The Ising Model on 3D Random Lattices: A Monte Carlo Study

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Abstract

We report single-cluster Monte Carlo simulations of the Ising model on three-dimensional Poissonian random lattices with up to 128,000 $\approx 50^3$ sites which are linked together according to the Voronoi/Delaunay prescription. For each lattice size quenched averages are performed over 96 realizations. By using reweighting techniques and finite-size scaling analyses we investigate the critical properties of the model in the close vicinity of the phase transition point. Our random lattice data provide strong evidence that, for the available system sizes, the resulting effective critical exponents are indistinguishable from recent high-precision estimates obtained in Monte Carlo studies of the Ising model and $\phi^4$ field theory on three-dimensional regular cubic lattices.
1 Introduction

Experimental studies of the critical behavior of real materials are often confronted with the influence of impurities and inhomogeneities. For a proper interpretation of the measurements it is, therefore, important to develop a firm theoretical understanding of the effect of such random perturbations. In many situations the typical time scale of thermal fluctuations in the idealized “pure” system is clearly separated from the time scale of the impurity dynamics, such that to a very good approximation the impurities can be treated as quenched (frozen), random disorder.

The importance of the effect of quenched, random disorder on the critical behavior of a physical system can be quite generally classified by the critical exponent of the specific heat of the pure system, $\alpha_p$. The Harris criterion\(^1\) asserts that for $\alpha_p > 0$ quenched, random disorder is a relevant perturbation, leading to a different critical behavior than in the pure case. In particular one expects\(^2\) in the disordered system that $\nu \geq 2/D$, where $\nu$ is the correlation length exponent and $D$ the dimension of the system. Assuming hyperscaling to be valid, this implies $\alpha = 2 - D\nu \leq 0$. For $\alpha_p < 0$ disorder is irrelevant, and in the marginal case $\alpha_p = 0$ no prediction can be made. For the case of (non-critical) first-order phase transitions it is known that the influence of quenched, random disorder can lead to a softening of the transition.\(^3\)

Many theoretical and numerical studies have been devoted to quenched, random site-dilution (SD), bond-dilution (BD), or more general random-bond (RB) systems. Since for the three-dimensional (3D) Ising model it is well known that $\alpha_p \approx 0.1 > 0$, quenched, random disorder should be relevant for this model. This has indeed been verified by a variety of techniques: Monte Carlo (MC) simulations for SD\(^4–6\) and BD,\(^7,8\) high-temperature series (HTS) expansions for BD\(^9\) and field theoretical renormalization group studies.\(^10–13\) For an excellent review and an extensive list of experimental, theoretical and numerical estimates in the last two decades, see Ref..\(^14\) As a result consensus has been reached that, while the critical exponent ratios $\gamma/\nu = 2 - \eta$ and $\beta/\nu$ are hardly distinguishable from the pure model, the correlation length exponent $\nu$ clearly signals the disordered fixed point,

$$\gamma/\nu = 1.966(6), \ \beta/\nu = 0.517(3), \ \nu = 0.6304(13) \ \text{ (pure)}^{15},$$

$$\gamma/\nu = 1.970(3), \ \beta/\nu = 0.515(2), \ \nu = 0.678(10) \ \text{ (disordered)}^{13},$$

and, moreover, satisfies the bound $\nu \geq 2/D = 2/3$ in the disordered case.
Recently also the predicted softening effect at first-order phase transitions has been confirmed for 3D \( q \)-state Potts models with \( q \geq 3 \) using MC\textsuperscript{16–18} and HTS\textsuperscript{19} techniques. The overall picture is even better in two dimensions (2D) where several models with \( \alpha_p > 0 \) (SD Baxter model,\textsuperscript{20} SD Baxter-Wu model,\textsuperscript{21} 3- and 4-state RB Potts model,\textsuperscript{22} Ashkin-Teller model\textsuperscript{23}) and the marginal (\( \alpha_p = 0 \)) 2D Ising model\textsuperscript{24–28} have been investigated. Also the softening of the first-order transitions for the 2D models with \( q \geq 5 \) has been confirmed.\textsuperscript{29–32} For a recent review, see Ref.\textsuperscript{33}

In this paper we study another type of quenched, random disorder, namely connectivity disorder, a generic property of random lattices whose local coordination numbers vary randomly from site to site. Physically the concept of random lattices plays an important role in an idealized description of the statistical geometry of random packings of particles.\textsuperscript{34–36} A prominent example is the crystallization process in liquids, and many statistical properties of random lattices have been studied in this context.\textsuperscript{37} From a more technical point of view, random lattices provide a convenient tool to discretize spaces of non-trivial topology without introducing defects or any kind of anisotropy.\textsuperscript{38–40} This desirable property has been exploited in a great variety of fields, ranging from diffusion limited aggregation\textsuperscript{41} and growth models for sandpiles\textsuperscript{42} over the statistical mechanics of membranes and strings\textsuperscript{43} to quantum field theory and quantum gravity.\textsuperscript{38–40,44,45} The preserved rotational, or more generally Poincaré, invariance suggests that spin systems or field theories defined on random lattices should reach the infinite-volume or continuum limit faster than on regular lattices. Conceptually, however, such an approach does only make sense as long as the critical properties of the considered system are not modified by the irregular lattice structure. In view of the quite general Harris criterion this is a non-trivial question (in particular due to the inherent spatial correlations of the disorder in this case), at least for systems characterized by \( \alpha_p \geq 0 \).

Specifically we considered 3D Poissonian random lattices of Voronoi/Delaunay type, and performed an extensive computer simulation study of the Ising model for lattices varying in size from \( N = 2\,000 \approx 13^3 \) to 128,000 \( \approx 50^3 \) sites. For each system size quenched averages over the connectivity disorder are approximated by averaging over 96 independent realizations. We concentrated on the close vicinity of the transition point and applied finite-size scaling (FSS) techniques\textsuperscript{46} to extract the critical exponents and the (weakly universal) “renormalized charges” \( U_2^* \) and \( U_4^* \). To achieve the desired accu-
racy of the data in reasonable computer time we applied the single-cluster algorithm\textsuperscript{47,48} to update the spins and furthermore made extensively use of the reweighting technique.\textsuperscript{49}

Previous studies of connectivity disorder focused mainly on 2D where pronounced effects were observed in MC simulations of \( q \)-state Potts models on quenched, random graphs provided by models of quantum gravity (modified universality classes for \( q = 2 \) and 4,\textsuperscript{50–52} and softening for \( q = 10 \).\textsuperscript{52,53}\textsuperscript{52,53}) For 2D random lattices of Voronoi/Delaunay type, on the other hand, no influence was seen in simulations for \( q = 2 \)\textsuperscript{54,55} and \( q = 8 \).\textsuperscript{56} The main difference between these two types of random lattices is the highly fractal structure\textsuperscript{57} of random gravity graphs which suggests a stronger “degree of disorder”. A similar dependence on the “degree of disorder” was recently observed for several aperiodic perturbations.\textsuperscript{58}

The rest of the paper is organized as follows. In Sect. 2, we describe some properties of the random lattices used in the simulations, and in Sect. 3 we define the model and give a few simulation details including estimates of autocorrelation times. The quantities measured are defined in Sect. 4, where also their theoretically expected FSS behavior is recalled. In Sect. 5, we present the FSS analysis of our data close to the transition point which yields estimates for the critical exponents. Finally, in Sect. 6 we present a discussion of our main results and close with a few concluding remarks.

## 2 Random lattices

**A. Voronoi/Delaunay random lattices:** The notion of “a random lattice” is by no means unique and, in fact, many different types of random lattices have been considered in the recent literature.\textsuperscript{41,50,52,53,59} In this paper we concentrate on so-called Poissonian Voronoi/Delaunay random lattices which, in arbitrary dimensions, are defined as follows\textsuperscript{38,44,45}:

1. Draw \( N \) sites \( x_i \) at random in a unit volume (square in 2D, cube in 3D, \ldots).
2. Associate with each site \( x_i \) a Voronoi cell, \( c_i \equiv \{ x \mid d(x, x_i) \leq d(x, x_j) \forall j \neq i \} \), which consists of all points \( x \) that are closer to the center site \( x_i \) than to any other site. Here \( d(x, y) \) denotes the usual Euclidean distance. This yields an irregular tessellation of the unit volume with \( D \)-dimensional Voronoi cells (polygons in 2D, polyhedra in 3D, \ldots).
3. Construct the dual Delaunay lattice by linking the center sites of all Voronoi cells which share a common face.

The first step approximates the Poissonian process. In the second step we always assume periodic boundary conditions, i.e., construct lattices of toroidal topology. Using this construction the number of nearest neighbors of the Delaunay lattice, the local coordination number \( q \), varies randomly from site to site. This constitutes the special type of quenched, random connectivity disorder we are investigating in this work.

The actually employed construction of the random lattices follows loosely the method of Ref.\(^60\) and is described in the Appendix. Following this method we succeeded to reduce the complexity of the lattice construction for all practical purposes from order \( N^2 \), as expected for the most straightforward implementation, to order \( N \), up to a small overhead \( \propto N^2 \) resulting from the initial calculation of the distances between all pairs of two sites. The actually measured CPU times as a function of \( N \) are shown in Fig. 1.

B. Random lattice properties: To test our random lattice construction we measured several quantities which characterize the topology and geometry of the lattices. These quantities are exactly known in the limit \( N \to \infty \). “Topological” quantities are the number of links \( N_l \), the number of triangles \( N_\triangle \), and the number of tetrahedra \( N_\tau \), which according to Euler’s theorem should satisfy for a torus topology in any number of dimensions and for any number of sites \( N \) the relation

\[
N - N_l + N_\triangle - N_\tau = 0. \quad (1)
\]

We have of course explicitly tested that this topological constraint is satisfied for all realizations, which is quite a sensitive confirmation that the lattice construction works properly. The measured averages of \( q, N_l, N_\triangle, \) and \( N_\tau \) over the 96 realizations used in the simulations are collected in Table 1. The error bars are computed from the fluctuations among the 96 realizations. We see that the analytically known \( N \to \infty \) limits are approached very rapidly. The only exception is perhaps our smallest lattice with \( N = 2000 \) sites where deviations from the infinite-volume limit are clearly visible. For \( N \geq 4000 \) in particular the average number of nearest neighbors, \( \overline{q} \), is fully consistent with the theoretical value of

\[
\overline{q} = 2 + \frac{48}{35} \pi^2 = 15.5354 \ldots \quad (2)
\]
Table 1: The average coordination number $\overline{q}$ and the total number of simplices normalized by the number of sites. The error bars are computed from the fluctuations among the 96 realizations.

| $N$  | $\overline{q}$  | $N_l/N$  | $N_\Delta/N$  | $N_\tau/N$ |
|------|-----------------|----------|----------------|------------|
| 2000 | 15.544(4)       | 7.7721(19) | 13.5441(37)   | 6.7721(19) |
| 4000 | 15.532(3)       | 7.7661(16) | 13.5321(32)   | 6.7661(16) |
| 8000 | 15.537(2)       | 7.7683(10) | 13.5366(21)   | 6.7683(10) |
| 16000| 15.535(2)       | 7.76744(76)| 13.5348(15)   | 6.76744(76)|
| 32000| 15.534(1)       | 7.76706(55)| 13.5342(11)   | 6.76706(55)|
| 64000| 15.5351(7)      | 7.76755(36)| 13.53507(72)  | 6.76755(36)|
| 128000| 15.5349(5)     | 7.76743(26)| 13.53486(53)  | 6.76743(26)|
| exact| $2 + \frac{46}{35} \pi^2$ | $1 + \frac{24}{35} \pi^2$ | $\frac{46}{35} \pi^2$ | $\frac{24}{35} \pi^2$ |

As “geometric” quantities we measured the average volumes of the simplices, i.e., the average link length $\overline{t} = \frac{\sum_{i=1}^{N_t} ((\sum_{i \in \tau} l_i)/6)}{N_\tau}$, the average triangle area $\overline{\Delta} = \frac{\sum_{i=1}^{N_\Delta} \Delta_i}{N_\Delta}$, and the average volume of a tetrahedron $\overline{\tau} = \frac{\sum_{i=1}^{N_\tau} \tau_i}{N_\tau}$. Notice that the average link length is defined in such a way that we first average over the sides of a given tetrahedron and then over all the tetrahedra. This is thus the average link length per tetrahedron, and not the mean link length averaged over the whole lattice, $\overline{t}^{(N)} = \frac{\sum_{i=1}^{N_t} l_i}{N_t}$, which is always larger due to the fluctuations in the number of tetrahedra from realization to realization. Our results normalized with an appropriate power of the density $\rho = N/V$ are displayed in Table 2. From $N = 4000$ on the numerical results are again fully consistent with the analytical predictions as $N \to \infty$. Since all numbers agree very well with the analytical predictions we can be quite sure that our lattice construction works properly and that we have picked a typical sample of random lattices.

The (normalized) probability densities $P(q)$ of the coordination numbers $q$ are shown in Fig. 2 for the lattices with $N = 64\,000$ and $N = 128\,000$ sites. The average coordination number is $\overline{q} = 2 + \frac{46}{35} \pi^2 = 15.5354\ldots$. Due to the long tail of $P(q)$ for large values of $q$, the maximum of $P(q)$ is found...
Table 2: Average simplex volumes properly normalized to natural units.

| \(N\) | \(\bar{v}^{(N)}/\rho^{-1/3}\) | \(\bar{v}/\rho^{-1/3}\) | \(\Delta/\rho^{-2/3}\) | \(\bar{\tau}/\rho^{-1}\) |
|------|-----------------|-----------------|-----------------|-----------------|
| 2000 | 1.28540(22)     | 1.23670(17)     | 0.597004(95)    | 0.147666(41)    |
| 4000 | 1.28566(18)     | 1.23717(13)     | 0.597362(70)    | 0.147797(34)    |
| 8000 | 1.28569(13)     | 1.23711(10)     | 0.597312(55)    | 0.147747(23)    |
| 16000| 1.285623(75)    | 1.237091(55)    | 0.597306(31)    | 0.147766(17)    |
| 32000| 1.285554(68)    | 1.237027(49)    | 0.597289(24)    | 0.147775(12)    |
| 64000| 1.285537(40)    | 1.237002(32)    | 0.597272(19)    | 0.1477630(78)   |
| 128000| 1.285541(28)   | 1.237012(20)    | 0.597275(11)    | 0.147766(58)    |
| exact| 1.237033...     | 0.597286...     | 0.1477600...    |
| exact| \((\frac{3}{4\pi})^{1/3} \frac{5}{3^{2/3}} \pi \Gamma(\frac{1}{3})\) | \((\frac{3}{4\pi})^{2/3} \frac{5^{2/3}}{3^{4/3}} \pi \Gamma(\frac{2}{3})\) | \(\frac{35}{24\pi\pi}\) |

at the next smaller integer number \(q = 15\), which occurs with a probability of 12.03%.

3 The model and simulation techniques

A. Model: We simulated the standard partition function of the Ising model,

\[ Z = \sum_{\{s_i\}} e^{-KE}; \quad E = -\sum_{\langle ij \rangle} s_i s_j; \quad s_i = \pm 1, \tag{3} \]

where \(K = J/k_B T > 0\) is the inverse temperature in natural units, and \(\langle ij \rangle\) denotes the nearest-neighbor links of our three-dimensional toroidal random lattices. In (3) we have adopted the convention used in Ref.\textsuperscript{55} and assigned to each link the same weight.

Another interesting option would be to assign to each link a weight depending on the geometrical properties of the Voronoi/Delaunay construction such as the length of the links or suitably defined areas of the associated tesselation. In addition to the connectivity disorder this would introduce also a (correlated) disorder of random-bond type. In order not to mix up these two quenched disorder types, we decided to concentrate in this study exclusively on the effect of the connectivity disorder.
B. Simulation: The finite-size scaling (FSS) analysis is performed on the basis of seven different lattice sizes with $N = 2000, 4000, 8000, 16000, 32000, 64000,$ and $128000$ sites. For later use we adopt the notation for regular lattices and define a linear lattice size $L$ by $L = N^{1/3}$. The linear sizes of the lattices thus vary from $L \approx 12.6$ to $L \approx 50.4$. To investigate the dependence of thermal averages on different realizations we performed MC simulations, for each lattice size, on the 96 randomly chosen random lattice realizations discussed in Sect. 2.

For the update of the Ising spins we employed Wolff’s single-cluster (1C) algorithm. Various tests, in particular for the Ising model on two- and three-dimensional regular lattices, have clearly demonstrated that critical slowing down can be significantly reduced with this non-local update algorithm. These tests also showed that in particular in three dimensions the single-cluster algorithm is more efficient than the original multiple-cluster formulation of Swendsen and Wang.

All runs were performed in the vicinity of the critical point $K_c$. As a first rough guess of $K_c$ we used the mean-field bound $K_c \geq K_{MF} = 1/\bar{q} \approx 0.064$. Due to the large average coordination number of 3D random lattices the mean-field approximation should be better than for the simple cubic (SC) lattice where $K_c^{(SC)} = 1/6$ is about $1.33$ times smaller than the actual $K_c^{(SC)} \approx 0.2216546$. Therefore, by applying the same correction factor to the random lattice mean-field estimate we expect that $K_c$ is bounded from above by $K_c \leq 0.085$. This heuristic argument thus suggests that $0.064 \leq K_c \leq 0.085$, such that $K_c \approx 0.075$ should be a reasonable a priori guess. Once good simulation points $K_0$ were estimated for the two smallest lattices with $N = 2000$ and $4000$ sites by determining for a few realizations the location of the specific-heat and susceptibility maxima, we used FSS extrapolations (assuming $\nu \approx 0.63$) to predict $K_0$ for the larger lattices.

Depending on the lattice size from 30000 to 180000 clusters were discarded to reach equilibrium from an initially completely ordered state. Primary observables are the energy per spin, $e = E/N$, and the magnetization per spin, $m = M/N = \Sigma_i s_i/N$, which were measured every $n_{\text{flip}}$ cluster flip and recorded in a time-series file. The average cluster size $\langle |C| \rangle$ is an estimator for the reduced susceptibility in the high-temperature phase, $\chi_{\text{red}} = N\langle m^2 \rangle$, and therefore scales with the lattice size according to $L^{\gamma/\nu}$, where $\gamma$ and $\nu$ are the usual critical exponents. Since in three dimensions $\gamma/\nu = 2 - \eta \approx 2$, we have to perform $n_{\text{flip}} \propto L$ cluster flips in order to flip, on
Table 3: Monte Carlo parameters of the simulations. \( N \equiv L^3 \) is the lattice size, the hash mark symbol \( \# \) denotes the number of realizations, \( K_0 \) the inverse simulation temperature, \( n_{\text{therm}} \) the number of cluster flips during equilibration, and \( n_{\text{meas}} \) the number of measurements taken every \( n_{\text{flip}} \) cluster flip.

| \( N \)  | \( L \)  | \#  | \( K_0 \) | \( n_{\text{therm}} \) | \( n_{\text{meas}} \) | \( n_{\text{flip}} \) |
|-------|-------|----|-------|----------|----------|---------|
| 2000  | 12.6  | 96 | 0.0735| 30000    | 100000   | 4       |
| 4000  | 15.9  | 96 | 0.0735| 50000    | 100000   | 5       |
| 8000  | 20.0  | 96 | 0.0732| 66000    | 110000   | 6       |
| 16000 | 25.2  | 96 | 0.0729| 80000    | 100000   | 8       |
| 32000 | 31.7  | 96 | 0.0728| 100000   | 100000   | 10      |
| 64000 | 39.1  | 96 | 0.0726| 150000   | 100000   | 13      |
| 128000| 50.4  | 32 | 0.0725| 180000   | 100000   | 16      |
| 128000| 50.4  | 64 | 0.0725| 180000   | 100000   | 16      |

The average, approximately the same fraction of the total number of spins, \( N = L^3 \), for all lattice sizes. By adjusting the absolute scale of \( n_{\text{flip}} \) we made sure that for all lattice sizes the measurements were taken after about \( N/2 \) spin flips. Since for the single-cluster update algorithm the autocorrelation times scale only weakly with \( L \) (see the discussion below) one then expects that, with about the same number of measurements, the statistical accuracy is comparable for all lattice sizes. With this set-up we performed 100 000 measurements for each lattice size and realization (110 000 for \( N = 8000 \)). For more details on the employed statistics, see Table 3.

C. Update dynamics: A useful measure of the update dynamics is the integrated autocorrelation time \( \hat{\tau} \).\(^64\) To estimate \( \hat{\tau} \) for the measurements of the energy \( e \) and the magnetization \( m \) we used the fact that \( \hat{\tau} \) enters in the error estimate \( \epsilon^2 = \sigma^2 2\hat{\tau}/n_{\text{meas}} \) for the mean value \( \overline{O} \) of \( n_{\text{meas}} \) correlated measurements with variance \( \sigma^2 = \langle O; O \rangle \equiv \langle O^2 \rangle - \langle O \rangle^2 \), and determined \( \epsilon^2 \) by blocking procedures. Using 100 blocks of 1000 measurements each we obtained the results compiled in Table 4 where all results are averaged over the 96 randomly chosen realizations. We see that the integrated autocorrelation
Table 4: Average cluster size $[⟨|C|⟩]_{av}$ and autocorrelation times of energy and magnetization at the simulation point $K_0$, where $τ = f \cdot \hat{τ}$ and $f = n_{flip}⟨|C|⟩/N$. The tau’s are obtained from a blocking analysis on the basis of 100 block.

| $N$ | $[⟨|C|⟩]_{av}$ | $[\hat{τ}_e]_{av}$ | $[τ_e]_{av}$ | $[f]_{av} \cdot [\hat{τ}_e]_{av}$ | $[\hat{τ}_m]_{av}$ | $[τ_m]_{av}$ | $[f]_{av} \cdot [\hat{τ}_m]_{av}$ |
|-----|----------------|-------------------|-------------|----------------|----------------|-------------|----------------|
| 2000 | 246.9(1.9)     | 2.529(40)         | 1.245(19)   | 1.249          | 2.434(38)      | 1.200(20)   | 1.202          |
| 4000 | 453.3(3.9)     | 2.443(42)         | 1.375(20)   | 1.384          | 2.412(38)      | 1.360(20)   | 1.367          |
| 8000 | 740.1(5.5)     | 2.702(42)         | 1.494(22)   | 1.500          | 2.594(40)      | 1.436(22)   | 1.440          |
| 16000| 1084.0(9.5)    | 2.955(43)         | 1.593(21)   | 1.602          | 2.730(38)      | 1.475(22)   | 1.480          |
| 32000| 1962(17)       | 2.741(50)         | 1.666(24)   | 1.681          | 2.572(44)      | 1.567(23)   | 1.577          |
| 64000| 2078(24)       | 3.177(55)         | 1.718(27)   | 1.728          | 2.795(42)      | 1.516(23)   | 1.520          |
| 128000| 3521(87)      | 3.92(11)          | 1.705(42)   | 1.725          | 3.157(72)      | 1.385(42)   | 1.390          |
| 128000| 4325(41)      | 3.399(72)         | 1.824(31)   | 1.838          | 2.964(60)      | 1.593(29)   | 1.602          |

Times for the measurements of $e$ and $m$ are of the order of $\hat{τ}_e \approx 2.5$ – 3.5 and $\hat{τ}_m \approx 2.4$ – 3.0. Since completely uncorrelated data correspond to $\hat{τ} = 0.5$, our thermal sample thus effectively consists of about 15 000 – 20 000 uncorrelated measurements for each of the 96 realizations. This amounts to a total of $(1.5 – 2.0) \times 10^6$ effectively uncorrelated measurements for each lattice size.

While this properly characterizes the effective statistics of our simulations, the numbers for $\hat{τ}$ of a single-cluster simulation are not yet well suited for a comparison with other update algorithms or with single-cluster simulations on regular lattices. To get a comparative work-estimate, the usual procedure is to convert the $\hat{τ}$ by multiplying with a factor $f = n_{flip}⟨|C|⟩/N$ to a scale where, on the average, measurements are taken after every spin has been flipped once (similar to, e.g., Metropolis simulations). For quenched, random systems this procedure is not unique due to the necessary average over realizations. In Table 4 we therefore present both options, $[τ]_{av} \equiv [f \cdot \hat{τ}]_{av}$ and also $[f]_{av} \cdot [\hat{τ}]_{av}$. The differences between the two averaging prescriptions are, however, extremely small.

The numbers in Table 4 obtained in this way are very similar to results for the regular simple cubic (SC) lattice. By fitting a power law, $τ \propto L^z$, to the data for the five largest lattices we obtain $[τ_e]_{av} = 0.82(6)L^{0.20(2)}$, and $[τ_m]_{av} = 1.07(8)L^{0.10(2)}$, respectively. The dynamic critical exponents $z$ for the random lattice simulations should be compared with $z_e = 0.28(2)$ and $z_χ = 0.14(2)$ for the SC lattice.
4 Observables and finite-size scaling

From the time series of $e$ and $m$ it is straightforward to compute in the FSS region various quantities at nearby values of $K_0$ by standard reweighting methods. For the estimation of the statistical (thermal) errors, for each of the 96 realizations the time-series data was split into 100 bins, which were jack-knifed to decrease the bias in the analysis of reweighted data. The final values are averages over the 96 realizations which will be denoted by square brackets $[\ldots]_{av}$, and the error bars are computed from the fluctuations among the realizations. Note that these errors contain both, the average thermal error for a given realization and the theoretical variance for infinitely accurate thermal averages which is caused by the variation of the quenched, random geometry of the 96 lattices.

From the time series of the energy measurements we can compute by reweighting the average energy, specific heat, and energetic fourth-order parameter,

\[
u(K) = \frac{\langle E \rangle_{av}}{N}, \quad (4)
\]
\[
C(K) = K^2 N[\langle e^2 \rangle - \langle e \rangle^2]_{av}, \quad (5)
\]
\[
V(K) = [1 - \frac{\langle e^4 \rangle}{3(\langle e^2 \rangle^2)}]_{av}. \quad (6)
\]

Similarly we can derive from the magnetization measurements the average magnetization, the susceptibility, and the magnetic cumulants,

\[
m(K) = \langle |m| \rangle_{av}, \quad (7)
\]
\[
\chi(K) = K N[\langle m^2 \rangle - \langle |m| \rangle^2]_{av}, \quad (8)
\]
\[
U_2(K) = [1 - \frac{\langle m^4 \rangle}{3\langle |m| \rangle^2}]_{av}, \quad (9)
\]
\[
U_4(K) = [1 - \frac{\langle m^4 \rangle}{3\langle m^2 \rangle^2}]_{av}. \quad (10)
\]

Further useful quantities involving both the energy and magnetization are

\[
\frac{d[\langle |m| \rangle]_{av}}{dK} = [\langle |m| E \rangle - \langle |m| \rangle \langle E \rangle]_{av}, \quad (11)
\]
\[
\frac{d \ln[\langle |m| \rangle]_{av}}{dK} = \left[ \frac{\langle |m| E \rangle}{\langle |m| \rangle} - \langle E \rangle \right]_{av}, \quad (12)
\]
\[
\frac{d \ln[\langle m^2 \rangle_{av}]}{dK} = \left[ \frac{\langle m^2 E \rangle}{\langle m^2 \rangle} - \langle E \rangle \right]_{av}. \tag{13}
\]

In the infinite-volume limit these quantities exhibit singularities at the transition point. In finite systems the singularities are smeared out and scale in the critical region according to

\[
C = C_{\text{reg}} + L^{\alpha/\nu} f_C(x)[1 + \ldots], \tag{14}
\]
\[
\langle |m| \rangle_{av} = L^{-\beta/\nu} f_m(x)[1 + \ldots], \tag{15}
\]
\[
\chi = L^{\gamma/\nu} f_{\chi}(x)[1 + \ldots], \tag{16}
\]
\[
\frac{d\langle |m| \rangle_{av}}{dK} = L^{(1-\beta)/\nu} f_{m'}(x)[1 + \ldots], \tag{17}
\]
\[
\frac{d \ln[\langle |m|^p \rangle_{av}]}{dK} = L^{1/\nu} f_p(x)[1 + \ldots], \tag{18}
\]
\[
\frac{dU_2}{dK} = L^{1/\nu} f_{U_2}(x)[1 + \ldots], \tag{19}
\]

where \(C_{\text{reg}}\) is a regular background term, \(\nu, \alpha, \beta, \) and \(\gamma\), are the usual critical exponents, \(f_i(x)\) are FSS functions with

\[
x = (K - K_c)L^{1/\nu} \tag{20}
\]

being the scaling variable, and the brackets \([1 + \ldots]\) indicate corrections-to-scaling terms which become unimportant for sufficiently large system sizes \(L\).

5 Results

By applying standard reweighting techniques to each of the 96 time-series data we first determined the temperature dependence of \(C_i(K), \chi_i(K), \ldots, i = 1, \ldots, 96\), in the neighborhood of the simulation point \(K_0\). By estimating the valid reweighting range for each of the realizations we made sure that no systematic errors crept in by this procedure. Once the temperature dependence is known for each realization, we can easily compute the disorder average, e.g., \(C(K) = \sum_{i=1}^{96} C_i(K)/96\), and then determine the maxima of the averaged quantities, e.g., \(C_{\text{max}}(K_{\text{max}}) = \max_K C(K)\). The locations of the maxima of \(C, \chi, dU_2/dK, dU_4/dK, d[\langle |m| \rangle_{av}]/dK, d \ln[\langle |m| \rangle]_{av}/dK, \) and
provide us with seven sequences of pseudo-transition points $K_{\text{max},i}(L)$ which all should scale according to $K_{\text{max},i}(L) = K_c + a_i L^{-1/\nu} + \ldots$. In other words, the scaling variable $x = (K_{\text{max},i}(L) - K_c) L^{1/\nu} = a_i + \ldots$ should be constant, if we neglect the small higher-order corrections indicated by \ldots. To give an idea on how these sequences approach $K_c$ we show below in Table 6 besides the estimates for $K_c$ also the values of $a_i$, $i = 1, \ldots, 7$ (see also Fig. 4).

It should be emphasized that while the precise estimates of $a_i$ do depend on the value of $\nu$, the qualitative conclusion that $x \approx \text{const.}$ for $K_{\text{max},i}$ does not require any a priori knowledge of $\nu$ or $K_c$. Using this information we have thus several possibilities to extract unbiased estimates of the critical exponents $\nu$, $\alpha/\nu$, $\beta/\nu$, and $\gamma/\nu$ from least-squares fits assuming the FSS behaviors (14)-(19). Once $\nu$ is estimated we can then use $K_{\text{max},i}(L) = K_c + a_i L^{-1/\nu} + \ldots$ to extract also $K_c$ and $a_i$.

**A. Critical exponent $\nu$:** Let us thus start with the correlation length exponent $\nu$ for which we can obtain $4 \times 7 = 28$ different estimates by considering the scaling of $d \ln[\langle |m| \rangle_{\text{av}}]/dK$, $d \ln[\langle m^2 \rangle_{\text{av}}]/dK$, $dU_2/dK$, and $dU_4/dK$ at the seven sequences of pseudo-transition points $K_{\text{max},i}(L)$. Of course, these estimates are statistically not uncorrelated, but they are differently affected by corrections to the leading FSS behavior. To test the importance of these corrections to scaling we first estimated $\nu$ from fits using all available lattice sizes, and then compared with the results of fits where the two smallest sizes were successively discarded. As a result we did not observe any systematic improvement by omitting the smallest lattices. In fact, already for the fits using all sizes, we obtained goodness-of-fit parameters $Q \geq 0.3$ for about 80% of the 28 fits. The only unacceptable fit was that of $d \ln[\langle m^2 \rangle_{\text{av}}]/dK$ at the $K_{\text{max},i}$ sequence with $Q = 0.01$. Here we omitted the $N = 2000$ data point which improves the goodness to $Q = 0.22$.

This analysis clearly shows that with our data there is no need to include corrections-to-scaling terms in the fits which would necessitate non-linear fitting procedures which usually tend to be quite unstable. Of course, such a conclusion depends on the accuracy of the data and only shows that the correction terms are so small that they cannot be resolved within our accuracy. This is quite remarkable because in the pure 3D Ising model study of Ref. for lattices of size $L = 8$ to 128 (using a different FSS technique) confluen-
ent corrections to scaling $\propto L^{-\omega}$ with $\omega = 0.87(9)$ could be resolved. One possible explanation is that random lattices with large average coordination number and preserved rotational invariance indeed reach the infinite-volume limit earlier than their regular counterparts. This is quite conceivable since the magnitude of the correction term does not only depend on the exponent but also on its amplitude, and the latter is a non-universal quantity which does definitely depend on the lattice structure.

Having now 28 estimates of $1/\nu$ from fits with $Q \geq 0.15$ we proceeded as follows. We computed arithmetic and error weighted averages for the four subgroups of estimates and finally over the total set of estimates. The main results are collected in Table 5. Because of the neglected cross-correlations between the fit results, in particular the error estimate of the weighted average should be taken with some care. As our best estimates we therefore quote throughout this paper in all tables in the lines labeled “final” always the weighted mean and quite conservatively the smallest error among all available fits (making thus the plausible assumption that the error of the weighted average is never larger than the error of the most accurate fit result that contributes to this average). As a general trend we see in Table 5 that the fits of $d\ln[\langle |m|^p \rangle]_{av}/dK$ yield more accurate results than those of $dU_{2p}/dK$. We also observe, however, that the partial averages over the fits of $dU_{2p}/dK$, $p = 1, 2$ and $d\ln[\langle |m|^p \rangle]_{av}/dK$, $p = 1, 2$, are only marginally consistent, even though in all cases the goodness-of-fit was high. We thus have no reason based on statistical arguments to favor one or the other group of fits and therefore take as our final value the weighted average over all 28 estimates which yields

$$1/\nu = 1.5875 \pm 0.0012, \quad \nu = 0.62992 \pm 0.00048,$$

with the minimal error coming from the fit of $d\ln[\langle |m|^2 \rangle]_{av}/dK$ at the maximum locations of $d[\langle |m| \rangle]_{av}/dK$.

If we only average the results of the fits of the maxima of $d\ln[\langle |m| \rangle]_{av}/dK$, $d\ln[\langle m^2 \rangle]_{av}/dK$, $dU_2/dK$, and $dU_4/dK$, we obtain basically the same final estimate with a slightly larger error bar:

$$1/\nu = 1.5878 \pm 0.0015, \quad \nu = 0.62980 \pm 0.00060.$$  \hspace{1cm} (22)

To give also a visual impression of the quality of these fits, they are shown in Fig. 3. This reconfirms the weighted average (21) over all 28 fits, and our
Table 5: Fit results for the critical exponent $1/\nu$.

| quantity | type | $1/\nu$ |
|----------|------|---------|
| $dU_2/dK$ | at maximum | 1.5818(33) |
|          | average   | 1.5783(32) |
|          | weighted av. | 1.5795(12) |
|          | final     | 1.5795(27) |
| $dU_4/dK$ | at maximum | 1.5797(42) |
|          | average   | 1.5763(34) |
|          | weighted av. | 1.5774(13) |
|          | final     | 1.5774(25) |
| $dU_2/p$ | average p = 1 and 2 | 1.5773(23) |
|          | weighted av. p = 1 and 2 | 1.5785(09) |
|          | final     | 1.5785(25) |
| $d\ln[\langle |m| \rangle_{av}]_{dK}$ | at maximum | 1.5886(16) |
|          | average   | 1.5888(08) |
|          | weighted av. | 1.5889(06) |
|          | final     | 1.5889(14) |
| $d\ln[\langle m^2 \rangle_{av}]_{dK}$ | at maximum | 1.5894(15) |
|          | average   | 1.5902(07) |
|          | weighted av. | 1.5898(06) |
|          | final     | 1.5898(12) |
| $d\ln[\langle m^p \rangle_{av}]_{dK}$ | average p = 1 and 2 | 1.5895(06) |
|          | weighted av. p = 1 and 2 | 1.5894(05) |
|          | final     | 1.5894(12) |
| $dU_2p/dK_{max}$ and $d\ln[\langle m^p \rangle_{av}]_{dK_{max}}$ | average (4 fits) | 1.5849(25) |
|          | weighted av. (4 fits) | 1.5875(10) |
|          | final (4 fits) | 1.5875(15) |
| $dU_2p/dK$ and $d\ln[\langle m^p \rangle_{av}]_{dK}$ | average (28 fits) | 1.5834(16) |
|          | weighted av. (28 fits) | 1.5875(04) |
|          | final (28 fits) | 1.5875(12) |
Final estimate for the correlation length exponent is thus

\[ \nu = 0.6299 \pm 0.0005. \quad (23) \]

For comparison, for Ising models on regular simple cubic (SC) lattices Ferrenberg and Landau\(^6\) obtained \( \nu = 0.6289(8) \), Blöte \textit{et al.}\(^6\) concluded that \( \nu = 0.6301(8) \), Ballesteros \textit{et al.}\(^6\) found\(^7\) \( \nu = 0.6294(5) \), and in a recent study of the \( \phi^4 \) lattice field theory Hasenbusch\(^7\) estimated \( \nu = 0.6296(3) \). Within error bars all three estimates are in perfect agreement with our result (23) for random lattices.

**B. Critical coupling \( K_c \):** Having determined the critical exponent \( \nu \), it is straightforward to obtain estimates of the critical coupling \( K_c \) from linear least-squares fits to

\[ K_{\text{max}} = K_c + a_i L^{-1/\nu}, \quad (24) \]

where \( K_{\text{max}} \) are the seven pseudo-transition points discussed earlier. Here we found a significant improvement of the quality of the fits if the smallest lattice size with \( N = 2000 \) was excluded. This can also be inspected visually in Fig. 4, where the data and fits are shown. We see a systematic trend that the \( N = 2000 \) data lie a little bit too low. In Table 6 we therefore display the fit results over the six lattice sizes \( N = 4000 - 128000 \). By using the same averaging procedure as before we arrive at the final estimate

\[ K_c = 0.0724 249 \pm 0.0000 040. \quad (25) \]

Of course, in principle this estimate is biased by our estimate of \( \nu \). We have checked, however, that the dependence on \( \nu \) is extremely weak. If we repeat the fits with \( 1/\nu = 1.5875 \pm 3\epsilon_{1/\nu} \), where \( \epsilon_{1/\nu} = 0.0012 \) is the error on the estimate of \( 1/\nu \) in (21), we obtain a variation in the estimate for \( K_c \) by only \( \pm 2 \) in the last digit, which is much smaller than the statistical error in (25).

**C. Critical exponent \( \gamma \):** The exponent ratio \( \gamma/\nu \) can be obtained from fits to the FSS behavior (16) of the susceptibility. By monitoring the quality of the fits we decided to discard the \( N = 2000 \) data for the \( K_{\text{max}C} \) and \( K_{\text{max}dm/dK} \) sequences (which led to \( Q \) values of 0.05 and 0.02, respectively). The fits collected in Table 7 then all have \( Q \geq 0.25 \). The final result is

\[ \gamma/\nu = 1.9576 \pm 0.0013, \quad (26) \]
Table 6: *Fit results for the critical coupling $K_c$, using the FSS ansatz $K_{\text{max},i} = K_c + a_i L^{-1/\nu}$ and our best estimate of $1/\nu = 1.5875(12)$. The fit range is always from $N = 4000$ to $128000$.*

| $K_{\text{max}}$ of | $K_c$         | $a$       | $Q$  |
|---------------------|--------------|-----------|------|
| $C$                 | 0.0724282(40)| 0.1010(12)| 0.63 |
| $\chi$              | 0.0724210(40)| 0.0182(11)| 0.87 |
| $dU_4/dK$           | 0.0724221(54)| -0.0326(14)| 0.65 |
| $dU_2/dK$           | 0.0724238(44)| 0.0002(12)| 0.89 |
| $d[\langle|m|\rangle]_{av}/dK$ | 0.0724277(40)| 0.0525(11)| 0.82 |
| $d\ln[\langle|m|\rangle]_{av}/dK$ | 0.0724250(43)| -0.0063(12)| 0.83 |
| $d\ln[\langle m^2 \rangle]_{av}/dK$ | 0.0724253(44)| -0.0146(12)| 0.76 |

| average             | 0.0724247(10)|          |      |
| weighted av.        | 0.0724249(16)|          |      |
| final               | 0.0724249(40)|          |      |

which should be compared with the estimates for regular SC lattices of $\gamma/\nu = 1.970(14)$ in Ref.,$^{68} \gamma/\nu = 1.9630(30)$ in Ref.$^{69} \gamma/\nu = 1.9626(6)[6]$ in Ref.$^{67}$ and $\gamma/\nu = 1.9642(4)[5]$ in Ref.$^{71}$

For the exponent $\eta$, the estimate (26) implies

$$\eta = 2 - \gamma/\nu = 0.0424 \pm 0.0013,$$

and, by using our value (21) for $1/\nu$, we derive

$$\gamma = 1.2332 \pm 0.0018.$$  

**D. Critical exponent $\beta$:** The exponent ratio $\beta/\nu$ can be either obtained from the FSS behavior of $[\langle|m|\rangle]_{av}$ or $d[\langle|m|\rangle]_{av}/dK$, Eqs. (15) or (17). In the first case, the sequences $K_{\text{max},x}$ and $K_{\text{max,}dm,4K}$ yield poor $Q$ values ($\leq 0.01$) if the $N = 2000$ are included in the fits. If we discard the smallest lattice in these two cases, all fits shown in Table 7 are characterized by $Q \geq 0.10$. The final estimate is then

$$\beta/\nu = 0.51587 \pm 0.00082,$$
Table 7: Fit results for the critical exponents $\gamma/\nu$, $\beta/\nu$, and $(1 - \beta)/\nu$. The superscripts * and # at the $Q$ values indicate that these fits start at $N = 4000$ and $N = 8000$, respectively. The other fits use all data from $N = 2000$ to 128000.

| at $K_{\text{max}}$ of | $\gamma/\nu$ | $Q$ | $\beta/\nu$ | $Q$ | $(1 - \beta)/\nu$ | $Q$ |
|------------------------|--------------|-----|-------------|-----|------------------|-----|
| $C$                    | 1.9357(37)   | 0.49* | 0.5089(11)  | 0.10 | 1.0690(32)       | 0.82# |
| $\chi$                 | 1.9551(13)   | 0.36  | 0.5237(82)  | 0.85* | 1.0663(14)       | 0.30 |
| $dU_4/dK$              | 1.9641(25)   | 0.35  | 0.5180(41)  | 0.27  | 1.0707(39)       | 0.20 |
| $dU_2/dK$              | 1.9581(13)   | 0.59  | 0.5173(22)  | 0.31  | 1.0713(19)       | 0.15 |
| $d\langle|m|\rangle_{av}/dK$ | 1.9476(19) | 0.29* | 0.5109(95)  | 0.10* | 1.0683(24)       | 0.65# |
| $d\ln\langle|m|\rangle_{av}/dK$ | 1.9603(13) | 0.76  | 0.5140(19)  | 0.28  | 1.0657(29)       | 0.90# |
| $d\ln\langle m^2 \rangle_{av}/dK$ | 1.9626(13) | 0.77  | 0.5128(22)  | 0.30  | 1.0664(30)       | 0.69# |
| average                | 1.9548(38)   | 0.5151(19) | 1.0682(84) |
| weighted av.           | 1.9575(59)   | 0.51587(49) | 1.0679(86) |
| final                  | 1.9576(13)   | 0.51587(82) | 1.0680(14) |

and, by using our estimate for $1/\nu$ in (21),

$$\beta = 0.32498 \pm 0.00077.$$  \hspace{1cm} (30)

If hyperscaling is valid, the estimate (29) would imply $\gamma/\nu = D - 2\beta/\nu = 1.9683(17)$, which however turns out to be only barely consistent with the direct estimate (26) of $\gamma/\nu$.

The FSS of $d\langle|m|\rangle_{av}/dK$ is less well behaved. Here we had to discard for the $K_{\text{max}C}$, $K_{\text{max}dm/dK}$, $K_{\text{max}dnm/dK}$, and $K_{\text{max}dnm^2/dK}$ sequences both the $N = 2000$ and $N = 4000$ data in order to guarantee that all fits entering the average have a goodness-of-fit parameter $Q \geq 0.15$. We then obtain

$$(1 - \beta)/\nu = 1.0680 \pm 0.0014,$$  \hspace{1cm} (31)

and by inserting the estimate (21) for $1/\nu$,

$$\beta/\nu = 0.5194 \pm 0.0026,$$  \hspace{1cm} (32)

and

$$\beta = 0.3272 \pm 0.0014.$$  \hspace{1cm} (33)

Recent MC estimates for regular SC lattices are $\beta/\nu = 0.518(7)$ in Ref.\textsuperscript{68} and $\beta/\nu = 0.5185(15)$ in Ref.\textsuperscript{69}.
E. Critical exponent $\alpha$: Due to the regular background term $C_{\text{reg}}$ in the FSS behavior (14), the specific heat is usually among the most difficult quantities to analyse.\(^{72}\) We tried non-linear fits to the ansatz $C = C_{\text{reg}} + aL^{\alpha/\nu}$, but for most sequences of pseudo-transition points the errors on the parameters of this fit turned out to be large. We therefore fixed the exponent $\alpha/\nu$ at the value one would expect if hyperscaling is valid, $\alpha/\nu = 2/\nu - D = 0.1102 \pm 0.0015$, (35)

and tested if linear two-parameter fits yield acceptable goodness-of-fit values. The results are shown in Fig. 5. We see that over the whole range of lattice sizes the expected linear behavior is satisfied. The quantitative analysis reveals some deviations for the two smallest lattice sizes, but for the fits starting with $N = 8,000$ we obtained for all seven sequences of pseudo-transition points goodness-of-fits parameters $Q \geq 0.5$.

F. Binder parameters $U_2$ and $U_4$: It is well known\(^{73}\) that the $U_{2p}(K)$ curves for different lattice sizes $L$ should intersect around $(K_c, U^*_2)$ with slopes $U''_{2p} \equiv dU_{2p}/dK \propto L^{1/\nu}$, where $U^*_2$ is the (weakly universal) “renormalized charge”. In Fig. 6 we show $U_2$ and $U_4$ as a function of $N \equiv L^3$ for 5 $K$-values around $K_c \approx 0.072425$. At our best estimate of $K_c$, both cumulants seem indeed to be almost independent of the lattice size. Taking as final estimate the weighted mean value (i.e., a least-squares fit to a constant) over the results for $N = 8,000 - 128,000$, we obtain

$$U^*_2 = 0.58706 \pm 0.00044,$$  \hspace{1cm} (36)

$$U^*_4 = 0.4647 \pm 0.0012.$$  \hspace{1cm} (37)

The variation due to the uncertainty in $K_c$ is about twice the statistical error at fixed $K$ (0.00080 for $U_2$ and 0.0020 for $U_4$). For comparison, for the standard nearest-neighbor Ising model on a SC lattice, Ferrenberg and Landau\(^{68}\) estimated $U^*_4 \approx 0.47$, by combining results for three different spin models belonging to the Ising universality class, Blöte \textit{et al.}\(^{69}\) derived $U^*_4 = 0.4652(4)$, and Ballesteros \textit{et al.} obtained $U^*_4 = 0.4656(4)$\(^{4}\). For the $\phi^4$ lattice field theory Hasenbusch\(^{71}\) extracted $U^*_4 \approx 0.46555(9)$. 

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6 Concluding remarks

We have performed a detailed finite-size scaling analysis of single-cluster Monte Carlo simulations of the Ising model on three-dimensional Poissonian random lattices of Voronoi/Delaunay type. At first sight our use of different quantities to estimate the same critical exponent might appear redundant, since the various estimates are, of course, not independent in a statistical sense. Their consistency, however, gives confidence that corrections to the asymptotic scaling behavior are very small and can safely be neglected. Our estimates for the exponents $\nu$, $\beta/\nu$, and $\gamma/\nu$ are all consistent with the best numerical estimates for the three-dimensional Ising model and $\phi^4$ field theory on regular lattices – at a very high level of accuracy which is comparable with the best estimates coming from field theoretical techniques, cf. Table 8.74 While our exponent ratio $\gamma/\nu$ would also be compatible with recent estimates for the 3D Ising disordered fixed point, our estimate for $U^*_4$ is more consistent with the pure Ising model estimates. The cleanest result yields the critical exponent $\nu$, where our result agrees within error bars with all previously derived estimates for the pure model but is clearly incompatible with the disordered fixed point value. We thus obtain strong evidence that, for the

Table 8: Recent estimates of critical parameters of the pure and disordered 3D Ising model (SC = simple cubic lattice, RG = renormalization group, SD = site-dilution, RIM = random Ising model).

| method      | $\nu$         | $\gamma/\nu$ | $U^*_4$  |
|-------------|---------------|--------------|----------|
| SC$^{68}$   | 0.6289(8)     | 1.970(14)    | 0.47     |
| SC$^{69}$   | 0.6301(8)     | 1.9630(30)   | 0.4652(4)|
| SC$^{67}$   | 0.6294(5)[5]  | 1.9626(6)[6] | 0.4656(4)[4] |
| SC $\phi^{471}$ | 0.6296(3)[4] | 1.9642(4)[5] | 0.4655(9) |
| RG$^{15}$   | 0.6304(13)    | 1.966(6)     | –        |
| This work   | 0.6299(5)     | 1.9576(13)   | 0.4647(12)|
| SD SC$^{6}$ | 0.6837(24)[29]| 1.9626(36)[9]| 0.449(5)[2]|
| RIM-RG$^{13}$ | 0.678(10)    | 1.970(3)     | –        |
| RIM-RG$^{11}$ | 0.675        | 1.951        | –        |
considered lattice sizes up to \( N = 128000 \approx 50^3 \), the Ising model on three-dimensional Poissonian random lattices of Voronoi/Delaunay type behaves effectively as on regular lattices.

Of course, we cannot exclude the possibility that on much larger length scales (lattice sizes) the scaling behavior may change. Such a late crossover is conceivable in the case of weak disorder, where the asymptotic critical behavior governed by a “disordered” fixed point may show up only in the extremely close vicinity of criticality, that is at extremely large system sizes in a finite-size scaling analysis. Even though the qualitative scaling behavior is expected to be universal, quantitative properties of the crossover point such as its location should depend on the strength of the disorder via non-universal amplitudes. In order to obtain for the random lattices a rough estimate of the strength \( S \) of the local connectivity disorder we have computed the relative variance of the local coordination numbers which may be viewed as a measure for the size of effective temperature variations over the lattice. From the probability density \( P(q) \) displayed in Fig. 2 it is straightforward to obtain

\[
S \equiv \frac{(q - \overline{q})^2}{\overline{q}^2} = 0.0461, \tag{38}
\]

with \( \overline{q} = 2 + 48\pi^2/35 = 15.5354 \ldots \). Similarly, for two-dimensional Poissonian Voronoi/Delaunay random lattices one finds \( S = 0.0491 \) with \( \overline{q} = 6 \). The relative variance (38) can be compared with the fluctuations of the number \( B \) of active bonds per site in bond-diluted models. Here \( B \) follows a binomial distribution, \( P(B) = \binom{2D}{B} p^B (1 - p)^{2D - B} \), where \( D \) is the dimension and \( p \) denotes the probability for a bond to be active (such that \( p = 1 \) corresponds to the pure model), and one obtains

\[
S = \frac{(B - \overline{B})^2}{\overline{B}^2} = \frac{1}{2D} \frac{1 - p}{p}, \tag{39}
\]

with \( \overline{B} = 2Dp \). By equating Eqs. (38) and (39) and solving for the dilution parameter \( p \) one can thus determine an associated bond-dilution model with the same local disorder fluctuations as for the random lattices. For the three-dimensional case this yields \( p = 0.7834 \), and in two dimensions one finds \( p = 0.8358 \). In the terminology of three-dimensional bond-diluted Ising\(^7\)–\(^9\) and \( q \)-state Potts\(^17\),\(^19\) models such a value of \( p \) belongs to the weak dilution regime where some influence of the disordered fixed point can be observed, but it is still difficult to clearly disentangle it. For site-diluted models the
corresponding $p$-value is presumably higher, in particular for weak dilution, since all bonds around a vacant site are non-active. In the latter models, of course, the dilution parameter $p$ can easily be tuned to study more accessible regions.

In view of the very high quality of our fits based on the leading FSS ansatz only we must conclude that in the case of Voronoi/Delaunay random lattices very much larger system sizes would be necessary to observe the expected crossover to the critical behavior associated with the disordered fixed point. This was clearly outside the scope of the present study and its computer budget which was equivalent to several years of fast workstation CPU time. Instead of further increasing the system size, an alternative and more promising route for future studies could be a systematic variation of the random lattice construction by modifying the Poissonian nature of the site distribution such as to achieve larger values of $S$ (corresponding to smaller values of $p$), or an investigation of the present random lattices coupled to a model with a larger critical exponent $\alpha$ where the expected crossover should set in for reasonable lattice sizes already for a moderate degree of local disorder.

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Appendix: Random lattice construction

The employed algorithm for the random lattice construction works as follows. Adapting the method described in Ref.~\textsuperscript{60} we first draw randomly $N$ sites uniformly distributed in a unit volume, thereby approximating a Poissonian distribution. For alternative distributions discussed in the literature see, e.g., Refs.~\textsuperscript{41,59} In the second step we link the sites according to the Voronoi/Delaunay prescription. We start by picking the first site that we drew and locate all its nearest neighbors, keeping them stored in an array. Then we proceed to the second site and search for all its nearest neighbors too; once finished with the second site we keep repeating the procedure until we have done it for all the sites. Of course, with this method we locate a given link twice, but the simplicity of its implementation pays off.

Starting from a given site, $x_1$, the linking procedure works as follows. Its nearest neighbor, $x_2$, is located from within the few hundred sites forming, or belonging to, the “cloud-of-neighbors” around $x_1$. We will comment later the issue of how to determine a cloud-of-neighbors for a given site. Notice that some care must be exercised when approaching the boundaries of the lattice to ensure the periodic boundary conditions. Afterwards a third site, $x_3$, is searched for, and a triangle is constructed. In order to locate this third site, we draw circumferences going through $x_1$, $x_2$, and the few hundred sites belonging to the cloud-of-neighbors of $x_1$. We pick as $x_3$ the site for which the radius of the circumference is the smallest. From the triangle $\Delta(x_1, x_2, x_3)$ we proceed to locate a fourth site, $x_4$, linked to $x_1$ and build a tetrahedron, $\tau(x_1, x_2, x_3, x_4)$. When a triangle still has not been used to build a tetrahedron from it, it is termed “active” and a logical flag is turned “on”. When already used, it is renamed to “inactive” and the flag is turned “off”.

To construct a tetrahedron from a triangle we split the volume in two half spaces: one “above” and the other “below” the plane lying on the triangle $\Delta(x_1, x_2, x_3)$. Let us suppose that we search for $x_4$ in the half space “above” the triangle. In order to determine $x_4$, we draw spheres going through $x_1$, $x_2$, $x_3$ and the sites belonging to the cloud-of-neighbors of $x_1$ placed in the half space in which we are working. If we happen to find several trial sites for which their distance to the circumcenter of the triangle $\Delta(x_1, x_2, x_3)$ is smaller than the radius of the circumscribed circle of $\Delta(x_1, x_2, x_3)$, then we pick as $x_4$ the site for which the radius of the circumscribed sphere of
τ(x₁, x₂, x₃, x₄) is the biggest. If, on the contrary, all the trial sites lie at a distance from the circumcenter of the triangle △(x₁, x₂, x₃) greater than the radius of the circumscribed circle of △(x₁, x₂, x₃), then we pick as x₄ the site for which the radius of the circumscribed sphere of τ(x₁, x₂, x₃, x₄) is the smallest. From the newly built tetrahedron τ(x₁, x₂, x₃, x₄), we can take two “active” triangles, △(x₁, x₂, x₄) and △(x₁, x₃, x₄) to continue building tetrahedra from triangles, and then triangles from tetrahedra. The closer neighbors of a given site x₁ are all found when there is no “active” triangle left connected to the site.

When describing how to locate x₁’s nearest neighbor, x₂, or how to find x₃ afterwards we emphasized that we only search from within the sites forming a cloud-of-neighbors around x₁. Its meaning is that before starting the linking procedure we set up an array for each site in the lattice containing the sites forming its cloud. A given site will belong to the cloud of, say, x₁ if it lies within a sphere centered in x₁. The radius of the sphere is chosen such that, on the average, the number of sites within the sphere is three times of an a priori upper limit to the maximum number of links that a site is likely to have in a finite Voronoi/Delaunay random lattice. To implement an efficient search of the sites which will belong to the cloud-of-neighbors of a given site, we subdivided the unit volume into smaller boxes. The optimal box size should be large enough to ensure that nearest neighbors will be located in the same box or at least in one of the 26 surrounding boxes, but small enough to minimize the time needed for testing all trial sites in a box.
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In Refs.67 and 71 the error bars are given separately for purely statistical and more systematic errors.

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74 In the summarizing Table 8 we confine ourselves to those critical parameters which are usually measured or calculated independently. Other exponents are often derived by scaling or hyperscaling relations which does not add any new information.
Figures

3D Poissonian Voronoi/Delaunay random lattices

Figure 1: The total CPU time spent in constructing a three-dimensional Poissonian random lattice according to the Voronoi/Delaunay prescription versus the number of sites $N$. The circles show the fraction of time needed to set up the “cloud-of-neighbors”.
Figure 2: (a) The probability density $P(q)$ of the coordination numbers $q$. The average is $\bar{q} = 2 + 48\pi^2/35 = 15.5354\ldots$. (b) The same data as in (a) on a logarithmic scale.
Figure 3: FSS fits to extract $1/\nu$.

Figure 4: FSS fits of the pseudo-transition points $K_{\text{max}, i}$ with $1/\nu = 1.5875$ fixed, yielding a combined estimate of $K_c = 0.0724249(40)$. 
Figure 5: FSS behavior of the specific heat, assuming $\alpha/\nu = 2/\nu - D = 0.1750$. 
Figure 6: FSS behavior of the magnetic cumulants. The central value of $K$ is our best estimate (25) for the inverse critical temperature. For the neighboring curves the $K$ values vary by about one statistical error bar.