Walk/Zeta Correspondence

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Abstract
Our previous work presented explicit formulas for the generalized zeta function and the
generalized Ihara zeta function corresponding to the Grover walk and the positive-support
version of the Grover walk on the regular graph via the Konno–Sato theorem, respectively.
This paper extends these walks to a class of walks including random walks, correlated random
walks, quantum walks, and open quantum random walks on the torus by the Fourier analysis.

Keywords  Zeta function · Quantum walk · Correlated random walk · Random walk · Open
quantum random walk · Torus

1 Introduction
In our previous paper [12], we studied a relation between the Grover walk and the zeta
function based on the Konno–Sato theorem [16] and called this relation “Grover/Zeta Cor-
respondence”. More precisely, we gave explicit formulas for the generalized zeta function
and the generalized Ihara zeta function corresponding to the Grover walk with F-type and
the positive-support version of the Grover walk with F-type on the vertex-transitive regular
graph by the Konno–Sato theorem, respectively. The Grover walk is one of the most well-
investigated quantum walks (QWs) inspired by the famous Grover algorithm. The QW is
a quantum counterpart of the correlated random walk (CRW) which has the random walk

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(RW) as a special model. In fact, the CRW is the RW with memory. As for the QW, see [13, 18, 19, 22] and as for the CRW and the RW, see [14, 21], for example.

In this paper, we extend the Grover walk with F-type and the positive-support version of the Grover walk with F-type to a class of walks with both F- and M-types by using not the Konno–Sato theorem but a method of the Fourier transform for the case of the $d$-dimensional torus. Our class contains QWs and CRWs. Moreover, we can treat the open quantum random walk (OQRW) which has the CRW as a special model. Concerning the OQRW, see [3, 4], for example. We call this kind of the zeta function the walk-type zeta function and call such a relationship “Walk/Zeta Correspondence”, corresponding to the above mentioned “Grover/Zeta Correspondence”. For the convenience of readers, we give a brief review of Grover/Zeta Correspondence presented by our previous paper [12] in Appendix A.

The rest of this paper is organized as follows. In Sect. 2, we define the $2d$-state discrete time walk on the $d$-dimensional torus. Moreover, we explain a method of the Fourier transform for the walk. Section 3 introduces the walk-type zeta function and presents our main results. In Sect. 4, we consider walks on the one-dimensional torus and give important models such as QWs, CRWs, RWs, and OQRWs. In Sect. 5, we treat specific examples for QWs and CRWs. Section 6 is devoted to the two-dimensional model. Finally, in Sect. 7, we deal with the higher-dimensional model. One of the interesting future problems might be to extend the torus to a suitable class of graphs for which we can use a similar method based on the Fourier analysis.

2 Walk on Torus

First we introduce the following notation: $\mathbb{Z}$ is the set of integers, $\mathbb{Z}_{\geq}$ is the set of non-negative integers, $\mathbb{Z}_{>}$ is the set of positive integers, $\mathbb{R}$ is the set of real numbers, and $\mathbb{C}$ is the set of complex numbers.

In this section, we give the definition of the $2d$-state discrete time walk on the $d$-dimensional torus ($d \geq 1$) with $N^d$ vertices, denoted by $T_N^d$, where $N \in \mathbb{Z}_{>}. Then we note that $T_N^d = (\mathbb{Z} \mod N)^d$. The discrete time walk is defined by using a shift operator and a coin matrix which will be mentioned below.

Let $f : T_N^d \to \mathbb{C}^{2d}$. For $j = 1, 2, \ldots, d$ and $x \in T_N^d$, the shift operator $\tau_j$ is defined by

$$(\tau_j f)(x) = f(x - e_j),$$

where $\{e_1, e_2, \ldots, e_d\}$ denotes the standard basis of $T_N^d$.

Let $A = [a_{ij}]_{i,j=1,2,\ldots,2d}$ be a $2d \times 2d$ matrix with $a_{ij} \in \mathbb{C}$ for $i, j = 1, 2, \ldots, 2d$. We call $A$ the coin matrix. If $a_{ij} \in [0, 1]$ and $\sum_{i=1}^{2d} a_{ij} = 1$ for any $j = 1, 2, \ldots, 2d$, then the walk is a CRW. In particular, when $a_{i1} = a_{i2} = \cdots = a_{i2d}$ for any $i = 1, 2, \ldots, 2d$, this CRW becomes a RW. If $A$ is unitary, then the walk is a QW. So our class of walks contains RWs, CRWs, and QWs as special models.

To describe the evolution of the walk, we decompose the $2d \times 2d$ coin matrix $A$ as

$$A = \sum_{j=1}^{2d} P_j A,$$

where $P_j$ denotes the orthogonal projection onto the one-dimensional subspace $\mathbb{C} \eta_j$ in $\mathbb{C}^{2d}$. Here $\{\eta_1, \eta_2, \ldots, \eta_{2d}\}$ denotes a standard basis on $\mathbb{C}^{2d}$. 
The discrete time walk associated with the coin matrix $A$ on $T_N^d$ is determined by the $2dN^d \times 2dN^d$ matrix

$$M_A = \sum_{j=1}^d \left( P_{2j-1} A \tau_j^{-1} + P_{2j} A \tau_j \right).$$

The state at time $n \in \mathbb{Z}_+$ and location $x \in T_N^d$ can be expressed by a $2d$-dimensional vector:

$$\Psi_n(x) = \left[ \Psi_n^1(x) \Psi_n^2(x) \cdots \Psi_n^{2d}(x) \right] \in \mathbb{C}^{2d}.$$  

For $\Psi_n : T_N^d \longrightarrow \mathbb{C}^{2d}$ ($n \in \mathbb{Z}_+$), from Eq. (1), the evolution of the walk is defined by

$$\Psi_{n+1}(x) = (M_A \Psi_n)(x) = \sum_{j=1}^d \left( P_{2j-1} A \Psi_n(x + e_j) + P_{2j} A \Psi_n(x - e_j) \right).$$

This equation means that the walker moves at each step one unit to the $-x_j$-axis direction with matrix $P_{2j-1} A$ or one unit to the $x_j$-axis direction with matrix $P_{2j} A$ for $j = 1, 2, \ldots, d$. Moreover, for $n \in \mathbb{Z}_+$ and $x = (x_1, x_2, \ldots, x_d) \in T_N^d$, the $2d \times 2d$ matrix $\Phi_n(x_1, x_2, \ldots, x_d)$ is given by

$$\Phi_n(x_1, x_2, \ldots, x_d) = \sum_{\Xi_n} (l_1, l_2, \ldots, l_{2d-1}, l_{2d}),$$

where the $2d \times 2d$ matrix $\Xi_n (l_1, l_2, \ldots, l_{2d-1}, l_{2d})$ is the sum of all possible paths in the trajectory of $l_{2j-1}$ steps $-x_j$-axis direction and $l_{2j}$ steps $x_j$-axis direction and $\sum_{\Xi_n}$ is the summation over $(l_1, l_2, \ldots, l_{2d-1}, l_{2d}) \in (\mathbb{Z}_+)^{2d}$ satisfying

$$l_1 + l_2 + \cdots + l_{2d-1} + l_{2d} = n, \quad x_j = -l_{2j-1} + l_{2j} \quad (j = 1, 2, \ldots, d).$$

For example, when $d = 2, n = 2$, and $(x_1, x_2) = (0, 0)$, we see

$$\Phi_2(0, 0) = \Xi_2 (1, 1, 0, 0) + \Xi_2 (0, 0, 1, 1),$$

and

$$\Xi_2 (1, 1, 0, 0) = (P_1 A)(P_2 A) + (P_2 A)(P_1 A),$$

$$\Xi_2 (0, 0, 1, 1) = (P_3 A)(P_4 A) + (P_4 A)(P_3 A).$$

Here we put

$$\Phi_0(x_1, x_2, \ldots, x_d) = \begin{cases} I_{2d} & \text{if} \quad (x_1, x_2, \ldots, x_d) = (0, 0, \ldots, 0), \\ O_{2d} & \text{if} \quad (x_1, x_2, \ldots, x_d) \neq (0, 0, \ldots, 0), \end{cases}$$

where $I_n$ is the $n \times n$ identity matrix and $O_n$ is the $n \times n$ zero matrix. Then, for the walk starting from $(0, 0, \ldots, 0)$, we obtain

$$\Psi_n(x_1, x_2, \ldots, x_d) = \Phi_n(x_1, x_2, \ldots, x_d) \Psi_0(0, 0, \ldots, 0) \quad (n \in \mathbb{Z}_+).$$

We call $\Phi_n(x) = \Phi_n(x_1, x_2, \ldots, x_d)$ matrix weight at time $n \in \mathbb{Z}_+$ and location $x \in T_N^d$ starting from $\theta = (0, 0, \ldots, 0)$.  

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When we consider the walk on not $T^d_N$ but $\mathbb{Z}^d$, we add the superscript “($\infty$)” to the notation like $\Psi^{(\infty)}$ and $\Xi^{(\infty)}$ which will be used in Sects. 3 and 4.

This type is moving shift model called M-type here. Another type is flip-flop shift model called F-type whose coin matrix is given by

$$A^{(f)} = (I_d \otimes \sigma) A,$$

where $\otimes$ is the tensor product and

$$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For example, when $d = 3$, we have

$$I_3 \otimes \sigma = \begin{bmatrix} \sigma & 0_2 & 0_2 \\ 0_2 & \sigma & 0_2 \\ 0_2 & 0_2 & \sigma \end{bmatrix}.$$

The F-type model is also important, since it has a central role in the Konno–Sato theorem. When we distinguish $A$ (M-type) from $A^{(f)}$ (F-type), we write $A$ by $A^{(m)}$.

For time $n \in \mathbb{Z}_\geq$ and location $x \in T^d_N$, we define the measure $\mu_n(x)$ by

$$\mu_n(x) = \|\Psi_n(x)\|_{C^{2d}}^p,$$

where $\| \cdot \|_{C^{2d}}^p$ denotes the standard $p$-norm on $C^{2d}$. That is,

$$\mu_n(x) = \sum_{j=1}^{2d} |\Psi^j_n(x)|^p.$$

As for CRWs and QWs, we take $p = 1$ and $p = 2$, respectively. Then CRWs and QWs satisfy

$$\sum_{x \in T^d_N} \mu_n(x) = \sum_{x \in T^d_N} \mu_0(x),$$

for any time $n \in \mathbb{Z}_\geq$. However, we do not necessarily impose such a condition for the walk we consider here. For example, the two-dimensional positive-support version of the Grover walk (introduced in Sect. 6) does not satisfy the condition. In this meaning, our walk is a generalized version for the usual walk.

To consider the zeta function, we use the Fourier analysis. To do so, we introduce the following notation: $\mathbb{K}_N = \{0, 1, \ldots, N - 1\}$ and $\mathbb{K}_N = \{0, 2\pi/N, \ldots, 2\pi(N - 1)/N\}$. For $f : \mathbb{K}_N^d \rightarrow \mathbb{C}^{2d}$, the Fourier transform of the function $f$, denoted by $\hat{f}$, is defined by the sum

$$\hat{f}(k) = \frac{1}{N^{d/2}} \sum_{x \in \mathbb{K}_N^d} e^{-2\pi i \langle x, k \rangle / N} f(x),$$

where $k = (k_1, k_2, \ldots, k_d) \in \mathbb{K}_N^d$. Here $\langle x, k \rangle$ is the canonical inner product of $\mathbb{K}_N^d$, i.e., $\langle x, k \rangle = \sum_{j=1}^d x_j k_j$. Then we see that $\hat{f} : \mathbb{K}_N^d \rightarrow \mathbb{C}^{2d}$. Moreover, we should remark that

$$f(x) = \frac{1}{N^{d/2}} \sum_{k \in \mathbb{K}_N^d} e^{2\pi i \langle x, k \rangle / N} \hat{f}(k),$$

$$\hat{f}(k) = \frac{1}{N^{d/2}} \sum_{x \in \mathbb{K}_N^d} e^{-2\pi i \langle x, k \rangle / N} f(x).$$
where \( x = (x_1, x_2, \ldots, x_d) \in \mathbb{K}_N^d \). By using
\[
\tilde{k}_j = \frac{2\pi k_j}{N} \in \tilde{\mathbb{K}}_N, \quad \tilde{k} = (\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_d) \in \tilde{\mathbb{K}}_N^d,
\] (6)
we can rewrite Eqs. (4) and (5) in the following way:
\[
\hat{g}(\tilde{k}) = \frac{1}{N^{d/2}} \sum_{x \in \mathbb{K}_N^d} e^{-i \langle x, \tilde{k} \rangle} g(x),
\]
\[
g(x) = \frac{1}{N^{d/2}} \sum_{\tilde{k} \in \tilde{\mathbb{K}}_N^d} e^{i \langle x, \tilde{k} \rangle} \hat{g}(\tilde{k}),
\] (7)
for \( g : \mathbb{K}_N^d \rightarrow \mathbb{C}^{2d} \) and \( \hat{g} : \tilde{\mathbb{K}}_N^d \rightarrow \mathbb{C}^{2d} \). In order to take a limit \( N \rightarrow \infty \), we introduced the notation given in Eq. (6). We should note that as for the summation, we sometimes write “\( k \in \mathbb{K}_N^d \)” instead of “\( \tilde{k} \in \tilde{\mathbb{K}}_N^d \)”, for example,
\[
g(x) = \frac{1}{N^{d/2}} \sum_{k \in \mathbb{K}_N^d} e^{i \langle x, k \rangle} \hat{g}(k),
\]
instead of Eq. (7).

From the Fourier transform and Eq. (2), we have
\[
\hat{\Psi}_{n+1}(k) = \hat{M}_A(k) \hat{\Psi}_n(k),
\]
where \( \Psi_n : T_N^d \rightarrow \mathbb{C}^{2d} \) and \( 2d \times 2d \) matrix \( \hat{M}_A(k) \) is determined by
\[
\hat{M}_A(k) = \sum_{j=1}^{d} \left( e^{2\pi ik_j/N} P_{2j-1}A + e^{-2\pi ik_j/N} P_{2j}A \right).
\]
By using notations in Eq. (6), we have
\[
\hat{M}_A(\tilde{k}) = \sum_{j=1}^{d} \left( e^{i\tilde{k}_j} P_{2j-1}A + e^{-i\tilde{k}_j} P_{2j}A \right).
\] (8)

Next we will consider the following eigenvalue problem for \( 2dN^d \times 2dN^d \) matrix \( M_A \):
\[
\lambda \Psi = M_A \Psi,
\] (9)
where \( \lambda \in \mathbb{C} \) is an eigenvalue and \( \Psi (\in \mathbb{C}^{2dN^d}) \) is the corresponding eigenvector. Noting that Eq. (9) is closely related to Eq. (2), we see that Eq. (9) is rewritten as
\[
\lambda \Psi(x) = (M_A \Psi)(x) = \sum_{j=1}^{d} \left( P_{2j-1}A \Psi(x + e_j) + P_{2j}A \Psi(x - e_j) \right),
\] (10)
for any \( x \in \mathbb{K}_N^d \). From the Fourier transform and Eq. (10), we have
\[
\hat{\lambda} \hat{\Psi}(k) = \hat{M}_A(k) \hat{\Psi}(k),
\]
for any \( k \in \mathbb{K}_N^d \). Then the characteristic polynomials of \( 2d \times 2d \) matrix \( \hat{M}_A(k) \) for fixed \( k(\in \mathbb{K}_N^d) \) is

\[
\det \left( \lambda I_{2d} - \hat{M}_A(k) \right) = \prod_{j=1}^{2d} \left( \lambda - \lambda_j(k) \right),
\]

(11)

where \( \lambda_j(k) \) are eigenvalues of \( \hat{M}_A(k) \). Similarly, the characteristic polynomials of \( 2dN^d \times 2dN^d \) matrix \( \hat{M}_A \) is

\[
\det \left( \lambda I_{2dN^d} - \hat{M}_A \right) = \prod_{j=1}^{2d} \prod_{k \in \mathbb{K}_N^d} \left( \lambda - \lambda_j(k) \right).
\]

Thus we have

\[
\det \left( \lambda I_{2dN^d} - u\hat{M}_A(\tilde{k}) \right) = \prod_{j=1}^{2d} \prod_{k \in \mathbb{K}_N^d} \left( 1 - u\lambda_j(\tilde{k}) \right).
\]

(12)

Moreover, from Eq. (8), we have the following important formula.

**Lemma 2**

\[
\det \left( I_{2d} - u\hat{M}_A(\tilde{k}) \right) = \det \left( I_{2d} - u \sum_{j=1}^{d} \left( e^{i\tilde{k}_j} P_{2j-1}A + e^{-i\tilde{k}_j} P_{2j} A \right) \right).
\]
3 Walk-Type Zeta Function

For our setting in the previous section, we define the walk-type zeta function by

$$\zeta\left(A, T^d_N, u\right) = \det\left(I_{2dN^d} - uMA\right)^{-1/N^d}. \tag{13}$$

In general, for a $d_c \times d_c$ coin matrix $A$, we put

$$\zeta\left(A, T^d_N, u\right) = \det\left(I_{2dN^d} - uMA\right)^{-1/N^d}.$$

We should remark that the walk-type zeta function becomes the generalized zeta function $\zeta\left(T^d_N, u\right)$ in [12] for the Grover walk (F-type). See also Appendix A. So we write the walk-type zeta function with a coin matrix $A$ as $\zeta\left(A, T^d_N, u\right)$. Furthermore, our walk is defined on the “site” $x \in T^d_N$, on the other hand, the walk in [12] is defined on the “arc” (i.e., directed edge). However both of the walks are same for the torus case.

We here briefly explain zeta functions related to our walk. As for the zeta function, see [12, 16], for example. Starting from $p$-adic Selberg zeta functions, Ihara [8] introduced the Ihara zeta functions of graphs, and showed that the reciprocals of the Ihara zeta functions of regular graphs are explicit polynomials. Recently, the Ihara zeta function of a finite graph was extended to an infinite graph. Clair [7] computed the Ihara zeta function for the infinite grid by using elliptic integrals and theta functions. Chinta et al. [6] obtained a generalized version of the determinant formula for the Ihara zeta function associated to finite or infinite graphs.

By Lemma 1 and Eqs. (12) and (13), we compute

$$\zeta\left(A, T^d_N, u\right)^{-1} = \exp\left[\frac{1}{N^d} \sum_{\tilde{k} \in \mathbb{Z}^d_N} \log \left\{ \det\left(I_{2d} - u\tilde{M}_A(\tilde{k})\right) \right\} \right].$$

So we have

$$\zeta\left(A, T^d_N, u\right)^{-1} = \exp\left[\frac{1}{N^d} \sum_{\tilde{k} \in \mathbb{Z}^d_N} \log \left\{ \det\left(I_{2d} - u\tilde{M}_A(\tilde{k})\right) \right\} \right].$$

Sometimes we write $\sum_{k \in \mathbb{Z}^d_N}$ instead of $\sum_{\tilde{k} \in \mathbb{Z}^d_N}$ as follows:

$$\zeta\left(A, T^d_N, u\right)^{-1} = \exp\left[\frac{1}{N^d} \sum_{k \in \mathbb{Z}^d_N} \log \left\{ \det\left(I_{2d} - u\tilde{M}_A(\tilde{k})\right) \right\} \right].$$

Noting $\tilde{k}_j = 2\pi k_j / N$ $(j = 1, 2, \ldots, d)$ and taking a limit as $N \to \infty$, we obtain

$$\lim_{N \to \infty} \zeta\left(A, T^d_N, u\right)^{-1} = \exp\left[\int_{[0, 2\pi]^d} \log \left\{ \det\left(I_{2d} - u\tilde{M}_A(\Theta^{(d)})\right) \right\} d\Theta^{(d)}_{unif}\right].$$
if the limit exists. We should note that when we take a limit as \( N \to \infty \), we assume that the limit exists throughout this paper. Here \( \Theta^{(d)} = (\theta_1, \theta_2, \ldots, \theta_d) (\in [0, 2\pi)^d) \) and \( d\Theta_{\text{unif}}^{(d)} \) denotes the uniform measure on \([0, 2\pi)^d\), that is,

\[
d\Theta_{\text{unif}}^{(d)} = \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_d}{2\pi}.
\]

Therefore we obtain one of our main results.

**Theorem 1**

\[
\bar{\zeta} \left( A, T_N^d, u \right)^{-1} = \exp \left[ \frac{1}{N^d} \sum_{\vec{k} \in \mathbb{Z}^d_N} \log \left\{ \det \left( \mathbf{F}(\vec{k}, u) \right) \right\} \right],
\]

\[
\lim_{N \to \infty} \bar{\zeta} \left( A, T_N^d, u \right)^{-1} = \exp \left[ \int_{[0,2\pi)^d} \log \left\{ \det \left( \Theta^{(d)}, u \right) \right\} d\Theta_{\text{unif}}^{(d)} \right],
\]

where

\[
F(\mathbf{w}, u) = I_{2d} - u \tilde{M}_A(\mathbf{w}),
\]

with \( \mathbf{w} = (w_1, w_2, \ldots, w_d) \in \mathbb{R}^d \).

Furthermore, we define \( C_r(A, T_N^d) \) by

\[
\bar{\zeta} \left( A, T_N^d, u \right) = \exp \left( \sum_{r=1}^{\infty} \frac{C_r(A, T_N^d)}{r} u^r \right).
\]

(14)

Sometime we write \( C_r(A, T_N^d) \) by \( C_r \) for short. Combining Eqs. (13) with (14) implies

\[
\det \left( I_{2dN^d} - u M_A \right)^{-1/N^d} = \exp \left( \sum_{r=1}^{\infty} \frac{C_r}{r} u^r \right).
\]

Thus we get

\[
- \frac{1}{N^d} \log \left\{ \det \left( I_{2dN^d} - u M_A \right) \right\} = \sum_{r=1}^{\infty} \frac{C_r}{r} u^r.
\]

(15)

It follows from Lemma 1 that the left-hand of Eq. (15) becomes

\[
- \frac{1}{N^d} \log \left\{ \det \left( I_{2dN^d} - u M_A \right) \right\} = - \frac{1}{N^d} \log \left\{ \det \left( I_{2dN^d} - u \tilde{M}_A \right) \right\}
\]

\[
= - \frac{1}{N^d} \sum_{j=1}^{2d} \sum_{\vec{k} \in \mathbb{Z}^d_N} \log \left\{ 1 - u \lambda_j(\vec{k}) \right\}
\]

\[
= \frac{1}{N^d} \sum_{j=1}^{2d} \sum_{\vec{k} \in \mathbb{Z}^d_N} \sum_{r=1}^{\infty} \frac{(\lambda_j(\vec{k}))^r}{r} u^r.
\]

By this and the right-hand of Eq. (15), we have

\[
C_r(A, T_N^d) = \frac{1}{N^d} \sum_{j=1}^{2d} \sum_{\vec{k} \in \mathbb{Z}^d_N} (\lambda_j(\vec{k}))^r
\]
This is rewritten as
\[
C_r(A, T_N^d) = \frac{1}{N^d} \sum_{j=1}^{2d} \sum_{\tilde{k} \in \mathbb{Z}_N^d} (\lambda_j(\tilde{k}))^r.
\] (16)

Noting \( \tilde{k}_j = 2\pi k_j / N \) \((j = 1, 2, \ldots, d)\) and taking a limit as \( N \to \infty \), we get
\[
\lim_{N \to \infty} C_r(A, T_N^d) = \sum_{j=1}^{2d} \int_{[0,2\pi]^d} \lambda_j \left( \Theta^{(d)} \right)^r d\Theta^{(d)}_{unif}.
\] (17)

Let \( \text{Tr}(A) \) denote the trace of a square matrix \( A \). Therefore by definition of \( \text{Tr} \) and Eqs. (16) and (17), we obtain

**Proposition 1**
\[
C_r(A, T_N^d) = \frac{1}{N^d} \sum_{\tilde{k} \in \mathbb{Z}_N^d} \text{Tr} \left( \left( \bar{M}_A(\tilde{k}) \right)^r \right),
\]
\[
\lim_{N \to \infty} C_r(A, T_N^d) = \int_{[0,2\pi]^d} \text{Tr} \left( \left( \bar{M}_A(\Theta^{(d)}) \right)^r \right) d\Theta^{(d)}_{unif}.
\] (18)

Furthermore, noting that
\[
\int_{[0,2\pi]^d} e^{i(m_1\theta_1 + m_2\theta_2 + \cdots + m_d\theta_d)} d\Theta^{(d)}_{unif} = 0,
\]
for \((m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d\) with \((m_1, m_2, \ldots, m_d) \neq (0, 0, \ldots, 0)\), we have
\[
\int_{[0,2\pi]^d} \left( \bar{M}_A(\Theta^{(d)}) \right)^r d\Theta^{(d)}_{unif} = \sum_{*} \Xi_r^*(l_1, l_2, \ldots, l_{2d-1}, l_{2d}),
\] (19)
where \( \sum_{*} \) is the summation over \((l_1, l_2, \ldots, l_{2d-1}, l_{2d})\) satisfying
\[
l_1 + l_2 + \cdots + l_{2d-1} + l_{2d} = r, \quad l_{2j-1} = l_{2j} \quad (j = 1, 2, \ldots, 2d).
\]

From Eqs. (3), (18), and (19), we obtain one of our main results.

**Theorem 2**
\[
\lim_{N \to \infty} C_r(A, T_N^d) = \text{Tr} \left( \Phi^{(\infty)}_r(0) \right),
\]
where \( 0 = (0, 0, \ldots, 0) \).

An interesting point is that \( \Phi^{(\infty)}_r(0) \) is the return “matrix weight” at time \( r \) for the walk on not \( T_N^d \) but \( \mathbb{Z}^d \). We should remark that in general \( \text{Tr}(\Phi^{(\infty)}_r(0)) \) is not the same as the return probability at time \( r \) for the walk (which will be briefly explained in Sects. 4 and 5).

To understand the derivation of Theorem 2 well, we consider the one-dimensional model with a coin matrix \( A \) as
\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.
\]
In a similar way, we can extend this argument to general \(d\)-dimensional model. We begin with \(r = 1, 2\) cases for \((\hat{M}_A(\theta))'\). Then we have

\[
\hat{M}_A(\theta) = e^{i\theta} P_1 A + e^{-i\theta} P_2 A,
\]

\[
(\hat{M}_A(\theta))^2 = e^{2i\theta} (P_1 A)^2 + \{(P_1 A)(P_2 A) + (P_2 A)(P_1 A)\} + e^{-2i\theta} (P_2 A)^2.
\]

Thus we see

\[
\int_0^{2\pi} \hat{M}_A(\theta) \frac{d\theta}{2\pi} = \int_0^{2\pi} \left(e^{i\theta} P_1 A + e^{-i\theta} P_2 A\right) \frac{d\theta}{2\pi} = O_2,
\]

\[
\int_0^{2\pi} \left(\hat{M}_A(\theta)^2\right) \frac{d\theta}{2\pi} = \int_0^{2\pi} \left(e^{2i\theta} (P_1 A)^2 + \{(P_1 A)(P_2 A) + (P_2 A)(P_1 A)\} + e^{-2i\theta} (P_2 A)^2\right) \frac{d\theta}{2\pi}
\]

\[
= (P_1 A)(P_2 A) + (P_2 A)(P_1 A)
\]

\[
= \Xi_2^{(\infty)}(1, 1) = \Phi_2^{(\infty)}(0).
\]

A similar argument implies that for \(l = 1, 2, \ldots\), we have

\[
\int_0^{2\pi} \left(\hat{M}_A(\theta)^l\right) \frac{d\theta}{2\pi} = \begin{cases} O_2 & \text{if } r = 2l - 1, \\ \Xi_2^{(\infty)}(l, l) & \text{if } r = 2l. \end{cases}
\]

By definition of \(\Phi_r^{(\infty)}(0)\), we see

\[
\Phi_r^{(\infty)}(0) = \begin{cases} O_2 & \text{if } r = 2l - 1, \\ \Xi_2^{(\infty)}(l, l) & \text{if } r = 2l. \end{cases}
\]

So we immediately get

\[
\int_0^{2\pi} \left(\hat{M}_A(\theta)^l\right) \frac{d\theta}{2\pi} = \Phi_r^{(\infty)}(0).
\]

Thus we have the desired conclusion for the one-dimensional case:

\[
\lim_{N \to \infty} C_r(A, T_N^1) = \int_0^{2\pi} \text{Tr} \left(\left(\hat{M}_A(\theta)\right)^l\right) \frac{d\theta}{2\pi} = \text{Tr} \left(\Phi_r^{(\infty)}(0)\right).
\]

On the other hand, we obtained the following expression \(\Phi_r^{(\infty)}(0)\) for \(r = 2l\) (see Konno [13, 14], for example).

**Lemma 3** We consider the walk with a coin matrix \(A\) on \(\mathbb{Z}\). Assume that \(a_{11}a_{12}a_{21}a_{22} \neq 0\). Then for \(l = 1, 2, \ldots\), we have

\[
\Phi_r^{(\infty)}(0) = \Xi_2^{(\infty)}(l, l)
\]

\[
= (a_{11}a_{22})^l \sum_{m=1}^{l-1} \left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^m \binom{l-1}{m-1}^2 \left[\frac{l-m}{a_{11}m} Q_1 + \frac{l-m}{a_{22}m} Q_2 + \frac{1}{a_{12}} Q_3 + \frac{1}{a_{21}} Q_4\right],
\]

where

\[
Q_1 = P_1 A, \quad Q_2 = P_2 A, \quad Q_3 = \sigma P_1 A, \quad Q_4 = \sigma P_2 A.
\]
and
\[ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \]

We note that
\[ Q_1 = \begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 & 0 \\ a_{21} & a_{22} \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0 & d_0 \\ a_{11} & a_{12} \end{bmatrix}, \quad Q_4 = \begin{bmatrix} a_{21} & a_{22} \\ 0 & 0 \end{bmatrix}. \]

Moreover, a direct computation gives
\[
\text{Tr} \left( \left( \frac{l - m}{a_{11}m} Q_1 + \frac{l - m}{a_{22}m} Q_2 + \frac{l - m}{a_{12}} Q_3 + \frac{1}{a_{21}} Q_4 \right) \right) = \frac{2l}{m}. \quad (20)
\]

So finally Lemma 3 implies

**Lemma 4** We consider the walk with a coin matrix A on \( T_{N}^1 \). Assume that \( a_{11}a_{12}a_{21}a_{22} \neq 0 \). Then for \( l = 1, 2, \ldots \), we have

\[
\lim_{N \to \infty} C_{2l}(A, T_{N}^1) = \text{Tr}\left( \Phi_{2l}^{(\infty)}(0) \right) = \text{Tr}\left( \Xi_{2l}^{(\infty)}(l, l) \right) = 2l \left( \frac{a_{11}a_{22}}{a_{11}a_{22}} \right)^l \sum_{m=1}^{l} \frac{1}{m} \left( l - 1 \right) m^{-1} \left( \frac{a_{12}a_{21}}{a_{11}a_{22}} \right)^m = 2l \left( \frac{a_{11}a_{22}}{a_{11}a_{22}} \right)^{l-1} \left( \frac{a_{12}a_{21}}{a_{11}a_{22}} \right) \frac{1}{2} \text{F}_1\left( 1 - l, 1 - l; 2, \frac{a_{12}a_{21}}{a_{11}a_{22}} \right).
\]

Here we introduced the hypergeometric series \( \text{F}_1(a, b; c; z) \) (see Andrews et al. [1], for example). In particular, we used the following relation:

\[
\sum_{k=1}^{n} \frac{1}{k} \left( \frac{n - 1}{k - 1} \right)^2 z^{k-1} = \text{F}_1\left( 1 - n, 1 - n; 2, z \right).
\]

If we take \( z = 1 \), then we have

\[
\sum_{k=1}^{n} \frac{1}{k} \left( \frac{n - 1}{k - 1} \right)^2 = \text{F}_1\left( 1 - n, 1 - n; 2, 1 \right) = \frac{\Gamma(2)\Gamma(2n)}{\Gamma(n+1)^2} = \frac{(2n-1)!}{(n!)^2}.
\]

where \( \Gamma(z) \) is the gamma function. Thus we get the following result will be used in Corollary 6 for the RW case.

\[
2n \sum_{k=1}^{n} \frac{1}{k} \left( \frac{n - 1}{k - 1} \right)^2 = \binom{2n}{n}. \quad (21)
\]

In the next section, we will give explicit expressions of \( \lim_{N \to \infty} C_r(A, T_{N}^1) \) for RWs, CRWs, and QWs by using Lemma 4.
4 One-Dimensional Case

This section deals with walks on the one-dimensional torus $T^1_N$ whose $2 \times 2$ coin matrix $A^{(m)}$ (M-type) or $A^{(f)}$ (F-type) as follows:

\[ A^{(m)} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad A^{(f)} = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}, \]

since

\[ A^{(f)} = (I_1 \otimes \sigma) A^{(m)} = \sigma A^{(m)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \]

The one-dimensional walk is the first example, so we will explain it in detail. Put $k = k_1$ and $\tilde{k} = \tilde{k}_1$. In this case, we take

\[ P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \]

Thus we immediately get

\[ \hat{M}_{A^{(m)}}(\tilde{k}) = e^{i\tilde{k}P_1 A^{(m)}} + e^{-i\tilde{k}P_2 A^{(m)}} = \begin{bmatrix} e^{i\tilde{k}a_{11}} & e^{i\tilde{k}a_{12}} \\ e^{-i\tilde{k}a_{21}} & e^{-i\tilde{k}a_{22}} \end{bmatrix}, \]

(22)

\[ \hat{M}_{A^{(f)}}(\tilde{k}) = e^{i\tilde{k}P_1 A^{(f)}} + e^{-i\tilde{k}P_2 A^{(f)}} = \begin{bmatrix} e^{i\tilde{k}a_{21}} & e^{i\tilde{k}a_{22}} \\ e^{-i\tilde{k}a_{11}} & e^{-i\tilde{k}a_{12}} \end{bmatrix}. \]

(23)

By these equations, we have

\[ \det \left( I_2 - u \hat{M}_{A^{(s)}}(\tilde{k}) \right) = 1 - \text{Tr} \left( \hat{M}_{A^{(s)}}(\tilde{k}) \right) u + \det \left( \hat{M}_{A^{(s)}}(\tilde{k}) \right) u^2 \quad (s \in \{m, f\}). \]

From Theorem 1, we obtain

**Proposition 2**

\[ \bar{\zeta} \left( A^{(s)}, T^1_N, u \right)^{-1} = \exp \left[ \frac{1}{N} \sum_{k=0}^{N-1} \log \left( 1 - \text{Tr} \left( \hat{M}_{A^{(s)}}(\tilde{k}) \right) u + \det \left( \hat{M}_{A^{(s)}}(\tilde{k}) \right) u^2 \right) \right], \]

\[ \lim_{N \to \infty} \bar{\zeta} \left( A^{(s)}, T^1_N, u \right)^{-1} = \exp \left[ \int_0^{2\pi} \log \left( 1 - \text{Tr} \left( \hat{M}_{A^{(s)}}(\theta) \right) u + \det \left( \hat{M}_{A^{(s)}}(\theta) \right) u^2 \right) \frac{d\theta}{2\pi} \right], \]

for $s \in \{m, f\}$.

From now on, we consider QWs, CRWs, and RWs and apply Proposition 2 and Lemma 4 to their models.

(a) QW case.

One of the typical classes for $2 \times 2$ coin matrix $A^{(m)}$ (M-type) or $A^{(f)}$ (F-type) is as follows:

\[ A^{(m)} = \begin{bmatrix} \cos \xi & \sin \xi \\ \sin \xi & -\cos \xi \end{bmatrix}, \quad A^{(f)} = \begin{bmatrix} \sin \xi & -\cos \xi \\ \cos \xi & \sin \xi \end{bmatrix} \quad (\xi \in [0, 2\pi]). \]

When $\xi = \pi/4$, the QW becomes the so-called Hadamard walk which is one of the most well-investigated model in the study of QWs like the Grover walk.

From Proposition 2, we have
Corollary 1
\[
\bar{\zeta} \left( A^{(s)}, T_N^1, u \right)^{-1} = \exp \left[ \frac{1}{N} \sum_{k=0}^{N-1} \log \left( F^{(s)} (\bar{k}, u) \right) \right],
\]
\[
\lim_{N \to \infty} \bar{\zeta} \left( A^{(s)}, T_N^1, u \right)^{-1} = \exp \left[ \int_0^{2\pi} \log \left( F^{(s)} (\theta, u) \right) d\theta \right].
\]
for \( s \in \{m, f\} \), where
\[
F^{(m)} (w, u) = 1 - 2i \cos \xi \sin w \cdot u - u^2,
\]
\[
F^{(f)} (w, u) = 1 - 2 \sin \xi \cos w \cdot u + u^2.
\]
Furthermore, Lemma 4 implies

Corollary 2
\[
\lim_{N \to \infty} C_{2l} (A^{(m)}, T_N^1) = 2l \left( -\cos^2 \xi \sum_{m=1}^{l} \frac{1}{m} \left( l - 1 \right)^2 \left( -\tan^2 \xi \right)^m \right)
\]
\[
= 2l \left( -\cos^2 \xi \right)^{l-1} \left( \sin^2 \xi \right) \frac{1}{2} F_1 \left( 1, 1 - l; 2; -\tan^2 \xi \right),
\]
\[
\lim_{N \to \infty} C_{2l} (A^{(f)}, T_N^1) = 2l \left( \sin \xi \right)^{2l} \left( -\cot^2 \xi \right)^{l-1} \left( \cos^2 \xi \right) \frac{1}{2} F_1 \left( 1, 1 - l; 2; -\cot^2 \xi \right),
\]
\[
\lim_{N \to \infty} C_{2l-1} (A^{(s)}, T_N^1) = 0 \quad (s \in \{m, f\}),
\]
for \( l = 1, 2, \ldots \) and \( \xi \in (0, \pi/2) \).

(b) CRW case.
One of the typical classes for \( 2 \times 2 \) coin matrix \( A^{(m)} \) (M-type) or \( A^{(f)} \) (F-type) is as follows:
\[
A^{(m)} = \begin{bmatrix}
\cos^2 \xi & \sin^2 \xi \\
\sin^2 \xi & \cos^2 \xi
\end{bmatrix}, \quad A^{(f)} = \begin{bmatrix}
\sin^2 \xi & \cos^2 \xi \\
\cos^2 \xi & \sin^2 \xi
\end{bmatrix} \quad (\xi \in [0, 2\pi)). \quad (24)
\]
By Proposition 2, we have

Corollary 3
\[
\bar{\zeta} \left( A^{(s)}, T_N^1, u \right)^{-1} = \exp \left[ \frac{1}{N} \sum_{k=0}^{N-1} \log \left( F^{(s)} (\bar{k}, u) \right) \right],
\]
\[
\lim_{N \to \infty} \bar{\zeta} \left( A^{(s)}, T_N^1, u \right)^{-1} = \exp \left[ \int_0^{2\pi} \log \left( F^{(s)} (\theta, u) \right) d\theta \right].
\]
for \( s \in \{m, f\} \), where
\[
F^{(m)} (w, u) = 1 - 2 \cos^2 \xi \cos w \cdot u + \cos(2\xi)u^2,
\]
\[
F^{(f)} (w, u) = 1 - 2 \sin^2 \xi \cos w \cdot u - \cos(2\xi)u^2.
\]
Moreover, Lemma 4 gives

**Corollary 4**

\[
\lim_{N \to \infty} C_{2l}(A^{(m)}, T_N^1) = 2l (\cos \xi)^{4l} \sum_{m=1}^{l} \frac{1}{m} \left( \frac{l-1}{m-1} \right)^2 (\tan^4 \xi)^m
\]

\[
= 2l (\cos \xi)^{4l-1} (\sin^4 \xi) F_1(1-l, 1-l; 2; \tan^4 \xi),
\]

\[
\lim_{N \to \infty} C_{2l}(A^{(f)}, T_N^1) = 2l (\sin \xi)^{4l} \sum_{m=1}^{l} \frac{1}{m} \left( \frac{l-1}{m-1} \right)^2 (\cot^4 \xi)^m
\]

\[
= 2l (\sin \xi)^{4l-1} (\cos^4 \xi) F_1(1-l, 1-l; 2; \cot^4 \xi),
\]

\[
\lim_{N \to \infty} C_{2l-1}(A^{(s)}, T_N^1) = 0 \quad (s \in \{m, f\}),
\]

for \(l = 1, 2, \ldots\) and \(\xi \in (0, \pi/2)\).

When we consider an extreme case \(\xi = \pi/2\), i.e.,

\[
A^{(m)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A^{(f)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]

then Corollary 4 for \(A^{(f)}\) (F-type) corresponds to results obtained by Komatsu et al. \[10\]. Furthermore, if we take \(2 \times 2\) coin matrix \(A^{(m)}\) (M-type) or \(A^{(f)}\) (F-type) as follows,

\[
A^{(m)} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}, \quad A^{(f)} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},
\]

where \(a + c = b + d = 1\) and \(a, b, c, d \in [0, 1]\), then Corollary 4 for \(A^{(f)}\) (F-type) corresponds to results in Komatsu et al. \[11\].

(c) RW case.

The RW is a special case of the CRW. The \(2 \times 2\) coin matrix \(A^{(m)}\) (M-type) or \(A^{(f)}\) (F-type) for the RW is expressed in the following:

\[
A^{(m)} = \begin{bmatrix} \cos^2 \xi & \cos^2 \xi \\ \sin^2 \xi & \sin^2 \xi \end{bmatrix}, \quad A^{(f)} = \begin{bmatrix} \sin^2 \xi & \sin^2 \xi \\ \cos^2 \xi & \cos^2 \xi \end{bmatrix} \quad (\xi \in [0, 2\pi]).
\]

Then a random walker moves at each step one unit to the left with probability \(\cos^2 \xi\) or one unit to the right with probability \(\sin^2 \xi\). From Proposition 2, we have

**Corollary 5**

\[
\bar{\xi}(A^{(s)}, T_N^1, u)^{-1} = \exp \left[ \frac{1}{N} \sum_{k=0}^{N-1} \log \left( F^{(s)}(\bar{k}, u) \right) \right],
\]

\[
\lim_{N \to \infty} \bar{\xi}(A^{(s)}, T_N^1, u)^{-1} = \exp \left[ \int_{0}^{2\pi} \log \left( F^{(s)}(\theta, u) \right) d\theta \right],
\]

for \(s \in \{m, f\}\), where

\[
F^{(m)}(w, u) = 1 - \left( 2i \cos^2 \xi \sin w + e^{-i\theta} \right) u,
\]

\[
F^{(f)}(w, u) = 1 + \left( 2i \cos^2 \xi \sin w - e^{i\theta} \right) u.
\]
Furthermore, by using Lemma 4 and Eq. (21), we get

**Corollary 6**

\[ \lim_{N \to \infty} C_{2l}(A^{(s)}, T_N^1) = 2l \left( \cos \xi \sin \xi \right)^{2l} \sum_{m=1}^{l} \frac{1}{m} \left( \frac{l}{m} - 1 \right)^2 = (\cos \xi \sin \xi)^{2l} \frac{2l}{l}, \]

\[ \lim_{N \to \infty} C_{2l}(A^{(s)}, T_N^1) = 0, \]

for \( s \in \{m, f\} \), \( l = 1, 2, \ldots \) and \( \xi \in (0, \pi/2). \)

We should note that \( \lim_{N \to \infty} C_{2l}(A^{(s)}, T_N^1) \) is nothing but the return probability of the RW at time \( 2l \). However, the corresponding value for the CRW in Corollary 4 is not the same as the return probability of the CRW at time \( 2l \). In general, such a correspondence is limited to the case of the RW.

When \( \xi = \pi/4 \) (symmetric RW), then \( 2 \times 2 \) coin matrices \( A^{(m)} \) (M-type) and \( A^{(f)} \) (F-type) become

\[ A^{(m)} = A^{(f)} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \]

Therefore we have

**Corollary 7**

\[ \bar{z} \left( A^{(s)}, T_N^1, u \right)^{-1} = \exp \left[ \frac{1}{N} \sum_{k=0}^{N-1} \log (1 - \cos \tilde{k} \cdot u) \right], \]

\[ \lim_{N \to \infty} \bar{z} \left( A^{(s)}, T_N^1, u \right)^{-1} = \exp \left[ \int_0^{2\pi} \log (1 - \cos \theta \cdot u) \frac{d\theta}{2\pi} \right], \]

\[ \lim_{N \to \infty} C_{2l}(A^{(s)}, T_N^1) = \left( \frac{1}{2} \right)^{2l} \frac{2l}{l}, \]

\[ \lim_{N \to \infty} C_{2l-1}(A^{(s)}, T_N^1) = 0, \]

for \( s \in \{m, f\} \) and \( l = 1, 2, \ldots \).

Next we consider a generalized version of our walk on \( T_N^1 \), whose \( 3 \times 3 \) coin matrix \( A^{(m)} \) (M-type) or \( A^{(f)} \) (F-type) is defined as follows:

\[ A^{(m)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad A^{(f)} = \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}. \]

In this case, we take the projections \( \{P_0, P_1, P_2\} \) by

\[ P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]

Similarly, we define a \( 3N \times 3N \) matrix

\[ M_A = P_1 A \tau^{-1} + P_0 A + P_2 A \tau. \]
Then the walker moves at each step one unit to the left with $P_1$ or one unit to the right with $P_2$ or stays at each step with $P_0$. Thus we get
\[
\det \left( I_3 - u \tilde{M}_A(\tilde{k}) \right) = \det \left( I_3 - u \times \left( e^{i\tilde{k}} P_1 A + P_0 A + e^{-i\tilde{k}} P_2 A \right) \right).
\]

A typical example is the three-state Grover walk whose $3 \times 3$ coin matrix $A^{(m)}$ (M-type) or $A^{(f)}$ (F-type) is defined by
\[
A^{(m)} = \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}, \quad A^{(f)} = \frac{1}{3} \begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}. \tag{25}
\]

Here $A^{(m)}$ is the $3 \times 3$ Grover matrix. In general, the $n \times n$ Grover matrix $G^{(n)} = \left[ G_{ab}^{(n)} \right]_{a,b=1,2,\ldots,d}$ is defined by
\[
G_{aa}^{(n)} = \frac{2}{n} - 1, \quad G_{ab}^{(n)} = \frac{2}{n} \quad (a \neq b).
\]

We should note that $G^{(n)}$ is unitary. The walk defined by the Grover matrix is called the 
Grover walk. Then, $A^{(m)}$ and $A^{(f)}$ in Eq. (25) are unitary, so the walks determined by them become QWs. In a similar fashion, we obtain

**Corollary 8**

\[
\bar{\zeta} \left( A^{(s)}(t), T_N^1, u \right)^{-1} = (1 + (-1)^\gamma(s) u) \exp \left[ \frac{1}{N} \sum_{k=0}^{N-1} \log \left( F^{(s)}(\tilde{k}, u) \right) \right], \tag{26}
\]

\[
\lim_{N \to \infty} \bar{\zeta} \left( A^{(s)}(t), T_N^1, u \right)^{-1} = (1 + (-1)^\gamma(s) u) \exp \left[ \frac{2\pi}{\pi} \log \left( F^{(s)}(\theta, u) \right) d\theta \right], \tag{27}
\]

for $s \in \{m, f\}$, where $\gamma(s) = 1$ for $s = m$, $\gamma(s) = 0$ for $s = f$, and
\[
F^{(s)}(w, u) = 1 - \frac{(-1)^\gamma(s)}{3} \left[ 1 + 2 \cos w + \gamma(s)(1 - \cos w) \right] u + u^2.
\]

Remark that the leading factor $(1 + (-1)^\gamma(s) u)$ of the right-hand side of Eqs. (26) and (27) corresponds to localization of the three-state Grover walk on $\mathbb{Z}$ (see Konno [13], for example). Localization means that limsup for time $n$ of the probability that the walker returns to the starting location at time $n$ is positive.

In the final part of this section, we consider the OQRW on $T_N^1$, whose dynamics is defined by $4 \times 4$ matrix determined by two $2 \times 2$ matrices $B$ and $C$. Here $B$ and $C$ satisfy
\[
B^* B + C^* C = I_2,
\]

where $*$ means the adjoint operator. The OQRW was introduced by Attal et al. [3, 4]. Put
\[
B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}.
\]

Let $2 \times 2$ matrix $\rho_n(x)$ denote the state at time $n \in \mathbb{Z}_+$ and location $x \in T_N^1$ for OQRW. The evolution of OQRW is determined by
\[
\rho_{n+1}(x) = B \rho_n(x + 1) B^* + C \rho_n(x - 1) C^*. \tag{28}
\]
The measure at time \( n \in \mathbb{Z}_+ \) and location \( x \in T^1_N \) is defined by

\[
\mu_n(x) = \text{Tr} (\rho_n(x)) \,.
\]

From now on, we consider a 2x2 matrix \( \rho_n(x) \) as the following four-dimensional vector:

\[
\rho_n(x) = \begin{bmatrix} \rho_{11}^n(x) \\ \rho_{12}^n(x) \\ \rho_{21}^n(x) \\ \rho_{22}^n(x) \end{bmatrix} \in \mathbb{C}^4.
\]

In this setting, Eq. (28) can be rewritten as

\[
\rho_{n+1}(x) = \tilde{P}_B \rho_n(x + 1) + \tilde{P}_C \rho_n(x - 1),
\]

where

\[
\tilde{P}_B = B \otimes \bar{B} = \begin{bmatrix} |b_{11}|^2 & b_{11}b_{12} & b_{11}b_{12} & |b_{12}|^2 \\ b_{11}b_{21} & |b_{12}|^2 & b_{12}b_{21} & |b_{22}|^2 \\ b_{11}b_{21} & b_{12}b_{21} & |b_{21}|^2 & b_{21}b_{22} \\ |b_{21}|^2 & b_{21}b_{22} & b_{21}b_{22} & |b_{22}|^2 \end{bmatrix},
\]

and

\[
\tilde{P}_C = C \otimes \bar{C} = \begin{bmatrix} |c_{11}|^2 & c_{11}c_{12} & c_{11}c_{12} & |c_{12}|^2 \\ c_{11}c_{21} & |c_{12}|^2 & c_{12}c_{21} & c_{12}c_{22} \\ c_{11}c_{21} & c_{12}c_{21} & |c_{21}|^2 & c_{21}c_{22} \\ c_{21}c_{22} & c_{21}c_{22} & c_{21}c_{22} & |c_{22}|^2 \end{bmatrix}.
\]

Here each component of \( \bar{A} \) is the complex conjugate of that of \( A \) for a matrix \( A \). The Fourier transform of \( \rho_n(x) \), denoted by \( \hat{\rho}_n(\tilde{k}) \), is defined by the sum

\[
\hat{\rho}_n(\tilde{k}) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{K}_N} e^{-i\tilde{k}x} \rho_n(x).
\]

Then we have

\[
\hat{\rho}_{n+1}(\tilde{k}) = \hat{M}_{B,C}(\tilde{k}) \hat{\rho}_n(\tilde{k}),
\]

where

\[
\hat{M}_{B,C}(\tilde{k}) = e^{i\tilde{k} \tilde{P}_B} + e^{-i\tilde{k} \tilde{P}_C}.
\]

Note that \( \hat{M}_{B,C}(\tilde{k}) \) is a counterpart of \( \hat{M}_A(\tilde{k}) \) in Eqs. (22) and (23), that is,

\[
\hat{M}_{A(s)}(\tilde{k}) = e^{i\tilde{k} P_1 A(s)} + e^{-i\tilde{k} P_2 A(s)} \quad (s \in \{m, f\}).
\]

One of the typical model (see [17], for example) is given by

\[
B = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.
\]

In this case, Eqs. (30) and (31) imply

\[
\tilde{P}_B = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tilde{P}_C = \frac{1}{3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{bmatrix}.
\]
By using these, we get
\[
\tilde{M}_{B,C}(\tilde{k}) = \frac{1}{3} \begin{bmatrix}
2 \cos \tilde{k} e^{i\tilde{k}} & e^{i\tilde{k}} & e^{i\tilde{k}} \\
-e^{-i\tilde{k}} & 2 \cos \tilde{k} & 0 \\
e^{-i\tilde{k}} & -e^{-i\tilde{k}} & 2 \cos \tilde{k}
\end{bmatrix}.
\]

Similarly, by computing \( \det \left( \mathbf{I}_4 - u \tilde{M}_{B,C}(\tilde{k}) \right) \), we have

**Corollary 9**

\[
\bar{\zeta} (A, T_N^1, u)^{-1} = \exp \left[ \frac{1}{N} \sum_{k=0}^{N-1} \log \left( \tilde{F}(\tilde{k}, u) \right) \right],
\]

\[
\lim_{N \to \infty} \bar{\zeta} (A, T_N^1, u)^{-1} = \exp \left[ \int_0^{2\pi} \log \left( \tilde{F}(\theta, u) \right) \frac{d\theta}{2\pi} \right],
\]

where

\[
\tilde{F}(w, u) = 1 - \frac{8 \cos w}{3} u + \frac{8 \cos^2 w + 1}{3} u^2 - \frac{16}{27} \cos w \left( 2 \cos^2 w + 1 \right) u^3 + \frac{4}{81} \cos^2 w \left( 4 \cos^2 w + 5 \right) u^4.
\]

Finally, we deals with a relation between OQRW and CRW. To do so, we take

\[
B = \begin{bmatrix} b_{11} & 0 \\ b_{21} & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & c_{12} \\ 0 & c_{22} \end{bmatrix}.
\]

Then assumption \( B^*B + C^*C = I_2 \) gives

\[
|b_{11}|^2 + |b_{21}|^2 = |c_{12}|^2 + |c_{22}|^2 = 1.
\]

Moreover, by Eqs. (30) and (31), we have

\[
\tilde{P}_B = \begin{bmatrix} |b_{11}|^2 & 0 & 0 \\ b_{11}b_{21} & 0 & 0 \\ b_{11}b_{21} & 0 & 0 \end{bmatrix}, \quad \tilde{P}_C = \begin{bmatrix} 0 & 0 & 0 & |c_{12}|^2 \\ 0 & 0 & 0 & c_{12}c_{22} \\ 0 & 0 & 0 & c_{12}c_{22} \end{bmatrix}.
\]

For the state of this model, we can reduce from four-dimensional vector to two-dimensional vector denoted by

\[
\rho_n^{(r)}(x) = \begin{bmatrix} \rho_{n11}^{(11)}(x) \\ \rho_{n22}^{(11)}(x) \end{bmatrix} \in \mathbb{C}^2.
\]

It follows from Eq. (29) that

\[
\rho_n^{(r)}(x + 1) = \tilde{P}_B^{(r)} \rho_n^{(r)}(x + 1) + \tilde{P}_C^{(r)} \rho_n^{(r)}(x - 1),
\]

where

\[
\tilde{P}_B^{(r)} = \begin{bmatrix} |b_{11}|^2 & 0 \\ |b_{21}|^2 & 0 \end{bmatrix}, \quad \tilde{P}_C^{(r)} = \begin{bmatrix} 0 & |c_{12}|^2 \\ 0 & |c_{22}|^2 \end{bmatrix}.
\]
Therefore we have
\[ \tilde{\rho}_{n+1}(\vec{k}) = \tilde{M}_{B,C}(\vec{k})\tilde{\rho}_n(\vec{k}), \]
where
\[ \tilde{M}_{B,C}(\vec{k}) = e^{i\vec{k}}\tilde{P}_B^{(r)} + e^{-i\vec{k}}\tilde{P}_C^{(r)}. \]  
(33)

Here we introduce the corresponding coin matrix \( A \) for the CRW as follows:
\[ A = \begin{bmatrix} b_{11} \left| b_{21} \right|^2 & c_{12} \left| c_{22} \right|^2 \\ b_{21} \left| b_{21} \right|^2 & c_{22} \left| c_{22} \right|^2 \end{bmatrix} \]
with \( |b_{11}|^2 + |b_{21}|^2 = |c_{12}|^2 + |c_{22}|^2 = 1 \). Remark that \( A = \tilde{P}_B + \tilde{P}_C \). In this case, we see
\[ \tilde{P}_B^{(r)} = AP_1, \quad \tilde{P}_C^{(r)} = AP_2, \]
where
\[ P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \]

Thus Eq. (33) becomes
\[ \tilde{M}_{B,C}(\vec{k}) = e^{i\vec{k}}AP_1 + e^{-i\vec{k}}AP_2. \]  
(34)

We recall the expression in Eq. (32), that is,
\[ \tilde{M}_{A(s)}(\vec{k}) = e^{i\vec{k}}P_1A^{(s)} + e^{-i\vec{k}}P_2A^{(s)} \quad (s \in \{m, f\}). \]  
(35)

So this walker defined by Eq. (34) moves at each step one unit to the left with \( AP_1 \) or to the right with \( AP_2 \). On the other hand, the previous walker defined by Eq. (35) moves at each step one unit to the left with \( P_1A^{(s)} \) or to the right with \( P_2A^{(s)} \) for \( s \in \{m, f\} \). This is the connection between OQRW and CRW we wanted to mention.

In this model, following Eq. (24), we put
\[ A = \begin{bmatrix} |b_{11}|^2 & |c_{12}|^2 \\ |b_{21}|^2 & |c_{22}|^2 \end{bmatrix} = \begin{bmatrix} \cos^2 \xi & \sin^2 \xi \\ \sin^2 \xi & \cos^2 \xi \end{bmatrix} = (A^{(m)}). \]

Then Eq. (34) implies
\[ \tilde{M}_{B,C}^{(r)}(\vec{k}) = \begin{bmatrix} e^{i\vec{k}}\cos^2 \xi & e^{-i\vec{k}}\sin^2 \xi \\ e^{i\vec{k}}\sin^2 \xi & e^{-i\vec{k}}\cos^2 \xi \end{bmatrix}. \]

In a similar way, we obtain

**Corollary 10**

\[ \bar{\zeta}(A, T_N^1, u)^{-1} = \exp \left[ \frac{1}{N} \sum_{k=0}^{N-1} \log \left( F \left( \vec{k}, u \right) \right) \right], \]
\[ \lim_{N \to \infty} \bar{\zeta}(A, T_N^1, u)^{-1} = \exp \left[ \int_0^{2\pi} \log \left( F \left( \theta, u \right) \right) \frac{d\theta}{2\pi} \right], \]

where
\[ F(w, u) = 1 - 2\cos^2 \xi \cos w \cdot u + \cos(2\xi)u^2. \]
We should note that this result is the same as $A^{(m)}$ case in Corollary 3. Furthermore, for this model, we change Lemma 3 as follows:

$$Q_1 = AP_1, \quad Q_2 = AP_2, \quad Q_3 = AP_2 \sigma, \quad Q_4 = AP_1 \sigma,$$

that is,

$$Q_1 = \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix}, \quad Q_3 = \begin{bmatrix} a_{12} & 0 \\ a_{22} & 0 \end{bmatrix}, \quad Q_4 = \begin{bmatrix} 0 & a_{11} \\ 0 & a_{21} \end{bmatrix}.$$

Noting that Eq. (20) is also correct for this case, we can confirm that Lemma 4 holds. Therefore we have the same result as $A^{(m)}$ case in Corollary 4.

### 5 Example

This section is devoted to specific examples of QWs and CRWs with M-type in the previous section for a better understanding of Walk/Zeta Correspondence. Other models can be considered in a similar fashion.

(a) QW case.

We deal with the Hadamard walk ($\xi = \pi/4$) with M-type whose $2 \times 2$ matrix $A^{(m)}$ is given by

$$A^{(m)} = \begin{bmatrix} \cos(\pi/4) & \sin(\pi/4) \\ \sin(\pi/4) & -\cos(\pi/4) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

From Corollary 2, we obtain

$$C_{2l}(0) = \lim_{N \to \infty} C_{2l}(A^{(m)}, T_N^1)$$

$$= l \left(-\frac{1}{2}\right)^{l-1} {}_2F_1 \left(1 - l, 1 - l; 2; -1 \right)$$

$$= l \left(-\frac{1}{2}\right)^{l-1} \sum_{m=1}^{l} \frac{1}{m} (-1)^{m-1} \left(\frac{l-1}{m-1}\right)^2.$$

For example, by using these, we get

$$C_2(0) = 1, \quad C_4(0) = -\frac{1}{2}.$$

It follows from Theorem 2 that

$$C_{2l}(0) = \text{Tr} \left(\Phi^{(\infty)}_{2l}(0)\right).$$

Next we compute $2 \times 2$ matrix $\Phi^{(\infty)}_{2l}(0)$ by using notations in Lemma 3 as follows.

$$\Phi^{(\infty)}_{2}(0) = Q_1 Q_2 + Q_2 Q_1 = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

$$\Phi^{(\infty)}_{4}(0) = (Q_1 Q_2 + Q_2 Q_1)^2 + Q_1^2 Q_2^2 + Q_2^2 Q_1^2 = \frac{1}{4} \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},$$

where

$$Q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad Q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$
So we have

\[ C_2(0) = \text{Tr} \left( \Phi_2(\infty)(0) \right) = 1, \quad C_4(0) = \text{Tr} \left( \Phi_4(\infty)(0) \right) = -\frac{1}{2}. \]

Therefore we can confirm that Corollary 2 holds for \( l = 1 \) and \( l = 2 \).

On the other hand, the return probability \( \mu_{2l}(0) \) for the Hadamard walk starting from the origin at time \( 2l \) depends on the initial state \( \varphi = T [\alpha, \beta] \in \mathbb{C}^2 \) with \( |\alpha|^2 + |\beta|^2 = 1 \), where \( T \) is the transposed operator. Then

\[ \mu_{2l}(0) = \| \Phi_{2l}(\infty)(0) \varphi \|_{\mathbb{C}^2}. \]

For example, if we take \( \varphi = T [1/\sqrt{2}, i/\sqrt{2}] \), then we obtain

\[ \mu_0(0) = 1, \quad \mu_2(0) = \frac{1}{2}, \quad \mu_4(0) = \mu_6(0) = \frac{1}{8}, \ldots. \]

Remark that it is known in Konno [15] that

\[ \mu_{4m}(0) = \mu_{4m+2}(0) = \frac{1}{2^{4m+1}} \binom{2m}{m}^2 \quad (m \geq 1). \]

Therefore we see that in general \( \mu_{2l}(0) \) is not the same as \( C_{2l}(0) \).

(b) CRW case.

As in the case of the Hadamard walk, we consider a CRW \((\xi = \pi/6)\) with M-type whose \( 2 \times 2 \) coin matrix \( A^{(m)} \) is given by

\[ A^{(m)} = \begin{bmatrix} \cos^2(\pi/6) & \sin^2(\pi/6) \\ \sin^2(\pi/6) & \cos^2(\pi/6) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}. \]

By Corollary 4, we get

\[ C_{2l}(0) = \lim_{N \to \infty} C_{2l}(A^{(m)}, T_N^l) \]

\[ = \frac{3^{2(l-1)} l}{2^{4l-1}} \sum_{m=1}^{l} \frac{1}{m} \left( \frac{1}{9} \right)^{m-1} \left( \frac{l-1}{m-1} \right)^2. \]

For example, from using these, we have

\[ C_2(0) = \frac{1}{8}, \quad C_4(0) = \frac{19}{128} = 0.1484 \ldots. \]

Next we calculate \( 2 \times 2 \) matrix \( \Phi_{2l}(\infty)(0) \) by using notations in Lemma 3 like the Hadamard walk case:

\[ \Phi_2(\infty)(0) = Q_1 Q_2 + Q_2 Q_1 = \frac{1}{16} \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \]

\[ \Phi_4(\infty)(0) = (Q_1 Q_2 + Q_2 Q_1)^2 + Q_1^2 Q_2^2 + Q_2^2 Q_1^2 = \left( \frac{1}{16} \right)^2 \begin{bmatrix} 19 & 33 \\ 33 & 19 \end{bmatrix}, \]

where

\[ Q_1 = \frac{1}{4} \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}, \quad Q_2 = \frac{1}{4} \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}. \]
Thus we get
\[ C_2(0) = \text{Tr} \left( \Phi_2^{(\infty)}(0) \right) = \frac{1}{8}, \quad C_4(0) = \text{Tr} \left( \Phi_4^{(\infty)}(0) \right) = \frac{19}{128}. \]
Therefore we can confirm that Corollary 4 is valid for \( l = 1 \) and \( l = 2 \).

On the other hand, the return probability \( \mu_{2l}(0) \) for the CRW starting from the origin at time \( 2l \) depends on the initial state \( \varphi = T[\alpha, \beta] \in [0, 1]^2 \) with \( \alpha + \beta = 1 \). Then
\[ \mu_{2l}(0) = \| \Phi_{2l}^{(\infty)}(0)\varphi \|_{\mathbb{R}^2}. \]
For instance, if we take \( \varphi = T[1/2, 1/2] \), then we have
\[ \mu_0(0) = 1, \quad \mu_2(0) = \frac{1}{4}, \quad \mu_4(0) = \frac{13}{64} = 0.2031, \ldots. \]
Thus we see that in general \( \mu_{2l}(0) \) is different from \( C_{2l}(0) \) as in the case of the Hadamard walk.

### 6 Two-Dimensional Case

This section treats walks on the two-dimensional torus \( T^2_N \) whose \( 4 \times 4 \) coin matrix \( A^{(m)} \) (M-type) or \( A^{(f)} \) (F-type) as follows:

\[
A^{(m)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \quad A^{(f)} = \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix},
\]

since
\[
A^{(f)} = (I_2 \otimes \sigma) A^{(m)}. \]

In this case, we take
\[
P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\]

Similarly, we define a \( 4N^2 \times 4N^2 \) matrix
\[
M_A = P_1A\tau_1^{-1} + P_2A\tau_1 + P_3A\tau_2^{-1} + P_4A\tau_2.
\]
Then the walker moves at each step one unit to the left with \( P_1 \) or to the right with \( P_2 \) or to the down with \( P_3 \) or to the up with \( P_4 \). Thus we get
\[
\det \left( I_4 - u\tilde{M}_A(\tilde{k}) \right) = \det \left( I_4 - u \times \left( e^{i\tilde{k}_1} P_1A + e^{-i\tilde{k}_1} P_2A + e^{i\tilde{k}_2} P_3A + e^{-i\tilde{k}_2} P_4A \right) \right).
\]
A typical example is the two-dimensional Grover walk whose $4 \times 4$ coin matrix $A^{(m)}$ (M-type) or $A^{(f)}$ (F-type) is defined by

$$A^{(m)} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}, \quad A^{(f)} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}.$$  

Here $A^{(m)}$ is the $4 \times 4$ Grover matrix. Then we have

**Corollary 11**

$$\bar{\zeta} \left( A^{(s)}, T_N^2, u \right)^{-1} = (1 - u^2) \exp \left[ \frac{1}{N^2} \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-1} \log \left( F^{(s)}(\tilde{k}_1, \tilde{k}_2, u) \right) \right],$$  

$$\lim_{N \to \infty} \bar{\zeta} \left( A^{(s)}, T_N^2, u \right)^{-1} = (1 - u^2) \exp \left[ \int_0^{2\pi} \int_0^{2\pi} \log \left( F^{(s)}(\theta_1, \theta_2, u) \right) \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \right],$$  

for $s \in \{m, f\}$, where

$$F^{(m)}(w_1, w_2, u) = 1 + (\cos w_1 + \cos w_2) u + u^2,$$

$$F^{(f)}(w_1, w_2, u) = 1 - (\cos w_1 + \cos w_2) u + u^2.$$

Note that the leading factor $(1 - u^2)$ of the right-hand side of Eqs. (36) and (37) corresponds to localization of the four-state Grover walk on $\mathbb{Z}^2$ (see [9], for example). Moreover, $F^{(s)}(w_1, w_2, u)$ for $s \in \{m, f\}$ are the same as the corresponding results in Asano et al. [2].

Another typical example is the two-dimensional Fourier walk whose $4 \times 4$ coin matrix $A^{(m)}$ (M-type) or $A^{(f)}$ (F-type) is determined by

$$A^{(m)} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & 1 & -1 \end{bmatrix}, \quad A^{(f)} = \frac{1}{2} \begin{bmatrix} 1 & i & -1 & -i \\ 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & 1 & -1 & 1 \end{bmatrix}.$$  

Here $A^{(m)}$ is the $4 \times 4$ Fourier matrix. In general, the $n \times n$ Fourier matrix $F^{(n)} = [F^{(n)}_{ab}]_{a,b=1,2,\ldots,d}$ is defined by

$$F^{(n)}_{ab} = \frac{1}{\sqrt{n}} \omega_n^{(a-1)(b-1)} \quad (\omega_n = \exp(2\pi i/n)).$$

We should remark that $F^{(n)}$ is unitary. The walk defined by the Fourier matrix is called the **Fourier walk**. Then, $A^{(m)}$ and $A^{(f)}$ are unitary, so the walks determined by them become QWs. In a similar fashion, we obtain

**Corollary 12**

$$\bar{\zeta} \left( A^{(s)}, T_N^2, u \right)^{-1} = \exp \left[ \frac{1}{N^2} \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-1} \log \left( F^{(s)}(\tilde{k}_1, \tilde{k}_2, u) \right) \right],$$  

$$\lim_{N \to \infty} \bar{\zeta} \left( A^{(s)}, T_N^2, u \right)^{-1} = \exp \left[ \int_0^{2\pi} \int_0^{2\pi} \log \left( F^{(s)}(\theta_1, \theta_2, u) \right) \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \right].$$
for \( s \in \{m, f\} \), where
\[
F^{(m)}(w_1, w_2, u) = 1 - \frac{1 + i}{2} (\cos w_1 + \sin w_1 + \cos w_2 + \sin w_2) u \\
- \frac{1 - i}{2} (1 - \cos(w_1 - w_2)) u^2 + \frac{1 + i}{2} (\cos w_1 + \sin w_1 + \cos w_2 + \sin w_2) u^3 - iu^4,
\]
\[
F^{(f)}(w_1, w_2, u) = 1 - (\cos w_1 - \cos w_2) u \\
+ \frac{1 - i}{2} (1 - \cos(w_1 - w_2)) u^2 + i (\cos w_1 - \cos w_2) u^3 - iu^4.
\]

Note that \( F^{(s)}(w_1, w_2, u) \) for \( s \in \{m, f\} \) are the same as the corresponding results in [2].

Finally, we consider the two-dimensional positive-support version of the Grover walk whose \( 4 \times 4 \) coin matrix \( A^{(m)} \) (M-type) or \( A^{(f)} \) (F-type) is determined by
\[
A^{(m)} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad A^{(f)} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.
\]

Here the positive support \( A^{+} = [A^{+}_{ab}] \) of a real matrix \( A = [A_{ab}] \) is defined as follows:
\[
A^{+}_{ab} = \begin{cases} 1 & \text{if } A_{ab} > 0, \\ 0 & \text{if } A_{ab} \leq 0. \end{cases}
\]

So the positive support \( G^{(n),+} \) of the \( n \times n \) Grover matrix \( G^{(n)} \) for \( n \geq 2 \) is
\[
G^{(n),+}_{ab} = \begin{cases} 1 & \text{if } a \neq b, \\ 0 & \text{if } a = b. \end{cases}
\]

We should remark that this model is neither QW nor CRW. However, we can apply our method to it. Thus we have the following results.

**Corollary 13**
\[
\lim_{N \to \infty} \xi(A^{(m)}, T^2_N, u)^{-1} = \exp \left[ \frac{1}{N^2} \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-1} \log \left( F^{(m)}(\tilde{k}_1, \tilde{k}_2, u) \right) \right]
\]
\[
\lim_{N \to \infty} \xi(A^{(m)}, T^2_N, u)^{-1} = \exp \left[ \int_0^{2\pi} \int_0^{2\pi} \log \left( F^{(m)}(\theta_1, \theta_2, u) \right) \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \right]
\]
\[
\xi(A^{(f)}, T^2_N, u)^{-1} = (1 - u^2) \exp \left[ \frac{1}{N^2} \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-1} \log \left( F^{(f)}(\tilde{k}_1, \tilde{k}_2, u) \right) \right]
\]
\[
\lim_{N \to \infty} \xi(A^{(f)}, T^2_N, u)^{-1} = (1 - u^2) \exp \left[ \int_0^{2\pi} \int_0^{2\pi} \log \left( F^{(f)}(\theta_1, \theta_2, u) \right) \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \right],
\]
for \( s \in \{m, f\} \), where
\[
F^{(m)}(w_1, w_2, u) = 1 - 2 (1 + 2 \cos w_1 \cos w_2) u^2 - 4 (\cos w_1 + \cos w_2) u^3 - 3u^4,
\]
\[
F^{(f)}(w_1, w_2, u) = 1 - 2 (\cos w_1 + \cos w_2) u + 3u^2.
\]

We should note that our result for F-type in Corollary 13 corresponds to Eq. (10) in Clair [7].
7 Higher-Dimensional Case

In this section, we consider walks on the $d$-dimensional torus $T^d_N$ ($d \geq 3$) with $2d \times 2d$ coin matrix $A^{(m)}$ (M-type) or $A^{(f)}$ (F-type). To do so, we introduce the $n$-th elementary symmetric polynomial $e^{(n)}_j(x_1, x_2, \ldots, x_n)$ as follows:

$$
e^{(n)}_j(x_1, x_2, \ldots, x_n) = \sum_{T \subset [n], |T| = j} \prod_{t \in T} x_t \quad (j = 1, 2, \ldots, n),$$

where $[n] = \{1, 2, \ldots, n\}$. For example,

$$
e^{(2)}_1(x_1, x_2) = x_1 + x_2, \quad e^{(2)}_2(x_1, x_2) = x_1x_2,$$

$$
e^{(3)}_1(x_1, x_2, x_3) = x_1 + x_2 + x_3, \quad e^{(3)}_2(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3,$$

$$
e^{(3)}_3(x_1, x_2, x_3) = x_1x_2x_3.$$

Moreover we put

$$
e^{(n, \cos)}_j(\mathbf{k}) = e^{(n)}_j(\cos k_1, \cos k_2, \ldots, \cos k_n),
\quad e^{(n, \cos)}_j(\Theta^{(n)}) = e^{(n)}_j(\cos \theta_1, \cos \theta_2, \ldots, \cos \theta_n),$$

for $j = 1, 2, \ldots, n$.

A typical example is the $d$-dimensional Grover walk with $2d \times 2d$ coin matrix $A^{(m)}$ (M-type) or $A^{(f)}$ (F-type), where $A^{(m)}$ is the $2d \times 2d$ Grover matrix and $A^{(f)} = (I_2 \otimes \sigma)A^{(m)}$.

First we consider F-type case, since we can obtain the result on the general $d$-dimensional torus. In fact, Theorem 1 gives

**Corollary 14**

$$
\zeta \left( A^{(f)}, T^d_N, u \right)^{-1} = (1 - u^2)^{d-1} \exp \left[ \frac{1}{N^d} \sum_{j=1}^{d} \sum_{k_j=0}^{N-1} \log \left\{ F^{(f)}(\mathbf{k}, u) \right\} \right],
$$

$$
\lim_{N \to \infty} \zeta \left( A^{(f)}, T^d_N, u \right)^{-1} = (1 - u^2)^{d-1} \exp \left[ \int_{[0, 2\pi]^d} \log \left\{ F^{(f)}(\Theta^{(d)}, u) \right\} d\Theta^{(d)}_{unif} \right],
$$

where

$$
F^{(f)}(w, u) = 1 - \frac{2}{d} e^{(d, \cos)}_1(w) u + u^2.
$$

Remark that the leading factor $(1 - u^2)^{d-1}$ for $d \geq 3$ corresponds to localization of the Grover walk on $\mathbb{Z}^d$ (see [9], for example). Komatsu et al. [12] obtained the same result by not our method based on the Fourier analysis but the Konno–Sato theorem (including $d = 2$ case) and called “Grover/Generalized-Zeta Correspondence”.

On the other hand, as for M-type case, we do not get the result on the general $d$-dimensional torus. Indeed, Theorem 1 for $d = 3$ case implies the following result. However it is not a simple form compared with the corresponding F-type.
Corollary 15

$$\zeta(A^{(m)}, T^3_N, u)^{-1} = (1 - u^2) \exp \left[ \frac{1}{N^3} \sum_{j=1}^{3} \sum_{k_j=0}^{N-1} \log \left\{ F^{(m)}(\mathbf{k}, u) \right\} \right],$$

$$\lim_{N \to \infty} \zeta(A^{(m)}, T^3_N, u)^{-1} = (1 - u^2) \exp \left[ \int_{[0,2\pi)^3} \log \left\{ F^{(m)}(\Theta^{(3)}, u) \right\} d\Theta^{(3)}_{unif} \right],$$

where

$$F^{(m)}(w, u) = 1 + \frac{4}{3} e^{(3,\cos)}_1(w) u + \left( 2 + \frac{4}{3} e^{(3,\cos)}_2(w) \right) u^2 + \frac{4}{3} e^{(3,\cos)}_1(w) u^3 + u^4.$$  

Another typical example is the positive-support version of the $d$-dimensional Grover walk with $2d \times 2d$ coin matrix $A^{(m)}$ (M-type) or $A^{(f)}$ (F-type), where $A^{(m)}$ is the positive-support of the $2d \times 2d$ Grover matrix and $A^{(f)} = (I_d \otimes \sigma) A^{(m)}$ like $d = 2$ case. The situation in this example is similar to that of the previous one. So, we first deal with F-type case, since we can obtain the result for the general $d$-dimensional torus. From Theorem 1, we have

Corollary 16

$$\zeta(A^{(f)}, T^d_N, u)^{-1} = (1 - u^2)^{d-1} \exp \left[ \frac{1}{N^d} \sum_{j=1}^{d} \sum_{k_j=0}^{N-1} \log \left\{ F^{(f)}(\mathbf{k}, u) \right\} \right],$$

$$\lim_{N \to \infty} \zeta(A^{(f)}, T^d_N, u)^{-1} = (1 - u^2)^{d-1} \exp \left[ \int_{[0,2\pi)^d} \log \left\{ F^{(f)}(\Theta^{(d)}, u) \right\} d\Theta^{(d)}_{unif} \right],$$

where

$$F^{(f)}(w, u) = 1 - 2 e^{(d,\cos)}_1(w) u + (2d - 1) u^2.$$  

This result is consistent with the $d$-dimensional torus $T^d_N$ case for Theorem 1.3 in Chinta et al. [6]. Komatsu et al. [12] got the same result by not our method based on the Fourier analysis but the Konno–Sato theorem (including $d = 2$ case) and called “Grover(Positive Support)/Generalized-Ihara-Zeta Correspondence”.

As in the case of the Grover walk, for M-type case, we do not have the result on the general $d$-dimensional torus. Actually, Theorem 1 for $d = 3$ case gives the following result which is a complicated form compared with the corresponding F-type.

Corollary 17

$$\zeta(A^{(m)}, T^3_N, u)^{-1} = \exp \left[ \frac{1}{N^3} \sum_{j=1}^{3} \sum_{k_j=0}^{N-1} \log \left\{ F^{(m)}(\mathbf{k}, u) \right\} \right],$$

$$\lim_{N \to \infty} \zeta(A^{(m)}, T^3_N, u)^{-1} = \exp \left[ \int_{[0,2\pi)^3} \log \left\{ F^{(m)}(\Theta^{(3)}, u) \right\} d\Theta^{(3)}_{unif} \right],$$

where
\[ F^{(m)}(w, u) = 1 - \left( 3 + 4 e_2^{(3, \cos)}(w) \right) u^2 - 8 \left( e_1^{(3, \cos)}(w) + 2 e_3^{(3, \cos)}(w) \right) u^3 \]
\[ - 3 \left( 3 + 4 e_2^{(3, \cos)}(w) \right) u^4 - 8 e_1^{(2, \cos)}(w) u^5 - 5 u^6. \]

**Appendix A: Grover/Zeta Correspondence**

In Appendix A, we briefly review our previous work on Grover/Zeta Correspondence based on the Konno–Sato theorem in [12]. We assume that all graphs are simple. Here we consider the Grover walk with F-type and the positive-support version of the Grover walk with F-type on a graph.

Let \( G = (V(G), E(G)) \) be a connected graph (without multiple edges and loops) with the set \( V(G) \) of vertices and the set \( E(G) \) of unoriented edges \( uv \) joining two vertices \( u \) and \( v \). Moreover, let \( n = |V(G)| \) and \( m = |E(G)| \) be the number of vertices and edges of \( G \), respectively. For \( uv \in E(G) \), an arc \((u, v)\) is the oriented edge from \( u \) to \( v \). Let \( D_G \) be the symmetric digraph corresponding to \( G \). Set \( D(G) = \{(u, v), (v, u) \mid uv \in E(G)\} \). For \( e = (u, v) \in D(G) \), set \( u = o(e) \) and \( v = t(e) \). Furthermore, let \( e^{-1} = (v, u) \) be the inverse of \( e = (u, v) \). For \( v \in V(G) \), the degree \( \deg_G v = \deg v = d_v \) of \( v \) is the number of vertices adjacent to \( v \) in \( G \). If \( \deg_G v = k \) (constant) for each \( v \in V(G) \), then \( G \) is called \( k \)-regular. A path \( P \) of length \( n \) in \( G \) is a sequence \( P = (e_1, \ldots, e_n) \) of \( n \) arcs such that \( e_i \in D(G) \), \( t(e_i) = o(e_{i+1}) \) (1 \( \leq i \leq n - 1 \)). If \( e_i = (v_{i-1}, v_i) \) for \( i = 1, \ldots, n \), then we write \( P = (v_0, v_1, \ldots, v_{n-1}, v_n) \). Put \( |P| = n \), \( o(P) = o(e_1) \) and \( t(P) = t(e_n) \). Also, \( P \) is called an \((o(P), t(P))-path\). We say that a path \( P = (e_1, \ldots, e_n) \) has a backtracking if \( e_{i+1}^{-1} = e_i \) for some \( i \) (1 \( \leq i \leq n - 1 \)). A \((v, w)\)-path is called a \( v \)-cycle (or \( v \)-closed path) if \( v = w \). Let \( B^r \) be the cycle obtained by going \( r \) times around a cycle \( B \). Such a cycle is called a multiple of \( B \). A cycle \( C \) is reduced if both \( C \) and \( C^2 \) have no backtracking. The Ihara zeta function of a graph \( G \) is a function of a complex variable \( u \) with \(|u| \) sufficiently small, defined by

\[ Z(G, u) = \exp \left( \sum_{r=1}^{\infty} \frac{N_r}{r} u^r \right), \]

where \( N_r \) is the number of reduced cycles of length \( r \) in \( G \). Let \( G \) be a connected graph with \( n \) vertices \( v_1, \ldots, v_n \). The adjacency matrix \( A = A(G) = (a_{ij}) \) is the square matrix such that \( a_{ij} = 1 \) if \( v_i \) and \( v_j \) are adjacent, and \( a_{ij} = 0 \) otherwise. The following result was obtained by Ihara [8] and Bass [5].

**Theorem 3** \[\text{Ihara [8], Bass [5]}\] \textit{Let \( G \) be a connected graph. Then the reciprocal of the Ihara zeta function of \( G \) is given by}

\[ Z(G, u)^{-1} = (1 - u^2)^{\gamma - 1} \det \left( I - uA(G) + u^2(D - I) \right). \]

\textit{Here} \( \gamma \) \textit{is the Betti number of} \( G \), and \( D = (d_{ij}) \) \textit{is the diagonal matrix with} \( d_{ii} = \deg v_i \) \textit{and} \( d_{ij} = 0, i \neq j \), \textit{where} \( V(G) = \{v_1, \ldots, v_n\} \).
Let \( G = (V(G), E(G)) \) be a connected graph with \( n \) vertices and \( x_0 \in V(G) \) a fixed vertex. Then the generalized Ihara zeta function \( \zeta(G, u) \) of \( G \) is defined by

\[
\zeta(G, u) = \exp \left( \sum_{r=1}^{\infty} \frac{N_r^0}{r} u^r \right),
\]

where \( N_r^0 \) is the number of reduced \( x_0 \)-cycles of length \( r \) in \( G \). A graph \( G \) is called vertex-transitive if there exists an automorphism \( \phi \) of the automorphism group \( \text{Aut}(G) \) of \( G \) such that \( \phi(u) = v \) for each \( u, v \in V(G) \). Note that if \( G \) is a vertex-transitive graph with \( n \) vertices, then

\[
\zeta(G, u) = Z(G, u)^{1/n}.
\]

Moreover, the Laplacian of \( G \) is given by

\[
\Delta_n = \Delta(G) = D - A(G).
\]

A formula for the generalized Ihara zeta function of a vertex-transitive graph is given in Chinta et al. [6] as follows.

**Theorem 4** Chinta et al. [6] Let \( G \) be a vertex-transitive \((q + 1)\)-regular graph with spectral measure \( \mu_\Delta \) for the Laplacian \( \Delta \). Then

\[
\zeta(G, u)^{-1} = (1 - u^2)^{(q-1)/2} \exp \left[ \int \log(1 - (q + 1 - \lambda)u + qu^2) d\mu_\Delta(\lambda) \right].
\]

Let \( G \) be a connected graph with \( n \) vertices and \( m \) edges. Put \( V(G) = \{v_1, \ldots, v_n\} \) and \( d_j = d_{v_j} = \text{deg} v_j, \; j = 1, \ldots, n \). Then the Grover matrix \( U = U(G) = (U_{ef})_{e, f \in D(G)} \) of \( G \) is defined by

\[
U_{ef} = \begin{cases} 
2/d_{t(f)}(= 2/d_{o(e)}) & \text{if } t(f) = o(e) \text{ and } f \neq e^{-1}, \\
2/d_{t(f)} - 1 & \text{if } f = e^{-1}, \\
0 & \text{otherwise}.
\end{cases}
\]

The discrete-time quantum walk with the matrix \( U \) as a time evolution matrix is the Grover walk with F-type on \( G \). Let \( G \) be a connected graph with \( n \) vertices and \( m \) edges. Then the \( n \times n \) matrix \( P_n = P(G) = (P_{uv})_{u, v \in V(G)} \) is given by

\[
P_{uv} = \begin{cases} 
1/\text{deg}_G u & \text{if } (u, v) \in D(G), \\
0 & \text{otherwise}.
\end{cases}
\]

Note that the matrix \( P(G) \) is the transition probability matrix of the simple random walk on \( G \). We introduce the positive support \( F^+ = (F_{ij}^+) \) of a real matrix \( F = (F_{ij}) \) as follows.

\[
F_{ij}^+ = \begin{cases} 
1 & \text{if } F_{ij} > 0, \\
0 & \text{otherwise}.
\end{cases}
\]

Ren et al. [20] showed that the Perron-Frobenius operator (or edge matrix) of a graph is the positive support \((U^T)U^+\) of the transpose of its Grover matrix \( U \), i.e.,

\[
Z(G, u)^{-1} = \det \left( I_{2m} - u(U^T)U^+ \right) = \det \left( I_{2m} - uU^+ \right).
\]

The Ihara zeta function of a graph \( G \) is just a zeta function on the positive support of the Grover matrix of \( G \). That is, the Ihara zeta function corresponds to the positive-support
version of the Grover walk (defined by the positive support of the Grover matrix \( U^+ \)) with F-type on \( G \).

Now we propose a new zeta function of a graph. Let \( G \) be a connected graph with \( m \) edges. Then we define a zeta function \( \overline{Z}(G, u) \) of \( G \) satisfying

\[
\overline{Z}(G, u)^{-1} = \det(I_{2m} - uU).
\]

In other words, the new zeta function corresponds to the Grover walk (defined by the Grover matrix \( U \)) with F-type on \( G \).

In this setting, Konno and Sato [16] presented the following result which is called the Konno–Sato theorem.

**Theorem 5** Konno and Sato [16] Let \( G \) be a connected vertex-transitive \((q + 1)\)-regular graph with \( n \) vertices and \( m \) edges. Then

\[
\overline{Z}(G, u)^{-1} = \det(I_{2m} - uU^+) = (1 - u^2)^{m-n} \det((1 + u^2)I_n - 2up_n)
\]

\[
= (1 - u^2)^{m-n} \det\left( (1 - 2u + u^2)I_n + \frac{2u}{q+1} \Delta_n \right).
\]

\[
Z(G, u)^{-1} = \det(I_{2m} - uU) = (1 - u^2)^{m-n} \det((1 + qu^2)I_n - (q + 1)up_n)
\]

\[
= (1 - u^2)^{m-n} \det\left( (1 - (q + 1)u + qu^2)I_n + u\Delta_n \right).
\]

Here we give a weight function \( w : D(G) \times D(G) \rightarrow \mathbb{C} \) as follows.

\[
w(e, f) = \begin{cases} 
\frac{2}{\deg t(e)} & \text{if } t(e) = o(f) \text{ and } f \neq e^{-1}, \\
\frac{2}{\deg t(e)} - 1 & \text{if } f = e^{-1}, \\
0 & \text{otherwise}.
\end{cases}
\]

For a cycle \( C = (e_1, e_2, \ldots, e_r) \), put

\[
w(C) = w(e_1, e_2) \cdots w(e_{r-1}, e_r)w(e_r, e_1).
\]

We define a generalized zeta function with respect to the Grover matrix of a graph. Let \( G = (V(G), E(G)) \) be a connected graph and \( x_0 \in V(G) \) a fixed vertex. Then the generalized zeta function \( \overline{\zeta}_G(u) \) of \( G \) is defined by

\[
\overline{\zeta}(G, u) = \exp\left( \sum_{r=1}^{\infty} \frac{N_r^0}{r} u^r \right),
\]

where

\[
N_r^0 = \sum \{w(C) \mid C : an \ x_0 \text{-cycle of length } r \text{ in } G \}.
\]

We should remark that if \( G \) is a vertex-transitive graph with \( n \) vertices, then

\[
\overline{\zeta}(G, u) = \overline{Z}(G, u)^{1/n}.
\]

Then the following result for a series of finite vertex-transitive \((q + 1)\)-regular graphs was given in our previous work [12], which is called Grover/Zeta Correspondence.
Theorem 6  Grover/Zeta Correspondence [12] Let $\{G_m\}_{m=1}^{\infty}$ be a series of finite vertex-transitive $(q + 1)$-regular graphs with $\lim_{m \to \infty} |V(G_m)| = \infty$. Then

$$\lim_{m \to \infty} \zeta(G_m, u)^{-1} = (1 - u^2)^{(q-1)/2} \exp \left[ \int \log \left\{ (1 + u^2) - 2u \lambda \right\} d\mu_P(\lambda) \right],$$

where $d\mu_P(\lambda)$ and $d\mu_\Delta(\lambda)$ are the spectral measures for the transition operator $P$ and the Laplacian $\Delta$.

We should note that the fourth formula in Theorem 6 is nothing but Theorem 1.3 in Chinta et al. [6] (see also Theorem 4 in Appendix A).

Next the following result for the generalized zeta function and the generalized Ihara zeta function of the $d$-dimensional torus $T^d$ was shown in our previous work [12], which is also called Grover/Zeta Correspondence ($T^d_N$ case). Note that $|E(T^d_N)| = dN^d$ and $T^d_N$ is a vertex-transitive $2d$-regular graph.

Theorem 7  Grover/Zeta Correspondence ($T^d_N$ case [12]) Let $T^d_N$ ($d \geq 2$) be the $d$-dimensional torus with $N^d$ vertices. Then we have

$$\lim_{N \to \infty} \zeta(T^d_N, u)^{-1} = (1 - u^2)^{d-1} \exp \left[ \int_{[0,2\pi)^d} \log \left\{ (1 + u^2) - \frac{2u}{d} \sum_{j=1}^{d} \cos \theta_j \right\} d\Theta_{unif}^{(d)} \right],$$

$$\lim_{N \to \infty} \zeta(T^d_N, u)^{-1} = (1 - u^2)^{d-1} \exp \left[ \int_{[0,2\pi)^d} \log \left\{ (1 + (2d - 1)u^2) - 2u \sum_{j=1}^{d} \cos \theta_j \right\} d\Theta_{unif}^{(d)} \right],$$

where $\int_{[0,2\pi)^d}$ is the $d$-th multiple integral and $d\Theta_{unif}^{(d)}$ is the uniform measure on $[0, 2\pi)^d$.

The first result on $\lim_{N \to \infty} \zeta(T^d_N, u)^{-1}$ is the same as that on $\lim_{N \to \infty} \zeta(A^f, T^d_N, u)^{-1}$ for the $d$-dimensional Grover walk with F-type in Corollary 14 (Sect. 7). The second result on $\lim_{N \to \infty} \zeta(T^d_N, u)^{-1}$ is the same as that on $\lim_{N \to \infty} \zeta(A^f, T^d_N, u)^{-1}$ for the positive-support version of the $d$-dimensional Grover walk with F-type in Corollary 16 (Sect. 7).

Specially, in the case of $d = 2$, the following result can be derived.

Corollary 18  Let $T^2_N$ be the 2-dimensional torus with $N^2$ vertices. Then we have

$$\lim_{N \to \infty} \zeta(T^2_N, u)^{-1} = (1 - u^2) \exp \left[ \int_{0}^{2\pi} \int_{0}^{2\pi} \log \left\{ (1 + u^2) - u \sum_{j=1}^{2} \cos \theta_j \right\} d\theta_1 \frac{d\theta_2}{2\pi} \right],$$

$$\lim_{N \to \infty} \zeta(T^2_N, u)^{-1} = (1 - u^2) \exp \left[ \int_{0}^{2\pi} \int_{0}^{2\pi} \log \left\{ (1 + 3u^2) - 2u \sum_{j=1}^{2} \cos \theta_j \right\} d\theta_1 \frac{d\theta_2}{2\pi} \right].$$
The first result on $\lim_{N \to \infty} \zeta(T_N^2, u)^{-1}$ is equivalent to that on $\lim_{N \to \infty} \zeta(A^{(f)}, T_N^2, u)^{-1}$ for the two-dimensional Grover walk with F-type in Corollary 11 (Sect. 6). The second result on $\lim_{N \to \infty} \zeta(T_N^2, u)^{-1}$ is equivalent to that on $\lim_{N \to \infty} \zeta(A^{(f)}, T_N^2, u)^{-1}$ for the positive-support version of the two-dimensional Grover walk with F-type in Corollary 13 (Sect. 6).

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