GLOBAL STABILITY OF THE DENGUE DISEASE TRANSMISSION MODELS

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ABSTRACT. In this paper, we further investigate the global stability of the dengue transmission models. By using persistence theory, it is showed that the disease of system uniformly persists when the basic reproduction number is larger than unity. By constructing suitable Lyapunov function methods and LaSalle Invariance Principle, we show that the unique endemic equilibrium of the model is always globally asymptotically stable as long as it exists.

1. Introduction. Dengue fever is a mosquito-borne tropical disease caused by dengue virus. It has been one of the most important public health problems in the tropical and subtropical developing countries and regions. The number of cases of dengue fever has increased dramatically since 1960s, with between 50 and 528 million people infected yearly [6, 1]. There are currently no licensed vaccines or specific therapeutics, and substantial vector control efforts have not stopped its rapid emergence and global spread [18]. The contemporary worldwide distribution of the risk of dengue virus infection and its global dynamics is still poorly known [10].

Mathematical models have been developed in the literatures [7, 8, 17, 20, 2, 15] to gain insights into the transmission dynamics of dengue in a community. Recently, a deterministic model for the transmission dynamics of a strain of dengue disease has been investigated by Garba, et al. [9]. The basic reproduction number $\mathcal{R}_0$ of the models has been obtained. By rigorous analysis, they have shown that the deterministic model with standard incidence formulation undergoes the phenomenon of backward bifurcation while the model with mass action formulation removes this phenomenon. By constructing Lyapunov function, they only have showed that the disease free equilibrium $E_0$ of the proposed mass action models is globally asymptotically stable in positively-invariant region for $\mathcal{R}_0 \leq 1$. However, the global stability of the endemic equilibria for the models with the mass action formulation was not investigated in this paper. In fact, it is a good subject to investigate the global stability of the nontrivial equilibrium of the infectious disease because of its
practical importance in biology [15], which can determine the long-term dynamics exhibited by a given nonlinear dynamical system. It is also generally believed that when the nontrivial equilibrium of the model is unique in the feasible region, it is globally asymptotically stable. To demonstrate global stability of the equilibria in some nonlinear dynamical system, some methods are applied and developed in literatures[16, 14, 12, 13, 3, 4, 5]. Especially, it is more suitable to construct Lyapunov function (or functionals) for higher-dimensional systems to show that the equilibrium of the model is globally asymptotically stable.

Motivated by the works of Li et al.[14] and Kalajdzievska [12], in the present note, using Lyapunov function methods, we show that the endemic equilibrium for the dengue epidemic model with the mass action formulation is globally asymptotically stable if \( \Re_0 \geq 1 \) in the paper [9]. The rest of this paper is organized as follows. Section 2 states the global stability of equilibrium for the dengue model without vaccination. The global stability of equilibrium for the vaccination dengue model is given in Section 3. Finally, concluding remarks are addressed in Section 4.

2. The global stability for the dengue model without vaccination. In this section, we mainly investigate the global stability for the existing endemic equilibrium of the dengue model without vaccination in the literature [9].

Let \( S_H(t), E_H(t), I_H(t), R_H(t) \) respectively, denote susceptible humans, exposed humans, infectious humans and recovered humans. Let \( S_V(t), E_V(t), I_V(t) \), respectively, denote susceptible mosquitoes, exposed mosquitoes, infectious mosquitoes. Since the variable equation for the recovered humans in the dengue model does not appear explicitly in other variable equations, we only consider the following the dengue model without vaccination in paper [9].

\[
\begin{cases}
S'_H = \Pi_H - \lambda_H^n S_H - \mu_H S_H, \\
E'_H = \lambda_H^n S_H - (\sigma_H + \mu_H)E_H, \\
I'_H = \sigma_H E_H - (\tau_H + \mu_H + \delta_H)I_H, \\
S'_V = \Pi_V - \lambda_V^n S_V - \mu_V S_V, \\
E'_V = \lambda_V^n S_V - (\sigma_V + \mu_V)E_V, \\
I'_V = \sigma_V E_V - (\mu_V + \delta_V)I_V,
\end{cases}
\]  

(2.1)

where

\[
\begin{align*}
\lambda_H^n &= C_H V (\eta_V E_V + I_V), \\
\lambda_V^n &= C_H V (\eta_H E_H + I_H).
\end{align*}
\]  

(2.2)

The parameters \( \Pi_H, \mu_H, \sigma_H, \delta_H, \tau_H, \Pi_V, \mu_V, \sigma_V, \delta_V \) are positive constants and have the same description as those of paper [9].

By using the next generation matrix method, the basic reproduction number \( \Re_0^n \) of the system (2.1) is given by

\[
\Re_0^n = \sqrt{\frac{C_H^2 \Pi_V \mu_H (\eta_H Q_2 + \sigma_H) Q_4 + \sigma_V)}{\Pi_H \mu_V Q_1 Q_2 Q_3 Q_4}},
\]

where,

\[
Q_1 = \sigma_H + \mu_H, Q_2 = \tau_H + \mu_H + \delta_H, Q_3 = \sigma_V + \mu_V, \text{ and } Q_4 = \mu_V + \sigma_V.
\]  

(2.3)

Let
Theorem 2.1. The dengue model with mass action incidence, given by (2.1) with (2.2), has the disease-free equilibrium $E_0$ if $R_0^m \leq 1$, and has a unique endemic $E^*(S^*_H, E^*_H, I^*_H, S^*_V, E^*_V, I^*_V)$ for $R_0^m > 1$ in the feasible region $\mathcal{D}$.

Here, the positive equilibrium $S^*_H, E^*_H, I^*_H, S^*_V, E^*_V, I^*_V$ satisfy the following equations:

$$
\begin{cases}
\Pi_H - \lambda_H^m S_H - \mu_H S_H = 0, \\
\lambda_H^m S_H - (\sigma_H + \mu_H)E_H = 0, \\
\sigma_H E_H - (\tau_H + \mu_H + \delta_H)I_H = 0, \\
\Pi_V - \lambda_V^m S_V - \mu_V S_V = 0, \\
\lambda_V^m S_V - (\sigma_V + \mu_V)E_V = 0, \\
\sigma_V E_V - (\mu_V + \delta_V)I_V = 0.
\end{cases}
$$

(2.4)

Now we first prove that the disease can persist by showing that the system (2.1) is uniformly persistent. Some methods and techniques have been recently employed by other authors [11, 19] to show the nonlinear system is uniformly persistent. We investigate our system (2.1) with terminology used in Hofbauer and So [11]. Let $(X,d)$ be metric space and $f : X \rightarrow X$ be continuous with a closed subspace $Y$ such that $X/Y$ is forward invariant under $f$. It is assumed that $X$ has a global attractor $A$. Let $M$ be the maximal compact invariant set in $Y$. Then $f$ is uniformly persistent (with respect to $Y$) i.e. there exists a positive constant number $\varepsilon > 0$ such that $\liminf_{t \rightarrow \infty} d(f(x(t)), Y) > \varepsilon$ for all $x \in X/Y$ if and if is isolated in $A$ and $W^s(M) = \{x \in X : f(x(t)) \rightarrow M \text{ as } t \rightarrow \infty\}$ [11]. Let $X = \mathbb{R}_+^6$ and $Y = \partial \mathbb{R}_+^6$, the boundary of $X$.

Theorem 2.2 If $R_0^m > 1$, then system (2.1) is uniformly persistent.

Proof. For any solution of system (2.1) $X = (S_H, E_H, I_H, S_V, E_V, I_V) \in \mathbb{R}_+^6$, let $\mathbb{T}$ define the map induced by system (2.1). It is easy to follow that if the initial conditions in system (2.1) is positive, then the solutions $(S_H(t), E_H(t), I_H(t), S_V(t), E_V(t), I_V(t)) > 0$ for $t > 0$. Therefore, $X \in X/Y$ is positively invariant for the system (2.1). Obviously, it is easy to obtain that system (2.1) has a global attractor and the only invariant set in $Y$ is $\{E_0\}$, which is isolated in $\{(S_H(t), E_H(t), I_H(t), S_V(t), E_V(t), I_V(t)) \in \mathbb{R}_+^6 : S_H + E_H + I_H \leq \frac{\Pi_H}{\mu_H}, S_V + E_V + I_V \leq \frac{\Pi_V}{\mu_V}\}$.

Now we show that

$W^s(E_0) \cap X_0 = \emptyset$,

where $X_0 = X/Y$.

Suppose on the contrary that there exists a solution $(S_H(t), E_H(t), I_H(t), S_V(t), E_V(t), I_V(t))$ with the positive initial conditions such that

$$
\lim_{t \rightarrow \infty} S_H(t) = \frac{\Pi_H}{\mu_H}, \quad \lim_{t \rightarrow \infty} S_V(t) = \frac{\Pi_V}{\mu_V},
$$

$$
\lim_{t \rightarrow \infty} E_H(t) = 0, \quad \lim_{t \rightarrow \infty} E_V(t) = 0,
$$

$$
\lim_{t \rightarrow \infty} I_H(t) = 0, \quad \lim_{t \rightarrow \infty} I_V(t) = 0.
$$

\[D = \left\{(S_H, E_H, I_H, S_V, E_V, I_V) \in \mathbb{R}_+^6 : S_H + E_H + I_H + R_H \leq \frac{\Pi_H}{\mu_H}, S_V + E_V + I_V \leq \frac{\Pi_V}{\mu_V} \right\}.
\]
Then, for any sufficiently small positive number \( \varepsilon > 0 \), there exists \( t_0 > 0 \), such that
\[
S_H(t) \geq \frac{\Pi_H}{\mu_H} - \varepsilon, \quad S_V(t) \geq \frac{\Pi_V}{\mu_V} - \varepsilon,
\]
for \( \forall t \geq t_0 \).

From the equations of (2.1), for any \( t > t_0 \), we have
\[
\begin{align*}
E'_H &\geq \lambda_H^{m} \left( \frac{\Pi_H}{\mu_H} - \varepsilon \right) - (\sigma_H + \mu_H)E_H, \\
I'_H &\geq \sigma_H E_H - (\tau_H + \mu_H + \delta_H)I_H, \\
E'_V &\geq \lambda_V^{m} \left( \frac{\Pi_V}{\mu_V} - \varepsilon \right) - (\sigma_V + \mu_V)E_V, \\
I'_V &\geq \sigma_V E_V - (\mu_V + \delta_V)I_V,
\end{align*}
\]
Consider the following matrix defined by
\[
J = \begin{pmatrix}
-(\sigma_H + \mu_H) & 0 & C_{HV} \eta_V \left( \frac{\Pi_H}{\mu_H} - \varepsilon \right) & C_{HV} \left( \frac{\Pi_H}{\mu_H} - \varepsilon \right) \\
\sigma_H & -(\tau_H + \mu_H + \delta_H) & 0 & 0 \\
C_{HV} \eta_H \left( \frac{\Pi_V}{\mu_V} - \varepsilon \right) & C_{HV} \left( \frac{\Pi_V}{\mu_V} - \varepsilon \right) & -(\sigma_V + \mu_V) & 0 \\
0 & 0 & \sigma_V & -(\mu_V + \delta_V)
\end{pmatrix}
\]
and \( J = H - V \) where \( H \) and \( V \) are determined by
\[
H = \begin{pmatrix}
0 & 0 & C_{HV} \eta_V \left( \frac{\Pi_H}{\mu_H} - \varepsilon \right) & C_{HV} \left( \frac{\Pi_H}{\mu_H} - \varepsilon \right) \\
0 & 0 & 0 & 0 \\
C_{HV} \eta_H \left( \frac{\Pi_V}{\mu_V} - \varepsilon \right) & C_{HV} \left( \frac{\Pi_V}{\mu_V} - \varepsilon \right) & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
\[
V = \begin{pmatrix}
-(\sigma_H + \mu_H) & 0 & 0 & 0 \\
\sigma_H & -(\tau_H + \mu_H + \delta_H) & 0 & 0 \\
0 & 0 & -(\sigma_V + \mu_V) & 0 \\
0 & 0 & \sigma_V & -(\mu_V + \delta_V)
\end{pmatrix}
\]
Consider the following linear system
\[
\begin{align*}
X'_H &= \lambda_H^{m} \left( \frac{\Pi_H}{\mu_H} - \varepsilon \right) - (\sigma_H + \mu_H)X_H, \\
Y'_H &= \sigma_H X_H - (\tau_H + \mu_H + \delta_H)Y_H, \\
X'_V &= \lambda_V^{m} \left( \frac{\Pi_V}{\mu_V} - \varepsilon \right) - (\sigma_V + \mu_V)X_V, \\
Y'_V &= \sigma_V X_V - (\mu_V + \delta_V)I_V,
\end{align*}
\]  \hspace{1cm} (2.6)
where \( \lambda_H^{m} = C_{HV} \eta_V X_V + Y_V \). \( T \) is denoted the map induced by system (2.6).
Noticing that each entry of \( T \) is positive and it follows from \( R_0^m > 1 \) that the special radius of \( T \) is larger than unity. Let \( U(t) = (X_H(t), Y_H(t), X_V(t), Y_V(t)) \) be a solution of (2.6) through the initial condition \((E_{H0}, I_{H0}, E_{V0}, I_{V0})\) at \( t = t_0 \).
Hence solutions of (2.6) are unbounded. As a result, \( E_i(t), i = H,V \) and \( I_i(t), i = H,V \) also become unbounded large as \( t \to \infty \). We obtain a contradiction and conclude that \( W^*(E_0) \cap X_0 \neq \emptyset \). Therefore, system (2.1) is uniformly persistent with respect to \( Y \) by Theorem 4.1 of paper [11]. i.e. there exists \( \varepsilon_0 > 0 \) such that
\[ \liminf_{t \to \infty} E_H \geq \varepsilon_0, \liminf_{t \to \infty} I_H \geq \varepsilon_0, \liminf_{t \to \infty} E_V \geq \varepsilon_0, \liminf_{t \to \infty} I_V \geq \varepsilon_0 \] for any solution \( (E_H(t), I_H(t), S_V(t), E_V(t), I_V(t)) \) with positive initial condition. \( \Box \)

Now, using Lyapunov function methods, we investigate the global stability of the unique endemic equilibrium \( E^* \) for system (2.1). We firstly establish the following result:

**Theorem 2.3.** If \( R_0^m > 1 \), the unique endemic equilibrium \( E^*(S^*_H, E^*_H, I^*_H, S^*_V, E^*_V, I^*_V) \) of the dengue model (2.1) is globally stable in the feasible region \( \mathcal{D} \).

**Proof.** Let \( \frac{S_H}{S_H^*} = x, \frac{E_H}{E_H^*} = y, \frac{I_H}{I_H^*} = z, \frac{S_V}{S_V^*} = m, \frac{E_V}{E_V^*} = n, \frac{I_V}{I_V^*} = h. \) System (2.1) can be rewritten as follows:

\[
\begin{align*}
    x' &= x \left[ \frac{\Pi_H}{S_H^*} \left( \frac{1}{x} - 1 \right) - C_HV_H^* \eta_V E_V^* (n - 1) - C_HV_I_V^* (h - 1) \right], \\
    y' &= y \left[ C_HV_H^* E_H^* S_H^* \left( \frac{xy}{y - 1} \right) + C_HV_I_V^* S_V^* \left( \frac{xh}{y - 1} \right) \right], \\
    z' &= z \left( \sigma_H E_H^* \left( \frac{y}{z} - 1 \right) \right), \\
    m' &= m \left[ \frac{\Pi_V}{S_V^*} \left( \frac{1}{m} - 1 \right) - C_HV_H^* \eta_H E_H^* (y - 1) - C_HV_I_H^* (z - 1) \right], \\
    n' &= n \left[ \frac{C_HV_H^* E_H^* S_V^*}{E_V^*} \left( \frac{ym}{n} - 1 \right) + \frac{C_HV_I_H^* S_V^*}{E_V^*} \left( \frac{zm}{n} - 1 \right) \right], \\
    h' &= h \left( \frac{\sigma_V E_V^*}{I_V^*} \left( \frac{n}{h} - 1 \right) \right). 
\end{align*}
\]

It is obvious that the endemic equilibrium \( E^* \) of (2.1) corresponds to the positive equilibrium \( \tilde{E}^*(1; 1; 1; 1; 1; 1) \) of (2.7), and that the global stability of \( E^* \) is the same as that of \( \tilde{E}^*(1; 1; 1; 1; 1; 1) \), so we will discuss the global stability of the equilibrium \( \tilde{E}^*(1; 1; 1; 1; 1; 1) \) of system (2.7) instead of \( E^* \).

Consider the following Lyapunov function:

\[
W_1(x, y, z, m, n, h) = S_H^*(x - 1 - \ln x) + a_1 E_H^*(y - 1 - \ln y) + a_2 I_H^*(z - 1 - \ln z) + a_3 S_V^*(m - 1 - \ln m) + a_4 E_V^*(n - 1 - \ln n) + a_5 I_V^*(h - 1 - \ln h),
\]

where \( a_1, a_2, a_3, a_4, a_5 > 0 \) are constants to be determined. Noticing that \( W_1(t) \) has a global minimum at \( \tilde{E}^* \). By directly calculating the derivation of \( W_1(t) \) along the solutions of system (2.1), we have

\[
\frac{W_1(t)}{dt} \bigg|_{(2.1)} = S_H^*(x - 1) \frac{x'}{x} + a_1 E_H^*(y - 1) \frac{y'}{y} + a_2 I_H^*(z - 1) \frac{z'}{z} + a_3 S_V^*(m - 1) \frac{m'}{m} + a_4 E_V^*(n - 1) \frac{n'}{n} + a_5 I_V^*(h - 1) \frac{h'}{h} \\
= \Pi_H \left( 2 - x + \frac{1}{x} \right) - C_HV_H^* \eta_V E_V^* \left( xy - n + 1 \right) - C_HV_S_H^* I_V^* \left( hx - h + x + 1 \right) + a_1 C_HV_H^* S_H^* E_V^* \left( xy - n + 1 \right) + a_1 C_HV_S_H^* I_V^* \left( xh - h + x + 1 \right) + a_2 C_HV_H^* E_H^* (y - z + 1) + a_3 \Pi_V \left( 2 - m - \frac{1}{m} \right) - a_3 C_HV_H^* E_V^* \left( my - m + y + 1 \right)
\]
Positive constants $a_1, a_2, a_3, a_4$ and $a_5$ are chosen as

$$a_1 = 1, \quad a_2 = \frac{a_3 C_H V H^* S_H^*}{\sigma H E_H^*}, \quad a_3 = a_4 = \frac{S_H^* (\eta V E_H^* + I_V^*)}{S_V^* (\eta H E_H^* + I_H^*)}, \quad a_5 = \frac{C_H V S_H^* I_V^*}{\sigma V E_V^*}.$$  

It can be verified that $a_1, a_2, a_3, a_4, a_5$ satisfy the following relations

$$C_H V \eta V E_V^* S_H^* - C_H V \eta V E_V^* S_H^* = 0,$$

$$C_H V I_V^* S_H^* - C_H V I_V^* S_H^* = 0,$$

$$C_H V I_V^* S_H^* - a_5 \sigma V E_H^* = 0,$$

$$C_H V \eta V E_V^* S_H^* - a_4 C_H V \eta H E_H^* S_H^* - a_4 C_H V I_H^* S_H^* + a_5 \sigma V E_V^* = 0,$$

$$a_1 C_H V \eta V E_V^* S_H^* + a_1 C_H V I_V^* S_H^* + a_5 \sigma H E_H^* - a_3 C_H V \eta H E_H^* S_H^* = 0,$$

$$a_2 \sigma H E_H^* - a_3 C_H V I_H^* S_H^* = 0,$$

$$a_4 C_H V \eta H E_H^* S_H^* - a_5 C_H V \eta H E_H^* S_H^* = 0,$$

$$a_3 C_H V I_H^* S_H^* - a_5 C_H V I_H^* S_H^* = 0.$$  

It is easy to verify that system (2.10) is compatible. Thus, system (2.9) can be reduced the following

$$\frac{dW}{dt}(2.11) = (2 \Pi_H + 2a_3 \Pi_V + a_3 C_H V I_H^* S_H^* + a_5 \sigma V E_V^*) - \Pi_H \frac{1}{x} + (C_H V \eta V E_V^* S_V^* + C_H V I_V^* S_H^*)$$

$$- \Pi_H x - C_H V \eta V E_V^* S_H^* \frac{x n}{1} - C_H V I_V^* S_H^* \frac{x n}{1} - a_5 \Pi V \frac{1}{x} + (\frac{\Pi V S_H^* (\eta V E_V^* + I_V^*)}{S_V^* (\eta H E_H^* + I_H^*)} \frac{1}{m} - a_3 C_H V \eta H E_H^* S_V^* \frac{y m}{n})$$

$$a_3 C_H V I_H^* S_V^* \frac{y m}{1} - \sigma V E_V^* \frac{y m}{1}.$$

Equation (2.11) can be rewritten as the following forms

$$\frac{dW}{dt}(2.12) = b_1 (2 - x - \frac{1}{x}) + b_2 (2 - m - \frac{1}{m}) + b_3 (4 - \frac{1}{m} - x - \frac{y m}{n})$$

$$+$
follows from (2.12) that obviously, each term in brackets of Eq.(2.12) is negative definite. Therefore, it using the arithmetic mean is greater than or equal to the geometric mean, thus, 

\[ \Pi_H - C_{HV}\eta_V E_V^* S_H^H - C_{HV} I_H^* S_H^H = \mu_H S_H^H, \quad \Pi_V = \mu_V S_V^\ast + C_{HV}(\eta_V E_V^* + I_V^*) S_V^\ast, \]

it follows from Eq. (2.13) that 

\[ b_1 = \mu_H S_H^H > 0, \quad b_2 = \frac{S_H^H(\eta_V E_V^* + I_V^*)\mu_V}{\eta_V E_V^* + I_V^*} > 0, \quad b_3 = a_4 C_{HV}\eta_V E_V^* S_H^H - b_4, \]

\[ b_5 = C_{HV} I_H^* I_V^* - b_4, \quad b_6 = a_2 \sigma_H E_H^* + b_4 - C_{HV} S_H^H I_V^*. \]

To assure that \( b_1, b_5 \) and \( b_6 \) are all nonnegative, \( b_4 \) satisfy the following inequality:

\[
\max\{0, C_{HV} S_H^H I_V^* - a_2 \sigma_H E_H^*\} \leq b_4 \leq \min\{a_4 C_{HV}\eta_V E_V^* S_H^H, C_{HV} S_H^H I_V^*\}. 
\]

In fact, \( b_4 \) always exists. For example, we may choose

\[ b_4 = \frac{a_4 C_{HV}\eta_V E_V^* S_H^H + C_{HV} S_H^H I_V^* - |a_4 C_{HV}\eta_V E_V^* S_H^H - C_{HV} S_H^H I_V^*|}{2}. \]

Thus, we can assure that the existence of \( b_i \geq 0, (i = 1, 2, 3, 5, 6) \) in (2.12). Using the arithmetic mean is greater than or equal to the geometric mean, thus, obviously, each term in brackets of Eq. (2.12) is negative definite. Therefore, it follows from (2.12) that \( \frac{dW(t)}{dt} \leq 0 \). Furthermore, \( \frac{dW(t)}{dt} \mid_{2(1)} = 0 \) if and only if \{\( (x, y, z, m, n, h) : x = y = z = m = n = h = 1 \} \} which corresponds to the set \( D_0 = \{ (S_H, E_H, I_H, S_V, E_V, I_V) : S_H = S_H^*, E_H = E_H^*, I_H = I_H^*, S_V = S_V^*, E_V = E_V^*, I_V = I_V^* \} \} \subset D \). From Theorem 2.2 and the characteristic of the system (2.1), we can see that the maximum invariant set of (2.1) on the set \( D \) is the singleton \( \{ E^* \} \). Therefore, the endemic equilibrium \( E^* \) of system (2.1) is globally stable in \( D \) by LaSalle Invariable Principle.
The vaccination dengue model.

3. Where the Lyapunov function:

\[ W_2(S_H, E_H, I_H, S_V, E_V, I_V) = a_0(S_H - S_H^0 - S_H^0 \ln \frac{S_H}{S_H^0}) + a_1 E_H + a_2 I_H \]

\[ + a_3(S_V - S_V^0 - S_V^0 \ln \frac{S_V}{S_V^0}) + a_4 E_V + a_5 I_V, \]

where

\[ a_0 = \left( (\mu_v + \delta_v) + \sigma_v \right) C_{HV} S_H^0, \]

\[ a_1 = \pi V \mu H (\mu_v + \delta_v) C_{HV}(\eta_v (\mu_v + \delta_v) + \sigma_v)(\eta_H (\tau_H + \mu_H + \delta_H) + \sigma_H), \]

\[ a_2 = \pi V \mu H (\mu_H + \delta_H) C_{HV}(\eta_v (\mu_v + \delta_v) + \sigma_v), \]

\[ a_3 = \left( \mu_H + \delta_H \right) C_{HV} S_V, \]

\[ a_4 = \mu V \mu H (\sigma_H + \mu_H)(\tau_H + \mu_H + \delta_H)(\mu_v + \delta_v) C_{HV}(\eta_v (\mu_v + \delta_v) + \sigma_v)(1 - R_0^m), \]

\[ a_5 = \mu V \mu H (\sigma_H + \mu_H)(\tau_H + \mu_H + \delta_H)(\mu_v + \mu_v)(1 - R_0^m). \]

Thus, we establish the following result

**Theorem 2.4.** If \( R_0^m < 1 \), the disease free equilibrium \( E_0 \left( \frac{\Pi_H}{\mu_H}, 0, 0, \frac{\Pi_V}{\mu_V}, 0, 0 \right) \) of the dengue model (2.1) is globally stable in the feasible region \( D \).

3. The vaccination dengue model. In this section, we investigate the global stability of the endemic equilibrium for the dengue model with vaccination in paper [9]. We still denote \( S_H(t), E_H(t), I_H(t), P_H(t) \), respectively, susceptible humans, exposed humans, infectious humans and vaccinated humans. Let \( S_V(t), E_V(t), I_V(t) \), respectively, denote susceptible mosquitoes, exposed mosquitoes, infectious mosquitoes. This extended model include a population of vaccinated individuals. Since it is assumed that the vaccine is imperfect, vaccinated individuals acquire infection at a rate \( \lambda_H^m(1 - \epsilon) \), where \( \epsilon \) is vaccine efficiency. It is assumed that the vaccine wanes at a rate \( \omega \) and susceptible individuals is vaccinated at a rate \( \xi \). We consider the following vaccination dengue model

\[
\begin{align*}
S_H' &= \Pi_H + \omega P_H - \lambda_H^m S_H - (\mu_H + \xi) S_H, \\
E_H' &= \xi S_H - \lambda_H^m (1 - \epsilon) P_H - (\omega + \mu_H) P_H, \\
E_H' &= \lambda_H [S_H + (1 - \epsilon) P_H] - (\sigma_H + \mu_H) E_H, \\
I_H' &= \sigma_H E_H - (\tau_H + \mu_H + \delta_H) I_H, \\
S_V' &= \Pi_V - \lambda_v^m S_V - \mu_v S_V, \\
E_V' &= \lambda_v^m S_V - (\sigma_v + \mu_v) E_V, \\
I_V' &= \sigma_v E_V - (\mu_v + \delta_v) I_V.
\end{align*}
\]

where

\[
\begin{align*}
\lambda_H^m &= C_{HV}(\eta_v E_V + I_V), \\
\lambda_v^m &= C_{HV}(\eta_H E_H + I_H).
\end{align*}
\]

Firstly, the vaccination reproduction number of the model (3.1), denoted by \( R^m_{vac} \), is given by
\[ \mathcal{R}_{vac}^m = \sqrt{\frac{C_{HV}^2 \Pi_V (\eta_H K_4 + \sigma_H) (\eta_V K_6 + \sigma_V) [S_H^0 + P_H^0 (1 - \varepsilon)]}{\mu_V K_3 K_4 K_5 K_6}}, \]

where,

\[ K_3 = \mu_H + \sigma_H, \quad K_4 = \mu_H + \delta_H + \tau_H, \quad K_5 = \mu_V + \sigma_V, \quad K_6 = \mu_V + \delta_V, \]

\[ S_H^0 = \frac{(\mu_H + \omega) \Pi_H}{\mu_H (\mu_H + \omega + \xi)}, \quad P_H^0 = \frac{\xi \Pi_H}{\mu_H (\mu_H + \omega + \xi)}. \]

Let

\[ \Gamma_0^\text{vac} = \left\{(S_H, P_H, E_H, I_H, S_V, E_V, I_V) \in \mathbb{R}_+^8 : N_H \leq \frac{\Pi_H}{\mu_H}, N_V \leq \frac{\Pi_V}{\mu_V} \right\}, \]

\[ \Gamma_{vac} = \left\{(S_H, P_H, E_H, I_H, S_V, E_V, I_V) \in \Gamma_0^\text{vac} : S_H \leq S_H^0, P_H \leq P_H^0, S_V \leq \frac{\Pi_V}{\mu_V} \right\}. \]

It is easy to verify that the region \( \Gamma_{vac} \) is positively invariant and attracting.

**Theorem 3.1.** The vaccination dengue model with mass action incidence, given by (3.1) with (3.2), has only the disease-free equilibrium if \( \mathcal{R}_{vac}^m \leq 1 \), and has a unique endemic \( E_{vac}^* \) for \( \mathcal{R}_{vac}^m > 1 \) in the feasible region \( \Gamma_{vac} \).

Let \( E_{vac}^*(S_H^*, P_H^*, E_H^*, I_H^*, S_V^*, E_V^*, I_V^*) \) be the endemic equilibrium of the model (3.1), where \( S_H^*, P_H^*, E_H^*, I_H^*, S_V^*, E_V^*, \) and \( I_V^* \) satisfy the following equations:

\[
\begin{align*}
\Pi_H + \omega P_H - \lambda_H m S_H - (\mu_H + \xi) S_H &= 0, \\
\xi S_H - \lambda_H (1 - \varepsilon) P_H - (\omega + \mu_H) P_H &= 0, \\
\lambda_H (S_H + (1 - \varepsilon) P_H) - (\sigma_H + \mu_H) E_H &= 0, \\
\sigma_H E_H - (\tau_H + \mu_H + \delta_H) I_H &= 0, \\
\Pi_V - \lambda_V m S_V - \mu_V S_V &= 0, \\
\lambda_V S_V - (\sigma_V + \mu_V) E_V &= 0, \\
bs \sigma_V E_V - (\mu_V + \delta_V) I_V &= 0.
\end{align*}
\]

Using the same method as proving the persistence of (2.1). Then we firstly give the following result.

**Theorem 3.2.** If \( \mathcal{R}_{vac}^m > 1 \), then system (3.1) is uniformly persistent.

Now, constructing Lyapunov function methods, we shall investigate the global stability of the unique endemic equilibrium \( E_{vac}^* \) for system (3.1). Thus, we establish the following conclusion

**Theorem 3.3.** If \( \mathcal{R}_{vac}^m > 1 \), the endemic equilibrium \( E_{vac}^*(S_H^*, P_H^*, E_H^*, I_H^*, S_V^*, E_V^*, I_V^*) \) of the vaccination dengue model (3.1) is globally stable in the feasible region \( \Gamma_{vac} \).

**Proof.** Let \( \frac{S_H}{S_H} = x, \frac{P_H}{P_H} = y, \frac{E_H}{E_H} = z, \frac{I_H}{I_H} = f, \frac{S_V}{S_V} = m, \frac{E_V}{E_V} = g, \frac{I_V}{I_V} = h \), using Eq.(3.3), system (3.1) can be written as follows

\[
x' = x \left[ \frac{\Pi_H}{S_H} \left( \frac{1}{x} - 1 \right) + \frac{\omega P_H}{S_H} \left( \frac{y}{x} - 1 \right) - C_{HV} \eta_V E_V (g - 1) - C_{HV} I_V (h - 1) \right],
\]

(3.4)
\[ y' = y \left[ \frac{\xi S_t^x}{P_H^x} \left( \frac{x}{y} \right) - 1 \right] - C_{HV} \eta V (1 - \varepsilon) E_V^* (g - 1) - C_{HV} (1 - \varepsilon) I_V^* (h - 1) \],
\[ z' = z \left[ \frac{C_{HV} \eta H S_t^x E_V^*}{E_H^*} \left( \frac{xg}{z} - 1 \right) + \frac{C_{HV} S_t^x I_V^*}{E_H^*} \left( \frac{xh}{z} - 1 \right) + \frac{C_{HV} \eta H S_t^x E_V^*}{E_H^*} \left( \frac{yg}{z} - 1 \right) + \frac{C_{HV} \eta H S_t^x E_V^*}{E_H^*} \left( \frac{yh}{z} - 1 \right) \right],
\[ f' = \frac{\sigma_H E_V^*}{I_V^*} \left( \frac{z}{f} - 1 \right),
\[ m' = m \left[ \frac{\Pi_V}{S_V^*} \left( \frac{1}{m} - 1 \right) - C_{HV} \eta H E_H^* (z - 1) - C_{HV} I_V^* (f - 1) \right],
\[ g' = g \left[ \frac{C_{HV} \eta H E_H^* S_V^*}{E_V^*} \left( \frac{zm}{g} - 1 \right) + \frac{C_{HV} I_V^* S_V^*}{E_V^*} \left( \frac{f m}{g} - 1 \right) \right],
\[ h' = h \left[ \frac{\sigma_V E_V^*}{I_V^*} \left( \frac{g}{h} - 1 \right) \right]. \]

Thus, the endemic equilibrium \( E_{vac}^* \) of (3.1) corresponds to the positive equilibrium \( E_{vac}^*(1; 1; 1; 1; 1; 1; 1) \) of (3.4), and that the global stability of \( E_{vac}^* \) is the same as that of \( E_{vac}^*(1; 1; 1; 1; 1; 1; 1) \), so we will discuss the global stability of the equilibrium \( E_{vac}^*(1; 1; 1; 1; 1; 1; 1) \) of system (3.4) instead of \( E_{vac}^* \).

Considering the following Lyapunov function:
\[ W_3(t) = S_t^x (x - 1 - \ln x) + a_1 P_H^x (y - 1 - \ln y) + a_2 E_H^* (z - 1 - \ln z) + a_3 I_H^* (f - 1 - \ln f) + a_4 S_V^* (m - 1 - \ln m) + a_5 E_V^* (g - 1 - \ln g) + a_6 I_V^* (h - 1 - \ln h), \]

where \( a_1, a_2, a_3, a_4, a_5, a_6 > 0 \) are constants to be determined. Note that \( W_3(t) \) is a Lyapunov function and has a global minimum at \( E_{vac}^* \). By directly calculating the derivation of \( W_3(t) \) along the solution of (3.4), we have
\[ W_3(t)_{(3.4)} = S_t^x (x - 1) + a_1 P_H^x (y - 1 + \frac{y'}{y}) + a_2 E_H^*(z - 1 + \frac{z'}{z}) + a_3 I_H^* (f - 1 + \frac{f'}{f}) + a_4 S_V^* (m - 1 + \frac{m'}{m}) + a_5 E_V^* (g - 1 + \frac{g'}{g}) + a_6 I_V^* (h - 1 + \frac{h'}{h}) \]
\[ = \Pi_H^x (2 - x - \frac{1}{x}) + \omega P_H^x (y - x - \frac{y}{x} - 1) - C_{HV} \eta V S_t^x E_V^* (xg - x - g + 1) - C_{HV} S_t^x \eta H S_t^x (xh - x - h + 1) + a_1 c S_t^x (x - y - \frac{x}{y} + 1) - a_1 C_{HV} \eta V (1 - \varepsilon) \]
\[ \times P_H^x E_V^* (yg - y + 1) - a_1 C_{HV} (1 - \varepsilon) P_H^x (yh - y + 1) + a_2 C_{HV} \eta H S_t^x E_V^* (xg - x - \frac{xg}{z} + 1) + a_2 C_{HV} \eta H S_t^x I_V^* (x - \frac{z}{x} + 1) + a_2 C_{HV} \eta H S_t^x I_V^* (xh - \frac{z}{x} + 1) + a_2 C_{HV} \eta H S_t^x I_V^* \]
\[ \times \frac{xh}{z} + 1 + a_2 C_{HV} \eta H \eta V (1 - \varepsilon) \]
\[ \times \frac{yg}{z} + 1 + a_2 C_{HV} (1 - \varepsilon) P_H^x (yh - \frac{yh}{z} + 1) + a_3 \sigma_H E_H^* (z - f - \frac{z}{f} + 1) + a_4 \Pi_V (2 - m - \frac{1}{m}) - a_4 C_{HV} \eta H E_H^* S_V^* (mz - m - \frac{1}{m} - 1) - a_4 C_{HV} I_H^* S_V^* (mf - m - f + 1) + a_5 C_{HV} \eta H E_H^* S_V^* (zm - g - \frac{zm}{g}) \]
The positive constants \( a \) are chosen as follows:

\[ a_1 = a_2 = 1, \quad a_3 = \frac{C_{HV} \eta E^*_V[I^*_V]}{S^*_V(\eta h E^*_H + I^*_H)}, \quad a_4 = a_5 = \frac{(\eta h E^*_V + I^*_V)[S^*_H + (1 - \varepsilon)P^*_H]_I^*_H}{S^*_V(\eta h E^*_H + I^*_H)}, \quad a_6 = \frac{C_{HV} I^*_V[S^*_H + (1 - \varepsilon)P^*_H]}{\sigma_V E^*_V}. \]  

(3.6)

These constants satisfy the following relations

\[ \begin{cases} 
-C_{HV} \eta V S^*_H E^*_V + a_2 C_{HV} \eta V S^*_H E^*_V = 0, \\
C_{HV} \eta V S^*_H E^*_V + a_1 C_{HV} \eta V (1 - \varepsilon)P^*_H E^*_V - a_5 C_{HV} (\eta h E^*_H + I^*_H) S^*_V + a_6 \sigma_V E^*_V = 0, \\
-a_1 C_{HV} (1 - \varepsilon)P^*_H I^*_V + a_2 C_{HV} (1 - \varepsilon)P^*_H I^*_V = 0, \\
-C_{HV} S^*_H I^*_V + a_2 C_{HV} S^*_H I^*_V = 0, \\
C_{HV} S^*_H I^*_V + a_1 C_{HV} (1 - \varepsilon)P^*_H I^*_V - a_6 \sigma_V E^*_V = 0, \\
-a_1 C_{HV} \eta V (1 - \varepsilon)P^*_H E^*_V + a_2 C_{HV} \eta V (1 - \varepsilon)P^*_H E^*_V = 0, \\
-a_2 C_{HV}[S^*_H + (1 - \varepsilon)P^*_H](\eta h E^*_V + I^*_V) + a_3 \sigma_H E^*_H + a_4 C_{HV} \eta h E^*_H S^*_V = 0, \\
-a_3 \sigma_H E^*_H + a_4 C_{HV} I^*_H S^*_V = 0, \\
-a_4 C_{HV} \eta h E^*_H S^*_V + a_5 C_{HV} \eta h E^*_H S^*_V = 0, \\
-a_4 C_{HV} I^*_H S^*_V + a_5 C_{HV} I^*_H S^*_V = 0.
\]  

(3.8)
Thus, system (3.6) can be reduced to the following

$$
\dot{W}(t)_{(3.4)} = (2\Pi_H + \omega P^*_H + \xi S^*_H + a_3\sigma_H E^*_H + 2a_4\Pi_V + a_6\sigma_V E^*_V) + [-\Pi_H - \omega p^*_H
+ C_H V S^*_H(\eta V E^*_V + I^*_V) + \xi S^*_H x - \Pi_H - \omega p^*_H - \xi S^*_H + C_H V (1-\varepsilon)
\times P^*_H(\eta V E^*_V + I^*_V)]y - \omega p^*_H \frac{y}{x} - \xi S^*_H \frac{x}{y} - C_H V \eta V S^*_H E^*_V \frac{yg}{z} - C_H V S^*_H I^*_V \frac{zh}{z}
- C_H V \eta V (1-\varepsilon) P^*_H I^*_V \frac{yh}{z} - a_3\sigma_H E^*_H \frac{z}{f} + [-a_4\Pi_V
+ a_4 C_H V (\eta V E^*_H + I^*_H) S^*_V]m - a_4\Pi_V \frac{1}{m} - a_5 C_H V \eta V E^*_H S^*_V \frac{zm}{g}
- a_5 C_H V I^*_H S^*_V \frac{fm}{g} - a_6\sigma_V E^*_V g \frac{h}{h}
}
$$

(3.9)

To prove $\dot{W}(t)_{(3.4)} \leq 0$ in Eq.(3.9), we rewrite above $\dot{W}(t)_{(3.4)}$ as the following forms

$$
\dot{W}(t)_{(3.4)} = b_1(2-x - \frac{1}{x}) + b_2(2-m - \frac{1}{m}) + b_3(2 - \frac{x}{y} - \frac{y}{x}) + b_4(3-y - \frac{1}{x} - \frac{x}{y})
+ b_5(4 - \frac{1}{m} - \frac{1}{x} - \frac{yg}{z} - \frac{zm}{g}) + b_6(5 - \frac{1}{m} - \frac{1}{x} - \frac{xh}{z} - \frac{zm}{g} - \frac{g}{h})
+ b_7(5 - \frac{1}{m} - \frac{1}{x} - \frac{yg}{z} - \frac{zm}{g}) + b_8(6 - \frac{1}{m} - \frac{1}{x} - \frac{yh}{z} - \frac{zm}{g} - \frac{g}{h} - \frac{y}{f})
+ b_9(5 - \frac{1}{m} - \frac{1}{x} - \frac{xg}{z} - \frac{fz}{g} - \frac{y}{f} - \frac{g}{h} - \frac{g}{f} - \frac{y}{f})
+ b_{10}(6 - \frac{1}{m} - \frac{1}{x} - \frac{xg}{z} - \frac{fz}{g} - \frac{y}{f} - \frac{g}{h} - \frac{g}{f} - \frac{y}{f})
+ b_{11}(7 - \frac{1}{m} - \frac{1}{x} - \frac{y}{z} - \frac{f}{g} - \frac{g}{h} - \frac{g}{f} - \frac{y}{f} - \frac{g}{f})

(3.10)

where the parameters $b_i \geq 0, (i = 1, 2, \ldots, 12)$, are constants to be determined.

Let the coefficients for the same terms between Eq.(3.9) and Eq.(3.10) be equal, which yields the following equations

$$
\begin{aligned}
-b_1 &= -\Pi_H - \omega p^*_H + C_H V S^*_H(\eta V E^*_V + I^*_V) + \xi S^*_H, \\
-(b_1 + b_4 + b_9 + b_{10} + b_{11} + b_{12}) &= -\Pi_H, \\
-b_2 &= -a_4\Pi_V + a_4 C_H V (\eta V E^*_H + I^*_H) S^*_V, \\
-(b_2 + b_5 + b_6 + b_7 + b_8 + b_9 + b_{10} + b_{11} + b_{12}) &= -a_4\Pi_V, \\
-b_3 &= -\omega p^*_H, \\
-(b_3 + b_4 + b_7 + b_8 + b_9 + b_{10} + b_{11} + b_{12}) &= -\xi S^*_H, \\
-b_5 &= -a_5 C_H V \eta V E^*_H S^*_V, \\
-(b_5 + b_9) &= -C_H V \eta V S^*_H E^*_V, \\
-(b_5 + b_6 + b_7 + b_8) &= -a_5 C_H V I^*_H S^*_V, \\
2(b_1 + b_2 + b_3) + 3b_4 + 4b_5 + 5b_6 + 6b_7 + 6b_8 + 7b_9 + 8b_{10} + 9b_{11} + 10b_{12} &= 2\Pi_H + \omega P^*_H + \xi S^*_H + a_3\sigma_H E^*_H + 2a_4\Pi_V + a_6\sigma_V E^*_V.
\end{aligned}
$$

(3.11)
The equality that
\[-\Pi H - \omega p^*_H + C_{HV}S^*_H(\eta V E^*_V + I^*_V) + \xi S^*_H = -\mu H S^*_H\]
imply that equations (3.11) has the following equivalent one:

\[
\begin{aligned}
    b_1 &= \mu H S^*_H, \\
    b_2 &= \frac{\mu V S^*_V}{C_{HV}S^*_V(\eta V E^*_H + I^*_H)}, \\
    b_3 &= \omega p^*_H, \\
    b_4 &= \mu H P^*_H, \\
    b_5 &= C_{HV}\eta V S^*_H E^*_V - a_5 C_{HV}I^*_H S^*_V + b_{10} + b_{11} + b_{12}, \\
    b_6 &= C_{HV}S^*_H I^*_V - b_{10}, \\
    b_7 &= C_{HV}\eta V(1 - \varepsilon) P^*_H E^*_V - b_{11}, \\
    b_8 &= C_{HV}(1 - \varepsilon) P^*_H I^*_V - b_{12}, \\
    b_9 &= a_5 C_{HV}I^*_H S^*_V - (b_{10} + b_{11} + b_{12}).
\end{aligned}
\quad (3.12)
\]

To assure that \( b_5, b_6, b_7, b_8 \) and \( b_9 \) are all nonnegative, \( b_{10}, b_{11}, \) and \( b_{12} \) should satisfy the following inequalities:

\[
\begin{aligned}
    a_5 C_{HV}I^*_H S^*_V - C_{HV}\eta V S^*_H E^*_V &\leq b_{10} + b_{11} + b_{12} = a_5 C_{HV}I^*_H S^*_V, \\
    0 &\leq b_{10} \leq C_{HV}S^*_H I^*_V, \\
    0 &\leq b_{11} \leq C_{HV}\eta V(1 - \varepsilon) P^*_H E^*_V, \\
    0 &\leq b_{12} \leq C_{HV}(1 - \varepsilon) P^*_H I^*_V. 
\end{aligned}
\quad (3.13)
\]

Obviously,

\[
C_{HV}\eta V S^*_H E^*_V + C_{HV} S^*_H I^*_V + C_{HV} \eta V(1 - \varepsilon) P^*_H E^*_V + C_{HV}(1 - \varepsilon) P^*_H I^*_V \geq a_5 C_{HV}I^*_H S^*_V.
\]

To obtain the existence of \( b_{10}, b_{11}, \) and \( b_{12} \) in (3.12), we consider the following cases

**Case (1).** If

\[
C_{HV}S^*_H I^*_V + C_{HV} \eta V(1 - \varepsilon) P^*_H E^*_V + C_{HV}(1 - \varepsilon) P^*_H I^*_V \leq a_5 C_{HV}I^*_H S^*_V,
\]

we choose that

\[
b_{10} = C_{HV}S^*_H I^*_V, b_{11} = C_{HV} \eta V(1 - \varepsilon) P^*_H E^*_V, b_{12} = C_{HV}(1 - \varepsilon) P^*_H I^*_V.
\]

**Case (2).** If

\[
C_{HV}S^*_H I^*_V > a_5 C_{HV}I^*_H S^*_V,
\]

we choose that

\[
b_{10} = a_5 C_{HV}I^*_H S^*_V, b_{11} = b_{12} = 0.
\]

**Case (3).** If

\[
C_{HV}\eta V(1 - \varepsilon) P^*_H E^*_V > a_5 C_{HV}I^*_H S^*_V,
\]

we choose that

\[
b_{10} = 0, b_{11} = C_{HV} \eta V(1 - \varepsilon) P^*_H E^*_V, b_{12} = 0.
\]

**Case (4).** If

\[
C_{HV}(1 - \varepsilon) P^*_H I^*_V > a_5 C_{HV}I^*_H S^*_V,
\]
we choose that
\[ b_{10} = 0, b_{11} = 0, b_{12} = C_{HV}(1 - \varepsilon)P^*_H I^*_V. \]

**Case (5).** If
\[
\begin{cases}
C_{HV} S^*_H I^*_V + C_{HV} \eta_V (1 - \varepsilon) P^*_H E^*_V > a_5 C_{HV} I^*_H S^*_V, \\
C_{HV} S^*_H I^*_V \leq a_5 C_{HV} I^*_H S^*_V, \\
C_{HV} \eta_V (1 - \varepsilon) P^*_H E^*_V \leq a_5 C_{HV} I^*_H S^*_V,
\end{cases}
\]
we choose that
\[ b_{10} = a_5 C_{HV} I^*_H S^*_V - C_{HV} \eta_V (1 - \varepsilon) P^*_H I^*_V, b_{11} = C_{HV} \eta_V (1 - \varepsilon) P^*_H E^*_V, b_{12} = 0. \]

**Case (6).** If
\[
\begin{cases}
C_{HV} S^*_H I^*_V + C_{HV} (1 - \varepsilon) P^*_H I^*_V > a_5 C_{HV} I^*_H S^*_V, \\
C_{HV} S^*_H I^*_V \leq a_5 C_{HV} I^*_H S^*_V, \\
C_{HV} (1 - \varepsilon) P^*_H I^*_V \leq a_5 C_{HV} I^*_H S^*_V,
\end{cases}
\]
we choose that
\[ b_{10} = a_5 C_{HV} I^*_H S^*_V - C_{HV} (1 - \varepsilon) P^*_H I^*_V, b_{11} = 0, b_{12} = C_{HV} (1 - \varepsilon) P^*_H I^*_V. \]

**Case (7).** If
\[
\begin{cases}
C_{HV} \eta_V (1 - \varepsilon) P^*_H E^*_V + C_{HV} (1 - \varepsilon) P^*_H I^*_V > a_5 C_{HV} I^*_H S^*_V, \\
C_{HV} (1 - \varepsilon) P^*_H I^*_V \leq a_5 C_{HV} I^*_H S^*_V, \\
C_{HV} (1 - \varepsilon) P^*_H I^*_V \leq a_5 C_{HV} I^*_H S^*_V,
\end{cases}
\]
we choose that
\[ b_{10} = a_5 C_{HV} I^*_H S^*_V - C_{HV} (1 - \varepsilon) P^*_H I^*_V, b_{11} = C_{HV} (1 - \varepsilon) P^*_H I^*_V. \]

**Case (8).** If
\[
\begin{cases}
C_{HV} S^*_H I^*_V + C_{HV} \eta_V (1 - \varepsilon) P^*_H E^*_V + C_{HV} (1 - \varepsilon) P^*_H I^*_V > a_5 C_{HV} I^*_H S^*_V, \\
C_{HV} S^*_H I^*_V + C_{HV} \eta_V (1 - \varepsilon) P^*_H E^*_V \leq a_5 C_{HV} I^*_H S^*_V, \\
C_{HV} S^*_H I^*_V + C_{HV} (1 - \varepsilon) P^*_H I^*_V \leq a_5 C_{HV} I^*_H S^*_V, \\
C_{HV} \eta_V (1 - \varepsilon) P^*_H E^*_V + C_{HV} (1 - \varepsilon) P^*_H I^*_V \leq a_5 C_{HV} I^*_H S^*_V.
\end{cases}
\]
we can choose that
\[ b_{10} = C_{HV} S^*_H I^*_V, b_{11} = C_{HV} \eta_V (1 - \varepsilon) P^*_H E^*_V, b_{12} = a_5 C_{HV} I^*_H S^*_V - C_{HV} S^*_H I^*_V - C_{HV} \eta_V (1 - \varepsilon) P^*_H E^*_V. \]

Therefore, from the above discussion, we can assure that the existence of \( b_i \geq 0 (i = 1, \cdots, 12) \) in (3.12). It is easy to show that \( \frac{dW_3(t)}{dt} \mid_{(3.4)} \leq 0 \). Furthermore, \( \frac{dW_3(t)}{dt} \mid_{(3.4)} = 0 \) if and only if \( \{(x, y, z, m, n, h) : x = y = z = f = m = g = h = 1\} \), which corresponds to the set \( \Gamma_{\text{vac}}^0 = \{(S_H; P_H; E_H; I_H; S_V; E_V; I_V) : S_H = S^*_H, P_H = P^*_H, E_H = E^*_H, I_H = I^*_H, S_V = S^*_V, E_V = E^*_V, I_V = I^*_V \} \subset \Gamma_{\text{vac}} \). It is easy to see that the maximum invariant set of (3.1) on the set \( \Gamma_{\text{vac}} \) is the singleton \( \{E^*_\text{vac}\} \), then the endemic equilibrium \( E^*_\text{vac} \) is globally stable in \( \Gamma_{\text{vac}} \) by LaSalle Invariable Principle. \( \square \)
4. **Concluding remarks.** In this paper, by constructing Lyapunov function methods and LaSalle Invariance Principle, we show the global stability of the endemic equilibrium in two dengue transmission models in paper [9]. Our results show that the unique endemic equilibrium of the models with mass incidence is always globally asymptotically stable as long as it exists. These results can further enrich the dynamics of dengue virus transmission. In contrast, authors in paper [9] have showed that the dengue model with standard incidence rate exhibits the backward bifurcation. Therefore, these obtained results show that the incidence rate (the rate of new infection) have played an important role in analyzing the disease transmission dynamical behavior. In this paper, it is also showed that it is important to construct Lyapunov function (or functionals) methods and LaSalle Invariance Principle in analyzing the global asymptotic stability of some higher-dimensional dynamical systems.

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**REFERENCES**

[1] S. Bhatt, P. W. Gething, O. J. Brady, et al., The global distribution and burden of dengue, *Nature*, 496 (2013), 504–507.

[2] L. Cai, S. Guo, X. Li and M. Ghosh, Global dynamics of a dengue epidemic mathematical model, *Chaos, Solitons and Fractals*, 42 (2009), 2297–2304.

[3] L. Cai, M. Martcheva and X. Li, epidemic models with age of infection, indirect transmission and incomplete treatment, *Discrete and Continuous Dynamical Systems Series B*, 18 (2013), 2239–2265.

[4] L. Cai, M. Martcheva and X. Li, Competitive exclusion in a vector-host epidemic model with distributed delay, *Journal of Biological Dynamics*, 7 (2013), 47–67.

[5] L. Cai, X. Li and M. Ghosh, Global dynamics of a mathematical model for HTLV-I infection of CD4+ T-cells, *Appl. Math. Model.*, 35 (2011), 3587–3595.

[6] CDC, Centers for disease control and prevention, Dengue Homepage, [http://www.cdc.gov/Dengue/epidemiology/index.html](http://www.cdc.gov/Dengue/epidemiology/index.html)

[7] L. Esteva and C. Vargas, Coexistence of different serotypes of dengue virus, *J. Math. Biol.*, 46 (2003), 31–47.

[8] Z. Feng and X. Jorge Velasco-Hernandez, Competitive exclusion in a vector-host model for the dengue fever, *J. Math. Biol.*, 35 (1997), 523–544.

[9] S. M. Garba, A. B. Gumel and M. R. Abu Bakar, Backward bifurcations in dengue transmission dynamics, *Math. Biosci.*, 215 (2008), 11–25.

[10] S. B. Halstead, Pathogenesis of dengue: Challenges to molecular biology, *Science*, 239 (1988), 476–481.

[11] J. Hofbauer and J. W.-H. So, Uniform persistence and repellers for maps, *Proceedings of the American Mathematical Society*, 107 (1989), 1137–1142.

[12] D. Kalajdzievska and M. Y. Li, Modeling the effects of carriers on transmission dynamics of infectious diseases, *Math. Biosci. Eng.*, 8 (2011), 711–722.

[13] M. Y. Li and J. S. Muldowney, A geometric approach to global-stability problems, *SIAM J. Math. Anal.*, 27 (1996), 1070–1083.

[14] J. Li, Y. Xiao, F. Zhang and Y. Yang, An algebraic approach to proving the global stability of a class of epidemic models, *Nonlinear Anal. RWA.*, 13 (2012), 2006–2016.

[15] Z. Ma, Y. Zhou, W. Wang and Z. Jin, *Mathematical Models and Dynamics of Infectious Diseases*, China Sciences Press, Beijing, 2004.

[16] C. C. McCluskey, Complete global stability for an SIR epidemic model with delay-distributed or discrete, *Nonlinear Anal: Real World Appl.*, 11 (2010), 55–59.

[17] P. Pongsumpun and I. M. Tang, Transmission of dengue hemorrhagic fever in an age structured population, *Math. Comput. Model.*, 37 (2003), 949–961.
[18] A. J. Tatem, S. I. Hay and D. J. Rogers, Global traffic and disease vector dispersal, *Proc. Natl. Acad. Sci. USA.*, 103 (2006), 6242–6247.
[19] H. R. Thieme, Persistence under relaxed point-dissipativity (with application to an endemic model), *SIAM J. Math. Anal.*, 24 (1993), 407–435.
[20] H. M. Yang and C. P. Ferreira, Assessing the effects of vector control on dengue transmission, *Appl. Math. Computat.*, 198 (2008), 401–413.

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