The Worldsheet Conformal Field Theory of the Fractional Superstring

Keith R. Dienes

Department of Physics, McGill University
3600 University St., Montréal, Québec H3A-2T8 Canada

Abstract

Two of the important unresolved issues concerning fractional superstrings have been the appearance of new massive sectors whose spacetime statistics properties are unclear, and the appearance of new types of “internal projections” which alter or deform the worldsheet conformal field theory in a highly non-trivial manner. In this paper we provide a systematic analysis of these two connected issues, and explicitly map out the effective post-projection worldsheet theories for each of the fractional-superstring sectors. In this way we determine their central charges, highest weights, fusion rules, and characters, and find that these theories are isomorphic to those of free worldsheet bosons and fermions. We also analyze the recently-discovered parafermionic “twist current” which has been shown to play an important role in reorganizing the fractional-superstring Fock space, and find that this current can be expressed directly in terms of the primary fields of the post-projection theory. This thereby enables us to deduce some of the spacetime statistics properties of the surviving states.

*E-mail address: dien@hep.physics.mcgill.ca.
1 Introduction

Over the past two years there has been considerable activity in a possible new class of string theories known as fractional superstrings [1–7]: these are non-trivial generalizations of superstrings and heterotic strings, and have the important property that their critical spacetime dimensions are less than ten. This reduction in the critical dimension is accomplished by replacing the worldsheet supersymmetry of the traditional superstring or heterotic string by a $K$-fractional supersymmetry: such symmetries relate worldsheet bosons not to worldsheet fermions, but rather to worldsheet $\mathbb{Z}_K$ parafermions of fractional spin $2/(K + 2)$. One then finds that the corresponding critical spacetime dimension of the theory is given by

$$ D_c = 2 + \frac{16}{K}, \quad K \geq 2. \quad (1.1) $$

Thus while the choice $K = 2$ reproduces the ordinary $D_c = 10$ superstring (with $\mathbb{Z}_2$ “parafermions” reducing to ordinary Majorana fermions), the choices $K = 4, 8, 16$ yield new theories with $D_c = 6, 4, 3$ respectively.

For $K > 2$, however, these new worldsheet theories are substantially more difficult to study than those of ordinary superstrings, since the fundamental worldsheet fractional superconformal algebra is necessarily non-local for $K > 2$, with branch cuts (rather than poles) appearing in the various parafermionic operator product expansions. This implies, for example, that the corresponding mode algebras involve neither commutation nor anti-commutation relations, but rather the more difficult generalized commutation relations; moreover, in many cases this also gives rise to non-trivial braiding relations for the underlying conformal fields.

It is primarily due to such non-linear complications on the worldsheet that fractional superstrings appear to exhibit qualitatively new features in spacetime, as compared to the usual superstrings and heterotic strings. Understanding these new features is thus of paramount importance, not only for demonstrating the internal consistency of the fractional superstring, but also as a means of shedding further light on the general but as yet poorly understood relationship between worldsheet string symmetries and spacetime physics. Therefore, in order to gain some insight into their spacetime physics, fractional superstrings have been studied from a variety of different perspectives; a non-technical review is given in [8]. One straightforward approach in analyzing these theories is essentially “bottom-up”, and attempts to derive the resulting spacetime physics of the fractional superstring (including the spacetime partition functions) by starting from the original worldsheet parafermionic theory described above and constructing the complete resulting Fock space of states. While this approach has proven successful for understanding the properties of certain low-lying states [9], difficulties arise at the higher mass levels of the fractional-superstring spectra. Another approach is essentially “top-down”, and starts by constructing the unique modular-invariant partition functions of the forms that these theories must
have; by analyzing these partition functions in great detail one can then obtain information about the underlying worldsheet physics from which they might ultimately be derived. This approach has also been successful to a large extent, and has shed light on many of the intrinsically new features of the fractional superstrings. Both complimentary approaches have uncovered but left unresolved, however, two crucial aspects of these new string theories which have no analogues in the traditional super- or heterotic string theories: (1) the appearance of extra unusual massive sectors (the so-called “B-sectors”) which contain spacetime particles whose physical roles are unclear; and (2) the appearance of new so-called “internal projections” which, unlike the traditional GSO projection, remove a sufficient number of states from the fractional superstring spectra to actually diminish the effective central charges of these theories. In fact, as we will see, these internal projections have the net effect of changing or deforming the underlying worldsheet parafermionic conformal field theories (CFT’s) upon which the fractional superstring is originally built, with the surviving states filling out the Fock spaces corresponding to new worldsheet CFT’s whose properties and worldsheet representations are as yet unknown. It is towards developing a better understanding of these connected issues that this paper is addressed, for these are the features which ultimately reflect the non-linearities of the worldsheet theory and which contain much of the new physics of the fractional superstring. While we certainly do not have complete resolutions to these puzzles, our results provide the first clues concerning both the effective worldsheet conformal field theory which survives the internal projections in each of the fractional superstring sectors, and the spacetime statistics properties of the surviving states.

In particular, our main results may be briefly summarized as follows. Whereas the original worldsheet conformal field theory of the $K$-fractional superstring in light-cone gauge has central charge $c = 48/(K + 2)$ and consists of a tensor product of $D_c - 2 = 16/K$ coordinate bosons tensored together with $16/K$ copies of the $Z_K$ parafermion theory,

$$\text{CFT} = \left( D_{c=2=16/K} X^\mu \right) \otimes \left\{ D_{c=2=16/K} \otimes_{\mu=1} (Z_K \text{PF})^\mu \right\}, \quad (1.2)$$

we find that the internal projections reduce this theory down to the smaller $c = 24/K$ conformal field theory

$$\text{new CFT} = \left( D_{c=2=16/K} X^\mu \right) \otimes \left( c = \frac{8}{K} \text{ theory} \right) \quad (1.3)$$

where this $c = 8/K$ component is a certain non-tensor-product theory whose central charge, highest weights, fusion rules, and characters render it isomorphic to a tensor product of $8/K$ bosons compactified on circles of certain radii (or to a single $c = 1/2$ Ising model in the $K = 16$ case). Specifically, this means for each relevant value of $K$, these $c = 8/K$ post-projection theories have the same central charges, highest
weights, fusion rules, and characters as those of $8/K$ free compactified bosons — even though (as we shall demonstrate) the post-projection CFT's cannot be represented in this simple manner. Moreover, we find that this “isomorphism” between our post-projection theories and these compactified-boson theories holds for all sectors of the fractional superstring, including the extra massive $B$-sectors, and that the only difference between these new massive sectors and the more traditional Neveu-Schwarz-like and Ramond-like sectors is an apparent change in the compactification radius of the bosons in the isomorphic theory. We also analyze the so-called parafermionic “twist current” which plays a crucial role in reorganizing the Fock space during the internal projections, and surprisingly find that this current can be represented as a certain primary field in the resulting post-projection theory. This then enables us to identify some of the spacetime statistics properties of the states surviving the internal projections, and to express all of our results in terms of an effective compactification lattice.

This paper is organized as follows. In Section 2 we first provide a non-technical overview of these two fundamental issues which confront the fractional superstring, followed by a more technical introduction in which we discuss the appearance of these new features and summarize some recent results upon which our work is based. We then proceed in Section 3 to examine the effects of these new internal projections on the better-understood Ramond- and Neveu-Schwarz-like sectors of the fractional superstring (the so-called “$A$-sectors”), and deduce many of the properties of the new $A$-sector conformal field theories which emerge after these internal projections have acted. In particular, we find that we are able to explicitly construct a mapping between the sectors of the pre-projection and post-projection conformal field theories in the $A$-sectors, and this allows us to obtain a set of minimal constraints (i.e., the central charges, highest weights, fusion rules, and characters) that these post-projection conformal field theories must satisfy. In Section 4 we repeat our analysis for the $B$-sectors, and in Section 5 we demonstrate that the post-projection CFT’s for both the $A$- and $B$-sectors closely resemble those of free worldsheet compactified bosons. We then rewrite our results in such a way that this isomorphism is manifest, and in Section 6 we use this reformulation to analyze the parafermionic “twist current”. In particular, we will find that we can express this twist current directly in language of the isomorphic compactified-boson theory, and this in turn will enable us to understand some of the spacetime statistics properties of the fractional-superstring sectors. We then close in Section 7 with a summary of our results, and with comments regarding the possibility of constructing a unified worldsheet conformal field theory capable of simultaneously describing all of the post-projection fractional-superstring sectors.

Our primary motivation is to discover various properties of the post-projection CFT’s in both the $A$- and $B$-sectors, and as such our goals are two-fold. First, we seek to demonstrate that the new $B$-sectors are consistent with the internal projections, and from the vantage point of the post-projection theory, we will see that these new sectors closely resemble (rather than differ from) the $A$-sectors whose properties
are better understood. This in itself should be of great importance in ultimately demonstrating the consistency of these new string theories. We will also find, however, that we cannot yet construct suitable representations for these post-projection CFT’s in either the $A$- or $B$-sectors, for some technical issues having to do with spacetime statistics remain as yet unresolved. Our second goal, therefore, is to somewhat broadly set forth a set of minimal conditions that these CFT’s and their appropriate representations must ultimately satisfy. In doing so, we will be following almost exclusively the “top-down” approach discussed earlier, exploiting the partition-function evidence as much as possible in order to provide insight into these post-projection CFT’s. Our results can then hopefully serve as a guide in any future “bottom-up” construction.

2 Massive Sectors and Internal Projections

In Sects. 2.1 and 2.2 we first provide a non-technical overview of the two fundamental issues which currently confront the fractional superstring. Sect. 2.3 then contains a more technical review of fractional superstrings and their constituent parafermion theories.

2.1 Massive Sectors

Fractional superstrings (like ordinary bosonic strings, superstrings, and heterotic strings) have spacetime particle spectra consisting of various infinite towers of states: each tower represents the Fock space of states built upon a unique vacuum state, and each of these various vacuum states corresponds to a certain primary field in the underlying worldsheet conformal field theory. The spacetime (mass)$^2$ of each vacuum state is of course related to the highest weight of the corresponding primary field via $m^2 = h - c/24$ where $h$ is this highest weight and where $c$ is the central charge of the underlying conformal field theory; similarly, the states in each tower have values of $m^2$ differing from those of their vacuum state only by integers. We will refer to each of these towers as a (conformal field theory) “sector” of the theory, avoiding the more traditional definition of a string-theory “sector” in terms of the toroidal boundary conditions of worldsheet fields. What will interest us here is the appearance of new massive fractional-superstring sectors which have no analogues in ordinary superstrings.

In order to specify the sense in which these sectors are new, it proves instructive to recall the case of the ordinary superstring in $D = 10$. The underlying lightcone worldsheet conformal field theory of the usual superstring has central charge $c = 12$, and consists of a tensor product of eight free bosons and eight free Majorana fermions (each of the latter being equivalent to a $c = 1/2$ Ising model). While each of the eight bosonic CFT’s has only one sector (the identity sector with $h = 0$), each Ising-model factor has three sectors (the identity sector $[1]$ with $h = 0$, the fermion
sector $[\psi]$ with $h = 1/2$, and the spin-field sector $[\sigma]$ with $h = 1/16$). There are thus, \textit{a priori}, a total of $\binom{10}{2} = 45$ possible sectors in the superstring, where we are not distinguishing the \textit{order} of the eight Ising-model factors. Not all of these potential sectors contribute spacetime particles to the physical spectrum, however. As is well-known, the four vacuum states $1^7\psi, 1^5\psi^3, 1^3\psi^5,$ and $1^1\psi^7$ are the so-called Neveu-Schwarz (NS) vacuum states which contribute to the superstring spectrum, and the four towers of states respectively built upon these vacua together comprise the so-called NS sector of the theory. All particles in these sectors are spacetime bosons, and have $m^2 \geq 0$ with $m^2 \in \mathbb{Z}$. (By contrast, the five complementary vacuum states $1^8, 1^6\psi^2, \ldots, \psi^8$, along with their infinite towers of states, do not appear in the superstring spectrum and are said to have been removed by the GSO projection.) Likewise, there is a fifth vacuum state which contributes to the spectrum of the ordinary superstring: this is the massless Ramond vacuum state $\sigma^8$, with all states in the corresponding tower comprising the so-called Ramond sector of the theory. All excitations in this sector are spacetime fermions with $m^2 \in \mathbb{Z}$, and these are in fact the spacetime superpartners of the particles arising in the four NS sectors (thus rendering the ordinary superstring spacetime-supersymmetric). The crucial observation, however, is that none of the remaining 35 potential “mixed” $1/\sigma, \psi/\sigma$, or $1/\psi/\sigma$ vacuum-state combinations contributes to the physical spectrum of states of the ordinary $D = 10$ superstring.

For the more general fractional superstrings with $K > 2$, this is no longer the case: there are a variety of fundamentally new sectors which contribute states to the spacetime spectrum and which must therefore be considered. For each value of $K \geq 2$, the light-cone worldsheet conformal field theory of the fractional superstring consists of tensor products of $D_c - 2 = 16/K$ pairs of free bosons and $\mathbb{Z}_K$ parafermions, and for $K > 2$ these $\mathbb{Z}_K$ parafermion theories contain in principle \textit{many} different independent sectors. These sectors still fall, however, into certain groups: the first group (to be collectively denoted $\{1\}$) contains the higher-$K$ analogues of the Ising-model sectors $[1]$ and $[\psi]$, the second group (to be collectively denoted $\{\sigma\}$) contains the analogues of the Ising-model sector $[\sigma]$, while a third group (to be denoted $\{\phi\}$) contains those parafermionic sectors having no analogues at all in the $K = 2$ special case. It is thus possible to classify the resulting light-cone tensor-product sector combinations which actually contribute to the spacetime spectrum of the fractional superstring, distinguishing those which either are or are not analogues of those encountered in the ordinary superstring. First, for example, there are various the NS-like combinations $\{1\}^{D_c-2}$: those which have integer values of $m^2$ again contribute to the resulting physical spectrum, while the others experience a GSO projection and do not appear. Second, there are the various Ramond-like combinations $\{\sigma\}^{D_c-2}$: those having integer values of $m^2$ again contribute to the spacetime spectrum, while the others are projected away. These NS and Ramond sectors, for

\footnote{Note that since this worldsheet theory is a tensor product, spacetime Lorentz invariance implies invariance under permutations of the transverse dimensions.}
example, together yield the massless supergravity multiplet in the fractional superstring, and are in fact complete superpartners of each other at all mass levels. These are the so-called “A-sectors” of the fractional superstring.

There are, however, two other kinds of sectors which contribute to the fractional-superstring spectrum. The first (the so-called “B-sectors”) are the higher-$K$ analogues of the 35 “mixed” superstring sectors: these all turn out to have the equally-mixed form $(\{1\}\{\sigma\})^{(D_c - 2)/2}$, and contain only states with masses $m^2 \in \mathbb{Z} + \frac{1}{2}$ with $m^2 > 0$ (i.e., states at the Planck scale). The second class consists of sectors (the so-called “C-sectors”) built exclusively from the individual parafermionic $[\phi]$ sectors: these new sectors all take the form $\{\phi\}^{D_c - 2}$, and thus have no analogues in the ordinary superstring. Like the above “mixed” sectors, however, they also a priori contribute states to the fractional-superstring spectrum, with masses $m^2 \in \mathbb{Z} + \frac{3}{4}$ with $m^2 > 0$. Determining the spacetime statistics of the particles in these new sectors is highly non-trivial, for the vacuum states upon which these sectors are built are not of the standard NS or Ramond variety. These $B$- and $C$-sectors are the “massive sectors” whose physical properties we seek to understand.

### 2.2 Internal Projections

The second fundamental issue which has remained unresolved concerns the appearance of new types of “internal projections” which, like the GSO projection discussed above, prevent certain states in the Fock space of the worldsheet CFT from appearing in the actual physical spacetime spectrum. These new internal projections are, however, quite different from the GSO projections. As we have seen, the GSO projections remove states from the physical spectrum only by eliminating the contributions from entire towers of states: any given tower, including the vacuum state as well the infinite Fock space of states it generates, will either fully contribute to the physical spectrum or suffer a complete GSO projection and not appear at all. Indeed, such an all-or-nothing, tower-by-tower projection is the only way in which the resulting spacetime spectrum can still be consistent with the underlying conformal field theory that gave rise to it, for any sectors which survive the GSO projection are guaranteed to be the intact highest-weight sectors of that underlying conformal field theory.

In the fractional superstring theories, however, a new type of “internal projection” appears whose action is far more subtle. Rather than project out entire towers of states, these new projections project away only some of the states in each individual tower, leaving behind a set of states which clearly cannot be interpreted as the complete Fock space of the original underlying worldsheet conformal field theory. On the face of it, this would seem to render the spacetime spectra of the fractional superstrings hopelessly inconsistent with any underlying worldsheet-theory interpretation.

Remarkably, however, evidence suggests that the residual states which survive the internal projections in each tower precisely recombine to fill out the complete
Fock space of a different underlying conformal field theory. Thus, whereas the GSO projection merely removed certain highest-weight sectors of the worldsheet conformal field theory, these new internal projections appear to actually change the underlying conformal field theory itself. This turns out to be a profound alteration. Since the central charges of the new (post-projection) conformal field theories are smaller than those of the original tensor-product parafermion theories, these internal projections must clearly remove exponentially large numbers of states from each of the mass levels of the original Fock space; indeed, it is only in this drastic yet highly non-trivial manner that the fractional-superstring spacetime spectrum can remain consistent with an underlying two-dimensional worldsheet theory interpretation. Such a delicate projection clearly has no analogue in the ordinary superstring, and perhaps more closely resembles the BRST projection which enables unitary minimal models to be constructed from free bosons in the Feigin-Fuchs construction.

Verifying that the internal projection indeed leaves behind a self-consistent Fock space is of course a difficult task, and to date the evidence for this has only been obtained via the “top-down” approach — i.e., through the partition functions and the implied degeneracies of states. In fact, it is only in this manner that the internal projections are evident; there does not presently exist any formulation of these internal projection in terms of, for example, a projection operator constructed out of world-sheet fields. Therefore, we too shall be forced to follow this “top-down” approach, and indeed we will see that this is sufficient to determine the central charges, highest weights, fusion rules, and characters of the light-cone worldsheet conformal field theory that remains after the internal projections. Actually constructing a suitable representation for this conformal field theory in terms of worldsheet fields remains an open question, however, and we shall discuss some of the difficulties at various points throughout this paper.

2.3 Technical Review of Fractional Superstrings and Parafermions

We now provide a more technical review of fractional superstrings and their constituent parafermion theories, stressing only those aspects which will be necessary for later sections. Complete details concerning the basic ideas behind fractional superstrings can be found in Refs. [1–4].

We begin by outlining some basic facts concerning the $\mathbb{Z}_K$ parafermion theories [10] which are ultimately the building blocks of the fractional superstrings. The $\mathbb{Z}_K$ parafermion theory can be defined as the coset theory $SU(2)_K/U(1)$ derived from the $SU(2)_K$ Wess-Zumino-Witten theory [11] after modding out by a free $U(1)$ boson, but for our purposes we can simply think of these $\mathbb{Z}_K$ parafermion theories as a set of conformal field theories with central charges

$$c_\phi = \frac{2K - 2}{K + 2},$$

(2.1)

and with primary fields $\phi^I_m$ labeled and organized by their $SU(2)$ quantum numbers.
$j$ and $m$, where $j \in \mathbb{Z}/2$, $j - m \in \mathbb{Z}$, $|m| \leq j$, and $0 \leq j \leq K/2$. These theories thus contain only a finite number of fundamental fields, this number growing with increasing $K$. We shall generally define $\ell \equiv 2j$ and $n \equiv 2m$, in terms of which the conformal dimensions (or highest weights or spins) of the fields $\phi^j_m$ are given by

$$h^\ell_n \equiv \Delta[\phi^j_m] = \frac{\ell(\ell + 2)}{4(K + 2)} - \frac{n^2}{4K} \quad (2.2)$$

and their fusion rules are given by

$$[\phi^j_{m_1}] \otimes [\phi^j_{m_2}] = \sum_{J = |j_1 - j_2|}^{J_{\text{max}}} [\phi^J_{m_1 + m_2}] \quad (2.3)$$

with $J_{\text{max}} \equiv \min(j_1 + j_2, K - j_1 - j_2)$. Here the sectors $[\phi^j_m]$ include the primary fields $\phi^j_m$ and their descendants. The presence of the “reflection” symmetry $j \leftrightarrow K/2 - j$ allows us to consistently identify the fields $\phi^j_m$ for values outside the range $|m| \leq j$ via

$$\phi^j_m = \phi^j_{m + K} = \phi^{K/2 - j}_{-(K/2 - m)} \quad (2.4)$$

Note that according to (2.3) and (2.4), those fields with $j, m \in \mathbb{Z}$ constitute a closed subalgebra for even $K \geq 2$.

The complete worldsheet field content of the fractional superstring is then taken to consist of a tensor product of $D_c$ copies of this $\mathbb{Z}_K$ parafermion conformal field theory, tensored together with $D_c$ uncompactified (coordinate) bosons $X^\mu$. Indeed, there exists a fractional supersymmetry relating each coordinate boson to its corresponding $\mathbb{Z}_K$ parafermion theory, with fractional supercurrent $J = \epsilon^\mu \partial X^\mu + \eta$. Here $\epsilon \equiv \phi^1_0$, and $\eta$ is defined to be that parafermion descendent of $\epsilon$ which appears in the $\epsilon^\mu(z) \epsilon_\mu(w)$ OPE with conformal dimension $1 + h^0_0$. The spacetime Lorentz indices $\mu = 0, 1, ..., D_c - 1$ are understood to be contracted with the Minkowski metric. The critical dimensions $D_c$ of these theories depend on $K$, and can be determined by a variety of arguments (see Refs. [1, 2] for details) yielding the result quoted in (1.1).

Our cases of interest are thus $K = 2, 4, 8, \text{and} \ 16$ (yielding $D_c = 10, 6, 4, \text{and} \ 3$ respectively).

With this tensor-product formulation, it is easy to see that the fractional superstring reduces to the ordinary superstring in the special case $K = 2$. The $\mathbb{Z}_2$ “parafermion” theory has central charge $c_\phi = 1/2$, and contains three independent fields $\{\phi^0_0, \phi^1_0, \phi^{1/2}_0\}$ with conformal dimensions $h = 0, 1/2, \text{and} \ 1/16$ respectively. Thus the $K = 2$ special case of the $\mathbb{Z}_K$ parafermion theory is readily identified as the Ising model, with the above three “parafermion” fields identified respectively as the identity $1$, the Majorana fermion field $\psi$, and the spin field $\sigma$. (A fourth field $\phi^{1/2}_{-1/2}$

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* This tensor-product formulation is to be distinguished from the “chiral algebra” formulation of fractional superstrings [12]; a summary of the possible relation between the two can be found in [1] and [3].
corresponds to the conjugate spin field $\sigma^\dagger$.) The worldsheet fractional supercurrent then reduces to ordinary supercurrent $J = \psi^\mu \partial X^\mu$, since $\phi_0^1 = \psi$ and no such parafermion field $\eta$ exists in the $\mathbb{Z}_2$ theory. The NS (spacetime bosonic) sectors of the superstring of course have vacuum states corresponding to the primary fields $\phi_0^0 = 1$ or $\phi_0^1 = \psi$ in each direction, and the Ramond (spacetime fermionic) sectors correspond only to the spin-field $\phi_{\pm 1/2}^{1/2} = \sigma$ in each direction. Excitations in all sectors are affected via $\phi_0^1 = \psi$, which (according to the fusion rules) does not mix NS or Ramond sectors with each other.

In analogous fashion, certain fields in the $\mathbb{Z}_K$ parafermion theories play special roles in the fractional superstring. As is clear from the fusion rules (2.3), the $\phi_0^0$ field continues to function as the identity for all $K \geq 2$, whereas $\phi_{\pm K/4}^{K/4}$ fields are the $K > 2$ analogues of the Ising-model spin field $\sigma$. The $\epsilon \equiv \phi_0^1$ field is the analogue of the Ising-model field $\psi$, and thus serves to generate Fock-space excitations in the fractional superstring. It is therefore possible to group many of the parafermion fields into classes depending on the spacetime statistics of the vacua they produce. All of the fields which close into each other under repeated fusings of $\epsilon$ with itself and with the identity $\phi_0^0$ correspond to the various NS-like subsectors of the theory, and all states in the towers built upon these vacua are spacetime bosonic. From the fusion rules (2.3), we see that this set of NS-like primary fields $\phi_m^j$ are those with $m = 0$; these were the fields collectively denoted as $\{1\}$ in Sect. 2.1, producing the various light-cone NS vacua $\{1\}_{Dc-2}$. Similarly, the fields which close into each other under repeated fusings of $\epsilon$ with the spin fields $\phi_{\pm K/4}^{K/4}$ correspond to the Ramond-like subsectors of the fractional superstring, yielding towers of states which are spacetime fermionic. These fields are of course those with $m = \pm K/4$, denoted collectively as $\{\sigma\}$ in Sect. 2.1 and producing the various transverse Ramond-like vacua $\{\sigma\}_{Dc-2}$. Note that since the $m$-quantum numbers of these fields are additive modulo $K/2$ under the fusion rules (2.3), this assignment reproduces the expected spacetime boson/fermion selection rules:

$$B \otimes B = B, \quad B \otimes F = F, \quad F \otimes F = B.$$  \tag{2.5}

Obtaining (2.5) is an important consistency check on our identification of the spacetime statistics of our parafermion states.

In order to most directly observe the presence of new sectors and projections in these fractional-superstring theories for $K > 2$, let us construct their one-loop partition functions $Z_K(\tau)$. We shall concentrate on only those closed string theories in which both the left- and right-moving theories exhibit a $K$-fractional supersymmetry [this is the so-called $(K,K)$ fractional superstring, the generalization of the ordinary Type-II superstring]. As usual, each worldsheet coordinate boson field $X^\mu$ contributes to the partition function $Z(\tau)$ a factor

$$Z_{\text{boson}} = \frac{1}{\sqrt{\tau_2} |\eta|^2},$$  \tag{2.6}

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where $\eta(\tau) \equiv q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ is the Dedekind $\eta$-function with $q \equiv \exp(2\pi i \tau)$ and $\tau_2 \equiv \text{Im} \tau$; similarly, the contribution from each (chiral) parafermion field $\phi^\ell_m$ is given by [13]

$$(Z^j_m)_{\text{parafermion}} = \eta c^2m$$  \hspace{1cm} (2.7)$$

where the $c^\ell_n(\tau)$ are the so-called “string functions” [14]:

$$c^\ell_n(\tau) = q^{h^\ell_n+[4(K+2)]^{-1}} \sum_{r,s=0}^{\infty} (-1)^{r+s} q^{r(r+1)/2+s(s+1)/2+rs(K+1)} \times$$

$$\times \left\{ q^{r(j+m)+s(j-m)} - q^{K+1-2j+r(K+1-j-m)+s(K+1-j+m)} \right\}$$  \hspace{1cm} (2.8)$$

and where the $h^\ell_n$ are the highest weights given in (2.2). We thus see that the string functions take the form $c^\ell_n = q^{H^\ell_n}(1 + \ldots)$ where only non-negative integer powers of $q$ appear within the parentheses and where

$$H^\ell_n = h^\ell_n - \frac{1}{24} (1 + c_\phi) .$$  \hspace{1cm} (2.9)$$

The string functions exhibit the symmetries

$$c^\ell_n = c^\ell_{-n} = c^{K-\ell}_{-n} = c^{\ell+2K}_n ;$$  \hspace{1cm} (2.10)$$

and one conventionally defines the useful combinations

$$d^\ell_n \equiv c^\ell_n + c^{K-\ell}_n$$  \hspace{1cm} (2.11)$$

for $K \in 4\mathbb{Z}$ and $\ell, n \in 2\mathbb{Z}$. These string functions $c^\ell_n$ can be viewed, of course, as $K > 2$ generalizations of the Ising-model characters $\chi_0$, $\chi_{1/2}$, and $\chi_{1/16}$, and indeed for $K = 2$ we find

$$K = 2 : \hspace{1cm} \eta c^0_0 = \chi_0 = \frac{1}{2} \left( \sqrt{\frac{\vartheta_3}{\eta}} + \sqrt{\frac{\vartheta_4}{\eta}} \right)$$

$$\eta c^2_0 = \chi_{1/2} = \frac{1}{2} \left( \sqrt{\frac{\vartheta_3}{\eta}} - \sqrt{\frac{\vartheta_4}{\eta}} \right)$$

$$\eta c^1_1 = \chi_{1/16} = \sqrt{\frac{\vartheta_2}{2\eta}} .$$  \hspace{1cm} (2.12)$$

We have written these Ising-model characters in terms of the three Jacobi $\vartheta$-functions for later convenience.

From (2.6) and (2.7), therefore, we expect the total partition function for the $(K,K)$ fractional superstring to take the form

$$Z^j_K(\tau) = \tau_2^{1-Dc/2} \sum (c)^{Dc-2}(c)^{Dc-2} = \tau_2^{1-Dc/2} \sum_{m,n} a_{mn} q^m q^n$$  \hspace{1cm} (2.13)$$
where the first summation is over all of the (as-yet undetermined) contributing sectors of the theory, and where ‘c’ schematically represents the corresponding string-function factors. The quantities \( a_{mn} \) will then represent the net number of states (spacetime bosonic minus fermionic) with left- and right-energies \( m \) and \( n \) respectively. In order
to determine the contributing sectors, we simply demand combinations of string functions in (2.13) which not only render \( Z_K(\tau) \) modular invariant, but also yield \( a_{mn} = 0 \) for all \( n < 0 \) (absence of physical tachyons) and \( a_{00} \neq 0 \) for at least one sector (corresponding to the existence of a graviton in the spacetime spectrum). It turns out that this is sufficient\(^1\) to yield the following set of unique solutions [1–3] for each relevant value of \( K \geq 2 \):

\[
\begin{align*}
Z_2 &= \tau_2^{-4} |A_2|^2 \\
Z_4 &= \tau_2^{-2} \left\{ |A_4|^2 + 3 |B_4|^2 \right\} \\
Z_8 &= \tau_2^{-1} \left\{ |A_8|^2 + |B_8|^2 + 2 |C_8|^2 \right\} \\
Z_{16} &= \tau_2^{-1/2} \left\{ |A_{16}|^2 + |C_{16}|^2 \right\}
\end{align*}
\]

with

\[
\begin{align*}
A_2 &= 8 (c_0^0)^7 (c_0^0) + 56 (c_0^0)^5 (c_0^3) + 56 (c_0^0)^3 (c_0^5) + 8 (c_0^0)(c_0^7) - 8 (c_1^0)^8 \\
A_4 &= 4 (c_0^0 + c_0^4)^3 (c_0^3) - 4 (c_0^2)^4 - 4 (c_2^4)^3 (c_0^2)^3 \\
B_4 &= 16 (c_0^0 + c_0^4)(c_0^3)(c_0^4)^2 + 8 (c_0^0 + c_0^4)^2 (c_2^4)(c_2^4) - 8 (c_0^3)^2 (c_2^2)^2 \\
A_8 &= 2 (c_0^0 + c_0^8)(c_0^2 + c_0^8) - 2 (c_0^4)^2 - 2 (c_4^8)^2 + 8 (c_0^4 c_4^8) \\
B_8 &= 4 (c_0^0 + c_0^8)(c_0^4) + 4 (c_0^2 + c_0^8)(c_0^4) - 4 (c_0^4 c_4^8) \\
C_8 &= 4 (c_0^2 + c_0^8)(c_0^2 + c_0^8) - 4 (c_2^4)^2 \\
A_{16} &= c_0^2 + c_0^{14} - c_0^8 + c_8^2 + 2 c_8^{14} \\
C_{16} &= 2 c_2^2 + 2 c_4^{14} - 2 c_8^4.
\end{align*}
\]

Note from (2.3) that these combinations of string functions each a priori have \( q \)-
expansions of the forms

\[
A_K \sim q^K (1 + \ldots) , \quad B_K \sim q^{K/2} (1 + \ldots) , \quad C_K \sim q^{3K/4} (1 + \ldots) ,
\]

from which we deduce that only the \( A \)-type sectors contain massless states. Indeed, the string-function combinations \(+ (D_c - 2)(c_0^0)^{D_c - 3}(c_0^2)\) and \(- (D_c - 2)(c_K^{K/2})^{D_c - 2}\)
within each of the \( A_K \) combinations above correspond to the (chiral) massless NS and

\(^1\) In the original derivations (see [1–3]), what is actually imposed is the stronger no-tachyon condition \( a_{mn} = 0 \) if \( m < 0 \) or \( n < 0 \). This condition thus prevents the appearance of not only physical (on-shell) tachyons, but also unphysical (or off-shell) tachyons — indeed, such a restriction is required for physically sensible theories of the Type-II [or \( (K, K) \)] variety. It is this stronger condition which is ultimately responsible for both the vanishing and the uniqueness of the partition functions obtained.
Ramond vacuum states respectively; note that they appear with the appropriate signs and normalizations for bosonic spacetime vectors and fermionic spacetime spinors. It is by tensoring the left- and right-moving (or holomorphic and anti-holomorphic) sectors together that these chiral states produce a massless $N = 2$ supergravity multiplet.

For $K = 2$, we can re-express $A_2$ in terms of the equivalent $\vartheta$-functions to find that $A_2 = \frac{1}{4} \eta^{-12} J$ where $J \equiv \vartheta_3^4 - \vartheta_2^4 - \vartheta_4^4$; thus $Z_2$ is indeed recognized as the partition function of the ordinary superstring, and the Jacobi identity $J = 0$ (or $A_2 = 0$) indicates the vanishing of $Z_2$ (i.e., the exact level-by-level cancellation of bosonic and fermionic states). This is of course the partition-function reflection of the spacetime supersymmetry of this theory. Remarkably, however, this property extends to higher $K$ as well, for it can be proven [3] that each of the combinations listed in (2.13) vanishes identically as a function of $q$:

$$A_K = B_K = C_K = 0 \quad \text{for all } K \geq 2 \ .$$

(2.17)

These resulting new identities, which are the higher-$K$ generalizations of the $K = 2$ Jacobi identity $A_2 = 0$, can therefore be taken as evidence of spacetime supersymmetry in the fractional superstrings. This is of course consistent with the appearance of the massless supergravity multiplet in the fractional superstring spectrum, as noted above.

We have now reached the point where the two fundamental issues confronting us are evident. Recalling (2.7) and the discussion above (2.5), we see that it is a straightforward matter to recognize whether each term within each $A_K$ arises from a sector which has a spacetime bosonic (NS) vacuum state of the form

$$\text{NS} : \prod_{i=1}^{D_{c-2}} \phi^{j_i}_0 \ ,$$

or arises from a spacetime fermionic (Ramond) vacuum state of the form

$$\text{R} : \prod_{i=1}^{D_{c-2}} \phi^{j_i}_{\pm K/4} \ .$$

(2.18)

Thus, we can separate each of these expressions $A_K$ into cancelling bosonic and fermionic pieces

$$A_K = A^b_K - A^f_K \ ,$$

(2.20)

where

$$K = 2 : \ A^b_2 \equiv 8(c_0^1) (c_0^2) + 56(c_0^0)(c_0^2)^5 + 56(c_0^0)(c_0^3)^5 + 8(c_0^0)(c_0^3)^7$$

$$A^f_2 \equiv 8(c_1^1)^8$$

$$K = 4 : \ A^b_4 \equiv 2(d_0^0)^3 d_0^2 - \frac{1}{4} (d_0^3)^4$$

$$A^f_4 \equiv (d_0^1)^8$$

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\[ A^f_4 \equiv \frac{1}{4}(d_2^4) - 2(d_2^0)^3 d_2^2 \]

For \( K = 8 \):
\[ A^b_8 \equiv 2d_0^0 d_0^2 - \frac{1}{2}(d_0^4)^2 \]
\[ A^f_8 \equiv \frac{1}{2}(d_4^1)^2 - 2d_0^2 d_4^2 \]
\[ K = 16 : \]
\[ A^b_8 \equiv d_0^2 - \frac{1}{2}d_0^8 \]
\[ A^f_8 \equiv \frac{1}{2}d_8^8 - d_8^2 . \quad (2.21) \]

For \( K > 2 \), we have made use of the identities (2.10) and written \( A^b,f_4^K \) in terms of the \( d_n^\ell \) combinations (2.11).

For the \( B \)- and \( C \)-type sectors, however, the situation is not as clear. It is readily seen from (2.15) that the \( B \)- and \( C \)-type terms arise from sectors with vacuum states corresponding to primary-field combinations of the forms

\[ B \equiv \prod_{i=1}^{(D_n-2)/2} (\phi^0_0 \phi^k_\pm K/4) \]
\[ C \equiv \prod_{i=1}^{D_n-2} \phi^k_\pm K/8 , \quad (2.22) \]

and thus these sectors have vacua which do not even appear in the ordinary superstring (indeed, no \( B_2 \) or \( C_2 \) terms appear in the \( K = 2 \) partition function). These \( B_K \) and \( C_K \) terms do appear, however, in the partition functions for \( K > 2 \), forced upon us by modular invariance. Although (2.16) indicates that these sectors contain only massive (i.e., Planck-scale) states, arguments suggest \[3\] that these sectors play a crucial role in the ultimate consistency of the fractional superstring. It is therefore necessary, as a first step, to distinguish those \( B \) and \( C \)-sector states which are bosonic and fermionic — i.e., to achieve a splitting of the \( B_K \) and \( C_K \) partition-function expressions into cancelling terms in analogy to that indicated in (2.20) for the \( A_K \)-type sectors. Recent results \[1, 7\] indicate, however, that the appropriate \( B_K \)-sector splitting is as follows:

\[ B^b_4 \equiv 2d_0^0 d_0^2 d_0^1 d_2^1 - \frac{1}{4}(d_0^2 d_2^2)^2 = 4q^{1/2}(1 + ...) \]
\[ B^f_4 \equiv \frac{1}{4}(d_2^2 d_0^2)^2 - 2d_0^2 d_0^4 d_0^1 d_0^2 = 4q^{1/2}(1 + ...) \]
\[ B^b_8 \equiv 2d_0^0 d_0^4 - \frac{1}{2}d_0^4 d_1^4 = 2q^{1/2}(1 + ...) \]
\[ B^f_8 \equiv \frac{1}{2}d_1^4 d_0^4 - 2d_0^2 d_0^8 = 2q^{1/2}(1 + ...) , \quad (2.23) \]

whereas the corresponding \( C_K \)-sector splitting indicates that these sectors contain no physical states whatsoever:

\[ C^b_8 \equiv 2d_0^0 d_0^2 - \frac{1}{2}(d_2^2)^2 = 0 \]
\[ C^f_8 \equiv \frac{1}{2}(d_4^1)^2 - 2d_0^2 d_2^2 = 0 \]
\[ C^b_{16} \equiv d_2^2 - \frac{1}{2}d_1^2 = 0 \]
\[ C^f_{16} \equiv \frac{1}{2}d_8^8 - d_8^2 = 0 . \quad (2.24) \]
This absence of states will be discussed below and in Sect. 5.3. In Ref. [7] a proposal was made whereby the $B_K$-sector splitting in (2.23) could be understood in terms of the ordering of the parafermion $m$-quantum numbers — e.g.,

\[
\text{bosonic } \iff \quad m_i = (0, K/4, 0, K/4, \ldots, 0, K/4) \\
\text{fermionic } \iff \quad m_i = (K/4, 0, K/4, 0, \ldots, K/4, 0) \tag{2.25}
\]

where each factor (0 or $K/4$) is repeated $(D_c - 2)/2 = 8/K$ times. Indeed, this assignment reproduces the proper statistics selection rules (2.5) not only within the $B$-sectors, but also between the $A$- and $B$-sectors. The necessity for such an ordered assignment, however, illustrates that the $B$-sectors appear to break permutation invariance amongst the $D_c - 2$ different copies of the fundamental boson plus $\mathbb{Z}_K$ parafermion theory. This loss of permutation invariance is thus inconsistent with the original formulation of the fractional superstring in which the worldsheet conformal field theory is taken to be a tensor product.

This problem is connected with the second fundamental issue, the “internal projections”. The expressions in (2.21) are written in terms of the string functions $c_n^\ell$ and $d_n^\ell$, each of which is the character of a particular sector of the joined boson plus parafermion system. Hence these expressions serve as a means of tallying the degeneracies $g_n^{b,f}$ of spacetime bosonic or fermionic states at each mass level $n$ in the parafermion plus boson Fock space:

\[
A_K^{b,f} = \sum_{n=0}^{\infty} g_n^{b,f} q^n. \tag{2.26}
\]

As such, it is easy to interpret the expressions $A_2^b$ and $A_2^f$ for the $K = 2$ superstring case: the summations within each of these separate expressions in (2.21) simply represent the added contributions from each of the bosonic or fermionic sectors in the theory. For $K > 2$, however, such an interpretation for each $A_K^b$ and $A_K^f$ becomes more difficult, since the contributions of certain parafermionic sectors appear to be subtracted rather than added. We shall refer to those parafermionic sectors whose contributions are subtracted as “internal projection sectors”; note that their presence is, like that of the $B$- and $C$-sectors, forced upon us by modular invariance. It is straightforward to verify that these internal projection sectors themselves contain only massive (i.e., Planck-scale) states, so the subtractions they introduce into the net state degeneracies $g_n^{b,f}$ in (2.26) appear only for $n \geq 1$. Thus the states at the massless levels of the bosonic and fermionic $A$-sectors (including the supergravity multiplet) are unaffected.

The effects of these internal projections are nevertheless quite profound. The GSO projection, for example, is what has prevented certain sectors from ever appearing within our partition function expressions, so we may say that effectively an entire tower of states has been projected out of the spectrum. These internal projections, on the other hand, seem to subtract one sector from a different one, leaving behind a set
of state degeneracies $g_{n}^{b,f}$ in (2.26) which may or may not be physically sensible from a worldsheet conformal field theory point of view. It can easily be verified that despite the internal projection, the values of each $g_{n}^{b,f}$ within (2.26) are all non-negative, so it remains to discover whether the expressions $A_{K}^{b,f}$ can themselves be consistently understood as the characters (or even the sum of characters) of the highest-weight sectors of some new conformal field theory:

$$A_{K}^{b,f} = \sum_{h} \chi_{h}. \quad (2.27)$$

It turns out, however, that the expressions $A_{K}^{b,f}$ all pass this first non-trivial test, for it can be shown [3] that

$$A_{K}^{b,f}(\tau) = (D_{c} - 2) \left( \frac{\eta}{\vartheta_{2}(\tau)} \right)^{D_{c} - 2} \left[ \chi_{1/16} \right]^{D_{c} - 2} = (D_{c} - 2) \left[ \frac{\vartheta_{2}(\tau)}{2 \eta^{3}(\tau)} \right]^{(D_{c} - 2)/2} \quad (2.28)$$

where $\chi_{1/16}$ is the Ising-model character given in (2.12). For $K = 2$, of course, this relation is manifestly true as a consequence of (2.12). For $K > 2$, however, this is a truly remarkable result, indicating that despite our original conformal-field-theoretic formulation in terms of bosons and parafermions, the numbers of states surviving the internal projections at each mass level of the theory are precisely those of $D_{c} - 2$ copies of a single boson plus fermion, or $(X, \psi)$, theory! This in turn implies that exponentially large numbers of states are being projected out of the spectrum by these new internal projections for $K > 2$, for while the original boson/parafermion theory in light-cone gauge has total central charge

$$c = (D_{c} - 2) (1 + c_{\phi}) = \frac{48}{K + 2}, \quad (2.29)$$

the post-projection theory in light-cone gauge has the smaller central charge

$$c' = (D_{c} - 2) (1 + c_{\psi}) = \frac{24}{K}. \quad (2.30)$$

Indeed, only for $K = 2$ are these two central charges equal.

A similar situation exists for the $B$-sectors, for it is readily seen from (2.23) that analogous internal projections appear within the expressions $B_{K}^{b,f}$ for both $K = 4$ and $K = 8$. The coefficients within a $q$-expansion of these expressions are again all positive, however, and remarkably there again exists a simple identity [4] analogous to (2.28):

$$B_{K}^{b}(\tau) = B_{K}^{f}(\tau) = (D_{c} - 2) \left[ \frac{\vartheta_{2}(\lambda \tau)}{2 \eta^{3}(\tau)} \right]^{(D_{c} - 2)/2} \quad (2.31)$$

where

$$\lambda \equiv (\Delta[\epsilon])^{-1} = (h_{0}^{2})^{-1} = \frac{1}{2} (K + 2) = \begin{cases} 3 & \text{for } K = 4 \\ 5 & \text{for } K = 8. \end{cases} \quad (2.32)$$
Thus, whereas $A_{K}^{b,f}$ are related to the Dedekind $\eta$-function and Jacobi $\vartheta$-functions whose arguments were the usual torus modular parameter $\tau$, we find that our $B$-sector characters $B_{K}^{b,f}$ are given by similar expressions in which the arguments of the $\vartheta$-functions are now rescaled by factors of $\lambda$, where $\lambda$ is in general the inverse spin of the parafermion field $\epsilon \equiv \phi_{1}^{0}$ in the $\mathbb{Z}_{K}$ parafermion theory. As we shall see in Sect. 4, Eq. (2.31) indicates that the numbers of states surviving the internal projections at each mass level of the $B$-sectors of the theory are precisely those of $D_{c} - 2$ copies of a single boson plus fermion theory, where this “$B$-sector fermion” is now formulated on a rescaled internal momentum lattice. This momentum-lattice rescaling for the fermion does not alter its central charge contribution, however, and thus (2.31) also implies the post-projection central-charge reduction (2.30). This is of course necessary for the consistency of the internal projections in both the $A$- and $B$-sectors. We remark in passing that it is gratifying to observe the original parafermion spin re-emerging in so non-trivial a manner, for the internal projections in $A$-sector seemed, by projecting each copy of our fundamental parafermion theory down to the Ising model (i.e., to a free fermion), to have erased all knowledge of the fractional spins which were the original starting point in the fractional-superstring construction. We also note that these internal projections provide a natural explanation for the absence of physical states in the fractional-superstring $C$-sectors (2.24): these states are not GSO-projected, but rather internally projected out of the spectrum. This will be discussed in Sect. 5.3.

Despite the appearances of (2.28) and (2.31), we cannot simply declare that the post-projection conformal field theory for the $A_{K}$ and $B_{K}$ sectors is that of $D_{c} - 2$ coordinate bosons tensored together with $D_{c} - 2$ independent copies of the Ising model (or the “rescaled” Ising model); as we will see, the conformal field theory this would produce turns out to be far too large to correctly describe the fractional-superstring spectrum of states. Furthermore, as our above arguments suggest, this smaller post-projection CFT is not expected to have a simple tensor-product formulation. Therefore, in order to determine the correct post-projection conformal field theory for the $A$- and $B$-sectors, we must carefully map out the parafermionic sectors which survive the internal projections, and attempt to describe them in terms of the Fock space of some new Ising-like conformal theory with central charge (2.30). It is towards this endeavor that we now turn.

3 Internal Projections in the $A$-Sector

In this section we explore the conformal field theory (CFT) describing the $A$-sector of the fractional superstring in light-cone gauge after the internal projection. We will find that we can determine its spectrum of highest weights as well as its fusion rules, and although we cannot presently construct an adequate representation of this conformal field theory in terms of worldsheet fields, our analysis will provide a minimal set of constraints which this representation must satisfy and which can
hopefully serve as a guide towards its ultimate identification. Our starting point will be Eq. (2.28), and indeed by using this relation we have already seen that the central charge of this CFT must be given by (2.30). In this section we will more fully and systematically examine the consequences of the relation (2.28).

Our fundamental approach — and indeed our philosophy concerning these internal projections — can be described as follows. Although the pre-projection conformal field theory was uniquely identified (by construction) as a tensor product of $D_{c-2}$ free bosons and $D_{c-2}$ copies of the $\mathbb{Z}_K$ parafermion theory, the resulting internal projections indicate that the Fock space of this original CFT is far too large for describing what we know (via the “top-down” partition function analysis) to ultimately be the physical states in the fractional-superstring spectrum. Therefore, while our present description of the fractional superstring involves this large parafermionic worldsheet CFT in conjunction with an as-yet mysterious internal projection, it is expected (and in fact required for self-consistency) that there exist an alternative description starting directly with the smaller post-projection CFT of central charge (2.30) in which no internal projection appears. Such a description would clearly be preferable, for it is not presently known how the internal projection is to be implemented in terms of the original parafermion fields [4].

We begin by focusing on the spectrum of states of this post-projection CFT, for obtaining a clear description of its spectrum is a necessary prerequisite for its complete identification. Since this smaller CFT is in some sense embedded within the original larger parafermionic CFT, we expect its Fock space to be describable in two ways: as all of the states of the smaller Ising-like CFT, and as only certain selected states in the large parafermion CFT. In Sect. 3.1, therefore, we shall first introduce the Ising-like CFT which turns out (along with the $D_{c-2}$ coordinate bosons) to be the post-projection CFT for the $A$-sector. In Sect. 3.2 we shall then explicitly determine which of the sectors in the parafermion theory are those which ultimately survive this internal projection and yield this Ising-like CFT.

3.1 The Post-Projection CFT and the Ising Model

Our fundamental starting point, and indeed the sole indication of any relation between the $A$-sector post-projection CFT and the Ising model, comes from (2.28): this equation indicates that the post-projection CFT in the $A$-sector contains the Ramond spin state $\sigma^{D_{c-2}}$ whose character is $(\chi_{1/16})^{D_{c-2}}$. Since the $\sigma$ field is a primary field of the Ising model, it is straightforward to demonstrate that any such CFT containing the state $\sigma^{D_{c-2}}$ must also contain the identity “vacuum state" $1^{D_{c-2}}$ as well as the fermion-field state $\psi^{D_{c-2}}$. The first complication that we face, however, is that there in general exist many different self-consistent “Ising-like” conformal field theories which contain all three of these states.

To demonstrate this, let us consider the situations for different values of $D_c$. In the $K = 16$ case, we have $D_c - 2 = 1$, and thus it is of course clear that our light-cone
“Ising-like” CFT can be nothing but the Ising model itself, with three primary fields \( \{1, \psi, \sigma\} \) with highest weights \( h = \{0, 1/2, 1/16\} \) respectively, and with fusion rules of the general form

\[
[\varphi_1] \times [\varphi_1] = [1] \\
[\varphi_1] \times [\varphi_2] = [\varphi_2] \\
[\varphi_2] \times [\varphi_2] = [1] + [\varphi_1]
\]  

(3.1)

where we identify \( \{1, \varphi_1, \varphi_2\} \leftrightarrow \{1, \psi, \sigma\} \). For the \( K = 8 \) case, however, we have \( D_c - 2 = 2 \), and it turns out that there are two possible self-consistent CFT’s which incorporate the three states \( 1^2, \psi^2, \) and \( \sigma^2 \). The first such theory is a simple tensor product of the two Ising models: this tensor-product \( (\text{Ising})^2 \) theory has \( c = 1 \) and \( a \text{ priori} \) contains a total of \( 3 \times 3 = 9 \) “primary fields” \( \square \). The second such theory, however, is the so-called \( c = 1 \) Dirac-fermion CFT: this theory is a subset of \( (\text{Ising})^2 \), and represents the “diagonal” combination of two Ising models (as is particularly easy to see in a toroidal boundary-condition basis for the two Ising-model Majorana fermions). This Dirac-fermion CFT is equivalent to a free boson compactified on a circle of radius \( R = 1 \), and contains only the three primary fields corresponding to the three winding-mode sectors which exist at this radius. These primary fields, which we can denote \( 1, \varphi_1, \) and \( \varphi_2 \), have highest weights \( h = 0, 1/2, \) and \( 1/8 \) respectively, and this theory turns out to have the same fusion rules (3.1) as the Ising model itself. The relations between the three characters of the Ising model given in (2.12), and the three characters \( \chi^{(c=1)} \) of this Dirac-fermion CFT, are as follows:

\[
\chi^{(c=1)}_0 = (\chi_0)^2 + (\chi_{1/2})^2 \\
\chi^{(c=1)}_{1/2} = \chi_0 \chi_{1/2} \\
\chi^{(c=1)}_{1/8} = (\chi_{1/16})^2.
\]  

(3.2)

From these relations it is thus easy to see that the \( \sigma^2 \) state is contained within the \( h = 1/4 \) sector of the Dirac fermion theory, while the \( 1^2 \) and \( \psi^2 \) states are contained within the \( h = 0 \) (vacuum sector) of this theory.

A similar situation exists for the \( K = 4 \) case with \( D_c - 2 = 4 \). Here again there are several different Ising-like CFT’s containing the three states \( 1^4, \psi^4, \) and \( \sigma^4 \); examples include a tensor product of four \( c = 1/2 \) Ising models, or a tensor product of two \( c = 1 \) Dirac fermion theories. There exists again, however, an even smaller \( c = 2 \) diagonal combination of the two Dirac fermion theories which, like the \( c = 1 \) Dirac

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* For \( c \geq 1 \) CFT’s, there are of course an infinite number of fields which are primary with respect to the Virasoro algebra. We are here adopting a somewhat looser terminology when speaking of such \( c \geq 1 \) theories, grouping these primary fields according to their highest weights (mod 1), and referring to this collected set of primary fields (as well as all of their Virasoro descendants) as a single “sector” as though it were generated by a single primary field. The associated fusion rules and characters must then be understood in this context.
theory above, contains only three primary field sectors. These “fields” have highest weights \( h = \{0, 1/2, 1/4\} \) respectively, and this theory too has the fusion rules (3.1).

The characters corresponding to these three primary fields are given by

\[
\begin{align*}
\chi_0^{(c=2)} &= \left[\chi_0^{(c=1)}\right]^2 + \left[\chi_1^{(c=1)}\right]^2 = \frac{1}{2} \left[ (\chi_0 + \chi_{1/2})^4 + (\chi_0 - \chi_{1/2})^4 \right] \\
\chi_1^{(c=2)} &= \chi_0^{(c=1)} \chi_1^{(c=1)} = \frac{1}{8} \left[ (\chi_0 + \chi_{1/2})^4 - (\chi_0 - \chi_{1/2})^4 \right] \\
\chi_{1/4}^{(c=2)} &= \left[\chi_1^{(c=1)}\right]^2 = (\chi_{1/16})^4. \quad (3.3)
\end{align*}
\]

Once again we see that the \( \sigma^4 \) state is contained within the \( h = 1/4 \) sector of this theory, while the \( 1^4 \) and \( \psi^4 \) states are contained within the vacuum sector with \( h = 0 \).

Finally, even in the \( K = 2 \) superstring case, there are \textit{a priori} a variety of choices. The largest is of course a tensor product of eight Ising models, and the smallest, analogously, is again a certain \( c = 4 \) diagonal combination of the two \( c = 2 \) diagonal theories above. This latter theory again contains three primary field sectors, one with highest weight \( h = 0 \) and two with \( h = 1/2 \), and has the same fusion rules as the Ising model itself. Its three characters are given by

\[
\begin{align*}
\chi_0^{(c=4)} &= \frac{1}{2} \left[ (\chi_0 + \chi_{1/2})^8 + (\chi_0 - \chi_{1/2})^8 \right] \\
\chi_{1/2}^{(c=4)} &= \frac{1}{16} \left[ (\chi_0 + \chi_{1/2})^8 - (\chi_0 - \chi_{1/2})^8 \right] \\
\tilde{\chi}_{1/2}^{(c=4)} &= (\chi_{1/16})^8. \quad (3.4)
\end{align*}
\]

Note that in this case we now have \textit{two} characters with \( h = 1/2 \); these characters are in fact equal as functions of \( q \) as a result of the Jacobi identity \( \chi_{1/2}^{(c=4)} = \tilde{\chi}_{1/2}^{(c=4)} \).

Choosing the appropriate Ising-like CFT from among these various possibilities for each value of \( K \) is of paramount importance, for while they all share the same ground state for each value of \( K \), they each have drastically different sets of primary fields and fusion rules. Our guide shall be the \( K = 2 \) case, for in the ordinary \( D_c = 10 \) superstring we know precisely which Ising-like CFT is ultimately the one required for self-consistency: it is the smallest or “minimal” CFT mentioned above, containing only three sectors. In the usual parlance, the sector corresponding to \( \chi_{1/2}^{(c=4)} \) is of course the Ramond sector \( \sigma^8 \), while the two sectors corresponding to \( \chi_0^{(c=4)} \) and \( \chi_{1/2}^{(c=4)} \) are the NS(\( ^+ \)) sectors of the superstring with odd and even \( G \)-parities respectively. The NS(\( ^- \)) sector with odd \( G \)-parity of course contains spacetime tachyons (since it has \( h = 0 \)), and it is removed by the GSO-projection. Thus, from (3.4), we see that the subsectors \( 1^{\psi^4} D_c^{1-2-i} \) with even \( i \) are projected out of the spectrum (as we discussed in Sect. 2.1), while those with odd \( i \) remain. That the Ramond and NS(\( ^+ \)) sectors are are the only two surviving sectors of the superstring can be easily seen from the superstring partition function, which takes the simple form:

\[
Z_{K=2} = 64 \tau_2^{-4} |\eta|^{-16} \left| \chi_{1/2}^{(c=4)} - \tilde{\chi}_{1/2}^{(c=4)} \right|^2 = 0. \quad (3.5)
\]
Thus, while the ordinary superstring could equivalently be described in terms of any of the larger Ising-like $c = 4$ theories considered above, one would have to assume all of the extraneous sectors thus introduced to be “GSO-projected” out of the spectrum so that (3.5) and the corresponding minimal $c = 4$ theory are ultimately obtained. This $c = 4$ theory is of course nothing but the $SO(8)$ Wess-Zumino-Witten (WZW) model, with all states forming representations of the transverse Lorentz group $SO(8)$.

The fact that there are no mixed Ramond/NS sectors in the superstring means, as we have already noted, that it is straightforward to identify the spacetime spin-statistics of all of the states in the Fock space of the superstring, and indeed all states transform under $SO(8)$ with either integer or half-integer spins. As we have seen in Sect. 2.3, however, the analogous situation exists in the $A$-sectors of the fractional superstring: all of the states in the $A$-sectors arise from vacuum states of the NS or Ramond types given in (2.18) and (2.19) respectively, and have well-determined spacetime spin-statistics. Therefore, we shall assume that the appropriate $A$-sector light-cone CFT’s implied by (2.28) for each value of $K$ are the “minimal” or diagonal Ising-model combinations discussed above:

$$K \geq 2 : \quad \left( c' = \frac{24}{K} \right) \text{CFT} = \left\{ \left. D c_{-2} = 16/K \right|_{\mu=1} X^\mu \right\} \otimes \left\{ \left. c = \frac{8}{K} \right| \text{minimal theory} \right\};$$

(3.6)

indeed, these “minimal” Ising-like CFT’s are equivalent to $SO(D_{c-2})_1$ WZW models. These choices (3.6) are thus not only free of the mixed Ramond/NS sectors and in agreement with the ordinary superstring for the $K = 2$ special case, but are also (as we shall find below) the only ones consistent with the parafermionic CFT’s and their internal projections. We caution, however, that (3.6) does not indicate how this CFT is to ultimately be represented in terms of an appropriately chosen set of worldsheet fields so that a spacetime string theory with the correct spacetime statistics properties might be constructed. These issues will be discussed in Sects. 3.3 and 6.

### 3.2 The Post-Projection CFT in Parafermion Language

Given the post-projection CFT in (3.6), we now seek to understand how it arises as the result of internal projections between certain parafermion sectors in the original $c = 48/(K + 2)$ worldsheet theory.

As a consequence of this decreasing of the central charge induced by the internal projections, not all of the parafermion sectors which have played a role in the fractional superstring prior to the internal projection can be expected to survive to play an analogous role in the residual post-projection CFT. To illustrate this point, let us consider the lowest parafermion vacuum state in the $K = 4$ theory. This state $(\phi_0^0)^4$ is tachyonic, with $(\text{mass})^2 = -1/3$, and although this state (along with the entire Fock space of states built upon it) ultimately suffers a GSO-projection and does not appear in the modular-invariant partition function (2.14), this state still serves as the ultimate ground state of the parafermionic worldsheet theory. More
precisely, in conformal-field-theoretic language, this field \( (\phi_0^0)^4 \) serves as the identity field in the \( c = 48/(K+2) = 8 \) parafermionic tensor-product theory, and has highest weight \( h = 4h_0^0 = 0 \). From (2.28), however, we have determined that the post-projection CFT has central charge \( c' = 24/K = 6 \), and therefore the ground state of the post-projection CFT should only have (mass)\(^2\) = \(-c'/24 = -1/K = -1/4\). This implies that the state \( (\phi_0^0)^4 \), although the true ground state of the parafermionic tensor-product CFT, is clearly not the ground state of the smaller post-projection Ising-like theory — indeed, it is effectively not even in this post-projection theory at all. Similar conclusions hold for the \( K = 8 \) and \( K = 16 \) cases as well.

Which parafermion state actually serves, then, as the true ground state of the post-projection CFT for each general value of \( K \)? We are of course guaranteed that such a state exists, for all states in the Ising-like post-projection CFT must also arise as states in the original parafermionic CFT. In the language of the minimal Ising-like model with \( c = 8/K \), this ground state is of course easily identified as the state \( 1^{D_c-2} \) within the \( h = 0 \) sector, but in the language of the larger parafermion theory with central charge \( c = 48/(K+2) \), this ground state must appear as some excited state with highest weight \( H_0 \) and (mass)\(^2\) = \(-1/K \). [Indeed, only the quantity (mass)\(^2\) is “invariant” under change of the CFT being used to describe the spectrum.] Thus, since in general (mass)\(^2\) = \( H - c/24 \), we obtain

\[
H_0 = \frac{c}{24} - \frac{1}{K} = \frac{2}{K + 2} - \frac{1}{K},
\] (3.7)

and for each \( K \) this can be rewritten in terms of the highest weights \( h_0^\ell \) of the parafermion theory as

\[
H_0 = (D_c - 3) h_0^0 + h_2^2.
\] (3.8)

Thus we see that for each value of \( K \), it is the state

\[
\text{effective vacuum : } (\phi_0^0)^{D_c-3} \phi_1^1, \quad \text{(mass)}^2 = -1/K
\] (3.9)

which serves as the effective vacuum of the post-projection CFT in the \( A \)-sector. Note that \( h_0^\ell = h_{-\ell} \) even though \( \phi_m^j \neq \phi_{-m}^j \); it is for this reason that there exists an ambiguity in identifying the sign of the \( m \)-quantum numbers of parafermion fields in this approach. Either choice of sign, however, yields a state at the same (mass)\(^2\) level. We also note, of course, that only for the \( K = 2 \) superstring case is this effective vacuum state in (3.9) equivalent to the original state \( (\phi_0^0)^{D_c-2} \); this follows as a consequence of (2.4). For other values of \( K \), it is the inequality of these two states which reflects the reduction in the central charge induced by the internal projection. Indeed, as a general rule, the less negative the (mass)\(^2\) of the ground state of a conformal field theory, the smaller its central charge.

One point deserves special emphasis: the fact that (3.9) serves as the effective vacuum of the post-projection CFT in the \( A \)-sector does not imply that there exist spacetime particles of (mass)\(^2\) = \(-1/K + n \), \( n \in \mathbb{Z} \) in the fractional-superstring
spectrum. Indeed, we know from the partition functions (2.14) that no such states exist in the $A$-sectors either before or after the internal projections: the $A$-sectors contain only states with $m^2 \in \mathbb{Z}$. The state (3.9), along with the entire tower of states built from it, therefore experiences a GSO projection and fails to appear in the spectrum in the same way the original vacuum state $(\phi_0^0)^{D_c-2}$ failed to appear; it is only the effect of the internal projection that the latter vacuum state has been replaced by the former.

Although the state in (3.9) serves as the effective vacuum of the internally projected $A$-sector, the entire parafermionic sector built upon this vacuum state cannot by itself comprise the corresponding sector of the internally projected theory, for we still must incorporate the internal projection. Specifically, this means that there must be at least one other sector in the parafermion theory whose Fock space of states must be subtracted or removed from those in the above vacuum-state sector in order to produce a residual Fock space appropriate for a CFT with $c = 24/K$. Such a subtraction would be analogous to those appearing in (2.21) for $K > 2$.

It is a simple matter to determine these various potential “projection sectors”. Since the states from two sectors can be subtracted from each other in this way only if they share the same (mass)$^2$ values, any such possible other “projection sectors” would have to have vacuum states with highest weights $H'_0$ differing from $H_0$ by integers. For $H'_0 = H_0 + 1$, we find that there exist in general two such possible states, for this value of $H'_0$ can be written in terms of the parafermionic highest weights in two distinct ways:

$$H'_0 = (D_c - 3) h_0^{K/2} + h_2^{K/2} = (D_c - 4) h_0^0 + h_0^2 + h_{K-2}^K.$$  \hfill (3.10)

[The second expression in (3.10) is of course appropriate only for the cases in which $D_c \geq 4$ — i.e., for $K = 4$ and $K = 8$.] We thus have, at this mass level, the two possible projection sectors

$$(\text{mass})^2 = 1 - \frac{1}{K} : \begin{align*}
(\phi_0^{\pm K/4})^{D_c-3 \phi_{\pm 1}^{K/4}}, & \quad (\phi_0^0)^{D_c-4 \phi_0^{K/2}} \phi_{\pm (K/2-1)}.
\end{align*} \hfill (3.11)$$

It turns out that we need not consider any higher values of $H'_0$, however, for all other solutions can be generated from those above. Since $h_n^{K-\ell} - h_n^{\ell} = \frac{1}{2}(K/2 - \ell)$, we see that $\phi^j_m$ and $\phi^{K/2-j}_m$ have highest weights differing by integers when $K \in 4\mathbb{Z}$ and $j \equiv \ell/2 \in \mathbb{Z}$. Thus, for each of the potential projection sectors listed above, there are others in which each $\phi^j_m$ is replaced by $\phi^{K/2-j}_m$, and these variations indeed fill out the entire space of solutions for all values of $H'_0$. We shall not explicitly list these other options, but they will be included in what follows. Thus, we expect that for each value of $K$, some linear combination of all of these sectors will yield the $h = 0$ sectors of our minimal Ising-like post-projection CFT’s; the presence of negative coefficients in these linear combinations will then indicate those sectors serving as projection sectors.
It is a relatively straightforward matter to determine these linear combinations, for the Ising-like characters $\chi^{(c=8/K)}_0$ of these $h = 0$ sectors have already been determined in (3.3) and (3.4), and the chiral character of $16/K$ free uncompactified bosons is simply $\eta^{-16/K}$. Thus, we seek a linear combination of the characters corresponding to each of the above potential sectors which reproduces these Ising-like characters. Remarkably, such a combination exists for each value of $K$. Let us define the three quantities:

\begin{align*}
K = 4 : & \quad U_4 \equiv \frac{1}{2}(d^0_3)^2d_2^2 + \frac{3}{2}(d^0_0)^2d_0^2d_2^4 - \frac{1}{4}(d^2_0)^3d_2^2 \\
K = 8 : & \quad U_8 \equiv d^0_0d_2^8 + d^2_0d_2^8 - \frac{1}{2}d_0^8d_2^8 \\
K = 16 : & \quad U_{16} \equiv d_2^8 - \frac{1}{2}d_2^8,
\end{align*}

where the $d^\ell_n$ functions were defined in (2.11). Then we indeed find

$$U_K = \eta^{-16/K}\chi^{(c=8/K)}_0(3.13)$$

where the $\chi^{(c=1/2)}_h$ characters are of course those of the Ising model. Thus, we see that it is always the first of the sectors listed in (3.11) which serves as the projection sector in each of the relevant cases, whereas the second sector (when present) merely contributes to the Fock space prior to the internal projection. Note that while each of the linear combinations on the left sides of the above equation involves a subtraction, as required for central charge reduction, no minus signs ever appear within the expressions on the right sides. Also note that the coefficients in (3.12) are always integers when these equations are expressed directly in terms of the individual string functions $c^\ell_n$ (rather than the combinations $d^\ell_n$). Thus, we indeed have a proper mapping between the internally projected parafermion theory and the $h = 0$ sector of the minimal Ising-like theory for each value of $K$. Furthermore, just as in the $K = 2$ superstring case, this $h = 0$ sector is GSO-projected out of the fractional superstring spectrum.

Having thus isolated the $h = 0$ ground state of the post-projection CFT, we now proceed to the next-lowest state. The next-to-lowest vacuum state in the minimal Ising-like CFT for $K > 2$ is clearly that with $h = 1/K$; note that this state $\sigma^{D, -2-1/16}$ thus sits higher than any of the $1^{D, -2-i/16}$ states with $h = i/16$ which would have appeared in the non-minimal Ising-like theories. Mapping this $h = 1/K$ sector back to the parafermion theory is, however, a trivial matter in this case, for we recognize that this $h = 1/K$ sector is indeed the $A$-sector which was our starting point in (2.28). Thus, in analogy to (3.13), we have the unique solutions

$$A^b_K = A^f_K = \frac{16}{K} \eta^{-16/K}\chi^{(c=8/K)}_1(3.14)$$

where the linear combinations $A^b_K$ and $A^f_K$ are given in (2.21). Eq. (3.14) is, of course, merely a rewriting of (2.28), and just as in the $K = 2$ superstring case, this $h = 1/K$
sector indeed survives all GSO-projections and contributes to the physical spectrum of the fractional superstring.

It now remains only to fill out the one remaining sector of the $c = 8/K$ minimal Ising-like CFT: this is the sector with $h = 1/2$. Proceeding as above, we find that the relevant vacuum state must correspond to a parafermion state of highest weight

$$H_1 = \frac{2}{K+2} + \frac{1}{2} - \frac{1}{K},$$

and this can be uniquely rewritten in terms of the parafermion highest weights as

$$H_1 = (D_c - 3) h_{K/2}^{K/2} + h_{K/2-2}^{K/2}.$$  

We therefore see that the corresponding state in the parafermion theory is

$$(\text{mass})^2 = \frac{1}{2} - \frac{1}{K} : \ (\phi_{K/4}^{K/4} D_c^{-3} \phi_{K/4}^{K/4} \phi_{(K/4-1)}^{K/4}) .$$

The potential projection sectors in this case can be obtained by starting with highest weight $H'_1 = H_1 + 2$, for

$$H'_1 = (D_c - 3) h_{K/2}^{K/2} + h_{K/2-2}^{K/2} = (D_c - 4) h_{K/2}^{K/2} + h_{K/2}^{K/2} + h_{K/2-2}^{K/2}.$$  

(The second expression is again appropriate only for cases with $D_c \geq 4$.) Thus, we have in general the two possible projection sectors

$$(\text{mass})^2 = \frac{5}{2} - \frac{1}{K} : \ (\phi_{K/2}^{K/2} D_c^{-3} \phi_{K/2-1}^{K/2} \phi_{K/4}^{K/4} \phi_{(K/4-1)}^{K/4} \phi_{K/4}^{K/4}) , \ (\phi_{K/2}^{K/2} D_c^{-4} \phi_{K/2}^{K/2} \phi_{K/4}^{K/4} \phi_{(K/4-1)}^{K/4} \phi_{K/4}^{K/4}) ;$$  

and all other potential projection sectors (some of which have $H'_1 = H_1 + 1$) can be obtained by substituting $\phi_{m}^{j} \leftrightarrow \phi_{m}^{K/2-j}$. Constructing linear combinations of the corresponding string functions in order to determine which of these sectors are indeed projection sectors, we find that we can once again reproduce the characters of the $h = 1/2$ sectors of the $c = 8/K$ minimal Ising-like models. Specifically, defining

$$K = 4 : \quad V_4 \equiv \frac{1}{2} d_0 d_1 d_2 d_3 - \frac{3}{2} d_0^2 d_2^2 - \frac{1}{2} d_0^2 d_2^3 \quad V_8 \equiv \frac{1}{2} d_1^4 - d_2^2 d_4^2 - d_4^2 d_6^2$$

$$K = 16 : \quad V_{16} \equiv \frac{1}{2} d_6^4 - d_6^4$$

we find

$$V_K = \frac{16}{K} \eta^{-16/K} \chi^{(c=8/K)}_{1/2} .$$

Thus, we see that in this case, all of the sectors in (3.19) serve as projection sectors. Note that unlike the $K = 2$ superstring case, this $h = 1/2$ sector is once again completely GSO-projected out of the resulting fractional-superstring spectrum. This, of
course, agrees completely with the fractional-superstring partition functions obtained in (2.14).

This, then, completes the partition-function mapping between the parafermionic theory and the post-projection minimal Ising-like theory described in Sect. 3.1. Specifically, collecting our results, we have found

\[
U_K = \eta^{-16/K} \chi_0^{(c=8/K)} \\
A_K^b = A_K^f = (D_c - 2) \eta^{-16/K} \chi_1^{(c=8/K)} \\
V_K = (D_c - 2) \eta^{-16/K} \chi_{1/2}^{(c=8/K)}. 
\]

Note the factors of \(D_c - 2\) which precede the characters of the two “excited” sectors with \(h > 0\). The suggestive appearance of (3.22) thus leads us to conclude that the post-projection \(A\)-sector CFT for \(K > 2\) is indeed the minimal \(c = 8/K\) Ising-like CFT tensored together with \(D_c - 2 = 16/K\) free bosons. It is in fact straightforward to verify that there exist no other string-function combinations which, together with internal projections, could produce the other Ising-like characters which would have arisen in any of the other non-minimal Ising-like models discussed in Sect. 3.1. It is this fact which is the ultimate justification for our selection of the minimal Ising-like CFT’s in (3.18).

### 3.3 Spacetime Statistics

The above description, however, is still incomplete, for while our fractional-superstring characters have been related to the characters of the minimal Ising-like CFT, the difficult task of finding a proper worldsheet formulation or representation of this post-projection CFT still remains. One indication of this difficulty can already be seen from the above results. Note from (3.22) that we have related both \(A_K^b\) and \(A_K^f\) to the same character of the same minimal Ising-like CFT. Although \(A_K^b\) and \(A_K^f\) are of course equal when expressed as functions of \(q\) (since in our supersymmetric theory we have \(A_K = A_K^b - A_K^f = 0\)), we do not expect these quantities to be equal when expressed in terms of conformal field theory characters, for these two distinct sectors (one spacetime bosonic and the other spacetime fermionic) cannot both be expected to originate from the same vacuum state or set of primary fields in our underlying post-projection CFT. In the \(K = 2\) superstring case, this problem does not arise, for although we find \(A_2^f = 8\eta^{-8} \chi_{1/2}^{(c=4)}\) [where this is the \(h = 1/K\) character in the second line of (3.22), corresponding to the Ramond sector], we in fact also find \(A_2^b = 8\eta^{-8} \chi_{1/2}^{(c=4)}\) [where this is the \(h = 1/2\) character in the third line of (3.22), corresponding to the NS sector with positive \(G\)-parity]; indeed, for the \(K = 2\) special case these two characters are equal due to the Jacobi identity \(\chi_{1/2}^{(c=4)} = \chi_{1/2}^{(c=4)}\). Thus, for \(K = 2\), we see that we should really identify \(A_2^b\) with the \(V_2\) sector, and it is for this reason that the \(\chi_{1/2}^{(c=8/K)}\) sector survives the GSO-projection in the \(K = 2\) superstring case even though it fails to do so for the \(K > 2\) fractional-superstring
cases. For $K > 2$, however, we see that neither the bosonic expression $A^b_K$ nor the fermionic expression $A^f_K$ can be identified with $V_K$, since both have the highest weight $h = 1/K$.

The question then arises as to whether it is $A^b_K$ or $A^f_K$ (or some linear combination of the two) which is to be identified with the $h = 1/K$ sector of the Ising-like CFT. This can be answered unambiguously by employing modular transformations, however, for although two distinct expressions such as $A^b_K$ and $A^f_K$ may be equal as functions of $q$, they need not transform identically when expressed as functions of their underlying characters. It is, however, a simple matter to determine the transformation properties of the characters on the right sides of (3.22), and using the definitions for $U_K$, $A^b_K$, $A^f_K$, and $V_K$ in terms of the parafermionic string functions, we can similarly determine the transformation properties of these four fractional-superstring quantities which appear on the left sides of (3.22). We find, expectedly, that $U_K$ and $V_K$ transform precisely as do their corresponding counterparts in (3.22), but surprisingly it is only the linear combination

$$\tilde{A}_K \equiv \frac{1}{2} \left( A^b_K + A^f_K \right)$$

which transforms in the same way as $\chi^{(c=8/K)}_i$ for all $K > 2$. Specifically, under $T : \tau \rightarrow \tau + 1$ and $S : \tau \rightarrow -1/\tau$ we have

$$\begin{pmatrix} U_K \\ A_K \\ V_K \end{pmatrix} \rightarrow T^{(K)}_{ij} \begin{pmatrix} U_K \\ A_K \\ V_K \end{pmatrix}; \quad \begin{pmatrix} U_K \\ A_K \\ V_K \end{pmatrix} \rightarrow S (-i\tau)^{-8/K} S^{(K)}_{ij} \begin{pmatrix} U_K \\ A_K \\ V_K \end{pmatrix}$$

where

$$T^{(K)}_{ij} = \exp [2\pi i (h_i - 1/K)] \delta_{ij}$$

and

$$S^{(4)}_{ij} = S^{(8)}_{ij} = \frac{1}{4} \begin{pmatrix} 2 & 2 & 2 \\ 4 & 0 & -4 \\ 2 & -2 & 2 \end{pmatrix}; \quad S^{(16)}_{ij} = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -1 \\ 1 & -\sqrt{2} & 1 \end{pmatrix}.$$ (3.26)

Note, as a check, that the $S_{ij}$ matrices must in general diagonalize the fusion rules of a given CFT \[15\], so that the fusion rules

$$[\phi_i] \times [\phi_j] = \sum_k N_{ijk} [\phi_k]$$

can always be obtained from the $S_{ij}$ matrices via the Verlinde formula

$$N_{ijk} = \sum_n S_{in} S_{jn} S_{nk} S_{1n}$$

\[1\] A trivial example of this fact appears even for the $K = 2$ case: the expressions $\chi^{(c=4)}_{1/2}$ and $\tilde{\chi}^{(c=4)}_{1/2}$ in (3.4) are equal as functions of $q$, but transform quite differently under the $S : \tau \rightarrow -1/\tau$ modular transformation. Indeed, only their difference is truly modular-invariant.
where $i = 1$ corresponds to the identity sector with $h = 0$. Substituting (3.26) into (3.28) indeed reproduces (3.1) for each value of $K$, with $\{1, \phi_1, \phi_2\} \leftrightarrow \{U_K, K \frac{K}{16} V_K, K \frac{K}{16} A_K\}$. Thus, it is only the linear combination (3.23) which can legitimately be identified with the $h = 1/K$ sector of the $c = 8/K$ minimal Ising-like CFT:

\[
U_K = \eta^{-16/K} \chi_0^{(c=8/K)} \\
\tilde{A}_K = (Dc - 2) \eta^{-16/K} \chi_{1/K}^{(c=8/K)} \\
V_K = (Dc - 2) \eta^{-16/K} \chi_{1/2}^{(c=8/K)} .
\] (3.29)

The above result implies that the $h = 1/K$ sector of our minimal $c = 8/K$ Ising-like CFT is neither purely spacetime bosonic nor fermionic for $K > 2$, but instead consists of states of both varieties! This alone indicates the difficulty of formulating worldsheet representations of our minimal Ising-like theory which naturally yield these appropriate spacetime properties for $K > 2$. In the $K = 2$ superstring case, for example, we established that $A_2^b$ is to be identified with the $h = 1/K$ sector, and that $A_2^b \equiv V_2$ is to be identified with the $h = 1/2$ sector. This is consistent with our understanding that for $K = 2$, the $h = 1/K$ sector is built upon the Ramond vacuum state $\sigma^8$: this state produces the necessary cuts for the worldsheet fermions and supercurrent, and serves as the fundamental spinor which allows all resulting spacetime particles to transform under the transverse eight-dimensional Lorentz group as spacetime fermions. Likewise, the $h = 0, 1/2$ sectors are built upon vacuum states of the form $1^{i\psi_8 - i}$, and these have natural spacetime bosonic interpretations. For $K > 2$, however, we require a representation for our minimal Ising-like CFT (3.6) which yields mixed spacetime statistics properties for the $h = 1/K$ sector, simultaneously containing states which are vectors and spinors under $SO(D)$. Thus, unlike the $K = 2$ superstring case, we no longer anticipate that this minimal Ising-like CFT can be represented simply as a tensor product of of worldsheet bosons and fermions. This issue will be discussed further in Sect. 6, where we shall obtain more detailed information concerning the spacetime statistics properties of the individual states which contribute to these $h = 1/K$ sectors.

For $K > 2$, then, we see that the post-projection CFT of the fractional superstring is isomorphic to the minimal Ising-like CFT in (3.6) — meaning that these two theories share the same central charges, highest weights, fusion rules, and characters — but that a new representation of this CFT in terms of worldsheet fields will be necessary in order to adequately describe the spacetime statistics properties of the resulting sectors.

4 Internal Projections in the $B$-Sector

We now turn to the remaining sectors of the fractional superstring, namely the massive $B$-sectors discussed in Sect. 2. In particular, we seek to subject these $B$-
sectors to an analysis analogous to that employed for the $A$-sectors in the previous section, and our corresponding starting point will be the result (2.31). Note that we will be focusing here exclusively on the $K = 4$ and $K = 8$ fractional superstrings, for (2.14) indicates that the $K = 2$ and $K = 16$ theories contain no $B$-type sectors.

In order to interpret (2.31) in terms of the characters of some post-projection CFT, we now must interpret the scaling of $\tau$, for CFT characters are generally defined only in terms of an unscaled $\tau$:

$$\chi_h(\tau) \equiv \text{Tr}_h \exp(2\pi i H \tau).$$

(4.1)

Here $H$ is the Hamiltonian of the two-dimensional CFT, and the trace is over the (appropriately defined) sector of highest weight $h$. Thus, we see that a scaling of $\tau$ can generally be reinterpreted as a rescaling of the energies in our underlying two-dimensional worldsheet theory. More specifically, let us recall the character $\chi^{(c=1)}_{1/8}$ of the $c = 1$ Dirac theory introduced in (3.2):

$$\chi^{(c=1)}_{1/8} \equiv \vartheta_2(\tau)/2\eta = \eta^{-1} \sum_{n \in \mathbb{Z}} q^{(n+1/2)^2/2}, \quad q \equiv \exp(2\pi i \tau).$$

(4.2)

This expression, which is the square of the Ising-model character $\chi_{1/16}$, is usually interpreted as the character of a single worldsheet complex fermion with periodic (Ramond) space boundary conditions on the torus: the $\eta$-denominator then represents the contribution to the trace from the infinite tower of states built upon each vacuum state in the Ramond sector, while the summation tallies the contributions from these (infinitely many) vacuum states. Note that these vacuum states have worldsheet momenta $P = n + 1/2$ with $n \in \mathbb{Z}$, and thus they together form an internal one-dimensional momentum lattice with lattice spacing one; the energies of these vacuum states are then generally given by $H = P^2/2$. Thus, by analogy, we see that the expression $\vartheta_2(\lambda \tau)/2\eta(\tau)$ corresponds to a complex Ramond fermion formulated on a rescaled momentum lattice with lattice spacing $\sqrt{\lambda}$; the energy scale for the infinite oscillator tower of states built upon each of these momentum vacuum states is then unchanged. Note that only this partial rescaling (momentum lattice rescaled but oscillator tower untouched) yields a consistent theory, for only this partial rescaling is equivalent to a change in the radius of compactification of the $c = 1$ boson to which these fermion theories are equivalent. (The connections of this $B$-sector theory to that of a compactified boson are discussed in Sect. 5.)

Thus, (2.31) indicates that the $B$-sector internal projections appear to reduce our original parafermion theories down to those of fermions formulated on rescaled momentum lattices (or “rescaled Ising models”), in the same way that the $A$-sector internal projections appeared to reduce these theories down to those of fermions on unscaled momentum lattices (or “unscaled Ising models”). This result implies that the central charges of the post-projection CFT’s in the $B$-sectors are the same as those of the $A$-sectors, for this partial rescaling (or radius change) does not alter
the underlying central charge. Obtaining the same central charge in all sectors is of course necessary if the fractional superstring is ultimately to be described in terms of a single post-projection CFT. This in turn implies that the parafermionic “effective vacuum state” for the $B$-sectors must be the same as that of the $A$-sectors given in (3.9), yet another indication that the behavior of the internal projections in the $B$-sectors is consistent with that of the projections of the $A$-sectors.

We still must determine, however, whether our $B$-sector post-projection CFT’s are isomorphic to the “minimal” combination of $D_c - 2$ rescaled Ising models, or perhaps to some other larger combination. In fact, since no $B$-sectors appear in the ordinary $K = 2$ superstring case, we have no guide as to whether the “minimal” assumption is correct in the one special case (that of the ordinary superstring) whose underlying CFT is well-understood. Hence, we shall adopt a somewhat different approach from that of Sect. 3 in analyzing the post-projection CFT’s of the $B$-sectors. Recall that in Sect. 3, we started by guessing that the minimal Ising-like theories were in fact our post-projection CFT’s for the $A$-sectors; this in turn implied the existence of three sectors with highest weights $h = 0, 1/K, 1/2$ whose characters $U_K, A_K, V_K$ we eventually constructed out of parafermionic string functions. These characters were then found to form a closed system under modular transformations, mixing under the $S$ modular transformation according to matrices $S_{ij}^{(K)}$ from which the originally assumed CFT fusion rules (3.1) were verified via (3.28). We shall apply this same procedure here, therefore, but in reverse. Starting with the characters $B_{K}^{b,f}$ in (2.23) expressed as linear combinations of string functions, we shall take modular transformations in order to fill out the complete system of characters in our $B$-sector post-projection CFT. The string-function combinations which comprise this system will thus be the $B$-sector analogues of the three $A$-sector characters $U_K, A_K, V_K$, and will thereby provide (as before) a mapping between the sectors of our smaller post-projection CFT and the larger (but internally projected) original parafermion theory. From this set of characters we will then be able to infer the relevant spectrum of highest weights in our post-projection CFT, as well as its complete set of fusion rules.

It is a straightforward matter to take the modular transformations of $B_{K}^{b,f}$ in (2.23), for the modular transformations of the individual string functions $c_n^\ell$ are well-known. We find the following results. For the $B$-sectors, we now find that our post-projection CFT has nine sectors for each value of $K$, since there are nine linearly independent combinations of string functions required for closure under modular transformations. In each case, however, we can simplify matters by grouping these nine characters into three groups of three, for we will see that each such group by itself resembles the three-sector minimal Ising-like theory. Thus, we choose a notation in advance which reflects this analogy: we will denote these three minimal Ising-like theories as copies $(a)$, $(b)$, and $(c)$, and denote the three sectors within each copy (in analogy with those for the $A$-sector) as $X_K, \bar{B}_K$, and $Y_K$. Then our nine string-function combinations are as follows. Recalling the definition $d_n^\ell \equiv c_n^\ell + c_n^{K-\ell}$, we find
Similarly, for $K = 8$, we have the combinations:

$$X_8^{(a)} \equiv d_0^0 d_2^2 - d_0^2 d_2^0 \quad (h = 0)$$
$$\tilde{B}_8 \equiv \tilde{B}_8^{(a)} \equiv \frac{1}{2}(B_8^h + B_8^i) \quad (h = 5/8)$$
$$Y_8^{(a)} \equiv d_0^0 d_2^0 - d_0^2 d_2^0 \quad (h = 5/2)$$
$$X_8^{(b)} \equiv d_0^0 d_2^2 - d_0^2 d_2^2 \quad (h = 2/5)$$
$$\tilde{B}_8^{(b)} \equiv d_0^0 d_4^1 - d_0^1 d_4^0 \quad (h = 1/40)$$
$$Y_8^{(b)} \equiv d_4^1 d_2^0 - d_2^0 d_4^0 \quad (h = 9/10)$$
$$X_8^{(c)} \equiv d_0^0 d_2^4 - d_0^4 d_2^0 \quad (h = 8/5)$$
$$\tilde{B}_8^{(c)} \equiv d_0^0 d_4^4 - d_0^4 d_4^0 \quad (h = 9/40)$$
$$Y_8^{(c)} \equiv d_2^4 d_4^0 - d_2^0 d_4^4 \quad (h = 1/10) \quad (4.4)$$

Next to each string-function combination in (4.3) and (4.4) we have indicated the highest weight of the corresponding sector in the post-projection CFT. These highest weights are readily determined by expanding each corresponding string-function combination in the form $q^h \sum_{n=0}^{\infty} a_n q^n$. Since the quantity $\ell$ must then be the (mass)$^2$ of the vacuum state in the corresponding sector, and since this in turn must equal $h - c/24$ (where $c$, the central charge of the post-projection CFT, is $24/K$ for both the $A$- and $B$-sectors), we have $h = \ell + 1/K$.

There are several things to note about the expressions in (4.3) and (4.4). We have already seen that there are internal projections acting within the expressions $B_K^{h,f}$, and this was the basis for central charge reduction in the $B$-sectors. It is now clear that indeed each of the parafermion sectors listed in these equations contains an analogous internal projection as well; this is of course required for self-consistency, since these different sectors are parts of the same post-projection CFT with reduced central charge $c = 24/K$. Thus, just as for the $A$-sector expressions $A_K$, $U_K$, and
Each of the above sets of nine quantities can be viewed as providing a mapping between the our residual smaller $B$-sector post-projection CFT and the larger original parafermion CFT with an internal projection. In this vein, note that once again it is the $h = 0$ sector which serves as the “effective vacuum” sector of the post-projection CFT, with the characters corresponding to the same state (3.9) appearing within $X_K^{(n)}$ for each value of $K$. Thus the $A$-sectors and $B$-sectors indeed share the same effective vacuum state at $(mass)^2 = -1/K$, indicating (as claimed earlier) that the internal projections in the $A$- and $B$-sectors appear to be consistent with each other. Note, however, that these are nevertheless different internal projections, for they combine the parafermionic projection and non-projection sectors in different ways in order to produce the different $A$- and $B$-sector post-projection CFT’s.

The above sets of string-function combinations form closed systems under modular transformations. Indeed, under the $T$ modular transformation, the nine combinations in (4.3) and (4.4) respectively transform according to (3.25), whereas under the $S$ transformation these sets separately transform into themselves with the following mixing matrices:

\[
S_i^{(K=4)} = \frac{1}{6} \begin{pmatrix}
1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 \\
1 & 0 & -1 & 2 & 0 & -2 & 1 & 0 & -1 \\
1 & -2 & 1 & 2 & -2 & 2 & 1 & -2 & 1 \\
2 & 4 & 2 & 1 & 1 & 1 & -1 & -2 & -1 \\
4 & 0 & -4 & 2 & 0 & -2 & -2 & 0 & 2 \\
2 & -4 & 2 & 1 & -1 & 1 & -1 & 2 & 1 \\
4 & 8 & 4 & -4 & -4 & 1 & 2 & 1 & 0 \\
4 & 0 & -4 & -4 & 0 & 4 & 1 & 0 & -1 \\
4 & -8 & 4 & -4 & 4 & -4 & 1 & -2 & 1
\end{pmatrix}
\quad (4.5)
\]

and

\[
S_i^{(K=8)} = \frac{1}{2} \begin{pmatrix}
r & r & r & r & r & r & r & r & r \\
2r & 0 & -2r & 2r & 0 & -2r & 0 & -2r & 0 \\
r & -r & r & r & -r & r & r & -r & r \\
2r & 2r & 2r & a & a & a & -b & -b & -b \\
4r & 0 & -4r & 2a & 0 & -2a & -2b & 0 & 2b \\
2r & -2r & 2r & a & a & a & -b & -b & -b \\
2r & 2r & 2r & -b & -b & -b & a & a & a \\
4r & 0 & -4r & -2b & 0 & 2b & 2a & 0 & -2a \\
2r & -2r & 2r & -b & -b & -b & a & -a & a
\end{pmatrix}
\quad (4.6)
\]

where $r \equiv 1/\sqrt{5}$, $a \equiv (1 - r)/2$, and $b \equiv (1 + r)/2$. These matrices are each written in the basis formed by the relevant nine string-function combinations, with the order of each basis taken to be the same as the order in which these combinations are listed within (4.3) and (4.4). Note that these matrices square to $1$, as required. Indeed, if we examine the smaller $3 \times 3$ blocks which together comprise these $9 \times 9$ matrices, we
see that these submatrices are each of the same general form (3.26) which we found for the \( A \)-sectors in Sect. 3 (up to renormalizations of our nine quantities). This is our first indication that our nine-sector theory can be decomposed into three copies of a theory resembling the three-sector minimal Ising-like theories of the \( A \)-sectors.

In order to rigorously define the manner in which this theory can be viewed as three such copies, we now proceed to determine the fusion rules of this theory. Recall that these fusion rules can be determined from the \( S_{ij}^{(K)} \)-matrices (1.5) and (4.6) via the formula (3.28). \textit{A priori}, however, there will be 36 linearly independent non-trivial fusion rules for each of these nine-sector systems. Let us therefore first organize these fusion rules in a coherent fashion. We already suspect that we have three copies of an Ising-like theory; these copies are labeled \((a), (b), \) and \((c)\). Furthermore, \textit{within} each copy, we expect that \( X_K \) plays the role of the identity \( 1 \), while \( \tilde{B}_K \) plays the role of the spin field \( \sigma \), and \( Y_K \) plays the role of the Majorana fermion \( \psi \). We therefore expect two kinds of fusion rules: those which indicate how any two copies of this Ising-like theory fuse together, and those which indicate how the fields \textit{within} each copy fuse together. More specifically, we must determine the fusions in \( \{X_K, \tilde{B}_K, Y_K\} \) space as well as those in \( \{a, b, c\} \) space. Let \([\phi^{(s)}] \) generically indicate a sector in copy \((s)\) of the Ising-like theory \((s = a, b, c)\). Then our results are as follows. For \( K = 4 \), the fusion rules \textit{between} the copies are as follows:

\[
K = 4 : \\
[\phi^{(a)}] \times [\phi^{(s)}] = [\phi^{(s)}] \quad (s = a, b, c) \\
[\phi^{(b)}] \times [\phi^{(b)}] = 4 [\phi^{(a)}] + [\phi^{(b)}] + 2 [\phi^{(c)}] \\
[\phi^{(b)}] \times [\phi^{(c)}] = 2 [\phi^{(b)}] + 2 [\phi^{(c)}] \\
[\phi^{(c)}] \times [\phi^{(c)}] = 4 [\phi^{(a)}] + 2 [\phi^{(b)}] + [\phi^{(c)}] ,
\]  

\[ (4.7) \]

whereas for \( K = 8 \) we find instead:

\[
K = 8 : \\
[\phi^{(a)}] \times [\phi^{(s)}] = [\phi^{(s)}] \quad (s = a, b, c) \\
[\phi^{(b)}] \times [\phi^{(b)}] = 2 [\phi^{(a)}] + [\phi^{(c)}] \\
[\phi^{(b)}] \times [\phi^{(c)}] = [\phi^{(b)}] + [\phi^{(c)}] \\
[\phi^{(c)}] \times [\phi^{(c)}] = 2 [\phi^{(a)}] + [\phi^{(b)}] .
\]  

\[ (4.8) \]

\textit{Within} each Ising-like copy, however, we indeed find that our fusion rules are the usual Ising-model fusion rules (3.3), with \( \{1, \varphi_1, \varphi_2 \} \sim \{X_K, Y_K, \tilde{B}_K\} \). Thus we see that we indeed have three distinct “copies” of an Ising-like theory in both the \( K = 4 \) and \( K = 8 \) cases (in the sense that they have the same internal fusion rules as the Ising-like theories of Sect. 3.1), and these cases differ only in the manner in which these three different copies are sewn (or fused) together. For example, putting these two types of fusion rules together yields the fusion rule:

\[
[\tilde{B}_K^{(c)}] \times [\tilde{B}_K^{(c)}] = \sum_{s=a,b,c} n_s \left( [X_K^{(s)}] + [Y_K^{(s)}] \right) 
\]  

\[ (4.9) \]
with \(\{n_a, n_b, n_c\} = \{4, 2, 1\}\) for \(K = 4\) and \(\{2, 1, 0\}\) for \(K = 8\). Note, however, that these individual copies bear no other relation to the minimal Ising-like theories of Sect. 3.1: they do not individually close under fusion as do the latter minimal theories, nor do their highest weights correspond.

The fusion rules (4.7) and (4.8) are of course associative, as is guaranteed by construction since they were obtained via (3.28) from \(S\)-matrices satisfying \(S^2 = 1\). The fact that some fusion-rule coefficients are greater than one suggests that some of these sectors actually appear in the post-projection CFT with multiplicities exceeding one. This would in turn suggest that there exist conserved quantum numbers according to which these multiple sectors might be distinguished, and we will see in Sect. 5 that this is indeed the case.

### 5 Relation to Compactified Bosons

In Sects. 3 and 4 we derived some of the minimal conditions (central charges, highest weights, and fusion rules) that must be satisfied by the post-projection CFT’s for both the fractional superstring \(A\)- and \(B\)-sectors. In this section we will demonstrate that all of these constraints can be reformulated naturally as the properties of the CFT’s of worldsheet bosons compactified on circles of certain radii, thereby providing a uniform language for discussing the \(A\)- and \(B\)-sectors on the same footing. In particular, we will be able to provide a direct mapping between the various \(B\)-sectors listed in (4.3) and (4.4) and the different winding-mode sectors of the compactified-boson theory, thereby explaining the appearance of these additional sectors and yielding explicit \(B\)-sector analogues of the identities (3.29). Moreover, the compactified-boson theory will provide a useful additional quantum number (namely the \(U(1)\) charge \(\alpha\)) through which the infinite number of individual states contributing to the \(A\)- and \(B\)-sectors might be distinguished. The results of this section will also prove vital in Sect. 6, where we will consider the problem of spacetime statistics at the level of individual \(\alpha\)-states through the use of the so-called “twist current”.

We remind the reader, however, that the relation between our post-projection CFT’s and the compactified-boson theories is only an isomorphism which holds at the level of their central charges, highest weights, fusion rules, and characters. Indeed, as we have seen (and as will become even clearer in Sect. 6), our post-projection CFT’s cannot ultimately be represented in terms of such free worldsheet bosonic fields, and an alternative representation remains to be found.

In order to establish conventions and notation, we begin in Sect. 5.1 by reviewing the compactified-boson CFT and its associated characters. The reformulation of our above results for the \(A\)- and \(B\)-sectors will then be given in Sect. 5.2. Finally, Sect. 5.3 contains comments concerning the fractional-superstring \(C\)-sectors, all of whose states are removed by the internal projections.
5.1 Compactified Bosons and their Chiral Characters

Let us consider a free (chiral) bosonic field \( \phi(z) \), normalized in the usual fashion so that
\[
\langle \phi(z) \phi(w) \rangle = -\ln(z - w), \quad T(z) = -\frac{1}{2} : [\partial \phi(z)]^2 : ,
\]
and compactified on a circle of radius \( R \) so that \( \phi \approx \phi + 2\pi R \). It is then straightforward to demonstrate that this conformal field theory has \( c = 1 \), with primary fields \( i\partial \phi \) of weight \( h = 1 \) and \( \epsilon^{(\alpha)} \equiv \exp(i\alpha \phi) \) of weight \( \alpha^2/2 \). The field \( \epsilon^{(0)} \equiv 1 \) thus serves as the identity, and the fusion rules for this theory take the form
\[
[\epsilon^{(\alpha)}] \times [\epsilon^{(\beta)}] = [\epsilon^{(\alpha + \beta)}] \quad (5.2)
\]
where we understand the sector \( [\epsilon^{(0)}] \) to include \( 1, i\partial \phi \), and all of its descendents. Thus \( \alpha \) appears as a conserved quantum number under fusion — indeed, it is the charge of the primary field \( \epsilon^{(\alpha)} \) with respect to the \( U(1) \) current \( i\partial \phi \).

The above results are independent of the radius of compactification. If the radius of compactification is infinite, however, so that \( R \to \infty \), then \( \alpha \) is entirely unconstrained, whereas for finite \( R \) we find that \( \alpha \) is restricted to the values \( \alpha_{m\ell} = m/2R + \ell R \), \( m, \ell \in \mathbb{Z} \). (5.3)

Here \( m \) and \( \ell \) respectively represent the boson momentum- and winding-mode quantum numbers of the corresponding state \( |m, \ell\rangle \equiv \epsilon^{(\alpha_{m\ell})}(0)|0\rangle \), and it is clear that this selected set of permitted values for \( \alpha_{m\ell} \) yields a consistent subalgebra of our fusion rules for any \( R \). We may rewrite this set of values as follows. Restricting ourselves to those radii for which this conformal field theory is rational — namely, \( R^2 \in \mathbb{Q} \) — we can without loss of generality write \( R = \sqrt{a/(2b)} \) where \( a, b \) are positive, relatively prime integers, and we define \( N \equiv 2ab \). We then see that
\[
\alpha_{m\ell} = \frac{mb + \ell a}{\sqrt{N}}, \quad (5.4)
\]
and since \( r \equiv mb + \ell a \in \mathbb{Z} \), we find that allowed values for \( \alpha_{m\ell} \) for arbitrary radius \( R \) are simply \( r/\sqrt{N}, r \in \mathbb{Z} \). The conformal dimensions for these primary fields \( \epsilon^{(\alpha)} \) are therefore given by
\[
h_{m\ell} = \frac{(\alpha_{m\ell})^2}{2} = \frac{r^2}{2N} = \frac{(mb + \ell a)^2}{2N}. \quad (5.5)
\]

It is now a simple matter to construct the set of chiral characters corresponding to this compactified-boson theory. The contribution arising from a single sector \( \alpha_{m\ell} \)
is as usual:

\[ Z_\alpha \equiv Z_{(m, \ell)} = \eta^{-1} q^{\mathcal{h}_{m\ell}} = \eta^{-1} q^{N(r/N)^2/2}; \quad (5.6) \]

here the factor of \( \eta^{-1} \) reflects the contribution from the infinite tower of states built upon the vacuum \( |m, \ell\rangle \). Thus, writing \( r = Nn + k \) where \( n \in \mathbb{Z} \) and \( 0 \leq k < N \), we see that we can construct \( N \) distinct compactified-boson characters

\[ \chi_{N,k} \equiv \eta^{-1} \sum_{n \in \mathbb{Z}} q^{N(n+k/N)^2/2}, \quad 0 \leq k < N \quad (5.7) \]

by adding together those contributions \( Z_\alpha \) from sectors \( \alpha = (m, \ell) \) satisfying

\[ \sqrt{N} \alpha = mb + \ell a = k \pmod{N}. \quad (5.8) \]

Explicitly, we have

\[ \chi_{N,k} = \sum_{\sqrt{N} \alpha = k \pmod{N}} Z_\alpha = \sum_{mb + \ell a = k \pmod{N}} Z_{(m, \ell)}. \quad (5.9) \]

It is for this reason that the number of “characters” \( \chi_{N,k} \) in our compactified-boson theory is finite, even though the number of primary fields is infinite.†

These compactified-boson characters \( \chi_{N,k} \) satisfy a number of identities. First, they transform covariantly under the modular group, \( \chi_{N,k}(M\tau) = \sum_{k'} M_{kk'} \chi_{N,k'}(\tau) \) where

\[ S_{kk'} = \frac{1}{\sqrt{N}} \exp \left( 2\pi i \frac{kk'}{N} \right), \quad T_{kk'} = \exp \left[ 2\pi i \left( \frac{k^2}{2N} - \frac{1}{24} \right) \right] \delta_{kk'}. \quad (5.10) \]

Furthermore, we observe that \( \chi_{N,k} = \chi_{N,-k} = \chi_{N,N+k} \), and thus the set of truly independent characters is simply \( \{ \chi_{N,k}, 0 \leq k \leq N/2 \} \). Using (5.7), we can also define the corresponding (classical Jacobi-Riemann) \( \Theta \)-functions via

\[ \Theta_{N,k} \equiv \eta \chi_{N,k} = \sum_{n=-\infty}^{\infty} q^{N(n+k/N)^2/2} = q^{k^2/2N} \prod_{n=1}^{\infty} \left( 1 + q^{N(n-k)^2} \right) \left( 1 + q^{N(n-k)^2} \right) (1 - q^{Nn}) . \quad (5.11) \]

* This expression assumes the absence of null states in the corresponding Verma module, which in turn requires \( h_{m\ell} \notin \mathbb{Z}/4 \). At the values of compactification radius \( R \) that we will be dealing with, however, this is indeed the case for all sectors with non-zero highest weights. The identity sector \([1] \), by contrast, always contains a null state at level one (corresponding to the primary field \( i\partial \phi \) with \( h = 1 \)). In this case (5.6) represents the sum of the contributions from both \([1]\) and \([i\partial \phi]\).

† Recall, in this context, the footnote in Sect. 3.1. The compactified-boson theory thus provides us with a means of explicitly distinguishing the infinitely many primary fields in these \( c \geq 1 \) theories via the \( \alpha \) quantum number.
These $\Theta$-functions are then related to the Jacobi $\vartheta$-functions as follows:

\[
\Theta_{N,k} = \begin{cases} 
\vartheta_2(N\tau) & \text{if } k = N/2 \\
\vartheta_3(N\tau) & \text{if } k = 0 \\
\vartheta_4(N\tau) & \text{if } k = N/4 \text{ and } N \in 4\mathbb{Z} .
\end{cases}
\]

In particular, note the appearance of $\vartheta$-functions with scaled arguments.

As required, these chiral characters $\chi_{N,k}$ can be combined with their complex conjugates in order to produce the full modular-invariant partition function corresponding to a compactified boson at radius $R$:

\[
Z(R) = \frac{1}{\eta^2} \sum_{m,\ell \in \mathbb{Z}} q^{(m/2R - \ell R)^2/2} q^{(m/2R + \ell R)^2/2} .
\]

Specifically, defining $a, b,$ and $N$ as above, we then find a pair of integers $a', b'$ so that $\det \begin{pmatrix} a & b \\ a' & b' \end{pmatrix} = 1$, and define $s \equiv a'b + b'a \pmod{N}$, with $0 \leq s < N$. Note that while $a', b'$ are not uniquely determined by this procedure, $s$ is uniquely defined. Then it is a simple matter to see that (5.13) is given by

\[
Z(R) = \sum_{k=0}^{N-1} \chi_{N,k}(q) \overline{\chi_{N,sk}(q)} ,
\]

which demonstrates that $\chi_{N,k}$ are indeed the proper chiral characters of the compactified-boson theory.

We conclude this brief review with an important comment concerning the identification of the physical radius $R$ on the basis of a set of chiral characters $\{\chi_{N,k}, 0 \leq k < N\}$. It turns out that there are three distinct types of ambiguities which may arise, only some of which shall concern us. First, of course, there is the duality transformation $R \to 1/(2R)$ which is an exact symmetry of the compactified-boson theory: physically this interchanges momentum-modes and winding-modes, and mathematically we see that $N$ and $\alpha$ are invariant under this transformation, while $(a,m) \leftrightarrow (b,\ell)$. Thus, a given value of $N$ corresponds to either $R$ or $1/(2R)$. This type of ambiguity shall not concern us, however, since both radii correspond to the same theory with the same set of physical states.

The second ambiguity is more subtle. Note that multiplying the radius $R$ of the compactified boson theory by an integer $n$ (or equivalently dividing its dual $\tilde{R}$ by $n$) produces a theory related to the original theory by the introduction of additional momentum-mode sectors and the simultaneous removal of corresponding winding-mode sectors (or vice versa). For integer $n$, however, some of these additional winding-mode sectors effectively replace the momentum-mode states which had been removed, with the result that the entire original theory is “embedded” in the new (larger) theory and in fact comprises a self-consistent subset of this larger theory. In terms of the above characters, we find that for $R \to R' \equiv nR$, we have $N \to N' \equiv n^2N$, 

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with a particular sector $k$ of highest weight $h = k^2/(2N)$ in the original theory now described in the new theory as that with $k' \equiv nk$. Linear combinations of the characters $\chi_{N',k}$ in the $N'$ system can then reproduce each of the characters $\chi_{N,k}$ of the original system. As an explicit example which will be relevant later, let us consider the cases where $R = n$ with $n \in \mathbb{Z}$; these theories then have $N = 4n^2$. Those theories corresponding to smaller values of $n$ can therefore be equivalently described as closed subsets of those corresponding to larger values, and the relations between their associated characters are:

\[ \begin{align*}
\chi_{4,0} &= \chi_{16,0} + \chi_{16,8} = \chi_{36,0} + \chi_{36,12} + \chi_{36,24} = \ldots \\
\chi_{4,1} &= \chi_{16,2} + \chi_{16,6} = \chi_{36,3} + \chi_{36,9} + \chi_{36,15} = \ldots \\
\chi_{4,2} &= \chi_{16,4} = \chi_{36,6} + \chi_{36,18} + \chi_{36,30} = \ldots 
\end{align*} \]

We will require that our results be invariant under this type of integer radius rescaling.

Finally, there exists a third type of ambiguity in identifying the radius $R$ on the basis of the chiral characters alone: as we have seen in (5.14), the radius is also determined in part by the manner in which these chiral characters are joined in the full left/right partition function. In the absence of a full partition function, therefore, we shall generally assume that a given set of characters $\{\chi_{N,k}\}$ corresponds to that radius for which the corresponding partition function is diagonal. This then amounts to the choice $R = \sqrt{N}/2$, which will be used for discussion purposes. None of our results, however, will depend on this particular choice.

### 5.2 The Post-Projection CFT’s as Compactified Bosons

We now turn to the fundamental issue: that of relating our $A$- and $B$-sector post-projection CFT’s to those of compactified bosons.

As expected from two-dimensional boson/fermion equivalence, there exists a natural relationship between our Ising-like theories formulated on rescaled momentum lattices, and free bosons compactified on circles of arbitrary radius. Explicitly, from (5.12), we have in general

\[ \vartheta_2(\lambda \tau) = 2 \Theta_{4\lambda,\lambda}(\tau), \]

and thus it is evident that $\vartheta_2(\lambda \tau)$ is indeed the character of a certain winding-mode sector of a boson compactified on a circle of radius $R = \sqrt{\lambda}$. Further integer scalings of $R$ are of course also possible. This result indicates that it is possible to relate the fractional-superstring sectors found in previous sections to their compactified-boson counterparts for $K = 4$ and $K = 8$. (For $K = 16$ our light-cone post-projection CFT is only that of the $A$-sector: a single coordinate boson tensored with the $c = 1/2$ Ising model. Thus of course no bosonized description is possible.)

Let us concentrate first on the $A$-sectors. For the $K = 8$ case, we have found that the post-projection CFT in the $A$-sector contains three sectors denoted $\tilde{A}_8$, $U_8$, and $V_8$, and as discussed in Sect. 3, this theory is indeed that of the Dirac fermion.

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The Dirac fermion theory, however, is equivalent to that of a boson compactified on a circle of radius $R = 1$. Thus we can easily relate their corresponding characters:

$$
U_8 = \eta^{-2} \chi_{4,0} \\
\tilde{A}_8 = 2\eta^{-2} \chi_{4,1} \\
V_8 = \eta^{-2} \chi_{4,2}.
$$

(5.17)

Similarly, for the $K = 4$ case, the post-projection CFT again contains three sectors, and as discussed this theory is the “minimal” Ising-like diagonal subset of the $(\text{Dirac})^2$ theory. Using the characters given in Sect. 3, we find that this too can be easily re-expressed in terms of products of the compactified-boson characters:

$$
U_4 = \eta^{-4} \left[ (\chi_{4,0})^2 + (\chi_{4,2})^2 \right] \\
\tilde{A}_4 = 4\eta^{-4} (\chi_{4,1})^2 \\
V_4 = 2\eta^{-4} \chi_{4,0} \chi_{4,2}.
$$

(5.18)

The pattern of these products of characters thus indicates the “diagonal” nature of this subset theory relative to the full two-boson tensor-product theory. An important point to which we will return in Sect. 7, however, is the fact that these fractional-superstring sectors are related to products of the same $N = 4$ compactified-boson theory regardless of the value of $K$.

We now turn, however, to the corresponding $B$-sectors — indeed, it is for interpreting the $B$-sectors that the compactified-boson language is especially appropriate. In Sect. 4, it was demonstrated that for each value of $K$ these $B$-sector CFT’s each contain nine sectors: these could be organized into three groups of three each, with corresponding characters denoted $\tilde{B}_K^{(s)}, X_K^{(s)}, Y_K^{(s)}$ where the index $s = a, b, c$ denotes the particular group. The explicit definitions of these characters in terms of the parafermionic string functions were given in (4.3) and (4.4). For the $K = 8$ case, we find that these can now be simply identified as the characters of the $N = 20$ compactified-boson system:

$$
X_8^{(a)} = \eta^{-2} \chi_{20,0} \\
\frac{1}{2} \tilde{B}_8^{(a)} = \eta^{-2} \chi_{20,5} \\
Y_8^{(a)} = \eta^{-2} \chi_{20,10} \\
\frac{1}{2} X_8^{(b)} = \eta^{-2} \chi_{20,4} \\
\frac{1}{2} \tilde{B}_8^{(b)} = \eta^{-2} (\chi_{20,1} + \chi_{20,9}) \\
\frac{1}{2} Y_8^{(b)} = \eta^{-2} \chi_{20,6} \\
\frac{1}{2} X_8^{(c)} = \eta^{-2} \chi_{20,8} \\
\frac{1}{2} \tilde{B}_8^{(c)} = \eta^{-2} (\chi_{20,3} + \chi_{20,7}) \\
\frac{1}{2} Y_8^{(c)} = \eta^{-2} \chi_{20,2}.
$$

(5.19)
These results thus constitute new identities relating the characters of $Z_K$ parafermions and the characters of compactified bosons; they are the $B$-sector analogues of the identities (5.29) found for the $A$-sectors.  

Similarly, for the $K = 4$ case, it was shown in Sect. 4 that the post-projection CFT in the $B$-sector is equivalent to a certain combination of two scaled-lattice fermions with $\lambda = 3$ (tensored together with, of course, four uncompactified coordinate bosons). Thus, the $K = 4$ characters can now also be simply identified as products of the characters of the $N = 12$ compactified-boson system, and indeed we find the results

\[
\begin{align*}
\frac{1}{4} X_4^{(a)} &= \eta^{-4} \left( (\chi_{12,0})^2 + (\chi_{12,6})^2 \right) \\
\frac{1}{4} \tilde{B}_4^{(a)} &= \eta^{-4} (\chi_{12,3})^2 \\
\frac{1}{4} Y_4^{(a)} &= \eta^{-4} \chi_{12,0}\chi_{12,6} \\
\frac{1}{4} X_4^{(b)} &= \eta^{-4} \left( (\chi_{12,2})^2 + (\chi_{12,4})^2 \right) \\
\frac{1}{4} \tilde{B}_4^{(b)} &= \eta^{-4} (\chi_{12,1} + \chi_{12,5})^2 \\
\frac{1}{8} Y_4^{(b)} &= \eta^{-4} \chi_{12,2}\chi_{12,4} \\
\frac{1}{8} X_4^{(c)} &= \eta^{-4} \left( \chi_{12,0}\chi_{12,4} + \chi_{12,2}\chi_{12,6} \right) \\
\frac{1}{8} \tilde{B}_4^{(c)} &= \eta^{-4} (\chi_{12,1} + \chi_{12,5}) \chi_{12,3} \\
\frac{1}{8} Y_4^{(c)} &= \eta^{-4} \chi_{12,0}\chi_{12,2} + \chi_{12,4}\chi_{12,6}.
\end{align*}
\]

Once again these results represent a set of new character identities.

The above results provide especially natural interpretations for the $A$- and $B$-sector fusion rules (3.1), (4.7), and (4.8) found in Sects. 3 and 4. Recall that a given character $\chi_{N,k}$ corresponds to that sector of the compactified-boson Fock space consisting of vacuum states with $U(1)$ charges satisfying (5.9). Thus, we see that the Ising-model fusion rules in the $K = 8$ $A$-sector arise naturally from (5.17) as the result of $U(1)$ charge conservation (5.2) in the compactified-boson theory, provided that we substitute $\chi_{N,k} \rightarrow \frac{1}{2} (\chi_{N,k} + \chi_{N,-k})$ (5.21) in each identity above. While such substitutions do not affect the validity of our identities as functions of $q$ (since $\chi_{N,k} = \chi_{N,-k}$), we now see that they are necessary in order to accurately reflect the physical states which contribute to each sector. [This substitution is analogous to that in Sect. 3.3, where it was found that $A_{K}^{b,f}$ in (3.22) should be replaced by $A_{K}$.] Similar results hold for the $K = 8$ $B$-sector. Additionally, for the $K = 4$ case, we see that $U(1)$ charge conservation naturally yields the $K = 4$ fusion rules provided that we also symmetrize our above two-boson expressions:

\[
\chi_{N,k_1}\chi_{N,k_2} \rightarrow \frac{1}{2} \left( \chi_{N,k_1}^{(1)}\chi_{N,k_2}^{(2)} + \chi_{N,k_2}^{(1)}\chi_{N,k_1}^{(2)} \right) \quad (5.22)
\]
where the superscripts indicate the relevant boson system. Thus, all of the fusion rules determined in Sects. 3 and 4 can be understood as the result of $U(1)$ charge conservation in the isomorphic compactified-boson theory, and the multiplicities found in (4.7) and (4.8) are now easily accounted for. Indeed, for the $K = 8$ $B$-sector, we see that Ising copy $(a)$ consists of those compactified-boson sectors with $\alpha = 0 \pmod{5}$, whereas copy $(b)$ contains sectors with $\alpha = \pm 1 \pmod{5}$, and copy $(c)$ contains sectors with $\alpha = \pm 2 \pmod{5}$. The $K = 4$ $B$-sector is then the similar “minimal” combination of two $N = 12$ theories which preserves this same fusion-rule structure. The major difference between the $A$- and $B$-sectors is the fact that the $B$-sector radius depends on the Kac-Moody level $K$ of the theory, whereas the $A$-sector radius is fixed:

$$R_A = 1, \quad R_B = \sqrt{\frac{1}{2}(K + 2)}.$$  

5.3 Internal Projections and the $C$-Sectors

Until now we have had little to say about the $C$-sectors of the fractional superstring. Before concluding this section, therefore, we shall briefly discuss the conformal-field-theoretic significance of the fact that the $C$-sectors contain no physical states.

Although we have seen in (2.17) that $A_K$, $B_K$, and $C_K$ all vanish, this was interpreted as the result of spacetime supersymmetry: for the $A$- and $B$-sectors, this does not imply the absence of bosonic and fermionic states, but merely the equality of their numbers. It is for this physical reason that one cannot simply “ignore” the $B$-sectors, or use the result $B = 0$ to claim that no such terms need appear in the partition function. According to (2.24), however, the $C$-sectors contain no states of either spacetime statistics, for all of the physical states in this sector are apparently removed by the internal projections. One wonders, therefore, whether it is consistent to set each $C_K$ to zero in the partition functions (2.14), and to ignore the $C$-sectors altogether.

This question can also be phrased more mathematically. The $B$- and $C$-sectors were originally discovered via the construction of the modular-invariant partition functions (2.14), for it was found via the modular transformation properties of the parafermionic string functions that the terms $B_K$ and $C_K$ are necessary in order for each $Z_K(\tau)$ to be invariant under $S: \tau \to -1/\tau$ and $T: \tau \to \tau + 1$. However, since $A_K$, $B_K$, and $C_K$ are each vanishing modular functions of $\tau$, each of these terms is clearly modular invariant by itself. In what sense, then, does modular invariance require the presence of the $B$- or $C$-sectors when forming $Z_K$?

On a mathematical level, the answer to this question concerns not the modular group per se, but rather the representations of the modular group. In general, starting from a given set of characters $\chi_i$, one determines the matrices $S_{ij}$ and $T_{ij}$ which describe their mixings under the $S$ and $T$ modular transformations. These matrices clearly form a representation of the modular group, and satisfy the modular-group
defining relations $S^2 = (ST)^3 = 1$. One then takes products of these characters in order to construct a partition function which is not merely modular invariant as a function of $\tau$, but which is also consistent with the original modular group representation specified by $S_{ij}$ and $T_{ij}$. Physically, this is tantamount to saying that our partition functions, though modular invariant, must also be consistent with an underlying CFT interpretation.

This then explains the difference between the $A$ or $B$-sectors and the $C$-sectors. With respect to the pre-projection CFT, namely the parafermion CFT with its associated string-function characters, only the full partition functions $Z_K$ are self-consistent, and one cannot drop any individual term. The internal projections, however, not only remove the $C$-sector states: they also change the underlying worldsheet CFT’s. With respect to these new post-projection CFT’s, therefore, it is self-consistent to drop the $C_K$ expressions entirely. Indeed, one of the important properties of these smaller post-projection CFT’s is precisely that they furnish us with alternative representation matrices $S_{ij}$ and $T_{ij}$ in terms of which the $C$-sectors are completely decoupled under modular transformations. These are of course the matrices (5.10) associated with the various chiral compactified-boson characters.

Thus our compactified-boson picture is fully consistent with the absence of states in the $C$-sectors, furnishing us with representation matrices $S_{ij}$ and $T_{ij}$ with respect to which the absence of $C$-sector terms in the partition function causes no inconsistency. One unfortunate consequence of this picture, however, is the fact that the $A$- and $B$-sector theories are apparently decoupled, corresponding to different compactified-boson theories with different radii of compactification. Several approaches towards dealing with this problem will be outlined in Sect. 7.

6 Spacetime Statistics and the Twist Current

We now turn to the so-called “twist currents” introduced independently in [6] and [7]. As we shall see, these twist currents turn out to be of far-reaching importance in both the pre-projection and post-projection CFT’s.

These twist currents appear in a number of different ways. First, as noted in [6, 7], they effect a reorganization of the fractional-superstring Fock space during the internal projections, playing a role reminiscent of that played by the “screening operators” in the Coulomb gas construction of the Virasoro minimal models from a free boson theory. Furthermore, as demonstrated in [6], they serve as the basis of an alternative derivation of the fractional-superstring partition functions (2.14). Here, however, we shall be using these currents as tools for determining some of the spacetime statistics properties of the various sectors of our post-projection CFT’s.

In particular, we will begin by showing that the actions of the twist currents are symmetries of the entire post-projection CFT’s which we have constructed in Sects. 3 and 4. This suggests that although these currents have thus far been constructed only in terms of the primary fields (i.e., the parafermion fields) of the pre-projection
CFT, they might be described directly in terms of the primary fields of the post-projection compactified-boson CFT as well. We shall show that this is indeed the case, and using the bosonized formalism of Sect. 5 we shall find that these twist currents are in fact isomorphic to certain primary fields in the compactified-boson theory. This will in turn allow us to cast our fractional-superstring partition functions in lattice-like language (i.e., as a sum over the sites of an internal shifted momentum lattice) in such a way that spacetime supersymmetry is naturally incorporated, no internal projection remains, and the spacetime statistics of the sectors are automatically taken into account in the usual way by a lattice shift vector (or “statistics vector”). These results, while indicating a certain self-consistency for the fractional-superstring spacetime statistics assignments, will also dramatically illustrate some of the technical difficulties involved in constructing representations of these post-projection CFT’s. These results can then hopefully serve as a guide in any future construction.

6.1 The Twist Current as Parafermion Primary Field

We begin by briefly reviewing the twist current and its properties. In determining how our original parafermionic worldsheet CFT is projected down to smaller effective CFT’s in both the $A$- and $B$-sectors, we saw that it was necessary to build a mapping between the respective highest-weight sectors of these two theories. In particular, this entailed determining those $\mathcal{Z}_K$ parafermion sectors which contributed directly (i.e., additively) to the fractional-superstring Fock space, and those which served as corresponding “projection sectors” whose states were subtracted from (rather than added to) this Fock space. It turns out that there exists a simple rule for determining which sectors serve as projection sectors, given those that serve as non-projection sectors. Looking at the definitions of $A^{b,f}_K$ and $B^{b,f}_K$ given in (2.21) and (2.23) for $K > 2$, we see that replacing $d^\ell_n \to d^\ell_{n+K/2}$ in each string-function combination $d^\ell_n$ maps terms occurring with positive coefficients to those with negative coefficients and vice versa. (Recall that $d^\ell_{n+K} = d^\ell_n$.) Indeed, this pattern is even slightly more involved, for this substitution transforms the non-projection sectors of the spacetime bosonic expressions $A^K_b$ and $B^K_b$ into the projection sectors of the spacetime fermionic expressions $A^K_f$ and $B^K_f$, and vice versa. Thus this substitution not only exchanges projection sectors with non-projection sectors, but induces a spacetime-statistics flip as well.

What operator in the parafermionic tensor-product CFT could have this effect under fusion? From the fusion rules (2.23), it is a simple matter to see that this operation $d^\ell_n \to d^\ell_{n+K/2}$ is the result of a fusion with the parafermion field $\phi^0_{K/4}$. Thus we introduce the general “twist current” \[ \Psi_K \equiv \bigotimes_{\mu=1}^{D_{n-2}} (\phi^0_{K/4})^\mu \]  
which has this effect under fusion in both the $A$- and $B$-sectors. This operator is
called a twist current for several reasons, among them the fact that the parafermionic fields $\phi^0_m$ are in general often referred to as the “currents” in the $\mathbb{Z}_K$ parafermion theory, and the fact that currents of this form (once tensored together with a suitable anti-holomorphic counterpart) can be used in general to “twist” a modular-invariant partition function in order to generate new modular-invariant combinations. Note that this current is not a simple generalization of the spin-field $\sigma^8 = (\phi^{1/2}_1 / 2 \pm 1/2)^8$ which would seem to play this role in the $K = 2$ case; indeed, for $K = 2$, this twist current in (6.1) does not even exist (since there are no fields $\phi^0_{1/2}$ with $j - m \notin \mathbb{Z}$). Rather, we will see in Sect. 6.2 that the current (6.1) appears as a generalization of the appropriate $K = 2$ twist current only when it is expressed in terms of the primary fields of the post-projection CFT.

Let us now investigate the general properties of this parafermionic twist current $\Psi_K$, starting from the fusion rules (2.3). Since each factor of $\phi^0_{K/4}$ in (6.1) corresponds to a different spacetime dimension and hence functions independently under the fusion rules, we shall concentrate on only a given single component $\phi^0_{K/4} = \phi^K_{-K/4}$.

It is simple to see under the fusion rules that $(\phi^0_{K/4})^2 = \phi^0_{K/2}$, and thus in general $(\phi^0_{K/4})^2$ is not the identity. Rather, operating on a given parafermion field $\phi^i_m$, we find that

$$(\phi^0_{K/4})^2 : \phi^i_m \leftrightarrow \phi^{K/2 - j}_m.$$  

(6.2)

Thus, under $(\phi^0_{K/4})^2$, only the combination of sectors $[\phi^i_m] + [\phi^{K/2 - j}_m]$ is invariant.

Remarkably, however, all of the parafermion sectors which appear in our fractional superstrings do so in precisely these combinations, and indeed all of the partition functions for $K > 2$ can be expressed solely in terms of the $d^{\ell}_n \equiv c^{\ell}_n + c^{K-\ell}_n$ combinations. This is ultimately a consequence of the fact that the pre-projection fractional-superstring worldsheet theory is constructed using only that subset of the $\mathbb{Z}_K$ parafermion theory involving the integer-spin fields $\phi^i_m$. Thus, for all sectors appearing in our fractional superstrings, we find that $(\phi^0_{K/4})^2$ functions as the identity, and indeed we can associate

$$(\Psi_K)^2 \iff 1.$$  

(6.3)

This is clear for the $A_K$ and $B_K$ sectors in (2.21) and (2.23), where we explicitly have

$$\Psi_K : \begin{cases} A^{b,f}_K \leftrightarrow - A^{f,b}_K, \\ B^{b,f}_K \leftrightarrow - B^{f,b}_K, \end{cases}$$

(6.4)

which implies

$$\Psi_K : \begin{cases} \tilde{A}_K \rightarrow - \tilde{A}_K, \\ \tilde{B}_K \rightarrow - \tilde{B}_K. \end{cases}$$

(6.5)

The relative minus sign introduced by each application of $\Psi_K$ is simply the reflection of the exchange of projection and non-projection sectors.
In Sects. 3 and 4 we found that ˜\(A_K\) and ˜\(B_K\) each correspond to merely one sector of their respective fractional-superstring post-projection CFT’s. It is thus natural to ask whether the action of \(\Psi_K\) appears as a general symmetry of the entire post-projection theories. We have already determined that the two remaining sectors of our \(A\)-sector post-projection theory are \(U_K\) and \(V_K\), however, and their character definitions are given in (3.12) and (3.20). Under the twist current, then, we indeed find

\[
\Psi_K : \quad U_K \leftrightarrow -V_K, 
\]

and thus in analogous fashion we interpret \(U_K\) and \(V_K\) as corresponding to sectors of opposite spacetime statistics. (We stress that unlike \(A^b_K\) and \(A^f_K\), these additional sectors \(U_K\) and \(V_K\) are not spacetime superpartners of each other; in particular, they have different highest weights and contain states at unequal mass levels.) We therefore see that the action of \(\Psi_K\) is in fact a symmetry of the entire \(A\)-sector post-projection CFT:

\[
\Psi_K : \quad \tilde{A}_K \leftrightarrow -\tilde{A}_K \\
U_K \leftrightarrow -V_K. 
\]  

We find similar results for the \(B\)-sector post-projection CFT, which was shown in Sect. 4 to contain a total of nine sectors. These nine sectors were denoted \(U^{(s)}_K\), \(\tilde{B}^{(s)}_K\), and \(V^{(s)}_K\) for \(s = a, b, c\), and their characters were given in terms of string functions in (4.3) and (4.4). Under the twist current, we likewise find

\[
\Psi_K : \quad \tilde{B}^{(s)}_K \leftrightarrow -\tilde{B}^{(s)}_K \\
X^{(s)}_K \leftrightarrow -Y^{(s)}_K \quad (s = a, b, c), 
\]

and thus we see that the effect of the twist current in the \(B\)-sectors is completely analogous to that for the \(A\)-sectors, with each \(B\)-sector Ising-like copy transforming under \(\Psi_K\) precisely as do the minimal Ising-like \(A\)-sector theories, and with projection and non-projection sectors interchanged by \(\Psi_K\).

The results (6.7) and (6.8) strongly suggest that sectors paired by the action of \(\Psi_K\) contain states of opposite spacetime statistics, with the sectors \(\tilde{A}_K\) and \(\tilde{B}^{(a)}_K\) containing states of both spacetime statistics. What all of these statistics actually are, though, is largely irrelevant from the point of view of spacetime physics, for most of these extra sectors are of course ultimately GSO-projected out of the fractional-superstring spacetime spectrum in a manner consistent with modular invariance. Indeed, only \(\tilde{A}_K\) and \(\tilde{B}^{(a)}_K\) actually contribute particles to the fractional-superstring spectrum, and these are presumably bosonic and fermionic. We shall seek to understand the spacetime statistics of these states in more detail in Sect. 6.3.

### 6.2 The Twist Current as a Compactified-Boson Primary Field

In Sect. 6.1 the twist current was constructed and analyzed in terms of the underlying parafermionic CFT’s which were the original basis of the fractional-superstring
worldsheet theory. We have just demonstrated, however, that the action of this current is a symmetry of the entire post-projection CFT, suggesting that this twist current is feature of the original parafermionic CFT which survives the internal projections. What will interest us here, therefore, is the construction of an operator directly in the isomorphic post-projection compactified-boson CFT which has the same effect. We will find that this is indeed possible, and that the twist current is in fact isomorphic to a certain primary field in the compactified-boson CFT. This will thereby enable us to interpret the effects of the twist current on the various $A$- and $B$-sectors in terms of the properties [such as the fusion rules and $U(1)$ charge conservation] of the isomorphic compactified-boson theory. This is of paramount importance if we are to understand the fractional-superstring post-projection CFT (and the associated twist current) without reference to its original parafermionic formulation.

Our line of attack will be first to determine the effect of the current $\Psi_K$ on the characters of the compactified-boson theory, and only subsequently to express this operator directly in terms of the primary fields of the compactified-boson theory. Let us first concentrate on the $K = 8$ case, which turns out to be somewhat simpler. We have already seen in (5.17) how the three characters $\tilde{A}_8$, $U_8$, and $V_8$ of the $A$-sector theory are related to those of the $N = 4$ compactified boson theory, and in terms of these latter characters, the action (5.7) of the twist current immediately implies:

$$
\Psi_8 : \quad \chi_{4,1} \leftrightarrow -\chi_{4,1}, \quad \chi_{4,0} \leftrightarrow -\chi_{4,2} .
$$

It is thus evident in this bosonized language that the action of this current is simply:

$$
\Psi_8 : \quad \chi_{N,k} \longrightarrow -\chi_{N,N/2+k}
$$

where we recall that $\chi_{N,\pm k} = \chi_{N,N+k}$. However, this form is not quite general enough for our purposes. For example, recall from (5.15) that these three $N = 4$ characters could also generally be written as linear combinations of the $N = 4n^2$ characters, where $n \in \mathbb{Z}$. If we had we chosen to write our fractional-superstring characters $\tilde{A}_8$, $U_8$, and $V_8$ in terms of the characters of any of these $N = 4n^2$ systems for $n > 1$, the above action of $\Psi_8$ would not have the desired effect. We therefore require a general action invariant under any integral radius rescaling. The arguments just above (5.15), however, indicate that if $R$ is rescaled by a factor of $n$, then $N$ is rescaled by a factor of $n^2$ but $k$ is rescaled only by a factor of $n$; indeed, (5.13) is the statement that for $R \in \mathbb{Z}$, the quantities

$$
\sqrt{N-1} \sum_{p=0}^{\sqrt{N-1}} \chi_{N,k+p\sqrt{N}} , \quad 0 \leq k < N
$$

are invariant under $N \rightarrow n^2 N$, $k \rightarrow nk$. Thus, in order for our twist-current action to be invariant under these radius rescalings, we must modify (6.10):

$$
\Psi_8 : \quad \chi_{N,k} \longrightarrow -\chi_{N,\sqrt{N}+k} .
$$
This result is consistent with the requirement $k \in \mathbb{Z}$, for $\sqrt{N}$ is always an integer for $N = 4n^2$.

The above action (6.12) thus describes the parafermionic twist current for the $A$-sectors, whose characters, we recall, can be formulated in terms of $\vartheta$-functions with unscaled arguments (i.e., with scaling parameter $\lambda = 1$). Let us now proceed to the $B$-sector, which for $K = 8$ has scaling parameter $\lambda = 5$. Here our mapping to compactified-boson characters in (5.19) proves especially fruitful, for in terms of these characters we see that the twist current $\Psi_8$ again has the simple action $\chi_{N,k} \rightarrow -\chi_{N,\sqrt{5}N+k}$. Rewriting this result so that it is invariant under integer radius-rescalings, we therefore have our result for the $B$-sector:

$$\Psi_8 : \chi_{N,k} \rightarrow -\chi_{N,\sqrt{5}N+k}.$$  \hspace{1cm} (6.13)

Thus, it is clear that (6.12) and (6.13) for both the $A$- and $B$-sectors can be generally written together in the $K = 8$ case as

$$\Psi_8 : \chi_{N,k} \rightarrow -\chi_{N,\sqrt{5}N+k}.$$  \hspace{1cm} (6.14)

This then reproduces the action of $\Psi_8$ at the level of the partition-function characters in both the $A$- and $B$-sectors, with the minus sign representing the projection/non-projection sector interchange.

We now turn to the $K = 4$ case, which is slightly more difficult because we are now dealing with bilinears of compactified-boson characters. In the $A$-sector, we know from the identities (5.18) that our current must transform

$$\Psi_4 : \begin{array}{c}
(\chi_{4,1})^2 \leftrightarrow - (\chi_{4,1})^2 \\
(\chi_{4,0})^2 + (\chi_{4,2})^2 \leftrightarrow -2 \chi_{4,0} \chi_{4,2}
\end{array}.$$  \hspace{1cm} (6.15)

Because these expressions are now bilinears of compactified-boson characters, however, there are two possible means of effecting these transformations. The first is via the rule

$$\Psi_4 : \chi_{N,k} \rightarrow \frac{1}{\sqrt{2}} \left( e^{i\pi/4} \chi_{N,k} - e^{-i\pi/4} \chi_{N,\sqrt{N}-k} \right),$$  \hspace{1cm} (6.16)

which is again invariant under integral radius rescalings. This rule, however, has the undesirable property that phases are introduced as the coefficients of characters, and although this causes no problem for the particular combinations of character bilinears that we face, this rule is clearly not conducive to a general interpretation. Thus, we instead adopt a second possibility:

$$\Psi_4 : \chi_{N,k_1} \chi_{N,k_2} \rightarrow -\frac{1}{2} \left( \chi_{N,k_1} \chi_{N,\sqrt{N}+k_2} + \chi_{N,\sqrt{N}+k_1} \chi_{N,k_2} \right),$$  \hspace{1cm} (6.17)

which has the advantage that all phases are manifestly avoided for all combinations of character bilinears. Using (5.20), it is easy to check that this rule applies for the
$B$-sectors of the $K = 4$ theory as well, with $\sqrt{N}$ replaced by $\sqrt{5N}$. We thus have the general twist-current rule for the $K = 4$ case:

$$\Psi_4 : \chi_{N,k_1} \chi_{N,k_2} \longrightarrow -\frac{1}{2} \left( \chi_{N,k_1} \chi_{N,\sqrt{5N}+k_2} + \chi_{N,\sqrt{5N}+k_1} \chi_{N,k_2} \right). \quad (6.18)$$

Having thus rather simply described the action of the currents in terms of the relevant characters of the compactified-boson theory, we now turn to the next task: we would like to mimic the situation for the twist current in our original parafermionic CFT, and express this current directly in terms of the underlying primary fields of the compactified-boson theory. The above actions for $\Psi$ could then be understood as the by-products of the compactified-boson fusion rules. This can be done rather straightforwardly. For the $K = 8$ case, we have seen that the general action of the twist current is given in (6.14), where the minus sign reflects the exchange of projection and non-projection sectors. In the post-projection theory, however, we do not have any internal projections remaining, and thus at the level of this underlying CFT we see that our current must simply transform the primary-field sectors contributing to $\chi_{N,k}$ into those that contribute to $\chi_{N,\sqrt{5N}+k}$. It is a simple matter to determine the primary fields which effect this change under fusion. The original sectors are parametrized by their $U(1)$ charges $\alpha$, and we see from (5.9) that these charges all satisfy $\sqrt{N\alpha} = k \pmod{N}$. The sectors contributing to $\chi_{N,\sqrt{5N}+k}$, however, have charges $\alpha'$ satisfying $\sqrt{N\alpha'} = k + \sqrt{N} \pmod{N}$, and thus in order to effect this transformation we must increase the charge of each sector by an amount $\Delta\alpha \equiv \alpha' - \alpha$ satisfying

$$\Delta\alpha = \sqrt{\lambda} \pmod{N}. \quad (6.19)$$

Since (5.2) indicates that this $U(1)$ charge is a conserved quantum number under fusion, we see that the effect of the current $\Psi_8$ can thus be understood simply as the result of fusion with any of the primary fields $e^{i(N\alpha + \sqrt{\lambda})}$, $p \in \mathbb{Z}$. Choosing the simplest case $p = 0$, then, we have the result:

$$\Psi_8 = e^{i\sqrt{\lambda}} = e^{i\sqrt{\lambda}\phi}. \quad (6.20)$$

(This result is to be interpreted not as a strict equality, of course, but rather as an isomorphic relation.) Similar arguments for the $K = 4$ case yield the analogous result in the two-boson system:

$$\Psi_4 = \frac{1}{2} \left( 1_{(1)} \otimes e^{i\sqrt{\lambda}\phi_2} + e^{i\sqrt{\lambda}\phi_1} \otimes 1_{(2)} \right). \quad (6.21)$$

where $1_{(n)}$ indicates the identity field in the conformal field theory of the $n^{th}$ compactified boson $\phi_n$.

Given these results, it is straightforward to verify that the action of $(\Psi_8)^2$ is in accordance with (6.13), for $(\Psi_8)^2$ indeed transforms the set of sectors with charges satisfying $\sqrt{N\alpha} = k \pmod{N}$ into itself for each value of $k$. This follows trivially as a consequence of the fact that $2\sqrt{\lambda N} = 0 \pmod{N}$, so that the effective value
of $k$ is increased under $(\Psi_8)^2$ by $\Delta k = N$. According to (5.4), this increase in $k$ is tantamount to a shift in the one-dimensional boson momentum lattice by one full lattice spacing, and thus we see that $\Psi_8$ itself shifts this momentum lattice by one half-lattice spacing. This in turn implies that bosonic and fermionic states are shifted by half-lattice spacings relative to each other, an observation we shall discuss below. Note, however, that since (6.20) effects a simple lattice translation $\Delta \alpha$, it squares to one only by acting on precisely those lattices with lattice spacing $2\Delta \alpha$.

The $K = 4$ case is analogous but more complicated. The current (6.21) also does not manifestly square to 1 when acting on arbitrary two-dimensional lattices; rather, we find

$$
(\Psi_4)^2 = \frac{1}{4} \left( 1 \otimes e^{i2\sqrt{\lambda} \phi_2} + e^{i2\sqrt{\lambda} \phi_1} \otimes 1 \right) + \frac{1}{2} e^{i\sqrt{\lambda} \phi_1} \otimes e^{i\sqrt{\lambda} \phi_2},
$$

(6.22)

and while the first two terms are again simple lattice-preserving translations, the third term represents a diagonal lattice translation which in general does not preserve the lattice. The sectors (5.20) of our $B$-sector CFT’s for $K = 4$ consist of only those special two-dimensional lattices for which this third term is lattice-preserving, however, and thus we find $(\Psi_4)^2 = 1$ when acting on the entire lattice.

### 6.3 Lattices and Spin-Statistics

We close this section by discussing the relationship between the momentum lattices appearing in these $K > 2$ fractional superstring theories, and the spacetime statistics of the corresponding states. We will find a close similarity with the analogous situation for ordinary $K = 2$ superstrings, as well as some crucial differences. Indeed, these differences will illustrate quite dramatically that our post-projection theories are ultimately not equivalent to compactified-boson theories, and that despite the isomorphism between the two which has been discussed in previous sections, an alternative non-bosonic representation for our post-projection CFT’s remains to be found.

In the $K = 8$ case, we have seen that the twist current $\Psi_8$ amounts to a translation of the one-dimensional lattice by a half-lattice spacing, so that bosonic and fermionic states are shifted by half-lattice spacings relative to each other. This observation provides a natural explanation for the fact, discussed in Sect. 3.3, that the $\tilde{A}_8$ and $\tilde{B}_8 \equiv \tilde{B}_8^{(a)}$ sectors contain states of both bosonic and fermionic spacetime statistics. Indeed, (5.17) and (5.19) tell us that

$$
\tilde{A}_8 \equiv \frac{1}{2} \left( A_8^b + A_8^f \right) = \eta^{-2} (\chi_{4,1} + \chi_{4,-1})
$$

$$
\tilde{B}_8 \equiv \frac{1}{2} \left( B_8^b + B_8^f \right) = \eta^{-2} (\chi_{20,5} + \chi_{20,-5}),
$$

(6.23)

yet we see that in general the lattices corresponding to $\chi_{N,\sqrt{\lambda}N/2}$ and $\chi_{N,-\sqrt{\lambda}N/2}$ are shifted by exactly a half-lattice spacing relative to each other. Thus, we can
separately identify the bosonic and fermionic sectors with the lattice sites of $U(1)$ charges $\alpha^b, \alpha^f$ satisfying

$$K = 8: \quad \alpha^b = + \frac{1}{2} \sqrt{\lambda} \quad (\text{mod } 2\sqrt{\lambda})$$
$$\alpha^f = - \frac{1}{2} \sqrt{\lambda} \quad (\text{mod } 2\sqrt{\lambda})$$

while maintaining consistency with the twist current interpretation. Thus we see that $\tilde{A}_8$ and $\tilde{B}_8$ need not each correspond to a single vacuum state at all: rather, they each correspond to an infinite number of vacuum ground states which can naturally be divided into two classes according to (6.24).

A similar situation exists for the $K = 4$ case. Here our identities (5.18) and (5.20) tell us that

$$\tilde{A}_4 \equiv \frac{1}{2}(A^b_4 + A^f_4) = \eta^{-4} \left[ \chi^{(2)}_{4,1} \chi^{(2)}_{4,-1} + \chi^{(2)}_{4,1} \chi^{(2)}_{4,-1} + \chi^{(1)}_{4,-1} \chi^{(1)}_{4,1} \right]$$
$$\tilde{B}_4 \equiv \frac{1}{2}(B^b_4 + B^f_4) = \eta^{-4} \left[ \chi^{(2)}_{12,3} \chi^{(2)}_{12,-3} - \chi^{(2)}_{12,3} \chi^{(2)}_{12,-3} + \chi^{(1)}_{12,-3} \chi^{(2)}_{12,3} \right]$$

where the superscripts explicitly indicate the two dimensions of the momentum lattice. Under the action of the $K = 4$ twist current (6.21), however, we see that the first two terms and the last two terms in each line of (6.25) are transformed into each other. There thus again exists a natural boson/fermion identification consistent with the twist current:

$$K = 4: \quad \tilde{\alpha}^b = \left( \pm \frac{1}{2} \sqrt{\lambda}, \pm \frac{1}{2} \sqrt{\lambda} \right) \quad (\text{mod } 2\sqrt{\lambda})$$
$$\tilde{\alpha}^f = \left( \pm \frac{1}{2} \sqrt{\lambda}, \mp \frac{1}{2} \sqrt{\lambda} \right) \quad (\text{mod } 2\sqrt{\lambda}).$$

These identifications of bosonic and fermionic states in fact satisfy a number of other properties reminiscent of so-called “lattice strings” (among which is the ordinary $K = 2$ superstring). As is well-known, lattice strings have states whose two dimensional (worldsheet) left-moving and right-moving momenta together form a $\frac{1}{2}(D-2), \frac{1}{2}(D-2)$-dimensional “shifted Lorentzian lattice” $\Lambda$: this means that any state $\tilde{\alpha} = (\tilde{\alpha}^{\text{left}} | \tilde{\alpha}^{\text{right}})$ can be written as $\tilde{\alpha} = \tilde{L} + \tilde{S}$ where $\tilde{S}$ is a constant shift-vector known as a “statistics vector”, and where the set of vectors $\tilde{L}$ forms a true lattice $\Lambda$, with inner product $\tilde{L}_1 \cdot \tilde{L}_2 \equiv \tilde{L}^{\text{left}}_1 \cdot \tilde{L}^{\text{left}}_2 - \tilde{L}^{\text{right}}_1 \cdot \tilde{L}^{\text{right}}_2$. (The shifted lattice $\Lambda$ is not a true lattice because $\tilde{\alpha}^{\text{right}}$ is not in $\Lambda$.) The spacetime spin-statistics of a given state $\tilde{\alpha} \in \Lambda$ are then generally discernible by computing $(\tilde{\alpha} - \tilde{S}) \cdot \tilde{S} = \tilde{L} \cdot \tilde{S}$:

$$\tilde{L} \cdot \tilde{S} \in \{ \mathbb{Z}, \mathbb{Z} + 1/2 \} \quad \text{bosonic}$$
$$\tilde{L} \cdot \tilde{S} \in \{ \mathbb{Z}, \mathbb{Z} + 1/2 \} \quad \text{fermionic}.$$
the interaction $1 + 2 \rightarrow 3$ the momentum of a final state $\vec{\alpha}_3$ is given not by the sum $\vec{\alpha}_1 + \vec{\alpha}_2 \notin \Lambda$, but rather by $\vec{\alpha}_1 + \vec{\alpha}_2 + \vec{S} \in \Lambda$.

For example, for the ordinary ten-dimensional (Type IIA) superstring, the $(4+4)$-dimensional lattice formed by the GSO-surviving Ramond and NS states is

$$\Lambda_{K=2} = \Lambda_L \otimes \Lambda_R$$

where the left- and right-moving lattices are given by

$$\Lambda_L = \Lambda_R = \left\{ n_1, n_2, n_3, n_4 \right\} \oplus \left\{ n_1 - \frac{1}{2}, n_2 - \frac{1}{2}, n_3 - \frac{1}{2}, n_4 - \frac{1}{2} \right\}$$

with $n_i \in \mathbb{Z}$ and $\sum n_i = \text{odd}$. The shift vector can then be taken to be

$$\vec{S}_{K=2} = (1, 0, 0, 0 | 1, 0, 0, 0)$$

whereupon the resulting lattice $\Lambda_2 - \vec{S}$ is a true lattice, with the spacetime bosonic and fermionic states $\vec{\alpha}$ respectively distinguished according to (6.27). From this we see that the chiral twist current $\Psi_2$ of the ordinary superstring takes the simple form

$$\Psi_2 = e^{i \sqrt{\lambda} \phi_i / 2} \otimes e^{i \sqrt{\lambda} \phi_i / 2} \otimes e^{i \sqrt{\lambda} \phi_i / 2} \otimes e^{i \sqrt{\lambda} \phi_i / 2}$$

where we are now explicitly indicating the chirality of the bosons $\phi_i$ (and where we of course have only an “A-sector” with $\lambda = 1$); this result is to be compared with (6.20) and (6.21), where a similar chirality is understood. Indeed, the partition function (3.3) of the ordinary superstring can now be recast in the usual fashion as a sum over lattice sites:

$$Z_2 = \tau_2^{-4} |\eta|^{-24} \sum_{\vec{\alpha} \in \Lambda_2} q^{(\vec{\alpha}^{\text{left}})^2 / 2} q^{(\vec{\alpha}^{\text{right}})^2 / 2} \exp \left[ 2\pi i (\vec{\alpha} - \vec{S}) \cdot \vec{S} \right],$$

and each three-point vertex is associated with an operator contributing the worldsheet momentum (6.31). The necessity for such a momentum insertion for the superstring in light-cone gauge was first perceived by Mandelstam in the early 1970’s [16], and indeed this operator contributes the conformal dimension $h = 1/2$ (that of the worldsheet supercharge) needed to produce a dimension-one vertex operator for consistent string emission.

For the fractional superstrings, the statistics assignments (6.24) and (6.26) permit a similar lattice interpretation. In particular, for the A-sectors (i.e., for $\lambda = 1$), these states together fill out the $(8/K + 8/K)$-dimensional shifted lattice

$$K > 2 : \quad \Lambda_K = \left\{ n_1 \pm \frac{1}{2}, ..., n_{8/K} \pm \frac{1}{2} \right\} \otimes \left\{ n_1 \pm \frac{1}{2}, ..., n_{8/K} \pm \frac{1}{2} \right\},$$

where $n_i \in \mathbb{Z}$ and where each sign is chosen independently. We can then simply take the shift vectors

$$\vec{S}_{K=4} = \left( \frac{1}{2}, \frac{1}{2} | \frac{1}{2}, \frac{1}{2} \right)$$

$$\vec{S}_{K=8} = \left( \frac{1}{2} | \frac{1}{2} \right)$$
whereupon the resulting set of states \( \Lambda_s \equiv \{ \vec{\alpha} - \vec{S} \} \) forms a true Lorentzian lattice, and the assignments (6.24) and (6.26) now follow directly as a consequence of (6.27). The \( B \)-sectors of these strings also have similar lattice descriptions, of course, with the relevant lattices scaled by a factor of \( \sqrt{\lambda} \). Thus we can write the fractional-superstring partition functions for each value of \( K \) in a manner analogous to (6.32):

\[
|A_K|^2 = 4 |\eta|^{-48/K} \sum_{\vec{\alpha} \in \Lambda_K} q^{(\vec{\alpha}_{\text{left}})^2/2} \bar{q}^{(\vec{\alpha}_{\text{right}})^2/2} \exp \left[ 2\pi i (\vec{\alpha} - \vec{S}) \cdot \vec{S} \right]
\]

\[
|B_K|^2 = 4 |\eta|^{-48/K} \sum_{\vec{\alpha} \in \sqrt{\lambda} \Lambda_K} q^{(\vec{\alpha}_{\text{left}})^2/2} \bar{q}^{(\vec{\alpha}_{\text{right}})^2/2} \exp \left[ 2\pi i (\vec{\alpha}/\sqrt{\lambda} - \vec{S}) \cdot \vec{S} \right].
\]

(6.35)

This result thus naturally incorporates spacetime supersymmetry, the internal projections, the spin-statistics assignments, and the actions of the twist currents in a natural manner. Indeed, these results suggest that the analogous fractional-superstring three-point vertex operator in light-cone gauge should correspond to a worldsheet momentum insertion \( \vec{S}_K \) with conformal dimension \( h = 1/K \). Note that with such an insertion, the spacetime statistics factor \((-1)^F\) is indeed a conserved quantum number in all three-point interactions, further suggesting that our assignments (6.24) and (6.26) are correct.

This analogy between the superstring lattice and these fractional-superstring lattices must not be pushed too far, however; fractional superstrings are not equivalent to lattice strings, and cannot be described in terms of free worldsheet bosons via a simple lattice-type construction. The simplest illustration of this is the fact that the heuristic fractional-superstring lattices described above are not self-dual; this failure follows as a trivial consequence of the rescaling of the \( B \)-sector lattices. Although the self-duality of an underlying lattice implies invariance of the partition function under the \( S \) modular transformation, it is certainly not the case that any \( S \)-invariant partition function corresponds to an underlying self-dual lattice: non-trivial counter-examples include orbifold-compactified string theories, which have no underlying lattice formulations. The lattice description presented here is thus meant only to demonstrate the consistency of our twist-current assignments (6.24) and (6.26), and their similarity to those of the ordinary superstring.

Indeed, this lattice description also illustrates quite dramatically why the fractional-superstring post-projection CFT’s cannot ultimately be represented in terms of free bosons. This isomorphism between our post-projection CFT’s and those of free bosons has thus far lead us to a lattice description in which each surviving fractional-superstring state of highest weight \( h \) is associated with a lattice site \( \vec{\alpha} \) satisfying \( h = \vec{\alpha} \cdot \vec{\alpha}/2 \) in such a way that the fusion rules of the post-projection CFT are equivalent to vector addition for \( \vec{\alpha} \). Our spacetime statistics assignments (6.24) and (6.26) are then the only ones consistent with the twist current, and manifestly imply spacetime supersymmetry. However, for \( K > 2 \) we cannot take the next step, and actually associate each lattice site \( \vec{\alpha} \) with the worldsheet boson field \( \exp(i \vec{\alpha} \cdot \phi) \).
The problem may be seen as follows. For the ordinary superstring, (chiral) lattice sites \( \vec{\alpha} \in \Lambda_{L,R} \) generally correspond to spacetime bosons or fermions depending on whether their lattice coordinates \( \alpha_i \) are integer or half-integer. This is consistent with their interpretation as arising from the fields \( \exp(i\vec{\alpha} \cdot \vec{\phi}) \) in the free-boson theory, for each of the primary fields \( \exp(\pm i\phi_i/2) \) is equivalent to a tensor product of two Ising-model spin fields \( \sigma \), and these spin fields create the necessary worldsheet cuts to alter the boundary conditions of worldsheet fermions and produce fermionic spacetime statistics. For \( K > 2 \), however, the statistics assignments in (6.24) and (6.26) clearly preclude any such free-boson representation; indeed, only the fermionic states in the \( A \)-sectors appear representable in this manner. Therefore an alternative representation for our light-cone worldsheet theory is needed, one which is consistent not only with these spacetime statistics assignments, but more generally with transverse \( (D_c - 2) \)-dimensional spacetime Lorentz invariance. Presumably the new massive \( B \)-sector states (with their additional rescaled lattice sites) will be important in this regard.

7 Summary and Concluding Remarks

In this paper we have taken the first steps towards a construction of the post-projection worldsheet conformal field theory of the fractional superstring for both the \( A \)-sectors and the \( B \)-sectors. By explicitly demonstrating how the internal projections rearrange the original parafermionic Fock space of the fractional superstring at the level of the corresponding characters, we have determined the central charges, the complete spectrum of highest weights, and the corresponding fusion rules of these post-projection conformal field theories: for the \( A \)-sectors these results are summarized in (3.29) with the characters of the \( A \)-subsectors given in (2.21), (3.12), and (3.20); and for the \( B \)-sectors the characters and highest weights are given in (4.3) and (4.4), with the fusion rules given in (4.7) and (4.8). We then demonstrated that all of these results have a natural reformulation in terms of worldsheet compactified bosons [Eqs. (5.17)–(5.20)], and this reformulation ultimately enabled us not only to demonstrate that the twist current is a symmetry of the entire post-projection CFT for both the \( A \) - and \( B \)-sectors [Eqs. (6.7) and (6.8)], but also to construct this current directly in terms of the primary fields of the post-projection CFT’s [Eqs. (6.20) and (6.21)]. This in turn permitted us to find a consistent separation of spacetime states into those that are spacetime bosonic or fermionic for both the \( A \) - and \( B \)-sectors [Eqs. (6.24) and (6.26)], and to subsequently express our fractional-superstring partition functions in lattice language [Eq. (6.35)] in a manner consistent with spacetime supersymmetry, the internal projections, and the twist-current spacetime statistics.

* Note that there are in fact two states at each lattice site, so that the massless states (i.e., those with \( \alpha^2 = 2/K \)) indeed have the correct multiplicities to fill out \( SO(D_c - 2) \) vector and spinor representations.
interpretation. Our results also exposed some of the outstanding problems that confront a “bottom-up” construction of suitable representations for these post-projection CFT’s, and taken together they can therefore be viewed as providing a set of minimal constraints that any future construction must satisfy.

As indicated at the end of Sect. 5.3, however, one of the results of our analysis is that the $A$- and $B$-sectors after the internal projections appear to be described as fully independent CFT’s with completely uncoupled fusion rules. A natural question then arises as to whether the $A$- and $B$-sector post-projection CFT’s can be described together as different subsectors of a single, larger post-projection CFT with central charge $c = 24/K$; indeed, we generally expect there to be interactions or couplings between $A$- and $B$-sector states, and these interactions should emerge naturally in a properly formulated larger CFT. We shall therefore conclude by briefly describing several different ideas with might ultimately prove useful in determining this single, larger light-cone CFT. We note, however, that it is of course conceivable that no such single worldsheet CFT exists for the light-cone version of the fractional superstring, and that the passage from a covariant formulation to light-cone gauge contains new features rendering the $A$- and $B$-sector subtheories apparently separate.

The first idea involves the compactification of bosons on orbifolds. We have seen in (5.23) that the $A$-sector post-projection CFT’s can be formulated as tensor products of bosons compactified on circles of radius $R_A = 1$, while the $B$-sectors correspond to products of bosons compactified on circles of radius $R_B = \sqrt{\lambda} = \sqrt{\frac{1}{2}(K + 2)}$. Thus, any single CFT which is to contain these theories as separate subtheories must simultaneously have characters corresponding to bosons of radius $R_A$ as well as characters corresponding to a different radius $R_B$. One group of theories which has this property is that of a boson compactified on an orbifold. For example, the partition function of a single boson compactified on a $\mathbb{Z}_2$ orbifold of general radius $R$ is given by

$$Z_{\text{orb}}(R) = \frac{1}{2} \left[ Z_{\text{circ}}(R) + 2 Z_{\text{circ}}(\sqrt{2}) - Z_{\text{circ}}(1/\sqrt{2}) \right]$$ (7.1)

where $Z_{\text{circ}}(R)$ is the partition function of the circle-compactified boson given in (5.13) and (5.14). In (7.1), the first term is the contribution from the untwisted sector, while

$$Z_{\text{twist}} \equiv Z_{\text{circ}}(\sqrt{2}) - \frac{1}{2} Z_{\text{circ}}(1/\sqrt{2}) = \frac{1}{2} |\chi_{8,0} - \chi_{8,4}|^2 + 2 |\chi_{8,1}|^2 + 2 |\chi_{8,3}|^2$$ (7.2)

is the contribution from the $\mathbb{Z}_2$-twisted sector. Other more complicated types of orbifold theories have partition functions similar to these [17]. Thus, we see that the use of orbifolds allows us to effectively combine circle-compactified theories of different radii, at least at the character (or partition-function) level. Furthermore, note that the untwisted sector has a radius which equals the radius of the orbifold, while the effective radii of the twisted sectors are constants which turn out to depend only on the type of orbifold being considered. This suggests that the post-projection CFT of the $K$-fractional superstring might consist of an $8/K$-fold tensor product of
$c = 1$ orbifold theories at radius $R_{\text{orb}} = \sqrt{\frac{1}{2}(K+2)}$ (along with the $D_c - 2 = 16/K$ original coordinate bosons): the $B$-sectors would then correspond to the \textit{untwisted} orbifold sectors, and the $A$-sectors would correspond to the \textit{twisted} orbifold sectors. This is certainly consistent with the observation that the $A$-sector is always a tensor product of the same $R = 1$ theory for each value of $K$, whereas the radius of the $B$-sector theory varies with $K$. Indeed, for the case of the $\mathbb{Z}_2$ orbifold, we find that the contribution $Z_{\text{twist}}$ in (7.2) can be rewritten in terms of \textit{unscaled} Jacobi $\vartheta$-functions regardless of the value of $R_{\text{orb}}$: \begin{equation}
abla_{\text{twist}} = \frac{1}{2} |\eta|^{-2} \left( |\vartheta_2 \vartheta_3| + |\vartheta_2 \vartheta_4| + |\vartheta_3 \vartheta_4| \right). \tag{7.3} \end{equation}

The problem with this $\mathbb{Z}_2$ orbifold approach, however, is readily apparent from (7.3): the $\mathbb{Z}_2$ orbifold theory does not contain the proper “diagonal” characters $\chi^{(c=8/K)}_h$ which (as we have found) comprise the $A$-sector CFT. Similar difficulties arise for the other types of $c = 1$ orbifold theories as well. Therefore, since these circle- and orbifold-compactified theories have been proven to completely span the space of unitary $c = 1$ CFT’s \cite{17, 18}, these observations indicate that this $c = 8/K$ component of the desired unified CFT will not be a simple $8/K$-fold tensor product of $c = 1$ theories.

Another character-based approach towards determining the desired larger CFT can be formulated by avoiding the assumption that these $c = 8/K$ component theories can be represented as tensor-products of $c = 1$ theories, and by instead working directly with the $A$- and $B$-sector characters determined in Sects. 3 and 4. Recall that we found a total of twelve such characters for each value of $K$: these were the $A$-sector characters $\{U_K, \tilde{A}_K, V_K\}$, as well as the $B$-sector characters $\{X^{(s)}_K, \tilde{B}^{(s)}_K, Y^{(s)}_K\}$ for $s = a, b, c$. While each of these sets of characters is separately closed under modular transformations, our goal is of course to construct a larger set of characters which not only contains these two smaller sets but which also introduces a non-trivial mixing between them. (In the language of Sect. 5.3, this is tantamount to constructing a new set of larger representation matrices $S_{ij}$ and $T_{ij}$ so that the $A$- and $B$-sectors are \textit{both} necessary for a consistent partition function.) We already have one indication of how these sectors should mix, however. Recall that the original linear combinations $A_K \equiv A^b_K - A^f_K$ and $B_K \equiv B^b_K - B^f_K$ [which are “orthogonal” to $\tilde{A}_K \equiv \frac{1}{2}(A^b_K + A^f_K)$ and $\tilde{B}_K \equiv \frac{1}{2}(B^b_K + B^f_K)$] close into each other under modular transformations — indeed, it is due to such mixings that the $B$-sectors originally appeared in the modular-invariant partition functions (2.14). This suggests that our twelve-member character set should be enlarged through the addition of the two extra linear combinations $A_K$ and $B_K$, or equivalently by treating the four characters $A^{b,f}_K$ and $B^{b,f}_K$ as completely independent. This is certainly consistent with our spacetime statistics assignments, according to which these four characters correspond to separate and distinguishable Fock spaces.

\* This approach was developed in collaboration with P.C. Argyres.
Moreover, this method of introducing a non-trivial coupling between the $A$- and $B$-sectors does not require the introduction of further additional sectors, since the resulting set of fourteen characters is again closed under the modular group. However, the fundamental problem one encounters in attempting to interpret these fourteen characters as those of a single CFT is the presence of two vacuum characters, $U_K$ and $X_K^{(a)}$, both with $h = 0$. Presumably some linear combination of these characters corresponds to the true vacuum state of the unified single CFT (assuming that such a CFT exists), but a quick check demonstrates that there do not appear to be any such combinations which simultaneously satisfy a number of other physical criteria.

A third approach to this problem might be to avoid the characters altogether, and to start with the bosonic lattice formulation presented in Sect. 6.3: we would then attempt to non-trivially combine the lattice of $A$-sector states with the rescaled lattice of $B$-sector states. However, since the $B$-sector lattice scaling factor $\sqrt{\lambda}$ is irrational, the $A$- and $B$-sector lattices are incommensurate, and their direct sum does not form a lattice (with or without a shift vector). Indeed, filling out the remaining lattice sites necessary to form a true lattice yields a set of lattice points which is dense in each lattice direction, and the physics of such a “lattice” is unclear. Alternatively, one might assume a coupling between the $A$-sector and $B$-sector theories which takes place only through the states corresponding to the origin $\vec{\alpha} = 0$, for this is the one lattice point which these theories have in common. In this scenario, then, $A$-sector states and $B$-sector states would be coupled only if the sums of the internal momenta of the $A$- and $B$-sector states each separately vanish. Unfortunately, this type of restricted coupling is highly unusual, and certainly has no analogue in the ordinary superstring. Whether a consistent theory can be formulated with such a restricted coupling in light-cone gauge remains to be seen.

Finally, regardless of whether there exists a single unified light-cone CFT for the fractional superstring, there of course remains the issue we have faced from the beginning: that of properly constructing our worldsheet CFT(s) in terms of worldsheet fields in a manner consistent with spacetime $(D_c - 2)$-dimensional Lorentz invariance and proper spacetime statistics. While our results concerning the twist current provide valuable clues as to what the statistics of these CFT sectors are expected to be, such results should emerge naturally in a proper formulation of the worldsheet CFT. In particular, this entails finding a representation of our light-cone CFT’s so that the states in our various sectors have well-defined transformation properties under the transverse Lorentz group $SO(D_c - 2)$.

Thus, our results concerning the post-projection worldsheet conformal field theories of the fractional superstring constitute only the first steps in their eventual construction, and many issues remain to be resolved before the consistency of the fractional superstring — both on the worldsheet and in spacetime — is demonstrated. Work in all of these areas is continuing.

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