Quantum field theory with a fundamental length: 
A general mathematical framework

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Abstract

We review and develop a mathematical framework for nonlocal quantum field theory (QFT) with a fundamental length. As an instructive example, we re-examine the normal ordered Gaussian function of a free field and find the primitive analyticity domain of its $n$-point vacuum expectation values. This domain is smaller than the usual future tube of local QFT, but we prove that in difference variables, it has the same structure of a tube whose base is the $(n-1)$-fold product of a Lorentz invariant region. It follows that this model satisfies Wightman-type axioms with an exponential high-energy bound which does not depend on $n$, contrary to the claims in the literature. In our setting, the Wightman generalized functions are defined on test functions analytic in the complex $l$-neighborhood of the real space, where $l$ is an $n$-independent constant playing the role of a fundamental length, and the causality condition is formulated with the use of an analogous function space associated with the light cone. In contrast to the scheme proposed by Br"uning and Nagamachi [J. Math. Phys. 45 (2004) 2199] in terms of ultra-hyperfunctions, the presented theory obviously becomes local as $l$ tends to zero.

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I. Introduction

Since the seminal works of Jaffe [1] and Meyman [2], it has been well recognized that the localizability condition imposes restrictions on the high-energy behavior of quantum fields. Specifically, the momentum-space expectation values of local fields have less than exponential growth. This has suggested a possible way of constructing a self-consistent theory of nonlocal interactions by going beyond this boundary. At the present time, nonlocal field theories of this kind continue to attract interest because of their interplay with string theory and holographic models, see, e.g., [3, 4, 5, 6] for a discussion of this and related issues. It is also important that these theories provide an alternative to the use of the idealized concept of local commutativity (also called microcausality) and are aimed at finding a more physically motivated implementation of causality.

Iofa and Fainberg [7, 8] were first to show that the Wightman axiomatic approach [9] can be extended to nonlocal field theories with an exponential growth of the off-shell-mass amplitudes. In this case, a field must be averaged with analytic test functions in order to yield a well defined operator, and this makes the formulation [9] of local commutativity impossible because it uses test functions of compact support. In [7, 8], the microcausality axiom was replaced (for the case of a single scalar field) by the condition of symmetry of the analytic Wightman functions under the permutations of their arguments. Subsequently it was shown [10] that a natural way of formulating causality in nonlocal field theories with analytic test functions is by using a suitably adapted notion of carrier of an analytic functional. In [10], it was also proved that such a formulation implies the symmetry of the Wightman functions when their domain of analyticity is nonempty.

In [7, 8, 10], it was assumed that the $n$-point vacuum expectation values of nonlocal fields grow in momentum space not faster than

$$\exp\left\{ \ell \sum_{j=1}^{n-1} |q_j| \right\}, \quad (1)$$

where $q_j$ is the momentum conjugate to the relative coordinate $\xi_j = x_j - x_{j+1}$ and $\ell$ is an $n$-independent constant playing the role of a fundamental length. This assumption is equivalent to saying that, in the coordinate representation, the vacuum expectation values, considered as generalized functions of the variables $\xi_j$, are defined on the test function space $A_\ell(\mathbb{R}^{4(n-1)})$ whose elements are analytic in the complex $\ell$-neighborhood of the real space $\mathbb{R}^{4(n-1)}$ and decrease rapidly at infinity. (The exact definition of this
space is given in Sec. III.) More recently, Brüning and Nagamachi [11] proposed to replace the uniform bound (1) with a weaker condition. Simply stated, this condition means that the growth of the vacuum expectation values in every variable $q_j$ at fixed $q_i$, $i \neq j$, is not faster than $\exp(\ell|q_j|)$. The formal definition [11] is given in terms of a special class of generalized functions called ultra-hyperfunctions. It can be rewritten in our notation as the requirement that the $n$-point vacuum expectation value considered as a generalized function of the $\xi_j$’s is defined on each of the spaces

$$A_\infty(\mathbb{R}^{4(j-1)}) \otimes A_\ell(\mathbb{R}^{4}) \otimes A_\infty(\mathbb{R}^{4(n-j-1)}), \quad j = 1, \ldots, n - 1.$$  \hspace{1cm} (2)

Brüning and Nagamachi claimed that such a modification is necessary to make the theory applicable to the nonlocal model $e^{g\phi^2}: (x)$, where $\phi$ is a free scalar field and $::$ stands for the normal ordering.

In the present paper, we show that this claim results from an imperfect description of the analyticity domain of the corresponding Wightman functions. We prove that the toy model $e^{g\phi^2}: (x)$ completely fits in the original framework [7, 8, 10]. So, this example does not give any grounds for a more complicated scheme with mixed-type spaces. Our main aim here is to present a development of the ideas proposed in [7, 8, 10] and give the rigorous and up-to-date formulation of a general mathematical framework for treating quantum field models with a fundamental length. We also show that the presented theory becomes local in the limit $\ell \to 0$, whereas the existence of a local limit for the modified scheme [11] is problematic.

In Sec. II, we describe exactly the analyticity domains of the Wightman functions of the field $e^{g\phi^2}$: and show that these domains are considerably larger than those found in [11] and have a simpler structure. Making use of this result, we prove in Sec. III that the $n$-point vacuum expectation values of this field, when considered as a generalized function of the difference variables $\xi_j$, are well defined on the spaces $A_\ell(\mathbb{R}^{4(n-1)})$, where $\ell = \sqrt{g/6}$. In Sec. IV, we show that these generalized functions satisfy the quasilocality condition introduced in [10]. A general mathematical framework for quantum field theory (QFT) with a fundamental length is discussed in detail in Sec. V, where we also compare our approach with that of Ref. [11] and reveal some essential differences in their consequences. Sec. VI contains concluding remarks.
II. Wightman functions of the field : \( \exp g\phi^2 : (x) \)

Let \( \phi \) be a free neutral scalar field of mass \( \mu \geq 0 \) in Minkowski space. We recall some simple but important properties of its two-point vacuum expectation value

\[
\langle \Psi_0, \phi(x_1)\phi(x_2)\Psi_0 \rangle = \Delta_+ (x_1 - x_2) = \frac{1}{(2\pi)^3} \int \theta(p^0)\delta(p^2 - \mu^2)e^{-ip(x_1-x_2)}dp. \tag{3}
\]

The function \( \Delta_+ (z) \), \( z = x + iy \), is analytic in the domain \( \mathbb{R}^4 + i\mathbb{V}^- \), where \( \mathbb{V}^- \) is the lower light cone \( \{ y \in \mathbb{R}^4 : y^2 = (y^0)^2 - y^2 > 0, y^0 < 0 \} \), and satisfies the inequality

\[
|\Delta_+ (x + iy)| \leq \frac{1}{4\pi^2} \cdot \frac{1}{y^2}, \quad y \in \mathbb{V}^- . \tag{4}
\]

Indeed, this function is Lorentz invariant and hence we can assume, without loss of generality, that \( y = (-\tau, 0, 0, 0) \), \( \tau > 0 \). Then,

\[
|\Delta_+ (x^0 - i\tau, x)| = \frac{1}{(2\pi)^3} \left| \int \frac{e^{-i\omega(p)(x^0 - i\tau)}}{2\omega(p)} e^{ipx}dp \right| \\
\leq \frac{1}{4\pi^2} \int_0^\infty \frac{e^{-\tau \sqrt{s^2 + \mu^2}} s^2 ds}{\sqrt{s^2 + \mu^2}} \leq \frac{1}{4\pi^2} \int_0^\infty e^{-\tau s} s^2 ds = \frac{1}{4\pi^2} \cdot \frac{1}{\tau^2}.
\]

We will use the estimate (4) to describe properties of the vacuum expectation values of the field

\[
: \exp g\phi^2 : (x) = \sum_{r=0}^{\infty} \frac{g^r}{r!} : \phi^{2r} : (x). \tag{5}
\]

As shown by Rieckers [12], the field (5) can be implemented in the Hilbert space of \( \phi \) as an operator-valued generalized function with test functions in any Gelfand-Shilov space \( S^\alpha \), where \( \alpha < 1 \). Rieckers argued that this field obeys all Wightman axioms, except for locality, and noted that its two-point function is well defined even on a larger space \( S^{1,\lambda} \), where \( \lambda \) depends on \( g \). We shall consider the \( n \)-point vacuum expectation values

\[
\mathcal{W}_n(x_1, \ldots, x_n) = \langle \Psi_0, : \exp g\phi^2 : (x_1) \ldots : \exp g\phi^2 : (x_n) \Psi_0 \rangle, \tag{6}
\]

where \( n \geq 2 \). We first find the analyticity domains of the corresponding Wightman functions \( W(\xi_1, \ldots, \xi_{n-1}) \) depending on the relative coordinates \( \xi_j = x_j - x_{j+1} \). These functions exist by the translation invariance of (6) and satisfy

\[
W(x_1 - x_2, \ldots, x_n - x_{n-1}) = \mathcal{W}_n(x_1, \ldots, x_n). \tag{7}
\]

We let \( \zeta_j = \xi_j + i\eta_j \) denote the complex difference variables and use the notation

\[
\mathbb{V}_l^- = \{ \eta \in \mathbb{R}^4 : \eta^2 > l^2, \eta^0 < 0 \}. \tag{8}
\]
Theorem 1. The Wightman function $W(\zeta_1, \ldots, \zeta_{n-1})$ of the field $\exp g\phi^2$ is analytic in the tube $(\mathbb{R}^4 + i\mathbb{V}^-)^{n-1}$, where $\ell = \sqrt{g/6}$.

Proof. From the Wick theorem it follows that the $n$-point vacuum expectation values of any normal ordered entire function of the free field $\phi$ are representable as formal power series in the variables

$$w_{ij} = \Delta_+(x_i - x_j), \quad 1 \leq i < j \leq n. \tag{9}$$

Let $R$ be a multi-index whose components $r_{ij}, 1 \leq i < j \leq n$, ranges over the set $\mathbb{Z}_+$ of nonnegative integers. If

$$\varphi(x) = \sum_{r=0}^{\infty} \frac{d_r}{r!} : \phi^r : (x), \tag{10}$$

then the formal power expansion has the form

$$\langle \Psi_0, \varphi(x_1) \ldots \varphi(x_n) \Psi_0 \rangle = \sum_R \frac{D_R}{R!} w^R, \tag{11}$$

where

$$w^R = \prod_{1 \leq i < j \leq n} w^r_{ij}, \quad R! = \prod_{1 \leq i < j \leq n} r_{ij}!.$$ 

An analysis carried out by Jaffe [13] shows that the coefficients $D_R$ in (11) are related to the initial coefficients $d_r$ by

$$D_R = \prod_{j=1}^{n} d_{R_j}, \tag{12}$$

where $R_j = r_{1j} + \ldots + r_{j-1,j} + r_{j,j+1} + \ldots + r_{jn}$ is the occurrence number of $x_j$ in the monomial $w^R$.

We consider the terms of the series on the right-hand side of (11) as functions of the complex variables $\zeta_j = \xi_j + i\eta_j$ lying in the domain $\mathbb{R}^4 + i\mathbb{V}^-$, where every monomial $w^R$ is analytic. In the case under study

$$d_r = \begin{cases} \frac{g^{r/2}r!}{(r/2)!} & \text{for even } r, \\ 0 & \text{for odd } r, \end{cases}$$

and by the Stirling formula these coefficients satisfy

$$d_r^2 \leq (2g)r! \tag{13}.$$ 

Now we define $w_{ij}$ and $r_{ij}$ for $i > j$ by setting $w_{ij} = w_{ji}$ and $r_{ij} = r_{ji}$. We also set $r_{jj} = 0$ for all $j = 1, \ldots, n$. Note that then $R_j = \sum_{i=1}^{n} r_{ij}$ is the sum of elements of the
\(j\)-th column of the symmetric matrix \((r_{ij})\). Moreover,
\[
\left| w^R \right|^2 = \prod_{i,j=1, i \neq j}^{n} \frac{|w_{ij}|r_{ij}}{r_{ij}!}.
\]
Using (12), (13) and taking into account that \(R_j!/r_{1j}! \ldots r_{nj}!\) is a multinomial coefficient, we obtain
\[
\left| \frac{D_R}{R!} w^R \right|^2 \leq \prod_{j=1}^{n} \left( (2g)^R_j \prod_{i=1, i \neq j}^{n} \frac{|w_{ij}|r_{ij}}{r_{ij}!} \right) \leq (2g)^{|R|} \prod_{j=1}^{n} \left( \sum_{i=1, i \neq j}^{n} |w_{ij}| \right)^{R_j},
\]
where \(2|R| = \sum_j R_j\). We note that if all the vectors \(\text{Im} \, \zeta_j\) belong to \(\mathbb{V}_l^-\), where \(l > 0\), then the sum on the right-hand side of (14) is uniformly bounded by a constant independent of \(n\). Indeed, \(z_i - z_j = \zeta_i + \cdots + \zeta_{j-1}\) for any \(j > i + 1\) and hence
\[
\text{Im}(z_i - z_j) \in (j-i)\mathbb{V}_l^- = \mathbb{V}_{(j-i)l}^- \quad \text{for all} \ j > i,
\]
because the set \(\mathbb{V}_l^-\) is convex. From (14) it follows that for such arguments,
\[
|\Delta_+(z_i - z_j)| \leq \frac{1}{4\pi^2 l^2} \cdot \frac{1}{(j-i)^2}, \quad j > i.
\]
Therefore,
\[
\sum_{i=1, i \neq j}^{n} |w_{ij}| = \sum_{j=1}^{i-1} |\Delta_+(z_i - z_j)| + \sum_{i=j+1}^{n} |\Delta_+(z_j - z_i)| \leq \frac{1}{2\pi^2 l^2} \sum_{k=1}^{n-1} \frac{1}{k^2}
\]
everywhere in the specified region of the complex variables. Combining (14) and (16) and taking into account that \(\sum_{k=1}^{\infty} (1/k^2) = \pi^2/6\), we obtain
\[
\left| \frac{D_R}{R!} w^R \right| \leq \left( \frac{g}{\pi^2 l^2} \sum_{k=1}^{n-1} \frac{1}{k^2} \right)^{|R|} \leq \left( \frac{g}{6l^2} \right)^{|R|}.
\]
We conclude that the formal series representing the Wightman functions \(W(\zeta_1, \ldots, \zeta_{n-1})\) converge absolutely if the imaginary parts of all \(\zeta_j\) belong to \(\mathbb{V}_l^-\), where \(l > \ell = \sqrt{g/6}\). The convergence is uniform in this region and hence the limit functions are analytic in \((\mathbb{R}^4 + i\mathbb{V}_l^-)^{n-1}\). The theorem is proved.

Remark 1. In the simplest case of zero mass, it is easy to obtain an explicit expression for the two-point function of the field (15). Indeed, we have
\[
\langle \Psi_0, : \exp g\phi^2 : (x_1) : \exp g\phi^2 : (x_2) \Psi_0 \rangle = \sum_{m=0}^{\infty} \frac{(2m)!}{(ml)!^2} (g \Delta_+(x_1 - x_2))^{2m}.
\]
This point seems to be overlooked in [11].
If \( z \in \mathbb{C} \) and \( 4|z| < 1 \), then
\[
\sum_{m=0}^{\infty} \frac{(2m)!}{(m!)^2} z^m = \frac{1}{\sqrt{1 - 4z}}.
\]

Furthermore, if \( \mu = 0 \), then \( \Delta_+(\zeta) = -\frac{1}{4\pi^2} \cdot \frac{1}{\zeta^2} \) and the estimate (11) is exact. Considering that \( |\zeta^2| \geq (\Im \zeta)^2 \) for all \( \Im \zeta \in \mathbb{V}^- \), we deduce that in this case, the two-point function (18) has the form
\[
\frac{1}{\sqrt{1 - 4g^2 \Delta_+(\zeta)^2}} = \frac{\zeta^2}{\sqrt{(\zeta^2)^2 - \ell_0^2}},
\]
where \( \ell_0 = \frac{1}{\pi} \sqrt{\frac{g}{2}} \)

and its primitive analyticity domain is the tube \( \mathbb{R}^4 + i\mathbb{V}_{\ell_0}^- \). Note that \( \ell = (\pi/\sqrt{3})\ell_0 \approx 1.8 \ell_0 \).

**Remark 2.** The analyticity domain found for \( W(\zeta_1, \ldots, \zeta_{n-1}) \) in [11], p. 2225, has the form
\[
\bigcup_{i=1}^{n-1} \mathcal{V}_{R,\epsilon,i},
\]
where
\[
\mathcal{V}_{R,\epsilon,i} = \{ \zeta \in \mathbb{C}^{4(n-1)} : -\Im \zeta_i \in \mathbb{V}_+ + (\ell_0 + \epsilon, 0), -\Im \zeta_j \in \mathbb{V}_+ + (R, 0), j \neq i \}
\]
and \( \epsilon > 0 \) can be taken arbitrarily small, but \( R \) is an undetermined constant depending on \( \epsilon \).

### III. The appropriate test functions

As a simple application of the above theorem, we specify the test function spaces that are adequate to the generalized functions [15]. Let \( \ell \) be a positive number or \( \infty \) and let \( |y| = \max_{1 \leq j \leq d} |y_j| \). We denote by \( A_{\ell}(\mathbb{R}^d) \) the topological vector space of functions analytic in the tubular domain \( \{ x + iy \in \mathbb{C}^d : |y| < \ell \} \) and such that all the norms
\[
\| f \|_{L,N} = \sup_{|y| \leq \ell} \sup_{x \in \mathbb{R}^d} (1 + |x|)^N |f(x + iy)|, \quad l < \ell, \quad N = 0, 1, 2, \ldots,
\]
(19)

are finite. Let \( \lambda = 1/(e\ell) \) and \( \lambda = 0 \) for the special case \( \ell = \infty \). The space \( A_{\ell}(\mathbb{R}^d) \) coincides with the space \( S^{1,\lambda}(\mathbb{R}^d) \) which by definition [14] consists of all infinitely differentiable functions on \( \mathbb{R}^d \) with the property that, for each \( \lambda > \lambda \),
\[
(1 + |x|)^N |\partial^\kappa f(x)| < C_N \lambda^{n|\kappa|} \kappa^\kappa,
\]
(20)
where \( N \) ranges over \( \mathbb{Z}_+ \), the multi-index \( \kappa \) ranges over \( \mathbb{Z}_+^d \), and \( C_{N\kappa} \) is a constant depending on \( f \). As shown in [13], the Taylor series expansion of any function belonging to \( S^{1,\lambda}(\mathbb{R}^d) \) defines an element of \( A_\ell(\mathbb{R}^d) \). Conversely, using Cauchy’s integral formula, one can easily verify that the restriction of any element of \( A_\ell(\mathbb{R}^d) \) to the real space \( \mathbb{R}^d \) belongs to \( S^{1,\lambda}(\mathbb{R}^d) \) and the map \( A_\ell(\mathbb{R}^d) \to S^{1,\lambda}(\mathbb{R}^d) \) is continuous. In what follows, the notation \( A_\ell(\mathbb{R}^d) \) is more convenient for brevity.

**Remark 3.** The space \( A_\infty(\mathbb{R}^d) = \bigcap_{\lambda > 0} S^{1,\lambda}(\mathbb{R}^d) \) consists of the entire analytic functions that decrease faster than any inverse power of \( |x| \) as \( |x| \to \infty \) and it coincides with the space \( T(T(\mathbb{R}^d)) \) in the notation used in [11]. The elements of its dual space \( T(T(\mathbb{R}^d))' = A'_\infty(\mathbb{R}^d) \) are said to be tempered ultra-hyperfunctions. The space \( A_\ell(\mathbb{R}^d) \) was denoted in [11] by \( T(T(O)) \), where \( T(O) = \mathbb{R}^d + iO \) and \( O = (−\ell, \ell)^d \).

According to Gelfand and Shilov [14], \( S^{1,\lambda}(\mathbb{R}^d) = A_\ell(\mathbb{R}^d) \) is a complete metrizable nuclear space. Therefore, it belongs to the class FS of Fréchet-Schwartz spaces. In particular, \( A_\ell(\mathbb{R}^d) \) is a Montel space, and hence, reflexive. These facts can be used to derive its other properties useful for applications. As a simple example, we prove in Appendix that the space \( S^0(\mathbb{R}^d) \), which is the Fourier transform of the space \( D(\mathbb{R}^n) \) of smooth functions of compact support, is dense in \( A_\ell(\mathbb{R}^d) \) for any \( \ell \). A fortiori, \( A_\infty(\mathbb{R}^d) \) is dense in \( A_\ell(\mathbb{R}^d) \). In the textbook [14] it is shown that the Fourier transformation \( f(x) \to \hat{f}(p) = \int f(x)e^{ipx}dx \) is an isomorphism of \( S^{1,\lambda}(\mathbb{R}^d) \) onto the space \( S_{1,\lambda}(\mathbb{R}^d) \) consisting of all smooth functions \( g(p) \) with the finite norms

\[
\|g\|_{l,N} = \max_{|\kappa| \leq N, p \in \mathbb{R}^d} |\partial^\kappa g(p)| \exp \left\{ l \sum_{j=1}^d |p_j| \right\}, \quad l < 1/(\epsilon \lambda). \tag{21}
\]

**Theorem 2.** Considered as a generalized function of the difference variables, the \( n \)-point vacuum expectation value of \( \exp \tilde{g}\phi^2 \) is well defined on the test function space \( A_\ell(\mathbb{R}^4(n-1)) \), where \( \ell = \sqrt{g/6} \).

**Proof.** The tempered distribution \( u^R(\xi) \) in [11] is the boundary value of the function \( u^R(\xi + i\eta) \) analytic in the tube \( (\mathbb{R}^4 + iV^-)^{n-1} \). Therefore, for each \( \eta \in (V^-)^{n-1} \) such that \( |\eta| < \ell \), we have

\[
(u^R, f) = \int u^R(\xi + i\eta) f(\xi + i\eta) \, d\xi, \quad f \in A_\ell(\mathbb{R}^4(n-1)). \tag{22}
\]

\(^4\)In Ref. [10], this space was denoted by \( \mathfrak{A}_\ell(\mathbb{R}^d) \).
Let $\eta_j = (-l, 0, 0, 0)$ for all $j$. From (17) and the definition (19), it follows that

$$
\left| \frac{D_R}{R!} (w^R, f) \right| \leq C \left\| f \right\|_{l, 4n-3} \left( \frac{g}{2^l l^2} \sum_{k=1}^{n-1} \frac{1}{k^2} \right)^{|R|}.
$$

(23)

where $C = \int_{R^{(n-1)}} (1 + |\xi|)^{-4n+3} d\xi$. Because $\sum_{k=1}^{n-1} (1/k^2)$ is strictly less than $\pi^2/6$, there exists an $l < \ell$ such that the number in the square brackets is less than 1. Hence the number series $\sum_R (D_R/R!)(w^R, f)$ converges absolutely for each $f \in A_\ell(R^{n-1})$ and defines a linear functional on $A_\ell(R^{n-1})$ whose continuity is ensured by the factor $\left\| f \right\|_{l, 4n-3}$ on the right-hand side of (23). The theorem is proved.

**Corollary.** The generalized function (6) is well defined on the space $A_{\ell_n}(R^{4n})$, where $\ell_n = \ell(n - 1)/2$.

**Proof.** We let $t$ denote the linear transformation

$$
(x_1, \ldots, x_n) \rightarrow (X, \xi_1, \ldots, \xi_{n-1}),
$$

(24)

where

$$
X = \frac{1}{n} (x_1 + \cdots + x_n), \quad \xi_j = x_j - x_{j+1}.
$$

Then, we have

$$
(W_n, f) = (W, f_t), \quad f_t(\xi) = \int_{R^4} f(t^{-1}(\xi, X)) \, dX,
$$

which is a rigorous form of the formal relation (7). The inverse transformation $t^{-1}$ is written as

$$
x_j = X - \frac{1}{n} \sum_{m=1}^{j-1} m \xi_m + \frac{1}{n} \sum_{m=1}^{n-j} m \xi_{n-m}.
$$

(25)

If the $\xi_j$’s in (25) are changed for $\xi_j + i\eta_j$, where $|\eta_j| < \ell$, then the variables $x_j$ gain imaginary parts $y_j$ satisfying

$$
|y_j| < \frac{l}{n} \left( \sum_{m=1}^{j-1} m + \sum_{m=1}^{n-j} m \right) \leq \frac{\ell}{n} \sum_{m=1}^{n-1} m = \ell \frac{n-1}{2}.
$$

(26)

Hence, if $f \in A_{\ell_n}(R^{4n})$, then $f_t$ is analytic in the domain $\{ \xi + i\eta \in C^{4(n-1)} : |\eta| < \ell \}$.

Let $l < \ell$. Using the inequalities $|\xi| \leq 2|x|$ and $|X| \leq |x|$, we get

$$
\left\| f_t \right\|_{l,N} \leq \sup_{|\eta| < \ell} \sup_{\xi} (1 + |\xi|)^N \int_{R^4} |f(t^{-1}(\xi + i\eta, X))| \, dX
\leq C \sup_{|y| < l(n-1)/2} \sup_{x} (1 + |x|)^{N+5} |f(x + iy)| = C \left\| f \right\|_{l(n-1)/2, N+5}.
$$

(27)
where $C = 2^N \int_{\mathbb{R}^d} (1 + |X|)^{-5} dX$. Therefore, $f_t \in A_\ell(\mathbb{R}^{4(n-1)})$ and the map $A_\ell_n(\mathbb{R}^{4n}) \to A_\ell(\mathbb{R}^{4(n-1)}): f \to f_t$ is continuous.

Remark 4. The formulas (25) and (26) exhibit a simple geometric fact. Namely, if we have an ordered set of points in $\mathbb{R}^d$ and the spacing between adjacent points is not greater than $\ell$, then the distance of any point from the geometric center of the set does not exceed $\ell(n - 1)/2$.

IV. Quasilocality

The microcausality axiom [9] expresses the idea of independence of field measurements performed at spacetime points separated by a spacelike interval. Explicitly,

$$[\varphi(f), \psi(g)]\Psi = 0$$

for any observable fields $\varphi$ and $\psi$, for all states $\Psi$ in their common domain and for any test functions $f, g$ whose supports are spacelike to each other. In the simplest case of a scalar local field $\varphi(x)$ this amounts to saying that for any $k \leq n$, the support of

$$\langle \Psi_0, \varphi(x_1)\ldots\varphi(x_k)\varphi(x_{k+1})\ldots\varphi(x_n)\Psi_0 \rangle$$

$$- \langle \Psi_0, \varphi(x_1)\ldots\varphi(x_{k+1})\varphi(x_k)\ldots\varphi(x_n)\Psi_0 \rangle$$

is contained in the wedge

$$\overline{V}_{(k,k+1)} = \{ x \in \mathbb{R}^{4n} : (x_k - x_{k+1})^2 \geq 0 \}. \quad (29)$$

In [10], it was argued that a natural generalization of this axiom to the nonlocal QFT with test functions in $A_\ell(\mathbb{R}^4)$ is the condition of continuity of the functional (28) in the topology of an analogous space associated with the wedge (29). We recall the definitions introduced there.

For each set $O \subset \mathbb{R}^d$, we define a space $A_\ell(O)$ in the following way. Let $O^l$ be the complex $l$-neighborhood of $O$, consisting of those points $z \in \mathbb{C}^d$ for which there is $x \in O$ such that $|z - x| \equiv \max_j |z_j - x_j| < \ell$. The space $A_\ell(O)$ consists of all functions analytic in $O^l$ for which the norms

$$\|f\|_{O,l,N} = \max_{\|\kappa\| \leq N} \|z^\kappa f(z)\|, \quad l < \ell, \quad N = 0, 1, 2, \ldots, \quad (30)$$
are finite. We note that \( A_\ell(O) \) coincides with \( A_\ell(\bar{O}) \), where \( \bar{O} \) is the closure of \( O \). Using the relations

\[
\max_{\kappa \leq N} |z^\kappa| = \max(1, |z|^N), \quad (1 + |z|)^N \leq 2^N \max(1, |z|^N),
\]

it is easy to verify that in the particular case \( O = \mathbb{R}^d \), this definition is equivalent to the definition of \( A_\ell(\mathbb{R}^d) \) given above. If \( O_2 \subset O_1 \), then there is a natural continuous injective map \( A_\ell(O_1) \to A_\ell(O_2) \) called “restriction morphism”. Let \( f|O_2 \) be the image of \( f \) under this morphism. Then the relation \( f|O_3 = (f|O_2)|O_3 \) holds for any \( O_3 \subset O_2 \subset O_1 \) and so we have a presheaf of vector spaces.

Now we shall show that the vacuum expectation values of the field (5), considered as generalized functions of the difference variables \( \xi_j = x_j - x_{j+1} \), satisfy the quasilocality condition introduced in [10]. The proof proceeds in the same manner as the derivation of an analogous theorem [15] for those Wick power series whose limits are defined on the space \( S^1 = \bigcup_{\lambda > 0} S^{1,\lambda} \) and are localizable in the sense of hyperfunctions. In this derivation, a key role is played by the Bargmann-Hall-Wightman theorem [9] which shows that the function \( \Delta^+ (z) \) can be analytically continued to an extended tube \( T_{\text{ext}} \). This tube is obtained by applying all complex Lorentz transformations to the primitive analyticity domain \( \mathbb{R}^4 + i\mathbb{V}^- \). The continued function is invariant under the complex Lorentz group \( L_+ (\mathbb{C}) \) and, in particular, under the total reflection \( z \to -z \). We note that the transposition \( x_k \) and \( x_{k+1} \) induces the transformation

\[
\xi_{k-1} \to \xi_{k-1} + \xi_k, \quad \xi_k \to -\xi_k \quad \xi_{k+1} \to \xi_{k+1} + \xi_k
\]

of the difference variables. (Strictly speaking, here we mean that \( n \neq 2 \) and \( 1 < k < n-1 \). For \( n = 2 \) we have simply the reflection \( \xi_1 \to -\xi_1 \). For \( n > 2 \) and \( k = 1, k = n-1 \) we have, respectively, \( \xi_1 \to -\xi_1, \xi_2 \to \xi_2 + \xi_3 \) and \( \xi_{n-2} \to \xi_{n-2} + \xi_{n-1}, \xi_{n-1} \to -\xi_{n-1} \).)

The functional (28) written in these variables looks as follows

\[
W(\xi_1, \ldots, \xi_{k-1}, \xi_k, \xi_{k+1}, \ldots, \xi_{n-1}) - \quad W(\xi_1, \ldots, \xi_{k-1} + \xi_k, -\xi_k, \xi_k + \xi_{k+1}, \ldots, \xi_{n-1}).
\]

Below we use an extension of the estimate (4) to \( T_{\text{ext}} \).
Lemma. For any $l > 0$, the function $\Delta_+(z)$ is analytic in the domain \( \{z = x + iy \in \mathbb{C}^4 : x^2 < -l^2 < y^2\} \) and satisfies the inequality

\[
|\Delta_+(x + iy)| \leq \frac{1}{2\pi^2} \cdot \frac{1}{y^2 - x^2}
\]

everywhere in this domain.

Proof. Suppose first that $y^2 > 0$. Because $\Delta_+$ is Lorentz invariant, we may assume without loss of generality that the vectors $y$ and $x$ have the form $y = (y^0, 0, 0, 0)$ and $x = (x^0, x^1, 0, 0)$, where $x^1 < 0$. This can be achieved by a boost with a subsequent rotation. Next we use the invariance under $L_+(\mathbb{C})$ and apply the transformation $\Lambda \colon (z^0, z^1, z^2, z^3) \to (iz^1, iz^0, z^2, z^3)$ belonging to this group. The vector $\text{Im } \Lambda z = (x^1, x^0, 0, 0)$ lies in $\mathbb{V}^-$ and $(\text{Im } \Lambda z)^2 = -x^2$. Using (4), we obtain the inequality $4\pi^2|\Delta_+(x + iy)| \leq \min[1/y^2, 1/(-x^2)]$, which implies (33). Now let $y^2 < 0$. Then, we may assume that $y = (0, y)$ and $x$ has the previous form. In this case, $(\text{Im } \Lambda z)^2 = -x^2 - (y^2)^2 - (y^3)^2 \geq -y^2 - x^2$ and we see that (33) holds again. Finally, let $y$ be lightlike. By using a real Lorentz transformation, it can be made such that $y^2 < \varepsilon^2$, where $\varepsilon$ is arbitrarily small. Then $(\text{Im } \Lambda z)^2 > -\varepsilon^2 - x^2$ and $\text{Im } \Lambda z \in \mathbb{V}^-$ if $\varepsilon < l$. Hence, in this case too, $z = x + iy$ is a point of analyticity at which the function $\Delta_+(z)$ satisfies (33). The lemma is proved.

**Theorem 3.** Let $W(\xi_1, \ldots, \xi_{n-1})$ be the Wightman function of $\varphi = \exp g \varphi^2$ defined by (6) and (7). Then, every functional $\Psi(\varphi)$ extends continuously to the space $A_{2\ell}(V(k))$, where $\ell = \sqrt{g/6}$ and $V(k) = \{\xi \in \mathbb{R}^{4(n-1)} : \xi_k^2 > 0\}$.

Proof. We note that

\[
\langle \Psi_0, \varphi(x_1) \ldots \varphi(x_{k+1}) \varphi(x_k) \ldots \varphi(x_n) \Psi_0 \rangle = \sum_R \frac{D_R}{R!} \tilde{w}^R, \tag{34}
\]

where $\tilde{w}_{ij}$ is the previous system (9) except for $\tilde{w}_{k,k+1} = \Delta_+(x_{k+1} - x_k)$. Indeed, let $\tau$ denote the transposition $(1, \ldots, k, k + 1, \ldots, n) \to (1, \ldots, k + 1, k, \ldots, n)$. Define $R'$ by $r'_{i,j} = r_{\tau i, \tau j}$ for $i \neq k$, $j \neq k + 1$ and $r'_{k,k+1} = r_{k,k+1}$. Clearly, we have the equality

\[
\prod_{i<j} w^R_{i,j} = \prod_{i<j} \tilde{w}^R_{i,j}. \]

The map of multi-indices $R \to R'$ is bijective. Furthermore, $R'! = R!$ and $D_{R'} = D_R$ because $R'_j = R_{\tau j}$. Thus (34) does hold. Since the analytic function $\Delta_+(z)$ is invariant
under the total reflection, the distribution $\tilde{w}^R(x)$ is a boundary value of the same holomorphic function $w^R(z)$ but from the cone
\[ \{ y = \text{Im } z \in \mathbb{R}^n : y_i - y_j \in \mathbb{V}^-, \ 1 \leq i < j \leq n, i \neq k, j \neq k + 1; \ y_k - y_{k+1} \in \mathbb{V}^+ \}. \]

Let $0 < l < \ell$ and let $\eta$ be a vector in $\mathbb{R}^{4(n-1)}$ such that $\eta_j = (-l, 0, 0, 0)$ for $j < k - 1$, $j > k + 1$ and $\eta_{k-1} = \eta_{k+1} = (-2l, 0, 0, 0)$, $\eta_k = (l, 0, 0, 0)$. Then all the differences $y_i - y_j = \eta_k + \cdots + \eta_{j-1}$, $i < j$, are in the suitable cones and hence the value of the distribution $\tilde{w}^R(\xi)$ at a test function $f \in A_{2\ell}(\mathbb{R}^{4(n-1)})$ can be written as
\[ (\tilde{w}^R, f) = \int w^R(\xi + i\eta) f(\xi + i\eta) \, d\xi, \quad (35) \]

Using the invariance of $\Delta_+(z)$ under the total reflection and the estimate (4), we obtain
\[ |\tilde{w}^R(\xi + i\eta)| \leq \frac{1}{(2\pi l)^{2|R|}} \prod_{i<j} \frac{1}{n^{2r_{ij}}}, \quad (36) \]

where
\[ n_{ij} = |\eta_i^0 + \cdots + \eta_{j-1}^0|/l \]

Next we apply the trick used in the proof of Theorem 1, squaring the expression (36) and passing to the symmetric matrix $(r_{ij})$. Using (12), (13), and the multinomial theorem, we obtain
\[ \left| \frac{D_R}{R!} \tilde{w}^R(\xi + i\eta) \right|^2 \leq \frac{(2g)^{2|R|}}{(2\pi l)^{4|R|}} \prod_{j=1}^{n} \left( \sum_{i=1}^{n} \frac{1}{n^{2r_{ij}}} \right)^{R_j}, \quad (37) \]

where $n_{ij} = |\eta_i^0 + \cdots + \eta_{j-1}^0|/l$ for $i > j$. It is easy to see that, for every fixed $j$, the system of nonzero integers $n_{ij}$, where $1 \leq i \leq n$, $i \neq j$, contains at most two identical integers. Therefore, we arrive at the inequality
\[ \left| \frac{D_R}{R!} (\tilde{w}^R, f) \right| \leq C \| f \|_{2l, 4n-3} \left[ \frac{g}{(\pi l)^2} \sum_{k=1}^{n-1} \frac{1}{k^2} \right]^{R_j}. \quad (38) \]

Taking $l$ sufficiently close to $\ell$, we infer that the series $\sum_R (D_R/R!) (\tilde{w}^R, f)$ converges absolutely for every $f \in A_{2\ell}(\mathbb{R}^{4(n-1)})$ and defines a continuous linear functional on this space.

Now let $\eta_k(t) = t(l, 0, 0, 0)$, where $-1 \leq t \leq 1$, and let the other $\eta_j(t)$, $j \neq k$, be as before. From the above, it is clear that the value of the functional (32) at a test function $f \in A_{2\ell}(\mathbb{R}^{4(n-1)})$ can be written as
\[ \sum_R \frac{D_R}{R!} \int_{R^{4(n-1)}} \left[ w^R(\xi + i\eta(-1)) f(\xi + i\eta(-1)) - w^R(\xi + i\eta(1)) f(\xi + i\eta(1)) \right] \, d\xi. \quad (39) \]
In extending continuously this functional to \( A_{2\ell}(V_{(k)}) \), there is no problem with the contribution to the integral from the region

\[
V^l_{(k)} = \{ \xi \in \mathbb{R}^{4(n-1)} : \xi^2_k > -l^2 \},
\]

because the surfaces \( V^l_{(k)} + i\eta(\pm 1) \) lie in the complex \( 2\ell \)-neighborhood of the wedge \( V_{(k)} \). The surfaces \( \mathcal{C}V^l_{(k)} + i\eta(\pm 1) \), together with

\[
\sigma^l_{(k)} = \{ \zeta \in \mathbb{C}^{4(n-1)} : (\text{Re} \zeta)^2 = -l^2, \text{Im} \zeta = \eta(t), -1 \leq t \leq 1 \},
\]

compose the piecewise smooth boundary of a \((4n-3)\)-dimensional flat surface in \( \mathbb{C}^{4(n-1)} \) which, by Lemma, is contained in the analyticity domain of \( w^R(\zeta)f(\zeta) \). Considering that \( w^R(\zeta) \) is bounded and \( f(\zeta) \) decreases rapidly as \( |\text{Re} \zeta| \to \infty \) and applying the Stokes theorem to the closed form \( \omega^R_f = w^R(\zeta)f(\zeta) d\zeta_1 \wedge \cdots \wedge d\zeta_{n-1}, \) we obtain

\[
\int_{\mathcal{C}V^l_{(k)}} [w^R(\xi + i\eta(-1)) f(\xi + i\eta(-1)) - w^R(\xi + i\eta(1)) f(\xi + i\eta(1))] d\xi = \int_{\sigma^l_{(k)}} \omega^R_f,
\]

where the surface \( \sigma^l_{(k)} \) is properly oriented. More explicitly, using the local coordinates \( \tilde{\zeta} = (\xi_1, \ldots, \xi_{k-1}, \xi_{k+1} \ldots \xi_{n-1}) \), \( \xi_k = (\xi^1_k, \xi^2_k, \xi^3_k) \), and \( t \), we have

\[
\int_{\sigma^l_{(k)}} \omega^R_f = I_+ + I_-,
\]

where

\[
|I_\pm| = l \left| \int_{-1}^{1} dt \int_{\mathbb{R}^{4(n-2)}} d\xi \int_{\xi^2 > l^2} w^R(\xi + i\eta(t)) f(\xi + i\eta(t)) \big|_{\xi^2_k = \pm \sqrt{\xi^2 - l^2}} d\xi_k \right|. \tag{40}
\]

By Lemma,

\[
|w_{k,k+1}(\xi, t)| = |\Delta_+(\xi_k + i\eta_k(t))| \leq \frac{1}{(2\pi l)^2} \quad \text{for} \quad \xi^2_k = -l^2, -1 \leq t \leq 1. \tag{41}
\]

For all other \( i < j \), the vector \( \eta_i(t) + \cdots + \eta_{j-1}(t) \) lies in \( V^- \) and

\[
|w_{ij}(\xi, t)| = |\Delta_+(\xi_i + \cdots + \xi_{j-1} + i\eta_i(t) + \cdots + i\eta_{j-1}(t))| \leq \frac{1}{(2\pi l)^2} \frac{1}{n_{ij}^2}, \quad -1 \leq t \leq 1, \tag{42}
\]

because the minimum of \( |\eta^0_i(t) + \cdots + \eta^0_{j-1}(t)| \) occurs at \( t = 1 \). From (41) and (42), it follows that

\[
\left| \int_{\sigma^l_{(k)}} \omega^R_f \right| \leq C' \|f\|_{V_{(k)}, 2l, 4n-3} \frac{1}{(2\pi l)^2 |R|} \prod_{i < j} \frac{1}{n_{ij}^{2r_{ij}}}, \tag{43}
\]

14
\[ C'' = 2 \int_{\mathbb{R}^{4(n-2)}} (1 + |\xi|)^{-4n+7} |\xi| \int_{\mathbb{R}^3} (1 + |\xi_k|)^{-4} \, d\xi_k. \]
Therefore, an analog of the inequality (38) with \( \|f\|_{V(k),2|4n-3} \) instead of \( \|f\|_{2|4n-3} \) holds for \( (D_R/R!) \int_{\sigma(k)} \omega_1^R \) as well as for \( (D_R/R!) \int_{V(k)} \omega_1^R \). We conclude that the series

\[
\sum \frac{D_R}{R!} \left( \int_{V(k)}^{\eta(-1)} + \int_{V(k)}^{\eta(1)} + \int_{\sigma(k)} \right) \omega_1^R
\]

converges absolutely for any \( f \in A_{2\ell}(V(k)) \) and defines a continuous linear functional which coincides with (32) on \( A_{2\ell}(\mathbb{R}^{4n-1}) \). Theorem 3 is thus proved.

### V. Wightman-type axioms for quasilocal fields

In this section we discuss general characteristic properties of the vacuum expectation values in the nonlocal QFT with an elementary length introduced via the presheaf of spaces \( A_{\ell}(O) \). For simplicity, we restrict our consideration to the case of a neutral scalar field \( \phi \). Its \( n \)-point vacuum expectation value \( \langle \Psi_0, \phi(x_1) \ldots \phi(x_n) \Psi_0 \rangle \) is denoted by \( \mathcal{W}_n(x_1, \ldots x_n) \). We hold the usual assumption that these generalized functions have no worse than polynomial growth in position space or, what is the same, that their Fourier transforms \( \hat{\mathcal{W}}_n \) have finite order of singularity. We take the test function space \( A_{\infty} \) to be the initial domain of definition of the field, just as was done in Refs. [7, 8], where, however, a different characterization of this space was used. The proper orthochronous Poincaré group \( \mathcal{P}_+^\parallel \) acts on the space \( A_{\infty}(\mathbb{R}^{4n}) \) by

\[
f_{(a,\Lambda)}(x_1, \ldots, x_n) = f(\Lambda^{-1}(x_1 - a), \ldots, \Lambda^{-1}(x_n - a)), \quad (a, \Lambda) \in \mathcal{P}_+^\parallel.
\]

We assume that the vacuum expectation values have the following characteristic properties.

(a.1) (Initial domain of definition)

\[ \mathcal{W}_n \in A_{\infty}'(\mathbb{R}^{4n}) \quad \text{for} \quad n \geq 1. \]

(a.2) (Hermiticity)

\[ \overline{\langle \mathcal{W}_n, f \rangle} = \langle \mathcal{W}_n, f^\dagger \rangle \quad \text{for each} \quad f \in A_{\infty}(\mathbb{R}^{4n}), \quad \text{with} \quad f^\dagger(x_1, \ldots, x_n) = \overline{f(x_n, \ldots, x_1)}. \]

(a.3) (Positive definiteness)

\[ \sum_{k,m=0}^N (\mathcal{W}_{k+m}, f_k^\dagger \otimes f_m) \geq 0, \]
where $W_0 = 1$, $f_0 \in \mathbb{C}$, and $\{f_1, \ldots, f_N\}$ is an arbitrary finite set of test functions such that $f_k \in A_\infty(\mathbb{R}^{4k})$, $k = 1, \ldots, N$.

(a.4) (Poincaré covariance)

$$(W_n, f) = (W_n, f_{(a, \Lambda)}) \quad \text{for each } f \in A_\infty(\mathbb{R}^{4n}) \text{ and } (a, \Lambda) \in \mathcal{P}^1_+.$$  

This is equivalent to the existence of Lorentz-invariant functionals $W_n \in A'_\infty(\mathbb{R}^{4(n-1)})$ such that

$$W_n(x_1, \ldots, x_n) = W_n(x_1 - x_2, \ldots, x_n - x_{n-1}), \quad n \geq 1.$$  

(a.5) (Spectral condition)

$$\text{supp } \hat{W}_n \subset \overline{V^+} \times \cdots \times \overline{V^+}.$$  

(a.6) (Cluster decomposition property)

If $a$ is a spacelike vector, then for each $f \in A_\infty(\mathbb{R}^{4k})$ and for each $g \in A_\infty(\mathbb{R}^{4m})$,

$$(W_{k+m}, f \otimes g_{(\lambda a, I)}) \rightarrow (W_k, f)(W_m, g) \quad \text{as } \lambda \rightarrow \infty.$$  

(a.7.1) (Quasilocalizability)

There exists $\ell < \infty$ such that every functional $W_n$ has a continuous extension to the space $A_\ell(\mathbb{R}^{4(n-1)})$.

(a.7.2) (Quasilocality)

For any $n \geq 2$ and $1 \leq k \leq n - 1$, the difference

$$W_n(\xi_1, \ldots, \xi_{k-1}, \xi_k, \xi_{k+1}, \ldots, \xi_{n-1})$$

$$- W_n(\xi_1, \ldots, \xi_{k-1} + \xi_k, -\xi_k, \xi_{k+1}, \ldots, \xi_{n-1})$$

(with the above specification for $k = 1, n - 1$) has a continuous extension to the space $A_\ell(V_k)$, where $V_k = \{\xi \in \mathbb{R}^{4(n-1)}: \xi_k^2 > 0\}$.

The listed properties of vacuum expectation values are different from those of the Wightman functions in local QFT [9, 10, 17] in two aspects. First, the space $A_\infty$ substitutes for the Schwartz space $S$ of infinitely differentiable functions of fast decrease. It should be emphasized that the role of $A_\infty$ is completely auxiliary because the true domain of definition of the vacuum expectation values is specified by condition (a.7.1). This
condition means, intuitively, that the correlation functions are localizable with respect to the relative coordinates at scales larger than \( \ell \). The reason is that all observable quantities are expressed in these variables. The space \( A_\infty \) may be replaced with other test function space which is invariant under linear transformations of the coordinates and is dense in every \( A_\ell, \ell < \infty \), and for which a kernel theorem holds. For instance, we may use \( S^0 = \mathcal{D} \) and this gives a system of axioms which is completely equivalent to the system listed above. Such a choice is quite convenient because the space \( \mathcal{D} \) is customary for physicists theoreticians and occupies a prominent place in functional analysis.

On the other hand, \( A_\infty \) is maximal among the spaces suitable for the role of an initial domain of definition of fields in nonlocal quantum field theory. The second and main difference of our approach from the traditional formalism [9, 16, 17] of local QFT is that the local commutativity axiom is replaced with the weaker quasilocality condition \((a.7.2)\) which means, intuitively, that causality principle is obeyed at spacetime scales large compared to the fundamental length \( \ell \).

Although the space \( A_\ell(\mathbb{R}^{d(n-1)}) \) is not Lorentz invariant, conditions \((a.7.1)\) and \((a.7.2)\) can be given an invariant form [18]. By condition \((a.4)\), the functional \( W_n \) is well defined on every space obtained from \( A_\ell(\mathbb{R}^{d(n-1)}) \) by a Lorentz transformation. All the spaces \( \Lambda A_\ell(\mathbb{R}^{d(n-1)}), \Lambda \in L^+ \), are contained in \( S^1(\mathbb{R}^{d(n-1)}) \) and their linear span can be given the inductive limit topology induced by the injections \( \Lambda A_\ell(\mathbb{R}^{d(n-1)}) \rightarrow S^1(\mathbb{R}^{d(n-1)}) \). Then \( W_n \) extends continuously to the resulting Lorentz invariant space. Analogously, condition \((a.7.2)\), together with condition \((a.4)\), implies that the functional \((44)\) has a continuous extension to the linear span in \( S^1(V(k)) \) of the spaces \( \Lambda A_\ell(V(k)), \Lambda \in L^+ \), and we thereby obtain a relativistically invariant implementation of causality.

In the original Wightman’s formulation [9], local commutativity is expressed as a property of the distributions \( W_n(x_1, \ldots, x_n) \) with respect to the permutations of spacelike separated arguments \( x_k \). For comparison, let us derive the corresponding consequence of condition \((a.7.2)\). Let \( \pi \) be a permutation of the indices 1, \ldots, \( n \). It acts on \( \mathbb{R}^{d(n)} \) by the rule
\[
\pi(x_1, \ldots, x_n) = (x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})
\]
and acts on \( f(x) \) in an analogous manner \((\pi f)(x) = f(\pi^{-1}x)\). We let \( \pi W_n \) denote the “permuted” vacuum expectation value \( \langle \Psi_0, \varphi(x_{\pi 1}) \cdots \varphi(x_{\pi n}) \Psi_0 \rangle \). In more exact terms,
\[
(\pi W_n, f) = (W_n, \pi^{-1} f), \quad f \in A_\infty(\mathbb{R}^{d(n)}).
\]
Theorem 4. Condition (a.7.1) implies that for any \( n \geq 1 \), the functional \( W_n \) has a continuous extension to the space \( A_\ell_n(\mathbb{R}^{4n}) \), where \( \ell_n = \ell(n - 1)/2 \). Furthermore, from condition (a.7.2) it follows that for any permutation \( \pi \), the difference \( W_n - \pi W_n \) extends continuously to the space \( A_\ell_n(V_\pi) \), where

\[
V_\pi = \bigcup_{k,m} V_{(k,m)}, \quad V_{(k,m)} = \{ x \in \mathbb{R}^{4n} : (x_k - x_m)^2 > 0 \},
\]

and the union ranges over all pairs of indices such that \( k < m \) and \( \pi^{-1}k > \pi^{-1}m \).

Proof. The first statement was already proved in deriving Corollary of Theorem 2 and it suffices to verify the second statement for a transposition \( \pi_k : (1, \ldots, k, k + 1, \ldots, n) \to (1, \ldots, k + 1, k, \ldots, n) \). If \( f \in A_\infty(\mathbb{R}^{4n}) \), then \( (W_n - \pi_k W_n, f) \) is equal to the value of functional (44) at the test function \( f_t(\xi) = \int_{\mathbb{R}^4} f(t^{-1}(\xi, X)) \, dX \), where \( t \) is defined by (24). Let us show that \( f \to f_t \) is a continuous map from \( A_\ell_n(V_{(k,k+1)}) \) into \( A_\ell(V_k) \). Let \( l < \ell \) and let \( \zeta \in V_l(\xi) \), i.e., there exists \( \bar{\xi} \in V_k \) such that \( |\zeta - \bar{\xi}| < l \). Suppose that \( \bar{x} \) is related to \( \bar{\xi} \) by (25) and \( z \) is related to \( \zeta \) in a similar way. Clearly, \( \bar{x} \in V_{(k,k+1)} \) and, for every \( j = 1, \ldots, n \), we have

\[
|z_j - \bar{x}_j| \leq \frac{1}{n} \left( \sum_{m=1}^{j-1} m|\zeta_m - \bar{\xi}_m| + \sum_{m=1}^{n-j} m|\zeta_{n-m} - \bar{\xi}_{n-m}| \right) < \frac{2n - 1}{2} = l_n.
\]

Thus, the point \( t^{-1}(\zeta, X) \) belongs to \( V_{(k,k+1)} \) for any \( X \in \mathbb{R}^4 \). This enables one to obtain an estimate analogous to (27), namely,

\[
\| f_t \|_{V_{(k,k+1)}, l, N} \leq C \| f \|_{V_{(k,k+1)}, l_n, N+5},
\]

which completes the proof.

Theorems 2 and 3 show that the vacuum expectation values calculated from the formula (11) satisfy conditions (a.7.1) and (a.7.2). Deriving these theorems, we used not the explicit form of the function \( \exp g \phi^2 \) : but the restriction (13) on the determining coefficients. For this reason a more general statement is true.

Theorem 5. Let \( \phi \) be a free neutral scalar field in Minkowski space and let

\[
\varphi(x) = \sum_{r=0}^{\infty} \frac{d_r}{r!} : \phi^r : (x),
\]

where the coefficients \( d_r \) are real and subject to the condition

\[
d_r^2 \leq C(2g)^r r!.
\]
Then, the vacuum expectation values of \( \varphi(x) \) have all properties (a.1) – (a.7), and the conditions (a.7.1) and (a.7.2) are satisfied with \( \ell = 2\sqrt{g/6} \).

Proof. Conditions (a.7.1) and (a.7.2) are fulfilled by Theorems 2 and 3 and the first of them even with the constant \( \sqrt{g/6} \). Condition (a.1) is obviously satisfied by Corollary of Theorem 2 because \( A_\infty \) is contained in every \( A_{k_n} \). The Hermiticity property (a.2) is obvious from the reality of \( d_r \). The positive definiteness (a.3) is a direct consequence of the fact that the Wightman functions of the Wick polynomials \( \sum_{r=0}^N (d_r/r!) : \phi^r : (x) \) have this property, which, in turn, is apparent from the definition of the Wick monomials : \( \phi^r : (x) \) via a limiting procedure. Property (a.4) follows immediately from the Lorentz invariance of \( \Delta_+(x) \). The spectral condition (a.5) is fulfilled by the Paley-Wiener-Schwartz theorem (see, e.g., Theorem 2.8 in Ref. [9] or Theorem B.8 in Ref. [17]) which shows that every term of the series defining \( \hat{W}_n \) has support in \( \bar{V}^+ \times \cdots \times \bar{V}^+ \). Hence, the sum of this series also has this support property because \( \hat{A}_\infty \subset D \). It remains to show that the cluster decomposition property (a.6) also holds. We will prove even a stronger version of this property.

Let \( a \) be a spacelike vector in \( \mathbb{R}^4 \) and let \( h \) be an arbitrary element of \( A_\infty(\mathbb{R}^{4n}) \), where \( n = k + m \). We set
\[
h_{\lambda a}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = h(x_1, \ldots, x_k, x_{k+1} - \lambda a, \ldots, x_n - \lambda a), \quad \lambda > 0.
\]
and claim that the vacuum expectation values of \( \varphi(x) \) satisfy the estimate
\[
|\langle W_{k+m} - W_k \otimes W_m, h_{\lambda a} \rangle| \leq C_a \|h\|_{n,4(n+1)} \frac{1}{1 + \lambda^2},
\]
where the norm of \( h \) is defined by (19). Setting \( h = f \otimes g \), we see that (46) implies (a.6) and moreover the limit converges no worse than \( 1/\lambda^2 \). To prove (46), we note that
\[
W_{k+m} - W_k \otimes W_m = \sum_R' \frac{D_R}{R!} w^R,
\]
where the prime indicates that the summation ranges over all matrices \( R \) with the property that \( r_{ij} \neq 0 \) for at least one pair of indices such that \( i \leq k, j > k \). Let \((i_0, j_0)\) be such a pair and let
\[
U_{i_0 j_0} = \sum_{\{R: r_{i_0 j_0} \neq 0\}} \frac{D_R}{R!} w^R.
\]
We estimate this expression, setting \( \text{Im} z_j = ((j - 1) l, 0, 0, 0) \) for all \( j = 1, \ldots, n \) and assuming that the Lorentz square of \( x_{i_0} - x_{j_0} = \text{Re}(z_{i_0} - z_{j_0}) \) is less than \( -L^2 \). By
Lemma in Sec. IV, we have
\[ |\Delta_+(z_{i_0} - z_{j_0})| \leq \frac{1}{2\pi^2} \cdot \frac{1}{l^2(j_0 - i_0)^2 + L^2} \leq \frac{2n^2l^2}{l^2 + L^2} \cdot \frac{1}{\pi^2l^2} \cdot \frac{1}{(j_0 - i_0)^2}. \]  
(48)

Let \( l > \ell \) and \( l^2 + L^2 > 2n^2l^2 \). We proceed as in the proof of Theorem 1, using (48) and (15) for other pairs of indices, and conclude that for \((x_{i_0} - x_{j_0})^2 \leq -L^2\),
\[ |U_{i_0j_0}(z)| \leq C_0 \frac{2n^2l^2}{l^2 + L^2}, \]  
(49)
where \( C_0 \) is a constant dominating the total sum \( \sum_R |D_R w_R|/R! \).

With \( y = \text{Im} \, z \) taken as stated above, we have the representation
\[ (U_{i_0j_0}, h_{\lambda a}) = I_< + I_>, \quad \text{where} \quad I_\| = \int_{(x_{i_0} - x_{j_0})^2 \leq -\lambda^2 a^2/2} U_{i_0j_0}(x + iy) h_{\lambda a}(x + iy) dx. \]
Choosing \( l \) so that \( \ell < l < \ell n/(n - 1) \) and using (49) with \( L^2 = \lambda^2 a^2/2 \), we obtain
\[ |I_<| \leq C_1 \| h \|_{n\ell, 4n+1} \frac{1}{1 + \lambda^2}. \]

From similarity considerations, it is clear that the distance of the point \(-\lambda a\) from the set \( \{ x \in \mathbb{R}^4 : x^2 \geq -\lambda^2 a^2/2 \} \) increases linearly with \( \lambda \). Because of this, there is \( \epsilon_a > 0 \) such that \( |x_{i_0} - x_{j_0} + \lambda a| > \epsilon_a \lambda \) for all points in the range of integration for \( I_> \). Therefore,
\[ |I_>| \leq C_0 \int_{|x_{i_0} - x_{j_0}| > \epsilon_a \lambda} |h_{\lambda a}(x + iy)| dx = C_0 \int_{|x_{i_0} - x_{j_0}| > \epsilon_a \lambda} |h(x + iy)| dx \\
\leq C_0 \int_{|x_{i_0} - x_{j_0}| > \epsilon_a \lambda} \frac{\| h \|_{n\ell, N}}{(1 + |x|)^{\lambda N}} dx \quad \text{for any} \quad N \geq 4n + 1. \]  
(50)

Using the elementary inequality \( |x_{i_0} - x_{j_0}| \leq 2|x| \) and setting \( N = 4(n + 1) \), we continue the estimate (50) as follows
\[ |I_>| \leq C_0 \int_{|x_{i_0} - x_{j_0}| > \epsilon_a \lambda} \frac{8\| h \|_{n\ell, 4(n+1)}}{(2 + |x_{i_0} - x_{j_0}|)^3(1 + |x|)^{4n+1}} dx \leq C_2 \| h \|_{n\ell, 4(n+1)} \frac{1}{1 + \lambda^3}. \]

We see that the term \( I_> \) is negligible compared to \( I_< \) and
\[ |(U_{i_0j_0}, h_{\lambda a})| \leq C'_2 \| h \|_{n\ell, 4(n+1)} \frac{1}{1 + \lambda^2}. \]

Next we choose another pair of indices \((i_1, j_1)\) such that \( i_1 \leq k, j_1 > k \) and consider
\[ U_{i_1j_1} = \sum_{\{R : r_{i_1j_1} \neq 0, r_{i_0j_0} = 0\}} \frac{D_R}{R!} w_R. \]
Using the same line of reasoning, we conclude that \((U_{ijji}, h_{\lambda a})\) satisfy an analogous bound. After a finite number of steps we exhaust the sum on the right-hand side of (47) and arrive at (46), which completes the proof.

Thus the nonlocal field : \(\exp g\phi^2 : (x)\) completely fits into the framework proposed in Refs. [7] [8] [10]. It should be stressed that the theory defined by assumptions (a.1) - (a.7) becomes local in the limit \(\ell \to 0\). The injective limit \(\text{inj lim}_{\ell \to 0} A_\ell(\mathbb{R}^d)\) coincides with the space \(S^1(\mathbb{R}^d)\), and the notion of support can be correctly defined for the continuous linear functionals on this space. This can be done by adapting the definitions used in the Sato-Martineau theory of hyperfunctions to the case of functionals of polynomial growth at infinity. The point is that if \(K_1\) and \(K_2\) are compact subsets of \(\mathbb{R}^d\), then the spaces \(S^1(K_i) = \text{inj lim}_{\ell \to 0} A_\ell(K_i), i = 1, 2\), obey the following structural relation

\[
S^1(K_1 \cap K_2) = S^1(K_1) + S^1(K_2),
\]

(51)

which is equivalent to the relation

\[
S^1(K_1 \cap K_2)' = S^1(K_1)' \cap S^1(K_2)',
\]

(52)

for the dual spaces. As shown in Ref. [19], these relations can be extended to compact sets in the radial compactification \(\hat{\mathbb{R}}^d\) of \(\mathbb{R}^d\). Because of this, for every \(v \in S^1(\mathbb{R}^d)\)', there exists a unique minimal compact set \(K \subset \hat{\mathbb{R}}^d\) such that \(v \in S^1(K)\}'. If Wightman functions \(W_n\) satisfy conditions (a.7.1) and (a.7.2) for each \(\ell > 0\), then from Theorem 4 it follows that they are defined on the spaces \(S^1(\mathbb{R}^{4n})\). Moreover, then all axioms (a.1) - (a.6) are fulfilled with test functions of the class \(S^1\) because \(A_\infty(\mathbb{R}^d)\) is dense in \(S^1(\mathbb{R}^d)\). Theorem 4 also implies that \(W_n - \pi W_n\) admits a continuous extension to the space \(S^1(V_\pi)\) in this limit, i.e., the quasilocality condition (a.7.2) turns into local commutativity understood in the sense of hyperfunctions.

Remark 5. In the axiomatic scheme proposed in [11], the role of condition (a.7.1) is played by an assumption which is equivalent to saying that every functional \(W_n\) has a continuous extension to each of the spaces \(\mathcal{V}_\pi\), where the tensor product is endowed with the projective topology. This assumption allows an exponential growth of order 1 and of arbitrary type when more than one of the momentum-space variables \(q_j\) tend to infinity together. Because such is the case for any \(\ell\), it is unlikely that the scheme [11] becomes local in the limit \(\ell \to 0\). In place of condition (a.7.2), a weaker condition was used in [11], namely, that the functionals (28) extend continuously to the spaces denoted
there by \( T(W_k^\ell) \). In our notation, this means that there exist continuous extensions to \( A_\infty(\mathbb{R}^{4(k-1)}) \otimes A_\ell(V_{(k,k+1)}) \otimes A_\infty(\mathbb{R}^{4(n-k-1)}) \), but this condition does not turn into local commutativity as \( \ell \to 0 \). In particular, there is seemingly no analogue of Theorem 4.

VI. Conclusion

The analysis performed in this paper supports the view [10] that the presheaf of spaces \( A_\ell(O) \) associated with sets \( O \subset \mathbb{R}^d \) provides a suitable framework for constructing quantum field theory with a fundamental length. Our approach agrees with the idea that localizability at large scales should be understood as a property of the correlation functions with respect to the relative coordinates, because observable quantities are expressed in these variables. It also meets the requirement that the theory must become local as the fundamental length tends to zero. The space \( A_\infty(\mathbb{R}^d) \) corresponding to the ultra-hyperfunctions can be used to construct an invariant domain of fields in the Hilbert space of states, but it plays an auxiliary role. The situation is somewhat similar to that with the Wick exponential of the massless scalar field or of the ghost field in a two dimensional spacetime, because the operator realization of these models is possible only under further restrictions [20, 21] on test functions besides those required by the Wightman functions. The reason was there in the occurrence of infrared singularities and in the lack of positivity. In the case of nonlocal field \( \exp g \phi^2 \), the reason is more prosaic and has to do with the fact that \( A_\ell(\mathbb{R}^d) \) is not invariant under linear transformations of \( \mathbb{R}^d \). The space \( A_\infty(\mathbb{R}^d) \) is its maximal invariant subspace, but one may also use smaller invariant dense subspaces and \( S^0(\mathbb{R}^d) \) in particular.

If an investigation of physically relevant models related, e.g., to string theory will give a good reason, then conditions (a.7.1) and (a.7.2) can be weakened by assuming that the nonlocality parameter increases with \( n \). Condition (a.7.2) might simply be replaced with the requirement that, for any permutation \( \pi \), the difference \( \mathcal{W}_n - \pi \mathcal{W}_n \) has a continuous extension to the space \( A_{2\ell_n}(V_\pi) \), where \( \ell_n \) grows linearly with increasing \( n \). However, in our opinion, modifications in these conditions may not involve test functions of the class \( A_\infty \) even in respect to a part of variables because this leads to the lack of the local limit.

Any axiomatic scheme is interesting not by itself but by its physical consequences. It should be noted in this connection that the distinction between our formulation and that of [11] becomes quite essential in deriving the analyticity properties of scattering
amplitudes and in finding bounds on their high-energy behavior. Our approach admits an extension to the Lehmann-Symansik-Zimmermann formalism which play an important role in the scattering theory. In particular, let $R(x; x_1, \ldots, x_n)$ be the retarded product which is formally defined in local QFT by

$$
R(x; x_1, \ldots, x_n) = (-i)^n \sum_\pi \theta(x^0 - x^0_\pi) \theta(x^0_\pi - x^0_\pi^2) \cdots \theta(x^0_\pi^{n-1} - x^0_\pi^n) \left[ \varphi(x), \varphi(x_\pi), \ldots, \varphi(x_\pi^n) \right].
$$

Then a natural generalization of its support properties to the case of a nonlocal field is the requirement of the existence of a continuous extension to the space $A_{\ell_n}(\mathcal{V}^+)$, where

$$
\mathcal{V}^+ = \{ (x; x_1, \ldots, x_n) : (x - x_j) \in \overline{\mathcal{V}^+}, j = 1, \ldots, n \}.
$$

We conclude by noting that the condition a.7.2 is evidently stronger than the condition that the matrix elements of the commutator $[\varphi(x_1), \varphi(x_2)]$ have continuous extensions to the space $S^0(\mathcal{V}_{(1,2)})$. The latter is referred to as asymptotic commutativity [3] and even this weaker condition ensures the existence of the $CPT$-symmetry (with respect to the product of charge conjugation $C$, space inversion $P$, and time inversion $T$) and the normal spin-statistics connection for nonlocal fields, as has been shown in [22] with the use of the notion of analytic wave front set of a distribution. Detailed proofs of these theorems for the general case of a finite family of fields $\{ \phi_i \}, \ i = 1, \ldots, I$, transforming according to irreducible representations of the proper Lorentz group $L^\uparrow_+$ or its covering group $SL(2, \mathbb{C})$ are given in [23].

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**Appendix. A density theorem**

As well known, the Fourier transformation maps isomorphically the space $D(\mathbb{R}^d)$ of compactly supported smooth functions on $\mathbb{R}^d$ onto the space $S^0(\mathbb{R}^d)$ consisting of entire functions on $\mathbb{C}^d$ with the property that

$$
|z^\kappa g(z)| \leq C_\kappa e^{a |\text{Im} z|}, \quad \kappa \in \mathbb{Z}^d_+.
$$
where $C_\kappa$ and $a$ are constants depending on $g$. The notation $S^0(\mathbb{R}^d)$ indicates that this space is smallest among the Gelfand-Shilov spaces $S^\alpha(\mathbb{R}^d)$, $\alpha \geq 0$.

**Proposition.** The space $S^0(\mathbb{R}^d)$ is dense in $A_\ell(\mathbb{R}^d)$ for any $\ell \in (0, +\infty]$.

**Proof.** Since $S^0(\mathbb{R}^d)$ is an algebra under multiplication, it contains a function $g$ such that $g(x) \geq 0$ for all $x \in \mathbb{R}^d$ and \( \int g(x) \, dx = 1 \).

Let \( g_\nu(x) = \nu^d g(\nu x) \), $\nu = 1, 2, \ldots$ and let $f \in A_\ell(\mathbb{R}^d)$, \( f_\nu(z) = \int g_\nu(z - x) f(x) \, dx \).

The functions $g_\nu$ and $f$ are analytic in the domain $|\text{Im } z| < \ell$ and decrease rapidly as $\text{Re } z \to \infty$. Therefore for any $z_0$ whose imaginary part $y_0$ satisfies $|y_0| < \ell$, we have

\[
f_\nu(z_0) = \int g_\nu(x_0 - x) f(x + iy_0) \, dx.
\]

It follows that $f_\nu(z_0) \to f(z_0)$ as $\nu \to \infty$. Setting $l < \ell$ and using the definition and the inequality $(1 + |x_0|) \leq (1 + |x_0 - x|)(1 + |x|)$, we obtain

\[
\|f_\nu\|_{l,N} \leq \|f\|_{l,N} \int (1 + |x_0 - x|)^N g_\nu(x_0 - x) \, dx \leq C_N \|f\|_{l,N},
\]

where $C_N = \int (1 + |x|)^N g(x) \, dx$. Because every bounded set in a Montel space is relatively compact, we conclude that $f_\nu \to f$ in the topology of $A_\ell(\mathbb{R}^d)$.

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