ON BOUNDS OF MATRIX EIGENVALUES

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Abstract. In this paper, we give estimates for both upper and lower bounds of eigenvalues of a simple matrix. The estimates are sharper than the known results.

1. Introduction

As is well known, the eigenvalues of a matrix play an important role in solving linear systems [1, 3, 5], especially in the perturbation problems [2, 6]. The purpose of this note is to give a specific estimate of the eigenvalues.

Let $A = (a_{ij})$ be an $n \times n$ complex matrix with conjugate transpose $A^*$, $\overline{A}$ denote the conjugate, and $\text{tr}A$ represent the trace of matrix $A$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $A$, then

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq \|A\|^2 = \sum_{i,j=1}^{n} |a_{ij}|^2 = \text{tr}(AA^*),$$

where $\|A\|$ denotes the Frobenius norm of $A$. Let

$$\Re_A = \frac{A + A^*}{2},$$
$$\Im_A = \frac{A - A^*}{2i},$$

we call $\Re_A$ the Hermitian real part and $\Im_A$ the Hermitian imaginary part of $A$. Let

$$q_A = \|A\|^2 - \frac{\text{tr}(A)^2}{n},$$
$$\Delta_A = \frac{\|AA^* - A^*A\|^2}{2}.$$  

2. Main theorem

Theorem 2.1. Suppose $\lambda$ is an eigenvalue of an $n \times n$ complex matrix $A$ with geometric multiplicity $t$, then

$$\left| \lambda - \frac{\text{tr}(A)}{n} \right| \leq \sqrt{\frac{n-t}{(2n-t)t}} \sqrt{\frac{n-t}{n}q_A} + \sqrt{\frac{q_A^2}{n} - \frac{(2n-t)t}{n^2} \Delta_A}. \quad (2.1)$$

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Theorem 2.2. Suppose $\lambda_{R_A}, \lambda_{I_A}$ are the eigenvalues of an $n \times n$ complex matrices $R_A$ and $I_A$ with geometric multiplicity $t$, respectively, then

\[
\left| \lambda_{R_A} - \frac{\text{tr}(R_A)}{n} \right| \leq \sqrt{\frac{n-t}{nt}} q_{R_A}, \tag{2.2}
\]

\[
\left| \lambda_{I_A} - \frac{\text{tr}(I_A)}{n} \right| \leq \sqrt{\frac{n-t}{nt}} q_{I_A}. \tag{2.3}
\]

3. Proof of Theorem

Before giving the proof of Theorems 2.1 and 2.2 we present some lemmas.

Lemma 3.1 (see [4]). Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of an $n \times n$ complex matrix $A$, then

\[
\sum_{j=1}^{n} |\lambda_j|^2 \leq \|A\|^4 - \Delta_A.
\]

Lemma 3.2. Let $A$ be an $n \times n$ complex matrix, rank($A$) represent the rank of $A$. Then

\[
|\text{tr}(A)|^2 \leq \text{rank}(A) \sqrt{\|A\|^4 - \Delta_A}.
\]

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $A$, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Suppose that the number of nonzero eigenvalues is $k$. Without loss of generality, we can denote the nonzero eigenvalues of $A$ by $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then it is easily seen that

\[
k \leq \text{rank}(A).
\]

Now suppose that $R = \Lambda + M$ is a Schur triangular form of $A$, i.e., $A = U^*RU$, $U$ is unitary orthogonal, $\Lambda$ is diagonal and $M$ is upper triangular. From Lemma 3.1, we have

\[
\sum_{j=1}^{k} |\lambda_j|^2 \leq \sqrt{\|A\|^4 - \Delta_A}.
\]

Then

\[
|\text{tr}(A)|^2 = \left| \sum_{j=1}^{k} \lambda_j \right|^2 \leq k \sum_{j=1}^{k} |\lambda_j|^2 \leq \text{rank}(A) \sum_{j=1}^{k} |\lambda_j|^2 \leq \text{rank}(A) \sqrt{\|A\|^4 - \Delta_A}.
\]

This shows the validity of conclusion. \qed

Next, we provide the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $M = \lambda I - A$, where $I$ is the $n \times n$ identity matrix, $\lambda$ is the $t$ multiple eigenvalue of $A$. Then we have

\[
\text{rank}(M) = \text{rank}(\lambda I - A) \leq n - t,
\]

and the following equality

\[
\Delta_M = \frac{\|(\lambda I - A)(\bar{\lambda} I - A^*) - (\bar{\lambda} I - A^*)(\lambda I - A)\|}{2} = \Delta_A.
\]
From Lemma 3.2, we have
\[ |\text{tr}(M)|^2 \leq \text{rank}(M) \sqrt{\|M\|^4 - \Delta_M} \]
\[ \leq (n-t) \sqrt{\|M\|^4 - \Delta_M} \leq (n-t) \sqrt{\|\lambda I - A\|^4 - \Delta_A}. \] (3.1)

In addition, by simple manipulations, we obtain
\[ |\text{tr}(\lambda I - A)|^2 = \text{tr}(\lambda I - A)^2 \]
\[ = n^2 |\lambda|^2 - n \lambda \text{tr}(A^*) - n \lambda \text{tr}(A) + |\text{tr}(A)|^2 = n \sigma + |\text{tr}(A)|^2, \] (3.2)

where \( \sigma = n|\lambda|^2 - \lambda \text{tr}(A^*) - \lambda \text{tr}(A). \) Moreover,
\[ \|\lambda I - A\|^4 = (\text{tr}((\lambda I - A)(\lambda I - A)^*))^2 \]
\[ = \left( n|\lambda|^2 - \lambda \text{tr}(A^*) - \lambda \text{tr}(A) + \|A\|^2 \right)^2 = (\sigma + \|A\|^2)^2. \] (3.3)

Eliminating \( \sigma \) from the formulae (3.2) and (3.3), we get
\[ \|\lambda I - A\|^4 = \left( \frac{|\text{tr}(\lambda I - A)|^2 - |\text{tr}(A)|^2}{n} + \|A\|^2 \right)^2. \] (3.4)

Let \( s = \left| \lambda - \frac{\text{tr}(A)}{n} \right|^2, q_A = \|A\|^2 - \frac{|\text{tr}(A)|^2}{n}. \) Then
\[ |\text{tr}(\lambda I - A)|^2 = n^2 s, \] (3.5)

and
\[ \|\lambda I - A\|^4 = (ns + q_A)^2. \] (3.6)

By substituting the equalities (3.5) and (3.6) into (3.1), it follows that
\[ n^2 s \leq (n-t) \sqrt{(ns + q_A)^2 - \Delta_A}. \]

Consequently, by straightforward computations, we have
\[ s = \left| \lambda - \frac{\text{tr}(A)}{n} \right|^2 \leq \frac{n-t}{2(n-t)t} \left( \frac{n-t}{n} q_A + \sqrt{q_A^2 - \frac{(2n-t)t}{n^2} \Delta_A} \right). \]

The result follows immediately. \( \square \)

**Proof of Theorem 2.2.** Notice that
\[ \sqrt{\frac{n-t}{2(n-t)t}} \left( \frac{n-t}{n} q_A + \sqrt{q_A^2 - \frac{(2n-t)t}{n^2} \Delta_A} \right) \leq \sqrt{n-t} q_A. \]

Furthermore, the above equality holds if and only if \( \Delta_A = 0. \) In other words, \( A \) is normal, i.e., \( AA^* = A^* A. \) By Theorem 2.1 and taking into account that \( \Re A \) and \( \Im A \) are both normal matrices, we get the validity of Theorem 2.2. \( \square \)

**Remark.** In terms of estimates on bounds of the largest modulus eigenvalue \( |\lambda|_{\text{max}} \) of matrix \( A, \) the following inequality was given in 7-8,
\[ \frac{|\text{tr}(A)|}{n} \leq |\lambda|_{\text{max}} \leq \frac{|\text{tr}(A)|}{n} + \sqrt{\frac{n-1}{n} q_A}. \] (3.7)

We note that the estimates (2.1), (2.2), (2.3) are sharper than (3.7) in some extent. That is to say, the results presented in this paper improve the ones given in 7-8 partially, and can be taken as supplements to the conclusions known in 5-7-8, especially for the upper bound estimation of eigenvalues of a matrix.
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