Noncommutative analysis and quantum physics
I. States and ensembles

Arnold Neumaier
Institut für Mathematik, Universität Wien
Strudlhofgasse 4, A-1090 Wien, Austria
email: neum@cma.univie.ac.at
WWW: http://solon.cma.univie.ac.at/~neum/

Abstract. In this sequence of papers, noncommutative analysis is used to give a consistent axiomatic approach to a unified conceptual foundation of classical and quantum physics.

The present Part I defines the concepts of observables, states and ensembles, clarifies the logical relations and operations for them, and shows how they give rise to dynamics and probabilities.

States are identified with maximal consistent sets of weak equalities in the algebra of observables (instead of, as usual, with the rays in a Hilbert space). This leads to a concise foundation of quantum mechanics, free of undefined terms, separating in a clear way the deterministic and the stochastic features of quantum physics.

The traditional postulates of quantum mechanics are derived from well-motivated axiomatic assumptions. No special quantum logic is needed to handle the peculiarities of quantum mechanics. Foundational problems associated with the measurement process, such as the reduction of the state vector, disappear.

The new interpretation of quantum mechanics contains ‘elements of physical reality’ without the need to introduce a classical framework with hidden variables. In particular, one may talk about the state of the universe without the need of an external observer and without the need to assume the existence of multiple universes.

1991 MSC Classification: primary 81P10
1990 PACS Classification: 03.65.Bz

Keywords: axiomatization of physics, correspondence principle, deterministic, elements of physical reality, ensemble, event, expectation, flow of truth, foundation of quantum mechanics, Heisenberg picture, hidden variables, ideal measurement, induction, laws of nature, model and reality, philosophy of probability, preparation of states, quantum computing, quantum logic, quantum probability, quantum probability, Schrödinger picture, selection, spin, state, state of the universe, uncertainty relation, weak equality, weak equation
1 Introduction

“Look,” they say, “here is something new!” But no, it has all happened before, long before we were born.
(Good News Bible, Eccl. 1:10)

Do not imagine, any more than I can bring myself to imagine, that I should be right in undertaking so great and difficult a task. Remembering what I said at first about probability, I will do my best to give as probable an explanation as any other – or rather, more probable; and I will first go back to the beginning and try to speak of each thing and of all.
Plato, ca. 367 B.C. [39]

After more than eleven years of long and exciting, often also very frustrating investigations into the foundations of physics, one of the most foggy regions of the Platonic world, the fog finally cleared. In the light of the morning sun, the continent of quantum physics appears as a well-organized, comprehensible and beautiful part of the Platonic world of precise ideas.

The present paper is the first one of a series of papers that spell out the fruits of my journeys, designed to give a mathematically elementary and philosophically consistent foundation of modern theoretical physics, presented in the framework of noncommutative analysis. It is an attempt to reconsider, from a more modern point of view, HILBERT’s [14] sixth problem, the axiomatization of theoretical physics. It is an attempt only since at the present stage of development, I have not yet tried to achieve full mathematical rigor everywhere. (However, Parts I and II are completely rigorous, and in later parts it is explicitly mentioned where the standard of rigor is relaxed.)

One of the basic premises throughout my years of search was that the split between classical physics and quantum physics should be as small as possible. This is optimally realized in the set-up proposed here. Except in the examples, the formalism never distinguishes between the classical and the quantum situation. Thus it can be considered as a consequent implementation of BOHR’s correspondence principle. This also has didactical advantages for teaching: Students can be trained to be acquainted with the formalism by means of intuitive, primarily classical examples at first. Later, without having to unlearn anything, they can apply the same formalism to quantum phenomena.

The present Part I is concerned with giving a concise foundation by defining the concepts of observables, states and ensembles, clarifying the logical relations and operations for them, and showing how they give rise to the
traditional postulates of quantum mechanics, including dynamics and probabilities.

Much of what is done here is common wisdom in quantum mechanics; however, the new interpretation slightly shifts the meaning of some concepts, fixing them in a way such as to forbid certain identifications that gave rise to the riddles of quantum philosophy. Traditional interpretations always assumed without reflection the validity of the equation

\[ \text{ideal measurement} = \text{state of the system} = \text{pure prepared state}. \]

The new approach differentiates between the three concepts and gives each one a distinct meaning. In particular, we shall strictly distinguish between states, measurement and preparation. This helps to clarify the meaning of these concepts and reduces the danger of paradoxical conclusions.

By identifying states with maximal consistent sets of weak equalities in the algebra of observables (instead of, as usual, with the rays in a Hilbert space), a concise foundation of quantum mechanics is given, free of undefined terms. The deterministic and the stochastic features of quantum physics are separated in a clear way. No special quantum logic is needed to handle the peculiarities of quantum mechanics. Foundational problems associated with the measurement process, such as the reduction of the state vector, disappear.

The new interpretation of quantum mechanics contains elements of physical reality in the sense of Einstein, Podolsky & Rosen [9], without the need to introduce a classical framework with hidden variables (cf. my analysis of flaws in the traditional mathematical arguments against realism in [29]). In particular, one may talk about the state of the universe without the need of an external observer and without the need to assume the existence of multiple universes.

To motivate the new conceptual foundation and to place it into context, I found it useful to embed the formalism into my philosophy of physics, while strictly separating the mathematics by using the classical definition-example-theorem-proof exposition style. Though I present my view generally without using subjunctive formulations or qualifying phrases, I do not claim that this is the only way to understand physics. Thus I do not attempt to refute any of the alternative interpretations or any arguments why particular philosophical positions close to the one I maintain are not compelling. However, I do attempt to give a vivid picture of the particular philosophical view that gave me the vision to find the new foundation. And I believe that this view is consistent with the mathematical formalism of quantum mechanics and accommodates naturally a number of puzzling questions about the nature of the world.
Section 2 motivates and introduces the basic concepts of statements, selections, and states, and how they are realized in classical physics and quantum mechanics. This section also spells out the relation of the new interpretation to the traditional concepts of states and ideal measurement. Section 3 then discusses the properties of selections and states in more detail, and in particular shows that there are states that cannot be described by a single ideal measurement.

In Section 4 we discuss elementary aspects of the dynamics of physical systems, as far as relevant for the new interpretation. In Section 5, we look at the way ensembles and probabilities arise in the new interpretation, and in Section 6 we consider the implications about the uncertainties inherent in state preparation and measurement.

The Section 7 concludes the main exposition by relating the new interpretation to more general philosophical questions. We discuss a particular philosophical position, namely that quantum physics should be regarded to be as deterministic as classical physics, and that the stochastic features arise from the impossibility of preparing deterministic states, due to the uncertainty principle. We look at the borderline between objective existence and subjective judgment, and we reconsider the unreasonable effectiveness of mathematics in the natural sciences (Wigner [54]).

Subsequent parts of this sequence of papers will present a differential calculus based on Poisson algebras and its application to the dynamics of physical systems [31], the calculus of integration [32] and its application to equilibrium thermodynamics [33], a relativistic covariant Hamiltonian multiparticle theory, and its application to nonequilibrium thermodynamics and quantum field theory.

Acknowledgments. I’d like to thank Hermann Schichl, Tapio Schneider and Karl Svozil for discussions that led to improvements of the present text, and Waltraud Huyer for pointing out the need to allow nonhermitian observables to get Proposition 2.7(iv).

2 Statements and states

How wonderful are the things the Lord does! All who are delighted with them want to understand them.

(Good News Bible, Psalm 111:2)

All our scientific knowledge is based on past observation, and only gives rise to conjectures about the future. Mathematical consistency requires that
our choices are constrained by some formal laws. When we want to predict
something, the true answer depends on knowledge we do not have. We can
calculate at best approximations whose accuracy can be estimated using
statistical techniques (assuming that the quality of our models is good).
This implies that we must distinguish between quantities (formal concepts
of what can possibly be measured or calculated) and numbers (the results of
measurements and calculations themselves); those quantities that are con-
stant by the nature of the concept considered behave just like numbers.
Comparison with experiment exclusively concerns expectations of quantities,
numbers associated to the quantities in a precise way depending on the prepa-
ration of an experiment (cf. Sections 5 and 6).

In quantum mechanics (see, e.g., Jammer [17, 18], Jauch [19], von Neum-
mann [34], Messiah [27]), observables are identified with certain elements
of the algebra $E$ of bounded linear operators in a Hilbert space. However, the
Hilbert space has no operational physical interpretation; hence we shall drop
it from our main line of discussion, and resurrect it only to give examples.
On the other hand, the algebra $E$ is essential. Physicists are used to calculat-
ing with quantities that they may add and multiply without restrictions; if
the quantities are complex, the complex conjugate can also be formed. Thus
we take as primitive objects of our treatment a set $E$ of quantities, such that
the sum and the product of quantities is again a quantity, and there is an
operation generalizing complex conjugation.

Operations on quantities are required to satisfy a few simple rules; they
are called axioms since we take them as a formal starting point without
making any further demands on the nature of the symbols we are using. Our
axioms are motivated by the wish to be as general as possible while still
preserving the ability to manipulate quantities in the manner familiar from
matrix algebra. (Similar axioms for quantities have been proposed, e.g., by
Dirac [6] and Thirring [50].)

2.1 Definition.

(i) $E$ denotes a set whose elements are called quantities. For any two quan-
tities $f, g \in E$, the sum $f + g$, the product $fg$, and the conjugate $f^*$ are
also quantities, and the following axioms (O1)–(O5) are assumed to hold for
$f, g, h \in E$ and $\alpha \in \mathbb{C}$.

(O1) $\mathbb{C} \subseteq E$, i.e., complex numbers are special quantities, where addition,
multiplication and conjugation have their traditional meaning.

(O2) $(fg)h = f(gh), \quad \alpha f = f \alpha, \quad 0f = 0, \quad 1f = f.$

(O3) $(f + g) + h = f + (g + h), \quad f(g + h) = fg + fh, \quad f + 0 = f.$
(O4) $f^{**} = f$, $(fg)^* = g^* f^*$, $(f + g)^* = f^* + g^*$,
(O5) $f^* f = 0$ $\Rightarrow$ $f = 0$.

(ii) We introduce the traditional notation

$$-f := (-1)f, \quad f - g := f + (-g), \quad [f, g] := fg - gf,$$

$$f^0 := 1, \quad f^l := f^{l-1}f \quad (l = 1, 2, \ldots),$$

$$\text{Re} \ f = \frac{1}{2}(f + f^*), \quad \text{Im} \ f = \frac{1}{2i}(f - f^*).$$

(iii) A quantity $f \in \mathbb{E}$ is called Hermitian if $f^* = f$, and normal if $[f, f^*] = 0$. We refer to normal quantities as observables, and we denote by

$$A_{\text{obs}} = \{f \in A \mid [f, f^*] = 0\}$$

the observable part of a subset $A$ of $\mathbb{E}$.

(Note that every Hermitian quantity is normal and hence an observable. But our definition also allows certain nonhermitian observables.)

We shall see that, for the general, qualitative aspects of the theory there is no need to know any details of how to actually perform calculations with quantities; this is only needed if one wants to calculate specific properties for specific systems. In this respect, the situation is quite similar to the traditional axiomatic treatment of real numbers: The axioms specify the permitted ways to handle formulas involving these numbers; and this is enough to derive calculus, say, without the need to specify either what real numbers are or algorithmic rules for addition, multiplication and division. Of course, the latter are needed when one wants to do specific calculations but not while one tries to get insight into a problem. And as the development of pocket calculators has shown, the capacity for understanding theory and that for knowing the best ways of calculation need not even reside in the same person.

Note that we assume commutativity only between numbers and quantities. However, commutativity of the addition is a consequence of our other assumptions:

2.2 Proposition. For all quantities $f, g, h \in \mathbb{E}$,

$$(f + g)h = fh + gh, \quad f - f = 0, \quad f + g = g + f.$$

Proof. The right distributive law follows from

$$\begin{align*}
(f + g)h &= ((f + g)h)^{**} = (h^*(f + g)^*)^* = (h^*(f^* + g^*))^* \\
&= (h^*f^* + h^*g^*)^* = (h^*f^*)^* + (h^*g^*)^* \\
&= f^{**}h^{**} + g^{**}h^{**} = fh + gh.
\end{align*}$$
It implies $f - f = 1f - 1f = (1 - 1)f = 0f = 0$. From this, we may deduce that addition is commutative, as follows. The quantity $h := -f + g$ satisfies

$$-h = (-1)((-1)f + g) = (-1)(-1)f + (-1)g = f - g,$$

and we have

$$f + g = f + (h - h) + g = (f + h) + (-h + g) = (f - f + g) + (f - g + g) = g + f.$$  

Thus, in conventional terminology (see, e.g., Rickart [40]), $E$ is a nondegenerate $*$-algebra with unity, but not necessarily with a commutative multiplication. As the example $E = \mathbb{C}^{n \times n}$ (with complex numbers identified with the scalar multiples of the identity matrix) shows, $E$ may have zero divisors, and not every nonzero quantity need have an inverse. Therefore, in the manipulation of formulas, precisely the same precautions must be taken as in ordinary matrix algebra.

A statement is the assertion of some pieces of information available about a system in a given state or set of states. We base our new interpretation on the assumption that the elementary pieces of information are assertions of weak equality $f \approx g$ between quantities $f, g \in E$.

The logical structure of the theory is defined by axioms for valid inference; these state how valid statements may be combined to give further valid statements. Axioms (R1)–(R3) below express reflexivity, symmetry and transitivity, (R4) says that we may add arbitrary quantities to both sides of a weak equality, and (R5) says that we may multiply weak equalities from the left by arbitrary quantities. On the other hand, (R6) restricts the addition of weak equalities to those where all four terms are mutually commuting observables. This is an expression of von Neumann’s [34] observations that the sum of two observables $f, g$ has a natural interpretation in terms of $f$ and $g$ only when these and their conjugates all commute with each other. Thus weak equality is indeed a weaker concept than standard equality.

2.3 Definition.

(i) A statement is a relation $\approx$ on $E$. We say that $f, g \in E$ are weakly equal if $f \approx g$, and that $f \in E$ vanishes if $f \approx 0$. In particular, a single weak equality is considered to be a statement.

(ii) A statement $\approx$ is logically closed if, for all $f, g, h \in E$,

(R1) $f \approx f$. 

\[\Box\]
(R2) \( f \approx g \Rightarrow g \approx f \),
(R3) \( f \approx g, \, g \approx h \Rightarrow f \approx h \),
(R4) \( f \approx g \Rightarrow f + h \approx g + h \),
(R5) \( f \approx g \Rightarrow hf \approx hg \),
and, for all \( f, g, f', g' \in \mathbb{E}_{\text{obs}} \) commuting with each other and with their conjugates,
(R6) \( f \approx g, \, f' \approx g' \Rightarrow f + f' \approx g + g' \).

(Note that \( f \approx g \) usually does not imply \( f^* \approx g^* \)!)  

(iii) The **logical closure** of a family of statements \( \approx_l \) \((l \in L)\) is the logically closed statement \( \approx \) with the fewest weak equalities satisfying

\[
f \approx_l g \Rightarrow f \approx g
\]

for all \( l \) and all \( f, g \in \mathbb{E} \). We say that any subset of weak equalities in the logical closure can be **inferred** from the statements \( \approx_l \) \((l \in L)\). The statements \( \approx_l \) \((l \in L)\) are called **consistent** if \( 1 \approx 0 \) cannot be inferred from them.

Traditional quantum logic (**Birkhoff & von Neumann** [1], see also **Svozil** [49]) can be regarded as the theory of weak equalities \( e \approx 0 \) or \( e \approx 1 \) for orthogonal projectors \( e \). (We shall use these projectors in Section 5 for counting events.) However, there is no intrinsic reason in the quantum mechanical formalism why only these statements should be admissible.

With our definition, there is no special need for a quantum logic. The only logic to be used is the classical logic for handling inferences about statements, with the rules (R1)–(R6) and standard logical operations and quantors. This is satisfying since, indeed, classical logic is used in practice to handle virtually all applications of quantum mechanics.

### 2.4 Examples.

(i) **Classical physics.** Classical physics happens in a set \( \Omega \) called the *phase space*, and \( \mathbb{E} \) is an algebra of bounded functions \( f : \Omega \rightarrow \mathbb{C} \).

If \( B \) is an open subset of \( \mathbb{R}^m \) and \( p \) is a vector of commuting Hermitian observables \( p_j \) \((j = 1, \ldots, m)\), we denote by \( p \approx \in B \) ("\( p \) is weakly in \( B \)"") the statement that all quantities \( F(p) \in \mathbb{E} \) with bounded \( F : \mathbb{R}^m \rightarrow \mathbb{C} \) and support disjoint from \( B \) vanish. (If \( B \) is not open, \( p \approx \in B \) is taken to mean \( p \approx \in B' \) for all open \( B' \) containing \( B \).)

In the particular case where \( B \) is the open ball with center \( k \in \mathbb{R}^n \) and radius \( \varepsilon > 0 \), we denote this statement by \( \|p - k\| \approx < \varepsilon \) ("\( \|p - k\| \) is weakly..."
smaller than \( \varepsilon \)). It represents the assertion that a measurement of \( p \) would give with certainty a value that deviates from \( k \) by less than \( \varepsilon \).

(ii) Nonrelativistic quantum mechanics. In nonrelativistic quantum mechanics, \( \mathbb{E} \) is the algebra of bounded linear operators in the Hilbert space \( L^2(\Omega) \), where \( \Omega \) is the direct product of \( \mathbb{R}^n \) and a finite set \( S \) that takes care of spin, color, and similar indices. (In contrast to the classical case, \( \Omega \) is only ‘half’ of phase space!) The statements \( p \approx \epsilon \) and \( \|p - k\| \approx < \epsilon \) have precisely the same definition and interpretation as in the classical case.

(iii) One of the nontrivial traditional postulates of quantum mechanics, that the possible values an observable \( f \) may take are the elements of the spectrum \( \text{Spec} f \) of \( f \), is in the new interpretation a simple consequence of the trivial axioms (R1)–(R5); Let \( f \in \mathbb{E}, \lambda \in \mathbb{C} \).

If the weak equality \( f \approx \lambda \) can be deduced from a consistent statement then \( \lambda \in \text{Spec} f \).

Indeed, if this is not the case then \( g := (\lambda - f)^{-1} \) exists. By (R5), \( f \approx \lambda \) implies \( gf \approx g\lambda \), hence \( 1 = g(\lambda - f) = g\lambda - gf \approx 0 \) by (R4), contradiction. Note that this holds both in classical physics and in quantum mechanics (and more generally whenever \( \mathbb{E} \) is a Banach *-algebra [40]).

Because of (R4) we may restrict attention to weak equalities where one side is zero. We therefore define the concept of a selection that serves to characterize the set of vanishing quantities that can be inferred from a given set of statements. A selection can be thought of as containing all information deducible from partial knowledge about a system in a given state.

2.5 Definition.

(i) A selection is a nonempty subset \( \Pi \) of \( \mathbb{E} \) such that

- \( f \in \mathbb{E}, g \in \Pi \Rightarrow fg \in \Pi, \) (S1)
- \( f, g \in \Pi_{\text{obs}}, [f, g] = [f, g^*] = 0 \Rightarrow f + g \in \Pi. \) (S2)

A selection is called ideal if, in place of (S2), the stronger statement

- \( f, g \in \Pi \Rightarrow f + g \in \Pi, \) (S2a)

holds, and valid if \( 1 \notin \Pi \). We denote the set of selections by \( \mathbb{P} \).

(ii) A state is a maximal valid selection, i.e., a valid selection \( \Sigma \) such that

- \( \Sigma \subseteq \Pi \in \mathbb{P}, \Pi \neq \Sigma \Rightarrow 1 \in \Pi. \)

(iii) A set of selections is called consistent if their union is contained in a valid selection. Two selections \( \Pi, \Pi' \) are called orthogonal (and we write \( \Pi \perp \Pi' \)) if

- \( f \in \Pi, g \in \Pi' \Rightarrow fg^* = 0. \)
An inconsistent pair of orthogonal selections is called an **alternative**.

(iv) A statement \( \approx \) is called **true** in the state \( \Sigma \) if

\[
    f - g \in \Sigma \quad \text{for all } f, g \in E \text{ with } f \approx g,
\]

and **false** otherwise.

### 2.6 Proposition.

(i) The set of \( f \in E \) for which \( f \approx 0 \) can be inferred from a given family of statements is a selection.

(ii) If \( \Pi \) is a selection then the relation \( \approx \) defined by

\[
    f \approx g \iff f - g \in \Pi
\]

is a logically closed statement.

**Proof.** This follows directly from the definitions.

\( \square \)

A state contains all possible information about a system, as far as it is accessible within the framework of the theory. It asserts a maximal set \( \Sigma \) of vanishing objects, hence of weak equalities, that does not yet contain invalid information that would allow to conclude the equation \( 1 \approx 0 \), i.e., \( 1 \in \Sigma \). By maximality, adding any additional information, i.e., a statement \( f \approx g \) with \( f - g \not\in \Sigma \), would allow to deduce the invalid statement \( 1 \approx 0 \).

In traditional terminology, a maximal valid ideal selection is a maximal left ideal of \( E \). If \( E \) is the algebra of bounded linear operators in a separable Hilbert space then every set of the form

\[
    \Pi_\psi = \{ f \in E \mid f \psi = 0 \}
\]

for some vector \( \psi \neq 0 \) is a maximal left ideal. Conversely, all closed maximal left ideals have this form. Clearly \( \Pi_\psi \) depends only on the ray \( \mathbb{C} \psi \) spanned by \( \psi \). Thus, in this case, the closed ideal selections that are maximal among the valid ones are in one-to-one correspondence with the rays in the Hilbert space, i.e., with the traditional quantum mechanical pure states.

According to established quantum mechanical thinking (as codified for example in **von Neumann** [34]), an ideal measurement defines a quantum mechanical pure state. It corresponds to a set of weak equalities (assertions true in this ‘pure state’), maximal with respect to the restriction that it can (in principle) be verified by an **instantaneous** experiment. (See, e.g., **Wigner** [56, pp.284-288] for details on the instantaneous approximations involved in ideal measurements.)
However, as we shall see soon, our concept of a state is essentially different from the traditional pure state of quantum mechanics. This has very interesting consequences for the interpretation of quantum mechanics. In particular, it seems now conceivable that the simultaneous assertion of precise values for position and momentum are consistent (cf. Problem 3.7(iv) below).

This is the decisive difference. In the traditional interpretations, closed maximal left ideals (rays) are associated with pure states obtained by a mysterious process called an ‘ideal measurement’, supposedly achieved by ‘state reduction’ through contact with a classical measuring apparatus (ill-defined since ‘classical’ has no meaning in the formalism). And there are severe interpretational problems with ‘superpositions of pure states’ that cannot be given a classical meaning.

In the present interpretation, measurements no longer figure in the conceptual basis (though they can be idealized as being represented by certain ideal selections), and states are not related to measurement but to the maximal possible information that can be asserted without contradiction, namely that certain quantities are weakly equal.

Therefore, the relation of actual experimental measurements to ideal measurements is no longer a matter obscuring the foundations of physics, but a thermodynamical question concerning the interaction of a system with a macroscopic, dissipation-producing measuring apparatus (cf., e.g., Davies [5], Busch et al. [4], Zurek [57], Joos & Zeh [20], Ghirardi, Rimini & Weber [12]).

The new interpretation has no longer a place for mysteries since all concepts used in the interpretation have precise definitions. In particular, there is no state reduction, except approximately, as far as thermodynamical arguments apply. And what was before a ‘superposition of pure states’ is now simply a particular set of valid weak equations, without any spooky associations.

2.7 Proposition. Let $\mathbb{E}$ be an arbitrary $*$-algebra.

(i) The only invalid selection is $\Pi = \mathbb{E}$.

(ii) Every set of the form

$$\Pi = \mathbb{E}g = \{fg \mid f \in \mathbb{E}\},$$

is an ideal selection.

(iii) If $fg^* = 0$ then $\mathbb{E}f$ and $\mathbb{E}g$ are orthogonal selections. In particular, if $e^2 = e = e^*$ then $\mathbb{E}e$ and $\mathbb{E}(1-e)$ form an alternative.

(iv) If $\mathbb{E}$ is commutative, every selection is ideal.
Proof. (i) If $\Pi$ is invalid then $1 \in \Pi$ and by (S1) every $f \in E$ is in $\Pi$. Hence $\Pi = E$.

(ii) and (iii) are straightforward.

(iv) Since $E$ is commutative, every quantity is normal, hence $\Pi_{obs} = \Pi$. Thus, since any two quantities commute, (S2) implies (S2a), i.e., $\Pi$ is ideal. $\square$

2.8 Example. (Classical physics)

For a set $\Omega$, consider the algebra $E$ of bounded functions $f : \Omega \rightarrow \mathbb{C}$. It is easy to see that an ideal selection $\Pi$ is characterized by the set

$$\hat{\Pi} = \{ \omega \in \Omega \mid f(\omega) = 0 \text{ for all } f \in \Pi\}$$

of points annihilating $\Pi$. Indeed, given $\hat{\Pi}$, the ideal selection $\Pi$ can be reconstructed from this set as $\Pi = \hat{\Pi}$, where, for $W \subseteq \Omega$,

$$\hat{W} = \{ f \in E \mid f(\omega) = 0 \text{ for all } \omega \in W\}.$$ 

Conversely, any $\hat{W}$ is an ideal selection. The maximal ideals are obtained for $W = \{ \omega \}$,

$$\Pi_{\omega} = \{ f \in E \mid f(\omega) = 0\}$$

with $\omega \in \Omega$, in one-to-one correspondence with the points of $\Omega$. By Proposition 2.7, every selection $\Pi$ is ideal, and the states are just the maximal valid ideal selections. Thus the set $\Omega$ can be identified with the set of possible states, a selection $\Pi$ is equivalent to finding out that the system measured is in one of the states $\omega \in \hat{\Pi}$, and an ideal measurement is equivalent to finding the precise state of the system. This is typical for classical physics, based on a commutative $\ast$-algebra $E$.

2.9 Example. (Discretized quantum physics)

In the algebra $E = \mathbb{C}^{n \times n}$ of $n \times n$-matrices with complex entries, the ideal selections $\Pi$ are characterized by their row space

$$\hat{\Pi} = \{ \phi^* f \mid \phi \in \mathbb{C}^n, f \in \Pi\}.$$ 

Indeed, given $\hat{\Pi}$, the ideal selection $\Pi$ can be reconstructed from this vector space as $\Pi = \hat{\Pi}$, where, for vector spaces $W$ of row vectors of length $n$,

$$\hat{W} = \{ f \in \mathbb{C}^{n \times n} \mid \phi^* f \in W \text{ for all } \phi \in \mathbb{C}^n\}.$$ 

Conversely, any $\hat{W}$ is an ideal selection. The maximal ideal selections are obtained by choosing for $W$ a hyperplane,

$$W_\psi = \{ \phi \mid \phi \in \mathbb{C}^n, \phi^* \psi = 0\}$$
for some nonzero $\psi \in \mathbb{C}^n$. Hence they have the form

$$\Pi = \hat{W}_\psi = \{ f \in \mathbb{C}^{n \times n} \mid f\psi = 0 \}.$$ 

Since $\hat{W}_\psi$ does not change when $\psi$ is multiplied by a nonzero number, the maximal valid ideal selections (and hence the ideal measurements) are in one-to-one correspondence with the rays $\mathbb{C}\psi$ with $\psi \in \mathbb{C}^n \setminus \{0\}$. However, as we shall see in Section 3, there are many selections that are not ideal, and the maximal ideal selections are no longer states. This is typical for quantum physics (though there one generally has a Hilbert space in place of $\mathbb{C}^n$, and topological considerations modify the results a little.)

3 Properties of selections and states

There are more things in heaven and earth, Horatio, than are dreamt of in your philosophy.
W. Shakespeare, 1602 A.D. [44]

To prove in general the existence of states, we need Zorn’s Lemma (see, e.g., Kelley [21]), a well-known consequence of the axiom of choice.

3.1 Lemma. (Zorn)
Let $\mathcal{F}$ be a family of sets with the chain property,
(C) The union of every subfamily of $\mathcal{F}$, linearly ordered by inclusion, is contained in some element of $\mathcal{F}$.
Then $\mathcal{F}$ contains a maximal element.

3.2 Theorem.
(i) Every valid selection is contained in some state.
(ii) Every valid ideal selection is contained in a maximal one.

Proof. (i) We have to prove that for every valid selection $\Pi$ there is a maximal valid selection $\Sigma$ containing $\Pi$. But the form of the axioms (S1) and (S2) implies that the set of selections has the chain property. Hence the assertion follows from Zorn’s Lemma.
(ii) Similarly, the set of valid ideal selections has the chain property, and Zorn’s lemma applies.
Thus states always exist, though our proof is not constructive. In fact, Zorn’s Lemma is equivalent to the axiom of choice, and hence cannot be made constructive; so it would be desirable to find a constructive proof for the case of *-algebras of physical interest. On the other hand, this nonconstructiveness might possibly raise interesting decidability problems for statements about physics.

We now proceed to get further insight into the properties of selections and states.

3.3 Proposition.

(i) The intersection of an arbitrary set of selections is again a selection.
(ii) The intersection of an arbitrary set of ideal selections is again an ideal selection.
(iii) Every subset $A \subseteq \mathbb{E}$ is contained in a unique smallest selection $\bar{A}$, the intersection of all selections containing $A$. In particular, writing $\bar{f} = \{ f \}$, we have

\[ \bar{0} = \{0\}, \quad \bar{1} = \mathbb{E}. \]

(iv) For every subset $A \subseteq \mathbb{E}$, the **orthogonal complement** of $A$, defined by

\[ A^\perp = \{ f \mid fg^* = 0 \text{ for all } g \in A \} \]

is an ideal selection.

*Proof.* Straightforward. \qed

3.4 Proposition. Two selections $\Pi$ and $\Pi'$ satisfying $1 \in \Pi_{\text{obs}} + \Pi'_{\text{obs}}$ are inconsistent.

*Proof.* Let $\Pi''$ be a selection containing $\Pi$ and $\Pi'$. If $1 \in \Pi_{\text{obs}} + \Pi'_{\text{obs}}$ then there is a quantity $f \in \Pi_{\text{obs}}$ such that $1 - f \in \Pi'_{\text{obs}}$. Hence $f, 1 - f \in \Pi''_{\text{obs}}$. Since $[f, 1 - f] = [f, 1 - f^*] = 0$, (S2) implies $1 = f + (1 - f) \in \Pi''$. \qed

We now show that, in the quantum case, there are valid selections not covered by an ideal selection.

3.5 Theorem. Let $\Psi$ be a set of vectors in a Hilbert space $\mathbb{H}$ of dimension $> 1$ such that distinct vectors in $\Psi$ are neither parallel nor orthogonal. Then

\[ \Pi_1(\Psi) := \{ \phi \psi^* \mid \phi \in \mathbb{H}, \psi \in \Psi \} \]

is a valid selection in the algebra of bounded linear operators on $\mathbb{H}$. 

14
Proof. Clearly, (S1) is satisfied. Two nonzero operators $f = \phi\psi^*$ and $g = \phi'\psi'^*$ commute iff $\phi' = \phi$ and $\psi' = \psi$, and the assumptions on $\Psi$ then imply that $\psi' = \psi$. Hence (S2) also holds.

Note that it is easy to choose $\Psi$ in many ways such that the subspace spanned by $\Pi_1(\Psi)$ coincides with the space of all finite rank operators on $\mathbb{H}$. The corresponding sets $\Pi_1(\Psi)$ are valid selections not contained in a valid ideal selection. In particular, by Theorem 3.2, this implies that there are states not contained in a valid ideal selection (and hence not obtainable by an ideal measurement).

In the smallest noncommutative $\ast$-algebra $E = \mathbb{C}^{2 \times 2}$, describing a single quantum spin, it is even possible to describe all possible selections, and hence all possible states, explicitly.

3.6 Theorem.
(i) Every selection $\Pi \neq 0, 1$ of $E = \mathbb{C}^{2 \times 2}$ has the form $\Pi = \Pi_1(\Psi)$ with $\Psi \subseteq \mathbb{C}^2$ as in Theorem 3.5.
(ii) The maximal valid ideal selections are precisely the sets $E\psi = \Pi_1(\{\psi\})$ with $\psi \in \mathbb{C}^2 \setminus \{0\}$.
(ii) The alternatives are precisely the pairs $0, 1$ and the pairs $E\phi, E\psi$ with nonzero, orthogonal $\phi, \psi \in \mathbb{C}^2$.
(iv) The states of $E$ have the form $\Pi = \Pi_1(\Psi)$, where $\Psi$ contains exactly one nonzero vector from each pair of orthogonal one-dimensional subspaces of $\mathbb{C}^2$. In particular, no ideal selection is a state.

Proof. This is a straightforward consequence of Theorem 3.5 and Example 2.9.

The wish to extend this result to more general situations suggests the following open questions. For $E = \mathbb{C}^{2 \times 2}$, the answers to (i) and (ii) are affirmative, and in (iii), no additional condition is needed.

3.7 Problems.
(i) Is every state a union of maximal valid ideal selections?
(ii) Given a state $\Sigma$ and an ideal selection $\Pi \not\subseteq \Sigma$, is there always an ideal selection $\Pi' \subseteq \Sigma$ not consistent with $\Pi$?
(iii) Let $E$ be a $\ast$-algebra (with 1) of linear operators on a Hilbert space $\mathbb{H}$. If $\Psi$ is a set of vectors from $\mathbb{H}$ such that distinct vectors in $\Psi$ are neither
parallel nor orthogonal, which additional conditions must be imposed on $\Psi$ to ensure that

$$\Pi(\Psi) := \{ f \in E \mid f\psi = 0 \text{ for some } \psi \in \Psi \}$$

is a valid selection?

(iv) Are arbitrarily accurate measurements of position and momentum consistent? More precisely, let $p_\mu$ and $q_\mu$ denote the (Hermitian) components of the canonical position and momentum vectors, with canonical commutation relations

$$[p_\mu, p_\nu] = [q_\mu, q_\nu] = 0, \quad [q_\mu, p_\nu] = \delta_{\mu\nu}i\hbar.$$

Are the statements $\|p - k\| \approx \varepsilon$ and $\|q - x\| \approx \varepsilon'$ (defined in Example 2.4) consistent whenever $\varepsilon, \varepsilon' > 0$?

4 Dynamics

_God does not play dice with the universe._
Albert Einstein, 1927 A.D. [8]

In this section we discuss elementary aspects of the dynamics of physical systems, as far as relevant for the new interpretation. We shall have much more to say about dynamics in later parts of this series of papers.

The observations about a physical system change with time. The dynamics of a conservative system is described by a fixed (but system-dependent) one-parameter family $T_t (t \in \mathbb{R})$ of automorphisms of the *-algebra $E$, i.e., mappings $T_t : E \to E$ satisfying (for $f, g \in E$, $\alpha \in \mathbb{C}$, $s, t \in \mathbb{R}$)

$$T_t(\alpha) = \alpha, \quad T_t(f^*) = T_t(f)^*,$$

$$T_t(f + g) = T_t(f) + T_t(g), \quad T_t(fg) = T_t(f)T_t(g),$$

$$T_0(f) = f, \quad T_{s+t}(f) = T_s(T_t(f)).$$

For dissipative systems, a semigroup of mappings $T_t$, $t \geq 0$ with the same properties replaces the group of automorphisms.

In the Heisenberg picture of the dynamics, where states are fixed and quantities change with time, $f(t) := T_t(f)$ denotes the Heisenberg quantity associated with $f$ at time $t$. Note that $f(t)$ is uniquely determined by $f(0) = f$. Thus the dynamics is deterministic, independent of whether we are in a classical or in a quantum setting.
4.1 Examples. In nonrelativistic mechanics, conservative systems are described by a Hermitian quantity $H$, called the Hamiltonian.

(i) In classical mechanics, a Poisson bracket $\{\cdot, \cdot\}$ together with $H$ defines the Liouville superoperator $Lf = \{f, H\}$, and the dynamics is given by

$$T_t(f) = e^{tL}(f).$$

(ii) In quantum mechanics, the dynamics is instead given by

$$T_t(f) = e^{-it\hbar/H} e^{it\hbar/H}.$$  

Of course, weak equalities valid at some time need not be valid at other times. To see what happens, suppose that $f \approx g$ at time $t = 0$. Then, in the Heisenberg picture, $f(t) \approx g(t)$, i.e., $T_t(f) \approx T_t(g)$ at time $t$. Thus, if $\Pi$ is the selection of all quantities that can be inferred to vanish at time $t = 0$ then

$$\Pi(t) := \{T_t(f) \mid f \in \Pi\}$$

is the selection of all quantities that can be inferred to vanish at an arbitrary time $t$. In particular, a state $\Sigma$ at time $t = 0$ develops into the state

$$\Sigma(t) := \{T_t(f) \mid f \in \Sigma\}$$

at time $t$. This describes the Schrödinger picture of the state dynamics, where quantities are fixed and states change with time. Again the dynamics is deterministic.

In a famous paper, EINSTEIN, PODOLSKY & ROSEN [9] introduced the following criterion for elements of physical reality:

*If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity* and postulated that

*the following requirement for a complete theory seems to be a necessary one: every element of the physical reality must have a counterpart in the physical theory.*

Traditionally, elements of physical reality were thought to have to emerge in a classical framework with hidden variables. However, to embed quantum mechanics in such a framework is impossible under natural hypotheses (Kochen & Specker [22]).
It is therefore interesting to see that, in the present interpretation, each true weak equality is such an element of physical reality. In this sense, the new interpretation is a realistic interpretation of quantum mechanics.

In particular, one can talk about the state of the universe without the need of an external observer (WIGNER [55]) and without the need to assume the existence of multiple universes (EVERETT [10]). Instead of many worlds, the new interpretation suggests that there is a single world, but one with more facets to it than hitherto suspected.

Taking another look at the form of the Schrödinger dynamics, we see that the vanishing quantities (or equivalently the statements) behave just like the particles in an ideal fluid. We may therefore say that the Schrödinger dynamics describes the flow of truth in an objective, deterministic manner. On the other hand, the Schrödinger dynamics is completely silent about what is true. Thus, as in mathematics, where all truth is relative to the logical assumptions made (what is considered true at the beginning of an argument), in physics truth is relative to the initial values assumed (what is considered true at time $t = 0$).

In both cases, theory is about what is consistent, and not about what is real or true. The formalism enables us only to deduce truth from other assumed truths. But what is regarded as true is outside the formalism, may be quite subjective and may even turn out to be contradictory, depending on the acquired personal habits of self-critical judgment. And finding out what is ‘really’ true is highly restricted by the quantum mechanical uncertainty relations (see Section 6); thus different experts can only form more or less valid approximations to the real truth. This is very much in agreement with what we see in practice.

What we can possibly know as true are the laws of physics, general relationships that appear often enough to see the underlying principle (cf. the remarks about induction in the final section). But concerning states (i.e., in practice, boundary conditions) we are doomed to idealized, more or less inaccurate approximations of reality. WIGNER’s [54, p.5] expressed this by saying, the laws of nature are all conditional statements and they relate only to a very small part of our knowledge of the world.
5 Ensembles and probability

We may assume that words are akin to the matter which they describe; when they relate to the lasting and permanent and intelligible, they ought to be lasting and unalterable, and, as far as their nature allows, irrefutable and immovable – nothing less. But when they express only the copy or likeness and not the eternal things themselves, they need only be likely and analogous to the real words. *As being is to becoming, so is truth to belief.*

Plato, ca. 367 A.D. [39]

*Only love transcends our limitations. In contrast, our predictions can fail, our communication can fail, and our knowledge can fail. For our knowledge is patchwork, and our predictive power is limited. But when perfection comes, all patchwork will disappear.*

St. Paul, ca. 57 A.D. [37]

The stochastic nature of quantum mechanics is usually discussed in terms of *probabilities*. However, from a strictly logical point of view, this has the drawback that one gets into conflict with the traditional foundation of probability theory by Kolmogorov [23], which does not extend to the noncommutative case. Mathematical physicists (see, e.g., Parthasarathy [36], Meyer [28]) developed a far reaching quantum probability calculus based on Hilbert space theory. But their approach is highly formal, drawing its motivation from analogies to the classical case rather than from the common operational meaning.

Whittle [53] presents a much less known alternative approach to classical probability theory, equivalent to that of Kolmogorov, that treats *expectation* as the basic concept and derives probability from axioms for the expectation. (See the discussion in [53, Section 3.4] why, for historical reasons, this has remained a minority approach.) The approach via expectations is easy to motivate, leads quickly to interesting results, and extends without much trouble to the quantum world, yielding the ensembles (‘mixed states’) of traditional quantum physics.

The axioms we shall require for meaningful expectations are those trivially satisfied for weighted averages of a finite ensemble of observations. While this motivates the form of the axioms and the name ‘ensemble’ attached to the concept, there is no need at all to interpret expectation as an average; this is the case only in certain classical situations. In general, the expectation of a quantity $f$ is simply a value near which, based on the theory, we may expect the measured value for $f$. At the same time, the standard deviation
serves as a measure of the amount to which we may expect this nearness to deviate from exactness.

5.1 Definition.
(i) An **ensemble** is a mapping \( \mathcal{F} \) that assigns to each quantity \( f \in \mathbb{E} \) its expectation \( \mathcal{F} =: \langle f \rangle \in \mathbb{C} \) such that for all \( f, g \in \mathbb{E}, \alpha \in \mathbb{C}, \)

(P1) \( \langle 1 \rangle = 1, \quad \langle f^* \rangle = \langle f \rangle^*, \quad \langle f + g \rangle = \langle f \rangle + \langle g \rangle, \)

(P2) \( \langle \alpha f \rangle = \alpha \langle f \rangle, \)

(P3) If \( f \geq 0 \) then \( \langle f \rangle \geq 0, \)

(P4) If \( f_i \in \mathbb{E}, \ f_i \downarrow 0 \) then \( \inf \langle f_i \rangle = 0. \)

Here \( f_i \downarrow 0 \) means that the \( f_i \) converge to 0 and \( f_{l+1} \leq f_i \) for all \( l. \)

(ii) The number

\[ \text{cov}(f,g) := \text{Re}\langle (f - \mathcal{F})(g - \mathcal{G})^* \rangle \]

is called the **covariance** of \( f, g \in \mathbb{E}, \) and the number

\[ \sigma(f) := \sqrt{\text{cov}(f,f)} \]

the **standard deviation** of \( f \in \mathbb{E}. \)

This definition generalizes the expectation axioms of WHITTLE [53] for classical probability theory. Note that (P3) ensures that \( \sigma(f) \) is a nonnegative real number that vanishes if \( f \) is a constant quantity (i.e., a complex number).

To avoid technicalities about topology and order relations (discussed in a more detailed treatment in Part III [32]), we don’t use in this section the topological axiom (P4), and assume that \( \mathbb{E} \) is either an algebra of functions or the algebra of bounded linear operators on some Hilbert space, with partial order relation defined by \( f \leq g \) iff \( g - f \) is Hermitian and nonnegative resp. positive semidefinite. In both cases, for all quantities \( f, g, \)

\[ f^*f \geq 0, \quad ff^* \geq 0, \]

\[ g \geq 0 \quad \Rightarrow \quad g = g^* \quad \text{and} \quad f^*gf \geq 0. \]

5.2 Examples.
(i) **Finite probability theory.** In the commutative algebra \( \mathbb{E} = \mathbb{C}^n \) with pointwise multiplication and componentwise inequalities, every linear functional on \( \mathbb{E}, \) and in particular every ensemble, has the form

\[ \langle f \rangle = \sum_{k=1}^{n} p_k f_k \]
for certain constants \( p_k \). The ensemble axioms hold precisely when the \( p_k \) are nonnegative and add up to one; thus \( \langle f \rangle \) is a weighted average.

By Example 2.8 (applied to \( \Omega = \{1, \ldots, n\} \)), the states are precisely the sets \( \Pi_\omega = \{ f \in \mathbb{E} \mid f_\omega = 0 \} \) with \( \omega \in \Omega \), and the natural probability of a state \( \Pi_\omega \) in the ensemble defined by (4) is \( p_\omega \). All elementary probability theory with a finite number of events can be discussed in this setting.

Note that the probability can be recovered from the expectation by means of the formula

\[
 p_\omega = 1 - \sum_{k \neq \omega} p_k = 1 - \sup \left\{ \sum p_k f_k \mid f_\omega = 0, f \leq 1 \right\} = 1 - \sup \left\{ \langle f \rangle \mid f \in (\Pi_\omega)_{\text{obs}}, f \leq 1 \right\}. \tag{5}
\]

(ii) Quantum mechanical ensembles. In the algebra \( \mathbb{E} \) of bounded linear operators on a Hilbert space \( \mathbb{H} \), traditional quantum mechanics describes a pure ensemble (traditionally called a ‘pure state’, but this terminology conflicts with our new interpretation) by the expectation

\[
 \langle f \rangle := \psi^* f \psi,
\]

where \( \psi \in \mathbb{H} \) is a unit vector. And quantum thermodynamics describes an equilibrium ensemble by the expectation

\[
 \langle f \rangle := \text{tr} e^{-S/k} f,
\]

where \( k > 0 \) is the Boltzmann constant, and \( S \) is a Hermitian observable with \( \text{tr} e^{-S/k} = 1 \) called the entropy whose spectrum is discrete and bounded below. In both cases, the ensemble axioms are easily verified.

5.3 Proposition. For any ensemble,

(i) \( f \leq g \implies \langle f \rangle \leq \langle g \rangle \).

(ii) If \( f \) is Hermitian then \( \bar{f} = \langle f \rangle \) is real and

\[
 \sigma(f) = \sqrt{\langle (f - \bar{f})^2 \rangle}.
\]

Proof. (ii) follows directly from (P1), and (i) from (P1) and (P3). \( \square \)

The interpretation of probability has been surrounded by philosophical puzzles for a long time; Fine [11] is probably still the best discussion of the problems involved. Our definition generalizes the classical intuition of probabilities as weights in a weighted average and is modeled after the formula (5) for finite probability theory in Example 5.2(i).
In the special case when a well-defined counting process may be associated
with the statement whose probability is assessed, our exposition supports
the conclusion of Drieschner [7, p.73], “probability is predicted relative
frequency” (German original: “Wahrscheinlichkeit ist vorausgesagte relative
Häufigkeit”). More specifically, we assert that, for counting events, the prob-
ability carries the information of expected relative frequency (see Theorem
5.6(v) below).

To make this precise we need a precise concept of independent events that
may be counted. To motivate our definition, assume that we look at times
t_1,\ldots,t_N for the presence of an event of the sort we want to count. We
introduce observables \( e_t \) whose value is the amount added to the counter
at time \( t_t \). For correct counting, we need \( e_t \approx 1 \) if an event happened at
time \( t_t \), and \( e_t \approx 0 \) otherwise; thus \( e_t \) should have the two possible values
0 and 1 only. Since these numbers are precisely the Hermitian idempotents
among the constant quantities, this suggests to identify events with general
Hermitian idempotent quantities.

5.4 Definition.

(i) A quantity \( e \in \mathbb{E} \) satisfying
\[
e^2 = e = e^*
\]
is called an event. Two events \( e,e' \) are independent in an ensemble \( \langle \cdot \rangle \) if
they commute and satisfy
\[
\langle ee' \rangle = \langle e \rangle \langle e' \rangle.
\]

With any event we associate the ideal selection
\[
\Pi_e := \mathbb{E}(1 - e),
\]
the set of all quantities vanishing as a consequence of \( e \approx 1 \).

(ii) In a given ensemble, the number
\[
\text{pr}(\Pi) := 1 - \sup\{ \langle f \rangle \mid f \in \Pi_{\text{obs}}, f \leq 1 \}
\]
is called the probability of the selection \( \Pi \), and the number \( \langle e \rangle \) is called the
probability of the event \( e \).

5.5 Examples.

(i) Classical probability theory. In the algebra of bounded complex-
valued functions on a set \( \Omega \), every characteristic function \( e = \chi_M \) (with
\( \chi_M(x) = 1 \) if \( x \in M \), \( \chi_M(x) = 0 \) otherwise) is an event, and
\[
\Pi_e = \{ f \in \mathbb{E} \mid f(x) = 0 \text{ for } x \in M \},
\]
\[ \Pi_e^\perp = \{ f \in E | f(x) = 0 \text{ for } x \not\in M \}. \]

(ii) **Quantum probability theory.** In the algebra of bounded linear operators on a Hilbert space \( \mathbb{H} \), every unit vector \( \psi \in \mathbb{H} \) gives rise to an elementary event \( e = \psi \psi^* \), and

\[ \Pi_e = \{ f \in E | f\psi = 0 \}, \]
\[ \Pi_e^\perp = \{ \phi \psi^* | \phi \in \mathbb{H} \}. \]

(There are also other, nonelementary events.)

5.6 Theorem.

(i) For any event \( e \), its negation \( \neg e = 1 - e \) is also an event, with

\[ \Pi_{\neg e} = \mathbb{E}e = \Pi_e^\perp. \quad (6) \]

Moreover, if the weak equality \( e \approx \lambda \) (\( \lambda \in \mathbb{C} \)) can be deduced from a consistent statement then \( \lambda \in \{0, 1\} \).

(ii) For commuting events \( e, e' \), the observables

\[ e \land e' = ee', \quad e \lor e' = e + e' - ee' \]

are also events. Their probabilities satisfy

\[ \langle e \land e' \rangle + \langle e \lor e' \rangle = \langle e \rangle + \langle e' \rangle, \]
\[ \langle e \rangle + \langle \neg e \rangle = 1. \]

Moreover,

\[ \langle e \land e' \rangle = \langle e \rangle \langle e' \rangle \quad \text{for independent events } e, e'. \]

(iii) For any selection \( \Pi \), we have \( 0 \leq \text{pr}(\Pi) \leq 1 \).

(iv) For any event \( e \),

\[ \text{pr}(\Pi_e) = \langle e \rangle, \quad \text{pr}(\Pi_e^\perp) = 1 - \langle e \rangle. \]

In particular, \( 0 \leq \langle e \rangle \leq 1 \).

(v) For a family of events \( e_l \) (\( l = 1, \ldots, N \)) with constant probability \( \langle e_l \rangle = p \), the relative frequency

\[ q := \frac{1}{N} \sum_{l=1}^{N} e_l \]
satisfies
\[ \langle q \rangle = p; \]
if the events are independent,
\[ \sigma(q) = \sqrt{\frac{p(1-p)}{N}}. \]
becomes arbitrarily small as \( N \) becomes large (weak law of large numbers).

(We remark in passing that, with the operations \( \wedge, \vee, \neg \), the set of events in any commutative subalgebra of \( \mathbb{E} \) forms a Boolean algebra; see Stone [47].)

**Proof.**
(i) Clearly \( \neg e \) is Hermitian, and \( (\neg e)^2 = (1-e)^2 = 1 - 2e + e^2 = 1 - e = \neg e \). Hence \( 1 - e \) is an event. The left equality in (6) holds by definition. Since \((fe)(g(1-e))^* = f(e-e^2)g^* = 0\), we have \( \mathbb{E}e \subseteq \Pi_e^\perp \). But if \( f \in \Pi_e^\perp \) then \( f(1-e)^* = 0 \), hence \( f = fe \in \mathbb{E}e \), so that \( \mathbb{E}e \subseteq \Pi_e^\perp \). Hence (6) holds. Finally, suppose that \( e \approx \lambda \). Then \( 0 \approx \lambda - e \) by (R4), and by (R5),
\[ 0 \approx (1 - \lambda - e)(\lambda - e) = \lambda(1 - \lambda) - e + e^2 = \lambda - \lambda^2. \]
If \( \lambda \neq 0,1 \), we may multiply on the left by the complex number \( (\lambda - \lambda^2)^{-1} \) and find the contradiction \( 0 \approx 1 \). Hence \( \lambda \in \{0,1\} \).

(ii) Since \( e \) and \( e' \) commute, \( (ee')^* = e^*e' = e'e = ee' \), hence \( ee' \) is Hermitian; and it is idempotent since \( (ee')^2 = ee'ee' = e^2e^2 = ee' \). Finally, \( e + e' - ee' = 1 - (1-e)(1-e') = \neg(e \wedge \neg e') \) is an event. The assertions about expectations are immediate.

(iii) Since, by Proposition 5.3(i), \( \langle f \rangle \leq 1 \) for all \( f \leq 1 \), we have \( 0 \leq \text{pr}(\Pi) \), and since \( 0 \in \Pi \), we have \( \text{pr}(\Pi) \leq 1 \).

(iv) Let \( \Pi = \Pi_e^\perp \). If \( f \in \Pi_{\text{obs}} \) then \( f = ge \) for some \( g \in \mathbb{E} \), hence \( fe = ge^2 = ge = f \) and \( ef = e^*f^* = (fe)^* = f^* = f \). Now \( f \leq 1 \) implies \( 1 - f \geq 0 \), hence \( e - f = e^2 - efe = e^*(1-f)e \geq 0 \), so that \( \langle f \rangle \leq \langle e \rangle \). Therefore \( \text{pr}(\Pi) \geq 1 - \langle e \rangle \).

Equality holds since \( f = e \in \Pi_{\text{obs}} \) satisfies \( 1 - f = 1 - e = (1-e)^*(1-e) \geq 0 \), hence \( f \leq 1 \). Thus \( \text{pr}(\Pi_e^\perp) = 1 - \langle e \rangle \). Replacing \( e \) by \( \neg e \) and noting (i) gives \( \text{pr}(\Pi_e) = \langle e \rangle \).

(v) \( \langle q \rangle = p \) follows from
\[ \langle q \rangle = \frac{1}{N}(\langle e_1 \rangle + \ldots + \langle e_N \rangle) = \frac{1}{N}(p + \ldots + p) = p. \]

To get the expression for \( \sigma(q) \) when the events are independent, we first note that
\[ q^2 = \frac{1}{N^2} \left( \sum_j e_j \right) \left( \sum_k e_k \right) = N^{-2} \sum_{j,k} e_j e_k. \]
In the expectation of this sum we get $N^2 - N$ contributions of size $\langle e_j \rangle \langle e_k \rangle = p^2$ and $N$ contributions of size $\langle e_j^2 \rangle = \langle e_j \rangle = p$. Hence

\[
\langle q^2 \rangle = N^{-2}(Np + (N^2 - N)p^2),
\]

\[
\sigma(q)^2 = \langle (q - p)^2 \rangle = \langle q^2 \rangle - 2p\langle q \rangle + p^2 = p(1 - p)/N.
\]

Applied to ideal measurements in a pure ensemble, our recipe for the probability just gives the classical **squared probability amplitude** formula:

**5.7 Corollary.** Let $\phi$ be a unit vector in a Hilbert space $\mathbb{H}$. In the algebra $\mathbb{E}$ of bounded linear operators on $\mathbb{H}$, the probability that a maximal ideal selection of the form

\[
\Pi_\phi = \{ f \in \mathbb{C}^{n \times n} \mid f\phi = 0 \}
\]

is valid in a given ensemble is

\[
\text{pr}(\Pi_\phi) = \langle \phi \phi^* \rangle.
\]

In particular, for a pure ensemble described by the unit vector $\psi \in \mathbb{H}$, the probability that $\Pi_\phi$ is valid is

\[
\text{pr}(\Pi_\phi) = |\phi^*\psi|^2. \tag{7}
\]

**Proof.** By Example 5.5(iii), $\Pi_\phi = \Pi_e$ with $e = \phi \phi^*$, and by Theorem 5.6(iv), $\text{pr}(\Pi_\phi) = \langle e \rangle = \langle \phi \phi^* \rangle$. In particular, for a pure ensemble described by the unit vector $\psi \in \mathbb{H}$, we have $\text{pr}(\Pi_\phi) = \langle \phi \phi^* \rangle = \psi^*\phi^*\psi = |\phi^*\psi|^2$. \qed

Equation (7) replaces the traditional interpretation of $|\phi^*\psi|^2$ as the probability that after preparing a pure ensemble in ‘state’ $\psi$, an ideal measurement causes a ‘state reduction’ to the new pure ‘state’ $\phi$. Note that the new interpretation of $|\phi^*\psi|^2$ is completely within the formal framework of the theory and completely independent of the measurement process.
6 Uncertainty

The lot is cast into the lap; but its every decision is from the LORD.
King Solomon, ca. 1000 B.C. [46]

As the heavens are higher than the earth, so are my ways higher than your ways and my thoughts than your thoughts.
The LORD, according to Isaiah, ca. 540 B.C. [16]

Enough, if we adduce probabilities as likely as any others; for we must remember that I who am the speaker, and you who are the judges, are only mortal men, and we ought to accept the tale which is probable and enquire no further.
Plato, ca. 367 B.C. [39]

The common form and deterministic nature of the dynamics, independent of any assumption of whether the system is classical or quantum, implies that there is no difference in the causality of classical mechanics and that of quantum mechanics. Therefore, the differences between classical mechanics and quantum mechanics cannot lie in an assumed intrinsic indeterminacy of quantum mechanics contrasted to deterministic classical mechanics.

In the new interpretation of quantum mechanics, no new principle needs to be invoked. As in statistical physics, the stochastic nature of quantum mechanics can be explained simply by our inability to prepare experiments with a sufficient degree of sharpness to pin down the state of the system. A ‘prepared state’ is not really a state, in fact we usually know little with certainty, and never everything. Thus we need to describe the preparation of experiments in a stochastic language that permits the discussion of such uncertainties; in other words, we shall model prepared experiments by ensembles.

Formally, the essential difference between classical mechanics and quantum mechanics in the latter’s lack of commutativity. While in classical mechanics there is in principle no limit to the accuracy with which we can approximate a desired state, the quantum mechanical uncertainty relation for noncommuting observables puts severe limits on the ability to prepare microscopic ensembles. Here, preparation is defined as bringing the system into an ensemble such that certain specified weak equalities hold to an accuracy specified by an explicit standard deviation.

We now discuss the limits of the extent to which this can be done.
6.1 Theorem.

(i) The Cauchy–Schwarz inequality

\[ |\langle fg^* \rangle|^2 \leq \langle ff^* \rangle \langle gg^* \rangle \]

holds for all \( f, g \in E \).

(ii) The uncertainty relation

\[ \sigma(f)^2 \sigma(g)^2 \geq |\text{cov}(f, g)|^2 + \left| \frac{1}{2} \langle fg^* - gf^* \rangle \right|^2 \]

holds for all \( f, g \in E \).

(iii) For \( f, g \in E \),

\[ \text{cov}(f, g) = \frac{1}{2}(\sigma(f + g)^2 - \sigma(f)^2 - \sigma(g)^2), \quad (8) \]

\[ |\text{cov}(f, g)| \leq \sigma(f)\sigma(g), \quad (9) \]

\[ \sigma(f + g) \leq \sigma(f) + \sigma(g). \quad (10) \]

Proof. (i) For arbitrary \( \alpha, \beta \in \mathbb{C} \) we have

\[ 0 \leq \langle (\alpha f - \beta g)(\alpha f - \beta g)^* \rangle \]
\[ = \alpha \alpha^* \langle ff^* \rangle - \alpha \beta^* \langle fg^* \rangle - \beta \alpha^* \langle gf^* \rangle + \beta \beta^* \langle gg^* \rangle \]
\[ = |\alpha|^2 \langle ff^* \rangle - 2 \text{Re}(\alpha \beta^* \langle fg^* \rangle) + |\beta|^2 \langle gg^* \rangle. \]

We now choose \( \beta = \langle fg^* \rangle \), and obtain for arbitrary real \( \alpha \) the inequality

\[ 0 \leq \alpha^2 \langle ff^* \rangle - 2 \alpha |\langle fg^* \rangle|^2 + |\langle fg^* \rangle|^2 \langle gg^* \rangle. \quad (11) \]

Now \( \langle gg^* \rangle \geq 0 \) by (P3). If \( \langle gg^* \rangle > 0 \) we can choose \( \alpha = \langle gg^* \rangle \) and obtain

\[ 0 \leq \langle gg^* \rangle^2 \langle ff^* \rangle - \langle gg^* \rangle |\langle fg^* \rangle|^2. \]

After division by \( \langle gg^* \rangle \), we find that (i) holds. And if \( \langle gg^* \rangle = 0 \) then \( \langle fg^* \rangle = 0 \)

since otherwise a tiny \( \alpha \) produces a negative right hand side in (11). Thus (i) also holds in this case.

(ii) Since \((f - \bar{f})(g - \bar{g})^* - (g - \bar{g})(f - \bar{f}) = fg^* - gf^*\), it is sufficient to prove the uncertainty relation for the case of quantities \( f, g \) whose expectation vanishes. In this case,

\[ (\text{Re}(fg^*))^2 + (\text{Im}(fg^*))^2 = |\langle fg^* \rangle|^2 \leq \langle ff^* \rangle \langle gg^* \rangle = \sigma(f)^2 \sigma(g)^2. \]

The assertion follows since \( \text{Re}(fg^*) = \text{cov}(f, g) \) and

\[ i \text{Im}(fg^*) = \frac{1}{2}(\langle fg^* \rangle - \langle fg^* \rangle^*) = \frac{1}{2}(fg^* - gf^*). \]
(iii) Again, it is sufficient to consider the case of quantities $f, g$ whose expectation vanishes. Then
\[
\sigma(f + g)^2 = \langle (f + g)(f + g)^* \rangle = \langle ff^* \rangle + \langle fg^* + gf^* \rangle + \langle gg^* \rangle = \sigma(f)^2 + 2 \text{cov}(f, g) + \sigma(g)^2,
\]
and (8) follows. (9) is an immediate consequence of (ii), and (10) follows easily from (12) and (9).

In the classical case of commuting Hermitian quantities, the uncertainty relation just reduces to the well-known inequality (9) of classical statistics. For noncommuting Hermitian quantities, the uncertainty relation is stronger. In particular, we may deduce Heisenberg’s [13, 41] uncertainty relation
\[
\sigma(q)\sigma(p) \geq \frac{1}{2}\hbar
\]
for a pair of conjugate observables $p, q$, characterized by $[q, p] = i\hbar$ and Hermiticity. Thus no ensemble can be prepared where both $p$ and $q$ have arbitrarily small standard deviation. (More general noncommuting Hermitian observables $f, g$ may have some ensembles with $\sigma(f) = \sigma(g) = 0$, namely among those with $\langle fg \rangle = \langle gf \rangle$.)

We conclude that, similar to the case discussed by Schaller & Svozil [43] of a universe generated by a universal discrete computation, in a universe containing a conjugate pair of observables, an internal observer bound to the laws of this universe cannot investigate completely the detailed properties of the system. However, an external super-observer (viewing this universe as a kind of huge computer game and having access to the simulation code) might well know everything; at least, the present paper shows such a view to be consistent with the mathematics of quantum mechanics.

It is worthwhile to expand the understanding of the uncertainty relation by relating it to the restricted additivity of weak equalities in (R6).

6.2 Proposition. Let $p, q$ be Hermitian quantities satisfying $[q, p] = i\hbar$. Then, for any $k, x \in \mathbb{R}$ and any positive $\Delta p, \Delta q \in \mathbb{R},$
\[
\left( \frac{p - k}{\Delta p} \right)^2 + \left( \frac{q - x}{\Delta q} \right)^2 \geq \frac{\hbar}{\Delta p \Delta q},
\]

Proof. The quantities $b = (q - x) / \Delta q$ and $c = (p - k) / \Delta p$ are Hermitian and satisfy $[b, c] = [q, p] / \Delta q \Delta p = i\kappa$ where $\kappa = \hbar / \Delta q \Delta p$. Now the assertion follows from
\[
0 \leq (b + ic)^*(b + ic) = b^2 + c^2 + i[b, c] = b^2 + c^2 - \kappa.
\]
Because of Proposition 5.3, the left hand side of (13) cannot have arbitrarily small expectation. Example 2.4(iii) implies the even stronger statement that the possible values that a measurement of the left hand side of (13) can give are the odd positive multiples of the right hand side. (Indeed, this is the spectrum, since $a := \kappa^{-1/2}(b + ic)$ is a standard annihilation operator.) However, since $p$ and $q$ do not commute, it is not permitted to deduce from this that the summands $(p-k\Delta p)^2$ and $(q-x\Delta q)^2$ cannot be both small. Thus we see that noncommutativity together with the restricted additivity (R6) of weak equalities work together to avoid contradictions between the uncertainty principle and possibly precise knowledge of position and momentum; cf. Problem 3.7.

This points to a significant difference between preparation and measurement. The two concepts describe quite different activities: In an experiment, preparation always precedes measurement; in particular, experiments require a distinguished direction of time since the time reverse of an experiment rarely makes sense.

Moreover, measurement produces new knowledge (‘elements of physical reality’ in form of weak equations or inequalities) about a system, while preparation assumes statistical knowledge of past behavior of the components of a system (‘elements of physical probability’ in form of an ensemble) without any measurement of the prepared system.

Thus the preparation of systems is provably limited by the uncertainty relation for ensembles, while measuring systems seems not to be limited in the same way. As mentioned in Section 2, ideal measurements (corresponding to rays in Hilbert space, hence mathematically equivalent to pure ensembles and subject to the uncertainty relation) are adequate descriptors for (idealized) measurement processes only if these can be considered instantaneous (Wigner [56, pp.284-288]).

Therefore, the key to getting more complete information about any microscopic system seems to be that one measures properties of the system at a number of different times and reconstructs by statistical methods the most likely values of the variables of interest, even of conjugate variables like position and momentum.

An example for this are the particle tracks routinely reconstructed in high energy experiments (see, e.g., Bock et al. [2]) on the silent assumption that the particles have definite paths at all times. These paths are approximated by least squares techniques and provide highly accurate knowledge about
both position and momentum of the particles involved.

Thus, for sufficiently small subsystems and sufficiently large measuring devices, it seems possible in principle in the new framework to find out arbitrarily precisely what has happened, after the fact. But, since measurements influence the system, this is not possible in a way that would allow the precise prediction of the fate of such a system in the future, to prepare an ensemble that would violate the uncertainty relations.

To which extent one can pin down the ‘true’ state of a system is one of the challenges the new interpretation of quantum mechanics given here offers. And, considering the possible impact on clarifying the options in quantum computing, it might be a challenge with immense practical consequences.

One of the basic questions, not yet decided either theoretically or experimentally, is whether we can prepare a two-level quantum system (a single spin) sharp enough to ensure more than a single bit of information (the obvious limit in the classical case). As we have seen in Theorem 3.6 a potentially infinite amount of information is contained in a quantum spin state. And the prospective builders of quantum computers hope that one can exploit the quantum properties to beat the classical limitations on computing power. See, e.g., Shor [45], Braunstein [3].

However, such a highly informative quantum spin ensemble would have a very rugged dependence of the spin on the direction in which it is measured, and no one knows how one should prepare such an ensemble. Preparing the spin to have a fixed value in one particular direction only gives a probability distribution for the values in other directions, that decreases with the cosine of the angle between a measured direction and the prepared direction (Neumayer [29]). The question is whether a system can be prepared in a way that this probability distribution can be sharpened...

7 Knowledge and physical reality

_The man who thinks he knows something does not yet know as he ought to know._

St. Paul, ca. 57 A.D. [38]

Let me discuss here a philosophical statement about the nature of the world and its relation to what physicists do.

1. _States are_ (objective existence) and change in a deterministic fashion.

2. _Statements are assumed to be known_ (deliberate choice using subjective assessment) and change in a deterministic fashion.
3. **Ensembles are prepared** (belief calibrated by past observations) and give rise to stochastic indeterminacy based on incomplete knowledge, that for systems involving a pair of conjugate observables is unavoidable in principle.

4. **Measurements are performed** (subjective experience calibrated by training) and are stochastically distributed according to the results of the quantum mechanical formalism with input from 1.–3.

Expectations are primarily properties *not* of reality itself but of the ensemble assigned to a system. The latter depends on our assessment of reality, i.e., on the assumed preparation of the device. Thus ensembles only express our *conjectures* (or even prejudice) about reality, and must be brought into agreement with reality by *measurement* (finding something out about a system), *pattern recognition* (identifying a substance as hydrogen, say, by means of a few measurements, and implying then all properties of hydrogen for the whole substance), or *preparation of an experiment* (arranging subsystems whose properties are assumed to be known). In actual practice, ensembles are always abstractions from reality accurate only to a certain extent, and this accuracy is assessed by a measurement-assisted subjective interpretation of reality.

From a practical point of view, theory defines what an object is: A gas is considered as ideal gas, and a solid as a crystal, if it behaves, to our satisfaction, as a model of an ideal gas, or a crystal, predicts. And in preparing experiments one uses equipment supposed to produce a predictable environment; cf. Wigner’s [54, p.5] statement, *[In these] machines, the functioning of which he can foresee, [...] the physicist creates a situation in which all relevant coordinates are known so that the behavior of the machine can be predicted.*

Thus, in practice, one never ‘prepares a state’ by what, in the traditional foundation of quantum mechanics, is known as an ideal measurement; instead, ensembles are prepared by well-informed assumptions concerning one’s equipment. (And if experiments don’t give the expected results one usually first checks whether these assumptions were justified!)

Our knowledge about prepared ensembles is obtained only via the observed behavior in the past in similar situations. This is the operational meaning of the ensemble – it is an ensemble chosen on the basis of subjective knowledge about *past* situations that we hope is representative enough to tell us about *future* events.

We know that certain materials or machines reliably produce ensembles that depend only on variables that are accounted for in our theory and that are
either fixed or controllable. More precisely, we assume that we know this, on the basis of past experience, claims of manufacturers, occasional measurements and consistency checks, etc.. Our measure of reliability is a subjective sense of our satisfaction, or the satisfaction of others whom we trust, who checked that certain norms are satisfied. If we are careless or credulous, our subjective knowledge will be far off the mark, and the expectations based on it will simply not be matched by reality.

This interface between what we understand and what is, between model and reality, between theory and experiment, between calculated expectations and measurements always remains a subjective matter. It is ultimately based on trust in measurement devices, apparatus specifications, published data, etc., or perhaps rather based on trust in the people (including ourselves) responsible for them.

The strength of theoretical physics lies in the fact that it can ignore this subjective side by assuming ensembles to be given, which allows one to calculate expectations from well-defined assumptions. The weakness of theoretical physics lies in the impossibility to objectively verify these assumptions; comparison with reality always rests on trust in subjective aspects of observation and communication. Science is possible only because (and in as much as) it is possible to make these subjective aspects less influential by training people to adhere to high standards of precision, carefulness and truthfulness.

The new interpretation makes this gap between model and reality very explicit by giving precise concepts of states, ensembles and expectations. In this way it frees theoretical physics from philosophical riddles by a careful cut just at the point where objective expectations and their subjective interpretation interact; all these riddles are pushed to the subjective side of the cut.

In this sense, this paper can be viewed as a mathematical commentary on the statement of Margenau [26], Measurement is ... the contact of reason with nature. A three volume work [24, 48, 25] on the foundations of measurements gives a comprehensive survey of – partially successful – attempts to extend the realm of objectivity further by axiomatizing the measurement process in classical physics. The problems involved for the quantum case are well covered in the reprint collection of Wheeler & Zurek [52]. However, one cannot avoid making the transition to subjective judgment at some stage, and the setting proposed in the present paper has the great advantage of simplicity.

Since this gap between model and reality forms a built-in part of our axiomatic treatment, the latter gives a satisfactory account for the well-known
problem of induction. Nothing can guarantee that any model is true, even when truth is restricted to 'within a specified accuracy'. (Shall I prove that you live forever? You experienced all your past birthdays, without a single exception. Invoking Ockham’s razor [35, 15], *frustra fit per plura quod potest fieri per pauciora* – that we should opt for the most economic model explaining a regularity, we conclude that this will go on forever!! But, of course, this proves nothing.)

*I dreamt that I was in Hell, and that Hell is a place full of all those happenings that are improbable but not impossible. [...] every time that they have made an induction, the next instance falsifies it. This, however, happens only during the first hundred years of their damnation. After that, they learn to expect that an induction will be falsified, and therefore it is not falsified until another century of logical torment has altered their expectation. Throughout all eternity surprise continues, but each time at a higher logical level.*

B. Russell, 1954 A.D. [42]

But any existing regularity or structure in our world can be discovered by means of induction: diligent observation and good theory allows us to formulate the observed regularities as mathematical models. And why can we discover laws of nature? Because they can be formulated with few words and formulas – so a limited amount of plausible information allows us to guess correctly with almost certainty any law of limited complexity that actually exists. This is confirmed by results of Webb [51] that, at least in applications to machine learning – the automatic discovery of descriptions of massive sets of data from an accessible subset of data –, low complexity seems to be an essential element in the appropriateness of Ockham’s razor.

Thus induction works in physics not for logical reasons, but because nature is so highly structured. That the latter is the case follows from our overwhelming success in describing nature by means of concepts and laws of physics. If any method can be effective in describing complex structural patterns in nature, it must use mathematics, the science of exact concepts and their relations. Thus the unreasonable effectiveness of mathematics in the natural sciences (Wigner [54]) is explained by the plain assumption that nature possesses highly accurate laws of limited complexity that are universally valid and allow one to explain and predict so much about our universe.
8 Epilogue

The axiomatic foundation given here of the basic principles underlying theoretical physics suggest that, from a formal point of view, the differences between classical physics and quantum physics are only marginal (though in the quantum case, the lack of commutativity requires some care and causes deviations from classical behavior). In both cases, everything flows from the same assumptions simply by changing the realization of the axioms.

It is remarkable that, in the setting of Poisson algebras described and explored in later parts of this series of papers, this remains so even as we go deeper into the details of dynamics and thermodynamics.

References

[1] G. Birkhoff and J. von Neumann, The logics of quantum mechanics, Ann. Math. 37 (1936), 823-843.

[2] R.K. Bock, H. Grote, D. Notz and M. Regler, Data-analysis techniques for high-energy physics experiments, Cambridge Univ. Press, Cambridge 1990.

[3] S.L. Braunstein, Quantum computation: a tutorial, WWW-document, http://chemphys.weizmann.ac.il/~schmuel/comp/comp.html

[4] P. Busch, M. Grabowski and P.J. Lahti, Operational quantum physics, Springer, Berlin 1995.

[5] E.B. Davies, Quantum theory of open systems, Academic Press, London 1976.

[6] P.A.M. Dirac, Lectures on quantum field theory. Belfer Grad. School of Sci., New York 1966.

[7] M. Drieschner, Voraussage – Wahrscheinlichkeit – Objekt. Über die begrifflichen Grundlagen der Quantenmechanik. Lecture Notes in Physics, Springer, Berlin, 1979.

[8] A. Einstein, Conversation with Bohr and Ehrenfest at the Fifth Solvay conference in October, 1927; cf. http://solon.cma.univie.ac.at/~neum/contrib/dice.txt
The formulation used is from a letter of September 7, 1944, reprinted pp.
275-276 in: A.P. French (ed.), Einstein, a centenary volume, Harvard Univ. Press, Cambridge, Mass. 1979.

[9] A. Einstein, B. Podolsky and N. Rosen, Can the quantum-mechanical description of physical reality be considered complete? Phys. Rev. 47 (1935), 777-780. (Reprinted in [52].)

[10] H. Everett III, Relative state formulation of quantum mechanics, Rev. Mod. Phys. 29 (1957) 454-462. (Reprinted in [52].)

[11] T.L. Fine, Theory of probability; an examination of foundations. Acad. Press, New York 1973.

[12] G.C. Ghirardi, A. Rimini and T. Weber, Unified dynamics for microscopic and macroscopic systems, Phys. Rev. D 34 (1986), 470-491.

[13] W. Heisenberg, Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, Zeitschrift f. Physik 43 (1927), 172-198. (Engl. translation: Section I.3 in [52].)

[14] D. Hilbert, Mathematische Probleme, Bull. Amer. Math. Soc. 8 (1902), 437-479.

[15] R. Hoffmann, V.I. Minkin and B.K. Carpenter, Ockham’s Razor and Chemistry, HYLE Int. J. Phil. Chem 3 (1997), 3-28.
http://rz70.rz.uni-karlsruhe.de/ ed01/Hyle/Hyle3/hoffman.htm

[16] Isaiah 55:9, Holy Bible, New International Version, 1984.

[17] M. Jammer, The conceptual development of quantum mechanics, McGraw-Hill, New York 1966.

[18] M. Jammer, The philosophy of quantum mechanics: the interpretations of quantum mechanics in historical perspective, Wiley, New York 1974.

[19] J.M. Jauch, Foundations of quantum mechanics, Addison-Wesley, Reading, MA 1968.

[20] E. Joos and H.D. Zeh, The emergence of classical properties through interaction with the environment, Z. Phys. B 59 (1985), 223-243.

[21] J.L. Kelley, General topology, Van Nostrand, Princeton 1957.
[22] S. Kochen and E.P. Specker, The problem of hidden variables in quantum mechanics, J. Math. Mech. 17 (1967), 59-67. (Reprinted in C.A. Hooker, ed., The logico-algebraic approach to quantum mechanics, Vol. I: Historical evolution, Reidel, Dordrecht 1975.)

[23] A.N. Kolmogorov, Foundations of the theory of probability, Chelsea, New York 1950. (German original: Grundbegriffe der Wahrscheinlichkeitsrechnung, Springer, Berlin 1933.)

[24] D.H. Krantz, R.D. Luce, P. Suppes, and A. Tversky, Foundations of measurements, Vol. 1. Acad. Press, New York 1971.

[25] R. D. Lee et al., Foundations of measurements, Vol. 3. Acad. Press, San Diego 1990.

[26] H. Margenau, Philosophical problems concerning the meaning of measurement in physics, Ch. 8 in: Measurement definitions and theories (C.W. Churchman and P. Ratoosh, eds.), Wiley, New York 1959.

[27] A. Messiah, Quantum mechanics, Vol. 1, North-Holland, Amsterdam 1991; Vol. 2, North-Holland, Amsterdam 1976.

[28] P.-A. Meyer, Quantum probability for probabilists, 2nd. ed., Springer, Berlin 1995.

[29] A. Neumaier, On a realistic interpretation of quantum mechanics, Preprint (1999), quant-ph/9908071.

[30] A. Neumaier, Noncommutative analysis and quantum physics I. States and ensembles. Preprint (1999).

[31] A. Neumaier, Noncommutative analysis and quantum physics II. Differential calculus and dynamics. In preparation.

[32] A. Neumaier, Noncommutative analysis and quantum physics III. Integration and quantization. In preparation.

[33] A. Neumaier, Noncommutative analysis and quantum physics IV. Equilibrium thermodynamics. In preparation.

[34] J. von Neumann, Mathematische Grundlagen der Quantenmechanik. Springer, Berlin 1932.

[35] W. of Ockham, Philosophical Writings, (ed. by P. Boehner) Nelson, Edinburgh 1957.
[36] K.R. Parthasarathy, An introduction to quantum stochastic calculus, Birkhäuser, Basel 1992.

[37] St. Paul, 1. Cor 13:8-10, in: The New Testament. This is my paraphrase of a famous quote by Paul; for other renderings, see, e.g., http://solon.cma.univie.ac.at/~neum/christ/contrib/1cor13.html

[38] St. Paul, 1. Cor 8:2, in: Holy Bible, New International Version, 1984.

[39] Plato, Timaeus, Hackett Publishing, Indianapolis 1999. The quotes (Tim. 28-29) are from the Project Gutenberg Etext at ftp://metalab.unc.edu/pub/docs/books/gutenberg/etext98/tmeus11.txt (the first half of the document is a commentary, then follows the original in English translation)

[40] C.E. Rickart, General theory of Banach algebras. Van Nostrand, Princeton 1960.

[41] H.P. Robertson, The uncertainty principle, Phys. Rev. 34 (1929), 163-164. (Reprinted in [52].)

[42] B. Russell, The metaphysician’s nightmare, in: B. Russell, Nightmares of eminent persons, Allen & Unwin, London 1954. http://geocities.com/Athens/Delphi/2795/metaphysicians_nightmare.htm

[43] M. Schaller and K. Svozil, Automaton partition logic versus quantum logic, Int. J. Theor. Physics 34 (1995), 1741-1750.

[44] W. Shakespeare, Hamlet (Act 1, Scene 5), 1602. http://www-tech.mit.edu/Shakespeare/Tragedy/hamlet/hamlet.1.5.html

[45] P. Shor, Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer, SIAM J. Computing 26 (1997), 1484-1509.

[46] King Solomon, Proverbs 16:3, in: Holy Bible, New International Version, 1984.

[47] M.H. Stone, The theory of representations for Boolean algebras, Trans. Amer. Math. Soc. 40 (1936), 37-111.

[48] P. Suppes, D.H. Krantz, R.D. Luce, and A. Tversky, Foundations of measurements, Vol. 2. Acad. Press, San Diego 1989.

[49] K. Svozil, Quantum logic. Springer, Berlin 1998.
[50] W.E. Thirring, Course in mathematical physics. Vol. 4: Quantum mechanics of large systems. Springer, Berlin 1983.

[51] G.I. Webb, Further Experimental Evidence against the Utility of Occam’s Razor, J. Artif. Intell. Res. 4 (1996), 397-417.  
http://www.cs.washington.edu/research/jair/contents/v4.html

[52] J.A. Wheeler and W. H. Zurek, Quantum theory and measurement. Princeton Univ. Press, Princeton 1983.

[53] P. Whittle, Probability via expectation, 3rd ed., Springer, New York 1992.

[54] E.P. Wigner, The unreasonable effectiveness of mathematics in the natural sciences, Comm. Pure Appl. Math. 13 (1960), 1-14.

[55] E.P. Wigner, Remarks on the mind-body question, pp. 171-184 in: E.P. Wigner, Symmetries and reflections, Indiana Univ. Press 1967. (Reprinted in [52].)

[56] E.P. Wigner, Interpretation of quantum mechanics, Lecture Notes, 1976. Section II.2 in [52].

[57] W.H. Zurek, Decoherence and the transition from quantum to classical, Physics Today 44 (1991), 36-44.