The coagulation-fragmentation hierarchy with homogeneous rates and underlying stochastic dynamics

Kenji Handa
Department of Mathematics
Saga University
Saga 840-8502
Japan
e-mail: handak@cc.saga-u.ac.jp
FAX: +81-952-28-8501

Dedicated to Professor Hiroshi Sugita on the occasion of his 60th birthday

A hierarchical system of equations is introduced to describe dynamics of ‘sizes’ of infinite clusters which coagulate and fragmentate with homogeneous rates of certain form. We prove that this system of equations is solved weakly by correlation measures for stochastic dynamics of interval partitions evolving according to some split-merge transformations. Regarding those processes, a sufficient condition for a distribution to be reversible is given. Also, an asymptotic result for properly rescaled processes is shown to obtain a solution to a nonlinear equation called the coagulation-fragmentation equation.

1 Introduction

The phenomena of coagulation and fragmentation are studied in various contexts of natural sciences. Mathematically, they are considered to be ‘dual’ to each other at least in some naive sense or to be simply the time-reversal of each other. Hence, one naturally expects that the coagulation-fragmentation dynamics may lead to a nontrivial equilibrium in the course of time provided the occurrence of coagulation and fragmentation is prescribed in an appropriate manner. Many authors have examined such situations through a nonlinear equation called often the coagulation-fragmentation equation. It takes the form

\[
\frac{\partial}{\partial t} c(t, x) = \frac{1}{2} \int_0^x [K(y, x - y)c(t, y)c(t, x - y) - F(y, x - y)c(t, x)] \, dy \\
- \int_0^{\infty} [K(x, y)c(t, x)c(t, y) - F(x, y)c(t, x + y)] \, dy,
\]

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where \( t, x > 0 \), and the functions \( K \) and \( F \) are supposed to be given, nonnegative, symmetric and depending on the mechanisms of coagulation and fragmentation, respectively. In the literature \( c(t, x) \) represents the ‘density’ of clusters of size \( x \) (or particles with mass \( x \)) at time \( t \) and the equation (1.1) is derived heuristically by some physical arguments or rigorously for some restricted cases. (Among results of the latter kind for both nonzero \( K \) and \( F \), we refer [12].) However, (1.1) is not complete for the full description of coagulation-fragmentation phenomena since it usually emerges after certain contraction procedure such as ‘propagation of chaos’ or under intuitive assumptions of asymptotic independence among distributions of clusters.

In this paper we study a hierarchical system of equations, for a special case of which we establish a direct connection with an infinite-dimensional stochastic dynamics incorporating coagulation and fragmentation. For each \( k \in \mathbb{N} := \{1, 2, \ldots\} \), the \( k \)th equation of the hierarchy reads

\[
\frac{\partial}{\partial t} c_k(t, z_1, \ldots, z_k) = -\frac{1}{2} \sum_{l=1}^{k} \int_0^{z_l} K(y, z_l - y) c_{k+1}(t, z_1, \ldots, z_{l-1}, y, z_l - y, z_{l+1}, \ldots, z_k) \, dy \\
- \frac{1}{2} \sum_{l=1}^{k} \int_0^{z_l} F(y, z_l - y) \, dy \, c_k(t, z_1, \ldots, z_k) \\
- \sum_{l=1}^{k} \int_0^{\infty} K(z_l, y) c_{k+1}(t, z_1, \ldots, z_l, y, z_{l+1}, \ldots, z_k) \, dy \\
+ \sum_{l=1}^{k} \int_0^{\infty} F(z_l, y) c_k(t, z_1, \ldots, z_{l-1}, z_l + y, z_{l+1}, \ldots, z_k) \, dy \\
- \mathbf{1}_{\{k \geq 2\}} \sum_{l<m}^{k} K(z_l, z_m) c_k(t, z_1, \ldots, z_k) \\
+ \mathbf{1}_{\{k \geq 2\}} \sum_{l<m}^{k} F(z_l, z_m) c_{k-1}(t, z_1, \ldots, z_{l-1}, z_l + z_m, z_{l+1}, \ldots, z_m-1, z_m+1, \ldots, z_k),
\]

where \( \mathbf{1}_E \) stands in general for the indicator function of a set \( E \) and the sum \( \sum_{l<m}^{k} \) is taken over pairs \( (l, m) \) of integers such that \( 1 \leq l < m \leq k \). If the last two terms on the right side of (1.2) were absent, it is readily checked that the system of equations is satisfied by the direct products \( c^{\otimes k}(t, z_1, \ldots, z_k) := c(t, z_1) \cdots c(t, z_k) \) of a solution to (1.1). The equations (1.2) are considered to be much more informative in the sense that interactions among an arbitrary number of clusters are took into account.

In fact, a finite-system version of (1.2) has been discussed for a pure coagulation model (i.e. the case \( F \equiv 0 \)) by Escobedo and Pezzotti [14]. Their derivation of (1.2) with \( F \equiv 0 \) starts from a finite set of evolution equations satisfied by the so-called mass probability functions associated with a stochastic coagulation model known as the Marcus-Lushnikov process. As pointed out in [14] the situation is similar to the derivation of the BBGKY hierarchy in classic kinetic theory although the underlying microscopic dynamics for the BBGKY hierarchy is not stochastic.
but deterministic. There is an extensive literature discussing a stochastic dynamics which serves as a basis of an infinite system called the Boltzmann hierarchy, which is a thermodynamic limit of the BBGKY hierarchy. (See the monograph of Petrina [27] and the references therein.) It should be mentioned also that a number of articles have discussed stochastic interacting systems of finite particles to derive kinetic equations, a special case of which is (1.1), in the limit as the number of particles tends to infinity. (See e.g. a paper by Eibeck and Wagner [13] and the references therein. Also, for a systematic treatment in a general framework related to such issues, see Kolokoltsov’s monograph [23].) We intend to explore the ‘coagulation-fragmentation hierarchy’ (1.2) by dealing with stochastic infinite systems directly and derive (1.1) as a macroscopic equation for them through the limit under proper rescaling. Such a limit theorem is regarded as the law of large numbers for measure-valued processes and related to the propagation of chaos. (See Remarks at the end of §4.1 below.)

In the case where the mechanisms of coagulation and fragmentation together enjoy the detailed balance condition, i.e.,

$$K(x, y)M(x)M(y) = F(x, y)M(x + y)$$ (1.3)

for some function $M$, equilibrium behaviors of the solution $c(t, x)$ to (1.1) with respect to a stationary solution of the form $x \mapsto M(x)e^{-bx}$ have been studied by many authors. In particular, for the equation with $K$ and $F$ being positive constants, Aizenman and Bak [1] carried out detailed analysis such as a uniform rate for strong convergence to equilibrium. (See also Stewart and Dubovski [31].) Lauren¸ cot and Mischler [24] studied that convergence under certain assumptions for $K, F$ and $M$ and suitable conditions on the initial state. Such results include particularly an $H$-theorem, namely the existence of a Lyapunov functional of entropy type for the solution.

In what follows, we shall restrict the discussion to the case where

$$K(x, y) = xy\hat{H}(x, y), \quad F(x, y) = (x + y)\bar{H}(x, y)$$ (1.4)

for some homogeneous functions $\hat{H}$ and $\bar{H}$ of common degree $\lambda \geq 0$, namely,

$$\hat{H}(ax, ay) = a^\lambda \hat{H}(x, y), \quad \bar{H}(ax, ay) = a^\lambda \bar{H}(x, y) \quad (a, x, y > 0).$$ (1.5)

To avoid trivialities, we suppose also that $\hat{H}$ and $\bar{H}$ are not identically zero. Therefore, both $K$ and $F$ are necessarily unbounded but of polynomial growth at most. In case $\theta\hat{H} = \bar{H}$ for a constant $\theta > 0$, (1.3) holds for $M(x) = \theta/x$.

The coagulation and fragmentation phenomena have been discussed also in the probability literature. See e.g. Bertoin’s monograph [6] for systematic accounts of stochastic models and random operations describing either phenomenon. The choice (1.4) is mainly motivated by a coagulation-fragmentation process studied by Mayer-Wolf et al [26] and Pitman [28]. These papers concern the case where $\hat{H}$ and $\bar{H}$ are constants. Having the infinite-dimensional simplex

$$\Omega_1 = \{\mathbf{x} = (x_i)_{i=1}^\infty : \ x_1 \geq x_2 \geq \ldots \geq 0, \ \sum_i x_i = 1\}$$
as its state space, the process keeps the total sum 1 of cluster sizes fixed. The special case \( \tilde{H} = \tilde{H} \equiv \text{const.} \) corresponds to the Markov process explored in [32] and [33], which is associated with ‘the simplest split-merge operator’ originally introduced by A. Vershik in the context of analysis of the infinite dimensional symmetric group. This model was studied extensively in [11] in a deep and explicit connection with a discrete analogue generated by the random transposition, for which one may refer to [30] for instance. For that discrete model, the coagulation and fragmentation rates (1.4) with both \( \tilde{H} \) and \( \tilde{H} \) being constants, naturally emerge as transition probabilities. (See (2.2) in [11].) In these works it was shown that the celebrated Poisson-Dirichlet distribution with parameter \( \theta \) is a reversible distribution of the process, and much efforts were made to prove the uniqueness of a stationary distribution. In particular, Diaconis et al [11] succeeded in proving it for \( \theta = 1 \) by giving an effective coupling result with the discrete coagulation-fragmentation processes. (See also Theorem 1.2 in [30] and Theorem 7.1 in [18].) We also refer the reader to [5] for another result of interest on a unique stationary distribution for the model evolving with a different class of coagulation-fragmentations. The coagulation and fragmentation we will be concerned with are only binary ones. Cepeda [9] constructed stochastic models incorporating coagulation and multi-fragmentation on a larger state space than that of our models (i.e., \( \Omega \) defined below). See Introduction and the references in [9] for previous works and development in the study of related stochastic models.

By virtue of the homogeneity assumption on \( \tilde{H} \) and \( \tilde{H} \) we can consider the generalized process associated with (1.4) not only on \( \Omega_1 \) but also on the infinite-dimensional cone

\[
\Omega = \{ \mathbf{z} = (z_i)_{i=1}^\infty : z_1 \geq z_2 \geq \ldots \geq 0, \ 0 < \sum_i z_i < \infty \}.
\]

In fact, the major arguments in this paper exploit some ingredients from theory of point processes. The idea is that each \( \mathbf{z} = (z_i) \in \Omega \) can be identified with the locally finite point-configuration

\[
\xi = \sum_i 1_{\{z_i > 0\}} \delta_{z_i}
\]

on the interval \((0, \infty)\), where \(\delta_{z_i}\) is the delta distribution concentrated at \(z_i\). Indeed, in one of our main results, the notion of correlation measures will make us possible to reveal an exact connection between hierarchical equations (1.2) and the coagulation-fragmentation process with rates (1.4). As another result based on the point process calculus we will present a class of coagulation-fragmentation processes having the Poisson-Dirichlet distributions or certain variants (including the laws of gamma point processes) as their reversible distributions, clarifying what mathematical structures are responsible for this result. That structure will be described in terms of correlation functions together with Palm distributions, certain conditional laws for the point process. (See (2.14) and (2.15) below.) We mention also that the reversibility will play some key roles in discussing the existence of strong solutions to (1.2). As for the original equation (1.1), introducing rescaled models which depend on the scaling parameter \(N\), one can discuss its derivation from the associated measure-valued processes in which each point is assigned mass \(1/N\). Such a result is
formulated as a limit theorem for Markov processes as $N \to \infty$ and one of the key steps is to show the tightness of their laws, which is far from trivial because there is less restriction on grows of order of $K$ and $F$. As will turn out later our setting of the degrees of homogeneity plays an essential role to overcome difficulties of this sort.

The organization of this paper is as follows. In Section 2, we introduce the coagulation-fragmentation process associated with rates (1.4) and give an equivalent description of the model in terms of the corresponding point process. Section 3 discusses a weak version of (1.2), which will turn out to be satisfied by the correlation measures of our coagulation-fragmentation process. After some preliminary arguments are made for rescaled models in Section 4, a solution to (1.1) will be obtained from properly rescaled empirical measures in Section 5.

2 The coagulation-fragmentation process associated with split-merge transformations

2.1 Definition of the models

As mentioned in Introduction the rates $K$ and $F$ are supposed to be of the form (1.4) with $\tilde{H}$ and $\hat{H}$ being homogeneous functions of degree $\lambda \geq 0$ throughout. Notice that this is equivalent to the condition that $K$ and $F$ are homogeneous functions of degree $\lambda + 2$ and $\lambda + 1$, respectively, for some $\lambda \geq 0$. As far as coagulation rates are concerned, the homogeneity, though mathematically a strong condition, is satisfied typically by examples of kernels used in the physical literature as seen in Table 1 in [2] (although our framework excludes any of such examples). See also [17] and [8], which discuss the equation with homogeneous(-like) $K$. In the rest, the following two conditions are also imposed without mentioning:

(H1) $\tilde{H}$ is a symmetric, nonnegative measurable function on $(0, \infty)^2$ such that

$$\tilde{C} := \sup \{ \tilde{H}(u, 1 - u) : 0 < u < 1 \} \in (0, \infty).$$

(H2) $\hat{H}$ is a symmetric, nonnegative measurable function on $(0, \infty)^2$ such that

$$\hat{C} := \int_0^1 \hat{H}(u, 1 - u) du \in (0, \infty).$$

In general, a homogeneous function $H$ on $(0, \infty)^2$ is determined by its degree $\lambda$ and the function $h(u) := H(u, 1 - u)$ on $(0, 1)$ through the relation $H(x, y) = (x + y)^\lambda h(x + y)$. As for the fragmentation rate, the homogeneity (1.5) combined with (1.4) implies that the overall rate of fragmentation of an $x$-sized cluster is necessarily given by the power-law form:

$$\frac{1}{2} \int_0^x F(y, x - y) dy = \frac{1}{2} x \int_0^1 \hat{H}(ux, (1 - u)x) du = \frac{\hat{C}}{2} x^{2+\lambda}.$$

Such a situation is featured by the coagulation-fragmentation equation studied in [4], whose conditions for coagulation rates are also well adapted to our setting. (See
An example of discontinuous homogeneous function of degree (H2)) is satisfied if 0 ≤ a < λ (resp. a > −1).

Remark at the end of Section 5 for related discussions.) Also, [34] examined the interplay between degrees of coagulation and fragmentation in the context of stability analysis. At the beginning of Section 4 we will mention another role of such interplay between them in the study of rescaled processes.

Examples. (i) Consider $H(x, y) = (xy)^a(x^b + y^b)^c$, for which $\lambda = 2a + bc$. Without loss of generality, we can suppose that $b \geq 0$. Then $H$ satisfies (H1) (resp. (H2)) if $a \geq 0$ (resp. $a > -1$).

(ii) Let $b > 0$ and define $H(x, y) = (xy)^a|x^b - y^b|^c$, for which $\lambda = 2a + bc$. $H$ satisfies (H1) (resp. (H2)) if $a \geq 0$ and $c \geq 0$ (resp. $a > -1$ and $bc > -1$). Indeed, it is readily observed that $H(u, 1 - u)/u^a \rightarrow 1(u \downarrow 0)$ and $H(u, 1 - u)/|2u - 1|^{bc} \rightarrow b2^{-(2a+bc)}(u \rightarrow 1/2)$.

(iii) Given $\lambda \geq 0$, let $H(x, y) = (x^a + y^a)(x^{\lambda - a} + y^{\lambda - a})$. It follows that (H1) (resp. (H2)) is satisfied if $0 \leq a \leq \lambda$ (resp. $-1 < a < \lambda + 1$).

(iv) An example of discontinuous homogeneous function of degree $\lambda$ is $H(x, y) = x^{\lambda}1_{\{x \geq ay\}} + y^{\lambda}1_{\{y \geq ax\}}$, where $a > 0$. Another one is $H(x, y) = (xy)^{\lambda/2}1_{\{a, 1/a\}}(x/y)$ with $0 < a < 1$. For each example and any $\lambda \geq 0$, both (H1) and (H2) hold.

In order to define our coagulation-fragmentation process as a continuous-time Markov process on $\Omega$, suitable modifications to the formulation in [26] are made in the following manner. Let $\Omega$ be equipped with the product topology and $B(\Omega)$ (resp. $B(\Omega_1)$) be the Banach space of bounded Borel functions on $\Omega$ (resp. $\Omega_1$) with the sup norm $\| \cdot \|_\infty$. For $z = (z_i) \in \Omega$ put $|z| = \sum z_i$. A useful inequality is $\sum z_i^{1+a} \leq |z|^{1+a}$ for any $a \geq 0$, which is implied by $\sum(z_i/|z|)^{1+a} \leq \sum(z_i/|z|) = 1$. Define a bounded linear operator $\tilde{L}$ on $B(\Omega)$ by

$$
\tilde{L}\Phi(z) = \frac{1}{2} \sum_{i \neq j} K(z_i, z_j) (\Phi(M_{ij}z) - \Phi(z)) + \frac{1}{2} \sum_{i} \int_{0}^{z_i} dyF(y, z_i - y) \left( \Phi(S_{ij}(y)z) - \Phi(z) \right),
$$

(2.1)

where $M_{ij}z \in \Omega$ (resp. $S_{ij}(y)z \in \Omega$) is obtained from a sequence $z = (z_i)$ by merging $z_i$ and $z_j$ into $z_i + z_j$ (resp. by splitting $z_i$ into $y$ and $z_i - y$) and then by reordering. Noting that $M_{ij}z = z$ (resp. $S_{ij}(y)z = z$) whenever $z_i z_j = 0$ (resp. $z_i = 0$), we regard the sum $\sum_{i \neq j}$ (resp. $\sum_{i}$) in (2.1) as the sum taken over $i \neq j$ (resp $i$) such that $z_i z_j > 0$ (resp. $z_i > 0$). We adopt such convention for the same kind of expression throughout. The boundedness of $L$ is seen easily from (H1) and (H2) in view of alternative expression for (2.1)

$$
\tilde{L}\Phi(z) = \frac{1}{2} \sum_{i \neq j} \frac{z_i \cdot z_j}{|z|} \left( z_i + z_j \right)^\lambda \tilde{H} \left( \frac{z_i}{z_i + z_j}, \frac{z_j}{z_i + z_j} \right) (\Phi(M_{ij}z) - \Phi(z)) + \frac{1}{2} \sum_{i} \left( \frac{z_i}{|z|} \right)^2 \left( \frac{z_i}{|z|} \right) \int_{0}^{1} du\tilde{H}(u, 1 - u) \left( \Phi(S_{ij}(u)z) - \Phi(z) \right),
$$

(2.2)

where $S_{ij}(u)z := S_{ij}(az_i)z$. Indeed, it follows that $\|\tilde{L}\Phi\|_\infty \leq (\tilde{C} \vee \tilde{C})\|\Phi\|_\infty$. Here and in what follows we use the notation $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. The
standard argument (e.g., §2 of Chapter 4 in [15]) shows that \( \tilde{L} \) generates a continuous-time Markov jump process \( \{ \tilde{Z}(t) = (\tilde{Z}_i(t))_{i=1}^\infty : t \geq 0 \} \), say, on \( \Omega \). It is obvious that \( |\tilde{Z}(t)| = |\tilde{Z}(0)| \) for all \( t \geq 0 \) a.s. Similarly, \( L_1 \), the restriction of \( \tilde{L} \) on \( B(\Omega_1) \), namely,

\[
L_1 \Phi(x) = \frac{1}{2} \sum_{i \neq j} K(x_i, x_j) (\Phi(M_{ij}x) - \Phi(x)) + \frac{1}{2} \sum_{i} \int_0^{x_i} dyF(y, x_i - y) \left( \Phi(S_i^{\gamma}(x)) - \Phi(x) \right)
\]

\[
= \frac{1}{2} \sum_{i \neq j} x_i x_j \tilde{H}(x_i, x_j) (\Phi(M_{ij}x) - \Phi(x)) + \frac{1}{2} \sum_{i} x_i^{2+\lambda} \int_0^1 du \tilde{H}(u, 1 - u) (\Phi(S_i^\alpha x) - \Phi(x))
\]

(2.3)
generates a continuous-time Markov process \( \{ X(t) = (X_i(t))_{i=1}^\infty : t \geq 0 \} \) on \( \Omega_1 \). In the case where \( \tilde{H} \equiv 1 \) and \( \lambda = 0 \), (2.3) can be thought of as the generator of continuous-time version of a Markov chain studied in [26]. Moreover, the operator (2.3) is a special case (more specifically, the binary fragmentation case) of the generator considered in [9], although in order for the model to be defined also on \( \Omega \) we need homogeneity of \( K \), whereas in [9] certain continuity of \( K \) is imposed.

For each \( a > 0 \), define the dilation map \( D_a : \Omega \to \Omega \) by \( D_a(z) = az := (az_i) \). The relationship between \( Z(t) \) and \( X(t) \) described in the following lemma is fundamental.

**Lemma 2.1** (i) Suppose that a process \( \{ \tilde{Z}(t) : t \geq 0 \} \) generated by \( \tilde{L} \) is given. Then the \( \Omega_1 \)-valued process \( \{ X(t) = (X_i(t))_{i=1}^\infty : t \geq 0 \} \) defined by \( X_i(t) = \tilde{Z}_i(t)/|\tilde{Z}(0)| = \tilde{Z}_i(t)/|\tilde{Z}(t)| \) is a process generated by \( L_1 \).

(ii) Suppose that a \( (0, \infty) \)-valued random variable \( V \) and a process \( \{ (X_i(t))_{i=1}^\infty : t \geq 0 \} \) generated by \( L_1 \) are mutually independent. Then the \( \Omega \)-valued process \( \{ Z(t) = (Z_i(t))_{i=1}^\infty : t \geq 0 \} \) defined by \( Z_i(t) = VX_i(t) \) is a process generated by \( L \).

**Proof.** (i) Take \( \Phi \in B(\Omega) \) arbitrarily and denote by \( E[\cdot | Z(0) = z] \) the expectation with respect to the process generated by \( L \) starting from \( z \in \Omega \). Since \( v^{-1}(M_{ij}z) = M_{ij}v^{-1}z \) and \( |z|^{-1}(S_i^{\alpha z_i}z) = S_i^\alpha(|z|^{-1}z) \), we see from (2.2)

\[
\tilde{L}(\Phi \circ D_{1/v})(z) = (L_1 \Phi)(v^{-1}z), \quad z \in \Omega, v = |z|.
\]

Hence, for any \( t > 0 \), by Fubini’s theorem,

\[
E[\Phi(X(t))] - E[\Phi(X(0))]
= E[\Phi(|\tilde{Z}(0)|^{-1}\tilde{Z}(t))] - E[\Phi(|\tilde{Z}(0)|^{-1}\tilde{Z}(0))]
= \int_\Omega P(\tilde{Z}(0) \in dz) \left\{ E[(\Phi \circ D_{1/|z|})(\tilde{Z}(t))|\tilde{Z}(0) = z] - (\Phi \circ D_{1/|z|})(z) \right\}
= \int_\Omega P(\tilde{Z}(0) \in dz) \int_0^t ds E[\tilde{L}(\Phi \circ D_{1/|z|})(\tilde{Z}(s))|\tilde{Z}(0) = z]
= \int_0^t ds \int_\Omega P(\tilde{Z}(0) \in dz) E[L_1 \Phi(|z|^{-1}\tilde{Z}(s))|\tilde{Z}(0) = z]
= \int_0^t ds E[L_1 \Phi(|\tilde{Z}(0)|^{-1}\tilde{Z}(s))] = \int_0^t ds E[L_1 \Phi(X(s))].
\]

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This proves the first assertion.

(ii) Based on the relation $L_1(\Phi \circ D_v)(x) = \tilde{L}\Phi(vx)$ for $x \in \Omega_1$ and $v > 0$, the proof of the second assertion is very similar to that for (i). So we omit it. ■

We call $V$ in Lemma 2.1 a lifting variable. Roughly speaking, lifting a process on $\Omega_1$ generated by $L_1$ yields a process on $\Omega$ generated by $\tilde{L}$. We need to consider an unbounded operator $L\Phi(z) = |z|^{2+\lambda}\tilde{L}\Phi(z)$ or

$$L\Phi(z) = \frac{1}{2} \sum_{i \neq j} K(z_i, z_j) (\Phi(M_{ij}z) - \Phi(z)) + \frac{1}{2} \sum_i \int_{\Omega} dy F(y, z_i - y) \left( \Phi(S_i(y)z) - \Phi(z) \right).$$

The corresponding process $\{Z(t) : t \geq 0\}$ on $\Omega$ can be obtained from a process $\{\tilde{Z}(t) : t \geq 0\}$ generated by $\tilde{L}$ with the same initial law by a random time-change $Z(t) := \tilde{Z}(\|\tilde{Z}(0)\|^{2+\lambda} t)$.

This can be shown by general theory of Markov processes, e.g., Theorem 1.3 in Chapter 6 of [15], or more directly, by making the following observation: for any $\Phi \in B(\Omega)$ such that $L\Phi \in B(\Omega)$, by the optional sampling theorem

$$\Phi(Z(t)) - \Phi(Z(0)) - \int_0^t du L\Phi(Z(u))$$
$$= \Phi(\tilde{Z}(\|\tilde{Z}(0)\|^{2+\lambda} t)) - \Phi(\tilde{Z}(0)) - \int_0^t du \|\tilde{Z}(0)\|^{2+\lambda}\tilde{L}\Phi(\tilde{Z}(\|\tilde{Z}(0)\|^{2+\lambda} u))$$
$$= \Phi(\tilde{Z}(\|\tilde{Z}(0)\|^{2+\lambda} t)) - \Phi(\tilde{Z}(0)) - \int_0^{\|\tilde{Z}(0)\|^{2+\lambda} t} ds \tilde{L}\Phi(\tilde{Z}(s))$$

is a martingale.

2.2 Reformulation in terms of point processes

We proceed to reformulate the above-mentioned processes as Markov processes taking values in a space of point-configurations on $(0, \infty)$. To discuss in the setting of point processes, we need the following notation. Set $Z_+ = \{0, 1, 2, \ldots\}$ and let $\mathcal{N}$ be the set of $Z_+$-valued Radon measures on $(0, \infty)$. Each element $\eta$ of $\mathcal{N}$ is regarded as a counting measure associated with a locally finite point-configuration on $(0, \infty)$ with multiplicity. We equip $\mathcal{N}$ with the vague topology and use the notation $|\eta| := \int_{(0, \infty)} \nu(\eta dv)$ for $\eta \in \mathcal{N}$ and $\langle f, \nu \rangle := \int_{(0, \infty)} f(v) \nu(\nu dv)$ for a measure $\nu$ and a Borel function $f$ on $(0, \infty)$. Denote by $B_+(S)$ the set of nonnegative bounded Borel measurable functions on a topological space $S$. For simplicity, we set $B_+ = B_+((0, \infty))$ and use the notation $B_+^k$ instead of $B_+((0, \infty)^k)$ for $k = 2, 3, \ldots$. As mentioned roughly in Introduction, the subsequent argument is based on the one-to-one correspondence between $z = (z_i) \in \Omega$ and

$$\Xi(z) := \sum_i 1_{\{z_i > 0\}} \delta_{z_i} \in \{\eta \in \mathcal{N} : |\eta| < \infty\} =: \mathcal{N}_1.$$
Clearly, if $\eta = \Xi(z)$, then $|\eta| = |z|$ and $\eta([\epsilon, \infty)) \leq |z|/\epsilon$ for $\epsilon > 0$. Note that the map $\Xi : \Omega \rightarrow \mathcal{N}_1$ is bi-measurable. It follows that

$$\Xi(M_{ij}z) - \Xi(z) = \delta_{z_i+z_j} - \delta_{z_i} - \delta_{z_j} \quad \text{if} \quad z_i, z_j > 0, \quad (2.5)$$

and

$$\Xi(S^{(y)} z) - \Xi(z) = \delta_y + \delta_{z-y} - \delta_{z_i} \quad \text{if} \quad z_i > y > 0. \quad (2.6)$$

Thus, employing $\Xi(z)$ rather than $z$ itself enables us to avoid an unnecessary complication arising from reordering of the sequence.

Besides, owing to the map $\Xi$, the arguments below make use of some effective tools in theory of point processes, such as correlation measures and (reduced) Palm distributions. (See e.g. §13.1 of [10] for general accounts.) We shall give their definitions for a locally finite point process $\xi = \sum \delta_{z_i}$ on $(0, \infty)$. In what follows, the domain of integration will be suppressed as long as it is $(0, \infty)^k$ for some $k \in \mathbb{N}$, which should be clear from context. The first correlation measure $q_1$ is the mean measure of $\xi$, and for $k = 2, 3, \ldots$ the $k$th correlation measure $q_k$ is the mean measure of the modified product counting measure

$$\xi[k] := \sum_{i_1, \ldots, i_k(\neq)} \delta_{(z_{i_1}, \ldots, z_{i_k})} \quad (2.7)$$

on $(0, \infty)^k$, where $\sum_{i_1, \ldots, i_k(\neq)}$ indicates that the sum is taken over $k$-tuplets $(i_1, \ldots, i_k)$ such that $i_l \neq i_m$ whenever $l \neq m$. The entire system $\{q_1, q_2, \ldots\}$ of correlation measures determines uniquely the law of $\xi$. (The identity (2.13) below is the structure underlying this fact.) The density of $q_k$ is called the $k$th correlation function of $\xi$ if it exists. Furthermore, letting $\{P_{z_1, \ldots, z_k} : z_1, \ldots, z_k \in (0, \infty)\}$ be a family of Borel probability measures on $\mathcal{N}$, we call $P_{z_1, \ldots, z_k}$ the $k$th-order reduced Palm distribution of $\xi$ at $(z_1, \ldots, z_k)$ if for any $G \in B_+((0, \infty)^k \times \mathcal{N})$

$$E \left[ \sum_{i_1, \ldots, i_k(\neq)} G(Z_{i_1}, \ldots, Z_{i_k}, \xi) \right] = E \left[ \int \xi[k](dz_1 \cdots dz_k)G(z_1, \ldots, z_k, \xi) \right]$$

$$= \int q_k(dz_1 \cdots dz_k)E_{z_1, \ldots, z_k}(G(z_1, \ldots, z_k, \eta + \delta_{z_1} + \cdots + \delta_{z_k})]. \quad (2.8)$$

Here and throughout, $E_{z_1, \ldots, z_k}$ denotes the expectation with respect to $P_{z_1, \ldots, z_k}$, so that

$$E_{z_1, \ldots, z_k}[\Psi(\eta)] = \int_{\mathcal{N}} \Psi(\eta)P_{z_1, \ldots, z_k}(d\eta), \quad \Psi \in B_+(\mathcal{N}).$$

(Rigorously speaking, certain measurability of $P_{z_1, \ldots, z_k}$ in $(z_1, \ldots, z_k)$ is required just as in the definition of regular conditional distributions. See Proposition 13.1.IV of [10] for the case $k = 1$.) An intuitive interpretation for $P_{z_1, \ldots, z_k}$ is the conditional law of $\xi - \delta_{z_1} - \cdots - \delta_{z_k}$ given $\xi(\{z_1\}) \cdots \xi(\{z_k\}) > 0$. We call an equality of the type (2.8) the $k$th-order Plam formula for $\xi$.

To state the main result of this section, let us recall some known results on Poisson-Dirichlet point process, which is by definition the point-configuration associated with
Lemma 2.2

Let \( \mathcal{L} \) be the totality of bounded Borel functions on \((0, \infty)\) with compact support and let \( \mathcal{B}_+ = \mathcal{B}_+ \cap \mathcal{L} \). The support and the sup norm of a function \( f \) are denoted by \( \text{supp}(f) \) and \( \|f\|_\infty \), respectively. We now give a class of functions on \( \Omega \) for which our generators act in a tractable manner. Set

\[
\mathcal{B}_+ = \{ \phi \in \mathcal{B}_+ : \phi - 1 \in \mathcal{L}, \|\phi - 1\|_\infty < 1 \}.
\]

For each \( \phi \in \mathcal{B}_+ \) we can define a function \( \Pi_\phi \) on \( \Omega \) by \( \Pi_\phi(z) = \Pi_{i:z_i>0} \phi(z_i) \), noting that the right side is a finite product. By abuse of notation, we also write \( \Pi_\phi(\xi) \) for \( \Pi_\phi(z) \) when \( \xi = \Xi(z) \). Thus

\[
\Pi_\phi(\xi) = \exp \left( \sum_i 1_{\{z_i>0\}} \log \phi(z_i) \right) = e^{\log \phi(\xi)}.
\]

An important remark is that the class \( \{\Pi_\phi : \phi \in \mathcal{B}_+\} \) is measure-determining on \( \Omega \) because it includes all functions of the form \( z \mapsto \exp(-\langle f, \Xi(z) \rangle) \) with \( f \in \mathcal{B}_+ \).

Given a function on \( \Omega \), we regard it also as a function on \( \Omega_{\leq R} := \{z \in \Omega : |z| \leq R\} \) for any \( R > 0 \). It is clear that \( L \) restricted on \( B(\Omega_{\leq R}) \) is bounded.

Lemma 2.2

Let \( \phi \in \mathcal{B}_+ \) and set \( f = \phi - 1 \). Then for any \( z \in \Omega \)

\[
0 \leq \Pi_\phi(z) \leq (1 - \|f\|_\infty)^{-\xi(\text{supp}(f))},
\]

where \( \xi = \Xi(z) \). Moreover, \( \Pi_\phi \in B(\Omega_{\leq R}) \) for any \( R > 0 \) and

\[
L \Pi_\phi(z) = \frac{1}{2} \sum_{i \neq j} K(z_i, z_j) [\phi(z_i + z_j) - \phi(z_i)\phi(z_j)] \prod_{k \neq i, j} \phi(z_k)
\]

(2.12)
\[
\frac{1}{2} \sum_{i} \int_{0}^{z_{i}} dy F(y, z_{i} - y) \left[ \phi(y) \phi(z_{i} - y) - \phi(z_{i}) \right] \prod_{k \neq i} \phi(z_{k}) \\
= \frac{1}{2} \int \xi[2](dv_{1}dv_{2}) K(v_{1}, v_{2}) \left[ \phi(v_{1} + v_{2}) - \phi(v_{1}) \phi(v_{2}) \right] \Pi_{\phi}(\xi - \delta_{v_{1}} - \delta_{v_{2}}) \\
+ \frac{1}{2} \int \xi(dv) \int_{0}^{v} dy F(y, v - y) \left[ \phi(y) \phi(v - y) - \phi(v) \right] \Pi_{\phi}(\xi - \delta_{v}).
\]

Proof. Let \(\phi(0) = 1\) and \(f(0) = 0\) by convention. To prove (2.11) observe that
\[
\prod_{i} \phi(z_{i}) = \prod_{i} (1 + f(z_{i})) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i_{1}, \ldots, i_{k} \neq i} f(z_{i_{1}}) \cdots f(z_{i_{k}}). \tag{2.13}
\]
So, using the notation \(\binom{\alpha}{k} = \alpha(\alpha - 1) \cdots (\alpha - k + 1)/k!\), we get
\[
0 \leq \Pi_{\phi}(z) \leq 1 + \sum_{k=1}^{\infty} \frac{\|f\|_{k}^{k}}{k!} \sum_{i_{1}, \ldots, i_{k} \neq i} 1_{\{z_{i_{1}}, \ldots, z_{i_{k}} \in \text{supp}(f)\}}
= 1 + \sum_{k=1}^{\infty} \|f\|_{\infty}^{k} \binom{\xi(\text{supp}(f))}{k}
= (1 - \|f\|_{\infty})^{-\xi(\text{supp}(f))},
\]
and thus (2.11) follows. Putting \(\epsilon = \inf \text{supp}(f) > 0\), we get
\[
\xi(\text{supp}(f)) \leq \xi([\epsilon, \infty)) \leq \epsilon^{-1} |z|.
\]
This combined with (2.11) implies that \(\Pi_{\phi}\) is bounded on \(\Omega_{\leq R}\). (2.12) is verified by direct calculations with the help of (2.5) and (2.6).

2.3 Reversible cases

We demonstrate the power of point process calculus by proving an extension of the reversibility result due to Mayer-Wolf et al [26]. (As for the stationarity result, a proof based on the underlying Poisson process can be found in §7.3 of [18].) To this end, we recall that the \(k\)th correlation function \(q_{k}\) of the PD(\(\theta\)) process \(\xi(\theta)\) is given in (2.9) and that the \(k\)th-order reduced Palm distribution \(P_{x_{1}, \ldots, x_{k}}\) of \(\xi(\theta)\) at \((x_{1}, \ldots, x_{k}) \in \Delta_{k}^{\theta}\) is the law of \(\xi(\theta) \circ D_{1-x_{1}, \ldots, -x_{k}}^{-1}\). In particular, for any \((x_{1}, x_{2}) \in \Delta_{2}^{\theta}\)
\[
x_{1}x_{2}q_{2}(x_{1}, x_{2}) = \theta(x_{1} + x_{2})q_{1}(x_{1} + x_{2}) \tag{2.14}
\]
and
\[
P_{x_{1}, x_{2}} = P_{x_{1} + x_{2}}. \tag{2.15}
\]
As will be shown in the next theorem, these identities are responsible for the reversibility of PD(\(\theta\)) with respect to processes associated with bounded operators on
for any $\Phi$

\[(i) \quad \text{Since} \quad \text{Proof.}\]

$L$ the process generated by $Q$ suffices to verify for functions $\Phi = \Pi$ which are regarded as a sort of coagulation-fragmentation duality. Furthermore, it is any nonzero bounded, symmetric nonnegative function on $\{(x, y) \mid x, y > 0, x + y \leq 1\}$. (It should be noted that Theorem 12 in [26] proved essentially the symmetry of $L_1^{(Q, \theta)}$ with $Q \equiv \text{const.}$ with respect to $\text{PD}(\theta)$.) More generally, we consider

\[L_1^x \Phi(x) = \frac{1}{2} \sum_{i \neq j} K_1(x_i, x_j) (\Phi(M_{ij}x) - \Phi(x)) + \frac{\theta}{2} \sum_i x_i^2 \int_0^1 du Q(u x_i, (1 - u)x_i) (\Phi(S_i^u x) - \Phi(x)),\]

where $Q$ is any nonzero bounded, symmetric nonnegative function on $\{(x, y) \mid x, y > 0, x + y \leq 1\}$. (It should be noted that Theorem 12 in [26] proved essentially the symmetry of $L_1^{(Q, \theta)}$ with $Q \equiv \text{const.}$ with respect to $\text{PD}(\theta)$.) More generally, we consider

\[L_1^x \Phi(x) = \frac{1}{2} \sum_{i \neq j} K_1(x_i, x_j) (\Phi(M_{ij}x) - \Phi(x)) + \frac{1}{2} \sum_i x_i \int_0^1 du F_1(u x_i, (1 - u)x_i) (\Phi(S_i^u x) - \Phi(x)) \quad (2.16)\]

with $K_1$ and $F_1$ being symmetric nonnegative functions on $\{(x, y) \mid x, y > 0, x + y \leq 1\}$ such that $(x_i) \mapsto \sum_{i \neq j} K_1(x_i, x_j) 1_{\{x_i, x_j > 0\}}$ and $(x_i) \mapsto \sum x_i \int_0^1 du F_1(u x_i, (1 - u)x_i)$ are bounded functions on $\Omega_1$. We may and do suppose that $K_1(x, y) = 0$ whenever $xy = 0$.

**Theorem 2.3** (i) Let $X = (X_i)_{i=1}^\infty$ be a random element of $\Omega_1$ and suppose that the first and second correlation functions $q_1$ and $q_2$ of $\xi := \Xi(X)$ exist and satisfy

\[K_1(x_1, x_2)q_2(x_1, x_2)P_{x_1, x_2} = F_1(x_1, x_2)q_1(x_1 + x_2)P_{x_1 + x_2}, \quad (2.17)\]

a.e. $(x_1, x_2) \in \Delta_2^\circ$. Here, $P_{x_1, x_2}$ and $P_{x_1 + x_2}$ are the reduced Palm distributions of $\xi$ and the above equality is understood as the one between two measures on $\mathcal{N}$. Then the process generated by $L_1^x$ is reversible with respect to the law of $X$.

(ii) $\text{PD}(\theta)$ is a reversible distribution of the process generated by $L_1^{(Q, \theta)}$.

**Proof.** (i) Since $L_1^x$ is bounded, we only have to check the symmetry

\[E [\Phi(X)L_1^x \Psi(X)] = E [\Psi(X)L_1^x \Phi(X)]\]

for any $\Phi, \Psi \in B(\Omega_1)$. We will prove stronger equalities

\[\langle \langle \Phi, \Psi \rangle \rangle_{\text{coag}} := E \left[ \Phi(X) \sum_{i \neq j} K_1(X_i, X_j) \Psi(M_{ij}X) \right] = E \left[ \sum_i X_i \int_0^1 du F_1(u X_i, (1 - u)X_i) \Phi(S_i^u X) \Psi(X) \right] =: \langle \langle \Phi, \Psi \rangle \rangle,\]

which are regarded as a sort of coagulation-fragmentation duality. Furthermore, it suffices to verify for functions $\Phi = \Pi_\phi$ and $\Psi = \Pi_\psi$ with $\phi, \psi \in \bar{B}_+$. Thanks to the
first-order and second-order Palm formulae for $\xi = \sum 1_{\{X_i > 0\}}\delta_{X_i}$, similar calculations to (2.12) show that

$$\langle \langle \Phi, \Psi \rangle \rangle_{\text{coag}} = E \left[ \sum_{i \neq j} K_1(X_i, X_j)\phi(X_i)\phi(X_j)\psi(X_i + X_j) \prod_{k \neq i,j} (\phi(X_k)\psi(X_k)) \right]$$

$$= E \left[ \int \xi(x_1, x_2)K_1(x_1, x_2)\phi(x_1)\phi(x_2)\psi(x_1 + x_2)\Pi_{\phi\psi}(\xi - \delta_{x_1} - \delta_{x_2}) \right]$$

$$= \int \Delta_2 q_2(x_1, x_2)K_1(x_1, x_2)\phi(x_1)\phi(x_2)\psi(x_1 + x_2)E_{x_1, x_2}[\Pi_{\phi\psi}(\eta)]\,dx_1\,dx_2$$

and

$$\langle \langle \Phi, \Psi \rangle \rangle_{\text{frag}} = E \left[ \sum_i X_i \int_0^1 du F_1(uX_i, (1 - u)X_i)\phi(uX_i)\phi((1 - u)X_i)\psi(X_i) \prod_{k \neq i} (\phi(X_k)\psi(X_k)) \right]$$

$$= E \left[ \int \xi(\eta)v \int_0^1 du F_1(uv, (1 - u)v)\phi(uv)\phi((1 - u)v)\psi(v)\Pi_{\phi\psi}(\xi - \delta_{v}) \right]$$

$$= \int_0^1 q_1(v)\int_0^1 F_1(uv, (1 - u)v)\phi(uv)\phi((1 - u)v)\psi(v)E_v[\Pi_{\phi\psi}(\eta)]\,du\,dv.$$

By virtue of (2.17) we obtain the desired equality $\langle \langle \Phi, \Psi \rangle \rangle_{\text{coag}} = \langle \langle \Phi, \Psi \rangle \rangle_{\text{frag}}$ after the change of variables $uv = x_1, (1 - u)v = x_2$

(ii) This assertion is immediate by noting that (2.17) with

$$K_1(x_1, x_2) = x_1x_2Q(x_1, x_2) \quad \text{and} \quad F_1(x_1, x_2) = \theta(x_1 + x_2)Q(x_1, x_2)$$

is valid for the PD($\theta$) process because of (2.14) and (2.15). The proof of Theorem 2.3 is complete.

Remarks. (i) It would be interesting to investigate the class of (nonnegative unbounded) functions $Q$ for which the operator $L_{1}^{Q,\theta}$ defines a Markov process on $\Omega$. For example, if taking $Q(x, y) = (xy)^{-1}$ is allowed in that sense, the operator

$$L_{1}^{Q,\theta}\Phi(x) = \frac{1}{2} \sum_{i \neq j} (\Phi(M_{ij}x) - \Phi(x)) + \frac{\theta}{2} \sum_{i} \int_0^1 \frac{du}{u(1 - u)} (\Phi(S_{i}^{u}x) - \Phi(x))$$

could deserve further exploration in connection with e.g. ‘asymptotic frequency’ of some exchangeable fragmentation-coalescence process studied in [5]. It is pointed out that at least the unbounded coagulation operator in the above can be treated within the fame work of [9].

(ii) Alternative direction of generalization of the processes reversible with respect to PD($\theta$) is based on an obvious generalization of (2.14) and (2.15), i.e.,

$$x_1 \cdots x_{k+1}q_{k+1}(x_1, \ldots, x_{k+1}) = \theta^k(x_1 + \cdots + x_{k+1})q_1(x_1 + \cdots + x_{k+1})$$
and
\[ P_{x_1,\ldots,x_{k+1}} = P_{x_1+\ldots+x_{k+1}}, \]
in which \( k \in \mathbb{N} \) is arbitrary and \( (x_1, \ldots, x_{k+1}) \in \Delta_{k+1}^\theta \). The corresponding process on \( \Omega_1 \) incorporates multiple-coagulation and multiple-fragmentation. (cf. the transition kernel (2.5) in [13] or Example 1.8 in [23]. See also [9] for more general scheme for the multiple-fragmentation.) One of the simplest examples of the generator of such a process is
\[ L_{1,k}^{(\theta)} \Pi_\phi(x) \]
\[ := \sum_{i_1,\ldots,i_{k+1}(\neq)} x_{i_1} \cdots x_{i_{k+1}} \left[ \phi(x_{i_1} + \cdots + x_{i_{k+1}}) - \phi(x_{i_1}) \cdots \phi(x_{i_{k+1}}) \right] \prod_{j \neq i_1,\ldots,i_{k+1}} \phi(x_j) \]
\[ + \theta^k \sum_i x_i^{k+1} \int_{\Delta_k} du_1 \cdots du_k \left( \phi(u_1 x_i) \cdots \phi(u_k x_i) \phi((1-|u|) x_i) - \phi(x_i) \right) \prod_{j \neq i} \phi(x_j) \]
\[ = \int \xi^{[k+1]}(dv_1 \cdots dv_{k+1}) v_1 \cdots v_{k+1} \]
\[ \times \left[ \phi(v_1 + \cdots + v_{k+1}) - \phi(v_1) \cdots \phi(v_{k+1}) \right] \Pi_\phi(\xi - \delta_{v_1} - \cdots - \delta_{v_{k+1}}) \]
\[ + \theta^k \int \xi (dv) v^{k+1} \int_{\Delta_k} du_1 \cdots du_k \]
\[ \times \left[ \phi(u_1 v) \cdots \phi(u_k v) \phi((1-|u|) v) - \phi(v) \right] \Pi_\phi(\xi - \delta_v), \]
where \( |u| = u_1 + \cdots + u_k \) and \( \xi = \Xi(x) \). Clearly \( L_{1,1}^{(\theta)} = 2L_{1,1}^{(1,\theta)} \). The calculations in the proof of Theorem 2.3 is easily modified to prove that \( \text{PD}(\theta) \) is still a reversible distribution of the process generated by \( L_{1,k}^{(\theta)} \). The details are left to the reader.

(iii) The spectral gap of a suitable extension \( L_{1,k}^{(Q,\theta)} \) of \( L_{1,k}^{(Q,\theta)} \) vanishes. Indeed, letting \( \Psi_\delta(z) = \sum_i z_i^\delta \) for \( z = (z_i) \in \Omega \) and \( \delta > 0 \), we see, with the help of Lemma 6.4 in [20], that \( \Psi_\delta \) is square integrable with respect to \( \text{PD}(\theta) \) and that its variance \( \text{var}(\Psi_\delta) \) is given by
\[ \text{var}(\Psi_\delta) = \frac{\Gamma(\theta+1)\Gamma(2\delta)}{\Gamma(\theta+2\delta)} + \Gamma(\delta)^2 \left( \frac{\theta \Gamma(\theta+1)}{\Gamma(\theta+2\delta)} - \frac{\Gamma(\theta+1)^2}{\Gamma(\theta+\delta)^2} \right) =: \chi_1(\delta) + \chi_2(\delta), \]
where \( \Gamma(\cdot) \) is Gamma function. As \( \delta \downarrow 0 \), \( \text{var}(\Psi_\delta) \to \infty \) since \( \chi_1(\delta) \to \infty \) and
\[ \chi_2(\delta) = \frac{\theta \Gamma(\theta+1)\Gamma(\delta+1)^2}{\Gamma(\theta+2\delta)\Gamma(\theta+\delta)^2} \cdot \frac{\Gamma(\theta+\delta)^2 - \Gamma(\theta)\Gamma(\theta+2\delta)}{\delta^2} \]
\[ \to \frac{\theta^2}{\Gamma(\theta)^2} \left( \Gamma'(\theta)^2 - \Gamma(\theta)\Gamma''(\theta) \right) \quad \text{(by l’Hospital’s rule)}. \]
As for Dirichlet form \( \mathcal{E}(\Psi_\delta) := E\left[ \Psi_\delta(X)(-L_{1,k}^{(Q,\theta)}\Psi_\delta(X)) \right] \) in which \( X = (X_i)_{i=1}^\infty \) is \( \text{PD}(\theta) \)-distributed, by the dominated convergence theorem
\[ \mathcal{E}(\Psi_\delta) = \frac{1}{2} E \left[ L_{1,k}^{(Q,\theta)}(\Psi_\delta^2)(X) - 2\Psi_\delta(X)L_{1,k}^{(Q,\theta)}\Psi_\delta(X) \right] \]
\[
\frac{1}{4} E \left[ \sum_{i \neq j} X_i X_j Q(X_i, X_j) \{ \Psi_\delta(M_{ij}, X) - \Psi_\delta(X) \}^2 \right] \\
+ \frac{\theta}{4} E \left[ \sum_i X_i^2 \int_0^1 duQ(uX_i, (1-u)X_i) \{ \Psi_\delta(S_i^u X) - \Psi_\delta(X) \}^2 \right]
\]

\[
= \frac{1}{4} E \left[ \sum_{i \neq j} X_i X_j Q(X_i, X_j) \left\{ (X_i + X_j) - X_i^\delta - X_j^\delta \right\}^2 \right] \\
+ \frac{\theta}{4} E \left[ \sum_i X_i^2 \int_0^1 duQ(uX_i, (1-u)X_i) \left\{ (uX_i) - ((1-u)X_i) - X_i^\delta \right\}^2 \right]
\]

\[
\rightarrow \frac{1}{4} E \left[ \sum_{i \neq j} X_i X_j Q(X_i, X_j) \right] + \frac{\theta}{4} E \left[ \sum_i X_i^2 \int_0^1 duQ(uX_i, (1-u)X_i) \right] < \infty
\]
as \delta \downarrow 0. (In fact, the second equality in the above needs justification. This can be done by considering bounded functions \( \Psi_{\delta, \epsilon}(x) = \sum_{i} x_i^\delta \mathbf{1}_{\{x_i \geq \epsilon\}} \) on \( \Omega_1 \), taking the limit as \( \epsilon \downarrow 0 \) and applying Lebesgue’s convergence theorem.) Consequently, \( \mathcal{E}(\Psi_\delta)/\text{var}(\Psi_\delta) \to 0 \), and hence the exponential convergence to equilibrium does not hold for the process generated by \( L_{1,2}^{(Q, \delta)} \). It seems, however, that the exact speed of convergence is unknown even in the case \( Q \equiv \text{const.} \).

The equality (2.17) is thought of as a probabilistic counterpart of the detailed balance condition (1.3). It should be noted that (2.17) is equivalent to the validity of two equalities \( K_1(x_1, x_2) q_2(x_1, x_2) = F_1(x_1 + x_2) q_1(x_1 + x_2) \) and \( P_{x_1, x_2} = P_{x_2 + x_2} \), a.e.-(\( x_1, x_2 \)) \( \in \Delta_2 \) because of the triviality that \( P_{x_1, x_2}(N) = P_{x_1 + x_2}(N) = 1 \). The reader may wonder whether there is any distribution other than Poisson-Dirichlet distributions which enjoys the relation (2.17) for some explicit \( K_1 \) and \( F_1 \). The following examples are intended to give answers to that question by discussing certain deformations of PD(\( \theta \)).

**Examples.** Let \( (X_i)_{i=1}^\infty \) be PD(\( \theta \))-distributed and \( \phi \) be a nonnegative measurable function on \( (0, 1) \). Assume that \( 0 < a := E [\prod_i \phi(X_i)] < \infty \) and define a probability measure \( \tilde{P} \) on \( \Omega_1 \) by

\[
\tilde{P}(\bullet) = a^{-1} E \left[ \prod_i \phi(X_i) : (X_i)_{i=1}^\infty \in \bullet \right].
\]

It is not difficult to show that the first and second correlation functions \( \tilde{q}_1 \) and \( \tilde{q}_2 \) and the first-order and second-order reduced Palm distributions of \( \sum X_i \) under \( \tilde{P} \) are given in terms of those of the PD(\( \theta \)) process (namely, \( q_1, q_2, P_v \) and \( P_{x_1, x_2} \)) by

\[
\tilde{q}_1(v) = a^{-1} \phi(v) q_1(v) E_v \left[ \prod_i \phi(X_i) \right],
\]

\[
\tilde{q}_2(x_1, x_2) = a^{-1} \phi(x_1) \phi(x_2) q_2(x_1, x_2) E_{x_1, x_2} \left[ \prod_i \phi(X_i) \right],
\]
\(\tilde{P}_v(\bullet) = E_v\left[ \prod_i \phi(X_i) : \sum \delta_{X_i} \in \bullet \right] \left( E_v\left[ \prod_i \phi(X_i) \right] \right)^{-1}\)

and

\(\tilde{P}_{x_1,x_2}(\bullet) = E_{x_1,x_2}\left[ \prod_i \phi(X_i) : \sum \delta_{X_i} \in \bullet \right] \left( E_{x_1,x_2}\left[ \prod_i \phi(X_i) \right] \right)^{-1},\)

respectively. Notice that the above formula for \(\tilde{P}_v\) (resp. \(\tilde{P}_{x_1,x_2}\)) is valid in \(\tilde{q}_1(v)dv\)-a.e. (resp. \(\tilde{q}_1(x_1,x_2)dx_1dx_2\)-a.e.) sense, so that the denominator can be considered to be positive. Combining with (2.14) and (2.15), we obtain

\[x_1x_2\phi(x_1 + x_2)\tilde{q}_2(x_1, x_2) = \theta(x_1 + x_2)\phi(x_1)\phi(x_2)\tilde{q}_1(x_1 + x_2)\]

and \(\tilde{P}_{x_1,x_2} = \tilde{P}_{x_1+x_2}\). Here are two examples of \(K_1\) and \(F_1\).

(i) The above two identities show that (2.17) is satisfied by \(\sum \delta_{X_i}\) under \(\tilde{P}\) when we choose

\[K_1(x_1, x_2) = x_1x_2\phi(x_1 + x_2) \quad \text{and} \quad F_1(x_1, x_2) = \theta(x_1 + x_2)\phi(x_1)\phi(x_2)\]

In this case, (2.16) reads

\[L_1^i\Phi(x) = \frac{1}{2} \sum_{i \neq j} x_ix_j\phi(x_i + x_j) (\Phi(M_{ij}x) - \Phi(x)) + \frac{\theta}{2} \sum i x_i^2 \int_0^1 du \phi(ux_i)\phi((1-u)x_i) (\Phi(S_i^uy - \Phi(x)),\]

which defines a bounded operator whenever \(\phi\) is bounded. For example, fixing \(s \in (0,1)\) arbitrarily and setting \(\phi(u) = 1_{[0,s]}(u)\), we see easily that \(a = P(X_1 \leq s)\) and the associated reversible distribution \(\tilde{P}\) is identified with the conditional law of \((X_i)_{i=1}^\infty\) given that \(X_1 \leq s\).

(ii) For another choice

\[K_1(x_1, x_2) = x_1x_2 \quad \text{and} \quad F_1(x_1, x_2) = \theta(x_1 + x_2)\phi(x_1)\phi(x_2)/\phi(x_1 + x_2)\]

(2.16) becomes

\[L_1^i\Phi(x) = \frac{1}{2} \sum_{i \neq j} x_ix_j (\Phi(M_{ij}x) - \Phi(x)) + \frac{\theta}{2} \sum i x_i^2 \int_0^1 du \phi(ux_i)\phi((1-u)x_i) (\Phi(S_i^uy - \Phi(x)).\]

(Compare with the transition kernel studied in [26].) This operator is bounded for any uniformly positive, bounded function \(\phi\) on \((0,1)\). To check, we take \(\phi(u) = \exp(bu)\) with \(b\) being an arbitrary real number and then verify that \(\tilde{P} = PD(\theta)\) and \(L_1^i = L_1^{(1,\theta)}\). The distribution \(\tilde{P}\) for \(\phi(u) = \exp(bu^2)\) has been discussed as the equilibrium measure of a certain model in population genetics. (See [19] and the references therein.)
To explore an analogue of Theorem 2.3 for processes on \( \Omega \), we discuss the process generated by \( L \) in (2.4). The special case \( \theta \tilde{H} \equiv \tilde{H} \) has the generator

\[
L^{(H,\theta)} \Phi(z) := \frac{1}{2} \sum_{i \neq j} z_i z_j H(z_i, z_j) (\Phi(M_{ij} z) - \Phi(z)) \\
+ \frac{\theta}{2} \sum_i z_i^2 \int_0^1 du H(u z_i, (1 - u) z_i) (\Phi(S^n u z) - \Phi(z)),
\]

where \( H \) is a symmetric, nonnegative homogeneous function \( H \) of degree \( \lambda \geq 0 \) satisfying (H1) and \( \theta > 0 \). By conditioning (or cut-off) argument, we get

**Theorem 2.4** (i) Let \( Z = (Z_i)_{i=1}^{\infty} \) be a random element of \( \Omega \) and suppose that the first and second correlation functions \( r_1 \) and \( r_2 \) of \( \xi := \Xi(Z) \) exist and satisfy

\[
K(z_1, z_2) r_2(z_1, z_2) P_{z_1, z_2} = F(z_1, z_2) r_1(z_1 + z_2) P_{z_1 + z_2},
\]

a.e.\( (z_1, z_2) \in (0, \infty)^2 \). Here, \( P_{z_1, z_2} \) and \( P_{z_1 + z_2} \) are the reduced Palm distributions of \( \xi \) and the above equality is understood as the one between two measures on \( \mathcal{N} \). Then the process generated by \( L \) is reversible with respect to the law of \( Z \).

(ii) Let \( L^{(H,\theta)} \) be as in (2.18). Suppose that \( (X_i)_{i=1}^{\infty} \) is PD(\( \theta \))-distributed. Then, for any \( (0, \infty) \)-valued random variable \( V \) independent of \( X \), the law of an \( \Omega \)-valued random element \( (V X_i)_{i=1}^{\infty} \) is a reversible distribution of the process generated by \( L^{(H,\theta)} \).

**Proof.** (i) Let \( R > 0 \) be such that \( P(|Z| \leq R) > 0 \). First, consider the process generated by \( L \) with initial distribution \( P^{(R)}(\bullet) := P(Z \in \bullet \mid |Z| \leq R) \). Then it is clear that such a process lies in \( \Omega_{\leq R} \), and hence its generator \( L \) is essentially bounded. So, just as in the proof of Theorem 2.3 the proof of the reversibility with respect to \( P^{(R)} \) can reduce to verifying that the equalities corresponding to (2.19) hold for \( \xi = \Xi(Z) \) under the conditional law \( P^{(R)} \). It is not difficult to show that, under \( P^{(R)} \), \( \xi = \Xi(Z) \) has the first and second correlation functions

\[
r_1^{(R)}(z) := r_1(z) \mathbf{1}_{\{z \leq R\}} \frac{P_z(|\eta| \leq R - z)}{P(|Z| \leq R)},
\]

\[
r_2^{(R)}(z_1, z_2) := r_2(z_1, z_2) \mathbf{1}_{\{z_1 + z_2 \leq R\}} \frac{P_{z_1, z_2}(|\eta| \leq R - (z_1 + z_2))}{P(|Z| \leq R)}
\]

and the first order and second order reduced Palm distributions

\[
P_z^{(R)}(\bullet) = P_z(\bullet \mid |\eta| \leq R - z), \quad P^{(R)}_{z_1, z_2}(\bullet) = P_{z_1, z_2}(\bullet \mid |\eta| \leq R - (z_1 + z_2)).
\]

These formulas combined with (2.19) yield

\[
K(z_1, z_2) r_2^{(R)}(z_1, z_2) P^{(R)}_{z_1, z_2} = F(z_1, z_2) r_1^{(R)}(z_1 + z_2) P^{(R)}_{z_1 + z_2},
\]

which is sufficient to imply the reversibility of the process \( \{Z(t) : t \geq 0\} \) generated by \( L \) with initial distribution \( P^{(R)} \) for the aforementioned reason. Before taking the
limit as $R \rightarrow \infty$, we interpret the reversibility obtained so far in terms of conditional expectations as follows: for any $n \in \mathbb{N}$, $0 < t_1 < \cdots < t_n < T$ and $\Phi_1, \ldots, \Phi_n \in B(\Omega)$

$$E [\Phi_1(Z(t_1)) \cdots \Phi_n(Z(t_n))] | Z(0) | \leq R,$$

$$E [\Phi_1(Z(T-t_1)) \cdots \Phi_n(Z(T-t_n))] | Z(0) | \leq R.$$

By letting $R \rightarrow \infty$ the required reversibility has been proved.

(ii) Consider the lifted process $\{\tilde{Z}(t) = (VX_i(t))_{i=1}^\infty : t \geq 0\}$, where $\{X(t) = (X_i(t))_{i=1}^\infty : t \geq 0\}$ is generated by $L^{(H,\theta)}$, independent of $V$ and such that $X(0) = (X_i)_{i=1}^\infty$. By Lemma 2.1 (ii) $\{\tilde{Z}(t)\}$ is generated by $\tilde{L}$ with $\tilde{H} = H$ and $\tilde{H} = \theta H$.

Accordingly

$$Z(t) := \tilde{Z}([Z(0)]^{2+\lambda}t) = VX(V^{2+\lambda}t)$$

is a process generated by $L^{(H,\theta)}$ and clearly $Z(0) = (VX_i)_{i=1}^\infty$. Letting $\rho$ denote the law of $V$, we see from the reversibility of $\{X(t)\}$ proved in Theorem 2.3 (ii) that for any $n \in \mathbb{N}$, $0 < t_1 < \cdots < t_n < T$ and $\Phi_1, \ldots, \Phi_n \in B(\Omega)$

$$E [\Phi_1(Z(t_1)) \cdots \Phi_n(Z(t_n))]$$

$$= \int \rho(dv) E [\Phi_1(vX(v^{2+\lambda}t_1)) \cdots \Phi_n(vX(v^{2+\lambda}t_n))]$$

$$= \int \rho(dv) E [\Phi_1(vX(v^{2+\lambda}T - v^{2+\lambda}t_1)) \cdots \Phi_n(vX(v^{2+\lambda}T - v^{2+\lambda}t_n))]$$

$$= E [\Phi_1(Z(T-t_1)) \cdots \Phi_n(Z(T-t_n))].$$

This proves that the law of $Z(0) = (VX_i)_{i=1}^\infty$ is a reversible distribution of $\{Z(t) : t \geq 0\}$, a process generated by $L^{(H,\theta)}$. 

In fact, alternative proof of Theorem 2.4 (ii) exists and is based on (2.19) together with the following static result on the correlation measures and the reduced Palm distributions of ‘the lifted PD($\theta$) process’ $\sum \delta_{VX_i}$.

**Lemma 2.5** Let $(X_i)_{i=1}^\infty$ be PD($\theta$)-distributed and suppose that a $(0, \infty)$-valued random variable $V$ independent of $(X_i)_{i=1}^\infty$ is given. Then, for each $k \in \mathbb{N}$, the $k$th correlation function $r_k$ on $(0, \infty)^k$ of $\sum \delta_{VX_i}$ is given by

$$r_k(z_1, \ldots, z_k) = \frac{\theta^k}{z_1 \cdots z_k} \int_{(|z|, \infty)} \rho(dv) \left(1 - \frac{|z|}{v}\right)^{\theta-1},$$

where $|z| = z_1 + \cdots + z_k$ and $\rho$ is the law of $V$. Moreover, for any $z = (z_1, \ldots, z_k) \in (0, \infty)^k$ such that $P(V > |z|) > 0$, the expectation $E_{z_1, \ldots, z_k}$ with respect to the $k$th-order reduced Palm distribution of $\sum \delta_{VX_i}$ at $z = (z_1, \ldots, z_k)$ is characterized by the formula

$$E_{z_1, \ldots, z_k} \left[ \prod_i \phi(VX_i) \right] = \frac{\int_{(|z|, \infty)} \rho(dv) \left(1 - \frac{|z|}{v}\right)^{\theta-1} E \left[ \prod_i \phi((v - |z|)X_i) \right]}{\int_{(|z|, \infty)} \rho(dv) \left(1 - \frac{|z|}{v}\right)^{\theta-1}}.$$

(2.21)
in which \( \phi \in \mathcal{B}_+ \) is arbitrary.

In the special case where \( V \) has the gamma density
\[
\rho_{\theta,b}(v) := \Gamma(\theta)^{-1} \theta^{-1} e^{-b v} 1_{(0,\infty)}(v)
\]  
(2.22)
with \( b > 0 \), \( \sum \delta_{V X_i} \) is a Poisson point process on \((0, \infty)\) with mean measure \( \theta y^{-1} e^{-by} dy \).

Proof. Let \( f \in \mathcal{B}_+^k \) be arbitrary. By the assumed independence and the Palm formula for the \( \text{PD}(\theta) \) process \( \sum \delta_{X_i} \)
\[
E \left[ \sum_{i_1, \ldots, i_k(\neq)} f(V X_{i_1}, \ldots, V X_{i_k}) \prod_{j \neq i_1, \ldots, i_k} \phi(V X_j) \right]
\]
\[
= \int \rho(dv) E \left[ \sum_{i_1, \ldots, i_k(\neq)} f(v X_{i_1}, \ldots, v X_{i_k}) \prod_{j \neq i_1, \ldots, i_k} \phi(v X_j) \right]
\]
\[
= \int \rho(dv) \int_{\mathbb{R}^k} f(v x_1, \ldots, v x_k) \frac{\theta^k (1 - |x|)^{\theta - 1}}{x_1 \cdots x_k} dx_1 \cdots dx_k E \left[ \prod_j \phi(v (1 - |x|) X_j) \right]
\]
\[
= \int_{(0,\infty)^k} dz_1 \cdots dz_k f(z_1, \ldots, z_k)
\]
\[
\times \frac{\theta^k}{z_1 \cdots z_k} \int_{(|z|, \infty)} \rho(dv) \left( 1 - \frac{|z|}{v} \right)^{\theta - 1} E \left[ \prod_j \phi((v - |z|) X_j) \right].
\]
Taking \( \phi \equiv 1 \) yields (2.20). Therefore, the above equalities suffice to imply (2.21). The last assertion follows from
\[
\frac{\theta^k}{z_1 \cdots z_k} \int_{(|z|, \infty)} \rho_{\theta,b}(v) \left( 1 - \frac{|z|}{v} \right)^{\theta - 1} dv = \frac{\theta^k e^{-b|z|}}{z_1 \cdots z_k} = \prod_{i=1}^k \left( \frac{\theta}{z_i} e^{-b z_i} \right),
\]
which is nothing but the \( k \)th correlation function of a Poisson point process on \((0, \infty)\) with mean density \( \theta y^{-1} e^{-by} \).

We call the above-mentioned Poisson process the gamma point process with parameter \((\theta, b)\). It is worth noting that in view of (2.20) and (2.21) the equalities
\[
z_1 z_2 r_2(v_1, v_2) = \theta(z_1 + z_2) r_1(z_1 + z_2) \quad \text{and} \quad E_{z_1,z_2} = E_{z_1+z_2}
\]
which correspond to (2.14) and (2.15), respectively, hold true for any lifted \( \text{PD}(\theta) \) process, and as a result it satisfies also (2.19) with
\[
K(x,y) = xy H(x,y) \quad \text{and} \quad F(x,y) = \theta(x+y) H(x,y).
\]  
(2.23)
In the forthcoming section it will be shown that the time-dependent system of correlation measures of the process generated by \( L \) solves the hierarchical equation (1.2) weakly. In connection with Theorem 2.4 (ii), we remark that the system of the correlation functions \( \{ r_k \} \) given by (2.20) is verified directly to be a stationary solution to (1.2) with (2.23). In these calculations, merely the following structure is relevant:
\[
z_1 \cdots z_k r_k(z_1, \ldots, z_k) = \theta^k g(z_1 + \cdots + z_k)
\]
for some function \( g \) independent of \( k \) (although (2.20) shows us the exact form of \( g \)).
3 Hierarchical equations for correlation measures

The purpose of this section is to derive equations (1.2) as those describing the time evolution of the correlation measures associated with coagulation-fragmentation processes introduced in the previous section. As far as the reversible process generated by $L^1(Q,\partial)$ with $Q \equiv \text{const.}$ is concerned, such an attempt is found essentially in [26] for the purpose of showing the uniqueness of stationary distributions. In that paper, however, the stationary hierarchical equations (called ‘the basic relations’ on p.19) for the correlation functions are incorrect, overlooking a term coming from coagulation between clusters with specific sizes given (i.e., a term involving $p_{k-1}$). It is not clear that developing such an approach could make one possible to settle the uniqueness issue in much more general setting. More specifically, in the reversible case described in Theorem 2.3 (resp. Theorem 2.4), it seems reasonable to expect the existence of a functional of distributions on $\Omega_1$ (resp. $\Omega$) which decays under the time evolution governed by $L^1$ (resp. $L$). (Notice that the uniqueness of stationary distributions cannot hold for the processes on $\Omega$.)

Intending only to derive an infinite system of equations describing fully the time evolution of our model, we begin by introducing a weak version of (1.2), namely, the equation obtained by operating on a test function by its formal adjoint. Since $K$ and $F$ are unbounded, suitable integrability conditions must be required for solutions. Let $B_{+,c}^k$ denote the totality of functions in $B^k_+$ with compact support. Also, the abbreviated notation $z_k = (z_1, \ldots, z_k)$ and $dz_k = dz_1 \cdots dz_k$ are used in the integral expressions. A family of measures $\{c_k(t, dz_k) : t \geq 0, k \in \mathbb{N}\}$ is said to be admissible if the following two conditions are fulfilled:

(A1) Each $c_k(t, \cdot)$ is a locally finite measure on $(0, \infty)^k$.

(A2) For any $f \in B_{+,c}^k$, $g \in B_{+,c}^{k-1}$ and $l \in \{1, \ldots, k\}$, the three functions below are locally integrable on $[0, \infty)$:

\begin{align*}
(i) \quad t \mapsto & \int f(z_k)c_k(t, dz_k), \\
(ii) \quad t \mapsto & \int z_l z_{l+1} f(z_1, \ldots, z_{l-1}, z_l + z_{l+1}, z_{l+2}, \ldots, z_{k+1}) c_{k+1}(t, dz_{k+1}), \\
(iii) \quad t \mapsto & \begin{cases} 
 z_1 (z_1^{l+\lambda} \lor 1) c_1(t, dz_1) & (k = 1), \\
 z_l (z_l^{l+\lambda} \lor 1) g(z_1, \ldots, z_{l-1}, z_{l+1}, \ldots, z_k) c_k(t, dz_k) & (k \geq 2).
\end{cases}
\end{align*}

Given an admissible $\{c_k(t, dz_k) : t \geq 0, k \in \mathbb{N}\}$, we call it a weak solution of the hierarchical coagulation-fragmentation equation (1.2) with kernels $K$ and $F$ given by (1.4) if for any $t > 0$, $k \in \mathbb{N}$ and $f \in B_{+,c}^k$,

\begin{align}
\int f(z_k)c_k(t, dz_k) - \int f(z_k)c_k(0, dz_k) &= \int_0^t ds \int K(z_l, z_{l+1}) f(\text{Coag}_{l,l+1} z_k) c_{k+1}(s, dz_{k+1}) \\
&= \frac{1}{2} \sum_{l=1}^k \int_0^t ds \int K(z_l, z_{l+1}) f(\text{Coag}_{l,l+1} z_k) c_{k+1}(s, dz_{k+1})
\end{align}
\[-\frac{1}{2} \sum_{l=1}^{k} \int_0^t ds \int_0^{z_l} dy F(y, z_l - y) f(z_k) c_k(s, dz_k) \]
\[-\sum_{l=1}^{k} \int_0^t ds \int K(z_l, z_{l+1}) f(z_1, \ldots, z_l, z_{l+2}, \ldots, z_{k+1}) c_{k+1}(s, dz_{k+1}) \]
\[+ \sum_{l=1}^{k} \int_0^t ds \int \int_0^{z_l} dy F(y, z_l - y) f(z_1, \ldots, z_l, y, z_{l+1}, \ldots, z_k) c_k(s, dz_k) \]
\[-I_{(k \geq 2)} \sum_{l<m}^{k} \int_0^t ds \int K(z_l, z_m) f(z_k) c_k(s, dz_k) \]
\[+ I_{(k \geq 2)} \sum_{l<m}^{k} \int_0^t ds \int \int_0^{z_l} dy F(y, z_l - y) f(Frag^{(y)}_{l,m} z_{k-1}) c_{k-1}(s, dz_{k-1}), \]

where

\[\text{Coag}_{l,t+1} z_{k+1} = (z_1, \ldots, z_{t-1}, z_l + z_{t+1}, z_{t+2}, \ldots, z_k)\]

and

\[\text{Frag}^{(y)}_{l,m} z_{k-1} = (z_1, \ldots, z_{t-1}, y, z_{t+1}, \ldots, z_{m-1}, z_l - y, z_m, \ldots, z_{k-1})\]

By (1.4), the equation (3.1) actually takes a more specific form

\[\int f(z_k) c_k(t, dz_k) - \int f(z_k) c_k(0, dz_k) = \]
\[= \frac{1}{2} \sum_{l=1}^{k} \int_0^t ds \int z_l z_{l+1} H(z_l, z_{l+1}) f(Coag_{l,t+1} z_{k+1}) c_{k+1}(s, dz_{k+1}) \]
\[+ \frac{1}{2} \sum_{l=1}^{k} \int_0^t ds \int z_l \int_0^{z_l} dy \dot{H}(y, z_l - y) f(z_k) c_k(s, dz_k) \]
\[+ \sum_{l=1}^{k} \int_0^t ds \int z_l \int_0^{z_l} dy \dot{H}(y, z_l - y) f(z_1, \ldots, z_l, y, z_{l+1}, \ldots, z_k) c_k(s, dz_k) \]
\[-1_{(k \geq 2)} \sum_{l<m}^{k} \int_0^t ds \int z_l z_m H(z_l, z_m) f(z_k) c_k(s, dz_k) \]
\[+ 1_{(k \geq 2)} \sum_{l<m}^{k} \int_0^t ds \int z_l \int_0^{z_l} dy \dot{H}(y, z_l - y) f(Frag^{(y)}_{l,m} z_{k-1}) c_{k-1}(s, dz_{k-1}) \]
\[= I_1 - I_2 - I_3 + I_4 - I_5 + I_6. \] (3.3)

Obviously, (A1) ensures that two integrals on the left side of (3.1) is finite. Considering the terms on the right side, we prepare the following bounds: by homogeneity (1.5), (H1) and (H2) together

\[\dot{H}(x, y) = \dot{H} \left( \frac{x}{x+y}, \frac{x}{x+y} \right) \leq \dot{\mathcal{C}}(x+y)^{\lambda} \] (3.4)
\[\leq \dot{\mathcal{C}}(1+x)^{\lambda} (y^\lambda \vee 1) \leq \ddot{\mathcal{C}}(1+x)^{\lambda} (y^{1+\lambda} \vee 1) \] (3.5)
and
\[ \int_0^x dy \tilde{H}(y, x-y) = x \int_0^1 du \tilde{H}(ux, (1-u)x) = \tilde{C} x^{1+\lambda}. \tag{3.6} \]

**Lemma 3.1** Assume that \( \{c_k(t, dz_k) : t \geq 0, k \in \mathbb{N}\} \) is admissible. Then every term on the right side of (3.1) (or equivalently of (3.2)) is finite.

**Proof.** We discuss \( I_i \)'s on (3.3) instead of the terms on the right side of (3.1). It follows from (A2) (i) that \( I_2 \) and \( I_5 \) are finite. Also, \( I_1 \) is finite because of (3.4) and (A2) (ii). \( I_3 \) converges by (3.5) together with (A2) (iii), and similarly the finiteness of \( I_4 \) is due to (3.6) and (A2) (iii) with
\[ g(z_1, \ldots, z_{k-1}) = \sup_{y > 0} f(z_1, \ldots, z_{l-1}, y, z_l, \ldots, z_{k-1}). \]

Lastly, again by (A2) (i), \( I_6 \) is finite since each function
\[ h_{l,m}(z_1, \ldots, z_{k-1}) := z_l \int_0^{z_l} dy \tilde{H}(y, z_l - y) f(\text{Frag}^{(y)}_{l,m} z_{k-1}) \]
is an element of \( B_{k-1}^{k-1} \) for \( k \geq 2 \). Indeed, taking \( \epsilon > 0 \) so that \( f(z_1, \ldots, z_k) = 0 \) whenever \( 0 < z_l < \epsilon \), we see that \( h_{l,m}(z_1, \ldots, z_{k-1}) = 0 \) for any \( z_l \in (0, \epsilon) \), and analogously, taking \( R > 0 \) so that \( f(z_1, \ldots, z_k) = 0 \) whenever \( z_l > R \) or \( z_m > R \), we see that \( h_{l,m}(z_1, \ldots, z_{k-1}) = 0 \) for any \( z_l > 2R \). The proof of Lemma 3.1 is complete.

The main result of this section yields stochastic construction of a solution to (1.2) with kernels we are concerned with.

**Theorem 3.2** (i) Let \( \{Z(t) : t \geq 0\} \) be the process generated by \( L \) and suppose that \( E[|Z(0)|^k] < \infty \) for all \( k \in \mathbb{N} \). For each \( t \geq 0 \) and \( k \in \mathbb{N} \) denote by \( r_k(t, dz_k) \) the \( k \)th correlation measure of \( \Xi(Z(t)) \). Then \( \{r_k(t, dz_k) : t \geq 0, k \in \mathbb{N}\} \) is admissible and solves weakly the hierarchical coagulation-fragmentation equation (1.2) with kernels \( K \) and \( F \) given by (1.4).

(ii) Let \( \tilde{Q} \) and \( \check{Q} \) be symmetric, nonnegative bounded functions on \( \{(x,y) | x, y > 0, x+y \leq 1\} \) and set
\[ K_1(x,y) = xy \tilde{Q}(x,y) \quad \text{and} \quad F_1(x,y) = (x+y) \check{Q}(x,y). \]

Let \( \{X(t) : t \geq 0\} \) be the process generated by \( L_1^2 \) in (2.16). For each \( t \geq 0 \) and \( k \in \mathbb{N} \) denote by \( q_k(t, dx_k) \) the \( k \)th correlation measure of \( \Xi(X(t)) \). Then \( \{q_k(t, dx_k) : t \geq 0, k \in \mathbb{N}\} \) is admissible and solves weakly the hierarchical coagulation-fragmentation equation (1.2) with kernels \( K_1 \) and \( F_1 \).

Recalling the definition of correlation measures (cf. (2.7)), the proof of this theorem is basically done by calculating carefully \( L \Phi(z) \) or \( L_1^2 \Phi(z) \) for
\[ \Phi(z) = \sum_{i_1, \ldots, i_k \neq \emptyset} f(z_{i_1}, \ldots, z_{i_k}). \tag{3.7} \]
where \( f \in B^k_{+,c} \) is arbitrary. Here, we understand that \( f(z_i, \ldots, z_k) = 0 \) when \( z_i \cdots z_k = 0 \), so that

\[
\sum_{i_1, \ldots, i_k(\neq)} f(z_{i_1}, \ldots, z_{i_k}) = \int f(y_1, \ldots, y_k) \xi|^{k}(dy_1 \cdots dy_k),
\]

where \( \xi = \Xi(z) \). Although such function’s \( \Phi \) on \( \Omega \) may be unbounded, we can control its growth order as will be seen in the next lemma. For each \( a > 0 \) denote by \( \mathcal{F}_a \) the class of measurable functions \( \Psi \) on \( \Omega \) such that, for some constant \( C < \infty \), \( |\Psi(z)| \leq C|z|^a \) for all \( z \in \Omega \).

**Lemma 3.3** Let \( f \in B^k_{+,c} \) and \( \Phi \) be as in (3.7). Then \( \Phi \in \mathcal{F}_k \).

**Proof.** Define \( \epsilon = \inf\{\min\{z_1, \ldots, z_k\} : (z_1, \ldots, z_k) \in \text{supp}(f)\} \), which is strictly positive because \( \text{supp}(f) \) is assumed to be a compact subset of \((0, \infty)^k\). Therefore

\[
0 \leq \Phi(z) \leq \|f\|_{\infty} \sum_{i_1, \ldots, i_k} \frac{z_{i_1}}{\epsilon} \cdots \frac{z_{i_k}}{\epsilon} 1_{\{(z_{i_1}, \ldots, z_{i_k}) \in \text{supp}(f)\}} \leq \|f\|_{\infty} \frac{|z|^k}{\epsilon^k}.
\]

This proves Lemma 3.3. \qed

At the core of our proof of Theorem 3.2 is

**Lemma 3.4** Let \( f \in B^k_{+,c} \) and \( \Phi \) be as in (3.7). Then for each \( z = (z_i)_{i=1}^\infty \in \Omega \)

\[
L\Phi(z) = \frac{1}{2} \sum_{i_1, \ldots, i_k} K(z_{i_1}, z_{i_1+1}) f(C_{i_1, i_1+1}(z_{i_1}, \ldots, z_{i_1+1}))
\]

\[
-\frac{1}{2} \sum_{l=1}^{k} \int_0^{z_{l}} dy F(y, z_{l} - y) f(z_{l}, \ldots, z_{l})
\]

\[
- \sum_{l=1}^{k} K(z_{l}, z_{l+1}) f(z_{l}, \ldots, z_{l}, z_{l+2}, \ldots, z_{k+1})
\]

\[
+ \sum_{l=1}^{k} \int_0^{z_{l}} dy F(y, z_{l} - y) f(z_{l}, \ldots, z_{l-1}, y, z_{l+1}, \ldots, z_{k})
\]

\[
-1_{\{k \geq 2\}} \sum_{l<m} K(z_{l}, z_{m}) f(z_{l}, \ldots, z_{m})
\]

\[
+1_{\{k \geq 2\}} \sum_{l<m} \int_0^{z_{l}} dy F(y, z_{l} - y) f(C_{l, m}(z_{l}, \ldots, z_{k-1}))
\]

\[
= \frac{1}{2} \sum_{l=1}^{k} \Psi^{(1)}(z) - \frac{1}{2} \sum_{l=1}^{k} \Psi^{(2)}(z) - \sum_{l=1}^{k} \Psi^{(3)}(z)
\]

\[
+ \sum_{l=1}^{k} \Psi^{(4)}(z) - 1_{\{k \geq 2\}} \sum_{l<m} \Psi^{(5)}(z) + 1_{\{k \geq 2\}} \sum_{l<m} \Psi^{(6)}(z).
\]

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Moreover, $\Psi_l^{(1)} \in \mathcal{F}_{k+1}$, $\Psi_l^{(2)} \in \mathcal{F}_k$, $\Psi_l^{(3)} \in \mathcal{F}_{k+1+\lambda}$, $\Psi_l^{(4)} \in \mathcal{F}_{k+1+\lambda}$, $1_{\{k\geq 1\}} \Psi_l^{(5)} \in \mathcal{F}_k$ and $1_{\{k\geq 2\}} \Psi_{l,m}^{(6)} \in \mathcal{F}_{k+2}$. Thus $L\Phi$ belongs to the linear span $\mathcal{F}$ of $\cup_{a>0} \mathcal{F}_a$.

**Proof.** The main proof of (3.8) consists of almost algebraic calculations (which are completely independent of other arguments) and so it is postponed. We here prove only the assertions for $\Psi_l^{(i)} (i \in \{1, 2, 3, 4\})$ and $\Psi_{l,m}^{(j)} (j \in \{5, 6\})$. The arguments below are based on similar observations to those in the proof of Lemma 3.1. From Lemma 3.3, it is evident that $\Psi_l^{(2)}$ and $\Psi_{l,m}^{(5)}$ are elements of $\mathcal{F}_k$. Relying on the fact that the function $h_{l,m}$ in the proof of Lemma 3.1 belongs to $B_{+e}^{k-1}$, one can verify similarly that $\Psi_{l,m}^{(6)} \in \mathcal{F}_{k-1}$ for $k \geq 2$. As for $\Psi_l^{(1)}$, taking $\epsilon > 0$ and $R > \epsilon$ such that $[\epsilon, R]^k \supset \text{supp}(f)$ and using (3.4), we observe as in the proof of Lemma 3.3 that for $l \in \{1, \ldots, k\}$

$$|\Psi_l^{(1)}(z)| \leq \tilde{C} \sum_{i_1, \ldots, i_{k+1}(\neq)} z_{i_1}z_{i_{k+1}}(z_{i_1} + z_{i_{k+1}})^\lambda f(\text{Coag}_{l+1}(z_1, \ldots, z_{k+1}))$$

$$\leq \tilde{C} \|f\|_\infty \sum_{i_1, \ldots, i_{k+1}} z_{i_1}z_{i_{k+1}} R^\lambda \frac{z_{i_1}}{\epsilon} \frac{z_{i_1-1}}{\epsilon} \frac{z_{i_{k+1}}}{\epsilon} \frac{z_{i_{k+1}+1}}{\epsilon}$$

$$= \tilde{C} R^\lambda \|f\|_\infty \frac{|z|^{k+1}}{\epsilon^{k-1}},$$

by which $\Psi_l^{(1)} \in \mathcal{F}_{k+1}$. Analogously

$$|\Psi_l^{(2)}(z)| \leq \tilde{C} \sum_{i_1, \ldots, i_{k+1}(\neq)} z_{i_1}z_{i_{k+1}}(z_{i_1} + z_{i_{k+1}})^\lambda f(z_1, \ldots, z_{i_1}, z_{i_1+2}, \ldots, z_{k+1})$$

$$\leq \tilde{C} \sum_{i_1, \ldots, i_{k+1}(\neq)} R z_{i_{k+1}} |z|^\lambda f(z_1, \ldots, z_{i_1}, z_{i_1+2}, \ldots, z_{k+1})$$

$$\leq \tilde{C} R |z|^{1+\lambda} \sum_{i_1, \ldots, i_k(\neq)} f(z_{i_1}, \ldots, z_k),$$

which combined with Lemma 3.3 implies that $\Psi_l^{(3)} \in \mathcal{F}_{k+1+\lambda}$. Lastly, we shall show that $\Psi_l^{(4)} \in \mathcal{F}_{k+1+\lambda}$. For $k = 1$ we have by (3.6)

$$|\Psi_l^{(4)}(z)| \leq \tilde{C} \sum_i z_i^{2+\lambda}\|f\|_\infty \leq \tilde{C} |z|^{2+\lambda}\|f\|_\infty$$

and hence $\Psi_l^{(4)} \in \mathcal{F}_{2+\lambda}$. For $k \geq 2$, by noting that

$$\mathcal{F}_l(z_1, \ldots, z_{k-1}) := \sup_{y>0} f(z_1, \ldots, z_{l-1}, y, z_l, \ldots, z_{k-1})$$

belongs to $B_{+e}^{k-1}$ and observing that

$$|\Psi_l^{(4)}(z)| \leq \tilde{C} \sum_{i_1, \ldots, i_k(\neq)} z_i^{2+\lambda} \mathcal{F}_l(z_{i_1}, \ldots, z_{i_{l-1}}, z_{i_{l+1}}, \ldots, z_k)$$

$$\leq \tilde{C} |z|^{2+\lambda} \sum_{i_1, \ldots, i_{k-1}(\neq)} \mathcal{F}_l(z_{i_1}, \ldots, z_{i_{k-1}}),$$
we deduce from Lemma 3.3 that $\Psi^{(4)}_t \in \mathcal{F}_{k+1+\lambda}$. Consequently, we have shown that $L\Phi \in \mathcal{F}$.

\textbf{Proof of Theorem 3.2 (i).} Let $f \in B_{+,c}^k$ be arbitrary. First, we must show the admissibility of $r_k(t, dz_k)$. The conditions (A1) and (A2) (i) for $r_k(t, dz_k)$ are easily seen to hold by combining Lemma 3.3 with

$$E[|Z(t)|^a] = E[|Z(0)|^a] =: m_a < \infty$$

for all $a \geq 1$. (A2) (ii) can be verified by a similar bound to that for $|\Psi^{(1)}_t(z)|$ in the proof of Lemma 3.4. Moreover, (A2) (iii) follows from observations that

$$\int z_1(z_1^{1+\lambda} \lor 1)r_1(t, dz_1) \leq E \left[ \sum_i (Z_i(t) + Z_i(t)^{2+\lambda}) \right]$$

$$\leq E \left[ |Z(t)| + |Z(t)|^{2+\lambda} \right] = m_1 + m_{2+\lambda}$$

and that for $k \geq 2$ and $l \in \{1, \ldots, k\}$

$$\int z_l(z_l^{1+\lambda} \lor 1)g(z_1, \ldots, z_{l-1}, z_{l+1}, \ldots, z_k)r_k(t, dz_k)$$

$$\leq C \int (z_l + z_l^{2+\lambda})z_1 \cdots z_{l-1}z_{l+1} \cdots z_kr_k(t, dz_k)$$

$$\leq CE \left[ (|Z(t)| + |Z(t)|^{2+\lambda})|Z(t)|^{k-1} \right] = C (m_k + m_{1+\lambda+k})$$

where $C$ is a finite constant. Second, we claim that, for $\Phi$ given by (3.7) and $t > 0$

$$\int f(z_k)r_k(t, dz_k) - \int f(z_k)r_k(0, dz_k) = \int_0^t dsE[L\Phi(Z(s))]. \quad (3.9)$$

Define, for each $R > 0$, $\Phi^{(R)}(z) = 1_{\{|z| \leq R\}} \Phi(z)$. Then $\Phi^{(R)} \in B(\Omega)$ by Lemma 3.3 and clearly $L\Phi^{(R)}(z) = 1_{\{|z| \leq R\}} L\Phi(z)$. Furthermore, by virtue of Lemma 3.4, $1_{\{|z| \leq R\}} L\Phi(z)$ is bounded and so is $L\Phi^{(R)}$. These observations together imply that

$$E \left[ \Phi^{(R)}(Z(t)) \right] - E \left[ \Phi^{(R)}(Z(0)) \right] = \int_0^t dsE \left[ L\Phi^{(R)}(Z(s)) \right]$$

$$= \int_0^t dsE \left[ 1_{\{|Z(s)| \leq R\}} L\Phi(Z(s)) \right].$$

Noting that every moment of $|Z(t)| = |Z(0)|$ is finite by the assumption and that $L\Phi \in \mathcal{F}$, we get (3.9) by taking the limit as $R \to \infty$ with the help of Lebesgue’s convergence theorem.

Integrating the right side of (3.8) with respect to the law of $Z(s)$ and then plugging the resulting expression for the expectation $E[L\Phi(Z(s))]$ into (3.9) yield (3.1) with $r_k(s, dz_k)$ in place of $c_k(s, dz_k)$. We thus obtained the required equations for $\{r_k(t, dz_k) : t \geq 0, k \in \mathbb{N} \}$ and the proof of Theorem 3.2 (i) is complete, provided that the identity (3.8) is shown by calculations which are self-contained.
Proof of (3.8). Recalling the definition (2.4) of $L$, we now calculate $L\Phi(z)$ for $\Phi$ of the form (3.7). Observe that, for any $i \neq j$ such that $z_i z_j > 0$, the ‘coagulation difference’ $\Phi(M_{ij}z) - \Phi(z)$ equals

$$
\frac{1}{2} \sum_{i \neq j} K(z_i, z_j)(\Phi(M_{ij}z) - \Phi(z))
$$

$$(3.10)$$

$$
= \frac{1}{2} \sum_{l=1}^{k} \sum_{i_1, \ldots, i_{l+1}(\neq)} K(z_{i_l}, z_{i_{l+1}}) f(z_{i_1}, \ldots, z_{i_{l-1}}, z_{i_l} + z_{i_{l+1}}, z_{i_{l+2}}, \ldots, z_{i_{k+1}})
$$

$$
- \sum_{l=1}^{k} \sum_{i_1, \ldots, i_{l+1}(\neq)} K(z_{i_l}, z_{i_{l+1}}) f(z_{i_1}, \ldots, z_{i_{l-1}}, z_{i_l}, z_{i_{l+2}}, \ldots, z_{i_{k+1}})
$$

$$
-1_{\{k \geq 2\}} \sum_{l=1}^{k} \sum_{i_1, \ldots, i_{l+1}(\neq)} K(z_{i_l}, z_{i_{l+1}}) f(z_{i_1}, \ldots, z_{i_{l-1}}, z_{i_l}, z_{i_{l+2}}, \ldots, z_{i_{k+1}}).
$$

Similarly, for each $i \in \mathbb{N}$ with $z_i > 0$ and any $y \in (0, z_i)$, the ‘fragmentation difference’ $\Phi(S_{iy}^{(y)}z) - \Phi(z)$ is expressed as

$$
\sum_{l=1}^{k} \sum_{i_1, \ldots, i_{l+1}(\neq)} f(z_{i_1}, \ldots, z_{i_{l-1}}, y, z_{i_l}, \ldots, z_{i_{k+1}})
$$

$$
+ \sum_{l=1}^{k} \sum_{i_1, \ldots, i_{l+1}(\neq)} f(z_{i_1}, \ldots, z_{i_{l-1}}, z_i - y, z_{i_l}, \ldots, z_{i_{k+1}})
$$
\[
+ \ 1_{\{k \geq 2\}} \sum_{l<m} \sum_{i_1, \ldots, i_{k-2}(\neq)} f(\text{Frag}_{i,m}^{(y)}(z_{i_1}, \ldots, z_{i_{l-1}}, z_i, z_{i_l}, \ldots, z_{i_{k-2}})) \\
+ \ 1_{\{k \geq 2\}} \sum_{l<m} \sum_{i_1, \ldots, i_{k-2}(\neq)} f(\text{Frag}_{i,m}^{(z_{i}-y)}(z_{i_1}, \ldots, z_{i_{l-1}}, z_i, z_{i_l}, \ldots, z_{i_{k-2}})) \\
- \sum_{l=1}^{k} \sum_{i_1, \ldots, i_{k-1}(\neq)} f(z_{i_1}, \ldots, z_{i_{l-1}}, z_i, z_{i_l}, \ldots, z_{i_{k-1}}) \\
=: \Sigma_i^{(6)}(y, z) + \Sigma_i^{(7)}(y, z) + \Sigma_i^{(8)}(y, z) + \Sigma_i^{(9)}(y, z) - \Sigma_i^{(10)}(z)
\]

because for \(1 \leq l < m \leq k\)

\[
\text{Frag}_{i,m}^{(y)}(z_{i_1}, \ldots, z_{i_{l-1}}, z_i, z_{i_l}, \ldots, z_{i_{k-2}}) \\
= (z_{i_1}, \ldots, z_{i_{l-1}}, y, z_i, \ldots, z_{i_{m-2}}, z_i - y, z_{i_{m-1}}, \ldots, z_{i_{k-2}})
\]

and

\[
\text{Frag}_{i,m}^{(z_{i}-y)}(z_{i_1}, \ldots, z_{i_{l-1}}, z_i, z_{i_l}, \ldots, z_{i_{k-2}}) \\
= (z_{i_1}, \ldots, z_{i_{l-1}}, z_i - y, z_i, \ldots, z_{i_{m-2}}, y, z_{i_{m-1}}, \ldots, z_{i_{k-2}}).
\]

Here, \(\sum_{i_1, \ldots, i_{k-1}(\neq)}^{(i)^c}\) indicates the sum taken over \((k-1)\)-tuples \((i_1, \ldots, i_{k-1})\) of distinct positive integers which are different from \(i\). By noting two identities \(\Sigma_i^{(6)}(y, z) = \Sigma_i^{(7)}(z_i - y, z)\) and \(\Sigma_i^{(8)}(y, z) = \Sigma_i^{(9)}(z_i - y, z)\)

\[
\frac{1}{2} \sum_i z_i \int_0^{z_i} dy F(y, z_i - y)(\Phi(S_i^{(y)} - z) - \Phi(z)) \\
= \sum_{l=1}^{k} \sum_{i_1, \ldots, i_{k}(\neq)} z_{i_l} \int_0^{z_{i_l}} dy F(y, z_{i_l} - y) f(z_{i_1}, \ldots, z_{i_{l-1}}, y, z_{i_{l+1}}, \ldots, z_{i_k}) \\
+ 1_{\{k \geq 2\}} \sum_{l<m} \sum_{i_1, \ldots, i_{k}(\neq)} z_{i_l} \int_0^{z_{i_l}} dy F(y, z_{i_l} - y) f(\text{Frag}_{i,m}^{(y)}(z_{i_1}, \ldots, z_{i_{k-1}})) \\
- \frac{1}{2} \sum_{l=1}^{k} \sum_{i_1, \ldots, i_{k}(\neq)} z_{i_l} \int_0^{z_{i_l}} dy F(y, z_{i_l} - y) f(z_{i_1}, \ldots, z_{i_k}).
\]

Consequently, (3.8) is deduced from (3.10) and (3.11). \(\Box\)

**Proof of Theorem 3.2 (ii).** The proof is essentially the same as the proof of (i). We just give some comments. The admissibility is shown similarly by \(|X(0)| = 1\). Concerning the analogue of Lemma 3.4, (3.8) with \(K_1\) and \(F_1\) in place of \(K\) and \(F\), respectively, holds true since the proof of (3.8) itself is almost algebraic. Moreover, the assertions corresponding to the second half of Lemma 3.4 (i.e., the assertions for \(\Psi_i^{(i)}\)’s and \(\Psi_{i,m}^{(i)}\)’s) are also valid for \(\lambda = 0\) as is easily seen from the proof. We complete the proof of Theorem 3.2. \(\Box\)
So far we have discussed only weak solutions. The final result in this section shows, under a certain condition on the initial distribution, the existence of a strong solution to (1.2) with \( \dot{H} = \theta H \) for some constant \( \theta > 0 \), which ensures reversibility of the underlying process as was shown in Theorem 2.4 (ii). We will require also for the initial distribution to be regarded as a ‘perturbation’ from some reversible distribution, namely the law of a lifted \( \text{PD}(\theta) \) process. Recall that a lifted \( \text{PD}(\theta) \) process is of the form \( \sum \delta_{VX_i} \), where a \( (0, \infty) \)-valued random variable \( V \) and a \( \text{PD}(\theta) \)-distributed random element \( \{X_i\}_{i=1}^{\infty} \) of \( \Omega_1 \) are mutually independent.

**Proposition 3.5** Let \( L^{(H,\theta)} \) be as in (2.18) and \( \{Z(t) = (Z_i(t))_{i=1}^{\infty} : t \geq 0\} \) be a process generated by \( L^{(H,\theta)} \). Set \( \xi(t) = \Xi(Z(t)) \) and denote by \( \{T_t\}_{t \geq 0} \) the semigroup associated with \( \{\xi(t) : t \geq 0\} \). Suppose that the law of \( \xi(0) \) is absolutely continuous with respect to the law of a lifted \( \text{PD}(\theta) \) process \( \sum \delta_{VX} \).

(i) For any \( k \in \mathbb{N} \) and \( t \geq 0 \), the kth correlation function of \( \xi(t) \) is given by

\[
r_k(t, z_k) := r_k(z_1, \ldots, z_k) \int_{\mathbb{N}} P_{z_1, \ldots, z_k}(d\eta)(T_t\Psi^*)(\eta + \delta_z) \,,
\]

where \( r_k(z_1, \ldots, z_k) \) is the kth correlation function (2.20) of \( \sum \delta_{VX_i} \), \( P_{z_1, \ldots, z_k} \) is the kth-order reduced Palm distribution of \( \sum \delta_{VX_i} \) at \( z_k = (z_1, \ldots, z_k) \) characterized by (2.21) and \( \Psi^* \) is the density of the law of \( \xi(0) \) with respect to the law of \( \sum \delta_{VX_i} \). (It is understood that \( r_k(t, z_k) = 0 \) whenever \( P(V > |z_k|) = 0 \).)

(ii) Suppose additionally that \( E[|Z(0)|^k] < \infty \) for all \( k \in \mathbb{N} \). Then the family of nonnegative measurable functions \( \{r_k(t, z_k) : t \geq 0, k \in \mathbb{N}\} \) given in (i) solves the equation (1.2) with \( K(x, y) = xyH(x, y) \) and \( F(x, y) = \theta(x + y)H(x, y) \) in the following sense: for any \( k \in \mathbb{N} \) and \( t \geq 0 \)

\[
r_k(t, z_k) - r_k(0, z_k) - \int_0^t L_k(s, z_k)ds = 0, \text{ a.e. } z_k \in (0, \infty)^k,
\]

where \( L_k(s, z_k) \) is the right side of (1.2) with \( c_{k+1}, c_k, c_{k-1} \) and \( t \) replaced by \( r_{k+1}, r_k, r_k - 1 \) and \( s \), respectively.

**Proof.** (i) Let \( f \in B_+^k \) be arbitrary and \( \Phi \) be as in (3.7). Thus, by abuse of notation as in the previous section

\[
\Phi(\xi) = \sum_{i_1, \ldots, i_k \neq 0} f(z_{i_1}, \ldots, z_{i_k})
\]

for \( (z_i)_{i=1}^{\infty} \in \Omega \) and \( \xi = \Xi((z_i)_{i=1}^{\infty}) = \sum 1_{\{z_i > 0\}} \delta_{z_i} \). By the assumption of absolute continuity together with the reversibility implied by Theorem 2.4 (ii)

\[
E \left[ \sum_{i_1, \ldots, i_k \neq 0} f(Z_{i_1}(t), \ldots, Z_{i_k}(t)) \right] = E[\Phi(\xi(t))] = E[(T_t\Phi)(\xi(0))]
\]

\[
= \int_{\mathbb{N}} (T_t\Phi)(\eta) P(\xi(0) \in d\eta) = \int_{\mathbb{N}} (T_t\Phi)(\eta) \Psi^*(\eta) P \left( \sum_i \delta_{VX_i} \in d\eta \right)
\]

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\[
\int_N \Phi(\eta) (T_t \Psi^*)(\eta) \, P \left( \sum_i \delta_{VX_i} \in d\eta \right) \\
= \int f(z_k) r_k(z_k) \, dz_k \int_N P_{\eta_1, \ldots, \eta_k} (d\eta) (T_t \Psi^*)(\eta + \delta_{z_1} + \cdots + \delta_{z_k}),
\]
where the last equality is deduced from the Palm formula (2.8) combined with Lemma 2.5. This proves (3.12).

(ii) As an immediate consequence of Theorem 3.2 (i) we have

\[
\int f(z_k) \left[ r_k(t, z_k) - r_k(0, z_k) - \int_0^t \mathcal{L}_k(s, z_k) \, ds \right] \, dz_k = 0
\]
for all \( f \in B_{+}^{k,c} \). Replace \( f(z_k) \) by \( f(z_k)z_1 \cdots z_k \) to get

\[
\int f(z_k) \left[ r_k(t, z_k) - r_k(0, z_k) - \int_0^t \mathcal{L}_k(s, z_k) \, ds \right] z_1 \cdots z_k \, dz_k = 0. \tag{3.14}
\]

It is easily verified from the assumption on the moments of \(|Z(0)|\) that the signed measure

\[
\left[ r_k(t, z_k) - r_k(0, z_k) - \int_0^t \mathcal{L}_k(s, z_k) \, ds \right] z_1 \cdots z_k \, dz_k
\]
is expressed as a linear combination of (at most) 8 finite measures on \((0, \infty)^k\). Therefore, (3.14) implies that it must vanish and accordingly (3.13) holds.

**Example.** In the case where the lifted process \( \sum \delta_{VX_i} \) is a gamma point process with parameter \((\theta, b)\) (see at the end of Section 2), the absolute continuity assumption in Proposition 3.5 is satisfied e.g. when \( \xi(0) \) is a Poisson process on \((0, \infty)\) with mean density of the form \( e^{h(y)} \theta e^{-by}y^{-1} \) and \( \int |e^{h(y)} - 1| e^{-by}y^{-1} \, dy < \infty \). In that case, the density \( \Psi^* \) mentioned in Proposition 3.5 (i) is given by

\[
\Psi^*(\eta) = \exp \left[ \langle h, \eta \rangle - \theta \int (e^{h(y)} - 1) e^{-by}y^{-1} \, dy \right].
\]

(See e.g. Lemma 2.4 of [6].) Also, since the reduced Palm distributions of any Poisson process are identical with its law, (3.12) becomes

\[
r_k(t, z_k) = \frac{\theta^k}{z_1 \cdots z_k} e^{-b(z_1 + \cdots + z_k)} E \left[ (T_t \Psi^*)(\eta + \delta_{z_1} + \cdots + \delta_{z_k}) \right],
\]
where \( \eta \) is a gamma point process with parameter \((\theta, b)\).

## 4  Preliminary results for rescaled processes

### 4.1 Models with a scaling parameter

Both this section and the subsequent section are devoted to a derivation of the equation (1.1) from properly rescaled coagulation-fragmentation processes. In principle,
the procedure is similar to that in [12] although that paper assumes the conditions, among others, of the form
\[ K(x, y) = o(x) o(y) \] and \[ \int_0^x F(y, x - y)dy = o(x) \] as \( x, y \to \infty \)
for the rates. In our situation, these conditions are never met since by (1.4) and (1.5)
\[ K(x, y) = xy(x + y) \lambda \tilde{H} \left( \frac{x}{x + y}, \frac{y}{x + y} \right) \] and \[ \int_0^x F(y, x - y)dy = \tilde{C} x^{2+\lambda}. \]
However, it will turn out that our setting on the degrees of homogeneity of \( K \) and \( F \) provides with us certain effective ingredients to overcome difficulties due to such growth orders. More specifically, it will play an essential role in the proof of Proposition 4.5 below. In this connection we mention that [34] discussed the relation between occurrence of ‘steady-state solutions’ for coagulation-fragmentation equations and the degrees of homogeneity. According to the authors’ criterion based on analysis of the moments in several basic examples our setting on the degrees is in the region corresponding to systems for which steady states occur.

Let us specify the model we are concerned with in the rest of the paper. Following [12], we introduce a scaling parameter \( N = 1, 2, \ldots \) and modify the generator \( L \) in (2.4) by replacing \( K \) by \( K/N \). To be more precise, define
\[ L^N \Phi(z) = \frac{1}{2N} \sum_{i \neq j} z_i z_j \tilde{H}(z_i, z_j) \left( \Phi(M_{ij}z) - \Phi(z) \right) + \frac{1}{2} \sum_i z_i \int_0^{z_i} dy \tilde{H}(y, z_i - y) \left( \Phi(S_i(y) z) - \Phi(z) \right) \] (4.1)
and denote by \( \{ Z^N(t) = (Z^N_i(t))_{i=1}^\infty : t \geq 0 \} \) the process generated by \( L^N \). The rescaled process we will study actually is
\[ \xi^N(t) := \frac{1}{N} \Xi(Z^N(t)) = \frac{1}{N} \sum_i 1_{[0, \infty)}(Z^N_i(t)) \delta_{Z^N_i(t)}. \]
This process is regarded as a process taking values in \( \mathcal{M} \), the totality of locally finite measures on \( (0, \infty) \). \( \mathcal{M} \) is equipped with the vague topology. For \( a \geq 0 \), let \( \psi_a \) denote the power function \( \psi_a(y) = y^a \), so that \( \langle \psi_a, \zeta \rangle \) stands for the ‘\( a \)th moment’ of \( \zeta \in \mathcal{M} \). We consider \( c_0 \in \mathcal{M} \) such that for some \( \delta > 1 \)
\[ 0 < \langle \psi_1, c_0 \rangle < \infty \] and \[ \langle \psi_{2+\lambda+\delta}, c_0 \rangle < \infty. \] (4.2)
Note that (4.2) implies that \( \langle \psi_a, c_0 \rangle < \infty \) for any \( a \in [1, 2+\lambda+\delta] \). Concerning initial distributions of \( \{ Z^N(t) : t \geq 0 \} \), we suppose that
\[ E \left[ \left| Z^N(0) \right|^{2+\lambda+\delta} \right] < \infty \] for each \( N = 1, 2, \ldots \). (4.3)
It shall be required also that \( \xi^N(0) \) converges to \( c_0 \) in distribution as \( N \to \infty \). A typical case where such a convergence holds is given in the following lemma, which is stated in a general setting. In the rest, \( \text{Po}(m) \) stands for the law of a Poisson point process on \( (0, \infty) \) with mean measure \( m \in \mathcal{M} \).
Lemma 4.1 Let $\zeta \in \mathcal{M}$ be arbitrary. Assume that $\sum \delta_{Y^N_i}$ is $\text{Po}(N \zeta)$-distributed for each $N = 1, 2, \ldots$. Then $\eta^N := N^{-1} \sum \delta_{Y^N_i}$ converges to $\zeta$ in distribution as $N \to \infty$. Moreover, if $\langle \psi_1, \zeta \rangle < \infty$ and $\langle \psi_2, \zeta \rangle < \infty$ for some $a > 1$, then

$$E[\langle \psi_1, \eta^N \rangle^a] = E\left[\left(\frac{\sum Y^N_i}{N}\right)^a\right] \to \langle \psi_1, \zeta \rangle^a \quad \text{as} \quad N \to \infty. \quad (4.4)$$

In particular, $\sup_N E\left[(N^{-1} \sum Y^N_i)^a\right] < \infty$.

Since the proof of Lemma 4.1 is rather lengthy and not relevant to other parts of this paper, the proof will be given in Appendix. It is worth noting here that requiring the law of $\sum \delta_{Y^N_i}$ to be Poisson automatically implies that every correlation measure of it is the direct product of the mean measure.

By looking at the limit points of $\{\xi^N(t) : t \geq 0\}$ as $N \to \infty$ we intend to derive a weak solution to (1.1) with $K$ and $F$ given by (1.4), namely

$$\frac{\partial}{\partial t} c(t, x) \quad \text{(4.5)}$$

$$= \frac{1}{2} \int_0^x \left[ y(x - y) \hat{H}(y, x - y)c(t, x) - x \hat{H}(y, x - y)c(t, x) - x \hat{H}(y, x - y)c(t, x) \right] dy$$

$$- \int_0^\infty \left[ xy\hat{H}(x, y)c(t, x) - (x + y)\hat{H}(x, y)c(t, x + y) \right] dy.$$}

One may realize that in the case where both $\hat{H}$ and $\hat{H}$ are constants, $a$ and $b$, say, respectively, (4.5) can be transformed by considering the ‘size-biased version’ $c^*(t, x) := xc(t, x)$ into

$$x^{-1} \frac{\partial}{\partial t} c^*(t, x) = \frac{1}{2} \int_0^x \left[ ac^*(t, y)c^*(t, x - y) - bc^*(t, x) \right] dy$$

$$- \int_0^\infty \left[ ac^*(t, x)c^*(t, y) - bc^*(t, x + y) \right] dy.$$}

This equation is very similar to (1.1) with $K$ and $F$ being constants, the solution of which has been studied extensively in [1] and [31]. However, it is not clear whether or not there is any direct connection between solutions of these two equations. Turning to (4.5), the weak form with test functions $f \in B_c$ and initial measure $c_0$ reads

$$\int f(x)c(t, dx) - \int f(x)c_0(dx)$$

$$= \frac{1}{2} \int_0^t ds \int c(s, dx)c(s, dy)xy\tilde{H}(x, y)(\Box f)(x, y)$$

$$- \frac{1}{2} \int_0^t ds \int c(s, dx)x \int_0^x dy\tilde{H}(y, x - y)(\Box f)(y, x - y),$$

where $(\Box f)(x, y) = f(x + y) - f(x) - f(y)$. A rough idea for its derivation can be described as follows. For $f \in B_+ \cup B_c$ and $z \in \Omega$, set $\Phi_f(z) = \sum f(z)$, adopting the
Indeed, letting $s, t \geq 0$ and $\xi$ be replaced by $\hat{\xi}$, we have

\[ M_f^N(t) := \langle f, \xi^N(t) \rangle - \langle f, \xi^N(0) \rangle - \frac{1}{N} \int_0^t L^N \Phi_f(Z^N(s))ds \]

is a martingale and

\[ \langle f, \xi^N(t) \rangle - \langle f, \xi^N(0) \rangle - M_f^N(t) \]

where

\[ \xi^N(s) = \frac{1}{N^2} \sum_{i \neq j} \mathbb{1}_{[0, \infty)}(Z_i^N(s) Z_j^N(s)) \delta_{(Z_i^N(s), Z_j^N(s))} \cdot (\nabla Z_i^N(s)) \cdot (\nabla Z_j^N(s)). \]

(By virtue of Lemma 3.4 with $k = 1$, the integrability of $M_f^N(t)$ is ensured by (4.3).) As far as the limit as $N \to \infty$ is concerned, $\xi^N(s)$ in the right side of (4.6) can be replaced by $\xi^N(s)^{\otimes 2}$ under a suitable assumption on the convergence of $\xi^N(0)$. Indeed, letting $R > 0$ be such that $\text{supp}(f) \subset [0, R]$, we have by (3.4)

\[
\left| \int (\xi^N(s)^{[2]} - \xi^N(s)^{\otimes 2}) (dx dy) xy \bar{H}(x, y)(\nabla f)(x, y) \right|
\]

\[
\leq \frac{C R(2R)^{\lambda}}{N^2} \sum_i Z_i^N(s) \cdot 3 \|f\|_{\infty}
\]

Thus, at least at formal level, the derivation of a weak solution to (4.5) from $\xi^N(t)$ would reduce to proving that $M_f^N(t)$ vanishes in a suitable sense as $N \to \infty$.

To this end, we shall calculate the quadratic variation $\langle M_f^N \rangle(t)$. However, the square integrability of $M_f^N(t)$ is nontrivial under the condition (4.3). We will guarantee this in the next lemma by the cutoff argument as in the proof of Theorem 3.2. Also, the following inequality will be used to bound the quadratic variation: for any $s, t \geq 0$ and $a \geq 0$

\[ (s + t)^a \leq C_{1,a} (s^a + t^a), \]

where $C_{1,a} = 2^{a-1} \vee 1$. This inequality for $a > 1$ is deduced from Hölder’s inequality and the one for $0 < a \leq 1$ is implied by the identity

\[ (s + t)^a - s^a - t^a = a(a - 1) \int_0^s du \int_0^t dv (u + v)^{a-2}. \]
Lemma 4.2 Suppose that $E \left[ \left| Z^N(0) \right|^{2+\lambda} \right] < \infty$. For each $f \in B_c$, $\{M_f^N(t) : t \geq 0\}$ is a square integrable martingale with quadratic variation

$$\langle M_f^N \rangle(t) = \int_0^t \Gamma_f^N(Z^N(s))ds,$$

where $\Gamma_f^N : \Omega \to \mathbb{R}_+$ is defined to be

$$\Gamma_f^N(z) = \frac{1}{N^2} \left\{ L^N((\Phi_f)^2)(z) - 2\Phi_f(z)L^N\Phi_f(z) \right\}$$

$$= \frac{1}{2N^3} \sum_{i \neq j} z_i z_j \hat{H}(z_i, z_j) \{f(z_i + z_j) - f(z_i) - f(z_j)\}^2 \tag{4.9}$$

$$+ \frac{1}{2N^2} \sum_i z_i^2 \int_0^1 du \hat{H}(uz_i, (1-u)z_i) \{f(uz_i) + f((1-u)z_i) - f(z_i)\}^2$$

$$= \frac{1}{2} \Sigma_K^N(z) + \frac{1}{2} \Sigma_F^N(z).$$

Moreover, the following estimates hold:

$$\Sigma_K^N(z) \leq 18\hat{C}C_{1,\lambda} \frac{\|f\|_\infty^2}{N} \cdot |z| \sum_i \frac{z_i^{1+\lambda}}{N} = 18\hat{C}C_{1,\lambda} \frac{\|f\|_\infty^2}{N} \langle \psi_1, \xi^N \rangle \langle \psi_{1+\lambda}, \xi^N \rangle \tag{4.10}$$

and

$$\Sigma_F^N(z) \leq 9\hat{C} \frac{\|f\|_\infty^2}{N} \sum_i \frac{z_i^{2+\lambda}}{N} = 9\hat{C} \frac{\|f\|_\infty^2}{N} \langle \psi_{2+\lambda}, \xi^N \rangle, \tag{4.11}$$

where $\xi^N = N^{-1} \Xi(z) = N^{-1} \sum_1 (z_i > 0) \delta_{z_i}$.

Proof. For any $R > 0$, let $\Phi_f^{(R)}(z) = \Phi_f(z)1_{|z| \leq R}$ and observe that

$$M_f^N(t) := \frac{1}{N} \Phi_f^{(R)}(Z^N(t)) - \frac{1}{N} \Phi_f^{(R)}(Z^N(0)) - \frac{1}{N} \int_0^t L^N \Phi_f^{(R)}(Z^N(s))ds$$

$$= M_f^N(t)1_{|Z^N(0)| \leq R}$$

is a bounded martingale with quadratic variation

$$\langle M_f^{N,R} \rangle(t) = \frac{1}{N^2} \int_0^t \left[ L^N((\Phi_f^{(R)})^2)(Z^N(s)) - 2\Phi_f^{(R)}(Z^N(s))L^N\Phi_f^{(R)}(Z^N(s)) \right] ds$$

$$= \int_0^t \Gamma_f^N(Z^N(s))ds1_{|Z^N(0)| \leq R}.$$
This proves (4.10), whereas (4.11) is immediate from (3.6). The assumption together with these two estimates ensure the integrability of \( \int_0^t \Gamma_f^N(Z^N(s))ds \). Therefore, by the monotone convergence theorem

\[
E\left[(M_f^N(t))^2\right] = \lim_{R \to \infty} E\left[(M_f^{N,R}(t))^2\right] = \lim_{R \to \infty} E\left[\int_0^t \Gamma_f^N(Z^N(s))ds 1_{\{|z| \leq R\}}\right] = E\left[\int_0^t \Gamma_f^N(Z^N(s))ds\right] < \infty.
\]

Once the square integrability of \( M_f^N(t) \) is in hand, one can show further by a similar argument that \( (M_f^N(t))^2 - \int_0^t \Gamma_f^N(Z^N(s))ds \) is a martingale. The proof of Lemma 4.2 is complete.

Remarks. (i) A heuristic derivation of (1.1) based on the hierarchical structure discussed in the previous section is available: due to the rescaling \( K \mapsto K/N \) and \( \xi \mapsto \xi^N := \xi/N \), the equation solved weakly by the family \( \{r^N_k(t, dz_k) : t \geq 0, k \in \mathbb{N}\} \) of correlation measures of \( \xi^N(t) \) becomes

\[
\frac{\partial}{\partial t} r_k^N(t, z_1, \ldots, z_k) = \frac{1}{2} \sum_{l=1}^k \int_0^{z_l} K(y, z_l - y) r_{k+1}^N(t, z_1, \ldots, z_{l-1}, y, z_l - y, z_{l+1}, \ldots, z_k)dy
\]

\[
-\frac{1}{2} \sum_{l=1}^k \int_0^{z_l} F(y, z_l - y)dy r_k^N(t, z_1, \ldots, z_k)
\]

\[
-\sum_{l=1}^k \int_0^\infty K(z_l, y) r_{k+1}^N(t, z_1, \ldots, z_l, y, z_{l+1}, \ldots, z_k)dy
\]

\[
+\sum_{l=1}^k \int_0^\infty F(z_l, y) r_k^N(t, z_1, \ldots, z_l, y, z_{l+1}, \ldots, z_k)dy
\]

\[
-\frac{1_{\{k \geq 2\}}}{N} \sum_{l<m}^k K(z_l, z_m) r_k^N(t, z_1, \ldots, z_k)
\]

\[
+\frac{1_{\{k \geq 2\}}}{N} \sum_{l<m}^k F(z_l, z_m) r_{k-1}^N(t, z_1, \ldots, z_{l-1}, z_l + z_{l+1}, \ldots, z_{m-1}, z_m + z_{m+1}, \ldots, z_k).
\]

So, the last two terms would be expected to vanish in the limit as \( N \to \infty \) and the limits \( r_k \) of \( r_k^N \), if they exist, would solve the same equations as the ones satisfied by the direct products \( c \otimes^k (t, z_1, \ldots, z_k) = c(t, z_1) \cdots c(t, z_k) \) of a solution to (1.1). This procedure has been accomplished in [14] for a pure coagulation model.

(ii) We also give a remark on the asymptotic equivalence between the moment measures and the correlation measures of the rescaled process \( \xi^N(t) \). (cf. Lemma 1.16 in [23].) The reader is cautioned that our terminology ‘moment measure’ is in conflict with that of [23].) For each \( k \in \{2, 3, \ldots\} \), under the assumption that
\[ \sup_N E[|Z^N(0)/N|^{k-1}] < \infty, \] it holds that for any \( f \in B^{k}_+, c \)
\[
\left| E \left[ \frac{1}{N^k} \sum_{i_1,\ldots,i_k} f(Z^N_{i_1}(t),\ldots,Z^N_{i_k}(t)) \right] - E \left[ \frac{1}{N^k} \sum_{i_1,\ldots,i_k(\neq)} f(Z^N_{i_1}(t),\ldots,Z^N_{i_k}(t)) \right] \right| \leq \frac{C}{N}
\]
for some constant \( C \) independent of \( N \). Indeed, it is sufficient to verify this by assuming that \( f(z_1,\ldots,z_k) = z_1 \cdots z_k 1_{[\epsilon,R]}(z_1) \cdots 1_{[\epsilon,R]}(z_k) \) with \( 0 < \epsilon < R \), for which the above difference is dominated by a finite sum of expectations of the form
\[
E \left[ \frac{1}{N^k} \sum_{i_1,\ldots,i_j} Z^N_{i_1}(t)^{n_1} 1_{[\epsilon,R]}(Z^N_{i_1}(t)) \cdots Z^N_{i_j}(t)^{n_j} 1_{[\epsilon,R]}(Z^N_{i_j}(t)) \right],
\]
where \( j \in \{1,\ldots,k-1\}, n_1,\ldots,n_j \in \mathbb{N} \) is such that \( n_1 + \cdots + n_j = k \) and hence \( n_l \geq 2 \) for some \( l \in \{1,\ldots,j\} \). The desired bound follows by noting that
\[
Z^N_{i_l}(t)^{n_l} 1_{[\epsilon,R]}(Z^N_{i_l}(t)) \leq RZ^N_{i_l}(t)^{n_l-1} 1_{[\epsilon,R]}(Z^N_{i_l}(t))
\]
and observing that the above expectation is less than or equal to
\[
E \left[ \frac{R}{N^k} |Z^N(t)|^{k-1} \right] = \frac{R}{N} E \left[ \left( \frac{|Z^N(0)|}{N} \right)^{k-1} \right].
\]

### 4.2 Key estimates for the martingale

Lemma 4.2 implies that for some constant \( C_2 > 0 \) independent of \( N, t \) and \( f \)
\[
C_2 \| f \|^{2}_\infty E \left[ \langle M^N_f \rangle(t) \right] \leq \frac{1}{N} E \left[ \left( \frac{|Z^N(0)|}{N} \right) \int_0^t ds \sum_i \frac{Z^N_i(s)^{1+\lambda}}{N} \right] + \frac{1}{N} E \left[ \int_0^t ds \sum_i \frac{Z^N_i(s)^{2+\lambda}}{N} \right]. \tag{4.12}
\]
In fact, the two expectations on (4.12) are related to each other in such a way that
\[
E \left[ \frac{|Z^N(0)|}{N} \int_0^t ds \sum_i \frac{Z^N_i(s)^{1+\lambda}}{N} \right] \leq \left( tE \left[ \left( \frac{|Z^N(0)|}{N} \right)^{2+\lambda} \right] \right)^{\frac{\alpha}{\alpha+\gamma+\epsilon}} \left( E \left[ \int_0^t ds \sum_i \frac{Z^N_i(s)^{2+\lambda}}{N} \right] \right)^{\frac{\gamma+\epsilon}{\alpha+\gamma+\epsilon}}, \tag{4.13}
\]
which is a special case \( (\alpha = 1, \gamma = \lambda, \epsilon = 1) \) of the following lemma.

**Lemma 4.3** For arbitrary \( \alpha, \gamma \geq 0 \) and \( \epsilon > 0 \)
\[
E \left[ \left( \frac{|Z^N(0)|}{N} \right)^{\alpha} \int_0^t ds \sum_i \frac{Z^N_i(s)^{1+\gamma}}{N} \right] \leq \left( tE \left[ \left( \frac{|Z^N(0)|}{N} \right)^{1+\alpha+\gamma+\epsilon} \right] \right)^{\frac{\alpha}{\alpha+\gamma+\epsilon}} \left( E \left[ \int_0^t ds \sum_i \frac{Z^N_i(s)^{1+\gamma+\epsilon}}{N} \right] \right)^{\frac{\gamma+\epsilon}{\alpha+\gamma+\epsilon}}. \tag{4.14}
\]
Proof. Since (4.14) for $\gamma = 0$ is obvious, we may assume that $\gamma > 0$. Then put $p = 1 + \gamma/\epsilon$ and $q = 1 + \epsilon/\gamma$, which are mutually conjugate. By virtue of Hölder’s inequality with respect to the ‘weight’ $E[N^{-1}\int_0^t ds \sum_i \cdot]$

$$E \left[ \left( \frac{|Z^N(0)|}{N} \right)^\alpha \right] \leq E \left[ \left( \frac{|Z^N(0)|}{N} \right) \alpha \sum_i \left( \frac{Z_i^N(s)}{N} \right)^\frac{1}{p} \right]^\frac{1}{\alpha} \left( E \left[ \left( \frac{\int_0^t ds \sum_i Z_i^N(s)}{N} \right)^{1+\gamma - \frac{1}{q}} \right] \right)^\frac{1}{\gamma}.$$

Here, the final equality is due to $(1 + \gamma - \frac{1}{p})q = 1 + \gamma + \epsilon$. (4.14) has been obtained since $\alpha p + 1 = 1 + \alpha + \alpha \gamma/\epsilon$.

The expression in the right side of (4.13) motivates us to define

$$m_a = \sup_N E \left[ \left( \frac{|Z^N(0)|}{N} \right)^a \right] = \sup_N E \left[ \langle \psi_1, \xi^N(0) \rangle^a \right]$$

for $a \geq 0$. We will discuss under the condition that

$$m_{2+\lambda+\delta} < \infty. \quad (4.15)$$

This condition is stronger than (4.3) and valid in the case of Poisson processes considered in Lemma 4.1. So, we shall focus attention on estimation of the second term of (4.12). Before doing it, we prepare an elementary but technically important lemma.

Lemma 4.4 Let $A, B > 0$, a real number $C$ and $q > 1$ be given. If $A$ satisfies a ‘self-dominated inequality’ $A \leq BA^{1/q} + C$, then

$$A \leq B^{\frac{q}{q-1}} + \frac{q}{q-1}(C \lor 0).$$

Proof. Since the assertion is obvious when $C \leq 0$, we only have to consider the case $C > 0$. Denote by $\tilde{A}$ the right side of the required inequality. The equation $x = Bx^{1/q} + C$ for $x > 0$ has a unique root and $x > Bx^{1/q} + C$ for sufficiently large $x$. Therefore, it is enough to show that $\tilde{A} > B\tilde{A}^{1/q} + C$. Observe that for $s, t > 0$

$$(s + t)^{\frac{q}{q-1}} - s^{\frac{q}{q-1}} = \frac{1}{q} \int_0^t du (s + u)^{\frac{q}{q-1}} - \frac{1}{q} \int_0^t du s^{\frac{q}{q-1}} = \frac{1}{q} s^{\frac{q}{q-1}} - 1$$

and thus $(s + t)^{1/q} < s^{1/q} + s^{(1/q) - 1}t/q$. Plugging $s = B\tilde{A}^{\frac{q}{q-1}}$ and $t = \frac{q}{q-1}C$ gives

$$B\tilde{A}^{1/q} + C \leq B \left( B^{\frac{1}{q-1}} + B^{-1} \frac{q}{q-1} C \cdot \frac{1}{q} \right) + C = B^{\frac{q}{q-1}} + \frac{q}{q-1}C = \tilde{A}.$$
as desired.

The next proposition supplies the key to proceeding further.

**Proposition 4.5** Assume that (4.2) holds for some \( \delta > 1 \). Suppose that \( \sum \delta Z_i^N(0) \) has mean measure \( Nc_0 \) for each \( N = 1, 2, \ldots \) and that (4.15) holds. Then, for each \( t > 0 \), there exist constants \( \mathcal{C}(t), \mathcal{C}_0(t) \) and \( C^*(t) \) independent of \( N \) such that

\[
E \left[ \int_0^t ds \langle \psi_{2+\lambda+\delta}, \xi^N(s) \rangle \right] = E \left[ \int_0^t ds \sum_i Z_i^N(s)^{2+\lambda+\delta} \right] \leq \mathcal{C}(t) \tag{4.16}
\]

\[
E \left[ \int_0^t ds \langle \psi_{2+\lambda}, \xi^N(s) \rangle \right] = E \left[ \int_0^t ds \sum_i Z_i^N(s)^{2+\lambda} \right] \leq \mathcal{C}_0(t) \tag{4.17}
\]

and

\[
E \left[ \langle \psi, \xi^N(t) \rangle \right] = E \left[ \sum_i Z_i^N(t)^\delta \right] \leq C^*(t). \tag{4.18}
\]

**Proof.** Define bounded functions \( \Psi_\delta^R (z) = \sum_i z_i^\delta 1_{\{ |z| \leq R \}} \) on \( \Omega \) for all \( R > 0 \). By direct calculations

\[
L^N \Psi_\delta^R (z) = \frac{1}{2N} \sum_{i \neq j} z_i z_j \mathcal{H}(z_i, z_j) \left\{ (z_i + z_j)^\delta - z_i^\delta - z_j^\delta \right\} 1_{\{ |z| \leq R \}}
\]

\[
+ \frac{1}{2} \sum_i z_i^{2+\lambda+\delta} \int_0^t du \mathcal{H}(u, 1-u) \left\{ u^\delta + (1-u)^\delta - 1 \right\} 1_{\{ |z| \leq R \}}.
\]

A crucial point here is that by \( \delta > 1 \) and (4.8)

\[
\mathcal{C}_\delta := \int_0^1 du \mathcal{H}(u, 1-u) \left\{ 1 - u^\delta - (1-u)^\delta \right\} \in (0, \mathcal{C}).
\]

Taking expectation of the martingale

\[
\Psi_\delta^R (Z^N(t)) - \Psi_\delta^R (Z^N(0)) - \int_0^t ds L^N \Psi_\delta^R (Z^N(s))
\]

and then letting \( R \rightarrow \infty \) yield

\[
\frac{\hat{\mathcal{C}}_\delta}{2} E \left[ \int_0^t ds \sum_i Z_i^N(s)^{2+\lambda+\delta} \right] + E \left[ \sum_i Z_i^N(t)^\delta \right] = E \left[ \sum_i Z_i^N(0)^\delta \right] + \frac{1}{2N} E \left[ \int_0^t ds \sum_{i \neq j} Z_i^N(s)Z_j^N(s) \mathcal{H}(Z_i^N(s), Z_j^N(s)) \right.
\]

\[
\cdot \left\{ (Z_i^N(s) + Z_j^N(s))^\delta - Z_i^N(s)^\delta - Z_j^N(s)^\delta \right\}
\]

\[
\leq N \langle \psi, c_0 \rangle + \frac{\hat{\mathcal{C}}}{2N} E \left[ \int_0^t ds \sum_{i \neq j} Z_i^N(s)Z_j^N(s)(Z_i^N(s) + Z_j^N(s))^{\lambda+\delta} \right]
\]

\[
\leq N \langle \psi, c_0 \rangle + \frac{\hat{\mathcal{C}} C_{1,\lambda+\delta}}{2N} E \left[ \int_0^t ds \sum_{i \neq j} Z_i^N(s)Z_j^N(s) \left\{ Z_i^N(s)^{\lambda+\delta} + Z_j^N(s)^{\lambda+\delta} \right\} \right]
\]

\[
\leq N \langle \psi, c_0 \rangle + \frac{\hat{\mathcal{C}} C_{1,\lambda+\delta}}{N} E \left[ Z^N(0) \int_0^t ds \sum_i Z_i^N(s)^{1+\lambda+\delta} \right].
\]

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Note that all the expectations in the above are finite by (4.15). We have obtained
\[
\frac{\hat{C}_\delta}{2} E \left[ \int_0^t ds \sum_i Z_i^N(s)^{2+\lambda+\delta} \right] + E \left[ \sum_i \frac{Z_i^N(t)}{N} \right] \\
\leq \hat{C} C_{1,\lambda+\delta} E \left[ \frac{|Z^N(0)|}{N} \int_0^t ds \sum_i \frac{Z_i^N(s)^{1+\lambda+\delta}}{N} \right] + \langle \psi_\delta, c_0 \rangle.
\] (4.19)

In addition, Lemma 4.3 with \(\alpha = 1, \gamma = \lambda + \delta\) and \(\epsilon = 1\) reads
\[
E \left[ \frac{|Z^N(0)|}{N} \int_0^t ds \sum_i \frac{Z_i^N(s)^{1+\lambda+\delta}}{N} \right] \\
\leq \left( t E \left[ \frac{\left( \frac{|Z^N(0)|}{N} \right)^{2+\lambda+\delta}}{N} \right] \right)^{\frac{1}{1+\lambda+\delta}} \left( E \left[ \int_0^t ds \sum_i \frac{Z_i^N(s)^{2+\lambda+\delta}}{N} \right] \right)^{\frac{\lambda+\delta}{1+\lambda+\delta}}.
\] (4.20)

Set \(p = 1 + \lambda + \delta\) and \(q = 1 + 1/(\lambda + \delta)\). By combining the above two inequalities
\[
A_N := E \left[ \int_0^t ds \sum_i \frac{Z_i^N(s)^{2+\lambda+\delta}}{N} \right] \leq B_N (A_N)^{1/q} + \frac{2 \langle \psi_\delta, c_0 \rangle}{C_{\delta}},
\]
where
\[
B_N = \frac{2 \hat{C} C_{1,\lambda+\delta}}{C_{\delta}} t E \left[ \frac{\left( \frac{|Z^N(0)|}{N} \right)^{2+\lambda+\delta}}{N} \right]\,
\] (4.20)

Applying Lemma 4.4 and (4.15) and noting that \(\frac{q}{q-1} = p\), we obtain
\[
A_N \leq (B_N)^{\frac{q}{q-1}} + \frac{q}{q-1} \cdot \frac{2 \langle \psi_\delta, c_0 \rangle}{C_{\delta}} \\
\leq \left( \frac{2 \hat{C} C_{1,\lambda+\delta}}{C_{\delta}} \right)^{1+\lambda+\delta} \frac{2(1 + \lambda + \delta) \langle \psi_\delta, c_0 \rangle}{\hat{C}_\delta} \\
=: \hat{C}(t)
\]

This proves (4.16). The estimate (4.17) can be deduced from (4.16), Lemma 4.3
\((\alpha = 0, \gamma = 1 + \lambda\) and \(\epsilon = \delta\)) and (4.15). (4.18) follows from (4.19), (4.20), (4.15)
and (4.16). We complete the proof of Proposition 4.5.

**Corollary 4.6** Under the same assumptions as in Proposition 4.5, for each \(t > 0\),
there exists a constant \(C(t)\) independent of \(N \in \mathbb{N}\) and \(f \in B_c\) such that
\[
E \left[ \langle M^N_f \rangle(t) \right] \leq \frac{\|f\|_2^2}{N} C(t).
\] (4.21)

**Proof.** This is immediate from (4.12), (4.13), (4.15) and (4.17).

We end this section with a lemma which will be used in the next section. Recall
that for \(\gamma > 0\) the function \(\Psi_\gamma\) on \(\Omega\) is defined by \(\Psi_\gamma(z) = \sum z_i^\gamma\).
Lemma 4.7 Let \( z \in \Omega \) and put \( \xi^N = N^{-1} \Xi(z) = N^{-1} \sum 1_{\{z_i > 0\}} \delta_{z_i} \).

(i) For any \( f \in B_c \)

\[
\frac{1}{N} |L^N \Phi_f(z)| \\
\leq \hat{C} C_{1,\lambda} \|f\|_{\infty} \langle \psi_1, \xi^N \rangle \langle \psi_{1+\lambda}, \xi^N \rangle + \frac{\hat{C}}{2} \|f\|_{\infty} \langle \psi_{2+\lambda}, \xi^N \rangle 
\]

(4.22)

\[
\leq \hat{C} C_{1,\lambda} \|f\|_{\infty} \langle \psi_1, \xi^N \rangle \langle \psi_{1+\lambda}, \xi^N \rangle \frac{1}{2} + \frac{\hat{C}}{2} \|f\|_{\infty} \langle \psi_{2+\lambda}, \xi^N \rangle 
\]

(4.23)

provided that \( \langle \psi_{2+\lambda}, \xi^N \rangle < \infty \).

(ii) Let \( \gamma > 1 \) and assume that \( E \left[ |Z^N(0)|^{2+\lambda+\gamma} \right] < \infty \). Then

\[
\bar{M}^N_\gamma(t) := \frac{1}{N} \Psi_\gamma(Z^N(t)) - \frac{1}{N} \Psi_\gamma(Z^N(0)) - \frac{1}{N} \int_0^t L^N \Psi_\gamma(Z^N(s)) ds
\]

(4.24)

is a martingale. Moreover,

\[
\frac{1}{N} |L^N \Psi_\gamma(z)| \leq \hat{C} C_{1,\lambda+\gamma} \langle \psi_1, \xi^N \rangle \langle \psi_{1+\lambda+\gamma}, \xi^N \rangle + \frac{\hat{C}}{2} \langle \psi_{2+\lambda+\gamma}, \xi^N \rangle \langle \psi_{2+\lambda+\gamma}, \xi^N \rangle
\]

(4.25)

provided that \( \langle \psi_{2+\lambda+\gamma}, \xi^N \rangle \) < \( \infty \).

Proof. (i) (4.22) can be shown in just a similar way to (4.10) and (4.11) in view of (4.9) and

\[
\frac{1}{N} L^N \Phi_f(z) = \frac{1}{2N^2} \sum_{i \neq j} z_i z_j \bar{H}(z_i, z_j) \{ f(z_i + z_j) - f(z_i) - f(z_j) \} \\
+ \frac{1}{2N} \sum_i z_i^2 \int_0^1 du \bar{H}(uz, (1-u)z) \{ f(uz) + f((1-u)z) - f(z) \}.
\]

(4.23) is a consequence of Hölder’s inequality.

(ii) The proof is quite analogous to that of Lemma 4.2 in view of calculations at the beginning of the proof of Proposition 4.5 with \( \gamma \) in place of \( \delta \). So the details are left to the reader.

5 Derivation of the macroscopic equation

5.1 Tightness arguments

Before studying the limit of the rescaled processes discussed in Section 4, relative compactness of their laws must be argued. In fact, it will be convenient to consider, rather than \( \xi^N(t) \), the measure-valued process

\[
\mu^N(t) := \frac{1}{N} \sum_i Z_i^N(t) \delta_{Z_i^N(t)}.
\]
which takes values in $\mathcal{M}_f$, the space of finite measures on $(0, \infty)$, almost surely as long as $P(|Z^N(0)| < \infty) = 1$. Denote by $C_c$ the set of continuous functions on $(0, \infty)$ with compact support and set $C_{+c} = B_+ \cap C_c$. We begin the tightness argument by introducing a metric on $\mathcal{M}_f$ compatible with the weak topology by

$$d_w(\nu, \nu') = \sum_{k=0}^{\infty} 2^{-k} \left( |\langle h_k, \nu \rangle - \langle h_k, \nu' \rangle| \wedge 1 \right), \quad \nu, \nu' \in \mathcal{M}_f$$

where $h_0 \equiv 1$ and $\{h_1, h_2, \ldots\} \subset C_{+c}$ is as in the proof of Proposition 3.17 of [29]. (Alternatively, see A 7.7 of [21].) In particular, denoting the vague convergence by $\overset{v}{\rightharpoonup}$, we have, for $\eta, \eta_1, \eta_2, \ldots \in \mathcal{M}$, $\eta_n \overset{w}{\rightharpoonup} \eta$ iff $\langle h_k, \eta_n \rangle \to \langle h_k, \eta \rangle$ for all $k \in \mathbb{N}$. $(\mathcal{M}_f, d_w)$ is a complete, separable metric space. Let $\overset{w}{\rightharpoonup}$ stand for the weak convergence. Given $a, b, \gamma > 0$ arbitrarily, let

$$\mathcal{M}^\gamma_{a,b} = \{\nu \in \mathcal{M}_f : \langle 1, \nu \rangle \leq a, \langle \psi_\gamma, \nu \rangle \leq b\}.$$

As will be seen in the next two lemmas, it is a compact set in $\mathcal{M}_f$ and plays an important role. We introduce an auxiliary function $\varphi_R$ for each $R > 0$ by

$$\varphi_R(y) = \begin{cases} 1 & (y \leq R) \\ R + 1 - y & (R \leq y \leq R + 1) \\ 0 & (y \geq R + 1), \end{cases}$$

which is bounded and continuous on $(0, \infty)$.

**Lemma 5.1** Let $\mathcal{M}^\gamma_{a,b}$ be as above and $\nu_1, \nu_2, \ldots \in \mathcal{M}^\gamma_{a,b}$. Assume that $\nu_n \overset{w}{\rightharpoonup} \nu$ for some $\nu \in \mathcal{M}_f$. Then

(i) $\nu \in \mathcal{M}^\gamma_{a,b}$.

(ii) For any $\alpha \in (0, \gamma)$, $\langle \psi_\alpha, \nu_n \rangle \to \langle \psi_\alpha, \nu \rangle$ as $n \to \infty$.

**Proof** This is a special case of Lemma 4.1 in [12]. But we give a proof for completeness.

(i) It is clear that $\langle 1, \nu \rangle = \lim_{n \to \infty} \langle 1, \nu_n \rangle \leq a$. Also,

$$\langle \psi_\gamma \varphi_R, \nu \rangle = \lim_{n \to \infty} \langle \psi_\gamma \varphi_R, \nu_n \rangle \leq \liminf_{n \to \infty} \langle \psi_\gamma, \nu_n \rangle \leq b.$$

Letting $R \to \infty$, we get $\langle \psi_\gamma, \nu \rangle \leq b$. Hence $\nu \in \mathcal{M}^\gamma_{a,b}$.

(ii) Fix $\alpha \in (0, \gamma)$ arbitrarily. Observe that for each $R > 0$

$$|\langle \psi_\alpha, \nu_n \rangle - \langle \psi_\alpha, \nu \rangle| \leq |\langle \psi_\alpha \varphi_R, \nu_n \rangle - \langle \psi_\alpha \varphi_R, \nu \rangle| + \langle \psi_\alpha (1 - \varphi_R), \nu_n \rangle + \langle \psi_\alpha (1 - \varphi_R), \nu \rangle.$$

The first term on the right side converges to 0 as $n \to \infty$. As for the second and third terms, we have a uniform bound in $n$

$$\langle \psi_\alpha (1 - \varphi_R), \nu_n \rangle \leq \frac{\psi_\alpha}{\psi_\gamma} \cdot 1_{(R, \infty)} \cdot \psi_\gamma, \nu_n \leq R^{\alpha - \gamma} \langle \psi_\gamma, \nu_n \rangle \leq b R^{\alpha - \gamma}.$$
and similarly $\langle \psi_{\alpha}(1 - \varphi_R), \nu \rangle \leq bR^{\alpha - \gamma}$, which vanishes as $R \to \infty$. Therefore, $\langle \psi_{\alpha}, \nu_n \rangle \to \langle \psi_{\alpha}, \nu \rangle$ as $n \to \infty$.

Since we know from Lemma 5.1 (i) that $\mathcal{M}_{a,b}^\gamma$ are closed subsets of $(\mathcal{M}_f, d_w)$, the next lemma is regarded as a slight generalization of Lemma 4.2 in [12], which corresponding to the case $\gamma = 1$. Their proof will be arranged in an obvious manner.

**Lemma 5.2** For any $a, b, \gamma > 0$, $\mathcal{M}_{a,b}^{\gamma}$ is compact.

**Proof.** It is sufficient to prove that any sequence $\{\nu_n\} \subset \mathcal{M}_{a,b}^{\gamma}$ has a weakly convergent subsequence. Since $\sup_a \langle 1, \nu_n \rangle \leq a$, we can choose a subsequence $\{\nu_{n_k}\}$ for which $a_0 := \lim_{k \to \infty} \langle 1, \nu_{n_k} \rangle$ exists. In case $a_0 = 0$, $\nu_{n_k} \overset{w}{\to} 0$. In case $a_0 > 0$, we may assume further that $\langle 1, \nu_{n_k} \rangle > a_0/2$ for all $k$ and consider probability measures $\tilde{\nu}_k := \langle 1, \nu_{n_k} \rangle^{-1} \nu_{n_k}$. The family $\{\tilde{\nu}_k\}$ is tight because

$$\tilde{\nu}_k ([R, \infty)) = \frac{\langle 1_{[R, \infty)}, \nu_{n_k} \rangle}{\langle 1, \nu_{n_k} \rangle} \leq R^{-\gamma} \frac{\langle \psi_{\gamma} 1_{[R, \infty)}, \nu_{n_k} \rangle}{\langle 1, \nu_{n_k} \rangle} < R^{-\gamma} \frac{2b}{a_0}.$$ 

Taking its subsequence $\{\tilde{\nu}_{k_l}\}$ such that $\tilde{\nu}_{k_l} \overset{w}{\to} \tilde{\nu}$ as $l \to \infty$ for some $\tilde{\nu} \in \mathcal{M}_f$, we see that $\nu_{n_{k_l}} \overset{w}{\to} a_0 \tilde{\nu}$ as desired.

We now prove the compact containment property of the laws of $\{\mu^N(t) : t \geq 0\}$. Note the triviality $\langle \psi_{\alpha}, \mu^N(t) \rangle = \langle \psi_{\alpha+1}, \xi^N(t) \rangle$. More generally, for any $f \in B_c$ $\langle f, \mu^N(t) \rangle = \langle f^*, \xi^N(t) \rangle$, where $f^* \in B_c$ is defined to be $f^*(y) = yf(y)$.

**Proposition 5.3** Assume that (4.2) holds for some $\delta > 1$. Suppose that $\sum \delta Z^N_i(t)$ is $\text{Po}(Nc_0)$-distributed for each $N = 1, 2, \ldots$. Then for any $T > 0$ and $\epsilon \in (0, 1)$ there exist $a, b > 0$ such that

$$\inf_N P \left( \mu^N(t) \in \mathcal{M}_{a,b}^{\delta - 1} \text{ for all } t \in [0, T] \right) \geq 1 - \epsilon. \quad (5.1)$$

**Proof.** Lemma 4.1 with $\alpha = 2 + \lambda + \delta$ ensures the validity of (4.15), which enables us to apply Proposition 4.5 and Lemma 4.7 with $\gamma = \delta$. Since

$$\langle 1, \mu^N(t) \rangle = \langle \psi_1, \xi^N(t) \rangle = N^{-1}|Z^N(t)| = N^{-1}|Z^N(0)| = \langle \psi_1, \xi^N(0) \rangle$$

and

$$\langle \psi_{\delta-1}, \mu^N(t) \rangle = \langle \psi_{\delta}, \xi^N(t) \rangle = \langle \psi_{\delta}, \xi^N(0) \rangle + \frac{1}{N} \int_0^t L^N \Psi_\delta(Z^N(s)) ds + \tilde{M}_\delta^N(t),$$

Chebyshev’s inequality and Doob’s inequality for submartingale (e.g. Corollary 2.17, Chapter 2 in [15]) together yield

$$1 - P \left( \mu^N(t) \in \mathcal{M}_{a,b}^{\delta - 1} \text{ for all } t \in [0, T] \right) \leq P \left( \sup_{t \in [0, T]} \langle 1, \mu^N(t) \rangle > a \right) + P \left( \sup_{t \in [0, T]} \langle \psi_{\delta-1}, \mu^N(t) \rangle > b \right)$$

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\[
\begin{align*}
\leq P\left(\langle \psi_1, \xi^N(0) \rangle > a \right) + P\left(\langle \psi_\delta, \xi^N(0) \rangle > \frac{b}{3} \right) \\
+ P\left(\frac{1}{N} \int_0^T |L^N \Psi_\delta(Z^N(s))| \, ds > \frac{b}{3} \right) + P\left(\sup_{t \in [0,T]} |\tilde{M}^N_\delta(t)| > \frac{b}{3} \right) \\
= a^{-1} E\left[\langle \psi_1, \xi^N(0) \rangle \right] + 3b^{-1} E\left[\langle \psi_\delta, \xi^N(0) \rangle \right] \\
+ 3b^{-1} E\left[\frac{1}{N} \int_0^T |L^N \Psi_\delta(Z^N(s))| \, ds \right] + 3b^{-1} E\left[|\tilde{M}^N_\delta(T)| \right].
\end{align*}
\]

Therefore, the proof of (5.1) reduces to showing that the four expectations in the above are bounded in \( N \). The first two ones are finite and independent of \( N \):

\[ E\left[\langle \psi_1, \xi^N(0) \rangle \right] = \langle \psi_1, c_0 \rangle \quad \text{and} \quad E\left[\langle \psi_\delta, \xi^N(0) \rangle \right] = \langle \psi_\delta, c_0 \rangle. \]

For the third expectation we deduce from (4.25) and (4.20) that

\[ E_N := E\left[\frac{1}{N} \int_0^T |L^N \Psi_\delta(Z^N(s))| \, ds \right] \leq \hat{C} C_{1,\lambda+\delta} E\left[\int_0^T \langle \psi_1, \xi^N(0) \rangle \langle \psi_{1+\lambda+\delta}, \xi^N(s) \rangle \, ds \right] + \frac{\hat{C}_\delta}{2} E\left[\int_0^T \langle \psi_{2+\lambda+\delta}, \xi^N(s) \rangle \, ds \right] \]

\[ \leq \hat{C} C_{1,\lambda+\delta} \left(T \cdot E\left[\langle \psi_1, \xi^N(0) \rangle^{2+\lambda+\delta} \right]\right)^\frac{\lambda+\delta}{1+\lambda+\delta} \left(E\left[\int_0^T \langle \psi_{2+\lambda+\delta}, \xi^N(s) \rangle \, ds \right]\right)^\frac{1+\delta}{1+\lambda+\delta} + \frac{\hat{C}_\delta}{2} E\left[\int_0^T \langle \psi_{2+\lambda+\delta}, \xi^N(s) \rangle \, ds \right]. \]

Thanking to \( m_{2+\lambda+\delta} < \infty \) and (4.16), this is bounded in \( N \). Lastly, in view of (4.24)

\[ E\left[|\tilde{M}^N_\delta(T)| \right] \leq E\left[\langle \psi_\delta, \xi^N(T) \rangle \right] + E\left[\langle \psi_\delta, \xi^N(0) \rangle \right] + E_N \leq C^* T + \langle \psi_\delta, c_0 \rangle + E_N, \]

where the last inequality follows from (4.18). We complete the proof of Proposition 5.3 since we have already seen that \( \sup_N E_N < \infty \).

Let \( D([0, \infty), \mathcal{M}_f) \) denote the space of right continuous functions \( \nu(\cdot) \) from \([0, \infty)\) into \( \mathcal{M}_f \) with left limits. This space is equipped with a metric which induces the Skorohod topology. We now state the main result of this subsection, which establishes tightness of the rescaled processes. Roughly speaking, the reason why a stronger assumption on \( c_0 \) than the one in Proposition 5.3 is made here is that showing equicontinuity of \( t \mapsto \langle f, \xi^N(t) \rangle \) requires to dominate \( N^{-1} E[|L^N \Phi_f(Z^N(t))|] \) uniformly in \( t > 0 \).

Theorem 5.4 Assume (4.2) with \( \delta = 2 + \lambda \), i.e.,

\[ 0 < \langle \psi_1, c_0 \rangle < \infty \quad \text{and} \quad \langle \psi_{4+2\lambda}, c_0 \rangle < \infty. \]

Suppose that \( \sum \delta Z^N_{i(0)} \) is \( Po(Nc_0) \)-distributed for each \( N = 1, 2, \ldots \). Then the sequence \( \{P^N\}_{N=1}^\infty \) of the laws \( P^N \) of \( \{\mu^N(t) : t \geq 0\} \) on \( D([0, \infty), \mathcal{M}_f) \) is relatively compact.
Proof. Lemma 5.2 and Proposition 5.3 together imply that for arbitrarily fixed \( t \geq 0 \) the family of the laws of \( \mu^N(t) (N = 1, 2, \ldots) \) on \( \mathcal{M}_f \) is relatively compact. Therefore, for the same reasoning as in the proof of Theorem 2.1 in [13], which exploits Corollary 7.4 in Chapter 3 of [15], it is sufficient to prove that, for any \( T > 0 \) and \( \epsilon > 0 \), there exists \( 0 < \Delta < 1 \) such that

\[
\limsup_{N \rightarrow \infty} P \left( \max_{i} \sup_{t_i < T, s \in [t_i, t_{i+1}]} d_w(\mu^N(s), \mu^N(t_i)) > \frac{\epsilon}{2} \right) \leq \epsilon, \tag{5.2}
\]

where \( t_i = i\Delta \) \( (i = 0, 1, 2, \ldots) \). By (5.1) with \( \delta = 2 + \lambda \) we can find \( a,b > 0 \) such that

\[
\inf_N P \left( W_{a,b}^N(T) \right) \geq 1 - \frac{\epsilon}{2},
\]

where \( W_{a,b}^N(T) \) is the event that \( \mu^N(t) \in \mathcal{M}^{1+\lambda}_{a,b} \) for all \( t \in [0, T + 1] \). So, we only have to prove

\[
\limsup_{N \rightarrow \infty} P \left( \max_{i} \sup_{t_i < T, s \in [t_i, t_{i+1}]} d_w(\mu^N(s), \mu^N(t_i)) > \frac{\epsilon}{2} \right) \cap W_{a,b}^N(T) \leq \frac{\epsilon}{2}. \tag{5.3}
\]

It follows from (4.23) that, on \( W_{a,b}^N(T) \), for any \( t \in [0, T] \) and \( s \in [t, t + \Delta) \)

\[
|h_k, \mu^N(s)| - |h_k, \mu^N(t)| = |\langle h^*_k, \xi^N(s) \rangle - \langle h^*_k, \xi^N(t) \rangle|
\]

\[
\leq |M^N_k(s) - M^N_k(t)| + \frac{1}{N} \int_t^s \left| L^N \Phi_{h^*_k}(Z^N(u)) \right| du
\]

\[
\leq 2 \sup_{u \in [0, T + 1]} |M^N_k(u)| + \Delta \left( \tilde{C} C_{1,\lambda} a^{\frac{2+\lambda}{1+\lambda}} b^{\frac{\lambda}{1+\lambda}} + \frac{\tilde{C}}{2} b \right) \|h^*_k\|_{\infty}.
\]

Hence, taking \( k_0 \in \mathbb{N} \) such that \( \sum_{k=k_0+1}^{\infty} 2^{-k} < \epsilon/4 \) and then \( \Delta \) sufficiently small so that

\[
\Delta \left( \tilde{C} C_{1,\lambda} a^{\frac{2+\lambda}{1+\lambda}} b^{\frac{\lambda}{1+\lambda}} + \frac{\tilde{C}}{2} b \right) \|h^*_k\|_{\infty} < \frac{\epsilon}{8k_0}
\]

for all \( k \in \{1, \ldots, k_0\} \), we get

\[
P \left( \max_{i} \sup_{t_i < T, s \in [t_i, t_{i+1}]} d_w(\mu^N(s), \mu^N(t_i)) > \frac{\epsilon}{2} \right) \cap W_{a,b}^N(T)
\]

\[
\leq P \left( \sum_{k=1}^{k_0} 2^{-k} \left\{ 2 \sup_{u \in [0, T + 1]} |M^N_k(u)| + \frac{\epsilon}{8k_0} \right\} + \sum_{k=k_0+1}^{\infty} 2^{-k} > \frac{\epsilon}{2} \right)
\]

\[
\leq \sum_{k=1}^{k_0} P \left( 2 \sup_{u \in [0, T + 1]} |M^N_k(u)| + \frac{\epsilon}{8k_0} > \frac{\epsilon}{4k_0} \right)
\]

\[
\leq \frac{16k_0}{\epsilon} \sum_{k=1}^{k_0} E \left[ |M^N_k(T + 1)| \right],
\]

in which the last inequality is implied by Doob’s inequality. Since each expectation in the above sum converges to 0 as \( N \rightarrow \infty \) by Corollary 4.6, (5.3) and hence (5.2) are obtained. The proof of Theorem 5.4 is complete. 

5.2 Studying the limit laws

This section is devoted to the study of the weak limit of an arbitrary convergent subsequence \( \{P^N_k\}_{k=1}^\infty \), say, of the laws \( P^N \) of \( \{\mu^N(t) : t \geq 0\} \) on \( D([0, \infty), \mathcal{M}_f) \). Although our main concern will be proving that under the limit law the weak form of (4.5) is satisfied almost surely, some properties on the limit are shown in advance. Let \( C([0, \infty), \mathcal{M}_f) \) be the space of continuous functions from \([0, \infty)\) to \( \mathcal{M}_f \). According to §10 of Chapter 3 of [15] it is equipped with the metric

\[
d_{C}(\nu_1(\cdot), \nu_2(\cdot)) = \int_0^\infty e^{-u} \sup_{t \in [0, a]} \{d_w(\nu_1(t), \nu_2(t)) \wedge 1\} du,
\]

which gives the topology of uniform convergence on compact subsets of \([0, \infty)\).

**Lemma 5.5** If \( \{\mu^N(t) : t \geq 0\} \) converges to a process \( \{\mu(t) : t \geq 0\} \) in distribution on \( D([0, \infty), \mathcal{M}_f) \) as \( l \to \infty \), then \( \mu(\cdot) \in C([0, \infty), \mathcal{M}_f) \) a.s.

**Proof.** As in the proof of Lemma 5.1 in [13], we employ Theorem 10.2 in Chapter 3 of [15]. To this end, observe that if \( \mu^N(t) \neq \mu^N(t-) = N^{-1} \sum z_i \delta_{z_i} \) for some \( z \in \Omega \), the signed measure \( \mu^N(t) - \mu^N(t-) \) equals either

\[
\frac{1}{N}(z_i + z_j)\delta_{z_i + z_j} - \frac{1}{N}(z_i \delta_{z_i} + z_j \delta_{z_j}) \quad \text{for some } z_i, z_j > 0 \text{ with } i \neq j
\]

or

\[
\frac{1}{N}(y \delta_y + (z_i - y) \delta_{z_i - y}) - \frac{1}{N}z_i \delta_{z_i} \quad \text{for some } z_i > 0 \text{ and } y \in (0, z_i).
\]

This implies that

\[
d_w(\mu^N(t), \mu^N(t-)) \\
\leq \sup_{i \neq j} \sum_{k=0}^\infty 2^{-k} \min \left\{ 1, N^{-1} |h^*_k(z_i + z_j) - h^*_k(z_i) - h^*_k(z_j)| \right\} \\
\vee \sup_i \sup_{y \in (0, z_i)} \sum_{k=0}^\infty 2^{-k} \min \left\{ 1, N^{-1} |h^*_k(y) + h^*_k(z_i - y) - h^*_k(z_i)| \right\} \\
\leq \sum_{k=1}^\infty 2^{-k} \min \{1, 3N^{-1} \|h^*_k\|_\infty\} \to 0 \quad \text{as } N \to \infty,
\]

in which all the terms for \( k = 0 \) in the sums vanish because of \( h^*_0(y) = y \). So, the above mentioned theorem proves the assertion. \( \blacksquare \)

Given \( \nu \in \mathcal{M} \), we define two measures \( \nu^* \) and \( \nu_* \) on \((0, \infty)\) by \( \nu^*(dy) = y\nu(dy) \) and \( \nu_*(dy) = y^{-1}\nu(dy) \), respectively. For instance, \( \mu^N(t) = \xi^N(t)^* \) and conversely \( \xi^N(t) = \mu^N(t)^* \).

**Lemma 5.6** Suppose that the same assumptions as in Proposition 5.3 hold. If \( \{\mu^N(t) : t \geq 0\} \) converges to a process \( \{\mu(t) : t \geq 0\} \) in distribution on \( D([0, \infty), \mathcal{M}_f) \) as \( l \to \infty \), then

\[
P(\mu(0) = c_0^*, \langle 1, \mu(t) \rangle = \langle 1, \mu(0) \rangle \text{ for all } t \geq 0) = 1 \quad (5.4)
\]
and for each $T > 0$ there exist a constant $\bar{C}(T)$ such that

$$E \left[ \langle \psi_{\delta-1}, \mu(T) \rangle + \int_0^T \{ \langle 1, \mu(0) \rangle \langle \psi_{\lambda+\delta}, \mu(s) \rangle + \langle \psi_{1+\lambda+\delta}, \mu(s) \rangle \} \, ds \right] \leq \bar{C}(T). \quad (5.5)$$

Proof. By the assumption on initial distributions and Lemma 4.1 together

$$P(\mu(0) = c_0) = 1 \quad \text{or equivalently} \quad P(\mu(0) = c_0^*) = 1.$$ 

For each $N$, $P(\langle 1, \mu(t)^N \rangle = \langle 1, \mu^N(0) \rangle$ for all $t \geq 0) = 1$ and as is seen easily

$$\{ \nu(\cdot) \in D([0, \infty), \mathcal{M}_f) : \langle 1, \nu(t) \rangle = \langle 1, \nu(0) \rangle \text{ for all } t \geq 0 \}$$

is a closed subset of $D([0, \infty), \mathcal{M}_f)$. So, by the assumed convergence in distribution $P(\langle 1, \mu(t) \rangle = \langle 1, \mu(0) \rangle$ for all $t \geq 0) = 1$, and summarizing, we have shown (5.4).

We proceed to verification of (5.5). In estimating $E_N$ in the proof of Proposition 5.3 we have shown the existence of a constant $\tilde{C}_1(T)$ independent of $N$ such that

$$E \left[ \int_0^T \{ \langle 1, \mu_N(0) \rangle \langle \psi_{\lambda+\delta}, \mu_N(s) \rangle + \langle \psi_{1+\lambda+\delta}, \mu_N(s) \rangle \} \, ds \right] \leq \tilde{C}_1(T).$$

Combining this with (4.18), we obtain

$$E \left[ \left\langle \psi_{\delta-1}, \mu^N(T) \right\rangle \right]$$

$$+ \quad E \left[ \int_0^T \{ \langle 1, \mu^N(0) \rangle \langle \psi_{\lambda+\delta}, \mu^N(s) \rangle + \langle \psi_{1+\lambda+\delta}, \mu^N(s) \rangle \} \, ds \right] \leq C(T),$$

where $\bar{C}(T) = \tilde{C}_1(T) + C^*(T)$. Therefore, for any $a, R > 0$ and $l \in \mathbb{N}$

$$E \left[ a \wedge \left\langle \psi_{\delta-1} \varphi_R, \mu^{N_l}(T) \right\rangle \right]$$

$$+ \quad E \left[ \int_0^T \min \left\{ a, \langle 1, \mu^{N_l}(0) \rangle \langle \psi_{\lambda+\delta} \varphi_R, \mu^{N_l}(s) \rangle + \langle \psi_{1+\lambda+\delta} \varphi_R, \mu^{N_l}(s) \rangle \right\} \, ds \right] \leq \bar{C}(T). \quad (5.7)$$

It is not difficult to check that the function on $D([0, \infty), \mathcal{M}_f)$

$$\nu(\cdot) \mapsto \int_0^T \min \left\{ a, \langle 1, \nu(0) \rangle \langle \psi_{\lambda+\delta} \varphi_R, \nu(s) \rangle + \langle \psi_{1+\lambda+\delta} \varphi_R, \nu(s) \rangle \right\} \, ds$$

is bounded and continuous. Hence, letting $l \to \infty$ in (5.7) yields (5.7) with $\mu$ in placed of $\mu^{N_l}$. Since $a$ and $R$ are arbitrary, (5.5) holds true by virtue of the monotone convergence theorem.

We are in a position to state the main result of this section. It should be emphasized that the continuity of $K$ and $F$ is required only in the next theorem.
**Theorem 5.7** In addition to the assumptions of Theorem 5.4, assume that \( \hat{H} \) and \( \hat{H} \) are continuous. If \( \{\mu^N(t) : t \geq 0\} \) converges to a process \( \{\mu(t) : t \geq 0\} \) in distribution on \( D([0, \infty), \mathcal{M}_f) \) as \( l \to \infty \), then \( \{\xi^N(t) : t \geq 0\} \) converges in distribution on \( D([0, \infty), \mathcal{M}) \) to \( \{\xi(t) : t \geq 0\} \) defined by \( \xi(t) = \mu(t)_* \). Moreover, with probability 1, it holds that for any \( f \in \mathcal{B}_c \) and \( t \geq 0 \)

\[
\langle f, \xi(t) \rangle - \langle f, c_0 \rangle = \frac{1}{2} \int_0^t ds \int \xi(s) \otimes_2 (dx, dy) \hat{H}(x, y)(\Box f)(x, y) - \frac{1}{2} \int_0^t ds \int \xi(s)(dx) x \int_0^x dy \hat{H}(y, x-y)(\Box f)(y, x-y)
\]

with the integrals on the right side being absolutely convergent.

From the analytic view point, this theorem particularly implies the existence of a \( \mathcal{M} \)-valued weak solution to (4.5) with symmetric continuous homogeneous functions \( \hat{H} \) and \( \hat{H} \) of degree \( \lambda \geq 0 \) satisfying (H1) and (H2), respectively, and with initial measure \( c_0 \) such that \( 0 < \langle \psi_1, c_0 \rangle < \infty \) and \( \langle \psi_{1+2\lambda}, c_0 \rangle < \infty \). Unfortunately, the uniqueness of the solution has not been proved and accordingly the convergence of the laws of \( \{\mu^N(t) : t \geq 0\} \) \( (N = 1, 2, \ldots) \) has not been obtained. Concerning this point, some comments will be given at the end of this section.

**Proof of Theorem 5.7.** In order to prove the first half it is sufficient to verify continuity of the map \( \{\nu(t) : t \geq 0\} \mapsto \{\nu(t)_* : t \geq 0\} \) from \( D([0, \infty), \mathcal{M}_f) \) to \( D([0, \infty), \mathcal{M}) \).

But, that continuity follows from an equivalent condition to the convergence with respect to the Skorohod topology (e.g. Proposition 5.3 of Chapter 3 in [15]) in terms of the metric on the state space \( \mathcal{M}_f \) or \( \mathcal{M} \) together with continuity of the map \( \nu \mapsto \nu_* \) from \( \mathcal{M}_f \) to \( \mathcal{M} \). (cf. Problem 13 of Chapter 3 in [15].)

The proof of the last half is divided into five steps.

**Step 1.** As was sketched roughly in §4.1, the argument will be based on (4.6). So, for any \( f \in \mathcal{C}_c \) and \( t > 0 \), set for \( \nu(\cdot) \in D([0, \infty), \mathcal{M}_f) \)

\[
I_{t,f}(\nu(\cdot)) = \frac{1}{2} \int_0^t ds \int \nu(s) \otimes_2 (dx, dy) \hat{H}(x, y)(\Box f)(x, y) - \frac{1}{2} \int_0^t ds \int \nu(s)(dx) x \int_0^x dy \hat{H}(y, x-y)(\Box f)(y, x-y)
\]

provided that both integrals converge absolutely. Then, by the assumption and (4.23) we have almost surely

\[
\langle f, \xi^N(t) \rangle - \langle f, \xi^N(0) \rangle - I_{t,f}(\mu^N(\cdot)) = M^N_f(t) + D^N_f(t) \quad \text{for all } t \geq 0,
\]

where

\[
D^N_f(t) = \frac{1}{2} \int_0^t ds \int \left( \xi^N(s)[2] - \xi^N(s) \otimes_2 \right) (dx, dy) xy \hat{H}(x, y)(\Box f)(x, y)
\]

\[
= \frac{1}{2N^2} \int_0^t ds \sum_i Z^N_i(s)^2 \hat{H}(Z^N_i(s), Z^N_i(s)) \left\{ f(2Z^N_i(s)) - 2f(Z^N_i(s)) \right\}.
\]
By the same calculations as in §4.1 (see the observation after (4.6)) it is readily shown that for some constant $C_f'$ independent of $N \in \mathbb{N}$ and $T > 0$

$$E \left[ \sup_{t \in [0,T]} |D_f^N(t)| \right] \leq \frac{C_f'T}{N}.$$  

By combining this with (4.21) and using Doob’s inequality we get for any $\epsilon > 0$

$$P \left( \sup_{t \in [0,T]} |\langle f, \xi^N(t) \rangle - \langle f, \xi^N(0) \rangle - I_{t,f}(\mu^N(\cdot))| > \epsilon \right) \leq P \left( \sup_{t \in [0,T]} |M_f^N(t)| > \frac{\epsilon}{2} \right) + P \left( \sup_{t \in [0,T]} |D_f^N(t)| > \frac{\epsilon}{2} \right) \leq \left( \frac{2}{\epsilon} \right)^2 \frac{4||f||_{\infty}^2 C(T)}{N} + \frac{2}{\epsilon} \cdot \frac{C_f'T}{N}. \tag{5.9}$$

Therefore, the main task in the rest of the proof is to show, in a suitable sense, convergence of $I_{t,f}(\mu^{N_l}(\cdot))$ to $I_{t,f}(\mu(\cdot))$ as $l \to \infty$. But $I_{t,f}$ cannot be defined as a function on $D([0, \infty), \mathcal{M}_f)$ and we need to handle by a cut-off argument.

**Step 2.** For each $R > 0$ decompose $I_{t,f}$ in the form

$$I_{t,f}(\nu(\cdot)) = I_{t,f,R}^{(2)}(\nu(\cdot)) + \tilde{I}_{t,f,R}^{(2)}(\nu(\cdot)) - I_{t,f,R}^{(1)}(\nu(\cdot)) - \tilde{I}_{t,f,R}^{(1)}(\nu(\cdot)), \tag{5.10}$$

where

$$I_{t,f,R}^{(2)}(\nu(\cdot)) = \frac{1}{2} \int_0^t ds \int \nu(s) \otimes 2(dx dy) \tilde{H}(x,y)(\Box f)(x,y)\varphi_R(x+y),$$

$$\tilde{I}_{t,f,R}^{(2)}(\nu(\cdot)) = \frac{1}{2} \int_0^t ds \int \nu(s) \otimes 2(dx dy) \tilde{H}(x,y)(\Box f)(x,y)(1 - \varphi_R(x+y)),$$

$$I_{t,f,R}^{(1)}(\nu(\cdot)) = \frac{1}{2} \int_0^t ds \int \nu(s)(dx) \int_0^x dy \tilde{H}(y,x-y)(\Box f)(y,x-y)\varphi_R(x)$$

and

$$\tilde{I}_{t,f,R}^{(1)}(\nu(\cdot)) = \frac{1}{2} \int_0^t ds \int \nu(s)(dx) \int_0^x dy \tilde{H}(y,x-y)(\Box f)(y,x-y)(1 - \varphi_R(x)).$$

Of course $I_{t,f,R}^{(2)}$ and $I_{t,f,R}^{(1)}$ should be dominant for $R$ large. Putting $\delta = \lambda + 2$, we actually claim that

$$\sup_{t \in [0,T]} |\tilde{I}_{t,f,R}^{(2)}(\nu(\cdot))| \leq \frac{\tilde{C} C_{1,\lambda+\delta}}{R^\delta} \|\Box f\|_\infty \int_0^T \langle 1, \nu(s) \rangle \langle \psi_{\lambda+\delta}, \nu(s) \rangle ds \tag{5.11}$$

and

$$\sup_{t \in [0,T]} |\tilde{I}_{t,f,R}^{(1)}(\nu(\cdot))| \leq \frac{\tilde{C}}{2R^\delta} \|\Box f\|_\infty \int_0^T \langle \psi_{1+\lambda+\delta}, \nu(s) \rangle ds \tag{5.12}$$

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whenever each integral on the right side is finite. Indeed, by (3.4) and (4.7)

\[
|\tilde{I}^{(2)}_{t,f,R}(\nu(\cdot))| \leq \frac{\tilde{C}}{2} \int_0^t ds \int \nu(s)^{\alpha^2} (dx dy) (x + y)^{\lambda} |(\square f)(x,y)| 1_{\{x+y>R\}}
\]

\[
\leq \frac{\tilde{C}}{2} \|\square f\|_{\infty} \int_0^t ds \int \nu(s)^{\alpha^2} (dx dy) (x + y)^{\lambda+\delta} \frac{1_{\{x+y>R\}}}{R^\delta}
\]

\[
\leq \frac{\tilde{C} C_{1,\lambda+\delta}}{2R^\delta} \|\square f\|_{\infty} \int_0^t ds \int \nu(s)^{\alpha^2} (dx dy) (x^{\lambda+\delta} + y^{\lambda+\delta}),
\]

from which (5.11) is immediate. Similarly, by (3.6)

\[
|\tilde{I}^{(1)}_{t,f,R}(\nu(\cdot))| \leq \frac{\tilde{C}}{2} \|\square f\|_{\infty} \int_0^t ds \int \nu(s)(dx)x^{1+\lambda} \cdot 1_{\{x>R\}}
\]

\[
\leq \frac{\tilde{C}}{2} \|\square f\|_{\infty} \int_0^t ds \int \nu(s)(dx)x^{1+\lambda+\delta} \frac{1_{\{x>R\}}}{R^\delta}
\]

\[
\leq \frac{\tilde{C}}{2R^\delta} \|\square f\|_{\infty} \int_0^t ds \int \nu(s)(dx)x^{1+\lambda+\delta}
\]

and thus (5.12) is valid. Now consider (5.11) and (5.12) with \(\nu(\cdot)\) being a random element \(\mu^N(\cdot)\) or \(\mu(\cdot)\). Thanking to Chebyshev’s inequality, taking expectations and then using (5.6), (5.4) and (5.5) lead to

\[
P\left( \sup_{t\in[0,T]} \{|\tilde{I}^{(2)}_{t,f,R}(\mu^N(\cdot))| + |\tilde{I}^{(1)}_{t,f,R}(\mu^N(\cdot))| \} > \epsilon \right) \leq \frac{2\tilde{C} C_{1,\lambda+\delta} + \tilde{C}}{2\epsilon R^\delta} \|\square f\|_{\infty} \tilde{C}(T)
\]

(5.13)

and

\[
P\left( \sup_{t\in[0,T]} \{|\tilde{I}^{(2)}_{t,f,R}(\mu(\cdot))| + |\tilde{I}^{(1)}_{t,f,R}(\mu(\cdot))| \} > \epsilon \right) \leq \frac{2\tilde{C} C_{1,\lambda+\delta} + \tilde{C}}{2\epsilon R^\delta} \|\square f\|_{\infty} \tilde{C}(T).
\]

(5.14)

**Step 3.** Let \(f \in C_c\) be arbitrary. Clearly proving that

\[
P\left( \langle f, \xi(t) \rangle - \langle f, \xi(0) \rangle - I_{t,f}(\mu(\cdot)) = 0 \text{ for all } t \geq 0 \right) = 1
\]

(5.15)

is equivalent to showing that for any \(T > 0\) and \(\epsilon \in (0,1)\)

\[
P\left( \sup_{t\in[0,T]} |\langle f, \xi(t) \rangle - \langle f, \xi(0) \rangle - I_{t,f}(\mu(\cdot))| > 2\epsilon \right) = 0.
\]

(5.16)

We claim here that the latter can be reduced to establishing the inequality

\[
P\left( \sup_{t\in[0,T]} |\langle f, \xi(t) \rangle - \langle f, \xi(0) \rangle - I_{t,f,R}^{(2)}(\mu(\cdot)) + I_{t,f,R}^{(1)}(\mu(\cdot))\rangle \right) > \epsilon)
\]

(5.17)

\[
\leq \liminf_{t \to \infty} P\left( \sup_{t\in[0,T]} |\langle f, \xi^N_i(t) \rangle - \langle f, \xi^N_i(0) \rangle - I_{t,f,R}^{(2)}(\mu^N_i(\cdot)) + I_{t,f,R}^{(1)}(\mu^N_i(\cdot))\rangle \right) > \epsilon)
\]

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for each \( R > 0 \). Indeed, by (5.10) and (5.14)

\[
P \left( \sup_{t \in [0, T]} |\langle f, \xi(t) \rangle - \langle f, \xi(0) \rangle - I_{t,f}^{(2)}(\mu(\cdot))| > \epsilon \right) 
\]

\[
\leq P \left( \sup_{t \in [0, T]} |\langle f, \xi(0) \rangle - I_{t,f,R}^{(2)}(\mu(\cdot)) + I_{t,f,R}^{(1)}(\mu(\cdot))| > \epsilon \right) 
+ \frac{2\hat{C}C_{1,\lambda+\delta} + \hat{C}}{2\epsilon R^\delta} \|f\|_\infty \bar{C}(T).
\]

On the other hand, by (5.9) and (5.13)

\[
\liminf_{l \to \infty} P \left( \sup_{t \in [0, T]} |\langle f, \xi(t) \rangle - \langle f, \xi(0) \rangle - I_{t,f}^{(2)}(\mu(\cdot))| > \epsilon \right) 
\]

\[
\leq \liminf_{l \to \infty} P \left( \sup_{t \in [0, T]} |\langle f, \xi(0) \rangle - I_{t,f}^{(1)}(\mu(\cdot))| > \frac{\epsilon}{2} \right) 
+ \frac{2\hat{C}C_{1,\lambda+\delta} + \hat{C}}{2\epsilon R^\delta} \|f\|_\infty \bar{C}(T) 
\leq \frac{2\hat{C}C_{1,\lambda+\delta} + \hat{C}}{2\epsilon R^\delta} \|f\|_\infty \bar{C}(T).
\]

Therefore, this combined with (5.17) and (5.18) yields

\[
P \left( \sup_{t \in [0, T]} |\langle f, \xi(t) \rangle - \langle f, \xi(0) \rangle - I_{t,f}^{(1)}(\mu(\cdot))| > \epsilon \right) \leq \frac{2\hat{C}C_{1,\lambda+\delta} + \hat{C}}{\epsilon R^\delta} \|f\|_\infty \bar{C}(T)
\]

and the aforementioned claim follows since \( R > 0 \) is arbitrary.

**Step 4.** We shall prove (5.17) for arbitrarily fixed \( T, R > 0, \epsilon \in (0, 1) \) and \( f \in \{h_1^*, h_2^*, \ldots\} (\subseteq C_c) \) at least. But, because of the triviality that \( x > \epsilon \) if and only if \( 1 \wedge x > \epsilon \), (5.17) can be rewritten into

\[
P \left( 1 \wedge \Upsilon(\mu(\cdot)) > \epsilon \right) \leq \liminf_{l \to \infty} P \left( 1 \wedge \Upsilon(\mu_l^{N_l}(\cdot)) > \epsilon \right),
\]

where \( \Upsilon = \Upsilon_{T,f,R} \) is a Borel measurable function on \( D([0, \infty), M_f) \) defined by

\[
\Upsilon(\nu(\cdot)) = \sup_{t \in [0, T]} |\langle f, \nu(t) \rangle - \langle f, \nu(0) \rangle - I_{t,f,R}^{(2)}(\nu(\cdot))| + I_{t,f,R}^{(1)}(\nu(\cdot)).
\]

Furthermore, (5.19) can be reduced to showing that for any \( n \in \mathbb{N} \)

\[
P \left( 1 \wedge \Upsilon(\mu(\cdot)) > \epsilon \mid \langle 1, \mu(0) \rangle \leq a_n \right) 
\leq \liminf_{l \to \infty} P \left( 1 \wedge \Upsilon(\mu_l^{N_l}(\cdot)) > \epsilon \mid \langle 1, \mu_l^{N_l}(0) \rangle \leq a_n \right),
\]

where \( a_1, a_2, \ldots \) are such that \( \lim_{n \to \infty} a_n = \infty, a_n > \langle \psi_1, c_0 \rangle = E[\langle 1, \mu_l^{N_l}(0) \rangle] \) and \( \lim_{l \to \infty} P(\langle 1, \mu_l^{N_l}(0) \rangle \leq a_n) = P(\langle 1, \mu(0) \rangle \leq a_n) \) for each \( n \in \mathbb{N} \). Indeed, assuming
(5.20), we get
\[
P(1 \land \Upsilon(\mu(\cdot)) > \epsilon) 
= P(1 \land \Upsilon(\mu(\cdot)) > \epsilon \mid \{1, \mu(0) \leq a_n\}) \cdot P(\{1, \mu(0) \leq a_n\}) 
\leq \liminf_{l \to \infty} P\left(1 \land \Upsilon(\mu^N(\cdot)) > \epsilon \mid \{1, \mu^N(0) \leq a_n\}\right) \cdot P(\{1, \mu(0) \leq a_n\}) 
\leq \liminf_{l \to \infty} \frac{P\left(1 \land \Upsilon(\mu^N(\cdot)) > \epsilon\right)}{1 - a_n^{-1}\langle \psi_1, c_0 \rangle} \cdot P(\{1, \mu(0) \leq a_n\}),
\]
which tends to the right side of (5.19) as \( n \to \infty \). For any \( a > 0 \), set \( M_{\leq a} = \{\nu \in M_f : \langle 1, \nu \rangle \leq a\} \), which is regarded as a closed subspace of \( M_f \). Accordingly \( D([0, \infty), M_{\leq a}) \) is a closed subspace of \( D([0, \infty), M_f) \). Note that by (5.4)
\[
P(\mu(\cdot) \in D([0, \infty), M_{\leq a}) \mid \{1, \mu(0) \leq a\}) = 1
\]
and similarly
\[
P\left(\mu^N(\cdot) \in D([0, \infty), M_{\leq a}) \mid \{1, \mu^N(0) \leq a\}\right) = 1.
\]
By the assumption of convergence of \( \{\mu^N(t) : t \geq 0\} \) to \( \{\mu(t) : t \geq 0\} \) in distribution as \( l \to \infty \) together with \( \lim_{l \to \infty} P(\{1, \mu^N(0) \leq a_n\}) = P(\{1, \mu(0) \leq a_n\}) \) \((n = 1, 2, \ldots)\) it is not difficult to verify that for each \( n \in \mathbb{N} \) the sequence of the conditional laws of \( \{\mu^N(t) : t \geq 0\} \) on \( D([0, \infty), M_{\leq a_n}) \) given \( \{1, \mu^N(0)\} \leq a_n \) converges to the conditional law of \( \{\mu(t) : t \geq 0\} \) on \( D([0, \infty), M_{\leq a_n}) \) given \( \{1, \mu(0)\} \leq a_n \). Therefore, (5.20) is naturally expected to follow as a consequence of certain continuity of \( \Upsilon \) restricted on \( M_{\leq a_n} \). In fact, by virtue of Theorem 10.2 in Chapter 3 of [15] and Lemma 5.5 together, we can conclude (5.20) as soon as the continuity of \( \Upsilon \) restricted on \( M_{\leq a_n} \) with respect to the metric \( d_t \) (defined at the beginning of this subsection) is checked to hold. We show below more generally that continuity of \( \Upsilon \) on \( M_{\leq a} \) for any \( a > 0 \).

For this purpose, take an arbitrary sequence \( \{\nu_n(\cdot)\}_{n=1}^\infty \) of \( D([0, \infty), M_{\leq a}) \) and \( \nu(\cdot) \in D([0, \infty), M_{\leq a}) \) such that \( d_t(\nu_n(\cdot), \nu(\cdot)) \to 0 \) as \( n \to \infty \). Then our task here is to show that \( \Upsilon(\nu_n(\cdot)) \to \Upsilon(\nu(\cdot)) \) for any \( f \in \{h_1, h_2, \ldots\} \). From general inequalities of the form
\[
\sup_t |\phi_1(t)| - \sup_t |\phi_2(t)| \leq \sup_t |\phi_1(t) - \phi_2(t)| \leq \sup_t |\phi_1(t)| + \sup_t |\phi_2(t)|
\]
we deduce
\[
|\Upsilon(\nu_n(\cdot)) - \Upsilon(\nu(\cdot))| 
\leq 2 \sup_{t \in [0, T]} |\langle f_\ast, \nu_n(t) \rangle - \langle f_\ast, \nu(t) \rangle| 
+ \sup_{t \in [0, T]} |I^{(1)}_{t,f,R}(\nu_n(\cdot)) - I^{(1)}_{t,f,R}(\nu(\cdot))| 
+ \sup_{t \in [0, T]} |I^{(2)}_{t,f,R}(\nu_n(\cdot)) - I^{(2)}_{t,f,R}(\nu(\cdot))|
\]
\[
= : 2s_n + s_n^{(1)} + s_n^{(2)}.
\]
Letting \( i \in \mathbb{N} \) be such that \( f = h_\ast^i \) or \( f_\ast = h_i \), we have
\[
s_n = \sup_{t \in [0,T]} |\langle h_i, \nu_n(t) \rangle - \langle h_i, \nu(t) \rangle|,
\]
which converges to 0 as \( n \to \infty \) by \( \sup_{t \in [0,T]} d_w(\nu_n(t), \nu(t)) \to 0 \). As for \( s_n^{(1)} \) observe that
\[
2s_n^{(1)} \leq \int_0^T dt |\langle g, \nu_n(t) \rangle - \langle g, \nu(t) \rangle|,
\]
where \( g(x) = \int_0^x dy \hat{H}(y, x - y)(\Box h_\ast^i)(y, x - y)\varphi_R(x) \). Since the assumed continuity of \( \hat{H} \) and (3.6) together assure that \( g \) is a bounded continuous function on \( (0, \infty) \), \( \langle g, \nu_n(t) \rangle \to \langle g, \nu(t) \rangle \) for each \( t \geq 0 \). Furthermore,
\[
\sup_{t \in [0,T]} |\langle g, \nu_n(t) \rangle - \langle g, \nu(t) \rangle| \leq 2a\|g\|_\infty.
\]
So, the dominated convergence theorem proves that \( s_n^{(1)} \to 0 \) as \( n \to \infty \). Basically a similar strategy can be adopted to \( s_n^{(2)} \):
\[
2s_n^{(2)} \leq \int_0^T dt \left| \langle G, \nu_n(t) \rangle - \langle G, \nu(t) \rangle \right|,
\]
where \( G(x, y) := \hat{H}(x, y)(\Box h_\ast^i)(x, y)\varphi_R(x + y) \) is verified to be bounded and continuous on \( (0, \infty)^2 \) thanks to (3.4) as well as the assumption that \( \hat{H} \) is continuous. Here, we claim that \( \nu_n(t) \to \nu(t) \) a.e. \( \nu \to \nu(t) \). In this respect, we rely on a slight generalization of Theorem 2.8 in [7], which implies in particular that if a sequence \( \{p_n\} \) of probability measures on \( (0, \infty) \) converges weakly to \( p \), then \( p_n \to p \) a.e. Generalizing this assertion to finite measures is easy by considering the normalized measures and it follows that \( \langle G, \nu_n(t) \rangle \to \langle G, \nu(t) \rangle \) as \( n \to \infty \) for each \( t \geq 0 \). In addition,
\[
\sup_{t \in [0,T]} \left| \langle G, \nu_n(t) \rangle - \langle G, \nu(t) \rangle \right| \leq 2a\|G\|_\infty.
\]
Hence again by the dominated convergence theorem \( s_n^{(2)} \to 0 \). Consequently we have proved the continuity of \( \Upsilon = \Upsilon_{T,f,R} \) on each \( M_{\leq a} \) with respect to \( d_U \) for any \( T, R > 0 \) and \( f \in \{h_1^\ast, h_2^\ast, \ldots\} \). As was already discussed this implies (5.15) for those \( f \)’s.

Step 5. The remaining task is derivation of the weak form (5.8). By combining (5.15) for \( f = h_\ast^i \) \( (i = 1, 2, \ldots) \) with Lemma 5.6 (implying in particular (5.15) for \( f = h_0^\ast \)) and then recalling the relation \( \xi(t)^\ast = \mu(t) \), we have proved so far that, with probability 1, for any \( i \in \mathbb{Z}_+ \) and \( t \geq 0 \)
\[
\langle h_i, \mu(t) \rangle - \langle h_i, \epsilon_0^\ast \rangle = \frac{1}{2} \int_0^t ds \int \mu(s)\Box^2(\sigma(x)dy) \hat{H}(x, y)(\Box h_\ast^i)(x, y)
- \frac{1}{2} \int_0^t ds \int \mu(s)(dx) \int_0^x dy \hat{H}(y, x - y)(\Box h_\ast^i)(y, x - y)
\]
with the integrals on the right side being absolutely convergent. Since \( \{h_0, h_1, \ldots\} \) is measure-determining, the above equalities are regarded as an equality among finite
measures. In other words, one can replace $h_i$ by arbitrary bounded Borel functions $f$. In particular, replacing $h_i$ by $f_i$ with $f \in B_c$ being arbitrary, we obtain (5.8). The proof of Theorem 5.7 is complete.

**Remark.** As mentioned earlier the uniqueness of weak solutions of (4.5) has not been proved. There is quite an extensive literature concerning the existence and/or uniqueness of solutions to coagulation-fragmentation equations. Well-posedness in the sense of measure-valued solutions was studied in e.g. [17] (for coagulation equations) and [9] (for coagulation multiple-fragmentation equations). Below we take up a result in [4]. While it considers classical solutions, the setting of that paper is well adapted to a special case of our models. Given $\lambda \geq 0$, let $\tilde{H}(x+y) = 2(x+y)^\lambda$ or $F(x+y) = 2(x+y)^{\lambda+1}$. Then (1.1) is rewritten into

\[
\frac{\partial}{\partial t} c(t, x) = -x^{2+\lambda} c(t, x) + 2 \int_x^\infty y^{\lambda+1} c(t, y) dy + \frac{1}{2} \int_0^x K(y, x-y) c(t, y) c(t, x-y) dy - c(t, x) \int_0^\infty K(y, y) c(t, y) dy.
\]

This coincides with the equation (1.3) in [4] with $\alpha = \lambda + 2$ and $\nu = 0$. One assumption made on $K$ in [4] (cf. (1.5) there) is the following:

for some $C > 0$ and $0 \leq \sigma \leq \rho < \alpha(= \lambda + 2)$

\[
0 \leq K(x, y) \leq C \left[ (1 + x)^\rho (1 + y)^\sigma + (1 + x)^\sigma (1 + y)^\rho \right], \quad x, y > 0.
\]

Under our assumptions (1.4) and (H1) on $K$, this condition is fulfilled with $C = \tilde{C} C_{1,\lambda}$, $\rho = \lambda + 1$ and $\sigma = 1$. Therefore, Theorem 2.2 (i) in [4] implies, among other things, the existence of a unique nonnegative classical solution to (5.21) which is local in time. As argued in a closely related article [3], showing the existence of a global solution requires a priori bound for the moments of solutions. For such a purpose, an analogue to our calculations in §4.2 could be useful, although we will not pursue this point here.

**Appendix**

In this appendix we prove Lemma 4.1.

**Proof of Lemma 4.1.** By the assumption for any $f \in B_{+,c}$

\[
E \left[ \exp(-\langle f, \eta^N \rangle) \right] = E \left[ \exp \left( -\frac{1}{N} \sum_i f(Y_i^N) \right) \right] = \exp \left( -N \langle 1 - e^{-f/N}, \zeta \rangle \right).
\]

As $N \to \infty$ the most right side converges to $\exp(-\langle f, \zeta \rangle)$ by Lebesgue’s convergence theorem. This proves the first assertion. To prove the convergence (4.4) in case $a(=: n) \in \mathbb{N}$ it suffices to give the moment formula of the form

\[
E \left[ \left( \sum_i Y_i^N \right)^n \right] = n! \sum_{k=1}^n \frac{N_k}{k!} \sum \frac{\langle \psi_{n_1}, \zeta \rangle \cdots \langle \psi_{n_k}, \zeta \rangle}{n_1! \cdots n_k!},
\]

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where the inner summation is taken over \( k \)-tuples \((n_1, \ldots, n_k)\) of positive integers such that \( n_1 + \cdots + n_k = n \). But, the proof of the above formula clearly reduces to showing that for each \( R > 0 \)
\[
E \left[ \left( \sum_i Y_i^N 1_{[0,R]}(Y_i^N) \right)^n \right] = n! \sum_{k=1}^n \frac{N^k}{k!} \sum \frac{\langle \psi_{n_1}^{(R)}, \zeta \rangle \cdots \langle \psi_{n_k}^{(R)}, \zeta \rangle}{n_1! \cdots n_k!},
\]
where \( \psi_i^{(R)}(y) = y^i 1_{[0,R]}(y) \), and this version is derived by comparing the coefficients of \( t^n \) after expanding in \( t \) each side of
\[
E \left[ \exp \left( t \sum_i Y_i^N 1_{[0,R]}(Y_i^N) \right) \right] = \exp \left( N \int_{[0,R]} \zeta(dy)(e^{ty} - 1) \right).
\]

It remains to prove (4.4) in case \( a \in (1, \infty) \setminus \mathbb{N} \). For such an \( a \), put \( n = \lfloor a \rfloor \) and \( \alpha = a - \lfloor a \rfloor \in (0, 1) \). Combining the aforementioned moment formula with the identity \( y^\alpha = C_{3,a} \int ds s^{-(1+\alpha)}(1 - e^{-sy}) \) for \( y > 0 \) we deduce
\[
E \left[ \left( \sum_i Y_i^N \right)^a \right] = C_{3,a} \int ds \frac{ds}{s^{1+\alpha}} \left\{ E \left[ \left( \sum_i Y_i^N \right)^n \right] - E \left[ \left( \sum_i Y_i^N \right)^n \exp \left( -s \sum_i Y_i^N \right) \right] \right\}
\]
\[
= C_{3,a} n! \int ds \frac{ds}{s^{1+\alpha}} \left\{ \sum_{k=1}^n \frac{N^k}{k!} \sum \frac{\int y^{n_1} \zeta(dy) \cdots \int y^{n_k} \zeta(dy)}{n_1! \cdots n_k!} \right\}
\]
\[
- \sum_{k=1}^n \frac{N^k}{k!} \sum \frac{\int y^{n_1} e^{-sy} \zeta(dy) \cdots \int y^{n_k} e^{-sy} \zeta(dy)}{n_1! \cdots n_k!} E \left[ \exp \left( -s \sum_i Y_i^N \right) \right].
\]

Here, the second equality follows from the fact that under the transformed measure
\[
\hat{P}_s \left( \sum \delta_{Y_i^N} \in \bullet \right) := E \left[ \exp \left( -s \sum_i Y_i^N \right) \right] \left( \sum \delta_{Y_i^N} \in \bullet \right) E \left[ \exp \left( -s \sum_i Y_i^N \right) \right]^{-1}
\]
\( \sum \delta_{Y_i^N} \) is \( \text{Po}(e^{-sy} \zeta(dy)) \)-distributed. (See e.g. [6], p.80, Lemma 2.4.) For \( s > 0 \) set \( \Lambda(s) = \int (1 - e^{-sy}) \zeta(dy) \) so that \( \Lambda(s) \leq s \int y \zeta(dy) \) and
\[
E \left[ \exp \left( -s \sum_i Y_i^N \right) \right] = \exp (-N \Lambda(s)).
\]

By the change of variable \( u := Ns \) in the integral with respect to \( ds \)
\[
E \left[ \left( \frac{\sum_i Y_i^N}{N} \right)^a \right] = C_{3,a} n! \int \frac{du}{u^{1+\alpha}} \left\{ \sum_{k=1}^n \frac{N^{k-n}}{k!} \sum \frac{\int y^{n_1} \zeta(dy) \cdots \int y^{n_k} \zeta(dy)}{n_1! \cdots n_k!} \right\}
\]
\[
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Lastly, this integral converges to
\[\langle \psi, \zeta \rangle^a\] since as before
\[C_{3,\alpha} \int \frac{du}{u^{1+\alpha}} \left| \left( \int y\zeta(dy) \right)^n - \left( \int ye^{-\frac{uy}{N}} \zeta(dy) \right)^n \right| \]
\[\leq \int y \left( C_{3,\alpha} \int \frac{du}{u^{1+\alpha}} (1 - e^{-\frac{uy}{N}}) \right) \zeta(dy) \cdot n \left( \int y\zeta(dy) \right)^{n-1} \]
\[= \int y \left( \frac{y}{N} \right)^\alpha \zeta(dy) \cdot n \left( \int y\zeta(dy) \right)^{n-1} \to 0 \]

and by Lebesgue's convergence theorem
\[C_{3,\alpha} \int \frac{du}{u^{1+\alpha}} \left( \int ye^{-\frac{uy}{N}} \zeta(dy) \right)^n \left\{ 1 - \exp \left( -N\Lambda \left( \frac{u}{N} \right) \right) \right\} \]
\[ C_{3,\alpha} \int \frac{du}{u^{1+\alpha}} \left( \int y\zeta(dy) \right)^n \left\{ 1 - \exp \left( -u \int y\zeta(dy) \right) \right\} = \left( \int y\zeta(dy) \right)^{n+\alpha} = \langle \psi_1, \zeta \rangle^\alpha. \]

Consequently (4.4) holds and the proof of Lemma 4.1 is complete.

\textbf{References}

[1] Aizenman, M. and Bak, T. A., Convergence to equilibrium in a system of reacting polymers, Comm. Math. Phys. 65 (1979) 203–230.

[2] Aldous, D.J., Deterministic and stochastic models for coalescence (aggregation, coagulation): a review of the mean-field theory of probabilists, Bernoulli 5 (1999) 3–48.

[3] Banasiak, J. and Lamb, W., Analytic fragmentation semigroups and continuous coagulation-fragmentation equations with unbounded rates, J. Math. Anal. Appl. 391 (2012) 312–322.

[4] Banasiak, J., Lamb, W. and Langer, M., Strong fragmentation and coagulation with power-law rates, J. Eng. Math. (2013) 199–215.

[5] Berestycki, J., Exchangeable fragmentation-coalescence processes and their equilibrium measures, Electron. J. Probab. 9 (2004), Paper no. 25, pages 770–824.

[6] Bertoin, J., Random Fragmentation and Coagulation Processes, Cambridge University Press, New York, 2006.

[7] Billingsley, P., Convergence of probability measures. 2nd edition. John Wiley & Sons, Inc., New York, 1999.

[8] Cepeda, E., Well-posedness for a coagulation multiple-fragmentation equation, Differential Integral Equations 27 (2014) 105–136.

[9] Cepeda, E., Stochastic coalescence multi-fragmentation processes, Stoch. Proc. their Appl. 126 (2016) 360–391.

[10] Daley, D. J. and Vere-Jones, D., An introduction to the theory of point processes, Vol. II. General theory and structure. Second edition, Springer-Verlag, New York, 2008.

[11] Diaconis, P., Mayer-Wolf, E., Zeitouni, O. and Zerner, M. P. W., The Poisson-Dirichlet law is the unique invariant distribution for uniform split-merge transformations, Ann. Probab. 32 (2004) 915–938.
[12] Eibeck, A. and Wagner, W., Approximative solution of the coagulation-fragmentation equation by stochastic particle systems, Stochastic Anal. Appl. 18 (2000) 921–948.

[13] Eibeck, A. and Wagner, W., Stochastic interacting particle systems and non-linear kinetic equations, Ann. Appl. Probab. 13 (2003) 845–889.

[14] Escobedo, M. and Pezzotti, F., Propagation of chaos in a coagulation model, Math. Models Methods Appl. Sci. 23 (2013) 1143–1176.

[15] Ethier, S. N. and Kurtz, T. G., Markov processes: characterization and convergence, Wiley, New York, 1986.

[16] Feng, S., The Poisson-Dirichlet distribution and related topics, Models and asymptotic behaviors. Springer, Heidelberg, 2010.

[17] Fournier, N. and Laurençot, Ph., Well-posedness of Smoluchowski’s coagulation equation for a class of homogeneous kernels, J. Functional Analysis 233 (2006) 351–379.

[18] Goldschmidt, C., Ueltschi, D. and Windridge, P., Quantum Heisenberg models and their probabilistic representations, Entropy and the quantum II, 177–224, Contemp. Math., 552, Amer. Math. Soc., Providence, RI, 2011.

[19] Handa, K., Sampling formulae for symmetric selection, Electron. Commun. Probab. 10 (2005) 223–234 (electronic)

[20] Handa, K., The two-parameter Poisson-Dirichlet point process, Bernoulli, 15 (2009) 1082–1116.

[21] Kallenberg, O., Random measures, Akademie-Verlag, Berlin 1975.

[22] Kingman, J. F. C., Poisson processes, Oxford University Press, New York, 1993.

[23] Kolokoltsov, V. N. Nonlinear Markov processes and kinetic equations, Cambridge University Press, Cambridge, 2014.

[24] Laurençot, P. and Mischler, S., Convergence to equilibrium for the continuous coagulation-fragmentation equation, Bull. Sci. Math. 127 (2003) 179–190.

[25] Lehéricy, T., Une propriété caractéristique des processus de Poisson-Dirichlet, preprint, 2015. Available at https://hal.inria.fr/hal-01233285

[26] Mayer-Wolf, E., Zeitouni, O. and Zerner, M. P. W., Asymptotics of certain coagulation-fragmentation processes and invariant Poisson-Dirichlet measures, Electron. J. Probab. 7 (2002), 25 pp.
[27] Petrina, D. Ya., Stochastic dynamics and Boltzmann hierarchy, de Gruyter Expositions in Mathematics, 48, Walter de Gruyter GmbH & Co. KG, Berlin, 2009.

[28] Pitman, J., Poisson-Dirichlet and GEM invariant distributions for split-and-merge transformation of an interval partition, Combin. Probab. Comput. 11 (2002) 501–514.

[29] Resnick, S. I., Extreme values, regular variation, and point process, Springer, New York, 1987.

[30] Schramm, O., Compositions of random transpositions, Israel J. Math. 147 (2005) 221–243. cf. arXiv:math/0404356v3 (2007)

[31] Stewart, I.W. and Dubovski, P.B., Approach to equilibrium for the coagulation-fragmentation equation via a Lyapunov functional, Math. Meth. in the Appl. Sci. 19 (1996) 171–185.

[32] Tsilevich, N. V., Stationary random partitions of positive integers, Theory Probab. Appl., 44 (2000) 60–74.

[33] Tsilevich, N. V., On the simplest split-merge operator on the infinite-dimensional simplex, PDMI Preprint 03/2001. cf. arXiv:math/0106005

[34] Vigil, R.D. and Ziff, R.M., On the stability of coagulation-fragmentation population balances, J. Colloid Interface Sci., 133 (1989) 257–264.

[35] Watterson, G.A., The stationary distribution of the infinitely-many neutral alleles diffusion model, J. Appl. Probab. 13 (1976) 639–651; correction. ibid. 14 (1977) 897.