the corresponding parts of the spectrum of \( A^\varepsilon \) as statistically relevant and irrelevant. However, it does not seem feasible to precisely demarcate the border between the relevant and irrelevant spectra of \( A^\varepsilon \): indeed, the notion of Weyl sequences “supported further and further away from the origin” is not easily quantifiable. In this section we make an attempt to characterise this vague and somewhat speculative notion of statistically relevant spectrum. First, we introduce the so called statistically relevant limiting spectrum \( \text{SR-lim} \text{Sp}(A^\varepsilon) \) in the definition below, which is interpreted as the limit of the statistically relevant part of \( \text{Sp}(A^\varepsilon) \), and prove that it coincides exactly with \( \text{Sp}(A^\text{hom}) \), Theorem 5.2. The proof of the first part of the statement given in Section 5.1 is straightforward. The proof of the inverse statement provided in Section 5.2 is the main technical ingredient of the theorem, where we construct rather explicitly a sequence \( \psi^\varepsilon_L \) and a bound \( g(\varepsilon, L) = \bar{R}(\varepsilon, L, \lambda) \) (cf. (122)) satisfying Definition 5.1. (For \( \lambda \) in the Bloch spectrum \( \text{Sp}(-\Delta_{\Omega}) \) we have \( g(\varepsilon, L) = \bar{R}(\varepsilon, 1/L, \lambda) \), see (137). This is simply a technical detail, since the nature of the irrelevant spectrum has nothing to do with the Bloch spectrum.
In Section 5.3 we define a subset $\widehat{\text{Sp}}_\varepsilon(A^\varepsilon)$ of $\text{Sp}(A^\varepsilon)$ in terms of the quantity $\widehat{\mathcal{R}}(\varepsilon, L, \lambda)$ (and $\mathcal{R}(\varepsilon, 1/L, \lambda)$). The definition of $\widehat{\text{Sp}}_\varepsilon(A^\varepsilon)$ is quite similar to the one of $\mathcal{SR}$-lim $\text{Sp}(A^\varepsilon)$, however, $\widehat{\text{Sp}}_\varepsilon(A^\varepsilon)$ is defined on “$\varepsilon$-level”, rather than “in the limit”. We interpret $\widehat{\text{Sp}}_\varepsilon(A^\varepsilon)$ as the approximate statistically relevant spectrum of $A^\varepsilon$. The set $\widehat{\text{Sp}}_\varepsilon(A^\varepsilon)$ depends on the choice of $\widehat{\mathcal{R}}(\varepsilon, L, \lambda)$, thus obtaining a more optimal bound in the proof of Theorem 5.2 will provide a better approximation.

We prove the convergence of $\widehat{\text{Sp}}_\varepsilon(A^\varepsilon)$ to $\text{Sp}(A^{\text{hom}})$ in the sense of Hausdorff, Theorem 5.8. The term $\widehat{\mathcal{R}}(\varepsilon, L, \lambda)$ also characterises the closeness of $\widehat{\text{Sp}}_\varepsilon(A^\varepsilon)$ to $\text{Sp}(A^{\text{hom}})$, see (142).

Appendix D contains auxiliary constructions and statements required in the proof of Theorem 5.8. In particular, the terms $N_j^\varepsilon$, $G_j^\varepsilon$ and $B^\varepsilon$, whose rate of convergence to zero (which is beyond the scope of the present paper) determine the asymptotic behaviour of $\widehat{\mathcal{R}}(\varepsilon, L, \lambda)$, are introduced.

**Definition 5.1.** We say that $\lambda$ belongs to the statistically relevant limiting spectrum of $A^\varepsilon$ and denote this set by $\mathcal{SR}$-lim $\text{Sp}(A^\varepsilon)$ if there exists a constant $C > 0$ and a function $g(\varepsilon, L)$, $L \in \mathbb{N}$, satisfying

$$\lim_{L \to \infty} \lim_{\varepsilon \to 0} g(\varepsilon, L) = 0,$$

such for every small enough $\varepsilon > 0$, and every $L \in \mathbb{N}$ there exists $\psi_L^\varepsilon$ from the domain of $A^\varepsilon$ satisfying the following conditions,

$$\frac{\| (A^\varepsilon - \lambda) \psi_L^\varepsilon \|_{L^2(\mathbb{R}^d)}}{\| \psi_L^\varepsilon \|_{L^2(\mathbb{R}^d)}} \leq g(\varepsilon, L),$$

and for sufficiently large $L$ one has

$$\lim_{\varepsilon \to 0} \inf \frac{\| \psi_L^\varepsilon \|_{L^2(\mathbb{R}^d)}}{\| \psi_L^\varepsilon \|_{L^2(\mathbb{R}^d)}} \geq C.$$

Note that in order for Definition 5.1 to make sense we need to ensure that any $\lambda \in \mathcal{SR}$-lim $\text{Sp}(A^\varepsilon)$ is an accumulation point for $\text{Sp}(A^\varepsilon)$. This easily follows from (101), since $\text{dist}(\lambda, \text{Sp}(A^\varepsilon)) \leq \inf_L g(\varepsilon, L)$. In particular, there exists a sequence $\lambda^\varepsilon \in \text{Sp}(A^\varepsilon)$ such that $\lambda^\varepsilon \to \lambda$.

We now proceed to the first of the two main results of the present section.

**Theorem 5.2.** The statistically relevant limiting spectrum of $A^\varepsilon$ coincides with the spectrum of $A^{\text{hom}}$;

$$\mathcal{SR}$-lim $\text{Sp}(A^\varepsilon) = \text{Sp}(A^{\text{hom}}).$$

In order to illustrate the basic idea behind Definition 5.1 we compare it to the argument presented in Theorem 4.2 (assuming that $\lambda \not\in \text{Sp}(-\Delta_\mathcal{O})$). The bound (101) is sufficient for constructing from the restrictions $u^\varepsilon|_{\Box \xi^\varepsilon}$ a family of approximate eigenfunctions satisfying the bound (101) for an appropriate $g(\varepsilon, L)$, by using cut-off functions and a resolvent argument (cf. (72)). Most of the energy of these approximate eigenfunctions will be localised in the vicinity of the cubes $\Box \xi^\varepsilon$, whose location in space is in general “arbitrary”, hence the corresponding local spectral averages $\ell(\varepsilon^{-1} \xi^\varepsilon, \varepsilon^{-1} L, \lambda, \omega)$ (cf. Remark 4.1) will be controlled from above only by $\beta_\infty(\lambda)$. (Note that here the local spectral averages are calculated on the rescaled cubes $\varepsilon^{-1} \Box \xi^\varepsilon$.) However, if we assume that for every $L \in \mathbb{N}$ the cubes are centred at the origin, i.e. $\xi^\varepsilon = 0$, as one has in Definition 5.1 then by the ergodic theorem the local spectral averages will converge to $\beta(\lambda)$ (cf. Remark 3.8).

The above argument infers that in the periodic setting (which allows translations) one has $\lim_{\varepsilon \to 0} \text{Sp}(A^\varepsilon) = \mathcal{SR}$-lim $\text{Sp}(A^\varepsilon)$. 

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5.1 Proof of $\text{SR-} \lim \text{Sp}(\mathcal{A}^\varepsilon) \subset \text{Sp}(\mathcal{A}^\text{hom})$

Let $\lambda \in \text{SR-} \lim \text{Sp}(\mathcal{A}^\varepsilon)$. Without loss of generality we can assume that $\|\psi_L^\varepsilon\|_{L^2(\mathbb{R}^d)} = 1$. Then we can write

$$(A^\varepsilon - \lambda)\psi_L^\varepsilon = f_L^\varepsilon,$$

where $\|f_L^\varepsilon\|_{L^2(\mathbb{R}^d)} \leq g(\varepsilon, L).$  

(103)

For each $L$ the sequences $f_L^\varepsilon$ and $\psi_L^\varepsilon$ are bounded and, hence, converge up to a subsequence weakly stochastically two-scale as $\varepsilon \to 0$ to some $f_L, \psi_L \in L^2(\mathbb{R}^d \times \Omega)$ respectively. In particular,

$$\|f_L\|_{L^2(\mathbb{R}^d \times \Omega)} \leq \lim_{\varepsilon \to 0} g(\varepsilon, L).$$

By Theorem 2.8 applied to the identity

$$(A^\varepsilon + 1)\psi_L^\varepsilon = (\lambda + 1)\psi_L^\varepsilon + f_L^\varepsilon,$$

we have that $\psi_L = \psi_{L,0} + \psi_{L,1} \in V$ is the solution to

$$(A^\text{hom}_\varepsilon - \lambda)\psi_L = f_L.$$

(104)

It remains to prove that $\psi_L$ does not vanish as $L \to \infty$. Multiplying (103) by $\overline{\psi_L^\varepsilon}$ and integrating by parts we easily get

$$\|\varepsilon \nabla \psi_L^\varepsilon\|_{L^2(S^0_\varepsilon)} + \|\nabla \overline{\psi_L^\varepsilon}\|_{L^2(\mathbb{R}^d)} \leq C,$$

where $\overline{\psi_L^\varepsilon}$ is the extension of $\psi_L^\varepsilon|_{S^1_\varepsilon}$ onto $\mathbb{R}^d$ by Theorem 2.8. Moreover, from the same theorem we have

$$\|\overline{\psi_L^\varepsilon}\|_{L^2(\mathbb{R}^d)} \leq C.$$

From the last two bounds we have that up to a subsequence $\overline{\psi_L^\varepsilon}$ converges strongly in $L^2(K)$, for any bounded domain $K$, to some function $\overline{\psi_L} \in W^{1,2}(\mathbb{R}^d)$ (which a priori could be zero). Then, by the properties of stochastic two-scale convergence, $\chi_1 \overline{\psi_L^\varepsilon} \overset{2}{\to} 1_{\Omega \setminus \varnothing} \overline{\psi_L}$. On the other hand, $\chi_1 \overline{\psi_L^\varepsilon} \overset{2}{\to} 1_{\Omega \setminus \varnothing} \psi_L = 1_{\Omega \setminus \varnothing} \psi_{L,0}$. Thus we have

$$\overline{\psi_L^\varepsilon} \to \psi_{L,0} \text{ in } L^2(K) \text{ for any bounded domain } K.$$

(105)

If $\lambda \in \text{Sp}(\Delta_\varnothing)$ then $\lambda \in \text{Sp}(\mathcal{A}_{\text{hom}}^\varepsilon)$. Assume that $\lambda \notin \text{Sp}(\Delta_\varnothing)$. Then arguing exactly as at the beginning of Section 4.3, cf. (40)-(44), we conclude that for small enough $\varepsilon$

$$\|\psi_L^\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C\|\overline{\psi_L^\varepsilon}\|_{L^2(\mathbb{R}^d)} + C\|f_L^\varepsilon\|_{L^2(\mathbb{R}^d)}.$$  

Combining this with (102), passing to the limit via (105), and taking into account (100), we see that for large enough $L$

$$\|\psi_{L,0}\|_{L^2(\mathbb{R}^d)} \geq C > 0.$$  

It follows that $\|\psi_L\|_{L^2(\mathbb{R}^d)} \geq C > 0$ and, therefore, since $f_L$ in (104) vanishes as $L \to \infty$, we conclude that $\lambda \in \text{Sp}(\mathcal{A}_{\text{hom}}^\varepsilon)$.

5.2 Proof of $\text{SR-} \lim \text{Sp}(\mathcal{A}^\varepsilon) \supset \text{Sp}(\mathcal{A}_{\text{hom}}^\varepsilon)$

Following the general scheme suggested in [KS18], starting from a standard prototype $u_L$ of a Weyl sequence for the homogenised operator $\mathcal{A}_{\text{hom}}^\varepsilon$, we will construct for each element of the sequence and each $\varepsilon$ an approximate solution $u_{LC}^\varepsilon$ to the spectral problem for the operator $\mathcal{A}^\varepsilon$ maintaining control of the error.
5.2.1 Case $\lambda \in \text{Sp}(A_{\text{hom}}) \setminus \text{Sp}(-\Delta_D)$

A part of the argument and the initial construction we present below are similar to those contained in the proof of Theorem 4.5. Therefore, we will often refer to the relevant places in Section 4.5 to fill in some detail and reuse a number of formulae therein so as not be repetitive. At the same time, since we are no longer in the periodic setting of Section 4.5, we have no asymptotic bounds on the homogenisation correctors $N_j^\varepsilon$ and the terms $G_j^\varepsilon$ and $B^\varepsilon$ carrying the information about the microscopic structure of the composite, and so we will need to keep track of these quantities in the error bounds explicitly.

By the assumption we have that $\beta(\lambda) \geq 0$. Similarly to the proof of Theorem 4.5 we consider $u(x) := e^{ikx}$ with $k$ such that $A_{\text{hom}}^k \cdot k = \beta(\lambda)$ and denote

$$u_L := \frac{\eta_L u}{\|\eta_L u\|_{L^2(\mathbb{R}^d)}},$$

where $\eta_L$ is given by (50). Then $u_L$ satisfies

$$\|(-\nabla \cdot A_{\text{hom}}^k \nabla - \beta(\lambda)) u_L\|_{L^2(\mathbb{R}^d)} \leq \frac{C}{L} \tag{106}$$

and the bound (69). We also consider $u_\varepsilon^L$ and $\hat{v}_L$ defined by (70) and (72) respectively.

Further, let $p_j \in \mathcal{X}$, $j = 1, \ldots, d$, be the solution to the problem (9) with $\xi = e_j$. By Lemma D.3 we can assume that $p_j \in \mathcal{V}_{\text{pot}}^2$. Then for any cube $\Box^L$ and any $\varepsilon$ there exists a function $N_j^\varepsilon \in W^{1,2}(\Box^L)$ such that

$$\nabla N_j^\varepsilon = p_j^\varepsilon \text{ in } \Box^L, \quad \int_{\Box^L} N_j^\varepsilon = 0,$$

where $p_j^\varepsilon$ in the $\varepsilon$-realisation of $p_j$. It follows from the ergodic theorem that

$$\|N_j^\varepsilon\|_{L^2(\Box^L)} \to 0 \text{ as } \varepsilon \to 0. \tag{107}$$

We define

$$u_{\text{LC}}^\varepsilon := u_\varepsilon^L + \partial_j u_L N_j^\varepsilon.$$

Similarly to (73) we can estimate the corrector as follows,

$$\|u_{\text{LC}}^\varepsilon - u_\varepsilon^L\|_{L^2(\mathbb{R}^d)} = \|\partial_j u_L N_j^\varepsilon\|_{L^2(\mathbb{R}^d)} \leq \frac{C}{L^{d/2}} \sum_j \|N_j^\varepsilon\|_{L^2(\Box^L)} \tag{108}.$$

As in the proof of Theorem 4.5, we substitute $u_{\text{LC}}^\varepsilon$ and a test function $v \in W^{1,2}(\mathbb{R}^d)$ into the sesquilinear form associated with the operator $A^\varepsilon + 1$ in order to get the identity (74), and begin by analysing the first term on the right-hand side of (74) (recall the decomposition (75) and the bounds (76), (77)):

$$\int_{S_j^\varepsilon} A_1 \nabla u_{\text{LC}}^\varepsilon \cdot \nabla v = \int_{\mathbb{R}^d} \left( A_{\text{hom}}^1 \nabla u_L + [\chi_1^j A_1(e_j + \nabla N_j^\varepsilon) - A_1^\text{hom} e_j] \partial_j u_L + \chi_1^j N_j^\varepsilon \nabla \partial_j u_L \right) \cdot \nabla \varepsilon \tag{109}.$$

The following bound is straightforward,

$$\left| \int_{\mathbb{R}^d} \chi_1^j N_j^\varepsilon \nabla \partial_j u_L \cdot \nabla \varepsilon \right| \leq \frac{C}{L^{d/2}} \|\nabla \varepsilon\|_{L^2(\mathbb{R}^d)} \sum_j \|N_j^\varepsilon\|_{L^2(\Box^L)} \tag{110}.$$
In order to estimate the second term on the right-hand side of (109) we employ a similar argument as in (82)–(86), however, we cannot use the scaling argument as in the periodic case. Denote

\[ g_j := 1_{\Omega \setminus \mathcal{O}} A_1(e_j + p_j) - A_1^{\text{hom}} e_j \in Y_{\text{sol}}, \quad j = 1, \ldots, d, \]

and consider its \( \varepsilon \)-realisations

\[ g_j^{\varepsilon}(\cdot) := g_j(T_x / \varepsilon \omega) = \chi_{A_1}^\varepsilon (e_j + \nabla N_j^\varepsilon) - A_1^{\text{hom}} e_j. \]  

By Corollary D.5 there exist skew-symmetric tensor fields \( G_j^\varepsilon \in W^{1,2}(\square) \) such that \( g_j^\varepsilon = \nabla \cdot G_j^\varepsilon \), and

\[ \lim_{\varepsilon \to 0} \| G_j^\varepsilon \|_{L^2(\square)} = 0. \]  

Proceeding as in (85) we obtain

\[ \left| \int_{\mathbb{R}^d} [\chi_{A_1}^\varepsilon (e_j + \nabla N_j^\varepsilon) - A_1^{\text{hom}} e_j] \partial_j u_L \cdot \nabla \tilde{v}_0^\varepsilon \right| \leq C \frac{d}{L d} \| \nabla \tilde{v}_0^\varepsilon \|_{L^2(\mathbb{R}^d)} \sum_j \| G_j^\varepsilon \|_{L^2(\square)}. \]  

Now we address the second integral on the right-hand side of (74). Analogously to (87) and by the identity (88) we have

\[ \varepsilon^2 \int_{S_{\delta}^\varepsilon} \nabla u_{LC}^\varepsilon \cdot \nabla v = \int_{S_{\delta}^\varepsilon} \lambda u_L (\lambda b^\varepsilon + 1) \tilde{v}_0^\varepsilon \]

\[ + \varepsilon^2 \int_{S_{\delta}^\varepsilon} (- \lambda \nabla u_L \cdot \nabla b^\varepsilon \tilde{v}_0^\varepsilon + \lambda u_L \nabla b^\varepsilon \cdot \nabla \tilde{v}_0^\varepsilon + [\nabla u_L (1 + \lambda b^\varepsilon) + \nabla (\partial_j u_L N_j^\varepsilon)] \cdot \nabla v). \]  

We estimate the second integral on the right-hand side of (114) employing Lemma E.2 and the bounds (69) and (76), as follows:

\[ \left| \int_{S_{\delta}^\varepsilon} (- \lambda \nabla u_L \cdot \nabla b^\varepsilon \tilde{v}_0^\varepsilon + \lambda u_L \nabla b^\varepsilon \cdot \nabla \tilde{v}_0^\varepsilon + [\nabla u_L (1 + \lambda b^\varepsilon) + \nabla (\partial_j u_L N_j^\varepsilon)] \cdot \nabla v) \right| \]

\[ \leq C \varepsilon \left( \| \nabla \tilde{v}_0^\varepsilon \|_{L^2(\mathbb{R}^d)} + \| \varepsilon \nabla v \|_{L^2(\mathbb{R}^d)} \left( 1 + \frac{1}{L d \varepsilon^2} \sum_j \left( \| N_j^\varepsilon \|_{L^2(\square \varepsilon)} + \| \nabla N_j^\varepsilon \|_{L^2(\square \varepsilon)} \right) \right) \right). \]  

Combining (109), (110), (113)–(115), and (77) we arrive at

\[ a^\varepsilon(u_{LC}^\varepsilon, v) = \int_{\mathbb{R}^d} A_1^{\text{hom}} \nabla u_L \cdot \nabla \tilde{v}_0^\varepsilon + \int_{S_{\delta}^\varepsilon} \lambda u_L (\lambda b^\varepsilon + 1) \tilde{v}_0^\varepsilon + \int_{\mathbb{R}^d} u_{LC}^\varepsilon \tilde{v} + \mathcal{R}^\varepsilon, \]

where the remainder \( \mathcal{R}^\varepsilon \) satisfies

\[ |\mathcal{R}^\varepsilon| \leq C(\lambda) \sqrt{a^\varepsilon(v, v)} \left( \frac{1}{L d \varepsilon^2} \sum_j \left( \| N_j^\varepsilon \|_{L^2(\square \varepsilon)} + \| G_j^\varepsilon \|_{L^2(\square \varepsilon)} + \varepsilon \| \nabla N_j^\varepsilon \|_{L^2(\square \varepsilon)} \right) + \varepsilon \right). \]  

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(Here we emphasise the dependence of the constant on $\lambda^1$)

Proceeding as in the proof of Theorem 4.5 and recalling the definition of $\hat{u}_L^\varepsilon$ we arrive at

$$a^\varepsilon (u_L^\varepsilon - \hat{u}_L^\varepsilon, v) = \int_{\mathbb{R}^d} \left( (-\nabla : A_1^\text{hom} \nabla - \beta(\lambda)) u_L^\varepsilon \bar{v}^\varepsilon + \lambda^2 u_L((b) - b^\varepsilon) \bar{v}^\varepsilon + \partial_j u_L N^\varepsilon_j \bar{v}^\varepsilon \right) + \mathcal{R}^\varepsilon. \quad (118)$$

It remains to estimate the second term in the integral on the right-hand side of the latter. Since $(b) - b^\varepsilon$ converges weakly to zero in $L^2_{\text{loc}}(\mathbb{R}^d)$, by Lemma D.6 there exists a sequence of vector fields $B^\varepsilon \in W^{1,2}(\Box L; \mathbb{R}^d)$ such that $\nabla \cdot B^\varepsilon = (b) - b^\varepsilon$ and

$$\lim_{\varepsilon \to 0} \|B^\varepsilon\|_{L^2(\Box L)} \to 0. \quad (119)$$

Then, integrating by parts, we get

$$\left| \int_{\mathbb{R}^d} u_L((b) - b^\varepsilon) \bar{v}^\varepsilon \right| = \left| \int_{\mathbb{R}^d} \nabla(u_L \bar{v}^\varepsilon) \cdot B^\varepsilon \right| \leq C \left( \|\bar{v}^\varepsilon\|_{L^2(\mathbb{R}^d)} + \|
abla \bar{v}^\varepsilon\|_{L^2(\mathbb{R}^d)} \right) \|B^\varepsilon\|_{L^2(\Box L)}. \quad (120)$$

Now we can estimate the left hand side of (118) via (106), (108), (116), and (120) as follows,

$$|a^\varepsilon (u_L^\varepsilon - \hat{u}_L^\varepsilon, v)| \leq \mathcal{R}(\varepsilon, L, \lambda) \sqrt{a^\varepsilon(v, v)}, \quad (121)$$

where

$$\widehat{\mathcal{R}}(\varepsilon, L, \lambda) := \tilde{C}(\lambda) \left( \frac{1}{L^{d/2}} \sum_j \left( \|N^\varepsilon_j\|_{L^2(\Box L)} + \|G^\varepsilon_j\|_{L^2(\Box L)} + \varepsilon \|\nabla N^\varepsilon_j\|_{L^2(\Box L)} \right) + \frac{1}{L^{d/2}} \|B^\varepsilon\|_{L^2(\Box L)} + \frac{1}{L} + \varepsilon \right), \quad (122)$$

where as in (116) the dependence of the constant on $\lambda$ can be written explicitly$^2$.

From the convergences (107), (112), (119), and the fact that the term $\sum_j \|\nabla N^\varepsilon_j\|_{L^2(\Box L)}$ is bounded for fixed $L$ by the ergodic theorem, we infer that

$$\lim_{\varepsilon \to 0} \widehat{\mathcal{R}}(\varepsilon, L, \lambda) = \tilde{C}(\lambda) \frac{1}{L}, \quad (124)$$

$^1$It is not difficult to keep track of the dependence of the constants on $\lambda$. Omitting the detail, after some simplifications the constant can be written in the form

$$C(\lambda) := C_1 \left( 1 + |\beta(\lambda)| + \frac{\lambda^{3/2} (1 + \sqrt{|\beta(\lambda)|})}{\text{dist}(\lambda, \text{Sp}(-\Delta_\Omega))} \right), \quad (117)$$

where $C_1$ depends only on the parameters from Assumption 2.7. In particular, $C(\lambda)$ blows up as $\lambda$ approaches the spectrum of $-\Delta_\Omega$. It can be refined slightly to show that for small positive $\lambda$ the constant behaves as $L^{-1} + \sqrt{\lambda}$.

$^2$It can be shown that

$$\tilde{C}(\lambda) := \tilde{C}_1 \left( 1 + |\beta(\lambda)| + \lambda^2 (1 + \sqrt{|\beta(\lambda)|}) + \frac{\lambda^{3/2} (1 + \sqrt{|\beta(\lambda)|})}{\text{dist}(\lambda, \text{Sp}(-\Delta_\Omega))} \right), \quad (123)$$

where $\tilde{C}_1$ depends only on the parameters from Assumption 2.7, cf. (117), and is independent of $\varepsilon, L$ or $\lambda$. Again, as in the case with the constant $C(\lambda)$ in (117), a slight refinement shows that $\tilde{C}(\lambda)$ behaves as $L^{-1} + \sqrt{\lambda}$ for small positive $\lambda$. 

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and, consequently,\[
\lim_{L \to \infty} \lim_{\varepsilon \to 0} \tilde{R}(\varepsilon, L, \lambda) = 0. \tag{125}
\]
Substituting \( v = u_L^\varepsilon - \tilde{u}_L^\varepsilon \) in (121) and using the bound (108) yields\[
\| u_L^\varepsilon - \tilde{u}_L^\varepsilon \|_{L^2(\mathbb{R}^d)} \leq \tilde{R}(\varepsilon, L, \lambda) \tag{126}
\]
(for a sufficiently large \( \hat{C}_1 \) in (123)), and, hence\[
\frac{\| (A^\varepsilon - \lambda) \tilde{u}_L^\varepsilon \|_{L^2(\mathbb{R}^d)}}{\| \tilde{u}_L^\varepsilon \|_{L^2(\mathbb{R}^d)}} \leq |\lambda + 1| \tilde{R}(\varepsilon, L, \lambda). \tag{127}
\]
Finally, from the bounds (71) and (126) we infer that for sufficiently large \( L \) and sufficiently small \( \varepsilon \)
\[
\frac{\| \tilde{u}_L^\varepsilon \|_{L^2(\mathbb{R}^d)}}{\| u_L^\varepsilon \|_{L^2(\mathbb{R}^d)}} \leq \frac{\| u_L^\varepsilon \|_{L^2(\mathbb{R}^d)} - \tilde{R}(\varepsilon, L, \lambda)}{\| u_L^\varepsilon \|_{L^2(\mathbb{R}^d)} + \tilde{R}(\varepsilon, L, \lambda)} > C_2 > 0, \tag{128}
\]
where \( C_2 \) does not depend on \( \varepsilon, L \) or \( \lambda \).

It follows that \( \lambda \in \mathcal{S}R\lim \text{Sp}(A^\varepsilon) \) by setting \( \psi_L^\varepsilon = \tilde{u}_L^\varepsilon \) in Definition 5.1.

**Remark 5.3.** Notice that \( \tilde{R}(\varepsilon, L, \lambda) \) is well defined and satisfies (125) for all \( \lambda \in \mathbb{R}^d \setminus \text{Sp}(-\Delta_\mathcal{O}) \).

**Remark 5.4.** So far we were concerned with what happens with the error bound \( \tilde{R}(\varepsilon, L, \lambda) \) when \( L \) is fixed and \( \varepsilon \to 0 \), cf. (124). It is not true in general, however, that \( \tilde{R}(\varepsilon, L, \lambda) \) remains uniformly small if we fix (small) \( \varepsilon \) and let \( L \to \infty \). In particular, for each (large enough) \( L \) the bound (128) holds for small enough \( \varepsilon \), i.e. for \( \varepsilon \leq \varepsilon_L \), where, in principle, the sequence \( \varepsilon_L \) may be decreasing to zero as \( L \to \infty \). Denote by \( \mathcal{N}(\varepsilon, \lambda) \) the set of \( L \in \mathbb{N} \) for which the bound (128) holds for the given \( \varepsilon \). We have that every \( L \in \mathcal{N}(\varepsilon, \lambda) \) for all \( \varepsilon \leq \varepsilon_L \).

### 5.2.2 Case \( \lambda \in \text{Sp}(-\Delta_\mathcal{O}) \)

The argument we utilise in this case is similar to the one we used above but is more straightforward technically. Since \( \lambda \) is in the spectrum of \(-\Delta_\mathcal{O}\), for any \( \delta > 0 \) there exists \( \varphi_\delta \in W^{1,2}_0(\mathcal{O}), \| \varphi_\delta \|_{L^2(\mathcal{O})} = 1 \), such that
\[
(-\Delta_\mathcal{O} - \lambda) \varphi_\delta = f_\delta, \quad \text{where} \quad \| f_\delta \|_{L^2(\mathcal{O})} \leq \delta. \tag{129}
\]
Let \( \kappa = \kappa(x, \tilde{\omega}) \) be the function constructed in Proposition D.8. Then for a.e. \( \tilde{\omega} \) we have
\[
\varphi_\delta(T_{/\varepsilon}\omega)\kappa(\cdot/\varepsilon, \tilde{\omega}) \rightharpoonup 0 \text{ weakly in } L^2_{\text{loc}}(\mathbb{R}^d).
\]

Consider the functions
\[
u_\delta^\varepsilon(x) := \begin{cases} 
\varphi_\delta(T_{/\varepsilon}\omega)\kappa(x/\varepsilon, \tilde{\omega}), & x \in \varepsilon\mathcal{O}_\delta^\varepsilon \text{ such that } \varepsilon\mathcal{O}_\delta^\varepsilon \subset \square, \\
0, & \text{otherwise},
\end{cases}
\]
and
\[
\tilde{u}_\delta^\varepsilon := (A^\varepsilon + 1)^{-1}(\lambda + 1)u_\delta^\varepsilon. \tag{130}
\]
It is not difficult to see that the sequence $u^\varepsilon_\delta$ satisfies the following properties:

$$u^\varepsilon_\delta \rightharpoonup 0 \text{ weakly in } L^2(\mathbb{R}^d);$$

$$C^{-1} \leq \|u^\varepsilon_\delta\|_{L^2(\mathbb{R}^d)} \leq C, \quad \|\varepsilon \nabla u^\varepsilon_\delta\|_{L^2(\mathbb{R}^d)} \leq C; \quad (131)$$

$$(-\varepsilon^2 \Delta - \lambda) u^\varepsilon_\delta = f^\varepsilon_\delta \kappa(\cdot/\varepsilon, \dot{\omega}) \text{ on each } \varepsilon \mathcal{O}_\omega^k \subset \Box, \quad (132)$$

where we define $f^\varepsilon_\delta(x)$ to be equal to $f_\delta(T_x/\varepsilon \omega)$ in the inclusions satisfying $\varepsilon \mathcal{O}_\omega^k \subset \Box$, and zero otherwise. By the ergodic theorem for small enough $\varepsilon$ we have

$$\|f^\varepsilon_\delta\|_{L^2(\mathbb{R}^d)} \leq C\delta. \quad (133)$$

By Lemma D.6 there exists a sequence of vector fields $U^\varepsilon_\delta \in W^{1,2}(\Box^2; \mathbb{R}^d)$ such that

$$\nabla \cdot U^\varepsilon_\delta = u^\varepsilon_\delta \text{ in } \Box^2, \quad \|U^\varepsilon_\delta\|_{L^2(\mathbb{R}^d)} \to 0. \quad (134)$$

Following the argument from Section 5.2.1 we substitute $u^\varepsilon_\delta$ and a test function $v \in W^{1,2}(\mathbb{R}^d)$ into the sesquilinear form $a^\varepsilon(\cdot, \cdot)$. Using the decomposition (75), integrating by parts and utilising the identity (132) we obtain

$$a^\varepsilon(u^\varepsilon_\delta, v) = -\int_{\mathbb{R}^d} \left( \varepsilon^2 \Delta u^\varepsilon_\delta v_0 + \varepsilon^2 \nabla u^\varepsilon_\delta \cdot \nabla v_\delta + u^\varepsilon_\delta v \right) \, \rho \, \delta \cdot \nabla \rho \, \delta + \int_{\mathbb{R}^d} \left( (\lambda u^\varepsilon_\delta + f^\varepsilon_\delta \kappa^\varepsilon \varepsilon_\delta v_0 + \varepsilon^2 \nabla u^\varepsilon_\delta \cdot \nabla v_\delta + u^\varepsilon_\delta v \right) \, \rho \, \delta \cdot \nabla \rho \, \delta \cdot \eta \, \delta$$

$$= \int_{\mathbb{R}^d} \left( \lambda + 1 \right) u^\varepsilon_\delta \cdot \delta \cdot \nabla \delta \cdot \delta + f^\varepsilon_\delta \kappa^\varepsilon \varepsilon_\delta v_0 + \varepsilon^2 \nabla u^\varepsilon_\delta \cdot \nabla \varepsilon_\delta + u^\varepsilon_\delta v \right) \, \rho \, \delta \cdot \nabla \rho \, \delta \cdot \eta \, \delta$$

We estimate the last two terms on the right-hand side of the latter via the bounds (76), (131) and (133) as follows:

$$\left| \int_{\mathbb{R}^d} f^\varepsilon_\delta \kappa^\varepsilon \varepsilon_\delta v_0 \right| \leq C\delta \|\varepsilon \nabla v\|_{L^2(\mathbb{R}^d)}, \quad \left| \int_{\mathbb{R}^d} \varepsilon^2 \nabla u^\varepsilon_\delta \cdot \nabla \varepsilon_\delta \right| \leq C \varepsilon \|\nabla \varepsilon_\delta\|_{L^2(\mathbb{R}^d)}. \quad (135)$$

Since $u^\varepsilon_\delta$ is zero outside $\Box$ we have

$$\int_{\mathbb{R}^d} \lambda u^\varepsilon_\delta \cdot \delta \cdot \nabla \delta \cdot \eta_2 = \int_{\Box^2} \lambda U^\varepsilon_\delta \cdot \delta \cdot \nabla \delta \cdot \eta_2 = \int_{\Box^2} \lambda \left( \nabla \cdot U^\varepsilon_\delta \right) \delta \cdot \nabla \delta \cdot \eta_2 = -\int_{\Box^2} \lambda U^\varepsilon_\delta \cdot \nabla \delta \cdot \nabla \delta \cdot \eta_2$$

(recall the definition of the cut-off function (50)). Hence

$$\left| \int_{\mathbb{R}^d} \lambda u^\varepsilon_\delta \cdot \delta \cdot \nabla \delta \cdot \eta_2 \right| \leq C \lambda \|U^\varepsilon_\delta\|_{L^2(\Box^2)} \left( \|\delta\|_{L^2(\mathbb{R}^d)} + \|\nabla \delta\|_{L^2(\mathbb{R}^d)} \right). \quad (136)$$

Combining (77), (130), (135) and (136) we arrive at

$$|a^\varepsilon(u^\varepsilon_\delta - \delta \varepsilon_\delta, v)| \leq \mathcal{R}(\varepsilon, \delta, \lambda) \sqrt{a^\varepsilon(v, v)},$$

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where
\[ \tilde{R}(\varepsilon, \delta, \lambda) := C_3(\delta + \varepsilon + \lambda \| U_{\varepsilon\delta} \|_{L^2(\mathbb{R}^d)}) \] (137)
and the constant \( C_3 \) does not depend on \( \lambda, \varepsilon \) and \( \delta \). By (134) we have
\[ \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \tilde{R}(\varepsilon, \delta, \lambda) = 0. \]

Arguing as in the first part of the proof we infer
\[ \| u_{\varepsilon\delta} - \hat{u}_{\varepsilon\delta} \|_{L^2(\mathbb{R}^d)} \leq \tilde{R}(\varepsilon, \delta, \lambda), \] (138)
\[ \frac{\| \tilde{u}_{\delta}^{\varepsilon} \|_{L^2(\mathbb{R}^d)}}{\| \tilde{u}_{\delta}^{\varepsilon} \|_{L^2(\mathbb{R}^d)}} \geq C_4 > 0 \text{ for sufficiently small } \varepsilon \text{ and } \delta, \]
\[ \frac{\| (\mathcal{A}^{\varepsilon} - \lambda)\tilde{u}_{\delta}^{\varepsilon} \|_{L^2(\mathbb{R}^d)}}{\| \tilde{u}_{\delta}^{\varepsilon} \|_{L^2(\mathbb{R}^d)}} \leq |\lambda + 1| \tilde{R}(\varepsilon, \delta, \lambda). \] (139)

Denoting \( \psi_L = \hat{u}_{1/L}^{\varepsilon} \) we conclude that \( \lambda \in SR-\lim Sp(\mathcal{A}^{\varepsilon}) \) by definition. The proof is complete.

**Remark 5.5.** Similarly to Remark 5.4 the bound (138) holds for small enough \( \varepsilon \), i.e. for \( \varepsilon \leq \tilde{\varepsilon}_\delta \).

Denote by \( \tilde{N}(\varepsilon, \lambda) \) the set of all \( n \in \mathbb{N} \) such that the bound (138) holds for the given \( \varepsilon \) with \( \delta = \frac{1}{n} \).

We have that every \( n \) is contained in \( \tilde{N}(\varepsilon, \lambda) \) for all \( \varepsilon \leq \tilde{\varepsilon}_1/n \). Since \( \tilde{N}(\varepsilon, \lambda) \) depends on the choice of \( \varphi_\delta \), by fixing for each \( \lambda \in -\Delta_\varnothing \) a sequence \( \varphi_\delta, \delta = \frac{1}{n} \), satisfying (129) we fix \( \tilde{N}(\varepsilon, \lambda) \).

**Remark 5.6.** Notice that in the case \( \lambda \in Sp(-\Delta_\varnothing) \) we do not need a sequence of growing domains \( \Box^L \) in order to construct a sequence of approximate solutions to the spectral problem. A similar but simpler construction can be done in the periodic setting of [Zh04] for the eigenvalues of the microscopic (or Bloch) operator corresponding to the eigenfunctions with non-zero mean. In particular, the probabilistic construction of the marking function \( \kappa \) can be replaced by marking the periodic inclusions with \( +1 \) and \( -1 \) in a chessboard pattern.

### 5.3 Approximate statistically relevant spectrum

The definition of the statistically relevant limiting spectrum is not constructive, and hence does not guarantee a priori that the set \( SR-\lim Sp(\mathcal{A}^{\varepsilon}) \) is non-empty. As we have seen in the previous subsection, the construction of families of approximate eigenfunctions satisfying Definition 5.1 is quite non-trivial and involves additionally a number of technical results presented in the appendix. At the same time it provides explicit bounds for the “error” \( g(\varepsilon, L) \) in terms of the homogenisation corrector and other quantities encoding the stochastic information at the microscale, cf. (127) and (139). Availability of specific error bounds provides us with an opportunity to classify in a similar manner the spectrum of \( \mathcal{A}^{\varepsilon} \) itself (albeit with a degree of inaccuracy). In what follows, we introduce a subset of \( Sp(\mathcal{A}^{\varepsilon}) \), which may be dubbed an approximate statistically relevant spectrum, and prove its convergence in the sense of Hausdorff to the spectrum of \( \mathcal{A}^{\text{hom}} \), thus making a step towards quantification of the notion of the statistically relevant spectrum and its distance to \( Sp(\mathcal{A}^{\text{hom}}) \).

**Definition 5.7.** For each small enough \( \varepsilon \) we define a set
\[ \hat{Sp}_\varepsilon(\mathcal{A}^{\varepsilon}) \]
as a subset of $\text{Sp}(A^\varepsilon)$ which consists of such $\lambda$ that there exists a family of functions $\psi^\varepsilon_L$, $L \in \mathbb{N}(\varepsilon, \lambda) \cap \hat{N}(\varepsilon, \lambda)$, from the domain of $A^\varepsilon$ satisfying the estimates

$$\frac{\|(A^\varepsilon - \lambda)\psi^\varepsilon_L\|_{L^2(\mathbb{R}^d)}}{\|\psi^\varepsilon_L\|_{L^2(\mathbb{R}^d)}} \leq 2|\lambda + 1|(\hat{\mathcal{R}}(\varepsilon, L, \lambda) + \hat{\mathcal{R}}(\varepsilon, 1/L, \lambda)),$$

(140)

$$\frac{\|\psi^\varepsilon_L\|_{L^2(\mathbb{R}^d)}}{\|\psi^\varepsilon_L\|_{L^2(\mathbb{R}^d)}} \geq \min\{C_2, C_4\},$$

(141)

where $N(\varepsilon, \lambda)$, $\hat{N}(\varepsilon, \lambda)$, $\hat{\mathcal{R}}(\varepsilon, L, \lambda)$, $\hat{\mathcal{R}}(\varepsilon, \delta, \lambda)$, $C_2$ and $C_4$ are defined in the preceding subsections. It is assumed additionally that $\hat{\mathcal{R}}(\varepsilon, L, \lambda) = 0$ if $\lambda \in \text{Sp}(\Delta_\mathcal{O})$ and $\hat{\mathcal{R}}(\varepsilon, L, \lambda) = 0$ if $\lambda \in \mathbb{R} \setminus \text{Sp}(\Delta_\mathcal{O})$.

It is clear that the set $\hat{\text{Sp}}_\varepsilon(A^\varepsilon)$ depends on the definition of $\hat{\mathcal{R}}$ and $\hat{\mathcal{R}}$ and the choice of the constants (one could try to obtain sharper estimates, for example).

**Theorem 5.8.** The set $\hat{\text{Sp}}_\varepsilon(A^\varepsilon)$ converges to $\text{Sp}(A^{\text{hom}}) = SR \cdot \lim \text{Sp}(A^\varepsilon)$ in the sense of Hausdorff.

**Proof.** Assume that $\lambda \in \text{Sp}(A^{\text{hom}})$. Depending on whether $\lambda \in \text{Sp}(A^{\text{hom}}) \setminus \text{Sp}(\Delta_\mathcal{O})$ or $\lambda \in \text{Sp}(\Delta_\mathcal{O})$ we define $\psi^\varepsilon_L := \tilde{w}^\varepsilon_L$, see (72), or $\psi^\varepsilon_L := \tilde{u}^\varepsilon_{1/L}$, see (130), respectively. Then by (127), (128), (138), (139), Remarks 5.4 and 5.5 we have that (141) holds and

$$\frac{\|(A^\varepsilon - \lambda)\psi^\varepsilon_L\|_{L^2(\mathbb{R}^d)}}{\|\psi^\varepsilon_L\|_{L^2(\mathbb{R}^d)}} \leq |\lambda + 1|(\hat{\mathcal{R}}(\varepsilon, L, \lambda) + \hat{\mathcal{R}}(\varepsilon, 1/L, \lambda)).$$

It follows that for small enough $\varepsilon$ there exist $\lambda^\varepsilon \in \text{Sp}A^\varepsilon$ such that

$$|\lambda^\varepsilon - \lambda| \leq |\lambda + 1| \inf_L (\hat{\mathcal{R}}(\varepsilon, L, \lambda) + \hat{\mathcal{R}}(\varepsilon, 1/L, \lambda)),$$

(142)

and we immediately arrive at (140). Thus $\lambda^\varepsilon \in \hat{\text{Sp}}_\varepsilon(A^\varepsilon)$ and $\lambda^\varepsilon \to \lambda \in \mathbb{R}$ as $\varepsilon \to 0$.

Conversely, let $\lambda^\varepsilon$ be a sequence such that $\lambda^\varepsilon \in \hat{\text{Sp}}_\varepsilon(A^\varepsilon)$ and $\lambda^\varepsilon \to \lambda_0 \in \mathbb{R}$. Without loss of generality we can assume that $\lambda_0 \notin \text{Sp}(\Delta_\mathcal{O})$ and that $\|\psi^\varepsilon_L\|_{L^2(\mathbb{R}^d)} = 1$. Then we have

$$(A^\varepsilon - \lambda^\varepsilon)\psi^\varepsilon_L = f^\varepsilon_L,$$

where

$$\|f^\varepsilon_L\|_{L^2(\mathbb{R}^d)} \leq 2|\lambda^\varepsilon + 1|\hat{\mathcal{R}}(\varepsilon, L, \lambda^\varepsilon)$$

(143)

for small enough $\varepsilon$.

Next we need to ensure that the convergence property (125) is preserved when a fixed $\lambda$ is replaced by the converging sequence $\lambda^\varepsilon$. Consider $b = b_\lambda$ - the solution to (16). By Hilbert’s resolvent identity

$$b_{\lambda^\varepsilon} - b_{\lambda_0} = (\lambda^\varepsilon - \lambda_0)(-\Delta_\mathcal{O} - \lambda^\varepsilon)^{-1}(-\Delta_\mathcal{O} - \lambda_0)^{-1}1_{\mathcal{O}},$$

so we have the bound

$$\|b_{\lambda^\varepsilon} - b_{\lambda_0}\|_{L^2(\mathcal{O})} \leq \frac{|\lambda^\varepsilon - \lambda_0|P(\mathcal{O})^{1/2}}{d_{\lambda_0}d_{\lambda^\varepsilon}},$$

where we use the notation

$$d_{\lambda} := \text{dist}(\lambda, \text{Sp}(\Delta_\mathcal{O})).$$
In particular,
\[ |\langle b_{\lambda^\varepsilon} \rangle - \langle b_{\lambda_0} \rangle| \leq \|b_{\lambda^\varepsilon} - b_{\lambda_0}\|_{L^2(O)} P(O)^{1/2} \leq \frac{|\lambda^\varepsilon - \lambda_0| P(O)}{d_{\lambda_0} d_{\lambda^\varepsilon}}. \]  
(144)

Similarly, applying Hilbert’s resolvent identity to the realisation \( b_{\lambda}(T_x \omega) \) on each \( O_k^\omega \) and taking into account (14), we get
\[ \|b_{\lambda^\varepsilon}(T_x \omega) - b_{\lambda_0}(T_x \omega)\|_{L^2(O_k^\omega)} \leq |\lambda^\varepsilon - \lambda_0| |O_k^\omega|^{1/2} d_{\lambda_0} d_{\lambda^\varepsilon}. \]

After the rescaling we easily obtain that for small enough \( \varepsilon \)
\[ \|b_{\lambda^\varepsilon}(T_x \omega^\varepsilon) - b_{\lambda_0}(T_x \omega)\|_{L^2(O_k^\omega^\varepsilon)} \leq |\lambda^\varepsilon - \lambda_0| \frac{1}{\hat{C}(\lambda_0) L}. \]  
(145)

Since by the ergodic theorem \( \langle b_{\lambda_0} \rangle - \langle b_{\lambda^\varepsilon} \rangle \) converges to zero weakly in \( L^2(\square L) \), the bound (145) immediately implies that
\[ \langle b_{\lambda^\varepsilon} \rangle - \langle b_{\lambda_0} \rangle \rightharpoonup 0 \text{ weakly in } L^2(\square L). \]

By Lemma [D.6] there exists \( B_{\lambda^\varepsilon} \in W^{1,2}(\square L; \mathbb{R}^d) \) such that \( \nabla \cdot B_{\lambda^\varepsilon} = \langle b_{\lambda^\varepsilon} \rangle - \langle b_{\lambda_0} \rangle \) in \( L^2(\square L) \) and
\[ \|B_{\lambda^\varepsilon}\|_{L^2(\square L)} \rightarrow 0. \]

Notice that by the explicit construction in Lemma [D.6] the function \( B_{\lambda^\varepsilon} \) coincides with \( B^\varepsilon \) in (122) upon replacing \( \lambda \) by \( \lambda^\varepsilon \). Thus we have that
\[ \lim_{\varepsilon \rightarrow 0} \hat{R}(\varepsilon, L, \lambda^\varepsilon) = \hat{C}(\lambda_0) \frac{1}{L}, \]
(143)

It follows from (143) that for each \( L \) the sequence \( f_L^\varepsilon \) converges (up to a subsequence) weakly stochastically two-scale as \( \varepsilon \rightarrow 0 \) to some \( f_L \in L^2(\mathbb{R}^d \times \Omega) \) satisfying
\[ \|f_L\|_{L^2(\mathbb{R}^d \times \Omega)} \leq 2|\lambda_0 + 1| \hat{C}(\lambda_0) \frac{1}{L}. \]  
(146)

Arguing exactly as in the proof of Theorem 5.2 we conclude that up to a subsequence \( \psi_L^\varepsilon \rightharpoonup \psi_L \in L^2(\mathbb{R}^d \times \Omega) \), where \( \psi_L \) is the solution to
\[ (A_{\text{hom}} - \lambda) \psi_L = f_L \]

satisfying \( \|\psi_L\|_{L^2(\mathbb{R}^d)} \geq C > 0 \). Since the right-hand side of (146) vanishes as \( L \rightarrow 0 \) we conclude that \( \lambda \in \text{Sp}(A_{\text{hom}}) \).
Appendices

A  Probability framework

For each $j = 1, \ldots, d$, we denote by $\mathcal{D}_j$ the infinitesimal generator of the unitary group

$$U(0, \ldots, 0, x_j, 0, \ldots, 0), \quad j = 1, \ldots, d. \quad (147)$$

Its domain $\text{dom}(\mathcal{D}_j)$ is a dense linear subset of $L^2(\Omega)$ and consists of $f \in L^2(\Omega)$ for which the limit

$$\mathcal{D}_j f(\omega) := \lim_{x_j \to 0} \frac{f(T(0, \ldots, 0, x_j, 0, \ldots, 0) \omega) - f(\omega)}{x_j}$$

exists in $L^2(\Omega)$. Notice that $i \mathcal{D}_j$, $j = 1, \ldots, d$, are self-adjoint, pairwise commuting linear operators on $L^2(\Omega)$. We denote

$$\nabla_\omega := (\mathcal{D}_1, \ldots, \mathcal{D}_d).$$

Furthermore, we define

$$W^{1,2}(\Omega) := \bigcap_{j=1}^d \text{dom}(\mathcal{D}_j),$$

$$W^{k,2}(\Omega) := \{ f \in L^2(\Omega) : \mathcal{D}_1^{\alpha_1} \ldots \mathcal{D}_d^{\alpha_d} f \in L^2(\Omega), \; \alpha_1 + \cdots + \alpha_d = k \},$$

$$W^{\infty,2}(\Omega) := \bigcap_{k \in \mathbb{N}} W^{k,2}(\Omega),$$

$$C^{\infty}(\Omega) = \{ f \in W^{\infty,2}(\Omega) : \forall (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d \; \mathcal{D}_1^{\alpha_1} \ldots \mathcal{D}_d^{\alpha_d} f \in L^{\infty}(\Omega) \}.$$ 

It is known that $W^{\infty,2}(\Omega)$ is dense in $L^2(\Omega)$, $C^{\infty}(\Omega)$ is dense in $L^p(\Omega)$ for all $p \in [1, \infty)$ as well as in $W^{k,2}(\Omega)$ for all $k$, and $W^{1,2}(\Omega)$ is separable.

The following subspace of $W^{1,2}(\Omega)$ play an essential role in our analysis:

$$W^{1,2}_0(\mathcal{O}) := \{ v \in W^{1,2}(\Omega) : v(T_\omega x) = 0 \text{ on } \mathbb{R}^d \setminus \mathcal{O}_\omega \; \forall \omega \in \Omega \}.$$ 

Notice that as a consequence of ergodic theorem (Theorem 2.5) one has

$$W^{1,2}_0(\mathcal{O}) = \{ v \in W^{1,2}(\Omega) : 1_{\mathcal{O}} v = v \},$$

i.e. $W^{1,2}_0(\mathcal{O})$ consists of $W^{1,2}$-functions that vanish on $\Omega \setminus \mathcal{O}$. We also introduce the space

$$C^{\infty}_0(\mathcal{O}) := \{ v \in C^{\infty}(\Omega) : v = 0 \text{ on } \Omega \setminus \mathcal{O} \}.$$ 

It is not difficult to see that Assumption 2.7 implies that the assumptions of Lemmas 3.1 and 3.2 in [CCV18] hold true, and, hence, we have the following statement.

**Lemma A.1.** The space $C^{\infty}_0(\mathcal{O})$ is dense in $L^2(\Omega)$ and in $W^{1,2}_0(\mathcal{O})$. 

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One can equivalently define $D_j f$ as the function with the property

$$\int_{\Omega} g D_j f = - \int_{\Omega} f D_j g \quad \forall g \in C^\infty(\Omega).$$

Notice that for a.e. $\omega \in \Omega$ we have

$$D_j f(T_x \omega) = \frac{\partial}{\partial x_j} f(x, \omega),$$

where the latter is the distributional derivative of $f(\cdot, \omega) \in L^2_{\text{loc}}(\R^d)$. For a random variable $f \in L^2(\Omega)$ its realisation $f \in L^2_{\text{loc}}(\R^d, L^2(\Omega))$ is a $T$-stationary random field, i.e. $f(x + y, \omega) = f(x, T_y \omega)$. Moreover, there is a bijection between random variables from $L^2(\Omega)$ and $T$-stationary random fields from $L^2_{\text{loc}}(\R^d, L^2(\Omega))$. This can be extended to Sobolev spaces, where one has a higher regularity in $x$ for realisations. Namely, the following identity holds (see [DG16] for details):

$$W^{1,2}(\Omega) = \left\{ f \in W^{1,2}_{\text{loc}}(\R^d, L^2(\Omega)) : f(x + y, \omega) = f(x, T_y \omega) \quad \forall x, y, \text{a.e. } \omega \right\} = \left\{ f \in C^1(\R^d, L^2(\Omega)) : f(x + y, \omega) = f(x, T_y \omega) \quad \forall x, y, \text{a.e. } \omega \right\}.$$

Following [ZKO94] we define the spaces of potential and solenoidal vector fields $L^2_{\text{pot}}(\Omega)$ and $L^2_{\text{sol}}(\Omega)$. Namely, a vector field $f \in L^2_{\text{loc}}(\R^d)$ is called potential if it admits a representation $f = \nabla u$, $u \in W^{1,2}_{\text{loc}}(\R^d)$. A vector field $f \in L^2_{\text{loc}}(\R^d)$ is called solenoidal if

$$\int_{\R^d} f_j \frac{\partial \varphi}{\partial x_j} = 0 \quad \forall \varphi \in C^\infty_0(\R^d).$$

A vector field $f \in L^2(\Omega)$ is called potential (respectively, solenoidal), if almost all its realisations $f(T_x \omega)$ are potential (respectively, solenoidal) in $\R^d$. The spaces $L^2_{\text{pot}}(\Omega)$ and $L^2_{\text{sol}}(\Omega)$ are closed in $L^2(\Omega)$. Setting

$$V^2_{\text{pot}} := \{ f \in L^2_{\text{pot}}(\Omega), \langle f \rangle = 0 \}, \quad V^2_{\text{sol}} := \{ f \in L^2_{\text{sol}}(\Omega), \langle f \rangle = 0 \},$$

we have the following orthogonal decomposition ("Weyl’s decomposition")

$$L^2(\Omega) = V^2_{\text{pot}} \oplus V^2_{\text{sol}} \oplus \R^d.$$

Notice that $L^2_{\text{pot}}(\Omega)$ is the closure of the space $\{ \nabla \omega u, u \in W^{1,2}(\Omega) \}$ in $L^2(\Omega)$.

We define the following notion of stochastic two-scale convergence, which is a slight variation of the definition given in [ZP06]. We shall stay in the Hilbert setting ($p = 2$), as it suffices for our analysis.

Let $S$ be an open Lipschitz set in $\R^d$.

**Definition A.2.** Let $(T_x \omega)_{x \in \R^d}$ be a typical trajectory and $(u^\varepsilon)$ a bounded sequence in $L^2(S)$. We say that $(u^\varepsilon)$ weakly stochastically two-scale converges to $u \in L^2(S \times \Omega)$ and write $u^\varepsilon \overset{\text{s.t.s.}}{\rightharpoonup} u$, if

$$\lim_{\varepsilon \downarrow 0} \int_S u^\varepsilon(x) g(x, T_{-1} \varepsilon \omega) dx = \int_S \int_S u(x, \omega) g(x, \omega) dx dP(\omega) \quad \forall g \in C^\infty_0(S) \otimes C^\infty(\Omega).$$

If additionally $\|u^\varepsilon\|_{L^2(S)} \to \|u\|_{L^2(S \times \Omega)}$, we say that $(u^\varepsilon)$ strongly stochastically two-scale converges to $u$ and write $u^\varepsilon \overset{\text{s.s.t.s.}}{\rightharpoonup} u$. 

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In the above $\otimes$ is the usual tensor product.

In the next proposition we collect some properties of stochastic two-scale convergence, see [CCV18] for the proof.

**Proposition A.3.** Stochastic two-scale convergence has the following properties.

a. Let $(u^\varepsilon)\subseteq L^2(S)$ be a bounded sequence in $L^2(S \times \Omega)$ such that $u^\varepsilon \rightharpoonup u$.

b. If $u^\varepsilon \rightharpoonup u$ then $\|u\|_{L^2(S \times \Omega)} \leq \lim \inf_{\varepsilon \rightarrow 0} \|u^\varepsilon\|_{L^2(S)}$.

c. If $(u^\varepsilon) \subseteq L^2(S)$ is a bounded sequence with $u^\varepsilon \rightarrow u$ in $L^2(S)$ for some $u \in L^2(S)$, then $u^\varepsilon \rightharpoonup u$.

d. If $(v^\varepsilon) \subseteq L^\infty(S)$ is uniformly bounded by a constant and $v^\varepsilon \rightarrow v$ strongly in $L^1(S)$ for some $v \in L^\infty(S)$, and $(u^\varepsilon)$ is bounded in $L^2(S)$ with $u^\varepsilon \rightharpoonup u$ for some $u \in L^2(S \times \Omega)$, then $v^\varepsilon u^\varepsilon \rightharpoonup vu$.

e. Let $(u^\varepsilon)$ be a bounded sequence in $W^{1,2}(S)$. Then on a subsequence (not relabelled) $u^\varepsilon \rightharpoonup u^0$ in $W^{1,2}(S)$, and there exists $u^1 \in L^2(S, W^{1,2}(\Omega))$ such that 

$$
\nabla u^\varepsilon \rightharpoonup \nabla u^0 + \nabla_\omega u^1(\cdot, \omega).
$$

f. Let $(u^\varepsilon)$ be a bounded sequence in $L^2(S)$ such that $\varepsilon \nabla u^\varepsilon$ is bounded in $L^2(S, \mathbb{R}^d)$. Then there exists $u \in L^2(S, W^{1,2}(\Omega))$ such that on a subsequence 

$$
u^\varepsilon \rightharpoonup u, \quad \varepsilon \nabla u^\varepsilon \rightharpoonup \nabla_\omega u(\cdot, \omega).
$$

**B Measurability properties**

In this part of the appendix we collect a number of technical results on the measurability of various quantities being used throughout the paper.

For $q = (q_1, \ldots, q_d) \in \mathbb{Q}^d$ we define the set 

$$
O_q := \{\omega \in \mathcal{O} : \text{there exists } k_0 \in \mathbb{N} \text{ such that } \{0, q\} \subset O^k_\omega\}.
$$

From now on we reserve the index $k_0$ for the inclusion $O^k_\omega$ containing the origin (here, obviously, we assume that $\omega \in \mathcal{O}$). Notice that for a fixed $\omega \in \mathcal{O}$ the set of points $q \in \mathbb{Q}^d$ such that $\omega \in O_q$ is exactly $O^k_\omega \cap \mathbb{Q}^d$.

**Lemma B.1.** For every $q \in \mathbb{Q}^d$, $O_q \subset \Omega$ is measurable.

**Proof.** Notice that

$$
\omega \in O_q \quad \equiv \quad \text{There exists a polygonal line } L \text{ that connects } 0 \text{ and } q \text{ and consists of a finite set of straight segments with rational endpoints such that for all } l \in \mathbb{Q}^d \text{ on this line one has } T_l \omega \in \mathcal{O}.
$$

Since for each fixed $q \in \mathbb{Q}^d$ there is a countable number of lines satisfying the property (148), the set $O_q$ is measurable.
Remark B.2. It can be shown in the same way that the set
\[ \tilde{O}_q := \{(x, \omega) \in \mathbb{R}^d \times \Omega : \text{there exists } k_0 \in \mathbb{N} \text{ such that } \{x, x + q\} \subset O_{\omega}^{k_0}\}, \]
is measurable with respect to \( \mathcal{B}(\mathbb{R}^d) \times \mathcal{F} \), where \( \mathcal{B}(\mathbb{R}^d) \) is the Borel \( \sigma \)-algebra. We will use this fact in the proof of Proposition D.8 below.

We define the random variables
\[ \tilde{D}_i(\omega) := \inf \{ q_i : \omega \in O_q \}, \quad \omega \in \Omega, \quad i = 1, \ldots, d. \]
Notice that \( \tilde{D}_i = +\infty \) whenever \( \omega \notin \mathcal{O} \), and for \( \omega \in \mathcal{O} \) one has \( \tilde{D}_i = (D_{\omega}^{k_0})_i \). We denote by \( D \) the random vector
\[ D := -(\tilde{D}_1, \ldots, \tilde{D}_d)^T - d_1/4. \]

For \( \omega \in \mathcal{O} \) we introduce the notation
\[ P_\omega := O_{\omega}^{k_0} + D. \]

Note that by Assumption 2.7 we have \( P_\omega \subset \Box^{1/2} \).

Lemma B.3. The mapping \( (x, \omega) \mapsto \text{dist}(x, P_\omega) \) is measurable from \( \mathbb{R}^d \times \mathcal{O} \) to \( \mathbb{R} \) (on \( \mathbb{R}^d \times \mathcal{O} \) we take the product of Borel \( \sigma \)-algebra with \( \mathcal{F} \)).

Proof. The statement follows from the representation
\[ \text{dist}(x, P_\omega) = \inf_{q_1 \in \mathbb{Q}^d} (|x - D(\omega) - q_1| + \inf_{q_2 \in \mathbb{Q}^d} \{ \text{dist}(q_1, q_2) : \omega \in O_{q_2}\}), \quad (x, \omega) \in \mathbb{R}^d \times \mathcal{O}. \]

\[ \square \]

Lemma B.4. The set-valued mapping \( \mathcal{H} : \omega \mapsto P_\omega \) is measurable, where on the closed subsets of \( \Box \) we take the \( \sigma \)-algebra generated by the Hausdorff distance (topology) \( d_H \).

Proof. Recall that the Hausdorff topology on the closed subsets of a compact set is compact and thus separable. We take a closed ball around a fixed compact set \( K \subset \Box \) of radius \( r \)
\[ \overline{B}_H(K, r) := \{ U \subset \Box : U \text{ is compact}, d_H(K, U) \leq r\}. \]

It is sufficient to prove that the set
\[ \mathcal{H}^{-1}(\overline{B}_H(K, r)^c) = \{ \omega \in \Omega : d_H(P_\omega, K) > r\}, \]
is measurable. But this can be easily seen by using Lemma [B.3] and the representation
\[ \mathcal{H}^{-1}(\overline{B}_H(K, r)^c) = \{ \omega \in \Omega : \exists q \in \mathbb{Q}^d \text{ such that } \text{dist}(q, P_\omega) = 0 \text{ and } \text{dist}(q, K) > r, \]
or \( q \in K \) and \( \text{dist}(q, P_\omega) > r\).

\[ \square \]
Let \( \{ \varphi^l \}_{l \in \mathbb{N}} \subset C^\infty_0(\square) \) be a family of functions dense in \( W^{1,2}_0(\square) \). Further, let \( \rho \in C^\infty_0(\mathbb{R}^d) \) be non-negative with \( \int_{\mathbb{R}^d} \rho = 1 \), \( \text{supp} \, \rho \subset B_1(0) \), and denote \( \rho_\delta(x) = \delta^{-d} \rho(x/\delta) \). We define a characteristic function of the set \( O_{\omega}^{b,m} + D \) and a mapping from \( \Omega \) taking values in \( W^{1,2}_0(\square) \)

\[
\chi^m(x, \omega) := 1_{O_{\omega}^{b,m}(x-D)}, \quad \varphi^{l,m}(x, \omega) := \rho_{1/2m} \ast \left( \chi^m(x, \omega) \varphi^l(x) \right).
\]

Notice that for a.e. \( \omega \in \mathcal{O} \) one has \( \text{supp} \, \varphi^{l,m}(\cdot, \omega) \subset P_\omega \).

**Lemma B.5.** For every \( l, m \in \mathbb{N} \), the mapping \( \omega \mapsto \varphi^{l,m}(\cdot, \omega) \) taking values in \( W^{1,2}_0(\square) \) is measurable with respect to the Borel \( \sigma \)-algebra on \( W^{1,2}_0(\square) \).

**Proof.** First notice that

\[
\omega \mapsto \chi^m(\cdot, \omega) \varphi^l(\cdot),
\]

is a measurable mapping taking values in the set \( L^2(\square) \), with Borel \( \sigma \)-algebra. To check this notice that for each \( q \in \mathbb{Q}^d \) the set

\[
B_q := \{ \omega \in \Omega : q \in O_{\omega}^{b,m} \},
\]

is measurable (the proof is similar to that of Lemma B.1). Further, for \( \psi \in C^\infty_0(\mathbb{R}^d) \) we have that \( \| \psi - \chi^m \varphi^l \|_{L^2} \) can be written as a limit of Riemann sums and each Riemann sum can be written in terms of finite number of \( 1_{B_q} \) and values of function \( \varphi^l \). Thus \( \omega \mapsto \psi - \chi^m \varphi^l \|_{L^2} \) is measurable. Since the topology in \( L^2(\mathbb{R}^d) \) is generated by the balls \( B(\psi, r) \), where \( \psi \in C^\infty_0(\mathbb{R}^d) \) and \( r \in \mathbb{Q} \), we have that the mapping generated by (150) is measurable. The final claim follows by using the fact that the convolution is a continuous (and thus measurable) operator from \( L \) to \( W^{1,2} \).

Notice that by the construction we have that for a.e. \( \omega \in \mathcal{O} \) the family \( \{ \varphi^{l,m}(\cdot, \omega) \}_{l, m \in \mathbb{N}} \subset C^\infty(\mathbb{R}^d) \) is a dense subset of \( W^{1,2}_0(P_\omega) \) (cf. Lemma A.1). Let’s introduce the sets \( E_{L,U} \subset \mathcal{O} \) for some random variables \( L, U : \Omega \rightarrow \mathbb{R}^d_+ \):

\[
E_{L,U} := \{ \omega \in \mathcal{O} : -\Delta_{O_{\omega}^{b}} \text{ has an eigenvalue in the random interval } [L, U] \}.
\]

We also define the set \( S_{L,U} \subset W^{1,2}_0(P_\omega) \) as

\[
S_{L,U} := \{ \psi \in W^{1,2}_0(P_\omega) : \psi \text{ is an eigenfunction of } -\Delta_{P_\omega} \text{ whose eigenvalue is in } [L, U] \}.
\]

For every \( r \in \mathbb{R} \) and \( l, m \in \mathbb{N} \) we define the random variable

\[
X^{l,m}_r := \begin{cases} \frac{\| -\Delta \varphi^{l,m}(\cdot, \omega) - r \varphi^{l,m}(\cdot, \omega) \|_{W^{-1,2}(P_\omega)}}{\| \varphi^{l,m}(\cdot, \omega) \|_{L^2(P_\omega)}} & \text{if } \varphi^{l,m}(\cdot, \omega) \neq 0, \\ +\infty & \text{otherwise}. \end{cases}
\]

**Lemma B.6.** \( X^{l,m}_r \) is a measurable function for every \( r \in \mathbb{R} \) and \( l, m \in \mathbb{N} \).

**Proof.** We use Lemma B.5 and the fact that \( -\Delta \) is a continuous map from \( W^{1,2}(\mathbb{R}^d) \) to \( W^{-1,2}(\mathbb{R}^d) \) and \( \| \cdot \|_{W^{-1,2}(\mathbb{R}^d)} \) is a measurable mapping from \( W^{-1,2}(\mathbb{R}^d) \) to \( \mathbb{R} \), since

\[
\| \psi(\cdot, \omega) \|_{W^{-1,2}(P_\omega)} = \sup_{l, m \in \mathbb{N}} \left\{ \frac{\left\langle \psi(\cdot, \omega), \varphi^{l,m}(\cdot, \omega) \right\rangle_{W^{1,2}(P_\omega)}}{\| \varphi^{l,m}(\cdot, \omega) \|_{W^{1,2}(P_\omega)}} : \varphi^{l,m}(\cdot, \omega) \neq 0 \right\}.
\]

\(^{3}\)We make use of the natural embedding of \( W^{1,2}_0(P_\omega) \) into \( W^{1,2}(\mathbb{R}^d) \).
Lemma B.7. For measurable $L, U$, the set $E_{L,U}$ is measurable.

Proof. The claim follows by observing that

$$E_{L,U} = \bigcap_{n \in \mathbb{N}} \left\{ \omega : \inf_{l,m \in \mathbb{N}, r \in \mathbb{Q} \cap [L - \frac{1}{n}, U + \frac{1}{n}]} X^l_m = 0 \right\}.$$ 

Note that we use the intervals $[L - \frac{1}{n}, U + \frac{1}{n}]$ in the above in order to address the case $L = U$. □

Lemma B.8. Let $\Phi : \Omega \to L^2(\mathbb{R}^d)$ and $L, U : \Omega \to \mathbb{R}_0^+$ be random variables. Then the mapping

$$\omega \mapsto \begin{cases} \text{dist}_{L^2(\mathbb{R}^d)}(\Phi, S_{L,U}) & \text{if } \omega \in \mathcal{O}, \\ +\infty & \text{otherwise.} \end{cases}$$

is measurable.

Proof. The claim follows from the formula

$$\text{dist}_{L^2(\mathbb{R}^d)}(\Phi, S_{L,U}) = \limsup_{n \to \infty} \inf_{l,m \in \mathbb{N}} \left\{ \| \varphi^{l,m}(\cdot, \omega) - \Phi \|_{L^2(\mathbb{R}^d)} : X^l_m \leq \frac{1}{n} \text{ for some } r \in \mathbb{Q} \cap [L,U] \right\}.$$ □

Lemma B.9. For $\omega \in \mathcal{O}$ we denote by $\Lambda_1(\omega) < \Lambda_2(\omega) < \cdots < \Lambda_s(\omega) < \cdots$ the eigenvalues of $-\Delta_{\Omega_{\omega}}$. Then for every $s \in \mathbb{N}$ the mapping $\Lambda_s$ is measurable.

Proof. For $s = 1$ we use the following:

$$\{ \omega : \Lambda_1 \leq x \} = E_{0,x}, \ x \in \mathbb{R}.$$ 

For $s > 1$ we have

$$\{ \Lambda_s \leq x \} = \bigcup_{n \in \mathbb{N}} E_{\Lambda_{s-1} + \frac{1}{n}, x}, \ x \in \mathbb{R}.$$ □

Lemma B.10. For each $\varphi \in L^2(\mathbb{R}^d)$ and $s \in \mathbb{N}$ the mapping $\omega \mapsto P_{S_{[\Lambda_s, \Lambda_s]}} \varphi$, taking values in $L^2(\mathbb{R}^d)$, where $P_{S_{[\Lambda_s, \Lambda_s]}}$ is the $L^2$-orthogonal projection on $S_{[\Lambda_s, \Lambda_s]}$, is measurable.

Proof. For every $n \in \mathbb{N}$ we define the random variable

$$H^n_s := \inf_{l,m \in \mathbb{N}} \left\{ \| \varphi^{l,m}(\cdot, \omega) - \varphi \|_{L^2(\mathbb{R}^d)} : \text{dist}_{L^2(\mathbb{R}^d)}(\varphi^{l,m}(\cdot, \omega), S_{[\Lambda_s, \Lambda_s]}) < \frac{1}{n} \right\}.$$ 

We also define the random variables $P^n_s$ in the following way:

$$P^n_s(\omega) = \varphi^{l_n(\omega), m_n(\omega)}(\omega),$$ 

$$l_n(\omega) = \min_{l \in \mathbb{N}} \left\{ \exists m \in \mathbb{N} : \| \varphi^{l,m}(\cdot, \omega) - \varphi \|_{L^2(\mathbb{R}^d)} < H^n_s + \frac{1}{n} \right\},$$ 

$$m_n(\omega) = \min_{m \in \mathbb{N}} \left\{ \| \varphi^{l_n(\omega), m(\cdot, \omega)} - \varphi \|_{L^2(\mathbb{R}^d)} < H^n_s + \frac{1}{n} \right\}.$$ 

It is easy to see that for a fixed $\omega \in \mathcal{O}$

$$P_{S_{[\Lambda_s, \Lambda_s]}} \varphi = \lim_{n \to \infty} P^n_s,$$

where the convergence on the right-hand side is in $L^2(\mathbb{R}^d)$. □
Lemma B.11. For every $\omega \in \mathcal{O}$ and $s \in \mathbb{N}$ we define $N_s(\omega)$ as the dimension of $S_{[\Lambda_s, \Lambda_s]}$. Then $N_s$ is measurable. Moreover, there exist $\Psi_1^s(\omega), \Psi_2^s(\omega), \ldots, \Psi_{N_s}^s(\omega)$, measurable, taking values in $L^2(\mathbb{R}^d)$, such that $\Psi_1^s, \ldots, \Psi_{N_s}^s$ is an orthonormal basis in $S_{[\Lambda_s, \Lambda_s]}$ (in the sense of $L^2(\mathbb{R}^d)$), and $\Psi_{N_s+1}^s = \Psi_{N_s+2}^s = \ldots = 0$.

Proof. We take the sequence $\tilde{\varphi}^k$ and define the measurable functions $C_k^s = P_S_{[\Lambda_s, \Lambda_s]} \tilde{\varphi}^k$ taking values in $L^2(\mathbb{R}^d)$. Then we construct the sequence $\tilde{\Psi}^k_s$ by applying the Gramm-Schmidt orthogonalization process to the sequence $C_k^s$. Notice that for every $s \in \mathbb{N}$ and $\omega \in \mathcal{O}$ we have that there is at most finite number of $\tilde{\Psi}^k_s$ and the ones that are different from zero form the orthonormal basis in $S_{[\Lambda_s, \Lambda_s]}$. The claim follows by rearranging the sequence $\tilde{\Psi}^k_s$, taking into account that the set of all finite subsets of $\mathbb{N}$ is countable. \qed

Clearly, the sequence $\{\Psi^p_s\}_{s \in \mathbb{N}, p=1, \ldots, N_s}$ is measurable in an orthonormal basis in $L^2(P_\omega)$. We conclude this section by one useful lemma.

Lemma B.12. There exists $c > 0$ such that $\Lambda_1(\omega) \geq c$ for a.e. $\omega \in \mathcal{O}$.

Proof. The smallest eigenvalue $\Lambda_1(\omega)$ is the inverse of the optimal constant in the Poincaré inequality

$$
\int_{P_\omega} |u|^2 \, dx \leq C \int_{P_\omega} |\nabla u|^2 \, dx, \quad u \in W^{1,2}_0(P_\omega).
$$

It is well known that the constant can be chosen to be not greater than the diameter of $P_\omega$, and the latter is bounded by $1/2$. \qed

C Higher regularity of the corrector

In order to prove higher regularity of the corrector we need to recall special versions of two well-known results: the Poincaré-Sobolev inequality for perforated domains and the reverse Hölder’s inequality. We begin with the former.

The Poincaré-Sobolev inequality provided in Lemma C.1 is valid for a more general family of perforated domains $\mathbb{R}^d \setminus \mathcal{O}_\omega$ than considered in this paper. Namely, in Lemma C.1 we assume that $\mathcal{O}_\omega$ is the set of inclusions satisfying Assumption 2.7, but we do not make the assumption that $\mathcal{O}_\omega$ is generated by a dynamical system from a probability space. We still keep the notation $\mathcal{O}_\omega$ and $\mathcal{O}_\omega^k$, $k \in \mathbb{N}$, for the set of inclusions and the individual inclusions respectively. Also, the number of inclusions may be finite.

Lemma C.1. There exist $C > 0$ and $m > 1$ such that for any $f \in W^{1,p}_{\text{loc}}(\mathbb{R}^d \setminus \mathcal{O}_\omega)$, $R > 0$, one has

$$
\|f - c_R(x)\|_{L^p(B_R(x) \setminus \mathcal{O}_\omega)} \leq C R^{d(m(1 - \frac{1}{p}) + 1)} \|\nabla f\|_{L^p(B_{mR}(x) \setminus \mathcal{O}_\omega)}, \tag{151}
$$

where $p \leq q \leq dp/(d-p)$ and $c_R(x)$ is some constant that depend on the values of $f$ in $B_{mR}(x) \setminus \mathcal{O}_\omega$.

Proof. Part 1 of Assumption 2.7 implies that there exists a number $R_0 > 0$ (the largest number such that the box $Q_{R_0, \gamma}$ is inscribed in the ball $B_{R_0}$) such that for any inclusion $\mathcal{O}_\omega^k$ and any $x_0 \in \partial \mathcal{O}_\omega^k$ there exist local coordinates $y$ congruent to $x$ with the origin at $x_0$ such that the intersection of the set $\mathbb{R}^d \setminus \mathcal{O}_\omega$ with the rectangular box $Q_{R_0, \gamma} := (-R_0, R_0)^{d-1} \times (-\gamma R_0, \gamma R_0)$ is the subgraph of a Lipschitz continuous function $g$ with a Lipschitz constant $\gamma$ satisfying $g(0) = 0$,

$$
Q_{R_0, \gamma} \setminus \overline{\mathcal{O}}_\omega = \{y \in Q_{R_0, \gamma} | y_d < g(y_1, \ldots, y_{d-1})\}. \tag{152}
$$
Remark C.2. Obviously, the above property also hold for boxes $Q_{R,\gamma}$ with $0 < R < R_0$.

Consider the class of sets of the form (a scaled version of the set \(152\))

$$U := \{ y \in Q_{1,\gamma} \mid y_d < g(y_1, \ldots, y_{d-1}) \},$$  \hspace{1cm} (153)

where $g$ is as before. Clearly, the sets $U$ satisfy the cone condition uniformly (i.e. with the cone parameters depending only on $\gamma$). It is well know (see [BC07] and e.g. [A75]) that in this case the Poincaré inequality

$$\| f - \bar f \|_{L^p(U)} \leq C \| \nabla f \|_{L^p(U)},$$  \hspace{1cm} (154)

and the Sobolev inequality

$$\| f \|_{L^q(U)} \leq C \| f \|_{W^{1,p}(U)}, \quad p \leq q \leq dp/(d-p),$$  \hspace{1cm} (155)

hold for any $f \in W^{1,p}(U)$ with a constant $C$ that depends only on the cone parameters (i.e. on $\gamma$) and $p$, but not on the domain $U$.

Remark C.3. It is not difficult to prove the uniform Poincaré inequality \(154\) directly, if we replace $Q_{1,\gamma}$ in \(153\) with $Q_{1,\gamma'}$ for some $\gamma' > \gamma$. It follows by making a simple change of variables transforming $U$ into the fixed box $(-1,1)^{d-1} \times (-\gamma,0)$, applying the Poincaré inequality on the box and transforming back to $U$.

Combining \(154\) and \(155\) we obtain the Sobolev-Poincaré inequality

$$\| f - \bar f \|_{L^q(U)} \leq C \| \nabla f \|_{L^p(U)}, \quad p \leq q \leq dp/(d-p),$$

with $C$ independent of $U$. Observing that $B_1(0) \subset Q_{1,\gamma} \subset B_{m_0}(0)$, $m = m_0 := \sqrt{d-1 + \gamma^2}$, we have

$$\| f - \bar f \|_{L^q(B_1(0))} \leq \| f - \bar f \|_{L^q(U)} \leq C \| \nabla f \|_{L^p(B_{m_0}(0))}, \quad p \leq q \leq dp/(d-p).$$

Now, taking $x \in \partial \Omega^k_\omega$ and $R \leq R_0$, and using the scaling argument, we arrive at \(151\) with $m = m_0$, $c_R(x) = \int_{Q_{2R,\gamma}} f$, where $Q_{2R,\gamma}$ is now understood in the global coordinates, i.e. is an appropriately oriented box centred at $x$.

Next we argue that \(151\) holds (with different $m$ and $c_R(x)$) for a ball $B_R(x)$ with arbitrary $R$ and $x$. Assume first that $B_R(x) \subset \mathbb{R}^d \setminus \Omega_\omega$, then \(151\) obviously holds with $m = 1$ and $c_R(x) = \langle f \rangle_{B_R}$ (standard Poincaré-Sobolev inequality for a ball). For $R \leq R_0/2$ and $\emptyset \neq B_R \cap \Omega_\omega \neq B_R$ there exists $x_0 \in \partial \Omega^k_\omega$ such that $B_R(x_0) \subset B_{2R}(x_0)$, and we easily conclude that \(151\) holds when $m = 2m_0 + 1$ and $c_R(x) = \langle f \rangle_{Q_{2R,\gamma}} \setminus \Omega_\omega$, where $Q_{2R,\gamma}$ is centred at $x_0$ and appropriately oriented. Finally, consider the case $R > R_0/2$ and again $\emptyset \neq B_R \cap \Omega_\omega \neq B_R$. Let $\bar f \in W^{1,p}_{\text{loc}}(\mathbb{R}^d)$ be the extension of $f$ as per Theorem 2.8. We have

$$\| f - \bar f \|_{L^q(B_R \setminus \Omega_\omega)} \leq \| f - \bar f \|_{L^q(B_R)} \leq CR^{\left(\frac{1}{q} - \frac{1}{p} + 1\right)} \| \nabla \bar f \|_{L^p(B_R)} \leq CR^{\left(\frac{1}{q} - \frac{1}{p} + 1\right)} \| \nabla f \|_{L^p(B_{R+\sqrt{d}} \setminus \Omega_\omega)},$$

where we use the fact that for any $\Omega^k_\omega$ with $\Omega^k_\omega \cap B_R \neq \emptyset$ the corresponding “extension” set $B^k_\omega$ is contained in $B_{R+\sqrt{d}}$ (i.e. $c_R(x) = \int_{B_R} \bar f$ in this case). Thus \(151\) holds for any ball $B_R$ with $m = \max\{2m_0 + 1, 1 + \sqrt{d}/R_0\}$ and $c_R(x)$ specified above depending on the situation.

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We will need the following simplified version of the reverse Hölder’s inequality, see e.g. Theorem 1.10 and Remark 1.14 in [BF02], suitable for our needs.

**Lemma C.4.** Let \( k \geq 2 \), \( f \in L^q(\square_L^{x_0}), q > 1 \), for some cube \( \square_L^{x_0} \), and assume that

\[
\int_{\square_L^{x_0}/k} |f|^q \leq c \left( 1 + \left( \int_{\square_L^{x_0}} |f| \right)^q \right)
\]

for every cube \( \square_L^{x_0} \) contained in \( \square_L^{x_0} \). Then there exists a constant \( \epsilon = \epsilon(d,c,k) \) such that \( \forall p \) with \( q \leq p < q + \epsilon \) one has

\[
\left( \int_{\square_L^{x_0}/k} |f|^p \right)^{1/p} \leq c_p \left( 1 + \left( \int_{\square_L^{x_0}} |f| \right)^{1/q} \right)
\]

for \( \square_L^{x_0} \subset \square_L^{x_0} \), where \( c_p \) depends on \( p, q, d, c, k \).

We are ready to proceed with the proof of Theorem 4.16. Recall that \( \hat{N}_j \in W^{1,2}_{\text{per}}(\square_{x_M}^{M+1}) \), \( j = 1, \ldots, d \), denote the solutions to (63). Without loss of generality we may assume \( x_M = 0 \). To simplify the notation in what follows we assume that \( \hat{N}_j \) and \( \nabla \hat{N}_j \) are extended by zero inside the inclusions, and extended by periodicity to the whole \( \mathbb{R}^d \). We redefine the cut-off function as follows: let \( \eta \in C_0^\infty(B_2(0)) \) be such that \( 0 \leq \eta \leq 1 \), \( \eta = 1 \) in \( B_1(0) \), and set \( \eta_R(x) := \eta\left(\frac{x-x_0}{R}\right) \), for some \( x_0 \in \mathbb{R}^d \) and \( R > 0 \), which are assumed to be fixed for the time being. We apply Lemma C.1 with the perforated domain being given by the periodic extension of the set \( \square_0^{M+1} \). Let \( c_R(x_0) \) be as in (151) with \( f \) replaced by \( \hat{N}_j \). Multiplying (63) by the test function \( (\hat{N}_j - c_R(x_0))\eta_R^2 \) and integrating by parts we get

\[
\int_{B_{2R}(x_0)} A_1 \nabla \hat{N}_j \cdot \nabla \hat{N}_j \eta_R^2 = - \int_{B_{2R}(x_0)} (A_1 e_j \cdot \nabla \hat{N}_j \eta_R^2 + 2A_1(e_j + \nabla \hat{N}_j) \cdot \nabla \eta_R(\hat{N}_j - c_R(x_0))\eta_R),
\]

and, hence,

\[
\int_{B_{2R}(x_0)} |\nabla \hat{N}_j \eta_R|^2 \leq C \int_{B_{2R}(x_0)} \left( |\nabla \hat{N}_j \eta_R^2 + R^{-1}|\hat{N}_j - c_R(x_0)|(1 + |\nabla \hat{N}_j|) \right).
\]

We estimate the right-hand side of the latter term by term.

\[
\int_{B_{2R}(x_0)} |\nabla \hat{N}_j \eta_R|^2 \leq \int_{B_{2R}(x_0)} (\alpha^{-1}|\nabla \hat{N}_j \eta_R|^2 + \alpha \eta_R^2) \leq \int_{B_{2R}(x_0)} \alpha^{-1}|\nabla \hat{N}_j \eta_R|^2 + \alpha |B_{2R}|,
\]

for any \( \alpha > 0 \). Applying (151) and then Hölder’s inequality we get

\[
\int_{B_{2R}(x_0)} R^{-1}|\hat{N}_j - c_R(x_0)| \leq \int_{B_{2R}(x_0)} (1 + R^{-2}|\hat{N}_j - c_R(x_0)|^2)
\]

\[
\leq |B_{2R}| + CR^{-2} \left( \int_{B_{2mR}(x_0)} |\nabla \hat{N}_j|^{\frac{2d}{d+2}} \right)^{\frac{d+2}{2}} \leq |B_{2R}| + CR^{-1} \left( \int_{B_{2mR}(x_0)} |\nabla \hat{N}_j|^{\frac{2d}{d+1}} \right)^{\frac{d+1}{d}}.
\]

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Next we first apply Hölder’s inequality with the exponents \( q = \frac{2d}{d-1} \) and \( p = \frac{2d}{d+1} \) to the terms \( |\hat{N}_j - c_R(x_0)| \) and \( |\nabla \hat{N}_j| \) and then the inequality \((151)\) to the term containing \( |\hat{N}_j - c_R(x_0)| \) with the same exponents in order to obtain

\[
\int_{B_{2R}(x_0)} R^{-1}|\hat{N}_j - c_R(x_0)||\nabla \hat{N}_j| \leq CR^{-1}\left( \int_{B_{2mR}(x_0)} |\nabla \hat{N}_j|^{2d} \right)^{\frac{d+1}{d}}.
\]

Combining the above estimates in \((156)\), moving the term containing \( \alpha^{-1} \) to the left-hand side, choosing \( \alpha \) large enough, and observing that \( \Box^{R/\sqrt{d}} \subset B_R(x_0) \) and \( B_{2mR}(x_0) \subset \Box^{2mR} \), we arrive at

\[
\int_{\Box^{R/\sqrt{d}}} |\nabla \hat{N}_j|^2 \leq C|B_{2R}| + CR^{-1}\left( \int_{\Box^{2mR}} |\nabla \hat{N}_j|^{2d} \right)^{\frac{d+1}{d}}.
\]

Denoting

\[
z := |\nabla \hat{N}_j|^{\frac{2d}{d+1}},
\]

and dividing both side of the inequality \((157)\) by \( |\Box^{R/\sqrt{d}}| \), we obtain

\[
\int_{\Box^{R/\sqrt{d}}} z^{\frac{d+1}{d}} \leq C + C\left( \int_{\Box^{2mR}} z \right)^{\frac{d+1}{d}},
\]

where \( x_0 \) and \( R > 0 \) are arbitrary. Now we can apply the reverse Hölder’s inequality (Lemma \((C.4)\)) to conclude that there exists \( \epsilon > 0 \) depending only \( C \) and \( d \) such that for any \( \mu \) satisfying \( \frac{2d}{d^2} \leq \mu < \frac{d+1}{d} + \epsilon \) one has

\[
\left( \int_{\Box^{R/\sqrt{d}}} z^\mu \right)^{1/\mu} \leq C + C\left( \int_{\Box^{2mR}} z \right)^{\frac{d+1}{d}}.
\]

Rewriting this for \( \nabla \hat{N}_j \) we arrive at

\[
\left( \int_{\Box^{R/\sqrt{d}}} |\nabla \hat{N}_j|^p \right)^{1/p} \leq C + C\left( \int_{\Box^{2mR}} |\nabla \hat{N}_j|^2 \right)^{1/2},
\]

with \( 2 \leq p < 2 + \epsilon\frac{2d}{d+1} \). Choosing \( \Box^{R/\sqrt{d}} = \Box^{M+\kappa} \), observing that the corresponding \( \Box^{2mR} \) can be covered by finitely many periodically shifted copies of \( \Box^{M+\kappa} \), and utilizing the periodicity of the corrector we finally obtain \((65)\).

### D Auxiliary results for Theorem 5.2

Denote by \( \mathcal{H}_{\rho,N,\gamma}(\Box) \) the family of all closed, connected sets \( P \subset \Box^{1/2} \) that are \((\rho, N, \gamma)\) minimally smooth, and let \( L^2_{\text{pot}}(\Box) \) be the space of potential vector fields on \( \Box \). The direct product \( L^2_{\text{pot}}(\Box) \times \)
\(\mathcal{H}_{\rho,N,\gamma}(\square)\) is a measurable space with the product \(\sigma\)-algebra generated by distance on \(L^2(\square; \mathbb{R}^d)\) and the Hausdorff distance on \(\mathcal{H}_{\rho,N,\gamma}(\square)\). For a pair \((g, P) \in L^2_{\text{pot}}(\square) \times \mathcal{H}_{\rho,N,\gamma}(\square)\) let \(\varphi \in W^{1,2}(\square)\) be a potential of \(g\), i.e. \(g = \nabla \varphi\), let \(\tilde{\varphi} \in W^{1,2}(\square)\) be the harmonic extension of \(\varphi|_{\square \setminus P}\) to the whole \(\square\), and denote \(\tilde{g} := \nabla \tilde{\varphi}\). This construction defines a mapping \(\tilde{E} : L^2_{\text{pot}}(\square) \times \mathcal{H}_{\rho,N,\gamma}(\square) \to L^2_{\text{pot}}(\square)\) by setting \(\tilde{E}(g, P) = \tilde{g}\).

**Lemma D.1.** The mapping \(\tilde{E}\) is measurable.

**Proof.** Consider a converging sequence \((g_n, P_n) \to (g, P)\) in \(L^2_{\text{pot}}(\square) \times \mathcal{H}_{\rho,N,\gamma}(\square)\), i.e. \(g_n \to g\) in \(L^2(\square; \mathbb{R}^d)\) and \(P_n \to P\) in the Hausdorff distance. We take \(\varphi, \varphi_n \in W^{1,2}(\square), n \in \mathbb{N}\), such that \(g = \nabla \varphi, g_n = \nabla \varphi_n\) and \(\int_\square \varphi = \int_\square \varphi_n = 0\). Clearly, \(\varphi_n \to \varphi\) in \(W^{1,2}(\square)\). By Theorem 2.8 and Remark 2.9 the harmonic extensions \(\tilde{\varphi}_n\) of \(\varphi_n|_{P_n}\) to the whole of \(\square\) satisfy the bound

\[
\|\tilde{\varphi}_n\|_{W^{1,2}(\square)} \leq C \|\varphi_n\|_{W^{1,2}(\square)}, \quad \forall n \in \mathbb{N}.
\]

It follows that \(\tilde{\varphi}_n\) converges to some \(\tilde{\varphi} \in W^{1,2}(\square)\) strongly in \(L^2(\square)\) and weakly in \(W^{1,2}(\square)\). It is not difficult to see that \(\tilde{\varphi}|_P = \varphi|_P\). Multiplying the equation

\[
\Delta \tilde{\varphi}_n = 0 \quad \text{in} \ P_n
\]

by an arbitrary \(f \in C^\infty_0(P')\), where \(P' \subseteq P\), integrating by parts and passing to the limit, we see that

\[
0 = \lim_{n \to \infty} \int_{P'} \nabla \tilde{\varphi}_n \cdot \nabla f = \int_{P'} \nabla \tilde{\varphi} \cdot \nabla f.
\]

We infer that \(\tilde{\varphi}\) is harmonic in \(P\), and, hence, \(\tilde{\varphi} = \tilde{\varphi}\), where the latter denotes the harmonic extensions of \(\varphi|_P\).

It remains to prove the strong convergence of \(\nabla \tilde{\varphi}_n\) to \(\nabla \tilde{\varphi}\). To this end we multiply \((158)\) by the test function \(\tilde{\varphi}_n - \varphi_n\) and integrate by parts:

\[
\int_{P_n} \nabla \tilde{\varphi}_n \cdot \nabla (\tilde{\varphi}_n - \varphi_n) = 0. \quad (159)
\]

Then passing to the limit we obtain

\[
\lim_{n \to \infty} \int_{P_n} |\nabla \tilde{\varphi}_n|^2 = \lim_{n \to \infty} \int_{P_n} \nabla \tilde{\varphi}_n \cdot \nabla \varphi_n = \int_{P} \nabla \tilde{\varphi} \cdot \nabla \varphi.
\]

The last equality follows from a simple observation that \(\nabla \varphi_n|_{P_n}\) converges to \(\nabla \varphi|_P\) in \(L^2(\square)\). Similarly to \((159)\) we get

\[
\int_{P} |\nabla \tilde{\varphi}|^2 = \int_{P} \nabla \tilde{\varphi} \cdot \nabla \varphi,
\]

thus we conclude that \(\nabla \tilde{\varphi}_n \to \nabla \tilde{\varphi}\) in \(L^2(\square)\). Therefore, the mapping \(\tilde{E}\) is continuous and, hence, measurable. \(\Box\)

**Proposition D.2.** For every \(f \in V^2_{\text{pot}}(L^2_{\text{pot}}(\Omega))\) there exists the extension \(\tilde{f} \in V^2_{\text{pot}}(L^2_{\text{pot}}(\Omega))\) of \(f|_{\Omega \setminus \omega}\) into \(\mathcal{O}\) such that for a.e. realization \(\tilde{f}(T_x \omega)\) its potential \(\tilde{\varphi} \in W^{1,2}_{\text{loc}}(\mathbb{R}^d)\), \(\nabla \tilde{\varphi}(x) = \tilde{f}(T_x \omega)\), is harmonic on the set of inclusions \(\mathcal{O}_\omega\), and

\[
\|\tilde{f}\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega \setminus \omega)}. \quad (160)
\]
Proof. Let $f \in L^2_{\text{pot}}(\Omega)$. The mapping from $\mathcal{O}$ to $L^2_{\text{pot}}(\square)$ that to every $\omega \in \mathcal{O}$ assigns the vector field $\varphi_\omega(x) := f(T_x - D\omega) \in L^2_{\text{pot}}(\square)$ is measurable (this can be easily checked for an arbitrary $f \in L^2(\Omega; \mathbb{R}^d)$ first by checking it for simple functions and then using the density argument). By using Lemmata B.4 and D.1 we conclude that the mapping $\omega \mapsto \tilde{E}(\varphi_\omega, P_\omega) \in L^2_{\text{pot}}(\square)$ is measurable. We define
\[
\tilde{f}(\omega) = \begin{cases} 
\tilde{E}(\varphi_\omega, P_\omega)(D), & \omega \in \mathcal{O}; \\
\tilde{f}(\omega), & \text{otherwise}. 
\end{cases}
\]
Notice that thus defined $\tilde{f}$ is measurable, since $\tilde{E}(\varphi_\omega, P_\omega)$ is harmonic and, hence, infinitely smooth in $P_\omega$, and is an element of $L^2_{\text{pot}}(\Omega)$ by construction.

Now we prove (160). By Assumption 2.7 for a.e. $\omega$ the ball $B_\rho$ has common points with at most one inclusion $\mathcal{O}_\omega^k$, at the same time the cube $\square^2$ contains this inclusion together with its extension domain $\mathcal{B}_\rho^k$. Therefore, we can apply Theorem 2.8 to have
\[
\int_{B_\rho} |\tilde{f}(T_\omega \omega)|^2 \leq C_{\text{ext}} \int_{\square \setminus \mathcal{O}_\omega} |f(T_\omega \omega)|^2 = C_{\text{ext}} \int_{\square^2} |(f 1_{\mathcal{O} \setminus \mathcal{O}})(T_\omega \omega)|^2.
\]
Integrating over $\Omega$ and applying Fubini theorem we obtain
\[
|B_\rho| \int_\Omega |\tilde{f}|^2 \leq 2^d C_{\text{ext}} \int_{\Omega \setminus \mathcal{O}} |f|^2.
\]
Notice that by construction if $\varphi \in W^{1,2}_{\text{loc}}(\mathbb{R}^d)$ is such that $\nabla \varphi(x) = f(T_x \omega)$, then $\tilde{f}(T_x \omega) = \nabla \tilde{\varphi}(x)$, where $\tilde{\varphi}$ is the harmonic extension of $\varphi_{\mathbb{R}^d \setminus \mathcal{O}_\omega}$ into the set of inclusions.

It remains to show that $\tilde{f} \in \mathcal{V}^2_{\text{pot}}$ if $f \in \mathcal{V}^2_{\text{pot}}$. Let $\varphi^\varepsilon, \tilde{\varphi}^\varepsilon \in W^{1,2}(\square)$ be two sequences such that $\nabla \varphi^\varepsilon(x) = f(T_x / \varepsilon \omega)$ and $\nabla \tilde{\varphi}^\varepsilon(x) = \tilde{f}(T_x / \varepsilon \omega)$ in $\square$. By the ergodic theorem we have
\[
\int_\square \nabla \varphi^\varepsilon \rightarrow \int_\Omega f = 0 \text{ as } \varepsilon \rightarrow 0. \tag{161}
\]
Since by construction $\varphi^\varepsilon = \tilde{\varphi}^\varepsilon$ on $\square \setminus S_0^\varepsilon$, by applying Gauss-Ostrogradsky theorem we have
\[
\int_{\mathcal{O}_k} \nabla \tilde{\varphi}^\varepsilon = \int_{\partial \mathcal{O}_k^\varepsilon} \tilde{\varphi}^\varepsilon n = \int_{\partial \mathcal{O}_k^\varepsilon} \varphi^\varepsilon n = \int_{\mathcal{O}_k^\varepsilon} \nabla \varphi^\varepsilon, \forall \mathcal{O}_k^\varepsilon \subset \square,
\]
here $n$ is the outward unit normal vector. Therefore,
\[
\int_{\square \setminus K^\varepsilon} \nabla \varphi^\varepsilon = \int_{\square \setminus K^\varepsilon} \nabla \tilde{\varphi}^\varepsilon,
\]
where $K^\varepsilon := \bigcup_{\mathcal{O}_k \cap \square \neq \emptyset} \mathcal{O}_k^\varepsilon$. It is not difficult to see that $\int_{\square \setminus K^\varepsilon} (\nabla \varphi^\varepsilon - \nabla \tilde{\varphi}^\varepsilon) \rightarrow 0$. Thus we conclude via (161) that
\[
\int_\Omega \tilde{f} = \lim_{\varepsilon \rightarrow 0} \int_\square \nabla \tilde{\varphi}^\varepsilon = 0.
\]
\[\square\]
Lemma D.3. For every $f \in \mathcal{X}$ there exists the extension $\tilde{f} \in \mathcal{Y}^2_{\text{pot}}$, $\tilde{f} = f$ on $\Omega \setminus \mathcal{O}$, such that for a.e. realisation $\tilde{f}(T_x \omega)$ its potential $\tilde{\varphi} \in W^{1,2}_{\text{loc}}(\mathbb{R}^d)$, $\nabla \tilde{\varphi}(x) = \tilde{f}(T_x \omega)$, is harmonic on the set of inclusions $\mathcal{O}_\omega$, and

$$\|\tilde{f}\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega \setminus \mathcal{O})}.$$  

Proof. Let $f_k$ be a sequence from $\mathcal{Y}^2_{\text{pot}}$ such that $\|f - f_k\|_{L^2(\Omega \setminus \mathcal{O})}$ converges to zero. For each $k$ let $\tilde{f}_k \in \mathcal{Y}^2_{\text{pot}}$ be the extension of $f_k$ as in Proposition D.2. By (160) the sequence $\tilde{f}_k$ is Cauchy and, hence, converges in $L^2(\Omega)$ to some $\tilde{f} \in \mathcal{Y}^2_{\text{pot}}$. The fact that $\tilde{f} = f$ on $\Omega \setminus \mathcal{O}$ is obvious.

The remaining part of the statement can be proven analogously to the argument used to show that $\tilde{\varphi}$ is harmonic in $P$ in the proof of Lemma D.1

Lemma D.4. Let $w$ be a zero-mean solenoidal random vector field, $w \in \mathcal{Y}^2_{\text{sol}}(\Omega)$. Then there exists a random tensor field $W_{ikl}$, $i,k,l = 1, \ldots, d$, such that for fixed $i$ and $k$ $W_{ik}$ is a potential random vector field, i.e. $W_{ik} := (W_{ik1}, \ldots, W_{ikn})^T \in \mathcal{Y}^2_{\text{pot}}(\Omega)$, for fixed $l$ the field $W_{ikl}$ is antisymmetric, i.e. $W_{ikl} = -W_{kl}$, and

$$w = W_{ik}.$$  

Proof. We assume that $L^2(\Omega)$ (and it subspaces) is complex valued. Since the differentiation operators $iD_k$, $k = 1, \ldots, d$, are self-adjoint and commuting, by the Spectral Theorem there exists a measure space $(M, \mu)$ with $\mu$ a finite measure, a unitary operator $U : L^2(\Omega) \to L^2(M)$, and real-valued functions $a_k$ on $M$, which are finite a.e., so that $UiD_k \varphi = a_kU \varphi$.

It is easy to see that $\ker \nabla_w$ consists of constants, so $w \perp \ker \nabla_w$, since $w$ has mean zero. For any $f \in \ker \nabla_w$ we have $a_kUf = 0$, $k = 1, \ldots, n$, and vice versa. Therefore, $U(\ker \nabla_w) = \cap_k \ker a_k$ consists of all $L^2(\Omega)$ functions that vanish on $M \setminus \{|a| = 0\}$, where $a := (a_1, \ldots, a_n)^T$. Since $Uw \perp U(\ker \nabla_w)$, we conclude that

$$Uw = 0 \text{ on } \{|a| = 0\}. \quad (162)$$

Further, since $w$ is solenoidal, we have

$$\int_{\Omega} w \cdot \nabla \varphi = 0, \forall \varphi \in W^{1,2}(\Omega).$$

Applying the unitary operator $U$ we obtain

$$\int_{M} Uw \cdot a \psi = 0, \forall \psi \text{ such that } \psi, a \psi \in L^2(M).$$

Since $W^{1,2}(\Omega)$ is dense in $L^2(\Omega)$, the set $\{U \varphi : \varphi \in W^{1,2}(\Omega)\} = \{\psi : a \psi \in L^2(M)\}$ is dense in $L^2(M)$, hence

$$Uw \cdot a = 0 \text{ a.e.} \quad (163)$$

Define an antisymmetric in $ik$ tensor $\tilde{W}_{ikl} \in L^2(M)$ as follows

$$\tilde{W}_{ikl} := \begin{cases} a_l a_i Uw_k - a_k Uw_i / |a|^2, & \text{on } \{|a| > 0\}, \\ 0, & \text{otherwise.} \end{cases}$$
Notice that \( \tilde{W}_{ikl} = U w_k \) by (162), (163). We define \( W_{ikl} := U^{-1} \tilde{W}_{ikl} \). It only remains to prove that \( W_{ik} \in V^2_{\text{pot}}(\Omega) \), but this easily follows from already observed fact that any \( \varphi \in L^2(\Omega) \) satisfies \( U\varphi \cdot a = 0 \) a.e. , c.f. (163) (the argument is equally valid for non-zero mean solenoidal fields).

Indeed, then we have

\[
\int_{\Omega} W_{ikl} \varphi_l = \int_M a_i U w_k - a_k U w_i \varphi_l = \int_M a_i U w_k - a_k U w_i a \cdot U \varphi = 0.
\]

\( \square \)

**Corollary D.5.** Let \( g_j^\varepsilon, j = 1, \ldots, d, \) be as in (111). There exist skew-symmetric tensor fields \( G_j^\varepsilon \in W^{1,2}(\square^L; \mathbb{R}^{d \times d}) \), \( (G_j^\varepsilon)_{ik} = -(G_j^\varepsilon)_{ki} \), such that

\[
g_j^\varepsilon = \nabla \cdot G_j^\varepsilon \quad \text{(i.e.} \quad (g_j^\varepsilon)_k = \partial_i(G_j^\varepsilon)_{ik}) \tag{164}
\]

and

\[
\|G_j^\varepsilon\|_{L^2(\square^L)} \rightarrow 0. \tag{165}
\]

**Proof.** Applying Lemma D.4 to \( g_j \) we have \( g_j = W_{ikl} \) (we drop the index \( j \) on the right-hand side for simplicity). Since for fixed \( i, k \), \( W_{ik} \in V^2_{\text{pot}}(\Omega) \) we have that there exists a zero mean function \( (G_j^\varepsilon)_{ik} \in W^{1,2}(\square^L) \) such that \( W_{ik}(T_{x/\varepsilon}\omega) = \nabla(G_j^\varepsilon)_{ik}(x) \) (i.e. \( W_{ikl}(T_{x/\varepsilon}\omega) = \partial_l(G_j^\varepsilon)_{ik} \)). Obviously, \( G_j^\varepsilon \) is skew-symmetric and (164) holds. The convergence (165) follows from the following two facts: a) since \( \langle W_{ikl} \rangle = 0 \) \( \partial_l(G_j^\varepsilon)_{ik} \) converges to zero weakly in \( L^2(\square^L) \) by the ergodic theorem; b) each \( (G_j^\varepsilon)_{ik} \) is a zero-mean on \( \square^L \) function. \( \square \)

**Lemma D.6.** Let \( f^\varepsilon \) be a bounded in \( L^2(K) \) sequence converging weakly to zero as \( \varepsilon \rightarrow 0 \), where \( K \) is a bounded domain. Then there exists a sequence of zero-mean solution \( B^\varepsilon \in W^{1,2}(K; \mathbb{R}^d) \) of

\[
\nabla \cdot B^\varepsilon = f^\varepsilon \quad \text{in} \quad K
\]

such that

\[
\|B^\varepsilon\|_{L^2(K)} \rightarrow 0. \tag{167}
\]

Moreover,

\[
\|B^\varepsilon\|_{W^{1,2}(K)} \leq \|f^\varepsilon\|_{L^2(K)}. \tag{168}
\]

**Proof.** By extending \( f^\varepsilon \) into a cube containing \( K \) by zero and using appropriate scaling and translation argument one can see that it is sufficient to prove the statement of the lemma for the case \( K = \square^{2\pi} \). Consider the Fourier series for \( f^\varepsilon \):

\[
f^\varepsilon = \sum_{k \in \mathbb{Z}^d} f_k^\varepsilon \exp(ik \cdot x).
\]

Since \( f^\varepsilon \) converges weakly to zero we have that its Fourier coefficients also converge to zero,

\[
f_k^\varepsilon \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0, \quad k \in \mathbb{Z}^d. \tag{169}
\]

We define \( B^\varepsilon \) as

\[
B^\varepsilon := \frac{f_0^\varepsilon}{d} x + \sum_{k \in \mathbb{Z}^d, k \neq 0} -\frac{i k f_k^\varepsilon}{|k|^2} \exp(ik \cdot x).
\]

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By direct inspection $B_\varepsilon$ has zero mean, satisfies equation (166) and bound (168). Since $\sum_k |f_k^\varepsilon|^2$ is bounded uniformly in $\varepsilon$, then for an arbitrary $\delta > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{|k| \geq k_0} |f_k^\varepsilon|^2 \leq \frac{1}{k_0^2} \sum_{|k| \geq k_0} |f_k^\varepsilon|^2 < \frac{\delta^2}{4}$$

for all $\varepsilon$. Further, by (169) for small enough $\varepsilon$ we have

$$\frac{2^{d-\pi d+2}}{3d} |f_0^\varepsilon|^2 + \sum_{|k| < k_0, k \neq 0} \frac{|f_k^\varepsilon|^2}{|k|^2} < \frac{\delta^2}{4}.$$

We conclude that for small enough $\varepsilon$

$$\|B_\varepsilon\|_{L^2(K)} \leq \sqrt{2} \left( \frac{2^{d-\pi d+2}}{3d} |f_0^\varepsilon|^2 + \sum_{|k| < k_0, k \neq 0} \frac{|f_k^\varepsilon|^2}{|k|^2} \right)^{1/2} < \delta,$$

which implies (167).

\[\square\]

**Remark D.7.** In the case when $f_\varepsilon$ is the $\varepsilon$-realisation of a zero-mean function from $L^2(\Omega)$ one can prove the first part of the statement of Lemma D.6 (without the bound (168)) via a similar argument used in Lemma D.4 and Corollary D.5. Namely, for a zero mean $f \in L^2(\Omega)$ the potential fields $F_i := (F_{i1}, \ldots, F_{in})^T \in V_{\text{pot}}^2(\Omega)$, $i = 1, \ldots, d$, defined via

$$UF_{ik} := \begin{cases} a_i a_k U f \frac{|a|^2}{|a|^2}, & \text{on } \{|a| > 0\}, \\ 0, & \text{otherwise.} \end{cases}$$

satisfies $f = F_{i_0}$. Then the zero-mean field $B_\varepsilon = (B_1^\varepsilon, \ldots, B_n^\varepsilon)^T \in W^{1,2}(K; \mathbb{R}^d)$ is defined by $\nabla B_i^\varepsilon = f_i^\varepsilon$. Notice that thus defined $B_\varepsilon$ does not necessary coincide with the on in the above lemma. We prefer the method used in the lemma since, first, it applies to a sequence which is not necessary the $\varepsilon$-realisation of a zero-mean function from $L^2(\Omega)$, and, second, it provides the bound (168).

The next statement is required in the last part of the proof of Theorem 5.2, where we argue that $\text{Sp}(\Delta_{\tilde{O}}) \subset SR-\lim \text{Sp}(A^\varepsilon)$. The idea of a function that randomly takes values 1 and $-1$ on inclusions and zero outside so that it converges to zero weakly as $\varepsilon \to 0$ is quite natural, however the construction of such function is somewhat technical.

**Proposition D.8.** Let $(\Omega_1, F_1, P_1)$ be the probability space generated by the sequences $Z = (Z_j)_{j \in \mathbb{N}}$ of independent identically distributed random variables having uniform distribution in the set $\{-1, 1\}$, and let $(\tilde{\Omega}, \tilde{F}, \tilde{P}) := (\Omega \times \Omega_1, F \times F_1, P \times P_1)$. There exists a measurable mapping $\kappa : \mathbb{R}^d \times \tilde{\Omega} \to \{-1, 0, 1\}$ such that for a fixed $\tilde{\omega} \in \tilde{\Omega}$ on each $\tilde{O}_k^\varepsilon$ the function $\kappa(\cdot, \tilde{\omega})$ equals either 1 or $-1$ and is zero in $\mathbb{R}^d \setminus \tilde{O}_\varepsilon$, and for any $f \in L^p(\Omega)$, $1 \leq p < \infty$, for almost every $\tilde{\omega} \in \tilde{\Omega}$ the sequence of functions $f(T_{x/\varepsilon} \kappa(x/\varepsilon, \tilde{\omega}))$ converges weakly to zero in $L^p_{\text{loc}}(\mathbb{R}^d)$.
Proof. First we enumerate the inclusions of each realisation \( O_\omega \) in a “measurable way”. Note that the enumeration of inclusions in (2) is arbitrary and is not suitable for our purposes. Let \( \zeta : \mathbb{N} \to \mathbb{Q}^d \) be a bijection with \( \zeta(1) = 0 \) (the latter is a technical assumption and is not essential). We define a sequence of random variables \( X_n = X_n(\omega), n \in \mathbb{N} \), taking values in \( \mathbb{N}_0 \), in the following way. First we set

\[
X_1(\omega) = \begin{cases} 
1, & \text{if } T_\zeta(1) \omega \in O, \\
0, & \text{otherwise}.
\end{cases}
\]

Assuming that \( X_1, \ldots, X_{n-1} \) are already defined we put

\[
X_n(\omega) = \begin{cases} 
0, & \text{if } T_\zeta(n) \omega \notin O, \\
X_1(\omega), & \text{if } T_\zeta(n) \omega \in O \text{ and } \exists l < n \text{ such that } \zeta(l), \zeta(n) \in O_k \text{ for some } k, \\
\max\{X_1(\omega), \ldots, X_{n-1}(\omega)\} + 1, & \text{if } T_\zeta(n) \omega \in O \text{ and } \nexists l < n \text{ such that } \zeta(l), \zeta(n) \in O_k \text{ for some } k.
\end{cases}
\]

The measurability of \( X_n \) for each \( n \in \mathbb{N} \) can be proved similarly to Lemma 3.3. Thus, each inclusion \( O_k^\omega \) is assigned with a natural number, which is the value of \( X_n(\omega) \), where \( n \) is such that \( \zeta(n) \in O_k^\omega \). We relabel the inclusions \( O_k^\omega \) in such a way that \( k = X_n(\omega), \zeta(n) \in O_k^\omega \). One can also write \( k = X_{\zeta^{-1}(q)}(\omega), q \in \mathbb{Q}^d \cap O_k^\omega \). Note that in general labels change under the action of the dynamical system.

For \( r > 0 \) we define the measurable functions (with respect to the product \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}^d) \times \mathcal{F} \))

\[
\kappa_1^r(x, \omega, z) = \begin{cases} 
1, & \text{if } \exists q \in \mathbb{Q}^d \cap B_r(x) \text{ with } z_{X_{\zeta^{-1}(q)}} = 1, \\
0, & \text{otherwise},
\end{cases}
\]

\[
\kappa_2^r(x, \omega, z) = \begin{cases} 
-1, & \text{if } \exists q \in \mathbb{Q}^d \cap B_r(x) \text{ with } z_{X_{\zeta^{-1}(q)}} = -1, \\
0, & \text{otherwise}.
\end{cases}
\]

Now we define the marking function \( \kappa : \mathbb{R}^d \times \Omega \to \{-1, 0, 1\} \)

\[
\kappa(x, \omega, z) := \limsup_{n \to \infty} \kappa_1^1(x, \omega, z) + \limsup_{n \to \infty} \kappa_2^1(x, \omega, z).
\]

By construction \( \kappa \) is a measurable piecewise constant function that equals \(+1\) or \(-1\) on the closure of each inclusion \( O_k^\omega \) (which is the value of \( z_k \)) and is zero in \( \mathbb{R}^d \setminus \overline{O_\omega} \) (this description can be used as the definition of \( \kappa \), however, we need the preceding construction in order to ensure the measurability). We say that \( \kappa(x, \omega, z) \) marks inclusions by assigning them a value \(+1\) or \(-1\).

We denote by \( \mathcal{K}(\omega) \) the family of all marking functions for the set of inclusions \( O_k^\omega, k \in \mathbb{N} \), i.e. piecewise constant functions that equal \(+1\) or \(-1\) on the closure of each inclusion \( O_k^\omega \) and vanish in \( \mathbb{R}^d \setminus \overline{O_\omega} \). The above procedure defines a mapping \( \Upsilon_\omega : \Omega_1 \to \mathcal{K}(\omega) \). In fact, this mapping is a bijection. Indeed, its inverse \( \Upsilon_\omega^{-1} \) maps a given element \( \kappa = \kappa(\cdot, \omega) \in \mathcal{K}(\omega) \) to \( z \in \Omega_1 \) by the formula \( z_k = “\text{the value that } \kappa(\cdot, \omega) \text{ takes on } O_k^\omega” \), where, we remind, the inclusions are labelled according to the values of \( X_n(\omega) \). The relation between \( \Omega_1 \) and \( \mathcal{K}(\omega) \) is useful, since it allows us to introduce a dynamical system on \( \tilde{\Omega} \) naturally induced by \( T_x \) (notice that for \( \kappa(\cdot, \omega) \in \mathcal{K}(\omega) \) we have \( \kappa(\cdot + x, \omega) \in \mathcal{K}(T_x\omega)):

\[
\tilde{T}_x(\omega, z) := \tilde{T}_x(\omega, \Upsilon_\omega^{-1}(\kappa(\cdot, \omega))) = (T_x\omega, \Upsilon_{T_x\omega}^{-1}(\kappa(\cdot + x, \omega))).
\] (170)
Let us show first that \( \tilde{T}_x \) is indeed a dynamical system. Property (a) of Definition 2.1 is obvious. The measurability of \( \tilde{T}_x \) follows from the measurability of the mappings \( (x, \omega) \mapsto T_x \omega \) and \( (x, \omega) \mapsto Y^{-1}_{T_x \omega} (\kappa(\cdot + x, \omega)) \). The measurability of the latter can be shown by using Remark B.2 and an argument analogous to the proof of Lemma B.1 (since the value of \( z = Y^{-1}_{T_x \omega} (\kappa(\cdot + x, \omega)) \) can be determined via a countable number of operations). Finally, it is enough to prove the measure preserving property for the sets of the form \( F \times G \), where \( F \in \mathcal{F} \) and \( G \) is a cylinder set in \( \Omega_1 \). Denoting by \( \Pi_1 \) the projection onto \( \Omega_1 \) we have \( P_1(\Pi_1 \tilde{T}_x(\omega, G)) = P_1(G) \) for any \( \omega \in \Omega \). It remains to apply Fubini’s theorem.

It is not difficult to see that, in fact, the constructed marking function \( \hat{\kappa}(\cdot, \omega, z) \) is the realisation of the function \( \hat{\kappa}(\cdot) = \hat{\kappa}(\omega, z) := 1_\Omega z_1 \) (this is where the assumption \( \zeta(1) = 0 \) is used):

\[
\kappa(x, \omega, z) = \hat{\kappa}(\tilde{T}_x \tilde{\omega}). \tag{171}
\]

Next we prove that \( \tilde{T}_x \) is ergodic. We use the well known fact that ergodicity is equivalent to the mixing property. The following lemma is a continuous version of the result presented in [H13, Chapter 6].

**Lemma D.9.** Let \((\Omega, \mathcal{F}, P)\) be a probability space and \((T_x)_{x \in \mathbb{R}^d}\) a dynamical system on it. It is ergodic if and only if one of the following two equivalent conditions holds:

\[
\frac{1}{L^d} \int_{\square L} P(A \cap T_{-x} B) dx \rightarrow P(A)P(B), \text{ as } L \rightarrow \infty, \text{ for every } A, B \in \mathcal{F}; \tag{172}
\]

\[
\frac{1}{L^d} \int_{\Omega} f(g \circ T_x) dP dx \rightarrow \int_{\Omega} f dP \int_{\Omega} g dP, \text{ as } L \rightarrow \infty, \text{ for every } f, g \in L^2(\Omega). \tag{173}
\]

**Proof.** The equivalence of (172) and (173) is proved in a standard way: first for simple functions, and then approximating \( L^2 \)-functions with the simple ones. We prove that (172) is equivalent to ergodicity. Assume first that (172) holds. Then, taking \( A \in \mathcal{F} \) such that \( T_x A = A \) for all \( x \in \mathbb{R}^d \), and setting \( B = A \) in (172), we obtain that \( P(A) = P(A)^2 \), which implies \( P(A) = 0 \) or \( P(A) = 1 \). By Remark 2.3 we conclude that \( T_x \) is ergodic.

Now let us assume that \( T_x \) is ergodic. By the ergodic theorem for a.e. \( \omega \)

\[
\frac{1}{L^d} \int_{\square L} 1_B(T_x \omega) dx \rightarrow P(B), \text{ as } L \rightarrow \infty.
\]

Then, applying Fubini’s and the dominated convergence theorems, we conclude that

\[
\frac{1}{L^d} \int_{\square L} P(A \cap T_{-x} B) dx = \frac{1}{L^d} \int_{\square L} \int_{\Omega} 1_A (1_B \circ T_x) dP dx
\]

\[
= \int_{\Omega} 1_A \frac{1}{L^d} \int_{\square L} (1_B \circ T_x) dx dP \rightarrow \int_{\Omega} 1_A dP \int_{\Omega} 1_B dP = P(A)P(B), \text{ as } L \rightarrow \infty.
\]

**Remark D.10.** In order to show the validity of (173) it is enough to take \( f \) and \( g \) in the dense subset of \( L^2(\Omega) \). In the same way, by using Dynkin’s \( \pi - \lambda \) theorem, for (172) to hold it is sufficient to check it for every \( A, B \in \Pi \), where \( \Pi \) is any \( \pi \)-system that generates \( \mathcal{F} \).

**Proposition D.11.** The dynamical system \((\tilde{T}_x)_{x \in \mathbb{R}^d}, \) defined in (170), is ergodic.
Proof. We will show that (172) holds for $\tilde{T}_x$ and $(\tilde{\Omega}, \tilde{F}, \tilde{P})$. Following Remark D.10 we take $A, B \in \Pi$, where $\Pi$ is a $\pi$-system defined by

$$\Pi = \{ F \times G : F \in \mathcal{F}, G \text{ is a simple cylinder set in } \Omega_1 \},$$

where by simple cylinder set we understand the sets of the form $\{ z : z_j = 1 \}$ and $\{ z : z_j = -1 \}$, $j \in \mathbb{N}$.

By the ergodic theorem we have pointwise convergence

$$\lim_{M \to \infty} \frac{1}{M^d} \int_{\square^M} 1_{\mathcal{O}(T_x \omega)} dx = P(\mathcal{O}) \text{ for a.e. } \omega,$$

which, in turn, implies convergence in measure: for any $\delta > 0$

$$P \left\{ \left| \frac{1}{M^d} \int_{\square^M} 1_{\mathcal{O}(T_x \omega)} dx - P(\mathcal{O}) \right| > \delta \right\} \to 0 \text{ as } M \to \infty. \quad (174)$$

Denote by $\Omega^M_N$ the set of all $\omega \in \Omega$ such that the cube $\square^M$ contains at least $N$ inclusions from $\mathcal{O}_\omega$. Then (174) directly implies that

$$P(\Omega^M_N) \to 1 \text{ as } M \to \infty. \quad (175)$$

It follows from Assumption 2.7 that there exists $\rho' > 0$ such that every inclusion $\mathcal{O}^k_\omega$ contains a ball of radius $\rho'$. We cover the space $\mathbb{R}^d$ with a finite number of balls of radius $\rho'/2$ and choose the minimal $m \in \mathbb{N}$ such that each ball having a non-empty intersection with $\square^M$ contains at least one point from the set $\{ \zeta(n), n = 1, \ldots, m \}$. Such $m$ obviously exists and depends on $M$, and there exists $M^* = M^*(M)$ such that the cube $\square^{M^*}$ contains the set $\{ \zeta(n), n = 1, \ldots, m \}$. Thus, every inclusion $\mathcal{O}^k_\omega \subset \square^M$ is labelled by the elements of the sequence $X_n(\omega)$ before the sequence $\zeta(n)$ leaves the cube $\square^{M^*}$. It follows that for every $\mathcal{O}^k_\omega \subset \mathbb{R}^d \setminus \square^{M^*}$ we have $k \geq N$.

Let $A = F_1 \times G_1$, $B = F_2 \times G_2 \in \Pi$, where $G_1, G_2$ are simple cylinder sets obtained by fixing $z_i$ and $z_j$ components respectively, for some $i, j \in \mathbb{N}$. We set $N = \max\{i, j\}$ and denote $\tilde{\Omega}^M_N := \Omega^M_N \times \Omega_1$. Let us fix $x \in \mathbb{R}^d$ and consider an element $\tilde{\omega} = (\omega, z) \in A \cap \tilde{\Omega}^M_N \cap \tilde{T}_x \left( B \cap \tilde{\Omega}^M_N \right)$. We denote $\tilde{\omega}^* = (\omega^*, z^*) := \tilde{T}_x \tilde{\omega} \in B \cap \tilde{\Omega}^M_N$. According to the preceding considerations, since $\omega^* \in \Omega^M_N$ and we have that $z_j^*$ component of $z^*$ marks an inclusion $\mathcal{O}_{\omega^*}^j$, that is contained in the cube $\square^{M^*+1}$. Therefore, for $x \notin \square^{2M^*+1}$ the shifted inclusion $\mathcal{O}_{\omega^*}^j - x \subset \mathcal{O}_\omega$ lies outside $\square^{M^*}$. At the same time $\omega \in \Omega^M_N$, thus the mark of $\mathcal{O}_{\omega^*}^j - x$ determines (via the mapping $\mathcal{Y}_x^1 = \mathcal{Y}_x^{1-\omega^*}$) the value of $z_{j'}$ component of $z$ for some $j' > N \geq i$ (i.e. the components of $z$ determined by the sets $G_1$ and $G_2$ for such $\tilde{\omega}$ and $x$ are never the same). We infer that for $x \notin \square^{2M^*+1}$ the following identity holds:

$$\tilde{P} \left( A \cap \tilde{T}_x B \right) = \tilde{P} \left( A \cap \tilde{\Omega}^M_N \cap \tilde{T}_x \left( B \cap \tilde{\Omega}^M_N \right) \right) + \tilde{P} \left( \left( A \setminus \tilde{T}_x \tilde{\Omega}^M_N \right) \cup \left( \tilde{T}_x B \setminus \tilde{\Omega}^M_N \right) \right) + P \left( F_1 \cap \Omega^M_N \cap \tilde{T}_x \left( F_2 \cap \Omega^M_N \right) \right) P_1(G_1) P_2(G_2) + \tilde{P} \left( \left( A \setminus \tilde{T}_x \Gamma^M_N \right) \cup \left( \tilde{T}_x B \setminus \Gamma^M_N \right) \right).$$

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Thus, for a fixed $M$ (and, hence, fixed $M^*$) we have via (175) that
\[
\limsup_{L \to \infty} \frac{1}{L^d} \int_{\Delta L} \hat{P}\left( A \cap \hat{T}_x B \right) = \limsup_{L \to \infty} \frac{1}{L^d} \int_{\Delta L \setminus \Delta L^{2M^*+1}} \hat{P}\left( A \cap \hat{T}_x B \right)
\]
\[
= \limsup_{L \to \infty} \frac{1}{L^d} \int_{\Delta L \setminus \Delta L^{2M^*+1}} \left[ P\left( F_1 \cap \Omega_N^M \cap T_x \left( F_2 \cap \Omega_N^M \right) \right) P_1(G_1)P_2(G_2)
\]
\[+ \hat{P}\left( \left( A \setminus \hat{T}_x \Omega_N^M \right) \cup \left( \hat{T}_x B \setminus \Omega_N^M \right) \right) \right]
\]
\[
= \limsup_{L \to \infty} \frac{1}{L^d} \int_{\Delta L} \left[ P\left( F_1 \cap \Omega_N^M \cap T_x \left( F_2 \cap \Omega_N^M \right) \right) P_1(G_1)P_2(G_2)
\]
\[+ \hat{P}\left( \left( A \setminus \hat{T}_x \Omega_N^M \right) \cup \left( \hat{T}_x B \setminus \Omega_N^M \right) \right) \right]
\]
\[= P(F_1 \cap \Omega_N^M)P(F_2 \cap \Omega_N^M)P_1(G_1)P_2(G_2) + \limsup_{L \to \infty} \frac{1}{L^d} \int_{\Delta L} \hat{P}\left( \left( A \setminus \hat{T}_x \Omega_N^M \right) \cup \left( \hat{T}_x B \setminus \Omega_N^M \right) \right).\]

A completely analogous formula is valid for the limit inferior. Therefore, since $M$ is arbitrary, we conclude via (175) that
\[
\limsup_{L \to \infty} \frac{1}{L^d} \int_{\Delta L} \hat{P}\left( A \cap \hat{T}_x B \right) = P(F_1)P(F_2)P_1(G_1)P_2(G_2) = \hat{P}(A)\hat{P}(B),
\]
and, hence, by Lemma D.9 the dynamical system $\hat{T}_x$ is ergodic. \hfill \Box

It remains to prove the last part of the statement of Proposition D.8. Let $f \in L^p(\Omega), 1 \leq p < \infty$. We consider $f$ as an element of $L^p(\hat{\Omega})$ via the natural embedding. Then $f(T_x \omega)$ can be considered as the realisation of $f \in L^p(\hat{\Omega})$ by the dynamical system $\hat{T}_x$ on a fixed $\hat{\omega} = (\omega, z)$, where $z$ is an arbitrary fixed element of $\Omega$. Then $f(T_x/\varepsilon \omega)\kappa(x/\varepsilon, \hat{\omega})$ is simply an $\varepsilon$-realisation of $f(\omega)\hat{\kappa}(\hat{\omega})$, cf. (171), and, by the ergodic theorem for a.e. $\hat{\omega}$ we have the following weak convergence in $L^p_{\text{loc}}(\mathbb{R}^d)$:
\[
f(T_x/\varepsilon \omega)\kappa(x/\varepsilon, \hat{\omega}) \rightharpoonup \int_{\hat{\Omega}} f(\omega)\hat{\kappa}(\hat{\omega}) = \int_{\hat{\Omega}} f(\omega)1_{\hat{\Omega}} \int_{\hat{\Omega}_1} z_1 = 0.
\]
\hfill \Box

E Other technical statements

**Lemma E.1.** Let $1 \leq p < \infty$ and $X \subset \mathbb{R}^d$ be such that continuous functions with compact support are dense in $L^p(X)$. Let $(X^\varepsilon_k)_{k \in \mathbb{N}}, \varepsilon > 0$, be a family of subsets of $X$, which are mutually disjoint for every $\varepsilon$, and satisfy $\lim_{\varepsilon \to 0} \sup_{k \in \mathbb{N}} \text{diam } X^\varepsilon_k = 0$. We define the following local averaging operators on $L^p(X)$ associated with $(X^\varepsilon_k)_{k \in \mathbb{N}}$:
\[
P^\varepsilon f := \begin{cases} |X^\varepsilon_k|^{-1} \int_{X^\varepsilon_k} f, & \text{if } x \in X^\varepsilon_k, \\ f, & \text{otherwise}. \end{cases}
\]

Then $P^\varepsilon f \to f$ strongly in $L^p(X)$ as $\varepsilon \to 0$ for every $f \in L^p(X)$. 

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Proof. For \( f \in L^p(X) \) we take a sequence \( f_n \in C_c(\mathbb{R}^d) \) such that \( f_n \to f \) strongly in \( L^p(X) \). Because of the uniform continuity of \( f_n \) we obviously have \( P^\varepsilon f_n \to f_n \) strongly in \( L^p(X) \) as \( \varepsilon \to 0 \) for a fixed \( n \). For an arbitrary \( \delta > 0 \) and \( n \in \mathbb{N} \) such that
\[
\|f - f_n\|_{L^p(X)} \leq \delta/3.
\]
Take also \( \varepsilon_0 > 0 \) such that
\[
\|f_n - P^\varepsilon f_n\|_{L^p(X)} < \delta/3, \quad 0 < \varepsilon < \varepsilon_0.
\]
Using Hölder’s inequality on each \( X_k^\varepsilon \) we have
\[
\|P^\varepsilon f_n - P^\varepsilon f\|_{L^p(X_k^\varepsilon)} = \frac{1}{|X_k^\varepsilon|^{p-1}} \left| \int_{X_k^\varepsilon} f_n - \int_{X_k^\varepsilon} f \right|^p 
\leq \frac{1}{|X_k^\varepsilon|^{p-1}} |X_k^\varepsilon|^{p-1} \int_{X_k^\varepsilon} |f_n - f|^p = \|f_n - f\|_{L^p(X_k^\varepsilon)},
\]
and, hence,
\[
\|P^\varepsilon f_n - P^\varepsilon f\|_{L^p(X)} \leq \delta/3.
\]
Now by triangle inequality we arrive at
\[
\|f - P^\varepsilon f\|_{L^p(X)} \leq \|f - f_n\|_{L^p(X)} + \|P^\varepsilon f_n - P^\varepsilon f\|_{L^p(X)} + \|f_n - P^\varepsilon f_n\|_{L^p(X)} \leq \delta, \quad \forall \varepsilon < \varepsilon_0.
\]

Lemma E.2. The following bounds hold for the \( \varepsilon \)-realisations of \( b \),
\[
\|b^\varepsilon\|_{L^2(\square^L)} \leq \frac{CL^{d/2}}{d_\lambda}, \quad \|\varepsilon \nabla b^\varepsilon\|_{L^2(\square^L)} \leq CL^{d/2} \left( \frac{\lambda}{(d_\lambda)^2} + \frac{1}{d_\lambda} \right)^{1/2}.
\]

Proof. The rescaled function \( b^\varepsilon_\lambda(\varepsilon y) \) is the solution to
\[
(-\Delta_{\mathcal{O}_\lambda^k} - \lambda) b^\varepsilon_\lambda(\varepsilon \cdot) = 1_{\mathcal{O}_\lambda^k}(\varepsilon \cdot)
\tag{176}
\]
on every \( \mathcal{O}_\varepsilon^k \). By (14) we have
\[
\|b^\varepsilon_\lambda(\varepsilon \cdot)\|_{L^2(\mathcal{O}_\varepsilon^k)} \leq \frac{|\mathcal{O}_\varepsilon^k|^{1/2}}{d_\lambda}.
\]
Then, multiplying (176) by \( b^\varepsilon_\lambda(\varepsilon \cdot) \) and integrating by parts we easily get
\[
\|\nabla b^\varepsilon_\lambda(\varepsilon \cdot)\|_{L^2(\mathcal{O}_\varepsilon^k)}^2 = \int_{\mathcal{O}_\varepsilon^k} (\lambda|b^\varepsilon_\lambda(\varepsilon \cdot)|^2 + b^\varepsilon_\lambda(\varepsilon \cdot))^2 \leq \frac{|\lambda||\mathcal{O}_\varepsilon^k|}{(d_\lambda)^2} + \frac{|\mathcal{O}_\varepsilon^k|}{d_\lambda}.
\]
The statement of the lemma follows by a rescaling argument and taking the norm over \( \square^L \). 

Lemma E.3. Let \( U \) be a bounded open set in \( \mathbb{R}^d \) and let \( U_n, n \in \mathbb{N} \) be a sequence of open sets converging to \( U \) as \( n \to \infty \) in the Hausdorff metric. Assume that \( \lambda \) is uniformly away from the
spectra of the Dirichlet Laplacian operators on each of the sets $-\Delta U, U_n, n \in \mathbb{N}$. Let $f \in L^2(V)$, where the open set $V$ is large enough to contain $U$ and $U_n, n \in \mathbb{N}$. Then the sequence of solutions

$$-\Delta \phi_n - \lambda \phi_n = f \text{ in } U_n, \phi \in W^{1,2}_0(U_n),$$

converges to the solution of

$$- \Delta \phi - \lambda \phi = f \text{ in } U, \phi \in W^{1,2}_0(U),$$

(177)

weakly in $W^{1,2}(V)$ and strongly in $L^2(V)$ (we extend $\phi, \phi_n$, by zero outside $U, U_n$ respectively.)

Proof. Clearly, the solutions $\phi_n$ are uniformly bounded in $W^{1,2}(V)$, and hence, converge (up to a subsequence) weakly in $W^{1,2}(V)$ and strongly in $L^2(V)$ to some $\hat{\phi} \in W^{1,2}(V)$. It is not difficult to see that $\hat{\phi} \in W^{1,2}_0(U)$. Let $U'$ be an arbitrary open set such that $\overline{U'} \subset U$. Then for sufficiently large $n$ we have $U' \subset U_n$. Let $\psi$ be an arbitrary test function from $W^{1,2}_0(U')$ extended by zero outside $U'$. We have

$$\int_U (\nabla \phi_n \cdot \nabla \psi - \lambda \phi_n \psi) = \int_U f \psi$$

for sufficiently large $n$. Passing to the limit we get

$$\int_U (\nabla \hat{\phi} \cdot \nabla \psi - \lambda \hat{\phi} \psi) = \int_U f \psi.$$

By the density of $\psi$ from $W^{1,2}_0(U')$ in $W^{1,2}_0(U)$ we conclude that $\hat{\phi}$ is the solution of (177), i.e. $\hat{\phi} = \phi$.

Since we can apply the above argument to an arbitrary subsequence we see that the convergence holds for the entire sequence $\phi_n$. 

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