ON AN INDEX THEOREM BY BISMUT

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Abstract. In this note we give a proof of an index theorem by Bismut. As a consequence we obtain another proof of the Grothendieck–Riemann–Roch theorem in differential cohomology.

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1. Introduction

The differential Grothendieck–Riemann–Roch theorem [7, Theorem 6.19], [11, Corollary 8.26], [13, Theorem 1] (abbreviated as dGRR) is a lift of the classical Grothendieck–Riemann–Roch theorem to differential cohomology. It states that for a proper submersion \( \pi : X \to B \) with closed spin\(^c\) fibers of even relative dimension, the following diagram commutes.

\[
\begin{array}{ccc}
\hat{K}(X) & \xrightarrow{\hat{\text{ch}}} & \hat{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q}) \\
\text{ind}\downarrow & & \downarrow \int_{X/B} \hat{\text{Todd}}(\hat{\nabla}^{T V} X) \ast (\cdot) \\
\hat{K}(B) & \xrightarrow{\hat{\text{ch}}} & \hat{H}^{\text{even}}(B; \mathbb{R}/\mathbb{Q})
\end{array}
\] (1)

Here \( \hat{K} \) is differential \( K \)-theory [5, 15, 11] and \( \hat{H} \) is Cheeger–Simons differential characters [9, 1].

In [13] the proof of the dGRR is to reduce it to an index theorem by Bismut [4, Theorem 1.15]:

\[
\hat{\text{ch}}(\ker(D^E), \nabla^{\ker}(D^E)) + i_2(\tilde{\eta}(E)) = \int_{X/B} \hat{\text{Todd}}(T^V X, \hat{\nabla}^{T V} X) \ast \hat{\text{ch}}(E, \nabla^E).
\] (2)

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One can regard (2) as a lift of the local family index theorem \[5\] to differential characters. Bismut’s proof of (2) involves certain adiabatic limits arguments given in \[5, 10\] and an Atiyah-Patodi-Singer index theorem in differential characters \[9, Theorem 9.2\]. In this note we give a proof of (2), which is inspired by \[1\] and does not make use of the above results.

Section 2 contains the background material needed in this note, including basic properties of differential characters, the construction of the index bundle and of the Bismut–Cheeger eta form. Section 3 contains the main result of this note.

2. Background Materials

2.1. Cheeger–Simons differential characters. We recall some basic properties of differential characters with coefficients in \(\mathbb{R}/\mathbb{Q}\), and refer the details to \[9, 1\].

Let \(X\) be a smooth manifold and \(k \geq 1\). A degree \(k\) differential character \(f\) with coefficients in \(\mathbb{R}/\mathbb{Q}\) is a group homomorphism \(f : Z_{k-1}(X) \to \mathbb{R}/\mathbb{Q}\) with a fixed \(\omega_f \in \Omega^k(X)\) such that for all \(c \in C_k(X)\), \(f(\partial c) = \int_c \omega_f \mod \mathbb{Q}\). The abelian group of degree \(k\) differential characters is denoted by \(\hat{H}^k(X; \mathbb{R}/\mathbb{Q})\). Denote by \(\Omega^k_{\mathbb{Q}}(X)\) the group of closed \(k\)-forms with periods in \(\mathbb{Q}\). It is easy to see that \(\omega_f \in \Omega^k_{\mathbb{Q}}(X)\) and is uniquely determined by \(f \in \hat{H}^k(X; \mathbb{R}/\mathbb{Q})\). Thus one can define a map \(\delta_1 : \hat{H}^k(X; \mathbb{R}/\mathbb{Q}) \to \Omega^k_{\mathbb{Q}}(X)\) by \(\delta_1(f) = \omega_f\). The map \(i_2 : \Omega^{k-1}_{\mathbb{Q}}(X) \to \hat{H}^k(X; \mathbb{R}/\mathbb{Q})\) defined by \(i_2(\alpha)(z) = \int_z \alpha \mod \mathbb{Q}\) is injective. In the following diagram, every square and triangle commutes, and the two diagonal sequences are exact.

The maps \(\delta_1\) and \(\delta_2\) are called the curvature and the characteristic class in literatures respectively.
There is a unique ring structure for $\tilde{H}^*(X;\mathbb{R}/\mathbb{Q})$ [1] Corollary 32, denoted by $\ast$. For a fiber bundle $\pi : X \to B$ with closed oriented fibers, the “integration along the fiber”, denoted by $\int_{X/B}$, exists and is unique [1] Theorem 39).

2.2. Index bundle and Bismut–Cheeger eta form. In this subsection we recall the construction of the index bundle and of the Bismut–Cheeger eta form. Our basic reference is [2].

Let $E \to X$ be a complex vector bundle with a Hermitian metric $h$ and $\nabla$ a unitary connection on $E \to X$. Let $\pi : X \to B$ be a proper submersion of even relative dimension $n$, and $T^V X \to X$ the vertical tangent bundle which is assumed to have a metric $g^{TV X}$. A given horizontal distribution $T^H X \to X$ and a Riemannian metric $g^{TB}$ on $B$ determine a metric on $TX \to X$ by $g^{TX} := g^{TV X} + \pi^* g^{TB}$. If $\nabla^{TX}$ is the corresponding Levi-Civita connection on $TX \to X$, then $\nabla^{TV X} := P \circ \nabla^{TX} \circ P$ is a connection on $T^V X \to X$, where $P : TX \to T^V X$ is the orthogonal projection. Assume the vertical bundle $T^V X \to X$ has a spin$^c$-structure. Denote by $S^V X \to X$ the spinor$^c$ bundle associated to the characteristic Hermitian line bundle $L^V X \to X$ with a unitary connection $\nabla^{L^V X}$. Define a connection $\widehat{\nabla}^{TV X}$ on $S^V X \to X$ by $\widehat{\nabla}^{TV X} := \nabla^{TV X} \otimes \nabla^{L^V}$, where $\nabla^{TV X}$ also denotes the lift of $\nabla^{TV X}$ to the local spinor bundle. The Todd form $\text{Todd}(\widehat{\nabla}^{TV X})$ of $S^V X \to X$ is defined by

$$\text{Todd}(\widehat{\nabla}^{TV X}) := \hat{A} (\nabla^{TV X}) \wedge e^{\frac{1}{2} c_1 (\nabla^{L^V X})}.$$ 

The Bismut–Cheeger eta form $\tilde{\eta}(\mathcal{E}) \in \frac{Q_{\text{odd}}(B)}{\text{Im}(d)}$ associated to $\mathcal{E} := (E, h, \nabla)$ is defined as follows. Consider the infinite-rank superbundle $\pi_* E \to B$, where the fibers at each $b \in B$ is given by

$$(\pi_* E)_b := \Gamma(X_b, (S^V X \otimes E)|_{X_b}).$$

Recall that $\pi_* E \to B$ admits an induced Hermitian metric and a connection $\nabla^{\pi_* E}$ compatible with the metric [2] §9.2, Proposition 9.13]. For each $b \in B$, the canonically constructed Dirac operator

$$D^E_b : \Gamma(X_b, (S^V X \otimes E)|_{X_b}) \to \Gamma(X_b, (S^V X \otimes E)|_{X_b})$$

gives a family of Dirac operators, denoted by $D^E : \Gamma(X, S^V X \otimes E) \to \Gamma(X, S^V X \otimes E)$. Assume the family of kernels $\text{ker}(D^E_b)$ has locally constant dimension, i.e., $\text{ker}(D^E) \to B$ is a finite-rank Hermitian superbundle. Let $P : \pi_* E \to \text{ker}(D^E)$ be the orthogonal projection, $h^\text{ker}(D^E)$ be the Hermitian metric on $\text{ker}(D^E) \to B$ induced by $P$, and $\nabla^{\text{ker}(D^E)} := P \circ \nabla^{\pi_* E} \circ P$ be the connection on $\text{ker}(D^E) \to B$ compatible to $h^\text{ker}(D^E)$. 

The scaled Bismut superconnection $A_t : \Omega(B, \pi_* E) \to \Omega(B, \pi_* E)$ [3, Definition 3.2] (see also [2, Proposition 10.15] and [10, (1.4)]), is defined by

$$A_t := \sqrt{t} D_E + \nabla^{\pi_* E} - \frac{c(T)}{4\sqrt{t}},$$

where $c(T)$ is the Clifford multiplication by the curvature 2-form of the fiber bundle. The Bismut–Cheeger eta form $\tilde{\eta}(\mathcal{E})$ [5, (2.26)] (see also [10] and [2, Theorem 10.32]) is defined by

$$\tilde{\eta}(\mathcal{E}) := \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} \text{Str} \left( \frac{dA_t}{dt} e^{-A_t^2} \right) dt.$$

The local family index theorem states that

$$d\tilde{\eta}(\mathcal{E}) = \int_{X/B} \hat{\text{Todd}}(\hat{\nabla}^T V^X) \wedge \hat{\text{ch}}(\nabla^E) - \hat{\text{ch}}(\nabla^{\ker(D_E)}).$$

Let $f : \tilde{B} \to B$ be a smooth map. In [7, §2.3.2] the pullback of the above geometric data is studied and in particular the Bunke eta form ([6, Definition 2.2.16]) is shown to respect pullback. Since the Bismut–Cheeger eta form is a special case of the Bunke eta form, we have

$$\tilde{\eta}(f^* \mathcal{E}) = f^* \tilde{\eta}(\mathcal{E}).$$

One can prove (5) directly as in [7, §2.3.2].

3. Main result

In this section we prove the main result in this note. Let $E \to X$ be a Hermitian bundle with a Hermitian metric $h$ and $\nabla$ a unitary connection on $E \to X$. Throughout this section we write $\mathcal{E}$ for $(E, h, \nabla)$.

As in [13] it suffices to prove (2) in the special case where the family of kernels of the Dirac operators has constant dimension, i.e., $\ker(D_E) \to B$ is a superbundle. The general case of (2) follows from a standard perturbation argument as in [11, §7] and its proof is essentially the same as the special case.

**Proposition 1.** Let $\pi : X \to B$ be a proper submersion with closed spin$^c$ fibers of even relative dimension. For any $\mathcal{E}$, the differential character

$$f := \int_{X/B} \hat{\text{Todd}}(\hat{\nabla}^T V^X) \wedge \hat{\text{ch}}(E, \nabla) - \hat{\text{ch}}(\ker(D_E), \nabla^{\ker(D_E)})$$

is uniquely characterized by the following conditions:

1. naturality, i.e., respect pullback,
2. compatibility with curvature; i.e.,

$$\delta_1(f) = \int_{X/B} \hat{\text{Todd}}(\hat{\nabla}^T V^X) \wedge \hat{\text{ch}}(\nabla) - \hat{\text{ch}}(\nabla^{\ker(D_E)}),$$

3. compatibility with characteristic class; i.e., $\delta_2(f) = 0$. 

(4) compatibility with topological trivialization; i.e., for any \( k \in \mathbb{N} \) and \( \beta \in \Omega^k(X) \), \( i_2(\beta) \ast f = i_2(\beta \wedge \delta_1(f)) \).

(5) compatibility with topological trivialization of flat characters; i.e., for any fiber bundle \( \pi : B \to B' \) with closed fibers \( F \) the following diagram commutes:

\[
\begin{array}{ccc}
\Omega_{d=0}^{\text{odd}}(B) & \xrightarrow{i_2} & \hat{H}^{\text{even}}(B; \mathbb{R}/\mathbb{Q}) \\
\left\downarrow f_{B/B'} \right. & & \left\downarrow \right. f_{B/B'} \\
\Omega_{d=0}^{\text{odd}} - \text{dim}(F)(B') & \xrightarrow{i_2} & \hat{H}^{\text{even}} - \text{dim}(F)(B'; \mathbb{R}/\mathbb{Q})
\end{array}
\]

**Proof.** Note that condition 1 and 2 ensure the uniqueness of differential characteristic classes \([12, \text{Proposition 3.1}]\); condition 1, 2, 3 and 4 ensure the uniqueness of ring structure of differential characters \([11, \text{Corollary 32}]\) and condition 1, 2, 3, 5 ensure the uniqueness of integration along the fibers \([11, \text{Theorem 39}]\). Obviously \( f \) is natural (see the discussion at the end of Section 2). Thus the differential character (6) is unique in the sense that it satisfies condition 1 to 5. \( \square \)

**Remark 1.**

(1) Consider the term \( \hat{c}(\ker(D^E), \nabla^{\ker(D^E)}) \) in (6). Although it does not need condition 3, 4 and 5 in Proposition \([11]\) to guarantee its uniqueness, it automatically satisfies these conditions.

(2) One can also prove the uniqueness of (6) along the lines of \([12, \text{Proposition 3.1}], [11, \text{Theorem 31}]\) and \([11, \text{Theorem 39}]\).

To prove Bismut’s theorem; i.e., (2) holds, it remains to verify \( i_2(\tilde{\eta}(E)) \) satisfies the conditions 1 to 5 in Proposition \([11]\).

**Proposition 2.** Let \( \pi : X \to B \) be a proper submersion with closed spin\(^c\) fibers of even relative dimension. For any \( E \), the differential character \( i_2(\tilde{\eta}(E)) \in \hat{H}^{\text{even}}(B; \mathbb{R}/\mathbb{Q}) \), where \( \tilde{\eta}(E) \in \frac{\Omega_{d=0}^{\text{odd}}(B)}{\text{Im}(d)} \) is the Bismut–Cheeger eta form of \( E \), satisfies conditions 1 to 5 in Proposition \([11]\).

**Proof.** As in Remark \([11]\) we only need to verify that \( i_2(\tilde{\eta}(E)) \) satisfies condition 1 and 2 in Proposition \([11]\) because these two conditions alone guarantee the uniqueness of the value of \( i_2(\tilde{\eta}(E)) \) on each \( z \in Z_{\text{odd}}(B) \). Condition 3, 4 and 5 are not needed for its uniqueness, but of course it automatically satisfies these three conditions.

The naturality of \( i_2(\tilde{\eta}(E)) \) follows from \([5]\). The curvature that \( i_2(\tilde{\eta}(E)) \) is equal to \([7]\) follows from the commutativity of the lower triangle of \([3]\) and the local family index theorem \([11]\). \( \square \)

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