THE REPRESENTATION DIMENSION OF A CLASS OF TAME ALGEBRAS

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Abstract. We prove that, if $A$ is a strongly simply connected algebra of polynomial growth then $A$ is torsionless-finite. In particular, its representation dimension is at most three.

Among the most useful algebraic invariants are the homological dimensions, which are meant to measure how much an algebra or a module deviates from a situation considered to be ideal. Introduced by Auslander in the early seventies [9], the representation dimension of an Artin algebra was long left aside from the mainstream of the theory, until a marked renewal of interest about ten years ago. It measures the least global dimension of all endomorphism rings of those finitely generated modules which are both generators and cogenerators of the module category. Part of the reason for this new interest comes from the fact that Igusa and Todorov have shown that, if the representation dimension of an Artin algebra is at most three, then its finitistic dimension is finite [19]. Also, Iyama has proved that, for any Artin algebra $A$, the representation dimension $\text{rep.dim.} A$ is finite [20]. Since then, there have been several attempts to understand this invariant and to calculate it for classes of algebras, see for instance [2, 16, 17, 22] or the survey [26]. It was shown by Auslander that an Artin algebra is representation-finite if and only if its representation dimension is at most two [9]. Since Auslander’s expectation was that this dimension would measure how far an algebra is from being representation-finite, it is natural to ask whether tame algebras have representation dimension at most three. The answer to this question is known to be positive for some classes of tame algebras, such as special biserial algebras [18] and domestic self-injective algebras socle equivalent to a weakly symmetric algebra of euclidean type [13].

The objective of our paper is to prove that the representation dimension of a strongly simply connected algebra of polynomial growth (over an algebraically

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closed field) is at most three. These algebras form a nice class which has been extensively studied by Skowroński, de la Peña and others, see, for instance, the survey [7]. In particular, it is shown in [28] that strongly simply connected algebras of polynomial growth are multicoil algebras. We recall also that, as pointed out in [26], an algebra which is torsionless-finite (that is, such that any indecomposable projective module has only finitely many isomorphism classes of indecomposable submodules) has its representation dimension at most three. Our main result is the following theorem.

**Theorem A.** Let $A$ be a strongly simply connected algebra of polynomial growth. Then $A$ is torsionless-finite. In particular, rep.dim $A \leq 3$.

In the course of the proof, we found that the following result, of independent interest, was necessary.

**Theorem B.** Let $C$ be a tame concealed algebra of type distinct of $\tilde{A}$, and $(H_\lambda)_\lambda$ be an infinite family of pairwise non-isomorphic simple homogeneous $C$-modules. Suppose $M$ is a $C$-module such that $H_\lambda \subset M$ for all $\lambda$. Then $C[M]$ is wild.

The paper is organized as follows. After a short preliminary section, in which we fix the notations and recall facts about multicoil algebras, we prove our Theorem(B) in section 2 and Theorem(A) in section 3.

### 1. Multicoil algebras

#### 1.1. Notation

In this paper, $k$ denotes a fixed algebraically closed field. By an algebra $A$ is meant a basic, connected, associative finite dimensional $k$-algebra with an identity. Thus, there exists a connected bound quiver $(Q_A, I)$ and an isomorphism $A \simeq kQ_A/I$. Equivalently, $A$ may be viewed as a $k$-category, of which the object class $A_0$ is the set of points of $Q_A$ and the set of morphisms from $x$ to $y$ is the quotient of the $k$-vector space $kQ_A(x, y)$ of linear combinations of paths in $Q_A$ from $x$ to $y$ by $I(x, y) = I \cap kQ_A(x, y)$, see [15]. A full subcategory $C$ of $A$ is convex (in $A$) if, for any path $x_0 \to x_1 \to \cdots \to x_t$ in $A$ with $x_0, x_t \in C_0$, we have $x_i \in C_0$ for all $i$. The algebra $A$ is triangular if $Q_A$ is acyclic.

By $A$-module is meant a finitely generated right $A$-module. We denote by $\text{mod} A$ the category of $A$-modules and by $\text{ind} A$ a full subcategory consisting of a complete set of representatives of the isomorphism classes of indecomposable $A$-modules. We recall that, if $A \simeq kQ/I$, then an $A$-module $M$ is identified to a corresponding representation $(M(x)_{x \in Q_0}, M(\alpha)_{\alpha \in Q_1})$ of the bound quiver $(Q, I)$, see [3]. For a point $x \in Q_0$, we denote by $P_x$ (or $I_x$, or $S_x$) the indecomposable projective (or injective, or simple, respectively) $A$-module corresponding to $x$. The support of an $A$-module $M$ is the full subcategory $\text{Supp}M$ of $A$ with objects those $x \in A_0$ such that $M(x) \neq 0$. For a full subcategory $\mathcal{C}$ of $\text{mod} A$, we denote by $\text{add} \mathcal{C}$ the additive full subcategory with objects the direct sums of direct summands of objects in $\mathcal{C}$. If $\mathcal{C}$ contains a single module $M$, we write $\text{add} \mathcal{C} = \{M\}$.
add $M$. For two full subcategories $C, C'$ of $\text{ind} A$, the notation $\text{Hom}_A(C, C') = 0$ means that $\text{Hom}_A(M, M') = 0$ for all $M \in C, M \in C'$. We then denote by $C' \vee C$ the full subcategory of $\text{ind} A$ having as objects those of $C_0 \cup C_0$.

A path in $\text{ind} A$ from $M$ to $N$ (sometimes denoted as $M \rightsquigarrow N$) is a sequence of non-zero morphisms

$$(*)\quad M = M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_i = N$$

with all $M_i$ in $\text{ind} A$. A path ($*$) is a cycle if $M = N$ and at least one of the morphisms is not an isomorphism. An indecomposable module is directed if it lies on no cycle.

We use freely properties of the Auslander-Reiten translations $\tau_A = \text{DTr}$ and $\tau_A^{-1} = \text{TrD}$ and the Auslander-Reiten quiver $\Gamma(\text{mod} A)$ of $A$ for which we refer to [3, 25]. We identify points in $\Gamma(\text{mod} A)$ with the corresponding indecomposable $A$-modules, and components with the corresponding full subcategories of $\text{ind} A$.

A component $\Gamma$ is standard if the category $\Gamma$ is equivalent to the mesh category $k(\Gamma)$. For tubes, tubular extensions and coextensions, we refer to [25] and for tame algebras, we refer to [27].

1.2. One-point extensions. The one-point extension of an algebra $A$ by an $A$-module $M$ is the matrix algebra

$$A[M] = \begin{pmatrix} A & 0 \\ M & k \end{pmatrix}$$

with the usual addition and multiplication of matrices. The quiver of $A[M]$ contains $Q_A$ as a full convex subquiver and there is an additional (extension) point which is a source. The $A[M]$-modules are identified with triples $(V, X, \varphi)$, where $V$ is a $k$-vector space, $X$ an $A$-module and $\varphi: V \longrightarrow \text{Hom}_A(M, X)$ is a $k$-linear map. An $A[M]$-linear map $(V, X, \varphi) \longrightarrow (V', X', \varphi')$ is a pair $(f, g)$ where $f: V \longrightarrow V'$ is $k$-linear and $g: X \longrightarrow X'$ is $A$-linear such that $\varphi'f = \text{Hom}_A(M, g)\varphi$. The dual notion is that of one-point coextension.

A vector space category $[24, 25]$ $\mathcal{K}$ is a $k$-category together with a faithful $k$-linear functor $| \cdot |: \mathcal{K} \longrightarrow \text{mod} k$. The subspace category $\mathcal{U}(\mathcal{K})$ of $\mathcal{K}$ has as objects the triples $(V, X, \varphi)$ where $V$ is a $k$-vector space, $X$ an object in $\mathcal{K}$ and $\varphi: V \longrightarrow |X|$ is a $k$-linear monomorphism. A morphism $(V, X, \varphi) \longrightarrow (V', X', \varphi')$ is a pair $(f, g)$ where $f: V \longrightarrow V'$ is $k$-linear and $g: X \longrightarrow X'$ is a morphism in $\mathcal{K}$ such that $\varphi'f = |g|\varphi$.

If $A$ is an algebra and $M$ an $A$-module, one considers the vector space category $\text{Hom}_A(M, \text{mod} A)$ whose objects are of the form $\text{Hom}_A(M, X)$ with $X$ an $A$-module, and morphisms are of the form

$$\text{Hom}_A(M, f): \text{Hom}_A(M, X) \longrightarrow \text{Hom}_A(M', X'),$$

where $f: X \longrightarrow X'$ is $A$-linear. Then $|\text{Hom}_A(M, X)|$ is just the underlying $k$-vector space of $\text{Hom}_A(M, X)$. It is shown in [24] that $\mathcal{U}(\text{Hom}_A(M, \text{mod} A))$ is equivalent to the full subcategory of $\text{mod} A[M]$ consisting of the triples $(V, X, \varphi)$
without non-zero direct summands of the form \((k, 0, 0)\) or \((0, Y, 0)\) where 
\(\text{Hom}_A(M, Y) = 0\). We need essentially the following lemma.

**Lemma.** Let \(A\) be a tame algebra, and \(M\) be an \(A\)-module. If \(L\) is a submodule of \(M\) such that \(A[L]\) is wild, then \(A[M]\) is wild.

**Proof.** Since \(A\) is tame, while \(A[L]\) is wild, then the vector space category 
\(\mathcal{U}(\text{Hom}_A(L, \text{mod}A))\) is wild. Now the inclusion \(L \hookrightarrow M\) induces an epifunctor
\[\mathcal{U}(\text{Hom}_A(M, \text{mod}A)) \rightarrow \mathcal{U}(\text{Hom}_A(L, \text{mod}A)).\]
Since the latter is wild, then so is \(\mathcal{U}(\text{Hom}_A(M, \text{mod}A))\). Therefore, \(A[M]\) is wild. \(\square\)

1.3. **Coils.** Let \(A\) be an algebra, and \(\Gamma\) a standard component of \(\Gamma(\text{mod}A)\). Given a module \(X \in \Gamma\), called the pivot, the support \(\text{Supp}(X, -)|_\Gamma\) of the functor \(\text{Hom}_A(X, -)|_\Gamma\) is defined as follows. Let \(\mathcal{H}_X\) be the full subcategory of \(\text{ind}A\) consisting of the \(Y \in \Gamma\) such that \(\text{Hom}_A(X, Y) \neq 0\), and \(\mathcal{I}_X\) be the ideal of \(\mathcal{H}_X\) consisting of the morphisms \(f: Y \rightarrow Y'\) (with \(Y, Y'\) in \(\mathcal{H}_X\)) such that \(\text{Hom}_A(X, f) \neq 0\). Then \(\text{Supp}(X, -)|_\Gamma = \mathcal{H}_X/\mathcal{I}_X\). We define three admissible operations:

(ad1) Assume \(\text{Supp}(X, -)|_\Gamma\) consists of an infinite sectional path starting at \(X\)
\[X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots\]
Let \(t \geq 1\), \(D = \mathcal{T}_t(k)\) be the full \(t \times t\) lower triangular matrix algebra, and \(Y\) be the unique indecomposable projective-injective \(D\)-module. The modified algebra of \(A\) is \(A' = (A \times D)[X \oplus Y]\). If \(t = 0\), it is simply \(A' = A[X]\).

(ad2) Assume \(\text{Supp}(X, -)|_\Gamma\) consists of two sectional paths starting at \(X\), the first infinite and the second finite with at least one arrow
\[Y_1 \leftarrow \cdots \leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots\]
with \(t \geq 1\). The modified algebra is \(A' = A[X]\).

(ad3) Assume \(\text{Supp}(X, -)|_\Gamma\) consists of two parallel sectional paths, the first infinite and starting at \(X\), and the second finite with at least one arrow
\[
\begin{array}{cccccc}
Y_1 & \rightarrow & Y_2 & \rightarrow & \cdots & \rightarrow Y_t \\
\uparrow & & & & & \uparrow \\
X = X_0 & \rightarrow & X_1 & \rightarrow & \cdots & \rightarrow X_{t-1} \rightarrow X_t \rightarrow \cdots
\end{array}
\]
where \(t \geq 2\). In particular, \(X_{t-1}\) is injective. The modified algebra is \(A' = A[X]\).

In each case, \(t\) is the parameter of the operation, and the component \(\Gamma'\) of \(\Gamma(\text{mod}A')\) containing \(X\) is the modified component. We also consider the duals of
the above operations, denoted by \((\text{ad}1^*), (\text{ad}2^*)\) and \((\text{ad}3^*)\), respectively. These six operations are the admissible operations.

An Auslander-Reiten component \(\Gamma\) is a coil if there exists a sequence \(\Gamma_0, \Gamma_1, \cdots, \Gamma_m = \Gamma\) where \(\Gamma_0\) is a stable tube and, for each \(i, \Gamma_{i+1}\) is obtained from \(\Gamma_i\) by an admissible operation.

Let \(A\) be an algebra \(A\). A family \(\mathcal{R} = (\mathcal{R}_\lambda)_{\lambda \in \Lambda}\) of components of \(\Gamma(\text{mod}\,A)\) is weakly separating if the indecomposable \(A\)-modules not in \(\mathcal{R}\) split into two classes \(\mathcal{P}\) and \(\mathcal{Q}\) such that

1. The \(\mathcal{R}_\lambda\) are standard and pairwise orthogonal.
2. \(\text{Hom}_A(\mathcal{Q}, \mathcal{P}) = \text{Hom}_A(\mathcal{Q}, \mathcal{R}) = \text{Hom}_A(\mathcal{R}, \mathcal{P}) = 0\).
3. Any morphism from \(\mathcal{P}\) to \(\mathcal{Q}\) factors through \(\text{add}\,\mathcal{R}\).

If \(\mathcal{R}\) is a weakly separating family in \(\Gamma(\text{mod}\,A)\) consisting of stable tubes, then an algebra \(B\) is a coil enlargement of \(A\) using modules from \(\mathcal{R}\) if there exists a finite sequence of algebras \(A = A_0, A_1, \cdots, A_m = B\) such that, for each \(i, A_{i+1}\) is obtained from \(A_i\) by an admissible operation with pivot either on a stable tube of \(\mathcal{R}\) or on a coil of \(\Gamma(\text{mod}\,A_i)\) obtained from a stable tube of \(\mathcal{R}\) by means of the admissible operations done so far. The coil type \(c_B = (c_B^-, c_B^+)\) of \(B\) is a pair of functors \(\lambda \longrightarrow \mathbb{N}\) defined by induction on \(i\), for each \(\lambda \in \Lambda\), as follows.

- **(a)** \(c_A = (c_0^-, c_0^+)\) is such that \(c_0^-(\lambda) = c_0^+(\lambda)\) is the rank of the stable tube \(\mathcal{R}_\lambda\).
- **(b)** If \(c_{A_{i-1}} = (c_{i-1}^-, c_{i-1}^+)\) is known and \(t_i\) is the parameter of the operation from \(A_{i-1}\) to \(A_i\), then \(c_{A_i} = (c_i^-, c_i^+)\) is defined by:

\[
c_i^-(\lambda) = \begin{cases} 
    c_{i-1}^-(\lambda) + t_i + 1 & \text{if the operations is (ad1*), (ad2*), (ad3*) with pivot in the coil of } \Gamma(\text{mod}\,A_{i-1}) \\
    c_{i-1}^-(\lambda) & \text{otherwise}
\end{cases}
\]

and

\[
c_i^+(\lambda) = \begin{cases} 
    c_{i-1}^+(\lambda) + t_i + 1 & \text{if the operations is (ad1), (ad2), (ad3) with pivot in the coil of } \Gamma(\text{mod}\,A_{i-1}) \\
    c_{i-1}^+(\lambda) & \text{otherwise}
\end{cases}
\]

If all but at most finitely many values of each of \(c_B^-, c_B^+\) equal 1, we replace each sequence by a finite sequence containing at least two terms, and including all those exceeding 1.

**Theorem.** [8] Let \(A\) be an algebra having a weakly separating family of stable tubes \(\mathcal{R}\), and \(B\) be a coil enlargement of \(A\) using modules from \(\mathcal{R}\). Then

- **(a)** There is a unique maximal branch coextension \(B^-\) (or extension \(B^+\)) of \(A\) which is a full convex subcategory of \(B\), and having \(c_B^-\) as coextension type (or \(c_B^+\) as extension type, respectively).
(b) \( \text{ind } B = \mathcal{P}' \lor \mathcal{R}' \lor \mathcal{Q}' \), where \( \mathcal{R}' \) is a weakly separating family of \( \text{ind } B \) obtained from \( \mathcal{R} \) by the sequence of admissible operations, and separating \( \mathcal{P}' \) from \( \mathcal{Q}' \). Moreover, \( \mathcal{P}' \) consists of \( B^- \)-modules, while \( \mathcal{Q}' \) consists of \( B^+ \)-modules.

1.4. Tame coil enlargements. Let \( B \) be a coil enlargement of a tame concealed algebra. Its \textit{coil type} \( c_B = (c_B^- , c_B^+) \) is tame if each of the sequences \( c_B^- , c_B^+ \) is one of the following: \((p,q)\) with \(1 \leq p \leq q\), \((2,2,r)\) with \(r \geq 2\), \((2,3,3)\), \((2,3,4)\), \((2,3,5)\), \((2,4,4)\), \((2,3,6)\) or \((2,2,2,2)\). We have the following result.

\textbf{Corollary.} \[8\] (4.3) Let \( B \) be a coil enlargement of a tame concealed algebra. The following conditions are equivalent:

(a) \( B \) is tame.
(b) \( B \) is of polynomial growth.
(c) \( c_B \) is tame.
(d) \( B^- \) and \( B^+ \) are tame.

Moreover, \( B \) is domestic if and only if \( B^- \) and \( B^+ \) are tilted of euclidean type.

1.5. Multicoil algebras. An Auslander-Reiten component \( \Gamma \) is a \textit{multicoil} if it contains a full translation subquiver \( \Gamma' \) which is a disjoint union of coils such that no point in \( \Gamma \setminus \Gamma' \) belongs to a cyclical path.

An algebra \( A \) is a \textit{multicoil algebra} if, for any cycle \( M_0 \to M_1 \to \cdots \to M_t = M_0 \) in \( \text{ind } A \), all the \( M_i \) lie in one standard coil of a multicoil of \( \Gamma(\text{mod } A) \). The first part of the following theorem is \[5\] (4.6), the second part is \[28\] (4.1).

\textbf{Theorem.} \[5, 28\] Let \( A \) be a multicoil algebra, then \( A \) is of polynomial growth. If \( A \) is strongly simply connected, then the converse also holds.

If \( A \) is a multicoil algebra, then it is triangular (hence of finite global dimension) \[6\] (3.5). Also, any full convex subcategory of \( A \) is a multicoil algebra \[6\] (5.6).

1.6. Supports. Supports of indecomposable modules over multicoil algebras are characterised in the following lemma.

\textbf{Lemma.} Let \( M \) be an indecomposable module over a multicoil algebra \( A \), and let \( B = \text{Supp } M \). Then:

(a) If \( M \) is directed, then \( B \) is tame and tilted.
(b) If \( M \) is not directed, then \( B \) is a tame coil enlargement of a tame concealed algebra.

\textit{Proof.} Since \( B \) is a full subcategory of \( A \), which is tame, then \( B \) is also tame. Then (a) follows from \[3\] (IX.2.8, p. 366) and (b) follows from \[6\] (5.9). \( \square \)

Note that, if \( L \) is an \( A \)-submodule of \( M \), and \( x \in A_0 \) is such that \( L(x) \neq 0 \), then \( M(x) \neq 0 \). Hence, \( L \) is also a \( B \)-submodule of \( M \).
2. ONE-POINT EXTENSION OF TAME CONCEALED ALGEBRAS

2.1. In this section, we prove that, if \( C \) is a tame concealed algebra and \( M \) is a \( C \)-module containing an infinity of pairwise non-isomorphic simple homogeneous submodules, then \( C[M] \) is wild. We start with reduction lemmata.

**Lemma.** Let \( C \) be a tame concealed algebra, let \( M \) be a preinjective \( C \)-module, and \( H \) a simple homogeneous submodule of \( M \). Then:

(a) \( M \) is sincere.
(b) If \( C[M] \) is not wild, then \( \dim_k(\text{top } M) \leq 2 \). Then, either \( M \) is indecomposable, or \( M \simeq M_1 \oplus M_2 \), with \( M_1, M_2 \) indecomposables with simple top.

**Proof.** (a) This follows from the sincerity of \( H \).
(b) If \( \dim_k(\text{top } M) = d > 2 \), then \( C[M] \) contains a wild full subcategory of the form

\[
\begin{array}{c}
\alpha_1 \\
\vdots \\
\alpha_d
\end{array}
\]

a contradiction. The second statement follows from Nakayama’s lemma, and (1.2).

2.2. Let \( A = kQ/I \) be a bound quiver algebra. It is well-known (see, for instance, [3](III.2.2, p.77)) that, if \( M \) is an \( A \)-module, then \( (\text{top } M)(x) = M(x) \) is \( x \) is a source, and \( (\text{top } M)(y) = \sum \{ \text{Coker}(M(\alpha)) : \alpha : x \rightarrow y \} \), if \( y \) is not a source.

If \( M \) satisfies the hypothesis of (2.1)(b), and \( x \) is a source, we then have \( \dim_k M(x) \leq 2 \).

**Corollary.** Let \( C \) be a tame concealed algebra, let \( M \) be a preinjective \( C \)-module, and \( H \) a simple homogeneous submodule of \( M \). If \( C[M] \) is not wild, then:

(a) \( C \) has at most two sources.
(b) If \( C \) has just one source \( s \) and \( \dim_k H(s) = d \geq 2 \), then \( d = 2 \) and \( \text{top } H = \text{top } M \).
(c) If \( C \) has two sources \( s_1, s_2 \) then \( \dim_k H(s_1) = \dim_k H(s_2) = 1 \) and \( \text{top } H = \text{top } M \).

**Proof.** (a) If \( C \) has at least three sources \( s_1, s_2, s_3 \), the remark above implies that \( \text{top } M \) has \( S_{s_1} \oplus S_{s_2} \oplus S_{s_3} \) as a summand, and \( \dim_k(\text{top } M) \geq 3 \), a contradiction.
(b) Since \( H(s) \subset M(s) \), which is at most two-dimensionsl, then \( d = 2 \) and \( \text{top } H = \text{top } M = S^2 \).
(c) Clearly, \( S_{s_1} \oplus S_{s_2} \) is a summand of both \( \text{top } H \) and \( \text{top } M \). Since \( \dim_k(\text{top } M) \leq 2 \), then \( \text{top } M = S_{s_1} \oplus S_{s_2} \). We claim that \( \text{top } H = S_{s_1} \oplus S_{s_2} \). If this is not the case, then there exists \( a \notin \{ s_1, s_2 \} \) such that \( S_a \) is a summand of \( \text{top } H \). Hence \( (\text{top } H)(a) \neq 0 \). By the remark above, there exists an arrow \( \alpha : b \rightarrow a \) in \( C \).
such that $H(\alpha)$ is not surjective. On the other hand, since $S_a$ is not a summand of $\text{top} M$, then $M(\alpha)$ is surjective. Now there is a path from one of the $s_i$ (with $i \in \{1, 2\}$) to $a$ passing through $\alpha$, which we may assume, without loss of generality, to be of minimal length:

\[ s_i = b_i \rightarrow b_{i-1} \rightarrow \cdots \rightarrow b_1 = b \xrightarrow{\alpha} a \]

We claim that $\dim_k M(b) \geq 2$. Indeed, if this is not the case, then the inclusion $j: H \hookrightarrow M$ yields a commutative square

\[
\begin{array}{ccc}
H(b) & \xrightarrow{H(\alpha)} & H(a) \\
\downarrow{j_b} & & \downarrow{j_a} \\
M(b) & \xrightarrow{M(\alpha)} & M(a)
\end{array}
\]

and $M(b) \simeq H(b) \simeq k$ gives that $j_b$ is an isomorphism. Also, since $H$ is sincere and $M(\alpha)$ is surjective, we have $M(a) \simeq H(a) \simeq k$. So $H(\alpha)$ is an isomorphism, a contradiction to its non-surjectivity. Since $\dim_k M(s_i) = 1$, there exists an arrow $\beta: c \rightarrow d$ on the path (*) above such that $1 = \dim_k M(c) < \dim_k M(d)$. Therefore $\text{Coker} M(\beta) \neq 0$, hence $S_d$ is a summand of $\text{top} M$, a contradiction to $\text{top} M = S_{s_1} \oplus S_{s_2}$ and $d \notin \{s_1, s_2\}$. □

2.3. Lemma. Let $C$ be a tame concealed algebra of type $\tilde{D}_4$ and $(H_\lambda)_\lambda$ be an infinite family of pairwise non-isomorphic simple homogeneous modules. If $M$ is a module such that $H_\lambda \subset M$ for all $\lambda$, then $C[M]$ is wild.

Proof. Assume first $C$ to be non-schurian, then $C$ is given by the quiver

\[
\begin{array}{c}
p \bullet \\
\alpha' \Downarrow \alpha \blade{\beta' \beta \beta'} \\
c \blade{\gamma' \gamma} \bullet \blade{s}
\end{array}
\]

bound by $\alpha \alpha' + \beta \beta' + \gamma \gamma' = 0$. For $\lambda \in k$, $H_\lambda$ is given by

\[
\begin{array}{c}
k \xrightarrow{k} \\
\lambda \xrightarrow{k} \blade{1} \xrightarrow{k} \blade{1} \xrightarrow{k} \lambda \\
-\lambda \xrightarrow{k} \blade{1} \xrightarrow{k} \blade{1} \xrightarrow{k}
\end{array}
\]

Since $H_\lambda \subset M$ for any $\lambda$, then $M$ is preinjective and we may further assume that $\dim_k(\text{top} M) \leq 2$. Also, $\text{soc} H_\lambda = S_p$ is a summand of $\text{soc} M$ while $S_s$ is a summand of $\text{top} M$. A straightforward examination of the preinjective component
of $C$ shows that $M$ is indecomposable and is one of the four modules $I_p, \tau^2_CI_a, \tau^2_CI_b$ or $\tau^2_CI_c$. In the first case, $C[I_p]$ has quiver

$$
\begin{array}{c}
p \bullet \alpha' \beta' b \beta \bullet \gamma' c \gamma \bullet \\
\end{array}
$$

bound by $\lambda\alpha = \mu\alpha$, $\lambda\beta = \mu\beta$, $\lambda\gamma = \mu\gamma$, $\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0$. Changing the presentation (replacing $\lambda$ by $\lambda - \mu$) gives the same quiver bound by $\lambda\alpha = 0$, $\lambda\beta = 0$, $\lambda\gamma = 0$, $\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0$. This is a split extension [1] of the algebra given by the quiver

$$
\begin{array}{c}
p \bullet \alpha' \beta' b \beta \bullet \gamma' c \gamma \bullet \\
\end{array}
$$

bound by $\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0$, which is evidently wild.

In the second case, $C[\tau^2_CI_a]$ has quiver

$$
\begin{array}{c}
p \bullet \alpha' \beta' b \beta \bullet \gamma' c \gamma \bullet \\
\end{array}
$$

bound by $\lambda\beta = \mu\beta$, $\lambda\gamma = \mu\gamma$, $\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0$. It contains as full convex subcategory the wild hereditary algebra with quiver

$$
\begin{array}{c}
a \bullet \alpha \bullet \lambda \bullet \\
\end{array}
\begin{array}{c}
\end{array}
$$

The remaining cases follow by symmetry.

We may thus assume that $C$ is schurian. If $C[M]$ is not wild, then $C$ is hereditary and has one of the quivers

$$
\begin{array}{c}
p_1 \bullet a \bullet s_1 \\
p_2 \bullet \\
\end{array}
\begin{array}{c}
p_1 \bullet a \bullet s \\
p_2 \bullet \\
\end{array}
\begin{array}{c}
p_1 \bullet a \bullet s \\
p_2 \bullet \\
\end{array}
\begin{array}{c}
p_3 \bullet s \\
p_4 \bullet \\
\end{array}
$$
Recall that \( \text{soc} H_\lambda \) is a summand of \( \text{soc} M \) and \( \dim_k(\text{top} M) \leq 2 \). It is easily seen that in the first and third cases, no preinjective \( C \)-module satisfies these conditions. In the second case, however, we have the possibility \( M = \tau_C^k I_s \), but then \( C[M] \) is given by the quiver

\[
\begin{array}{c}
p_1 \cdot \\
p_2 \cdot \\
p_3 \cdot \\
\end{array}
\begin{array}{c}
\beta \\
\gamma \\
\delta \\
\end{array}
\begin{array}{c}
a \cdot \\
\alpha \cdot \\
s \cdot \\
\mu \cdot \\
\lambda \cdot \\
\end{array}
\end{array}
\]

bound by \( \lambda \alpha \beta = \mu \alpha \beta, \lambda \alpha \gamma = \mu \alpha \gamma, \lambda \alpha \delta = \mu \alpha \delta \). It contains the wild hereditary full subcategory

\[
\begin{array}{c}
a \cdot \\
\alpha \cdot \\
s \cdot \\
\mu \cdot \\
\lambda \cdot \\
\end{array}
\]

This establishes the lemma.

**Example.** Before proving our next result, we observe that the above lemma does not hold true for tame concealed algebras of type \( \widetilde{A} \). Indeed, let \( C \) to be the Kronecker algebra, that is, the hereditary algebra given by the quiver

\[
\begin{array}{c}
p \cdot \\
s \cdot \\
\end{array}
\begin{array}{c}
\beta \\
\alpha \\
\delta \\
\gamma \\
\end{array}
\end{array}
\]

Observe that the indecomposable injective \( C \)-module \( I_p \) at the point \( p \) contains all the indecomposable regular modules \( H_\lambda \) given by

\[
\begin{array}{c}
k \cdot \\
\lambda \cdot \\
k \cdot \\
\end{array}
\begin{array}{c}
1 \\
\end{array}
\]

Now, the one point extension \( C[I_p] \) is the radical square zero algebra given by the quiver

\[
\begin{array}{c}
p \cdot \\
s \cdot \\
\end{array}
\begin{array}{c}
\beta \\
\alpha \\
\delta \\
\gamma \\
\end{array}
\end{array}
\]

bounded by \( \alpha \delta = 0, \gamma \beta = 0 \) and \( \alpha \beta = \gamma \delta \), which is clearly tame.

**Remark.** Notice, however, that if \( C \) is tame concealed of type \( \widetilde{A} \) and schurian and \( M \) satisfies the conditions of (2.3), then \( C[M] \) is wild. Indeed, if this is the case then, because of (2.2), \( C \) has at most 2 sources, hence is hereditary with quiver
Here, soc\(H_\lambda = S_{p_1} \oplus S_{p_2}\) is a summand of soc\(M\) while top\(M = \text{top}H_\lambda = S_{s_1} \oplus S_{s_2}\). It is easily seen that no preinjective \(C\)-module satisfies these conditions.

2.4. Lemma. Let \(C\) be a tame concealed algebra of type \(\widetilde{D}_n\), with \(n \geq 5\), or \(\widetilde{E}\) and \(H\) be a simple homogeneous \(C\)-module. If \(M\) is preinjective such that \(H \subset M\), then \(C[M]\) is wild.

Proof. We claim that \(C[M]\) is wild. Indeed, the tubular type of \(C\) is of the form \((a, b, c)\) with at least one of \(a, b, c\) larger than or equal to 3. Taking then a one-point extension by a simple homogeneous module yields a wild algebra. This establishes our claim. Applying now (1.2), we get that \(C[M]\) is wild, as required.

2.5. We are now able to prove the main result of this section.

Theorem. Let \(C\) be a tame concealed algebra of type distinct of \(\widetilde{A}\), and \((H_\lambda)_{\lambda}\) be an infinite family of pairwise non-isomorphic modules. Suppose \(M\) is a \(C\)-module such that \(H_\lambda \subset M\) for all \(\lambda\). Then \(C[M]\) is wild.

Proof. This follows immediately from (2.3) and (2.4).

3. Torsionless finiteness and representation dimension.

3.1. Let \(A\) be an Artin algebra. An \(A\)-module \(M\) is called a generator (of \(\text{mod}A\)) if \(A_A \in \text{add}M\) and a cogenerator if \((DA)_A = \text{Hom}_k(A_A, k) \in \text{add}M\). Let \(A\) be a non-semisimple algebra. The representation dimension \(\text{rep.dim.}A\) of \(A\) is the infimum of the global dimensions of the algebras \(\text{End}_AM\), where \(M\) is a generator and a cogenerator of \(\text{mod}A\), see [9].

An Artin algebra \(A\) is called torsionless-finite if every indecomposable projective \(A\)-module has only finitely many isomorphism classes of indecomposable submodules. One defines cotorsionless-finite dually. An Artin algebra is torsionless-finite if and only if it is cotorsionless-finite [10]. We need essentially the following result (see [26]).

Theorem. If \(A\) is torsionless-finite, then \(\text{rep.dim.}A \leq 3\).
3.2. Lemma. Let $A$ be a (possibly wild) branch enlargement of a tame concealed algebra. Then $A$ is torsionless-finite. In particular, $\text{rep.dim.} A \leq 3$.

Proof. Using the description of $\text{mod}A$ in [21], we see that, if $P_A$ is an indecomposable projective $A$-module, then either $P_A$ is postprojective (in which case it clearly has only finitely many isomorphism classes of indecomposable modules) or it lies in an inserted tube $\Gamma$. But in this latter case, $P$ has only finitely many indecomposable submodules lying in $\Gamma$ and, using [11], there are only finitely many postprojective modules $X$ such that $\dim_k X \leq \dim_k P$. Since the other tubes in the same family as $\Gamma$ are orthogonal to $\Gamma$, the proof is complete. □

3.3. Corollary. Let $A$ be a tame quasi-tilted algebra, then $A$ is torsionless-finite. In particular, $\text{rep.dim.} A \leq 3$.

Proof. This follows from (3.2) and [29] □

Remark. This corollary is a particular case of the main result of [22]. Further, the same proof as in (3.2) and [4] give that if $A$ is iterated tilted of euclidean type, then $A$ is torsionless-finite, and so $\text{rep.dim.} A \leq 3$. This is a particular case of the main result of [16].

3.4. Lemma. Let $A$ be a strongly simply connected tame coil enlargement of a tame concealed algebra, then $A$ is torsionless-finite. In particular, $\text{rep.dim.} A \leq 3$.

Proof. By (1.3), $A$ contains as full convex subcategories a unique maximal branch coextension $A^-\rightarrow$ and a unique maximal branch extension $A^+$ of a tame concealed algebra, and $\text{mod}A$ contains a weakly separating family of coils $\mathcal{T}'$ such that $\text{ind}A = \mathcal{P}' \vee \mathcal{T}' \vee \mathcal{Q}'$ where $\mathcal{P}' \subset \text{ind}A^-$ and $\mathcal{Q}' \subset \text{ind}A^+$. Let $P_A$ be an indecomposable projective module. We have three cases:

(a) Assume $P \in \mathcal{P}'$, then $P$ is an indecomposable projective $A^-$-module. Because of (3.2), and the fact that $A^-$ is a branch enlargement of a tame concealed algebra, $P$ has only finitely many isomorphism classes of indecomposable submodules in $\text{mod}A^-$. Now, the indecomposable submodules of $P$ in $\text{mod}A$ and $\text{mod}A^-$ coincide.

(b) Assume $P \in \mathcal{Q}'$. For the same reason, $P$ has only finitely many isomorphism classes of indecomposable submodules in $\text{mod}A^+$, hence in $\text{mod}A$.

(c) Assume $P \in \mathcal{T}'$, and let $M = \text{rad}P$. There exists a sequence of full convex subcategories of $A$

$$A^- = A_0 \subset A_1 \subset \cdots \subset A_i = A$$

which are iterated one-point extensions of $A^-$, and an index $i$ such that $A_{i+1} = A_i[M]$, that is, $P$ is the unique indecomposable projective in $\text{mod}A_{i+1}$ which is not in $\text{mod}A_i$. Also, $M$ is the pivot of an admissible operation and then is indecomposable except perhaps in the case (ad1). Then, $M = M' \oplus M''$, where $M''$ is a directed indecomposable, while $M'$ is an indecomposable lying in a coil. Since $M''$ has only finitely many isomorphism classes of indecomposable
submodules and $A_{i+1}$ is of the form $(A'_i \times D)[M' \oplus M'']$, where $D$ is a triangular matrix algebra, and $A'_i$ is the full subcategory of $A_i$ with objects $(A'_i)_0 = (A_i)_0 \setminus D_0$, then it suffices to consider the submodules of $M'$. We may thus for simplicity assume that $M = M'$.

Let $M^-$ be the largest $A^-$-submodule of $M$. Assume that $M^-$ has infinitely many isomorphism classes of indecomposable $A^-$-submodules. Because of (3.2), $M^-$ has to be a preinjective $A^-$-module. By (2.5), $A^-[M^-]$ is wild. Since $A^-$ is a full convex subcategory of $A_i$, then $A_i[M^-]$ is wild. By (1.2), and the tameness of $A_i$, we get that $A_{i+1} = A_i[M]$ is wild. This is absurd, because $A_{i+1}$ is a full convex subcategory of $A$ which is tame. This shows that $M^-$ has only finitely many non-isomorphic indecomposable submodules. Since a submodule of $M$ is either a submodule of $M^-$, or lies in the coil containing $M$, we are done. □

3.5. COROLLARY. Let $A$ be a multicoil algebra, and $P$ be an indecomposable projective $A$-module lying in a coil $\Gamma$ of $\Gamma(\text{mod } A)$, then $P$ has only finitely many isomorphism classes of indecomposable submodules.

Proof. By [6](5.9), we can assume that $\Gamma$ is obtained from a stable tube over a tame concealed algebra $C$ by a sequence of admissible operations and the support algebra $B$ of $\Gamma$ is obtained from $C$ by the corresponding sequence of one-point extensions and coextensions. Since the $A$-submodules and the $B$-submodules of $P$ coincide, the result follows from (3.4). □

3.6. We are now able to state and prove the main result of this paper.

THEOREM. Let $A$ be a strongly simply connected algebra of polynomial growth, then $A$ is torsionless-finite. In particular, $\text{rep.dim.} A \leq 3$.

Proof. We may clearly assume that $A$ is representation-infinite. Let $P$ be an indecomposable projective $A$-module. By (1.6), if $P$ is not directed, then the support algebra $B$ of $P$ is a tame coil enlargement of a tame concealed algebra. By (3.4), $P$ has only finitely many isomorphism classes of indecomposable $B$-submodules, hence $A$-submodules. On the other hand, if $P$ is directed, then, by (1.6) again, $B$ is tame and tilted. Since, clearly, $B$ has a sincere directed indecomposable module (namely $P$), then, by [23], $B$ is domestic in at most two one-parameters. Moreover, by [14], $B$ is a full convex subcategory of $A$.

Let $\Gamma$ denote the component of $\Gamma(\text{mod } B)$ containing $P$, and $\Sigma$ denote the full subquiver of $\Gamma$ consisting of the indecomposable modules $X \in \Gamma$ such that there is a path $X \rightsquigarrow P$, and every such path is sectional. By [3](IX.2.6, p. 364), $\Sigma$ is a complete slice in $\Gamma(\text{mod } B)$. Note also that in $\Gamma(\text{mod } B)$, there are finitely many directed components preceding $\Gamma$ (actually, by [23], at most two). Since indecomposable modules lying in a directed component are uniquely determined by their composition factors [11], then $P$ has infinitely many isomorphism classes of indecomposable submodules if and only if there exists an infinite family $(T_\lambda)_\lambda$ of homogeneous tubes over a tame concealed algebra $C$, and an infinite family...
of homogeneous modules $H_\lambda \in \mathcal{T}_\lambda$ such that $H_\lambda \subset \text{rad} P$, for each $\lambda$. Clearly, then, all the $H_\lambda$ are contained in the largest $C$-submodule $R$ of $\text{rad} P$. Moreover, by the structure of homogeneous tubes, we can assume the $H_\lambda$ to be simple homogeneous. By (2.5), if this is the case, then $C[R]$ is wild, hence so is $B$. This shows that $P$ has only finitely many isomorphism classes of indecomposable $B$-submodules, hence $A$-submodules. □

3.7. **Remark.** As is seen from the example in Section 2, it is not true in general that multicoil algebras are torsionless-finite. However, notice that the algebra

$$
\begin{array}{ccc}
\beta & & \alpha \\
\delta & \cdot & \gamma
\end{array}
$$

bounded by $\alpha\delta = 0$, $\gamma\beta = 0$ and $\alpha\beta = \gamma\delta$ is tilted and so, by [2], has representation dimension at most 3.

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