On a hyperbolic system of equations in the problem of unsteady fluid motion

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Abstract. We consider a nonlinear system of the first-order partial differential equations with constant coefficients admitting a solution of the traveling wave type. Such systems are used in applied research to describe the dynamics of unsteady water movement in a pressure pipeline. In this paper, we propose methods for obtaining exact solutions in the case of an inhomogeneous equation of fluid motion under restrictions on the right-hand side in the form of a function. A special feature of the present research is reduction of the problem under study to the equations integrable in elementary functions.

1. Introduction
The following system of the first-order partial differential equations [1] is well-known in the theory of hydrodynamics:

\[
\frac{\partial p}{\partial t} + \alpha_0 \frac{\partial x}{\partial t} + \alpha_1 |x| \cdot x = 0, \quad (1)
\]

\[
\frac{\partial x}{\partial t} + \alpha_2 \frac{\partial p}{\partial t} = 0. \quad (2)
\]

This system describes the unsteady movement of water in a pressure pipeline. Here, \(\alpha_i > 0\) are the known coefficients \((i = 0, 2)\), associated with the inner diameter of the water pipeline, the transmitted media density and the pressure waves propagation velocity; \(p = p(l, t)\) and \(x = x(l, t)\) are unknown coefficients characterizing the pressure and mass rate, respectively, and depending on the time \(t\) and the spatial variable coordinate \(l\).

Even though the list of references addressing the solution of system (1), (2) is huge (see, for example, the surveys [2], [3] and references therein), methods for its solution remain an area of primary focus for many researchers. A detailed survey of methods for calculating hydraulic processes is given in [4, pp. 25-29]. Approximate methods for solving (1), (2) are based on using characteristic properties of these equations. The most popular techniques include numerical methods for modeling transient processes (the method of characteristics and the wave characteristic method) based on Eulerian and Lagrangian approaches [5], [6]. A defining property of hyperbolic equations is a finite velocity of the wave propagation in the area of integration (see the fundamental work [7]). Precisely this particular feature is used in the approach [4, p. 38], which is based on application of the Riemann invariants.
Nevertheless, unresolved issues associated with the performance and reliability of computing procedures for dynamic characteristics call for modification of existing and development of new methods of solution for system (1), (2). Consider new ways to obtain exact solutions for (1) with a nontrivial right-hand side that has restrictions imposed on its type.

2. Problem statement
Consider the case of an inhomogeneous equation of fluid motion, so that instead of (1) we have

\[ \frac{\partial p}{\partial t} + \alpha_0 \frac{\partial x}{\partial t} + \alpha_1 |x| \cdot x + f = 0, \]

where \( f = f(l, t) \) is a control action.

In this paper, we study the problem of searching \( x(l, t), p(l, t) \) from (2), (3) taking into account the traveling wave. Traveling wave solutions for partial differential equations are widely used in modeling the dynamics of systems of various physical nature (see for example, [8], [9]).

Introduce the ratio \( \lambda_k \), characterizing the propagation velocity of a traveling wave [10]. Note that for \( f(l, t) \equiv 0 \) system (1), (2) has a solution of the following form:

\[
x(l, t) = \frac{\left( \frac{k}{\lambda} \right)^2 - \alpha_0 \alpha_2}{\alpha_1 \alpha_2 \left( \frac{k}{\lambda} l - t - C_2 \right)}, \quad p(l, t) = \frac{\left( \frac{k}{\lambda} \right) \left( \left( \frac{k}{\lambda} \right)^2 - \alpha_0 \alpha_2 \right)}{\alpha_1 \alpha_2 \left( \frac{k}{\lambda} l - t - C_2 \right)} + C_1 \text{ by } x(l, t) > 0,
\]

\[
x(l, t) = -\frac{\left( \frac{k}{\lambda} \right)^2 - \alpha_0 \alpha_2}{\alpha_1 \alpha_2 \left( \frac{k}{\lambda} l - t - C_2^* \right)}, \quad p(l, t) = -\frac{\left( \frac{k}{\lambda} \right) \left( \left( \frac{k}{\lambda} \right)^2 - \alpha_0 \alpha_2 \right)}{\alpha_1 \alpha_2 \left( \frac{k}{\lambda} l - t - C_2^* \right)} + C_1 \text{ by } x(l, t) < 0,
\]

where \( C_1, C_2, C_2^* \) are integration constants, which are determined from the initial conditions.

Assuming that

\[ f(l, t) = F(y), \quad y = kl - \lambda t, \]

(4)

define the situations that help find the desired functions \( x(l, t) \equiv u(y), p(l, t) \equiv v(y) \) from (2), (3) explicitly.

3. The problem solution
Note that under condition (4) the equation (3) can be reduced to the following form

\[ au'(y) + \alpha_1 |u(y)| \cdot u(y) + F(y) = 0, \quad a = \frac{k^2}{\alpha_2 \lambda} - \alpha_0 \lambda. \]

(5)

We will search for solutions to (2), (3) assuming that

\[ F(y) = By^m, \]

(6)

where \( B \) is some number. Further, consider three cases for different values of \( m \).

3.1. Case \( m = 0 \)
Let \( m = 0 \) in (6). Then, instead of (5), we have

\[ au'(y) + \alpha_1 |u(y)| \cdot u(y) + B = 0. \]

(7)
Equation (7) can be written in the form:

\[ u'(y) + \frac{\alpha_1}{a} u^2(y) + \frac{B}{a} = 0 \quad \text{by} \quad u(y) > 0, \quad (8) \]
\[ u'(y) - \frac{\alpha_1}{a} u^2(y) + \frac{B}{a} = 0 \quad \text{by} \quad u(y) < 0. \quad (9) \]

Solving (8), (9) by separating variables, we obtain

\[ u(y) = \frac{\sqrt{B}}{\sqrt{\alpha_1}} \tan \left( \frac{\sqrt{B}(C_1 - y)}{a \sqrt{\alpha_1}} \right) \]

and

\[ u(y) = \frac{\sqrt{B} C_2 e^{-\frac{\sqrt{B}(C_1 - y)}{\alpha_1}}} {\sqrt{\alpha_1} C_2 e^{-\frac{\sqrt{B}(C_1 - y)}{\alpha_1}} - \sqrt{\alpha_1}} \hat{C}_2 = e^{\frac{2 \sqrt{B} C_2}{\alpha_1}}, \]

respectively. Moreover, from (2) we obtain

\[ v(y) = \frac{k u(y) - C_1}{\alpha_2 \lambda}. \quad (10) \]

Here and in what follows, \( C_1, C_2 \) are arbitrary constants associated with the initial conditions for (2), (3). Returning to the original variables \( l \) and \( t \), the solution to system (2), (3) takes the form

\[ x(l, t) = \frac{\sqrt{B} \hat{C}_2 e^{-\frac{\sqrt{B}(k l - \lambda t)}{\alpha_1}}} {\sqrt{\alpha_1} \hat{C}_2 e^{-\frac{\sqrt{B}(k l - \lambda t)}{\alpha_1}} - \sqrt{\alpha_1}} \]
\[ p(l, t) = \frac{k x(l, t) - C_1}{\alpha_2 \lambda}. \quad (11) \]

3.2. Case \( m = -2 \)

For \( m = -2 \), instead of (5), we have

\[ u'(y) + \frac{\alpha_1}{a} \left| u(y) \right| u(y) + \frac{B}{a y^2} = 0, \]

which, using the substitute

\[ u(y) = \frac{w(y)}{y}, \]

can be reduced to the equation

\[ w'(y) + \left( \frac{\alpha_1}{a} \left| \frac{w(y)}{y} \right| - \frac{1}{y} \right) w(y) + \frac{B}{a y} = 0 \]

for an unknown function \( w(y) \). As the result, the problem of finding \( w(y) \) is reduced to the set of equations with separable variables:

\[ \frac{adw}{\alpha_1 |w| \cdot w - aw + B} + \frac{dy}{y} = 0 \quad \text{by} \quad y > 0, \]
\[ \frac{adw}{-\alpha_1 |w| \cdot w - aw + B} + \frac{dy}{y} = 0 \quad \text{by} \quad y < 0. \]
For $y \neq 0$, each equation of this set has a solution of the following form

$$\frac{a}{\alpha_1} \int \frac{dw}{|w|} \cdot w - \frac{a}{\alpha_1} w + \frac{B}{\alpha_1} - \ln y = C_2$$

and

$$-\frac{a}{\alpha_1} \int \frac{dw}{|w|} \cdot w + \frac{a}{\alpha_1} w - \frac{B}{\alpha_1} - \ln y = C_2,$$

respectively. Returning to the original notation, we have $x(l,t) = \frac{w(y)}{y}$, and $p(l,t)$ has the form of (11). As before, $v(y)$ is defined by (10).

3.3. Case $m \neq 0$, $m \neq -2$, $\frac{m}{2m + 4} \in \mathbb{Z}$.

In this case, $x(l,t) = u(y)$, where $u(y)$ is a solution to the equation

$$au'(y) + \alpha_1 |u(y)| \cdot u(y) + By^m = 0. \tag{12}$$

Equation (12) is reduced to a special Riccati equation, which, in turn, is integrable in elementary functions under the condition for the power index $m$, so that the ratio $\frac{m}{2m + 4}$ is an integer number, i.e. $\frac{m}{2m + 4} \in \mathbb{Z}$ (a recommendation for the solution is given in [11]).

4. Conclusion

In this paper, we considered a system of hyperbolic differential equations that comprises the fluid motion and balance equations. The practical significance of this system is connected to calculating hydrodynamic processes in pressure pipeline systems. In particular, we studied an inhomogeneous equation of fluid motion with a special right-hand side and described situations when the original problem can be reduced to equations integrable in elementary functions.

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