GELFAND-ZETLIN BASIS, WHITTAKER VECTORS AND A BOSONIC FORMULA FOR THE $\mathfrak{sl}_{n+1}$ PRINCIPAL SUBSPACE

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Abstract. We derive a bosonic formula for the character of the principal space in the level $k$ vacuum module for $\mathfrak{sl}_{n+1}$, starting from a known fermionic formula for it. In our previous work, the latter was written as a sum consisting of Shapovalov scalar products of the Whittaker vectors for $U_{t^\pm 1}(\mathfrak{gl}_{n+1})$. In this paper we compute these scalar products in the bosonic form, using the decomposition of the Whittaker vectors in the Gelfand-Zetlin basis. We show further that the bosonic formula obtained in this way is the quasi-classical decomposition of the fermionic formula.

1. Introduction

One of the central results in the theory of Kac-Moody algebras is the Weyl formula for the characters of the irreducible representations. This formula can be interpreted “quasi-classically”. It means the following. Let $L_\chi$ be an integrable representation with the highest weight $\chi$. In $L_\chi$ there are some special vectors called the extremal vectors. They are labelled by the Weyl group and have the form $w \cdot v_\chi$, where $v_\chi$ is the highest weight vector, $w$ is an element of the Weyl group and $w \cdot v$ denotes the projective action. The Weyl formula reads as

$$\text{ch} L_\chi = \sum_{w \in W} C_w(\chi),$$

where $C_w(\chi)$ is interpreted as the character of $L_\chi$ in the vicinity of the extremal vector $w \cdot v_\chi$. In this interpretation we suppose $\chi$ is “big”, so that the extremal vector $w \cdot v_\chi$ is well-separated from other extremal vectors. To be more precise it means that, when $\chi \to \infty$, generically the character $L_\chi$ in the vicinity of $w \cdot v_\chi$ stabilizes and gives $C_w(\chi)$. The Weyl formula states that the quasi-classical decomposition is exact for finite $\chi$. One important point in the decomposition is that each term $C_w(\chi)$ is, up to a simple monomial, the inverse of a (possibly infinite) product of simple factors.

Now suppose that $\hat{g}$ is an affine Kac-Moody algebra, and $L_k$ be the vacuum representation of level $k$ with the highest weight vector $v_k$. Let $g = n_+ \oplus h \oplus n_-$ be the Cartan decomposition, and let $\hat{n}_+ = n_+ \otimes \mathbb{C}[t, t^{-1}] \subset \hat{g}$ be the nilpotent subalgebra. Set

$$V^k = U(\hat{n}_+) \cdot v_k \subset L_k$$

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and call it the principal subspace in \( L_k \). The quasi-classical formula for the character of \( V^k \) can also be written. For example, if \( g = \mathfrak{sl}_2 \), we have [FL]
\[
\text{ch } V^k \overset{\text{def}}{=} \text{Tr}_{V^k} q^{d_z^2/2}
\]
\[
= \sum_{m=0}^{\infty} \left( \frac{q^{2m+1}z}{(q)_m(q^{-2m+1}z)^m} \right) q^{kmz^k} m
\]
where \( d = t \frac{d}{dt} \) is the scaling operator, \( H \) is the generator of the Cartan subalgebra, and \((z)_m = \prod_{i=1}^{m}(1 - q^{i-1}z)\). In this formula the right hand side is understood as power series in \( z \). The \( m = 0,1,2, \cdots \) terms are the contributions from the extremal vectors \( v_k, e_{-1}^k v_k, e_{-3}^k e_{-1}^k v_k, \cdots \).

In general
\[
\text{ch } V^k = \sum_{\gamma \in Q^+} C_\gamma(k)
\]
where \( Q \) is the root lattice, and the subset \( Q^+ \) consists of linear combinations of the simple roots with non-negative integer coefficients.

In [FFJMM], it was proved that for \( g = \mathfrak{sl}_3 \)
\[
\text{ch } V^k \overset{\text{def}}{=} \sum_{d_1,d_2=0}^{\infty} q^{k(d_1^2+d_2^2-d_1d_2)} z_1^{kd_1} z_2^{kd_2} J_{d_1,d_2}(q, q^{2d_1-d_2} z_1, q^{2d_2-d_1} z_2),
\]
\[
J_{d_1,d_2}(q, z_1, z_2) = \frac{1}{(qz_1)_\infty (qz_2)_\infty (qz_1z_2)_\infty}
\]
\[
\times \frac{(qz_1^{-1}z_2^{-1})_{d_1+d_2}}{(q)_{d_1}(q)_{d_2} (qz_1^{-1})_{d_1} (qz_2^{-1})_{d_2} (qz_1^{-1}z_2^{-1})_{d_1} (qz_1^{-1}z_2^{-1})_{d_2}}
\]
The terms in this formula are still factorized but they have nontrivial factors in the numerators. On the other hand, in [FFJMM], we have also derived another expression for the same character, in which \( J_{d_1,d_2}(q, q^{2d_1-d_2} z_1, q^{2d_2-d_1} z_2) \) is split into 12 terms, each of which is a simple power in \( q, z_1, z_2 \) with a factorized denominator. We call such formula the “desingularization”. In general, \( C_\gamma(k) \) is complicated and cannot be factorized. However, in this paper, we show that at least a desingularization can be found for the character of the principal subspace for the vacuum module where \( g = \mathfrak{sl}_{n+1} \) (see Theorem 3.1 and Proposition 2.2). For \( g = \mathfrak{sl}_3 \) we have
\[
J_{d_1,d_2}(q, z_1, z_2) = \sum_{m=0}^{\min(d_1,d_2)} (-z_1)^m q^{-m(d_2-m)+m(m-1)/2}
\]
\[
\times \frac{1}{(qz_1)_m(qz_2)_m(qz_1z_2)_m(qz_1^{-1}z_2^{-1})_{d_1+m} (qz_1^{-1}z_2^{-1})_{d_2-m}}
\]
We call such a formula a bosonic formula. We note that in [FFJMM] bosonic formulas for more general modules over \( \hat{\mathfrak{n}}_+ \) are obtained in the case where \( g = \mathfrak{sl}_3 \), in which we used more terms than the case of the vacuum module in this paper.

Following some geometrical ideas from [BrTi], one naturally expects that in the desingularization of the \( \mathfrak{sl}_{n+1} \) formula the terms are labelled by some basis in the Verma modules of \( U_q(\mathfrak{g}_n^{\mathfrak{g}_{n+1}}) \) where \( q = v^2 \), actually by the Gelfand-Zetlin basis. Our proof goes as follows. In [FFJMM2], we managed
to rewrite the fermionic formula \( \text{FS} \) for \( \text{ch} V^k \) in terms of the eigenfunctions of the quantum difference Toda Hamiltonian. Such eigenfunctions were written by using the Whittaker vectors in the Verma modules for \( U_q (\mathfrak{g}_{n+1}) \). In this paper, we decompose the Whittaker vectors in the Gelfand-Zetlin basis. This decomposition produces the decomposition of the coefficients of the eigenfunctions. Moreover, each term of this decomposition has a factorized form. As a by-product we get some interesting fermionic formulas and their quasi-classical decompositions.

Fermionic formulas are statistical sums over configurations of particles with color and weight. A configuration of particles is determined by a set of non-negative integers \( \mathbf{m} = (m_{i,t})_{(i,t) \in S} \) which represents the number of particles with color \( i \) and weight \( t \). Given a function \( B(\mathbf{m}) \), the fermionic sum is of the form

\[
F(S, B) = \sum_{\mathbf{m}} \frac{q^{B(\mathbf{m})}}{\prod_{(i,t) \in S} (q)_{m_{i,t}}}
\]

See (1.2) for the case we study in this paper. Let us discuss the fermionic formula for the character \( \text{ch} V^k \) for \( \mathfrak{g} = \mathfrak{sl}_3 \). In this case we take \( S_k = \{1,2\} \times [1,k] \) for \( S \). In [FFJMM2] we have shown that the quasi-classical decomposition is valid in the following sense. Fix \((m_1, m_2, n_1, n_2) \in \mathbb{Z}_+^4 \), and consider the above sum with the restriction that

\[
\sum_{1 \leq t < k} m_{i,t} = m_i, \quad \sum_{1 < t \leq k} m_{i,t} = n_i.
\]

In the limit \( k \to \infty \) this sum approaches some rational function \( F_{1,k}(m_1, m_2, n_1, n_2) \). In [FFJMM2] we have shown that for finite \( k \geq 0 \), we have the equality

\[
F(S_k, B) = \sum_{m_1, m_2, n_1, n_2} F_{1,k}(m_1, m_2, n_1, n_2).
\]

We call this equality the quasi-classical decomposition. In this paper we consider the case where we take

\[
S_{k',k} = \{(i,t)|1 \leq t \leq k \delta_{i,1} + k' \delta_{i,2}\}.
\]

In the limit \( 1 \ll k' < k \), we have a similar decomposition:

\[
F(S_{k',k}, B) = \sum_{m_1, m_2, n_1, n_2, l_1} F_{1,k',k}(m_1, m_2, n_1, n_2, l_1).
\]

The restriction for the sum for \( F_{1,k',k}(m_1, m_2, n_1, n_2, l_1) \) is such that

\[
\sum_{1 \leq t < k'} m_{i,t} = m_i, \quad \sum_{1 < t \leq k'} m_{1,t} + \sum_{k' < t \leq k} m_{1,t} = n_1, \quad \sum_{1 < t \leq k'} m_{2,t} = n_2, \quad \sum_{k' < t \leq k} m_{1,t} = l_1.
\]

There are two remarkable features. First, the decomposition is exact for finite \( k \geq k' \geq 1 \). Therefore, if \( k = k' \), it gives another formula for \( \text{ch} V^k \). Second, each summand in this decomposition is factorized. In fact, summing up over \( m_1, m_2 \) we obtain (1.1), (1.2). We will derive such a decomposition for general \( \mathfrak{g} = \mathfrak{sl}_{n+1} \) by using the Drinfeld Casimir elements of smaller rank.

Finally, we note that our paper is inspired by [BrFi]. Actually we study the structure of the singular points on some moduli spaces by using the equivalent language from the representation theory of affine Lie algebras.
2. WHITTAKER VECTORS FOR $\mathfrak{gl}_{n+1}$

In this section we recall some known facts about Whittaker vectors for $\mathfrak{gl}_{n+1}$ and their Shapovalov scalar product, including the Toda recursion and fermionic formulas. We give their explicit formulas using the Gelfand-Zetlin basis of Verma modules.

2.1. Gelfand-Zetlin basis. Throughout the text, we consider the complex Lie algebra $\mathfrak{gl}_{n+1}$. Let $e_0, \cdots, e_n$ be a basis of the Cartan subalgebra orthonormal with respect to the invariant scalar product $(\ , \ )$. The simple roots and fundamental weights are expressed as $\alpha_i = e_{i-1} - e_i$, $\omega_i = e_0 + \cdots + e_{i-1}$, $1 \leq i \leq n$. We set $Q = \mathbb{Z}^{n}_{i=1} \omega_i$, $P = \mathbb{Z}^{n}_{i=0} \omega_i$, and $\rho = \sum_{i=1}^{n} \omega_i$.

Let $U_v(\mathfrak{gl}_{n+1})$ be the corresponding quantum group over $\mathbb{K} = \mathbb{C}(v)$, with generators $\{E_i, F_i\}_{1 \leq i \leq n}$, $\{v^{\pm \epsilon_i}\}_{0 \leq i \leq n}$ and standard defining relations. We set $K_i = v^{\epsilon_{i-1} - \epsilon_i}$. For $\lambda = \sum_{i=0}^{n} \lambda_i \epsilon_i \in P$, let $\mathcal{V}_\lambda$ be the Verma module over $U_v(\mathfrak{gl}_{n+1})$ generated by the highest weight vector $1^\lambda$ with defining relations

$$E_i 1^\lambda = 0 \ (1 \leq i \leq n), \quad v^\epsilon 1^\lambda = v^\lambda 1^\lambda \ (0 \leq i \leq n).$$

Recall that $\mathcal{V}_\lambda$ has a distinguished basis (known as the Gelfand-Zetlin basis) relative to the tower of subalgebras

$$(2.1) \quad A_0 \subset A_1 \subset \cdots \subset A_n,$$

where $A_k \simeq U_v(\mathfrak{gl}_{k+1})$ $(k = 0, \ldots, n)$ denotes the subalgebra of $U_v(\mathfrak{gl}_{n+1})$ generated by $\{E_i, F_i\}_{1 \leq i \leq k}$ and $\{v^{\pm \epsilon_i}\}_{0 \leq i \leq k}$. Each subspace of $\mathcal{V}_\lambda$ which is jointly invariant under $A_k$'s is one dimensional. Such subspaces are labeled by arrays of numbers

$$\lambda = \begin{pmatrix} \lambda_{0,n} & \lambda_{1,n} & \cdots & \lambda_{n-1,n} & \lambda_{n,n} \\ \lambda_{0,n-1} & \lambda_{1,n-1} & \cdots & \lambda_{n-2,n} & \lambda_{n-1,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{0,1} & \lambda_{1,1} & & & \\ & & \lambda_{0,0} & & \end{pmatrix}$$

(2.2)

called the Gelfand pattern. Here we set

$$\lambda_{k,i} = \lambda_k - m_{k,i},$$

and $m_{k,i}$ are non-negative integers satisfying

$$0 = m_{k,n} \leq m_{k,n-1} \leq m_{k,n-2} \leq \cdots \leq m_{k,k} \quad (0 \leq k \leq n).$$

In particular, we have

$$\lambda_{k,n} = \lambda_k.$$

For economy of space we shall also write $\lambda$ as

$$\lambda = (\lambda^{(n)}, \cdots, \lambda^{(0)}), \quad \lambda^{(i)} = \lambda_{0,i} \epsilon_0 + \cdots + \lambda_{i,i} \epsilon_i.$$

By choosing an appropriate generator $|\lambda\rangle = |\lambda^{(n)}, \cdots, \lambda^{(0)}\rangle$ of each subspace corresponding to (2.2), the action of Chevalley generators can be described explicitly. For this purpose it is convenient to extend the base field from $\mathbb{K}$ to $\mathbb{R}$ obtained by adjoining all elements of the form $\sqrt{f} \ (f \in \mathbb{K})$. 


We use the same symbols $U_v(\mathfrak{gl}_{n+1})$ (resp. $\mathcal{Y}^\lambda$) to denote $\mathcal{R} \otimes_K U_v(\mathfrak{gl}_{n+1})$ (resp. $\mathcal{R} \otimes_K \mathcal{Y}^\lambda$). Then the Chevalley generators act by the formula

$$v^s_i|\lambda\rangle = v^{h_i(\lambda)-h_{i-1}(\lambda)}|\lambda\rangle,$$

(2.5)

$$E_i|\lambda\rangle = \sum_{k=0}^{i-1} c_{k,i-1}(\lambda)|\lambda^{(k,i-1)}_+\rangle,$$

(2.6)

$$F_i|\lambda\rangle = \sum_{k=0}^{i-1} c_{k,i-1}(\lambda)|\lambda^{(k,i-1)}_-\rangle.$$

(2.7)

Here $h_i(\lambda) = \sum_{k=0}^i \lambda_{k,i}$, and $\lambda^{(k,i-1)}_\pm$ signifies the Gelfand pattern wherein $\lambda_{k,i-1}$ is replaced by $\lambda_{k,i-1} \pm 1$ while keeping all other $\lambda_j$’s unchanged. The coefficients $c_{k,i-1}(\lambda)$ have the factorized form

$$c_{k,i-1}(\lambda)^2 = \frac{\prod_{0 \leq l \leq i-2}[\lambda_{l,i-2} - \lambda_{k,i-1} - l + k - 1] \prod_{0 \leq l \leq i}[\lambda_{l,i} - \lambda_{k,i-1} - l + k - 1]}{\prod_{0 \leq l \leq i-(\lambda_{k,i-1})}[\lambda_{l,i} - \lambda_{k,i-1} - l + k - 1][\lambda_{l,i} - \lambda_{k,i-1} - l + k - 1]}.$$

(2.8)

Here and after, we use the symbols $[m] = (v^m - v^{-m})/(v - v^{-1})$, $[m]! = [m][m+1] \cdots [1]$, and $[m]_k = [m][m+1] \cdots [m+k-1]$.

2.2. **Whittaker vectors.** The Verma module carries an obvious grading $\mathcal{Y}^\lambda = \bigoplus_{\beta \in Q^+} (\mathcal{Y}^\lambda)_\beta$ where

$$\mathcal{Y}^\lambda = \{w \in \mathcal{Y}^\lambda \mid K_i w = v^{(\alpha_i,\lambda_{i-\beta})} w \ (1 \leq i \leq n)\}.$$

A Whittaker vector $\theta^\lambda = \sum_{\beta \in Q^+} \theta^\lambda_\beta$ is an element of a completion $\prod_{\beta \in Q^+} (\mathcal{Y}^\lambda)_\beta$ of the Verma module. It is uniquely defined by the conditions that $\theta^\lambda_0 = 1^\lambda$ and

$$E_i K_i^{-1} \theta^\lambda = \frac{1}{1 - v^2} \theta^\lambda \ (1 \leq i \leq n).$$

(2.9)

Let us give an explicit formula for $\theta^\lambda$ in terms of the Gelfand-Zetlin basis. For $i = 1, \ldots, n$ and parameters $\mu = (\mu_0, \ldots, \mu_i)$, $\nu = (\nu_0, \ldots, \nu_{i-1})$ satisfying $\mu_k - \nu_k \in \mathbb{Z}_{\geq 0}$, define

$$A_i(\mu, \nu)^2 = \frac{1}{\prod_{k=0}^{i-1} \mu_k!} \frac{1}{\prod_{0 \leq k < l \leq i-1} [\nu_k - \nu_l - k + l + 1]_{\mu_k - \nu_k}} \prod_{0 \leq k < l \leq i} [\nu_k - \mu_l - k + l]_{\mu_k - \nu_k}.$$

(2.10)

**Proposition 2.1.** In the Gelfand-Zetlin basis (2.2), the Whittaker vector $\theta^\lambda$ has the following representation:

$$\theta^\lambda = \sum_\lambda \left( \frac{1}{1 - v^2} \right)^{h(\lambda)} \prod_{i=1}^n C_i(\lambda^{(i)}_\beta, \lambda^{(i-1)}_{\beta}) |\lambda\rangle.$$

(2.11)
Here we have set
\[ \text{ht}(\lambda) = \sum_{0 \leq k \leq i \leq n-1} (\lambda_{k,n} - \lambda_{k,i}), \]
\[ C_i(\lambda^{(i)}, \lambda^{(i-1)}) = v_0^{n_i(\lambda^{(i)}, \lambda^{(i-1)})} A_i(\lambda^{(i)}, \lambda^{(i-1)}) \]
\[ p_i(\lambda^{(i)}, \lambda^{(i-1)}) = (i - 1) \left( \sum_{k=0}^{i-1} \lambda_{k,i-1} \left( \sum_{k=0}^{i-1} \lambda_{k,i} - \sum_{k=0}^{i-1} \lambda_{k,i} \right) \right) - \sum_{0 \leq k < l \leq i} \lambda_{k,i-1} \lambda_{l,i-1} + \sum_{0 \leq k < l \leq i} \lambda_{k,i} \lambda_{l,i} \]
\[ - \sum_{k=1}^{i-1} k(i-k)(\lambda_{k,i-1} - \lambda_{k,i}). \]
The sum ranges over all Gelfand patterns \([2.2] - [2.3]\) with fixed \(\lambda_0, \ldots, \lambda_n\).

**Proof.** The proof is a direct calculation using formulas \([2.5], [2.6]\) for the action of \(E_i, K_i\). The defining relations \([2.9]\) reduce to the identities
\[ \sum_{l=0}^{i} \frac{\prod_{k=0}^{i-1} (a_k - b_l)}{\prod_{k \neq l} (b_k - b_l)} \left( - \sum_{k=0}^{i} a_k + \sum_{k=0}^{i-1} b_k \right) = 1. \]

\(\square\)

2.3. **Scalar product.** The main object of our interest is the scalar product of the Whittaker and the dual Whittaker vectors. To define the latter, we consider the quantum group \(U_{v^{-1}}(\mathfrak{gl}_{n+1})\) with parameter \(v^{-1}\). Its generators are denoted by \(\{\tilde{E}_i, \tilde{F}_i\}_{1 \leq i \leq n}, \{\tilde{v}^\pm \}_{0 \leq i \leq n}\). Let \(\overline{\mathcal{V}}^\lambda\) be the Verma module over \(U_{v^{-1}}(\mathfrak{gl}_{n+1})\) generated by the highest weight vector \(\overline{\mathbf{1}}^\lambda\) with defining relations
\[ \tilde{E}_i \overline{\mathbf{1}}^\lambda = 0 \quad (1 \leq i \leq n), \quad \tilde{v}^i \overline{\mathbf{1}}^\lambda = v^{-\lambda_i} \overline{\mathbf{1}}^\lambda \quad (0 \leq i \leq n). \]
The dual Whittaker vector is defined similarly as an element \(\overline{\theta}^\lambda \in \prod_{\beta \in \mathcal{Q}^+} (\overline{\mathcal{V}}^\beta),\) imposing \(\overline{\theta}^0 = \overline{\mathbf{1}}^\lambda\) and
\[ \tilde{E}_i \tilde{K}_{i-1}^{-1} \overline{\theta}^\lambda = \frac{1}{1 - v^{-2}} \overline{\theta}^\lambda \quad (1 \leq i \leq n) \]
in place of \([2.9]\).

Let \(\sigma\) be the \(\mathcal{R}\)-linear anti-isomorphism of algebras given by
\[ \sigma : U_v(\mathfrak{gl}_{n+1}) \to U_{v^{-1}}(\mathfrak{gl}_{n+1}), \quad E_i \mapsto \widetilde{E}_i, \quad F_i \mapsto \widetilde{F}_i, \quad K_i \mapsto \tilde{K}_{i-1}^{-1}. \]
There is a unique non-degenerate \(\mathcal{R}\)-bilinear pairing \((, )\) \(\mathcal{V}^\lambda \times \overline{\mathcal{V}}^\lambda \to \mathcal{R}\) such that \((\mathbf{1}^\lambda, \overline{\mathbf{1}}^\lambda) = 1\) and
\[ (xw, w') = (w, \sigma(x)w') \]
for all \(x \in U_v(\mathfrak{gl}_{n+1})\) and \(w \in \mathcal{V}^\lambda, w' \in \overline{\mathcal{V}}^\lambda\). We call \([2.14]\) the Shapovalov pairing. The Gelfand-Zetlin basis \(\{|\lambda\}\} of \(\mathcal{V}^\lambda\) and \(\{\overline{|\lambda\}}\} of \(\overline{\mathcal{V}}^\lambda\) are orthonormal with respect to the Shapovalov pairing: \((|\lambda\rangle, \overline{|\lambda\}}\rangle) = \delta_{\lambda^{\lambda}, \lambda^{\lambda}}.
In \([FFJMM2]\), we considered the scalar product
\[ J^\lambda_\beta = J^\lambda_\beta [0, \infty) = v^{-(\beta, \beta)/2} (\lambda, \beta) (\overline{\theta}^\lambda_\beta, \overline{\theta}^\lambda_\beta). \]
We set $J^0_\beta = 0$ unless $\beta \in Q^+$. The notation $J^0_\beta[0, \infty)$ comes from the fact that the corresponding fermionic formula is related to the interval $[0, \infty)$ (see Theorem 3.2 in [FFJMM2] and Proposition 2.4 below).

In what follows, we choose the variables $z_i = q^{-(\lambda, \alpha_i)}$ and write

$$
J_{d_1, \cdots, d_n}(q, z_1, \cdots, z_n) = J^0_\beta[0, \infty) \text{ for } \beta = \sum_{i=1}^n d_i \alpha_i.
$$

These are rational functions in $q$ and $z_1, \cdots, z_n$.

The explicit formula (2.11) (and for its dual) yields the following expression for (2.16). Set

$$
z_{k,l} = \prod_{j=k+1}^l z_j.
$$

**Proposition 2.2.** We have

$$
J_{d_1, \cdots, d_n}(q, z_1, \cdots, z_n) = \sum_{m_0, i+1 + \cdots + m_{l-1, i-1} = d_i} \frac{(-1)^{n}}{\sum_{i=1}^n m_i \sum_{j=i+1}^n m_j} \times q^{p(m)} \prod_{i=1}^n \prod_{0 \leq k < i \leq m_i} \frac{1}{(q^{m_i} - z_{k,i})^{m_i}} \times \prod_{0 \leq k < l \leq n} \frac{1}{(q^{m_i} - z_{k,i} z_{l,i})^{m_i}},
$$

where

$$
p(m) = -\sum_{0 \leq k < i \leq l \leq n-1} m_{k,i} m_{l,i} + \sum_{0 \leq k < l \leq n-1} m_{k,i} m_{l,i-1} + \frac{1}{2} \sum_{0 \leq k < l \leq n-1} m_{k,i} (m_{k,i} - 1).
$$

The sum is taken over all non-negative integers $m_{k,i}$ satisfying (2.4) and $\sum_{k=0}^{l-1} m_{k,i} = d_i$.

**Example.** We have

$$n = 1 : J_{d_1}(q, z_1) = \frac{1}{(q^{d_1})(q^{z_1})},
$$

$$n = 2 : J_{d_1, d_2}(q, z_1, z_2) = \frac{1}{(q^{d_1, d_2})(q^{z_1})^{d_2}} \times \frac{1}{(q^{z_1})^{d_2}(q^{z_2})^{d_2} - m(q^{d_2 + m z_1})^{d_1 - m}}.
$$

The second formula can be further simplified to

$$J_{d_1, d_2}(q, z_1, z_2) = \frac{(q^{z_1 z_2})_{d_1 + d_2}}{(q^{d_1})_{d_2} (q^{z_1})_{d_2} (q^{z_2})_{d_1} (q^{z_1 z_2})_{d_1} (q^{z_1 z_2})_{d_2}}.
$$

1The present definition for $z_i$ is different from [FFJMM2] where $z_i = q^{-(\lambda, \alpha_i)}$ was used.
The existence of a factorized form is a specific (and rather accidental) feature of \( n = 1, 2 \). It does not hold for \( n \geq 3 \).

2.4. **Toda Hamiltonian and fermionic formula.** The quantity \( J_{d_1, \ldots, d_n}(q, z_1, \ldots, z_n) \) admits, besides the explicit formula (2.17), other ways of characterization. For completeness, we quote these facts from the literature adapting to the present notation.

The first is through the quantum difference Toda Hamiltonian of type \( A \). It is a \( q \)-difference operator which acts on functions \( f(y_1, \ldots, y_n) \):

\[
H f = \sum_{i=0}^{n} D_{i,n}^{-1} D_{i+1}(z_{i,n}(1 - y_i)f).
\]

Here \( D_i \) stands for the \( q \)-shift operator \((D_i f)(y_1, \ldots, qy_i, \ldots, y_n) = f(y_1, \ldots, qy_i, \ldots, y_n)\), and we set \( y_0 = 0, D_0 = D_{n+1} = 1 \).

**Proposition 2.3.** \( \text{[Sev, Et]} \) The generating series

\[
F(q, y_1, \ldots, y_n; z_1, \ldots, z_n) = \sum_{d_1, \ldots, d_n \geq 0} J_{d_1, \ldots, d_n}(q, z_1, \ldots, z_n) y_1^{d_1} \cdots y_n^{d_n}
\]

is an eigenfunction of the Toda Hamiltonian:

\[
H F = \left( \sum_{i=0}^{n} z_{i,n} \right) F.
\]

The second way is the fermionic formula. Here we restrict the general consideration in \( \text{[FFJMM2]} \) to the Cartan matrix of type \( A \). For a (possibly infinite) interval \([r, s]\), consider the sum

\[
I_{d_1, \ldots, d_n}(q, z_1, \ldots, z_n | r, s) = \sum_{\begin{subarray}{c} \sum_{t=1}^{n} l_t \geq d_t \end{subarray}} q^{\sum_{t=1}^{n} l_t - \min(t, t')} \left( \sum_{i=1}^{n} l_{t_i} l_{t_i'} - \sum_{i=1}^{n-1} l_{t_i} l_{t_i+1} \right) \prod_{i=1}^{n} z_i^{\sum_{t=r}^{s} d_{t_i}}.
\]

Then we have

**Proposition 2.4.** \( \text{[FFJMM2]} \) The following formula holds.

\[
J_{d_1, \ldots, d_n}(q, z_1, \ldots, z_n) = I_{d_1, \ldots, d_n}(q, z_1, \ldots, z_n | 0, \infty).
\]

3. **Character of the principal subspace**

Consider the affine Lie algebra \( \tilde{\mathfrak{sl}}_{n+1} = \mathfrak{sl}_{n+1}[t, t^{-1}] \oplus \mathbb{C}e \oplus \mathbb{C}d \). Let \( M^k \) be the integrable highest weight vacuum module of level \( k \in \mathbb{Z}_{\geq 0} \). Namely \( M^k \) is the irreducible highest weight \( \mathfrak{sl}_{n+1} \)-module generated by the highest weight vector \( w \), such that

\[
(x \otimes t^j)w = 0 \quad (x \in \mathfrak{sl}_{n+1}, j \geq 0).
\]

\(^{2}\)The definition is modified from that of \( \text{[FFJMM2]} \), (2.3), in order to match with the change of the definition of \( z_1 \).
and the canonical central element $c$ acts as the scalar $k$. Let $\hat{\mathfrak{n}}_+ = \mathfrak{n}_+ \otimes \mathbb{C}[t, t^{-1}]$ be the current algebra over the nilpotent subalgebra $\mathfrak{n}_+ \subset \mathfrak{sl}_{n+1}$. The $\hat{\mathfrak{n}}_+$-submodule generated from $w$,

$$V^k = U(\hat{\mathfrak{n}}_+)w \subset M^k$$

is called the principal subspace of $M^k$.

The following fermionic formula is known [FS] (see also [FJMMT]).

$$\text{ch } V^k = \sum_{l_1, \ldots, l_n \geq 0} q^{\sum_{t, t' \geq 0} \min(t, t') \left( \sum_{i=1}^n l_i L_{i, t'_{i+1}} - \sum_{i=1}^{n-1} t_i L_{i, t'_{i+1}} \right) \prod_{i=1}^n \prod_{l_{t, i}}^{k}(q)_{t, i}}.$$ 

In the notation of (2.20), we have

$$\text{ch } V^k = \sum_{d_1, \ldots, d_n \geq 0} I_{d_1, \ldots, d_n}(z_1, \ldots, z_n | 1, k).$$

The main result of the present note is the following bosonic formula, which generalizes a result of [FFJMM] for $n = 2$.

**Theorem 3.1.** The character of the principal subspace of the level $k$ vacuum module over $\hat{\mathfrak{sl}}_{n+1}$ is given by

$$\text{ch } V^k = \sum_{d_1, \ldots, d_n \geq 0} q^{k(\sum_{i=1}^n d_i^2 - \sum_{i=1}^{n-1} d_i) z_1 z_2 \ldots z_n} \times J_{d_1, \ldots, d_n}(q, q^{2d_1}, q^{d_1+2d_2}, \ldots, q^{d_1+2d_2+\cdots+2d_n} z_n),$$

where

$$J_{d_1, \ldots, d_n}(q, z_1, \ldots, z_n) = \prod_{0 \leq i < j \leq n} (q^{z_{i,j}})^{d_i - d_j} \cdot J_{d_1, \ldots, d_n}(q, z_1^{-1}, \ldots, z_n^{-1}),$$

and $J_{d_1, \ldots, d_n}(q, z_1, \ldots, z_n)$ is given by (2.17).

**Proof.** Writing $(q)_{\beta} = (q)_{d_1} \cdots (q)_{d_n}$ for $\beta = d_1 \alpha_1 + \cdots + d_n \alpha_n$, let us introduce the notation

$$J^r_{\beta}[r, s] = \sum_{r \geq t \geq s, \gamma_t = \beta, \sum_{r \geq t \geq s, \gamma_t = \beta}} q^{(1/2) \sum_{r \leq t \leq s} \min(t, t') (\gamma_t + \lambda + (\sum_{t < s} \gamma_t)) \prod_{t=r}^s (q)_{\gamma_t}}.$$ 

Then (2.20) can be written as

$$I_{d_1, \ldots, d_n}(q, z_1, \ldots, z_n | 1, k) = J^r_{\lambda \omega_1 + \cdots + \lambda \omega_n}[r, s],$$

and hence

$$\text{ch } V^k = \sum_{\beta \in Q^+} J^r_{\beta}[1, k].$$

The following formula was proved in [FFJMM2] (see (4.26), Theorem 4.13):

$$J^r_{\beta}[0, k] = \sum_{\alpha \in Q^+} J^{\alpha - \lambda - 2\rho}[0, \infty) \cdot J^{\lambda - \alpha}[0, \infty) \times q^{k((\alpha, \alpha)/2-(\lambda + \rho, \alpha))}.$$ 

Using the relation

$$J^r_{\beta}[r + 1, s + 1] = q^{(\beta, \beta)/2-(\lambda + \rho, \beta)} J^r_{\beta}[r, s],$$
we deduce that
\[ J^\lambda_\beta[1, k] = \sum_{\alpha \in Q^+} J^\alpha_{\alpha - \lambda - \rho}[0, \infty) J^\lambda_{\beta - \alpha}[1, \infty) \times q^{k((\alpha, \alpha)/2 - (\lambda + \rho, \alpha))}. \]

On the other hand, it is simple to show that
\[ \sum_{\gamma \in Q^+} J^\gamma_\gamma[1, \infty) = \frac{1}{\prod_{0 \leq i < j \leq n}(q_{z_{i,j}})^\infty}. \]

Summing (3.2) over \( \beta \), setting \( \alpha = \sum_{i=1}^n d_i \alpha_i \) and noting that \( q^{-(\lambda - \alpha + \rho, \alpha_i)} = q^{(\alpha, \alpha_i) z_i} \), we obtain the desired formula. \( \square \)

4. QUASI-CLASSICAL EXPANSION

In this section we extend the fermionic formula (2.20) to the setting corresponding to the tower of subalgebras (2.21), and discuss its ‘quasi-classical’ decomposition. In the following, we indicate by suffix \( k \) the quantities associated with the subalgebra \( A_k \cong U_v(\mathfrak{gl}_{k+1}) \); for instance, \( P_k = \oplus_{i=0}^k Z \varepsilon_i \) and \( Q_k^+ = \oplus_{i=1}^k Z_{\geq 0} \alpha_i \).

Let \( -\infty \leq r_1 \leq \cdots \leq r_n \leq \infty \) be a non-decreasing sequence of integers (possibly including \( \pm \infty \)), and set \( I = [r_1, \infty) \). Generalizing (3.1), we define for \( \lambda \in P_n \) and \( \beta \in Q_n^+ \)
\[
(4.1) \quad J \left( \frac{r_1}{Q_1^+}, \frac{r_2}{Q_2^+}, \ldots, \frac{r_n}{Q_n^+} \right| \lambda, \beta \right) = \sum_{\{\gamma_t \}_{t \in I}} \frac{1}{\prod_{t \in I}(q^t)^{\gamma_t}} q^{B(\{\gamma_t\}|\lambda)},
\]
\[
(4.2) \quad B(\{\gamma_t\}|\lambda) = \frac{1}{2} \sum_{t, t' \in I} \min(t, t')(\gamma_t, \gamma_{t'}) - (\lambda + \rho, \sum_{t \in I} t \gamma_t).
\]

The sum in (4.1) is taken over \( \gamma_t \in Q^+_n \) (\( t \in I \)) such that
\[
\sum_{t \in I} \gamma_t = \beta,
\]
\[
\gamma_t \in Q^+_n \text{ for } r_t \leq t < r_{t+1} \ (i = 1, \cdots, n).
\]

In the new notation we have \( J^\lambda_\beta[r, s] = J \left( \frac{r, s}{Q_n^+} \right| \lambda, \beta \right) \).

Recall that in the completion of \( U_v(\mathfrak{gl}_{n+1}) \) there is an element \( u \) which satisfies
\[
K_i u = u K_i, \quad E_i u = u K_i^2 E_i, \quad F_i u = u F_i K_i^{-2} \quad \text{for all } i = 1, \ldots, n.
\]

Up to multiplication by a simple factor, \( u \) is the Drinfeld Casimir element. On each weight component \( \mathcal{V}_\beta \) of the Verma module, \( u \) acts as the scalar \( q^{-((\beta, \beta))/2 - (\lambda + \rho, \beta)} \). In [FFJMM2], the fermionic formula (2.20) was derived by inserting \( u \) in the scalar product (2.15) which defines the Whittaker vectors and calculating it in two different ways. The same calculation can be repeated using ‘partial’ Drinfeld Casimir element. Namely let \( u_k \) denote the counterpart of \( u \) corresponding to the subalgebra \( A_k, \ k = 1, \cdots, n \).
Proposition 4.1. Let \( r_1 \leq \cdots \leq r_n \leq 0 \), and set \( r_{n+1} = 0 \). Then we have

\[
\nu^{-(\beta, \beta)}/2 + (\lambda, \beta) \left( \prod_{k=1}^n u_k^{r_k + r_{k+1}} \cdot \theta(\lambda, \beta) \right) = J \left( r_1, r_2, \cdots, r_n, \infty \right)_{Q^+_1} Q^+_n | \lambda, \beta \).
\]

Proof. The calculation is the same as in [FFJMM2], Theorem 3.1, and the proof following it. \( \square \)

For each \( k = 1, \cdots, n - 1 \), the Whittaker vector \( \theta(\lambda) \) admits the decomposition in terms of those for the lower rank subalgebra \( A_k \):

\[
\theta(\lambda) = \sum_{\lambda^{(k)} \vdash \lambda^{(k-1)}} (1 - q)^{-\sum_{i=0}^{k-1} \sum_{l=0}^{\lambda_{l,i-1}} - \sum_{l=0}^{k-1} (\lambda_{l,i} - \lambda_{l,i,k})}
\times \prod_{i=1}^n C_i(\lambda(i), \lambda(i-1)) \cdot \theta(\lambda(n), \cdots, \lambda(k) | \beta(k)),
\]

where \( \beta^{(k)} \) are defined by

\[
\beta^{(n)} = \beta,
\]

\[
(\lambda(i+1) - \beta(i+1)) |_{P_i} = (\lambda(i) - \beta(i)) \quad (\beta(i) \in Q^+_i, i = 1, \cdots, n - 1),
\]

where \( \epsilon_k |_{P_i} = \sum_{j=0}^i \delta_{j,k} \epsilon_k \) is the projection to \( P_i \). Note that from (4.7) we see that

\[
\beta(i) - \beta(i-1) = (\lambda_{0,i} - \lambda_{0,i-1})(\alpha_1 + \cdots + \alpha_i) + (\lambda_{1,i} - \lambda_{1,i-1})(\alpha_2 + \cdots + \alpha_i) + \cdots + (\lambda_{i-1,i} - \lambda_{i-2,i-1}) \alpha_i.
\]

Therefore, the sum \( \sum_{\lambda^{(n-1)}, \cdots, \lambda^{(1)}} \) is equivalent to the sum over partition of \( \beta \),

\[
\beta = \gamma^{(1)} + \cdots + \gamma^{(n)}
\]

where

\[
\gamma^{(i)} = \beta^{(i)} - \beta^{(i-1)} \in R^+_i \quad \text{def} \ Z_{\geq 0}(\alpha_1 + \cdots + \alpha_i) \oplus Z_{\geq 0}(\alpha_2 + \cdots + \alpha_i) \oplus \cdots \oplus Z_{\geq 0} \alpha_i.
\]

Note that \( \lambda^{(n)} = \lambda \) and other \( \lambda^{(i)} \) are determined by

\[
\lambda^{(i)} = \left( \lambda^{(i+1)} - \gamma^{(i+1)} \right) |_{P_i},
\]

Here \( \theta(\lambda^{(n)}, \cdots, \lambda^{(k)} | \beta(k)) \) is a weight component of a Whittaker vector with respect to the subalgebra \( A_k \approx U(\mathfrak{gl}_{k+1}) \) and its Verma module with highest weight \( \lambda^{(k)} \). It is so normalized that the coefficient of the vector \( |\lambda^{(n)}, \cdots, \lambda^{(0)}\rangle \) satisfying \( \lambda_{l,k} = \lambda_{l,k-1} = \cdots = \lambda_{l,I} \) for all \( 0 \leq l \leq k - 1 \) is 1.

Lemma 4.2. Formula (4.1) can be decomposed as

\[
J \left( r_1, r_2, \cdots, r_n, \infty \right)_{Q^+_1} Q^+_n | \lambda, \beta \rangle = \sum_{\lambda^{(n-1)}, \cdots, \lambda^{(1)}} J \left( r_1, \infty \right)_{Q^+_1} \lambda^{(1)} \lambda^{(1)} \lambda^{(1)} \prod_{i=2}^n d(\lambda(i), \gamma^{(i)} | r_i) A_i(\lambda(i), \lambda^{(i-1)}, \lambda^{(i-1)} \gamma^{(i)} \lambda^{(i-1)} \lambda^{(i-1)})^2,
\]

where
where $A_i(\mu, \nu)$ is given in \textcolor{red}{(2.10)}. The coefficients $d(\mu, \nu|r)$ have the factorized form
\begin{equation}
(4.10) \quad d(\mu, \nu|r) = v^{-(\nu, \mu)/2+(\mu, \nu)} q^{r((\nu, \mu)/2-(\mu, \nu))} (1-q)(1-q^{-1})^{-(\rho, \nu)}.
\end{equation}

**Proof.** The action of $\prod_{k=1}^{n} u_k^{-r_k+r_k+1}$ can be calculated by using the decomposition \textcolor{red}{(4.4)} and
\[ u_k \theta(\lambda(n), \cdots, \lambda(k)|\beta(k)) = q^{-(\beta(k), \beta(k))/2+(\lambda(k)+\rho, \beta(k))} \theta(\lambda(n), \cdots, \lambda(k)|\beta(k)). \]

Taking the scalar product with $\bar{\theta}_\lambda^i$ and simplifying the result, we obtain the assertion. \hfill \square

**Lemma 4.3.** We have
\begin{equation}
(4.11) \quad J \left( -\infty, \infty \mid Q_{n-1}^{+}, Q_{n}^{+} \mid \lambda(n), \gamma(n) \right) = d(\lambda(n), \gamma(n)|r) A_n(\lambda(n), \lambda(n-1))^2.
\end{equation}

**Proof.** Consider the decomposition \textcolor{red}{(4.4)} with $k = n - 1$ and apply $u_{n-1}^{-r_n} u_n^{-r_n}$. By the same computation as in the previous Lemma, we find
\[ J \left( r_{n-1}, r_n \mid Q_{n-1}^{+}, Q_{n}^{+} \mid \lambda(n), \beta(n) \right) = \sum_{\gamma(n) \in R^+, \beta(n-1) = \beta(n) - \gamma(n) \geq 0} J \left( r_{n-1}, \infty \mid Q_{n-1}^{+} \mid \lambda(n-1), \beta(n-1) \right) d(\lambda(n), \gamma(n)|r_n) A_n(\lambda(n), \lambda(n-1))^2. \]

Now let $r_{n-1} \rightarrow -\infty$. In this limit, only one term $\beta(n-1) = 0$ in the sum contributes. With this choice the factor $J$ in the right hand side is 1 and $\beta(n) = \gamma(n)$, hence we obtain the desired result. \hfill \square

Substituting \textcolor{red}{(4.11)} (with $n$ replaced by $i = 2, \cdots, n$) back into \textcolor{red}{(4.9)}, we arrive at the following result.

**Theorem 4.4.** Notation being as in \textcolor{red}{(4.7)}, we have
\begin{equation}
(4.12) \quad J \left( r_1, r_2, \cdots, r_n \mid Q_{1}^{+}, Q_{n}^{+} \mid \lambda, \beta \right) = \sum_{\gamma^{(1)} + \cdots + \gamma^{(n)} = \beta, \gamma^{(i)} \in R^+_i} J \left( r_1, \infty \mid Q_{1}^{+} \mid \lambda^{(1)}, \gamma^{(1)} \right) \times \prod_{i=2}^{n} J \left( -\infty, r_i \mid Q_{i-1}^{+} \mid r_{i}, \infty \mid \lambda^{(i)}, \gamma^{(i)} \right).
\end{equation}

This Theorem has the following interpretation. In formula \textcolor{red}{(4.1)}, let us consider the limiting situation where $r_1 \leq \cdots \leq r_n$. Imagine that we take the sum separately over the variables $\gamma_t$ taking $t$ to be ‘in the vicinity’ $I_t$ of each end point $r_i$, $i = 1, \cdots, n$. Then the contribution to \textcolor{red}{(4.2)} would become
\[ \sum_{i=1}^{n} B_i - \sum_{i=1}^{n-1} \left( \sum_{t \in I_i} t \gamma_t, \sum_{j > i} \sum_{t' \in I_j} \gamma_{t'} \right) \]
where $B_i$ stands for \textcolor{red}{(4.2)} with $t, t' \in I_i$. The corresponding sum, with $\sum_{t \in I_i} \gamma_t = \gamma^{(i)}$ being fixed, gives a summand in the right hand side of \textcolor{red}{(4.12)}. Theorem \textcolor{red}{4.4} tells that this ‘quasi-classical decomposition’ in fact gives an exact answer.
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