\textbf{$\tau^2$-STABLE TILTING COMPLEXES OVER WEIGHTED PROJECTIVE LINES}

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\textbf{Abstract.} Let $X$ be a weighted projective line and $\text{coh } X$ the associated category of coherent sheaves. We classify the tilting complexes $T$ in $D^b(\text{coh } X)$ such that $\tau^2 T \cong T$, where $\tau$ is the Auslander-Reiten translation in $D^b(\text{coh } X)$. As an application of this result, we classify the 2-representation-finite algebras which are derived-equivalent to a canonical algebra. This complements Iyama-Oppermann’s classification of the iterated tilted 2-representation-finite algebras. By passing to 3-preprojective algebras, we obtain a classification of the selfinjective cluster-tilted algebras of canonical-type. This complements Ringel’s classification of the selfinjective cluster-tilted algebras.

\section{Introduction}

Let $X$ be a weighted projective line over an algebraically closed field and

$$\tau: D^b(\text{coh } X) \to D^b(\text{coh } X)$$

be the Auslander-Reiten translation in the bounded derived category of $\text{coh } X$, see \cite{12} for definitions. The following objects, which are closely related to each other, are classified in this article:

(a) The $\tau^2$-stable tilting complexes in $D^b(\text{coh } X)$,

(b) the 2-representation-finite algebras which are derived equivalent to $\text{coh } X$ and

(c) the selfinjective cluster-tilted algebras of canonical type.

The interest in classifying the objects above has its origin in higher Auslander-Reiten theory which was introduced by Iyama in \cite{20}. As the name suggests, it is a higher-dimensional analog of classical Auslander-Reiten theory for finite dimensional algebras. Higher Auslander-Reiten theory can be developed in distinguished subcategories of $\text{mod } \Lambda$, nowadays called $n$-cluster-tilting subcategories. A subcategory $\mathcal{M}$ of $\text{mod } \Lambda$ is an $n$-\textit{cluster-tilting subcategory} if

$$\mathcal{M} = \{ N \in \text{mod } \Lambda \mid \text{Ext}^i_{\Lambda}(-,N)|_{\mathcal{M}} = 0 \text{ for } i \in \{1, \ldots, n-1\} \}$$

$$= \{ N \in \text{mod } \Lambda \mid \text{Ext}^i_{\Lambda}(N,-)|_{\mathcal{M}} = 0 \text{ for } i \in \{1, \ldots, n-1\} \}.$$

One of the most remarkable features of higher Auslander-Reiten theory is the existence of a functor $\tau_n: \mathcal{M} \to \mathcal{M}$ together with a natural isomorphism

$$\text{Ext}^n_{\Lambda}(X,Y) \cong D\text{Hom}_{\Lambda}(Y,\tau_n X) \quad \text{for all } X,Y \in \mathcal{M},$$

which is a higher analog of usual Auslander-Reiten duality.

The simplest class of algebras which have an $n$-cluster-tilting subcategory are the so-called $n$-representation-finite algebras, which where introduced by Iyama and Oppermann in \cite{21}. A finite dimensional algebra $\Lambda$ is said to be $n$-\textit{representation-finite} if $\Lambda$ has global dimension $n$ and there exists a $\Lambda$-module $M$ such that $\text{add } M$
is an \( n \)-cluster-tilting subcategory (in this case \( M \) is called a \( n \)-cluster-tilting module). For example, \( 1 \)-representation-finite algebras are precisely representation-finite hereditary algebras. In this sense, \( n \)-representation-finite algebras may be regarded as a higher analog of representation-finite hereditary algebras.

Now we explain what are the objects that we classify in this article, and how do they relate to each other. The \( 1 \)-representation-finite algebras were classified by Gabriel in [11]: they are precisely the algebras which are Morita-equivalent to the path algebras of quivers whose underlying graph is a Dynkin diagram of simply-laced type. It is then natural to study \( 2 \)-representation-finite algebras. Important structural results regarding \( 2 \)-representation finite algebras in terms of selfinjective quivers with potential have been obtained by Herschend and Iyama in [19] where they also have provided large classes of examples of such algebras. Following [16], we say that a finite dimensional algebra is piecewise hereditary if it is derived equivalent to a hereditary category \( \mathcal{H} \) or, equivalently, if it is isomorphic to the endomorphism algebra of a tilting complex in \( \text{D}^b(\mathcal{H}) \). From a homological point of view, the simplest kind of \( 2 \)-representation-finite algebras are the ones which are piecewise hereditary.

By a celebrated result of Happel [15, Thm. 3.1], it is known that there are only two kinds of hereditary categories (satisfying suitable finiteness conditions) which have a tilting object: the ones which are derived equivalent to \( \text{mod} \ H \) where \( H \) is a finite dimensional hereditary algebra, and the ones which are derived equivalent to \( \text{coh} \ X \) where \( X \) is a weighted projective line. We distinguish between piecewise hereditary algebras as follows: We say that a finite dimensional algebra \( \Lambda \) is iterated tilted if \( \text{mod} \ \Lambda \) is derived equivalent to \( \text{mod} \ H \) where \( H \) is a finite dimensional hereditary algebra. Similarly, we say that \( \Lambda \) is derived-canonical if \( \text{mod} \ \Lambda \) is derived equivalent to \( \text{coh} \ X \) for some weighted projective line \( X \).

Taking advantage of Ringel’s classification of the selfinjective cluster-tilted algebras [28], the \( 2 \)-representation-finite algebras which are iterated tilted were classified by Iyama and Oppermann in [22, Thm. 3.12]. Note that these algebras are derived equivalent to representation-finite hereditary algebras whose underlying quiver is of Dynkin type \( D \). In particular, there are no \( 2 \)-representation-finite algebras which are derived equivalent to a tame or wild hereditary algebra.

The following result is the main result of this article. It gives a classification of the \( 2 \)-representation-finite derived canonical algebras, and thus complements Iyama-Oppermann’s classification [22, Thm. 3.12].

**Theorem 1.1** (see Theorem 3.6). The complete list of all basic \( 2 \)-representation-finite derived-canonical algebras is given in Figures 1.1, 1.2 and 1.3. In this case, the corresponding weighted projective line has tubular type \( (2, 2, 2, 2; \lambda) \), \( (2, 4, 4) \) or \( (2, 3, 6) \).

Note that there are no \( 2 \)-representation-finite algebras which are derived equivalent to \( \text{coh} \ X \) for a weighted projective line \( X \) of wild type. It is important to note that in the case \( (2, 2, 2, 2; \lambda) \) all derived-canonical algebras are \( 2 \)-representation-finite. The classification of all derived-canonical algebras of type \( (2, 2, 2, 2; \lambda) \) is known, see for example Skowroński [24, Ex. 3.3], Barot-de la Peña [3, Fig. 1] and Meltzer in [27, Thm. 10.4.1]. Also, part 1 of Figure 1.2 already appeared in [19, Fig. 1].

We mention that there exist a notion of \( 2 \)-APR-(co)tilting, which is a higher analog of classical APR-(co)tilting, and that it preserves \( 2 \)-representation-finiteness,
Let $\tau : D^b(\text{coh} \mathcal{X}) \rightarrow D^b(\text{coh} \mathcal{X})$ be the Auslander-Reiten translation. We say that a sheaf $X \in D^b(\text{coh} \mathcal{X})$ is $\tau^2$-stable if $\tau^2 X \cong X$. Theorem 1.1 is a consequence of the following result, which gives a classification of the $\tau^2$-stable tilting sheaves over a weighted projective line.

**Theorem 1.2** (see Theorem 3.7). Let $\mathcal{X}$ be a weighted projective line and $T$ a basic tilting complex in $D^b(\text{coh} \mathcal{X})$. Then $T$ is $\tau^2$-stable if and only if $\text{End}_{D^b(\mathcal{X})}(T)$
is isomorphic to one of the algebras in Figures 1.1, 1.4 and 1.5. Moreover, this determines $T$ up to an autoequivalence of $\mathcal{D}^b(\text{coh} \, \mathcal{X})$. In this case, the corresponding weighted projective line has tubular type $(2, 2, 2; \lambda)$, $(2, 4, 4)$ or $(2, 3, 6)$.

A finite dimensional algebra is cluster-tilted of canonical type if it is isomorphic to the endomorphism algebra of a cluster-tilting object in the cluster category $\mathcal{C}_\mathcal{X}$ associated to a weighted projective line $\mathcal{X}$, see Section 2.4 for definitions.

By results of Keller [23] and Amiot [1], the basic cluster-tilted algebras of canonical type are 3-preprojective algebras of basic derived canonical algebras of global dimension at most 2. Moreover, they are Jacobian algebras of quivers with potential, see Section 2.4. As a consequence of Theorem 1.1, we obtain a classification of the selfinjective cluster-tilted algebras of canonical type. This complements Ringel’s classification [28].

**Theorem 1.3** (see Theorem 3.8). The complete list of all basic selfinjective cluster-tilted algebras of canonical type is given by the Jacobian algebras of the quivers with potential in Figures 1.6, 1.7 and 1.8. In this case, the corresponding weighted projective line has tubular type $(2, 2, 2; \lambda)$, $(2, 4, 4)$ or $(2, 3, 6)$.
The algebras listed in Theorem 1.3 already appeared in related contexts: Figure 1.6 is precisely the exchange graph of endomorphism algebras of cluster-tilting objects the cluster category associated to a weighted projective line of type $(2,2,2;\lambda)$, see [4]. In addition, Figures 1.7 and 1.8 appeared in [19, Figs. 3 and 2] respectively.

We conclude this section by fixing our conventions and notation. Throughout this article we work over an algebraically closed field $K$. If $\Lambda$ is a finite dimensional $K$-algebra, we denote by $\text{mod} \, \Lambda$ the category of finitely generated right $\Lambda$-modules.

If $\Lambda$ is a basic algebra, we denote its Gabriel quiver by $Q_\Lambda$. More generally, if $X$ is a basic object in a Krull-Schmidt $K$-linear category $\mathcal{H}$, we denote by $Q_X$ the Gabriel quiver of the algebra $\text{End}_\mathcal{H}(X)$. If $\mathcal{H}$ is an abelian category, we denote by $D^b(\mathcal{H})$
Figure 1.3. (Part 1 of 2) The basic 2-representation-finite derived-canonical algebras of type (2,3,6). Thick lines indicate 2-APR-(co)tilting. The algebras that arise as endomorphism algebras of tilting sheaves in coh $X$ are enclosed in a frame.
2. Preliminaries

2.1. Coherent sheaves over a weighted projective line. We recall the construction of the category of coherent sheaves over a weighted projective line together with its basic properties. We follow the exposition of [24].

Choose a parameter sequence $\lambda = (\lambda_1, \ldots, \lambda_t)$ of pairwise distinct points of $\mathbb{P}^t_K$ and a weight sequence $p = (p_1, \ldots, p_t)$ of positive integers. Without loss of generality, we assume that $t \geq 3$ and that for each $i \in \{1, \ldots, t\}$ we have $p_i \geq 1$. Moreover, we may also assume that $\lambda_1 = \infty$, $\lambda_2 = 0$ and $\lambda_3 = 1$. For convenience, we set $p := \text{lcm}(p_1, \ldots, p_t)$. We call the triple $X = (\mathbb{P}^t_K, \lambda, p)$ a weighted projective line of weight type $p$. 
Figure 1.4. Endomorphism algebras of $\tau^2$-stable basic tilting complexes in $\text{D}^b(\text{coh} X)$ for type $(2,4,4)$. All relations are commutativity or zero relations, cf. Figure 1.7. Thick lines indicate 2-APR-(co)tilting along orbits of the action of $\tau^2$, which is given by rotation by $\pi$. The algebras that arise as endomorphism algebras of tilting sheaves in $\text{coh} X$ are enclosed in a frame.

Figure 1.5. Endomorphism algebras of $\tau^2$-stable basic tilting complexes in $\text{D}^b(\text{coh} X)$ for type $(2,3,6)$. All relations are commutativity or zero relations, cf. Figure 1.8. Thick lines indicate 2-APR-(co)tilting along orbits of the action of $\tau^2$, which is given by counter-clockwise rotation by $2\pi/3$. The algebras that arise as endomorphism algebras of tilting sheaves in $\text{coh} X$ are enclosed in a frame.

The category $\text{coh} X$ of coherent sheaves over $X$ is defined as follows. Consider the rank 1 abelian group $\mathbb{L} = \mathbb{L}(\mathbf{p})$ with generators $\vec{x}_1, \ldots, \vec{x}_t, \vec{c}$ subject to the relations

$$p_1 \vec{x}_1 = \cdots = p_t \vec{x}_t = \vec{c}. $$
The element $\vec{c}$ is called the **canonical element of** $\mathbb{L}$. It follows that every $\vec{x} \in \mathbb{L}$ can be written uniquely in the form

$$\vec{x} = m\vec{c} + \sum_{i=1}^{t} m_i\vec{x}_i$$

where $m \in \mathbb{Z}$ and $0 \leq m_i < p_i$ for each $i \in \{1, \ldots, t\}$. Hence $\mathbb{L}$ is an ordered group with positive cone $\sum_{i=1}^{t} \mathbb{N}x_i$, and that for every $\vec{x} \in \mathbb{L}$ we have either $0 \leq \vec{x}$ or $\vec{x} \leq \vec{c} + \vec{\omega}$ where

$$\vec{\omega} := (t - 2)\vec{c} - \sum_{i=1}^{t} x_i$$

is the **dualizing element of** $\mathbb{X}$.

Next, consider the $\mathbb{L}$-graded algebra $K[x_1, \ldots, x_t]$ where $\deg x_i = \vec{x}_i$ for each $i \in \{1, \ldots, t\}$. When $t = 3$, we write $x = x_1$, $y = x_2$, $z = x_3$ and relabel the generators of $\mathbb{L}$ accordingly. Let $I = (f_3, \ldots, f_t)$ be the homogeneous ideal of $K[x_1, \ldots, x_t]$ generated by all the **canonical relations**

$$f_i = x_i^{p_i} - \lambda_i' x_i^{p_2} - \lambda_i'' x_i^{p_3}.$$
Consequently, we obtain an \( \mathbb{L} \)-graded algebra \( R = R(\lambda, p) := K[x_1, \ldots, x_t]/I \). Note that the group \( \mathbb{L} \) acts by degree shift on the category \( \text{gr}^L R \) of finitely generated \( \mathbb{L} \)-graded \( R \)-modules. Namely, given a \( \mathbb{L} \)-graded \( R \)-module \( M \) and \( \vec{x} \in \mathbb{L} \) we denote by \( M(\vec{x}) \) the \( R \)-module with grading \( M(\vec{x}) \vec{y} := M(\vec{x} + \vec{y}) \).

Let \( \text{gr}^L R \) be the category of finitely generated \( \mathbb{L} \)-graded \( R \)-modules. Note that \( \mathbb{L} \) acts on \( \text{gr}^L R \) by degree shift: given \( \vec{x} \in \mathbb{L} \) and \( M \in \text{gr}^L R \), we define \( M(\vec{x}) \in \text{gr}^L R \) to be the \( R \)-module with \( M \) with new grading \( M(\vec{x}) \vec{y} := M(\vec{x} + \vec{y}) \). The category \( \text{coh} \mathcal{X} \) is defined as the localization \( q_{\text{gr}^L R} \) of \( \text{gr}^L R \) by its Serre subcategory \( \text{gr}^L_0 R \) of finite dimensional \( \mathbb{L} \)-graded \( R \)-modules. We denote the image of a module \( M \) under the canonical quotient functor \( \text{gr}^L R \to q_{\text{gr}^L} R \) by \( \widetilde{M} \). It follows that the action of \( \mathbb{L} \) on \( \text{gr}^L R \) induces an action on \( \text{coh} \mathcal{X} \) given by \( \widetilde{M}(\vec{x}) := (M(\vec{x}))^{-} \). We call \( \mathcal{O} = \mathcal{O}_\mathcal{X} := \widetilde{R} \) the structure sheaf of \( \mathcal{X} \).
Theorem 2.1. [12, 24] Thm. 2.2] The category $\text{coh } X$ is connected, Hom-finite, $K$-linear and abelian. Moreover we have the following:

(a) The category $\text{coh } X$ is hereditary, i.e. we have $\text{Ext}^i_X(-,-) = 0$ for all $i \geq 2$.
(b) (Serre duality) Let $\tau : \text{coh } X \to \text{coh } X$ be the autoequivalence given by $E \mapsto E(\omega)$. Then, there is a bifunctorial isomorphism

$$D \text{Ext}^1_X(X,Y) \cong \text{Hom}_X(Y,\tau X).$$

We call $\tau$ the Auslander-Reiten translation of $\text{coh } X$.
(c) Let $\text{coh}_0 X$ be the full subcategory of $\text{coh } X$ of sheaves of finite length (=torsion sheaves). Also, let $\text{vect } X$ be the full subcategory of $\text{coh } X$ of sheaves with no non-zero torsion subsheaves (=vector bundles). Then, each $X \in \text{coh } X$ has a unique decomposition $X = X_+ \oplus X_0$ where $X_+ \in \text{vect } X$ and $X_0 \in \text{coh}_0 X$.
(d) The simple objects in $\text{coh}_0 X$ are parametrized by $\mathbb{P}_K^1$ as follows: For each $\lambda \in \mathbb{P}_K^1 \setminus \lambda$ there exist a unique simple sheaf $S_{\lambda}$ called the ordinary simple concentrated at $\lambda$, and for each $\lambda_i \in \lambda$ there exist $p_i$ exceptional (i.e. not ordinary) simple sheaves $S_{\lambda_1, \ldots, \lambda_p}$, defined by a short exact sequence

$$0 \longrightarrow \mathcal{O}(-m\vec{x}_i) \longrightarrow \mathcal{O}(1-m\vec{x}_i) \longrightarrow S_{\lambda_i,m} \longrightarrow 0$$

for $i \in \{1, \ldots, t\}$ and $m \in \{1, \ldots, p_i\}$.
(e) For each simple sheaf $S$ we have $\text{End}_X(S) \cong K$. If $S$ is an ordinary simple sheaf, then $\text{Ext}^1_X(S,S) \cong K$. If $S$ is an exceptional simple sheaf, then $\text{Ext}^1_X(S,S) = 0$.
(f) Let $\lambda \in \mathbb{P}_K^1$. The category $\mathcal{T}(\lambda)$ of all sheaves which have a finite filtration by simple sheaves concentrated at $\lambda$ form a standard tube. If $\lambda \notin \lambda$ then $\mathcal{T}(\lambda)$ has rank 1; if $\lambda = \lambda_i$, then $\mathcal{T}(\lambda)$ has rank $p_i$.
(g) Let $\vec{a}, \vec{b} \in \mathbb{L}$. Then $\text{Hom}_X(\mathcal{O}(\vec{a}), \mathcal{O}(\vec{b})) = R_{\vec{b}-\vec{a}}$. In particular, there is a non-zero morphism $\mathcal{O}(\vec{a}) \to \mathcal{O}(\vec{b})$ if and only if $\vec{b} - \vec{a} \geq 0$.

The complexity of the classification of indecomposable sheaves $\text{coh } X$ is controlled by its Euler characteristic

$$\chi(X) := 2 - \sum_{i=1}^t \left(1 - \frac{1}{p_i}\right).$$

Weighted projective lines of Euler characteristic zero will turn out to be our main concern in this article. An easy calculation shows that $\chi(X) = 0$ if and only if $p \in \{2,2,2,2,3,3,3,2,4,4,2,3,6\}$, if and only if the dualizing element $\omega$ has finite order $p = \text{lcm}(p_1, \ldots, p_t)$ in $\mathbb{L}$. In this case, we say that $X$ has tubular type and it follows that $\tau$ acts periodically on each connected component of the Auslander-Reiten quiver of $\text{coh } X$.

Let $K_0(\mathbb{X})$ be the Grothendieck group of $\text{coh } X$. There are two important linear forms $\text{rk}$ and $\text{deg}$ on $K_0(\mathbb{X})$. We refer the reader to [24 Sec. 2.2] for information on these numerical invariants. The rank $\text{rk}: K_0(\mathbb{X}) \to \mathbb{Z}$ is characterized by the property $\text{rk}(\mathcal{O}(\vec{x})) = 1$ for each $\vec{x}$ in $\mathbb{L}$. The degree $\text{deg}: K_0(\mathbb{X}) \to \mathbb{Z}$ is characterized by the property $\text{deg}(\mathcal{O}(\vec{x})) = \delta(\vec{x})$ where $\delta: \mathbb{L} \to \mathbb{Z}$ is the unique group homomorphism sending each $\vec{x}_i$ to $p/p_i$. Note that we have

$$\chi(X) = \frac{-\delta(\omega)}{p}.$$
Moreover, if a sheaf $X \in \text{coh} X$ satisfies $\deg(X) = \text{rk}(X) = 0$, then $X = 0$. The slope of a non-zero sheaf $X$ is defined as $\text{slope}(X) := \text{rk}(X)/\deg(X) \in \mathbb{Q} \cup \{\infty\}$.

**Proposition 2.2.** [24, Lemma. 2.5] For each non-zero $X \in \text{vect} X$ we have $\text{slope}(\tau X) = \text{slope}(X) + \delta(\vec{\omega})$.

A complex $T$ in $D^b(\text{coh} X)$ is called a tilting complex if $\text{Ext}^i_{D^b(X)}(T,T) = 0$ for all $i \neq 0$ and if the conditions $\text{Ext}^i_{D^b(X)}(T,X) = 0$ for all $i \in \mathbb{Z}$ imply that $X = 0$. Equivalently, $T$ is a tilting complex if and only if $T$ is rigid and the number of pairwise non-isomorphic indecomposable direct summands of $T$ equals $2 + \sum_{i=1}^t (p_i - 1)$, the rank of $K_0(X)$.

The vector bundle $T = T_\oplus := \bigoplus_{0 \leq \vec{x} \leq \vec{c}} \mathcal{O}(\vec{x})$

is a tilting sheaf whose endomorphism algebra is precisely $\Lambda = \Lambda(\lambda, \mathbf{p})$, the canonical algebra of type $(\lambda, \mathbf{p})$, see Figure 2.1 for an example. It follows that the bounded derived categories $D^b(\text{coh} X)$ and $D^b(\text{mod} \Lambda)$ are equivalent as triangulated categories.

**Proposition 2.3.** [25, Cor. 3.5]. Let $X$ be a weighted projective line of tubular type and $T$ a tilting sheaf in $\text{coh} X$. Then there exists an automorphism $F : D^b(\text{coh} X) \to D^b(\text{coh} X)$ such that $\tau F T \in \text{coh} X$ has a simple sheaf as a direct summand.

We say that $T$ is in normal position if $T_0 \neq 0$, see Theorem 2.1(c) and Theorem 2.3.

The result below collects further properties of $\text{coh} X$ which are needed to prove Theorem 3.5.

**Theorem 2.4.** [13, 24] Let $X$ be a weighted projective line of weight type $(p_1, \ldots, p_t)$ and $T$ a tilting sheaf in normal position. Then, the following statements hold:

(a) Let $q_i$ be the number of indecomposable direct summands of $T_0$ in $T(\lambda_i)$. Then, the perpendicular category $T_0^\perp := \{X \in \text{coh} X \mid \text{Hom}_X(T_0, X) = 0 \text{ and } \text{Ext}^1_X(T_0, X) = 0\}$
is equivalent to $\text{coh} \, Y$ where $Y$ is a weighted projective line of weight type $(p_1 - q_1, \ldots, p_t - q_t)$.

(b) If $X$ has tubular type, then $\chi(Y) > 0$. In this case, $\text{coh} \, Y$ is derived equivalent to $\text{mod} \, H$ for a tame hereditary algebra of extended Dynkin type $\Delta$, and the Auslander-Reiten quiver of $\text{vect} \, Y$ has shape $\mathbb{Z}\Delta$.

(c) The embedding $\text{coh} \, Y \cong T_0^\perp \subset \text{coh} \, X$ preserves line bundles and torsion sheaves. That is, we have $\text{vect} \, Y \cong (\text{vect} \, X \cap T_0^\perp)$ and $\text{coh}_0 \, Y \cong (\text{coh}_0 \, X \cap T_0^\perp)$.

(d) Let $\bar{x} \in \mathbb{L}$ be such that $T(\bar{x}) \cong T$. Then the functor $\tau(\bar{x}) : \text{coh} \, X \to \text{coh} \, X$ induces an action on $\text{coh} \, Y$ which acts freely on line bundles in $\text{coh} \, Y$.

(e) The sheaf $T_+$ is a tilting bundle in $\text{coh} \, Y$. If $X$ has tubular type, then $T_+$ contains a line bundle as a direct summand.

Proof. Statements (a) and (d) are shown in [13, Thm. 9.5 and Prop. 9.6].

The first claim is a straightforward computation. The remaining statements are shown for example in [24, Thm. 3.5, Cor. 3.6].

First, note that the group $L$ acts freely on line bundles in $\text{coh} \, X$ be degree shift. Moreover, this action preserves $\text{vect} \, X$ and $\text{coh}_0 \, X$. Let $\bar{x} \in \mathbb{L}$ be such that $T(\bar{x}) \cong T$. Since we have $T_0(\bar{x}) \cong T_0$, it follows that $(\bar{x})$ induces an action on $T_0^\perp \cong \text{coh} \, Y$. By part (c), this action acts freely on line bundles in $\text{coh} \, Y$.

The first claim follows since $T_+$ is also rigid in $T_0^\perp \subset \text{coh} \, X$ and the number of indecomposable direct summands of $T_+$ coincides with the rank of $K_0(Y)$. The second claim follows since $\chi(Y) > 0$, and hence every tilting bundle in $\text{coh} \, Y$ contains a line bundle as a direct summand, see [24, Cor. 3.7].

We have the following simple observation regarding $\tau^2$-stable rigid sheaves.

Lemma 2.5. Let $X$ be a weighted projective line and $X \in \text{coh} \, X$ be a $\tau^2$-stable rigid sheaf. Then, each indecomposable direct summand of $X$ is an exceptional simple sheaf.

Proof. Firstly, by Theorem [2.1] there are no rigid sheaves in an exceptional tube of rank 1. Secondly, since $X$ is a rigid $\tau^2$-stable sheaf, we have that

$$\text{Ext}^2_X(X, X) \cong D \text{Hom}_X(X, \tau X) \cong D \text{Hom}_X(\tau X, X) = 0.$$ 

Let $Y$ be an indecomposable direct summand of $X$. Then we have $\text{Hom}_X(\tau Y, Y) = 0$. This is happens if and only if $Y$ is an exceptional simple sheaf.

The Auslander-Reiten translation of $\text{coh} \, X$ extends to an autoequivalence

$$\tau : D^b(\text{coh} \, X) \to D^b(\text{coh} \, X).$$

Moreover, the autoequivalence $\tau := \tau[1]$ gives a Serre functor of $D^b(\text{coh} \, X)$.

Definition 2.6. A complex $X$ in $D^b(\text{coh} \, X)$ is $\tau^2$-stable if $\tau^2 X \cong X$.

The following result is a particular case of [25, Thm. 3.1]. It allows us to compute the endomorphism algebra of a tilting sheaf in a given weighted projective line in terms of a weighted projective line of smaller weights.

Theorem 2.7. Let $X$ be a weighted projective line of type $(p_1, \ldots, p_t)$. Let $T$ be a $\tau^2$-stable tilting sheaf in $\text{coh} \, X$, and suppose that the indecomposable direct summands of $T_0$ are exceptional simple sheaves at the points $\lambda_i, \ldots, \lambda_k \in \lambda$. We make the identification $\text{coh} \, Y \cong T_0^\perp$, see Proposition 2.4. Finally, let $E \in \text{coh} \, Y$
be the direct sum of all exceptional simple sheaves at the points \( \lambda_1, \ldots, \lambda_k \). Then, there is an isomorphism of algebras

\[
\text{End}_\mathbb{K}(T) \cong \text{End}_\mathbb{Y}(T_+ \oplus E) \cong \begin{bmatrix} \text{End}_\mathbb{Y}(T_+) & \text{Hom}_\mathbb{Y}(T_+, E) \\ 0 & \text{End}_\mathbb{Y}(E) \end{bmatrix}.
\]

Proof. Let \( r : \text{coh} \mathbb{X} \to \text{coh} \mathbb{Y} \) be the right adjoint of the inclusion \( \text{coh} \mathbb{Y} \cong T_0^+ \). It is easy to see that \( r \) induces a bijection between the indecomposable direct summands of \( T_0 \) and the exceptional simple sheaves in \( \text{coh} \mathbb{Y} \) at the points \( \lambda_1, \ldots, \lambda_k \). Then the result follows immediately from the proof of [23 Thm. 3.1].

2.2. Graded quivers with potential and their mutations. Quivers with potentials and their Jacobian algebras where introduced in [8] as a tool to prove several of the conjectures of [10] about cluster algebras in a rather general setting, see [9]. Their graded version was introduced in [2] in order to describe the effect of mutation of cluster tilting objects in generalized cluster categories at the level of the corresponding derived category.

Let \( Q = (Q_0, Q_1) \) be a finite quiver without loops or 2-cycles and \( d : Q_1 \to \mathbb{Z} \) a map called a degree function on the set of arrows of \( Q \). Then \( d \) induces a \( \mathbb{Z} \)-grading on the complete path algebra \( \bar{k}Q \) in an obvious way. We endow \( \bar{k}Q \) with the \( J \)-adic topology where \( J \) is the radical of \( \bar{k}Q \). A potential \( x \in \mathbb{Q} \) is a formal linear combination of cyclic paths in \( Q \); we are only interested in potentials which are homogeneous elements of \( \bar{k}Q \). For a cyclic path \( a_1 \cdots a_d \) in \( Q \) and \( a \in Q_1 \), let

\[
\partial_a(a_1 \cdots a_d) = \sum_{a_1 = a} a_{i+1} \cdots a_d a_1 \cdots a_{i-1}
\]

and extend it linearly and continuously to an arbitrary potential in \( Q \). The maps \( \partial_a \) are called cyclic derivatives.

Definition 2.8. A graded quiver with potential is a triple \((Q, W, d)\) where \((Q, d)\) is a \( \mathbb{Z} \)-graded finite quiver without loops and 2-cycles and \( W \) is a homogeneous potential for \( Q \). The graded Jacobian algebra of \((Q, W, d)\) is the \( \mathbb{Z} \)-graded algebra

\[
\text{Jac}(Q, W, d) \cong \frac{\bar{k}Q}{\partial(W)}
\]

where \( \partial(W) \) is the closure in \( \bar{k}Q \) of the ideal generated by the subset \( \{ \partial_a(W) \mid a \in Q_1 \} \).

For each vertex of \( Q \) there is a pair of well defined operations on the right-equivalence classes of graded quivers with potential called left and right mutations (see [8 Def. 4.2] for the definition of right-equivalence). Note that right equivalent quivers with potential have isomorphic Jacobian algebras.

Let \((Q, W, d)\) be graded quiver with potential with \( W \) homogeneous of degree \( d(W) \) and \( k \in Q_0 \). The non-reduced left mutation at \( k \) of \((Q, W, d)\) is the graded quiver with potential \( \hat{\mu}_k^L(Q, W, d) = (Q', W', \hat{d}') \) defined as follows:

(a) The quivers \( Q \) and \( Q' \) have the same set of vertices.
(b) All arrows of \( Q \) which are not adjacent to \( k \) are also arrows of \( Q' \) and of the same degree.
(c) Each path \( i \xrightarrow{a} k \xrightarrow{b} j \) in \( Q \) creates an arrow \([ba] : i \to j\) of degree \( d(a) + d(b) \) in \( Q' \).
(d) Each arrow \( a : i \to k \) of \( Q \) is replaced in \( Q' \) by an arrow \( a^* : k \to i \) of degree \(-d(a) + d(W)\).
(e) Each arrow $b : k \to j$ of $Q$ is replaced in $Q'$ by an arrow $b^* : j \to k$ of degree $-d(b)$.

(f) The new potential is given by
\[ W' = [W] + \sum_{i \to_k b \to j} [ba]a^*b^* \]
where $[W]$ is the potential obtained from $W$ by replacing each path $i \to_k b \to j$ which appears in $W$ with the corresponding arrow $[ba]$ of $Q'$. By [2, Thm. 6.6], there exists a graded quiver with potential $(Q'_{\text{red}}, W'_{\text{red}}, d'_{\text{red}})$ which is right equivalent to $(Q', W', d')$ and such that $Q'_{\text{red}}$ has neither loops or 2-cycles. The left mutation at $k$ of $(Q', W', d')$ is then defined as
\[ \mu_{k}^L(Q, W, d) := (Q'_{\text{red}}, W'_{\text{red}}, d'_{\text{red}}). \]

The right mutation at $k$ of $(Q, W, d)$ is defined almost identically, just by replacing (d) and (e) above by

- (d) Each arrow $a : i \to k$ of $Q$ is replaced in $Q'$ by an arrow $a^* : k \to i$ of degree $-d(a)$.

- (e) Each arrow $b : k \to j$ of $Q$ is replaced in $Q'$ by an arrow $b^* : j \to k$ of degree $-d(b) + d(W)$.

Finally, the following definition is very convenient for our purposes.

**Definition 2.9.** [19, Sec. 3] Let $(Q, W, d)$ be a graded quiver with potential with $d(W)$. Then the truncated Jacobian algebra is the degree zero part of $\text{Jac}(Q, W, d)$, which is given by the factor algebra
\[ \text{Jac}(Q, W, d) := \text{Jac}(Q, W)/\langle a \in Q \mid d(a) = 1 \rangle = \widehat{kQ}/\langle \partial_a(W) \mid d(a) = 1 \rangle. \]

Also, we say that $(Q, W, d)$ is algebraic if $\text{Jac}(Q, W, d)$ has global dimension at most 2 and the set
\[ \{ \partial_a(W) \mid d(a) = 1 \} \]
is a minimal set of generators of the ideal $\langle \partial_a(W) \mid d(a) = 1 \rangle$ of $\widehat{kQ}$.

Note that left and right mutation differ from each other at the level of the grading only.

### 2.3. 2-representation-finite algebras and 2-APR-tilting.

Let $\Lambda$ be a finite dimensional algebra of global dimension 2. Following [21, Def. 2.2], we say that $\Lambda$ is 2-representation-finite if there exist a finite dimensional $\Lambda$-module $M$ such that
\[ \text{add } M = \{ X \in \text{mod } \Lambda \mid \text{Ext}_1^\Lambda(M, X) = 0 \} = \{ X \in \text{mod } \Lambda \mid \text{Ext}_1^\Lambda(X, M) = 0 \}. \]

Such $\Lambda$-module $M$ is called a 2-cluster-tilting module. The functors
\[ \tau_2 := D \text{Ext}_1^\Lambda(-, \Lambda) : \text{mod } \Lambda \to \text{mod } \Lambda \]
and
\[ \nu_2 := \nu[-2] : \text{D}^b(\text{mod } \Lambda) \to \text{D}^b(\text{mod } \Lambda), \]
where $\nu : - \otimes_{\Lambda}^L \Lambda : \text{D}^b(\text{mod } \Lambda) \to \text{D}^b(\text{mod } \Lambda)$ is the Nakayama functor of $\text{D}^b(\text{mod } \Lambda)$ play an important role in the theory of 2-representation-finite algebras. Moreover, they are related by a functorial isomorphism $\tau_2 \cong H^0(\nu_2-)$. Note that $\tau_2$ induces a bijection between indecomposable non-projective objects in add $M$ and indecomposable non-injective objects in add $M$. 
Definition 2.10. [18] Def. 1.2 Let Λ be a 2-representation-finite algebra. We say that Λ is 2-homogeneous if each \( \tau_2 \)-orbit of indecomposable objects in \( \text{add} \ M \) consists of precisely two objects. This is equivalent to \( \nu_2^{-1}(\Lambda) \) being an injective \( \Lambda \)-module.

The class of 2-representation-finite algebras can be characterized in terms of the so-called 3-preprojective algebras.

Definition 2.11. [23] Let \( \Lambda \) be a finite dimensional algebra of global dimension at most 2. The complete 3-preprojective algebra of \( \Lambda \) is the tensor algebra
\[
\Pi_3(\Lambda) := \prod_{d \geq 0} \text{Ext}^2_\Lambda(D\Lambda, \Lambda) \otimes d.
\]

We have the following characterization of 2-representation-finite algebras.

Proposition 2.12. [19] Prop. 3.9 Let \( \Lambda \) be a finite dimensional algebra of global dimension at most 2. Then \( \Lambda \) is 2-representation-finite if and only if \( \Pi_3(\Lambda) \) is a finite dimensional selfinjective algebra.

Following [23], \( \Pi_3(\Lambda) \) can be presented as a graded Jacobian algebra for some quiver with potential obtained from \( \Lambda \). For this, let \( Q \) be the Gabriel quiver of \( \Lambda \) and let
\[
\Lambda \cong \mathbb{k}Q/\langle r_1, \ldots, r_s \rangle
\]
where \( \{r_1, \ldots, r_s\} \) is a minimal set of relations for \( \Lambda \). Consider the extended quiver
\[
\hat{Q} = Q \amalg \{r_i^s : t(r_i) \to s(r_i) \mid r_i : s(r_i) \to t(r_i)\}_{1 \leq i \leq s},
\]
i.e. \( \hat{Q} \) is obtained from \( Q \) by adding an arrow in the opposite direction for each relation in \( \Lambda \). We consider \( \hat{Q} \) as a graded quiver where the arrows in \( Q_1 \) have degree zero and the arrows \( r_i^s \) have degree one. Then we can define a homogeneous potential \( W \) in \( \hat{Q} \) of degree one by
\[
W := \sum_{i=1}^s r_i r_i^s.
\]

Theorem 2.13. [23] Thm. 6.10] Let \( \Lambda \) be a finite dimensional algebra of global dimension at most 2. Then there is an isomorphism of graded algebras between \( \text{Jac}(\hat{Q}, W, d) \) and \( \Pi_3(\Lambda) \).

A useful tool to construct 2-representation-finite algebras which are derived equivalent to a given one is 2-APR-tilting, which is a higher analog of usual APR-tilting. The notion of 2-APR-co-tilting is defined dually.

Definition 2.14. [21] Def. 3.14] Let \( \Lambda \) be a finite dimensional algebra of global dimension at most 2 and \( \Lambda = P \oplus Q \) any direct summand decomposition of \( \Lambda \) such that
\begin{enumerate}
\item \( \text{Hom}_\Lambda(Q, P) = 0. \)
\item \( \text{Ext}^i_\Lambda(\nu Q, P) = 0 \) for any \( 0 < i \neq 2. \)
\end{enumerate}
We call the complex
\[
T := (\nu_2^{-1} P) \oplus Q \in \text{D}^b(\text{mod} \Lambda)
\]
the 2-APR-tilting complex associated with \( P \).

In analogy with APR-tilting for hereditary algebras, 2-APR-tilting preserves 2-representation-finiteness.
Theorem 2.15. [21] Thm. 4.7] Let \( \Lambda \) be a 2-representation-finite algebra and \( T \) a 2-APR-tilting complex in \( D^b(\text{mod } \Lambda) \). Then the algebra \( \text{End}_{D^b(\Lambda)}(T) \) is also 2-representation-finite.

We can describe the effect of 2-APR-tilting using Theorem 2.13 as follows:

Theorem 2.16. [21] Sec. 3.3] Let \( \Lambda \) be a 2-representation-finite algebra and \( P \) an indecomposable projective \( \Lambda \)-module which corresponds to a sink \( k \) in the Gabriel quiver of \( \Lambda \) and let \( T \) be the associated 2-APR-tilting \( \Lambda \)-module. Also, let \((\tilde{Q}, W, d)\) be the graded quiver with potential associated to \( \Pi^3(\Lambda) \), see Theorem 2.13. Then there is an isomorphism of graded algebras

\[
\text{End}_\Lambda(T) \cong \text{Jac}(\tilde{Q}, W, d')
\]

where \( d' \) coincides with \( d \) on arrows not incident to \( k \), for an arrow \( a \in \tilde{Q} \) incident to \( k \) we have \( d'(a) = 1 \) if \( d(a) = 0 \), and we have \( d'(a) = 0 \) if \( d(a) = 1 \).

2.4. The cluster category of coh \( X \). Cluster categories associated with hereditary algebras were introduced in [7] in order to categorify the combinatorics of acyclic cluster algebras. The cluster category of a weighted projective line was studied in [6], [4] and [5]. For the point of view of this article, they arise as the categorical environment of 3-preprojective algebras of endomorphism algebras of tilting sheaves in coh \( X \).

The cluster category associated with coh \( X \) is by definition the orbit category

\[
C = C_X := D^b(\text{coh } X)/\tau[-1].
\]

Thus, the objects of \( C \) are bounded complexes of coherent sheaves over \( X \) and the morphism spaces are given by

\[
\text{Hom}_C(X,Y) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{D^b(X)}(X, \tau^i Y[-i])
\]

with the obvious composition rule. Note that \( \text{Hom}_C(X,Y) \) has a natural \( \mathbb{Z} \)-grading.

It is known [6] that \( C \) is a Hom-finite, Krull-Schmidt, \( K \)-linear triangulated category with the 2-Calabi-Yau property: There is a natural isomorphism

\[
D \text{Hom}_C(X,Y) \cong \text{Hom}_C(Y,X[2])
\]

for every \( X, Y \) in \( C \). It follows from [6] Prop. 2.1] that coh \( X \) is a complete system of representatives of isomorphism classes in \( C \) and that we have a natural isomorphism

\[
\text{Hom}_C(X,Y) \cong \text{Hom}_X(X,Y) \oplus \text{Ext}^1_X(X, \tau^{-1} Y)
\]

for \( X, Y \in \text{coh } X \). Recall that an object \( T \) in \( C \) is said to be rigid provided that \( \text{Hom}_C(T,T[1]) = 0 \); more strongly, if we have that

\[
\text{add } T = \{ X \in C \mid \text{Hom}_C(X,T[1]) = 0 \},
\]

then \( T \) is called a cluster-tilting object. Identifying isomorphism classes in coh \( X \) with those in \( C \), it follows that tilting (resp. rigid) sheaves in coh \( X \) are precisely cluster-tilting (resp. rigid) objects in \( C \). Moreover, we have the following description of the endomorphism algebras of cluster-tilting objects in \( C \).

Proposition 2.17. [1] Prop. 4.7] Let \( T \) be a tilting sheaf in coh \( X \). Then there is an isomorphism of graded algebras between \( \text{End}_C(T) \) and \( \Pi^3(\text{End}_X(T)) \).
The category $\mathcal{C}$ has a cluster structure in the sense of [7]. Moreover, mutation of cluster-tilting objects is compatible with mutations of tilting sheaves and mutation of Jacobian algebras, see [14, Secs. 1, 2.5] for example.

Finally, we have the following characterization of cluster-tilting objects with selfinjective endomorphism algebra.

**Proposition 2.18.** [19, Prop. 4.4] Let $T$ be a cluster-tilting object in $\mathcal{C}$. Then $T \cong T[2]$ if and only if $\text{End}_{\mathcal{C}}(T)$ is a selfinjective algebra.

3. Proofs of the main results

In this section we give the proofs of the main results of this article, see Theorems 3.6, 3.7 and 3.8.

Note that by the definition of the cluster category associated to $X$, we have a commutative diagram of functors

$$
\begin{array}{ccc}
\text{coh } X & \xrightarrow{\tau} & \text{coh } X \\
\downarrow & & \downarrow \\
\mathcal{C}_X & \xrightarrow{\tau} & \mathcal{C}_X
\end{array}
$$

where the vertical arrows correspond to the canonical projection functor. The following characterization can be easily deduced from known results.

**Proposition 3.1.** Let $T$ be a tilting complex in $D^b(\text{coh } X)$. Then, the following conditions are equivalent:

(a) The algebra $\text{End}_X(T)$ is a 2-representation-finite algebra.
(b) The algebra $\text{End}_{\mathcal{C}}(T)$ is a finite-dimensional selfinjective algebra.
(c) We have $T[2] \cong T$ in $\mathcal{C}_X$, and $\text{End}_{D^b(\text{coh } X)}(T)$ has global dimension at most 2.

Moreover, if $T \in \text{coh } X$, then any of the three equivalent conditions above is equivalent to $T$ being $\tau^2$-stable.

**Proof.** (i) is equivalent to (ii). Let $\Lambda := \text{End}_X(T)$. By Proposition 2.12, the algebra $\Lambda$ is 2-representation-finite if and only if $\Pi_3(\Lambda)$ is a selfinjective finite dimensional algebra. Moreover, Proposition 2.17 yields an isomorphism between $\Pi_3(\Lambda)$ and $\text{End}_{\mathcal{C}}(T)$. The claim follows. The equivalence between (ii) and (iii) is shown in Proposition 2.18.

Finally, let $T \in \text{coh } X$. We show that (i) is equivalent to $T$ being $\tau^2$-stable. Note that, by the definition of $\mathcal{C}$, the functors $[1]: \mathcal{C} \to \mathcal{C}$ and $\tau: \mathcal{C} \to \mathcal{C}$ are naturally isomorphic. Hence, we have $T[2] \cong \tau^2 T$ as objects of $\mathcal{C}$ and, since isomorphism classes in $\mathcal{C}$ and $\text{coh } X$ coincide, we have $T[2] \cong \tau^2 T$ in $\text{coh } X$. The claim follows. □

In the case of $\tau^2$-stable tilting sheaves we obtain further restrictions on their endomorphism algebras.

**Proposition 3.2.** Let $T$ be a tilting complex in $D^b(\text{coh } X)$. Then, $T$ is $\tau^2$-stable if and only if $\text{End}_{D^b(\text{coh } X)}(T)$ is a 2-homogeneous 2-representation-finite algebra.

**Proof.** Let $T$ be a tilting complex in $D^b(\text{coh } X)$ and set $\Lambda := \text{End}_{D^b(\text{coh } X)}(T)$. Then, we have $\tau^2 T \cong T$ if and only if $(\nu[-1])^2(\Lambda) \cong \Lambda$ which is equivalent to $\nu \Lambda \cong \nu_2^{-1} \Lambda$. Hence, to show that $\Lambda$ is a 2-homogeneous 2-representation-finite algebra,
see Definition 2.10, we only need to show that if $T$ is $\tau^2$-stable then $\Lambda$ has global dimension at most 2. Indeed, for each $i \geq 3$ we have

$$\text{Ext}^i_\Lambda(D\Lambda, \Lambda) \cong \text{Hom}_{D\Lambda}(\nu^{-1}\Lambda[2], \Lambda[i]) \cong D\text{Hom}_{D\Lambda}(\Lambda[i - 2], \Lambda) = 0.$$ 

Thus $\Lambda$ has global dimension at most 2 as required. \hfill \Box

The following result is crucial in our approach, as it allows to pass from $\tau^2$-stable tilting complex to $\tau^2$-stable tilting sheaves using 2-APR-(co)tilting. Recall that the effect of 2-APR-(co)tilting on the endomorphism algebras of basic tilting complexes can be described using mutations of graded quivers with potential, see Theorem 2.16

**Proposition 3.3.** Let $T$ be a basic tilting complex in $D^b(\text{coh} \, \mathbb{X})$ such that $\text{End}_{D^b(\mathbb{X})}(T)$ is a 2-representation-finite algebra. Then, there exists a $\tau^2$-stable tilting sheaf $E \in \text{coh} \, \mathbb{X}$ obtained by iterated 2-APR-tilting from $T$.

**Proof.** Since shifting does not change endomorphism algebras, we can assume that $T$ is concentrated in degrees $-\ell, \ldots, -1, 0$. Since $\text{coh} \, \mathbb{X}$ is hereditary, we have $T \cong T_0[-\ell] \oplus \cdots \oplus T_1[-1] \oplus T_0$ where each $T_i$ is a non-zero sheaf. We proceed by induction on $\ell$. The case $\ell = 0$ follows immediately from Proposition 3.1 so let $\ell > 0$. We claim that the complex

$$T' := (\tau^{-1}T_0)[1 - \ell] \oplus T_{\ell - 1}[1 - \ell] \oplus \cdots \oplus T_1[-1] \oplus T_0$$

is a 2-APR-tilting complex. Indeed, since there are no negative extensions between objects of $\text{coh} \, \mathbb{X}$, we have

$$\bigoplus_{i=0}^{\ell-1} \text{Hom}_{D^b(\mathbb{X})}(T_i[-i], T_\ell[-\ell]) = \bigoplus_{i=0}^{\ell-1} \text{Hom}_{D^b(\mathbb{X})}(T_i, T_\ell[i - \ell]) = 0.$$

Moreover, using the identity $\nu = \tau[1]$, we obtain

$$\bigoplus_{i=0}^{\ell-1} \text{Ext}^1_{D^b(\mathbb{X})}(\nu T_i[-i], T_\ell[-\ell]) \cong \bigoplus_{i=0}^{\ell-1} \text{Hom}_{D^b(\mathbb{X})}(\tau T_i[-i], T_\ell[-\ell])$$

$$\cong \bigoplus_{i=0}^{\ell-1} \text{Hom}_{D^b(\mathbb{X})}(\tau T_i, T_\ell[i - \ell]) = 0.$$

Finally, since $\text{End}_{D^b(\mathbb{X})}(T)$ has global dimension 2 we have that

$$\bigoplus_{i=0}^{\ell-1} \text{Ext}^2_{D^b(\mathbb{X})}(\nu T_i[-i], T_\ell[-\ell]) = 0$$

for all $j \geq 3$. This shows that $T'$ is a 2-APR tilting complex and, by Theorem 2.15, we have that $\text{End}_{D^b(\mathbb{X})}(T')$ is a 2-representation-finite algebra. Hence, by the induction hypothesis, by iterated 2-APR-tilting we can construct a $\tau^2$-stable tilting sheaf $E$ from $T$. \hfill \Box

Next, we determine which weighted projective lines can have $\tau^2$-stable tilting sheaves.

**Proposition 3.4.** Let $T \in \text{coh} \, \mathbb{X}$ be a $\tau^2$-stable tilting sheaf. Then $\mathbb{X}$ has tubular weight type $(2, 2, 2, 2)$, $(2, 4, 4)$ or $(2, 3, 6)$.
Proof. Since there are no tilting sheaves of finite length, we have $\text{slope}(T) \in \mathbb{Q}$. Moreover, as we have $\tau^2 T \cong T$, it follows from Proposition \ref{prop:tau} that $\delta(\omega) = 0$. Then, using equation \ref{eq:delta}, we have that $\chi(X) = 0$ hence $X$ has tubular type.

We recall if $X$ has tubular type, then the full subcategory of $\text{coh} X$ given of all sheaves of a fixed slope is equivalent to the category $\text{coh}_0 X$ of torsion sheaves over $X$ \cite[Thm. 3.10]{[24]}. Assume now that $X$ is an indecomposable summand of $T$ which belongs to a tube of odd period $2a + 1$, so we have $X \cong \tau(\tau^{2a} X)$. By hypothesis, $\tau^{2a} X$ is a direct summand of $T$. Hence, by Serre duality we have

$$0 = \text{Ext}^1_X(\tau^{2a} X, X) \cong D \text{Hom}_X(X, \tau(\tau^{2a} X)) = D \text{Hom}_X(X, X),$$

a contradiction. Hence every indecomposable summand of $T$ belongs to a tube of even period. This rules out weight type $(3,3,3)$. Therefore must have tubular weight type $(2,2,2)$, $(2,4,4)$ or $(2,3,6)$.

The following result gives a classification of the endomorphism algebras of basic $\tau^2$-stable tilting sheaves in $\text{coh} X$.

**Theorem 3.5.** Let $T$ be a basic $\tau^2$-stable tilting sheaf in $\text{coh} X$. Then $\text{End}_X(T)$ is isomorphic to one of the algebras indicated in Figures \ref{fig:1.1}, \ref{fig:3.1} or \ref{fig:3.2}.

**Proof.** First, suppose that $X$ has type $(2,2,2;\lambda)$. Since $\tau^2$ is the identity in $\text{coh} X$, all tilting sheaves are $\tau^2$-stable in this case. Their endomorphism algebras are known, see Skowroński \cite[Ex. 3.3]{[29]} (see also Figure \ref{fig:1.1}).

For the other cases, weight types $(2,4,4)$ and $(2,3,6)$, we rely on the following argument. Let $T$ be a $\tau^2$-stable tilting sheaf in $\text{coh} X$. By Proposition \ref{prop:tau} we can assume that $T$ is in normal position. By Lemma \ref{lem:indecomposable} every indecomposable direct summand of $T_0$ is an exceptional simple sheaf. Also, the perpendicular category $T_0^\perp$ is equivalent to a category of the form $\text{coh} Y$ where $Y$ is a weighted projective line with $\chi(Y) > 0$. Moreover, there exist a finite dimensional algebra $H$ of extended Dynkin type $\Delta$ such that $\text{coh} Y$ is derived equivalent to $\text{coh} Y$. In addition, we have $T_+ \in \text{vect} Y$, see Proposition \ref{prop:vector}. It follows that $\text{End}_X(T_+) = \text{End}_Y(T_+)$ is isomorphic to the endomorphism algebra of a preprojective $H$-module. Note that $\text{End}_Y(T_+)$ must admit an action of order $p/2$ which does not fix any line bundle summands of $T_+$, see by Proposition \ref{prop:vector} \cite{[11]}. Finally, Proposition \ref{prop:vector} yields an isomorphism of algebras

$$\text{End}_X(T) \cong \text{End}_Y(T_+ \oplus E) \cong \begin{bmatrix} \text{End}_Y(T_+) & \text{End}_Y(T_+ \oplus E) \\ 0 & \text{End}_Y(E) \end{bmatrix}$$

where $E$ is the direct sum of all regular simple modules in $\text{coh} Y$ in the exceptional tubes concentrated in the $\lambda_i$’s such that $q_i \neq 0$ (note that here $q_i$ must be either zero or $p_i/2$). It follows that $\text{End}_Y(E)$ is a semisimple algebra with $q_1 + \cdots + q_t$ simple modules.

Hence, to prove the theorem we only need to do the following:

(a) Take a vector bundle in $T_+ \in \text{coh} Y$ whose endomorphism algebra admits a symmetry of order 2 for $X$ of type $(2,4,4)$ or order 3 for $X$ of type $(2,3,6)$ not fixing any line bundles.

(b) Compute the algebra $\text{End}_X(T_+ \oplus E)$.

(c) Check if $\text{End}_X(T_+ \oplus E)$ is a 2-representation-finite algebra, see Proposition \ref{prop:2-rep}.
This process, although lengthy, is straightforward. We illustrate part of it for $X$ of weight type $(2, 4, 4)$. The case were $X$ has type $(2, 3, 6)$ is completely analogous. The cases we need to deal with are stated in Table 3.1.

In this case we have $\Delta = \tilde{A}_{4, 4}$. The only possibility for the Gabriel quiver of $\text{End}_Y(T_+)$ is a non-oriented cycle with 8 vertices. Moreover, it must have 4 arrows pointing in clockwise direction and 4 arrows pointing in counterclockwise direction. These are the quivers highlighted in the algebras $C_1$ and $C_2$ Figure 3.1. All of these algebras are 2-representation-finite algebras, as can be readily verified by checking that their 3-preprojective algebras are selfinjective, see Proposition 2.12. The reader can verify that they indeed arise by the procedure described in Theorem 2.7.

The cases $Y(1, 2, 4)$ and $Y(1, 2, 2)$ are completely analogous, The resulting 2-representation finite algebras correspond to $C_3$ and $C_4$ respectively in Figure 3.1.
Figure 3.2. Endomorphism algebras of $\tau^2$-stable tilting sheaves in normal position over a weighted projective line $X$ of weight type $(2, 3, 6)$. We have indicated $\text{End}_X(T_+)$ by gray arrows. Note that $\tau^2$ acts on each configuration by left rotation by $\pi/3$. The weight type of the reduced weighted projective line $Y$ is indicated for reference.

$Y(2, 2, 2)$ In this case we have $\Delta = \hat{D}_4$. The only possible endomorphism algebras of preprojective tilting $H$-modules are orientations of the Dynkin diagram of type $\hat{D}_6$

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or the canonical algebra of type $(2, 2, 2)$, see Happel-Vossieck’s list [17]. The only quivers which admit an action of order 2 which does not fixes any line bundle summand of $T_+$ are the ones highlighted in algebras $C_1^{\text{op}}$, $C_5$ and $C_6$ in Figure 3.1 corresponding to symmetric orientations of the Dynkin diagram above.
Table 3.1. Possible weight types for $\mathcal{Y}$ and the extended Dynkin type $\Delta$ of the associated hereditary algebra.

| $\mathcal{X}$ | $\mathcal{Y}$ | $\Delta$ |
|---------------|--------------|-----------|
| $\mathcal{X}(2, 4, 4)$ | $(1, 4, 4)$ | $\hat{A}_{4,4}$ |
| | $(1, 2, 4)$ | $\hat{A}_{2,4}$ |
| | $(1, 2, 2)$ | $\hat{A}_{2,2}$ |
| | $(2, 2, 2)$ | $\hat{D}_4$ |
| | $(2, 2, 4)$ | $\hat{D}_6$ |

| $\mathcal{X}(2, 3, 6)$ | $\mathcal{Y}$ | $\Delta$ |
|---------------|--------------|-----------|
| | $(1, 3, 6)$ | $\hat{A}(3, 6)$ |
| | $(1, 3, 3)$ | $\hat{A}(3, 3)$ |
| | $(2, 3, 3)$ | $\hat{E}_6$ |

Figure 3.3. Endomorphism algebras of preprojective tilting modules of type $\hat{D}_6$ with dimension vectors. The orientation of simple edges can be chosen arbitrarily and the relations indicate that the sum of all paths with the corresponding endpoints is zero.

$\mathcal{Y}(2, 2, 4)$ We have $\Delta = \hat{D}_6$. In this case (and only in this case), there are algebras in Happel-Vossieck’s list which have an action of order $p/2$ which do not extend to a 2-representation-finite algebra. One way to rule out these algebras before doing any computation is to determine the action induced by $\tau^2$ on the Auslander-Reiten quiver of vect $\mathcal{Y}$, which has shape $\mathbb{Z}D_6$, see Theorem 2.4(b) and [24, Table 1]. We prove below that this action is given by rotation along the horizontal axis of vect $\mathcal{Y}$, corresponding to the action given by degree shift by $\vec{y} + 2\vec{\omega}_Y \in \mathcal{L}(2, 2, 4)$. Taking this into account, according to [17] the possible endomorphism algebras of preprojective tilting $H$-modules are given in Figure 3.3. The only quivers in Figure 3.3 which are stable under rotation by $\pi$ along the horizontal axis of the Auslander-Reiten quiver...
of vect $\mathcal{Y}$ are the ones highlighted in algebras $C_{2}^\text{op}$, $C_{3}^\text{op}$, $C_{4}^\text{op}$, $C_{5}^\text{op}$, $C_{7}$, $C_{7}^\text{op}$ and $C_{8}$ in Figure 3.1.

Finally, let us prove that the action induced by $\tau^{2}$ on vect $\mathcal{Y}$ is indeed given by rotation along the horizontal axis. This also serves as an example of the method to compute $\text{End}_{X}(T)$ using Theorem 2.7.

Let $X$ be a weighted projective line of tubular type $(2, 4, 4)$. Let $X$ be an exceptional simple sheaf concentrated at $\lambda_{2}$ and set $T_{0} := X \oplus \tau^{2}X$. We write $L(2, 2, 4) = (\vec{x}, \vec{y}, \vec{z}, \vec{c})$ and $\vec{\omega} = \vec{\omega}_{\mathcal{Y}}$. Also, we put $R := R(2, 2, 4)$.

Let $T_{+} \in \text{vect} \mathcal{Y}$ be the tilting bundle indicated in Figure 3.4 and $E = S \oplus S'$ be the direct sum of the two exceptional simple sheaves in coh $\mathcal{Y}$ concentrated at the point $\lambda_{2}$. Put $T := T_{+} \oplus T_{0}$. By Theorem 2.7, we have an isomorphism of $K$-algebras

\[
\text{End}_{X}(T) \cong \begin{bmatrix}
\text{End}_{\mathcal{Y}}(T_{+}) & \text{Hom}_{\mathcal{Y}}(T_{+}, E) \\
0 & \text{End}_{\mathcal{Y}}(E) \cong K \times K
\end{bmatrix}.
\]

We need to compute $\text{End}_{\mathcal{Y}}(T_{+}, E)$. For this, recall from Theorem 2.1(d) that we have short exact sequences

\[(2) \quad 0 \longrightarrow \mathcal{O}_{\mathcal{Y}}(-\vec{y}) \longrightarrow \mathcal{O}_{\mathcal{Y}} \longrightarrow S \longrightarrow 0\]

and

\[(3) \quad 0 \longrightarrow \mathcal{O}_{\mathcal{Y}}(-2\vec{y}) \longrightarrow \mathcal{O}_{\mathcal{Y}}(-\vec{y}) \longrightarrow S' \longrightarrow 0.\]

We shall compute $\dim \text{Hom}_{\mathcal{Y}}(\mathcal{O}(\vec{z}), S)$ and $\dim \text{Hom}_{\mathcal{Y}}(\mathcal{O}(\vec{z}), S')$ by applying the functor $\text{Hom}_{\mathcal{Y}}(\mathcal{O}(\vec{z}), -)$. Before that, it is convenient to make some preliminary calculations.

Firstly, by Theorem 2.1(g) we have $\text{Hom}_{\mathcal{Y}}(\mathcal{O}(\vec{z}), \mathcal{O}) = 0$ and using Serre duality we obtain

\[
\text{Ext}_{\mathcal{Y}}^{1}(\mathcal{O}(\vec{z}), E) \cong D\text{Hom}_{X}(E, \mathcal{O}(\vec{z} + \vec{\omega})) = 0,
\]
since there are no non-zero morphisms from a torsion sheaf to a vector bundle.

Secondly, again by Theorem 2.1(g) and Serre duality we have

\[ \text{Ext}^1_Y(\mathcal{O}(\vec{z}), \mathcal{O}) \cong D \text{Hom}_Y(\mathcal{O}, \mathcal{O}(\vec{z} + \vec{\omega})) \cong R_{\vec{x} - \vec{y}} = 0. \]

Similarly, we have

\[ \text{Ext}^1_Y(\mathcal{O}(\vec{z}), \mathcal{O}(-\vec{y})) \cong D \text{Hom}_Y(\mathcal{O}(-\vec{y}), \mathcal{O}(\vec{z} + \vec{\omega})) = R_{\vec{z} + \vec{y} + \vec{x}} = R_{\vec{x}}. \]

In addition, we have

\[ \text{Hom}_Y(\mathcal{O}(\vec{z}), \mathcal{O}) = R_{-\vec{z}} = 0 \quad \text{Hom}_Y(\mathcal{O}(\vec{z}), \mathcal{O}(-\vec{y})) = R_{-\vec{y} - z} = 0. \]

Hence, applying the functor \( \text{Hom}_Y(\mathcal{O}_Y(\vec{z}), -) \) to the sequences (2) and (3) yields exact sequences

\[ 0 \rightarrow \text{Hom}_Y(\mathcal{O}(\vec{z}), S) \rightarrow \text{Ext}^1_Y(\mathcal{O}(\vec{z}), O(-\vec{y})) \rightarrow 0 \]

and

\[ 0 \rightarrow \text{Hom}_Y(\mathcal{O}(\vec{z}), S') \rightarrow \text{Ext}^1_Y(\mathcal{O}(\vec{z}), O(-2\vec{y})) \rightarrow \text{Ext}^1_Y(\mathcal{O}(\vec{z}), O(-\vec{y})) \rightarrow 0 \]

Then, by Theorem 2.1(g) we have

\[
\begin{align*}
dim \text{Hom}_Y(\mathcal{O}(\vec{z}), S) &= \dim \text{Ext}^1_Y(\mathcal{O}(\vec{z}), O(-\vec{y})) \\
&= \dim \text{Hom}_Y(\mathcal{O}(-\vec{y}), \mathcal{O}(\vec{z} + \vec{\omega})) \\
&= \dim R_{\vec{z} + \vec{y} + \vec{z}} \\
&= \dim R_{\vec{x}} = 1
\end{align*}
\]

and

\[
\begin{align*}
dim \text{Hom}_Y(\mathcal{O}(\vec{z}), S') &= \dim \text{Ext}^1_Y(\mathcal{O}(\vec{z}), O(-2\vec{y})) - \dim \text{Ext}^1_Y(\mathcal{O}(\vec{z}), O(-\vec{y})) \\
&= \dim \text{Hom}_Y(\mathcal{O}(-2\vec{y}), \mathcal{O}(\vec{z} + \vec{\omega})) - 1 \\
&= \dim R_{2\vec{y} + \vec{z} + \vec{\omega}} - 1 \\
&= \dim R_{\vec{x} + \vec{y}} - 1 = 0.
\end{align*}
\]

A similar argument shows that

\[ \text{Hom}_Y(\mathcal{O}(\vec{x} + \vec{\omega}), S) = 0 \]

and

\[ \dim \text{Hom}_Y(\mathcal{O}(\vec{x} + \vec{\omega}), S') = 1. \]

Proceeding in the same fashion, the reader can verify the equalities

\[
\begin{align*}
\dim \text{Hom}_Y(\mathcal{O}(\vec{x} + 3\vec{\omega}), S) &= \dim \text{Hom}_Y(\mathcal{O}(\vec{x} + 3\vec{\omega}), S') = 1, \\
\dim \text{Hom}_Y(\mathcal{O}(\vec{y} + 2\vec{\omega}), S) &= \dim \text{Hom}_Y(\mathcal{O}(\vec{y} + 2\vec{\omega}), S') = 1, \\
\dim \text{Hom}_Y(\mathcal{O}(\vec{z} + 2\vec{\omega}), S) &= \dim \text{Hom}_Y(\mathcal{O}(\vec{z} + 2\vec{\omega}), S') = 0, \\
\dim \text{Hom}_Y(\mathcal{O}, S) &= 1, \quad \dim \text{Hom}_Y(\mathcal{O}, S') = 0.
\end{align*}
\]

It remains to compute \( \dim \text{Hom}_Y(U, S) \) and \( \dim \text{Hom}_Y(U, S') \). We have an exact sequence

\[
\begin{array}{cccc}
0 & \rightarrow & \mathcal{O}(\vec{\omega}) & \rightarrow & U & \rightarrow & \mathcal{O}(\vec{z}) & \rightarrow & 0.
\end{array}
\]
Applying the contravariant functor $\text{Hom}_Y(-, S)$ yields an exact sequence

$$0 \rightarrow \text{Hom}_Y(\mathcal{O}(\vec{z}), S) \rightarrow \text{Hom}_Y(U, S) \rightarrow \text{Hom}_Y(\mathcal{O}(\vec{w}), S) \rightarrow \text{Ext}^1_Y(\mathcal{O}(\vec{z}), S) = 0.$$ 

We already know that $\dim \text{Hom}_Y(\mathcal{O}(\vec{z}), S) = 1$. Proceeding as before, applying the functor $\text{Hom}_Y(\mathcal{O}(\vec{w}), -)$ to the short exact sequence (2), we can show that $\dim \text{Hom}_Y(U, S) = 0$. Hence $\dim \text{Hom}_Y(U, S') = 1$. We can show that $\dim \text{Hom}_Y(U, S') = 1$ in a similar manner.

It follows, by a suitable change of basis of $\text{End}_Y(T)$, that the quiver with relations of $\text{End}_X(T)^{op}$ is given by

$$\begin{array}{cccccc}
\mathcal{O} & \rightarrow & \mathcal{O}(\vec{z}) & \rightarrow & S \\
\mathcal{O}(\vec{z} + 2\vec{w}) & \rightarrow & U & \leftarrow & \mathcal{O}(\vec{x} + 3\vec{w}) \\
S' & \leftarrow & \mathcal{O}(\vec{x} + \vec{w}) & \leftarrow & \mathcal{O}(\vec{y} + 2\vec{w})
\end{array}$$

where each relation is a zero relation or a commutative relation.

Using Proposition 2.12 it is easy to check that $\text{End}_X(T)$ is a 2-representation-finite algebra. Therefore $T$ is a $\tau^2$-stable tilting sheaf, see Proposition 3.1. Then, we can see in Figure 3.3 that the only action of order 2 on the Auslander-Reiten quiver of vect $Y$ which fixes $T_+$ given by rotation by $\pi$ along the horizontal axis, which can be interpreted as degree shift by $\vec{y} + 2\vec{z}_Y$. This concludes the proof of the theorem.

As a consequence of Theorem 3.5 we obtain the following classification results.

**Theorem 3.6.** Let $X$ be a weighted projective line and basic $T$ a tilting complex in $\text{D}^b(X) = \text{D}^b(\text{coh} X)$. Then $\text{End}_{\text{D}^b(X)}(T)$, is a 2-representation-finite algebras if and only if $\text{End}_{\text{D}^b(X)}(T)$ is one of the algebras in Figures 1.1, 1.2 and 1.3. Moreover, this determines $T$ up to an autoequivalence of $\text{D}^b(\text{coh} X)$.

**Proof.** The first claim follows immediately from Proposition 3.3 and Theorem 3.5 since 1.2 and 1.3 are all algebras that can be obtained by 2-APR-(co)tilting from the algebras in Figures 1.4 and 1.5. The second claim is a standard application of [26, Thm. 3.2].

**Theorem 3.7.** Let $T$ be a basic complex in $\text{D}^b(\text{coh} X)$. Then, $T$ is $\tau^2$-stable if and only if $\text{End}_{\text{D}^b(X)}(T)$ is one of the algebras in Figures 1.1, 1.4 or 1.5. Moreover, this determines $T$ up to an autoequivalence of $\text{D}^b(\text{coh} X)$.

**Proof.** By Theorem 3.6, the algebra $\text{End}_X(T)$ can be obtained from one of the algebras in Figures 1.1, 1.4 or 1.5 by iterated 2-APR-(co)-tilting. By Proposition 3.1 we have that $T$ is $\tau^2$-stable is a 2-homogeneous 2-representation-finite algebra. These are precisely the algebras in Figures 1.1, 1.4 or 1.5.

**Theorem 3.8.** Let $T$ be a basic $\tau^2$-stable tilting sheaf in $\text{coh} X$. Then the cluster-tilted algebra $\text{End}_X(T)$ is isomorphic to the Jacobian algebra associated to one of the quivers with potential in Figures 1.6, 1.7 or 1.8 and all of the Jacobian algebras associated to one of these quivers with potential arise in this way.
Proof. Let $T$ be a basic $\tau^2$-stable tilting sheaf in $\text{coh}X$ and set $\Lambda = \text{End}_X(T)$. It follows from Theorem 3.5 that $\Lambda$ is isomorphic to one of the algebras in Figures 1.1, 3.1 or 3.2. Then, by Proposition 2.17 there exist an isomorphism $\text{End}_C(T) \cong \Pi_3(\Lambda)$. By Theorem 2.13, we have that $\Pi_3(\Lambda)$ is isomorphic to the Jacobian algebra to one of the quivers with potential in Figures 1.6, 1.7 or 1.8.

Conversely, each Jacobian algebra associated to one of the quivers with potential in Figures 1.6, 1.7 or 1.8 is of the form $\Pi_3(\Lambda)$ for some $\Lambda$ in Figures 1.1, 3.1 or 3.2, see [19, Secs. 5.1, 9.2 and 9.3]. The theorem follows. □

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