A COUNTEREXAMPLE TO THE 2–JET DETERMINATION CHERN-MOSER THEOREM IN HIGHER CODIMENSION

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Abstract. One constructs an example of a generic quadratic submanifold of codimension 5 in $\mathbb{C}^9$ which admits a real analytic infinitesimal CR automorphism with homogeneous polynomial coefficients of degree 3.

1. Introduction

Let $M$ be a real-analytic submanifold of $\mathbb{C}^N$ of codimension $d$. Consider the set of germs of biholomorphisms $F$ at a point $p \in M$ such that $F(M) \subset M$. By the work of Chern and Moser [9], if the codimension $d = 1$, every such $F$ is uniquely determined by its derivatives at $p$ provided that its Levi map at $p$ is non-degenerate. See also Cartan and Tanaka ([8], [19]).

Theorem 1. [9] Let $M$ be a real-analytic hypersurface through a point $p$ in $\mathbb{C}^N$ with non-degenerate Levi form at $p$. Let $F$, $G$ be two germs of biholomorphic maps preserving $M$. Then, if $F$ and $G$ have the same 2-jets at $p$, they coincide.

Note that the result becomes false without any hypothesis on the Levi form (See for instance [6]). A generalization of this Theorem to real-analytic submanifolds $M$ of higher codimension $d > 1$ has been proposed by Beloshapka in [3] (and quoted many times by several authors) under the hypothesis that $M$ is Levi generating (or equivalently of finite type with 2 the only Hörmander number) with non-degenerate Levi map. Unfortunately, an error has been discovered and explained in [6]. In this short note, one constructs an example of a generic (Levi generating with non-degenerate Levi map) quadratic submanifold that admits an element in its stability group which has the same 2–jet as the identity map but is not the identity map. In addition, this example is Levi non-degenerate in the sense of Tumanov. One points out that if $M$ is strictly pseudoconvex, that is Levi non-degenerate in the sense of Tumanov with a positivity condition, then by a recent result of Tumanov [20], the 2-jet determination result holds in any codimension.

One also points out that finite jet determination problems for submanifolds has attracted much attention. One refers in particular to the papers of Zaitsev [21], Baouendi, Ebenfelt and Rothschild [2], Baouendi, Mir and Rothschild [7], Ebenfelt, Lamel and Zaitsev [12], Lamel and Mir [17], Juhlin
2. The Example

Let $M \subseteq \mathbb{C}^9$ be the real submanifold of (real) codimension 5 through 0 given in the coordinates $(z, w) = (z_1, \ldots, z_4, w_1, \ldots, w_5) \in \mathbb{C}^9$, by

\[
\begin{align*}
\text{Im } w_1 &= P_1(z, \bar{z}) = z_1 \bar{z}_2 + z_2 \bar{z}_1 \\
\text{Im } w_2 &= P_2(z, \bar{z}) = -iz_1 \bar{z}_2 + iz_2 \bar{z}_1 \\
\text{Im } w_3 &= P_3(z, \bar{z}) = z_3 \bar{z}_2 + z_4 \bar{z}_1 + z_2 \bar{z}_3 + z_1 \bar{z}_4 \\
\text{Im } w_4 &= P_4(z, \bar{z}) = z_1 \bar{z}_1 \\
\text{Im } w_5 &= P_5(z, \bar{z}) = z_2 \bar{z}_2
\end{align*}
\]

The matrices corresponding to the $P'_i$s are

\[
A_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad
A_2 = \begin{pmatrix}
0 & -i & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad
A_3 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \quad
A_4 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad
A_5 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Lemma 2. The following holds:

(1) the $A'_i$s are linearly independent,

(2) the $A'_i$s satisfy the condition of Tumanov, that is, there is $c \in \mathbb{R}^d$ such that $\det \sum c_j A_j \neq 0$.

Proposition 3. The real submanifold $M$ given by (1) is Levi generating at 0, that is, of finite type with 2 the only Hörmander number, and its Levi map is non-degenerate.

Proof. This follows for instance from Proposition 8, Lemma 3 and Remark 4 in [6]. \qed

Remark 4. The following identity between the $P'_i$s holds:

\[
P_1^2 + P_2^2 - 4P_4P_5 = 0.
\]

The following holomorphic vectors fields are in $\text{hol}(M, 0)$, the set of germs of real-analytic infinitesimal CR automorphisms at 0.

(1) $X := i(z_1 \frac{\partial}{\partial z_3} + z_2 \frac{\partial}{\partial z_4})$
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\[ (2) \quad Y := i(-iz_1 \frac{\partial}{\partial z_3} + iz_2 \frac{\partial}{\partial z_4}) \]
\[ (3) \quad Z := i(z_1 \frac{\partial}{\partial z_4}) \]
\[ (4) \quad U := i(z_2 \frac{\partial}{\partial z_3}) \]

Lemma 5. Let \( P = (P_1, \ldots, P_4) \). The following holds:

\[ (1) \quad X(P) = (0, 0, iP_1, 0, 0) \]
\[ (2) \quad Y(P) = (0, 0, iP_2, 0, 0) \]
\[ (3) \quad Z(P) = (0, 0, iP_4, 0, 0) \]
\[ (4) \quad U(P) = (0, 0, iP_5, 0, 0). \]

Lemma 6. The following identities hold:

\[ (1) \quad P_1(-Y(P)) + P_2(X(P)) = 0 \]
\[ (2) \quad P_1X(P) + P_2(Y(P) + P_3(-2Z(P)) + P_4(-2U(P)) = 0 \]
\[ (3) \quad P_2(-2Z(P)) + P_4(2Y(P)) = 0 \]
\[ (4) \quad P_2(-2U(P)) + P_5(2Y(P)) = 0 \]

With the help of the Lemmata, one obtains

Theorem 7. Let \( Y_0 = -Y, \quad Y_1 = 2Y, \quad Z_1 = -2Z, \quad U_1 = -2U. \)

The holomorphic vector field \( T \) defined by

\[ (3) \quad T = \frac{1}{2} w_1^2 Y_0 + \frac{1}{2} w_2^2 Y + w_1 w_2 X + w_2 w_5 Z_1 + w_2 w_4 U_1 + w_4 w_5 Y_1 \]
is in \( \text{hol}(M, 0) \).

Hence 2–jet determination does not hold for germs of biholomorphisms sending \( M \) to \( M \).

Remark 8. Notice that the bound for the number \( k \) of jets needed to determine uniquely any germ of biholomorphism sending \( M \) to \( M \) is

\[ k = (1 + \text{codim} \ M), \]

\( M \) beeing a generic (Levi generating with non-degenerate Levi map) real-analytic submanifold: see Theorem 12.3.11, page 361 in [1]. One points out that Zaitsev obtained the bound \( k = 2(1 + \text{codim} \ M) \) in [21].

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