Fidelity susceptibility for Lifshitz geometries via Lifshitz Holography

Davood Momeni
Department of Physics, College of Science, Sultan Qaboos University,
P.O. Box 36, P.C. 123, Al-Khowd, Muscat, Sultanate of Oman

Mir Faizal
Irving K. Barber School of Arts and Sciences,
University of British Columbia - Okanagan, 3333 University Way,
Kelowna, British Columbia V1V 1V7, Canada
Department of Physics and Astronomy, University of Lethbridge,
Lethbridge, Alberta, T1K 3M4, Canada

Aizhan Myrzakul, Ratbay Myrzakulov
Eurasian International Center for Theoretical Physics
and Department of General Theoretical Physics,
Eurasian National University, Astana 010008, Kazakhstan

Abstract

In order to analyze the fidelity susceptibility of non-relativistic field
theories, which are important in condensed matter systems, we gener-
alize the proposal to obtain the fidelity susceptibility holographically
to Lifshitz geometries. It will be argued that this proposal can be
used to study the fidelity susceptibility for various condensed matter
systems. To demonstrated this, we will explicitly use this proposal
to analyze the fidelity susceptibility for a non-relativistic many-body
system, and argue that the fidelity susceptibility of this theory can be
holographically obtained from a bulk Lifshitz geometry. In fact, using
a Einstein-Dilaton-Maxwell-AdS-Lifshitz theory, we explicitly demon-
strated that the fidelity susceptibility obtained from this bulk geometry
is equal to the fidelity susceptibility of a bosonic many-body system.

1 Introduction

It is known that the entropy of a black hole scales with its area. As black
holes are maximum entropy objects, this implies that the maximum entropy
of that certain region of space scales with the area of its boundary. This
observation has led to the development of the holographic principle, which
equates the number of degrees of freedom in a region of space to the number of degrees of freedom on the boundary surrounding that region of space [1, 2]. The AdS/CFT correspondence is a realization of the holographic principle as it is a duality between the string theory/supergravity in AdS spacetime and the field theory on its boundary [3]. As AdS/CFT correspondence is a duality between two very different theories, it seems from the AdS/CFT correspondence and the holographic principle that laws of physics are fundamentally just information theoretical processes. In fact, various studies done in different fields of science seem to indicate that the laws of physics are informational theoretical processes [4, 5]. So, the AdS/CFT can be used to obtain information theoretical information relating to a conformal field theory from the bulk geometry. The entanglement entropy of a field theory is a most important informational theoretical quantity relating to a conformal field theory. It has been demonstrated that the entanglement entropy of a conformal field theory can be holographically obtained from the bulk AdS spacetime, as it is dual a minimal surface in asymptotically AdS spacetime [6, 7].

It is also important to know how much information is retained in a system, and holographic entanglement entropy can be used to quantify this as it measures the loss of information in a system. However, it is also important to know the difficulty to obtain this information, and this can be quantified using complexity. As laws of physics can be understood in terms of information theoretical processes [4, 5], and complexity is an important informational theoretical quantity, complexity is expected to be an important physical quantity used in the laws of physics. In fact, complexity has been used to understand the behavior of condensed matter systems [8, 9] and molecular physics [10], quantum computing [11]. In fact, it has been argued that the information might not be ideally lost in a black hole, but it would be effectively lost, as it would not be possible obtain this information from a black hole due to its chaotic nature [12].

The complexity of a conformal field theory can also be obtained holographically, as the holographic complexity is dual to a volume in AdS spacetime [13, 14, 15, 16, 17, 18]. It has been demonstrated that the holographic complexity of a field theory can be related to the fidelity susceptibility of the boundary field theory. So, the fidelity susceptibility of a field theory can be holographically calculated using a maximal volume $V(\gamma_{\text{max}})$ in the AdS which ends on the time slice at the AdS boundary [19, 20].

As fidelity susceptibility is important to understand the behavior of condensed matter systems [21, 22, 23, 24, 25], it is important to generalize this proposal to non-relativistic field theories describing condensed matter systems. It may be noted that such non-relativistic condensed matter systems can be holographically analyzed using Lifshitz holography [26, 27, 28, 29, 30]. In the Lifshitz holography, the Lifshitz deformed of AdS can be related to the Lifshitz field theories, in which spacetime have different scaling behavior.
As Schrödinger invariant quantum system, which describe condensed matter system, the space and time scale differently, we will use Lifshitz holography to analyze such a system. It may be noted that a relation between the Lifshitz holography and Schrödinger invariant quantum system has been studied for certain system \cite{32,33}. As we will be studying a static case, we will be able to find a holographic relation between a Lifshitz system and a Schrödinger invariant quantum system.

2 Lifshitz Fidelity Susceptibility

In this section, motivating by \cite{31}, in this work we will now generalize the relativistic proposal to obtain the fidelity susceptibility from holographic complexity \cite{19,20} to Lifshitz geometries. So, if we consider one parameter a Lifshitz field theory, with states denoted by $|\Psi(\lambda)>$, then the inner product of two such states separated by an infinitesimally perturbation $\delta\lambda$, can be expressed as

$$<\Psi(\lambda)|\Psi(\lambda + \delta\lambda)> = 1 - \Xi_F(\delta\lambda)^2. \quad (1)$$

Now using the argument used in \cite{19,20}, it can be argued that $\Xi_F$ can be written as

$$\Xi_F = \int dm <O(x), O(x')>, \quad (2)$$

where $dm$ is a suitable integral measure for this system, and $O(x), O(x')$ are suitable local operator in the Lifshitz theory. Now it can again be argued that this quantity will be described holographically. However, as these operators are defined in a Lifshitz theory, and the holographic dual of such a field theory is described by Lifshitz gravity \cite{31}, we can use the argument of \cite{19,20}, to argue that $\Xi_F$ can also be obtained holographically from a Lifshitz-AdS geometry. In fact, as this quantity has to reduce to holographic complexity for $z = 1$ for usual theory, we can also calculate this quantity in Lifshitz geometries using the volume of maximum surfaces.

This can be done by defining $V(\gamma_{max})$ as a maximal volume in the Lifshitz deformation of the AdS spacetime, which ends on the time slice at the Lifshitz-AdS boundary. We can use this maximal volume in the Lifshitz geometry to define the holographic complexity in such a geometry as

$$F = \frac{V(\gamma_{max})}{8\pi RG}. \quad (3)$$

It may be noted that for $z = 1$ this maximal volume reduces to the usual maximal volume in AdS spacetime, and so this holographic complexity reduces to the usual holographic complexity for $z = 1$ \cite{19,20}. Now the Lifshitz holography reduces to the usual holography for $z = 1$, and it is known that there are divergences associated with such volumes for $z = 1$ \cite{34}. So, we
need to regularize this volume, before we can define the fidelity susceptibility for Lifshitz geometries. This will be done by subtracting the background Lifshitz-AdS geometry $V(\gamma_{\text{max}})_{\text{LAdS}}$ from the deformed Lifshitz-AdS geometry $V(\gamma_{\text{max}})_{\text{DLAdS}}$. So, we can define a regularized Lifshitz maximal volume

$$V(\gamma_{\text{max}}) = V(\gamma_{\text{max}})_{\text{DLAdS}} - V(\gamma_{\text{max}})_{\text{LAdS}}$$  \hspace{1cm} (4)$$

Now using this regularized maximal volume in the Lifshitz geometry, we can define the regularized holographic complexity of a non-relativistic boundary theory as

$$\Xi_F = F_{\text{DLAdS}} - F_{\text{LAdS}} = \frac{V(\gamma_{\text{max}})}{8\pi RG},$$  \hspace{1cm} (5)$$

where $R$ is the radius of the curvature of this Lifshitz-AdS geometry. This regularized holographic complexity is equal to fidelity susceptibility of the boundary field theory, and so it fidelity susceptibility of the boundary field theory can be holographically calculated from holographic complexity. It may be noted that for $z = 1$ this expression reduces to the usual expression for the regularized fidelity susceptibility \[35, 36\].

3 Boundary Bosonic System

Now we will use this proposal for holographically analyze a simple system of system of $N$ bosons, without any self-interaction. These bosons will be placed in a background uniform magnetic field $\vec{H}$ in the $z$ direction, with $\vec{H} = H \hat{e}_z$. So, even though the bosons do not interact with a dynamical field, they do interact with a background field. It is possible to holographically analyze such systems in a background field \[37, 38, 39, 40\]. We will analyze this proposal for such a simple system, to demonstrate how such a holographic correspondence can work, and so this proposal can be used for holographically analyzing more complicated systems. Now before we analyze the bulk Lifshitz geometry dual to such a system, we will calculate the fidelity susceptibility of this bosonic theory. The Hamiltonian for each of these bosonic particles is $H_i = (-i\vec{\nabla}_i - q\vec{A})^2/2m$, where $\vec{A} = \vec{\nabla} \times \vec{H}$, so the time-dependent Schrödinger equation for these bosonic particles can be written as

$$\sum \frac{(-i\vec{\nabla}_i - q\vec{A})^2}{2m}\Psi_{\text{tot}}(\vec{x}_1, \vec{x}_2, ..., \vec{x}_N; t) = i\frac{\partial}{\partial t}\Psi_{\text{tot}}(\vec{x}_1, \vec{x}_2, ..., \vec{x}_N; t).$$  \hspace{1cm} (6)$$

Now we can write the total wave function as

$$\Psi_{\text{tot}}(\vec{x}_1, \vec{x}_2, ..., \vec{x}_N; t) = \Pi_{i=1}^N \psi_i(\vec{x}_i; t).$$  \hspace{1cm} (7)$$
We need to find only ground state wave function $\Psi_{0}(\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_N; t)$. Choosing the gauge, $\vec{A} = (0,Hx,0)$, we can write $\vec{A} = \nabla \times \vec{H}$. Using this gauge for $\vec{A}$, Schrödinger equation can be expressed as

$$-\nabla_{i}^{2} \psi_{i}(\vec{x}_{i}; t) + 2iqHx \frac{\partial \psi_{i}(\vec{x}_{i}; t)}{\partial y} + q^{2}H^{2}x^{2} \psi_{i}(\vec{x}_{i}; t) = 2mE_{i} \psi_{i}(\vec{x}_{i}; t) \tag{8}$$

We can express $\psi_{i}(\vec{x}_{i}; t)$ as

$$\psi_{i}(\vec{x}_{i}; t) = e^{(k_{i}x + \beta_{i}y - E_{i}t)} \phi_{i}(x_{i}). \tag{9}$$

Now we can write

$$-\frac{d^{2}\phi_{i}(x_{i})}{dx_{i}^{2}} + \left(q^{2}H^{2}x_{i}^{2} - 2q\beta_{i}Hx_{i}\right)\phi_{i}(x_{i}) = \left(2mE_{i} - k_{i}^{2} - \beta_{i}^{2}\right)\phi_{i}(x_{i}). \tag{10}$$

It may be noted that this looks just like the Schrödinger equation for a simple harmonic oscillator, such that the coordinates have been shifted as $x_{i} \rightarrow \xi + \frac{\beta_{i}}{qH}$. So, we can express the Schrödinger equation for this system as

$$-\frac{1}{2m} \frac{d^{2}\phi_{i}(\xi_{i})}{d\xi_{i}^{2}} + \frac{1}{2}m\omega_{i}^{2}\xi_{i}^{2}\phi_{i}(\xi_{i}) = \left(E_{i} - \frac{k_{i}^{2}}{2m}\right)\phi_{i}(\xi_{i}), \tag{11}$$

where $\omega_{i} = \frac{qH}{m}$ denotes frequency (energy) of the system. Exact solution for this equation can be expressed in terms of Hermite functions. Now using the ground state wave function for a bosonic particle, $\phi_{i,0}(\xi_{i}) = \sqrt{\frac{qH}{\pi}} e^{-\frac{qH}{2m}\xi_{i}^{2}}$, $E_{i,0} = \frac{qH + k_{i}^{2}}{2m}$, the ground state wave function for whole system can be written as

$$\Psi_{0}(\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_N; t) = \left(\frac{qH}{\pi}\right)^{N/4} \Pi_{i=1}^{N} \phi_{0}(\xi_{i}) \times e^{i\left(k_{i,z_{i}} + \beta_{i}y_{i} - \frac{qH+k_{i}^{2}}{2m}\right)t} - \frac{qH}{4}(x_{i} - \frac{\beta_{i}}{qH})^{2}. \tag{12}$$

The fidelity susceptibility can be obtained by varying $H \rightarrow H + \delta H$, and computing the following inner product,

$$F = \langle \Psi_{0}(H)|\Psi_{0}(H + \delta H) \rangle = 1 - (\delta H)^{2}\Xi_{F} + \ldots \tag{13}$$

So, we expand $F$ in series up to the second order $\Xi_{F}$,

$$\langle \Psi_{0}(H)|\Psi_{0}(H + \delta H) \rangle = \Pi_{i=1}^{N} \int d^{3}x_{i} \Psi_{0}^{*}(H + \delta H) \Psi_{0}(H) \tag{14}$$

$$= \left(\int d^{3}x \psi^{*}(\vec{x}; t, H + \delta H) \psi(\vec{x}; t, H)\right)^{N}.$$ 

Thus, we can express the fidelity susceptibility of this system as

$$\Xi_{F} = \frac{N}{8qH^{3}} (qH + 4\beta^{2}). \tag{15}$$

Expression given in (15) is the exact fidelity susceptibility for a system of $N$ charged bosonic particles in a uniform magnetic field.
4 Lifshitz Bulk Dual

This boundary theory studied in the previous section is defined by non-relativistic Hamiltonian. Now we will study a theory with Lifshitz symmetry in the bulk theory \[42, 43, 44, 45\]. Now we will suppose that the Einstein-Dilaton-Maxwell-AdS-Lifshitz bulk action \[46, 47, 48, 49\]. We would like to point out that the Schrödinger symmetry is different from Lifshitz symmetry. However, a relation between the Lifshitz holography and certain Schrödinger invariant quantum systems has been studied \[32, 33\]. Thus, it will be interesting to note that we will demonstrate that the fidelity susceptibility obtained from this Lifshitz theory will match the results calculated in the previous section for the theory with Schrödinger symmetry. This occurs as we study a static case, and so, it would be interesting to analyze if such a match occurs for other holographic calculations between such theories, for a static case. Thus, we propose the following action for the bulk theory \[48, 49\],

\[
S_{\text{Bulk}} = \int_{\mathcal{M}} d^4x \sqrt{-g} \left[ \frac{(R - 2\Lambda)}{2\kappa^2} - \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + V(\phi) - \xi e^{\lambda \phi} (F^{\mu\nu} F_{\mu\nu}) \right].
\]  

(16)

where the potential is \( V(\phi) = V_0 e^{\gamma \phi} \) with parameters \( V_0 \) and \( \gamma \). Here \( \phi \) is non-minimally coupled with electromagnetic potential, and the electromagnetic field strength coupled to scalar field as \( \xi e^{\lambda \phi} (F^{\mu\nu} F_{\mu\nu}) \), such that \( \xi, \lambda \) are suitable constants.

The metric for a static, spherically symmetric solution in this Einstein-Dilaton-Maxwell-AdS-Lifshitz can be written as \[48, 49\]

\[
ds^2 = -e^{2\alpha(r)} B(r) dt^2 + \frac{dr^2}{B(r)} + r^2 d\sigma^2_{2,k},
\]

(17)

where \( \alpha(r), B(r) \) are function of \( r \). Here \( d\sigma^2_{2,k} \) is the metric for a topological two-dimensional surface parametrized by \( k = 0, \pm 1 \). This two-dimensional manifold is a sphere \( S_2 \) for \( k = 1 \), a torus \( T_2 \) for \( k = 0 \), and a compact hyperbolic manifold \( Y_2 \) for \( k = -1 \). Now we can choose \( k = 0 \) and write the planar Euclidean coordinates as \( d\sigma^2_{2,k} = dx^i dx_i, \quad i = 1, 2, x^i = \{x, y\} \).

It may be noted that due to Lifshitz scaling, \( \alpha(r) \propto \log r^z/2 \), where \( z \) is the Lifshitz parameter. So, the general form of the metric with Lifshitz symmetry can be written as

\[
ds^2 = - \left( \frac{r}{r_0} \right)^z B(r) dt^2 + \frac{dr^2}{B(r)} + r^2 dx_i dx^i.
\]

(18)

Here the function \( B(r) \) can be written as \[48, 49\]

\[
B(r) = \frac{2}{(2 + z)} \left[ \frac{V_0}{2} \right] + \left( \frac{r_+}{r} \right)^{1+z/2} \left( \left( \frac{2(\Lambda + \bar{Q}^2 \xi)}{(6 + z)} \right) r^2_+ \right.
\]

\[
- \left. \frac{2}{(2 + z)} \left( \frac{V_0}{2} \right) - \left( \frac{2(\Lambda + \bar{Q}^2 \xi)}{(6 + z)} \right) r^2. \right)
\]

(19)
This bulk theory is dual to the bosonic system, we have analyzed in the previous section.

5 Holographic Complexity

As we have obtained the fidelity susceptibility of the bosonic system in the previous section, we will holographically analyze it in this section. So, we will use generalization the fidelity susceptibility \[19, 20\] to a Lifshitz geometry given by Eq. (5). The fidelity susceptibility in such geometries depends on \( V(\gamma_{\text{max}}) \), and we can obtain \( V(\gamma_{\text{max}}) \) using

\[
V(\gamma_{\text{max}}) = \int_{r_{+}}^{r_{\infty}} \frac{r^2 dr}{\sqrt{B(r)}}
\]

where \( r_{+} \) is horizon, and \( r_{\infty} \) is an IR cutoff. Now we can use the Poincare coordinate \( w = \frac{r}{r_{+}} \) to evaluate integral (20) as

\[
V(\gamma_{\text{max}}) = r_{+}^3 \int_{1}^{\epsilon} \frac{dw}{w^4 \sqrt{b(w)}}
\]

where \( \epsilon \to 0 \) is an UV cutoff. We also have

\[
b(w) = b_1 w^{1+z/2} - \frac{b_{-2}}{w^2}.
\]

It may be noted as \( z = -\frac{4\tilde{Q}^2 \xi}{\Lambda + \tilde{Q}^2 \xi} \geq 3, \Lambda = -\frac{3}{16} \), so \( z = 4 \) is an interesting solution. In this case, the coefficients \( b_n \) are given as following

\[
b_1 = \left( \frac{\Lambda + \tilde{Q}^2 \xi}{5} \right) r_{+}^2 - \frac{1}{3} \left[ \frac{V_0}{2} \right],
\]

\[
b_{-2} = \frac{r_{+}^2 (\Lambda + \tilde{Q}^2 \xi)}{5}.
\]

Now we obtain

\[
V(\gamma_{\text{max}}) = r_{+}^3 \left( \frac{A}{3840(-b_{-2})^{9/2}} \right) + \frac{r_{+}^3}{2\epsilon^2 \sqrt{-b_{-2}}}
\]

here \( A = 640b_{-2}(b_1 - 3b_{-2}) \). Here the bulk charge \( \tilde{Q} \) is dual to the magnetic charge (strength) \( H \) of the boundary theory, and \( H \) varying smoothly. So, we can the volume (25) for \( O(\frac{1}{T^3}) \), and obtain,

\[
F_{\text{DAdS}} = \frac{\sqrt{5}r_{+}^2 \sqrt{-\xi} \left( 2L^2 \xi \tilde{Q} + 3 \right)}{48\pi G L^3 \xi^2 \tilde{Q}^3} - \frac{\sqrt{5}r_{+}^2 \sqrt{-\xi} \left( 2L^2 \xi \tilde{Q}^2 + 3 \right)}{32\pi G L^3 \xi^2 \tilde{Q}^3 r^2 \epsilon^2}.
\]

Now this equation contains both the finite and divergent parts of the holographic fidelity susceptibility. However, we can regularize it by subtracting...
it from the background AdS geometry, and obtain the regularized finite fidelity susceptibility:

\[ \Xi_F = F_{DLADS} - F_{LAdS} = \frac{\sqrt{5r^2_+ \sqrt{-\xi}} \left(2L^2 \xi \tilde{Q} + 3\right)}{48\pi G L^3 \xi^2 Q^3}. \]  

(27)

It may be noted that the regularized fidelity susceptibility calculated holographically in (27) is same as the fidelity susceptibility of the boundary theory obtained in (15). So, we can write

\[ \frac{\sqrt{5r^2_+ \sqrt{-\xi}} \left(2L^2 \xi \tilde{Q} + 3\right)}{48\pi G L^3 \xi^2 Q^3} = \frac{N}{8qH^2} (qH + 4\beta^2). \]  

(28)

Now if we holographically identify \( \tilde{Q} = H \), we can also identify many parameters in the bulk to boundary theories. In fact, from this identification, we obtain

\[ \frac{2\sqrt{5L^2 \xi \tilde{Q}^2 \sqrt{-\xi}}}{48\pi G L^3 \xi^2} = \frac{N}{8}, \]  

(29)

\[ \frac{3\sqrt{5r^2_+ \sqrt{-\xi}}}{48\pi G L^3 \xi^2} = \frac{N\beta^2}{2q}. \]  

(30)

So, the number of boundary quantum systems \( N \) and \( \beta^2 q^{-1} \) can be expressed as

\[ N_{\text{boundary}} = \left( -\frac{16\sqrt{5L^2 \xi \tilde{Q}^2 \sqrt{-\xi}}}{48\pi G L^3 \sqrt{-\xi}} \right)_{\text{Bulk}}, \]  

(31)

\[ \left( \frac{\beta^2}{q} \right)_{\text{boundary}} = \left( \frac{3}{8L^2 \xi} \right)_{\text{Bulk}}. \]  

(32)

It may be noted that \( \xi < 0 \), for this system. So, we have analyzed a system of bosonic particles in a magnetic field, and we obtained the fidelity susceptibility for this system. As this system was a non-relativistic system, it was expected to be dual to an AdS-Lifshitz spacetime. We have demonstrated that this theory is dual to an Einstein-Dilaton-Maxwell-AdS-Lifshitz, and the fidelity susceptibility calculated from the bulk using this theory matches with the fidelity susceptibility of the boundary theory.

6 Conclusion

In this paper, we propose that the fidelity susceptibility of a non-relativistic system can be obtained holographically from the holographic complexity of a Lifshitz-AdS theory. We use this proposal to holographically analyze
the fidelity susceptibility of non-relativistic field theories, and demonstrated that fidelity susceptibility of the bulk theory is the same as the boundary theory. So, using a Einstein-Dilaton-Maxwell-AdS-Lifshitz theory, we explicitly demonstrated that the fidelity susceptibility obtained from this bulk geometry is equal to the fidelity susceptibility of a bosonic many-body system.

It may be noted that as the boundary system considered explicitly in this paper described a simple system, it was possible to calculate the fidelity susceptibility for this system, both in the boundary and in the bulk. However, it is not always possible to calculate the fidelity susceptibility for the boundary and the bulk system. It would be difficult to analyze the strongly coupled field theories, and perform such calculations in the field theory side of the duality. However, it is known that a strongly coupled limit of the field theory can be holographically analyzed using a weakly coupled limit on the gravity side of the duality. Thus, for such non-relativistic systems, where such calculations cannot be performed on the field theory side of the duality, this holographic calculations can be performed using the gravitational side of the duality. As fidelity susceptibility is an very important quantity in condensed matter systems, and many condensed matter systems can be modeled using conformal field theories, it would be possible to use the formalism developed in this paper for analyzing such condensed matter systems.

It is possible to describe condensed matter systems like Weyl semi-metal as strongly coupled systems [50], it would be possible and interesting to use the results of this paper for holographically analyzing such fidelity susceptibility of such system. As we have generalized the fidelity susceptibility to Lifshitz geometries, and Lifshitz geometries can have important condensed matter applications [26, 27, 28, 29, 30], the results of this paper can have interesting condensed matter applications. So, it would be interesting to analyze realistic condensed matter systems, and then use the Lifshitz holography to understand the behavior of fidelity susceptibility for such condensed matter systems. It is also possible to study various interesting time-dependent generalizations of this solution. It is expected that such a time-dependent system on the boundary will be dual to some time-dependent Lifshitz bulk solution. It may be noted that fidelity susceptibility for time-dependent geometries has been studied [51], and it is expected that this formalism can be generalized to bulk geometries with Lifshitz symmetry. This would be interesting as such time-dependent systems are important in condensed matter physics [52]. It would be interesting to analyze such geometries, and use them to understand the boundary fidelity susceptibility. Thus, the proposal developed in this paper can be used to analyze interesting condensed matter systems, and calculate the fidelity susceptibility for such systems. We would like to point out that it would be interesting to extend this work to other geometries [54, 55], and calculate the fidelity susceptibility for such
geometries.

References

[1] G. ’t Hooft, [arXiv:gr-qc/9310026]
[2] L. Susskind, J. Math. Phys. 36, 6377 (1995)
[3] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998)
[4] K. H. Knuth, AIP Conf. Proc. 1305, 3 (2011)
[5] P. Goyal, Information 3, 567 (2012)
[6] S. Ryu and T. Takayanagi, Phys. Rev. Lett. 96, 181602 (2006)
[7] V. E. Hubeny, M. Rangamani and T. Takayanagi, JHEP 0707, 062 (2007)
[8] F. Barahona, J. Phys. A 15, 3241 (1982)
[9] M. Troyer and U. J. Wiese. Phys. Rev. Lett 94, 170201 (2005)
[10] J. Grunenberg, Phys. Chem. Chem. Phys. 13, 10136 (2011)
[11] M. Stanowski, Complicity 2, 78 (2011)
[12] S. W. Hawking, M. J. Perry and A. Strominger, Phys. Rev. Lett. 116, 231301 (2016)
[13] L. Susskind, Fortsch. Phys. 64, 24 (2016)
[14] L. Susskind, Fortsch. Phys. 64, 24 (2016)
[15] D. Stanford and L. Susskind, Phys. Rev. D 90, 12, 126007 (2014)
[16] D. Momeni, S. A. H. Mansoori and R. Myrzakulov, Phys. Lett. B 756, 354 (2016)
[17] M. Alishahiha, Phys. Rev. D 92, 126009 (2015)
[18] M. Alishahiha and A. Faraji Astaneh, Phys. Rev. D 96, no. 8, 086004 (2017)
[19] M. Miyaji, T. Numasawa, N. Shiba, T. Takayanagi, and K. Watanabe, Phy. Rev. Lett 115, 261602 (2015)
[20] N. S. Mazhari, D. Momeni, S. Bahamonde, M. Faizal and R. Myrzakulov, Phys. Lett. B 766, 94 (2017)
[21] M. Weber, F. F. Assaad and M. Hohenadler, Phys. Rev. B 94, 245138 (2016)

[22] V. Mukherjee, S. Montangero and R. Fazio, Phys. Rev. A 93, 062108 (2016)

[23] E. J. Konig, A. Levchenko and N. Sedlmayr, Phys. Rev. B 93, 235160 (2016)

[24] B. Damski, Sci. Rep. 5, 15779 (2015)

[25] R. Jafari, J. Phys. A: Math. Theor. 49, 185004 (2016)

[26] S. A. Hartnoll, Class. Quant. Grav. 26, 224002 (2009)

[27] C. J. McGreevy, Adv. High Energy Phys. 2010, 723105 (2010)

[28] P. Herzog, J. Phys. A 42, 343001 (2009)

[29] J. Gath, J. Hartong, R. Monteiro and N. A. Obers, JHEP 1304, 159 (2013)

[30] S. Sachdev, Ann. Rev. Cond. Matt. Phys. 3, 9 (2012)

[31] T. Griffin, P. Horava and C. M. Melby-Thompson, Phys. Rev. Lett. 110, 081602 (2013)

[32] J. Hartong, E. Kiritsis and N. A. Obers, Phys. Lett. B 746, 318 (2015)

[33] J. Hartong, E. Kiritsis and N. A. Obers, Phys. Rev. D 92, 066003 (2015)

[34] D. Carmi, R. C. Myers and P. Rath, JHEP 1703, 118 (2017)

[35] D. Momeni, M. Faizal, K. Myrzakulov and R. Myrzakulov, Phys. Lett. B 765, 154 (2017)

[36] N. S. Mazhari, D. Momeni, S. Bahamonde, M. Faizal and R. Myrzakulov, Phys. Lett. B 766, 94 (2017)

[37] D. T. Son, Phys. Rev. D 78, 046003 (2008)

[38] K. Balasubramanian and J. McGreevy, Phys. Rev. Lett. 101, 061601 (2008)

[39] Y. Nishida and D. T. Son, Phys. Rev. D 76, 086004 (2007)

[40] D. Martelli and Y. Tachikawa, JHEP 1005, 091 (2010)

[41] S. Sachdev, Quantum Phase Transitions, Cambridge University Press, Cambridge, UK (2000)
[42] X. Wang, J. Yang, M. Tian, A. Wang, Y. Deng and G. Cleaver, Phys. Rev. D91, 064018 (2015)

[43] G. Tallarita, Phys. Rev. D 89, 106005 (2014)

[44] T. Andrade, C. Keeler, A. Peach and S. F. Ross, Class. Quant. Grav. 32, 085006 (2015)

[45] P. Burda, R. Gregory and S. Ross, JHEP 1411, 073 (2014)

[46] M. K. Zangeneh, A. Dehyadegari, M. R. Mehdizadeh, B. Wang and A. Sheykhi, arXiv:1610.06352 [hep-th]

[47] J. Tarrio and S. Vandoren, JHEP 1109, 017 (2011)

[48] D. Momeni, R. Myrzakulov, L. Sebastiani and M. R. Setare, Int. J. Geom. Meth. Mod. Phys. 12, 1550015 (2015)

[49] M. H. Dehghani, R. Pourhasan and R. B. Mann, Phys. Rev. D 84, 046002 (2011)

[50] T. Bzdusek, A. Ruegg and M. Sigrist, Phys. Rev. B 91, 165105 (2015)

[51] D. Momeni, M. Faizal, S. Bahamonde and R. Myrzakulov, Phys. Lett. B 762, 276 (2016)

[52] F. Mahfouzi, B. K. Nikolic and N. Kioussis, Phys. Rev. B 93, 115419 (2016)

[53] D. T. Son, Phys. Rev. D 78, 046003 (2008)

[54] Y. F. Cai, S. Capozziello, M. De Laurentis and E. N. Saridakis, Rept. Prog. Phys. 79, 106901 (2016)

[55] S. Capozziello and M. De Laurentis, Phys. Rept. 509, 167 (2011)