A dynamic viscoelastic problem: Experimental and numerical results of a finite vibrating plate

A. Segade¹, J.R. Fernández²*, J.A. López-Campos¹, M. Masid² and J.A. Vilán¹

Abstract: In this paper, we numerically study a dynamic viscoelastic problem. The variational formulation is written as a linear parabolic variational equation for the velocity field. An existence and uniqueness result is recalled. Then, fully discrete approximations are introduced using the implicit Euler scheme and the finite element method, for which some a priori error estimates are derived, leading to the linear convergence of the algorithm under suitable additional regularity conditions. Finally, some one- and three-dimensional numerical simulations are presented to show the accuracy of the algorithm and the behaviour of the solution, including a comparison with an experimental study.

Subjects: Science; Mathematics & Statistics; Applied Mathematics; Mathematical Physics

Keywords: linear viscoelasticity; finite elements; a priori error estimates; numerical simulations; vibrating plate

2010 AMS subject classifications: 65M12; 65M60; 74D05; 74H15

ABOUT THE AUTHORS

A. Segade received his PhD degree in Industrial Engineering from the University of Vigo (Spain) in 2008. Currently, he is an associate professor at this university.

J.R. Fernández received his PhD degree in Applied Mathematics from the University of Santiago de Compostela (Spain) in 2002. Currently, he is an associate professor at the University of Vigo.

J.A. López-Campos received his BSc (Mechanical Engineering) from the University of Vigo in 2014. Currently, he is a PhD student in this university under the supervision of A. Segade and J. A. Vilán.

M. Masid received her BSc and MSc (both in Mathematics) degrees from the University of Santiago de Compostela in 2014. Currently, she is a PhD student, with a Marie Curie Fellow, in the Ecole Polytechnique Fédérale de Lausanne (Switzerland).

J.A. Vilán received his PhD degree in Industrial Engineering from the University of Santiago de Compostela (Spain) in 1998. Currently, he is an associate professor at the University of Vigo.

PUBLIC INTEREST STATEMENT

Dynamic problems involving viscoelastic materials appear usually in industry. These materials have been utilized in many engineering applications since they can be customized to meet a desired performance while maintaining low cost. An important issue concerning such materials is that they may exhibit time-dependent and inelastic deformations. In this work, we introduce an efficient algorithm for solving a linear case, proving error estimates which allow to show its convergence. Moreover, we compare the numerical results with those obtained experimentally in a vibrating plate.
1. Introduction

Dynamic and static problems for viscoelastic or elastic materials have been studied in numerous publications. For instance, we could refer the papers Berti and Naso (2011), Chirita (1997), Cocou (2002), Copetti and Fernández (2013), Day (1981), Fabrizio and Chirita (2004), Ionescu and Nguyen (2002), Jaruśek and Eck (1999), Marin (1996, 2010), Marin and Lupu (1998) and Pata (2006) where different problems and mathematical issues are treated, including effects like the contact with a rigid or deformable obstacle (with and without friction), adhesion and damage. Moreover, the numerical approximation of these problems were also done (see, e.g. Ahn & Stewart, 2009; Fernández & Santamarina, 2011; Garcia-Orden & Romero, 2011; Grob & Betsch, 2010; Hauret & Le Tallec, 2006; Konovalov, 2013; Keramat & Heidari, 2014; Larsson & Saedpanah, 2010; Shaw, Warby, & Whiteman, 1994).

These viscoelastic materials have been utilized in many engineering applications since they can be customized to meet a desired performance while maintaining low cost. An important issue concerning such materials is that they may exhibit time-dependent and inelastic deformations. The viscoelastic strain component consists of a recoverable-reversible part (elastic strain) and a recoverable-dissipative deformation part (inelastic strain).

In this paper, we assume that the material is linear for the sake of simplicity. The variational formulation is written in terms of the velocity field, leading to a linear parabolic variational equation. Then, its numerical analysis is performed, providing a priori error estimates for the fully discrete approximations. Finally, some numerical simulations are presented in one- and three-dimensional examples to demonstrate the accuracy of the algorithm and the behaviour of the solution. In order to check the applicability of the proposed model, an experimental study has been designed, related to the vibration of a metallic plate, comparing the measured results with those obtained from the numerical implementation. This work has application in the modelling of vibrating systems in order to better understand their behaviour and to optimize damping structures.

The outline of this paper is as follows. In Section 2, we describe the mathematical problem and derive its variational formulation. An existence and uniqueness result is recalled. Then, fully discrete approximations are introduced in Section 3 by using the finite element method for the spatial approximation and the implicit Euler scheme for the discretization of the time derivatives. An error estimate result is proved from which the linear convergence is deduced under suitable regularity assumptions. Finally, in Section 4, some one- and three-dimensional numerical examples are shown to demonstrate the accuracy of the algorithm and the behaviour of the solution.

2. Mechanical problem and its variational formulation

Denote by $\mathbb{S}^d$, $d = 1, 2, 3$, the space of second-order symmetric tensors on $\mathbb{R}^d$ and by “·” and $\| \cdot \|$ the inner product and the Euclidean norms on $\mathbb{R}^d$ and $\mathbb{S}^d$.

Let $\Omega \subset \mathbb{R}^d$ denote a domain occupied by a viscoelastic body with a Lipschitz boundary $\Gamma = \partial \Omega$ decomposed into two measurable parts $\Gamma_0$ and $\Gamma_F$, such that $\text{meas}(\Gamma_0) < 0$. Let $[0, T], T > 0$, be the time interval of interest. The body is being acted upon by a volume force with density $f_0$, it is clamped on $\Gamma_0$ and surface tractions with density $f_F$ act on $\Gamma_F$. Moreover, let $\nu = (\nu_i)_{i=1}^d$ be the outward unit normal vector.

Let $x \in \Omega$ and $t \in [0, T]$ be the spatial and time variables, respectively. In order to simplify the writing, in some places, we do not indicate the dependence of the functions on $x$ and $t$. Moreover, a dot above a variable represents its first derivative with respect to the time variable and two dots indicate its derivative of second order.

Let $\mathbf{u} = (u_i)_{i=1}^d \in \mathbb{R}^d$, $\mathbf{\sigma} = (\sigma_{ij})_{i,j=1}^d \in \mathbb{S}^d$ and $\mathbf{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))_{i,j=1}^d \in \mathbb{S}^d$ denote the displacement field, the stress tensor and the linearized strain tensor, respectively. We recall that
\[ \epsilon_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i,j = 1, \ldots, d. \]

The body is assumed linearly viscoelastic and it satisfies the following constitutive law (see, for instance, Duvaut & Lions, 1976),

\[ \sigma(x, t) = A \varepsilon(\dot{u}(x, t)) + B \varepsilon(u(x, t)), \quad x \in \Omega, \ t \in (0, T), \]

where \( A = (a_{ijkl}) \) and \( B = (b_{ijkl}) \) are the fourth-order viscous and elastic tensors, respectively.

We turn now to describe the boundary conditions.

On the boundary part \( \Gamma_D \) we assume that the body is clamped and thus the displacement field vanishes there (and so \( u = 0 \) on \( \Gamma_D \times (0, T) \)). Moreover, since the density of traction forces \( f_f \) is applied on the boundary part \( \Gamma_f \), it follows that \( \sigma v = f_f \) on \( \Gamma_f \times (0, T) \).

The mechanical problem of the dynamic deformation of a viscoelastic body is then written as follows.

**Problem P.** Find a displacement field \( u: \Omega \times [0, T] \to \mathbb{R}^d \) and a stress field \( \sigma: \Omega \times [0, T] \to \mathbb{S}^d \) such that,

\[ \sigma(x, t) = A \varepsilon(\dot{u}(x, t)) + B \varepsilon(u(x, t)) \quad \text{for a.e.} \quad x \in \Omega, \ t \in (0, T), \]

\[ \rho \ddot{u} - \text{Div} \sigma = f_0 \quad \text{in} \quad \Omega \times (0, T), \]

\[ u = 0 \quad \text{on} \quad \Gamma_D \times (0, T), \]

\[ \sigma v = f_f \quad \text{on} \quad \Gamma_f \times (0, T), \]

\[ u(0) = u_0, \quad \dot{u}(0) = v_0 \quad \text{in} \quad \Omega. \]

Here, \( \rho > 0 \) is the density of the material (which is assumed constant for simplicity), and \( u_0 \) and \( v_0 \) are initial conditions for the displacement and velocity fields, respectively. Moreover, \( \text{Div} \) represents the divergence operator for tensor-valued functions.

In order to obtain the variational formulation of Problem P, let us denote by \( H = [L^2(\Omega)]^d \), and define the variational spaces \( V \) and \( Q \) as follows,

\[ V = \{ w \in [H^1(\Omega)]^d; \ w = 0 \quad \text{on} \quad \Gamma_D \}, \]

\[ Q = \{ \tau = (\tau_{ij})_{i,j=1}^d \in [L^2(\Omega)]^{d \times d}; \ \tau_{ij} = \tau_{ji}, \quad i,j = 1, \ldots, d \}, \]

where for a Hilbert space \( X \), let \((\cdot, \cdot)_X\) and \( \| \cdot \|_X \) be the scalar product and norm in \( X \), respectively.

We will make the following assumptions on the problem data.

The viscosity tensor \( A(x) = (a_{ijkl}(x))_{i,j,k,l=1}^d \) : \( \tau \in \mathbb{S}^d \to A(x)(\tau) \in \mathbb{S}^d \) satisfies:

(a) \( a_{ijkl} = a_{klij} = a_{ikjl} \quad \text{for} \quad i,j,k,l = 1, \ldots, d. \)

(b) \( a_{ijkl} \in L^\infty(\Omega) \quad \text{for} \quad i,j,k,l = 1, \ldots, d. \)

(c) There exists \( m_A > 0 \) such that \( A(x) \tau \cdot \tau \geq m_A \| \tau \|^2 \quad \forall \ \tau \in \mathbb{S}^d, \ \text{a.e.} \ x \in \Omega. \)
The elastic tensor $B(x) = (b_{ijkl}(x))_{ijkl}^{d} : \tau \in \mathbb{S}^{d} \rightarrow B(x)(\tau) \in \mathbb{S}^{d}$ satisfies:

(a) $b_{ijkl} = b_{ijlk}$ for $i, j, k, l = 1, \ldots, d.$
(b) $b_{ijkl} \in L^{\infty}(\Omega)$ for $i, j, k, l = 1, \ldots, d.$
(c) There exists $m_{ij} > 0$ such that $B(x)\tau \cdot \tau \geq m_{ij} \|\tau\|^2$

\[ \forall \tau \in \mathbb{S}^{d}, \text{ a.e. } x \in \Omega. \]  

The following regularity is assumed on the density of volume forces and tractions:

\[ f_0 \in C([0, T]; H), \quad f_F \in C([0, T]; [L^{2}(\Gamma_F)])^d. \]  

Finally, we assume that the initial displacement and velocity satisfy

\[ u_0, v_0 \in V. \]  

Moreover, we denote by $V'$ the dual space of $V.$ We identify $H$ with its dual and consider the Gelfand triple

$V \subset H \subset V'.$

We use the notation $\langle \cdot, \cdot \rangle_{V' \times V}$ to denote the duality product and, in particular, we have

$\langle v, u \rangle_{V' \times V} = \langle v, u \rangle_H \quad \forall u \in V, \; v \in H.$

Using Riesz theorem, from (8) we can define the element $f(t) \in V'$ given by

\[ \langle f(t), w \rangle_{V' \times V} = \int_{\Omega} f_0(t) \cdot w \, dx + \int_{\Gamma_F} f_F(t) \cdot w \, d\Gamma \quad \forall w \in V, \]

and then $f \in C([0, T]; V').$

Plugging (1) into (2) and using the previous boundary conditions, applying Green’s formula, we derive the following variational formulation of Problem P, written in terms of the velocity field $v(t) = \dot{u}(t).$

**Problem VP.** Find a velocity field $v:[0, T] \rightarrow V$ such that $v(0) = v_0$ and for a.e. $t \in (0, T)$ and for all $w \in V,$

\[ \langle \rho \dot{v}(t), w \rangle_{V' \times V} + (A\epsilon(v(t)) + B\epsilon(u(t))), \epsilon(w))_{\Omega} = \langle f(t), w \rangle_{V' \times V}, \]  

where the displacement field $u(t)$ is given by

\[ u(t) = \int_{0}^{t} v(s) \, ds + u_0. \]  

The existence of a unique solution can be obtained proceeding as in Kuttler, Shillor and Fernández (2006), where the contact with a deformable obstacle is also considered and the damage and adhesion effects are included. Thus, we have the following.

**Theorem 2.1** Assume (6)–(9) hold. Then, there exists a unique solution $v$ to Problem VP. Furthermore, the solution satisfies

\[ v \in C([0, T]; V) \cap C^{1}([0, T]; H). \]
3. Fully discrete approximations and an a priori error analysis

In this section, we introduce a finite element algorithm for approximating solutions to variational problem VP. Its discretization is done in two steps. First, we consider the finite element spaces \( V^h \subset V \) and \( Q^h \subset Q \) given by

\[
V^h = \{ \mathbf{v}^h \in (C(\bar{\Omega}))^d; \mathbf{v}^h_h \in [P_1(T)]^d, \ T \in T^h, \ \mathbf{v}^h = \mathbf{0} \text{ on } \Gamma_D \},
\]

\[
Q^h = \{ \mathbf{q}^h \in L^2(\Omega)^d; \mathbf{q}^h_h \in [P_0(T)]^d, \ T \in T^h \},
\]

where \( \bar{\Omega} \) is assumed to be a polyhedral domain, \( T^h \) denotes a triangulation of \( \bar{\Omega} \) compatible with the partition of the boundary \( \Gamma = \partial \Omega \) into \( \Gamma_D \) and \( \Gamma_N \), and \( P_q(T), q = 0, 1 \), represents the space of polynomials of global degree less or equal to \( q \) in \( T \). Here, \( h > 0 \) denotes the spatial discretization parameter.

Secondly, the time derivatives are discretized using a uniform partition of the time interval \( [0, T] \), denoted by \( 0 = t_0 < t_1 < \cdots < t_N = T \), and let \( k \) be the time step size, \( k = T/N \). Moreover, for a continuous function \( f(t) \) we denote \( f_n = f(t_n) \) and, for the sequence \( \{z_n\}_{n=0}^N \) we denote by \( \delta z_n = (z_n - z_{n-1})/k \) its corresponding divided differences.

Using the well-known implicit Euler scheme, the fully discrete approximation of Problem VP is the following.

**Problem VP\(^{hk} \)**. Find a discrete velocity field \( \mathbf{v}^{hk} = (\mathbf{v}^{hk}_n)_{n=0}^N \subset V^h \) such that \( \mathbf{v}^{hk}_0 = \mathbf{v}^h_0 \) and for \( n = 1, \ldots, N \) and for all \( \mathbf{w}^h \in V^h \),

\[
(\rho \delta \mathbf{v}^{hk}_n, \mathbf{w}^h)_h + \left( A \mathbf{e}(\mathbf{v}^{hk}_n) + B \mathbf{e}(\mathbf{u}^{hk}_n), \mathbf{e}(\mathbf{w}^h) \right)_Q = (\mathbf{f}^h_n, \mathbf{w}^h)_{V^h \times V^h},
\]

where the discrete displacement field \( \mathbf{u}^{hk} = (\mathbf{u}^{hk}_n)_{n=0}^N \subset V^h \) is given by

\[
\mathbf{u}^{hk}_n = k \sum_{j=1}^{n} \mathbf{v}^{hk}_j + \mathbf{u}^h_0.
\]

Here, we note that the discrete initial conditions, denoted by \( \mathbf{u}^h_0 \) and \( \mathbf{v}^h_0 \), are given by

\[
\mathbf{u}^h_0 = P^h \mathbf{u}_0, \quad \mathbf{v}^h_0 = P^h \mathbf{v}_0,
\]

where \( P^h \) is the \( (L^2(\Omega))^d \)-projection operator over the finite element space \( V^h \), and \( \mathbf{f}^h_n = P^h \mathbf{f}_n \).

Using assumptions (6)–(9) and the classical Lax–Milgram lemma, it is easy to prove that Problem VP\(^{hk} \) has a unique discrete solution \( \mathbf{v}^{hk} \subset V^h \).

Our aim in this section is to derive some a priori error estimates for the numerical errors \( \mathbf{u}_n - \mathbf{u}^{hk}_n \) and \( \mathbf{v}_n - \mathbf{v}^{hk}_n \). For the sake of simplicity in the calculations developed below, we assume that \( \mathbf{f} \in C([0, T]; V^h) \) and so, \( \mathbf{f}^h_n = \mathbf{f}_n \). It is straightforward to extend the results presented in the rest of this section to a more general situation.

Thus, we have the following.

**Theorem 3.1**  Let assumptions (6)–(9) hold. If we denote by \( \mathbf{v} \) and \( \mathbf{v}^{hk} \) the respective solutions to Problems VP and VP\(^{hk} \), then there exists a positive constant \( C > 0 \), independent of the discretization parameters \( h \) and \( k \), such that, for all \( \mathbf{w}^h = (\mathbf{w}^h_j)_{j=0}^N \subset V^h \),
\[
\begin{align*}
\max_{0 \leq n \leq N} \|v_n - v_n^h\|^2 + \max_{0 \leq n \leq N} \|u_n - u_n^h\|^2 + C k \sum_{j=1}^N \|v_j - v_j^h\|^2 \\
\leq C k \sum_{j=1}^N \left( \|v_j - \delta v_j\|^2 + \|u_j - \delta u_j\|^2 + \|v_j - v_j^h\|^2 \right) \\
+ C \max_{0 \leq n \leq N} \|v_n - w_n^0\|^2 + C \left( \|u_0 - u_0^h\|^2 + \|v_0 - v_0^h\|^2 \right) \\
+ \frac{C}{k} \sum_{j=1}^{n-1} \|v_j - w_j^h - \left( v_{j+1} - w_{j+1}^h \right) \|^2.
\end{align*}
\] (17)

**Proof.** We subtract variational Equation (10), at time \( t = t_n \) and for \( w = w^h \in V^h \), and discrete variational Equation (14) to get, for all \( w^h \in V^h \),

\[
\left( \rho (\dot{v}_n - \delta v_n^h), w^h \right)_H + \left( \left( \alpha (v_n - v_n^h) + B \varepsilon (u_n - u_n^h), \varepsilon (w^h) \right)_Q = 0. \right.
\]

Therefore, we find that, for all \( w^h \in V^h \),

\[
\left( \rho (\dot{v}_n - \delta v_n^h), v_n - v_n^h \right)_H + \left( \left( \alpha (v_n - v_n^h) + B \varepsilon (u_n - u_n^h), \varepsilon (v_n - v_n^h) \right)_Q = 0. \right.
\]

= \left( \rho (\dot{v}_n - \delta v_n^h), v_n - w^h \right)_H + \left( \left( \alpha (v_n - v_n^h) + B \varepsilon (u_n - u_n^h), \varepsilon (v_n - w^h) \right)_Q \right.

Keeping in mind assumptions (6) and (7), it follows that

\[
\left( \alpha (v_n - v_n^h), \varepsilon (v_n - v_n^h) \right)_Q \geq C \|v_n - v_n^h\|^2,
\]

\[
\left( B \varepsilon (u_n - u_n^h), \varepsilon (v_n - v_n^h) \right)_Q \geq \left( B \varepsilon (u_n - u_n^h), \varepsilon (u_n - \delta u_n) \right)_Q \\
+ \frac{m \varepsilon}{2k} \left\{ \|u_n - u_n^h\|_V^2 + \|u_{n-1} - u_{n-1}^h\|_V^2 \right\},
\]

where \( \delta u_n = (u_n - u_{n-1})/k \), and since

\[
\left( \rho (\dot{v}_n - \delta v_n^h), v_n - v_n^h \right)_H = \left( \rho (\dot{v}_n - \delta v_n^h), v_n - v_n^h \right)_H \\
+ \frac{\rho}{2k} \left\{ \|v_n - v_n^h\|^2 - \|v_{n-1} - v_{n-1}^h\|^2 \right\},
\]

where \( \delta v_n = (v_n - v_{n-1})/k \), using several times Cauchy’s inequality \( ab \leq \alpha a^2 + \frac{1}{\alpha} b^2 \), \( a, b, c \in \mathbb{R} \) with \( \varepsilon > 0 \), and assumptions (6)–(9), we have, for all \( w^h \in V^h \),

\[
\frac{1}{2k} \left[ \|v_n - v_n^h\|^2 - \|v_{n-1} - v_{n-1}^h\|^2 \right] + \|u_0 - u_0^h\|^2 + \frac{1}{2k} \left[ \|u_n - u_n^h\|_V^2 - \|u_{n-1} - u_{n-1}^h\|_V^2 \right] \\
\leq C \left( \|v_n - v_n^h\|^2 + \|u_n - u_n^h\|^2 + \|v_n - v_n^h\|^2 + \|v_n - w^h\|^2 \right) \\
+ \|u_n - u_n^h\|^2 + \|\delta v_n - \delta v_n^h, v_n - w^h\|_H^2.
\]

Thus, by induction we find that, for all \( w^h \in \{w^h\}_{n=0}^N < V^h \),

\[
\left. \begin{array}{l}
\|v_n - v_n^h\|^2 + \|u_n - u_n^h\|^2 + C k \sum_{j=1}^n \|v_j - v_j^h\|^2 \\
\leq C k \sum_{j=1}^n \left( \|v_j - \delta v_j\|^2 + \|u_j - \delta u_j\|^2 + \|v_j - v_j^h\|^2 \right) \\
+ \|v_j - w_j^h\|^2 + \|u_j - u_j^h\|^2 + \left( \delta v_j - \delta v_j^h, v_j - w_j^h \right)_H \right) \\
+ C \left( \|v_0 - v_0^h\|^2 + \|u_0 - u_0^h\|^2 \right) .
\end{array} \right\} (18)
\]

Now, taking into account that
where we have, for all \( \mathbf{w}^h = (\mathbf{w}^h)^{n+1} \subseteq \mathbf{V}^h \),

\[
\| \mathbf{v}_n - \mathbf{v}_n^{\text{sh}} \|_2^2 + \| \mathbf{u}_n - \mathbf{u}_n^{\text{sh}} \|_2^2 + \frac{C}{h^2} \sum_{j=1}^{n} \| \mathbf{v}_j - \mathbf{v}_j^{\text{sh}} \|_2^2 + \| \mathbf{u}_j - \mathbf{u}_j^{\text{sh}} \|_2^2 + \frac{C}{h^2} \sum_{j=1}^{n-1} \| \mathbf{v}_j - \mathbf{v}_j^{\text{sh}} - \mathbf{v}_{j+1}^{\text{sh}} \|_2^2
\]

\[\leq C \left( \| \mathbf{v}_0 - \mathbf{v}_0^{\text{sh}} \|_H^2 + \| \mathbf{u}_0 - \mathbf{u}_0^{\text{sh}} \|_H^2 + \| \mathbf{v}_1 - \mathbf{v}_1^{\text{sh}} \|_H^2 + \| \mathbf{v}_n - \mathbf{v}_n^{\text{sh}} \|_H^2 \right) + \frac{C}{h^2} \sum_{j=1}^{n-1} \| \mathbf{v}_j - \mathbf{v}_j^{\text{sh}} - \mathbf{v}_{j+1}^{\text{sh}} \|_H^2.
\]

We will use now the following lemma which constitutes a discrete version of Gronwall’s inequality (see Barboteu, Fernández, & Hoarau-Mantel, 2005; Campo, Fernández, Kuttler, Shillor, & Viaño, 2006).

**Lemma 3.2.** Assume that \( \{E_n\}_{n=0}^N \) and \( \{G_n\}_{n=0}^N \) are two sequences of nonnegative real numbers satisfying, for a positive constant \( C_1 > 0 \) independent of \( G_n \) and \( E_n \),

\[
E_0 \leq C_1 G_0,
\]

\[
E_n \leq C_1 G_n + C_1 \sum_{j=1}^{n} k E_j, \quad n = 1, \ldots, N,
\]

where \( k \) is a positive constant. Then,

\[
\max_{0 \leq n \leq N} E_n \leq C \max_{0 \leq n \leq N} G_n,
\]

where \( C = C_1 (1 + C_1 T e^{C_1 T}) \) and \( T = Nk \).

Finally, if we define

\[
E_n = \| \mathbf{v}_n \mathbf{v}_n^{\text{sh}} \|_H^2 + \| \mathbf{u}_n \mathbf{u}_n^{\text{sh}} \|_H^2 + k \sum_{j=1}^{n} \| \mathbf{v}_j \mathbf{v}_j^{\text{sh}} \|_H^2,
\]

\[
E_0 = G_0 = \| \mathbf{v}_0 - \mathbf{v}_0^{\text{sh}} \|_H^2 + \| \mathbf{u}_0 - \mathbf{u}_0^{\text{sh}} \|_H^2,
\]

and \( G_n \), the remaining terms on the right-hand side of the previous estimates, applying Lemma 3.2 we derive the a priori error estimates (17). \( \square \)

We note that from estimates (17) we can derive the convergence order under suitable additional regularity conditions. For instance, if we assume that the continuous solution has the additional regularity:

\[
\mathbf{u} \in H^2(0, T; \mathbf{V}) \cap H^3(0, T; \mathbf{H}) \cap C^1([0, T]; [H^2(\Omega)]^d),
\]

(19)

then we have the following result.

**Corollary 3.3.** Let the assumptions of Theorem 3.1 still hold. Under the additional regularity conditions (19), it follows the linear convergence of the solution obtained by Problem VP\( k \); that is, there exists a positive constant \( C_1 \) independent of the discretization parameters \( h \) and \( k \), such that
\[ \max_{0 \leq n \leq N} \| v_n - v_n^h \|_{H^1} + \max_{0 \leq n \leq N} \| u_n - u_n^h \|_{V} \leq C(h + k). \]

Notice that this linear convergence is based on some well-known results concerning the approximation by the finite element method (see, for instance, Ciarlet, 1993), the discretization of the time derivatives and the following result (see Barboteu et al., 2005; Campo et al., 2006 for details),

\[ \frac{1}{k} \sum_{j=1}^{N-1} \| v_j - w_j^h - (v_{j+1} - w_{j+1}^h) \|_{H^1}^2 \leq Ch^2 \| u \|_{H^2(0,T;V)}. \]

4. Numerical results

In this final section, we present the numerical scheme which has been implemented in MATLAB (1D) and the commercial code ANSYS (3D) to obtain the solutions to Problem \( VP^k \) and then we show some numerical examples to demonstrate its accuracy and its behaviour.

4.1. Numerical scheme

Using the finite element spaces defined in (12) and (13), let \( n = 1, 2, \ldots, N \) and given \( u_{n-1}^h \) and \( v_{n-1}^h \in V^h \), the discrete velocity field \( v_n^h \), at time \( t = t_n \) is then obtained from Equation (14). That is, we solve the following problem, for all \( w^h \in V^h \),

\[ \langle \rho v_n^h, w^h \rangle_H + \left( k A e(v_n^h) + k^2 B e(v_n^h), e(w^h) \right)_Q = k \left( \langle \rho f_n, w^h \rangle_{0,T;V} + \langle \rho v_{n-1}^h, w^h \rangle_H - k \left( B e(u_{n-1}^h), e(w^h) \right)_Q \right). \]

Then, the discrete displacement field \( u_n^h \) is updated from Equation (15) as follows,

\[ u_n^h = u_{n-1}^h + k v_n^h. \]

We point out that this problem leads to a linear system which is solved using classical Cholesky's method. In the one-dimensional simulations, this numerical scheme was implemented on a 3.2 Ghz PC using MATLAB, and a typical run \( (h = k = 0.01) \) took about 0.303 s of CPU time. For the three-dimensional analysis, we used the commercial software ANSYS with a 2.86 Ghz PC, where the CPU time was 43 min and 35 s due to the large size of the finite element mesh and a time step \( k = 0.001 \).

4.2. A first example: Numerical convergence

As an academical example, in order to show the accuracy of the approximations, we consider the following simpler problem.

**Problem \( P^e \)** Find a displacement field \( u : [0,1] \times [0,1] \rightarrow \mathbb{R} \) and a stress field \( \sigma : [0,1] \times [0,1] \rightarrow \mathbb{R} \) such that,

\[ \sigma(x,t) = A \dot{u}(x,t) + B u(x,t) \quad \text{for a.e.} \quad x \in (0,1), \quad t \in (0,1), \]

\[ \dot{u} - \sigma_x = f_0 \quad \text{in} \quad (0,1) \times (0,1), \]

\[ u(0,t) = u(1,t) = 0 \quad \text{for all} \quad t \in (0,1), \]

\[ u(x,0) = x(x-1), \quad \dot{u}(x,0) = x(x-1) \quad \forall x \in [0,1], \]

where the volume force \( f_0 \) is given by

\[ f_0(x,t) = e^t(x^2 - x - 4). \]

We note that Problem \( P^e \) corresponds to Problem P with the following data:

\[ \Omega = (0,1), \quad \Gamma_D = \{0,1\}, \quad \Gamma_f = \emptyset, \quad T = 1, \quad \rho = 1, \quad A = 1, \quad B = 1, \]

and initial conditions
Moreover, the exact solution to Problem $P^k$ can be easily calculated and it has the form, for $(x, t) \in (0, 1) \times (0, 1)$,

$$u(x, t) = e^t x(x - 1).$$

To show the numerical convergence and the asymptotic behaviour of the algorithm, the numerical errors given by

$$\max_{0 \leq n \leq N} \left\{ \|v_n - v^h_n\|_H + \|u_n - u^h_n\|_V \right\},$$

are calculated and presented (multiplied by $10^2$) in Table 1 for several values of the discretization parameters $h$ and $k$. Finally, the evolution of the error depending on the parameter $h + k$ is plotted in Figure 1. We notice that the convergence of the algorithm is clearly observed, and the linear convergence, stated in Corollary 3.3, is achieved.

4.3. A second example: Comparison with experimental results

In the second example, we consider a three-dimensional setting. Then, the aim is to simulate a finite vibrating plate made of aluminium. The plate occupies the domain $\Omega = (0, 35) \times (0, 2) \times (0, 250)$ (where the unit is the millimetre), and we study its deformations during 0.35 s (i.e. $T = 0.35$ s). We assume no body forces acting in $\Omega$ and that the body is clamped on the part

| Table 1. Example 1: Numerical errors ($\times 10^2$) for some $h$ and $k$ |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|
| $h \downarrow k \rightarrow$ | 0.01 | 0.005 | 0.001 | 0.0005 | 0.0002 | 0.0001 | 0.00005 |
| 0.01 | 1.827738 | 1.686866 | 1.591623 | 1.581748 | 1.578811 | 1.578778 | 1.578863 |
| 0.005 | 1.087129 | 0.913951 | 0.806755 | 0.795544 | 0.789172 | 0.787406 | 0.787050 |
| 0.001 | 0.632006 | 0.356211 | 0.182805 | 0.168680 | 0.161344 | 0.159093 | 0.158005 |
| 0.0005 | 0.609513 | 0.316244 | 0.108765 | 0.091403 | 0.083069 | 0.080671 | 0.079546 |
| 0.0002 | 0.602984 | 0.303616 | 0.071276 | 0.047532 | 0.036561 | 0.032501 |
| 0.0001 | 0.602043 | 0.301746 | 0.063287 | 0.035640 | 0.021754 | 0.018280 | 0.015868 |
| 0.00005 | 0.601807 | 0.301276 | 0.061041 | 0.031645 | 0.016868 | 0.009140 |
\[ \Gamma_0 = \partial([0, 35] \times [0, 2] \times [0, 40]) \] (we used the boundary of the domain \([0, 35] \times [0, 2] \times [0, 40]\) in order to simulate real clamping conditions), meanwhile the rest of the boundary is assumed traction-free (see Figure 2). An initial displacement is prescribed moving point \(x = (17.5, 0, 210)\) 3.2 mm in the Y-direction. We note that no surface forces are then applied in this example and that we simulate the action of the sensor, used to obtain the experimental measures, with an additional mass of 13 g located at its position (see Figure 3).

The following data were used in the simulations:

\[
T = 0.35 \text{s}, \quad \Omega = (0, 35) \times (0, 2) \times (0, 250), \quad f_0 = 0 \text{N/m}^3, \quad f_f = 0 \text{N/m}^2, \\
\Gamma_0 = \partial((0, 35] \times [0, 2] \times [0, 40]), \quad \Gamma_f = \partial\Omega - \Gamma_0, \quad \dot{u}(0) = 0 \text{ m/s},
\]

where the initial displacement \(u_0\) is the deformation of the plate corresponding to the prescribed initial displacement.

Figure 2. Example 2: Simulation of a finite vibrating plate.

Figure 3. Example 2: Boundary conditions.
The elastic tensor $B$ is defined using Young’s modulus $E$ and Poisson’s ratio $\nu$; i.e.

$$ (B\tau)_{ij} = \frac{E \nu}{(1 + \nu)(1 - 2\nu)} \left( \sum_{k=1}^{3} \tau_{kk} \right) \delta_{ij} + \frac{E}{1 + \nu} \tau_{ij} \quad i,j = 1, \ldots, 3, $$

where $\tau = (\tau_{ij})_{i,j=1}^{3}$ and $(\delta_{ij})_{i,j=1}^{3}$ denote a three-dimensional symmetric tensor and the classical Kronecker delta, respectively. Moreover, since we are simulating a metallic plate made of
aluminium, we take values $E = 72,000$ MPa and $\nu = 0.3$. Moreover, its density $\rho$ is assumed to be $2,770$ Kg/m$^3$.

For the definition of the viscous part $A$, we employ a damping coefficient $\beta$ with respect to the elastic part, and so $A = \beta B$. In these simulations, we use the value $\beta = 1.0954 \times 10^{-4}$.

Using a finite element mesh composed of 35,611 nodes and 6,804 elements (hexahedra), which is shown in Figure 4, and time step $k = 10^{-3}$, Problem $VPh_k$ is now solved using the well-known commercial software ANSYS, which we checked it uses a similar implementation as described in the previous section (the unique difference is that it is based on a displacement-type formulation, as a result of plugging expression $v^n_h = (u^n_h - u^{n-1}_h)/k$ into the discrete variational Equation (14)).

Therefore, in Figure 5, the deformation of the plate is plotted at final time. As expected, due to the vibration process, a bending is produced in the Y-direction.

Moreover, the viscoelastic stresses (von Mises norm) are plotted in Figure 6. Obviously, due to the clamping conditions and since no external forces are applied, they concentrate around the clamped area.

From these results, we find a numerical frequency $\omega^n_h = 182.131$ rad/s and a numerical coefficient of relative damping $z^n_h = 0.0099754$.

Finally, our aim is to compare the above numerical results with those obtained in an experimental study. Then, a metallic plate is subjected to a prescribed initial displacement as we did with the numerical simulation (see Figure 7). As can be seen, the plate made of aluminium is clamped on a part of its right-hand side, and there is a sensor, which we simulated numerically as an added mass, and a prescribed initial displacement on the left-hand part.

Thus, we perform some experimental studies obtaining (in mean values) a natural frequency $\omega = 180.453$ rad/s and a coefficient of relative damping $z = 0.0098830$. As we can see, both values are similar with those obtained numerically. As an example, the results obtained in one of these experiments are plotted in Figure 8. There we show the evolution in time of the voltage measured by the sensor located at point $x = (17.5, 0, 210)$. As expected, there is a vibration of the plate and, due to the viscoelastic (dissipative) behaviour, there is a decrease in its amplitude.
Finally, in Figure 9, we plot the displacement in the Y-direction at point $x = (17.5, 0, 210)$ obtained experimentally (blue curve) and numerically (orange curve). As can be seen, both almost coincide.

Funding
The work of J.A. López, A. Segade and J.A. Vilán was partially supported by the programme of Grupos de Referencia Competitiva with reference [grant number GRC2015/016] (Xunta de Galicia, Spain). The work of J.R. Fernández was partially supported by the Ministerio de Economía y Competitividad under the research projects [grant numbers MTM2012-36452-C02-02 and MTM2015-66640-P] (with FEDER Funds).

Author details
A. Segade¹
E-mail: asegade@uvigo.es
J.R. Fernández²
E-mail: jose.fernandez@uvigo.es
ORCID ID: http://orcid.org/0000-0002-8533-1858
J.A. López-Campos¹
E-mail: joseangellopezcampos@uvigo.es
M. Masid²
E-mail: maria.masid@gmail.com
J.A. Vilán¹
E-mail: jvilan@uvigo.es

¹ Departamento de Ingeniería Mecánica, Máquinas y Motores Térmicos y Fluidos, Escola de Engenharía Industrial, Campus As Lagoas Marcosende s/n, 36310 Vigo, Spain.
² Departamento de Matemática Aplicada I, Universidade de Vigo, ETSI Telecomunicación, Campus As Lagoas Marcosende s/n, 36310 Vigo, Spain.

Citation information
Cite this article as: A. dynamic viscoelastic problem: Experimental and numerical results of a finite vibrating plate, A. Segade, J.R. Fernández, J.A. López-Campos, M. Masid & J.A. Vilán, Cogent Mathematics (2017), 4: 1282691.

References
Ahn, J., & Stewart, D. E. (2009). Dynamic frictionless contact in linear viscoelasticity. IMA Journal of Numerical Analysis, 29, 43–71.
Baboteu, M., Fernández, J. R., & Hoarau-Mantel, T. V. (2005). A class of evolutionary variational inequalities with applications in viscoelasticity. Mathematical Methods in the Applied Sciences, 15, 1595–1617.
Berti, A., & Naso, M. G. (2011). Unilateral dynamic contact of two viscoelastic beams. Quarterly of Applied Mathematics, 69, 477–507.
Campo, M., Fernández, J. R., Kuttler, K. L., Shillor, M., & Viano, J. M. (2006). Numerical analysis and simulations of a dynamic frictionless contact problem with damage. Computer Methods in Applied Mechanics and Engineering, 196, 476–488.
Chirolo, S. (1997). On Saint-Venant's principle in dynamic linear viscoelasticity. Quarterly of Applied Mathematics, 55, 139–149.
Clariet, P.G. (1993). Basic error estimates for elliptic problems. P.G. Clariet (Ed.), Handbook of Numerical Analysis (Vol. II, pp. 17–351).
Cocou, M. (2002). Existence of solutions of a dynamic Signorini’s problem with nonlocal friction in viscoelasticity. Zeitschrift für angewandte Mathematik und Physik ZAMP, 53, 1099–1109.

Copetti, M. I. M., & Fernández, J. R. (2013). A dynamic contact problem involving a Timoshenko beam model. Applied Numerical Mathematics, 63, 117–128.

Day, W. A. (1981). On the quasistatic approximation in dynamic linear viscoelasticity. Archive for Rational Mechanics and Analysis, 76, 265–282.

Duvaut, G., & Lions, J. L. (1976). Inequalities in mechanics and physics. Berlin: Springer-Verlag.

Fabrizio, M., & Chirita, S. (2004). Some qualitative results on the dynamic viscoelasticity of the Reissner-Mindlin plate model. Quarterly Journal of Mechanics & Applied Mathematics, 57, 59–78.

Fernández, J. R., & Santamarina, D. (2011). An a posteriori error analysis for dynamic viscoelastic problems. ESAIM: Mathematical Modelling and Numerical Analysis, 45, 925–945.

García-Orden, J. C., & Romero, I. (2011). Energy-Entropy-Momentum integration of discrete thermo-visco-elastic dynamics. European Journal of Mechanics - A/Solids, 32, 76–87.

Grob, M., & Betsch, F. (2010). Energy-momentum consistent finite element discretization of dynamic finite viscoelasticity. International Journal for Numerical Methods in Engineering, 81, 1341–1386.

Hauret, P., & Le Tallec, P. (2006). Energy-controlling time integration methods for nonlinear elastodynamics and low-velocity impact. Computer Methods in Applied Mechanics and Engineering, 195, 4890–4916.

Ionescu, I. R., & Nguyen, Q.-L. (2002). Dynamic contact problems with slip dependent friction in viscoelasticity. International Journal of Applied Mathematics and Computer Science, 12, 71–80.

Jarušek, J., & Eck, C. (1999). Dynamic contact problems with small Coulomb friction for viscoelastic bodies: Existence of solutions. Mathematical Models and Methods in Applied Sciences, 9, 11–34.

Keramat, A., & Heidari, K. (2014). Finite element based dynamic analysis of viscoelastic solids using the approximation of Volterra integrals. Finite Elements in Analysis and Design, 86, 89–100.

Konovalov, A. N. (2013). Completely conservative difference schemes for dynamic problems of linear elasticity and viscoelasticity. Differential Equations, 49, 857–868.

Kuttler, K. L., Shillor, M., & Fernández, J. R. (2006). Existence and regularity for dynamic viscoelastic adhesive contact with damage. Applied Mathematics & Optimization, 53, 31–66.

Larsson, S., & Saedpanah, F. (2010). The continuous Galerkin method for an integro-differential equation modeling dynamic fractional order viscoelasticity. IMA Journal of Numerical Analysis, 30, 964–986.

Marin, M. (1996). Some basic theorems in elastostatics of micropolar materials with voids. Journal of Computational and Applied Mathematics, 70, 115–126.

Marin, M. (2010). A domain of influence theorem for microstretch elastic materials. Nonlinear Analysis: Real World Applications, 11, 3446–3452.

Marin, M., & Lupu, M. (1998). On harmonic vibrations in thermoelasticity of micropolar bodies. Journal of Vibration and Control, 4, 507–518.

Pata, V. (2006). Exponential stability in linear viscoelasticity. Quarterly of Applied Mathematics, 64, 499–513.

Shaw, S., Warby, M. K., & Whiteman, J. W. (1994). Numerical techniques for problems of quasistatic and dynamic viscoelasticity. In The mathematics of finite elements and applications. Chichester: Wiley.