Characteristic Genera of Closed Orientable 3-Manifolds

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Abstract. A complete invariant defined for (closed connected orientable) 3-manifolds is an invariant defined for the 3-manifolds such that any two 3-manifolds with the same invariant are homeomorphic. Further, if the 3-manifold itself can be reconstructed from the data of the complete invariant, then it is called a characteristic invariant defined for the 3-manifolds. In a previous work, a characteristic lattice point invariant defined for the 3-manifolds was constructed by using an embedding of the prime links into the set of lattice points. In this paper, a characteristic rational invariant defined for the 3-manifolds called the characteristic genus defined for the 3-manifolds is constructed by using an embedding of a set of lattice points called the PDelta set into the set of rational numbers. The characteristic genus defined for the 3-manifolds is also compared with the Heegaard genus, the bridge genus and the braid genus defined for the 3-manifolds. By using this characteristic rational invariant defined for the 3-manifolds, a smooth real function with the definition interval (−1, 1) called the characteristic genus function is constructed as a characteristic invariant defined for the 3-manifolds.

1. Introduction

It is classically well-known\(^1\) that every closed connected orientable surface \(F\) is characterized by the maximal number, say \(n(\geq 0)\) of mutually disjoint simple loops \(\omega_i (i = 1, 2, \ldots, n)\) in \(F\) such that the complement \(F \setminus \bigcup_{i=1}^{n} \omega_i\) is connected. This number \(n\) is called the genus of \(F\). We consider the union \(L^0\) of \(n\) mutually disjoint 0-spheres \(S^0_i (i = 1, 2, \ldots, n)\) in the 2-sphere \(S^2\) (namely, the set of \(2n\) points in \(S^2\)) as an \(S^0\)-link with \(n\) components. Then the surface characterization stated above

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\(^1\)cf. B. von Kerékjártó [15].
is dual to the statement that the surface $F$ of genus $n$ is obtained as the 1-handle surgery manifold $\chi(L^0)$ of $S^2$ along an $S^0$-link $L^0$ with $n$ components. Let $\mathcal{M}^2$ be the set of (the unoriented types of) closed connected orientable surfaces, and $\mathbb{L}^0$ the set of (unoriented types of) $S^0$-links. Since any two $S^0$-links with the same number of components belong to the same type, we have a well-defined bijection

$$\alpha^0 : \mathcal{M}^2 \rightarrow \mathbb{L}^0$$

sending a surface $F \in \mathcal{M}^2$ to an $S^0$-link $L^0 \in \mathbb{L}^0$ such that $\chi(L^0) = F$. Further, let $\mathcal{X}^0$ be the set of non-negative integers, and $\mathbb{G}^0$ the set of (the isomorphism classes of) “the link groups” $\pi_1(S^2 \setminus L)$ of all $S^0$-links $L^0 \in \mathbb{L}^0$. Then we have further two natural bijections

$$\sigma^0 : \mathbb{L}^0 \rightarrow \mathcal{X}^0, \quad \pi^0 : \mathbb{L}^0 \rightarrow \mathbb{G}^0$$

such that $\sigma^0(L^0) = n$ and $\pi^0(L^0) = \pi_1(S^2 \setminus L)$ for an $S^0$-link $L^0$ with $n$ components, respectively, so that we have the composite bijections

$$g^0 = \sigma^0 \alpha^0 : \mathcal{M}^2 \rightarrow \mathcal{X}^0, \quad \pi^0 = \pi^0 \alpha^0 : \mathcal{M}^2 \rightarrow \mathbb{G}^0.$$

For every surface $F \in \mathcal{M}^2$, the number $g^0(F) = n$ is equal to the genus of $F$, and the group $\pi_1^0(F)$ is a free group of rank $2n - 1$ (if $n \geq 1$) or the trivial group $\{1\}$ (if $n = 0$). Thus, the genus $g^0(F)$ determines the $S^0$-link $\alpha^0(F)$, the group $\pi_1^0(F)$ and the surface $F$ itself. As we discussed in the paper [5], an analogous argument is possible for closed connected orientable 3-manifolds, although the existence of non-trivial links in the 3-sphere $S^3$ makes the classification complicated. Here, for convenience we explain an idea of this argument of [5] briefly. Let $\mathcal{M}$ be the set of (unoriented types of) closed connected orientable 3-manifolds. Let $\mathcal{L}$ be the set of (unoriented types of) links in $S^3$ (including the knots as one-component links). A lattice point of length $n$ is an element $x$ of $\mathbb{Z}^n$ for the natural number $n$ where $\mathbb{Z}$ denotes the set of integers.

In this paper, the empty lattice point $\emptyset$ of length 0 and the empty knot $\emptyset$ are also considered. Let $\mathcal{X}$ be the set of all lattice points. We have a canonical map

$$cl : \mathcal{X} \rightarrow \mathcal{L}$$

sending a lattice point $x$ to a closed braid diagram $cl(x)$, which is surjective by the Alexander theorem (cf. J. S. Birman [1]). It was shown in [5] that every well-order of the set $\mathcal{X}$ induces an injection

$$\sigma : \mathcal{L} \rightarrow \mathcal{X}$$

which is a right inverse of the map $cl$. In particular, by taking the canonical well-order which is explained in § 2, we consider the subset $\mathcal{L}^p \subset \mathcal{L}$ consisting of prime links as a well-ordered set with the order inherited from $\mathcal{X}$ by $\sigma$, where the two-component trivial link is excluded from $\mathcal{L}^p$. The length $\ell(L)$ of a prime link $L \in \mathcal{L}^p$ is the length $\ell(\sigma(L))$ of the lattice point $\sigma(L)$. Let $\mathbb{G}$ be the set of (isomorphism
types of) the link groups \( \pi_1(S^3 \setminus L) \) for all links \( L \) in \( S^3 \). Let \( \pi : \mathbb{L} \to \mathbb{G} \) be the map sending a link \( L \) to the link group \( \pi_1(S^3 \setminus L) \). Let \( \mathbb{L}^\pi \) be the subset of \( \mathbb{L}^p \) consisting of a \( \pi \)-minimal link, that is, a prime link \( L \) such that \( L \) is the initial element of the subset

\[
\{ L' \in \mathbb{L}^p \mid \pi_1(S^3 \setminus L') = \pi_1(S^3 \setminus L) \}.
\]

We are interested in this subset \( \mathbb{L}^\pi \) because it has a crucial property that the restriction of \( \pi \) to \( \mathbb{L}^\pi \) is injective. Since the restriction of \( \sigma \) to \( \mathbb{L}^\pi \) is also injective, we can consider \( \mathbb{L}^\pi \) as a well-ordered set by the order induced from the order of \( \chi \). In [4], we showed that the set

\[
\mathbb{L}^\pi(M) = \{ L \in \mathbb{L}^\pi \mid \chi(L, 0) = M \}
\]

is not empty for every 3-manifold \( M \in \mathbb{M} \), where \( \chi(L, 0) \) denotes the 0-surgery manifold of \( S^3 \) along \( L \) and we define \( \chi(L, 0) = S^3 \) when \( L \) is the empty knot \( \phi \). By R. Kirby’s theorem [16] on the Dehn surgeries of framed links, we note that the set \( \mathbb{L}^\pi(M) \) is defined in terms of only links so that any two \( \pi \)-minimal links in \( \mathbb{L}^\pi(M) \) are related by two kinds of Kirby moves and choices of orientations of \( S^3 \). Sending every 3-manifold \( M \) to the initial element of \( \mathbb{L}^\pi(M) \) induces an embedding

\[
\alpha : \mathbb{M} \to \mathbb{L}
\]

with \( \chi(\alpha(M), 0) = M \) for every 3-manifold \( M \in \mathbb{M} \), which further induces two embeddings

\[
\sigma_\alpha = \sigma \alpha : \mathbb{M} \to \mathbb{X}, \quad \pi_\alpha = \pi \alpha : \mathbb{M} \to \mathbb{G}.
\]

By a special feature of the 0-surgery, the \( S^0 \)-link \( \alpha(M) \cap S^2 \) in \( S^2 \) produces a surface \( \chi(\alpha(M) \cap S^2) \) naturally embedded in \( M \) with \( \alpha^0(\chi(\alpha(M) \cap S^2)) = \alpha(M) \cap S^2 \) for every 2-sphere \( S^2 \) in \( S^3 \) meeting the link \( \alpha(M) \) transversely. In this sense, the embedding \( \alpha \) is an extension of the embedding \( \alpha^0 \). In this construction, we can reconstruct the link \( \alpha(M) \), the group \( \pi_\alpha(M) \), and the 3-manifold \( M \) itself from the lattice point \( \sigma(M) \in \mathbb{X} \). Thus, we have constructed the embeddings \( \sigma, \sigma_\alpha \) and \( \pi_\alpha \) analogous to the embeddings \( \sigma, \sigma_\alpha \) and \( \pi_\alpha \), respectively. The length \( \ell(M) \) of a 3-manifold \( M \in \mathbb{M} \) is the length \( \ell(\sigma_\alpha(M)) \) of the lattice point \( \sigma_\alpha(M) \). In [14], the 3-manifolds of lengths \( \leq 10 \) are classified (see also [9, 11, 12]). In this process, the prime links and their exteriors of lengths \( \leq 10 \) have been earlier classified (See [6, 7, 8, 10]). In general, an invariant \( \text{Inv} \) defined for a family of topological objects is complete if any two members \( A \) and \( A' \) with \( \text{Inv}(A) = \text{Inv}(A') \) are homeomorphic. The complete invariant \( \text{Inv}(A) \) is a characteristic invariant if the object \( A \) can be reconstructed from data of \( \text{Inv}(A) \). For example, the group invariant \( \pi_\alpha(M) \) is a complete invariant defined for the 3-manifolds \( M \in \mathbb{M} \) taking the value in finitely presented groups and the lattice point \( \sigma_\alpha(M) \) is a characteristic invariant defined for the 3-manifolds \( M \in \mathbb{M} \) taking the value in lattice points. For an interval \( I \subset \mathbb{R} \), we put \( I_\mathbb{Q} = I \cap \mathbb{Q} \), where \( \mathbb{R} \) and \( \mathbb{Q} \) denote the sets of real numbers and rational numbers, respectively.
In this paper, we consider a lattice point set \( P\Delta \) called the \( P\Delta \) set such that
\[
\sigma(\mathcal{M}) \subset \sigma(\mathcal{L}) \subset P\Delta \subset X.
\]
An embedding \( g : P\Delta \to [0, +\infty) \) called the characteristic genus is constructed so that the image \( g(S) \) of every subset \( S \subset P\Delta \) containing the empty lattice point \( \emptyset \) and the zero lattice point \( 0 \in \mathbb{Z} \) (called a \( P\Delta \) subset) is a characteristic invariant defined for the set \( S \). By taking \( S = \sigma(\mathcal{L}) \), the characteristic genus \( g(L) \) defined for the prime links \( L \in \mathcal{L} \) is obtained. By taking \( S = \sigma(\mathcal{M}) \), the characteristic genus \( g(M) \) defined for the 3-manifolds \( M \in \mathcal{M} \) is obtained.

An explanation of the \( P\Delta \) set is made in §2. A construction of the embedding \( g \) is done in §3. In §4, some properties of the characteristic genera of the 3-manifolds are stated together with the calculation results of the 3-manifolds of lengths \( \leq 7 \). In particular, the characteristic genus \( g(M) \) for a 3-manifold \( M \) is compared with the Heegaard genus \( g_h(M) \), the bridge genus \( g_b(M) \) and the braid genus \( g_{br}(M) \). In §5, from the characterstic genus \( g \), we construct a smooth real function \( G_\sigma(t) \) with the definition interval \((-1, 1)\) for every \( P\Delta \) subset \( S \) which is a characteristic invariant defined for the set \( S \). By taking \( S = \sigma(\mathcal{L}) \), the characteristic prime link function \( G_{L'}(t) \) is obtained as a characteristic invariant defined for the prime link set \( \mathcal{L} \). By taking \( S = \sigma(\mathcal{M}) \), the characteristic genus function \( G_M(t) \) is obtained as a characteristic invariant defined for the 3-manifold set \( \mathcal{M} \).

Concluding this introductory section, we mention here some analogous invariants derived from different viewpoints. Y. Nakagawa defined in [18] a family of integer-valued characteristic invariants of the set of knots by using R. W. Ghrist’s universal template (although a generalization to oriented links appears difficult). Also, J. Milnor and W. Thurston defined in [17] a non-negative real-valued invariant defined for the closed connected 3-manifolds with the property that if \( N \to N \) is a degree \( n \geq 2 \) connected covering of a closed connected 3-manifold \( N \), then the invariant of \( N \) is \( n \) times the invariant of \( N \), so that it does not classify lens spaces.

2. The Range of the Prime Links in the Set of Lattice Points

To investigate the image \( \sigma(\mathcal{L}) \subset X \), we need some notations on lattice points in \([5, 6, 7, 8, 9, 10, 11, 12, 14]\). For a lattice point \( x = (x_1, x_2, \ldots, x_n) \) of length \( \ell((x)) = n \), we denote the lattice points \((x_1, x_2, x_1)\) and \(\langle |x_1|, |x_2|, \ldots, |x_n| \rangle\) by \(x^T\) and \(|x|\), respectively. Let \(|x|_N\) be a permutation \((|x_j_1|, |x_j_2|, \ldots, |x_j_n|)\) of the coordinates \(|x_j|\) \((j = 1, 2, \ldots, n)\) of \(|x|\) such that
\[
|x_j_1| \leq |x_j_2| \leq \cdots \leq |x_j_n|.
\]
Let \( \min |x| = \min_{1 \leq i \leq n} |x_i| \) and \( \max |x| = \max_{1 \leq i \leq n} |x_i| \). The dual lattice point of \( x \) is given by \( \delta(x) = (x'_1, x'_2, \ldots, x'_n) \) where \( x'_i = \text{sign}(x_i)(\max |x| + 1 - |x_i|) \) and \( \text{sign}(0) = 0 \) by convention.

Defining \( \delta^0(x) = x \) and \( \delta^n(x) = \delta(\delta^{n-1}(x)) \) inductively, we note that \( \delta^2(x) \neq x \) in general, but \( \delta^{n+2}(x) = \delta^n(x) \) for all \( n \geq 1 \). For a lattice point \( y = (y_1, y_2, \ldots, y_m) \)
of length $m$, we denote by $(x, y)$ the lattice point

$$(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m).$$

of length $n + m$. For an integer $m$ and a natural number $n$, we denote by $m^n$ the lattice point $(m, m, \ldots, m)$ of length $n$. Also, we take $-m^n = (-m)^n$. A reason why we do not consider $L$ but $L^p$ is because we can use the following lemma which is shown in [5]:

**Lemma 2.1.** We have $cl \beta(x) = cl \beta(y)$ in $L$ modulo split additions of trivial links if and only if $y$ is obtained from $x$ by a finite number of the following transformations:

1. $(x, 0) \leftrightarrow x$.
2. $(x, y, -y T) \leftrightarrow x$.
3. $(x, y) \leftrightarrow x$ when $|y| > \max |x|$.
4. $(x, y) \leftrightarrow (x, z, y)$ when $\min |y| > \max |z| + 1$ or $\min |z| > \max |y| + 1$.
5. $(x, \pm y, y, y) \leftrightarrow (x, y + 1, y, \pm(y + 1))$ when $y(y + 1) \neq 0$.
6. $(x, y) \leftrightarrow (y, x)$.
7. $x \leftrightarrow x T \leftrightarrow -x \leftrightarrow -x T$.
8. $x \leftrightarrow x'$ when $cl \beta(x)$ is a disconnected link and $cl \beta(x')$ is obtained from $cl \beta(x)$ by changing the orientation of a component of $cl \beta(x)$.

There is an algorithm to obtain $cl \beta(x')$ from $cl \beta(x)$ in (8).

The **canonical order** of $\mathbb{X}$ is a well-order determined as follows: Namely, the well-order in $\mathbb{Z}$ is defined by $0 < 1 < -1 < 2 < -2 < 3 < -3 < \ldots$, and this order of $\mathbb{Z}$ is extended to a well-order in $\mathbb{Z}^n$ for every $n \geq 2$ so that for $x_1, x_2 \in \mathbb{Z}^n$ we define $x_1 < x_2$ if we have one of the following conditions (1)-(3):

1. $|x_1| < |x_2|$ by the lexicographic order (on the natural number order).
2. $|x_1| = |x_2|$ and $|x_1| < |x_2|$ by the lexicographic order (on the natural number order).
3. $|x_1| = |x_2|$ and $x_1 < x_2$ by the lexicographic order on the well-order of $\mathbb{Z}$ defined above.

Finally, for any two lattice points $x_1, x_2 \in \mathbb{X}$ with $\ell(x_1) < \ell(x_2)$, we define $x_1 < x_2$.

For a subset $S \subset \mathbb{X}$ and a non-negative integer $n$, let

$$S^{(n)} = \{ x \in S | \ell(x) \leq n \}$$

and call it the $n$-**fragment** of $S$. 


The Delta set is the subset $\Delta$ of $\mathbb{X}$ consisting of $\emptyset$, $0$ and all lattice points $x$ of lengths $n \geq 2$ satisfying $x_1 = 1$ and

$$1 \leq \min |x| \leq \max |x| \leq \frac{n}{2}$$

An important property of the Delta set $\Delta$ is that the $n$-fragment $\Delta^{(n)}$ of the Delta set $\Delta$ is a finite set for every non-negative integer $n$.

In our argument, the special lattice point $a_n$ of length $n$ defined for every even integer $n = 2m \geq 4$ is important. This lattice point $a_n$ is defined inductively as follows: Let $a_4 = (1, -2, 1, -2)$. Assuming that $a_{n} = (a'_{n}, (-1)^{m-1}m)$ is defined, we define

$$a_{n} + 2 = (a'_{n}, (-1)^{m}(m + 1), (-1)^{m-1}m, (-1)^{m}(m + 1)).$$

It is noted that the $n$th coordinate of $a_n$ is $(-1)^{m-1}m$ and $\text{cl}(a_n)$ is a 2-bridge knot or a 2-bridge link according to whether $m$ is even or odd, respectively. The $P$Delta set $P\Delta$ is the subset of the Delta set $\Delta$ consisting of

$$\emptyset, 0, 1^2, a_n \ (\text{for any even } n \geq 4)$$

and all lattice points $x$ of lengths $n \geq 3$ satisfying $x_1 = 1$ and

$$1 \leq \min |x| \leq \max |x| < \frac{n}{2}.$$

A sublattice point of a lattice point $x$ is a lattice point $x'$ such that $x = (u, x', v)$ for some lattice points $u, v$ (which may be the empty lattice point). When we write $|x| = (e_1, 2e_2, \ldots, me_m)$ for $m = \max |x|$, the non-negative integer $e_k$ is called the exponent of $k$ in $x$ and denoted by $\exp_k(x)$.

The DeltaStar set $\Delta^*$ is the subset of $P\Delta$ consisting of

$$\emptyset, 0, 1^n \ (\text{for any } n \geq 2), a_n \ (\text{for any even } n \geq 4)$$

and all the lattice points $x = (x_1, x_2, \ldots, x_n) \ (n \geq 5)$ which have all the following conditions (1)-(8):

1. $x_1 = 1$, $2 \leq |x_n| \leq \max |x| < \frac{n}{2}$.
2. $\exp_k(x) \geq 2$ for every $k$ with $1 \leq k \leq \max |x|$.
3. Every lattice point obtained from $x$ by permuting the coordinates of $x$ cyclically is not of the form $(x', x'')$ where $1 \leq \max |x'| < \min |x''|$.
4. For every $i < n$, one of the following identities or inequality holds: $|x_i| - 1 = |x_{i+1}|$, $x_i = x_{i+1}$ or $|x_i| < |x_{i+1}|$.

Further restricted subsets of the present Delta set are called Delta sets in [5, 6, 8, 9, 11, 12, 14].
(5) For a sublattice point $x'$ of $x$ such that $|x'| = (k, (k + 1)^c, k)$ and $\exp_k x = 2$ for some $k, e \geq 1$ or such that $|x'| = (k^e, k + 1, k)$ and $\exp_k(x) = e + 1$ for some $k, e \geq 1$, then $x' = \pm(k, -\varepsilon(k + 1)^c, k), \pm(\varepsilon k^e, -(k + 1), k)$ or $\pm(k, -(k + 1), \varepsilon k^e)$ for some $\varepsilon = \pm1$, respectively. Further, if $e = 1$, then $\varepsilon = 1$.

(6) For a sublattice point $x'$ of $x$ with $|x'| = (k + 1, k^c, k + 1)$ for some $k, e \geq 1$, then $x' = \pm(k + 1, \varepsilon k^e, k + 1)$ for some $\varepsilon = \pm1$. Further if $e = 1$, then $\varepsilon = -1$.

(7) $x$ is the initial element of the set of the lattice points obtained from every lattice point of $\pm x, \pm x^T, \pm \delta(x)$ and $\pm \delta(x)^T$ by permuting the coordinates cyclically.

(8) $|x|$ is not of the form $(|x'|, k + 1, k, (k + 1)^c, k)$ or $(|x'|, k + 1, k^2, k + 1, k)$ for $e \geq 1, k \geq 2$ and $\max |x'| \leq k$.

The following lemma is important to our argument:

**Lemma 2.3.** $\sigma_o(M) \subset \sigma(L^p) \subset \Delta^* \subset P\Delta$.

This lemma means that the collections of the links $\cl\beta(x)$ and the 3-manifolds $\chi(\cl\beta(x, 0)$ for all lattice points $x \in P\Delta$ contain all the prime links and all the 3-manifolds, respectively.

**Proof of Lemma 2.3.** In [5], the inclusions $\sigma_o(M) \subset \sigma(L^p) \subset \Delta$ are shown except counting the property (8). In [8, Lemma 3.6], we showed that $\sigma(L^p)$ has (8). Then to complete the proof, it is sufficient to show that if $x \in \sigma(L^p)$ has $\ell(x) = n \geq 4$ and $\max |x| = 2$, then we have $x = a_n$. Since $x$ is in $\Delta$, we see that $|x|_{\Delta} = (1^2, 2^2, \ldots, m^2)$. By the transformations (1)-(7) in Lemma 2.1, we see that unless $|x| = |a_n|$, we can transform $x$ into a smaller lattice point $x'$. Then considering $x$ itself, we conclude that unless $x = a_n$, the lattice point $x$ is transformed into a smaller lattice point $x''$. \hfill \Box

The DeltaStar set $\Delta^*$ approximates the prime link lattice point set $\sigma(L^p)$, but they are different. For example, the lattice point $(1^2, 2, -1^2, 2) \in \Delta^*$ does not belong to the prime link subset $\sigma(L^p)$. In fact, the prime link $L = cl\beta(1^2, 2, -1^2, 2) = 63$ appears as a smaller lattice point $(1^2, 2, 1^2, 2)$ in the tables of [5, 8, 12, 14].

### 3. Embedding the PDelta Set into the Set of Rational Numbers

For a lattice point $x = (x_1, x_2, \ldots, x_n) \in P\Delta$ with $n \geq 2$, we define the rational numbers

$$
\tau(x) = \frac{1}{n^{-1}}(x_2 + x_3 n + \cdots + x_n n^{n-2}),
\quad g(x) = n + \tau(x).
$$

For example, we have

$$
\tau(1^2) = \frac{1}{2}, \quad g(1^2) = 2 + \frac{1}{2}.
$$
By convention, we put:
\[ \tau(\emptyset) = g(\emptyset) = 0, \quad \tau(0) = 0, \quad g(0) = 1. \]

The rational number \( g(x) \) is called the \textit{characteristic genus} or simply the \textit{genus} of \( x \), and \( \tau(x) \) the \textit{decimal part} of the characteristic genus \( g(x) \) or the \textit{decimal torsion} of \( x \). According to whether the last coordinate \( x_n \) is positive or negative, the lattice point \( x \) is called to be \textit{ending-positive} or \textit{ending-negative}, respectively.

We show the following theorem:

\textbf{Theorem 3.1.} The map \( x \mapsto g(x) \) induces an embedding
\[ g : P\Delta \to [0, 1)_Q \]
such that for every \( x = (x_1, x_2, \ldots, x_n) \in P\Delta \) with \( n \geq 3 \) we have the following properties (1)-(3):

(1) According to whether \( x \) is ending-positive or ending-negative, we respectively
\[ g(x) \in (n, n + 1/2)_Q \quad \text{or} \quad g(x) \in (n-1/2, n)_Q. \]
In particular, the length \( \ell(x) \) is equal to the maximal integer not exceeding the number \( g(x) + 1/2 \).

(2) The lattice point \( x \in P\Delta \) is reconstructed from the value of \( g(x) \).

(3) There are only finitely many \( x \in P\Delta \) with
\[ g(x) \in (n - 1/2, n + 1/2)_Q. \]

Here is a note on the values on \( \emptyset \), \( 0 \) and \( 1^2 \).

\textbf{Remark 3.2.} The values \( \tau(\emptyset) = g(\emptyset) = 0, \quad \tau(0) = 0 \) and \( g(0) \) are not definite values. For example, As another choice, by a geometric meaning on the braids, the zero lattice point \( 0 \) may be considered as the lattice point \((1, -1)\) where the values \( \tau(1, -1) = -1/2 \) and \( g(1, -1) = 2 - 1/2 = 1 + 1/2 \) are taken. On the other hand, the lattice points \((1, -1)\) and \( 1^2 \) are considered as exceptional ones in the sense that the characteristic genus does not determine the decimal torsion uniquely as follows:
\[ g(1, -1) = 2 - 1/2 = 1 + 1/2 \quad \text{and} \quad g(1^2) = 2 + 1/2 = 3 - 1/2. \]

\textbf{Proof of Theorem 3.1.} To show the first half of (1), first consider a lattice point \( x \in P\Delta \) with \( |x_i| < \frac{n}{2} \) for all \( i \). Then we have \( |x_i| \leq \frac{n-1}{2} \) and
\[ |\tau(x) - \frac{x_n}{n}| \leq \frac{n-1}{2} \cdot \frac{1}{n^{n-1}} (1 + n + \cdots + n^{n-3}) \]
\[ = \frac{n-1}{2} \cdot \frac{1}{n^{n-1}} \cdot n^{n-2} - 1 \cdot \frac{1}{n^{n-1}} < \frac{1}{2n}. \]
Further, for every prime link $\tau$ defined to be $n$ lattice points with length $x_i$ for all $x_i$. Hence
\[
\ell(n) = (\frac{a_n - 2}{n^3} + \frac{a_{n-1}}{n^2} + \frac{a_n}{n}) \leq \frac{n - 1}{2} \cdot \frac{1}{n^{n-1}} (1 + n + \cdots + n^{n-5})
\]
\[
= \frac{n - 1}{2} \cdot \frac{1}{n^{n-1}} \cdot \frac{n^{n-4} - 1}{n - 1} = \frac{1}{2n^3} - \frac{1}{2n^{n-1}} < \frac{1}{2n^3}.
\]
For the sign $\varepsilon$ of $a_n$, we have
\[
\frac{a_n - 2}{n^3} + \frac{a_{n-1}}{n^2} + \frac{a_n}{n} = \varepsilon(\frac{1}{2n^2} - \frac{n - 2}{n^2} + \frac{1}{2}) = \varepsilon(n - 1)(n + 1),
\]
so that
\[-\frac{1}{2n^3} < \varepsilon(n - 1)(n + 1) < \frac{1}{2n^3}.
\]
This shows that the assertion of (1) holds for the lattice points $a_n$.

To show that $g$ is an embedding, let $\ell(x) = n \geq 3$. Then $g(x)$ is distinct from $g(0) = 0$, $g(12) = 1$ and $g(1^2) = 1 + \frac{1}{2}$. If the value of $g(x)$ is given, then the length $n$ of $x$ is uniquely determined by (1). For $x' = (x'_1, x'_2, \ldots, x'_n) \in P\Delta$, assume that
\[
g(x) = g(x') = n + \frac{x'_1}{n^{n-1}} + \cdots + \frac{x'_n}{n}.
\]
If $\max |x| < \frac{n}{2}$ or $\max |x'| < \frac{n}{2}$, then we have inductively
\[
x'_i - x_i \equiv 0 \pmod{n} \text{ and } |x'_i - x_i| \leq |x'_i| + |x_i| < \frac{n}{2} + \frac{n}{2} = n
\]
for all $i (i = 1, 2, \ldots, n)$. Thus, we must have $x'_i - x_i = 0$ for all $i$ and $x = x'$. If $\max |x| = \frac{n}{2}$ or $\max |x'| = \frac{n}{2}$, then we obtain by definition and the argument above $x = x' = a_n$, showing (2). Since there are only finitely many lattice points with length $n$ in $P\Delta$, we have (3) by (1).

The decimal torsion and the characteristic genus of a prime link $L \in \mathbb{L}P$ is defined to be $\tau(L) = \tau(\sigma(L))$ and $g(L) = g(\sigma(L))$, respectively. Then $g(L) = \ell(L) + \tau(L)$. For the empty knot $\phi$, the trivial knot $O$ and the Hopf link $2_1$, we have
\[
\tau(\phi) = g(\phi) = 0, \tau(O) = 0, g(O) = 1, \tau(2_1^2) = 1, g(2_1^2) = 2 + \frac{1}{2}.
\]
Further, for every prime link $L$ with $\ell(L) \geq 3$, we have
\[
g(L) \in (\ell(L) - \frac{1}{2}, \ell(L) + \frac{1}{2}).\]
by Theorem 3.1. The decimal torsion and the characteristic genus of a 3-manifold \( M \in \mathbb{M} \) is defined to be \( \tau(M) = \tau(\sigma_n(M)) \) and \( g(M) = g(\sigma_n(M)) \), respectively, whose properties will be discussed in § 4.

It is also noted that there are many embeddings similar to \( g \). For example, for a lattice point \( x = (x_1, x_2, \ldots, x_n) \in \Delta \), we define the rational number

\[
g'(x) = n + \frac{x_2}{(n+1)^{n-1}} + \cdots + \frac{x_n}{n+1}.
\]

By convention, we have \( g'(\emptyset) = 0 \) and \( g'(0) = 1 \). The following embedding result is essentially a consequence of Theorem 3.1 and observed earlier in [8] (although the Delta set was taken as a smaller set).

**Corollary 3.3.** The map \( x \mapsto g'(x) \) induces an embedding

\[
g' : \Delta \to [0, +1) \mathbb{Q}
\]

such that for every \( x = (x_1, x_2, \ldots, x_n) \in \Delta \) with \( n \geq 2 \) we have the following properties (1)-(3):

1. \( |g'(x) - n| < \frac{1}{2} \).
2. The lattice point \( x \in \Delta \) is reconstructed from the value of \( g'(x) \).
3. There are only finitely many \( x \in \Delta \) with

\[
g'(x) \in (n - \frac{1}{2}, n + \frac{1}{2}) \mathbb{Q}.
\]

In fact, this corollary is shown by an analogous argument of Theorem 3.1 taking a lattice point \( x \) of length \( n \) as a lattice point \( (x, 0) \) of length \( n + 1 \). Our argument also goes well by using Corollary 3.2, but there is a demerit that the denominator of the rational value becomes further large.

In the forthcoming paper [13], a joint work with T. Tayama, a subset of the Delta set \( \Delta \), called the \( A\Delta \) set \( A\Delta \) which is different from the \( P\Delta \) set \( P\Delta \) discussed here, is discussed as a complex number version of this paper by representing every lattice point of \( A\Delta \) in the complex number plane with norm smaller than or equal to \( \frac{1}{2} \).

4. Properties of the Characteristic Genus of a 3-Manifold
Table 4.1: The characteristic genera of 3-manifolds with lengths up to 7

\[
\begin{array}{|c|c|c|}
\hline
M & x & g \\
\hline
M_{0,1} = \chi(\phi,0) = S^3 & \phi & 0 \\
M_{1,1} = \chi(0,0) = S^1 \times S^2 & 0 & 1 \\
M_{3,1} = \chi(3_1,0) & 1^4 & 3 + \frac{3}{4} = 3.4444444\ldots \\
M_{4,1} = \chi(4_1,0) & 1^4 & 4 + \frac{2}{5} = 4.328125 \\
M_{4,2} = \chi(4_2,0) & (1, -2, 1, -2) & 4 - \frac{1}{7} = 3.53125 \\
M_{5,1} = \chi(5_1,0) & 1^5 & 5 + \frac{1}{4} = 5.25 \\
M_{5,2} = \chi(5_2,0) & (1^2, -2, 1, -2) & 5 - \frac{1}{2} = 4.5 \\
M_{6,1} = \chi(6_1,0) & 1^6 & 6 + \frac{1}{3} = 6.1666666\ldots \\
M_{6,2} = \chi(6_2,0) & (1^3, 2, -1, 2) & 6 + \frac{2}{3} = 6.3333333\ldots \\
M_{6,3} = \chi(6_3,0) & (1^3, -2, 1, -2) & 6 - \frac{2}{3} = 5.1666666\ldots \\
M_{6,4} = \chi(6_4,0) & (1^2, 2, 1^2, 2) & 6 + \frac{1}{2} = 6.5 \\
M_{6,5} = \chi(6_5,0) & (1^2, -2, 1, -2^2) & 6 - \frac{1}{2} = 5.5 \\
M_{6,6} = \chi(6_6,0) & (1^2, -2, 1, -2^2) & 6 + \frac{1}{2} = 6.5 \\
M_{6,7} = \chi(6_7,0) & (1, -2, 1, -2, 1, -2) & 6 + \frac{1}{3} = 6.6666666\ldots \\
M_{6,8} = \chi(6_8,0) & (1, -2, 1, 3, -2, 3) & 6 + \frac{1}{3} = 6.6666666\ldots \\
M_{7,1} = \chi(7_1,0) & 1^7 & 7 + \frac{1}{4} = 7.25 \\
M_{7,2} = \chi(7_2,0) & (1^4, 2, -1, 2) & 7 + \frac{2}{3} = 7.6666666\ldots \\
M_{7,3} = \chi(7_3,0) & (1^4, -2, 1, -2) & 7 - \frac{2}{3} = 6.6666666\ldots \\
M_{7,4} = \chi(7_4,0) & (1^3, -2, 1^2, -2) & 7 + \frac{1}{3} = 7.3333333\ldots \\
M_{7,5} = \chi(7_5,0) & (1^3, -2, 1, -2^2) & 7 - \frac{1}{3} = 6.3333333\ldots \\
M_{7,6} = \chi(7_6,0) & (1^2, -2, 1, -2, 1, -2) & 7 + \frac{1}{2} = 7.5 \\
M_{7,7} = \chi(7_7,0) & (1^2, -2, 1, -2, 1, -2) & 7 - \frac{1}{2} = 6.5 \\
M_{7,8} = \chi(7_8,0) & (1^2, -2, 1, -2, 1, -2) & 7 + \frac{1}{2} = 7.5 \\
M_{7,9} = \chi(7_9,0) & (1^2, -2, 1, -3, 2, -3) & 7 + \frac{1}{3} = 7.6666666\ldots \\
M_{7,10} = \chi(7_10,0) & (1^2, -2, 1, -3, 2, -3) & 7 - \frac{1}{3} = 6.6666666\ldots \\
M_{7,11} = \chi(7_11,0) & (1, -2, 1, -2, 3, -2, 3) & 7 + \frac{1}{4} = 7.25 \\
\hline
\end{array}
\]

By the classification of [5], if $\ell(M) = 1, 2$, then we have $M = S^1 \times S^2, S^3$, respectively. The reason why $S^3$ occurs by $\ell(M) = 2$ is because we take $S^3$ as the 0-surgery manifold of $S^3$ along the Hopf link $2_1^3$ and we have $\sigma_0(S^3) = 1^2$. However, we can also take $S^3$ as the 3-manifold without 0-surgery of $S^3$ along a link. This is the reason why the empty lattice point $0 \in P \Delta \subset \mathbb{X}$ of length 0 and the empty knot $\phi \in L^p$ with bridge index 0 are introduced. We assume

$$\alpha(S^3) = \phi, \sigma_0(S^3) = 0, \ell(0) = 0, g(0) = 0,$$

so that $g(S^3) = 0$. Also, we have the group invariant $\pi_0(S^3) = \{1\}$ by introducing the trivial group $\{1\}$ to the set $G$ of link groups. Under this consideration, there is no 3-manifold $M \in M$ with $\ell(M) = 2$. Since $\sigma_0(M) \subset P \Delta$ and the $n$-fragment of $P \Delta$ for every $n$ is a finite set, there are only finitely many 3-manifolds with length
n for every \( n \geq 0 \). According to the canonical well-order of \( \mathbb{X} \), the 3-manifolds of length \( n \geq 1 \) are enumerated as follows:

\[
M_{n,1} < M_{n,2} < \cdots < M_{n,m_n}
\]

for a non-negative integer \( m_n \) depending only on \( n \). By the introduction of the empty knot \( \phi \in \mathbb{L}^P \), we put \( M_{0,1} = S^3 \). By [5], we reconstruct from the lattice point \( \sigma_\alpha(M_{n,1}) \) the link \( \alpha(M_{n,1}) \in \mathbb{L}^P \), the group \( \pi_\alpha(M_{n,1}) \in \mathbb{G} \) and the 3-manifold \( M_{n,1} \) itself. By (2) of Theorem 3.1, we reconstruct the lattice point \( \sigma_\alpha(M_{n,1}) \) from the characteristic genus \( g(M_{n,1}) \), so that we can construct from \( g(M_{n,1}) \) the lattice point \( \sigma_\alpha(M_{n,1}) \), the link \( \alpha(M_{n,1}) \), the group \( \pi_\alpha(M_{n,1}) \) and the 3-manifold \( M_{n,1} \) itself.

In [KTB] the lattice points of the 3-manifolds \( M_{n,1} \) together with the geometric structures for all \( n \leq 10 \) are listed. In the following table, the characteristic genera \( g(M_{n,1}) \) for all \( n \leq 7 \) are given together with the data of the lattice point \( \sigma_\alpha(M_{n,1}) \) and the link \( \alpha(M_{n,1}) \) identified with a knot or a link in D. Rolfsen's table [20], where it is noted that there is no 3-manifold of length 2 by the reason stated above and at this point the table is different from the tables of [5, 11, 12, 14].

For every 3-manifold \( M \in \mathbb{M} \) with \( M \neq S^3, S^1 \times S^3 \), we have \( \ell(M) \geq 3 \). Every 3-manifold \( M \in \mathbb{M} \) has a Heegaard splitting, i.e., a union of two handlebodies by pasting along the boundaries. The Heegaard genus, \( g_{br}(M) \) of \( M \) is the minimum of the genera of such handlebodies. The following lemma gives a relationship between a bridge presentation of a link \( L \in \mathbb{L} \) (see [3] for an explanation of bridge presentation) and Heegaard splittings of the Dehn surgery manifolds along \( L \).

**Lemma 4.2.** Let a link \( L \in \mathbb{L} \) have a \( g \)-bridge presentation. Then every Dehn surgery manifold \( M \) of \( S^3 \) along \( L \) admits a Heegaard splitting of genus \( g \).

**Proof.** Since \( S^3 \) is a union of two 3-balls \( B, B' \) pasting along the boundary spheres such that \( T = L \cap B \) and \( T' = L \cap B' \) are trivial tangles of \( g \) proper arcs in \( B \) and \( B' \), respectively. Let \( N(T) \) be a tubular neighborhood of \( T \) in \( B \), \( V = \text{cl}(B \setminus N(T)) \), and \( V' = B' \cup N(T) \). By construction, \( V \) and \( V' \) are handlebodies of genus \( g \) and forms a Heegaard splitting of \( S^3 \). To complete the proof, it suffices to show that the Dehn surgery from \( S^3 \) to \( M \) along \( L \) just changes \( V' \) into another handlebody \( V'' \), so that \( V \) and \( V'' \) forms a Heegaard splitting of \( M \) of genus \( g \). Since \( T' \) is a trivial tangle in \( B' \) of \( g \) proper arcs, there are \( g - 1 \) proper disks \( D_i \) \((i = 1, 2, \ldots, g - 1)\) in \( B' \) which split \( B' \) into a 3-manifold regarded as a tubular neighborhood \( N(T') \) of \( T' \) in \( B' \). Then the union \( N(L) = N(T) \cup N(T') \) is regarded as a tubular neighborhood of \( L \) in \( S^3 \). The Dehn surgery from \( S^3 \) to \( M \) along \( L \) just changes \( N(L) \) into the union of solid tori obtained from \( N(L) \) by the Dehn surgery without changing the boundary \( \partial N(L) \). Thus, we obtain the desired handlebody \( V'' \) by pasting along the disks corresponding to \( D_i \) \((i = 1, 2, \ldots, g - 1)\).

Let \( g_{br}(M) \) and \( g_{br}(M) \) denote respectively the bridge genus and the braid genus of \( M \), namely the minimal bridge index and the minimal braid index for links whose 0-surgery manifolds are \( M \). We define \( g_{br}(S^3) = g_{br}(S^3) = 0 \) by considering that \( S^3 \) is obtained from \( S^3 \) by the 0-surgery along the empty knot \( \phi \). The 3-manifold
$M$ with $\ell(M) \geq 3$ is ending-positive or ending-negative, respectively, according to whether $\sigma_o(M)$ is ending-positive or ending-negative. Then we have the following lemma:

**Lemma 4.3.** For every $M \in \mathfrak{M}$ with $\ell(M) \geq 3$, we have

$$2g_h(M) - 2 \leq 2g_b(M) - 2 \leq 2g_br(M) - 2 \leq \ell(M) < g(M) + \text{end}(M),$$

where end$(M)$ is 0 or $\frac{1}{2}$, respectively, according to whether $M$ is ending-positive or ending-negative.

**Proof.** By Lemmas 2.3 and 4.2, we have

$$g_h(M) \geq g_b(M) \geq g_br(M) \geq \ell(M) + 1.$$ 

By Theorem 3.1 (1), according to whether $M$ is ending-positive or ending-negative, the inequality $\ell(M) < g(M)$ or $\ell(M) < g(M) + \frac{1}{2}$ holds, respectively, from which the result follows. \qed

We show the following theorem:

**Theorem 4.4.** The characteristic genus $g(M)$ of every $M \in \mathfrak{M}$ is a characteristic invariant defined for $M$ and the properties (1)-(3) hold. (4) is obtained in Lemma 4.3.

We show the following theorem:

**Theorem 4.4.** The characteristic genus $g(M)$ of every $M \in \mathfrak{M}$ is a characteristic invariant defined for $M$ such that

$$g_b(S^3) = g_b(S^3) = g_br(S^3) = g(S^3) = \ell(S^3) = 0,$$
$$g_b(S^1 \times S^2) = g_b(S^1 \times S^2) = g_br(S^1 \times S^2) = g(S^1 \times S^2) = \ell(S^1 \times S^2) = 1$$

and every $M \in \mathfrak{M}$ with $M \neq S^3, S^1 \times S^2$ has the following properties:

(1) The 3-manifold $M$ itself, the lattice point $\sigma_o(M)$, the link $\alpha(M)$ and the group $\pi_o(M)$ are reconstructed from the value of $g(M)$.

(2) According to whether $M$ is ending-positive or ending-negative, the characteristic genus $g(M)$ belongs to $(n, n + \frac{1}{2})_\mathbb{Q}$ or $(n - \frac{1}{2}, n)_\mathbb{Q}$ for $n = \ell(M)$.

(3) There are only finitely many 3-manifolds $M \in \mathfrak{M}$ such that

$$g(M) \in (n - \frac{1}{2}, n + \frac{1}{2})_\mathbb{Q}.$$ 

(4) The inequalities

$$2g_h(M) - 2 \leq 2g_b(M) - 2 \leq 2g_br(M) - 2 \leq \ell(M) < g(M) + \text{end}(M)$$

hold, where end$(M)$ is 0 or $\frac{1}{2}$, respectively, according to whether $M$ is ending-positive or ending-negative.

**Proof.** By definition, we have the values of $S^3$ and $S^1 \times S^2$. By the property of $\sigma_o$ in [5] and Theorem 3.1, it is seen that $g(M)$ is a characteristic rational invariant defined for $\mathfrak{M}$ and the properties (1)-(3) hold. (4) is obtained in Lemma 4.3. \qed
The following corollary is direct from Theorem 4.5 (3).

**Corollary 4.5.** For any infinite subset $M' \subset M$, we have

$$\sup \{ \ell(M) | M \in M' \} = +\infty.$$  

For every integer $n > 1$, since there are infinitely many 3-manifolds $M \in M$ with $g_{br}(M) \leq n$, we see from Corollary 4.5 that there are lots of 3-manifolds $M \in M$ such that the difference $\ell(M) - g_{br}(M)$ is sufficiently large. However, exact calculations of the invariants $g_b(M), g_{br}(M), \ell(M)$ for most 3-manifolds are not known and remain as an open problem. Here are some elementary examples.

**Example 4.6.** (1) Let $M = \chi(3_1, 0) = M_{3,1}$ for the trefoil knot $3_1$. Since the braid index of $3_1$ is 2 and $M$ is not the lens space, we see from Table 4.1 that $g_b(M) = g_b(M) = g_{br}(M) = 2 < \ell(M)/2 + 1 = 2.5$ and $g(M) = 3 + \frac{4}{9} = 3.444\ldots$.

(2) Let $M = \chi(4_1^2, 0) = M_{4,1}$ for the $(2,4)$-torus link $4_1^2$. Since the braid index of $4_1^2$ is 2 and the first integral homology $H_1(M)$ has exactly 2 generators, we see from Table 4.1 that $g_b(M) = g_b(M) = g_{br}(M) = 2 < \ell(M)/2 + 1 = 3$ and $g(M) = 4 + \frac{21}{64} = 4.328\ldots$.

(3) Let $M = \chi(4_1, 0) = M_{4,2}$ for the figure eight knot $4_1$. Since the bridge index of $4_1$ is 2 and $M$ is not any lens space, we see that $g_b(M) = g_b(M) = 2$. If $M$ is obtained from a knot or link of braid index 2, then $M$ would be obtained from a $(2k + 1)$-half-twist knot $K(k)$ by 0-surgery. However, this is impossible because the Alexander polynomial of the homology handles $M$ and $M(k) = \chi(K(k), 0)$ are

$$A_H(t) = t^2 - 3t + 1, \quad A_{M(k)} = \frac{t^{2k+1} + 1}{t + 1}$$  

and they are distinct. These results and Table 4.1 mean that $g_b(M) = g_b(M) = 2 < g_{br}(M) = \ell(M)/2 + 1 = 3 < g(M) = 4 - \frac{15}{32} = 3.531\ldots$. 

We note here that the bridge genus behaves differently from the Heegaard genus, although $g_b(M) = g_b(M)$ in Example 4.6. For example, if $M$ is a lens space except $S^3$ and $S^1 \times S^2$, then we have $g_b(M) \geq 3$ whereas $g_b(M) = 1$. In fact, the first homology $H_1(M)$ is a non-trivial finite cyclic group. On the other hand, if $1 \leq g_b(M) \leq 2$, then $H_1(M)$ would be isomorphic to the infinite cyclic group $\mathbb{Z}$ or a direct double $\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$ for some $m \geq 0$, which is a contradiction. Concretely,
the pro-ective 3-space $M = P^3$ has $\sigma_\alpha(M) = (1^2, 2, 1^2, 2)$ (see [5, 14]) and hence $g_b(M) = 3$. By developing a similar consideration, S. Okazaki[19] has observed a linear independence on the Heegaard genus $g_b(M)$, the bridge genus $g_b(M)$ and the braid genus $g_{br}(M)$.

5. Constructing a Characteristic Smooth Real Function Defined for the PDelta Set

A PDelta subset is a subset $S$ of the PDelta set $P\Delta$ containing the lattice points $\emptyset$ and $0^3$. Let $a$ and $t$ be real numbers such that either $-1 \leq a \leq 1$ and $-1 < t < 1$ or $-1 < a < 1$ and $-1 \leq t \leq 1$. Then the linear fraction

$$B(t; a) = \frac{t - a}{1 - at}$$

is considered. If $|t| < 1$ and $|a| < 1$, then $|B(t; a)| < 1$, because we have

$$1 - |B(t; a)|^2 = \frac{(1 - t^2)(1 - a^2)}{(1 - at)^2}.$$ 

If $|a| = 1$ or $|t| = 1$, then it is easily checked that $|B(t; a)| = 1$. In fact, we have $B(t; \pm 1) = B(\mp 1, a) = \mp 1$.

Noting that the decimal torsions of $\emptyset$, $0$ and $1/2$ are not definite values as it is explained in Remark 3.2, we put the following definition for any $x \in P\Delta$:

$$G_x(t) = \begin{cases} B(t; \tau(x)) & (x \geq 3) \\ B(t; 1) = -1 & (x = 1^2) \\ B(t; -1) = 1 & (x = \emptyset, 0) \end{cases}$$

For every $n$-fragment $S^{(n)}$ of a PDelta subset $S \subset P\Delta$, the function

$$G_S^{(n)}(t) = \prod_{x \in S^{(n)}} G_x(t)$$

is called a finite Blaschke product\(^4\) whose zero’s are precisely the decimal torsions $\tau(x)$ for all $x \in S^{(n)}$ except $\emptyset$, $0$ and $1^2$. By the assumption of the set $S$, we have

$$G_S^{(0)}(t) = G_S^{(1)}(t) = 1.$$ 

Further, according to whether the lattice point $1^2$ belongs to $S$ or not, we have $G_S^{(2)}(t) = -1$ or $1$, respectively. For example, when we take $S = \mathbb{I}^p$, the functions $G_{\mathbb{I}^p}^{(n)}(t)$ for $n = 0, 1, 2, 3, 4, 5$ are calculated as follows:

\(^3\)This condition is imposed for simplicity.

\(^4\)See Blaschke [2]. The author thanks to K. Sakan for suggesting the Blaschke product.
We obtain the following theorem.

**Theorem 5.1.** For every $\text{PDelta subset } S$, the series function

$$G_S(t) = \sum_{n=0}^{+\infty} G_{S(n)}(t)t^n$$

is a smooth real function defined on the interval $(-1, 1)$ which is a characteristic invariant defined for the set $S$.

**Proof.** Since $|G_{S(n)}(t)| \leq 1$ for any $n$, we have

$$|G_S(t)| \leq \sum_{n=0}^{+\infty} |t|^n = \frac{1}{1-|t|}.$$

This means that the series $G_S(t)$ defined on $(-1, 1)$ is uniformly convergent in the wide sense. Using that the function $G_{S(n)}(t)$ ($t \in (-1, 1)$) is uniformly convergent in the wide sense, we see from the Weierstrass double series theorem that the series function $G_S(t)$ is a smooth real function defined on $(-1, 1)$. To see that the function $G_S(t)$ is characteristic for $S$, it suffices to see by induction on $n \geq 2$ that the set of the decimal torsions $\tau(x)$ for all lattice points $x \in S(n)$ except $0$ is determined by the function $G_S(t)$. According to whether $t^2$ is in $S$ or not, the second derivative $\frac{d^2}{dt^2}G_S(0)$ is $-2$ or $2$, respectively. Thus, $S(2)$ is determined by the function $G_S(t)$. Assume that all the lattice points of $S(n-1)$ $(n-1 \geq 2)$ are determined by the function $G_S(t)$. Let

$$G_S^{(n)}(t) = G_S(t) - \sum_{i=0}^{n-1} G^{(i)}S(t)t^i.$$

The function $G_{S(n)}(t)$ has the following splitting form:

$$G_{S(n)}(t) = G_{S(n)}(t) \cdot \tilde{G}(t) \cdot t^n,$$
where
\[ \tilde{G}(t) = 1 + \tilde{G}_S^{(n+1)}(t)t + \tilde{G}_S^{(n+2)}(t)t^2 + \tilde{G}_S^{(n+3)}(t)t^3 + \ldots \]
for some finite Blaschke products \( \tilde{G}_S^{(n+i)}(t) \) with
\[ G_S^{(n)}(t) \cdot \tilde{G}_S^{(n+i)}(t) = G_S^{(n+i)}(t) \]
for all \( i \) \((i = 1, 2, 3, \ldots)\). We show that the function \( \tilde{G}(t) \) has no zero’s in the interval \((-\frac{1}{2}, \frac{1}{2})\). In fact, we have
\[ |\tilde{G}(t)| \geq 1 - \sum_{i=1}^{\infty} |t|^i = \frac{1-2|t|}{1-|t|} > 0 \]
for any \( t \) with \( |t| < \frac{1}{2} \). This means that the decimal torsions \( \tau(x) \) for all lattice points \( x \in S^{(n)} \) except \( 0, 0 \) and \( 1^2 \) are characterized by the zero’s of the function \( \tilde{G}^{(n)}(t) \) in the interval \((-\frac{1}{2}, \frac{1}{2}) \setminus \{0\} \).

It is noted that the series function \( G_S(t) \) does not converge for \( t = \pm 1 \). This is because
\[ \lim_{n \to +\infty} |G_S^{(n)}(\pm 1)| = 1 \neq 0. \]
The function \( G_S(t) \) is called the characteristic genus function defined for the PDelta subset \( S \). For example, for \( S = \{\emptyset, 0\} \), we have
\[ G_S(t) = 1 + t + t^2 + t^3 + \ldots = \frac{1}{1-t}. \]
For \( S = \{\emptyset, 0, 1^2\} \), we have
\[ G_S(t) = 1 + t - (t^2 + t^3 + t^4 + \ldots) = 1 + t - \frac{t^2}{1-t}. \]
For a finite set \( S \) with the maximal length \( n \),
\[ G_S(t) = \sum_{i=0}^{n-1} G_S^{(i)}(t)t^i + G_S^{(n)}(t)\frac{t^n}{1-t}. \]

For the subset \( S = \sigma(L^p) \), we denote \( G_S^{(n)}(t) \) and \( G_S(t) \) by \( G_{L^p}^{(n)}(t) \) and \( G_{L^p}(t) \), respectively. The following corollary is direct from Theorem 5.1.

**Corollary 5.2.** The series function
\[ G_{L^p}(t) = \sum_{n=0}^{+\infty} G_{L^p}^{(n)}(t)t^n \]
\[ = 1 + t - t^2 - B(t, \frac{4}{9})t^3 - B(t, \frac{4}{9})B(t, \frac{21}{64})B(t, \frac{-15}{32})t^4 \]
\[ - B(t, \frac{4}{9})B(t, \frac{21}{64})B(t, \frac{-15}{32})B(t, \frac{156}{625})B(t, \frac{-234}{625})t^5 + \ldots \]
is a smooth real function defined on the interval $(-1, 1)$ which is a characteristic
invariant defined for the prime link set $\mathbb{L}_p$.

For example, let $\mathbb{L}(2, \ast)$ be the set of $(2, n)$-torus links regarding the $(2, 0)$-torus
link as the empty knot $\phi$. Since

$$\sigma(\mathbb{L}(2, \ast)) = \{1^n | n = 0, 1, 2, 3, \ldots \},$$

where $1^0 = \phi$, $1 = 0$ and $\tau(1^n) = \frac{1}{n-1} - \frac{1}{n^n}$ for $n \geq 3$, we have:

$$G_{\mathbb{L}(2, \ast)}(t) = 1 + t - t^2 - \sum_{n=3}^{+\infty} \left( \prod_{k=3}^{n} B \left( t, \frac{1}{k-1} - \frac{1}{k^k - k^{k-1}} \right) \right) t^n.$$

For the subset $S = \sigma_\alpha(M)$, we denote $G_S^{(n)}(t)$ and $G_S(t)$ by $G_M^{(n)}(t)$ and $G_M(t)$, respectively. Noting that the lattice point $1^2$ is excluded from $\sigma(M)$ (by the reason that the empty lattice point $\emptyset$ is introduced), we have the following corollary obtained from Theorem 5.1.

**Corollary 5.3.** The series function

$$G_M(t) = \sum_{n=0}^{+\infty} G_M^{(n)}(t) t^n$$

is a smooth real function defined on the interval $(-1, 1)$ which is a characteristic
invariant defined for the 3-manifold set $M$.

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