Replacing functors with enriched ones

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Abstract

We describe simple criteria under which a given functor is naturally equivalent to an enriched one. We do this for several bases of enrichment, namely (pointed) simplicial sets, (pointed) topological spaces and orthogonal spectra. We also describe a few corollaries, such as a result on simplicial Dwyer-Kan localizations that may be of independent interest.

1 Introduction

In this paper, we will investigate the question of when a given functor is equivalent to one that respects a certain enrichment. This question is motivated by examples of the following kind: An important result in Goodwillie’s calculus of functors is that any linear functor $L : \text{Sp} \rightarrow \text{Sp}$ is equivalent to the functor $X \mapsto X \otimes C_L$, where $C_L$ is the image of the sphere spectrum $S$ under $L$ and $\otimes$ denotes the smash product of spectra. To prove this result, one notes that the functors $L$ and $- \otimes C_L$ both preserve finite colimits and shifts. Since every finite spectrum is obtained from the sphere spectrum via finite colimits and shifts, the result follows if one can construct a natural transformation $X \otimes C_L \rightarrow L(X)$ extending the identification $C_L = L(S)$. Such a natural transformation can be constructed explicitly, but in a more general setting where the source or target of $L$ are different, writing down an explicit description of such a comparison map may become unwieldy. However, if one were to know that $L$ is a map of $\text{Sp}$-enriched categories, then by noting that $- \otimes C_L$ is an $\text{Sp}$-enriched left Kan extension in the diagram

one obtains the desired comparison from the universal property of enriched left Kan extensions. Unfortunately, if one works with $\text{Sp}$-enriched model categories and one uses Goodwillie’s constructions of the $n$-excisive and the $n$-reduced approximations [Goo05], then the (multi)linear functors $L$ that one obtains are generally not $\text{Sp}$-enriched. If, however, one can show that the obtained functor $L$ is weakly equivalent to an enriched one, then this still suffices to obtain the desired natural equivalence $- \otimes C_L \simeq L$. This illustrates that if one knows that a given functor is equivalent to an enriched one, then
one can often avoid explicit constructions to obtain maps and use universal properties instead, leading to cleaner and more conceptual arguments.

In this paper, we will describe criteria that ensure that a functor is equivalent to an enriched one for several bases of enrichment $V$: we will first consider the cases where $V$ is the category of simplicial sets or topological spaces, we will then consider their pointed counterparts $V = \text{sSet}_*, \text{Top}_*$, and finally, we will consider the case where $V$ is the category of orthogonal spectra. In the topological case, our main result is the following.

**Theorem A.** Let $\mathcal{C}$ be a small topological category and $\mathcal{N}$ a good topological model category, and assume either that $\mathcal{C}$ admits tensors by the unit interval or that $\mathcal{C}$ admits cotensors by the unit interval. Then an unenriched functor $\mathcal{C} \to \mathcal{N}$ is equivalent to a $\text{Top}$-functor if and only if it sends homotopy equivalences to weak equivalences.

By a topological (model) category, we mean a $\text{Top}$-enriched (model) category. For the analogous result in the context of simplicial categories, we refer the reader to Theorem 3.3.

The definition of a good topological model category is somewhat technical and given in Definition 2.23 below. However, most topological model categories that one encounters in nature satisfy the assumptions of Definition 2.23, meaning that Theorem A applies to them. Moreover, Theorem A can be strengthened to produce Quillen equivalences between certain model categories of unenriched and enriched functors, see Theorems 3.10 and 3.15.

As a byproduct of the simplicial version of this theorem, we obtain a result on the existence of simplicial (co)fibrant replacement functors in general simplicial model categories (see Corollary 3.14) and the following result on simplicial localizations.

**Theorem B.** Let $\mathcal{C}$ be a small simplicial category that either admits tensors by the standard simplex $\Delta[n]$ for every $n \geq 0$, or that admits cotensors by the standard simplex $\Delta[n]$ for every $n \geq 0$. Then the simplicial localization of the underlying category of $\mathcal{C}$ at the simplicial homotopy equivalences is $\text{DK}$-equivalent to $\mathcal{C}$.

A remarkable corollary of this result is the fact that the localization of the underlying category of the category of finite simplicial sets $\text{sSet}^{\text{fin}}$ at the set of simplicial homotopy equivalences returns the category $\text{sSet}^{\text{fin}}$ with its usual simplicial enrichment.

After treating the simplicial and topological case, we study which functors can be replaced by ones that respect an enrichment in $\text{sSet}_*$ or $\text{Top}_*$. The main result is that, under mild hypotheses, a functor is equivalent to a $\text{Top}_*$- or $\text{sSet}_*$-enriched one if and only if it is reduced; that is, it preserves the zero object up to weak equivalence. For precise statements, see Theorems 4.3 and 4.10. As a corollary of these results, we obtain new symmetric monoidal model categories of spectra in Examples 4.8 and 4.11: we obtain analogues of Lydakis’s stable category of simplicial functors [Lyd98] and of the $\mathcal{Y}$-spectra of [Man+01] where one works with unenriched functors as opposed to $\text{sSet}_*$- or $\text{Top}_*$-enriched functors.

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1In fact, the author is not aware of any natural example of a topological model category that is not good in the sense of Definition 2.23.

2The simplicial hom-sets of $\text{sSet}^{\text{fin}}$ generally have the wrong homotopy type; that is, the simplicial hom-set between two finite simplicial sets usually does not agree with the space of maps between the corresponding spaces. The fact that one can still recover the homotopy types of these “wrong” mapping spaces by localizing at the “wrong” maps strikes the author as remarkable.
Finally, we turn to the case of functors enriched in orthogonal spectra. We show that one can replace a \( \text{Top}_\ast \)-enriched functor with an \( \text{Sp} \)-enriched functor if and only if it satisfies a certain linearity condition. We call an \( \text{Sp} \)-enriched (model) category a spectral (model) category.

**Theorem C.** Let \( \mathcal{C} \) be a small spectral category that admits cotensors by \( S^{-1} \), let \( \mathcal{N} \) be a good spectral model category and assume that the hom-spaces of the underlying \( \text{Top}_\ast \)-category of \( \mathcal{C} \) are nondegenerately based. For any \( \text{Top}_\ast \)-enriched functor \( F : \mathcal{C} \to \mathcal{N} \), there exists a canonical map

\[
L \Sigma F(c) \to F(c^{S^{-1}}),
\]

and \( F \) is naturally equivalent to an \( \text{Sp} \)-enriched functor if and only if this map is a weak equivalence for every \( c \) in \( \mathcal{C} \).

Here \( L \Sigma \) denotes the left derived functor of the suspension functor, and the canonical map \( L \Sigma F(c) \to F(c^{S^{-1}}) \) is constructed in Construction 5.1. One can view the condition that the map \( L \Sigma F(c) \to F(c^{S^{-1}}) \) is a weak equivalence as a linearity condition: since cotensors by \( S^{-1} \) are a type of suspension, this map being a weak equivalence can be interpreted as saying that \( F \) preserves suspensions. The definition of a good spectral model category is given in Definition 2.31 below; like in the topological case, it is a condition that is satisfied in many examples that one encounters in nature. Finally, let us also mention that we upgrade the statement of Theorem C to a Quillen equivalence in Theorem 5.16. It should be noted that Pereira proved a similar result in [Per13, Thm. 10.15] for functors enriched in symmetric spectra, namely when \( \mathcal{N} \) is the category \( \text{Mod}_B \) and \( \mathcal{C} \) is a certain small full subcategory of \( \text{Mod}_A \), where \( A \) and \( B \) are commutative algebras in symmetric spectra.

In Goodwillie calculus, an important role is played by symmetric multilinear functors. Since the intended applications of this work lie in Goodwillie calculus, we also deduce analogues of our main results for (symmetric) functors of multiple variables in Sections 4.3 and 5.2.

The proofs of our main results all follow roughly the same strategy. Namely, in all cases we use (derived) enriched left Kan extensions to replace functors with enriched ones, which we construct either using bar constructions or enriched coends. Our results are then deduced through a careful analysis of these bar constructions and enriched coends.

**Overview of the paper**

We start by discussing some preliminaries on enriched (model) category theory and the bar construction in Section 2. Most results of this section are well-known, and we advise the reader to only skim this section at a first reading and refer back to it as necessary while reading the rest of the paper.

We then prove Theorem A and its simplicial version, namely Theorem 3.3, in Section 3. Here we also prove Theorem B, our result on simplicial localizations. The reader who is solely interested in this result can skip most of this paper and restrict their attention to Sections 3.1 and 3.3 and the necessary parts of Section 2.

In Section 4, we prove our main results in the setting of \( \text{sSet}_\ast \) and \( \text{Top}_\ast \)-enriched functors. We treat the case of (symmetric) functors of several variables separately in Section 4.3, since this requires some extra work.
Finally, we treat the case of spectrally enriched functors in Section 5. We prove Theorem C in Section 5.1 and generalize this result to the case of (symmetric) functors of several variables in Section 5.2.

**Notation and terminology**

Throughout this paper, whenever an adjunction is depicted as

\[ F : \mathcal{C} \rightleftarrows \mathcal{D} : G, \]

the left adjoint is always on the left.

Let \((\mathcal{V}, \otimes, 1)\) denote a closed symmetric monoidal category. For brevity, we will call a category enriched in \(\mathcal{V}\) a \(\mathcal{V}\)-category. Similarly, \(\mathcal{V}\)-enriched functors will be called \(\mathcal{V}\)-functors and \(\mathcal{V}\)-enriched natural transformations will be called \(\mathcal{V}\)-natural transformations.

Let \(\mathcal{C}\) and \(\mathcal{D}\) be \(\mathcal{V}\)-categories together with a \(\mathcal{V}\)-functor \(F : \mathcal{C} \to \mathcal{D}\), let \(x, y\) and \(z\) be objects of \(\mathcal{C}\) and let \(v\) be an object of \(\mathcal{V}\). We will use the following notation and terminology:

- The hom-objects of \(\mathcal{C}\) are denoted by \(\mathcal{C}(x, y)\).
- Composition in \(\mathcal{C}\) is denoted by \(c : \mathcal{C}(y, z) \otimes \mathcal{C}(x, y) \to \mathcal{C}(x, z)\).
- The tensor (if it exists) of \(x\) by \(v\) is denoted \(v \otimes x\).
- The cotensor (if it exists) of \(x\) by \(v\) is denoted \(x^v\).
- In case the relevant tensor in \(\mathcal{D}\) exists, we will write \(ac : \mathcal{C}(x, y) \otimes F(x) \to F(y)\) for the adjunct of \(\mathcal{C}(x, y) \to \mathcal{D}(F(x), F(y))\) and call it the *action* of \(\mathcal{C}(x, y)\) on \(F(x)\).
- The category of \(\mathcal{V}\)-functors \(\mathcal{C} \to \mathcal{D}\) and \(\mathcal{V}\)-natural transformations between them is denoted \(\text{Fun}(\mathcal{C}, \mathcal{D})\). Note that if \(\mathcal{V}\) is complete and \(\mathcal{C}\) is small, then \(\text{Fun}(\mathcal{C}, \mathcal{D})\) is (canonically) a \(\mathcal{V}\)-category.

Throughout this paper, we will mainly consider the following symmetric monoidal categories as bases of enrichment:

- The category \(\text{Top}\) of (compactly generated weak Hausdorff) topological spaces, endowed with the Cartesian product. A \(\text{Top}\)-category will be called a *topological* category and a \(\text{Top}\)-functor will be called *continuous*.
- The category \(\text{sSet}\) of simplicial sets, endowed with the Cartesian product. We will call \(\text{sSet}\)-categories and \(\text{sSet}\)-functors *simplicial* categories and *simplicial* functors, respectively.
- The category \(\text{Top}_\ast\) of pointed topological spaces, endowed with the smash product.
- The category \(\text{sSet}_\ast\) of pointed simplicial sets, endowed with the smash product.
- The category \(\text{Sp}\) of orthogonal spectra, endowed with the Day convolution product. An \(\text{Sp}\)-category will be called a *spectral* category and an \(\text{Sp}\)-functor will similarly be called a *spectral* functor.

All adjunctions between symmetric monoidal categories considered in this paper are **strong symmetric** monoidal, so we will simply call them *monoidal adjunctions*.

Since we will frequently change the base of enrichment, our categories are often decorated by subscripts indicating which enrichment on that category is considered. For example, if \(\mathcal{C}\) is a \(\text{Top}\)-enriched category, then its underlying simplicial category, obtained by taking the singular complex of each hom-space, is denoted \(\mathcal{C}_\Delta\), while its underlying ordinary category is denoted \(\mathcal{C}_0\). When considering functors between enriched categories,
we use the convention that the enrichment of the domain category indicates the enrichment of the functor. For example, if \( C \) and \( D \) are topological categories, then \( \text{Fun}(C, D) \) indicates the category of continuous functors, while \( \text{Fun}(C_0, D) \) indicates the category of ordinary functors from \( C \) to \( D \).

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**2 Preliminaries**

We will first discuss some preliminaries on enriched categories, model categories and bar constructions that will be used throughout the rest of the paper. Most of the definitions and facts discussed here are standard, although the notion of “goodness” discussed in Definitions 2.23, 2.28 and 2.31 is not. We advise the reader to only skim these preliminaries at a first reading and refer back to them as necessary while reading the rest of the paper.

**2.1 Enriched category theory**

Throughout this section, let \((V, \otimes, 1)\) be a closed symmetric monoidal category that is both complete and cocomplete. For the convenience of the reader, we will briefly discuss several concepts from enriched category theory that are used throughout this paper. We will assume that the reader is familiar with basic notions of enriched category theory, in particular \(V\)-categories, \(V\)-functors, \(V\)-natural transformations and \(V\)-adjunctions, the definition of \(V\)-(co)completeness, the construction of the \(V\)-category \(\text{Fun}(C, E)\) of \(V\)-functors between two \(V\)-categories \(C\) and \(E\), and the enriched Yoneda lemma.\(^3\) For an introduction to these concepts, we refer the reader to \([Rie14, \S3]\) and \([Kel82]\).

We will first recall the definitions of (co)tensors, (co)ends and enriched Kan extensions, and then discuss changing the base of enrichment along a symmetric monoidal adjunction.

**(Co)tensors**

**Definition 2.1.** Let \( C \) be a \( V\)-category and \( v \) an object in \( V \). We say that \( C \) admits tensors by \( v \) if for any object \( c \) of \( C \), there exists an object \( v \otimes c \) in \( C \) together with isomorphisms

\[
\mathcal{C}(v \otimes c, d) \cong V(v, \mathcal{C}(c, d))
\]

that are \(V\)-natural in \( d \). Dually, we say that \( C \) admits cotensors by \( v \) if for any object \( d \) of \( C \), there exists an object \( d^v \) in \( C \) together with isomorphisms

\[
\mathcal{C}(c, d^v) \cong V(v, \mathcal{C}(c, d))
\]

\(^3\)Since we define (co)tensors and enriched (co)ends below, which are often used to define \(V\)-(co)completeness and to construct the hom-objects in \(\text{Fun}(C, E)\), this assumption is somewhat circular. However, the purpose of this section is merely to recall the concepts from enriched category theory that play an important role throughout this paper and not to serve as a self-contained introduction.
that are \(\mathcal{V}\)-natural in \(c\).

Throughout this paper, we will use that (co)tensors, when they exist, can be made functorial. To make this precise, let us define the product of two \(\mathcal{V}\)-categories.

**Definition 2.2.** Given \(\mathcal{V}\)-categories \(\mathcal{E}\) and \(\mathcal{D}\), define the \(\mathcal{V}\)-category \(\mathcal{E} \otimes \mathcal{D}\) by

(a) defining the objects as \(\text{Ob}(\mathcal{E} \otimes \mathcal{D}) := \text{Ob}(\mathcal{E}) \times \text{Ob}(\mathcal{D})\), and

(b) defining the hom-objects by \((\mathcal{E} \otimes \mathcal{D})(c, d), (c', d') := \mathcal{E}(c, c') \otimes \mathcal{D}(d, d')\).

The identities and compositions are defined in the obvious way, making use of the symmetry of \(\mathcal{V}\).

**Remark 2.3.** This product can be promoted to a closed symmetric monoidal structure on the category \(\mathcal{V}\text{-Cat}\) of small \(\mathcal{V}\)-categories. The internal hom for this symmetric monoidal structure is given by the \(\mathcal{V}\)-category \(\text{Fun}(\mathcal{E}, \mathcal{D})\) of \(\mathcal{V}\)-functors (see [Kel82, §2.3] for details).

**Construction 2.4.** Let \(\mathcal{E}\) be a \(\mathcal{V}\)-category and suppose that \(\mathcal{U}\) is a class of objects in \(\mathcal{V}\) such that \(\mathcal{E}\) admits tensors by every \(u \in \mathcal{U}\). Let \(\mathcal{U}\) abusively also denote the full sub-\(\mathcal{V}\)-category of \(\mathcal{V}\) spanned by these objects. Then the tensors \(u \otimes c\) can be promoted to a \(\mathcal{V}\)-functor \(\mathcal{U} \otimes \mathcal{E} \to \mathcal{E}\) and the natural transformation \(\mathcal{E}(u \otimes c, d) \cong \mathcal{V}(u, \mathcal{E}(c, d))\) to a \(\mathcal{V}\)-natural transformation of functors \(\mathcal{U}^{\text{op}} \otimes \mathcal{E}^{\text{op}} \otimes \mathcal{E} \to \mathcal{V}\). This essentially follows by identifying \(\mathcal{U} \otimes \mathcal{E}\) with its image in \(\text{Fun}(\mathcal{U}^{\text{op}} \otimes \mathcal{E}^{\text{op}}, \mathcal{V})\) under the Yoneda embedding; see [Kel82, §1.10] for details. Of course, the dual statement holds for cotensors.

**((Colends)** In the following definition of enriched (co)ends, we will assume that \(\mathcal{E}\) is a \(\mathcal{V}\)-(co)complete category. For a definition that works for any \(\mathcal{V}\)-category \(\mathcal{E}\) and which is equivalent to the one given here when \(\mathcal{E}\) is \(\mathcal{V}\)-(co)complete, we refer the reader to [Kel82, §3.10].

**Definition 2.5.** Let \(\mathcal{E}\) be a small \(\mathcal{V}\)-category, \(\mathcal{E}\) a \(\mathcal{V}\)-(co)complete \(\mathcal{V}\)-category, and let a \(\mathcal{V}\)-functor \(F: \mathcal{E}^{\text{op}} \otimes \mathcal{E} \to \mathcal{E}\) be given. Then the coend \(\int^{\mathcal{E}} F(c, c)\) is defined as the coequalizer

\[
\int^{\mathcal{E}} F(c, c) := \text{coeq} \left( \bigsqcup_{c, c'} \mathcal{E}(c, c') \otimes F(c', c) \rightrightarrows \bigsqcup_{c} F(c, c) \right)
\]

in \(\mathcal{E}\), and the end \(\int_{c} F(c, c)\) as the equalizer

\[
\int_{c} F(c, c) := \text{eq} \left( \bigsqcup_{c} F(c, c) \rightrightarrows \bigsqcup_{c, c'} F(c, c')^{\mathcal{E}(c, c')} \right).
\]

**Example 2.6.** Suppose \(\mathcal{E}\) is a \(\text{sSet}_{\text{1-}}\)-cocomplete simplicial category and let \(X_{\bullet}\) be a simplicial object in \(\mathcal{E}\). Then the geometric realization of \(X_{\bullet}\) in \(\mathcal{E}\) is the coend

\[
|X_{\bullet}| = \int^{\Delta^{\text{op}}} \Delta^{n} \otimes X_{n}
\]

**Example 2.7.** Given two \(\mathcal{V}\)-functors of \(\mathcal{V}\)-categories \(F, G: \mathcal{E} \to \mathcal{D}\) with \(\mathcal{E}\) small, then the natural transformations object \(\text{Nat}(F, G)\) is defined as the end

\[
\text{Nat}(F, G) = \int_{c} \mathcal{D}(F(c), G(c)).
\]

These are the hom-objects of the \(\mathcal{V}\)-category \(\text{Fun}(\mathcal{E}, \mathcal{D})\).
There exists the following “Fubini theorem” for enriched (co)ends (cf. (3.63) of [Kel82]).

**Lemma 2.8.** Let $\mathcal{C}, \mathcal{D}$ be small $\mathcal{V}$-categories, $\mathcal{E}$ a $\mathcal{V}$-(co)complete $\mathcal{V}$-category and $F : \mathcal{C}^{op} \otimes \mathcal{C} \otimes \mathcal{D}^{op} \otimes \mathcal{D} \to \mathcal{E}$ a $\mathcal{V}$-functor. Then there are canonical isomorphisms

$$\int^c \int^d F(c, c, d, d) \cong \int^{c \otimes d} F(c, c, d, d) \cong \int^d \int^c F(c, c, d, d)$$

and

$$\int^c \int^d F(c, c, d, d) \cong \int^{c \otimes d} F(c, c, d, d) \cong \int_d \int^c F(c, c, d, d).$$

**Kan extensions** As in our definition of (co)ends, we will define enriched Kan extensions only for functors into a $\mathcal{V}$-(co)complete $\mathcal{V}$-category $\mathcal{E}$, favouring the explicit formula that one can write down in this case over the more general definition given in [Kel82, §4.1].

**Definition 2.9.** Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be $\mathcal{V}$-categories, let $\alpha : \mathcal{C} \to \mathcal{D}$ be a functor, and suppose that $\mathcal{C}$ is small and that $\mathcal{E}$ is $\mathcal{V}$-(co)complete. For a functor $F : \mathcal{C} \to \mathcal{E}$, we define the (enriched) left Kan extension $\text{Lan}_\alpha F : \mathcal{D} \to \mathcal{E}$ of $F$ along $\alpha$ by

$$\text{Lan}_\alpha F(d) = \int^c \mathcal{D}(ac, d) \otimes F(c)$$

and the (enriched) right Kan extension $\text{Ran}_\alpha F : \mathcal{D} \to \mathcal{E}$ of $F$ along $\alpha$ by

$$\text{Ran}_\alpha F(d) = \int_c F(c)^{\mathcal{D}(d, ac)}.$$

By [Kel82, Thm. 4.50], Kan extension provides left and right adjoints to the functor $\text{Fun}(\mathcal{D}, \mathcal{E}) \to \text{Fun}(\mathcal{C}, \mathcal{E})$ given by precomposition with $\alpha$.

**Proposition 2.10.** If in the hypotheses of Definition 2.9 we furthermore assume that $\mathcal{D}$ is small, then all (enriched) left and right Kan extensions assemble into functors $\text{Lan}_\alpha, \text{Ran}_\alpha : \text{Fun}(\mathcal{C}, \mathcal{E}) \to \text{Fun}(\mathcal{D}, \mathcal{E})$ that are left and right $\mathcal{V}$-adjoint to the precomposition functor $\alpha^* : \text{Fun}(\mathcal{D}, \mathcal{E}) \to \text{Fun}(\mathcal{C}, \mathcal{E})$, respectively.

**Change of base** Recall that an adjunction $\mathcal{V} \rightleftarrows \mathcal{W}$ between symmetric monoidal categories is called strong symmetric monoidal if the left adjoint is strong symmetric monoidal; note that in this case the right adjoint is automatically lax monoidal. All adjunctions between symmetric monoidal categories considered in this paper are strong symmetric monoidal, so we will simply call them monoidal adjunctions from now on.

Fix a monoidal adjunction $L : \mathcal{V} \rightleftarrows \mathcal{W} : U$ between (co)complete closed symmetric monoidal categories. Then any $\mathcal{W}$-category $\mathcal{D}$ has an underlying $\mathcal{V}$-category $U\mathcal{D}$ obtained by applying $U$ to each hom-object of $\mathcal{D}$. Similarly, for a $\mathcal{V}$-category $\mathcal{C}$ one can construct the free $\mathcal{W}$-category $L\mathcal{C}$ on $\mathcal{C}$ by applying $L$ to each hom-object. These define an adjunction

$$L : \mathcal{V}\text{-Cat} \rightleftarrows \mathcal{W}\text{-Cat} : U.$$  \hspace{1cm} (1)

When in the presence of a monoidal adjunction $L : \mathcal{V} \rightleftarrows \mathcal{W} : U$ and when given a $\mathcal{V}$-category $\mathcal{C}$ and a $\mathcal{W}$-category $\mathcal{D}$, we will write $\text{Fun}(\mathcal{C}, \mathcal{D})$ to denote the $\mathcal{W}$-category $\text{Fun}(L\mathcal{C}, U\mathcal{D})$ (which agrees with $\text{Fun}(\mathcal{C}, U\mathcal{D})$ as a $\mathcal{V}$-category by (1)).

If we are given a second $\mathcal{W}$-category $\mathcal{E}$, then the functor $\text{Fun}(\mathcal{D}, \mathcal{E}) \to \text{Fun}(U\mathcal{D}, U\mathcal{E})$ sending a $\mathcal{W}$-functor to its underlying $\mathcal{V}$-functor agrees with the functor $\text{Fun}(\mathcal{D}, \mathcal{E}) \to \text{Fun}(L\mathcal{D}, \mathcal{E})$ given by precomposition with the counit $LU\mathcal{C} \to \mathcal{C}$. In particular, Proposition 2.10 gives the following result.
Let \( C \) be a small \( \mathcal{W} \)-category and \( E \) a \( \mathcal{W} \)-(co)complete \( \mathcal{W} \)-category. Then the functor 
\[
\text{Fun}(C, E) \rightarrow \text{Fun}(U_C, E)
\]
sending a functor to its underlying \( \mathcal{V} \)-functor has both a left and right \( \mathcal{W} \)-adjoint, which are given by left and right Kan extension along \( U_C \), respectively.

Finally, we wish to mention the following useful fact (cf. [Rie14, Thm. 3.7.11]).

Lemma 2.12. Let \( L : \mathcal{V} \rightleftarrows \mathcal{W} : U \) be a monoidal adjunction between (co)complete closed symmetric monoidal categories. Then for any \( \mathcal{W} \)-(co)complete category \( E \), the underlying \( \mathcal{V} \)-category \( UE \) is \( \mathcal{V} \)-(co)complete. Moreover, tensors and cotensors by \( v \in \text{Ob}(\mathcal{V}) \) in \( UE \) are computed as tensors and cotensors by \( Lv \) in \( E \).

2.2 Model categories

We will assume that the reader is familiar with the basic theory of model categories as explained in e.g. [Hov99, Ch. 1] or [Hiro3, Chs. 7-8].

Enriched model categories  For the reader’s convenience, we briefly recall a few facts about enriched model categories. We wish to mention that in all the symmetric monoidal model categories considered throughout this paper, the unit is cofibrant. Since the theory of enriched model categories simplifies if this is the case, we have decided to make this part of our definition.

Definition 2.13. A symmetric monoidal model category \((\mathcal{V}, \otimes, \mathbb{1})\) is a (co)complete closed symmetric monoidal category together with a model structure on \( \mathcal{V} \) such that

(a) for any pair of cofibrations \( v \rightarrow v' \) and \( w \rightarrow w' \), the pushout-product map 
\[
v \otimes w' \cup_{v \otimes w} v' \otimes w \rightarrow v' \otimes w'
\]
is a cofibration which is trivial if one of \( v \rightarrow v' \) and \( w \rightarrow w' \) is, and

(b) the unit \( \mathbb{1} \) is cofibrant.

Definition 2.14. Given a symmetric monoidal model category \( \mathcal{V} \), a \( \mathcal{V} \)-model category is a \( \mathcal{V} \)-category \( N \) that is equipped with a model structure on its underlying category \( N_0 \) such that

(a) \( N \) is \( \mathcal{V} \)-(co)complete, and

(b) for any cofibration \( M \rightarrow N \) in \( N_0 \) and any fibration \( L \rightarrow K \) in \( N_0 \), the pullback-power map 
\[
N(N, L) \rightarrow N(N, K) \times_{N(M, K)} N(M, L)
\]
is a fibration in \( \mathcal{V} \), which is trivial if either \( M \rightarrow N \) or \( L \rightarrow K \) is.

Given two symmetric monoidal model categories \( \mathcal{V} \) and \( \mathcal{W} \), we define a monoidal Quillen pair as a (strong symmetric) monoidal adjunction \( \mathcal{V} \rightleftarrows \mathcal{W} \) that is also a Quillen pair. Given such a monoidal Quillen pair \( L : \mathcal{V} \rightleftarrows \mathcal{W} : U \) and a \( \mathcal{W} \)-model category \( N \), it follows from Lemma 2.12 and the fact that \( U \) preserves (trivial) fibrations that the underlying \( \mathcal{V} \)-category \( UN \) is a \( \mathcal{V} \)-model category. The main question that this paper is concerned with, is that for given a \( \mathcal{W} \)-category \( \mathcal{E} \), which \( \mathcal{V} \)-functors \( UE \rightarrow N \) are equivalent to a \( \mathcal{W} \)-functor \( \mathcal{E} \rightarrow N \)?
Definition 2.15. In a situation as above, we say that a \( V \)-functor \( F : \mathcal{U} \mathcal{C} \to \mathcal{N} \) is \textit{equivalent to a} \( W \)-functor if there exists a \( W \)-functor \( G : \mathcal{C} \to \mathcal{N} \) and a zigzag of \( V \)-natural transformations between \( F \) and \( UG \) that are pointwise weak equivalences.

Remark 2.16. What we call a monoidal Quillen pair above should actually be called a strong symmetric monoidal Quillen pair. However, since all monoidal adjunctions that we consider in this paper are strong symmetric, we will drop these adjectives.

Example 2.17. The category \( \mathbf{sSet} \) of simplicial sets can be equipped with the Kan-Quillen model structure [Qui67, §II.3]. This is a symmetric monoidal model category under the Cartesian product. A \( \mathbf{sSet} \)-model category will be called a \textit{simplicial model category}.

Example 2.18. Let \( \mathbf{Top} \) denote the category of compactly generated weak Hausdorff spaces. Then \( \mathbf{Top} \), endowed with the Quillen model structure [Qui67, §II.3], is a symmetric monoidal model category under the Cartesian product. A \( \mathbf{Top} \)-model category will be called a \textit{topological model category}. The singular complex and geometric realization form a monoidal Quillen equivalence

\[ |−| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : \text{Sing}, \]

hence any topological model category has an underlying simplicial model category.

Example 2.19. The categories \( \mathbf{sSet}_* \) and \( \mathbf{Top}_* \) have model structures in which the fibrations, cofibrations and weak equivalences are defined underlying in \( \mathbf{sSet} \) and \( \mathbf{Top} \), respectively. Both are symmetric monoidal model categories under the smash product, and the adjunctions

\[ (−)_+ : \mathbf{sSet} \rightleftarrows \mathbf{sSet}_* : U \quad \text{and} \quad (−)_+ : \mathbf{Top} \rightleftarrows \mathbf{Top}_* : U, \]

where \( (−)_+ \) adds a disjoint basepoint and \( U \) forgets the basepoint, are monoidal Quillen adjunctions. Moreover, the singular complex and geometric realization again form a monoidal Quillen equivalence

\[ |−| : \mathbf{sSet}_* \rightleftarrows \mathbf{Top}_* : \text{Sing}. \]

Orthogonal spectra In this paper, we will work with the category \( \mathbf{Sp} \) of orthogonal spectra as our preferred symmetric monoidal model category of spectra. Recall that the category of orthogonal spectra is defined as the enriched diagram category \( \text{Fun}(\mathcal{I}_\mathbb{S}, \mathbf{Top}_*) \), where \( \mathcal{I}_\mathbb{S} \) is the \( \mathbf{Top}_* \)-category whose objects are the natural numbers and whose hom-objects are given by \( \mathcal{I}_\mathbb{S}(m,n) = O(n)_+ \wedge_{O(n-m)} S^{n-m} \). The category \( \mathcal{I}_\mathbb{S} \) is symmetric monoidal, hence \( \mathbf{Sp} \) is closed symmetric monoidal under the Day convolution product \( \otimes \). A category enriched in \( \mathbf{Sp} \) will be called a \textit{spectral} category.

We consider \( \mathbf{Sp} \) as a model category by endowing it with the stable model structure of [Man+01]. This is a left Bousfield localization of the projective model structure compatible with the Day convolution product. In particular, the stable model structure on \( \mathbf{Sp} \) is a symmetric monoidal model category in the sense of Definition 2.13. The category of orthogonal spectra \( \mathbf{Sp} \) comes with a monoidal left Quillen functor \( \Sigma^\infty \) from \( \mathbf{Top}_* \), hence we can speak of the underlying \( \mathbf{Top}_* \)-category of a spectral category. An \( \mathbf{Sp} \)-model category in the sense of Definition 2.14 will be called a \textit{spectral} model category.

Recall that a model category is called \textit{stable} if it has a zero object and the suspension functor induces a self-equivalence on the homotopy category.
Lemma 2.20. Any spectral model category is stable.

Proof. A spectral model category $N$ has a zero object since it is (co)tensored over $\mathbf{Sp}$ and $\mathbf{Sp}$ has a zero object. To see that it is stable, note that we can compute suspensions as (derived) tensors with $S^1$. If we let $S^{-1}$ denote the image of 1 under the Yoneda embedding $S^p \to \mathbf{Sp}$, then we have a stable equivalence $S^{-1} \otimes S^1 \cong S^1 \otimes S^{-1} \to S^0$. In particular, we have natural equivalences $S^1 \otimes S^{-1} \otimes N \to N$ and $S^{-1} \otimes S^1 \otimes N \to N$ for any cofibrant $N$ in $N$. This shows that on the level of homotopy categories, (derived) tensors with $S^{-1}$ are inverse to the suspension functor.

The following fact, which follows from [SS03, Prop. 6.3], will be used several times in Section 5.

Lemma 2.21. Let $\mathcal{C}$ be a small spectral category. Then there exists a small spectral category $\widehat{\mathcal{C}}$ with the same set of objects and whose hom-objects are cofibrant, together with a map $\widehat{\mathcal{C}} \to \mathcal{C}$ that is the identity on objects and a trivial fibration on all hom-objects.

Left Bousfield localization  While left Bousfield localizations are not necessary to prove our main results, we will make use of them throughout this paper to upgrade results such as Theorem A and Theorem C to actual Quillen equivalences of model categories. We assume that the reader is familiar with the theory of left Bousfield localizations; we refer to [Law20, §10] for an introduction and to [Hiro3] for more details. Let us remark that throughout this paper, a left Bousfield localization of a model category $N$ is simply defined as a new model structure on $N$ with the same class of cofibrations but a larger class of weak equivalences. We will frequently use the following result.

Lemma 2.22. Let $L : M \rightleftarrows N : U$ be a Quillen pair such that

(a) $U$ creates weak equivalences, and

(b) for any cofibrant object $X$ in $M$ that is weakly equivalent to an object in the image of $U$, the unit $X \to ULX$ is a weak equivalence.

Moreover assume that there exists a left Bousfield localization $M'$ of $M$ in which the fibrant objects are exactly the fibrant objects of $M$ that are weakly equivalent to an object in the image of $U$. Then

$$L : M' \rightleftarrows N : U$$

is a Quillen equivalence.

Proof. We first show that $L : M' \rightleftarrows N : U$ is a Quillen pair. By Corollary A.2 and Remark A.3 of [Dug01], it suffices to show that $U$ preserves fibrant objects, which is true by definition. To see that this is a Quillen equivalence, it suffices to show that the derived adjunction gives an equivalence of homotopy categories. Note that $\text{Ho}(M')$ is the full subcategory of $\text{Ho}(M)$ on objects that are in the essential image of $U$ (cf. [Law20, Prop. 10.2]). This shows that the right derived functor $\text{Ho}(N) \to \text{Ho}(M')$ is essentially

---

4Strictly speaking, one of the hypotheses of [SS03, Prop. 6.3] is that all objects are small with respect to the whole category, which is not the case in $\mathbf{Sp}$. However, all objects are small with respect to levelwise closed inclusions, which is sufficient for the proof to go through.

5See [nLa22, Prop. 3.7] for a detailed proof of the fact that a map between fibrant objects in $M'$ is a fibration in $M'$ if and only if it is in $M$. 

10
surjective, while item (a) above ensures that this functor is conservative. Moreover, item (b) ensures that the unit of the derived adjunction is an isomorphism, hence \( \text{Ho}(N) \to \text{Ho}(\mathcal{M}') \) is an equivalence.

**Goodness** We will now discuss a notion for (pointed) topological and spectrally enriched model categories \( N \) that we call “goodness”, and which can be seen as ensuring that the enriched functor categories \( \text{Fun}(\mathcal{C}, N) \) are well-behaved even if the hom-objects of \( \mathcal{C} \) are not cofibrant. While the conditions of goodness may appear to be strong, we will show that they are satisfied in many natural situations. We warn the reader that the definitions given here are not standard and that some proofs are quite technical.

Recall that, given a \( \mathcal{V} \)-model category \( N \) and a small \( \mathcal{V} \)-category \( \mathcal{C} \), the projective model structure on \( \text{Fun}(\mathcal{C}, N) \) is defined (if it exists) as the model structure in which the fibrations and the weak equivalences are the pointwise ones. We will call a functor \( F: \mathcal{C} \to \mathcal{D} \) of \( \mathcal{V} \)-categories which is bijective on objects a weak equivalence if \( \mathcal{C}(c,d) \to \mathcal{D}(F(c),F(d)) \) is a weak equivalence in \( \mathcal{V} \) for all \( c,d \in \text{Ob}(\mathcal{C}) \).

**Definition 2.23.** A topological model category \( N \) is called good if

(a) for any small topological category \( \mathcal{C} \), the projective model structure on \( \text{Fun}(\mathcal{C}, N) \) exists, and

(b) for any weak equivalence \( \alpha: \mathcal{C} \to \mathcal{D} \) of small topological categories that is bijective on objects, the precomposition functor \( \alpha^*: \text{Fun}(\mathcal{D}, N) \to \text{Fun}(\mathcal{C}, N) \) is a right Quillen equivalence.

Let us state some direct consequences of these conditions.

**Proposition 2.24.** Let \( N \) be a good topological model category, \( \mathcal{C} \) a small topological category, \( F: \mathcal{C} \to N \) a projectively cofibrant functor and \( G \xrightarrow{\sim} G' \) a pointwise weak equivalence between functors \( \mathcal{C}^{op} \to \text{Top} \). Then \( \int^\mathcal{C} G \otimes F \to \int^\mathcal{C} G' \otimes F \) is a weak equivalence in \( N \). Similarly, if \( G \) is any functor \( \mathcal{C}^{op} \to \text{Top} \) and \( F \xrightarrow{\sim} F' \) is a pointwise weak equivalence between projectively cofibrant functors \( \mathcal{C} \to N \), then \( \int^\mathcal{C} G \otimes F \to \int^\mathcal{C} G \otimes F' \) is a weak equivalence.

**Proof.** Given a functor \( G: \mathcal{C}^{op} \to \text{Top} \), one can form the \( \text{Top} \)-category \( \mathcal{C} \rhd G \) with \( \text{Ob}(\mathcal{C} \rhd G) = \text{Ob}(\mathcal{C}) \amalg \{\ast\} \) and with hom-objects given by

\[
\begin{align*}
(\mathcal{C} \rhd G)(c,d) &= \mathcal{C}(c,d) & \text{for } c,d \in \text{Ob}(\mathcal{C}) \\
(\mathcal{C} \rhd G)(c,\ast) &= G(d) & \text{for } c \in \text{Ob}(\mathcal{C}) \quad b \\
(\mathcal{C} \rhd G)(\ast,d) &= \emptyset & \text{for } d \in \text{Ob}(\mathcal{C}) \amalg \{\ast\}.
\end{align*}
\]

The composition is given by composition in \( \mathcal{C} \) and the action of \( \mathcal{C} \) on \( G \). Let \( i_G: \mathcal{C} \hookrightarrow \mathcal{C} \rhd G \) denote the inclusion. It follows from the definition of left Kan extension that

\[
\int^\mathcal{C} G \otimes F \cong (\text{Lan}_{i_G} F)(\ast).
\]

Since \( \text{Lan}_{i_G} \) is left Quillen, it preserves pointwise equivalences \( F \xrightarrow{\sim} F' \) between projectively cofibrant functors. Evaluating at \( \ast \) yields the desired equivalence \( \int^\mathcal{C} G \otimes F \xrightarrow{\sim} \int^\mathcal{C} G \otimes F' \).
On the other hand, if we are given an equivalence \( G \xrightarrow{\sim} G' \), then we obtain an equivalence \( \alpha : C \triangleright G \xrightarrow{\sim} C \triangleright G' \). The goodness of \( N \) now implies that

\[
\int_C G \otimes F \cong (\text{Lan}_{i_C} F)(\ast) \xrightarrow{\sim} (\text{Lan}_a \text{Lan}_{i_C} F)(\ast) \cong (\text{Lan}_{i_C'} F)(\ast) \cong \int_C G' \otimes F
\]

is a weak equivalence for any projectively cofibrant \( F \).  

The special case \( C = \ast \) yields the following result on tensors in \( N \).

**Proposition 2.25.** Let \( N \) be a good topological model category, \( T \) a topological space and \( N \) a cofibrant object in \( N \). Then \( T \otimes - : N \to N \) preserves weak equivalences between cofibrant objects, while \( - \otimes N : \text{Top} \to N \) preserves all weak equivalences.

Goodness also implies the existence of projective model structures on simplicial diagram categories.

**Proposition 2.26.** Let \( N \) be a good topological model category. Then for any small simplicial category \( C \), the projective model structure on \( \text{Fun}(C, N) \) exists. Moreover, for any topological category \( D \), the forgetful functor \( \text{Fun}(D, N) \to \text{Fun}(\text{Sing}(D), N) \) is a right Quillen equivalence.

**Proof.** For the first part, note that \( \text{Fun}(C, N) = \text{Fun}(|C|, N) \). For the second part, recall from Lemma 2.11 that the forgetful functor can be identified with the precomposition functor \( \epsilon^* : \text{Fun}(D, N) \to \text{Fun}(|\text{Sing}(D)|, N) \), where \( \epsilon : |\text{Sing}(D)| \xrightarrow{\sim} D \) is the counit of the adjunction \( |-| \dashv \text{Sing} \). The result now follows from part (b) of Definition 2.23.  

As promised above, we will now show that many topological model categories appearing in nature are good.

**Proposition 2.27.**

(i) \( \text{Top} \) is a good topological model category.

(ii) For any good topological model category \( N \) and any small topological category \( C \), the projective model structure on \( \text{Fun}(C, N) \) is good.

(iii) For any small topological category \( C \) and any set of maps \( S \) in \( \text{Fun}(C, \text{Top}) \), the left Bousfield localization \( L_S \text{Fun}(C, \text{Top}) \) is good.

**Proof.** Item (i) is well known. Item (ii) follows since for any small topological category \( D \), one has the equivalence of categories \( \text{Fun}(D, \text{Fun}(C, N)) \simeq \text{Fun}(D \times C, N) \).

For item (iii), we leave it to the reader to verify that the projective model structure on \( \text{Fun}(D, L_S \text{Fun}(C, \text{Top})) \) exists. To verify item (b) of Definition 2.23, let \( D \to D' \) be a weak equivalence which is bijective on objects. Then

\[
\text{Fun}(D', \text{Fun}(C, \text{Top})) \to \text{Fun}(D, \text{Fun}(C, \text{Top}))
\]

right Quillen equivalence. Using this, one easily verifies the conditions of [Man+01, Lem. A.2.(iii)], proving that

\[
\text{Fun}(D', L_S \text{Fun}(C, \text{Top})) \to \text{Fun}(D, L_S \text{Fun}(C, \text{Top}))
\]

is also a right Quillen equivalence.  

\[ \square \]
We now define goodness for pointed topological categories. Recall that a basepoint $x_0$ of a topological space $X$ is nondegenerate if $(X, x_0)$ is an NDR-pair in the sense of [May99, §6.4]. Since constructions in $\text{Top}_*$ are generally only well-behaved with respect to nondegenerately based spaces, we will always assume that the hom-objects of our indexing categories are of this kind.

**Definition 2.28.** A pointed topological model category $N$ is called good if

(a) for any small $\text{Top}_*$-category $\mathcal{C}$ whose hom-spaces are nondegenerately based, the projective model structure on $\text{Fun}(\mathcal{C}, N)$ exists, and

(b) for any two small $\text{Top}_*$-categories $\mathcal{C}$ and $\mathcal{D}$ whose hom-objects are nondegenerately based and any weak equivalence $\alpha: \mathcal{C} \to \mathcal{D}$ that is bijective on objects, the precomposition functor $\alpha^*: \text{Fun}(\mathcal{D}, N) \to \text{Fun}(\mathcal{C}, N)$ is a right Quillen equivalence.

**Remark 2.29.** If $N$ is a good pointed topological model category, then its underlying $\text{Top}_*$-model category is also good. Namely, if $\mathcal{C}$ is a small topological category, then $\text{Fun}(\mathcal{C}, N) = \text{Fun}(\mathcal{C}_+, N)$, where $\mathcal{C}_+$ is obtained by adding disjoint basepoints to the hom-objects of $\mathcal{C}$. The result follows since disjoint basepoints are always nondegenerate.

One can formulate analogues of Propositions 2.24 to 2.27 in the setting of pointed topological model categories and prove them in the same way. For future reference, we state the analogue of Proposition 2.24.

**Proposition 2.30.** Let $N$ be a good $\text{Top}_*$-model category, $\mathcal{C}$ a small $\text{Top}_*$-category whose hom-spaces have nondegenerate basepoints, $F: \mathcal{C} \to N$ a projectively cofibrant functor and $G \xrightarrow{\sim} G'$ a pointwise weak equivalence between functors $\mathcal{C}^{\text{op}} \to \text{Top}_*$ that are pointwise nondegenerately based. Then $\int^\mathcal{C} G \otimes F \to \int^\mathcal{C} G' \otimes F$ is a weak equivalence in $N$. Similarly, if $G$ is a functor $\mathcal{C}^{\text{op}} \to \text{Top}_*$ that is pointwise nondegenerately based and $F \xrightarrow{\sim} F'$ is a pointwise weak equivalence between projectively cofibrant functors $\mathcal{C} \to N$, then $\int^\mathcal{C} G \otimes F \to \int^\mathcal{C} G \otimes F'$ is a weak equivalence.

Finally, let us define goodness for spectral model categories.

**Definition 2.31.** A spectral model category $N$ is called good if

(a) for any small spectral category $\mathcal{C}$, the projective model structure on $\text{Fun}(\mathcal{C}, N)$ exists, and

(b) for any weak equivalence $\alpha: \mathcal{C} \to \mathcal{D}$ of small spectral categories that is bijective on objects, the precomposition functor $\alpha^*: \text{Fun}(\mathcal{D}, N) \to \text{Fun}(\mathcal{C}, N)$ is a right Quillen equivalence.

**Remark 2.32.** If $N$ is a good spectral model category, then its underlying $\text{Top}_*$-model category is also good.

One can mimic the proofs of Propositions 2.24 and 2.25 to obtain analogous statements for good spectral model categories. Let us state the analogue of Proposition 2.25 for future reference.

**Proposition 2.33.** Let $N$ be a good spectral model category, $T$ an object in $\text{Sp}$ and $N$ a cofibrant object in $N$. Then $T \otimes - : N \to N$ preserves weak equivalences between cofibrant objects, while $- \otimes N: \text{Top} \to N$ preserves all weak equivalences.
One can also prove an analogue of Proposition 2.27, showing that many common examples of spectral model categories are good.

**Proposition 2.34.**

(i) \( \text{Sp} \) is a good spectral model category.

(ii) For any good spectral model category \( \mathcal{N} \) and any small \( \text{Sp} \)-category \( \mathcal{C} \) whose hom-objects are cofibrant, the projective model structure on \( \text{Fun}(\mathcal{C}, \mathcal{N}) \) is good.

(iii) Let \( \mathcal{C} \) be a small spectral category whose hom-objects are cofibrant and let \( S \) be a set of maps in \( \text{Fun}(\mathcal{C}, \text{Sp}) \). Then the enriched left Bousfield localization \( L_S \text{Fun}(\mathcal{C}, \text{Sp}) \) of the projective model structure at \( S \) is a good spectral model category.

**Proof.** For item (i), note that properties (a) and (b) of Definition 2.31 follow from [SS03, Thm. 7.2].

For item (ii), property (a) of Definition 2.31 follows as in Proposition 2.27. For property (b) of Definition 2.31, note that since \( \mathcal{C} \) has cofibrant hom-objects, one has by [Man+01, Prop. 12.3] that \( \mathcal{C} \otimes \mathcal{D} \simeq \mathcal{C} \otimes \mathcal{D}' \) for any weak equivalence \( \mathcal{D} \simeq \mathcal{D}' \) of spectral categories.

Finally, note for item (iii) that the spectrally enriched left Bousfield localization at a set \( S \) is the same as the unenriched left Bousfield localization at the set \( S' \) that consists of all shifts of maps in \( S \). The result now follows in the same way as in the proof of Proposition 2.27. ■

2.3 Bar constructions and derived Kan extensions

Given a \( \mathcal{V} \)-model category \( \mathcal{N} \) and a \( \mathcal{V} \)-functor \( \alpha : \mathcal{C} \to \mathcal{D} \) between small \( \mathcal{V} \)-categories, derived (enriched) left Kan extension is defined as the left derived functor of the ordinary (enriched) left Kan extension functor \( \text{Lan}_\alpha : \text{Fun}(\mathcal{C}, \mathcal{N}) \to \text{Fun}(\mathcal{D}, \mathcal{N}) \). One can obtain such a left derived functor by endowing these diagram categories with the projective model structures and noting that \( \text{Lan}_\alpha \) is left Quillen with respect to these. However, the projective model structures might not always exist, and the resulting description of the derived left Kan extension involves a projective cofibrant replacement functor, which is generally very inexplicit. In certain cases, one can use the bar construction to give an alternative construction of the derived (enriched) left Kan extension functor which circumvents the first problem and moreover produces an explicit formula. Throughout the rest of this section, \( \mathcal{N} \) will be a \( \mathcal{V} \)-model category and \( \alpha : \mathcal{C} \to \mathcal{D} \) will be a \( \mathcal{V} \)-functor between small \( \mathcal{V} \)-categories. Moreover, we will assume that the symmetric monoidal model category \( \mathcal{V} \) comes equipped with a monoidal Quillen pair

\[
L : \text{sSet} \rightleftarrows \mathcal{V} : U.
\]

In particular, any \( \mathcal{V} \)-model category has an underlying simplicial model category and one can speak of geometric realizations of simplicial objects in a \( \mathcal{V} \)-model category.

We wish to point out that all definitions and results of this section can be dualized to obtain statements about the cobar construction and derived enriched right Kan extensions.

**Definition 2.35.** Let \( \mathcal{E} \) be a \( \mathcal{V} \)-cocomplete \( \mathcal{V} \)-category, \( \mathcal{C} \) a small \( \mathcal{V} \)-category and let \( F : \mathcal{C} \to \mathcal{E} \), \( G : \mathcal{C}^{op} \to \mathcal{V} \) be \( \mathcal{V} \)-functors. The **bar construction** \( B(G, \mathcal{E}, F) \) is defined as the geometric
realization of the simplicial object $B_*(G, \mathcal{E}, F)$ in \mathcal{E} given by

$$B_n(G, \mathcal{E}, F) = \prod_{c_0, \ldots, c_n \in \text{Ob}(\mathcal{E})} G(c_n) \otimes \mathcal{E}(c_{n-1}, c_n) \otimes \cdots \otimes \mathcal{E}(c_0, c_1) \otimes F(c_0),$$

with the face and degeneracy maps as in [Rie14, Def. 9.1.1].

Let us record a few lemmas on the bar construction.

**Lemma 2.36.** $B(-, \mathcal{E}, F) : \text{Fun}(\mathcal{E}^{op}, \mathcal{V}) \to \mathcal{E}$ preserves colimits and tensors by objects of \mathcal{V}.

*Proof.* This follows since geometric realization commutes with colimits and tensors by objects of \mathcal{V}. ■

We will generally be interested in the case where we are given a \mathcal{V}-functor $\alpha : \mathcal{E} \to \mathcal{D}$ and $G$ is of the form $c \mapsto \mathcal{D}(ac, d)$, where $d$ is an object of \mathcal{D}. By letting $d$ vary, we obtain a functor $\mathcal{D} \to \mathcal{E}$ given by

$$d \mapsto B(\mathcal{D}(a-c, d), \mathcal{E}, F).$$

For brevity, we will denote this functor by $B(\mathcal{D}, \mathcal{E}, F)$; the map $\alpha$ should always be clear from the context. When we write $B(\mathcal{E}, \mathcal{E}, F)$, we always mean the case that $\alpha = \text{id}_\mathcal{E}$. In this case, there is a canonical map $B(\mathcal{E}, \mathcal{E}, F) \to F$ coming from the augmentation

$$B_0(\mathcal{E}, \mathcal{E}, F)(c) = \prod_d \mathcal{E}(d, c) \otimes F(d) \xrightarrow{\mathcal{E}} F(c).$$

**Lemma 2.37.** Let $F : \mathcal{E} \to \mathcal{N}$ be a \mathcal{V}-functor. Then the map $B(\mathcal{E}(-, c), \mathcal{E}, F) \to F(c)$, and hence $B(\mathcal{E}, \mathcal{E}, F) \to F$, is a natural weak equivalence.

*Proof.* This can be proved by showing that $B(\mathcal{E}(-, c), \mathcal{E}, F)$ has an “extra degeneracy”, cf. [Rie14, Ex. 4.5.7]. ■

**Lemma 2.38.** Let $F : \mathcal{E} \to \mathcal{N}$ be a \mathcal{V}-functor. Then $\text{Lan}_\alpha B(\mathcal{E}, \mathcal{E}, F) \cong B(\mathcal{D}, \mathcal{E}, F)$.

*Proof.* This follows from Lemma 2.8 and the fact that $\text{Lan}_\alpha(\mathcal{E}(c, -) \otimes E) \cong \mathcal{D}(ac, -) \otimes E$.

Since $B(\mathcal{E}, \mathcal{E}, F) \to F$ is a natural weak equivalence, we see that

$$B(\mathcal{E}, \mathcal{E}, -) : \text{Fun}(\mathcal{E}, \mathcal{N}) \to \text{Fun}(\mathcal{E}, \mathcal{N})$$

is a left deformation in the sense of [Shuo06, §3]. If one combines this functor with a cofibrant replacement functor $Q : \mathcal{N} \to \mathcal{N}$, then one can often show that $\text{Lan}_\alpha$ is homotopical on the image of $B(\mathcal{E}, \mathcal{E}, Q-)$ (cf. [Shuo06, Thm. 13.7]) and hence that $\text{Lan}_\alpha B(\mathcal{E}, \mathcal{E}, Q-)$ is a left derived functor of $\text{Lan}_\alpha$. It is important to note that for this to work, $Q$ needs to be a \mathcal{V}-functor and not just an ordinary functor.

**Lemma 2.39.** Suppose that the hom-objects of $\mathcal{E}$ and $\mathcal{D}$ are cofibrant and that $1 \to \mathcal{E}(c, c)$ is a cofibration for any $c$ in $\mathcal{E}$. Then, for any \mathcal{V}-model category $\mathcal{N}$ with a \mathcal{V}-enriched cofibrant replacement functor $Q$, the bar construction $B(\mathcal{D}, \mathcal{E}, Q-)$ is a left derived functor of $\text{Lan}_\alpha : \text{Fun}(\mathcal{E}, \mathcal{N}) \to \text{Fun}(\mathcal{D}, \mathcal{N})$.

*Proof.* This is shown in [Shuo06, Thm. 13.7]. ■
The following lemma is a variation on [Shu06, Prop. 23.6].

**Lemma 2.40.** Suppose that all hom-objects of \( \mathcal{C} \) are cofibrant and that \( \mathbf{1} \rightarrow \mathcal{C}(c, c) \) is a cofibration in \( \mathcal{V} \) for every \( c \) in \( \mathcal{C} \). Then for any pointwise cofibration \( F \rightarrow F' \) in \( \text{Fun}(\mathcal{C}^{op}, \mathcal{V}) \) and any pointwise cofibrant \( G \) in \( \text{Fun}(\mathcal{C}, \mathcal{N}) \), the map

\[
B_{\bullet}(F, \mathcal{C}, G) \rightarrow B_{\bullet}(F', \mathcal{C}, G)
\]

is a Reedy cofibration in \( \mathcal{N}^{\mathcal{V}^{op}} \).

**Proof.** In order to prove this result, we give a slightly different presentation of the bar construction. Given a set \( V \), we endow \( \mathcal{V} \), \( \mathcal{V} \times \mathcal{V} \) and \( \mathcal{N} \) with the model structure in which the (co)fibrations and weak equivalences are defined pointwise. We have a product \( \odot : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V} \times \mathcal{V} \) given by

\[
(X \odot Y)(s, t) = \bigsqcup_u X(u, s) \odot Y(t, u),
\]

endowing \( \mathcal{V} \times \mathcal{V} \) with a (non-symmetric) monoidal structure. The monoids for this are exactly \( \mathcal{V} \)-enriched categories whose set of objects is \( \mathcal{N} \). Given such a \( \mathcal{V} \)-category \( \mathcal{D} \), we will write \( \mathcal{D}^\odot \) for the simplicial object in \( \mathcal{V} \times \mathcal{V} \) given by

\[
(\mathcal{D}^\odot)_n := \mathcal{D}^{\odot n} = \underbrace{\mathcal{D} \odot \cdots \odot \mathcal{D}}_{\text{n-times}}.
\]

Observe that there are similar products \( \mathcal{V} \times \mathcal{V} \times \mathcal{N} \rightarrow \mathcal{N} \), \( \mathcal{V} \times \mathcal{V} \times \mathcal{V} \times \mathcal{N} \rightarrow \mathcal{N} \) and \( \mathcal{N} \times \mathcal{N} \rightarrow \mathcal{N} \), which we will also denote by \( \odot \). It follows immediately from the definition that all these products satisfy the pushout-product axiom (cf. part (a) of Definition 2.13). Taking \( S = \text{Ob}(\mathcal{C}) \), the bar construction can be rewritten as

\[
B_n(F, \mathcal{C}, G) = F \odot \mathcal{C}^{\odot n} \odot G.
\]

Under this identification, the \( n \)-th latching map

\[
B_n(F, \mathcal{C}, G) \cup_{L_n B_{\bullet}(F, \mathcal{C}, G)} L_n B_{\bullet}(F', \mathcal{C}, G) \rightarrow B_n(F', \mathcal{C}, G)
\]

is identified with the map

\[
F \odot \mathcal{C}^{\odot n} \odot G \cup_{F \odot L_n \mathcal{C}^{\odot} \odot G} F' \odot L_n \mathcal{C}^{\odot} \odot G \rightarrow F' \odot \mathcal{C}^{\odot n} \odot G,
\]

which is the pushout-product of \( F \rightarrow F' \) and \( L_n \mathcal{C}^{\odot} \odot G \rightarrow \mathcal{C}^{\odot n} \odot G \). Since \( G \) is pointwise cofibrant and \( F \rightarrow F' \) a pointwise cofibration, it follows that this map is a cofibration if \( L_n \mathcal{C}^{\odot} \rightarrow \mathcal{C}^{\odot n} \) is. The latter follows exactly as in the proof of [Shu06, Prop. 23.6]. \[\square\]

Note that for any functor \( a: \mathcal{C} \rightarrow \mathcal{D} \), the precomposition functor \( a^* : \text{Fun}(\mathcal{D}, \mathcal{N}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{N}) \) preserves (pointwise) weak equivalences, hence it is already right derived. In particular, if \( B(\mathcal{D}, \mathcal{C}, Q-) \) is a left derived functor of \( \text{Lan}_a \), then

\[
B(\mathcal{D}, \mathcal{C}, Q-) : \text{Ho}(\text{Fun}(\mathcal{C}, \mathcal{N})) \rightleftarrows \text{Ho}(\text{Fun}(\mathcal{D}, \mathcal{N})) : a^*
\]

is an adjunction by [Rie14, Thm. 2.2.11]. A diagram chase shows that the unit and counit of this adjunction are given by

\[
F \xleftarrow{\sim} \nu B(\mathcal{C}, \mathcal{C}, QF) \xrightarrow{\eta B(\mathcal{C}, \mathcal{C}, QF)} a^* B(\mathcal{D}, \mathcal{C}, QF)
\]

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and
\[ B(\mathcal{D}, \mathcal{C}, Q^*G) \xrightarrow{\text{Lan}_\alpha(v)} \text{Lan}_\alpha \xrightarrow{\epsilon_G} G, \]
respectively, where \( v: B(\mathcal{C}, \mathcal{C}, Q) \xrightarrow{\sim} F \) denotes the composition of the equivalence \( B(\mathcal{C}, \mathcal{C}, Q) \xrightarrow{\sim} Q \) of Lemma 2.37 and \( Q \xrightarrow{\sim} F \).

**Definition 2.41.** We call
\[ \eta_{B(\mathcal{C}, \mathcal{C}, Q)}: B(\mathcal{C}, \mathcal{C}, Q) \rightarrow \alpha^*B(\mathcal{D}, \mathcal{C}, Q) \]
and \( \epsilon_G \circ \text{Lan}_\alpha(v): B(\mathcal{D}, \mathcal{C}, Q^*G) \rightarrow G \)
the *derived unit* and *derived counit*, respectively.

# 3 Homotopy functors

In this part, we will describe conditions under which a functor can be replaced by one that respects an enrichment in simplicial sets or topological spaces; in particular, we will prove Theorem A. We start by describing such conditions in the simplicial case in Section 3.1, after which we will consider the topological case in Section 3.2. In Section 3.3, we show how to obtain our result on Dwyer-Kan localizations, namely Theorem B, as a corollary of the simplicial case.

**Remark 3.1.** There are also versions of the main results of Sections 3.1 and 3.2 for (symmetric) functors of several variables. We will not explicitly prove them here, but refer the reader to Section 4.3 and Remark 4.20 instead.

## 3.1 Simplicial functors

Throughout this section \( \mathcal{C} \) denotes a small simplicial category and \( N \) a simplicial model category. The underlying category of \( \mathcal{C} \) will be denoted by \( \mathcal{C}_0 \) whenever we need to distinguish it from \( \mathcal{C} \).

**Definition 3.2.** (i) Two maps \( f, g: C \rightarrow D \) in \( \mathcal{C} \) are *strictly homotopic* if there exists a 1-simplex \( H \in \mathcal{C}(C, D)_1 \) such that either \( d_0H = f \) and \( d_1H = g \), or \( d_1H = f \) and \( d_0H = g \).

(ii) Two maps \( f, g: C \rightarrow D \) in \( \mathcal{C} \) are *(simplicially) homotopic* if there exists a sequence of maps \( f_0, \ldots, f_n: C \rightarrow D \) with \( f_0 = f \) and \( f_n = g \) such that for every \( 0 \leq i < n \), the map \( f_i \) is strictly homotopic to \( f_{i+1} \).

(iii) A map in \( \mathcal{C} \) is a *(simplicial) homotopy equivalence* if up to homotopy, it has both a left and a right inverse.

(iv) A functor \( \mathcal{C}_0 \rightarrow N \) is called a *homotopy functor* if it sends homotopy equivalences to weak equivalences.

Note that any simplicial functor is automatically a homotopy functor since simplicial functors preserve (strict) homotopies. The goal of this section is to show that the converse of this statement holds up to natural equivalence if \( \mathcal{C} \) either admits tensors or cotensors by standard simplices.
**Theorem 3.3.** Let \( \mathcal{C} \) be a small simplicial category that either admits tensors by the standard simplex \( \Delta[n] \) for every \( n \geq 0 \), or that admits cotensors by the standard simplex \( \Delta[n] \) for every \( n \geq 0 \). Then for any simplicial model category \( \mathcal{N} \), every homotopy functor \( \mathcal{C} \to \mathcal{N} \) is weakly equivalent to a simplicial functor.

Note that by Lemma 2.11, one can produce a simplicial functor from any ordinary functor \( \mathcal{C} \to \mathcal{N} \) by left or right Kan extending along \( \mathcal{C} \to \mathcal{N} \), where we view \( \mathcal{C} \) as a discrete simplicial category. We will prove Theorem 3.3 by comparing a homotopy functor to its derived left or right Kan extension along \( \mathcal{C} \to \mathcal{N} \), which can be computed explicitly using the (co)bar construction by Lemma 2.39. Let \( \mathcal{U} \) denote the forgetful functor \( \text{Fun}(\mathcal{C}, \mathcal{N}) \to \text{Fun}(\mathcal{C}_0, \mathcal{N}) \) (which we often omit from notation) and \( \mathcal{L} \) its left Kan extension constructed using left Kan extension. Recall the definition of the derived (co)unit of the bar construction from Definition 2.41.

**Proposition 3.4.** Let \( \mathcal{C} \) be a small simplicial category that admits cotensors by the standard simplex \( \Delta[n] \) for every \( n \geq 0 \) and let \( \mathcal{N} \) be a simplicial model category with cofibrant replacement functor \( \mathcal{F} \). Then the derived unit

\[
\mathcal{F} \overset{\approx}{\dashv} \mathcal{B}(\mathcal{C}, \mathcal{C}_0, \mathcal{F}) \Rightarrow \mathcal{B}(\mathcal{C}, \mathcal{C}_0, \mathcal{QF})
\]

is a weak equivalence for any homotopy functor \( \mathcal{F} : \mathcal{C} \to \mathcal{N} \) and the derived counit

\[
\mathcal{B}(\mathcal{C}, \mathcal{C}_0, \mathcal{QUG}) \Rightarrow \mathcal{LUG} \overset{\approx}{\Rightarrow} \mathcal{G}
\]

is a weak equivalence for any simplicial functor \( \mathcal{G} : \mathcal{C} \to \mathcal{N} \).

**Remark 3.5.** By formally dualizing the statement of Proposition 3.4, one obtains a result about the bar construction in the case that \( \mathcal{C} \) admits tensors by the standard simplices.

Before proving Proposition 3.4, let us show how to derive Theorem 3.3 from it.

**Proof of Theorem 3.3.** First suppose that \( \mathcal{C} \) admits cotensors by the standard simplices and let \( \mathcal{F} : \mathcal{C} \to \mathcal{N} \) be a homotopy functor. Write \( \mathcal{Q} \) for a cofibrant replacement functor on \( \mathcal{N} \). Then \( \mathcal{F} \overset{\approx}{\dashv} \mathcal{B}(\mathcal{C}, \mathcal{C}_0, \mathcal{QF}) \Rightarrow \mathcal{B}(\mathcal{C}, \mathcal{C}_0, \mathcal{QF}) \) by Proposition 3.4.

If \( \mathcal{C} \) admits tensors by the standard simplices instead, then the result follows by replacing \( \mathcal{C} \) with \( \mathcal{C}^{op} \) and \( \mathcal{N} \) with \( \mathcal{N}^{op} \).

The proof of Proposition 3.4 will use the following lemmas about the bar construction and simplicial objects.

**Lemma 3.6.** Let \( \mathcal{F} : \mathcal{C}_0 \to \mathcal{N} \) and \( \mathcal{G} : \mathcal{C}_0^{op} \to \mathbf{sSet} \) be functors, and let \( \mathcal{G}_n \) denote the functor \( \mathcal{C}_0^{op} \to \mathbf{Set} \) that sends an object \( C \) to the set of \( n \)-simplices \( G(C)_n \) of \( G(C) \). Then \( B(G, \mathcal{C}_0, F) \) is isomorphic to the geometric realization of the simplicial object \( \Delta[n] \to B(G_n, \mathcal{C}_0, F) \) in \( \mathcal{N} \), and if moreover \( F \) is pointwise cofibrant, then this simplicial object is Reedy cofibrant.

**Proof.** The first statement follows since

\[
B(G, \mathcal{C}_0, F) \cong B \left( \int \Delta[n] \otimes \mathcal{G}_n, \mathcal{C}_0, F \right) \cong \int \Delta[n] \otimes B(G_n, \mathcal{C}_0, F)
\]

holds by Lemma 2.8. For the second statement, note that the latching object \( L_nB(G_n, \mathcal{C}_0, F) \) agrees with \( B(L_nG_n, \mathcal{C}_0, F) \) by Lemma 2.36. The map \( B(L_nG_n, \mathcal{C}_0, F) \to B(G_n, \mathcal{C}_0, F) \) is the inclusion of a summand into a coproduct of cofibrant objects, hence a cofibration.
For the following lemmas, recall that a simplicial object $X_\bullet$ in a model category is called **homotopically constant** if for any $f : [n] \to [m]$, the map $f^* : X_m \to X_n$ is a weak equivalence.

**Lemma 3.7.** Let $F : \mathcal{C} \to \mathcal{N}$ be a homotopy functor and suppose that $\mathcal{C}$ admits cotensors by the standard simplices $\Delta[n]$. Then for any object $c$ in $\mathcal{C}$, the simplicial object $[n] \mapsto F(c^{\Delta[n]})$ is homotopically constant.

**Proof.** By the two-out-of-three property and the fact that $F$ is a homotopy functor, it suffices to show that for every $n \geq 0$, the map $c^{\Delta[0]} \to c^{\Delta[n]}$ is a simplicial homotopy equivalence. Now note that the map $\Delta[n] \to \Delta[0]$ is a simplicial deformation retract by [Lam68, Beispiel I.5.4]. Since simplicial cotensors assemble into a simplicial functor by Construction 2.4, we see that $c^{\Delta[0]} \to c^{\Delta[n]}$ is indeed a simplicial homotopy equivalence. □

**Lemma 3.8.** Let $N_\bullet$ be a homotopically constant Reedy cofibrant simplicial object in $\mathcal{N}$. Then the canonical map $N_0 \to |N_\bullet|$ is a weak equivalence.

**Proof.** Let $cN_0$ denote the constant simplicial object on $N_0$. Then $cN_0 \to N_\bullet$ is a levelwise weak equivalence between Reedy cofibrant objects, hence $N_0 = |cN_0| \to |N_\bullet|$ is a weak equivalence. □

**Proof of Proposition 3.4.** We first show that the derived unit is a weak equivalence for any homotopy functor $F : \mathcal{C}_0 \to \mathcal{N}$. For this, it suffices to show that for any pointwise cofibrant $F$, the map $B(\mathcal{C}_0, \mathcal{C}_0, F) \to B(\mathcal{C}, \mathcal{C}_0, F)$ is a weak equivalence. Applying Lemma 3.6 to $G = \mathcal{C}(-, c) \cong \{\mathcal{C}_0(\mathcal{C}_0, \mathcal{C}_0, F)\}_{n \geq 0}$ yields that $B(\mathcal{C}, \mathcal{C}_0, F)$ is naturally isomorphic to the geometric realization of the Reedy cofibrant simplicial object

$$[n] \mapsto B(\mathcal{C}_0(\mathcal{C}_0, \mathcal{C}_0, F)).$$

By Lemma 3.8, the map $B(\mathcal{C}_0, \mathcal{C}_0, F) \to B(\mathcal{C}, \mathcal{C}_0, F)$ is a weak equivalence if this simplicial object is homotopically constant. Since $B(\mathcal{C}_0(\mathcal{C}_0, \mathcal{C}_0, F)) \simeq F(c^{\Delta[n]})$ by Lemma 2.37, it follows from Lemma 3.7 that this is indeed the case.

To see that the derived counit is a weak equivalence, it suffices to show that the counit of

$$B(\mathcal{C}, \mathcal{C}_0, Q-) : \text{Ho}(\text{Fun}(\mathcal{C}_0, \mathcal{N})) \cong \text{Ho}(\text{Fun}(\mathcal{C}, \mathcal{N})) : U$$

is a natural isomorphism. This follows from the fact that $U$ creates weak equivalences, that any simplicial functor is a homotopy functor and that the derived unit is a weak equivalence on homotopy functors. □

**Remark 3.9.** One can also use the ideas of [RSS01, §6] to show that homotopy functors $\mathcal{C} \to \mathcal{N}$ can be replaced by simplicial functors if $\mathcal{C}$ admits tensors or cotensors by standard simplices. To see this, let $R$ denote a simplicial Reedy fibrant replacement functor in $\mathcal{N}$ and $Q$ a simplicial Reedy cofibrant replacement functor in $\mathcal{N}$, assuming these exist. A homotopy functor $F : \mathcal{C} \to \mathcal{N}$ is then equivalent to the totalization of $[n] \mapsto RF(\Delta[n] \otimes -)$ or to the realization of $[n] \mapsto QF((-)^{\Delta[n]})$ if the relevant (co)tensors exist, and these functors can be shown to be simplicial functors. The advantage of this construction is that it also works for simplicial indexing categories $\mathcal{C}$ that are not small. However, to ensure the existence of simplicial Reedy (co)fibrant replacement functors, one needs to make extra assumptions about $\mathcal{N}$ that we don’t require in Theorem 3.3.
We will now show that Theorem 3.3 can be upgraded to a Quillen equivalence if the relevant model structures exist. Define (if it exists) the homotopy model structure \( \text{Fun}^{\text{ho}}(\mathcal{C}, N) \) on \( \text{Fun}(\mathcal{C}, N) \) to be left Bousfield localization of the projective model structure in which the fibrant objects are the pointwise fibrant homotopy functors.

**Theorem 3.10.** Let \( \mathcal{C} \) be a small simplicial category and \( N \) a simplicial model category. If the projective model structures on \( \text{Fun}(\mathcal{C}, N) \) and \( \text{Fun}(\mathcal{C}_0, N) \) exist, then

\[
L : \text{Fun}(\mathcal{C}_0, N) \rightleftharpoons \text{Fun}(\mathcal{C}, N) : U
\]

is a Quillen pair. If moreover \( \mathcal{C} \) admits either tensors by the standard simplices or cotensors by the standard simplices and if the homotopy model structure \( \text{Fun}^{\text{ho}}(\mathcal{C}_0, N) \) exists, then this Quillen pair becomes a Quillen equivalence between \( \text{Fun}^{\text{ho}}(\mathcal{C}_0, N) \) and the projective model structure on \( \text{Fun}(\mathcal{C}, N) \).

**Remark 3.11.** The projective model structures on \( \text{Fun}(\mathcal{C}, N) \) and \( \text{Fun}(\mathcal{C}_0, N) \) always exist if \( N \) is cofibrantly generated. Moreover, the homotopy model structure \( \text{Fun}^{\text{ho}}(\mathcal{C}_0, N) \) exists whenever \( N \) is left proper and combinatorial or cellular: In these cases, there always exists a set \( T \) of cofibrant objects in \( N \) such that \( N \to M \) is a weak equivalence if and only if \( \text{Map}(T, N) \to \text{Map}(T, M) \) is a weak equivalence for every \( T \in T \). The homotopy model structure is then obtained as the left Bousfield localization of \( \text{Fun}(\mathcal{C}_0, N) \) with respect to the maps

\[
\{ f^*: \mathcal{C}_0(d, -) \otimes T \to \mathcal{C}_0(c, -) \otimes T \mid f: c \to d \text{ is a homotopy equivalence in } \mathcal{C} \text{ and } T \in T \}.
\]

**Proof of Theorem 3.10.** It is clear that the forgetful functor \( U \) preserves pointwise (trivial) fibrations, hence that \( L \dashv U \) is a Quillen pair with respect to the projective model structures. For the second statement, let us first assume that \( \mathcal{C} \) admits cotensors by the standard simplicies. Then Proposition 3.4 shows that the derived unit of \( L \dashv U \) is a weak equivalence for any homotopy functor \( \mathcal{C}_0 \to N \), hence the result follows from Lemma 2.22. If, on the other hand, \( \mathcal{C} \) admits tensors by the standard simplices, then the proof of Lemma 2.22 only shows that \( L \dashv U \) is a Quillen pair. To see that it is a Quillen equivalence, note that since \( U \) preserves weak equivalences, its left and right derived functors agree. In particular, it suffices to show the left derived functor

\[
\text{Ho}(\text{Fun}(\mathcal{C}, N)) \to \text{Ho}(\text{Fun}^{\text{ho}}(\mathcal{C}_0, N))
\]

of \( U \) is an equivalence of categories. This follows since by the dual of Lemma 2.39, the cobar construction can be used to derive the right adjoint of \( U \) and by the dual of Proposition 3.4, the derived unit and counit of the resulting adjunction are weak equivalences.

**Example 3.12.** Let \( \text{sSet}^{\text{fin}} \) denote the category of finite simplicial sets, i.e. simplicial sets with finitely many non-degenerate simplices. Then the homotopy model structure exists on \( \text{Fun}(\text{sSet}^{\text{fin}}_0, \text{sSet}) \) by Remark 3.11, hence the category of simplicial functors \( \text{Fun}(\text{sSet}^{\text{fin}}, \text{sSet}) \) equipped with the projective model structure is Quillen equivalent to the homotopy model structure \( \text{Fun}^{\text{ho}}((\text{sSet}^{\text{fin}})_0, \text{sSet}) \). There are of course

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\(^a\)In [Lyd8], Lydakis considers functors between \( \text{sSet}^{\text{fin}} \) and \( \text{sSet} \) that preserve weak homotopy equivalences and calls them homotopy functors. Note that this terminology conflicts with ours, since in our terminology a homotopy functor only sends simplicial homotopy equivalences to weak homotopy equivalences.
many variations on this, such as a Quillen equivalence between \( \text{Fun}(sSet^{\text{fin}}, sSet) \) and \( \text{Fun}^{ho}(sSet^{\text{fin}}_0, sSet) \) to which we will come back in Example 4.8.

**Example 3.13.** Let \( \mathcal{C} \) be any small simplicial category. One can “freely” add tensors by the standard simplices in the following way: First identify \( \mathcal{C} \) with its image under the enriched Yoneda embedding \( \mathcal{C} \to sSet^\text{op} \); let us write \( c \) for the presheaf represented by an object \( c \) of \( \mathcal{C} \). The category \( sSet^\text{op} \) admits tensors by the standard simplices that are defined pointwise. If \( \tilde{\mathcal{C}} \) denote the full subcategory of \( sSet^\text{op} \) spanned by all objects of the form \( X \otimes c \), where \( X \) is a finite product of standard simplices, then \( \tilde{\mathcal{C}} \) is a simplicial category that admits tensors by standard simplices. Then Theorem 3.10 yields a right Quillen equivalence \( \text{Fun}(\tilde{\mathcal{C}}, sSet) \to \text{Fun}(\mathcal{C}_0, sSet) \), while the restriction \( \text{Fun}(\tilde{\mathcal{C}}, sSet) \to \text{Fun}(\mathcal{C}, sSet) \) is a right Quillen equivalence since \( \mathcal{C} \to \tilde{\mathcal{C}} \) is a DK-equivalence (cf. Definition 3.18).

In a cofibrantly generated simplicial model category, one can use the enriched small object argument to obtain simplicial fibrant and cofibrant replacement functors (cf. [Rie14, Thm. 13.2.1]). Theorem 3.3 allows us to obtain a similar result for simplicial model categories \( \mathcal{N} \) that are not necessarily cofibrantly generated, but at the cost of restricting to a small full subcategory of \( \mathcal{N} \). Given a full subcategory \( \mathcal{N}' \subseteq \mathcal{N} \), we define a cofibrant replacement functor on \( \mathcal{N}' \) as a functor \( F: \mathcal{N}' \to \mathcal{N} \) landing in the full subcategory of cofibrant objects together with a natural weak equivalence \( F \sim i \), where \( i \) denotes the inclusion \( \mathcal{N}' \hookrightarrow \mathcal{N} \). A fibrant replacement functor on \( \mathcal{N}' \) is defined dually.

**Corollary 3.14.** Let \( \mathcal{N} \) be a simplicial model category. For any small full subcategory \( \mathcal{N}' \) of \( \mathcal{N} \), there exist simplicial fibrant and cofibrant replacement functors on \( \mathcal{N}' \).

**Proof.** We will only treat the case of the cofibrant replacement functors, since the statement about fibrant replacement functors is obtained by applying this case to \( \mathcal{N}^\text{op} \). We may assume without loss of generality that \( \mathcal{N}' \) is closed under cotensors by the standard simplices \( \Delta[n] \), since otherwise we can enlarge \( \mathcal{N}' \) to make this the case.

Let \( Q \) be an (unenriched) cofibrant replacement functor on \( \mathcal{N} \). Applying \( B(\mathcal{N}', \mathcal{N}'_0, Q-) \) to the inclusion \( i: \mathcal{N}' \hookrightarrow \mathcal{N} \), we see by Proposition 3.4 that there is a natural equivalence

\[
B(\mathcal{N}', \mathcal{N}'_0, Qi) \sim i,
\]

while \( B(\mathcal{N}', \mathcal{N}'_0, Qi) \) is pointwise cofibrant by Lemma 2.40. \( \square \)

### 3.2 Continuous functors

We will now prove the topological counterparts of Theorems 3.3 and 3.10. Throughout this section, \( \mathcal{C} \) denotes a small topological category, \( \mathcal{N} \) a topological model category and \( \mathcal{C}_0 \) the underlying category of \( \mathcal{C} \). Note that just like a simplicial category, any topological category \( \mathcal{C} \) comes with an obvious notion of homotopy defined via paths in the hom-spaces, and we will call a functor \( \mathcal{C}_0 \to \mathcal{N} \) a homotopy functor if it takes homotopy equivalences to weak equivalences. Clearly any continuous functor is a homotopy functor.

Recall the definition of goodness from Definition 2.23. As before, we define (if it exists) the homotopy model structure \( \text{Fun}^{ho}(\mathcal{C}_0, \mathcal{N}) \) as the left Bousfield localization of the projective model structure on \( \text{Fun}(\mathcal{C}_0, \mathcal{N}) \) in which the fibrant objects are the pointwise fibrant homotopy functors.
**Theorem 3.15.** Let \( \mathcal{C} \) be a small topological category that admits either tensors or cotensors by the unit interval, and let \( \mathcal{N} \) be a good topological model category. Then a functor \( \mathcal{C}_0 \to \mathcal{N} \) is weakly equivalent to a continuous functor if and only if it is a homotopy functor. Moreover, the forgetful functor

\[
U : \text{Fun}(\mathcal{C}, \mathcal{N}) \to \text{Fun}^{ho}(\mathcal{C}_0, \mathcal{N})
\]

from the projective model structure to the homotopy model structure is a Quillen equivalence whenever the homotopy model structure exists.

**Proof.** Let \( \mathcal{C}_\Delta \) denote the simplicial category \( \text{Sing}(\mathcal{C}) \). If \( \mathcal{C} \) has (co)tensors by the unit interval \( I \), then it has (co)tensors by \( I^n \) for every \( n \geq 0 \). Since \( |\Delta[n]| \cong I^n \), it follows that \( \mathcal{C}_\Delta \) admits (co)tensors by the standard simplices.

By the goodness of \( \mathcal{N} \), we see that the projective model structures exist on \( \text{Fun}(\mathcal{C}_0, \mathcal{N}) \), \( \text{Fun}(\mathcal{C}_\Delta, \mathcal{N}) \) and \( \text{Fun}(\mathcal{C}, \mathcal{N}) \). Moreover, by left Kan extending along the maps \( \mathcal{C}_0 \to |\mathcal{C}_\Delta| \to \mathcal{C} \) and their composite, we obtain the Quillen pairs

\[
\begin{align*}
\text{Fun}(\mathcal{C}_\Delta, \mathcal{N}) & \leftarrow \text{Fun}(\mathcal{C}, \mathcal{N}) \leftrightarrow \text{Fun}(\mathcal{C}_0, \mathcal{N})
\end{align*}
\]

where the left adjoints are drawn on the left or on top. The right adjoints in this diagram are the forgetful functors. By Proposition 2.26, the top horizontal adjunction is a Quillen equivalence. Moreover, it is clear that a functor \( \mathcal{C} \to \mathcal{N} \) is a homotopy functor if and only if the underlying simplicial functor \( \mathcal{C}_\Delta \to \mathcal{N} \) is a homotopy functor, so the result now follows by applying Theorems 3.3 and 3.10 to \( \text{Fun}(\mathcal{C}_\Delta, \mathcal{N}) \).

Note that many common topological model categories \( \mathcal{N} \) are good, cellular and left proper, for example all left Bousfield localizations of categories of diagrams in \( \text{Top} \) (cf. Proposition 2.27). In these cases, the existence of the homotopy model structure \( \text{Fun}^{ho}(\mathcal{C}_0, \mathcal{N}) \) follows from the same argument as in Remark 3.11, hence Theorem 3.15 provides us with a Quillen equivalence \( \text{Fun}^{ho}(\mathcal{C}_0, \mathcal{N}) \to \text{Fun}(\mathcal{C}, \mathcal{N}) \) whenever \( \mathcal{C} \) admits tensors or cotensors by the unit interval.

**Example 3.16.** Let \( \mathcal{W} \) denote the category of pointed finite CW-complexes, viewed as a \( \text{Top} \)-enriched category, and write \( \mathcal{W}_0 \) for the underlying ordinary category. Then the homotopy model structure on \( \text{Fun}(\mathcal{W}_0, \text{Top}_*) \) exists and is Quillen equivalent to the projective model structure on \( \text{Fun}(\mathcal{W}, \text{Top}_*) \). We will see in Example 4.11 below that if one left Bousfield localizes the homotopy model structure further so that the fibrant objects become the linear functors, then one obtains a symmetric monoidal model category of spectra.

**Example 3.17.** Let \( \mathcal{C} \) be any small \( \text{Top} \)-enriched category. Analogously to Example 3.13, one can construct a category \( \tilde{\mathcal{C}} \) by freely adding tensors with the unit interval \( I \) to \( \mathcal{C} \). One then obtains right Quillen equivalences \( \text{Fun}(\mathcal{C}, \text{Top}) \leftarrow \text{Fun}(\tilde{\mathcal{C}}, \text{Top}) \rightarrow \text{Fun}(\mathcal{C}_0, \text{Top}) \).

### 3.3 Digression: Dwyer-Kan localization

Given a category \( \mathcal{C} \) together with a subcategory \( \mathcal{W} \subseteq \mathcal{C} \), Dwyer and Kan in the series of papers [DK80b; DK80a; DK83; DK87] developed methods to formally turn the maps in \( \mathcal{W} \).
into homotopy equivalences, producing a simplicial category $L(\mathcal{C}, \mathcal{W})$ called its simplicial localization. While their constructions have good formal properties, it is generally hard to compute these simplicial localizations explicitly. The main result of this section is to give an explicit description, using the results proved above, of this simplicial localization in certain situations.

We will call a simplicial category $\mathcal{C}$ equipped with an ordinary subcategory $\mathcal{W}$ a relative category. A functor of relative categories $(\mathcal{C}, \mathcal{W}) \to (\mathcal{D}, \mathcal{V})$ is a functor $F: \mathcal{C} \to \mathcal{D}$ such that $F(\mathcal{W}) \subseteq \mathcal{V}$.

Let us recall the definition of a DK-equivalence.

**Definition 3.18.** A functor $F: \mathcal{C} \to \mathcal{D}$ of simplicial categories is called a Dwyer-Kan equivalence or DK-equivalence for short if

(a) for any two objects $c, d \in \mathcal{C}$, the map $\mathcal{C}(c, d) \to \mathcal{D}(Fc, Fd)$ is a weak homotopy equivalence, and

(b) for any $d \in \mathcal{D}$, there exists a $c \in \mathcal{C}$ such that there is a simplicial homotopy equivalence $c \simeq d$.

**Theorem 3.19.** Let $\mathcal{C}$ be a simplicial category that either admits tensors by the standard simplices or cotensors by the standard simplices, and let $\mathcal{W} \subseteq \mathcal{C}_0$ denote the subcategory of simplicial homotopy equivalences. Then the simplicial localization $L(\mathcal{C}_0, \mathcal{W})$ of $\mathcal{C}_0$ with respect to $\mathcal{W}$ is DK-equivalent to $\mathcal{C}$.

The idea of the proof is to identify both $L(\mathcal{C}_0, \mathcal{W})$ and $\mathcal{C}$ with the full subcategory of $\text{Fun}^{\text{ho}}(\mathcal{C}_0, \text{sSet})^{\text{op}}$ spanned by the representable functors through a “derived Yoneda embedding”. To make this precise, we use Theorem 2.2 of [DK87].

**Proof.** If $\mathcal{C}$ is a small simplicial category and $\mathcal{W}$ is an ordinary subcategory, then by an argument similar to that of Remark 3.11, the category $\text{Fun}(\mathcal{C}, \text{sSet})$ can be endowed with a model structure in which the cofibrations are the projective ones and in which the fibrant objects are precisely the pointwise fibrant functors that send the maps of $\mathcal{W}$ to homotopy equivalences. Let us denote this model category by $\text{Fun}^{\mathcal{W}}(\mathcal{C}, \text{sSet})$. It is proved in [DK87, Thm. 2.2] that if $(\mathcal{C}, \mathcal{W}) \to (\mathcal{D}, \mathcal{V})$ is a functor of relative categories such that the induced Quillen pair

$$\text{Fun}^{\mathcal{W}}(\mathcal{C}, \text{sSet}) \rightleftarrows \text{Fun}^{\mathcal{V}}(\mathcal{D}, \text{sSet})$$

is a Quillen equivalence, then the induced functor $L(\mathcal{C}, \mathcal{W}) \to L(\mathcal{D}, \mathcal{V})$ on simplicial localizations is a DK-equivalence “up to retracts” (cf. [DK87, 1.3.(iii)]). In particular, if $\mathcal{C} \to \mathcal{D}$ is an isomorphism on objects, then this implies that $L(\mathcal{C}, \mathcal{W}) \to L(\mathcal{D}, \mathcal{V})$ is a DK-equivalence.

Now assume that $\mathcal{C}$ is a small simplicial category that either admits tensors or cotensors by the standard simplices, and let $\mathcal{W}$ be the full subcategory of simplicial homotopy equivalences. Then we have functors of relative categories

$$(\mathcal{C}, \emptyset) \to (\mathcal{C}, \mathcal{W}) \leftarrow (\mathcal{C}_0, \mathcal{W})$$

which induce right Quillen functors

$$\text{Fun}(\mathcal{C}, \text{sSet}) \leftarrow \text{Fun}^{\mathcal{W}}(\mathcal{C}, \text{sSet}) \to \text{Fun}^{\mathcal{W}}(\mathcal{C}_0, \text{sSet})$$

The result of Dwyer-Kan is phrased differently since they did not have the general machinery of left Bousfield localizations in combinatorial model categories available yet, but it comes down to the same thing.
The left-hand functor is a Quillen equivalence since it is the identity, while the right-hand functor is a Quillen equivalence by Theorem 3.10. We deduce from [DK87, Thm. 2.2] that there are DK-equivalences \( \mathcal{E} \simeq L(\mathcal{E}, \varnothing) \simeq L(\mathcal{E}, W) \simeq L(\mathcal{E}_0, W) \).

**Example 3.20.** One can apply Theorem 3.19 to all examples considered in Sections 3.1 and 3.2. For example, the simplicial localization of the category \( \mathcal{E}_0 \) constructed in Example 3.13 with respect to the simplicial homotopy equivalences recovers the original category \( \mathcal{E} \). Similarly, when applying Theorem 3.19 to the category of finite simplicial sets, one obtains the (to the author) surprising result that if one localizes \( \text{sSet}^{\text{fin}} \) at the simplicial homotopy equivalences, then one obtains the simplicial category \( \text{sSet}^{\text{fin}} \).

**Example 3.21.** Let \( \text{SC}^* \) denote the \( (\text{Top}_*, \longrightarrow) \)-enriched category of separable \( C^* \)-algebras. Since this category has cotensors by the unit interval, its underlying simplicial category admits cotensors by the standard simplices as explained in the proof of Theorem 4.10. In particular, the simplicial localization \( L(\text{SC}^*_0, \text{hoeq}) \) of the underlying category \( \text{SC}^*_0 \) at the homotopy equivalences is DK-equivalent to \( \text{SC}^* \), recovering Proposition 3.17 of [BJM17].

## 4 Reduced functors

We will now describe conditions under which one can replace a simplicial or continuous functor with one that is \( \text{sSet}_* \)- or \( \text{Top}_* \)-enriched. In Section 4.1, we describe the simplicial case, from which we deduce the topological case in Section 4.2. We will then extend these results to (symmetric) multifunctors in Section 4.3, which are a type of functor naturally occurring in Goodwillie calculus.

### 4.1 Pointed simplicial functors

Throughout this section \( \mathcal{C}_* \) denotes a small \( \text{sSet}_* \)-enriched category and \( \mathcal{N} \) a \( \text{sSet}_* \)-enriched model category. We will write \( \mathcal{C} \) for the underlying \( \text{sSet} \)-enriched category of \( \mathcal{C}_* \) obtained by forgetting the basepoints of all the hom-objects.

Note that in a \( \text{sSet}_* \)-enriched category \( \mathcal{C}_* \), any initial object \( X \) is automatically terminal and vice-versa, and that this is the case if and only if the basepoint of \( \mathcal{C}_*(X, X) \) is the identity map \( \text{id}_X \). We call an object that is both initial and terminal a **zero object**. If \( \mathcal{C} \) is a simplicial category that has a zero object \( * \),\(^8\) then there is a unique way of upgrading \( \mathcal{C} \) to a \( \text{sSet}_* \)-enriched category \( \mathcal{C}_* \) by defining the basepoint of \( \mathcal{C}(X, Y) \) to be the image of

\[
* \cong \mathcal{C}(*, X) \wedge \mathcal{C}(X, Y) \to \mathcal{C}(X, Y).
\]

In particular, a \( \text{sSet}_* \)-enriched model category is the same as a simplicial model category with a zero object. For this reason, we will call a \( \text{sSet}_* \)-enriched category with a zero object a **pointed simplicial category**, and a \( \text{sSet}_* \)-enriched model category a **pointed simplicial model category**. We will always denote zero objects by \( * \).

**Definition 4.1.** Let \( \mathcal{C} \) be a pointed simplicial category, \( \mathcal{N} \) a pointed simplicial model category and \( F : \mathcal{C} \to \mathcal{N} \) a (not necessarily enriched) functor. Then

\[^8\]This statement should be interpreted in the enriched sense and not just on the level of the underlying category, meaning that for every object \( X \) of \( \mathcal{C} \), there should be isomorphisms \( \mathcal{C}(*, X) \cong * \cong \mathcal{C}(X, *) \) of simplicial sets.
(i) the functor \( F \) is called \textit{pointed} if it preserves zero objects (i.e. \( F(\ast) \cong \ast \)), and

(ii) the functor \( F \) is called \textit{reduced} if the map \( F(\ast) \to \ast \) is a weak equivalence in \( N \).

\textbf{Remark 4.2.} A \( \text{sSet}_\ast \)-enriched functor is by definition the same as a simplicial functor that preserves the basepoints of all the hom-objects. Since the zero object determines the basepoints of the hom-objects through the map (2), a simplicial functor between pointed simplicial categories is a \( \text{sSet}_\ast \)-enriched functor if and only if it is pointed.

Clearly any pointed functor is a reduced functor. We will show that the converse holds up to natural equivalence; by the previous remark this means that any reduced functor \( F: C \to N \) can be replaced by a \( \text{sSet}_\ast \)-functor.

\textbf{Theorem 4.3.} Let \( C \) be a small pointed simplicial category and let \( N \) be a pointed simplicial model category. A simplicial functor \( \mathcal{C} \to N \) is weakly equivalent to a pointed simplicial functor if and only if it is reduced.

The proof strategy is the same as for Theorem 3.3. Let \( \mathcal{C}_+ \) denote the \( \text{sSet}_\ast \)-category obtained by adjoining a disjoint basepoint to all hom-objects of \( \mathcal{C} \), and let \( L \) denote the left adjoint of the forgetful functor

\[ U: \text{Fun}(\mathcal{C}_+, N) \to \text{Fun}(\mathcal{C}, N). \]

Note that \( L \) is given by left Kan extension along \( \mathcal{C}_+ \to \mathcal{C}_\ast \).

\textbf{Proposition 4.4.} Let \( \mathcal{C} \) be a small pointed simplicial category and let \( N \) be a pointed simplicial model category. Then

\[ B(\mathcal{C}_+, \mathcal{C}_+, F) \to B(\mathcal{C}_\ast, \mathcal{C}_+, F) \]

is a weak equivalence for any pointwise cofibrant reduced simplicial functor \( F: \mathcal{C} \to N \), and the composite

\[ B(\mathcal{C}_\ast, \mathcal{C}_+, UG) \to LUG \to G \]

is a weak equivalence for any pointwise cofibrant pointed simplicial functor \( G: \mathcal{C}_\ast \to N \).

\textbf{Remark 4.5.} We would like to call the two maps of this proposition the derived (co)units. However, since there might not exist a simplicial cofibrant replacement functor \( Q \) defined on the whole category \( N \), this would be slightly wrong terminology.

\textbf{Proof of Proposition 4.4.} For the first equivalence, note that the square

\[ \begin{array}{ccc}
* & \cong & \mathcal{C}_+(-, \ast) \\
\downarrow & & \downarrow \\
\ast & \Rightarrow & \mathcal{C}_\ast(-, d)
\end{array} \]

is a pushout. By Lemma 2.36 and Lemma 2.40, the square

\[ \begin{array}{ccc}
\ast & \cong & B(\ast, \mathcal{C}_+, F) \\
\downarrow & & \downarrow \\
\ast & \Rightarrow & B(\mathcal{C}_\ast, \mathcal{C}_+, F)
\end{array} \]

is a pullback.
is a pushout square in which the top horizontal arrow is a pointwise cofibration. Since the leftmost vertical map is a weak equivalence between pointwise cofibrant objects, we conclude that the rightmost vertical map is a weak equivalence, which was what we needed to show.

For the second map, consider the commutative triangle

\[
\begin{array}{c}
UB(\mathcal{A}_+, \mathcal{B}, UG) \\
\uparrow \\
B(\mathcal{B}, \mathcal{C}_+, UG)
\end{array}
\]

The vertical map is a weak equivalence by the above, while the diagonal map is a weak equivalence by Lemma 2.37, hence the horizontal map is a weak equivalence. Since the forgetful functor \(U\) creates weak equivalences, the desired result follows.

\[
\square
\]

**Proof of Theorem 4.3.** Clearly any pointed functor is reduced. For the converse, let \(F: \mathcal{C} \to \mathcal{N}\) be a reduced simplicial functor. By Corollary 3.14, we may assume without loss of generality that \(F\) is pointwise cofibrant, so by Proposition 4.4, \(B(\mathcal{A}_+, \mathcal{B}, F)\) is a pointed simplicial functor weakly equivalent to \(F\).

\[
\square
\]

Again, this result can be upgraded to a Quillen equivalence if the required model structures exist. The reduced model structure \(\text{Fun}^{\text{red}}(\mathcal{C}, \mathcal{N})\) is defined, if it exists, as the left Bousfield localization of the projective model structure on \(\text{Fun}(\mathcal{C}, \mathcal{N})\) in which the fibrant objects are the reduced pointwise fibrant functors.

**Theorem 4.6.** Let \(\mathcal{C}\) be a small pointed simplicial category and \(\mathcal{N}\) a simplicial model category. If the projective model structures on \(\text{Fun}(\mathcal{C}_+, \mathcal{N})\) and \(\text{Fun}(\mathcal{C}, \mathcal{N})\) exist, then

\[
L : \text{Fun}(\mathcal{C}, \mathcal{N}) \rightleftarrows \text{Fun}(\mathcal{C}_+, \mathcal{N}) : U
\]

is a Quillen pair. Furthermore, if the reduced model structure \(\text{Fun}^{\text{red}}(\mathcal{C}, \mathcal{N})\) exists, then this adjunction becomes a Quillen equivalence between \(\text{Fun}^{\text{red}}(\mathcal{C}, \mathcal{N})\) and the projective model structure on \(\text{Fun}(\mathcal{C}_+, \mathcal{N})\).

**Proof.** If the projective model structure on \(\text{Fun}(\mathcal{C}, \mathcal{N})\) exists, then we can use its cofibrant replacement functor \(Q\) to replace any functor \(F: \mathcal{C} \to \mathcal{N}\) with the pointwise cofibrant functor \(QF\). In particular, \(B(\mathcal{A}_+, \mathcal{B}, Q-)\) is a left derived functor of \(L: \text{Fun}(\mathcal{C}, \mathcal{N}) \to \text{Fun}(\mathcal{C}_+, \mathcal{N})\). The rest of the proof is analogous to that of Theorem 3.10, using Proposition 4.4 instead of Proposition 3.4.

\[
\square
\]

**Example 4.7.** Extending Example 3.12, we obtain a Quillen equivalence

\[
\text{Fun}^{\text{hored}}((\text{Set}_\text{fin}_0), \text{Set}_\text{fin}) \rightleftarrows \text{Fun}(\text{Set}_\text{fin}, \text{Set}_\text{fin})
\]

where \(\text{Fun}^{\text{hored}}((\text{Set}_\text{fin}_0), \text{Set}_\text{fin})\) denotes the left Bousfield localization of the projective model structure in which the fibrant objects are the pointwise fibrant homotopy functors that are also reduced.

**Example 4.8.** For brevity, let us write \(\mathcal{C}_0\) for \((\text{Set}_\text{fin})_0\). In the previous example, one can localize both sides of the Quillen equivalence further such that the fibrant objects become the pointwise fibrant functors that send weak homotopy equivalences to weak.
equivalences, that are reduced, and that are linear in the sense that for every finite pointed simplicial set $X$,
\[
\begin{array}{c}
F(X) \\ \\
\downarrow \\ \\
F(*)
\end{array}
\begin{array}{c}
F(CX) \\ \\
\downarrow \\ \\
F(\Sigma X)
\end{array}
\]
is a homotopy pullback square. Here $CX$ denotes the reduced cone on $X$ and $\Sigma X$ the suspension. On the left-hand side of (3), this localization $\text{Fun}^{\text{lin}}(\mathcal{C}_0, \text{sSet}_*)$ is obtained by localizing at the following two sets of maps:
\[
\{ \mathcal{C}_0(Y, -)_+ \to \mathcal{C}_0(X, -)_+ | X \to Y \text{ is a weak homotopy equivalence} \}, \\
\{ \mathcal{C}_0(\ast, -)_+ \cup_{\mathcal{C}_0(\Sigma X, -)_+} \mathcal{C}_0(CX, -)_+ \to \mathcal{C}_0(X, -)_+ | X \in \text{sSet}_* \}.
\]
Moreover, the Day convolution product on $\text{Fun}^{\text{lin}}(\mathcal{C}_0, \text{sSet}_*)$ endows this localization with the structure of a symmetric monoidal model category. If one localizes the right-hand side of (3) at the analogous sets of maps, then one obtains Lydakis’s stable model category $[\text{Lyd98}]$ together with a monoidal Quillen equivalence between $\text{Fun}^{\text{lin}}(\mathcal{C}_0, \text{sSet}_*)$ and Lydakis’s model category.

Finally, let us mention the following pointed analogue of Corollary 3.14.

**Corollary 4.9.** Let $\mathcal{N}$ be a pointed simplicial model category. For any small full subcategory $\mathcal{N}'$ of $\mathcal{N}$, there exist pointed simplicial fibrant and cofibrant replacement functors on $\mathcal{N}'$.

**Proof.** Assume without loss of generality that $\mathcal{N}'$ contains the zero object of $\mathcal{N}$. By Corollary 3.14, there exists a simplicial cofibrant replacement functor $Q: \mathcal{N}' \to \mathcal{N}$. It follows as in the proof of Corollary 3.14 that $B(\mathcal{C}_+ \mathcal{C}_+, Q)$ is a pointed simplicial cofibrant replacement functor $\mathcal{N}' \to \mathcal{N}$.

A pointed simplicial fibrant replacement functor $\mathcal{N}' \to \mathcal{N}$ is constructed by applying the previous construction to $\mathcal{N}'^{op}$. $
$

### 4.2 Pointed continuous functors

In this section we will deduce the topological analogue of the main result of Section 4.1. The proof strategy is similar as for Theorem 3.15, but note that it will be necessary to assume that the hom-spaces of the indexing category have nondegenerate basepoints.

We say that topological category $\mathcal{C}$ is **pointed** if it has a zero object. Similar as in Section 4.1, there is a unique way of upgrading a pointed topological category $\mathcal{C}$ to a $\text{Top}_*$-enriched category $\mathcal{C}_*$. Since a $\text{Top}_*$-enriched model category has a zero object by definition, we will call such a model category a **pointed topological model category**.

Pointed and reduced functors are defined as in Definition 4.1; that is, $F: \mathcal{C} \to \mathcal{N}$ is reduced if $F(*) \simeq *$ and pointed if $F(*) \cong *$. As in Remark 4.2, a pointed continuous functor $F: \mathcal{C} \to \mathcal{N}$ is the same as a $\text{Top}_*$-enriched functor $\mathcal{C}_* \to \mathcal{N}$. The definition of the reduced model structure $\text{Fun}^{\text{red}}(\mathcal{C}, \mathcal{N})$ is the same as in the simplicial case.

**Theorem 4.10.** Let $\mathcal{C}$ be a small pointed topological category whose hom-spaces have nondegenerate basepoints, and let $\mathcal{N}$ be a good pointed topological model category. Then a continuous functor $F: \mathcal{C} \to \mathcal{N}$ is reduced if $F(*) \simeq *$ and pointed if $F(*) \cong *$. As in Remark 4.2, a pointed continuous functor $F: \mathcal{C} \to \mathcal{N}$ is the same as a $\text{Top}_*$-enriched functor $\mathcal{C}_* \to \mathcal{N}$. The definition of the reduced model structure $\text{Fun}^{\text{red}}(\mathcal{C}, \mathcal{N})$ is the same as in the simplicial case. 


C → N is weakly equivalent to a pointed functor if and only if it is reduced. Moreover, the forgetful functor

\[ U : \text{Fun}(C_*, N) \to \text{Fun}^{\text{red}}(C, N) \]

from the projective model structure to the reduced model structure is a Quillen equivalence whenever the reduced model structure exists.

Proof. This is analogous to the proof of Theorem 3.15.

Example 4.11. Recall the category \( \mathcal{W} \) and the Quillen equivalence

\[ \text{Fun}^{\text{ho}}(\mathcal{W}_0, \text{Top}_*) \rightleftarrows \text{Fun}(\mathcal{W}_T, \text{Top}_*) \]

from example Example 3.16. Here we view \( \mathcal{W} \) as a \( \text{Top}_* \)-category and \( \mathcal{W}_T \) denotes its underlying topological category. As in Example 4.7, we obtain a Quillen equivalence

\[ \text{Fun}^{\text{hored}}(\mathcal{W}_0, \text{Top}_*) \rightleftarrows \text{Fun}(\mathcal{W}, \text{Top}_*) , \]

and we can localize both sides of this Quillen equivalence as in Example 4.8 such that the fibrant objects become the linear reduced homotopy functors. The localization on the right-hand side then becomes the absolute stable model category of \( \mathcal{W} \)-spectra defined in [Man+01, §17]. Let us denote the localization on the left-hand side by \( \text{Fun}^{\text{lin}}(\mathcal{W}_0, \text{Top}_*) \). As in Example 4.8, one can verify that the Day convolution makes \( \text{Fun}^{\text{lin}}(\mathcal{W}_0, \text{Top}_*) \) into a symmetric monoidal model category and that the Quillen equivalence with the absolute stable model category of \( \mathcal{W} \)-spectra is monoidal.

4.3 (Symmetric) multi-reduced functors

In this section, we extend the previous results to symmetric multifunctors. Such functors occur naturally when studying the layers of the Taylor tower in Goodwillie calculus, which is the main motivation for discussing them here. We will first treat the case of non-symmetric multifunctors and then lift the obtained results to the case of symmetric multifunctors.

Definition 4.12. Let \( \mathcal{V} \) be a closed symmetric monoidal category. A multifunctor is a \( \mathcal{V} \)-functor of the form

\[ \mathcal{C}^{\otimes n} \to \mathcal{D} , \]

where \( \mathcal{C} \) and \( \mathcal{D} \) are \( \mathcal{V} \)-categories and \( \mathcal{C}^{\otimes n} \) denotes

\[ \mathcal{C}^{\otimes n} = \mathcal{C} \otimes \cdots \otimes \mathcal{C} \]

(\( n \)-times) (cf. Definition 2.2).

Throughout this section we will work with topological categories. The results also hold in the simplicial case, in fact even under slightly more general assumptions. We will come back to this in Remark 4.19 below. To avoid an overload of parentheses, we shall write \( \mathcal{C}_*^{\wedge n} \) for \( (\mathcal{C}_*)^{\wedge n} \) and \( \mathcal{C}_+^{\wedge n} \) for \( (\mathcal{C}_+)^{\wedge n} \) throughout this section.

Note that if \( \mathcal{C}_* \) is a \( \text{Top}_* \)-category, then the underlying topological category of \( \mathcal{C}_*^{\wedge n} \) is not \( \mathcal{C}^{\wedge n} \). This means that we can’t apply the previous results directly and that we need a different definition of “reduced” in the case of a multifunctor.
Definition 4.13. Let \( \mathcal{C} \) be a pointed topological category, \( N \) a pointed simplicial model category and \( F: \mathcal{C}^{\times n} \to N \) a multifunctor. Then

(i) the multifunctor \( F \) is called (multi)pointed if \( F(c_1, \ldots , c_n) \cong * \) whenever \( c_i = * \) for some \( 1 \leq i \leq n \), and

(ii) the multifunctor \( F \) is called (multi)reduced if \( F(c_1, \ldots , c_n) \simeq * \) whenever \( c_i = * \) for some \( 1 \leq i \leq n \).

Remark 4.14. As in Remark 4.2, a \( \operatorname{Top}_* \)-functor \( \mathcal{C} \wedge n \to N \) is the same as a multipointed topological functor \( F: \mathcal{C} \times n \to N \).

The multireduced model structure \( \operatorname{Fun}^{\text{mred}}(\mathcal{C}^{\times n}, N) \) is defined, if it exists, as the left Bousfield localization of the projective model structure on \( \operatorname{Fun}(\mathcal{C}^{\times n}, N) \) in which the fibrant objects are the multireduced pointwise fibrant functors.

Theorem 4.15. Let \( \mathcal{C} \) be a small pointed topological category whose hom-spaces have nondegenerate basepoints, and let \( N \) be a good pointed topological category. Then a continuous multifunctor \( \mathcal{C}^{\times n} \to N \) is weakly equivalent to a multipointed functor if and only if it is multireduced. Moreover, the forgetful functor

\[ U: \operatorname{Fun}(\mathcal{C}_+^{\wedge n}, N) \to \operatorname{Fun}^{\text{mred}}(\mathcal{C}^{\times n}, N) \]

from the projective model structure to the multireduced model structure is a Quillen equivalence whenever the multireduced model structure exists.

Proof. The forgetful functor can be viewed as the restriction along

\[ (\mathcal{C}^{\times n})_+ \cong \mathcal{C}_+^{\wedge n} \to \mathcal{C}_+^{\wedge n}. \] (4)

In particular, left Kan extension along this map provides a left adjoint to the forgetful functor.

Now let \( F: \mathcal{C}^{\times n} \to N \) be multireduced and assume without loss of generality that \( F \) is cofibrant in the projective model structure on \( \operatorname{Fun}(\mathcal{C}^{\times n}, N) \). We will show that the left Kan extension of \( F \) along (4) is weakly equivalent to \( F \). In order to see this, note that this map factors as

\[ \mathcal{C}_+^{\wedge n} \xrightarrow{\eta_1} \cdots \xrightarrow{\eta_{k-1}} \mathcal{C}_+^{\wedge (n-k+1)} \wedge \mathcal{C}_+^{\wedge (k-1)} \xrightarrow{\eta_k} \mathcal{C}_+^{\wedge (n-k)} \wedge \mathcal{C}_+^{\wedge k} \xrightarrow{\eta_{n-k}} \cdots \xrightarrow{\eta_n} \mathcal{C}_+^{\wedge n}. \]

In particular, we can prove the claim for each of these maps separately, so it suffices to show that the left Kan extension of a reduced functor

\[ G: \mathcal{C}_+ \to \operatorname{Fun}(\mathcal{C}_+^{\wedge (n-k)} \wedge \mathcal{C}_+^{\wedge (k-1)}, N) \]

along \( \mathcal{C}_+ \to \mathcal{C}_+^{\wedge n} \) is weakly equivalent to \( G \). This was proved in Theorem 4.10.

The claim about the Quillen equivalence follows directly from Lemma 2.22. \( \blacksquare \)

We now extend this result to symmetric multifunctors.
**Definition 4.16.** Let \( \mathcal{V} \) be a closed symmetric monoidal category and let \( \mathcal{C} \) and \( \mathcal{D} \) be \( \mathcal{V} \)-categories. A **symmetric multifunctor** is a multifunctor \( F: C^{\otimes n} \to \mathcal{D} \) equipped with \( \mathcal{V} \)-natural isomorphisms
\[
\sigma_F: F(c_1, \ldots, c_n) \cong F(c_{\sigma(1)}, \ldots, c_{\sigma(n)})
\]
for every \( \sigma \in \Sigma_n \), satisfying \( (\tau \sigma)_F = \sigma_F \tau_F \). A symmetric \( \mathcal{V} \)-natural transformation is a \( \mathcal{V} \)-natural transformation between symmetric functors that is compatible with the symmetry, and the category of symmetric multifunctors and such natural transformations is denoted \( \text{Fun}^{\text{sym}}(C^{\otimes n}, \mathcal{D}) \).

**Remark 4.17.** In the definition above, it follows automatically that for the identity permutation \( \epsilon \in \Sigma_n \), one has \( \epsilon_F = \text{id} \).

In the proof of Theorem 4.18 below, we will use a different description of the category \( \text{Fun}^{\text{sym}}(C^{\otimes n}, \mathcal{N}) \), namely as the homotopy fixed points of \( \text{Fun}(C^{\otimes n}, \mathcal{N}) \). Note that \( \Sigma_n \) acts from the left on \( C^{\otimes n} \) by permuting the factors, giving a right \( \Sigma_n \)-action on \( \text{Fun}(C^{\otimes n}, \mathcal{N}) \). Define \( E\Sigma_n \) to be the groupoid whose objects are the elements of \( \Sigma_n \) and which has exactly one arrow between any pair of objects. \( \Sigma_n \) acts on this groupoid by multiplication from the right. The category \( \text{Fun}^{\text{sym}}(C^{\otimes n}, \mathcal{D}) \) is easily seen to be equivalent to the category of \( \Sigma_n \)-equivariant functors
\[
E\Sigma_n \to \text{Fun}(C^{\otimes n}, \mathcal{D})
\]
and \( \Sigma_n \)-equivariant \( \mathcal{V} \)-natural transformation between such functors.

In [BR14, Lem. 3.6], yet another description of \( \text{Fun}^{\text{sym}}(C^{\otimes n}, \mathcal{D}) \) is given. Namely, it is shown that \( \text{Fun}^{\text{sym}}(C^{\otimes n}, \mathcal{D}) \) is equivalent to \( \text{Fun}(\Sigma_n \lhd C^{\otimes n}, \mathcal{D}) \) for a certain \( \mathcal{V} \)-category \( \Sigma_n \lhd C^{\otimes n} \), namely the wreath product category (cf. [BR14, Def. 3.3]). In particular, we see that \( \text{Fun}^{\text{sym}}(C^{\otimes n}, \mathcal{D}) \) is an ordinary functor category, so the usual methods for constructing projective model structures apply to it. In the theorem below, the **symmetric multireduced model structure** \( \text{Fun}^{\text{symred}}(C^{\times n}, \mathcal{N}) \) is defined, if it exists, as the left Bousfield localization of the projective model structure on \( \text{Fun}^{\text{sym}}(C^{\times n}, \mathcal{N}) \) in which the fibrant objects are the symmetric multireduced pointwise fibrant functors.

**Theorem 4.18.** Let \( \mathcal{C} \) be a small pointed topological category whose hom-spaces have nondegenerate basepoints, and let \( \mathcal{N} \) be a good pointed topological category. Then a continuous symmetric multifunctor \( C^{\times n} \to \mathcal{N} \) is weakly equivalent (as symmetric multifunctors) to a symmetric multipointed functor if and only if it is multireduced. Moreover, the forgetful functor
\[
U: \text{Fun}^{\text{sym}}(C^{\land n}, \mathcal{N}) \to \text{Fun}^{\text{symred}}(C^{\times n}, \mathcal{N})
\]
from the projective model structure to the symmetric multireduced model structure is a Quillen equivalence whenever the symmetric multireduced model structure exists.

**Proof.** The main idea of this proof is that one can work underlying in \( \text{Fun}(C^{\times n}, \mathcal{N}) \) and \( \text{Fun}(C^{\land n}, \mathcal{N}) \), where we have already proved the result. Since \( C^{\land n} \to C^{\times n} \) is a \( \Sigma_n \)-!equivariant functor, the adjunction
\[
L : \text{Fun}(C^{\times n}, \mathcal{N}) \rightleftarrows \text{Fun}(C^{\land n}, \mathcal{N}) : U
\]
obtained by restriction and left Kan extension along this functor is \( \Sigma_n \)-equivariant. By identifying symmetric functors with \( \Sigma_n \)-equivariant functors out of \( E\Sigma_n \) as in (5), we obtain an adjunction
\[
L : \text{Fun}^{\text{sym}}(C^{\times n}, \mathcal{N}) \rightleftarrows \text{Fun}^{\text{sym}}(C^{\land n}, \mathcal{N}) : U
\]
where the functors are computed underlying as in (6) (hence the abuse of notation). Furthermore, note that the forgetful functor

\[
\text{Fun}^\text{sym}(\mathcal{C}^\times n, \mathcal{N}) \to \text{Fun}(\mathcal{C}^\times n, \mathcal{N})
\]

admits a right adjoint given by \( F \mapsto \prod_{\sigma \in \Sigma} \sigma^* F \). Since this right adjoint preserves pointwise fibrations, we see that the forgetful functor (8) preserves projectively cofibrant functors. This implies that the derived adjunction of (7) can be computed underlying as the derived adjunction of (6). By the proof of Theorem 4.15, the derived unit of this adjunction is a weak equivalence.

The claim about the Quillen equivalence again follows directly from Lemma 2.22. ■

Remark 4.19. If the relevant projective model structures exist, then the above proofs also go through for simplicial functors. However, if these projective model structures do not exist, then it is still true that any (symmetric) multireduced simplicial functor \( F \) is equivalent to a (symmetric) multipointed one. Namely, if one uses Corollary 4.9 to replace \( F \) with a pointwise cofibrant functor, then the proofs of Theorems 4.15 and 4.18 still go through if one uses the bar construction instead of left Kan extension everywhere.

Remark 4.20. One can also prove similar results for homotopy functors. Namely, if \( \mathcal{C} \) is a topological or simplicial category, then a multifunctor \( (\mathcal{C}_0)^{\times n} = (\mathcal{C}^{\times n})_0 \to \mathcal{N} \) is a homotopy functor if and only if it sends homotopy equivalences to weak equivalences in each variable separately. In particular, the analogue of Theorem 4.15 follows directly from Theorems 3.3, 3.10 and 3.15. This can then be upgraded to a statement about symmetric multifunctors by the same proof as Theorem 4.18.

## Linear functors

In the final part of this paper, we will study in which cases a given functor is equivalent to one that respects an enrichment in the category \( \text{Sp} \) of orthogonal spectra. In Section 5.1, under the assumption that the indexing category has cotensors by \( S^{-1} \), we describe a condition called strict linearity (cf. Definition 5.2) and show that a functor is equivalent to an \( \text{Sp} \)-functor if and only if it satisfies this condition. In particular, we prove Theorem C. We then prove an analogous statement for (symmetric) functors of multiple variables in Section 5.2.

### 5.1 Strict linearity and spectral functors

Recall the category of orthogonal spectra \( \text{Sp} \) described in Section 2.2. We will write \( S^{-n} \) for the image of \( n \in J_\Sigma \) under the Yoneda embedding \( J_\text{pp} \hookrightarrow \text{Sp} \)\(^9\). Given a spectral category \( \mathcal{C} \), we will write \( \mathcal{C}_s \) for its underlying \( \text{Top}_s \)-category.

**Construction 5.1.** Let \( \mathcal{C} \) be a spectral category that admits cotensors by \( S^{-1} \) and let \( \mathcal{N} \) be a spectral model category. Since \( S^{-k} \otimes S^{-1} = S^{-k-1} \), we see inductively that \( \mathcal{C} \) admits cotensors by \( S^{-n} \) for every \( n \geq 0 \). By Construction 2.4, these cotensors assemble into a functor \( J^\text{pp}_\Sigma \otimes \mathcal{C} \to \mathcal{C} \), where \( J^\text{pp}_\Sigma \) is the full spectral subcategory of \( \text{Sp} \) on the objects

\(^9\)To avoid awkward notation, we will write \( S \) or \( S_0 \) and not \( S^{-0} \) for the image of \( 0 \in \Sigma_\Sigma \) under the Yoneda embedding.
\{S^{-n}\}_{n \geq 0}$. By the enriched Yoneda lemma, the underlying \textbf{Top}_*\text{-category of } \mathcal{J}_S\text{ is } \mathcal{J}_S. In particular, restricting the cotensor functor to the underlying \textbf{Top}_*\text{-categories yields a functor } \mathcal{J}_S \land \mathcal{C}_* \to \mathcal{C}_*. Composing this functor with a \textbf{Top}_*\text{-functor } F: \mathcal{C}_* \to \mathcal{N}\) yields maps \(\mathcal{J}_S(0, n) \to \mathcal{N}(F(c), F(c^S^{-}))\) and hence, by adjunction, maps

\[
\sigma_n: \Sigma^n F(c) = \Sigma_S(0, n) \otimes F(c) \to F(c^S^{-n})
\]

that are natural in \(c\).

**Definition 5.2.** Let \(\mathcal{C}\) be a spectral category that admits cotensors by \(S^{-1}\), let \(\mathcal{N}\) be a spectral model category and assume that the hom-spaces of the underlying \(\mathcal{C}\)-functor to \(\mathcal{N}\) are nondegenerately based. Then a \(\mathcal{C}\)-functor \(\mathcal{N}\) is strictly linear if and only if it is strictly linear.

**Remark 5.3.** Using the associativity of the cotensor, one can show that \(\sigma_{n+1} = \sigma_1 \circ \Sigma \sigma_n\). In particular, if \(F: \mathcal{C}_* \to \mathcal{N}\) is strictly linear, then it follows inductively that \((\Sigma \Sigma^n) F(c) \to F(c^S^{-n})\) is a weak equivalence for every \(n \geq 0\).

**Example 5.4.** The spectral category \(\mathcal{J}_S\) from Construction 5.1 is dual to the full subcategory of \(\textbf{Sp}\) spanned by \(\{S^{-n}\}\), hence it admits cotensors by \(S^{-1}\). A functor \(X = \{X_n\}_{n \in \mathbb{N}}: \mathcal{J}_S \to \mathcal{N}\) is strictly linear if and only if for every \(n \geq 0\), the map \(\Sigma \Sigma X_n \to X_{n+1}\) is a weak equivalence. Since \(\mathcal{N}\) is stable by Lemma 2.20, this is equivalent to \(X_n \to \text{R}\Omega X_{n+1}\) being a weak equivalence. That is, strictly linear functors \(\mathcal{J}_S \to \mathcal{N}\) are the same as orthogonal \(\Omega\)-spectrum objects in \(\mathcal{N}\).

The main result of this section is the following.

**Theorem 5.5.** Let \(\mathcal{C}\) be a small spectral category that admits cotensors by \(S^{-1}\), let \(\mathcal{N}\) be a good spectral model category and assume that the hom-spaces of the underlying \textbf{Top}_*\text{-category \(\mathcal{C}_*\) of \(\mathcal{C}\)} are nondegenerately based. Then a \textbf{Top}_*\text{-functor } \mathcal{C}_* \to \mathcal{N}\text{ is equivalent to a spectral functor if and only if it is strictly linear.}

Let us start by showing that spectral functors are strictly linear. We will use the following lemmas.

**Lemma 5.6.** Let \(\mathcal{C}\) be a small spectral category and \(\mathcal{N}\) a good spectral model category. Denote the left derived functor of \(\Sigma\): \(\text{Fun}(\mathcal{C}, \mathcal{N}) \to \text{Fun}(\mathcal{C}, \mathcal{N})\) by \(\Sigma \Sigma P\) and the left derived functor of \(\Sigma\): \(\mathcal{N} \to \mathcal{N}\) by \(\Sigma \Sigma N\). For any \(F: \mathcal{C} \to \mathcal{N}\) and any \(c\) in \(\mathcal{C}\), one has \((\Sigma \Sigma P) F(c) \simeq (\Sigma \Sigma N) F(c)\).

**Proof.** The subtlety of this statement lies in the fact that projectively cofibrant functors \(F: \mathcal{C} \to \mathcal{N}\) need not be pointwise cofibrant. To see that nonetheless \((\Sigma \Sigma P) F(c) \simeq (\Sigma \Sigma N) F(c)\), note that by Lemma 2.21 there exists a weak equivalence of spectral categories \(\mathcal{C} \overset{\sim}{\longrightarrow} \mathcal{C}\) where \(\mathcal{C}\) has cofibrant hom-objects. By the goodness of \(\mathcal{N}\), restriction along this weak equivalence induces a Quillen equivalence

\[
L : \text{Fun}(\mathcal{C}, \mathcal{N}) \rightleftharpoons \text{Fun}(\mathcal{C}, \mathcal{N}) : U.
\]
In particular, one can write \( F \simeq LG \) for a projectively cofibrant functor \( G : \mathcal{C} \to \mathcal{N} \). Since \( LG \) is projectively cofibrant, we have \( (\mathcal{L}\Sigma_F)F(c) \simeq \Sigma LG(c) \). Because \( \hat{\mathcal{C}} \) has cofibrant hom-objects, the functor \( G \) is pointwise cofibrant, hence \( G(c) \) is a cofibrant replacement of \( F(c) \). Since \( \Sigma G \xrightarrow{\simeq} UL(\Sigma G) \simeq \Sigma ULG \), we have \( (\mathcal{L}\Sigma_N)F(c) \simeq \Sigma G(c) \simeq \Sigma LG(c) \). \[ \blacksquare \]

**Lemma 5.7.** The canonical maps \( \Sigma S^{-1} \to (S^{-1})S^{-1} \) and \( S^0 \to (S^{-1})S^{-1} \) are stable equivalences.

**Proof.** Note that cotensors by \( S^{-1} \) in \( \mathcal{S}p \) are the same as shifts. In particular, the map \( \Sigma S^{-1} \to (S^{-1})S^{-1} \), constructed as in Construction 5.1, is a stable equivalence by [Sch18, Prop. 3.1.25.(ii)] (taking \( G \) to be the trivial group, \( X = S^{-1} \) and \( V = \mathbb{R} \) in that proposition).

For the second map, note that the functor \( (\cdot)^{S^{-1}} = \text{sh} \) preserves stable equivalences by [Sch18, Prop. 3.1.25.(ii)] and that it is a right Quillen equivalence. In particular, we see that \( S^0 \to (S^{-1})S^{-1} \) is a stable equivalence if and only if its adjunct \( \text{id} : S^{-1} \to S^{-1} \) is a stable equivalence, which is the case since it is the identity map. \[ \blacksquare \]

**Proposition 5.8.** Let \( \mathcal{C} \) and \( \mathcal{N} \) be as in Theorem 5.5 and let \( F : \mathcal{C} \to \mathcal{N} \) be a spectral functor. Then \( F \) is strictly linear.

**Proof.** By the goodness of \( \mathcal{N} \) and the fact that strict linearity of functors \( \mathcal{C} \to \mathcal{N} \) is a homotopy invariant property, we may assume without loss of generality that \( F \) is projectively cofibrant. By Lemma 5.6, \( F \) is strictly linear if and only if for every \( c \in \mathcal{C} \), the map \( \sigma_1 : \Sigma F(c) \to F(c^{S^{-1}}) \) is a weak equivalence in \( \mathcal{N} \). We will denote the object \( S^{-n} \) of the spectral category \( \mathcal{I}_S \) defined in Construction 5.1 by \( n \). By construction, \( \sigma_1 \) agrees with the map \( S^1 \otimes F(c) \simeq \mathcal{I}_S(0,1) \otimes F(c) \to F(c^{S^{-1}}) \).

Since \( \mathcal{I}_S(1,0) \simeq S^{-1} \), \( \mathcal{I}_S(0,0) \simeq S^0 \) and \( \mathcal{I}_S(1,1) \simeq (S^{-1})S^{-1} \), we obtain a commutative diagram

\[
\begin{array}{ccc}
S^1 \otimes S^{-1} \otimes S^1 \otimes F(c) & \xrightarrow{id \otimes id \otimes ac} & S^1 \otimes S^{-1} \otimes F(c^{S^{-1}}) & \xrightarrow{c \otimes id} & (S^{-1})S^{-1} \otimes F(c^{S^{-1}}) \\
\downarrow{id \otimes c \otimes id} & & \downarrow{id \otimes ac} & & \downarrow{ac} \\
S^1 \otimes S^0 \otimes F(c) & \xrightarrow{id \otimes ac} & S^1 \otimes F(c) & \xrightarrow{\sigma_1} & F(c^{S^{-1}}).
\end{array}
\] (9)

The leftmost vertical map is a weak equivalence since \( F \) is projectively cofibrant and \( c : S^{-1} \otimes S^1 \to S^0 \) is a stable equivalence between cofibrant orthogonal spectra. Moreover, the bottom left horizontal map is an isomorphism. To see that the composition of the top right horizontal map and the rightmost vertical map is a weak equivalence, let \( \mathcal{I}_S \simeq \mathcal{I}_S \) be a cofibrant replacement in the sense of Lemma 2.21. In particular, the hom-object \( \mathcal{I}_S(1,1) \) is cofibrant. We then obtain the commutative diagram

\[
\begin{array}{ccc}
\mathcal{I}_S(0,1) \otimes \mathcal{I}_S(1,0) \otimes F(c^{S^{-1}}) & \xrightarrow{c \otimes id} & \mathcal{I}_S(1,1) \otimes F(c^{S^{-1}}) & \xrightarrow{ac} & F(c^{S^{-1}}) \\
\downarrow & & \downarrow & & \downarrow{id} \\
S^1 \otimes S^{-1} \otimes F(c^{S^{-1}}) & \xrightarrow{c \otimes id} & (S^{-1})S^{-1} \otimes F(c^{S^{-1}}) \otimes F(c^{S^{-1}}) & \xrightarrow{ac} & F(c^{S^{-1}})
\end{array}
\]

The leftmost vertical map is a weak equivalence since \( \mathcal{I}_S(0,1) \otimes \mathcal{I}_S(1,0) \to S^1 \otimes S^{-1} \) is a stable equivalence between cofibrant orthogonal spectra. Moreover, \( \mathcal{I}_S(0,1) \otimes \mathcal{I}_S(1,0) \to \mathcal{I}_S(1,1) \) is a stable equivalence between cofibrant orthogonal spectra since \( S^1 \otimes S^{-1} \to \mathcal{I}_S(1,1) \) is a stable equivalence between cofibrant orthogonal spectra. \[ \blacksquare \]
$(S^{-1})S^{-1}$ is a stable equivalence by Lemma 5.7. This shows that the top left horizontal map is a weak equivalence. Finally, to see that the top right horizontal map is a weak equivalence, consider the diagram

$$
S^0 \otimes F(c^{S^{-1}}) \longrightarrow \tilde{\Sigma}(1, 1) \otimes F(c^{S^{-1}}) \longrightarrow F(c^{S^{-1}})
$$

The unit map $S^0 \to \tilde{\Sigma}(1, 1)$ is a weak equivalence between cofibrant objects since $S^0 \to (S^{-1})S^{-1}$ is a weak equivalence by Lemma 5.7, hence the map $\tilde{\Sigma}(1, 1) \otimes F(c^{S^{-1}}) \to F(c^{S^{-1}})$ is a weak equivalence by the two-out-of-three property. In particular, the composition

$$S^1 \otimes S^{-1} \otimes F(c^{S^{-1}}) \overset{c \otimes \text{id}}{\longrightarrow} (S^{-1})S^{-1} \otimes F(c^{S^{-1}}) \overset{\sigma_1}{\longrightarrow} F(c^{S^{-1}})$$

is a weak equivalence, hence an application of the two-out-of-six property to diagram (9) shows that $\sigma_1 : \Sigma F(c) \to F(c^{S^{-1}})$ is a weak equivalence.

Like in the proofs of Theorems 3.3 and 4.3, we will show that any strictly linear functor is equivalent to a spectral one by carefully analyzing derived left Kan extensions. In our analysis, we will make use of (orthogonal) spectrum objects in $\mathcal{N}$. Recall that an (orthogonal) spectrum object in $\mathcal{N}$ is a $\textbf{Top}_*$-functor $\mathcal{J}_S \to \mathcal{N}$. The category $\textbf{Sp}(\mathcal{N}) := \text{Fun}(\mathcal{J}_S, \mathcal{N})$ of spectrum objects in $\mathcal{N}$ comes with the usual adjunction

$$\Sigma^\infty : \mathcal{N} \rightleftarrows \textbf{Sp}(\mathcal{N}) : \Omega^\infty,$$

where $\Sigma^\infty$ sends an object $N$ of $\mathcal{N}$ to the spectrum object $\{S^n \otimes N\}_{n \in \mathbb{N}}$ and where $\Omega^\infty$ evaluates a spectrum object at the 0-th level. Moreover, since $\mathcal{N}$ is a spectral model category, it comes with a second adjunction

$$| - |_S : \textbf{Sp}(\mathcal{N}) \rightleftarrows \mathcal{N} : \text{Sing}_S,$$

where the right adjoint $\text{Sing}_S$ sends an object $N$ of $\mathcal{N}$ to $\{N^{S^{-n}}\}_{n \in \mathbb{N}}$ while the left adjoint is given by $|\{M_n\}_{n \in \mathbb{N}}|_S = \int^S S^{-n} \otimes M_n$.

**Remark 5.9.** If $\mathcal{C}$ is a spectral category that admits cotensors by $S^{-1}$, then given a $\textbf{Top}_*$-functor $F : \mathcal{C} \to \mathcal{N}$, one can form the spectrum object $\{F(c^{S^{-n}})\}_{n \in \mathbb{N}}$. Since $F(c) \cong F(c^S) = \Omega^\infty\{F(c^{S^{-n}})\}$, we obtain a map

$$\{\Sigma^\infty F(c)\} \to \{F(c^{S^{-n}})\}_{n \in \mathbb{N}}$$

by adjunction. It follows by construction that this is levelwise the map $\sigma_a$ of Construction 5.1. In particular, if $F$ is projectively cofibrant, then by Remark 5.3 and Lemma 5.6, the functor $F$ is strictly linear if and only if the map (10) is a level equivalence for every $c$ in $\mathcal{C}$.

We will endow $\textbf{Sp}(\mathcal{N})$ with the projective model structure and call the weak equivalences of the projective model structure level equivalences. The two adjunctions described above are Quillen pairs with respect to the projective model structure on $\textbf{Sp}(\mathcal{N})$ since their right adjoints clearly preserve (trivial) fibrations.
Lemma 5.10. There is a natural isomorphism $|\Sigma^\infty N|_S \cong N$.

Proof. This follows since $N \cong N^0 = \Omega^\infty \text{Sing}_S(N)$. ■

Lemma 5.11. Let $\mathcal{C}_s$ be a small $\text{Top}_s$-category and let $F: \mathcal{C}_s \to \mathcal{N}$ and $G: \mathcal{C}_s^\otimes \to \mathcal{S}_p$ be $\text{Top}_s$-functors. Then there is a natural isomorphism

$$\int^{\mathcal{C}_s} G \otimes F \cong \left\{ \int^{\mathcal{C}_s} G_n \otimes F \right\}_{n \in \mathbb{N}}_{\mathcal{S}}$$

where $G_n$ denotes the functor $\mathcal{C}_s \to \mathcal{sSet}$ that sends $c$ to the $n$-th space of $G(c)$.

Proof. Note that if $X = \{X_n\}_{n \in \mathbb{N}}$ is an orthogonal spectrum, then $\int^{j_3} S^{-n} \otimes X_n \cong X$ by the enriched coYoneda lemma (see e.g. (3.72) of [Kel82]). In particular, we see that

$$\int^{\mathcal{C}_s} G \otimes F \cong \int^{\mathcal{C}_s} \int^{j_3} S^{-n} \otimes G_n \otimes F \cong \int^{j_3} S^{-n} \otimes \int^{\mathcal{C}_s} G_n \otimes F$$

by Lemma 2.8. ■

Lemma 5.12. Let $\mathcal{N}$ be a good spectral model category and let $\mathcal{C}_s$ be a small $\text{Top}_s$-category. Then for any projectively cofibrant $F: \mathcal{C}_s \to \mathcal{N}$ and any pointwise cofibrant $G: \mathcal{C}_s \to \mathcal{S}_p$, the spectrum object $\left\{ \int^{\mathcal{C}_s} G_n \otimes F \right\}_{n \in \mathbb{N}}$ is projectively cofibrant in $\mathcal{S}_p(\mathcal{N})$.

Proof. We need to show that if $\{L_n\}_{n \in \mathbb{N}} \to \{K_n\}_{n \in \mathbb{N}}$ is levelwise a trivial fibration, then any map $\left\{ \int^{\mathcal{C}_s} G_n \otimes F \right\}_{n \in \mathbb{N}} \to \{K_n\}_{n \in \mathbb{N}}$ lifts to a map to $\{L_n\}_{n \in \mathbb{N}}$. By adjunction, this is equivalent to constructing a lift in

$$\begin{array}{c}
\int^{j_3} L_n^{G_s} \\
F \\
\downarrow \\
\int^{j_3} K_n^{G_s}
\end{array}$$

hence the result follows if we can show that $\int^{j_3} L_n^{G_s} \to \int^{j_3} K_n^{G_s}$ is pointwise a trivial fibration. In particular, it suffices to show that for any cofibrant orthogonal spectrum $X = \{X_n\}_{n \in \mathbb{N}}$, the functor

$$\mathcal{S}_p(\mathcal{N}) \to \mathcal{N}; \quad \{N_n\}_{n \in \mathbb{N}} \mapsto \int^{j_3} N_n^{X_s}$$

is right Quillen. This follows since its left adjoint is the functor $\mathcal{N} \to \mathcal{S}_p(\mathcal{N})$ defined by $N \mapsto \{X_n \otimes N\}_{n \in \mathbb{N}}$, which is easily seen to be left Quillen. ■

Finally, let us mention the following criterion for determining whether $F: \mathcal{C}_s \to \mathcal{N}$ is strictly linear.

Lemma 5.13. Let $\mathcal{C}$ and $\mathcal{N}$ be as in Theorem 5.5, and let $p: \widehat{\mathcal{C}} \to \mathcal{C}$ be a cofibrant replacement of $\mathcal{C}$ in the sense of Lemma 2.21. Then $F: \mathcal{C}_s \to \mathcal{N}$ is strictly linear if and only if there exists a projectively cofibrant $G: \widehat{\mathcal{C}}_s \to \mathcal{N}$ such that $F \simeq \text{Lan}_p G$ and

$$\{\Sigma^n G\}_{n \in \mathbb{N}} \to \left\{ \int^{\widehat{\mathcal{C}_s}} \widehat{\mathcal{C}_s}(c, -) \otimes G(c) \right\}_{n \in \mathbb{N}}$$

is pointwise a level equivalence in $\mathcal{S}_p(\mathcal{N})$. 

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Proof. Note that \( p : \overset{\cdot}{\mathcal{C}} \rightarrow \mathcal{C} \) is a weak equivalence between \( \text{Top}_* \)-categories whose hom-spaces are nondegenerately based, hence the goodness of \( N \) implies that \( F \simeq \text{Lan}_p G \) for some projectively cofibrant \( G : \overset{\cdot}{\mathcal{C}} \rightarrow N \). Since strict linearity is a homotopy invariant property, it suffices to show that \( \text{Lan}_p G \) is strictly linear if and only if \((11)\) is a level equivalence. To see that this holds, note that

\[
\int_{\overset{\cdot}{\mathcal{C}}} \mathcal{C}(c,d)_n \otimes G(c) \cong \int_{\overset{\cdot}{\mathcal{C}}} \mathcal{C}(c,d_{S^n}) \otimes \text{Lan}_p G(c) \cong \text{Lan}_p G(d_{S^n})
\]

naturally in \( d \). In particular, we obtain the commutative diagram

\[
\begin{array}{ccc}
\{\Sigma^n G(d)\}_{n \in \mathbb{N}} & \longrightarrow & \{\int_{\overset{\cdot}{\mathcal{C}}} \mathcal{C}(c,d)_n \otimes G(c)\}_{n \in \mathbb{N}} \\
\downarrow & & \downarrow \\
\{\Sigma^n \text{Lan}_p G(d)\}_{n \in \mathbb{N}} & \cong & \{\text{Lan}_p G(d_{S^n})\}_{n \in \mathbb{N}} \\
\end{array}
\]

Using that \( \overset{\cdot}{\mathcal{C}} \simeq \mathcal{C} \) is a level equivalence on hom-objects, we see that the left-hand vertical arrow is a level equivalence since \( \Sigma^n G \) is projectively cofibrant. On the other hand, the right-hand vertical arrow is a level equivalence by Proposition 2.30. It follows from Remark 5.9 that \( \text{Lan}_p G \) is strictly linear if and only if \((11)\) is a level equivalence. \( \blacksquare \)

Remark 5.14. In light of the previous lemma, it seems reasonable to call a functor \( G : \overset{\cdot}{\mathcal{C}} \rightarrow N \) strictly linear if for any projectively cofibrant replacement \( G' : \overset{\cdot}{\mathcal{C}} \rightarrow N \), the map \((11)\) is pointwise a level equivalence in \( \text{Sp}(N) \). The proof of Lemma 5.13 then shows that under the Quillen equivalence \( \text{Lan}_p : \text{Fun}(\overset{\cdot}{\mathcal{C}},N) \rightleftarrows \text{Fun}(\mathcal{C},N) : U \), a functor \( F : \mathcal{C} \rightarrow N \) is strictly linear in the sense of Definition 5.2 if and only if \( UF \) is strictly linear sense described here.

Combining these lemmas, we prove the other direction of Theorem 5.5. Write

\[
L : \text{Fun}(\mathcal{C},N) \rightleftarrows \text{Fun}(\overset{\cdot}{\mathcal{C}},N) : U
\]

for the Quillen pair induced by restriction and left Kan extension along \( \Sigma^\infty \mathcal{C}_s \rightarrow \mathcal{C} \), where \( \Sigma^\infty \mathcal{C}_s \) denotes the category obtained by applying \( \Sigma^\infty \) to the hom-spaces of \( \mathcal{C}_s \).

Proposition 5.15. Let \( \mathcal{E} \) be a small spectral category that admits cotensors by \( S^{-1} \), let \( N \) be a good model category and assume that the hom-spaces of the underlying \( \text{Top}_* \)-category of \( \mathcal{E} \) are nondegenerately based. Then for any strictly linear and projectively cofibrant \( \text{Top}_* \)-functor \( F : \mathcal{E} \rightarrow N \), the unit \( F \rightarrow \text{ULF} \) is a pointwise weak equivalence.

Proof. Let \( \overset{\cdot}{\mathcal{C}} \simeq \mathcal{C} \) be a cofibrant replacement in the sense of Lemma 2.21. Restriction and left Kan extension gives us a diagram of Quillen pairs

\[
\begin{array}{ccc}
\text{Fun}(\overset{\cdot}{\mathcal{C}},N) & \rightleftarrows & \text{Fun}(\mathcal{C},N) \\
\downarrow & & \downarrow \\
\text{Fun}(\mathcal{C},N) & \rightleftarrows & \text{Fun}(\overset{\cdot}{\mathcal{C}},N)
\end{array}
\]

The horizontal adjunctions are Quillen equivalences by the goodness of \( N \). In particular, by Lemma 5.13 the proposition follows if we can show that for any projectively cofibrant \( G : \overset{\cdot}{\mathcal{C}} \rightarrow N \) such that the map \((11)\) is a level equivalence, the unit
\[ G \to U'L'G = \int^\mathcal{E}_\ast \hat{\mathcal{C}}(c, -) \otimes G(c) \] is a pointwise equivalence. By Lemmas 5.10 and 5.11, this map is obtained by applying \(|-|_S\) to
\[ \{\Sigma^n G(d)\}_{n \in \mathbb{N}} \to \left\{ \int^\mathcal{E}_\ast \hat{\mathcal{C}}(c, d)_n \otimes G(c) \right\}_{n \in \mathbb{N}} \]
This map is a level equivalence by assumption. Since \(G\) is projectively cofibrant and \(\mathcal{E}\) has cofibrant hom-objects, we see that \(G(d)\) and hence \(\Sigma^n G(d)\) are cofibrant. Moreover, \(\{\int^\mathcal{E}_\ast \hat{\mathcal{C}}(c, d)_n \otimes G(c)\}_{n \in \mathbb{N}}\) is cofibrant by Lemma 5.12, so we conclude that the unit \(G \to U'L'G\) is a pointwise equivalence.

**Proof of Theorem 5.5.** This follows from Proposition 5.8 and Proposition 5.15.

The linear model structure \(\text{Fun}^{\text{lin}}(\mathcal{E}_s, \mathcal{N})\) is defined, if it exists, as the left Bousfield localization of the projective model structure on \(\text{Fun}(\mathcal{E}_s, \mathcal{N})\) in which the fibrant objects are the strictly linear pointwise fibrant functors. As in the previous cases, we can upgrade Theorem 5.5 to a Quillen equivalence.

**Theorem 5.16.** Let \(\mathcal{C}\) be a small spectral category that admits cotensors by \(S^{-1}\), let \(\mathcal{N}\) be a good spectral model category and assume that the hom-spaces of the underlying \(\text{Top}_s\)-category of \(\mathcal{C}\) are nondegenerately based. Then the forgetful functor
\[ U: \text{Fun}(\mathcal{E}_s, \mathcal{N}) \to \text{Fun}^{\text{lin}}(\mathcal{E}_s, \mathcal{N}) \]
from the projective model structure to the linear model structure is a right Quillen equivalence whenever the linear model structure exists.

**Proof.** Write \(L\) for the left adjoint of the forgetful functor \(U\). It follows from the proof of Theorem 5.5 that the unit \(F \to ULF\) is a weak equivalence for any projectively cofibrant strictly linear functor \(F: \mathcal{E}_s \to \mathcal{N}\). By Lemma 2.22, \(L \Rightarrow U\) is a Quillen equivalence.

### 5.2 (Symmetric) multilinear functors

We will now extend the results of Section 5.1 to (symmetric) multifunctors. Recall the definition of a (symmetric) multifunctor from Definitions 4.12 and 4.16.

Let \(\mathcal{C}\) denote a spectral category that admits tensors by \(S^{-1}\) and \(\mathcal{N}\) a spectral model category. Given a multifunctor \(F: \mathcal{C}^\wedge m \to \mathcal{N}\), one can define maps
\[ \sigma_m^i: \Sigma^n F(c_1, \ldots, c_i, \ldots, c_n) \to F(c_1, \ldots, c_{i-1}^\Sigma^{-m}, \ldots, c_n) \]
in exactly the same way as in Construction 5.1.

**Definition 5.17.** Let \(\mathcal{C}\) be a spectral category that admits cotensors by \(S^{-1}\), let \(\mathcal{N}\) be a spectral model category and let \(F: \mathcal{C}^\wedge n \to \mathcal{N}\) be a (possibly symmetric) multifunctor. Then \(F\) is called strictly multilinear if for every \(n\)-tuple \(c_1, \ldots, c_n\) in \(\mathcal{C}\) and every \(1 \leq i \leq n\), the composition
\[ (\Sigma) F(c_1, \ldots, c_i, \ldots, c_n) \to \Sigma F(c_1, \ldots, c_i, \ldots, c_n) \xrightarrow{\sigma_i^1} F(c_1, \ldots, c_i^{\Sigma^{-1}}, \ldots, c_n) \]
is a weak equivalence in \(\mathcal{N}\).
We wish to show that strictly multilinear functors $\hat{C}^{\wedge n} \to N$ are equivalent to spectral functors $\mathcal{C}^{\wedge n} \to N$. However, care has to be taken, since the tensor product of two hom-objects of $\mathcal{C}$ might not be the derived tensor product.\footnote{To the author’s knowledge, the question of whether the tensor product of orthogonal spectra is fully homotopical or not is open (cf. [SS21, Question 1.5]). If it were fully homotopical, then this section could be significantly simplified.} For that reason, we need to work with a cofibrant replacement $\hat{\mathcal{C}}$ of $\mathcal{C}$ and consider spectral functors $\hat{\mathcal{C}}^{\wedge n} \to N$ instead.

Throughout the rest of this section, fix a good spectral model category $N$ and a small spectral category $\mathcal{C}$ that admits cotensors by $S^{-1}$ and whose underlying $\text{Top}_*$-category $\mathcal{C}$ has nondegenerately based hom-spaces. Moreover, fix a cofibrant replacement $\hat{\mathcal{C}} \simeq \mathcal{C}$ in the sense of Lemma 2.2.1.

The characterization given in Lemma 5.13 extends to the case of multifunctors, the proof of which we leave to the reader. Let $p$ denote the equivalence $\hat{C}^{\wedge n} \simeq C^{\wedge n}$.

**Lemma 5.18.** A multifunctor $F: \mathcal{C}^{\wedge n} \to N$ is strictly multilinear if and only if there exists a projectively cofibrant $G: \hat{C}^{\wedge n} \to N$ such that $\text{Lan}_p G \simeq F$ and for every $c_1, \ldots, c_n$ in $\hat{\mathcal{C}}$ and every $1 \leq i \leq n$, the map

$$\{\Sigma^m G(c_1, \ldots, c_n)\}_{m \in \mathbb{N}} \to \left\{ \int^{\hat{\mathcal{C}}} \hat{\mathcal{C}}(d, c_i)_m \otimes G(c_1, \ldots, c_{i-1}, d, c_{i+1}, \ldots, c_n) \right\}_{m \in \mathbb{N}}$$

(12)

is a level equivalence in $\text{Sp}(N)$.

In light of this lemma, we will call a functor $G: \hat{C}^{\wedge n} \to N$ strictly multilinear if for every projectively cofibrant replacement of $G$, the condition of Lemma 5.18 holds. Equivalently, $G$ is strictly multilinear if and only if the functor $\mathcal{C}^{\wedge n} \to N$ corresponding to $G$ under the Quillen equivalence $\text{Fun}(\hat{C}^{\wedge n}, N) \rightleftarrows \text{Fun}(\mathcal{C}^{\wedge n}, N)$ is strictly multilinear.

**Remark 5.19.** One can verify that $G: \hat{C}^{\wedge n} \to N$ is strictly multilinear if and only if for every $1 \leq i \leq n$ and for every $d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n$ in $\hat{\mathcal{C}}$, the functor

$$G': \hat{\mathcal{C}} \to N; \quad c \mapsto \hat{G}(d_1, \ldots, d_{i-1}, c, d_{i+1}, \ldots, d_n)$$

is strictly linear in the sense of Remark 5.14.

Any spectral multifunctor $\hat{\mathcal{C}}^{\wedge n} \to N$ has an underlying $\text{Top}_*$-multifunctor $\hat{C}^{\wedge n} \to N$ constructed by precomposing with $\Sigma^\infty(\hat{\mathcal{C}}^{\wedge n}) \cong (\Sigma^\infty \hat{\mathcal{C}})^{\wedge n} \to \hat{\mathcal{C}}^{\wedge n}$. The main result of this section in the non-symmetric case is the following.

**Theorem 5.20.** Let $N$ and $\hat{\mathcal{C}} \simeq \mathcal{C}$ be as above. Then a multifunctor $\hat{\mathcal{C}}^{\wedge n} \to N$ is equivalent to a spectral multifunctor if and only if it is multilinear.

**Proof.** Let a spectral multifunctor $G: \hat{\mathcal{C}}^{\wedge n} \to N$ be given. Let $1 \leq i \leq n$ and objects $d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n$ in $\hat{\mathcal{C}}$ be given and define the functor $G': \hat{\mathcal{C}} \to N; c \mapsto G(d_1, \ldots, d_{i-1}, c, d_{i+1}, \ldots, d_n)$. By Remarks 5.14 and 5.19, it suffices to show that the functor $\hat{\mathcal{C}} \to N$ corresponding to the underlying functor $\hat{\mathcal{C}} \to N$ of $G'$ under the Quillen equivalence $\text{Fun}(\hat{\mathcal{C}}, N) \rightleftarrows \text{Fun}(\hat{\mathcal{C}}, N)$ is strictly linear. This follows from Proposition 5.8.

For the converse, let $G: \mathcal{C}^{\wedge n}$ be a multilinear functor and suppose that it is projectively cofibrant. The unit of the left Kan extension of $G$ along $\Sigma^\infty(\hat{\mathcal{C}}^{\wedge n}) \to \hat{\mathcal{C}}^{\wedge n}$ is given by

$$G(c_1, \ldots, c_n) \to \int^{\hat{\mathcal{C}}^{\wedge n}} \hat{\mathcal{C}}(d_1, c_1) \otimes \ldots \otimes \hat{\mathcal{C}}(d_n, c_n) \otimes G(d_1, \ldots, d_n)$$

(13)
Using Lemma 2.8, one can rewrite the right-hand side as an $n$-fold coend over $\hat{\mathcal{C}}$. By iterating the proof of Proposition 5.15 $n$ times, it then follows that this map is a weak equivalence. ■

**Remark 5.21.** One can also prove that the unit (13) is weak equivalence using $n$-fold spectrum objects in $\mathcal{N}$; that is, functors $\mathcal{C}^\Lambda_n \rightarrow \mathcal{N}$. The result then follows by mimicking the proof of Proposition 5.15 but using $n$-fold spectrum objects instead of (1-fold) spectrum objects everywhere.

The multilinear model structures $\text{Fun}^\text{mlin}(\mathcal{C}^\Lambda_n, \mathcal{N})$ and $\text{Fun}^\text{mlin}(\hat{\mathcal{C}}^\Lambda_n, \mathcal{N})$ are defined, if they exist, as the left Bousfield localizations of the projective model structures in which the fibrant objects are the strictly multilinear pointwise fibrant functors.

**Theorem 5.22.** Let $\mathcal{N}$ and $\hat{\mathcal{C}} \xrightarrow{\sim} \mathcal{C}$ be as above. If the multilinear model structures $\text{Fun}^\text{mlin}(\mathcal{C}^\Lambda_n, \mathcal{N})$ and $\text{Fun}^\text{mlin}(\hat{\mathcal{C}}^\Lambda_n, \mathcal{N})$ exist, then there is a zigzag of Quillen equivalences

$$\text{Fun}^\text{mlin}(\mathcal{C}^\Lambda_n, \mathcal{N}) \rightleftharpoons \text{Fun}^\text{mlin}(\hat{\mathcal{C}}^\Lambda_n, \mathcal{N}) \rightleftharpoons \text{Fun}(\hat{\mathcal{C}}^\otimes_n, \mathcal{N}),$$

where the left adjoints are drawn on top.

**Proof.** To see that

$$\text{Fun}^\text{mlin}(\hat{\mathcal{C}}^\Lambda_n, \mathcal{N}) \rightleftharpoons \text{Fun}^\text{mlin}(\mathcal{C}^\Lambda_n, \mathcal{N})$$

is a Quillen equivalence, note that this is already a Quillen equivalence with respect to the projective model structures by the goodness of $\mathcal{N}$. Since a functor in $\text{Fun}(\mathcal{C}^\Lambda_n, \mathcal{N})$ is strictly multilinear if and only if the corresponding functor in $\text{Fun}^\text{mlin}(\hat{\mathcal{C}}^\Lambda_n, \mathcal{N})$ is, the result follows.

The other Quillen equivalence in the zigzag follows by applying Lemma 2.22 to the Quillen pair arising from restriction and left Kan extension along $\Sigma^\infty(\mathcal{C}^\Lambda_n) \rightarrow \hat{\mathcal{C}}^\otimes_n$. ■

By exactly the same procedure as in the proof of Theorem 4.18, one can deduce the following symmetric versions of Theorems 5.20 and 5.22.

**Theorem 5.23.** Let $\mathcal{N}$ and $\hat{\mathcal{C}} \rightarrow \mathcal{C}$ be as above. Then a symmetric multifunctor $\hat{\mathcal{C}}^\Lambda_n \rightarrow \mathcal{N}$ is equivalent (as symmetric multifunctors) to a symmetric spectral multifunctor if and only if it is multilinear.

The symmetric multilinear model structures $\text{Fun}^\text{symlin}(\mathcal{C}^\Lambda_n, \mathcal{N})$ and $\text{Fun}^\text{symlin}(\hat{\mathcal{C}}^\Lambda_n, \mathcal{N})$ are defined analogously to the multilinear model structures.

**Theorem 5.24.** Let $\mathcal{N}$ and $\hat{\mathcal{C}} \xrightarrow{\sim} \mathcal{C}$ be as above. If the symmetric multilinear model structures $\text{Fun}^\text{symlin}(\mathcal{C}^\Lambda_n, \mathcal{N})$ and $\text{Fun}^\text{symlin}(\hat{\mathcal{C}}^\Lambda_n, \mathcal{N})$ exist, then there is a zigzag of Quillen equivalences

$$\text{Fun}^\text{symlin}(\mathcal{C}^\Lambda_n, \mathcal{N}) \rightleftharpoons \text{Fun}^\text{symlin}(\hat{\mathcal{C}}^\Lambda_n, \mathcal{N}) \rightleftharpoons \text{Fun}^\text{sym}(\hat{\mathcal{C}}^\otimes_n, \mathcal{N}),$$

where the left adjoints are drawn on top.
| Reference | Details |
|-----------|---------|
| [BJM17]   | I. Barnea, M. Joachim, and S. Mahanta. “Model structure on projective systems of C∗-algebras and bivariant homology theories”. *New York J. Math.* 23 (2017), pp. 383–439. url: [http://nyjm.albany.edu:8000/j/2017/23_383.html](http://nyjm.albany.edu:8000/j/2017/23_383.html). |
| [BR14]    | G. Biedermann and O. Röndigs. “Calculus of functors and model categories, II”. *Algebr. Geom. Topol.* 14.5 (2014), pp. 2853–2913. doi: [10.2140/agt.2014.14.2853](https://doi.org/10.2140/agt.2014.14.2853). |
| [Dug01]   | D. Dugger. “Replacing model categories with simplicial ones”. *Trans. Amer. Math. Soc.* 353.12 (2001), pp. 5003–5027. doi: [10.1090/S0002-9947-01-02661-7](https://doi.org/10.1090/S0002-9947-01-02661-7). |
| [DK80a]   | W. G. Dwyer and D. M. Kan. “Calculating simplicial localizations”. *J. Pure Appl. Algebra* 18.1 (1980), pp. 17–35. doi: [10.1016/0022-4049(80)90113-9](https://doi.org/10.1016/0022-4049(80)90113-9). |
| [DK87]    | W. G. Dwyer and D. M. Kan. “Equivalences between homotopy theories of diagrams”. In: *Algebraic topology and algebraic K-theory (Princeton, N.J., 1983)*. Vol. 113. Ann. of Math. Stud. Princeton Univ. Press, Princeton, NJ, 1987, pp. 180–205. |
| [DK83]    | W. G. Dwyer and D. M. Kan. “Function complexes for diagrams of simplicial sets”. *Nederl. Akad. Wetensch. Indag. Math.* 45.2 (1983), pp. 139–147. |
| [DK80b]   | W. G. Dwyer and D. M. Kan. “Simplicial localizations of categories”. *J. Pure Appl. Algebra* 17.3 (1980), pp. 267–284. doi: [10.1016/0022-4049(80)90049-3](https://doi.org/10.1016/0022-4049(80)90049-3). |
| [Goo03]   | T. G. Goodwillie. “Calculus. III. Taylor series”. *Geom. Topol.* 7 (2003), pp. 645–711. doi: [10.2140/gt.2003.7.645](https://doi.org/10.2140/gt.2003.7.645). |
| [Hiro3]   | P. S. Hirschhorn. *Model categories and their localizations*. Vol. 99. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003, pp. xvi+457. ISBN: 0-8218-3279-4. doi: [10.1090/surv/099](https://doi.org/10.1090/surv/099). |
| [Hov99]   | M. Hovey. *Model categories*. Vol. 63. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1999, pp. xii+209. ISBN: 0-8218-1359-5. |
| [Kel82]   | G. M. Kelly. *Basic concepts of enriched category theory*. Vol. 64. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge-New York, 1982, p. 245. ISBN: 0-521-28702-2. Reprint available at [http://www.tac.mta.ca/tac/reprints/articles/10/tr10abs.html](http://www.tac.mta.ca/tac/reprints/articles/10/tr10abs.html). |
| [Lam68]   | K. Lamotke. *Semisimpliziale algebraische Topologie*. Die Grundlehren der mathematischen Wissenschaften, Band 147. Springer-Verlag, Berlin-New York, 1968, pp. viii+285. ISBN: 3-540-04140-0. |
| [Law20]   | T. Lawson. *An introduction to Bousfield localization*. Feb. 2020. arXiv: [2002.03888](https://arxiv.org/abs/2002.03888). Version 1. |
| [Lyd98]   | M. Lydakis. *Simplicial functors and stable homotopy theory*. May 6, 1998. Unpublished preprint, available at [http://users.math.uoc.gr/~mlydakis/papers/sf.pdf](http://users.math.uoc.gr/~mlydakis/papers/sf.pdf). |
| [Man+01]  | M. A. Mandell et al. “Model categories of diagram spectra”. *Proc. London Math. Soc. (3)* 82.2 (2001), pp. 441–512. doi: [10.1112/S0024611501012692](https://doi.org/10.1112/S0024611501012692). |
[May99] J. P. May. *A concise course in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999, pp. x+243. ISBN: 0-226-51183-9.

[nLa22] nLab authors. *Bousfield localization of model categories*. [http://ncatlab.org/nlab/show/Bousfieldlocalizationofmodelcategories]. Aug. 2022. Revision 59.

[Per13] L. A. Pereira. “Goodwillie calculus and algebras over a spectral operad”. PhD thesis. Massachusetts Institute of Technology, 2013. URL: [http://hdl.handle.net/1721.1/82440](http://hdl.handle.net/1721.1/82440).

[Qui67] D. G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967, pp. iv+156. ISBN: 978-3-540-03914-3. doi: 10.1007/BFb0097438.

[RSS01] C. Rezk, S. Schwede, and B. Shipley. “Simplicial structures on model categories and functors”. *Amer. J. Math.* 123.3 (2001), pp. 551–575. doi: 10.1353/ajm.2001.0019.

[Rie14] E. Riehl. *Categorical homotopy theory*. Vol. 24. New Mathematical Monographs. Cambridge University Press, Cambridge, 2014, pp. xviii+352. ISBN: 978-1-107-04845-4. doi: 10.1017/CBO9781107261457.

[SS21] S. Sagave and S. Schwede. “Homotopy invariance of convolution products”. *Int. Math. Res. Not. IMRN* 8 (2021), pp. 6246–6292. doi: 10.1093/imrn/rnz334.

[Sch18] S. Schwede. *Global homotopy theory*. Vol. 34. New Mathematical Monographs. Cambridge University Press, Cambridge, 2018, pp. xviii+828. ISBN: 978-1-108-42581-0. doi: 10.1017/9781108349161.

[SS03] S. Schwede and B. Shipley. “Equivalences of monoidal model categories”. *Algebr. Geom. Topol.* 3 (2003), pp. 287–334. doi: 10.2140/agt.2003.3.287.

[Shu06] M. Shulman. *Homotopy limits and colimits and enriched homotopy theory*. Oct. 2006. arXiv: math/0610194. Version 3.

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