CALABI PROBLEM FOR MANIFOLDS WITH EDGE-CONE SINGULARITIES

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ABSTRACT. In this note, we propose a new approach to solving the Calabi problem on manifolds with edge-cone singularities of prescribed angles along complex hypersurfaces. It is shown how the classical approach of Aubin-Yau in deriving \textit{a priori} estimates for the complex hessian can be made to work via adopting a \textit{good reference metric} and studying equivalent equations with different reference metrics. This further allows extending much of the methods used in the smooth setting to the edge setting. These results generalise to the case of multiple hypersurfaces with possibly normal crossing.

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1. Introduction

The study of problems around cone-edge singularities, in particular, the problems around finding Kähler-Einstein metrics prescribed edge-cone behaviour, has received quite a bit of attention in the past years. One reason is the key rôle such metrics play in the approach taken by Chen, Donaldson, and Sun in [5, 6, 7], (and also other attempts for solving the same problem in [18]), in proving the relation between $K$-stability and the existence of Kähler-Einstein metrics on Fano manifolds. In the present work we shall show how one can use the geometry of edge-cone manifolds by constructing edge metrics with curvature bounded from below to obtain the estimates needed for solving the Calabi problem. Other than finding Kähler-Einstein metrics, this approach allows prescribing a wide class of Ricci forms. Since in most constructions and proofs it is straightforward to see how they should be modified for the case of divisors with possibly normal crossing, in the rest of this work, in order to keep the statements and proofs clearer, we confine ourselves to the case of one smooth hypersurface.

By Kähler metrics with \textit{edge} or \textit{edge-cone} singularities we mean a Kähler metric with conical singularity along a complex hypersurface, that is, a metric which asymptotically resembles a cone on $\mathbb{C}$ of total angle $2\pi\tau$ in the directions normal to the hypersurface, and is smooth in the tangential directions. Examples of such metrics were already known as they arise as orbifold metrics. More generally, one may construct such metrics as follows. Let $(M^n, \omega_0)$ Kähler manifolds, where $\omega_0$ is smooth. Assume that $D^{n-1} \subset M^n$ is a complex hypersurface and that $s$ is a holomorphic section of an hermitian line bundle $(L, h)$ which vanishes of order zero along $D$. Then, the following metric

\begin{equation}
\omega_\tau := \omega_0 + adf^2 |s|^2 h
\end{equation}

is an edge-cone metric along $D$ of angle $\tau$ when $0 < a \ll 1$. This statement, along with the rest of results can be generalised to the case wherein $D$ consists of a union of irreducible divisors, $D_j$, with at most normal crossing. Indeed, to the best of the author’s knowledge, in the works prior to the

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present work the metric $\omega_\tau$, thus defined, has been taken as the conical background metric in order to do the analysis.

Various approaches to the study of the Calabi problem on manifolds with edge-cone singularities have proved effective. Ricci-flat edge-cone Kähler metrics were proved to exist under suitable topological conditions by Brendle in [2] provided that $\tau \leq \frac{1}{2}$. There, the classical method used by Aubin and Yau in [1, 21] was used. The fact that the classical calculations can be adapted to this situation is due to the fact that the reference metric $\omega_\tau$ has bounded curvature away from the divisor $D$ when $\tau \leq \frac{1}{2}$. More specifically, when $\tau > \frac{1}{2}$, the curvature of the reference metric (1.1) might become unbounded from below. This possible lack of lower bound in the curvature has been one main obstacle in deriving laplacian estimates and higher order regularity results since the approach of Aubin and Yau for deriving estimates on the complex hessian depends on the existence of a lower bound on the bisectional curvature of the reference metric. Further, the lack of of a lower bound on the Ricci curvature of the reference metric means the lack of a lower bound on the laplacian of the Ricci potential, a yet another quantity the finiteness of whose lower bound is needed in deriving the laplacian estimate for the potential in the approach of Aubin-Yau.

Another work, which covers the case of larger angles as well is that of Jeffres, Mazzeo and Rubinstein [13]. There, in order to derive the laplacian estimate in the absence of a lower bound on curvature of $\omega_\tau$ a corollary of the Chern-Lu inequality has been used which, rather than the lower bound of the curvature of the background metric requires an upper bound on it. Along with the observation that the curvature of $\omega_\tau$ is always bounded from above, the existence of Kähler-Einstein metrics are proved in [13]. Using a different approach to the Chern-Lu inequality, X.-X. Chen et al have also derived the laplacian estimate in the Fano case in [6]. This has been further refined by Yao in [20].

In a different direction, in [11] one finds a clever adaptation of the classical calculations to the edge-cone setting by using an auxiliary function to control the behaviour of the possibly unbounded curvature terms in the study of the complex Monge-Ampère equation. The interested reader is referred to Chapter 7 of the survey article by Rubinstein [16], wherein an extensive treatment of the laplacian estimates in the works mentioned above can be found.

However, one core idea in the present work is the observation that indeed the Pogorelov-Aubin-Yau approach can be used with very little modification if one observes that within the same edge-cone cohomology class there always exist metrics with lower bound on their bisectional curvature, and that with respect to this reference metric, the Ricci potential has the correct behaviour.

This allows us to show the following.

**Theorem 1.1.** (a) [Calabi’s first problem] Let $\tilde{\omega} \in c_1(M) - (1 - \tau)c_1(L) \in H^{1,1}(M, \mathbb{R})$ be a closed real (1,1)-form which is of class $C^\alpha$ on $M \sim D$, such that $(\text{tr}_{\omega_\tau, \tilde{\omega}})^+ = o(|s|^{-2\tau})$ for some $\epsilon > 0$, and in the vicinity that $\tilde{\omega}$ is generated by a local potential of the Hölder class $C^\alpha_\theta$ for some exponent $\alpha$. Then, there is a potential $\phi$, which belongs to the class $C^2_\tau$ for some exponent $\theta$, determined uniquely up to a constant, such that

$$\omega_\tau + dd^c \phi = \tilde{\omega} + 2\pi(1 - \tau)[D]$$

(b) [Calabi’s second problem] Assume that $c(M) - (1 - \tau)c_1(L) = \mu[\omega_0]$, where, either i) $\mu \leq 0$, or ii) we have an $L^\infty$ bound on the potential $\phi$. Then, there is a potential, belonging to the class $C^2_\tau$ for some exponent $\theta$, such that

$$\omega_{\tau, \phi} = \mu \omega_\tau + 2\pi(1 - \tau)[D]$$
for $\omega_{\tau, \phi} = \omega_{\tau} + dd^c \phi$. Further, the potential $\phi$ is unique up to addition of a constant when $\mu = 0$, and is unique otherwise.

As in the smooth case, the solution of the Calabi problem relies on solving a complex Monge-Ampère equation, but this time with an edge Kähler reference metric. This connection, which calls for a bit more careful handling than in the smooth case, will be clarified in §3.

**Theorem 1.2.** Assume that the metric $\omega_{\tau}$ is as defined in (1.1), and that for some exponent $\alpha \in (0, 1)$, $f \in C^0_\alpha$ is a function such that $(\Delta f)^{-1} = o(|s|^{-2\tau}_h)$. Then, there exists a solution $\phi$ to the following equation:

\[
(\omega_{\tau} + dd^c \phi)^n = e^f \omega_{\tau}^n; \quad \int e^f \omega_{\tau}^n = \int \omega_{\tau}^n
\]

which is unique up to a constant and belongs to the edge-cone Hölder space $C^{2, \theta}_\tau$ for some exponent $\theta$.

In order to solve the equation, we also use idea of approximating the edge-cone metric by a family of smooth metrics similar to what is done in [4, 11]. An important observation that allows deriving estimates independent of the upper bound on the scalar curvature of the reference metric is also borrowed from an earlier work of Păun [15].

The proof of the theorems above relies on the following proposition which allows us to explicitly construct a good reference metric.

**Proposition 1.3.** Let $\omega_0$ be a smooth Kähler metric and let $\tau \in (0, 1)$ and $\tau' \in (\tau, 1)$ be two real numbers. Then, for sufficiently small positive constant $c > 0$, the following $(1,1)$-form

\[
\tilde{\omega} := \omega_{\tau} - cdd^c |s|^{2\tau}_{h} = \omega_0 + add^c |s|^{2\tau}_{h} - cdd^c |s|^{2\tau'}_{h}
\]

is an edge-cone Kähler metric of cone angle $\tau$, equivalent to the following edge-cone metric:

\[
\omega_{\tau} := \omega_0 + add^c |s|^{2\tau}_{h}
\]

in such a way that the curvature of $\tilde{\omega}$ is bounded from below. Further, the parameter $c$ can be chosen to be sufficiently small so that the metrics $\omega_{\tau}$ and $\tilde{\omega}_{\tau}$ are arbitrarily close with respect to the Hölder norm $C^{2, \theta}_\tau$.

Having proved the proposition above, we have to study how the equation and its components, in particular, the Ricci potential, transform under this change of the reference metric and then prove that we can indeed derive the estimates. The proof of this proposition and the study of how the equation transform under the change of the reference metric is the subject of §4.

Besides opening way for estimates previously known in the smooth case, which we shall hopefully explore elsewhere, one advantage of this approach is that this method is based on the geometry of the edge-cone metrics and we, therefore, hope that such observations about conical metrics will be of interest in their own right. Further, this approach allows a wide class of preassigned Ricci forms to be realised by edge-cone Kähler metrics.

2. Edge-cone functional spaces, Linear elliptic theory

In this section, we shall introduce the notation and basic concepts we shall frequently make use of in the rest of this note. For the definition of spaces we follow [10], where the linear elliptic theory for edge-cone metrics is developed.
Let us for the sake of clarity work on $\mathbb{C}^n$ and assume that the edge-cone singularity occurs along the divisor $\sum_{j=1}^{k} (1 - \tau_j) |z_j| = 0$. The edge-cone Kähler form we consider as our model on $\mathbb{C}^n$ is the following:

$$\omega_R := \dd c \left( \sum_{j=1}^{k} |z_j|^{2\tau_j} + \sum_{j=k+1}^{n} |z_j|^2 \right) = \sum_{j=1}^{k} i \tau_j |z_j|^{2\tau_j} - 2 d\bar{z}_j \wedge d\bar{z}_j + \sum_{j=k+1}^{n} i d\bar{z}_j \wedge d\bar{z}_j$$

To keep our notation simpler, in the following definitions we shall only state the case of a single divisor along $|z_1| = 0$. The definitions can be extended to the case of multiple divisors with possible normal crossing in the obvious way.

**Definition 2.1.** Consider the Kähler space $\mathbb{C}_\tau \times \mathbb{C}^{n-1}$. For a given function $f(z^1, \ldots, z^n)$ define the associated function $\tilde{f}(\xi, \ldots, z^n)$ where $\xi = |z_1|^{\tau - 1}$. Then, $f$ is said to be of class $C^{\alpha}_\tau$ provided that $\tilde{f} \in C^{\alpha}$.

There is another change of variable which we shall use which is compatible with the picture one has of in the case of conical angle $\tau = \frac{1}{p}$, for $p \in \mathbb{N}$, on $\mathbb{C}_{\frac{1}{p}}$, which can be uniformised via the map

$$\mathbb{C}_{\frac{1}{p}} \to \mathbb{C}$$

$$z \mapsto z^\frac{1}{p} = w$$

The advantage of this transformation, unlike transformation $z \mapsto \xi$, is that change of variable to $w$ is a -local- bi-holomorphism, and hence we can calculate geometric quantities such as curvature in $\zeta$. It is important to observe that $w$ and $\xi$, defined above, only disagree in the angle variable. This, in particular, means that the pull-backs of the euclidean metric by the two transformations define equivalent distances. This allows us to define the Hölder spaces using either transformation.

Further, in either case, the function $|z|^2$, which is the Kähler potential of the flat metric on $\mathbb{C}$, is sent to the function $|z^\frac{1}{p}|^2$. We have defined the spaces $C_{\tau}^{k,\alpha}$ in such a way that functions such as $|z|^\frac{1}{p}$ will belong to them.

From this picture it should be clear that there is a more intrinsic way of defining the Hölder spaces $C^{\alpha,\beta}$ which is the content of the following definition. We shall indeed make use of this equivalence later.

**Definition 2.2.** For a given function $f$, define the semi-norm $[f]^\tau_\alpha$ as follows:

$$[f]^\tau_\alpha := \sup_{x,y} \frac{|f(x) - f(y)|}{d(\tau(x,y)^\alpha}$$

Also, define the $C^{\alpha}_\tau$ norm of the function $f$ to be $\|f\|_{\alpha,\tau} = \|f\|_0 + [f]^\tau_\alpha$, and let $C^{\alpha}_\tau$ designate the space of all functions having finite $\|\cdot\|_{\alpha,\tau}$ bound.

Using the $C^{\alpha}_\tau$ spaces for functions, we can now define $C^{\alpha}_\tau$ forms and thereby define the space $C^{2,\alpha}_\tau$.

**Definition 2.3.** Let $\sigma$ be a $(1,0)$-form. We say define:

$$\|\sigma\|_{\alpha,\beta} := \sup$$

Following Donaldson’s definition of edge-cone Hölder spaces we recall the following definitions.

**Definition 2.4.** We say that a function $f$ is $C^{\alpha}(S_\tau)$ provided that $f$ is $\alpha$-Hölder continuous on the the sector $\arg \zeta \in [0, 2\pi\tau)$ with the two rays $\tau = 0$ and $\tau = 2\pi\tau$ identified outside of the origin. The spaces $C^{k,\alpha}$ are defined similarly.
3. Setting up the equation

In this section, we shall see how to reduce the proof of Theorem 1.1 can be reduced to Theorem 1.2. To this end, we study the behaviour of various components of the complex Monge-Ampère equation written with respect to the reference metrics (1.1) and (1.5). Here we will assume some of the results in §4.

In the case of smooth background Kähler metric $\omega_0$ on a manifold which satisfies $\mu \omega_0 \in c_1(M)$, when one tries to find a Kähler-Einstein metric with constant $\mu \in \{-1, 0, 1\}$, the function $f$ on the right hand side of the complex Monge-Ampère equation $(\omega_0 + dd^c \phi)^n = e^{f - \mu \phi} \omega_0^n$ can be obtained by solving the following equation for $f$:

$$\varrho(\omega_0) - \mu \omega_0 = dd^c f$$

However, as we shall see in §4 the Ricci-form of the family of metrics $\omega_0 + dd^c|s|^{2r}$ has already been well-known as a difficulty in the theory of edge-cone Kähler spaces as certain estimates depend on the lower bound of the curvature of the reference metric. But This, in particular, implies that we cannot assume a laplacian bound on $f$.

To make this idea more quantitative, we recall that by a edge-cone Kähler-Einstein metric, $\omega$, of cone angle $\tau$ we mean one that satisfies $\varrho(\omega_0) = \mu \omega_0 + 2\pi(1 - \tau)[D]$. Similar to the smooth case, we are lead to the following equation for $f$:

$$dd^c f = \varrho(\tilde{\omega}_0) - \mu \tilde{\omega}_0 - 2\pi(1 - \tau)[D]$$

(3.1)

wherein $\omega_0$ is an arbitrary edge-cone background metric.

First, as we shall see in §4, the curvature of the reference metric $\tilde{\omega}_0$ becomes unbounded from above at the rate of $|s|^{2r-4}$, more precisely, we have:

$$\tilde{\varrho}(\mathfrak{g} r) \geq C g_{\tau} \otimes g_{\tau} |s|^{2r-4}$$

wherein $\otimes$ denotes the Kalkurni-Nomizu product of tensors. As a result, the Ricci form of the metric also behaves asymptotically as:

$$\varrho(\tilde{\omega}_0) \geq C g_{\tau} |s|^{2r-4}$$

Therefore, in terms of the asymptotic behaviour close to the divisor in §1 the term $\varrho(\tilde{\omega}_0)$ dominates the metric. As a result, the form $dd^c f$ is bounded from below by a multiple of the Kähler form $\tilde{\omega}_0$, and the laplacian $\Delta f$ is bounded from below.

Similarly, assume that instead of a Kähler-Einstein metric we seek to realise a prescribed Ricci form $\varrho$. The potential $f$ will then haveto satisfy the following:

$$dd^c f = \varrho(\tilde{\omega}_0) - \tilde{\varrho}$$

Again, we see that so long as $(\text{tr}_{\omega_0} \tilde{\varrho})^+ = O(|s|^{-2r})$ we can find a $\tau' \in (\tau, 1)$ so that the Ricci form $\tilde{\varrho}(\tilde{\omega}_0)$ will have blow up at a higher rate and will, thereby, dominate the behaviour and will make $dd^c f$ a current bounded from below. In particular, this means that $\Delta f$ will be bounded from below which is what we need for the laplacian estimate.

Having made these observations, one can follow the usual way of reducing the statement of Theorem 1.1 to that of Theorem 1.2.
4. CHOOSING A GOOD METRIC AND APPROXIMATION BY SMOOTH FAMILY

In deriving the laplacian estimates, it is evident in the calculations of the Pogorelov-Aubin-Yau approach that the estimate depends on the lower bound of the bisectional curvature. However, once we add a potential of the form $|s|^{2\tau}_h$ to a smooth background metric in the vicinity of the zero locus of $s$ the curvature might become unbounded from below, for $\tau > \frac{1}{2}$. Outside of the divisor, the curvature of such metric is always bounded from above, however, as it has been shown in the appendix of [13].

Here, we shall show how one could perturb the metric in the same -edge- cohomology class so that the curvature of the metric will become bounded from below. Indeed, the perturbed metric and the metric $\omega_\tau$ are close in a suitable Hölder norm. Hence, bounding the laplacian with respect to one of them will suffice for deriving a priori bounds on the complex hessian.

The smallness of the parameter $c$ guarantees that the $(1,1)$-form thus obtained is positive definite and therefore a Kähler metric. Obviously, we can always scale the metric $h$ defined on the line bundle so that the expression $\tilde{\omega} = \omega_0 + dd^c|s|^{2\tau}_h - dd^c|s|^{2\tau'}_h$ is positive definite. Indeed we shall assume this from now on and will drop the coefficients $a$ and $c$. It is easy to observe in the proof that the correction $|s|^{2\tau'}_h$ to the potential does not change the curvature properties of the metric so much at the points away from the divisor.

Since we are going to construct the solution as the limit of a sequence of smooth solutions, each of which is obtained by solving with respect to a smooth reference metric, we need the following lemma which allows proving uniform estimate for a sequence of smooth approximations of $\tilde{\omega}_\epsilon$.

**Lemma 4.1.** The family $\{\omega_\epsilon\}_\epsilon$ defined as:

$$
\tilde{\omega}_\epsilon = \omega_0 + dd^c\left((|s|^{2\tau}_h + \epsilon)^\tau - (|s|^{2\tau}_h + \epsilon)^{\tau'}\right)
$$

is a smooth family approximating $\tilde{\omega}$. Further, elements of $\tilde{\omega}_\epsilon$ have a uniform (in the parameter $\epsilon$) lower bound on their curvature.

Indeed, what we shall prove is that in curvature of these metrics satisfy $R^c(v, \bar{v}, w, \bar{w}) \geq C|v|_\tau|w|_\tau$ for some constant $C$. The following proof essentially shows both Proposition [13] and Lemma [11]. Indeed the lemma is just a small but useful observation about the content of Proposition [13]. The rest of this section is dedicated to the proof of the proposition.

**4.1. Proof of Proposition [13].** Let us notice that a function of the form $\eta = |s|^{2\tau'}_h$, where $\tau' \in (\tau', 1)$, belongs to $C^{2,\alpha}_\tau$ for some exponent $\alpha$ in the sense defined before. In particular, $\eta$ is a valid Kähler potential to be added to the metric. Let us first prove that $\tilde{\omega}$ is indeed a metric equivalent to $\omega_\tau$. It will suffice to show that the complex hessian of the correction potential, $dd^c|s|^{2\tau'}_h$, is bounded, as a $(1,1)$-form, when measured with respect to the metric $\omega_\tau$. Just as it is done already in [13], by calculating the complex hessian of the potential $|s|^{2\tau'}_h$ in the special coordinates we see that the expression

$$
\|\tilde{\omega} - \omega_\tau\|^2_{\omega_\tau} = g^{\alpha \beta} g^{\mu \nu} \left(\frac{|s|^{2\tau'}_h}{\omega_\tau}\right)_{\alpha \beta} \left(\frac{|s|^{2\tau'}_h}{\omega_\tau}\right)_{\mu \nu}
$$

consists of terms of the form $g^{\alpha \beta} g^{\mu \nu} M_{\alpha \beta} M_{\mu \nu} (|s|^{1|4\tau'}_h)$, which are finite, and when all indices are equal to 1, other terms which are dominated by $g^{11} g^{11} M^2 |3|^{1|4\tau'-4}|$. Upon noticing that $g^{11} = O(|3|^{2-2\tau})$, and that $\tau' > \tau$, one concludes that $\|\tilde{\omega}\|^2_{\omega_\tau} = O(|3|^{2(\tau - \tau')})$ which is finite and, indeed, tends to zero as the points approach the divisor.
In order to study the curvature tensor we shall use the following lemma which will simplify our calculations. This observation seems to have been first stated in [19], in the proof of Lemma 4.3 in [19], and also used in the proof of the existence of an upper bound on the curvature of edge-cone metrics in [13]. The version we state below as well as the proof may be found in [4], where it appears as Lemma 4.1.

**Lemma 4.2.** Assume that for \( j \in \{1, ..., n_0\} \) the \( D_j \)'s are irreducible divisors with at most normal crossing with associated line bundles \( L_j \) with hermitian metrics \( h_j \) and defining sections of the corresponding line bundles \( s_j \). Let \( p \in \cap D_j \). Then, there is a neighbourhood of \( p_0 \) in which for any point \( p \) there is a choice of local coordinates for the manifold \( M \) and trivialisations \( \theta_j \) for the line bundles \( L_j \) on an open set \( U \) so that

1. the hypersurfaces locally correspond to flat hyperplanes: \( U \cap D_j = [z_j = 0] \),
2. if the hermitian metric \( h_j \) is represented by \( e^{-\phi} \) in the trivialisation \( \theta_j \), then at the point \( p \) we have
   \[
   \phi_j(0) = 0, \quad d\phi_j(p) = 0, \quad \frac{\partial \phi}{\partial \bar{z}^\alpha \partial z^\beta}(p) = 0.
   \]

Further, all higher derivatives of \( \phi \) are bounded uniformly when the point varies on a compact subset of \( U \).

**Remark 4.3.** It is probably worth mentioning that the above lemma will be used to estimate the rate of blow-up of quantities in terms of \( \bar{z}_1 \). However, in order for such an estimate to make sense one has to also notice that although the coordinates chosen do depend on the point \( p \), all the coordinates chosen, in particular \( \bar{z}_1 \), are uniformly equivalent as the point \( p \) varies on a compact set. In particular, it is well-defined to speak of the rate of blow-up or the rate of vanishing in terms of powers of \( \bar{z}_1 \).

By bounds on the curvature we mean \( R(v, \bar{v}, w, \bar{w}) = R_{\alpha\beta\gamma\delta} v^\alpha \bar{v}^\beta w^\gamma \bar{w}^\delta \) when \( v = \partial_\alpha v^\alpha \) and \( w = \partial_\alpha w^\alpha \) are of unit norm with respect to the edge-cone metric \( \omega_R \). Since in our adopted coordinate system the metric satisfies \( g_{11} \approx |\bar{z}_1|^{2\tau-2} \), we have that \( |\bar{v}_1|, |w_1| \leq C |\bar{z}_1|^{1 - \frac{1}{2\tau}} \). Therefore, in studying the terms appearing in the curvature tensor, we shall consider only the once that persist as -potentially- infinite terms after multiplying \( |\bar{z}_1|^2 - 2\tau \).

We have to prove that the family of metrics \( \bar{\omega}_\epsilon = \omega_0 + d\epsilon (|s|_h^2)^{\tau} + \frac{1}{N} (\epsilon + |s|_h^2)^{\tau} \) has a uniform lower bound on the curvature tensor. The coefficient \( \frac{1}{N} \) is added to make sure that \( \bar{\omega}_\epsilon \) stays positive definite. It is easy to see that the metric \( h \) on the line bundle \( L \) can always be scaled so that the positivity condition holds for the following family of metrics defined in \([11]\). Therefore, without loss of generality, we shall assume from now on that \( N = 1 \). One can find the curvature tensor \( R^e_{\alpha\beta\gamma\delta} \) by differentiating the metric. However, one may note that since the components of the curvature tensor, written in coordinates, are combinations of various powers of \( \epsilon + |s|^2 \) and \( |s|^2 \), it will suffice to show the claim when the parameter \( \epsilon \) is zero and that will prove the proposition for the entire range of the parameter \( \epsilon \).

In the coordinate system constructed in Lemma 4.2 we can write \( |s|^2 = a^\tau |\bar{z}_1|^{2\tau} \), and also \( |s|^2 = a^\tau |\bar{z}_1|^{2\tau} \). Let us keep make the following substitution in order to keep the notation simpler: \( K := a^\tau \), \( M := a^\tau \). Evidently, in the special coordinates of Lemmas 4.1,2 we have at the point \( p \) that \( M(p) = K(p) = 1 \), \( dK(p) = dM(p) = 0 \), \( K_{\alpha\beta} = M_{\alpha\beta} = 0 \), for \( \alpha, \beta = 1 ... n \). Since we have taken \( \epsilon = 0 \), we have:

\[
\bar{g}_{\alpha\beta} = g_{\alpha\beta}^0 + (K|\bar{z}_1|^{2\tau} - M|\bar{z}_1|^{2\tau}),_{\alpha\beta}
\]
By differentiating directly we obtain:

\[
\hat{g}_{\alpha\beta} = g_{\alpha\beta}^0 + K_{\alpha\beta}|\beta|^{2r} + \tau \delta_{1\beta}K_{\alpha\beta} |\beta|^{2r-2}|\gamma| + \tau \delta_{1\alpha}K_{\beta\gamma}|\beta|^{2r-2} \tilde{\gamma} + \tau^2 \delta_{1\alpha}\delta_{1\beta}K_{\beta\gamma}|\beta|^{2r-3}
\]

(4.2)

\[ - M_{\alpha\beta}|\beta|^{2r} - \tau \delta_{1\beta}M_{\alpha\beta}|\beta|^{2r-2} \tilde{\gamma} - \tau \delta_{1\alpha}M_{\beta\gamma}|\beta|^{2r-2} \tilde{\gamma} - \tau^2 \delta_{1\alpha}\delta_{1\beta}M_{\beta\gamma}|\beta|^{2r-2}
\]

Which, in the coordinates of Lemma 4.2 simplifies to the following

\[
\hat{g}_{\alpha\beta} = g_{\alpha\beta}^0 + K_{\alpha\beta}|\beta|^{2r} + \tau \delta_{1\beta}K_{\alpha\beta}|\beta|^{2r-2} - M_{\alpha\beta}|\beta|^{2r} - \tau^2 \delta_{1\alpha}\delta_{1\beta}M_{\beta\gamma}|\beta|^{2r-2}
\]

We now have for the first derivatives of the metric that

\[
\hat{\tilde{g}}_{\alpha\beta,\gamma} = g_{\alpha\beta,\gamma}^0 + K_{\alpha\beta,\gamma}|\beta|^{2r} + \tau \delta_{1\beta}K_{\alpha\beta,\gamma}|\beta|^{2r-2} \tilde{\gamma} + \tau \delta_{1\alpha}K_{\beta\gamma,\beta}|\beta|^{2r-2} \tilde{\gamma} + \tau^2 \delta_{1\alpha}\delta_{1\beta}K_{\beta\gamma,\beta}|\beta|^{2r-3}
\]

\[ + M_{\alpha\beta,\gamma}|\beta|^{2r} - \tau \delta_{1\beta}M_{\alpha\beta,\gamma}|\beta|^{2r-2} \tilde{\gamma} - \tau \delta_{1\alpha}M_{\beta\gamma,\beta}|\beta|^{2r-2} \tilde{\gamma} - \tau^2 \delta_{1\alpha}\delta_{1\beta}M_{\beta\gamma,\beta}|\beta|^{2r-2} \tilde{\gamma}
\]

(4.3)

and in the coordinates of Lemma 4.2 this simplifies to:

\[
\hat{\tilde{g}}_{\alpha\beta,\gamma} = g_{\alpha\beta,\gamma}^0 + K_{\alpha\beta,\gamma}|\beta|^{2r} + \tau \delta_{1\beta}K_{\alpha\beta,\gamma}|\beta|^{2r-2} \tilde{\gamma} + \tau \delta_{1\alpha}K_{\beta\gamma,\beta}|\beta|^{2r-2} \tilde{\gamma} + \tau^2 \delta_{1\alpha}\delta_{1\beta}K_{\beta\gamma,\beta}|\beta|^{2r-3}
\]

\[ + M_{\alpha\beta,\gamma}|\beta|^{2r} - \tau \delta_{1\beta}M_{\alpha\beta,\gamma}|\beta|^{2r-2} \tilde{\gamma} - \tau \delta_{1\alpha}M_{\beta\gamma,\beta}|\beta|^{2r-2} \tilde{\gamma} - \tau^2 \delta_{1\alpha}\delta_{1\beta}M_{\beta\gamma,\beta}|\beta|^{2r-2} \tilde{\gamma}
\]

(4.4)

And similarly, the expression for \( g_{\alpha\beta,\gamma\delta} \) in the coordinates of Lemma 4.2 is:

\[
\hat{\tilde{g}}_{\alpha\beta,\gamma\delta} = g_{\alpha\beta,\gamma\delta}^0 + K_{\alpha\beta,\gamma\delta}|\beta|^{2r} + \tau \delta_{1\beta}K_{\alpha\beta,\gamma\delta}|\beta|^{2r-2} \tilde{\gamma} + \tau \delta_{1\alpha}K_{\beta\delta,\beta\gamma}|\beta|^{2r-2} \tilde{\gamma} + \tau^2 \delta_{1\alpha}\delta_{1\beta}K_{\beta\delta,\beta\gamma}|\beta|^{2r-3}
\]

\[ + M_{\alpha\beta,\gamma\delta}|\beta|^{2r} - \tau \delta_{1\beta}M_{\alpha\beta,\gamma\delta}|\beta|^{2r-2} \tilde{\gamma} - \tau \delta_{1\alpha}M_{\beta\delta,\beta\gamma}|\beta|^{2r-2} \tilde{\gamma} - \tau^2 \delta_{1\alpha}\delta_{1\beta}M_{\beta\delta,\beta\gamma}|\beta|^{2r-2} \tilde{\gamma}
\]

(4.5)

Also, for the second derivative terms we have in the special coordinates that:

\[
\hat{\tilde{g}}_{\alpha\beta,\gamma\delta} = g_{\alpha\beta,\gamma\delta}^0 + K_{\alpha\beta,\gamma\delta}|\beta|^{2r} + \tau \delta_{1\beta}K_{\alpha\beta,\gamma\delta}|\beta|^{2r-2} \tilde{\gamma} + \tau \delta_{1\alpha}K_{\beta\delta,\beta\gamma}|\beta|^{2r-2} \tilde{\gamma} + \tau^2 \delta_{1\alpha}\delta_{1\beta}K_{\beta\delta,\beta\gamma}|\beta|^{2r-3}
\]

\[ + M_{\alpha\beta,\gamma\delta}|\beta|^{2r} - \tau \delta_{1\beta}M_{\alpha\beta,\gamma\delta}|\beta|^{2r-2} \tilde{\gamma} - \tau \delta_{1\alpha}M_{\beta\delta,\beta\gamma}|\beta|^{2r-2} \tilde{\gamma} - \tau^2 \delta_{1\alpha}\delta_{1\beta}M_{\beta\delta,\beta\gamma}|\beta|^{2r-2} \tilde{\gamma}
\]

\[ - \tau^2 (1-\tau) \delta_{1\alpha}\delta_{1\beta}\delta_{1\gamma}\delta_{1\delta} K_{\beta\delta,\beta\gamma}|\beta|^{2r-3} \tilde{\gamma} + \tau (1-\tau) \delta_{1\alpha}\delta_{1\beta}\delta_{1\gamma}\delta_{1\delta} K_{\beta\delta,\beta\gamma}|\beta|^{2r-3} \tilde{\gamma}
\]

\[ + \tau^2 (1-\tau) \delta_{1\alpha}\delta_{1\beta}\delta_{1\gamma}\delta_{1\delta} K_{\beta\delta,\beta\gamma}|\beta|^{2r-3} \tilde{\gamma} + \tau^2 (1-\tau) \delta_{1\alpha}\delta_{1\beta}\delta_{1\gamma}\delta_{1\delta} K_{\beta\delta,\beta\gamma}|\beta|^{2r-3} \tilde{\gamma}
\]

(4.6)
Which in the special coordinate system adopted at a given point becomes:
\[
\bar{g}_{\alpha\beta,\gamma\delta} = \frac{g^{0}_{\alpha,\beta,\gamma,\delta}}{1} + K_{\bar{\alpha}\bar{\beta},\bar{\gamma}\bar{\delta}}|1|^{2r} + \tau(\delta_{1,\beta}K_{\bar{\alpha}\gamma\delta} + \delta_{1,\delta}K_{\bar{\alpha}\beta\gamma})|1|^{2r-2} - 1
\]
\[
+ \frac{g^{1}_{\alpha,\beta,\gamma,\delta}}{1} + \frac{g^{2}_{\alpha,\beta,\gamma,\delta}}{1} + \frac{g^{3}_{\alpha,\beta,\gamma,\delta}}{1} + \frac{g^{4}_{\alpha,\beta,\gamma,\delta}}{1}
\]
\[
(4.7)
\]

In order to study the behaviour of the terms with the inverse of the metric close to the divisor in its coordinate representation we shall need the following observation.

**Lemma 4.4.** For the inverse matrix \(g^{\mu\nu}\) we have:

- \(g^{11} = \frac{1}{\tau^2}|1|^{2(1-\tau)}\left(1 + \frac{\tau^2}{2}\right)|1|^{2(\tau-1)} + O(|1|^{2(1-\tau)})\),
- \(g^{\mu1} = O(|1|^{2(1-\tau)}), \) for \(\mu \neq 1\).

**Proof.** We shall derive the asymptotic behaviour of the elements of \(g^{\bar{\gamma}}\) by studying the components \(\frac{\det g^{\bar{\gamma}}_{ij}}{\det g}\), where \(g^{\bar{\gamma}}_{ij}\) is the minor obtained by removing the \(i\)-th row and \(j\)-th column. Since in the coordinate systems introduced in Lemma 4.2 the only unbounded component of the metric tensor will be \(g^{\bar{\gamma}}_{11} = 2(\tau - 1)O(|1|^{2(\tau - 1)}) + \) bounded terms, we see by expanding the determinant in the first row that:

\[
\det g = [\tau^2|1|^{2(1-\tau)}] \det g^{\bar{\gamma}}_{11} + \text{bounded terms}
\]

and that \(\det g^{\bar{\gamma}}_{11}\) is bounded. Therefore, \(\frac{\det g^{\bar{\gamma}}_{11}}{\det g} = \frac{\det g^{\bar{\gamma}}_{11}}{\tau^2|1|^{2(1-\tau)} - \tau^2|1|^{2(\tau - 1)}}\), where \(a_j\)'s are bounded functions. One can now expand the quotient as

\[
\frac{g^{11}}{\det g} = \frac{1}{\tau^2|1|^{2(1-\tau)}} \left[1 + \frac{\tau^2}{2}\right]|1|^{2(\tau-1)} + \frac{\tau^2}{\tau^2|1|^{2(1-\tau)}}\det g^{\bar{\gamma}}_{11}K|1|^{2(1-\tau)}
\]

(4.8)

As it will become clear in the calculations of the curvature tensor, it is indeed the first order expansion of the fraction that is of importance for our purpose. Indeed, when we do not have the extra term \(|1|^{2\tau}\) in the potential, the term that could produce the negative infinity in the curvature is the term \(\frac{\det g^{\bar{\gamma}}_{11}}{\det g}|1|^{2(1-\tau)}\). However, in our case this latter term is dominated by the term \(|1|^{2(\tau-\tau)}\).

For components \(g^{11}, \ i \neq 1\) we can also derive a similar expression, the difference being that we do not need first order expansion, the signs of the terms appearing are not known, and do not play a role in the unboundedness of curvature tensor.

Having found the derivatives and the inverse of the metric (as a matrix in local coordinates) we can now turn to finding the curvature. Let us recall that we have the following formula for the curvature of a Kähler metric:

\[
\bar{R}_{\alpha\beta,\gamma\delta} = -g_{\alpha\beta,\gamma\delta} + g^{\mu\nu}\frac{\partial{g_{\alpha\nu}}}{\partial{g^{\beta\mu}}}g_{\mu\nu,\gamma\delta}
\]

(4.9)
It is not hard to see that the only terms in the second derivative, $\dd g_{\alpha\beta,\gamma\delta}$, which need to be considered are the following: $\delta_1\delta_1\delta_1\delta_1 \left( \tau^2 (1 - \tau)^2 K |b^1|^{2r - 4} - \tau^2 (1 - \tau)^2 M |b^1|^{2r - 4} \right)$. This term is non-zero only when all indices are equal to 1. In the first order expression $g^{\mu\nu} g_{\alpha\nu,\gamma} g_{\mu\beta,\delta}$ one can first notice that $g^{11} = O(|b^1|^{2-2r})$. So, the only term in the product $g_{\alpha\nu,\gamma} g_{\mu\beta,\delta}$ that might stay unbounded after multiplying into the relevant components of $v$ and $w$ is the following

$$\begin{align*}
(\tau^2 (\tau - 1))& \delta_1\delta_1\delta_1\delta_1 \left( \tau^2 (\tau - 1)^2 K |b^1|^{2r - 4} - \tau^2 (\tau - 1)^2 M |b^1|^{2r - 4} \right) \\
& = \tau^2 (\tau - 1)^2 K |b^1|^{2r - 4} - \tau^2 (\tau - 1)^2 M |b^1|^{2r - 4} \\
& \tag{4.10}
\end{align*}$$

and that only when $\alpha = \beta = \gamma = \delta = 1$. It takes a straightforward verification to see that after multiplying by relevant components of $v$ and $w$, the only potentially unbounded terms are in $R_{1111}$.

Let us study the terms in $R_{\alpha\beta\gamma\delta} v^\alpha w^\beta \bar{w}^\gamma \bar{w}^\delta$ separately. First, we take

$$\begin{align*}
& - g_{\alpha\beta,\gamma\delta} v^\alpha w^\beta \bar{w}^\gamma \bar{w}^\delta \\
& \text{By directly inspecting the terms in (4.10) we see that the only unbounded term in (4.10) is the highest order term} \\
& \left( - \tau^2 (1 - \tau)^2 K |b^1|^{2r - 4} + \tau^2 (1 - \tau)^2 M |b^1|^{2r - 4} \right) v^1 \bar{v}^1 w^1 \bar{w}^1. \\
& \text{ Behaviour of the term } g^{\mu\nu} g_{\alpha\nu,\gamma} g_{\mu\beta,\delta} v^\alpha w^\beta \bar{w}^\gamma \bar{w}^\delta \text{ can similarly be understood as follows: all the terms appearing in the product are bounded except the term} \\
& g^{\mu\nu} \delta_1\delta_1\delta_1\delta_1 \delta_1 \delta_1 \left( \tau^2 (1 - \tau) K |b^1|^{2r - 4} \right) \\
& \tag{4.11}
\end{align*}$$

which, using (1.8) can be written as

$$\begin{align*}
& \frac{1}{\tau^2} \left( 1 + \frac{\tau^2}{\tau^2} |b^1|^{2(r' - \tau)} + O(|b^1|^{2(1 - \tau)}) \right) \left[ \tau^4 (1 - \tau)^2 |b^1|^{2r - 4} \\
& - 2\tau^2 \tau^2 (1 - \tau) (1 - \tau') |b^1|^{4r' - 2r - 4} + \tau^2 (1 - \tau)^2 |b^1|^{2r - 4} v^1 \bar{v}^1 w^1 \bar{w}^1 \right] \\
& = \left( \tau^2 (1 - \tau)^2 |b^1|^{2r - 4} - 2\tau^2 (1 - \tau) (1 - \tau') |b^1|^{6r' - 4r - 4} + \frac{\tau^4}{\tau^2} (1 - \tau)^2 |b^1|^{4r' - 2r - 4} \\
& + \tau^2 (1 - \tau)^2 |b^1|^{2r - 4} - 2\tau^2 (1 - \tau) (1 - \tau') |b^1|^{6r' - 4r - 4} + \frac{\tau^6}{\tau^4} (1 - \tau)^2 |b^1|^{4r' - 2r - 4} \\
& + O(|b^1|^{-2}) \right) v^1 \bar{v}^1 w^1 \bar{w}^1. \\
& \text{Having obtained these expressions for the individual terms appearing in the components of the curvature tensor, we can now put them together to obtain the following expression for two unit vectors } v, w \\
& R_{1111} v^1 \bar{v}^1 w^1 \bar{w}^1 = O(1) + \left[ - \tau^2 (1 - \tau)^2 |b^1|^{2r - 4} + \tau^2 (1 - \tau)^2 |b^1|^{2r - 4} + \tau^2 (1 - \tau)^2 |b^1|^{2r - 4} - 2\tau^2 (1 - \tau) (1 - \tau') |b^1|^{2r - 4} + \frac{\tau^4}{\tau^2} (1 - \tau)^2 |b^1|^{4r' - 2r - 4} \\
& + \tau^2 (1 - \tau)^2 |b^1|^{2r - 4} - 2\tau^2 (1 - \tau) (1 - \tau') |b^1|^{6r' - 4r - 4} + \frac{\tau^6}{\tau^4} (1 - \tau)^2 |b^1|^{4r' - 2r - 4} + O(|b^1|^{-2}) \right] v^1 \bar{v}^1 w^1 \bar{w}^1. \\
& \text{As one may observe, starting from the lowest power, the terms with } |b^1|^{2r - 4}, \text{ cancel, and the next lowest power, } |b^1|^{2r - 4}, \text{ has a positive coefficient: } \tau^2 (1 - \tau)^2 + \tau^2 (1 - \tau)^2 - 2\tau^2 (1 - \tau) (1 - \tau') = \right] v^1 \bar{v}^1 w^1 \bar{w}^1.
\end{align*}$$
$\tau'^2 (\tau' - \tau)^2 > 0$. This means we can disregard the terms with larger exponents altogether and the behaviour of the curvature is dominated by the positive term $(\tau' - \tau)^2 |b^{-1}|^{2\tau' - 4}$. In particular, this means as $|z_1| \to 0$, the $R_{1111} v^1 w^1 \bar{w}^1$ blows up in the positive direction and is bounded from below.

Finally, it should be noted that the components of the curvature tensor other than the $1\bar{1}1\bar{1}$ component are bounded when multiplied by the corresponding elements of $g^{\alpha \bar{\beta}}$. □

**Remark 4.5.** Although we have not detailed this here, but one can repeat similar calculations for a metric of the form $\omega_0 + dd^c (|s|^2 h^\pm \epsilon |s|^{2\tau'})$, $\epsilon > 0$, and observe that for such metric the curvature indeed blows up in the negative or the positive direction -depending on the sign before $\epsilon$- at the rate of $|s|^{2\tau' - 4}$. In particular, this gives an example of a smooth edge-cone metric whose curvature becomes unbounded from either below or above close to the divisor at a rate faster than $|s|^2 h$. This also means one can construct such edge-cone metrics even when $\tau < \frac{1}{2}$. We find it worthwhile to emphasise that this phenomenon is not exclusive to higher dimensions. In an identical fashion it is indeed possible to construct cone metrics on $\mathbb{C}$ whose curvatures are unbounded from below or above.

### 5. Proof of the main results—Solving the equation

Having made the observations in §3 and 4, we can now prove Theorem 1.2.

**Proof of Theorem 1.2.** In order to do so, we shall approximate the equation by a family of equations with smooth components. In this section we first establish uniform a priori estimates which will allow taking limit of the family of solutions.

As we have mentioned before, the way we solve the equation with edge reference metric is by approximating the edge metric by a family of smooth metrics and deriving estimates independent of the parameter of the sequence. We also take a family of smooth functions $\{f_\epsilon\}_\epsilon$ approximating the source term $f$. That is, we solve the following family of equations

$$ (\tilde{\omega}_\epsilon + dd^c \phi_\epsilon)^n = \tilde{\omega}_\epsilon^n e^{\epsilon} $$

It is not hard to see that the right hand side converges in $L^p(\omega_0)$ for some $p$ depending on the angle. The fundamental theorem of Kołodziej [14] on the stability in $L^p$ of the Monge–Ampère operator comes to our aid to guarantee that since the right hand side converges in $L^p$, the potentials obtained as solutions converge to the unique Hölder continuous solution. This also takes care of the $L^\infty$ estimate automatically.

Just as in the classical case, in order to derive estimates on the complex hessian, we derive an upper bound on the laplacian. This is the content of Theorem 5.1.

Finally, we need to derive an estimate on the modulus of continuity of the second derivative, namely, the $C^{2,\theta}$ estimates à la Evans and Krylov. This will be done in the following section. One can then repeat the usual method based on taking a sequence of solutions, $\{\phi_{\epsilon j}\}_j$, and if necessary pass to a subsequence and by evoking the uniform estimates prove that there is a solution as $\epsilon \to 0^+$. This will thus conclude the proof of Theorem 1.2. □

Without mentioning, the functions and metric appearing below are assumed to be the ones corresponding to the $\epsilon$-approximation of the equation.

**Theorem 5.1.** Let $\phi$ be a $C^3$ solution of Equation 5.1. Then, we have the following a priori bound:

$$ \|dd^c \phi\|_{\omega_0} \leq C $$

**Proof.**

(5.1) $$(\tilde{\omega}_\epsilon + dd^c \phi_\epsilon)^n = \tilde{\omega}_\epsilon^n e^{\epsilon}$$

It is not hard to see that the right hand side converges in $L^p(\omega_0)$ for some $p$ depending on the angle. The fundamental theorem of Kołodziej [14] on the stability in $L^p$ of the Monge–Ampère operator comes to our aid to guarantee that since the right hand side converges in $L^p$, the potentials obtained as solutions converge to the unique Hölder continuous solution. This also takes care of the $L^\infty$ estimate automatically.

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Without mentioning, the functions and metric appearing below are assumed to be the ones corresponding to the $\epsilon$-approximation of the equation.

**Theorem 5.1.** Let $\phi$ be a $C^3$ solution of Equation 5.1. Then, we have the following a priori bound:

$$ \|dd^c \phi\|_{\omega_0} \leq C $$

(5.2) $$(\tilde{\omega}_\epsilon + dd^c \phi_\epsilon)^n = \tilde{\omega}_\epsilon^n e^{\epsilon}$$
where \( C = C \left( \| \phi \|_{L^\infty}, \inf R^\epsilon (., ., .), \inf f_\epsilon, \inf (\Delta_\omega f_\epsilon) ^- \right) \).

**Proof.** Other than a few observations, the argument is very similar to the classical argument. The reader can consult §[17]. We shall drop the subscript \( \epsilon \) in the rest of the proof. Let us set \( \Delta \) and \( \Delta' \) to be the laplacian associated to \( \tilde{\omega} \) and \( \bar{\omega} + dd^c \phi \) respectively. In the rest of this section, we drop the subscript \( \epsilon \). Let us start with the following well-known inequality:

\[
(5.3) \quad \Delta' \log (n + \Delta \phi) \geq \frac{1}{n + \Delta \phi} \left[ \Delta f + \sum_{\alpha, \beta} \left( - R_{\alpha \bar{\alpha} \beta \bar{\beta}} + R_{\alpha \bar{\alpha} \beta \bar{\beta}} \frac{1 + \phi_{\bar{\alpha} \bar{\alpha}}}{1 + \phi_{\beta \bar{\beta}}} \right) \right]
\]

The reader can refer to §3.2 of [17] for example. Using the symmetries of the curvature tensor, we now rewrite the expressions containing curvature terms as

\[
\sum_{\alpha, \beta} \left( - R_{\alpha \bar{\alpha} \beta \bar{\beta}} + R_{\alpha \bar{\alpha} \beta \bar{\beta}} \frac{1 + \phi_{\bar{\alpha} \bar{\alpha}}}{1 + \phi_{\beta \bar{\beta}}} \right) = 2 \sum_{\alpha < \beta} \left( \frac{1 + \phi_{\bar{\alpha} \bar{\alpha}}}{1 + \phi_{\beta \bar{\beta}}} + \frac{1 + \phi_{\beta \bar{\beta}}}{1 + \phi_{\alpha \bar{\alpha}}} - 2 \right) R_{\alpha \bar{\alpha} \beta \bar{\beta}}
\]

In the original way the laplacian estimate was derived, it depended on the upper bound of the scalar curvature and the lower bound of the bi-sectional curvature. This observation, which is already used in [15], allows dropping the former requirement.

Let \( C \) be a positive constant so that \( R_{\alpha \bar{\alpha} \beta \bar{\beta}} u^\alpha \bar{u}^\alpha v^\beta \bar{v}^\beta \geq - C |u_\omega| v |\omega| \). Then, upon noticing that the terms \( 1 + \phi_{\bar{\alpha} \bar{\alpha}} + 1 + \phi_{\beta \bar{\beta}} - 2 \) are all non-negative, we deduce that

\[
(5.4) \quad \Delta' \log (n + \Delta \phi - C_2 \phi) \geq \frac{1}{n + \Delta \phi} \left[ \Delta f - \sum_{\alpha, \beta} C \left( \frac{1 + \phi_{\bar{\alpha} \bar{\alpha}}}{1 + \phi_{\beta \bar{\beta}}} + \frac{1 + \phi_{\beta \bar{\beta}}}{1 + \phi_{\alpha \bar{\alpha}}} - 2 \right) \right] - C_2 n + C_2 \sum_\alpha \frac{1}{1 + \phi_{\alpha \bar{\alpha}}}
\]

In the expression above, for a function \( v, v^- := \min \{v, 0\} \).

After choosing \( C_2 \) to be sufficiently large, it will require a standard application of the maximum principle to conclude the argument. We deduce that the quantity \( \Delta \phi \) is bounded by a constant \( C_4 = C_4(\Delta f)^-, \inf_{\alpha, \beta} R_{\alpha \bar{\alpha} \beta \bar{\beta}}, \| \phi \|_0 \). All of these quantities are finite by the fact that we have chosen a reference metric whose curvature is bounded from below, along with the observations in Propositions [13] that guarantee the the laplacian of the Ricci potential is bounded from below. \( \square \)

Having proved the bounds on the complex hessian of the solution, we can now turn to proving the H"older continuity of the second derivative using a version of the so-called Evans-Krylov theory to deduce that the solution \( \phi_0 \), obtained as the limit of \( \{ \phi_\epsilon \}_\epsilon \), belongs to the edge H"older space \( C^{2, \theta}_\tau \) for some exponent \( \theta \). This will be the the subject of the next section.

### 5.1. Change of the reference metric redux: Evans-Krylov theory

The \( C^{2, \theta}_\tau \) estimates have been studied before. In [7] such an estimate has been derived for K"ahler-Einstein metrics. Also, in [3] the Evans-Krylov theory has been extended to the edge-cone setting with the restriction of \( \tau \leq \frac{2}{3} \), and in [13] for \( \tau \in (0, 1) \). Later, an interesting argument based on approximating the cone angle by rational numbers and using the geometry of rational cone angles was developed in [11]. We assume the validity Evans-Krylov theory on \( \mathbb{C}_\tau \times \mathbb{C}^{n-1} \), which is what has been established in [3][13][9][11].

So, we first localise the equation as detailed in the next paragraph. In the backdrop there is indeed a local change of the background metric which allows considering the equation on the flat edge model. This is necessary since in the edge case, unlike the smooth case, the second derivatives
of the reference metric might not be bounded, so taking derivatives of $\omega$ in its current form might not help with the proof.

In the case of the edge metrics, the equation that the metric satisfies is the following

\[ g(\omega_\phi) = \mu \omega_\phi + 2\pi (1 - \tau)[D]. \]

In the unit ball in $C_\tau \times C^{n-1}$ the twisted Kähler-Einstein equation can be written as

\[ -dd^c \log(dd^c w)^n = dd^c \log |z_1|^{2-2\tau} + \mu dd^c w \]

Similar to the smooth case, we obtain the following equation for $w$:

\[ \log \det(w,\alpha\bar{\beta}) = \log |z_1|^{2\tau - 2} - \mu w + \mathcal{H} \]

for some pluri-harmonic function $\mathcal{H}$.

Of course since we have no boundary conditions, there are infinitely many choices of a pluri-harmonic function $\mathcal{H}$ which in general satisfy no uniformity of any sort. Noting the fact that this equation is satisfied locally by all Kähler-Einstein potentials, it comes as little surprise that with no boundary conditions prescribed and an undetermined source term, $\mathcal{H}$, Equation 5.6 has too many degrees of freedom. It might, at the face value, seem like by doing so we have lost a great deal of information. However, when one has readily obtained a bound on the complex hessian of the solution $\phi$ on the manifold, it translates to the fact that $dd^c w$ can be assumed to be bounded. Further, since $\phi$, and hence $w$, are bounded in $L^\infty$, along with the fact that the form $dd^c w$ is bounded, we see from the equation that $\|\mathcal{H}\|_{L^\infty}$ is a priori bounded. Then, since $\mathcal{H}$ is a pluri-harmonic, and in particular a harmonic function, it is a well-known fact that its oscillation is bounded in terms of the oscillation of $\mathcal{H}$. (One way to prove this fact is by an application of the mean value theorem. For such a proof, see [12]). Although, since we take $\partial_{\kappa\bar{\lambda}}$ derivatives, and $\mathcal{H}_{\kappa\bar{\lambda}} = 0$, we only need the bound on the first derivative of $\mathcal{H}$.

Now, in the global equation, (1.4), the Ricci potential, $f$, is bounded, and in our local equation $\omega_\tau$ is equivalent to the standard edge-cone metric, $\omega_\mathcal{H}$. This, along with the fact that the potential $w$ is bounded allows us to immediately see that the oscillation of $\mathcal{H}$ is bounded, which, in turn, gives a bound on all higher derivatives of $\mathcal{H}$. We can summarise these observations as the following lemma:

Now one may apply the Evans-Krylov theory on the flat space to (5.6) and establish the membership of $w$, and hence $\phi$, in $C^{2,\theta}_\tau$ for some $\theta \in (0,1)$.

In the case of prescribed Ricci form in Theorem 1.2 there is another consideration to be taken into account, namely the regularity of the Ricci potential when the problem is localised. It is not hard to see that the Ricci potential stays Hölder continuous after the change of the background metric and also after localising the problem. Assuming the Hölder continuity of the Ricci potential when the equation is localised with the flat reference metric, we can apply the result in [9]. Amongst the Evans-Krylov estimates derived in the edge-cone setting, the result in this reference is the only one which does not require differentiating the equation, and hence, does not require the boundedness of derivatives of the Ricci potential. We may summarise this as it is stated below.

**Theorem 5.2.** Assume that in Equation (1.4) the function $\phi$ satisfies:

\[ ||\phi||_{L^\infty}, \ ||dd^c \phi||_{\omega_\tau}, \ ||f||_{C^\theta_\tau} \leq C \]

for some constant $C > 0$.

We notice that the condition on the Hölder continuity of $f$ is satisfied in particular when the prescribed Ricci form, $\tilde{g}$, can be locally realised by Hölder continuous potentials, which is one of the conditions is Theorem part (a).
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