Generalization and new proof for almost everywhere convergence to imply local convergence in measure

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Abstract

With a new proof approach we prove in a more general setting the classical convergence theorem that almost everywhere convergence of measurable functions on a finite measure space implies convergence in measure. Specifically, we generalize the theorem for the case where the codomain is a separable metric space and for the case where the limiting map is constant and the codomain is an arbitrary topological space.

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1. Introduction

It is a classical result that, if \( f, f_1, f_2, \ldots \) are measurable \( \mathbb{C} \)-valued functions on a finite measure space and if \( f_n \to f \) almost everywhere, then \( f_n \to f \) in measure. The importance of the convergence theorem is fully aware. It would be useful (and also intellectually amusing) to prove the convergence theorem when the codomain of the maps \( f, f_1, f_2, \ldots \) is a metric space or even an arbitrary topological space. This task is not trivial; for example, the usual proof approach, for \( f \) constant, cannot deal with the case when \( \mathbb{C} \) is replaced with an arbitrary topological space.

We give a new proof for the convergence theorem that, to a certain extent, allows of the aforementioned generalization. At the same time,
although an application of our generalization, for purposes such as a probabilistic or statistical one, is in a sense immediate for “well-behaved” maps as a probability measure is a suitably scaled finite measure, we provide a counterexample showing that the result does not necessarily hold if the measurability of the involved maps is undecided; difficulty in proving measurability is not unusual in applications such as in the context of asymptotic statistical inference, e.g. establishing the measurability of a nonlinear least squares estimator in $\mathbb{R}^k$ for some integer $k \geq 1$ (Lemma 2 in Jennrich [3]).

2. Preliminaries

Throughout, we fix a finite measure space $(\Omega, \mathcal{F}, \mathcal{M})$.

Following the convention of probability theory, we will in general write for simplicity a set of the form $\{\omega | g(\omega) \text{ has a given property}\}$ as $\{g \text{ has the property}\}$. When written in juxtaposition with a set function, in particular a measure or an outer measure, the set $\{g \text{ has the property}\}$ will simply take the form $(g \text{ has the property})$.

If $S$ is a topological space with $\mathcal{B}_S$ the Borel sigma-algebra, if $f_n : \Omega \to S$ is $(\mathcal{F}, \mathcal{B}_S)$-measurable for all $n \in \mathbb{N}$, and if $f : \Omega \to S$ is constant, then, regarding the convergence of the sequence $(f_n)$ to $f$, the involved notion of closeness to $f$ is understood in terms of the open subsets of $S$ that contain (the point of $S$ identified with) $f$. For example, the definition of convergence in $\mathcal{M}$-measure is to be paraphrased in this case as “for every open $G \subset S$ containing the constant identified with $f$, we have $\mathcal{M}((\omega \in \Omega | f_n(\omega) \in G)) = \mathcal{M}(f_n \in G) \to 0$.”

3. Results

Given a sequence of subsets of $\Omega$, we can partition the space $\Omega$ into the limit inferior of the sequence and the limit superior of the sequence of the complements of the subsets of $\Omega$; this observation is the fundamental proof idea.

**Theorem 1**: Let $S$ be a topological space; let $f, f_n : \Omega \to S$ be measurable-$(\mathcal{F}, \mathcal{B}_S)$ for all $n \in \mathbb{N}$; let $f_n \to f$ almost everywhere with respect to $\mathcal{M}$. i) If $S$ is in particular a separable metric space, then $f_n \to f$ in $\mathcal{M}$-measure; ii) if $f$ is in particular a constant map, then $f_n \to f$ in $\mathcal{M}$-measure.

**Proof**: Let $\epsilon > 0$. 
For i), let \( d \) be the separable metric on \( S \times S \). Since \( d \) is continuous with respect to the product \( d \)-topology, and since the countable base property of \( S \) ensures that the map \((f_n, f)\) is measurable with respect to the Borel sigma-algebra generated by the product \( d \)-topology for all \( n \in \mathbb{N} \), the function \( d(f_n, f) \) is measurable for all \( n \in \mathbb{N} \).

Let \( N \in \mathbb{N} \) whenever \( \liminf_{n \to \infty} \{ d(f_n, f) \leq \varepsilon \} \) is empty; otherwise, let \( N := \inf_{\omega \in \Omega} \inf J \), where, for every \( \omega \in \Omega \), the inner infimum extends over all \( J \in \mathbb{N} \) such that \( d(\omega_j, f(\omega)) \leq \varepsilon \) for all \( j \geq J \). Then, for all \( n \geq N \) we have

\[
0 \leq \mathbb{M}(d(f_n, f) > \varepsilon) = \mathbb{M}(d(f_n, f) > \varepsilon, d(f_m, f) \leq \varepsilon \text{ for sufficiently large } m) + \mathbb{M}(d(f_n, f) > \varepsilon, \limsup_{m \to \infty} d(f_m, f) > \varepsilon) \leq 0 + \mathbb{M}(\limsup_{m \to \infty} d(f_m, f) > \varepsilon) = \mathbb{M}(\Omega) - \mathbb{M}(d(f_m, f) \leq \varepsilon \text{ for sufficiently large } m) \leq \mathbb{M}(\Omega) - \mathbb{M}(f_n \to f \text{ pointwise}) = 0;
\]

the last equality follows from the convergence assumption.

For ii), we rewrite the partition of the finite measure space \( \Omega \) used above as

\[
\{ \liminf_{n \to \infty} \{ f_n \in G \}, \limsup_{n \to \infty} \{ f_n \notin G \} \}
\]

with \( G \ni f \) being a given open subset of \( S \). Then ii) follows from the main argument above with the apparent slight modification. \( \square \)

**Remark**: For the breadth of some application possibilities of Theorem 1, we recall that many familiar function spaces can be made a separable metric space, e.g., the space \( \mathbb{R}^\infty \) (equipped with a usual product metric), the spaces \( L_p(\mathbb{R}^n) \) (equipped with the metric induced by the usual \( L_p \)-norm) for \( 1 \leq p < +\infty \) and \( n \in \mathbb{N} \), the space \( C \) of all the \( \mathbb{R} \)-valued continuous functions on \([0, 1]\) (equipped with the uniform metric), and the space \( D \) of all the \( \mathbb{R} \)-valued cádlág functions on \([0, 1]\) (equipped with

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1 By a cádlág function we mean a function that has left limit and is right continuous everywhere.
the Skorokhod metric, which is a metric derived from the uniform metric), are separable metric spaces.

If the involved maps \( f, f_n \) are not all measurable, or if the measurability is not obvious, then one may try to circumvent the measurability issue via the outer measure obtained by taking for every \( A \subset \Omega \) the infimum of the set \( \{ M(B) \mid B \supset A, B \in \mathcal{F} \} \) and consider the convergence modes in terms of the \( M \)-outer measure. The convergence modes with respect to the \( M \)-outer measure reduce to the usual modes, respectively, whenever measurability is available.

However, even with the \( M \)-outer measure, the first conclusion of Theorem 1 does not necessarily hold in the presence of a measurability issue. Indeed, a consideration over rational translations of a usual Vitali set \( V \) (which is not Lebesgue measurable) in \([0, 1]\), whose elements are the components of a tuple of the Cartesian product \( \times_{A \in [0,1]/R} A \) where the product extends over all the elements of the quotient space \([0,1]/R\) with respect to the equivalence relation \( R \subset [0,1]^2 \) defined by declaring that \( xRy \) iff \( x - y \in \mathbb{Q} \), with full outer measure would lead to a counterexample. Here we certainly acknowledge the axiom of choice. Although the counterexample thus obtained is somewhat of a routine nature, for clarity we still elaborate and highlight a possible construction:

**Proposition 1**: There are some Borel finite measure space and some sequence of nonmeasurable functions from the measure space to \( \mathbb{R} \) that converges pointwise but not in the outer measure.

**Proof**: Consider the unit interval \([0, 1]\) equipped with Lebesgue measure \( L \) restricted to the Borel subsets of \([0, 1]\).

It is (well-)known that there is some Vitali set \( V \subset [0,1] \) such that \( L^*(V) = 1 \). Let \( \{ q_n \}_{n \in \mathbb{N}} = \mathbb{Q} \cap [0,1] \); for each \( n \in \mathbb{N} \), let \( V_n \) be obtained from a rational translation \( V + q_n \) of \( V \) so that \( \{ V_n \} \) is a partition of \([0, 1]\). Then each \( V_n \) is not Lebesgue measurable, and each \( V_n \) has Lebesgue-outer measure \( L^*(V_n) = 1 \).

Let \( f_n := \mathbb{1}_{V_n} \) on \([0, 1]\) for each \( n \in \mathbb{N} \); then each \( f_n \) is not Lebesgue measurable, and \( f_n \to 0 \) pointwise. In particular, (if informative) we have \( f_n \to 0 \) almost everywhere with respect to both \( L \) and \( L^* \).

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2 For the separability of \( L_p(\mathbb{R}^n) \), there is a proof given in Brezis [2]; for the separability of each of the other cases, there is a proof contained in Billingsley [1].
However, we have $\mathbb{L}'(|f_n| > \varepsilon) = \mathbb{L}'(V_n) = 1$ for all $n \in \mathbb{N}$ and all $0 < \varepsilon < 1$; so the sequence $(f_n)$ does not converge in $\mathbb{L}'$ to the zero function.

Since the measure space [0, 1] considered in the above proof can be viewed as the probability space describing the uniform distribution concentrated on [0, 1], Proposition 1 has a probabilistic interpretation and hence is not terribly artificial.

Proposition 1 prevents one from quickly generalizing Theorem 1, which assumes the absence of a measurability issue, to cover the case where a measurability issue is of concern.

It would be interesting to ask to what extent Theorem 1 still persists under indefinite measurability.

References

[1] Billingsley, P. (1999). *Convergence of Probability Measures*, second edition. John Wiley & Sons, Chichester.

[2] Brezis, H. (2011). *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, New York.

[3] Jennrich, R. I. (1969). Asymptotic properties of nonlinear least squares estimators. *The Annals of Mathematical Statistics* 40 633–643.

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