Application of Exp-function method to Sawada-Kotere equation with variable coefficients

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Abstract: In this paper, the Exp-function method is used to obtain generalized solitonary solutions and periodic solutions of Sawada-Kotere equation with variable coefficients. It is shown that the Exp-function method, with the help of symbolic computation, provides a powerful mathematical tool for solving nonlinear evolution equations with variable coefficients in mathematical physics.

1. Introduction

In the past few decades, the nonlinear evolution equations (NLEEs) are widely used to describe many important phenomena and dynamic processes in physics, mechanics, biology, etc. By the aid of exact solutions, when they exist, the phenomena modeled by these NLEEs can be better understood. They can also help to analyze the stability of these solutions and to check numerical analysis for these NLEEs. With the development of soliton theory, many methods which are used to find solution of NLEEs have been proposed, such as the inverse scattering method[1], the Bäcklund transformation[2], Hirota method[3], the mapping method[4], the tanh function method[5], homogeneous balance method[6], extended tanh function method[7], the Jacobian elliptic function method[8], and so on.

However, as far as we known, most of aforementioned methods are related to the constant-coefficient models. Recently, the investigations of variable-coefficient nonlinear equations have been paid attention to by many researchers[9-13].

He and Wu[14] proposed a straightforward and concise method, called Exp-function method, to obtain generalized solitonary solutions and periodic solutions of NLEEs. The solution procedure for this method, by the help of Mathematica, is of utter simplicity and this method can be easily extended to all kinds of NLEEs. Some illustrative examples in Refs.[15-17] show that this method is very effective to search for various solitonary and periodic solutions of nonlinear equations. Recently, Exp-function method have been applied to finding generalized solutions of nonlinear differential-difference equations by Zhu[18-19].

In this letter, we deduce abundant solitonary solutions extend the Exp-function method for variable coefficient Sawada-Kotere equation as the following form

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\[ u_t + f(t)u^2u_x + g(t)uu_{xx} + \alpha(t)uu_{xxx} + \beta(t)u_{xxxx} = 0 \quad (1) \]

Where \( f(t) \), \( g(t) \), \( \alpha(t) \), \( \beta(t) \) are all arbitrary functions of indicated variable.

2. Basic idea of the Exp-function method
Using the transformation
\[ u = U(\eta) \quad \eta = kx + \int \phi(t)dt \quad (2) \]

where \( k \) is a constant, \( \phi(t) \) is an integral function of \( t \) to be determined later, Eq.(1) becomes
\[ \phi(t)U' + kf(t)U^2 + k^2 g(t)U'U'' + k^3 \alpha(t)UU'' + k^4 \beta(t)U''' = 0 \quad (3) \]

where prime denotes the differential with respect to \( \eta \).

According to the Exp-function method, we assume that the solution of Eq.(3) can be expressed in the following form
\[ U(\eta) = \sum_{n=-c}^{d} a_n \exp(n\eta) \sum_{m=-q}^{p} b_m \exp(m\eta) \quad (4) \]

where \( c, d, p \) and \( q \) are positive integers which are unknown to be further determined, \( a_n \) and \( b_m \) are unknown constants. Eq.(4) can be re-written in an alternative form [17] as follow
\[ U(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_d \exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_q \exp(-q\eta)} \quad (5) \]

In order to determine values of \( c \) and \( p \), we balance the linear term of highest order in Eq.(3) with the highest nonlinear term[14,17]. By simple calculation, we have
\[ U''' = \frac{c_1 \exp[(3p + c)\eta] + \cdots}{c_2 \exp[3p\eta] + \cdots} \quad (6) \]

and
\[ UU''' = \frac{c_3 \exp[(7p + 2c)\eta] + \cdots}{c_4 \exp[9p\eta] + \cdots} = \frac{c_3 \exp[(30p + 2c)\eta] + \cdots}{c_4 \exp[32p\eta] + \cdots} \quad (7) \]

where \( c_i \) are determined coefficients only for simplicity. Balancing highest order of Exp-function in Eq.(6) and (7), we have
\[ 30p + 2c = 31p + c \quad (8) \]

which lead to the result
\[ p = c \quad (9) \]

Similarly to determine values of \( d \) and \( q \), we balance the linear term of lowest order in Eq.(3)
\[ U''' = \frac{\cdots + d_4 \exp[-(31q + d)\eta]}{\cdots + d_2 \exp[-32d\eta]} \quad (10) \]

and
\[ U U'' = \cdots + d_1 \exp[-(7q + 2d)\eta] + \cdots + d_4 \exp[-32d\eta] + \cdots \]  

(11)

where \( d_i \) are determined coefficients only for simplicity. Balancing highest order of Exp-function in Eq.(10) and (11), we have

\[-(30q + 2d) = 31q + d\]  

(12)

which leads to the result

\[ q = d \]  

(13)

Application to the Sawada-Kotere equation with variable coefficients

We can freely choose the values of \( c, d, p, q \). For simplicity, we set \( p = c = 1, d = q = 1 \). Eq.(5) becomes

\[ \exp(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)} \]  

(14)

Substituting Eq.(14) into Eq.(3) and collecting coefficients of power of \( \exp(n\eta) \), yields a set of algebraic equations for \( a_1, a_0, a_{-1}, b_1, b_0, b_{-1} \) and \( \phi(t) \). With the aid of Mathematica, then set each coefficients to zero, solving the system of algebraic equations, we obtain

\[ a_0 = \frac{b_0}{b_1}\left(3k^2b_1 + \delta_1^2a_1\right), \quad a_{-1} = \frac{a_1b_0^2}{4b_1^2}, \quad b_{-1} = \frac{b_0^2}{4b_1}, \quad \beta(t) = \frac{\beta(t)}{\alpha(t)} = \delta_1, \]

\[ f(t) = \delta_2, \quad g(t) = \delta_3, \quad \delta_1 = 1 + 10\delta_2, \]

\[ \phi(t) = -\frac{k(a_1^2\delta_2 + a_1b_1k^2 + b_1^2k^4\delta_1)}{b_1^2}\alpha(t), \]  

(15)

Where \( a_1, b_1, b_0, b_{-1} \) are arbitrary constants and \( \delta_1, \delta_2, \delta_3 \) are constants too. Substituting Eq.(15) into Eq.(14) yields the following generalized solitary solution of Eq.(1)

\[ \varphi(t) = \exp\left[k\left(\delta_2^2 + a_1^2k^2 + b_1^2k^4\delta_1\right)\right] \]  

(16)

If we choose \( b_1 = 1, b_0 = 2 \), we have

\[ u = a_1 + \frac{6k^2}{\delta_2}\left(\exp\left[k\left(\delta_2^2 + a_1^2k^2 + b_1^2k^4\delta_1\right)\right] + 2 + \exp\left[k\left(\delta_2^2 + a_1^2k^2 + b_1^2k^4\delta_1\right)\right]\right) \]  

(17)
we set \( g_1 = \frac{3k^2}{2\delta_2} \). \( p=k/2 \)

Re-writer Eq.(17) in the form

\[
 u = a_1 + \frac{4g_1}{e^{[-2p\sigma_1 \square + 4p^2a_1 + 16\sigma_1 \sigma_2]}\int \sigma(\tau) d\tau} + e^{[-2p\sigma_1 \square + 4p^2a_1 + 16\sigma_1 \sigma_2]}\int \sigma(\tau) d\tau \nonumber
\]

(18)

Express in another form

\[
 u = a_1 + g_1 \sec h^2 (px - \int p(a_1^2f(t) + 4p^2a_1\alpha(t) + 16p^4\beta(t))) dt
\]

(19)

This is a ball-type soliton solution.

And we have

\[
 u = a_1 + g_1 \tanh^2 (px - \int p(a_1^2f(t) + 4p^2a_1\alpha(t) + 16p^4\beta(t))) dt
\]

(20)

This is a kink-type soliton solution.

When \( k \) is an imaginary number, the obtained solitonic solution can be converted into periodic solution [14,17]. We write \( k=iK \). Using the transformation

\[
 \int \left[ Kx - K(\delta_2a_1^2 - K^2a_1 + 16\delta_1K^4) \right] + i \sin[Kx - K(\delta_2a_1^2 - K^2a_1 + 16\delta_1K^4)] dt
\]

(21)

Then Eq.(16) become

\[
 u = \frac{a_1}{b_1} - \frac{3K^2b_0/\delta_2}{(b_1 + b_0^2)\cos[Kx - K(\delta_2a_1^2 - K^2a_1 + 16\delta_1K^4)] + b_0 + (b_1 + b_0^2)i\sin[Kx - K(\delta_2a_1^2 - K^2a_1 + 16\delta_1K^4)]}
\]

(22)

If we search for a periodic solution or compact-like solution, the imaginary part in Eq.(22) must be zero, that require that

\[
 b_1 - \frac{b_0^2}{4b_1} = 0
\]

(23)

From Eq.(23) we obtain

\[
 b_0 = \pm 2b_1
\]

(24)

Substituting Eq.(24) into (22) yields two periodic solutions

\[
 u = \frac{a_1}{b_1} - \frac{3K^2/\delta_2}{\cos[Kx - K(\delta_2a_1^2 - K^2a_1 + 16\delta_1K^4)] + 1}
\]

(25)

And

\[
 u = \frac{a_1}{b_1} + \frac{3K^2/\delta_2}{\cos[Kx - K(\delta_2a_1^2 - K^2a_1 + 16\delta_1K^4)] - 1}
\]

(26)
There are periodic solutions.

3. Summary and discussion
In this Letter, we apply the Exp-function method to obtain generalized solitornary solutions and periodic solutions of Sawada-Kotere equation with variable coefficients. This method can also be extended to other NLEEs with variable coefficients. The Exp-function method is a promising and powerful new method for NLEEs arising in mathematical physics. Its applications are worth further studying.

References
[1] Cardner C S, Kruskal J M and Miura R M 1976 Phys. Rev. Lett. 19 1095
[2] Wahlquist H D and Estabrook F B 1973 Phys. Rev. Lett. 31 1386
[3] Hirota R 1971 Phys. Rev. Lett. 27 1192
[4] LOU S Y and Ni G J 1989 J. Math. Phys. 30 1614
[5] Malfliet W 1992 Am. J. Phys. 60 650
[6] Wang M L, Zhou Y B and Li Z B 1996 Phys. Lett. A. 216 67
[7] Fan E G 2001 Phys. Lett. A. 282 18
[8] Liu S K, Fu Z T and Liu S D 2001 Phys. Lett. A. 289 69
[9] Lou S Y and Ruan H Y 1992 Acta. Phys. Sin 41 182
[10] Sabry R, Zahran M A and Fan E G 2004 Phys. Lett. A. 326 93
[11] Li H M 2001 Chinese. Phys. Lett. 19 745
[12] Li H M 2002 Commun. Theor. Phys. 37 561
[13] Liu Q and Zhu J M 2006 Phys. Lett. A. 352 233
[14] He J H and Wu X H 2006 Chaos. Soliton. Fract. 30 700
[15] Wu X H and He J H 2007 Solitary solutions, periodic solutions and compacton-like solutions using the Exp-function method, Comput. Math. Appl. In Press
[16] Wu X H and He J H 2007 EXP-function method and its application to nonlinear equations, Chaos. Soliton. Fract. In Press
[17] He J H and Abdou M A 2007 New periodic solutions for nonlinear evolution equations using Exp-function method Chaos. Soliton. Fract. In Press
[18] Zhu S D 2007 Int.J. Nonlinear. Sci. 8 461
[19] Zhu S D 2007 Int.J. Nonlinear. Sci. 8 465