Black Hole Thermodynamics in Modified Gravity

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We analyze the thermodynamics of a non-rotating and rotating black hole in a modified theory of gravity that includes scalar and vector modifications to general relativity, which results in a modified gravitational constant \( G = G_N(1 + \alpha) \) and a new gravitational charge \( Q = \sqrt{\alpha} G_N M \). The influence of the parameter \( \alpha \) alters the non-rotating black hole’s lifetime, temperature and entropy profiles from the standard Schwarzschild case. The thermodynamics of a rotating black hole is analyzed and it is shown to possess stable, cold remnants. The thermodynamic properties of a vacuum solution regular at \( r = 0 \) are investigated and the solution without a horizon called a “gray hole” is not expected to possess an information loss problem.

PACS numbers:

I. INTRODUCTION

There is a motivation to modify general relativity. One of the main reasons is the discrepancy between the observed dynamics of galaxies and clusters of galaxies and the amount of luminous matter these galaxies and clusters contain [1,2]. This is usually explained by postulating the existence of dark matter. To date, no dark matter particle candidates have been detected in laboratory experiments or in satellite missions. An alternate resolution to the problem of the galaxy and galaxy cluster dynamics is a modification of the laws of gravitation on scales where Newtonian gravity has not been extensively tested. One such framework called MOG (MOdified Gravity) [4] has been able to explain the dynamics of galaxies and galaxy clusters without the need for dark matter in the present epoch of the universe [4,7].

In the MOG formulation, the field content of general relativity has been increased to include scalar fields and a massive vector field. Also called the Scalar-Tensor-Vector Gravity (STVG) theory, it has been used to describe the growth of structure, the matter power spectrum and the cosmic microwave background (CMB) acoustical power spectrum data in the early universe [8]. Solar system experiments are also in accordance with MOG [9], and the dynamics of the Bullet Cluster has been explained without dark matter [10].

In the following, we will analyze the thermodynamics of the black hole (BH) solutions derived from the MOG field equations for constant \( G = G_N(1 + \alpha) \), where \( \alpha \) is a free parameter [12,13]. The gravitational repulsive vector field \( \phi_{\mu} \) is treated as massless. The fits to galaxy rotation curves and clusters have been performed with the universal values \( \alpha = 8.89 \) and \( \mu = 0.042 \) kpc\(^{-1} \), where \( \mu \) is the range parameter for the modified Newtonian acceleration law. The mass corresponding to this value of \( \mu \) is \( m_{\phi} = 2.6 \times 10^{-28} \) eV, and for the size of black holes this mass can be ignored.

II. THERMODYNAMICS OF STATIC BLACK HOLES IN MOG

The first solution we will analyze is a static spherically symmetric solution to the field equations. The metric can be written as [12]:

\[
\begin{align*}
 ds^2 &= \left( 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} \right) dt^2 \\
 &\quad - \left( 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} \right)^{-1} dr^2 - r^2 d\Omega^2, 
\end{align*}
\]

where the usual Newtonian constant \( G_N \) gets modified to \( G = G_N(1 + \alpha) \). Here, \( Q = \sqrt{\alpha} G_N M \) represents the gravitational charge corresponding to the vector field \( \phi_{\mu} \).

This metric exhibits two horizons at

\[
r_{\pm} = GM \pm \sqrt{G^2 M^2 - GQ^2} = G_N M (1 + \alpha \pm \sqrt{1 + \alpha}) .
\]

The temperature can be readily calculated via the surface gravity method evaluated at the outer horizon,

\[
T = \frac{\kappa}{2\pi}, \quad \kappa = \frac{1}{2} \frac{d\Omega_{00}}{dr} (r = r_{+}),
\]

which gives

\[
T = \frac{1}{2\pi G_N M} \left( \frac{1}{(1 + \sqrt{1 + \alpha}) (1 + \alpha + \sqrt{1 + \alpha})} \right).
\]

This temperature can also be obtained by using the standard expression

\[
T = \frac{r_{+} - r_{-}}{4\pi r_{+}^2}
\]

when \( \alpha = 0 \), the usual Schwarzschild BH temperature is obtained, \( T = \frac{r_{+} - r_{-}}{8\pi G_N M} \). We shall call this the Schwarzschild-MOG black hole solution. We note that so long as \( \alpha > 0 \), there is no value for which \( T = 0 \). This implies that there are no remnants, and the black hole
evaporation continues until a time when quantum gravity takes over \cite{10}. The temperature is plotted in Fig. 1. Solutions for \( \alpha < 0 \) are not considered, since it would result in a complex gravitational charge \( Q \), an undefined temperature, as well as a negative gravitational constant.

\[ C = -2\pi M^2(1 + \sqrt{1 + \alpha})(1 + \alpha + \sqrt{1 + \alpha}) < 0 \quad (6) \]

and so is unstable for all \( M \).

The radiative power of the black hole is

\[ \frac{dM}{dt} = 4\pi R_+^2 \sigma T^4, \quad (7) \]

where \( \sigma \) is the Stefan-Boltzmann constant. The lifetime is thus

\[ \tau_\alpha = -\int_{M_{\text{BH}}}^{0} \frac{dM}{\pi G N M_{\text{BH}}^4} \]

\[ = \frac{4\pi^3}{3\sigma}(1 + \sqrt{1 + \alpha})^4(1 + \alpha + \sqrt{1 + \alpha})^2 G_N^2 M_{\text{BH}}^3. \quad (8) \]

Figure 2 shows the ratio of the lifetime for various values of \( \alpha \) to that of a Schwarzschild BH (\( \alpha = 0 \)). The dependence on \( \alpha \) clearly influences the lifetime of each black hole. A value of \( \alpha = 2 \) increases the value by roughly a factor of 20. A two-fold increase can be obtained with as small a value as \( \alpha = 0.312 \).

\[ \frac{\tau_\alpha}{\tau_0} \quad (9) \]

which is the Schwarzschild BH solution.

The entropy of the black hole can be derived from the temperature by

\[ S_T = \int_0^{M_{\text{BH}}} \frac{dM}{T} \]

\[ = \pi G_N M_{\text{BH}}^2 \left( \sqrt{1 + \alpha} + 1 + \alpha \right) (\sqrt{1 + \alpha} + 1). \quad (10) \]

Written in this form, it is apparent that it is related to the entropy of the Schwarzschild BH as given by the area-entropy law. Defining \( A_+ = 4\pi R_+^2 \) and using Eq. (2), we find

\[ \frac{S_A}{4G} = \pi G_N M_{\text{BH}}^2 \left( 1 + \sqrt{1 + \alpha} \right)^2, \quad (11) \]
which is slightly different from the result in Eq. (11) by the amount:
\[
\Delta S = S_T - S_A = \pi G_N M_{BH}^2 \alpha (1 + \sqrt{1 + \alpha})
\]  
(12)

In this sense, MOG can be seen as introducing a correction term to the Bekenstein-Hawking bound \[14\] [15]. In both cases, the entropy reduces to the familiar Schwarzschild BH value \( S = 4\pi G_N M_{BH}^2 \) when \( \alpha = 0 \).

### III. THERMODYNAMICS OF ROTATING BLACK HOLES IN MOG

In this section, we will analyze a rotating black hole solution in MOG. We shall call this the Kerr-MOG solution, as it reduces to the usual Kerr solution in the limit \( \alpha = 0 \). The Kerr-MOG metric can be written as \[12\]
\[
ds^2 = \frac{\Delta}{\rho^2} (dt - a \sin^2 \theta \, d\phi)^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2
\]  
(13)
which describes black holes with horizon radii:
\[
r_{\pm} = G_N (1 + \alpha) M \left[ 1 \pm \sqrt{1 - \frac{\alpha^2}{G_N^2 (1 + \alpha)^2 M^2}} - \frac{\alpha}{1 + \alpha} \right]
\]  
(14)
with \( \Delta = r^2 - 2GMr + a^2 + \alpha G_N GM^2 \) and \( \rho^2 = r^2 + a^2 \cos^2 \theta \). The ergosphere is located at
\[
r_e = G_N (1 + \alpha) M \left[ 1 + \sqrt{1 - \frac{\alpha^2 \cos^2 \theta}{G_N^2 (1 + \alpha)^2 M^2}} - \frac{\alpha}{1 + \alpha} \right]
\]  
(15)
This solution is again algebraically identical to the Kerr-Newman metric with \( Q = \sqrt{\alpha G_N M} \), although here \( Q \) is not a free parameter. The temperature for this Kerr-MOG black hole solution is determined by
\[
T = \frac{\kappa}{2\pi}, \quad \kappa = \frac{r_+ - r_-}{2(\rho_+^2 + a^2)}
\]  
(16)
We obtain
\[
T = \frac{1}{2\pi G_N M} \beta, \quad \beta = \frac{\alpha + 1 - \frac{\alpha^2}{G_N^2 M^2}}{(1 + \alpha)^2}
\]  
(17)
where
\[
\beta = \sqrt{\frac{\alpha + 1 - \frac{\alpha^2}{G_N^2 M^2}}{(1 + \alpha)^2}}
\]  
(18)
There is an extremal configuration for these black holes when \( r_+ = r_- \), or
\[
1 - \frac{\alpha^2}{G_N^2 M^2 (1 + \alpha)^2} - \frac{\alpha}{1 + \alpha} = 0
\]  
(19)

\[ \implies \]
\[
M_{\text{ext}} = \frac{1}{G_N \sqrt{1 + \alpha}}.
\]  
(19)

It can be shown that the temperature \[17\] vanishes for this value as well, implying that this is a zero-temperature remnant. Corresponding temperature profiles are plotted in Figure 3 and the generic behaviour of the heat capacity is shown in Figure 4. The effect for increasing \( \alpha > 0 \) is to shift the temperature curve to the left, resulting in small remnants. Furthermore, the maximum temperature of the Kerr-MOG black hole is correspondingly decreased.

### IV. REGULAR SCHWARZSCHILD-MOG BLACK HOLE

A classical solution that is regular at \( r = 0 \) has also been obtained in \[12\]. The metric function is given by
\[
ds^2 = f(r) dt^2 - \frac{dr^2}{f(r)} - r^2 d\Omega^2,
\]  
(20)
where
\[
f(r) = 1 - \frac{2GMr^2}{(r^2 + \alpha G_N GM^2)^{3/2}} + \frac{\alpha G_N M^2 r^2}{(r^2 + \alpha G_N M^2)^2}.
\]  
(21)
FIG. 4: Kerr-MOG heat capacity for $\alpha = 1$ and $a = 1$. The black hole is thermodynamically-unstable when $C < 0$, and reaches a maximum temperature when $C \rightarrow \pm \infty \ (M_{\text{max}} \approx 0.9$ in this case). For $M < M_{\text{max}}$, the heat capacity is positive and the black hole evaporates to a stable cold remnant.

It describes a Schwarzschild-MOG black hole for large $r$ and the metric becomes asymptotically flat in the limit $r \to \infty$. For small $r$, it describes a (anti)deSitter core (depending on the value of $\alpha$) with the effective cosmological constant

$$\Lambda = \frac{3}{G_N M^2} \left( \frac{\alpha^{1/2} - 2}{\alpha^{3/2}(1 + \alpha)} \right). \quad (22)$$

The metric function $f(r)$ approaches 1 in the limit $r = 0$. Horizons are obtained as the roots of $f(r) = 0$, which depend on a critical value $\alpha = \alpha_{\text{crit}}$. When $\alpha < \alpha_{\text{crit}}$, there are two horizons $r = r_{\pm}$, but when $r > \alpha_{\text{crit}}$, there is no horizon. Figure 4 shows the behavior of $f(r)$ for a normalized mass $M = 1$ and $G_N = 1$, in which the three phases of solutions are visible.

In order to evaluate the temperature, we proceed by the surface gravity method. First, the derivative of the metric function is given by

$$f'(r) = -\frac{4(1 + \alpha)Mr}{\sigma^{3/2}} + \frac{6(1 + \alpha)Mr^3}{\sigma^{5/2}}$$

$$+ \frac{2\alpha(1 + \alpha)M^2r}{\sigma^2} - \frac{4\alpha(1 + \alpha)M^2r^3}{\sigma^3}, \quad (23)$$

where $\sigma = r^2 + \alpha(1 + \alpha)M^2$ and the derivative is evaluated at the outer horizon $r = r_+$. The corresponding real, positive roots of $f(r) = 0$ are given in Table I. Evaluating the temperature $T = f'(r = r_+)/4\pi$ for the values therein, one observes that $T \sim 1/M$ for all values of $r$. This means that the temperature diverges as the black hole evaporates, despite the regular nature of the solution. Thus, the divergent behavior of the temperature $T$ in the singular Schwarzschild BH solution is retained for the regular black hole solution with an outer horizon.

| $\alpha$ | $r_-$ | $r_+$ |
|---------|-------|-------|
| 0.1     | 0.163M | 2.067M |
| 0.2     | 0.135M | 2.117M |
| 0.3     | 0.438M | 2.148M |
| 0.4     | 0.676M | 2.152M |
| 0.5     | 0.904M | 2.120M |

TABLE I: Inner and outer horizons $r_{\pm}$ for the regular Schwarzschild-MOG BH for various values of $\alpha > 0$.

V. CONCLUSIONS

We have analyzed the thermodynamics of black holes in a modified theory of gravity in which the field content of general relativity was increased to include scalar fields and a vector field. The addition of these new fields alters the vacuum solutions of general relativity.
thermodynamical properties of the vacuum solutions corresponding to a spherically symmetric black hole and a rotating black hole were analyzed, and it was found that the entropy area law gets changed by increasing the size of the MOG parameter $\alpha$.

The thermodynamics of a vacuum solution in MOG, with and without horizons, which is regular at $r = 0$ [12] was investigated. The “gray hole” without a horizon is expected to emit particles and radiation, so that it will not suffer from the black hole information loss paradox [10,18]. It is also expected that its entropy will be less than the entropy of black holes of a similar size and mass, which signifies that it is not a maximum entropy object. We anticipate that without a horizon the gray hole will not emit the standard Hawking radiation of black holes.

Additional interesting and testable phenomenological characteristics can be extracted from this solution, including quasinormal modes and gravitational wave signatures of binary inspirals. The latter is of particular interest due to the definitive fingerprint of MOG in a black hole’s shadow [13]. If similar effects are as pronounced in gravitational waves, then they should be detectable in the upcoming run of LIGO [19]. These analyses are currently underway by the authors [20].

Acknowledgements

JRM thanks the generous hospitality of the Perimeter Institute for Theoretical Physics, where this work was commenced. Research at the Perimeter Institute for Theoretical Physics is supported by the Government of Canada through industry Canada and by the Province of Ontario through the Ministry of Research and Innovation (MRI).

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