On global behavior of mappings with integral constraints

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Received: 3 January 2022 / Revised: 14 March 2022 / Accepted: 26 March 2022 / Published online: 19 April 2022
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Abstract
This article is devoted to the study of mappings with branch points whose characteristics satisfy integral-type constraints. We have proved theorems concerning their local and global behavior. In particular, we established the equicontinuity of families of such mappings inside their definition domain, as well as, under additional conditions, equicontinuity of the families of these mappings in its closure.

Mathematics Subject Classification Primary 30C65; Secondary 31A15 · 31B25

1 Introduction
This article is devoted to the study of mappings satisfying upper bounds for the distortion of the modulus of families of paths, see, for example, [1, 5, 12–14, 22, 35]. In particular, here we are dealing with mappings whose characteristics satisfy the so-called conditions of integral type, see, for example, [21, 22, 30]. The main purpose of the present manuscript is to study the local behavior of mappings having branch points whose characteristics are bounded only on the average. It is worth noting some of the previous results in this direction. In particular, in [21], homeomorphisms with similar conditions were investigated, and in [30], mappings with branch points having a general characteristic \( Q \). Unfortunately, the most general case when the mappings are not homeomorphisms and also do not have a common majorant has been overlooked. Note that the most interesting applications related to the study of the Beltrami equation and the Dirichlet problem for it are associated with the absence of a general majorant for maximal complex characteristics, (see, e.g., [2, 11]).

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Let us move on to definitions and formulations of results. Given $p \geq 1$, $M_p$ denotes the $p$-modulus of a family of paths, and the element $dm(x)$ corresponds to the Lebesgue measure in $\mathbb{R}^n$, $n \geq 2$, see [36]. In what follows, we usually write $M(\Gamma)$ instead of $M_n(\Gamma)$. For the sets $A, B \subset \mathbb{R}^n$ we set, as usual,

$$\text{diam } A = \sup_{x, y \in A} |x - y|, \quad \text{dist } (A, B) = \inf_{x \in A, y \in B} |x - y|.$$ 

Sometimes we also write $d(A)$ instead of diam $A$ and $d(A, B)$ instead of dist $(A, B)$, if no misunderstanding is possible. For given sets $E$ and $F$ and a given domain $D$ in $\mathbb{R}^n = \mathbb{R}^n \cup \{\infty\}$, we denote by $\Gamma(E, F, D)$ the family of all paths $\gamma : [0, 1] \to \mathbb{R}^n$ joining $E$ and $F$ in $D$, that is, $\gamma(0) \in E$, $\gamma(1) \in F$ and $\gamma(t) \in D$ for all $t \in (0, 1)$. Everywhere below, unless otherwise stated, the boundary and the closure of a set are understood in the sense of the extended Euclidean space $\overline{\mathbb{R}^n}$. Let $x_0 \in \overline{D}$, $x_0 \neq \infty$,

$$S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}, \quad S_i = S(x_0, r_i), \quad i = 1, 2,$$

$$A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}.$$  \hfill (1.1)

Everywhere below, unless otherwise stated, the closure $\overline{A}$ and the boundary $\partial A$ of a set $A$ are understood in the topology of the space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$. Let $Q : \mathbb{R}^n \to \mathbb{R}^n$ be a Lebesgue measurable function satisfying the condition $Q(x) \equiv 0$ for $x \in \mathbb{R}^n \setminus D$. Given $p \geq 1$, a mapping $f : D \to \overline{\mathbb{R}^n}$ is called a ring $Q$-mapping at a point $x_0 \in \overline{D}\setminus\{\infty\}$ with respect to $p$-modulus, if the condition

$$M_p(f(\Gamma(S_1, S_2, D))) \leq \int_{A \cap D} Q(x) \cdot \eta^p(|x - x_0|) \, dm(x)$$ \hfill (1.2)

holds for all $0 < r_1 < r_2 < d_0 := \sup_{x \in D} |x - x_0|$ and any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r) \, dr \geq 1.$$ \hfill (1.3)

The inequalities of the form (1.2) were established for many well-known classes of mappings. So, for quasiconformal mappings and mappings with bounded distortion, they hold for $p = n$ and some $Q(x) \equiv K = \text{const}$ (see, for example, [12,Theorem 7.1] and [36,Definition 13.1]). Such inequalities also hold for many mappings with unbounded characteristic, in particular, for homeomorphisms belonging to the class $W_{1,p}^{1,p}$, $p > n - 1$, the inner dilatation of the order $\alpha := \frac{p}{p-n+1}$ is locally integrable (see, for example, [14,Theorems 8.1, 8.5] and [23,Corollary 2], [24,Theorem 9, Lemma 5]).

The concept of a set of capacity zero, used below, can be found in [13,Section 2.12] and is therefore omitted. A mapping $f : D \to \mathbb{R}^n$ is called discrete if the preimage $\{f^{-1}(y)\}$ of each point $y \in \mathbb{R}^n$ consists of isolated points, and open if the image of
any open set $U \subset D$ is an open set in $\mathbb{R}^n$. Let us formulate the main results of this manuscript. In what follows, $h$ denotes the so-called chordal metric defined by the equalities

$$h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}}, \quad x \neq \infty \neq y, \quad h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}. \quad (1.4)$$

For a given set $E \subset \mathbb{R}^n$, we set

$$h(E) := \sup_{x, y \in E} h(x, y). \quad (1.5)$$

The quantity $h(E)$ in (1.5) is called the chordal diameter of the set $E$. For given sets $A, B \subset \mathbb{R}^n$, we put

$$h(A, B) = \inf_{x \in A, y \in B} h(x, y),$$

where $h$ is a chordal metric defined in (1.4).

Given a domain $D \subset \mathbb{R}^n$, a number $M_0 > 0$, and a strictly increasing function $\Phi: \mathbb{R}^+ \to \mathbb{R}^+$ let us denote by $\mathfrak{F}_{M_0}^D(D)$ the family of all open discrete mappings $f: D \to \mathbb{R}^n \setminus E$ for which there exists a function $Q = Q_f(x): D \to [0, \infty]$ such that (1.2)–(1.3) hold for any $x_0 \in D$ with $p = n$ and, in addition,

$$\int_D \Phi(Q(x)) \frac{dm(x)}{(1 + |x|^2)^n} \leq M_0 < \infty. \quad (1.6)$$

Let $(X, d)$ and $(X', d')$ be metric spaces with distances $d$ and $d'$, respectively. A family $\mathfrak{F}$ of mappings $f: X \to X'$ is said to be equicontinuous at a point $x_0 \in X$, if for every $\epsilon > 0$ there is $\delta = \delta(\epsilon, x_0) > 0$ such that $d'(f(x), f(x_0)) < \epsilon$ for all $f \in \mathfrak{F}$ and $x \in X$ with $d(x, x_0) < \delta$. The family $\mathfrak{F}$ is equicontinuous if $\mathfrak{F}$ is equicontinuous at every point $x_0 \in X$.

**Remark 1.1** There are several different definitions of equicontinuous families of mappings, including a similar definition in which $\delta$ is independent on $x_0$. For this reason, let us call the family $\mathfrak{F}$ of mappings $f: X \to X'$ uniformly equicontinuous in $X$, if for every $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that $d'(f(x), f(x_0)) < \epsilon$ for all $f \in \mathfrak{F}$ and $x \in X$ with $d(x, x_0) < \delta$. The family $\mathfrak{F}$ is equicontinuous if $\mathfrak{F}$ is equicontinuous at every point $x_0 \in X$.

An analogue of the following statement was established for homeomorphisms in [21,Theorem 4.1], and for mappings whose corresponding function $Q$ is fixed, in [30,Theorem 1]. In the form given below, the indicated statement seems to be the most interesting from the point of view of applications to the problem of compactness of solutions of the Beltrami equations and the Dirichlet problem (see, for example, [2,Theorem 2]) and [11,Theorem 1]).
Theorem 1.1 Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $\text{cap} E > 0$. If
\[
\int_{\delta_0}^{\infty} \frac{d\tau}{\tau \left[ \Phi^{-1}(\tau) \right]^{1/n-1}} = \infty
\]
holds for some $\delta_0 > \tau_0 := \Phi(0)$, then $\mathcal{S}_{M_0,E}(D)$ is equicontinuous in $D$.

Note that the statement of Theorem 1.1 is much simpler and more elegant for the similar class of mappings satisfying the relation (1.2) for $p \in (n-1, n)$. Given $p \geq 1$, a domain $D \subset \mathbb{R}^n$, a number $M_0 > 0$ and a strictly increasing function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ let us denote by $\mathcal{S}_{M_0,p}(D)$ the family of all open discrete mappings $f : D \to \mathbb{R}^n$ for which there exists a function $Q = Q_f(x) : D \to [0, \infty]$ such that (1.2)–(1.3) hold for any $x_0 \in D$ and, in addition, (1.6) holds. The following statement is true.

Theorem 1.2 Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $p \in (n-1, n)$. If the relation
\[
\int_{\delta_0}^{\infty} \frac{d\tau}{\tau \left[ \Phi^{-1}(\tau) \right]^{1/p-1}} = \infty
\]
holds for some $\delta_0 > \tau_0 := \Phi(0)$, then $\mathcal{S}_{M_0,p}(D)$ is equicontinuous in $D$.

Remark 1.2 In Theorem 1.1, the equicontinuity of the corresponding family of mappings should be understood in the sense of mappings between metric spaces $(X, d)$ and $(X', d')$, where $X$ is a domain $D$, and $d$ is a Euclidean metric in $D$, besides, $X' = \mathbb{R}^n$, and $h$ is a chordal metric defined in (1.4). At the same time, in Theorem 1.2 the space $X$ remains the same, and the space $X'$ is an usual Euclidean $n$-dimensional space with the Euclidean metric $d'$.

Remark 1.3 Note that, under bounded functions $Q$ and for $p \in (n-1, n)$, the mappings in (1.2) are local quasi-isometries, see [3, Theorem 3]. However, note that the class of mappings involved in Theorem 1.2 does not belong to the class of Lipschitz or bi-Lipschitz mappings.

To substantiate this fact, consider the following mapping $f$, given in the unit ball (see, e.g., [32,§ 7], cf. [25]). Let $Q(x) = \log \frac{e}{|x|}$, $x \in \mathbb{B}^n$, let $q_0(r) := \log \frac{e}{r}$, and let $n - 1 < p < n$. Observe that $\int_{\varepsilon_0}^{\varepsilon_1} \frac{dt}{t^{p-1} q_0^\frac{1}{p-1}(t)} < \infty$ for any $\varepsilon_0 \in (0, 1)$ and any $\varepsilon \in (0, \varepsilon_0)$, in addition,
\[
\int_{\varepsilon_0}^{\varepsilon_1} \frac{dt}{t^{p-1} q_0^\frac{1}{p-1}(t)} = \infty.
\]
Set
\[
f(x) = \frac{x}{|x|} \rho(|x|), \quad x \in \mathbb{B}^n \setminus \{0\}, \quad f(0) := 0,
\]
where

$$
\rho(|x|) = \left(1 + \frac{n - p}{p - 1} \int_{|x|}^{1} \frac{dt}{t^{p-1}q_0^{-\frac{p-1}{p-n}}} \right)^{\frac{p-1}{p-n}}.
$$

Observe that, \( f \in ACL \) and, besides that, \( f \) is differentiable in \( \mathbb{R}^n \) almost everywhere. Set \( l(f'(x)) := \min_{|h|=1} |f'(x)h|, \| f'(x) \| := \max_{|h|=1} |f'(x)h|, J(x, f) := \det f'(x), \)

\[
K_{1,p}(x, f) = \begin{cases} \frac{|J(x, f)|}{l(f'(x))^{\frac{p}{p-1}}}, & J(x, f) \neq 0, \\ 1, & f'(x) = 0, \\ \infty, & \text{otherwise} \end{cases},
\]

\[
K_{0,p}(x, f) = \begin{cases} \frac{\| f'(x) \|^p}{|J(x, f)|}, & J(x, f) \neq 0, \\ 1, & f'(x) = 0, \\ \infty, & \text{otherwise} \end{cases}
\]

By the technique used in [14, Proposition 6.3]

\[
|J(x, f)| = \delta_r(x) \cdot \delta_r(x), \quad \| f'(x) \| = \max\{\delta_r(x), \delta_r(x)\}
\]

\[
l(f'(x)) = \min\{\delta_r(x), \delta_r(x)\},
\]

where

\[
\delta_r(x) = \frac{|f(x)|}{|x|}, \quad \delta_r(x) = \frac{|\partial f(x)|}{|x|}.
\]

Thus,

\[
\| f'(x) \| = \left(1 + \frac{n - p}{p - 1} \int_{|x|}^{1} \frac{dt}{t^{p-1}q_0^{-\frac{p-1}{p-n}}} \right)^{\frac{p-1}{p-n}} \frac{1}{|x|},
\]

\[
l(f'(x)) = \left(1 + \frac{n - p}{p - 1} \int_{|x|}^{1} \frac{dt}{t^{p-1}q_0^{-\frac{p-1}{p-n}}} \right)^{\frac{p-1}{p-n}} \frac{1}{|x|^{\frac{p-1}{p-n}}q_0^{-\frac{p-1}{p-n}}(|x|)}
\]

and

\[
|J(x, f)| = \left(1 + \frac{n - p}{p - 1} \int_{|x|}^{1} \frac{dt}{t^{p-1}q_0^{-\frac{p-1}{p-n}}} \right)^{\frac{(n-1)p}{p-n}} \frac{1}{|x|^{n-1+\frac{n-1}{p-n}}q_0^{-\frac{p-1}{p-n}}(|x|)}.
\]
Observe that, $f \in W^{1,p}_{\text{loc}}$ for $p > n - 1$. Indeed, $\|f'(x)\|$ is bounded outside of some neighborhood of the origin, in addition, for sufficiently small $r > 0$ and some $C > 0$, we obtain that $\|f'(x)\| \leq C/|x|$. Besides that, by the Fubini theorem
$$\int_{B(0,r)} \|f'(x)\|^p \, dm(x) \leq C^p \omega_{n-1} \cdot \int_0^r dr < \infty.$$ Let $K$ be any compact set in $\mathbb{B}^n$. Then there is $R > 0$ such that $K \subset B(0, R)$. Due to the above, we obtain that
$$\int_K \|f'(x)\|^p \, dm(x) \leq \int_{B(0,r)} \|f'(x)\|^p \, dm(x) + \int_{B(0,R) \setminus B(0,r)} \|f'(x)\|^p \, dm(x) < \infty,$$
therefore, $f \in W^{1,p}_{\text{loc}}$. Observe that $K_{I,p}(x, f) = q_0(|x|)$. Applying Theorem 2 in [23] for $n \geq 2$ and $\varphi(t) = t^p$, we obtain that $f$ satisfies the inequality (1.2) with $Q(x) = q_0(|x|)$ at any point $x_0 \in \mathbb{B}^n$. For $n = 2$, we may take $Q(x) = K_0^{p-1}(x, f) = q_0(|x|)$, see e.g. [24, Theorem 9, Lemma 5]. By direct calculations, one can verify that $q_0(|x|) = \log \frac{1}{|x|} \in L^1(\mathbb{B}^n)$. It may also be established that the function $Q$ satisfies relation (1.8) for some $M_0 > 0$ and for $\Phi(\tau) = e^\tau$. Note, however, that the mapping $f$ is not Lipschitz at the origin, which can be directly verified using L’Hospital’s rule (see [25, Example]). Observe that, since $f$ is not Lipschitz, it is not a mapping with a bounded length distortion, as well, see [15, Lemma 2.7].

A separate research topic is the equicontinuity of families of mappings in the closure of a domain. Results of this kind were obtained for fixed characteristics in some of our papers. In particular, in [31] we considered the case of fixed domains between which the mappings act, and in [33, 34] we considered the case when the mapped domain can change. Note that the classical results on the equicontinuity of quasiconformal mappings in the closure of a domain were obtained by Näkki and Palka, see e.g. [19, Theorem 3.3]. Let us formulate the main results related to some more general case.

Let $I$ be a fixed set of indices and let $D_i, i \in I$, be some sequence of domains. Following [19, Sect. 2.4], we say that a family of domains $\{D_i\}_{i \in I}$ is equi-uniform with respect to $p$-modulus if for any $r > 0$ there exists a number $\delta > 0$ such that the inequality
$$M_p(\Gamma(F^*, F, D_i)) \geq \delta$$
holds for any $i \in I$ and any continua $F, F^* \subset D_i$ such that $h(F) \geq r$ and $h(F^*) \geq r$. Given $p \geq 1$, a number $\delta > 0$, a number $M_0 > 0$, a domain $D \subset \mathbb{R}^n$, $n \geq 2$, a continuum $A \subset D$ and a strictly increasing function $\Phi: \mathbb{R}^+ \to \mathbb{R}^+$ denote $\delta_{M_0}^{D, \Phi, A, p, \delta}(D)$ the family of all homeomorphisms $f : D \to \mathbb{R}^n$ for which there exists a function $Q = Q_f(x) : D \to [0, \infty]$ such that: 1) relations (1.2)–(1.3) hold for any $x_0 \in \overline{D}$, 2) the relation (1.6) holds and 3) the relations $h(f(A)) \geq \delta$ and $h(\mathbb{R}^n \setminus f(D)) \geq \delta$ hold. The following statement is true.
Theorem 1.3 Let \( p \in (n-1, n] \), a domain \( D \) is locally connected at any \( x_0 \in \partial D \) and domains \( D'_f = f(D) \) are equi-uniform with respect to \( p \)-modulus over all \( f \in \mathcal{A}^{M_0}_{\Phi, \delta, p, E}(D) \). If (1.7) holds for some \( \delta_0 > \tau_0 := \Phi(0) \), then any \( f \in \mathcal{A}^{M_0}_{\Phi, \delta, p, E}(D) \) has a continuous extension in \( \overline{D} \) and, besides that, the family \( \mathcal{A}^{M_0}_{\Phi, \delta, p, E}(\overline{D}) \) of all extended mappings \( \overline{f} : \overline{D} \to \mathbb{R}^n \) is equicontinuous in \( \overline{D} \).

As usual, we use the notation

\[
C(f, x) := \{ y \in \mathbb{R}^n : \exists x_k \in D : x_k \to x, f(x_k) \to y, k \to \infty \}.
\]

A mapping \( f \) between domains \( D \) and \( D' \) is called \textit{closed} if \( f(E) \) is closed in \( D' \) for any closed set \( E \subset D \) (see, e.g., [37, Section 3]). Any open discrete mapping is boundary preserving, i.e. \( C(f, \partial D) \subset \partial D' \), where \( C(f, \partial D) = \bigcup_{x \in \partial D} C(f, x) \) (see e.g. [37, Theorem 3.3]).

Given \( p \geq 1 \), a domain \( D \subset \mathbb{R}^n \), a set \( E \subset \mathbb{R}^n \), a strictly increasing function \( \Phi: \mathbb{R}^+ \to \mathbb{R}^+ \), a number \( \delta > 0 \) and a number \( M_0 > 0 \) denote by \( \mathcal{A}^{M_0}_{\Phi, \delta, p, E}(D) \) the family of all open discrete and closed mappings \( f : D \to \mathbb{R}^n \setminus E \) such that: 1) relations (1.2)–(1.3) hold for any \( x_0 \in \overline{D} \), 2) the relation (1.6) holds and 3) there exists a continuum \( K_f \subset D'_f \) such that \( h(K_f) \geq \delta \) and \( h(f^{-1}(K_f), \partial D) \geq \delta > 0 \). The following statement is true.

Theorem 1.4 Let \( p \in (n-1, n] \), a domain \( D \) is locally connected at any point \( x_0 \in \partial D \) and, besides that, domains \( D'_f = f(D) \) are equi-uniform with respect to \( p \)-modulus over all \( f \in \mathcal{A}^{M_0}_{\Phi, \delta, p, E}(D) \). Let \( \text{cap} \, E > 0 \) for \( p = n \), and let \( E \) is any closed set for \( n-1 < p < n \). If (1.7) holds for some \( \delta_0 > \tau_0 := \Phi(0) \), then any \( f \in \mathcal{A}^{M_0}_{\Phi, \delta, p, E}(D) \) has a continuous extension in \( \overline{D} \) and, besides that, the family \( \mathcal{A}^{M_0}_{\Phi, \delta, p, E}(\overline{D}) \) of all extended mappings \( \overline{f} : \overline{D} \to \mathbb{R}^n \) is equicontinuous in \( \overline{D} \).

Remark 1.4 In Theorems 1.3 and 1.4, the equicontinuity should be understood in terms of families of mappings between metric spaces \( (X, d) \) and \( (X', d') \), where \( X = \overline{D} \), \( d \) is a chordal metric \( h \), \( X' = \mathbb{R}^n \) and \( d' \) is a chordal (spherical) metric \( h \), as well.

Theorems 1.3 and 1.4 admit a natural generalization to the case of ”bad” boundaries, when the maps do not have a continuous extension to points of the boundary of the domain in the usual sense, however, this extension holds in the sense of the so-called prime ends. Let us recall several important definitions associated with this concept. In the following, the next notation is used: the set of prime ends corresponding to the domain \( D \), is denoted by \( E_D \), and the completion of the domain \( D \) by its prime ends is denoted \( \overline{D}_P \). The definition of prime ends used below corresponds to the definition given in [7], and therefore is omitted. Consider the following definition, which goes back to Nåkki [18], see also [8]. We say that the boundary of the domain \( D \) in \( \mathbb{R}^n \) is \textit{locally quasiconformal}, if each point \( x_0 \in \partial D \) has a neighborhood \( U \) in \( \mathbb{R}^n \), which can be mapped by a quasiconformal mapping \( \varphi \) onto the unit ball \( \mathbb{B}^n \subset \mathbb{R}^n \) so that \( \varphi(\partial D \cap U) \) is the intersection of \( \mathbb{B}^n \) with the coordinate hyperplane.
A sequence of cuts \( \sigma_m, m = 1, 2, \ldots \), is called regular, if \( \sigma_m \cap \sigma_{m+1} = \emptyset \) for \( m \in \mathbb{N} \) and, in addition, \( d(\sigma_m) \to 0 \) as \( m \to \infty \). If the end \( K \) contains at least one regular chain, then \( K \) will be called regular. We say that a bounded domain \( D \) in \( \mathbb{R}^n \) is regular, if \( D \) can be quasiconformally mapped to a domain with a locally quasiconformal boundary whose closure is a compact in \( \mathbb{R}^n \), and, besides that, every prime end in \( D \) is regular. Note that space \( D_P = D \cup E_D \) is metric, which can be demonstrated as follows. If \( g: D_0 \to D \) is a quasiconformal mapping of a domain \( D_0 \) with a locally quasiconformal boundary onto some domain \( D \), then for \( x, y \in D_P \) we put:

\[
\rho(x, y) := |g^{-1}(x) - g^{-1}(y)|,
\]

where the element \( g^{-1}(x), x \in E_D \), is to be understood as some (single) boundary point of the domain \( D_0 \). The specified boundary point is unique and well-defined by [7,Theorem 2.1, Remark 2.1], cf. [18,Theorem 4.1]. It is easy to verify that \( \rho \) in (1.12) is a metric on \( D_P \), and that the topology on \( D_P \), defined by such a method, does not depend on the choice of the map \( g \) with the indicated property. The analogs of Theorems 1.3 and 1.4 for the case of prime ends are as follows.

**Theorem 1.5** Let \( p \in (n-1, n] \) and let \( D \) be a regular domain. Assume that \( D_f = f(D) \) are bounded equi-uniform domains with respect to \( p \)-modulus over all \( f \in M_0^{A,p,\delta}(D) \), which are domains with a locally quasiconformal boundary, as well. If (1.7) holds for some \( \delta_0 > \tau_0 := \Phi(0) \), then any \( f \in M_0^{A,p,\delta}(D) \) has a continuous extension in \( D_P \) and, besides that, the family \( M_0^{A,p,\delta}(D) \) of all extended mappings \( f : D_P \to \mathbb{R}^n \) is equicontinuous in \( D_P \).

**Theorem 1.6** Let \( p \in (n-1, n] \) and let \( D \) be a regular domain. Assume that domains \( D_f = f(D) \) are bounded equi-uniform domains with respect to \( p \)-modulus over all \( f \in M_0^{A,p,\delta,\varepsilon}(D) \), which are domains with a locally quasiconformal boundary, as well. Let \( \text{cap} E > 0 \) for \( p = n \), and let \( E \) is any closed domain whenever \( n-1 < p < n \). If (1.7) holds for some \( \delta_0 > \tau_0 := \Phi(0) \), then any \( f \in M_0^{A,p,\delta,\varepsilon}(D) \) has a continuous extension in \( D_P \) and, besides that, the family \( M_0^{A,p,\delta,\varepsilon}(D) \) of all extended mappings \( f : D_P \to \mathbb{R}^n \) is equicontinuous in \( D_P \).

**Remark 1.5** In Theorems 1.5 and 1.6, the equicontinuity should be understood in terms of families of mappings between metric spaces \( (X, d) \) and \( (X', d') \), where \( X = D_P \), \( d \) is one of the possible metrics, corresponding to the topological space \( D_P \), \( X' = \mathbb{R}^n \) and \( d' \) is a chordal (spherical) metric in \( \mathbb{R}^n \).

2 Auxiliary lemmas

The key point related to the proof of the main statements of the article is related to the connection between conditions (1.6)–(1.7) and the divergence of an integral of
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a special form (see, for example, [21, Lemma 3.1]). Given a Lebesgue measurable function \( Q : \mathbb{R}^n \to [0, \infty] \) and a point \( x_0 \in \mathbb{R}^n \) we set

\[
q_{x_0}(t) = \frac{1}{\omega_{n-1} t^{n-1}} \int_{S(x_0,t)} Q(x) \, dH^{n-1},
\]

(2.1)

where \( H^{n-1} \) denotes \((n - 1)\)-dimensional Hausdorff measure. The following lemma is of particular importance.

**Lemma 2.1** Let \( 1 \leq p \leq n \), and let \( \Phi : [0, \infty] \to [0, \infty] \) be a strictly increasing convex function such that the relation

\[
\int_{\delta_0}^{\infty} \frac{d\tau}{\tau \left[ \Phi^{-1}(\tau) \right]^{\frac{1}{p-1}}} = \infty
\]

(2.2)

holds for some \( \delta_0 > \tau_0 := \Phi(0) \). Let \( \Omega \) be a family of functions \( Q : \mathbb{R}^n \to [0, \infty] \) such that

\[
\int_{D} \Phi(Q(x)) \frac{dm(x)}{(1 + |x|^2)^n} \leq M_0 < \infty
\]

(2.3)

for some \( 0 < M_0 < \infty \). Now, for any \( 0 < r_0 < 1 \) and for every \( \sigma > 0 \) there exists \( 0 < r_* = r_*(\sigma, r_0, \Phi) < r_0 \) such that

\[
\int_{\varepsilon}^{r_0} \frac{dt}{\varepsilon^{\frac{1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(t)} \geq \sigma, \quad \varepsilon \in (0, r_*),
\]

(2.4)

for any \( Q \in \Omega \).

**Proof** Using the substitution of variables \( t = r/r_0 \), for any \( \varepsilon \in (0, r_0) \) we obtain that

\[
\int_{\varepsilon}^{r_0} \frac{dr}{\varepsilon^{\frac{1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(r)} \geq \int_{\varepsilon}^{r_0} \frac{dr}{r q_{x_0}^{\frac{1}{p-1}}(r)} = \int_{\varepsilon/r_0}^{1} \frac{dt}{t q_{x_0}^{\frac{1}{p-1}}(tr_0)} = \int_{\varepsilon/r_0}^{1} \frac{dt}{tq_0^{\frac{1}{p-1}}(t)},
\]

(2.5)

where \( q_0(t) \) is the average integral value of the function \( \widetilde{Q}(x) := Q(r_0x + x_0) \) over the sphere \( |x| = t \), see the ratio (2.1). Then, according to [21, Lemma 3.1],

\[
\int_{\varepsilon/r_0}^{1} \frac{dt}{tq_0^{\frac{1}{p-1}}(t)} \geq \frac{M_*^{\varepsilon/r_0)}{n } \int_{\varepsilon/r_0}^{1} \frac{d\tau}{\tau \left[ \Phi^{-1}(\tau) \right]^{\frac{1}{p-1}}},
\]

(2.6)
where

\[ M_\ast (\varepsilon/r_0) = \frac{1}{\Omega_n (1 - (\varepsilon/r_0)^n)} \int_{A(0, \varepsilon/r_0, 1)} \Phi(Q(r_0 x + x_0)) \, dm(x) \]

\[ = \frac{1}{\Omega_n (r_0^n - \varepsilon^n)} \int_{A(x_0, \varepsilon, r_0)} \Phi(Q(x)) \, dm(x) \]

and \( A(x_0, \varepsilon, r_0) \) is defined in (1.1) for \( r_1 := \varepsilon \) and \( r_2 := r_0 \). Observe that \(|x| \leq |x - x_0| + |x_0| \leq r_0 + |x_0| \) for any \( x \in A(x_0, \varepsilon, r_0) \). Thus

\[ M_\ast (\varepsilon/r_0) \leq \frac{\beta(x_0)}{\Omega_n (r_0^n - \varepsilon^n)} \int_{A(x_0, \varepsilon, r_0)} \Phi(Q(x)) \frac{dm(x)}{(1 + |x|^2)^n}, \]

where \( \beta(x_0) = (1 + (r_0 + |x_0|^2)^2)^n \). Therefore,

\[ M_\ast (\varepsilon/r_0) \leq \frac{2\beta(x_0)}{\Omega_n r_0^n} M_0 \]

for \( \varepsilon \leq r_0/\sqrt{2} \), where \( M_0 \) is a constant in (2.3). Observe that

\[ M_\ast (\varepsilon/r_0) > \Phi(0) > 0, \]

because \( \Phi \) is increasing. Now, by (2.5) and (2.6) we obtain that

\[ \int_{\varepsilon}^{r_0} \frac{dr}{r^{n-1} q_{x_0}^{-1}(r)} \geq \frac{1}{n} \int_{\Phi^{-1}(\tau)^{1/n}}^{\Phi^{-1}(\tau_0)^{1/n}} \frac{d\tau}{2\beta(x_0) M_0 \Omega_n r_0^n}. \]

(2.7)

The desired conclusion follows from (2.7) and (2.2). □

Recall that a pair \( E = (A, C) \), where \( A \) is an open set in \( \mathbb{R}^n \), and \( C \) is a compact subset of \( A \), is called condenser in \( \mathbb{R}^n \). Given \( p \geq 1 \), a quantity

\[ \text{cap}_p E = \text{cap}_p (A, C) = \inf_{u \in W_0(E)} \int_A |\nabla u|^p \, dm(x), \]

where \( W_0(E) = W_0 (A, C) \) is a family of all nonnegative absolutely continuous on lines (ACL) functions \( u : A \to \mathbb{R} \) with compact support in \( A \) and such that \( u(x) \geq 1 \) on \( C \), is called \( p\)-capacity of the condenser \( E \). We write \( \text{cap} E \) for \( \text{cap}_p E \). We also need the following statement given in [20, Proposition II.10.2].
Proposition 2.1 Let $E = (A, C)$ be a condenser in $\mathbb{R}^n$ and let $\Gamma_E$ be the family of all paths of the form $\gamma : [a, b) \to A$ with $\gamma(a) \in C$ and $|\gamma| \cap (A \setminus F) \neq \emptyset$ for every compact set $F \subset A$. Then $\cap E = M(\Gamma_E)$.

Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $f : D \to \mathbb{R}^n$ (or $f : D \to \overline{\mathbb{R}}^n$) be a discrete open mapping, $\beta : [a, b) \to \mathbb{R}^n$ be a path, and $x \in f^{-1}(\beta(a))$. A path $\alpha : [a, c) \to D$ is called a maximal $f$-lifting of $\beta$ starting at $x$, if (1) $\alpha(a) = x$; (2) $f \circ \alpha = \beta|_{[a,c)}$; (3) for $c < c' \leq b$, there is no paths $\alpha' : [a, c') \to D$ such that $\alpha = \alpha'|_{[a,c)}$ and $f \circ \alpha' = \beta|_{[a,c')}$. Let $f : D \to \mathbb{R}^n$ be a discrete open mapping, $\beta : [a, b) \to \mathbb{R}^n$ be a path, and $x \in f^{-1}(\beta(a))$. Then $\beta$ has a maximal $f$-lifting starting at $x$ (see [20, Corollary II.3.3]).

In what follows, we set $a/\infty = 0$ for $a \neq \infty$, $a/0 = \infty$ for $a > 0$ and $0 \cdot \infty = 0$. One of the most important statements allowing us to connect the study of mappings in (1.2) with the conditions (1.6)–(1.7) is the following proposition. The principal points related to its proof were indicated during the establishment of Lemma 1 in [27]; however, for the sake of completeness of presentation, we will establish it in full in the text.

Proposition 2.2 Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, let $x_0 \in \overline{D}\setminus\{\infty\}$, let $Q : D \to [0, \infty]$ be a Lebesgue measurable function and let $f : D \to \overline{\mathbb{R}}^n$ be an open discrete mapping satisfying relations (1.6)–(1.7) at a point $x_0$. If $0 < r_1 < r_2 < \sup_{x \in D} |x - x_0|$, then

$$M_p(f(\Gamma(S(x_0, r_1), S(x_0, r_2), D))) \leq \frac{\omega_{n-1}}{I_{p-1}},$$  \hspace{1cm} (2.8)

where

$$I = I(x_0, r_1, r_2) = \int_{r_1}^{r_2} \frac{dr}{r^{n-1} \int_0^1 q_x(r)}.$$  \hspace{1cm} (2.9)

If, in addition, $x_0 \in D$, $0 < r_1 < r_2 < r_0 = \text{dist}(x_0, \partial D)$ and $E = \left( B(x_0, r_2), B(x_0, r_1) \right)$, then

$$\text{cap}_p f(E) \leq \frac{\omega_{n-1}}{I_{p-1}},$$  \hspace{1cm} (2.10)

where $f(E) = \left( f(B(x_0, r_2)), f(B(x_0, r_1)) \right)$.

Proof We may consider that $I \neq 0$, since (2.8) and (2.10) are obvious, in this case. We also may consider that $I \neq \infty$. Otherwise, we may consider $Q(x) + \delta$ instead of $Q(x)$ in (2.8) and (2.10), and then pass to the limit as $\delta \to 0$. Let $I \neq \infty$.

Let us first prove the relation (2.8). If $0 < I < \infty$, then $q_{x_0}(r) \neq 0$ for a.e. $r \in (r_1, r_2)$. Set

$$\psi(t) = \begin{cases} 1/[t^{n-1} q_{x_0}^{1/(n-1)}(t)], & t \in (r_1, r_2), \\ 0, & t \notin (r_1, r_2). \end{cases}$$
In this case, by Fubini’s theorem,

$$\int_A Q(x) \cdot \psi^p(|x - x_0|) \, dm(x) = \omega_{n-1} I,$$

(2.11)

where $A = A(x_0, r_1, r_2)$ is defined in (1.1). Observe that a function $\eta_1(t) = \psi(t)/I$, $t \in (r_1, r_2)$, satisfies (1.3) because $\int_{r_1}^{r_2} \eta_1(t) \, dt = 1$. Now, by the definition of $f$ in (1.2)

$$M_p(f(\Gamma(S(x_0, r_1), S(x_0, r_2), D))) \leq \int_A Q(x) \cdot \eta_1^p(|x - x_0|) \, dm(x) = \frac{\omega_{n-1}}{I^{p-1}}.
$$

(2.12)

The first part of Proposition 2.2 is established. Let us prove the second part, namely, relation (2.10). Let $\Gamma_E$ and $\Gamma_{f(E)}$ be families of paths in the sense of the notation of Proposition 2.1. By this proposition

$$\operatorname{cap}_p f(E) = \operatorname{cap}_p(f(B(x_0, r_2)), f(B(x_0, r_1))) = M_p(\Gamma_{f(E)}).
$$

(2.13)

Let $\Gamma^*$ be a family of all maximal $f$-liftings of $\Gamma_{f(E)}$ starting in $B(x_0, r_1)$. Arguing similarly to the proof of Lemma 3.1 in [28], one can show that $\Gamma^* \subset \Gamma_E$. In what follows, we say that $\Gamma_1$ is minorized by $\Gamma_2$ and write $\Gamma_2 \prec \Gamma_1$ if every $\gamma \in \Gamma_1$ has a subpath which belongs to $\Gamma_2$. As known, see e.g. [36, Theorem 6.4], cf. [20, Proposition II.1.5], the condition $\Gamma_1 \succ \Gamma_2$ implies that $M_p(\Gamma_1) \leq M_p(\Gamma_2)$. Observe that $\Gamma_{f(E)} \succ f(\Gamma^*)$, and $\Gamma_E \succ \Gamma(S(x_0, r_2 - \delta), S(x_0, r_1), D)$ for sufficiently small $\delta > 0$. By (2.12), we obtain that

$$M_p(\Gamma_{f(E)}) \leq M_p(f(\Gamma^*)) \leq M_p(f(\Gamma_E))$$

$$\leq M_p(f(\Gamma(S(x_0, r_1), S(x_0, r_2 - \delta), A(r_1, r_2 - \delta, x_0))))$$

$$\leq \frac{\omega_{n-1}}{I^{p-1}} \cdot \left( \int_{r_1}^{r_2-\delta} \frac{dt}{t^{p-1} \frac{1}{q_{x_0}}(t)} \right)^{p-1}.
$$

(2.14)

Observe that a function $\tilde{\psi}(t) := \psi_{(r_1, r_2)} = \frac{1}{t^{p-1} \frac{1}{q_{x_0}}(t)}$ is integrable on $(r_1, r_2)$, because $I \neq \infty$. Hence, by the absolute continuity of the integral, we obtain that

$$\int_{r_1}^{r_2-\delta} \frac{dt}{t^{p-1} \frac{1}{q_{x_0}}(t)} \to \int_{r_1}^{r_2} \frac{dt}{t^{p-1} \frac{1}{q_{x_0}}(t)}$$

(2.15)
as $\delta \to 0$. By (2.14) and (2.15), we obtain that

$$M_p(\Gamma_{f(E)}) \leq \frac{\omega_{n-1}}{(r_2^2 - r_1^2) \int \frac{dt}{r_1^{n-1} q_x(r)^{p-1}}(t)}^{p-1}. \quad (2.16)$$

Combining (2.13) and (2.16), we obtain (2.10).

The next lemma contains an application of the previous Lemma 2.1 to mapping theory.

**Lemma 2.2** Let $D$ be a domain in $\mathbb{R}^n$, let $1 \leq p \leq n$, let $\Phi : [0, \infty] \to [0, \infty]$ be a strictly increasing convex function and let $x_0 \in D$. Denote by $\mathcal{R}_{\Phi, p}^{M_0}(D)$ the family of all discrete open mappings for which there exists a Lebesgue measurable function $Q = Q_f(x) : \mathbb{R}^n \to [0, \infty]$, $Q(x) \equiv 0$ for $x \in \mathbb{R}^n \setminus D$, satisfying (1.2)–(1.3) for any $x_0 \in D$, and, in addition, (2.3) holds for some $0 < M_0 < \infty$. Let $0 < r_1 < r_2 < d_0 = \text{dist}(x_0, \partial D)$, and let $E = (B(x_0, r_2), B(x_0, r_1))$ be a condenser. If the relation (2.2) holds for some $\delta_0 > \tau_0 := \Phi(0)$, then

$$\text{cap}_p f(E) \to 0$$

as $r_1 \to 0$ uniformly over $f \in \mathcal{R}_{\Phi, p}^{M_0}(D)$.

**Proof** By Proposition 2.2

$$\text{cap}_p f(E) \leq \frac{\omega_{n-1}}{I^{p-1}}, \quad (2.17)$$

where $\omega_{n-1}$ denotes an area of the unit sphere $S^{n-1} := S(0, 1)$ in $\mathbb{R}^n$, $I := \int_{r_1}^{r_2} \frac{dr}{r^{n-1} q_{x_0}(r)}$ and $q_{x_0}$ is defined in (2.1). The rest of the statement follows by Lemma 2.1. \qed

**3 Proof of Theorems 1.1 and 1.2**

The following statement was proved for $p = n$ in [13, Lemma 3.11] (see also [20, Lemma 2.6, Ch. III]).

**Proposition 3.1** Let $F$ be a compact proper subset of $\overline{\mathbb{R}^n}$ with $\text{cap} F > 0$. Then for every $a > 0$ there exists $\delta > 0$ such that

$$\text{cap}(\overline{\mathbb{R}^n} \setminus F, C) \geq \delta$$

for every continuum $C \subset \overline{\mathbb{R}^n} \setminus F$ with $h(C) \geq a$. 
Proof of Theorem 1.1 largely uses the classical scheme used in the quasiregular case, as well as applied by the author earlier, see, for example, [13,Theorem 4.1], [20,Theorem 2.9.III], [1,Theorem 8], [28,Lemma 3.1] and [26,Lemma 4.2].

Let \( x_0 \in D, \varepsilon_0 < d(x_0, \partial D) \), and let \( E = (A, C) \) be a condenser, where \( A = B(x_0, \varepsilon_0) \) and \( C = \overline{B(x_0, \varepsilon)} \). As usual, \( \varepsilon_0 := \infty \) for \( D = \mathbb{R}^n \). Let \( a > 0 \). Since \( \text{cap} \ E > 0 \), by Proposition 3.1 there exists \( \delta = \delta(a) > 0 \) such that

\[
\text{cap} \left( \mathbb{R}^n \setminus F, E \right) \geq \delta
\]  

for any continuum \( C \subset \mathbb{R}^n \setminus E \) such that \( h(C) \geq a \). On the other hand, by Lemma 2.2 there exists a function \( \alpha \) and a number \( \varepsilon_1 > 0 \) such that

\[
\text{cap} \ f(E) \leq \alpha(\varepsilon), \ \varepsilon \in (0, \varepsilon_1),
\]

for any \( f \in \mathcal{F}_{M_0,E}(D) \). Then, for a number \( \delta = \delta(a) \) there exists \( \varepsilon_* = \varepsilon_*(a) \) such that

\[
\text{cap} \ f(E) \leq \delta, \ \varepsilon \in (0, \varepsilon_*(a)).
\]  

By (3.2), we obtain that

\[
\text{cap} \left( \mathbb{R}^n \setminus E, f(B(x_0, \varepsilon)) \right) \leq \text{cap} \left( f(B(x_0, \varepsilon_0)), f(B(x_0, \varepsilon)) \right) \leq \delta
\]

for \( \varepsilon(0, \varepsilon_*(a)) \). Now, by (3.1), \( h(f(B(x_0, \varepsilon))) < a \). Finally, for any \( a > 0 \) there is \( \varepsilon_* = \varepsilon_*(a) \) such that \( h(f(B(x_0, \varepsilon))) < a \) for \( \varepsilon \in (0, \varepsilon_*(a)) \). Theorem is proved. \( \square \)

To prove Theorem 1.2, we need the following statement (see [16,(8.9)]).

Proposition 3.2 \ Given a condenser \( E = (A, C) \) and \( 1 < p < n \),

\[
\text{cap}_p E \geq n \Omega_p \left( \frac{n-p}{p-1} \right)^{p-1} \left[ m(C) \right]^{\frac{n-p}{n}},
\]

where \( \Omega_n \) denotes the volume of the unit ball in \( \mathbb{R}^n \), and \( m(C) \) is the \( n \)-dimensional Lebesgue measure of \( C \).

The basic lower estimate of capacity of a condenser \( E = (A, C) \) in \( \mathbb{R}^n \) is given by

\[
\text{cap}_p E = \text{cap}_p (A, C) \geq \left( b_n \frac{(d(C))^p}{(m(A))^{1-n+p}} \right)^\frac{1}{p-1}, \ \ p > n - 1,
\]  

where \( b_n \) depends only on \( n \) and \( p \) and \( d(C) \) denotes the diameter of \( C \) (see [9,Proposition 6], cf. [12,Lemma 5.9]).
Proof of Theorem 1.2 is based on the approach used in the proof of Lemma 2.4 in [4]. Let $0 < r_0 < \text{dist}(x_0, \partial D)$. Consider a condenser $E = (A, C)$ with $A = B(x_0, r_0)$, $C = B(x_0, \varepsilon)$. By Lemma 2.2, there is a function $\alpha = \alpha(\varepsilon)$ and $0 < \varepsilon'_0 < r_0$ such that $\alpha(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and, in addition,

$$\text{cap}_p f(E) \leq \alpha(\varepsilon)$$

for any $\varepsilon \in (0, \varepsilon'_0)$ and $f \in \mathcal{F}_{M_0, p}(D)$. Applying Proposition 3.2, one obtains

$$\alpha(\varepsilon) \geq \text{cap}_p f(E) \geq n \Omega_n \left( \frac{n-p}{p-1} \right)^{p-1} [m(f(C))]^{\frac{n-p}{n}}$$

where $\Omega_n$ denotes the volume of the unit ball in $\mathbb{R}^n$, and $m(C)$ stands for the $n$-dimensional Lebesgue measure of $C$. In other words,

$$m(f(C)) \leq \alpha_1(\varepsilon),$$

where $\alpha_1(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. The last relation implies the existence of a number $\varepsilon_1 \in (0, 1)$, such that

$$m(f(C)) \leq 1, \quad (3.4)$$

where $C = B(x_0, \varepsilon_1)$.

Further reasoning is related to the repeated application of Lemma 2.2. Consider one more condenser in this respect. Let $E_1 = (A_1, C_\varepsilon)$, $A_1 = B(x_0, \varepsilon_1)$, and $C_\varepsilon = B(x_0, \varepsilon), \varepsilon \in (0, \varepsilon_1)$. By Lemma 2.2 there is a function $\alpha_2(\varepsilon)$ and a number $0 < \varepsilon'_0 < \varepsilon_1$ such that

$$\text{cap}_p f(E_1) \leq \alpha_2(\varepsilon)$$

for any $\varepsilon \in (0, \varepsilon'_0)$, where $\alpha_2(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. On the other hand, according to (3.3),

$$\left( c_1 \left( \frac{d(f(B(x_0, \varepsilon))))^p}{m(f(B(x_0, \varepsilon_1))))^{1-n+p}} \right)^{\frac{1}{n-1}} \leq \text{cap}_p f(E_1) \leq \alpha_2(\varepsilon). \quad (3.5)$$

By (3.4) and (3.5), one gets

$$d(f(B(x_0, \varepsilon))) \leq \alpha_3(\varepsilon), \quad (3.6)$$

where $\alpha_3(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. The proof of Theorem 1.2 is completed, since the mapping $f \in \mathcal{F}_{M_0, p}(D)$ participating in (3.6) is arbitrary. $\square$
4 Proof of Theorems 1.3–1.6

The proofs of these theorems are conceptually close to the proofs of Theorems 1–4 in [33] and use the same approach. Let's start with the following very useful remark (see, for example, [33, Remark 1]).

Remark 4.1 Let us show that, for a given domain $D_i$, the relation (1.11) implies the so-called strong accessibility of its boundary with respect to $p$-modulus (see also [17, Theorem 6.2]). Let $i \in I$, let $x_0 \in \partial D_i$ and let $U$ be some neighborhood of $x_0$. We may assume that $x_0 \neq \infty$. Let $\varepsilon_1 > 0$ be such that $V := B(x_0, \varepsilon_1)$ and $V \subset U$. If $\partial U \neq \emptyset$ and $\partial V \neq \emptyset$, put $\varepsilon_2 := \text{dist} (\partial U, \partial V) > 0$. Let $F$ and $G$ be continua in $D_i$ such that $F \cap \partial U \neq \emptyset \neq F \cap \partial V$ and $G \cap \partial U \neq \emptyset \neq G \cap \partial V$. From the last relations it follows that $h(F) \geq \varepsilon_2$ and $h(G) \geq \varepsilon_2$. By the equi-uniformity of $D_i$ with respect to $p$-modulus, we may find $\delta = \delta(\varepsilon_2) > 0$ such that $M_p(\Gamma(F, G, D_i)) \geq \delta > 0$. In particular, for any neighborhood $U$ of $x_0$, there is a neighborhood $V$ of the same point, a compact set $F$ in $D_i$ and a number $\delta > 0$ such that $M_p(\Gamma(F, G, D_i)) \geq \delta > 0$ for any continuum $G \subset D_i$ such that $G \cap \partial U \neq \emptyset \neq G \cap \partial V$. This property is called a strong accessibility of $\partial D_i$ at the point $x_0$ with respect to $p$-modulus. Thus, this property is established for any domain $D_i$ which is an element of some equi-uniform family $\{D_i\}_{i \in I}$.

Proof of Theorem 1.3 The equicontinuity of the family $S_{\Phi, A, p, \delta}^{M_0}(D)$ inside the domain $D$ follows from [21, Theorem 4.1] for $p = n$ and Theorem 1.2 for $p \neq n$. Put $f \in S_{\Phi, A, p, \delta}^{M_0}(D)$ and $Q = Q_f(x)$. Set

$$Q'(x) = \begin{cases} Q(x), & Q(x) \geq 1 \\ 1, & Q(x) < 1 \end{cases}.$$

Observe that $Q'(x)$ satisfies (1.6) up to a constant. Indeed,

$$\int_D \Phi(Q'(x)) \frac{dm(x)}{(1 + |x|^2)^n} = \int_{\{x \in D: Q(x) < 1\}} \Phi(Q'(x)) \frac{dm(x)}{(1 + |x|^2)^n}$$

$$+ \int_{\{x \in D: Q(x) \geq 1\}} \Phi(Q'(x)) \frac{dm(x)}{(1 + |x|^2)^n} \leq M_0 + \Phi(1) \int_{\mathbb{R}^n} \frac{dm(x)}{(1 + |x|^2)^n} = M_0' < \infty.$$

Thus, by [29, Theorem 2] and Remark 4.1, a mapping $f \in S_{\Phi, A, p, \delta}^{M_0}(D)$ has a continuous extension to $\overline{D}$ for $p = n$. Let $p \neq n$. By Lemma 2.1,

$$\int_0^{r_0} \frac{dt}{t^{n-1} q_{x_0}^{'(t)}} = \infty,$$

where $q_{x_0}^{'(t)} = \frac{1}{\omega_{n-1} |t^{n-1}|} \int_{S(x_0, t)} Q'(x) d\mathcal{H}^{n-1}$. In this case, a continuous extension of the mapping $f$ to $\partial D$ can be established similarly to Theorem 1 in [29]. Note that a
rigorous proof of this fact was given in [6, Theorem 1.2] for the case when the domains $D$ and $f(D)$ have compact closures, and its proof in an arbitrary case can be presented completely by analogy.

It remains to show that the family $\mathcal{F}_{\Phi, A, p, \delta}(D)$ is equicontinuous at $\partial D$. Suppose the opposite. Then there is $x_0 \in \partial D$ for which $\mathcal{F}_{\Phi, A, p, \delta}(D)$ is not equicontinuous at $x_0$.

Due to the additional application of the inversion $\varphi(x) = \frac{x}{|x|^2}$, we may assume that $x_0 \neq \infty$. Then there is a number $a > 0$ with the following property: for any $m = 1, 2, \ldots$ there is $x_m \in D$ and $f_m \in \mathcal{F}_{\Phi, A, p, \delta}(D)$ such that $|x_0 - x_m| < 1/m$ and, in addition, $h(f_m(x_m), f_m(x_0)) \geq a$. Since $f_m$ has a continuous extension at $x_0$, we may find a sequence $x'_m \in D$, $x'_m \to x_0$ as $m \to \infty$ such that $h(f_m(x'_m), f_m(x_0)) \leq 1/m$.

Thus,

$$h(f_m(x_m), f_m(x'_m)) \geq a/2 \quad \forall m \in \mathbb{N}. \quad \text{(4.1)}$$

Since $f_m$ has a continuous extension to $\partial D$, we may assume that $x_m \in D$. Since $D$ is locally connected at the point $x_0$, there is a sequence of neighborhoods $V_m$ of the point $x_0$ with $h(V_m) \to 0$ as $m \to \infty$ such that the sets $D \cap V_m$ are domains and $D \cap V_m \subset B(x_0, 2^{-m})$. Without loss of the generality of reasoning, going to subsequences, if necessary, we may assume that $x_m, x'_m \in D \cap V_m$. Join the points $x_m$ and $x'_m$ by a path $\gamma_m : [0, 1] \to \mathbb{R}^n$ such that $\gamma_m(0) = x_m$, $\gamma_m(1) = x'_m$ and $\gamma_m(t) \in V_m$ for $t \in (0, 1)$, see Fig. 1.

We denote by $C_m$ the image of the path $\gamma_m(t)$ under the mapping $f_m$. From the relation (4.1) it follows that

$$h(C_m) \geq a/2 \quad \forall m \in \mathbb{N}, \quad \text{(4.2)}$$

where $h(C_m)$ denotes the chordal diameter of the set $C_m$.
Let $\varepsilon_0 := \text{dist} (x_0, A)$. Without loss of the generality of reasoning, one may assume that the continuum $A$ participating in the definition of the class $\mathfrak{R}_{\Phi, \delta, p, E}^M_0 (D)$ lies outside the balls $B(x_0, 2^{-m})$, $m = 1, 2, \ldots$, and $B(x_0, \varepsilon_0) \cap A = \emptyset$. In this case, the property of connected sets that lie neither inside nor outside the given set implies the relation

$$\Gamma_m > \Gamma (S(x_0, 2^{-m}), S(x_0, \varepsilon_0), D), \quad (4.3)$$

see e.g. [10, Theorem 1.1.5.46]. Using Proposition 2.2 and by (2.4), (4.3), we obtain that

$$M_p (f_m (\Gamma_m)) \leq \frac{\omega_{n-1}}{\omega_{n-1} r_0} \frac{\int_{r_0} dr}{\int_{2^{-m}}^{n-1} \frac{dr}{\mu_m (r)}} \rightarrow 0, \quad m \rightarrow \infty, \quad (4.4)$$

where $q_{m,0} (t) = \frac{1}{\omega_{n-1} r_0} \int_{S(x_0, t)} Q_m (x) dH^{n-1}$ and $Q_m$ corresponds to the function $Q$ of $f_m$ in (1.2). On the other hand, observe that $f_m (\Gamma_m) = \Gamma (C_m, f_m (A), D'_m)$. By the condition of the lemma, $h(f_m (A)) \geq \delta$ for any $m \in \mathbb{N}$. Therefore, by (4.2) $h(f_m (A)) \geq \delta_1$ and $h(C_m) \geq \delta_1$, where $\delta_1 := \min [\delta, \alpha/2]$. Taking into account that the domains $D'_m := f_m (D)$ are equi-uniform with respect to $p$-modulus, we conclude that there exists $\sigma > 0$ such that

$$M_p (f_m (\Gamma_m)) = M_p (\Gamma (C_m, f_m (A), D'_m)) \geq \sigma \quad \forall \ m \in \mathbb{N},$$

which contradicts the condition (4.4). The resulting contradiction indicates that the assumption about the absence of equicontinuity of $\mathfrak{R}_{\Phi, \delta, p, E}^M_0 (\overline{D})$ was wrong. The resulting contradiction completes the proof. □

**Proof of Theorem 1.4** The equicontinuity of the family $\mathfrak{R}_{\Phi, \delta, p, E}^M_0 (D)$ inside the domain $D$ follows from Theorem 1.1 for $p = n$ and Theorem 1.2 for $p \neq n$. The possibility of continuous extension of any mapping $f \in \mathfrak{R}_{\Phi, \delta, p, E}^M_0 (D)$ to $\partial D$ is established in the same way as at the beginning of the proof of Theorem 1.3, and therefore the proof of this fact is omitted.

It remains to show that the family $\mathfrak{R}_{\Phi, \delta, p, E}^M_0 (D)$ is equicontinuous at $\partial D$. Suppose the opposite. Then there is $x_0 \in \partial D$ for which $\mathfrak{R}_{\Phi, \delta, p, E}^M_0 (D)$ is not equiuniform at $x_0$. Due to the additional application of the inversion $Q(x) = \frac{x}{|x|^2}$, we may assume that $x_0 \neq \infty$. Then there is a number $a > 0$ with the following property: for any $m = 1, 2, \ldots$ there is $x_m \in \overline{D}$ and $f_m \in \mathfrak{R}_{\Phi, \delta, p, E}^M_0 (D)$ such that $|x_0 - x_m| < 1/m$ and, in addition, $h(f_m (x_m), f_m (x_0)) \geq a$. Since $f_m$ has a continuous extension at $x_0$, we may assume that $x_m \in D$. Besides that, we may find a sequence $x'_m \in D$, $x'_m \to x_0$ as $m \to \infty$ such that $h(f_m (x'_m), f_m (x_0)) \leq 1/m$. Now, the relation (4.1) holds. Since the domain $D$ is locally connected at the point $x_0$, there is a sequence of neighborhoods $V_m$ of the point $x_0$ with $h(V_m) \to 0$ for $m \to \infty$ such that the sets $D \cap V_m$ are domains and $D \cap V_m \subset B(x_0, 2^{-m})$. Without loss of the generality of reasoning,
going to subsequences, if necessary, we may assume that \( x_m, x_m' \in D \cap V_m \). Join the points \( x_m \) and \( x_m' \) by a path \( \gamma_m : [0, 1] \to \mathbb{R}^n \) such that \( \gamma_m(0) = x_m, \gamma_m(1) = x_m' \) and \( \gamma_m(t) \in V_m \) for \( t \in (0, 1) \), see Fig. 2.

We denote by \( C_m \) the image of the path \( \gamma_m \) under the mapping \( f_m \). It follows from the relation (4.1) that a condition (4.2) is satisfied, where \( h \) denotes a chordal diameter of the set.

By the definition of the family of mappings \( \mathcal{R}_{\Phi, \delta, p, E}^M (D) \), for any \( m = 1, 2, \ldots \), any \( f_m \in \mathcal{R}_{\Phi, \delta, p, E}^M (D) \) and any domain \( D'_m := f_m(D) \) there is a continuum \( K_m \subset D'_m \) such that \( h(K_m) \geq \delta \) and \( h(f_m^{-1}(K_m), \partial D) \geq \delta > 0 \). Since, by the hypothesis of the lemma, the domains \( D'_m \) are equi-uniform with respect to \( p \)-modulus, by (4.2) we obtain that

\[
M_p(\Gamma(K_{m}, C_{m}, D'_m)) \geq b \tag{4.5}
\]

for any \( m = 1, 2, \ldots \) and some \( b > 0 \). Let \( \Gamma_m \) be a family of all paths \( \beta : [0, 1) \to D'_m \) such that \( \beta(0) \in C_m \) and \( \beta(t) \to p \in K_m \) as \( t \to 1 \). Recall that a path \( \alpha : [a, b) \to \mathbb{R}^n \) is called a (total) \( f \)-lifting of a path \( \beta : [a, b) \to \mathbb{R}^n \) starting at \( x_0 \), if \( (f \circ \alpha)(t) = \beta(t) \) for any \( t \in [a, b) \). Let \( \Gamma^*_m \) be a family of all total \( f_m \)-liftings \( \alpha : [0, 1) \to D \) of \( \Gamma_m \) starting at \( \gamma_m \) (such a family is well-defined by [37, Theorem 3.7]). Since the mapping \( f_m \) is closed, we obtain that \( \alpha(t) \to f_m^{-1}(K_m) \) as \( t \to b - 0 \), where \( f_m^{-1}(K_m) \) denotes the pre-image of \( K_m \) under \( f_m \). Since \( \mathbb{R}^n \) is a compact metric space, the set \( C^\delta := \{ x \in D : h(x, \partial D) \geq \delta \} \) is compact in \( D \) for any \( \delta > 0 \) and, besides that, \( f_m^{-1}(K_m) \subset C^\delta \). In this case, we may assume that \( \text{dist}(x_0, E^\delta) \geq \varepsilon_0 \) by decreasing \( \varepsilon_0 \). By the property of connected sets that lie neither inside nor outside the given set, we obtain that

\[
\Gamma^*_m > \Gamma(S(x_0, 2^{-m}), S(x_0, \varepsilon_0), D), \tag{4.6}
\]

see e.g. [10, Theorem 1.1.5.46]. Using Proposition 2.2 and by (2.4), (4.6), we obtain that

\[
M_p(f_m(\Gamma^*_m)) \leq M_p(f_m(\Gamma(S(x_0, 2^{-m}), S(x_0, \varepsilon_0), D)))
\]

\[
\leq \frac{\omega_{n-1}}{\int_{S(x_0, r)} Q_m(x) d\mathcal{H}^{n-1}} \to 0, \quad m \to \infty, \tag{4.7}
\]

where \( q_{m,x_0}(t) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{S(x_0, t)} Q_m(x) d\mathcal{H}^{n-1} \) and \( Q_m \) corresponds to \( f_m \) in (1.2). Observe that \( f_m(\Gamma^*_m) = \Gamma_m \) and \( M_p(\Gamma_m) = M_p(\Gamma(K_m, C_m, D'_m)) \), so that

\[
M_p(f_m(\Gamma^*_m)) = M_p(\Gamma(K_m, C_m, D'_m)). \tag{4.8}
\]
However, the relations (4.7) and (4.8) together contradict (4.5). The resulting contradiction indicates that the original assumption (4.1) was incorrect, and therefore the family of mappings $\mathcal{F}_{\Phi_1, \delta, p, E}(D)$ is equicontinuous at every point $x_0 \in \partial D$. \qed

**Proof of Theorem 1.5** The equicontinuity of the family $\mathcal{F}_{M_0, A, p, \delta}(D)$ inside the domain $D$ follows from [21, Theorem 4.1] for $p = n$ and Theorem 1.2 for $p \neq n$. The existence of a continuous extension of each $f \in \mathcal{F}_{M_0, A, p, \delta}(D)$ to a continuous mapping in $\overline{D}$ follows from [32, Lemma 3]. In particular, the strong accessibility of $D'_f = f(D)$ with respect to $p$-modulus follows by Remark 4.1.

Let us show the equicontinuity of the family $\mathcal{F}_{M_0, A, p, \delta}(D)$ at $ED$, where $ED$ denotes the space of prime ends in $D$. Suppose the contrary, namely, that the family $\mathcal{F}_{M_0, A, p, \delta}(D)$ is not equicontinuous at some point $P_0 \in ED$. Then there is a number $a > 0$, a sequence $P_k \in \overline{D}_p$, $k = 1, 2, \ldots$, and elements $f_k \in \mathcal{F}_{M_0, A, p, \delta}(D)$ such that $d(P_k, P_0) < 1/k$ and

$$h(f_k(P_k), f_k(P_0)) \geq a \quad \forall \quad k = 1, 2, \ldots. \quad (4.9)$$

Since $f_k$ has a continuous extension to $\overline{D}_p$, for any $k \in \mathbb{N}$ there is $x_k \in D$ such that $d(x_k, P_k) < 1/k$ and $h(f_k(x_k), f_k(P_k)) < 1/k$. Now, by (4.9) we obtain that

$$h(f_k(x_k), f_k(P_0)) \geq a/2 \quad \forall \quad k = 1, 2, \ldots. \quad (4.10)$$

Similarly, since $f_k$ has a continuous extension to $\overline{D}_p$, there is a sequence $x'_k \in D$, $x'_k \to P_0$ as $k \to \infty$ for which $h(f_k(x'_k), f_k(P_0)) < 1/k$ for $k = 1, 2, \ldots$. Now, it follows from (4.10) that

$$h(f_k(x_k), f_k(x'_k)) \geq a/4 \quad \forall \quad k = 1, 2, \ldots. \quad (4.11)$$

where $x_k$ and $x'_k$ belong to $D$ and converge to $P_0$ as $k \to \infty$, see Fig. 3. \qed
By [7,Lemma 3.1], cf. [8,Lemma 2], a prime end $P_0$ of a regular domain $D$ contains a chain of cuts $\sigma_k$ lying on spheres $S_k$ centered at some point $x_0 \in \partial D$ and with Euclidean radii $r_k \to 0$ as $k \to \infty$. Let $D_k$ be domains associated with the cuts $\sigma_k$, $k = 1, 2, \ldots$. Since the sequences $x_k$ and $x_k'$ converge to the prime end $P_0$ as $k \to \infty$, we may assume that $x_k$ and $x_k' \in D_k$ for any $k = 1, 2, \ldots$. Let us join the points $x_k$ and $x_k'$ by a path $\gamma_k$, completely lying in $D_k$. One can also assume that the continuum $A$ from the definition of the class $\tilde{S}^M_{\Phi, A, p, \delta}(\overline{D})$ does not intersect with any of the domains $D_k$, and that $\text{dist}(\partial D, A) > \varepsilon_0$.

We denote by $C_k$ the image of the path $\gamma_k$ under the mapping $f_k$. It follows from the relation (4.11) that

$$h(C_k) \geq \frac{a}{4} \quad \forall k \in \mathbb{N},$$

where $h$ is a chordal diameter of the set.

Let $\Gamma_k$ be a family of all paths joining $|\gamma_k|$ and $A$ in $D$. By [10,Theorem 1.1.5.46],

$$\Gamma_k > \Gamma(S(x_0, r_k), S(x_0, \varepsilon_0), D).$$

Using Proposition 2.2 and by (2.4), (4.13) we obtain that

$$M_p(f_k(\Gamma_k)) \leq M_p(f_k(\Gamma(S(x_0, r_k), S(x_0, \varepsilon_0), D)))$$

$$\leq \frac{\omega_{n-1}}{r_0^{p-1}} \int_{r_k}^{r_0} \frac{dr}{(r_0)^{p-1} q_{kx_0}(r)}\rightarrow 0, \quad k \to \infty,$$

where $q_{kx_0}(t) = \frac{1}{\omega_{n-1} t^{n-1}} \int_{S(x_0, t)} Q_k(x) \, d\mathcal{H}^{n-1}$ and $Q_k$ corresponds to the function $Q$ of $f_k$ in (1.2).
On the other hand, note that \( f_k(\Gamma_k) = \Gamma(C_k, f_k(A), D'_k) \), where \( D'_k = f_k(D) \).

Since by the hypothesis of the lemma \( h(f_k(A)) \geq \delta \) for any \( k \in \mathbb{N} \), by (4.12), \( h(f_k(A)) \geq \delta_1 \) and \( h(C_k) \geq \delta_1 \), where \( \delta_1 := \min(\delta, a/4) \). Using the fact that the domains \( D'_k \) are equi-uniform with respect to \( p \)-modulus, we conclude that there is \( \sigma > 0 \) such that

\[
M_p(f_k(\Gamma_k)) = M_p(\Gamma(C_k, f_k(A), D'_k)) \geq \sigma \quad \forall k \in \mathbb{N},
\]

which contradicts condition (4.14). The resulting contradiction indicates that the assumption of the absence of an equicontinuity of the family \( \mathcal{R}_{\Phi,A,p,\delta}(D) \) was wrong. The resulting contradiction completes the proof of the theorem. \( \square \)

**Proof of Theorem 1.6** The equicontinuity of the family \( \mathcal{R}_{\Phi,q,p,E}(D) \) inside the domain \( D \) follows from [21, Theorem 4.1] for \( p = n \) and Theorem 1.2 for \( p \neq n \). The existence of a continuous extension of each \( f \in \mathcal{R}_{\Phi,q,p,E}(D) \) to a continuous mapping in \( \overline{D} \) follows from [32, Lemma 3]. In particular, the strong accessibility of \( D'_f = f(D) \) with respect to \( p \)-modulus follows by Remark 4.1.

It remains to show that the family \( \mathcal{R}_{\Phi,q,p,E}(D) \) is equicontinuous at \( E_D := \overline{D} \setminus D \). Suppose the opposite. Arguing as in the proof of Theorem 1.5, we construct two sequences \( x_k \) and \( x'_k \) in \( D \), converging to the prime end \( P_0 \) as \( k \to \infty \), for which a relation (4.11) holds. Let us join the points \( x_k \) and \( x'_k \) of the path \( \gamma_k : [0, 1] \to \mathbb{R}^n \) such that \( x'_k \in D \), \( \gamma_k(0) = x_k \), \( \gamma_k(1) = x'_k \) and \( \gamma_k(t) \in D \) for \( t \in (0, 1) \). Denote by \( C_k \) the image of \( \gamma_k \) under the mapping \( f_k \). It follows from the relation (4.11) that

\[
h(C_k) \geq a/4 \quad \forall k = 1, 2, \ldots \tag{4.15}
\]

By [7, Lemma 3.1], cf. [8, Lemma 2], a prime end \( P_0 \) of a regular domain \( D \) contains a chain of cuts \( \sigma_k \) lying on spheres \( S_k \) centered at some point \( x_0 \in \partial D \) and with Euclidean radii \( r_k \to 0 \) as \( k \to \infty \). Let \( D_k \) be domains associated with the cuts \( \sigma_k \), \( k = 1, 2, \ldots \). Since the sequences \( x_k \) and \( x'_k \) converge to the prime end \( P_0 \) as \( k \to \infty \), we may assume that \( x_k \) and \( x'_k \) in \( D_k \) for any \( k = 1, 2, \ldots \).

By the definition of the family \( \mathcal{R}_{\Phi,q,p,E}(D) \), for every \( f_k \in \mathcal{R}_{\Phi,q,p,E}(D) \) and any domain \( D'_k := f_k(D) \) there is a continuum \( K_k \subset D'_k \) such that \( h(K_k) \geq \delta \) and

\[
h(f^{-1}(K_k), \partial D) \geq \delta > 0.
\]

Since, by the condition of the lemma, the domains \( D'_k \) are equi-uniform with respect to \( p \)-modulus, by (4.15) we obtain that

\[
M_p(\Gamma(K_k, C_k, D'_k)) \geq b \tag{4.16}
\]

for any \( k = 1, 2, \ldots \) and some \( b > 0 \). Let \( \Gamma_k \) be a family of all paths \( \beta : [0, 1] \to D'_k \), where \( \beta(0) \in C_k \) and \( \beta(t) \to p \in K_k \) as \( t \to 1 \). Let \( \Gamma^* \) be a family of all total liftings \( \alpha : [0, 1] \to D \) of \( \Gamma_k \) under the mapping \( f_k \) starting at \( \gamma_k \). Such a family is well-defined by [37, Theorem 3.7]. Since \( f_k \) is closed, \( \alpha(t) \to f_k^{-1}(K_k) \) as \( t \to 1 \), where \( f_k^{-1}(K_k) \) denotes the pre-image of \( K_k \) under \( f_k \). Since \( \overline{\mathbb{M}}^n \) is a compact metric space, the set \( C_\delta := \{ x \in D : h(x, \partial D) \geq \delta \} \) is compact in \( D \) for any \( \delta > 0 \) and,
besides that, \( f_k^{-1}(K_k) \subset C_\delta \). In this case, we may assume that \( \text{dist}(x_0, E_\delta) \geq \epsilon_0 \) by decreasing \( \epsilon_0 \). By [10, Theorem 1.1.5.46],

\[
\Gamma_k^* > \Gamma(S(x_0, r_k), S(x_0, \epsilon_0), D).
\]

(4.17)

Using Proposition 2.2 and by (2.4), (4.17) we obtain that

\[
M_p(f_k(\Gamma_k^*)) \leq M_p(f_k(\Gamma(S(x_0, r_k), S(x_0, \epsilon_0), D))) \leq \omega_{n-1} \int_{r_k}^{r_0} \frac{dr}{r^{p-1} q_{kx_0}^{-1}(r)} \to 0, \quad k \to \infty,
\]

(4.18)

where \( q_{kx_0}(t) = \frac{1}{\omega_{n-1} t^{n-1}} \int_{S(x_0, t)} Q_k(x) d\mathcal{H}^{n-1} \) and \( Q_k \) corresponds to the function \( Q \) of \( f_k \) in (1.2). Observe that \( f_k(\Gamma_k^*) = \Gamma_k \) and, simultaneously, \( M_p(\Gamma_k) = M_p(\Gamma(K_k, C_k, D_k')) \). Now

\[
M_p(f_k(\Gamma_k^*)) = M_p(\Gamma(K_k, C_k, D_k')).
\]

(4.19)

Combining (4.18) and (4.19), we obtain a contradiction with (4.16). The resulting contradiction indicates that the initial assumption (4.1) was incorrect, and, therefore, the family of mappings \( \mathcal{M}_{\Phi, \delta, p, E}(D) \) is equicontinuous at any point \( x_0 \in E_D \). \( \square \)

**Data availability** The datasets generated and/or analysed during the current study are available from the corresponding author on reasonable request.

**Conflict of interest** The author has no financial or proprietary interests in any material discussed in this article.

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