FINITE ORDER $q$-INVARIANTS OF IMMERSIONS OF SURFACES INTO 3-SPACE

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Abstract. Given a surface $F$, we are interested in $\mathbb{Z}/2$ valued invariants of immersions of $F$ into $\mathbb{R}^3$, which are constant on each connected component of the complement of the quadruple point discriminant in $Imm(F, \mathbb{R}^3)$. Such invariants will be called “$q$-invariants.” Given a regular homotopy class $A \subseteq Imm(F, \mathbb{R}^3)$, we denote by $V_n(A)$ the space of all $q$-invariants on $A$ of order $\leq n$. We show that if $F$ is orientable, then for each regular homotopy class $A$ and each $n$, $\dim(V_n(A)/V_{n-1}(A)) \leq 1$.

1. Introduction

Let $F$ be a closed surface. Let $Imm(F, \mathbb{R}^3)$ denote the space of all immersions of $F$ into $\mathbb{R}^3$ and let $I_0 \subseteq Imm(F, \mathbb{R}^3)$ denote the space of all generic immersions.

Definition 1.1. A function $f : I_0 \to \mathbb{Z}/2$ will be called a “$q$-invariant” if whenever $H_t : F \to \mathbb{R}^3$ ($0 \leq t \leq 1$) is a generic regular homotopy with no quadruple points, then $f(H_0) = f(H_1)$.

Definition 1.2. Let $I_n \subseteq Imm(F, \mathbb{R}^3)$ denote the space of all immersions whose unstable self intersection consists of precisely $n$ generic quadruple points, and let $I = \bigcup_{n=0}^{\infty} I_n$.

Definition 1.3. Given a $q$-invariant $f : I_0 \to \mathbb{Z}/2$ we extend it to $I$ as follows: For $i \in I_n$ let $i_1, ..., i_{2^n} \in I_0$ be the $2^n$ generic immersions that may be obtained by slightly deforming $i$. Define

$$f(i) = \sum_{k=1}^{2^n} f(i_k).$$

For any $q$-invariant, we will always assume without mention that it is extended to the whole of $I$ as in Definition 1.3.

The following relation clearly holds:

Proposition 1.4. Let $f$ be a $q$-invariant. Let $i \in I_n$, $n \geq 1$, and let $p \in \mathbb{R}^3$ be one of its $n$ quadruple points. Then: $f(i) = f(i_1) + f(i_2)$ where $i_1, i_2 \in I_{n-1}$ are the two immersions that may be obtained by slightly deforming $i$ in a small neighborhood of $p$.

(Or equivalently, since we are in $\mathbb{Z}/2$, $f(i_2) = f(i_1) + f(i)$.)

Definition 1.5. A $q$-invariant $f$ will be called “of finite order” if $f|_{I_n} \equiv 0$ for some $n$.

The “order” of a finite order $q$-invariant $f$ is defined as the minimal $n$ such that $f|_{I_{n+1}} \equiv 0$.

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An example of a $q$-invariant of order 1 is the invariant $Q$ which is defined by the property that if $H_t : F \to \mathbb{R}^3$ $(0 \leq t \leq 1)$ is a generic regular homotopy in which $m$ quadruple points occur, then $Q(H_1) = Q(H_0) + m \mod 2$. In other words $Q$ is defined by the property that $Q|_{t_1} \equiv 1$. It was proved in [N] that $Q$ indeed exists for any surface $F$.

There are $M = 2^{2-\chi(F)}$ regular homotopy classes (i.e. connected components) in $\text{Imm}(F, \mathbb{R}^3)$. Given a regular homotopy class $A \subseteq \text{Imm}(F, \mathbb{R}^3)$, we may repeat all our definitions with $A$ in place of $\text{Imm}(F, \mathbb{R}^3)$. Let then $V_n(A)$ (respectively $V_n$) denote the space of all $q$-invariants on $A$ (respectively $\text{Imm}(F, \mathbb{R}^3)$) of order $\leq n$. $V_n(A)$ and $V_n$ are vector spaces over $\mathbb{Z}/2$, and $V_n = \bigoplus_{\alpha=1}^{M} V_n(A_\alpha)$ where $A_1, \ldots, A_M$ are the regular homotopy classes in $\text{Imm}(F, \mathbb{R}^3)$. More precisely, a function $f : I_0 \to \mathbb{Z}/2$ is a $q$-invariant of order $\leq n$ iff for every $1 \leq \alpha \leq M$, $f|_{I_0 \cap A_\alpha}$ is a $q$-invariant of order $\leq n$. And so studying $q$-invariants on $\text{Imm}(F, \mathbb{R}^3)$ is the same as studying $q$-invariants on the various regular homotopy classes.

The purpose of this work is to prove the following:

**Theorem 1.6.** If $F$ is orientable then $\dim( V_n(A)/V_{n-1}(A) ) \leq 1$ for any $A$ and $n$.

By [N] $\dim( V_1(A)/V_0(A) ) \geq 1$ for any $A$ (for all surfaces, not necessarily orientable) and so we get:

**Corollary 1.7.** If $F$ is orientable then $\dim( V_1(A)/V_0(A) ) = 1$ for any $A$.

Since as mentioned, $V_n = \bigoplus_{\alpha=1}^{M} V_n(A_\alpha)$, we get:

**Corollary 1.8.** If $F$ is orientable of genus $g$ then $\dim(V_n/V_{n-1}) \leq 2^{2g}$ for every $n$, and $\dim(V_1/V_0) = 2^{2g}$.

## 2. General $q$-Invariants

The results in this section will not assume that the $q$-invariant $f$ is of finite order.

**Theorem 2.1** (The 10 Term Relation). Let $i : F \to \mathbb{R}^3$ be any immersion whose non-stable self intersection consists of one generic quintuple point, and some finite number of generic quadruple points. Let the quintuple point be located at $p \in \mathbb{R}^3$ and let $S_1, \ldots, S_5$ be the five sheets passing through $p$. Let $i^1_k$ and $i^2_k$ ($k = 1, \ldots, 5$) be the two immersions obtained from $i$ by slightly pushing $S_k$ away from $p$ to either side. Then for any $q$-invariant $f$:

$$\sum_{k=1}^{5} \sum_{l=1}^{2} f(i^l_k) = 0.$$  

**Proof.** Starting with $i$, take $S_1$ and push it slightly to one side. Then take $S_2$ and push it away on a much smaller scale. What we now have is an immersion $j$ where sheets $S_2, \ldots, S_5$ create a little tetrahedron, and $S_1$ passes outside this tetrahedron. We define the following regular homotopy $H_t : F \to \mathbb{R}^3$ beginning and ending with $j$, we describe it in four steps: (a) $S_1$ sweeps to the other side of the tetrahedron. In this step four quadruple points occur. (b) $S_2$ sweeps across the
triple point of sheets $S_3, S_4, S_5$. This results in the vanishing of the tetrahedron and its inside-out reappearance. One quadruple point occurs here. (c) $S_1$ sweeps back to its place. Four more quadruple points occur. (d) $S_2$ sweeps back to its place. One more quadruple point occurs.

All together we have ten quadruple points, and say the $m$th quadruple point occurs at time $t_m$. It is easy to verify that the ten immersions $H_{t_1},...,H_{t_{10}}$ are precisely (equivalent to) the ten immersions $i^l_k$ ($l = 1, 2, k = 1, ..., 5$). Also, $f(H_{t_m}) = f(H_{t_m-\varepsilon}) + f(H_{t_m+\varepsilon})$ and so:

$$\sum_{kl} f(i^l_k) = \sum_{m=1}^{10} f(H_{t_m}) = \sum_{m=1}^{10} (f(H_{t_m-\varepsilon}) + f(H_{t_m+\varepsilon})).$$

But $f(H_{t_m+\varepsilon}) = f(H_{t_m+1-\varepsilon})$ (where $m + 1$ means $(m + 1) \mod 10$) and so this sum is 0.

\[\square\]

**Proposition 2.2.** Let $B(1) \subseteq \mathbb{R}^3$ be the unit ball. Let $D_1(1),...,D_4(1) \subseteq F$ be four disjoint discs which will each be parameterized as the unit disc, and let $D(1) = \bigcup_{k=1}^{4} D_k(1)$. Let $i \in I$ and assume $i^{-1}(B(1)) = D(1)$ and $i_{\mid D(1)}$ maps each $D_k(1)$ linearly onto some $L_k \cap B(1)$ where $L_k$ is a plane through the origin, and $L_1,...,L_4$ are in general position. Let $i': D(1) \to B(1)$ be an immersion of the same sort as $i_{\mid D(1)}$ but with planes $L'_1,...,L'_4$.

For $0 \leq r \leq 1$ let $B(r) \subseteq B(1)$ and $D_k(r) \subseteq D_k(1)$ be the ball and discs of radius $r$ and let $D(r) = \bigcup_{k=1}^{4} D_k(r)$.

Then: There exists an immersion $j: F \to \mathbb{R}^3$ satisfying:

1. $j$ is regularly homotopic to $i$ via a regular homotopy that fixes $F - D(1)$.
2. $j^{-1}(B(\frac{1}{2})) = D(\frac{1}{2})$
3. $j_{\mid D(\frac{1}{2})} = i'_{\mid D(\frac{1}{2})}$
4. $f(j) = f(i)$ for any $q$-invariant $f$.

**Proof.** Slightly perturb $i$ if necessary so that the eight planes $L_k, L'_k$ will all be in general position. We define a regular homotopy $H_t$ from $i$ to an immersion $\tilde{i}$ as follows: Say $a$ is the point in $D_1(1)$ which is mapped to the origin. Keeping $a$ and $F - D_1(1)$ fixed, we isotope $D_1(1)$ within $B(1)$ to get $\tilde{i}$ with $\tilde{i}^{-1}(B(\frac{1}{2})) = D(\frac{1}{2})$ and $\tilde{i}_{\mid D(\frac{1}{2})} = i'_{\mid D(\frac{1}{2})}$.

Let $i^1, i^2$ be the two immersions obtained from $i$ by slightly pushing $D_1(1)$ off of the origin, and let $\tilde{i}^1, \tilde{i}^2$ be the corresponding slight deformations of $\tilde{i}$. $H_t$ induces regular homotopies $H^l_t$ ($l = 1, 2$) from $i^l$ to $\tilde{i}^l$, and such that $H^l_t_{\mid D_1(1)}$ avoids the origin.

Now, the only triple point of $\{L_2, L_3, L_4\}$ is the origin, and $H^l_{\mid D_1(1)}$ is an isotopy which avoids the origin, and so $H^l_t$ will have no quadruple point, and so $f(i^l) = f(\tilde{i}^l)$ ($l = 1, 2$). And so (By Proposition [14]) $f(i) = f(i^1) + f(i^2) = f(\tilde{i}^1) + f(\tilde{i}^2) = f(\tilde{i})$.

We now repeat this process in the ball $B(\frac{1}{2})$ and with $D_2(\frac{1}{2})$, obtaining an immersion $\tilde{i}$ with $\tilde{i}_{\mid D_1(\frac{1}{2}) \cup D_2(\frac{1}{2})} = i'_{\mid D_1(\frac{1}{2}) \cup D_2(\frac{1}{2})}$. After four iterations we get the desired $j$.

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### 3. $q$-Invariants of Order $n$

We now prove the following theorem, which clearly implies Theorem [16] (our main theorem):


Theorem 3.1. Assume $F$ is orientable and let $f$ be a $q$-invariant of order $n$.

Then for any regular homotopy class $A \subseteq \text{Imm}(F, \mathbb{R}^3)$, $f$ is constant on $I_n \cap A$.

Proof. Let $i \in I$ and $p \in \mathbb{R}^3$ a quadruple point of $i$. A ball $B \subseteq \mathbb{R}^3$ centered at $p$ as in Proposition 2.2, i.e. such that $i^{-1}(B)$ is a union of four disjoint discs intersecting in $B$ as four planes, will be called “a good neighborhood for $i$ at $p$.”

For $i \in I_n$ let $p_1, \ldots, p_n \in \mathbb{R}^3$ be the $n$ quadruple points of $i$ in some order, and let $B_1, \ldots, B_n$ be disjoint good neighborhoods for $i$ at $p_1, \ldots, p_n$. We define $\pi_k(i): F \to \partial B_k$ as follows: Push each one of the four discs in $B_k$ slightly away from $p_k$ into the preferred side determined by the orientation of $F$. We now have a map that avoids $p_k$. Define $\pi_k(i)$ as the composition of this map with the radial projection $\mathbb{R}^3 - \{p_k\} \to \partial B_k$.

Let $d_k(i)$ denote the degree of the map $\pi_k(i)$.

Let the symmetric group $S_n$ act on $\mathbb{Z}^n$ by $\sigma(a_1, \ldots, a_n) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$, and let $\widehat{\mathbb{Z}}^n = \mathbb{Z}^n/S_n$. Let the class of $(a_1, \ldots, a_n)$ in $\widehat{\mathbb{Z}}^n$ be denoted by $[a_1, \ldots, a_n]$. For $i \in I_n$ we define $d(i) \in \widehat{\mathbb{Z}}^n$ by $d(i) = [d_1(i), \ldots, d_n(i)]$.

We break our proof into two steps. Step 1: If $i, j \in I_n \cap A$ and $d(i) = d(j)$ then $f(i) = f(j)$.

Step 2: For any $(a_1, \ldots, a_n) \in \mathbb{Z}^n$, there are immersions $i, j \in I_n \cap A$ with $d(i) = [a_1, a_2, \ldots, a_n]$, $d(j) = [a_1 + 1, a_2, \ldots, a_n]$ and $f(i) = f(j)$. The theorem clearly follows from these two claims.

Proof of Step 1: By composing $i$ with an isotopy $U_i: \mathbb{R}^3 \to \mathbb{R}^3$ we may assume that $p_1, \ldots, p_n \in \mathbb{R}^3$ are the quadruple points of both $i$ and $j$ and that $d_k(i) = d_k(j)$ for each $1 \leq k \leq n$. Let $B_1, \ldots, B_n$ be disjoint good neighborhoods for both $i$ and $j$ at $p_1, \ldots, p_n$. By composing $i$ with an isotopy $V_i: F \to F$ we may further assume that $i^{-1}(B_k) = j^{-1}(B_k)$ for every $k$. We name the four discs in $F$ corresponding to $p_k$ by $D^{kl}$, $l = 1, \ldots, 4$.

Using Proposition 2.2 we may now change $i$ such that (for smaller $B_k$’s) we will have $i|_{D^{kl}} = j|_{D^{kl}}$ for all $1 \leq k \leq n, 1 \leq l \leq 4$. The process of Proposition 2.2 indeed does not change $d_k(i)$, since the slightly pushed discs appearing in the definition of $\pi_k(i)$ can follow the regular homotopy of Proposition 2.2 and this will induce a homotopy between the corresponding $\pi_k(i)$’s.

So we may assume $i|_{P^{kl}} = j|_{P^{kl}}$ for all $1 \leq k \leq n, 1 \leq l \leq 4$. We will now show that there exists a regular homotopy from $i$ to $j$ such that each $D^{kl}$ moves only within its image in $\mathbb{R}^3$, and $F - \bigcup_{kl} D^{kl}$ moves only within $\mathbb{R}^3 - \bigcup_k B_k$. We will then be done since such a regular homotopy cannot change $f(i)$. Indeed, no sheet will pass $p_1, \ldots, p_n$ and so the only singularities that might be relevant are the quadruple points occurring in $\mathbb{R}^3 - \bigcup_k B_k$. But whenever such a quadruple point occurs, then we will have $n + 1$ quadruple points all together, and so since $f$ is of order $n$, $f(i)$ will not change. (Proposition 1.4.)

To show the existence of the above regular homotopy, we construct the following handle decomposition of $F$. Our discs $D^{kl}$, $(1 \leq k \leq n, 1 \leq l \leq 4)$ will be the 0-handles. If $g$ is the genus of $F$ we will have $2g + 4n - 1$ 1-handles as follows: $2g$ 1-handles will have both ends glued to $D^{11}$ such that $D^{11}$ with these $2g$ handles will decompose $F$ in the standard way. Then choose an ordering of the discs $D^{kl}$ with $D^{11}$ first, and connect each two consecutive discs with a 1-handle. The complement of the 0- and 1-handles is one disc which will be the unique 2-handle.

We first define our regular homotopy on the union of 0- and 1-handles. Take a 1-handle $h$ of the first type. Since $i$ and $j$ are regularly homotopic, their restrictions to the annulus $D^{11} \cup h$ are also
regularly homotopic. We can construct such a regular homotopy of $D^{11} \cup h$ fixing $D^{11}$ and avoiding $\bigcup_k B_k$.

Next consider the 1-handles of the second type. Take the 1-handle $h$ connecting $D^{11}$ to the second disc in our ordering, call it $D'$. Then if $i|_h$ and $j|_h$ are not regularly homotopic relative the gluing of $h$ to $D^{11} \cup D'$, then we perform one full rotation of $D'$, as to make them regularly homotopic. (This will require a motion of the next 1-handle too.) Again we perform all regular homotopies while avoiding $\bigcup_k B_k$. We can now go along the chain of 1-handles of the second type, and regularly homotope them one by one as we did the first one. At each step we might need to move the next 0-handle and 1-handle, but we never need to change what we have already done.

So far we have constructed the desired regular homotopy on the union of 0- and 1-handles. By [S] this regular homotopy may be extended to the whole of $F$ (still avoiding $\bigcup_k B_k$.) And so, if we denote our 2-handle by $D$, we are left with regularly homotoping $i|_D$ to $j|_D$ (relative $\partial D$.) Since $d_k(i) = d_k(j)$ for all $k$, these maps are homotopic in $\mathbb{R}^3 - \bigcup_k B_k$. By [S] they are also regularly homotopic in $\mathbb{R}^3 - \bigcup_k B_k$, since the obstruction to that would lie in $\pi_2(SO_3) = 0$.

**Proof of Step 2:** Take any immersion $i' \in I_n \cap A$ with $d(i') = [a_1, ..., a_n]$ and let $p_1, ..., p_n \in \mathbb{R}^3$ be the quadruple points of $i'$, ordered such that $d_k(i') = a_k, 1 \leq k \leq n$. (Clearly any $[a_1, ..., a_n] \in \widetilde{\mathbb{Z}}^n$ may be realized within any regular homotopy class.) Take a disc in $F$ which is away from the $p_k$’s and start pushing it (i.e. regularly homotoping it) into its preferred side directing it towards $p_1$. Avoid any of the $p_k$’s on the way, and so the immersion $i$ we will get just before arriving at $p_1$, will still have $d_k(i) = a_k$ for all $k$. We then pass $p_1$ creating a quintuple point, and continue to the other side arriving at an immersion $j$ which is again in $I_n$. Clearly $d_1(j) = a_1 + 1$ and $d_k(j) = a_k$ for $k \geq 2$. We will now use Step 1 and the 10 term relation (Theorem [2.1]) to show that $f(i) = f(j)$. Indeed, let us name the five sheets of our quintuple point by $S_1, ..., S_5$ where $S_1$ is the sheet coming from the disc that we pushed into $p_1$. Let $i^1_m (m = 1, ..., 5)$ denote the immersion obtained by pushing $S_m$ into its non-preferred side, and $i^2_m$ the immersion obtained by pushing $S_m$ into its preferred side. Then $i = i^1_1$ and $j = i^2_1$. Recall that $\pi_1(i^1_m)$ is constructed by pushing all four sheets involved in the quadruple point at $p_1$ into their preferred side. And so for each $1 \leq m \leq 5$, $\pi_1(i^1_m)$ has one sheet pushed into the non-preferred side and four sheets into the preferred side, and so $d_1(i^1_m)$ are all equal to each other. And, for each $1 \leq m \leq 5$, $\pi_1(i^2_m)$ has all five sheets pushed into the preferred side and so also $d_1(i^2_m)$ are all equal to each other. Clearly all this has no effect on $d_k$ for $k \geq 2$, and so we have $d(i^1_m) = d(i)$ and $d(i^2_m) = d(j)$ for all $1 \leq m \leq 5$. And so by step 1, $f(i^1_m) = f(i)$ and $f(i^2_m) = f(j)$ for all $1 \leq m \leq 5$. And so by the 10 term relation, $0 = \sum_m f(i^1_m) = 5f(i) + 5f(j) = f(i) + f(j)$ i.e. $f(i) = f(j)$.

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**References**

[N] T. Nowik: “Quadruple points of regular homotopies of surfaces in 3-manifolds.” - Preprint -
http://www.math.columbia.edu/~tahl/quadruple.ps

[O] T. Ozawa: “Finite order topological invariants of plane curves.” J. Knot Theory Ramification 8 (1999) no. 1, 33–47.

[S] S. Smale: “A classification of immersions of the two-sphere.” Trans. Amer. Math. Soc. 90 (1958) 281–290.
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