A REMARK ON THE RIGIDITY OF THE FIRST CONFORMAL STEKLOV EIGENVALUE

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ABSTRACT. We show rigidity of the first conformal Steklov eigenvalue on annuli and Möbius bands. The proof relies, among others, on uniqueness results due to Fraser–Schoen, a compactness theorem of the second named author, and recent work of the authors on asymptotic control of Steklov eigenvalues in gluing constructions.

For a compact and connected surface $\Sigma$ with smooth, non-empty boundary that is endowed with a smooth metric $g$, we denote by $\sigma_1(\Sigma, g)$ the first positive Steklov eigenvalue. This is the smallest non-zero eigenvalue of the Dirichlet-to-Neumann operator given by $T u = \partial_\nu \hat{u}$ for $u \in C^\infty(\partial \Sigma)$, where $\hat{u} \in C^\infty(\Sigma)$ denotes the harmonic extension of $u$ to all of $\Sigma$, and $\nu$ is the outward-pointing normal vector field along $\partial \Sigma$. We also write $L_g(\partial \Sigma)$ for the length of the boundary of $\Sigma$.

We show the following rigidity result for the first Steklov eigenvalue in a conformal class on annuli and Möbius bands.

**Theorem 1.** Let $\Sigma$ be an annulus or a Möbius band endowed with a flat metric $g$ then there is $\omega: \Sigma \to \mathbb{R}$ such that

$$\sigma_1(\Sigma, e^{2\omega} g) L_{e^{2\omega} g}(\partial \Sigma) > 2\pi.$$  

Note that

$$\sigma_1(\mathbb{D}^2, \xi) L_{\xi}(\partial \mathbb{D}^2) = 2\pi,$$

where $\xi$ is the standard metric. Moreover, the metric $\xi$ maximizes the first Steklov eigenvalue among all metrics on the flat disk thanks to Weinstock’s inequality, [Wei54].

It is conjectured that Theorem 1 holds for any $\Sigma$ not diffeomorphic to the disk. We remark that the analogous result for the first Laplace eigenvalue on closed surfaces is known by work of the second author [Pet14], see also [Pet15] for the higher dimensional case.

Recall that $\Phi: \Sigma \to \mathbb{B}^N$ with $\Phi(\partial \Sigma) \subset \partial \mathbb{B}^N$ is a free boundary harmonic map if $\Delta \Phi = 0$ in $\Sigma \setminus \partial \Sigma$ and $\partial_\nu \Phi$ is parallel to $\Phi$ along $\partial \Sigma$. Variationally these arise as critical points of the Dirichlet energy under the constraint of mapping $\partial \Sigma$ to $\partial \mathbb{B}^N$. Such free boundary harmonic maps also appear in the theory of maximizing Steklov eigenvalues on surfaces in a fixed conformal class [FS16, Pet19]. As a consequence of the work in [Pet19] and Theorem 1 we also obtain

**Theorem 3.** Let $\Sigma$ be an annulus or a Möbius band endowed with a flat metric $g$ then there there is a smooth function $\omega: \Sigma \to \mathbb{R}$ such that

$$\sigma_1(\Sigma, e^{2\omega} g) L_{e^{2\omega} g}(\partial \Sigma) \geq \sigma_1(\Sigma, e^{2\tau} g) L_{e^{2\tau} g}(\partial \Sigma)$$
for any smooth $\tau: \Sigma \to \mathbb{R}$. In particular, there is a free boundary harmonic map $\Phi: (\Sigma, g) \to \mathbb{B}^N$ by first eigenfunctions. Here, $N \leq 3$ if $\Sigma$ is an annulus and $N \leq 4$ if $\Sigma$ is a Möbius band.

The bounds on $N$ follow from \[KKP14\] Corollary 1.4.1, see also \[FS16\] Theorem 2.3 and Theorem 2.4.

We remark that it follows from our work \[MP20\] that for any compact $\Sigma$ with smooth non-empty boundary, that is not diffeomorphic to a disk, there is an entire family of specifically degenerating conformal classes for which the analogue of Theorem 1 holds, see Theorem 1\] below for the corresponding version for annuli. In the case of annuli and Möbius bands the strong rigidity results by Fraser–Schoen allow us to propagate this to all conformal classes. These rigidity results, stated in Theorem 5\] below, do not hold for surfaces of larger topological complexity. In fact, it is known that for sufficiently large number of boundary components $k$ and genus zero there are at least two non-isometric free boundary minimal surfaces in $\mathbb{B}^N$, see \[FS16\] \[GL20\] \[MP20\] for a family with area approaching $4\pi$ as $k \to \infty$ and \[Ke17\] for a family of with area approaching $2\pi$ as $k \to \infty$.

In a similar direction Karpukhin–Stern, also relying on \[GL20\], have recently obtained versions of Theorem 1\] and Theorem 3\] for a very different set of conformal classes, \[KS20\]. They show the analogous result holds for some $\Sigma \subset (S, g)$, where $S$ is a closed surface and $\Sigma$ has many boundary components.

Our proof of Theorem 1\] combines several ingredients. The existence of maximizing metrics for the normalized first Steklov eigenvalue in a conformal class under the gap assumption \[2\] proved by the second named author \[Pet19\], the connection of extremal metrics for the Steklov problem and free boundary minimal surfaces, a rigidity result by Fraser–Schoen for free-boundary minimal annuli and Möbius bands \[FS16\], the asymptotic computation of the Steklov eigenvalues of $\mathbb{B}_1 \setminus B_{\varepsilon}$ \[Dit04\], and the glueing construction from the recent work of the authors \[MP20\].

We begin by recalling the connection between extremal metrics for Steklov eigenvalues and free-boundary minimal surfaces. For simplicity we only state a simplified version sufficient for our purposes.

Theorem 4 (\[FS16\] Proposition 5.2]). Let $g$ be a smooth metric on $\Sigma$ such that
\[
\sigma_1(\Sigma, g)L_g(\partial \Sigma) \geq \sigma_1(\Sigma, h)L_h(\partial \Sigma)
\]
for any metric $h \in U$ where $U$ is an open neighborhood of $g$ in the $C^{\infty}$-topology\[3\]. Then there is a branched, conformal, minimal immersion $\Phi: (\Sigma, g) \to \mathbb{B}^N$ with free boundary by first eigenfunction for some $N \geq 2$.

Next, we state the rigidity results by Fraser–Schoen.

Theorem 5 (\[FS16\] Theorem 1.2 and Theorem 1.4]). Let $\Phi: \Sigma \to \mathbb{B}^N$ be a minimal, free boundary immersion by first Steklov eigenfunctions. If $\Sigma$ is an annulus, then $\Sigma$ is homothetic to the critical catenoid. If $\Sigma$ is a Möbius band, then $\Sigma$ is homothetic to the critical Möbius band.

We have the following existence and compactness result by the second named author.

\[In \[FS16\] it is assumed that $g$ is globally maximizing but the proof does not use this.\]
Theorem 6 ([Pet19] Theorem 2]). Let \((\Sigma, g)\) be a compact surface with non-empty boundary such that
\[
\sup_\omega (\sigma_1(\Sigma, e^{2\omega} g)) L_{e^{2\omega} g}(\partial \Sigma) > 2\pi,
\]
then there is a smooth function \(\tau : \Sigma \to \mathbb{R}\) such that
\[
\sigma_1(\Sigma, e^{2\tau} g) L_{e^{2\tau} g}(\partial \Sigma) = \sup_\omega (\sigma_1(\Sigma, e^{2\omega} g)) L_{e^{2\omega} g}(\partial \Sigma)).
\]
Moreover, for any sequence \((g_k)_{k \in \mathbb{N}}\) of metrics with
\[
\liminf_{k \to \infty} \sup_\omega (\sigma_1(\Sigma, e^{2\omega} g_k)) L_{e^{2\omega} g_k}(\partial \Sigma) > 2\pi,
\]
the sequence \((\tau_k)_{k \in \mathbb{N}}\) as in (7) is smoothly precompact.

We remark that the last item is not explicitly stated in [Pet19] Theorem 2]. Instead it easily follows from the characterization of these maximizing metrics in terms of free boundary harmonic maps. Along a sequence enjoying (8) there can not be any bubbling of these harmonic maps. This is handled in [Pet19] even under much weaker assumptions.

We also have the following comparison result for Steklov eigenvalues.

Theorem 9 ([Dit04], see also [GP17, Example 4.2.5.]). For \(\varepsilon > 0\) sufficiently small, we have
\[
\sigma_1(B_1 \setminus B_\varepsilon L(\partial(B_1 \setminus B_\varepsilon)) > 2\pi.
\]

We need the analogous result for Möbius bands. Let \(M_\varepsilon\) the Möbius band obtained as follows. We glue together two copies \(A_1\) and \(A_2\) of \(B_1 \setminus B_\varepsilon\) along \(\partial B_\varepsilon\) and identify points by the involution given by \(\iota(x_1) = -x_2\), where \(x_1 \in A_1\) and \(x_2\) denotes the point with the same coordinates as \(x_1\) but in \(A_2\). Note that the metric on \(M_\varepsilon\) is only Lipschitz, but it can easily be approximated by a sequence of smooth metrics such that the length of the boundary and the first Steklov eigenvalue converge.

Proposition 10. We have that
\[
\sigma_1(M_\varepsilon L(\partial M_\varepsilon)) > 2\pi
\]
for \(\varepsilon > 0\) sufficiently small.

The argument is analogous to (and in fact easier than) the proof of Theorem 9. We record it below for the convenience of the reader.

Proof. Note that there is a second isometric involution on \(A_1 \cup A_2\) given by \(\tau(x_1) = x_2\) in the notation above. Since \(\tau\) is an isometric involution it acts on any eigenspace of the Dirichlet-to-Neumann operator and splits these into \(\pm 1\)-eigenspaces. Note that the +1 eigenspace corresponds to eigenfunctions on \(A_\varepsilon = B_1 \setminus B_\varepsilon\) of the Steklov problem with Neumann boundary conditions along \(\partial B_\varepsilon\). Analogously the \(-1\) eigenspace corresponds to eigenfunctions of the Steklov problem on \(A_\varepsilon\) with Dirichlet boundary conditions along \(\partial B_\varepsilon\). These eigenvalues can be computed explicitly as follows. Write \(u_k = (A_k r^k + A_{-k} r^{-k})T(k\theta)\) for \(T\) either cos or sin. Then it easily checked that for \(u_k\) to be a Dirichlet eigenfunction we need to have that \(A_k = -A_{-k} \varepsilon^{-2k}\), which leads to \(\sigma = k \varepsilon^{-2k+1} > k\). Additionally, there is an eigenfunction given by \(\log(r/\varepsilon)\), which is not \(\iota\) invariant. Analogously, \(u_k\) is a Neumann eigenfunction if and only if \(A_k = A_{-k} \varepsilon^{-2k}\).
The corresponding eigenvalue is given by \( k \frac{\varepsilon - 2k - 1}{\varepsilon - 2k + 1} \rightarrow k \) as \( \varepsilon \rightarrow 0 \). However, for \( k = 1 \) none of these eigenfunctions is \( \iota \) invariant.

We also need that a similar result holds near the other end of the moduli space. This result is much more subtle and was only very recently obtained in much greater generality by the authors in [MP20, Theorem 1.3]. While this is not explicitly stated there it easily follows from the specific construction, which we briefly recall now.

Let \( \mathbb{D}^2 \) be the flat disk and

\[
\Omega_\varepsilon = \left\{ (x, y) \in \mathbb{R}^2 : \varepsilon \exp\left( -\frac{1}{\varepsilon^\alpha} \right) \leq y \leq \varepsilon, \ -\frac{y^2}{2} \leq x \leq \frac{y^2}{2} \right\}
\]

for some \( \alpha \in \left( \frac{1}{3}, \frac{1}{2} \right) \). We then obtain \( \Sigma_\varepsilon \) by glueing \( \Omega_\varepsilon \) along the two sides \( \{ y = \varepsilon \} \) and \( \{ y = \varepsilon \exp\left( -\frac{1}{\varepsilon^\alpha} \right) \} \) to \( \partial \mathbb{D}^2 \) along disjoint intervals of the corresponding length in the boundary in conformal coordinates. We denote by \( g_\varepsilon \) the canonically induced metric from this construction. Note that \( \Sigma_\varepsilon \) can be an annulus or a Möbius band depending on the orientations chosen in the glueing procedure. In [MP20] we study sharp asymptotics of the first Steklov in this construction taking into account an additional dilation parameter on \( \Omega_\varepsilon \). As \( \varepsilon \rightarrow 0 \) this recovers the eigenvalues of the flat disk. The additional dilation parameter is used to obtain good control on the error term, which in particular leads to the following result.

**Theorem 11** ([MP20, Theorem 1.3]). There are \( \omega_\varepsilon : \Sigma_\varepsilon \rightarrow \mathbb{R} \) such that

\[ \sigma_1(\Sigma_\varepsilon, e^{2\omega_\varepsilon} g_\varepsilon) L_{e^{2\omega_\varepsilon} g_\varepsilon} (\partial \Sigma_\varepsilon) > \sigma_1(\mathbb{D}^2, \xi) L_\xi (\partial \mathbb{D}^2) = 2\pi \]

for \( \varepsilon \) sufficiently small.

We use the following simple lemma to control the conformal class of the surfaces \( \Sigma_\varepsilon \).

**Lemma 12.** Let \( \phi : [0, r] \times [0, 1] \rightarrow [0, R] \times [0, 1] \) be conformal embedding, smooth in the interior, such that \( \phi([0, r] \times \{ i \}) \subset [0, R] \times \{ i \} \), for \( i = 0, 1 \). Then we have that \( r \leq R \).

**Proof.** Since \( \phi \) is conformal, we have that

\[ \phi^*(dx^2 + dy^2) = e^{2\omega}(dx^2 + dy^2) \]

for a function \( \omega \) that is smooth in the interior. By assumption, we have that

\[
1 \leq \left( L(\phi([x] \times [0, 1])) \right)^2 = \left( \int_0^1 e^{\omega(x,y)} \, dy \right)^2 \leq \int_0^1 e^{2\omega(x,y)} \, dy,
\]

where we have used Jensen’s inequality and note that this remains valid also if \( \phi([x] \times [0, 1]) \) is not a rectifiable curve. This implies that

\[
r = \int_0^r dx \leq \int_0^r \int_0^1 e^{2\omega(x,y)} \, dy \, dx = \text{area}(\phi) \leq R \]

We write \( A_r = \overline{B_1} \setminus B_r \subset \mathbb{R}^2 \) and note that any compact annulus is conformal to some \( A_r \) for a unique \( r \in (0, 1) \). Another way to characterize the conformal class of \( A_r \) is to parametrize its universal covering on \((-\infty, \infty) \times [0, 1]\) and require the deck transformation group to be generated by \((x, y) \mapsto (x + R, y)\) for \( R > 0 \), which uniquely determines \( R \). The exponential map provides an explicit conformal map realizing this and one easily finds that \( R = \frac{2\pi}{\log(1/r)} \).
Lemma 13. Assume that $\Sigma_\varepsilon$ is an annulus and let $\Phi: B_1 \setminus B_r \to \Sigma_\varepsilon$ be a conformal homeomorphism, which is smooth in the interior. Then we have that $r \geq 1 - O(\varepsilon)$ as $\varepsilon \to 0$.

Proof. Let
\begin{equation}
\phi_0: [0, r_0] \times [0, 1] \to \Omega_\varepsilon \subseteq \left[ -\frac{\varepsilon^2}{2}, \frac{\varepsilon^2}{2} \right] \times \left[ \varepsilon \exp\left( -\frac{1}{\varepsilon^a} \right), \varepsilon \right]
\end{equation}
be a conformal homeomorphism which is smooth in the interior and on the boundary maps vertices to vertices\(^2\) such that we have that $\phi_0([0, r_0] \times \{0\}) \subset \left[ -\frac{\varepsilon^2}{2}, +\frac{\varepsilon^2}{2} \right] \times \{ \varepsilon \exp\left( -\frac{1}{\varepsilon^a} \right) \}$ and $\phi_0([0, r_0] \times \{1\}) \subset \left[ -\frac{\varepsilon^2}{2}, +\frac{\varepsilon^2}{2} \right] \times \{ \varepsilon \}$. Also note that the inclusion of $\Omega_\varepsilon$ into the rectangle in (14) is isometric, in particular conformal. After scaling the codomain by $\varepsilon - \varepsilon \exp\left( -\frac{1}{\varepsilon^a} \right) \sim \varepsilon$ we can apply Lemma 12 to $\phi_0$ and find that
\begin{equation}
r_0 \leq C \varepsilon
\end{equation}
for some fixed constant $C > 0$.

We now conformally parametrize the universal covering $\tilde{\Sigma}_\varepsilon$ of $\Sigma_\varepsilon$ by $(-\infty, \infty) \times [0, 1]$ with deck transformations generated by $(x, y) \mapsto (x + R, y)$ for $R > 0$, which uniquely determines $R$. Using $\phi_0$ we then find a conformal embedding $\phi: [0, r_0^{-1}] \times [0, 1] \to \Sigma_\varepsilon$
given explicitly by
\[ \phi(x, y) = \phi_0(r_0 y, r_0 x) \in \Omega_\varepsilon \subseteq \Sigma_\varepsilon. \]
We then lift $\phi$ to a map
\[ \tilde{\phi}: [0, r_0^{-1}] \times [0, 1] \to [0, R] \times [0, 1] \subset \tilde{\Sigma}_\varepsilon \]
with $\tilde{\phi}([0, r_0^{-1}] \times \{i\}) \subset [0, R] \times \{i\}$. Lemma 12 applied to $\tilde{\phi}$ gives that
\[ r_0^{-1} \leq R, \]
from which the claim easily follows thanks to (15) and recalling that $R = \frac{2\pi}{\log(1/r)}$. \hfill \Box

The very same argument applied to the orientation covering also gives the corresponding result for Möbius bands.

Corollary 16. Assume that $\Sigma_\varepsilon$ is a Möbius band and let $\tilde{\Sigma}_\varepsilon$ be the orientation covering of $\Sigma_\varepsilon$. Let $\Phi: B_1 \setminus B_r \to \tilde{\Sigma}_\varepsilon$ be a conformal homeomorphism, which is smooth in the interior. Then we have that $r \to 1$ as $\varepsilon \to 0$.

Proof of Theorem 7. We give the proof for $\Sigma$ an annulus, the case of Möbius bands is completely analogous.

Consider the functional $\sigma_1: (0, 1) \to (0, \infty)$ given by
\[ \sigma_1(r) = \sup_{\omega} \sigma_1(A_r, e^{2\omega} \xi) L_{e^{2\omega} \xi}(\partial A_r), \]
where $\xi$ denotes the canonical flat metric on $A_r$. We want to show that $\sigma_1(r) > 2\pi$ for any $r \in (0, 1)$.

\(^2\)Note that even though the Möbius group only acts simply three-transitively on the boundary, the additional parameter $r_0$ allows us to do this.
For \( r \) sufficiently small, we have thanks to Theorem 10 that
\[
\sigma_1(r) > 2\pi.
\]
Similarly, we have from Theorem 11 and Lemma 13 that
\[
\sigma_1(r) > 2\pi
\]
for \( r \) sufficiently close to 1. Moreover, we also know that
\[
\lim_{r \to 0} \sigma_1(r) = \lim_{r \to 1} \sigma_1(r) = 2\pi,
\]
see \([FS16, Proposition 4.4]\).

Let \( r^\ast \in (0,1) \) be chosen such that \( A_{r^\ast} \) is conformal to the critical catenoid. Suppose now towards a contradiction that there is some \( s \in (0,1) \) such that \( \sigma_1(s) \leq 2\pi \). We assume that \( s > r^\ast \). The argument for the other case is identical using (17) instead of (18). Then, thanks to (18) and (19), we claim there has to be \( t \in (s,1) \) such that
\[
\sigma_1(t) = \max_{r \in (s,1)} \sigma_1(r) > 2\pi.
\]
Indeed, this follows immediately from the observation that \( \sigma_1 \) when restricted to any of the open sets \( \{ r \in (0,1) : \sigma_1(r) > 2\pi + \delta \} \) with \( \delta > 0 \) is continuous thanks to Theorem 6. Moreover, also thanks to Theorem 6 we have \( \omega: A_t \to \mathbb{R} \) such that
\[
\sigma_1(A_t, e^{2\omega}(\partial A_t)) \geq \sigma_1(A_t, e^{2\tau}(\partial A_t)) L_{\omega,\omega}(\partial A_t)
\]
for any smooth function \( \tau: A_t \to \mathbb{R} \). It then follows immediately that \( e^{2\omega} \) is a local maximum of the normalized first Steklov eigenvalue and hence induced by a branched, free-boundary minimal immersion into \( \mathbb{B}^N \) by first eigenfunctions thanks to Theorem 4. But by the uniqueness of the critical catenoid, see Theorem 5 above, this is impossible for \( t \neq r^\ast \).

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