Evaluation of Brauer elements over local fields

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Abstract

We study the evaluation maps given by elements of the Brauer group of varieties over local fields. We show constancy of the aforementioned maps in several interesting cases.

1 Introduction

Manin, in trying to understand the failure of the Hasse principle combined global class field theory with the Brauer group of a scheme in order to introduce the Brauer-Manin set of a variety over a number field [26]. This gave birth to the theory of Brauer-Manin obstruction, which has evolved a lot since then and has become an important tool in the study of rational points.

We briefly recall the main points and refer the reader to [32 §5] for more details on the Brauer-Manin obstruction and its variants or to [38 §2] for a more recent report. Let $X$ be a smooth, proper, geometrically irreducible variety over a number field $k$. Let $\text{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m)$ denote the cohomological Brauer group of $X$. By functoriality of the Brauer group, for any field $F$ containing $k$ we can induce by specialization an evaluation map $\text{ev}_{\text{sf}} : X(F) \to \text{Br}(F)$. Combining the evaluation maps at all the completions of $k$ we get the Brauer-Manin pairing:

$$\text{Br}(X) \times X(\mathbb{A}_k) \to \mathbb{Q}/\mathbb{Z}$$

where $\mathbb{A}_k$ is the ring of adeles of $k$. We denote by $X(\mathbb{A}_k)^{\text{Br}(X)}$ the adelic points orthogonal to $\text{Br}(X)$. If we diagonally embed $X(k)$ into $X(\mathbb{A}_k)$, by the global reciprocity law we have the following chain of inclusions

$$X(k) \subseteq X(\mathbb{A}_k)^{\text{Br}(X)} \subseteq X(\mathbb{A}_k)$$

If $X(\mathbb{A}_k) \neq \emptyset$ and $X(\mathbb{A}_k)^{\text{Br}(X)} = \emptyset$, we say that there is Brauer-Manin obstruction to the Hasse principle on $X$. If for any $X$ in a class of varieties $\mathcal{C}$, we have
that $X(\mathbb{A}_k)^{\text{Br}}(X) \neq \emptyset$ implies $X(k) \neq \emptyset$, we say that the Brauer-Manin obstruction to the Hasse principle for elements of $\mathcal{C}$ is the only one.

Because of conjectures of Bombieri and Lang, it was never expected that the Brauer-Manin obstruction would explain all the failures of the Hasse principle for smooth and proper varieties. Indeed the first example with $X(\mathbb{A}_k)^{\text{Br}}(X) \neq \emptyset$ and $X(k) = \emptyset$ was a bielliptic surface given by Skorobogatov [31]. Even the more refined étale Brauer-Manin obstruction is insufficient to explain all the failures of the Hasse principle as shown by Poonen [29]. The above notwithstanding, we have the following two important conjectures. The first conjecture was formulated by Colliot-Thélène in [10] and states that if $X$ is rationally connected, then $X(k)$ is dense in $X(\mathbb{A}_k)^{\text{Br}}(X)$. This generalises the same conjecture for geometrically rational surfaces by Colliot-Thélène and Sansuc, first asked as a question in [11]. The second conjecture is by Skorobogatov in [33] and states that the Brauer-Manin obstruction to the Hasse principle is the only one for K3 surfaces. Nowadays, there is a large body of literature around the Brauer-Manin obstruction and it has become clear that is important to understand the various evaluation maps given by elements of the Brauer group.

The crucial part of the calculation of the Brauer-Manin obstruction is the calculation of the local evaluation map for each completion of the number field $k$. In this note we apply deep results of Kato ([20],[21]) to prove the constancy of the local evaluation map in the good reduction case for arbitrary Brauer elements. This generalises previous known results of Colliot-Thélène and Skorobogatov [12] by removing the assumption that the order of the element is coprime to the residual characteristic. Our approach also recovers many previous results and gives a uniform treatment of all local evaluation maps. Let $K$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$ and let $X$ be a smooth, proper, geometrically irreducible variety over $K$. Assume that there is a model $\mathcal{X}/\mathcal{O}_K$ which is regular with geometrically integral fibres. The novel idea of our approach is the definition of a subgroup $B$ of $\text{Br}(X)$. We show that the relevant evaluation maps are constant for elements of this subgroup. The definition of $B$ uses Kato’s Swan conductor [20], and depends on the model. In some applications we can show that $B$ is the whole Brauer group. For example we obtain a relatively simple proof of the following.

**Theorem A**

Let $p$ be an odd prime. Suppose that $\mathcal{X}/\mathbb{Z}_p$ is smooth, proper with geometrically integral fibres, and either the special fibre is separably rationally connected or the generic fibre is a K3 surface. Then

$$\text{ev}_{\text{eff}} : X(\mathbb{Q}_p) \to \text{Br}(\mathbb{Q}_p)$$

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is a constant map, for any $\mathcal{A} \in \text{Br}(X)$.

Note that we actually have more general results in Proposition 7 for rationally connected varieties and in Proposition 8 for $K3$ surfaces. Proposition 8 already appears in [7], cf. Remark 9.

There are some cases where we can replace the assumption on the existence of a good model, with an assumption on the Galois action on $\ell$-adic cohomology.

**Theorem B**

Let $p$ be an odd prime. Suppose that $X$ is the Kummer surface of an abelian surface over $\mathbb{Q}_p$, and the Galois representation on $\ell$-adic cohomology is unramified for some (any) $\ell \neq p$. Then

$$\text{ev}_{\mathcal{A}} : X(\mathbb{Q}_p) \to \text{Br}(\mathbb{Q}_p)$$

is a constant map, for any $\mathcal{A} \in \text{Br}(X)$.

We have various results of the above kind in §4. Note that we formulate the results in the introduction in their simplest form, for ease of exposition. The reader can find more general statements in the main body of the text.

We want to emphasize that besides giving a quick and uniform proof of existing results, our approach has a wide range of applicability. For example most of the results of §5 cannot be obtained by combining existing results in the literature. As a sample, we can show the following (see Remark after Corollary 21)

**Proposition C**

Suppose that $X / K$ is a del Pezzo surface which splits over an unramified extension of $K$ and admits a regular proper model with geometrically integral special fibre. Then

$$\text{ev}_{\mathcal{A}} : X(K) \to \text{Br}(K)$$

is a constant map, for any $\mathcal{A} \in \text{Br}(X)$.

In a different spirit, we also have a result which could be quite useful in computations. Note the interesting feature that the precision needed for the computation depends only on the ground field.

**Proposition D**

Suppose that $X / \mathbb{Q}_p$ admits a smooth proper model $\mathcal{X} / \mathbb{Z}_p$ with geometrically integral fibres. Let $\mathcal{A} \in \text{Br}(X)[p]$. Then

$$\text{ev}_{\mathcal{A}} : X(\mathbb{Q}_p) \to \text{Br}(\mathbb{Q}_p)$$

factors through $\mathcal{X}(\mathbb{Z}_p/p^2)$ for $p$ odd, and through $\mathcal{X}(\mathbb{Z}_p/p^3)$ for $p$ even.

Finally, we note that our results have direct implications for a question Swinnerton-Dyer asked the authors of [12], see [12, Introduction]. That question was related
to his work on density of rational points on certain surfaces, and was the main motivation for [12].

The outline is as follows. In section 2 we fix notation and state some preliminaries, while in section 3 we prove our main Theorem. Sections 4 and 5 consist of applications in the case of good and bad reduction respectively.

Relation to other work. In case that there exists a smooth proper model and \( X \) is geometrically simply connected, the constancy of \( \text{ev}_\mathcal{A} \) for elements of order prime to \( p \) follows from [12, Prop. 2.4]. Under the same assumptions, for elements of order a power of \( p \), one can use [7, Lem. 7.2, Lem. 7.3] in order to show constancy of the evaluation map. Our main result, Theorem 3 does not use any results from [7], but its important consequence, Proposition 6 does (for the case \( \ell = p \)).

Our approach, on the one hand, gives a short natural proof of our main new result, Theorem A, and, on the other hand, leads to a number of other results including Theorem B and Propositions C and D above (see also Propositions 11, 13, 19 and 23 below).

2 Preliminaries

We will use the following notation. Given an abelian group \( A \) and a positive integer \( n \) we denote by \( A[n] \) the subgroup of elements annihilated by \( n \) and by \( A[n^{\infty}] \) the subgroup of elements annihilated by a power of \( n \). Given a field \( k \), we denote by \( \overline{k} \) a separable algebraic closure of \( k \). Given a variety \( V \) over \( k \), we denote by \( \overline{V} \) or \( V_\overline{k} \) the variety \( V \otimes_k \overline{k} \) over \( \overline{k} \).

- \( K \) is a finite extension of \( \mathbb{Q}_p \), with ring of integers \( \mathcal{O}_K \) and residue field \( F \). We denote by \( \pi \) a uniformizer of \( \mathcal{O}_K \), and by \( e \) the absolute ramification index of \( K \).
- \( \mathcal{X} \) is a faithfully flat, regular, finite type scheme over \( \mathcal{O}_K \), with geometrically integral fibres.
- \( X/K \) is the generic fibre of \( \mathcal{X}/\mathcal{O}_K \).
- \( Y/F \) is the special fibre of \( \mathcal{X}/\mathcal{O}_K \).
- \( \ell \) is a prime number (the case \( \ell = p \) is allowed).
We suppose that \( X(K) \neq \emptyset \) and we identify \( \text{Br}(K) \) with \( \text{Br}_0(X) \). We remind the reader that by definition \( \text{Br}_0(X) := \text{Im}(\text{Br}(K) \to \text{Br}(X)) \).

- For a discretely valued field \( T \), we denote by \( T_{nr} \) its maximal unramified extension, and we define \( \text{Br}_u(T) := \ker(\text{Br}(T) \to \text{Br}(T_{nr})) \).

- Let \( R = \mathcal{O}_{X,Y} \) be the local ring of \( X \) at \( Y \). Note that \( R \) is a discrete valuation ring. We denote by \( R_{h} \) the henselization of \( R \) and by \( R_{h,nr} \) the maximal unramified extension of \( R_{h} \). Hence \( R_{h,nr} \) is a strict henselisation of \( R \).

- We set \( R_{h,nr,b} := R_{h} \otimes_{\mathcal{O}_K} \mathcal{O}_{K_{nr}} \). Equivalently \( R_{h,nr,b} \) is the direct limit of the extensions of \( R_{h} \) that correspond to the various extensions \( k(Y)/F(Y) \) where \( k/F \) is a finite field extension and \( k(Y) \) denotes the function field of \( Y \otimes_F k \).

- We denote by \( L, L^h, L_{nr,b}^h, L_{nr}^h \) the fraction field of \( R, R^h, R_{nr,b}^h, R_{nr}^h \) respectively.

- We denote by \( F(Y), \overline{F(Y)}, \overline{F(Y)} \) the residue field of \( R^h, R_{nr,b}^h, R_{nr}^h \) respectively.

Note that \( F(Y) \) is the function field of \( Y \) and \( \overline{F(Y)} \) is the function field of \( Y \otimes_F F \).

We will also use some notation from [20]. In particular, see [20, (1.2)] for the definition of \( H^1_q(k) \) and \( H^2_q(k) \), when \( k \) is a field. Moreover, see [20, (1.4)] for the definition of the injective maps \( \lambda_\pi : H^{q-1}_r(k) \oplus H^q_r(k) \to H^q(K) \) when \( K \) is the fraction field of a henselian discrete valuation ring with a chosen uniformiser \( \pi \) and residue field \( k \) (cf. [21, Thm 3]). In this last case Kato also defines an increasing filtration \( \{ \text{fil}_n H^q(K) \}_{n \geq 0} \) (see [20, §2]). We remind the reader that for any field \( k \) we have \( H^1(k) = \text{Hom}_{\text{cont}}(\text{Gal}(k^{ab}/k), \mathbb{Q}/\mathbb{Z}) \) and \( H^2(k) = \text{Br}(k) \), see [20, (1.2)].

Let \( i \geq 1 \). We have the following commutative diagram:

\[
\begin{array}{ccc}
H^1_{\overline{\ell}}(F(Y)) \oplus H^2_{\overline{\ell}}(F(Y)) & \to & H^1_{\overline{\ell}}(\overline{F(Y)}) \oplus H^2_{\overline{\ell}}(\overline{F(Y)}) \\
\downarrow & & \downarrow \\
\text{Br}(X) \to \text{Br}(L) & \to & \text{Br}(L^h) \\
\downarrow & & \downarrow \\
\text{Br}(L^h) & \to & \text{Br}(L_{nr,b}^h) \to \text{Br}(L_{nr}^h)
\end{array}
\]

The vertical maps are injective, and we identify their sources with their images, see [20, (1.6)] and [21, Thm 3]. Note that by [20, Prop. 6.1 (1)] and [13]...
Prop. 1.4.3 (iii)] we have that $H^1(F(Y)) \oplus H^2(F(Y)) = Br_u(L^h)$. Moreover it follows from the definitions in [20 (1.4)] that $H^2(F(Y)) \subseteq Br(R^h)$. With a little more work one can show that we actually have equality, but we will not use this fact in the sequel.

For the convenience of the reader we briefly recall some definitions from [20].

Let $K$ denote a henselian discrete valuation field with valuation ring $\mathcal{O}_K$ and with residue field $F$. Let $\pi$ be a uniformiser of $\mathcal{O}_K$. Kato defines an increasing filtration \( \{\text{fil}_nH^q(K)\}_{n \geq 0} \) of the group $H^q(K)$ by

\[
\chi \in \text{fil}_nH^q(K) \iff \{\chi, 1 + \pi^{n+1}T\} = 0 \text{ in } V^{q+1}(\mathcal{O}_K[T])
\]

where $V^{q+1}(\mathcal{O}_K[T])$ denotes a certain direct limit of cohomology groups and \( \{,\} \) is induced by cup product. For the exact definitions we refer the reader to [20 (1.2), (1.7) and (2.1)]. The filtration is exhaustive and the swan conductor of $\chi$, denoted by $\text{sw}(\chi)$, is defined as the minimum integer $n$ such that $\chi \in \text{fil}_nH^q(K)$.

3 The main result

We keep the notation of the previous section. Before stating the main Theorem we need some preparatory results and definitions. Denote by $\alpha$ the map

$\alpha : Br(L) \to Br(L^h)$

and by abuse of notation denote by the same letter the map it induces $Br_u(L) \to Br_u(L^h)$. We have the following lemma

Lemma 1. \( \alpha^{-1}(H^2(F(Y))) \subseteq Br(R) \).

Proof. By [1 Thm. 3.3] we have the following commutative diagram with exact rows

\[
\begin{array}{cccc}
0 & \to & Br(R) & \to & Br_u(L) & \to & H^1(F(Y), \mathbb{Q}/\mathbb{Z}) & \to & 0 \\
& & \downarrow & & \alpha & \downarrow & \text{id} & \downarrow & \\
0 & \to & Br(R^h) & \to & Br_u(L^h) & \to & H^1(F(Y), \mathbb{Q}/\mathbb{Z}) & \to & 0
\end{array}
\]

The result follows from the above diagram since $H^2(F(Y)) \subseteq Br(R^h)[\ell]$. \hfill \square
We will now define various subgroups of $\text{Br}(X)$.

Let

$$M_\ell := \text{Br}(X) \cap \alpha^{-1}(H^1_\ell(F(Y)) \oplus H^2_\ell(F(Y))).$$

We have an induced map

$$M_\ell \to H^1_\ell(\mathcal{F}(Y)) \oplus H^2_\ell(\mathcal{F}(Y))$$

and we denote by $\tau_\ell$ the composition of the map above with the projection to the first factor. Hence we have the following map:

$$\tau_\ell : M_\ell \to H^1_\ell(\mathcal{F}(Y)).$$

In a similar way we define a map:

$$\tau'_\ell : M_\ell \to H^1_\ell(F(Y)).$$

We now define $\ell B \subseteq \ell M \subseteq \text{Br}(X)$ as follows:

$$\ell M := \bigcup_{i \geq 0} M_{i\ell},$$

$$\ell B := \bigcup_{i \geq 0} \ker(\tau_{i\ell}).$$

We caution the reader that what Kato denotes by $K$ and $F$ in our case are $L^h$ and $F(Y)$ respectively. The filtration of Kato on $\text{Br}(L^h)$ (see [20, §2]) induces a filtration on $\text{Br}(X)$ and $\text{Br}(L)$. With respect to this filtration we have the following important result.

**Proposition 2.**

1. $\ell M[\ell^\infty] = \text{fil}_0(\text{Br}(X))[\ell^\infty].$

2. If $\ell \neq p$, then $\ell M[\ell^\infty] = \text{Br}(X)[\ell^\infty].$

**Proof.** This follows from [20 Prop. 6.1 (1)] cf. last paragraphs of Preliminaries. \qed

Denote by $B$ the subgroup of $\text{Br}(X)$ generated by the $iB$ as $\ell$ varies through all the primes. Note that $B$ depends on the regular model with geometrically integral fibres $\mathcal{X}/\mathcal{O}_K$. When we want to be explicit about this dependence we will denote $B$ by $\mathcal{X}_K \text{Br}(X)$. We can now state and prove our main result.
Theorem 3. Let $\mathcal{A} \in B$. Then

$$\text{ev}_{\mathcal{A}} : \mathcal{X}(\mathcal{O}_K) \to \text{Br}(K)$$

is a constant map.

Proof. We may assume that $\mathcal{A} \in \ell M$, for some $\ell$. Fix $i$ with $\mathcal{A} \in \ker(\tau_{\ell^i})$. Let $\chi \in H^1_{\ell^i}(F)$ denote a character of the Galois group of $F$ of order $\ell^i$ and denote by $\phi : H^1_{\ell^i}(F) \to H^1_{\ell^i}(F(Y))$ the natural map. It follows from the definitions that the image of $\chi$ in $\text{Br}(K)$ is the cyclic algebra $B = (\chi, \pi)$, and $\alpha(B)$ is the image of $\phi(\chi)$ in $\text{Br}(L^h)$. Since $Y$ is geometrically integral, $\phi$ is injective and hence $\alpha(B)$ has order $\ell^i$. Now, by the inflation-restriction sequence we have

$$\ker(H^1_{\ell^i}(F(Y)) \to H^1_{\ell^i}(\mathcal{F}(Y)) = H^1(G, \mathbb{Z}/\ell)$$

where $G = \text{Gal}(\mathcal{F}(Y)/F(Y)) \cong \mathbb{Z}$. Therefore $\ker(H^1_{\ell^i}(F(Y)) \to H^1_{\ell^i}(\mathcal{F}(Y))$ is the subgroup generated by $\alpha(B)$. As $\tau_{\ell^i}(\mathcal{A}) \in \ker(H^1_{\ell^i}(F(Y)) \to H^1_{\ell^i}(\mathcal{F}(Y))$ by assumption, we deduce that $\mathcal{A} - n\mathcal{B} \in \alpha^{-1}(H^1_{\ell^i}(F(Y)))$ for some $n \in \mathbb{Z}$. By Lemma 1 we have that $\mathcal{A} - n\mathcal{B} \in \text{Br}(R)$. Hence $\mathcal{A} - n\mathcal{B} \in \text{Br}(R) \cap \text{Br}(X) = \text{Br}(\mathcal{X})$ where the last equality follows from [13, Thm. 3.7.6]. Since $\text{ev}_{\mathcal{A}}$ is identically zero for $\mathcal{C} \in \text{Br}(\mathcal{X})$ it follows that $\text{ev}_{\mathcal{A}}$ sends everything to $n\mathcal{B}$.

To state our next result and make the link to [12] more explicit, denote by $\beta$ the map

$$\beta : \text{Br}(L) \to \text{Br}(L^h_{nr,b}).$$

Corollary 4. Let $\mathcal{A} \in \text{Br}(X)$. Suppose that $\mathcal{A} \in \ker(\beta)$. Then

$$\text{ev}_{\mathcal{A}} : \mathcal{X}(\mathcal{O}_K) \to \text{Br}(K)$$

is constant.

Proof. We can suppose that the order of $\mathcal{A}$ is a power of $\ell$. By [20, Prop. 6.1 (1)], we have that $\mathcal{A} \in \text{fil}_0(\text{Br}(X))$. Therefore we have that $\mathcal{A} \in \ell M$ by Proposition 2. It is then clear from the definitions that $\mathcal{A} \in \ell B$, and so we conclude by Theorem 3.

\[\Box\]
Remark 5. Note that \( \ker(Br(X) \to Br(X \otimes_K K_{nr})) \subseteq \ker(\beta) \) and so this corollary recovers \([12, \text{Lemma 2.2 (ii)}]\), where they additionally assume that \( \mathcal{X} / \mathcal{O}_K \) is proper. Note also that in the proof of \([12, \text{Prop. 2.3}]\) it is shown that if \( H^1(X, \mathcal{O}_X) = 0 \) and the Neron-Severi group \( \text{NS}(X \otimes_K \overline{K}) \) is torsion-free then \( \ker(Br(X) \to Br(X \otimes_K K_{nr})) = Br_1(X) \).

We can combine Theorem 3 with two results from \([7]\) to obtain the following.

Proposition 6. Suppose that \( \mathcal{X} / \mathcal{O}_K \) is smooth and that the maximal pro-\( \ell \) quotient of the geometric fundamental group of \( Y \) is trivial. Let \( \mathcal{A} \in Br(X) \) and suppose one of the following:

(i) \( \mathcal{A} \in \ell \mathcal{M} \);

(ii) \( \mathcal{A} \in Br(X)\ell \) and \( \text{fil}_0(\text{Br}(X))\ell = Br(X)\ell \);

(iii) \( \mathcal{A} \in Br(X)\ell \) and \( \ell \neq p \);

(iv) \( \mathcal{A} \in Br(X)\ell, \ell = p, H^0(Y, \Omega_Y^1) = 0 \) and \( e < p - 1 \).

Then

\[
\text{ev}_{\mathcal{A}} : \mathcal{X}(\mathcal{O}_K) \to Br(K)
\]

is a constant map.

Proof. Cases (ii) and (iii) follow from Proposition 2 and case (i). Case (iv) follows from \([7, \text{Lemma 7.2}]\) and case (ii). It remains to prove case (i). Fix \( i \) with \( \mathcal{A} \in M_\ell \).

By Proposition \([7, \text{Prop. 3.1}]\) for the case \( \ell = p \) and by \([12, \text{proof of Prop. 2.4}]\) for the case \( \ell \neq p \), we see that \( \tau_{\ell}(\mathcal{A}) \) lies in \( H^1(Y \otimes F, \mathbb{Z}/\ell) \), which is trivial by assumption. Therefore \( \mathcal{A} \in \ell B \) and we are done by Theorem 3.

\[\square\]

4 Applications to good reduction

We keep the notation and assumptions of the previous section. In this section we will use some more notions from \([20]\). In particular see \([20, \text{pg. 121}]\) for the definition of \( \text{sw}_p(\mathcal{A}) \), when \( S \) is a normal irreducible scheme with function field \( L, p \in S^1 \) (i.e. \( p \in S \) and \( \dim(\mathcal{O}_{S,p}) = 1 \)), and \( \mathcal{A} \in Br(L) \).
Our first result concerns rationally connected varieties. In respect to the relation of the hypothesis to the Brauer-Manin obstruction we note the following which follows from [22 IV Thm 3.11]: if $k$ is a number field and $X/k$ is smooth, proper and rationally connected, then there is a finite set of places $T$ containing the archimedean places such that $X \otimes_k k_v$ admits a smooth, proper model with separably rationally connected special fibre for any $v \notin T$.

**Proposition 7.** Suppose that $\mathcal{X}/\mathcal{O}_K$ is smooth and proper, and the special fibre is separably rationally connected. Then

$$\text{ev}_\mathcal{A}: X(K) \to \text{Br}(K)$$

is a constant map, for any $\mathcal{A} \in \text{Br}(X)$.

**Proof.** We can assume that the order of $\mathcal{A}$ is a power of a prime $\ell$. The special fibre is geometrically simply connected, see e.g. [15, Cor. 3.6]. Since $\mathcal{X}/\mathcal{O}_K$ is proper, we can conclude by Proposition 6 once we show that $\text{fil}_0(\text{Br}(X))[\ell^\infty] = \text{Br}(X)[\ell^\infty]$ in the case $\ell = p$. Note that by [22 Cor. IV. 3.8] and its proof we have that $H^0(Y, \Omega^1_Y) = H^0(Y, \Omega^2_Y) = 0$. Hence it follows from [7, Lemma 7.1] that $\mathcal{A} \in \text{fil}_0(\text{Br}(X))$.

Our next result concerns $K3$ surfaces.

**Proposition 8.** Suppose that $\mathcal{X}/\mathcal{O}_K$ is smooth and proper, $X/K$ is a $K3$ surface and $e < p - 1$. Then

$$\text{ev}_\mathcal{A}: X(K) \to \text{Br}(K)$$

is a constant map, for any $\mathcal{A} \in \text{Br}(X)$.

**Proof.** We can assume that the order of $\mathcal{A}$ is a power of a prime $\ell$. Under our assumptions $\text{Pic}(\mathcal{X}) = \text{Pic}(X)$ and $\Omega_{\mathcal{X}/\mathcal{O}_K}$ is locally free. These together with the fact that $\omega_X$ is trivial and that the formation of exterior powers of the sheaf of relative differentials commutes with base change, imply that $\Omega^2_{\mathcal{X}/\mathcal{O}_K}$ is also trivial. Hence $\omega_Y$ is trivial as well. Moreover, since $\mathcal{X}/\mathcal{O}_K$ is smooth and proper it follows from [23 Cor. VI. 4.2] that the Betti numbers of the generic and the special fibre are equal. Hence $Y$ is a $K3$ surface by the classification of surfaces in characteristic $p$, see e.g. [3]. Therefore $H^0(Y, \Omega^1_Y) = 0$ by a theorem of Rudakov and Šafarevič [24], see also [25]. Hence the result follows from Proposition 6 since $\mathcal{X}/\mathcal{O}_K$ is proper.
Remark 9. The above result is already in [7, Remark 7.5]. They showed it by combining some of their results with results from [12] as explained in the Relation to other work paragraph of the introduction.

The next corollary is interesting as it does not explicitly mention the existence of a good model in its assumptions.

Corollary 10. Suppose that $X$ is the Kummer surface of an abelian surface $A$ over $K$, the $\text{Gal}(\overline{K}/K)$-representation $H^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified for some (any) $\ell \neq p$ and $e < p - 1$. Then

$$\text{ev}_A : X(K) \to \text{Br}(K)$$

is a constant map, for any $A \in \text{Br}(X)$.

Proof. By Proposition 8, it suffices to show the existence of $\mathcal{X}/\mathcal{O}_K$ which is smooth, proper and has $X$ as generic fibre. This follows from [27, Thm. 4.1].

We record a result for Enriques surfaces.

Proposition 11. Suppose that $\mathcal{X}/\mathcal{O}_K$ is smooth and proper and that the generic fibre is an Enriques surface. Let $A \in \text{Br}(X)$ have odd order. Then

$$\text{ev}_A : X(K) \to \text{Br}(K)$$

is a constant map.

Proof. It is well known that the geometric fundamental group of the generic fibre is isomorphic to $\mathbb{Z}/2$, see e.g. [2, Lemma VIII.15.1]. By [30, Thm. X. 3.8] the geometric fundamental group of the special fibre is a quotient of $\mathbb{Z}/2$. Under our assumptions $\text{Pic}(\mathcal{X}) = \text{Pic}(X)$ and $\omega_X^2$ is trivial. By the same reasoning as in the proof of Proposition 8 it follows that $\omega_Y^2$ is trivial. Moreover the Betti numbers of the generic and the special fibre are equal, since $\mathcal{X}/\mathcal{O}_K$ is smooth and proper [28, Cor. VI. 4.2]. Hence $Y$ is an Enriques surface by the classification of surfaces in characteristic $p$, see e.g. [3]. We can assume that the order of $A$ is a power of an odd prime $\ell$. Since $\mathcal{X}/\mathcal{O}_K$ is proper, we can conclude by Proposition 6 once we show that $\text{fil}_0(\text{Br}(X))[\ell^\infty] = \text{Br}(X)[\ell^\infty]$ in the case $\ell = p$. We have that $H^0(Y, \Omega_Y^2) = H^0(Y, \Omega_Y^2) = 0$ by [14, Prop. 1.1.4.1]. Hence it follows from [7, Lemma 7.1] that $A \in \text{fil}_0(\text{Br}(X))$. 

\[\square\]
Remark 12. If $X$ is an Enriques surface then $\text{Br}(\bar{X}) \cong \mathbb{Z}/2$, and so an element of odd order in $\text{Br}(X)$ is necessarily algebraic. I do not know of examples where $H^1(K, \text{Pic}X) \cong \text{Br}_1(X)/\text{Br}_0(X)$ has odd torsion.

We have the following for smooth surfaces in $\mathbb{P}^3$.

**Proposition 13.** Suppose that $\mathcal{X}/\mathcal{O}_K$ is smooth and proper. Assume that the special fibre is isomorphic to a surface in $\mathbb{P}^3$ and $e < p - 1$. Then

$$\text{ev}_A : X(K) \rightarrow \text{Br}(K)$$

is a constant map, for any $A \in \text{Br}(X)$.

**Proof.** The special fibre is geometrically simply connected and it follows from Bott vanishing for $\mathbb{P}^3$ that $H^0(Y, \Omega_Y) = 0$. Therefore we are done by Proposition 6. \qed

The assumption for the crystalline cohomology in the next result, is not easy to check in general. However we note that we do get a new proof of the constancy of the evaluation maps for some classes of $K3$ surfaces, since such surfaces are known to have torsion free crystalline cohomology.

**Proposition 14.** Assume that $K = \mathbb{Q}_p$ and $p \neq 2$.

Suppose that $\mathcal{X}/\mathcal{O}_K$ is smooth and proper, and the generic fibre $X/K$ is one of the following:

1. geometrically Kummer;
2. smooth complete intersection of dimension at least 2;
3. isomorphic to an $n$-dimensional subvariety of $\mathbb{P}^r$, with $2n - r \geq 1$.

Assume moreover that $H^1_{\text{cris}}(Y/W)$ and $H^2_{\text{cris}}(Y/W)$ are torsion-free. Let $A \in \text{Br}(X)$. Then

$$\text{ev}_A : X(K) \rightarrow \text{Br}(K)$$

is a constant map.
Proof. Take an embedding $K \to \mathbb{C}$ and let $X_{\mathbb{C}} = X \otimes_K \mathbb{C}$. Then it is well known that $X_{\mathbb{C}}$ is simply connected and has first Betti number equal to 0, see e.g. [35, Thm. 1 and Thm. 2] and [23, Thm. 3.2.1 and Cor. 3.2.2]. This implies that $X$ is geometrically simply connected ([30, Cor. X. 1.8]) and that $H^1_{dR}(X) = 0$. The fact that $H^1_{dR}(X) = 0$ and our assumptions on the crystalline cohomology imply that $H^1_{dR}(Y) = 0$ (see [19, §1.3]). Therefore it follows from the Hodge spectral sequence for $Y$ and [16, Cor. 2.5] that $H^0(Y, \Omega_Y) = 0$. We can conclude by Proposition 6.

Remark 15. If $X$ is a smooth complete intersection of dimension at least 3, then $Br(X) = Br_0(X)$ by [9, Prop. A.1].

Before we state the next Proposition, we need some preparation. Suppose that $\mathcal{X}/\mathcal{O}_K$ is smooth and let $p \in \mathcal{X}$ be the generic point of $Y$. Let $P_0 \in \mathcal{X}(\mathcal{O}) \subset \mathcal{X}$ and let $A = \mathcal{O}_{\mathcal{X}, P_0}$. We will need the following lemma (cf. [4, §3.2]).

Lemma 16. Let $\mathcal{X} \to \mathcal{X}$ denote the blow-up of $\mathcal{X}$ at $P_0$. Let $\mathcal{Z}$ be the strict transform of $Y$. Then

(i) $\mathcal{Z}/\mathcal{O}_K$ is smooth;

(ii) the map $\mathcal{Z}(\mathcal{O}_K) \to \mathcal{X}(\mathcal{O}_K)$ surjects to the elements of $\mathcal{Z}(\mathcal{O}_K)$ that reduce to $P_0$;

(iii) for any integer $i \geq 1$, the map $\mathcal{Z}(\mathcal{O}_K) \to \mathcal{Z}(\mathcal{O}_K/\mathcal{O}_K^{i+1})$ factors through the map $\mathcal{Z}(\mathcal{O}_K) \to \mathcal{X}(\mathcal{O}_K/\mathcal{O}_K^{i+1})$.

Proof. It follows from [37, Thm. 3.1], [17, Prop. 6.1.5] and [36, Lemma 01V8] that $\mathcal{Z}/\mathcal{O}_K$ is smooth.

Let $s \in \mathcal{X}(\mathcal{O}_K)$ be an element that reduces to $P_0$. It is easy to see that $s$ lifts uniquely to an element of $\mathcal{Z}(\mathcal{O}_K)$ by the universal property of blowing-up, see e.g. [18, Prop. II. 7.14]. In order to show (ii) and (iii) we can work locally and replace $\mathcal{X}$ by Spec $(A)$. We will follow the proof of [18, Prop. II. 7.14]. Let $\pi, a_1, \ldots, a_m$ be a regular system of parameters of $A$. These define a closed embedding of $\mathcal{X}$ in $\mathbb{P}^m_A$. The exceptional divisor $E$ is isomorphic to $\mathbb{P}^m_A$ and the intersection of $E$ with the strict transform of $Y$ corresponds to the hyperplane in $E$ with first coordinate equal to 0. Following through the definitions we see that $s$ lifts to an element of $\mathcal{Z}(\mathcal{O}_K)$ which has coordinates $\left(\frac{s(a_1)}{\pi}, \ldots, \frac{s(a_m)}{\pi}\right)$ in $A^m_A$. We can deduce parts (ii) and (iii) from this explicit description.
Let’s show part (iii). All morphisms that follow are ring homomorphisms. Note that $s$ is a section of the structure map $\mathcal{O}_K \to A$, and that $\pi, a_1, \ldots, a_m$ generate $m$, the maximal ideal of $A$.

We have

$$\mathcal{O}_K \to A \xrightarrow{s} \mathcal{O}_K,$$

that $F = \mathcal{O}_K / \pi = A / m$, $s(\pi) = \pi$ and $\text{val}(s(a_i)) \geq 1$.

Let $\tilde{s}, \tilde{\tau} \in \mathcal{X}(\mathcal{O}_K)$ denote the lifts of $s, \tau \in \mathcal{X}(\mathcal{O}_K)$, where $\tau$ also reduces to $P_0$. We have to show the following: If $s, \tau$ have the same image in $\mathcal{X}(\mathcal{O}_K / \pi^{i+1})$ then $\tilde{s}, \tilde{\tau}$ have the same image in $\mathcal{X}(\mathcal{O}_K / \pi^i)$. From the explicit description of the lifts, this readily translates to showing the following

$$s(a_j) \equiv \tau(a_j) \mod \pi^{i+1} \Rightarrow \frac{s(a_j)}{\pi} \equiv \frac{\tau(a_j)}{\pi} \mod \pi^i$$

for all $1 \leq j \leq m$. It is easy to see that the last displayed implication holds.

We now recollect some things from [20, §7 and §8]. Let $A = \mathcal{O}_\mathcal{Y}, P_0$ as before. We use the notation of [20] §7. In that notation $k = F$ and we caution the reader that the $K$ appearing there denotes the function field of $\mathcal{X}$ and so it is different from the $K$ in this paper. We will need the special case $q = 2, r = 1, p = p_1 = \pi A$.

Note that an element $\mathcal{A} \in \text{Br}(X)$ satisfies the assumptions at the beginning of [20] §7. Moreover since $F$ is a finite field we have that $\Omega_F^i$ is trivial for $i > 0$. Therefore if $\text{sw}_p(\mathcal{A}) \geq 1$ then $\mathcal{A}$ is not strongly clean with respect to $A$ (see [20] Def. 7.4 for the definition of strongly clean).

We can now state and prove our next result, which can be useful in actual computations. Note that the $n$ appearing in the statement of the Proposition depends only on $K$.

**Proposition 17.** Suppose that $\mathcal{X} / \mathcal{O}_K$ is smooth. Let $n$ be the smallest integer that is greater than $\frac{e - r - 1}{\pi}$ Let $\mathcal{A} \in \text{Br}(X)[p^i]$ Then

$$\text{ev}_{\mathcal{A}} : \mathcal{X}(\mathcal{O}_K) \to \text{Br}(K)$$

factors through $\mathcal{X}(\mathcal{O}_K / \pi^{n+1})$.

**Proof.** Let $p \in \mathcal{X}$ be the generic point of $Y$. If $\text{sw}_p(\mathcal{A}) = 0$ then by [7] Prop. 3.1 $\text{ev}_{\mathcal{A}}$ factors through $\mathcal{X}(\mathcal{O}_K / \pi)$ and we are done. Hence we suppose that $\text{sw}_p(\mathcal{A}) \geq 1$. Let $P_0 \in \mathcal{X}(F) \subset \mathcal{X}$ and let $A = \mathcal{O}_\mathcal{Y}, P_0$. Let $\mathcal{X} \to \mathcal{X}$ denote the blow-up of $\mathcal{X}$ at $P_0$. The residue field of $A$ is a finite field and as explained before
the Proposition it follows that $A$ is not strongly clean with respect to $A$. Therefore
if $v \in \mathcal{X}$ denotes the generic point of the exceptional divisor, it follows from [20, Thm 8.1] that $\text{sw}_v(A) < \text{sw}_p(A)$. Let $\mathcal{X}$ be $\mathcal{X}$ minus the strict transform of $Y$.

If $\text{sw}_v(A) = 0$ then by [7, Prop. 3.1] applied to $\mathcal{X}$, the map $\text{ev}_A$ is constant on
the elements of $\mathcal{X}(\mathcal{O}_K)$ with the same image in $\mathcal{X}(\mathcal{O}_K/\pi)$. Moreover, it follows
from Lemma [16] that if $s_1, s_2 \in \mathcal{X}(\mathcal{O}_K)$ reduce to $P_0$ and have the same image
in $\mathcal{X}(\mathcal{O}_K/\pi^2)$, then their lifts in $\mathcal{X}(\mathcal{O}_K)$ have the same image in $\mathcal{X}(\mathcal{O}_K/\pi)$. Therefore $\text{ev}_A$ is constant on the elements of $\mathcal{X}(\mathcal{O}_K)$ that reduce to $P_0$ and have
the same image in $\mathcal{X}(\mathcal{O}_K/\pi^2)$. If $\text{sw}_v(A) \neq 0$ then we repeat the argument
starting with a point in $\mathcal{X}(F)$. If this process stops after $n + te - 1$ times then the
Proposition follows in a similar vein to the above, by using Lemma [16] repeatedly (in the end we apply the same argument for each point in $\mathcal{X}(F)$). Therefore it
suffices to show that $\text{sw}_p(A) \leq n + te - 1$.

By [20, proof of Lemma 2.4], in particular by [20, (2.4.1)], it follows by an
easy induction that every element of $1 + \pi^{n+ke}R$ is a $p^k$-power in $R^*$ for every
integer $k \geq 1$, where $R$ is the ring denoted $(\mathcal{O}_K[T])^{(h)}$ in [20]. We remind the
reader that Kato’s $K$ in [20] is $L^h$ in our notation. In particular $1 + \pi^{n+te}T$ is a
$p'$-power. Since the cup product is bilinear and $p' \mathcal{A} = 0$ it follows from [20, Def.
2.1 and Def. 2.3] that $\text{sw}_p(A) \leq n + te - 1$.

\begin{remark}
In contrast, if the order of $\mathcal{A}$ is coprime to $p$ then it is well-known
that $\text{ev}_\mathcal{A}$ factors through $\mathcal{X}(\mathcal{O}_K/\pi)$, see e.g. [12, proof of Prop. 2.4].
\end{remark}

5 Applications to bad reduction

In this section we assume that $\mathcal{X}/\mathcal{O}_K$ is regular and $X/K$ is smooth, projective
and geometrically integral. Let $Y = \mathcal{X} \otimes F$ be the special fibre. We assume that the irreducible components of $Y$ are geometrically integral. Note that an element of $\mathcal{X}(\mathcal{O}_K)$ will intersect the smooth locus of a unique irreducible component of $Y$. We thus have a map $\mathcal{X}(\mathcal{O}_K) \to \text{IrredComp}(Y)$. Our first result in this section is also an illustration of the robustness of our approach.

\begin{proposition}
Assume that $\mathcal{A} \in \ker(\text{Br}(X) \to \text{Br}(X \otimes_K K_{nr}))$. Then

$$\text{ev}_\mathcal{A} : \mathcal{X}(\mathcal{O}_K) \to \text{Br}(K)$$

factors through $\text{IrredComp}(Y)$
\end{proposition}
Proof. Let \( \text{IrredComp}(Y) = \{ Y_1, \ldots, Y_n \} \).

Fix \( 1 \leq i \leq n \) and set \( X' = X - \bigcup_{j \neq i} Y_j \). By Corollary 4 and the remark below it, we see that the map \( X'(\mathcal{O}_K) \to \text{Br}(K) \) given by evaluating at \( \mathcal{A} \) is constant. This is what we had to prove.

\[ \square \]

Remark 20. The above gives a new proof of part of the main result in [5], see [5, Thm. 1]

Corollary 21. Let \( I = \text{Gal}(\overline{K}/K_{nr}) \) and assume that \( H^1(I, \text{Pic}(\overline{X})) = 0 \). Let \( \mathcal{A} \in \text{Br}_1(X) \). Then

\[ \text{ev}_{\mathcal{A}} : X'(\mathcal{O}_K) \to \text{Br}(K) \]

factors through \( \text{IrredComp}(Y) \).

Proof. The condition \( H^1(I, \text{Pic}(\overline{X})) = 0 \) and the fact that \( I \) has cohomological dimension 1, imply via the Hochschild-Serre spectral sequence that \( \text{Br}_1(X \otimes_K K_{nr}) \) is trivial. Therefore it follows immediately that

\[ \text{Br}_1(X) = \text{ker}(\text{Br}(X) \to \text{Br}(X \otimes_K K_{nr})) \]

and we can conclude by Proposition 19.

\[ \square \]

Remark 22. For example let \( X \) be a del Pezzo surface which admits a regular proper model such that the special fibre is geometrically integral. If \( X \) splits over an unramified extension of \( K \) or more generally if \( \text{Br}(X \otimes_K K_{nr}) \) is trivial then the map

\[ \text{ev}_{\mathcal{A}} : X(K) \to \text{Br}(K) \]

is constant for any \( \mathcal{A} \in \text{Br}(X) \). See [6, §4] for conditions on the special fibre that ensure that \( \text{Br}(X \otimes_K K_{nr}) \) is trivial.

We also record the following result for curves, which is potentially useful for computations in specific cases. We note that the assumptions are the necessary ones for the argument to go through.
Proposition 23. Suppose that $\mathcal{X}/\mathcal{O}_K$ is regular and proper, $X/K$ is a smooth, geometrically integral curve of genus $g$ and $Y \otimes F$ is irreducible with a unique singular point $P$. We assume that there is only one tangent direction to $\overline{Y}$ at $P$, and that $P$ has multiplicity $r$ with $g = \frac{r(r-1)}{2}$.

Let $\mathcal{A} \in Br(X)$ have order coprime to $p$. Then

$$\text{ev}_{\mathcal{A}} : X(K) \to Br(K)$$

is a constant map.

Proof. We may assume that the order of $\mathcal{A}$ is a power of $\ell$ for some prime $\ell \neq p$. The arithmetic genus of $\overline{Y}$ equals $g$. From our assumptions it follows that the normalisation of $\overline{Y}$ is the projective line and that the inverse image of $P$ in the normalisation consists of one point. Therefore $\overline{Y} - P$ is isomorphic to the affine line. It is well-known that $\mathbb{A}^1_F$ has no $\ell$ coverings, cf. [30, Cor. XIII. 2.12]. The result now follows from Proposition 6 applied to $\mathcal{X} - P$.

Let us explain the potential usefulness of the next result, which is basically a reformulation of Proposition 6 convenient for applications. Calculating fundamental groups of open subvarieties is usually a difficult task. However a recent result in [8] allows us to do this in the case of the smooth locus for a wide class of varieties, see the remark below. In particular some of the hypothesis in the following Proposition will be automatically satisfied.

Proposition 24. Suppose that $\mathcal{X}/\mathcal{O}_K$ is proper but not necessarily regular, $X/K$ is smooth, geometrically integral and $Y$ is geometrically integral. Assume that the maximal pro-$\ell$ quotient of $\pi_1(Y_F - \text{Sing}(Y_F))$ is trivial.

Let $\mathcal{A} \in Br(X)[\ell^\infty]$ and assume one of the following:

1. $\ell \neq p$;
2. $\ell = p$ and $\text{sw}_p(\mathcal{A}) = 0$ where $p \in \mathcal{X}$ is the generic point of $Y$.

Let $\mathcal{X}' = \mathcal{X} - \text{Sing}(Y)$. Then

$$\text{ev}_{\mathcal{A}} : \mathcal{X}'(\mathcal{O}_K) \to Br(K)$$

is a constant map.

Proof. This follows from Proposition 6 applied to $\mathcal{X}'/\mathcal{O}_K$. $\square$
Remark 25. 1) Globally $F$-regular varieties were introduced in [34], and they include many classes of varieties. The following result is related to the assumptions of the previous Proposition. Suppose that $Y$ is projective globally $F$-regular and the codimension of the singular locus of $Y$ is at least 2. Then by [8, Cor. 5.7 and Cor. 4.18] we have that $N = \left| \pi_1(Y - \text{Sing}(Y)) \right| \leq \frac{1}{s(R)}$ where $s(R)$ is the $F$-signature of a section ring of $Y$ with respect to an ample sheaf and moreover $N$ is coprime to $p$.

2) In favourable cases, we might be able to show that $X'(O_K) = X(K)$ by looking at the equations of $X$.

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