Asynchronous finite differences in most probable distribution with finite numbers of particles

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For a discrete function \( f(x) \) on a discrete set, the finite difference can be either forward and backward. If \( f(x) \) is a sum of two such functions \( f(x) = f_1(x) + f_2(x) \), the first order difference of \( \Delta f(x) \) can be grouped into four possible combinations, in which two are the usual synchronous ones \( \Delta^f f_1(x) + \Delta^f f_2(x) \) and \( \Delta^b f_1(x) + \Delta^b f_2(x) \), and other two are asynchronous ones \( \Delta^f f_1(x) + \Delta^b f_2(x) \) and \( \Delta^b f_1(x) + \Delta^f f_2(x) \), where \( \Delta^f \) and \( \Delta^b \) denotes the forward and backward difference respectively. Thus, the first order variation equation \( \Delta f(x) = 0 \) for this function \( f(x) \) gives at most four different solutions which contain both true and false one. A formalism of the discrete calculus of variations is developed to single out the true one by means of comparison of the second order variations, in which the largest value in magnitude indicates the true solution, yielding the exact form of the distributions for Boltzmann, Bose and Fermi system without requiring the numbers of particle to be infinitely large.

When there is only one particle in the system, all distributions reduce to be the Boltzmann one.

Keywords: statistical distribution, discrete calculus of variations, most probable distribution

I. INTRODUCTION

This paper aims at solving a long time and also fundamental problem in statistical physics. For the historical side which was the initial motivation of the present study, we are familiar with a difficulty of possible existence of a derivation of Bose-Einstein (or simply Bose), Fermi-Dirac (or simply Fermi) and Boltzmann-Maxwell (or simply Boltzmann) distribution without use of the Stirling approximation of the factorials, within the method of most probable distribution (MPD) in statistical physics. Some feels lukewarm with the use of the Stirling approximation, and some thinks it a serious defect, but no one considers it comfortable. The comments of Tolman in 1938 on the problem are worthy of mention. [1] The first part of the comments is that, with the help of the approximation, we can use the calculus of continuous functions instead of that of the discrete ones. The original sentences include: "To carry out the analysis, let us take the numbers of molecules \( n_i \) ... as being large enough, not only to permit the foregoing use of Stirling’s approximation for their factorials, but also to justify us in treating the numbers themselves as continuous variables in applying the calculus of variations." (p.79), and "This was done in order to employ the usual methods of the calculus of variations as applied to continuous variables." (p.80). The second part of the comments is that, the approximations are evidently worse sometimes. The original sentences include: "And, from a computational point of view, it will be remembered that our previous calculations of the molecular distributions to be expected at equilibrium were actually carried out for simplicity, with a somewhat unsatisfactory introduction of the Stirling approximation for factorial numbers, which, as emphasized by Fowler, might even involve the use of that approximation for integers as small as zero or one." (p. 481), and "The derivations which we have given for these relations have been obtained with the help of the Stirling approximation for factorials, and this may be regarded to some extent as a defect, ... ". (p.373). The similar comments can also be found in vast literature, among them I like to mention two [2, 3].

For the present side which urges the present study, we are impressed by the recent experimental realization of the single-atom heat engine [4]. The single particle exists in a thermal distribution whose width is proportional to its temperature, which clearly is the Boltzmann distribution [4]. We know that in the classical limit, all Bose, Fermi and Boltzmann distributions are all reduced to the Boltzmann one, but whether it is so in the limit of single atoms poses a contemporary problem.

The method of MPD is the most common way used to derive various statistical distributions in mathematics, physics, chemistry, materials science and computational science, etc. The functions in statistics are usually are defined on a discrete lattice rather than a continuous interval, so we must be able to deal with the differences, difference quotients and sums of discrete functions, instead of the differentials, derivatives and integrations of the continuous functions. However, once manipulating the discrete calculus, we immediately run into a problem: Given a discrete function or functional, the difference can be forward, backward, central, and even more complicated.
combinations of these differences, so that the first order derivatives or variational lead to many possibilities which can not be all true. In statistical physics, we are familiar with Boltzmann, Bose and Fermi system of many particles, in which there are a huge amount of distributions compatible with the constraint conditions. To obtain the MPD, the routine manner is to render the discrete problem into a continuous one with use of the Stirling approximation of the factorials, which is more accurate as the variables of factorial are larger. In the thermodynamic limit of an infinitely large number of particles, the obtained MPD is the true one. However, the fetal shortcoming of the routine manner lies in that the variables of factorial are frequently small, and they can even be one or two. To note that in mathematics, the Stirling approximation can be accurate when the variables can be very small, but in statistical physics, we always mean that \( \ln n! \approx n(\ln n - 1) \). It is understandable for the exact form of distributions for Boltzmann, Bose and Fermi system of finite numbers of particles has posed a formidable problem for long time. Nowadays, the statistical mechanics for finite number of particles attracts much attention. In present work, I report an exact discrete calculus of variations to give the exact distributions for the Boltzmann, Bose and Fermi system of finite numbers of particles.

In section II, we present a new procedure of discrete calculus of variations. In section III and IV, by use of this procedure, we show how to derive the Bose, Fermi and Boltzmann distribution without invoking the Stirling approximation, where the section IV focuses on the distributions with a few number of particles. In final section V, a brief conclusion is given.

II. A NEW DISCRETE CALCULUS OF VARIATIONS

At first, given a function \( f(x) \) of variable \( x \) on a discrete lattice. The first order finite differences of the function \( f(x) \) can be grouped into two groups. The first group is the forward difference: \( (\Delta^f)^1 f(x) = f(x + h) - f(x) \) \((h > 0)\), and the second group is the backward difference \( (\Delta^b)^1 f(x) = f(x) - f(x - h) \neq (\Delta^f)^1 f(x) \). This positive and finite constant \( h \) has different meanings in different situations. In numerical calculations of continuous functions, \( h \) may be determined by the computational accuracy. For the discrete function, \( h \) can not be arbitrarily chosen, and especially in discrete calculus of variations, \( h \) can only take the smallest distance between two nearest sites of the lattice. Other finite differences such as central one, \( (\Delta^{central})^1 f(x) = f(x + h/2) - f(x - h/2) \) and the general eccentric one \( (\Delta^{eccentric})^1 f(x) = f(x + (h - \varepsilon)) - f(x - \varepsilon), \varepsilon \in (0, h) \) are also mathematically definable, but irrelevant to our problems. For our functions are defined on the simple cubic lattice whose points lie at positions \((x_1, x_2, x_3, ..., x_k)\) in the \( k \)-dimensional Cartesian space, where \( x_1, x_2, x_3, ..., x_k \) are integers, and \( h \) is the lattice constant which can be taken as unit, \( h = 1 \), for convenience. In consequence, the second order differences belong to two groups respectively: the forward one, \( (\Delta^f)^2 f(x) = f(x + 2h) - f(x + h) - f(x) + f(x - h) = f(x + 2h) - 2f(x + h) + f(x) \), and the backward one \( (\Delta^b)^2 f(x) = f(x) - f(x - h) - f(x) + f(x - 2h) = f(x) - 2f(x - h) + f(x - 2h) \). A very important simple function appears as \( (\Delta^f)^1 f(x) = (\Delta^b)^1 f(x) \), we denote these functions by \( L \), which implies \( l(x) = \text{const.} \), or \( l(x) \) is linear in \( x \) once \( l(x) \in L \). In our treatment, the meaningful function \( l(x) \) can not be separated from \( f(x) \) which satisfies \( (\Delta^b)^1 f(x) \neq (\Delta^f)^1 f(x) \).

Secondly, we can introduce the sum of two functions \( f(x) = f_1(x) + f_2(x) \), and so forth, the sum of many functions. Once \( f(x) = f_1(x) + f_2(x) \), a sum of two functions and either of them is purely an \( l(x) \), the first order of differences \( \Delta f \) actually means four combinations:

\[
\begin{align*}
(\Delta^f)^1 f_1(x) + (\Delta^f)^1 f_2(x), & \text{ denoted by 1f2f, group 1,} \\
(\Delta^f)^1 f_1(x) + (\Delta^b)^1 f_2(x), & \text{ denoted by 1f2b, group 2,} \\
(\Delta^b)^1 f_1(x) + (\Delta^f)^1 f_2(x), & \text{ denoted by 1b2f, group 3,} \\
(\Delta^b)^1 f_1(x) + (\Delta^b)^1 f_2(x), & \text{ denoted by 1b2b, group 4.}
\end{align*}
\]

These combinations can be grouped into four distinct types, the usual \textit{synchronous finite differences} 1f2f and 1b2b, and the \textit{asynchronous finite differences} 1f2b and 1b2f, which were simply overlooked before. Accordingly, the second order of differences \( (\Delta^f)^2 f(x) \) for these four groups, 1f2f, 1f2b, 1b2f and 1b2b are given by, respectively,

\[
\begin{align*}
(\Delta^f)^2 f_1(x) + (\Delta^f)^2 f_2(x), & \text{ for 1f2f,} \\
(\Delta^f)^2 f_1(x) + (\Delta^b)^2 f_2(x), & \text{ for 1f2b,} \\
(\Delta^b)^2 f_1(x) + (\Delta^f)^2 f_2(x), & \text{ for 1b2f,} \\
(\Delta^b)^2 f_1(x) + (\Delta^b)^2 f_2(x), & \text{ for 1b2b.}
\end{align*}
\]
Similarly, we can define higher order of differences \((\Delta^j f) \ (j = 3, 4, 5, \ldots)\) for these four groups, but these cases do not interest us for the present.

Thirdly, we define the difference quotients of different orders. For a function, we have forward difference quotients of different orders \((\Delta^j f)/h, \ (\Delta^j f)/h^2, \ldots\), and backward difference quotients of different orders are, \((\Delta^j f)/h, \ (\Delta^j f)/h^2, \ldots\), respectively. With \(h = 1\), we have \((\Delta^j f)/h^i = (\Delta^j f)/h\) and \((\Delta^j f)/h^i = (\Delta^j f)/h\) \((i = 1, 2, 3, \ldots)\), and the difference quotients of second order for the sum of two functions \(f(x) = f_1(x) + f_2(x)\) are given in Eqs. \(24\) - \(26\).

Fourthly, we consider one discrete function \(\Psi(n)\) defined, for convenience, on the interval of semi-positive integers \(n \in \mathbb{Z}^+\), which has the local maxima and minima, subject to some equality constraints \(\psi = (\psi_1, \psi_2, \psi_3, \ldots) = 0\) and the constraints belong to \(L\). For finding the local maxima and minima of the function \(\Psi(n)\), we construct a functional \(\Phi\) which behaves as a function,

\[
\Phi = \Psi \{n\} + \alpha \cdot \psi,
\]

where \(\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)\) are Lagrange multipliers each of which \(\alpha_i\) companies a constraint condition \(\psi_i = 0\). The local maxima and minima satisfy,

\[
\delta \Phi = 0.
\]

Since the discreteness of the function \(\Psi(n)\), the smallest finite change of \(n\) is \(\Delta n = 1\), and we have accordingly two differences (or difference quotients) of the function \(\Psi\{n\}\) in the following:

\[
(\delta f) \Psi \{n\} \delta n = \Psi \{n + 1\} - \Psi \{n\} = (\Delta f) \Psi \{n\} \text{ for the forward difference and }
\]

\[
\left(\delta^i f\right) \Psi \{n\} / \delta n = \Psi \{n + 1\} - \Psi \{n - 1\} = (\Delta^i f) \Psi \{n\} \text{ for the backward difference.}
\]

In consequence, two relations \(n^\mu = n^\mu (\alpha) \ (\mu = 1, 2)\) solve the equation \(\Phi\), which contains both spurious and true one. Then how to determine the true one?

Assuming that all relations \(n^\mu = n^\mu (\alpha)\) have different values of the second order differences, and denoting \(n^1 = n^1 (\alpha)\) that solves the forward difference variation as \((\Delta f) \Psi \{n^1\} + \alpha \cdot \Delta^f \Psi \{n^1\} = 0\), the another solution \(n^2 = n^2 (\alpha)\) solves the backward difference variation as \((\Delta^b f) \Psi \{n^2\} + \alpha \cdot \Delta^b \Psi \{n^2\} = 0\). If only one solution leads to the maximal distribution, the true solution is obtained, the simplest case. Now we deal with complicated case. Assuming that \((\Delta f)^2 \Psi \{n^1\} < 0 \text{ and } (\Delta^b f)^2 \Psi \{n^2\} \neq (\Delta f)^2 \Psi \{n^1\}\) and \(n^2 (\alpha)\) can give maximal value for \(\Psi(n)\). If \(|(\Delta f)^2 \Psi \{n^1\}| > |(\Delta^b f)^2 \Psi \{n^2\}|\), the solution \(n^1 = n^1 (\alpha)\) is true, and vice versa.

In other words, the second order difference of the true solution take largest value among two maximal values \(\{|(\Delta f)^2 \Psi \{n^1\}|, |(\Delta^b f)^2 \Psi \{n^2\}|\}\).

Once the discrete function \(\Psi(n)\) is the sum of two functions \(\Psi_1(n)\) and \(\Psi_2(n)\), \(\Psi(n) = \Psi_1(n) + \Psi_2(n)\) and either of them belongs to \(L\), there are four relations \(n^\xi = n^\xi (\alpha) \ (\xi = 1, 2, 3, 4)\), solving, respectively,

\[
(\Delta f) \Psi_1 \{n^1\} + (\Delta f) \Psi_2 \{n^2\} + \alpha \cdot \Delta^f \Psi \{n^1\} = 0, \text{ for } 1f2f, \quad \text{(5a)}
\]

\[
(\Delta f) \Psi_1 \{n^2\} + (\Delta f) \Psi_2 \{n^3\} + \alpha \cdot \Delta^f \Psi \{n^2\} = 0, \text{ for } 1f2b, \quad \text{(5b)}
\]

\[
(\Delta^b f) \Psi_1 \{n^1\} + (\Delta^b f) \Psi_2 \{n^3\} + \alpha \cdot \Delta^b \Psi \{n^1\} = 0, \text{ for } 1b2f, \quad \text{(5c)}
\]

\[
(\Delta^b f) \Psi_1 \{n^2\} + (\Delta^b f) \Psi_2 \{n^4\} + \alpha \cdot \Delta^b \Psi \{n^2\} = 0, \text{ for } 1b2b. \quad \text{(5d)}
\]

In similar manner, we deal with only the complicated case. The true solution \(n^\xi = n^\xi (\alpha)\) must be that whose second order difference \(\Delta^2 \Psi \{n^\xi\}\) takes the largest value in magnitude among all solutions from four groups, 1f2f, 1f2b, 1b2f and 1b2b. I.e.,

\[
|\Delta^2 \Psi \{n^\xi\}| = \max \{|\Delta^2 \Psi \{n^1\}|, |\Delta^2 \Psi \{n^2\}|, |\Delta^2 \Psi \{n^3\}|, |\Delta^2 \Psi \{n^4\}|\}. \quad \text{(6)}
\]

Five immediate remarks follow. 1. In our procedure above, the correct solution and the true solution differ; and the former refers to its solving the first order equation \(\Phi\) and the latter refers to its maximizing the second order variations in magnitude. In contrast, the conventional procedure accepts all solutions once they are stationary. 2. Once the second order variations give infinite values in magnitude, the corresponding solution is acceptable. 3. If there are solutions which have no difference up to second differences such that we can not identify which is true, higher order differences can be invoked; and our procedure can be easily generalized for the discrete function \(\Psi(n)\) that is the sum of more functions. 4. If the constraint function \(\psi\) is nonlinear in \(n\), the problem must be treated in similar manner. 5. Our procedure is quite general, not limited to method of MPD, but the present application is limited to it.
It is worth to emphasize: a proper formalism of discrete calculus of variations must be taken account of both the synchronous finite differences and asynchronous finite differences, allowing for solution groups of finite differences based on all possible combinations of forward and backward difference. In the following section, I will illustrate the fundamental importance of the asynchronous finite differences with a detailed derivation of the exact form of the Bose distribution, and briefly discuss the Boltzmann and Fermi distribution.

III. ASYNCHRONOUS FINITE DIFFERENCES AND EXACT DISTRIBUTIONS

Considering a system of $N$ noninteracting, indistinguishable particles confined to a space of volume $V$ and sharing a given energy $E$. Let $\varepsilon_i$ denote the energy of $i$-th level and $\varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \ldots$, and $g_i$ denote the degeneracy of the level. In a particular situation, we may have $n_1$ particles in the first level $\varepsilon_1$, $n_2$ particles in the second level $\varepsilon_2$, and so on, defining a distribution set $\{n_i\}$. The number of the distinct microstates in set $\{n_i\}$ is then given by,

$$\Omega \{n_i\} = \prod_i \frac{(n_i + g_i - 1)!}{n_i! (g_i - 1)!}.$$  \hspace{1cm} (7)

The Bose functional $f$ is with two Lagrange multipliers $\alpha$ and $\beta$,

$$f = \sum_i (\ln(n_i + g_i - 1)! - \ln(n_i)! - \ln(g_i - 1)!) - \alpha \left( \sum_i n_i - N \right) - \beta \left( \sum_i n_i \varepsilon_i - E \right).$$  \hspace{1cm} (8)

The variational $\delta f$ is,

$$\delta f = \sum_i \left( \delta \ln(n_i + g_i - 1)! - \delta \ln(n_i)! - \delta \ln(g_i - 1)! \right) - \delta n_i (\alpha + \beta \varepsilon_i)$$

$$= \sum_i \delta n_i \left\{ \frac{\delta \ln(n_i + g_i - 1)!}{\delta n_i} - \frac{\delta \ln(n_i)!}{\delta n_i} - (\alpha + \beta \varepsilon_i) \right\}.$$  \hspace{1cm} (9)

Since the independence of variables $n_i$, $\delta f = 0$ leads to,

$$\frac{\delta \ln(n_i + g_i - 1)!}{\delta n_i} - \frac{\delta \ln(n_i)!}{\delta n_i} - (\alpha + \beta \varepsilon_i) = 0.$$  \hspace{1cm} (10)

Thus, there are essentially two functions $\Psi_1 = \ln(n_i + g_i - 1)!$ and $\Psi_2 = \ln n_i!$, and $-n_i (\alpha + \beta \varepsilon_i)$ is clearly linear in $n_i$ and can be a part of $\Psi_1$ or $\Psi_1$. For $\ln(n_i + g_i - 1)!$ and $\ln n_i!$. The smallest forward and backward differences are given in Table II. Four combinations constructing $\frac{\Delta \ln(n_i + g_i - 1)!}{\Delta n_i} - \frac{\Delta \ln n_i!}{\Delta n_i}$ are summarized in Table II. Accordingly, we have four solutions presented in Table III, and all satisfy $\delta f = 0$. These distributions are mutually different when $n_i \sim 1$, though all of them converge to the same one when $n_i \gg 1$,

$$n_i \approx \frac{g_i}{e^{\alpha + \beta \varepsilon_i} - 1}. \hspace{1cm} (11)$$

The second order variations of $\delta^2 \ln \Omega \{n_i\}$ for the four solutions are explicitly shown in the Table IV, which are crucial for us to identify the true solution among all possible ones in Table III. It is easily to verify that one combination Table II is right for we have, from results in Table IV,

\begin{align*}
\frac{n_i}{n_i - 1} \frac{n_i + g_i}{n_i + g_i + 1} &> \frac{n_i}{n_i - 1} \frac{n_i + g_i - 2}{n_i + g_i - 1} > 0, \quad (n_i > 1) \hspace{1cm} (12a) \\
\frac{n_i}{n_i - 1} \frac{n_i + g_i}{n_i + g_i + 1} &> \frac{n_i + 2}{n_i + 1} \frac{n_i + g_i}{n_i + g_i + 1} > 0, \quad (n_i > 1) \hspace{1cm} (12b) \\
\frac{n_i}{n_i - 1} \frac{n_i + g_i}{n_i + g_i + 1} &> \frac{n_i + 2}{n_i + 1} \frac{n_i + g_i - 2}{n_i + g_i - 1} > 0, \quad (n_i > 1). \hspace{1cm} (12c)
\end{align*}
Due to the independence of variables \( n_i \), care, which will be done in next section. So far, we reproduce the Bose distribution with \( n_i \geq 2 \), nevertheless, 

\[
n_i = \frac{g_i}{e^{\alpha + \beta \varepsilon_i} - 1}, \quad (n_i \geq 2).
\]

Next, let us briefly consider Boltzmann and Fermi system. For the Boltzmann system, the number of the distinct microstates in set \( \{ n_i \} \) is then given by, 

\[
\Omega \{ n_i \} = \prod_i \frac{(g_i)^{n_i}}{n_i!}.
\]

The variational calculation of \( \delta f \) is with two Lagrange multipliers \( \alpha \) and \( \beta \),

\[
\delta f = \sum_i \{ \delta n_i (\ln g_i - \alpha - \beta \varepsilon_i) - \delta \ln n_i! \} = \sum_i (\delta n_i (\ln g_i - \alpha - \beta \varepsilon_i) - \delta \ln n_i!).
\]

Due to the independence of variables \( n_i \), the first order variational equation \( \delta f = 0 \) amounts to following single equation,

\[
\frac{\delta f}{\delta n_i} = (\ln g_i - \alpha - \beta \varepsilon_i) - \frac{\delta \ln n_i!}{\delta n_i} = 0, \quad \text{i.e.,} \quad \frac{\delta \ln n_i!}{\delta n_i} = \ln g_i - \alpha - \beta \varepsilon_i,
\]

in which there is one function \( \ln n! \).

Through the second order variations, we can easily find that the exact distribution for the Boltzmann system \( n_i \geq 1 \) appears at backward difference of the function \( \ln n! \) as \( (\Delta^b)^i \ln n! = \ln n_i!/\delta n_i \) in Eq. (16), and the result is,

\[
n_i = g_i e^{- (\alpha + \beta \varepsilon_i)}, \quad (n_i \geq 1),
\]

which takes the same form as the conventional one.

For the Fermi system, the number of the distinct microstates in set \( \{ n_i \} \) is then given by,

\[
\Omega \{ n_i \} = \prod_i \frac{g_i!}{n_i!(g_i - n_i)!}.
\]

The variational calculation of \( \delta f \) is with two Lagrange multipliers \( \alpha \) and \( \beta \),

\[
\delta f = -\sum_i \{ \delta \ln n_i! + \delta \ln (g_i - n_i)! \} + \delta n_i (\alpha + \beta \varepsilon_i).
\]

Due to the independence of variables \( n_i \), the first order variational equation \( \delta f = 0 \) amounts to following single equation,

\[
\frac{\delta \ln (g_i - n_i)!}{\delta n_i} + \frac{\delta \ln n_i!}{\delta n_i} + (\alpha + \beta \varepsilon_i) = 0, \quad \text{i.e.,} \quad \frac{\delta \ln (g_i - n_i)!}{\delta n_i} + \frac{\delta \ln n_i!}{\delta n_i} = -\alpha - \beta \varepsilon_i,
\]
TABLE IV: The column and row give the forward/backward finite differences for $\ln(n_i + g_i - 1)!$ and $\ln n_i!$, respectively, and then forming $\Delta^2 \ln(n_i + g_i - 1)! - \Delta^2 \ln n_i!$ accordingly.

| $\Psi_1$ forward | $\Psi_2$ forward | $\Psi_1$ backward | $\Psi_2$ backward |
|------------------|-------------------|-------------------|-------------------|
| $\Psi_1$ forward | $-\ln\left(\frac{n_i + g_i + 1}{n_i + 1}\right)$ for 1f2f | $-\ln\left(\frac{n_i + g_i + 1}{n_i + 1}\right)$ for 1f2b | $-\ln\left(\frac{n_i + g_i + 1}{n_i + 1}\right)$ for 1b2f | $-\ln\left(\frac{n_i + g_i + 1}{n_i + 1}\right)$ for 1b2b |

in which there are two functions, and the first is $\ln(g_i - n_i)!$ and the second is $\ln n_i!$.

Also through the second order variations, the exact distribution for the Fermi system $n_i \geq 2$ appears at $\left(\Delta^f\right) \ln(g_i - n_i)! + \left(\Delta^b\right) \ln n_i! = -\ln(g_i - n_i) + \ln n_i \ln \left(\frac{g_i}{n_i}\right)$, which takes the same form as the conventional one, as well.

Clearly, our procedure puts no requirement on the total number of particle $N$ to be very large, though $n_i \geq 2$ is for Bose and Fermi system. However, the distributions with $n_i = 1$ for Fermi and Bose system can actually be obtained in similar manner, which will be discussed in next section.

### IV. Exact Distributions with Finite Numbers of Particles

The distributions with $n_i = 0$ are physically irrelevant. This is because two Lagrange multipliers $\alpha$ and $\beta$ are determined by two constraints,

$$N = \sum_{i=0}^{i_{\text{max}}} n_i, \quad E = \sum_{i=0}^{i_{\text{max}}} n_i \varepsilon_i, \quad (22)$$

in which there is no position for a term with $n_i = 0$. In general, we have $n_0 \geq n_1 \geq ... \geq n_{i_{\text{max}}}$ from distributions (13), (17), and (21). It is not necessary to assume $n_{i_{\text{max}}} = 1$, but it is convenient to assume so then its explicit dependence on $(\alpha, \beta, \varepsilon_i, g_i)$ is what we want to obtain. Introducing the probability $\rho_i$ via,

$$\rho_i \equiv \frac{n_i}{N} = \frac{n_i}{\sum_{i=0}^{i_{\text{max}}} n_i}, \quad (23)$$

we can prove that $n_{i_{\text{max}}} = 1$ is physically insignificant because its probability is completely ignorable due to the extremely high energy level.

First, let us deal with the possible exact distribution for Bose system with $n_{i_{\text{max}}} = 1$, from Eqs. (7)-(10). The first and second order variations of four solutions groups, and the results are presented in Table V. Clearly, there are two

TABLE V: Results for second order variations $\Delta^2 \ln(n_i + g_i - 1)! - \Delta^2 \ln n_i!$ for $n_i = 0$ and $n_i = 1$, respectively. The results with $n_i = 0$ are practically useless, and are given only for reference. For $n_i = 0$ second order variation runs into $\ln(-1)$ that is undefinable within the real numbers domain for both solutions, 1f2b and 1b2b, and we can not judge whether these two solutions are true or not.

| Solution | $\delta f = 0$ | $\delta^2 f$ |
|----------|----------------|--------------|
| 1f2f     | $\frac{g_{i_{\text{max}}}}{2 \exp(\alpha + \beta \varepsilon_{i_{\text{max}}}) - 1}$ | $-\ln\left(\frac{g_{i_{\text{max}}}^{1/2}}{g_i^{1/2}}\right)$ |
| 1f2b     | $\frac{g_{i_{\text{max}}}}{\exp(\alpha + \beta \varepsilon_{i_{\text{max}}}) - 1}$ | $-\infty$ |
| 1b2f     | $\frac{g_{i_{\text{max}}}}{2 \exp(\alpha + \beta \varepsilon_{i_{\text{max}}}) - 1}$ | $-\ln\left(\frac{g_{i_{\text{max}}}^{1/2}}{g_i^{1/2}}\right)$ |
| 1b2b     | $\frac{g_{i_{\text{max}}}}{\exp(\alpha + \beta \varepsilon_{i_{\text{max}}})}$ | $-\infty$ |

solutions, 1f2b and 1b2b, that are satisfactory for the second order variations are negatively infinitely large. Two solutions from the 1f2b and 1b2b with $n_i = 1$ are, respectively, from the results in Table V,

$$1 = \frac{g_{i_{\text{max}}}}{\exp(\alpha + \beta \varepsilon_{i_{\text{max}}}) - \delta}, \quad \delta = 1 \text{ for 1f2b, and } \delta = 0 \text{ for 1b2b}. \quad (24)$$
It appears to be an arbitrariness in my procedure for there are two possible solutions, 1f2b and 1b2b, and it seems to be no principle to exclude one of them. However, there must be two solutions to account for two different situations. On one hand, if there is only one particle in the system given $\alpha$ and $\beta$, no body can tell it is a Fermion or a Boson for there is no other particle to exchange with it within the system. In consequence, the distribution function must be Boltzmannian \(17\), which is what the solution 1b2b implies. On the other hand, if there are other particles in other energy levels, which can exchange with the particle in the highmost energy level $i_{\text{max}}$ and $n_{i_{\text{max}}} = 1$, the particle can be identified as a Boson. In consequence, we must take a non-Boltzmannian distribution, i.e., we are forced to choose another solution 1f2b in Table V,

$$n_i = \frac{g_i}{e^{\alpha+\beta e_i} - 1}, \quad (n_i \geq 1).$$ \(25\)

It is the exact form of the Bose distribution.

In similar manner, we can deal with Fermi system with $n_i = 1$, and we can also take usual distribution,

$$n_i = \frac{g_i}{e^{\alpha+\beta e_i} + 1}, \quad (n_i \geq 1),$$ \(26\)
as the exact form of the Fermi distribution.

Together with the exact form of the Boltzmann distribution \(17\), we can conclude that the usual forms of the distribution functions for Boltzmann, Bose and Fermi system are actually the exact ones. However, there is a fundamental difference between our procedure and the usual one is that ours is applicable to finite number of particles while the usual one holds for very large number of particles. Clearly, when there is only one particle in the system which keeps thermal contact with the heat bath and also keeps averagely one particle in it, the particle obeys the Boltzmann distribution.

V. CONCLUSIONS AND DISCUSSIONS

In contrast to the continuous calculus of variations, the discrete one possesses some peculiarities. A new discrete calculus is developed such that the finite difference of the discrete function can not be treated in the synchronously forward or backward, but must exhaust all possible combinations. By use of the new discrete calculus, we carefully examine the statistical distributions for Bose, Fermi and Boltzmann system, and demonstrate that usual form of the distribution functions holds true exactly for the number of particles is large than one. When there is only one particle in it, there is no way to distinguish it as Boson or Fermion, our theory automatically presents a result that the distribution is the Boltzmann one, which is also compatible with the recent experiment on the single particle engine.

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