Physical origin of the power-law tailed statistical distributions

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Starting from the BBGKY hierarchy, describing the kinetics of nonlinear particle system, we obtain the relevant entropy and stationary distribution function. Subsequently, by employing the Lorentz transformations we propose the relativistic generalization of the exponential and logarithmic functions. The related particle distribution and entropy represents the relativistic extension of the classical Maxwell-Boltzmann distribution and of the Boltzmann entropy respectively and define the statistical mechanics presented in [Phys. Rev. E 66, 056125 (2002)] and [Phys. Rev. E 72, 036108 (2005)]. The achievements of the present effort, support the idea that the experimentally observed power law tailed statistical distributions in plasma physics, are enforced by the relativistic microscopic particle dynamics.

I. INTRODUCTION

In plasma physics, the power-law tails in the particle population, has been systematically observed in the last fifty years. For instance the cosmic ray spectrum

\[ f_i \propto \chi(\beta E_i - \beta \mu) , \] (1)

which extends over 13 decades in energy, from \(10^8\) eV to \(10^{20}\) eV, and spans 33 decades in particle flux, from \(10^{-29}\) to \(10^4\) units, obeys the Boltzmann law of classical statistical mechanics i.e.

\[ \chi(x) \sim \exp(-x) , \] (2)

for low energies, while for high energies this spectrum presents power law fat tails i.e.

\[ \chi(x) \sim x^{-1/\kappa} , \] (3)

the spectral index \(\kappa\) being close to 0.32-0.37. The above spectrum was approached for the first time in 1968, by using a different distribution from the Boltzmann one. In his proposal Vasyliunas heuristically identified the function \(\chi(x)\) with the Student distribution function which presents power-law tails \[\rho\]. In the last 40 years a vast amount of literature has been produced, regarding the so called kappa-plasmas, based on the Vasyliunas distribution. Up to now, several physical mechanisms have been explored in order to furnish theoretical support to the experimentally observed power-law-tailed distribution functions. However there is currently an intense debate regarding the theoretical foundations of the non-Boltzmannian distributions.

In the last years, after noting that the power-law tails are placed in the high energy region, and then regards relativistic particles, the question has been posed whether the solution of the problem, i.e. the theoretic determination of the function \(\chi(x)\) and consequently of the related distribution and entropy, can be explained by invoking the basic principles of special relativity.

The present paper, deals with the relativistic statistical theory \[\rho\], predicting for the function \(\chi(x)\), the very simple form i.e.

\[ \chi(x) = \exp_\kappa(-x) , \] (4)

with

\[ \exp_\kappa(x) = (\sqrt{1 + \kappa^2 x^2} + \kappa x)^{1/\kappa} . \] (5)

The parameter \(\kappa < 1\) is the reciprocal of light speed in a dimensionless form while the function \(\exp_\kappa(x)\) represents the relativistic generalization of the ordinary exponential which recovers in the classical limit \(\kappa \to 0\).

In the last few years various authors have considered the foundations of the statistical theory based on the distribution function involving the generalized exponential \(\exp_\kappa(x)\), e.g. the H-theorem and the molecular chaos hypothesis \[\rho\], the thermodynamic stability \[\rho\], the Lesche stability \[\rho\], the Legendre structure of the ensued thermodynamics \[\rho\] and quantum versions of the theory \[\rho\], the geometrical structure of the theory \[\rho\], various mathematical aspects of the theory \[\rho\], etc. On the other hand specific applications to physical systems have been considered, e.g. the cosmic rays \[\rho\], relativistic \[\rho\] and classical \[\rho\] plasmas in presence of external electromagnetic fields, the relaxation in relativistic plasmas under wave-particle interactions \[\rho\], anomalous diffusion \[\rho\], particle kinetics in the presence of temperature gradients \[\rho\], particle systems in external conservative force fields \[\rho\], stellar distributions in astrophysics \[\rho\], quark-gluon plasma formation \[\rho\], quantum hadrodynamics models \[\rho\], the fracture propagation \[\rho\], etc. Other applications regard dynamical systems at the edge of chaos \[\rho\], fractal systems \[\rho\], field theories \[\rho\], the random matrix theory \[\rho\], the error theory \[\rho\], the game theory \[\rho\], the information theory \[\rho\], etc. Also applications to economic systems have been considered e.g. to study the personal income distribution \[\rho\], to model deterministic heterogeneity in tastes and product differentiation \[\rho\] etc. Finally in \[\rho\], some historical
II. KINETIC EQUATION

Let us consider the most general relativistic equation imposing the particle conservation during the evolution of a many body system. That equation is the first equation of the BBGKY hierarchy i.e.

\[ p^\nu \partial_\nu f - m F^\nu = \int \frac{d^3p' d^3p_1 d^3p'_1}{p^\nu p_1^\nu p'_1^\nu} G [f' \otimes f'_1 - f \otimes f_1], \]

(6)

and describes, through the one-particle correlation function or distribution function \( f = f(x, p) \), a relativistic particle system in presence of an external force field. The streaming term as well as the Lorentz invariant integrations in the collision integral, have the standard forms of the relativistic kinetic theory.

The two particle correlation function, here denoted by \( f \otimes f_1 \), appearing in Eq. (6), at the moment remains an unknown, two variable function. We recall that in classical kinetics, the two-particle correlation function, according to the molecular chaos hypothesis, is the ordinary product of \( f \) and \( f_1 \) i.e. \( f \otimes f_1 = f \times f_1 \). Therefore the composition law \( f \otimes f_1 \) can be viewed as a relativistic generalized product of \( f \) and \( f_1 \), isomorphic to the ordinary product.

Following standard lines of kinetic theory, we note that in stationary conditions, the collision integral in Eq. (6) vanishes and then it follows that \( f \otimes f_1 = f' \otimes f'_1 \). More in general it holds

\[ L(f \otimes f_1) = L(f' \otimes f'_1), \]

(7)

\( L(x) \) being an arbitrary function, and this relationship expresses a conservation law for the particle system. On the other hand a conservation law has form

\[ \Lambda(f) + \Lambda(f_1) = \Lambda(f') + \Lambda(f'_1), \]

(8)

\( \Lambda(f) \) being the collision invariant of the system. Therefore we can pose

\[ L(f \otimes f_1) = \Lambda(f) + \Lambda(f_1). \]

(9)

From the definition of the correlation function and taking into account of the indistinguishability of the particles, it follows that \( f \otimes 1 = 1 \otimes f = f \) and this implies that \( \Lambda(1) = 0 \) and \( L(f) = \Lambda(f) \). After taking into account the later relationship, Eq. (9) becomes

\[ \Lambda(f \otimes f_1) = \Lambda(f) + \Lambda(f_1). \]

(10)

Consequently the collision invariant \( \Lambda(f) \) permits us to determine univocally the correlation function \( f \otimes f_1 \) as follows

\[ f \otimes f_1 = \Lambda^{-1}(\Lambda(f) + \Lambda(f_1)). \]

(11)

In relativistic kinetics, the collision invariant \( \Lambda(f) \), unless an additive constant, is proportional to the microscopic relativistic invariant \( I \). Then we can pose

\[ \Lambda(f) = -\beta I + \beta \mu, \]

(12)

\( \beta \) and \( \beta \mu \) being two arbitrary constants. In presence of an external electromagnetic field \( A^\nu \), the more general microscopic relativistic invariant \( I \), has a form proportional to

\[ I = (p^\nu + q A^\nu/c) U_\nu - mc^2, \]

(13)

\( U_\nu \) being the hydrodynamic four-vector velocity with \( U_\nu U^\nu = c^2 \). Finally, after inversion of Eq. (12), the stationary distribution is obtained as follows

\[ f = \Lambda^{-1}(-\beta I + \beta \mu). \]

(14)

It is remarkable to note that Eq. (12) follows from the variational equation

\[ \frac{\delta}{\delta f} \left[ S - \beta \int d^3p \left[ I f + \beta \mu \int d^3p f \right] \right] = 0, \]

(15)

where the functional \( S \), unless an arbitrary additive constant, is given by

\[ S = -\int d^3p \left[ \int \Lambda(f) df \right], \]

(16)

\( \int \Lambda(f) df \) being the indefinite integral of \( \Lambda(f) \). The variational equation (15) represents the maximum entropy principle. The constants \( \beta \) and \( \beta \mu \) are the Lagrange multipliers while the functional \( S \), defined though Eq. (16), is the system entropy.

We stress that the function \( \Lambda(f) \) defines univocally both the stationery distribution (14) and the entropy (16) of the system as well as the two-particle correlation function (11). In classical statistical mechanics, it is well known that \( \Lambda(f) = \ln(f) \) so that the two particle correlation function becomes \( f \otimes f_1 = f \times f_1 \), while (14) and (16) reduces to the exponential distribution and Boltzmann entropy respectively.

In the next section, in order to develop a relativistic statistical mechanics, we will determine the function \( \Lambda(f) \) within the special relativity, starting from the Lorentz transformations.
III. LORENTZ TRANSFORMATIONS

In the present section we will show that the function $\Lambda^{-1}(f)$ emerges within the special relativity as the relativistic generalization of the ordinary exponential of classical physics.

Let us consider in the one-dimension frame $S$ two identical particles $A$ and $B$, of rest mass $m$. We suppose that the two particles move with velocity $v_A$ and $v_B$ respectively. The momenta of the two particles are indicated with $p_A = p(v_A)$ and $p_B = p(v_B)$, while their energies are indicated with $E_A = E(v_A)$ and $E_B = E(v_B)$ respectively.

In classical physics, in the rest frame $S'$ of particle $B$, the momentum of the particle $B$ is $p'_B = 0$ while the momentum of the particle $A$ is given by the Galilei transformation formula

$$p'_A = p_A - p_B.$$ (17)

After introducing in place of $p$ the dimensionless momentum $q = p/\gamma$, we note that the exponential function $\exp(q)$, permits us to write the Galilei additivity law [17], in the following factorized form

$$\exp(q'_A) = \exp(q_A) \exp(-q_B).$$ (18)

The Galilei relativity principle, imposes the equivalence of all the inertial frames. According to this principle, the inverse Galilei transformation must have the same structure of the direct transformation (18) except for the substitutions $q'_A \leftrightarrow q_A$ and $q_B \rightarrow -q_B$. This requirement is satisfied thanks to the following property of the exponential function

$$\exp(x) \exp(-x) = 1.$$ (19)

We consider now the two particles in the rest frame $S'$ of particle $B$, within the special relativity. The velocity, momentum and energy of the particle $B$ are $v'_B = 0$, $p'_B = 0$ and $E'_B = mc^2$ respectively. In $S'$ the velocity of particle $A$ is given by the formula

$$v'_A = \frac{v_A - v_B}{1 - v_A v_B / c^2}.$$ (20)

defining the relativistic velocity composition law, which follows directly from the kinematic Lorentz transformations. In the same frame $S'$ the momentum and energy of particle $A$ are given by the dynamic Lorentz transformations

$$p'_A = \gamma(v_B) p_A - c^2 v_B \gamma(v_B) E_A,$$ (21)

$$E'_A = \gamma(v_B) E_A - v_B \gamma(v_B) p_A,$$ (22)

$$\gamma(v_B) = \left(1 - v_B^2 / c^2\right)^{-1/2}$$ being the Lorentz factor. After taking into account the expression of the momentum $p_B = m v_B \gamma(v_B)$ and of the energy $E_B = m c^2 \gamma(v_B)$ of the particle $B$ the latter transformations become

$$p'_A = p_A E_B / mc^2 - E_A p_B / mc^2,$$ (23)

$$E'_A = E_A E_B / mc^2 - p_A p_B / m.$$ (24)

Let us introduce in place of the dimensional variables $(v, p, E)$ the dimensionless variables $(u, q, \mathcal{E})$ through

$$\frac{v}{u} = \frac{p}{mq} = \sqrt{\frac{E}{m\mathcal{E}}} = |\kappa|c = v_* < c.$$ (25)

From its definition, it follows that $\kappa$ can be viewed as the Einstein $\beta$ factor related to the velocity $v_*$. The condition $v_* < c$, implies that $-1 < \kappa < +1$.

The dynamic Lorentz transformations for the dimensionless momentum and energy variables become

$$q'_A = \kappa^2 q_A \mathcal{E}_B - \kappa^2 q_B \mathcal{E}_A,$$ (26)

$$\mathcal{E}'_A = \kappa^2 \mathcal{E}_A \mathcal{E}_B - q_A q_B,$$ (27)

while the classical limit $c \rightarrow \infty$ is replaced by the limit $\kappa \rightarrow 0$.

For a particle at rest it results $E(0) = mc^2$ and then $\mathcal{E}(0) = 1/\kappa^2$. Then $1/\kappa^2$ represents the dimensionless rest energy of the particle. Alternatively $1/\kappa$ can be viewed as the refractive index of a medium in which the light speed is $v_*$.

From the Lorentz invariance it follows easily the energy-momentum dispersion relation

$$\kappa^4 \mathcal{E}^2 - \kappa^2 q^2 = 1.$$ (28)

After expressing in the right hand side of Eq. (26), the energy in terms of the momentum $\mathcal{E} = \sqrt{1 + \kappa^2 q^2 / \kappa^2}$, we obtain the momentum relativistic additivity law as follows

$$q'_A = q_A \sqrt{1 + \kappa^2 q_B^2} - q_B \sqrt{1 + \kappa^2 q_A^2},$$ (29)

which in the classical limit $\kappa \rightarrow 0$ reproduces the ordinary additivity law (17), of classical physics.

The Galilei relativity principle, holding both in classical physics and in special relativity, imposes the equivalence of all the inertial frames. According to this principle, the inverse transformation of (29) must have the same structure of the direct transformations (29) except for the substitutions $q'_A \leftrightarrow q_A$ and $q_B \rightarrow -q_B$ i.e.

$$q_A = q'_A \sqrt{1 + \kappa^2 q_B^2} + q_B \sqrt{1 + \kappa^2 q_A^2},$$ (30)

It is easy to verify that (30) follows directly from (29) and viceversa.

The Lorentz transformation for the relativistic momenta (29), representing the momenta additivity law in special relativity, has the important feature that the contributions of the two particles, appearing in the right hand side of the equation, are not factorized. Spontaneously the question emerges at this point, whether new variables exist, able to factorize the contribution of the two particles in the right hand side of the relativistic additivity law (29).

It is easy to verify that the new variable is given by function

$$\exp_\kappa(q) = \left(\sqrt{1 + \kappa^2 q^2} + \kappa q\right)^{1/\kappa},$$ (31)
so that the Lorentz transformation (29) assumes the following factorized form
\[
\exp_\kappa(q_A') = \exp_\kappa(-q_B) \exp_\kappa(q_A) .
\] (32)

On the other hand Galilei relativity principle imposes, for the inverse Lorentz transformation, the following factorized form
\[
\exp_\kappa(q_A) = \exp_\kappa(q_B) \exp_\kappa(q_A') .
\] (33)

By comparison of the above direct and inverse Lorentz transformations, it obtains the property
\[
\exp_\kappa(q) \exp_\kappa(-q) = 1 ,
\] (34)

which can be verified easily, by direct inspection of the definition (33).

It is remarkable that the function \(\exp_\kappa(q)\) emerges in one-particle special relativity as the variable, able to factorize the momentum Galilei transformation, and represents the relativistic generalization of the ordinary exponential which factorize the momentum Galilei transformation in classical physics. Clearly in the classical limit \(\kappa \to 0\), \(\exp_\kappa(q)\) reduces to \(\exp(q)\).

The inverse function of \(\exp_\kappa(q)\) indicated by \(\ln_\kappa(q)\) and defined through \(\ln_\kappa(\exp_\kappa(q)) = \exp_\kappa(\ln_\kappa(q)) = 1\), represents the relativistic generalization of the ordinary logarithm, which recovers in the classical limit \(\kappa \to 0\), and is given by
\[
\ln_\kappa(q) = \frac{q^\kappa - q^{-\kappa}}{2\kappa} .
\] (35)

The property (34) of \(\exp_\kappa(q)\), enforced by the Galilei relativity principle, transforms into the following property of \(\ln_\kappa(x)\)
\[
\ln_\kappa(1/x) = - \ln_\kappa(x) ,
\] (36)

holding also for the ordinary logarithm of classical physics.

IV. RELATIVISTIC STATISTICAL MECHANICS

In the previous section it has been shown that the functions \(\exp_\kappa(q)\) and \(\ln_\kappa(q)\) emerge as the relativistic generalizations of the ordinary exponential and logarithm functions of classical physics. Therefore in the following we pose
\[
\Lambda(x) = \ln_\kappa(x) ,
\] (37)
\[
\Lambda^{-1}(x) = \exp_\kappa(x) .
\] (38)

After taking into account of the property
\[
\frac{d}{dx} x \ln_\kappa(x) = \frac{1}{\gamma} \ln_\kappa(\epsilon x) ,
\] (39)

with
\[
\gamma = \frac{1}{\sqrt{1 - \kappa^2}} ,
\] (40)
\[
\epsilon = \exp_\kappa(\gamma) ,
\] (41)

the entropy (10) simplifies as
\[
S = -\gamma \int d^3p f \ln_\kappa(f/\epsilon) .
\] (42)

It is worth stressing that the latter relationship defines the relativistic entropy, as proportional to the mean value of \(-\ln_\kappa(f/\epsilon)\), like in the case of classical statistical mechanics where the Boltzmann entropy, \(S = -\int d^3p f \ln(f/\epsilon)\), is proportional to the mean value of \(-\ln(f/\epsilon)\). Clearly the entropy (12), in the classical limit reduces to the Boltzmann entropy. The constant \(\gamma\), given by (10), represents the Lorentz factor related to the velocity \(v_s\) appearing in (25), and in the classical limit, approaches the unity. On the other hand the constant \(\epsilon\) given by (11), represent a relativistic generalization of the Napier number \(e\), which recovers in the classical limit.

The entropy (12) can be written explicitly as follows
\[
S = \frac{1}{2\kappa} \int d^3p \left( \frac{f^{1-\kappa}}{1 - \kappa} - \frac{f^{1+\kappa}}{1 + \kappa} \right) ,
\] (43)
while the related stationary distribution (13) assumes the form
\[
f = \exp_\kappa(-\beta I + \beta \mu) ,
\] (44)
and reduces to the Maxwell-Boltzmann distribution, in the classical \(\kappa \to 0\) limit.

The distribution (14), in the global rest frame where \(U^\nu = (c, 0, 0, 0)\) and in absence of external forces i.e. \(A^\nu = 0\), simplifies as
\[
f = \exp_\kappa(-\beta E + \beta \mu) ,
\] (45)
\(E\) being the relativistic kinetic energy. This distribution at low energies \((E \to 0)\) reduces to the classical Maxwell-Boltzmann distribution i.e. \(f \propto \exp(-\beta E)\), while at relativistic energies \((E \to +\infty)\) presents power-law tails i.e.
\[
f \propto E^{-1/\kappa} ,
\] (46)
in accordance with the experimental evidence. We recall that the first experimental validation of the distribution (45), concerns cosmic rays and has been considered in ref. [4]. Recently a computer validation of the same distribution, has been considered in refs. [34, 35], where the relaxation in relativistic plasmas under wave-particle interaction, has been simulated numerically.

Finally, after posing \(\Lambda(f) = \ln_\kappa(f)\), in Eq. (11), the relativistic two-particle correlation function assumes the form
\[
f \otimes f_1 = \exp_\kappa(\ln_\kappa f + \ln_\kappa f_1) ,
\] (47)
which in the \(\kappa \to 0\) classical limit, reduces to \(ff_1\), as dictated by the molecular chaos hypothesis. Consequently the relationship (47) can be viewed as defining the relativistic extension of the molecular chaos hypothesis.
V. CONCLUSIONS

Let us consider the probability distribution function

\[ f = \xi_1 \exp\left(-\xi_2 \beta \left[ E - \mu \right]\right), \quad (48) \]

\( \xi_1 \) and \( \xi_2 \) being two arbitrary constants. In the expression of the distribution function \( E \) is the relativistic microscopic energy, while \( \beta \) and \( \beta \mu \) are the Lagrange multipliers. The Maximum Entropy Principle asserts that the distribution \((48)\) can be obtained, by maximizing the entropy

\[ S = -\frac{\gamma}{\xi_2} \int d^3 p \ f \ln(f/\epsilon \xi_1), \quad (49) \]

with \( \gamma = 1/\sqrt{1-\kappa^2} \) and \( \epsilon = \exp(\gamma) \), under the constraints imposing the conservation of the norm of \( f \) and of the mean value of the \( E \). It is remarkable the above distribution and entropy are linked through the Maximum Entropy Principle, independently on the particular values of the arbitrary constants \( \xi_1 \) and \( \xi_2 \).

In kinetic theory, customarily it is posed

\[ \xi_1 = \xi_2 = 1, \quad (50) \]

in order to simplify the expression of the distribution function and of the two particle correlation function, appearing in kinetic equation. This choice, is made naturally, in the present paper, being the starting point of our presentation the kinetic equation \((50)\).

On the other hand, in statistical mechanics, in order to simplify the expression of the entropy, it is posed

\[ \xi_1 = 1/\epsilon, \quad (51) \]

\[ \xi_2 = \gamma. \quad (52) \]

This latter choice has been made for instance in the ref. \[4, 5\] where the starting point for the presentation of the theory was the entropy functional.
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