BIHOLOMORPHIC MAPPING ON THE BOUNDARY I

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Abstract. We present a new proof of Chern-Ji’s mapping theorem on a strongly pseudoconvex domain with differentiable spherical boundary. We show that a proper holomorphic self mapping of a strongly pseudoconvex domain with the real analytic boundary is biholomorphic.

0. Introduction and Preliminaries

We shall show that a bounded domain $D$ is biholomorphic to an open ball $B^{n+1}$ whenever the boundary $bD$ is locally biholomorphic to the boundary of an open ball $B^{n+1}$.

Theorem 1. Let $D$ be a simply connected bounded domain in $\mathbb{C}^{n+1}$ with differentiable spherical boundary $bD$. Suppose that there is a biholomorphic mapping $\phi \in H(U \cap D) \cap C^1(U \cap \overline{D})$ for a connected open neighborhood $U$ of a point $p \in bD$ satisfying $\phi(U \cap bD) \subset bB^{n+1}$.

Then the mapping $\phi$ is analytically continued to a biholomorphic mapping from $D$ onto $B^{n+1}$.

Our result is a new proof of a weaker version of Chern-Ji’s mapping theorem [CJ]. The main steps of our proof come as follows: We show that the inverse mapping $\phi^{-1}$ is analytically continued on the unit ball $B^{n+1}$ to be a locally biholomorphic mapping

$$\varphi : B^{n+1} \to D.$$  

We show that the mapping $\varphi$ is a proper holomorphic mapping onto a universal covering Riemann domain over $D$. Thus the mapping $\varphi$ is a biholomorphic mapping whenever $D$ is simply connected.

We shall study on a proper holomorphic mapping $\phi$ between strongly pseudoconvex bounded domains $D, D'$ with real analytic boundaries $bD, bD'$.

Theorem 2. Let $D, D'$ be strongly pseudoconvex bounded domains in $\mathbb{C}^{n+1}$ with real analytic boundaries $bD, bD'$ and $\phi : D \to D'$ be a proper holomorphic mapping. Then the mapping $\phi$ is locally biholomorphic. If $D = D'$, then the mapping $\phi$ is a biholomorphic self mapping.

Our result is a new proof of a weaker version of Pinchuk’s mapping theorem [P]. The main steps of our proof come as follows: We show that the mapping $\phi$ is analytically continued along any path on $bD$ as a locally biholomorphic mapping when $bD$ is nonspherical so that the mapping $\phi : D \to D'$ are locally biholomorphic. From the study of Theorem 1 we show that the same is true when $bD$ is spherical so that the mapping $\phi : D \to D'$ are locally biholomorphic. For the case of $D = D'$, we show that the boundary $bD$ is necessarily spherical whenever the claim is not
true. Then we show that there is a sequence of automorphisms \( \phi_j \in Aut(D) \) and a sequence of points \( p_j \) on a compact subset \( K \subset D \) such that

\[
\phi_j (p_j) \to bD
\]

whenever the boundary \( bD \) is spherical and the claim is not true. We apply Wong-Rosay Theorem so that the domain \( D \) is biholomorphic to an open ball \( B^{n+1} \). Then we obtain a contradiction that the mapping \( \phi \) induces a nonautomorphic proper self mapping of an open ball \( B^{n+1} \) whenever the boundary \( bD \) is spherical and the claim is not true.

We remark that Theorem 1 is a weaker version of the following theorem:

**Theorem 3** (cf. Chern-Ji [CJ].) Let \( D \) be a simply connected bounded domain in \( \mathbb{C}^{n+1} \) with continuous spherical boundary \( bD \). Suppose that there is a biholomorphic mapping

\[
\phi \in H(U \cap D) \cap C(U \cap \overline{D})
\]

for a connected open neighborhood \( U \) of a point \( p \in bD \) satisfying

\[
\phi(U \cap bD) \subset bB^{n+1}.
\]

Then the mapping \( \phi \) is analytically continued to a biholomorphic mapping from \( D \) onto \( B^{n+1} \).

We remark that Theorem 2 is a weaker version of the following theorem:

**Theorem 4** (cf. Pinchuk [Pi].) Let \( D, D' \) be strongly pseudoconvex bounded domains in \( \mathbb{C}^{n+1} \) with the boundaries \( bD, bD' \) of class \( C^2 \) and \( \phi: D \to D' \) be a proper holomorphic mapping. Then the mapping \( \phi \) is locally biholomorphic. If \( D = D' \), then the mapping \( \phi \) is a biholomorphic self mapping.

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### 0.1. Canonical normalizing mapping

Let \( M \) be a nondegenerate analytic real hypersurface in \( \mathbb{C}^{n+1} \). For each point \( p \in M \), there is a complex tangent hyperplane \( H_p \subset T_pM \) so that there is a unit tangent vector \( v_p \in T_pM \) perpendicular to the complex tangent hyperplane \( H_p \) with respect to the usual riemannian metric in \( \mathbb{C}^{n+1} = \mathbb{R}^{2n+2} \). Then we can take a unique distinguished chain \( \gamma_p \) tangential to the direction \( v_p \) and passing through the complex tangent hyperplane \( H_p \) at the point \( p \) on \( M \)(cf. [Pa3]). Further, there is a distinguished normal parametrization on the chain \( \gamma_p \) having the same values up to order 2 of the straight real line to the direction \( v_p \) with the usual euclidean parametrization. Therefore, we can take a distinguished normalizing mapping \( \mu_p \) to Moser normal form

\[
\mu_p: M \to \mu_p(M)
\]

sending the germ \( M \) at the point \( p \) to a normal form such that \( \mu_p(p) \) is the origin and \( \mu_p(\gamma_p) \) is on the straightened chain of the normal form.

We take a local orientation near the point \( p \in M \) so that the tangent vector \( v_p \) extends to a smooth unit vector field \( v \) on \( M \cap U \) for an open neighborhood \( U \) of the point \( p \) such that \( v_q \in T_qM \) is a unit tangent vector perpendicular to the
complex tangent hyperplane $H_q$ for each $q \in M \cap U$. Then we obtain a family of the distinguished normalizing mapping $\mu_q$ for $q \in M \cap U$:

$$\mu_q : M \to \mu_q(M)$$

associated the unit tangent vector $v_q \in T_qM \setminus H_q$. The distinguished normalizing mapping $\mu_q$ shall be called the canonical normalizing mapping associated Moser normal form.

**Lemma 5.** Let $p_j \in M$ be a sequence of points converging to a point $p \in M$. Then there is a positive real number $\delta > 0$ such that

1. the mapping $\mu_{p_j}$ and its inverse $\mu_{p_j}^{-1}$ are analytically continued respectively on $B(p_j; \delta)$ and $B(0; \delta)$ as a biholomorphic mapping,
2. the real hypersurface $\mu_{p_j}(M)$ is analytically continued on $B(0; \delta)$ by its defining equation,
3. the sequence $\mu_{p_j}^{-1}$ uniformly converges to $\mu_p^{-1}$ on $B(0; \delta)$ as a biholomorphic mapping,
4. the sequence $\mu_{p_j}(M)$ uniformly converges on $B(0; \delta)$ to the real hypersurface $\mu_p(M)$.

Let $H$ be the local automorphism group at the origin of the real hyperquadric $v = \langle z, z \rangle$.

The isotropy subgroup $\text{Aut}_p(M)$ is naturally identified to the isotropy subgroup $\text{Aut}_0(\mu_p(M))$ by the following relation:

$$\mu_p \circ \phi \circ \mu_p^{-1} \in \text{Aut}_0(\mu_p(M)) \text{ for } \phi \in \text{Aut}_p(M).$$

Note that every biholomorphic mapping $\varphi$ between real hypersurfaces in normal form is faithfully represented by a natural group action of the isotropy subgroup $H$ (cf. [Pa3]) such that

$$\varphi = Ne \text{ for } e \in H.$$ 

Because $\mu_p(M)$ is in normal form, there is a natural identification of a local automorphism $\phi \in Aut_p(M)$ to an element

$$(U_\phi, a_\phi, \rho_\phi, r_\phi) \in H,$$

where $U_\phi, a_\phi, \rho_\phi, r_\phi$ are the normalizing parameters (cf. [Pa3]) of the mapping

$$\mu_p \circ \phi \circ \mu_p^{-1} \in \text{Aut}_0(\mu_p(M)).$$

**Lemma 6.** If the isotropy subgroup $\text{Aut}_p(M)$ is compact, then $\text{Aut}_p(M)$ is isomorphic to the subgroup

$$\{(U_\phi, a_\phi, \rho_\phi, r_\phi) \in H : \phi \in \text{Aut}_p(M)\}$$

as a Lie group.
0.2. Preliminary Lemmas. We have lemmas on the automorphism of $B^{n+1}$.

**Lemma 7.** Let $p, q$ be two distinct points on $bB^{n+1}$ and $\phi \in \text{Aut} \left( bB^{n+1} \right)$ be a local automorphism of $bB^{n+1}$ such that 
\[
\phi(p) \neq q.
\]
Then there is a unique decomposition
\[
\phi = \psi \circ \varphi
\]
where
\[
\varphi \in \text{Aut}_p \left( bB^{n+1} \right), \quad \psi \in \text{Aut}_q \left( bB^{n+1} \right)
\]
and the local automorphism $\psi$ acts trivially on the complex tangent hyperplane of $bB^{n+1}$ at the fixed point $q$.

**Proof.** Note that the isotropy subgroup $\text{Aut}_q \left( bB^{n+1} \right)$ acts on $bB^{n+1} \setminus q$ transitively. Further, there is a unique element $\psi \in \text{Aut}_q \left( bB^{n+1} \right)$ for each point displacement on $bB^{n+1} \setminus q$ by requiring the element $\psi$ acts trivially on the complex tangent hyperplane of $bB^{n+1}$ at the point $q$ (cf. [Pa1]).

Let’s put $p' = \phi(p)$. Since $p' \neq q$, we take a unique automorphism $\psi \in \text{Aut}_q \left( bB^{n+1} \right)$ such that
\[
\psi(p') = p.
\]
Then $\varphi \equiv \psi \circ \phi \in \text{Aut}_p \left( bB^{n+1} \right)$ so that
\[
\phi = \psi^{-1} \circ \varphi.
\]
This completes the proof. \(\square\)

**Lemma 8.** Let $p, q$ be two distinct points on $bB^{n+1}$ and $\phi \in \text{Aut} \left( bB^{n+1} \right)$ be a local automorphism of $bB^{n+1}$ such that 
\[
p' \equiv \phi(p) \neq q.
\]
Then there is a unique decomposition
\[
\phi = \varphi \circ \psi
\]
where
\[
\varphi \in \text{Aut}_{p'} \left( bB^{n+1} \right), \quad \psi \in \text{Aut}_q \left( bB^{n+1} \right)
\]
and the local automorphism $\psi$ acts trivially on the complex tangent hyperplane of $bB^{n+1}$ at the fixed point $q$.

**Proof.** By Lemma 8, there is a decomposition
\[
\phi^{-1} = \psi \circ \varphi
\]
where
\[
\varphi \in \text{Aut}_{p'} \left( bB^{n+1} \right), \quad \psi \in \text{Aut}_q \left( bB^{n+1} \right)
\]
and the local automorphism $\psi$ acts trivially on the complex tangent hyperplane of $bB^{n+1}$ at the fixed point $q$. Hence we obtain
\[
\phi = \varphi^{-1} \circ \psi^{-1}
\]
where
\[ \varphi^{-1} \in \text{Aut}_{p'}(bB^{n+1}), \quad \psi^{-1} \in \text{Aut}_{q}(bB^{n+1}). \]
This completes the proof. \(\square\)

Lemma 9. Let \(\phi_j\) be a sequence of automorphisms of \(B^{n+1}\). Suppose that the sequence \(\phi_j\) converges to a holomorphic mapping \(\lambda\) uniformly on every compact subset of \(B^{n+1}\). Then the mapping \(\lambda\) is either a constant mapping or an automorphism of \(B^{n+1}\).

Proof. Note that the mapping \(\lambda\) satisfies
\[ \lambda(B^{n+1}) \subset \overline{B^{n+1}} \]
and a complex line is mapped to a complex line under the biholomorphic automorphism of the unit ball \(B^{n+1}\).

Suppose that there is a complex line \(\pi\) such that
\[ \pi \cap B^{n+1} \neq \emptyset \quad \text{and} \quad \lambda(\pi \cap B^{n+1}) = \emptyset. \]
Note that \(bB^{n+1}\) is strongly pseudoconvex so that, by the condition \(\|\cdot\|\) there is a point \(q \in bB^{n+1}\) satisfying
\[ q = \lambda(\pi) \cap \overline{B^{n+1}}. \]
Then we obtain
\[ \lambda|_{\pi \cap B^{n+1}} = q. \]
Let \(x\) be an interior point of \(\pi \cap B^{n+1}\) and \(p'\) be an arbitrary point of \(B^{n+1}\) so that we take a complex line \(\pi'\) passes through \(x\) and \(p'\). Since \(B^{n+1}\) is strongly pseudoconvex, the maximum modulus theorem of one complex variable yields
\[ \lambda|_{\pi' \cap B^{n+1}} = q \]
so that the mapping \(\lambda\) is a constant mapping.

Suppose that the mapping \(\lambda\) is not a constant mapping. Then, for a complex line \(\pi\) satisfying
\[ \pi \cap B^{n+1} \neq \emptyset, \]
there is a real number \(\varepsilon > 0\) such that
\[ |\phi_j(\pi \cap B^{n+1})| \geq \varepsilon \]
where \(|\phi_j(\pi \cap B^{n+1})|\) is the area of the analytic disk
\[ \phi_j(\pi \cap B^{n+1}). \]
We take a point \(p \in \pi \cap bB^{n+1}\) so that
\[ \phi_j(p) \rightarrow p' \in bB^{n+1}. \]
Then we take a point \(p'' \in bB^{n+1}\) such that
\[ p'' \notin \{\phi_j(p) : j \in \mathbb{N}^+\}, \]
if necessary, passing to a subsequence. We have the following decomposition
\[ \phi_j = \varphi_j \circ \psi_j. \]
where
\[ \varphi_j \in \text{Aut}_{p'}(bB^{n+1}) \quad \text{and} \quad \psi_j \in \text{Aut}_{p''}(bB^{n+1}) \]
where the automorphisms \( \psi_j \) act trivially on the complex tangent hyperplane at the fixed point \( p'' \). Then we obtain
\[ U_{\psi_j} = \text{id}_{n \times n}, \quad \rho_{\psi_j} = 1. \]
By the condition 3, there is a real number \( e > 0 \) such that
\[ |a_{\psi_j}| \leq e, \quad |r_{\psi_j}| \leq e. \]
By the condition 2, there is a real number \( e > 0 \), if necessary, increasing \( e \), such that
\[ |a_{\psi_j}| \leq e. \]
Since the mapping \( \lambda \) is not a constant mapping, there is a real number \( e > 0 \), if necessary, increasing \( e \), such that
\[ e^{-1} \leq |\rho_{\varphi_j}| \leq e, \quad |r_{\varphi_j}| \leq e. \]
Since \( bB^{n+1} \) is strongly pseudoconvex, we have
\[ |U_{\varphi_j}| = 1. \]
Then the Jacobian determinant \( \det \varphi'_j \) is uniformly bounded from the zero on an open neighborhood of the point \( p \in bB^{n+1} \). By Hurwitz theorem, the mapping \( \lambda \) is locally biholomorphic and, further, the mapping \( \lambda \) is one-to-one. Hence the mapping \( \lambda \) is an automorphism of the unit ball \( B^{n+1} \). This completes the proof.

We have lemmas on the chain of \( bB^{n+1} \).

**Lemma 10.** Let \( \gamma : [0, 1] \to bB^{n+1} \) be a chain-segment on \( bB^{n+1} \). Then there is a complex line \( \pi \) such that
\[ \gamma[0, 1] \subseteq \pi \cap bB^{n+1}. \]

**Proof.** We take a point \( p \in \gamma[0, 1] \). Then the chain-segment \( \mu_p \circ \gamma[0, 1] \) on \( \mu_p(bB^{n+1}) \) is on a complex line \( \pi' \) (cf. [Pa3]). Since the canonical normalizing mapping \( \mu_p \) of the sphere \( bB^{n+1} \) is a fractional linear mapping (cf. [Pa3]), we take
\[ \pi = \mu_p^{-1}(\pi') \]
so that
\[ \gamma[0, 1] \subseteq \pi \cap bB^{n+1}. \]
This completes the proof.

**Lemma 11.** Let \( \pi \) be a complex line such that
\[ \pi \cap B^{n+1} \neq \emptyset. \]
Then the circle \( \pi \cap bB^{n+1} \) is a chain on \( bB^{n+1} \).
Proof. We take a point \( p \in \pi \cap bB^{n+1} \). Then we obtain

\[
\mu_p(\pi) \cap \mu_p(bB^{n+1}) = \mu_p(\pi \cap bB^{n+1}) \neq \emptyset.
\]

Since the canonical normalizing mapping \( \mu_p \) of the sphere \( bB^{n+1} \) is a fractional linear mapping (cf. [Pa3]), \( \mu_p(\pi) \) is a complex line so that \( \mu_p(\pi) \cap \mu_p(bB^{n+1}) \) is a chain (cf. [Pa3]). Thus the circle

\[
\pi \cap bB^{n+1} = \mu_p^{-1}(\mu_p(\pi)) \cap \mu_p(bB^{n+1})
\]

is a chain as well. This completes the proof.

Lemma 12. Let \( \gamma \) be a chain passing through a point \( p \in bB^{n+1} \) and \( \delta_\gamma \) be an analytic disk such that

\[
\gamma = \pi \cap bB^{n+1} \quad \text{and} \quad \delta_\gamma = \pi \cap B^{n+1}
\]

where \( \pi \) is a complex line. Let \( \theta_\gamma \) be the angle between the tangent vector of \( \gamma \) at the point \( p \) and a unit vector \( v_p \) perpendicular to the complex tangent hyperplane at the point \( p \in bB^{n+1} \) and \( |\delta_\gamma| \) be the area of the analytic disk \( \delta_\gamma \). Then

\[
|a_\gamma| \equiv |\tan \theta_\gamma| \to \infty
\]

if and only if

\[
|\delta_\gamma| \to 0.
\]

Proof. We easily see that

\[
\theta_\gamma \to \frac{\pi}{2}
\]

if and only if

\[
|a_\gamma| \equiv |\tan \theta_\gamma| \to \infty.
\]

Note that the complex line \( \pi \) would be on the complex tangent hyperplane of \( bB^{n+1} \) at the point \( p \) if

\[
\theta_\gamma = \frac{\pi}{2}.
\]

Since \( bB^{n+1} \) is strongly pseudoconvex and \( B^{n+1} \) is strongly convex, the complex tangent hyperplane of \( bB^{n+1} \) at the point \( p \) has no intersection to \( B^{n+1} \). Thus we easily see

\[
\theta_\gamma \to \frac{\pi}{2}
\]

if and only if

\[
|\delta_\gamma| \to 0.
\]

This completes the proof.

We may require the following well-known results in this article (cf. [Kr], [Ra], [Bo]).

Lemma 13 (Lewy, Pinchuk). Let \( D, D' \) be domains with strongly pseudoconvex real analytic boundaries \( bD, bD' \) and \( U \) be a connected open neighborhood of a point \( p \in bD \). Suppose that there is a holomorphic mapping \( \phi \) on \( U \cap D \) such that

\[
\phi \in H(U \cap D) \cap C^1(U \cap \overline{D}), \quad \phi(U \cap bD) \subset bD'
\]

and the induced mapping \( \phi : U \cap bD \to bD' \) is CR diffeomorphic. Then the mapping \( \phi \) is analytically continued on \( U \), if necessary, shrinking \( U \).
Lemma 14 (Lewy). Let $D$ be a domain with a strongly pseudoconvex boundary $bD$ and $U$ be an open connected neighborhood of a point $p \in bD$. Suppose that there is a holomorphic mapping $\phi$ on $U \cap bD$. Then there is an open neighborhood $V$ of the point $p$ such that the mapping $\phi$ is analytically continued onto $V \cap D$.

Lemma 15 (Wong, Rosay). Let $D$ be a strongly pseudoconvex bounded domain. Suppose that there is a compact set $K \subset D$ and a sequence $p_j \in K$ and automorphisms $\phi_j \in \text{Aut}(D)$ such that $\phi_j(p_j) \to bD$. Then the domain $D$ is biholomorphic to an open unit ball $B^{n+1}$.

Lemma 16 (Bell-Catlin, Diederich-Fornaess). Let $D, D'$ be strongly pseudoconvex bounded domains with the boundaries $bD, bD'$ of class $C^\infty$ and $\phi : D \to D'$ be a proper holomorphic mapping. Then $\phi : D \to D'$ is a locally biholomorphic mapping and the induced mapping $\phi : bD \to bD'$ is a locally CR diffeomorphism.

1. Analytic Continuation on a Sphere

1.1. Analytic continuation with finiteness. Let $D$ be a domain in $\mathbb{C}^{n+1}, n \geq 1$, with real analytic boundary $bD$. The boundary $bD$ shall be called spherical if, for each point $p \in bD$, there is a connected open neighborhood $U$ of the point $p$ and a biholomorphic mapping $\phi$ on $U$ such that

$$\phi(U \cap bD) \subset bB^{n+1}.$$  

Note that a domain $D$ with spherical real analytic boundary is necessarily strongly pseudoconvex.

Lemma 17. Let $p$ be a point of $bB^{n+1}$ and $U$ be a connected open neighborhood of the point $p$. Suppose that there is a biholomorphic mapping $\phi$ on $U$ such that

$$\phi(U \cap bB^{n+1}) \subset bB^{n+1}.$$  

Then the mapping $\phi$ is analytically continued on an open neighborhood of the closed ball $\overline{B^{n+1}}$.

Proof. Note that each local automorphism $\varphi \in \text{Aut}_p(bB^{n+1})$ for any point $p \in bB^{n+1}$ is necessarily birational such that $\varphi$ is analytically continued on an open neighborhood of $\overline{B^{n+1}}$ as a biholomorphic mapping (cf. [Pa1]).

Let’s put $q = \phi(p)$. We take a point $r \in bB^{n+1}$ such that $r \neq p, r \neq q$. Then we take an automorphism

$$\psi \in \text{Aut}_r(bB^{n+1})$$

satisfying

$$\psi(q) = p$$

so that

$$\varphi \equiv \psi \circ \phi \in \text{Aut}_p(bB^{n+1}).$$
Then the mapping $\psi^{-1} \circ \varphi$ is an automorphism of $B^{n+1}$ and an analytic continuation of the mapping $\phi$ such that
\[
\phi = \psi^{-1} \circ \varphi \quad \text{on} \quad U.
\]
This completes the proof. \qed

**Theorem 18.** Let $D$ be a domain in $\mathbb{C}^{n+1}$ with spherical real analytic boundary $bD$. Suppose that there is a connected open neighborhood $U$ of a point $p \in bD$ and a biholomorphic mapping $\phi$ on $U$ such that $\phi(U \cap bD) \subset bB^{n+1}$. Then the mapping $\phi$ is analytically continued along any path on $bD$ as a local biholomorphic mapping.

**Proof.** Suppose that the assertion is not true. Then there would be a path $\gamma : [0, 1] \to bD$ such that $\gamma(0) \in U \cap bD$ and the germ of a biholomorphic mapping $\phi$ at the point $\gamma(0)$ is analytically continued along the subpath $\gamma[0, \tau]$ with all $\tau < 1$ as a local biholomorphic mapping, but not the whole path $\gamma[0, 1]$.

Since $bD$ is spherical, by definition, there exist a connected open neighborhood $V$ of the point $\gamma(1)$ and a biholomorphic mapping $\psi$ on $V$ such that $\psi(V \cap bD) \subset bB^{n+1}$.

We take $\lambda \in [0, 1)$ such that $\gamma(\tau) \in V \cap bD$ for all $\tau \in [0, \lambda]$, and we take a sufficiently small connected open neighborhood $W$ of the point $\gamma(\lambda)$ such that $\phi$ is analytically continued on $W \subset V$ along the path $\gamma[0, \lambda]$ and
\[
\psi(W) \cap bB^{n+1} \neq \emptyset.
\]
Then we have
\[
\psi(\psi(W) \cap bB^{n+1}) \subset bB^{n+1}
\]
where
\[
\psi = \phi \circ \varphi^{-1}.
\]
By Lemma 18, $\psi$ is analytically continued on an open neighborhood of $bB^{n+1}$ as a local biholomorphic mapping. By abuse of notation, the mapping $\psi \circ \varphi$ is biholomorphic on the open set $V$ such that
\[
\psi \circ \varphi = \phi \quad \text{on} \quad W.
\]
Thus the germ $\psi \circ \varphi$ at the point $\gamma(1)$ is an analytic continuation of the germ $\phi$ at the point $\gamma(0)$ along the path $\gamma[0, 1]$ as a local biholomorphic mapping. This contradiction completes the proof. \qed

**Lemma 19.** Let $D$ be a bounded domain in $\mathbb{C}^{n+1}$ with spherical real analytic boundary $bD$ such that the fundamental group $\pi_1(bD)$ is finite. Suppose that there is a connected open neighborhood $U$ of a point $p \in bD$ and a biholomorphic mapping $\phi$ on $U$ such that $\phi(U \cap bD) \subset bB^{n+1}$. Then $\phi$ is analytically continued to a biholomorphic mapping from $D$ onto $B^{n+1}$.

**Proof.** By Lemma 18, the mapping $\phi$ is analytically continued along any path on $bD$ as a local biholomorphic mapping. Let $E$ be the path space of $bD$ pointed at the point $p$ mod homotopy so that $E$ is a universal covering of $bD$ with a natural CR structure and a CR projection $\varphi : E \to bD$. Then there is a unique CR lift $\psi : E \to bB^{n+1}$ as the analytic continuation of the biholomorphic mapping $\phi$. Note
that \( \psi : E \to bB^{n+1} \) is an open mapping because \( \phi \) and \( \varphi \) are both locally CR diffeomorphisms.

Since \( bD \) is finitely connected, \( E \) is necessarily compact so that the mapping \( \psi : E \to bB^{n+1} \) is surjective. Further, the mapping \( \psi : E \to bB^{n+1} \) is a simple covering map because \( \psi \) is locally a CR diffeomorphism and the sphere \( bB^{n+1} \) is simply connected. Hence there exists a locally biholomorphic mapping \( \lambda : bB^{n+1} \to bD \) defined by

\[
\lambda = \varphi \circ \psi^{-1} \quad \text{on} \quad bB^{n+1}.
\]

By Hartogs extension theorem, the mapping \( \lambda \) uniquely extends to the open ball \( B^{n+1} \) as a local biholomorphic mapping and, further, the extension is smooth up to the boundary. Hence \( \lambda \) is well defined as a locally biholomorphic mapping on an open neighborhood of \( \overline{B^{n+1}} \) by Lemma 13.

We obtain a proper mapping \( \lambda : B^{n+1} \to D \) so that \( \lambda \) is a globally branched covering and the branched locus of \( \lambda \) cannot be bounded by \( bD \). Since \( \lambda^{-1} = \phi \) is locally biholomorphic on \( bD \), \( \lambda : B^{n+1} \to D \) and \( \lambda : bB^{n+1} \to bD \) are finite coverings respectively of \( D \) and \( bD \). Since the closed ball \( \overline{B^{n+1}} \) has the fixed point property, the mapping \( \lambda : B^{n+1} \to D \) is globally one-to-one. Otherwise, there would be a nontrivial deck transform of \( B^{n+1} \) which is continuous on \( \overline{B^{n+1}} \) without a fixed point. Hence the mapping \( \lambda^{-1} : D \to B^{n+1} \) is biholomorphic with \( \lambda^{-1} = \phi \) on \( bD \). This completes the proof.

Theorem 20. Let \( D \) be a bounded domain in \( \mathbb{C}^{n+1} \) with a connected spherical real analytic boundary \( bD \). Suppose that there is a biholomorphic mapping \( \phi \) on a connected open neighborhood \( U \) of a point \( p \in bD \) satisfying

\[
\phi(U \cap bD) \subset bB^{n+1}
\]

such that the analytic continuation of \( \phi \) on the boundary \( bD \) yields finitely many germs at each point on \( bD \). Then \( D \) is necessarily simply connected and the mapping \( \phi \) is analytically continued to a biholomorphic mapping from \( D \) onto \( B^{n+1} \).

Proof. We claim that there is a finite covering space \( E_1 \) of \( bD \) with a natural CR structure and a CR projection \( \varphi_1 : E_1 \to bD \) and a local CR diffeomorphism \( \psi_1 : E_1 \to bB^{n+1} \) satisfying the relation \( \psi_1 = \phi \circ \varphi_1 \). Then the desired result follows from this claim by the same argument in the proof of Lemma 13.

Let \( E \) be the path space of \( bD \) pointed at the point \( p \in bD \) mod homotopy so that \( E \) is a universal covering of \( bD \) with a natural CR structure and a CR projection \( \varphi : E \to bD \). Then there is a unique CR lift \( \psi : E \to bB^{n+1} \):

\[
\begin{array}{ccc}
E & \xrightarrow{\psi} & bB^{n+1} \\
\downarrow \varphi & \swarrow & \\
bD & \xrightarrow{\phi} & bB^{n+1}
\end{array}
\]

satisfying the relation \( \psi = \phi \circ \varphi \). Note that \( \psi : E \to bB^{n+1} \) is an open mapping because \( \phi \) and \( \varphi \) are both local CR diffeomorphisms. Let \( F \) be the image of the mapping \( \psi \) such that \( F = \psi(E) \). Then \( F \) is an open subset of \( bD \).

Suppose that \( bF \neq \emptyset \). Then we take a point \( p \in bF \) and a sequence \( p_j \in F \) such that \( p_j \to p \). Thus there exists a point \( q_j \in bD \) and germs of biholomorphic mappings \( \phi_j \) such that

\[
\phi_j(q_j) \to p
\]
where $\phi_j$ are analytic continuations of the mapping $\phi$ on $bD$. Since $bD$ is compact, there exist a point $q \in bD$ and a subsequence $q_{m_j}$ of $q_j$ such that $q_{m_j} \to q$. Further, by passing to a subsequence, if necessary, we may assume that

$$\phi^* = \phi_{m_j} \quad \text{for all} \quad m_j$$

because the analytic continuation of the mapping $\phi$ yields only finitely many germs at the point $q \in bD$. Hence we obtain

$$\phi^*(q) = \lim_{j \to \infty} \phi_{m_j}(q_{m_j}) = p \in F.$$ 

This contradiction implies that $bF = \emptyset$, i.e., $\psi(E) = bB^{n+1}$.

For each point $q \in bB^{n+1}$, there is a subset $X_q \subset E$ such that $X_q = \{p \in E : \psi(p) = q\}$. Since $\psi$ is a CR diffeomorphism, $X_q$ is necessarily a discrete set on $E$. Then we define a subset $Y_q \subset bD$ such that $Y_q = \{\phi(p) \in bD : p \in X_q\}$.

Suppose that $Y_q$ has an accumulation point $y \in bD$. Then there is a sequence of point $p_j \in bD$ satisfying

$$p_j \to y \quad \text{and} \quad p_j \neq y,$$

and biholomorphic mappings $\phi_j$ such that

$$\phi_j(p_j) = q$$

where the mapping $\phi$ at the point $p_j$ is the analytic continuation of the mapping $\phi$. Because the analytic continuation of $\phi$ yields only finitely many germs at the point $y$, we can take a subsequence of $\phi$ and a biholomorphic mapping $\phi^*$ such that $\phi^* = \phi_{m_j}$. Then we have

$$\phi^*(y) = \lim_{j \to \infty} \phi_{m_j}(p_{m_j}) = q.$$ 

Since $\phi^*$ is locally biholomorphic, it is impossible that $\phi^*(p_{m_j}) = q = \phi^*(y)$ and $p_{m_j} \to y, p_{m_j} \neq y$ at the same time. Thus we find that the set $Y_q$ is finite.

Therefore, the analytic continuation of the mapping $\phi$ on $bD$ is mapped to a point of $bB^{n+1}$ only at finitely many points of $bD$. Since the analytic continuation of the mapping $\phi$ yields finitely many germs at each point on $bD$, only finitely many germs of the analytic continuation of the mapping $\phi$ on $bD$ are mapped to each point of $bB^{n+1}$. Then, by the compactness of $bB^{n+1}$, we obtain a finite covering space $E_1$ of $bD$ satisfying all conditions in the claim. This completes the proof. □

1.2. First Dogginal Lemma.

**Lemma 21** (First Scaling Lemma). Let $p$ be a point of the boundary $bB^{n+1}$ and $p_j$, $j \in \mathbb{N}^+$, be a sequence of points of $bB^{n+1}$ such that $p_j \neq p$ for all $j$ and $p_j \to p$ as $j \to \infty$ to a direction transversal to the complex tangent hyperplane at the point $p \in bB^{n+1}$. Let $\varepsilon_j$ be the euclidean distance between the two points $p_j$ and $p$, and $\delta_j$ be the analytic disk

$$\delta_j = \pi_j \cap B^{n+1}$$
where \( \pi_j \) is the complex line passing through the two points \( p_j \) and \( p \). Suppose that there is a sequence \( p_j' \) of points of \( bB^{n+1} \) satisfying
\[
p_j' \to p' \in bB^{n+1},
\]
and a sequence of biholomorphic automorphisms \( \phi_j \in Aut\left(B^{n+1}\right) \) satisfying
\[
\phi_j(p_j') = p_j
\]
such that the sequence \( \phi_j \) converges to a constant mapping and the area \( |\phi_j^{-1}(\delta_j)| \) of the analytic disks
\[
|\phi_j^{-1}(\delta_j)|
\]
is bounded from the below, i.e., there is a real number \( c > 0 \) satisfying
\[
|\phi_j^{-1}(\delta_j)| \geq c.
\]
Then there is a subsequence \( \phi_{m_j} \) and a sequence of local automorphisms \( \sigma_j \in Aut_{p_m_j}\left(B^{n+1}\right) \) such that
\[
U_{\sigma_j} = id_{n \times n}, \quad \rho_{\sigma_j} = \varepsilon_{m_j}, \quad a_{\sigma_j} = 0, \quad r_{\sigma_j} = 0
\]
and the composition
\[
\sigma_j^{-1} \circ \phi_{m_j} : B^{n+1} \to B^{n+1}
\]
uniformly converges to an automorphism of the unit ball \( B^{n+1} \).

Proof. Note that there is a subsequence \( \phi_{m_j} \) which converges to the point \( p \in bB^{n+1} \) uniformly on every compact subset of the unit ball \( B^{n+1} \). We take a point \( p' \in bB^{n+1} \) such that \( p' \neq p \) and
\[
p' \notin \{p_{m_j} : j \in \mathbb{N}^+\},
\]
if necessary, passing to a subsequence. Then, by Lemma 8, there is a unique decomposition of the automorphism \( \phi_{m_j} \) such that
\[
\phi_{m_j} = \varphi_j \circ \psi_j
\]
where
\[
\varphi_j \in Aut_{p_m_j}\left(bB^{n+1}\right), \quad \psi_j \in Aut_{p'}\left(bB^{n+1}\right)
\]
and the local automorphism \( \psi_j \) acts trivially on the complex tangent hyperplane of \( bB^{n+1} \) at the fixed point \( p' \).

Let \( U_{\psi_j}, \rho_{\psi_j}, a_{\psi_j}, r_{\psi_j} \) be the normalizing parameters of the local automorphism \( \psi_j \). Since the local automorphism \( \psi_j \) acts trivially on the complex tangent hyperplane of \( bB^{n+1} \) at the fixed point \( p' \), we obtain
\[
U_{\psi_j} = id_{n \times n}, \quad \rho_{\psi_j} = 1.
\]
Since the sequence \( \phi_{m_j} \) uniformly converges to the point \( p \), there is a real number \( e > 0 \) such that
\[
(5) \quad |a_{\psi_j}| \leq e, \quad |r_{\psi_j}| \leq e.
\]

Let \( \pi_j \) be the complex line passing through the two points \( p \) and \( p_{m_j} \). Since the points \( p_j \) converges to the point \( p \) to a direction transversal to the complex tangent hyperplane at the point \( p \in bB^{n+1} \), the analytic disks
\[
\delta_j = \pi_j \cap B^{n+1}
\]
is uniformly bounded in their area from the below by a positive number.
Let $U_\varphi, \rho_\varphi, a_\varphi, r_\varphi$ be the normalizing parameters of the local automorphism $\varphi_j$. Note that the analytic disk $\phi_m^{-1}(\delta_j)$ is mapped by $\phi_m$ onto the analytic disk $\delta_j$, where the areas of the analytic disks in both classes $|\phi_m^{-1}(\delta_j)|$ and $|\delta_j|$ are bounded from the below. Since the chain $\phi_m^{-1}(b\delta_j)$ is mapped by $\phi_m$ to the chain $b\delta_j$, there is a real number $e > 0$, if necessary, increasing $e$, such that

$$|a_\varphi| \leq e,$$

(6)

Since the area of the analytic disks $\delta_j$ is bounded by a positive number, the point $p$ is attracted to the center of the local automorphism $Aut_{p_m}(bB^{n+1})$. Then, by passing to a subsequence, if necessary and increasing $e$, there is a real number $e > 0$ such that

$$|r_\varphi| \leq e$$

and

$$\rho_\varphi \to 0 \quad \text{as} \quad j \to \infty.$$

Since $bB^{n+1}$ is strongly pseudoconvex, we obtain

$$|U_\varphi| = 1.$$

Let $\eta_j$ be a local automorphism in $Aut_{p_m}(bB^{n+1})$ defined by the normalizing parameters

$$U_{\eta_j} = id_{n \times n}, \quad \rho_{\eta_j} = \rho_{\varphi_j}, \quad a_{\eta_j} = 0, \quad r_{\eta_j} = 0.$$

Then the composition

$$\varphi'_j = \eta_j^{-1} \circ \varphi_j \in Aut_{p_m}(bB^{n+1})$$

has the same normalizing parameters of the mapping $\varphi_j$ except for $\rho_{\varphi'_j} = 1$, i.e.,

$$U_{\varphi'_j} = U_{\varphi_j}, \quad \rho_{\varphi'_j} = 1, \quad a_{\varphi'_j} = a_{\varphi_j}, \quad r_{\varphi'_j} = r_{\varphi_j}.$$ 

Therefore, by passing to a subsequence, if necessary, the sequence

$$\tau_j = \eta_j^{-1} \circ \phi_m : B^{n+1} \to B^{n+1}$$

converges by Hurwitz theorem to a locally biholomorphic mapping. Since $\tau_j$ are automorphism of $B^{n+1}$, the sequence $\tau_j$ converges to an automorphism of $B^{n+1}$.

Let $q \in \overline{B^{n+1}}$ be the limit point of the sequence

$$\tau_j \left( p_{m_j} \right) \to q \in \overline{B^{n+1}}.$$

We set

$$\lambda_j = \frac{\varepsilon_{m_j}}{\rho_{\varphi_j}}.$$

The sequence $\lambda_j$ converges to the distance between the two distinct points $q$ and $p \in bB^{n+1}$ so that there is a real number $e > 0$ such that

$$e^{-1} \leq |\lambda_j| \leq e,$$

(8)

if necessary, increasing $e$. 

Let \( \sigma_j \) be a local automorphism in \( \text{Aut}_{p,m_j}(bB^{n+1}) \) defined by the normalizing parameters
\[
U_{\sigma_j} = id_{n \times n}, \quad \rho_{\sigma_j} = \varepsilon_{m_j}, \quad a_{\sigma_j} = 0, \quad r_{\sigma_j} = 0.
\]
Then the composition
\[
\varphi''_j = \sigma_j^{-1} \circ \varphi_j \in \text{Aut}_{p,m_j}(bB^{n+1})
\]
has the same normalizing parameters of the mapping \( \varphi_j \) except for \( \rho_{\varphi''_j} = \lambda_j^{-1} \), i.e.,
\[
U_{\varphi''_j} = U_{\varphi_j}, \quad \rho_{\varphi''_j} = \lambda_j^{-1}, \quad a_{\varphi''_j} = a_{\varphi_j}, \quad r_{\varphi''_j} = r_{\varphi_j}.
\]
Therefore, by the estimates \([8],[9],[10],[11]\), the sequence
\[
\tau_j = \sigma_j^{-1} \circ \varphi_{m_j} : B^{n+1} \to B^{n+1}
\]
uniformly converges to an automorphism of \( B^{n+1} \). This completes the proof. \( \square \)

**Lemma 22.** Let \( D \) be a bounded domain in \( \mathbb{C}^{n+1} \) with spherical real analytic boundary \( bD \). Suppose that there is a connected open neighborhood \( U \) of a point \( p \in bD \) and a biholomorphic mapping \( \varphi \) on \( U \) such that \( \varphi(U \cap bD) \subset bB^{n+1} \) and the inverse mapping \( \varphi^{-1} \) on \( bB^{n+1} \) is analytically continued along every chain of \( bB^{n+1} \). Then the mapping \( \varphi \) is analytically continued to a biholomorphic mapping from \( D \) onto \( B^{n+1} \).

**Proof.** The chain on \( bB^{n+1} \) is characterized to be the intersection of a complex line on \( bB^{n+1} \). Thus the chains on \( bB^{n+1} \) form a continuous family so that the analytic continuity of the inverse mapping \( \varphi^{-1} \) along every chain on \( bB^{n+1} \) is equivalent to the analytic continuity along any path on \( bB^{n+1} \).

Note that the inverse mapping \( \varphi^{-1} \) is analytically continued along any path on \( bB^{n+1} \) and, by Hartogs extension theorem, the branching locus of a proper mapping cannot be bounded by \( bD \). Since \( bB^{n+1} \) is simply connected, by the monodromy theorem, the mapping \( \varphi^{-1} \) is analytically continued to, by abuse of notation, a locally biholomorphic proper mapping \( \varphi^{-1} : B^{n+1} \to D \) such that \( \varphi^{-1}(bB^{n+1}) = bD \). By the fixed point property of the closed ball \( B^{n+1} \), the proper mapping \( \varphi^{-1} : B^{n+1} \to D \) is globally one-to-one so that the mapping \( \varphi^{-1} : B^{n+1} \to D \) is biholomorphic. This completes the proof. \( \square \)

**Lemma 23.** Let \( D \) be a domain in \( \mathbb{C}^{n+1} \) with spherical real analytic boundary \( bD \) and \( U \) be a connected open neighborhood of a point \( p \in bD \). For a chain \( \gamma \) on \( bD \) passing through the point \( p \) and tangential to the direction with an angle \( \theta_{\gamma} \) with respect to a unit vector \( v_p \) at the point \( p \) perpendicular to the complex tangent hyperplane, we denote \( |a_\gamma| = |\tan \theta_{\gamma}| \). Then there is a real number \( e > 0 \) such that every chain \( \gamma \) passing through the point \( p \) with \( |a_\gamma| \geq e \) is the boundary of a nonsingular analytic disk \( \delta_\gamma \subset U \cap D \) such that
\[
\gamma = \delta_\gamma \cap bD.
\]

**Proof.** We take a biholomorphic mapping \( \varphi \) on the open set \( U \), if necessary, shrinking \( U \) such that
\[
\varphi(U \cap bD) \subset bB^{n+1}.
\]
Every chain $\lambda$ on $bB^{n+1}$ is an intersection with a complex line $\pi_\lambda$ such that

$$\lambda = \pi_\lambda \cap bB^{n+1}.$$ 

For a sufficiently large real number $e > 0$, each chain $\gamma$ with $|a_\gamma| \geq e$ is obtained by the relation

$$\gamma = \phi^{-1}(\lambda)$$

where the chain $\lambda$ on $bB^{n+1}$ satisfies the condition

$$\pi_\lambda \cap B^{n+1} \subset \phi(U \cap D).$$

Then we take

$$\delta_{\gamma} = \phi^{-1}(\pi_\lambda \cap B^{n+1}).$$

This completes the proof.

**Lemma 24 (First Dogginal Lemma).** Let $D$ be a bounded domain in $\mathbb{C}^{n+1}$ with spherical real analytic boundary $bD$ and $\phi$ be a biholomorphic mapping on a connected open neighborhood $U$ of a point $p \in bB^{n+1}$ satisfying

$$\phi(U \cap bB^{n+1}) \subset bD.$$ 

Suppose that there is a chain-segment $\gamma : [0, 1] \to bB^{n+1}$ such that $\gamma(0) \in U \cap bB^{n+1}$ and the mapping $\phi$ is analytically continued along the subpath $\gamma[0, \tau]$ for all $\tau < 1$, but not the whole path $\gamma[0, 1]$ as a local biholomorphic mapping. Let $\pi$ be the complex line containing the chain-segment $\gamma[0, 1]$. Then there is an open neighborhood $V$ along the path $\gamma[0, 1]$ such that

1. $\gamma[0, \tau] \subset V$ for all $\tau < 1$,
2. $bV \cap \pi \cap B(\gamma(1); \delta)$ is an angle for a sufficient small $\delta > 0$, which contains the chain-segment $\gamma[0, 1]$,
3. $bV \cap bB^{n+1} \cap B(\gamma(1); \delta)$ is slanted paraboloid for a sufficiently small $\delta > 0$, which smoothly touches the complex tangent hyperplane at the point $\gamma(1)$,
4. the mapping $\phi$ is analytically continued on $V$ as a local biholomorphic mapping.

**Proof.** By the analytic continuation of the mapping $\phi$ along the subpath $\gamma[0, \tau]$ for all $\tau < 1$, there is a path $\phi \circ \gamma : [0, 1) \to bD$. Then we consider the following sequences

$$p_j = \gamma \left(1 - \frac{1}{j}\right), \quad \text{for } j \in \mathbb{N}^+,$$

$$p'_j = \phi \circ \gamma \left(1 - \frac{1}{j}\right), \quad \text{for } j \in \mathbb{N}^+.$$ 

Since $bD$ is compact, there is a subsequence $p'_{m_j}$ and a point $p' \in bD$ such that

$$p'_{m_j} \to p'.$$

By Theorem $\text{[8]}$, the mapping $\phi^{-1}$ is analytically continued along the path $\phi \circ \gamma[0, 1] \subset bD$.

Let $\varphi_j$ be the analytic continuation of the mapping $\phi^{-1}$ at the point $p'_{m_j}$ along the path $\phi \circ \gamma[0, 1] \subset bD$. By Theorem $\text{[8]}$, there is an open neighborhood $W$ of the point $p'$ such that $\varphi_j$ is locally biholomorphic on an open neighborhood of $W \cap bD$. 

By Lemma 14, we may assume that $\varphi_j$ is holomorphic on $W \cap D$, if necessary, shrinking $W$.

Since $bD$ is spherical, there is an open neighborhood $W$ of the point $p'$, if necessary, shrinking $W$, and a biholomorphic mapping $\varphi$ on $W$ such that

$$\varphi (W \cap bD) \subset bB^{n+1}.$$

Then, by Lemma 17, the compositions

$$\phi_j \equiv \varphi_j \circ \varphi^{-1} : \varphi (W) \cap bB^{n+1} \subset bB^{n+1}$$

are analytically continued, by abuse of notation, to automorphisms $\phi_j$ of the unit ball $B^{n+1}$. Without loss of generality, we may assume that the sequence $\phi_j$ converges to a holomorphic mapping uniformly on every compact subset of $B^{n+1}$.

We set

$$p''_{m_j} \equiv \varphi (p'_{m_j}) \in bB^{n+1} \quad \text{and} \quad p'' \equiv \varphi (p')$$

so that

$$p''_{m_j} \to p''.$$ 

Hence the relation

$$\phi_j \left( p''_{m_j} \right) = p_{m_j}$$

yields

$$\phi_j \left( p''_{m_j} \right) \to p.$$

By Lemma 8, the mapping $\phi_j$ converges to either a constant mapping or an automorphism of $B^{n+1}$. We claim that the sequence $\phi_j$ converges to the point $p$ uniformly on every compact subset of $B^{n+1}$. Otherwise, the sequence $\phi_j$ converges to an automorphism of $B^{n+1}$ so that the sequence $\phi_j = \varphi_j \circ \varphi^{-1}$ converges to a biholomorphic mapping on an open neighborhood $W$ of the point $p'$. Then there is a real number $\delta$ such that the mapping $\varphi_j$ and its inverse $\varphi_j^{-1}$ are analytically continued respectively on

$$B \left( p'_{m_j}; \delta \right) \quad \text{and} \quad B \left( p_{m_j}; \delta \right)$$

as a locally biholomorphic mapping. Thus the mapping $\phi = \varphi_j^{-1}$ is biholomorphic on $B \left( p_{m_j}; \delta \right)$ for every point $p_{m_j} \to p$. This is impossible by the hypothesis on the mapping $\phi$. Thus, by Lemma 8, the sequence $\phi_j$ converges to the point $p$ on every compact subset of $B^{n+1}$.

Let $\pi_j$ be the complex line passing through the two points $p$ and $p_{m_j}$, and $\delta_j$ be the analytic disk

$$\delta_j = \pi_j \cap B^{n+1}.$$ 

Let $\epsilon_j$ be the euclidean length between the two points $p$ and $p_{m_j}$. Note that the area $|\phi_j^{-1} (\delta_j)|$ of the analytic disk

$$\phi_j^{-1} (\delta_j)$$
is bounded from the below. Otherwise, by Lemma 2, the mapping \( \phi = \varphi^{-1} \) is analytically continued over the point \( p \) as a locally biholomorphic mapping. Therefore, by First Scaling Lemma, there is a sequence \( \sigma_j \in Aut_{p_m}(bB^{n+1}) \) such that

\[
U_{\sigma_j} = id_{n \times n}, \quad \rho_{\sigma_j} = \varepsilon_j, \quad a_{\sigma_j} = 0, \quad r_{\sigma_j} = 0
\]

and the sequence

\[
\psi_j \equiv \sigma_j^{-1} \circ \phi_j
\]

converges to an automorphism of \( B^{n+1} \).

Note that there is a real number \( \delta > 0 \) such that the mapping \( \varphi \) and its inverse \( \varphi^{-1} \) are biholomorphically continued respectively on

\[
B \left( p'_m; \delta \right) \quad \text{and} \quad B \left( p''_m; \delta \right),
\]

if necessary, passing to a subsequence. Further, there is a real number \( \delta > 0 \) such that the mapping \( \psi_j \) and its inverse \( \psi_j^{-1} \) are biholomorphically continued respectively to

\[
B \left( p'_m; \delta \right) \quad \text{and} \quad B \left( p_m; \delta \right),
\]

if necessary, shrinking \( \delta \). Then the mapping

\[
\phi = \varphi_j^{-1} = \varphi^{-1} \circ \psi_j^{-1} \circ \sigma_j^{-1}
\]

is biholomorphically continued on the open neighborhood

\[
\sigma_j \left( B \left( p_m; \delta \right) \right).
\]

For a canonical normalizing mapping \( \mu_{p_m} \), we obtain

\[
\sigma'_j \equiv \mu_{p_m} \circ \sigma_j \circ \mu^{-1}_{p_m} : \begin{cases} z^* = \sqrt{\varepsilon_j} z \\ w^* = \varepsilon_j w \end{cases}
\]

Since \( p_m \to p \), by Lemma 5, there is a real number \( \delta > 0 \) such that the mapping \( \mu_{p_m} \) and its inverse \( \mu^{-1}_{p_m} \) are biholomorphically continued respectively to

\[
B \left( p_m; \delta \right) \quad \text{and} \quad B \left( 0; \delta \right).
\]

Hence the mapping

\[
\phi \circ \mu^{-1}_{p_m} = \varphi^{-1} \circ \psi_j^{-1} \circ \mu^{-1}_{p_m} \circ \sigma'_j^{-1}
\]

is biholomorphically continued on

\[
\sigma'_j \left( B \left( 0; \delta \right) \right).
\]

Since \( p_m \to p \), by Lemma 5, the canonical normalizing mapping \( \mu_{p_m} \) uniformly converges to the mapping \( \mu_p \) so that the mapping \( \phi \) is biholomorphically continued near the point \( p_m \) on

\[
\mu^{-1}_{p_m} \circ \sigma'_j \left( B \left( 0; \delta \right) \right)
\]

Therefore, the analytically continued region of the mapping \( \phi \) along the chain \( \gamma(0,1) \subset bB^{n+1} \) contains an open set along the chain \( \gamma[0,1] \) which touches to the point \( \gamma(1) \) by an edge shape transversal to \( bB^{n+1} \) and by a slanted paraboloid shape on \( bB^{n+1} \). This completes the proof. \( \square \)
1.3. **Doggaebi variety on a sphere.** Let $\phi$ be a biholomorphic mapping on an open neighborhood $U$ of a point $p \in bD$ satisfying $\phi(U \cap bD) \subset bB^{n+1}$. Let $L \subset bB^{n+1}$ be the singular locus of the analytic continuation of the mapping $\phi^{-1}$, which shall be called the Doggaebi variety associated to the mapping $\phi$.

**Lemma 25** (Second Scaling Lemma). Let $p$ be a point of the boundary $bB^{n+1}$ and $p_j, j \in \mathbb{N}^+$, be a sequence of points of $bB^{n+1}$ such that $p_j \neq p$ for all $j$ and $p_j \to p$ as $j \to \infty$ to a direction tangential to the complex tangent hyperplane at the point $p \in bB^{n+1}$. Let $\varepsilon_j$ be the euclidean distance between the two points $p_j$ and $p$, and $\delta_j$ be the analytic disk

$$\delta_j = \pi_j \cap B^{n+1}$$

where $\pi_j$ is the complex line passing through the two points $p_j$ and $p$. Suppose that there is a sequence $p'_j$ of points of $bB^{n+1}$ satisfying

$$p'_j \to p' \in bB^{n+1},$$

and a sequence of biholomorphic automorphisms $\phi_j \in \text{Aut}(B^{n+1})$ satisfying

$$\phi_j(p'_j) = p_j$$

such that the area $|\phi_j^{-1}(\delta_j)|$ of the analytic disks

$$\phi_j^{-1}(\delta_j)$$

is bounded from the below, i.e., there is a real number $c > 0$ satisfying

$$|\phi_j^{-1}(\delta_j)| \geq c.$$  

Then there is a subsequence $\phi_{m_j}$ and a sequence of local automorphisms $\sigma_j \in \text{Aut}_{p_{m_j}}(bB^{n+1})$ such that

$$U_{\sigma_j} = \text{id}_{n \times n}, \quad \rho_{\sigma_j} = \varepsilon_{m_j}^2, \quad a_{\sigma_j} = 0, \quad r_{\sigma_j} = 0$$

and the composition

$$\sigma_j^{-1} \circ \phi_{m_j} : B^{n+1} \to B^{n+1}$$

uniformly converges to an automorphism of the unit ball $B^{n+1}$.

**Proof.** Note that there is a subsequence $\phi_{m_j}$ which converges to the point $p \in bB^{n+1}$ uniformly on an open neighborhood of the closed ball $\overline{B^{n+1}}$. We take a point $p'' \in bB^{n+1}$ such that $p'' \neq p$ and

$$p'' \notin \{p_{m_j} : j \in \mathbb{N}^+\},$$

if necessary, passing to a subsequence. Then, by Corollary 8, there is a unique decomposition of the automorphism $\phi_{m_j}$ such that

$$\phi_{m_j} = \varphi_j \circ \psi_j$$

where

$$\varphi_j \in \text{Aut}_{p_{m_j}}(bB^{n+1}), \quad \psi_j \in \text{Aut}_{p''}(bB^{n+1})$$

and the local automorphism $\psi_j$ acts trivially on the tangent hyperplane of $bB^{n+1}$ at the fixed point $p'$. 
Let $U_{\psi_j}, \rho_{\psi_j}, a_{\psi_j}, r_{\psi_j}$ be the normalizing parameters of the local automorphism $\psi_j$. Since the local automorphism $\psi_j$ acts trivially on the tangent hyperplane of $bB^{n+1}$ at the fixed point $p''$, we obtain

$$U_{\psi_j} = id_{n \times n}, \quad \rho_{\psi_j} = 1.$$  

Since the sequence $\phi_{m_j}$ uniformly converges to the point $p$, there is a real number $e > 0$ such that

$$|a_{\psi_j}| \leq e, \quad |r_{\psi_j}| \leq e.$$  

Let $\pi_j$ be the complex line passing through the two points $p$ and $p_{m_j}$, and $\delta_j$ be the analytic disks $\delta_j = \pi_j \cap B^{n+1}$. Let $\sigma_j$ be a local automorphism in $\text{Aut}_{p_{m_j}} \left( bB^{n+1} \right)$ defined by the normalizing parameters

$$U_{\sigma_j} = id_{n \times n}, \quad \rho_{\sigma_j} = \frac{e_{m_j}}{2}, \quad a_{\sigma_j} = 0, \quad r_{\sigma_j} = 0.$$  

Since the points $p_j$ converges to the point $p$ to a direction tangential to the complex tangent hyperplane at the point $p \in bB^{n+1}$, the area $|\sigma_{j}^{-1}(\delta_j)|$ of the analytic disk

$$\sigma_{j}^{-1}(\delta_j)$$

is bounded from below.

Let $U_{\varphi_j}, \rho_{\varphi_j}, a_{\varphi_j}, r_{\varphi_j}$ be the normalizing parameters of the local automorphism $\varphi_j$. Note that the analytic disk

$$\phi_{m_j}^{-1}(\delta_j)$$

is mapped by $\sigma_{j}^{-1} \circ \phi_{m_j}$ onto the analytic disk

$$\sigma_{j}^{-1}(\delta_j)$$

where the areas of the analytic disks in both classes are bounded from below. Further, the normalizing parameters $a_{\varphi_j'}, r_{\varphi_j'}$ of the composition $\varphi_j' = \sigma_{j}^{-1} \circ \varphi_j$ is the same value of $a_{\varphi_j}, r_{\varphi_j}$, i.e.,

$$a_{\varphi_j'} = a_{\varphi_j}, \quad r_{\varphi_j'} = r_{\varphi_j}$$

so that there is a real number $e > 0$, if necessary, increasing $e$, such that

$$|a_{\varphi_j}| \leq e.$$  

Since the area of the analytic disks $\sigma_{j}^{-1}(\delta_j)$ is bounded by a positive number, the point $p$ should be attracted to the center of the local automorphism $\text{Aut}_{p_{m_j}} \left( bB^{n+1} \right)$. Hence, by passing to a subsequence, if necessary and increasing $e$, there is a real number $e > 0$ such that

$$|r_{\varphi_j}| \leq e$$  

and

$$\rho_{\varphi_j} \to 0 \quad \text{as} \quad j \to \infty.$$  

Since $bB^{n+1}$ is strongly pseudoconvex, we obtain

$$|U_{\varphi_j}| = 1.$$
We set 
\[ \lambda_j = \frac{\varepsilon_{m_j}}{\rho_{\varphi_j}} \]
so that there is a real number \( e > 0 \) such that
\[ e^{-1} \leq |\lambda_j| \leq e, \] (12)
if necessary, increasing \( e \). Then the composition
\[ \varphi_j' \equiv \sigma_j^{-1} \circ \varphi_j \in \text{Aut}_{p_{m_j}}(bB^{n+1}) \]
has the same normalizing parameters of the mapping \( \varphi_j \) except for \( \rho_{\varphi_j'} = \lambda_j^{-1} \), i.e.,
\[ U_{\varphi_j'} = U_{\varphi_j}, \quad \rho_{\varphi_j'} = \lambda_j^{-1}, \quad a_{\varphi_j'} = a_{\varphi_j}, \quad r_{\varphi_j'} = r_{\varphi_j}. \]
Therefore, by the estimates 9, 10, 11, 12, the sequence
\[ \tau_j \equiv \sigma_j^{-1} \circ \phi_{m_j} : B^{n+1} \to B^{n+1} \]
uniformly converges to an automorphism of \( B^{n+1} \). This completes the proof. \( \Box \)

**Theorem 26.** Let \( D \) be a bounded domain in \( \mathbb{C}^{n+1} \) with spherical real analytic boundary \( bD \). Suppose that there is a biholomorphic mapping \( \phi \) on a connected open neighborhood \( U \) of a point \( p \in bD \) satisfying
\[ \phi(U \cap bD) \subset bB^{n+1}. \]
Then the Doggaebi variety \( L \) associated to the mapping \( \phi \) is a finite subset of \( bB^{n+1} \) such that the inverse mapping \( \phi^{-1} \) is analytically continued along any piecewise chain curve on \( bB^{n+1} \setminus L \) as a locally biholomorphic mapping.

**Proof.** By First Dogginal Lemma, the singular locus of the analytic continuation of the inverse mapping \( \phi^{-1} \) on \( bB^{n+1} \) is an integral manifold of the subdistribution of the complex tangent hyperplanes on \( bB^{n+1} \) in its smooth part. Thus the mapping \( \phi^{-1} \) cannot be branched on \( bB^{n+1} \) by a branching locus passing through the boundary \( bB^{n+1} \). Otherwise, the intersection of the nontrivial branch of the mapping \( \phi^{-1} \) to the boundary \( bB^{n+1} \) would be a nontrivial complex submanifold on \( bB^{n+1} \). Hence the singular locus of the analytic continuation of the mapping \( \phi^{-1} \) is well defined so that the inverse mapping \( \phi^{-1} \) is analytically continued on \( bB^{n+1} \setminus L \) as a locally biholomorphic mapping.

We take a chain-segment \( \gamma : [0, 1] \to bB^{n+1} \) with \( \gamma(1) = p \in L \) such that a germ of the mapping \( \phi^{-1} \) at the point \( \gamma(0) \) is analytically continued along all subpath \( \gamma[0, \tau] \) with \( \tau < 1 \), but not the whole path \( \gamma[0, 1] \). Note that the distribution of the complex tangent hyperplanes on \( bB^{n+1} \) is maximally nonintegrable. Thus, by First Dogginal Lemma, the singular locus \( L \) of the analytic continuation of the mapping \( \phi^{-1} \) cannot bound the open region of the analytic continuation of the mapping \( \phi^{-1} \). Then the mapping \( \phi^{-1} \) is analytically continued on the opposite side of the complex tangent hyperplane \( H_p \) at the point \( p \in bB^{n+1} \). Thus there is a complex line \( \pi \) passing through the point \( p \) and an open neighborhood \( U \) of the point \( p \) such that
\[ \pi \cap bB^{n+1} \]
is a chain on \( bB^{n+1} \) satisfying
\[ L \cap U \cap \pi \cap bB^{n+1} = \{ p \}. \]
We claim that, if necessary, shrinking $U$, 
\[ L \cap U \cap bB^{n+1} = \{ p \} \]
so that the singular locus $L$ is a finite set on $bB^{n+1}$. By First Dogginal Lemma, there is a sequence $p_j \in bB^{n+1} \setminus L$ such that $p_j \to p$ and the sequence $p_j$ converges to the point $p$ to a direction tangential to the complex tangent hyperplane at the point $p \in L \subset bB^{n+1}$. Since $bD$ is compact, there is a subsequence $p_{m_j}$ and a point $p' \in bD$ such that
\[ p_{m_j}' \equiv \phi^{-1}(p_{m_j}) \to p'. \]
Let $\phi_j$ be the germ of the mapping $\phi$ at the point $p_{m_j}' \in bD$ such that
\[ \phi_j(p_{m_j}') = p_{m_j}. \]
Since $bD$ is spherical, there is an open neighborhood $W$ of the point $p'$ and a biholomorphic mapping $\psi$ on $W$ such that
\[ \psi(W \cap bD) \subset bB^{n+1}. \]
Then the compositions $\varphi_j \equiv \phi_j \circ \psi^{-1}$ satisfy
\[ \varphi_j(\psi(W) \cap bB^{n+1}) \subset bB^{n+1}, \]
if necessary, shrinking $W$ so that, by abuse of notation, the mapping $\varphi_j$ is an automorphism of the unit ball $B^{n+1}$.

Let $\pi_j$ be the complex line passing through the points $p_{m_j}$ and $p$, and $\delta_j$ be the analytic disk
\[ \delta_j = \pi_j \cap B^{n+1}. \]
Note that the area $|\varphi_j^{-1}(\delta_j)|$ of the analytic disk $\varphi_j^{-1}(\delta_j)$ is bounded from the below. Otherwise, $p \in bB^{n+1} \setminus L$. Further, the sequence $\varphi_j$ converges to the point $p$ uniformly on every compact subset of $B^{n+1}$, if necessary, passing to a subsequence. Otherwise, the sequence $\varphi_j$ converges to an automorphism of the unit ball $B^{n+1}$ so that $p \in bB^{n+1} \setminus L$. Let $\varepsilon_j$ be the euclidean length between the two points $p$ and $p_{m_j}$. Then, by Second Scaling Lemma, there is a sequence of local automorphisms $\sigma_j \in Aut_{p_{m_j}}(bB^{n+1})$ such that
\[ U_{\sigma_j} = id_{n \times n}, \quad \rho_{\sigma_j} = \varepsilon_j^2, \quad a_{\sigma_j} = 0, \quad r_{\sigma_j} = 0 \]
and the composition
\[ \tau_j \equiv \sigma_j^{-1} \circ \varphi_j : B^{n+1} \to B^{n+1} \]
uniformly converges to an automorphism of the unit ball $B^{n+1}$. Thus there is a positive real number $\delta > 0$ such that the mapping $\tau_j$ and its inverse $\tau_j^{-1}$ are analytically continued respectively on
\[ B(p_{m_j}', \delta) \quad \text{and} \quad B(p_{m_j}; \delta) \]
where
\[ p_{m_j}' = \psi(p_{m_j}). \]
Therefore the mapping \( \phi^{-1} \) at the point \( p_{mj} \):
\[
\phi^{-1} = \phi^{-1}_j = \psi^{-1}_j \circ \tau^{-1}_j \circ \sigma^{-1}_j
\]
is analytically continued to the open neighborhood
\[
\sigma_j \left(B \left(p_{mj} ; \delta \right)\right).
\]
By the canonical normalizing mapping \( \mu_{p_{mj}} \), we obtain
\[
\sigma'_j \equiv \mu_{p_{mj}} \circ \sigma_j \circ \mu^{-1}_{p_{mj}} : \begin{cases}
z^* = \varepsilon_j z \\
w^* = \varepsilon_j^2 w
\end{cases}.
\]
Hence the mapping \( \phi^{-1} \) at the point \( p_{mj} \) is analytically continued to the open neighborhood
\[
\mu^{-1}_{p_{mj}} \circ \sigma'_j \left(B (0; \delta)\right),
\]
if necessary, shrinking \( \delta \). Since \( \varepsilon_j \) is the euclidean length between the two points \( p \) and \( p_{mj} \), the mapping \( \phi^{-1} \) is analytically continued on an open region which touches to the point \( p \) to a converging direction to the sequence \( p_j \) by an edge shape tangential to the complex tangent hyperplane \( H_p \) at the point \( p \) and a \( \sqrt{|x|} \) curve shape normal to \( H_p \) on \( bB^{n+1} \).

Therefore, the singular locus \( L \) of the analytic continuation of the mapping \( \phi^{-1} \) is isolated to the direction of the complex tangent hyperplane at each point of \( L \). Since the boundary \( bB^{n+1} \) is compact, the singular locus \( L \) is a finite subset of \( bB^{n+1} \). This completes the proof.

**Lemma 27.** Let \( D \) be a bounded domain in \( \mathbb{C}^{n+1} \) with spherical real analytic boundary \( bD \). Then every chain on \( bD \) is extended each direction infinitely in its euclidean length.

**Proof.** Suppose that the assertion is not true. Then there would be a path \( \gamma : [0, 1] \to bD \) such that the subpath \( \gamma[0, \tau] \) is a chain-segment for all \( \tau < 1 \) but the whole path \( \gamma[0, 1] \) is not a chain segment.

Since \( bD \) is spherical, by definition, there exist a connected open neighborhood \( U \) of the point \( \gamma(1) \) and a biholomorphic mapping \( \phi \) on \( U \) such that \( \phi(U \cap bD) \subset bB^{n+1} \). There is a unique closed circle \( \chi \) on \( bB^{n+1} \) such that \( \chi \) is a chain on \( bB^{n+1} \) and \( \phi \circ \gamma[0, \tau] \subset \chi \) for all \( \tau < 1 \)(cf. [Pa3]).

Then the inverse image \( \phi^{-1} (\chi \cap \phi(U)) \) under the biholomorphic mapping \( \phi \) is a chain on \( bB^{n+1} \)(cf. [Pa3]) such that
\[
\gamma[0, 1] \cap U \subset \phi^{-1} (\chi \cap \phi(U)) .
\]

This is a contradiction. This completes the proof.

**Theorem 28.** Let \( D \) be a bounded domain in \( \mathbb{C}^{n+1} \) with spherical real analytic boundary \( bD \). Suppose that there is a biholomorphic mapping \( \phi \) on a connected open neighborhood \( U \) of a point \( p \in bD \) satisfying
\[
\phi(U \cap bD) \subset bB^{n+1}.
\]
Let \( E \) be the path space of \( bD \) pointed at the point \( p \in bD \) mod homotopic relation so that \( E \) is a universal covering of \( bD \) with a natural CR structure and a natural
CR covering map $\psi : E \to bD$. Then there is a unique CR equivalence $\varphi : E \to bB^{n+1} \setminus L$ commuting the diagram
\[
\begin{array}{c}
E \\
\downarrow \psi \\
bD \\
\overset{\psi}{\rightarrow} \\
\phi \\
bB^{n+1} \setminus L
\end{array}
\]
where $L \subset bB^{n+1}$ is the Doggaebi variety associated to the mapping $\phi$.

Proof. We obtain the set $bB^{n+1} \setminus L$ when the mapping $\phi$ is analytically continued along chains on $bD$. Since the mapping $\varphi : E \to bB^{n+1}$ is naturally induced by the analytic continuation of the mapping $\phi$ on $bD$ (cf. Theorem 18), we obtain
\[
bB^{n+1} \setminus L \subset \varphi(E)
\]
Let $\gamma : [0, 1] \to bD$ be a path such that $\gamma(0) = p$. Then, for a sufficiently small $\varepsilon > 0$, there is an $\varepsilon$ open neighborhood $V$ of the path $\gamma[0, 1]$ such that the mapping $\phi$ is analytically continued on $V$ as a local biholomorphic mapping. Then we take a piecewise chain curve $\eta : [0, 1] \to bD$ (cf. [Pa3]) such that
\[
\eta(0) = \gamma(0), \quad \eta(1) = \gamma(1)
\]
and
\[
\eta[0, 1] \subset V.
\]
Further, we may take a continuous function $\Gamma : [0, 1] \times [0, 1] \to V$ such that
\[
\Gamma(0, \tau) = \gamma(\tau), \quad \Gamma(1, \tau) = \eta(\tau) \quad \text{for } \tau \in [0, 1]
\]
and $\Gamma(\cdot, \tau) : [0, 1] \to V$ is a chain-segment for all $\tau \in [0, 1]$. Note that every chain-segment in this construction is finite in its euclidean length. By Lemma 27, the mapping $\phi$ is analytically continued along the whole path $\gamma[0, 1]$.

Then the image of $E$ under the mapping $\varphi$ satisfies
\[
\varphi(E) \subset bB^{n+1} \setminus L
\]
which yields
\[
\varphi(E) = bB^{n+1} \setminus L.
\]
Since the mapping $\phi^{-1}$ is analytically continued on $bB^{n+1} \setminus L$, the mapping $\varphi : E \to bB^{n+1}$ is a CR equivalence. This completes the proof.

Corollary 29. Let $D$ be a bounded domain with spherical real analytic boundary $bD$. Then there is a natural embedding
\[
\text{Aut}(D) \subset \text{Aut}(b^n + 1).
\]
Proof. Let $E$ be the path space of $bD$ pointed at the point $p \in bD$ mod homotopy so that $E$ be a universal covering of $bD$ with a natural CR structure and a natural CR covering map $\psi : E \to bD$. Then we take a biholomorphic mapping $\phi$ on an open neighborhood $U$ of a point $p \in bD$ such that
\[
\phi(U \cap bD) \subset bB^{n+1}.
\]
Let $L$ be the Doggaebi variety associated to the mapping $\phi$. Then, by Theorem 28, the analytic continuation of the mapping $\phi$ yields a natural CR equivalence
\[
\text{Aut}(E) \simeq \text{Aut}(bB^{n+1} \setminus L).
\]
Note that
\[ \text{Aut}(D) = \text{Aut}(bD) \supset \text{Aut}(E) \simeq \text{Aut}(bB^{n+1}\setminus L) \subset \text{Aut}(bB^{n+1}) = \text{Aut}(B^{n+1}). \]
This completes the proof. \( \square \)

**Lemma 30.** Let \( D, D' \) be bounded domains in \( \mathbb{C}^{n+1} \) with spherical real analytic boundaries \( bD, bD' \), and \( \phi : D \to D' \) be a proper holomorphic mapping. Suppose that there is an open neighborhood \( U \) of a point \( p \in bD \) such that the mapping \( \phi \) is analytically continued on \( U \). Then the mapping \( \phi \) is analytically continued along any path on \( bD \) as a locally biholomorphic mapping so that \( \phi : D \to D' \) is locally biholomorphic.

**Proof.** We may obtain the result by the boundary regularity of Lemma 16 and the transformation formula for a complex Monge-Ampere equation. Here we may give an independent proof. Note that
\[ \phi(U \cap bD) \subset bD' \]
so that, by shrinking \( U \), if necessary, the mapping \( \phi \) is biholomorphic on \( U \). By shrinking \( U \), if necessary, we take biholomorphic mappings \( \phi, \phi' \) respectively on \( U, \phi(U) \) such that
\[ \phi(U \cap bD) \subset bB^{n+1}, \]
\[ \phi'(U \cap bD') \subset bB^{n+1}. \]
Then there are Doggaebi varieties \( L, L' \) on \( bB^{n+1} \) such that the inverses \( \varphi^{-1}, \varphi'^{-1} \) are analytically continued respectively on \( bB^{n+1}\setminus L \) and \( bB^{n+1}\setminus L' \) as a local biholomorphic mapping. Note that the composition \( \psi = \varphi' \circ \phi \circ \varphi^{-1} \) satisfies the relation
\[ \psi \left( \varphi(U) \cap bB^{n+1} \right) \subset bB^{n+1}. \]
By abuse of notation, the mapping \( \psi \) is an automorphism of \( bB^{n+1} \) which comments the following diagram:
\[
\begin{array}{ccc}
bB^{n+1}\setminus L & \psi \downarrow & bB^{n+1}\setminus L' \\
\downarrow \kappa & & \downarrow \kappa' \\
bD & \phi \downarrow & bD' \\
\end{array}
\]
where \( \kappa : bB^{n+1}\setminus L \to bD \) is the analytic continuation of the mapping \( \varphi^{-1} \) and \( \kappa' : bB^{n+1}\setminus L' \to bD' \) is the analytic continuation of the mapping \( \varphi'^{-1} \) on the boundary \( bB^{n+1} \).

We claim
\[ L' \subset \psi(L) \]
so that the mapping \( \phi \) is analytically continued along any path on \( bD \) as a locally biholomorphic mapping. Otherwise, there is a point \( q' \in L' \) such that
\[ q' \in \psi(L), \]
From the proof of First Scaling Lemma, there is a sequence \( p'_j \in bD' \) with \( p'_j \to p' \in bD' \) and a sequence \( \varphi'_j \) of the analytic continuation of the mapping \( \varphi' \) at the point \( p'_j \) with \( q'_j \equiv \varphi'_j(p'_j) \to q' \in \psi'(L) \) such that there is an open neighborhood \( V \) of the point \( p' \) such that the mapping \( \varphi'_j \) is biholomorphic on \( V \cap D' \) and the sequence
\[
\begin{array}{ccc}
bB^{n+1}\setminus L & \psi \downarrow & bB^{n+1}\setminus L' \\
\downarrow \kappa & & \downarrow \kappa' \\
bD & \phi \downarrow & bD' \\
\end{array}
\]
$\varphi_j'$ converges to the point $r$ uniformly on every compact subset of $V \cap D'$. Since $\psi$ is an automorphism of the unit ball $B^{n+1}$, we obtain

$$\lambda_j \equiv \kappa \circ \psi^{-1} \circ \varphi_j' : V \cap D' \to D$$

and

$$\phi \circ \lambda_j = id \quad \text{on} \ V \cap D'.$$

We set

$$p_j \equiv \lambda_j (p_j') \in bD.$$

Since $bD$ is compact, there is a subsequence $p_{m_j} \in bD$ and a point $p \in bD$ such that

$$p_{m_j} \to p.$$

Since $\phi$ is a proper mapping, the mapping $\phi$ is a globally finite covering so that there is an open neighborhood $W$ of the point $p$ such that

$$\lambda_{m_j} = \phi^{-1} \quad \text{on} \ \phi(W) \cap D.$$

Hence we obtain

$$\psi = \varphi_{m_j}' \circ \phi \circ \kappa.$$

Since the mapping $\psi$ is an automorphism of the unit ball $B^{n+1}$, there is a sequence $q_j \in bB^{n+1}$ such that

$$\psi (q_j) = q_j'.$$

Hence we obtain

$$\kappa (q_{m_j}) = p_{m_j}.$$

Note that there is a real number $\delta > 0$ such that

$$B (r; 2\delta) \cap bB^{n+1} \cap \psi (L) = \emptyset.$$

Passing to a subsequence, if necessary, we may assume

$$q_{m_j} \in \psi^{-1} (B (r; \delta)) \cap bB^{n+1}$$

so that there is a point $q \in bB^{n+1} \setminus L$ satisfying

$$q_{m_j} \to q.$$

Hence we set

$$\chi_{m_j} \equiv \psi^{-1} \circ \varphi_{m_j}' \circ \lambda_{m_j}^{-1}$$

and

$$\kappa \circ \chi_{m_j} = id.$$

Finally, we obtain

$$\varphi_{m_j}' = \psi \circ \chi_{m_j} \circ \lambda_{m_j}.$$

Note that there is a real number $\delta > 0$ that the mappings $\lambda_{m_j}$ and the inverse mappings $\lambda_{m_j}^{-1}$ are analytically continued respectively on

$$B (p_{m_j}'; \delta) \quad \text{and} \quad B (p_{m_j}; \delta),$$
and the mappings \( \chi_{m_j} \) and the inverse mappings \( \chi_{m_j}^{-1} \) are analytically continued respectively on
\[
B \left( p_{m_j}; \delta \right) \quad \text{and} \quad B \left( q_{m_j}; \delta \right),
\]
and the mapping \( \psi \) and the inverse mapping \( \psi^{-1} \) are analytically continued respectively on
\[
B \left( q_{m_j}; \delta \right) \quad \text{and} \quad B \left( q'_{m_j}; \delta \right),
\]
Hence the mapping \( \varphi'_{m_j} \) and the inverse mappings \( \varphi'_{m_j}^{-1} = \kappa' \) are analytically continued respectively on
\[
B \left( p'_{m_j}; \delta \right) \quad \text{and} \quad B \left( q'_{m_j}; \delta \right),
\]
as a locally biholomorphic mapping. Since \( q'_{m_j} \to q' \), the mapping \( \kappa' \) is analytically continued to the point \( q' \) as a locally biholomorphic mapping so that
\[
q' \in bB^{n+1} \setminus L'.
\]
This is a contradiction so that
\[
L' \subset \psi(L).
\]
This completes the proof. \( \square \)

2. Bounded Domains with Spherical Boundaries

2.1. Differentiable spherical boundary. Let \( D \) be a domain in \( \mathbb{C}^{n+1}, n \geq 1 \), with a differentiable boundary \( bD \). The boundary \( bD \) shall be called spherical if, for each point \( p \in bD \), there is a connected open neighborhood \( U \) of the point \( p \) and a biholomorphic mapping \( \phi \) on \( U \cap D \) such that
\[
(13) \quad \phi \in H \left( U \cap D \right) \cap C^1 \left( U \cap \overline{D} \right), \quad \phi \left( U \cap bD \right) \subset bB^{n+1}
\]
and the induced mapping \( \phi : U \cap bD \to bB^{n+1} \) is CR diffeomorphic.

Lemma 31. Let \( U \) be a connected open neighborhood of a point \( p \in bB^{n+1} \) and \( \phi \) be a biholomorphic mapping on \( U \cap B^{n+1} \) such that
\[
\phi \in H \left( U \cap B^{n+1} \right) \cap C^1 \left( U \cap \overline{B^{n+1}} \right), \quad \phi \left( U \cap bB^{n+1} \right) \subset bB^{n+1}
\]
and the induced mapping \( \phi : U \cap bB^{n+1} \to bB^{n+1} \) is CR diffeomorphic. Then the mapping \( \phi \) is analytically continued to an automorphism of the unit ball \( B^{n+1} \).

Proof. By Lemma 13, the mapping \( \phi \) is biholomorphic on \( U \), if necessary, shrinking \( U \). Then, by Lemma 17, we obtain the desired result. This completes the proof. \( \square \)

The chain on \( bB^{n+1} \) is defined to be the intersection on \( bB^{n+1} \) by a complex line. The family of chains on \( bB^{n+1} \) leaves invariant under the action of biholomorphic automorphisms of \( B^{n+1} \). We define the chain on the spherical differentiable boundary \( bD \) of a domain \( D \) to be the inverse image of the chain on \( bB^{n+1} \) under the mapping \( \overline{D} \). By Lemma 31, the chain on the spherical differentiable boundary \( bD \) is well defined so that a chain of a spherical differentiable boundary is mapped to a chain of another spherical differentiable boundary under any induced CR diffeomorphism.
Lemma 32. Let $D$ be a domain in $\mathbb{C}^{n+1}$ with spherical differentiable boundary $bD$. Suppose that there is a connected open neighborhood $U$ of a point $p \in bD$ and a biholomorphic mapping $\phi$ on $U \cap D$ such that

$$\phi \in H \left( U \cap D \right) \cap C^1 \left( U \cap \overline{D} \right), \quad \phi(U \cap bD) \subset bB^{n+1}.$$ 

Then the mapping $\phi$ is analytically continued along any path on $bD$ as a local biholomorphic mapping.

Proof. Suppose that the assertion is not true. Then there would be a path $\gamma : [0, 1] \to bD$ such that $\gamma(0) \in U \cap bD$ and the germ of a biholomorphic mapping $\phi$ at the point $\gamma(0)$ is analytically continued along the subpath $[0, \tau]$ with all $\tau < 1$ as a local biholomorphic mapping, but not the whole path $\gamma[0, 1]$.

Thus we reduce the proof to a local problem near the point $\gamma(1) \in bD$. Then, by the definition, there is a connected open neighborhood $V$ of the point $\gamma(1)$ and a biholomorphic mapping $\varphi$ on $V \cap D$ such that

$$\varphi \in H \left( V \cap D \right) \cap C^1 \left( V \cap \overline{D} \right), \quad \varphi(V \cap bD) \subset bB^{n+1}$$

and the induced mapping $\varphi : U \cap bD \to bB^{n+1}$ is CR diffeomorphic.

Then we consider the mapping

$$\phi \circ \varphi^{-1} \in H \left( \varphi(V \cap D) \right) \cap C^1 \left( \varphi(V \cap \overline{D}) \right), \quad \phi \circ \varphi^{-1}(\varphi(V \cap bD)) \subset bB^{n+1}$$

and the curve $\varphi \circ \gamma[0, 1] \cap \varphi(V \cap bD) \subset bB^{n+1}$.

By Lemma 33, the remaining part of the proof repeats the proof of Lemma 32. This completes the proof.

Lemma 33 (First Dogginal Lemma). Let $D$ be a bounded domain in $\mathbb{C}^{n+1}$ with spherical differentiable boundary $bD$ and $\phi$ be a biholomorphic mapping on $U \cap D$ for a connected open neighborhood $U$ of a point $p \in bB^{n+1}$ satisfying

$$\phi \in H \left( U \cap B^{n+1} \right) \cap C^1 \left( U \cap \overline{B^{n+1}} \right), \quad \phi(U \cap bB^{n+1}) \subset bD.$$ 

Suppose that there is an injective path $\gamma : [0, 1] \to bB^{n+1}$ such that $\gamma[0, 1] \subset bB^{n+1}$ is a chain-segment satisfying

$$\gamma(0) \in U \cap bB^{n+1}$$

and the mapping $\phi$ is analytically continued along the subpath $\gamma[0, \tau]$ for all $\tau < 1$, but not the whole path $\gamma[0, 1]$ as a local biholomorphic mapping. Let $\pi$ be the complex line containing the chain-segment $\gamma[0, 1]$. Then there is an open neighborhood $V$ along the path $\gamma[0, 1]$ such that

1. $\gamma[0, \tau] \subset V$ for all $\tau < 1$,
2. $bV \cap \pi \cap B(\gamma(1) ; \delta)$ is an angle for a sufficient small $\delta > 0$, which contains the chain-segment $\gamma[0, 1]$,
3. $bV \cap bB^{n+1} \cap B(\gamma(1) ; \delta)$ is paraboloid for a sufficiently small $\delta > 0$, which smoothly touches the complex tangent hyperplane at the point $\gamma(1)$,
4. the mapping $\phi$ is analytically continued on $V \cap bB^{n+1}$ as a local biholomorphic mapping

$$\phi \in H(V \cap B^{n+1}) \cap C^1 \left( V \cap \overline{B^{n+1}} \right).$$
Proof. By the analytic continuation of the mapping $\phi$ along the subpath $\gamma[0, \tau]$ for all $\tau < 1$, there is a path $\phi \circ \gamma : [0, 1) \to bD$. Then we consider the following sequences
\[
p_j = \gamma \left( 1 - \frac{1}{j} \right), \quad \text{for } j \in \mathbb{N}^+,
\]
\[
p_j' = \phi \circ \gamma \left( 1 - \frac{1}{j} \right), \quad \text{for } j \in \mathbb{N}^+.
\]
Since $bD$ is compact, there is a subsequence $p_{m_j}'$ and a point $p' \in bD$ such that $p_{m_j}' \to p'$.

Thus we reduce the proof to a local problem near the point $p' \in bD$. Then, by the definition, there is a connected open neighborhood $W$ of the point $p'$ and a biholomorphic mapping $\phi$ on $W \cap D$ such that
\[
\phi \in H (W \cap D) \cap C^1 (W \cap \overline{D}), \quad \phi (W \cap bD) \subset bB^{n+1}
\]
and the induced mapping $\phi : U \cap bD \to bB^{n+1}$ is CR diffeomorphic.

Then we consider the mapping
\[
\phi^{-1} \circ \phi^{-1} \in H (\phi (W \cap D)) \cap C^1 (\phi (W \cap \overline{D})), \quad \phi^{-1} \circ \phi^{-1} (\phi (W \cap bD)) \subset bB^{n+1}.
\]
By Lemma 33, the remaining part of the proof repeats the proof of Lemma 24. This completes the proof.

By Lemma 31 and Lemma 33, we obtain the following result by the same argument of the proof of Theorem 28.

**Theorem 34.** Let $D$ be a bounded domain in $\mathbb{C}^{n+1}$ with spherical differentiable boundary $bD$. Suppose that there is a biholomorphic mapping $\phi$ on $U \cap D$ for a connected open neighborhood $U$ of a point $p \in bD$ satisfying
\[
\phi (U \cap bD) \subset bB^{n+1}.
\]
Let $E$ be the path space of $bD$ pointed at the point $p \in bD$ mod homotopy so that $E$ be a universal covering of $bD$ with a natural CR structure and a natural CR covering map $\psi : E \to bD$. Then there is a unique CR equivalence $\varphi : E \to bB^{n+1} \setminus L$ commuting the diagram
\[
\begin{array}{ccc}
E & \xrightarrow{\varphi} & bB^{n+1} \setminus L \\
\downarrow \psi & & \downarrow \phi \\
bD & \xrightarrow{\phi} & bB^{n+1} \setminus L
\end{array}
\]
where $L \subset bB^{n+1}$ is the Doggaebi variety associated to the mapping $\phi$.

2.2. **Second Dogginal Lemma.** We have examined the analytic continuation of a biholomorphic mapping $\phi$ from the spherical differentiable boundary $bD$ of a bounded domain $D$ into the boundary $bB^{n+1}$ of the unit ball $B^{n+1}$. From now on, we shall examine the analytic continuation of the mapping $\phi$ into the domain $D$.

**Lemma 35.** Let $D$ be a bounded domain in $\mathbb{C}^{n+1}$ with spherical differentiable boundary $bD$. Suppose that there is a biholomorphic mapping $\phi$ on a connected open neighborhood $U$ of a point $p \in bD$ satisfying
\[
\phi (U \cap bD) \subset bB^{n+1}.
\]
Then the inverse mapping $\phi^{-1}$ is analytically continued to a locally biholomorphic mapping from $B^{n+1}$ into $D$.

**Proof.** By Theorem 34 there is a Doggaebi variety $L$ on $bB^{n+1}$ such that the mapping $\phi^{-1}$ is uniquely analytically continued on the set $bB^{n+1}\setminus L$. Then the mapping $\phi^{-1}$ is analytically continued to a holomorphic mapping on the unit ball $B^{n+1}$. Hence we have a holomorphic mapping $\varphi : B^{n+1} \to D$ which extends to the mapping $\phi^{-1} : bB^{n+1}\setminus L \to bD$. Note that the zero set of the determinant of the Jacobian matrix $\varphi'$ of the mapping $\varphi$ cannot be located on the set $L$ on the boundary $bB^{n+1}$. Thus the mapping $\varphi : B^{n+1} \to D$ is locally biholomorphic. This completes the proof.

**Lemma 36.** Let $D$ be a bounded domain in $\mathbb{C}^{n+1}$ with spherical differentiable boundary $bD$ and $\phi$ be a biholomorphic mapping on an open neighborhood $U$ of a point $r \in bD$ satisfying

$$\phi \in H(U \cap D) \cap C^1(U \cap \overline{D}) \quad \text{and} \quad \phi(U \cap bD) \subset bB^{n+1}.$$

Let $L$ be the Doggaebi variety associated to the mapping $\phi$ and $\varphi : B^{n+1} \to D$ be a locally biholomorphic mapping to be an analytic continuation of the mapping $\phi^{-1}$. Suppose that there is a line segment $\gamma : [0, 1] \to D$ with

$$p = \gamma(1) \quad \text{and} \quad p_j = \gamma \left( 1 - \frac{1}{j} \right)$$

and the germ of a locally biholomorphic mapping $\phi = \varphi^{-1}$ at the point $p_1 \equiv \gamma(0)$ is analytically continued along the subpath $\gamma[0, \tau]$ with all $\tau < 1$ as a locally biholomorphic mapping. Let $p'_j \equiv \phi(p_j) \in B^{n+1}$ be the sequence obtained by the analytic continuation of the mapping $\phi = \varphi^{-1}$ along the path $\gamma[0, 1)$. Then there is a subsequence $p'_{m_j}$ and a point $p' \in L \subset bB^{n+1}$ such that the subsequence $p'_{m_j}$ converges to the point $p'$ to a direction transversal to the complex tangent hyperplane at the point $p' \in bB^{n+1}$.

**Proof.** Since the closed ball $\overline{B^{n+1}}$ is compact, there is a subsequence $p'_{m_j}$ which converges to a point $p' \in B^{n+1}$. Since the mapping $\varphi$ is locally biholomorphic on $B^{n+1}$, the point $p'$ should be on the boundary $bB^{n+1}$. Further, since $\varphi = \phi^{-1} : bB^{n+1}\setminus L \to bD$ is locally biholomorphic, we obtain

$$p'_{m_j} \to p' \in L.$$

Let $\pi_p$ be the complex line containing the line segment $\gamma[0, 1]$. Since $p'_{m_j} \equiv \phi(p_{m_j})$, the analytic continuation of the mapping $\phi = \varphi^{-1}$ on the complex line $\pi_p$ yields a complex curve $\varphi^{-1}(\pi_p \cap D)$ which touches the point $p' \in L \subset bB^{n+1}$. Since the mapping $\varphi : B^{n+1} \to D$ is locally biholomorphic up to the boundary subset $bB^{n+1}\setminus L$ and the boundary $bB^{n+1}$ is strongly pseudoconvex, the extension

$$\varphi^{-1}(\pi_p \cap \overline{D})$$

of the complex curve $\varphi^{-1}(\pi_p \cap D)$ is transversal to the boundary $bB^{n+1}$ near the point $p' \in L \subset bB^{n+1}$.

We claim that the extension $\varphi^{-1}(\pi_p \cap \overline{D})$ is transversal to the complex tangent hyperplane at the point $p' \in L$. We take a path $\lambda : [0, 1] \to bB^{n+1}$ such that
$p' = \lambda(1)$ and
\[ \lambda[0, 1] \subset \varphi^{-1}(\pi_p \cap bD) \cap bB^{n+1}. \]

Note that there are finitely many closed paths $\gamma_j$ on $bD$ such that
\[ \pi_p \cap bD = \bigcup_j \gamma_j. \]

Since $bD$ is strongly pseudoconvex, each path $\gamma_j$ is transversal to the complex tangent hyperplane on $bD$ at each point.

We consider the following sequence
\[ q_j = \varphi \circ \lambda \left( 1 - \frac{1}{j} \right) \in \pi_p \cap bD, \quad q'_j = \lambda \left( 1 - \frac{1}{j} \right) \quad \text{for} \quad j \in \mathbb{N}^+ \]
such that
\[ q'_j \to p' \quad \text{as} \quad j \to \infty. \]

Since the set $\pi_p \cap bD$ is compact, there is a subsequence $q_{m_j}$ and a point $q \in \pi_p \cap bD$ such that
\[ q_{m_j} \to q \quad \text{as} \quad j \to \infty. \]

We consider the following sequence
\[ q'_j = \varphi \circ \lambda \left( 1 - \frac{1}{j} \right) \in \pi_p \cap bD, \quad q''_j = \lambda \left( 1 - \frac{1}{j} \right) \quad \text{for} \quad j \in \mathbb{N}^+ \]
such that
\[ q''_j \to q' \quad \text{as} \quad j \to \infty. \]

Let $\phi_{m_j}$ be the germ of the analytic continuation of the mapping $\phi = \varphi^{-1}$ at the point $q_{m_j}$. By Theorem 18 and Lemma 14, we may assume that there is an open neighborhood $W$ of the point $q$ such that the mapping $\phi_{m_j}$ is holomorphic on $W \cap bD$.

Since $bD$ is spherical, there is a biholomorphic mapping $\psi$ on $W$, if necessary, shrinking $W$, such that
\[ \psi \in H(W \cap bD) \cap C^1(W \cap bD) \quad \text{and} \quad \psi(W \cap bD) \subset bB^{n+1}. \]

We may assume that
\[ q'' = \psi(q) \neq p' \quad \text{and} \quad q'''_{m_j} = \psi(q_{m_j}). \]

Then, by Lemma 17, the composition
\[ \eta_j \equiv \phi_{m_j} \circ \psi^{-1} \in \text{Aut}(B^{n+1}) \]
are automorphisms of the unit ball $B^{n+1}$. Note that we have the condition
\[ \eta_j \left( q''_{m_j} \right) = q'''_{m_j} \to p' \]
and the sequence $\eta_j$ converges to the point $p'$ uniformly on every compact subset of $B^{n+1}$, if necessary, passing to a subsequence. Otherwise, $p' \notin L$.

Let $\pi'_j$ be the complex line passing through the two points $p'$ and $q'''_{m_j}$, and $\delta'_j$ be the analytic disk
\[ \delta'_j = \pi'_j \cap B^{n+1}. \]

The area $|\eta_j^{-1}(\delta'_j)|$ of the analytic disk $\eta_j^{-1}(\delta'_j)$ is bounded from the below. Otherwise, we would have $p' \notin L$. Then we decompose the mapping $\eta_j$ as follows
\[ \eta_j = \mu_j \circ \nu_j \]
where
\[ \mu_j \in \text{Aut}_{q'''_{m_j}}(bB^{n+1}), \quad \nu_j \in \text{Aut}_{p'}(bB^{n+1}). \]
Then, by First and Second Scaling Lemmas, we take a sequence of local automorphisms \( \sigma_j \in Aut_{q_{m_j}} (bB^{n+1}) \) defined by the normalizing parameters
\[
U_{\sigma_j} = id_{n \times n}, \quad \rho_{\sigma_j} = \rho_{\mu_j}, \quad a_{\sigma_j} = 0, \quad r_{\sigma_j} = 0.
\]
Then the following sequence
\[
\beta_j = \sigma_j^{-1} \circ \eta_j
\]
converges to an automorphism of the unit ball \( B^{n+1} \), if necessary, passing to a subsequence.

Let \( \pi_j \) be the complex line containing the tangent vector of the path
\[
\psi (\varphi \circ \lambda [0, 1] \cap W) \subset bB^{n+1}
\]
at the point \( q_{m_j}' \), and \( \delta_j \) be the analytic disk
\[
\delta_j = \pi_j \cap B^{n+1}.
\]
Clearly, the area \( |\delta_j| \) of the analytic disk \( \delta_j \) is bounded from the below. Note that, by the mapping \( \sigma_j \in Aut_{q_{m_j}} (bB^{n+1}) \), the area \( |\sigma_j^{-1}(\delta_j')| \) of the analytic disk \( \sigma_j(\delta_j') \) is bounded from the below whenever the area \( |\delta_j'| \) of the analytic disk \( \delta_j' \) is bounded from the below. Thus the area \( |\eta_j(\delta_j)| \) of the analytic disk
\[
\eta_j (\delta_j) = \sigma_j \circ \beta_j (\delta_j)
\]
is bounded from the below. Since the analytic disk \( \eta_j (\delta_j) \) is the intersection of \( B^{n+1} \) and the complex line containing the tangent vector of the path \( \lambda [0, 1] \) at the point
\[
q_{m_j}' = \lambda \left( 1 - \frac{1}{m_j} \right).
\]
Thus the path \( \lambda [0, 1] \) is transversal to the complex tangent hyperplane at the point \( p' = \lambda (1) \). Therefore, the point \( p_j' \) approaches to the point \( p' = L \) to a direction transversal to the complex tangent hyperplane at the point \( p' \in bB^{n+1} \). This completes the proof. \( \square \)

**Lemma 37** (Second Dogginal Lemma). Let \( D \) be a bounded domain in \( \mathbb{C}^{n+1} \) with spherical differentiable boundary \( bD \) and \( \phi \) be a biholomorphic mapping on an open neighborhood \( U \) of a point \( r \in bD \) satisfying
\[
\phi \in H (U \cap D) \cap C^1 (U \cap \overline{D}) \quad \text{and} \quad \phi (U \cap bD) \subset bB^{n+1}.
\]
Let \( L \) be the Doggaebi variety associated to the mapping \( \phi \) and \( \varphi : B^{n+1} \to D \) be a locally biholomorphic mapping to be an analytic continuation of the mapping \( \phi^{-1} \). Suppose that there is a line segment \( \gamma : [0, 1] \to D \) with
\[
p = \gamma (1) \quad \text{and} \quad p_j = \gamma \left( 1 - \frac{1}{j} \right)
\]
and the germ of a locally biholomorphic mapping \( \phi = \varphi^{-1} \) at the point \( p_{i} \equiv \gamma (0) \) is analytically continued along the subpath \( \gamma [0, \tau] \) with all \( \tau < 1 \) as a locally biholomorphic mapping. Let \( p_j' \equiv \phi (p_j) \in B^{n+1} \) be the sequence obtained by the analytic continuation of the mapping \( \phi = \varphi^{-1} \) along the path \( \gamma [0, 1] \). Suppose that there is a point \( p' \in L \subset bB^{n+1} \) such that the sequence \( p_j' \) converges to the point \( p' \) to a direction transversal to the complex tangent hyperplane at the point \( p' \in bB^{n+1} \).
Let $\pi_p$ be the complex line containing the line segment $\gamma[0,1]$. Then there is a distinguished complex hyperplane $H_p \subset T_p \mathbb{C}^{n+1}$ at the point $p \in D$ satisfying
$$T_p \mathbb{C}^{n+1} = H_p \oplus \pi_p$$
and an open neighborhood $V$ of the line segment $\gamma[0,1)$ such that
1. $bV \cap \pi_p \cap B(p;\delta)$ is an angle for a sufficient small $\delta > 0$, which contains the path $\gamma[0,1]$,
2. $bV \cap R_p \cap B(p;\delta)$ is a slanted paraboloid for a sufficiently small $\delta > 0$, which smoothly touches the complex hyperplane $H_p$ at the point $p$,
3. the germ of a locally biholomorphic mapping $\phi = \varphi^{-1}$ at the point $p_1 \equiv \gamma(0)$ is analytically continued on $V$ as a locally biholomorphic mapping, where $R_p$ is a real hyperplane containing the line segment $\gamma[0,1]$ and $H_p \subset R_p$.

Proof. We take a sequence of automorphisms $\phi_j \in \text{Aut} \left( B^{n+1} \right)$ such that
$$\phi_j (p_1) = p'_j \rightarrow p' \in L.$$ Then the composition $\tau_j \equiv \varphi \circ \phi_j : B^{n+1} \rightarrow D$ forms a normal family so that there is a subsequence $\tau_{m_j} \equiv \varphi \circ \phi_{m_j}$ which converges to a holomorphic mapping $\tau : B^{n+1} \rightarrow D$ uniformly on every compact subset of $B^{n+1}$.

We claim that $\tau$ is a locally biholomorphic mapping. We take a point $q \in bB^{n+1}\setminus L$ and $q'_{m_j} = \tau_{m_j} (q) \in bD$ such that
$$q'_{m_j} = \tau_{m_j} (q) \rightarrow q' \in bD,$$
if necessary, passing to a subsequence. Since $bD$ is spherical, there is an open neighborhood $W$ of the point $q'$ and a biholomorphic mapping $\psi$ on $W \cap bD$ such
$$\psi \in H (W \cap D) \cap C^1 (W \cap \overline{D}) \quad \text{and} \quad \psi (W \cap bD) \subset bB^{n+1}.$$ Then we consider the composition $\chi_{m_j} \equiv \psi \circ \varphi \circ \phi_{m_j}$ so that, by Lemma [17] and by abuse of notation,
$$\chi_{m_j} \in \text{Aut} \left( B^{n+1} \right).$$ Suppose that the assertion is not true. Then the sequence $\chi_{m_j}$ converges to a boundary point $\psi (q') \in bB^{n+1}$ uniformly on every compact subset of $B^{n+1}$, if necessary, passing to a subsequence. Hence the sequence $\tau_{m_j} \equiv \varphi \circ \phi_{m_j}$ would converge to a boundary point $q' \in bD$ uniformly on every compact subset of $B^{n+1}$. By the way, note that
$$\tau_{m_j} (p'_j) = p_{m_j} \rightarrow p \in D.$$ This is a contradiction. Hence the holomorphic mapping $\tau$ is locally biholomorphic. Thus there is a real number $\delta > 0$ such that the mapping $\tau_{m_j}$ and the inverse $\tau_{m_j}^{-1}$ are analytically continued to a biholomorphic mapping respectively on
$$B (p_{m_j};\delta), \quad B (p'_j;\delta).$$ Let $\varepsilon_j$ be the euclidean length between the two points $p'$ and $p'_j$. By First Scaling Lemma, there is a sequence of automorphisms
$$\sigma_j \in \text{Aut}_{p'} \left( bB^{n+1} \right)$$
such that, for a subsequence $\phi_{m_j}$, 
\[ U_{\sigma_j} = id_{n \times n}, \quad \rho_{\sigma_j} = \varepsilon_{m_j}, \quad a_{\sigma_j} = 0, \quad r_{\sigma_j} = 0 \]
and the composition 
\[ \eta_j \equiv \sigma_j^{-1} \circ \phi_{m_j} \]
uniformly converges to an automorphism of the unit ball $B^{n+1}$. Then there is a real number $\delta > 0$ such that the mapping $\eta_j$ and the inverse $\eta_j^{-1}$ are analytically continued respectively on 
\[ B(p'_m; \delta), \quad B\left(p''_{m_j}; \delta\right) \]
where 
\[ p''_{m_j} = \eta_j\left(p'_m\right) \]
Then we obtain 
\[ \varphi = \tau_{m_j} \circ \eta_j^{-1} \circ \sigma_j^{-1} \]
Thus the mapping $\phi = \varphi^{-1}$ is analytically continued onto the open set 
\[ \tau_{m_j} \circ \eta_j^{-1} \left( \sigma_j \left( B\left(p''_{m_j}; \delta\right) \right) \right) \]
which is centered at the point $p_{m_j}$. For a canonical normalizing mapping $\mu_{p'}$, we obtain 
\[ \sigma_j' \equiv \mu_{p'} \circ \sigma_j \circ \mu_{p'}^{-1} : \begin{cases} z^* = \sqrt{\varepsilon_{m_j}} z \\ w^* = \varepsilon_{m_j} w \end{cases} \]
We set 
\[ \mu_{p'} \left(p''_{m_j}\right) \to p''' \in \mu_{p'} \left(B^{n+1}\right) \]
so that, shrinking $\delta > 0$, if necessary, the mapping $\phi = \varphi^{-1}$ is analytically continued onto the open set 
\[ \tau_{m_j} \circ \eta_j^{-1} \circ \mu_{p'} \left( \sigma_j' \left( B\left(p'''_j; \delta\right) \right) \right) \]
which is centered at the point $p_{m_j}$.

Since $\varepsilon_{m_j}$ is the euclidean distance between the two points $p$ and $p_{m_j}$, the analytically continued region of the mapping $\phi = \varphi^{-1}$ along the line segment $\gamma[0, 1] \subset D$ contains an open set along the line segment $\gamma[0, 1]$ which touches to the point $p = \gamma(1)$ by an edge shape on the complex line $\pi_p$ and by a slanted paraboloid shape on the real hyperplane $R_p$. This completes the proof. \[ \square \]

**Lemma 38.** Let $D$ be a bounded domain in $\mathbb{C}^{n+1}$ with spherical differentiable boundary $bD$ and $\phi$ be a biholomorphic mapping on an open neighborhood $U$ of a point $r \in bD$ satisfying 
\[ \phi \in H \left(U \cap D\right) \cap C^1 \left(U \cap \overline{D}\right) \quad \text{and} \quad \phi \left(U \cap bD\right) \subset bB^{n+1}. \]
Let $L$ be the Doggaebi variety associated to the mapping $\phi$ and $\varphi : B^{n+1} \to D$ be a locally biholomorphic mapping to be an analytic continuation of the mapping $\phi^{-1}$. Let $H_p$ be the complex hyperplane at the point $p \in D$ in Lemma 37 and $\pi_p$ be a complex line satisfying 
\[ T_p \mathbb{C}^{n+1} = H_p \oplus \pi_p. \]
Suppose that, along a line segment $\gamma : [0, 1] \to \pi_p \cap D$ with $p = \gamma (1)$, the germ of a locally biholomorphic mapping $\phi = \varphi^{-1}$ at the point $\gamma (0)$ is analytically continued along the subpath $[0, \tau]$ with all $\tau < 1$ as a locally biholomorphic mapping such that the limit point

$$\lim_{\tau \to 1} \phi \circ \gamma (\tau)$$

is the point $p' \in L \subset bB^{n+1}$ in Lemma 37. Then there is an open neighborhood $V$ of the line segment $\gamma [0, 1)$ such that

1. $bV \cap \pi_p \cap B (p; \delta)$ is an angle for a sufficient small $\delta > 0$, which contains the path $\gamma [0, 1]$,  
2. $bV \cap R_p \cap B (p; \delta)$ is a slanted paraboloid for a sufficiently small $\delta > 0$, which smoothly touches the complex hyperplane $H_p$ at the point $p$,  
3. the germ of a locally biholomorphic mapping $\phi = \varphi^{-1}$ at the point $\gamma (0)$ is analytically continued on $V$ as a locally biholomorphic mapping,  

where $R_p$ is a real hyperplane containing the line segment $\gamma [0, 1]$ and

$$H_p \subset R_p.$$  

Proof. By Lemma 38, $\phi \circ \gamma (\tau)$ approaches to the point $p' \in L$ to a direction transversal to the complex tangent hyperplane at the point $p'$ as $\tau \to 1$. Then there is a complex hyperplane $H_p'$ satisfying the conditions in Lemma 37.

Note that the complex hyperplane $H_p$ is determined by the mapping $\varphi : B^{n+1} \to D$ and the complex tangent hyperplane $H_{p'}$ at the point $p' \in L \subset bB^{n+1}$, but $H_p$ does not depend on the approaching direction of $\phi \circ \gamma (\tau) \to p'$ as $\tau \to 1$. Thus $H'_p = H_p$. This completes the proof.

2.3. Biholomorphic equivalence.

Lemma 39. Let $D$ be a bounded domain in $\mathbb{C}^{n+1}$ with spherical differentiable boundary $bD$ and $G$ be the path space of $D$ mod homotopic relation so that $G$ is a universal covering Riemann domain with a natural complex structure and a natural holomorphic covering map $\kappa : G \to D$. Suppose that there is a biholomorphic mapping

$$\phi \in H (U \cap D) \cap C^1 (U \cap D)$$

for a connected open neighborhood $U$ of a point $p \in bD$ satisfying

$$\phi (U \cap bD) \subset bB^{n+1}.$$  

Then the analytic continuation of the inverse mapping $\phi^{-1}$ is lifted to a holomorphic mapping $\psi$ from $B^{n+1}$ onto $G$ such that $\psi : B^{n+1} \to G$ is a proper locally biholomorphic mapping, i.e., a finite covering of $G$, satisfying the following relation:

$$\begin{array}{c}
G \\
\psi \hspace{1cm} \phi \hspace{1cm} \kappa \\
\downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
B^{n+1} \rightarrow \hspace{1cm} \rightarrow \hspace{1cm} \rightarrow \\
\varphi \hspace{1cm} \varphi \hspace{1cm} \varphi \\
\hspace{1cm} \hspace{1cm} \hspace{1cm} \\
D \\
\end{array}$$

Proof. By Theorem 35, there is a locally biholomorphic mapping

$$\varphi : B^{n+1} \to D$$
such that \( \varphi \) is the analytic continuation of the mapping \( \phi^{-1} \) and
\[
\varphi = \phi^{-1} \quad \text{on} \quad bB^{n+1} \setminus L
\]
where \( L \) is the Doggaebi variety associated to the mapping \( \phi \).

A piecewise line segment curve is a path \( \gamma : [0, 1] \to D \) consisting of finitely many line segments. Fix a point \( b \in U \cap D \). We shall show that the mapping \( \phi \) is analytically continued along any piecewise line segment curve. Suppose that \( \gamma : [0, 1] \to D \) be a piecewise line segment curve with \( \gamma(0) = b \) such that the germ of the mapping \( \phi \) at the point \( \gamma(0) \) is analytically continued along all subpath \( \gamma[0, \tau] \) with \( \tau < 1 \), but not the whole path \( \gamma[0, 1] \). Let \( \pi_p \) be a complex line satisfying, for sufficiently small \( \varepsilon > 0 \),
\[
\gamma[1 - \varepsilon, 1] \subset \pi_p.
\]
Then, by Second Dogginal Lemma, there is a distinguished complex hyperplane \( H_p \) at the point \( p = \gamma(1) \) satisfying
\[
T_p C^{n+1} = H_p \oplus \pi_p
\]
and an open neighborhood \( V \) containing the line segment \( \gamma[0, \tau] \) with all \( \tau < 1 \) such that
1. \( bV \cap \pi_p \cap B(p; \varepsilon) \) is an angle for a sufficient small \( \varepsilon > 0 \), which contains the path \( \gamma[0, 1] \),
2. \( bV \cap R_p \cap B(p; \varepsilon) \) is a slanted paraboloid for a sufficiently small \( \varepsilon > 0 \), which smoothly touches the complex hyperplane \( H_p \) at the point \( p \),
3. the germ of a locally biholomorphic mapping \( \phi = \varphi^{-1} \) at the point \( \gamma(0) \) is analytically continued on \( V \) as a locally biholomorphic mapping,

where \( R_p \) is a real hyperplane satisfying, for sufficiently small \( \varepsilon > 0 \),
\[
\gamma[1 - \varepsilon, 1] \subset R_p \quad \text{and} \quad H_p \subset R_p.
\]
Thus we can find a sequence of points \( p_j \in V \cap B(p; \varepsilon) \) and a sequence of germs \( \phi_j \) of the analytic continuation of the mapping \( \phi \) at the point \( p_j \) such that the sequence \( p_j \) converges to the point \( p \) to a direction tangential to the complex hyperplane \( H_p \) at the point \( p \). We set
\[
p_j' = \phi_j(p_j) \in B^{n+1}.
\]
Then there is a point \( p' \in L \subset bB^{n+1} \) such that \( p_j' \to p' \) to a direction tangential to the complex tangent hyperplane at the point \( p' \in bB^{n+1} \). We take a sequence of automorphisms \( \varphi_j \in Aut(B^{n+1}) \) such that
\[
\varphi_j(p_j') = p_j' \to p' \in L.
\]
Then the composition \( \tau_j \equiv \varphi \circ \varphi_j : B^{n+1} \to D \) forms a normal family so that there is a subsequence \( \tau_{m_j} \equiv \varphi \circ \varphi_{m_j} \) which converges to a holomorphic mapping \( \tau : B^{n+1} \to D \) uniformly on every compact subset of \( B^{n+1} \).

We claim that \( \tau \) is a locally biholomorphic mapping. We take a point \( q \in bB^{n+1} \setminus L \) and \( q_{m_j}' = \tau_{m_j}(q) \in bD \) such that
\[
q_{m_j}' = \tau_{m_j}(q) \to q' \in bD,
\]
if necessary, passing to a subsequence. Since \( bD \) is spherical, there is an open neighborhood \( W \) of the point \( q' \) and a biholomorphic mapping \( \psi \) on \( W \cap D \) satisfying
\[
\psi \in H(W \cap D) \cap C^1(W \cap \overline{D}) \quad \text{and} \quad \psi(W \cap bD) \subset bB^{n+1}.
\]
Then we consider the composition \( \chi_{m_j} \equiv \psi \circ \varphi \circ \varphi_{m_j} \) so that, by Lemma 17 and by abuse of notation,

\[
\chi_{m_j} \in \text{Aut} \left( B^{n+1} \right).
\]

Suppose that the assertion is not true. Then the sequence \( \chi_{m_j} \) converges to a boundary point \( \psi(q') \in bB^{n+1} \) uniformly on every compact subset of \( B^{n+1} \), if necessary, passing to a subsequence. Hence the sequence \( \tau_{m_j} \equiv \varphi \circ \varphi_{m_j} \) would converge to a boundary point \( q' \in bD \) uniformly on every compact subset of \( B^{n+1} \).

By the way, note that

\[
\tau_{m_j}(p'_1) = p_{m_j} \to p \in D.
\]

This is a contradiction. Hence the holomorphic mapping \( \tau \) is locally biholomorphic. Thus there is a real number \( \delta > 0 \) such that the mapping \( \tau_{m_j} \) and the inverse \( \tau_{m_j}^{-1} \) are analytically continued to a biholomorphic mapping respectively on

\[
B \left( p_{m_j}; \delta \right), \quad B \left( p'_1; \delta \right).
\]

Let \( \varepsilon_j \) be the euclidean length between the two points \( p' \) and \( p'_j \). By Second Scaling Lemma, there is a sequence of automorphisms

\[
\sigma_j \in \text{Aut}_{p'} \left( bB^{n+1} \right)
\]

such that, for a subsequence \( \varphi_{m_j} \),

\[
U_{\sigma_j} = id_{n \times n}, \quad \rho_{\sigma_j} = \varepsilon_{m_j}^2, \quad a_{\sigma_j} = 0, \quad r_{\sigma_j} = 0
\]

and the composition

\[
\eta_j \equiv \sigma_j^{-1} \circ \varphi_{m_j}
\]

uniformly converges to an automorphism of the unit ball \( B^{n+1} \). Then there is a real number \( \delta > 0 \) such that the mapping \( \eta_j \) and the inverse \( \eta_j^{-1} \) are analytically continued respectively on

\[
B \left( p'_1; \delta \right), \quad B \left( p''_{m_j}; \delta \right)
\]

where

\[
p''_{m_j} = \eta_j \left( p'_{m_j} \right).
\]

Then we obtain

\[
\varphi = \tau_{m_j} \circ \eta_j^{-1} \circ \varphi_{m_j}^{-1}.
\]

Thus the mapping \( \phi = \varphi^{-1} \) is analytically continued onto the open set

\[
\tau_{m_j} \circ \eta_j^{-1} \left( \sigma_j \left( B \left( p''_{m_j}; \delta \right) \right) \right)
\]

which is centered at the point \( p_{m_j} \). For a canonical normalizing mapping \( \mu_{p'} \), we obtain

\[
\sigma'_j \equiv \mu_{p'} \circ \sigma_j \circ \mu_{p'}^{-1} : \begin{cases} z^* = \varepsilon_{m_j} z \\ w^* = \varepsilon_{m_j}^2 w \end{cases}
\]

We set

\[
\mu_{p'} \left( p''_{m_j} \right) \to p''' \in \mu_{p'} \left( B^{n+1} \right)
\]
so that, shrinking $\delta > 0$, if necessary, the mapping $\phi = \varphi^{-1}$ is analytically continued onto the open set
\[ \tau_{m_j} \circ \eta_j^{-1} \circ \mu p' (\sigma_j' (B(p''; \delta))) \]
which is centered at the point $p_{m_j}$.

Since the sequence $p_{m_j}$ converges to $p$ to a direction tangential to the complex hyperplane $H_p$ at the point $p$, and $\varepsilon_{m_j}$ is the euclidean distance between the two points $p$ and $p_{m_j}$, the analytically continued region of the mapping $\phi = \varphi^{-1}$ along the sequence $p_{m_j} \in V \cap D$ contains an open set along the converging direction of the sequence $p_{m_j}$ by an edge shape tangential to the complex hyperplane $H_p$ at the point $p$ and a $\sqrt{|x|}$ curve shape normal to $H_p$.

Thus there is a complex line $\pi'_p$ satisfying $\pi'_p \subset H_p$ and $p \in \pi'_p$, and an open neighborhood $W$ of the point $p$ such that the mapping $\phi$ is analytically continued on $W \cap \pi'_p$.

Then we take a line segment $\gamma : [0, 1] \to \pi'_p \subset H_p$ such that $\gamma : [0, 1) \subset W \cap (\pi'_p \setminus p)$ and $\gamma(1) = p$.

Note that
\[ \lim_{\tau \to 1} \phi \circ \gamma(\tau) = p' \in L \subset bB^{n+1}. \]

By Second Dogginal Lemma, the distinguished complex tangent hyperplane $H_p$ satisfies
\[ T_pC^{n+1} = H_p \oplus \pi'_p \]
so that
\[ \pi'_p \cap H_p = \{p\}. \]

This is a contradiction so that the germ of the mapping $\phi$ at the point $b \in U \cap D$ is analytically continued along the whole path $\gamma[0, 1]$. Hence the mapping $\phi$ is analytically continued along any piecewise line segment curve on $D$ as a locally biholomorphic mapping.

We claim that the mapping $\phi : U \cap D \to B^{n+1}$ is analytically continued along any path on $D$ as a locally biholomorphic mapping. Let $\gamma : [0, 1] \to D$ be a path on $D$. Then we take a piecewise line segment curve $\lambda : [0, 1]$ and a continuous function $\Gamma : [0, 1] \times [0, 1] \to D$ such that
\[ \gamma(0) = \lambda(0), \quad \gamma(1) = \lambda(1) \]
\[ \Gamma(0, \tau) = \gamma(\tau), \quad \Gamma(1, \tau) = \lambda(\tau) \quad \text{for all } \tau \in [0, 1] \]
and the path $\Gamma(\cdot, \tau) : [0, 1] \to D$ for each $\tau \in [0, 1]$ is a piecewise line segment curve on $D$. Clearly, the mapping $\phi$ is analytically continued along the whole path $\gamma[0, 1]$ so that the mapping $\phi$ is analytically continued along any path on $D$ as a locally biholomorphic mapping.

Let $G$ be the path space of $D$ pointed by a point of $U \cap D$ mod homotopic relation so that $G$ is a universal covering of $D$ with a natural complex structure and a natural holomorphic covering map $\kappa : G \to D$. By the analytic continuation of the mapping $\phi : U \cap D \to B^{n+1}$ on $D$, the mapping $\varphi : B^{n+1} \to D$ has its natural
proper locally biholomorphic lift \( \psi : B^{n+1} \to G \). Since the mapping \( \psi : B^{n+1} \to G \) is proper and locally biholomorphic, the mapping \( \psi \) is a finite covering of \( G \). This completes the proof. \( \square \)

**Theorem 40.** Let \( D \) be a bounded domain in \( \mathbb{C}^{n+1} \) with spherical differentiable boundary \( bD \) such that the fundamental group \( \pi_1(D) \) is finite. Suppose that there is a biholomorphic mapping

\[
\phi \in H \left( U \cap D \right) \cap C^1 \left( U \cap \overline{D} \right)
\]

for a connected open neighborhood \( U \) of a point \( p \in bD \) satisfying

\[
\phi \left( U \cap bD \right) \subset bB^{n+1}.
\]

Then \( D \) is necessarily simply connected and the mapping \( \phi \) is analytically continued to a biholomorphic mapping from \( D \) onto \( B^{n+1} \).

**Proof.** Let \( G \) be the path space of \( D \) pointed by a point of \( U \cap D \) mod homotopic relation with a natural complex structure and a natural locally biholomorphic covering map \( \kappa : G \to D \). Then, by Lemma 39, there is a locally biholomorphic finite covering lift \( \psi : B^{n+1} \to G \) satisfying

\[
\phi = \kappa \circ \psi
\]

where \( \varphi : B^{n+1} \to D \) is an analytic continuation of the inverse mapping \( \phi^{-1} \).

Since the fundamental group \( \pi_1(D) \) is finite, the mapping \( \kappa : G \to D \) is a finite covering. Thus the mapping

\[
\varphi = \kappa \circ \psi : B^{n+1} \to D
\]

is a locally biholomorphic finite covering. Therefore, the analytic continuation of the inverse mapping \( \phi^{-1} \) on the boundary \( bB^{n+1} \) yields finitely many germs at each point of \( bB^{n+1} \). By Lemma 20, the Doggaebi variety \( L \) associated to the mapping \( \phi \) is empty so that the mapping

\[
\varphi = \phi^{-1} : bB^{n+1} \to bD
\]

is also a locally biholomorphic finite covering. Thus the mapping

\[
\varphi : \overline{B^{n+1}} \to \overline{D}
\]

is a finite covering mapping. Then the fixed point property of the close ball \( \overline{B^{n+1}} \) implies that the mapping \( \varphi : \overline{B^{n+1}} \to \overline{D} \) is a simple cover. Otherwise, a nontrivial deck transformation of the closed ball \( \overline{B^{n+1}} \) yields a continuous function on \( \overline{B^{n+1}} \) with no fixed point. Therefore, the analytic continuation of the mapping \( \phi : U \cap D \to B^{n+1} \) is analytically continued to a biholomorphic mapping

\[
\varphi^{-1} : D \to B^{n+1}.
\]

This completes the proof. \( \square \)
3. Locally Biholomorphic Mappings

3.1. Estimates of normalizing parameters.

**Lemma 41.** Let $M$ be a nonspherical analytic real hypersurface in $\mathbb{C}^{n+1}$ and $\text{Aut}_p(M)$ be the isotropy subgroup at a point $p \in M$. Suppose that there is a real number $c \geq 1$ satisfying

$$\sup_{\varphi \in \text{Aut}_p(M)} |U_\varphi| \leq c < \infty.$$ 

Then there is a real number $e \geq 1$ satisfying

$$|a_\phi| \leq e, \quad e^{-1} \leq |\rho_\phi| \leq e, \quad |r_\phi| \leq e$$

for every local automorphism $\phi \in \text{Aut}_p(M)$.

**Proof.** The real hypersurface $\mu_p(M)$ in normal form is expanded as follows:

$$v = \langle z, z \rangle + \sum_{k=4}^{\infty} F_k(z, \overline{z}, u)$$

where

$$F_k(\nu z, \nu \overline{z}, \nu^2 u) = \nu^k F_k(z, \overline{z}, u).$$

Since $M$ is nonspherical, not all $F_k(z, \overline{z}, u)$ are zero. Then we make an estimate of the normalizing parameters $a_\phi, \rho_\phi, r_\phi$ for $\phi \in \text{Aut}_p(M)$ (cf. [Pa3]). This completes the proof.

**Theorem 42.** Let $M$ be a nonspherical strongly pseudoconvex analytic real hypersurface in a complex manifold. Then the local isotropy subgroup $\text{Aut}_p(M)$ is compact for every point $p \in M$.

**Proof.** Because the situation is local, we may assume by taking a coordinate chart, if necessary, that the complex manifold is $\mathbb{C}^{n+1}$. Then the pseudoconvexity of $M$ leads to

$$|U_\varphi| = 1 \quad \text{for} \quad \varphi \in \text{Aut}_p(M).$$

By Lemma 41, we obtain the desired result. This completes the proof.

**Lemma 43.** Let $M, M'$ be nonspherical analytic real hypersurfaces in $\mathbb{C}^{n+1}$ and $p, p'$ be points respectively of $M, M'$ such that the two germs $M$ at $p$ and $M'$ at $p'$ are biholomorphically equivalent. Suppose that the isotropy subgroup $\text{Aut}_p(M)$ is compact. Then there is a real number $\delta_p > 0$ such that each germ of a biholomorphic mapping $\phi$ sending the germ $M$ at $p$ to the germ $M'$ at $p'$ is analytically continued to the open ball $B(p; \delta_p)$.

**Proof.** We take a biholomorphic mapping $\phi$ on a connected open neighborhood $U$ of the point $p \in M$ such that

$$\phi(U \cap M) \subset M'.$$

Then every germ of a biholomorphic mapping sending the germ $M$ at $p$ to the germ $M'$ at $p'$ is one of the following

$$\phi \circ \varphi \quad \text{for} \quad \varphi \in \text{Aut}_p(M).$$

Then the compactness of the group $\text{Aut}_p(M)$ leads to the desired result. This completes the proof.
Lemma 44. Let $M$ be a nonspherical analytic real hypersurface in $\mathbb{C}^{n+1}$ such that the isotropy subgroup $\text{Aut}_p(M)$ is compact at every point $p \in M$. Then, for each compact subset $K \subset M$, there is a real number $e \geq 1$ satisfying

$$e^{-1} \leq |U_\phi| \leq e, \quad |a_\phi| \leq e, \quad e^{-1} \leq |\rho_\phi| \leq e, \quad |r_\phi| \leq e$$

for every point $p \in K$ and every local automorphism $\phi \in \text{Aut}_p(M)$.

Proof. The real hypersurface $\mu_p(M)$ in normal form is expanded as follows:

$$v = \langle z, z \rangle + \sum_{|I|,|J| \geq 2, k \geq 1} \lambda_{IJk}(p) z^I \overline{z}^J u^k.$$

Suppose that the normalization $N_{e, e} = (U, a, \rho, r) \in H$, transforms the real hypersurface $\mu_p(M)$ to a real hypersurfaces expanded as follows:

$$v = \langle z, z \rangle + \sum_{|I|,|J| \geq 2, k \geq 1} \eta_{IJk}(p; U, a, \rho, r) z^I \overline{z}^J u^k.$$

Note that the element $(U_\phi, a_\phi, \rho_\phi, r_\phi)$ for $\phi \in \text{Aut}_p(M)$ is characterized by the following equalities:

$$\lambda_{IJk}(p) = \eta_{IJk}(p; U, a, \rho, r), \quad |I|, |J| \geq 2, k \geq 1.$$

Then, since the algebraic subset of $(U, a, \rho, r)$ is characterized by finitely many equalities, there is an integer $K$ such that the following equalities

$$(14) \quad \lambda_{IJk}(p) = \eta_{IJk}(p; U, a, \rho, r), \quad |I|, |J|, k \leq K$$

characterize the element $(U_\phi, a_\phi, \rho_\phi, r_\phi)$ for $\phi \in \text{Aut}_p(M)$.

Since the isotropy subgroup $\text{Aut}_p(M)$ is compact, there is a real number $e_p \geq 1$ such that

$$e_p^{-1} \leq |U_\phi| \leq e_p, \quad |a_\phi| \leq e_p, \quad e_p^{-1} \leq |\rho_\phi| \leq e_p, \quad |r_\phi| \leq e_p$$

for every element $\phi \in \text{Aut}_p(M)$. Thus the algebraic set defined by the equalities in $(4)$ is bounded. Note that the boundedness of the algebraic set is preserved on an open neighborhood of the point $p$ so that there are a real number $\delta_p > 0$ and a real number $c_p \geq 1$ satisfying

$$c_p^{-1} \leq |U_\phi| \leq c_p, \quad |a_\phi| \leq c_p, \quad c_p^{-1} \leq |\rho_\phi| \leq c_p, \quad |r_\phi| \leq c_p$$

for every point $p \in B(p; \delta_p) \cap M$ and every local automorphism $\phi \in \text{Aut}_p(M)$.

Since the subset $K$ is compact, there are finitely many points $p_1, \ldots, p_l$ such that

$$K \subset \bigcup_{k=1}^l B(p_k; \delta_{p_k}) \cap M.$$

Then we take

$$e = \max \left\{ e_{p_k} : 1 \leq k \leq l \right\}.$$

This completes the proof. \qed

Lemma 45. Let $M$ be nonspherical analytic real hypersurfaces in $\mathbb{C}^{n+1}$ such that the isotropy subgroup $\text{Aut}_p(M)$ is compact at every point $p \in M$. Then, for each compact subset $K \subset M$, there is a real number $\delta > 0$ such that each germ of a local automorphism $\phi \in \text{Aut}_p(M), p \in K$, is analytically continued to the open ball $B(p; \delta)$. 
Proof. By the construction of the normalizing map $\mu_p$, the real number $\delta_p$ depend only on the point $p \in M$. Hence there is a real number $\delta_p > 0$ for each $p \in M$ such that, for every $q \in B(p; \delta_p) \cap M$, the mapping $\mu_q$ is biholomorphic on $B(q; \delta_p)$ and the inverse $\mu_q^{-1}$ is biholomorphic on $B(0; \delta_p)$. Since $K$ is a compact subset, there are finitely many points $p_1, \ldots, p_l$ such that

$$K \subset \bigcup_{k=1}^l B(p_k; \delta_{p_k}) \cap M.$$ 

Then we take

$$\delta_1 = \max \{\delta_{p_k} : 1 \leq k \leq l\}$$

so that $\mu_p$ is biholomorphic on $B(p; \delta_1)$ and $\mu_p^{-1}$ is biholomorphic on $B(0; \delta_1)$ for every $p \in K$.

By Lemma 44, there is a real number $\delta_2$ such that every local automorphism $\varphi \in Aut_0(\mu_p(M))$ for every $p \in K$ is biholomorphically continued to the neighborhood $B(0; \delta_2)$. Then we take

$$\delta = \min \{\delta_1, \delta_2\}.$$

This completes the proof. \qed

Lemma 46. Let $M$ be a nonspherical analytic real hypersurface and $\gamma : [0, 1] \to M$ be a chain-segment. Let $M'$ be a nonspherical analytic real hypersurface in Moser-Vitushkin normal form (cf. [Pa1]). Suppose that there is a connected open neighborhood $U$ of the point $\gamma(0)$ and a biholomorphic mapping $\phi$ on $U$ such that

$$\phi(U \cap M) \subset M'$$

and the image $\phi(U \cap \gamma[0, 1])$ is on the straightened chain of $M'$. Then the mapping $\phi$ is analytically continued along the whole chain-segment $\gamma[0, 1]$ as a local biholomorphic mapping.

Proof. Suppose that the assertion is not true. Then there is a real number $\lambda$, $0 < \lambda \leq 1$, such that the mapping $\phi$ is analytically continued along all the subpath $\gamma[0, \tau], \tau < \lambda$, but not the whole path $\gamma[0, \lambda]$, as a local biholomorphic mapping.

Since $\gamma[0, 1]$ is a chain-segment, there is a biholomorphic mapping $\varphi$ on an open neighborhood $V$ of the point $\gamma(\lambda)$ such that the image $\varphi(V \cap \gamma[0, 1])$ is on the straightened chain of $M'$. Then we take a point $p \in \varphi(V \cap \gamma[0, \lambda])$ and an open neighborhood $W$ of the point $p$ such that the mapping $\phi$ is analytically continued on $W$ along the chain-segment $\gamma[0, 1]$ and $W \subset V$ so that

$$\phi \circ \varphi^{-1} (\varphi(W) \cap M') \subset M'$$

and the mapping $\phi \circ \varphi^{-1}$ maps the straightened chain $\varphi(W \cap \gamma[0, 1])$ of $M'$ onto the straightened chain of $M'$.

Note that the composition $\psi = \phi \circ \varphi^{-1}$ is necessarily analytically continued along the whole straightened chain of $M'$ (cf. [Pa3]). Then, by abuse of notation, the composition $\psi \circ \varphi$ is the analytic continuation of the mapping $\phi$ on the point $\gamma(\lambda)$ along the chain-segment $\gamma[0, 1]$. This is a contradiction. This completes the proof. \qed
3.2. Analytic continuation on a real hypersurface. We defined a canonical normalization \( \nu_p \) of a nondegenerate analytic real hypersurface \( M \) at a point \( p \in M \) to Moser-Vitushkin normal form (cf. [Pa1]) by the same way of the canonical normalization \( \mu_p \) to Moser normal form.

**Lemma 47.** Let \( M, M' \) be nonspherical analytic real hypersurfaces in complex manifolds such that \( M' \) is compact and the isotropy subgroup \( \text{Aut}_p(M) \) is compact at every point \( p \in M \). Let \( \gamma : [0,1] \to M \) be a chain-segment and \( \phi \) be a biholomorphic mapping on a connected open set \( U \) of the point \( \gamma(0) \) such that

\[
\phi(U \cap M) \subset M'.
\]

Then the mapping \( \phi \) is analytically continued along the whole chain-segment \( \gamma[0,1] \) as a local biholomorphic mapping.

**Proof.** Suppose that the assertion is not true. Then, without loss of generality, we may assume that the mapping \( \phi \) is analytically continued along all the subpath \( \gamma[0,\tau) \), \( \tau < 1 \), but not the whole chain-segment \( \gamma[0,1] \) as a local biholomorphic mapping. By the analytic continuation of the mapping \( \phi \), there is a path \( \phi \circ \gamma : [0,1) \to M' \). Then we consider the following sequences

\[
p_j \equiv \gamma(1 - \frac{1}{j}), \quad j \in \mathbb{N}^+, \n\]

\[
p'_j \equiv \phi \circ \gamma(1 - \frac{1}{j}), \quad j \in \mathbb{N}^+.
\]

Since \( M' \) is compact, passing to a subsequence, say \( m_j \), there is a point \( p' \in M' \) such that \( p'_m \to p' \) as \( j \to \infty \) so that the closure

\[
\{p'_m \equiv \phi \circ \gamma(1 - \frac{1}{m_j}), \quad j \in \mathbb{N}^+\}
\]

is a compact subset in a coordinate chart of the complex manifold.

Without loss of generality, we may assume that the chain \( \gamma[0,1] \subset M \) in a coordinate chart of the complex manifold. By abuse of notation, we assume that \( M, M' \) are in \( \mathbb{C}^{n+1} \). Let \( \nu_p \) be the canonical normalization of a nondegenerate real hypersurface \( M \) at a point \( p \in M \) to Moser-Vitushkin normal form.

Then the mapping \( \phi \) yields the following germs of a biholomorphic mapping:

\[
\varphi_j \equiv \nu_{p'_m} \circ \phi \circ \nu_{p_m}^{-1} : \nu_{p_m}(M) \to \nu_{p'_m}(M').
\]

Since \( \gamma : [0,1] \to M \) is a chain-segment, there is a sequence of local automorphisms \( \sigma_j \in \text{Aut}_{p_m}(M) \) such that the mapping \( \nu_{p'_m} \circ \phi \circ \sigma_j \) sends the germ of the chain \( \gamma \) at the point \( p_m \) onto the straightened chain of the real hypersurface \( \mu_{p_m}(M') \) in Moser-Vitushkin normal form. By Lemma 46, the composition \( \psi_j = \nu_{p'_m} \circ \phi \circ \sigma_j \) is analytically continued along the whole chain-segment \( \gamma[0,1] \). Therefore, there is a real number \( \delta > 0 \) such that, by abuse of notation, the mapping \( \psi_j \) is biholomorphic on the open neighborhood \( B(p_m; \delta) \).

By Lemma 45, the set \( \{ \nu_{p_m} \circ \sigma_j \circ \nu_{p_m}^{-1} : j \in \mathbb{N}^+ \} \) is relatively compact. Thus, by passing to a subsequence and shrinking \( \delta > 0 \), if necessary, we may assume that
the composition $\tau_j = \nu_{p_m j} \circ \sigma_j \circ \nu_{p_m j}^{-1}$ and its converse $\tau_j^{-1}$ are biholomorphically continued on the open neighborhood $B(0; \delta)$.

Since $p_{m_j} \to \gamma(1)$ and $p'_{m_j} \to p'$, by passing to a subsequence and shrinking $\delta > 0$, if necessary, we may assume that the canonical normalizations $\nu_{p_{m_j}}, \nu_{p'_{m_j}}$ and their inverses $\nu_{p_{m_j}}^{-1}, \nu_{p'_{m_j}}^{-1}$ are biholomorphically continued respectively on the open neighborhoods

$$B \left( p_{m_j}; \delta \right), \quad B \left( p'_{m_j}; \delta \right), \quad B(0; \delta), \quad B(0; \delta).$$

Then, by abuse of notation, the composition

$$\chi_j = \nu_{p'_{m_j}}^{-1} \circ \psi_j \circ \nu_{p_{m_j}}^{-1} \circ \tau_j \circ \nu_{p_{m_j}}$$

is biholomorphic on the open neighborhood $B \left( p_{m_j}; \delta \right)$, if necessary, by shrinking $\delta > 0$.

Note that the mapping $\chi_j$ is a local biholomorphic continuation of the germ of the mapping $\phi$ at the point $p_{m_j}$ for each $j \in \mathbb{N}^+$. Thus we take an integer $K$ such that

$$\gamma(1) \in B \left( p_{m_K}; \delta/2 \right)$$

so that the mapping $\chi_K$ is an analytic continuation of the mapping $\phi$ on the point $\gamma(1)$ along the chain-segment $\gamma[0, 1]$. This contradiction completes the proof. □

**Lemma 48.** Let $M, M'$ be nonspherical analytic real hypersurfaces in complex manifolds such that $M'$ is compact and the isotropy subgroup $\text{Aut}_p(M)$ is compact at every point $p \in M$. Suppose that there is a biholomorphic mapping $\phi$ on a connected open set $U$ of a point $p \in M$ such that

$$\phi(U \cap M) \subset M'.$$

Then the biholomorphic mapping $\phi$ is analytically continued along any path on $M$ as a local biholomorphic mapping.

**Proof.** For each point $p \in M$, we make a biholomorphically equivalent deformation of the real hypersurfaces $\mu_p(M)$ in normal form continuously to a real hyperquadric by using the scaling mapping

$$\begin{cases} z^* = \lambda z, \\ w^* = \lambda^2 w \end{cases}, \quad \lambda \in \mathbb{R}^+. $$

Because the chain is characterized by an order differential equation, the continuous family of chains on the real hyperquadric is continuously deformed by the parameter $\lambda$ on a real hypersurface biholomorphic to $M$ near the point $p$(cf. [Pa3]).

Let $\gamma: [0, 1] \to M$ be a path on $M$ such that $\gamma(0) \in U \cap M$. Then, for each $\tau \in [0, 1]$, there is a real number $\varepsilon_\tau > 0$, a point $p_\tau \in M$ and a continuous function

$$\Gamma_\tau: [0, 1] \times ([0, 1] \cap (\tau - \varepsilon_\tau, \tau + \varepsilon_\tau)) \to M$$

such that

1. $\Gamma_\tau(\cdot, \sigma) : [0, 1] \to M$ is a chain-segment for each $\sigma \in [0, 1] \cap (\tau - \varepsilon_\tau, \tau + \varepsilon_\tau)$,
2. $\Gamma_\tau(0, \sigma) = \gamma(\sigma)$ for each $\sigma \in [0, 1] \cap (\tau - \varepsilon_\tau, \tau + \varepsilon_\tau)$,
3. $\Gamma_\tau(1, \sigma) = p_\tau$ for all $\sigma \in [0, 1] \cap (\tau - \varepsilon_\tau, \tau + \varepsilon_\tau)$. 


Note that the family \(\{[0, 1] \cap (\tau - \varepsilon_\tau, \tau + \varepsilon_\tau) : \tau \in [0, 1]\}\) is an open covering of the compact set \([0, 1]\). Thus there is a finite subcover
\[
\{[0, 1] \cap (\tau_j - \varepsilon_\tau_j, \tau_j + \varepsilon_\tau_j) : \tau_j \in [0, 1], \ j = 1, \cdots, m\}\.
\]
Then, by Lemma 47, the biholomorphic mapping \(\phi\) is analytically continued along the whole path \(\gamma[0, 1]\) as a local biholomorphic mapping. This completes the proof.

3.3. Holomorphic mapping on the boundary.

**Lemma 49.** Let \(Q\) be a real hyperquadric defined by
\[
v = (z, z) \equiv z_1z_1 + \cdots + z_nz_n
\]
and \(\phi\) be a polynomial mapping as follows:
\[
\phi : \left\{ \begin{array}{l}
z^* = f(z, w) \\
w^* = g(z, w)
\end{array} \right.
\]
where, for \(m \geq 2\),
\[
\begin{align*}
f(\mu z, \mu^2 w) &= \mu^m f(z, w) \\
g(\mu z, \mu^2 w) &= \mu^{2m} g(z, w).
\end{align*}
\]
Suppose that
\[
\phi(Q) \subset Q.
\]
Then \(\phi \equiv 0\).

**Proof.** The mapping \(\phi = (f, g)\) yields the identity
\[
\Im g(z, u + i(z, z)) = (f(z, u + i(z, z)), f(z, u + i(z, z))).
\]
Suppose that \(m\) is even, i.e., \(m = 2k \geq 2\). We may consider \(z, \overline{z}, u\) as independent variables in the identity \([13]\). Taking \(\overline{z} = 0\) in the equality \([13]\) yields
\[
g(z, u) - \overline{g}(0, u) = 2i\langle f(z, u), f(0, u) \rangle.
\]
Thus we can put
\[
g(z, u) = 2i\langle f(z, u), f(0, u) \rangle + \Re g(0, u) - i\langle f(0, u), f(0, u) \rangle.
\]
Note that
\[
f(0, u) = f(0, 1)u^\overline{z}, \quad g(0, u) = g(0, 1)u^m.
\]
Then the identity \([16]\) yields
\[
\begin{align*}
2\Re \left\{ \langle f(z, u + i(z, z)), f(0, 1) \rangle (u + i(z, z))^{\overline{z}} \right\} \\
+ \Re g(0, 1)3(u + i(z, z))^m \\
- \langle f(0, 1), f(0, 1) \rangle \Re (u + i(z, z))^m \\
= \langle f(z, u + i(z, z)), f(z, u + i(z, z)) \rangle.
\end{align*}
\]
Let \(p(z, \overline{z})\) be a polynomial of the variables \(z, \overline{z}\) satisfying
\[
p(\mu z, \nu \overline{z}) = \mu^l \nu^m p(z, \overline{z}).
\]
Then the polynomial \(p(z, \overline{z})\) is said to be of type \((l, m)\). Collecting the terms of type \((l, 1)\), \(l = 0, 1, \cdots, m - 1\), yields
\[
mlu^{-1} \langle z, z \rangle \{\langle f(z, u), f(0, u) \rangle - \langle f(0, u), f(0, u) \rangle + \Re g(0, u) \} = 0.
\]
Hence we obtain
\[ (f(z, u), f(0, u)) - \langle f(0, u), f(0, u) \rangle + \Re g(0, u) = 0, \]
so that
\[ \langle f(z, u), f(0, u) \rangle = \langle f(0, u), f(0, u) \rangle \]
\[ \Re g(0, u) = 0. \]
Thus, from the identity 17, we obtain
\[ g(z, u) = i\langle f(0, u), f(0, u) \rangle \]
by which the identity 16 yields
\[ \langle f(0, 1), f(0, 1) \rangle \Re (u + i\langle z, z \rangle)^m = \langle f(z, u + i\langle z, z \rangle), f(z, u + i\langle z, z \rangle) \rangle. \]
Note that
\[ \langle f(z, u + i\langle z, z \rangle), f(z, u + i\langle z, z \rangle) \rangle \geq 0, \]
but
\[ \Re (u + i\langle z, z \rangle)^m = u_m - \frac{m(m - 1)}{2} u^{m-2} \langle z, z \rangle^2 + \cdots. \]
Thus collecting terms of type (2, 2) yields
\[ -\frac{m(m - 1)}{2} \langle f(0, 1), f(0, 1) \rangle u^{m-2} \langle z, z \rangle^2 \]
\[ = \sum_{\alpha, \beta} \frac{z^\alpha z^\beta}{2} \left( \frac{\partial^2 f}{\partial z^\alpha \partial z^\beta} \right)(0, u)^2 + \frac{mu}{2} (u + i\langle z, z \rangle)^2 (f(0, u), f(0, u)) \]
\[ \geq 0. \]
Since \( m \geq 2 \), we obtain
\[ \langle f(0, 1), f(0, 1) \rangle = 0 \]
which yields
\[ \langle f(z, u + i\langle z, z \rangle), f(z, u + i\langle z, z \rangle) \rangle = 0, \]
i.e.,
\[ f(z, w) = g(z, w) = 0. \]
Suppose that \( m \) is odd, i.e., \( m = 2k + 1 \geq 3 \). We may consider \( z, \bar{z}, u \) as independent variables in the identity 16. Then, by the condition \( \bar{z} \) taking \( \bar{z} = 0 \) in the equality 16 yields
\[ g(z, u) = \overline{\overline{g}(0, u)} \]
since
\[ f(0, u) = 0. \]
Thus we can put
\[ g(z, u) = g(0, u), \quad g(0, 1) \in \mathbb{R}. \]
Then the identity 16 yields
\[ g(0, 1) \Im (u + i\langle z, z \rangle)^m = \langle f(z, u + i\langle z, z \rangle), f(z, u + i\langle z, z \rangle) \rangle. \]
Note that
\[ \langle f(z, u + i(z, z)), f(z, u + i(z, z)) \rangle \geq 0, \]
but
\[ \Im(u + i(z, z))^m = m u^{m-1}(z, z) - \frac{m(m-1)(m-2)}{2} u^{m-3}(z, z)^3 + \cdots. \]

Thus collecting terms of type $(1, 1)$ yields
\[ mg(0, 1)u^{m-1}(z, z) = \left| \sum_{\alpha} z^\alpha \left( \frac{\partial f}{\partial z^\alpha} \right)(0, u) \right|^2 \geq 0. \]

Collecting terms of type $(3, 3)$ yields
\[ -\frac{m(m-1)(m-2)}{2} g(0, 1)u^{m-3}\langle z, z \rangle^3 = \left| \sum_{\alpha, \beta, \gamma} z^\alpha z^\beta z^\gamma \frac{\partial^3 f}{\partial z^\alpha \partial z^\beta \partial z^\gamma}(0, u) \right|^2 + \frac{m-1}{2} \left| \sum_{\alpha} z^\alpha \left( \frac{\partial f}{\partial z^\alpha} \right)(0, u) \right|^2 u^{-2}\langle z, z \rangle^2 \geq 0. \]

Since $m \geq 3$, we obtain
\[ g(0, 1)\langle z, z \rangle = 0 \]
by which the identity \[13\] yields
\[ \langle f(z, u + i(z, z)), f(z, u + i(z, z)) \rangle = 0, \]
i.e.,
\[ f(z, w) = g(z, w) = 0. \]

This completes the proof. \[\square\]

Lemma 50. Let $M, M'$ be strongly pseudoconvex analytic real hypersurfaces in $\mathbb{C}^{n+1}$. Let $\phi$ be a holomorphic mapping on an open neighborhood $U$ of a point $p \in M$ such that
\[ \phi(U \cap M) \subset M'. \]

Then the mapping $\phi$ is either a constant mapping or a biholomorphic mapping on $U$, if necessary, shrinking $U$.

Proof. We take $q = \phi(p)$ so that
\[ \varphi \equiv \mu_q \circ \phi \circ \mu_p^{-1} : \mu_p(M) \to \mu_q(M'). \]
Then the mapping $\varphi$ is a holomorphic mapping on an open neighborhood $V$ of the origin satisfying
\[ \varphi(\mu_p(M) \cap V) \subset \mu_q(M'). \]
The mapping $\varphi = (f, g)$ in $\mathbb{C}^n \times \mathbb{C}$ is decomposed as follows:

$$f(z, w) = \sum_{k=1}^{\infty} f_k(z, w), \quad g(z, w) = \sum_{k=1}^{\infty} g_k(z, w),$$

where

$$f_m(\mu z, \mu^2 w) = \mu^m f_m(z, w), \quad g_m(\mu z, \mu^2 w) = \mu^m g_m(z, w).$$

We assume that $\mu_p(M), \mu_q(M')$ are defined respectively by the equations

$$v = F(z, \overline{z}, u), \quad v = \langle z, z \rangle + F^*(z, \overline{z}, u)$$

where

$$F(z, \overline{z}, u) = \langle z, z \rangle + O\left(|z|^4\right), \quad F^*(z, \overline{z}, u) = O\left(|z|^4\right).$$

Then we obtain the following identity near the origin

$$\Re g(z, w) = \langle z, z \rangle \Re g(0, 1).$$

By the condition $F(z, \overline{z}, u) = \langle z, z \rangle + O\left(|z|^4\right), F^*(z, \overline{z}, u) = o\left(|z|^4\right)$, the identity \ref{identity} yields

$$\Im g_2(z, w) = 0.$$
Hence the hypothesis necessarily comes to
\[ f_l(z, w) = 0, \quad l = 1, \cdots, m - 1, \]
\[ g_l(z, w) = 0, \quad l = 1, \cdots, 2m - 1. \]
Then the identity yields
\[ \Im g_{2m}(z, u + i(z, z)) = (f_m(z, u + i(z, z)), f_m(z, u + i(z, z))). \]
Note that the polynomial mapping \( \varphi_m \equiv (f_m, g_{2m}), m \geq 2, \) satisfies
\[ \varphi_m(Q) \subset Q. \]
By Lemma \( \ref{lem:analytic_continuation} \), \( \varphi_m \equiv 0 \) so that
\[ f_l(z, w) = 0, \quad l = 1, \cdots, m, \]
\[ g_l(z, w) = 0, \quad l = 1, \cdots, 2m. \]
This completes the induction so that \( \varphi \equiv 0, \) i.e., the mapping \( \varphi \) is a constant mapping. This completes the proof.

3.4. Proper holomorphic mappings.

Lemma 51. Let \( D, D' \) be strongly pseudoconvex bounded domains with nonspherical real analytic boundaries \( bD, bD' \) such that the boundaries \( bD, bD' \) are both simply connected. Suppose that there is a biholomorphic mapping \( \varphi \) on a connected open neighborhood \( U \) of a point \( p \in bD \) such that
\[ \varphi(U \cap bD) \subset bD'. \]
Then the mapping \( \varphi \) is analytically continued to a biholomorphic mapping from \( D \) onto \( D' \).

Proof. Since \( bD, bD' \) are simply connected, by Lemma \( \ref{lem:analytic_continuation} \), the mappings \( \varphi, \varphi^{-1} \) are both analytically continued, by abuse of notation, respectively to a biholomorphic mapping
\[ \varphi : D \to D \]
and
\[ \varphi^{-1} : D' \to D. \]
Then, by the identity theorem, the mapping \( \varphi \) is biholomorphic. This completes the proof.

Lemma 52. Let \( D, D' \) be strongly pseudoconvex bounded domains with nonspherical real analytic boundaries \( bD, bD' \) such the boundary \( bD' \) is simply connected and the closed set \( D' \) satisfies the fixed point property. Suppose that there is a biholomorphic mapping \( \varphi \) on a connected open neighborhood \( U \) of a point \( p \in bD \) such that
\[ \varphi(U \cap bD) \subset bD'. \]
Then the mapping \( \varphi \) is analytically continued to a biholomorphic mapping from \( D \) onto \( D' \).
Proof. Since $bD'$ is simply connected, by Lemma 48, the inverse mapping $\phi^{-1}$ is analytically continued to a locally biholomorphic proper mapping

$$\varphi : D' \to D.$$  

Note that the mappings $\varphi : D' \to D$ and $\varphi : bD' \to bD$ are both covering maps. We claim that the covering is simple. Otherwise, there is a nontrivial deck transformation of $D'$ yields a continuous self mapping of $D'$ with no fixed point. This is a contradiction. Therefore, the mapping $\varphi : D' \to D$ is biholomorphic. This completes the proof.  

**Lemma 53.** Let $D, D'$ be strongly pseudoconvex bounded domains with nonspherical real analytic boundaries $bD, bD'$ such that the domain $D$ and the boundary $bD'$ are both simply connected. Suppose that there is a biholomorphic mapping $\phi$ on a connected open neighborhood $U$ of a point $p \in bD$ such that

$$\phi(U \cap bD) \subset bD'.$$

Then the mapping $\phi$ is analytically continued to a biholomorphic mapping from $D$ onto $D'$. 

Proof. Since $bD'$ is simply connected, by Lemma 48, the inverse mapping $\phi^{-1}$ is analytically continued to a locally biholomorphic proper mapping

$$\varphi : D' \to D.$$  

Note that the mapping $\varphi : D' \to D$ is a covering map. Since $D$ is simply connected, the covering is simple. Therefore, the mapping $\varphi : D' \to D$ is biholomorphic. This completes the proof.  

**Theorem 54.** Let $D, D'$ be strongly convex bounded domains with real analytic boundaries $bD, bD'$. Suppose that there is a biholomorphic mapping $\phi$ on a connected open neighborhood $U$ of a point $p \in bD$ such that

$$\phi(U \cap bD) \subset bD'.$$

Then the mapping $\phi$ is analytically continued to a biholomorphic mapping from $D$ onto $D'$. 

Proof. Note that a strongly convex bounded domain is homeomorphic to an open ball $B^{n+1}$. Suppose that the boundaries $bD, bD'$ are spherical. Then we take a biholomorphic mapping $\varphi$ on $U$, if necessary, shrinking $U$, such that

$$\varphi(U \cap bD) \subset bB^{n+1}.$$  

Then, by Lemma 48, the mapping $\varphi$ and the composition $\psi = \varphi \circ \phi^{-1}$ are analytically continued, by abuse of notation, to biholomorphic mappings as follows:

$$\varphi : D \to B^{n+1} \quad \text{and} \quad \psi : D' \to B^{n+1}.$$  

Thus the composition $\psi^{-1} \circ \varphi : D \to D'$ is a biholomorphic mapping and the analytic continuation of the mapping $\phi$.

Suppose that the boundaries $bD, bD'$ are nonspherical. Then, by Lemma 53, the mapping $\phi$ is analytically continued to a biholomorphic mapping from $D$ onto $D'$. This completes the proof.
Lemma 55. Let $D, D'$ be strongly pseudoconvex bounded domains with real analytic boundaries $bD, bD'$. Suppose that there is a proper holomorphic mapping $\phi : D \to D'$. Then there is an open neighborhood $U$ of a point $p \in bD$ such that the mapping $\phi$ is analytically continued on $U$ and

$$\phi(U \cap bD) \subset bD'.$$

Proof. We may apply the boundary regularity of Lemma 16 so that the mapping $\phi : D \to D'$ is holomorphic on an open neighborhood of $D$. This completes the proof.

Lemma 56. Let $\phi : B^{n+1} \to B^{n+1}$ be a proper holomorphic mapping. Then $\phi \in \text{Aut}(B^{n+1})$.

Proof. By Lemma 55, there is an open neighborhood $U$ of a point $p \in bB^{n+1}$ such that $\phi$ is analytically continued on $U$ and

$$\phi(U \cap bB^{n+1}) \subset bB^{n+1}.$$

By Lemma 17, $\phi \in \text{Aut}(B^{n+1})$. This completes the proof.

Lemma 57. Let $D, D'$ be strongly pseudoconvex bounded domains in $\mathbb{C}^{n+1}$ with real analytic boundaries $bD, bD'$. Suppose that there is a proper holomorphic mapping $\phi : D \to D'$. Then the mapping $\phi : D \to D'$ is a locally biholomorphic mapping so that $\phi : \overline{D} \to \overline{D}'$ is a covering map.

Proof. By Lemma 55, there is a point $p \in bD$ and an open neighborhood $U$ of the point $p$ such that the mapping $\phi$ is analytically continued to $U$ and

$$\phi(U \cap bD) \subset bD'.$$

By Lemma 64, the mapping $\phi$ is biholomorphic on $U$, if necessary, shrinking $U$.

Suppose that the boundaries $bD, bD'$ are spherical. Then, by Lemma 30, the mapping $\phi : D \to D'$ is a locally biholomorphic mapping.

Suppose that the boundary $bD, bD'$ are nonspherical. Then, by Lemma 48, the mapping $\phi : D \to D'$ is analytically continued on an open neighborhood of the boundary $bD$ to be locally biholomorphic. Thus the mapping $\phi$ is a locally biholomorphic mapping. This completes the proof.

Theorem 58. Let $D$ be a strongly pseudoconvex bounded domain with real analytic boundary $bD$. Suppose that there is a proper holomorphic self mapping $\phi : D \to D$. Then $\phi$ is a biholomorphic automorphism of $D$.

Proof. By Lemma 57, the mapping $\phi : \overline{D} \to \overline{D}$ is a self covering map. We claim that $\phi : D \to D$ is a simple covering. Otherwise, there would be an integer $m > 1$ such that, for every $p \in \overline{D}$, $m$ is the order of the set

$$\{q \in \overline{D} : \phi(q) = p\}.$$

Then we define the $k$ times composition $\phi^k$ of the mapping $\phi$ as follows

$$\phi^k \equiv \phi \circ \cdots \circ \phi : \overline{D} \to \overline{D}.$$

Note that the inverse image of each point $p \in \overline{D}$ under the mapping $\phi^k : \overline{D} \to \overline{D}$, $k \in \mathbb{N}^+$, is a set of order $m^k$. 

We claim that, for a given real number $\delta > 0$, there is a point $q \in bD$ and a compact subset $K \subset D$ and a subsequence $\phi^{m_k}$ such that

$$\phi^{-m_k} (B(q; \delta) \cap D) \cap K \neq \emptyset \quad \text{for all} \ k \in \mathbb{N}^+.$$ 

Otherwise, for every compact subset $K \subset D$, there is an integer $l$ such that

$$\phi^{-k} (D \setminus K) \subset D \setminus K \quad \text{for all} \ k \geq l.$$

Since $\phi^k : D \to D$ is a finite covering map, we have

$$D = \phi^{-k} (K \cup (D \setminus K)) = \phi^{-k} (K) \cup \phi^{-k} (D \setminus K) \cup \phi^{-k} (K \setminus (D \setminus K)) \cup \phi^{-k} (K \setminus (D \setminus K)) \cup \phi^{-k} (K \setminus (D \setminus K)).$$

so that

$$K \subset \phi^{-k} (K) \quad \text{for all} \ k \geq l.$$

Note that the inverse image of $\phi^{-k} (K)$ is a union of $m_k$ disconnected compact subsets. We may assume that $K$ is connected so that $K$ is in a compact subset of the union $\phi^{-k} (K)$. This is impossible. Thus, with such a point $q \in bD$, we take an accumulation point $q' \in D$ of the set

$$\lim_{k \to \infty} \phi^{-m_k} ((q)) = \lim_{k \to \infty} \{p \in bD : \phi^{m_k} (p) = q\}.$$

Suppose that $bD$ is spherical. Then there are open neighborhoods $U, U'$ respectively of $q, q'$ and biholomorphic mappings $\psi, \psi'$ respectively on $U, U'$ such that

$$\psi(U \cap bD) \subset bB^{n+1}$$

$$\psi'(U' \cap bD) \subset bB^{n+1}.$$ 

By Lemma 17, the compositions

$$\varphi_k \equiv \psi' \circ \phi^{-k} \circ \psi^{-1} : \psi(U \cap D) \to \psi'(U' \cap D)$$

is analytically continued to, by abuse of notation, automorphisms $\varphi_k \in Aut(B^{n+1})$. By the construction, a subsequence $\varphi_{m_k}$ must converges to the point $\psi'(q')$ uniformly on every compact subset of $B^{n+1}$. Thus there is a sequence of real numbers $\delta_k \searrow 0$ such that

$$\phi^{m_k} : B(q'; \delta_k) \cap D \to U \cap D$$

and

$$\phi^{m_k} = \psi^{-1} \circ \varphi^{-1}_{m_k} \circ \psi' \quad \text{on} \ B(q'; \delta_k) \cap D.$$ 

Then we take a point $p \in U \cap D$ such that there is a sequence $q_k$ satisfying

$$q_k \in \phi^{-m_k} ((p)) \cap B(q'; \delta_k)$$

and, for a compact subset $K \subset D$, there is a sequence $p_k$ satisfying

$$p_k \in \phi^{-m_k} ((p)) \cap K.$$ 

Therefore, there is a sequence of deck transformations $\phi_k \in Aut(D)$ of the covering map $\phi^{m_k} : D \to D$ such that

$$\phi_k (p_k) = q_k \to q' \in bD.$$ 

Since $p_k \in K$, by Lemma 18, there is a biholomorphic mapping

$$\sigma : D \to B^{n+1}.$$ 

Then the composition

$$\iota \equiv \sigma \circ \phi \circ \sigma^{-1} : B^{n+1} \to B^{n+1}$$
would be a proper self mapping, but not an automorphism of the unit ball $B^{n+1}$. This is a contradiction to Lemma \ref{lem10} that every proper self mapping of the unit ball $B^{n+1}$ is an automorphism of the unit ball $B^{n+1}$.

Suppose that $bD$ is nonspherical. We take a sequence $p_k \in bD$ such that 
$$p_k \in \phi^{-k} (\{q\}) \subset bD$$
and
$$|p_k - q'| = \min \{|p - q'| : p \in \phi^{-k} (\{q\})\} .$$
Let $\mu_{p_k}$ be the canonical normalizing mapping at the point $p_k \in bD$. Since $p_k \rightarrow q'$, there is an open neighborhood $W$ of the point $q'$ such that the mapping $\mu_{p_k}$ is analytically continued on $W$. Then we define a function $\varepsilon_k (p)$ for a sufficiently large $k$ and a point $p \in W \cap bD$ such that
$$\varepsilon_k (p) \equiv \sum_{j=1}^{n} |z_j \circ \mu_{p_k} (p)|^2 + |w \circ \mu_{p_k} (p)|$$
where $z_j, j = 1, \cdots, n, w$ are the coordinate functions of $\mathbb{C}^{n+1}$ such that $z_j, j = 1, \cdots, n$, are for the complex tangent hyperplane and $w$ are for the complex line normal to the complex tangent hyperplane of $\mu_{p_k} (W \cap bD)$ at the origin. Then we take a sequence $q_k$ such that
$$q_k \in \phi^{-k} (\{q\}) \subset bD$$
and
$$\varepsilon_k (q_k) = \min \{\varepsilon_k (p) : p \in \phi^{-k} (\{q\})\} .$$
Let $\pi_k$ be a complex line containing $p_k$ and $q_k$. Then we take a subsequence $\pi_{m_k}$ so that $\pi_{m_k}$ converges to a complex line $\pi_{q'} \subset T_{q'} \mathbb{C}^{n+1}$. In other words, the a subsequence $q_{m_k}$ converges to the point $q'$ to a direction. Then we obtain
$$\varphi_k \equiv \mu_{p_k} \circ \phi^{-k} \circ \mu_{q}^- : \mu_{q} (bD) \rightarrow \mu_{p_k} (bD) .$$
Then we define a biholomorphic mapping
$$\sigma_k : \begin{cases} z^* = \sqrt{\varepsilon_{m_k}}z \\
 w^* = \varepsilon_{m_k}w \end{cases}$$
where
$$\varepsilon_{m_k} = \varepsilon_{m_k} (q_{m_k}) .$$
Then we take the composition
$$\kappa_k \equiv \sigma^{-1}_k \circ \varphi_{m_k} : \mu_{q} (bD) \rightarrow \sigma^{-1}_k \circ \mu_{p_{m_k}} (bD) .$$
Note that there is a real number $\delta > 0$ such that the germs of real hypersurfaces $\mu_{q} (bD)$ and $\sigma^{-1}_k \circ \mu_{p_{m_k}} (bD)$ are analytically continued to the open neighborhood $B (0; \delta)$. Since the boundary $bD$ is nonspherical, by Lemma \ref{lem10}, the mapping $\kappa_k$ is biholomorphic on $B (0; \delta)$, if necessary, shrinking $\delta > 0$. Further, by the construction, the limit of the sequence $\kappa_k$ cannot be a constant mapping on $B (0; \delta)$ so that, by Theorem \ref{thm01}, the limit of the sequence $\kappa_k$ would be biholomorphic on $B (0; \delta)$.

By the way, the sequence of real hypersurfaces $\sigma^{-1}_k \circ \mu_{p_{m_k}} (bD)$ converges to a real hyperquadric uniformly on an open neighborhood of the origin. Therefore, the germ of the boundary $bD$ at the point $q'$ is spherical so that the boundary $bD$ is spherical(cf. \cite{Pa3}). This is a contradiction to the fact that the boundary $bD$ is nonspherical. This completes the proof.\hfill $\Box$
3.5. **Locally realizable CR manifolds.** Let $M$ be a CR manifold of CR dimension $n$ and CR codimension 1 with a CR structure $(D, I)$ where $D$ is $2n$ dimensional smooth subbundle of the tangent bundle $TM$ and $I$ is an automorphism on $D$ such that

$$I^2V = -V \quad \text{for} \quad V \in \Gamma D.$$ 

For each point $p \in M$, there is a local coordinate chart $(U, \varphi)$ such that

$$\varphi(U) \subset \mathbb{R}^{2n+1}.$$ 

Then $M$ shall be called locally realizable CR manifold if there is an open neighborhood $U$ of each point $p \in M$ and CR functions $f_1, \ldots, f_{n+1}$ on $\varphi(U)$ satisfying

$$df_1 \wedge \cdots \wedge df_{n+1} \neq 0.$$ 

Let $z_j$, $j = 1, \ldots, n+1$, be the coordinate functions of $\mathbb{C}^{n+1}$. Then we obtain a local embedding $\sigma$ of $U \subset M$ into $\mathbb{C}^{n+1}$ such that

$$f_j \equiv z_j \circ \sigma \circ \varphi^{-1}.$$ 

The CR manifold $M$ shall be called a locally analytically realizable CR manifold when $\sigma(U)$ is real analytic by a local embedding $\sigma$ on an open neighborhood $U$ of each point $p \in M$. Note that $M$ is either spherical or nonspherical (cf. [Pa3]).

**Lemma 59.** Let $M$ be a connected locally analytically realizable CR manifold. Suppose that there is a nontrivial CR mapping $\varphi$ on an open neighborhood $U$ of a point $p \in M$ such that

$$\varphi(U) \subset bB^{n+1}.$$ 

Then the mapping $\varphi$ is CR continued along any path on $M$ as a locally CR diffeomorphic mapping.

**Proof.** Since $M$ is connected, $M$ is necessarily spherical (cf. [Pa3]). Suppose that the assertion is not true. Then there is a path $\gamma : [0, 1] \to M$ with $\gamma(0) = p$ such that the mapping $\varphi$ is CR continued along all subpath $\gamma[0, \tau]$ with $\tau < 1$, but not the whole path $\gamma[0, 1]$. Then we take a CR embedding $\sigma(V) \subset \mathbb{C}^{n+1}$ of an open neighborhood $V$ of the point $\gamma(1)$ such that $\sigma(V)$ is a spherical analytic real hypersurface and

$$\varphi \circ \sigma^{-1} : \sigma(V) \to bB^{n+1}$$

is a locally CR diffeomorphism. Without loss of generality, we may assume that $\gamma[0, 1] \subset V$. Hence there is an open neighborhood $W$ of the point $\sigma(\gamma(0))$ and a biholomorphic mapping $\phi$ on $W$ such that

$$\phi = \varphi \circ \sigma^{-1} \quad \text{on} \quad \sigma(V) \cap W.$$ 

Then the mapping $\phi$ is analytically continued along the whole path $\sigma(\gamma[0, 1])$. From, by abuse of notation, the mapping $\phi$ at an open neighborhood $W'$ of the point $\sigma(\gamma(1))$ and a CR embedding $\sigma$ of an open neighborhood $V'$ of the point $\gamma(1)$, we obtain the CR mapping

$$\phi \circ \sigma : \sigma^{-1}(W' \cap \sigma(V')) \to bB^{n+1}$$

which is a CR continuation of the CR mapping $\varphi$. This completes the proof. \qed
**Theorem 60.** Let $M$ be a connected locally analytically realizable CR manifold. Suppose that $M$ is compact and there is a nontrivial CR mapping $\varphi$ on an open neighborhood $U$ of a point $p \in M$ such that

$$\varphi(U) \subset bB^{n+1}.$$ 

Then there is a finite subset $L \subset bB^{n+1}$ such that the inverse CR mapping $\varphi^{-1}$ is CR continued along any path on $bB^{n+1}\backslash L$ as a locally CR diffeomorphic mapping.

**Proof.** Since $M$ is connected, $M$ is necessarily spherical (cf. [Pa3]). Let $q_j \in bB^{n+1}$ be a sequence converging to a point $q \in bB^{n+1}$ and $\phi_j$ be a sequence of the CR continuation of the inverse mapping $\varphi^{-1}$ at the point $q_j$. We set

$$q_j' = \phi_j(q_j) \in M.$$ 

Since $M$ is compact, there is a subsequence $q_{m_j}$ and a point $q' \in M$ such that

$$q_{m_j} \to q'.$$

Then we take a CR embedding $\sigma$ of an open neighborhood $V$ of the point $q' \in M$ such that $\sigma(V) \subset \mathbb{C}^{n+1}$ is a spherical analytic real hypersurface. Then we apply the same argument as in the previous sections so that the singular locus $L$ is a finite subset of $bB^{n+1}$ and the inverse CR mapping $\varphi^{-1}$ is CR continued along any path on $bB^{n+1}\backslash L$. This completes the proof.

**Lemma 61.** Let $M, M'$ be connected locally analytically realizable CR manifolds with positive definite Levi form. Suppose that $M, M'$ are nonspherical and $M'$ is compact, and there is a nontrivial CR mapping $\varphi$ on an open neighborhood $U$ of a point $p \in M$ such that

$$\varphi(U) \subset M'.$$

Then the mapping $\varphi$ is CR continued along any path on $M$ as a locally CR diffeomorphic mapping.

**Proof.** Since $M, M'$ are locally analytically realizable and the Levi forms of $M, M'$ are positive definite, the isotropy subgroups $Aut_p(M), Aut_{p'}(M')$ are compact for every point $p \in M, p' \in M'$.

Let $q_j \in M$ be a sequence converging to a point $q \in M$ and $\varphi_j$ be a sequence of the CR continuation of the mapping $\varphi$ at the point $q_j$. We set

$$q_j' = \varphi_j(q_j) \in M'.$$

Since $M'$ is compact, there is a subsequence $q_{m_j}'$ and a point $q' \in M'$ such that

$$q_{m_j}' \to q'.$$

Then we take CR embeddings $\sigma, \sigma'$ respectively of open neighborhoods $V, V'$ respectively of the points $q \in M, q' \in M'$ such that $\sigma(V), \sigma'(V') \subset \mathbb{C}^{n+1}$ are nonspherical analytic real hypersurface. Then we apply the same argument as in the previous subsection. This completes the proof.

**Theorem 62.** Let $M, M'$ be connected locally analytically realizable CR manifolds with positive definite Levi form. Suppose that $M, M'$ are compact and nonspherical, and there is a nontrivial CR mapping $\varphi$ on an open neighborhood $U$ of a point $p \in M$ such that

$$\varphi(U) \subset M'.$$
Then the maximal CR extension of the mapping \( \varphi \) is a CR equivalence between the natural universal covering spaces of \( M, M' \) to be the pointed path spaces respectively of \( M, M' \) mod homotopic relation.

**Proof.** By Lemma \( \text{[61]} \) the mapping \( \varphi \) is CR continued along any path on \( M \) as a locally CR diffeomorphic mapping. Note that \( M \) is compact and

\[
\varphi^{-1}(\varphi(U)) \subset M
\]

so that we apply Lemma \( \text{[61]} \) to the inverse mapping \( \varphi^{-1} \). Thus the inverse mapping \( \varphi^{-1} \) is CR continued along any path on \( M' \) as a locally CR diffeomorphic mapping. Thus the CR continuation of the mapping \( \phi \) induces a CR equivalence between the natural universal coverings of \( M, M' \), which are the path spaces mod homotopic relation respectively over \( M, M' \) with the natural CR structure. This completes the proof. \( \square \)

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