Existence of Wormholes in Einstein-Kalb-Ramond space time

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Abstract

In recent, Kar.S et.al [ Phys Rev D 67,044005 (2003) ] have obtained static spherically symmetric solutions of the Einstein-Kalb-Ramond field equations. We have shown that their solutions, indeed, represent Wormholes.

In Einstein-Cartan theory, the symmetric Christoffel connection is modified with the introduction of an anti-symmetric tensorial term, known as space time torsion, which is presumed to have a direct relation with spin [1]. It has been shown that the massless anti-symmetric tensor Kalb-Ramond field, \( B_{\mu\nu} \) equivalent to torsion is an inherent feature in the low energy effective string action. Several authors [2] have shown that the presence of Kalb-Ramond field in the background space time may lead to various interesting astrophysical and cosmological phenomena such as cosmic optical activity, neutrino lecility flip, parity violation etc. Recently, Kar et.al [3] have carried out the most general study of the existence carried out of possible spherical symmetric solutions of the Einstein-Kalb-Ramond field equations. They have studied gravitational lensing and perihelion precession in these space-time. They have also shown that for a special case, one can get wormhole for a real Kalb-Ramond field. In this article, we have shown that their general solutions, indeed, always represent wormholes. According to the formation in [3], the action is given by

\[
S = \int d^4 x \sqrt{-g} \left[ \frac{\mathcal{R}(g)}{k} - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right]
\]  

(1)

where \( \mathcal{R}(g) \) is Ricci scalar curvature and \( H_{\mu\nu\lambda} \) is Kalb-Ramond field strength and \( k = 8\pi G \).

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The field equations are
\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = kT_{\mu\nu} \] (2)
\[ D_{\mu}H^{\mu\nu\lambda} = \frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-g}H^{\mu\nu\lambda}) = 0 \] (3)

\( T_{\mu\nu} \) is a symmetric two tensor which is analogous to the energy momentum tensor and is given by
\[ T_{\mu\nu} = \frac{1}{4}(3g_{\nu\rho}H_{\alpha\beta\mu}H^{\alpha\beta\rho} - \frac{1}{2}g_{\mu\nu}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}) \] (4)

The asymmetric property of \( H_{\mu\nu\lambda} \) implies it has only four independent components. According to Sengupta and Sur [4], the only nonzero component is \( H_{023} \). We denote \( H_{023} = [h(r)]^2 \).

Kar et al. [3] have considered the general spherical symmetric metric structure as
\[ ds^2 = B(r)dt^2 - \frac{dr^2}{A(r)} - r^2d\Omega^2 \] (5)

From the gravitational field equations, they have obtained the solutions as [3]
\[ B(r) = 1 + \frac{c_1}{r} + \frac{bc_1}{6r^3} - \frac{b^2c_1}{6r^4} + \frac{6bc_1^3 + 3b^2c_1}{40r^5} + \ldots \] (6)
\[ A(r) = 1 + \frac{c_1}{r} - \frac{b}{r^2} - \frac{bc_1}{2r^3} + \frac{b^2c_1}{3r^4} + \frac{1}{4r^5}(bc_1^3 + \frac{b^2c_1}{6}) + \ldots \] (7)
\[ h(r) = \sqrt{\frac{\bar{k}}{k}r^2}[1 - \frac{c_1}{r} + \frac{c_1^2}{r^2} - \frac{1}{r^3}(c_1^3 + \frac{bc_1}{6}) + \ldots] \] (8)

where \( \bar{k} = \frac{3k}{4} \) and \( b, c_1 \) are arbitrary constants.

They have noticed that if \( c_1 = 0 \), then \( B(r) = 1, A(r) = 1 - \frac{b}{r^2} \) which represent a wormhole.

Now, we shall show that general solutions (6) and (7) always represent wormholes. To show this we rewrite the metric into the Morris-Thorne Canonical form [5]
\[ ds^2 = e^{2f(r)}dt^2 - \frac{1}{[1 - b(r)/r]}dr^2 - r^2d\Omega^2 \] (9)

where, \( r \in (-\infty, +\infty) \).

To represent a wormhole, one must impose the following conditions on the metric (9) as [6]:
1) The redshift function, \( f(r) \) must be finite for all values of \( r \). This means no horizon exists in the space time.
2) The shape function, \( b(r) \) must obey the following conditions at the throat \( r = r_0 \) :
\( b(r_0) = r_0 \) and \( b'(r_0) < 1 \) [ these are known as Flair-out conditions ].

3) \( \frac{b(r)}{r} < 1 \) for \( r > r_0 \) i.e. out of throat.

4) The space time is asymptotically flat i.e. \( \frac{b(r)}{r} \to 0 \) as \( |r| \to \infty \).

Here, the red-shift function and shape function take the form

\[
2f = \ln[1 + \frac{c_1}{r} + \frac{bc_1}{6r^3} - \ldots]
\]

\[
b(r) = -c_1 + \frac{b}{r} - \frac{bc_1}{2r^2} + \frac{bc_1^2}{3r^3} - \ldots
\]

The throat of the wormhole occurs at \( r = r_0 \) where \( r_0 \) satisfies the equation \( b(r) = r \).

Suppose \( \frac{1}{r} = y \), then \( b(r) = r \) implies

\[
g(y) = -c_1y + by^2 - \frac{bc_1}{2}y^3 + \frac{bc_1^2}{3}y^4 - \ldots - 1 = 0
\]

Since \( \frac{1}{r_0} \) is a root of equation (12), then by standard theorem of algebra, either \( g(y) > 0 \) for \( y > \frac{1}{r_0} \) and \( g(y) < 0 \) for \( y < \frac{1}{r_0} \) or \( g(y) < 0 \) for \( y > \frac{1}{r_0} \) and \( g(y) > 0 \) for \( y < \frac{1}{r_0} \). Let us take the first possibility and one can note that for \( y = \frac{1}{r} < \frac{1}{r_0} \) i.e. \( r > r_0 \), \( g(y) < 0 \), in other words, \( b(r) < r \). But when \( y = \frac{1}{r} > \frac{1}{r_0} \) i.e. \( r < r_0 \), \( g(y) > 0 \), this means, \( b(r) > r \), which violates the wormhole structure given in equation (9).

Here \( e^{2f(r)} \equiv B(r) \) has no zero for \( r \geq r_0 > 0 \) (as one can not assume \( r < r_0 \)) because (1) at \( r \to \infty \), \( B \to 1 \), so \( e^{2f(r)} \neq 0 \) at \( r \to \infty \) (2) if \( B(r) = 0 \) at \( r = r_0 \), then \( B(r_o) - A(r_0) = 0 \) i.e.

\[
\frac{b}{r_0^2} - \frac{bc_1}{3r_0^3} + \frac{bc_1^2}{6r_0^4} + \ldots = 0
\]

But, \( A(r_0) \equiv [1 - \frac{b(r_0)}{r_0}] = 0 \) implies

\[
1 + \frac{c_1}{r_0} - \frac{b}{r_0^2} + \frac{bc_1}{2r_0^3} - \frac{bc_1^2}{3r_0^4} + \frac{1}{4r_0^5}(bc_1^3 + \frac{b^2c_1}{6})\ldots = 0
\]

One can note that the above two equations could not hold simultaneously.

Hence \( B(r_0) \neq 0 \). In other words, no horizon exists in the space time. Also, \( \frac{b(r)}{r} \to 0 \) as \( |r| \to \infty \). Thus the space-time with the solutions (6) and (7) describes static spherically symmetric wormholes.
Retaining a few terms, the shape of the wormhole takes the form:

\[ 0.4 \quad 0.5 \quad 0.6 \quad 0.7 \quad 0.8 \quad 0.9 \]

\[ 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \ r \]

Figure 1: Shape of the wormhole for \( c_1 = -1 \) and \( b = -1 \)

The asymptotical wormhole mass reads

\[ M = \lim_{r \to \infty} \frac{1}{2} b(r) = -\frac{1}{2} c_1 \] \hspace{1cm} (15)

The axially symmetric embedded surface \( z = z(r) \) shaping the Wormhole's spatial geometry is a solution of

\[ \frac{dz}{dr} = \pm \frac{1}{\sqrt{r b(r)} - 1} \] \hspace{1cm} (16)

By the definition of Wormhole, we can note that at the value \( r = r_0 \) (the wormhole throat radius) equation (16) is divergent i.e. embedded surface is vertical there. According to Morris and Thorne [5], the 'r' co-ordinate is ill-behaved near the throat, but proper radial distance

\[ l(r) = \pm \int_{r_0}^{r} \frac{dr}{\sqrt{1 - \frac{b(r)}{r}}} \] \hspace{1cm} (17)

must be well behaved everywhere i.e. we must require that \( l(r) \) is finite throughout the space-time.
For this Model,

\[ l(r) = \pm \int_{r_0}^{r} \frac{dr}{\sqrt{1 - \frac{1}{r^2}\left[-c_1 \frac{b}{r^2} - \frac{bc_1}{3r^3} + \ldots\right]}} \] (18)

Though it is not possible to get the explicit form of the integral but one can see that the above integral is a convergent integral i.e. proper length should be finite.

To summarize, we have shown that the static spherically symmetric solutions of the Kalb-Ramond field equations obtained by Kar et.al always represent Wormholes.

According to Morris-Thorne [5], to keep a wormhole open, the stress energy tensor of matter violates the null energy conditions. As a result, the energy density of matter may be seen as negative by some observer. Since maximum Kalb-Ramond energy density is negative [3], it is clear that one can always construct wormhole supported by Kalb-Ramond field. Finally, if we take the parameter \( c_1 < 0 \), then asymptotic mass \( M' \) of the Kalb-Ramond wormhole is positive i.e. a distant observer could not see any difference of gravitational nature between Wormhole and a compact mass 'M'.

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