On rough traces of BV functions

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Abstract

In metric measure spaces, we study boundary traces of BV functions in domains equipped with a doubling measure and supporting a Poincaré inequality, but possibly having a very large and irregular boundary. We show that the trace exists in the ordinary sense in a certain part of the boundary, and that this part is sufficient to determine the integrability of the rough trace, as well as the possibility of zero extending the function to the whole space as a BV function.

1 Introduction

Boundary traces of Sobolev and BV (bounded variation) functions are a relevant concept for example in the study of Dirichlet boundary value problems. Classical treatments of boundary traces, or traces for short, of BV functions in Euclidean spaces can be found e.g. in [2, Chapter 3] and [8, Chapter 5]. Traditionally, one considers domains with a Lipschitz boundary. On the other hand, traces can be studied also in more general domains, and in abstract metric measure spaces \((X, d, \mu)\). Usually one assumes that the measure \(\mu\) satisfies a doubling property and that the space supports a Poincaré inequality. In such a setting it is natural to define the trace as follows: given a function \(u\) defined on an open set \(\Omega \subset X\), the number \(Tu(x) \in \mathbb{R}\) is the boundary trace of \(u\) at \(x \in \partial \Omega\) if

\[
\lim_{r \to 0} \int_{\Omega \cap B(x, r)} |u - Tu(x)| \, d\mu = 0.
\]

In [20, 21], various properties of traces of BV functions in metric spaces were shown. Despite being far more general than the classical setting of Lipschitz domains, the theory in these papers still required rather strong regularity of \(\Omega\) and especially of its boundary. If one assumes very little or no regularity of \(\Omega\), the ordinary boundary trace might not exist on the entire boundary, but the following rough trace always exists:

\[
T_*u(x) := \sup \{t \in \mathbb{R} : \theta^*(\{u > t\}, x) > 0\}, \quad x \in \partial \Omega.
\]

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Here we denote by $\theta^*$ the upper measure density; see Section 2 for definitions. Rough traces have been studied by Maz’ya [22, Section 9.5] in Euclidean spaces, and by Buffa–Miranda [7] in metric spaces.

In the current paper, we are interested in considering open sets $\Omega \subset X$ that have some regularity but significantly less than those considered [20, 21]. This means in particular that the boundary $\partial \Omega$ can be very large and that the (ordinary) trace $Tu$ might only exist on a part of it. Nonetheless, this part turns out to be enough to largely determine the behavior of the rough trace $T^*u$ on the entire boundary. We will assume that, in a suitable sense, $\Omega$ is equipped with a doubling measure and supports a Poincaré inequality locally near its boundary. We abbreviate this by saying that $\Omega$ is PLB, see Definition 3.5.

Our main result is the following. We define $\Omega_\beta$ to be the set where the lower measure density of $\Omega$ is at least a constant $\beta > 0$, and $\Sigma_\beta \Omega := \Omega_{\beta} \cap (X \setminus \Omega_{\beta})$.

**Theorem 1.1.** Suppose $\Omega \subset X$ is PLB and let $u \in BV(\Omega)$. Then

1. the trace $Tu(x)$ exists at $\mathcal{H}$-a.e. $x \in \partial \Omega \cap \Omega_\beta$;
2. the integrals of the boundary traces satisfy
   \[
   \int_{\partial \Omega \cap \{\theta^*(\Omega, \cdot) > 0\}} |T^*u| d\mathcal{H} = \int_{\partial \Omega \cap \Omega_\beta} |Tu| d\mathcal{H} ;
   \]
   and
3. the zero extension of $u$ from $\Omega$ to $X$ is in $BV(X)$ if and only if
   \[
   \int_{\Sigma_\beta \Omega} |Tu| d\mathcal{H} < \infty .
   \]

In essence, the theorem says that the trace exists in the ordinary sense on a part of the boundary, namely $\partial \Omega \cap \Omega_\beta$, and that this is enough to determine the integrability of the rough trace, as well as the zero extension property of the function. We will prove the three different parts of the theorem in Sections 3, 4, and 5. In Section 5 we also examine the possibility of zero extending $u$ as a BV function without adding any total variation.

## 2 Preliminaries

In this section we introduce the basic notation, definitions, and assumptions that are employed in the paper.

Throughout this paper, $(X, d, \mu)$ is a complete metric space that is equipped with a metric $d$ and a Borel regular outer measure $\mu$ satisfying a doubling property, meaning that there exists a constant $C_d \geq 1$ such that

\[
0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty
\]

for every open ball $B(x, r) := \{ y \in X : d(y, x) < r \}$, with $x \in X$ and $r > 0$. We assume that $X$ consists of at least 2 points. When a property holds outside of a set of $\mu$-measure zero, we say that it holds at $\mu$-a.e. point.
All functions defined on $X$ or its subsets will take values in $[-\infty, \infty]$. As a complete metric space equipped with a doubling measure, $X$ is proper, that is, closed and bounded sets are compact. A function $u$ defined in an open set $\Omega \subset X$ is said to be in the class $L^{1}_{\text{loc}}(\Omega)$ if it is in $L^{1}(\Omega')$ for every open $\Omega' \subset \Omega$. Here $\Omega' \subset \Omega$ means that $\overline{\Omega'}$ is a compact subset of $\Omega$. Other local spaces of functions are defined analogously.

By a curve we mean a rectifiable continuous mapping from a compact interval of the real line into $X$. The length of a curve $\gamma$ is denoted by $\ell_{\gamma}$. We will assume every curve to be parametrized by arc-length (see e.g. [10, Theorem 3.2]), so we have $\gamma: [0, \ell_{\gamma}] \to X$. A nonnegative Borel function $g$ on $X$ is an upper gradient of a function $u$ on $X$ if for all nonconstant curves $\gamma$, we have

$$|u(x) - u(y)| \leq \int_{\gamma} g \, ds := \int_{0}^{\ell_{\gamma}} g(\gamma(s)) \, ds,$$

(2.1)

where $x$ and $y$ are the end points of $\gamma$. We interpret $|u(x) - u(y)| = \infty$ whenever at least one of $|u(x)|$, $|u(y)|$ is infinite. Upper gradients were originally introduced in [13].

We say that a domain $\Omega \subset X$ is $M$-uniform, with constant $M \geq 1$, if for every $x, y \in \Omega$ there exists a curve $\gamma: [0, \ell_{\gamma}] \to \Omega$ with $\ell_{\gamma} \leq M d(x, y)$, $\gamma(0) = x$, $\gamma(\ell_{\gamma}) = y$, and such that for all $t \in [0, \ell_{\gamma}]$ we have

$$\text{dist}(\gamma(t), X \setminus \Omega) \geq M^{-1} \min\{t, \ell_{\gamma} - t\}.$$  

(2.2)

We say that a domain is uniform if it is $M$-uniform for some $1 \leq M < \infty$.

Let $1 \leq p < \infty$. (We will work almost exclusively with $p = 1$.) The $p$-modulus of a family of curves $\Gamma$ is defined by

$$\text{Mod}_{p}(\Gamma) := \inf \int_{X} \rho^{p} \, d\mu,$$

where the infimum is taken over all nonnegative Borel functions $\rho$ such that $\int_{\gamma} \rho \, ds \geq 1$ for every curve $\gamma \in \Gamma$. A property is said to hold for $p$-almost every curve if it fails only for a curve family with zero $p$-modulus. If $g$ is a nonnegative $\mu$-measurable function on $X$ and (2.1) holds for $p$-almost every curve, then we say that $g$ is a $p$-weak upper gradient of $u$. By only considering curves $\gamma$ in a set $A \subset X$, we can talk about a function $g$ being a ($p$-weak) upper gradient of $u$ in $A$.

Given an open set $\Omega \subset X$, we let

$$\|u\|_{N^{1,p}(\Omega)} := \left( \int_{\Omega} |u|^{p} \, d\mu + \inf \int_{\Omega} g^{p} \, d\mu \right)^{1/p},$$

where the infimum is taken over all $p$-weak upper gradients $g$ of $u$ in $\Omega$. Then we define the Newton-Sobolev space

$$N^{1,p}(\Omega) := \{u: \|u\|_{N^{1,p}(\Omega)} < \infty\},$$

which was first introduced in [24]. For any $u \in N^{1,p}(\Omega)$ the quantity $\|u\|_{N^{1,p}(\Omega)}$ agrees with the classical Sobolev norm, see e.g. [4, Corollary A.4]. It is known that
for every $u \in N_{1,loc}^{1,p}(\Omega)$ there exists a minimal $p$-weak upper gradient of $u$ in $\Omega$, which we always denote by $g_u$, satisfying $g_u \leq g \mu$-almost everywhere in $\Omega$ for every $p$-weak upper gradient $g \in L_{loc}^p(\Omega)$ of $u$ in $\Omega$, see [4, Theorem 2.25].

For any open sets $\Omega, \Omega_0 \subset X$, the space of Newton-Sobolev functions with zero boundary values is defined as

$$N_{0}^{1,p}(\Omega, \Omega_0) := \{u|_{\Omega \cap \Omega_0} : u \in N_{1,loc}^{1,p}(\Omega_0) \text{ and } u = 0 \text{ in } \Omega_0 \setminus \Omega\}. \quad (2.3)$$

This space is a subspace of $N^{1,p}(\Omega \cap \Omega_0)$, and it can be understood to be a subspace of $N_{0}^{1,p}(\Omega, X)$. We let $N_{0}^{1,p}(\Omega) := N_{0}^{1,p}(\Omega, X)$.

The $p$-capacity of a set $A \subset X$ is defined by

$$\text{Cap}_p(A) := \inf \|u\|_{N_{1,loc}^{1,p}(X)}^p, \quad (2.4)$$

where the infimum is taken over all functions $u \in N_{1,loc}^{1,p}(X)$ such that $u \geq 1$ in $A$.

For any set $A \subset X$ and $0 < R < \infty$, the Hausdorff content of codimension one is defined by

$$H_R(A) := \inf \left\{ \sum_{j=1}^{\infty} \mu(B(x_j, r_j)) : A \subset \bigcup_{j=1}^{\infty} B(x_j, r_j), r_j \leq R \right\}. \quad (2.5)$$

We also allow finite coverings by interpreting $\mu(B(x, 0))/0 = 0$. The codimension one Hausdorff measure of $A \subset X$ is then defined by

$$H(A) := \lim_{R \to 0} H_R(A).$$

We also define “centered” versions $\hat{H}_R$ and $\hat{H}$ in the same way, but with the additional requirement that $x_j \in A$. By the doubling property of $\mu$, these are comparable to $H_R$ and $H$.

By [11, Theorem 4.3, Theorem 5.1] we know that for every $A \subset X$,

$$\text{Cap}_1(A) = 0 \text{ if and only if } H(A) = 0. \quad (2.5)$$

We will assume throughout the paper that $X$ supports a $(1,1)$-Poincaré inequality, meaning that there exist constants $C_P > 0$ and $\lambda \geq 1$ such that for every ball $B(x, r)$, every $u \in L_{loc}^1(X)$, and every upper gradient $g$ of $u$, we have

$$\int_{B(x, r)} |u - u_{B(x, r)}| \, d\mu \leq C_P \int_{B(x, \lambda r)} g \, d\mu,$$

where

$$u_{B(x, r)} := \int_{B(x,r)} u \, d\mu := \frac{1}{\mu(B(x, r))} \int_{B(x, r)} u \, d\mu.$$

Next we present the definition and basic properties of functions of bounded variation on metric spaces, following Miranda Jr. [23]. Given an open set $\Omega \subset X$ and a function $u \in L_{loc}^1(\Omega)$, we define the total variation of $u$ in $\Omega$ by

$$\|Du\|_{(\Omega)} := \inf \left\{ \liminf_{i \to \infty} \int_{\Omega} g_{u_i} \, d\mu : u_i \in N_{1,loc}^{1,1}(\Omega), u_i \to u \text{ in } L_{loc}^1(\Omega) \right\}. \quad (2.6)$$
where each \( g_{u_i} \) is the minimal 1-weak upper gradient of \( u_i \) in \( \Omega \). We say that a function \( u \in L^1(\Omega) \) is of bounded variation, and denote \( u \in BV(\Omega) \), if \( \|Du\|_*(\Omega) < \infty \). For an arbitrary set \( A \subset X \), we define
\[
\|Du\|(A) := \inf \{\|Du\|(W): A \subset W, W \subset X \text{ is open} \}.
\]

In [23], local Lipschitz constants were used in place of upper gradients, but the theory can be developed similarly with either definition. It is sometimes also required that \( u_i \in \text{Lip}_{\text{loc}}(\Omega) \) instead of \( u_i \in N^{1,1}_{\text{loc}}(\Omega) \), but for us this does not make a difference, since \( \text{Lip}_{\text{loc}}(\Omega) \) is dense in \( N^{1,1}_{\text{loc}}(\Omega) \), see [4, Theorem 4.57].

If \( u \in L^1_{\text{loc}}(\Omega) \) and \( \|Du\|_*(\Omega) < \infty \), then \( \|Du\|_*(\cdot) \) is a Borel regular outer measure on \( \Omega \) by [23, Theorem 3.4]. A \( \mu \)-measurable set \( E \subset X \) is said to be of finite perimeter if \( \|D\chi_E\|_*(X) < \infty \), where \( \chi_E \) is the characteristic function of \( E \). The perimeter of \( E \) in \( \Omega \) is also denoted by
\[
P(E, \Omega) := \|D\chi_E\|_*(\Omega).
\]

We define the lower and upper densities of a set \( E \subset X \) at a point \( x \in X \) as follows:
\[
\theta_*(E, x) := \liminf_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} \quad \text{and} \quad \theta^*(E, x) := \limsup_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))}.
\]
The measure-theoretic interior of a set \( E \subset X \) is defined by
\[
I_E := \{x \in X: \theta^*(X \setminus E, x) = 0\},
\]
and the measure-theoretic exterior by
\[
O_E := \{x \in X: \theta^*(E, x) = 0\}.
\]
The measure-theoretic boundary is defined as the set of points \( x \in X \) at which both \( E \) and its complement have strictly positive upper density:
\[
\partial^* E := \{x \in X: \theta^*(E, x) > 0 \text{ and } \theta^*(X \setminus E, x) > 0\}.
\]
Note that the space \( X \) is always partitioned into the disjoint sets \( I_E, O_E, \) and \( \partial^* E \). We also let
\[
E_b := \{x \in X: \theta_*(E, x) \geq b\}, \quad b > 0.
\] (2.7)
The strong boundary \( \Sigma_b E \), for \( 0 < b \leq 1/2 \), is defined as \( \Sigma_b E := E_b \cap (X \setminus E)_b \).

For an open set \( \Omega \subset X \) and a \( \mu \)-measurable set \( E \subset X \) with \( P(E, \Omega) < \infty \), we know that
\[
\mathcal{H}(\partial^* E \setminus \Sigma_\gamma E) \cap \Omega = 0
\] (2.8)
for some number \( \gamma = \gamma(C_d, C_P, \lambda) > 0 \), see [1, Theorem 5.4]. We also know that for any Borel set \( A \subset \Omega \),
\[
P(E, A) = \int_{\Sigma_\gamma E \cap A} \theta_E d\mathcal{H},
\] (2.9)
where $\theta_E: \Omega \to [\alpha, C_d]$ with $\alpha = \alpha(C_d, C_P, \lambda) > 0$, see [1, Theorem 5.3] and [3, Theorem 4.6]. In particular, $P(E, \Omega) < \infty$ implies that $\mathcal{H}(\partial^*E \cap \Omega) < \infty$. Federer’s characterization of sets of finite perimeter states that the converse is also true; more precisely, if $E \subset X$ is a $\mu$-measurable set such that $\mathcal{H}(\partial^*E \cap \Omega) < \infty$, then $P(E, \Omega) < \infty$, see [19, Theorem 1.1]. See also Federer [9, Section 4.5.11] for the original Euclidean result.

The strong boundary can also be used to characterize sets of finite perimeter, as follows.

**Theorem 2.10 ([17, Theorem 1.1]).** Let $\Omega \subset X$ be an open set and let $E \subset X$ be a $\mu$-measurable set with $\mathcal{H}(\Sigma_\beta E \cap \Omega) < \infty$, where $0 < \beta \leq 1/2$ only depends on the doubling constant of the measure and the constants in the Poincaré inequality. Then $P(E, \Omega) < \infty$.

Throughout this paper, we will use $\beta$ to denote the constant from this theorem; we can assume that $\beta \leq \gamma$. Now combining Theorem 2.10 and (2.9), we obtain that for every open $\Omega \subset X$ and $\mu$-measurable $E \subset X$, we have

$$P(E, \Omega) \leq C\mathcal{H}(\Sigma_\beta E \cap \Omega)$$

for some constant $C \geq 1$ depending only on the constants $C_d, C_P, \lambda$. By combining (2.8) and (2.9), we also get

$$\mathcal{H}(\Sigma_\beta E \cap \Omega) \leq C P(E, \Omega).$$

In total, we have

$$\frac{1}{C} \mathcal{H}(\Sigma_\beta E \cap \Omega) \leq P(E, \Omega) \leq C \mathcal{H}(\Sigma_\beta E \cap \Omega) \quad (2.11)$$

for some constant $C \geq 1$ depending only on the constants $C_d, C_P, \lambda$.

For a function $u$ defined on an open set $\Omega \subset X$, we will often abbreviate superlevel sets in the form

$$\{ u > t \} := \{ x \in \Omega : u(x) > t \}, \quad t \in \mathbb{R}.$$

The following coarea formula is given in [23, Proposition 4.2]: if $\Omega \subset X$ is open and $u \in L^1_{\text{loc}}(\Omega)$, then

$$\|Du\|(\Omega) = \int_{-\infty}^{\infty} P(\{u > t\}, \Omega) \, dt. \quad (2.12)$$

The integral should be understood as an upper integral; however if either side is finite, then both sides are finite and the integrand is measurable.

The lower and upper approximate limits of a function $u$ on an open set $\Omega$ are defined respectively by

$$u^\wedge(x) := \sup \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(B(x, r) \cap \{ u < t \})}{\mu(B(x, r))} = 0 \right\}$$
and
\[ u^\vee(x) := \inf \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(B(x, r) \cap \{u > t\})}{\mu(B(x, r))} = 0 \right\} \] (2.13)
for \( x \in \Omega \). We use the usual convention that the supremum and infimum of an empty set are \(-\infty\) and \(\infty\), respectively. The jump set of \( u \) is then defined by
\[ S_u := \{ u^\wedge < u^\vee \}. \]

Since we understand \( u^\wedge \) and \( u^\vee \) to be defined only on \( \Omega \), also \( S_u \) is understood to be a subset of \( \Omega \). It is straightforward to check that \( u^\wedge \) and \( u^\vee \) are always Borel functions.

The following Lebesgue point result was proved in [16, Theorem 3.5]: if \( \Omega \subset X \) is open and \( u \in BV(\Omega) \), then for \( \mathcal{H}\text{-a.e. } x \in \Omega \) we have
\[ \lim_{r \to 0} \int_{B(x,r)} |u - u^\vee(x)| \, d\mu = 0. \] (2.14)
(Equally well we could replace \( u^\vee(x) \) by \( u^\wedge(x) \).

By [3, Theorem 5.3], the variation measure of a BV function can be decomposed into the absolutely continuous and singular part, and the latter into the Cantor and jump part, as follows. Given an open set \( \Omega \subset X \) and \( u \in BV(\Omega) \), we have for any Borel set \( A \subset \Omega \)
\[ \|Du\|(A) = \|Du\|^a(A) + \|Du\|^s(A) \]
\[ = \|Du\|^a(A) + \|Du\|^c(A) + \|Du\|^j(A) \]
\[ = \int_A a \, d\mu + \|Du\|^c(A) + \int_{A \setminus S_u} \int_{u^\wedge(x)} \int_{u^\vee(x)} \theta_{\{u > t\}}(x) \, dt \, d\mathcal{H}(x), \] (2.15)
where \( a \in L^1(\Omega) \) is the density of the absolutely continuous part and the functions \( \theta_{\{u > t\}} \in [\alpha, C_d] \) are as in (2.9).

Throughout this paper we assume that \( (X, d, \mu) \) is a complete metric space that is equipped with the doubling measure \( \mu \) and supports a \((1,1)\)-Poincaré inequality.

### 3 Existence of boundary traces

In this section we study existence results for boundary traces on a certain part of the boundary. The symbol \( \Omega \subset X \) always denotes an arbitrary open set. Recall that when \( u \) is a function defined on \( \Omega \), we denote
\[ \{u > t\} := \{x \in \Omega : u(x) > t\} \subset \Omega, \quad t \in \mathbb{R}. \]

We start with the definitions of the boundary traces, or traces for short, that we will study throughout the paper.
Definition 3.1. Let $u$ be a $\mu$-measurable function on $\Omega$.

The rough trace of $u$ at $x \in \partial \Omega$ is

$$T^* u(x) := \sup \{ t \in \mathbb{R} : \theta^* (\{ u > t \}, x) > 0 \}.$$ 

As before, we interpret the supremum of an empty set to be $-\infty$.

We define a second version of the rough trace at $x \in \partial \Omega$ by

$$T^*_\beta u(x) := \sup \{ t \in \mathbb{R} : \theta^* (\{ u > t \}, x) \geq \beta \},$$

where $0 < \beta \leq 1/2$ is the constant from Theorem 2.10.

Finally, if for $x \in \partial \Omega$ there exists $b \in \mathbb{R}$ such that

$$\lim_{r \to 0} \int_{\Omega \cap B(x, r)} |u - b| d\mu = 0,$$

then we say that $T u(x) := b$ is an (ordinary) trace of $u$ at $x$.

Remark 3.2. In the classical definition of the rough trace given in [22, Section 9.5], in the supremum it is required that $x \in \partial^* \{ u > t \}$, but we replace this with the weaker requirement $\theta^* (\{ u > t \}, x) > 0$, since we wish to consider also points $x \in \partial \Omega$ where $\Omega$ has density one. Note that if $u \in BV(\Omega)$, it follows from the coarea formula (2.12) that

$$P(\{ u > t \}, \Omega) < \infty \quad \text{for a.e. } t \in \mathbb{R}.$$ 

In the classical definition, it is additionally required that $\{ u > t \}$ has finite perimeter in the whole space, that is $P(\{ u > t \}, X) < \infty$. We do not wish to require this, since we consider very general open sets $\Omega$ that could have infinite perimeter in $X$, and then the super-level sets $\{ u > t \}$ often also have infinite perimeter in $X$. However, our definition coincides with the classical one $\mathcal{H}$-almost everywhere on the boundary under the assumptions that $P(\Omega, X) < \infty$ and $\mathcal{H}(\partial \Omega \setminus \partial^* \Omega) = 0$, which are assumed in the classical theory.

It is straightforward to check that all three traces are Borel functions on $\partial \Omega$ (in the case of $T u$, more precisely it is Borel on the subset of $\partial \Omega$ where it is defined), and so integrals with respect to the Borel outer measure $\mathcal{H}$ are well-defined. In terms of comparisons between the traces, clearly $T^*_\beta u(x) \leq T u(x)$ for all $x \in \partial \Omega$. If $x \in \partial \Omega$ such that $\theta^*(\Omega, x) > 0$ and $T u(x)$ exists, then it is easy to check that $T^*_u(x) = T^*_\beta u(x) = T u(x)$.

We start by giving a simple example demonstrating that in completely general open sets $\Omega$, the (ordinary) trace $T u$ might not exist at any point of the boundary $\partial \Omega$. For this reason, in our main results we will assume that $\Omega$ has some regularity, formulated in terms of local doubling and Poincaré conditions.

Example 3.3. Let $C \subset [0, 1]$ be the ternary Cantor set. Let $\Omega := (0, 1) \setminus C$. Then $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$, where each $\Omega_j$ consists of $2^{j-1}$ open intervals of length $3^{-j}$. Let

$$u := \sum_{j=1}^{\infty} b_j \chi_{\Omega_j}, \quad b_j := \begin{cases} j & \text{for } j \text{ odd}, \\ -j & \text{for } j \text{ even}. \end{cases}$$
Note that
\[ \|u\|_{L^1(\Omega)} = \frac{1}{2} \sum_{j=1}^{\infty} j \left( \frac{2}{3} \right)^j < \infty. \]

Moreover \( \|Du\|_{L^1(\Omega)} = 0 \), since \( u \) is locally constant. Thus \( u \in BV(\Omega) \).

Note that \( \partial \Omega = C \). Now for every \( x \in C \) and every \( t > 0 \), for every odd \( j > t \) we have that \( B(x, 3^{-j+1}) \) contains an interval of length \( 3^{-j} \) and belonging to \( \{u > t\} \).

Denoting the 1-dimensional Lebesgue measure by \( L^1 \), we get
\[ \limsup_{r \to 0} \frac{L^1(B(x, r) \cap \{u > t\})}{L^1(B(x, r))} \geq \frac{1}{4} \]
and similarly
\[ \limsup_{r \to 0} \frac{L^1(B(x, r) \cap \{u < -t\})}{L^1(B(x, r))} \geq \frac{1}{4}. \]

Thus for every \( x \in C \), we have \( T_u(x) = \infty \) while the trace \( Tu(x) \) fails to exist. Moreover, due to the alternating between negative and positive values in \( b_j \), the trace would still fail to exist with any reasonable definition that would allow \( Tu(x) \) to take the values \( \pm \infty \).

In the rest of the work, most of the time we will consider open sets \( \Omega \) that satisfy certain regularity near the boundary, at least locally. In order to define such \( \Omega \), we will consider subsets \( A \subset \overline{\Omega} \) as metric spaces in their own right (including the case \( A = \Omega \)).

**Definition 3.4.** For any \( A \subset \overline{\Omega} \), we define the metric measure space \((A, d, \mu_A)\) as follows. The metric \( d \) is simply inherited from \( X \). When \( A \subset \Omega \), we equip it with the measure \( \mu \) restricted to subsets of \( A \). This restriction is a Borel regular outer measure on \( A \) by [14, Lemma 3.3.11]. When \( A \subset \overline{\Omega} \) is Borel, we equip it with the zero extension of \( \mu \) from \( \Omega \cap A \) to \( A \), denoted by \( \mu_A \). That is, for every \( D \subset A \) we have \( \mu_A(D) = \mu(D \cap \Omega) \). By [14, Lemma 3.3.16] we know that \( \mu_A \) is a Borel regular outer measure on \( A \). We denote a ball in the space \( A \) by \( B_A(x, r) := B(x, r) \cap A \).

We also denote by \( H_{A,R}, R > 0, \) and \( H_A \) the codimension one Hausdorff content and measure in the space \((A, d, \mu_A)\).

**Definition 3.5.** We say that the open set \( \Omega \) is PLB (1-Poincaré space locally near its boundary) if for \( H \)-a.e. \( x \in \partial \Omega \cap \Omega_\beta \) (recall (2.7)) there exists an open set \( W \subset \Omega \) such that \( (W, d, \mu_W) \) is a metric space for which \( \mu_W \) is doubling and \( W \) supports a \((1,1)\)-Poincaré inequality, and \( x \in V \) for some relatively open set \( V \subset \overline{\Omega} \) with \( V \cap \Omega \subset W \).

Briefly, we will sometimes say that “\((W, d, \mu_W)\) satisfies doubling and \((1,1)\)-Poincaré”. Given sets \( V \) and \( W \) as above, note that
\[ \partial \Omega \cap V = \partial W \cap V. \quad (3.6) \]

**Remark 3.7.** There is a wide range of domains \( \Omega \) that satisfy doubling and Poincaré globally (more precisely, \((\Omega, d, \mu_\Omega)\) satisfies doubling and \((1,1)\)-Poincaré), but for us
it will be enough to assume the significantly weaker local conditions as in Definition 3.5. In this way, we include more open sets $\Omega$ in our theory, and even in cases where doubling and Poincaré hold globally in $\Omega$, the local conditions are often much easier to check. In particular, for many Euclidean domains $\Omega$ it is easy to check that for $H$-a.e. $x \in \partial \Omega \cap \Omega_j$, for sufficiently small $r > 0$ the set $W := B(x, r) \cap \Omega$ has a Lipschitz boundary and is thus a uniform domain. (Note that in the Euclidean space $(\mathbb{R}^n, d_{\text{euc}}, L^n)$, the codimension one Hausdorff measure $H$ is comparable to the $n-1$-dimensional Hausdorff measure $H^{n-1}$.) By [6, Theorem 4.4], the space $(W, d, \mu)$ then satisfies doubling and $(1, 1)$-Poincaré. We will use these facts in Examples 3.14 and 4.7.

**Lemma 3.8.** Let $V$ and $W$ be two sets as in Definition 3.5. Let $A \subset V$. Then $\mathcal{H}(A) = 0$ implies that also $\mathcal{H}(A) = 0$.

Here $\mathcal{H}(A)$ is the “centered” version of $\mathcal{H}$, that is, in the definition the coverings are required to be centered in the set $A$.

Note that from Definition 3.5 we get $V \subset \overline{W}$, so then also $A \subset \overline{W}$.

**Proof.** Suppose $\mathcal{H}(A) = 0$. Let

$$V_\delta := \{ x \in V : d(x, \overline{\Omega} \setminus V) > \delta \}, \quad \delta > 0.$$  

We have $V = \bigcup_{j=1}^\infty V_1/j$, and so it is enough to prove that $\mathcal{H}(V_\delta \cap A) = 0$ for an arbitrary but fixed $\delta > 0$.

Fix also $0 < \varepsilon < \delta$. We find a covering $\{B(x_j, r_j)\}_{j=1}^\infty$ of $V_\delta \cap A$ in the space $\overline{W}$ such that $r_j < \varepsilon$ and

$$\sum_{j=1}^\infty \frac{\mu(B(x_j, r_j))}{r_j} < \varepsilon.$$  

By the doubling property of $\mu$, we can assume that $x_j \in V_\delta \cap A$. Thus $B(x_j, r_j) \subset V \subset \overline{W}$, so that $\overline{W} \cap B(x_j, r_j) = \overline{\Omega} \cap B(x_j, r_j)$. Then by (3.6), we have $W \cap B(x_j, r_j) = \Omega \cap B(x_j, r_j)$. Thus

$$\mu(B(x_j, r_j)) = \mu(W \cap B(x_j, r_j)) = \mu(\Omega \cap B(x_j, r_j)) = \mu(B(x_j, r_j)).$$  

Thus also

$$\hat{\mathcal{H}}(V_\delta \cap A) \leq \sum_{j=1}^\infty \frac{\mu(B(x_j, r_j))}{r_j} < \varepsilon.$$  

Since $\varepsilon > 0$ was arbitrary, we get $\hat{\mathcal{H}}(V_\delta \cap A) = 0$. \hfill $\square$

**Lemma 3.9.** Suppose $A \subset \overline{\Omega}$ such that $\theta_\ast(\Omega, x) > 0$ for all $x \in A$. Then $\mathcal{H}(A) = 0$ if and only if $\hat{\mathcal{H}}(A) = 0$.

**Proof.** First suppose that $\mathcal{H}(A) = 0$. Fix $\varepsilon > 0$. There exists a covering $\{B(x_k, r_k)\}_{k=1}^\infty$ of $A$ with $r_k < \varepsilon/2$ and

$$\sum_{k=1}^\infty \frac{\mu(B(x_k, r_k))}{r_k} < \varepsilon/2.$$  

(3.10)
We can assume that every ball in the covering intersects \( A \). Thus for every \( k \in \mathbb{N} \) we find a point \( y_k \in B(x_k, r_k) \cap A \). In particular \( y_k \in \Omega \), and \( \{ B_\Omega(y_k, 2r_k) \}_{k=1}^\infty \) is a covering of \( A \) in the space \( \Omega \), with \( B(y_k, 2r_k) \subset B(x_k, 3r_k) \). Thus we get

\[
\hat{\mathcal{H}}_{\Omega, \varepsilon}(A) \leq \sum_{k=1}^\infty \frac{\mu(B_\Omega(y_k, 2r_k))}{2r_k} \leq \sum_{k=1}^\infty \frac{\mu(B(x_k, 3r_k))}{r_k} < \varepsilon
\]

by (3.10). It follows that \( \hat{\mathcal{H}}_{\Omega}(A) = 0 \).

Conversely, suppose that \( \hat{\mathcal{H}}_{\Omega}(A) = 0 \). For each \( j \in \mathbb{N} \), let

\[
A_j := \left\{ x \in A : \inf_{0 < r \leq 1/j} \frac{\mu(\Omega \cap B(x, r))}{\mu(B(x, r))} \geq \frac{1}{j} \right\}.
\]

Then \( A = \bigcup_{j=1}^\infty A_j \), and of course \( \hat{\mathcal{H}}_{\Omega}(A_j) = 0 \) for all \( j \in \mathbb{N} \). Fix \( j \in \mathbb{N} \) and fix \( 0 < \varepsilon < 1/j \). There exists a covering \( \{ B_\Omega(x_k, r_k) \}_{k=1}^\infty \) of \( A_j \) with \( x_k \in A_j \), \( r_k < \varepsilon \), and

\[
\sum_{k=1}^\infty \frac{\mu(B_\Omega(x_k, r_k))}{r_k} < \frac{\varepsilon}{j}.
\]

Now \( \{ B(x_k, r_k) \}_{k=1}^\infty \) is a covering of \( A_j \) in \( X \), and so

\[
\mathcal{H}_\varepsilon(A_j) \leq \sum_{k=1}^\infty \frac{\mu(B(x_k, r_k))}{r_k} \leq j \sum_{k=1}^\infty \frac{\mu(\Omega \cap B(x_k, r_k))}{r_k} = j \sum_{k=1}^\infty \frac{\mu(B_\Omega(x_k, r_k))}{r_k} < \varepsilon.
\]

In conclusion, \( \mathcal{H}(A_j) = 0 \). It follows that

\[
\mathcal{H}(A) \leq \sum_{j=1}^\infty \mathcal{H}(A_j) = 0.
\]

The statement of the following lemma is quite close to the definition of the total variation (2.6); the idea is simply that we can have \( u_i \in N^{1,1}(\Omega) \) and convergence in \( L^1(\Omega) \), instead of only \( u_i \in N^{1,1}_{1,1}(\Omega) \) and convergence in \( L^1_{1,1}(\Omega) \).

**Lemma 3.11.** Let \( u \in BV(\Omega) \). Then there exists a sequence of functions \( u_i \in N^{1,1}(\Omega) \) with \( u_i \to u \) in \( L^1(\Omega) \) and

\[
\|Du\|(\Omega) = \lim_{i \to \infty} \int_\Omega g_{u_i} \, d\mu,
\]

where each \( g_{u_i} \) is the minimal 1-weak upper gradient of \( u_i \) in \( \Omega \).

**Proof.** This is given as part of [20, Corollary 6.7]. Note that the “strong relative isoperimetric inequality” mentioned there is proved in [19, Corollary 5.6].

Now we prove the existence of the trace \( Tu \) on a part of the boundary. This gives Claim (1) of Theorem 1.1.
**Proposition 3.12.** Suppose \( \Omega \) is PLB. Let \( u \in BV(\Omega) \). Then the trace \( Tu(x) \) exists for \( \mathcal{H} \text{-a.e. } x \in \partial \Omega \cap \Omega_\beta \).

**Proof.** By definition, we can cover \( \mathcal{H} \)-almost all of \( \partial \Omega \cap \Omega_\beta \) by relatively open sets \( V \subset \Omega \) such that \( V \subset \overline{W} \) for open sets \( W \subset \Omega \) for which each \((W, d, \mu_W)\) satisfies doubling and \((1, 1)\)-Poincaré. Since \( X \) is Lindelöf and thus so are its subsets, we find a countable collection of these sets \( \{W_j\}_{j=1}^\infty \) such that the corresponding sets \( V_j \) cover \( \mathcal{H} \)-almost all of \( \partial \Omega \cap \Omega_\beta \). Fix \( V_j \) and \( W_j \) and denote them \( V \) and \( W \). By Lemma 3.8, among subsets of \( V \), being a null set with respect to \( \mathcal{H} \) implies being a null set with respect to \( \mathcal{H} \) and by Lemma 3.9, among subsets of \( \Omega_\beta \), the measures \( \mathcal{H}_\beta \) and \( \mathcal{H} \) have the same null sets. Thus it is enough to prove that \( Tu(x) \) exists for \( \mathcal{H} \text{-a.e. } x \in \partial \Omega \cap V \).

Denote by \( \overline{u} \) the zero extension of \( u \) from \( W \) to \( \overline{W} \). The function \( \overline{u} \) remains measurable in the space \( \overline{W} = (\overline{W}, d, \mu_{\overline{W}}) \), see [14, Lemma 3.3.18]. Clearly \( \|\overline{u}\|_{L^1(\overline{W})} = \|u\|_{L^1(W)} \). By Lemma 3.11, we find a sequence of functions \( u_i \in N^{1,1}(W) \) such that \( u_i \rightarrow u \) in \( L^1(W) \) and

\[
\|Du\|(W) = \liminf_{i \rightarrow \infty} \int_W g_{ui} \, d\mu.
\]

Since \((W, d, \mu_W)\) satisfies doubling and \((1, 1)\)-Poincaré, we know that Lipschitz functions \( \text{Lip}(W) \) are dense in \( N^{1,1}(W) \), see [4, Theorem 5.1]. Thus we can in fact assume that \( u_i \in \text{Lip}(W) \). Now we can extend every \( u_i \) to a Lipschitz function on \( \overline{W} \), still denoted by \( u_i \). By [6, Lemma 5.11], the zero extension of \( u_i \) to \( \overline{W} \), still denoted by the same symbol, is the minimal \( 1 \)-weak upper gradient of \( u_i \) in \( \overline{W} \). Now \( u_i \rightarrow \overline{u} \) in \( L^1(\overline{W}) \) and so

\[
\|Du\|(W) = \liminf_{i \rightarrow \infty} \int_W g_{ui} \, d\mu = \liminf_{i \rightarrow \infty} \int_{\overline{W}} g_{ui} \, d\mu_{\overline{W}} \geq \|D\overline{u}\|((\overline{W})).
\]

Thus we have \( \overline{u} \in BV(\overline{W}) \) with \( \|D\overline{u}\|((\overline{W}) = \|Du\|(W) \), so it follows that \( \|D\overline{u}\|(\partial W) = 0 \). By (3.6) we have that \( \partial W \cap V = \partial \Omega \cap V \). Thus in total, we have

\[
\overline{u} \in BV(\overline{W}) \quad \text{with} \quad \|D\overline{u}\|(\partial \Omega \cap V) = 0. \tag{3.13}
\]

By [5, Proposition 3.3 & 3.6] we know that \((\overline{W}, d, \mu_{\overline{W}})\) also satisfies doubling and \((1, 1)\)-Poincaré. Since \( \|D\overline{u}\|(\partial \Omega \cap V) = 0 \), by the decomposition (2.15) we also know that \( \mathcal{H}((S_{\overline{u}} \cap \partial \Omega \cap V) = 0 \). By the Lebesgue point result (2.14), for \( \mathcal{H} \text{-a.e. } x \in \overline{W} \setminus S_{\overline{u}} \) we have

\[
\lim_{r \rightarrow 0} \int_{B(x, r)} |u - \overline{u}'(x)| \, d\mu = \lim_{r \rightarrow 0} \int_{B_{\overline{W}}(x, r)} |\overline{u} - \overline{u}'(x)| \, d\mu_{\overline{W}} = 0.
\]

In particular, this now holds for \( \mathcal{H} \text{-a.e. } x \in \partial \Omega \cap V \). Thus we can choose \( Tu(x) := \overline{u}'(x) \).

In the proof above, we used Lemma 3.9 to compare the measures \( \mathcal{H} \) and \( \mathcal{H} \); of course it is more natural to formulate results in terms of the latter. In [20] it was
noted that these two measures have the same null sets on \( \partial \Omega \) if \( \Omega \) satisfies a measure density condition, meaning that there is a constant \( c_m > 0 \) such that
\[
\mu(B(x, r) \cap \Omega) \geq c_m \mu(B(x, r))
\]
for all \( x \in \partial \Omega \) and \( r \in (0, \operatorname{diam}(\Omega)) \). We wish to avoid such an assumption that of course rules out e.g. domains with exterior cusps. Instead, we have considered the part of the boundary where the measure density condition is satisfied in an asymptotic sense, namely \( \partial \Omega \cap \Omega^\beta \). Concerning the rest of the boundary, we first observe that the trace might not exist.

**Example 3.14.** Let \( X = \mathbb{R}^2 \). For \( A \subset \mathbb{R}^2 \) and \( x \in \mathbb{R}^2 \), we define
\[
A + x := \{ y + x : y \in A \}.
\]
Define \( \Omega \subset \mathbb{R}^2 \) as a union of rectangles “piled on top of each other”, as follows. Define
\[
A_j := [0, 2^{-j(j+1)}] \times [0, 2^{-j^2}] + c_j
\]
with
\[
c_j := \left( -2^{-j(j+1)-1}, \sum_{k=1}^{j-1} 2^{-k^2} \right), \quad j \in \mathbb{N}.
\]
Then let \( A := \bigcup_{j=1}^{\infty} A_j \) and \( \Omega := \text{int}(A) \). Consider the point
\[
x_0 := \left( 0, \sum_{k=1}^{\infty} 2^{-k^2} \right) \in \partial \Omega.
\]
Equip the space with the weighted Lebesgue measure
\[
d\mu := w \, d\mathcal{L}^2, \quad \text{with} \quad w(x) = \frac{1}{2\pi} |x - x_0|^{-1}.
\]
It is straightforward to check that \( w \) is a Muckenhoupt \( A_1 \)-weight, and thus \( \mu \) is doubling and supports a \((1,1)\)-Poincaré inequality, see e.g. [12, Chapter 15] for these concepts. We will show that \( x_0 \in \partial^* \Omega \) but \( \theta^*(\Omega, x_0) = 0 \). First note that
\[
\mu(B(x_0, r)) = \frac{1}{2\pi} \int_0^r \frac{2\pi t}{t} \, dt = r, \quad r > 0.
\]
Denote by \( Q_j \) the square of side length \( 2^{-j(j+1)} \) located at the top of the rectangle \( A_j \). Note that the ball \( B(x_0, 2^{-j(j+1)+1}) \) contains \( Q_j \). Thus for every \( j \in \mathbb{N} \),
\[
\frac{\mu(B(x_0, 2^{-j(j+1)+1}) \cap \Omega)}{\mu(B(x_0, 2^{-j(j+1)+1}))} \geq \frac{\mu(Q_j)}{2^{-j(j+1)+1}} \geq \frac{1}{2\pi} \frac{\mathcal{L}^2(Q_j)}{2^{-j(j+1)+1}} \geq \frac{1}{2\pi} \frac{2^{-j(j+1)} \times 2^{-j(j+1)} / 2^{-j(j+1)+1}}{2^{-j(j+1)+1}} = \frac{1}{8\pi}.
\]
This combined with the obvious fact that \( \theta^*(\mathbb{R}^2 \setminus \Omega, x_0) \geq \theta_*(\mathbb{R}^2 \setminus \Omega, x_0) \geq 1/2 \) tells us that \( x_0 \in \partial^* \Omega \). On the other hand, we let \( S_j := \sum_{k=j+1}^{\infty} 2^{-k^2} \) and then we estimate

\[
\mu(A_j) \leq 2^{-j(j+1)} \times \int_{S_j}^{S_j + 2^{-j^2}} t^{-1} \, dt \\
\leq 2^{-j(j+1)} \times \log \left( \frac{(S_j + 2^{-j^2})/S_j}{(S_j + 2^{-j^2})/S_j} \right) \\
\leq 2^{-j(j+1)} \times \log \left( 1 + 2^{-j^2}/2^{-(j+1)^2} \right) \\
\leq 2^{-j(j+1)} \times \log \left( 1 + 2^{3j} \right).
\]

Thus for every \( j \in \mathbb{N} \), we have

\[
\frac{\mu(B(x_0, 2^{-j^2}) \cap \Omega)}{\mu(B(x_0, 2^{-j^2}))} \leq \frac{1}{2^{-j^2}} \sum_{k=j}^{\infty} \mu(A_k) \\
\leq \frac{1}{2^{-j^2}} \sum_{k=j}^{\infty} 2^{-k(k+1)} \times \log \left( 1 + 2^{3k} \right) \\
\leq \frac{2}{2^{-j^2}} 2^{-j(j+1)} \times \log \left( 1 + 2^{3j} \right) \\
\to 0 \quad \text{as} \quad j \to \infty.
\]

This tells us that \( \theta_*(\Omega, x_0) = 0 \), so in particular \( x_0 \in \partial^* \Omega \setminus \Sigma_{\beta} \Omega \).

Next note that for every \( x \in \partial \Omega \setminus \{x_0\} \), we can choose \( \delta > 0 \) so small that \( B(x, \delta) \) only intersects at most two of the rectangles that make up \( \Omega \). Then \( W := B(x, \delta) \cap \Omega \) is obviously a Lipschitz domain. As noted in Remark 3.7, \((W, d, \mu)\) is then a metric space equipped with a doubling measure and supporting a \((1,1)\)-Poincaré inequality. Thus \( \Omega \) is PLB.

Define a function

\[
u := \sum_{k=1}^{\infty} (-1)^k k \chi_{A_k}.
\]

Then by (3.16), we get

\[
\|u\|_{L^1(\Omega)} \leq \sum_{k=1}^{\infty} k \mu(\chi_{A_k}) \leq \sum_{k=1}^{\infty} k \cdot 2^{-k(k+1)} \times \log \left( 1 + 2^{3k} \right) < \infty,
\]

and

\[
\|Du\|_1(\Omega) \leq \sum_{k=1}^{\infty} 2k \cdot 2^{-k(k+1)} < \infty.
\]

Thus \( u \in BV(\Omega) \). Now, for any even numbers \( j \in \mathbb{N} \) and \( k \geq j \), we have

\[
\frac{\mu(B(x_0, 2^{-k(k+1)+1}) \cap \{u \geq j\})}{\mu(B(x_0, 2^{-k(k+1)+1}))} \geq \frac{\mu(Q_k)}{2^{-k(k+1)+1}} \geq \frac{1}{8\pi} \quad \text{by (3.15)}.
\]
Thus \( \theta^*(\{u \geq j\}, x_0) \geq (8\pi)^{-1} \) for all even \( j \in \mathbb{N} \), and so \( T_\beta u(x_0) = \infty \). On the other hand, we have \( \theta_*\{u > t\}, x_0 = 0 \) for all \( t \in \mathbb{R} \) since we proved that in fact \( \theta_*{(\Omega, x_0)} = 0 \), and so \( T_\beta u(x_0) = -\infty \). Finally, clearly \( T_u(x_0) \) does not exist, nor would it exist with any reasonable definition allowing for the possibility that \( T_u(x_0) \) take the values \( \pm \infty \). To estimate \( \mathcal{H}(\{x_0\}) \) it is of course enough to consider coverings consisting of one ball, and then it is straightforward to show that

\[
\frac{1}{4} \leq \mathcal{H}(\{x_0\}) \leq 1. \tag{3.17}
\]

This example demonstrates that there can be a significant part of the boundary, in terms of \( \mathcal{H} \)-measure, where \( \theta^*(\Omega, \cdot) > 0 \) but \( \theta_*(\Omega, \cdot) = 0 \). On this part, the trace \( T_u \) might not exist but the rough trace \( T_\beta u \) is of course defined. We will see in the next sections that the values of \( T_u \) on this part of the boundary are nonetheless controlled by the part of the boundary where \( T_u \) exists. (If there is a part of \( \partial \Omega \) where even \( \theta^*(\Omega, \cdot) = 0 \), there we have \( T_u \equiv -\infty \) and so the rough trace is not interesting.)

4 Integrability of the boundary trace

In this section we study the integrability of the rough trace \( T_u \). As usual, \( \Omega \subset X \) always denotes an arbitrary open set.

In a bounded Lipschitz domain \( \Omega \subset \mathbb{R}^n \), by Lusin’s theorem the boundary trace \( T_u \) has the following continuity: for every \( \varepsilon > 0 \) there exists a relatively open set \( G \subset \partial \Omega \) such that \( \mathcal{H}^{n-1}(G) < \varepsilon \) and \( T_u|_{\partial \Omega \setminus G} \) is continuous. In more general domains in metric spaces, it is difficult to obtain a similar result, already because \( \partial \Omega \) could have infinite \( \mathcal{H} \)-measure. Nonetheless, we get the following kind of continuity.

**Proposition 4.1.** Let \( u \in BV(\Omega) \). Then for \( \mathcal{H} \)-a.e. \( x \in \partial \Omega \setminus \Omega_\beta \), we have that if \( T_u(x) > -\infty \) and \( \varepsilon > 0 \), then

\[
\mathcal{H}(B(x, \varepsilon) \cap \{T_\beta u \geq \min\{T_u(x) - \varepsilon, 1/\varepsilon\}\}) = \infty.
\]

Note that we take the minimum because it can happen that \( T_u(x) = \infty \), in which case we of course interpret \( T_u(x) - \varepsilon = \infty \). The above condition means that close to \( x \) there is a very large subset of \( \partial \Omega \) where \( T_\beta u \) is not much less than \( T_u(x) \) (though conversely, recall that always \( T_\beta u \leq T_u \)).

**Proof.** Since \( X \) and then also \( \partial \Omega \) is Lindelöf, we find a countable collection of balls \( \{B_j\}_{j=1}^{\infty} \) such that every \( x \in \partial \Omega \) is contained in some ball \( B_j \) with arbitrarily small radius. By the coarea formula (2.12), we can take a countable, dense subset \( \{q_k\}_{k \in \mathbb{N}} \) of \( \mathbb{R} \) such that

\[
P(\{u > q_k\}, \Omega) < \infty \quad \text{for all} \quad k \in \mathbb{N}.
\]

For each \( j, k \in \mathbb{N} \), if \( P(\{u > q_k\}, B_j) < \infty \), then define

\[
N_{j,k} := B_j \cap (\partial^*\{u > q_k\} \setminus \Sigma_\beta\{u > q_k\}).
\]

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Otherwise let \( N_{j,k} := \emptyset \). Then define the exceptional set

\[
N := \bigcup_{j,k=1}^{\infty} N_{j,k}.
\]

We have \( \mathcal{H}(N) = 0 \) by (2.8); recall that we assume \( \beta \leq \gamma \).

Now fix \( x \in \partial \Omega \setminus (\Omega_\beta \cup N) \). Suppose \( T_u(x) \in (-\infty, \infty] \) and let \( \varepsilon > 0 \). Denote \( b := \min\{T_u(x) - \varepsilon, 1/\varepsilon\} \in (-\infty, \infty) \). Suppose that for some fixed \( \varepsilon > 0 \),

\[
\mathcal{H}(B(x, \varepsilon) \cap \{ T_u \geq b \}) < \infty.
\]

(4.2)

Note that for every point \( y \in \partial \Omega \) with \( \theta_{\ast}(\{u > b\}, y) \geq \beta \), we have \( y \in \{T_u \geq b\} \). Thus

\[
\mathcal{H}(B(x, \varepsilon) \cap \partial \Omega \cap \{ \theta_{\ast}(u > b) \geq \beta \}) < \infty.
\]

We find \( j, k \in \mathbb{N} \) such that \( B_j \subset B(x, \varepsilon) \) and

\[
b = \min\{T_u(x) - \varepsilon, 1/\varepsilon\} < q_k < T_u(x).
\]

Then

\[
\mathcal{H}(B_j \cap \partial \Omega \cap \Sigma_{\beta}\{u > q_k\}) \leq \mathcal{H}(B_j \cap \partial \Omega \cap \{ \theta_{\ast}(u > q) \geq \beta \}) < \infty.
\]

By (2.9), also

\[
\mathcal{H}(B_j \cap \Omega \cap \Sigma_{\beta}\{u > q_k\}) < \infty,
\]

and clearly \( B_j \cap \Sigma_{\beta}\{u > q_k\} \setminus \Omega = \emptyset \). Then \( P(\{u > q_k\}, B_j) < \infty \) by Theorem 2.10. Since \( x \notin \Omega_{\beta} \), necessarily \( \theta_{\ast}\{u > q_k\} < \beta \), and since \( x \notin N \), in fact \( \theta_{\ast}\{u > q_k\} = 0 \). This implies that \( T_u(x) \leq q_k \). This contradicts the fact that \( q_k < T_u(x) \). Thus (4.2) is false, proving the claim. \( \square \)

For nice domains we get the following version.

**Proposition 4.3.** Suppose \( \Omega \) is PLB and let \( u \in \text{BV}(\Omega) \). Then for \( \mathcal{H}\text{-a.e. } x \in \partial \Omega \setminus \Omega_{\beta} \), we have that if \( T_u(x) > -\infty \) and \( \varepsilon > 0 \), then

\[
\mathcal{H}(B(x, \varepsilon) \cap \Omega_{\beta} \cap \{ Tu \geq \min\{T_u(x) - \varepsilon, 1/\varepsilon\}\}) = \infty.
\]

**Proof.** By Proposition 4.1, for \( \mathcal{H}\text{-a.e. } x \in \partial \Omega \setminus \Omega_{\beta} \) we have that if \( T_u(x) > -\infty \) and \( \varepsilon > 0 \), then

\[
\mathcal{H}(B(x, \varepsilon) \cap \{ T_{\beta}u \geq \min\{T_u(x) - \varepsilon, 1/\varepsilon\}\}) = \infty.
\]

Note that necessarily \( \{ T_{\beta}u \geq \min\{T_u(x) - \varepsilon, 1/\varepsilon\}\} \subset \Omega_{\beta} \). By Proposition 3.12, in the set \( \partial \Omega \cap \Omega_{\beta} \) we have that \( Tu \) exists and thus \( T_{\beta}u = Tu \) \( \mathcal{H}\text{-almost everywhere} \) in this set, completing the proof. \( \square \)

**Remark 4.4.** Clearly, the continuity property of Propositions 4.1 and 4.3 does not generally hold at points \( x \in \partial \Omega \cap \Omega_{\beta} \). A counterexample is given simply by any domain with \( \mathcal{H}(\partial \Omega) < \infty \), for example any bounded Lipschitz domain \( \Omega \subset \mathbb{R}^n \).
Lemma 4.5. Let \( u \) be a \( \mu \)-measurable function on \( \Omega \). Then
\[
|T_*|u|(x)| \geq |T_*u(x)| \quad \text{for every } x \in \partial \Omega.
\]

Proof. We can assume that \( \theta^*(\Omega, x) > 0 \). If \( T_*u(x) \geq 0 \), the claim is obvious. If \( T_*u(x) := b \in (-\infty, 0) \), let \( \delta > 0 \) and note that
\[
\theta^*(\{u > b + \delta/2\}, x) = 0
\]
and so
\[
\theta^*(\{u < b + \delta\}, x) \geq \theta^*(\{u \leq b + \delta/2\}, x) = \theta^*(\Omega, x) > 0.
\]
Thus
\[
\theta^*(\{|u| > |b| - \delta\}, x) \geq \theta^*(\{-u > |b| - \delta\}, x) = \theta^*(\{u < b + \delta\}, x) > 0.
\]
Thus \( T_*|u|(x) \geq |b| - \delta \), and since \( \delta > 0 \) was arbitrary,
\[
T_*|u|(x) \geq |b| = |T_*u(x)|.
\]
The case \( b = -\infty \) is similar. \( \Box \)

Now we can prove the following result on the integrability of the trace; this gives Claim (2) of Theorem 1.1.

Proposition 4.6. Suppose \( \Omega \) is PLB and let \( u \in BV(\Omega) \). Then
\[
\int_{\partial \Omega \cap \Omega_\beta} |T_*|u| \mathcal{H} = \int_{\partial \Omega \setminus \{\theta^*(\Omega, \cdot) > 0\}} |T_*|u| \mathcal{H} = \int_{\partial \Omega \setminus \{\theta^*(\Omega, \cdot) > 0\}} |T_*u| \mathcal{H}
\]

Proof. First note that the trace \( Tu(x) \) exists at \( \mathcal{H} \)-a.e. \( x \in \partial \Omega \cap \Omega_\beta \) by Proposition 3.12, and so the integral on the left-hand side is well-defined. It is also easy to check that whenever \( Tu(x) \) exists, we have \( |Tu(x)| = T|u|(x) \). Thus to prove the first equality, we can assume that \( u \geq 0 \).

Let \( N \) be the exceptional set of Proposition 4.3. Consider a point
\[
x \in \partial \Omega \cap \{\theta^*(\Omega, \cdot) > 0\} \setminus (\Omega_\beta \cup N).
\]
If \( T_*u(x) \in (0, \infty] \), then Proposition 4.3 with the choice \( \varepsilon = \min\{T_*u(x)/2, 1\} \) gives
\[
\mathcal{H}(B(x, \varepsilon) \cap \Omega_\beta \cap \{Tu \geq \varepsilon\}) = \infty.
\]
This means that
\[
\int_{\partial \Omega \cap \Omega_\beta} Tu \mathcal{H} = \infty,
\]
and since \( T_*u = Tu \mathcal{H} \)-almost everywhere in \( \partial \Omega \cap \Omega_\beta \), then also
\[
\int_{\partial \Omega \setminus \{\theta^*(\Omega, \cdot) > 0\}} T_*u \mathcal{H} = \infty,
\]
and so the first equality holds. The other option is that for every $x \in \partial \Omega \cap \{\theta^*(\Omega, \cdot) > 0\} \setminus (\Omega_\beta \cup N)$, we have $T_* u(x) = 0$. Then

$$
\int_{\partial \Omega \setminus \{\theta^*(\Omega, \cdot) > 0\}} T_* u \, d\mathcal{H} = \int_{\partial \Omega \setminus (\Omega_\beta \cup N)} T_* u \, d\mathcal{H} = \int_{\partial \Omega \setminus \Omega_\beta} T_* u \, d\mathcal{H} \quad \text{since } \mathcal{H}(N) = 0
$$

$$
= \int_{\partial \Omega \setminus \Omega_\beta} T u \, d\mathcal{H} \quad \text{by Proposition 3.12.}
$$

This completes the proof of the first equality.

To prove the second equality, note that by Lemma 4.5 and the first equality, we have

$$
\int_{\partial \Omega \cap \{\theta^*(\Omega, \cdot) > 0\}} |T u| \, d\mathcal{H} \geq \int_{\partial \Omega \cap \{\theta^*(\Omega, \cdot) > 0\}} |T_* u| \, d\mathcal{H}.
$$

The opposite inequality is obvious, since at $\mathcal{H}$-a.e. $x \in \partial \Omega \cap \Omega_\beta$ we have $T u(x) = T_* u(x)$.

Example 4.7. Let $X = \mathbb{R}^2$ equipped with the weighted Lebesgue measure

$$
d\mu := w \, d\mathcal{L}^2, \quad \text{with} \quad w(x) := \frac{1}{2\pi} |x|^{-1}.
$$

Just as in Example 3.14, we have that $(X, d_{euc}, \mu)$ satisfies doubling and $(1, 1)$-Poincaré. Denote the origin by 0. Let $x_j := (2^{-j}, 0)$, $r_j := 2^{-2j}$, and

$$
\Omega := B(0, 1) \setminus \left\{0\right\} \cup \bigcup_{j=3}^{\infty} B(x_j, r_j).
$$

It is straightforward to check that $\Omega$ is a uniform domain; recall the definition from (2.2). Thus by [6, Theorem 4.4], $(\Omega, d_{euc}, \mu)$ satisfies doubling and $(1, 1)$-Poincaré.

Now clearly $\partial \Omega = \partial \Omega \cap \Omega_\beta$ and $0 \in \partial \Omega$, but $\theta^*(X \setminus \Omega, 0) = 0$ so in particular $0 \notin \Sigma_\beta \Omega$. Let $u := 1$ on $\Omega$, so that $u \in BV(\Omega)$ and $T u = T_* u = 1$ on $\partial \Omega$. However,
by (3.17) we have $\mathcal{H}({\{0\}}) > 0$. Denoting the one-dimensional Hausdorff measure by $\mathcal{H}^1$, we have

$$\int_{\partial\Omega^c\cap\Omega_j} |Tu| d\mathcal{H} = \mathcal{H}({\{0\}}) + \sum_{j=3}^{\infty} \mathcal{H}(\partial B_j)$$

$$\leq \mathcal{H}({\{0\}}) + \sum_{j=3}^{\infty} w(2^{-j-1}, 0) \cdot \pi \cdot \mathcal{H}^1(\partial B_j)$$

$$\leq \mathcal{H}({\{0\}}) + \sum_{j=3}^{\infty} \frac{1}{2\pi} \cdot 2^{j+1} \cdot \pi \cdot 2^{-2j} < \infty.$$

Now

$$\int_{\Sigma_\beta\Omega} |Tu| d\mathcal{H} = \sum_{j=3}^{\infty} \mathcal{H}(\partial B_j) < \int_{\partial\Omega^c\cap\Omega_j} |Tu| d\mathcal{H}.$$  

Thus in Claim (2) of Theorem 1.1, we cannot replace $\partial\Omega^c\cap\Omega_j$ with $\Sigma_\beta\Omega$.

## 5 Zero extension

In this section we give a characterization of those functions $u \in \text{BV}(\Omega)$ that can be zero extended to the whole space as BV functions. We also study the possibility of doing this without adding any total variation.

As before, $\Omega \subset X$ always denotes an arbitrary open set.

**Lemma 5.1.** Let $u$ be a measurable function on $\Omega$. Let $A \subset \partial\Omega$ be a Borel set. Then

$$\mathcal{H}(\{\theta_*(\{u > t\}, \cdot) \geq \beta\} \cap A) = \mathcal{H}(\{T_\beta u > t\} \cap A) \quad \text{for a.e. } t \in \mathbb{R}.$$

**Proof.** For every $t \in \mathbb{R}$ we have

$$\{T_\beta u > t\} \subset \{\theta_*(\{u > t\}, \cdot) \geq \beta\} \cap \partial\Omega,$$

because if $x \in \partial\Omega$ with $\theta_*(\{u > t\}, x) < \beta$, then $T_\beta u(x) \leq t$.

Choose $a \in [-\infty, \infty]$ such that $\mathcal{H}(\{T_\beta u > t\} \cap A) < \infty$ for all $t > a$ and $\mathcal{H}(\{T_\beta u > t\} \cap A) = \infty$ for all $t < a$. There are at most countably many $t > a$ for which $\mathcal{H}(\{T_\beta u = t\} \cap A) > 0$. Consider a number $t > a$ outside this countable set. If $x \in A$ such that $T_\beta u(x) \neq t$ and $\theta_*(\{u > t\}, x) \geq \beta$, then $T_\beta u(x) > t$. Conversely if $T_\beta u(x) > t$, then $\theta_*(\{u > t\}, x) \geq \beta$ by (5.2). Thus

$$\mathcal{H}(\{\theta_*(\{u > t\}, \cdot) \geq \beta\} \cap A) = \mathcal{H}(\{T_\beta u > t\} \cap A) \quad \text{for a.e. } t > a.$$

Then consider $t < a$. Now $\mathcal{H}(\{T_\beta u > t\} \cap A) = \infty$. By (5.2) we get

$$\{T_\beta u > t\} \cap A \subset \{\theta_*(\{u > t\}, \cdot) \geq \beta\} \cap A,$$

and so also

$$\mathcal{H}(\{\theta_*(\{u > t\}, \cdot) \geq \beta\} \cap A) = \infty.$$

Thus, the claim holds also for almost every (in fact, every) $t < a$. $\square$
Now we prove the following characterization, which gives Claim (3) of Theorem 1.1.

**Proposition 5.3.** Suppose $\Omega$ is PLB and let $u \in \text{BV}(\Omega)$. Then the zero extension of $u$ from $\Omega$ to the whole space $X$ belongs to $\text{BV}(X)$ if and only if

$$\int_{\Sigma_{\beta}} |T u| \, d\mathcal{H} < \infty.$$

**Proof.** For numbers $a, b \geq 0$, we write $a \approx b$ if $C^{-1} a \leq b \leq C a$ for some constant $C \geq 1$ depending only on the doubling constant of $\mu$ and the constants in the Poincaré inequality (that is, the doubling and Poincaré that hold globally in the space $X$). Denote by $u$ also the zero extension; obviously $u \in L^1(X)$. First assume that $u \geq 0$. Now by the coarea formula (2.12), we get the following; note that we use upper integrals since measurability is not clear, and this is also the reason for the second “≈”:

$$\|Du\|(X) = \int_{(-\infty, \infty)}^* P(\{u > t\}, X) \, dt$$

$$= \int_{(0, \infty)}^* P(\{u > t\}, X) \, dt$$

$$\approx \int_{(0, \infty)}^* \mathcal{H}(\Sigma_{\beta}\{u > t\}) \, dt \quad \text{by (2.11)}$$

$$\approx \int_{(0, \infty)}^* \mathcal{H}(\Sigma_{\beta}\{u > t\} \cap \Omega) \, dt + \int_{(0, \infty)}^* \mathcal{H}(\Sigma_{\beta}\{u > t\} \cap \partial \Omega) \, dt$$

$$\approx \int_{(0, \infty)}^* P(\{u > t\}, \Omega) \, dt + \int_{(0, \infty)}^* \mathcal{H}(\Sigma_{\beta}\{u > t\} \cap \partial \Omega) \, dt \quad \text{by (2.11)}$$

$$= \|Du\|(\Omega) + \int_{(0, \infty)}^* \mathcal{H}(\Sigma_{\beta}\{u > t\} \cap \partial \Omega \cap \Omega_{\beta}) \, dt;$$

(5.4)

the last inequality follows from the coarea formula (2.12) and since obviously $\Sigma_{\beta}\{u > t\} \cap \partial \Omega \subset \Omega_{\beta}$. Now let $V$ and $W$ be two sets as in Definition 3.5. By [5, Proposition 3.3 & 3.6], we have that $(\overline{W}, d, \mu_{\overline{W}})$ satisfies doubling and (1,1)-Poincaré. Denote by $\overline{u}$ the zero extension of $u$ from $W$ to $\overline{W}$. By the coarea formula (2.12) and (3.13), we have

$$\int_{(0, \infty)}^* \mathcal{H}_{\overline{W}}(\partial^*\{\overline{u} > t\} \cap V \cap \partial \Omega) \, dt \approx \|D\overline{u}\|(V \cap \partial \Omega) = 0.$$

Thus for a.e. $t > 0$ we have

$$\mathcal{H}_{\overline{W}}(\partial^*\{\overline{u} > t\} \cap V \cap \partial \Omega) = 0.$$

Fix such $t$. For $\mathcal{H}_{\overline{W}}$-a.e. $x \in V \cap \partial \Omega$, we have $x \notin \partial^*\{\overline{u} > t\}$ and so either

$$\lim_{r \to 0} \frac{\mu(B(x, r) \cap W \cap \{u > t\})}{\mu(B(x, r) \cap W)} = 0 \quad \text{or} \quad \lim_{r \to 0} \frac{\mu(B(x, r) \cap W \cap \{u \leq t\})}{\mu(B(x, r) \cap W)} = 0.$$
For small enough \( r > 0 \) we have \( B(x, r) \cap \Omega \subset V \cap \Omega \subset W \). Thus

\[
\lim_{r \to 0} \frac{\mu(B(x, r) \cap \Omega \cap \{u > t\})}{\mu(B(x, r))} = 0 \quad \text{or} \quad \lim_{r \to 0} \frac{\mu(B(x, r) \cap \Omega \cap \{u \leq t\})}{\mu(B(x, r))} = 0.
\]

Now if \( x \notin (X \setminus \Omega)_{\beta} \), then either

\[
\liminf_{r \to 0} \frac{\mu(B(x, r) \cap \{u > t\})}{\mu(B(x, r))} < \beta \quad \text{or} \quad \liminf_{r \to 0} \frac{\mu(B(x, r) \cap \{u \leq t\})}{\mu(B(x, r))} < \beta,
\]

and so \( x \notin \Sigma_{\beta}\{u > t\} \). Thus for a.e. \( t > 0 \) we have

\[
\mathcal{H}(\Sigma_{\beta}\{u > t\} \cap V \cap \partial\Omega \setminus (X \setminus \Omega)_{\beta}) = 0.
\]

By Lemma 3.8 and Lemma 3.9, we have in fact

\[
\mathcal{H}(\Sigma_{\beta}\{u > t\} \cap \partial\Omega \cap \Omega_{\beta} \setminus (X \setminus \Omega)_{\beta}) = 0.
\]

As noted in the first paragraph of the proof of Proposition 3.12, we can cover \( \mathcal{H} \)-almost all of \( \partial\Omega \cap \Omega_{\beta} \) by countably many such sets \( V \). It follows that

\[
\mathcal{H}(\Sigma_{\beta}\{u > t\} \cap \partial\Omega \cap \Omega_{\beta} \setminus (X \setminus \Omega)_{\beta}) = 0 \quad \text{for a.e.} \quad t > 0.
\]

From (5.4) we now get

\[
\|Du\|(X) \approx \|Du\|(\Omega) + \int_{(0, \infty)}^* \mathcal{H}(\Sigma_{\beta}\{u > t\} \cap \Sigma_{\beta}\Omega) \, dt.
\]

Here

\[
\int_{(0, \infty)}^* \mathcal{H}(\Sigma_{\beta}\{u > t\} \cap \Sigma_{\beta}\Omega) \, dt = \int_0^\infty \mathcal{H}(\{\theta_u(\{u > t\}, \cdot) \geq \beta\} \cap \Sigma_{\beta}\Omega) \, dt
\]

\[
= \int_0^\infty \mathcal{H}(\{T_{\beta}u > t\} \cap \Sigma_{\beta}\Omega) \, dt \quad \text{by Lemma 5.1}
\]

\[
= \int_{\Sigma_{\beta}\Omega} T_{\beta}u \, d\mathcal{H} \quad \text{by Cavalieri’s principle}
\]

\[
= \int_{\Sigma_{\beta}\Omega} Tu \, d\mathcal{H} \quad \text{by Proposition 3.12}.
\]

In total,

\[
\|Du\|(X) \approx \|Du\|(\Omega) + \int_{\Sigma_{\beta}\Omega} Tu \, d\mathcal{H}. \quad (5.5)
\]

Now we drop the assumption \( u \geq 0 \). For a general \( u \in BV(\Omega) \), note first that whenever the trace \( Tu(x) \) exists at a point \( x \in \partial\Omega \), then we have

either \( Tu(x) = Tu_+(x) \) and \( Tu_-(x) = 0 \), or \( Tu(x) = -Tu_-(x) \) and \( Tu_+(x) = 0 \).

In both cases we get

\[
|Tu(x)| = Tu_+(x) + Tu_-(x). \quad (5.6)
\]
Using the coarea formula (2.12), it is easy to check that \(\|Du\|(X) = \|Du_+\|(X) + \|Du_-\|(X)\), and similarly with \(X\) replaced by \(\Omega\). Now we have
\[
\|Du\|(X) = \|Du_+\|(X) + \int_{\Sigma^\beta \Omega} Tu_+ d\mathcal{H} + \|Du_-\|(X) + \int_{\Sigma^\beta \Omega} Tu_- d\mathcal{H} \quad \text{by (5.5)}
\]
\[
= \|Du\|(\Omega) + \int_{\Sigma^\beta \Omega} |Tu| d\mathcal{H} \quad \text{by (5.6)}.
\]
This proves the result.

Now we have proved all the claims of our main Theorem 1.1:

Proof of Theorem 1.1. Claim (1) is given by Proposition 3.12, Claim (2) by Proposition 4.6, and Claim (3) by Proposition 5.3.

Remark 5.7. In essence, the strategy of our proofs was the following: we divided the boundary \(\partial \Omega\) into two parts, \(\partial \Omega \cap \Omega^\beta\) and \(\partial \Omega \setminus \Omega^\beta\). In the first part, we were able to show the existence of the (ordinary) trace. In the second part, we could control the relevant quantities by using the new version of Federer’s characterization, Theorem 2.10.

Consider a function \(u \in N^{1,p}_0(\Omega)\); recall the definition from (2.3). Interpreting \(u\) to be defined on the whole space, we have \(g_u = 0\) \(\mu\)-almost everywhere in \(X \setminus \Omega\), and \(\|u\|_{N^{1,p}(\Omega)} = \|u\|_{N^{1,p}(X)}\); see [4, Corollary 2.21 & Proposition 2.38]. In this sense, for Newton-Sobolev functions with zero boundary values, zero extension to the whole space adds no energy. In this section we have considered the possibility of zero extending BV functions to the whole space, but possibly adding total variation on the boundary \(\partial \Omega\). For example, if \(u = \chi_{\Omega} = \chi_{B(0,1)}\) in \(\mathbb{R}^n\), zero extension adds total variation on the boundary \(\partial \Omega\). Now we study the possibility of extending without adding any total variation. Recall the definition of \(u^\vee\) from (2.13).

Theorem 5.8. Let \(\Omega \subset \Omega_0 \subset X\) be open sets. Let \(u \in \text{BV}(\Omega)\). Define the zero extension
\[
u_0 := \begin{cases} u & \text{in } \Omega, \\ 0 & \text{in } \Omega_0 \setminus \Omega. \end{cases}
\]
Then the following are equivalent:

(1) \(u_0 \in \text{BV}(\Omega_0)\) with \(\|Du_0\|(\Omega_0 \setminus \Omega) = 0\) and \(u_0^\vee(x) = 0\) for \(\mathcal{H}\)-a.e. \(x \in \Omega_0 \cap \partial \Omega\).

(2) For \(\mathcal{H}\)-a.e. \(x \in \Omega_0 \cap \partial \Omega\), we have
\[
\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r) \cap \Omega} |u| d\mu = 0.
\]

(3) For \(\mathcal{H}\)-a.e. \(x \in \Omega_0 \cap \partial \Omega\), we have
\[
\liminf_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r) \cap \Omega} |u| d\mu = 0. \quad (5.9)
\]
In conclusion, it follows that \( \mathcal{H}(S_{u_0} \cap \Omega_0 \setminus \Omega) = 0 \). By Theorem\( \mathcal{H} \)-a.e. \( x \in \Omega_0 \setminus \Omega \), and in particular for \( \mathcal{H} \)-a.e. \( x \in \Omega_0 \cap \partial \Omega \), that

\[
\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r) \cap \Omega} |u| d\mu = \lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u_0 - u_0^\gamma(x)| d\mu = 0.
\]

(2) \(\Rightarrow\) (3): This is obvious.

(3) \(\Rightarrow\) (1): Fix \( x \in \Omega_0 \cap \partial \Omega \) such that (5.9) holds. If \( t < 0 \), then

\[
\liminf_{r \to 0} \frac{\mu(B(x,r) \setminus \{u_0 > t\})}{\mu(B(x,r))} = \liminf_{r \to 0} \frac{\mu(B(x,r) \setminus \Omega \setminus \{u > t\})}{\mu(B(x,r))} \leq \liminf_{r \to 0} \frac{1}{|t| \mu(B(x,r))} \int_{B(x,r) \cap \Omega} |u| d\mu = 0.
\]

If \( t > 0 \), then

\[
\liminf_{r \to 0} \frac{\mu(B(x,r) \cap \{u_0 > t\})}{\mu(B(x,r))} = \liminf_{r \to 0} \frac{\mu(B(x,r) \cap \Omega \cap \{u > t\})}{\mu(B(x,r))} \leq \liminf_{r \to 0} \frac{1}{|t| \mu(B(x,r))} \int_{B(x,r) \cap \Omega} |u| d\mu = 0.
\]

In both cases it follows that \( x \notin \Sigma_\beta \{u_0 > t\} \). Clearly this is true also for every \( x \in \Omega_0 \setminus \overline{\Omega} \). In conclusion, \( \mathcal{H}(\Sigma_\beta \{u_0 > t\} \cap \Omega_0 \setminus \Omega) = 0 \) for every \( t \neq 0 \). By the coarea formula (2.12) we know that

\[
P(\{u_0 > t\}, \Omega) = P(\{u > t\}, \Omega) < \infty \quad \text{for a.e. } t \in \mathbb{R}.
\]

For such \( t \neq 0 \), by (2.9) we now have

\[
\mathcal{H}(\Sigma_\beta \{u_0 > t\} \cap \Omega_0) = \mathcal{H}(\Sigma_\beta \{u_0 > t\} \cap \Omega) \leq \alpha^{-1} P(\{u_0 > t\}, \Omega) < \infty.
\]

By Theorem 2.10 it follows that \( P(\{u_0 > t\}, \Omega_0) < \infty \). Since \( \mathcal{H}(\partial^* \{u_0 > t\} \cap \Omega_0 \setminus \Omega) = 0 \), by (2.9) we have \( P(\{u_0 > t\}, \Omega_0 \setminus \Omega) = 0 \). Again by the coarea formula,

\[
\|Du_0\|\Omega_0) = \int_{-\infty}^{\infty} P(\{u_0 > t\}, \Omega_0) dt = \int_{-\infty}^{\infty} P(\{u_0 > t\}, \Omega) dt = \|Du_0\|\Omega).
\]

It follows that \( u_0 \in BV(\Omega_0) \) with \( \|Du_0\|\Omega_0 \) = 0. From (5.10) and (5.11) we also get \( u_0^\gamma(x) = 0 \) for \( \mathcal{H} \)-a.e. \( x \in \Omega_0 \cap \partial \Omega \).

Conditions (1) and (2) of the above theorem were shown to be equivalent already in [20, Theorem 6.1] as well as in [18, Theorem 4.5]. By exploiting the new version of Federer’s characterization (Theorem 2.10), we have been able to prove that (3) is equivalent as well. The proofs given in the above references as well as in Theorem 5.8 follow along the lines of [15], where a characterization of Newton-Sobolev functions with zero boundary values was proved. We close by giving the following new version of this characterization. Recall the definition of the \( p \)-capacity from (2.4).
Theorem 5.12. Suppose $\Omega \subset X$ is open and bounded. Let $1 \leq p < \infty$ and let $u \in N^{1,p}(\Omega)$. Then the following are equivalent:

1. $u \in N_0^{1,p}(\Omega)$.
2. For $\text{Cap}_p$-almost every $x \in \partial \Omega$, we have
   \[ \lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r) \cap \Omega} |u| \, d\mu = 0. \]  
3. For $\text{Cap}_p$-almost every $x \in \partial \Omega$, we have
   \[ \liminf_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r) \cap \Omega} |u| \, d\mu = 0. \]  

(5.13)

Proof. (1) $\iff$ (2): This is shown in [15, Theorem 1.1]; note that the “strong relative isoperimetric inequality” mentioned in the statement of that theorem is proved in [19, Corollary 5.6].

(2) $\Rightarrow$ (3): This is trivial.

(3) $\Rightarrow$ (1): We have $u \in N^{1,p}(\Omega) \subset N^{1,1}(\Omega)$, since $\Omega$ is bounded. From the definition of the total variation (2.6) it follows that $N^{1,1}(\Omega) \subset \text{BV}(\Omega)$, and so we have $u \in \text{BV}(\Omega)$. By [4, Proposition 2.46], the condition (5.13) holds also for $\text{Cap}_1$-almost every $x \in \partial \Omega$. Then by (2.5), it holds for $\mathcal{H}$-a.e. $x \in \partial \Omega$. Now by Theorem 5.8, we get $u_0 \in \text{BV}(X)$ with $\|Du_0\|(X \setminus \Omega) = 0$. After this, we can follow almost verbatim the proof given in [15, Theorem 1.1] (p. 521). \hfill \Box

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