CONSTRUCTING DOUBLY-POINTED HEEGAARD DIAGRAMS COMPATIBLE WITH (1,1) KNOTS

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Abstract. A (1,1) knot $K$ in a 3-manifold $M$ is a knot that intersects each solid torus of a genus 1 Heegaard splitting of $M$ in a single trivial arc. Choi and Ko developed a parameterization of this family of knots by a four-tuple of integers, which they call Schubert’s normal form. This article presents an algorithm for constructing a doubly-pointed Heegaard diagram compatible with $K$, given a Schubert’s normal form for $K$. The construction, coupled with results of Ozsváth and Szabó, provides a practical way to compute knot Floer homology groups for (1,1) knots. The construction uses train tracks, and its method is inspired by the work of Goda, Matsuda and Morifuji.

1. Introduction

Some objects of 3-dimensional topology can be usefully characterized by objects of 2-dimensional topology. A significant example is a Heegaard splitting, which is a closed 3-manifold obtained by identifying the boundaries of two genus $g$ handlebodies by a homeomorphism. The homeomorphism is completely determined by two sets $\alpha, \beta$ of $g$ curves in the splitting surface $\Sigma$, and we study the 3-manifold in terms of the Heegaard diagram $(\Sigma, \alpha, \beta)$. The utility of Heegaard diagrams extends, by a slight modification, from manifolds to knots in manifolds. By marking two points $x, y \in \Sigma - \alpha - \beta$ in a given Heegaard diagram $(\Sigma, \alpha, \beta)$ for a 3-manifold $M$, we specify a unique knot $K \subset M$. The knot is comprised of the union of the two trivial arcs on either side of $\Sigma$ that join $x$ and $y$ while avoiding meridian disks bounded by $\alpha, \beta$. We say $(\Sigma, \alpha, \beta, x, y)$ is a doubly-pointed Heegaard diagram compatible with $K$. This article addresses the problem of constructing a genus 1 doubly-pointed Heegaard diagram which is compatible with a given knot in the three-sphere.

The motivation behind this question comes from the knot Floer homology developed by Ozsváth and Szabó and, independently, Rasmussen. Knot Floer homology is a robust invariant; it produces the Alexander polynomial, the Seifert genus, fiberedness, a concordance invariant, unknot detection, and...
information about Dehn surgeries on a knot. For an introduction to the theory, see [OS06]. The basic input for knot Floer homology is a Heegaard knot diagram, and Ozsváth and Szabó showed that, in the case of doubly-pointed genus 1 Heegaard diagrams, the computation of knot Floer homology is combinatorial. Although Manolescu, Ozsváth and Sarkar found a combinatorial algorithm to calculate the Floer homology of any knot, the invariants are computationally more accessible given a genus 1 doubly-pointed Heegaard diagram for a knot as opposed to a multi-point and multi-curve Heegaard diagram that are used in the more general setting.

Goda, Matsuda and Morifuji recognized that the set of knots admitting genus 1 doubly-pointed Heegaard diagrams correspond to \((g, b)\) knots with \(g = b = 1\) [GMM05]. A \((1, 1)\) knot \(K\) in a 3-manifold \(M\) is a knot that intersects each solid torus of a genus 1 Heegaard splitting of \(M\) in a single trivial arc. The \((1, 1)\) knots form a large family. The torus knots and 2-bridge knots are proper subsets of the set of \((1, 1)\) knots, and Fujii has shown that every Laurent polynomial satisfying the properties of an Alexander polynomial appears as the Alexander polynomial of some \((1, 1)\) knot [Fuj96]. Goda, Matsuda and Morifuji constructed numerous examples of genus 1 doubly-pointed Heegaard diagrams compatible with \((1, 1)\) knots whose Floer homology was unknown, including several knots up to 10 crossings and certain pretzels [GMM05]. In each example, the attaching circle \(\alpha\) remains fixed while \(\beta\) is constructed via a sequence of steps beginning from a genus 1 doubly-pointed Heegaard diagram compatible with the trivial knot. Each step involves an isotopy of one of the trivial arcs and a simultaneous deformation of \(\beta\), both of which are described by careful illustrations. Using Schubert form, a parameterization of \((1, 1)\) knots due to Choi and Ko [CK03], we are able to generalize Goda, Matsuda and Morifuji’s examples to obtain an algorithm
which constructs a genus 1 doubly-pointed Heegaard diagram for any (1,1) knot with a given a Schubert form (Theorem 3).

Rasmussen defined an integer 4-tuple parameterization \( K(p, q, r, s) \) of genus 1 doubly-pointed Heegaard diagrams [Ras], and Doyle developed a computer program based on [GMM05] to compute the knot Floer homology groups up to relative Maslov grading for a given 4-tuple [Doy05]. We use the complementarity between the (1,1) knot trivial arcs and Heegaard diagram attaching circles to define an integer 4-tuple parameterization of genus 1 doubly-pointed Heegaard diagrams. Given a diagram resulting from our construction, we compute this parameterization, which we call Heegaard diagram Schubert form \( HS(r', s', t', \rho') \) (Theorem 4). Using the correspondence below, which follows by direct inspection, we obtain the Rasmussen parameters required for Doyle’s algorithm:

\[
HS(r', s', t', \rho') = K(2r' + s' + t', r', s', -\rho' \mod 2r' + s' + t').
\]

Hedden observed that each step in the construction of Theorem 3 either preserves or increases the total rank of the Floer homology [Hed]. In particular, the absolute Maslov grading of the generators is unchanged throughout the construction of the Heegaard diagram. This continuity in the algorithm extends from the absolute grading of the unknot in the first step to the absolute Maslov grading for the knot whose Heegaard diagram appears in the final step. This feature makes it possible to take advantage of Doyle’s software and calculate the knot Floer homology groups exactly, including the absolute Maslov grading. The results of this paper and Doyle’s software were employed in [Ord06], to investigate the Floer invariants for satellite (1,1) knots.

The paper is organized as follows. Section 2 summarizes the basic terminology of Heegaard diagrams, Schubert form, and train tracks, which offer a particularly convenient way to characterize curves in a surface. Section 3 defines Schubert form in terms of train tracks. In Section 4 we divide a (1,1) knot into increments by which the Heegaard diagram construction of Section 5 proceeds. Section 6 computes the Heegaard diagram Schubert form. Examples 1–3 demonstrate all results in the simplest nontrivial case, the trefoil.
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2. Preliminaries

2.1. Heegaard diagrams. A Heegaard splitting is a closed 3-manifold \(H_\alpha \cup_h H_\beta\) obtained by identifying the boundaries of two genus \(g\) handlebodies \(H_\alpha\) and \(H_\beta\) by a homeomorphism \(h : \partial H_\beta \to \partial H_\alpha\). Let \(\alpha \subset \partial H_\alpha\) and \(\beta \subset \partial H_\beta\) be sets of \(g\) meridional curves of \(H_\alpha\) and \(H_\beta\), respectively. The homeomorphism \(h\) is determined (up to isotopy) by the attaching circles \(h(\beta) \subset \partial H_\alpha\), (see, for example, Singer [Sin33]). A Heegaard diagram \((\Sigma, \alpha, \beta)\) consists of the splitting surface \(\Sigma = \partial H_\alpha = \partial H_\beta\) and the curves \(\alpha\) and \(h(\beta)\) in \(\Sigma\).

Given a Heegaard diagram \((\Sigma, \alpha, \beta)\) for a 3-manifold \(M\), we specify a knot \(K \subset M\) uniquely by marking two points \(x, y \in \Sigma - \alpha - \beta\). To identify \(K\), let \(D_\alpha \subset H_\alpha\) be a set of pairwise disjoint properly embedded disks, the boundaries of which correspond to the set of meridians \(\alpha\). A properly embedded arc \(\gamma\) in a 3-manifold \((M, \partial M)\) is trivial if there is an embedded disk \(D\) such that \(\partial D \cap \gamma = \gamma\) and \(\partial D \cap \partial M\) is the arc \(\partial D \setminus \text{int}(\gamma)\). There is a unique (up to isotopy) trivial arc \(t_\alpha \subset H_\alpha - D_\alpha\) spanning \(x, y\), since the complement of \(D_\alpha\) in \(H_\alpha\) is homeomorphic to a ball. Define \(t_\beta\) similarly. If the closed curve \(t_\alpha \cup t_\beta\) has knot type \(K\), the \((\Sigma, \alpha, \beta, x, y)\) is a called a doubly-pointed Heegaard diagram compatible with \(K\).

Figure 1 depicts a genus 1 doubly-pointed Heegaard diagram compatible with the trefoil (in the three-sphere).

2.2. \((1,1)\) knots. Doll defined a knot \(K\) in a 3-manifold \(M\) to be genus-\(g\) bridge-\(b\), or, simply \((g, b)\) if there is a genus \(g\) Heegaard splitting of \(M\) such that \(K\) intersects each handlebody in \(b\) trivial arcs [Dol92]. Call \((H_\alpha, t_\alpha) \cup_h (H_\beta, t_\beta)\) a \((1,1)\) decomposition of a \((1,1)\) knot \(K \subset M\) if \(H_\alpha \cup_h H_\beta\) is a genus 1 Heegaard splitting of \(M\), \(t_\alpha \subset H_\alpha\) and \(t_\beta \subset H_\beta\) are properly embedded trivial arcs, and \(K = t_\alpha \cup t_\beta\).

Choi and Ko [CK03] demonstrated that every \((1,1)\) knot is represented by an integer 4-tuple \((r, s, t, \rho)\), where \(r, s,\) and \(t\) are non-negative. In their presentation, called Schubert’s normal form (or simply Schubert form), they isotope the arc \(t_\beta\) of a \((1,1)\) decomposition into the torus \(\partial H_\alpha\). Then they prove that such an arc in the torus is isotopic to an arc in the form depicted in Figure 2, where opposite ends of the cylinder are identified by a \(\frac{2\pi \rho}{2r+s+t+1}\)-rotation. The parameters \(r, s, t\) count the number of strands in each parallel class. The arc \(t_\alpha\) belongs to a “standard” meridian disk of the solid torus \(H_\alpha\), as shown in Figure 2. The symbol \(S(r, s, t, \rho)\) denotes the knot type of \(t_\alpha \cup t_\beta\). Note that Schubert form is not unique; for example, \(S(0, 0, 2, 2)\) and \(S(1,1,0,-1)\) are identical trefoils.
2.3. **Train tracks.** Thurston originally introduced train tracks to study geodesic laminations of hyperbolic surfaces [Thu80], but we use the terminology merely in terms of the combinatorics of doubly-pointed Heegaard diagrams in the torus. The following definitions are drawn from Penner and Harer [PH92], and we also rely, in part, on the exposition by Masur, Mosher, and Schleimer [MMS].

A **train track with stops** (or simply a track) is a graph $\ tau$ that is properly embedded in a surface $\Sigma$ and satisfies the following conditions. Edges of a train track, which are called **branches**, are assumed to be smoothly embedded, and at each vertex the incident edges are mutually tangent. A vertex from which branches emanate in two directions (“incoming” and “outgoing”) is called a **switch**, and a **stop** is a vertex from which branches emanate in one direction. Each vertex of a train track is either a switch in the interior of $\Sigma$ or a stop in $\partial \Sigma$. There is at most one stop in each boundary component of the surface, and the one-sided tangent vector of each branch incident on a stop is transverse to $\partial \Sigma$. We assume that at each switch exactly three branches meet, a pair of small branch ends to one side and one large branch end to the other.

Let $B = B(\tau)$ be the set of all branches of a track $\tau$. A **tie neighborhood** $N(\tau) \subset \Sigma$ of $\tau$ is a union of rectangles $\{R_b : b \in B\}$, each of which is foliated by vertical intervals, called ties, that are transverse to $\tau$. At each switch, the upper and lower thirds of the vertical side of the large branch end rectangle is identified with the two vertical sides of the small branch end rectangles. It is required that each component of $\Sigma \setminus N(\tau)$ has at least one corner and the following index has a negative value on the component:

$$\chi - \frac{1}{4} \# \text{ outward-pointing corners} + \frac{1}{4} \# \text{ inward-pointing corners},$$

where $\chi$ is the Euler characteristic. In particular, no branch is homotopic to another branch or a boundary component.

If $\gamma$ is a curve contained in $N(\tau)$ and transverse to the ties, then $\gamma$ is said to be **carried** by $\tau$, and we write $\gamma \prec \tau$. A **transverse measure** $w$ on $\tau$ is a function which assigns a non-negative number, called a **weight**, to each branch in such a way that the sums of the weights on the incoming and outgoing branches at each switch are equal. Given an ordering on the branch set $B = \{b_1, \ldots, b_n\}$, the following notation will be convenient: $(w_1, \ldots, w_n) := (w(b_1), \ldots, w(b_n))$. A collection of curves $\gamma \prec \tau$ defines a transverse measure on $\tau$, $w_{\gamma}(b) := |\gamma \cap t|$, where $t$ is any tie of $R_b$. We call $w_{\gamma}$ the counting measure of $\gamma$ on $\tau$. Conversely, to every non-negative integer-valued function $w$ on $B$ satisfying the switch condition, there is a collection of curves $\gamma \prec \tau$ with $w = w_{\gamma}$. An **extension** of a train track $\tau$ is a train track $\sigma$ which contains $\tau$ as a subset, and we write $\tau \prec \sigma$ (we also say $\tau$ is a **subtrack** of $\sigma$). The **support** of an arc $\gamma \prec \sigma$ is the smallest subtrack $\text{supp} \gamma \prec \sigma$ that carries $\gamma$.  


Two measured train tracks \((\tau, w), (\tau', w')\) are equivalent if they are related by a sequence of isotopies and the following local moves. A left splitting or right splitting replaces a branch with large ends (a large branch) by a branch with small ends (a small branch). Suppose that \(w\) is a transverse measure on the train track \(\tau\) before splitting at the branch \(b_1\), shown in Figure 3, bottom center. If \(w_2 \geq w_3\) (\(w_2 \leq w_3\)), then \(w\) induces a transverse measure \(w'\) on the train track \(\tau'\) resulting from the left (right) splitting of \(\tau\) at \(b_1\), and \(w'_1 = w_2 - w_3\) (\(w'_1 = w_3 - w_2\)), shown in Figure 3 bottom left (right). A slide alters a branch having one large and one small end (a mixed branch) according to Figure 3 top. For a slide move, \(w'_1 = w_2 + w_3\). All other branch weights are unchanged. The inverse of a slide is a slide, and the inverse of a splitting is called a fold.

3. Schubert train track

Let \(\Sigma\) be the closed oriented surface of genus 1. Fix once and for all an oriented meridian-longitude pair \(\mu, \lambda \subset \Sigma\) and a point \(x \in \mu \setminus \lambda\). Let \(y := \mu \cap \lambda\). We will represent \(\Sigma\) by the fundamental polygon \(P = \lambda \mu \lambda^{-1} \mu^{-1}\), which is a rectangle with horizontal homologous to \(\lambda\), oriented from right to left, and verticals homologous to \(\mu\), oriented from bottom to top. Suppose \(\hat{\Sigma} = \Sigma - D_x - D_y\), where \(D_x, D_y \subset \Sigma\) are small disks whose boundary contains \(x, y\), respectively.

**Definition 1.** The Schubert train track \(\sigma\) denotes one of three train tracks with stops \(\sigma_{-\cdot}, \sigma_{-}\), \(\sigma_{+}\) in \(\hat{\Sigma}\) that appear in Figure 4. The boundary component \(D_x\) (\(D_y\)) and stop \(x\) (\(y\)) are depicted by the small black (white) circle.

**Lemma 1.** The trivial arc \(t_{\beta}\) of a \((1, 1)\) knot with Schubert form \(S(r, s, t, \rho)\) is carried by \(\sigma\), where \(\sigma\) is the Schubert train track \(\sigma_{-}\) if \(\rho < -r\), \(\sigma_{-}\) if \(-r \leq \rho < 0\), and \(\sigma_{+}\) if \(\rho \geq 0\). The counting measure \(w = w_{t_{\beta}}\) of \(t_{\beta}\) on \(\sigma\) is \((w_1, w_2, w_3, w_4, w_5) = (r, s, t, |\rho|, 1)\).
Proof. From the definition of Schubert form $S(r, s, t, \rho)$, it follows that $t_\beta \prec \theta$, where $\theta$ is the train track with stops $\theta_-$ if $\rho < 0$ and $\theta_+$ if $\rho \geq 0$, as shown in Figure 5.

Moreover, the counting measure $v = v_{t_\beta}$ of $t_\beta$ on $\theta$ is $(v_1, \ldots, v_5) = (r, s, t, |\rho|, 1)$. The equivalence of the measured train tracks $(\theta, v)$ and $(\sigma, w)$, results from the following sequence of train track moves taking $\theta$ to $\sigma$.

First, apply slide moves at the two mixed branches of $\theta$ that are adjacent to branches $b_3$ and $b_4$. The resulting train track $\theta'$ has a single large branch, which, for the case $\rho < 0$, appears in Figure 6, left. Suppose $v'$ is the counting measure of $t_\beta$ on $\theta'$ induced by $v$. If $\rho \geq 0$, then $v'$ induces a measure $v''$ on the left splitting of $\theta'$ at its large branch, and the resulting
train track is $\sigma_+$. If $\rho < 0$, $v'$ induces $v''$ on the right splitting, and the resulting train track is $\theta''$ shown in Figure 6 center. Apply slide moves at the two branches indicated in the figure by the large arrows, so that the result is $\theta''$ that is shown in Figure 6 right. If $\rho < -r$, the measure $v''$ on $\theta''$, induced by $v''$, induces the measure $w$ on the left splitting, which results in $\sigma_-$. If $-r \leq \rho < 0$, $v''$ induces the measure $w$ on the right splitting, which results in $\sigma_-$. □

The Schubert train track has ten switches and over a dozen branches, yet for $t_\beta \prec \sigma$, a component of $t_\beta \setminus (\mu \cup \lambda)$ belongs to one of at most five parallel classes of subarcs in the fundamental polygon $P$. The following lemma simplifies $\sigma$ in a way that will facilitate the parameterization of $t_\beta$ taken up in the next section.

Lemma 2. If trivial arc $t_\beta$ of a $(1,1)$ knot with Schubert form $S(r,s,t,\rho)$ is carried by $\sigma'$, where $\sigma'$ is the train track with terminals $\sigma'_{-}$ if $\rho < -2s-r-1$, $\sigma'_{-} \prec -2r-s-1 \leq \rho < -r$, $\sigma'_0$ if $-r \leq \rho < t$, and $\sigma'_+ \prec \rho \geq t$, as depicted in Figure 7. The counting measure $w' = w'_\beta$ of $t_\beta$ on $\sigma'$ is $(w'_1, \ldots, w'_6) =$

\[
\begin{align*}
(r + s, r, 3r + s + t + 1, r, -2r - s - \rho - 1, 1) & \quad \text{if } \rho < -2r - s - 1, \\
(r, -r - \rho - 1, r, r + t - \rho, 2r + s + \rho + 1, 1) & \quad \text{if } -2r - s - 1 \leq \rho < -r, \\
(3r + s + \rho + 1, r, r + \rho, r, t - \rho, 1) & \quad \text{if } -r \leq \rho < t, \\
(r, r + t, r, 3r + s + t + 1, \rho - t, 1) & \quad \text{if } \rho \geq t.
\end{align*}
\]

Proof. We show that the measured train tracks $(\sigma, w)$ and $(\sigma', w')$ are equivalent. If $\rho \geq t$, then $t_\beta \prec \sigma_+$, according to Lemma 1. Apply a sequence of

![Figure 7. Folded Schubert train track $\sigma'$](image-url)
slide and fold moves so as to collapse the complementary regions labeled 1–6, in that order, that appear in Figure 8. The resulting train track with induced branch weights appears in Figure 8 to the right. Split the large branch adjacent to the two branches with weights $r + t$ and $r + \rho$. If $\rho \geq t$, then a left splitting of the branch results in $\sigma'_0$, otherwise $\rho < t$ and a right splitting yields $\sigma'_0$. The induced measure $w'$ on $\sigma'$ is easily obtained. The other cases follow similarly, and they are omitted. □

**Figure 8.** Folding $\sigma_+$.

### 4. Bridge sequence

The meridians $\mu, \lambda$ divide $t_\beta$ into subarcs, which we denote by $d_1, \ldots, d_n$, each of which is properly embedded in the fundamental polygon. Index the subarcs according to their order in $t_\beta$ from $y$ to $x$. If $t_\beta \prec \sigma'$, then Figure 7 shows the number $n$ of subarcs is $\Sigma_{i=1}^n w'_i$, where $w'_i$ is the counting measure of $t_\beta$ on $\sigma'$. Parameterize $P$ by the half-open interval $[0, 2n + 4) \subset \mathbb{R}$, such that $p \in P$ is integral if and only if $p$ is an endpoint of $d_k$, for some $i \in \{1, \ldots, n\}$ or $p$ is identified with one of the basepoints $x, y$ in the torus $\Sigma$. Let $s_{2i-1}, s_{2i}$ denote the endpoints of $d_i$, $i = 1, \ldots, n$. The “bridge” $t_\beta = d_1 \cup \cdots \cup d_n$ is uniquely determined (up to isotopy) by the sequence of endpoints $s_1, s_3, \ldots, s_{2n-1}$.

**Definition 2.** The *bridge sequence* of a $(1, 1)$ knot in Schubert form is the sequence of integers $(s_1, s_3, \ldots, s_{2n-1})$.

Let $y_1, y_2, y_3, y_4 \in P$ denote the points homologous to $y$, and $x_1, x_2 \in P$ the points homologous to $x$, the order chosen to agree with the parameterization of $P$. Figure 7 and Lemma 2 determine the integer representatives of $y_1, y_2, x_1, y_3, y_4, x_2$, and they appear in Table 1. The initial and final points of $t_\beta$ follow similarly:

1. $s_1 = \begin{cases} y_3 & \text{if } \rho < -r, \\ y_2 & \text{if } \rho \geq -r, \end{cases}$
2. $s_{2n} = x_2$.

The branches $b_1, \ldots, b_4$ of $\sigma'$ are ordered so as to facilitate the statement of the following Proposition 1, which computes the bridge sequence explicitly.
in terms of Schubert form. Define the center \( p_i \) of branch \( b_i \), \( i = 1, \ldots, 4 \) to be as in Table 2; that is, \( p_i \) is a point in \( \{y_1, y_2, x_1, y_3, y_4, x_2\} - \{s_1, s_{2n}\} \) that is equidistant in \( P \) from the ends of each subarc carried by \( b_i \).

### Table 1.

| \( \rho < -r \) | \( \rho \geq -r \) |
|-----------------|-----------------|
| \( y_1 \) | 0 |
| \( y_2 \) | 2r + \( \rho + 1 \) |
| \( x_1 \) | 4r + \( s + \rho + 2 \) |
| \( y_3 \) | 6r + \( s + t + \rho + 3 \) |
| \( y_4 \) | 8r + \( s + t + 2\rho + 4 \) |
| \( x_2 \) | 10r + \( s + t + 2\rho + 5 \) |

### Table 2.

| \( \rho < -2r - s - 1 \) | \( -2r - s - 1 \leq \rho < -r \) | \( -r \leq \rho < t \) | \( t \leq \rho \) |
|--------------------------|----------------------------|----------------|--------|
| \( p_1 \) | \( y_2 \) | \( y_1 \) | \( x_1 \) |
| \( p_2 \) | \( x_1 \) | \( y_2 \) | \( x_1 \) |
| \( p_3 \) | \( y_4 \) | \( x_1 \) | \( y_3 \) |
| \( p_4 \) | \( y_1 \) | \( y_4 \) | \( y_1 \) |

**Proposition 1.** Let \( S = S(r, s, t, \rho) \) be the Schubert form of a \((1, 1)\) knot. If \( w' \) is the counting measure of \( t_\beta \) on \( \sigma' \), then the terms of the bridge sequence of \( S \) are

\[
 s_{2i - 1} = (\psi \phi)^{i-1}(s_1), \quad i = 1, \ldots, n
\]

where \( \psi, \phi \) are permutations given by the cycle notation below:

\[
\phi = \prod_{i=1}^{4} \prod_{j=1}^{w_i'} \left( \frac{p_i - j}{p_i + j} \right) \prod_{k=1}^{w_2'} \left( \frac{p_1 - w_1' - k}{p_2 + w_2' + k} \right),
\]

\[
\psi = \prod_{i=1}^{y_2 - y_1 - 1} \left( y_1 + i \right) \prod_{j=1}^{y_3 - y_2 - 1} \left( y_4 + j \right),
\]

and \( a = \bar{a} \mod 2n + 4 \).

Note that the initial term \( s_1 \) from which the proposition computes the bridge sequence is given by (1) and Table 1; hence, the bridge sequence is completely specified by the proposition.

**Proof.** The \( i \)th term \( s_{2i - 1} \) of the bridge sequence corresponds to the initial endpoint of the subarc \( d_i \subset t_\beta \). For \( i < n \), the next term, \( s_{2i + 1} \), in the bridge sequence is the (unique) point in \( P \) that is homologous (in \( \Sigma \)) to the final endpoint \( s_{2i} \) of \( d_i \). We will show that \( \phi \) transposes the endpoints of \( d_i \), and \( \psi \) transposes homologous (integer) pairs in \( P \), hence

\[
 s_{2i + 1} = \psi \phi (s_{2i - 1}), \]

and the bridge sequence is completely specified by the proposition.
and the bridge sequence is obtained from \( s_1 \) by iterating \( \psi \phi \) as claimed.

Consider \( \phi \). There are five parallel classes of arcs among \( d_1, \ldots, d_5 \), and they correspond to the branches \( b_1, \ldots, b_5 \) of the train track \( \sigma' \). For \( i = 1, \ldots, 4 \), the branch \( b_i \) is centered on \( p_i \), and the endpoints of an arc carried by \( b_i \) lie at a distance \( j = 1, \ldots, w'_i \) to either side of \( p_i \). Given that \( P = [0, 2n + 4) \), the two points at distance \( j \) from \( p_i \) are \( p_i \pm j \), where \( \sigma = a \mod 2n + 4 \). Hence, the 2-cycle \( (p_i - j \quad p_i + j) \) transposes endpoints of arcs carried by the branch \( b_i \), \( i = 1, \ldots, 4 \). As for \( b_5 \), one endpoint is adjacent to the end of \( b_1 \) preceeding \( p_1 \) (in \( P \)), and the other is adjacent the end of \( b_2 \) succeeding \( p_2 \). Therefore, one endpoint of an arc carried by \( b_5 \) is less than \( p_1 \) by \( w'_1 - k \), and the other is greater than \( p_2 \) by \( w'_2 + k \) for some \( k = 1, \ldots, w'_5 \); the permutation \( (p_1 - w'_1 - k \quad p_2 + w'_2 + k) \) transposes the two. Multiplying all the above transpositions results in the permutation \( \phi \), as required.

The “gluing” permutation \( \psi \) is also a product of disjoint transpositions. The two sequences of integers \( y_1 + 1, y_1 + 2, \ldots, y_2 - 1 \) and \( y_3 + 1, y_3 + 2, \ldots, y_4 - 1 \) are identified in \( \lambda \subset \Sigma \), in reverse order, and the homologous pairs are transposed by cycles \( (y_1 + i \quad y_4 - i) \), for \( i = 1, \ldots, y_2 - y_1 - 1 \). Similarly, the integer sequences \( y_2 + 1, y_2 + 2, \ldots, y_3 - 1 \) and \( y_4 + 1, y_4 + 2, \ldots, y_1 - 1 \) are identified in \( \mu \subset \Sigma \), in reverse order. The corresponding transpositions are \( (y_2 + j \quad y_4 - j) \), for \( j = 1, \ldots, y_3 - y_2 - 1 \). (There is no single point identified with one of \( y_1, \ldots, y_4 \); they have been excluded from the permutation \( \psi \) because they are already all identified with \( s_1 \).) The product of the transpositions associated with \( \lambda \) and \( \mu \) is \( \psi \), as claimed. \( \square \)

**Example 1** (Trefoil bridge sequence). A Schubert form for the trefoil is \( (r, s, t, \rho) = (0, 0, 2, 2) \). From Lemma 2 we have \( (w'_1, \ldots, w'_5) = (0, 2, 0, 3, 0) \), with “centers” given by Table 2 \( (p_1, p_2, p_3, p_4) = (x_1, y_3, y_4, y_1) \). From Table 1 basepoints \( y_1, y_2, x_1, y_3, y_4, x_2 \) take positions 0, 3, 4, 7, 10, 13, respectively, hence \( (p_1, p_2, p_3, p_4) = (4, 7, 10, 0) \). The gluing permutation is

\[
\psi = \prod_{i=1}^{2} (0 + i \quad 10 - i) \prod_{j=1}^{3} (10 + j \quad 7 - j) \\
= (1\ 9)(2\ 8)(11\ 6)(12\ 5)(13\ 4),
\]

where \( \sigma = a \mod 2n + 4 = a \mod 14 \), since \( n = \Sigma_{i=1}^{5} w'_i = 5 \). There are only two branches of positive weight, \( b_2 \) and \( b_4 \), and the subarc permutation is

\[
\phi = \prod_{j=1}^{2} (7 + j \quad 7 - j) \prod_{j=1}^{3} (0 - j \quad 0 + j) \\
= (6\ 8)(5\ 9)(13\ 1)(12\ 2)(11\ 3).
\]

Since \( \rho \geq -r \), the initial term \( s_1 \) is \( y_2 = 2r + \rho + 1 = 3 \). Going to the end of the first subarc, we meet \( \phi(3) = 11 \), which is homologous to \( s_3 = \psi(11) = 6 \).

Thus, the bridge sequence begins 3, 6. Continuing in this way, alternating
\(\phi\) and \(\psi\) a total of \(n = 5\) times, we obtain the bridge sequence \((3, 6, 2, 5, 1)\). See Figure 10.

5. Attaching sequence

Let \(t_\beta\) be the bridge of a \((1, 1)\) knot in Schubert form \(S\), whose bridge sequence is \((s_1, s_3, \ldots, s_{2n-1})\). By \(t_\beta^i\), we denote the \(i\)th partial bridge \(d_1 \cup \cdots \cup d_i\), for \(i = 1, \ldots, n\). Define \(t_\beta^0 := s_1\). The same parameterization of \(P\) developed in the previous section to characterize \(t_\beta\) also serves to characterize an oriented attaching circle \(\gamma \subset \Sigma - t_\beta^1\).

**Definition 3.** Let \(i\) be an integer such that \(0 \leq i \leq n\). An attaching sequence for \(t_\beta^i\) is a sequence of half-integer points \((c_1, \ldots, c_m)\) in \(P\) for which there is an oriented simple closed curve \(\gamma \subset \Sigma - t_\beta^i\) with the property that \(\gamma \setminus P\) consists of \(m\) arcs, and the tail endpoint \(c_j^\prime\) of the \(j\)th arc satisfies the inequality \(c_j - \frac{1}{2} < c_j^\prime < c_j + \frac{1}{2}\) for all \(j = 1, \ldots, m\).

An attaching sequence \((c_1, \ldots, c_m)\) determines the curve \(\gamma\) uniquely (up to isotopy), and we will therefore use the attaching sequence to represent the curve. Extend the gluing map \(\psi\) defined in Proposition 1 to the half integers:

\[
\psi = \prod_{i=1}^{2y_2-2y_1-1} \left( y_1 + \frac{i}{2} \right) \prod_{j=1}^{2y_3-2y_2-1} \left( y_4 + \frac{j}{2} \right)
\]

Finally, we use interval notation \((a, b)\) to denote the clockwise-oriented segment in \(P\), from \(a\) to \(b\):

\[
(a, b) = \begin{cases} 
{x \in P : a < x < b} & \text{if } a \leq b, \\
{x \in P : b < x \text{ or } x < a} & \text{if } b < a.
\end{cases}
\]

**Proposition 2.** Let \((s_1, s_3, \ldots, s_{2n-1})\) be the bridge sequence of a \((1, 1)\) knot in Schubert form \(S\). Suppose \(\gamma = (c_1, c_2, \ldots, c_m)\) is an attaching sequence for \(t_\beta^i\). Let \(a = s_{2i-1}\) and \(b = \phi(s_{2i-1})\). Define

\[
\phi_{S,i}(\gamma) = (f_{S,i}(c_1), \ldots, f_{S,i}(c_j), \ldots, f_{S,i}(c_m)),
\]

where

\[
f_{S,i}(c_j) = \begin{cases} 
c_j, & \text{if } c_j \in (a, b) \text{ and } \psi(c_{j+1}) \in (b, a), \\
c_j, & \text{if } c_j \in (b, a) \text{ and } \psi(c_{j+1}) \in (a, b), \\
c_j, & \text{else,}
\end{cases}
\]

for \(j = j \mod m\), and \(i = 1, \ldots, n\). Then \(\phi_{S,i}(\gamma)\) is an attaching sequence for \(t_\beta^{i+1}\), and the curve represented by \(\phi_{S,i}(\gamma)\) is isotopic to \(\gamma\).

**Proof.** We will apply a “finger move” along the subarc of \(t_\beta\) from \(a\) to \(b\), and then observe that the result has attaching sequence \(\phi_{S,i}(\gamma)\), as defined.

Let \(g_1, \ldots, g_p \subset \gamma\) be arcs, each of which intersects \(P\) exactly in its two endpoints, one of lies in \((a, b)\) and the other in \((b, a)\). Cut the fundamental
polygon along $t^{i+1}_\beta$ and $\gamma - g_1 - \cdots - g_p$. Among the regions remaining, there is one, $R$, that contains $a,b$. Take a simple path in the boundary of $R$ that connects $a$ and $b$, and push it just off the boundary to obtain an arc $u$ isotopic to $d_{i+1} = t^{i+1}_\beta - t^i_\beta$. The arc $u$ is disjoint from $t^i_\beta$ and intersects $\gamma$ in $p$ points, with one intersection point in each of $g_1, \ldots, g_p$. Relabeling if necessary, index $g_1, \ldots, g_p$ by the order in which $u$ meets them, from $b$ back to $a$.

![Figure 9. Finger move.](image)

For each $k = 1, \ldots, p$, take an $\epsilon/k$-radius tube $U_k$ around $u$, where $0 < \epsilon \ll 1$. The intersection $g_k \cap U_k$ divides $U_k$ into two simply connected regions, of which one, call it $D_k$, is disjoint from $g_{k+1}, \ldots, g_p$ and contains $b$. The boundary of $D_k$ consists of $g_k \cap U_k$ joined with the path $\partial U_k \cap D_k$. Let $g'_k$ be the path obtained by replacing the subarc $(g_k \cap U_k)$ of $g_k$ by $\partial U_k \cap D_k$. The curve $\gamma'$ obtained by replacing $g_k$ by $g'_k$ in $\gamma$ for all $k = 1, \ldots, p$, is isotopic to $\gamma$ because $g_k$ and $g'_k$ are isotopic. By construction, $\gamma'$ is disjoint from the partial bridge $t^{i+1}_\beta = t^i_\beta \cup u$. See Figure 9.

An arc $g_k$ that crosses $u$ from left to right corresponds to $c_j \in (a,b)$ for which $\psi(c_j) \in (b,a)$. In this case, $g'_k$ passes around $u$ (at a distance of $\epsilon/k$) from left to right; $g'_k$ is comprised of three arcs whose endpoints are $c_j, w - \frac{1}{2}; \psi(b - \frac{1}{2}), \psi(b + \frac{1}{2});$ and $b + \frac{1}{2}, \psi(c_{j+1})$. Replacing $g_k$ by $g'_k$, takes $c_j$ to the three initial points of the above arcs, as required by the first case in the definition of $f_{S,i}$. If $g_k$ crosses $u$ from right to left, then the second case results. Arcs of $\gamma$ that do not cross $u$ are unchanged by $f_{S,i}$ and this is the content of the third and final case. Having demonstrated that in all cases the term $c_j$ of $\gamma$ is replaced by $f_{S,i}(c_j)$ in $\gamma$, the proof is complete.

**Theorem 3.** Let $K$ be a $(1,1)$ knot with $(1,1)$ decomposition $(H_\alpha, t_\alpha) \cup_h (H_\beta, t_\beta)$ and Schubert form $S = S(r,s,t,\rho)$. Let $\Sigma = \partial H_\alpha$ and $\{x,y\} = \partial t_\alpha = \partial t_\beta$. Suppose $s_1, s_3, \ldots, s_{2n-1}$ is the bridge sequence of $S$. Let

$$\alpha = \begin{cases} -\rho - \frac{1}{2} & \text{if } \rho < -r, \\ 2r + \rho + \frac{1}{2} & \text{if } \rho \geq -r. \end{cases}$$
Define $\beta = \beta_0$, where
\[
\beta_0 = \begin{cases} 
-\rho + \frac{1}{2} & \text{if } \rho < -r, \\
2r + \rho + \frac{3}{2} & \text{if } \rho \geq -r,
\end{cases}
\]
for $i = 0, \ldots, n-1$. Then $(\Sigma, \alpha, \beta, x, y)$ is a genus 1 doubly-pointed Heegaard diagram compatible with $K$.

**Proof.** As defined, $\alpha$ and $\beta_0$ correspond to the half-integers $y_2 - \frac{1}{2}$ and $y_2 + \frac{1}{2}$, which belong to $\lambda$ and $\mu$, respectively. As (one-term) attaching sequences, they represent simple closed curves parallel to $\mu$ and $\lambda$; that is, $y_2 - \frac{1}{2}$ and $y_2 + \frac{1}{2}$ are meridian curves for $H_\alpha$ and $H_\beta$, respectively. Thus, $(\Sigma, \alpha, \beta_0)$ is a Heegaard diagram for the $S^3$. By Proposition 2, $f_{S,i}(\beta_i)$ is isotopic to $\beta_i$ for all $i = 0, \ldots, n$. Therefore $\beta$ is isotopic to $\beta_0$, and $(\Sigma, \alpha, \beta)$ is also a Heegaard diagram for $S^3$.

According to Schubert form, the trivial arc $t_\alpha$ belongs to the meridian disk bounded by $\mu$. Since $\mu$ and $\alpha$ are parallel in $\Sigma$, we can define $D_\alpha \subset H_\alpha$ to be the parallel meridian disk with $\partial D_\alpha = \alpha$, for which $D_\alpha \cap t_\alpha = \emptyset$. From Proposition 2, it follows that the curve $\beta_i$ is disjoint from $t_\beta^i$ for all $i = 0, \ldots, n$. Thus $\beta = \beta^n$ is disjoint from $t_\beta = t_\beta^n$, and any properly embedded disk $D_\beta \subset H_\alpha$ with $\partial D_\beta = \beta$ is disjoint from $t_\beta$. In conclusion, the unique isotopy classes of trivial arcs between $x$ and $y$ in $H_\alpha - D_\alpha$ and $H_\beta - D_\beta$ correspond to $t_\alpha$ and $t_\beta$, respectively, and $(\Sigma, \alpha, \beta, x, y)$ is a doubly-pointed Heegaard diagram compatible with $K = t_\alpha \cup t_\beta$. \qed

**Definition 4.** An attaching sequence $c_1, \ldots, c_m$ for $t_\beta^i$ is reduced if for all $j = 1, \ldots, m$, $(c_j, \psi(c_{j+1})) \cap t_\beta = \emptyset$ and $(\psi(c_{j+1}), c_j) \cap t_\beta = \emptyset$, where $j = j$ mod $m$. A pair of terms $c_j, c_{j+1}$ that does not meet the above conditions is called an inessential pair.

**Lemma 3.** Suppose $\gamma$ is an attaching sequence for $t_\beta^i$. If $\gamma'$ is the sequence obtained by removing every inessential pair from $\gamma$, then $\gamma'$ is also an attaching sequence for $t_\beta^i$, and $\gamma$ and $\gamma'$ are isotopic as curves.

**Proof.** Let $c_1, \ldots, c_m$ be the terms of the attaching sequence $\gamma$. Suppose $c_j, c_{j+1}$ is an inessential pair, for which, say, $(c_j, \psi(c_{j+1})) \cap t_\beta = \emptyset$. Then $d_j \cup (c_j, \psi(c_{j+1}))$ bounds a disk in $\Sigma - t_\beta^i$, where $d_j \subset \gamma$ is the properly embedded subarc with endpoints $c_j, \psi(c_{j+1})$. Among all such disks there is an innermost disk whose interior is disjoint from $\gamma$. Cancel the inessential arc by an isotopy taking it across the disk, thereby removing the points of intersection corresponding to the inessential pair. Let $\gamma'$ be the result of having applied the same isotopy to the next innermost disk, and so on, until all inessential arcs are removed. Then $\gamma'$ is represented by attaching sequence $c'$ and $\gamma'$ is isotopic to $\gamma$. \qed

From now on we assume that an attaching sequence is reduced. For otherwise we may take any given attaching sequence and successively remove
inessential pairs until it is reduced. By Lemma 3, the result is an attaching sequence for the same curve.

Example 2 (Trefoil attaching sequence). Schubert form, bridge sequence, and the permutations $\phi$ and $\psi$ were obtained in Example 1 as

$$S = S(0,0,2,2),$$
$$s_1, s_3, \ldots, s_{2n-1} = 3, 6, 2, 5, 1,$$
$$\phi = (6\ 8)(5\ 9)(13\ 1)(12\ 2)(11\ 3),$$
$$\psi = (1\ 9)(2\ 8)(11\ 6)(12\ 5)(13\ 4).$$

The extended gluing permutation is

$$\psi = (0.5\ 9.5)(1\ 9)(1.5\ 8.5)(2\ 8)(2.5\ 7.5)(10.5\ 6.5)$$
$$= (11\ 6)(11.5\ 5.5)(12\ 5)(12.5\ 4.5)(13\ 4)(13.5\ 3.5).$$

Given that $\rho \geq -r$, we have

$$\alpha = 2.5, \quad \beta_0 = 3.5.$$ 

Therefore $\alpha$ meets $P$ at 2.5 and $\psi(2.5) = 7.5$. To determine $\beta_1 = f_{S,1}(\beta_0)$, note that the first bridge arc has endpoints $a = s_1 = 3$ and $b = \phi(s_1) = 11$. Since $\beta_0$ meets $P$ at 3.5 and $\psi(3.5) = 13.5$, which belong to $(a, b)$ and $(b, a)$, respectively, $f_{S,1}(\beta_0) = f_{S,1}(3.5) = (3.5, \psi(w - .5), w + .5) = (3.5, 6.5, 11.5)$. However, the pair 11.5, 3.5 is inessential since the interval $(11.5, \psi(3.5)) = (11.5, 13.5)$ is disjoint from $t^1_\beta$, which meets $P$ in the points $\{0, 3, 7, 10, 11\}$. The reduced attaching sequence of $t^1_\beta$ is

$$\beta_1 = 6.5.$$ 

For the next bridge arc, $a = 6$ and $b = 8$. Here $f_{S,2}(\beta_1) = f_{S,2}(6.5) = (6.5, 2.5, 8.5)$, since $6.5 \in (a, b)$ and $\psi(6.5) = 10.5 \in (b, a)$. This attaching sequence is reduced;

$$\beta_2 = 6.5, 2.5, 8.5.$$ 

For $i = 3$, $a = 2$ and $b = 12$. Here $f_{S,3}(6.5) = 6.5$, $f_{S,3}(2.5) = (2.5, 5.5, 12.5)$, and $f_{S,3}(8.5) = 8.5$. The sequence is reduced;

$$\beta_3 = 6.5, 2.5, 5.5, 12.5, 8.5.$$ 

The computations for $i = 4, 5$ follow similarly, and the results are

$$\beta_4 = 6.5, 2.5, 5.5, 1.5, 9.5, 12.5, 9.5,$$
$$\beta_5 = 6.5, 2.5, 5.5, 1.5, 4.5, 13.5, 9.5, 13.5, 9.5.$$ 

See Figure.
6. Heegaard Diagram Schubert Form

Definition 5. Let $\sigma \subset \hat{\Sigma}$ be the Schubert train track, and define $\alpha \subset \hat{\Sigma} \setminus \mu$ to be a meridian parallel to $\mu$. Suppose $w = w_3$ is the counting measure of a curve $\beta \prec \sigma$, and $(w_1, \ldots, w_5) = (r', s', t', \rho', 0)$. If $(\Sigma, \alpha, \beta, x, y)$ is a doubly-pointed Heegaard diagram compatible with a knot $K$, then we say $HS(r', s', t', \rho')$ is a Heegaard diagram Schubert form of $K$, where the sign $\epsilon = \pm 1$ is positive if $\sigma$ equals $\sigma_+$, and negative otherwise.

We note that Cattabriga and Mulazzani defined an integer 4-tuple parameterization $K(a, b, c, r)$ of (1,1) knots in terms of Heegaard diagrams and Dunwoody manifolds [CM04]. Inspection reveals that the Heegaard diagram Schubert form $HS(r', s', t', \rho')$ corresponds to $K(r', s', t', \rho' - s')$.

Definition 6. The intersection numbers of an attaching sequence $\gamma = c_1, \ldots, c_m$ is a 4-tuple of non-negative integers $(a, b, c, d)$, where
\[
\begin{align*}
a &= |\{i : y_2 \leq c_i < y_3 \text{ and } y_4 \leq c_{i+1} < 2 y_2\}|, \\
b &= |\{i : y_2 \leq c_i < x_1 \text{ or } x_2 \leq c_i < 2 y_2\}|, \\
c &= |\{i : x_1 \leq c_i < y_3 \text{ or } y_4 \leq c_i < x_2\}|, \\
d &= |\{i : y_1 \leq c_i < y_2 \text{ or } y_3 \leq c_i < y_4\}|
\end{align*}
\]
and $\bar{j} = j \mod m$. 

Figure 10. Attaching circle construction for the trefoil.
Intersection number $a$ counts the number of subarcs of $\gamma$ that intersect $P$ exactly in their endpoints, both of which belong to $(y_2, y_3)$ And $b$, $c$, $d$ represent the geometric intersection numbers $|\gamma \cap (y_2, x_1)|$, $|\gamma \cap (x_1, y_3)|$, $|\gamma \cap (y_1, y_2)|$, respectively.

**Theorem 4.** If $(a, b, c, d)$ are intersection numbers of a reduced attaching sequence $\beta$ for the bridge $t_\beta$ of a nontrivial $(1,1)$ knot $K$ in Schubert form $S(r, s, t, \rho)$, then a Heegaard diagram Schubert form $HS(r', s', t', \rho')$ of $K$ is provided by one of the following cases:

i. If $s \neq 0 \neq t$, or $2a \leq b$ and $2a \leq c$, then

$$(r', s', t', \rho') = \begin{cases} (a, -2a + b, -2a + c, -d) & \text{if } \rho < -r, \\ (a, -2a + b, -2a + c, -2a + d) & \text{if } \rho \geq -r. \end{cases}$$

ii. If $s = 0 < t$ and $2a > b$, then

$$(r', s', t', \rho') = \begin{cases} (-a + b, -2a + c, 2a - b, 2a - 2b - d) & \text{if } \rho < -r, \\ (-a + b, -2a + c, 2a - b, -2b + d) & \text{if } \rho \geq -r. \end{cases}$$

iii. If $s > 0 = t$ and $2a > c$, then

$$(r', s', t', \rho') = \begin{cases} (-a + c, 2a - c, -2a + b, a - c - d) & \text{if } \rho < -r, \\ (-a + c, 2a - c, -2a + b, d) & \text{if } \rho \geq -r. \end{cases}$$

iv. If $s = 0 = t$ and $2a > c \geq b$, then $(r', s', t', \rho') =$

$$\begin{cases} (-a + c, b - c - b - 2a, b - 2a, 2a - c - d - b - 2a) & \text{if } \rho < -r, \\ (-a + c, 2a - c, b - c - 2a - c, -b + d + 2a - c) & \text{if } \rho \geq -r, \end{cases}$$

where $\bar{x} = x \mod b - c$.

v. If $s = 0 = t$ and $2a > b > c$, then $(r', s', t', \rho') =$

$$\begin{cases} (-a + b, -b + c - 2a - b, 2a - b, 2a - 2b + c - d - 2a - b) & \text{if } \rho < -r, \\ (-a + b, -b + c - 2a - b, 2a - b, -b + d + 2a - b) & \text{if } -r \leq \rho < 0, \\ (-a + b, -2a + b, -b + c - 2a + b, -b + d + 2a + b) & \text{if } \rho > 0, \end{cases}$$

where $\bar{x} = x \mod -b + c$.

**Proof.** Case i. According to Lemma 4, $\beta < \sigma$. The only branch arc of $\sigma$ with boundary in $(y_2, y_3)$ in the fundamental polygon is $b_1$ (see Figure 4), hence

$$(3) \quad a = w_1 = r',$$

where $w = w_\beta$ is the counting measure of $\beta$ on $\sigma$. Intersection numbers $b$, $c$, and $d$ equal the sum of $w$ over the branches of $\sigma$ that are transverse to the intervals $(y_2, x_1)$, $(x_1, y_3)$, and $(y_1, y_2)$, respectively. Using the switch conditions on $\sigma$, which determine the value of $w$ on $B(\sigma)$ in terms of $w_1, \ldots, w_5$,
Lemma 4. Let be the measure 
\[ \sigma = 2i \]
for which the solution is 
\[ \tau = \begin{cases} 
\tau_1 & \text{if } s > 0 \text{ and } t > 0, \text{ or } 2a \leq b \text{ and } 2a \leq c, \\
\tau_2 & \text{if } s = 0 < t \text{ and } 2a > b, \\
\tau_3 & \text{if } s > 0 = t \text{ and } 2a > c, \\
\tau_4 & \text{if } s = 0 = t \text{ and } 2a > c \geq b, \\
\tau_5 & \text{if } s = 0 = t \text{ and } 2a > b > c, 
\end{cases} \]
and \( \tau_i \) is the train track \( \tau_i^{--} \) if \( \rho < -r \), \( \tau_i^- \) if \( -r \leq \rho < 0 \), and \( \tau_i^+ \) if \( \rho > 0 \) for \( i = 1, \ldots, 5 \).

We obtain the required \( (r', s', t', \rho') \) as the solution to the system of linear equations (3)-(6).

Case ii. Lemma 4 implies that \( \beta \prec \tau_2 \), where \( \tau_2 \) is defined by Figure 11. The intersection numbers correspond to a system of linear equations in terms of \( v_1, \ldots, v_5 \), where \( v = v_\beta \) is the counting measure of \( \beta \) on \( \tau_2 \),

\[
\begin{align*}
a &= v_1 + v_2, \\
b &= 2v_1 + v_2, \\
c &= 2v_1 + 2v_2 + v_3, \\
d &= \begin{cases} 
v_2 + v_4 - v_5 & \text{if } \rho < -r, \\
v_1 + v_2 - v_4 & \text{if } -r \leq \rho < 0, \\
v_1 + 2v_2 + v_4 & \text{if } \rho > 0, 
\end{cases}
\end{align*}
\]

for which the solution is

\[
(v_1, \ldots, v_4) = \begin{cases} 
(-a + b, 2a - b, -2a + c, -2a + b + d) & \text{if } \rho < -r, \\
(-a + b, 2a - b, -2a + c, -b - d) & \text{if } -r \leq \rho < 0, \\
(-a + b, 2a - b, -2a + c, -2a + d) & \text{if } \rho > 0.
\end{cases}
\]

Lemma 5 determines the required Heegaard diagram Schubert form from the measure \( v \).

Cases iii-v. These follow from computations similar to Case ii, and they are omitted. \(\square\)
Figure 11. Maximal extensions $\tau = \tau_2, \tau_3, \tau_4, \tau_5$.

Proof. Given that $\beta$ is an attaching sequence for $t_\beta$, $\beta$ determines (up to isotopy) an attaching circle (that we denote by $\beta$ also) that is disjoint from $t_\beta$. Let $N = N(supp t_\beta)$ be a tie neighborhood of the support $supp t_\beta < \sigma$. Since $\beta \cap t_\beta = \emptyset$, we may assume that $\beta \cap N$ consists of properly embedded
arcs in $N$, each of which passes between distinct vertical sides of $N$. Furthermore, we take $\beta$ to be transverse to the ties of $N$ since $N$ is comprised of rectangles.

Given that $K$ is a nontrivial knot, the twist parameter $\rho$ is nonzero. Suppose $\rho > 0$.

If $s > 0$ and $t > 0$, then a component of $\Sigma - N$ is a polygon with at most three vertical sides. If an arc of $\beta \setminus N$ passes between two vertical sides of $N$ that are adjacent to a horizontal side in the frontier of $N$, then the arc is carried by $\text{supp} t_\beta$. Otherwise there is a boundary arc of $\partial \Sigma$ separating the vertical sides in the frontier of $N$ (this case corresponds to $r = 0$), and the arc is carried by the extension $\text{supp} t_\beta < \sigma$. In either case $\beta \prec \sigma$.

If $s = 0 = t$, then there is a component of $\sigma - N$ with four vertical sides, and there may be an arc of $\beta \setminus N$ which is not parallel to a horizontal or terminal side. The hypothesis that $\beta$ is reduced implies, however, that each arc of $\beta \setminus (N \cup P)$ passes between distinct vertical sides of $N$ or components of $P \setminus N$, which we call passages. Each component of $\Sigma \setminus (N \cup P)$ has at most three vertical sides or passages. Thus, $\beta$ is carried by the extension $\text{supp} t_\beta < \tau'$ obtained by adding branches between those vertical sides and passages not already connected by a horizontal side, provided we join all branches incident on a single passage. See Figure 12. Let $w = w_\beta$ be the counting measure of $\beta$ on $\tau'$. From the definition of the intersection numbers and the switch conditions, we deduce $a = w_1$, $b = w_1 + w_2$, and $c = 2w_1 + w_3$. If $2a \leq b$ ($2a > b$), then $w_1 \leq w_2$ ($w_1 > w_2$), and $\beta$ is carried by the right (left) splitting of $\tau'$ at branch $b_6$. The right splitting is equivalent to $\sigma$ whereas the left splitting is the train track $\tau'_2$ as required.

If $s > 0 = t$, a similar argument shows that $\beta$ is carried by the extension $\text{supp} t_\beta < \tau''$ shown in Figure 12. In this case $2a \leq b$. If $2a \leq c$, then $\beta$ is carried by $\sigma$, which is the left splitting of $\tau''$ at $b_6$. Otherwise $2a > c$, and $\beta$ is carried by $\tau_3^+$, which is the right splitting at $b_6$.

If $s = 0 = t$, then each component of $\Sigma \setminus (N \cup P)$ still has at most three vertical sides or passages, and $\beta$ is carried by the extension $\tau'''$ (Figure 12, right) which is defined in a manner analogous to $\tau'$ and $\tau''$. If $2a \leq b$ and $2a \leq c$ then $\beta$ is carried by the track resulting from splitting $\tau'''$ right at $b_6$.

Figure 12. Extensions $\tau'$, $\tau''$, $\tau'''$ for $\rho > 0$. 
and left at $b_7$, which is equivalent to $\sigma$. If $2a > c > b$, then a left splitting at $b_6$ results in a train track equivalent to $\tau^+_4$, which carries $\beta$. If $2a > b > c$, then $\beta$ is carried by the train track obtained by splitting $\tau''$ right at $b_7$, which is equivalent to $\tau^+_5$.

In the remaining cases, $\rho < -r$ and $-r \leq \rho < 0$, similar arguments demonstrate that $\beta$ is carried by $\tau$ as claimed. \hfill \Box

**Lemma 5.** If $v = v_\beta$ is the counting measure of $\beta$ on the train track $\tau \neq \sigma$, where $\beta$ is an attaching circle of a $(1,1)$ knot $K$ with Schubert form $S(r,s,t,\rho)$, then a Heegaard diagram Schubert form $HS(r',s',t',\rho')$ for $K$ is as follows. Let $\overline{x}$ denote $x \mod v_2$.

If $\tau = \tau_2$, then

$$(r', s', t', \rho') = \begin{cases} (v_1, v_3, v_2, -v_1 - v_2 - v_4) & \text{if } \rho < 0, \\ (v_1, v_3, v_2, -2v_1 + v_4) & \text{if } \rho > 0. \end{cases}$$

If $\tau = \tau_3$, then

$$(r', s', t', \rho') = \begin{cases} (v_1, v_3, v_2, -v_4 - v_3 - v_1) & \text{if } \rho < -r, \\ (v_1, v_3, v_2, 2v_1 + v_3 - v_4) & \text{if } -r \leq \rho < 0, \\ (v_1, v_3, v_2, 2v_1 + v_3 + v_4) & \text{if } \rho \geq -r. \end{cases}$$

If $\tau = \tau_4$, then

$$(r', s', t', \rho') = \begin{cases} (v_1, v_2 - v_2 - v_3, v_2 - v_3, -v_2 - v_3 - v_4) & \text{if } \rho < -r, \\ (v_1, v_3, v_2 - v_3, -v_2 + v_3 - v_4) & \text{if } -r \leq \rho < 0, \\ (v_1, v_3, v_2 - v_3, v_2 + v_3 + v_4) & \text{if } \rho > 0. \end{cases}$$

If $\tau = \tau_5$, then

$$(r', s', t', \rho') = \begin{cases} (v_1, v_2 - v_3, v_3 - v_3 - v_4) & \text{if } \rho < -r, \\ (v_1, v_2 - v_3, v_3 - v_2 + v_3 - v_4) & \text{if } -r \leq \rho < 0, \\ (v_1, v_2 - v_3, v_2 - v_3, v_2 + v_2 - v_3 + v_4) & \text{if } \rho > 0. \end{cases}$$

**Proof.** Suppose $\tau = \tau_2$ and $\rho > 0$. Apply an isotopy sliding the basepoint $y$ once around $\Sigma$ along a circular path parallel to $\mu$ and with the same orientation as $\mu$. See Figure 13. Split the large branch adjacent to $b_4$, and the result is a subtrack of $\sigma$ on which the counting measure of $\beta$ takes the

![Figure 13. Isotopy of $\tau = \tau_2^+$.](image-url)
values $w_1 = v_1$, $w_2 = v_3$, $w_3 = v_2$, and $w_4 = v_4 - 2v_1$, as required. The same isotopy treats cases with $\rho < 0$.

If $\tau = \tau_3$, slide the basepoint $y$ once around $\Sigma$ along a path parallel to $\mu$ and against its orientation. The Heegaard diagram Schubert form follows by a splitting as in the above case.

If $\tau = \tau_4$, then arcs of $\beta$ carried by $b_2$ wrap $v_3/v_2$-times around a meridian $\gamma$ parallel to $\mu$ with the same orientation. The case $\rho > 0$ is shown in Figure 14 left, where, for sake of clarity, we have isotoped the basepoint $y$ toward the center of the fundamental polygon. In this case, if we apply a $-\lfloor v_3/v_2 \rfloor$ Dehn twist to $\beta$ along $\gamma$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$, then the result $\beta'$ is carried by the train track $\tau'$ shown in Figure 14 center. Since $\gamma$ is disjoint from $\alpha$, $x$, and $y$, and the twist is integral, the doubly-pointed Heegaard diagrams $(\Sigma, \alpha, \beta, x, y)$ and $(\Sigma, \alpha, \beta', x, y)$ are equivalent; i.e., they represent the same knot in $S^3$. If $v'$ is the counting measure of $\beta'$ induced by $v$, then $(v'_1, \ldots, v'_4) = (v_1, v_2, v_3, v_4)$, where $v = x \mod v_2$. By isotoping $\tau'$ into a neighborhood of $\sigma$, as shown in Figure 14, we obtain the Heegaard diagram Schubert form $HS(v'_1, v'_2, v'_3, v'_4) = HS(v_1, v_3, 2v_2 + v_3 + v_4)$, as required. The case $\rho < 0$ follows similarly. If $\rho < -\rho$, an additional twist is required for Schubert form; after a $-\lfloor v_3/v_2 \rfloor$ Dehn twist the given Heegaard diagram Schubert form is obtained, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to $x$.

**Figure 14.** Dehn twist of $\tau = \tau_4^+$ along $\gamma$.

If $\tau = \tau_5$ and $\rho < -\rho$ ($\rho \geq -r$), then a positive $\lfloor v_3/v_2 \rfloor$ ($\lceil v_3/v_2 \rceil$) Dehn twist along $\gamma$ takes $\beta$ to a neighborhood of $\sigma$. The required Heegaard diagram Schubert parameters follow by an argument similar to the one applied in the $\tau_4$ case.

**Example 3** (Trefoil Heegaard diagram Schubert form). From Example 2 an attaching sequence for the trefoil with Schubert form $S(0,0,2,2)$ is

$$\beta = (6.5,2.5,5.5,1.5,4.5,13.5,9.5,13.5,9.5).$$

Given the trefoil basepoints $(y_1, y_2, x_1, y_3, y_4, x_2) = (0,3,4,7,10,13)$ from Example 1 the intersection numbers of $\beta$ are $(a, b, c, d) = (1,2,3,4)$. Since $2a \leq b$, $2a \leq c$, and $\rho \geq -r$, a Heegaard diagram Schubert form for the trefoil is

$$HS(s', t', \rho') = (1,0,1,2).$$
To verify that this corresponds to a Heegaard diagram compatible with the trefoil, notice that an isotopy sliding $y$ in the direction of the path $\mu \lambda^{-1}$ takes $\beta$ into the shape of Figure 1.

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