On the Areas of
Cyclic and Semicyclic Polygons

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Abstract
We investigate the “generalized Heron polynomial” that relates the squared area of an \( n \)-gon inscribed in a circle to the squares of its side lengths. For a \((2m + 1)\)-gon or \((2m + 2)\)-gon, we express it as the defining polynomial of a certain variety derived from the variety of binary \((2m - 1)\)-forms having \( m - 1 \) double roots. Thus we obtain explicit formulas for the areas of cyclic heptagons and octagons, and illuminate some mysterious features of Robbins’ formulas for the areas of cyclic pentagons and hexagons. We also introduce a companion family of polynomials that relate the squared area of an \( n \)-gon inscribed in a circle, one of whose sides is a diameter, to the squared lengths of the other sides. By similar algebraic techniques we obtain explicit formulas for these polynomials for all \( n \leq 7 \).

1 Introduction
Heron of Alexandria (c. 60 BC) is credited with the formula that relates the area \( K \) of a triangle to its side lengths \( a \), \( b \), and \( c \):

\[
K = \sqrt{s(s - a)(s - b)(s - c)}
\]

where \( s = (a + b + c)/2 \) is the semiperimeter. For polygons with more than three sides, the side lengths do not in general determine the area, but they do

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if the polygon is convex and cyclic (inscribed in a circle). Brahmagupta, in the seventh century, gave the analogous formula for a convex cyclic quadrilateral with side lengths $a$, $b$, $c$, and $d$:

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where $s = (a+b+c+d)/2$. See [2] for an elementary proof.

Robbins [7] found a way to generalize these formulas. First, drop the requirement of convexity and consider the square of the (signed) area $K$ of a possibly self-intersecting oriented cyclic polygon. For this purpose we can define the area enclosed by a closed curve $C$ to be $\oint_C x \, dy$. Second, express the relation between $K^2$ and the side lengths as a polynomial equation with integer coefficients. Given a cyclic polygon, one can permute its edges within its circumscribed circle without changing its area, so the polynomial will be symmetric in the side lengths, and in fact it can be written in terms of $16K^2$ and the elementary symmetric functions $\sigma_i$ in the squares of the side lengths. For instance, the Heron and Brahmagupta formulas can be written

$$16K^2 - 4\sigma_2 + \sigma_1^2 - \epsilon \cdot 8\sqrt{\sigma_4} = 0$$

in which $\epsilon$ is 0 for a triangle, 1 for a convex quadrilateral, and $-1$ for a non-convex quadrilateral. Hence all cyclic quadrilaterals satisfy the polynomial equation $(16K^2 - 4\sigma_2 + \sigma_1^2)^2 - 64\sigma_4 = 0$. The general result is the following.

**Theorem 1.** [7] For each $n \geq 3$, there is a unique (up to sign) irreducible polynomial $\alpha_n$ with integer coefficients, homogeneous in $n+1$ variables with the first variable having degree 2 and the rest having degree 1, such that $\alpha_n(16K^2, a_1^2, \ldots, a_n^2) = 0$ whenever $a_1, \ldots, a_n$ are the side lengths of a cyclic $n$-gon and $K$ is its area.

The polynomials $\alpha_n$ are now known in the literature as *generalized Heron polynomials*. For certain sets of $n$ side lengths, as shown in [7], one can find up to $\Delta_n$ distinct squared areas, where

$$\Delta_n = \frac{n}{2} \left( \left\lfloor \frac{n-1}{2} \right\rfloor \right) - 2^{n-2}.$$  

Hence one expects that $\alpha_n$ has degree $\Delta_n$ in its first variable. This conjecture of Robbins, and two others made in [7], have recently been established. We summarize them in Theorem [2].
**Theorem 2.** The polynomial $\alpha_n$ is monic in $16K^2$ and has total degree $2\Delta_n$. If $n$ is even, then $\alpha_n = \beta_n \beta^*_n$ where $\beta_n$ is a polynomial in $16K^2$, $\sigma_1, \ldots, \sigma_{n-1}$, $\sqrt{\sigma_n}$, where $\sqrt{\sigma_n} = a_1 \cdots a_n$ and $\beta^*_n$ is $\beta_n$ with $\sqrt{\sigma_n}$ replaced by $-\sqrt{\sigma_n}$.

See [3] or [5] for the degree, [8] or [1] for monicity, and [8] for the factorization when $n$ is even. Robbins’ main interest, however, and the motivation for our research, was to find reasonably explicit formulas for all $\alpha_n$ and $\beta_n$.

In [7], Robbins found formulas for $\alpha_5$ and $\beta_6$ that have a curious form. To present them concisely, we introduce the crossing parity $\epsilon$ of a cyclic $n$-gon. Assume that the $n$-gon has vertices $v_1, \ldots, v_n$ in the complex plane and circumcenter $0$. For odd $n$ let $\epsilon = 0$, and for even $n$ let $\epsilon = \pm 1$ be

$$\epsilon = \text{sign}[(1 - v_2/v_1)(1 - v_3/v_2) \cdots (1 - v_1/v_n)],$$

which is well-defined because the product is real and nonzero if all edges have positive length. (This definition agrees with the previous definition of $\epsilon$ for $n \in \{3, 4\}$.) Now assume $n \in \{5, 6\}$. Define $u_2 = -4K^2$, and make the substitutions

$$t_1 = \sigma_1,$$
$$t_2 = -\sigma_2 + \frac{1}{4}t_1^2 - u_2,$$
$$t_3 = \sigma_3 + \frac{1}{2}t_1 t_2 - \epsilon \cdot 2\sqrt{\sigma_6},$$
$$t_4 = -\sigma_4 + \frac{1}{4}t_2^2 + \epsilon \cdot t_1 \sqrt{\sigma_6},$$
$$t_5 = \sigma_5 + \epsilon \cdot t_2 \sqrt{\sigma_6}. \tag{1}$$

Then, for any cyclic pentagon or hexagon of the given crossing parity, the cubic polynomial $u_2 + t_3 z + t_4 z^2 + t_5 z^3$ has a double root, so its discriminant vanishes:

$$t_4^2 t_5^2 - 4 u_2 t_4^3 - 4 t_3^2 t_5 + 18 u_2 t_3 t_4 t_5 - 27 u_2^2 t_5^2 = 0.$$

When the $t_i$ are expanded, this discriminant is a polynomial of degree $\Delta_5 = 7$ in $u_2$, and hence in $16K^2$. Multiplying it by $2^{18}$ makes it monic in $16K^2$ and yields $\alpha_5$, $\beta_6$, or $\beta^*_6$ according to whether $\epsilon$ is 0, +1, or −1.

In [3] we generalize this construction. Fix $n$ and the crossing parity $\epsilon$, and let $m = [(n - 1)/2]$. We introduce auxiliary quantities $u_2, \ldots, u_m$, with $u_2 = -4K^2$, and inductively define certain polynomial expressions $t_i$ in the $\sigma_j$ and $u_j$ with $j \leq i$. For $n = 5$ or 6, these definitions reduce to [11]. Corollary then says that the polynomial

$$P_n(z) = u_2 + \cdots + u_m z^{m-2} + t_{m+1} z^{m-1} + \cdots + t_{2m+1} z^{2m-1}$$
is divisible by the square of a polynomial of degree \( m - 1 \). In other words, for any values of the \( t_i \) and \( u_j \) coming from a cyclic \( n \)-gon, \( P_n(z) \) has \( m - 1 \) double roots over \( \mathbb{C} \) (counting with multiplicity, and including roots at infinity). Such polynomials form a variety of codimension \( m - 1 \), defined locally by \( m - 1 \) equations. So, if we regard \( u_3 \) through \( u_m \) as indeterminates and expand each \( t_i \) in terms of the \( \sigma_j \) and \( u_j \), we can in principle eliminate the \( m - 2 \) unwanted quantities \( u_3, \ldots, u_m \) and recover a single polynomial, which is \( \alpha_{2m+1}, \beta_{2m+2}, \) or \( \beta_{2m+2}^* \) depending on \( \epsilon \). In §4 we carry out this program for \( m = 3 \) to obtain formulas for \( \alpha_7 \) and \( \alpha_8 \), the generalized Heron polynomials for cyclic heptagons and octagons.

There is another family of area polynomials susceptible to the same analysis. Call an \((n + 1)\)-gon \textit{semicyclic} if it is inscribed in a circle with one of its sides being a diameter. Its squared area satisfies a polynomial relation with the squares of the lengths of the other \( n \) sides; the degree in the squared area turns out to be

\[
\Delta'_n = \frac{n}{2} \left( \frac{n-1}{2} \right) = \Delta_n + 2^{n-2}.
\]

**Theorem 3.** For each \( n \geq 2 \), there exists a unique monic irreducible polynomial \( \alpha'_n \) with integer coefficients, homogeneous in \( n + 1 \) variables with the first variable having degree 2 and the rest having degree 1, such that \( \alpha'_n(16K^2, a_1^2, \ldots, a_n^2) = 0 \) whenever \( a_1, \ldots, a_n \) are the lengths of the sides of a semicyclic \((n + 1)\)-gon excluding a diameter, and \( K \) is its area. The total degree of \( \alpha'_n \) is \( 2\Delta'_n \).

The proof that \( \alpha'_n \) exists and is unique (without assuming monicity) follows the proof of Theorem 1 in [7] almost verbatim, and the argument in [5] shows that \( \alpha'_n \) is monic. We establish the degree by an elementary argument in §5 which is independent of the rest of this paper.

Cyclic and semicyclic polygons are similar in many ways. For instance, just as the polygon of largest area one can make with \( n \) given side lengths is convex and cyclic, the polygon of largest area one can make with \( n \) given side lengths and one free side is convex and semicyclic. We will adduce many algebraic similarities in the following sections. For now we just observe that the polynomial \( \alpha'_3 \), which can be worked out by hand, also takes the form of a discriminant: if \( u_2 = -4K^2 \), then

\[
\alpha'_3 = 16 \text{discr}_z(z^3 + \sigma_1 z^2 + (\sigma_2 + u_2)z + \sigma_3).
\]
2 The Main Identity

All our area formulas are based on a generating function identity that relates the symmetric functions $\sigma_i$ in the squared side lengths to certain quantities $\tau_j$ that arise in Robbins’ proofs of the pentagon and hexagon formulas. The identity, Theorem 4, holds for both cyclic and semicyclic polygons and for both odd and even $n$.

Suppose we have a cyclic $n$-gon or semicyclic $(n + 1)$-gon inscribed in a circle of radius $r$ centered at the origin in the complex plane. Let its vertices be $v_1, \ldots, v_n$ and $v_{n+1} = \delta v_1$, where $\delta = 1$ for a cyclic $n$-gon and $\delta = -1$ for a semicyclic $(n + 1)$-gon. Introduce the vertex quotients $q_i = v_{i+1}/v_i$ for $i = 1, \ldots, n$, and let $\tau_0, \tau_1, \ldots, \tau_n$ be the elementary symmetric functions of the $q_i$. Then $\tau_0 = 1$ and $\tau_n = q_1q_2\cdots q_n = \delta$. Elementary geometry yields the equations

\begin{align*}
a_i^2 &= r^2(2 - q_i - q_i^{-1}), \quad 1 \leq i \leq n, \quad (3) \\
16R^2 &= -r^4(q_1 + \cdots + q_n - q_1^{-1} - \cdots - q_n^{-1})^2 \\
&= -r^4(\tau_1 - \delta \tau_{n-1})^2. \quad (4)
\end{align*}

Using (3) one can express each $\sigma_i$ in terms of $r$ and the $\tau_i$. Let $g(y) = y^2 + (x/r^2 - 2)y + 1$. Observe that $x$ is one of the values $a_i^2$ exactly when $g(y)$ has one of the vertex quotients $q_i$ as a root, or in other words, when $g(y)$ has a common root with the polynomial $f(y) = \prod_{i=1}^{n}(y - q_i) = \sum_{i=0}^{n}(-1)^i\tau_iy^{n-i}$. Hence the resultant of $f(y)$ and $g(y)$ is a constant times

$$h(x) = \prod_{i=1}^{n}(x - a_i^2) = \sum_{i=0}^{n}(-1)^i\sigma_ix^{n-i},$$

and the coefficient of $x^n$ reveals that the constant is $\delta r^{-2n}$. By expanding the resultant, one finds that each $\sigma_i$ is $r^{2i}$ times a quadratic polynomial in the $\tau_i$. A particularly simple example is

$$\sigma_n = \delta(-1)^n r^{2n}(\tau_0 - \tau_1 + \tau_2 - \cdots \pm \tau_n)^2.$$ 

If $n$ is even and $\delta = 1$, then $\sqrt{\sigma_n}$ is expressible in terms of $r^2$, the $\tau_i$, and the crossing parity $\epsilon$:

$$\sqrt{\sigma_n} = |v_1 - v_2| \cdots |v_n - v_{n+1}| = r^n |1 - q_1| \cdots |1 - q_n|$$
$$= r^n \epsilon(1 - q_1) \cdots (1 - q_n) = r^n \epsilon(\tau_0 - \tau_1 + \tau_2 - \cdots + \tau_n). \quad (5)$$
So far we are following [7] except for the addition of the semicyclic case.

Consider now the involution that reflects the polygon in the real axis. This operation preserves the squared area and the side lengths, but it replaces each $q_i$ with $q_i^{-1}$ and hence replaces each $\tau_i$ with $\delta \tau_{n-i}$. Because each $\sigma_i$ is a quadratic form in the $\tau_j$ preserved by the involution, it can be uniquely decomposed into a two parts: a quadratic form in symmetric linear combinations of the $\tau_j$, and a quadratic form in antisymmetric linear combinations of the $\tau_j$. When we perform this decomposition on the whole generating function $\sum_i (-x)^i \sigma_i$, each part factors in a surprising way, which our main identity records.

To write the identity explicitly, we need the following linear combinations of the $\tau_j$, for $0 \leq k \leq n/2$:

$$d_k = \sum_{i=0}^{k} (-1)^i \binom{n-1-2k+i}{i} (\tau_{k-i} - \tau_{n-k+i}),$$

$$e_k = \sum_{i=0}^{k} (-1)^i \left[ \binom{n-2k+i}{i} + \binom{n-2k+i-1}{i-1} \right] (\tau_{k-i} + \tau_{n-k+i}).$$

Let $D(x) = \sum d_i x^i$ and $E(x) = \sum e_i x^i$.

**Theorem 4 (Main Identity).** For a cyclic $n$-gon or semicyclic $(n+1)$-gon of radius $r$, with $\delta = 1$ or $-1$ respectively, the symmetric functions $\sigma_i$ of the squared side lengths and $\tau_i$ of the vertex quotients are related by

$$\delta \cdot \sum_{i=0}^{n} (-x)^i \sigma_i = \frac{1}{4} E(r^2 x^2) + (r^2 x - \frac{1}{4}) D(r^2 x^2).$$

**Proof.** When the $\sigma_i$ are expanded in terms of the $\tau_j$, both sides of the main identity become polynomials in $r^2 x$, so we may assume $r = 1$. The left-hand side is then

$$\delta \cdot \sum_{i=0}^{n} (-x)^i \sigma_i = \delta x^n h(x^{-1}) = x^n \text{Res}(f, g)$$

where $f(y) = \sum_{i=0}^{n} (-1)^i \tau_i y^{n-i}$ and $g(y) = (y-1)^2 + x^{-1}y$.

We calculate the resultant using its $PGL(2)$-invariance and other standard properties [11]. Make the change of variable $y = (z-1)/(z+1)$ so that the roots of $g$ are related by $z \mapsto -z$ instead of $y \mapsto y^{-1}$. We obtain the
polynomials
\[ f^*(z) = (z + 1)^n f \left( \frac{z}{z+1} \right) = \sum_{i=0}^{n} (-1)^i \tau_i (z - 1)^{n-i} (z + 1)^i, \]
\[ g^*(z) = (z + 1)^2 g \left( \frac{z}{z+1} \right) = x^{-1}(z^2 + 4x - 1), \]
and the transformation has determinant 2, so
\[ x^n \text{Res}(f, g) = 2^{-2n} x^n \text{Res}(f^*, g^*) = 2^{-2n} f^* \left( \sqrt{1 - 4x} \right) f^* \left( \sqrt{1 - 4x} \right). \]
Write \( f^*(z) = f_0(z^2) + z f_1(z^2) \), separating even and odd powers of \( z \). Then
\[ x^n \text{Res}(f, g) = 2^{-2n} \left[ f_0(1 - 4x)^2 - (1 - 4x)f_1(1 - 4x)^2 \right], \]
which explains the form of the main identity. It remains to evaluate \( f_0(1 - 4x) \) and \( f_1(1 - 4x) \). We consider only \( f_0 \), as \( f_1 \) is similar but simpler.

It helps to introduce the Fibonacci polynomials \( F_n(x) = \sum_{i=0}^{|n/2|} (-1)^i \tau_i x^{n-2i} \), which count compositions of \( n \) by 1’s and 2’s. They satisfy the recurrence
\[ F_n(x) = F_{n-1}(x) + xF_{n-2}(x), \quad n \geq 1, \] (8)
with \( F_0(x) = 1 \) and \( F_n(x) = 0 \) for \( n < 0 \), and have generating function
\[ F(x; t) = \sum_n F_n(x)t^n = (1 - xt^2)^{-1}. \]
The generating functions for the \( d_k \) and \( e_k \) can then be written
\[ D(x) = \sum_{i=0}^{|n/2|} (\tau_i - \tau_{n-i})x^i F_{n-2i-1}(-x), \]
\[ E(x) = \sum_{i=0}^{|n/2|} (\tau_i + \tau_{n-i})x^i \left( F_{n-2i}(-x) - xF_{n-2i-2}(-x) \right). \]
To evaluate \( f_0(1 - 4x) \), first rewrite \( f_0(z^2) = \frac{1}{2} \left( f^*(z) + f^*(-z) \right) \) in terms of the sums \( \tau_i + \tau_{n-i} \). If we let \( \theta = \frac{1}{2} \) if \( 2i = n \) and \( \theta = 1 \) otherwise, then
\[ f_0(z^2) = \sum_{i=0}^{|n/2|} (-1)^{n-i} \theta (\tau_i + \tau_{n-i}) (1 - z^2)^i \sum_j \binom{n-2i}{2j} z^{2j}. \]
Next, we want to substitute \( z^2 = 1 - 4x \) and evaluate the sum over \( j \). Writing \( m = n - 2i \), we can compute the generating function

\[
\sum_{m \geq 0} t^m \sum_{j} \binom{m}{2j} (1 - 4x)^j = \frac{1 - t}{1 - 2tx + 4tx^2} = (1 - t)F(-x; 2t)
\]

by interchanging sums and simplifying. Thus we get

\[
f_0(1 - 4x) = \sum_{i=0}^{[n/2]} (-1)^{i+n} \theta(\tau_i + \tau_{n-i})(4x)^i 2^{n-2i} \left( F_{n-2i}(-x) - \frac{1}{2} F_{n-2i-1}(-x) \right).
\]

The recurrence \( \square \) shows that \( \theta \left( F_{n-2i}(-x) - \frac{1}{2} F_{n-2i-1}(-x) \right) \) is equivalent to \( \frac{1}{2} \left( F_{n-2i}(-x) - x F_{n-2i-2}(-x) \right) \), and so

\[
f_0(1 - 4x) = (-1)^n 2^{n-1} E(x).
\]

Likewise \( f_1(1 - 4x) = (-1)^n 2^{n-1} D(x) \), and the identity follows. \( \square \)

# 3 Consequences of the Main Identity

The main identity tells us how to generalize the definition of the quantities \( t_i \) and \( u_j \) that were so useful in simplifying the pentagon and hexagon formulas. Cyclic \( n \)-gons have \( d_0 = \tau_0 - \tau_n = 0 \) and \( e_0 = \tau_0 + \tau_n = 2 \) by \( \square \) and \( \square \), so the expansions of \( E(r^2x)^2 \) and \( D(r^2x)^2 \) include linear terms in the \( e_k \) but not in the \( d_k \). The substitutions that replace the \( \sigma_i \) with the \( t_i \) will first isolate and then eliminate the variables \( e_k \), and leave us with expressions relating the \( t_i \) and \( u_j \) to the radius \( r \) and the variables \( d_k \). The main identity will then express the algebraic relationship among the \( t_i \) and \( u_j \) as the factorization of a single polynomial \( P_n(z) \).

**Corollary 5.** Given a cyclic \( n \)-gon of crossing parity \( \epsilon \) and radius \( r \), let \( m = [(n - 1)/2] \) and let \( u_j = r^{2j} \sum_{i=1}^{j-1} \left( \frac{1}{4} d_i - d_{i-1} \right) d_{j-i} \) for \( j \geq 1 \). Inductively define \( t_0 = -2 \) and

\[
t_j = (-1)^{j+1} \sigma_j + \sum_{1 \leq i,j,i \leq m} \frac{t_i t_{j-i} t_{j-i}}{4} + \begin{cases} -u_j, \\ \epsilon \cdot (-1)^m t_{j-m-1} \sqrt{\sigma_n}, \end{cases} \quad \text{if } j \leq m,
\]

\[
t_j = 0\text{ otherwise}, \quad \text{if } j > m,
\]

for \( j = 1, \ldots, 2m + 1 \). Then \( t_j = -e_j r^{2j} \) for \( 0 \leq j \leq m \), and the polynomial \( P_n(z) = u_2 + u_4 z + \cdots + u_m z^{m-2} + t_{m+1} z^{m-1} + \cdots + t_{2m+1} z^{2m-1} \) factors as \( (\frac{1}{4} - r^2 z) z^{-1} D(r^2 z)^2 \).
By (6) and (4), we have
\[ u_2^2 = \frac{1}{4} r^4 d_1^2 = \frac{1}{4} r^4 (\tau_1 - \tau_{n-1})^2 = -4K^2, \]
and \( u_1 = 0 \) by definition. Thus the \( t_j \) and \( u_j \) in Corollary 5 agree with those defined in (11).

**Proof of Corollary 5.** We have \( e_0 = \tau_0 + \tau_n = 2 \) by the definition (7), so \( t_0 = -e_0 r_0^0 \). Now let \( 1 \leq j \leq m \). The coefficient of \( x^j \) in \( \frac{1}{4} E(r^2x)^2 \) is
\[
\sum_{i=0}^{j} \frac{e_i e_j - i}{4} = r^2j e_j + \sum_{i=1}^{j-1} \frac{t_i t_{j-i}}{4}
\]
by induction on \( j \). The coefficient of \( x^j \) in \( (r^2x - \frac{1}{4}) D(r^2x)^2 \) is \( -u_j \), so the equation \( t_j = -e_j r^{2j} \) follows by comparing coefficients of \( x^j \) in the main identity.

For \( j > m \) we must consider the coefficient \( r^{2m+2} e_{m+1} \) of \( x^{m+1} \) in \( E(r^2x) \). If \( n = 2m + 1 \), then \( E \) has degree \( m \) by definition so this coefficient is zero. But if \( n = 2m + 2 \), then the coefficient is
\[
r^{2m+2} e_{m+1} = r^n \left( 2\tau_{m+1} + \sum_{i=1}^{m+1} (-1)^i 2(\tau_{m+1-i} + \tau_{m+1+i}) \right)
\]
\[
= 2(-1)^{m+1} \epsilon \sqrt{\sigma_n}
\]
by (7) and (5). So for \( m < j \leq 2m + 1 \), the coefficient of \( x^j \) in \( \frac{1}{4} E(r^2x)^2 \) is
\[
r^{2j} \sum_{i=j-m}^{m} \frac{e_i e_{j-i}}{4} + r^{2j} \frac{e_{j-m-1} e_{m+1}}{2} = \sum_{i=j-m}^{m} \frac{t_i t_{j-i}}{4} + t_{j-m-1}(-1)^m \epsilon \sqrt{\sigma_n},
\]
and this holds whether \( n \) is odd or even because \( \epsilon = 0 \) when \( n \) is odd. Thus, by the main identity, \( -t_j \) is the coefficient of \( x^j \) in \( (r^2x - \frac{1}{4}) D(r^2x)^2 \) for \( j = m + 1, \ldots, 2m + 1 \).

We now see that \( (r^2x - \frac{1}{4}) D(r^2x)^2 \), a polynomial of degree \( 2m + 1 \) whose two lowest terms vanish, is exactly \( -x^2 P_n(x) \).

There is a geometric argument that Corollary 5 contains enough information to recover \( \alpha_n \). To simplify the explanation, assume \( n = 2m + 1 \) and \( m \geq 2 \). The nonzero polynomials of degree up to \( 2m - 1 \) that have a squared factor of degree \( m - 1 \) naturally form a projective variety of codimension \( m - 1 \) in \( \mathbb{P}^{2m-1} \), which is irreducible because it is the image of \( \mathbb{P}^1 \times \mathbb{P}^{m-1} \) under a regular map. Hence the affine variety \( X_m \subset \mathbb{A}^{2m} \) of such polynomials
(now including the zero polynomial), which has the same ideal, is also irreducible. The substitutions that write $t_{m+1}, \ldots, t_{2m+1}$ in terms of the $\sigma_i$ and $u_j$ amount to a morphism $f : \mathbb{A}^{3m} \to \mathbb{A}^{2m}$, which is a product bundle with fiber $\mathbb{A}^m$. This is because, for any point $(u_2, \ldots, u_m, t_{m+1}, \ldots, t_{2m+1})$ in the range and any given values of $\sigma_1, \ldots, \sigma_m$, the values of $\sigma_{m+1}, \ldots, \sigma_{2m+1}$ are uniquely determined as polynomial functions of the other variables. Hence $f^{-1}(X_m) \approx X_m \times \mathbb{A}^{m}$ is irreducible and of codimension $m - 1$. Finally, when we apply the projection $\pi : \mathbb{A}^{3m} \to \mathbb{A}^{2m+2}$ that eliminates the $m - 2$ variables $u_3, \ldots, u_m$, the closure of the image $\pi(f^{-1}(X_m))$ is an irreducible variety of codimension at least 1 that contains $V(\alpha_n)$, so it must equal $V(\alpha_n)$.

Corollary 5 therefore reduces the problem of finding $\alpha_n$ to two subproblems: finding the defining equations of the variety $X_m$, and then, after expanding $t_{m+1}, \ldots, t_{2m+1}$ in terms of the $\sigma_i$ and $u_j$, eliminating the $m - 2$ variables $u_3, \ldots, u_m$.

The application of the main identity to semicyclic polygons is similar but slightly different. In this case $e_0 = \tau_0 + \tau_n = 0$ and $d_0 = \tau_0 - \tau_n = 2$, so the main identity involves linear terms in the $d_k$ but not the $e_k$. Our definitions of $t_i$ and $u_j$ are therefore designed to extract and eliminate the variables $d_k$. Again we can distill the relationship among the $t_i$ and $u_j$ to the factorization of a polynomial $P_n'(z)$. This time, due to the factor $(r^2 x - \frac{1}{4})$ in the main identity, the expression for $t_i$ explicitly includes $r^2$, so there remains one more unwanted variable to eliminate for a given $n$.

**Corollary 6.** Given a semicyclic $(n+1)$-gon of radius $r$, let $m = \lfloor (n-1)/2 \rfloor$ and let $u_j = r^{2j} \sum_{i=1}^{j-1} e_i e_{j-i}/4$ for $1 \leq j \leq m$. Inductively define $t_0 = -2$ and

$$t_j = (-1)^{j+1} \sigma_j + \sum_{1 \leq i \leq m}^{1 \leq j-i \leq m} \frac{t_i t_{j-i}}{4} - r^2 \sum_{0 \leq i-1 \leq m}^{0 \leq j-i \leq m} t_{i-1} t_{j-i} + \begin{cases} -u_j, & \text{if } j \leq m, \\ 0, & \text{if } j > m, \end{cases}$$

for $j = 1, \ldots, n$. Then $t_j = -d_j r^{2j}$ for $0 \leq j \leq m$, and the polynomial

$$P_n'(z) = u_2 + u_3 z + \cdots + u_m z^{m-2} + t_{m+1} z^{m-1} + \cdots + t_n z^{n-2}$$

is the square of $E(r^2 z)/2z$. In particular, $t_n = 0$ if $n$ is odd.

Once again $u_2 = \frac{1}{4} r^4 e_1^2 = \frac{1}{4} r^4 (\tau_1 + \tau_{n-1})^2 = -4 K^2$ by (4), since now $\delta = -1$. 

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Proof. As in Corollary 5, the claims follow from equating coefficients of $x^j$ in the main identity and inducting on $j$ to evaluate $t_j$ for $0 \leq j \leq m$. If $n = 2m + 1$, the degree of $E(x)$ is just $m$, so $t_n = 0$.

The polynomial $P_n'(z)$ contains $m - 1$ unwanted variables, namely $r^2$ and $u_3, \ldots, u_m$. If $n = 2m + 1$, then $P_n'(z)$ is a polynomial of degree $2m - 2$ that is a square, which gives rise to $m - 1$ equations in its coefficients, and we have the additional equation $t_n = 0$. If $n = 2m + 2$, then $P_n'(z)$ is a square of degree $2m$, which yields $m$ equations. In either case Corollary 6 holds enough information, in principle, to derive the area formula $\alpha_n$. As before, one can make this claim precise using some algebraic geometry.

4 Explicit Formulas

In this section we apply the results of §3 to produce area formulas for cyclic heptagons and octagons, and also semicyclic quadrilaterals, pentagons, hexagons, and heptagons.

Because the degree of the generalized Heron polynomial $\alpha_n$ is exponential in $n$, and the number of terms could be exponential in $n^2$, there is some question as to what constitutes an explicit formula. Our formulas have concise descriptions, and if a polygon is given with exact (for instance, rational) side lengths, the polynomial satisfied by its area can be computed exactly using standard operations such as evaluating determinants.

First let us apply Corollary 5 to the cases $n \in \{7, 8\}$. It gives us a binary quintic form

$$x^5P_n(y/x) = u_2x^5 + u_3x^4y + t_4x^3y^2 + t_5x^2y^3 + t_6xy^4 + t_7y^5$$

whose coefficients are polynomials in $u_2, u_3, \sigma_1, \ldots, \sigma_7$ and perhaps $\sqrt{\sigma_8}$, and which, when its coefficients are evaluated for any cyclic $n$-gon, has two linear factors over $\mathbb{C}$ of multiplicity two. The condition for a quintic form $Q$ to factor in this way is given by the vanishing of a certain covariant $C$, which in the notation of transvectants is

$$C = 2Q(H, i)^{(2)} + 25H(Q, i)^{(2)} + 6Qi^2, \quad H = (Q, Q)^{(2)}, \quad i = (Q, Q)^{(4)}.$$ 

Here $(f, g)^{(d)} = \sum_{i=0}^{d}(-1)^i\binom{d}{i}(\partial^df/\partial x^i\partial y^{d-i})(\partial^dg/\partial x^{d-i}\partial y^i)$. This fact about quintics is presumably classical, but we have not yet found a reference. In any case, $C$ is a form of degree 9 in $\{x, y\}$ whose coefficients are forms of
degree 5 in the coefficients of the original quintic, so its coefficients give us
ten degree-5 polynomials in $u_2, u_3, t_4, t_5, t_6, t_7$ that must vanish. These same
ten polynomials can be obtained as the Gröbner basis, with a graded term
ordering, for the ideal of the variety of quintic forms that factor as a linear
form times the square of a quadratic.

To obtain the desired relation between $u_2$ and the $\sigma_i$, we must expand
the coefficients of $C$ as polynomials in $u_3$ and then eliminate $u_3$. We can do
this most explicitly using resultants with respect to $u_3$. The two simplest
coefficients of $C$ are

\[
F = u_3^2t_4^3 - 4u_2t_4^4 - 4u_2^2t_4t_5 + 18u_2u_3t_4^2t_5 - 27u_2^2t_4t_5^2 \\
+ (8u_3^4 - 42u_2u_3^2t_4 + 36u_2^2t_4^2 + 54u_2^2u_3t_5 - 80u_2^3t_6)t_7 \\
+ (8u_2u_3^3 - 30u_2^2u_3t_4 + 50u_2^3t_5)t_7,
\]

of total degree 18, and

\[
G = u_3^2t_4^3t_5 - 4u_2t_4^3t_5 - 4u_2^3t_5^2 + 18u_2u_3t_4^2t_5^2 - 27u_2^2t_5^3 \\
+ (2u_3^2t_4 - 8u_2u_3^2t_4 - 6u_2u_3^2t_5 + 36u_2^2t_4t_5 - 8u_2^3u_3t_6)t_6 \\
+ (16u_3^4 - 74u_2u_3^2t_4 + 40u_2^2t_4^2 + 110u_2^2u_3t_5 - 200u_2^3t_6)t_7,
\]

of total degree 19. Let $P \mapsto \tilde{P}$ denote the operation of expanding the $t_i$ in
terms of $u_2, u_3, \sigma_1, \ldots, \sigma_n$ as specified by Corollary 5. This operation
preserves total degree. Both $\tilde{F}$ and $\tilde{G}$ have degree 6 in $u_3$. Their resultant
with respect to $u_3$ therefore has total degree $6 \times 19 = 114$, and it must have
the polynomial $\alpha_7$ of total degree $2\Delta_7 = 76$ as a factor.

The resultant $\text{Res}(\tilde{F}, \tilde{G})$ seems to be too large to compute and factor
explicitly, but we can describe its unwanted factors as follows with a little
computer assistance. First observe that every term in $F$ and $G$ is divisible
by either $u_2$ or $u_3$, and hence the same is true of $\tilde{F}$ and $\tilde{G}$. It follows that
$\text{Res}(\tilde{F}, \tilde{G})$ is divisible by $u_2$. In fact $u_2^7 \mid \text{Res}(\tilde{F}, \tilde{G})$, as we will see in Lemma 7
below. Next, consider the polynomials

\[
F_1 = 4u_3^3 - 15u_2u_3t_4 + 25u_2^2t_5, \\
G_1 = 7u_3^2t_4 - 20u_2t_4^2 - 5u_2u_3t_5 + 100u_2^2t_6,
\]

which are closely related to the coefficients of $t_7$ in $F$ and $G$. Specifically,
$F_1 = (2u_2)^{-1}\partial F/\partial t_7$ and $G_1 = u_2^{-1}(2u_3F_1 - \partial G/\partial t_7)$. We will show that
$\text{Res}(\tilde{F}_1, \tilde{G}_1)$ divides $\text{Res}(\tilde{F}, \tilde{G})$. 

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First we claim that if $F_1 = G_1 = 0$, then $F = G = 0$. The ideal $\langle F_1, G_1 \rangle$ does not contain $F$ and $G$, but by some easy calculations, it does contain $u_2F, u_3F, u_2G,$ and $u_3G$. If $F_1 = G_1 = 0$, then all four of these polynomials vanish; so if either $u_2 \neq 0$ or $u_3 \neq 0$, we must have $F = G = 0$, while if $u_2 = u_3 = 0$, we already know that $F = G = 0$. This establishes the claim. It follows that $\tilde{F}_1 = \tilde{G}_1 = 0$ implies $\tilde{F} = \tilde{G} = 0$. Consequently, wherever $\text{Res}(\tilde{F}_1, \tilde{G}_1)$ vanishes, so does $\text{Res}(\tilde{F}, \tilde{G})$. Algebraically, this means that every irreducible factor of $\text{Res}(\tilde{F}_1, \tilde{G}_1)$ divides $\text{Res}(\tilde{F}, \tilde{G})$. The resultant of $\tilde{F}_1$ and $\tilde{G}_1$ with respect to $u_3$ is simple enough to compute explicitly. It has total degree 30, and it factors as $u_2^3$ times an irreducible polynomial in $\mathbb{Q}[u_2, \sigma_1, \ldots, \sigma_7]$ of total degree 24.

Thus, not only does $\text{Res}(\tilde{F}_1, \tilde{G}_1)$ divide $\text{Res}(\tilde{F}, \tilde{G})$, but $u_2^2\text{Res}(\tilde{F}_1, \tilde{G}_1)$ does also. The quotient by the latter polynomial has total degree $114 - 8 - 30 = 76 = 2\Delta$, so it must be a scalar multiple of the desired polynomial $\alpha_7, \beta_8, \text{ or } \beta_8^2$; there are no more unwanted factors. The scalar can be computed by setting $\sigma_2, \ldots, \sigma_7$ to zero, and we find that

$$\frac{2^{1015}5\text{Res}(\tilde{F}, \tilde{G}, u_3)}{u_2^2\text{Res}(\tilde{F}_1, \tilde{G}_1, u_3)}$$

is $\alpha_7, \beta_8, \text{ or } \beta_8^2$, according to whether the crossing parity $\epsilon$ is 0, +1, or −1. It remains only to prove the following lemma.

**Lemma 7.** With the definitions above, $u_2^2 | \text{Res}(\tilde{F}, \tilde{G}, u_3)$.

**Sketch of proof.** By direct calculation on a computer, $u_2^2$ divides $\text{Res}(F, G, u_3)$ but $u_2^3$ does not. The only component of $V(F, G)$ lying on the hyperplane $u_2 = 0$ is the linear variety $V(u_2, u_3)$, so $V(F)$ and $V(G)$ must intersect with multiplicity 7 along $V(u_2, u_3) \subset \mathbb{A}^6$. Now, assuming $n = 7$ for definiteness, pull back via the projection $\pi : \mathbb{A}^9 \to \mathbb{A}^6$ that maps $(u_2, u_3, \sigma_1, \ldots, \sigma_7) \mapsto (u_2, u_3, t_4, \ldots, t_7)$. Because $\pi$ is smooth, the intersection multiplicity of $V(\tilde{F})$ and $V(\tilde{G})$ along $\pi^{-1}V(u_2, u_3) = V(u_2, u_3) \subset \mathbb{A}^9$ is also 7. For fixed generic values of $\sigma_1, \ldots, \sigma_7$, we therefore have $u_2^2 | \text{Res}(\tilde{F}, \tilde{G}, u_3)$. We conclude that this divisibility holds globally as well. \hfill \Box

For the rest of this section, we turn our attention to semicyclic $(n+1)$-gons with $n = 3, 4, 5, \text{ and } 6$. To state the area formulas most cleanly we introduce a notion of parity for semicyclic polygons. Let $n$ be even, and observe that the quantities $e_1 = \tau_1 + \tau_{n-1} = \sum q_i - \sum q_i^{-1}$ and $\frac{1}{2}e_{n/2} = \sum (-1)^i \tau_i = \prod (1 - q_i)$
are both pure imaginary. (Compute their complex conjugates using \(q_i = q_i^{-1}\).) Hence their product is real. Let \(\epsilon \in \{-1, 0, +1\}\) be its sign. Then we have

\[
\epsilon |K| \overline{\sigma_n} = \epsilon \cdot \frac{1}{4} r^2 |r_1 + \tau_{n-1}| \cdot |v_1 - v_2| |v_2 - v_3| \cdots |v_n - v_1|
\]

\[
= \epsilon \cdot \frac{1}{4} r^2 |r_1 + \tau_{n-1}| \cdot r^n |1 - q_1| \cdots |1 - q_n|
\]

\[
= r^{n+2} \cdot \frac{1}{4} e_1 \cdot \frac{1}{2} e_{n/2}.
\]

Define \(w = 2\epsilon |K| \overline{\sigma_n} = \epsilon \sqrt{v_2 t_n}\) for \(n\) even, and let \(w = 0\) for \(n\) odd. Our formulas for \(\alpha'_4\) and \(\alpha'_6\) will factor when written in terms of \(w\) rather than \(\sigma_n\). We do not know whether this type of factorization occurs in general.

For \(n \in \{3, 4\}\), it is simplest to use the main identity directly. Defining \(e_2 = 0\) if \(n = 3\), we have

\[
\frac{1}{4} E(r^2 x)^2 = \frac{1}{4} r^4 e_1 x^2 + \frac{1}{2} r^6 e_1 e_2 x^3 + \frac{1}{4} r^8 e_2^2 x^4
\]

so, by Theorem \[\text{1} \] the cubic \(1 - \sigma_1 x + (\sigma_2 + u_2)x^2 - (\sigma_3 - 2w)x^3\) factors as \(-(r^2 x - \frac{1}{4})D(r^2 x)^2\). In particular, its discriminant vanishes. Replacing \(x\) by \(-x^{-1}\), we recover equation \[\text{2} \] for \(n = 3\), and for \(n = 4\) we have factored \(\alpha'_4\) as the product of two discriminants \(\beta'_4\) and \((\beta'_4)^*\) corresponding to \(\epsilon = +1\) and \(\epsilon = -1\) respectively.

For larger \(n\) we need Corollary \[\text{6} \] For \(n = 5\), it says that \(u_2 + t_3 z + t_4 z^2 + t_5 z^3\) is the square of the linear polynomial \(E(r^2 z)/(2z)\), which yields the two equations \(t_3^2 - 4u_2 t_4 = 0\) and \(t_5 = 0\). Their degrees in \(r^2\) are 6 and 5 respectively, so their resultant with respect to \(r^2\) has the correct total degree \(2\Delta'_5 = 30\). (Remember that \(r^2\) has degree 1.) It remains only to scale the resultant to be monic in \(-4u_2\), and we get

\[
\alpha'_5 = \frac{1}{4} \text{Res}(t_3^2 - 4u_2 t_4, t_5, r^2).
\]

For \(n = 6\), Corollary \[\text{6} \] gives us the factorization

\[
u_2 + t_3 z + t_4 z^2 + t_5 z^3 + t_6 z^4 = \frac{1}{4} r^4 (e_1 + e_2 r^2 z + e_3 r^4 z^2)^2.
\]

Using \(w = \frac{1}{4} r^8 e_1 e_3\), we derive the equations

\[
u_2 t_5 - t_3 w = 0,
\]

\[
u_2 + t_3 z + (t_4 - 2w) z^2 = \frac{1}{4} r^4 (e_1 + e_2 r^2 z)^2,
\]

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the second of which implies that \( t_3^2 - 4u_2(t_4 - 2w) = 0 \). Thus we can form the resultant of \( t_3^2 - 4u_2(t_4 - 2w) \) and \( u_2t_5 - t_3w \) to eliminate \( r^2 \) and obtain a multiple of the desired area formula. The resultant is small enough to compute and factor symbolically, and we obtain

\[
\alpha' = (\beta'_6)(\beta'_6)^* \]

where

\[
\beta'_6 = \frac{\text{Res}(t_3^2 - 4u_2(t_4 - 2\sqrt{u_2t_6}), u_2t_5 - t_3\sqrt{u_2t_6}, r^2)}{4u_2^6},
\]

and \((\beta'_6)^*\) is \(\beta'_6\) with the opposite sign on \(\sqrt{u_2t_6}\).

5 Degree Calculations

In this section we show by elementary means that the homogeneous polynomials \(\alpha_n\) and \(\alpha'_n\) have degrees \(2\Delta_n\) and \(2\Delta'_n\) respectively. First we explain why the degrees cannot be smaller. In [7], Robbins shows that \(\deg(\alpha_n) \geq 2\Delta_n\) by constructing \(\Delta_n\) cyclic \(n\)-gons with generically different squared areas from a given set of edge lengths. He takes the edge lengths to be nearly equal if \(n\) is odd, and adds a much shorter edge if \(n\) is even. For semicyclic polygons, we can take the edge lengths to be nearly equal if \(n\) is even; the argument of [7] then yields the desired number \(\Delta'_n\) of semicyclic \(n\)-gons.

Suppose now that \(n\) is odd. It is not necessary (and in fact not possible) to construct \(\Delta'_n\) inequivalent semicyclic polygons with given positive real edge lengths \(a_j\). It suffices instead to construct \(\Delta_n\) configurations \((r, q_1, \ldots, q_n)\) of complex numbers satisfying

\[
a_j^2 = r^2(2 - q_j - q_j^{-1}), \quad j = 1, \ldots, n;
\]

and \(q_1 \cdots q_n = -1\), since it is from these equations, together with the relation \(16K^2 = -r^4(\sum q_j - \sum q_j^{-1})^2\), that one derives the existence and uniqueness of the irreducible polynomial \(\alpha'_n\). In our configurations \(r\) is always real and positive, but sometimes \(r < \min\{a_j/2\}\), in which case the \(q_j\) are negative real numbers instead of complex numbers of norm 1. The plan is to regard each \(q_j\) as a function of \(r\) by choosing a branch of equation (9), and then find values of \(r\) such that \(q_1 \cdots q_n = -1\).

Let \(n = 2m + 1\), let the first \(2m\) edge lengths be large and nearly equal, and let \(a_n = 2\). To find configurations with \(r > \max\{a_j/2\}\), choose arbitrarily whether \(0 < \arg q_n < \pi\) (the short edge goes “forward”) or \(-\pi < \arg q_n < 0\) (“backward”), and likewise choose a set of \(k < m\) of the long edges to
go backward. Then there exist $m - k$ semicyclic polygons with the given edge lengths and edge directions whose angle sums $\sum \arg q_j$ are $\pi, 3\pi, \ldots, (2m - 2k - 1)\pi$. (Apply the Intermediate Value Theorem to $\sum \arg q_j$ as $r$ varies from $\max\{a_j/2\}$ to $\infty$.) The total number of such configurations is

$$\sum_{k=0}^{m-1} 2^k \binom{2m}{k} (m-k) = m \binom{2m}{m}.$$ 

To find configurations with $r < \min\{a_j/2\} = 1$, choose the branch $q_j < -1$ for exactly $m$ of the long edges, and choose the branch $q_j > -1$ for the other long edges. Let $\epsilon_j = +1$ or $\epsilon_j = -1$ respectively. As $r \to 0$, the product $q_1 \cdots q_m \to \prod_{j=1}^{2m} a_j^{2\epsilon_j}$, and hence $q_1 \cdots q_m$ approaches 0 if $q_n > -1$ or $-\infty$ if $q_n < -1$. By choosing the branch for $q_n$ according to whether $q_1 \cdots q_m$, evaluated at $r = 1$, is greater or less than 1, we guarantee that $q_1 \cdots q_m = -1$ for some intermediate value of $r$. Thus we obtain another $\frac{1}{2}(2m)$ configurations. (The factor of $1/2$ is present because inverting every $q_j$ preserves the radius and the squared area; it corresponds to reversing the orientation.) The total number of configurations is therefore at least

$$(m + \frac{1}{2}) \binom{2m}{m} = \frac{n}{2} \binom{n - 1}{\lceil \frac{n-1}{2} \rceil} = \Delta'_n.$$ 

To establish matching upper bounds on the degrees of $\alpha_n$ and $\alpha'_n$, we proceed indirectly. First we revive an argument of Möbius from the 19th century [6], which produces a polynomial of degree $\Delta_n$ that relates $r^2$ for a cyclic polygon to the squared side lengths. Hence there are generically at most $\Delta_n$ circumradii for a given set of edge lengths. For generic side lengths (in particular, no two equal) and a radius $r$ that admits a solution $(q_1, \ldots, q_n)$ to the system of equations (9) and $q_1 \cdots q_m = 1$, the solution is unique up to inverting all the $q_j$. (Any other solution would differ by inverting a proper subset of the $q_j$, so those $q_j$ would need to have product $\pm 1$.) Thus, because $r$ and the $q_j$ determine the area, there are generically at most $2\Delta_n$ possible signed areas, so $\deg(\alpha_n) \leq 2\Delta_n$. The same argument applied to semicyclic polygons will yield $\deg(\alpha'_n) \leq 2\Delta'_n$.

Given a cyclic polygon with circumradius $r$ and side lengths $2y_1, \ldots, 2y_n$, let $\theta_j = \sin^{-1}(y_j/r)$ be half the angle subtended by the $j^{th}$ side. Then, for some choice of signs $\epsilon_2, \ldots, \epsilon_n$ (namely, $\epsilon_j$ is $+1$ or $-1$ according to whether the $j^{th}$ side goes “forward” or “backward” relative to the first side), the sum
\( \theta_1 + \varepsilon_2 \theta_2 + \cdots + \varepsilon_n \theta_n \) is a multiple of \( \pi \). Therefore

\[
\prod_{\varepsilon_j = \pm 1} r^n \sin(\theta_1 + \varepsilon_2 \theta_2 + \cdots + \varepsilon_n \theta_n) = 0. \tag{10}
\]

The factors of \( r \) make this a polynomial relation over \( \mathbb{Q} \) between \( r^2 \) and the squared side lengths. To see why, introduce the variables \( x_j = r \cos \theta_j = (r^2 - y_j^2)^{1/2} \) and rewrite (10) as

\[
\prod_{\varepsilon_1, \ldots, \varepsilon_n} \frac{1}{2i} \left[ \prod_{j=1}^{n} (x_j + i\varepsilon_j y_j) - \prod_{j=1}^{n} (x_j - i\varepsilon_j y_j) \right] = 0 \tag{11}
\]

where \( \varepsilon_1 = 1 \). The left-hand side of (11) has a great deal of symmetry. Obviously, flipping the sign of \( y_j \) is equivalent to negating \( \varepsilon_j \). Flipping the sign of any \( x_j \) is equivalent to flipping \( \varepsilon_j \) and negating each product over \( j \). If \( j = 1 \), we can restore the condition \( \varepsilon_1 = 1 \) by flipping every \( \varepsilon_j \) and negating every bracket. All these operations just permute and possibly negate all the \( 2^{n-1} \) bracketed factors, so they leave the overall expression unchanged. Therefore, in the expansion of (11), each \( x_j \) occurs only to even powers, and hence each \( x_j^2 \) can be replaced by \( r^2 - y_j^2 \). Likewise each \( y_j \) occurs only to even powers. Thus we obtain a polynomial equation \( M(r^2, y_1^2, \ldots, y_n^2) = 0 \).

The remaining part of Möbius’ argument uses series expansion to find the degrees of the leading and trailing terms of \( M \). Fix the \( \varepsilon_j \), and rewrite the bracketed factor of (11) as

\[
\prod_{j=1}^{n} \left( \sqrt{r^2 - y_j^2} + i\varepsilon_j y_j \right) - \prod_{j=1}^{n} \left( \sqrt{r^2 - y_j^2} - i\varepsilon_j y_j \right).
\]

To find the term of highest degree in \( r \), expand around \( r = \infty \); the highest terms cancel, so the degree is \( n - 1 \). To find the term of lowest degree, expand around \( r = 0 \) to get

\[
\prod_{j=1}^{n} i y_j \left( 1 + \varepsilon_j - \frac{r^2}{2y_j^2} - \cdots \right) - \prod_{j=1}^{n} i y_j \left( 1 - \varepsilon_j - \frac{r^2}{2y_j^2} - \cdots \right).
\]

Its initial term has degree \( \min(k, n - k) \) in \( r^2 \), where \( k \) is the number of \( \varepsilon_j \) equal to \(-1\). Therefore \( M \) is a power of \( r^2 \) times a polynomial in \( r^2 \) of degree

\[
2^{n-1} \frac{n-1}{2} - \sum_{k=0}^{n-1} \binom{n-1}{k} \min(k, n - k),
\]

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which simplifies to $\Delta_n$. We can factor out the unwanted power of $r^2$ because it was not needed to make equation (10) hold.

For semicyclic polygons, the signed sum of the $\theta_j$ is an odd multiple of $\pi/2$. Equation (10) therefore becomes

$$\prod_{\epsilon_j = \pm 1} r^n \cos(\theta_1 + \epsilon_2 \theta_2 + \cdots + \epsilon_n \theta_n) = 0,$$

which expands to a polynomial relation $M'(r^2, y_1^2, \ldots, y_n^2) = 0$. Using series expansion again, one finds that $M'$ is monic of degree $n2^{n-2}$ in $r^2$, and its lowest nonzero term has the same degree as that of $M$. Hence $M'$ is a power of $r^2$ times a polynomial whose degree in $r^2$ is $\Delta_n + 2^{n-2} = \Delta'_n$.

6 Specializations

Corollaries 5 and 6, which relate the generalized Heron polynomials $\alpha_n$ and $\alpha'_n$ to the polynomials $P_n(z)$ and $P'_n(z)$, allow us to understand and factor certain specializations of $\alpha_n$ and $\alpha'_n$. With a little extra work one can describe some of the factors explicitly. In this section we offer two such results concerning cyclic $n$-gons with $n$ odd.

Let $n = 2m + 1 \geq 5$, and consider the constant term of $\alpha_n$; that is, let $u_2 = 0$. Then $P_n(z)$ has $m - 1$ double roots if and only if either $(P_n|_{u_2=0})/z$ has $m - 1$ double roots, or $u_3 = 0$ and $(P_n|_{u_2=u_3=0})/z^2$ has $m - 2$ double roots. Geometrically, the projective variety

$$X = \{ [u_2 : \cdots : u_m : t_{m+1} : \cdots : t_{2m+1}] \mid P_n(z) \text{ factors as } (b_0 + b_1z)(c_0 + c_1z + \cdots + c_{m-1}z^{m-1})^2 \}$$

intersects the hyperplane $\{u_2 = 0\}$ in two irreducible components, one corresponding to $b_0 = 0$ and one corresponding to $c_0 = 0$. The second component has intersection multiplicity two because $X$ is tangent to $\{u_2 = 0\}$ along it.

Chasing through the geometric interpretation of $\alpha_n$ (after Corollary 5), we find that $\alpha_n|_{u_2=0}$, as a polynomial in $\sigma_1, \ldots, \sigma_n$, is an irreducible polynomial times the square of another irreducible. The factors are not necessarily irreducible as polynomials in the side lengths $a_i$, however.

**Proposition 8.** If $n$ is odd, the constant term of $\alpha_n$ factors as

$$\alpha_n|_{16K^2=0} = \gamma^2_n \prod_{a_i \pm a_2 \pm \cdots \pm a_n}$$
where the product is over all $2^{n-1}$ sign patterns.

**Proof.** Heron’s formula takes care of the case $n = 3$, so we may assume $n \geq 5$ and apply the analysis above. By Corollary 5, cyclic $n$-gons satisfy

$$P_n(z) = \left(\frac{1}{4} - r^2 z\right)\left(D(r^2 z)/z\right)^2,$$

so the factor $\gamma_n^2$ corresponds to $d_1 = 0$, and the other factor corresponds to $[\frac{1}{4} : -r^2] = [0 : 1]$ and represents projective solutions at $r^2 = \infty$. The presence of the linear factors $a_1 \pm a_2 \pm \cdots \pm a_n$ in the constant term was proved in [3], and they correspond to solutions with $r^2 = \infty$: As a signed sum of edge lengths approaches zero, the polygon can degenerate to a chain of collinear line segments, which has zero area and infinite circumradius. (One can easily construct a curve of solutions to the equations (3) tending to any such point at infinity.) For $n \geq 3$, the product of these $2^{n-1}$ linear factors is symmetric in the $a_i^2$, so by irreducibility, no other factors can appear.

The same kind of analysis applies to $\alpha_n$ when the side length $a_n$ goes to zero, and so $t_n = \sigma_n = 0$. It’s geometrically clear that the result should be divisible by $\alpha_{n-1}^2$ (as the $n^{th}$ side shrinks to zero, it can go either “forward” or “backward”), and the algebra confirms it. The intersection of $X$ with the hyperplane $t_n = 0$ includes a component of multiplicity two where

$$t_{n-1} = -\sigma_{2m} + \frac{1}{4}m^2 = 0$$

and $P_n(z)$, considered as degree $2m - 3$, has $m - 2$ double roots. Substituting the solutions $t_m = \pm 2\sqrt{\sigma_{2m}}$ back into the definitions of $t_{m+1}$ through $t_{2m-1}$, we recover the definitions of the $t_j$ for $n = 2m$ and $\epsilon = \pm 1$ (see Corollary 5), and observe that $u_m$ becomes $t_m$. Thus $P_n(z)$ specializes to $P_{n-1}(z)$, and so $\alpha_n|_{a_n=0}$ is divisible by $(\beta_{n-1})^2(\beta_n^{*})^2 = \alpha_{n-1}^2$.

The other component of $X \cap \{t_n = 0\}$ corresponds to solutions in which the leading coefficient $r^2$ of the linear factor $(\frac{1}{4} - r^2 z)$ vanishes. We can describe these solutions explicitly.

**Proposition 9.** If $n \geq 5$ and $n$ is odd, then

$$\alpha_n|_{a_n=0} = \alpha_{n-1}^2 \prod (16K^2 + (a_1^2 \pm a_2^2 \pm \cdots \pm a_{n-1}^2)^2),$$

the product taken over all sign patterns with $(n - 1)/2$ minus signs.
Proof. It suffices to show that all the factors in the product are present; the result then follows by comparing degrees (both sides being monic in $16K^2$).

Fix generic positive real values for $a_1, \ldots, a_{n-1}$ and signs $\epsilon_1, \ldots, \epsilon_{n-1} \in \{-1, +1\}$ with $\epsilon_1 = +1$ and $\sum \epsilon_j = 0$. Recall that cyclic polygons satisfy

$$q_j^2 + (a_j^2/r^2 - 2)q_j + 1 = 0, \quad 1 \leq j \leq n,$$

and $q_1q_2 \cdots q_n = 1$, and any solution to these $n + 1$ equations also satisfies $\alpha_n$ with the squared area given by equation (3). For sufficiently small positive values of $a_n$, the method of §5 shows that there exist solutions $(r^2, q_1, \ldots, q_n)$ with $r^2 < 0$ and $q_j \approx (a_j^2/r^2)^{\epsilon_j}$ for $1 \leq j < n$; furthermore $a_n^2/r^2$ tends to a constant. Equation (3) now implies

$$\lim_{a_n \to 0} 16K^2 = -\left(\sum_{\epsilon_j = +1} a_j^2 - \sum_{\epsilon_j = -1} a_j^2\right)^2,$$

so the set of solutions with $a_n^2 = r^2 = 0$ includes all points on the hypersurface $16K^2 + (\sum_{j=1}^{n-1} \epsilon_j a_j^2)^2 = 0$. \hfill \Box

7 Conclusions

The generalized Heron polynomials $\alpha_n$ for cyclic $n$-gons and $\alpha'_n$ for semicyclic $(n + 1)$-gons are defined implicitly by $n + 2$ equations in $n + 1$ unknowns: $n$ equations (3) relating the side lengths to the vertex quotients $q_1, \ldots, q_n$ and the radius $r$; equation (4) which expresses the squared area $K^2$, or equivalently $u_2 = -4K^2$, in the same way; and the equation $q_1q_2 \cdots q_n = \delta$. Our analysis eliminates the variables $q_j$ (and in the cyclic case, $r$ also) at the cost of introducing $[(n - 5)/2]$ unwanted quantities $u_3, \ldots, u_m$. The reduction in the number of auxiliary variables allows us, for small $n$, to eliminate them by ad hoc means and obtain formulas for $\alpha_n$ and $\alpha'_n$.

The quantities $u_2, \ldots, u_m$ appear on equal footing in this analysis, so we could equally well eliminate all but $u_k$ for some $k > 2$ and obtain a polynomial relation, presumably of total degree $k\Delta_n$ or $k\Delta'_n$, between $u_k$ and the squares of the side lengths. Unfortunately, we do not yet have a geometric interpretation for $u_3$ or the higher $u_k$.

For large $n$ the goal of eliminating $u_3, \ldots, u_m$ seems rather distant, but Corollaries 5 and 6 still illuminate aspects of the polynomials $\alpha_n$ and $\alpha'_n$. In particular, Corollary 5 establishes a close relationship between $\alpha_{2m+1}$ and
generalizing those between Heron’s and Brahmagupta’s formulas and between Robbins’ pentagon and hexagon formulas.

It may be of some interest to know how our main results were obtained. Robbins solved many combinatorial and algebraic problems in his lifetime by what he called the “Euler method”: calculate examples, using a computer if convenient; discover a general pattern; and prove it, with hints from further calculations if necessary. His work on generalized Heron polynomials took this approach but was somewhat frustrated by lack of data from which to generalize. As explained in [7], Robbins first found $\alpha_5$ and the closely related polynomial $\alpha_6$ by interpolating from several dozen numerical examples, and he rewrote them concisely in terms of variables $t_i$ (slightly different from ours) by interacting with a computer algebra system. Neither step is particularly feasible for $\alpha_7$, whose expansion in terms of the symmetric functions $\sigma_k$ has almost a million coefficients. It was possible, however, to evaluate certain specializations of $\alpha_7$ and $\alpha_8$ by interpolation, and to conjecture Propositions 8 and 9. For instance, we discovered that the constant term of $\alpha_7$ (the specialization $u_2 = 0$) is divisible by the square of the discriminant of

$$t_4 + t_5z + t_6z^2 + t_7z^3;$$

where the $t_i$ are defined as in Corollary 4 but with $u_2 = u_3 = 0$. Unfortunately, the hidden presence of $u_3$ made it difficult to guess the rest of $\alpha_7$.

We introduced semicyclic polygons and their area polynomials $\alpha'_n$ in an effort to obtain more data to study. In particular, we noticed that the mysterious cubic discriminant so prominent in $\alpha_5$ appeared already in the simpler polynomial $\alpha'_3$, and we hoped that whatever new phenomenon arose in $\alpha_7$ would also appear in $\alpha'_5$, which we could compute by interpolation. In fact $\alpha_7$ turns out to be rather different from $\alpha'_5$, but it was the struggle to simplify $\alpha'_5$ that led us to manipulate the relations between the $\sigma_i$ and $\tau_j$ by hand and thence to discover the main identity (Theorem 4). In the end, the most crucial calculations turned out to be those we did on the blackboard.

**Postscript**

David Robbins (1942–2003) had an exceptional ability to see and communicate the simple essence of complicated mathematical issues, and to discover elegant new results about seemingly well-understood problems. He taught and inspired a long sequence of younger mathematicians including the two
surviving authors. His interest in cyclic polygons began at age 13 when he derived a version of Heron’s formula. In the early 1990’s he discovered the area formulas for cyclic pentagons and hexagons. When diagnosed with a terminal illness in the spring of 2003, he chose to work on this topic once again. Sadly, he did not live to see the discovery of the main identity or the heptagon formula. This paper is dedicated to the memory of our friend and colleague, whose loss is keenly felt.

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