Regression Adjustments under Covariate-Adaptive Randomizations with Imperfect Compliance

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Abstract

We study regression adjustments with additional covariates in randomized experiments under covariate-adaptive randomizations (CARs) when subject compliance is imperfect. We develop a regression-adjusted local average treatment effect (LATE) estimator that is proven to improve efficiency in the estimation of LATEs under CARs. Our adjustments can be parametric in linear and nonlinear forms, nonparametric, and high-dimensional. Even when the adjustments are misspecified, our proposed estimator is still consistent and asymptotically normal, and their inference method still achieves the exact asymptotic size under the null. When the adjustments are correctly specified, our estimator achieves the minimum asymptotic variance. When the adjustments are parametrically misspecified, we construct a new estimator which is weakly more efficient than linearly and nonlinearly adjusted estimators, as well as the one without any adjustments. Simulation evidence and empirical application confirm efficiency gains achieved by regression adjustments relative to both the estimator without adjustment and the standard two-stage least squares estimator.

Keywords: Randomized experiment, Covariate-adaptive randomization, High-dimensional data, Local average treatment effects, Regression adjustment.

JEL codes: C14, C21, I21
1 Introduction

Randomized experiments have seen increasing use in economic research. In existing experiments, one of the popular randomization methods applied by economists to achieve balance between treatment and control is covariate-adaptive randomization (CAR) (Bruhn and McKenzie (2009)). In CAR modelling, units are randomly assigned to treatment and control within strata formed by a few key pre-treatment variables. Recent studies in economics using CAR include Burchardi, Gulesci, Lerva, and Sulaiman (2019), Anderson and McKenzie (2021), and de Mel, McIntosh, Sheth, and Woodruff (2022).

With rare exceptions, experimental subject compliance with assignment is almost always incomplete. For example, recent experimental studies under CARs and with incomplete compliance include Blattman and Dercon (2018) and Dupas, Karlan, Robinson, and Ubfal (2018). When there is incomplete compliance, the local average treatment effects (LATEs) – the average treatment effects among those who comply with the assignment, can be identified, as formulated in the seminal work by Imbens and Angrist (1994).

The central theme of this paper is estimation and inference of the LATEs under CARs. In practice, researchers usually run two-stage least squares (2SLS) with treatment status as instrumental variable for estimation, and use heteroscedasticity-robust standard errors for inference. The validity of this approach is first established by Angrist and Imbens (1994) for natural experiments or randomized experiments under completely random sampling. However, different from those experiments in observational studies or under complete randomization, the treatment statuses generated under CARs are cross-sectionally dependent due to its randomization scheme. When there are no additional covariates, Ansel, Hong, and Li (2018) show the 2SLS robust standard error is inconsistent and usually conservative. Such a phenomenon is common for inference under CARs and has been discovered for other causal parameters such as average treatment effect (ATE; see Bugni, Canay, and Shaikh, 2018, 2019) and quantile treatment effect (QTE; see Zhang and Zheng, 2020). With additional covariates, in practice, researchers simply include them as exogenous controls in the 2SLS estimation. Such a practice may lead to degradation of the estimation precision, which is known as the Freedman’s critique (Freedman, 2008a, 2008b). These two issues raise questions of how to compute the non-conservative standard error for LATE under CARs and use the additional covariates to improve the estimation precision.

To tackle these problems, we propose a regression-adjusted LATE estimator in this paper. Our method takes into account of the dependence structure that arises from CAR and is thus able to avoid the conservatism, achieving the exact asymptotic size under the null. It is also weakly more efficient than both the estimator without regression adjustment and the one obtained by the standard 2SLS. In addition, it is robust to adjustment misspecification and easy to implement.

This paper makes four main contributions. The first is to propose a regression-adjusted estimator of the LATE and connect it with the standard 2SLS estimator. We show that the usual 2SLS estimator with additional covariates is a special case of our estimator when CAR
achieves strong balance, and thus, identifies LATEs. This complements the recent discussion by Blandhol, Bonney, Mogstad, and Torgovitsky (2022) for observational studies. Second, we show the weighted average of the fully saturated (with strata dummies) 2SLS estimators, including the fully saturated estimator without additional covariates recently proposed by Bugni and Gao (2021), is a special case of our adjusted estimator as well.

The second contribution of the paper is to develop an asymptotic theory for our regression-adjusted LATE estimator under high-level conditions required for the regression adjustments. Our analysis follows a new asymptotic framework that was recently established by Bugni et al. (2018) to study ATE estimators under CARs, which accounts for the cross-sectional dependence caused by the randomization. We prove that, even when the adjustments are misspecified, our proposed estimator is still consistent and asymptotic normal and that their inference method still achieves the exact asymptotic size under the null. When the adjustments are correctly specified, our estimator achieves the minimum asymptotic variance.

In our third contribution, we investigate efficiency gains brought by regression adjustments in parametric (both linear and nonlinear), nonparametric, and high-dimensional forms. When adjustments are linear, we drive the most efficient estimator among all linearly adjusted LATE estimators, and in particular, show that it is weakly more efficient than the estimator without adjustment and the standard 2SLS estimator. We also derive a nonlinearly adjusted LATE estimator. We further construct a new estimator which combines the linearly and nonlinearly adjusted estimators, and show it is weakly more efficient than both as well as the one without any adjustments, i.e., the fully saturated estimator proposed by Bugni and Gao (2021). We further study nonparametric and high-dimensional adjustments and provide conditions under which they are weakly more efficient than all the other adjusted estimators considered in this paper and are as if the correctly specified regression adjustments are used.

The final contribution of the paper is to provide simulation evidence and empirical support for the efficiency gains achieved by our regression-adjusted LATE estimator. We compare it with both the one without any adjustment and the one obtained by the standard 2SLS and confirm sizable efficiency gains that can be achieved by regression adjustments. In the empirical application, we revisit the experiment with a CAR design in Dupas et al. (2018). We find that by just using the same two covariates adopted in that paper, over nine outcome variables, the standard errors of our adjusted LATE estimator are on average around 7% lower than those without adjustments. For some outcome variables, regression adjustments can reduce the standard errors by about 15%. Compared with the 2SLS estimators, the standard errors of our estimators are generally smaller as well, although by a smaller margin.

Our paper is related to several lines of research. Bugni et al. (2018, 2019); Hu and Hu (2012); Ma, Hu, and Zhang (2015); Ma, Qin, Li, and Hu (2020); Olivares (2021); Shao and Yu (2013); Shao, Yu, and Zhong (2010); Zhang and Zheng (2020); Ye (2018); Ye and Shao (2020) studied inference of either ATEs or
QTEs under CARs without considering additional covariates. Bloniarz, Liu, Zhang, Sekhon, and Yu (2016); Fogarty (2018); Lin (2013); Lu (2016); Lei and Ding (2021); Li and Ding (2020); Liu, Tu, and Ma (2020); Liu and Yang (2020); Negi and Wooldridge (2020); Ye, Yi, and Shao (2021); Zhao and Ding (2021) studied the estimation and inference of ATEs using a variety of regression methods under various randomization. Jiang, Phillips, Tao, and Zhang (2021) examined regression-adjusted estimation and inference of QTEs under CARs. We contribute to the literature by studying the LATE estimators in the context of CARs and regression adjustment.

In a recent work, Bugni and Gao (2021) considered inference of the LATE estimators in CARs, but they did not use covariates in addition to the stratum indicator for regression adjustments. In one section of their paper, Ansel et al. (2018) also studied inference of a LATE estimator via 2SLS with covariates under CARs, which can be viewed as a linearly adjusted LATE estimator. We complement their works by proposing a novel LATE estimator that include 2SLS as a special case but also other nonlinear adjustments, deriving the optimal coefficient for the (potentially misspecified) linear adjustment, and developing a procedure to further improve the initial (potentially misspecified) linear and nonlinear adjustments. The final adjusted estimator of LATE is guaranteed to be weakly more efficient than the initially adjusted and the unadjusted estimator, the latter of which is just Bugni and Gao’s 2021 fully saturated estimator. It is also guaranteed to be weakly more efficient than the 2SLS estimator with control variables which is commonly used in empirical researches. We further study the nonparametric and high-dimensional adjustments which are completely new to these two papers. Ren and Liu (2021) studied the regression-adjusted LATE estimator in completely randomized experiments for a binary outcome using the finite population asymptotics. We differ from their work by considering the regression-adjusted estimator in covariate-adaptive randomizations for a general outcome using the superpopulation asymptotics.

Our paper also connects to a vast literature on the estimation and inference in randomized experiments, including Hahn, Hirano, and Karlan (2011); Athey and Imbens (2017); Abadie, Chingos, and West (2018); Tabord-Meehan (2021); Bai, Shaikh, and Romano (2021); Bai (2020); Jiang, Liu, Phillips, and Zhang (2021) among many others. Based on pilot experiments, Tabord-Meehan (2021) and Bai (2020) devise optimal randomization designs that may produce an ATE estimator with the lowest variance. It is possible to extend their works to LATE. Alternatively, we provide a way to improve efficiency in the data analysis stage, which is practically useful for researchers who are unable to run pilot experiments due to budget constraints or the fact that they analyze someone else’s datasets or conduct subsample analyses. Furthermore, although the optimal designs can ensure the covariates used in the randomization are well balanced, their updated versions obtained in future waves are usually not, as pointed out by Bruhn and McKenzie (2009). Our methodology can therefore complement the ‘optimal’ randomization by using these updated covariates to further improve the efficiency. In general, experimenters may miss the opportunity to achieve ‘optimal’ in the design stage. Our method gives them a second chance to improve the precision of their estimators in the
data analysis stage.

The rest of the paper is organized as follows. Section 2 lays out the rest of our setup, and Section 3 introduces our regression-adjusted LATE estimator \( \hat{\tau} \). We establish the asymptotic properties (asymptotic normality etc) of \( \hat{\tau} \) in Section 4. We then examine efficiency of \( \hat{\tau} \) in contexts of parametric, nonparametric and high-dimensional models in Sections 5, 6 and 7, respectively. We conduct Monte Carlo simulations in Section 8 and an empirical application in Section 9. Section 10 concludes. Some implementation details for sieve and Lasso regressions and the proofs of the theoretical results are included in Appendix.

2 Setup

Let \( Y_i \) denote the observed outcome of interest for individual \( i \); write \( Y_i = Y_i(1)D_i + Y_i(0)(1 - D_i) \), where \( Y_i(1), Y_i(0) \) are the potential outcomes for individual \( i \)’s hypothetical treated and untreated outcome, respectively, and \( D_i \) is a binary random variable indicating whether individual \( i \) received the treatment \( (D_i = 1) \) or not \( (D_i = 0) \) in the actual study. One could link \( D_i \) to the treatment status \( A_i \) in the following way: \( D_i = D_i(1)A_i + D_i(0)(1 - A_i) \), where \( D_i(1), D_i(0) \) are binary random variables.

Consider a CAR with \( n \) individuals; that is, a researcher can observe the data \( \{Y_i, D_i, A_i, S_i, X_i\}_{i=1}^n \). The support of vectors \( \{X_i\}_{i=1}^n \) is denoted \( \text{Supp}(X) \). Define \( [n] = \{1, 2, \ldots, n\} \), \( p(s) = \mathbb{P}(S_i = s) \), \( n(s) = \sum_{i \in [n]} 1\{S_i = s\} \), \( n_1(s) = \sum_{i \in [n]} A_i 1\{S_i = s\} \), \( n_0(s) = n(s) - n_1(s) \), \( S^{(n)} := (S_1, \ldots, S_n) \), \( X^{(n)} := (X_1, \ldots, X_n) \), and \( A^{(n)} := (A_1, \ldots, A_n) \). We make the following assumptions on the data generating process (DGP) and the treatment assignment rule.

**Assumption 1.**

(i) \( \{Y_i(1), Y_i(0), D_i(0), D_i(1), S_i, X_i\}_{i=1}^n \) is i.i.d. over \( i \). For each \( i \), We allow \( X_i \) and \( S_i \) to be dependent.

(ii) \( \{Y_i(1), Y_i(0), D_i(0), D_i(1), X_i\}_{i=1}^n \downarrow A^{(n)}|S^{(n)} \).

(iii) Suppose that \( p(s) \) is fixed w.r.t. \( n \) and positive for every \( s \in S \).

(iv) Let \( \pi(s) \) denote the propensity score for stratum \( s \). Then, \( c < \min_{s \in S} \pi(s) \leq \max_{s \in S} \pi(s) < 1 - c \) for some constant \( c \in (0, 0.5) \) and \( \frac{B_n(s)}{n(s)} = o_p(1) \) for \( s \in S \), where \( B_n(s) = \sum_{i=1}^n (A_i - \pi(s)) 1\{S_i = s\} \).

(v) Suppose \( \mathbb{P}(D(1) = 0, D(0) = 1) = 0 \).

(vi) \( \max_{s = 0, 1} \mathbb{E}(|Y_i(a)|^q|S_i = s) \leq C < \infty \) for \( q \geq 4 \).

Several remarks are in order. First, Assumption 1(ii) implies the treatment assignment \( A^{(n)} \) are generated only based on strata indicators. Second, Assumption 1(iii) imposes the sizes of strata are balanced. Third, Bugni et al. (2018) show that Assumption 1(iv) holds under several covariate-adaptive treatment assignment rules such as simple random sampling (SRS), biased-coin
design (BCD), adaptive biased-coin design (WEI), and stratified block randomization (SBR). For completeness, we briefly repeat their descriptions below. Note we only require $B_n(s)/n(s) = o_p(1)$, which is weaker than the assumption imposed by Bugni et al. (2018) but the same as that imposed by Bugni et al. (2019) and Zhang and Zheng (2020). Fourth, Assumption 1(v) implies there are no defiers. Last, Assumption 1(iv) is a standard moment condition.

**Example 1** (SRS). Let $\{A_i\}_{i=1}^n$ be drawn independently across $i$ and of $\{S_i\}_{i=1}^n$ as Bernoulli random variables with success rate $\pi$, i.e., for $k = 1, \cdots, n$,

$$\mathbb{P}(A_k = 1|\{S_i\}_{i=1}^n, \{A_j\}_{j=1}^{k-1}) = \mathbb{P}(A_k = 1) = \pi.$$  

Then, Assumption 1(iv) holds.

**Example 2** (WEI). This design was first proposed by Wei (1978). Let $n_{k-1}(S_k) = \sum_{i=1}^{k-1} \mathbb{1}\{S_i = S_k\}$, $B_k(S_k) = \sum_{i=1}^{k-1} \left(A_i - \frac{1}{2}\right) \mathbb{1}\{S_i = S_k\}$, and

$$\mathbb{P}(A_k = 1|\{S_i\}_{i=1}^k, \{A_j\}_{j=1}^{k-1}) = f\left(\frac{2B_{k-1}(S_k)}{n_{k-1}(S_k)}\right),$$

where $f(\cdot): [-1, 1] \mapsto [0, 1]$ is a pre-specified non-increasing function satisfying $f(-x) = 1 - f(x)$. Here, $\frac{B_k(S_k)}{n_k(S_k)}$ and $B_0(S_1)$ are understood to be zero. Then, Bugni et al. (2018) show that Assumption 1(iv) holds with $\pi(s) = \frac{1}{2}$. Recently, Hu (2016) generalized the adaptive biased-coin design to multiple treatment values and unequal target factions.

**Example 3** (BCD). The treatment status is determined sequentially for $1 \leq k \leq n$ as

$$\mathbb{P}(A_k = 1|\{S_i\}_{i=1}^k, \{A_j\}_{j=1}^{k-1}) = \begin{cases} \frac{1}{2} & \text{if } B_k(S_k) = 0 \\ \lambda & \text{if } B_k(S_k) < 0 \\ 1 - \lambda & \text{if } B_k(S_k) > 0, \end{cases}$$

where $B_k(S_k)$ is defined as above and $\frac{1}{2} < \lambda \leq 1$. Then, Bugni et al. (2018) show that Assumption 1(iv) holds with $\pi = \frac{1}{2}$.

**Example 4** (SBR). For each stratum, $|\pi(s)n(s)|$ units are assigned to treatment and the rest are assigned to control. Then obviously Assumption 1(iv) holds as $n(s) \rightarrow \infty$.

### 3 Estimation

In an imperfectly complied study there are four types of individuals: complier ($D_i(0) = 0, D_i(1) = 1$), always taker ($D_i(0) = 1, D_i(1) = 1$), never taker ($D_i(0) = 0, D_i(1) = 0$) and defier ($D_i(0) = 1, D_i(1) = 0$). We are interested in estimating the local average treatment effect (LATE) which is denoted $\tau$ and defined as

$$\tau = \mathbb{E}[Y(1) - Y(0)|D(1) > D(0)].$$
that is, we are interested in the ATE for the compliers (Angrist and Imbens (1994)).

In this section, we propose a regression-adjusted LATE estimator for \( \tau \) and connect it to the standard 2SLS estimator and the estimators developed in recent literature. We further show that those estimators are special cases of our estimator.

Define \( \mu^D(a, s, x) = \mathbb{E}[D(a)|S = s, X = x] \) and \( \mu^Y(a, s, x) = \mathbb{E}[Y(D(a))|S = s, X = x] \) for \( a = 0, 1 \), where \( Y(D(a)) = Y(1)D(a) + Y(0)(1 - D(a)) \). For \( a = 0, 1 \), we suppose that working models \( \mu^D(a, s, x) \) and \( \mu^Y(a, s, x) \) are employed, which may differ from the true conditional expectations \( \mu^D(a, s, x) \) and \( \mu^Y(a, s, x) \), respectively. We will also suppose that estimation is based on the working model; let \( \hat{\mu}^D(a, s, x) \) and \( \hat{\mu}^Y(a, s, x) \) be the estimated functions.

In CAR, the propensity score is usually known or can be consistently estimated by \( \hat{\pi}(s) = \frac{n_1(s)}{n(s)} \) because

\[
\hat{\pi}(s) - \pi(s) = \frac{1}{n(s)} \sum_{i=1}^{n} (A_i - \pi(s))1\{S_i = s\} = \frac{B_n(s)}{s} = o_p(1). \tag{3.1}
\]

Then, the proposed estimator of LATE is

\[
\hat{\tau} = \left( \frac{1}{n} \sum_{i \in [n]} \Xi_{H,i} \right)^{-1} \left( \frac{1}{n} \sum_{i \in [n]} \Xi_{G,i} \right), \tag{3.2}
\]

where

\[
\Xi_{H,i} = \frac{A_i(D_i - \hat{\mu}^D(1, S_i, X_i))}{\hat{\pi}(S_i)} - \frac{(1 - A_i)(D_i - \hat{\mu}^D(0, S_i, X_i))}{1 - \hat{\pi}(S_i)} + \hat{\mu}^D(1, S_i, X_i) - \hat{\mu}^D(0, S_i, X_i), \tag{3.3}
\]

\[
\Xi_{G,i} = \frac{A_i(Y_i - \hat{\mu}^Y(1, S_i, X_i))}{\hat{\pi}(S_i)} - \frac{(1 - A_i)(Y_i - \hat{\mu}^Y(0, S_i, X_i))}{1 - \hat{\pi}(S_i)} + \hat{\mu}^Y(1, S_i, X_i) - \hat{\mu}^Y(0, S_i, X_i). \tag{3.4}
\]

It is worth noting that, as \( \hat{\pi} \) is the consistent estimator for the propensity score, our proposed LATE estimator \( \hat{\tau} \) is consistent for \( \tau \) even when the the working models \( \mu^D(a, s, x) \) and \( \mu^Y(a, s, x) \) are misspecified.

This estimator takes the form of doubly robust moments (see Robins, Rotnitzky, and Zhao (1994), Robins and Rotnitzky (1995), Scharfstein, Rotnitzky, and Robins (1999), Robins, Rotnitzky, and van der Laan (2000), Hirano and Imbens (2001), Frölich (2007), Wooldridge (2007), Rothe and Firpo (2019) etc; see Sloczyński and Wooldridge (2018) and Seaman and Vansteelandt (2018) for recent reviews). To the best of our knowledge, we are the first to apply doubly robust methods to study the LATEs under CARs. Our analysis takes into account the cross-sectional dependence caused by the randomization and is different from the double robustness literature that mostly focuses on the observational data.

For different choices of working models, (3.2) can be interpreted as 2SLS with or without interaction effects. First, with additional covariate \( X_i \), empirical researchers usually run a 2SLS
regression of $Y_i$ on $D_i, X_i, \{1\{S_i = s\}\}_{s \in S}$ using $A_i$ as the IV and use the coefficient of $D_i$, denoted as $\hat{\tau}^{2sls}$, as the point estimate for the LATE. Then, by the indirect least squares, we have

$$\hat{\tau}_{1}^{2sls} = \frac{\hat{\Pi}_1}{\hat{\Pi}_2}, \quad (3.5)$$

where

$$\hat{\Pi}_1 = \frac{\sum_{i \in [n]} A_i(Y_i - X_i^T \hat{\theta}_x - \hat{\theta}_S)}{n_1} - \frac{\sum_{i \in [n]} (1 - A_i)(Y_i - X_i^T \hat{\theta}_x - \hat{\theta}_S)}{n_0},$$

$$\hat{\Pi}_2 = \frac{\sum_{i \in [n]} A_i(D_i - X_i^T \hat{\beta}_x - \hat{\beta}_S)}{n_1} - \frac{\sum_{i \in [n]} (1 - A_i)(D_i - X_i^T \hat{\beta}_x - \hat{\beta}_S)}{n_0},$$

($\hat{\beta}_x, \hat{\beta}_S$) are the estimators of $X_i$ and $1\{S_i = s\}$ from the OLS regression of $D_i$ on $A_i, X_i, \{1\{S_i = s\}\}_{s \in S}$, ($\hat{\theta}_x, \hat{\theta}_S$) are the estimators of $X_i$ and $1\{S_i = s\}$ from the OLS regression of $Y_i$ on $A_i, X_i, \{1\{S_i = s\}\}_{s \in S}$, $\hat{\beta}_S$ and $\hat{\theta}_S$ are $\hat{\beta}_x$ and $\hat{\theta}_x$, respectively, when $S_i = s$, and $n_1$ and $n_0$ are the numbers of treated and control units, respectively. When the covariate-adaptive randomization achieves strong balance (such as BCD and SBR) so that $\pi(S_i) = \pi(S_i) = \pi$, $n_1 = n\pi$, and $n_0 = n(1 - \pi)$ such a 2SLS estimator is a special case of our estimator with

$$\hat{\mu}^D(1, S_i, X_i) = \hat{\mu}^D(0, S_i, X_i) = X_i^T \hat{\beta}_x + \hat{\beta}_S \quad \text{and} \quad \hat{\mu}^Y(1, S_i, X_i) = \hat{\mu}^Y(0, S_i, X_i) = X_i^T \hat{\theta}_x + \hat{\theta}_S.$$

Second, our framework allows for the assignment probability $\pi(\cdot)$ to be varying across strata. Therefore, we can follow Bugni and Gao (2021) and run 2SLS with a full set of interactions by regressing $Y_i$ on $\{D_i1\{S_i = s\}, X_i1\{S_i = s\}, 1\{S_i = s\}\}_{s \in S}$ using $\{A_i1\{S_i = s\}\}_{s \in S}$ as IVs. The estimators of $D_i1\{S_i = s\}$, denoted as $\hat{\tau}_{2}^{2sls}(s)$, can be written as

$$\hat{\tau}_{2}^{2sls}(s) = \frac{\hat{\Pi}_1(s)}{\hat{\Pi}_2(s)},$$

where

$$\hat{\Pi}_1(s) = \frac{\sum_{i \in [n]} A_i(Y_i - X_i^T \hat{\theta}_{x,s} - \hat{\theta}_s)1\{S_i = s\}}{n_1(s)} - \frac{\sum_{i \in [n]} (1 - A_i)(Y_i - X_i^T \hat{\theta}_{x,s} - \hat{\theta}_s)1\{S_i = s\}}{n_0(s)},$$

$$\hat{\Pi}_2(s) = \frac{\sum_{i \in [n]} A_i(D_i - X_i^T \hat{\beta}_{x,s} - \hat{\beta}_s)1\{S_i = s\}}{n_1(s)} - \frac{\sum_{i \in [n]} (1 - A_i)(D_i - X_i^T \hat{\beta}_{x,s} - \hat{\beta}_s)1\{S_i = s\}}{n_0(s)},$$

($\hat{\beta}_{x,s}, \hat{\beta}_S$) are the estimators of $X_i$ and intercept from the OLS regression of $D_i$ on $A_i, X_i, 1$ using observations in stratum $S_i = s$, ($\hat{\theta}_{x,s}, \hat{\theta}_s$) are the estimators of $X_i$ and intercept from the OLS regression of $Y_i$ on $A_i, X_i, 1$ using observations in stratum $S_i = s$, and $n_1(s)$ and $n_0(s)$ are the
numbers of treated and control units in stratum $s$. Then, the LATE estimator can be written as a weighted average of $\hat{\tau}_2^{2sls}(s)$, i.e.,

$$\hat{\tau}_2^{2sls} = \sum_{s \in S} \hat{\Pi}_0(s) \hat{\tau}_2^{2sls}(s),$$

where $\hat{\Pi}_0(s) = \frac{\hat{p}(s)\hat{H}_2(s)}{\sum_{s \in S} \hat{p}(s)\hat{H}_2(s)}$, $\hat{p}(s) = n(s)/n$, and $n(s)$ is the number of units in stratum $s$. This is a reasonable estimator for LATE because when there is no $X$, this estimator reduces to the fully saturated estimator proposed by Bugni and Gao (2021). In addition, with covariates $X$, we have

$$\hat{\tau}_2^{2sls} = \sum_{s \in S} \hat{\Pi}_0(s) \hat{\tau}_2^{2sls}(s) = \frac{\sum_{i \in [n]} \left[ A_i (Y_i - X_i^\top \hat{\theta}_x,S_i - \hat{\theta}_b) - (1-A_i)(Y_i - X_i^\top \hat{\theta}_x,S_i - \hat{\theta}_b) \right]}{\sum_{i \in [n]} \left[ A_i (D_i - X_i^\top \hat{\beta}_x,S_i - \hat{\beta}_b) - (1-A_i)(D_i - X_i^\top \hat{\beta}_x,S_i - \hat{\beta}_b) \right]},$$

which is a special case of our estimator defined in (3.2) with

$$\hat{\mu}^D(1,S_i,X_i) = \hat{\mu}^D(0,S_i,X_i) = X_i^\top \hat{\beta}_x,S_i + \hat{\beta}_S$$

and $\hat{\mu}^Y(1,S_i,X_i) = \hat{\mu}^Y(0,S_i,X_i) = X_i^\top \hat{\theta}_x,S_i + \hat{\theta}_b$.

This also means Bugni and Gao’s (2021) fully saturated estimator is a special case of our estimator without additional covariates.\(^2\)

### 4 Asymptotic Properties

We first make the following high-level assumptions on the regression adjustments.

**Assumption 2.**

(i) For $a = 0,1$ and $s \in S$, define

$$\Delta^Y(a,s,X_i) := \hat{\mu}^Y(a,s,X_i) - \pi^Y(a,s,X_i), \quad \Delta^D(a,s,X_i) := \hat{\mu}^D(a,s,X_i) - \pi^D(a,s,X_i), \quad \text{and} \quad I_a(s) := \{i \in [n]: A_i = a, S_i = s\}.$$ 

Then, for $a = 0,1$, $b = D, Y$, we have

$$\max_{s \in S} \frac{\sum_{i \in I_1(s)} \Delta^b(a,s,X_i)}{n_1(s)} - \frac{\sum_{i \in I_0(s)} \Delta^b(a,s,X_i)}{n_0(s)} = o_p(n^{-1/2}).$$

(ii) For $a = 0,1$ and $b = D, Y$, $\frac{1}{n} \sum_{i=1}^n \Delta^b(a,S_i,X_i) = o_p(1)$.

(iii) Suppose $\max_{a=0,1,s \in S} \mathbb{E}[(\hat{\mu}^2(a,S_i,X_i)|S_i = s) \leq C < \infty$ for some constant $C$.

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\(^1\)Indeed in this case, we have $\hat{\tau}_2^{2sls}(s)$ equals $\hat{\beta}_{sat}(s)$ in Bugni and Gao (2021) and $\hat{\Pi}_0(s) = \frac{n(s)}{n} \left( \frac{n_{AD}(s)}{n_{11}(s)} - \frac{n_{AD}(s) - n_{AD}(s)}{n_{01}(s)} \right) / (\sum_{s \in S} \frac{n(s)}{n} \left( \frac{n_{AD}(s)}{n_{11}(s)} - \frac{n_{AD}(s) - n_{AD}(s)}{n_{01}(s)} \right))$, where $n_{AD}(s) = \sum_{i \in [n]} A_i D_i 1\{S_i = s\}$ and $n_D(s) = \sum_{i \in [n]} D_i 1\{S_i = s\}$. This $\hat{\Pi}_0(s)$ is exactly $\hat{P}(S = s|C)$ defined by Bugni and Gao (2021).

\(^2\)Ansel et al. (2018) also use cluster-wise regressions to construct their estimator, which is similar to, but not exactly the same as $\hat{\tau}_2^{2sls}$. 

9
Assumption 2 is mild. Consider a linear working model \( \mathbf{Y}(a, s, X_i) = X_i^\top \beta_{a,s} \), where the coefficient \( \beta_{a,s} \) may vary across treatment statuses and strata. Its estimator \( \hat{\mu}^{Y}(a, s, X_i) \) can be written as \( X_i^\top \hat{\beta}_{a,s} \) where \( \hat{\beta}_{a,s} \) is an estimator of \( \beta_{a,s} \). Then, Assumption 2(i) requires

\[
\max_{s \in S, a = 0, 1} \left| \left( \frac{1}{n_1(s)} \sum_{i \in I_1(s)} X_i - \frac{1}{n_0(s)} \sum_{i \in I_0(s)} X_i \right)^\top \left( \hat{\beta}_{a,s} - \beta_{a,s} \right) \right| = o_p(n^{-1/2}),
\]  

which holds whenever \( \hat{\beta}_{a,s} \xrightarrow{p} \beta_{a,s} \). Similar remark applies to Assumption 2(ii).

In order to write down the limit distribution of \( \hat{\tau} \), we need to introduce additional notation. For \( a = 0, 1 \), let \( D_i = \{ Y_i(1), Y_i(0), D_i(1), D_i(0), X_i \} \),

\[
\begin{align*}
\hat{\mu}^{Y}(a, S, X_i) & := \mathbf{Y}(a, s, X_i) - E \left[ \mathbf{Y}(a, s, X_i) | S_i \right], \\
W_i & := Y_i(D_i(1)) = Y_i(1)D_i(1) + Y_i(0)(1 - D_i(1)), \\
Z_i & := Y_i(D_i(0)) = Y_i(1)D_i(0) + Y_i(0)(1 - D_i(0)), \\
\hat{\tau} & := W_i - E[W_i | S_i], \\
\hat{\Xi}_1(D_i, S_i) & := \left( 1 - \frac{1}{\pi(S_i)} \right) \hat{\mu}^{Y}(1, S, X_i) - \hat{\mu}^{Y}(0, S, X_i) + \frac{\hat{W}_i}{\pi(S_i)} \\
& \quad - \hat{\tau} \left[ \left( 1 - \frac{1}{\pi(S_i)} \right) \hat{\mu}^{D}(1, S, X_i) - \hat{\mu}^{D}(0, S, X_i) + \frac{\hat{D}_i(1)}{\pi(S_i)} \right], \\
\hat{\Xi}_0(D_i, S_i) & := \left( \frac{1}{1 - \pi(S_i)} - 1 \right) \hat{\mu}^{Y}(0, S, X_i) + \hat{\mu}^{Y}(1, S, X_i) - \frac{\hat{Z}_i}{1 - \pi(S_i)} \\
& \quad - \hat{\tau} \left[ \left( \frac{1}{1 - \pi(S_i)} - 1 \right) \hat{\mu}^{D}(0, S, X_i) + \hat{\mu}^{D}(1, S, X_i) - \frac{\hat{D}_i(0)}{1 - \pi(S_i)} \right], \\
& \quad \text{and} \quad \Xi_2(S_i) := E[W_i - Z_i | S_i] - E[W_i - Z_i] - \hat{\tau} \left[ E[D_i(1) - D_i(0) | S_i] - E[D_i(1) - D_i(0)] \right].
\end{align*}
\]

Then, Theorem 4.1 below shows that \( \hat{\tau} \) is asymptotically normal in which the asymptotic variance, denoted as \( \sigma^2 \), can be written as

\[
\sigma^2 = \frac{\sigma_1^2 + \sigma_0^2 + \sigma_2^2}{\mathbb{P}(D(1) > D(0))^2},
\]

where

\[
\sigma_1^2 = E(\pi(S_i) \Xi_1^2(D_i, S_i)), \quad \sigma_0^2 = E(1 - \pi(S_i)) \Xi_0^2(D_i, S_i), \quad \text{and} \quad \sigma_2^2 = E \Xi_2^2(S_i).
\]
Next, we propose an estimator $\hat{\sigma}^2$ of $\sigma^2$. Recall $\Xi_{H,i}$ defined in (3.3). We define $\hat{\sigma}^2$ as

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \left[ A_i \tilde{\Xi}_1^n(D_i, S_i) + (1 - A_i) \tilde{\Xi}_0^n(D_i, S_i) + \tilde{\Xi}_2^n(S_i) \right] \left( \frac{1}{n} \sum_{i=1}^{n} \Xi_{H,i} \right)^2,$$

where

$$\tilde{\Xi}_1(D_i, s) := \tilde{\Xi}_1(D_i, s) - \frac{1}{n_1(s)} \sum_{i \in I_1(s)} \tilde{\Xi}_1(D_i, s),$$

$$\tilde{\Xi}_0(D_i, s) := \tilde{\Xi}_0(D_i, s) - \frac{1}{n_0(s)} \sum_{i \in I_0(s)} \tilde{\Xi}_0(D_i, s),$$

$$\tilde{\Xi}_2(s) := \frac{1}{n_1(s)} \sum_{i \in I_1(s)} (Y_i - \hat{\tau} D_i) \left( (1 - \frac{1}{\pi(s)}) \hat{\mu}^Y(1, s, X_i) - \hat{\mu}^Y(0, s, X_i) + \frac{Y_i}{\pi(s)} \right) \left( 1 - \frac{1}{\pi(s)} \hat{\mu}^D(1, s, X_i) - \hat{\mu}^D(0, s, X_i) + \frac{D_i}{\pi(s)} \right) \hat{\tau} \right],$$

$$\tilde{\Xi}_1(D_i, s) := \left[ (1 - \frac{1}{\pi(s)}) \hat{\mu}^Y(1, s, X_i) \right] - \hat{\tau} \left[ \frac{1}{1 - \pi(s)} \hat{\mu}^D(0, s, X_i) + \frac{D_i}{1 - \pi(s)} \right].$$

Theorem 4.1. (i) Suppose Assumptions 1 and 2 hold, then

$$\sqrt{n} (\hat{\tau} - \tau) \sim N(0, \sigma^2) \quad \text{and} \quad \hat{\sigma}^2 \xrightarrow{p} \sigma^2.$$

(ii) In addition, if the working models are correctly specified, i.e., $\overline{\mu}(a, s, x) = \mu(a, s, x)$ for all $(a, b, s, x) \in \{0, 1\} \times \{D, Y\} \times SX$ where $SX$ is the joint support of $(S, X)$, then the asymptotic variance $\sigma^2$ achieves the minimum.

Theorem 4.1(i) establishes limit distribution of our adjusted LATE estimator and gives a consistent estimator of its asymptotic variance. Such a variance depends on the working model $\overline{\mu}(a, s, x)$ for $(a, b) \in \{0, 1\} \times \{D, Y\}$. Theorem 4.1(ii) further shows the asymptotic variance is minimized when the working model is correctly specified. In fact, in this case, the asymptotic variance of $\hat{\tau}$ is

$$\sigma^2 = (\mathbb{P}(D_i(1) > D_i(0)))^{-2} \mathbb{E} \left[ \frac{(W_i - \mu^Y(1, S_i, X_i) - \tau(D_i(1) - \mu^D(1, S_i, X_i)))^2}{\pi(S_i)} \right].$$
\[ + \mathbb{E} \left[ \frac{(Z_i - \mu^Y(0, S_i, X_i) - \tau(D_i(0) - \mu^D(0, S_i, X_i)))^2}{1 - \pi(S_i)} \right] \]
\[ + \mathbb{E}(\mu^Y(1, S_i, X_i) - \mu^Y(0, S_i, X_i) - \tau(\mu^D(1, S_i, X_i) - \mu^D(0, S_i, X_i)))^2 \],

which coincides with the semiparametric efficiency bound for LATE under SRS as derived by Frölich (2007). It means for such a randomization scheme, our adjusted estimate is semiparametrically efficient. For other randomization schemes, it is still unknown whether the semiparametric efficiency bound for LATE will remain unchanged. To derive the semiparametric efficiency bound for LATE under general CAR is outside the scope of this paper, and thus, is left for future research.

Second, when there are no adjustments so that \( \mu^Y(\cdot) \) and \( \mu^D(\cdot) \) are just zero, we have
\[
\sigma^2 = \sum_{s \in S} \frac{\var(W - \tau D(1)|S = s)}{\mathbb{P}(D(1) > D(0))^2} + \frac{\sum_{s \in S} \frac{\var(Z - \tau D(0)|S = s)}{1 - \pi(s)}}{\mathbb{P}(D(1) > D(0))^2}.
\]

As previously noted, in this case, our estimator coincides with Bugni and Gao’s (2021) fully saturated estimator. Indeed, we can verify, by some tedious calculation, that \( \sigma^2 \) defined above is the same as the asymptotic variance of the fully saturated estimator derived by Bugni and Gao (2021).³ Then, Bugni and Gao (2021) have shown that our estimator without adjustments is weakly more efficient than the strata fixed effects and two sample IV estimators. In the next section, we show that, with adjustments, we can further improve the efficiency, even when the working models are potentially misspecified.

5 Parametric Adjustments

In this section, we consider estimating \( \mu^b(a, s, x) \) for \( a = 0, 1, s \in S \), and \( b = D, Y \) via parametric regressions. Note we do not require the parametric model to be correctly specified. Suppose
\[
\mu^Y(a, S_i, X_i) = \sum_{s \in S} 1\{S_i = s\} \Lambda^Y_{a,s}(X_i, \theta_{a,s}) \quad \text{and} \quad \mu^D(a, S_i, X_i) = \sum_{s \in S} 1\{S_i = s\} \Lambda^D_{a,s}(X_i, \beta_{a,s}),
\]
(5.1)

where \( \Lambda^b_{a,s}(\cdot) \) for \( (a, b, s) \in \{0, 1\} \times \{D, Y\} \times S \) is a known function of \( X_i \) up to some finite-dimensional parameter (i.e., \( \theta_{a,s} \) and \( \beta_{a,s} \)). The researchers have the freedom to choose the functional forms of \( \Lambda^b_{a,s}(\cdot) \), the parameter values of \( (\theta_{a,s}, \beta_{a,s}) \), and the ways they are estimated. In fact, as the parametric models are potentially misspecified, different estimation methods of the same model can lead to distinctive pseudo true values. We will discuss several detailed examples in Sections 5.1, 5.2, and 5.3 below. Here, we first focus on the general setup.

³Derivation is available upon request.
Define the estimators of \((\theta_{a,s}, \beta_{a,s})\) as \((\hat{\theta}_{a,s}, \hat{\beta}_{a,s})\), and thus, the corresponding feasible parametric regression adjustments as

\[
\hat{\mu}^Y(a, s, X_i) = \Lambda^Y_{a,s}(X_i, \hat{\theta}_{a,s}) \quad \text{and} \quad \hat{\mu}^D(a, s, X_i) = \Lambda^D_{a,s}(X_i, \hat{\beta}_{a,s}).
\]  

\[\text{Assumption 3.} \quad \begin{align*}
(i) & \text{ Suppose } \max_{a=0,1,s \in S} \|\hat{\theta}_{a,s} - \theta_{a,s}\|_2 \rightarrow p 0 \text{ and } \max_{a=0,1,s \in S} \|\hat{\theta}_{a,s} - \theta_{a,s}\|_2 \rightarrow p 0. \\
(ii) & \text{ There exist a positive random variable } L_i \text{ and a positive constant } C > 0 \text{ such that for all } a = (0,1) \text{ and } s \in S,
\end{align*}
\]

\[
\begin{align*}
||\theta_0 \Lambda^Y_{a,s}(X_i, \theta_{a,s})||_2 & \leq L_i, \quad ||\Lambda^Y_{a,s}(X_i, \theta_{a,s})||_2 \leq L_i \\
||\theta_\beta \Lambda^D_{a,s}(X_i, \beta_{a,s})||_2 & \leq L_i, \quad ||\Lambda^D_{a,s}(X_i, \beta_{a,s})||_2 \leq L_i,
\end{align*}
\]

and \(E(L_i^q | S_i = s) \leq C \) for some \(q \geq 2\).

Assumption 3(i) means our estimators are consistent. Assumption 3(ii) means the parametric models are smooth in their parameters, which is true for many widely used regression models such as linear, logit, and probit regressions. This restriction can be further relaxed to allow for non-smoothness under less intuitive entropy conditions.

\[\text{Theorem 5.1. Suppose Assumption 3 holds. Then, } \overrightarrow{\mu}^b(a, s, X_i) \text{ and } \hat{\mu}^b(a, s, X_i) \text{ defined in (5.1) and (5.2), respectively, satisfy Assumption 2.}\]

Theorem 5.1 generalizes the intuition in (4.1) to general parametric models. It means Assumption 2 holds for parametric models as long as the parameters are consistently estimated.

5.1 Optimal Linear Adjustments

Suppose, for \(a = 0,1\) and \(s \in S\), \(\overrightarrow{\mu}^Y(a, s, X) = \Psi^T_{t,a,s} b_{a,s}\) and \(\overrightarrow{\mu}^D(a, s, X) = \Psi^T_{b,a,s} b_{a,s}\), where \(\Psi_{t,a,s} = \Psi_s(X_i)\) is a function and the functional form can vary across \(s \in S\). The restriction that the function \(\Psi_s(\cdot)\) does not depend on \(a = 0,1\) is innocuous as if it does, we can stack them up and denote \(\Psi_{t,a,s} = (\Psi^T_{t,a,s}(X_i), \Psi^T_{b,a,s}(X_i))^T\). Similarly, it is also innocuous to impose that the function \(\Psi_s(\cdot)\) is the same for modeling \(\overrightarrow{\mu}^Y(a, s, X)\) and \(\overrightarrow{\mu}^D(a, s, X)\).

The asymptotic variance of the adjusted LATE estimator \(\hat{\tau}\) is denoted as \(\sigma^2\), which depends on \((\overrightarrow{\mu}^Y(a, s, X), \overrightarrow{\mu}^D(a, s, X))\), and thus, depends on \((t_{a,s}, b_{a,s})\). The following theorem characterizes the optimal linear coefficients that minimize the asymptotic variance of \(\hat{\tau}\) over all possible \((t_{a,s}, b_{a,s})\).

Let

\[
\Theta^* = \left( (\theta^{*}_{a,s}, \beta^{*}_{a,s})_{a=0,1,s \in S} : (\theta^{*}_{a,s}, \beta^{*}_{a,s})_{a=0,1,s \in S} \in \arg\min_{(t_{a,s}, b_{a,s})_{a=0,1,s \in S}} \sigma^2((t_{a,s}, b_{a,s})_{a=0,1,s \in S}) \right)
\]
Assumption 4. Suppose \( \mathbb{E}(|\Psi_{i,s}|^2 | S_i = s) \leq C < \infty \) for some constant \( C \). Denote \( \tilde{\Psi}_{i,s} = \Psi_{i,s} - \mathbb{E}(\Psi_{i,s} | S_i = s) \) for \( s \in S \). Then, there exist constants \( 0 < c < C < \infty \) such that

\[
c < \lambda_{\min}(\mathbb{E}(\tilde{\Psi}_{i,s} \tilde{\Psi}_{i,s}^\top)) \leq \lambda_{\max}(\mathbb{E}(\tilde{\Psi}_{i,s} \tilde{\Psi}_{i,s}^\top)) \leq C,
\]

where for a generic symmetric matrix \( A \), \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \) denote the minimum and maximum eigenvalues of \( A \).

Assumption 4 requires the regressor \( \Psi_{i,s} \) does not contain stratum invariant regressors such as the constant term. In fact, (3.3) and (3.4) imply that our estimator is numerically invariant to stratum-specific location shift because by definition,

\[
\sum_{i=1}^{n} \left( \frac{A_i}{\pi(S_i)} - 1 \right) 1\{S_i = s\} = 0 \quad \text{and} \quad \sum_{i=1}^{n} \left( \frac{1 - A_i}{1 - \pi(S_i)} - 1 \right) 1\{S_i = s\} = 0.
\]

Theorem 5.2. Suppose Assumptions 1 and 4 hold. Then, we have

\[
\Theta^* = \left\{ (\theta_{a,s}^*, \beta_{a,s}^*)_{a=0,1,s\in S} : \sqrt{\frac{\pi(s)}{1-\pi(s)}} (\theta_{1,s}^* - \tau \beta_{1,s}^*) + \sqrt{\frac{1-\pi(s)}{\pi(s)}} (\theta_{0,s}^* - \tau \beta_{0,s}^*) \right\},
\]

where

\[
\theta_{LP,a,s} = \left[ \mathbb{E}(\tilde{\Psi}_{i,s} \tilde{\Psi}_{i,s}^\top | S_i = s) \right]^{-1} \left[ \mathbb{E}(\tilde{\Psi}_{i,s} Y_i \mid D_i(a) | S_i = s) \right],
\]

\[
\beta_{LP,a,s} = \left[ \mathbb{E}(\tilde{\Psi}_{i,s} \tilde{\Psi}_{i,s}^\top | S_i = s) \right]^{-1} \left[ \mathbb{E}(\tilde{\Psi}_{i,s} D_i(a) | S_i = s) \right].
\]

The optimality result in Theorem 5.2 rely on two key restrictions: (1) the regressor \( \Psi_{i,s} \) is the same for treated and control units and (2) both the adjustments \( \bar{p}^Y(a,s,X) \) and \( \bar{p}^D(a,s,X) \) are linear. The first restriction is innocuous as we can stack up regressors for treated and control units as previously mentioned. But it indeed rules out the scenario that some regressors are just available for treated or control units, but not both. The second restriction means it is possible to have nonlinear adjustments that are more efficient. We will come back to this point in Sections 5.2, 5.3, and 6.

In view of Theorem 5.1, the optimal linear coefficients are not unique. In order to achieve the optimality, we only need to consistently estimate one point in \( \Theta^* \). For the rest of the section, we choose \( (\theta_{LP,a,s}^*, \beta_{LP,a,s}^*) \) with the corresponding optimal linear adjustments

\[
\bar{p}^Y(a,s,X_i) = \Psi_{i,s}^\top \theta_{LP,a,s}^* \quad \text{and} \quad \bar{p}^D(a,s,X_i) = \Psi_{i,s}^\top \beta_{LP,a,s}^*.
\]
We estimate \((\theta_{a,s}^{LP}, \beta_{a,s}^{LP})\) by \((\hat{\theta}_{a,s}^{LP}, \hat{\beta}_{a,s}^{LP})\), where
\[
\hat{\Psi}_{i,a,s} := \Psi_{i,s} - \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \Psi_{i,s} \\
\hat{\theta}_{a,s}^{LP} := \left( \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \hat{\Psi}_{i,a,s} \hat{\Psi}_{i,a,s}^T \right)^{-1} \left( \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \hat{\Psi}_{i,a,s} Y_i \right) \\
\hat{\beta}_{a,s}^{LP} := \left( \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \hat{\Psi}_{i,a,s} \hat{\Psi}_{i,a,s}^T \right)^{-1} \left( \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \hat{\Psi}_{i,a,s} D_i \right). \tag{5.5}
\]

Then, the feasible linear adjustments can be defined as
\[
\hat{\mu}^Y(a, s, X_i) = \Psi_{i,a,s}^T \hat{\theta}_{a,s}^{LP} \quad \text{and} \quad \hat{\mu}^D(a, s, X_i) = \Psi_{i,a,s}^T \hat{\beta}_{a,s}^{LP}. \tag{5.6}
\]

Suppose \(S = \{1, \cdots, S\}\) for some integer \(S > 0\). It is clear that \(\hat{\theta}_{a,s}^{LP}\) and \(\hat{\beta}_{a,s}^{LP}\) are the coefficients of \(\Psi_{i,s} 1\{S_i = s\} 1\{A_i = a\}\) in the following two linear OLS regressions:
\[
Y_i \sim \Psi_{i,1} 1\{S_i = 1\} 1\{A_i = 1\} + \cdots + \Psi_{i,S} 1\{S_i = S\} 1\{A_i = 1\} \\
+ \Psi_{i,1} 1\{S_i = 1\} 1\{A_i = 0\} + \cdots + \Psi_{i,S} 1\{S_i = S\} 1\{A_i = 0\} \\
+ 1\{S_i = 1\} 1\{A_i = 1\} + \cdots + 1\{S_i = S\} 1\{A_i = 1\} \\
+ 1\{S_i = 1\} 1\{A_i = 0\} + \cdots + 1\{S_i = S\} 1\{A_i = 0\} \tag{5.7}
\]
\[
D_i \sim \Psi_{i,1} 1\{S_i = 1\} 1\{A_i = 1\} + \cdots + \Psi_{i,S} 1\{S_i = S\} 1\{A_i = 1\} \\
+ \Psi_{i,1} 1\{S_i = 1\} 1\{A_i = 0\} + \cdots + \Psi_{i,S} 1\{S_i = S\} 1\{A_i = 0\} \\
+ 1\{S_i = 1\} 1\{A_i = 1\} + \cdots + 1\{S_i = S\} 1\{A_i = 1\} \\
+ 1\{S_i = 1\} 1\{A_i = 0\} + \cdots + 1\{S_i = S\} 1\{A_i = 0\}. \tag{5.8}
\]

**Theorem 5.3.** Suppose Assumptions 1 and 4 holds. Then,
\[
\{\bar{\mu}^Y(a, s, X_i)\}_{b = D, Y, a = 0, 1, s \in S} \quad \text{and} \quad \{\bar{\mu}^D(a, s, X_i)\}_{b = D, Y, a = 0, 1, s \in S}
\]
defined in (5.4) and (5.6), respectively, satisfy Assumption 2. Denote the adjusted LATE estimator with adjustment \(\{\bar{\mu}^Y(a, s, X_i)\}_{b = D, Y, a = 0, 1, s \in S}\) defined in (5.6) as \(\hat{\tau}^{LP}\). Then, all the results in Theorem 4.1(i) hold for \(\hat{\tau}^{LP}\). In addition, \(\hat{\tau}^{LP}\) is the most efficient among all linearly adjusted LATE estimators, and in particular, weakly more efficient than the LATE estimator with no adjustments.

The 2SLS estimators \(\hat{\tau}_1^{2SLS}\) and \(\hat{\tau}_2^{2SLS}\) mentioned in Section 3 are also linearly adjusted estimators. However, Theorem 5.3 shows our \(\hat{\tau}^{LP}\) is weakly more efficient than both by letting \(\Psi_{i,s} = X_i\). In general, \(\hat{\tau}^{LP}\) and the 2SLS estimators are not asymptotically equivalent because the optimal linear adjustments (\(\hat{\mu}^D(a, s, x)\) and \(\hat{\mu}^Y(a, s, x)\) defined in (5.6)) can be different for treated and control groups while those for 2SLS estimators are forced to be the same.
5.2 Linear and Logistic Regressions

It is also possible to consider a linear model for $p^Y(a, s, X_i)$ and a logistic model for $p^D(a, s, X_i)$, i.e.,

$$p^Y(a, s, X_i) = \hat{\Psi}_{i,s}^T \hat{\beta}_{a,s}$$ and $$p^D(a, s, X_i) = \lambda(\hat{\Psi}_{i,s}^T h_{a,s}),$$

where $\hat{\Psi}_{i,s} = (1, \Psi_{i,s})$, $\Psi_{i,s} = \Psi_s(X_i)$ and $\lambda(u) = \exp(u)/(1 + \exp(u))$ is the logistic CDF. As the model for $p^D(a, s, X_i)$ is non-linear, the optimality result established in the previous section does not apply. We can consider fitting the linear and logistic models by OLS and MLE, respectively, and call this method the OLS-MLE adjustment. Specifically, define

$$\hat{\mu}^Y(a, s, X_i) = \hat{\Psi}_{i,s}^T \hat{\beta}_{a,s}^{OLS}$$ and $$\hat{\mu}^D(a, s, X_i) = \lambda(\hat{\Psi}_{i,s}^T \hat{\beta}_{a,s}^{MLE}),$$ (5.9)

where

$$\hat{\beta}_{a,s}^{OLS} = \left( \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \Psi_{i,s} \Psi_{i,s}^T \right)^{-1} \left( \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \Psi_{i,s} Y_i \right)$$ and

$$\hat{\beta}_{a,s}^{MLE} = \arg \max_b \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \left[ D_i \log(\lambda(\Psi_{i,s}^T b)) + (1 - D_i) \log(1 - \lambda(\Psi_{i,s}^T b)) \right].$$ (5.10)

It is clear that $\hat{\beta}_{a,s}^{OLS}$ and $\hat{\beta}_{a,s}^{MLE}$ are the OLS and MLE estimates of the coefficient of $\Psi_{i,s} 1\{S_i = s\} 1\{A_i = a\}$ in the following linear and logistic regressions, respectively,

$$Y_i \sim \Psi_{i,1} 1\{S_i = 1\} 1\{A_i = 1\} + \cdots + \Psi_{i,S} 1\{S_i = S\} 1\{A_i = 1\}$$
$$+ \Psi_{i,1} 1\{S_i = 1\} 1\{A_i = 0\} + \cdots + \Psi_{i,S} 1\{S_i = S\} 1\{A_i = 0\}$$
$$D_i \sim \Psi_{i,1} 1\{S_i = 1\} 1\{A_i = 1\} + \cdots + \Psi_{i,S} 1\{S_i = S\} 1\{A_i = 1\}$$
$$+ \Psi_{i,1} 1\{S_i = 1\} 1\{A_i = 0\} + \cdots + \Psi_{i,S} 1\{S_i = S\} 1\{A_i = 0\}.$$ (5.11)

In OLS and MLE methods, we do allow the regressor $\hat{\Psi}_{i,s}$ to contain the constant term. Suppose $\hat{\beta}_{a,s}^{OLS} = (\hat{h}_{a,s}, \hat{\beta}_{a,s}^{OLS})^T$, where $\hat{h}_{a,s}$ is the coefficients of the constant terms in $\hat{\Psi}_{i,s}$. Then, because our adjusted LATE estimator is invariant to the stratum-specific location shift of the adjustment term, using $\hat{\mu}^Y(a, s, X_i) = \hat{\Psi}_{i,s}^T \hat{\beta}_{a,s}^{OLS} = \hat{h}_{a,s} + \Psi_{i,s} \hat{\beta}_{a,s}^{OLS}$ and $\hat{\mu}^Y(a, s, X_i) = \Psi_{i,s}^T \hat{\beta}_{a,s}^{OLS}$ produce the exact same LATE estimator. In addition, we can obtain $\hat{\beta}_{a,s}^{OLS}$ via the OLS regression

$$Y_i \sim \Psi_{i,1} 1\{S_i = 1\} 1\{A_i = 1\} + \cdots + \Psi_{i,S} 1\{S_i = S\} 1\{A_i = 1\}$$
$$+ \Psi_{i,1} 1\{S_i = 1\} 1\{A_i = 0\} + \cdots + \Psi_{i,S} 1\{S_i = S\} 1\{A_i = 0\}$$
$$+ 1\{S_i = 1\} 1\{A_i = 1\} + \cdots + 1\{S_i = S\} 1\{A_i = 1\}$$
$$+ 1\{S_i = 1\} 1\{A_i = 0\} + \cdots + 1\{S_i = S\} 1\{A_i = 0\}.$$
which is exactly the same as (5.7). This implies \( \hat{\theta}^{OLS}_{a,s} = \hat{\theta}^{LP}_{a,s} \). On the other hand, because the logistic regression is nonlinear, the non-intercept part of \( \hat{\beta}^{MLE}_{a,s} \) does not equal \( \hat{\beta}^{LP}_{a,s} \).

The limits of \( \tilde{\theta}^{OLS}_{a,s} \) and \( \tilde{\beta}^{MLE}_{a,s} \) are defined as

\[
\theta^{OLS}_{a,s} = \left( \mathbb{E}(\hat{\Psi}_{i,s} \hat{\Psi}_{i,s}^\top | S_i = s) \right)^{-1} \left( \mathbb{E}(\hat{\Psi}_{i,s} Y_i(D_i(a)) | S_i = s) \right)
\]

and

\[
\hat{\beta}^{MLE}_{a,s} = \arg\max_b \mathbb{E} \left( \left[ D_i(a) \log(\lambda(\hat{\Psi}_{i,s}^\top b)) + (1 - D_i(a)) \log(1 - \lambda(\hat{\Psi}_{i,s}^\top b)) \right] | S_i = s \right),
\]

which imply that

\[
\mathbb{P}^Y(a,s,X_i) = \hat{\Psi}_{i,s}^\top \theta^{OLS}_{a,s} \quad \text{and} \quad \mathbb{P}^D(a,s,X_i) = \lambda(\hat{\Psi}_{i,s}^\top \beta^{MLE}_{a,s}).
\]  

(5.11)

**Assumption 5.** (i) For \( a = 0, 1 \) and \( s \in S \), suppose \( \mathbb{E}(\hat{\Psi}_{i,s} \hat{\Psi}_{i,s}^\top | S_i = s) \) is invertible and

\[
\mathbb{E} \left( \left[ D_i(a) \log(\lambda(\hat{\Psi}_{i,s}^\top b)) + (1 - D_i(a)) \log(1 - \lambda(\hat{\Psi}_{i,s}^\top b)) \right] | S_i = s \right)
\]

has \( \beta^{MLE}_{a,s} \) as its unique maximizer.

(ii) There exists constant \( C < \infty \) such that \( \max_{a=0,1,s \in S} \mathbb{E}||\hat{\Psi}_{i,s}||_q^2 \leq C < \infty \) for some \( q \geq 2 \).

**Theorem 5.4.** Suppose Assumptions 1 and 5 hold. Then,

\[
\{\mathbb{P}^Y(a,s,X_i)\}_{b=D,Y,a=0,1,s \in S} \quad \text{and} \quad \{\bar{\mu}^b(a,s,X_i)\}_{b=D,Y,a=0,1,s \in S}
\]

defined in (5.11) and (5.9), respectively, satisfy Assumption 2. Denote the adjusted LATE estimator with adjustment \( \{\mathbb{P}^D(a,s,X_i)\}_{b=D,Y,a=0,1,s \in S} \) as \( \hat{\tau}^{LG} \). Then, all the results in Theorem 4.1(i) hold for \( \hat{\tau}^{LG} \).

Several remarks are in order. First, the OLS and MLE estimates are not optimal in the sense that they minimize the asymptotic variance of the corresponding LATE estimator. Second, the OLS-MLE adjustment is not necessarily less efficient than the optimal linear adjustment studied in Section 5.1 as the regression model for \( \mathbb{P}^D(a,s,X_i) \) is nonlinear. In fact, as Theorem 4.1 shows, if the adjustments are correctly specified, then the adjusted LATE estimator can achieve the global minimum asymptotic variance. Compared with the linear probability model considered in Section 5.1, the logistic model is expected to be less misspecified, especially when the regressor \( \Psi_i \) contains technical terms of \( X_i \) such as interactions and quadratic terms. Third, we will further justify the above intuition in Section 6 below, in which we let \( \Psi_{i,s} \) be the sieve basis functions with an increasing dimension and show that the OLS-MLE method can consistently estimate the correct specification. Fourth, one theoretical shortcoming of the OLS-MLE adjustment is that, unlike the optimal linear adjustment, it is not guaranteed to be more efficient than no adjustment. We address this issue in Section 5.3 below.
5.3 Further Efficiency Improvement

Let \( \theta_{a,s}^{OLS} = (h_{a,s}^{OLS}, \beta_{a,s}^{OLS}) \), where \( h_{a,s}^{OLS} \) is the intercept. If \( \beta_{a,s}^{MLE} \) were known, the OLS-MLE adjustment can be viewed as a linear adjustment. Specifically, denote

\[
\Phi_{i,s} := (\Psi_{i,s}, \lambda(\Psi_{i,s}^\top \beta_{i,s}^{MLE}), \lambda(\Psi_{i,s}^\top \beta_{i,s}^{MLE}))^\top
\]

\( (5.12) \)

\[
t_{a,s}^{LG} := a \begin{pmatrix} \theta_{1,s}^{OLS} \\ 0 \\ 0 \end{pmatrix} + (1-a) \begin{pmatrix} \theta_{0,s}^{OLS} \\ 0 \\ 0 \end{pmatrix}, \quad b_{a,s}^{LG} := a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (1-a) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},
\]

where \( d_\psi \) is the dimension of \( \psi_{i,s} \). Then, the OLS-MLE adjustment can be written as

\[
\mu_Y(a,s,X_i) = \Phi_{i,s}^\top t_{a,s}^{LG} + h_{a,s}^{OLS}
\]

and

\[
\mu_D(a,s,X_i) = \Phi_{i,s}^\top b_{a,s}^{LG}.
\]

Because our estimator is invariant to stratum-level location shift of adjustments, the OLS-MLE adjustments and the linear adjustments

\[
\tilde{\mu}_Y(a,s,X_i) = \Phi_{i,s}^\top \tilde{t}_{a,s}^{LG} + h_{a,s}^{OLS} \quad \text{and} \quad \tilde{\mu}_D(a,s,X_i) = \Phi_{i,s}^\top \tilde{b}_{a,s}^{LG}.
\]

produce the same estimator. Similarly, we can replicate no adjustments and the optimal linear adjustments with \( \Phi_{i,s} \) defined in (5.12) as regressors by letting

\[
\tilde{\mu}_Y(a,s,X_i) = \Phi_{i,s}^\top t_{a,s}^{LP} \quad \text{and} \quad \tilde{\mu}_D(a,s,X_i) = \Phi_{i,s}^\top b_{a,s}^{LP},
\]

with \( (t_{a,s}, b_{a,s}) = (0,0) \) and \( (t_{a,s}, b_{a,s}) = (t_{a,s}^{LP}, b_{a,s}^{LP}) \), respectively, where

\[
t_{a,s}^{LP} := a \begin{pmatrix} \theta_{1,s}^{LP} \\ 0 \\ 0 \end{pmatrix} + (1-a) \begin{pmatrix} \theta_{0,s}^{LP} \\ 0 \\ 0 \end{pmatrix}, \quad b_{a,s}^{LP} := a \begin{pmatrix} \beta_{1,s}^{LP} \\ 0 \\ 0 \end{pmatrix} + (1-a) \begin{pmatrix} \beta_{0,s}^{LP} \\ 0 \\ 0 \end{pmatrix}.
\]

Based on Theorem 5.2, we can further improve all three types of adjustments by setting the linear coefficients of \( \Phi_{i,s} \) as

\[
\theta_{a,s}^{LP*} := \left( \mathbb{E} \tilde{\Phi}_{i,s} \tilde{\Phi}_{i,s}^\top | S_i = s \right)^{-1} \left( \mathbb{E} \tilde{\Phi}_{i,s} Y_i(D_i(a)) | S_i = s \right),
\]

\[
\beta_{a,s}^{LP*} := \left( \mathbb{E} \tilde{\Phi}_{i,s} \tilde{Y}_{i,s} | S_i = s \right)^{-1} \left( \mathbb{E} \tilde{\Phi}_{i,s} D_i(a) | S_i = s \right),
\]

where \( \tilde{\Phi}_{i,s} = \Phi_{i,s} - \mathbb{E}(\Phi_{i,s}|S_i = s) \). The final linear adjustments with \( \theta_{a,s}^{LP*} \) and \( \beta_{a,s}^{LP*} \) are

\[
\tilde{\mu}_Y(a,s,X_i) = \Phi_{i,s}^\top \theta_{a,s}^{LP*} \quad \text{and} \quad \tilde{\mu}_D(a,s,X_i) = \Phi_{i,s}^\top \beta_{a,s}^{LP*}.
\]

(5.13)
Because $\beta_{a,s}^{MLE}$ is unknown, we can replace it by its estimate proposed in Section 5.2, i.e., define

$$\hat{\Phi}_{i,s} := (\Psi_{i,s}, \lambda(\Psi_{i,s}^\top \hat{\beta}_{a,s}^{MLE}), \lambda(\Psi_{i,s}^\top \hat{\beta}_{0,s}^{MLE}))^\top$$

and

$$\hat{\Phi}_{i,a,s} = \hat{\Phi}_{i,s} - \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \hat{\Phi}_{i,s}.$$

Then, we define the estimators of $\theta_{LP}^*_{a,s}$ and $\beta_{LP}^*_{a,s}$ as

$$\hat{\theta}_{LP}^*_{a,s} := \left( \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \hat{\Phi}_{i,a,s} \hat{\Phi}_{i,a,s}^\top \right)^{-1} \left( \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \hat{\Phi}_{i,a,s} Y_i \right),$$

$$\hat{\beta}_{LP}^*_{a,s} := \left( \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \hat{\Phi}_{i,a,s} \hat{\Phi}_{i,a,s}^\top \right)^{-1} \left( \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \hat{\Psi}_{i,a,s} D_i \right). \quad (5.14)$$

The corresponding feasible adjustments are

$$\hat{\mu}^Y(a, s, X_i) = \hat{\Phi}_{i,s}^\top \hat{\theta}_{LP}^*_{a,s}$$

and

$$\hat{\mu}^D(a, s, X_i) = \hat{\Phi}_{i,s}^\top \hat{\beta}_{LP}^*_{a,s}. \quad (5.15)$$

**Assumption 6.** Suppose Assumption 4 holds for $\Phi_{i,s}$ defined in (5.12).

**Theorem 5.5.** Suppose Assumptions 1, 5, and 6 hold. Then, the sets

$$\{\hat{\mu}^b(a, s, X_i)\}_{b=D,Y,a=0,1,s \in S}$$

and

$$\{\hat{\mu}^b(a, s, X_i)\}_{b=D,Y,a=0,1,s \in S}$$

defined in (5.13) and (5.15), respectively, satisfy Assumption 2. Denote the LATE estimator with regression adjustments $\{\hat{\mu}^b(a, s, X_i)\}_{b=D,Y,a=0,1,s \in S}$ defined in (5.15) as $\hat{\tau}_{LP}^*$. Then, all the results in Theorem 4.1(i) holds for $\hat{\tau}_{LP}^*$. In addition, $\hat{\tau}_{LP}^*$ is weakly more efficient than $\hat{\tau}^L$, $\hat{\tau}^L$, and the LATE estimator with no adjustments.

Theorem 5.5 shows that by refitting OLS-MLE adjustment in a linear regression with optimal linear coefficients, we can further improve the efficiency of the adjusted LATE estimator. As a byproduct, $\hat{\tau}_{LP}^*$ is guaranteed to be weakly more efficient than the LATE estimator without any adjustments.

### 6 Nonparametric Adjustments

In this section, we consider the nonparametric regression as the adjustments for our LATE estimator. Specifically, we use linear and logistic sieve regressions to estimate the true specifications $\mu^Y(a, s, X_i)$ and $\mu^D(a, s, X_i)$, respectively. For implementation, the nonparametric adjustment is exactly the same as OLS-MLE adjustment studied in Section 5.2. Theoretically, we will let the regressors $\Psi_{i,s}$ in (5.9) be sieve basis functions whose dimensions will diverge to infinity as sample size increases. For notation simplicity, we suppress the subscript $a$ and denote the sieve regressors as $\Psi_{i,n} \in \mathbb{R}^{h_n}$, where the dimension $h_n$ can diverge with the sample size and the corresponding sieve estimators as $\hat{\theta}_{NP}^{a,s}$ and $\hat{\beta}_{NP}^{a,s}$ as (5.10) where $\Psi_{i,s}$ is replaced by $\Psi_{i,n}$. The corresponding feasible
regression adjustments are

\[ \hat{\mu}^Y(a, s, X_i) = \Psi_{i,n}^T \hat{\beta}_{a,s}^{NP} \quad \text{and} \quad \hat{\mu}^D(a, s, X_i) = \lambda(\Psi_{i,n}^T \hat{\beta}_{a,s}^{NP}), \]  

(6.1)

and the corresponding adjusted LATE estimator is denoted as \( \hat{\tau}^{NP} \).

**Assumption 7.**

(i) There exist constants \( 0 < c < C < \infty \) such that with probability approaching one,

\[ \kappa_1 \leq \lambda_{\min}\left( \frac{1}{n_\alpha(s)} \sum_{i \in I_\alpha(s)} \Psi_{i,n} \Psi_{i,n}^T \right) \leq \lambda_{\max}\left( \frac{1}{n_\alpha(s)} \sum_{i \in I_\alpha(s)} \Psi_{i,n} \Psi_{i,n}^T \right) \leq \kappa_2 \]

and

\[ \kappa_1 \leq \lambda_{\min}\left( E(\Psi_{i,n} \Psi_{i,n}^T | S_i = s) \right) \leq \lambda_{\max}\left( E(\Psi_{i,n} \Psi_{i,n}^T | S_i = s) \right) \leq \kappa_2. \]

(ii) For \( a = 0, 1 \), there exists an \( h_n \times 1 \) vector \( \theta_{a,s}^{NP} \) and \( \beta_{a,s}^{NP} \) such that for

\[ R^Y(a, s, x) = E(Y_i(D_i(a))|S_i = s, X_i = x) - \lambda(\Psi_{i,n} \theta_{a,s}^{NP}) \quad \text{and} \quad \]

\[ R^D(a, s, x) = P(D_i(a) = 1|S_i = s, X_i = x) - \lambda(\Psi_{i,n} \beta_{a,s}^{NP}), \]

we have \( \sup_{a=0,1,b \in \{D,Y\},s \in S,X \in \supp(X)} |R^b(a, s, x)| = o(1), \)

\[ \sup_{a=0,1,b \in \{D,Y\},s \in S,X \in \supp(X)} \frac{1}{n_\alpha(s)} \sum_{i \in I_\alpha(s)} (R^b(a, s, X_i))^2 = O_p\left( \frac{h_n \log(n)}{n} \right), \]

and

\[ \sup_{a=0,1,b \in \{D,Y\},s \in S} E((R^b(a, s, X_i))^2 | S_i = s) = O\left( \frac{h_n \log(n)}{n} \right). \]

(iii) For \( a = 0, 1 \), there exists a constant \( c \in (0, 0.5) \) such that

\[ c \leq \inf_{a=0,1,s \in S,X \in \supp(X)} P(D_i(a) = 1|S_i = s, X_i = x) \]

\[ \leq \sup_{a=0,1,s \in S,X \in \supp(X)} P(D_i(a) = 1|S_i = s, X_i = x) \leq 1 - c. \]

(iv) Suppose \( E(\Phi^2_{i,n,k} | S_i = s) \leq C < \infty \) for some constant \( C > 0 \), \( \max_{i \in |n|} ||\Psi_{i,n}||_2 \leq \zeta(h_n), \)

\( \zeta^2(h_n) h_n \log(n) = o(n) \), and \( h_n^2 \log^2(n) = o(n) \), where \( \Phi_{i,n,k} \) denotes the kth coordinate of \( \Psi_{i,n} \).

**Theorem 6.1.** Suppose Assumptions 1 and 7 hold. Then, \( \{\hat{\mu}^b(a, s, X_i)\}_{b=D,Y,a=0,1,s \in S} \) defined in (6.1) and \( \hat{\tau}^{NP}(a, s, X) = \hat{\mu}^b(a, s, X) \) satisfy Assumption 2. Then, all the results in Theorem 4.1(i) hold for \( \hat{\tau}^{NP} \). In addition, \( \hat{\tau}^{NP} \) achieves the minimum asymptotic variance characterized in Theorem 4.1(ii).

The OLS-MLE and nonparametric adjustments are numerically identical if the same set of re-
gressors are used. Theorem 6.1 then shows that the OLS-MLE adjustment with technical regressors performs well because it can closely approximate the correct specification. Under the asymptotic framework that the dimension of the regressors diverges to infinity and the approximation error converges to zero, the OLS-MLE adjustment can be viewed as the nonparametric adjustment, which achieves the minimum asymptotic variance of the adjusted LATE estimator.

7 High-Dimensional Adjustments

In this section, we consider the case that the regressor $\Psi_{i,n} \in \mathbb{R}^{p_n}$ is high-dimensional so that $p_n \gg n$. In this case, we can no longer use the OLS-MLE (nonparametric) adjustment method. Instead, we need to regularize the least squares and logistic regressions. Specifically, let

$$\hat{\mu}^Y(a, s, X_i) = \Psi_{i,n}^T \hat{\beta}_{a,s}^{HD}$$
and
$$\hat{\mu}^D(a, s, X_i) = \lambda(\Psi_{i,n}^T \hat{\beta}_{a,s}^{HD}),$$

and the corresponding adjusted LATE estimator is denoted as $\hat{\tau}^{HD}$, where

$$\hat{\beta}_{a,s}^{HD} = \arg\min_b \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \left( Y_i - \Psi_{i,n}^T t \right)^2 + \frac{\theta_{n,a}(s)}{n_a(s)} ||\Omega^Y t||_1,$$

$$\hat{\beta}_{a,s}^{HD} = \arg\min_b \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \left( D_i \log(\lambda(\Psi_{i,n}^T b)) + (1 - D_i) \log(1 - \lambda(\Psi_{i,n}^T b)) \right) + \frac{\theta_{n,a}(s)}{n_a(s)} ||\hat{\Omega}^D b||_1,$$

$\{\theta_{n,a}(s)\}_{a=0,1,s \in S}$ are tuning parameters, and $\hat{\Omega}^b = \text{diag}(\hat{\omega}_1^b, \ldots, \hat{\omega}_{p_n}^b)$ is a diagonal matrix of data-dependent penalty loadings for $b = D, Y$. We provide more detail about $\hat{\Omega}^b$ in Section A.

We maintain the following assumptions for Lasso and logistic Lasso regressions.

Assumption 8. (i) For $a = 0, 1$. Suppose

$$\mathbb{E}(Y|(D_i(a))|X_i, S_i = s) = \Psi_{i,n}^T \theta_{a,s}^{HD} + R^Y(a, s, X_i) \quad \text{and} \quad \mathbb{P}(D_i(a) = 1|X_i, S_i = s) = \lambda(\Psi_{i,n}^T \beta_{a,s}^{HD}) + R^D(a, s, X_i)$$

such that $\max_{a=0,1,s \in S} \max(||\hat{\theta}_{a,s}^{HD}||_0, ||\hat{\beta}_{a,s}^{HD}||_0) \leq h_n$.

(ii) Suppose $\sup_{a \in [n]} ||\Psi_{i,n}||_\infty \leq \zeta_n$ and $\sup_{h \in [p_n]} \mathbb{E}(||\Psi_{i,n,h}^q||_1) S_i = s < \infty$ for $q > 2$.

(iii) Suppose

$$\max_{a=0,1, b = D, Y, s \in S} \frac{1}{n_a(s)} \sum_{i \in I_a(s)} (R^b(a, s, X_i))^2 = O_p(h_n \log(p_n)/n),$$

$$\max_{a=0,1, b = D, Y, s \in S} \mathbb{E}((R^b(a, s, X_i))^2 | S_i = s) = O(h_n \log(p_n)/n),$$

and

$$\sup_{a=0,1, b = D, Y, s \in S, x \in X} |R^b(a, s, X)| = O(\sqrt{\frac{\zeta_n^2 h_n^2 \log(p_n)\log(p_n)}{n}}).$$
(iv) Suppose \( \frac{\log(p_n)\xi^2 h_n^2}{n} \to 0 \) and \( \frac{\log^2(p_n)\log^2(n)h_n^2}{n} \to 0 \).

(v) There exists a constant \( c \in (0, 0.5) \) such that
\[
c \leq \inf_{a=0,1,s \in S} \mathbb{P}(D_i(a) = 1 | S_i = s, X_i = x) \leq \sup_{a=0,1,s \in S} \mathbb{P}(D_i(a) = 1 | S_i = s, X_i = x) \leq 1 - c.
\]

(vi) Let \( \ell_n \) be a sequence that diverges to infinity. Then, there exist two constants \( \kappa_1 \) and \( \kappa_2 \) such that with probability approaching one,
\[
0 < \kappa_1 \leq \inf_{a=0,1,s \in S, ||v||_0 \leq h_n \ell_n} \frac{v^T \left( \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \Psi_{i,n} \Psi_{i,n}^T \right) v}{||v||_2^2} \leq \kappa_2 < \infty,
\]
and
\[
0 < \kappa_1 \leq \inf_{a=0,1,s \in S, ||v||_0 \leq h_n \ell_n} \frac{v^T \mathbb{E}(\Psi_{i,n} \Psi_{i,n}^T | S_i = s)v}{||v||_2^2} \leq \kappa_2 < \infty,
\]
where \( ||v||_0 \) denotes the number of nonzero components in \( v \).

(vii) For \( a = 0, 1 \), let \( \varrho_{n,a}(s) = c \sqrt{n_a(s)}F_N^{-1}(1 - 0.1/(\log(n_a(s))4p_n)) \) where \( F_N(\cdot) \) is the standard normal CDF and \( c > 0 \) is a constant.

Assumption 8 is standard in the literature and we refer interested readers to Belloni, Chernozhukov, Fernández-Val, and Hansen (2017) for more discussion.

**Theorem 7.1.** Suppose Assumptions 1 and 8 hold. Then, \( \{\hat{\mu}^b(a,s,X_i)\}_{b=D,Y,a=0,1,s \in S} \) defined in (7.1) and \( \bar{\mu}^b(a,s,X) = \mu^b(a,s,X) \) satisfy Assumption 2. Then, all the results in Theorem 4.1(i) hold for \( \hat{\tau}^{HD} \). In addition, \( \hat{\tau}^{HD} \) achieves the minimum asymptotic variance characterized in Theorem 4.1(ii).

Based on the approximate sparsity, Lasso can consistently estimate the correct specification, which then cause the adjusted LATE estimator to achieve the minimum variance, due to Theorem 4.1(ii).
8 Simulations

8.1 Data Generating Processes

Three data generating processes (DGPs) are used to assess the finite sample performance of the estimation and inference methods introduced in the paper. Suppose that

\[ Y_i(1) = a_1 + \alpha(X_i, Z_i) + \varepsilon_{1,i} \]
\[ Y_i(0) = a_0 + \alpha(X_i, Z_i) + \varepsilon_{2,i} \]
\[ D_i(0) = \{ b_0 + \gamma(X_i, Z_i) > c_0\varepsilon_{3,i} \} \]
\[ D_i(1) = \begin{cases} 
1\{ b_1 + \gamma(X_i, Z_i) > c_1\varepsilon_{4,i} \} & \text{if } D_i(0) = 0 \\
1 & \text{otherwise} 
\end{cases} \]
\[ D_i = D_i(1) A_i + D_i(0)(1 - A_i) \]
\[ Y_i = Y_i(1) D_i + Y_i(0)(1 - D_i) \]

where \( \{X_i, Z_i\}_{i=1}^n, \alpha(\cdot, \cdot), \{a_i, b_i, c_i\}_{i=0,1} \) and \( \{\varepsilon_{j,i}\}_{j=4,i\in[n]} \) are separately specified as follows.

(i) Let \( Z_i \) be i.i.d. according to standardized Beta(2, 2), \( S_i = \sum_{j=1}^{4} 1\{Z_i \leq g_j\} \), and \( (g_1, g_2, g_3, g_4) = (-0.25\sqrt{20}, 0, 0.25\sqrt{20}, 0.5\sqrt{20}) \). \( X_i := (X_{1,i}, X_{2,i})^\top \), where \( X_{1,i} \) follows a uniform distribution on \([-2, 2]\), \( X_{2,i} := Z_i + N(0, 1) \), and \( X_{1,i} \) and \( X_{2,i} \) are independent. Further define

\[ \alpha(X_i, Z_i) = X_{1,i}^2 - X_{2,i}^2 + Z_i^2 \]
\[ \gamma(X_i, Z_i) = 0.5X_{1,i}^2 - 0.5X_{2,i}^2 - 0.5Z_i^2, \]
\[ a_1 = 2, a_0 = 1, b_1 = 1.3, b_0 = -1, c_1 = c_0 = 3, \text{ and } (\varepsilon_{1,i}, \varepsilon_{2,i}, \varepsilon_{3,i}, \varepsilon_{4,i})^\top \sim \text{i.i.d } N(0, \Sigma) \], where

\[
\Sigma = \begin{pmatrix}
1 & 0.5 & 0.5^2 & 0.5^3 \\
0.5 & 1 & 0.5 & 0.5^2 \\
0.5^2 & 0.5 & 1 & 0.5 \\
0.5^3 & 0.5^2 & 0.5 & 1
\end{pmatrix}
\]

(ii) Let \( Z \) be i.i.d. according to uniform\([-2, 2]\), \( S_i = \sum_{j=1}^{4} 1\{Z_i \leq g_j\} \), and \( (g_1, g_2, g_3, g_4) = (-1, 0, 1, 2) \). Let \( X_i := (X_{1,i}, X_{2,i})^\top \), where \( X_{1,i} \) follows a uniform distribution on \([-2, 2]\), \( X_{2,i} \) follows a standard normal distribution, and \( X_{1,i} \) and \( X_{2,i} \) are independent. Further define

\[ \alpha(X_i, Z_i) = X_{1,i} \cdot X_{2,i} + Z_i^2 + Z_i \cdot X_{1,i} \]
\[ \gamma(X_i, Z_i) = 0.5X_{1,i}^2 - 0.5X_{2,i}^2 - 0.5Z_i^2, \]
\[ a_1 = 2, a_0 = 1, b_1 = 1, b_0 = -1, c_1 = c_0 = 3, \text{ and } (\varepsilon_{1,i}, \varepsilon_{2,i}, \varepsilon_{3,i}, \varepsilon_{4,i})^\top \text{ are defined in DGP(i).} \]
(iii) Let $Z$ be i.i.d. according to standardized Beta(2, 2), $S_i = \sum_{j=1}^4 1\{Z_i \leq g_j\}$, and $(g_1, g_2, g_3, g_4) = (-0.25\sqrt{20}, 0, 0.25\sqrt{20}, 0.5\sqrt{20})$. Let $X_i := (X_{1,i}, \cdots, X_{20,i})\top$, where $X_i \overset{i.i.d.}{\sim} N(0_{20 \times 1}, \Omega)$ where $\Omega$ is the Toeplitz matrix

$$
\Omega = \begin{pmatrix}
1 & 0.5 & 0.5^2 & \cdots & 0.5^{19} \\
0.5 & 1 & 0.5 & \cdots & 0.5^{18} \\
0.5^2 & 0.5 & 1 & \cdots & 0.5^{17} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0.5^{19} & 0.5^{18} & 0.5^{17} & \cdots & 1
\end{pmatrix}
$$

Further define $\alpha(X_i, Z_i) = \sum_{k=1}^{20} X_{k,i} \beta_k + Z_i$, $\gamma(X_i, Z_i) = \sum_{k=1}^{20} X_{k,i}\top \gamma_k - Z_i$, with $\beta_k = \sqrt{6}/k^2$ and $\gamma_k = -2/k^2$. Moreover, $a_1 = 2, a_0 = 1, b_1 = 2, b_0 = -1, c_1 = c_0 = \sqrt{7}$, and $(\varepsilon_{1,i}, \varepsilon_{2,i}, \varepsilon_{3,i}, \varepsilon_{4,i})\top$ are defined in DGP(i).

For each data generating process, we consider the following four randomization schemes as in Zhang and Zheng (2020) with $\pi(s) = 0.5$ for $s \in S$:

(i) SRS: Treatment assignment is generated as in Example 1.

(ii) WEI: Treatment assignment is generated as in Example 2 with $f(x) = (1 - x)/2$.

(iii) BCD: Treatment assignment is generated as in Example 3 with $\lambda = 0.75$.

(iv) SBR: Treatment assignment is generated as in Example 4.

We compute the true LATE effect $\tau_0$ using Monte Carlo simulations, with sample size being 10000 and the number of Monte Carlo simulations being 1000. We test the true hypothesis

$$H_0 : \tau = \tau_0$$

by the test described in Theorem 4.1 in order to gauge the size of the test. The power is investigated by the hypothesis

$$H_0 : \tau = \tau_0 + 1.$$

All the tests are carried out at 5% level of significance.

### 8.2 Estimators for Comparison

For DGP(i)-(ii), we consider the following estimators.

(i) NA: the estimator with no adjustments, i.e., setting $\hat{\mu}^b(a, s, x) = \hat{\mu}^{b}(a, s, x) = 0$ for $b = D, Y$, $a = 0, 1$, all $s$ and all $x$. 

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(ii) 2SLS: the two-stage least squares (2SLS) estimator of \( \tau \). That is, run the following IV regression

\[
Y_i = \tau D_i + X_i^\top u + \sum_{k=1}^{4} v_k 1\{S_i = k\} + \text{err}_i
\]

where \( D_i \) is instrumented by \( A_i \). We use the IV heteroskedasticity-robust standard error for inference.

(iii) LP: the optimal linear estimator with \( \Psi_{i,s} = X_i \) and the pseudo true values being estimated by \( \hat{\theta}_{\text{LP}}^{a,s} \) and \( \hat{\beta}_{\text{LP}}^{a,s} \) defined in (5.5).

(iv) LG: the OLS-MLE estimator with \( \Psi_{i,s} = X_i \), and the pseudo true values being estimated by \( \hat{\theta}_{\text{OLS}}^{a,s} \) and \( \hat{\beta}_{\text{MLE}}^{a,s} \) defined in (5.10).

(v) F: the further efficiency improving estimator with \( \Psi_{i,s} = X_i \), and the pseudo true values being estimated by \( \hat{\theta}_{\text{LP}^*}^{a,s} \) and \( \hat{\beta}_{\text{LP}^*}^{a,s} \) defined in (5.14).

(vi) NP: the nonparametric estimator outlined in Section 6. For DGP(i), the regressors are

\[
\Psi_{i,n} = \left(1, X_{1,i}, X_{2,i}, X_{1,i} 1\{X_{1,i} > t_1\}, X_{2,i} 1\{X_{2,i} > t_2\}, X_{1,i} 1\{X_{1,i} > t_1\} X_{2,i} 1\{X_{2,i} > t_2\}\right)^\top
\]

where \( t_1 \) and \( t_2 \) are the sample medians of \( \{X_{1,i}\}_{i \in [n]} \) and \( \{X_{2,i}\}_{i \in [n]} \), respectively. For DGP(ii), the regressors are

\[
\Psi_{i,n} = \left(1, X_{1,i}, X_{2,i}, X_{1,i} 1\{X_{1,i} > t_1\}, X_{2,i} 1\{X_{2,i} > t_2\}, X_{1,i} X_{2,i}\right)^\top.
\]

The pseudo true values are estimated by \( \hat{\theta}_{\text{NP}}^{a,s} \) and \( \hat{\beta}_{\text{NP}}^{a,s} \) defined in (6.1).

For GDP(iii), we consider the estimator with no adjustments (NA), and the high-dimensional lasso estimators \( \hat{\theta}_{\text{HD}}^{a,s} \) and \( \hat{\beta}_{\text{HD}}^{a,s} \) defined in (7.1) with \( \Psi_{i,n} = X_i \). The implementation details are given in Section A.

### 8.3 Simulation Results

Table 1 presents the empirical sizes and powers of the true null \( H_0 : \tau = \tau_0 \) and false null \( H_0 : \tau = \tau_0 + 1 \), respectively, under DGPs (i)-(iii). Note that none of the working models is correctly specified. Consider DGP (i). When \( N = 200 \), only the NA estimator is slightly under-sized while all other estimators have sizes close to the nominal level 5%. This confirms that our estimation and inference procedures are robust to misspecification. In terms of power, the NA estimator has the lowest power, corroborating the belief that one should carry out the regression adjustment whenever covariates correlate with the potential outcomes. Power of the 2SLS estimator is also
Table 1: NA, 2SLS, LP, LG, F, NP, HD stand for the no-adjustment, 2SLS, optimal linear, OLS-MLE, further efficiency improving, nonparametric and high-dimensional Lasso estimators, respectively.

| Methods | $N = 200$ | $N = 400$ |
|---------|-----------|-----------|
|         | SRS   | WEI   | BCD   | SBR   | SRS   | WEI   | BCD   | SBR   |
| A.1: Size |        |        |       |       |        |        |       |       |
| NA      | 0.033 | 0.036 | 0.034 | 0.033 | 0.042 | 0.042 | 0.040 | 0.042 |
| 2SLS    | 0.036 | 0.039 | 0.036 | 0.035 | 0.043 | 0.045 | 0.041 | 0.043 |
| LP      | 0.045 | 0.048 | 0.046 | 0.042 | 0.049 | 0.046 | 0.046 | 0.048 |
| LG      | 0.045 | 0.048 | 0.046 | 0.041 | 0.049 | 0.046 | 0.045 | 0.048 |
| F       | 0.056 | 0.059 | 0.055 | 0.054 | 0.054 | 0.056 | 0.051 | 0.056 |
| NP      | 0.064 | 0.063 | 0.055 | 0.057 | 0.062 | 0.058 | 0.058 | 0.059 |
| A.2: Power |       |        |       |       |        |        |       |       |
| NA      | 0.082 | 0.084 | 0.084 | 0.083 | 0.181 | 0.169 | 0.155 | 0.153 |
| 2SLS    | 0.085 | 0.090 | 0.089 | 0.086 | 0.184 | 0.164 | 0.163 | 0.152 |
| LP      | 0.174 | 0.168 | 0.171 | 0.170 | 0.308 | 0.277 | 0.279 | 0.257 |
| LG      | 0.171 | 0.162 | 0.167 | 0.167 | 0.301 | 0.275 | 0.280 | 0.254 |
| F       | 0.242 | 0.246 | 0.245 | 0.248 | 0.450 | 0.396 | 0.402 | 0.406 |
| NP      | 0.337 | 0.343 | 0.333 | 0.336 | 0.553 | 0.577 | 0.563 | 0.544 |
| B.1: Size |        |        |       |       |        |        |       |       |
| NA      | 0.029 | 0.031 | 0.030 | 0.031 | 0.041 | 0.042 | 0.043 | 0.041 |
| 2SLS    | 0.032 | 0.031 | 0.032 | 0.033 | 0.041 | 0.043 | 0.044 | 0.042 |
| LP      | 0.042 | 0.040 | 0.044 | 0.038 | 0.046 | 0.047 | 0.046 | 0.048 |
| LG      | 0.042 | 0.039 | 0.043 | 0.037 | 0.046 | 0.047 | 0.046 | 0.048 |
| F       | 0.052 | 0.048 | 0.048 | 0.045 | 0.049 | 0.049 | 0.053 | 0.051 |
| NP      | 0.062 | 0.053 | 0.056 | 0.053 | 0.050 | 0.051 | 0.052 | 0.049 |
| B.2: Power |       |        |       |       |        |        |       |       |
| NA      | 0.192 | 0.189 | 0.189 | 0.190 | 0.311 | 0.320 | 0.321 | 0.316 |
| 2SLS    | 0.194 | 0.193 | 0.193 | 0.194 | 0.313 | 0.322 | 0.325 | 0.320 |
| LP      | 0.301 | 0.290 | 0.299 | 0.297 | 0.456 | 0.463 | 0.466 | 0.457 |
| LG      | 0.294 | 0.285 | 0.294 | 0.293 | 0.454 | 0.461 | 0.463 | 0.455 |
| F       | 0.341 | 0.343 | 0.346 | 0.345 | 0.513 | 0.522 | 0.520 | 0.515 |
| NP      | 0.430 | 0.430 | 0.436 | 0.433 | 0.667 | 0.674 | 0.664 | 0.663 |
| C.1: Size |        |        |       |       |        |        |       |       |
| NA      | 0.048 | 0.043 | 0.046 | 0.048 | 0.040 | 0.047 | 0.045 | 0.047 |
| HD      | 0.049 | 0.044 | 0.044 | 0.041 | 0.047 | 0.050 | 0.049 | 0.051 |
| C.2: Power |       |        |       |       |        |        |       |       |
| NA      | 0.172 | 0.170 | 0.170 | 0.177 | 0.224 | 0.237 | 0.235 | 0.239 |
| HD      | 0.426 | 0.436 | 0.445 | 0.439 | 0.739 | 0.745 | 0.748 | 0.746 |

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worse; this is probably due to its failure to incorporate the dependence arises under CARs. Powers of the LP, LG, F and NP estimators are much better. In particular, power of the LP estimator is slightly higher than that of the LG estimator even though a logistic model is less misspecified. This shows some robustness of the LP estimator. The power of the F estimator is considerably higher, which is consistent with our theory that the F estimator is weakly more efficient than the NA, LP and LG estimators. The NP estimator enjoys the highest power as a nonparametric model could approximate the true specification very well. When the sample size is increased to 400, all the sizes and powers of the estimators improve, and all the observations continue to hold. Moreover, the similar patterns exist in DGP (ii).

We now consider the DGP (iii). In this high-dimensional setting, only the NA and HD estimators are feasible. When \( N = 200 \), both estimators have the correct sizes but the HD estimator has considerably higher power. When \( N = 400 \), the sizes of these two estimators remain relatively unchanged, while their powers improve with a diverging gap.

8.4 Practical Recommendation

We suggest using estimation method F which is guaranteed to be weakly more efficient than simple 2SLS, LP, and LG. We can include linear, quadratic and interaction terms of original covariates as regressors \((\Psi_{i,s})\).

9 Empirical Application

Banking the unbanked is considered to be the first step toward broader financial inclusion – the focus of the World Bank’s Universal Financial Access 2020 initiative.\(^4\) In a field experiment with a CAR design, Dupas et al. (2018) examined the impact of expanding access to basic saving accounts for rural households living in three countries: Uganda, Malawi, and Chile. In particular, apart from the intent-to-treat effects for the whole sample, they also studied the local average treatment effects for the households who actively used the accounts. This section presents an application of our regression adjusted estimators to the same dataset to examine the LATEs of opening bank accounts on savings – a central outcome of interest in their study.

We focus on the experiment conducted in Uganda. The sample consists of 2,160 households who were randomized with a CAR design. Specifically, within each of 41 stratum formed by gender, occupation, and bank branch, half of households were randomly allocated to the treatment group, the other half to the control one. Households in the treatment group were then offered a voucher to open bank accounts with no financial costs. However, not every treated household ever opened and used the saving accounts for deposit. In fact, among the treated households, only 41.87% of them opened the accounts and made at least one deposit within 2 years. Subject compliance is

\(^4\)https://www.worldbank.org/en/topic/financialinclusion/brief/achieving-universal-financial-access-by-2020
therefore imperfect in this experiment.

The randomization design apparently satisfies statements (i), (ii), and (iii) in Assumption 1. The target fraction of treated households is 1/2. Because \( \sup_{s \in S} \left\{ \frac{D_n(s)}{\text{D}(s)} \right\} \approx 0.056 \), it is plausible to claim that Assumption 1(iv) is also satisfied. Since households in the control group need to pay for the fees of opening accounts while the treated ones bear no financial costs, no-defiers statement in Assumption 1(v) holds plausibly in this case.

One of the key analyses in Dupas et al. (2018) is to estimate the treatment effects on savings for active users – households who actually opened the accounts and made at least one deposit within 2 years. We follow their footprints to estimate the same LATEs at savings balance. Specifically, for each item in the savings balance, we estimate the LATEs on savings for active users by the methods “NA”, “2SLS”, “LP”, “LG”, “F”, and “NP”. To maintain comparability, for each outcome variable, we keep \( X_i \) similar to those used in Dupas et al. (2018) for all the adjusted estimators.

Table 2 presents the LATE estimates and their standard errors (in parentheses) estimated by these methods. These results lead to four observations. First, consistent with the theoretical and simulation results, the standard errors for the LATE estimates with regression adjustments are lower than those without adjustments. This observation holds for all the outcome variables and all the regression adjustment methods. Over nice outcome variables, the standard errors estimated by regression adjustments are on average around 7% lower than those without adjustment. In particular, when the outcome variable is total informal savings, the standard errors obtained via the further improvement adjustment – “F” method is about 14.9% lower than those without adjustment. This means that regression adjustments, just with the same two covariates used in Dupas et al. (2018), can achieve sizable efficiency gains in estimating the LATEs.

Second, the standard errors for the regression-adjusted LATE estimates are mostly lower than those obtained by the usual 2SLS procedure. Especially, when the outcome variable is savings in friends/family, the standard errors estimated by the optimal linear adjustment – “LP” method is around 6.9% lower than those obtained by the two-stage least square. This means that, compared with our regression-adjusted methods, the two-stage least square is less efficient to estimate the LATEs under CAR.

Third, the standard errors for the LATE estimates with regression adjustments are similar in size. This implies that all the regression adjustments achieve close efficiency gain in this case.

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5 Savings balance includes savings in formal financial intuitions, mobile money, cash at home or in secret place, savings in ROSCA/VSLA, savings with friends/family, other cash savings, total formal savings, total informal savings, and total savings (See Dupas et al. (2018) for details). Our analysis uses these variables obtained from the first follow-up survey.

6 The description of these estimators is similar to that in Section 8. Except for savings in formal financial institutions, mobile money, and total formal savings, \( X_i \) includes baseline value for outcome of interest and dummy for missing observations. For savings in formal financial institutions, mobile money, and total formal savings, since their baseline values are all zero, we set \( X_i \) as baseline value of total savings and dummy for missing observations. For the nonparametric adjustment “NP”, we choose the bases \{1, \( X_{i1} \), \( X_{i1} - q_{x1,0.3} \) \} for \( X_{i1} > q_{x1,0.3} \), \( X_{i1} - q_{x1,0.5} \) \} for \( X_{i1} > q_{x1,0.5} \), \( X_{i1} - q_{x1,0.8} \) \} for \( X_{i1} > q_{x1,0.8} \) \} where \( X_{i1} \) denotes the continuous \( X \) variable. \( q_{x1,0.3} \), \( q_{x1,0.5} \), and \( q_{x1,0.8} \) are 30%th, 50%th, and 80%th quantiles of \( \{ X_{i1} \} \in [n] \).
Finally, as in Dupas et al. (2018), for the households who actively use bank accounts, we find that reducing the cost of opening a bank account can significantly increase their savings in formal institutions. We also observe the evidence of crowd-out – mainly moving cash from saving at home to saving in bank.

Table 2: Impacts on Saving Stocks in 2010 US Dollars

| Y          | n     | NA     | 2SLS   | LP     | LG     | F      | NP     |
|------------|-------|--------|--------|--------|--------|--------|--------|
| Formal fin. inst. | 1968  | 20.558 | 20.824 | 22.192 | 22.070 | 23.033 | 22.367 |
|            |       | (3.067) | (3.042) | (3.028) | (3.018) | (3.057) | (3.024) |
| Mobile     | 1972  | -0.208 | -0.197 | -0.351 | -0.349 | -0.363 | -0.374 |
|            |       | (0.223) | (0.223) | (0.217) | (0.216) | (0.215) | (0.215) |
| Total formal | 1966  | 20.399 | 20.678 | 21.894 | 21.758 | 22.737 | 22.228 |
|            |       | (3.089) | (3.064) | (3.053) | (3.041) | (3.084) | (3.070) |
| Cash at home | 1971  | -10.826 | -8.785 | -9.285 | -9.153 | -8.978 | -8.135 |
|            |       | (5.003) | (4.559) | (4.506) | (4.443) | (4.482) | (4.390) |
| ROSCA/VSLA | 1975  | -1.933 | -2.575 | -1.714 | -1.769 | -1.609 | -1.769 |
|            |       | (1.971) | (1.866) | (1.784) | (1.841) | (1.823) | (1.841) |
| Friends/family | 1974  | -3.621 | -3.472 | -2.436 | -2.539 | -2.525 | -2.540 |
|            |       | (2.040) | (1.997) | (1.860) | (1.946) | (1.873) | (1.946) |
| Other cash | 1965  | 0.027  | 0.040  | 0.033  | 0.033  | 0.033  | 0.033  |
|            |       | (0.046) | (0.045) | (0.044) | (0.044) | (0.044) | (0.044) |
| Total informal savings | 1960  | -17.643 | -15.714 | -16.894 | -16.805 | -16.177 | -17.973 |
|            |       | (6.200) | (5.454) | (5.329) | (5.298) | (5.276) | (5.352) |
| Total savings | 1952  | 2.787  | 5.355  | 6.061  | 6.023  | 6.820  | 5.639  |
|            |       | (7.290) | (6.502) | (6.340) | (6.301) | (6.292) | (6.283) |

Notes: The table reports the LATE estimates of opening bank accounts on saving stocks. NA, 2SLS, LP, LG, F, NP, stand for the no-adjustment, 2SLS, optimal linear, OLS-MLE, further efficiency improving, nonparametric estimators, respectively. n is the number of households. Standard errors are in parentheses.
10 Conclusion

In this paper, we address the problem of estimation and inference of local average treatment effects under covariate-adaptive randomizations using regression adjustments. We first propose a regression-adjusted LATE estimator under CARs. We then derive its limit theory and show that, even under the potential misspecification of adjustments, our estimator maintains its consistency and its inference method still achieves an asymptotic size equal to the nominal level under the null. When the adjustment is correctly specified, our LATE estimator achieves minimum asymptotic variance. We also examine the efficiency gains brought by regression adjustments in parametric (both linear and nonlinear), nonparametric, and high-dimensional forms. When the adjustment is parametrically misspecified, we construct a new estimator by combining the linear and nonlinear adjustments. This new estimator is shown to be weakly more efficient than all the parametrically adjusted estimators, including the one without any adjustment. Simulations and empirical application confirm efficiency gains that materialize from regression adjustments relative to both the estimator without adjustment and the standard two-stage least squares estimator.

A Implementation Details for Sieve and Lasso Regressions

Sieve regressions. We provide more details on the sieve basis. Recall \( \Psi_{i,n} \equiv (b_{1,n}(x), \cdots, b_{h,n}(x))^T \), where \( \{b_{h,n}(\cdot)\}_{h \in [h_n]} \) are \( h_n \) basis functions of a linear sieve space, denoted as \( B \). Given that all the elements of vector \( X \) are continuously distributed, the sieve space \( B \) can be constructed as follows.

1. For each element \( X(l) \) of \( X \), \( l = 1, \cdots, d_x \), where \( d_x \) denotes the dimension of vector \( X \), let \( B_l \) be the univariate sieve space of dimension \( J_n \). One example of \( B_l \) is the linear span of the \( J_n \) dimensional polynomials given by

\[
B_l = \left\{ \sum_{k=0}^{J_n} \alpha_k x^k, x \in \text{Supp}(X(l)), \alpha_k \in \mathbb{R} \right\};
\]

Another example is the linear span of \( r \)-order splines with \( J_n \) nodes given by

\[
B_l = \left\{ \sum_{k=0}^{r-1} \alpha_k x^k + \sum_{j=1}^{J_n} b_j [\max(x-t_j,0)]^{r-1}, x \in \text{Supp}(X(l)), \alpha_k, b_j \in \mathbb{R} \right\},
\]

where the grid \(-\infty = t_0 \leq t_1 \leq \cdots \leq t_{J_n} \leq t_{J_n+1} = \infty \) partitions \( \text{Supp}(X(l)) \) into \( J_n + 1 \) subsets \( I_j = [t_j, t_{j+1}] \cap \text{Supp}(X(l)), j = 1, \cdots, J_n - 1, I_0 = (t_0, t_1) \cap \text{Supp}(X(l)), \) and \( I_{J_n} = (t_{J_n}, t_{J_n+1}) \cap \text{Supp}(X(l)) \).

2. Let \( B \) be the tensor product of \( \{B_l\}_{l=1}^{d_x} \), which is defined as a linear space spanned by the functions \( \prod_{l=1}^{d_x} g_l \), where \( g_l \in B_l \). The dimension of \( B \) is then \( K \equiv d_x J_n \) if \( B_l \) is spanned by
We refer interested readers to Hirano, Imbens, and Ridder (2003) and Chen (2007) for more details about the implementation of sieve estimation. Given the sieve basis, we can compute the \( \{\hat{\mu}(a, s, X_i)\}_{a=0,1,b=D,Y,s\in S} \) following (6.1).

**Lasso regressions.** We follow the estimation procedure and the choice of tuning parameter proposed by Belloni et al. (2017). We provide details below for completeness. Recall \( \theta_{n,a}(s) = c\sqrt{n_a(s)F_N^{-1}(1-1/(p_n \log(n_a(s))))} \). We set \( c = 1.1 \) following Belloni et al. (2017). We then implement the following algorithm to estimate \( \hat{\theta}_{a,s}^{HD} \) and \( \hat{\beta}_{a,s}^{HD} \):

(i) Let \( \hat{\sigma}_h^{Y,(0)} = \frac{1}{n_a(s)} \sum_{i \in I_a(s)} (Y_i - \hat{Y}_{a,s})^2 \Psi_{i,n}^2 \) and \( \hat{\sigma}_h^{D,(0)} = \frac{1}{n_a(s)} \sum_{i \in I_a(s)} (D_i - \hat{D}_{a,s})^2 \Psi_{i,n}^2 \) for \( h \in [p_n] \), where \( \hat{Y}_{a,s} = \frac{1}{n_a(s)} \sum_{i \in I_a(s)} Y_i \) and \( \hat{D}_{a,s} = \frac{1}{n_a(s)} \sum_{i \in I_a(s)} D_i \). Estimate

\[
\hat{\theta}_{a,s}^{HD,0} = \arg \min_t \frac{-1}{n_a(s)} \sum_{i \in I_a(s)} \left( Y_i - \Psi_{i,n}^T t \right)^2 + \frac{\theta_{n,a}(s)}{n_a(s)} \sum_{h \in [p_n]} \hat{\sigma}_h^{Y,(0)} |t_h|,
\]

\[
\hat{\beta}_{a,s}^{HD,0} = \arg \min_b \frac{-1}{n_a(s)} \sum_{i \in I_a(s)} \left( D_i \log(\lambda(\Psi_{i,n}^T b)) + (1 - D_i) \log(1 - \lambda(\Psi_{i,n}^T b)) \right)
\]

\[
+ \frac{\theta_{n,a}(s)}{n_a(s)} \sum_{h \in [p_n]} \hat{\sigma}_h^{D,(0)} |b_h|.
\]

(ii) For \( k = 1, \ldots, K \), obtain \( \hat{\sigma}_h^{Y,(k)} = \sqrt{\frac{1}{n} \sum_{i \in [n]} (\Psi_{i,n,h} \hat{\epsilon}_i^{Y,(k)})^2} \), where \( \hat{\epsilon}_i^{Y,(k)} = Y_i - \Psi_{i,n}^T \hat{\theta}^{HD,k-1}_{a,s} \) and \( \hat{\sigma}_h^{D,(k)} = \sqrt{\frac{1}{n} \sum_{i \in [n]} (\Psi_{i,n,h} \hat{\epsilon}_i^{D,(k)})^2} \), where \( \hat{\epsilon}_i^{D,(k)} = D_i - \lambda(\Psi_{i,n}^T \hat{\beta}^{HD,k-1}_{a,s}) \). Estimate

\[
\hat{\theta}_{a,s}^{HD,k} = \arg \min_t \frac{-1}{n_a(s)} \sum_{i \in I_a(s)} \left( Y_i - \Psi_{i,n}^T t \right)^2 + \frac{\theta_{n,a}(s)}{n_a(s)} \sum_{h \in [p_n]} \hat{\sigma}_h^{Y,(k-1)} |t_h|,
\]

\[
\hat{\beta}_{a,s}^{HD,k} = \arg \min_b \frac{-1}{n_a(s)} \sum_{i \in I_a(s)} \left( D_i \log(\lambda(\Psi_{i,n}^T b)) + (1 - D_i) \log(1 - \lambda(\Psi_{i,n}^T b)) \right)
\]

\[
+ \frac{\theta_{n,a}(s)}{n_a(s)} \sum_{h \in [p_n]} \hat{\sigma}_h^{D,(k-1)} |b_h|.
\]

(iii) Let \( \hat{\theta}_{a,s}^{HD} = \hat{\theta}_{a,s}^{HD,K} \) and \( \hat{\beta}_{a,s}^{HD} = \hat{\beta}_{a,s}^{HD,K} \).

**B Proof of Theorem 4.1**

Let

\[
G := \mathbb{E} \left[ (Y(1) - Y(0)) \left( D(1) - D(0) \right) \right],
\]

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\( H := \mathbb{E}[D(1) - D(0)] \),

\[
\hat{G} := \frac{1}{n} \sum_{i \in [n]} \left[ \frac{A_i(Y_i - \hat{\mu}^Y(1, S_i, X_i))}{\hat{\pi}(S_i)} - \frac{(1 - A_i)(Y_i - \hat{\mu}^Y(0, S_i, X_i))}{1 - \hat{\pi}(S_i)} + \hat{\mu}^Y(1, S_i, X_i) - \hat{\mu}^Y(0, S_i, X_i) \right],
\]

\[
\hat{H} := \frac{1}{n} \sum_{i \in [n]} \left[ \frac{A_i(D_i - \hat{\mu}^D(1, S_i, X_i))}{\hat{\pi}(S_i)} - \frac{(1 - A_i)(D_i - \hat{\mu}^D(0, S_i, X_i))}{1 - \hat{\pi}(S_i)} + \hat{\mu}^D(1, S_i, X_i) - \hat{\mu}^D(0, S_i, X_i) \right].
\]

Then, we have

\[
\sqrt{n}(\hat{\tau} - \tau) = \sqrt{n} \left( \frac{\hat{G}}{\hat{H}} - \frac{G}{H} \right) = \frac{1}{H} \sqrt{n}(\hat{G} - G) - \frac{G}{HH} \sqrt{n}(\hat{H} - H) = \frac{1}{H} \left[ \sqrt{n}(\hat{G} - G) - \tau \sqrt{n}(\hat{H} - H) \right]. \tag{B.1}
\]

Next, we divide the proof into four steps. In the first step, we obtain the linear expansion of \( \sqrt{n}(\hat{G} - G) \). Based on the same argument, we can obtain the linear expansion of \( \sqrt{n}(\hat{H} - H) \). In the second step, we obtain the linear expansion of \( \sqrt{n}(\hat{\tau} - \tau) \) and then prove the asymptotic normality. In the third step, we show the consistency of \( \hat{\tau} \). In the fourth step, we show that when \( \bar{a}(a, s, x) = \mu(a, s, x) \) for all \((a, s, x) \in \{0, 1\} \times \text{Supp}(SX)\), the asymptotic variance \( \sigma^2 \) achieves the minimum.

**Step 1.** We have

\[
\sqrt{n}(\hat{G} - G) = \frac{1}{\sqrt{n}} \sum_{i \in [n]} \left[ \frac{A_i(Y_i - \hat{\mu}^Y(1, S_i, X_i))}{\hat{\pi}(S_i)} - \frac{(1 - A_i)(Y_i - \hat{\mu}^Y(0, S_i, X_i))}{1 - \hat{\pi}(S_i)} + \hat{\mu}^Y(1, S_i, X_i) - \hat{\mu}^Y(0, S_i, X_i) \right] - \sqrt{n}G
\]

\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ \hat{\mu}^Y(1, S_i, X_i) - \frac{A_i \hat{\mu}^Y(1, S_i, X_i)}{\hat{\pi}(S_i)} \right] + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ \frac{(1 - A_i) \hat{\mu}^Y(0, S_i, X_i)}{1 - \hat{\pi}(S_i)} - \hat{\mu}^Y(0, S_i, X_i) \right]
\]

\[
+ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{A_i Y_i}{\hat{\pi}(S_i)} - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{(1 - A_i) Y_i}{1 - \hat{\pi}(S_i)} - \sqrt{n}G
\]

\[
= R_{n,1} + R_{n,2} + R_{n,3},
\]

where

\[
R_{n,1} := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ \hat{\mu}^Y(1, S_i, X_i) - \frac{A_i \hat{\mu}^Y(1, S_i, X_i)}{\hat{\pi}(S_i)} \right],
\]

\[
R_{n,2} := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ \frac{(1 - A_i) \hat{\mu}^Y(0, S_i, X_i)}{1 - \hat{\pi}(S_i)} - \hat{\mu}^Y(0, S_i, X_i) \right],
\]

\[
R_{n,3} := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ \frac{A_i Y_i}{\hat{\pi}(S_i)} - \frac{(1 - A_i) Y_i}{1 - \hat{\pi}(S_i)} \right] - \sqrt{n}G.
\]
\[ R_{n,2} := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ \frac{(1 - A_i)\mu^Y(0, S_i, X_i)}{1 - \pi(S_i)} - \hat{\mu}^Y(0, S_i, X_i) \right], \]
\[ R_{n,3} := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i Y_i \frac{1}{\pi(S_i)} - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{(1 - A_i)Y_i}{1 - \pi(S_i)} - \sqrt{n} G. \]

Lemma J.1 shows that
\[ R_{n,1} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( 1 - \frac{1}{\pi(S_i)} \right) A_i \hat{\mu}^Y(1, S_i, X_i) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (1 - A_i) \hat{\mu}^Y(1, S_i, X_i) + o_p(1), \]
\[ R_{n,2} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \frac{1}{1 - \pi(S_i)} - 1 \right) (1 - A_i) \hat{\mu}^Y(0, S_i, X_i) - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i \hat{\mu}^Y(0, S_i, X_i) + o_p(1), \]
\[ R_{n,3} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\pi(S_i)} \hat{W}_i A_i - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1 - A_i}{1 - \pi(S_i)} \hat{Z}_i + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \mathbb{E}[W_i - Z_i|S_i] - \mathbb{E}[W_i - Z_i] \right). \]

This implies
\[
\sqrt{n}(\hat{G} - G) = \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ \left( 1 - \frac{1}{\pi(S_i)} \right) \hat{\mu}^Y(1, S_i, X_i) - \frac{\hat{W}_i}{\pi(S_i)} \right] A_i \right. \\
+ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ \left( \frac{1}{1 - \pi(S_i)} - 1 \right) \hat{\mu}^Y(0, S_i, X_i) + \frac{\hat{Z}_i}{1 - \pi(S_i)} \right] (1 - A_i) \right\} + o_p(1).
\]
(B.2)

Similarly, we can show that
\[
\sqrt{n}(\hat{H} - H) = \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ \left( 1 - \frac{1}{\pi(S_i)} \right) \hat{\mu}^D(1, S_i, X_i) - \frac{\hat{D}_i(1)}{\pi(S_i)} \right] A_i \right. \\
+ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ \left( \frac{1}{1 - \pi(S_i)} - 1 \right) \hat{\mu}^D(0, S_i, X_i) + \frac{\hat{D}_i(0)}{1 - \pi(S_i)} \right] (1 - A_i) \right\} + o_p(1),
\]
(B.3)

where \( \hat{D}_i(a) = D_i(a) - \mathbb{E}(D_i(a)|S_i) \) for \( a = 0, 1 \) and \( \hat{\mu}^D(0, s, X_i) = \pi^D(0, s, X_i) - \mathbb{E}(\pi^D(0, S_i, X_i)|S_i = s) \).

Combining (B.1), (B.2), and (B.3), we obtain the linear expansion for \( \hat{\tau} \) as
\[
\sqrt{n}(\hat{\tau} - \tau) = \frac{1}{H} \left[ \sqrt{n}(\hat{G} - G) - \tau \sqrt{n}(\hat{H} - H) \right]
\]
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shows. This further implies \( \hat{\mu}(1, S_i, X_i) \rightarrow \mu \), and the three terms are asymptotically independent, where

\[
\Xi_1(D_i, S_i) = \left[ \left( 1 - \frac{1}{\pi(S_i)} \right) \hat{\mu}^Y(1, S_i, X_i) - \hat{\mu}^Y(0, S_i, X_i) + \frac{\tilde{W}_i}{\pi(S_i)} \right] \\
- \tau \left[ \left( 1 - \frac{1}{\pi(S_i)} \right) \hat{\mu}^D(1, S_i, X_i) - \hat{\mu}^D(0, S_i, X_i) + \frac{\hat{D}_i(1)}{\pi(S_i)} \right],
\]

\[
\Xi_0(D_i, S_i) = \left[ \left( \frac{1}{1 - \pi(S_i)} - 1 \right) \hat{\mu}^Y(0, S_i, X_i) + \hat{\mu}^Y(1, S_i, X_i) - \frac{\hat{Z}_i}{1 - \pi(S_i)} \right] \\
- \tau \left[ \left( \frac{1}{1 - \pi(S_i)} - 1 \right) \hat{\mu}^D(0, S_i, X_i) + \hat{\mu}^D(1, S_i, X_i) - \frac{\hat{D}_i(0)}{1 - \pi(S_i)} \right],
\]

\[
\Xi_2(S_i) = \left( \mathbb{E}[W_i - Z_i|S_i] - \mathbb{E}[W_i - Z_i] \right) - \tau \left( \mathbb{E}[D_i(1) - D_i(0)|S_i] - \mathbb{E}[D_i(1) - D_i(0)] \right).
\]

**Step 2.** Lemma J.2 implies that

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Xi_1(D_i, S_i) A_i \rightarrow N(0, \sigma^2_1), \quad \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Xi_0(D_i, S_i)(1 - A_i) \rightarrow N(0, \sigma^2_0), \quad \text{and} \quad \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Xi_2(S_i) \rightarrow N(0, \sigma^2_2),
\]

and the three terms are asymptotically independent, where

\[
\sigma^2_1 = \mathbb{E}\pi(S_i)\Xi^2_1(D_i, S_i), \quad \sigma^2_0 = \mathbb{E}(1 - \pi(S_i))\Xi^2_0(D_i, S_i), \quad \text{and} \quad \sigma^2_2 = \mathbb{E}\Xi^2_2(S_i).
\]

This further implies \( \hat{H} \rightarrow^{p} H \) and

\[
\sqrt{n}(\hat{\tau} - \tau) \rightarrow N \left( 0, \frac{\sigma^2_1 + \sigma^2_0 + \sigma^2_2}{H^2} \right),
\]

**Step 3.** We aim to show the consistency of \( \hat{\sigma}^2 \). First note that

\[
\frac{1}{n} \sum_{i=1}^{n} \Xi_{H,i} = \hat{H} \rightarrow^{p} H = \mathbb{E}(D_i(1) - D_i(0)).
\]

In addition, Lemma J.3 shows.

\[
\frac{1}{n} \sum_{i=1}^{n} \hat{\Xi}^2_1(D_i, S_i) \rightarrow^{p} \sigma^2_1, \quad \frac{1}{n} \sum_{i=1}^{n} \hat{\Xi}^2_0(D_i, S_i) \rightarrow^{p} \sigma^2_0, \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^{n} \hat{\Xi}^2_2(D_i, S_i) \rightarrow^{p} \sigma^2_2.
\]
This implies $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$.

**Step 4.** Let

$$
\nu^Y(a, S_i, X_i) = E(Y_i(D_i(a))|S_i, X_i) - E(Y_i(D_i(a))|S_i)
$$

and

$$
\nu^D(a, S_i, X_i) = E(D_i(a)|S_i, X_i) - E(D_i(a)|S_i).
$$

(B.4)

Also recall that $W_i = Y_i(D_i(1))$, $Z_i = Y_i(D_i(0))$, $\mu^Y(a, S_i, X_i) = E(Y_i(D_i(a))|S_i, X_i)$. Then, we have

$$
E\pi(S_i)\Xi^2_1(D_i, S_i) = E\left\{ \frac{(W_i - \mu^Y(1, S_i, X_i) - \tau(D_i(1) - \mu^D(1, S_i, X_i)))^2}{\pi(S_i)} \right\}
$$

$$
+ E\left\{ \pi(S_i) \left[ \frac{\nu^Y(1, S_i, X_i) - \tilde{\mu}^Y(1, S_i, X_i) - \tau(\nu^D(1, S_i, X_i) - \tilde{\mu}^D(1, S_i, X_i))}{\pi(S_i)} \right] ^2 \right\}.
$$

Similarly, we have

$$
E(1 - \pi(S_i))\Xi^2_0(D_i, S_i) = E\left\{ \frac{(Z_i - \mu^Y(0, S_i, X_i) - \tau(D_i(0) - \mu^D(0, S_i, X_i)))^2}{1 - \pi(S_i)} \right\}
$$

$$
+ E\left\{ (1 - \pi(S_i)) \left[ \frac{\nu^Y(0, S_i, X_i) - \tilde{\mu}^Y(0, S_i, X_i) - \tau(\nu^D(0, S_i, X_i) - \tilde{\mu}^D(0, S_i, X_i))}{1 - \pi(S_i)} \right] ^2 \right\}.
$$

Last, we have

$$
E\Xi^2_2(S_i) = E(\mu^Y(1, S_i, X_i) - \mu^Y(0, S_i, X_i) - \tau(\mu^D(1, S_i, X_i) - \mu^D(0, S_i, X_i)))^2
$$

$$
- E(\nu^Y(1, S_i, X_i) - \nu^Y(0, S_i, X_i) - \tau(\nu^D(1, S_i, X_i) - \nu^D(0, S_i, X_i)))^2
$$

Let

$$
\sigma_x^2 = (P(D_i(1) > D_i(0)))^{-2} \left\{ E \left[ \frac{(W_i - \mu^Y(1, S_i, X_i) - \tau(D_i(1) - \mu^D(1, S_i, X_i)))^2}{\pi(S_i)} \right] \right\}
$$

$$
+ E \left[ \frac{(Z_i - \mu^Y(0, S_i, X_i) - \tau(D_i(0) - \mu^D(0, S_i, X_i)))^2}{1 - \pi(S_i)} \right] \right\}
$$

$$
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$$
+ \mathbb{E}(\mu^Y(1, S_i, X_i) - \mu^Y(0, S_i, X_i) - \tau(\mu^D(1, S_i, X_i) - \mu^D(0, S_i, X_i)))^2 \right) \right),

where does not depend on the working models \( \hat{\mu}^b(a, S_i, X_i) \) for \( a = 0, 1 \) and \( b = D, Y \). Then, we have

\[
\begin{align*}
(\mathbb{P}(D_i(1) > D_i(0)))^2 & (\sigma^2 - \sigma^{2}_{\ast}) \\
= & \mathbb{E} \left\{ \pi(S_i) \left[ \frac{\nu^Y(1, S_i, X_i) - \hat{\mu}^Y(1, S_i, X_i) - \tau(\nu^D(1, S_i, X_i) - \hat{\mu}^D(1, S_i, X_i))}{\pi(S_i)} \right. \\
& + \left. \hat{\mu}^Y(1, S_i, X_i) - \hat{\mu}^Y(0, S_i, X_i) - \tau(\hat{\mu}^D(1, S_i, X_i) - \hat{\mu}^D(0, S_i, X_i)) \right] \right\}^2 \\
& \mathbb{E} \left\{ (1 - \pi(S_i)) \left[ \frac{\nu^Y(0, S_i, X_i) - \hat{\mu}^Y(0, S_i, X_i) - \tau(\nu^D(0, S_i, X_i) - \hat{\mu}^D(0, S_i, X_i))}{1 - \pi(S_i)} \right. \\
& - \left. \hat{\mu}^Y(0, S_i, X_i) - \hat{\mu}^Y(0, S_i, X_i) - \tau(\hat{\mu}^D(0, S_i, X_i) - \hat{\mu}^D(0, S_i, X_i)) \right] \right\}^2 \\
= & \mathbb{E} \left[ \pi(S_i) \left( \frac{A_1(S_i, X_i)}{\pi(S_i)} + B_i \right)^2 + (1 - \pi(S_i)) \left( \frac{A_0(S_i, X_i)}{1 - \pi(S_i)} - B_i \right)^2 - (A_1(S_i, X_i) - A_0(S_i, X_i) + B_i)^2 \right] \\
= & \mathbb{E} \left( \sqrt{\frac{\pi(S_i)}{1 - \pi(S_i)}} A_1(S_i, X_i) + \sqrt{\frac{1 - \pi(S_i)}{\pi(S_i)}} A_0(S_i, X_i) \right)^2 \geq 0,
\end{align*}
\]

where

\[
A_a(S_i, X_i) = \nu^Y(a, S_i, X_i) - \hat{\mu}^Y(a, S_i, X_i) - \tau(\nu^D(a, S_i, X_i) - \hat{\mu}^D(a, S_i, X_i)) \quad a = 0, 1,
\]

\[
B_i = \hat{\mu}^Y(1, S_i, X_i) - \hat{\mu}^Y(0, S_i, X_i) - \tau(\hat{\mu}^D(1, S_i, X_i) - \hat{\mu}^D(0, S_i, X_i)).
\]

The equal sign holds when \( A_a(S_i, X_i) = 0 \) for \( i = 1, \ldots, n \) and \( a = 0, 1 \), which can be achieved when \( \pi^b(a, s, x) = \hat{\mu}^b(a, s, x) \) for all \( (a, b, s, x) \in \{0, 1\} \times \{D, Y\} \times \mathcal{X} \).

### C Proof of Theorem 5.1

The proof is divided into two steps. In the first step, we show Assumption 2(i). In the second step, we establish Assumptions 2(ii) and 2(iii).

**Step 1.** Recall

\[
\Delta^Y(a, s, X_i) = \hat{\mu}^Y(a, s, X_i) - \bar{\mu}^Y(a, s, X_i) = \Lambda^Y_{a, s}(X_i, \hat{\theta}_{a, s}) - \Lambda^Y_{a, s}(X_i, \theta_{a, s}),
\]

and \( \{X^s_i\}_{i \in [n]} \) is generated independently from the distribution of \( X_i \) given \( S_i = s \), and so is independent of \( \{A_i, S_i\}_{i \in [n]} \). Let \( M_{a, s}(\theta_1, \theta_2) = \mathbb{E}[\Lambda_{a, s}(X_i, \theta_1) - \Lambda_{a, s}(X_i, \theta_2)|S_i = s] = \mathbb{E}[\Lambda_{a, s}(X^s_i, \theta_1) - \Lambda_{a, s}(X^s_i, \theta_2)|S_i = s] \).
We have
\[
\left| \sum_{i \in I_1(s)} \Delta Y(a, s, X_i) \right| \leq \sum_{i \in I_0(s)} \Delta Y(a, s, X_i) \leq o_p(n^{-1/2}).
\]

To see the last equality, we note that, for any \( \varepsilon > 0 \), with probability approaching one (w.p.a.1), we have
\[
\max_{s \in S} \| \hat{\theta}_{a,s} - \theta_{a,s} \| \leq \varepsilon.
\]

Therefore, on the event \( A_n(\varepsilon) \equiv \{ \max_{s \in S} \| \hat{\theta}_{a,s} - \theta_{a,s} \| \leq \varepsilon, \min_{s \in S} n_1(s) \geq \varepsilon n \} \) we have
\[
\left| \sum_{i \in I_1(s)} \Delta Y(a, s, X_i) - M_{a,s}(\theta_{a,s}, \hat{\theta}_{a,s}) \right| = o_p(n^{-1/2}).
\]

By Assumption 3, \( \mathcal{F} \) is a VC-class with a fixed VC index and envelope \( L_i \). In addition,
\[
\sup_{f \in \mathcal{F}} \mathbb{E} f^2 \leq \mathbb{E} L_i^2(\theta_1 - \theta_2)^2 \leq C \varepsilon^2.
\]

Therefore, for any \( \delta > 0 \) we have
\[
\mathbb{P} \left( \left| \sum_{i \in I_1(s)} \Delta Y(a, s, X_i) - M_{a,s}(\theta_{a,s}, \hat{\theta}_{a,s}) \right| \geq \delta n^{-1/2} \right) \\
\leq \mathbb{P} \left( \left| \sum_{i \in I_1(s)} \Delta Y(a, s, X_i) - M_{a,s}(\theta_{a,s}, \hat{\theta}_{a,s}) \right| \geq \delta n^{-1/2}, A_n(\varepsilon) \right) + \mathbb{P}(A_n^c(\varepsilon))
\]

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\[
\begin{align*}
\leq & \ E \left[ \mathbb{P} \left( \left| \frac{\sum_{i \in I_1(s)} [X_i - M_{a,s}(\theta_{a,s}, \hat{\theta}_{a,s})]}{n_1(s)} \right| \geq \delta n^{-1/2}, A_n(\varepsilon) \right| \{A_i, S_i\}_{i \in [n]} \right] \\
& + \mathbb{P}(A_n^c(\varepsilon)) \\
\leq & \sum_{s \in S} E \left[ \mathbb{P} \left( \left| \frac{\sum_{i \in I_1(s)} [X_i - M_{a,s}(\theta_{a,s}, \hat{\theta}_{a,s})]}{n_1(s)} \right| \geq \delta n^{-1/2}, \{A_i, S_i\}_{i \in [n]} \right| 1\{n_1(s) \geq n\varepsilon\} \right] + \mathbb{P}(A_n^c(\varepsilon)) \\
\leq & \sum_{s \in S} E \left[ n^{1/2} \mathbb{E} \left( \left| \frac{\sum_{i \in I_1(s)} [X_i - M_{a,s}(\theta_{a,s}, \hat{\theta}_{a,s})]}{n_1(s)} \right| \geq \delta \right) 1\{n_1(s) \geq n\varepsilon\} \right] + \mathbb{P}(A_n^c(\varepsilon)).
\end{align*}
\]

By Chernozhukov, Chetverikov, and Kato (2014, Corollary 5.1),

\[
\begin{align*}
n^{1/2}E \left[ \left| \frac{\sum_{i \in I_1(s)} [X_i - M_{a,s}(\theta_{a,s}, \hat{\theta}_{a,s})]}{n_1(s)} \right| \geq \delta \right] 1\{n_1(s) \geq n\varepsilon\} \\
& \leq C \left( \sqrt{\frac{n}{n_1(s)}} \varepsilon^2 + n^{1/2} \varepsilon^{1/q-1}(s) \right) 1\{n_1(s) \geq n\varepsilon\} \\
& \leq C(\varepsilon^{1/2} + n^{1/q-1/2} \varepsilon^{1/q-1}).
\end{align*}
\]

Therefore,

\[
\begin{align*}
E \left\{ \left| \frac{\sum_{i \in I_1(s)} [X_i - M_{a,s}(\theta_{a,s}, \hat{\theta}_{a,s})]}{n_1(s)} \right| \geq \delta \right\} \leq C \varepsilon^{1/2} + n^{1/q-1/2} \varepsilon^{1/q-1} / \delta.
\end{align*}
\]

By letting \(n \to \infty\) followed by \(\varepsilon \to 0\), we have

\[
\lim_{n \to \infty} \mathbb{P} \left( \left| \frac{\sum_{i \in I_1(s)} [X_i - M_{a,s}(\theta_{a,s}, \hat{\theta}_{a,s})]}{n_1(s)} \right| \geq \delta n^{-1/2} \right) = 0,
\]

Therefore,

\[
\begin{align*}
\left| \frac{\sum_{i \in I_1(s)} [X_i - M_{a,s}(\theta_{a,s}, \hat{\theta}_{a,s})]}{n_1(s)} \right| = o_p(n^{-1/2}).
\end{align*}
\]

For the same reason, we have

\[
\begin{align*}
\left| \frac{\sum_{i \in I_0(s)} [X_i - M_{a,s}(\theta_{a,s}, \hat{\theta}_{a,s})]}{n_0(s)} \right| = o_p(n^{-1/2}),
\end{align*}
\]

and (C.1) holds.
Specifically, we have

\[
\frac{1}{n} \sum_{i=1}^{n} \Delta^Y(a, s, X_i) = \frac{1}{n} \sum_{i=1}^{n} \sum_{s \in S} 1\{S_i = s\} (\Lambda^Y_{a,s}(X_i, \hat{\theta}_{a,s}) - \Lambda^Y_{a,s}(X_i, \theta_{a,s}))
\]

\[
\leq \left( \frac{1}{n} \sum_{i=1}^{n} L_i \right) \max_{s \in S} |\hat{\theta}_{a,s} - \theta_{a,s}|_2 = o_p(1).
\]

This verifies Assumption 2(ii). Assumption 2(iii) holds by Assumption 3(iii).

\section*{D Proof of Theorem 5.2}

Following Step 4 in the proof of Theorem 4.1, we have

\[
\sigma^2((a, s)_{a=0,1,s \in S}) = \frac{\sigma_S^2 + V((a, s)_{a=0,1,s \in S})}{\mathbb{P}(D_1 > D_i(0))^2},
\]

where \(\sigma_S^2\) does not depend on \((a, s)_{a=0,1,s \in S}\) and

\[
V((a, s)_{a=0,1,s \in S}) = \mathbb{E} \left( \left( \frac{\pi(S_i)}{1 - \pi(S_i)} A_1(S_i, X_i) + \frac{1 - \pi(S_i)}{\pi(S_i)} A_0(S_i, X_i) \right)^2 \right)
\]

\[
= \sum_{s \in S} p(s) \mathbb{E} \left( \left( \frac{\pi(s)}{1 - \pi(s)} A_1(s, X_i) + \frac{1 - \pi(s)}{\pi(s)} A_0(s, X_i) \right)^2 \right) \mathbb{E}\left[|S_i = s| \right]
\]

where for \(a = 0, 1\),

\[
A_a(s, x) = \nu^Y(a, s, x) - \hat{\nu}^Y(a, s, x) - \tau(\nu^D(a, s, x) - \hat{\nu}^D(a, s, x))
\]

\[
= (\nu^Y(a, s, x) - \nu^D(a, s, x)) - H_{i,s}(t_{a,s} - \tau b_{a,s}),
\]

and \((\hat{\nu}^Y(a, s, x), \hat{\nu}^D(a, s, x))\) and \((\nu^Y(a, s, x), \nu^D(a, s, x))\) are defined in (4.2) and (B.4), respectively. Specifically, we have

\[
\hat{\nu}^Y(a, s, x) = \Psi_{i,s}^T t_{a,s}, \quad \hat{\nu}^D(a, s, x) = \Psi_{i,s}^T b_{a,s}, \quad \text{and} \quad \Psi_{i,s} = \Psi_{i,s} - \mathbb{E}(\Psi_{i,s}|S_i = s).
\]

In order to minimize \(V((a, s)_{a=0,1,s \in S})\), it suffices to minimize

\[
\mathbb{E} \left( \left( \frac{\pi(s)}{1 - \pi(s)} A_1(s, X_i) + \frac{1 - \pi(s)}{\pi(s)} A_0(s, X_i) \right)^2 \right) \mathbb{E}\left[|S_i = s| \right]
\]

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for each $s \in S$. In addition, we have

$$E \left( \left( \frac{\pi(s)}{1 - \pi(s)} A_1(s, X_i) + \frac{1 - \pi(s)}{\pi(s)} A_0(s, X_i) \right)^2 \right) \mid S_i = s$$

$$= E \left( (\bar{y}_{i,s} - \bar{y}_{i,s}^\top \gamma_s)^2 \right) \mid S_i = s,$$

where

$$\bar{y}_{i,s} = \sqrt{\frac{\pi(s)}{1 - \pi(s)}} (\nu^Y(1, s, X_i) - \tau \nu^D(1, s, X_i)) + \sqrt{\frac{1 - \pi(s)}{\pi(s)}} (\nu^Y(0, s, X_i) - \tau \nu^D(0, s, X_i))$$

and

$$\gamma(s) = \sqrt{\frac{\pi(s)}{1 - \pi(s)}} (\pi_i - \tau b_{i,s}) + \sqrt{\frac{1 - \pi(s)}{\pi(s)}} (t_{0,s} - \tau b_{0,s}).$$

By solving the first order condition, we find that

$$\Theta^* = \left( \frac{(\theta_{a,s}, \beta_{a,s})_{a=0,1,s \in S} :}{\sqrt{\frac{\pi(s)}{1 - \pi(s)}} (\theta_{1,s} - \tau \beta_{1,s}) + \sqrt{\frac{1 - \pi(s)}{\pi(s)}} (\theta_{0,s} - \tau \beta_{0,s})} \right)$$

$$= \left( \frac{(\theta_{a,s}, \beta_{a,s})_{a=0,1,s \in S} :}{\sqrt{\frac{\pi(s)}{1 - \pi(s)}} (\theta_{1,s}^L - \tau \beta_{1,s}^L) + \sqrt{\frac{1 - \pi(s)}{\pi(s)}} (\theta_{0,s}^L - \tau \beta_{0,s}^L)} \right),$$

where

$$\theta_{a,s}^L = [E(\tilde{y}_{i,s} \tilde{y}_{i,s}^\top | S_i = s)]^{-1} [E(\tilde{y}_{i,s} \nu^Y(a,s, X_i) | S_i = s)]$$

$$= [E(\tilde{y}_{i,s} \tilde{y}_{i,s}^\top | S_i = s)]^{-1} [E(\tilde{y}_{i,s} E(Y_i(a) | D_i(a)) | S_i = s)]$$

$$= [E(\tilde{y}_{i,s} \tilde{y}_{i,s}^\top | S_i = s)]^{-1} [E(\tilde{y}_{i,s} Y_i(a) | D_i(a)) | S_i = s)].$$

Similarly, we have

$$\beta_{a,s}^L = [E(\tilde{y}_{i,s} \tilde{y}_{i,s}^\top | S_i = s)]^{-1} [E(\tilde{y}_{i,s} D_i(a) | S_i = s)].$$

This concludes the proof.
E  Proof of Theorem 5.3

In order to verify Assumption 2, by Theorem 5.1, it suffices to show that \( \hat{\theta}_{a,s}^{LP} \xrightarrow{p} \theta_{a,s}^{LP} \) and \( \hat{\beta}_{a,s}^{LP} \xrightarrow{p} \beta_{a,s}^{LP} \). We focus on the former with \( a = 1 \). Let \( \{W_i^s, X_i^s\}_{i \in [n]} \) be generated independently from the joint distribution of \((Y_i(D_i(1)), X_i)\) given \( S_i = s \) and denote \( \Psi_i^s = \Psi_s(X_i^s) \), \( \Psi_i^{s,\text{\tiny \hat{a}}}_1 = \Psi_s(X_i^s) - \frac{1}{n_1(s)} \sum_{i=N(s)+1}^{N(s)+n_1(s)} \Psi_s(X_i^s) \), and \( \Psi_i^{s,\text{\tiny \hat{a}}}_0 = \Psi_s(X_i^s) - \frac{1}{n_0(s)} \sum_{i=N(s)+n_1(s)+1}^{N(s)+n_1(s)+n_0(s)} \Psi_s(X_i^s) \). Then, we have

\[
\hat{\theta}_{1,s}^{LP} \xrightarrow{d} \left( \frac{1}{n_1(s)} \sum_{i=N(s)+1}^{N(s)+n_1(s)} \Psi_i^{s,\text{\tiny \hat{a}}}_1 \Psi_i^{s,\text{\tiny \hat{a}}}_1^T \right)^{-1} \left( \frac{1}{n_1(s)} \sum_{i=N(s)+1}^{N(s)+n_1(s)} \Psi_i^{s,\text{\tiny \hat{a}}}_1 W_i^s \right).
\]

As \( \frac{1}{n_1(s)} \sum_{i=N(s)+1}^{N(s)+n_1(s)} \Psi_i^{s,\text{\tiny \hat{a}}}_1 \Psi_i^{s,\text{\tiny \hat{a}}}_1^T \xrightarrow{p} \mathbb{E} \Psi_i^{s,\text{\tiny \hat{a}}}_1 \Psi_i^{s,\text{\tiny \hat{a}}}_1^T = \mathbb{E}(\Psi_s(X_i^s)) = \mathbb{E}(\Psi_s(X_i^s)|S_i = s) \) by the standard LLN, we have

\[
\left( \frac{1}{n_1(s)} \sum_{i=N(s)+1}^{N(s)+n_1(s)} \Psi_i^{s,\text{\tiny \hat{a}}}_1 \Psi_i^{s,\text{\tiny \hat{a}}}_1^T \right) \xrightarrow{p} \mathbb{E} \Psi_i^{s,\text{\tiny \hat{a}}}_1 \Psi_i^{s,\text{\tiny \hat{a}}}_1^T = \mathbb{E}(\Psi_s(X_i^s)|S_i = s),
\]

\[
\left( \frac{1}{n_1(s)} \sum_{i=N(s)+1}^{N(s)+n_1(s)} \Psi_i^{s,\text{\tiny \hat{a}}}_1 W_i^s \right) \xrightarrow{p} \mathbb{E} \Psi_i^{s,\text{\tiny \hat{a}}}_1 W_i^s = \mathbb{E}(\Psi_i^{s,\text{\tiny \hat{a}}}_1 Y_i(D_i(1)|S_i = s)).
\]

Last, Assumption 4 implies \( \mathbb{E}(\Psi_i^{s,\text{\tiny \hat{a}}}_1 \Psi_i^{s,\text{\tiny \hat{a}}}_1^T|S_i = s) \) is invertible, this means

\[
\hat{\theta}_{1,s}^{LP} \xrightarrow{p} \left[ \mathbb{E}(\Psi_i^{s,\text{\tiny \hat{a}}}_1 \Psi_i^{s,\text{\tiny \hat{a}}}_1^T|S_i = s) \right]^{-1} \mathbb{E}(\Psi_i^{s,\text{\tiny \hat{a}}}_1 Y_i(D_i(1)|S_i = s)) = \theta_{1,s}^{LP}.
\]

Similarly, we can show that \( \hat{\theta}_{a,s}^{LP} \xrightarrow{p} \theta_{a,s}^{LP} \) and \( \hat{\beta}_{a,s}^{LP} \xrightarrow{p} \beta_{a,s}^{LP} \) for \( a = 0, 1 \) and \( s \in S \). Therefore, Assumption 2 holds, and thus, all the results in Theorem 4.1 hold for \( \hat{\tau}^{LP} \). Then, the optimality result in the second half of Theorem 5.3 is a direct consequence of Theorem 5.2.
Proof of Theorem 5.4

Let \( \{D_i^s(1), X_i^s\}_{i \in [n]} \) be generated independently from the joint distribution of \((D_i(1), X_i)\) given \( S_i = s, \Psi_{i,s} = \Psi_s(X_i^s), \) and \( \Psi_s^* = (1, \Psi_{i,s}^T)^T. \) Then, we have, pointwise in \( b, \)

\[
\frac{1}{n_1(s)} \sum_{i \in I_1(s)} \left[ D_i \log(\lambda(\hat{\Psi}_{i,s}^T b)) + (1 - D_i) \log(1 - \lambda(\hat{\Psi}_{i,s}^T b)) \right] \\
\overset{\text{d}}{=} \frac{1}{n_1(s)} \sum_{i = N(s) + 1}^{N(s) + n_1(s)} \left[ D_i^s(1) \log(\lambda(\hat{\Psi}_{i,s}^{s,T} b)) + (1 - D_i^s(1)) \log(1 - \lambda(\hat{\Psi}_{i,s}^{s,T} b)) \right] \\
\xrightarrow{p} \mathbb{E} \left[ D_i(1) \log(\lambda(\hat{\Psi}_{i,s}^T b)) + (1 - D_i(1)) \log(1 - \lambda(\hat{\Psi}_{i,s}^T b)) | S_i = s \right].
\]

As the logistic likelihood function is concave in \( b, \) the pointwise convergence in \( b \) implies uniform convergence, i.e.,

\[
\sup_b \left| \frac{1}{n_1(s)} \sum_{i \in I_1(s)} \left[ D_i \log(\lambda(\hat{\Psi}_{i,s}^T b)) + (1 - D_i) \log(1 - \lambda(\hat{\Psi}_{i,s}^T b)) \right] \\
- \mathbb{E} \left[ D_i(1) \log(\lambda(\hat{\Psi}_{i,s}^T b)) + (1 - D_i(1)) \log(1 - \lambda(\hat{\Psi}_{i,s}^T b)) | S_i = s \right] \xrightarrow{p} 0.
\]

Then, by the standard proof for the extremum estimation, we have \( \hat{\beta}_{a,s}^{MLE} \xrightarrow{p} \beta_{a,s}^{MLE}. \) Similarly, we can show that \( \hat{\theta}_{a,s}^{OLS} \xrightarrow{p} \theta_{a,s}^{OLS}. \) The verifies Assumption 3(i). Assumptions 3(ii) and 3(iii) follow Assumption 5(ii). Then, the desired results hold due to Theorem 5.1.

Proof of Theorem 5.5

We note that the adjustments proposed in Theorem 5.5 are still parametric. Specifically, we have

\[
\overline{\mu}^Y (a, s, X_i) = \Lambda_{a,s}^Y (X_i, \{\beta_{1,s}^{MLE}, \beta_{0,s}^{MLE}, \theta_{a,s}^{LP*}\}), \\
\overline{\mu}^D (a, s, X_i) = \Lambda_{a,s}^D (X_i, \{\beta_{1,s}^{MLE}, \beta_{0,s}^{MLE}, \beta_{a,s}^{LP*}\}), \\
\hat{\mu}^Y (a, s, X_i) = \Lambda_{a,s}^Y (X_i, \{\hat{\beta}_{1,s}^{MLE}, \hat{\beta}_{0,s}^{MLE}, \hat{\theta}_{a,s}^{LP*}\}), \quad \text{and} \\
\hat{\mu}^D (a, s, X_i) = \Lambda_{a,s}^D (X_i, \{\hat{\beta}_{1,s}^{MLE}, \hat{\beta}_{0,s}^{MLE}, \hat{\beta}_{a,s}^{LP*}\}),
\]

where

\[
\Lambda_{a,s}^Y (X_i, \{b_1, b_0, t^*_a\}) = \left( \begin{array}{c} \Psi_{i,s}^T \\ \lambda(\hat{\Psi}_{i,s}^{T_b} b_1) \\ \lambda(\hat{\Psi}_{i,s}^{T_b} b_0) \end{array} \right)^T t^*_a \quad \text{and} \quad \Lambda_{a,s}^D (X_i, \{b_1, b_0, b_a^*\}) = \left( \begin{array}{c} \Psi_{i,s}^T \\ \lambda(\hat{\Psi}_{i,s}^{T_b} b_1) \\ \lambda(\hat{\Psi}_{i,s}^{T_b} b_0) \end{array} \right)^T b_a^*.
\]
Therefore, in view of Theorem 5.1, to verify Assumption 2, it suffices to show that $\hat{\theta}_{a,s}^{LP*} \overset{p}{\to} \theta_{a,s}^{LP*}$ and $\hat{\beta}_{a,s}^{LP*} \overset{p}{\to} \beta_{a,s}^{LP*}$, as we have already shown the consistency of $\hat{\beta}_{a,s}^{MLE}$ in Theorem 5.4. We focus on $\hat{\theta}_{a,s}^{LP*}$. Recall $\Phi_{i,a,s} = \dot{\Phi}_{i,a,s} - \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \Phi_{i,a,s}$, where

$$\Phi_{i,a,s} = \begin{pmatrix} \Psi_{i,s} \\ \lambda(\tilde{\Phi}_{i,a,s}^{\beta_{a,s}^{MLE}}) \\ \lambda(\tilde{\Phi}_{i,a,s}^{\beta_{a,s}^{MLE}}) \end{pmatrix}.$$ 

We first show that

$$\frac{1}{n_a(s)} \sum_{i \in I_a(s)} \Phi_{i,a,s} \Phi_{i,a,s}^T = \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \Phi_{i,a,s}^T \Phi_{i,a,s} + o_p(1). \quad (G.1)$$

Let $v, u \in \mathbb{R}^d$ be two arbitrary vectors such that $\|u\|_2 = \|v\|_2 = 1$. Then, we have

$$\left| v^T \left[ \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \left( \Phi_{i,a,s}^T \Phi_{i,a,s} - \Phi_{i,a,s} \Phi_{i,a,s}^T \right) \right] u \right|$$

$$= \left| \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \left[ (v^T \dot{\Phi}_{i,a,s})(u^T \dot{\Phi}_{i,a,s}) - (v^T \dot{\Phi}_{i,a,s})(u^T \dot{\Phi}_{i,a,s}) \right] \right|$$

$$= \left| \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \left[ v^T (\Phi_{i,a,s} - \dot{\Phi}_{i,a,s}) (u^T \dot{\Phi}_{i,a,s}) + (v^T \dot{\Phi}_{i,a,s}) u^T (\dot{\Phi}_{i,a,s} - \dot{\Phi}_{i,a,s}) \right] \right|$$

$$\leq \frac{1}{n_a(s)} \sum_{i \in I_a(s)} ||\dot{\Phi}_{i,a,s} - \dot{\Phi}_{i,a,s}||_2 (||\dot{\Phi}_{i,a,s}|| + ||\dot{\Phi}_{i,a,s}||_2)$$

$$\leq \frac{C}{n_a(s)} \sum_{i \in I_a(s), a' = 0,1} \left( 1 + \||\dot{\Psi}_{i,s}||_2 + \frac{1}{n_a(s)} \sum_{i \in I_a(s)} ||\dot{\Psi}_{i,s}||_2 \right)^2$$

$$\times \left\{ \sum_{a' = 0,1} \left[ ||\dot{\beta}_{a,s}^{MLE} - \dot{\beta}_{a,s}^{MLE}||_2 \right] \right\} \left\{ \sum_{a' = 0,1} \left[ ||\beta_{a',s}^{MLE}||_2 + ||\hat{\beta}_{a',s}^{MLE}||_2 \right] \right\}$$

$$= o_p(1),$$

where the first inequality is by the Hölder’s inequality, the second inequality is by the facts that

$$||\dot{\Phi}_{i,a,s} - \dot{\Phi}_{i,a,s}||_2$$

$$\leq \left\| \dot{\Psi}_{i,s} - \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \dot{\Psi}_{i,s} \right\|_2 \left\{ \sum_{a' = 0,1} \left[ ||\dot{\beta}_{a,s}^{MLE} - \dot{\beta}_{a,s}^{MLE}||_2 \right] \right\}$$

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\[
\leq C \left( 1 + \|\Psi_{i,s}\|_2 + \left\| \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \Psi_{i,s} \right\|_2 \right) \left\{ \sum_{a=0,1} \left[ \|\hat{\beta}_{a',s}^{MLE} - \beta_{a',s}^{MLE}\|_2 \right] \right\}
\]

and

\[
\|\hat{\Phi}_{i,a,s}\| + \|\hat{\Phi}_{i,a,s}\|_2 
\leq C \left( 1 + \|\Psi_{i,s}\|_2 + \left\| \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \Psi_{i,s} \right\|_2 \right) \left\{ \sum_{a'=0,1} \left[ \|\hat{\beta}_{a',s}^{MLE}\|_2 + \|\beta_{a',s}^{MLE}\|_2 \right] \right\},
\]

and the last equality is by the fact that

\[
\|\hat{\beta}_{a,s}^{MLE} - \beta_{a,s}^{MLE}\|_2 = o_p(1).
\]

As the inequality holds for arbitrary \(u,v\), it implies (G.1). Similarly, we can show that

\[
\frac{1}{n_a(s)} \sum_{i \in I_a(s)} \hat{\Phi}_{i,a,s} Y_i = \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \hat{\Phi}_{i,a,s} Y_i + o_p(1). \tag{G.2}
\]

Following the same argument in the proof of Theorem 5.3, we can show that

\[
\left[ \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \hat{\Phi}_{i,a,s} \hat{\Phi}_{i,a,s}^\top \right]^{-1} \left[ \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \hat{\Phi}_{i,a,s} Y_i \right] \xrightarrow{p} \theta^{LP*}_{a,s}. \tag{G.3}
\]

In addition, by Assumption 6, with probability approaching one, there exists a constant \(c > 0\) such that

\[
\lambda_{\min} \left( \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \hat{\Phi}_{i,a,s} \hat{\Phi}_{i,a,s}^\top \right) \geq c.
\]

Combining (G.1), (G.2), and (G.3), we can show that

\[
\hat{\theta}_{a,s}^{LP*} = \left[ \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \hat{\Phi}_{i,a,s} \hat{\Phi}_{i,a,s}^\top \right]^{-1} \left[ \frac{1}{n_a(s)} \sum_{i \in I_a(s)} \hat{\Phi}_{i,a,s} Y_i \right] \\
\xrightarrow{p} \theta^{LP*}_{a,s}.
\]

Similarly, we have \(\hat{\beta}_{a,s}^{LP*} \xrightarrow{p} \beta_{a,s}^{LP*}\), which implies all the results in Theorem 4.1 hold for \(\hat{\tau}^{LP*}\). The optimality result in the second half of the theorem is a direct consequence of Theorem 5.2.
H Proof of Theorem 6.1

We focus on verifying Assumption H for $\tilde{\mu}^D(a, s, X_i)$. The proof for $\tilde{\mu}^Y(a, s, X_i)$ is similar, and thus, is omitted. Following the proof of Theorem 5.4, we note that, for each $\alpha = 0, 1$ and $s \in \mathcal{S}$, the data in cell $I_a(s) (\{D^s_i(a), X^s_i\}_{i \in [n]})$ can be viewed as i.i.d. following the joint distribution of $(D_i(a), X_i)$ given $S_i = s$ conditionally on $\{A_i, S_i\}_{i \in [n]}$. Then, following the standard logistic sieve regression in Hirano et al. (2003), we have

$$\max_{a=0,1,s \in \mathcal{S}} ||\tilde{\beta}_{a,s}^{NP} - \beta_{a,s}^{NP}||_2 = O_p \left( \sqrt{h_n/n_a(s)} \right).$$

Then, we have

$$\left| \frac{\sum_{i \in I_1(s)} \Delta^D(a, s, X_i)}{n_1(s)} - \frac{\sum_{i \in I_0(s)} \Delta^D(a, s, X_i)}{n_0(s)} \right| \leq \left| \frac{\sum_{i \in I_1(s)} (\lambda(\Psi_{i,n}^T \hat{\beta}_{a,s}^{NP}) - \lambda(\Psi_{i,n}^T \beta_{a,s}^{NP}))}{n_1(s)} - \frac{\sum_{i \in I_0(s)} (\lambda(\Psi_{i,n}^T \hat{\beta}_{a,s}^{NP}) - \lambda(\Psi_{i,n}^T \beta_{a,s}^{NP}))}{n_0(s)} \right|$$

$$+ \frac{1}{n_1(s)} \sum_{i \in I_1(s)} R^D(a, s, X_i) - E(R^D(a, s, X_i) | S_i = s)$$

$$+ \frac{1}{n_0(s)} \sum_{i \in I_0(s)} R^D(a, s, X_i) - E(R^D(a, s, X_i) | S_i = s)$$

$$:= I + II + III.$$ (H.1)

To bound term $I$ in (H.1), we define $M_{a,s}(\beta_1, \beta_2) = E(\lambda(\Psi_{i,n}^T \beta_1) - \lambda(\Psi_{i,n}^T \beta_2) | S_i = s) = E(\lambda(\Psi_{i,n}^T \beta_1) - \lambda(\Psi_{i,n}^T \beta_2))$, where $\Psi_{i,n} = \Phi(X_i^s)$. Then, we have

$$I \leq \left| \frac{\sum_{i \in I_1(s)} (\lambda(\Psi_{i,n}^T \hat{\beta}_{a,s}^{NP}) - \lambda(\Psi_{i,n}^T \beta_{a,s}^{NP}) - M_{a,s}(\hat{\beta}_{a,s}^{NP}, \beta_{a,s}^{NP}))}{n_1(s)} \right|$$

$$+ \left| \frac{\sum_{i \in I_0(s)} (\lambda(\Psi_{i,n}^T \hat{\beta}_{a,s}^{NP}) - \lambda(\Psi_{i,n}^T \beta_{a,s}^{NP}) - M_{a,s}(\hat{\beta}_{a,s}^{NP}, \beta_{a,s}^{NP}))}{n_0(s)} \right|$$

$$:= I_1 + I_2.$$ (H.2)

Following the argument in the proof of Theorem 5.1, in order to show $I_1 = o_p(n^{-1/2})$, we only need to show

$$E(||P_{n_1(s)} - P||_F | \{A_i, S_i\}_{i \in [n]} | 1\{n_1(s) \geq n \varepsilon, n_0(s) \geq n \varepsilon\} = o(1),$$

where $\varepsilon$ is an arbitrary but fixed constant, and for an arbitrary but fixed constant $C > 0$,

$$\mathcal{F} := \{\lambda(\Phi^T \beta_1) - \lambda(\Phi^T \beta_{a,s}^{NP}) : \beta_1 \in \mathbb{R}^n, ||\beta_1 - \beta_{a,s}^{NP}||_2 \leq C \sqrt{h_n/n_a(s)}\}.$$
Furthermore, we note that $F$ has a bounded envelope, is of the VC-type with VC-index upper bounded by $2h_n$, and has
\[ \sup_{f \in F} \mathbb{E}(f^2|\{A_i, S_i\}_{i \in [n]}) \leq \frac{Ch_n}{n_\alpha(s)}. \]

Therefore, by Chernozhukov et al. (2014, Corollary 5.1),
\[ n^{1/2} \mathbb{E} \left[ \left| \sum_{i=1}^n \left[ \left( \sum_{j=1}^{n_{1}(s)} R_i(a, s, X_i^s) - \mathbb{E}(R_i(a, s, X_i^s)) \right) - \mathbb{E} \left( \sum_{j=1}^{n_{1}(s)} R_i(a, s, X_i^s) \right) \right] \right| \right] \leq C \sqrt{\frac{n_{1}(s)}{n}} \left( \sqrt{\frac{h_2^2 \log(n)}{n_\alpha(s)}} + \frac{h_2 \log(n)}{\sqrt{n_{1}(s)}} \right). \]

By the Chebyshev’s inequality as $\mathbb{E}(R_i(a, s, X_i^s)) = \mathbb{E}((R_i(a, s, X_i^s))^2 | S_i = s) = o(1)$. Similarly, we have $III = o_p(n^{-1/2})$. Combining the bounds of $I$, $II$, $III$ with (H.1), we have
\[ \left| \frac{\sum_{i \in I_1(s)} \Delta_i(a, s, X_i)}{n_{1}(s)} - \frac{\sum_{i \in I_0(s)} \Delta_i(a, s, X_i)}{n_0(s)} \right| = o_p(n^{-1/2}), \]
which verifies Assumption 2(i).

To verify Assumption 2(ii), we note that
\[ \left| \frac{1}{n} \sum_{i=1}^n \Delta_i(a, s, X_i) \right| \leq \frac{1}{n} \sum_{i=1}^n \sum_{s \in S} 1\{S_i = s\} \left[ \lambda(\Psi_{i,n}^{\top} \hat{\beta}_{NP}^{s,n}) - \lambda(\Psi_{i,n}^{\top} \beta_{a,s}) \right] + \frac{1}{n} \sum_{i=1}^n \left| R_i(a, S_i, X_i) \right| \]
\[ \leq \max_{s \in S} \left[ \frac{1}{n} \sum_{i=1}^n \left( \Psi_{i,n}^{\top} \hat{\beta}_{NP}^{s,n} - \beta_{a,s} \right)^2 \right]^{1/2} + o_p(1) = o_p(1). \]

Last, Assumption 2(iii) holds by Assumption 1(vi).

I Proof of Theorem 7.1

We focus on verifying Assumption 2 for $\hat{\mu}^D(a, s, X_i)$. The proof for $\hat{\mu}^Y(a, s, X_i)$ is similar, and thus, is omitted. Following the proof of Theorem 5.4, we note that, for each $a = 0, 1$ and $s \in S$, the data in cell $I_a(s)$ ($\{D_i(a), X_i^s\}_{i \in [n]}$) can be viewed as i.i.d. following the joint distribution of $(D_i(a), X_i)$ given $S_i = s$ conditionally on $\{A_i, S_i\}_{i \in [n]}$. Then, following the standard logistic Lasso
max_{a=0,1,s\in S} \|\hat{\beta}_{a,s}^{HD} - \beta_{a,s}^{HD}\|_2 = O_p\left(\sqrt{h_n \log(p_n) / n_a(s)}\right) \quad \text{and} \quad \max_{a=0,1,s\in S} \|\hat{\beta}_{a,s}\|_0 = O_p(h_n).

Then, we have

\[
\frac{\left|\sum_{i\in I_1(s)} \Delta^D(a, s, X_i) - \sum_{i\in I_0(s)} \Delta^D(a, s, X_i)\right|}{n_1(s)} \\
\le \left|\sum_{i\in I_1(s)} (\lambda(\Psi_{i,n}^T \hat{\beta}_{a,s}^{HD}) - \lambda(\Psi_{i,n}^T \hat{\beta}_{a,s}^{HD}))\right| \frac{n_0(s)}{n_1(s)} \\
+ \frac{1}{n_1(s)} \sum_{i\in I_1(s)} R^D(a, s, X_i) - E(R^D(a, s, X_i)|S_i = s) \\
+ \frac{1}{n_0(s)} \sum_{i\in I_0(s)} R^D(a, s, X_i) - E(R^D(a, s, X_i)|S_i = s) \\
:= I + II + III. \tag{I.1}
\]

To bound term $I$ in (I.1), we define $M_{a,s}(\beta_1, \beta_2) = E(\lambda(\Psi_{i,n}^T \beta_1)|S_i = s) = E(\lambda(\Psi_{i,n}^T \beta_1) - \lambda(\Psi_{i,n}^T \beta_2))$, where $\Psi_{i,n}^s = \Phi(X_i^s)$. Then, we have

\[
I \le \left|\sum_{i\in I_1(s)} (\lambda(\Psi_{i,n}^T \hat{\beta}_{a,s}^{HD}) - \lambda(\Psi_{i,n}^T \hat{\beta}_{a,s}^{HD}) - M_{a,s}(\hat{\beta}_{a,s}^{HD}, \hat{\beta}_{a,s}^{HD}))\right| \frac{n_1(s)}{n_0(s)} \\
+ \left|\sum_{i\in I_0(s)} (\lambda(\Psi_{i,n}^T \hat{\beta}_{a,s}^{HD}) - \lambda(\Psi_{i,n}^T \hat{\beta}_{a,s}^{HD}) - M_{a,s}(\hat{\beta}_{a,s}^{HD}, \hat{\beta}_{a,s}^{HD}))\right| \frac{n_0(s)}{n_0(s)} \\
:= I_1 + I_2.
\]

Following the argument in the proof of Theorems 5.1 and 6.1, in order to show $I_1 = o_p(n^{-1/2})$, we only need to show

\[
E(|\mathbb{P}_{n_1(s)} - \mathbb{P}|_{\mathcal{F}}\{A_i, S_i\}_{i\in [n]}|n_1(s) \ge n \varepsilon, n_0(s) \ge n \varepsilon} = o(1),
\]

where $\varepsilon$ is an arbitrary but fixed constant, and for an arbitrary but fixed constant $C > 0$,

\[
\mathcal{F} := \{\lambda(\Psi^T \beta_1) - \lambda(\Psi^T \beta_{a,s}^{HD}) : \beta_1 \in \mathbb{R}^p, ||\beta_1 - \beta_{a,s}^{HD}||_2 \le C \sqrt{h_n \log(p_n) / n_a(s)}, ||\beta_1||_0 \le Ch_n\}.
\]

Furthermore, we note that $\mathcal{F}$ has a bounded envelope and

\[
\sup_Q N(\mathcal{F}, c_2, \varepsilon||F||_{Q,2}) \le \left(\frac{c_1 p_n}{\varepsilon}\right)^{c_2 h_n},
\]

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where $c_1, c_2$ are two fixed constants, $N(\cdot)$ is the covering number, $e_Q(f, g) = \sqrt{Q|f - g|^2}$, and the supremum is taken over all discrete probability measures $Q$. Last, we have

$$\sup_{f \in F} \mathbb{E}(f^2|\{A_i, S_i\}_{i \in [n]}) \leq \frac{C h_n(\log(p_n))}{n(\sigma)}.$$  
Therefore, by Chernozhukov et al. (2014, Corollary 5.1),

$$n^{1/2} \mathbb{E} \left[ ||\hat{P}_{n}(s) - P||_F|\{A_i, S_i\}_{i \in [n]} \right] 1\{n_1(s) \geq n\varepsilon \} \leq C \left( \sqrt{\frac{n}{n_1(s)}} \left( \frac{h_n(\log(p_n))}{\sqrt{n_1(s)}} \right) \right) \mathbb{1}\{n_1(s) \geq n\varepsilon, n_0(s) \geq n\varepsilon \} \to 0.$$

The bounds for $I_2$, $II$ and $III$ can be established following the same argument as in the proof of Theorem 6.1. We omit the detail for brevity. This leads to Assumption 2(i).

To verify Assumption 2(ii), we note that

$$\left| \frac{1}{n} \sum_{i=1}^{n} \Delta^D(a, s, X_i) \right| \leq \left| \frac{1}{n} \sum_{i=1}^{n} \sum_{s \in S} 1\{S_i = s\} \left( \lambda(\Psi_{i,n}^T \beta^{HD}) - \lambda(\Psi_{i,n}^T \beta_{a,s}^{HD}) \right) \right| + \frac{1}{n} \sum_{i=1}^{n} |R^D(a, S_i, X_i)|$$

$$\leq \max_{s \in S} \left[ \frac{1}{n} \sum_{i=1}^{n} \left( \Psi_{i,n}^T \beta^{HD} - \beta_{a,s}^{HD} \right)^2 \right]^{1/2} + o_p(1) = o_p(1).$$

Last, Assumption 2(iii) holds by Assumption 1(vi).

**J Technical Lemmas Used in the Proof of Theorem 4.1**

**Lemma J.1.** Suppose assumptions in Theorem 4.1 hold. Then, we have

$$R_{n,1} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( 1 - \frac{1}{\pi(S_i)} \right) A_i \tilde{\mu}^Y(1, S_i, X_i) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (1 - A_i) \tilde{\mu}^Y(1, S_i, X_i) + o_p(1),$$

$$R_{n,2} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \frac{1}{1 - \pi(S_i)} - 1 \right) (1 - A_i) \tilde{\mu}^Y(0, S_i, X_i) - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i \tilde{\mu}^Y(0, S_i, X_i) + o_p(1),$$

$$R_{n,3} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\pi(S_i)} \tilde{W}_i A_i - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1 - A_i}{1 - \pi(S_i)} \tilde{Z}_i + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\mathbb{E}[W_i - Z_i|S_i] - \mathbb{E}[W_i - Z_i]) + o_p(1),$$

where for $a = 0, 1$,

$$\tilde{\mu}^Y(a, S_i, X_i) := \tilde{\pi}^Y(a, S_i, X_i) - \tilde{\pi}^Y(a, S_i), \quad \tilde{\pi}^Y(a, S_i) := \mathbb{E} [\tilde{\pi}^Y(a, S_i, X_i)|S_i],$$

$$W_i := Y_i(1)D_i(1) + Y_i(0)(1 - D_i(1)), \quad Z_i := Y_i(1)D_i(0) + Y_i(0)(1 - D_i(0)),$$

$$\tilde{W}_i := W_i - \mathbb{E}[W_i|S_i], \quad \tilde{Z}_i := Z_i - \mathbb{E}[Z_i|S_i].$$
Proof. We have

\[ R_{n,1} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ \tilde{\mu}^Y(1, S_i, X_i) - \frac{A_i \tilde{\mu}^Y(1, S_i, X_i)}{\pi(S_i)} \right] \]

\[ = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i - \frac{\tilde{\pi}(S_i)}{\pi(S_i)} \mu^Y(1, S_i, X_i) \]

\[ = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i - \frac{\hat{\pi}(S_i)}{\pi(S_i)} \mu^Y(1, S_i, X_i) \]

\[ = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i - \frac{\hat{\pi}(S_i)}{\pi(S_i)} \left[ \mu^Y(1, S_i, X_i) - \tilde{\mu}^Y(1, S_i, X_i) + \mu^Y(1, S_i, X_i) \right] \]

\[ = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i - \frac{\hat{\pi}(S_i)}{\pi(S_i)} \mu^Y(1, S_i, X_i) - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i \tilde{\mu}^Y(1, S_i, X_i) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mu^Y(1, S_i, X_i), \]

where the last equality is due to

\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i \tilde{\mu}^Y(1, S_i) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mu^Y(1, S_i). \]

Consider the first term of (J.1).

\[
\left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i - \frac{\hat{\pi}(S_i)}{\pi(S_i)} \Delta^Y(1, S_i, X_i) \right| = \left| \frac{1}{\sqrt{n}} \sum_{s \in S} \sum_{i=1}^{n} A_i \Delta^Y(1, s, X_i) 1\{S_i = s\} - \sum_{s \in S} \sum_{i=1}^{n} \Delta^Y(1, s, X_i) 1\{S_i = s\} \right|
\]

\[
= \frac{1}{\sqrt{n}} \left| \sum_{s \in S} \frac{1}{\pi(s)} \sum_{i=1}^{n} A_i \Delta^Y(1, s, X_i) 1\{S_i = s\} - \sum_{s \in S} \sum_{i=1}^{n} \Delta^Y(1, s, X_i) 1\{S_i = s\} \right|
\]

\[
= \frac{1}{\sqrt{n}} \left| \sum_{s \in S} \sum_{i \in I_1(s)} \Delta^Y(1, s, X_i) \frac{n(s)}{n_1(s)} - \sum_{s \in S} \sum_{i \in I_0(s) \cup I_1(s)} \Delta^Y(1, s, X_i) \right|
\]

\[
= \frac{1}{\sqrt{n}} \left| \sum_{s \in S} \sum_{i \in I_1(s)} \Delta^Y(1, s, X_i) \frac{n_0(s)}{n_1(s)} - \sum_{s \in S} \sum_{i \in I_0(s)} \Delta^Y(1, s, X_i) \right|
\]

\[
= \frac{1}{\sqrt{n}} \left| \sum_{s \in S} n_0(s) \left[ \sum_{i \in I_1(s)} \Delta^Y(1, s, X_i) - \sum_{i \in I_0(s)} \Delta^Y(1, s, X_i) \right] \right|
\]

\[
\leq \frac{1}{\sqrt{n}} \sum_{s \in S} n_0(s) \left| \sum_{i \in I_1(s)} \Delta^Y(1, s, X_i) - \sum_{i \in I_0(s)} \Delta^Y(1, s, X_i) \right| = o_p(1)
\]
where the last equality is due to Assumption 2. Thus

\[ R_{n,1} = -\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{A_i}{\hat{\pi}(S_i)} \hat{\mu}^Y(1, S_i, X_i) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \hat{\mu}^Y(1, S_i, X_i) + o_p(1) \]

\[ = -\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{A_i}{\hat{\pi}(S_i)} \hat{\mu}^Y(1, S_i, X_i) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i \hat{\mu}^Y(1, S_i, X_i) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (1 - A_i) \hat{\mu}^Y(1, S_i, X_i) + o_p(1) \]

\[ = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(1 - \frac{1}{\hat{\pi}(S_i)}\right) A_i \hat{\mu}^Y(1, S_i, X_i) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (1 - A_i) \hat{\mu}^Y(1, S_i, X_i) + o_p(1). \quad (J.2) \]

In addition, we note that

\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(1 - \frac{1}{\hat{\pi}(S_i)}\right) A_i \hat{\mu}^Y(1, S_i, X_i) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(1 - \frac{1}{\hat{\pi}(S_i)}\right) A_i \hat{\mu}^Y(1, S_i, X_i) \]

\[ + \sum_{s \in S} \left(1 - \frac{1}{\hat{\pi}(s)}\right) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i \hat{\mu}^Y(1, s, X_i) 1\{S_i = s\}. \]

Note that under Assumption 1(i), conditional on \{S^{(n)}, A^{(n)}\}, the distribution of

\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i \hat{\mu}^Y(1, s, X_i) 1\{S_i = s\} \]

is the same as the distribution of the same quantity where units are ordered by strata and then ordered by \(A_i = 1\) first and \(A_i = 0\) second within strata. To this end, define \(N(s) := \sum_{i=1}^{n} 1\{S_i < s\}\) and \(F(s) := \mathbb{P}(S_i < s)\). Furthermore, independently for each \(s \in S\) and independently of \(S^{(n)}, A^{(n)}\), let \(\{X_i^s : 1 \leq i \leq n\}\) be i.i.d with marginal distribution equal to the distribution of \(X_i|S = s\). Define

\[ \hat{\mu}^b(a, s, X_i^s) := \hat{\mu}(a, s, X_i^s) - \mathbb{E} [\hat{\mu}(a, s, X_i^s)|S_i = s] \]

Then, we have, for \(s \in S\),

\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i \hat{\mu}^Y(1, s, X_i) 1\{S_i = s\} \overset{d}{=} \frac{1}{\sqrt{n}} \sum_{i=N(s)+1}^{N(s)+n_1(s)} \hat{\mu}^Y(1, s, X_i^s). \]

In addition, we have

\[ \mathbb{E} \left[ \left( \frac{1}{\sqrt{n}} \sum_{i=N(s)+1}^{N(s)+n_1(s)} \hat{\mu}^Y(1, s, X_i^s) \right)^2 \bigg| S^{(n)}, A^{(n)} \right] = \left[ \frac{n_1(s)}{n} \text{Var}(\hat{\mu}^Y(a, s, X_i^s)) \right] \]

\[ \leq \frac{n_1(s)}{n} \mathbb{E} \left[ (\hat{\mu}^Y(a, s, X_i))^2 \big| S_i = s \right] = O_p(1), \]

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which implies

$$\max_{s \in S} \left| \frac{1}{\sqrt{n}} \sum_{i=N(s)+1}^{N(s)+n_1(s)} \mu_Y(1, s, X_i^s) \right| = O_p(1).$$

Combining this with the facts that \(\max_{s \in S} |\pi(s) - \pi(s)| = o_p(1)\) and \(\min_{s \in S} \pi(s) > c > 0\) for some constant \(c\), we have

$$\frac{1}{\sqrt{n}} \sum_{s \in S} \left( \frac{1}{\pi(s)} - \frac{1}{\pi(s)} \right) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i \mu_Y(1, s, X_i) 1\{S_i = s\} = o_p(1)$$

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( 1 - \frac{1}{\pi(S_i)} \right) A_i \hat{\mu}_Y(1, S_i, X_i) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( 1 - \frac{1}{\pi(S_i)} \right) A_i \hat{\mu}^2_Y(1, S_i, X_i) + o_p(1).$$

Therefore, we have

$$R_{n,1} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( 1 - \frac{1}{\pi(S_i)} \right) A_i \hat{\mu}_Y(1, S_i, X_i) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (1 - A_i) \hat{\mu}_Y^2(1, S_i, X_i) + o_p(1).$$

The linear expansion of \(R_{n,2}\) can be established in the same manner.

For \(R_{n,3}\), note that

$$Y_i = Y_i(1) [D_i(1)A_i + D_i(0)(1 - A_i)] + Y_i(0) [1 - D_i(1)A_i - D_i(0)(1 - A_i)]$$

$$= [Y_i(1)D_i(1) - Y_i(0)D_i(1)] A_i + [Y_i(1)D_i(0) - Y_i(0)D_i(0)] (1 - A_i) + Y_i(0).$$

Then

$$A_i Y_i = \left[ Y_i(1)D_i(1) + Y_i(0)(1 - D_i(1)) \right] A_i,$$

$$A_i Y_i = \left[ Y_i(1)D_i(0) + Y_i(0)(1 - D_i(0)) \right] (1 - A_i),$$

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{A_i Y_i}{\pi(S_i)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\pi(S_i)} \left[ Y_i(1)D_i(1) + Y_i(0)(1 - D_i(1)) \right] A_i =: \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\pi(S_i)} W_i A_i,$$

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{(1 - A_i) Y_i}{1 - \pi(S_i)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{Y_i(1)D_i(0) + Y_i(0)(1 - D_i(0))}{1 - \pi(S_i)} (1 - A_i) =: \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{Z_i(1 - A_i)}{1 - \pi(S_i)}.$$

Thus we have

$$R_{n,3} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{A_i Y_i}{\pi(S_i)} - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{(1 - A_i) Y_i}{1 - \pi(S_i)} - \sqrt{n} G$$

$$= \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\pi(S_i)} W_i A_i - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1 - A_i}{1 - \pi(S_i)} Z_i \right\}.$$
where the second equality is by (J.5). Therefore, we have

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\pi(S_i)} \mathbb{E}[W_i|S_i] A_i = \sum_{s \in S} \frac{\mathbb{E}[W|S = s]}{\sqrt{n} \mathbb{E}[W|S = s]} n(s),
\]  

We now consider the second term on the RHS of (J.3). First note that

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\pi(S_i)} \mathbb{E}[W_i|S_i] A_i = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\pi(S_i)} \mathbb{E}[W_i|S_i] A_i - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\hat{\pi}(S_i) - \pi(S_i)}{\pi(S_i) \pi(S_i)} \mathbb{E}[W_i|S_i] A_i,
\]

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\pi(S_i)} \mathbb{E}[W_i|S_i] A_i
\]

\[
= \sum_{s \in S} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\pi(s)} \mathbb{E}[W_i|S_i = s] A_i \{S_i = s\}
\]

\[
= \sum_{s \in S} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\mathbb{E}[W_i|S_i = s]}{\pi(s)} (A_i - \pi(s)) \{S_i = s\} + \sum_{s \in S} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\pi(s)} \mathbb{E}[W_i|S_i = s] \pi(s) \{S_i = s\}
\]

\[
= \sum_{s \in S} \frac{\mathbb{E}[W|S = s]}{\pi(s) \sqrt{n}} \sum_{i=1}^{n} (A_i - \pi(s)) \{S_i = s\} + \sum_{s \in S} \frac{\mathbb{E}[W|S = s]}{\sqrt{n}} \sum_{i=1}^{n} 1 \{S_i = s\}
\]

\[
= \sum_{s \in S} \frac{\mathbb{E}[W|S = s]}{\pi(s) \sqrt{n}} \mathbb{E}[W|S = s] B_n(s) + \sum_{s \in S} \frac{\mathbb{E}[W|S = s]}{\sqrt{n}} n(s),
\]

and

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\hat{\pi}(S_i) - \pi(S_i)}{\pi(S_i) \pi(S_i)} \mathbb{E}[W_i|S_i] A_i = \sum_{s \in S} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\hat{\pi}(s) - \pi(s)}{\pi(s) \pi(s)} \mathbb{E}[W_i|S_i = s] A_i \{S_i = s\}
\]

\[
= \sum_{s \in S} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{B_n(s)}{n(s) \pi(s) \pi(s)} \mathbb{E}[W_i|S_i = s] A_i \{S_i = s\}
\]

\[
= \sum_{s \in S} \frac{B_n(s) \mathbb{E}[W|S = s]}{\sqrt{n} \mathbb{E}[W|S = s]} \sum_{i=1}^{n} A_i \{S_i = s\}
\]

\[
= \sum_{s \in S} \frac{B_n(s) \mathbb{E}[W|S = s]}{\sqrt{n} \mathbb{E}[W|S = s]} n(s)
\]

\[
= \sum_{s \in S} \frac{B_n(s) \mathbb{E}[W|S = s]}{\sqrt{n} \pi(s)} n(s),
\]

where the second equality is by (3.1). Therefore, we have

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\pi(S_i)} \mathbb{E}[W_i|S_i] A_i = \sum_{s \in S} \frac{\mathbb{E}[W|S = s]}{\sqrt{n}} n(s).
\]
Similarly, we have
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1 - A_i}{1 - \hat{\pi}(S_i)} \mathbb{E}[Z_i | S_i] = \sum_{s \in S} \frac{\mathbb{E}[Z | S = s]}{\sqrt{n}} n(s)
\]

Then, we have
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\hat{\pi}(S_i)} \mathbb{E}[W_i | S_i] A_i - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1 - A_i}{1 - \hat{\pi}(S_i)} \mathbb{E}[Z_i | S_i] - \sqrt{n}G
\]
\[
= \sum_{s \in S} \frac{\mathbb{E}[W | S = s]}{\sqrt{n}} n(s) - \sum_{s \in S} \frac{\mathbb{E}[Z | S = s]}{\sqrt{n}} n(s) - \sqrt{n}G
\]
\[
= \sum_{s \in S} \sqrt{n} \left( \frac{n(s)}{n} - p(s) \right) \mathbb{E}[W - Z | S = s] + \sum_{s \in S} \sqrt{n}p(s) \mathbb{E}[W - Z | S = s] - \sqrt{n}G
\]
\[
= \sum_{s \in S} \sqrt{n} \left( \frac{n(s)}{n} - p(s) \right) \mathbb{E}[W - Z | S = s] + \sqrt{n} \mathbb{E}[W - Z] - \sqrt{n}G
\]
\[
= \sum_{s \in S} \frac{n(s)}{\sqrt{n}} \mathbb{E}[W - Z | S = s] - \sqrt{n} \mathbb{E}[W - Z]
\]
\[
= \frac{1}{\sqrt{n}} \sum_{s \in S} \sum_{i=1}^{n} \left( 1 \{ S_i = s \} \mathbb{E}[W_i - Z_i | S_i = s] \right) - \sqrt{n} \mathbb{E}[W - Z]
\]
\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbb{E}[W_i - Z_i | S_i] - \sqrt{n} \mathbb{E}[W - Z]
\]
\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \mathbb{E}[W_i - Z_i | S_i] - \mathbb{E}[W_i - Z_i] \right)
\]
(J.7)

Combining (J.3) and (J.7), we have
\[
R_{n,3} = \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\hat{\pi}(S_i)} \tilde{W}_i A_i - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1 - A_i}{1 - \hat{\pi}(S_i)} \tilde{Z}_i \right\} + \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \mathbb{E}[W_i - Z_i | S_i] - \mathbb{E}[W_i - Z_i] \right) \right\}
\]
\[
= \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\hat{\pi}(S_i)} \tilde{W}_i A_i - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1 - A_i}{1 - \hat{\pi}(S_i)} \tilde{Z}_i \right\}
\]
\[
+ \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \mathbb{E}[W_i - Z_i | S_i] - \mathbb{E}[W_i - Z_i] \right) \right\} + o_p(1),
\]
where the second equality holds because
\[
\left( \frac{1}{\pi(s)} - \frac{1}{\hat{\pi}(s)} \right) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \tilde{W}_i A_i 1 \{ S_i = s \} = o_p(1)
\]
and
\[
\left( \frac{1}{\pi(s)} - \frac{1}{\bar{\pi}(s)} \right) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \tilde{Z}_i (1 - A_i) 1\{S_i = s\} = o_p(1)
\]
due to the same argument used in the proofs of \(R_{n,1}\). \hfill \Box

**Lemma J.2.** Under the assumptions in Theorem 4.1, we have

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Xi_1(D_i, S_i) A_i \sim N \left( 0, \mathbb{E} \pi(S_i) \Xi_1^2(D_i, S_i) \right),
\]

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Xi_0(D_i, S_i) (1 - A_i) \sim N \left( 0, \mathbb{E} (1 - \pi(S_i)) \Xi_0^2(D_i, S_i) \right), \quad \text{and}
\]

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Xi_2(S_i) \sim N(0, \mathbb{E} \Xi_2^2(S_i)),
\]

and the three terms are asymptotically independent.

**Proof.** Note that under Assumption 1(i), conditional on \(\{S^{(n)}, A^{(n)}\}\), the distribution of

\[
\left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Xi_1(D_i, S_i) A_i, \quad \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Xi_0(D_i, S_i)(1 - A_i) \right)
\]

is the same as the distribution of the same quantity where units are ordered by strata and then ordered by \(A_i = 1\) first and \(A_i = 0\) second within strata. To this end, define \(N(s) := \sum_{i=1}^{n} 1\{S_i < s\}\) and \(F(s) := \mathbb{P}(S_i < s)\). Furthermore, independently for each \(s \in S\) and independently of \(\{S^{(n)}, A^{(n)}\}\), let \(\{D_i^s : 1 \leq i \leq n\}\) be i.i.d with marginal distribution equal to the distribution of \(D|S = s\). Then, we have

\[
\left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Xi_1(D_i^s, s) A_i, \quad \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Xi_0(D_i^s, s)(1 - A_i) \right) \bigg| S^{(n)}, A^{(n)}
\]

\[
\overset{d}{=} \left( \frac{1}{\sqrt{n}} \sum_{s \in S} \sum_{i=N(s)+1}^{N(s)+n(s)} \Xi_1(D_i^s, s), \quad \frac{1}{\sqrt{n}} \sum_{s \in S} \sum_{i=N(s)+1}^{N(s)+n(s)} \Xi_0(D_i^s, s) \right) \bigg| S^{(n)}, A^{(n)}
\]

In addition, since \(\Xi_2(S_i)\) is a function of \(\{S^{(n)}, A^{(n)}\}\), we have, arguing along the line of a joint distribution being the product of a conditional distribution and a marginal distribution,

\[
\left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Xi_1(D_i, S_i) A_i, \quad \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Xi_0(D_i, S_i)(1 - A_i), \quad \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Xi_2(S_i) \right)
\]

\[
\overset{d}{=} \left( \frac{1}{\sqrt{n}} \sum_{s \in S} \sum_{i=N(s)+1}^{N(s)+n(s)} \Xi_1(D_i^s, s), \quad \frac{1}{\sqrt{n}} \sum_{s \in S} \sum_{i=N(s)+1}^{N(s)+n(s)} \Xi_0(D_i^s, s), \quad \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Xi_2(S_i) \right).
\]
Define $\Gamma_{a,n}(u, s) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor un \rfloor} \Xi_a(D_i, s)$ for $a = 0, 1, s \in S$. We have

$$
\frac{1}{\sqrt{n}} \sum_{s \in S} \sum_{i = N(s) + 1}^{N(s) + n_1(s)} \Xi_1(D_i, s) = \sum_{s \in S} \left[ \Gamma_{1,n} \left( \frac{N(s) + n_1(s)}{n} \right) - \Gamma_{1,n} \left( \frac{N(s) + 1}{n} \right) , s \right],
$$

$$
\frac{1}{\sqrt{n}} \sum_{s \in S} \sum_{i = N(s) + n_1(s) + 1}^{N(s) + n(s)} \Xi_0(D_i, s) = \sum_{s \in S} \left[ \Gamma_{0,n} \left( \frac{N(s) + n(s)}{n} \right) - \Gamma_{0,n} \left( \frac{N(s) + n_1(s) + 1}{n} \right) , s \right].
$$

In addition, the partial sum process (w.r.t. $u \in [0, 1]$) is stochastic equicontinuity and

$$
\left( \frac{N(s)}{n}, \frac{n_1(s)}{n} \right) \rightarrow_{p} (F(s), \pi(s)p(s)).
$$

Therefore,

$$
\left( \frac{1}{\sqrt{n}} \sum_{s \in S} \sum_{i = N(s) + 1}^{N(s) + n_1(s)} \Xi_1(D_i, s), \frac{1}{\sqrt{n}} \sum_{s \in S} \sum_{N(s) + n_1(s) + 1}^{N(s) + n(s)} \Xi_0(D_i, s), \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Xi_2(S_i) \right)
$$

$$
= \left( \sum_{s \in S} \left[ \Gamma_{1,n} \left( F(s) + p(s)\pi(s), s \right) - \Gamma_{1,n} \left( F(s), s \right) \right], \sum_{s \in S} \left[ \Gamma_{0,n} \left( F(s) + p(s)\pi(s), s \right) - \Gamma_{0,n} \left( F(s) + p(s), s \right) \right] , \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Xi_2(S_i) \right) + o_p(1)
$$

and by construction,

$$
\sum_{s \in S} \left[ \Gamma_{1,n} \left( F(s) + p(s)\pi(s), s \right) - \Gamma_{1,n} \left( F(s), s \right) \right],
$$

$$
\sum_{s \in S} \left[ \Gamma_{0,n} \left( F(s) + p(s)\pi(s), s \right) - \Gamma_{0,n} \left( F(s) + p(s), s \right) \right],
$$

and

$$
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Xi_2(S_i)
$$

are independent. Last, we have

$$
\sum_{s \in S} \left[ \Gamma_{1,n} \left( F(s) + p(s)\pi(s), s \right) - \Gamma_{1,n} \left( F(s), s \right) \right] \sim \mathcal{N} \left( 0, \mathbb{E}(\pi(S_i))\Xi_1^2(D_i, S_i) \right)
$$

$$
\sum_{s \in S} \left[ \Gamma_{1,n} \left( F(s) + p(s)\pi(s), s \right) - \Gamma_{1,n} \left( F(s), s \right) \right] \sim \mathcal{N} \left( 0, \mathbb{E}(1 - \pi(S_i))\Xi_0^2(D_i, S_i) \right)
$$

$$
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Xi_2(S_i) \sim \mathcal{N} \left( 0, \mathbb{E}\Xi_2^2(\Xi_2) \right).
$$

This implies the desired result.
Lemma J.3. Suppose assumptions in Theorem 4.1 hold. Then,
\[ \frac{1}{n} \sum_{i=1}^{n} A_i \hat{\xi}_1^2(D_i, S_i) \xrightarrow{p} \sigma_1^2, \quad \frac{1}{n} \sum_{i=1}^{n} (1 - A_i) \hat{\xi}_1^2(D_i, S_i) \xrightarrow{p} \sigma_0^2, \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^{n} \hat{\xi}_2^2(D_i, S_i) \xrightarrow{p} \sigma_2^2. \]

Proof. To derive the limit of \( \frac{1}{n} \sum_{i=1}^{n} A_i \hat{\xi}_1^2(D_i, S_i) \), we first define
\[
\hat{\xi}_1^*(D_i, s) = \left( 1 - \frac{1}{\bar{\pi}(s)} \right) \bar{\mu}^Y(1, s, X_i) - \bar{\mu}^Y(0, s, X_i) + \frac{Y_i}{\bar{\pi}(s)} - \tau \left( 1 - \frac{1}{\bar{\pi}(s)} \right) \bar{\mu}^D(1, s, X_i) - \bar{\mu}^D(0, s, X_i) + \frac{D_i}{\bar{\pi}(s)} \right] \]
and
\[
\hat{\xi}_1(D_i, s) = \left( 1 - \frac{1}{\bar{\pi}(s)} \right) \bar{\mu}^Y(1, s, X_i) - \bar{\mu}^Y(0, s, X_i) + \frac{Y_i}{\bar{\pi}(s)} - \hat{\tau} \left( 1 - \frac{1}{\bar{\pi}(s)} \right) \bar{\mu}^D(1, s, X_i) - \bar{\mu}^D(0, s, X_i) + \frac{D_i}{\bar{\pi}(s)} \right]\]

Then, we have
\[
\left[ \frac{1}{n_1(s)} \sum_{i \in I_1(s)} (\hat{\xi}_1^*(D_i, s) - \hat{\xi}_1(D_i, s))^2 \right]^{1/2} \leq \left[ \frac{1}{n_1(s)} \sum_{i \in I_1(s)} (\hat{\xi}_1^*(D_i, s) - \hat{\xi}_1(D_i, s))^2 \right]^{1/2} + \left[ \frac{1}{n_1(s)} \sum_{i \in I_1(s)} (\hat{\xi}_1(D_i, s) - \hat{\xi}_1(D_i, s))^2 \right]^{1/2} \leq \frac{\bar{\pi}(s) - \pi(s)}{\bar{\pi}(s) \pi(s)} \left\{ \left[ \frac{1}{n_1(s)} \sum_{i \in I_1(s)} (\bar{\mu}^Y(1, s, X_i))^2 \right]^{1/2} + \left[ \frac{1}{n_1(s)} \sum_{i \in I_1(s)} W_i^2 \right]^{1/2} \right\} + \left( |\hat{\tau} - \tau| + \frac{|\hat{\tau} \bar{\pi}(s) - \tau \pi(s)|}{\bar{\pi}(s) \pi(s)} \right) \left\{ \left[ \frac{1}{n_1(s)} \sum_{i \in I_1(s)} (\bar{\mu}^D(1, s, X_i))^2 \right]^{1/2} + |\hat{\tau}| \left[ \frac{1}{n_1(s)} \sum_{i \in I_1(s)} D_i^2(1) \right]^{1/2} \right\} + \left( \frac{1}{\bar{\pi}(s)} - 1 \right) \left\{ \left[ \frac{1}{n_1(s)} \sum_{i \in I_1(s)} (\Delta Y(1, s, X_i))^2 \right]^{1/2} + |\hat{\tau}| \left[ \frac{1}{n_1(s)} \sum_{i \in I_1(s)} (\Delta D(1, s, X_i))^2 \right]^{1/2} \right\} + \left[ \frac{1}{n_1(s)} \sum_{i \in I_1(s)} (\Delta Y(0, s, X_i))^2 \right]^{1/2} + \left[ \frac{1}{n_1(s)} \sum_{i \in I_1(s)} (\Delta D(0, s, X_i))^2 \right]^{1/2} = o_p(1),
\]

where the second inequality holds by the triangle inequality and the fact that when \( i \in I_1(s), A_i = 1, Y_i = W_i, \) and \( D_i = D_i(1), \) and the last equality is due to Assumption 2(ii) and the facts
that \( \hat{\pi}(s) \xrightarrow{p} \pi(s) \) and \( \tilde{\tau} \xrightarrow{p} \tau \). This further implies

\[
\frac{1}{n_1(s)} \sum_{i \in I_1(s)} \left( \hat{\Xi}_1^*(D_i, s) - \tilde{\Xi}_1^*(D_i, s) \right) \xrightarrow{p} 0,
\]

and thus,

\[
\left[ \frac{1}{n_1(s)} \sum_{i \in I_1(s)} \hat{\Xi}_1^2(D_i, s) \right]^{1/2} = \left[ \frac{1}{n_1(s)} \sum_{i \in I_1(s)} \left( \hat{\Xi}_1^*(D_i, s) - \frac{1}{n_1} \sum_{i \in I_1(s)} \hat{\Xi}_1^*(D_i, s) \right)^2 \right]^{1/2} + o_p(1).
\]

Next, following the same argument in the proof of Lemma J.2, we have

\[
\frac{1}{n_1(s)} \sum_{i \in I_1(s)} \hat{\Xi}_1^*(D_i, s) \overset{d}{=} \frac{1}{n_1(s)} \sum_{i = N(s) + 1}^{N(s) + n_1(s)} \left\{ \left[ 1 - \frac{1}{\pi(s)} \right] \bar{Y}(1, s, X_i^s) - \bar{Y}(0, s, X_i^s) + \frac{W_i^s}{\pi(s)} \right\}
\]

\[
- \tau \left[ \left( 1 - \frac{1}{\pi(S_i)} \right) \bar{D}(1, S_i, X_i) - \bar{D}(0, S_i, X_i) + \frac{D_i(1)}{\pi(S_i)} \right] \xrightarrow{p} \mathbb{E} \left\{ \left[ 1 - \frac{1}{\pi(S_i)} \right] \bar{Y}(1, S_i, X_i) - \bar{Y}(0, S_i, X_i) + \frac{W_i}{\pi(S_i)} \right\}
\]

\[
- \tau \left[ \left( 1 - \frac{1}{\pi(S_i)} \right) \bar{D}(1, S_i, X_i) - \bar{D}(0, S_i, X_i) + \frac{D_i(1)}{\pi(S_i)} \right] |S_i = s \}
\]

This implies

\[
\left[ \frac{1}{n_1(s)} \sum_{i \in I_1(s)} \left( \hat{\Xi}_1^*(D_i, s) - \frac{1}{n_1} \sum_{i \in I_1(s)} \hat{\Xi}_1^*(D_i, s) \right)^2 \right]^{1/2}
\]

\[
= \left[ \frac{1}{n_1(s)} \sum_{i \in I_1(s)} \left( \hat{\Xi}_1^*(D_i, s) - \mathbb{E} \left\{ \left[ 1 - \frac{1}{\pi(S_i)} \right] \bar{Y}(1, S_i, X_i) - \bar{Y}(0, S_i, X_i) + \frac{W_i}{\pi(S_i)} \right\} \right) \right]^{1/2} + o_p(1)
\]

\[
= \left[ \frac{1}{n_1(s)} \sum_{i \in I_1(s)} \left( \left[ 1 - \frac{1}{\pi(S_i)} \right] \bar{Y}(1, S_i, X_i) - \bar{Y}(0, S_i, X_i) + \frac{W_i}{\pi(S_i)} \right) \right]^{1/2} + o_p(1)
\]

\[
= \left[ \frac{1}{n_1(s)} \sum_{i \in I_1(s)} \left( \left[ 1 - \frac{1}{\pi(S_i)} \right] \bar{Y}(1, S_i, X_i) - \bar{Y}(0, S_i, X_i) + \frac{W_i}{\pi(S_i)} \right) \right]^{1/2} + o_p(1)
\]
Last, following the same argument in the proof of Lemma J.2, we have

\[
\frac{1}{n_1(s)} \sum_{i \in I_1(s)} \left( \left( 1 - \frac{1}{\pi(S_i)} \right) \hat{\mu}^Y(1, S_i, X_i) - \hat{\mu}^Y(0, S_i, X_i) + \frac{\hat{W}_i}{\pi(S_i)} \right) - \tau \left( 1 - \frac{1}{\pi(S_i)} \right) \left( \hat{\mu}^D(1, S_i, X_i) - \hat{\mu}^D(0, S_i, X_i) + \frac{\hat{D}_i(1)}{\pi(S_i)} \right) \right]^{1/2} \rightarrow \sum_{i=1}^{N(s)+n_1(s)} \left( \left( 1 - \frac{1}{\pi(s)} \right) \hat{\mu}^Y(1, s, X_i^s) - \hat{\mu}^Y(0, s, X_i^s) + \frac{\hat{W}_i^s}{\pi(s)} \right) - \tau \left( 1 - \frac{1}{\pi(s)} \right) \left( \hat{\mu}^D(1, s, X_i^s) - \hat{\mu}^D(0, s, X_i^s) + \frac{\hat{D}_i^s(1)}{\pi(s)} \right) \right]^{1/2},
\]

where \( \hat{W}_i^s = W_i^s - \mathbb{E}(W_i|S_i = s) \) and \( \hat{D}_i^s(1) = D_i^s(1) - \mathbb{E}(D_i(1)|S_i = s) \) and the last convergence is due to the fact that conditionally on \( S^{(n)}, A^{(n)}, \{X_i^s, \hat{W}_i^s, \hat{D}_i^s(1)\}_{i \in I_1(s)} \) is a sequence of i.i.d. random variables so that the standard LLN is applicable. Combining all the results above, we have shown that

\[
\frac{1}{n} \sum_{i=1}^{n} A_i \hat{\Xi}_2^2(D_i, S_i) = \sum_{s \in \mathcal{S}} \frac{n_1(s)}{n} \sum_{i \in I_1(s)} \left( \left( 1 - \frac{1}{\pi(S_i)} \right) \hat{\mu}^Y(1, S_i, X_i) - \hat{\mu}^Y(0, S_i, X_i) + \frac{\hat{W}_i}{\pi(S_i)} \right) - \tau \left( 1 - \frac{1}{\pi(S_i)} \right) \left( \hat{\mu}^D(1, S_i, X_i) - \hat{\mu}^D(0, S_i, X_i) + \frac{\hat{D}_i(1)}{\pi(S_i)} \right) \right] \rightarrow \sum_{s \in \mathcal{S}} p(s) \pi(s) \mathbb{E}(\hat{\Xi}_2^2(D_i, S_i)|S_i = s) = \mathbb{E}(\pi(S_i)\mathbb{E}(\hat{\Xi}_2^2(D_i, S_i)|S_i) = \sigma_1^2.
\]

For the same reason, we can show that

\[
\frac{1}{n} \sum_{i=1}^{n} (1 - A_i) \hat{\Xi}_2^2(D_i, S_i) \rightarrow \sigma_0^2.
\]

Last, by the similar argument, we have

\[
\frac{1}{n} \sum_{i=1}^{n} \hat{\Xi}_2^2(S_i) = \sum_{s \in \mathcal{S}} \frac{n(s)}{n} \hat{\Xi}_2^2(s) = \sum_{s \in \mathcal{S}} \frac{n(s)}{n} (\mathbb{E}(W_i - \tau D_i(1)|S_i = s) - \mathbb{E}(Z_i - \tau D_i(0)|S_i = s))^2 + a_p(1)
\]

\[
= \sum_{s \in \mathcal{S}} \frac{n(s)}{n} \hat{\Xi}_2^2(s) + a_p(1)
\]

\[
\rightarrow \sum_{s \in \mathcal{S}} p(s) \Xi_2^2(s) = \mathbb{E}(\Xi_2^2(S_i)) = \sigma_2^2.
\]
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