Instability of single- and double-periodic waves in the fourth-order nonlinear Schrödinger equation

N. Sinthuja · S. Rajasekar · M. Senthilvelan

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Abstract In this work, we investigate the instability of single- and double-periodic waves of a fourth-order nonlinear Schrödinger equation, which describes the propagation of ultra-short pulses in a high-speed, long-distance optical fiber transmission system. The single- and double-periodic solutions of this fourth-order nonlinear Schrödinger equation are derived in terms of Jacobian elliptic functions such as \(dn\), \(cn\) and \(sn\). From the spectral problem, we compute Lax and stability spectrum for different values of elliptic modulus parameter. We then calculate the instability rate of single- and double-periodic waves for different values of elliptic modulus and system parameters. We also highlight certain novel features come out from our studies. In the case of single-periodic waves, the instability rate for the \(dn\) periodic wave is larger when compared to the \(cn\) periodic one. Also, our results reveal that the instability growth rate is higher for the single-periodic waves when compared to the double-periodic waves. Further, the width and height (maximal instability rate) of the instability rate of double-periodic waves increase when we increase the system parameter value. This, in effect, leads to faster evolution of the periodic waves (single and double) with a higher growth rate.

Keywords Fourth-order nonlinear Schrödinger equation · Modulational instability · Single-periodic waves · Double-periodic waves

1 Introduction

The nonlinear Schrödinger (NLS) equation is one among the completely integrable systems that describes the dynamics of deep ocean waves, propagation of pulse in optical fibers and several other phenomena from the nonlinear dynamics perspective \([1]\). It is shown that rogue waves (RWs) \([2–7]\) can be modelled by rational solutions of the NLS equation. Nonlinear Schrödinger equation plays a major role in understanding the appearance of complex RWs in the ocean. The rogue or freak waves that appear in the ocean are unpredictable, and when they appear, they come up three times larger than the ambient waves and produce catastrophic effects. Initially, efforts have been made to study the optical RWs (optical analogs of sea waves) on a constant background. The appearance of RW is related to the modulational instability (MI) of the wave background \([8,9]\). MI represents the forma-
tion of a train of localized waves due to the breakup of carrier waves. This occurs due to the exponentially increasing amplitude of the wave as a consequence of the competition between dispersive effects and nonlinearity [10–17]. Recently, it has been realized that the mathematical studies on RWs on periodic wave background will be more useful in analyzing the dynamics of RWs on the surface of the ocean, where the surface wave appears periodic. Employing Darboux method and imposing periodic waves as the seed solution, RW solutions have been constructed on a background of periodic wave for the NLS equation [18,19].

Since the dynamics of RWs depends on the MI characteristics [20–24], studies have also been initiated to investigate the MI of the periodic waves [6,25,26]. The stability of periodic standing waves has been computed for the NLS equation through Floquet–Bloch decomposition method [27]. Later, the periodicity in both space and time (double-periodic) has been taken into account. RWs on the top of this doubly periodic environment were also investigated for the NLS equation [28]. Very recently, Pelinovsky studied the instability of the double-periodic waves of the NLS equation numerically and found that the instability rate of the standing periodic waves is greater than the waves that are periodic, both spatially and temporally [29].

The NLS equation accurately simulates the propagation of light pulses since in this model only the lower-order dispersive (second-order) term and a nonlinearity term were augmented. Subsequently, the fundamental NLS equation was expanded to include self-frequency shift, self-steepening and higher-order dispersion terms up to fifth order. These terms are included to study the dynamics of the propagation of femto/pico second LASER pulses in optical fibers and examine the higher-order nonlinear effects [30–39]. The utility of these extended NLS equations can be seen not only in water waves and optics but also in other branches of physics [40,41]. Many of these higher-order NLS equations are integrable and different kinds of localized solutions have also been constructed for these equations [31,42]. It has been shown that the third- and fifth-order (odd order) operators in this family of equations give different features (introducing velocity or rotation in the (x-t) plane) to the solutions, whereas even ordered operators (4th and 6th, etc.) do not. Instead, they change the MI characteristics and influence the phase of the solutions. For more details, one may refer to the works [43,44]. Motivated by the above facts, in this work, we investigate MI of the fourth-order NLS equation. We consider a fourth-order NLS equation [36,43,44],

$$i\partial_t q + \alpha(q_{xx} + 2 |q|^2 q) + \gamma(q_{xxxx} + 6 \bar{q} q_x^2 + 4 |q_x|^2 q + 8 |q|^2 q_{xx} + 2 q^2 \bar{q}_{xx} + 6 |q|^4 q) = 0,$$

in which \( \bar{q} = \bar{q}(x,t) \) is the complex conjugation of \( q = q(x,t) \) and it represents the wave envelope. In Eq. (1), \( \alpha \) and \( \gamma \) are real parameters and the subscripts denote partial differentiation with respect to that variable. The restriction \( \gamma = 0 \), in Eq. (1), gives the NLS equation. Porsezian et al. [45] discovered Eq. (1) in connection with integrability facets of the one-dimensional Heisenberg spin chain problem. Differing from the work [46] the coefficient \( \gamma \) in Eq. (1), denoting the higher-order elements, is arbitrarily chosen. Fourth-order NLS Equation (1) has two free parameters which makes certain studies more complex. The higher-order nonlinear terms provide certain new physical insights.

Ankiewicz et al. [1,21,47,48] have constructed soliton, RW and breather solutions of Eq. (1). Extended NLS Equation (1) also admits one- and two-breather solutions, and these solutions degenerate into second-order RWs and few other localized solutions under appropriate conditions [49]. In Ref. [50], studies have also been carried out on the instability of Eq. (1). Feng et al. have presented the instability of standing waves for the mixed dispersion fourth-order NLS equation [51]. Recently, RWs on the periodic [52] and double-periodic wave backgrounds [36,53] have also been studied for Eq. (1).

Chowdury et al. have studied the growth rate of breather solutions of Eq. (1) [44]. They have shown that for large values of the higher-order system parameter, the maximal growth rate is higher when compared to the lower-order NLS equation. Recently, Pelinovsky computed the instability rate of single- and double-periodic waves of the NLS equation [29]. Motivated by these recent works, in this article, we investigate the spectral stability problem and MI of Eq. (1). For this purpose, we consider periodic wave solutions (single- and double-periodic in terms of Jacobi elliptic functions) instead of breather solutions [44] and utilize the Floquet–Bloch method [29]. We aim to study how the instability rate changes when we vary the system and elliptic modulus parameters. From the out-
come, we analyze the instability differences that occur in lower-order NLS and higher-order NLS equations. To our knowledge, the instability of periodic waves (in terms of elliptic functions) for Eq. (1) has not been studied.

To carry out our task, initially, we determine the eigenvalues of single-periodic wave solutions (periodic in x) by solving the Lax pair equations. Then, we compute the values of certain unknown parameters that appear in the solution. We then add a linear perturbation term to the periodic wave solutions. Using this perturbation, we derive two linearized equations whose solutions describe the linear stability/instability of single-periodic waves. We then add a linear perturbation to this solution and study the stability spectrum from the solutions of the perturbed part.

The Lax pair associated with fourth-order NLS Eq. (1) reads

\[ \Phi_x = \tilde{M}(\lambda, q) \Phi, \quad \tilde{M}(\lambda, q) = \begin{pmatrix} \lambda & q \\ -\bar{q} & -\lambda \end{pmatrix}, \] (2a)

\[ \Phi_t = \tilde{N}(\lambda, q) \Phi, \quad \tilde{N}(\lambda, q) = i \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix}, \] (2b)

where \( \Phi = (\Phi_1, \Phi_2)^T \), and

\[ A_1 = \frac{|q|^2}{2} + \lambda^2 + \gamma (3 |q|^4 + \bar{q}q_{xx} + q\bar{q}_{xx} - |q_x|^2 - 2\lambda q\bar{q}_{x} + 2\lambda q_x \bar{q} + 4 |q|^2 \lambda^2 + 8|q|^4), \]

\[ B_1 = \frac{q_x}{2} + \lambda q + \gamma (8\lambda^3 q + 4\lambda^2 q_x + 6 |q|^2 q_x + q_{xxx} + 2\lambda q_{xx} + 4\lambda |q|^2 q), \]

\[ C_1 = \frac{\bar{q}_x}{2} - \bar{\lambda} \bar{q} + \gamma (-8\lambda^3 \bar{q} + 4\lambda^2 \bar{q}_x + 6 |q|^2 \bar{q}_x + \bar{q}_{xxx} - 2\lambda \bar{q}_{xx} - 4\lambda |q|^2 \bar{q}). \] (2c)

In Eq. (2), the spectral parameter \( \lambda \) is a complex one and \( \alpha = \frac{1}{2} \). One can unambiguously verify that the compatibility condition of linear Eq. (2), say \( \tilde{M}_t - \tilde{N}_x + [\tilde{M}, \tilde{N}] = 0 \), where the \([, ]\) represents the commutator matrix, gives fourth-order NLS Eq. (1).

2.1 Single-periodic wave solutions and eigenvalues of Eq. (1)

We seek periodic wave solutions of Eq. (1) in the following manner, that is

\[ q(x, t) = f(x) e^{2ibt}, \] (3)

where \( f(x) \) is a real periodic function (which we intend to find in terms of \( dn \) and \( cn \) Jacobian elliptic functions) and the term \( b \) in the exponential function corresponds to a real constant. Inserting Eq. (3) and its derivatives in Eq. (1), we arrive at

\[ f_{xx} + 2 |f|^2 f - 4bf + 2\gamma (f_{xxxx} + 6\bar{f}f_x^2 + 4 |f_x|^2 f + 8 |f|^2 f_{xx} + 2 f^2 \bar{f}_{xx} + 6 |f|^4 f) = 0. \] (4)
One can check that Eq. (4) admits $dn$ and $cn$ solutions for the following values of $b$:

$$
\begin{align*}
  f(x) &= dn(x, k), \quad b = \frac{2 - k^2}{4} + \frac{\nu(6 - 6k^2 + k^4)}{2}, \\
  f(x) &= cn(x, k), \quad b = \frac{2k^2 - 1}{4} + \frac{\nu(1 - 6k^2 + 6k^4)}{2},
\end{align*}
$$

in which the parameter $k$ is nothing but elliptic modulus and it varies from 0 to 1. For $k = 0$, the $dn$ solution gives a constant background wave $q(x, t) = e^{\frac{1}{2}(1 + 6\nu)}$, whereas the $cn$ solution attains the zero background. For $k = 1$, both the solutions take solitary profile (which is normalized), $q(x, t) = \text{sech}(x) e^{\frac{1}{2}(1 + 6\nu)}$.

The functions reported in Eqs. (5) and (6) are also satisfy for following two nonlinear ordinary differential equations, namely

$$
\begin{align*}
  \frac{d^2 f}{dx^2} + 2|f|^4 f &= b_0 f, \\
  \left| \frac{df}{dx} \right|^2 + |f|^4 &= b_0 |f|^2 + b_1,
\end{align*}
$$

where $b_0$ and $b_1$ are real constants. In $dn$ solution (5), we consider $b_0 = 2 - k^2$ and $b_1 = k^2 - 1$, while for the $cn$ solution (6), we consider $b_0 = 2k^2 - 1$ and $b_1 = k^2(1 - k^2)$.

Since the eigenvalue plays an important role in computing the instability of both $dn$- and $cn$-periodic waves, in the following, we determine for which eigenvalues the constructed waves emerge as the solutions satisfying Lax pair (2).

Now, we compute the rate of instability of single-periodic waves given in Eq. (3) from spectral problem (2a) and (2b).

Let us generalize solution (3) in the form

$$
\Phi_1(x, t) = \vartheta_1(x) e^{\Gamma_1 i b_1 t}, \quad \Phi_2(x, t) = \vartheta_2(x) e^{\Gamma_2 i b_2 t},
$$

with

$$
\begin{align*}
  A_1 &= \frac{|f|^2}{2} + \lambda^2 + \gamma(3|f|^4 + \tilde{f}f_{xx} \\
       &+ f_{xx} - |f|^2 - 2\lambda f\tilde{f}_x \\
       &+ 2\lambda f_x\tilde{f} + 4|f|^2\lambda^2 + 8\lambda^4), \\
  B_1 &= \frac{\tilde{f}_x}{2} + \lambda f + \gamma(8\lambda^3 f + 4\lambda^2 f_x \\
       &+ 6|f|^2 f_x + f_{xxx} + 2\lambda f_{xx} \\
       &+ 4\lambda |f|^2 f), \\
  C_1 &= \frac{\tilde{f}_x}{2} - \lambda \tilde{f} + \gamma(-8\lambda^3 \tilde{f} + 4\lambda^2 \tilde{f}_x \\
       &+ 6|f|^2 \tilde{f}_x + \tilde{f}_{xxx} - 2\lambda \tilde{f}_{xx} \\
       &- 4\lambda |f|^2 \tilde{f}).
\end{align*}
$$

We may say $\lambda$ is associated with the eigenvalue problem’s Lax spectrum, Eq. (9a), if $\theta \in [\omega_1, \omega_2]$. Since $f(x)$ is a periodic function, we can impose the condition $f(x + \omega_1) = f(x)$ with the fundamental period $\omega_1 > 0$. In this case, we can represent the bounded solutions of linear Eq. (9a) as [25] (using Floquet’s theorem)

$$
\tilde{\vartheta}(x) = \tilde{\vartheta}(x) e^{i\theta x},
$$

where $\tilde{\vartheta}(x + \omega_1) = \tilde{\vartheta}(x)$ and $\theta \in [-\pi/\omega_1, \pi/\omega_1]$. Above-bounded solutions (10) are periodic and anti-periodic when we consider $\theta = 0$ and $\theta = \pm \pi/\omega_1$, respectively, [25].

From spectral problem, Eq. (9b), a set of nonlinear algebraic equations can be obtained. A nonzero solution for these equations can be obtained only when the determinant of the coefficient matrix is zero. The determinant provides a connection between $\Gamma$ and $\lambda$ in the form

$$
\Gamma^2 + J(\lambda) = 0,
$$

with $J(\lambda) = 4\tilde{J}(\lambda)$ and

$$
\tilde{J}(\lambda) = \lambda^4 - 2b\lambda^2 + b^2 + 2d.
$$

The parameter $d$ is given by

$$
|f_x|^2 + |f|^4 - 4b |f|^2 = 8d.
$$
Fig. 1 Lax and stability spectrum of $dn$-periodic waves given in Eq. (5) with $\Lambda = 2\Gamma$. Sub-figures a, c, e and g are Lax spectrum for $k = 0.2, 0.4, 0.8$ and $0.9$ which are computed from the spatial part of the Lax pair. Sub-figures b, d, f and h are stability spectrum for the same $k$ values which are computed from the temporal part of the Lax pair.
Fig. 2 Lax spectrum (computed from the spatial part of the Lax pair) and stability spectrum (computed through the temporal part of the Lax pair) of \(cn\)-periodic waves of (6) for \(\Lambda = 2\Gamma\): (a, b) \(k = 0.2\), (c, d) \(k = 0.7\), (e, f) \(k = 0.9\) and (g, h) \(k = 0.94\).
Now comparing Eq. (13) with the second equation in (7), we can fix $b_0 = 4b$ and $b_1 = 8d$.

The polynomial $\tilde{J}(\lambda)$ can be rewritten in the form

$$\tilde{J}(\lambda) = \lambda^4 - \frac{1}{2}(f_1^2 + f_2^2)\lambda^2 + \frac{1}{16}(f_1^2 - f_2^2),$$

where we have considered $4b = f_1^2 + f_2^2$ and $8d = -f_1^2 f_2^2$. The parameters $b$ and $d$ can be rewritten in terms of $b_0$ and $b_1$, by choosing $b_0 = f_1^2 + f_2^2$, $b_1 = -f_1^2 f_2^2$. Two positive roots of the polynomial $\tilde{J}(\lambda)$ are

$$\lambda_1 = \frac{1}{2}(f_1 + f_2), \quad \lambda_2 = \frac{1}{2}(f_1 - f_2).$$

Using Eq. (15), we can rewrite Eq. (14) in the form

$$\tilde{J}(\lambda) = (\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2).$$

Upon substituting Eqs. (3) and (8) in Eq. (2a), we can obtain the value of $\lambda$. The spectral problem now takes the form,

$$\left( \frac{d}{d\theta} + i\theta - \hat{f} \frac{d}{d\theta} - d\right) \begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{pmatrix} = \lambda \begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{pmatrix},$$

where $\theta \in (0, \frac{2\pi}{m_1})$. To compute the Lax spectrum we have to determine the eigenvalues of Eq. (17). For this, we solve Eq. (17) numerically using Floquet–Bloch decomposition method [18].

2.2 Linear perturbation to single-periodic wave (3)

Upon perturbing solution (3) linearly,

$$q(x, t) = [f(x) + g(x, t)] e^{2ibt},$$

and inserting the latter expression into Eq. (1) and leaving out the quadratic terms in $g$, we end up at

$$i\xi g_t - 2bg + g_{x x x x} + 4[f^2 g + 2f^2 g + \gamma(18|f|^4 g + 12|f|^2 f^2 g + 6f^2 g + 4|f|^2 g + 12f_{x x} g + 4f_{x x} g + 8f_{x x} g + 8f_{x x} g e^{ibt}] = 0,$$

(19a)

$$-i\tilde{g}_t - 2b\tilde{g} + \tilde{g}_{x x x x} + 4[f^2 \tilde{g} + 2f^2 \tilde{g} + \gamma(18|f|^4 \tilde{g} + 12|f|^2 f^2 \tilde{g} + 6f^2 \tilde{g} + 4|f|^2 \tilde{g} + 12f_x \tilde{g} + 4f_x \tilde{g} + 8f_x \tilde{g} + 8f_x \tilde{g} g_{x x} + 4f_x \tilde{g} + 4f_{x x} \tilde{g} + 8|f|^2 \tilde{g}_{x x} + 2f^2 g_{x x x} + 2f^2 g_{x x x x}) = 0.$$ 

(19b)

We assume a simple nontrivial separable solution to Eq. (19) in the form

$$g(x, t) = m_1(x) e^{it}, \quad \tilde{g}(x, t) = m_2(x) e^{it},$$

where $m_1$ and $m_2$ are functions of $x$. $\Lambda$ represents the spectral parameter and $\Lambda = (m_1, m_2)^T$. Using Eqs. (19) and (20) we rewrite the spectral stability problem as

$$i\Lambda \sigma_3 m + \begin{pmatrix} \bar{s}_{11} & s_{21} \\ s_{21} & \bar{s}_{11} \end{pmatrix} m = 0, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(21a)

with

$$s_{11} = -2b + 4|f|^2 + \partial_{xx} + \gamma(18|f|^4 + 4|f|^2 + 12f_{x x} \partial_x + 4f_x \partial_x + 8f_{x x} + 8|f|^2 \partial_{xx} + \partial_{xxxx}).$$

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\[ s_{21} = 2f^2 + \gamma (12f^2 f^2 + 6f_x^2 + 4f_x \partial_x + 8ff_x + 2f^2 \partial_{xx}). \]  

(21b)

In Eq. (21), \( \Lambda \) belongs to the stability spectrum, given that \( m \) is real and eigenvalue problem (9) for the Lax spectrum is related to the eigenvalue \( \lambda \), \( \vartheta_1^2 \) and \( \vartheta_2^2 \) are the bounded squared eigenfunctions that determines the eigenfunctions of Eq. (21a), which are also bounded, that is \( m_1 = \vartheta_1^2 \) and \( m_2 = -\vartheta_2^2 \). Moreover, we also have a relation \( \Lambda = 2\Gamma \) during this process, where \( \Gamma \) calculates the eigenvalues. If \( Re(\Lambda) > 0 \) for \( \lambda \) in the Lax spectrum, the single-periodic waves given in Eq. (3) are said to be spectrally unstable. In the eigenvalue spectrum (on the \( \Lambda \) plane), if \( Re(\Lambda) > 0 \) and it passes through the origin (0,0) and if it is horizontal to the imaginary axis, then the periodic waves are said to be modulationally unstable. It has been pointed out in the literature that RWs will degenerate into propagating algebraic solitons instead of localization, when the periodic waves are unstable spectrally and stable modulationally (see Ref. [54]). On the other hand, if the spectrum with \( Re(\Lambda) > 0 \) in the \( \Lambda \) plane intersects the origin and it vertically reaches the imaginary axis, then the corresponding RWs reduce into an algebraic propagating soliton [25].

Now, we analyze the modulational and spectral stability of the single-periodic waves. Substituting the relation \( \Lambda = 2\Gamma \) in the expression \( \Gamma^2 + 4\bar{J}(\lambda) = 0 \), we obtain \( \Lambda^2 + 16\bar{J}(\lambda) = 0 \). The eigenspectrum of the perturbed part and the unperturbed part is connected by this relation. This enables us to calculate the growth rate of instability from the relation \( \Lambda = \pm 4i\sqrt{\bar{J}(\lambda)} \). Hence, we can inspect the change in the growth rate of instability of both \( dn\)- and \( cn\)-periodic waves by varying \( k \).

The Lax and stability spectrum of Eq. (9) is shown in Fig. 1 for different values of \( k \) with \( \gamma = 0.2 \) by considering the \( dn\)-periodic wave with \( f_1 = 1 \) and \( f_2 = \sqrt{1-k^2} \). Using Eq. (17) for \( k = 0.2 \), we compute the Lax spectrum, which depends on the \( \lambda \) plane whose outcome is demonstrated in Fig. 1a. The unstable spectrum is calculated from the expression, \( \Lambda = \pm 4i\sqrt{\bar{J}(\lambda)} \), and is displayed in Fig. 1b, where \( \bar{J}(\lambda) \) is given in Eq. (16). Figure 1c, e, g is the Lax spectrum for \( k = 0.4 \), 0.8 and 0.9, respectively. The corresponding stability spectrum is displayed in Fig. 1d, f, h. This figure shows how the eigenvalues evolve in the Lax and stability spectrum. For \( k = 0.2 \), the eigenvalues are placed nearby at the origin (0,0), which is shown in Fig. 1a. Further, when we increase the value of \( k \) to 0.9 the eigenvalues are shifted from zero to \( (-1, 1) \), which is shown in Fig. 1g. In the stability spectrum, Fig. 1d, f, h, the finite segment line that occurs in real part \( (Re(\Lambda)) \) touches the origin. From this, we conclude that \( dn\)-periodic wave solution (5) is both spectrally and modulationally unstable.

For the \( cn\)-periodic wave, we take \( f_1 = 1 \) and \( f_2 = i\sqrt{1-k^2} \). The Lax spectrum of Eq. (9a) for the \( cn\)-periodic wave is given in Fig. 2a, c, e, g for \( \gamma = 0.2 \) with \( k = 0.2, 0.7, 0.9 \) and 0.94, respectively. Figure 2b, d, f, h represents the stability spectrum (unstable) of \( cn\)-periodic waves. Compared to stability spectrum one may visualize in Fig. 2a, c, e, g there are appreciable changes in the Lax spectrum for each \( k \) value. The stability spectrum is similar, but it has a narrow line with a small size eight-band occurring on the origin (0,0) for \( k = 0.2 \) and the size of the eight-band increase when we increase the value of \( k \) from 0.2 to 0.9 which has been demonstrated in Fig. 2d, f. The main difference that we notice in these figures is that the eight bands are connected (intersects) with purely imaginary bands for the first three \( k \) values, that are 0.2, 0.7 and 0.9 (Fig. 2b, d, f). In contrast, they do not intersect for \( k = 0.94 \) (Fig. 2h). In Fig. 2, the bifurcation can be predicted by checking the eigenvalues. From the outcome, we conclude that the \( cn\)-periodic wave given in Eq. (6) is spectrally and modulationally unstable.

Figure 3 shows the rate of the instability of single-periodic waves for different elliptic modulus \( (k) \) values. In Fig. 3, we plot \( Re(\Lambda) \) against \( \theta \) where \( \theta \) is the Floquet parameter which we vary from 0 to \( \pi/\omega_1 \). The largest maximal instability is obtained for \( k = 0 \) which is shown in Fig. 3a. When we increase the value of \( k \) from 0 to 1 the largest maximal instability decreases. For \( k = 1 \) it vanishes. Figure 3b represents the rate of instability of the \( cn\)-periodic waves, which is given in Eq. (6). The maximal instability decreases when we vary the value of \( k \) from 0 to 0.9. In other words, the highest maximal instability is observed for \( k = 0.8 \). Thus, the largest maximal instability of \( dn\)-periodic wave is larger than that of \( cn\)-periodic wave. The maximal instability rate of single-periodic wave is larger for the fourth-order NLS equation when compared to the standard NLS equation (see Ref. [29]).
3 Instability rates of double-periodic waves of (1)

In this section, we evaluate the rate of instability of waves that are periodic both spatially and temporally and compare the rate of instability of these double-periodic waves with single-periodic waves. Equation (1) admits the following form of double-periodic solution

\[ q(x, t) = \left[ f(x, t) + i \Delta(t) \right] e^{i\theta(t)}. \]  

In Eq. (22), the first term \( f(x, t) \) is periodic in both \( x \) and \( t \), whereas the second term \( \Delta(t) \) has a double period in time \( t \). The term \( \theta(t) \) in the exponential is a real function, which is constant in the variable \( x \). The solutions of this kind for Eq. (1) have already been reported [55], and they read as

\[ q(x, t) = \sqrt{\nu} \left[ \text{cn}(Bt, k) \sqrt{\nu} \text{sn}(Bt, k) \right] e^{i(xt + \tau)} e^{i\xi t}, \]

\[ \tau = \sqrt{1 - k^2} \sqrt{\nu}, \]

\[ q(x, t) = \frac{\text{dn}(Bt, k) \sqrt{\nu}}{\sqrt{1 - k^2}} e^{i(xt + \tau)} e^{i\xi t}, \]

\[ \tau = \sqrt{1 - k^2} \sqrt{\nu}, \]  

where \( \nu = 1 + k \) and the constants \( B \) and \( \xi \) that appear in Eqs. (23) and (24) are related to the system parameter \( \gamma \) through the relations, \( B = 1 + 8\nu \) and \( \xi = 1 + (8 - \frac{2}{k^2})\gamma \)[55].

Equations (23) and (24) also satisfy the following identities

\[ q(x, t) = \Phi(x, t) e^{i\Phi_1(x, t)}, \]

\[ \Phi(x + \omega_1, t) = \Phi(x, t + \omega_2) = \Phi(x, t), \]  

where \( \omega_1 > 0 \) and \( \omega_2 > 0 \) are, respectively, the fundamental period in space and time coordinate and one should consider \( 2b = \xi \) for Eq. (23) and \( 2b = \xi k \) for Eq. (24).

The double-periodic solutions given in (23) and (24) can be generalized to the form given in (25). Here, \( \Phi \) represents the solution to linear equation (2) along with

\[ \Phi_1(x, t) = \theta_1(x, t) e^{\eta x + (\Gamma + ib)t}, \]

\[ \Phi_2(x, t) = \theta_2(x, t) e^{\eta x + (\Gamma - ib)t}, \]  

where \( \eta, \Gamma \) are spectral parameters and \( \theta = (\theta_1, \theta_2)^T \). Substituting Eq. (26) into Lax pair Eq. (2) we obtain

the following set of equations, namely

\[ \theta_{x} + \eta \theta = \left( \frac{\lambda}{-\Phi} \right) \theta, \]  

\[ \theta_{t} + \Gamma \theta = i \left( \hat{A}_1 - b \hat{B}_1 \right) \theta, \]  

where

\[ \hat{A}_1 = \frac{|\Phi|^2}{2} + \lambda^2 + \gamma(3|\Phi|^4 + \Phi_{xx} + \Phi_{xxx}) \]

\[ - |\Phi_{x}|^2 - 2\lambda \Phi \Phi_{x} + 2\lambda \Phi_{x} \Phi \]

\[ + 4|\Phi|^2 \lambda^2 + 8\lambda^4, \]

\[ \hat{B}_1 = \frac{\Phi_{xx}}{2} + \lambda \Phi + \gamma(8\lambda^3 \Phi + 4\lambda^2 \Phi_{x} + 6 |\Phi|^2 \Phi_{x}) \]

\[ + \Phi_{xxx} + 2\lambda \Phi_{xx} + 4\lambda \Phi^2 \Phi, \]

\[ \hat{C}_1 = \frac{\Phi_{xx}}{2} - \lambda \Phi + \gamma(-8\lambda^3 \Phi + 4\lambda^2 \Phi_{x} + 6 |\Phi|^2 \Phi_{x}) \]

\[ + \Phi_{xxx} - 2\lambda \Phi_{xx} - 4\lambda |\Phi|^2 \Phi. \]  

The parameters \( \eta \) and \( \Gamma \) are independent of space \( (x) \) and time \( (t) \) coordinates. Upon considering the function \( q(x, t) \) in Eq. (25), the compatibility condition of the system of linear Eqs. (2a) and (2b) satisfies fourth-order NLS Eq. (1). The spectral parameters \( \eta \) and \( \Gamma \) can be fixed in terms of \( \lambda \) using the periodicity condition, \( \theta(x + \omega_1, t) = \theta(x, t + \omega_2) = \theta(x, t) \) (Floquet theorem). The condition which defines the Lax spectrum is that the eigenvalue \( \lambda \) corresponds to an admissible set whereby the solution given in Eq. (26) is bounded in \( x \) with \( \eta = i\theta, \theta \in [-\frac{\pi}{2\omega_1}, \frac{\pi}{2\omega_1}] \). The associated eigenvalue \( \lambda \) is calculated using eigenvalue solver technique with the help of spectral problem (27a) for every \( t \) which is real \( (\theta(x + \omega_1, t) = \theta(x, t)) \). The stability spectrum defined by the condition \( \theta(x, t + \omega_2) = \theta(x, t) \) can be obtained by solving spectral problem (27b) for \( \Gamma \) with the help of \( \lambda \) in the Lax spectrum when all \( x \) is real. In other words, the construction of Lax spectrum and stability spectrum of double-periodic waves depends on \( x \) and \( t \). The time coordinate \( t \) is fixed when \( x \) having a bounded value, and the space coordinate \( x \) is fixed when \( t \) has a bounded value from which we can compute the Lax and stability spectrum. The separation between space \( (x) \) and time \( (t) \) coordinates arises since we examine the stability of double-periodic waves. Otherwise, a similar procedure can be followed to determine the perturbed eigenvalue \( \Lambda = 2\Gamma \) as in
the study of instability of the single-periodic wave with $q = \Phi$. The spectral parameter $\Gamma$ is defined by the range $\text{Im}(\Gamma) = |\frac{\pi}{ao}|$. Here $\text{Re}(\Gamma)$ determines the rate of instability ($\text{Re}(\Lambda)$) through the relation $\Lambda = 2\Gamma$.

In Fig. 4, we plot the surface plots of $q(x, t)$ of the double-periodic solutions. Figure 4a represents the amplitude of double-periodic waves (23) for $k = 0.999$. In Fig. 4b, we plot the amplitude of double-periodic waves (24) for the same elliptic modulus ($k$) value. From these figures, one may notice that the solutions given in Eqs. (23) and (24) represent phase-repeated and phase-alternating wave patterns, respectively.

If the spectrum $\text{Re}(\Lambda) > 0$ in the Lax spectrum with eigenvalue $\lambda$, then the double-periodic wave solution (Eq. (25)) is said to be spectrally unstable. The Lax spectrum for eigenvalue problem (27a) and stability spectrum of the double-periodic solutions for different $k$ values are displayed in Fig. 5. Figure 5a, c, e, g represents the Lax spectrum on the $\lambda$ plane for $k = 0.4$, $k = 0.7$, $k = 0.9$ and $k = 0.999$, respectively. The stability spectrum of the solution (Eq. (23)) is demonstrated in Fig. 5b, d, f, h for the same values of $k$. Here we notice that for every $k$ value (0 to 1), the unstable spectrum can be found at the strip’s boundary $\text{Im}(\Lambda)$. Thus, the double-periodic wave given in Eq. (23) is spectrally unstable. Solution (23) of the fourth-order NLS equation converges to the double-periodic solution of the standard NLS equation when $\gamma = 0$. The stability spectrum of NLS equation is demonstrated in Fig. 6. Figure 6a–d shows the stability of double-periodic solution for four different $k$ values (0.4, 0.7, 0.9 and 0.999). These four figures help us to compare the stability spectrum of Eq. (1) with NLS equation. We can see the following stability differences in Figs. 6 (NLS equation) and 5 (fourth-order NLS equation). The presence of eigenvalues (spectral line) on the $\text{Re}(\Lambda)$ in the stability spectrum of Fig. 6 is smaller compared to Fig. 5. Also, the spectral line on the $\text{Im}(\Lambda)$ is increased in Fig. 5b, d compared to NLS case, Fig. 6a, b. Further, the boundary value of the spectral lines in both the $\text{Re}(\Lambda)$ and $\text{Im}(\Lambda)$ axis is larger when compared to the NLS equation. This spectral line changes because of the presence of higher-order dispersion parameter $\gamma$.

Figure 7 is the same as Fig. 5. Figure 7a, b represents the Lax and stability spectrum of (24) for $k = 0.2$, whereas Fig. 7c, d is drawn for $k = 0.5$, Fig. 7e, f is drawn for $k = 0.7$, and Fig. 7g, h is depicted for $k = 0.9$. The Lax spectrum has three types of bands (upper, center, lower) on the $\lambda$ plane, of which two bands (upper and lower) connect both the imaginary and real axis (Fig. 7a, c, e, g), whereas the third (center) band occupies the real axis alone. The spectrum which is unstable on the $\Lambda$ plane includes two symmetrically inverted heart-shaped bands having a boundary at $\text{Im}(\Lambda) = \pm \frac{2\pi}{ao}$. The two bands are connected with the pure imaginary band. When $k \to 1$ (for $k \approx 0.9$), these two bands narrow down and the stability spectrum appears similar to Fig. 5d. This is due to the fact that double-periodic solutions (23) and (24) converge to one kind of Akhmediev breather (AB) solution for $k = 1$. For the choice $\gamma = 0$, double-periodic solution (24) of Eq. (1) becomes the solution of NLS equation. The corresponding stability spectrum is given in Fig. 8. Figure 8a–d exposes the stability spectrum for $k = 0.2$, 0.5, 0.7 and 0.9. Here also, we can see the stability differences between in the NLS equation (Fig. 8) and the fourth-order NLS equation (Fig. 7). In the NLS equation, the two symmetrically inverted heart-shaped bands (eight-band) intersect [29] at the origin (0, 0), whereas in the fourth-order NLS equation the two symmetrically inverted heart-shaped bands (like eight-band) do not intersect at the origin (it goes away from the origin). Also, the spectral line on the $\text{Re}(\Lambda)$ in the...
Fig. 5 Lax spectrum (computed from the spatial part of the Lax pair) and stability spectrum (computed through the temporal part of the Lax pair) of double-periodic waves (23) with $\gamma = 0.01$: (a, b) Lax and stability spectrum for $k = 0.4$, (c, d) Lax and stability spectrum for $k = 0.7$, (e, f) Lax and stability spectrum for $k = 0.9$, (g, h) Lax and stability spectrum for $k = 0.999$.
stability spectrum of Fig. 8 is small when compared to Fig. 7. For example, in Fig. 8a, the presence of eigenvalues in the symmetrically inverted heart-shaped bands is placed in the region nearby zero ($Re(\Lambda) \approx -0.1$ to 0.1) on the $Re(\Lambda)$ axis. In the $Im(\Lambda)$ axis also the range turns out to be small ($Im(\Lambda) \approx -0.4$ to 0.4). But in fourth-order NLS, Eq. (1), the presence of eigenvalue ranges is increased in both the axes, which is confirmed in Fig. 7b. Figure 8b–d also gives the same kind of observation when compared to Fig. 7d, f, h. This is because of the presence of higher-order dispersion parameter $\gamma$.

Instability rate of the fourth-order NLS equation for the aforementioned two double-periodic waves is illustrated in Fig. 9 for four different values of $k$. Here $Re(\Lambda)$ and the Floquet parameter $\theta$ ($\theta \in [0, \frac{\pi}{\omega_1}]$) are plotted against each other with $\eta = i\theta$. The instability rate is highest when $k = 0.999$, that is, at the Akhmediev breather (when $k = 1$). The double-periodic wave given in Eq. (23) can be visualized in Fig. 9a. The unstable spectrum begins at the same value as the cutoff $\theta$, and it extends to $\theta = \frac{\pi}{\omega_1}$. When we decrease the value of $k$ from a higher value to a lower value, the maximal instability decreases. For $k = 0$ the double-periodic solutions converge to the NLS soliton. In Fig. 9b, we display the rate of instability of double-periodic wave (24), where we can see the rate of instability is high for the choice $k \to 0$; at the time, the double-periodic wave converges to the $cn$-periodic waves. The highest instability rate is obtained for $k = 0.999$ ($k \approx 1$), that is, when it reduces to the Akhmediev breather solution. Here, we can observe that, for $k = 1$, the instability rate reaches its maximum level. The instability rate decreases when we decrease the value of $k$ from higher to lower (0.999 to 0.2). For both the double-periodic waves, the instability rate attains its maximal level to 1 (because of its unity). To understand the instability rate difference between NLS and (1), we slightly increase the system parameter $\gamma$ value from 0.02 to 0.08 and then to 0.2. The outcome is illustrated in Fig. 9c–f. As one can see, in Fig. 9e, while we increase the value of $\gamma$, the instability rate vanishes for the $k$ values 0.3, 0.4 and 0.7. Differing from this, the instability rate of double-periodic wave (24) is not vanishing for the $k$ values 0.2, 0.5, 0.7, which is displayed in Fig. 9d. The outcome also confirms that the instability rate increases when the value of system parameter $\gamma$ is increased. In both the solutions, the largest instability rate gradually increases, and the width of the instability band changes when we increase the system parameter $\gamma$ value.
Fig. 7 Lax spectrum and stability spectrum of double-periodic wave (24) with $\gamma = 0.01$: (a, b) Lax and stability spectrum for $k = 0.2$, (c, d) Lax and stability spectrum for $k = 0.5$, (e, f) Lax and stability spectrum for $k = 0.7$ and (g, h) Lax and stability spectrum for $k = 0.9$
4 Conclusion

In this paper, we have analyzed the rates of instability of two types (single and double) of periodic waves for a fourth-order NLS equation. We have divided the method of computing instability rate of the single-periodic waves ($dn$ and $cn$) into two parts. In the first part, we have obtained the eigenvalues and periodic wave solutions by solving the Lax pair equations analytically. We have also found the values of certain undetermined parameters that exist in the considered solution. In the second part, we have introduced a linear perturbation term in the periodic wave solutions and derived a system of linearized equations whose solutions describe the stability/instability of these waves. By relating the variables of these equations to another spectral parameter $\Lambda$ we have studied the instability of the solutions. We have also investigated the instability of the waves, which has periodicity both in space and time for the considered equation. By considering the solutions in the periodic form and invoking Floquet theory, we have obtained the Lax spectrum (fixed $t$) and stability spectrum (fixed $x$) of these periodic waves. We therefore demonstrated the instability of double-periodic solutions through this simple algorithm and estimated the rate of instability of it for two different values of $\gamma$. Our investigations reveal that the instability of single-periodic wave is higher when compared to double-periodic wave solutions. We have also observed that the stability spectrum (presence of eigenvalue) range and instability rate increase when the value of system parameter $\gamma$ is increased. This corresponds to the faster evolution of the periodic waves (single and double) with a higher growth rate. It has been shown that the width and height (maximal of the instability rate) of the instability rate of double-periodic waves of the NLS equation is smaller when compared to the fourth-order NLS equation. The properties of MI can affect the width of the optical RWs, see for example Ref. [50]. When the instability growth rate increases, the width of the RW increases first and then decreases, where the growth rate increasing phenomena make the existence time of the RW shorter. If the perturbation grows to a comparable size, it may affect the desired result. For example, if we give a small perturbation (periodic perturbation) in the fiber optics experiment, RWs will arise due to MI. There will be a change in the RW appearance for a larger perturbation value as well. From this, we can conclude that a larger perturbation will affect the real applications. Our results will be useful in such studies, particularly in the optical RW. Through this study, we have outlined the properties of
Fig. 9 Rate of instability for the double-periodic waves for different elliptic modulus values: a, c, e instability rate for Eq. (23) with three different values of $\gamma$, which are 0.02, 0.08 and 0.2 (pink, red, green, blue and black colors denote the different values of $k$, which is 0.3, 0.4, 0.7, 0.9 and 0.999, respectively) and b, d, f instability rate for Eq. (24) with $\gamma = 0.02$, $\gamma = 0.08$ and $\gamma = 0.2$ (pink, red, green, blue and black colors denote the different values of $k$, which are 0.2, 0.5, 0.7, 0.9 and 0.999, respectively). (Color figure online)

MI gain for two different periodic wave backgrounds for fourth-order NLS Eq. (1).

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Data availability The data that support the findings of this study are available within the article.

Declarations

Conflict of interest The authors have no competing interests.

Ethics approval This research did not involve human or animal subjects.

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