Calculation of an $A = 3$ bound-state matrix element in pionless effective field theory

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In this paper, we establish a general framework for calculating pionless ($\pi$EFT) matrix elements between $A = 3$ bound-states up to next-to-leading-order. This framework is useful for $\pi$EFT calculations of electroweak observables, such as $^3$He magnetic moments and $^3$H $\beta$ decay. Starting from a Bethe-Salpeter equation, we prove that for a bound-state, the three-nucleon wave-function normalization can be expressed diagrammatically in a way that is equivalent to the unit operator between two identical three-nucleon bound-states. This diagrammatic form of the identity matrix element is the foundation for constructing an $A = 3$ matrix element of a general operator. We show that this approach can be used to calculate the energy difference between $^3$H and $^3$He due to the Coulomb interaction, and to calculate the NLO corrections to the $^3$H and $^3$He scattering amplitudes due to effective range corrections.

I. INTRODUCTION

Low-energy electroweak interactions in light nuclear systems ($d$, $^3$H, $^3$He) take part in many scenarios, such as $\beta$-decay, Big Bang nucleosynthesis and stellar evolution. The fundamental theory of physics at low energies is Quantum Chromodynamics (QCD), but, unfortunately, a direct calculation of low-energy nuclear observables is not possible due to the non-perturbative character of QCD in the nuclear regime. One way to overcome this problem is to use effective field theory (EFT). EFT is a simple, renormalizable and model-independent theoretical method for describing low-energy reactions. The prerequisite for describing a physical process using EFT is that its transfer momentum, $Q$, is small compared to the physical cutoff, $\Lambda_{\text{cut}}$, which is frequently related to the lightest exchange particle or lowest lying excitation not included in the theory. The EFT then has to preserve all symmetries of the fundamental theory, and the resulting Lagrangian includes only the relevant degrees of freedom, while heavier excitations are integrated out of the theory. Thus, one can obtain observables organized as a power series in $Q/\Lambda_{\text{cut}}$.

The so-called pionless EFT ($\pi$EFT) is an EFT approach to light nuclei that is particularly useful at the low energies that are of interest for astrophysical processes, i.e., $Q \sim 10$ MeV $\ll m_\pi = 140$ MeV. In addition, the strong interaction characterizing QCD at low energies leads to a scale separation between the nucleon-nucleon scattering length $a$ and the range of the interaction $R$. $\pi$EFT exploits this ratio as an expansion parameter. Thus, $\pi$EFT at leading order is a quantum field theoretical formulation of the zero-range limit, in which the range of the interaction is taken to zero. As a consequence, a three-body force is needed at leading order for the description of three-nucleon systems within this framework, a feature directly related to the well-known Efimov effect. Since the binding energies of nuclei with $A \leq 3$ are small ($\text{i.e.}, E_B < 10$ MeV), those nuclei can indeed be described using $\pi$EFT.

The Coulomb interaction in light nuclei is an additional complication: the Coulomb interaction is nonperturbative at low momenta $\lesssim 10$ MeV, but should be perturbative in nuclei where the typical momenta are much higher. $^3$He is the lightest and therefore the simplest nucleus to test the combination of $\pi$EFT and the Coulomb interaction, and many recent works have discussed this problem. In particular, it was shown that while at leading order (LO) $^3$He is described correctly within $\pi$EFT, at next-to-leading order (NLO) the results are inconclusive, and some approaches have shown the need for additional, isospin-dependent, three-body forces. Then additional three-nucleon observables are needed to obtain predictive power within $\pi$EFT at NLO.

Most of these $\pi$EFT studies formulate this field theory using scattering equations, imposing a momentum cutoff $\Lambda$ on the resulting integral equations. This method is completely trivial when studying scattering problems, and is intuitively presented using Feynman diagrams. However, many well-measured nuclear properties are just matrix elements of scattering operators between the wave functions of the bound nuclei.

These wave-functions are related to the residues of the scattering amplitudes at the binding energy pole and thus a solution of homogeneous scattering equations. This approach, however, requires careful studying of the normalization of the way function, and in addition loses the intuitive diagrammatic representation.

In the last few years, pioneering studies of $A = 3$ nuclear properties within $\pi$EFT have been accomplished. König et al. have calculated the binding energy difference between $^3$H and $^3$He that originates from the Coulomb interaction, treating the Coulomb interaction as perturbation. Vanasse et al. have calculated the perturbative NLO corrections to the $^3$H and $^3$He scat-
tering amplitudes, as well the effective range corrections to three-nucleon binding energy [13]. In 2017, the Nuclear Physics with Lattice Quantum Chromo Dynamics (NPLQCD) collaboration calculated the triton $\beta$-decay [16], to calibrated weak low-energy constant (LEC) $L_{1,\Lambda}$.

The goal of this paper is to provide a general diagrammatic approach to the calculation of matrix elements between nuclear wave-functions, obtained in $\pi$EFT at next-to-leading order. Our motivation and purpose are to lay the groundwork for future calculations of electroweak properties of $A = 3$ nuclei. This is accomplished in several steps. A Hubbard-Stratonovich (H-S) transformation on the $\pi$EFT Hamiltonian transforms the problem into a Hamiltonian of single nucleons and dibaryons, whose interactions are tuned to reproduce the physical scattering lengths and effective ranges of two nucleons (Section II). In this way the $A = 3$ bound-states’ energies and wave-function are found using a non-relativistic coupled channels Bethe-Salpeter equation. The normalization procedure of this Bethe-Salpeter equation is used to form a diagrammatic representation of a normalization operator (Section III). This is then generalized to any operator connecting $A = 3$ $\pi$EFT leading-order eigenstates (Section IV). We use this approach to calculate the properties of the $A = 3$ dibaryon system in the spin-singlet and -triplet channels. We use a formulation of $\pi$EFT with dynamical dibaryon fields $t$ and $s$. The fields $t$ and $s$ have the quan-

FIG. 1: The dressed dibaryon propagator. The bare dibaryon propagator is dressed by nucleon bubbles to all orders.

The covariant derivative is:

$$D_\mu = \partial_\mu + i e A_\mu \hat{Q},$$

where $e$ is the electric charge and $\hat{Q}$ is the charge operator, coupled to the electromagnetic field, $A_\mu$.

The bare dibaryon propagator arising from eq. (1) is

$$i D_{t,s}^{bare}(p_0, p) = -i \left[ p_0 - \frac{p^2}{4 M} - \sigma_{t,s} \right]^{-1}. \quad (4)$$

We use a power counting that is appropriate for systems with a scattering length $a$ that is large compared to the range of the interaction $R$. The full dibaryon propagator (Fig. 1) is therefore defined as the geometric sum of nucleon bubbles connected by bare dibaryon propagators (see Refs. [6] [7] for more details):

$$i D_{t,s}^{full}(p_0, p) = i D_{t,s}^{bare}(p_0, p) \times \sum_n \left( D_{t,s}^{bare}(p_0, p) i \tilde{L}_B (-2 i y_{t,s})^2 \right)^n, \quad (5)$$

where $\tilde{L}_B$ denotes the two-nucleon loop integral evaluated using the so-called power divergence subtraction (PDS) scheme (see [3] [5]). The full (unrenormalized) propagator becomes then

$$i D_{t,s}^{full}(p_0, p) = -i \left[ p_0 - \frac{p^2}{4 M} - \sigma_{t,s} \right]^{-1} + \frac{M y_{t,s}^2}{4 \pi} (\sqrt{-M p_0 + p^2/4} - i e - \mu), \quad (6)$$

where $\mu$ denotes a renormalization scale introduced through the PDS scheme.

The coupling constants can be obtained by matching to the effective range exansioner:

$$y_{t,s} = \frac{\sqrt{8 \pi}}{M \sqrt{\rho_{t,s}}}, \quad (7)$$

$$\sigma_{t,s} = \frac{2}{M \rho_{t,s}} \left( \frac{1}{a_{t,s}} - \mu \right), \quad (8)$$

where $\rho_{t,s}$ is the effective range and $a_{t,s}$ is the scattering length. Given that, eq. (6) becomes:
The dibaryon propagator shown above has two poles. One corresponds to the physical bound-state (virtual) pole that results from the large scattering length in the triplet (singlet) channel. The other pole is a spurious pole that results from the large scattering length in the triplet channel. The other pole is a spurious pole whose energy scale lies beyond the breakdown scale of the EFT. We expand the propagator in eq. (9) in powers of the effective range since the spurious pole causes problems in calculations for few-body systems. Through this expansion, we can also isolate the pieces that are dependent and independent of the effective range. Accordingly, we define the LO dibaryon propagator as:

\[
i D_{t,s}^{\text{LO}}(p_0, p) = i \frac{4\pi}{M y_t^{2}} \left( \frac{1}{a_{t,s}^i} - \sqrt{-M p_0 + \frac{p^2}{4}} \right)^{-1} + \frac{\rho_{t,s}^i}{2} \left( \frac{p^2}{4} - M p_0 \right) \right]^{-1}. \tag{9}
\]

In the case of a bound-state, we expand the triplet propagator near the deuteron pole. Up to NLO, the triplet propagator up is given by \[13:\]

\[
i D_{t,s}^{\text{NLO}}(p_0, p) = i \frac{4\pi}{M y_t^{2}} \left( \frac{1}{a_{t,s}^i} - \sqrt{-M p_0 + \frac{p^2}{4}} \right)^{-1} \times \left[ 1 + \frac{\rho_{t,s}^i}{2} \left( \frac{p^2}{4} - M p_0 - \frac{\gamma_t^2}{4} \right) \right]. \tag{10}
\]

For the singlet channel, the singlet propagator up to NLO is given by:

\[
i D_{s}^{\text{NLO}}(p_0, p) = i \frac{4\pi}{M y_s^{2}} \left( \frac{1}{a_{s}^i} - \sqrt{-M p_0 + \frac{p^2}{4}} \right)^{-1} \times \left[ 1 + \frac{\rho_{s}^i}{2} \left( \frac{p^2}{4} - M p_0 - \frac{\gamma_s^2}{4} \right) \right]. \tag{11}
\]

The long range properties of the deuteron wave-function are set by its residue, given by:

\[
Z_d = \frac{1}{1 - \gamma_t \rho_t} \approx 1.690(3), \tag{13}
\]

where \(\gamma_t\) is the deuteron binding momentum.

In the effective range expansion (ERE), the order by order expansion of \(Z_d\) is given by:

\[
Z_d^{\text{LO}} = 1, \quad Z_d^{\text{NLO}} = 1 + \gamma_t \rho_t \approx 1.408, \tag{14}
\]

where \(\gamma_t\) and \(\rho_t\) values are given in Tab. [I]

| Parameter | Value | Parameter | Value |
|-----------|-------|-----------|-------|
| \(\gamma_t\) | 45.701 MeV [18] | \(\rho_t\) | 1.765 fm [19] |
| \(a_s\) | -23.714 fm [20] | \(\rho_s\) | 2.73 fm [21] |
| \(a_p\) | -7.8063 fm [22] | \(\rho_C\) | 2.794 fm [22] |

TABLE I: Experimental two-body parameters

This result for the perturbative expansion of the \(Z\)-factor is based on the matching of the parameters in the EFT to the effective range expansion (ERE). At NLO, the parameters can also be chosen to fix the pole position and residue of the triplet two-body propagator to the deuteron values. This parameterization is known as the \(Z\)-parameterization and is advantageous because it reproduces the correct residue about the deuteron pole at NLO, instead of being approached perturbatively, order-by-order, as in ERE-parameterization [23, 27].

A. The proton-proton dibaryon

The expressions for the proton-proton (\(pp\)) dibaryon propagator up to NLO introduced below are based on Refs. [8, 24, 28].

At LO, the \(pp\) propagator contains an infinite series of ladder diagrams of Coulomb photon exchanges. The proton-proton propagator is given by [8]:

\[
i D_{pp}^{\text{LO}}(p_0, p) = i \frac{4\pi}{M y_{pp}^2} \left( \frac{1}{a_{s}^i} + 2\kappa \Phi (\kappa \rho') \right)^{-1}, \tag{15}
\]

where \(a_p\) denotes the proton-proton scattering length in the modified effective range expansion (recall that S-wave proton-proton scattering can only occur in the spin-singlet channel),

\[
\kappa = \frac{\alpha M}{2}, \tag{16}
\]

\(\alpha\) is the fine-structure constant \(\alpha \sim 1/137\), and

\[
p' = i \sqrt{\frac{p^2}{4} - M p_0}, \tag{17}
\]

with

\[
\Phi(x) = \psi(ix) + \frac{1}{2ix} - \log (ix), \tag{18}
\]

and \(\psi\) is the logarithmic derivative of the \(\Gamma\)-function.

The NLO correction to the \(pp\) propagator, results in a single NLO insertion into the LO \(pp\) propagator ampli-
The Faddeev equation for scattering, we set:

\[ D^{NLO}_{pp}(p_0, p) = D^{LO}_{pp}(p_0, p) \left[ 1 - \frac{\rho_C}{2} \left( \frac{p'^2 - \alpha q M}{a_p + 2\kappa \Phi(k/p')} \right) \right], \]

where \( \rho_C \) is the proton-proton effective range.

### III. THE THREE-NUCLEON SYSTEM

In this section, we review the derivation of the Faddeev equation for nucleons and its projections on the quantum numbers relevant for \( \text{H} \) \((n-d)\) and \( \text{He} \) \((p-d)\) at LO. The derivation of the Faddeev equation is based on Refs. \cite{6,7,9,10,12,14,29}. Three-nucleon S-wave scattering can occur in two channels: Either the quartet channel, in which the spin of the neutron and the deuteron are coupled to \( S = 3/2 \), or the doublet channel, in which the spins of the three nucleons are coupled to a total spin of \( 1/2 \). The spin-singlet dibaryon can now appear in the intermediate state, which leads to two coupled amplitudes that differ in the type of the outgoing dibaryon.

#### A. n-d scattering and the \( ^3\text{H} \) bound-state

The doublet channel in \( n-d \) scattering contains three coupled amplitudes, as shown in Fig. 2. For the \( n-d \) scattering, we set: \( a_{nn} = a_{np} = a_s \) and \( S_{np} = S_{nn} = S \). The Faddeev equation for \( n-d \) scattering can be written

\[ T(E, k, p) = M \rho_0(E, k, p) \]

\[ - My_s^2 \int D_s(E, p') T(E, k, p') K_0(p', p, E) \frac{p'^2}{2\pi^2} dp' \]

\[ + 3My_s y_t \int D_t(E, p') T(E, k, p') K_0(p', p, E) \frac{p'^2}{2\pi^2} dp' , \]

\[ S(E, k, p) = -3My_t y_s K_0 \]

\[ - My_s^2 \int D_s(E, p') S(E, k, p') K_0(p', p, E) \frac{p'^2}{2\pi^2} dp' \]

\[ + 3My_s y_t \int D_t(E, p') S(E, k, p') K_0(p', p, E) \frac{p'^2}{2\pi^2} dp' , \]

where

\[ K_0(k, p, E) = \frac{1}{2\pi k} Q_0 \left( \frac{p^2 + k^2 - ME}{pk} \right) , \]

with the 0th Legendre function of the second kind:

\[ Q_0(a) = \frac{1}{2} \int_{-1}^{1} \frac{1}{x + a} dx , \]

and the redefined propagator

\[ D_{t,s}(E, p) = D_{t,s} \left( E - \frac{p^2}{2M} \cdot p \right) . \]

Here, \( E \) denotes the total energy of the three-body system.

\#EFT is renormalizable, i.e., theory has no dependence on the ultraviolet cutoff (\( \Lambda \)). However, numerical and theoretical solutions of the integral equations eqs. (20) and (21) reveal a strong dependence on this cutoff. To overcome this problem, one needs to add a three-body force counterterm at LO, to restore renormalizability \cite{6, 7}.

Formally, this three-body force term is obtained by adding:

\[ \mathcal{L}_3 = M \frac{H(\Lambda)}{3\Lambda^2} \left( y_s^2 N^\dagger (\vec{t} \cdot \vec{\sigma}) (\vec{t} \cdot \vec{\sigma}) N + y_t^2 N^\dagger (\vec{s} \cdot \vec{\tau}) (\vec{s} \cdot \vec{\tau}) N \right. \]

\[ - y_t y_s \left[ N^\dagger (\vec{t} \cdot \vec{\sigma}) (\vec{s} \cdot \vec{\tau}) N + h.c. \right] , \]

\[ (25) \]

to the two-body Lagrangian (eq. (1)) and modifying the nucleon exchange term to contain the three-body force

\[ K_0(k, p, E) \rightarrow K_0(k, p, E) + \frac{H(\Lambda)}{\Lambda^2} , \]

where \( H(\Lambda) \) is the three-body force.

Equations (20) and (21) can be written in matrix form: \( t^{LO}(E, k, p) = B_0(E, k, p) + t^{LO}(E, k, p) \otimes \tilde{K}(p', p, E) \), \( (27) \)

where for n-d scattering:

\[ t^{nd}(E, k, p) = \left( \frac{T(E, k, p)}{S(E, k, p)} \right) , \]

\[ (28) \]
and we have defined the operation:

\[ A(\ldots, p) \otimes B(p, \ldots) = \int A(\ldots, p) B(p, \ldots) \frac{dp}{2\pi^d} . \tag{29} \]

The inhomogeneous part of the integral equation is given by:

\[ B_0^{ud}(E, k, p) = \left[ K_0(k, p, E) + \frac{H}{\Lambda^2} \right] \times \left( \frac{M y_i^2}{3 y_i y_s} \right) . \tag{30} \]

The kernel is,

\[ \hat{K}^{ud}(p', E) = K_0(p', p, E) \times \left( \frac{M y_i^2}{3 y_i y_s} \right) \times \left( \frac{D_t(E, p')}{D_s(E, p')} \right) + \frac{H(\Lambda)}{\Lambda^2} \times \left( \frac{M y_i^2}{3 y_i y_s} \right) \times \left( \frac{D_t(E, p')}{D_s(E, p')} \right) . \tag{31} \]

### 1. The Faddeev equation for the bound-state

The above sections describe the Faddeev equation for the three-nucleon system at an arbitrary energy. For energies close to the three-nucleon binding energy, i.e., when \( E \sim E_B \), the scattering amplitude takes the form

\[ t(E, k, p) = \frac{B^1(k) B(p)}{E - E_B} + \mathcal{R}(E, k, p) , \tag{32} \]

where the \( B(E, k) \) are what we call amputated wave functions or vertex factors, whereas the \( \mathcal{R}(E, k, p) \) are terms that are regular at \( E = E_B \), and thus can be neglected for \( E \to E_B \) \[39\]. By substituting eq. (32) into eq. (27), eq. (27) becomes

\[ \mathcal{B}^3_H(p) = \mathcal{B}^3_H(p') \otimes \hat{K}^3_H(p', p, E_{3H}) , \tag{33} \]

where \( \hat{K}^3_H(p', p, E_{3H}) = \hat{K}^{ud}(p', p, E_{3H}) \), i.e., the homogeneous integral equation has the form of the non-relativistic Bethe-Salpeter equation \[31\] \[32\], with \( E_{3H} \), the triton binding energy.

Specifically, for the case of the \(^3\)H bound-state, we express the amplitude as

\[ \mathcal{B}^3_H(p) = \left( \frac{\Gamma^3_H(p)}{\Gamma^3_H(p')} \right) , \tag{34} \]

where \( \Gamma_t, \Gamma_s \) denote the two bound-state amplitudes that have a spin-triplet or spin-singlet dibaryon, respectively.

For the triton, one needs to solve the integral equation:

\[
\begin{align*}
&\left( \frac{\Gamma^3_H(p)}{\Gamma^3_H(p')} \right) = \\
&\left[ K_0(p', p, E_{3H}) \left( -M y_i^2 D_t(E_{3H}, p') + 3 M y_i y_s D_s(E_{3H}, p') \right) \\
&+ \frac{H(\Lambda)}{\Lambda^2} \left( -M y_i^2 D_t(E_{3H}, p') + 3 M y_i y_s D_s(E_{3H}, p') \right) \\
&\otimes \left( \frac{\Gamma^3_H(p')}{\Gamma^3_H(p')} \right) ,
\end{align*}
\]

which can be written in compact form:

\[ \Gamma^3_H(p) = \sum_{\nu=t,s} M y_i y_{\nu} \left[ a_{\mu,\nu} K_0(p', p, E_{3H}) + b_{\mu,\nu} \frac{H(\Lambda)}{\Lambda^2} \right] \times \left( \frac{D_{\nu}(E_{3H}, p') \Gamma^3_H(p')} {\Gamma^3_H(p')} \right) , \tag{36} \]

where \( \mu = t, s \) are the different triton channels and \( y_{\mu,\nu} \) are the nucleon-dibaryon coupling constants for the different channels. The \( a_{\mu,\nu} \) and \( b_{\mu,\nu} \) are a result of \( n - d \) doublet-channel projection (see, for example, Ref. \[39\]), for example:

\[
\begin{align*}
&\hat{a}_{tt} = \frac{4}{3} \left[ (\sigma^t)^\alpha_\beta (P_t^\dagger)_{\gamma\delta} (P_t^\dagger)^{\delta\beta} (\sigma^t)^\gamma_\alpha \right] = -1 \quad (37a) \\
&\hat{a}_{ts} = \frac{4}{3} \left[ (\sigma^t)^\alpha_\beta (P_t^\dagger)_{\gamma\delta} (P_s^\dagger)^{\delta\beta} (\tau^A)_{\alpha\beta} \right]_{c,d=2} = 3 \quad (37b) \\
&\hat{a}_{st} = \frac{4}{3} \left[ (\tau^A)^{\alpha\beta} (P_t^\dagger)^{\delta\beta} (P_s^\dagger)^{\delta\alpha} (\sigma^t)^\gamma_\alpha \right]_{a,d=2} = 3 \quad (37c) \\
&\hat{a}_{ss} = \frac{4}{3} \left[ (\tau^A)^{\alpha\beta} (P_t^\dagger)^{\delta\beta} (P_t^\dagger)^{\delta\alpha} (\tau^B)^{cc} \right]_{a,e=2} = -1, \quad (37d)
\end{align*}
\]

where \( i, j \) are the different spin projections and \( A, B \) are the isospin projections, the same as those in eq. (1).

### B. \( p-d \) scattering at LO

In this subsection, we rederive the Faddeev equations for the \( p-d \) scattering for the doublet channel, similarly to the \( n-d \) scattering, where the quartet channel, which is of higher orders, is not relevant for this work.

The isospin partner of \(^3\)H, \(^3\)He, contains one neutron and two protons, so the Coulomb interaction should be taken into account for accurately describing this system. The photon Lagrangian of the Coulomb interaction retains only contributions from the Coulomb photon that generate a static Coulomb potential between two charged particles, defined as \[12\]:

\[
\mathcal{L}_{\text{photon}} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{\xi} \left( \partial_\mu A^\mu - \eta_\mu \eta_\nu \partial_\nu A^\mu \right)^2 , \tag{38}
\]
where $F^{\mu\nu}$ is the electromagnetic tensor, $A^\mu$ is the electromagnetic four-potential, $\eta^\mu = (1, \mathbf{0})$ is the unit timelike vector and the parameter $\xi$ determines the choice of gauge. For convenience, we introduce the Feynman rule corresponding to the Coulomb photon propagator:

$$iD_{\text{photon}}(k) = \frac{i}{k^2 + \lambda^2},$$

where $\lambda$ is an artificial small photon mass, added to regulate the singularity of the propagator when the momentum transfer vanishes [12].

Naively, proton-deuteron ($p-d$) scattering should contain an infinite sum of photon exchanges [10]. The typical momentum scale for the $^3$He bound-state is $Q \geq \sqrt{2ME_B}/A$ and the Coulomb parameter $\eta$ [8] is defined as:

$$\eta(Q) = \frac{\alpha M}{2Q}. \quad (40)$$

Therefore, for $^3$He, $Q \approx 70\text{MeV}$ and $\eta(Q) \ll 1$, the Coulomb interaction can be treated as a perturbation, which entails only one-photon exchange diagrams. The Coulomb diagrams that contribute to $p-d$ scattering are shown in Fig. 3 while the fine-structure constant $\alpha \sim 1/137$ can be used as an additional expansion parameter.

The power counting for the diagrams shown in Fig. 3 was discussed in Refs. [9]-[14], whereas in Ref. [15] it was shown that diagram (e) is of a higher order than diagrams (a)-(d), and need not be taken into account at NLO. Diagram (f) is the contribution from the non-perturbative proton-proton propagator, which affects the $^3$H-$^3$He binding energy difference, as discussed in detail in [15], and will be shown later.

1. The doublet channel

The doublet channel in $p-d$ scattering contains three coupled amplitudes as shown in Fig. 4. In contrast to the triton, for the $p-d$ scattering the spin-singlet dibaryon has two distinct isospin projections, i.e., the $np$ and $pp$ spin-singlet states [12]. The Faddeev equations for $p-d$ scattering, at LO, can be written as:

$$t^{pd}(E, k, p) = B_0^{pd}(E, k, p) + t^{pd}(E, k, p') \otimes \left[ \hat{K}^{pd}(p', p, E) + \hat{K}_0^{C}(p', p, E) \right], \quad (41)$$

where the three individual components of the amplitude $t$ are

$$t^{pd}(E, k, p) = \begin{pmatrix} T(E, k, p) \\ S(E, k, p) \\ P(E, k, p) \end{pmatrix}, \quad (42)$$

and:

$$B_0^{pd}(E, k, p) = \left[ K_0(k, p, E) + \frac{H}{\Lambda^2} \right] \times \left( \begin{array}{c} M y_0^2 \\ -y_t y_s \\ -2y_t y_s \end{array} \right) + \left( \begin{array}{c} M y_0^2 \left[ K_0^p(k, p, E) + K_0^C(k, p, E) \right] \\ -M y_t y_s K_0^p(k, p, E) \\ -2M y_t y_s K_0^b(k, p, E) \end{array} \right), \quad (43)$$

$$\hat{K}^{pd}(p', p, E) = MK_0(p', p, E) \begin{pmatrix} -y_t^2 D_t(E, p') \\ y_t y_s D_s(E, p') \\ y_t D_t(E, p') \end{pmatrix} + \frac{M H(\Lambda)}{\Lambda^2} \begin{pmatrix} -\frac{y_t^2}{3} D_t(E, p') \\ y_t y_s D_s(E, p') \\ -\frac{1}{3} y_t^2 D_s(E, p') \end{pmatrix}, \quad (44)$$

and

$$\hat{K}_0^{C}(p', p, E) = MK_0^{C}(p', p, E) \times \begin{pmatrix} -y_t^2 D_t(E, p') \\ y_t y_s D_s(E, p') \\ y_t D_t(E, p') \end{pmatrix} + \frac{M H(\Lambda)}{\Lambda^2} \begin{pmatrix} -\frac{y_t^2}{3} D_t(E, p') \\ y_t y_s D_s(E, p') \\ -\frac{1}{3} y_t^2 D_s(E, p') \end{pmatrix}. \quad (45)$$

FIG. 3: The possible one-photon exchange diagrams.
where
\[
K^C(p', p, E) = \begin{pmatrix}
K^C_{0}(p', p, E) + K^C_{1}(p', p, E) & K^C_{2}(p', p, E) & K^C_{3}(p', p, E) \\
K^C_{0}(p', p, E) & -K^C_{1}(p', p, E) + K^C_{2}(p', p, E) & K^C_{3}(p', p, E) \\
K^C_{0}(p', p, E) & K^C_{1}(p', p, E) & 0
\end{pmatrix}
\] (46)

and where:
\[
D_{pp}(E, p) = D_{pp}\left(E - \frac{p^2}{2M}, p\right)
\] (47)
is the Coulomb propagator [8,34].

The different one-photon exchange diagrams contributing to the Coulomb interaction are:
\[
K^C_{0}(p', p, E) = \frac{M\alpha}{2p'p}Q_0\left(\frac{-p^2 + p^2 + \lambda^2}{2p'p}\right) \times \left[\frac{\arctan\left(\frac{p' + p}{\sqrt{3p^2 - 4ME}}\right) - \arctan\left(\frac{2p' + p}{\sqrt{3p^2 - 4ME}}\right)}{p' - p}\right]
\] (48)
for Fig. 3(a),
\[
K^C_{0}(p', p, E) = K^C_{0}(p', p, E) = \alpha K_0(p', p, E)
\]
\[
\times \frac{1}{4\pi(p' - p)} \log\left\{\frac{2ME - 2p^2 + 2p'p - 2p'^2}{-p'\sqrt{4ME - 3p'^2 + 2p^2} + p + 3p^2 + 4ME - 3p'^2 + 2p'p - 3p^2 + 2ME - p'^2 - p^2}\right\}
\] (49)
for Fig. 3(b), and
\[
K^C_{0}(p', p, E) = K^C_{0}(p', p, E) = \alpha K_0(p', p, E)
\]
\[
\times \frac{1}{4\pi(p' - p)} \log\left\{\frac{2ME - 2p^2 + 2p'p - 2p'^2}{-p'\sqrt{4ME - 3p'^2 + 2p^2} + p + 3p^2 + 4ME - 3p'^2 + 2p'p - 3p^2 + 2ME - p'^2 - p^2}\right\}
\] (50)
for Fig. 3(c and d), where \(K_0\) was defined in eq. (22).

C. \(^3\)He bound-state amplitude and three-body force

The above section provides all the information necessary to solve the homogeneous Faddeev equations for \(^3\)He, similarly to those corresponding to \(^3\)H. For \(^3\)He, the homogeneous part of eq. (41) can be written as:

\[
\Gamma_{\mu}^{3\text{He}}(p) = \sum_{\nu=t,s,pp} M_{\mu\nu}y_{\nu} \left[ a_{\mu\nu}' K_0(p', p, E_{3\text{He}}) + b_{\mu\nu}' H(\Lambda) \frac{1}{\Lambda^2} \right] + a_{\mu\nu}' K^C_{\mu\nu}(p', p, E_{3\text{He}}) \otimes D_{\nu}(E_{3\text{He}}, p') \Gamma_{\nu}^{3\text{He}}(p'),
\] (51)

where \(\mu = t, s, pp\) are the different channels of \(^3\)He and and \(K^C_{\mu\nu}\) is the \(\mu, \nu\) index of \(K^C\) (eq. 46). Notice that for the \(p - d\) doublet-channel projection, the electromagnetic interaction does not couple to isospin eigenstates

\[\Gamma_{\mu}(p) = \sum_{\nu=3} M_{\mu\nu}y_{\nu} \left[ a_{\mu\nu}' K_0(p', p, E_{3\text{He}}) + b_{\mu\nu}' H(\Lambda) \frac{1}{\Lambda^2} \right] + a_{\mu\nu}' K^C_{\mu\nu}(p', p, E_{3\text{He}}) \otimes D_{\nu}(E_{3\text{He}}, p') \Gamma_{\nu}(p'),\]

such that:

\[
d'_{is} = \frac{4}{3} \left[ (\sigma^i)_{\beta}^\beta (P^i)^{\gamma}_t (P^A)^{\delta\eta}_t (\hat{1} \cdot i\delta^{A,3})_{d\alpha} \right]_{e, d=1} = 3,
\] (52a)

\[
d'_{ipp} = \frac{4}{3} \left[ (\sigma^i)_{\beta}^\beta (P^i)^{\gamma}_t (P^A)^{\delta\eta}_t (\hat{1} \cdot i\delta^{A,3})_{d\alpha} \right]_{e, d=1} = 3.
\] (52b)

The three-body force \(H(\Lambda)\) has no isospin dependence, i.e., \(H(\Lambda)^{3\text{H}} = H(\Lambda)^{3\text{He}}\). Therefore, it is possible to calculate the binding energy of \(^3\)He using the three-body force \(H(\Lambda)\) obtained in the triton system. Similar to Ref. [13], we find the binding energy that solves eq. (51) numerically, using the three-body force known from \(^3\)H at LO with a large range of binding energies. To evaluate the effect of the Coulomb interaction, we calculated the \(^3\)He binding energy for two cases: the full \(^3\)He Faddeev equations, as presented in this section, and the case of \(\alpha = 0\), with \(a_{mp} \neq a_{pp}\). The numerical results, as shown in Fig. 7, imply that the major contribution to the \(^3\)He
FIG. 4: Diagrammatic form of the homogeneous part of $p - d$ scattering that includes a three-body force. The double lines denote the dibaryon propagators $D_t$ (solid), $D_{np}$ (dashed) and $D_{pp}$ (doted). The red bubbles (T) represent the triplet channel (T=0, S=1), the green bubbles (S) represent the singlet channel (T=1, S=0) with an np dibaryon, while the blue bubbles (P) represent the singlet channel (T=1, S=0) with a pp dibaryon. The blue squares represent the three-body force.

binding energy originates from the isospin breaking (i.e., $a_{np} \neq a_{np}$ and the difference between eqs. (31) and (44), as discussed in Refs. [12, 14]) and not from the Coulomb diagrams (eq. (46)). In Section IV, we introduce the form of a general matrix element and use both the Coulomb interaction (eq. (46)) and the scattering length difference for the perturbative calculation of the $^3$He-$^3$He binding energy presented in Fig. 7.

Note that from now on we will use the numerical binding energies $E_{^3\text{He}}(\Lambda)$ at LO as the binding energy of $^3\text{He}$ rather than the experimental $E_{^3\text{He}} = 7.72$ MeV.

IV. NORMALIZATION OF THE THREE-NUCLEON AMPLITUDE

In this section, we define the expression that gives the normalization of the three-nucleon (i.e., $^3\text{H}$ and $^3\text{He}$) bound-state amplitude in the form of the non-relativistic Bethe-Salpeter (BS) equation. This normalization, as introduced in Refs. [32, 35, 36], is found to have a diagrammatic representation, enabling the calculation of the normalization operator as a sum over all the possible connections between two identical three-nucleon amplitudes.

A. The non-relativistic Bethe-Salpeter wave-function normalization

The three-nucleon homogeneous integral equation (eq. (33)) was found to have the same form as the non-relativistic bound-state Bethe-Salpeter equation (eq. (A-5)):

$$\Gamma(p) = M g^2 K_0(p, p', E) D(E, p') \otimes \Gamma(p') .$$  (53)
The normalization condition for the equation is given in Appendix A and in [35–37].

This is thus a representation of the normalization operator, $\hat{Z}$, such that:

$$\hat{Z}^{-1} = \int \frac{d^4p}{(2\pi)^4} d^4p' \Gamma(p) S(-p_0, -\mathbf{p}) D(E + p_0, \mathbf{p})$$

$$\times \frac{\partial}{\partial E} \left[ I(E, p, p') - M y^2 K_0(p, p', E) \right]_{E=E_B}$$

$$\times D(E + p_0, \mathbf{p}') S(-p_0, -\mathbf{p}') \Gamma(p').$$

Carrying out the angular and energy integrations gives

$$\hat{Z}^{-1} = \int \frac{p^2 dp}{2\pi^2} \int \frac{p'^2 dp'}{2\pi^2} \Gamma(p) D(E, p)$$

$$\times M^2 y^2 \left\{ \frac{1}{4\pi \sqrt{3p^2 - 4EM}} \frac{2\pi^2}{p'^2} \delta(p - p')$$

$$- \frac{1}{2 \left[ p'^2 (p^2 - 2EM) + (p^2 - EM)^2 + p'^4 \right]} \right\} D(E, p') \Gamma(p'),$$

with:

$$\hat{I}(E, p, p') = \frac{2\pi^2}{p'^2} \delta(p - p') D^{-1}(E, p),$$

and $S(p_0, \mathbf{p})$ as the one-nucleon propagator:

$$S(E, \mathbf{p}) = \frac{1}{p_0 - \mathbf{p}/\sqrt{2M}}.$$  (57)

\[B. \ The \ normalization \ of \ ^3\text{He}, ^3\text{He} \ wave-functions\]

The homogeneous part of the Faddeev equation of both $^3\text{H}$ and $^3\text{He}$ has the form of a non-relativistic BS equation, which couples different channels.

Using eq. (36), the normalization condition that determines the wave-function factor $Z^{3\text{H}}$ has the form:

$$1 = Z^{3\text{H}} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \sum_{\mu,\nu=t,s} \Gamma^{3\text{H}}(p) D_{\mu}(E_{3\text{H}} + p_0, p)$$

$$\times \left\{ \frac{\partial}{\partial E} \left[ \hat{I}_{\mu\nu}(E, p, p') - \hat{K}^{3\text{H}}_{\mu\nu}(p, p', E) \right]_{E=E_{3\text{H}}} \right\}$$

$$\times D_{\nu}(E_{3\text{H}} + p_0, p') \Gamma^{3\text{H}}(p').$$  (58)

We rewrite the above equation in terms of the wave-functions $\psi^3\text{H}_\mu(p)$ and obtain

$$1 = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \sum_{\mu,\nu=t,s} \psi^3\text{H}_\mu(p)$$

$$\times \left\{ \frac{\partial}{\partial E} \left[ \hat{I}_{\mu\nu}(E, p, p') - \hat{K}^{3\text{H}}_{\mu\nu}(p, p', E) \right]_{E=E_{3\text{H}}} \right\} \psi^3\text{H}_\nu(p').$$  (59)

We recall that $\psi^3\text{H}$ is the normalized three-nucleon wave-function

$$\langle \psi^3\text{H}_\mu | p \rangle = \sqrt{Z^{3\text{H}}} \int dp_0 D_{\mu}(E_{3\text{H}} + p_0, p) \Gamma^{3\text{H}}_\mu(p) S(-p_0, -p),$$

and

$$\hat{I}_{\mu\nu}(E, p, p') = \frac{2\pi^2}{p'^2} \delta(p - p') D_{\mu}(E, p)^{-1} \delta_{\mu, \nu},$$

$$\hat{K}^{3\text{H}}_{\mu\nu}(p, p', E) = M y_\mu y_\nu a_{\mu\nu} K_0(p', p, E),$$

where $\delta_{\mu, \nu}$ is the Kronecker delta.

For $^3\text{He}$, the normalization condition that determines the wave-function factor $Z^{3\text{He}}$ has the form:

$$1 = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \sum_{\mu,\nu=t,s,pp} \psi^{3\text{He}}_\mu(p)$$

$$\times \left\{ \frac{\partial}{\partial E} \left[ \hat{I}_{\mu\nu}(E, p, p') - \hat{K}^{3\text{He}}_{\mu\nu}(p, p', E) \right]_{E=E_{3\text{He}}} \right\} \psi^{3\text{He}}_\nu(p'),$$  (63)

where:

$$\langle \psi^{3\text{He}}_\mu | p \rangle = \sqrt{Z^{3\text{He}}} \int dp_0 D_{\mu}(E_{3\text{He}} + p_0, p)$$

$$\times \Gamma^{3\text{He}}_\mu(p) S(-p_0, -p),$$  (64)

$$\hat{K}^{3\text{He}}_{\mu\nu}(p, p', E) = M y_\mu y_\nu a'_{\mu\nu} \left[ K_0(p', p, E) + K^C_{\mu\nu}(p', p, E) \right],$$  (65)

and $K^C_{\mu\nu}(p', p, E)$ is the $\mu, \nu$ index of the matrix $K^C$ (eq. (46)).

\[C. \ The \ diagrammatic \ form \ of \ the \ normalization\]

The implication of the one-body unit operator is turning a single nucleon operator into two one-nucleon propagators under the assumption of energy and momentum conservation in the center-of-mass system:
\[ \sum_{i}^A p^i = \sum_{i}^A p'^i = 0, \]
\[ \sum_{i}^A p_0^i = \sum_{i}^A p_0'^i = E, \]  
(66)

where \( i, j \) are the different nucleon indexes, \( p_i, (p_i') \) refers to the one-nucleon incoming (outcoming) momentum and \( p_0^i(p_0'^i) \) refers to the \( i \)'s nucleon incoming (outcoming) energy.

The Jacobi momentum \( p \) is defined as the relative momentum between the dimer and the one-nucleon of the incoming (outcoming) three-nucleon wave-function, \( p(p') = \frac{1}{2} [p(p') - (-p(-p'))] \) and \( E \) is the total three-nucleon energy.

Let us note that an energy derivative acting on a single nucleon propagator that contains the energy \( E \) can be written as two propagators:
\[ \frac{\partial}{\partial E} S(E, p) = - \int \frac{d^3p'}{(2\pi)^3} S(E, p) \times S(E, p') (2\pi)^3 \delta(p - p'). \]  
(67)

Therefore, the normalization operator for eq. (53) can be written as a multiplication of the one-nucleon propagators and the corresponding delta functions, under the assumption of energy and momentum conservation:
\[ \frac{\partial}{\partial E} S(E, p) = - \int \frac{d^3p'}{(2\pi)^3} \left[ S(E, p) \times S(E, p') \right] (2\pi)^3 \delta(p - p'). \]  

By performing the energy integration, eq. (68) becomes:
\[ \frac{\int d^3p \int d^3p'}{(2\pi)^6} \Gamma(p) D(E, p) \times \frac{\partial}{\partial E} S(E, p) \]
\[ \times M^2 y^2 \left[ \frac{-i}{4\pi(p - p')} \log \left( \frac{i\sqrt{3p'^2 - 4EM} - 2p - p'}{i\sqrt{3p'^2 - 4EM} - p - 2p'} \right) \right] \frac{1}{p'^2 - (2p^2 - 2EM)^2 + (p^2 - EM)^2 + 2p'^4} \]
\[ \times D(E, p') \Gamma(p'), \]  
(69)

which is identical to eq. (54).

Figure 5 shows in detail the two topologies of the normalization diagrams. For the case in which the normalization insertion connects the two dimers in the three-nucleon systems, it is proportional to \( \frac{\partial}{\partial E} \hat{\mathcal{I}} \) (Fig. 5 (a)). For the case in which the one-nucleon exchange propagator connects both one of the dimer nucleons and the single nucleon, the diagram is proportional to \( \frac{\partial}{\partial E} \hat{\mathcal{K}} \) (Fig. 5 (b)).

Note that since for both \(^3\)H and \(^3\)He, \( \hat{\mathcal{K}}_{\mu\nu} \) is not diagonal, eqs. (58) and (63) involve different channels.

V. THREE-NUCLEON MATRIX ELEMENTS IN \( \#\text{EFT} \)

In this section, we present the general method for calculating three-nucleon matrix elements in \( \#\text{EFT} \). This method is used in this work to calculate three-nucleon electroweak observables, as well as the \(^3\)He energy shift perturbatively and the NLO contribution to the three-nucleon wave-functions.

A. The general form of an \( A = 3 \) matrix element

In Section [IV], we showed that the three-nucleon normalization can be written as:
\[ 1 = \sum_{\mu, \nu} \psi_{\mu}^i(p) \otimes \left\{ \frac{\partial}{\partial E} [\hat{\mathcal{I}}_{\mu\nu}(E, p, p')] - \hat{\mathcal{K}}_{\mu\nu}(p, p', E) \right\}_{E=E_i} \otimes \psi_{\nu}^i(p'), \]  
(70)
which are a result of $N - d$ doublet-channel projection (eqs. (34) and (32)). Note that we are considering here one-body operators that do not have additional momentum dependence. However, the formulas given here could easily be extended also to this case.

Equation (72) can be generalized to any operator, $O_{j,i}$, between the initial (i) and final (j) $A = 3$ bound-state wave-functions ($\psi_{i,j}$), whose matrix element is evaluated as

$$
\langle O_{j,i}(q_0, q) \rangle = \langle S, S', I', I', E' | O_{j,i}(q_0, q) | S, S, I, I, E \rangle ,
$$

(75)

where:

- $S$ denotes the total spin ($\frac{1}{2}$) of the three-nucleon system.
- $S_z, S'_z$ denote the initial and final spin projections, respectively.
- $I$ denotes the total isospin ($\frac{1}{2}$) of the three-nucleon.
- $I_z, I'_z$ denote the initial and final isospin projections, respectively.
- $q$ is the momentum transfer of such an operator (assuming that for the initial state, the three-nucleon total momentum is zero).

The energy transfer is defined as: $q_0 = E' - E$.

Therefore, a general operator that connects two three-nucleon bound-states with $I = \frac{1}{2}, S = \frac{3}{2}$, factorizes into the following parts:

$$
O_{j,i} = O^J O^T O_{j,i}(q_0, q),
$$

(76)

where $O^J$, the spin part of the operator whose total spin is $J$, and $O^T$, the isospin part of the operator, depend on the initial and final quantum numbers. The spatial part of the operator, $O_{j,i}(q_0, q)$, is a function of the three-nucleon wave-function’s binding energies ($E_i, E_j$) and the energy and momentum transfer ($q_0, q$, respectively).

The observable associated with the above matrix element is also related to a reduced matrix element between $A = 3$ bound-state wave functions:

$$
\langle ||O_{j,i}(q_0, q)|| \rangle = \langle S, I, I', E', q | O^J O^T O_{j,i}(q_0, q) | S, I, E \rangle .
$$

In the next subsection, we write explicitly the reduce matrix element term for a general one-body operator. Note that the amplitude $\Gamma^i(p)$ (and as a result, $\psi^i(p)$) still carries implicit spin and isospin indices, $S_z$ and $I_z$, respectively. We calculate the reduced matrix element shown above by performing the spin algebra with the afore-mentioned spin and isospin projectors and the spin- and isospin part of the operator under consideration for one particular choice of external spin projections. Then we use the Wigner-Eckhart theorem to combine this matrix element with a Clebsch-Gordan coefficient to obtain the reduced matrix element.
B. Matrix elements of one-body operators

The one-body normalization operator, $O_{\mu\nu}^{\text{norm}}(E)$ (eq. (72)), is a result of $N-d$ doublet-channel projection (eqs. (37) and (52)). For the case that the one-body spin and isospin operators are combinations of Pauli matrices, the general matrix element will be a result of the different $N-d$ doublet-channel projections coupled to a spin-isospin operator. To evaluate the reduced matrix element of a general one-body operator, one needs to calculate explicitly one component of the spin operator, $O_{J^0}$. For an operator whose spin part is proportional to $\sigma\sigma\sigma$, for example, the zero-component of $\langle O_{J^0} \rangle$, $\langle O_{J^0} \rangle_0$ is given by:

$$\langle O_{1Bj,i}(q_0,q) \rangle_0 = \sum_{\mu,\nu} y_{\mu} y_{\nu} \langle \psi_j | y_{\mu} y_{\nu} \left( d_{\mu\nu}^{ij} \hat{I}(q_0,q) + a_{\mu\nu}^{ij} \hat{K}(E,q_0,q) \right) | \psi_i \rangle ,$$  

(77)

where $\hat{I}(E,q_0,q)$ and $\hat{K}(E,q_0,q)$ represent all the possible connections between two three-nucleon wave-functions ($\psi_j, \psi_i$) that contain a one-body insertion of momentum and energy transfer without a Coulomb interaction. The spatial parts that do not contain a one-nucleon exchange are denoted by $\hat{I}(E,q_0,q)$, and the spatial parts that do contain a one-nucleon exchange are denoted by $\hat{K}(q_0,q)$; the full expressions for $\hat{I}(E,q_0,q)$ and $\hat{K}(E,q_0,q)$ are given in Appendix B. $a_{\mu\nu}^{ij}$ and $d_{\mu\nu}^{ij}$ are a result of the $N-d$ doublet-channel projection coupled to $O_{1B}^\mu O_{1B}^\nu$. $K_{C\mu\nu}(E,q_0,q)$ are the diagrams that contain a one-photon interaction in addition to the energy and momentum transfer. A derivation of an analytical expression for these diagrams is too complex, so they were calculated numerically only.

Figure 6 shows all possible diagrams of a one-body insertion of momentum and energy transfer between two three-nucleon wave-functions that contain a one-nucleon exchange.

The one-body reduced matrix element, $\langle \langle O_{1Bj,i}(E,q_0,q) \rangle \rangle$, can be easily calculated as a function of the three-nucleon quantum total spin and isospin numbers, using the Wigner-Eckart theorem. Since these calculations are not dependent on the spatial structure of the three-nucleon wave function, one can isolate the spin and isospin matrix elements in terms of the three-nucleon quantum numbers such that (again for the zero component):

$$\langle \langle O_{1Bj,i}(q_0,q) \rangle \rangle_0 = \frac{\sqrt{2}}{\frac{1}{2}S_z J_0} \sum_{\mu,\nu} \langle \psi_j | y_{\mu} y_{\nu} \left( d_{\mu\nu}^{ij} \hat{I}(q_0,q) + a_{\mu\nu}^{ij} \hat{K}(q_0,q) \right) | \psi_i \rangle ,$$  

(78)
that can be written as:

\[
\langle \| O_{j,i}^{J_R}(q_0, q) \| \rangle = \frac{1}{2} \| O^J \| \frac{1}{2} \langle \frac{1}{2}. I'_z | O^T | I_z, \frac{1}{2} \rangle \\
\times \sum_{\mu, \nu} \langle \psi_{\mu}^j | y_{\mu} y_{\nu} \{ d_{\mu i}^{ij} \} | \psi_{\nu}^i \rangle,
\]

such that for \( i = j \):

\[
d_{\mu i}^{ii} = \delta_{\mu, \nu},
\]

\[
a_{\mu i}^{ii} = \begin{cases} a_{\mu i} & i = 3 \text{ H} \\
a_{\mu i}' & i = 3 \text{ He} \end{cases}
\]

where \( K_{\mu \nu}(q_0, q) = 0 \) for \( ^3\text{H} \).

The reduced matrix element of the spin part of the operator, \( \langle \frac{1}{2} \| O^J \| \frac{1}{2} \rangle \), is a function of the initial and final total spin of the \( A = 3 \) nucleon wave-function. For the case that \( O^J = \sigma \), the reduced matrix element, \( \langle \frac{1}{2} \| O^J \| \frac{1}{2} \rangle \), is calculated using the Wigner-Eckart theorem such that:

\[
\langle \frac{1}{2} \| \sigma \| \frac{1}{2} \rangle = 2 \langle \frac{1}{2} \| s \| \frac{1}{2} \rangle = \sqrt{6}.
\]

**C. Two-body matrix element**

In contrast to the normalization operator given in eq. (72), which contains only one-body interactions, a typical EFT electroweak interaction contains also the following two-body interactions up to NLO:

\[
t^t t, s^t \bar{s}, (s^t t + h.c),
\]

under the assumption of energy and momentum conservation. The diagrammatic form of the different two-body interactions, given in Tab. III is a result of the Hubbard-Stratonovich transformation of a four-nucleon interaction vertex (see, for example, Refs. [34, 38] and Appendix C).
### TABLE II: The Feynman rules for the two-body interactions.

For all Feynman rules, a capital letter indicates the isospin index, while a small letter indicates the spin index.

| Diagrammatic Structure | Feynman Rule |
|------------------------|--------------|
| (6a) $s^1(N^TP_sN)+h.c.$ | $\frac{1}{\sqrt{2\pi\rho_s}}(\mu - \frac{1}{a_s})\left[\frac{1}{\sqrt{8}}\sigma^2T^2A + h.c\right]$ |
| (6b) $s^1(N^TP_sN)+h.c.$ | $\frac{1}{2\pi\rho_s}(\mu - \frac{1}{a_s})^2\left[\frac{1}{\sqrt{8}}\sigma^2T^2A + h.c\right]$ |
| (7) $s^1s$ | $\frac{1}{2\pi\rho_s}(\mu - \frac{1}{a_s})^2$ |

Note that diagrams 1a, 2a, 4a and 6a are proportional to $\mu$, in contrast to the other diagrams, which are proportional to $\mu^2$ and therefore can be neglected.

### D. Deuteron normalization and the matrix element in pionless EFT

Deuteron normalization and the matrix element

The calculation of matrix elements is significantly harder in the three-body sector than in the two-body sector due to the more complicated structure of three-body diagrams \[6, 7\]. However, a closer look at the deuteron wave-function normalization reveals that the deuteron wave-function normalization can also be written in the same manner as discussed above, since

$$Z_d^{-1} = i\frac{\partial}{\partial E} \frac{1}{iD_t(E,p)}\bigg|_{E=E, p=0}, \quad Z_d = \frac{1}{1 - \gamma_t\rho_t}, \quad (84)$$

where the energy derivative of $\frac{1}{iD_t(E,p)}$ is equivalent to the addition of a one-nucleon propagator, as discussed in Section \[14\]. Hence, a general deuteron matrix element that contains energy and momentum transfer (such as the deuteron magnetic moment) can be written as the sum over all possible connections \[5\]:

$$\langle S, p' || O_{j,i}(q_0, q) || S, p \rangle = \langle 1 || O^j || 1 \rangle \left(\frac{M^2}{8\pi\gamma_t}\right)^{-1} \langle p' || \tilde{T}(q_0, q) || p \rangle. \quad (85)$$

For the case that $O^j = 1$ and $q_0, q = 0$, eq. \[85\] gives the deuteron form factor, $F_C(0)$, which is equal to 1:

$$\langle || O_{j,i}(q_0, q) || \rangle = \left(\frac{M^2}{8\pi\gamma_t}\right)^{-1} \tilde{T}(E_d, 0, 0) = \left(\frac{M^2}{8\pi\gamma_t}\right)^{-1} M^2 = 1. \quad (86)$$

This matrix element form, which is very similar to the general three-body matrix element (eq. \[78\]), implies that in the case of bound-state matrix elements, our wave-function approach can be applied in the two- and the three-nucleon systems, consistently.

### E. Example: $^3$He-$^3$H binding energy difference with a perturbative Coulomb

In this subsection, we apply the formalism introduced above to the so-called Coulomb energy shift in the three-nucleon system. We define the Coulomb-induced energy shift, $\Delta E$, as \[12, 14, 15\]:

$$-E_{^3\text{He}} = -E_{^3\text{H}} + \Delta E. \quad (87)$$

The energy difference between $^3$H and $^3$He due to the Coulomb interaction can be calculated perturbatively at LO as a matrix element of one-photon exchange diagrams (Fig. 3 a-d) and the $pp$ propagator (diagram f) between two triton bubbles, as described in detail in \[14, 15\]. In our notation, these Coulomb interactions can be treated as a special case of a general matrix element, despite the fact that the Coulomb interaction does not conserve the three-nucleon isospin. This representation is possible since we divided the contribution to the energy shift into a one-body (1B) term and a two-body (2B) term. The one-body term originates from the one-photon exchange diagrams being calculated as a one-body inter-
action between two $^3$H bound-state wave-functions and does not affect the three-nucleon isospin. The two-body term originates from the difference between the proton-proton propagator and the spin singlet propagator (which is a two-body operator).

In terms of eq. (75), $\Delta E$ has the form:

$$
\Delta E(\Lambda) = Z_{^3\text{H}}^H \sum_{\mu,\nu=t,s} \psi_\mu^3(p) \otimes c_{\mu\nu} K_{\mu\nu}^C(p, p', E_{^3\text{H}}) \otimes \psi_\nu^3(p') + Z_{^3\text{H}}^H \sum_{\mu=t,s} \left[ \Gamma_{\mu}^3(p) D_{\mu}(E_{^3\text{H}}, p) \right] \otimes \left[ a_{\mu s} K_0(p, p', E_{^3\text{H}}) + b_{\mu s} \frac{H(\Lambda)}{\Lambda^2} \right] \otimes \left\{ [D_{pp}(E_{^3\text{H}}, p') - D_{s}(E_{^3\text{H}}, p')] \Gamma_{s}^3(p') \right\} .
$$

(88)

Using the fact that:

$$
\Gamma_{s}^3(p') = \sum_{\mu=t,s} \left[ \Gamma_{\mu}^3(p) D_{\mu}(E_{^3\text{H}}, p) \right] \otimes \left[ a_{\mu s} K_0(p, p', E_{^3\text{H}}) + b_{\mu s} \frac{H(\Lambda)}{\Lambda^2} \right],
$$

(89)

eq (88) becomes:

$$
\Delta E(\Lambda) = \sum_{\mu,\nu=t,s} \psi_\mu^3(p) \otimes c_{\mu\nu} K_{\mu\nu}^C(p, p', E_{^3\text{H}}) \otimes \psi_\nu^3(p') + \sum_{\mu=t,s} \psi_\mu^3(p) \otimes \left[ \frac{D_{pp}(E_{^3\text{H}}, p) - D_{s}(E_{^3\text{H}}, p)}{D_{s}(E_{^3\text{H}}, p)^2} \times \frac{\delta(p - p')}{p'^2} \frac{2\pi^2 \delta_{\mu s} \delta_{\nu s}}{(90)} \right)
$$

where $Z_{^3\text{H}}^H$ is the $^3\text{H}$ normalization, and

$$
O_{\mu\nu}^{(1B)}(E, p, p') = c_{\mu\nu} K_{\mu\nu}^C(p, p', E) \delta(q - p + p'),
$$

(91)

$$
O_{\mu\nu}^{(2B)}(E, p, p') = \left[ \frac{D_{pp}(E, p) - D_{s}(E, p)}{D_{s}(E, p)^2} \times \frac{\delta(p - p')}{p'^2} \frac{2\pi^2 \delta_{\mu s} \delta_{\nu s}}{2\pi^2} \right),
$$

(92)

where $K_{\mu\nu}^C(p, p', E)$ is given in eq. (46) and $c_{\mu\nu} = a_{\mu\nu}$ under the assumption that $\Gamma_{x} = \Gamma_{np}, \Gamma_{pp}$; $D_{pp}(E, p), D_{s}(E, p)$ were defined in Section I.

Figure 7 shows that summing over all possible one- and two-body Coulomb diagrams (eq. (88)) is consistent with the non-perturbative calculation presented in Subsection III C. Both calculations reproduce the predictions presented in Ref. 35, and this result serves as a test of the numerical calculation presented here.

VI. EFT AT NEXT-TO-LEADING ORDER: PERTURBATIVE CORRECTION TO THE THREE-NUCLEON MATRIX ELEMENTS

The components needed for a consistent calculation of an $A = 3$ matrix element up to NLO (i.e., retaining terms of order $\frac{\Lambda}{\Lambda_{\text{cut}}}$) are the interaction operator, $O_{ij}(q_0, q)$, and the bound-state amplitudes up to this order. In this section, we present how the NLO contributions to the three-nucleon bound-state amplitude can be calculated using the method presented in Section V. Specifically, we follow the NLO bound-state calculation of Vanasse et al. 13,26, except that we consider $y_i \neq y_s$.

In our notation, we distinguish between the NLO correction to the scattering matrix, $(E, k, p)$, and the NLO correction to the bound-state scattering amplitude $(B(k))$, which is the homogeneous solution of the Faddeev equations.

A. The NLO correction to the full scattering amplitude

In this subsection, we use the formalism introduced in section V to calculate the NLO correction to the full scattering amplitude.

For simplicity, in a similar manner to that presented in section IV, we first write the NLO correction for the case that the $t$-matrix contains only one channel, i.e., $t(E, k, p) = T(E, k, p)$, and then extend this formalism for $^3$H and $^3$He.

The full $t$-matrix can be expanded order-by-order:

$$
T(E, k, p) = T^{(1)}(E, k, p) + T^{(1)}(E, k, p) + \ldots,
$$

(93)
FIG. 7: Predictions for the $E_{3\Lambda}$, $E_{3\Lambda}$ binding energies as a function of the cutoff $\Lambda$. The dashed-dotted line is the $^{3}\text{He}$ binding energy calculated using the non-perturbative solution where $\alpha = 0$ and $a_{np} \neq a_{pp}$. The solid line is the $^{3}\text{He}$ binding energy calculated using the non-perturbative solution (subsection III C). The points represent the binding energy predicted using perturbation theory for the one-body term (dots) and one- and two-body term (crosses) (eq. (88)). The short-dashed line is the experimental value $E_{3\Lambda} = 7.72$ MeV, and the long-dashed line is the experimental value $E_{3\Lambda} = 8.48$ MeV.

By using the STM equation (eq. [20]), eq. [95] becomes:

$$T^{(1)}(E,k,p) = \int \frac{p^2 dp'}{2\pi^2} T^{LO}(E,k,p')D^{LO}(E,p')T^{LO}(E,p',p)$$

$$= - \frac{2\pi}{My^2} \rho \int \frac{p^2 dp'}{2\pi^2} T^{LO}(E,k,p') \frac{3p^2/4 - EM - 1/a_2^2}{(\sqrt{3}p^2/4 - EM - 1/a_2^2)^2} \times T^{LO}(E,p',p),$$

(96)

where $\rho$ is the effective range, $a_2$ is the dibaryon scattering length, $K_0$ is defined in eq. (22),

$$\Delta(E,p) = D^{NLO}(E,p) - D^{LO}(E,p),$$

(97)

and $D^{NLO}(E,p)$ is defined in eq. (11).

FIG. 8: The $t$-matrix describing a bound-state up to NLO (red bubbles). The LO $t$-matrix is the result of the LO homogeneous Faddeev equation - eq. (27). The NLO correction to the $t$-matrix includes the effective range $\rho$ (black dot). The double lines are the propagators of the two dibaryon fields and the blue square is the three-body force.

B. The NLO corrections to the three-nucleon bound-state pole position

Following Vanasse et al. [13], we use eq. (95) to predict the NLO correction to the three-nucleon binding energy. We extend the method developed by Ji, Phillips and Plat-ter [39] to include complications due to the isospin. The scattering amplitude possesses a pole at the binding en-
ergy and can be written as:
\[
T(E, k, p) = T^{\text{LO}}(E, k, p) + T^{(1)}(E, k, p)
\]
\[
= \frac{Z^{\text{LO}}(k, p) + Z^{(1)}(k, p)}{E - (E_B + \Delta E_B)} + R_0(E, k, p) + R_1(E, k, p),
\]
where \(Z^{\text{LO}}, Z^{(1)}\) are the residue vector functions and \(\Delta E_B\) is the NLO correction to the binding energy. Both \(R_0(E, k, p)\) and \(R_1(E, k, p)\) are regular at \(E = E_B\), so they can be neglected. At the first order in ERE (NLO) of eq. (98), one finds that (99):
\[
T^{(1)}(E, k, p) = \frac{Z^{(1)}(k, p)}{E - E_B} + \Delta E_B \left(\frac{Z^{\text{LO}}(k, p)}{E - E_B}\right)^2,
\]
where \(Z^{\text{LO}}\) is defined around the pole \((E \to E_B)\) from eq. (98) as:
\[
Z^{\text{LO}}(k, p) = \lim_{E \to E_B} (E - E_B) T^{\text{LO}}(E, k, p).
\]

For \(E \to E_B\), \(\Delta E_B\) is given by:
\[
\Delta E_B = \lim_{E \to E_B} \frac{T^{(1)}(E, k, p)(E - E_B)^2}{Z^{\text{LO}}(k, p)}.
\]

It might seem that the binding energy correction \(\Delta E_B\) depends on the incoming and outgoing momenta \((k, p)\). However, we would expect the NLO binding energy, \(E_B^{\text{NLO}} = E_B + \Delta E_B\), to depend on the cutoff \(\Lambda\) only, similarly to LO (as shown in Fig. 7), so it is essential to examine its momentum dependence. Since for a bound-state (eq. (32)):
\[
T(E, k, p) = \Gamma(k)\Gamma(p) \frac{1}{E - E_B},
\]
eq (95) becomes:
\[
T^{(1)}(E, k, p)(E - E_B)^2 = 
\Gamma^{\text{LO}}(k)\Gamma^{\text{LO}}(p) D^{\text{LO}}(E, p') \otimes O^{(1)}(E, p', p'')
\]
\[
\otimes D^{\text{LO}}(E, p'') \Gamma^{\text{LO}}(p) = 
\Gamma^{\text{LO}}(k)\psi^{\text{LO}}(k') \otimes O^{(1)}(E, p', p'') \otimes \psi^{\text{LO}}(p'') \Gamma^{\text{LO}}(p),
\]
where \(\psi^{\text{LO}}(p)\) is the three-nucleon wave-function (eq. (60) for \(3^\text{H}\) and eq. (64) for \(3^\text{He}\)) and \(D^{\text{LO}}(E, p)\) is the dibaryon propagator at LO (eq. (10)). Since \(Z^{\text{LO}}(k, p) = \lim_{E \to E_B} T^{\text{LO}}(k)\Gamma^{\text{LO}}(p)\), substituting eq. (103) into eq. (101) yields:
\[
\Delta E_B = \psi^{\text{LO}}(p') \otimes O^{(1)}(E, p', p'') \otimes \psi^{\text{LO}} = f(\Lambda),
\]
which is a function of the cutoff \(\Lambda\) only, i.e., it has no dependence on the momenta \(k\) and \(p\).

### C. NLO three-body force

From eq. (104), we find that the NLO correction to \(3^\text{H}\) has a cutoff dependence that needs to be removed [41]. Similarly to the LO case, this \(\Lambda\)-dependence is removed by adding a term that includes an NLO correction to the LO three-body force, \(H^{(1)}(\Lambda)\), such that \(T^{(1)}\) becomes:
\[
T^{(1)}(E, k, p) = -T^{\text{LO}}(E, k, p') D^{\text{LO}}(E, p')
\]
\[
\otimes \left\{ \frac{My^2}{2} \left[ K_0(p', p'', E) + \frac{H(\Lambda)}{A^2} \right] \right\}
\]
\[
\otimes [\Delta(E, p'') + \Delta(E, p')] \right\} \otimes D^{\text{LO}}(E, p'') T^{\text{LO}}(E, p'', p)
\]
\[
- T^{\text{LO}}(E, k, p') D^{\text{LO}}(E, p') \otimes \frac{H^{(1)}(\Lambda)}{A^2} My^2
\]
\[
\otimes D^{\text{LO}}(E, p'') T^{\text{LO}}(E, p'', p). \]

Using the STM equation (eq. (20)) yields:
\[
T^{(1)}(E, k, p) = -\frac{\rho}{My^2} \int \frac{p'dp'}{\pi} T^{\text{LO}}(E, k, p')
\]
\[
\times \left\{ \frac{3p^2/4 - EM - 1/a^2}{\sqrt{3p^2/4 - EM - 1/a^2}} \right\} T^{\text{LO}}(E, p', p)
\]
\[
- T^{\text{LO}}(E, k, p') D^{\text{LO}}(E, p') \otimes \frac{H^{(1)}(\Lambda)}{A^2} My^2 \otimes D^{\text{LO}}(E, p'') \times T^{\text{LO}}(E, p'', p).
\]

Using eq. (105), \(\Delta E_B\) is now given by:
\[
\Delta E_B(\Lambda) = -\frac{1}{2} My^2 \psi^{\text{LO}}(p') \otimes \left\{ \left[ K_0(p', p'', E) + \frac{H(\Lambda)}{A^2} \right] \right\}
\]
\[
\times [\Delta(E, p'') + \Delta(E, p')] \right\} \otimes \psi^{\text{LO}}(p'')
\]
\[
- My^2 \psi^{\text{LO}}(p') \otimes \frac{H^{(1)}(\Lambda)}{A^2} \otimes \psi^{\text{LO}}(p'').
\]

Let us now consider the three-nucleon case and set eq. (107) to zero for \(3^\text{H}\) [13], with
\[
\mathcal{B}(p) = \mathcal{B}^{3\text{H}}(p)
\]
and
\[
\psi(p) = \psi^{3\text{H}}(p) = \left( \frac{\psi_1^{3\text{H}}(p)}{\psi_2^{3\text{H}}(p)} \right).
\]

The NLO correction to the \(3^\text{H}\) binding energy is given by:
\[
\Delta E_B(\Lambda) = \sum_{\mu, \nu} \psi^{\text{LO}}(p) \otimes O^{(1)}_{\mu \nu}(E_{3\text{H}}, p, p') \otimes \psi^{\text{LO}}(p').
\]
with

\[
O^{(1)}_{\mu} (E_3H, p, p') = M y_\mu y_{\nu} \left\{ \frac{1}{2} \left[ a_{\mu \nu} K_0(p, p', E_3H) + b_{\mu \nu} \frac{H(\Lambda)}{\Lambda^2} \right] \right. \\
\times \left[ \Delta_\mu (E_3H, p) + \Delta_\nu (E_3H, p') \right] + \left. b_{\mu \nu} \frac{H^{(1)}(\Lambda)}{\Lambda^2} \right\}.
\]

(111)

Therefore, we find that the NLO three-body force has the form:

\[
- \frac{H^{(1)}(\Lambda)}{\Lambda^2} = M \sum_{\mu, \nu = t, s} \psi^3_H(p) \\
\otimes \left\{ \frac{1}{2} y_\mu y_{\nu} \left[ a_{\mu \nu} K_0(p, p', E_3H) + b_{\mu \nu} \frac{H(\Lambda)}{\Lambda^2} \right] \right. \\
\times \left[ \Delta_\mu (E_3H, p) + \Delta_\nu (E_3H, p') \right] \right. \\
\times \left\left. \left. b_{\mu \nu} \otimes \psi^3_H(p') \right\right\} \right\} \right. \\
\left[ M \sum_{\mu, \nu = t, s} y_\mu y_{\nu} \psi^3_H(p) \otimes b_{\mu \nu} \otimes \psi^3_H(p') \right]^{-1}. \quad (112)
\]

Using the fact that:

\[
\Gamma^3_H(p') = M \sum_{\mu = t, s} y_\mu y_{\nu} \psi^3_H(p) \otimes \left[ a_{\mu \nu} K_0(p, p', E_3H) + b_{\mu \nu} \frac{H(\Lambda)}{\Lambda^2} \right] \otimes \psi^3_H(p'),
\]

(113)

and

\[
\Gamma^3_H(p) = M \sum_{\nu = t, s} y_\mu y_{\nu} \left[ a_{\mu \nu} K_0(p, p', E_3H) + b_{\mu \nu} \frac{H(\Lambda)}{\Lambda^2} \right] \otimes \psi^3_H(p') \right. \\
\left. \otimes \psi^3_H(p'). \quad (114)
\]

Equation (112) becomes:

\[
- \frac{H^{(1)}(\Lambda)}{\Lambda^2} = \frac{1}{2} M \sum_{\mu, \nu = t, s} \psi^3_H(p) \times \left\{ \frac{\Delta_\mu (E_3H, p) + \Delta_\nu (E_3H, p')}{D_\mu (E_3H, p) + D_\nu (E_3H, p')} \right\} \times \left[ 2 \pi^2 \frac{\delta(p - p')}{p^2} \right] \otimes \psi^3_H(p') \right. \\
\left. \times \left[ M \sum_{\mu, \nu = t, s} y_\mu y_{\nu} \psi^3_H(p) \otimes b_{\mu \nu} \otimes \psi^3_H(E_3H, p') \right]^{-1}. \quad (115)
\]

A comparison of the analytical [39] and the numerical results of the NLO three-body force, \(H^{(1)}(\Lambda)\) of eq. (112) reveals that they are in good agreement, as shown in Fig. 9. The diagrammatic representation of \(\Delta E_B(\Lambda)\) is given in Appendix D.

D. NLO corrections to the three-body wave-function

The full (non-perturbative in \(\alpha\)) Faddeev equations for \(^3\)He (at LO in ERE) consist of two parts - the strong part and the Coulomb interaction part:

\[
\Gamma^3_{\mu}(p) = \sum_{\nu = t, s} M y_\mu y_{\nu} \times \left[ a'_{\mu \nu} K_0(p', p, E_3He) + b'_{\mu \nu} \frac{H(\Lambda)}{\Lambda^2} + c'_{\mu \nu} K_{\mu \nu}^C(p', p, E_3He) \right] \otimes D_{\nu}(E_3He, p') \Gamma^3_{\nu}(E_3He, p') \\
+ M y_\mu y_{\nu} \left[ a'_{\mu \nu} K_0(p', p, E_3He) + b'_{\mu \nu} \frac{H(\Lambda)}{\Lambda^2} \right] \\
\otimes D_{\nu}(E_3He, p') \Gamma^3_{\nu}(E_3He, p'). \quad (116)
\]

Faddeev equations for \(^3\)He at LO are:

\[
\Gamma^3_{\mu}(p) = \sum_{\nu = t, s} M y_\mu y_{\nu} \left[ a_{\mu \nu} K_0(p', p, E_3H) + b_{\mu \nu} \frac{H(\Lambda)}{\Lambda^2} \right] \otimes D_{\nu}(E_3H, p') \Gamma^3_{\nu}(E_3H, p'). \quad (117)
\]

Using eq. (116), the \(^3\)He-\(^3\)H binding energy difference, defined in subsection \(\psi_E\), is a function of the Coulomb part of eq. (116), using \(^3\)H wave-functions and assuming that \(\psi_s(E, p) = \psi_{nn}(E, p) = \psi_{np}(E, p) = \psi(E, p)_{pp}\).

This implies that the \(^3\)He-\(^3\)H binding energy difference
can be written as a first-order perturbation in $\alpha$:

$$\Delta E(\Lambda) = \sum_{\mu,\nu} \psi_0(\mu) \otimes \mathcal{O}_{\mu \nu}^C(E, p, p') \otimes \psi_0(\nu'),$$  \hspace{1cm} (118)

where $\psi_0(\mu) = \psi_0^0(\mu)$ is the three-nucleon wave-function without the Coulomb interaction, and $\mathcal{O}_{\mu \nu}^C(E, p, p')$ are the Coulomb parts of eq. (116):

$$\mathcal{O}_{\mu \nu}^C(E, p, p') = c_{\mu \nu} K_{\mu \nu}^C(p, p, E)$$

$$+ \left[ a_{\mu \nu} K_0(p, p', E) + b_{\mu \nu} \frac{H(\Lambda)}{\Lambda^2} \right]$$

$$\times \left[ \frac{D_{pp}(E, p) - D_s(E, p)}{D_s(E, p)^2} \right] \delta_{\nu, s}. \hspace{1cm} (119)$$

The NLO correction to the binding energy can also be written as a first-order perturbation in $Q/\Lambda_{\text{cut}}$:

$$\Delta E_B(\Lambda) = \sum_{\mu, \nu} \psi_L^{(1)}(\mu) \otimes \mathcal{O}_{\mu \nu}^{(1)}(E, p, p') \otimes \psi_L^{(1)}(\nu')$$

$$= Z_L^LO \sum_{\mu, \nu} \left[ \Gamma_L^{(1)}(\mu) D_{\mu \nu}^{LO}(E, p) \right] \otimes \mathcal{O}_{\mu \nu}^{(1)}(E, p, p')$$

$$\otimes \left[ D_{\nu \nu}^{LO}(E, p') \Gamma_{\nu \nu}^{(1)}(p') \right], \hspace{1cm} (120)$$

where $\mu, \nu$ are the different dibaryon channels and $\mathcal{O}_{\mu \nu}^{(1)}(E, p, p')$ (defined in eq. (111)) is the NLO correction to the binding energy in terms of the different dibaryon channels. Since eqs. (118) and (120) have the same form, we can define the homogeneous scattering amplitude up to NLO such that for $^3\text{H}$:

$$\Gamma_\mu^{NLO}(p) = \Gamma_\mu^{LO}(p) + \Gamma_\mu^{(1)}(p)$$

$$\sum_{\nu=t, s} \left[ a_{\mu \nu} K_0(p, p', E_{3\text{H}}) + b_{\mu \nu} \frac{H(\Lambda)}{\Lambda^2} + \mathcal{O}_{\mu \nu}^{(1)}(E_{3\text{H}}, p, p') \right]$$

$$\otimes D_{\nu \nu}^{LO}(E_{3\text{H}}, p') \Gamma_{\nu \nu}^{LO}(p'), \hspace{1cm} (121)$$

and for $^3\text{He}$,

$$\Gamma_\mu^{NLO}(p) = \Gamma_\mu^{LO}(p) + \Gamma_\mu^{(1)}(E_{3\text{He}}, p)$$

$$\sum_{\nu=t, s, pp} \left\{ a'_{\mu \nu} [K_0(p, p', E_{3\text{He}}) + K_{\mu \nu}^C(p, p', E_{3\text{He}})]$$

$$+ b'_{\mu \nu} \frac{H(\Lambda)}{\Lambda^2} + \mathcal{O}_{\mu \nu}^{(1)}(E_{3\text{He}}, p, p')] \right\} \otimes D_{\nu \nu}^{LO}(E_{3\text{He}}, p') \Gamma_{\nu \nu}^{LO}(p'), \hspace{1cm} (122)$$

which are no longer Bethe-Salpeter equations, therefore, the Bethe-Salpeter normalization condition is not valid.

Having defined the NLO correction for the bound-state scattering amplitude, $\Gamma_\mu$, it is now possible to define the general form of a three-nucleon matrix element (such as an electroweak (EW) interaction) up to NLO:

$$\langle \mathcal{O}_{\text{EW}}^{(1)} \rangle + \langle \mathcal{O}_{\text{EW}}^{(1)} \rangle = \sum_{\mu, \nu} \langle \psi_\mu \otimes \mathcal{O}_{\mu \nu}^{LO} \otimes \psi_\nu \rangle$$

$$+ \langle \psi_\mu \otimes \mathcal{O}_{\mu \nu}^{LO} \otimes \psi_\nu + \langle \psi_\mu \otimes \mathcal{O}_{\mu \nu}^{LO} \otimes \psi_\nu \rangle,$$  \hspace{1cm} (123)

where:

$$\psi_\mu^{(1)}(p) = \sqrt{Z_1} \left\{ [D_{\mu \nu}^{NLO}(E, p) - D_{\mu \nu}^{LO}(E, p)] \Gamma_{\mu \nu}^{(1)}(p)$$

$$+ D_{\mu \nu}^{LO}(E, p) \Gamma_{\mu \nu}^{(1)}(p) \right\}, \hspace{1cm} (124)$$

where $Z_1$ is the NLO correction to the three-nucleon normalization, which is determined by the $\Lambda=3$ form factor, as will be discussed next.

### E. The NLO normalization

Charge conservation puts strong constraints on the zero-momentum limit of the electric form factor. In this subsection, we, therefore, want to relate the three-nucleon charge form factor to the three-nucleon normalization procedure discussed here. Following Ref. [3], we expand the charge form factor of the deuteron up to NLO

$$F_C = F_C^{LO} + F_C^{(1)}, \hspace{1cm} (125)$$

where for the deuteron:

$$F_C^{LO}(0) = Z_d^{LO} \lim_{q \to 0} \frac{4\gamma_t}{q} \arctan \left( \frac{q}{4\gamma_t} \right) = 1. \hspace{1cm} (126)$$

Up to NLO, one finds that:

$$F_C(0) = Z_d^{NLO} \lim_{q \to 0} \frac{4\gamma_t}{q} \arctan \left( \frac{q}{4\gamma_t} \right)$$

$$- Z_d^{LO} \lim_{q \to 0} \gamma_t \rho_t \frac{4\gamma_t}{q} \arctan \left( \frac{q}{4\gamma_t} \right). \hspace{1cm} (127)$$

We can rewrite this as:

$$\left( Z_d^{LO} + Z_d^{(1)} \right) F_C^{(0)} - Z_d^{LO} \gamma_t \rho_t F_C^{(0)} = 1$$

$$\rightarrow Z_d^{(1)} - \gamma_t \rho_t = 0. \hspace{1cm} (128)$$

From eq. (127), it is easy to show that up to NLO:

$$Z_d^{NLO} - \gamma_t \rho_t = 1 \rightarrow Z_d^{NLO} = 1 + \gamma_t \rho_t, \hspace{1cm} (129)$$

which equals 1.408, as discussed in Section [1].

Similarly, the $A = 3$ NLO normalization is obtained from the $^3\text{H}$ and $^3\text{He}$ form factor up to NLO [3, 20, 42]. Based on eq. (178), it is easy to show that at LO, the
A = 3 form factor is given by:

\[ F_C^{(0)}(q) = \sum_{\mu,\nu} \psi_{\mu}^{LO}(E', p') \otimes \mathcal{O}_{\mu\nu}^{FC(1B)}(q) \otimes \psi_{\nu}^{LO}(E, p), \]

where:

\[ \mathcal{O}_{\mu\nu}^{FC(1B)}(q) = y_{\mu} y_{\nu} \left\{ d_{\mu\nu}^{ii}(\bar{q}_0, q) + a_{\mu\nu}^{ii} \left[ \bar{K}(q_0, q) + \bar{K}_C(q_0, q) \right] \right\}, \]

and \( d_{\mu\nu}^{ii}, a_{\mu\nu}^{ii} \) were defined in eqs. \( 80 \) and \( 81 \).

Based on eq. \( 123 \), up to NLO, \( F_C(0) \) is given by:

\[ F_C^{NLO}(0) = F_C^{(0)}(0) + F_C^{(1)}(0) = \sum_{\mu,\nu} \psi_{\mu}^{LO}(E, p') \otimes \mathcal{O}_{\mu\nu}^{FC(1B)(0)} \otimes \psi_{\nu}^{LO}(E, p) \]

\[ + \frac{1}{2} \left[ \psi_{\mu}^{(1)}(E, p') \otimes \mathcal{O}_{\mu\nu}^{FC(1B)(0)} \otimes \psi_{\nu}^{LO}(E, p) \right] \]

\[ + \psi_{\mu}^{LO}(E, p') \otimes \mathcal{O}_{\mu\nu}^{FC(1B)(0)} \otimes \psi_{\nu}^{(1)}(E, p) \]

\[ + \psi_{\mu}^{LO}(E, p') \otimes \mathcal{O}_{\mu\nu}^{FC(2B)(0)} \otimes \psi_{\nu}^{LO}(E, p) = F_C^{(0)}(0) = 1, \]

where:

\[ \mathcal{O}_{\mu\nu}^{FC(1B)(0)} = \mathcal{O}_{\mu\nu}^{norm}(E), \]

and \( i = ^3\text{H}, ^3\text{He}. \)

Since the two-body term is a result of the \( A_0 \) photons, which couple only to the triplet channel, the two-body term can be written as \( 122 \):

\[ \mathcal{O}_{\mu\nu}^{FC(2B)(0)} = \frac{2\pi^2}{p'^2} \delta(p - p') \delta_{\mu,\nu} \delta_{t,\bar{t}}. \]

By substituting eqs. \( 133 \) and \( 134 \) in eq. \( 132 \), one finds that the NLO correction to the triton form factor, \( F_C^{(1)}(0) \), is given by:

\[ F_C^{(1)}(0) = \frac{1}{2} \sum_{\nu=t,s} \left\{ \psi_{\mu}^{(1)}(p) \otimes \mathcal{O}_{\mu\nu}^{norm}(E_{3\text{He}}) \otimes \psi_{\nu}^{LO}(p') \right\} \]

\[ + \psi_{\mu}^{LO}(p) \otimes \mathcal{O}_{\mu\nu}^{norm}(E_{3\text{He}}) \otimes \psi_{\nu}^{(1)}(p') \]

\[ - \frac{2}{3} \psi_t^{LO}(p) \otimes \frac{2\pi^2}{p'^2} \delta(p - p') \otimes \psi_t^{LO}(p') = 0, \]

and similarly for \(^3\text{He}\):

\[ F_C^{(1)}(0) = \frac{1}{2} \sum_{\nu=3,\bar{3}} \left\{ \psi_{\mu}^{(1)}(p) \otimes \mathcal{O}_{\mu\nu}^{norm}(E_{3\text{He}}) \otimes \psi_{\nu}^{LO}(p') \right\} \]

\[ + \psi_{\mu}^{LO}(p) \otimes \mathcal{O}_{\mu\nu}^{norm}(E_{3\text{He}}) \otimes \psi_{\nu}^{(1)}(p') \]

\[ - \frac{2}{3} \psi_t^{LO}(p) \otimes \frac{2\pi^2}{p'^2} \delta(p - p') \otimes \psi_t^{LO}(p') = 0, \]

where for both \(^3\text{H}\) and \(^3\text{He}\), \( \psi_{\mu}^{(1)}(p) \) is the normalized NLO correction to the three-nucleon wave-function \([43]\).

The expressions for the NLO corrections to the triton and \(^3\text{He}\) homogeneous scattering amplitude (\(\Gamma_s\)) are given in Appendix C.

VII. SUMMARY AND OUTLOOK

In this paper, we have established a perturbative and consistent framework for calculating an \( A = 3 \) bound-state matrix element in pionless effective field theory up to NLO. Our method is using field theoretically defined bound-state amplitudes and a diagrammatic expansion related to the operators whose matrix elements are calculated.

We showed that matrix elements could be calculated diagrammatically by summing all the possible insertions of the transition operator between two nuclear amplitudes. At LO, that is consistent with the Bethe-Salpeter normalization condition, and its diagrammatic representation is equivalent to the sum of all the possible insertions of a one-nucleon propagator between two identical \( A = 3 \) bound-state wave functions.

For the Coulomb interaction, we have shown that summing over all the one- and two-body photon exchange diagrams perturbatively yields the energy difference between \(^3\text{H}\) and \(^3\text{He}\), which is the equivalent of solving the non-perturbative Faddeev equations for \(^3\text{He}\).

We have tested the correct renormalization of our perturbative calculation by an analysis of the residual cutoff dependence of the matrix elements, up to very large cutoffs, significantly larger than the breakdown scale of the EFT. The numerical results for the RG invariance reproduce theoretical predictions and serve therefore as an additional test of the calculation and our approach.

At NLO, we showed that a consistent diagrammatic expansion is just the sum of all the possible diagrams with a single NLO perturbation insertion.

The LO and NLO three-nucleon (\(^3\text{H}\) and \(^3\text{He}\)) amplitudes were calculated as solutions of the homogeneous Faddeev equations. At NLO, these solutions require a recalibration of the three-body force (\(H^i(\Lambda), H^s(\Lambda)\)) reproducing the results of Refs. \([13,39]\). We were able to reduce the regularization effects at a small cutoff by taking the natural coupling \(y_t \neq y_s\). This improved sig-
nificantly the comparison of the analytical solution to the NLO three-body force.

Using this diagrammatic approach, we can now calculate a wide range of electroweak interactions of $A = 3$, such as $\beta$ decay of $^3\text{H}$ into $^3\text{He}$, $A = 3$ magnetic moments, etc., up to NLO.

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APPENDIX A - NORMALIZATION OF THREE-BODY BETHE-SALPETER WAVE-FUNCTIONS

The three-nucleon Faddeev equations (eqs. [27] and [41]) have the same form as the non-relativistic BS equation [32, 35–37]:

\[ M = V - VG_{BS}M = V - M G_{BS}V, \]  

(A-1)

where $M$ is the scattering matrix, $V$ is the two-body interaction kernel and $G_{BS}$ is the free two-body propagator.

\[ \begin{array}{c|c|c|}
\hline
M & V & V \hline
\end{array} \]

FIG. A.1: Diagrammatic representation of the two-body BS equation for the scattering matrix $M$.

From eq. (A-1) we find:

\[ V = M + MG_{BS}V, \]  

(A-2)

and upon substituting $V$ into eq. (A-1), we get:

\[ M = V - MG_{BS}M - MG_{BS}VG_{BS}M. \]  

(A-3)

For a bound-state, $M$ has the form:

\[ M = \frac{|\mathcal{B}\rangle \langle \mathcal{B}|}{E - E_B} + \mathcal{R}, \]  

(A-4)

where $|\mathcal{B}\rangle$ is the wave-function amplitude and $\mathcal{R}$ is a regular part that is finite at $E = E_B$ and therefore can be neglected for $E \to E_B$. Substituting eq. (A-4) into (eq. (A-1)) and equating residues at $E = E_B$ yields the wave equation for $|\mathcal{B}\rangle$:

\[ |\mathcal{B}\rangle = -VG_{BS}|\mathcal{B}\rangle. \]  

(A-5)
Substituting eq. (A-4) into eq. (A-3), multiplying the resulting equation by $E - E_B$ and taking the limit $E \to E_B$, one finds that:

$$
|\langle B|\rangle = -\lim_{E \to E_B} \frac{|\langle B|G_{BS}(1 + V_{BS})|\rangle}{E - E_B} (A-6)
$$

$$
\Rightarrow 1 = -\lim_{E \to E_B} \frac{|\langle B|G_{BS}(1 + V_{BS})|\rangle}{E - E_B} = -\lim_{E \to E_B} \frac{|\langle B|G_{BS}(G_{BS}^{-1} + V)|G_{BS}|\rangle}{E - E_B}.
$$

From eq. (A-5), $\lim_{E \to E_B} \langle B|G_{BS}(1 + V_{BS})|\rangle = 0$, so the RHS of eq. (A-6) is of the form $0/0$, so one can use the l'Hôpital's rule to evaluate the limit (which equals -1) explicitly:

$$
\lim_{E \to E_B} \frac{\partial}{\partial E} \frac{|\langle B|G_{BS}(G_{BS}^{-1} + V)|G_{BS}|\rangle}{E - E_B} = \lim_{E \to E_B} \frac{\partial}{\partial E} (\langle B|G_{BS}(G_{BS}^{-1} + V)|G_{BS}|\rangle) =
$$

$$
\langle B|G_{BS}G_{BS}^{-1}G_{BS} + G_{BS}(G_{BS}^{-1})'G_{BS} + G_{BS}G_{BS}^{-1}G_{BS}^{}|\rangle_{E = E_B}
$$

$$
+ \langle B|G_{BS}V_{BS}G_{BS}^{} + G_{BS}V'G_{BS}^{} + G_{BS}V'G_{BS}^{}|\rangle_{E = E_B} =
$$

$$
\langle B|G_{BS} + G_{BS}(G_{BS}^{-1})'G_{BS} + G_{BS}'|\rangle_{E = E_B} + \langle B| - G_{BS} + G_{BS}'V'G_{BS} - V'G_{BS}^{}|\rangle_{E = E_B} =
$$

$$
\langle B|G_{BS}(G_{BS}^{-1})'G_{BS} + G_{BS}'V'G_{BS}^{}|\rangle_{E = E_B} = \langle B|G_{BS}\frac{\partial}{\partial E} (G_{BS}^{-1} + V)|G_{BS}|\rangle_{E = E_B} = -1, \quad (A-7)
$$

where the terms proportional to $\frac{\partial}{\partial E} |\langle B|\rangle$ vanish due to the BS equation [36].

According to our notation, $G_{BS} = -D(E, p)$, the two-body propagator, and $V = M \gamma^2 K_0(p, p', E)$, the one-nucleon exchange matrix, such that the three-nucleon normalization condition is:

$$
1 = \langle B|G_{BS}\frac{\partial}{\partial E} (-G_{BS} - V)|G_{BS}|\rangle_{E = E_B}, \quad (A-8)
$$

APPENDIX B - THE GENERAL FORM OF AN $A = 3$ MATRIX ELEMENT IN THE CASE OF MOMENTUM AND ENERGY TRANSFER

For the general case of momentum and energy transfer, the spatial parts of the matrix element have the form:

$$
\hat{I}(E, q_0, q) = \frac{M^2}{4\pi(q - p + p')},
$$

$$
\left\{ \log \left[ q \left( \sqrt{4EM - 3p'^2 - 2p + p'} \right) + (p' - p) \left( \sqrt{4EM - 3p'^2 - 2p - p'} \right) + q^2 + 2q_0M \right] - \sqrt{4EM - q^2 + 2qp - 4q_0M - 3p^2} \right\}
$$

$$
\delta(q_0 - E + E')\delta(p' - p) \frac{2\pi^2}{p'^2} (B-1)
$$

and:

$$
\hat{K}(E, q_0, q) = \frac{M^2}{2pp'\left[ q(2(p + p') - q) - 2Mq_0 \right]} \times
$$

$$
\left\{ \log \left[ (-EM + p^2 + pp' + p'^2) \left( 2EM - 2Mq_0 - 2(p^2 - pp' + p'^2) + 2q(p + p') - q^2 \right) \right] - \log \left[ (-EM + p^2 - pp' + p'^2) \left( 2EM - 2Mq_0 - 2(p^2 + pp' + p'^2) + 2q(p + p') - q^2 \right) \right]. \right\}
$$

$$
\delta(q_0 - E + E')\delta(p' - p) \frac{2\pi^2}{p'^2} (B-2)
$$
APPENDIX C - THE HUBBARD-STRATONOVICH TRANSFORMATION WITH TWO-BODY ELECTROWEAK INTERACTION

In this appendix, we present the Hubbard-Stratonovich (H-S) transformation for a π/EFT Lagrangian with an electroweak interaction.

The two-body Lagrangian with electroweak interaction has the form:

\[ \mathcal{L} = \mathcal{L}_{\text{strong}} + \mathcal{L}_{\text{electroweak}}, \]  

(C-1)

where \( \mathcal{L} \) is two-body Lagrangian [41]:

\[ \mathcal{L} = N^\dagger \left( i\partial_0 + \frac{\nabla^2}{2M} \right) N - \sum_\mu C_0^\mu \phi^\dagger_\mu \phi_\mu - \frac{C_2 M^\mu}{2} \left[ \phi^\dagger_\mu \mathcal{O}_D \phi_\mu + \text{h.c} \right], \]  

(C-2)

where:

\[ (N^T P_{t,s} N) = \phi_{t,s}, \]

(C-3)

\[ \mathcal{O}_D = \left( i\partial_0 + \frac{\nabla^2}{4M} \right), \]

(C-4)

and (see for example [3]):

\[ C_0^\mu = \frac{4\pi}{M} \left( -\mu + \frac{1}{\alpha_\mu} \right) \]

(C-5)

\[ C_2^\mu = C_2^\mu = \frac{4\pi}{M} \left( -\mu + \frac{1}{\alpha_\mu} \right) \frac{\rho_\mu}{2}. \]

(C-6)

\( \mathcal{L}_{\text{electroweak}} \) is the electroweak part of the π/EFT Lagrangian:

\[ \mathcal{L}_{\mu \text{electroweak}} \propto A_\mu = A_\mu^{1B} + A_\mu^{2B}. \]  

(C-7)

the two-body part of the electroweak current, \( A_\mu^{2B} \), has the form:

\[ A_\mu = \sum_{\mu, \nu} L_{\mu\nu} \psi^\dagger_\mu \psi_\nu, \]  

(C-8)

where \( L_{\mu\nu} \) is the LEC that couples the two two-nucleon fields, e.g., for the weak interaction \( L_{\mu\nu} = L_{ts} = L_{1,A} \). In order to find the right H-S transformation for \( \mathcal{L} \), we assume that after applying the H-S transformation, \( \mathcal{L} \) is of the form:

\[ \mathcal{L}^{H-S}_{\text{electroweak}} = \sum_{\mu, \nu = t,s} \left( -\alpha_\mu \left( \mu^\dagger \phi_\mu + \text{h.c} \right) - \mu^\dagger \beta_{\mu\nu} \gamma_{\mu\nu} \phi^\dagger_\nu \phi_\nu + \gamma_{\mu\nu} \left[ \mu^\dagger \nu + \text{h.c} \right] \right), \]  

(C-9)

where the H-S transformation is defined such that:

\[ \int dt \int ds \exp \left( -\mathcal{L}^{H-S}_{\text{electroweak}} \right) = \exp \left[ -\sum_{\mu = t,s} A_\mu \phi^\dagger_\mu \phi_\mu + B_\mu \mathcal{O}_D \phi^\dagger_\mu \phi_\mu + L_{\mu,\nu} \left( \psi^\dagger_\mu \psi_\nu + \text{h.c} \right) \right]. \]  

(C-10)

By setting:

\[ A_\mu = -C_0^\mu \]

(C-11)

\[ B_\mu = -\frac{C_2^\mu M}{2} \]

(C-12)
and 
\[ \alpha_\mu = y_\mu \]  
\[ \beta_\mu = \mathcal{O}_D - \sigma_\mu \]  
we get that:

\[ \gamma_{\mu\nu} = \frac{L_{\mu\nu}}{\sqrt{C_2^\mu C_2^\nu M}^2} + c \left( \frac{C_2^\mu}{C_0^\mu \sqrt{M C_2^\nu}} \right) \]  
\[ \gamma'_{\mu\nu} = \frac{L_{\mu\nu}}{\sqrt{C_2^\mu C_2^\nu M^2}^2} - \frac{2c}{C_0^\mu} \left( \frac{C_0^\nu C_2^\mu + C_0^\mu C_2^\nu}{C_0^\mu C_0^\nu \sqrt{M^2 C_2^\mu C_2^\nu}} \right) \]  
where \( c \) is an arbitrary constant that has to be determined by the original Lagrangian.

**APPENDIX D - THE NLO CORRECTIONS TO TRITON AND \(^3\)HE BOUND-STATE AMPLITUDES**

**D.I. The triton channel**

1. The NLO correction to a triton homogeneous wave function

The NLO corrections \(^3\)H scattering amplitude are constructed from the effective range expansion and from an additional 3-body force. For the remainder of this section, we assume the energy to be close to the triton binding energy (see Section VI). Based on eq. (121),

\[ \Gamma^{(1)}_\mu (p) = \sum_{\mu,\nu=t,s} \left[ \mathcal{O}^{(1)}_{\mu\nu}(E_{^3\text{H}}, p, p') \otimes D_{\nu}^{\text{LO}}(E_{^3\text{H}}, p') \Gamma_{\nu}^{\text{LO}}(p') \right], \]  

where:

\[ \mathcal{O}^{(1)}_{\mu\nu}(E_{^3\text{H}}, p, p') = M y_\mu y_\nu \left\{ \frac{1}{2} \left[ a_{\mu\nu} K_0(p, p', E_{^3\text{H}}) + b_{\mu\nu} \frac{H^{(1)}(\Lambda)}{\Lambda^2} \right] \times \left[ \Delta_\mu(E_{^3\text{H}}, p) + \Delta_\nu(E_{^3\text{H}}, p') \right] + b_{\mu\nu} \frac{H^{(1)}(\Lambda)}{\Lambda^2} \right\} \]  

and \( H^{(1)}(\Lambda) \) is calculated numerically by setting:

\[ \Delta E_{B}(\Lambda) = \sum_{\mu,\nu=t,s} \psi_{^3\text{H}}^\mu(p) \otimes \mathcal{O}^{(1)}_{\mu\nu}(E, p, p') \otimes \psi_{^3\text{H}}^\nu(p') = 0. \]  

The diagrammatic representation of \( \Delta E_{B}(\Lambda) \) for the case of \(^3\)H, is shown in Fig. D.1

**FIG. D.1**: The NLO correction for \(^3\)H binding energy. The double lines are propagators of the two intermediate auxiliary fields, \( D_t \) (solid) and \( D_{np} \) (dashed). The red bubbles (\( \Gamma_1 \)) represent the triplet channel (\( T=0, S=1 \)), the green bubbles represent (\( \Gamma_s \)) the singlet channel (\( T=1, S=0 \)). The black circles denote the NLO correction to the dibaryon propagator, while the blue squares denote the NLO correction to the three-body force (\( H^{(1)}(\Lambda) \)).

**D.II. \(^3\)He - correction to the three-body force and the wave-function normalization**

The prediction of \( H^{(1)}(\Lambda) \) for \(^3\)H (see subsection VI C) enables us to calculate the NLO corrections to \(^3\)He as well. Similarly to the LO calculation, we are using the three-body force to determine the NLO correction to \(^3\)He binding
energy, by assuming that $H^{(1)}(\Lambda)$ has no isospin dependence \[13\]. The diagrammatic representation of $\Delta E_B(\Lambda)$ for the case of $^3\text{He}$, is shown in Fig. D.2

1. NLO binding energy and NLO three-body force

Similarly to the $^3\text{H}$, the correction to the binding energy of $^3\text{He}$ is a function of $\Lambda$ only \[13\]:

$$\Delta E_B(\Lambda) = \sum_{\mu,\nu=t,s,pp} \psi^{^3\text{He}}(p) \otimes O_{\mu\nu}^{(1)}(E^{^3\text{He}},p,p') \otimes \psi^{^3\text{He}}(p'), \quad (D-4)$$

$$O_{\mu\nu}^{(1)}(E^{^3\text{He}},p,p') = M y_{\mu} y_{\nu} \left\{ \frac{1}{2} \left[ a'_{\mu\nu} K_0(p,p',E^{^3\text{He}}) + a'_{\mu\nu}^C K^{C}(p,p',E^{^3\text{He}}) + b'_{\mu\nu} \frac{2H(\Lambda)}{\Lambda^2} \right] \right\} + \Delta_\nu(E^{^3\text{He}},p,p') \right\} + \frac{\alpha Q_0}{2pp'} \right\} \times (\delta_{\mu,t,\nu,t} + 3\delta_{\mu,s,\nu,s}), \quad (D-5)$$

where $\alpha Q_0 \left( \frac{p^2 + p'^2 + \lambda^2}{2pp'} \right)$ originates from diagram (f) in Fig. 3.

In contrast to $^3\text{H}$ and to $^3\text{He}$ at LO, the numerical result of eq. (D-5) reveals that $\Delta E_B$ for $^3\text{He}$ diverges with the cutoff $\Lambda$ (see Ref. \[13\]) and does not vanish. This contradicts the assumption that the addition of an isospin independent $H^{NLO}(\Lambda)$ to $T^{NLO}$ removes the cutoff dependence of $\Delta E_B$ for both $^3\text{H}$ and $^3\text{He}$. The solution to this issue is obtained by defining a different three-body force for $^3\text{He}$, such that:

$$E^{^3\text{He}}_{NLO} = E^{^3\text{He}}_{LO}(\Lambda) + \Delta E_B(\Lambda) = 7.72 \text{ MeV}, \quad (D-6)$$

which equals to the experimental binding energy of $^3\text{He}$, where $E^{^3\text{He}}_{LO}(\Lambda)$ is shown in Fig. 7.

Accordingly, the new three-body force, $H^\alpha(\Lambda)$, is defined and can be calculated numerically as:

$$\frac{H^\alpha(\Lambda)}{\Lambda^2} = \left[ \frac{7.72 \text{ MeV} - E^{^3\text{He}}_{LO}(\Lambda)}{\Lambda^2} \right] - \sum_{\mu,\nu=t,s,pp} \psi^{^3\text{He}}(p) \otimes O_{\mu\nu}^{(1)}(E^{^3\text{He}},p,p') \otimes \psi^{^3\text{He}}(p') \right] \times \left[ \sum_{\mu,\nu=t,s,pp} \psi^{^3\text{He}}(p) \otimes b'_{\mu\nu}(E^{^3\text{He}},p,p') \otimes \psi^{^3\text{He}}(p') \right]^{-1} \quad (D-7)$$

while its analytical form is given in Refs. \[13\], \[39\].

The diagrammatic representation of $\Delta E_B(\Lambda)$ for the case of $^3\text{He}$, is shown in Fig. D.2

D.III. $^3\text{He}$ - the NLO correction to the scattering amplitude

These equations are similar to those giving the NLO corrections for $^3\text{H}$. However, for $^3\text{He}$ additional contributions are resulting from NLO Coulomb diagrams \[13\].

For $^3\text{He}$, we have:

$$\Gamma^{(1)}_\mu(p) = \sum_{\mu,\nu=t,s} \left[ O_{\mu\nu}^{(1)}(E^{^3\text{He}},p,p') \right] \otimes D_{\nu}^{LO}(E^{^3\text{He}},p') \Gamma^{LO}_\nu(p'), \quad (D-8)$$
FIG. D.2: The NLO correction for $^3$He binding energy. The double lines are propagators of the two intermediate auxiliary fields, $D_t$ (solid) and $D^{np}_t$ (dashed) and $D^{pp}$ (dotted). The red bubbles ($\Gamma_t$) represent the triplet channel (T=0, S=1), the green bubbles represent ($\Gamma_s$) the singlet channel (T=1, S=0) with an np dibaryon, while the blue bubbles ($\Gamma_{pp}$) represent the singlet channel (T=1, S=0) with pp dibaryon. The black circles denote the NLO correction to the dibaryon propagator, while the blue squares denote the NLO correction to the three-body force ($H^{(1)}(\Lambda) + H^{(\alpha)}(\Lambda)$).

where:

$$O_{\mu\nu}^{(1)}(E_{^3\text{He}}, p, p') =$$

$$M y_\mu y_\nu \left\{ \frac{1}{2} \left[ a'_{\mu\nu} K_0(p, p', E_{^3\text{He}}) + c'_{\mu\nu} K^{C}_\mu(p, p', E_{^3\text{He}}) + b'_{\mu\nu} \frac{H(\Lambda)}{\Lambda^2} \right] \right.$$ 

$$\times \left[ \Delta_{\nu}(E_{^3\text{He}}, p) + \Delta_{\nu}(E_{^3\text{He}}, p') \right] + b'_{\mu\nu} \frac{H^{(1)}(\Lambda)}{\Lambda^2} + b'_{\mu\nu} \frac{H^{(\alpha)}(\Lambda)}{\Lambda^2} \right\} +$$

$$\alpha Q_0 \left( \frac{p^2 + p'^2 + \lambda^2}{2pp'} \right) \times (\delta_{\mu,t}\delta_{\nu,t} + 3\delta_{\mu,s}\delta_{\nu,s}) \ . \quad (D-9)$$

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