A two-dimensional resistor network model for transition-edge sensors with normal metal features

Daikang Yan\textsuperscript{1,2}, Lisa M Gades\textsuperscript{1}, Tejas Guruswamy\textsuperscript{1}, Antonino Miceli\textsuperscript{1,2}, Umeshkumar M Patel\textsuperscript{1} and Orlando Quaranta\textsuperscript{1,2}\textsuperscript{\circledast}

\textsuperscript{1}Argonne National Laboratory, Argonne, IL 60439 United States of America
\textsuperscript{2}Northwestern University, Evanston, IL 60208 United States of America

E-mail: oquaranta@anl.gov

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Abstract

Transition-edge sensors (TESs) can be used in high-resolution photon detection, exploiting the steep slope of the resistance in the superconducting-to-normal transition edge. Normal metal bars on the TES film are commonly used to engineer its transition shape, namely the dependence of resistance on temperature and current. This problem has been studied in one-dimension, however until now, there have been no predictive models of the influence of two-dimensional (2D) normal metal features on the TES transition shape. In this work, we approach this problem by treating the TES as a 2D network of resistors, the values of which are based on the two-fluid model. We present a study of the behavior of devices with different 2D geometric features. Our 2D network model is capable of predicting how typical TES geometry parameters, such as number of bars, bar spacing, and overall dimensions, influence device behavior and thus is a powerful tool to guide the engineering of new TES devices.

Keywords: transition-edge sensor, 2D resistors network, two-fluid model, transition shape

(Some figures may appear in colour only in the online journal)

1. Introduction

Transition-edge sensors (TESs) are typically made of low-temperature superconductor materials or multilayers, and have been widely used and studied as bolometers and calorimeters. In the small signal, linearized limit, the signal response of a TES can be expressed as a set of electrothermal differential equations and solved analytically [1]. However, the TES response and specifically resistance as a function of current and temperature \( R(I, T) \) is usually nonlinear over the full superconducting transition width [2, 3]. Knowledge of the full nonlinear \( R(I, T) \) function is necessary to correctly describe the device response to large signals as well as its saturation and dynamic range. Multiple models have been used to describe the nonlinear transition shape of TESs [3, 4], including the resistively and capacitively shunted junction model [5] for small dimension devices, that treats the TES as a weak link between superconducting leads; and the two-fluid model [6, 7] that describes the TES current as a superposition of a superconductive (Cooper pair) and a normal (quasiparticle) current. The latter has shown good quantitative agreement with the current–voltage (\( I-V \)) curves measured from a 350 \( \mu \)m \( \times \) 350 \( \mu \)m Mo-Cu bilayer TES [7]. However, it is also known that the geometry of noise mitigating normal metal bars/banks patterned on the superconducting film influences the \( R(I, T) \) transition shape [8, 9]; this is not accounted for in the simple two-fluid model. The effect of normal metal bars has been studied in a one-dimensional (1D) scheme utilizing the Usadel equation [8], but has not been addressed in two dimensions (2D). In this work, we present a resistor network model to calculate the TES transition shape given 2D normal metal features of arbitrary geometry. The current distributions and \( R(I, T) \) surfaces for some example TESs are solved, and dependence on the normal metal features is shown.
The 2D resistor network model of a TES is represented as a 2D normal metal banks heating effect and any potential non-uniform cooling in the metal features are present we assume these units are completely in the normal state due to the proximity effect. In the case of a unit where no normal metal features are present (in blue in figure 1, labeled ‘transition region’), the resistance \( R \) depends on both the total current \( I \) flowing through the unit, and its temperature \( T \). The net current flowing through a unit is calculated as \( I = (I_e^2 + I_e^3)^{1/2} \) where \( I_e \) are the current components in the respective directions, while the temperature is assumed to be the same across the TES, ignoring the uneven Joule heating effect and any potential non-uniform cooling in the film. The \( R(I, T) \) relation for each transition region unit is defined by the two-fluid model and will be introduced in section 2.2.

The resistor network must obey a matrix equation representing Ohm’s Law

\[
[S] \cdot \vec{V}_\text{node} = \vec{I}_\text{node}. \tag{1}
\]

For an \( m \times n \) 2D network, \( \vec{V}_\text{node} \) is an \( (m+1)(n+1) \) column vector that consists of the node voltages, while \( \vec{I}_\text{node} \) is an \( (m+1)(n+1) \) column vector that consists of the net node currents. \([S]\) is an \( (m+1)(n+1) \times (m+1)(n+1) \) matrix consisting of the electrical conductances between each node and is constructed based on the values of the resistors in the network. By Kirchhoff’s current law, the net current at each node must be zero, except for the two nodes representing the contacts with the bias leads, where the net current is \( \pm I_{bias} \). Here the sign defines whether the current is flowing in or out of the node (figure 1).

Matrix \([S]\) is singular, and is of order \((m+1)(n+1) \times (m+1)(n+1) - 1\). Physically, this is because the node voltages are relative to an arbitrary ground voltage. This is resolved by fixing the voltage of one node. Here we set the voltage of the node connected to the ground to be zero (although it can be any arbitrary value). This changes the ground node Kirchhoff equation from

\[
\vec{S} \cdot \vec{V}_\text{node} = -I_{bias}, \tag{2a}
\]

to

\[
[0, 0, \ldots, 1, \ldots 0] \cdot \vec{V}_\text{node} = 0. \tag{2b}
\]

where the \( \vec{S} \) vector is all zero except the value one at the position that multiplies with \( V_{\text{ground}} \). The modified Kirchhoff equation set is now linearly independent, therefore the node voltages can be solved by

\[
\vec{V}_\text{node} = [S]^{-1} I_{node}. \tag{3}
\]

The current between any two nodes can then be calculated by dividing the voltage difference by the connecting resistance.

Figure 2 illustrates a simple network consisting of two four-terminal units. Following the steps introduced above, its node voltages can be calculated as equation (4).

The geometry of the TES determines the 2D network and the matrix \([S]\). Figure 1 shows the network model of a 150 \( \mu \text{m} \times 150 \mu \text{m} \) TES film with parallel normal metal
banks and three perpendicular normal metal bars. The width of the banks and bars is \( \sim 10 \mu m \), therefore the TES is divided into \( 5 \mu m \times 5 \mu m \) squares, providing sufficient resolution to accurately represent the geometry. The units representing the normal metal use resistor values of \( R_{\text{metal}} = 8.94 \text{ m\Omega} \), and the transition region units use a normal state resistance of \( R_n = 23.54 \text{ m\Omega} \) in the two-fluid model calculation. These numbers are based on [9], but with the four-terminal unit configuration the resistor values are doubled relative to the measured sheet resistance, to ensure that any square section of the overall network has the correct total resistance. Theoretically, the resistance of the superconducting leads is zero. However, to avoid null values in the simulation, a very small value of 1 n\( \Omega \) is used instead.

2.2. Resistance–current relation

The two-fluid model defines the resistance of a superconductor in the transition region \( R(I, T) \) as:

\[
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4 \\
V_5 \\
V_6
\end{bmatrix} = \begin{bmatrix}
\frac{2}{R_a} & -\frac{1}{R_a} & 0 & -\frac{1}{R_a} & 0 & 0 \\
\frac{2}{R_a} + \frac{2}{R_b} & -\frac{1}{R_b} & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{R_a} & 2 & 0 & 0 & -\frac{1}{R_b} \\
-\frac{1}{R_a} & 0 & 0 & \frac{2}{R_b} & -\frac{1}{R_a} & 0 \\
0 & -\frac{1}{R_a} & -\frac{1}{R_b} & 0 & \frac{2}{R_a} + \frac{2}{R_b} & -\frac{1}{R_b} \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}^{-1} \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
l_{\text{bias}}
\end{bmatrix}
\]

\[(4)\]

\[
R(I, T) = \left[ 1 - \frac{c_1 I_0}{I} \left( 1 - \frac{T}{T_c} \right)^{3/2} \right] c_R R_n
\]

\[
I > c_1 I_0 \left( 1 - \frac{T}{T_c} \right)^{3/2} \quad \text{(5a)}
\]

\[
R(I, T) = 0, \quad I < c_1 I_0 \left( 1 - \frac{T}{T_c} \right)^{3/2} \quad \text{(5b)}
\]

This model has shown good agreement with experimental results when applied to the entire TES [4, 7]. In this equation, \( T_c \) is the critical temperature, \( I_0 \) is the critical current at zero temperature, and \( c_1 \) and \( c_R \) are unitless coefficients. \( T \) is the temperature, assumed to be the same for all the units, and \( I \) is the current passing through each unit. In the case of our 2D network model, the current \( I \) depends on the distribution of \( R \).

Because \( R \) and \( I \) are mutually dependent, and because this relationship is nonlinear, as suggested by equation (5a), the solution to equation (3) can only be obtained by numerical methods. We choose to solve equation (3) iteratively with Newton-Krylov method, using finite-differences to estimate function derivatives. To avoid non-convergence of the numerical solver in the zero-derivative region when \( I < I_0 \) in equation (5b), \( R(I, T) \) is modified to have a constant slope as a function of current below 5% \( R_n \), as shown in figure 3. In order to evaluate the influence of this modification, we simulated via the 2D resistor network the \( R(I, T) \) of a TES with no normal metal structures, and compared it to the prediction of the two-fluid model applied to an equivalent single-body TES. The two results should be effectively identical, with a uniform current distribution through the 2D network, resulting from a uniformly segmented resistance. The only difference between the two approaches arises from the modification in the 2D network model of the two-fluid \( R(I, T) \) below 5% \( R_n \). The comparison shows that the difference caused by this modification is below 5% \( R_n \) total (\( R_n \) total is the total normal resistance of the device), which we deemed negligible. With this point verified, we believe this 2D network model is capable of describing how typical TES geometry parameters, such as number of bars, bar spacing, and
Table 1. The geometry and model parameters of the seven TESs simulated.

| Geometry         | 1/2-sq. three-bar | 1/2-sq. two-bar | one-sq. four-bar | one-sq. three-bar | two-sq. five-bar | two-sq. four-bar | two-sq. three-bar |
|------------------|-------------------|----------------|-----------------|------------------|-----------------|-----------------|------------------|
| Number of units  |                   |                |                 |                  |                 |                 |                  |
| between bars     | 4 5 4             | 6 5            |                 |                  |                 |                 |                  |
| $T_c$ (mK)       | 71.3 73.5         | 72.8           |                 | 73.4             | 73.0            | 73.9            | 76.3             |
| $c_I$            |                   |                |                 | 0.79             |                 |                 |                  |
| $c_R$            | 0.25, 0.5, 1      |                |                 |                  |                 |                 |                  |
| $G$ (pW K$^{-1}$)| 100               |                |                 |                  |                 |                 |                  |
| $T_{bath}$ (mK)  | 55                |                |                 |                  |                 |                 |                  |

Figure 4. The 2D layout of the TESs under study: (a) 106 $\mu$m $\times$ 212 $\mu$m (1/2-sq.) TESs are made with three bars and two bars; (b) 150 $\mu$m $\times$ 150 $\mu$m (one-sq.) TESs are made with four bars and three bars; (c) 212 $\mu$m $\times$ 106 $\mu$m (two-sq.) TESs are made with five bars, four bars, and three bars.

overall dimensions, influence device behavior. The results of these simulations for some specific TES designs, chosen to be similar to those of [10], are described in the following section.

3. Results and discussion

3.1. Simulation parameters

To explore the influence of some common TES design parameters, we decided to focus on a set of seven different designs labeled with the following scheme: $x$-sq. $y$-bars, where $x$ represents the aspect ratio (width/height) of the device and $y$ represents the number of bars present. It should be expected that changing the width of the device results in a different total TES resistance, and therefore a different trajectory on the $R(I, T)$ surface under voltage bias, while changing the number of bars may affect the current distribution in the device, defining a different shape for the $R(I, T)$ surface. The specific designs under examination are based on those described in [10] and illustrated in figure 4. We also used the material and device parameters measured in [10] (see table 1), providing means to validate the model against the experimental measurements presented in that work.

However, not all the parameters required here have an immediate equivalent in that paper—in particular, the measured two-fluid model parameters of [10] are those appropriate to an entire TES including bilayer and normal metal features, while in the 2D resistor network both units with and without normal metal features are present at the same time, therefore some assumptions had to be made and are described below.

In table 1, the critical temperature of a unit resistor has been defined as dependent on the overall design. This approximates the lateral proximity effect induced by the normal metal features, as our model does not directly account for this effect. The critical temperature of each unit is the $T_c$ of the corresponding device measured in [10].

The transition unit $I_0$, on the other hand, is assumed to be the same for all TESs. Experimental data suggest that the critical current at zero temperature for a TES is roughly proportional to the spacing between the normal metal bars [10]. This is likely because in the superconducting state, the current meanders around the bars, and the width of the current path is the bar spacing. For devices with the same bar spacing, the $I_0$ variation is small; therefore, we chose one TES design to obtain the unit $I_0$ value: the 1/2-sq. two-bar device with a total $I_0$ of 7.8 mA and five inter-bar units gives a per-unit $I_0 = 1.48$ mA. This value is applied to all other simulated devices.

According to equation (5a), the value of $c_R$ should be unity when the TES is in the normal state, to obtain the correct total normal state resistance, and $c_I$ should be unity when $T = 0$. However, they have both been experimentally observed to have a smaller value when the TES is biased in the transition [7, 10], and theoretical considerations based on the phase-slip model of the superconducting transition also suggest $c_R$ should vary with temperature [6, 11]. However, without any detailed model of the dynamics of phase-slip lines in 2D films, or experimental measurements of the parameters those models might require, we kept $c_R$ and $c_I$ as fixed parameters in each of our simulations. Instead, we simply carried out separate calculations with manually chosen values of $c_I = 0.79$ (as measured in [10]) and $c_R = 0.25, 0.5, 1$, to account for its variation. Any one calculation presented here is only strictly valid, therefore, for the region of the transition where $c_I$ and $c_R$ are close to these chosen values. Until a model for these two-fluid parameters is developed and we can identify those regions, we present our results over the full transition width.

3.2. Simulation results

Figure 5 shows the simulated resistance and current distribution of the one-sq. three-bar devices under different biases. Operating the device at a fixed temperature of $T = 72$ mK, as the TES bias current $I_{bias}$ increases from zero, several phenomena develop in the simulation. Initially the
current flows through the device encountering no resistance (or very little, due to the approximation described previously—figure 3). This is because, although there are normal metal (resistive) regions present, a lower resistance path that meanders around the bars is present. Further increasing $I_{\text{bias}}$ will cause the current through a unit to increase, along with the total resistance along the meander path. Under this scenario, the current will still meander around the bars, because the total resistance of that long path is still below the equivalent resistance that the current would experience if it would go via the shorter, direct path intersecting the bars. Figures 5(a) and (b) illustrate this for a device biased at $\sim12\% R_n^{\text{total}}$. Further increasing $I_{\text{bias}}$ will increase the transition unit resistance to a point where the direct path now has lower resistance, and so it is preferred by the current. This is illustrated in figures 5(c) and (d), at $\sim44\% R_n^{\text{total}}$. This behavior indicates that the current flow pattern in a TES with bars is dependent on the bias position in the transition. Similar current distribution dependence on bias has been reported in [9], although obtained via a different modeling and measurement technique.

With the resistance and current distributions calculated in figure 5, the Joule heating effect in the TES film can be addressed quantitatively. Given the unitary resistances of a few m$\Omega$ as shown in figures 5(a), (c), with the Wiedemann–Franz Law, the thermal conductivity among units can be estimated to be several $\mu$W K$^{-1}$. Given the unitary currents of a couple $\mu$A as shown in figures 5(b), (d), the Joule heating power differences between adjacent units are on the order of $\mu$W, leading to a few nK temperature variation within the TES film. Compared to the $\sim1$ mK transition width of a typical TES, this temperature difference caused by uneven Joule heating can be ignored. It is therefore reasonable, in first approximation, to assume that the units in the 2D model have the same temperature.

Repeating this calculation at different values of $I_{\text{bias}}$ and $T$ generates a 3D map of the $R(I, T)$ function for this device, as shown in figure 5(a) for a two-sq. three-bar device, and figure 5(b) for a two-sq. five-bar device. Comparing the two $R(I, T)$ surfaces shows (figure 6) that when biased beyond $\sim20\% R_n^{\text{total}}$, the transition shapes of the two devices are about the same. This is because although the two devices have a different number of bars and different bar spacing, at those biases the current is flowing through the TES uniformly, irrespective of the bar layout. Conversely, while biased below $\sim20\% R_n^{\text{total}}$, the device with the larger bar spacing (two-sq. three-bar) shows a sharper transition; this is due to the wider current path available between bars. It can therefore support a larger critical current, making the transition width narrower. This difference is more evident in the comparison of the thermal sensitivity $\alpha$, and current sensitivity $\beta$, defined as:

$$\alpha = \frac{\partial \log R}{\partial \log T} \bigg|_{(T_0, I_0)}, \quad \beta = \frac{\partial \log R}{\partial \log I} \bigg|_{(T_0, I_0)},$$

shown in figures 6(c) and (d), respectively. $(T_0, I_0)$ are the TES bias points represented by the black trajectories on the respective $R(I, T)$ surfaces in figures 6(a), (b). These trajectories have been
The $\sqrt{1+\frac{2}{\beta}}/\alpha$ values at 20% $R_n^{\text{total}}$ for the seven TESs. The unfilled marks show experimental data for devices with different shapes and number of bars; the solid marks show simulated results for these devices.

Correctly calculating the changes in TES transition steepness due to either the device dimensions (i.e. $R_n^{\text{total}}$ and consequently $I_0$) and/or the number and spacing of bars (i.e. $I_{0}^{\text{total}}$) [10]. These results demonstrate the predictive power of this model.

The previously described simulations all use a fixed value for $c_R = 0.5$, chosen as an average value at low bias among those reported for these kinds of devices in [10]. However, $c_R$ varies through the transition, and it is smaller at lower bias points and increases with bias. Equation (5a) shows that a smaller $c_R$ will result in a lower resistance in the transition region units, and consequently the current will prefer to meander around the normal metal bars until higher biases, and vice versa for higher values of $c_R$. Varying $c_R$ can consequently generate a family of curves of the kind showed in figure 8 for a given device. Due to the lack of experimental data on the dependence of $c_R$ on the bias and the difficulty in estimating the materials parameters that determine this phenomenological parameter (for example, the charge imbalance relaxation length [7]), for now its influence on TES transition shape can only be evaluated qualitatively. Figure 8 shows the values of $\alpha$ throughout the transition for the one-sq. three-bar device when using $c_R = 1$, 0.5, and 0.25. Based on the $c_R$ values measured in [10], this plot is divided into three regions; each region provides $\alpha$ values with a $c_R$ best estimated at that bias. The actual dependence of $\alpha$ on % $R_n^{\text{total}}$, i.e. the actual shape of the TES transition, can likely be approximated by a combination of these three curves. The model will be fairly straightforwardly improved if a quantitative model for $c_R$ and $c_I$ can be developed.

4. Conclusions

While the engineering of TES transition shapes has been performed experimentally through the manipulation of 2D features, a predictive 2D model has until now been lacking. In
this paper, we present a 2D resistor network model which can calculate the current distribution and overall $R(I, T)$ surface for TES devices with arbitrary geometry including normal metal features. The TES is divided into 4-terminal units, with resistances based on the superconducting two-fluid model and calculated self-consistently based on the temperature and net current, allowing for the calculation of the current distribution and total resistance of the device. The model has been used to simulate the transition shape and the $I$–$V$ characteristics of a series of previously experimentally measured devices of varying dimensions and normal metal features. The simulations show how the normal metal features force the current flowing through the TES to meander around them at lower biases, while at higher biases the current tends to flow more uniformly through the entire width of the device, independently of the specific bar arrangement. The simulations also show how the different bar designs affect the steepness of the TES transition, replicating phenomena experimentally observed, such as the dependence of $\alpha$ on the number of bars. Our model shows qualitative agreement with experimental results, and therefore represents a powerful tool to guide the design of TESs with normal metal features and other non-uniform geometries. In the future, more complex effects such as the lateral proximity effect could be implemented in this 2D model. It may also be possible to study the noise mitigating mechanisms of the normal metal features.

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ORCID iDs

Daikang Yan © https://orcid.org/0000-0003-4358-7959
Orlando Quaranta © https://orcid.org/0000-0002-7918-3991

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