BLOW-UP FOR SEMILINEAR WAVE EQUATIONS WITH TIME-DEPENDENT DAMPING IN AN EXTERIOR DOMAIN

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(Communicated by Hongjie Dong)

Abstract. We consider the semilinear wave equation with time-dependent damping
\[ \partial_{tt}u - \Delta u + \mu(1 + t)^{-\beta} \partial_t u = |u|^p, \quad (t, x) \in (0, \infty) \times D^c, \]
where \( D^c = \mathbb{R}^N \setminus D \), \( D \) is the closed unit ball in \( \mathbb{R}^N \), \( N \geq 2 \), \( \mu > 0 \), \( p > 1 \) and \(-1 < \beta < 1\). The considered equation is investigated under the boundary conditions:
\[ \partial u / \partial n^+(t, x) = b(t)f(x) \quad \text{on} \quad (0, \infty) \times \partial D, \]
where \( n^+ \) is the outward (relative to \( D^c \)) unit normal on \( \partial D \). General blow-up results are established for the considered problems. Moreover, for a certain class of functions \( b \), the critical exponent in the sense of Fujita is obtained.

1. Introduction. We are concerned with the question of blow-up of solutions to the exterior Cauchy problems for the semilinear wave equation with time-dependent damping
\[
\begin{align*}
\partial_{tt}u - \Delta u + a(t) \partial_t u &= |u|^p & \text{in} & \ (0, \infty) \times D^c, \\
\partial_t u(t, x) &= b(t)f(x) & \text{on} & \ (0, \infty) \times \partial D, \\
(u(0, x), \partial_t u(0, x)) &= (u_0(x), u_1(x)) & \text{in} & \ D^c,
\end{align*}
\]
and
\[
\begin{align*}
\partial_{tt}u - \Delta u + a(t) \partial_t u &= |u|^p & \text{in} & \ (0, \infty) \times D^c, \\
\partial (t, x) &= b(t)f(x) & \text{on} & \ (0, \infty) \times \partial D, \\
(u(0, x), \partial_t u(0, x)) &= (u_0(x), u_1(x)) & \text{in} & \ D^c,
\end{align*}
\]
where \( p > 1 \), \( a(t) = \mu(1 + t)^{-\beta} \), \( \mu > 0 \), \(-1 < \beta < 1\), \( D \) is the closed unit ball in \( \mathbb{R}^N \), \( N \geq 2 \), \( D^c \) is its complement and \( n^+ \) is the outward (relative to \( D^c \)) unit normal on \( \partial D \). We shall assume that the initial data \( u_i, i = 0, 1 \), are such that local solutions exist for problems (1.1)–(1.2).

2010 Mathematics Subject Classification. Primary: 35B33, 35L71; Secondary: 35L05.

Key words and phrases. Semilinear wave equation, time-dependent damping, exterior domain, blow-up, critical exponent.

B. Samet is supported by Researchers Supporting Project RSP-2019/4, King Saud University, Riyadh, Saudi Arabia.

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When $D = \emptyset$ and $(\mu, \beta) = (1, 0)$, problem (1.1) reduces to
\[
\begin{align*}
\partial_{tt} u - \Delta u + \partial_t u &= |u|^p \quad \text{in } (0, \infty) \times \mathbb{R}^N, \\
(u(0, x), \partial_t u(0, x)) &= (u_0(x), u_1(x)) \quad \text{in } \mathbb{R}^N.
\end{align*}
\] (1.3)

In [11], the authors proved that
\begin{itemize}
  \item If $\int_{\mathbb{R}^N} u_i(x) \, dx > 0$, $i = 0, 1$, and $1 < p < 1 + \frac{2}{N}$, then problem (1.3) admits no global solutions.
  \item If $1 + \frac{2}{N} < p \leq \frac{N}{N-2}$ for $N \geq 3$, and $p > 1 + \frac{2}{N}$ for $N = 1, 2$, then for small initial data, problem (1.3) admits a unique global solution in a certain functional space.
\end{itemize}

It should be pointed out that the number $1 + \frac{2}{N}$ is the Fujita critical exponent for the semilinear heat equation $\partial_t u - \Delta u = |u|^p$ in $(0, \infty) \times \mathbb{R}^N$ (see Fujita [1]).

When $D = \emptyset$, problem (1.1) reduces to
\[
\begin{align*}
\partial_{tt} u - \Delta u + a(t) \partial_t u &= |u|^p \quad \text{in } (0, \infty) \times \mathbb{R}^N, \\
(u(0, x), \partial_t u(0, x)) &= (u_0(x), u_1(x)) \quad \text{in } \mathbb{R}^N.
\end{align*}
\] (1.4)

In [8], problem (1.4) was studied in the case $\mu > 0$ and $0 \leq \beta < 1$. Namely, the global existence of solutions was established for small data, in the case
\begin{equation}
1 + \frac{2}{N} < p < \frac{N + 2}{[N - 2]^+} := \begin{cases} 
\infty & \text{if } N = 1, 2, \\
\frac{N + 2}{N - 2} & \text{if } N \geq 3.
\end{cases}
\end{equation}

Moreover, by applying a blow-up lemma for a certain ordinary differential inequality, it was shown that, if
\[
\int_{\mathbb{R}^N} u_i(x) \, dx \geq 0, \ i = 0, 1, \quad \int_{\mathbb{R}^N} (u_0 + u_1)(x) \, dx > 0
\]
and
\[
1 + \frac{2\beta}{N} \leq p \leq 1 + \frac{1 + \beta}{N} \quad (p > 1 \text{ if } \beta = 0),
\]
then (1.4) admits no global solutions. Note that in [8], it was conjectured that $1 + \frac{2}{N}$ is still critical for this problem. Later, problem (1.4) was investigated in [6] when $\mu > 0$ and $-1 < \beta < 1$. Using the weighted energy method and a test function approach [13], it was shown that the Fujita exponent is still critical even in the time-dependent damping case.

In [2] (see also [3]), problem (1.1) in the case $N = 2$, $(\mu, \beta) = (1, 0)$ and $b \equiv 0$, was investigated. For $p > 2$, it was shown that for small initial data, the corresponding problem admits a unique global solution in a certain functional space. In [10], the same problem was studied in the $N$-dimensional case, $N \geq 2$. In particular, it was proved that, if $N \leq 5$ and
\[
1 + \frac{4}{N + 2} \leq p \leq 1 + \frac{2}{N - 2}, \quad 2 < p < \infty \text{ if } N = 2,
\]
then a unique global solution exists in a certain functional space for small initial data. In [5], it was proved that, when $N \geq 3$, $1 + \frac{2}{N}$ belongs to the blow-up case.

In [9], a blow-up result was derived in the case $1 < p < 1 + \frac{2}{N}$.

Recently, motivated by [12], we studied in [4] problem (1.1) in the case $N \geq 2$, $(\mu, \beta) = (1, 0)$, $b \equiv 1$ and $f \in C(\partial D)$, $f \geq 0$, $f \neq 0$. Using a test function approach,
we proved that the critical exponent in this case is given by

\[
p^* := \begin{cases} \infty & \text{if } N = 2, \\ 1 + \frac{2}{N - 2} & \text{if } N \geq 3. \end{cases}
\]

Motivated by the above cited works, problems (1.1)-(1.2) are investigated in this paper. To better state our main results, let us provide the definitions of solutions. First, following [6], we introduce the nonnegative function \( g \), which is the solution of the initial value problem for the ordinary differential equation

\[
\begin{align*}
&g'(t) + a(t)g(t) = 1, \quad t > 0, \\
g(0) = \ell,
\end{align*}
\]

where

\[
\ell = \int_0^\infty \exp \left( -\int_0^t a(s) \, ds \right) \, dt.
\]

Next, suppose that \( u \) is a global solution to problem (1.1). Multiplying the first equation in (1.1) by \( g(t) \), we find that \( u \) solves the Cauchy problem

\[
\begin{align*}
&\partial_t (g(t)u) - \Delta (g(t)u) - \partial_t (g'(t)u) + u_t = g(t)|u|^p \quad \text{in } (0, \infty) \times D^c, \\
&\left. \frac{\partial u}{\partial n} \right|_{\partial D} = b(t) f(x) \quad \text{on } (0, \infty) \times \partial D, \\
&(u(0), \partial_t u(0, x)) = (u_0(x), u_1(x)) \quad \text{in } D^c.
\end{align*}
\]

Similarly, if \( u \) is a global solution to problem (1.2), then \( u \) solves the Cauchy problem

\[
\begin{align*}
&\partial_t (g(t)u) - \Delta (g(t)u) - \partial_t (g'(t)u) + u_t = g(t)|u|^p \quad \text{in } (0, \infty) \times D^c, \\
&\left. \frac{\partial u}{\partial n^+} \right|_{\partial D} = b(t) f(x) \quad \text{on } (0, \infty) \times \partial D, \\
&(u(0), \partial_t u(0, x)) = (u_0(x), u_1(x)) \quad \text{in } D^c.
\end{align*}
\]

Hence, we define global weak solutions for problems (1.1)-(1.2) as follows.

Let \( \Phi_D \) be the set of functions \( \varphi \in C^2([0, \infty) \times \overline{D}^c) \) satisfying

(i) \( \varphi|_{\partial D} \equiv 0 \).

(ii) \( \varphi(t, \cdot) \equiv 0, \ t \geq T, \ \text{for some } T > 0 \).

(iii) \( \varphi(\cdot, x) \equiv 0, \ |x| \geq R, \ \text{for some } R > 1 \).

Definition 1.1. Let \( u_0, u_1 \in L^1_{loc}(\overline{D}^c), \ f \in L^1(\partial D) \) and \( b \in L^1_{loc}([0, \infty)) \). We say that \( u \) is a global weak solution to problem (1.1) if

(a) \( u \in L^p_{loc}([0, \infty) \times \overline{D}^c) \).

(b) For all function \( \varphi \in \Phi_D \), there holds

\[
\begin{align*}
&\int_{(0, \infty) \times \overline{D}^c} g(t)|u|^p \varphi \, dx \, dt + \int_{D^c} (u_0(x) + \ell u_1(x)) \varphi(0, x) \, dx \\
&\quad - \ell \int_{D^c} u_0(x) \partial_t \varphi(0, x) \, dx - \int_{(0, \infty) \times \partial D} \frac{\partial \varphi}{\partial n^+} g(t)b(t)f(x) \, dS_x \, dt \\
&= \int_{(0, \infty) \times \overline{D}^c} g(t)u \partial_t \varphi \, dx \, dt - \int_{(0, \infty) \times \overline{D}^c} u \partial_t \varphi \, dx \, dt \\
&\quad + \int_{(0, \infty) \times \overline{D}^c} g'(t)u \partial_t \varphi \, dx \, dt - \int_{(0, \infty) \times \overline{D}^c} g(t)u \Delta \varphi \, dx \, dt.
\end{align*}
\]
Let $\Phi_N$ be the set of functions $\varphi \in C^2([0, \infty) \times \overline{D^c})$ satisfying

(i) $\frac{\partial \varphi}{\partial n} \equiv 0$.
(ii) $\varphi(t, \cdot) \equiv 0$, $t \geq T$, for some $T > 0$.
(iii) $\varphi(\cdot, x) \equiv 0$, $|x| \geq R$, for some $R > 1$.

**Definition 1.2.** Let $u_0, u_1 \in L^1_{\text{loc}}(\overline{D^c})$, $f \in L^1(\partial D)$ and $b \in L^1_{\text{loc}}([0, \infty))$. We say that $u$ is a global weak solution to problem (1.2) if

(a) $u \in L^p_{\text{loc}}([0, \infty) \times \overline{D^c})$.
(b) For all function $\varphi \in \Phi_N$, there holds

\[
\int_{(0, \infty) \times D^c} g(t)|u|^p \varphi \, dx \, dt + \int_{D^c} (u_0(x) + \ell u_1(x)) \varphi(0, x) \, dx \\
- \ell \int_{D^c} u_0(x) \partial t \varphi(0, x) \, dx + \int_{(0, \infty) \times \partial D} g(t)b(t)f(x) \varphi \, dS_x \, dt \\
= \int_{(0, \infty) \times D^c} g(t)u \partial t \varphi \, dx \, dt - \int_{(0, \infty) \times D^c} u \partial_x \varphi \, dx \, dt \\
+ \int_{(0, \infty) \times D^c} g'(t)u \partial_t \varphi \, dx \, dt - \int_{(0, \infty) \times D^c} g(t)u \Delta \varphi \, dx \, dt.
\]

Our main results are the following. Let $F$ be the harmonic function in $D^c$ given by

\[
F(x) = \begin{cases} 
\ln |x| & \text{if } N = 2, \\
(1 - |x|^{2-N}) & \text{if } N \geq 3.
\end{cases}
\]

We first consider the case $b \equiv 0$.

**Theorem 1.3.** Let $N \geq 2$, $-1 < \beta < 1$, $\mu > 0$, $u_0, u_1 \in L^1_{\text{loc}}(\overline{D^c})$ and $b \equiv 0$. Suppose that

\[
(u_0 + \ell u_1)F \in L^1(\overline{D^c}) \quad \text{and} \quad \int_{D^c} (u_0(x) + \ell u_1(x)) F(x) \, dx > 0.
\]

If

\[
1 < p < 1 + \frac{2}{N} \quad (N \geq 3) \quad \text{and} \quad 1 < p \leq 2 \quad (N = 2),
\]

then problem (1.1) admits no global weak solutions.

**Remark 1.** (i) Let us consider problem (1.1) in the case $N = 2$, $(\mu, \beta) = (1, 0)$ and $b \equiv 0$. From Theorem 1.3, under condition (1.10), if $p = 2$, then problem (1.1) admits no global weak solutions. This completes the blow-up result obtained by Laia and Yin [5], where it was shown that in the $N$-dimensional case, $N \geq 3$, the number $1 + \frac{2}{N}$ belongs to the blow-up case.

(ii) At this time, we do not know whether the exponent $1 + \frac{2}{N} \quad (N \geq 3)$ belongs to the blow-up case or not for problem (1.1).

**Theorem 1.4.** Let $N \geq 2$, $-1 < \beta < 1$, $\mu > 0$, $u_0, u_1 \in L^1_{\text{loc}}(\overline{D^c})$ and $b \equiv 0$. Suppose that

\[
u_0 + \ell u_1 \in L^1(D^c) \quad \text{and} \quad \int_{D^c} (u_0(x) + \ell u_1(x)) \, dx > 0.
\]

If

\[
1 < p < 1 + \frac{2}{N},
\]

then problem (1.2) admits no global weak solutions.
Remark 2. We do not know whether the exponent $1 + \frac{2}{N}$ belongs to the blow-up case or not for problem (1.2).

Next, we consider the case $b \neq 0$.

**Theorem 1.5.** Let $N \geq 2$, $p > 1$, $-1 < \beta < 1$, $\mu > 0$, $u_0, u_1 \in L^1_{\text{loc}}(\mathbb{T}^D)$, $u_0, u_1 \geq 0$, $f \in L^1(\partial D)$, $\int_{\partial D} f(x) dS_x > 0$ and $b \in L^1_{\text{loc}}([0, \infty))$, $b \geq 0$, $b \neq 0$.

(I) If $N = 2$ and

$$\limsup_{T \to \infty} \left( \int_0^T (1 + t)^{\beta} b(t) \, dt \right) T^{-\frac{(\beta+1)(2-p)}{p-1}} (\ln T)^{-1} = +\infty,$$  \hfill (1.14)

then problem (1.1) admits no global weak solutions.

(II) If $N \geq 3$ and

$$\limsup_{T \to \infty} \left( \int_0^T (1 + t)^{\beta} b(t) \, dt \right) T^{-\beta+1} (\frac{2}{p} + \frac{1}{p'-1}) = +\infty,$$  \hfill (1.15)

then problem (1.1) admits no global weak solutions.

**Theorem 1.6.** Let $N \geq 2$, $p > 1$, $-1 < \beta < 1$, $\mu > 0$, $u_0, u_1 \in L^1_{\text{loc}}(\mathbb{T}^D)$, $u_0, u_1 \geq 0$, $f \in L^1(\partial D)$, $\int_{\partial D} f(x) dS_x > 0$ and $b \in L^1_{\text{loc}}([0, \infty))$, $b \geq 0$, $b \neq 0$. If (1.15) holds, then problem (1.2) admits no global weak solutions.

Further, we discuss the special case when $b$ belongs to set of functions

$$B := \{ b \in L^1_{\text{loc}}([0, \infty)) : \text{ess inf } b > 0 \}.$$

**Theorem 1.7.** Let $N \geq 2$ and $-1 < \beta < 1$.

(I) If $N = 2$, $\mu > 0$, $u_0, u_1 \in L^1_{\text{loc}}(\mathbb{T}^D)$, $u_0, u_1 \geq 0$, $f \in L^1(\partial D)$, $\int_{\partial D} f(x) dS_x > 0$ and $b \in B$, then for all $p > 1$, problems (1.1)–(1.2) admits no global weak solutions.

(II) If $N \geq 3$, $\mu > 0$, $u_0, u_1 \in L^1_{\text{loc}}(\mathbb{T}^D)$, $u_0, u_1 \geq 0$, $f \in L^1(\partial D)$, $\int_{\partial D} f(x) dS_x > 0$ and $b \in B$, then for all

$$1 < p < 1 + \frac{2}{N-2},$$

problems (1.1)–(1.2) admit no global weak solutions.

(III) If $N \geq 3$ and

$$p > 1 + \frac{2}{N-2},$$

then for all $\mu > 0$ and $-1 < \beta < 1$, problems (1.1)–(1.2) admit global solutions for some $u_0, u_1 \geq 0$, $f > 0$ and $b \in B$.

**Remark 3.** (i) By Theorem 1.7, we observe that in the case $\int_{\partial D} f(x) dS_x > 0$ and $b \in B$, the exponent $p^*$ given by (1.5) is critical for problems (1.1)–(1.2).

(ii) In the case $N \geq 3$, we do not know whether the critical exponent $p^* = 1 + \frac{2}{N-2}$ belongs to the blow-up case or not for problems (1.1)–(1.2).

The rest of the paper is organized as follows. In Section 2, we establish some preliminary estimates that will be used later in the proofs of our main results. The case $b \equiv 0$ is discussed in Section 3, where we prove Theorems 1.3 and 1.4. In Section 4, we discuss the case $b \neq 0$, and prove Theorems 1.5–1.7.
2. Preliminary estimates. In this section, we provide some estimates that will be used later in the proofs of our main results. For the proof of the following result, see [6].

Lemma 2.1. Let $-1 < \beta < 1$. There exists a constant $C > 0$ such that

$$C^{-1} |a(t)|^{-1} \leq g(t) \leq C|a(t)|^{-1}, \quad t \geq 0.$$  

Let $0 < T < \infty$. We define the sets

$$Q_T = [0, 2T] \times D^c \quad \text{and} \quad \Gamma_T = [0, 2T] \times \partial D.$$  

Moreover, we introduce the functions

$$\vartheta(t) = \left( \frac{t}{T} \right)^\omega, \quad t \geq 0 \quad \text{and} \quad \Xi(x) = \left[ \xi \left( \frac{|x|}{T^{2r}} \right) \right]^\omega, \quad x \in \mathbb{R}^N,$$  

where $\omega > 1$, $r > 0$ and $\xi \in C^\infty([0, \infty); \mathbb{R})$ is a function satisfying

$$0 \leq \xi \leq 1; \quad \xi(\sigma) = \begin{cases} 1 & \text{if } 0 \leq \sigma \leq 1, \\ 0 & \text{if } \sigma \geq 2. \end{cases}$$

2.1. Estimates related to problem (1.1). Given $0 < T < \infty$, we introduce the function

$$D(t, x) = \vartheta(t)\psi(x), \quad (t, x) \in [0, \infty) \times \overline{D^c},$$  

where

$$\psi(x) = F(x)\Xi(x), \quad x \in \overline{D^c}$$

and $F$ is given by (1.9). One can easily show that

$$D \in \Phi_D, \quad T \gg 1.$$  

For the proofs of the following two lemmas, we refer to [4].

Lemma 2.2. We have

$$\int_{D^c} \psi(x) \, dx = \begin{cases} O \left( T^{2r} \ln T \right) (T \to \infty) & \text{if } N = 2, \\ O \left( T^{Nr} \right) (T \to \infty) & \text{if } N \geq 3. \end{cases}$$

Lemma 2.3. Let $m > 1$. There holds

$$\int_{D^c} [\psi(x)]^{\frac{m}{m-1}} |\Delta \psi|^{\frac{m}{m-1}} \, dx = \begin{cases} O \left( T^{\frac{m+1}{m-1}} \ln T \right) (T \to \infty) & \text{if } N = 2, \\ O \left( T^r (N - \frac{m}{m-1}) \right) (T \to \infty) & \text{if } N \geq 3. \end{cases}$$

Lemma 2.4. Let $-1 < \beta < 1$ and $m > 1$. There holds

$$\int_{Q_T} g(t) |\partial_t D|^{\frac{m}{m-1}} \, dx \, dt = \begin{cases} O \left( T^{\beta - \frac{m+1}{m-1} + 2r} \ln T \right) (T \to \infty) & \text{if } N = 2, \\ O \left( T^{\beta - \frac{m+1}{m-1} + Nr} \right) (T \to \infty) & \text{if } N \geq 3. \end{cases}$$

Proof. By the definition of the function $D$, one has

$$\int_{Q_T} g(t) |\partial_t D|^{\frac{m}{m-1}} \, dx \, dt = \left( \int_0^{2T} g(t) |\vartheta(t)|^{\frac{m}{m-1}} |\vartheta''(t)|^{\frac{m}{m-1}} \, dt \right) \left( \int_{D^c} \psi(x) \, dx \right).$$  

(2.1)

On the other hand,

$$\int_0^{2T} g(t) |\vartheta(t)|^{\frac{m}{m-1}} |\vartheta''(t)|^{\frac{m}{m-1}} \, dt = T^{-\frac{2m}{m-1}} \int_0^{2T} g(t) \left[ \xi \left( \frac{t}{T} \right) \right]^{\frac{m}{m-1}} |\vartheta \left( \frac{t}{T} \right)|^{\frac{m}{m-1}} \, dt,$$
where
\[ \theta(s) = \omega(\omega - 1)|\xi'(s)|^2 + \omega \xi(s)\xi''(s), \quad s \geq 0. \]

Hence, using Lemma 2.1 and the change of variable \( s = \frac{t}{T} \), one obtains
\[
\int_0^{2T} g(t)|\bar{\vartheta}(t)|^{\frac{m-1}{m}} |\bar{\vartheta}''(t)|^{\frac{m}{m-1}} dt = T^{-\frac{m+1}{m-1}} \int_1^2 g(Ts)|\xi(s)|^{\omega - \frac{2m}{m-1}} |\theta(s)|^{\frac{m}{m-1}} ds \\
\leq CT^{-\frac{m+1}{m-1}} \int_1^2 (1 + Ts)^\beta ds \leq CT^{\beta - \frac{m+1}{m-1}}.
\]

Here and below, we denote by \( C \) a positive constant, whose value may change from line to line. Hence,
\[
\int_0^{2T} g(t)|\bar{\vartheta}(t)|^{\frac{m-1}{m}} |\bar{\vartheta}''(t)|^{\frac{m}{m-1}} dt = O \left( T^{\beta - \frac{m+1}{m-1}} \right), \quad \text{as} \ T \to \infty. \tag{2.2}
\]

Combining (2.1) with (2.2) and using Lemma 2.2, the desired estimate follows. \( \square \)

**Lemma 2.5.** Let \(-1 < \beta < 1 \) and \( m > 1 \). There holds
\[
\int_{Q_T} [g(t)]^{\frac{m-1}{m}} D^{\frac{1}{m-1}} |\partial_t D|^{\frac{m}{m-1}} dx dt = \begin{cases} O \left( T^{2r-\frac{\beta+1}{m-1} \ln T} \right) (T \to \infty) & \text{if} N = 2, \\
O \left( T^{N-\frac{\beta+1}{m-1}} \right) (T \to \infty) & \text{if} N \geq 3. 
\end{cases}
\]

**Proof.** We have
\[
\int_{Q_T} [g(t)]^{\frac{m-1}{m}} D^{\frac{1}{m-1}} |\partial_t D|^{\frac{m}{m-1}} dx dt \\
= \left( \int_0^{2T} [g(t)]^{\frac{m-1}{m}} |\bar{\vartheta}(t)|^{\frac{m}{m-1}} |\bar{\vartheta}''(t)|^{\frac{m}{m-1}} dt \right) \left( \int_{D^n} \psi(x) dx \right). \tag{2.3}
\]

On the other hand,
\[
\int_0^{2T} [g(t)]^{\frac{m-1}{m}} |\bar{\vartheta}(t)|^{\frac{m}{m-1}} |\bar{\vartheta}''(t)|^{\frac{m}{m-1}} dt \\
= CT^{-\frac{m}{m-1}} \int_T^{2T} [g(Ts)]^{\frac{m-1}{m}} \left[ \xi'(\frac{t}{T}) \right]^{\omega - \frac{m}{m-1}} |\xi'(\frac{t}{T})|^{\frac{m}{m-1}} dt.
\]

Using Lemma 2.1 and the change of variable \( s = \frac{t}{T} \), for \( T \gg 1 \), there holds
\[
\int_0^{2T} [g(t)]^{\frac{m-1}{m}} |\bar{\vartheta}(t)|^{\frac{m}{m-1}} |\bar{\vartheta}''(t)|^{\frac{m}{m-1}} dt \\
= CT^{-\frac{1}{m-1}} \int_1^2 [g(Ts)]^{\frac{m}{m-1}} |\xi(s)|^{\omega - \frac{m}{m-1}} |\xi'(s)|^{\frac{m}{m-1}} ds \\
\leq CT^{-\frac{1}{m-1}} \int_1^2 (1 + Ts)^\beta ds \leq CT^{\beta - \frac{m+1}{m-1}},
\]

which yields
\[
\int_0^{2T} [g(t)]^{\frac{m-1}{m}} |\bar{\vartheta}(t)|^{\frac{m}{m-1}} |\bar{\vartheta}''(t)|^{\frac{m}{m-1}} dt = O \left( T^{\beta - \frac{m+1}{m-1}} \right), \quad \text{as} \ T \to \infty. \tag{2.4}
\]

Combining (2.3) with (2.4) and using Lemma 2.2, the desired estimate follows. \( \square \)
Combining (2.5) with (2.6) and using Lemma 2.3, we obtain the desired estimate.

Using Lemma 2.5, we obtain the desired estimate.

Estimates related to problem 2.2.

Lemma 2.7. Let \(-1 < \beta < 1\) and \(m > 1\). There holds

\[
\int_{Q_T} |g(t)|^{\frac{1}{m-1}} |g'(t)|^{\frac{m}{m-1}} D^{\frac{1}{m-1}} |\partial_t D|^{\frac{m}{m-1}} dx dt
= \begin{cases} O \left( T^{2r-\frac{\beta+1}{m-1}} \ln T \right) (T \to \infty) & \text{if } N = 2, \\
O \left( T^{N-\frac{\beta+1}{m-1}} \right) (T \to \infty) & \text{if } N \geq 3.
\end{cases}
\]

Proof. By (1.6), we have \(|g'(t)| = |a(t)g(t) - 1|, \ t > 0\). Using Lemma 2.1, we obtain

\(|g'(t)|^m \leq C, \ t > 0\).

Therefore,

\[
\int_{Q_T} |g(t)|^{\frac{1}{m-1}} |g'(t)|^{\frac{m}{m-1}} D^{\frac{1}{m-1}} |\partial_t D|^{\frac{m}{m-1}} dx dt \leq C \int_{Q_T} |g(t)|^{\frac{1}{m-1}} D^{\frac{1}{m-1}} |\partial_t D|^{\frac{m}{m-1}} dx dt.
\]

Using Lemma 2.5, we obtain the desired estimate. \(\square\)

Lemma 2.7. Let \(-1 < \beta < 1\) and \(m > 1\). There holds

\[
\int_{Q_T} g(t) D^{\frac{1}{m-1}} |\Delta D|^{\frac{m}{m-1}} dx dt = \begin{cases} O \left( T^{\beta+1-\frac{2r}{m-1}} \ln T \right) (T \to \infty) & \text{if } N = 2, \\
O \left( T^{\beta+1+r(N-\frac{2m}{m-1})} \right) (T \to \infty) & \text{if } N \geq 3.
\end{cases}
\]

Proof. We have

\[
\int_{Q_T} g(t) D^{\frac{1}{m-1}} |\Delta D|^{\frac{m}{m-1}} dx dt = \left( \int_{0}^{2T} g(t) \vartheta(t) dt \right) \left( \int_{D^c} |\psi(x)|^{\frac{1}{m-1}} |\Delta \psi|^{\frac{m}{m-1}} dx \right). \quad (2.5)
\]

On the other hand, using the change of variable \(s = \frac{t}{T}\) and Lemma 2.1, for \(T \gg 1\), we have

\[
\int_{0}^{2T} g(t) \vartheta(t) dt = \int_{0}^{2T} g(t) \left[ \xi \left( \frac{t}{T} \right) \right]^{\omega} dt = T \int_{0}^{2} g(Ts) |\xi(s)|^{\omega} ds
\leq T \int_{0}^{2} (1 + Ts)^{\beta} ds \leq CT^{\beta+1},
\]

which yields

\[
\int_{0}^{2T} g(t) \vartheta(t) dt = O \left( T^{\beta+1} \right), \quad \text{as } T \to \infty. \quad (2.6)
\]

Combining (2.5) with (2.6) and using Lemma 2.3, we obtain the desired estimate. \(\square\)

2.2. Estimates related to problem (1.2). Given \(0 < T < \infty\), we introduce the function

\[
\mathcal{N}(t, x) = \vartheta(t) \Xi(x), \quad (t, x) \in [0, \infty) \times \overline{D}^c.
\]

One can easily show that

\[
\mathcal{N} \in \Phi_N, \quad T \gg 1.
\]

Lemma 2.8. We have

\[
\int_{D^c} \Xi(x) \, dx = O \left( T^{Nr} \right), \quad \text{as } T \to \infty.
\]
Proof. Let $T \gg 1$. We have

$$
\int_{D^c} \Xi(x) \, dx = \int_{|x|>1} \left[ \xi \left( \frac{|x|^2}{T^2r} \right) \right] \omega \, dx.
$$

Using the change of variable $x = T^r y$, $x \in D^c$, we get

$$
\int_{D^c} \Xi(x) \, dx = T^{Nr} \int_{T^{-r}<|y|<\sqrt{2}} \left[ \xi(|y|^2) \right] \omega \, dy \leq T^{Nr} \int_{0<|y|<\sqrt{2}} \left[ \xi(|y|^2) \right] \omega \, dy = CT^{Nr},
$$

which yields the desired estimate. \hfill \Box

Lemma 2.9. Let $m > 1$. There holds

$$
\int_{D^c} \left| \Xi(x) \right|^{-\frac{1}{m}} \left| \Delta \Xi \right| \frac{m}{m-1} \, dx = O \left( T^{Nr - \frac{m}{m-1}} \right), \quad \text{as } T \to \infty.
$$

Proof. We have

$$
\int_{D^c} \left| \Xi(x) \right|^{-\frac{1}{m}} \left| \Delta \Xi \right| \frac{m}{m-1} \, dx = \int_{D^c} \left[ \xi \left( \frac{|x|^2}{T^2r} \right) \right] \frac{m}{m-1} \left| \Delta \right| \frac{m}{m-1} \, dx.
$$

Using the change of variable $x = T^r y$, $x \in D^c$, we get

$$
\int_{D^c} \left| \Xi(x) \right|^{-\frac{1}{m}} \left| \Delta \Xi \right| \frac{m}{m-1} \, dx = T^{Nr - \frac{m}{m-1}} \int_{1<|y|<\sqrt{2}} \left[ \xi(|y|^2) \right] \frac{m}{m-1} \left| \Delta \right| \frac{m}{m-1} \, dy. \quad (2.7)
$$

On the other hand,

$$
\Delta \left[ \xi(|y|^2) \right] \omega = \left[ \xi(|y|^2) \right] \omega - 2 \eta(y), \quad 1 < |y| < \sqrt{2},
$$

where

$$
\eta(y) = \omega \left( \omega - 1 \right) \left| \nabla \Psi(|y|^2) \right|^2 + \Psi(|y|^2) \Delta \Psi(|y|^2). \quad (2.8)
$$

Hence,

$$
\int_{1<|y|<\sqrt{2}} \left[ \xi(|y|^2) \right] \frac{m}{m-1} \left| \Delta \right| \frac{m}{m-1} \, dy = \int_{1<|y|<\sqrt{2}} \left[ \xi(|y|^2) \right] \omega - \frac{2m}{m-1} \eta(y) \frac{m}{m-1} \, dy < \infty.
$$

Combining (2.7) with (2.8), the desired estimate follows. \hfill \Box

Lemma 2.10. Let $-1 < \beta < 1$ and $m > 1$. There holds

$$
\int_{Q_T} g(t)|N|^{-\frac{1}{m+1}} \left| \partial_t N \right| \frac{m}{m-1} \, dx \, dt = O \left( T^{\beta - \frac{m+1}{m-1} + Nr} \right), \quad \text{as } T \to \infty.
$$

Proof. By the definition of the function $N$, we have

$$
\int_{Q_T} g(t)|N|^{-\frac{1}{m+1}} \left| \partial_t N \right| \frac{m}{m-1} \, dx \, dt = \left( \int_0^{2T} g(t) \left| \partial \varphi(t) \right| \frac{m}{m-1} \left| \partial^\prime \varphi(t) \right| \frac{m}{m-1} \, dt \right) \left( \int_{D^c} \Xi(x) \, dx \right).
$$

Using (2.2) and Lemma 2.8, the desired estimate follows. \hfill \Box

Lemma 2.11. Let $-1 < \beta < 1$ and $m > 1$. There holds

$$
\int_{Q_T} [g(t)]^{-\frac{1}{m+1}} N^{-\frac{1}{m+1}} \left| \partial_t N \right| \frac{m}{m-1} \, dx \, dt = O \left( T^{N r - \frac{\beta+1}{m-1}} \right), \quad \text{as } T \to \infty.
$$
Proof. We have
\[
\int_{Q_T} [g(t)]^{\frac{1}{\beta-1}} N^{\frac{1}{\beta-1}} |\partial_t N|^{\frac{m}{\beta}} dx dt = \left( \int_0^{2T} [g(t)]^{\frac{1}{\beta-1}} [\vartheta(t)]^{\frac{1}{\beta-1}} |\vartheta'(t)|^{\frac{m}{\beta}} dt \right) \left( \int_{D^c} \Xi(x) dx \right).
\]
Using (2.4) and Lemma 2.8, the desired estimate follows.

Using a similar argument as that in the proof of Lemma 2.6, we obtain the following estimate.

Lemma 2.12. Let \(-1 < \beta < 1\) and \(m > 1\). There holds
\[
\int_{Q_T} [g(t)]^{\frac{1}{\beta-1}} |g'(t)|^{\frac{m}{\beta}} N^{\frac{1}{\beta-1}} |\partial_t N|^{\frac{m}{\beta}} dx dt = O \left( T^{N-\beta+1} \right), \quad \text{as } T \to \infty.
\]

Lemma 2.13. Let \(-1 < \beta < 1\) and \(m > 1\). There holds
\[
\int_{Q_T} g(t)N^{\frac{1}{\beta-1}} |\Delta N|^{\frac{m}{\beta}} dx dt = O \left( T^{\beta+1+r(N-\frac{2m}{\beta})} \right), \quad \text{as } T \to \infty.
\]

Proof. We have
\[
\int_{Q_T} g(t)N^{\frac{1}{\beta-1}} |\Delta N|^{\frac{m}{\beta}} dx dt = \left( \int_0^{2T} g(t) \vartheta(t) dt \right) \left( \int_{D^c} [\Xi(x)]^{\frac{1}{\beta-1}} |\Delta \Xi|^{\frac{m}{\beta}} dx \right).
\]
Using (2.6) and Lemma 2.9, the desired estimate follows.

3. The case \(b \equiv 0\). In this section, problems (1.1)–(1.2) are investigated in the case \(b \equiv 0\). Namely, we prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. The proof is based on a rescaled test-function argument (see [7] for a general account of these methods). Suppose that \(u \in L^p_{loc}((0,\infty) \times \overline{D})\) is a global weak solution to problem (1.1). Taking \(\varphi = D\) in (1.7), one obtains
\[
\int_{(0,\infty) \times D^c} g(t)|u|^p D dx dt + \int_{D^c} (u_0(x)+\ell u_1(x)) D(0,x) dx - \ell \int_{D^c} u_0(x) \partial_t D(0,x) dx
\]
\[
\leq \int_{(0,\infty) \times D^c} g(t)|u| |\partial_t D| dx dt + \int_{(0,\infty) \times D^c} |u||\partial_t D| dx dt
\]
\[
+ \int_{(0,\infty) \times D^c} |g'(t)||u||\partial_t D| dx dt + \int_{(0,\infty) \times D^c} g(t)|u||\Delta D| dx dt.
\]
Since \(D(t,\cdot) \equiv 0, t \geq 2T,\) and \(\partial_t D(0,\cdot) \equiv 0,\) one deduces that
\[
\int_{Q_T} g(t)|u|^p D dx dt + \int_{D^c} (u_0(x)+\ell u_1(x)) D(0,x) dx
\]
\[
\leq \int_{Q_T} g(t)|u| |\partial_t D| dx dt + \int_{Q_T} |u||\partial_t D| dx dt
\]
\[
+ \int_{Q_T} |g'(t)||u||\partial_t D| dx dt + \int_{Q_T} g(t)|u||\Delta D| dx dt. \quad (3.1)
\]
Using the \(\varepsilon\)-Young inequality, \(\varepsilon > 0,\) one gets
\[
\int_{Q_T} g(t)|u||\partial_t D| dx dt \leq \varepsilon \int_{Q_T} g(t)|u|^p D dx dt + C \int_{Q_T} g(t) D^{\frac{1}{\beta-1}} |\partial_t D|^{\frac{m}{\beta}} dx dt. \quad (3.2)
\]
Similarly, one obtains
\[
\int_{Q_T} |u| |\partial_t D| dx dt \leq \varepsilon \int_{Q_T} g(t) |u|^p D dx dt + C \int_{Q_T} |g(t)| \Delta D^{\frac{1}{m}} |\partial_t D|^{\frac{p}{m}} dx dt,
\]
(3.3)
\[
\int_{Q_T} |g'(t)| |u| |\partial_t D| dx dt \leq \varepsilon \int_{Q_T} g(t) |u|^p D dx dt + C \int_{Q_T} |g(t)| \Delta D^{\frac{1}{m}} |\partial_t D|^{\frac{p}{m}} dx dt
\]
and
\[
\int_{Q_T} g(t) |u| |\Delta D| dx dt \leq \varepsilon \int_{Q_T} g(t) |u|^p D dx dt + C \int_{Q_T} |g(t)| \Delta D^{\frac{1}{m}} |\partial_t D|^{\frac{p}{m}} dx dt.
\]
(3.5)

Next, using (3.1)–(3.5) and taking \( \varepsilon = \frac{1}{2} \), we get
\[
\int_{D^c} (u_0(x) + \ell u_1(x)) D(0,x) dx \leq C \left( \int_{Q_T} g(t) |u|^p D dx dt + \int_{Q_T} |g(t)| \Delta D^{\frac{1}{m}} |\partial_t D|^{\frac{p}{m}} dx dt \right) + \frac{1}{2} \left( \int_{Q_T} g(t) |u|^p D dx dt + \int_{Q_T} |g(t)| \Delta D^{\frac{1}{m}} |\partial_t D|^{\frac{p}{m}} dx dt \right)
\]
(3.6)
\[
:= C (I_1(T) + I_2(T) + I_3(T) + I_4(T)).
\]

Further, using Lemma 2.4 with \( m = p \), we obtain
\[
I_1 = O \left( T^{\beta - \frac{p+1}{p-1} + N r} (\ln T)^{\delta_N} \right), \quad \text{as } T \to \infty,
\]
(3.7)
where
\[
\delta_N = \begin{cases} 1 & \text{if } N = 2, \\ 0 & \text{if } N \geq 3. 
\end{cases}
\]

Using Lemmas 2.5–2.6, for \( i = 2, 3 \), there holds
\[
I_i = O \left( T^{N r - \frac{p+1}{p-1}} (\ln T)^{\delta_N} \right), \quad \text{as } T \to \infty.
\]
(3.8)

Lemma 2.7 with \( m = p \) yields
\[
I_4 = O \left( T^{\beta+1+r(N-\frac{2p}{p-1})} (\ln T)^{\delta_N} \right), \quad \text{as } T \to \infty.
\]
(3.9)

Hence, combining (3.6)–(3.9), for \( T \gg 1 \), one obtains
\[
\int_{D^c} (u_0(x) + \ell u_1(x)) D(0,x) dx \leq C \left( T^{\beta+1+r(N-\frac{2p}{p-1})} + T^{N r - \frac{p+1}{p-1} + T^{\beta+1+r(N-\frac{2p}{p-1})}} \right) (\ln T)^{\delta_N}.
\]

Observe that (since \( \beta < 1 \))
\[
\beta - \frac{p+1}{p-1} < - \frac{\beta+1}{p-1}.
\]

On the other hand, for \( r = \frac{\beta+1}{2} > 0 \) (since \( \beta > -1 \)), one has
\[
N r - \frac{\beta+1}{p-1} = \beta + 1 + r \left( N - \frac{2p}{p-1} \right) = (\beta+1) \left( \frac{N}{2} - \frac{1}{p-1} \right).
\]

Hence, with such \( r \), one obtains
\[
\int_{D^c} (u_0(x) + \ell u_1(x)) D(0,x) dx \leq CT^{(\beta+1)\left( \frac{N}{2} - \frac{1}{p-1} \right)} (\ln T)^{\delta_N}, \quad T \gg 1,
\]
that is,
\[
\int_{D^c} (u_0(x) + \ell u_1(x)) \left[ \xi \left( \frac{|x|^2}{T^2r} \right)^\omega \right] F(x) \, dx \leq CT^{(\beta + 1)}(\frac{N}{2} - \frac{p}{p-\ell})(\ln T)^{\beta N}, \quad T \gg 1.
\]
Passing to the limit as \( T \to \infty \) in the above inequality, using (1.10), the dominated convergence theorem and (1.11), we obtain
\[
0 < \int_{D^c} (u_0(x) + \ell u_1(x)) F(x) \, dx \leq 0,
\]
which is a contradiction. This completes the proof of Theorem 1.3. \( \square \)

Now, we prove Theorem 1.4.

**Proof of Theorem 1.4.** Suppose that \( u \in L^p_{\text{loc}}([0, \infty) \times D^c) \) is a global weak solution to problem (1.2). Taking \( \varphi = N \) in (1.8), one obtains
\[
\int_{Q_T} g(t)|u|^pN \, dx \, dt + \int_{D^c} (u_0(x) + \ell u_1(x))N(0, x) \, dx
\leq \int_{Q_T} g(t)|u|\partial_t N \, dx \, dt + \int_{Q_T} |u|\partial_t N \, dx \, dt + \int_{Q_T} g'(t)|u|\partial_t N \, dx \, dt
+ \int_{Q_T} g(t)|u|\Delta N \, dx \, dt.
\]
Following the same argument used in the proof of Theorem 1.3, one deduces that
\[
\int_{D^c} (u_0(x) + \ell u_1(x))N(0, x) \, dx
\leq C \left( \int_{Q_T} g(t)N^{\frac{p}{p-\ell}}|\partial_t N|^{\frac{p}{p-\ell}} \, dx \, dt + \int_{Q_T} |g(t)|^{\frac{p}{p-\ell}}N^{\frac{1}{p-\ell}}|\partial_t N|^{\frac{p}{p-\ell}} \, dx \, dt
+ \int_{Q_T} g'(t)|N^{\frac{1}{p-\ell}}|g(t)|^{\frac{p}{p-\ell}}N^{\frac{1}{p-\ell}}|\partial_t N|^{\frac{p}{p-\ell}} \, dx \, dt + \int_{Q_T} g(t)|N^{\frac{1}{p-\ell}}|\Delta N|^{\frac{p}{p-\ell}} \, dx \, dt \right)
:= C(J_1(T) + J_2(T) + J_3(T) + J_4(T)). \tag{3.10}
\]
Further, using Lemmas 2.10–2.13 with \( m = p \), one obtains
\[
J_1 = O \left( T^{\beta + \frac{p}{p-\ell} - N} \right), \quad \text{as} \ T \to \infty, \tag{3.11}
J_i = O \left( T^{N r_i - \frac{\beta + 1}{p-\ell}} \right), \quad \text{as} \ T \to \infty, \quad i = 2, 3, \tag{3.12}
J_4 = O \left( T^{\beta + 1 + \frac{p}{p-\ell} - N} \right), \quad \text{as} \ T \to \infty. \tag{3.13}
\]
Next, using (3.10)–(3.13) and taking \( r = \frac{\beta + 1}{2} \), there holds
\[
\int_{D^c} (u_0(x) + \ell u_1(x)) \left[ \xi \left( \frac{|x|^2}{T^2r} \right)^\omega \right] \, dx \leq CT^{(\beta + 1)}(\frac{N}{2} - \frac{p}{p-\ell}), \quad T \gg 1.
\]
Passing to the limit as \( T \to \infty \) in the above inequality, using (1.12)–(1.13), we obtain a contradiction. This completes the proof of Theorem 1.4. \( \square \)
4. The case $b \neq 0$. In this section, problems (1.1)–(1.2) are investigated in the case $b \neq 0$. Namely, we prove Theorems 1.5–1.7.

Proof of Theorem 1.5. (I) Suppose that $u \in L^p_{\text{loc}}([0, \infty) \times \overline{\mathcal{D}})$ is a global weak solution to problem (1.1). Taking $\varphi = \mathcal{D}$ in (1.7), one obtains

$$\int_{Q_T} g(t)|u|^p \mathcal{D} dx dt + \int_{\mathcal{D}^T} (u_0(x) + \ell u_1(x)) \mathcal{D}(0, x) dx - \int_{T} \frac{\partial \mathcal{D}}{\partial n^+} g(t)b(t)f(x) dS_t dt$$

$$\leq \int_{Q_T} g(t)|u| \partial_t \mathcal{D} dx dt + \int_{Q_T} |u||\partial_t \mathcal{D}| dx dt + \int_{Q_T} |g'(t)||u||\partial_t \mathcal{D}| dx dt$$

$$+ \int_{Q_T} g(t)|u| \Delta \mathcal{D} dx dt. \quad (4.1)$$

On the other hand, one has

$$\frac{\partial \mathcal{D}}{\partial n^+}(t, x) = \begin{cases} -\varphi(t) & \text{if } N = 2, \\ (2 - N)\varphi(t) & \text{if } N \geq 3. \end{cases} \quad (4.2)$$

Hence, using Lemma 2.1 and (4.2), there holds

$$- \int_{T} \frac{\partial \mathcal{D}}{\partial n^+} g(t)b(t)f(x) dS_t dt = C \left( \int_0^{2T} \varphi(t)g(t)b(t) dt \right) \left( \int_{\partial \mathcal{D}} f(x) dS_x \right)$$

$$= C \left( \int_0^{2T} \left[ \xi \left( \frac{t}{T} \right) \right]^{\omega} g(t)b(t) dt \right) \left( \int_{\partial \mathcal{D}} f(x) dS_x \right)$$

$$\geq C \left( \int_0^T \left[ \xi \left( \frac{t}{T} \right) \right]^{\omega} g(t)b(t) dt \right) \left( \int_{\partial \mathcal{D}} f(x) dS_x \right) \quad (4.3)$$

Combining (4.1) with (4.3) and using the fact that $u_i \geq 0$, $i = 0, 1$, and $\mathcal{D} \geq 0$, one deduces that

$$\int_{Q_T} g(t)|u|^p \mathcal{D} dx dt + \int_0^T (1 + t)^{\beta} b(t) dt$$

$$\leq \int_{Q_T} g(t)|u| \partial_t \mathcal{D} dx dt + \int_{Q_T} |u||\partial_t \mathcal{D}| dx dt + \int_{Q_T} |g'(t)||u||\partial_t \mathcal{D}| dx dt$$

$$+ \int_{Q_T} g(t)|u| \Delta \mathcal{D} dx dt.$$

Next, proceeding as in the proof of Theorem 1.3, there holds

$$\int_0^T (1 + t)^{\beta} b(t) dt \leq C T^{\beta+1}(\frac{2}{\beta+1} - \frac{2}{\beta}) (\ln T)^{\delta_1}, \quad T \gg 1, \quad (4.4)$$

which yields for $N = 2$,

$$\left( \int_0^T (1 + t)^{\beta} b(t) dt \right) T^{(\beta+1)(\frac{2}{\beta+1} - \frac{2}{\beta})} (\ln T)^{-1} \leq C, \quad T \gg 1.$$

Passing to the supremum limit as $T \to \infty$ in the above inequality and using (1.14), a contradiction follows.
(II) For $N \geq 3$, using (4.4), one obtains
\[
\left( \int_0^T (1 + t)^\beta \, b(t) \, dt \right) T^{-(\beta+1)\left(\frac{2}{p} - \frac{1}{p-1}\right)} \leq C, \quad T \gg 1.
\] (4.5)
Passing to the supremum limit as $T \to \infty$ in the above inequality and using (1.15), we reach a contradiction. This completes the proof of Theorem 1.5.

Now, we prove Theorem 1.6.

Proof of Theorem 1.6. The proof is similar to that of Theorem 1.5. Namely, if $u \in L^p_{loc}([0, \infty) \times \mathbb{D}^c)$ is a global weak solution to problem (1.2), taking $\varphi = N$ in (1.8), one obtains
\[
\int_{Q_T} g(t)|u|^pN \, dx \, dt + C \int_0^T (1 + t)^\beta \, b(t) \, dt \\
\leq \int_{Q_T} g(t)|u||\partial_t N| \, dx \, dt + \int_{Q_T} |u||\partial_t N| \, dx \, dt + \int_{Q_T} |g'(t)||u||\partial_t N| \, dx \, dt \\
+ \int_{Q_T} g(t)|u||\Delta N| \, dx \, dt,
\]
which yields (4.5) that contradicts (1.15).}

Finally, we prove Theorem 1.7.

Proof of Theorem 1.7. (I) Let $N = 2$ and
\[
m = \text{ess inf } b > 0.
\]
For $T > 0$, one has
\[
\left( \int_0^T (1 + t)^\beta \, b(t) \, dt \right) T^{\frac{(\beta+1)(2-p)}{p-1}}(\ln T)^{-1} \\
\geq \frac{m}{\beta + 1} \left( (1 + T)^{\beta+1} - 1 \right) T^{\frac{(\beta+1)(2-p)}{p-1}}(\ln T)^{-1} \]
\[
= \frac{m}{\beta + 1} \left( \left( \frac{1}{T} + 1 \right)^{\beta+1} - \frac{1}{T^{\beta+1}} \right) T^{\frac{\beta+1}{p-1}}(\ln T)^{-1}.
\] (4.6)
Since $\frac{\beta+1}{p-1} > 0$ for all $p > 1$, there holds
\[
\lim_{T \to \infty} \left( \int_0^T (1 + t)^\beta \, b(t) \, dt \right) T^{\frac{(\beta+1)(2-p)}{p-1}}(\ln T)^{-1} = +\infty.
\]
Hence, using part (I) of Theorem 1.5, one deduces that, for all $p > 1$, problem (1.1) admits no global weak solutions. Similarly, for all $p > 1$, using (4.6), one has
\[
\lim_{T \to \infty} \left( \int_0^T (1 + t)^\beta \, b(t) \, dt \right) T^{-(\beta+1)\left(\frac{2}{p} - \frac{1}{p-1}\right)} \\
= \lim_{T \to \infty} \left( \int_0^T (1 + t)^\beta \, b(t) \, dt \right) T^{\frac{(\beta+1)(2-p)}{p-1}} = +\infty.
\]
Hence, using Theorem 1.6, one deduces that, for all $p > 1$, problem (1.2) admits no global weak solutions.
(II) Let $N \geq 3$. For $T > 0$, one has
\[
\left( \int_0^T (1 + t)^\beta b(t) \, dt \right) T^{-(\beta + 1)} \left( \frac{2}{p} - \frac{1}{p+\tau} \right) \geq \frac{m}{\beta + 1} T^{(\beta + 1)} \left( \frac{2}{p+\tau} - \frac{\tau}{2} \right).
\]
Hence, if
\[
p < 1 + \frac{2}{N-2},
\]
on one obtains
\[
\lim_{T \to \infty} \left( \int_0^T (1 + t)^\beta b(t) \, dt \right) T^{-(\beta + 1)} \left( \frac{2}{p} - \frac{1}{p+\tau} \right) = +\infty.
\]
Therefore, using part (II) of Theorem 1.5 and Theorem 1.6, one deduces that, for all $p < 1 + \frac{2}{N-2}$, problems (1.1)–(1.2) admit no global weak solutions.

(III) Let $N \geq 3$, $\mu > 0$, $-1 < \beta < 1$ and
\[
p > 1 + \frac{2}{N-2}.
\]
We take
\[
u(t, x) = \varepsilon |x|^\delta, \quad (t, x) \in [0, \infty) \times D^c,
\]
where
\[
\varepsilon = \left[ \frac{2 (\beta (N - 2) p - N)}{(p - 1)^2} \right]^{\frac{1}{p+\tau}} \quad \text{and} \quad \delta = -\frac{2}{p-1}.
\]
Then, for all $(t, x) \in (0, \infty) \times D^c$, one obtains
\[
\partial_t u - \Delta u + a(t) \partial_x u = -\varepsilon \Delta (|x|^\delta) = -\varepsilon \Delta (N + \delta - 2) |x|^\delta - 2
\]
\[
= \left[ \frac{2 ((N - 2) p - N)}{(p - 1)^2} \right]^{\frac{1}{p+\tau}} \left[ \frac{2 ((N - 2) p - N)}{(p - 1)^2} \right] |x|^{\frac{2p}{p+\tau}}
\]
\[
= \left[ \frac{2 ((N - 2) p - N)}{(p - 1)^2} \right]^{\frac{1}{p-1}} |x|^{\frac{2p}{p-1}} = \varepsilon^p |x|^\delta = |u|^p.
\]
On the other hand, for $(t, x) \in (0, \infty) \times \partial D$, one has
\[
u(t, x) = \varepsilon \quad \text{and} \quad \frac{\partial u}{\partial n^+}(t, x) = -\varepsilon \delta.
\]
Hence, $u$ solves problem (1.1) with $(u_0(x), u_1(x)) = (\varepsilon |x|^\delta, 0)$, $f \equiv \varepsilon$ and $b \equiv 1 \in B$. Moreover, $u$ solves problem (1.2) with $(u_0(x), u_1(x)) = (\varepsilon |x|^\delta, 0)$, $f \equiv -\varepsilon \delta$ and $b \equiv 1$. This completes the proof of Theorem 1.7. \hfill \Box

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Received July 2019; revised December 2019.

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