Completely integrable curve flows on Adjoint orbits

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Dedicated to Professor S. S. Chern on his 90th Birthday

Abstract

It is known that the Schrödinger flow on a complex Grassmann manifold is equivalent to the matrix non-linear Schrödinger equation and the Ferapontov flow on a principal Adjoint $U(n)$-orbit is equivalent to the $n$-wave equation. In this paper, we give a systematic method to construct integrable geometric curve flows on Adjoint $U$-orbits from flows in the soliton hierarchy associated to a compact Lie group $U$. There are natural geometric bi-Hamiltonian structures on the space of curves on Adjoint orbits, and they correspond to the order two and three Hamiltonian structures on soliton equations under our construction. We study the Hamiltonian theory of these geometric curve flows and also give several explicit examples.

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1. Introduction

There are several natural geometric flows on Adjoint orbits that are known to be equivalent to soliton equations. The first example is the Heisenberg ferromagnetic model (HFM) for \( \gamma : \mathbb{R}^2 \to S^2 \),

\[
\gamma_t = \gamma \times \gamma_{xx}, \quad \text{(HFM)}
\]

where \( \times \) is the cross product in \( \mathbb{R}^3 \). It was proved in [FT] that the HFM is equivalent to the non-linear Schrödinger equation (NLS):

\[
q_t = (q_{xx} + 2 |q|^2 q). \quad \text{(NLS)}
\]

The second example is the Schrödinger flow on the Hermitian symmetric space \( \text{Gr}(k, \mathbb{C}^n) \). Recall that the Schrödinger flow on a Kähler manifold \( M \) is the evolution equation on the space of maps from \( \mathbb{R} \) to \( M \):

\[
\gamma_t = j_\gamma (\nabla_{\gamma_x} \gamma_x),
\]

where \( j \) is the complex structure and \( \nabla \) is the Levi-Civita connection of the Kähler metric on \( M \). The Adjoint \( U(n) \)-orbit \( M \) at

\[
a = \frac{1}{2} \begin{pmatrix} i \text{Id}_k & 0 \\ 0 & -i \text{Id}_{n-k} \end{pmatrix} \quad \text{(1.1)}
\]
equipped with the induced metric from the inner product \( \langle u_1, u_2 \rangle = -\text{tr}(u_1 u_2) \) on \( u(n) \) is isometric to the Hermitian symmetric space \( \text{Gr}(k, \mathbb{C}^n) \), and \( j_x = \text{ad}(x) \) is the complex structure on \( \text{Gr}(k, \mathbb{C}^n) \). Terng and Uhlenbeck showed in [TU2] that the Schrödinger flow on \( \text{Gr}(k, \mathbb{C}^n) \) is

\[
\gamma_t = j_\gamma (\nabla_{\gamma_x} \gamma_x) = [\gamma, \gamma_{xx}], \quad \text{(1.2)}
\]

and is equivalent to the matrix non-linear Schrödinger equation (MNLS) for maps \( q \) from \( \mathbb{R}^2 \) to the space \( M_{k \times (n-k)} \) of \( k \times (n-k) \) complex matrices:

\[
q_t = (q_{xx} + 2qq^*q), \quad \text{(MNLS)}
\]

where \( q^* = \bar{q}^t \). Note that when \( n = 2 \), \( M \) is isometric to the round sphere. If we identify \( su(2) \) with \( \mathbb{R}^3 \) in the usual way, then \( [\xi, \eta] \) corresponds to the cross product \( \xi \times \eta \). So the Schrödinger flow (1.2) on \( S^2 \) is the HFM.

The third example is due to Ferapontov. Let \( \mathcal{U} \) denote the Lie algebra of a compact Lie group \( U \). An element \( a \in \mathcal{U} \) is called regular (singular respectively) if the Adjoint \( U \)-orbit \( M_a \) at \( a \) in \( \mathcal{U} \) is a principal (singular respectively) orbit. Let \( \langle \cdot, \cdot \rangle \) denote an Ad-invariant inner product on \( \mathcal{U} \), \( \mathcal{U}_a \) the isotropy subalgebra of \( a \), and \( \mathcal{U}_a^\perp \) the orthogonal complement of \( \mathcal{U}_a \) in \( \mathcal{U} \) with respect to \( \langle \cdot, \cdot \rangle \). If \( a \in \mathcal{U} \) is regular, then it is known that ([Te1])

\[
\begin{align*}
\diamond & \quad \text{the normal bundle } \nu(M_a) \text{ is flat}, \\
\diamond & \quad \nu(M_a)_x = \mathcal{U}_x, \text{ which is the maximal abelian subalgebra of } \mathcal{U} \text{ containing } x, \\
\diamond & \quad \text{given any } b \in \mathcal{U}_a, \text{ the map } \hat{b} \text{ defined by }
\end{align*}
\]

\[
\hat{b}(gag^{-1}) = gbg^{-1}, \quad g \in U, \quad \text{(1.3)}
\]

is a well-defined parallel normal field of \( M_a \).

Ferapontov proved in [F4] that a solution \( u : \mathbb{R}^2 \to \mathcal{U}_a^\perp \) of the \( n \)-wave equation associated to \( \mathcal{U} \),

\[
u_t = \text{ad}(b) \text{ad}(a)^{-1}(u_x) + [u, \text{ad}(b) \text{ad}(a)^{-1}(a)], \quad \text{(1.4)}
\]
gives rise to a solution of the following curve flow on \( M_a \):

\[
\gamma_t = (\hat{b}(\gamma))_x = -A_{b(\gamma)}(\gamma_x). \quad \text{(1.5)}
\]
Here $A_v$ is the shape operator along $v$.

The MNLS and the $n$-wave equation (1.4) are flows in the $U(n)$- and $U$-hierarchy of soliton flows respectively. The $U$-hierarchy of soliton flows is obtained by restricting the ANKS-ZS hierarchy to an invariant submanifold associated to the reality condition given by the real group $U$. One goal of this paper is to give a systematic method to construct geometric curve flows on an Adjoint $U$-orbit for each flow in the $U$-hierarchy that includes all three examples given above.

We review the construction of the $U$-hierarchy next. Let $a \in \mathcal{U}$, and $S(\mathbb{R}, \mathcal{U}^\perp_a)$ the space of maps from $\mathbb{R}$ to $\mathcal{U}^\perp_a$ that decay rapidly at infinity. The $U$-hierarchy defined by $a$ is a collection of commuting Hamiltonian flows on $S(\mathbb{R}, \mathcal{U}^\perp_a)$ (cf. [TU1]). When $a$ is regular, the $U$-hierarchy is parametrized by $(b, j)$ with $b \in \mathcal{U}_a$ and $j$ positive integer. The $(b, j)$-flow is

$$u_t = (Q_{b,j}(u))_x + [u, Q_{b,j}(u)], \quad (1.6)$$

where $Q_{b,j}(u)$ are $U$-valued maps determined by the following conditions:

$$(Q_{b,j}(u))_x + [u, Q_{b,j}(u)] = [Q_{b,j+1}(u), a], \quad Q_{b,0}(u) = b, \quad (1.7)$$

and

$$\sum_{j=0}^{\infty} Q_{b,j}(u)\lambda^{-j} \text{ is conjugate to } b \text{ as an asymptotic expansion.} \quad (1.8)$$

These conditions imply that $Q_{b,j}(u)$ is a polynomial in $u, \partial_x u, \ldots, \partial_x^{-1} u$ and for $j \geq 1$

$$Q_{b,j}(u) \in S(\mathbb{R}, \mathcal{U}) \quad \text{if } u \in S(\mathbb{R}, \mathcal{U}^\perp_a). \quad (1.9)$$

For more detail see [Sa] and [TU1].

When $a$ is singular, the $U$-hierarchy is the collection of $(a, j)$-flows:

$$u_t = (Q_{a,j}(u))_x + [u, Q_{a,j}(u)]. \quad \text{Recall that a } G \text{-valued connection 1-form } w = Adx + Bdt \text{ is flat if } dw = -w \wedge w \text{ or equivalently,}$$

$$A_t - B_x = [A, B].$$

The recursive formula (1.7) implies that $u$ is a solution of the $(b, j)$-flow (1.6) if and only if

$$\theta_\lambda = (a \lambda + u) dx + (b \lambda^j + Q_{b,1}(u)\lambda^{j-1} + \cdots + Q_{b,j}(u)) dt \quad (1.10)$$

is a flat $\mathcal{U}_\mathbb{C}$-valued connection 1-form over the $(x, t)$ plane for all $\lambda \in \mathbb{C}$. The 1-form $\theta_\lambda$ is called a Lax pair of the $(b, j)$-flow.

For example:

- If $U = SU(2)$ and $a = \frac{1}{2} \text{ diag}(i, -i)$, then $\mathcal{U}_a^\perp = \left\{ \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix} \left| q \in \mathbb{C} \right. \right\}$. Identify $S(\mathbb{R}, \mathcal{U}_a^\perp)$ with $S(\mathbb{R}, \mathbb{C})$. The $(a, 2)$-flow in the $SU(2)$-hierarchy is the NLS.

- Let $a \in \mathcal{U}$ be a regular element, and $b \in \mathcal{U}_a$. The $(b, 1)$-flow in the $U$-hierarchy on $S(\mathbb{R}, \mathcal{U}_a^\perp)$ is the $n$-wave equation (1.4).

- Let $a \in u(n)$ be as in (1.1). Then $\mathcal{U}_a^\perp$ is the space of $\begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}$ with $q$ a $k \times (n-k)$ complex matrix. Identify $u$ with $q$. The $(a, 2)$-flow of the $U(n)$-hierarchy on $S(\mathbb{R}, \mathcal{U}_a^\perp)$ is the MNLS.

- If $U/K$ is a Hermitian symmetric space, then (cf. [H], [W]) there exists $a \in \mathcal{K}$ such that $\mathcal{U}_a = \mathcal{K}$, ad$(a)^2 = -\text{Id}$ on $\mathcal{U}_a^\perp$, and $\mathcal{U} = \mathcal{U}_a + \mathcal{U}_a^\perp$ is a Cartan decomposition. The Adjoint $U$-orbit at $a$ in $\mathcal{U}$ is an isometric embedding of the Hermitian symmetric space $U/K$ into
Euclidean space $\mathcal{U}$. A direct computation shows that the $(a, 2)$-flow in the $U$-hierarchy on $S(\mathbb{R}, U_a^\perp)$ is

$$u_t = [a, u_{xx}] - \frac{1}{2} [u, [u, [a, u]]]. \tag{1.11}$$

When $U/K$ is the complex Grassmannian $\text{Gr}(k, \mathbb{C}^n)$, then $a$ is given by (1.1) and equation (1.11) is the MNLS with $u = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}$.

It is well-known in soliton theory (cf. [TU2]) that there are two Poisson operators on $S(\mathbb{R}, U_a^\perp)$ so that flows in the $U$-hierarchy on $S(\mathbb{R}, U_a^\perp)$ are commuting Hamiltonian flows with respect to both Poisson structures. The first Poisson operator is

$$J_a(v) = [v, a]. \tag{1.12}$$

The second Poisson operator $P_u$ is defined as follows:

$$P_u(v) = v_x + \pi_1([u, v]) + [u, h], \tag{1.13}$$

where

$$h(x) = -\int_{-\infty}^{x} \pi_0([u(s), v(s)])ds,$$

and $\pi_0, \pi_1$ are the orthogonal projection of $\mathcal{U}$ onto $\mathcal{U}_a$ and $\mathcal{U}_a^\perp$ respectively. Moreover, for each $k$,

$$(J_k)_a(v) = J_a(J_a^{1} P_u)^k = (P_u J_a^{1})^k J_a \tag{1.14}$$

is also a Poisson operator on $S(\mathbb{R}, U_a^\perp)$.

The Hamiltonian of the $(b, j)$-flow with respect to $J_a$ is $F_{b,j} : S(\mathbb{R}, U_a^\perp) \to \mathbb{R}$ defined by

$$F_{b,j}(u) = -\frac{1}{j+1} \int_{-\infty}^{\infty} \langle Q_{b,j+2}(u), a \rangle \, dx. \tag{1.15}$$

In other words

$$\nabla F_{b,j}(u) = \pi_1(Q_{b,j+1}(u)), \tag{1.16}$$

and the $(b, j)$-flow is

$$u_t = J_a(\nabla F_{b,j}(u)). \tag{1.17}$$

It follows from the definition of $P_u$, the recursive formula (1.7), and (1.16) that the $(b, j)$-flow can also be written as

$$u_t = (J_k)_a(\nabla F_{b,j-k}(u)), \quad k \leq j. \tag{1.17}$$

So the $(b, j)$-flow is Hamiltonian with respect to $J_k$ for all $k \leq j$.

Next we explain how to construct geometric curve flows on Adjoint orbits from flows in the $U$-hierarchy. Let $u$ be a solution of the $(b, j)$-flow in the $U$-hierarchy on $S(\mathbb{R}, U_a^\perp)$. Since its Lax pair $\theta_\lambda$ defined by (1.10) is flat for all $\lambda$, $\theta_0 = u dx + Q_{b,j}(u) dt$ is a flat $\mathcal{U}$-valued connection 1-form. Hence there exists $g : \mathbb{R}^2 \to U$ such that

$$g^{-1} g_x = u, \quad g^{-1} g_t = Q_{b,j}(u). \tag{1.18}$$

Set

$$\gamma(x, t) = g(x, t) a g(x, t)^{-1}. \tag{1.19}$$
Then $\gamma(\cdot, t)$ is a family of curves on the Adjoint $U$-orbit $M_a$ in $U$. Take the $t$ derivative of (1.19) to get

$$\gamma_t = g[Q_{b,j}(u), a]g^{-1}. \quad (1.20)$$

We will show that when $M_a$ is a Hermitian symmetric space and $(b, j) = (a, 2)$, the right hand side of (1.20) is equal to

$$[\gamma, \gamma_{xx}] = [\gamma, \nabla_{\gamma_x} \gamma_x].$$

This implies that solutions of MNLS and (1.11) give rise to solutions of the Schrödinger flow (1.2) on $Gr(k, C^n)$ and Hermitian symmetric space respectively. For $j = 1$, the $(b, 1)$ flow is the $n$-wave equation, and the right hand side of (1.20) is equal to

$$(gbg^{-1})_x = (\dot{b}(\gamma))_x = -A_{b(\gamma)}(\gamma_x).$$

So solutions of the $n$-wave equation give rise to solutions of the Ferapontov flow (1.5). However, some natural questions come up in this construction:

1. Since the solution $g$ of (1.18) is only unique up to left multiplication, the corresponding solution $\gamma$ of (1.20) is unique up to conjugation. Can we normalize the curves in $M_a$ so that the correspondence between $u$ and $\gamma$ is unique and preserves the flow?
2. Is (1.20) a geometric curve flow on $M_a$? In other word, can the right hand side of (1.20) be written as some geometric quantity $H_{b,j}(\gamma)$?
3. Can the procedure of constructing solutions of (1.20) from solutions of the $(b, j)$-flow be reversed?
4. Is there a Hamiltonian formulation of the flow (1.20)? If yes, what is its relation to the Hamiltonian theory of the $(b, j)$-flow?

When the $(b, j)$-flow is the MNLS, Terng and Uhlenbeck chose a normalization for curves on $M_a$ at $-\infty$ and were able to answer the above questions in a satisfactory way ([TU2]). In this paper we generalize their results to any flows in the $U$-hierarchy. To do this, we need to recall the development map constructed in [TU2] next.

Let $M_a$ be the Adjoint $U$-orbit at $a$ in $U$, and $C_a(\mathbb{R}, M_a)$ the space of all smooth curves $\gamma : \mathbb{R} \rightarrow M_a$ such that $\lim_{x \rightarrow -\infty} \gamma(x) = a$ and $\gamma_x \in \mathcal{S}(\mathbb{R}, U)$. The following results were proved in [TU2]:

(i) Given $\gamma \in C_a(\mathbb{R}, M_a)$, there exists a unique $g : \mathbb{R} \rightarrow U$ such that

$$\gamma = gag^{-1}, \quad \lim_{x \rightarrow -\infty} g(x) = e, \quad g_x \in \mathcal{S}(\mathbb{R}, U_a^\perp),$$

where $e \in U$ is the identity element.

(ii) The development map $\Phi : C_a(\mathbb{R}, M_a) \rightarrow \mathcal{S}(\mathbb{R}, U_a^\perp)$ defined by $\Phi(\gamma) = g^{-1}g_x$ is a bijection.

(iii) Since $\Phi$ is a bijection, $\Phi^*(J_k)$ is a Poisson operator on $C_a(\mathbb{R}, M_a)$. Moreover, $\Phi^*(J_2)$ is a zero order Poisson operator, and $\Phi^*(J_k)$ $(k \geq 3)$ is a order $(k - 2)$ non-local Poisson operator. Here a non-local Poisson operator is said to have order $k$ if it involves derivatives up to order $k$ and antiderivatives. Note that although $J_k$ has order $k$, the pullback $\Phi^*(J_k)$ has order $k - 2$.

Next we give a geometric interpretation of $\Phi^*(J_2)$ and $\Phi^*(J_3)$. There is a natural zero order Poisson operators on $C_a(\mathbb{R}, M_a)$ obtained by identifying the Adjoint $U$-orbit $M_a$ as the coadjoint orbit at $\ell_a \in U^\ast$, where $\ell_a(x) = \langle x, a \rangle$. So $M_a$ is equipped with the coadjoint orbit symplectic form and the corresponding Poisson operator at $y \in M_a$ is $-\text{ad}(y)$. Hence it induces a natural zero order Poisson operator on $C_a(\mathbb{R}, M_a)$:

$$J_\gamma(v) = [v, \gamma]. \quad (1.21)$$

In fact, $J = \Phi^*(J_2)$.

The Poisson operator $\Lambda = \Phi^*(J_3)$, which is non-local and of first order, can be described geometrically as follows: Given $v \in T(C_a(\mathbb{R}, M_a))_\gamma$, there exists a unique vector field $\eta$ along $\gamma$.
normal to \( M \) such that \( (v + \eta)_x \) is tangential. Then the Poisson operator \( \Lambda = \Phi^*(J_3) \) is given by \( \Lambda_\gamma(v) = (v + \eta)_x \). Moreover, \( \Phi^*(J_k)_\gamma = (\Lambda_\gamma J_{\gamma}^{-1})^{k-2} J_\gamma \).

When \( a \) is regular, the Poisson operator \( \Lambda \) is the same as the Poisson operator constructed by Ferapontov in \([F1]\) using the Dirac reduction of the Poisson operator \( d_x \) on \( C(\mathbb{R}, \mathcal{U}) \) to \( C(\mathbb{R}, M_a) \).

Below are some of our results:

(i) the curve flow on \( M_a \) corresponding to the \((b, j)\)-flow under the development map \( \Phi \) is

\[
\gamma_t = -(\Lambda_\gamma J_{\gamma}^{-1})^{j-1}(A_{b(\gamma)}(\gamma_2)) = (\Lambda_\gamma J_{\gamma}^{-1})^{j-1}(\nabla H_b(\gamma)),
\]

where

\[
H_b(\gamma) = \int_{-\infty}^{\infty} \langle \gamma(x), b \rangle dx.
\]

(ii) The Poisson operators \( \Lambda, J \) and the constant of motions \( F_{b,j} \circ \Phi \) for (1.22) can be expressed in geometric terms.

(iii) The curve flow (1.22) is Hamiltonian with respect to \( J \) and \( \Lambda \), and is completely integrable.

(iv) If \( M_a \) is isometric to a Hermitian symmetric space, then the curve flow on \( M_a \) corresponding to the \((a, 2)\)-flow (1.11) under \( \Phi \) is the Schrödinger flow on \( M_a \).

(v) If \( M_a \) is a principal Adjoint \( U \)-orbit in \( \mathcal{U} \) and \( f : \mathcal{U} \to \mathbb{R} \) is a polynomial invariant under the Adjoint action, then the curve flow

\[
\gamma_t = (\nabla f(\gamma))_x
\]

with constraint \( \gamma(x, t) \in M_a \), is the Ferapontov flow (1.5) on \( M_a \) with \( b = \nabla f(a) \). Moreover, the collection of curve flows

\[
\gamma_t = (\Lambda_\gamma J_{\gamma}^{-1})^{j-1}(\nabla h(\gamma)),
\]

with \( h : \mathcal{U} \to \mathbb{R} \) a polynomial invariant under the Adjoint \( U \)-action and \( j \geq 1 \), is a hierarchy of commuting Hamiltonian flows on \( C_a(\mathbb{R}, M_a) \) with respect to both Poisson operators \( J \) and \( \Lambda \).

Next we explain how to construct the hierarchy of commuting flows associated to a symmetric space. Let \( \sigma \) be the involution on \( U \) such that \( K \) is the fixed point of \( \sigma \), and \( \mathcal{P} \) the \(-1\) eigenspace of \( d\sigma \). Then \( U/K \) is a symmetric space. It is known that (cf. \([TU1]\)):

(i) The subspace \( S(\mathbb{R}, \mathcal{U}^1_a \cap K) \) is invariant under all \((b, j)\)-flows with odd \( j \). The collection of these restricted flows is called the \( U/K \)-hierarchy.

(ii) If \( k \) is odd, then the restriction of the Poisson operator \( J_k \) to the invariant submanifold \( S(\mathbb{R}, \mathcal{U}^1_a \cap K) \) is again a Poisson operator. The odd flows in the \( U/K \)-hierarchy are Hamiltonian with respect to these Poisson structures.

Let \( a \in \mathcal{P} \), and \( M_a \) and \( N_a \) denote the Adjoint \( U \)-orbit in \( \mathcal{U} \) and Adjoint \( K \)-orbit in \( \mathcal{P} \) at \( a \) respectively. So \( N_a \subset M_a \), and \( C(\mathbb{R}, N_a) \) is a submanifold of \( C_a(\mathbb{R}, M_a) \). We will show that \( C_a(\mathbb{R}, N_a) \) is invariant under the curve flow (1.22) if \( j \) is odd. Moreover, if \( k \) is odd, then \( \Phi^*(J_k) \) induces a Poisson operator on \( C_a(\mathbb{R}, N_a) \).

This paper is organized as follows: We write down curve flows on Adjoint \( U \)-orbits corresponding to flows in the \( U \)-hierarchy under the development map as geometric flows, and express the corresponding Poisson operators and constant of motions in geometric terms in section 2. We construct Bäcklund transformations and finite type solutions of these geometric curve flows in section 3, and consider the curve flows corresponding to flows in the \( U/K \)-hierarchy in section 4. Finally, we study the curve flow (1.5) on a principal orbit of the isotropy representation of a symmetric space as a hydrodynamic system in section 5.
2. Integrable curve flows on Adjoint orbits

Let $M_a$ denote the Adjoint $U$-orbit at $a$ in $U$. In this section, we write down the curve flows, Poisson structures, and commuting Hamiltonians on $C_a(\mathbb{R}, M_a)$ corresponding to soliton flows in the $U$-hierarchy via the development map in geometric terms.

If $f : X \to Y$ is a diffeomorphism and $w$ is a symplectic form on $Y$, then the pull back $f^*(w)$ is a symplectic form on $X$ and $f : (X, f^*(w)) \to (Y, w)$ is a symplectic diffeomorphism. If $g$ and $h$ are Riemannian metrics on $X$ and $Y$ respectively, then there exists section $B$ of $L(TX, TX)$ that relates the metrics $g$ and $f^*(g)$ on $X$ as follows:

$$f^*(g)_{x}(v_{1}, v_{2}) = h_{f(x)}(df_x(v_1), df_x(v_2)) = g_x(B_x(v_1), v_2).$$

Let $J$ be the Poisson operator corresponding to $w$ on $Y$, i.e.,

$$w_y(v_1, v_2) = h_y(J_{y}^{-1}(v_1), v_2).$$

A direct computation shows that the Poisson operator $f^*(J)$ corresponding to $f^*(w)$ is

$$f^*(J)_x = df_x^{-1} \circ J_{f(x)} \circ df_x \circ B_x^{-1}. \quad (2.1)$$

The gradient $\nabla H$ of $H : Y \to \mathbb{R}$ is defined by

$$dH_y(v) = h_y(\nabla H(y), v).$$

The Hamiltonian flow for $H$ on $Y$ with respect to $w$ is

$$\frac{dy}{dt} = J_y(t)(\nabla H(y(t))). \quad (2.2)$$

The Hamiltonian equation for $H \circ f : X \to \mathbb{R}$ with respect to $f^*(J)$ is

$$\frac{dx}{dt} = f^*(J)_x(\nabla (H \circ f)(x)) = df_x^{-1}(J_{f(x)}(\nabla H(f(x))). \quad (2.3)$$

It is clear that the Hamiltonian flow for $H \circ f$ on $X$ maps to the Hamiltonian flow for $H$ on $Y$. Equations (2.2) and (2.3) can be viewed as the same equation written in different coordinate systems.

Let $\Phi : C_a(\mathbb{R}, M_a) \to S(\mathbb{R}, U_a^\perp)$ be the development map on the Adjoint $U$-orbit $M_a$ given in section 1. To compute $\Phi^*(J_k)$, we need to specify the metrics. The space $S(\mathbb{R}, U_a^\perp)$ is equipped with the $L^2$ inner product

$$\langle u_1, u_2 \rangle = \int_{-\infty}^{\infty} \langle u_1(x), u_2(x) \rangle \, dx.$$ 

The tangent space of $C_a(\mathbb{R}, M_a)$ at $\gamma$ is the space of vector fields $\xi$ tangent to $M_a$ along $\gamma$ and $\xi_x \in S(\mathbb{R}, U)$. Let $ds^2$ denote the $L^2$ metric on $C_a(\mathbb{R}, M_a)$ defined by

$$ds^2_\gamma(\xi_1, \xi_2) = \int_{-\infty}^{\infty} \langle \xi_1(x), \xi_2(x) \rangle \, dx$$

for tangent fields $\xi_1, \xi_2$ along $\gamma$.

The following is proved in [TU2]:

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2.1 Proposition. Let $\xi \in T(C_a(\mathbb{R}, M_a))_\gamma$. Then:

(i) There exist $g : \mathbb{R} \to U$ and $v : \mathbb{R} \to U_a^\perp$ so that $\lim_{x \to -\infty} g(x) = e$, $\gamma = gag^{-1}$, $u = g^{-1}g_x = \Phi(\gamma)$, and $\xi = gvg^{-1}$.

(ii) There exists a unique vector field $\eta$ along $\gamma$ such that $\eta(x)$ is normal to $M_a$, $\lim_{x \to -\infty} \eta(x) = 0$, and $(\xi + \eta)_x$ is tangent to $M_a$.

(iii) $d\Phi_\gamma(\xi) = P_uJ_a^{-1}(v)$, \hspace{1cm} (2.4)

where $J_a$ and $P_u$ are the Poisson operators on $S(\mathbb{R}, U_a^\perp)$ defined by (1.12) and (1.13) respectively.

(iv) Let $J_3 = (P_uJ_a^{-1})^3J_a$ be the Poisson operator defined by (1.14), and $\Lambda = \Phi^*(J_3)$. Then

$$\Lambda_\gamma(\xi) = g(P_u(v))g^{-1} = (\xi + \eta)_x,$$ \hspace{1cm} (2.5)

where $\eta$ and $g$ are given in (ii).

A direct computation implies

2.2 Proposition (\cite{TU2}). Let $J_2$ be the Poisson operator on $S(\mathbb{R}, U_a^\perp)$ defined by $(J_2)_u = P_uJ_a^{-1}J_u$, $J$ the natural Poisson operator on $C_a(\mathbb{R}, M_a)$ given by (1.21), and $\Phi$ the development map. Then $\Phi^*(J_2) = J$.

When $M_a$ is a principal Adjoint $U$-orbit, we show below that the Poisson operator $\Lambda$ can be written in terms of geometric invariants of $M_a$ as a submanifold of the Euclidean space $U$.

2.3 Theorem. Let $T$ be a maximal abelian subalgebra of $U$, $\{a_1, \ldots, a_k\}$ an orthonormal basis of $T$, and $a = a_1$ regular. Let $M_a$ be the Adjoint $U$-orbit at $a$ in $U$, and $\Lambda = \Phi^*(J_3)$ the induced Poisson operator on $C_a(\mathbb{R}, M_a)$. Then

$$\Lambda_\gamma(\xi) = \nabla_{\gamma_x}\xi - \sum_{i=1}^k h_iA_{\xi_i}(\gamma_x),$$ \hspace{1cm} (2.6)

where $h_i(x) = -\int_{-\infty}^x \langle \Pi(\xi(s), \gamma_x(s)), \hat{a}_i(\gamma(s)) \rangle ds$ for $1 \leq i \leq k$, $\hat{a}_i$ is the parallel normal field on $M_a$ defined by (1.3), and $\Pi$ is the second fundamental form of $M_a$ as a submanifold of $U$.

PROOF. Let $u = \Phi(\gamma)$, and $\xi, v, g$ as in Proposition 2.1. So we have

$$\gamma = gag^{-1}, \hspace{0.5cm} u = g^{-1}g_x, \hspace{0.5cm} \xi = gvg^{-1}.$$ 

By Proposition 2.1 (iv) and the formula (1.13) for $P_u$, we have

$$\Lambda_\gamma(\xi) = gP_u(v)g^{-1} = g(v_x + [u, v][1 + [u, h]]g^{-1},$$

where $h_x = -[u, v][0]$, and $[u, v][0]$ and $[u, v][1]$ denote the projection of $[u, v]$ onto $U_a = T$ and $U_a^\perp = T^\perp$ respectively.

Given $\xi \in U$, let $p_x(\xi)$ and $p_x^\perp(\xi)$ denote the orthogonal projection of $\xi$ onto $T(M_a)_x$ and $\nu(M_a)_x$ respectively. Let $\nabla$ denote the Levi-Civita connection of the induced metric on $M_a$. Then

$$(\nabla_{\gamma_x}\xi)(x) = p_{\gamma(x)}((gvg^{-1})_x) = p_{\gamma(x)}(g(v_x + [u, v])g^{-1})$$

$$= g(v_x + [u, v][1]g^{-1} = g(v_x + [u, v][1])g^{-1}.$$ 

But

$$(ghg^{-1})_x = g(h_x + [u, h])g^{-1}.$$ 

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Since $h(x) \in \mathcal{T}$ and $a_1, \ldots, a_k$ is a basis of $\mathcal{T}$, there exist $h_1, \ldots, h_k$ such that
\[
h(x) = \sum_{i=1}^{k} h_i(x)a_i.
\]
Therefore
\[
g_{hx}g^{-1} = \left(\sum_{i=1}^{k} (h_i)_xa_i\right)g^{-1} = \sum_{i=1}^{k} (h_i)_x\hat{a}_i(\gamma(x)).
\]
Let $\nabla^\perp$ denote the induced normal connection on $\nu(M)$. Then
\[
\nabla^\perp_{\gamma_x}(ghg^{-1}) = g_{hx}g^{-1} = \sum_{i=1}^{k} (h_i)_x\hat{a}_i.
\]
Definition of the second fundamental form implies that
\[
\Pi(\xi, \gamma_x) = p^\perp_{\gamma_x}(gvg^{-1})_x = g[u, v]o g^{-1} = -g_{hx}g^{-1}
\]
\[
= -\sum_{i=1}^{k} (h_i)_x\hat{a}_i
\]
\[
= \sum_{i=1}^{k} \langle \Pi(\xi, \gamma_x), \hat{a}_i \rangle \hat{a}_i.
\]
So $(h_i)_x = -\langle \Pi(\xi, \gamma_x), \hat{a}_i(\gamma(x)) \rangle$. But $\lim_{x \to -\infty} \eta(x) = 0$. So $h_i$ is given by the formula in the Proposition. ■

2.4 Remark. Let $M^n$ be a submanifold of $\mathbb{R}^{n+k}$ with flat normal bundle, and $(e_{n+1}, \ldots, e_{n+k})$ a parallel orthonormal normal frame field on $M$. Note that $\xi$ is a tangent vector of $C(\mathbb{R}, M)$ at $\gamma$ iff $\xi$ is a vector field along $\gamma$ tangent to $M$. Let $Z_\gamma$ be the operator on $TC(\mathbb{R}, M)_\gamma$ defined by
\[
Z_\gamma(\xi) = \nabla_{\gamma_x}\xi + \sum_{\alpha=n+1}^{n+k} h_\alpha A_{\kappa_\alpha}(\gamma)(\gamma_x), \text{ where}
\]
\[
h_\alpha(x) = \int_{-\infty}^{x} \langle \Pi(\xi(s), \gamma_x(s)), e_\alpha(\gamma(s)) \rangle ds \text{ for } n+1 \leq \alpha \leq n+k,
\]
and $\nabla$ is the Levi-Civita connection of the induced metric on $M$. Ferapontov proved in [F1] that $Z$ is a Poisson operator on $C(\mathbb{R}, M)$. The normal bundle of a principal Adjoint $U$-orbit $M_a$ is flat. Proposition 2.3 proves that $\Phi^*(J_3) = \Lambda = Z$.

2.5 Theorem ([TU2]). Let $J_k$ be the Poisson structures on $S(\mathbb{R}, U^+_a)$ defined by (1.14), and $L_{k-2} = \Phi^*(J_k)$ the induced Poisson structures on $C_a(\mathbb{R}, M_a)$. Then
\[
(L_j)_\gamma = J_\gamma(J^{-1}_\gamma \Lambda_\gamma)^j = (\Lambda_\gamma J^{-1}_\gamma)^j J_\gamma.
\]

It follows from (2.3) that the flow on $C_a(\mathbb{R}, M_a)$ corresponding to the $(b, j)$-flow in the $U$-hierarchy on $S(\mathbb{R}, U^+_a)$ is
\[
\gamma_t = d\Phi^{-1}_\gamma([Q_{b, j+1}(u), a])
\]
Propositions 2.1 and (1.17) imply that
2.6 Proposition. The curve flow on $M$ corresponding to the $(b, j)$-flow in the $U$-hierarchy on $S(\mathbb{R}, U^\perp_a)$ is

$$\gamma_t = g[Q_{b, j}(u), a]g^{-1} = [gQ_{b, j}(u)g^{-1}, \gamma],$$

(2.10)

where $\gamma = gag^{-1}$, $\lim_{x \to -\infty} g(x, t) = e$, and $u = g^{-1}g_x = \Phi(\gamma)$. Moreover, (2.10) is the Hamiltonian flow for $F_{b, j-k-2} \circ \Phi$ with respect to $L_k = \Phi^*(J_{k+2})$.

Next we want to express (2.10) and $F_{b, j} \circ \Phi$ in geometric terms. This follows from

2.7 Proposition. Let $\gamma, u, g$ be as in Proposition 2.6. Then $gQ_{b, j}(u)g^{-1}$ can be expressed in terms of geometric invariants of $M$ along $\gamma$.

PROOF. We prove this proposition by induction. The recursive formula (1.7) implies

$$Q_{b, 1}(u) = \text{ad}(b) \text{ad}(a)^{-1}(u).$$

A direct computation gives

$$\gamma_x = g[u, a]g^{-1},$$
$$gug^{-1} = J_{\gamma}^{-1}(\gamma_x),$$
$$gQ_{b, 1}(u)g^{-1} = J_{\gamma}^{-1}(g[Q_{b, 1}(u), a]g^{-1}) = J_{\gamma}^{-1}(g[u, b]g^{-1}).$$

The parallel normal field $\hat{b}$ defined by (1.3) is $\hat{b}(gag^{-1}) = gb^{-1}$. The shape operator for $M$ along $\hat{b}$ is

$$A_{\hat{b}}(\gamma)(\gamma_x) = -\langle \hat{b}(\gamma) \rangle_x = -(gb^{-1})_x = -g[u, b]g^{-1}.$$

Hence

$$gQ_{b, 1}(u)g^{-1} = -J_{\gamma}^{-1}(A_{\hat{b}}(\gamma)(\gamma_x)) = J_{\gamma}^{-1}(\hat{b}(\gamma)_x).$$

(2.11)

Suppose we have expressed $gQ_{b, j}(u)g^{-1}$ geometrically. Let $\pi_0, \pi_1$ be the orthogonal projection onto $U_a$ and $U^\perp_a$ respectively, and $p_x$ and $p^\perp_x$ the orthogonal projection of $U$ onto $U_x = \nu(M_a)_x$ and $U^\perp_x = T(M_a)_x$ respectively. The recursive formula (1.7) and Proposition 2.1 imply that

$$g\pi_1(Q_{b, j+1}(u))g^{-1} = J_{\gamma}^{-1}\Lambda_\gamma(g\pi_1(Q_{b, j}(u))g^{-1}),$$

(2.12)

and

$$(g\pi_0(Q_{b, j+1}(u))g^{-1})_x = -g\pi_0([u, \pi_1(Q_{b, j+1}(u))])g^{-1}$$
$$= -p_{gag^{-1}}^{-1}([gug^{-1}, g\pi_1(Q_{b, j+1}(u))g^{-1}])$$
$$= -p_{gag^{-1}}^{-1}([J_{\gamma}^{-1}(\gamma_x), g\pi_1(Q_{b, j+1}(u))g^{-1}]).$$

(2.13)

The induction hypothesis and (2.12) imply that $g\pi_1(Q_{b, j+1}(u))g^{-1}$ can be expressed geometrically. Then (2.13) implies that $g\pi_0(Q_{b, j+1}(u))g^{-1}$ can also expressed as geometric terms.

Use (2.11), (2.12) to get

$$g[Q_{b, j}(u), a]g^{-1} = (\Lambda_\gamma J_{\gamma}^{-1})^{j-1}(\hat{b}(\gamma)_x) = -(\Lambda_\gamma J_{\gamma}^{-1})^{j-1}(A_{\hat{b}}(\gamma)(\gamma_x)).$$

(2.14)

The next theorem follows from (2.14) and Proposition 2.6.
\textbf{2.8 Theorem.} Suppose \( a \in \mathcal{U} \) is regular, and \( b \in \mathcal{U}_a \). Then the curve flow on \( C_a(\mathbb{R}, M_a) \) corresponding to the \((b,1)\)-flow in the \( U \)-hierarchy is

\[
\gamma_t = (\dot{b} \circ \gamma)_x = -A_{\dot{b}(\gamma)}(\gamma_x),
\]

where \( A_{\dot{b}(\gamma)} \) is the shape operator along \( \dot{b}(\gamma) \). In general, the curve flow on \( C_a(\mathbb{R}, M_a) \) corresponding to the \((b,j)\)-flow is

\[
\gamma_t = (\Lambda_\gamma J^{-1}_\gamma)^{j-1}(b(\gamma))_x = -(\Lambda_\gamma J^{-1}_\gamma)^{j-1}(A_{\dot{b}(\gamma)}(\gamma_x)).
\]

Moreover,

(i) \( F_{b,j} \circ \Phi \) can be expressed in terms of shape operators of \( M_a \) and the Poisson operators \( J \) and \( \Lambda \),

(ii) the flow (2.16) is the Hamiltonian flow of \( F_{b,j-k} \circ \Phi \) on \( C_a(\mathbb{R}, M_a) \) with respect to \( L_{k-2} = \Phi^*(J_k) \), where \( F_{b,j} \) is the functional defined by (1.15) and \( \Phi \) is the development map.

\textbf{2.9 Example.} Let \( a \in \mathcal{U} \) be a regular element. The curve flow (2.16) on the Adjoint orbit \( M_a \) corresponding to the \((b,2)\)-flow is

\[
\gamma_t = -(J^{-1}_\gamma(A_{\dot{b}(\gamma)}(\gamma_x)))_x.
\]

When \( b = a \) (without the assumption that \( a \) is regular), we have

\textbf{2.10 Corollary.} For \( a \in \mathcal{U} \), the curve flow on \( C_a(\mathbb{R}, M_a) \) corresponding to the \((a,j)\)-flow in the \( U \)-hierarchy is

\[
\gamma_t = (\Lambda_\gamma J^{-1}_\gamma)^{j-1}(\gamma_x).
\]

Since (2.16) is the flow corresponding to the \((b,j)\)-flow under the development map \( \Phi \) and \( u \) is a solution of the \((b,j)\)-flow, we get that \( \gamma(\cdot, t) = \Phi^{-1}(u(\cdot, t)) \) is a solution of (2.16). Proposition 2.6 states that

\[
\gamma_t = g[Q_{b,j}(u), a]g^{-1},
\]

where \( g \) is the solution of

\[
g^{-1}g_x = u, \quad \lim_{x \to -\infty} g(x, t) = e.
\]

On the other hand, we have seen in section 1 that if \( k : \mathbb{R}^2 \to U \) solves

\[
k^{-1}k_x = u, \quad k^{-1}k_t = Q_{b,j}(u),
\]

then

\[
\tilde{\gamma} = kak^{-1}
\]

also satisfies (2.18). Next we show that \( g \) also satisfies (2.20).

\textbf{2.11 Proposition.} Let \( u \) be a solution of the \((b,j)\)-flow on \( S(\mathbb{R}, U_a^+) \), \( \theta_\lambda \) the corresponding Lax pair (1.10), and \( k \) the trivialization of the Lax pair \( \theta_0 \), i.e.,

\[
k^{-1}dk = u \ dx + Q_{b,j}(u) \ dt, \quad k(0,0) = e.
\]

Then there exists a constant \( c \in U \) such that \( \lim_{x \to -\infty} k(x,t) = c \),

\[
\gamma = c^{-1}kak^{-1}c
\]

is a solution of the curve flow (2.16) on \( C_a(\mathbb{R}, M_a) \), and \( \Phi(\gamma(\cdot, t)) = u(\cdot, t) \).
PROOF. For fix \( t \), let \( g(\cdot, t) \) be the map from \( \mathbb{R} \) to \( U \) satisfying (2.19). Claim that \( g^{-1}g_t = Q_{b,j}(u) \). Too see this, we note that because both \( g \) and \( k \) satisfy \( g^{-1}g_x = k^{-1}k_x = u \), there exists \( C(\cdot) \) such that \( g(x, t) = C(t)k(x, t) \).

Compute directly to get
\[
g^{-1}g_t = k^{-1}k_t + k^{-1}C^{-1}C_t k = Q_{b,j}(u) + k^{-1}C^{-1}C_t k.
\] (2.21)

Since \( \lim_{x \to -\infty} g(x, t) = e \), \( \lim_{x \to -\infty} g_t(x, t) = 0 \). Recall that \( Q_{b,j}(u) \in \mathcal{S}(\mathbb{R}, U) \) if \( u(\cdot, t) \in \mathcal{S}(\mathbb{R}, U_u^+) \) ((1.9)). Hence
\[
\lim_{x \to -\infty} Q_{b,j}(u)(x, t) = 0.
\]

By (2.21), we conclude
\[
\lim_{x \to -\infty} k^{-1}C^{-1}C_t k = 0.
\] (2.22)

But
\[
\| k^{-1}C^{-1}C_t k \| = \| C^{-1} \|.
\] (2.23)

It follows from (2.22) and (2.23) that \( C^{-1}C_t = 0 \). Hence \( C(t) \) is a constant, \( g = C k \), and we have proved the claim. 

2.12 Corollary. Let \( u \) be a solution of the \( (b, j) \) flow, \( k, c \) as in Proposition 2.11, and \( g = c^{-1}k \). Then the gauge transformation of the Lax pair \( \theta_\lambda \) ((1.10)) by \( g \),
\[
\gamma \lambda \; dx + (gb^{-1}\lambda^j + gQ_{b,1}(u)g^{-1}\lambda^{j-1} + \cdots + gQ_{b,j-1}(u)g^{-1}\lambda) \; dt
\]
is a Lax pair of the curve flow (2.16).

2.13 Example. The curve flow (2.15) on \( M_a \) has a Lax pair
\[
\gamma \lambda \; dx + \hat{b}(\gamma) \lambda \; dt.
\]

2.14 Example ([F4]). Let \( M_a \) be a principal Adjoint \( U(n) \)-orbit in \( u(n) \). If \( k \) is a natural number, then \( b = i^{k-1}a^k \in u(n)_a \), and the curve flow (1.5) on \( M_a \) corresponding to the \( (b, 1) \)-flow becomes
\[
\gamma_t = i^{k-1}(\gamma^k)_x
\]
with \( \gamma(x, t) \in M_a \). Its Lax pair is \( \gamma \lambda \; dx + i^{k-1}\gamma^k \lambda \; dt \).

2.15 Example. Let \( a \in \mathcal{U} \) be regular, and \( b \in \mathcal{U}_a \). It is known (cf. [Te1]) that there exists a polynomial \( f : \mathcal{U} \to \mathbb{R} \) invariant under the Adjoint action such that \( \nabla f(a) = b \). Since \( f \) is Ad-invariant, \( \nabla f \) is equivariant. Hence \( \nabla f \vert M_a \) is a parallel normal field of \( M_a \). So the Ferapontov flow (1.5) for \( b = \nabla f(a) \) becomes
\[
\gamma_t = (\nabla f(\gamma))_x, \quad \gamma(x, t) \in M_a.
\] (2.24)

Its Lax pair is \( \gamma \lambda \; dx + \nabla f(\gamma) \lambda \; dt \). Since the \( (b, 1) \)-flow (1.4) is a completely integrable Hamiltonian system with respect to the Poisson operator \( J_2 \) and \( J_3 \), and (2.24) is the flow corresponding to the \( (b, 1) \)-flow under the development map \( \Phi \), the curve flow (2.24) is completely integrable with respect to the Poisson operators \( J \) and \( \Lambda = \Phi^*(J_3) \). Moreover, the flows
\[
\gamma_t = (\Lambda \gamma J_\gamma^{-1})^j((\nabla h(\gamma))_x),
\]
with \( h : \mathcal{U} \to \mathbb{R} \) a polynomial invariant under the Adjoint \( U \)-action and \( j \geq 0 \) an integer, are commuting Hamiltonian flows on \( C_a(\mathbb{R}, M_a) \).
2.16 Example. Let $U/K$ be a compact Hermitian symmetric space. Then there exists $a \in K$ such that $\mathcal{U}_a = K$ and $\text{ad}(a)^2 = -\text{Id}$ on $\mathcal{P} = \mathcal{U}_a^\perp$. Moreover, the Adjoint $U$-orbit $M_a$ at $a$ in $\mathcal{U}$ is an isometric embedding of the Hermitian symmetric space $U/K$. In fact, the induced metric on $M_a$ is the standard Kähler metric on $U/K$. Use condition (1.7) and (1.9) and a direct computation to see that the $(a, 2)$-flow in the $U$-hierarchy on $S(\mathbb{R}, \mathcal{U}_a^\perp)$ is (1.11):

$$u_t = [a, u_{xx}] - \frac{1}{2}[u, [u, [a, u]]].$$

The corresponding curve flow (2.17) on $M_a$ is

$$\gamma_t = \Lambda_\gamma J_{\gamma}^{-1}(\gamma_x). \tag{2.25}$$

Claim that this flow is the Schrödinger flow on the Hermitian symmetric space $U/K$:

$$\gamma_t = j_\gamma(\nabla_{\gamma_x} \gamma_x) = [\gamma, \nabla_{\gamma_x} \gamma_x]. \tag{2.26}$$

To see this, let $g$ be the solution of

$$g^{-1}g_x = u, \quad \lim_{x \to -\infty} g(x, t) = e.$$ 

Set $\gamma = gag^{-1}$. Then

$$\gamma_x = g[u, a]g^{-1}.$$ 

We compute the right hand side of (2.25) as follows:

$$\Lambda_\gamma J_{\gamma}^{-1}(\gamma_x) = \Lambda_\gamma (J_{\gamma}^{-1}(g[u, a]g^{-1})) = \Lambda_\gamma (gug^{-1}).$$

It follows from (1.13) and (2.5) that

$$\Lambda_\gamma (gug^{-1}) = gu_xg^{-1}.$$ 

But

$$\nabla_{\gamma_x} \gamma_x = p(g[u, a]g^{-1})_x = p(g[u, a]g^{-1} + g[u, [u, a]]g^{-1}), \tag{2.27}$$

where $p(v)$ denotes the orthogonal projection of $v$ onto $TM_a$ and $\nabla$ is the Levi-Civita connection of the induced metric on $M_a$. But

$$(TM_a)_\gamma = g\mathcal{U}_a^\perp g^{-1} = g\mathcal{P}g^{-1}, \quad \nu(M_a)_\gamma = g\mathcal{U}_a g^{-1} = gKg^{-1}.$$ 

Since $M_a = U/U_a = U/K$ is a symmetric space,

$$[K, K] \subset K, \quad [K, \mathcal{P}] \subset \mathcal{P}, \quad [\mathcal{P}, \mathcal{P}] \subset K.$$ 

So $[\mathcal{P}, [\mathcal{P}, K]] \subset K$, which implies that $g[u, [u, a]]g^{-1}$ is normal to $M_a$. Hence

$$\nabla_{\gamma_x} \gamma_x = g[u_x, a]g^{-1}.$$ 

So we have

$$j_\gamma(\nabla_{\gamma_x} \gamma_x) = [\gamma, g[u_x, a]g^{-1}] = g[a, [u_x, a]]g^{-1} = gu_xg^{-1}.$$ 

Here we use the fact that $-\text{ad}(a)^2 = -\text{Id}$. Therefore we have proved the claim, i.e., (2.25) is the Schrödinger flow (2.26) on $M_a$. 

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Next we claim that the right hand side of (2.26) is also equal to $[\gamma, \gamma_{xx}]$. This is done by a direct computation: Since $\gamma_x = g[u, a]g^{-1}$,

$$
\gamma_{xx} = (g[u, a]g^{-1})_x = g([u, [u, a]] + [u_x, a])g^{-1}.
$$

Hence

$$
[\gamma, \gamma_{xx}] = g([a, [u, [u, a]]] + [a, [u_x, a]])g^{-1}.
$$

Since $\text{ad}(a)^2 = \text{Id}$ on $U_a^\perp$, $[a, [u_x, a]] = u_x$. Jacobi-identity implies that

$$
[a, [u, [u, a]]] = -[u, [[u, a], a]] = -[[u, a], [a, u]] = -[u, -u] + 0 = 0.
$$

So $[\gamma, \gamma_{xx}] = gu_xg^{-1}$, and the flow (2.25) becomes

$$
\gamma_t = [\gamma, \nabla_x \gamma_x] = [\gamma, \gamma_{xx}], \quad \gamma(x, t) \in M_a.
$$

Equation (2.26) has a Lax pair

$$
\gamma_\lambda \ dx + (\gamma_\lambda^2 + [\gamma, \gamma_x]\lambda \ dt).
$$

2.17 Example. Let $U/K$ be a compact Hermitian symmetric space and let $a$ be as in Example 2.16. Then we know that the $(a, 2)$-flow in the $U$-hierarchy on $S(\mathbb{R}, U_a^\perp)$ is (1.11)

$$
u_t = [a, u_{xx}] - \frac{1}{2}[u, [u, u_x]].
$$

We would like to write this equation out explicitly for each irreducible compact symmetric space involving classical groups.

(i) As was pointed out in the introduction, the equation in case of the Grassmannian $Gr(k, \mathbb{C}^n)$ is the MNLS.

(ii) We consider the Grassmannian $Gr(2, \mathbb{R}^{n+2}) = SO(n + 2)/SO(2) \times SO(n)$. We have $U = SO(n + 2)$. The element $a$ in $so(n + 2)$ has the matrix

$$
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
$$

in the upper left corner and elsewhere zeros. Its centralizer $U_a$ is $so(2) \times so(n)$ and $U_a^\perp$ is the set of matrices of the form

$$
\begin{pmatrix}
0 & 0 & x_1 & \cdots & x_n \\
0 & 0 & y_1 & \cdots & y_n \\
-x_1 & -y_1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-x_n & -y_n & 0 & \cdots & 0
\end{pmatrix}
$$

where $X = (x_1, \ldots, x_n)$, $Y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. One can clearly identify $U_a^\perp$ with $\mathbb{C}^n$ by mapping the above matrix onto $Z = X + iY$. Under this identification the complex structure $\text{ad}(a)$ on $U_a^\perp$ corresponds to the usual complex structure on $\mathbb{C}^n$. The equation (1.11) becomes the following system of equations

$$
\begin{cases}
X_t = -Y_{xx} + (X \cdot Y)X - \frac{1}{2}(3X \cdot X + Y \cdot Y)Y, \\
Y_t = X_{xx} + \frac{1}{2}(X \cdot X + 3Y \cdot Y)X - (X \cdot Y)Y,
\end{cases}
$$

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where \(X \cdot Y\) is the standard inner product on \(\mathbb{R}^n\).

(iii) We now consider the Hermitian symmetric space \(SO(2n)/U(n)\). We have \(U = SO(2n)\). The element \(a\) is

\[
a = \frac{1}{2} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}
\]

whose centralizer \(\mathcal{U}_a\) is \(u(n)\). The embedding of \(U(n)\) into \(SO(2n)\) can be described on the level of Lie algebras as follows: \(Z = A + iB\) in \(u(n)\) where \(A\) and \(B\) are real matrices (\(A\) skew and \(B\) symmetric) is mapped to

\[
\begin{pmatrix} A & B \\ -B & A \end{pmatrix}
\]

in \(so(2n)\). Hence \(\mathcal{U}_a^\perp\) is the set of matrices

\[
\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}
\]

where \(X\) and \(Y\) are both skew. The space \(\mathcal{U}_a^\perp\) can be identified with \(\Lambda^2(\mathbb{C}^n)\) by associating the matrix

\[
\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}
\]

to the form in \(\Lambda^2(\mathbb{C}^n)\) whose matrix with respect to the canonical basis of \(\Lambda^2(\mathbb{C}^n)\) is \(X + iY\). Notice that the complex structure \(\text{ad}(a)\) on \(\mathcal{U}_a^\perp\) and the standard one on \(\Lambda^2(\mathbb{C}^n)\) correspond under this identification. The equation (1.11) now becomes

\[
\begin{align*}
X_t &= (-Y)_{xx} + [X, [X, Y]] + 2Y^3 + YX^2 + X^2Y, \\
Y_t &= X_{xx} + [Y, [X, Y]] - 2X^3 - XY^2 - Y^2X.
\end{align*}
\]

(iv) We finally consider the Hermitian symmetric space \(Sp(n)/U(n)\). The group \(U\) is therefore \(Sp(n)\). Recall that the Lie algebra \(sp(n)\) consists of matrices of the form

\[
\begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}
\]

where \(A\) and \(B\) are complex \(n \times n\) matrices, \(\bar{A}^t = -A\) and \(B^t = B\). The element \(a\) is

\[
a = \frac{1}{2} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.
\]

The stabilizer \(\mathcal{U}_a\) is \(u(n)\). The embedding of \(U(n)\) into \(Sp(n)\) can be described on the level of Lie algebras as follows: \(Z = A + iB\) in \(u(n)\), where \(A\) and \(B\) are real matrices (\(A\) skew and \(B\) symmetric), is mapped to

\[
\begin{pmatrix} A & -B \\ B & A \end{pmatrix}
\]

in \(sp(n)\). Notice that this means that \(\mathcal{U}_a = u(n)\) consists of the real matrices in \(sp(n)\). It is therefore clear that \(\mathcal{U}_a^\perp\) consists of the purely imaginary matrices in \(sp(n)\). In other words, \(\mathcal{U}_a^\perp\) is the set of matrices of the form

\[
i \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}
\]
where $X$ and $Y$ are real symmetric matrices. The space $\mathcal{U}_a^+$ can be identified with the space $S^2(\mathbb{C}^n)$ of symmetric two-forms on $\mathbb{C}^n$ by associating the matrix

\[
\begin{pmatrix}
X & Y \\
Y & -X
\end{pmatrix}
\]

with the symmetric form whose matrix with respect to the canonical basis is $X + iY$. Notice that the complex structure $ad(a)$ on $\mathcal{U}_a^+$ and the standard one on $S^2(\mathbb{C}^n)$ correspond under this identification. The equation (1.11) becomes the system

\[
\begin{cases}
X_t = (-Y)_{xx} - [X, [X, Y]] - 2Y^3 - YX^2 - X^2Y,
Y_t = X_{xx} - [Y, [X, Y]] + 2X^3 + XY^2 + Y^2X,
\end{cases}
\]

where $X, Y$ are real symmetric $n \times n$ matrices.

We recall the following result proved in [Te2] concerning the operator $P_u$ (Remark 4.1 of [Te2]):

2.18 Proposition. Let $v \in S(\mathbb{R}, \mathcal{U}_a^+)$. Suppose there exists a smooth map $\tilde{v} : \mathbb{R} \to \mathcal{U}$ such that $\pi_1(\tilde{v}) = v$, $\lim_{x \to -\infty} \tilde{v}(x) = 0$, and $(\tilde{v})_x + [u, \tilde{v}] \in \mathcal{U}_a^+$. Then $P_u(v) = (\tilde{v})_x + [u, \tilde{v}]$.

Next we prove that the Hamiltonian for the curve flow (1.5) with respect to $\Lambda$ is the height function.

2.19 Proposition. Let $M_a$ be the principal Adjoint $U$-orbit at $a$ in $\mathcal{U}$, $b \in \nu(M_a)_a$, and $\hat{b}$ the parallel normal field on $M_a$ defined by (1.6). Then the Hamiltonian flow for

\[
H_b(\gamma) = \int_{-\infty}^{\infty} \langle \gamma(x), b \rangle dx
\]

with respect to $\Lambda = \Phi^*(J_3)$ is the Ferapontov flow (2.15):

\[
\gamma_t = (\hat{b}(\gamma))_x.
\]

PROOF. Given $x \in M_a$, let $p_x$ and $p_x^\perp$ denote the orthogonal projection of $\mathcal{P}$ onto $T(M_a)_x$ and $\nu(M_a)_x$ respectively. Recall that

\[
\nu(M_a)_{gag^{-1}} = g\mathcal{U}_a g^{-1}, \quad T(M_a)_{gag^{-1}} = g\mathcal{U}_a^+ g^{-1}.
\]

The gradient of $H_b$ at $\gamma$ is the vector field along $\gamma$ defined as follows: Suppose $\gamma = gag^{-1}$, $\lim_{x \to -\infty} g(x) = e$, $g^{-1}g_x = u \in \mathcal{U}_a^+$ as before. Then

\[
\nabla H_b(\gamma) = p_\gamma(b) = p_{gag^{-1}}(g(g^{-1}bg)g^{-1}) = g\pi_1(g^{-1}bg)g^{-1},
\]

where $\pi_1$ is the projection onto $\mathcal{U}_a^+$. We use (2.5) and Proposition 2.18 to compute $\Lambda(\nabla H_b(\gamma))$ next. Let $\tilde{v} = g^{-1}bg - b$.

Then $\pi_1(\tilde{v}) = \pi_1(g^{-1}bg)$ and $\lim_{x \to -\infty} \tilde{v}(x) = 0$. But

\[
(\tilde{v})_x + [u, \tilde{v}] = (g^{-1}bg - b)_x + [u, g^{-1}bg - b] = -[u, b] \in \mathcal{U}_a^+.
\]

By Proposition 2.18, we have

\[
P_u(\pi_1(g^{-1}bg)) = -[u, b].
\]

Therefore the Hamiltonian vector field for $H_b$ with respect to $\Lambda$ is

\[
\Lambda_x(\nabla H_b(\gamma)) = gP_u(\pi_1(g^{-1}bg))g^{-1} = g[u, b]g^{-1} = (\hat{b}(\gamma))_x.
\]

This completes the proof.  

3. Bäcklund transformations and finite type solutions

It is well-known that the \((b, j)\)-flow on \(S(\mathbb{R}, \mathcal{U}_a^\perp)\) has Bäcklund transformations, explicit soliton solutions, and finite type solutions. So we can use the development map \(\Phi\) and Proposition 2.11 to get similar results for the curve flow (2.16) on the Adjoint \(U\)-orbit \(M_a\). We explain the construction for the group \(SU(n)\). Other compact groups can be worked out in a similar manner.

Let \(\gamma\) be a solution of the curve flow (2.16) on \(M_a\). Given \(z \in \mathbb{C}\) and a complex linear subspace \(V\) of \(\mathbb{C}^n\), we will construct a new solution \(\tilde{\gamma}\) of (2.16) associated to \(\gamma, z, V\). This is done using the Bäcklund transformation of the solution \(u\) of the \((b, j)\)-flow corresponding to \(\gamma\) under the development map \(\Phi\). We give a quick review of the algorithm next. Let \(u\) be a solution of the \((b, j)\)-flow in the \(U\)-hierarchy, and \(E(x, t, \lambda)\) the trivialization of the Lax pair \(\theta_\lambda\) given by (1.10) normalized at the origin, i.e., \(E\) is the solution of

\[
\begin{align*}
E^{-1}E_x &= a\lambda + u, \\
E^{-1}E_t &= b\lambda^j + Q_{b,1}(u)\lambda^{j-1} + \cdots + Q_{b,j}(u), \\
E(0, 0, \lambda) &= e.
\end{align*}
\]

We call \(E\) the frame of the solution \(u\). Let \(\pi\) be the Hermitian projection of \(\mathbb{C}^n\) onto \(V\), and \(\pi^\perp = I - \pi\). Let

\[
f_{z,\pi}(\lambda) = I + \frac{\bar{z} - z}{\lambda - \bar{z}} \pi^\perp.
\]

Then \(f\) satisfies the \(SU(n)\)-reality condition, i.e.,

\[
f_{z,\pi}(\bar{\lambda})^*f_{z,\pi}(\lambda) = I.
\]

(Here \(X^* = \bar{X}^t\)). Set

\[
\tilde{V}(x, t) = E(x, t, z)^*(V),
\]

and \(\bar{\pi}(x, t)\) the Hermitian projection of \(\mathbb{C}^n\) onto \(\tilde{V}(x, t)\). Then (cf. [TU1])

\begin{enumerate}
\item \(\tilde{u} = u + (z - \bar{z})[\bar{\pi}, a]\) is a solution of the \((b, j)\)-flow,
\item \(\tilde{E}(x, t, \lambda) = f_{z,\pi}(\lambda)E(x, t, \lambda)f_{\bar{z},\bar{\pi}(x, t)}(\lambda)^{-1}\) is the frame for the solution \(\tilde{u}\).
\end{enumerate}

The transformation \(u \mapsto \tilde{u}\) is a Bäcklund transformation.

By Proposition 2.11, there exists a constant \(c \in SU(n)\) such that

\[
\lim_{x \to -\infty} \tilde{E}(x, t, 0) = c
\]

for all \(t\). Then

\[
\tilde{\gamma} = c^{-1}\tilde{E}(x, t, 0)a\tilde{E}(x, t, 0)^{-1}c
\]

is a new solution of the curve flow (2.16). For example, if we start with the constant solution \(\gamma(x, t) \equiv a\), then the corresponding solution of the \((b, j)\)-flow is the vacuum solution \(u \equiv 0\). Since the frame of \(u = 0\) is \(E(x, t, \lambda) = e^{a\lambda x + b\lambda^t}\), the solution \(\tilde{u}\) and \(\tilde{\gamma}\) are explicit given by the above formulas, which are 1-soliton solutions. If we apply Bäcklund transformation repeatedly, we get the \(N\)-soliton solutions of the \((b, j)\)-flow and of the curve flow (2.16) on \(M_a\).

The algorithm of obtaining finite type solutions of the \((b, j)\)-flow is also well-known: Fix a positive integer \(k\), solve

\[
(\xi_1, \ldots, \xi_k) : \mathbb{R}^2 \to \mathcal{U}_a^\perp \times \mathcal{U} \times \cdots \times \mathcal{U}
\]

for the following two compatible ordinary differential equations:

\[
\begin{align*}
\sum_{i=0}^k (\xi_i) x \lambda^{-i} &= \left[a\lambda + \xi_1, \sum_{i=0}^k \xi_i \lambda^{-i}\right], \\
\sum_{i=0}^k (\xi_i) t \lambda^{-i} &= \left[b\lambda^j + Q_{b,1}(\xi_1)\lambda^{j-1} + \cdots + Q_{b,j}(\xi_1), \sum_{i=0}^k \xi_i \lambda^{-i}\right],
\end{align*}
\]

(3.1)
with $\xi_0 = a$. Equate coefficients of $\lambda^i$ of (3.1) to get a system of compatible ODE in the $x$ and $t$ variables. So the initial value problem for (3.1) has a unique solution. If $(\xi_1, \ldots, \xi_k)$ is a solution of (3.1), then $u = \xi_1$ is a solution of the $(b, j)$-flow. Such a solution is called a finite type solution. To obtain the corresponding solution of the curve flow (2.16), we first solve $k : \mathbb{R}^2 \to U$ for

$$k^{-1}dk = u \, dx + Q_{b,j}(u) \, dt.$$  

By Proposition 2.11 there exists a constant $c$ such that $\lim_{x \to -\infty} k(x, t) = c$ and $\gamma = c^{-1}kak^{-1}c$ is the solution of the curve flow (2.16) corresponding to the finite type solution $u$ under the development map $\Phi$.

4. The $U/K$-hierarchy and corresponding curve flows

Let $U$ be a compact Lie group, $\sigma : U \to U$ be a group involution, $K$ the fixed point set of $\sigma$, and $K, P$ the $\pm 1$ eigenspaces of $d\sigma$ on $U$. Then

$$U = K + P, \quad [K, K] \subset K, \quad [K, P] \subset P, \quad [P, P] \subset K.$$ 

In particular, $\text{Ad}(k)(P) \subset P$ for all $k \in K$. The quotient $U/K$ is a symmetric space, $U = K + P$ is a Cartan decomposition of $U/K$, and the $\text{Ad}(K)$ representation on $P$ is the isotropy representation of $U/K$. As reviewed in section 1, if $a \in P$, then $S(\mathbb{R}, U_a^\perp \cap K)$ is invariant under the odd flows in the $U$-hierarchy, and if $k$ is odd, then the restriction of Poisson operator $J_k$ to $S(\mathbb{R}, K \cap U_a^\perp)$ is again a Poisson operator and the odd flows in the $U/K$-hierarchy are Hamiltonian with respect to $J_k$.

The curve flow corresponding to the flows in the $U/K$-hierarchy can be obtained by restricting the curve flow on the Adjoint orbit to a submanifold. To see this, let $a \in P$, $N_a$ the $\text{Ad}(K)$-orbit at $a$ in $P$, and $M_a$ the Adjoint $U$-orbit in $U$. Since $N_a \subset M_a$, $C_a(\mathbb{R}, N_a) \subset C_a(\mathbb{R}, M_a)$.

Suppose $b \in U_a^\perp \cap P$, $j$ is odd, and $u : \mathbb{R}^2 \to K \cap U_a^\perp$ is a solution of the $(b, j)$-flow in the $U/K$-hierarchy. Let $\theta_\lambda$ be the Lax pair (1.10) of the $(b, j)$-flow. Since $j$ is odd and $u \in K$, $Q_{b,j}(u) \in K$. Hence

$$u \, dx + Q_{b,j}(u) \, dt$$

is a $K$-valued flat connection 1-form. So $k, c$ of Proposition 2.11 lie in $K$ and $\gamma = c^{-1}kak^{-1}c$ lies in the submanifold $N_a$ of $M_a$. Hence we have

4.1 Theorem. Let $U/K$ be a symmetric space, $U = K + P$ a Cartan decomposition, and $a \in P$ such that the $\text{Ad}(K)$-orbit $N_a$ at $a$ in $P$ is a principal orbit. Let $M_a$ be the $\text{Ad}(U)$-orbit at $a$ in $U$. Then

(i) the development map $\Phi$ maps $C_a(\mathbb{R}, N_a)$ isomorphically onto $S(\mathbb{R}, K \cap U_a^\perp)$,  
(ii) if $j$ is odd, then the curve flow (2.16) on $C_a(\mathbb{R}, M_a)$ leaves $C_a(\mathbb{R}, N_a)$ invariant, and the restriction of (2.16) to $C_a(\mathbb{R}, N_a)$ corresponds to the $(b, j)$-flow in the $U/K$-hierarchy. Moreover, if $k$ is odd, the restricted flow on $C_a(\mathbb{R}, N_a)$ is Hamiltonian with respect to $i^*(L_k) = i^*\Phi^*(J_k+2)$,  
where $i : C_a(\mathbb{R}, N_a) \to C_a(\mathbb{R}, M_a)$ is the inclusion.

4.2 Example. Let $U = K + P$ be a Cartan decomposition of the rank $k$ symmetric space $U/K$, $a \in P$ such that $N_a$ is a principal $K$-orbit in $P$, and $b \in U_a$. It is known (cf. [Te1]) that there exists a polynomial $f : P \to \mathbb{R}$ invariant under the $\text{Ad}(K)$-action such that $\nabla f(a) = b$. Since $f$ is $K$-invariant, $\nabla f \mid N_a$ is a parallel normal field. So the curve flow $\gamma_t = (\nabla f(\gamma))_x$ becomes

$$\gamma_t = (\nabla f(\gamma))_x \quad (4.1)$$

on $N_a$. Since (4.1) corresponds to the $(b, 1)$-flow in the $U/K$-hierarchy under $\Phi$ and the $(b, 1)$-flow is completely integrable with respect to $J_3$, (4.1) is completely integrable with respect to the
Poisson operator $\Lambda$. By Proposition 2.19, the restriction of $H_b$ to $C_a(\mathbb{R}, N_a)$ is the Hamiltonian of (4.1) with respect to $\Lambda$. The higher order conserved quantities of (4.1) are the restrictions of $F_{b,2k+1} \circ \Phi$ to $C_a(\mathbb{R}, N_a)$. Moreover, the construction of Bäcklund transformations and finite type solutions work in a similar way by requiring $f(\lambda)$ and $\xi(\lambda) = \sum_{i=0}^{N} \xi_i \lambda^{-i}$ in section 3 to satisfy the extra reality condition given by the involution $\sigma$:

$$\sigma(f(-\lambda)) = f(\lambda), \quad d\sigma_e(\xi(-\lambda)) = \xi(\lambda).$$

4.3 Example. Consider the rank one symmetric space

$$U/K = SO(n + 1)/SO(n) = S^n.$$ The Cartan decomposition is $so(n + 1) = so(n) + \mathcal{P}$, where $\mathcal{K} = so(n)$, and

$$\mathcal{P} = \left\{ \left( \begin{array}{cc} 0 & -v^t \\ v & 0 \end{array} \right) \middle| v \in \mathbb{R}^n \right\}.$$ Let $e_{ij}$ be the elementary matrix, and

$$a = e_{21} - e_{12}.$$ The $SO(n)$-orbit in $\mathcal{P}$ at $a$ under the isotropy representation of $S^n$ is the standard unit sphere $S^{n-1}$. Below we compute the third flow in the $S^n$-hierarchy and the corresponding curve flow on $S^{n-1}$. A direct computation shows that $\mathcal{K} \cap \mathcal{U}_a^\perp$ and $\mathcal{P} \cap \mathcal{U}_a^\perp$ are spanned by

$$k_\alpha = e_{\alpha 2} - e_{2 \alpha}, \quad p_\alpha = e_{\alpha 1} - e_{1 \alpha}, \quad 3 \leq \alpha \leq n + 1$$

respectively. Henceforth in this example, we assume $3 \leq \alpha, \beta \leq n + 1$. Let $u = \sum_{\alpha=3}^{n+1} u_\alpha k_\alpha$, and $Q_j = Q_{a,j}(u)$. We use (1.7) and (1.8) to compute the $Q_j$’s. Note that $Q_0 = a$ and $Q_1 = u$. The reality conditions for the $S^n$-hierarchy are

$$A(\lambda) = A(\lambda), \quad A(\lambda)^t + A(\lambda) = 0, \quad I_{n,1}^{-1} A(-\lambda) I_{n,1} = A(\lambda),$$

(4.2)

where $I_{n,1} = \text{diag}(1, \ldots, 1, -1)$. It is known (cf. [TU1]) that $\sum_{j=0}^{\infty} Q_j \lambda^{-j}$ satisfies the reality conditions (4.2). Hence $Q_i$ is in $\mathcal{K}$ for odd $i$ and in $\mathcal{P}$ for even $i$. In particular, $Q_2 \in \mathcal{P}$. Write $Q_2 = y_0 a + \sum_{\alpha=3}^{n+1} y_\alpha p_\alpha$. Then

$$[Q_2, a] = (Q_1)_x + [u, Q_1] = u_x$$

implies that

$$y_\alpha = -(u_\alpha)_x.$$ To compute $y_0$, we use condition (1.8) to conclude that

$$\text{tr}(a + u \lambda^{-1} + Q_2 \lambda^{-2} + \cdots)^2 \sim \text{tr}(a^2)$$
as an asymptotic expansion. Compare the coefficients of $\lambda^{-2}$ in the above equation to get

$$\text{tr}(aQ_2 + Q_2 a + u^2) = 0.$$ This implies $y_0 = -\frac{\| u \|^2}{2}$. Hence

$$Q_2 = -\frac{\| u \|^2}{2} a - \sum (u_\alpha)_x p_\alpha.$$
We know $Q_3 \in K$, and
\[(Q_2)_x + [u, Q_2] = [Q_3, a].\] (4.3)

Let $Q_3 = \sum_{i,j=1}^n y_{ij} e_{ij}$. Then (4.3) implies that
\[y_{2\alpha} = (u_\alpha)_{xx} + \frac{\|u\|^2}{2} u_\alpha.\]

Compare the coefficients of $e_{\alpha\beta}$ for $\alpha, \beta \geq 3$ in
\[(Q_3)_x + [u, Q_3] = [Q_4, a]\]
to get
\[(y_{\alpha\beta})_x = -u_\alpha(u_\beta)_x + u_\beta(u_\alpha)_x.\]

The right hand side is equal to $((u_\alpha(u_\beta)_x + u_\beta(u_\alpha)_x)_x$. By (1.9), $Q_{b,j}(u) \in S(\mathbb{R}, \mathcal{U})$. So
\[y_{\alpha\beta} = -u_\alpha(u_\beta)_x + u_\beta(u_\alpha)_x.\]

Hence
\[Q_3 = \sum_{\alpha} \left( -(u_\alpha)_{xx} - \frac{\|u\|^2}{2} u_\alpha\right) k_\alpha + \sum_{\alpha, \beta} y_{\alpha\beta} e_{\alpha\beta}.\] (4.4)

Now we compute the third flow
\[u_t = (Q_3)_x + [u, Q_3]\] (4.5)
directly. The coefficient of $e_{2\alpha}$ of the left hand side of (4.5) is $-(u_\alpha)_t$, and of the right hand side is
\[\left( (u_\alpha)_{xx} + \frac{\|u\|^2}{2} u_\alpha \right)_x - \sum_\beta u_\beta (-u_\beta(u_\alpha)_x + u_\alpha(u_\beta)_x) \]
\[= (u_\alpha)_{xxx} + \frac{\|u\|^2}{2} (u_\alpha)_x + \sum_\beta u_\beta(u_\beta)_x u_\alpha + \|u\|^2 (u_\alpha)_x - u_\alpha \sum_\beta u_\beta(u_\beta)_x.\]

So the third flow is the vector modified KdV equation:
\[(u_\alpha)_t = -(u_\alpha)_{xxx} - \frac{3}{2} \|u\|^2 (u_\alpha)_x,\] (4.6)

or equivalently
\[u_t = -\left( u_{xxx} + \frac{3}{2} \|u\|^2 u_x \right), \quad (\text{vmKdV})\]

where $u : \mathbb{R}^2 \to \mathbb{R}^{n-1}$.

The Ad(K)-orbit at $a$ in $\mathcal{P}$ is the standard sphere $S^{n-1}$ in $\mathcal{P}$. By Proposition 2.6, the curve flow on $S^{n-1}$ corresponding to the first flow is the translation flow $\gamma_t = \gamma_x$. The curve flow on $S^{n-1}$ corresponding to the third flow in the $S^n$-hierarchy is
\[\gamma_t = g(Q_3(u), a) g^{-1},\] (4.7)
where $\gamma = gag^{-1}$ and $g^{-1}g_x = u$. Substitute (4.4) into the right hand side of (4.7) to get

$$\gamma_t = -\sum_{a=3}^{n+1} \left( (u_\alpha)_{xx} + \frac{\|u\|^2}{2} u_\alpha \right) gp_\alpha g^{-1}. \tag{4.8}$$

A direct computation gives

$$\gamma_x = \sum_\alpha u_\alpha gp_\alpha g^{-1},$$

$$\gamma_{xx} = \sum_\alpha (u_\alpha)_x gp_\alpha g^{-1} - \|u\|^2 gag^{-1},$$

$$\gamma_{xxx} = \sum_\alpha ((u_\alpha)_{xx} - \|u\|^2 u_\alpha) gp_\alpha g^{-1} - 3 \sum_\alpha u_\alpha (u_\alpha)_x gag^{-1}.$$ 

We can express the right hand side of (4.8) in terms of $\gamma$ and its $x$-derivatives to get

$$\gamma_t = -\left( \gamma_{xxx} + 3\langle \gamma_x, \gamma_{xx} \rangle \gamma + \frac{3}{2} \| \gamma_x \|^2 \gamma_x \right).$$

5. Weakly non-linear hydrodynamic systems

Let $M$ be a principal orbit of the isotropy representation of a symmetric space $U/K$, and $v$ a parallel normal field. Ferapontov noted in [F2] and [F4] that the curve flow (1.5)

$$\gamma_t = (v(\gamma))_x = -A_{v(\gamma)}(\gamma_x)$$

is a weakly non-linear hydrodynamic system on $M$. In this section, we study this curve flow on principal orbits of the isotropy representation of a symmetric space as a hydrodynamic system.

First we review some definitions and results on hydrodynamic systems. Given a smooth map $v = (v_{ij}) : \mathbb{R}^n \to gl(n, \mathbb{R})$, the first order quasilinear system for $u = (u_1, \ldots, u_n)^t : \mathbb{R}^2 \to \mathbb{R}^n$,

$$(u_i)_t = \sum_{j=1}^n v_{ij}(u)(u_j)_x, \tag{5.1}$$

is called a hydrodynamic system on $\mathbb{R}^n$. The system (5.1) is said to be diagonalizable in an open subset $O$ of $\mathbb{R}^n$ if there exist local coordinates $(y_1, \ldots, y_n)$ in $O$ such that the system (5.1) is of the form

$$(y_i)_t = v_i(y)(y_i)_x.$$ 

It is called weakly non-linear or linearly degenerate on $O$ if

(i) the eigenvalues of $(v_{ij}(x))$ have constant multiplicities $m_1, \ldots, m_s$ (so the corresponding eigenvalue functions $\lambda_1, \ldots, \lambda_s$ are smooth),

(ii) if $\xi$ is an eigenvector of $v = (v_{ij})$ with eigenvalue $\lambda_j$, then $d\lambda_j(\xi) = 0$.

A hydrodynamic system on a manifold $M$ is a first order quasilinear system for $\gamma : \mathbb{R}^2 \to M$ of the form:

$$\gamma_t = P(\gamma)(\gamma_x), \tag{5.2}$$

where $P$ is a smooth section of $L(TM, TM)$. It is easy to see that this system in local coordinates looks like a hydrodynamic system on an open subset of $\mathbb{R}^n$. System (5.2) is called diagonalizable (weakly non-linear respectively) if locally it is diagonalizable (weakly non-linear respectively).
Recall that a submanifold $M$ of $\mathbb{R}^{n+k}$ is called \textit{isoparametric} if the normal bundle is flat and the principal curvatures along any parallel normal field are constant. By definition of weak non-linearity, the flow (1.5) on an isoparametric submanifold is a weakly non-linear hydrodynamic system. Since a principal orbit $M$ of the isotropy representations of a symmetric space is isoparametric, (1.5) on $M$ is a weakly non-linear hydrodynamic system.

Let $\mathbb{R}^n$ be equipped with the standard inner product, and $\nabla$ the Levi-Civita connection. Then $\nabla_u v$ is a Poisson operator on $\mathcal{S}(\mathbb{R}, \mathbb{R}^n)$. Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, the Hamiltonian equation of the functional

$$F(u) = \int_{-\infty}^{\infty} f(u(x)) \, dx$$

with respect to $\nabla_u v$ is

$$u_t = \nabla_v (\nabla f(u)).$$

Write (5.3) in the standard coordinates of $\mathbb{R}^n$ to get

$$u_t = (\nabla f(u))_x,$$

i.e.,

$$(u_i)_t = \sum_{j=1}^n f_{u_i u_j}(u)(u_j)_x, \quad 1 \leq i \leq n. \tag{5.4}$$

So (5.4) is a hydrodynamic Hamiltonian system on $\mathbb{R}^n$. Dubrovin and Novikov investigated the systems (5.4) on $\mathbb{R}^n$, and obtained many remarkable results (cf. [DN]). Novikov conjectured that if (5.3) is diagonalizable then it is a completely integrable Hamiltonian system. Tsarev proved this conjecture and gave a complete classification of such systems (see [Ts]). Moreover, all constant of motions of (5.3) are of zero order. We remark that the boundary conditions of the Poisson operator $\nabla_u v$ are not taken into account in these results.

Let $M$ be a principal orbit in $\mathcal{P}$ of the isotropy representation of the symmetric space $U/K$. We have seen in Example 4.2 that the curve flow (1.5) is of the form (5.5) is of the form

$$\gamma_t = (\nabla f(\gamma))_x \tag{5.5}$$

on $M$ for some $K$-invariant polynomial $f : \mathcal{P} \to \mathbb{R}$. Note that (5.5) on $\mathcal{P}$ is a hydrodynamic system and is Hamiltonian with respect to $\nabla_u v$. Although $C_a(\mathbb{R}, M)$ is invariant under the flow (5.5), the restriction of $\nabla_u v$ to $C_a(\mathbb{R}, M)$ is not a Poisson structure. However, we have shown that (5.5) is a completely integrable Hamiltonian system with respect to the Poisson operator $\Lambda$ on $C_a(\mathbb{R}, M)$, its Hamiltonian is of zero order (given by the height function $H_b$, and it has infinitely many higher order conserved quantities. Ferapontov ([F3]) noted that (5.5) is non-diagonalizable if $U/K$ is $SU(3)/SO(3)$. Below we use submanifold geometry to prove directly that (5.5) is non-
diagonalizable on any irreducible isoparametric submanifold.

First we need to review some isoparametric theory (cf. [Te1]). Let $M^n \subset \mathbb{R}^{n+k}$ be an isoparametric submanifold. Then there exist smooth subbundles $E_1, \ldots, E_p$ of $TM$ and parallel normal fields $v_1, \ldots, v_p$ such that

(i) $TM = \bigoplus_{i=1}^p E_i$,
(ii) if $v$ is a parallel normal vector field on $M$ then the shape operator

$$A_v | E_i = \langle v, v_i \rangle \text{id}_{E_i},$$

(iii) there exists a parallel normal field $v$ such that

$$\langle v, v_1 \rangle, \langle v, v_2 \rangle, \ldots, \langle v, v_p \rangle$$

are distinct; in particular, $E_1, \ldots, E_p$ are eigenspaces of $A_v$.

The $E_i$'s and $v_i$'s are called \textit{curvature distributions} and \textit{curvature normals} of $M$. An isoparametric submanifold $M$ of $\mathbb{R}^{n+k}$ is \textit{irreducible} if $M$ is not a product of two lower dimensional isoparametric submanifolds. It is known that

(i) if $\langle v_i, v_j \rangle = 0$ for all $i \neq j$, then $M$ is a product of spheres,
(ii) principle orbits of isotropy representations of irreducible symmetric spaces are irreducible isoparametric submanifolds.
5.1 Proposition. Let \( X : M^n \to \mathbb{R}^{n+k} \) be a compact, irreducible, isoparametric submanifold with \( k \geq 2 \), \( v_1, \ldots, v_p \) the curvature normals of \( M \), and \( v \) a parallel normal field on \( M \) such that \( \langle v_1, v \rangle, \ldots, \langle v_p, v \rangle \) are distinct. Then the hydrodynamic system \( \gamma_t = (v(\gamma))_x \) on \( M \) is not diagonalizable.

PROOF. Suppose \( \gamma_t = (v(\gamma))_x = -A_v(\gamma x) \) is diagonalizable. Then there is a local coordinate system \( y \) such that
\[
I = \sum_{i=1}^{n} a_i(y)^2 dy_i^2,
\]
\[
A_v(X_{y_i}) = \lambda_i X_{y_i}, \quad 1 \leq i \leq n
\]
for some smooth functions \( a_i \) and constants \( \lambda_i \). Set \( e_i = X_{y_i}/a_i \). Then \( e_1, \ldots, e_n \) is a local orthonormal tangent frame on \( M \). Since \( E_1, \ldots, E_p \) are eigenspaces of \( A_v \), we may arrange the indices \( i \) so that
\[
\begin{align*}
\{ e_1, \ldots, e_n \} & \quad \text{span} \ E_1 \\
\{ e_{n+1}, \ldots, e_{n+p} \} & \quad \text{span} \ E_2, \\
\cdots & \quad \cdots \\
\{ e_{n+p-1}, \ldots, e_{n+k} \} & \quad \text{span} \ E_p.
\end{align*}
\]
Set
\[
I_j = \{ i_{j-1} + 1, \ldots, i_j \}.
\]
The dual coframe of \( e_i \) is \( w_i = a_i \ dy_i \). Let \( e_{n+1}, \ldots, e_{n+k} \) be a parallel normal frame on \( M \). Write
\[
d e_i = \sum_{j=1}^{n+k} w_{ji} e_j, \quad 1 \leq i \leq n + k.
\]
It follows from elementary local submanifold geometry that we have
\[
\begin{align*}
w_{\alpha\beta} &= 0, \quad n + 1 \leq \alpha, \beta \leq n + k \\
w_{i\alpha} &= \lambda_{i\alpha} w_i, \quad 1 \leq i \leq n, \quad n + 1 \leq \alpha \leq n + k, \\
w_{ij} &= \left( \frac{(a_i)_{y_j}}{a_j} \right) dy_i - \left( \frac{(a_j)_{y_i}}{a_i} \right) dy_j, \quad 1 \leq i \neq j \leq n, \\
d w_{AB} &= - \sum_{C=1}^{n+k} w_{AC} \wedge w_{CB}, \quad 1 \leq A, B \leq n + k,
\end{align*}
\]
and
\[
v_m = \sum_{\alpha=n+1}^{n+k} \lambda_{i\alpha} e_{i}, \quad \text{where} \ i \in I_m
\]
The Codazzi equation \( dw_{i\alpha} = \sum_{j=1}^{n} w_{ij} \wedge w_{j\alpha} \) implies that
\[
\left( \frac{(a_i)_{y_j}}{a_j} \right) = \left( \frac{\lambda_{i\alpha} a_i}{\lambda_{j\alpha} a_j} \right) = \frac{\lambda_{i\alpha}}{\lambda_{j\alpha}} \left( \frac{a_i}{a_j} \right) \quad (5.6)
\]
for all \( 1 \leq i \neq j \leq n \) and \( n + 1 \leq \alpha \leq n + k \). If \( e_i \in E_k, e_j \in E_m \) and \( k \neq m \), then since \( v_1, \ldots, v_p \) are distinct, there exists \( \alpha \) such that \( \lambda_{i\alpha} \neq \lambda_{j\alpha} \). It follows from equation (5.6) that
\[
\text{If} \ r \neq s, \ i \in I_r, \text{ and } j \in I_s, \text{ then } w_{ij} = 0. \quad (5.7)
\]
Next we claim that $\langle v_r, v_s \rangle = 0$ if $r \neq s$. To see this, let $i \in I_r$ and $j \in I_s$. The Gauss equation implies that

$$dw_{ij} = -\sum_{m=1}^{n} w_{im} \wedge w_{mj} + \sum_{\alpha=n+1}^{n+k} w_{i\alpha} \wedge w_{j\alpha}.$$ 

By (5.7), the left hand side and the first term of the right hand side of the above equation are zero. So

$$\sum_{\alpha=n+1}^{n+k} w_{i\alpha} \wedge w_{j\alpha} = 0.$$ 

But

$$\sum_{\alpha=n+1}^{n+k} w_{i\alpha} \wedge w_{j\alpha} = \sum_{\alpha} \lambda_{i\alpha} \lambda_{j\alpha} w_i \wedge w_j = \langle v_r, v_s \rangle w_i \wedge w_j,$$

which proves the claim. So $M$ is the product of standard spheres. This contradicts the assumption that $M$ is irreducible. Hence $\gamma_t = (v(\gamma))_x$ is not diagonalizable.  \[\blacksquare\]
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