Ultra-small time-delay estimation via a weak measurement technique with post-selection

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Abstract
Weak measurement is a novel technique for parameter estimation with higher precision. In this paper we develop a general theory for the parameter estimation based on a weak measurement technique with arbitrary post-selection. The weak-value amplification model and the joint weak measurement model are two special cases in our theory. Applying the developed theory, time-delay estimation is investigated in both theory and experiments. The experimental results show that when the time delay is ultra-small, the joint weak measurement scheme outperforms the weak-value amplification scheme, and is robust against not only misalignment errors but also the wavelength dependence of the optical components. These results are consistent with theoretical predictions that have not been previously verified by any experiment.

Keywords: weak measurement, post-selection, phase estimation

1. Introduction
As an advanced technique that provides higher sensitivity for parameter estimation, the weak measurement scheme has attracted much attention in recent years [1–5]. Commonly, the weak measurement technique with post-selection involves two physical systems, called the ancillary system and pointer. These systems are weakly interacting, which can be described by a coupling parameter. To estimate this extremely small parameter with higher sensitivity, the ancillary system is post-selected by two orthogonal states, and a subsequent measurement on the pointer system provides suitable information for estimating the coupling parameter.

Two special weak measurement techniques with post-selection, i.e. weak-value amplification (WVA) [1] and joint weak measurement (JWM) [6], have been investigated in the greatest detail. The main characteristic of these techniques is that the post-selection can improve the estimation precision in the presence of technical restrictions [7–9]. In the WVA scheme, the measurement result is recorded when the ancillary system is successfully post-selected by one of the post-selected states, which is chosen almost orthogonal to the initial state. This leads the post-selection probability to be extremely low. Using the recorded results, the coupling parameter is estimated by simply averaging the measurement results [10]. Investigations show that WVA is useful in suppressing some technical noises and realistic limitations, such as detector saturation, correlated noises, and angular beam jitter [9]. Thus, it is easier to reach the quantum standard limit with common experimental devices in the WVA scheme [11]. Moreover, in the scenarios of phase-shift and time-delay measurements, the WVA technique may provide at least a three order of magnitude improvement of estimation precision over standard interferometry [10] when similar alignment errors are taken into account [12, 13]. Of course, there are also disadvantages, i.e. the restrictions on the estimation precision due to systematic errors and alignment errors...
[10, 12], and the limitations on efficiency due to the cost of the large number of unselected events after the post-selection operations [14].

To overcome the disadvantages in the WVA scheme, a natural approach is to collect the unselected data [15, 16]. This leads the so-called JWM scheme [6], in which all measurement results are recorded when the events are nearly equiprobably post-selected by the two orthogonal post-selected states. Using the measurement results, the coupling parameter is estimated by employing the maximum-likelihood estimation method. In [6] it was proved that by carrying out the JWM on all events, one can not only increase the efficiency, but also remove systematic errors and alignment errors. In particular, the JWM technique has no ultimate precision limit [6, 8], which is very useful in practice.

In this paper, we develop a general theory for the weak measurement technique with arbitrary post-selection. Since the post-selecteds involved in the WVA and JWM schemes are extremely unbalanced and almost balanced, respectively, they form two special cases in our general model. Our work is distinguished from previous works concerning the WVA regime [17, 18]. With the proposed general theory, the ultra-small time-delay estimation is investigated. In particular, a novel experiment for time-delay estimation, making use of the weak measurement technique with post-selection, is first presented5. Our experiment exactly verifies the theoretical predictions in [8], and we find that when the time delay is ultra-small the JWM technique outperforms the WVA technique, and is robust against not only misalignment errors but also the wavelength dependence of the optical components.

This paper is structured as follows. In section 2, a general theory for parameter estimation based on the weak measurement technique with arbitrary post-selection is presented and is then applied to time-delay estimation. In section 3, an ultra-small time-delay estimation experiment is presented. Finally, the conclusions are drawn in section 4.

2. Theory

In this section, we develop a general theory for parameter estimation based on the weak measurement technique with arbitrary post-selection. We then apply this theory to ultra-small time-delay estimation.

2.1. General theory of weak measurement based parameter estimation

Consider a weak measurement scenario involving a two-level ancillary system prepared in state $|\varphi_0\rangle$ and a pointer with a continuous degree of freedom in state $|\phi\rangle = \int d\phi\phi(p)|p\rangle$, where $p$ is a continuous variable and $\phi(p)$ is the corresponding wave function. The interaction between the system

\[ U_{\text{int}} = e^{-ip\hat{\phi}}, \]

and the pointer is described by an unitary operator

\[ \hat{A} = \cos(g\hat{p})\mathbb{1} - i\sin(g\hat{p})\hat{\phi}. \]

where $\hat{A}$ acts on the system with eigenvalues of 1 and $-1$, $\hat{\phi}$ acts on the pointer, and $g$ is the coupling parameter which indicates the coupling strength. This operator can be expressed as [20]:

\[ U_{\text{int}} = e^{-ip\hat{\phi}} = \cos(g\hat{p})\mathbb{1} - i\sin(g\hat{p})\hat{\phi}. \]

After the interaction, an initial product state

\[ |\Psi\rangle = |\varphi_0\rangle \otimes |\phi\rangle \]

evolves to

\[ |\Psi\rangle = U_{\text{int}}|\varphi_0\rangle|\phi\rangle = [\cos(g\hat{p})|\varphi_0\rangle - i\sin(g\hat{p})\hat{\phi}|\varphi_0\rangle]|\phi\rangle. \]

The ancillary system is then post-selected by two orthogonal states, i.e. $|\varphi_1\rangle$ and $|\varphi_2\rangle$, which converts the pointer state to

\[ |\phi_j\rangle = \langle \varphi_j|\varphi\rangle|\Psi\rangle = \langle \varphi_j|\varphi\rangle|\varphi_0\rangle|\phi\rangle + \langle \varphi_j|\varphi\rangle|\varphi_1\rangle|\phi\rangle + \langle \varphi_j|\varphi\rangle|\varphi_2\rangle|\phi\rangle, \]

where $j = 1, 2$, and $A_{x_j} \equiv \langle \varphi_j|A|\varphi_j\rangle$ is the well known weak value [1]. Consequently, the probability distribution of $p$ associated with the pointer becomes

\[ P_j(p) = |\langle \varphi_j|\varphi\rangle|^2 P_0(p)\zeta_j(p, g), \]

where $P_0(p) \equiv |\langle \varphi|\varphi\rangle|^2$ is the initial probability distribution of the pointer, and $\zeta_j(p, g) \equiv \cos^2(gp) + \sin^2(gp)|A_{x_j}|^2 + 2\cos(gp)\sin^2(gp)|A_{x_j}|$ is the part that changes the shape of the initial probability distribution.

To achieve an unbiased estimate of $g$, we apply the maximum-likelihood estimation method [21]. First, we construct the log-likelihood estimator as

\[ L(g) = \sum_j \int dp Q_j(p) \log P_j(p), \]

where $Q_j(p)(j = 1, 2)$ is the probability distribution of observing $p$ in the experiment, which in principle converges to $P_j(p)$ as the number of measured events is increased. Then, the maximum-likelihood estimate of $g$ can be achieved by solving the likelihood equation $\frac{\partial L(g)}{\partial g} = 0$ [22]. In our case, it yields

\[ \frac{\partial L(g)}{\partial g} = \sum_j \int dp Q_j(p) \frac{\partial \log \zeta_j(g, p)}{\partial g} \]

\[ = \sum_j \int dp Q_j(p) \times p\sin^2(gp)|A_{x_j}|^2 + 2\cos(gp)\sin^2(gp)|A_{x_j}| \cos^2(gp)\sin^2(gp)|A_{x_j}| \]

\[ = 0. \]

5 After finishing this manuscript, we became aware of the similar experimental set-up presented in [19].
As the weak measurement technique requires $gp \ll 1$ and $|A_w|gp \ll 1$ [23], equation (7) can be simplified as

$$\sum_{j} \int dpQ_j(p) \left( -2 \text{Im} A_w p^3 g^2 + (|A_w|^2 - 1) p^2 g + \text{Im} A_w p \right) = 0,$$  \hspace{1cm} (8)

Interestingly, equation (8) shows that prior knowledge about $P_0(p)$ is unnecessary.

Generally, the analytical solutions of equation (8) are difficult to obtain. Fortunately, under the condition of $|A_w|gp \ll 1$, equation (8) can be approximately simplified to a quartic equation

$$Ag^4 + Bg^3 + Cg^2 + Dg + E = 0,$$  \hspace{1cm} (9)

where

$$A = \sum_{j} \int dpQ_j(p)p^4 [2(|A_{w,j}|^2 - 1)] \text{Im} A_{w,j}],$$

$$B = \sum_{j} \int dpQ_j(p)p^4 [4(\text{Im} A_{w,j})^2 - (|A_{w,j}|^2 - 1)^2],$$

$$C = \sum_{j} \int dpQ_j(p)p^3 [\text{Im} A_{w,j}(1 - 3|A_{w,j}|^2)],$$

$$D = \sum_{j} \int dpQ_j(p)p^2 [(|A_{w,j}|^2 - 1) - 2 \text{Im} A_{w,j}],$$

and

$$E = \sum_{j} \int dpQ_j(p)p \text{Im} A_{w,j}.$$

As $g$ is extremely small, one may obtain the first-order approximate solution

$$g_{\text{ext}}^{(1)} = - \frac{E}{D} = \frac{\sum \text{Im} A_{w,j} \langle p \rangle_j}{\sum [2(\text{Im} A_{w,j})^2 - |A_{w,j}|^2 + 1] \langle p^2 \rangle_j},$$  \hspace{1cm} (10)

where

$$g_{\text{ext}}^{(1)} = - \frac{E}{D} = \frac{\sum \text{Im} A_{w,j} \langle p \rangle_j}{\sum [2(\text{Im} A_{w,j})^2 - |A_{w,j}|^2 + 1] \langle p^2 \rangle_j},$$

and

$$\langle p^2 \rangle_j = \int dpQ_j(p)p^2.$$  \hspace{1cm} (11)

Equation (10) provides an appropriate estimate of $g$ with very high precision.

### 2.2. Time-delay estimation

As an application of our developed theory, we consider the time-delay estimation described in figure 1. In this scheme, an ultra-small time delay of $\tau$ between two orthogonal polarizations, which are denoted as horizontal ($H$) and vertical ($V$), involves a thin birefringent crystal. The corresponding unitary operator is given by

$$\hat{U}_{\text{int}}(\tau) = e^{-i\hat{D}_{\text{xy}}\omega},$$  \hspace{1cm} (11)

where $\hat{D}_{\text{xy}} = |H \rangle \langle V| - |V \rangle \langle H|$ is the $Z$ operator acting on the polarization states and $\omega$ represents the frequencies of the photons. In this scenario the parameters $g$ and $p$ in the general model are replaced by time delay $\tau$ and frequency $\omega$, respectively, and the photon polarizations and frequencies are chosen as the ancillary system variable and the pointer variable, respectively.

Given an initial state of $|\phi_1\rangle = \frac{|H\rangle + |V\rangle}{\sqrt{2}}$ and post-selected states of $|\phi_1\rangle = \frac{|H\rangle + e^{i\phi}|V\rangle}{\sqrt{2}}$ and $|\phi_2\rangle = \frac{|H\rangle - e^{i\phi}|V\rangle}{\sqrt{2}}$, we get the weak values

$$A_{w1} = \frac{|\langle \phi_1 | \hat{A} | \phi_1 \rangle|}{|\langle \phi_1 \rangle|} = 1 - e^{-i\phi} e^{-i\phi} = i \tan \phi,$$

$$A_{w2} = \frac{|\langle \phi_2 | \hat{A} | \phi_2 \rangle|}{|\langle \phi_2 \rangle|} = 1 + e^{-i\phi} e^{-i\phi} = -i \cot \phi.$$  \hspace{1cm} (12)

The post-selection probabilities are given by

$$P_{f1} = |\langle \phi_{1,j} | \phi_{f1} \rangle|^2 \int d\omega P_0(\omega) \zeta_1(\omega, \tau),$$

$$= \cos^2 \phi \left[ 1 + \left( \tan^2 \phi - 1 \right) \omega_0^2 \tau^2 + 2 \tan \phi \omega_0 \tau \right];$$

$$P_{f2} = |\langle \phi_{2,j} | \phi_{f2} \rangle|^2 \int d\omega P_0(\omega) \zeta_2(\omega, \tau),$$

$$= \sin^2 \phi \left[ 1 + \left( \cot^2 \phi - 1 \right) \omega_0^2 \tau^2 - 2 \cot \phi \omega_0 \tau \right],$$  \hspace{1cm} (13)

where $\omega_0 \equiv \int P_0(\omega) d\omega$ represents the initial average frequency of light before the interaction. In practice, $P_{f1}$ and $P_{f2}$ can be estimated from experimental data by $\int Q_1(\omega) d\omega$ and $\int Q_2(\omega) d\omega$, respectively.

Inserting equation (12) into equation (10) gives

$$g_{\text{ext}}^{(1)} = \frac{\sin^2 \phi (\omega_1 - \omega_2)}{\tan \phi \omega_1^2 + \cot \phi \omega_2^2},$$  \hspace{1cm} (14)

We note that estimating $\tau$ with equation (14) requires prior knowledge of $\phi$, which is, however, not necessary in high precision to achieve a high signal-to-noise ratio (S/N) [6]. Further discussion can be found in section 3.3.

Now we consider applications of the proposed theory model in two specific schemes, i.e. the JWM scheme and the WVA scheme. In the JWM scheme, the light intensities detected by two spectrometers are nearly equal, which requires $\phi \approx \frac{\pi}{2}$, the pre-selected and post-selected states can be seen in figure 2. The post-selection probabilities of the two
output ports are
\[ P_{11} = \cos^{2} \frac{\phi}{2} \left( 1 + 2 \tan^{2} \frac{\phi}{2} \omega_{0} \tau \right), \]
\[ P_{12} = \sin^{2} \frac{\phi}{2} \left( 1 - 2 \cot^{2} \frac{\phi}{2} \omega_{0} \tau \right). \] (15)

In the JWM scheme, the weak value is not anomalous, thus one cannot observe a significant shift of the pointer’s wave function, as it is the case in WVA. In fact, the measured shift exists in the probability difference \[ \left( 19 \right). \]

Then, equation (14) can be simplified as
\[ \tau_{\text{JW}} \approx \frac{P_{1} - P_{1}}{\Delta \omega^{2}}. \] (16)

Employing equation (16) the time delay \( \tau \) can be estimated without being precisely aware of \( \phi \). We note that this result is in conflict with the formula given in [6], which is
\[ \tau = \frac{1}{4} \left( \langle \omega \rangle_{2} - \langle \omega \rangle_{1} \right) \left( \frac{1}{\Delta \omega^{2}} - \frac{P_{1} - P_{1}}{\Delta \omega^{2}} \right). \]

In section 3 we will further verify the correctness of our formula through experiments.

Next we consider the application of the proposed theory model in the WVA technique. In the WVA scheme, most of the light output comes from port 1, which requires \( \phi \approx 0 \) as shown in figure 2. The post-selection probabilities of the two output ports are
\[ P_{11} \approx \left( 1 - \frac{\phi^{2}}{4} \right) \left( 1 + \phi \omega_{0} \tau \right), \]
\[ P_{12} \approx \frac{\phi^{2}}{4} \left( 1 - \frac{4}{\phi^{2}} \omega_{0} \tau \right). \] (17)

In this case, equation (14) gives
\[ \tau_{\text{WVA}} = \frac{\langle \omega \rangle_{2} - \langle \omega \rangle_{1}}{\Delta \omega^{2} \left( \langle \omega \rangle_{1} + \langle \omega \rangle_{2} - 2 \omega_{0} \langle \omega \rangle_{2} \right)}, \] (18)

where
\[ \langle \omega \rangle_{j} = \frac{\int Q_{j}(\omega) \omega^{2} d\omega}{\int Q_{j}(\omega) d\omega} = \langle \omega^{2} \rangle_{j} / P_{f_{j}}, \]

and
\[ \langle \omega^{2} \rangle_{j} = \frac{\int Q_{j}(\omega) \omega^{2} d\omega}{\int Q_{j}(\omega) d\omega} = \langle \omega^{2} \rangle_{j} / P_{f_{j}}. \]

In equation (18), we have \( \langle \omega \rangle_{1} - \langle \omega \rangle_{2} \approx \omega_{0} - \langle \omega \rangle_{2} \equiv \delta \omega \) and \( \langle \omega^{2} \rangle_{1} + \langle \omega^{2} \rangle_{2} - 2 \omega_{0} \langle \omega \rangle_{2} \approx 2 \langle \Delta \omega^{2} \rangle \). Recalling that \( A_{w2} = -1 \cot(\phi/2) \approx -i \frac{2}{\phi} \), we obtain
\[ \delta \omega \approx 2 \gamma \text{Im} \Delta A_{w2} \langle \Delta \omega^{2} \rangle, \] (19)

which is the familiar formula derived in the WVA scheme [12].

3. The experiment

In this section, we perform an experiment to demonstrate the time-delay estimation based on the theory model presented in section 2.2. Making use of the experimental results, we study the performances of the JWM and WVA schemes, so that we can verify the theoretical predictions about the advantages of JWM presented in [6].
3.1. Experimental set-up

The experimental set-up is described in figure 3, in which a thin birefringent crystal is used to induce the time delay. We note that it is very difficult to produce and manipulate this kind of plate in practice. Following the ideas in [13], we use a double-plate system by placing their optical axes (OAs) perpendicular to each other to equivalently realize this effect. Restricted by the laboratory conditions, we use binary compound zero-order half-wave plates (HWPs) in our experiment. These plates are built by combing two multi-order wave plates and aligning the fast axis of one plate with the slow axis of the other to obtain a zero-order phase delay. In contrast to the true zero-order plates used in [13], they are insensitive to temperature changes. We place both plates perpendicular to the light velocity to cancel their phase delay. The delay is realized by pivoting the second plate around its OA by a tiny angle $\theta$. This pivot increases the optical path of the second HWP, which makes the double-HWP system equivalent to a very thin plate of the same material with the OA orienting as the tilted axis. The relationship between the time delay $\tau$ and the tilt angle $\theta$ is

$$\tau = \frac{(n_e - n_o) \theta^2}{2cn},$$

where $n_e$, $n_o$, and $n$ are the refractive indices of quartz for extraordinary light, ordinary light, and average light, respectively, $h$ is the thickness of the plate, $c$ is speed of light speed in a vacuum, and $\lambda$ is the wavelength of the light. For detailed derivations, see appendix A and [24].

Finally, in order to finish the post-selection, the photons sequentially enter a quarter-wave plate (QWP) and a polarization beam splitter (PBS). Initially, the OA of the QWP is perpendicular to the axis of the Glan–Tylor prism and the PBS splits the photons to polarization states $|H\rangle$ and $|V\rangle$. Then, the PBS is rotated by an angle of $\gamma = \pi/4 - \frac{\phi}{2}$ according to the GT prism’s OA orientation to post-select final states of $|\phi_{j1}\rangle = \cos \gamma |H\rangle + \sin \gamma |V\rangle$ and $|\phi_{j2}\rangle = \sin \gamma |H\rangle - \cos \gamma |V\rangle$.

In our setting, when $\phi \approx \frac{\pi}{2}$, the light intensities that go out from the two output ports are nearly the same, which corresponds to the JWM proposal [6]. In contrast to $\phi \approx 0$, almost all of the light goes out from one output port, which corresponds to the WVA scheme. In the following, we will present the experimental results of these two schemes to verify the theory derived in section 2.

3.2. Joint weak measurement versus weak-value amplification

Firstly, we consider the JWM technique where the post-selection is almost balanced. In this scheme, we set $\phi \approx \left(\frac{\pi}{2} + 0.071\right)$ rad (by rotating the PBS with an angle of $-0.071$ rad), and the post-selection probabilities for each output port can be calculated by equation (15). In figure 4, according to the tilted angle we applied, different estimations about the time delay are shown, where the theoretical predictions drawn by the blue line are calculated by equation (20). Among these estimations, the blue triangles show the complete solutions to equation (8), while the red circles and black crosses are the approximate solutions given by equations (14) and (16), respectively. Compared to equation (14), equation (16) does not require prior knowledge of $\phi$, as this parameter is estimated by the light intensity difference of the two output ports. However equation (16) provides an estimation with less accuracy, because environmental effects, such as temperature fluctuation, make the output intensity of the LED unstable.

Then, we consider the WVA technique where the post-selection is extremely unbalanced. In this scheme, we set $\phi \approx 0.03$ (by rotating the PBS for $\pi/4 - 0.03$ rad), and the post-selection probabilities for each output port, which can be calculated by equation (17), are extremely unbalanced. Figure 5 shows different estimations for the time delay according to the tilted angle we chose, where the theoretical predictions drawn by blue line are calculated by equation (20). Among these estimations, the blue triangles are the complete solutions to equation (8), while red circles are the approximate solutions given by equation (14).

We can find that the red circles fit well with the theory when $\tau$ is less than $0.5 \times 10^{-17}$ but the deviation increases along with the growing $\tau$, because the effect of the wavelength dependence of the QWP was not taken into account in equation (18) (see appendix B for details). For more discussion on this phenomenon, we refer the reader to [26–28].

3.3. Discussion

Utilizing the imaginary part of the weak value provides a large amplification factor for parameter estimation. For this

![Figure 4. Relation of $\theta$ and $\tau$ in the JWM scheme. The blue line presents the theoretical prediction from equation (20), the blue triangles, red circles, and black crosses are estimates of $\tau$ from the experimental data, calculated by the most rigorous equation, (8), the first-order approximation (14), and the simplified approximation (16), respectively.](image-url)
purpose, a QWP for adding a circular component on the polarization is required. However, as is pointed out in [13], the QWP involves a large uncertainty to the polarization, which is difficult to compensate because of the wide spectrum of the light source.

There are two possible ways to solve this problem: get rid of the QWP with the cost of much smaller weak value, or apply the JWM method to suppress the wavelength-dependence effect.

In our experimental set-up (see figure 3), removing the QWP reverts to a similar set-up to that in [13]. Alternatively, we can collect the data from both output sides and apply equation (10) for parameter estimation. When the condition $|A_0|/\omega_0 \tau \ll 1$ is satisfied, the estimation fits quite well to the theoretical prediction (see figure 5).

In [13], the uncertainty of measuring $\alpha$ (equal to $\omega_0\tau$ in our paper) depends on the spectrometer resolution $\Delta(\delta\lambda)$ and the post-selection parameter $\beta$ (equal to $2\gamma$ in our paper). The uncertainty of $\alpha$, which is denoted as $\Delta\alpha$, can be derived as [13]:

$$\Delta\alpha = \frac{\lambda_0}{2\Delta\lambda} \cdot \frac{(\alpha^2 + \beta^2)^2}{\alpha^2} \cdot \Delta(\delta\lambda). \quad (21)$$

According to this formula, optimal precision can be achieved when $\beta \approx \alpha$, but it is very impractical because the value of $\alpha$ is unknown before it is measured. Alternatively, the authors pointed out that for measuring small $\alpha$ one can set $\beta = 0$ and obtain $\Delta\alpha \approx 0.1 \alpha$ when $\Delta(\delta\lambda) = 0.1$ nm. However, an important factor, the misalignment of $\beta$, has not been considered. In fact, it sets a limitation on the minimum detectable phase when using WVA [10]. Assuming that the actual value of $\beta$ is $\epsilon$ when we set it to be 0, and consider

Figure 5. Relation of $\theta$ and $\tau$ in the WVA scheme. The blue line presents the theoretical prediction from equation (20), the blue triangles and red circles are the estimates of $\tau$ from experimental data, calculated by the most rigorous equation, (8), and the first-order approximation (14), respectively.

Figure 6. S/N in the time-delay estimation experiment with JWM. Red circles: $\phi = 0.03$; blue stars: $\phi = 0.05$; black crosses: $\phi = 0.08$.

$$\alpha_{\min} = \Delta\alpha$$ as the minimum detectable value of the measured phase, we can derive from equation (21) that

$$\alpha_{\min} = \frac{\lambda_0}{2\Delta\lambda} \Delta(\delta\lambda)$$

where we define $C \equiv \frac{\lambda_0}{2\Delta\lambda} \Delta(\delta\lambda)$ for simplicity. By setting $\Delta(\delta\lambda) = 0.1$ nm, which is assumed in [13] and practically applied in our experiment, we obtain $\alpha_{\min} \approx 3.7\epsilon$.

On the other hand, by using the JWM scheme proposed in [6] and demonstrated in section 3.2, the uncertainty of the measured phase is insensitive to the alignment error. In our experiment, we set $\phi = 0.03$. The exact value of $\phi$ can be estimated by maximizing equation (8), but it is not necessary to do so to achieve high precision. In figure 6 we calculate the S/N when estimating $\tau$ by equation (10) and setting $\phi = 0.03$, 0.05, and 0.08 rad respectively. The S/N drops slightly even when our prediction about $\phi$ has a deviation of 0.05 rad, and remains higher than 10 dB for $\alpha > 0.002$. This shows that the estimation results are insensitive to $\phi$ in a balanced post-selection scheme. In contrast, for WVA in an unbalanced post-selection scheme, for a typical deviation of $\epsilon = 0.0027$ [13], the S/N drops down to 0 dB when $\alpha = 0.01$, much worse than applying the JWM method.

As is predicted by the theory [6] and presented in figure 6, the S/R of the JWM scheme remains stable when $\alpha$ grows, because the estimation error increases along with the signal. As in the WVA scheme, the S/N increases along with the signal [25], which implies that at some point the WVA scheme will achieve a higher S/N than the JWM scheme. In that case, one can adopt the WVA scheme for better performance, however, the disadvantage of the QWP must be overcome by, e.g., applying a broadband wave plate. This is beyond the scope of our current work.

Finally, it is useful to compared JWM and WVA to other schemes for practical purposes. In [10] the authors compared...
the WVA and a standard interferometric scheme for phase-shift estimation, and proved that when similar alignment errors are taken into account, the estimation precision of WVA is at least three orders of magnitude higher than the standard interferometric scheme. Combining this with the above discussion, it is reasonable to expect that JWM has the highest precision for estimating extremely small time delays among these three schemes.

4. Conclusion

In summary, we present a general theory for parameter estimation based on weak measurement with arbitrary post-selection. Applying this theory, we study the time-delay estimation in both theory and experiment, with particular interest in two specific schemes, i.e. the WVA and JWM schemes. Through the experimental results, we find that the JWM scheme outperforms the WVA scheme when the time delay is ultra-small. These results support the theoretical predictions presented in [6], which had not been previously verified by any experiment. Furthermore, the JWM technique is robust against not only misalignment errors but also the wavelength dependence of the optical components. The JWM technique appears to be an advanced parameter estimation approach for achieving higher precision with convenient laboratory devices.

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Appendix A. Phase retardation of a compound binary zero-order wave plate

When light propagates through a uniaxial crystal plate, the phase delay between the ordinary light and extraordinary light is

$$\delta = \frac{2\pi}{\lambda} (n_e - n_o) L \sin^2 \eta,$$

where $\lambda$ is the wavelength, and $n_e$ and $n_o$ stand for refractive indices of ordinary and extraordinary light, respectively. $L = \frac{h}{\cos \eta}$ means the path length propagating in the birefringent crystal. $\eta$ is the angle between the normal line and the OA, $\eta$ is the angle between the light velocity in the crystal and the normal line, and $h$ is the thickness of the plate. For simplicity, we establish the Cartesian coordinates showed in figure 7 to calculate $C = L \sin^2 \eta$ in the crystal, which we call the effective path length in the following. The origin is coincident with the point of incidence. The $x$-$z$ plane is on the surface of the plate and and the OA is paralleled to the $z$-axis. It is obvious that $L^2 = x^2 + y^2 + z^2$, and

$$C = L \sin^2 \eta = \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + z^2}}.$$  \hspace{1cm} (A2)

The exit point is $(x, h, z)$ and the effective path length could be obtained by replacing $h$ by $y$. In our experiment, the pivot is around the OA, we define the azimuth angle and elevation angle as $\xi$ and $\psi$, respectively, so

$$x^2 + h^2 = h^2 / \cos^2 \xi,$$

$$z^2 + h^2 = h^2 / \cos^2 \psi,$$

and we can deduce that

$$x^2 + y^2 + z^2 = \frac{h^2 (1 - \sin^2 \xi \sin^2 \psi)}{\cos^2 \xi \cos^2 \psi}. \hspace{1cm} (A3a)$$

Inserting equations (A2) and (A3a) into equation (A1), we can obtain the general expression of the phase retardance of oblique incidence on a crystal plate:

$$\delta = \frac{2\pi}{\lambda} (n_e - n_o) C = \frac{2\pi (n_e - n_o) h \cos \psi}{\sqrt{h^2 (1 - \sin^2 \xi \sin^2 \psi)}}. \hspace{1cm} (A4)$$

The compound binary plate is built by combining two multi-order wave plates to obtain an zero-order phase delay by aligning the fast axis of one plate with the slow axis of the other. The retardance could be expressed as

$$\delta = \frac{2\pi}{\lambda} (n_e - n_o) (h_1 - h_2). \hspace{1cm} (A5)$$

Figure 8 shows the Cartesian coordinates when light goes through the second plate, while figure 7 still shows the incidence on the first plate. The origin overlaps with the exit point of the first plate. Since the OA of the second plate is
perpendicular to that of the first plates, the coordinate system is rotating by 90°. Let the subscripts denote the two plates. We neglect the refringence between the two plates and, assuming that $\xi$ in figure 8 is approximately equal to $\psi$ in figure 7, we can derive

$$
z_1/h_1 = x_2/h_2, \quad x_1/h_1 = -z_2/h_2, \quad (A6)$$

The phase delay of the compound plate is

$$
\delta = \frac{2\pi}{\lambda}(n_e - n_o)(C1 - C2), \quad (A7)
$$

where

$$
C1 = \frac{s_1^2 + h_1^2}{h_1^2 + h_2^2 + s_1^2}, \quad C2 = \frac{h_2(s_1^2 + h_1^2)}{h_1^2 + h_2^2 + s_1^2}. \quad (A8)
$$

Combining with equation (A2) and taking into account the relationship of the angles, we find the general expression of the phase retardation of the compound plate:

$$
\delta = \frac{2\pi}{\lambda}(n_e - n_o)(h_1 \cos \psi - h_2 \cos \xi)/[\sqrt{1 - \sin^2 \xi \sin^2 \psi}], \quad (A9)
$$

When $\psi = 0$ and $\xi$ is small, equation (A9) becomes

$$
\delta = \frac{2\pi}{\lambda}(n_e - n_o)(h_1 / \cos \xi - h_2 / \cos \xi) \approx \frac{2\pi}{\lambda}(n_e - n_o)(h_1 - h_2) + \xi^2(h_1 + h_2)/2, \quad (A10a)
$$

and when $\xi = 0$ and $\psi$ is small, equation (A9) approximates to

$$
\delta = \frac{2\pi}{\lambda}(n_e - n_o)(h_1 \cos \psi - h_2 / \cos \psi) \approx \frac{2\pi}{\lambda}(n_e - n_o)[h_1 - h_2] - \psi^2(h_1 + h_2)/2. \quad (A10b)
$$

In equation (A10), the first term is the plate’s retardance of vertical incidence and the second term is the phase delay of oblique incidence. Considering the relationship between the incidence angle $\theta$, $\psi$, and $\xi$, we could rewrite the second term as

$$
\Delta \delta = \frac{\pm \pi}{\lambda}(n_e - n_o)(h_1 + h_2)\theta^2 / \lambda h_1 h_2, \quad (A11)
$$

When $\psi = 0$ it has the positive value and a negative value can be obtained when $\xi = 0$. In the range of our light emitting diode spectrum, $n_e - n_o$ is almost constant, in other words, the optical length in the crystal is insensitive to the variation of wavelength. With the canceling effect of the two wave plates in our experiment, the time delay induced by the pivot is

$$
\tau = \frac{\Delta \delta \lambda}{2\pi c} = \frac{\pm (n_e - n_o)(h_1 + h_2)\theta^2}{2\lambda h_1 h_2}, \quad (A12)
$$

where $c$ is the the speed of light in a vacuum.

### Appendix B. Effect of the wavelength-dependent quarter-wave plate

The post-selection in our experiment consists of a QWP and a linear PBS. For using ideal QWPs equation (12) is established, however, in practice optical components are wavelength-dependent. A QWP with its OA at a 45° angle and a central angular frequency of $\omega_0$ can be described by a Jones matrix as

$$
U_{\lambda/4} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} e^{-i\omega_0} & 0 \\ 0 & e^{i\omega_0} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \cos \omega_0 & -i \sin \omega_0 \\ -i \sin \omega_0 & \cos \omega_0 \end{pmatrix}, \quad (B1)
$$

where $\tau_0 = \pi/4\omega_0$ and $\omega$ is the angular frequency of the light.

Denoting the linear polarization state selected by PBS as $|\psi\rangle$, the post-selection of $|\psi\rangle$ can be divided as

$$
\langle \psi'\rangle = \langle \psi | U_{\lambda/4} | \psi \rangle = \langle \psi | \langle U_{\lambda/4} | U_{\lambda/4} | \psi \rangle | \psi \rangle. \quad (B2)
$$

Obviously, $|\psi'\rangle = U_{\lambda/4}^+ |\psi\rangle$ is frequency-dependent. Combining with equation (B1) and the following expressions, i.e.,

$$
|\psi_{j1}\rangle = \cos \gamma |H\rangle + \sin \gamma |V\rangle
$$

and

$$
|\psi_{j2}\rangle = \sin \gamma |H\rangle - \cos \gamma |V\rangle,
$$

we can derive

$$
\begin{align*}
|\psi'_{j1}\rangle &= (\cos \omega_0 \cos \gamma + i \sin \omega_0 \sin \gamma) |H\rangle \\
&+ (\cos \omega_0 \sin \gamma + i \sin \omega_0 \cos \gamma) |V\rangle,
\end{align*}
$$

and

$$
\begin{align*}
|\psi'_{j2}\rangle &= (\cos \omega_0 \sin \gamma - i \sin \omega_0 \cos \gamma) |H\rangle \\
&+ (- \cos \omega_0 \cos \gamma + i \sin \omega_0 \sin \gamma) |V\rangle.
\end{align*}
$$

Consequently, the weak values can be calculated as

$$
A_{w1} = \frac{|\psi'_{j1}\rangle \langle A | \psi\rangle}{|\langle \psi'_{j1}\rangle |^2} = \frac{(\cos \omega_0 \cos \gamma + i \sin \omega_0 \sin \gamma)}{(\cos \omega_0 \sin \gamma - i \sin \omega_0 \cos \gamma)}, \quad (B3)
$$

$$
A_{w2} = \frac{|\psi'_{j2}\rangle \langle A | \psi\rangle}{|\langle \psi'_{j2}\rangle |^2} = \frac{(\cos \omega_0 \sin \gamma + i \sin \omega_0 \cos \gamma)}{(\cos \omega_0 \sin \gamma - i \sin \omega_0 \cos \gamma)}, \quad (B4)
$$

In this case, $A_{w1}$ cannot be taken outside the integral of equation (8). Comparing the results shown in figures 4 and 5 we can see that in the balanced post-selection scheme, equation (10) provides a good approximation, while in the unbalanced post-selection scheme the wavelength dependence of the QWP involves an non-negligible deviation.
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