Revisiting additivity violation of quantum channels

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Abstract

We prove additivity violation of minimum output entropy of quantum channels by straightforward application of ε-net argument and Lévy’s lemma. The additivity conjecture was disproved initially by Hastings. Later, a proof via asymptotic geometric analysis was presented by Aubrun, Szarek and Werner, which uses Dudley’s bound on Gaussian process (or Dvoretzky’s theorem with Schechtman’s improvement). In this paper, we develop another proof along Dvoretzky’s theorem in Milman’s view showing additivity violation in broader regimes than the existing proofs. Importantly, Dvoretzky’s theorem works well with norms to give strong statements but these techniques can be extended to functions which have norm-like structures - positive homogeneity and triangle inequality. Then, a connection between Hastings’ method and ours is also discussed. Besides, we make some comments on relations between regularized minimum output entropy and classical capacity of quantum channels.

1 Preliminary

1.1 Introduction

Existence of quantum channels which show additivity violation of minimum output entropy was proven by Hastings [Has09], which is stated as follows. There exists a quantum channel Φ such that, denoting the complex conjugate of Φ by Φ,

\[ S_{\min}(\Phi \otimes \Phi) < S_{\min}(\Phi) + S_{\min}(\Phi) \]  (1.1)

Here, \( S_{\min}(\cdot) \) is the minimum output entropy of quantum channel, which is defined as

\[ S_{\min}(\Phi) = \min_{\rho} S(\Phi(\rho)) \]  (1.2)

where \( S(\cdot) \) is the von Neumann entropy and \( \rho \) runs over all the pure states which are rank-one projections; general quantum states (mixed states) are written by positive Hermitian matrices of trace one but we can assume that input states are pure states because the function \( S(\cdot) \) is concave. Historically, the additivity question was made in [KR01]. Note that \( \leq \) is obvious in (1.1).

One of consequences of the additivity violation of minimum output entropy is that Holevo capacity is not in general additive either. Holevo capacity \( \chi(\cdot) \) is defined as

\[ \chi(\Phi) = \max_{\{\rho_i, p_i\}} \left[ S \left( \Phi \left( \sum_i p_i \rho_i \right) \right) - \sum_i p_i S(\Phi(\rho_i)) \right] \]  (1.3)

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Here, \( \{p_i, \rho_i\} \) runs over all possible ensembles, where probabilities \( \{p_i\} \) are assigned to quantum states \( \{\rho_i\} \) \cite{SW97, Hol98}. One form of additivity violation of Holevo capacity can be stated as

\[
\chi (\Phi \otimes 2) > 2\chi (\Phi)
\]  

(1.4)

This is deduced by getting the additivity violation of minimum output entropy for two identical channels from (1.1) via the result in \cite{FW07}, and using the equivalence relation between the two additivity or non-additivity properties \cite{Sho04}. (The latter technique is extended in Section 6 to show a similar statement for regularized quantities.) This, in turn, implies that

\[
C(\Phi) = \lim_{n \to \infty} \frac{1}{n} \chi (\Phi^{\otimes n}) = \lim_{m \to \infty} \frac{1}{2m} \chi (\Phi^{\otimes 2m}) \geq \frac{1}{2} \chi (\Phi^{\otimes 2}) > \chi (\Phi)
\]

(1.5)

Here, \( C(\cdot) \) is the classical capacity and this operational quantity is defined in an asymptotic form as in the first equality. On the other hand, Holevo capacity is written in a one-letter formula as in (1.3) and it gives the classical capacity when we do not use entangled inputs \cite{SW97, Hol98}. These two quantities had been conjectured to be identical but now we know that they are different in general as one can see in (1.5). Therefore, this in particular implies that entanglement inputs can increase the classical capacity of some channels. We refer interested readers to \cite{Hol06}.

Soon after the Hasting’s paper \cite{Has09} was publicized in 2008, several papers followed to give rigorous proofs and generalize the result \cite{FKM10, BH10, FK10}. Moreover, in 2010, Aubrun, Szarek and Werner found another proof in \cite{ASW11} via the Dudley’s bound on Gaussian process \cite{Dud67, JM78} (or Dvoretzky’s theorem with Schechtman’s improvement \cite{Sch89}). The original Dvoretzky’s theorem can be found in \cite{Dvo61}. In fact, a year before, they proved in \cite{ASW10} additivity violation of p-Renyi entropy for \( p > 1 \) via Dvoretzky’s theorem in Milman’s version \cite{Mil71, FLM77}, but it was not strong enough to prove additivity violation of minimum output entropy as was written in \cite{ASW11}. (The additivity violation of p-Renyi entropy for \( p > 1 \) itself was first proven by Hayden and Winter in 2007 \cite{HW08}, and later by Collins and Nechita \cite{CN11} via free probability.) Also, Additivity violation for \( p \) close to 0 was proven in \cite{CHL+08}. Note that our problem corresponds to the case \( p = 1 \). Interestingly, no concrete counterexample has been found yet for \( p = 1 \) whereas a counterexample for \( p > 2 \) was explicitly constructed in \cite{GHP10}, many years after the counterexample for \( p > 4.79 \) was found in \cite{WH02}. Also, we must mention a recent paper \cite{BCN13} where they proved a rather large additivity violation and the smallest output dimension could be as small as 183 while the dimensions of input and environment are infinite. Their method is based on free probability.

In this paper, we show that additivity violation of minimum output entropy can be proven by the standard method via \( \epsilon \)-net argument and Lévy’s Lemma, which in fact is very similar to the Milman’s view on Dvoretzky’s theorem. Interestingly, this pair of techniques was used in \cite{HLW06} to show existence of strongly entangled subspaces, which finally lead to the additivity violation of Ranyi entropy for \( p > 1 \) \cite{HW08}. However, their estimate was not strong enough to prove the additivity violation of minimum output entropy. On the other hand, our new approach gives an improved estimate which makes it possible. Historically, approximating the von Neumann entropy by using the Hilbert-Schmidt distance from the maximally mixed state, which was introduced in \cite{BH10} (perhaps originally from \cite{Has09} via Taylor expansion), fitted into asymptotic geometric analysis argument in \cite{ASW11}. Moreover we suggest that its norm-like structures - almost positive homogeneity and triangle inequality - actually put our problem into the framework of the Milman’s view. The technical discussion on this issue is written in Section 3 after stating additivity violation in Section 2. One can see that our result is stronger than all the existing proofs \cite{Has09, FKM10, BH10, FK10, ASW11} in the sense that we can prove the additivity violation asymptotically as long as the dimensions of input and output are proportional to each other and proportionally larger than or equal to square of the dimension of environment; there is no restriction on
the ratios. We make some analysis on our method and compare it to Hastings’ in Section 4. Our proof method can be applied to random unitary channels, which is briefly studied in Section 5.

Besides, there is an open problem:

$$C(\Phi \otimes \Omega) = C(\Phi) + C(\Omega)$$

for different channels \(\Phi\) and \(\Omega\). In Section 6, we provide a proof with the widely-known fact that additivity violation of regularized minimum output entropy implies that of classical capacity.

1.2 Channel

A (quantum) state is a positive Hermitian operator of trace one, and a (quantum) channel is a completely positive and trace-preserving map on the states. We denote the set of unit vectors in \(\mathbb{C}^d\) by \(S_{\mathbb{C}^d}\) and the linear maps on \(\mathbb{C}^d\) by \(L(\mathbb{C}^d)\). Also, we denote the dual of vector \(x \in \mathbb{C}^d\) by \(x^*\), where, in the bra-ket notation, \(x = |x\rangle\) and \(x^* = \langle x|\), which we don’t use in this paper.

In Stinespring’s picture \([Sti55]\), channels are identified as isometries:

$$V : \mathbb{C}^l \rightarrow \mathbb{C}^k \otimes \mathbb{C}^n$$

and channels are written for \(x \in S_{\mathbb{C}^l}\) as

$$\Phi : L(\mathbb{C}^l) \rightarrow L(\mathbb{C}^k) \quad xx^* \mapsto \text{Tr}_{\mathbb{C}^n}[V xx^* V^*]$$

Moreover, through this embedding picture, we can identify quantum channels as \(l\)-dimensional subspaces \(E \subseteq \mathbb{C}^k \otimes \mathbb{C}^n\) such that

$$\Phi_E : L(E) \rightarrow L(\mathbb{C}^k) \quad xx^* \mapsto \text{Tr}_{\mathbb{C}^n}[xx^*] = XX^*$$

for \(x \in \tilde{E} = E \cap S_{\mathbb{C}^k \otimes \mathbb{C}^n}\). Here, partial trace is understood by the following identification between linear spaces:

$$\mathbb{C}^k \otimes \mathbb{C}^n = \mathcal{M}_{k,n}(\mathbb{C}) \quad x = X$$

In what follows, we use the lower and upper cases of the same letter to represent this identification in \((1.13)\). Importantly, by applying Schmidt decomposition to \(x \in \mathbb{C}^k \otimes \mathbb{C}^n\) we know that \(\text{Tr}_{\mathbb{C}^n}[xx^*]\) and \(\text{Tr}_{\mathbb{C}^k}[xx^*]\) share the same non-zero eigenvalues. So, we always assume safely that \(k \leq n\). In this case, we say that the dimensions of input, output and environment are \(l, k\) and \(n\), respectively, although the spaces of environment and output are interchangeable for the additivity problem of minimum output entropy \([Hol05, KMNR07]\).

We give some definitions here. To define the complex conjugate of channel \(\Phi_E\), we fix some isometry \(V\) in \((1.9)\) such that its image is \(E\) and then define the channel \(\bar{\Phi}_E\) by \(\bar{V}\). This definition is unique only up to rotations, but this does not cause a problem in our paper. Since we identify channels as subspaces we define random quantum channels as follows. Fix an \(l\)-dimensional subspace \(E_0\) in \(\mathbb{C}^k \otimes \mathbb{C}^n\) and generate random subspaces \(UE_0\) with \(U \in U(kn)\) where \(U\) is chosen randomly according to the Haar measure on the unitary group.
2 Additivity violation

First, the canonical Bell state on $\mathbb{C}^d \otimes \mathbb{C}^d$ is defined as

$$b_d = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} e_i \otimes e_i$$

(2.1)

where $\{e_i\}$ is the canonical basis in $\mathbb{C}^d$. The Bell state flips matrices with transpose and in particular:

$$(U \otimes \bar{U}) b_d = \left((U \bar{U})^T \otimes I\right) b_d = b_d$$

(2.2)

for $U \in U(d)$. This property ensures a rather large eigenvalue of $(\Phi_E \otimes \bar{\Phi}_E)(b_l b_l^*)$. This idea is originated from [HW08] where $l$ divides $kn$ but their proof is easily adapted to any $l \leq kn$, which is written below. (A proof on this property through graphical calculus was given in [CN10], in which the exact limit eigenvalue distribution of $(\Phi \otimes \bar{\Phi})(b_l b_l^*)$ was also calculated with random isometry in the picture of (1.9).) This, in turn, implies the following lemma.

**Lemma 2.1.** For any channel $\Phi_E$, let $l = an$ with $a > 0$ and we have

$$S_{\min} (\Phi_E \otimes \bar{\Phi}_E) \leq 2 \log k - a \log \left(k\right) + \frac{2a}{k}$$

(2.3)

for large enough $k$.

**Proof.** Firstly, we get a lower bound for the largest eigenvalue of

$$(\Phi_E \otimes \bar{\Phi}_E)(b_l b_l^*) = \text{Tr}_{\mathbb{C}^n \otimes \mathbb{C}^n} \left[(V \otimes \bar{V}) b_l b_l^* (V^* \otimes V^T)\right]$$

(2.4)

by projecting them to the one-dimensional subspace of the Bell state as in [HW08]:

$$b_k^* (\Phi_E \otimes \bar{\Phi}_E)(b_l b_l^*) b_k \geq |b_l^* (V^* \otimes V^T) (b_k \otimes b_n)|^2 = \frac{l}{kn} |b_l^* b_l|^2 = \frac{l}{kn}$$

(2.5)

Indeed, for $l \times d$ matrix $A$,

$$(I_d \otimes A) \sum_{i=1}^{d} e_i \otimes e_i = \sum_{i=1}^{d} e_i \otimes \left(\sum_{j=1}^{l} A_{j,i} f_j\right) = \sum_{j=1}^{l} \left(\sum_{i=1}^{d} A_{j,i} e_i\right) \otimes f_j = (A^T \otimes I_l) \sum_{j=1}^{l} f_j \otimes f_j$$

(2.6)

where $\{f_j\}_{j=1}^{l}$ is the canonical basis in $\mathbb{C}^l$.

Secondly, the bound in the statement of theorem is derived from the largest possible entropy under this constraint.

$$- \frac{a}{k} \log \left(\frac{a}{k}\right) - \left(1 - \frac{a}{k}\right) \log \left(1 - \frac{a}{k}\right) \leq \frac{a}{k} \log k - \frac{a}{k} \log a + \left(1 - \frac{a}{k}\right) \left(2 \log k - \log \left(1 - \frac{a}{k}\right)\right)$$

$$\leq 2 \log k - \frac{a \log k}{k} + \frac{a}{k} \left[2 - \log a - \frac{2a}{k}\right]$$

(2.7)

for large enough $k$ so that the bound $\log(1 - \frac{a}{k}) \geq -\frac{2a}{k}$ holds. 

Next, the following approximating bound of the von Neumann entropy around the maximally mixed state was introduced in [BH10].
Lemma 2.2. For any state $\rho$ on $\mathbb{C}^k$,
\[
\log k - S(\rho) \leq k \cdot \|\rho - \tilde{I}_k\|_2^2
\]
(2.8)
where $\tilde{I}_k = I_k / k$, the identity on $\mathbb{C}^k$ normalized to be trace-one.

The bound can be seen easily from the concavity of $S(\cdot)$ in particular around the maximally mixed state. This idea extremely fits into asymptotic geometric analysis as was pointed out in [ASW11]. Also, it fits even better to our method because the function in (3.1), which is made out of Lemma 2.2, almost shows positive homogeneity and triangle inequality.

We are now ready to state the main theorem. In our statement, the result holds asymptotically when $l$ and $n$ are proportionally larger than or equal to $k^2$, whereas all the previous results set $l = n$ or require some constraints on the ration $l/n$.

**Theorem 2.3 (Main theorem).** Suppose $l = an$ and $k^2 = \alpha n$ for $a, \alpha > 0$, i.e., $l, n \sim k^2$. Then, we observe additivity violation of minimum output entropy when $k$ is large enough. Moreover, the statement holds even if $\alpha \to 0$, i.e., in the regime where $l \sim n \gg k^2$.

**Proof.** Take $\theta$ and $\epsilon$ as in Theorem 3.5. For example, set $\theta = 1/4$ and
\[
\epsilon = \sqrt{\frac{8a}{3} \log \left(1 + \frac{2}{\theta}\right)}
\]
(2.9)
Then, Theorem 3.5 implies that there exists some constant $C > 0$ and subspace $E$ such that
\[
\max_{x \in \tilde{E}} \|\Phi_E (xx^*) - \tilde{I}_k\|_2 \leq \frac{C}{k}
\]
for sufficiently large $k$. Therefore, by using Lemma 2.1 and Lemma 2.2 we have
\[
S_{\min} (\Phi_E \otimes \Phi_E) \leq 2 \log k - \frac{a \log k}{k} + 2a \cdot \left[\log k - \frac{C^2}{k}\right] \leq S_{\min} (\Phi_E) + S_{\min} (\Phi_E)
\]
(2.11)
for large enough $k$. Note that $S_{\min} (\Phi_E) = S_{\min} (\Phi_E)$.

\[\square\]

3 Technical part

Define a function $f : \mathbb{C}^k \otimes \mathbb{C}^n \to \mathbb{R}$ as
\[
f(x) = \left\|XX^* - \text{Tr}[XX^*] \tilde{I}_k\right\|_2
\]
(3.1)
Here, again we use the upper and lower cases to show the identification in (1.13). The following lemma describes the typical behavior of $f(x)$ for $x \in S_{\mathbb{C}^k \otimes \mathbb{C}^n}$ when it is chosen uniformly random. The result shows why we need $n \gtrsim k^2$ for additivity violation in our framework for general quantum channels, and it seems impossible to improve the orders.

**Lemma 3.1.** For $x \in S_{\mathbb{C}^k \otimes \mathbb{C}^n}$ uniformly distributed,
\[
\mathbb{E}[f] \leq \frac{1}{\sqrt{n}}, \quad \text{med}(f) \leq \frac{1}{\sqrt{n}} \left(1 + \frac{3}{\sqrt{k}}\right)
\]
(3.2)
Here, $\mathbb{E}(\cdot)$ and med$(\cdot)$ are mean and median respectively.

**Comment:** for our main theorem, one only needs $\text{med}(f) \lesssim k^{-1}$ assuming $n \gtrsim k^2$. 

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Proof. By the Jensen’s inequality
\[ (\mathbb{E}[f(x)])^2 \leq \mathbb{E}[(f(x))^2] = \mathbb{E}\left[\text{Tr}\left((XX^*)^2\right)\right] - \frac{1}{k^2} \] (3.3)

Further, we see that there exist \( \alpha, \beta > 0 \) and
\[ \mathbb{E}\left[\text{Tr}\left((XX^*)^2\right)\right] = \text{Tr}\left[\mathbb{E}[xx^* \otimes xx^*](P_k \otimes I_n^2)\right] = \text{Tr}\left[(\alpha I_{k^2n^2} + \beta P_{kn})(P_k \otimes I_n^2)\right] \] (3.4)

where \( P_k \) and \( P_{kn} \) are swapping matrices on \( \mathbb{C}^k \otimes \mathbb{C}^k \) and \( (\mathbb{C}^k \otimes \mathbb{C}^n) \otimes (\mathbb{C}^k \otimes \mathbb{C}^n) \), respectively. Here, for the last equality we used the Schur’s lemma as in [BHI0] because the expectation is invariant for \( U \otimes U \) with \( U \in U(kn) \). Since \( \alpha = \beta = \frac{1}{kn(kn+1)} \), we get a bound:
\[ \mathbb{E}[f(x)] \leq \sqrt{\frac{k+n}{kn+1}} - \frac{1}{k} \leq \frac{1}{\sqrt{n}} \] (3.5)

For the second statement, the standard argument proceeds as follows.
\[ |\mathbb{E}(f) - \text{med}(f)| \leq \mathbb{E}|f - \text{med}(f)| \]
\[ \leq 2\sqrt{\frac{\pi}{8}} \int_0^{+\infty} \exp\left\{ -\frac{(kn-1)^2}{4} \right\} \, d\varepsilon = \sqrt{\frac{\pi}{8}} \cdot \sqrt{\frac{4\pi}{kn-1}} \leq \frac{3}{\sqrt{kn}} \] (3.6)

Here, we applied Jensen’s inequality and Lemma [A.3] for the first two inequalities. Note that in this calculation we set \( L = 2 \) in (A.3), by using the bound (3.13) with \( \|X\|_\infty, \|Y\|_\infty \leq 1 \), which works for all \( \varepsilon > 0 \). The second last equality comes from the identity for the Gaussian distribution.

All the existing papers on additivity violation of minimum output entropy via measure concentration argument use large deviation bounds similar to the one in Theorem 3.3. Especially the order \( \exp\{-n\} \), instead of \( \exp\{-n/k\} \), is important for their proofs. To this end, they essentially rectify the concerned functions on the unit spheres and apply the Lévy’s lemma. However, in this rectifying process, one needs another large deviation bound from random matrix theory or some extra efforts. In Theorem 3.3, we avoid this complication and prove the desired bound directly via the Lévy’s Lemma (see Lemma [A.3]). For this purpose, we need a small lemma before going on to the theorem:

**Lemma 3.2.** Let \( x \in S_{\mathbb{C}^k \otimes \mathbb{C}^n} \) be such that \( f(x) \leq \text{med}(f) \). Then, we have for \( k \geq 9 \)
\[ \|X\|_\infty \leq \frac{1}{\sqrt{k}} + \frac{\sqrt{k}}{n} \] (3.8)

**Proof.** The condition \( f(x) \leq \text{med}(f) \) implies via Lemma 3.1 that
\[ \|XX^*\|_\infty \leq \|XX^* - \tilde{I}_k\|_2 + \|\tilde{I}_k\|_\infty \leq \frac{1}{\sqrt{n}} \left( 1 + \frac{3}{\sqrt{k}} \right) + \frac{1}{k} \leq \left( \frac{1}{\sqrt{k}} + \frac{\sqrt{k}}{n} \right)^2 \] (3.9)

This lemma gives the desired large deviation bound in a straightforward way:

**Theorem 3.3.** Let \( k^2 = \alpha^2 n \) with \( \alpha > 0 \). Then, for \( x \in S_{\mathbb{C}^k \otimes \mathbb{C}^n} \) uniformly distributed,
\[ \Pr\{\text{f}(x) > h(k, \alpha, \varepsilon) + \text{med}(f)\} < \sqrt{\frac{\pi}{8}} \exp\left\{ -\varepsilon^2 \left( n - \frac{1}{k} \right) \right\} \] (3.10)

for all \( \varepsilon > 0 \). Here,
\[ h(k, \alpha, \varepsilon) = \frac{2\varepsilon(1 + \alpha + \varepsilon)}{k} \] (3.11)
Proof. We follow the notations in Theorem A.3. Let $A = \{ x \in S_{C^k \otimes C^n} : f(x) \leq \text{med}(f) \}$ and then Lemma 3.2 shows that for $x \in A^c$ with $\varepsilon = \frac{\delta}{\sqrt{k}}$, we have
\[
\|X\|_\infty \leq \frac{1 + \alpha + \varepsilon}{\sqrt{k}}
\] (3.12)

Note that the fact that $\| \cdot \|_\infty \leq \| \cdot \|_2$ implies that difference in the Hilbert-Schmidt norm bounds that in infinity norm. Hence, we can set an upper bound of the Lipschitz constant on $A^{c/\sqrt{k}}$ to be twice as large as (3.12). Indeed, for $x, y \in C^k \otimes C^n$,
\[
|f(x) - f(y)| \leq \|XX^* - YY^*\|_2 \leq (\|X\|_\infty + \|Y\|_\infty) \|X - Y\|_2
\] (3.13)

This trick on the Lipschitz constant with $\| \cdot \|_\infty$ was used in [ASW11] and originally from [BH10]. Therefore, applying Lemma A.3 completes the proof; $\varepsilon L$ in (A.3) is replaced by
\[
\frac{\varepsilon}{\sqrt{k}} \cdot \frac{2(1 + \alpha + \varepsilon)}{\sqrt{k}}
\] (3.14)

which is what we want as $h(k, \alpha, \varepsilon)$. \hfill \square

The following lemma brings our problem back to Milman’s view of Dvoretzky’s theorem. We define a $\theta$-net to be a subset of the unit sphere such that any point on the sphere finds a point in the subset within distance $\theta$. This approximation technique works well not only with norms, as in Milman’s view, but also with functions having more or less positive homogeneity and triangle inequality:

Lemma 3.4. Let $E$ be an $l$-dimensional subspace in $C^k \otimes C^n$ and $\tilde{E}$ the unit sphere in it. Then, we can construct a $\theta$-net on $\tilde{E}$, denoted by $N_\theta$, so that the following statements hold.
\[
|N_\theta| \leq \left(1 + \frac{2}{\theta} \right)^{2l}
\] (3.15)
\[
\max_{x \in \tilde{E}} f(x) \leq \frac{1}{1 - \theta^2 - 2\theta} \cdot \max_{x \in N_\theta} f(x)
\] (3.16)

for $\theta > 0$ such that RHS of (3.16) is positive, in particular $0 < \theta \leq \frac{1}{4}$.

Proof. Since the first bound is well-known, for example see [Pis89], we only prove the second statement. For any $v \in \tilde{E}$ there exists $x \in N_\theta$, $y \in \tilde{E}$ and $0 \leq \delta \leq \theta$ such that $v = x + \delta y$. Then,
\[
f(x + \delta y) \leq \left\|XX^* - \bar{I}_k\right\|_2 + \delta^2 \left\|YY^* - \bar{I}_k\right\|_2 + \delta \left\|XX^* + XY^* - \text{Tr}[XY^* + YX^*] \bar{I}_k\right\|_2
\] (3.17)
\[
\leq \max_{x \in N_\theta} f(x) + (\delta^2 + 2\delta) \cdot \max_{x \in \tilde{E}} f(x)
\] (3.18)

Indeed, since $\text{Tr}_{C^n}[xy^* + yx^*] = XY^* + YX^*$, we firstly write
\[
xy^* + yx^* = \alpha zz^* + \beta ww^*
\] (3.19)

for some $z, w \in \tilde{E}$ and $\alpha, \beta \in \mathbb{R}$, and secondly, we get
\[
(*) \leq |\alpha| \left\|ZZ^* - \text{Tr}[ZZ^*] \bar{I}_k\right\|_2 + |\beta| \left\|WW^* - \text{Tr}[WW^*] \bar{I}_k\right\|_2
\] (3.20)
\[
\leq (|\alpha| + |\beta|) \max_{x \in \tilde{E}} f(x) \leq 2 \max_{x \in \tilde{E}} f(x)
\] (3.21)
Here, we used the following bound:

$$|\alpha| + |\beta| = \|xy^* + yx^*\|_1 \leq \|xy^*\|_1 + \|yx^*\|_1 = 2$$

(3.22)

This completes the proof. 

\[\square\]

**Theorem 3.5.** Suppose we have random \(l\)-dimensional subspaces \(E \subset \mathbb{C}^k \otimes \mathbb{C}^n\) where \(k^2 = \alpha^2 n\). For any \(\epsilon > 0\) and \(0 < \theta \leq \frac{1}{4}\), if we choose \(l_n\) such that

$$\frac{l_n}{n} \leq \frac{3\epsilon^2}{8 \log (1 + \frac{2}{\theta})}$$

(3.23)

then there exists a subspace \(E\) such that

$$\max_{x \in E} f(x) \leq \frac{1}{1 - \theta^2 - 2\theta} \left[ h(k, \alpha, \epsilon) + \frac{2\alpha}{k} \right]$$

(3.24)

for large enough \(n\), where the function \(h(\cdot, \cdot, \cdot)\) is defined in (3.11). Moreover, the statement holds as \(\alpha \to 0\), i.e., it holds in the regime where \(n \gtrsim k^2\). In particular, we can fix \(k\) and take \(n \to \infty\).

**Comment:** An important message of this theorem is that the RHS of (3.24) is bounded by \(C(\alpha, \theta, \epsilon)\) where \(C(\alpha, \theta, \epsilon)\) is some constant depending on \(\alpha, \theta\) and \(\epsilon\).

**Proof.** Fix a subspace \(E_0\) of dimension \(l_n\) and construct a \(\theta\)-net on \(\tilde{E}_0 = E_0 \cap S_{\mathbb{C}^k \otimes \mathbb{C}^n}\), which we denote by \(N_\theta\). We calculate

$$\Pr_{U \in \mathcal{U}(kn)} \left\{ f(Ux) > h(k, \alpha, \epsilon) + \frac{2\alpha}{k}, \quad \exists x \in UN_\theta \right\}$$

(3.25)

$$\leq |N_\theta| \cdot \Pr_{U \in \mathcal{U}(kn)} \left\{ f(Ux_0) > h(k, \alpha, \epsilon) + \text{med}(f), \quad \text{for fixed } x_0 \in N_\theta \right\}$$

(3.26)

$$= \exp \left\{ 2l_n \log \left( 1 + \frac{2}{\theta} \right) \right\} \times \sqrt{\frac{\pi}{8}} \exp \left\{ -\epsilon^2 n \left( 1 - \frac{1}{kn} \right) \right\}$$

(3.27)

Here, we used the first statement of Lemma 3.4 and Theorem 3.3. Since \(1 - \frac{1}{kn} > \frac{3}{4}\) in our regime, the condition (3.23) implies that (3.27) is smaller than one for large enough \(n\). Hence there exists \(U \in \mathcal{U}(kn)\) such that

$$\max_{x \in UN_\theta} f(x) \leq h(k, \alpha, \epsilon) + \frac{2\alpha}{k}$$

(3.28)

Therefore, for this \(U\), set \(E = UE_0\) where \(UN_\theta\) constitutes a \(\theta\)-net for \(\tilde{E} = U\tilde{E}_0\) so that the second statement of Lemma 3.4 completes the proof.

\[\square\]

4 Hastings’ proof and ours

An important step in our proof can be seen in (3.16) where the bound over the whole domain (subspace) can be set to be, for example, half as large as the bound only over the net if one properly chooses \(\theta > 0\). We emphasize here that choice of \(\theta\) is independent of \(k\). If we had thought of this problem by using the Lipschitz constant, the correction would be an additive term instead of a multiplicative constant. Since the Lipschitz constant is at best proportional to \(\frac{1}{\sqrt{k}}\), the additive correction would be proportional to \(\frac{\theta}{\sqrt{k}}\). However, we need a bound proportional to \(\frac{1}{k}\). Hence \(\theta\) must be proportional to \(\frac{1}{\sqrt{k}}\), which would give an
unwanted $k$-dependent factor in (3.23). Therefore it is crucial in our method to use “positive homogeneity
and triangle inequality” of function $f$ in order to get the bound (3.16).

In this kind of problems, one of useful approaches is to ask “how much of the domain can be approx-
imated by one point”. In our proof, it is $\exp \{-2n \log (1 + \frac{2}{3})\}$ when $l = n$. We dare to say that this
corresponds to (37), derived from (34), in the supplementary information of [Has09]. This idea of him is
roughly stated as follows, hoping that there is not misunderstanding.

One can decompose uniformly distributed $z \in S_{C^n}$ as
\begin{equation}
    z = \omega x + \sqrt{1 - |\omega|^2} y
\end{equation}
where $x$ is fixed and $y$ is uniformly distributed on $S_{x^\perp} \cong S_{C^n-1}$. Also, note that $|\omega|^2$ has the law of Beta
distribution. Via this decomposition, we have
\begin{align*}
    \Phi(z^*z) &\approx |\omega|^2 \Phi(xx^*) + (1 - |\omega|^2) \Phi(yy^*) \\
    \Phi(zz^*) &\approx |\omega|^2 \Phi(xx^*) + (1 - |\omega|^2) \tilde{I}_k
\end{align*}
(4.2)
The second approximation is assumed because generically channels send random inputs to a neighborhood
of $\tilde{I}_k$ although we need a careful analysis for this statement. For ex-ample, see [FKM10], where the
important idea tubal neighborhood was reformulated as TUBE. However we believe that we arrive at the
same goal, or at least get convinced, if we look at (4.2) in the Hilbert-Schmidt norm. First, we have
\begin{align*}
    \|\Phi(zz^*) - \tilde{I}_k\|_2 &\geq |\omega|^2 \|\Phi(xx^*) - \tilde{I}_k\|_2
\end{align*}
(4.3)
but $|\omega|^2 > \frac{1}{2}$ occurs with probability $\exp \{-n \log 2\}$ because $|\omega|^2$ has the law of the beta distribution
$B(2, 2n - 2)$. This means that any fixed point $x$ approximates other points of measure $\exp \{-n \log 2\}$ in
such a way that $(\ast)$ is at least half as large as $(\ast\ast)$. So, assuming that there exists an input $xx^*$ which gives
a large value in $(\ast)$ we get a contradiction because if we take random quantum channels $(\ast)$ is likely to be
small with the large deviation bound as in Theorem 3.3; we just set parameters to get proper constants
which result in a contradiction.

Therefore, the connection between those two methods can be stated as follows. Hastings’ method
considers how much part of the domain can be approximated by unwanted points to get a contradiction.
Our method focuses on desired points in the domain and use $\epsilon$-net argument to prove the result. Inter-
estingly, then, both methods result in similar estimates as written above. On the other hand, however,
our estimate for this approximation in the domain is made only from the norm-like properties whereas
Hastings’ involves probabilistic arguments.

5 Random unitary channel

We briefly discuss on a class of channels called random unitary channels:
\begin{equation}
    \Phi(\rho) = \frac{1}{k} \sum_{i=1}^{k} U_i \rho U_i^*
\end{equation}
(5.1)
where $U_i \in U(n)$. Through these channels, input states will be rotated by $U_i$ with equal probability. To
construct random channels in this class, we take $U_i$ with respect to the Haar measure independently.

It may seem obvious that additivity violation holds for this class too, but since this class forms a
measure-zero set in the general channels it is not rigorously obvious. However, this class of channels are
very close to the one considered in Hastings’ paper [Has09], so additivity violation for this class may be
deduced from it. If one wants to use our method, one can use the measure concentration argument on
\begin{equation}
    S_{C^n} \times \cdots \times S_{C^n}
\end{equation}
(5.2)
See 6.5.2 of [MSS06] for details where one can find that this product space forms a normal Lévy’s family.
6 Regularized minimum output entropy

We define the regularized minimum output entropy of channels as follows.

\[ S_{\text{min}}(\Phi) = \lim_{n \to \infty} \frac{1}{n} S_{\text{min}}(\Phi^\otimes n) \]  

(6.1)

The limit exists from the following property:

\[ S_{\text{min}}(\Phi^\otimes (m+n)) \leq S_{\text{min}}(\Phi^\otimes m) + S_{\text{min}}(\Phi^\otimes n) \]  

(6.2)

We think that it may be a good idea to investigate the following additivity question:

\[ \bar{S}_{\text{min}}(\Phi \otimes \Omega) \overset{?}{=} \bar{S}_{\text{min}}(\Phi) + \bar{S}_{\text{min}}(\Omega) \]  

(6.3)

to understand better the question of additivity of classical capacity in (1.6). This is because the former problem can be analyzed by eigenvalues of output states while the latter needs the geometry of output states. In fact, this eigenvalue approach lead us to discovery of additivity violation of Holevo capacity somehow but it did not happen. This is why, we suggest that the question (6.3) should be asked first. In fact, Theorem 6.1 supports this idea. We state and prove a widely known fact which shows a relation between \( C(\cdot) \) and \( \bar{S}_{\text{min}}(\cdot) \) by extending the proof method in \[ \text{Sho04} \].

**Theorem 6.1.** Additivity violation of regularized minimum output entropy will imply additivity violation of classical capacity.

**Proof.** Suppose there are some channels \( \Phi \) and \( \Omega \) such that

\[ \lim_{n \to \infty} \frac{1}{n} S_{\text{min}}(\Phi^\otimes n \otimes \Omega^\otimes n) < \lim_{n \to \infty} \frac{1}{n} S_{\text{min}}(\Phi^\otimes n) + \lim_{n \to \infty} \frac{1}{n} S_{\text{min}}(\Omega^\otimes n) \]  

(6.4)

Then, by Lemma 6.2, there are channels \( \tilde{\Phi} \) and \( \tilde{\Omega} \) such that

\[ \log(k_1k_2) - \lim_{n \to \infty} \frac{1}{n} \chi(\tilde{\Phi}^\otimes n \otimes \tilde{\Omega}^\otimes n) < \log k_1 - \lim_{n \to \infty} \frac{1}{n} \chi(\tilde{\Phi}^\otimes n) + \log k_2 - \lim_{n \to \infty} \frac{1}{n} \chi(\tilde{\Omega}^\otimes n) \]  

(6.5)

where \( k_1 \) and \( k_2 \) are output dimensions of \( \Phi \) and \( \Omega \), respectively. 

To complete the above proof we need to show Lemma 6.2. To this end, we introduce the following definitions. For the additive group \( \mathbb{Z}_k = \{0, 1, \ldots, k-1\} \) we define the discrete Weyl operators on \( \mathbb{C}^k \):

\[ W_z = U^z V_y \quad \text{where} \quad z = (x, y) \in \mathbb{Z}_k \times \mathbb{Z}_k \]  

(6.6)

Here, \( U \) and \( V \) are defined as

\[ U e_r = e_{r+1} \quad \text{and} \quad V e_r = \exp\{2\pi i r/k\} \cdot e_r \quad (r = 0, \ldots, k-1) \]  

(6.7)

where \( \{e_0, \ldots, e_{k-1}\} \) is the canonical basis of \( \mathbb{C}^k \).

**Lemma 6.2.** Take two channels

\[ \Phi : L(\mathbb{C}^l) \to L(\mathbb{C}^k) \quad \text{and} \quad \Omega : L(\mathbb{C}^{l'}) \to L(\mathbb{C}^{k'}) \]  

(6.8)

then there exist channels \( \tilde{\Phi} \) and \( \tilde{\Omega} \) such that

\[ \chi(\tilde{\Phi}^\otimes m \otimes \tilde{\Omega}^\otimes n) = \log(k^m(k')^n) - S_{\text{min}}(\Phi^\otimes m \otimes \Omega^\otimes n) \quad \text{for} \quad \forall m, n \in \mathbb{N} \cup \{0\} \]  

(6.9)
Proof. Define a channel $\tilde{\Phi} : L(C^{k^2} \otimes C^l) \to L(C^k)$ such that
\[
\tilde{\Phi}(\rho) = \sum_{z \in \mathbb{Z}_k \times \mathbb{Z}_k} W_z \Phi\left((e_z^* \otimes I)\rho(e_z \otimes I)\right) W_z^*
\] (6.10)
Here, $e_z = e_z \otimes e_y$ so that $\{e_z\}$ is the canonical basis of $C^{k^2} = C^k \otimes C^k$. We also define $\tilde{\Omega}$ in a similar way. Suppose
\[
S_{\min}\left(\tilde{\Phi}^{\otimes m} \otimes \tilde{\Omega}^{\otimes n}\right) = S\left(\tilde{\Phi}^{\otimes m} \otimes \tilde{\Omega}^{\otimes n}(\rho_0)\right)
\] for some $\rho_0 \in L((C^l)^{\otimes m} \otimes (C^l')^{\otimes n})$. Then, think of states
\[
E\left(z^{(m)}\right) \otimes E\left(z^{(n)}\right) \otimes \rho_0
\] (6.12)
Here, $z^{(m)} = (z_1, \ldots, z_m)$ are strings of $\mathbb{Z}_k \times \mathbb{Z}_k$ of length $m$, and $z^{(n)} = (z_1', \ldots, z_n')$ of $\mathbb{Z}_k' \times \mathbb{Z}_k'$ of length $n$ so that
\[
E\left(z^{(m)}\right) = e_z e_z^* \otimes \cdots \otimes e_z e_z^*\] (6.13)
and $E(z^{(n)})$ is defined similarly. Note that combinations of these two strings amount to $k^{2m}(k')^{2n}$. Then, we claim that the ensemble of states made from all the possible strings with equal probability leads us to our conclusion. First,
\[
S \left(\frac{1}{k^{2m}(k')^{2n}} \sum_{(z^{(m)}, z^{(n)})} \tilde{\Phi}^{\otimes m} \otimes \tilde{\Omega}^{\otimes n}\left(E\left(z^{(m)}\right) \otimes E\left(z^{(n)}\right) \otimes \rho_0\right)\right) = \log(k^m(k')^n)
\] (6.14)
Secondly, for each $(z^{(m)}, z^{(n)})$,
\[
S\left(\tilde{\Phi}^{\otimes m} \otimes \tilde{\Omega}^{\otimes n}\left(E\left(z^{(m)}\right) \otimes E\left(z^{(n)}\right) \otimes \rho_0\right)\right) = S\left(\tilde{\Phi}^{\otimes m} \otimes \tilde{\Omega}^{\otimes n}(\rho_0)\right)
\] (6.15)
\[
= S_{\min}\left(\tilde{\Phi}^{\otimes m} \otimes \tilde{\Omega}^{\otimes n}\right) = S_{\min}\left(\tilde{\Phi}^{\otimes m} \otimes \tilde{\Omega}^{\otimes n}\right)
\] (6.16)
Here, the last equality is from the concavity of $S(\cdot)$. Therefore,
\[
\chi\left(\tilde{\Phi}^{\otimes m} \otimes \tilde{\Omega}^{\otimes n}\right) = \log(k^m(k')^n) - S_{\min}\left(\tilde{\Phi}^{\otimes m} \otimes \tilde{\Omega}^{\otimes n}\right) \quad \text{for } \forall m, n \in \mathbb{N} \cup \{0\}
\] (6.17)
Indeed, the RHS is the upper bound for the Holevo capacity, which has been achieved by the ensemble. \[\square\]

7 Concluding remark

In this paper, we developed concise proofs on additivity violation of minimum output entropy of quantum channels. In regimes where dimensions of input and output are proportional to each other and proportionally larger than or equal to square of dimension of environment, we proved that asymptotically the violation is typical. Nevertheless, there are some interesting questions left. 1) Is the pair - a quantum channel and its complex conjugate - the best for the violation? 2) Is the violation a phenomenon for bipartite systems? Through the project in [CFN12], I feel that the first question is true for the random quantum channels. For the second question, weak form of additivity is proven in [Mon13]. Also, Hastings conjectured in [Has09] that the additivity holds for quantum channels of the form $\Phi \otimes \bar{\Phi}$. A positive mathematical evidence for this conjecture was found in [CFN12]. These results naively suggest that additivity violation may be a concept for bipartite systems. More researches should be done to answer these questions.
A Results from asymptotic geometric analysis

In this appendix, we collect results in asymptotic geometric analysis which we need. We refer interested readers to [MS86].

Let $X$ be a space with metric $\rho$ and Borel probability measure $\mu$. Then, $(X_r, \rho_r, \mu_r)$ with $r \in \mathbb{N}$ is called a normal Lévy family with constants $c_1, c_2 > 0$ if

$$1 - \mu(A_r^\varepsilon) \leq c_1 \exp\{-c_2 \varepsilon^2 r\}$$

(A.1)

for all $A_r^\varepsilon$ with $\varepsilon > 0$ and $r \in \mathbb{N}$. Here, $A_r^\varepsilon \subseteq X$ is defined for Borel sets $A_r \subseteq X_r$ with $\mu(A_r) \geq 1/2$ in the following way:

$$A_r^\varepsilon = \{x \in X_r : \rho(x, A_r) \leq \varepsilon\}$$

(A.2)

The unit spheres forms a normal Lévy’s family; see, for example, 2.2 of [MS86]:

**Theorem A.1.** The unit sphere $S^{r+1} \subset \mathbb{R}^{r+1}$ with the geodesic metric and the uniform measure is a normal Lévy’s family with $c_1 = \sqrt{\frac{\pi}{8}}$ and $c_2 = \frac{1}{2}$.

Based on this result, we state Lévy’s lemma [Lév51] in our view that behavior of the Lipschitz constant outside $A_r^\varepsilon$ does not matter:

**Theorem A.2** (Lévy’s Lemma in our view). For $S_{C^k \otimes C^n} = S^{2kn-1} \subset \mathbb{R}^{2kn}$ with $k \in \mathbb{N}$ fixed, take a sequence of continuous functions $f_n : S_{C^k \otimes C^n} \rightarrow \mathbb{R}$ in the Hilbert-Schmidt norm, and define $A_n = \{x \in S_{C^k \otimes C^n} : f_n(x) \leq \text{med}(f_n)\}$. Suppose there exist $\varepsilon > 0$ and $L > 0$ such that the Lipschitz constant of $f_n$ is upper-bounded by $L$ on $A_n^\varepsilon \setminus A^\varepsilon$. Then,

$$\mu \{x \in S_{C^k \otimes C^n} : f_n(x) > \text{med}(f_n) + \varepsilon L\} \leq \sqrt{\frac{\pi}{8}} \exp\{-\varepsilon^2 (kn - 1)\}$$

(A.3)

**Proof.** The proof is identical to the one for the usual Lévy’s lemma:

$$\mu \{x \in S_{C^k \otimes C^n} : f_n(x) > \text{med}(f_n) + \varepsilon L\} \leq \mu \left(S^{2kn-1} \setminus A_n^\varepsilon\right) = 1 - \mu(A_n^\varepsilon)$$

(A.4)

Indeed, $x \in A_n^\varepsilon$ implies $f_n(x) \leq \text{med}(f_n) + \varepsilon L$. Note that we switched metric from the geodesic distance to the Hilbert-Schmidt distance where the former is always larger than the latter. 

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