Thermodynamics of $osp(1|2)$ Integrable Spin Chain: Finite Size Correction

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Abstract

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The thermodynamic Bethe ansatz (TBA) equation for an integrable spin chain related to the Lie superalgebra $osp(1|2)$ is analyzed. The central charge determined by low temperature asymptotics of the specific heat can be expressed by the Rogers dilogarithmic function, and identified to be 1. Solving the TBA equation numerically, we evaluate the several thermodynamic quantities. The excited state TBA equation is also discussed.

KEYWORDS: central charge, finite size correction, Lie superalgebra, $osp(1|2)$, quantum transfer matrix, Rogers dilogarithm, string hypothesis, thermodynamic Bethe ansatz, $T$-system

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1 Introduction

Recently, thermodynamics of quantum integrable spin chains related to Lie superalgebras received much attentions. In particular, several people [1, 2, 3, 4, 5] studied thermodynamic Bethe ansatz (TBA) equations [3] related to $sl(r|s)$. On the other hand, study on $osp(r|2s)$ case has begun only recently in refs. [7, 8] (cf. ref. [9]), in which we deal with the simplest $osp(1|2)$ model [10, 11]. Namely, we have derived the $Y$-system from the $osp(1|2)$ version of the $T$-system [12], and transformed it into the TBA equation and excited state TBA equation from the point of view of the quantum transfer matrix (QTM) method [13]. As for the largest eigenvalue sector of the dressed vacuum form (DVF), this TBA equation coincides with the one [16] from the traditional string hypothesis [14, 15]. On the other hand, the excited state TBA equation from the second largest eigenvalue sector, which characterizes the correlation length, is difficult to derive by the string hypothesis. The purpose of this paper is applications of our previous results [4, 5] to the calculation of physical quantities.

The Hamiltonian [16] of the $osp(1|2)$ integrable spin chain, which we deal with in this paper is given by

$$H = J \sum_{j=1}^{L} \left( P_{j,j+1}^{g} + \frac{2}{3} E_{j,j+1} \right), \quad (1.1)$$

where we assume the periodic boundary condition. For precise definition of (1.1), see refs. [16, 8]. In this paper, we consider the case $J = -1$, which corresponds to the antiferromagnetic regime.

In section 2, we consider the finite temperature correction of the DVF $T_{1}^{(1)}(v)$ from the $T$-system [12] and calculate the central charge from the low temperature asymptotics of the specific heat (see, refs. [17, 18, 19, 20, ...].
The central charge is expressed in terms of the Rogers dilogarithm function, and reproduces the conjecture \( c = 1 \) from the root density method. Moreover we solve the TBA equation numerically and evaluate the thermodynamic quantities such as the free energy, the internal energy, the specific heat and the entropy, which are depicted in fig. [1]. To the author’s knowledge, this is the first evaluations of the physical quantities of \( osp(1|2) \) model at finite temperature.

Above calculation corresponds to the largest eigenvalue sector of the DVF. In section 3 we comment on the excited state TBA equations, which have singularities from zeros of fused QTMs. For comparison, we also briefly mention a finite temperature correction from the traditional string hypothesis in Appendix.

\section{Analysis of the TBA Equation}

In our previous papers [7, 8], we proposed the TBA equation of the \( osp(1|2) \) model

\[
\log Y_m(v) = -\frac{\pi \beta m_1}{\cosh \pi v} + K \ast \log \left\{ \frac{(1 + Y_{m+1})(1 + Y_{m-1})}{1 + Y^{-1}_m} \right\}(v),
\]

(2.1)

where \( m \in \mathbb{Z}_{\geq 1}, Y_0(v) := 0, \beta = 1/T \) (\( T \): temperature; in this paper, we set the Boltzmann constant to 1.), \( K(v) \) is the kernel defined by

\[
K(v) = \frac{1}{2 \cosh \pi v},
\]

(2.2)

and \( A \ast B(v) \) denotes the convolution

\[
A \ast B(v) = \int_{-\infty}^{\infty} dw A(v - w)B(w).
\]

(2.3)
Through this TBA equation, the logarithm of the largest eigenvalue of the QTM $T_1^{(1)}(v)$ (See, ref.[8] for the definition of $T_m^{(1)}(v)$.) can be expressed as

$$\log T_1^{(1)}(0) = \beta \left( \frac{4\pi}{3\sqrt{3}} - 1 \right) + G \log (1 + Y_1)(0), \quad (2.4)$$

and the free energy per site $f$ is written as

$$f = -T \log T_1^{(1)}(0). \quad (2.5)$$

Here the kernel $G(v)$ is defined by

$$G(v) = \frac{2 \sinh \frac{4\pi v}{3}}{\sqrt{3} \sinh 2\pi v}. \quad (2.6)$$

Note that $-4\pi/(3\sqrt{3}) + 1 =: E_{gs}$ in eq. (2.4) is the ground state energy $\mathbb{I}$ of (1.1).

Following ref. [17], we shall analyze the TBA equation (2.1). In the second term of the right hand side of (2.1), we find that the factor $Y_m^{-1}(v)$ is not relevant for the calculation of the finite temperature correction of $T_1^{(1)}(v)$. To avoid this, we have to modify (2.1). After some manipulation, one can rewrite (2.1) as

$$\log Y_m(v) = -\frac{4\pi \beta \delta_{m1} \sinh \frac{4\pi v}{3}}{\sqrt{3} \sinh 2\pi v} + G \log \left\{ \frac{(1 + Y_m)(1 + Y_{m-1})}{1 + Y_m} \right\} (v), \quad (2.7)$$

where $m \in \mathbb{Z}_{\geq 1}$, $Y_0(v) := 0$. We introduce the scaling function in the low temperature limit

$$y_{m,\pm}(v) = \lim_{\beta \to \infty} Y_m \left( \pm \frac{3}{2\pi} \left( v + \log \frac{4\pi \beta}{\sqrt{3}} \right) \right). \quad (2.8)$$

From the modified TBA equation (2.7), we find that $y_{m,\pm}(v)$ satisfies the following non-linear integral equation

$$\log y_{m,\pm}(v) = -e^{-v} \delta_{m1} + \tilde{G} \log \left\{ \frac{(1 + y_{m+1,\pm})(1 + y_{m-1,\pm})}{1 + y_{m,\pm}} \right\} (v), \quad (2.9)$$
where \( m \in \mathbb{Z}_{\geq 1}, y_{0,\pm}(v) := 0 \) and

\[
\tilde{G}(v) = \frac{3}{2\pi} G\left(\frac{3}{2\pi}v\right). \tag{2.10}
\]

We shall divide \( T^{(1)}_1(v) \) into the ground state and the finite temperature correction parts, \( T^{gs}_1(v) \) and \( T^{fn}_1(v) \), respectively,

\[
T^{(1)}_1(v) = T^{gs}_1(v)T^{fn}_1(v). \tag{2.11}
\]

By using the functions (2.8), one can derive the asymptotic behavior of the finite temperature correction part for large \( \beta \):

\[
\log T^{fn}_1(v) = G \ast \log(1 + Y_1)(v)
\]

\[
= \frac{\sqrt{3}}{\pi} \int_{-\infty}^{\infty} dw \frac{\sinh \frac{4\pi}{3} \left( v - \frac{3}{2\pi} \left( w + \log \frac{4\pi\beta}{\sqrt{3}} \right) \right)}{\sinh 2\pi \left( v - \frac{3}{2\pi} \left( w + \log \frac{4\pi\beta}{\sqrt{3}} \right) \right)} 
\times \log \left( 1 + Y_1 \left( \frac{3}{2\pi} \left( w + \log \frac{4\beta}{\sqrt{3}} \right) \right) \right) 
+ \frac{\sqrt{3}}{\pi} \int_{-\infty}^{\infty} dw \frac{\sinh \frac{4\pi}{3} \left( v + \frac{3}{2\pi} \left( w + \log \frac{4\pi\beta}{\sqrt{3}} \right) \right)}{\sinh 2\pi \left( v + \frac{3}{2\pi} \left( w + \log \frac{4\pi\beta}{\sqrt{3}} \right) \right)} 
\times \log \left( 1 + Y_1 \left( -\frac{3}{2\pi} \left( w + \log \frac{4\beta}{\sqrt{3}} \right) \right) \right)
\]

\[
= \frac{3}{4\pi^2\beta} \left\{ e^{\frac{2\pi}{3}} \int_{-\infty}^{\infty} dw e^{-w} \log(1 + y_{1,+}(w)) 
+ e^{-\frac{2\pi}{3}} \int_{-\infty}^{\infty} dw e^{-w} \log(1 + y_{1,-}(w)) \right\} + o\left(\frac{1}{\beta}\right). \tag{2.12}
\]

Taking account of the relation \( \tilde{G}(-v) = \tilde{G}(v) \), we can derive the following relation from (2.9):

\[
\int_{-\infty}^{\infty} dv e^{-v} \log(1 + y_{1,\pm}(v)) = \frac{1}{2} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} dv \left\{ \log(1 + y_{m,\pm}(v)) \frac{d}{dv} \log y_{m,\pm}(v) 
- \log y_{m,\pm}(v) \frac{d}{dv} \log(1 + y_{m,\pm}(v)) \right\}
\]
\[ \frac{1}{2} \sum_{m=1}^{\infty} \int_{y_{m,-}(-\infty)}^{y_{m,+}(\infty)} dy \left\{ \frac{\log(1+y)}{y} - \frac{\log y}{1+y} \right\} = \sum_{m=1}^{\infty} \left\{ L \left( \frac{y_{m,+}(\infty)}{1+y_{m,+}(\infty)} \right) - L \left( \frac{y_{m,-}(\infty)}{1+y_{m,-}(\infty)} \right) \right\}, \tag{2.13} \]

where we assume that \( y_{m,\pm}(v) \) are non-decreasing functions on \( v \in \mathbb{R} \) and \( L(v) \) is the Rogers dilogarithm function

\[ L(x) = -\frac{1}{2} \int_{0}^{x} dy \left\{ \frac{\log(1-y)}{y} + \frac{\log y}{1-y} \right\} \text{ for } 0 \leq x \leq 1. \tag{2.14} \]

Substituting (2.13) into (2.12), we obtain

\[ \log T_{1}^{\text{fn}}(v) = \frac{3}{4\pi^2 \beta} \left( e^{\frac{2\pi}{v}} \sum_{m=1}^{\infty} \left\{ L \left( \frac{y_{m,+}(\infty)}{1+y_{m,+}(\infty)} \right) - L \left( \frac{y_{m,+}(-\infty)}{1+y_{m,+}(-\infty)} \right) \right\} + e^{-\frac{2\pi}{v}} \sum_{m=1}^{\infty} \left\{ L \left( \frac{y_{m,-}(\infty)}{1+y_{m,-}(\infty)} \right) - L \left( \frac{y_{m,-}(-\infty)}{1+y_{m,-}(-\infty)} \right) \right\} \right) + o(\frac{1}{\beta}). \tag{2.15} \]

In our case, both \( y_{m,+}(v) \) and \( y_{m,-}(v) \) behave in the same manner, thus we set \( y_{m}(v) := y_{m,\pm}(v) \). For small \( T \), the leading term of the specific heat \( C \)

\[ C = -\frac{\partial}{\partial T} \left( T^2 \frac{\partial}{\partial T} \left( \frac{f}{T} \right) \right) = \frac{\partial}{\partial T} \left( T^2 \frac{\partial}{\partial T} \log T_{1}^{\text{fn}}(0) \right) \tag{2.16} \]

is proportional \([19, 20]\) to the central charge \( c \):

\[ C = \frac{\pi c T}{3v_{F}} + o(T), \tag{2.17} \]

where the Fermi velocity is \( v_{F} = 2\pi/3 \) \([16]\). Thus, we can express the central charge as

\[ c = \frac{6}{\pi^2} \sum_{m=1}^{\infty} \left\{ L \left( \frac{y_{m}(\infty)}{1+y_{m}(\infty)} \right) - L \left( \frac{y_{m}(-\infty)}{1+y_{m}(-\infty)} \right) \right\}. \tag{2.18} \]

Above expression is widely seen in the model related to rank one algebras (see for example, refs. \([21, 17, 22, 18]\)).
Now we shall evaluate the limit $y_m(\pm \infty)$. For $v \to \infty$, (2.9) reduces to the constant $Y$-system: \[ y_m = \frac{(1 + y_{m+1})(1 + y_{m-1})}{1 + y_m} \quad \text{for} \quad m \in \mathbb{Z}_{\geq 1}, \tag{2.19} \]
where $y_0 := 0$. Thus, we expect (see, eq. (4.1) in ref. [8])
\[ \lim_{v \to \infty} y_m(v) = \frac{m(m + 3)}{2}. \tag{2.20} \]
The divergence from the first term in rhs of eq. (2.9) in the limit $v \to -\infty$ is expected to be canceled by lhs if
\[ y_1(v) \to +0 \quad \text{for} \quad v \to -\infty. \tag{2.21} \]
Then remaining $y_m(v)$ for $m \in \mathbb{Z}_{\geq 2}$ obey the constant $Y$-system (2.19). Thus we expect $y_m(-\infty)$ is given as the solution of (2.19) with $m \to m - 1$:
\[ \lim_{v \to -\infty} y_m(v) = \frac{(m - 1)(m + 2)}{2} \quad \text{for} \quad m \in \mathbb{Z}_{\geq 2}. \tag{2.22} \]
Using these relations (2.18), (2.20), (2.22) and the fact $L(1) = \pi^2/6$, we finally arrive at $c = 1$. This agrees with the conjecture [16] by the root density method.

Now we shall solve the TBA equation (2.7) numerically to analyze the thermodynamic quantities at finite temperatures. To analyze this equation (2.7) numerically, one needs to truncate it as a finite number (we call this number $m_t$) of nonlinear integral equations since it is composed of an infinite number of nonlinear integral equations. The validity of this approximation can be verified from the fact that the numerical results are almost independent of $m_t$ when $m_t$ is more than a certain large number.

Figure 1 is numerical results of temperature dependence of some important thermodynamic quantities such as the free energy (2.5), the specific heat (2.16), the entropy $S = \frac{\partial}{\partial T}(T \log T_1^{(1)}(0))$ and the internal energy...
Figure 1: Temperature dependence of fundamental thermodynamic quantities: (a) Free energy \( f(T) \), (b) Specific heat \( C(T) \), (c) Internal energy \( U(T) \) and (d) Entropy \( S(T) \).

\[
U = T^2 \frac{\partial}{\partial T} (\log T_1^{(1)}(0)).
\]

These results agree with well-known values at the both special limits: (i) at the low temperature limit, the free energy and the internal energy agree with the ground state energy \([16] \) \( E_{gs} \approx -1.4184 \) and the specific heat shows proportionate increase with respect to the temperature, which leads the central charge \( c = 1 \), (ii) at the high temperature limits, the entropy agrees with the value \( \log 3 \approx 1.09861 \) derived from the fact that the present model has three states.
Comment on the Excited State

The excited state TBA equations are characterized by zeros of DVF $T_m^{(1)}(v)$ of the fused QTM and a phase factor. They have the following form (cf. ref. [8]).

$$
\log Y_m(v) = -\frac{\pi \beta \delta_{m1}}{\cosh \pi v} + K * \log \left\{ \frac{(1 + Y_{m+1})(1 + Y_{m-1})}{1 + Y_m^{-1}} \right\}(v) \\
+ \log \left\{ \frac{\prod_{\{z_{m+1}\}} \tanh \frac{\pi}{2} (v - z_{m+1}) \prod_{\{z_{m-1}\}} \tanh \frac{\pi}{2} (v - z_{m-1})}{\prod_{\{z_m\}} \tanh \frac{\pi}{2} (v - z_m)} \right\} \\
+\{1 - (-1)^m\} \{1 - (-1)^\zeta_\infty\} \frac{\pi i}{4},
$$

(3.1)

where $m \in \mathbb{Z}_{\geq 1}$; $Y_0(v) := 0$; $\zeta_\infty = \lim_{N \to \infty} (N - n)$; $N$: the Trotter number; $n$: a quantum number in the Bethe ansatz equation (BAE); $\{z_m\}$ are zeros of $T_m^{(1)}(v)$ in the physical strip $\text{Im} v \in [-1/2, 1/2]$, which can also be characterized as

$$
Y_m(z_m \pm \frac{i}{2}) = -1.
$$

(3.2)

For the second largest eigenvalue case ($n = N - 1$, $\{z_m\} = \{\pm x_m | m \in \mathbb{Z}_{\geq 1}\}$: there is a misprint in the phase factor of eq.(5.3) in ref. [8]), we have to consider zeros of $T_m^{(1)}(v)$, which enter into the physical strip for all $m \in \mathbb{Z}_{\geq 1}$.

In addition, the second largest eigenvalue is conjectured to be described by the same distribution pattern (see, Fig.3 and Fig.5 in ref. [8]) of the root of the BAE at any finite temperatures. On the other hand, for the third largest eigenvalue case, our numerical study for finite $N$ indicates that the distribution pattern of the roots of the BAE may vary with temperature. This suggests a possibility of the occurrence of the level crossing. We have also observed numerically that only the zeros of $T_1^{(1)}(v)$ appear in the physical strip for one of the distribution patterns for the third largest eigenvalue.

\*\*This misprint was corrected in math-ph/9912014 v3.\*\*
Numerical studies about such excited states are also a crucial problem. However to solve the excited state TBA equation, we have to analyze not only infinite number of unknown functions but also their zeros which are determined by subsidiary condition (3.2). To make numerical results converge, one needs to determine all the zeros in the physical strip accurately. For this reason the truncation approach available for solving the TBA equation have no use for the excited state case.

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Appendix: Finite Temperature Correction from the String Hypothesis

We can also confirm the central charge \( c = 1 \) based on the string hypothesis (see, refs. [23, 24, 21, 22]). In this case, the \( Y \)-function in TBA (2.7) corresponds to the ratio of the particle density \( \rho^p_m(v) \) and the hole density \( \rho^h_m(v) \): \( Y_m(v) = \rho^h_m(v)/\rho^p_m(v) \). From our previous results (eq. (15) in ref. [7]), the particle and hole densities satisfy the following relation.

\[
\frac{2\delta_{m1} \sinh \frac{4\pi v}{\sqrt{3}}}{\sqrt{3} \sinh 2\pi v} = \rho^p_m(v) + \rho^h_m(v) + G*\rho^h_m(v) - G*\rho^h_{m-1}(v) - G*\rho^h_{m+1}(v), \quad (A.1.1)
\]

where we define \( \rho^h_0 = 0 \). To consider the low temperature behavior of (2.7), we set \( Y_m(v) = \exp(\beta \epsilon_m(v)) \) and shift the variable \( v \to \frac{3}{2\pi}(v + \log \frac{4\pi \beta}{\sqrt{3}}) \). For \( T \ll 1 \), (2.7) can be expressed in terms of the function \( \varphi_m(v) := \beta \epsilon_m(\frac{3}{2\pi}(v \) + \(
\)
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\[ \log \frac{4\pi\beta}{\sqrt{3}} \),

\[ e^{-v}\delta_{m1} = \log(1 + \exp(-\varphi_m(v))) - \log(1 + \exp(\varphi_m(v))) \]

\[ -\tilde{G} * \log(1 + \exp(\varphi_m(v)))(v) + \tilde{G} * \log(1 + \exp(\varphi_{m-1})(v) \]

\[ + \tilde{G} * \log(1 + \exp(\varphi_{m+1}))(v). \]  

(A.1.2)

This is effectively the same equation as (2.9). Requiring the same shift for (A.1.1) and comparing the result with the \( v \)-derivative of (A.1.2), we find the asymptotics of the particle and hole densities in the limit \( v \to \infty \):

\[ \rho^p_m(v) \simeq \frac{3}{4\pi^2} f(\beta\epsilon_m(v)) \frac{d}{dv} \epsilon_m(v) \]

\[ \rho^h_m(v) \simeq \frac{3}{4\pi^2} (1 - f(\beta\epsilon_m(v))) \frac{d}{dv} \epsilon_m(v), \]

(A.1.3)

where \( f(\varphi) = (1 + e^\varphi)^{-1} \) is the Fermi distribution function. Using (A.1.3) and taking into account the contribution from the negative large \( v \) derived by the shift \( v \to -\frac{3}{2\pi}(v + \log \frac{4\pi\beta}{\sqrt{3}}) \), we find the entropy per site (eq. (21) in ref. [1]) have the following asymptotics

\[ S = -\frac{3}{2\pi^4 \beta} \sum_{m=1}^{\infty} \int_{\varphi_m(-\infty)}^{\varphi_m(\infty)} d\varphi \{ f(\varphi) \log f(\varphi) + (1 - f(\varphi)) \log(1 - f(\varphi)) \} \]

\[ = \frac{3}{\pi^2 \beta} \sum_{m=1}^{\infty} \{ L(1 - f(\varphi_m(\infty))) - L(1 - f(\varphi_m(-\infty))) \} \].  

(A.1.4)

Using the fact \( C = T(\partial S/\partial T) \), eq. (2.17) and the identification \( y_m(v) = \lim_{T \to 0} \exp(\varphi_m(v)) \), we can reconfirm the central charge \( c = 1 \).

References

[1] P. Schlottmann: Phys. Rev. B36 (1987) 5177.

[2] F. H. L. Essler and V. E. Korepin: Int. J. Mod. Phys. B8 (1994) 3243.
[3] G. Jüttner, A. Klümper and J. Suzuki: Nucl. Phys. B512 (1998) 581.

[4] H. Frahm: Nucl. Phys. B559 (1999) 613.

[5] H. Saleur: Nucl. Phys. B578 (2000) 552.

[6] C. N. Yang and C. P. Yang: J. Math. Phys. 10 (1969) 1115.

[7] K. Sakai and Z. Tsuboi: Mod. Phys. Lett. A14 (1999) 2427; math-ph/9911010.

[8] K. Sakai and Z. Tsuboi: Int. J. Mod. Phys. A15 (2000) 2329; math-ph/9912014.

[9] Z. Tsuboi: A note on the osp(1|2s) thermodynamic Bethe ansatz equation (2000) submitted.

[10] P. P. Kulish: J. Sov. Math. 35 (1986) 2648.

[11] V. V. Bazhanov and A. G. Shadrikov: Theor. Math. Phys. 73 (1988) 1302.

[12] Z. Tsuboi: J. Phys. A: Math. Gen. 32 (1999) 7175.

[13] A. Klümper: Ann. Physik 1 (1992) 540.

[14] M. Takahashi: Prog. Theor. Phys. 46 (1971) 401.

[15] M. Gaudin: Phys. Rev. Lett. 26 (1971) 1301.

[16] M. J. Martins: Nucl. Phys. B450 (1995) 768.

[17] A. Klümper and P. Pearce: Physica A183 (1992) 304.
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[18] A. Kuniba, T. Nakanishi and J. Suzuki: Int. J. Mod. Phys. A9 (1994) 5267.

[19] H. W. J. Blöte, J. L. Cardy and M. P. Nightingale: Phys. Rev. Lett. 56 (1986) 742.

[20] I. Affleck: Phys. Rev. Lett. 56 (1986) 746.

[21] V. V. Bazhanov and N. Yu. Reshetikhin: Int. J. Mod. Phys. A4 (1989) 115.

[22] A. Kuniba: Nucl. Phys. B389 (1993) 209.

[23] H. M. Babujan: Nucl. Phys. B215 (1983) 317.

[24] A. M. Tsvelick and P. B. Wiegmann: Adv. Phys. 32 (1983) 453.