Semiclassical approach to chaotic quantum transport

Sebastian Müller¹, Stefan Heusler², Petr Braun²,³ and Fritz Haake²

¹ Cavendish Laboratory, University of Cambridge, J J Thomson Avenue, Cambridge CB3 0HE, UK
² Fachbereich Physik, Universität Duisburg-Essen, 47048 Duisburg, Germany
³ Institute of Physics, Saint-Petersburg University, 198504 Saint-Petersburg, Russia
E-mail: sm645@cam.ac.uk

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Abstract. We describe a semiclassical method to calculate universal transport properties of chaotic cavities. While the energy-averaged conductance turns out governed by pairs of entrance-to-exit trajectories, the conductance variance, shot noise and other related quantities require trajectory quadruplets; simple diagrammatic rules allow us to find the contributions of these pairs and quadruplets. Both pure symmetry classes and the crossover due to an external magnetic field are considered.
1. Introduction

Mesoscopic cavities show universal transport properties—such as conductance, conductance fluctuations, or shot noise—provided the classical dynamics inside the cavity is fully chaotic. Here chaos may be due to either implanted impurities or bumpy boundaries. A phenomenological description of these universal features is available through random-matrix theory (RMT) by averaging over ensembles of systems (whose Hamiltonians are represented by matrices) [1]. For systems with impurities, one can alternatively average over different disorder potentials. However, experiments show that even individual cavities show universal behaviour faithful to these averages.

We propose a semiclassical explanation of universal transport through individual chaotic cavities, based on the interfering contributions of close classical trajectories. This approach generalizes earlier work in [2]–[5] and is inspired by recent progress for universal spectral statistics [6]–[9]. Our semiclassical procedure often turns out to be technically easier than RMT; transport properties are evaluated through very simple diagrammatic rules.

We consider a two-dimensional cavity accommodating chaotic classical motion. Two (or more) straight leads are attached to the cavity and carry currents. We shall mostly consider electronic currents, but most of the following ideas apply to transport of light or sound as well, minor modifications apart.
The channels exactly determine the absolute values of the initial and final angles of incidence of the Fermi velocity, and a magnetic field producing magnetic actions of the order of the latter case, we shall also interpolate between both pure symmetry classes by accounting for a weak with time reversal invariance (‘orthogonal case’) and without that symmetry (‘unitary case’). For free; we shall mostly consider of the openings is large in the sense that $w_i^2 \gg L\lambda_F$, $i=1,2$. In this regime, we will determine transport properties to all orders in $1/N$, in contrast to much of the previous literature which was restricted to the lowest orders. However, we will neglect corrections related to the finite value of the ratio of the Ehrenfest time $T_E = (1/\lambda)\ln \frac{L}{\lambda_F}$ to the dwell time of trajectories inside the cavity, $T_D \propto \frac{L^2}{\lambda_F}$. (Here $\lambda$ is the Lyapunov exponent, not to be confused with the wavelength $\lambda_F$, $v_F$ is the Fermi velocity, and $w = w_1 + w_2$.) These corrections play an important role in the regime considered; for their treatment see [10]–[12].

Following Landauer and Büttiker [13, 14], we view transport as scattering between leads and deal with amplitudes $t_{a_1a_2}$ for transitions between channels $a_1$ and $a_2$. These amplitudes form an $N_1 \times N_2$ matrix $t = \{t_{a_1a_2}\}$. Each $t_{a_1a_2}$ can be approximated semiclassically, by the van Vleck approximation for the propagator, as a sum over trajectories connecting the channels $a_1$ and $a_2$ [15],

$$t_{a_1a_2} \sim \frac{1}{\sqrt{T_H}} \sum_{a_0,a_1\rightarrow a_2} A_a e^{iS_a/\hbar}.$$  

The channels exactly determine the absolute values of the initial and final angles of incidence $\theta_1, \theta_2$ of the contributing trajectories; again both positive and negative angles are possible. In (1), $T_H$ denotes the so-called Heisenberg time, i.e., the quantum time scale $2\pi\hbar \tilde{\rho}$ associated to the mean level density $\tilde{\rho}$. The Heisenberg time diverges in the semiclassical limit like $T_H \propto V^{3/2}/\hbar$. This is the Lyapunov exponent, not to be confused with the wavelength $\lambda_F$, $v_F$ is the Fermi velocity, and $w = w_1 + w_2$.) These corrections play an important role in the regime considered; for their treatment see [10]–[12].

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Within the framework just delineated, we will evaluate, for individual fully chaotic cavities,

1. the mean conductance $\langle \text{tr} (tt^\dagger) \rangle$ (actually, the conductance is given by $\frac{e^2}{\hbar} \langle \text{tr} (tt^\dagger) \rangle$, taking into account two possible spin orientations; we prefer to express the result in units of $\frac{e^2}{\hbar}$),
2. the conductance variance $\langle (\text{tr} tt^\dagger)^2 \rangle - \langle \text{tr} (tt^\dagger) \rangle^2$,
3. the mean shot noise power $\langle \text{tr} (tt^\dagger - tt^\dagger tt^\dagger) \rangle$, in units of $\frac{2e^3|V|}{\hbar}$. 

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4. for a cavity with three leads, correlations between the currents flowing from lead 1 to lead 2 and from lead 1 to lead 3 depending on the corresponding transition matrices $t^{(1\rightarrow 2)}$, $t^{(1\rightarrow 3)}$ as 
\[ \langle \text{tr}(t^{(1\rightarrow 2)} t^{(1\rightarrow 2)^T} t^{(1\rightarrow 3)} t^{(1\rightarrow 3)^T}) \rangle, \]
5. the conductance covariance at two different energies which characterizes the so-called Ericson fluctuations.

Here the angular brackets signify an average over an energy interval sufficient to smooth out the fluctuations of the respective physical property. We will see that after such an averaging each of these quantities takes a universal form in agreement with RMT, without any need for an ensemble average. To show this, we shall express the transition amplitudes as sums over trajectories as in (1). The above observables then turn into averaged sums over pairs or quadruplets of trajectories, which will be evaluated according to simple and universal diagrammatic rules. We shall first derive and exploit these rules for the orthogonal and unitary cases and then generalize to the interpolating case (weak magnetic field).

Due to the unitarity of the time evolution, we could equivalently express transport properties through reflection amplitudes and trajectories starting and ending at the same lead. For the average conductance we have checked explicitly that the same result is obtained, meaning that our approach preserves unitarity.

2. Mean conductance

We first consider the mean conductance and propose to show that individual chaotic systems are faithful to the random-matrix prediction [1, 16]

\[ \langle G(E) \rangle = \langle \text{tr}(tt^\dagger) \rangle = \begin{cases} N_1 N_2 & \text{unitary case,} \\ N & \\ N_1 N_2 & \text{orthogonal case.} \end{cases} \] (2)

In the semiclassical approximation (1), the average conductance becomes a double sum over trajectories $\alpha, \beta$ connecting the same channels $a_1$ and $a_2$,

\[ \langle \text{tr}(tt^\dagger) \rangle = \frac{1}{T_H} \left\langle \sum_{a_1, a_2} \sum_{\alpha} A_\alpha A_{\beta}^* e^{i(S_\alpha - S_\beta)/\hbar} \right\rangle. \] (3)

Due to the phase factor $e^{i(S_\alpha - S_\beta)/\hbar}$, the contributions of most trajectory pairs oscillate rapidly in the limit $\hbar \to 0$, and vanish after averaging over the energy. Systematic contributions can only arise from pairs with action differences $\Delta S = S_\alpha - S_\beta$ of the order of $\hbar$.

2.1. Diagonal contribution

The simplest such pairs involve identical trajectories $\alpha = \beta$, with a vanishing action difference [6, 17]. These ‘diagonal’ pairs contribute

\[ \langle \text{tr}(tt^\dagger) \rangle |_{\text{diag}} = \frac{1}{T_H} \left\langle \sum_{a_1, a_2} \sum_{\alpha} |A_\alpha|^2 \right\rangle. \] (4)
The foregoing single-trajectory sum may be evaluated using the following rule established by Richter and Sieber \[2\]: summation over trajectories connecting fixed channels is equivalent to integration over the dwell time \( T \) as

\[
\sum_{\alpha:a_1 \to a_2} |A_{\alpha}|^2 = \int_0^\infty dT e^{-\left(\frac{N}{T_H}\right)T} = \frac{T_H}{N}.
\] (5)

Here, the integrand \( e^{-\left(\frac{N}{T_H}\right)T} \) can be understood as the survival probability, i.e., the probability for the trajectory to remain inside the cavity up to the time \( T \). The factor \( \frac{N}{T_H} = \frac{2mVF(w_1+w_2)}{\Omega_1^2} \) is the classical escape rate (with \( m \) denoting the mass). Due to \( \Omega \propto mL^2 \), the inverse of that rate is the typical dwell time \( T_D \) mentioned in the introduction.

Finally, summing over all \( N_1 \) possible choices for \( a_1 \) and over the \( N_2 \) possibilities for \( a_2 \), one finds \[2, 17\]

\[
\langle \text{tr}(tt^\dagger) \rangle_{\text{diag}} = \frac{N_1N_2}{N^2}.
\] (6)

Equation (6) reproduces the RMT result for the unitary case, and gives the leading term in the orthogonal case.

2.2. Richter/Sieber pairs

For the orthogonal case, Richter and Sieber attributed the next-to-leading order to another family of trajectory pairs. In the following, we shall describe these pairs in a language adapted to an extension to higher orders in \( \frac{1}{N} \). In each Richter/Sieber pair (see figure 1), the trajectory \( \alpha \) contains a ‘2-encounter’ wherein two stretches are almost mutually time-reversed; in configuration space it looks like either a small-angle self-crossing or a narrow avoided crossing. We demand that these two stretches come sufficiently close such that their motion is mutually linearizable. The three pieces of \( \alpha \) outside the encounter will be referred to as ‘links’\(^4\). The partner trajectory \( \beta \) is distinguished from \( \alpha \) only by differently connecting these links inside the 2-encounter. Along the links, however, \( \beta \) is practically indistinguishable from \( \alpha \); in particular, the entrance and exit angles of \( \alpha \) and \( \beta \) (defined by the in- and out-channels) coincide\(^5\). The initial and final links are traversed in the same sense of motion by \( \alpha \) and \( \beta \), while for the middle link the velocities are opposite. Obviously, such Richter/Sieber pairs \( \alpha, \beta \) can exist only in time-reversal invariant systems. The two trajectories in a Richter/Sieber pair indeed have nearly the same action, with the action difference originating mostly from the encounter region.

We stress that inside a Richter/Sieber pair, the encounter stretches and the leads must be separated by links of positive durations \( t_1, t_2, t_3 > 0 \). For the ‘inner’ loop with duration \( t_2 \) the

\(^4\) In our previous papers \[9, 18, 19\], we used the term ‘loop’ to refer to the comparatively long orbit pieces connecting encounter stretches to one another or to the openings. We decided to replace this expression by ‘link’ which is more appropriate since the beginning and the end of such a piece may be far removed from each other (in the case of the initial and final link and of links between different encounters).

\(^5\) Following Richter and Sieber we find the semiclassical estimate for a conductance component \( \langle |ta_1a_2|^2 \rangle \) between two given in- and out-channels and demand therefore that all contributing trajectories have the same in- and out-angles. An alternative \[11\] is to replace summation over channels in the formula for the transport property by integration; then the channel numbers of the contributing trajectories found through an additional saddle point approximation will not be integer.
Figure 1. Scheme of a Richter/Sieber pair. The trajectories $\alpha$ (full line) and $\beta$ (dashed line) connect the same channels $a_1$ and $a_2$, and differ only inside a 2-encounter (in the box). A Poincaré section $\mathcal{P}$ intersects the encounter stretches at the points $x_{P_1}$ and $x_{P_2}$ in phase space whose configuration-space location is highlighted by two dots. $\mathcal{P}$ divides the encounter in two parts with durations $t_s \sim \frac{1}{4} \ln \frac{c|s|}{\lambda}$ and $t_u \sim \frac{1}{2} \ln \frac{|u|}{\lambda}$. The relevant trajectories are in reality much longer than depicted here. In the absence of a potential they consist of a huge number of straight segments reflected at the boundary; in particular both encounter stretches are in fact two broken lines piecewise close and almost parallel to each other.

reasons were worked out in previous publications dealing with periodic orbits ([20, 21], and [9, 22] for more complicated encounters): essentially, $\beta$ is obtained from $\alpha$ by switching connections between four points where the encounter stretches begin and end; to have four such points the stretches must be separated by a non-vanishing link. If the widths of the openings $w_1, w_2$ are larger than the transversal size of the relevant encounters ($\sqrt{L/\lambda_F}$, see below), the initial and final links must be non-vanishing as well. If one of these links were absent, both stretches would have to ‘stick out’ of the same opening. The encounter would then enter and exit the cavity through the same opening and thus be irrelevant for the conductance.

Encounters have an important effect on the survival probability [4] (again under the assumption of large openings). The trajectory $\alpha$ is exposed to the ‘danger’ of getting lost from the cavity only during the three links and on the first stretch of the encounter. If the first stretch remains inside the cavity, the second stretch, being close to the first one (up to time reversal) must remain inside as well. If we denote the duration of one encounter stretch by $t_{\text{enc}}$, the total ‘exposure time’ is thus given by $T_{\text{exp}} = t_1 + t_2 + t_3 + t_{\text{enc}}$; it is shorter than the dwell time $T$ which includes a second summand $t_{\text{enc}}$ representing the second encounter stretch. Consequently, the survival probability $e^{-(N/T_0)T_{\text{exp}}}$ exceeds the estimate $e^{-(N/T_0)T}$. In brief, encounters hinder the loss of a trajectory to the leads.

To describe the phase-space geometry of a 2-encounter, we consider a Poincaré section $\mathcal{P}$ orthogonal to the first encounter stretch in an arbitrary phase-space point $x_{P_1}$. This section must also intersect the second stretch in a phase-space point $x_{P_2}$ almost time-reversed with respect to $x_{P_1}$. In figure 1, the configuration-space locations of $x_{P_1}$ and $x_{P_2}$ are highlighted by two dots.
For a hyperbolic, quasi two-dimensional system, the small phase space separation between the time-reversed $T\mathbf{x}_{P2}$ of $\mathbf{x}_{P2}$ and $\mathbf{x}_{P1}$ can be decomposed as $[23]–[25]$

$$T\mathbf{x}_{P2} - \mathbf{x}_{P1} = s\mathbf{e}^s(\mathbf{x}_{P1}) + u\mathbf{e}^u(\mathbf{x}_{P1}),$$

(7)

where $\mathbf{e}^s(\mathbf{x}_{P1})$ and $\mathbf{e}^u(\mathbf{x}_{P1})$ are the so-called stable and unstable directions at $\mathbf{x}_{P1}$. If $P$ moves along the trajectory, following the time evolution of $\mathbf{x}_{P1}$, the unstable component $u$ will grow exponentially while the stable component $s$ shrinks exponentially. For times large compared with the ballistic time ($L/v$ with $v$ the velocity) the rate of growth (or shrinking) is given by the Lyapunov exponent $\lambda$,

$$u(t) \sim u(0)e^{\lambda t}, \quad s(t) \sim s(0)e^{-\lambda t}. \quad (8)$$

By our definition of a 2-encounter, the stable and unstable components are confined to ranges $-c < s < c$, $-c < u < c$, with $c$ a small phase-space separation. The exact value of $c$ will be irrelevant, except that it should be small enough to allow mutual linearization of motion along the encounter stretches. As a consequence, the time between $P$ and the end of the encounter is $t_u \sim \frac{1}{\lambda} \ln \frac{c}{|u|}$, i.e., the time the unstable component needs to grow from $u$ to $\pm c$. Likewise the time between the beginning of the encounter and $P$ reads $t_s \sim \frac{1}{\lambda} \ln \frac{c}{|s|}$. Both times sum up to the encounter duration

$$t_{enc} = t_u + t_s \sim \frac{1}{\lambda} \ln \frac{c^2}{|su|}. \quad (9)$$

A glance at figure 1 shows that the times $t_{P1}$, $t_{P2}$ of the piercing points $\mathbf{x}_{P1}$, $\mathbf{x}_{P2}$ (measured from the beginning of the trajectory) are now given by

$$t_{P1} = t_1 + t_u, \quad t_{P2} = t_1 + t_{enc} + t_2 + t_s. \quad (10)$$

Finally, the stable and unstable coordinates determine the action difference as $[24, 25]$ (see also $[9, 20]$)

$$\Delta S = su. \quad (11)$$

The encounters relevant for the transport phenomena have action differences of order $\hbar$ and thus durations $t_{enc} \sim \frac{1}{\lambda} \ln \frac{c^2}{|\Delta S|}$ of the order of the Ehrenfest time $T_E = \frac{1}{\lambda} \ln \frac{c^2}{\hbar}$. We can now also justify the above estimate for the spatial width of the relevant encounters, e.g., for the minimal distance $d$ between the encounter stretches in the partner trajectory with an avoided crossing. In this case (11) boils down to $\Delta S \sim m\lambda d^2 \sim \frac{m\gamma d^2}{L} \propto \frac{\hbar d^2}{L^2} [26, 27]$ which is of the order $\hbar$ if $d \sim \sqrt{L\lambda}$.

With this input, we can determine the average number of 2-encounters inside trajectories $\alpha$ of a given dwell time $T$. In ergodic systems, the probability for a trajectory to pierce through a fixed Poincaré section $P$ in a time interval $(t_{P2}, t_{P2} + dt_{P2})$ with stable and unstable separations from $\mathbf{x}_{P1}$ inside $(s, s + ds) \times (u, u + du)$ is uniform, and given by the Liouville measure $\frac{1}{\Omega} dt_{P2} ds du$. To count all 2-encounters inside $\alpha$, we have to integrate this density over $t_{P2}$ (to get all piercings $\mathbf{x}_{P2}$ through a given $P$) and over $t_{P1}$ (to get all possible $\mathbf{x}_{P1}$ and thus all possible sections $P$). When integrating over $t_{P1}$, we weigh the contribution of each encounter with the corresponding

6 Our treatment can easily be extended to $f > 2$, see [21] and the appendices of [9, 22].
The encounter integral

\[ T \int \frac{1}{\Omega t_{\text{enc}}(s, u)} ds du \]

This density is normalized such that integration over all \( s, u \) belonging to a given interval of \( \Delta S = su \) yields the number of 2-encounters of \( \alpha \) giving rise to action differences within that interval.

To find what Richter/Sieber pairs contribute to the conductance \( (3) \), we replace the sum over \( \beta \) by a sum over 2-encounters inside \( \alpha \) or, equivalently, an integral over \( w(s, u) \). The additional approximation \( A_\beta \approx A_\alpha \) yields

\[ \langle \text{tr}(t^{\dagger}) \rangle_{\text{RS}} = \frac{1}{T_H} \left[ \sum_{a_1, a_2} \int ds du \sum_{\alpha, a_1 \rightarrow a_2} |A_{a_1}|^2 w(s, u) e^{isu/\hbar} \right]. \]

Next, we employ the Richter/Sieber rule to do the sum over \( \alpha \) by integrating over the dwell time \( T \), with the integrand involving the (modified) survival probability \( e^{-(N/T_H)\tau_{\text{exp}}} = e^{-(N/T_H)(t_1 + t_2 + t_3 + t_{\text{enc}})} \).

The integral over \( T \) may be transformed into an integral over the duration of the final link. Moreover, summation over all channels \( a_1 = 1 \ldots N_1, a_2 = 1 \ldots N_2 \) yields a factor \( N_1 N_2 \). We are thus led to

\[ \langle \text{tr}(t^{\dagger}) \rangle_{\text{RS}} = \frac{N_1 N_2}{T_H} \left[ \int_0^\infty dt_1 dt_2 dt_3 \int ds du \frac{1}{\Omega t_{\text{enc}}(s, u)} e^{-(N/T_H)(t_1 + t_2 + t_3 + t_{\text{enc}}(s, u))} e^{isu/\hbar} \right]. \]

This integral factors into three independent integrals over the link durations,

\[ \int_0^\infty dt e^{-(N/T_H)t} = \frac{T_H}{N}, \]

and an integral over the stable and unstable separations within the encounter,

\[ I = \left[ \int ds du \frac{1}{\Omega t_{\text{enc}}(s, u)} e^{-(N/T_H)t_{\text{enc}}(s, u)} e^{isu/\hbar} \right]. \]

The encounter integral \( I \) can be evaluated if we expand the exponential as \( e^{-(N/T_H)t_{\text{enc}}(s, u)} = 1 - \frac{N}{T_H} t_{\text{enc}}(s, u) + \ldots \). As shown in \( [9] \), the constant term yields \( \langle \int_0^c ds \int_{-c}^c du \frac{1}{\Omega t_{\text{enc}}(s, u)} e^{isu/\hbar} \rangle = \frac{2\hbar k}{\Omega} \sin^2 \frac{\delta}{\hbar} \) which oscillates rapidly as \( \hbar \rightarrow 0 \) and therefore vanishes after averaging. In the

\[ \sum_{\beta} \langle \text{tr}(t^{\dagger}) \rangle_{\beta} \]

For long trajectories, the derivative \( \frac{\partial}{\partial a} \) in \( A_\alpha \) [15] is proportional to the so-called stretching factor \( \Lambda_\alpha \), i.e., the factor by which an initial separation along the unstable direction grows until the end of the trajectory \( \alpha \). This factor can be written as a product of the (time-reversal invariant) stretching factors of the individual links and encounter stretches. Since \( \beta \) contains practically the same links and stretches, we have \( \Lambda_\beta \approx \Lambda_\alpha \). All other factors almost coincide as well (see \( [20, 22, 25] \) for the Maslov index), entailing \( A_\beta \approx A_\alpha \).
In the semiclassical limit, the value of \( I \) is solely determined by the linear term for which the denominator \( t_{\text{enc}}(s,u) \) cancels out,

\[
I = -\frac{N}{\Omega T_H} \left( \int ds du \, e^{iu/h} \right) = -\frac{N}{T_H^2},
\]

where we use \( \int ds du \, e^{iu/h} \to 2\pi h \) and \( T_H = \frac{\Omega}{2\pi h} \); all further terms vanish compared to the linear one like \( \frac{N_{\text{enc}}}{T_H} \sim \frac{T_{\text{enc}}}{T_H} \) and will thus be neglected. Since all occurrences of \( T_H \) in equations (14), (15) and (17) mutually cancel, we can formulate the following ‘diagrammatic rule’: Each link yields a factor \( \frac{1}{N} \) and each encounter a factor \( -N \). The result still has to be multiplied with the number of channel combinations \( N_1 N_2 \). Altogether, the contribution of Richter/Sieber pairs to the average conductance is hence determined as

\[
\langle \text{tr}(t^+) \rangle_{\text{RS}} = -\frac{N_1 N_2}{N^2}.
\]

Our present treatment differs from Richter’s and Sieber’s original paper [2] in two points: first, we exclude encounters which stick out of the opening. (In [2], encounters were described through self-crossings in configuration space, and only these crossings, approximately in the center of the encounter, had to be located inside the cavity.) Second, we take into account that encounters hinder the escape into the leads. While these two corrections mutually cancel for Richter/Sieber pairs they will presently turn out of crucial importance for higher-order contributions to the mean conductance, as well as for other observables like shot noise.

### 2.3. Diagrammatic rules for all orders in \( \frac{1}{N} \)

To proceed to all orders in \( \frac{1}{N} \), we must consider pairs of trajectories differing by their connections inside arbitrarily many encounters. Each of these encounters may involve arbitrarily many stretches. We shall speak of an \( l \)-encounter whenever \( l \) stretches of a trajectory come close in phase space. In time-reversal invariant systems we must also allow for encounters whose stretches are almost mutually time-reversed. As a consequence, the resulting conductance will depend on the symmetry class: we shall see all higher-order contributions to mutually cancel in the unitary case but to yield the \( 1/N \) expansion of the RMT result (2) in the orthogonal case.

A list of examples is displayed in figure 2, where for later convenience we did not draw the cavity and formally joined the initial and final points of the trajectories together. These examples illustrate the simplest among infinitely many topologically different families of trajectory pairs.

The individual families are characterized (i) by the numbers \( v_l \) of \( l \)-encounters in which the two partners differ. These numbers can be assembled into a ‘vector’ \( \vec{v} = (v_2, v_3, v_4, \ldots) \), and determine the overall number of encounters \( V(\vec{v}) = \sum_{l=2}^{\infty} v_l \) and the total number of encounter stretches \( L(\vec{v}) = \sum_{l=2}^{\infty} lv_l \). The number of links exceeds the number of stretches by one and reads \( L(\vec{v}) + 1 \). Further characteristics of our families of trajectory pairs are (ii) the order in which the encounters are traversed by the trajectory \( \alpha \), (iii) the mutual orientation of the encounter stretches (i.e., \( \Rightarrow \) versus \( \Leftarrow \), or \( \Rightarrow \Rightarrow \) versus \( \Leftarrow \Leftarrow \)), and (iv) the reconnections leading to the partner trajectory \( \beta \). We stress that \( \beta \) must be a single connected trajectory; reconnections leading to, e.g., one trajectory and one periodic orbit as in figure 3 must be excluded.
Figure 2. Families of trajectory pairs ($\alpha$, $\beta$) differing in one 2-encounter (a), two 2-encounters (b)–(f) or in one 3-encounter (g)–(j). In contrast to figure 1 the cavity is not depicted, and the initial and final points of the trajectories are joined together (junction symbolized by two dots and intervening bar, $\cdot|\cdot$). The same orbit pair may result from joining the beginning and end of different trajectory pairs (like c and d; e and f; or h, i and j). Arrows indicate the directions of motion inside the encounters, and highlight those links which are traversed by $\alpha$ and $\beta$ with opposite sense of motion. Note that all families apart from (b) and (g) involve almost mutually time-reversed encounter stretches and thus require time-reversal invariance (TRI). Together with the diagonal pairs, the ones shown here reproduce the mean conductance up to order $N_1N_2/N_3$.

All families of trajectory pairs contribute to the conductance according to the same rules as do Richter/Sieber pairs: each link yields a factor $1/N$, each encounter gives a factor $-N$, and we have to multiply with the number of channel combinations $N_1N_2$. To prove this assertion, we place a Poincaré section $P_\sigma$ (with $\sigma = 1 \ldots V$) across each of the $V$ encounters. Similar as for Richter/Sieber pairs, we characterize each $l$-encounter by $l-1$ stable coordinates $s_{\sigma j}$ (with $\sigma = 1 \ldots V$ and $j = 1 \ldots l-1$), and by $l-1$ unstable coordinates $u_{\sigma j}$ [9], measuring the phase-space separations between the points where the $l$-encounter stretches pierce through $P_\sigma$. All $V$ encounters are thus characterized by $\sum_{l\geq2}(l-1)v_l = L - V$ stable coordinates, and the same number of unstable coordinates. As shown in [9], these coordinates determine the action difference as $\Delta S = \sum_{\sigma,j} s_{\sigma j}u_{\sigma j}$. Each encounter lasts as long as the absolute values of all
coordinates remain below the bound $c$. Consequently, the duration of an encounter is determined by the largest stable and unstable coordinates, the first to reach $c$. In analogy to (9), the $\sigma$th encounter thus has the duration

$$t_{\text{enc}}^\sigma(s, u) \sim \frac{1}{\lambda} \ln \frac{c^2}{\max_j |s_{\sigma j}| \times \max_j |u_{\sigma j}|}.$$ (19)

Again, the trajectory $\alpha$ may get lost from the cavity only during the links and on the first stretch of each encounter. If it survives on that first stretch, it cannot escape on the remaining stretches of the same encounter, since these are close to the first. The exposure time $T_{\text{exp}}$ is thus obtained as the sum of all link and encounter durations $T_{\text{exp}} = \sum_{i=1}^{L+1} t_i + \sum_{\sigma=1}^{V} t_{\text{enc}}^\sigma$. The survival probability $e^{-\left(\frac{N}{T_{\text{H}}}\right)T_{\text{exp}}}$ results, again larger than the naive estimate $e^{-\left(\frac{N}{T_{\text{H}}}\right)T_{\text{H}}}$.

We proceed to investigating the statistics of encounters for a given family of trajectory pairs. Generalizing the treatment of subsection 2.2, we first keep all $V$ Poincaré sections $\mathcal{P}_\sigma$ fixed: each $\mathcal{P}_\sigma$ is placed orthogonal to $\alpha$ at a phase-space point traversed at a fixed time. The further $L-V$ piercings through these sections may be considered statistically independent; the probability density for their occurrence at given times and with given stable and unstable coordinates thus reads $\frac{1}{\Omega^L-V}$. To account for all encounters, we must integrate $\frac{1}{\Omega^L-V}$ over all times for the $L-V$ later piercings. Moreover, to include all possible $\mathcal{P}_\sigma$, we must integrate over the times of the $V$ phase-space points chosen as the origin of a section $\mathcal{P}_\sigma$. Since all sets of piercings within the duration $t_{\text{enc}}^\sigma$ belong to the same encounter, the latter integral weighs each encounter with a factor $t_{\text{enc}}^\sigma$. Exactly as for Richter/Sieber pairs, this factor must subsequently be divided out. To simplify our calculation, we replace the integral over altogether $L$ times of piercings by an integral over the durations $t_1, t_2, \ldots, t_L$ of all links except the final one. This replacement is permissible because all piercing times may be written as functions of $t_1, t_2, \ldots, t_L$, with the Jacobian of the transformation equal to unity. Here, the duration $t_{L+1}$ of the final link does not show up, since that link does not precede any encounter stretch or piercing point. The integral
goes over positive \( t_i \), by the same reasoning as for 2-encounters\(^8\); moreover, the cumulative duration of the first \( L \) links and all encounter stretches must be smaller than the dwell time \( T \), to allow for a non-vanishing final link. We thus obtain the following density of stable and unstable coordinates

\[
w(s, u) = \int_{s_{\sigma,j}, u_{\sigma,j}^+}^{s_{\sigma,j}^-, u_{\sigma,j}^-} dt_1 \ldots dt_L \frac{1}{\Omega_L - V \prod_{\sigma=1}^V \tau_{\text{enc}}^\sigma(s, u)}.
\]

The weight \( w(s, u) \) is normalized similarly as in subsection 2.2: integration over \( s_{\sigma,j}, u_{\sigma,j} \) corresponding to a given interval of \( \Delta S = \sum_{\sigma,j} s_{\sigma,j} u_{\sigma,j} \) leads to the number of partner trajectories \( \beta \) of a given \( \alpha \), with action difference inside that interval, and with the pair \((\alpha, \beta)\) belonging to the family considered.

We can now evaluate the contribution of an arbitrary family of trajectory pairs to the average conductance \((3)\). We again approximate \( A_\beta \approx A_\alpha \), and write the sum over \( \beta \) as an integral over \( w(s, u) \),

\[
\langle \text{tr}(u^\dagger) \rangle_{\text{fam}} = \frac{1}{T_H} \left( \sum_{a_1, a_2} \int_{a_1, a_2} dL^{-V}s dL^{-V}u \sum_{\alpha, a_1, a_2} |A_\alpha|^2 \frac{e^{i\sum_{\sigma,j} s_{\sigma,j} u_{\sigma,j}/\hbar}}{\Omega_L - V \prod_{\sigma=1}^V \tau_{\text{enc}}^\sigma(s, u)} \right).
\]

As before, the sum over \( a_1, a_2 \) leads to a multiplication with the number of channel combinations \( N_1 N_2 \). The sum over \( \alpha \) can be done using the (modified) Richter/Sieber sum rule, and leads to integration over the duration \( T_{L+1} \) of the final link, with an integrand involving the survival probability \( e^{-(N/T_H)T_{\text{enc}}} = e^{-(N/T_H)(\sum_{\sigma=1}^{V} \tau_{\text{enc}}^\sigma + V \sum_{\sigma=1}^V \tau_{\text{enc}}^\sigma)} \). Together with the integrals over the remaining \( L \) link durations in \( w(s, u) \), equation \((20)\), all links are now treated equally, and we obtain

\[
\langle \text{tr}(u^\dagger) \rangle_{\text{fam}} = \frac{N_1 N_2}{T_H} \prod_{i=1}^{L+1} \left( \int dt_i e^{-(N/T_H)t_i} \right) \times \prod_{\sigma=1}^V \left( \int ds_{\sigma,1} \ldots ds_{\sigma,l-1} \int du_{\sigma,1} \ldots du_{\sigma,l-1} e^{i\sum_{\sigma,j} s_{\sigma,j} u_{\sigma,j}/\hbar} e^{-(N/T_H)\tau_{\text{enc}}^\sigma(s, u)} \right).
\]

Just like in \((14)\) the integral factorizes into several integrals, one for each link and each encounter: each link gives \( \frac{T_H}{N} \), and the encounter integral is determined by the linear term in the series expansion of the exponential \( e^{-(N/T_H)\tau_{\text{enc}}^\sigma(s, u)} \). The encounter integral is slightly changed because the piercing probability \( \frac{1}{\Omega_L-1} \), and the simple integral \( \int ds \int du e^{isu/\hbar} \rightarrow 2\pi\hbar \) now both appear in the \((L-1)\)-fold power. Each encounter thus yields \( -\frac{N}{T_H} \left( \frac{2\pi\hbar}{\Omega_L^2} \right)^{L-1} = -\frac{N}{T_H} \). Again, all powers of \( T_H \) mutually cancel. We thus find the same diagrammatic rules as above with a factor \( \frac{1}{N} \) from each of the \( L+1 \) links, a factor \(-N\) from each of the \( V \) encounters, and a factor \( N_1 N_2 \) representing the possible channel combinations. The contribution of each family is therefore given by

\[
\langle \text{tr}(u^\dagger) \rangle_{\text{fam}} = (-1)^{V(f)} \frac{N_1 N_2}{N^{L(b)-V(f)+1}}.
\]

\(^8\) Note that the initial and final links must have positive duration also in case they lead to stretches of a parallel encounter. Otherwise all parallel stretches of that encounter would enter or leave the cavity through the same lead. This is impossible, since the trajectories cannot enter or leave the cavity several times.
To obtain the overall conductance, we must sum over all families. If we let \( N(\vec{v}) \) denote the number of families associated to \( \vec{v} \), equations (6) and (23) imply

\[
\langle \text{tr}(tt^\dagger) \rangle = \frac{N_1 N_2}{N} \left( 1 + \sum_{\vec{v}} (-1)^{V(\vec{v})} \frac{1}{N^{L(\vec{v})-V(\vec{v})}} N(\vec{v}) \right) = \frac{N_1 N_2}{N} \left( 1 + \sum_{m=1}^{\infty} \frac{c_m}{N^m} \right).
\]  

(24)

Here, each coefficient \( c_m \) is determined by the families with given \( m = L - V \),

\[
c_m = \sum_{\vec{v}} (-1)^{V(\vec{v})} N(\vec{v}).
\]  

(25)

Our task has thus been reduced to counting families of trajectory pairs and evaluating \( c_m \).

For the lowest orders \( m \), the counting is easily done. We have already seen that for time-reversal invariant systems the next-to-leading contribution to conductance originates from the Richter/Sieber family of trajectory pairs differing in one 2-encounter, see figure 1 or figure 2(a). This family has \( L = 2, V = 1 \), thus \( m = 1 \). Hence, it gives rise to a coefficient \( c_1 = -1 \) in the orthogonal case. In the unitary case the absence of such trajectory pairs implies \( c_1 = 0 \).

The following coefficient \( c_2 \) is determined by pairs of trajectories which either differ in two 2-encounters (i.e., \( L = 4, V = 2 \)) and thus contribute with a positive sign, or differ in one 3-encounter (i.e., \( L = 3, V = 1 \)) and contribute with a negative sign. The relevant families are sketched in figure 2. In the unitary case, there is only one family of the first type (figure 2(b)), and one family of the second type (figure 2(g)). Both contributions mutually cancel, i.e., \( c_2 = 0 \).

In the orthogonal case, we must allow for encounters with mutually time-reversed stretches. We then find five families contributing with a positive sign (figures 2(b)–(f)) and four families contributing with a negative sign (figures 2(g)–(j)). All contributions sum up to \( c_2 = 1 \).

The higher coefficients \( c_m \) require more involved combinatorial methods to which we now turn.

2.4. Combinatorics

In [9], we found a systematic way for summing contributions of families of trajectory pairs that differ in arbitrarily many encounters. In particular, we obtained a recursion for the numbers \( N(\vec{v}) \). Since the treatment of [9] was geared towards spectral statistics of closed systems, it was formulated for pairs of periodic orbits rather than pairs of open trajectories. It can, however, be easily adapted to open trajectories. We just have to turn trajectory pairs into orbit pairs by joining the initial and final points as in figure 2, or cut orbits in order to form trajectories.

The topology of orbit pairs \((A, B)\) was described by ‘structures’. To define these structures, we numbered the encounter stretches of \( A \) in their order of traversal, starting from an arbitrary reference stretch; the links of \( A \) were numbered as well, with the first link preceding the first encounter stretch. Then, each structure is characterized by (i) a vector \( \vec{v} \) as above, (ii) a way of distributing the numbered stretches among the encounters, (iii) fixing the mutual orientation of stretches inside each encounter, and (iv) shifting connections inside the encounters to form a partner orbit \( B \).

With this definition, each family of trajectory pairs corresponds to one structure of orbit pairs. We only have to glue together the initial and final points of the trajectories as in figure 2,
and keep the first stretch of $\alpha$ (the one following the initial point) as the first stretch of $A$. Thus, each of the pictures in figure 2 represents one structure of orbit pairs, and the numbers of orbit pair structures and of trajectory pair families both equal $\mathcal{N}(\vec{v})$.

To illustrate this relation, we consider the two families of trajectory pairs depicted in figures 2(e) and (f). If we join the initial and final points for any of these families, we obtain orbit pairs of one and the same topology. But still the resulting structures are different, because in figure 2(e) the first encounter stretch (the one following the initial point of the trajectory) precedes a stretch of the same encounter, i.e., the two antiparallel encounters respectively involve the stretches $(1, 2)$ and $(3, 4)$. A different choice of the initial stretch as in figure 2(f) means that the two encounters comprise the stretches $(1, 4)$ and $(2, 3)$. Indeed, structures of orbit pairs and families of trajectory pairs are one-to-one.

For later convenience, we refer to the structure involving one antiparallel encounter (figure 2(a)) as the *Sieber/Richter structure* (it was proposed by these authors in [8], see also [26, 27]), to the structure involving two parallel 2-encounters (figure 2(b)) as *ppi*, to the two structures involving a parallel and an antiparallel 2-encounter (figures 2(c) and (d)) as *api*, and to the structures involving two antiparallel encounters (figures 2(e) and (f)) as *aas* [18]. The structure involving one parallel 3-encounter (figure 2(g)) will be called *pc*, and the three structures of the type $\rightarrow\rightarrow\nleftarrow$ (figures 2(h)–(j)) *ac*.

To formulate our recursion for $\mathcal{N}(\vec{v})$, we now denote by $\mathcal{N}(\vec{v}, l)$ the number of structures of orbit pairs (or families of trajectory pairs) related to $\vec{v}$, for which the first stretch belongs to an $l$-encounter. We had established the identity

$$\mathcal{N}(\vec{v}, l) = \frac{l v_l}{L(\vec{v})} \mathcal{N}(\vec{v})$$

which has the following intuitive interpretation (not to be confused with the proof in [9]): the probability that the first stretch forms part of an $l$-encounter is given by the overall number $l v_l$ of stretches belonging to $l$-encounters, divided by the overall number $L(\vec{v})$ of stretches in all encounters; to obtain $\mathcal{N}(\vec{v}, l)$, we have to multiply $\mathcal{N}(\vec{v})$ with that probability. Incidentally, the definition of $\mathcal{N}(\vec{v}, l)$ implies

$$\sum_{l \geq 2} \mathcal{N}(\vec{v}, l) = \mathcal{N}(\vec{v}).$$

The relevant recursion relation reads [9]

$$\mathcal{N}(\vec{v}, 2) - \sum_{k \geq 2} \mathcal{N}(\vec{v}^{[k,2\rightarrow k+1]}, k + 1) = \left(\frac{2}{\beta} - 1\right) \mathcal{N}(\vec{v}^{[2\rightarrow 1]}),$$

with $\beta = 2$ and 1 respectively referring to the unitary and orthogonal case. The symbol $\vec{v}^{[k,2\rightarrow k+1]}$ denotes the vector obtained from $\vec{v}$ if we reduce $v_k$ and $v_2$ by one, and increase $v_{k+1}$ by one; likewise $\vec{v}^{[2\rightarrow 1]}$ is obtained from $\vec{v}$ if we reduce $v_2$ by one. In general, the list on the left-hand side of the arrow contains the sizes of ‘removed’ encounters, whereas the right-hand side contains the sizes of ‘added’ encounters.

---

9 Equation (27) is a special case of equations (42) and (54) in [9], with $l = 2$. To understand the equivalence, note that in [9] we allowed for ‘vectors’ $\vec{v}$ including a non-vanishing component $v_i$, which may formally be interpreted as a number of ‘1-encounters’. We moreover showed that $\mathcal{N}(\vec{v}', 1) = \mathcal{N}(\vec{v}^{[1\rightarrow 1]})$ (see the paragraph preceding equation (58) of [9]). Applying this relation to $\vec{v}' = \vec{v}^{[2\rightarrow 1]}$, one sees that the $\mathcal{N}(\vec{v}^{[2\rightarrow 1]}, 1)$ appearing in equation (54) of [9] coincides with $\mathcal{N}(\vec{v}^{[2\rightarrow 1]})$. New Journal of Physics 9 (2007) 12 (http://www.njp.org/)
To turn (27) into a recursion for the coefficients $c_m$, we multiply with $(-1)^{V(\vec{v})}$ and sum over all $\vec{v}$ with fixed $m = M(\vec{v}) \equiv L(\vec{v}) - V(\vec{v}),$

$$M(\vec{v}) = m \sum_{\vec{v}} (-1)^{V(\vec{v})} \mathcal{N}(\vec{v}, 2) - \sum_{k \geq 2} \sum_{\vec{v}} (-1)^{V(\vec{v})} \mathcal{N}(\vec{v}^{|k,2\rightarrow k+1|}, k + 1)$$

$$= \left( \frac{2}{\beta} - 1 \right) \sum_{\vec{v}} (-1)^{V(\vec{v})} \mathcal{N}(\vec{v}^{|2\rightarrow 1|}).$$

In each of the foregoing sums we can replace the summation variable $\vec{v}$ by $\vec{v}' \equiv \vec{v}^{[k,2\rightarrow k+1]}$ or $\vec{v}'' \equiv \vec{v}^{|2\rightarrow 1|}$. Given that trajectory pairs associated with $\vec{v}'$ have one encounter and one encounter stretch less than those of $\vec{v}$, we then have to sum over $\vec{v}'$ with $M(\vec{v}') = L(\vec{v}') - V(\vec{v}') = (L(\vec{v}) - 1) - (V(\vec{v}) - 1) = m$. By definition, we should restrict ourselves to $\vec{v}'$ with $v_{k+1}^* > 0$; however, that restriction may be dropped since $\vec{v}'$ with $v_{k+1}^* = 0$ have $\mathcal{N}(\vec{v}', k + 1) = 0$. In contrast, trajectory pairs associated to $\vec{v}''$ have one encounter and two stretches less than those associated to $\vec{v}$. The pertinent sum runs over $\vec{v}''$ with $M(\vec{v}'') = L(\vec{v}'') - V(\vec{v}'') = (L(\vec{v}) - 2) - (V(\vec{v}) - 1) = m - 1$. Using $(-1)^{V(\vec{v})} = -(1)^{V(\vec{v})} = -(1)^{V(\vec{v}'')} = -(1)^{V(\vec{v}'')}$ we can rewrite (28) as

$$M(\vec{v}) = m \sum_{\vec{v}} (-1)^{V(\vec{v})} \mathcal{N}(\vec{v}, 2) + \sum_{k \geq 2} \sum_{\vec{v}} (-1)^{V(\vec{v})} \mathcal{N}(\vec{v}', k + 1) = - \left( \frac{2}{\beta} - 1 \right) \sum_{\vec{v}} (-1)^{V(\vec{v})} \mathcal{N}(\vec{v}').$$

(29)

The left-hand side now boils down to $\sum_{\vec{v}} M(\vec{v}) (-1)^{V(\vec{v})} \sum_{k \geq 1} \mathcal{N}(\vec{v}, k + 1) = \sum_{\vec{v}} M(\vec{v}) (-1)^{V(\vec{v})} \mathcal{N}(\vec{v}) = c_m$, while the right-hand side reads $- \left( \frac{2}{\beta} - 1 \right) c_{m - 1}$. We thus end up with a recursion for $c_m$, $m \geq 2$,

$$c_m = - \left( \frac{2}{\beta} - 1 \right) c_{m - 1} \begin{cases} 0 & \text{unitary case}, \\ -c_{m - 1} & \text{orthogonal case}. \end{cases}$$

(30)

For the unitary case we conclude that all off-diagonal contributions to the average conductance mutually cancel; the remaining diagonal term, $\frac{N_1 N_2}{N}$, reproduces the random-matrix result. For the orthogonal case an initial condition is provided by the coefficient $c_1 = -1$, originating from Richter/Sieber pairs; hence $c_m = (-1)^m$. The anticipated mean conductance (2) is recovered through (24) as the geometric series

$$\langle \text{tr}(u^T) \rangle = \frac{N_1 N_2}{N} \left( 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{N^m} \right) = \frac{N_1 N_2}{N + 1}.$$ 

(31)

We have thus shown for both symmetry classes that the energy-averaged conductance of individual chaotic cavities takes the universal form predicted by RMT as an ensemble average.
3. Conductance variance

Experiments with chaotic cavities also reveal universal conductance fluctuations. In particular, the conductance variance agrees with the random-matrix prediction [1]

\[
\langle G(E)^2 \rangle - \langle G(E) \rangle^2 = \langle (\text{tr}(tt^\dagger))^2 \rangle - \langle \text{tr}(tt^\dagger) \rangle^2 = \begin{cases} 
\frac{N_1^2 N_2^2}{N^2(N^2-1)} & \text{unitary case,} \\
2N_1N_2(N_1+1)(N_2+1) & \text{orthogonal case.}
\end{cases}
\] (32)

Once more, the semiclassical limit offers itself for an explanation of such universality. With the van Vleck approximation for the transition amplitudes (1), the mean squared conductance turns into a sum over quadruplets of trajectories,

\[
\langle (\text{tr}(tt^\dagger))^2 \rangle = \frac{1}{T_H} \left( \sum_{a_1} \sum_{c_1} \sum_{a_2} \sum_{c_2} A_\alpha A_\beta^* A_\gamma A_\delta^* e^{i(S_\alpha - S_\beta + S_\gamma - S_\delta)/\hbar} \right). \tag{33}
\]

Here \(a_1, c_1 = 1 \ldots N_1\) and \(a_2, c_2 = 1 \ldots N_2\) are channel indices. The trajectories \(\alpha\) and \(\beta\) lead from the same ingoing channel \(a_1\) to the same outgoing channel \(a_2\), whereas \(\gamma\) and \(\delta\) connect the ingoing channel \(c_1\) to the outgoing channel \(c_2\). We can expect systematic contributions to the quadruple sum over trajectories only from quadruplets with action differences \(\Delta S \equiv S_\alpha - S_\beta + S_\gamma - S_\delta\) of the order of \(\hbar\).

3.1. Diagonal contributions

The leading contribution to (33) originates from ‘diagonal’ quadruplets with pairwise coinciding trajectories either as \((\alpha = \beta, \gamma = \delta)\), or as \((\alpha = \delta, \beta = \gamma)\); both scenarios imply vanishing action differences. The first scenario \(\alpha = \beta, \gamma = \delta\) obviously leads to \(\beta\) connecting the same channels as \(\alpha\), and \(\delta\) connecting the same channels as \(\gamma\), as required in (33); this holds regardless of the channel indices \(a_1, c_1, a_2, c_2\). The second scenario \((\alpha = \delta, \beta = \gamma)\) brings about admissible quadruplets only if all trajectories connect the same channels, i.e., both the ingoing channels \(a_1 = c_1\) and the outgoing channels \(a_2 = c_2\) coincide. The contribution of these diagonal quadruplets to (33) may thus be written as the following double sum over trajectories \(\alpha\) and \(\gamma\)

\[
\langle (\text{tr}(tt^\dagger))^2 \rangle_{\text{diag}} = \frac{1}{T_H} \left( \sum_{a_1} \sum_{a_2} \sum_{c_1} \sum_{c_2} |A_\alpha|^2 |A_\gamma|^2 + \sum_{a_1} \sum_{a_2} \sum_{c_1} \sum_{c_2} |A_\alpha|^2 |A_\gamma|^2 \right). \tag{34}
\]

The sum over channels just yields the number of possible channel combinations as a factor, namely \(N_1^2 N_2^2\) for the first scenario and \(N_1 N_2\) for the second one. Doing the sums over \(\alpha\) and \(\gamma\) with the Richter/Sieber rule (5) we get

\[
\langle (\text{tr}(tt^\dagger))^2 \rangle_{\text{diag}} = \frac{N_1^2 N_2^2 + N_1 N_2}{N^2}. \tag{35}
\]
Figure 4. (a) Schematic graph of \textit{d-quadruplets}, with the hatched area a ‘black box’ containing any number of encounters; one of the dashed partner trajectories shares initial and final points with $\alpha$; the second partner trajectory similarly related to $\gamma$. (b)–(h) \textit{d-quadruplets} responsible for the leading-order contribution to the conductance variance. The diagrams (b), (f)–(h) containing encounters with antiparallel stretches exist only in the orthogonal case. A diagram may have a ‘twin’ obtained by reflection in a horizontal line; the number of symmetric versions of each diagram is indicated by a multiplier underneath. (i) Schematic graph of \textit{x-quadruplets}, encounters suppressed: one of the partners shares initial link with $\gamma$ and final link with $\alpha$, the second one connects initial link of $\alpha$ with final link of $\gamma$. (j) An \textit{x-quadruplet} involving one 2-encounter.

The larger one of the two summands, $\frac{N_2^2}{N_2}$, is cancelled by the squared diagonal contribution to the mean conductance.

In recent works on Ehrenfest-time corrections \cite{11, 12} the diagonal approximation was extended to include trajectories which slightly differ close to the openings. The relation of the methods used in these papers to our present approach is not fully settled yet; further investigation about this relation is desirable.

3.2. \textit{Trajectory quadruplets differing in encounters}

Off-diagonal contributions arise from quadruplets of trajectories differing in encounters; see figure 4 for examples. Each trajectory pair ($\alpha, \gamma$) typically contains a huge number of encounters, where stretches of $\alpha$ and/or $\gamma$ come close to each other (up to time reversal). Partner trajectories $\beta$, $\delta$ can be obtained by switching connections within some encounters. Together, $\beta$ and $\delta$ go through the same links as $\alpha$ and $\gamma$, and traverse each $l$-encounter exactly $l$ times, just like the pair ($\alpha, \gamma$). Consequently, the cumulative action of ($\beta, \delta$) is close to the one of ($\alpha, \gamma$), with a small action difference $\Delta S = (S_\alpha + S_\gamma) - (S_\beta + S_\delta)$ originating from the intra-encounter reconnections.
Different quadruplet families are distinguished by the number of $l$-encounters, the mutual orientation of encounter stretches, their distribution among $\alpha$ and $\gamma$, and the reconnections leading to $\beta$ and $\delta$. Similar to the diagonal quadruplets, some families of quadruplets involve one partner trajectory whose initial and final links practically coincide with those of $\alpha$ and the other one whose initial and final links coincide with those of $\gamma$. When these families are depicted schematically with encounters suppressed they all look the same and in fact like diagonal quadruplets (see figure 4(a)), for which reason we shall refer to them as ‘$d$-families’; examples are depicted in figures 4(b)–(h). In analogy to the diagonal quadruplets, $d$-quadruplets contribute with altogether $N_1^2 N_2^2 + N_1 N_2$ channel combinations. Of these, $N_1^2 N_2^2$ arise when $\alpha$ and $\beta$ start and end alike since then the four channel indices involved are unrestricted; when $\alpha$ and $\beta$ start and end alike, $N_1 N_2$ combinations arise since then the four channel indices involved are unrestricted; when $\alpha = v$ and $l = 1$ may be chosen as $\beta$, a form $\gamma$ may be chosen as $\delta$, the integration range $\nu = (v_2, v_3, v_4, \ldots)$. Again, $\nu$ determines the total number of encounters $V(\nu)$ and the number of encounter stretches $L(\nu)$. The overall number of links is now given by $L(\nu) + 2$, since there is one link preceding each of the $L(\nu)$ encounter stretches, and the two final links of $\alpha$ and $\gamma$ which do not precede any encounter stretch. Similarly as for trajectory pairs, we can determine a density $w(s, u)$ of stable and unstable separations; this density will be normalized such that integration over all $s, u$ belonging to an interval $(\Delta s, \Delta s + d\Delta s)$ of action differences $\Delta s$ yields the number of pairs $\beta, \delta$ differing from given $\alpha, \gamma$ such that the quadruplet $(\alpha, \beta, \gamma, \delta)$ belongs to a given family and the action difference is inside that interval. Using the same arguments as in subsection 2.3, one finds $w(s, u)$ as an integral over $\{\Omega^{L-V} \prod_{\alpha=1}^{V} \sigma_{\text{enc}}(s, u)\}^{-1}$, with the integration running over the durations of all links, except the final links of $\alpha$ and $\gamma$. The integration range must be restricted such that all links (including the final ones) have positive durations. To evaluate the contribution of one family to the quadruplet sum in (33), we may now replace the summation over $\beta$ and $\delta$ by integration over $w(s, u)$,

$$\langle (\text{tr}(\Pi^+))^2 \rangle_{\text{fam}} = \frac{1}{T_H^4} \int_{\Omega^{L-V}} \int d^{L-V} s \, d^{L-V} u \sum_{\alpha \rightarrow \gamma} \sum_{\sigma_1 \rightarrow \sigma_2} |A_{\sigma_1}|^2 |A_{\gamma}|^2 w(s, u) e^{i \sum_{\alpha} x_{\alpha} \mu_{\alpha}/\hbar}. \quad (36)$$

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The sums over \( \alpha, \gamma \) can be performed using the Richter/Sieber rule to ultimately get further integrals over the durations of the final links of \( \alpha \) and \( \gamma \), with an integrand involving the survival probability \( \exp\left(-\frac{N}{\hbar} \left( \sum_{i=1}^{L+2} t_i + \sum_{\sigma=1}^{V} t_{\text{enc}}^\sigma (s, u) \right) \right) \). We thus meet with link and encounter integrals of the same type as for the mean conductance. All powers of \( T_1 \) mutually again cancel, and we are left with a factor \( \frac{1}{N} \) from each of the \( L(\vec{v}) + 2 \) links and a factor \( -N \) from each of the \( V(\vec{v}) \) encounters which altogether give \( \frac{(-1)^m}{N_{L(\vec{v})-V(\vec{v})}} \). The summation over \( a_1, c_1, a_2, c_2 \) yields the number of channel combinations mentioned.

If we denote by \( N_d(\vec{v}), N_x(\vec{v}) \) the numbers of \( d- \) and \( x- \)families associated to \( \vec{v} \), the sum over all families with fixed \( L(\vec{v}) - V(\vec{v}) = m \) involves the subsums

\[
\begin{align*}
  d_m &= \sum_{\vec{v}} (-1)^{V(\vec{v})} N_d(\vec{v}) \\
  x_m &= \sum_{\vec{v}} (-1)^{V(\vec{v})} N_x(\vec{v})
\end{align*}
\]

which allow to write the yield of all families as

\[
\langle (\text{tr}(tt^\dagger))^2 \rangle = (N_1^2 N_2^2 + N_1 N_2) \left( \frac{1}{N^2} + \sum_{m=1}^{\infty} \frac{d_m}{N^{m+2}} \right) + N N_1 N_2 \sum_{m=1}^{\infty} \frac{x_m}{N^{m+2}},
\]

with \( D \) arising from \( d- \)families (including the diagonal contribution) and \( X \) from \( x- \)families.

The coefficients \( d_m, x_m \) are obtained by counting families of quadruplets. That counting is an elementary task for small \( m \). The coefficient \( d_1 \) accounts for families of \( d- \)quadruplets differing in one 2-encounter \((L = 2, V = 1, m = 1)\). In the unitary case there are no such families, i.e., \( d_1 = 0 \). In the orthogonal case, we must consider quadruplets with one partner trajectory differing from \( \alpha \) in a 2-encounter, and one partner trajectory identical to \( \gamma \), see figure 4(b); the quadruplet thus contains one Richter/Sieber pair and one diagonal pair. A similar family of quadruplets involves one partner trajectory identical to \( \alpha \), and one partner trajectory differing from \( \gamma \) in a 2-encounter. We thus have \( d_1 = -2 \).

The following coefficient \( d_2 \) is determined by \( d- \)quadruplets differing in two 2-encounters or in one 3-encounter, the latter quadruplets contributing with a negative sign. Some of these quadruplets fall into two pairs contributing to the average conductance. Quadruplets consisting of one diagonal pair and one pair contributing to the coefficient \( c_2 \) of the average conductance (see figure 2) yield a contribution \( 2c_2 \) to \( d_2 \) (i.e., 0 in the unitary case and 2 in the orthogonal case); the factor 2 arises because either \( \alpha \) or \( \gamma \) may belong to the diagonal pair. In the orthogonal case, there is one further family of quadruplets consisting of two Richter/Sieber pairs. Finally, we must reckon with quadruplets that do not fall into two pairs contributing to the mean conductance, as depicted in figures 4(c)–(e), for the unitary case. Two further families are obtained by ‘reflection’, i.e., interchanging \( \alpha \) and \( \gamma \) in figures 4(d) and (e). Taking into account the negative sign for figure 4(e) and its reflected version, the respective contributions sum up to 1. In the orthogonal case, the additional families in figures 4(f)–(h) and the reflected versions of figures 4(g) and (h) yield a further summand 1. Altogether, we thus find \( d_2 = 1 \) in the unitary case and \( d_2 = 2 + 1 + 1 + 1 = 5 \) in the orthogonal case.
The most important family of $x$-quadruplets, see figure 4(j), involves a parallel encounter between one stretch of $\alpha$ and one stretch of $\gamma$. This family, discovered in [3] for quantum graphs, gives rise to a coefficient $x_1 = -1$ for systems with or without time-reversal invariance.

With the coefficients $d_1$, $d_2$, $x_1$, the conductance variance (38) can be evaluated up to corrections of order $O(\frac{1}{N})$. The result

$$
\langle (\text{tr}(tt^+)^2) \rangle - \langle \text{tr}(tt^+) \rangle^2 = \begin{cases} 
\frac{N_1^2 N_2^2}{N^4} + O\left(\frac{1}{N}\right) & \text{unitary case,} \\
\frac{2N_1^2 N_2^2}{N^4} + O\left(\frac{1}{N}\right) & \text{orthogonal case,}
\end{cases}
$$

(39)

coincides with the random-matrix prediction (32). We note that equation (39) could ultimately be attributed only to the quadruplets shown in figures 4(c)–(h), since all other contributions mutually cancel. (In particular, the contributions proportional to $N_1^2 N_2^2$ from all $d$-quadruplets that consist of two pairs contributing to the conductance are cancelled by the squared average conductance. The term proportional to $N_1 N_2$ in the diagonal approximation is compensated by the contribution of $x$-quadruplets as in figure 4(j).)

To go beyond equation (39) (and to show that no terms were missed in equation (39)), we must systematically count families of $d$- and $x$-quadruplets with arbitrarily many encounters. Similar to the case of conductance this can be done by establishing relations between families of trajectory quadruplets and structures of periodic orbit pairs. For details see appendix A; the results differ for the two universality classes.

In the unitary case we find

$$
d_m = \begin{cases} 
0 & \text{if } m \text{ odd,} \\
1 & \text{if } m \text{ even,}
\end{cases} \quad x_m = \begin{cases} 
-1 & \text{if } m \text{ odd,} \\
0 & \text{if } m \text{ even.}
\end{cases}
$$

(40)

The total contributions of all $d$- and $x$-families (per channel combination) now read

$$
D = \frac{1}{N^2} + \sum_{m=1}^{\infty} \frac{d_m}{N^{m+2}} = \frac{1}{N^2 - 1}, \quad X = \sum_{m=1}^{\infty} \frac{x_m}{N^{m+2}} = -\frac{1}{N(N^2 - 1)}.
$$

(41)

The resulting conductance variance

$$
\langle (\text{tr}(tt^+)^2) \rangle - \langle \text{tr}(tt^+) \rangle^2 = \frac{N_1^2 N_2^2}{N^2(N^2 - 1)}
$$

(42)

agrees with the random-matrix prediction (32).

In the orthogonal case we have

$$
d_m = (-1)^m \frac{3^m + 1}{2}, \quad x_m = (-1)^m \frac{3^m - 1}{2}.
$$

(43)

$^{10}$In counting orders we assume that all numbers of channels are of the same order of magnitude.
The contributions of \(d\)- and \(x\)-families per channel combination now read

\[
D = \frac{1}{N^2} + \sum_{m=1}^{\infty} \frac{d_m}{N^{m+2}} = \frac{N + 2}{N(N + 1)(N + 3)}, \quad X = \sum_{m=1}^{\infty} \frac{x_m}{N^{m+2}} = -\frac{1}{N(N + 1)(N + 3)}
\]

and determine the variance in search as

\[
\langle (\text{tr}(tt^\dagger))^2 \rangle - \langle \text{tr}(tt^\dagger) \rangle^2 = \frac{2N_1N_2(N_1 + 1)(N_2 + 1)}{N(N + 1)^2(N + 3)},
\]

again in agreement with (32). Thus, we have once more verified the universal behaviour of individual chaotic cavities.

### 4. Shot noise

Our reasoning can be extended to a huge class of observables which are quartic in the transmission amplitudes and thus determined by \(d\)- and \(x\)-quadruplets as well. For a first example, we consider shot noise: due to the discreteness of the elementary charge, the current flowing through a mesoscopic cavity fluctuates in time as \(I(t) = I + \delta I(t)\) where \(I\) denotes the average current. These current fluctuations, the so-called shot noise, remain in place even at zero temperature. They are usually characterized through the power [1]

\[
P = 4 \int_0^\infty \delta I(t_0) \delta I(t_0 + t) \, dt,
\]

where the overline indicates an average over the reference time \(t_0\).

Using our semiclassical techniques, we proceed to showing that for chaotic cavities, the energy-averaged power of shot noise takes a universal form. Again, our treatment applies to individual cavities and yields an expansion to all orders in the inverse number of channels. That expansion turns out convergent and summable to a simple expression which subsequently to our prediction was checked to agree with RMT by Savin and Sommers [30]. Previously, higher orders in \(1/N\) were not known from RMT, apart from the special case \(N = 1\).

Following Büttiker [14], we express the power of shot noise through the transition matrices \(t\)

\[
\langle P \rangle = \langle \text{tr}(tt^\dagger) - \text{tr}(tt^\dagger tt^\dagger) \rangle;
\]

here, \(P\) is averaged over the energy and measured in units \(\frac{2e^2|V|}{\pi\hbar}\) depending on the voltage \(V\). While the average conductance \(\langle \text{tr}(tt^\dagger) \rangle\) was already evaluated in section 2, the quartic term turns into a quadruple sum over trajectories similar to the conductance variance

\[
\langle (\text{tr}(tt^\dagger tt^\dagger)) \rangle = \left( \sum_{a_1, c_1} t_{a_1 c_1}^* t_{c_1 a_1}^* t_{c_1 c_2}^* t_{a_1 c_2} \right) = \frac{1}{T^2} \sum_{a_1, c_1} \sum_{a_2, c_2} A_{a_1} A_{a_2}^* A_{c_1} A_{c_2}^* e^{i(S_{a_1} - S_{a_2} + S_{c_1} - S_{c_2})/\hbar};
\]

\[11\text{This definition, as well as the treatment of three-lead correlation in the following section, follows the conventions of [1, 28], and differs by a factor 2 from [29].}

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the trajectories $\alpha, \beta, \gamma, \delta$ must now connect the ingoing channels $a_1, c_1$ to the outgoing channels $a_2, c_2$ as indicated in the summation prescription.

As a consequence, the possible channel combinations for $d$- and $x$-families of quadruplets are changed relative to the conductance variance. In the present case, $d$-quadruplets, with one partner trajectory coinciding at its beginning and end with $\alpha$, and the other partner trajectory doing the same with $\gamma$, contribute only if either the ingoing or the outgoing channels coincide. If the ingoing channels coincide, the partner trajectory connecting the same points as $\alpha$ is of the type $a_1 = c_1 \rightarrow a_2$ and may be taken as $\beta$, whereas the trajectory connecting the same points as $\gamma$ has the form $a_1 = c_1 \rightarrow c_2$ and may be chosen as $\delta$. If the outgoing channels coincide, similar arguments apply, with $\beta$ and $\delta$ interchanged. Thus, $d$-quadruplets contribute only for $N_1 N_2^2 + N_2^2 N_2 = NN_1 N_2$ channel combinations. In this sense, they take the role played by $x$-quadruplets in case of the conductance variance.

In turn, $x$-quadruplets now contribute for all channel combinations. Moreover, if both the ingoing and the outgoing channels coincide, either of the two partner trajectories may be chosen as $\beta$ or $\delta$, meaning that the corresponding channel combinations have to be counted for a second time. Thus, $x$-quadruplets now contribute for altogether $N_1^2 N_2^2 + N_1 N_2$ channel combinations, like $d$-quadruplets in case of the conductance variance.

We can simply interchange the multiplicity factors in our formula for $\langle (\text{tr} (tt^\dagger)) \rangle$, equation (38), to get

$$\langle \text{tr} (tt^\dagger) \rangle = NN_1 N_2 D + (N_1^2 N_2^2 + N_1 N_2) X$$

and thus

$$\langle P \rangle = \begin{cases} N_1^2 N_2^2 / N (N^2 - 1) & \text{unitary case,} \\ N_1 (N_1 + 1) N_2 (N_2 + 1) / N (N + 1) (N + 3) & \text{orthogonal case.} \end{cases}$$

Equation (50) extends the known random-matrix result [1],

$$\langle P \rangle = \begin{cases} N_1^2 N_2^2 / N^3 + O \left( \frac{1}{N} \right) & \text{unitary case,} \\ N_1^2 N_2^2 / N^3 + N_1 N_2 (N_1 - N_2)^2 / N^4 + O \left( \frac{1}{N} \right) & \text{orthogonal case.} \end{cases}$$

to all orders in $1/N$, for individual chaotic cavities. We can, moreover, give an intuitive interpretation for the terms in (51). The diagonal contributions to $\langle \text{tr} (tt^\dagger) \rangle$ and $\langle \text{tr} (tt^\dagger tt^\dagger) \rangle$ both read $N_1 N_2 / N$ and therefore mutually cancel. The leading contribution, $-N_1 N_2 / N^2$, arises from $d$-quadruplets differing in a single 2-encounter (see figure 4(j)). In the unitary case, there are no terms of order 1, since all related families require time reversal invariance (TRI). In the orthogonal case, Richter/Sieber pairs yield a contribution $-N_1 N_2 / N^2$ to $\langle \text{tr} (tt^\dagger) \rangle$, from which we have to subtract two contributions to $\langle \text{tr} (tt^\dagger tt^\dagger) \rangle$, the term $-2 N_1 N_2 / N^3$ accounting for $d$-quadruplets differing in a single antiparallel 2-encounter (see figure 4(b)), and a term $4 N_1^2 N_2^2 / N^4$ arising from $x$-quadruplets contributing to $x_2 = 4$. The latter $x$-quadruplets may differ in two 2-encounters, as in figures 5(a).
Figure 5. Families of $x$-quadruplets with $m = L - V = 2$, contributing to the next-to-leading order of shot noise for time-reversal invariant systems.

Figure 6. A cavity with three leads. If a voltage $V$ is applied between lead 1 and leads 2 and 3, one observes currents $I^{(1\rightarrow 2)}$ and $I^{(1\rightarrow 3)}$. As explained in the text, correlations between these currents are again determined by families of $d$- and $x$-quadruplets of trajectories.

and (b), or in one 3-encounter, as in figure 5(c). From the examples in figure 5, further families are obtained by interchanging $\alpha$ and $\gamma$, interchanging the two leads (for figures 5(a) and (c)), or interchanging the pairs $(\alpha, \gamma)$ and $(\beta, \delta)$ (for figure 5(b)). Each of the figures 5(a)–(c) therefore represents altogether four families, whose contributions indeed sum up to $x_2 = 4(-1)^2 + 4(-1)^2 + 4(-1) = 4$. Together with the contributions mentioned before, they combine to (51).12

5. Current correlations in cavities with three leads

Another interesting experimental setting involves a chaotic cavity with three leads, respectively supporting $N_1$, $N_2$ and $N_3$ channels; see figure 6. The second and the third lead are kept at the same potential, and a voltage is applied between these leads and the first one. Consequently, currents $I^{(1\rightarrow 2)}$, $I^{(1\rightarrow 3)}$ flow from the first lead to the second and third one. We shall be interested in the fluctuations $\delta I^{(1\rightarrow 2)}$, $\delta I^{(1\rightarrow 3)}$ of these currents around the corresponding averages values, and study correlations between $\delta I^{(1\rightarrow 2)}$ and $\delta I^{(1\rightarrow 3)}$ [28, 29].

12In [12], trajectory quadruplets where the encounter directly touches the lead are shown to become relevant when the mean dwell time is of the order of the Ehrenfest time.
This setting is similar to the famous *Hanbury-Brown and Twiss experiment* [31] in quantum optics: there, light from some source (corresponding to the first lead) was detected by two photomultipliers (corresponding to the second and third lead). Similar work on Fermions began somewhat later [28, 29] but the precise form of the correlation function is as yet unknown.

Our semiclassical reasoning can easily be extended to fill this gap. The two currents depend on the matrices $t^{(1-2)}$, $t^{(1-3)}$ containing the transition amplitudes between channels of the first and the second and third lead; these matrices have the sizes $N_1 \times N_2$ and $N_1 \times N_3$. As shown in [28, 29], correlations between $\delta I^{(1-2)}$ and $\delta I^{(1-3)}$ are determined by the transition amplitudes as

\[
4 \int_0^\infty \delta I^{(1-2)}(t_0) \delta I^{(1-3)}(t_0 + t) \, dt = -\langle \text{tr}(t^{(1-2)} t^{(1-2)\dagger} t^{(1-3)} t^{(1-3)\dagger}) \rangle
\]  

(in units of $\frac{2e^3}{\hbar}$). Using the semiclassical expression for the transition amplitudes, we are again led to a sum over quadruplets of trajectories

\[
\langle \text{tr}(t^{(1-2)} t^{(1-2)\dagger} t^{(1-3)} t^{(1-3)\dagger}) \rangle = \left\langle \sum_{a_1,a_2} t_{a_1,a_2}^{(1-2)} t_{a_1,a_2}^{(1-2)\dagger} t_{a_1,a_2}^{(1-3)} t_{a_1,a_2}^{(1-3)\dagger} \right\rangle
\]

\[= \frac{1}{T_H} \left\langle \sum_{a_1,a_2} \sum_{a_3} A_\alpha A_\beta A_\gamma A_\delta e^{i(S_\alpha - S_\beta + S_\gamma - S_\delta)/\hbar} \right\rangle,
\]

with $a_1$, $c_1$, $a_2$, $c_2$ labelling channels of the first, second and third lead, as indicated by the subscript. The trajectories $\alpha$, $\beta$, $\gamma$, $\delta$ must connect these channels as $\alpha(a_1 \rightarrow a_2)$, $\beta(c_1 \rightarrow a_2)$, $\gamma(c_1 \rightarrow c_3)$, $\delta(a_1 \rightarrow c_3)$.

The contribution of each family of trajectory quadruplets can be evaluated similarly to the conductance variance or shot noise. Since a particle can leave the cavity through any of the three leads, the *escape rate* depends on the overall number of channels $N = N_1 + N_2 + N_3$. Again, integration brings about factors $\frac{1}{N}$ and $-N$ for each link and each encounter. Only the numbers of channel combinations are changed. *x-quadruplets* as in figure 6(a) contribute for all $N_1^2 N_2 N_3$ possible choices of $a_1, a_2, c_1, c_2$. For any of these choices, partner trajectories connecting the initial point of $\gamma(c_1 \rightarrow c_3)$ to the final point of $\alpha(a_1 \rightarrow a_2)$, and the initial point of $\alpha$ to the final point of $\gamma$ are of the form $c_1 \rightarrow a_2$ and $a_1 \rightarrow c_3$ and can be chosen as $\beta$ and $\delta$, respectively. In contrast, *d-quadruplets* as in figure 6(b) contribute only for the $N_1 N_2 N_3$ combinations with coinciding ingoing channels $a_1 = c_1$. For these combinations the partner trajectory coinciding at its ends with $\alpha$ is of the type $a_1 = c_1 \rightarrow a_2$ and can be taken as $\beta$ whereas the partner trajectory coinciding at its ends with $\gamma$ has the form $a_1 = c_1 \rightarrow c_3$ and can be chosen as $\delta$. With these numbers of channel combinations, the current correlations in a 3-lead geometry are obtained as

\[
\langle \text{tr}(t^{(1-2)} t^{(1-2)\dagger} t^{(1-3)} t^{(1-3)\dagger}) \rangle = N_1 N_2 N_3 D + N_1^2 N_2 N_3 X
\]

\[
= \begin{cases} 
\frac{N_1 N_2 N_3 (N_2 + N_3)}{N(N^2 - 1)} & \text{unitary case}, \\
\frac{N_1 N_2 N_3 (N_2 + N_3 + 2)}{N(N + 1)(N + 3)} & \text{orthogonal case}.
\end{cases}
\]
6. Ericson fluctuations

Another interesting quantum signature of chaos are so-called Ericson fluctuations, which have first been discovered experimentally in nuclear physics. In compound-nucleus reactions with strongly overlapping resonances, universal fluctuations in the correlation of two scattering cross sections at different energies have been observed. A first interpretation in terms of RMT was provided by Ericson [32] and further theoretical investigations have been reported in [33, 34].

Later on, the relation between classical chaotic scattering and Ericson fluctuations in single-particle quantum mechanics has been discussed [35]. Theoretical work shows that, e.g. the photoionization cross section of Rydberg atoms in external fields displays universal correlations once the underlying classical dynamics is chaotic [36].

Chaotic transport through a ballistic cavity displays Ericson fluctuations in the covariance of the conductance at two different energies,

\[ \langle C(E, \epsilon) \rangle = \left( G(E)G \left( E + \frac{\epsilon N}{2\pi \rho} \right) \right) - \langle G \rangle^2, \]

with \( G(E) = \text{tr}(tt^\dagger) \). Here, the difference between the two energies was made dimensionless by referral to the energy scale \( \frac{N}{2\pi \rho} \) proportional to the number of channels and to the mean level spacing. Similarly as for the conductance variance, the semiclassical approximation (1) for \( t(E), t(E') \) where \( E' = E + \frac{\epsilon N}{2\pi \rho} \) leads to a quadruple sum over trajectories,

\[ \langle \text{tr} tt^\dagger(E) \text{tr} tt^\dagger(E') \rangle = \left\langle \sum_{a_1, a_2, c_1, c_2} \sum_{\alpha, \beta, \gamma, \delta} A^*_\alpha A^*_\beta A^*\gamma A^*\delta e^{i(S_\alpha(E) - S_\beta(E) + S_\gamma(E') - S_\delta(E'))/\hbar} \right\rangle, \]

the only difference to (33) being that the trajectories \( \gamma \) and \( \delta \) have to be taken at energy \( E' = E + \frac{\epsilon N}{2\pi \rho} \). Using \( \frac{\partial S_\gamma}{\partial E} = T_\gamma, \frac{\partial S_\delta}{\partial E} = T_\delta \), and \( T_H = 2\pi \hbar \rho \), the phase factor can be cast into the form

\[ e^{i(S_\alpha(E) - S_\beta(E) + S_\gamma(E') - S_\delta(E'))/\hbar} \approx e^{i(S_\alpha(E) - S_\beta(E) + S_\gamma(E) - S_\delta(E))/\hbar} \times e^{i(N/T_H)\epsilon(T_\gamma - T_\delta)}, \]

i.e., the quadruple sum in (56) differs from (33) by an additional factor depending on the difference between the dwell times of \( \gamma \) and \( \delta \).

The latter difference may be written as a sum over links (with durations \( t_i \)) and encounters (with durations \( t^\sigma_{\text{enc}} \)),

\[ T_\gamma - T_\delta = \sum_{i=1}^{L+2} \eta_i t_i + \sum_{\sigma=1}^{V} \eta_\sigma t^\sigma_{\text{enc}}, \]

Here, the integer numbers \( \eta_i \) and \( \eta_\sigma \) characterize the individual links and encounters. (Note the distinction between links and encounters by Latin and Greek subscripts.) Each link occurs twice in the quadruplet, once in one of the original trajectories \( \alpha, \gamma \) and then in one of the partner trajectories \( \beta, \delta \). The number \( \eta_i = 0, \pm 1 \) gives the difference between the numbers of times the
As a consequence, our diagrammatic rules are modified to yield the factor $\alpha$ to connect the initial point of $\gamma$ and not by $\delta$, $i = 0$ if it is traversed either by both or none of the two trajectories, and $i = -1$ if it is traversed only by $\delta$. Similarly, $\eta_\sigma$ gives the difference between the numbers of traversals of the $\sigma$th encounter by $\gamma$ and $\delta$. For an $l$-encounter, $\eta_\sigma$ may range between $-l$ and $l$.

When evaluating the contribution of each family of quadruplets, equation (36), we simply have to add a phase factor $e^{i(N/T)\eta_\sigma\eta_i}$ for each link, and a phase factor $e^{i(N/T)\eta_\sigma\epsilon_{\text{enc}}}$ for each encounter. The link and encounter integrals are thus replaced by

$$
\int_0^\infty dt e^{-(N/T)\eta_i} e^{i(N/T)\eta_i\epsilon_{\text{enc}}} = \frac{T_H}{N(1 - i\eta_i)},
$$

and

$$
\int d^{l-1}s d^{l-1}u \frac{1}{\Omega^{l-1}T_{\text{enc}}(s,u)} e^{-(N/T)\eta_\sigma\epsilon_{\text{enc}}(s,u)} e^{i(N/T)\eta_\sigma\epsilon_{\text{enc}}(s,u)} e^{i\sum_j x_j\eta_\sigma H_j / \hbar} = -\frac{N(1 - i\eta_\sigma\epsilon)}{T_H^{l-1}}.
$$

As a consequence, our diagrammatic rules are modified to yield a factor $\frac{1}{N(1 - i\eta_\sigma\epsilon)}$ for each link and a factor $-N(1 - i\eta_\sigma\epsilon)$ for each encounter.

We must, however, be aware that the numbers $\eta_i, \eta_\sigma$ depend on which of the two partner trajectories is labelled as $\beta$ and which is labelled as $\delta$. Each family of $d$- or $x$-quadruplets hence comes with two different sets of numbers $\{\eta_i, \eta_\sigma\}$, depending on the combinations of channels considered. For each $d$-family we have to keep into account $N_1^2 N_2^2$ channel combinations with $\delta$ coinciding at its ends with $\gamma$; all these ‘$\delta\gamma$-type’ combinations give rise to the same $\{\eta_i, \eta_\sigma\}$ and to the same link and encounter factors. In addition, we must consider $N_1 N_2$ combinations of the ‘$\delta\alpha$-type’ with $\delta$ coinciding at its ends with $\alpha$, and a different set of $\{\eta_i, \eta_\sigma\}$.

For each $x$-family we would, in principle, have to distinguish between $N_1 N_2^2$ combinations with coinciding ingoing channels, and $\delta$ coinciding at its beginning with $\alpha$ and at its end with $\gamma$, and $N_2^2 N_2$ combinations with coinciding outgoing channels, and $\delta$ coinciding at its beginning with $\gamma$ and at its end with $\alpha$. Such caution is, however, unnecessary for reasons of symmetry. Each $x$-family is accompanied by another one which is topologically mirror-symmetrical, with left and right in figure 4 interchanged. In this family (which may be identical with the original one), initial points turn into final ones, and vice versa, implying that $\beta$ and $\delta$ are interchanged. Since both families are taken into account simultaneously, ‘mistakes’ like always choosing $\delta$ to connect the initial point of $\alpha$ to the final point of $\gamma$, are automatically compensated.

We can thus write the conductance covariance as

$$
C(\epsilon) = N_1^2 N_2^2 \sum_{m=0}^\infty \frac{d_{\text{m}}^{(\delta\gamma)}}{N_{m+2}} + N_1 N_2 \sum_{m=0}^\infty \frac{d_{\text{m}}^{(\delta\alpha)}}{N_{m+2}} + N_1 N_2 N \sum_{m=1}^\infty \frac{x_m}{N_{m+2}} - \langle G \rangle^2.
$$

Here the coefficients $x_m(\epsilon), d_{\text{m}}^{(\delta\gamma)}(\epsilon), d_{\text{m}}^{(\delta\alpha)}(\epsilon)$ are the overall contributions of the $x$-quadruplets and the two mentioned groups of $d$-quadruplets, with $m = L(\bar{v}) - V(\bar{u}) - 2$ (and thus $m = 0$ for the diagonal quadruplets) and the denominator $N^{-m-2}$ dropped. The squared averaged conductance is $\epsilon$-independent and is determined by equation (2).

The leading contribution to the conductance covariance corresponds to dropping in (61) all coefficients but $x_1, d_0^{(\delta\gamma)}, d_0^{(\delta\alpha)}, d_1^{(\delta\gamma)}, d_1^{(\delta\alpha)}$. For the conductance variance, we have seen that the contributions of $d$-quadruplets that fall into pairs ($\alpha, \beta$) and ($\gamma, \delta$) contributing to conductance cancel with the squared average conductance. The same remains valid here, since for these pairs
\( \gamma \) and \( \delta \) traverse the same links and encounters, and all \( \eta_i \) and \( \eta_\sigma \) vanish. Again, the contributions of diagonal quadruplets \( \alpha = \delta, \beta = \gamma \) and \( \chi \)-quadruplets as in figure 4(j) mutually compensate; compared to the variance both kinds of quadruplets receive the same additional factors due to one link with \( \eta_i = 1 \) (the trajectory \( \beta = \gamma \) and the lower left link in figure 4(j)) and one link with \( \eta_i = -1 \) (the trajectory \( \alpha = \delta \) and the upper left link in figure 4(j)).

Like for the conductance variance, the leading contribution to the covariance thus originates from the \( d \)-families in figures 4(c)–(h) which contain encounters between \( \alpha \) and \( \gamma \) and thus do not fall into pairs relevant for conductance. The family of figure 4(c) involves a link with \( \eta_i = -1 \) (the upper central one) and a link with \( \eta_i = 1 \) (immediately below), and thus yields

\[
\frac{N_1^2 N_2^2 (-N)^2}{N^6 (1+i\epsilon)(1-i\epsilon)} = \frac{N_1^2 N_2^2}{N^4 (1+i\epsilon)}.
\]

The same holds for the family in figure 4(f), which requires TRI. In contrast, the contributions of figures 4(d), (e), (g) and (h) remain independent of \( \epsilon \), since additional factors from links with \( \eta_i = -1 \) and encounters with \( \eta_\sigma = -1 \) mutually compensate; the same applies for the families represented by ‘\times 2’ in figure 4, with \( (\alpha, \beta) \) and \( (\gamma, \delta) \) interchanged and the signs of \( \eta_i \) and \( \eta_\sigma \) flipped. As for the variance of conductance, the contributions of figures 4(d), (e), (g) and (h) thus mutually cancel, both in the orthogonal case and in the unitary case (where only figures 4(d) and (e) may exist). Altogether, we now obtain

\[
\langle G(E)G(E+\epsilon N/2\pi \rho) \rangle = \left\{ \begin{array}{ll}
\frac{N_1^2 N_2^2}{N^4 (1+\epsilon^2)} + \mathcal{O}\left(\frac{1}{N}\right) & \text{unitary case,} \\
\frac{2N_1^2 N_2^2}{N^4 (1+\epsilon^2)} + \mathcal{O}\left(\frac{1}{N}\right) & \text{orthogonal case.}
\end{array} \right.
\]

The Lorentzian form of (62) confirms the random-matrix predictions of [32].

Higher orders in \( \frac{1}{N} \), not known from RMT, can be accessed by straightforward computer-assisted counting of families of quadruplets differing in a larger number of encounters, or in encounters with more stretches. To do so, we generated permutations which describe possible structures of orbit pairs (see [9]). We then ‘cut’ through these pairs as described in appendix A to obtain quadruplets of trajectories and determined the corresponding \( \eta_i, \eta_\sigma \). The final result can be written as

\[
\langle G(E)G(E+\epsilon N/2\pi \rho) \rangle = \left\{ \begin{array}{ll}
\frac{N_1^2 N_2^2}{N^4 (1+\epsilon^2)} \left\{ \frac{1}{(1+\epsilon^2)} + \frac{1+3\epsilon^2+21\epsilon^4+5\epsilon^6+2\epsilon^8}{N^2 (1+\epsilon^2)^5} \right\} + \mathcal{O}\left(\frac{1}{N}\right) & \text{unitary case;} \\
\frac{2N_1^2 N_2^2}{N^4 (1+\epsilon^2)} + \frac{2N_1 N_2}{N^3 (1+\epsilon^2)} - \frac{2N_1^2 N_2^2 (5+12\epsilon^2+3\epsilon^4)}{N^5 (1+\epsilon^2)^3} - \frac{4N_1 N_2}{N^4 (1+\epsilon^2)^3} + \frac{2N_1^2 N_2^2 (18+78\epsilon^2+177\epsilon^4+48\epsilon^6+11\epsilon^8)}{N^6 (1+\epsilon^2)^5} + \mathcal{O}\left(\frac{1}{N^3}\right) & \text{orthogonal case.}
\end{array} \right.
\]

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In the unitary case the $x$-type contribution cancels in all orders with the $d^{(\delta \alpha)}$-contribution; for that reason the overall result is proportional to $N_1^2 N_2^2$.

7. Quantum transport in the presence of a weak magnetic field

7.1. Changed diagrammatic rules

Our methods can also be applied to the case of a weak magnetic field, with a magnetic action of the order of $\hbar$. The necessary modifications were introduced in [37] for the spectral form factor; see also [19]. As in [37], we will obtain results interpolating between the orthogonal case (without a magnetic field) and the unitary case, where the magnetic field is strong enough to fully break TRI. We shall assume that the field is too weak to influence the classical motion, meaning that we have to deal with the same families of trajectory pairs as in the orthogonal case. However, the action of each trajectory is increased by an amount proportional to the integral of the vector potential $A$ along that trajectory, i.e., by

$$\Theta_\alpha = \int_a^c A(q) \cdot dq.$$  (64)

for the trajectory $\alpha$. When we evaluate the average conductance, the action difference inside each pair of trajectories $\alpha$ and $\beta$ is thus increased by $\Theta_\alpha - \Theta_\beta$. This additional term may be neglected for pairs of trajectories where all encounters are parallel. For these pairs, all links and stretches of $\beta$ are close in phase space to links and stretches of $\alpha$, and therefore receive almost the same magnetic action.

The situation is different for pairs where $\alpha$ and $\beta$ traverse links or stretches with opposite sense of motion. Since the magnetic action changes sign under time reversal, such orbit pairs have significant magnetic action differences $\Theta_\alpha - \Theta_\beta$. These differences can be split into contributions from the individual links and encounters. Let us first consider links. If $\beta$ contains the time-reversed of the $i$th link of $\alpha$, it must obtain the negative of the corresponding magnetic action $\Theta_i$. The difference $\Theta_\alpha - \Theta_\beta$ then receives a contribution $2\Theta_i$. Therefore we may write the contribution of each link as $2\mu_i/\Theta_\alpha$ with $\mu_i = 1$ if the link changes direction on $\beta$ and $\mu_i = 0$ otherwise.

Consider now the contribution of encounters. We assume that in the original trajectory $\alpha$ the encounter $\sigma$ had $\nu_\sigma$ stretches traversed in some direction (arbitrarily chosen as ‘positive’) meaning that the remaining $l_\sigma - \nu_\sigma$ stretches were traversed in the opposite, ‘negative’ direction; in the trajectory partner $\beta$ these numbers will generally change to $\nu'_\sigma, l_\sigma - \nu'_\sigma$ consequently. Denoting the magnetic action accumulated on a single stretch traversed in a positive direction by $\Theta_\sigma$ we see that the encounter $\sigma$ yields $2\mu_\sigma \Theta_\sigma$ to the magnetic action difference, with $\mu_\sigma = \nu_\sigma - \nu'_\sigma$. The overall magnetic contribution to the action difference now reads

$$\Theta_\alpha - \Theta_\beta = \sum_{i=1}^{L+1} 2\mu_i \Theta_i + \sum_{\sigma=1}^{V} 2\mu_\sigma \Theta_\sigma$$  (65)

and yields a phase factor

$$\prod_{i=1}^{L+1} e^{i2\mu_i \Theta_i/\hbar} \prod_{\sigma=1}^{V} e^{i2\mu_\sigma \Theta_\sigma/\hbar},$$  (66)
where we again distinguish between links and encounters only through Latin versus Greek subscripts.

To handle this additional phase factor, we show that for fully chaotic (in particular, ergodic and mixing) dynamics, the magnetic action may effectively be seen as a random variable \([37]\). For fully chaotic systems, any point on any trajectory can be located everywhere on the energy shell, with a uniform probability given by the Liouville measure. Moreover, phase-space points following each other after times larger than a certain classical ‘equilibration’ time \(t_{cl}\) can be seen as uncorrelated. We will therefore split each link or encounter stretch into pieces of duration \(t_{cl}\).

These pieces have different magnetic actions. Let us consider the probability density for these actions. Since positive and negative contributions to the magnetic action are equally likely, the expectation value for the action of an orbit piece must be equal to zero. The width \(W\) (i.e., the square root of the variance \(W^2\)) must be proportional to the vector potential and therefore to the magnetic field \(B\). Since the magnetic actions of the individual pieces are uncorrelated, the central limit theorem then implies that the magnetic actions of links with \(K \equiv \frac{t_i}{t_{cl}} \gg 1\) obey a Gaussian probability distribution with the width \(\sqrt{KW}\), i.e.,

\[
P(\Theta_i) = \frac{1}{\sqrt{2\pi KW^2}} e^{-\Theta_i^2/(2KW^2)}.
\]

The phase factor arising from a link averages to \(\int d\Theta_i P(\Theta_i) e^{i2\mu_i \Theta_i / \hbar} = e^{-\mu_i h t_i}\), depending on the system-specific parameter \(b = \frac{2KW^2}{\hbar t_i} = \frac{2W^2}{\hbar t_{cl}} \propto \frac{B^2}{\hbar^2}\) and on \(\mu_i = \mu^2_i \in [0, 1]\). Similarly, the phase factor associated with the \(\sigma\)th encounter averages to \(e^{-\mu^2_\sigma b t_{enc}}\) \([37]\). Links and stretches traversed in opposite directions by \(\alpha\) and \(\beta\) thus lead to exponential suppression factors in the contributions of trajectory pairs.

These factors have to be taken into account when evaluating the average conductance, starting from (22). The link integrals are changed into

\[
\int_0^\infty dt_i e^{-(N/T_H) h t_i} e^{-\mu_i h t_i} = \frac{T_H}{N(1 + \mu_i \xi)},
\]

with \(\xi \equiv \frac{T_H}{N} b \propto \frac{B^2}{\hbar}\), whereas for each encounter we find an integral

\[
\left\langle \int d^{l-1}s \, d^{l-1}u \frac{1}{\Omega^{l-1}t_{enc}^\alpha(s,u)} e^{-(N/T_H) t_{enc}^\alpha(s,u)} e^{-\mu^2_i b t_{enc}^\alpha(s,u)/\hbar} e^{i \sum_j s_{\alpha j} u_{\alpha j} / \hbar} \right\rangle = -\frac{N(1 + \mu^2_\sigma \xi)}{T_H^{l-1}}.
\]

Since the \(T_H\)s again mutually cancel, our diagrammatic rules are changed to give a factor \(\frac{1}{N_1 N_2 \mu_i \xi}\) for each link and a factor \(-N(1 + \mu^2_\sigma \xi)\) for each encounter; the arising product has to be multiplied with the number of channel combinations, i.e., \(N_1 N_2\) for the average conductance.

The same rules carry over to the conductance variance, shot noise, and correlations in a three-lead geometry. In these cases, \(\mu_i\) is equal to 1 if the \(i\)th link of the pair \((\alpha, \gamma)\) is reverted in \((\beta, \delta)\), and \(\mu_\sigma\) counts the stretches of the \(\sigma\)th encounter of \((\alpha, \gamma)\) which are reverted in \((\beta, \delta)\); the sign of \(\mu_\sigma\) is fixed as above.
7.2. Mean conductance

For the average conductance, the diagonal contribution, \( \frac{N_1 N_2}{N} \), remains unaffected by the magnetic field. The contribution of Richter/Sieber pairs, \( -\frac{N_1 N_2}{N} \), obtains an additional factor \( \frac{1}{1 + \xi} \), since one of the three links of \( \alpha \) in figure 1 or figure 2(a) is traversed by \( \beta \) in opposite sense. The next order originates from trajectory pairs as in figures 2(b)–(j) where arrows indicate the direction of motion inside the encounters and highlight those links which are traversed by \( \alpha \) and \( \beta \) with opposite sense of motion. The contributions of the families in figures 2(b), (c), (d), (g), (i) and (j) remain unchanged: in figures 2(b) and (g) no links or encounter stretches are reverted; for figures 2(c), (d), (i) and (j) the number of links with \( \mu_i = 1 \) and encounters with \( \mu_2 = 1 \) coincide, meaning that the \( \xi \)-dependent factors mutually compensate. The six above families cancel mutually due to the negative sign for figures 2(g), (i) and (j). The contributions of figures 2(e), (f) and (h) obtain a factor \( \frac{1}{(1 + \xi)^2} \) from two reverted links; due to the negative sign of figure 2(h), they sum up to \( \frac{N_1 N_2}{N^3 (1 + \xi)^2} \). We thus find

\[
\langle \text{tr}(tt^\dagger) \rangle = \frac{N_1 N_2}{N} \left\{ 1 - \frac{1}{N(1 + \xi)} + \frac{1}{N^2(1 + \xi)^2} \right\} + O\left( \frac{1}{N^2} \right). \tag{70}
\]

Counting further families of trajectory pairs with the help of a computer program one is able to proceed to rather high orders in \( \frac{1}{N} \). We then find

\[
\langle \text{tr}(tt^\dagger) \rangle = \frac{N_1 N_2}{N} \left\{ 1 - \frac{1}{N(1 + \xi)} + \frac{1}{N^2(1 + \xi)^2} - \frac{1}{N^3} \frac{1 + 2\xi + 13\xi^2 + 4\xi^3 + \xi^4}{(1 + \xi)^5} \right. \\
+ \left. \frac{1}{N^4} \frac{1 + 2\xi + 49\xi^2 + 4\xi^3 + \xi^4}{(1 + \xi)^6} \right\} + O\left( \frac{1}{N^5} \right). \tag{71}
\]

As expected, equations (70) and (71) interpolate between the results for the orthogonal case, reached for \( B \to 0 \) and thus \( \xi \to 0 \), and the unitary case, formally reached for \( B \to \infty \) and thus \( \xi \to \infty \). The convergence to the unitary result is non-trivial: the contributions of the families in figures 2(c), (d), (i) and (j) are not affected by a magnetic field, because all \( \xi \)-dependent factors cancel. These contributions thus survive in the limit \( \xi \to \infty \) (i.e., when the magnetic action becomes much larger than \( \hbar \), but the trajectory deformations due to Lorentz force can still be disregarded), but vanish in the unitary case (i.e., when the magnetic field is strong enough to considerably deform the trajectories). The agreement between the limit \( \xi \to \infty \) and the unitary result implies that the contributions of all such families must sum to zero, for all orders in \( \frac{1}{N} \). Order by order, (71) coincides with the results of [38], where the individual coefficients were given as (rather involved) random-matrix integrals.

7.3. Conductance variance, shot noise and three-lead correlations

For observables determined by families of trajectory quadruplets, it is convenient to first evaluate the overall contributions of \( d \)- and \( x \)-families per channel combination. These contributions, denoted by \( D \) and \( X \), now depend on the parameter \( \xi \). The contribution of \( d \)-families reads

\[
D = \frac{1}{N^2} - \frac{2}{N^3(1 + \xi)} + \frac{1}{N^4} \left[ 1 + \frac{4}{(1 + \xi)^2} \right] \\
+ \frac{1}{N^5} \left[ -\frac{10}{1 + \xi} - \frac{4}{(1 + \xi)^3} - \frac{2(1 + 4\xi)^2}{(1 + \xi)^5} + \frac{2(1 + 9\xi)}{(1 + \xi)^4} \right] + O\left( \frac{1}{N^6} \right), \tag{72}
\]
generalizing our previous results (41) and (44) for the unitary and orthogonal cases. The leading term, originating from diagonal quadruplets, remains unaffected by the magnetic field. The second term is due to quadruplets as in figure 4(b). Since in these quadruplets, one link of (α, γ) is time-reversed in (β, δ), the corresponding contribution is proportional to \( \frac{1}{N^6} \). It is easy to check that the third term correctly accounts for \( d \)-quadruplets differing in two 2-encounters, or in one 3-encounter; compare subsection 3.2 and figure 4. The higher-order terms were again generated by a computer program.

In the overall contribution of \( x \)-families,

\[
X = -\frac{1}{N^3} + \frac{4}{N^4(1 + \xi)} + \frac{1}{N^5} \left[ -1 - \frac{10}{(1 + \xi)^2} - \frac{2(1 + 4\xi)}{(1 + \xi)^4} \right] + O \left( \frac{1}{N^6} \right),
\]

the term \(-\frac{1}{N^3}\) accounts for \( x \)-quadruplets differing in a parallel 2-encounter, figure 4(j). These quadruplets are not affected by the magnetic field. All families responsible for the second term, figures 5(a)–(c), display a Lorentzian field dependence \( \frac{1}{N^3(1 + \xi)} \); while figures 5(a) and (c) contain one time-reversed link and only encounters with \( \mu_3 = 0 \), figure 5(b) involves two time-reversed links and one encounter with \( \mu_3^2 = 1 \). The remaining terms were again found with the help of a computer.

With these values of \( D \) and \( X \), we obtain, writing out only terms up to \( O(N^{-1}) \),

1. the conductance variance

\[
\langle (\text{tr}(t^+))^2 \rangle - \langle (\text{tr}(t^+))^2 \rangle = \frac{N_1N_2^2}{N^4} \left( 1 + \frac{1}{(1 + \xi)^2} \right) + \frac{2N_1^2N_2^2(5 + 8\xi + 4\xi^2)}{N^5(1 + \xi)^3} + \frac{2N_1N_2}{N^3(1 + \xi)} + O \left( \frac{1}{N^2} \right),
\]

2. the power of shot noise

\[
\langle \text{tr}(t^+) \rangle - \langle \text{tr}(t^+)^2 \rangle = \frac{N_1N_2^2}{N^3} + \frac{N_1N_2(N_1 - N_2)^2}{N^4(1 + \xi)} + \frac{N_1^2N_2^2(13 + 32\xi + 16\xi^2 + 4\xi^3 + \xi^4)}{N^5(1 + \xi)^4} + \frac{3N_1N_2}{N^3(1 + \xi)^2} + O \left( \frac{1}{N^2} \right),
\]

3. and current correlations for a cavity with three leads

\[
\langle \text{tr}(t^{(1\rightarrow 2)}t^{(1\rightarrow 2)})t^{(1\rightarrow 3)}t^{(1\rightarrow 3)^\dagger} \rangle = \frac{N_1N_2N_3(N_2 + N_3)}{N^3} + \frac{2N_1N_2N_3(N_1 - N_2 - N_3)}{N^4(1 + \xi)} + \frac{N_1N_2N_3[(N_2 + N_3)(1 + \xi)^2(5 + 2\xi + \xi^2) - 2N_1(4 + 10\xi + 3\xi^2)]}{N^5(1 + \xi)^4} + O \left( \frac{1}{N^2} \right).
\]

At least the higher orders in \( \frac{1}{N} \) are new results. In particular, for the power of shot noise, we do not only obtain the previously known cancellation of the second term at \( N_1 = N_2 = \frac{N}{2} \), but also a new field dependence due to the third term,

\[
\langle \text{tr}(t^+) \rangle - \langle \text{tr}(t^+)^2 \rangle = \frac{N}{16} + \frac{1}{N} \left( \frac{1 + 8\xi + 4\xi^2 + 4\xi^3 + \xi^4}{16(1 + \xi)^4} \right) + O \left( \frac{1}{N^2} \right).
\]
7.4. Ericson fluctuations

When studying Ericson fluctuations in a weak magnetic field, we have to deal with two parameters (apart from the channel numbers): the scaled energy difference $\epsilon$ and the parameter $\xi$ proportional to the squared magnetic field. Our diagrammatic rules are then changed in a straightforward way. Each link yields a factor $\frac{1}{N(1+\mu_i\xi-i\eta_\sigma\epsilon)}$, whereas each encounter gives $-N(1+\mu_i^2\xi-i\eta_\sigma\epsilon)$, to be multiplied with the number of channel combinations.

As in the orthogonal and unitary cases, the leading contribution can be attributed to the quadruplets in figures 4(c) and (f); all other contributions of the same or lower order, including the remaining families in figure 4, mutually cancel. Quadruplets as in figure 4(c) do not feel the magnetic field, and thus yield $\frac{N_1^2 N_2^2}{N^4(1+\epsilon^2)}$ as shown in section 6; here we dropped lower-order corrections due to the case of coinciding channels. For the family of quadruplets depicted in figure 4(f), the two links with $\eta_i = \pm 1$ connecting the two encounters are reversed inside $(\beta, \delta)$ and thus have $\mu_i = 1$. We obtain factors $\frac{1}{N(1+i\xi)}$ and $\frac{1}{N(1+i\xi)}$ from these two links, $\frac{1}{N}$ from each of the four remaining link, and $-N$ from the encounter. Multiplication with the number of channel combination yields a contribution $\frac{N_1^2 N_2^2}{N^4((1+i\xi)^2+\epsilon^2)}$. Ericson fluctuations in a weak magnetic field are therefore determined as

$$\left\langle G(E)G\left(E + \frac{\epsilon N}{2\pi \rho}\right)\right\rangle - \left\langle G\right\rangle^2 = \frac{N_1^2 N_2^2}{N^4} \left\{ \frac{1}{1+\epsilon^2} + \frac{1}{(1+i\xi)^2+\epsilon^2} \right\} + O\left(\frac{1}{N}\right). \quad (78)$$

8. Conclusions

A semiclassical approach to transport through chaotic cavities is established. We calculate mean and variance of the conductance, the power of shot noise, current fluctuations in cavities with three leads, and the covariance of the conductance at two different energies. These observables are dealt with for systems with and without a magnetic field breaking TRI, as well as in the crossover between these scenarios caused by a weak magnetic field leading to a magnetic action of the order of $\bar{\hbar}$. In contrast to RMT, our results apply to individual chaotic cavities, and do not require any averaging over ensembles of systems. Moreover, we go to all orders in the inverse number of channels (neglecting, however, Ehrenfest corrections).

Transport properties are expressed as sums over pairs or quadruplets of classical trajectories. These sums draw systematic contributions from pairs and quadruplets whose members differ by their connections in close encounters, and almost coincide in the intervening links. The contributions arising from the topologically different families of quadruplets or pairs are evaluated using simple and general diagrammatic rules, summarized in table 1.

We note that in the regime considered, corrections connected to the ratio of the Ehrenfest time $T_E$ (i.e. the duration of the relevant encounters) to the dwell time $T_D$ are of importance. Previous work [2, 4, 5] has already motivated a treatment of these corrections. Here, the semiclassical approach helped to settle questions controversial in the random-matrix literature [10]–[12]. As shown in [11, 12], the leading contributions to the average conductance and the power of shot noise become proportional to powers of $e^{-T_E/T_D}$, ultimately arising from the exponential decay of the survival probability. In contrast, the conductance variance turned out to be independent of $T_E$ [11]. It would be desirable to extend our reasoning on the one hand, to include the Ehrenfest
Table 1. Diagrammatic rules determining chaotic quantum transport. The table shows link and encounter contributions for all observables discussed in the present paper. For the simplest cases (conductance, conductance variance, shot noise, 3-lead correlations), we have also listed the summed-up contributions of trajectory pairs and $d$- and $x$-quadruplets and the numbers of channel combinations.

| Contribution of each link | Simplest case |       |       |
|--------------------------|--------------|-------|-------|
|                          | With energy diff. $\propto \epsilon$ and squ. magn. field $\propto \xi$ | $N \frac{1}{N(1 + \mu_i \xi - i\eta_i \epsilon)}$ |       |
| Contribution of each encounter | Simplest case |       |       |
|                          | With energy diff. $\propto \epsilon$ and squ. magn. field $\propto \xi$ | $-N \frac{1}{N(1 + \mu_2^2 \xi - i\eta_2 \epsilon)}$ |       |
| Total contribution per channel combination | Trajectory pairs |       |       |
|                          | Unitary case | $\frac{1}{N}$ |       |
|                          | Orthogonal case | $\frac{1}{N+1}$ |       |
|                          | $d$-quadruplets | Unitary case | $\frac{1}{N^2 - 1}$ |
|                          | Orthogonal case | $\frac{1}{N(N+1)(N+3)}$ |       |
|                          | $x$-quadruplets | Unitary case | $\frac{1}{N(N^2 - 1)}$ |
|                          | Orthogonal case | $\frac{1}{N(N+1)(N+3)}$ |       |
| Number of channel combinations | Trajectory pairs |       |       |
|                          | Conductance | $N_1 N_2$ |       |
|                          | Variance of conductance | $N_1^2 N_2^2 + N_1 N_2$ |       |
|                          | Shot noise | $N_1 N_2^2$ |       |
|                          | 3-lead correlations | $N_1 N_2 N_3$ |       |
|                          | $d$-quadruplets |       |       |
|                          | Variance of conductance | $N_1^2 N_2 N$ |       |
|                          | Shot noise | $N_1^2 N_2^2 + N_1 N_2$ |       |
|                          | 3-lead correlations | $N_1^2 N_2 N_3$ |       |
|                          | $x$-quadruplets |       |       |
|                          | Variance of conductance | $N_1^2 N_2 N$ |       |
|                          | Shot noise | $N_1^2 N_2^2 + N_1 N_2$ |       |
|                          | 3-lead correlations | $N_1^2 N_2 N_3$ |       |

corrections to all orders, and, on the other hand, to go beyond the limitation of relatively wide openings $w \gg \sqrt{L\lambda_F}$, as discussed in section 2.2.

The door is open for a semiclassical treatment of many more transport phenomena, such as quantum decay [39], weak antilocalization [40], parametric correlations [19, 41], the full counting statistics of two-port cavities, and cavities with more leads. Extensions to the symplectic symmetry class, along the lines of [9, 42], and to the seven new symmetry classes [43] (relevant e.g. for normal-metal/superconductor heterostructures or quantum chromodynamics) should be within reach. A generalization to quasi one-dimensional wires would finally lead to a semiclassical understanding of dynamical localization.

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Appendix A. Trajectory quadruplets versus orbit pairs

In this appendix, we will establish a combinatorial method for counting families of trajectory quadruplets appearing in the theory of conductance variance and shot noise. We will see that trajectory quadruplets can be glued together to form orbit pairs, and orbit pairs can be cut into quadruplets of trajectories. In contrast to the case of trajectory pairs, see figure 2, we shall now need two cuts.

Our approach will be purely topological; e.g., an orbit pair \((A, B)\) is regarded just as a pair of directed closed lines with links coinciding in \(A\) and \(B\) but differently connected in the encounters. Similarly, within each quadruplet \((\alpha, \beta, \gamma, \delta)\) we can assume that the links of \(\alpha, \gamma\) exactly coincide with those of \(\beta, \delta\). Mostly, we can even think of the quadruplets as black boxes with two left ports \(a_1, c_1\) and and two right ports \(a_2, c_2\). Regardless of the actual number of encounters inside, an \(x\)-quadruplet can then be treated like a ‘dressed’ 2-encounter: connections \(a_1\rightarrow a_2, c_1\rightarrow c_2\) of the quadruplet are replaced by \(a_1\rightarrow c_2, c_1\rightarrow a_2\) in the partner pair, exactly as if a single 2-encounter existed between the trajectories \(\alpha, \gamma\) of the quadruplet. On the other hand, no change in the connections occurs between the ports in a \(d\)-quadruplet, hence it is topologically equivalent to a pair of dressed links.

We shall consider both the unitary and the orthogonal case. In each case, we will use two slightly different methods to relate trajectory quadruplets and orbit pairs. This will allow us to express the quantities \(x_m\) and \(d_m\) defined in (37) through the auxiliary sums

\[
A_m = \sum_{\vec{v}} (-1)^{V(\vec{v})} (L(\vec{v}) + 1)N(\vec{v}), \quad B_m = \sum_{\vec{v}} (-1)^{V(\vec{v})}N(\vec{v}, 2),
\]

(A.1)

where \(N(\vec{v})\) and \(N(\vec{v}, 2)\) are numbers of structures of orbit pairs (see subsection 2.4) and we have \(M(\vec{v}) \equiv L(\vec{v}) - V(\vec{v})\); these auxiliary sums will be determined recursively in appendix A3 below.

A.1. Unitary case

To illustrate method I, let us consider a \(d\)-quadruplet \((\alpha, \beta, \gamma, \delta)\), as on the left-hand side of figure A.1, and merge \(\alpha\) and \(\gamma\) into one ‘orbit’ \(A\). We connect the final point of \(\alpha\) to the initial point of \(\gamma\) and the final point of \(\gamma\) to the initial point of \(\alpha\), as shown on the right-hand side. Likewise, \(\beta\) and \(\delta\) can be glued together to an ‘orbit’ \(B\).\(^{13}\) The connection lines added are the same for \((\alpha, \gamma)\) and for \((\beta, \delta)\): one connection line joins the coinciding final links of \(\alpha\) and \(\beta\) with the coinciding initial links of \(\gamma\) and \(\delta\), whereas the second one joins the final links of \(\gamma\) and \(\delta\) with the initial links of \(\alpha\) and \(\beta\). The orbits \(A\) and \(B\) differ in the same encounters as \((\alpha, \gamma)\) and

\(^{13}\)As mentioned, \(\beta\) and \(\delta\) may be interchanged if the ingoing and outgoing channels coincide. This has no impact on the present considerations. The naming of partner trajectories as \(\beta\) and \(\delta\) in all figures will be arbitrary.
Figure A.1. Left-hand side: schematic picture of a $d$-quadruplet of trajectories $\alpha, \gamma$ (full lines), $\beta, \delta$ (dashed lines). Right-hand side: outside the ‘bubble’, we added connection lines joining $(\alpha, \gamma)$ and $(\beta, \delta)$ into periodic orbits $A$ and $B$ with the initial link indicated by ‘1.’. The additional lines of $A$ and $B$ coincide.

To fix one structure for the orbit pair $A, B$, we have to single out one link as the ‘first’ and choose as such the link of $A$ created by merging the final link of $\gamma$ with the initial link of $\alpha$ (indicated by ‘1.’ in figure A.1).

We can revert the above procedure, to obtain families of $d$-quadruplets from structures of orbit pairs. We first have to cut both orbits inside the ‘initial’ link. This leads to a trajectory pair with $L(\mathbf{v}) + 1$ rather than $L(\mathbf{v})$ links. We then have $L(\mathbf{v}) + 1$ choices for placing a second cut in any of these links. In each case, we end up with a trajectory quadruplet. Within this quadruplet, the trajectories following the first cut through $A$ and $B$ are labelled by $\alpha$ and $\beta$; the remaining ones are called $\gamma$ and $\delta$. In this way, each of the $\mathcal{N}(\mathbf{v})$ structures of orbit pairs related to a given $\mathbf{v}$ gives rise to $L(\mathbf{v}) + 1$ families of $d$-quadruplets with the same $\mathbf{v}$. The quantities $\mathcal{N}_d(\mathbf{v})$ and $d_m$ characterizing the $d$-families in (37) thus become accessible as

$$\mathcal{N}_d(\mathbf{v}) = (L(\mathbf{v}) + 1)\mathcal{N}(\mathbf{v}), \quad (A.2)$$

$$d_m = \sum_{\mathbf{v}} (-1)^{V(\mathbf{v})}(L(\mathbf{v}) + 1)\mathcal{N}(\mathbf{v}) = A_m. \quad (A.3)$$

Let us discuss a few examples. According to (A.3) the coefficient $d_1$ is determined by orbit pairs with $m = 1$ (i.e., only one 2-encounter). Since in the unitary case there are no such orbit pairs, we have $d_1 = 0$. The following coefficient $d_2$ is determined by orbit pairs with $m = 2$. In the unitary case there are only two such structures, $ppi$ and $pc$ (figures 2(b) and (g)). All quadruplets responsible for the coefficient $d_2$ can be obtained by making two cuts through these orbit pairs, one through the initial link which may be chosen arbitrarily. The second cut can go through any link; in particular, there are two possibilities for the second cut in the initial link, before and after the first cut. That means $L + 1 = 5$ possible positions of the second cut for $ppi$. These lead to the quadruplets as in figures 4(c) and (d), the reflected version of figure 4(d), and quadruplets where either $\alpha$ or $\gamma$ contain two 2-encounters and the other trajectory contains none. For $pc$ there are four possible positions for the second cut, corresponding to figure 4(e), its reflected version, and quadruplets where either $\alpha$ or $\gamma$ contain the full 3-encounter. All quadruplet families related to a given structure make the same contributions $(-1)^V$ to the coefficient $d_2$, i.e., 1 for those obtained from $ppi$ and -1 for those obtained from $pc$; we again see that $d_2 = 5 - 4 = 1$.

To explain method II, let us now consider $x$-quadruplets as on the left-hand side in figure A.2. On the right-hand side, $\alpha$ and $\gamma$ are again merged into a periodic orbit $A$, and $\beta$ and $\delta$ are once
Figure A.2. Left-hand side: schematic picture of an x-quadruplet of trajectories \( \alpha, \gamma \) (full lines), \( \beta, \delta \) (dashed lines). Right-hand side: outside the ‘bubble’, we added connection lines joining (\( \alpha, \gamma \)) and (\( \beta, \delta \)) into periodic orbits \( \mathcal{A} \) and \( \mathcal{B} \). The additional lines of \( \mathcal{A} \) and \( \mathcal{B} \) differ from each other, and can be viewed as an additional 2-encounter.

more merged into \( \mathcal{B} \), by connection lines leading from the end of one trajectory to the beginning of the other one. In contrast to the first scenario, the pair (\( \mathcal{A}, \mathcal{B} \)) has one further 2-encounter between these lines, with different connections for the two partner orbits. To fix one structure for the latter orbit pair, we take the initial link of \( \alpha \) as the ‘first’ link of the orbit pair. This link is preceded by a ‘final’ stretch, which must belong to the added 2-encounter. We must therefore reckon with orbit pairs associated to the vector \( \vec{v}^{[\rightarrow 2]} \) and whose final stretches belong to a 2-encounter. In the notation of Subsection 2.4, the number of structures of such orbit pairs is given by \( \mathcal{N}(\vec{v}^{[\rightarrow 2]}, 2) \).

Each of these structures can be turned back into one family of x-quadruplets, by cutting out the added 2-encounter. Consequently, there is a one-to-one relation between x-families and the structures of orbit pairs considered. The number of x-families related to \( \vec{v} \) and the coefficients \( x_m \) defined in (37) are given by

\[
\mathcal{N}_x(\vec{v}) = \mathcal{N}(\vec{v}^{[\rightarrow 2]}, 2),
\]

\[
x_m = \sum_{\vec{v}} (-1)^{V(\vec{v})} \mathcal{N}(\vec{v}^{[\rightarrow 2]}, 2).
\]

Rather than over \( \vec{v} \), we may sum over \( \vec{v}' \equiv \vec{v}^{[\rightarrow 2]} \) with \( L(\vec{v}') - V(\vec{v}') = L(\vec{v}) + 2 - (V(\vec{v}) + 1) = m + 1 \). While the latter sum should be restricted to \( \vec{v}' \) with \( v'_2 > 0 \), that restriction may be ignored since \( \vec{v}' \) with \( v'_2 = 0 \) have \( \mathcal{N}(\vec{v}', 2) = 0 \). Using \( (-1)^{V(\vec{v})} = -(-1)^{V(\vec{v})} \) and dropping the primes we can express the coefficient \( x_m \) as

\[
x_m = -\sum_{\vec{v}} (-1)^{V(\vec{v})} \mathcal{N}(\vec{v}, 2) = -B_m + 1.
\]

For instance, the coefficient \( x_1 \) is determined by orbit pairs with \( L(\vec{v}) - V(\vec{v}) = 2 \). Only one such structure (\( \text{ppi} \)) contains 2-encounters and yields a contribution \(-1\), whereas \( \text{pc} \) involves only one 3-encounter. We thus find \( x_1 = -1 \), as already seen previously.

Taken together, equations (A.3) and (A.6) indeed relate the conductance variance to structures of orbit pairs.
Figure A.3. Sketches of orbit pairs \((A, B)\). Inside the bubbles: quadruplets of trajectories obtained by cutting \((A, B)\) in the initial link (indicated by ‘1.’), and in one further link. In (a) both links are traversed by \(A\) and \(B\) with the same sense of motion, and the trajectory quadruplet is of type \(d\). In (b), the second cut is placed in a link traversed with opposite senses of motion, and the resulting quadruplet is of type \(x\).

A.2. Orthogonal case

For time-reversal invariant systems, we must now consider pairs of orbits \(A, B\) differing in encounters whose stretches are either close or almost mutually time-reversed. The sense of traversal of an orbit now being arbitrary we fix the direction of \(B\) such that \(B\) traverses the ‘initial’ link of \(A\) in the same direction.

We start with method I, i.e., we cut an orbit pair \((A, B)\) inside links, first inside the initial link and afterwards in an arbitrary link. We have to distinguish two cases, respectively leading to \(d\)- and \(x\)-quadruplets. First assume that the second cut is placed in a link traversed by \(A\) and \(B\) with the same sense of motion; see figure A.1 or figure A.3(a). As in the unitary case we then obtain a \(d\)-quadruplet of trajectories; this quadruplet is highlighted by a grey ‘bubble’ in figures A.1 and A.3(a), without depicting the encounters. Inside this quadruplet, \(\alpha\) is defined as the trajectory following the cut through the initial link (marked by ‘1.’ in figures A.1–A.4).

Now assume that the second cut is placed in a link traversed by \(A\) and \(B\) with opposite senses of motion. The resulting quadruplet, shown in the grey bubble in figure A.3(b), resembles an \(x\)-quadruplet, apart from the directions of motion. To obtain a true \(x\)-quadruplet, one has to revert the directions of motion of two trajectories in figure A.3(b), such that all trajectories point in the same direction as \(\alpha\) (i.e., the trajectory following the cut inside the initial link).

We hence obtain the following relation between orbit pairs and trajectory quadruplets: by cutting inside links, each of the \(\mathcal{N}(\vec{v})\) structures of orbit pairs related to \(\vec{v}\) can be turned into a family of trajectory quadruplets in \(L(\vec{v}) + 1\) possible ways. Through such cuts, all \(\mathcal{N}_d(\vec{v})\) \(d\)-families and all \(\mathcal{N}_x(\vec{v})\) \(x\)-families are obtained exactly once, since each \(d\)- or \(x\)-family could be inserted inside the bubble in figure A.3(a) or figure A.3(b), respectively. We thus have

\[
(L(\vec{v}) + 1)\mathcal{N}(\vec{v}) = \mathcal{N}_d(\vec{v}) + \mathcal{N}_x(\vec{v})
\]  

(A.7)

and summation as in the unitary case leads to

\[
A_m = d_m + x_m
\]  

(A.8)

with \(A_m\) defined in (A.1) and \(d_m, x_m\) defined in (37).
Figure A.4. Sketches of orbit pairs \( (A, B) \) with one 2-encounter singled out. This 2-encounter contains the ‘final’ encounter stretch, and is either (a) parallel in both \( A \) and \( B \), (b) parallel in \( A \) and antiparallel in \( B \), (c) antiparallel in \( A \) and parallel in \( B \), or (d) antiparallel in both \( A \) and \( B \).

We now turn to method II (cutting inside 2-encounters). We consider orbit pairs \( A, B \) containing one more 2-encounter compared with the quadruplets in question, assuming that the final encounter stretch of the orbits belongs to the ‘added’ 2-encounter; the number of these structures is \( \mathcal{N}(\vec{v}[\rightarrow 2], 2) \). Cutting out the added 2-encounter will create all possible quadruplets associated to \( \vec{v} \), some of them, as we shall see, in several copies.

The added 2-encounter can be either parallel or antiparallel in the orbit \( A \) as well as in its partner \( B \). This leads to four different possibilities, depicted by arrows on white background in figures A.4(a)–(d). First suppose that the encounter in question is antiparallel in both \( A \) and \( B \), as in figure A.4(d). This is possible only if the ports of this encounter are connected in the same way in \( A \) and \( B \), up to time reversal; one can easily check that all other connections would lead to either \( A \) or \( B \) decomposing into several disjoint orbits. The connections outside the encounter, as depicted in the bubble in figure A.4(d), thus resemble \( d \)-quadruplets and can be turned into true \( d \)-quadruplets if we revert the sense of motion on some trajectories. Any \( d \)-family could be substituted for the bubble in figure A.4(d). Thus, cutting through 2-encounters of the kind in figure A.4(d) produces all possible families of \( d \)-quadruplets.

In the three other cases, figures A.4(a)–(c), the remaining connections have to be of type \( x \), up to the sense of motion on some trajectories, since connections of type \( d \) would lead to decomposing orbits. To better understand these cases, it is helpful to view the corresponding \( x \)-quadruplets as ‘dressed’ 2-encounters. Then, the orbit pairs of figures A.4(a)–(c) are topologically equivalent to the simple diagrams in figure 2. The orbit pair in figure A.4(a) is of the type \( ppi \)

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14 Rather than reverting directions of motion, we could also identify the initial points of all trajectories inside the bubble with ingoing leads, and the final points with outgoing leads, loosing the identification of the two sides of our bubble with the two openings of the cavity. This would entail a different mapping between orbit pairs and trajectory quadruplets, but not affect the following results.
whereas the pairs in figures A.4(b) and (c) are both of the type api. (In figure A.4(b) the initially parallel encounter is identified with the added encounter, and the initially antiparallel encounter is identified with the \( x \)-quadruplet, whereas the situation is opposite in figure A.4(c).) Any family of type \( x \) could be substituted for the bubbles in each of the figures A.4(a)–(c), and can therefore be obtained by cutting three different structures of orbit pairs.

We have thus seen that by cutting through 2-encounters of the \( \mathcal{N}(\vec{v}^{[2]}), 2 \) structures of orbit pairs considered, we obtain each family of \( d \)-quadruplet once and once only, whereas each \( x \)-family is produced by three different structures. We therefore have

\[
\mathcal{N}(\vec{v}^{[2]}), 2) = \mathcal{N}_d(\vec{v}) + 3\mathcal{N}_x(\vec{v}), \tag{A.9}
\]

and, by summing over \( \vec{v} \) as in appendix A1,

\[
-B_{m+1} = d_m + 3x_m. \tag{A.10}
\]

Equations (A.8) and (A.10) form a system of equations for the coefficients \( d_m \), \( x_m \), with the solution

\[
d_m = \frac{3}{2}A_m + \frac{1}{2}B_{m+1}, \tag{A.11a}
\]

\[
x_m = -\frac{1}{2}A_m - \frac{1}{2}B_{m+1}. \tag{A.11b}
\]

A.3. Recursion relations

We must calculate the auxiliary sums \( A_m, B_m \) defined in (A.1) for both the unitary and the orthogonal cases. We start from the recursion for the number of structures \( \mathcal{N}(\vec{v}) \) already used to evaluate the average conductance in subsection 2.4

\[
\mathcal{N}(\vec{v}, 2) - \sum_{k \geq 2} \mathcal{N}(\vec{v}[k,2 \rightarrow k+1], k + 1) = \left( \frac{2}{\beta} - 1 \right) \mathcal{N}(\vec{v}^{[2]}); \tag{A.12}
\]

see equation (27). This time, (A.12) has to be multiplied not with \( (-1)^{V(\vec{v})} \), but with \( (-1)^{V(\vec{v})} L(\vec{v}) = -\left(-1\right)^{V(\vec{v}^{[2]} \rightarrow k+1)}(L(\vec{v}[k,2 \rightarrow k+1]) + 1) = -\left(-1\right)^{V(\vec{v}^{[2]} \rightarrow k+1)}(L(\vec{v}^{[2]} \rightarrow 1) + 2) \). If we subsequently sum over all \( \vec{v} \) with \( M(\vec{v}) = m \), our recursion turns into

\[
\sum_{\vec{v}} \left\{ (-1)^{V(\vec{v})} L(\vec{v}) \mathcal{N}(\vec{v}, 2) + \sum_{k \geq 2} (-1)^{V(\vec{v}^{[k,2 \rightarrow k+1]} \rightarrow k+1)}(L(\vec{v}[k,2 \rightarrow k+1]) + 1) \mathcal{N}(\vec{v}[k,2 \rightarrow k+1], k + 1) \right\} = -\left( \frac{2}{\beta} - 1 \right) \sum_{\vec{v}} (-1)^{V(\vec{v}^{[2]} \rightarrow 1)}(L(\vec{v}^{[2]} \rightarrow 1) + 2) \mathcal{N}(\vec{v}). \tag{A.13}
\]

Changing the summation variables as in subsection 2.4, we find

\[
\sum_{\vec{v}} \left\{ (-1)^{V(\vec{v})} L(\vec{v}) \mathcal{N}(\vec{v}, 2) + \sum_{k \geq 2} (-1)^{V(\vec{v})} (L(\vec{v} + 1) \mathcal{N}(\vec{v}, k + 1) \right\} = -\left( \frac{2}{\beta} - 1 \right) \sum_{\vec{v}} (-1)^{V(\vec{v})} (L(\vec{v} + 2) \mathcal{N}(\vec{v}). \tag{A.14}
\]
Using \( \mathcal{N}(\vec{v}, 2) + \sum_{k \geq 2} \mathcal{N}(\vec{v}, k + 1) = \mathcal{N}(\vec{v}) \), we can simplify the left-hand side to
\[
\sum_{\vec{v}}^{M(\vec{v})=m} (-1)^{V(\vec{v})} (L(\vec{v}) + 1) \mathcal{N}(\vec{v}) - \sum_{\vec{v}}^{M(\vec{v})=m} (-1)^{V(\vec{v})} \mathcal{N}(\vec{v}, 2) = A_m - B_m,
\]
whereas the right-hand side turns into \(-\left(\frac{2}{\beta} - 1\right)(A_{m-1} + c_{m-1})\) with \( c_{m-1} = \sum_{\vec{v}}^{M(\vec{v})=m-1} (-1)^{V(\vec{v})} \mathcal{N}(\vec{v}) \), see equation (25). We thus find a first relation between \( A_m \) and \( B_m \),
\[
A_m - B_m = -\left(\frac{2}{\beta} - 1\right) (A_{m-1} + c_{m-1}). \tag{A.15}
\]

A second relation between \( A_m \) and \( B_m \) follows from the recursion [9] \(^{15}\)
\[
\mathcal{N}(\vec{v}, 3) - \sum_{k \geq 2} \mathcal{N}(\vec{v}^{[k,3-k+2]}, k + 2) = 2 \left(\frac{2}{\beta} - 1\right) \mathcal{N}(\vec{v}^{[3-2]}, 2) + \frac{2}{\beta} (L(\vec{v}^{[3-1]}) + 1) \mathcal{N}(\vec{v}^{[3-1]}), \tag{A.16}
\]
which has to be multiplied with \((-1)^{V(\vec{v})} = (-1)^{V(\vec{v}^{[k,3-k+2]})} = (-1)^{V(\vec{v}^{[3-2]})} = (-1)^{V(\vec{v}^{[3-1]})}\) and summed over all \( \vec{v} \) with \( M(\vec{v}) = m \). It is easy to see that the changed vectors \( \vec{v}^{[k,2-k+1]}, \vec{v}^{[3-2]}, \vec{v}^{[3-1]} \) in the arguments of \( \mathcal{N} \) have \( M(\vec{v}^{[k,3-k+2]}) = M(\vec{v}^{[3-2]}) = M(\vec{v}^{[3-1]}) = m - 2 \). Again transforming the sums over \( \vec{v} \) into sums over the arguments of \( \mathcal{N} \), we are led to
\[
\sum_{\vec{v}}^{M(\vec{v})=m} (-1)^{V(\vec{v})} \left( \mathcal{N}(\vec{v}, 3) + \sum_{k \geq 2} \mathcal{N}(\vec{v}, k + 2) \right) = 2 \left(\frac{2}{\beta} - 1\right) \sum_{\vec{v}}^{M(\vec{v})=m-1} (-1)^{V(\vec{v})} \mathcal{N}(\vec{v}, 2)
- \frac{2}{\beta} \sum_{\vec{v}}^{M(\vec{v})=m-2} (-1)^{V(\vec{v})} (L(\vec{v}) + 1) \mathcal{N}(\vec{v}). \tag{A.17}
\]
Equation (A.17) can be simplified if we use \( \mathcal{N}(\vec{v}, 3) + \sum_{k \geq 2} \mathcal{N}(\vec{v}, k + 2) = \mathcal{N}(\vec{v}) - \mathcal{N}(\vec{v}, 2) \), and recall the definitions of \( A_m, B_m \) and \( c_m \), to get
\[
c_m - B_m = 2 \left(\frac{2}{\beta} - 1\right) B_{m-1} - \frac{2}{\beta} A_{m-2}. \tag{A.18}
\]

A.4. Results

The two relations between \( A_m \) and \( B_m \), equations (A.15) and (A.18) yield a recursion for both quantities. In the unitary case, \( \frac{2}{\beta} - 1 = 0 \), \( c_m = 0 \), the two equations simplify to
\[
A_m - B_m = 0, \quad -B_m = -A_{m-2}. \tag{A.19}
\]

\(^{15}\)Equation (A.16) follows from the special case \( l = 3 \) of equations (42), (54) in [9]. To understand the equivalence to [9], we need the identity \( \mathcal{N}(\vec{v}^{[3-1]}, 1) \equiv \mathcal{N}(\vec{v}^{[3-1]}) \)? (26) \( \equiv \frac{L(\vec{v}^{[3-1]})}{L(\vec{v}^{[3-1]})} \mathcal{N}(\vec{v}^{[3-1]}), \) following from (\( \ast \)) \( \mathcal{N}(\vec{v}, 1) = \mathcal{N}(\vec{v}^{[1-1]}) \) (see the footnote in subsection 2.4), equation (26) of the present paper, and (\( \ast \ast \)) the fact that \( v_l = 0 \) and thus \( v_l^{[3-1]} = 1 \).
i.e., $A_m$ and $B_m$ coincide and depend only on whether $m$ is even or odd. Since in the unitary case $d_m = A_m$, $x_m = B_m$, see (A.3), and since the initial values $d_1 = 0$, $d_2 = 1$ are already established, we come to the expressions (40) for $x_m$, $d_m$.

In the orthogonal case $\frac{2}{\beta} - 1 = 1$, $c_m = (-1)^m$, equations (A.15) and (A.18) take the form

$$A_m - B_m = -A_{m-1} + (-1)^m,$$  \hspace{1cm} (A.20a)

$$(-1)^m - B_m = 2B_{m-1} - 2A_{m-2}.$$ \hspace{1cm} (A.20b)

Eliminating $B_m$ and $B_{m-1}$ in (A.20b) with the help of (A.20a), we find a recursion for $A_m$,

$$A_m = -3A_{m-1}.$$ \hspace{1cm} (A.21)

An initial condition is provided by $A_1 = -3$, which accounts for the Richter/Sieber family of trajectory pairs with one 2-encounter, $V = 1$, $L = 2$ and $m = 1$. We thus obtain

$$A_m = (-3)^m, \quad B_m \stackrel{(A.20a)}{=} A_m + A_{m-1} + (-1)^{m-1} = (-1)^m (2 \cdot 3^{m-1} - 1).$$ \hspace{1cm} (A.22)

After that, using (A.11a) and (A.11b), we arrive at the result (43) of the main text.

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