Variational Convergence Analysis With Smoothed-TV Interpretation

Erdem Altuntac
Institute for Numerical and Applied Mathematics, University of Göttingen, Lotzestr. 16-18, D-37083, Göttingen, Germany
E-mail: e.altuntac@math.uni-goettingen.de

Abstract.
The problem of minimization of the least squares functional with a Fréchet differentiable, lower semi-continuous, convex penalizer $J$ is considered to be solved. The penalizer maps the functions of Banach space $V$ into $\mathbb{R}^+$, $J : V \to \mathbb{R}^+$. To be more precise, we also assume that some given measured data $f^\delta$ is defined on a compactly supported domain $Z \subset \mathbb{R}^+$ and in the class of Hilbert space, $f^\delta \in H = \mathcal{L}^2(Z)$. Then the general Tikhonov cost functional, associated with some given linear, compact and injective forward operator $T : V \to \mathcal{L}^2(Z)$, is formulated as

$$F_\alpha(\varphi, f^\delta) : V \times \mathcal{L}^2(Z) \to \mathbb{R}^+,$$

$$(\varphi, f^\delta) \mapsto F_\alpha(\varphi, f^\delta) := \frac{1}{2}\|T\varphi - f^\delta\|_{\mathcal{L}^2(Z)}^2 + \alpha J(\varphi).$$

Convergence of the regularized optimum solution $\varphi_{\alpha(\delta)} \in \arg\min_{\varphi \in V} F_\alpha(\varphi, f^\delta)$ to the true solution $\varphi^\dagger$ is analysed by means of Bregman distance.

First part of this work aims to provide some general convergence analysis for generally strongly convex functional $J$ in the cost functional $F_\alpha$. In this part the key observation is that strong convexity of the penalty term $J$ with its convexity modulus implies norm convergence in the Bregman metric sense. We also study the characterization of convergence by means of a concave, monotonically increasing index function $\Psi : [0, \infty) \to [0, \infty)$ with $\Psi(0) = 0$. In the second part, this general analysis will be interepreted for the smoothed-TV functional, $J^\text{TV}_\beta(\varphi) := \int_{\Omega} \sqrt{\| \nabla \varphi(x) \|^2 + \beta} dx$, where $\Omega$ is a compact and convex domain. To this end, a new lower bound for the Hessian of $J^\text{TV}_\beta$ will be estimated. The result of this work is applicable for any strongly convex functional.

Keywords. convex regularization, Bregman distance, smoothed total variation.

1. Introduction
As alternative to well established Tikhonov regularization, [33, 34], studying convex variational regularization with some general penalty term $J$ has become important over the last decade. Introducing a new image denoising method named as total variation, [36], is commencement of such study. Application and analysis of the method have been widely carried out in the communities of inverse problems and optimization,
Problem of finding the optimum minimizer for a general Tikhonov type functional is formulated below

\[ \varphi_{\alpha(\delta)} \in \arg \min_{\varphi \in \mathcal{V}} \left\{ \frac{1}{2} \| T\varphi - f^\delta \|^2_H + \alpha J(\varphi) \right\}. \]  

(1.1)

Here, \( J : \mathcal{V} \rightarrow \mathbb{R}_+, \) is the convex penalty term and it is smooth in the Fréchet derivative sense with the regularization parameter \( \alpha > 0 \) before it.

This work aims to utilize convex analysis together with Bregman distance as two fundamental concepts to arrive at convergence and convergence rates in convex regularization strategy. In particular, it will be observed that the strong convexity provides new quantitative analysis for the Bregman distance which also implies norm convergence. We will interprete this observation for the smoothed-TV functional, \( J^\text{TV}_\beta, \)

\[ J^\text{TV}_\beta(\varphi) := \int_\Omega \sqrt{\| \nabla \varphi(x) \|^2 + \beta} \, dx. \]

Eventually, it will be shown that the strong convexity of \( J^\text{TV}_\beta \) requires the solution to be in the class of the Sobolev space \( \mathcal{W}^{1,2}. \)

We rather focus on a posteriori strategy for the choice of regularization parameter \( \alpha = \alpha(\delta, f^\delta) \). and this does not require any a priori knowledge about the true solution. We always work with the given perturbed data \( f^\delta \) and introduce the rates according to the perturbation amount \( \delta. \) Under this a posteriori strategy and the assumed deterministic noise model, \( f^\delta \in B_\delta(f^{\dagger}), \) in the measurement space, the following rates will be able to be quantified;

(i) \( T\varphi_{\alpha(\delta,f^\delta)} \in B_{\alpha(\delta)}(T\varphi^{\dagger}); \) the discrepancy between \( T\varphi_{\alpha(\delta,f^\delta)} \) and \( T\varphi^{\dagger} \) by the rate of \( o(\delta). \)

(ii) \( D_f(\varphi_{\alpha(\delta,f^\delta)}, \varphi^{\dagger}) \leq O(\Psi(\delta)); \) upper bound for the Bregman distance \( D_f, \) which will immediately imply the desired norm convergence \( \| \varphi_{\alpha(\delta,f^\delta)} - \varphi^{\dagger} \|_V. \)

(iii) \( \varphi_{\alpha(\delta,f^\delta)} \in B_{\alpha(\Psi(\delta))}(\varphi^{\dagger}); \) convergence of the regularized solution \( \varphi_{\alpha(\delta,f^\delta)} \) to the true solution \( \varphi^{\dagger} \) by the rate of the index function \( O(\Psi(\delta)). \)

2. Notations and prerequisite knowledge

2.1. Functional analysis notations

Let \( \mathcal{C}(\Omega) \) be the space of continuous functions on a compact domain \( \Omega \) with its Lipschitz boundary \( \partial \Omega. \) Then, the function space \( \mathcal{C}^k(\Omega) \) is defined by

\[ \mathcal{C}^k(\Omega) := \{ \varphi \in \mathcal{C}(\Omega) : D^\sigma(\varphi) \in \mathcal{L}^p(\Omega), \forall \sigma \in \mathbb{N} \text{ with } |\sigma| \leq k \}. \]
We will also need to work with Sobolev spaces. We define Sobolev space for $p \geq 1$ by,

$$\mathcal{W}^{k,p}(\Omega) := \{ \phi \in L^p(\Omega) : D^\sigma(\phi) \in L^p(\Omega), \forall \sigma \in \mathbb{N} \text{ with } |\sigma| \leq k \}.$$  

We also denote another Sobolev function space with zero boundary value by

$$\mathcal{W}^{k,p}_0(\Omega) := \{ \phi \in C(\Omega) | D^\sigma(\phi) \in L^p(\Omega), \forall \sigma \in \mathbb{N} \text{ with } |\sigma| \leq k, \text{ and } \phi(x) = 0 \text{ for } x \in \partial \Omega \}.$$  

It is also worthwhile to recall the density argument, [21, Subsection 5.2.2], in $\mathcal{W}^{k,p}(\Omega)$,

$$\overline{C^\infty_0(\Omega)} = \mathcal{W}^{k,p}_0(\Omega).$$

In this work, we focus on the total variation (TV) of a $C^1$ class function. TV of a function defined over the compact domain $\Omega$ is given below.

**Definition 2.1** ($TV(\phi, \Omega)$). [37, Definition 9.64] Over the compact domain $\Omega$, total variation of a function $TV(\phi, \Omega)$ is defined by the following variational form

$$TV(\phi, \Omega) := \sup_{\phi \in C^1_c(\Omega)} \left\{ \int_{\Omega} \phi(x) \text{div } \Phi(x) \, dx : ||\Phi||_{\infty} \leq 1 \right\} \tag{2.1}$$

Total variation type regularization targets the reconstruction of bounded variation (BV) class of functions that are defined by

$$BV(\Omega) := \{ \phi \in L^1(\Omega) : TV(\phi, \Omega) < \infty \} \tag{2.2}$$

with the norm

$$||\phi||_{BV} := ||\phi||_{L^1} + TV(\phi, \Omega). \tag{2.3}$$

BV function spaces are Banach spaces, [39]. Furthermore, if a function $\phi$ is in the class of Sobolev space $\mathcal{W}^{1,1}$ it is also in the space of $BV(\Omega)$, (see [1] and [39, Proposition 8.13]). By the result in [1, Theorem 2.1], it is known that one can arrive, with a proper choice of $\Phi \in C^1_c(\Omega)$, at the following formulation from (2.1),

$$TV(\phi) = \int_{\Omega} ||\nabla \phi(x)||_2 \, dx \approx \int_{\Omega} (||\nabla \phi(x)||_2^2 + \beta)^{1/2} \, dx = J_{\beta}^{TV}(\phi), \tag{2.4}$$

where $0 < \beta < 1$ is fixed. We also refer to [12, 14, 17, 36, 40] where (2.4) has appeared.

### 2.2. Some motivation for general regularization theory

For the given linear, injective and compact forward operator $T : \mathcal{V} \rightarrow \mathcal{H}$, over some compact and convex domain $\Omega$, we formulate the following smooth, convex variational minimization,

$$\arg \min_{\phi \in \mathcal{V}} \left\{ \frac{1}{2} ||T \phi - f^\delta||_\mathcal{H}^2 + \alpha J(\phi) \right\} \tag{2.5}$$

with its penalty $J : \mathcal{V} \rightarrow \mathbb{R}_+$, and the regularization parameter $\alpha > 0$. Another dual minimization problem to (2.5) is given by

$$J(\phi) \rightarrow \min_{\phi \in \mathcal{V}}, \text{ subject to } ||T \phi - f^\delta||_\mathcal{H} \leq \delta. \tag{2.6}$$
Following from the problem (2.5), in what follows, the general Tikhonov type cost functional $F_{\alpha} : V \times H \to \mathbb{R}_+$ with 2−convex penalty term $J : V \to \mathbb{R}_+$ is then formulated by

$$F_{\alpha}(\varphi, f^\delta) := \frac{1}{2} \|T\varphi - f^\delta\|_H^2 + \alpha J(\varphi). \quad (2.7)$$

In the Hilbert scales, it is known that the solution of the penalized minimization problem (2.5) equals to the solution of the constrained minimization problem (2.6), [11, Subsection 3.1]. The regularized solution $\varphi_{\alpha(\delta)}$ of the problem (2.5) satisfies the following first order optimality conditions,

$$0 = \nabla F_{\alpha}(\varphi_{\alpha(\delta)})$$

$$0 = T^* (T\varphi_{\alpha(\delta)} - f^\delta) + \alpha(\delta) \nabla J(\varphi_{\alpha(\delta)})$$

$$T^* (f^\delta - T\varphi_{\alpha(\delta)}) = \alpha(\delta) \nabla J(\varphi_{\alpha(\delta)}). \quad (2.8)$$

The choice of regularization parameter $\alpha(\delta, f^\delta)$ in this work does not require any a priori knowledge about the true solution. We always work with perturbed data $f^\delta$ and introduce the rates according to the perturbation amount $\delta$. Throughout stability analysis here, we consider the classical deterministic noise model

$$f^\delta \in B_\delta(f^\perp), \text{ i.e., } \|f^\perp - f^\delta\| \leq \delta.$$ 

2.3. Bregman distance as a vital tool for the norm convergence

**Definition 2.2. [Bregman distance][10]** Let $P : V \to \mathbb{R} \cup \{\infty\}$ be a convex functional and smooth in the Fréchet derivative sense. Then, for $u \in V$, Bregman distance associated with the functional $P$ is defined by

$$D_P(u, u^*) = P(u) - P(u^*) - \langle \nabla P(u^*), u - u^* \rangle. \quad (2.9)$$

Following formulation emphasizes the functionality of the Bregman distance in proving the norm convergence of the minimizer of the convex minimization problem to the true solution.

**Definition 2.3. [Total convexity][9, Definition 1]**

Let $P : V \to \mathbb{R} \cup \{\infty\}$ be a Fréchet differentiable convex functional. Then $P$ is called totally convex in $u^* \in V$, if,

$$P(u) - P(u^*) - \langle \nabla P(u^*), u - u^* \rangle \to 0 \Rightarrow \|u - u^*\|_V \to 0.$$

It is said that $P$ is $q$-convex in $u^* \in V$ with a $q \in [2, \infty)$, if for all $M > 0$ there exists a $c^* > 0$ such that for all $\|u - u^*\|_V \leq M$ we have

$$P(u) - P(u^*) - \langle \nabla P(u^*), u - u^* \rangle \geq c^* \|u - u^*\|_V^q. \quad (2.10)$$
Variational Convergence Analysis

Throughout our norm convergence estimations, we refer to this definition for the case of 2−convexity. We will also study different formulations of the Bregman distance. Common usage of the Bregman distance is to associate it with the penalty term \( J \) appears in the problem (2.5). Here, we also make use of different examples of the Bregman distance.

Remark 2.4. [Examples of the Bregman distance] Let \( \varphi_{\alpha(\delta)}, \varphi^\dagger \in \mathcal{V} \) be the regularized and the true solutions of the problem (2.5) respectively. Then we give the following examples of the Bregman distance;

- Bregman distance associated with the cost functional \( F_\alpha \):
  \[
  D_{F_\alpha}(\varphi_{\alpha(\delta)}, \varphi^\dagger) = F_\alpha(\varphi_{\alpha(\delta)}, f^\delta) - F_\alpha(\varphi^\dagger, f^\delta) - \langle \nabla F_\alpha(\varphi^\dagger, f^\delta), \varphi_{\alpha(\delta)} - \varphi^\dagger \rangle,
  \]
  (2.11)

- Bregman distance associated with the penalty \( J \):
  \[
  D_J(\varphi_{\alpha(\delta)}, \varphi^\dagger) = J(\varphi_{\alpha(\delta)}) - J(\varphi^\dagger) - \langle \nabla J(\varphi^\dagger), \varphi_{\alpha(\delta)} - \varphi^\dagger \rangle.
  \]
  (2.12)

Composite form of the classical Bregman distance brings another formulation of it named as symmetrical Bregman distance, [24, Definition 2.1], and defined by

\[
D^{\text{sym}}_{\mathcal{P}}(u, u^*) := D_{\mathcal{P}}(u, u^*) + D_{\mathcal{P}}(u^*, u).
\]
(2.13)

Inherently, symmetric Bregman distance is also useful for showing norm convergence as established below.

Proposition 2.5. [24, as appears in the proof of Theorem 4.4] Let \( \mathcal{P} : \mathcal{V} \to \mathbb{R}_+ \cup \{\infty\} \) be a smooth and \( q \)-convex functional. Then there exists positive constant \( c^* > 0 \) such that for all \( ||u - u^*||_{\mathcal{V}} \leq M \) we have

\[
D^{\text{sym}}_{\mathcal{P}}(u, u^*) = \langle \nabla \mathcal{P}(u^*) - \nabla \mathcal{P}(u), u^* - u \rangle \\
\geq c^* ||u - u^*||^2_{\mathcal{V}}.
\]
(2.14)

Proof. Proof is a straightforward result of the estimation in (2.10) and the symmetrical Bregman distance definition given by (2.13).

In Definition 2.3 by the estimation in (2.10), it has been stated that the norm convergence is guaranteed in the presence of some positive real valued constant to bound the Bregman distance, given by (2.9), from below. It is possible to derive an alternative estimation to (2.10), or to well known Xu-Roach inequalities in [41], in the case of \( q = 2 \), by making further assumption about the functional \( \mathcal{P} \) which is strong convexity with modulus \( c, \) [5, Definition 10.5]. Below, we formulate the first result of this work which is the base of our \( \mathcal{L}^2 \) norm estimations in the analysis. We introduce another notation before giving our formulation. From some reflexive Banach space \( \mathcal{V} \) to \( \mathbb{R} \), let \( A, B : \mathcal{V} \to \mathbb{R} \) and \( A, B \in \mathcal{L}(\mathbb{R}) \). Then \( A > B \) means that \( \langle h, (A - B)h \rangle \geq 0 \) for all \( h \in \mathcal{V} \).
Proposition 2.6. Over the compact and convex domain $\Omega$, let $\mathcal{P} : \mathcal{L}^2(\Omega) \subset V \to \mathbb{R} \cup \{\infty\}$ be some strongly convex and twice continuously differentiable functional. Then the Bregman distance $D_{\mathcal{P}}$ can be bounded below by

$$D_{\mathcal{P}}(u,v) \geq c ||u - v||^2_{\mathcal{L}_2^2(\Omega)},$$

where the modulus of convexity $c > 0$ satisfies $\frac{1}{2} \mathcal{P}'' \succ cI$.

Proof. Let us begin with considering the Taylor expansion of $\mathcal{P}$,

$$\mathcal{P}(u) = \mathcal{P}(v) + \langle \mathcal{P}'(v), u - v \rangle + \frac{1}{2} \langle \mathcal{P}''(v)(u - v), u - v \rangle + o(||u - v||^2_{\mathcal{L}_2^2(\Omega)}),$$

where $o(||u - v||^2_{\mathcal{L}_2^2(\Omega)}) := R^2(u - v)$ is the remainder given in the integral form by

$$R^2(u - v) = \frac{1}{6} \int_0^1 \mathcal{P}'''(v + t(u - v)) \cdot ((1 - t)(u - v))^2 (u - v)dt.$$ 

Then the Bregman distance reads

$$D_{\mathcal{P}}(u,v) = \mathcal{P}(u) - \mathcal{P}(v) - \langle \mathcal{P}'(v), u - v \rangle$$

$$= \langle \mathcal{P}'(v), u - v \rangle + \frac{1}{2} \langle \mathcal{P}''(v)(u - v), u - v \rangle + o(||u - v||^2_{\mathcal{L}_2^2(\Omega)}) - \langle \mathcal{P}'(v), u - v \rangle$$

$$= \frac{1}{2} \langle \mathcal{P}''(v)(u - v), u - v \rangle + o(||u - v||^2_{\mathcal{L}_2^2(\Omega)}) .$$

Since $\mathcal{P}$ is strictly convex and $o(||u - v||^2_{\mathcal{L}_2^2(\Omega)}) > 0$, due to strong convexity, one eventually obtains that

$$D_{\mathcal{P}}(u,v) \geq c ||u - v||^2_{\mathcal{L}_2^2(\Omega)},$$

where $c$ is the modulus of convexity.

2.4. Choice of regularization parameter with Morozov’s discrepancy principle

We are also concerned with asymptotic properties of the regularization parameter $\alpha$ for the Tikhonov-regularized solution obtained by Morozov’s discrepancy principle. Morozov’s discrepancy principle (MDP) serves as an a posteriori parameter choice rule for the Tikhonov type cost functionals (2.7) and has certain impact on the convergence of the regularized solution for the problem in (2.5) with some general convex penalty term $J$. As has been introduced in [2, Theorem 3.10] and [3], we will make use of the following set notations in the theorem formulations that are necessary to prove the norm convergence of the solution $\varphi_{\alpha(\delta,f)}$ to the true solution $\varphi^*$ for the problem (2.5).
\[ S := \{ \alpha : \| T \varphi_{\alpha} - f^\delta \|_{L^2(Z)} \leq \tau \delta \text{ for some } \varphi_{\alpha} \in \arg\min_{\varphi \in V} \{ F_{\alpha}(\varphi, f^\delta) \} \}, \quad (2.18) \]
\[ S := \{ \alpha : \tau \delta \leq \| T \varphi_{\alpha} - f^\delta \|_{L^2(Z)} \text{ for some } \varphi_{\alpha} \in \arg\min_{\varphi \in V} \{ F_{\alpha}(\varphi, f^\delta) \} \}, \quad (2.19) \]

where \( 1 \leq \tau \leq \overline{\tau} \) are fixed. Analogously, as well known from \([20, \text{ Eq. (4.57) and (4.58)}]\), \([31, \text{ Definition 2.3}]\), in order to obtain tight rates of convergence of \( \| \varphi_{\alpha} - \varphi^\dagger \| \) we are interested in such a regularization parameter \( \alpha(\delta, f^\delta) \), with some fixed \( 1 \leq \underline{\tau} \leq \overline{\tau} \), that
\[ \alpha(\delta, f^\delta) \in \{ \alpha > 0 \mid \underline{\tau} \delta \leq \| T \varphi_{\alpha(\delta, f^\delta)} - f^\delta \|_{L^2(Z)} \leq \overline{\tau} \delta, \text{ for all given } (\delta, f^\delta) \}. \quad (2.20) \]

3. Variational Convergence Analysis

Due to sophisticated nature of the TV penalty term in convex/non-convex minimization problems, variational inequalities in convergence analysis for the minimization problems in the form of (2.5) is useful. The title name of this section solely expresses the duty of the variational inequalities in convergence analysis. As alternative to well established Tikhonov regularization, \([33, 34]\), studying convex regularization strategy has been initiated by introducing a new image denoising method named as total variation, \([36]\). Particularly, formulating the minimization problem as variational problem and estimating convergence rates with considering source conditions in variational inequalities has also become popular recently, \([11, 23, 24, 25, 32]\) and references therein.

Recall the facts that classical deterministic noise model \( f^\delta \in B_\delta(f^\dagger) \) and the 2-convexity of the penalty term of our minimization problem (2.5) are taken into account throughout our analysis. Under some \textit{a posteriori} strategy together with the aforementioned assumptions, we will quantify the following rates;

(i) \( T \varphi_{\alpha(\delta, f^\delta)} \in B_{\alpha(\delta)}(T \varphi^\dagger) \); the discrepancy between \( T \varphi_{\alpha(\delta, f^\delta)} \) and \( T \varphi^\dagger \) by the rate of \( o(\delta) \).

(ii) \( D_J(\varphi_{\alpha(\delta, f^\delta)}, \varphi^\delta) \leq O(\Psi(\delta)) \); upper bound for the Bregman distance \( D_J \), which will immediately imply the desired norm convergence \( \| \varphi_{\alpha(\delta, f^\delta)} - \varphi^\dagger \|_V \).

(iii) \( \varphi_{\alpha(\delta, f^\delta)} \in B_{\alpha(\Psi(\delta))}(\varphi^\dagger) \); convergence of the regularized solution \( \varphi_{\alpha(\delta, f^\delta)} \) to the true solution \( \varphi^\dagger \) by the rate of the index function \( O(\Psi(\delta)) \).

3.1. Choice of the regularization parameter with Morozov’s discrepancy principle

We are also concerned with asymptotic properties of the regularization parameter \( \alpha \) for the Tikhonov-regularized solution obtained by Morozov’s discrepancy principle. Morozov’s discrepancy principle (MDP) serves as an \textit{a posteriori} parameter choice rule for the Tikhonov type cost functionals (2.7) and has certain impact on the convergence of the regularized solution for the problem in (2.5) with some general convex penalty term \( J \). As has been introduced in \([2, \text{ Theorem 3.10}]\) and \([3]\), we will make use of
the following set notations in the theorem formulations that are necessary to prove the norm convergence of the solution $\varphi_{\alpha(\delta, f^\delta)}$ to the true solution $\varphi^\dagger$ for the problem (2.5).

$$
\overline{\mathcal{S}} := \{\alpha : \| T\varphi_{\alpha(\delta)} - f^\delta \|_{L^2(\Omega)} \leq \tau \delta \text{ for some } \varphi_{\alpha(\delta)} \in \arg \min_{\varphi \in \mathcal{V}} \{ F_\alpha(\varphi, f^\delta) \} \}, \quad (3.1)
$$

$$
\underline{\mathcal{S}} := \{\alpha : \tau \delta \leq \| T\varphi_{\alpha(\delta)} - f^\delta \|_{L^2(\Omega)} \text{ for some } \varphi_{\alpha(\delta)} \in \arg \min_{\varphi \in \mathcal{V}} \{ F_\alpha(\varphi, f^\delta) \} \}, \quad (3.2)
$$

where $1 \leq \tau \leq \overline{\tau}$ are fixed. Analogously, as well known from [20, Eq. (4.57) and (4.58)], [31, Definition 2.3], in order to obtain tight rates of convergence of $\| \varphi_{\alpha(\delta)} - \varphi^\dagger \|$ we are interested in such a regularization parameter $\alpha(\delta, f^\delta)$, with some fixed $1 \leq \underline{\tau} \leq \overline{\tau}$, that

$$
\alpha(\delta, f^\delta) \in \{ \alpha > 0 \mid \tau \delta \leq \| T\varphi_{\alpha(\delta, f^\delta)} - f^\delta \|_{L^2(\Omega)} \leq \overline{\tau} \delta \} \text{ for all given } (\delta, f^\delta). \quad (3.3)
$$

### 3.2. Variational inequalities for norm convergence

Convergence rates results for some general operator $T$ can be obtained by formulating variational inequality which uses the concept of index functions. A function $\Psi : [0, \infty) \to [0, \infty)$ is called index function if it is continuously defined, monotonically increasing and $\Psi(0) = 0$.

**Assumption 3.1.** [Variational Inequality][23, Eq. 1], [27, Eq 1.5], [29, Eq 2] There exists some constant $\gamma \in (0, 1)$ and an index function $\Psi$, for all $\varphi \in \mathcal{D}(T)$, such that

$$
\gamma \| \varphi - \varphi^\dagger \|_{L^2(\Omega)}^2 \leq J(\varphi) - J(\varphi^\dagger) + \Psi \left( \| T\varphi - T\varphi^\dagger \|_{L^2(\Omega)} \right). \quad (3.4)
$$

**Lemma 3.2.** For the cost functional defined by

$$
F_\alpha(\varphi, f^\delta) := \frac{1}{2} \| T\varphi - f^\delta \|_{L^2(\Omega)}^2 + \alpha J(\varphi),
$$

with some Fréchet differentiable and convex penalty term $J : \mathcal{V} \to \mathbb{R}$, that is defined on a Hilbert space $\mathcal{V}$, $J : \mathcal{V} \to \mathbb{R}$, let $\varphi_{\alpha} \in \arg \min_{\varphi \in \mathcal{V}} \{ F_\alpha(\varphi, f^\delta) \}$. Then for all $\varphi \in \mathcal{D}(T) \subset \mathcal{V}$ and any regularization parameter $\alpha > 0$,

$$
\alpha(\nabla J(\varphi), \varphi_{\alpha} - \varphi) \leq \langle T^*(T\varphi - f^\delta), \varphi - \varphi_{\alpha} \rangle. \quad (3.5)
$$

**Proof.** Since $\varphi_{\alpha}$ is the minimum of the cost functional $F_\alpha$ then, it is hold that $F_\alpha(\varphi_{\alpha}, f^\delta) \leq F_\alpha(\varphi, f^\delta)$ for all $\varphi \in \mathcal{D}(T) \subset \mathcal{V}$ and $\alpha > 0$. Now, recall the Bregman distance formulation associated with the cost functional in (2.11).

$$
0 \leq D_{F_\alpha}(\varphi_{\alpha}, \varphi) = F_\alpha(\varphi_{\alpha}) - F_\alpha(\varphi) - \langle \nabla F_\alpha(\varphi), \varphi_{\alpha} - \varphi \rangle \\
\leq - \langle \nabla F_\alpha(\varphi), \varphi_{\alpha} - \varphi \rangle \\
= \langle \nabla F_\alpha(\varphi), \varphi - \varphi_{\alpha} \rangle \quad (3.6)
$$
Variational Convergence Analysis

We, by the definition of the cost functional $F_\alpha$ in (2.7), have that

$$0 \leq \langle \mathcal{T}^* (\mathcal{T} \varphi - f^\delta) + \alpha \nabla J(\varphi), \varphi - \varphi_\alpha \rangle,$$

which yields the assertion. □

It is also an immediate consequence of MDP, see [3, Remark 2.7], that

$$||\mathcal{T} \varphi_\alpha(\delta) - \mathcal{T} \varphi^\dagger||_{L^2(\mathcal{Z})} \leq (\tau + 1)\delta.$$ (3.8)

We use this observation to formulate the following theorem. The first assertion below is an expected result for minimization problems given by (2.5), see e.g. [27, Lemma 1].

**Theorem 3.3.** Under the same assumption in Lemma 3.2 together with $||\mathcal{T} \varphi^\dagger - f^\delta||_{L^2(\mathcal{Z})} \leq \delta$, then we, for any $\alpha > 0$, have that

$$J(\varphi_\alpha) - J(\varphi^\dagger) \leq \frac{\delta^2}{2\alpha}.$$ (3.9)

Moreover, for $\alpha(\delta, f^\delta) \in \overline{S}$, the Bregman distance $D_J$ is bounded above by

$$D_J(\varphi_\alpha(\delta, f^\delta), \varphi^\dagger) \leq \frac{\delta^2}{\alpha(\delta, f^\delta)} \left( \frac{3}{2} + \tau \right).$$ (3.10)

**Proof.** Since $\varphi_\alpha$, for any $\alpha > 0$, is the minimizer of the cost functional $F_\alpha$, then

$$F_\alpha(\varphi_\alpha, f^\delta) = \frac{1}{2}||\mathcal{T} \varphi_\alpha - f^\delta||^2_{L^2(\mathcal{Z})} + \alpha J(\varphi_\alpha)$$

$$\leq \frac{1}{2}||\mathcal{T} \varphi^\dagger - f^\delta||^2_{L^2(\mathcal{Z})} + \alpha J(\varphi^\dagger) = F_\alpha(\varphi^\dagger, f^\delta),$$

which is in other words,

$$\alpha(J(\varphi_\alpha) - J(\varphi^\dagger)) \leq \frac{1}{2}||\mathcal{T} \varphi^\dagger - f^\delta||^2_{L^2(\mathcal{Z})} - \frac{1}{2}||\mathcal{T} \varphi_\alpha - f^\delta||^2_{L^2(\mathcal{Z})}.$$ (3.11)

By the assumed deterministic noise model $||\mathcal{T} \varphi^\dagger - f^\delta||_{L^2(\mathcal{Z})} \leq \delta$ and the fact that $||\mathcal{T} \varphi_\alpha - f^\delta||_{L^2(\mathcal{Z})} > 0$, one obtains the first assertion

$$J(\varphi_\alpha) - J(\varphi^\dagger) \leq \frac{\delta^2}{2\alpha}.$$ 

Regarding second assertion, since $\alpha(\delta, f^\delta) \in \overline{S}$, by the definition in (3.1), $||\mathcal{T} \varphi_\alpha(\delta, f^\delta) - f^\delta||_{L^2(\mathcal{Z})} \leq \overline{\tau} \delta$. From the formulation of Bregman distance (2.12) and Lemma 3.2, we obtain
\[ D_J(\varphi_\alpha(\delta,f^\delta), \varphi^\dagger) \leq |J(\varphi_\alpha(\delta,f^\delta)) - J(\varphi^\dagger)| + |\langle \nabla J(\varphi^\dagger), \varphi_\alpha(\delta,f^\delta) - \varphi^\dagger \rangle| \]

\[ \leq \frac{\delta^2}{2\alpha(\delta,f^\delta)} + \frac{\delta}{\alpha(\delta,f^\delta)} \| T\varphi_\alpha(\delta,f^\delta) - T\varphi^\dagger \|_{L^2(Z)}. \]

Hence, the observation in (3.8) yields the second assertion.

Obtaining tight rates of convergence with an \textit{a posteriori} strategy for the choice of regularization parameter \( \alpha = \alpha(\delta,f^\delta) \) is the aim of this chapter. Henceforth, we will show the impact of this strategy on the convergence and convergence rates by associating it with the index function \( \Psi \) that has appeared in Assumption 3.1. In [27, Eq (3.2)], a reasonable index function has been introduced. We, in analogous with that function in the regarding work, introduce

\[ \Phi(\delta,f^\delta) := \frac{\delta^2}{2\Psi(\delta)}. \quad (3.12) \]

From this index function, it is possible to be able to formulate an improved counterpart of the result in [27, Corollary 1]. Firstly, we give a preliminary estimate result based on the variational inequality.

**Lemma 3.4. [27, Lemma 2]** Let, for some \( \alpha, \varphi_\alpha \in \arg\min_{\varphi \in V} \{ F_\alpha(\varphi, f^\delta) \} \) satisfy Assumption (3.1). Then

\[ \| T\varphi_\alpha - T\varphi^\dagger \|_{L^2(Z)}^2 \leq 4\delta^2 + 4\alpha \Psi \left( \| T\varphi_\alpha - T\varphi^\dagger \|_{L^2(Z)} \right), \]

where \( \varphi^\dagger \in D(T) \) is the true solution for the problem (2.5).

We are now ready to introduce our result which is comparable with [27, Corollary 1]. In our formulation, we still follow \textit{a posteriori} rule of choice of the regularization parameter \( \alpha = \alpha(\delta,f^\delta) \in \overline{S} \) as has been introduced in (3.1).

**Corollary 3.5.** Under the same assumption in Lemma 3.4, if the regularization parameter \( \alpha(\delta,f^\delta) \in \overline{S} \) is chosen as

\[ \alpha(\delta,f^\delta) := \Phi(\delta,f^\delta), \quad (3.13) \]

then we have

\[ \| T\varphi_{\alpha(\delta,f^\delta)} - T\varphi^\dagger \|_{L^2(Z)} \leq \delta \sqrt{6 + 2\overline{\tau}}, \quad (3.14) \]

where fixed \( 1 \leq \overline{\tau} \) satisfies \( \| T\varphi_{\alpha(\delta,f^\delta)} - f^\delta \|_{L^2(Z)} \leq \overline{\tau}\delta. \)
Variational Convergence Analysis

Proof. By the defined index function in (3.12) and the result in Lemma 3.4, we immediately obtain,

\[
\| \mathcal{T} \varphi_{\alpha(\delta,f^\delta)} - \mathcal{T} \varphi^\dagger \|_{L^2(Z)}^2 \leq 4 \delta^2 + 4 \alpha(\delta,f^\delta) \Psi \left( \| \mathcal{T} \varphi_{\alpha(\delta,f^\delta)} - \mathcal{T} \varphi^\dagger \|_{L^2(Z)} \right)
\]

\[
= 4 \delta^2 + 2 \frac{\delta^2}{\Psi(\delta)} \Psi \left( \frac{\delta}{\delta} \| \mathcal{T} \varphi_{\alpha(\delta,f^\delta)} - \mathcal{T} \varphi^\dagger \|_{L^2(Z)} \right)
\]

\[
= 4 \delta^2 + 2 \delta \| \mathcal{T} \varphi_{\alpha(\delta,f^\delta)} - \mathcal{T} \varphi^\dagger \|_{L^2(Z)} \leq 8 \delta^2 (\tau + 1) = \delta^2 (6 + 2\tau).
\]

\[
\| \mathcal{T} \varphi_{\alpha(\delta,f^\delta)} - f^\delta \|_{L^2(Z)} \geq \tau \delta.
\]

With the introduced index function in (3.12), it is essential to be able to find lower bound for the regularization parameter \( \alpha \).

Corollary 3.6. Suppose that, for a chosen regularization parameter \( \alpha(\delta,f^\delta) \in \Sigma \) that is defined in (3.2), the regularized solution \( \varphi_{\alpha(\delta,f^\delta)} \) to the problem (2.5) satisfies the variational inequality in Assumption 3.1. Then the regularization parameter can be bounded below as such,

\[
\frac{1}{2} \left( \tau - 1 \right)^2 \frac{\tau^2 - 1}{\tau^2 + 1} \Phi(\delta, f^\delta) \leq \alpha(\delta, f^\delta).
\]

(3.15)

Proof. Since \( \| \mathcal{T} \varphi_{\alpha(\delta,f^\delta)} - f^\delta \|_{L^2(Z)} \geq \tau \delta \) and the regularized solution \( \varphi_{\alpha(\delta,f^\delta)} \) satisfies the assertion in Assumption 3.1, we immediately obtain,

\[
\frac{\tau^2 \delta^2}{2} \leq \frac{\| \mathcal{T} \varphi_{\alpha(\delta,f^\delta)} - f^\delta \|^2_{L^2(Z)}}{2} \leq \frac{\delta^2}{2} + \alpha \left( J(\varphi^\dagger) - J(\varphi_{\alpha(\delta)}) \right)
\]

\[
\leq \frac{\delta^2}{2} + \alpha \Psi \left( \| \mathcal{T} \varphi_{\alpha(\delta,f^\delta)} - \mathcal{T} \varphi^\dagger \|_{L^2(Z)} \right),
\]

and this follows up

\[
\delta^2 \leq \frac{2\alpha}{\tau^2 - 1} \Psi \left( \| \mathcal{T} \varphi_{\alpha(\delta,f^\delta)} - \mathcal{T} \varphi^\dagger \|_{L^2(Z)} \right).
\]

(3.16)

We plug this into the bound in Lemma 3.4 with the abbreviation \( p_\alpha := \| \mathcal{T} \varphi_{\alpha(\delta,f^\delta)} - \mathcal{T} \varphi^\dagger \|_{L^2(Z)} \)

\(^8\)By the given index function \( \Phi \) in (3.12) and since \( \alpha(\delta,f^\delta) := \Phi(\delta,f^\delta) \), the equation follows.

\(^9\)See [27, Proposition 1].

\(^{10}\)Since \( \alpha(\delta,f^\delta) \in \Sigma \), then \( \| \mathcal{T} \varphi_{\alpha(\delta,f^\delta)} - f^\delta \|_{L^2(Z)} \leq \tau \delta \). This, by the triangle inequality, implies that \( \| \mathcal{T} \varphi_{\alpha(\delta,f^\delta)} - \mathcal{T} \varphi^\dagger \|_{L^2(Z)} \leq (\tau + 1)\delta \).
\[ p^2 \alpha \leq 4\delta^2 + 4\alpha \Psi(p_\alpha) \leq \frac{8\alpha}{\tau^2 - 1} \Psi(p_\alpha) + 4\alpha \Psi(p_\alpha) = 4\alpha \Psi(p_\alpha) \frac{\tau^2 + 1}{\tau^2 - 1}. \] (3.17)

Note that
\[ \tau \delta \leq ||T\varphi_{\alpha(\delta,f^\delta)} - f^\delta||_{L^2(Z)} \leq p_\alpha + \delta \]
which implies
\[ (\tau - 1)\delta \leq p_\alpha. \] (3.18)

Hence, from (3.17),
\[ \frac{1}{2}(\tau - 1)(\tau^2 - 1) \frac{\tau^2 + 1}{\tau^2 - 1} \Phi(\delta, f^\delta) \leq \alpha. \] (3.19)

**Theorem 3.7.** Suppose that the regularized solution \( \varphi_{\alpha(\delta,f^\delta)} \) to the problem (2.5) obeys Assumption 3.1, for some regularization parameter \( \alpha(\delta,f^\delta) \) satisfying
\[ \tau \delta \leq ||T\varphi_{\alpha(\delta,f^\delta)} - f^\delta||_{L^2(Z)} \leq \tau \delta, \]
where \( 1 \leq \tau \leq \tau \) are fixed and with the lower bound in Corollary 3.6. Then, by the second assertion (3.10) in Theorem 3.3, the Bregman distance \( D_J \) can be bounded by
\[ D_J(\varphi_{\alpha(\delta,f^\delta)}, \varphi^\dagger) \leq O(\Psi(\delta)). \] (3.20)

**Proof.** Corollary 3.6 and the index function defined by (3.12) provide the result
\[ D_J(\varphi_{\alpha(\delta,f^\delta)}, \varphi^\dagger) \leq \frac{\delta^2}{\frac{1}{2}(\tau - 1)^2 \frac{\tau^2 + 1}{\tau^2 - 1} \Phi(\delta, f^\delta)} \left( \frac{3}{2} + \tau \right), \]
\[ = \frac{4\Psi(\delta)(\tau^2 + 1)}{(\tau - 1)^3(\tau + 1)} \left( \frac{3}{2} + \tau \right). \]
Theorem 3.8. Let $T : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\mathcal{Z})$ be the compact and linear operator. Over the compact and convex domain $\Omega$, let $\varphi_{\alpha(\delta, f^\delta)} \in \mathcal{L}^2(\Omega)$ satisfy the assumption of Lemma 3.2 and Assumption 3.1. If the regularization parameter $\alpha(\delta, f^\delta) \in \mathcal{S}$ is chosen as $\alpha(\delta, f^\delta) := \Phi(\delta, f^\delta)$ where $\Phi$ is defined by (3.12) with some given noisy measurement $f^\delta \in \mathcal{B}_\delta(f^1)$, then one can find the following upper bound for the symmetric Bregman distance,

$$D_J(\varphi_{\alpha(\delta, f^\delta)}, \varphi^\dagger) \leq D_J^{\text{sym}}(\varphi_{\alpha(\delta, f^\delta)}, \varphi^\dagger) \leq \frac{1}{\epsilon} \left( \frac{1}{\gamma} + 1 \right) \Psi(\delta) + \epsilon \Psi^2(\delta) \|T^*\|^2(\tau^2 + 1),$$

where the coefficients are arbitrarily chosen as $\epsilon \in \mathbb{R}_+$, $\gamma \in (0, 1]$, and $\tau \geq 1$. Furthermore, if the smooth penalty term $J : \mathcal{L}^2(\Omega) \to \mathbb{R}$ is 2–convex, then this upper bound implies,

$$\|\varphi_{\alpha(\delta, f^\delta)} - \varphi^\dagger\|^2_{\mathcal{L}^2(\Omega)} \leq O(\Psi(\delta)). \quad (3.21)$$

Proof. By the definition of $D_J^{\text{sym}}$ in (2.13), it suffices to prove the last inequality. First, observe that,

$$D_J^{\text{sym}}(\varphi_{\alpha(\delta, f^\delta)}, \varphi^\dagger) = \langle \nabla J(\varphi_{\alpha(\delta, f^\delta)}), \varphi_{\alpha(\delta, f^\delta)} - \varphi^\dagger \rangle.$$  

We will bound each inner product separately. The regularized solution $\varphi_{\alpha(\delta, f^\delta)}$, for the regularization parameter $\alpha(\delta, f^\delta) := \Phi(\delta, f^\delta)$ where $\Phi$ is defined by (3.12), satisfies the first order optimality condition given in (2.8) as well as the variational inequality in Assumption 3.1. So,

$$\|\nabla J(\varphi_{\alpha(\delta, f^\delta)}), \varphi_{\alpha(\delta, f^\delta)} - \varphi^\dagger \| = \left| \frac{1}{\alpha(\delta, f^\delta)} \langle T^*(f^\delta - T \varphi_{\alpha(\delta, f^\delta)}), \varphi_{\alpha(\delta)} - \varphi^\dagger \rangle \right| \leq \frac{1}{\alpha(\delta, f^\delta)} \|T^*\| \|T \varphi_{\alpha(\delta, f^\delta)} - f^\delta\|_{\mathcal{L}^2(\mathcal{Z})} \|\varphi_{\alpha(\delta, f^\delta)} - \varphi^\dagger\|_{\mathcal{L}^2(\Omega)} \leq \frac{\epsilon}{2\alpha^2(\delta, f^\delta)} \|T^*\|^2 \delta^2 \tau^2 + \frac{1}{2\epsilon} \|\varphi_{\alpha(\delta, f^\delta)} - \varphi^\dagger\|^2_{\mathcal{L}^2(\Omega)} \leq \frac{\epsilon}{2\alpha^2(\delta, f^\delta)} \|T^*\|^2 \delta^2 \tau^2 + \frac{1}{2\epsilon} \left( \frac{\delta^2}{2\alpha(\delta, f^\delta)} + \Psi(\delta) \right).$$

The assertion in Lemma 3.2, with the regularization parameter $\alpha(\delta, f^\delta) > 0$, brings the following bound

$q$By Young’s inequality and since $\alpha(\delta, f^\delta) \in \mathcal{S}$.
\[ |\langle \nabla J(\varphi^\dagger), \varphi_{\alpha(\delta,f^\delta)} - \varphi^\dagger \rangle | \leq \left| \frac{1}{\alpha(\delta,f^\delta)} \langle T^*(f^\dagger - f^\delta), \varphi^\dagger - \varphi_{\alpha(\delta,f)} \rangle \right| \]
\[ \leq \frac{\delta}{\alpha(\delta,f^\delta)} \|T^*\| \| \varphi_{\alpha(\delta,f^\delta)} - \varphi^\dagger \|_{L^2(\Omega)}. \]
\[ \leq \| \frac{\epsilon}{2\alpha^2(\delta,f^\delta)} \|T^*\|^2 \delta^2 + \frac{1}{2\epsilon} \| \varphi_{\alpha(\delta,f^\delta)} - \varphi^\dagger \|_{L^2(\Omega)}^2 \]
\[ \leq \frac{\epsilon}{2\alpha^2(\delta,f^\delta)} \|T^*\|^2 \delta^2 + \frac{1}{2\epsilon} \left( \frac{1}{\gamma} \frac{\delta^2}{2\alpha(\delta,f^\delta)} + \Psi(\delta) \right). \]

Since the regularization parameter is chosen as \( \alpha(\delta,f^\delta) := \Phi(\delta,f^\delta) \), see (3.12), then

\[ D_{\alpha(\delta,f^\delta)}(\varphi_{\alpha(\delta,f^\delta)}, \varphi^\dagger) \leq \frac{1}{2\epsilon} \left( \frac{1}{\gamma} \frac{\delta^2}{2\alpha(\delta,f^\delta)} + \Psi(\delta) \right) + \frac{\epsilon}{2\alpha^2(\delta,f^\delta)} \|T^*\|^2 \delta^2(\tau^2 + 1). \]

With the additional assumption on \( J \) which is 2−convexity, then the norm convergence of \( \| \varphi_{\alpha(\delta,f^\delta)} - \varphi^\dagger \|_{L^2(\Omega)} \) is obtained due to (2.10).

4. Convex Regularization for the Smoothed-TV

In this section, we give the specific interpretation of the general convex regularization for the 2−convex, see (2.10) in Definition 2.3, smoothed-TV functional. To this end, we state the following minimization problem

\[ \varphi_{\alpha(\delta)} \in \arg \min_{\varphi \in W^{1,2}(\Omega)} \left\{ \frac{1}{2} \| T\varphi - f^\delta \|_{L^2}^2 + \alpha J_{TV}^\beta(\varphi) \right\}, \tag{4.1} \]

where the smoothed-TV penalty, \([14, 17]\), is defined by

\[ J_{TV}^\beta(\varphi) := \int_{\Omega} \sqrt{\| \nabla \varphi(x) \|_2^2 + \beta} \, dx. \]

Existence of the solution for the problem (4.1) has been studied extensively in \([1, 26, 38]\). Moreover, an existence and uniqueness theorem for the minimizer of quadratic functionals with different type of convex integrands has been established in \([15, \text{Theorem 9.5-2}]\). As has been given by the Minimal Hypersurfaces problem in \([19]\), the minimizer of the problem (4.1) exists on the Hilbert space \( W^{1,2}(\Omega) \).

Unlike in the available literature \([1, 4, 6, 12, 13, 14, 17, 18, 40]\), we will arrive at a new lower bound for the Bregman distance particularly associated with the smoothed-TV functional \( J_{TV}^\beta \). We will achieve this by means of the strong convexity of the regarding functional.

\[ \text{By Young's inequality and since } \alpha(\delta,f^\delta) \in \overline{\mathbb{S}}. \]
By the choice of $\Psi = \Phi$ and since $\|\nabla \varphi(x)\|_2^2 + \beta dx$, we can calculate the Hessian in the direction $\Psi$.

Theorem 4.2. [Smoothed-TV functional is strongly convex] For any $\beta > 0$, the functional $J_{\beta}^{TV} : W^{1,1}(\Omega) \to \mathbb{R}_+$, that is defined by $J_{\beta}^{TV}(\varphi) := \sqrt{\|\nabla \varphi(x)\|_2^2 + \beta dx}$, is convex.

Before the Hessian of $J_{\beta}^{TV}$, we first calculate the Fréchet derivative of it in the direction $\Phi \in C_c(\Omega)$,

$$
\frac{d}{dt} J_{\beta}^{TV}(\varphi + t\Phi) \big|_{t=0} = \int_{\Omega} \frac{(\nabla \varphi(x) + t\nabla \Phi(x)) \nabla \Phi(x)}{\|\nabla \varphi(x) + t\nabla \Phi(x)\|_2^2 + \beta} dx \big|_{t=0}
$$

$$
= \int_{\Omega} \frac{\nabla \varphi(x) \nabla \Phi(x)}{\|\nabla \varphi(x)\|_2^2 + \beta} dx.
$$

$$
= \int_{\Omega} \nabla^* \left( \frac{\nabla \varphi(x)}{\|\nabla \varphi(x)\|_2^2 + \beta} \right) \Phi(x) dx,
$$

(4.2)

where $\nabla^*$ represents the adjoint of the gradient operator which is $\nabla^* = -\text{div}$.

Theorem 4.2. [Smoothed-TV functional is strongly convex] For any $\varphi \in W^{1,2}(\Omega)$ defined over the compactly supported domain $\Omega \subset \mathbb{R}^3$ and for the smoothed-TV functional $J_{\beta}^{TV} : W^{1,2}(\Omega) \to \mathbb{R}_+$,

$$
J_{\beta}^{TV}(\varphi) := \int_{\Omega} \sqrt{\|\nabla \varphi(x)\|_2^2 + \beta} dx,
$$

where $\beta > 0$ is fixed, the Hessian of of $J_{\beta}^{TV}$, which is $(J_{\beta}^{TV})''(\varphi)(\Phi, \Phi)$, can be bounded below by some functional $l : W^{1,2}(\Omega) \to \mathbb{R}_+$ satisfying

$$(J_{\beta}^{TV})''(\varphi)(\Phi, \Phi) \geq l(\varphi)\|\nabla \Phi\|_{L^2(\Omega)}^2.$$

Proof. In (4.2), we, in the direction $\Phi \in C_c(\Omega)$, have calculated that

$$(J_{\beta}^{TV})'(\varphi)(\Phi) = \int_{\Omega} \frac{\nabla \varphi(x) \nabla \Phi(x)}{\|\nabla \varphi(x)\|_2^2 + \beta} dx.$$

Following from here, we can calculate the Hessian in the direction $\Psi \in W^{1,2}(\Omega)$,

$$(J_{\beta}^{TV})''(\varphi)(\Phi, \Phi) = \frac{d}{ds} (J_{\beta}^{TV})'(\varphi + s\Psi)(\Phi) \big|_{s=0}
$$

$$
= \frac{d}{ds} \int_{\Omega} \frac{(\nabla \varphi(x) + s \nabla \Psi(x)) \nabla \Phi(x)}{\|\nabla \varphi(x) + s \nabla \Psi(x)\|_2^2 + \beta} dx \big|_{s=0},
$$

which is

$$
\frac{d}{ds} (J_{\beta}^{TV})'(\varphi + s\Psi)(\Phi) \big|_{s=0} = \int_{\Omega} \frac{\nabla \Psi(x) \nabla \Phi(x)(\|\nabla \varphi(x)\|_2^2 + \beta) - \|\nabla \varphi(x) \nabla \Phi(x)\|^2}{(\|\nabla \varphi(x)\|_2^2 + \beta)^{3/2}} dx.
$$

By the choice of $\Psi = \Phi$ and since $\|\nabla \varphi \nabla \Phi\|_2^2 \leq \|\nabla \varphi\|_2^2 \|\nabla \Phi\|_2^2$, we get

$$
\frac{d}{ds} (J_{\beta}^{TV})'(\varphi + s\Psi)(\Phi) \big|_{s=0} = \int_{\Omega} \frac{\nabla \varphi(x) \nabla \Phi(x)(\|\nabla \varphi(x)\|_2^2 + \beta)}{(\|\nabla \varphi(x)\|_2^2 + \beta)^{3/2}} dx.
$$
\[
(J^\text{TV}_\beta)^\prime\prime[\varphi](\Phi, \Phi) \geq \int_\Omega \frac{\|\nabla \Phi(x)\|_2^2\|\nabla \varphi(x)\|^2 + \beta}{(\|\nabla \varphi(x)\|_2^2 + \beta)^{3/2}} dx - \int_\Omega \frac{\beta\|\nabla \Phi(x)\|_2^2}{(\|\nabla \varphi(x)\|_2^2 + \beta)^2} dx.
\]

Now, let us abbreviate
\[
m_\beta(\varphi) := \frac{\beta}{(\|\nabla \varphi\|_2^2 + \beta)^2}.
\]
Hence, we have
\[
(J^\text{TV}_\beta)^\prime\prime[\varphi](\Phi, \Phi) \geq \inf_{x \in \Omega} \{m_\beta(\varphi)\} \|\nabla \Phi\|_{L^2(\Omega)}^2,
\]
which is the desired result by defining, \(l : W^{1,2}(\Omega) \to \mathbb{R}_+\),
\[
l(\varphi) := \inf_{x \in \Omega} \{m_\beta(\varphi)\}.
\]

Combining this result together with our early Proposition 2.6 yields a new lower bound for the Bregman distance particularly associated with the smoothed-TV term \(J^\text{TV}_\beta\) that is formulated below.

**Corollary 4.3.** Under the same assumption of Theorem 4.2, and for any \(\varphi, \psi \in W^{1,2}(\Omega)\), the Bregman distance associated with the strongly convex smoothed-TV functional
\[
J^\text{TV}_\beta(\varphi) := \int_\Omega \sqrt{\|\nabla \varphi(x)\|^2 + \beta} dx,
\]
can be bounded below by some \(l : W^{1,2}(\Omega) \to \mathbb{R}_+\) as such
\[
l(\varphi)\|\nabla \Phi\|_{L^2(\Omega)}^2\|\varphi - \psi\|_{L^2(\Omega)}^2 \leq D_\text{TV}(\varphi, \psi),
\]
where \(\Phi \in C^1(\Omega) \cap W^{1,2}(\Omega)\) satisfies
\[
(J^\text{TV}_\beta)^\prime\prime[\varphi](\Phi, \Phi) \geq l(\varphi)\|\nabla \Phi\|_{L^2(\Omega)}^2.
\]

**Proof.** In the proof of Proposition 2.6, we set \(u := \psi\) and \(v := \varphi\). This setting has no impact on the proof since \(\|u - v\| = \|v - u\|\). By this setting and following the calculations in the regarding proof, and also by Theorem 4.2, we associate the necessary lower bound with \((J^\text{TV}_\beta)^\prime\prime(\varphi)[\Phi, \Phi]\) for \(\varphi \in W^{1,2}(\Omega)\). \(\square\)
Corollary 4.4. Let the regularized solution \( \varphi_{\alpha(\delta,f^\delta)} \in W^{1,2}(\Omega) \) of the problem (4.1) satisfy Assumption 3.1. Then under the same assumptions of Theorem 4.2 and Corollary 4.3, for a posteriori rule for the choice of regularization parameter \( \alpha(\delta,f^\delta) \),

\[
||\varphi_{\alpha(\delta,f^\delta)} - \varphi^\dagger||_{L^2(\Omega)}^2 \rightarrow O(\Psi(\delta))
\]
as \( \delta \rightarrow 0 \) due to the estimation (2.10) of Definition 2.3, Theorem 3.3, Theorem 3.8, and Theorem 3.7.

Remark 4.5. Note that the term \( O(\Psi(\delta)) \) in Corollary 4.4 also contains the term \( \frac{1}{l(\varphi)} \) where \( l : W^{1,2}(\Omega) \rightarrow \mathbb{R}_+ \) is defined in Theorem 4.2 as well as in Corollary 4.3.

Acknowledgement

The author is indebted to D. Russell Luke for the valuable help in the formulation of Proposition 2.6 and is also grateful to Thorsten Hohage for the strategic discussion both on the development of the general theory and on the correct interpretation of the general analysis for the smoothed-TV functional.
References

[1] R. Acar and C. R. Vogel. Analysis of bounded variation penalty methods for ill-posed problems. Inverse Problems, 10, 6, 1217 - 1229, 1994.

[2] S. Anzengruber and R. Ramlau. Morozov’s discrepancy principle for Tikhonov-type functionals with nonlinear operators. Inverse Problems, 26, 025001 (17pp), 2010.

[3] S. Anzengruber and R. Ramlau. Convergence rates for Morozov’s discrepancy principle using variational inequalities. Inverse Problems, 27, 105007 (18pp), 2011.

[4] M. Bachmayr and M. Burger. Iterative total variation schemes for nonlinear inverse problems. Inverse Problems, 25, 105004 (26pp), 2009.

[5] H. H. Bauschke and P. L. Combettes. Convex analysis and monotone operator theory in Hilbert spaces. Springer New York, 2011.

[6] J. M. Bardsley and A. Luttman. Total variation-penalized Poisson likelihood estimation for ill-posed problems. Adv. Comput. Math., 31:25-59, 2009.

[7] M. Bergounioux. On Poincaré-Wirtinger inequalities in space of functions of bounded variation. Control Cybernet., 40, 4, 921-29, 2011.

[8] T. Bonesky, K. S. Kazimierski, P. Maass, F. Schöpfer and T. Schuster. Minimization of Tikhonov Functionals in Banach Spaces. Abstr. Appl. Anal., Art. ID 192679, 19 pp, 2008.

[9] K. Bredies. A forward-backward splitting algorithm for the minimization of non-smooth convex functionals in Banach space. Inverse Problems 25, 1, 015005, 20 pp, 2009.

[10] L. M. Bregman. The relaxation method of finding the common points of convex sets and its application to the solution of problems in convex programming. Z. Vyčisl. Mat. i Mat. Fiz., 7, 620 - 631, 1967.

[11] M. Burger and S. Osher. Convergence rates of convex variational regularization. Inverse Problems, 20, 5, 1411 - 1421, 2004.

[12] A. Chambolle and P. L. Lions. Image recovery via total variation minimization and related problems. Numer. Math. 76, 167 - 188, 1997.

[13] T. F. Chan and K. Chen. An optimization-based multilevel algorithm for total variation image denoising. Multiscale Model. Simul. 5, 2, 615-645, 2006.

[14] T. Chan, G. Golub and P. Mulet. A nonlinear primal-dual method for total variation-baes image restoration. SIAM J. Sci. Comp 20: 1964-1977, 1999.

[15] P. G. Ciarlet. Linear and nonlinear functional analysis with applications. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2013.

[16] D. Colton and R. Kress. Inverse Acoustic and Electromagnetic Scattering Theory. Springer Verlag Series in Applied Mathematics, 93, Third Edition, 2013.

[17] D. Dobson and O. Scherzer. Analysis of regularized total variation penalty methods for denoising. Inverse Problems, 12, 5, 601 - 617, 1996.

[18] D. C. Dobson and C. R. Vogel. Convergence of an iterative method for total variation denoising. SIAM J. Numer. Anal., 34, 5, 1779 - 1791, 1997.

[19] I. Ekeland. On the variational principle. J. Math. Anal. Appl., 47, 324 - 353, 1974.

[20] H. W. Engl, M. Hanke and A. Neubauer. Regularization of Inverse Problems. Math. Appl., 375., Kluwer Academic Publishers Group, Dordrecht, 1996.

[21] L. C. Evans. Partial differential equations. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998.

[22] J. M. Fowkes, N. I. M. Gould and C. L. Farmer. A branch and bound algorithm for the global optimization of Hessian Lipschitz continuous functions. J. Glob. Optim., 56, 1792 - 1815, 2013.

[23] M. Grasmair. Generalized Bregman distances and convergence rates for non-convex regularization methods. Inverse Problems 26, 11, 115014, 16pp, 2010.

[24] M. Grasmair. Variational inequalities and higher order convergence rates for Tikhonov regularisation on Banach spaces. J. Inverse Ill-Posed Probl., 21, 379-394, 2013.

[25] M. Grasmair, M. Haltmeier and O. Scherzer. Necessary and sufficient conditions for linear
convergence of $l^1$-regularization. Comm. Pure Appl. Math. 64(2), 161-182, 2011.

[26] M. Hintermüller, C.N. Rautenberg and J. Hahn. Functional-analytic and numerical issues in splitting methods for total variation-based image reconstruction. Inverse Problems 30, 055014(34pp), 2014.

[27] B. Hofmann and P. Mathé. Parameter choice in Banach space regularization under variational inequalities. Inverse Problems 28, 104006 (17pp), 2012.

[28] B. Hofmann and M. Yamamoto. On the interplay of source conditions and variational inequalities for nonlinear ill-posed problems. Appl. Anal. 89, 11, 1705 - 1727, 2010.

[29] T. Hohage and F. Weidling. Verification of a variational source condition for acoustic inverse medium scattering problems. Inverse Problems, 31, 075006 (14pp), 2015.

[30] V. Isakov. Inverse problems for partial differential equations. Second edition. Applied Mathematical Sciences, 127. Springer, New York, 2006.

[31] A. Kirsch. An Introduction to the Mathematical Theory of Inverse Problems. Second edition. Applied Mathematical Sciences, 120. Springer, New York, 2011.

[32] D. A. Lorenz. Convergence rates and source conditions for Tikhonov regularization with sparsity constraints. J. Inv. Ill-Posed Problems, 16, 463-478, 2008.

[33] A. N. Tikhonov. On the solution of ill-posed problems and the method of regularization. Dokl. Akad. Nauk SSSR, 151, 501-504, 1963.

[34] A. N. Tikhonov and V. Y. Arsenin. Solutions of Ill-posed Problems. Translated from the Russian. Preface by translation editor Fritz John. Scripta Series in Mathematics. V. H. Winston & Sons, Washington, D.C.: John Wiley & Sons, New York-Toronto, Ont.-London, xiii+258 pp, 1977.

[35] R.T. Rockafellar and R. J.-B. Wets. Variational Analysis. Fundamental Principles of Mathematical Sciences, 317. Springer-Verlag, Berlin, 1998.

[36] L. I. Rudin, S. J. Osher and E. Fatemi. Nonlinear total variation based noise removal algorithms. Physica D, 60, 259-268, 1992.

[37] O. Scherzer, M. Grasmair, H. Grossauer, M. Haltmeier F. Lenzen. Variational methods in imaging. Applied Mathematical Sciences, 167, Springer, New York, 2009.

[38] S. Setzer. Operator splittings, Bregman methods and frame shrinkage in image processing. Int. J. Comput. Vis. 92, 265-80, 2011.

[39] C. R. Vogel. Computational Methods for Inverse Problems. Frontiers Appl. Math. 23, 2002.

[40] C. R. Vogel and M. E. Oman. Iterative methods for total variation denoising. SIAM J. Sci. Comput., 17, 1, 227-238, 1996.

[41] Z. B. Xu and G. F. Roach. Characteristic inequalities of uniformly convex and uniformly smooth Banach spaces. J. Math. Anal. Appl., 157, 1, 189-210, 1991.