How much random a random network is: a random matrix analysis

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We analyze complex networks under random matrix theory framework. Particularly, we show that \( \Delta_3 \) statistic, which gives information about the long range correlations among eigenvalues, provides a qualitative measure of randomness in networks. As networks deviate from the regular structure, \( \Delta_3 \) follows random matrix prediction of linear behavior, in semi-logarithmic scale with the slope of \( 1/\pi^2 \), for the longer scale.

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Triggered by recently available data on large real world networks (e.g. structure of the Internet or molecular networks in the living cell) combined with increasing computer power, an avalanche of qualitative research on network structure and dynamics are currently stimulating diverse scientific fields such as web, nervous systems, cellular metabolism, scientific collaborations, Internet, human language etc. [1, 2, 3]. Results obtained so far span areas from the prevention of computer viruses to the stability and diversity of systems such as Internet, regulatory circuits of genome and ecosystems. These real world networks have several universal features, like small diameter, large clustering coefficient, scale-free degree distribution, assortative or disassortative mixing of the nodes, module structures [4], etc. Irrespective of real world networks having one or more above mentioned features, one thing is common in all of them, and that is the existence of some amount of randomness or disorder in the connections structure. According to many recent studies, randomness in connections is one of the most important and desirable ingredient for the proper functionality or the efficient performance of systems having underlying network structure [1, 2, 3]. For instance, information processing in brain is considered to be highly influenced by random connections among different modular structure [3], other examples of recent studies suggest the importance of randomness in various fields, such as evolution of language capacities of different species [6], evolution of cooperation in game theory [7], etc. On one hand these studies emphasize on the randomness in systems, on the other hand importance of structure or regularity is known for functional performance [4].

Seminal work of Watts and Strogatz shows that a very small amount of randomness has drastic impact on the diameter of network, leading to the so called small-world (SW) phenomenon [4]. Starting with a regular one dimensional lattice, random rewiring of connections leads to the SW behavior for very small rewiring probability \( p \). This simple rewiring scheme preserves the regular lattice structure yielding high clustering coefficients, while increases in the number of random connections among distant nodes resulting in the decrease of diameter of the network. Though this model network captures two most important characteristics of the real world networks, the whole structure of the network still remains a very regular one-d lattice type, whereas real world networks are far more complex with a large number of random connections or seemingly random connections. The question arises whether one can identify or characterize the level of randomness in the complex networks? There could be various possible ways to look into the problem. Recently L. da F. Costa has attempted to characterize randomness by looking for regular patterns in networks [10]. The purpose of present work is to have a qualitative measure of randomness in networks using eigenvalue fluctuation statistics of underlying adjacency matrix.

Eigenvalues distributions have been studied profusely in the literature [11, 12]. They have certain signatures of the underlying network structures, like for complete random network it follows Wigner semi-circular law, for scale-free it follows triangular shape and for the small-world networks spectral distribution has multi-peaks structure [12, 13]. In this paper we are interested in comparing the randomness in the network structure and except for the above mentioned typical network structures density distributions do not contain much information about the network structures, particularly it is not able to address the issue of “how much randomness”.

Our earlier contributions [13, 14, 15] have shown that the spacing distributions of complex random networks follow universal behavior of random matrix theory (RMT). In this paper we characterize the level of randomness in networks by using spectral rigidity test of RMT. Range for which \( \Delta_3 \) statistics follows random matrix prediction increases with the increase in random connections in the network.

RMT was proposed by Wigner to explain the statistical properties of nuclear spectra [16]. Later this theory was successfully applied in the study of different complex systems including disordered systems, quantum chaotic systems, spectra of large complex atoms, etc [17]. More recently, RMT has been applied successfully to analyze time-series data of stock-market [18, 19], atmosphere [20], human EEG [21], and many more [17, 22]. Nearest neighbor spacing distribution (NNSD) of the eigenvalues fol-

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lows two universal properties depending upon the underlying correlations among the eigenvalues. For correlated eigenvalues, the NNSD follows Wigner-Dyson formula of Gaussian orthogonal ensemble (GOE) statistics of RMT; whereas, the NNSD follows Poisson statistics of RMT for uncorrelated eigenvalues.

We denote the eigenvalues of the adjacency matrix of network by \( \lambda_i, \; i = 1, \ldots, N \), where \( N \) is the size of the network and \( \lambda_i > \lambda_j \forall i \). Eigenvalues are unfolded using the technique described in [13, 14, 17]. Using the unfolded spectra \( \{ \bar{\lambda}_i \} \), we calculate the nearest-neighbor spacings as

\[
s^{(i)} = \bar{\lambda}_{i+1} - \bar{\lambda}_i;
\]

The NNSD \( P(s) \) for the case of GOE statistics

\[
P(s) = \frac{\pi}{2} s \exp \left( -\frac{\pi s^2}{4} \right). \tag{1}
\]

The NNSD reflects only local correlations among the eigenvalues. The spectral rigidity, measured by the \( \Delta_3 \)-statistic of RMT, provides information about the long-range correlations among the eigenvalues and is more sensitive test for RMT properties of the matrix under investigation [16, 23]. In the following, we describe the procedure to calculate this quantity.

The \( \Delta_3 \)-statistic measures the least-square deviation of the spectral staircase function representing the cumulative density \( N(\bar{\lambda}) \) from the best straight line fitting for a finite interval \( L \) of the spectrum, i.e.,

\[
\Delta_3(L; x) = \frac{1}{L} \min_{a,b} \int_x^{x+L} [N(\bar{\lambda}) - a\bar{\lambda} - b]^2 d\bar{\lambda} \tag{2}
\]

where \( a \) and \( b \) are obtained from a least-square fit. Average over several choices of \( x \) gives the spectral rigidity \( \Delta_3(L) \). For GOE, \( \Delta_3(L) \) depends logarithmically on \( L \), i.e.,

\[
\Delta_3(L) \simeq \frac{1}{\pi^2} \ln L. \tag{3}
\]

Following we describe the method to generate networks with different randomness. Starting with one dimension ring lattice of \( N \) nodes, in which every node is connected to its \( k/2 \) nearest neighbors, we randomly rewire each connection of the lattice with the probability \( p \) such that self and multiple connections are excluded. Thus \( p = 0 \) corresponds to a regular network, and \( p = 1 \) gives a completely random network. The typical small-world behavior is observed around \( p_c = 0.002 \) [13]. Our earlier papers [13, 15] show that at this value of \( p = p_c \) NNSD follows universal GOE prediction of RMT. Universal GOE behavior suggests that there exists a minimal amount of randomness in the networks yielding to the short range correlations among the corresponding eigenvalues. With the increase in \( p \), obviously randomness increases in the network, but NNSD is not able to provide this information of enhancement of randomness.

As discussed in the introduction, NNSD only gives the information about the correlations among the neighboring eigenvalues. It does not tell anything about the long range correlations among the eigenvalues. We probe long range correlations among the eigenvalues using \( \Delta_3 \) statistic, which tells that how closely the network follows ideal behavior of GOE of random matrices. For different values of \( p \geq p_c \), we construct several networks of size \( N = 2000 \) and average degree \( k = 20 \) With increase in the value of \( p \), number of random rewired connections increases. We study the spectral rigidity of networks generated for various \( p \) and then present the results for ensemble average of networks for each \( p \) value.

We start with the network generated for \( p = p_c = 0.002 \), below this value of \( p \) spectral properties of networks can not be modeled by GOE statistics of RMT (see our earlier papers [13, 14]). At \( p = p_c \), there exists minimal amount of randomness sufficient to create correlations among the eigenvalues yielding GOE statistics of
NNSD. Figure 1 plots spectral distribution of several networks generated by Watts-Strogatz algorithm with different $p \geq p_c$ values. As $p$ increases, local regular structure destroys and random connections among nodes increase.

Figure 2 plots NNSD distribution and $\Delta_3$ statistic of eigenvalues spectra of these networks. $\Delta_3(L)$ is calculated following Eq. (2) respectively. It can be seen from the plots in Figs. 2(a)-(d), that after $p = p_c$, NNSD remains same with universal GOE statistics, and we do not infer anything more about the randomness in the network. Though NNSD does not address the question of how much randomness, it contains a very important information; that various other aspects of RMT can now be applied to study these networks.

Figures 2(e)-(h) plot $\Delta_3$ statistic of several networks with different $p$ values. As it is seen from the figures $\Delta_3$ statistic has consistent different behavior for different $p$ values. Figs 2(a)-(d) are plotted in the increasing order of the $p$ values, with (a) being plotted at the transition to the small-world behavior. It can be seen from Figures 1(a) - (c), that the density distribution of eigenvalues for $p$ values ($0.002$-$0.02$) are very similar with typical multi-peaks type of structure shown by small world networks. One can infer from these figures that which network has high random connections and which one has less. NNSD plots are also same for these networks. But interesting information comes out when we look at the spectral rigidity plots.

Figure 2(e) shows that the spectral rigidity follows GOE predictions upto range $L \sim 30$. As $p$ is increased, the scale for which spectral rigidity follows GOE prediction, in general, increases. Figure 2(f) is plotted for $p = 0.01$, for this value of $p$, $\Delta_3(L)$ statistic following RMT prediction for very large scale ($L \sim 135$). For $N = 2000$, we can not have meaningful $\Delta_3$ statistic for the larger $L$ (see text). Hence for $p > 0.05$, $p = 0.1$ $0.2$ and $1.0$: $\Delta_3(L)$ statistic following RMT prediction for the scale $L \sim 135$. NNSD follows RMT predictions, $\Delta_3(L)$ statistic following RMT prediction only upto $L \sim 30$. (b) $p = 0.01$ : spectral density (1(b)) and NNSD show similar behavior as for the previous $p$ value while $\Delta_3(L)$ statistic following RMT prediction for larger scale, upto $L \sim 40$. (c) $p = 0.02$ : $\Delta_3(L)$ statistic following RMT prediction upto $L \sim 75$. (d) $p = 0.05$ : for such a small value of $p$, $\Delta_3(L)$ statistic following RMT prediction for very large scale ($L \sim 135$). For $N = 2000$, we can not have meaningful $\Delta_3(L)$ statistic following RMT prediction for the scale $L \sim 135$. More generally, for $p > 0.05$, $\Delta_3(L)$ statistic does not follow RMT predictions, is increased, local regular structure destroys and random connections among nodes increase.

Figure 2 plots $\Delta_3(L)$ statistic following RMT prediction for larger scale, characteristic of small-world networks. One can not infer from these figures that which network has high random connections and which one has less. NNSD plots are also same for these networks. But interesting information comes out when we look at the spectral rigidity plots.

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We study complex networks under random matrix theory framework and use $\Delta_3$ statistics to provide a measure of deviation from the regular structure. Starting with a regular lattice, i.e. a network with complete regular structure, some connections are randomly rewired with probability $p$. Larger values of $p$ indicate more deviation from regular structure in the network. In this manner several networks are generated with different deviations from the regular structure. According to the RMT $: \Delta_3$ statistic, which measures long range correlations among the eigenvalues, provides insight about the randomness in the corresponding matrix. For a complete random matrix, eigenvalues are correlated till very long range. We use this result of eigenvalues correlation statistics to distinguish the randomness among the networks, and show that $\Delta_3$ can be used as a qualitative measure of randomness. With the increase in the value of $p$, the length scale for which $\Delta_3$ follows GOE prediction of RMT also increases. In the semi-logarithmic plots, the slope of $\Delta_3$ (Eq. 3) matches exactly the GOE predictions of $\sim 1/\pi^2$.

Interestingly, for the networks generated with $p \sim 0.05$, $\Delta_3$ statistic already starts following GOE prediction for the scale as long as for the networks with $p \sim 1$. According to the RMT, this tells that the eigenvalues of the networks generated with $p \sim 0.05$ are as much correlated as for the networks generated with $p \sim 1$, which are complete random networks. We interpret this result as following: for $p \sim 0.05$ there is some kind of spreading of randomness over the network yielding the correlations among the eigenvalues as large as for the networks with $p = 1$.

To conclude, continuing our RMT analysis of complex networks, this paper uses $\Delta_3$ statistic to study the randomness in the networks. It is shown that RMT is particularly suitable to study this important aspects of complex networks, thus on one hand enhancing the applicability of RMT, on the other hand opening a completely new domain to the complex networks studies. We hope that the complexity of the systems having network structures can be addressed under the RMT framework, and thus helping to understand the dynamical behavior and robustness of such networks better.

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