Abstract. This paper concerns with numerical approximations of solutions of second order fully nonlinear partial differential equations (PDEs). A new notion of weak solutions, called moment solutions, is introduced for second order fully nonlinear PDEs. Unlike viscosity solutions, moment solutions are defined by a constructive method, called vanishing moment method, hence, they can be readily computed by existing numerical methods such as finite difference, finite element, spectral Galerkin, and discontinuous Galerkin methods with “guaranteed” convergence. The main idea of the proposed vanishing moment method is to approximate a second order fully nonlinear PDE by a higher order, in particular, a fourth order quasilinear PDE. We show by various numerical experiments the viability of the proposed vanishing moment method. All our numerical experiments show the convergence of the vanishing moment method, and they also show that moment solutions coincide with viscosity solutions whenever the latter exist.

Key words. Fully nonlinear PDEs, Monge-Ampère type equations, moment solutions, vanishing moment method, viscosity solutions, finite element method, mixed finite element method, spectral and discontinuous Galerkin methods.

AMS subject classifications. 65N30, 65M60, 35J60, 35K55, 53C45

1. Introduction. Fully nonlinear PDEs are those PDEs which depend nonlinearly on the highest order derivatives of unknown functions. Fully nonlinear PDEs arise from many areas in science and engineering such as kinetic theory, materials science, differential geometry, general relativity, optimal control, mass transportation, image processing and computer vision, meteorology, semigeostrophic fluid dynamics. They constitute the most difficult class of differential equations to analyze analytically and to approximate numerically, see [13, 46, 45, 15, 50, 59] and references therein.

The general first order fully nonlinear PDE has the form

\[ F(\nabla u(x), u(x), x) = 0 \quad x \in \Omega \subset \mathbb{R}^n. \] (1.1)

The best known examples include Eikonal equation

\[ |\nabla u(x)| = f(x) \quad x \in \Omega, \]

and the general Hamilton-Jacobi equation [46, 25]

\[ H(\nabla u(x)) = 0 \quad x \in \Omega. \]

The general second order fully nonlinear PDE, which will be the focus of this paper, takes the form

\[ F(D^2 u(x), \nabla u(x), u(x), x) = 0 \quad x \in \Omega. \] (1.2)
where and throughout this paper $D^2u(x)$ denotes the Hessian matrix of $u$ at $x$. The best known examples are the Monge-Ampère type equations \[46, 51, 57\]
\[
\det(D^2u(x)) = f(\nabla u(x), u(x), x) \quad x \in \Omega,
\]
and the Bellman equations \[46, 45\]
\[
\sup_{\theta \in \Theta} L_\theta(D^2u, \nabla u, u, x) = 0,
\] (1.3)
where $\det(D^2u(x))$ stands for the determinant of $D^2u(x)$ and $L_\theta$ is a given family of second order linear differential operators.

For the first order fully nonlinear PDEs, tremendous progresses have been made in the past three decades. A revolutionary viscosity solution theory has been established (cf. \[26, 24, 25, 45\]) and wealthy amount of efficient and robust numerical methods and algorithms have been developed and implemented (cf. \[10, 20, 27, 58, 67, 71, 72\]). However, for second order fully nonlinear PDEs, the situation is strikingly different. On one hand, there have been enormous advances in theoretical analysis in the past two decades after the introduction of the notion of viscosity solutions by M. Crandall and P. L. Lions in 1983 (cf. \[11, 13, 12, 25, 46, 51\]). On the other hand, in contrast to the success of the PDE analysis, numerical solutions for general second order fully nonlinear PDEs (except in the case of Bellman type PDEs, see below for details) is mostly an untouched area, and computing viscosity solutions of second order fully nonlinear PDEs has been impracticable. There are several reasons for this lack of progress. Firstly, the strong nonlinearity is an obvious one. Secondly, the conditional uniqueness (i.e., uniqueness holds only in certain class of functions) of solutions is difficult to handle numerically. Lastly and most importantly, the notion of viscosity solutions, which is not variational, has no equivalence at the discrete level.

To see the above points, let us consider the following model Dirichlet problem for the Monge-Ampère equation:

\[
\begin{align*}
\det(D^2u) &= f &\text{in } \Omega, \\
u &= g &\text{on } \partial \Omega.
\end{align*}
\] (1.4-1.5)

It is well-known that for non-strictly convex domain $\Omega$ the above problem does not have classical solutions in general even $f$, $g$ and $\partial \Omega$ are smooth (see \[46\]). Classical result of A. D. Aleksandrov states that the Dirichlet problem with $f > 0$ has a unique generalized solution in the class of convex functions (cf. \[11, 17\]). Major progress on analysis of problem \[1.4-1.5\] has been made later by using the viscosity solution concept and machinery (cf. \[13, 25, 51\]). We recall that a convex function $u \in C^0(\overline{\Omega})$ satisfying $u = g$ on $\partial \Omega$ is called a viscosity subsolution (resp. viscosity supersolution) of \[1.4\] if for any $\varphi \in C^2$ there holds $\det(D^2\varphi(x_0)) \leq f(x_0)$ (resp. $\det(D^2\varphi(x_0)) \geq f(x_0)$) provided that $u - \varphi$ has a local maximum (resp. a local minimum) at $x_0 \in \Omega$. $u \in C^0(\overline{\Omega})$ is called a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

First, the reason to restrict the admissible set to be the set of convex functions is that the Monge-Ampère equation is elliptic only in that set \[51\]. It should be noted that in general the Dirichlet problem \[1.4-1.5\] may have other (nonconvex) solutions besides the unique convex solution, multiple solutions are often expected for the Monge-Ampère type PDEs and for second order fully nonlinear PDEs. It is easy to see that if one discretizes \[1.4\] straightforwardly using the finite difference method,
one immediately loses control on which solution the numerical scheme approximates even assuming that the nonlinear discrete problem has solutions. Second, the situation is even worse if one tries to formulate a Galerkin type method (such as the finite element method and the spectral Galerkin method), because there is no variational or weak formulation to start with. In fact, this is clear from the definition of viscosity solutions. It is not defined by the traditional integration by parts approach, instead, it is defined by a “differentiation by parts” (a terminology coined by L. C. Evans [33, 34]) approach. Although the “differentiation by parts” approach has worked remarkably well for establishing the viscosity solution theory for second order fully nonlinear PDEs in the past two decades, it is extremely difficult (if all possible) to mimic it at the discrete level. Third, regardless which method is used, one can easily envisage that the anticipated algebraic problem from the discretization of a fully nonlinear PDE such as the Monge-Ampère equation must be very difficult to solve due to the nonuniqueness of solutions and very strong nonlinearity.

Nevertheless, a few recent numerical attempts and results have been known in the literature. In [66] Oliker and Prussner proposed a finite difference scheme for computing Aleksandrov measure induced by $D^2u$ (and obtained the solution $u$ of (1.4) as a by-product) in 2-d. The scheme is extremely geometric and difficult to use and to generalize to other second order fully nonlinear PDEs. In [7] Barles and Souganidis showed that any monotone, stable and consistent finite difference scheme converges to the correct solution provided that there exists a comparison principle for the limiting equation. Their result provides a guideline for constructing convergent finite difference methods although it did not address how to construct such a scheme. Very recently, Oberman [65] was able to construct some wide stencil finite difference schemes which fulfill the criterions listed in [7] for the Monge-Ampère type equations. In [5] Baginski and Whitaker proposed a finite difference scheme for Gauss curvature equation (see §4 in 2-d by mimicking the unique continuation method (used to prove existence of the PDE) at the discrete level. Finally, in a series of papers [28, 29, 30, 31] Dean and Glowinski proposed an augmented Lagrange multiplier method and a least squares method for problem (1.4)–(1.5) and the Pucci’s equation (cf. [13, 46]) in 2-d by treating the Monge-Ampère equation and Pucci’s equation as a constraint and using a variational criterion to select a particular solution. Numerical experiments results were reported in [66, 65, 5] 28, 29, 30, 31, however, convergence analysis was not addressed except in [65].

In addition, we like to remark that there is a considerable amount of literature available on using finite difference methods to approximate viscosity solutions of second order fully nonlinear Bellman type PDE (1.3) arisen from stochastic optimal control. See [7, 8, 34, 50]. Due to the special nonlinearity of the Bellman type PDEs, the approach used and the methods proposed in those papers unfortunately could not be extended to other types of second order fully nonlinear PDEs since the construction of those methods critically relies on the linearity of the operators $L_\theta$.

The first goal of this paper is to introduce a new weak solution concept and a method to construct such a solution for second order fully nonlinear PDEs, in particular, for the Monge-Ampère type equations. These new weak solutions are called moment solutions and the method to construct such a moment solution is called the vanishing moment method. The crux of this new method is that we approximate a second order fully nonlinear PDE by a sequence of higher order (in particular, fourth order) quasilinear PDEs. The limit of the solution sequence of the higher order PDEs, if exists, is defined as a moment solution of the original second order
fully nonlinear PDE. Hence, moment solutions are constructive by nature. The second
goal of this paper is to present a number of numerical methods for computing moment
solutions of second order fully nonlinear PDEs, and to present extensive numerical
experiment results to demonstrate the convergence and effectiveness of the proposed
vanishing moment methodology. Indeed, one of advantages of the vanishing moment
method is that it allows one to use wealthy amount of existing numerical methods
and algorithms as well as computer codes for fourth order linear and quasilinear
PDEs to solve second order fully nonlinear PDEs. The third and the last goal of
this paper is to show using numerical studies that the notion of moment solutions
generalizes the notion of viscosity solutions in the sense that the former coincides
with the later whenever the later exists. These numerical studies indeed motivate
us to give a rigorous convergence analysis of the vanishing moment method for the
Monge-Ampère equation in two spatial dimensions [36].

The remainder of the paper is organized as follows. In §2 we introduce the
abstract framework of moment solutions and the vanishing moment method for gen-
eral second order fully nonlinear PDEs. In §3, we propose two classes of numerical
discretization methods and briefly discuss solution algorithms. In §4 we apply the ab-
stract framework to several classes of second order fully nonlinear PDEs which include
the Monge-Ampère type equations, Pucci’s extremal equations, the infinite Laplace
equation, and second order parabolic fully nonlinear PDEs. In §5, we present many
2-d and 3-d numerical experiment results to demonstrate the convergence and effec-
tiveness of the vanishing moment methodology, and provide numerical evidences of
the agreement of moment solutions and viscosity solutions whenever the latter exists.
The paper is concluded by a summary and some conclusions in §6.

2. Vanishing moment method and the notion of moment solutions.

2.1. Preliminaries. Standard space notation will be adopted throughout this
paper, we refer to [46, 57] for their exact definitions. Ω denotes a generic bounded
domain in \( \mathbb{R}^n \). \((\cdot, \cdot)\) and \(\langle \cdot, \cdot \rangle\) are used to denote the \(L^2\)-inner products on \(\Omega\) and on
\(\partial \Omega\), respectively. We assume \(n \geq 2\), except in §3 and §5 where we restrict \(n = 2, 3\)
when we develop numerical methods and perform numerical experiments.

Since the notion of viscosity solutions has been and will continue to be referred
many times, and it is closely related to the notion of moment solutions to be described
later in this paper, for readers’ convenience, we briefly recall its definition and history
here and refer to [13, 25, 26, 34] for detailed discussions.

**Definition 2.1.** Suppose \(F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}\) is continuous (nonlinear)
function.

(i) A function \(u \in C^0(\Omega)\) is called a viscosity subsolution of (1.1) if, for every
\(C^1\) function \(\varphi = \varphi(x)\) such that \(u - \varphi\) has a local maximum at \(x^0 \in \Omega\), there
holds

\[ F(\nabla \varphi(x^0), \varphi(x^0), x^0) \leq 0. \]

(ii) A function \(u \in C^0(\Omega)\) is called a viscosity supersolution of (1.1) if, for every
\(C^1\) function \(\varphi = \varphi(x)\) such that \(u - \varphi\) has a local minimum at \(x^0 \in \Omega\), there
holds

\[ F(\nabla \varphi(x^0), \varphi(x^0), x^0) \geq 0. \]

(iii) A function \(u \in C^0(\Omega)\) is called a viscosity solution of (1.1) if it is both a
viscosity subsolution and a viscosity supersolution.
It should be pointed out that the above definition is a modern definition of viscosity solutions for \( (1.1) \). It can be regarded as a “differentiation by parts” definition (cf. [34]). However, viscosity solutions were first introduced differently by a vanishing viscosity procedure (cf. [26]), that is, equation \( (1.1) \) is approximated by the second order quasilinear PDEs
\[
-\epsilon \Delta u^\epsilon + F(\nabla u^\epsilon, u^\epsilon, x) = 0,
\]
and \( \lim_{\epsilon \to 0^+} u^\epsilon \), if exists, is called a viscosity solution of \( (1.1) \). It was later proved that the two definitions are equivalent for equation \( (1.1) \) (cf. [24]).

Another important reason to favor the modern “differentiation by parts” definition is that the definition and the notion of viscosity solutions can be readily extended to second order fully nonlinear PDEs.

**Definition 2.2.** Suppose \( F: \mathbb{R}^{n \times n} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) is continuous (nonlinear) function.

(i) A function \( u \in C^0(\Omega) \) is called a viscosity subsolution of \( (1.2) \) if, for every \( C^2 \) function \( \varphi = \varphi(x) \) such that \( u - \varphi \) has a local maximum at \( x^0 \in \Omega \), there holds
\[
F(D^2 \varphi(x^0), \nabla \varphi(x^0), \varphi(x^0), x^0) \leq 0.
\]

(ii) A function \( u \in C^0(\Omega) \) is called a viscosity supersolution of \( (1.2) \) if, for every \( C^2 \) function \( \varphi = \varphi(x) \) such that \( u - \varphi \) has a local minimum at \( x^0 \in \Omega \), there holds
\[
F(D^2 \varphi(x^0), \nabla \varphi(x^0), \varphi(x^0), x^0) \geq 0.
\]

(iii) A function \( u \in C^0(\Omega) \) is called a viscosity solution of \( (1.2) \) if it is both a viscosity subsolution and a viscosity supersolution.

As it is known now, a successful theory of viscosity solutions has been established for second order fully nonlinear PDEs in the past two decades (cf. [13, 25, 51]). On the other hand, it should be noted that the phrase “viscosity solution” loses its original meaning in this theory since it has nothing to do with the vanishing viscosity method in the case of second order fully nonlinear PDEs. We recall that to establish the existence of viscosity solutions the technique used to substitute for the vanishing viscosity method in the theory is the classical Perron’s method (cf. [25, 13]). To the best of our knowledge, viscosity solutions of second order fully nonlinear PDEs were never defined and/or constructed by a limiting process like one described above for the Hamilton-Jacobi equation.

**2.2. General framework of the vanishing moment method.** For the reasons and difficulties explained in §1 as far as we can see, it is unlikely (at least very difficult if all possible) that one can directly approximate viscosity solutions of general second order fully nonlinear PDEs such as Monge-Ampère type equations using any available numerical methodology (finite difference method, finite element method, spectral method, meshless method etc.). From computational point of view, the notion of viscosity solutions is a “bad” notion for second order fully nonlinear PDEs because it is not constructive nor variational, so one has no handle on how to compute such a solution.

In searching for a “better” notion of weak solutions for second order fully nonlinear PDEs, we are inspired by the following simple but crucial observation: the essence
of the vanishing viscosity method for the Hamilton-Jacobi equation and the original notion of viscosity solutions is to approximate a lower order fully nonlinear PDE by a sequence of higher order quasilinear PDEs. This observation then suggests us to apply the above principle to second order fully nonlinear PDE (1.2), this is exactly what we are going to do in this paper. That is, we approximate equation (1.2) by the following higher order quasilinear PDEs:

$$G_\varepsilon(D^r u^\varepsilon) + F(D^2 u^\varepsilon, \nabla u^\varepsilon, x) = 0 \quad (r \geq 3, \varepsilon > 0),$$  \hfill (2.1)

where \( \{G_\varepsilon\} \) is a family of suitably chosen linear or quasilinear differential operators of order \( r \). The above approximation then naturally leads to the next definition.

**Definition 2.3.** Suppose that \( u^\varepsilon \) solves (2.1) for each \( \varepsilon > 0 \), we call \( \lim_{\varepsilon \to 0^+} u^\varepsilon \) a moment solution of (1.2) provided that the limit exists. We also call this limiting process the vanishing moment method.

Clearly, the above definition is a loose definition since the operator \( G_\varepsilon \) is not specified, nor is the meaning of the limit, but they will become clear later in this section. We note that the reason to use the terminology “moment solution” will also be explained later in this section, and the notion of moment solutions and the vanishing moment method are clearly in the spirit of the (original) notion of viscosity solution and the vanishing viscosity method [26].

To establish a complete theory of moment solutions and vanishing moment method for second order fully nonlinear PDEs, there are many issues we must address. For instance,

- How to choose the operator \( G_\varepsilon \)?
- What additional boundary condition(s) should \( u^\varepsilon \) satisfy?
- Does the limit \( \lim_{\varepsilon \to 0^+} u^\varepsilon \) always exist? If it does, what is the rate of convergence?
- How do moment solutions relate to viscosity solutions?
- How to solve (2.1) numerically?
- Error estimates, nonlinear solvers, computer implementations.

As expected, we do not have answers for all the questions now, nor do we intend to address all of them in this paper. Instead, the focus of this paper is to develop the framework for moment solutions and the vanishing viscosity method, and to present numerical evidences to show effectiveness of the method and to justify the proposed approach. On the other hand, we do plan to address all theoretical issues in forthcoming papers [36, 63].

Regarding to the first issue, although the choices for \( G_\varepsilon \) are abundant and flexible, the following are some guidelines for choosing a good operator \( G_\varepsilon \).

- \( G_\varepsilon \) must be a linear or quasilinear operator.
- \( G_\varepsilon \to 0 \) in some reasonable sense as \( \varepsilon \to 0^+ \).
- \( G_\varepsilon(D^r u) \) is better to be elliptic, in particular, when PDE (1.2) is elliptic.
- Equation (2.1) should be relatively easy to solve numerically.

Since an elliptic operator is necessarily of even order, so guideline (c) above implies that \( r \) must be an even number in (2.1). Hence, the lowest order of equation (2.1) is \( r = 4 \). When talking about fourth order elliptic operators, the biharmonic operator stands out immediately. So we let

$$G_\varepsilon(D^4 v) := -\varepsilon \Delta^2 v,$$

then equation (2.1) becomes

$$-\varepsilon \Delta^2 u^\varepsilon + F(D^2 u^\varepsilon, \nabla u^\varepsilon, x) = 0.$$  \hfill (2.2)
After the differential operator $G_{\varepsilon}$ is chosen, next we need to take care the boundary conditions. Here we only consider Dirichlet problem for (1.2). Suppose that
\[ u = g \quad \text{on} \quad \partial \Omega, \]  
(2.3)
it is obvious that we need to impose
\[ u_{\varepsilon} = g \quad \text{or} \quad u_{\varepsilon} \approx g \quad \text{on} \quad \partial \Omega. \]  
(2.4)
Moreover, since (2.2) is a fourth order PDE, in order to uniquely determine $u_{\varepsilon}$ we need to impose an additional boundary condition for $u_{\varepsilon}$. Mathematically, many boundary conditions can be used for this purpose. Physically, any additional boundary condition will introduce a “boundary layer”, so a better choice would be one which minimizes the boundary layer. Here we proposed to use one of following three boundary conditions
\[ \Delta u_{\varepsilon} = \varepsilon^2 \quad \text{or} \quad \frac{\partial \Delta u_{\varepsilon}}{\partial n} = \varepsilon^2 \quad \text{or} \quad D^2 u_{\varepsilon} \cdot n = \varepsilon^2 \quad \text{on} \quad \partial \Omega. \]  
(2.5)
In particular, the first two boundary conditions, which are natural boundary conditions, have an advantage in PDE convergence analysis [36, 63]. Another valid boundary condition is the Neumann boundary condition $\frac{\partial u_{\varepsilon}}{\partial n} = \varepsilon^2$ on $\partial \Omega$. But since this is an essential boundary condition, it produces a larger boundary layer than the above three boundary conditions. The rationale for picking the above boundary conditions is that we implicitly impose an extra boundary condition $\varepsilon^m \Delta u_{\varepsilon} + u_{\varepsilon} = g + \varepsilon^{m+2}$ on $\partial \Omega$ for equation (2.2), which is a higher order perturbation of the original Dirichlet boundary condition $u_{\varepsilon} = g$ on $\partial \Omega$. Intuitively, we expect and hope that the extra boundary condition converges to the original Dirichlet boundary condition as $\varepsilon$ tends to zero. We note that $m$ can be any positive integer, and power 2 is used for convenience and it can be replaced by any positive integer.

We now remark that when $n = 2$ in mechanical applications $u_{\varepsilon}$ often stands for the vertical displacement of a plate and $D^2 u_{\varepsilon}$ is the moment tensor, and in the weak formulation, the biharmonic term becomes $-\varepsilon(D^2 u_{\varepsilon}, D^2 v)$ which should vanish as $\varepsilon \to 0^+$. This is the very reason why we call $\lim_{\varepsilon \to 0^+} u_{\varepsilon}$, if exists, a moment solution and call the limiting process the vanishing moment method.

In summary, we propose to approximate the second order fully nonlinear Dirichlet problem (1.2), (2.3) by the fourth order quasilinear boundary value problems (2.2), (2.4), (2.5). Since we expect $u_{\varepsilon} \in W^{m,p}(\Omega)$ for $m \geq 2, p \geq 2$, so the convergence $\lim_{\varepsilon \to 0^+} u_{\varepsilon}$ in Definition 2.3 can be understood in $H^2$-topology or in $H^1$-topology or even in $L^2$-topology. To distinguish these different limits, we introduce the following refined definition of Definition 2.3.

**Definition 2.4.** Suppose that $u_{\varepsilon} \in H^2(\Omega)$ solves problem (2.2), (2.4), (2.5). $\lim_{\varepsilon \to 0^+} u_{\varepsilon}$ is called respectively a sub-weak, weak and strong moment solution to problem (1.2), (2.3) if the convergence holds in $L^2$, $H^1$- and $H^2$-topology.

**Remark 2.1.** Since sub-weak and weak moment solutions do not have second order weak derivatives, they are very hard (if all possible) to identify. On the other hand, since strong moment solutions do have second order weak derivatives, naturally they are expected to satisfy the PDE (1.2) almost everywhere in $\Omega$ and to fulfill the boundary condition (2.3) pointwise on $\partial \Omega$ (cf. [36, 63]).

3. **Discretization and solution methods.** The vanishing moment method reduces the problem of solving (1.2), (2.3) to a problem of solving (2.2), (2.4), (2.5).
for each fixed \( \varepsilon > 0 \). Since (2.2) is a nonlinear biharmonic equation, one can use any of the wealth amount of existing numerical methods for biharmonic problems to discretize the equation. Although other types of numerical methods are applicable, here we focus on Galerkin type methods such as finite element methods, mixed finite element methods, discontinuous and spectral Galerkin methods [18, 9, 22, 8]. Throughout this section, we assume \( n = 2, 3 \).

3.1. Finite element methods in 2-d. In the two-dimensional case many finite element methods, such as confirming Argyris, Bell, Bogner–Fox–Schmit and Hsieh–Clough–Tocher elements and nonconforming Adini, Morley, and Zienkiewicz elements, were extensively developed in 60’s and 70’s for the biharmonic problems. A beautiful theory of plate finite element methods was also established (cf. [18]). Naturally, one would want to solve problem (2.2), (2.4), (2.5) by using and adapting these well-known plate finite element methods. That is exactly what we are going to do next. For the sake of presentation clarity, here we only discuss the confirming finite element methods, and refer to [63] for a detailed development of nonconfirming finite element methods for problem (2.2), (2.4), (2.5).

The variational formulation for (2.2), (2.4), (2.5) is defined as: Find \( u^\varepsilon \in H^2(\Omega) \) with \( u^\varepsilon = g \) a.e. on \( \partial \Omega \) such that for any \( v \in H^2(\Omega) \cap H^1_0(\Omega) \) there holds
\[
-\varepsilon (\Delta u^\varepsilon, \Delta v) + (F(D^2 u^\varepsilon, \nabla u^\varepsilon, u^\varepsilon, x), v) = -\varepsilon^3 \langle \frac{\partial v}{\partial n} \rangle.
\]

Let \( T_h \) be a quasiuniform triangular or rectangular mesh with mesh size \( h \in (0, 1) \) for the domain \( \Omega \subset \mathbb{R}^2 \). Let \( U^h_g \subset H^2(\Omega) \) denote one of confirming finite element spaces (as mentioned above) whose functions take the boundary value \( g \) at all nodes on \( \partial \Omega \). Then our finite element method is defined as: Find \( u^\varepsilon_h \in U^h_g \) such that
\[
-\varepsilon (\Delta u^\varepsilon_h, \Delta v_h) + (F(D^2 u^\varepsilon_h, \nabla u^\varepsilon_h, u^\varepsilon_h, x), v_h) = -\varepsilon^3 \langle \frac{\partial v_h}{\partial n} \rangle, \quad \forall v_h \in U^h_0.
\]

In [65] we shall present several numerical experiment results for the Monge-Ampère type equations to show the excellent performance of the Argyris finite element method. Convergence and error analysis of the above scheme and other finite element schemes will be presented in forthcoming papers (also see [63]).

3.2. Mixed finite element methods in 2-d and 3-d. Along with the theory of plate finite element methods, another beautiful theory of mixed finite element methods was also extensively developed in '70s and '80s for the biharmonic problems in 2-d (cf. [9, 18, 35]). It is interesting to point out that all these 2-d mixed finite element methods can be easily generalized to solving 3-d biharmonic problems and general fourth order quasilinear PDEs (cf. [32, 41, 42]).

Because the Hessian matrix \( D^2 u^\varepsilon \) appears in (2.2) in a nonlinear fashion, to design a mixed method we are “forced” to introduce \( \sigma^\varepsilon := D^2 u^\varepsilon \) (not \( v^\varepsilon := \Delta u^\varepsilon \) alone) as additional variables so the mixed method simultaneously seeks \( u^\varepsilon \) and \( \sigma^\varepsilon \). This observation then excludes the usage of the popular family of Ciarlet-Raviart mixed finite element methods (originally designed for the biharmonic problems) [18, 19], on the other hand, the observation suggests to try Hermann-Miyoshi mixed elements [35, 53, 60, 61, 68] and Hermann-Johnson mixed elements [35, 53, 55] both use \( \sigma^\varepsilon \) as additional variables.

To define Hermann-Miyoshi type mixed finite element methods, we first derive the following mixed variational formulation for problem (2.2), (2.4), (2.5): Find
\((u^\varepsilon, \sigma^\varepsilon) \in V_g \times W_\varepsilon\) such that
\[
(\sigma^\varepsilon, \mu) + (\nabla u^\varepsilon, \text{div} \mu) = \sum_{i=1}^{n-1} \left( \frac{\partial g}{\partial \tau_i}, \mu n \cdot \tau_i \right) \quad \forall \mu \in W_0, \tag{3.3}
\]
\[
\varepsilon (\text{div} \sigma^\varepsilon, \nabla v) + (F(\sigma^\varepsilon, \nabla u^\varepsilon, u^\varepsilon, x), v) = (f, v) \quad \forall v \in V_0, \tag{3.4}
\]
where \(\tau_i, i = 1, 2, \cdots, (n-1)\) denote the \((n-1)\) tangential directions at each point on \(\partial \Omega\), \(\frac{\partial g}{\partial \tau_i}\) denotes the tangential derivative of \(g\) along \(\tau_i\), and
\[
V_g := \{ v \in H^1(\Omega); v|_{\partial \Omega} = g \}, \quad V_0 := \{ v \in H^1(\Omega); v|_{\partial \Omega} = 0 \},
\]
\[
W_\varepsilon := \{ \mu \in [H^1(\Omega)]^{n \times n}; \mu_{ij} = \mu_{ji}, \mu n \cdot n|_{\partial \Omega} = \varepsilon^2 \},
\]
\[
W_0 := \{ \mu \in [H^1(\Omega)]^{n \times n}; \mu_{ij} = \mu_{ji}, \mu n \cdot n|_{\partial \Omega} = 0 \}.
\]

Let \(T_h\) be a quasiuniform triangular or rectangular mesh if \(n = 2\) and be a quasiuniform tetrahedral or 3-d rectangular mesh if \(n = 3\) for the domain \(\Omega\). Let \(V^h \subset H^1(\Omega)\) be the Lagrange finite element space consisting of continuous piecewise polynomials of degree \(k\) \((k \geq 2)\) associated with the mesh \(T_h\). Let
\[
V_g^h := V^h \cap V_g, \quad V_0^h := V^h \cap V_0, \quad W_\varepsilon^h := [V^h]^{n \times n} \cap W_\varepsilon, \quad W_0^h := [V^h]^{n \times n} \cap W_0.
\]

Based on the variational formulation (3.3)–(3.4) we define our (Hermann-Miyoshi type) mixed finite element methods as follows: Find \((u_h^\varepsilon, \sigma_h^\varepsilon) \in V_g^h \times W_\varepsilon^h\) such that
\[
(\sigma_h^\varepsilon, \mu_h) + (\nabla u_h^\varepsilon, \text{div} \mu_h) = \sum_{i=1}^{n-1} \left( \frac{\partial g}{\partial \tau_i}, \mu_h n \cdot \tau_i \right) \quad \forall \mu_h \in W_0^h, \tag{3.5}
\]
\[
\varepsilon (\text{div} \sigma_h^\varepsilon, \nabla v_h) + (F(\sigma_h^\varepsilon, \nabla u_h^\varepsilon, u_h^\varepsilon, x), v_h) = (f, v_h) \quad \forall v_h \in V_0^h. \tag{3.6}
\]

Similarly, we can define variants of the above scheme as those proposed in [68] as well as Hermann-Johnson type mixed methods. In [65] we shall present several numerical experiment results for the above scheme applying to the Monge-Ampère type equations. Convergence and error analysis of the above scheme and other mixed finite element schemes will be presented in a forthcoming paper (also see [63]).

Remark 3.1. Besides the finite element and mixed finite element discretization methods, one can also approximate problem (2.2), (2.4), (2.5) by discontinuous Galerkin methods [2, 21, 22, 47, 37, 38, 62] and spectral Galerkin methods [8, 14, 69]. It should be pointed out that these methods are dimension-independent, hence, can be used in both 2-d and 3-d cases. We refer to [63] for a detailed exposition.

3.3. Remarks on second order fully nonlinear parabolic equations. By adopting the method of line approach, generalizations of the numerical methods discussed in previous subsections to the corresponding parabolic equations (4.11) and (4.12) are standard (cf. [32], [38] and references therein). Assuming that an implicit time stepping method such as the backward Euler and the Crank-Nicolson schemes will be used for time discretization, then at each time step we only need to solve a fully nonlinear elliptic equation of the form (2.2). As a result, all numerical methods discussed in [3.4], [3.5] immediately apply. On the other hand, it should be pointed out that the convergence and error analysis of all fully discrete schemes are expected to be harder, in particular, establishing error estimates which depend on \(\varepsilon^{-1}\) polynomially instead of exponentially will be very challenging (cf. [10], [11], [42], [39], [44], [43]).
3.4. Remarks on nonlinear solvers and preconditioning. After equations (2.2) and (4.11) are discretized by any of above discretization methods, we get a strong nonlinear algebraic system to solve. To the end, one has to use one or another iterative solution method to do the job. In all numerical experiments to be given in §5, we use preconditioned Newton iterative methods as our nonlinear solvers. A few fixed point iterations might be needed to generate initial guess for Newton type iterative methods. Another strategy which we are currently investigating is the following “multi-resolution” strategy: first compute a numerical solution using a relatively large $\varepsilon$, then use the computed solution as an initial guess for the Newton method at the finer resolution $\varepsilon$. Regarding to preconditioning, we use the simple ILU preconditioner in all simulations of §5. We plan to use more sophisticated multigrid and Schwarz (or domain decomposition) preconditioners when the amount of computations becomes intensive and large in 3-d.

4. Applications. In this section, we shall apply the vanishing moment methodology outlined in the previous section to several classes of specific second order fully nonlinear PDEs.

4.1. Monge-Ampère type equations. Monge-Ampère type equations refer to a class of second order fully nonlinear PDEs of the form (cf. [13, 12, 17, 46, 51])

$$F(D^2u^0, Du^0, u^0, x) := \det(D^2u^0) - f(\nabla u^0, u^0, x) = 0,$$

(4.1)

Note that from now on we shall always use $u^0$ to denote a solution of a second order fully nonlinear PDE we intend to solve. Equation (4.1) reduces to the classical Monge-Ampère equation

$$\det(D^2u^0) = f(x)$$

if $f(\nabla u^0, u^0, x) = f(x) > 0$, and to Gauss curvature equation

$$\det(D^2u^0) = K(1 + |\nabla u^0|^2)^{\frac{n+2}{2}}$$

if $f(\nabla u^0, u^0, x) = K(1 + |\nabla u^0|^2)^{\frac{n+2}{2}}$. Where the constant $K$ is a prescribed Gauss curvature. Monge-Ampère type equations are the best known second order fully nonlinear PDEs, they arise in differential geometry and applications such as mass transportation and meteorology. It is well-known that Monge-Ampère type equations are elliptic only in the set of convex functions (cf. [13, 51]). So their viscosity solutions are defined as convex functions in the sense of Definition 2.2.

The vanishing moment approximation (2.2) to (4.1) reads as:

$$-\varepsilon \Delta^2 u^\varepsilon + \det(D^2u^\varepsilon) = f(\nabla u^\varepsilon, u^\varepsilon, x) \quad (\varepsilon > 0).$$

(4.2)

For each fixed $\varepsilon > 0$, this is a quasilinear fourth order PDE with Hessian type nonlinearity. It is complemented by boundary conditions (2.4) and (2.5) (or (2.5)2).

For the classical Monge-Ampère equation, it can be shown that

- For each fixed $\varepsilon > 0$, problem (4.2), (2.4), (2.5)1 (or (2.5)2) has a unique weak solution in $W^{3,2}(\Omega)$ for $n = 2, 3$.
- $\det(D^2u^\varepsilon) > 0$ and $\Delta u^\varepsilon > 0$ in $\Omega$ for $\varepsilon > 0$ when $n = 2, 3$.
- $u^\varepsilon$ is convex in $\Omega$ for $\varepsilon > 0$ when $n = 2$.
- The Dirichlet problem for the classical Monge-Ampère equation has a unique convex strong moment solution, which coincides with the unique convex viscosity solution of the same problem when $n = 2$. 
The above results immediately imply that in the two dimensions the vanishing moment method indeed works for the classical Monge-Ampère equation and the notion of moment solutions and the notion of viscosity solutions are equivalent in this case.

Remark 4.1. Recall that when \( n = 2 \), the Dirichlet problem (1.4)–(1.5) has at most two solutions (see [22]). An amazing numerical discovery to be given in §5 is that if we restrict \( \varepsilon \) in (4.2) to \( \varepsilon < 0 \), then \( \lim_{\varepsilon \to 0} u^\varepsilon \) also exists and the limit is nothing but the other solution solution of problem (1.4)–(1.5) which is concave (see Figures 5.1–5.6)!

### 4.2. Pucci’s equations

Pucci’s extremal equations are referred to the following two families of fully nonlinear PDEs (cf. [46, 13])

\[
M_\alpha[u] := \alpha \Delta u + (1 - n\alpha)\lambda_n(D^2 u^0) = f(x), \quad (4.3)
\]

\[
m_\alpha[u] := \alpha \Delta u + (1 - n\alpha)\lambda_1(D^2 u^0) = f(x), \quad (4.4)
\]

for \( 0 < \alpha \leq \frac{1}{n} \). Where \( \lambda_n(D^2 u^0) \) and \( \lambda_1(D^2 u^0) \) denote the maximum and minimum eigenvalues of the Hessian matrix \( D^2 u^0 \). In the 2-d case, the above equations can be rewritten in terms of \( \Delta u^0 \) and \( \det(D^2 u^0) \) (cf. [31]).

The vanishing moment approximations to (4.3) and (4.4) are defined as

\[
-\varepsilon \Delta^2 u^\varepsilon + \alpha \Delta u^\varepsilon + (1 - n\alpha)\lambda_n(D^2 u^\varepsilon) = f(x), \quad (4.5)
\]

\[
-\varepsilon \Delta^2 u^\varepsilon + \alpha \Delta u^\varepsilon + (1 - n\alpha)\lambda_1(D^2 u^\varepsilon) = f(x), \quad (4.6)
\]

which should be complemented by boundary conditions (2.4) and (2.5) (or (2.5)2).

### 4.3. Infinite Laplace equation

The infinite Laplace equation refers to the following degenerate quasilinear PDE:

\[
F(D^2 u^0, Du^0, u^0, x) := \Delta_\infty u^0 = 0, \quad (4.7)
\]

where

\[
\Delta_\infty u^0 := \langle D^2 u^0 \nabla u^0, \nabla u^0 \rangle = D^2 u^0 \nabla u^0 \cdot \nabla u^0.
\]

\( \Delta_\infty u^0 \) can be regarded as the limit of the \( p \)-Laplacian \( \Delta_p u^0 := \text{div} (|\nabla u^0|^{p-2} \nabla u^0) \) as \( p \to \infty \), it also can be derived as the Euler-Lagrange equation of the \( L^\infty \) functional

\[
I(v) := \text{ess sup}_{x \in \Omega} |\nabla v(x)|,
\]

whose minimizers are often called “absolute minimizers” [3]. Besides its mathematical appeals, the infinite Laplace equation also arises from image processing, geography, and geology applications [4, 15]. Although the infinite Laplace equation is only a degenerate quasilinear PDE, not a fully nonlinear PDE, it is very difficult to solve numerically. This is because the infinite Laplace equation does not have classical solutions in general [3], and since it is not in divergence form, its weak solutions are defined and understood in the viscosity sense. We refer to [44] for recent developments on finite difference approximations of the infinite Laplace equation.

Here we propose the following vanishing moment approximation for (4.7):

\[
-\varepsilon \Delta^2 u^\varepsilon + \Delta_\infty u^\varepsilon = 0, \quad (4.8)
\]

which is complemented by boundary conditions (2.4) and (2.5) (or (2.5)2). In §5 we shall present numerical results which show that the vanishing moment approximation
exactly converges to the unique viscosity solution of the Dirichlet problem for (4.7). This is another example which shows that the notion of moment solutions and the notion of viscosity solutions coincide.

**Remark 4.2.** It is easy to see that the above vanishing moment method also applies to the \( p \)-Laplacian equation \(-\Delta_p u^0 = f\) for \( 1 \leq p < \infty \).

### 4.4. Second order fully nonlinear parabolic PDEs

We first like to note that there are several different versions of legitimate parabolic generalizations to elliptic PDE (1.2) (cf. \[57, 70\]). In this paper, we shall only consider the following widely studied (and it turns out to be the “easiest”) class of second order fully nonlinear parabolic PDEs:

\[
F(D^2 u^0, \nabla u^0, u^0, x, t) - u_t^0 = 0, \tag{4.9}
\]

assuming that \( F(D^2 u^0, \nabla u^0, u^0, x, t) \) is elliptic. Clearly, this is the most natural parabolic generalization to equation (1.2). For example, the corresponding parabolic Monge-Ampère type equation reads as

\[
\det(D^2 u^0) - u_t^0 = f(\nabla u^0, u^0, x, t) \geq 0. \tag{4.10}
\]

In the past two decades the viscosity solution theory has been well developed for equations (4.9) and (4.10), see \[57, 70, 52\]. On the other hand, numerical approximation to these fully nonlinear parabolic PDEs is a completely untouched area. To the best of our knowledge, no numerical result (in fact, no attempt) is known in the literature.

Similarly, we can define the vanishing moment method and the notion of moment solutions for initial and initial-boundary value problems for (4.9), and then ask the same questions as we did in §2.2. We leave this as an exercise to interested readers and refer to \[63\] for a detailed exposition.

Following the derivation of §2.2, we propose the following vanishing moment approximations to (4.9) and (4.10), respectively,

\[
F(D^2 u^0, \nabla u^0, u^0, x, t) - \varepsilon \Delta^2 u^0 - u_t^0 = 0, \tag{4.11}
\]

\[
\det(D^2 u^0) - \varepsilon \Delta^2 u^0 - u_t^0 = f(\nabla u^0, u^0, x, t), \tag{4.12}
\]

each of the above equations is a fourth order quasilinear parabolic PDEs.

### 5. Numerical experiments

In this section, we shall present a number of numerical experiment results obtained by using the vanishing moment method together with the numerical methods proposed in \[3\]. Both 2-d and 3-d tests will be presented. All the 3-d tests are obtained by a Hermann-Miyoshi type mixed finite element method, while the 2-d tests are computed by using both the Argyris (plate) finite element method and the Hermann-Miyoshi mixed finite element method.

#### 5.1. Two-dimensional numerical experiments

The numerical solutions of the first seven tests are computed using the Argyris finite element method.

**Test 1:** In this test we solve the Monge-Ampère problem (1.4)–(1.5) on the unit square \( \Omega = (0, 1)^2 \) with the following data:

\[
f(x, y) \equiv 1, \quad g(x, y) \equiv 0.
\]

We remark that problem (1.4)–(1.5) has a unique convex viscosity solution but does not have a classical solution (cf. \[51, 31\]).
Recall that the vanishing moment approximation of (1.4)–(1.5) is problem (4.2), (2.4), (2.5) with the above $f$ and $g$. We discretize problem (4.2), (2.4), (2.5) using the Argyris plate element as described in §3.1. Figure 5.1 displays the computed (moment) solutions using $\varepsilon = 10^{-3}$ (left graph) and $\varepsilon = -10^{-3}$ (right graph). Clearly, the vanishing moment approximations correctly capture the convex viscosity solution (left graph) and the concave viscosity solution (right graph). Hence, the moment solutions coincide with the viscosity solutions (see [36] for a rigorous proof).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig5.1}
\caption{Computed (moment) solutions of Test 1: Graph on left corresponds to $\varepsilon = 10^{-3}$ and graph on right corresponds to $\varepsilon = -10^{-3}$.}
\end{figure}

To have a better view of the convexity of the computed solution, we also plot selected cross sections of the left figure in Figure 5.1. The cross sections clearly show that the computed solution is a convex function. In particular, there is no visible smear at the boundary.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig5.2}
\caption{x-cross sections (left figure) of the left graph in Fig. 5.1 at $x = 0.1, 0.3, 0.5, 0.7, 0.9$ (indicated respectively by asterisk, circle, plus sign, square, and triangle); y-cross sections (right figure) of the left graph in Fig. 5.1 at $y = 0.1, 0.3, 0.5, 0.7, 0.9$ (indicated respectively by asterisk, circle, plus sign, square, and triangle).}
\end{figure}

\textbf{Test 2:} The only difference between this test and Test 1 is that the datum
functions are now chosen as

\[ f(x, y) = (1 + (x^2 + y^2))e^{(x^2+y^2)}, \quad g(x, y) = \begin{cases} e^{y^2/2} & \text{if } x = 0, \\ e^{x^2/2} & \text{if } y = 0, \\ e^{(1+x^2)/2} & \text{if } y = 1, \\ e^{(1+y^2)/2} & \text{if } x = 1, \end{cases} \]

so that \( u^0(x, y) = \frac{1}{2}e^{(x^2+y^2)} \) is an exact solution of problem (1.4)–(1.5). Clearly, \( u^0 \) is a convex function, hence \( u^0 \) must be the unique convex viscosity solution of problem (1.4)–(1.5) (cf. [51]).

Figure 5.3 shows the computed (moment) solutions using \( \varepsilon = 10^{-3} \) (left graph) and \( \varepsilon = -10^{-3} \) (right graph). Again, the vanishing moment approximations correctly capture the convex viscosity solution \( u^0 \) (left graph) and the concave viscosity solution (right graph), hence the moment solutions coincide with the viscosity solutions (see [36] for a rigorous proof).

**Fig. 5.3.** Computed (moment) solutions of Test 2: Graph on left corresponds to \( \varepsilon = 10^{-3} \) and graph on right corresponds to \( \varepsilon = -10^{-3} \).

**Test 3:** Similar to Test 2, the only difference between this test and Test 1 is that the datum functions are now chosen as

\[ f(x, y) = \frac{1}{x^2 + y^2}, \quad g(x, y) = \begin{cases} \frac{2\sqrt{2}}{3} y^2 & \text{if } x = 0, \\ \frac{2\sqrt{2}}{3} x^2 & \text{if } y = 0, \\ \frac{2\sqrt{2}}{3} (1 + x^2)^{1/3} & \text{if } y = 1, \\ \frac{2\sqrt{2}}{3} (1 + y^2)^{1/3} & \text{if } x = 1, \end{cases} \]

so that \( u^0(x, y) = \frac{2\sqrt{2}}{3}(x^2 + y^2)^{1/3} \) is the unique convex viscosity solution of problem (1.4)–(1.5).

Figure 5.4 displays the computed (moment) solutions using \( \varepsilon = 10^{-3} \) (left graph) and \( \varepsilon = -10^{-3} \) (right graph). As expected, the vanishing moment approximations correctly capture the convex viscosity solution \( u^0 \) (left graph) and the concave viscosity solution (right graph), hence the moment solutions coincide with the viscosity solutions (see [36] for a rigorous proof).

**Test 4:** Again, the only difference between this test and Test 1 is that the datum functions are now chosen as

\[ f(x, y) = (1 - x - y)^2 \quad g \equiv 0. \]
MOMENT SOLUTIONS FOR 2nd ORDER FULLY NONLINEAR PDEs

Fig. 5.4. Computed (moment) solutions of Test 3: Graph on left corresponds to $\epsilon = 10^{-3}$ and graph on right corresponds to $\epsilon = -10^{-3}$.

On the other hand, mathematically there is a significant difference between these two test problems. Note that $f(x,y) = 0$ on the line $x + y = 1$ in the domain $\Omega = (0, 1)^2$. Hence, problem (1.4)–(1.5) is known as a degenerate Monge-Ampère problem (cf. [51]).

Figure 5.5 displays the computed (moment) solutions using $\epsilon = 10^{-3}$ (left graph) and $\epsilon = -10^{-3}$ (right graph). Once again, the vanishing moment approximations correctly capture the convex viscosity solution (left graph) and the concave viscosity solution (right graph), hence the moment solutions coincide with the viscosity solutions (see [36] for a rigorous proof). In addition, our numerical result shows that the vanishing moment method is robust with respect to the degeneracy of the underlying PDE.

Test 5: Once again, the only difference between this test and Test 1 is that the datum functions are now chosen as

$$f(x, y) = x^2 - y^2 \quad g \equiv 0.$$  

Mathematically, the difference between this test problem and Test 1 is even more dramatic because not only $f(x, y) = 0$ on the line $x - y = 0$ but also $f$ changes sign
(hence the PDE changes type) in \( \Omega \). To the best of our knowledge, there is no viscosity solution theory for this type Monge-Ampere problems in the literature. However, the vanishing moment method seems to work well for this problem. Our numerical results indicate existence of both convex and concave moment solutions.

Figure 5.6 displays the computed convex (moment) solution using \( \varepsilon = 10^{-3} \) (left graph) and the computed concave (moment) solution using \( \varepsilon = -10^{-3} \) (right graph).

![Figure 5.6](image)

**Fig. 5.6.** Computed (moment) solutions of Test 5: Graph on left corresponds to \( \varepsilon = 10^{-3} \) and graph on right corresponds to \( \varepsilon = -10^{-3} \).

Again, to have a better view of the convexity of the computed solution, we also plot selected cross sections of the left figure in Figure 5.6. The cross sections clearly show that the computed solution is a convex function. In particular, there is no visible smear at the boundary.

![Figure 5.7](image)

**Fig. 5.7.** x-cross sections (left figure) of the left graph in Fig. 5.6 at \( x = 0.1, 0.3, 0.5, 0.7, 0.9 \) (indicated respectively by asterisk, circle, plus sign, square, and triangle); y-cross sections (right figure) of the left graph in Fig. 5.6 at \( y = 0.1, 0.3, 0.5, 0.7, 0.9 \) (indicated respectively by asterisk, circle, plus sign, square, and triangle).

**Test 6:** In this test we solve the following Gauss curvature (or \( K \)-surface) equation
MOMENT SOLUTIONS FOR 2nd ORDER FULLY NONLINEAR PDEs

\[ \det(D^2 u^0) = K(1 + |\nabla u^0|^2)^2 \quad \text{in } \Omega := (-0.57, 0.57)^2, \]
\[ u^0 = x^2 + y^2 - 1 \quad \text{on } \partial \Omega, \]

where \( K > 0 \) is a given constant Gauss curvature. Note that the above problem is a special case of problem (4.1), (2.3) with \( f(\nabla u^0, u^0, x, y) = K(1 + |\nabla u^0|^2)^{\frac{n+2}{2}}, \ n = 2, \)
and \( g(x, y) = x^2 + y^2 - 1. \)

It was proved by Guan [48] that there exists \( K^* > 0 \) such that for each \( K \in [0, K^*) \) problem (5.1)–(5.2) (with more general Dirichlet data) has a unique convex viscosity solution. Theoretically, it is very difficult to give an accurate estimate for the curvature upper bound \( K^* \). However, hand, this offers an ideal opportunity for numerical analysts to help and to contribute. It turns out that the vanishing moment method proposed in this paper works very well for such a problem, hence it might provide a useful tool and answer to the challenge.

Since we are only interested in convex solutions of the Gauss curvature equation, so we restrict \( \varepsilon > 0 \) in (4.2). Figure 5.8 displays the computed convex (moment) solution using \( \varepsilon = 10^{-3} \) and \( K = 0.1, 1, 2, 2.1, \) respectively. We note that our com-

\[ \begin{array}{cccc}
\text{Surface vs. height} & \text{Map vs. light} & \text{Surface vs. height} & \text{Map vs. light} \\
\text{Min: -0.671} & \text{Max: -0.354} & \text{Min: -0.671} & \text{Max: -0.354} \\
\text{Min: -0.639} & \text{Max: -0.354} & \text{Min: -0.639} & \text{Max: -0.354} \\
\end{array} \]

\text{FIG. 5.8. Computed (moment) solutions of Test 6: } \varepsilon = 10^{-3} \text{ and } K = 0.1, 1, 2, 2.1. \text{ Graphs are arranged row-wise.}

puter code stops producing a convergent numerical solution for \( K = 2.2 \). Hence we conjecture that \( K^* \approx 2.1 \) for the above test problem.

\textbf{Test 7:} In this test, we solve problem (4.7), (2.3) over the domain \( \Omega = (-\frac{1}{2}, \frac{1}{2})^2 \).
with the following boundary datum function
\[
g(x, y) = \begin{cases} 
(\frac{1}{2})^{\frac{3}{4}} - y^{\frac{3}{4}} & \text{if } x = -\frac{1}{2}, \\
x^{\frac{3}{4}} - \left(\frac{1}{2}\right)^{\frac{3}{4}} & \text{if } y = -\frac{1}{2}, \\
(\frac{1}{2})^{\frac{3}{4}} - y^{\frac{3}{4}} & \text{if } x = \frac{1}{2}, \\
x^{\frac{3}{4}} - \left(\frac{1}{2}\right)^{\frac{3}{4}} & \text{if } y = \frac{1}{2},
\end{cases}
\]
so that \(u^0(x, y) = x^{\frac{3}{4}} - y^{\frac{3}{4}}\) is the unique viscosity solution (cf. \([3]\)).

We remark that this is an important example in the theory of absolutely minimizing functions since \(u^0\) is the least regular absolutely minimizing function known in the case of the Euclidean norm (cf. \([3]\) and the references therein). It is easy to check that \(u^0\) is a Hölder continuous function with exponent \(\frac{1}{3}\). However, it is not twice differentiable on the axes. We also note that (see §4.3) the infinite Laplace equation (4.7) is only a second order (degenerate) quasilinear (instead fully nonlinear) PDE, however, the complicate and nondivergence structure makes the infinite Laplace equation very difficult to analyze theoretically and to compute numerically.

Figure 5.9 displays the computed (moment) solution using \(\varepsilon = 10^{-3}\) (left graph) and the exact solution \(u^0\) (right graph). Once again, the vanishing moment approximation correctly captures the viscosity solution \(u^0\), hence, the moment solution coincides with the viscosity solution (see \([36]\) for a rigorous proof).

The numerical solutions of the next two tests are obtained by using the Hermann-Miyoshi mixed finite element method with piecewise quadratic shape functions.

**Test 8:** This test is a re-run of Test 1 but using the quadratic Hermann-Miyoshi mixed finite element method. Figure 5.10 is the counterpart of Figure 5.1. We remark that the mixed method also produces an approximation to the Hessian matrix \(D^2u^\varepsilon\), which is not shown here. Clearly, the numerical results of Test 1 and Test 8 have the same accuracy. However, it should be noted that the mixed method runs about 20 times faster than the Argyris method on this test problem.

**Test 9:** This test solves, using the quadratic Hermann-Miyoshi mixed finite method, the Monge-Ampère problem (1.4)–(1.5) on the unit square \(\Omega = (0, 1)^2\) with
Fig. 5.10. Computed (moment) solutions of Test 8: Graph on left is the computed $u^\varepsilon_h$ with $\varepsilon = 10^{-3}$ and graph on right is the computed $u^\varepsilon_h$ with $\varepsilon = -10^{-3}$.

Fig. 5.11. Computed (moment) solutions of Test 9 by the quadratic Hermann-Miyoshi mixed method. $\varepsilon = 10^{-3}$.

the following data:

$$f(x, y) = \frac{4}{(4 - x^2 - y^2)^2}, \quad g(x, y) = \sqrt{4 - x^2 - y^2}$$

so that $u^0 = \sqrt{4 - x^2 - y^2}$ is an exact (convex) solution. We note that problem (1.4)–(1.5) has exact two solutions, one is convex and the other is concave (cf. [23]). Figure 5.11 displays the computed (moment) solution $u^\varepsilon_h$ using $\varepsilon = 10^{-3}$ (left graph) and its error (right graph). As expected, the vanishing moment approximation correctly captures the convex viscosity solution $u^0$. Hence the moment solution coincides with the viscosity solution (see [36] for a rigorous proof). Again, we remark that the mixed method also gives an approximation to the Hessian matrix $D^2 u^\varepsilon$, which is not shown here, and the mixed method runs about 20 times faster than the Argyris method for solving this test problem.

5.2. Three-dimensional numerical experiments. In this subsection we present two numerical tests on computing moment (and viscosity) solutions of the Monge-Ampère problem (1.4)–(1.5) in the unit cube $\Omega = (0, 1)^3$. Numerical approximations of fully nonlinear PDEs in 3-d is known to be very difficult. To the best of our knowl-
edge, no 3-d numerical results are given in the literature for the Monge-Ampère type
fully nonlinear PDEs.

**Test 10:** Consider the Monge-Ampère problem \((1.4)-(1.5)\) on the unit cube \(\Omega = (0,1)^3\) with the following data:

\[
f(x,y,z) = (1 + x^2 + y^2 + z^2) \exp\left(\frac{x^2 + y^2 + z^2}{2}\right), \quad g(x,y,z) = \exp\left(\frac{x^2 + y^2 + z^2}{2}\right).
\]

It is easy to verify that \(u^0 = \exp\left(\frac{x^2 + y^2 + z^2}{2}\right)\) is a unique exact (convex) solution. We compute this solution using the vanishing moment method combined with a generalized Hermann-Miyoshi type mixed finite element method using linear shape functions.

Figure 5.12 displays color plots of five x-slices of the computed (moment) solution \(u^0_h\) (left graph) and its corresponding error (right graph). Figure 5.13 displays color plots of five z-slices of the computed (moment) solution \(u^0_h\) (left graph) its corresponding error (right graph). As expected, the vanishing moment approximation correctly captures the convex viscosity solution \(u^0\). Hence the moment solution coincides with the viscosity solution.

**Fig. 5.12.** x-slices of the computed (moment) solution of Test 10 by a generalized linear Hermann-Miyoshi mixed method. \(\varepsilon = 10^{-3}\).

**Fig. 5.13.** z-slices of the computed (moment) solution of Test 10 by a generalized linear Hermann-Miyoshi mixed method. \(\varepsilon = 10^{-3}\).
**Test 11:** Our last numerical test solves the 3-dimensional generalization of the test problem in Test 1. That is, we assume \( u \) satisfies the Monge-Ampère problem (1.4)–(1.5) in \( \Omega = (0, 1)^3 \) with the data
\[
f(x, y, z) \equiv 1, \quad g(x, y, z) \equiv 0.
\]
We remark that the above problem has a unique convex viscosity solution but does not have a classical solution (cf. [51, 31]). There is no explicit solution formula for the boundary value problem.

Figure 5.14 displays color plots of x-slices (left graph) and z-slices (right graph) of the computed (moment) solution \( u^\varepsilon_h \) using a generalized linear Hermann-Miyoshi mixed method. Once again, the vanishing moment approximation correctly captures the convex viscosity solution \( u^0 \). Hence the moment solution coincides with the viscosity solution.

**6. Conclusions.** In this paper we introduce a new notion of weak solutions, called moment solutions, through a constructive limiting process, called the vanishing moment method, for second order fully nonlinear PDEs. The notion of moment solutions and the vanishing moment method are exactly in the same spirit as the original notion of viscosity solutions and the vanishing viscosity method proposed by M. Crandall and P. L. Lions in [26] for the Hamilton-Jacobi equations, which is based on the idea of approximating a fully nonlinear PDE by a higher order quasilinear PDE. We first present a general framework of the vanishing moment method and the notion of moment solutions in §2. We then apply the general framework to several classes of PDEs including the Monge-Ampère type equations, Pucci’s extremal equations, the infinite Laplace equation, and second order fully nonlinear parabolic PDEs. We then propose two classes of numerical methods to discretize the fourth order “regularized/perturbed” vanishing moment approximation equations. Finally, we present a number of numerical experiments using the vanishing moment methodology together with the proposed numerical methods to demonstrate convergence and effectiveness of the vanishing moment method, as well as the relationship between the notion of moment solutions and the notion of viscosity solution for second order fully nonlinear PDEs.

This paper provides a practical and systematic methodology/approach, which can be backed by rigorous PDE and numerical theories, for approximating second order
fully nonlinear PDEs. As a by-product, the moment solution theory will provide some insights to our understanding of the viscosity solution theory, and might provide a logical and natural generalization/extension for the notion of viscosity solution, especially, in the cases where there are no theories or the existing viscosity solution theory fails (such as Monge-Ampère equations of sub elliptic and hyperbolic types [16], and systems of second order fully nonlinear PDEs.)

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MOMENT SOLUTIONS FOR 2rd ORDER FULLY NONLINEAR PDEs 23

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