Non-Hermitian Floquet topological phases with arbitrarily many real-quasienergy edge states

Longwen Zhou and Jiangbin Gong

Department of Physics, National University of Singapore, Singapore 117551, Republic of Singapore

(Dated: 2018-07-04)

Topological states of matter in non-Hermitian systems have attracted a lot of attention due to their intriguing dynamical and transport properties. In this study, we propose a periodically driven non-Hermitian lattice model in one-dimension, which features rich Floquet topological phases. The topological phase diagram of the model is derived analytically. Each of its non-Hermitian Floquet topological phases is characterized by a pair of integer winding numbers, counting the number of real $0$- and $\pi$-quasienergy edge states at the boundaries of the lattice. Non-Hermiticity induced Floquet topological phases with unlimited winding numbers are found, which allow arbitrarily many real $0$- and $\pi$-quasienergy edge states to appear in the complex quasienergy bulk gaps in a well-controlled manner. We further suggest to probe the topological winding numbers of the system by dynamically imaging the stroboscopic spin textures of its bulk states.

I. INTRODUCTION

Floquet topological states of matter appear in systems described by time-periodic Hamiltonians. Being intrinsically out-of-equilibrium, these states are characterized by topological invariants, bulk-edge relations and classification schemes that are either similar to or distinct from their static counterparts [1–31]. Under properly designed driving fields, Floquet topological phases with many edge states can be engineered [28,32], resulting in intriguing transport signatures [33,34]. Especially, recipes and prototypical models were discovered, which allow the generation of unlimited number of degenerate $0/\pi$ edge modes [35,36], chiral edge states [37] and linked nodal loops [38] in well-controlled manners. Experimentally, Floquet topological states have also been found in cold atom, photonic, phononic and acoustic systems [39–45].

In more general situations, the physical setups used to realize Floquet topological phases are usually subject to gain and loss or environment-induced dissipations [46]. These processes could be effectively described by introducing a non-Hermitian part to the Hamiltonian of the Floquet system. The dynamics of the system, now induced effectively by a time-periodic non-Hermitian Hamiltonian would then be non-unitary, and the quasienergies of the corresponding Floquet operator (i.e., time evolution operator over a complete driving period) could also be complex. How will these non-Hermitian effects change a Floquet topological state, and whether non-Hermiticity induced Floquet topological phases with new features could appear in these systems are largely unexplored [47,48]. In some previous studies, periodic drivings have been employed to stabilize stroboscopic dynamics [49] and reveal signatures of exceptional points (EPs) [51–57]. In another early work, losses have been introduced as dissipative probes to the topological invariants of a Hermitian lattice model [58]. This idea motivates further theoretical analysis and experimental realizations of non-Hermitian quantum walks [59–65], together with the detection of their topological invariants [66].

In this manuscript, we further reveal the richness of Floquet topological phases in non-Hermitian systems. We first propose a periodically driven non-Hermitian lattice model, which could be realizable in various types of quantum simulators. Under periodic boundary conditions (PBC), the phase diagram of the model is obtained by deriving the general conditions for band touching points (BTPs) to appear in its complex Floquet quasienergy spectrum. Each of the phases is further shown to be characterized by a pair of integer topological winding numbers. Under open boundary conditions (OBC), these winding numbers predict the number of topological edge states appear at zero and $\pi$-quasienergies in the Floquet spectrum. Remarkably, our simple model shows that it is possible to have a non-Hermitian Floquet system with unlimited winding numbers and arbitrarily many topological edge states with purely real quasienergies. Finally, we propose to image the stroboscopic spin textures of our system, whose patterns have direct connections with the winding numbers of each of its non-Hermitian Floquet topological phases.

II. THE MODEL

In this study, we focus on a tight-binding lattice model with a ladder geometry, which is subjected to piecewise time-periodic quenches. The geometry of the lattice is illustrated in Fig. [1] with two sublattices A and B in each of its unit cells.

The time-dependent Hamiltonian of the lattice is given by:

$$\hat{H}(t) = \begin{cases} \hat{H}_1, \quad t \in [\ell T, \ell T + \frac{T}{2}] \\ \hat{H}_2, \quad t \in [\ell T + \frac{T}{2}, \ell T + T] \end{cases}$$

Here $\hat{H}_1 = \sum_n \left[ \hat{r}_x (|n+1\rangle\langle n| - h.c.) + 2\gamma |n\rangle\langle n| \right] \otimes \sigma_z$, $\hat{H}_2 = \sum_n \left[ \hat{r}_y (|n+1\rangle\langle n| + h.c.) + 2\mu |n\rangle\langle n| \right] \otimes \sigma_x$, $\ell \in \mathbb{Z}$, $T$ is the driving period, $n \in \mathbb{Z}$ is the unit cell index and the lattice constant has been set to 1. The system parameters $r_x, r_y, \mu$ and $\gamma$ all take real values. The Pauli matrices $\sigma_x, \sigma_y$ act on sublattice degrees of freedom A and B.

In the first half of a driving period, the lattice Hamiltonian contains intercell hoppings $\pm r_x$ between A,B sublattices in nearest neighbor unit cells, and asymmetric couplings $\pm 2\gamma$ between sublattices A,B in the same unit cell, which introduce

**Footnotes**

[1] zhoulw13@u.nus.edu
[2] phygj@nus.edu.sg
non-Hermitian effects. In experimental setups like coupled-resonator waveguides, such asymmetric couplings may be realized by the asymmetric scattering between a clockwise and a counterclockwise wave component within each resonator. In the second half of a driving period, the lattice Hamiltonian is Hermitian, with hoppings $2\mu$ and $r_s$ between sublattices A and B in the same unit cell and among nearest neighbor unit cells, respectively. Putting together, the Floquet operator describing the evolution of the system over a complete driving period is given by

$$
\hat{U} = e^{-i\Sigma k_n} e^{-i[-\mu k + \gamma y r_s]},
$$

and $k \in [-\pi, \pi]$ is the quasimomentum. Without periodic drivings, topological phases in similar non-Hermitian lattice models have been explored in several studies. In the following, we will show that the periodic quenches considered in this manuscript make the system much richer in realizing non-Hermitian topological states with real quasienergies, with the possibility of reaching phases with arbitrarily large topological invariants and induced solely by non-Hermitian effects.

III. BULK PROPERTIES

To reveal the richness of Floquet topological phases in the periodically driven non-Hermitian lattice (PDNLH) model introduced in Eq. (1), we first analyze the bulk quasienergy spectrum and eigenstates of its Floquet operator $U(k)$ in the following subsections. By investigating the gapless condition of the quasienergy spectrum, we obtain the trajectories of Floquet BTPs in the parameter space, which form the boundaries of different non-Hermitian Floquet topological phases. We further introduce a pair of integer winding numbers, which could uniquely characterize each of the Floquet topological phases in the phase diagram.

A. Floquet spectrum and band-touching points

The Floquet spectrum of $U(k)$ as defined in Eq. (3) has two quasienergy bands $\pm E(k)$, with $E(k)$ being an eigenvalue of the effective Floquet Hamiltonian $H(k) = -i\ln U(k)$. Since $H(k)$ is non-Hermitian, we in general have $E(k) \neq E^*(k)$. But similar to Hermitian Floquet systems, the real part of $E(k)$ is a phase factor, which is defined modulus 2$\pi$ and take values in the first quasienergy Brillouin zone $(-\pi, \pi)$. Therefore the Floquet spectrum of $U(k)$ could have two gaps around $E(k) = 0$ and $E(k) = \pi$ in the complex quasienergy plane.

From Eq. (6), it is not hard to see that $E(k)$ satisfies the equation:

$$
\cos[E(k)] = \cos[h_x(k)] \cos[h_y(k) + iy].
$$

Then if the quasienergy gap closes at $E(k) = 0$ or $E(k) = \pi$, we will have $\cos[E(k)] = 1$ or $\cos[E(k)] = -1$, respectively. Plugging Eqs. (4) and (5) into the right hand side of Eq. (6), we find the gapless conditions to be $\mu + r_s \cos(k) = m\pi \pm \arccos[\frac{1}{\cos(y)}]$ and $r_s \sin(k) = n\pi$, where $m, n$ are integers of the same parity (opposite parities) if the gap closes at $E(k) = 0$ or $E(k) = \pi$. Combining these conditions together, we find the equation for trajectories of Floquet BTPs (i.e., BTPs of the complex quasienergy spectrum) in the parameter space as:

$$
\frac{1}{r_x^2} \left( m^2 \pi^2 \pm \arccos \left( \frac{1}{\cos(y)} \right) - \mu \right)^2 + \frac{n^2 \pi^2}{r_y^2} = 1,
$$

where $m, n \in \mathbb{Z}$. These trajectories form boundaries separating different non-Hermitian Floquet topological phases, as will be discussed in the following subsections.

B. Phase boundaries

Before presenting explicit examples of the phase diagram at finite values of the non-Hermitian coupling strength $\gamma$, let’s first discuss two limiting cases. In the Hermitian limit ($\gamma = 0$), Eq. (7) reduces to $\frac{(m\pi - \mu)^2}{r_x^2} + \frac{n^2 \pi^2}{r_y^2} = 1$. The topological phases in the corresponding Hermitian Floquet system have been explored in Ref. [66], with very rich Floquet topological states.
In the opposite limit (γ → ∞), Eq. (7) is simplified to
\[ \frac{(\pi \pm \gamma) \mu}{r} + \pi^2 \gamma = 1. \] In this case, Floquet BTPs appearing at zero and π quasienergies tend to coincide with each other in the parameter space, resulting in simpler phase boundary structures. For completeness, we present examples of phase boundary diagrams near these two limits in the Appendix A.

For a general non-Hermitian coupling γ ≠ 0, the phase boundaries are split and deformed from the Hermitian limit by an amount \( \pm \arccos \frac{1}{\cosh(\gamma)} \). Note that these changes cannot be eliminated by tuning the strength of intracell hopping \( \mu \), which indicates their genuine non-Hermitian origin. In Fig. 2, we present an example of the phase boundary diagram with \( \mu = 0 \) at a fixed \( \gamma = \arccosh(\sqrt{2}) \). Floquet BTPs forming the red solid (blue dashed) lines are related to spectrum gap closings at quasienergy zero (π). We further notice that for any given integers \( m \) and \( n \), the Floquet BTPs related to the two branches \( \pm \arccos \frac{1}{\cosh(\gamma)} \) of phase boundaries are always due to gap closings at the same quasienergy (either zero or π). In between these two branches, new Floquet topological phases that are absent in the Hermitian limit emerge. In the next subsection, we will give a unique characterization of the phase formed in each closed patch on the phase boundary diagram through a pair of topological invariants.

**C. Topological winding number and phase diagram**

The symmetry classification of Hermitian Floquet topological states has been established in several studies [24,25]. In one dimension, all Floquet topological states in Hermitian systems are symmetry protected. One important class of these topological phases is protected by a chiral symmetry. Each one-dimensional Floquet topological phase with chiral symmetry is characterized by a pair of integer winding numbers, defined in two symmetric time frames of the system’s Floquet operator [24]. For non-Hermitian Floquet systems, however, a general symmetry classification scheme has not yet been formulated. Our study here provides useful resources for the establishment of such a framework.

The Floquet operator \( U(k) \) of our PDNLH model in Eq. (3) has chiral symmetry in two symmetric time frames. These frames are obtained by shifting the starting time of the evolution forward or backward over half of the driving period. The resulting Floquet operators in these symmetric time frames are given by:

\[
U_1(k) = e^{-i\frac{\mu h_0(k)}{\gamma} \sigma_z} e^{-i(h_0(k) + i\gamma) \sigma_y} e^{-i\frac{\mu h_0(k)}{\gamma} \sigma_z},
\]

\[
U_2(k) = e^{-i\frac{\mu h_0(k)}{\gamma} \sigma_z} e^{-i(h_0(k) + i\gamma) \sigma_y} e^{-i\frac{\mu h_0(k)}{\gamma} \sigma_z}.
\]

Note that \( U_1(k) \) and \( U_2(k) \) are different from \( U(k) \) by similarity transformations, and therefore sharing with it the same complex quasienergy spectrum. However, these Floquet operators in different time frames are not unitarily equivalent due to the non-Hermiticity of our system. The chiral symmetry of \( U_1(k) \) and \( U_2(k) \) is described by a unitary transformation \( \Gamma = \Gamma^* = \Gamma^{-1} \), under which

\[
\Gamma U_\alpha(k) \Gamma = U^{-1}_\alpha(k), \quad \alpha = 1, 2.
\]

It is not hard to see that \( \Gamma = \sigma_z \) satisfies this condition, and therefore describes the chiral symmetry of our PDNLH model.

With the chiral symmetry, we can introduce winding numbers for \( U_1(k) \) and \( U_2(k) \) following the same routine as in Hermitian Floquet systems [24]. Using the Euler formula and Eq. (6), we can rewrite \( U_1(k) \) and \( U_2(k) \) as

\[
U_\alpha(k) = \cos[E(k)] - i[n_{x\alpha}(k) \sigma_x + n_{y\alpha}(k) \sigma_y],
\]

where \( \alpha = 1, 2 \), and the components \( n_{x\alpha}(k) \) and \( n_{y\alpha}(k) \) are given by:

\[
n_{1x}(k) = \sin[h_0(k)] \cos[h_0(k) + i\gamma],
\]

\[
n_{1y}(k) = \sin[h_0(k) + i\gamma],
\]

\[
n_{2x}(k) = \sin[h_0(k)],
\]

\[
n_{2y}(k) = \cos[h_0(k)] \sin[h_0(k) + i\gamma].
\]

Notably, due to the chiral symmetry, both the real and imaginary parts of vectors \( n_x(k) = [n_{1x}(k), n_{1y}(k)] \) and \( n_y(k) = [n_{2x}(k), n_{2y}(k)] \) are constrained to move in a two dimensional plane under the change of \( k \). Therefore they have well defined winding numbers around the origin of the plane, given by

\[
W_\alpha = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{k \, n_{x\alpha}(k) d n_{y\alpha}(k) - n_{y\alpha}(k) d n_{x\alpha}(k)}{n_{y\alpha}^2(k) + n_{x\alpha}^2(k)}
\]

for \( \alpha = 1, 2 \). Even though the integrand of these winding numbers have non-vanishing imaginary parts, it has been shown...
that these imaginary parts in general have no windings under the integral of quasimomentum $k$ over the first Brillouin zone [73]. Therefore both $W_1$ and $W_2$ are real and take integer values. Combining these winding numbers allows us to introduce another pair of winding numbers $W_0$ and $W_\pi$, defined as

$$W_0 = \frac{W_1 + W_2}{2}, \quad W_\pi = \frac{W_1 - W_2}{2}. \quad (17)$$

These winding numbers will be used to characterize the non-Hermitian Floquet topological phases of our system.

As an example, we present in Fig. 3 the calculations of $W_0$ (blue circles) and $W_\pi$ (red stars) along two trajectories (dotted lines) in the phase boundary diagram Fig. 2. We see that in both cases, the winding numbers only take integer values, as suggested by our theory. Furthermore, the value of $W_0$ ($W_\pi$) has a quantized jump every time when a trajectory of Floquet BTPs related to gap closings at quasienergy zero ($\pi$) is crossed in the parameter space. These observations indicate that the trajectories of Floquet BTPs described by Eq. (7) are indeed boundaries of different non-Hermitian Floquet topological phases characterized by different values of winding numbers ($W_0, W_\pi$). Comparing Fig. 3(a) with the phase diagram Fig. 1 reported in Ref. [36], which studied the Floquet topological phases of our model in its Hermitian limit, we also find new topological phases in the range of system parameters $n\pi - \arccos\left(\frac{1}{\cosh(\gamma)}\right) < r_x < n\pi + \arccos\left(\frac{1}{\cosh(\gamma)}\right)$ for all $n \in \mathbb{Z}$, which are solely induced by non-Hermitian effects ($\gamma \neq 0$). Furthermore, these non-Hermiticity induced Floquet topological phases could possess arbitrarily large winding numbers. For example, in the range of parameters $r_x \in (n\pi - \arccos\left(\frac{1}{\cosh(\gamma)}\right), n\pi + \arccos\left(\frac{1}{\cosh(\gamma)}\right))$, $r_x \in (0, \pi)$ and for any $\gamma \in (0, \infty)$, we have $(W_0, W_\pi) = (n, -n)$ for any $n \in \mathbb{Z}^*$.

To demonstrate the later point, we present in Fig. 4 the calculation of $(W_0, W_\pi)$ as defined in Eq. (17) versus the hopping amplitude $r_x$, with other system parameters fixed at $\mu = 0$, $r_y = \frac{\pi}{2}$ and $\gamma = \arccosh(\sqrt{2})$. The linear and unbounded growth of $W_0$ and $W_\pi$ are clearly seen with the increase of $r_x$. Note that the non-Hermitian topological phases appearing in the region $r_x \in (n\pi - \arccos\left(\frac{1}{\cosh(\gamma)}\right), n\pi + \arccos\left(\frac{1}{\cosh(\gamma)}\right))$ for any $n \in \mathbb{Z}$ are absent in the Hermitian limit ($\gamma = 0$) of the system.

To give a more global view of the topological phases that can appear in our system, we show in Fig. 5 the results of $W_0$ and $W_\pi$ in the range of system parameters $(r_x, r_y) \in [0, 3\pi] \times [0, 3\pi]$, with $\mu = 0$ and the non-Hermitian coupling strength $\gamma = \arccosh(\sqrt{2})$. In panels 5(a) and 5(b), each color corresponds to a range of system parameters $(r_x, r_y)$ in which $W_0$ or $W_\pi$ takes the same value, as also specified in the figure. The trajectories of Floquet BTPs corresponding to gap closings at quasienergies zero (solid lines) and $\pi$ (dashed lines) are also denoted in the figure. They are found to match precisely the boundaries across which the values of $W_0$ or $W_\pi$ take quantized changes. Therefore we conclude that the trajectories of Floquet BTPs in the parameter space, as predicted by Eq. (7), are indeed boundaries of different non-Hermitian Floquet topological phases characterized by different values of winding numbers $(W_0, W_\pi)$.

The next question to ask concerns the physical implication of these winding numbers. In the following two sections, we try to address this issue from two complementary perspectives: the edge states and bulk dynamics.
In one dimensional chiral symmetric Hermitian Floquet systems, the winding numbers $W_0$ and $W_1$ have a direct connection with the number of degenerate edge state pairs $n_0$ and $n_1$ at quasienergies zero and $\pi$ [24], i.e.,

$$n_0 = |W_0|, \quad n_1 = |W_1|.$$  \hspace{1cm} (18)

This relation belongs to the family of bulk-edge correspondence in topological insulators. Experimentally, it also provides a useful route to detect topological invariants of bulk materials through quantized transport along their boundaries, and to distinguish topologically distinct states of matter by imagining bound states appearing at their interfaces.

In non-Hermitian systems, however, the relation between bulk invariants and edge states becomes a subtle issue due to the existence of EPs, which could make the Hamiltonian matrix of the system defective (algebraic multiplicity ≠ geometric multiplicity), with its spectrum extremely sensitive to boundary conditions [46]. Cases regrading the breakdown of bulk-edge correspondence and its reparation in non-Hermitian systems have been explored in several studies, with still unsettled debates over the literature [68, 72, 73, 76, 82]. The PDNHL model we introduced in Eq. (1) provides an example, in which the bulk-edge relation (18) for Hermitian Floquet systems still holds, with all topological edge states taking real quasienergies.

To show this, we studied the quasienergy spectrum of Floquet operator $\hat{U}$ as defined in Eq. (2). Numerically, they are obtained by solving the eigenvalue equation $\hat{U}|\psi\rangle = e^{iE\tau}|\psi\rangle$ under OBC. As an example, we present in Fig. 6 the Floquet spectrum $E$ versus hopping amplitude $r_x$ at fixed values of hopping amplitude $r_z = \frac{3}{2}$ and non-Hermitian coupling strength $\gamma = \arccosh(\sqrt{2})$ in a lattice of $N = 150$ unit cells. In Fig. 6(c), the red solid and blue dashed lines correspond to phase boundaries obtained from the Eq. (7) for Floquet BTPs. We see that the number of edge state pairs $(n_0, n_1)$, as denoted in Fig. 6(c), changes across each of the phase boundaries. Furthermore, refer to Fig. 5, we find that $(n_0, n_1)$ matches exactly the absolute values of winding number $(|W_0|, |W_1|)$ in each of the corresponding non-Hermitian Floquet topological phases. Therefore the relation (18) is verified for the example considered here. With the increase of $r_x$, the bulk-edge relation (18) and unlimited winding numbers as shown in Fig. 4 then allow arbitrarily many zero and $\pi$ topological edge states to appear with purely real quasienergies. This gives the first example of generating many topological edge states in non-Hermitian systems following the Floquet engineering approach.

In Appendix B, we present an example of the quasienergy spectrum $E$ versus $r_z$ for which the bulk-edge relation (18) again survives the test. In more general situations (e.g., along the direction $r_x = r_z$ in Fig. 2), the verification of Eq. (18) in our PDNHL is more demanding due to the complicated configuration of BTPs in the phase diagram, which can result
in many topological phase transitions and fluctuations of the edge state numbers in a relatively small parameter window. A more systematic understanding of these situations may require statistical-type treatments, and will be left for future studies.

Note in passing that in certain regions of the Floquet spectrum, e.g., \( \pi - \text{arccosh} \left( \frac{1}{2} \right) < r_c < 1 + \text{arccos} \left( \frac{1}{\cosh(\gamma)} \right) \) of Fig. 3, it seems that there are no bulk quasienergy gaps at \( E = \pm \pi \). But on the complex quasienergy plane \( \text{Re}E - \text{Im}E \), the edge states at \( E = \pm \pi \) are still surrounded by complex spectrum gaps. To further clarify this point, we show the quasienergy spectrum at \( r_c = \frac{\pi}{3}, \gamma = \text{arccosh} \left( \frac{\sqrt{2}}{2} \right) \) with \( r_x = \frac{\pi}{3}, r_y = \frac{\pi}{2}, \gamma = \frac{3\pi}{5} \) and \( 2\pi \) in Fig. 7. In all these examples, we see that edge states at \( E = 0 \) and \( \pm \pi \) indeed have real quasienergies and surrounded by gaps in the complex quasienergy plane. The survival of these real-quasienergy edge states in the vicinity of a dissipative bulk may have further applications in achieving robust state manipulations against environmental effects.

V. STROBOSCOPIC SPIN TEXTURES

Besides employing edge states and bulk-edge correspondence, it is also helpful to find a direct probe to the topological winding numbers of our PDNLH model. In this section, we suggest to achieve this by imaging the stroboscopic spin textures of the system in its dynamics following a sudden quench. The detections of these spin textures are already available in certain quantum simulators like ultracold atoms [83].

In Ref. [84], a dynamical classification of topological quantum phases was introduced by imaging the time-averaged spin textures of the system under consideration following a quantum quench. For a system prepared at a trivial state and quenched to a topologically nontrivial Hamiltonian at the beginning of its time evolution, it was found that the spin expectation value of the system vanishes on the so-called band inversion surfaces (BISs) after averaging over a long time [84]. For a one dimensional system described by a chiral symmetric Bloch Hamiltonian \( \hat{h}(k) = h_x(k)\sigma_x + h_y(k)\sigma_y \) after the quench, the BISs are formed by quasi-momenta \( k \in \{ -\pi, \pi \} \) at which

\[
\frac{\langle \sigma_j(k) \rangle}{\tau} = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau dt \langle \psi(k, t)|\sigma_j|\psi(k, t) \rangle = 0, \quad j = x, y.
\]  

Here \( |\psi(k, t)\rangle \) is the state of the system evolved to time \( t \) following the quench. Notably, when one of the averaged spin component \( \langle \sigma_j(k) \rangle \) (e.g., \( j = x \)) vanishes on a BIS, with the same sign on its two sides, the other averaged spin component (e.g., \( j = y \)) will in general change its sign as a function of \( k \) from one side to the other side of the BIS. A pair of such “turnovers” of the same spin component along the same direction in the Brillouin zone, e.g., one of \( \langle \sigma_j(k) \rangle \) changing form negative to positive with the increase of \( k \) around two different BISs, is related to a quantized winding number \( +1 \) or \( -1 \) of the postquench Hamiltonian \( \hat{h}(k) \) [84].

The formalism proposed in Ref. [84] is applicable to both closed and open quantum systems. Therefore, it should
also be useful to extract the topological invariants of a non-Hermitian system in its non-unitary dynamics. To do this, we introduce the stroboscopic time-averaged spin textures of our PDNLH model as

$$\langle \sigma_j(k,m) \rangle_\alpha = \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \langle \sigma_j(k,m) \rangle_\alpha, \quad j = x, y. \quad (20)$$

The stroboscopic spin expectation value $\langle \sigma_j(k,mT) \rangle_\alpha$ is given by

$$\langle \sigma_j(k,mT) \rangle_\alpha = \frac{\langle \phi_0 | U^\alpha_0(k)^m \sigma_j U^\alpha_m(k) | \phi_0 \rangle}{\langle \phi_0 | U^\alpha_0(k)^m U^\alpha_m(k) | \phi_0 \rangle}, \quad \alpha = 1, 2. \quad (21)$$

Here $|\phi_0\rangle$ is the initial state, and $U_\alpha(k)$ is the Floquet operator in symmetric time frame $\alpha$ as defined by Eqs. (8) and (9). The normalization factor $\langle \phi_0 | U^\alpha_0(k)^m U^\alpha_m(k) | \phi_0 \rangle$ is introduced since the dynamics is not unitary.

To image the bulk topological invariant of our PDNLH model, we consider dynamics following a sudden quench at the initial time $t = 0$ from the prequench Hamiltonian $h_0(k) = \sigma_z$, to the postquench Hamiltonian $h_{\alpha}(k) = i \ln U_\alpha(k)$, executed separately in the two symmetric time frames $\alpha = 1, 2$. The initial states in all cases are chosen to be the ground states of $h_0(k)$ at all $k \in [-\pi, \pi]$, and the spin textures in postquench dynamics are obtained numerically from Eqs. (20) and (21).

Computation examples of $\langle \sigma_j(k,mT) \rangle_{1,2}$ and $\langle \sigma_j(k) \rangle_{1,2}$ for two hopping amplitudes $r_1 = \frac{\pi}{2}$ and $r_2 = \frac{\pi}{12}$, with other system parameters fixed at $\mu = 0$, $r_2 = \frac{\pi}{2}$ and $\gamma = \arccosh(\sqrt{2})$ are presented in Figs. 8 and 9. The results for all $\langle \sigma_j(k) \rangle_\alpha$ are averaged over $M = 300$ driving periods.

The results at $r_1 = \frac{\pi}{2}$ are presented in Fig. 8. The patterns of stroboscopic spin expectation values $\langle \sigma_j(k,m) \rangle_{1,2}$ and $\langle \sigma_j(k,m) \rangle_{1,2}$ versus the quasimomentum $k$ and the number of evolution periods $m$ are shown in the top and middle panels. In the bottom panels of Fig. 8 we observe a pair of “turnovers” in the stroboscopic time averaged spin textures $\langle \sigma_j(k) \rangle_{1,2}$ at the center and edge of the Brillouin zone $k \in [-\pi, \pi]$, whereas $\langle \sigma_j(k) \rangle_{1,2}$ reach zero at these BISs while retaining the same sign on both sides of them (negative in time frame 1 and positive in time frame 2). Therefore we can identify winding numbers $W_1 = 1$ and $W_2 = 1$ from the behaviors of $\langle \sigma_x(k) \rangle_1$ and $\langle \sigma_x(k) \rangle_2$ in the two symmetric time frames, respectively.

Eq. (17) then yields $(W_0, W_0) = (1, 0)$ for the winding numbers of this non-Hermitian Floquet topological phase, which are consistent with the winding numbers presented in the phase diagram Fig. 5 at the same system parameters.

The results for $r_1 = \frac{5\pi}{2}$ are presented in Fig. 9. The patterns of stroboscopic spin expectation values $\langle \sigma_j(k,m) \rangle_{1,2}$ and $\langle \sigma_j(k,m) \rangle_{1,2}$ versus the quasimomentum $k$ and the number of evolution periods $m$ are shown in the top and middle panels. In the bottom panels of Fig. 9 we observe one pair (five pairs) of “turnovers” in the stroboscopic time averaged spin texture $\langle \sigma(k) \rangle_{1,2}$ at two (ten) different quasimomenta, whereas $\langle \sigma(k) \rangle_{1,2}$ reach zero at these quasimomenta but retaining the same sign on both sides of them only in the time frame $\alpha = 2$. Therefore we can identify winding numbers $W_1 = 1$ and $W_2 = 5$ from the behaviors of $\langle \sigma_x(k) \rangle_1$ and $\langle \sigma_x(k) \rangle_2$ in the two symmetric time frames, respectively.

Eq. (17) then yields $(W_0, W_0) = (3, -2)$ for the winding numbers of this non-Hermitian Floquet topological phase, which are again consistent with the winding numbers presented in the phase diagram Fig. 5 at the same system parameters. Therefore, the connection between spin textures and winding numbers is verified to be applicable also in large winding number regimes.

For system parameters sitting in regular regions of the phase diagram [i.e., $(r_x, r_y) \in (0, \pi - \arccosh(1/cosh \gamma)) \times (0, \infty)$ and $(r_x, r_y) \in (0, \infty) \times (0, \pi)$], we have checked and found the consistency between the winding numbers extracted from the time-averaged spin textures $\langle \sigma(k) \rangle_{1,2}$ and those obtained theoretically from Eq. (16). Therefore we conclude that the time-averaged spin textures, proposed first in Ref. [84], can also be a useful tool to image the bulk invariants of non-Hermitian Floquet topological phases. Experimentally, the measurement of these spin textures is already available in cold atom systems [83], providing direct signatures of topological invariants from bulk state dynamics as complementary to the detection of edge states.

More generally, it is interesting to know whether the time-averaged spin textures studied here can also be used to image the topological invariants of other non-Hermitian systems in different symmetry classes and at different physical dimensions. These questions are beyond the scope of this manuscript and we will leave it for future explorations.
VI. SUMMARY

In this work, we explored Floquet topological phases in a periodically driven non-Hermitian lattice model. We found the bulk phase diagram of the system analytically, with phase boundaries formed by Floquet BTPs of complex quasienergies versus system parameters. Each of the phases in the diagram is characterized by a pair of topological winding numbers, which take integer values and predict the number of topological edge states at 0 and π-quasienergies of the Floquet spectrum. Along certain regular directions of the phase diagram, we also found Floquet topological phases with unlimited winding numbers and arbitrarily many real-quasienergy edge states induced solely by the non-Hermiticity of the system. These edge states are also robust to (non-)Hermitian perturbations which do not break the chiral symmetry of the system. We further suggested a dynamical approach to extract the bulk topological winding numbers of the system by investigating its stroboscopic spin textures [84]. A simple connection was observed between the number of times of spin flips over the Brillouin zone in long time limit and the winding numbers, suggesting a promising route to detect these non-Hermitian Floquet topological phases in cold atom systems or other quantum simulators.

By setting \( \mu = 0 \) in the model considered in this manuscript, more asymmetry could appear in its phase diagram, resulting in other possible non-Hermitian Floquet topological phases. A thorough exploration of these situations will be left for future study. An extension of the model studied here to two-dimension may also allow the appearance of anomalous chiral edge states traversing both the 0- and π-quasienergy gaps of the Floquet spectrum. The topological characterization of these non-Hermitian anomalous edge states in Floquet systems and their possible bulk-edge correspondence are also interesting topics for future explorations.

ACKNOWLEDGEMENT

J.G. is supported by the Singapore NRF grant No. NRF-NRFI2017-04 (WBS No. R-144-000-378-281) and the Singapore Ministry of Education Academic Research Fund Tier I (WBS No. R-144-000-353-112).

Appendix A: Phase boundary diagram: more examples

In this appendix, we give two more examples of the phase boundary diagrams formed by the trajectories of Floquet BTPs in the \( r_x-r_y \) parameter space. The other system parameters are chosen as \( \mu = 0, \gamma = 0.1 \) and \( \mu = 0, \gamma = 5 \) for the two examples presented here. All the results are obtained from Eq. (7) in the main text.

In the first example as shown in Fig. A.1, the system is close to its Hermitian limit, with a small non-Hermitian coupling \( \gamma = 0.1 \). Compared with the phase diagram of the corresponding Hermitian Floquet system (see Fig. 1 of Ref. [46]), each phase boundaries now splits into a pair of closely spaced trajectories formed by Floquet BTPs. In between, new Floquet topological phases induced by non-Hermiticity appear, as discussed in the main text. But major parts of the phase diagram are still dominated by Floquet topological phases carried over from the system in its Hermitian limit.

In the second example as shown in Fig. A.2, the system is close to a non-Hermitian limit with a relatively large non-
Hermitian coupling $\gamma = 5$. New phase boundaries generated by non-Hermitian effects are now shifted in a way, such that each trajectory of Floquet BTPs related to a gap closing at quasienergy zero is almost overlapped with another trajectory of Floquet BTPs related to a gap closing at quasienergy $\pi$. This means that crossing these phase boundaries from one side to the other, we will encounter phase transitions accompanied by spectrum gap closings at both zero and $\pi$ quasienergies simultaneously, which is very different from the situations of the system in its Hermitian limit. Therefore, a large non-Hermitian coupling $\gamma$ could introduce strong modifications to the topological phases of our PDNHL model.

**Appendix B: Floquet spectrum and edge states under open boundary conditions: more examples**

In this appendix, we present more examples of the Floquet spectrum of the PDNHL model under OBC.

In Fig. [B.1](#fig:b1) we presented the Floquet spectrum $E$ versus hopping amplitude $r_x$ at fixed values of hopping amplitude $r_y = \frac{\pi}{2}$ and non-Hermitian coupling strength $\gamma = \text{arccosh}(\sqrt{2})$ in a lattice of $N = 300$ unit cells. In Fig. [B.1](#fig:b1)(c), the red solid and blue dashed lines correspond to phase boundaries obtained from the Eq. (7) for Floquet BTPs. We see that the number of edge state pairs $(n_0, n_\pi)$, as denoted in Fig. [B.1](#fig:b1)(c), changes across each of the phase boundaries. Furthermore, refer to Fig. [B.2](#fig:b2) we find that $(n_0, n_\pi)$ matches exactly the absolute value of winding number $(|W_0|, |W_\pi|)$ in each of the corresponding non-Hermitian Floquet topological phases. Therefore the relation (18) is verified for the example considered here. In Fig. [B.2](#fig:b2) we further showed the quasienergy spectrum at $r_x = \frac{\pi}{2}$, $\gamma = \text{arccosh}(\sqrt{2})$ with $r_y = \pi, 2\pi, 3\pi$ and $4\pi$ in the complex quasienergy plane. In all these examples, we see that edge states at $E = 0$ and $\pm\pi$ have real quasienergies and are surrounded by gaps in the complex quasienergy plane.

![Fig. B.1](#fig:b1) The Floquet spectrum of $\hat{U}$ versus hopping amplitude $r_x$ under OBC. The lattice has $N = 300$ unit cells. Other system parameters are fixed at $\mu = 0$, $r_y = \frac{\pi}{2}$ and $\gamma = \text{arccosh}(\sqrt{2})$. Panels (a) and (b) show the real and imaginary parts of the quasienergy $E$. Panel (c) shows the absolute values of quasienergy $E$. Red solid (blue dashed) lines represent phase boundaries at which the spectrum gaps close at quasienergy zero ($\pi$). They are obtained from Eq. (7) in the main text for Floquet BTPs. The integers in light blue in panel (c) denote the number of degenerate edge state pairs at quasienergies zero ($n_0$) and $\pi$ ($n_\pi$). Their values are equal to the absolute values of winding numbers $W_0$ and $W_\pi$ as shown in Fig. [B.1](#fig:b1)b of the main text, respectively.

![Fig. B.2](#fig:b2) The Floquet spectrum of $\hat{U}$ at separate values of $r_x$, shown in the complex quasienergy plane. The lattice has $N = 300$ unit cells. Other system parameters are fixed at $\mu = 0$, $r_y = \frac{\pi}{2}$ and $\gamma = \text{arccosh}(\sqrt{2})$. Dots at $\text{Im}E = 0$ and $|\text{Re}E| = 0$ ($|\text{Re}E| = \pi$) represent degenerate edge states with real quasienergy zero ($\pi$). Other dots denote bulk states with complex quasienergies. Panel (a): $r_x = \pi$, there is one pair of edge states at quasienergy $E = 0$, and no edge states at quasienergy $E = \pi$. Panel (b): $r_x = 2\pi$, there is one pair of edge states at quasienergy $E = 0$, and two pairs of edge states at quasienergy $E = \pi$. Panel (c): $r_x = 3\pi$, there is three pairs of edge states at quasienergy $E = 0$, and two pairs of edge states at quasienergy $E = \pi$. Panel (d): $r_x = 4\pi$, there are three pairs of edge states at quasienergy $E = 0$, and four pairs of edge states at quasienergy $E = \pi$.

---

[1] T. Oka and H. Aoki, Phys. Rev. B 79, 081406 (2009).
[2] N. H. Lindner, G. Refael, and V. Galitski, Nat. Phys. 7, 490 (2011).
[3] J. P. Dahlhaus, J. M. Edge, J. Tworzydlo, and C. W. J. Beenakker, Phys. Rev. B 84, 115133 (2011).
[4] T. Kitagawa, T. Oka, A. Brataas, L. Fu, and E. Demler, Phys. Rev. B 84, 235108 (2011).
[5] H. Wang, D. Y. H. Ho, W. Lawton, J. Wang, and J. Gong, Phys.
[67] S. Malzard, C. Poli, and H. Schomerus, Phys. Rev. Lett. **115**, 200402 (2015).

[68] T.E. Lee, Phys. Rev. Lett. **116**, 133903 (2016).

[69] S. Weimann, M. Kremer, Y. Plotnik, Y. Lumer, S. Nolte, K.G. Makris, M. Segev, M.C. Rechtsman, A. Szameit, Nat. Mater. **16**, 433 (2017).

[70] D. Leykam, K. Y. Bliokh, C. Huang, Y. Chong, F. Nori, Phys. Rev. Lett. **118**, 040401 (2017).

[71] L. Zhou, Q.h. Wang, H. Wang, J. Gong, arXiv:1711.10741 (2017).

[72] H. Shen, B. Zhen, L. Fu, Phys. Rev. Lett. **120**, 146402 (2018).

[73] C. Yin, H. Jiang, L. Li, R. Lu, S. Chen, Phys. Rev. A **97**, 052115 (2018).

[74] J.W. Ryu, N. Myoung, H.C. Park, Phys. Rev. B **96**, 125421 (2017); J. W. Ryu, N. Myoung, H. C. Park, Sci. Rep. **7**, 8746 (2017).

[75] C. Yuce, Phys. Rev. A **97**, 042118 (2018).

[76] S. Yao, Z. Wang, arXiv:1803.01876 (2018).

[77] S. Lieu, Phys. Rev. B **97**, 045106 (2018).

[78] Y.C. Hu, T.L. Hughes, Phys. Rev. B **84**, 153101 (2011).

[79] K. Esaki, M. Sato, K. Hasebe, M. Kohmoto, Phys. Rev. B **84**, 205128 (2011).

[80] Y. Xiong, Journal of Physics Communications **2**, 035043 (2018).

[81] V. M. Martinez Alvarez, J. E. Barrios Vargas, and L. E. F. Foa Torres, Phys. Rev. B **97**, 121401 (2018).

[82] Z. Gong, Y. Ashida, K. Kawabata, K. Takasan, S. Higashikawa, M. Ueda, arXiv:1802.07964 (2018).

[83] W. Sun, C. Yi, B. Wang, W. Zhang, B. C. Sanders, X. Xu, Z. Wang, J. Schmiedmayer, Y. Deng, X. Liu, S. Chen, and J. Pan, arXiv:1804.08226 (2018).

[84] L. Zhang, L. Zhang, S. Niu, and X. Liu, arXiv:1802.10061 (2018).