Two-dimensional weakly interacting Fermi gas in a magnetic field: Level splitting

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Abstract

We apply the diagram-technique formalism beyond the Hartree-Fock approximation to a two-dimensional nearly ideal electron gas in a weak perpendicular magnetic field. The case of an almost completely filled upper Landau level (for its filling factor \( \nu_0 \) holds \( (1 - \nu_0) \ll 1 \)) with a quantum number \( N_0 \gg 1 \) is considered. We uncover two regimes of renormalization by electron-electron interactions. In the first regime, where \( N_0^{1/2}(1 - \nu_0) \ll 1 \), these interactions lead to a splitting of the Landau levels. In the second regime, where \( N_0^{1/2}(1 - \nu_0) \gg 1 \), apart of the splitting, a renormalization of the bare Zeeman splitting occurs. The intermediate case \( N_0^{1/2}(1 - \nu_0) \approx 1 \) cannot be studied within our approach. The applicability of the Fermi-liquid description is investigated for both regimes.
I. INTRODUCTION

In the Fractional Quantum Hall Effect (FQHE), the Hall conductivity $\sigma_{xy}$, measured as a function of the magnetic field $H$, reveals plateaus at some fractional values of the filling factor $\nu$. A similar behavior of $\sigma_{xy}$ at integer filling factors is known as the Integer Quantum Hall Effect (IQHE). However, after the work of Laughlin, it became clear that in the FQHE, in contrast to the IQHE, electron-electron interactions play an important part. An essential advance in understanding of the FQHE was reached through the composite-fermion concept, further developed in Ref. for the filling factor $\nu = 1/2$. There, the problem of a two-dimensional (2D) interacting electron gas in a strong perpendicular magnetic field was reduced to the investigation of composite fermions in an effective magnetic field. Since exactly at $\nu = 1/2$ the field is gauged out, composite fermions were assumed to form a Fermi liquid. Such a system should, in an effective magnetic field, exhibit the de Haas-Shubnikov effect, the extreme form of which is the IQHE. Thus, the fractional Hall plateaus of the original electron system were interpreted as the integer ones of the transformed fermions. However, the question crucial for such an approach, whether a 2D system of fermions in a magnetic field can indeed be treated in the framework of the usual Fermi-liquid theory, deserves a thorough investigation.

The concept of the Fermi liquid makes it possible to pass from interacting fermions to non-interacting quasiparticles. The analysis in Refs. shows that interacting three-dimensional (3D) electrons in a sufficiently weak magnetic field can be described essentially as the ideal Fermi gas, after one changes all the properties of particles of the ideal Fermi gas to those of quasiparticles. On the other hand, the energy spectrum of a noninteracting 2D electron gas in a perpendicular magnetic field is known to consist of discrete and dispersionless Landau levels (LL’s). The renormalization of this spectrum should differ drastically from that in the 3D case and in systems with dispersion in the spectrum. The reason is that the LL’s of noninteracting 2D electrons in a magnetic field are highly degenerate. According to the basic concepts of quantum mechanics, even weak interactions between the particles, can lift this degeneracy, sometimes resulting in nontrivial structures of the renormalized LL’s.

Thus, the singularity in the density of states in 2D is reflected also in other physical quantities, whereas the singularity in 3D is smoothed because of the presence of the third momentum along the direction of the magnetic field. More rigorously, according to Ref., the self-energy part of the one-particle Green function $\Sigma$ depends on the magnetic field $H$ not only via the usual operator $\hat{p} - (e/c)\mathbf{A}$, where $\mathbf{A}$ is the vector potential of the magnetic field, but also via a contribution of the order $(\hbar\omega_c/\mu)^{d/2}$, where $d$ is the spatial dimension of the system, the cyclotron frequency $\omega_c = eH/mc$, and $\mu$ is the Fermi energy. For $d = 3$ and a weak magnetic field, the contribution of the second kind is small and could be disregarded in comparison with the first one, which is of the order of $\hbar\omega_c/\mu$. For a 2D system, both terms are equally important and the applicability of the Fermi-liquid description in a weak magnetic field is questionable. To check this, we will find all essential contributions to $\Sigma$ and obtain the renormalized spectrum.

Some insight into the problems discussed above can be obtained with the help of a rather simple model of interparticle interactions. In this paper, the case of short-range (contact) interactions is considered. An analogous analysis for such interactions was already performed for a 3D electron gas, for a 3D electron gas in a weak magnetic field, and for a 2D electron
All these systems are examples of the traditional Landau Fermi liquid.

The question whether the Fermi-liquid theory is appropriate for the description of the 2D electron gas with contact interparticle interactions in a weak magnetic field, was investigated for the case of $N_0$ completely filled LL's\cite{11}. The number $N_0$ is determined by the Fermi energy $\mu$, $N_0 = \mu/\hbar \omega_c \gg 1$. For a field $H$ (or density of electrons $n$), such that the upper LL $N_0$ is only partly filled, its filling factor $\nu_0 = nhc/2eHN_0 \lesssim 1$, the problem is still open. The case of incomplete filling is very interesting from the experimental point of view as well as for theoretical study. In the present work, we investigate how a deviation of $\nu_0$ from unity changes the spectrum of the interacting system, treating $1 - \nu_0$ as a small parameter.

We find the spectrum for two regimes. In the first regime, where $N_0^{1/2}(1 - \nu_0) \ll 1$, a splitting of the LL’s occurs. In the second regime, where $1/N_0^{1/2} \ll 1 - \nu_0 \ll 1$, the splitting is accompanied by a shift of the Zeeman-split LL’s. The intermediate case $N_0^{1/2}(1 - \nu_0) \approx 1$ is the most complex and we were not able to study it.

The renormalization of the spectrum has an unusual character, compared with all known cases. Such features of the bare spectrum, as the absence of dispersion and the equal spacing between the neighbouring Landau levels, lead to resonances which split the bare spectrum. The renormalization of the Zeeman-split LL’s by long-range Coulomb interactions is a well-known effect for strong\cite{12} and intermediate\cite{13} magnetic fields. This result is of a similar origin with the second regime, being a manifestation of the asymmetry in interactions of particles having opposite spin projections with the electrons from the Fermi sphere.

It is noteworthy that there is a certain similarity between the renormalization in our system and that of fermion systems with a dispersion in the bare spectrum: We will find criteria which determine when the system under consideration belongs to Fermi liquids. They provide conditions for the bare spectrum near the LL $N_0$ to be weakly renormalized, so that classification of the LL’s according to their indices is preserved, i.e. the LL’s do not overlap with each other.

The spectrum obtained shows that electron-electron interactions do not break the translational symmetry and thus partially preserve the initial degeneracy of the spectrum. In contrast, introducing impurities into the sample leads to a broadening of the bare LL’s and the formation of bands of delocalized states\cite{14}.

It is worth mentioning that in our work, we study the spectrum of quasiparticle-type excitations whose physical properties are strongly affected by electron-electron interactions. Quantities like the heat capacity and the spin paramagnetism are sensitive to it. In contrast, the cyclotron effective mass and the zero-field plasma frequency are not changed by these interactions\cite{15}.

This paper is arranged as follows. In Section II, we introduce the model. In Section III, we detect the effect of splitting, crucial for our system, within second order of perturbation theory. Then, with the aid of general rules formulated in Section IV, we select important diagrams for the LL’s in Section V. In Sections VI and VII, respectively, the renormalization of LL’s near and further away from the LL $N_0$ is studied. Section VIII contains the summary.
II. NEARLY IDEAL FERMI GAS

In this work we study the properties of a 2D system of electrons in a uniform magnetic field. We consider the Hamiltonian

\[ \hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}, \]

where the Hamiltonian of the noninteracting electrons (apart from the Zeeman term), \( \hat{H}_0 \), is given by

\[ \hat{H}_0 = \frac{1}{2m} \int dr \hat{\psi}_+^\sigma(r) \left[-i\hbar \nabla + eA(r)/c\right]^2 \hat{\psi}_\sigma(r). \]

In Eq. (2.2), summation over the spin quantum number \( \sigma \) is implied, \( \hat{\psi}_+^\sigma(r) \) and \( \hat{\psi}_\sigma(r) \) are the creation and annihilation operators, \( A(r) \) is the vector potential of an external magnetic field; \( m \) is the effective mass of the electron and \(-e\) is its charge.

We use the term “nearly ideal” for a Fermi gas in which the short-range repulsive pair-interaction potential \( U(r) \) with range \( a \) simultaneously satisfies two conditions

\[ ak_f \ll 1, \quad \frac{mU_0}{\hbar^2} \ln \left( \frac{1}{ak_f} \right) \ll 1. \]

To disregard corrections to the bare vertex from the Cooper channel in the case of a weak magnetic field, more severe condition \( mU_0/\hbar^2 \ln (N_0) \ll 1 \) should be imposed on the potential \( U(r) \). In Eq. (2.3), \( U_0 = \int U(r) dr \) and \( p_f = \hbar k_f \) is the Fermi momentum. The first of conditions (2.3) means that one has a dilute system and the second assures the applicability of the Born approximation in 2D. Because of the short-range nature of the potential, only interactions between particles with opposite spins are important:

\[ \hat{H}_{\text{int}} = U_0 \int dr \hat{\psi}_+^\uparrow(r) \hat{\psi}_+^\downarrow(r) \hat{\psi}_\downarrow(r) \hat{\psi}_\uparrow(r). \]

We adopt the Landau gauge for which the single-particle wave functions are characterized by two quantum numbers, the number \( j \) of the LL and one (conserved) projection of the momentum \( p \):

\[ \phi_{jp}(r_x, r_y) = (2^j j! \sqrt{\pi})^{-1/2} e^{ipy} e^{-(r_x+p)^2/2} H_j(r_x + p), \]

where \( H_n \) is a Hermite polynomial, \( l_H = \sqrt{\hbar c/eH} \) is the magnetic length. Hereinafter, the dimensionless variables \( r_x/l_H, p/l_H/\hbar \) will be used.

We employ the temperature diagram-technique formalism, which is due to Gibbs averaging capable of dealing with systems with degeneracy. The single-particle Green function in a coordinate representation, expanded with respect to the basis \( \phi_{jp} \), is

\[ G_0(r, r'; \omega) = \sum_{j, p, \sigma} \frac{\phi_{jp}(r) \phi_{jp}^*(r')} {i\omega - \epsilon_{j, \sigma}}, \]

where \( \omega = \pi T(2k+1) \) is a fermion Matsubara frequency. The bare one-particle energies are the energies of the LL’s \( \epsilon_{j, \sigma} = \hbar \omega_e(j + 1/2) + g\mu_B \sigma \) with the Bohr magneton \( \mu_B = e\hbar/2m_0c \).
and $m_0$ is the free mass of the electron. As it will become clear in the following, the features
of the spectrum we will obtain do not depend on the exact value of the $g$ factor as long as
$g \neq 0, 2$.

In the following, we will sometimes omit the spin index $\sigma = \pm 1/2$ keeping only the
number of the LL. For example, for the case of complete filling ($\nu_0 = 1$), each spin-split LL
is either filled or not, so there exists a spin symmetry. It is obvious that in this case, all the
effects of electron-electron interactions cannot depend on spin either. The dependence on
spin will be given explicitly for all equations where it is important. In this case, the notation
$N_0^+ + j$ and $N_0^- + j$ will be used for LL’s with the level index $N_0 + j$ and the respective
quantum number of spin $\sigma = 1/2$ and $\sigma = -1/2$. We denote by $N_0^+$ the incompletely filled
LL corresponding to the chemical potential, suggesting that the LL $N_0^+$ and all LL’s below
it become completely filled in the limit $T \to 0$.

After we have introduced $\hat{H}_{\text{int}}$ and $G_0$, we proceed with the diagram technique in the
standard manner. Expanding the annihilation and creation operators in Eq. (2.4) in
functions $\phi_{jp}(r)$, we can analyze the interaction of the electrons in terms of their transitions
between of the LL’s (cf. Fig. 1). It is convenient to redefine $G_0$ and $\hat{H}_{\text{int}}$ attributing all
four functions $\phi_{jp}(r)$ appearing in $\hat{H}_{\text{int}}$ to the interaction vertex. As a result, only the
denominators are left in sum (2.6). Conditions (2.3) assure the applicability of the Born
approximation for a dilute 2D system of electrons without magnetic field, with the vertex
of order $V_0 \sim \left(\frac{mU_0}{\hbar^2}\right)(\hbar\omega_c)^2$. The presence of a magnetic field immediately changes the
vertex. After switching on a weak magnetic field, an additional characteristic length $l_H$ is
introduced such that $a \ll 1/k_f \ll l_H$. The non-interacting electrons in a magnetic field
are described by the functions (2.5), four of which enter in the vertex of interaction. We
consider the contact interaction, $U(r) \sim \delta(r)$, so that $\phi^4(r)$ is replaced with $\phi^4(0)$, and we
arrive at the following estimate for the vertex in a magnetic field:

$$V \approx \left(\frac{mU_0}{\hbar^2}\right) (\hbar\omega_c)^2.$$  \hspace{1cm} (2.7)

The parameter of perturbation theory is $V/\hbar\omega_c \ll 1$.

The spectrum of one-particle excitations is determined by the poles of the of the analytical
continuation of the temperature Green function $G^{-1}(\omega) = G_0^{-1}(\omega) - \Sigma(\omega)$. We will obtain
the self-energy part of the one-particle Green function $\Sigma(\omega)$ by summing up the series of the
most important diagrams $\Sigma^{(n)}$. In the following sections, we derive the series by selecting the
most essential contributions among all the diagrams arising in a given order $n$. In order to
find out whether the Fermi-liquid description is correct for the system under consideration,
we derive the expression for $\Sigma(\omega)$ as an expansion in small parameters, $1-\nu_0$, $1/N_0$ and $V/\hbar\omega$.
We restrict ourselves to determining the dependencies of the diagrams on powers of these
parameters, disregarding numerical coefficients and weaker functions of the parameters. In
this way, we derive the Dyson equation and from it the spectrum. It is worth emphasizing,
however, that in spite of the approximations made in estimating of $\Sigma^{(n)}$, this procedure is
non-perturbative since the initial spectrum is degenerate (cf. Conclusions).
III. SECOND ORDER OF PERTURBATION THEORY

Any system of fermions belongs to the class of Fermi-liquid systems if near the Fermi momentum \( p_f \) its Green function is given by

\[
G(\omega) = \frac{Z}{(\omega - \frac{p_f}{m^*}|p_f - p|)} ,
\]

where \( m^* \) and \( Z \) are the renormalized effective mass and the discontinuity of the momentum distribution of the particles at the Fermi surface, respectively. In Eq. (3.1), \( \omega \) is measured from the chemical potential \( \mu \). For a 2D or 3D Fermi gas without magnetic field or a 3D Fermi gas with magnetic field and contact interaction, the first-order diagram gives rise to a trivial shift of \( \mu \). The parameters of the Fermi-liquid theory, \( Z \) and \( m^*/m \), are determined by the second-order diagram (Fig. 2). They differ from those for the ideal Fermi gas \( Z = 1, m^*/m = 1 \) in second order of the parameter \( (mU_0/\hbar^2) \), according to Ref. [1].

Similarly, in the case of a 2D Fermi gas in a magnetic field, the first-order diagram gives only a constant contribution to the self-energy \( \Sigma \) and simply shifts the chemical potential \( \mu \). We shall disregard contributions of this kind, taking for \( \mu \) its shifted value. In contrast, the second-order diagram leads to a renormalization which drastically differs from the one in the absence of a magnetic field. In order to understand this effect and obtain the characteristic values of the parameters involved, let us start from the case of \( H = 0 \) and then recover the corresponding result in a finite magnetic field.

The second-order diagram determines the probability of a particle with momentum \( p \) to decay. Its value at \( H = 0 \) (all notations are introduced in Fig. 2) is given by:

\[
\Sigma^{(2)}(\omega, p) \propto U_0^2 \int \frac{dp_1 dp_2 dp_3}{(2\pi \hbar)^4} \delta(\omega + \epsilon_1 - \epsilon_2 - \epsilon_3) \delta(p + p_1 - p_2 - p_3) ,
\]

where the \( \delta \)-functions account for energy and momentum conservations. One writes \( \delta(r) = \int \exp(iqr) dq/2\pi \), integrates over the angles in \( r \)- and \( p_1 \)-spaces, and obtains four Bessel functions using the representation \( \int_0^{2\pi} \exp(i\alpha \sin \varphi) d\varphi/2\pi = J_0(\alpha) \). The momenta \( p, p_i \) in the magnetic field are given by \( p_i^2 = 2m\hbar\omega(\nu_i^4 - 1/2) \), so that the integration over \( p_i^2 \) becomes a summation \( \int dp_i^2 \to \sum N_i 2m\hbar\omega \epsilon \) in the limit of a weak magnetic field, \( N_0 \gg 1 \). One obtains the second-order contribution to \( \Sigma \):

\[
\Sigma^{(2)}(\omega, N) = \left( \frac{mU_0}{\hbar^2} \right)^2 \frac{m}{(2\pi)^2} \frac{(\hbar\omega\epsilon)^3}{\hbar^2} \sum_{N_1,N_2,N_3} I(N, N_1, N_2, N_3) \times
\]

\[
\times \frac{\nu(N_1)[1 - \nu(N_2)][1 - \nu(N_3)] + [1 - \nu(N_1)]\nu(N_2)\nu(N_3)}{\omega + \epsilon(N_1) - \epsilon(N_2) - \epsilon(N_3)} ,
\]

where \( \nu(N_i) \) is the filling factor of the LL with number \( N_i \). We use the short-cut notations \( N_i \) for \( N_i^1 \) and \( N_i^3 \) in Eqs. (3.3), (3.4), and (3.7). The value \( I(N, N_1, N_2, N_3) \) is given by

\[
I(N, N_1, N_2, N_3) = \int_0^\infty r dr J_0(\rho r)J_0(\rho r_1)J_0(\rho r_2)J_0(\rho r_3) ,
\]

where \( \alpha = \sqrt{2(N_i^4 - 1/2)/\hbar^2} \). As in the 3D case, the term \( \nu(N_1)[1 - \nu(N_2)][1 - \nu(N_3)] \) in (3.3) accounts for a process where the interaction causes the initial particle to jump from
the LL $N > N_0$ to the LL $N_2$ above the Fermi surface, $N_2 > N_0$, so that a particle below the Fermi surface in the LL $N_1 < N_0$, moves above the Fermi surface to the LL $N_3 > N_0$ (Figs. 2a, 2b). The term $[1 - \nu(N_1)]\nu(N_2)\nu(N_3)$ accounts for the analogous process when there is a hole in the initial state (Figs. 2c, 2d). In a first step, we choose all particles and holes in intermediate states such that there is no particle in the LL $N_i$, $N_i \leq N_0^\uparrow$, and no hole in the LL $N_k$, $N_k > N_0^\uparrow$, in any intermediate state. The processes with a particle involved in the LL $N_0^\uparrow$ will be discussed later.

The way we arrive at Eqs. (3.3) and (3.4), rather qualitative, could be easily reformulated rigorously in the second order of perturbation theory. In such an analysis, the Bessel functions would naturally appear in Eq. (3.4) as this function represents the Green function in the coordinate space in the quasiclassical limit. However, such an analysis becomes rather sophisticated for higher-order diagrams whereas our simple approach will be generalized in Section IV to estimate a diagram of any order.

Although the integral (3.4) can be exactly expressed through a complete elliptic integral of the second kind, already the simplest dimensionality analysis gives an estimate of its value. This is sufficiently accurate for our analysis (only the weak function $\ln N_0$ is missing at $N, N_i \approx N_0$): $I \approx l_\text{H}^2/N_0$.

The essential property of $\Sigma^{(2)}$ for 2D electrons in a magnetic field is that the internal energies are summed in Eq. (3.3), not integrated as in the case of vanishing magnetic field. This is because the spectrum of the system is discrete. As a result, the denominator of Eq. (3.3) contains terms with $N + N_1 = N_2 + N_3$. They describe resonances when the energy of the initial particle (hole) is exactly equal to the energy of the system in the intermediate state, one hole (particle) and two particles (holes) in Fig. 2. One distinguishes these resonant processes from non-resonant ones, for which the denominator in Eq. (3.3) differs from that of the bare Green function by an integer of $\hbar\omega_c$. The total $\Sigma^{(2)}$ is given by $\Sigma^{(2)} = \Sigma^{(2)}_{\text{res}} + \Sigma^{(2)}_{\text{nonres}}$. Introducing in Eq. (3.3) a new index $j$ instead of $N_1$, $j = N_2 + N_3 - N - N_1$, one performs the summation over indices $N_2$ and $N_3$:

$$\Sigma^{(2)}(\omega, N) \propto \frac{U_0}{l_\text{H}^2} 2\hbar\omega_c \frac{A_j^2}{\mu} \sum_j \frac{A_j^2}{\omega - \epsilon(N) - j\hbar\omega_c}. \quad (3.5)$$

The coefficients $A_j^2$ contain the filling factors $\nu_i$ from Eq. (3.3) and some combinatorial factors (cf. Eq. (3.6) and the discussion below).

The physical effects considered in the present paper are due to resonant terms in the self-energy part. These terms have nonzero weight (poles) on the plane $\omega$ due to degeneracy of the system, and not the branch cut as for a system with a continuous spectrum:

$$\Sigma^{(2)}_{\text{res}}(\omega, N) \propto \frac{C^2}{\omega - \epsilon(N)}. \quad (3.6)$$

Here $C^2 \approx k(k - 1)V^2A_0^2/2N_0$, where $V$ is given by Eq. (2.7), and the combinatorial factor $k(k - 1)/2$ accounts for the number of possible resonances for the LL with index $N$ such that $k = N - N_0$ for a particle and $k = N_0 - N + 1$ for a hole.

The Green function $G^{(2)}(\omega, N) = 1/[\omega - \epsilon(N) - \Sigma^{(2)}_{\text{res}}(\omega, N)]$, corresponding to $\Sigma^{(2)}_{\text{res}}$, is given by
As a result of resonance, the initial LL with energy \( \epsilon_0 = \epsilon(N) \) splits up into two sublevels with energies \( \epsilon_\pm = \epsilon(N) \pm C \) and with the corresponding residues \( 1/2 \). With the use of the diagram technique, one is thus able to describe the situation, typical for any degenerate system, when an external interaction – in the present case interparticle interactions – leads to a splitting of the initially degenerate LL with a partial lifting of the degeneracy.

So far we neglected the possibility of a particle to move onto the vacant place in the LL \( N_0^\uparrow \). Let us now consider the dependence of \( \Sigma^{(2)} \) on the filling factors \( \nu(N_i) \) in Eqs. (3.3), (3.6). Recalling that \( \nu(N_i) = 1 \) for \( N_i < N_0^\uparrow \), \( \nu(N_i) = 1 - \nu_0 \) for \( N_i = N_0^\uparrow \), and \( \nu(N_i) = 0 \) for \( N_i > N_0^\uparrow \), one concludes that processes not involving a particle in the LL \( N_0^\uparrow \) (cf. Fig. 3a), can exist for all values of \( 1 - \nu_0 \). They are symmetric with respect to spin. The characteristic order of renormalization of the energy by these processes is \( C_l \sim V/N_0^{1/2} \). We shall call them “large” processes. Processes in which a particle is created in the LL \( N_0^\uparrow \), contain an additional small parameter in the numerator of Eq. (3.6). For them, the renormalization of energy is of the order \( C_s \sim V[(1 - \nu_0)/N_0]^{1/2} \). Such “small” processes occur only when \( \nu_0 \) is different from 1: for \( \nu_0 = 1 \) only “large” processes survive. Examples of “small” and “large” processes for the LL \( N_0^\downarrow +2 \) are given in Fig. 3. The “small” processes are symmetric with respect to the spin of the initial particle: Electrons or holes in a LL with spin down, \( N_i^\downarrow \), can directly interact with the electrons and holes in the LL \( N_0^\uparrow \), giving rise to processes where only these two LL’s, \( N_i^\downarrow \) and \( N_0^\uparrow \), are involved (cf. Fig. 3b). For a LL with spin up, this is forbidden because of the interaction chosen.

The poles of the terms in \( \Sigma^{(2)} \), for a 3D system without and with magnetic field and a 2D system without magnetic field\(^8\), determine the damping of quasiparticles \( \gamma \sim (mU_0/\hbar^2)^2(p - p_f)^2/m \). Using \( p_f(p - p_f)/m = \hbar \omega_c \), this result is also recovered from Eq. (3.3) in the limit \( H \to 0 \). However, when \( H \neq 0 \), the square root is taken from \( V^2/N_0 \) due to the resonance. That is, the renormalization of the spectrum described by the terms \( \Sigma^{(2)} \) is proportional to the first power of the parameter of perturbation theory (cf. (3.7)), in contrast to the usual situation. Analogously, instead of a factor \( (p - p_f)^2/m \), characterizing the damping of quasiparticle with momentum \( p \) remote from the Fermi surface, the splitting of the LL \( N_0 \pm k \) is proportional to the first power of analogous parameter \( k \). The resonant terms do not renormalize the total filling factor of each LL. Similarly to the traditional theory of the Fermi liquid\(^8\), it is the non-resonant processes \( \Sigma^{(2)}_{nres} \) which renormalize the filling factors of LL’s\(^8\). The corrections to the bare values are of second order in the perturbation parameter.

Thus, already in second order of perturbation theory, one finds that the renormalization of our system should be essentially different from that of the known systems with a dispersion in the spectrum. In the next sections, we shall consider the higher orders of perturbation theory, concentrating on the contributions to \( \Sigma \) which determine the first-order corrections to the spectrum.
IV. RULES FOR ESTIMATES

Having understood the character of renormalization of the spectrum in second-order perturbation theory – its splitting with partial lifting of degeneracy – we proceed to higher orders of perturbation theory. In this section we formulate the general rules, which hold for any LL with $N_i \gg 1$, according to which the most essential diagrams should be selected.

In calculating a given diagram $\Sigma^{(n)}$, one has to sum over all independent frequencies and integrate over all independent momenta. These two procedures are independent of one another and will be shown to produce a small parameter $1 - \nu_0$ or $1/N_0$.

Indeed, as a result of the summation, there will appear a product of different filling factors $\nu_i$ and $1 - \nu_j$, so that $\nu_i (1 - \nu_j)$ corresponds to a propagator of a hole in the $i$th LL (an electron in the $j$th LL). From all possible processes, one should select only those with the minimal power of $1 - \nu_0$. In general, the renormalized values of the filling factors should be taken. However, we are searching for the correction to the spectrum of the first order in the parameter $V/\hbar \omega_c$. Since the corrections to the filling factors are of second order, it is permissible to use the bare filling factors.

The parameter of perturbation theory $V/\hbar \omega_c$ was introduced in Eq. (2.7) of Section II. The $n$th power of $V$, having the dimensionality of energy, enters each diagram of $n$th order. To preserve the proper dimensionality of this correction to the self-energy $\Sigma^{(n)}$, the denominator should have the dimensionality of the $(n-1)$th power of energy. Applying the Luttinger algorithm for the determination of the self-energy part, one finds

$$\Sigma^{(n)}(\omega, N) \propto \frac{V^n}{T^p \Pi_i (\omega - k_i \hbar \omega_c - \epsilon_N)^{s_i}}$$

with integers $k_i$, $s_i$, $p$ such that $\sum_i s_i + p = n - 1$. The second-order contribution, Eq. (3.6), also has the form of Eq. (4.1). According to Ref. (2), each cut of a diagram contributes a factor $(\sum_i \epsilon_i)$ to the denominator in Eq. (4.1), where $\epsilon_i$ is the energy of an electron in this cut (for a hole $-\epsilon_i$ should be taken). A cut containing one external frequency $\omega$ contributes a power of $(\omega - k_i \hbar \omega_c - \epsilon_N)$, where the integer $k$ denotes the difference from the bare energy $(\omega - \epsilon_N)$. These factors can be identical for different cuts; this is accounted for by integers $s_i$. In principle, powers of $T$ could also occur in the denominator of Eq. (4.1) from the third order of perturbation theory on. A power of $T$ could occur each time two propagators (the incoming and the outgoing) with the external frequency $\omega$ are present in a cut of a diagram. So, one could expect that the cut A-B of the diagram shown in Fig. 4 gives rise to the first power of $1/T$ in Eq. (4.1) on condition that $\epsilon_1^4 + \epsilon_2^4 = \epsilon_3^1 + \epsilon_4^4$. Such terms would give diverging contributions to $\Sigma$ at $T \to 0$. However, powers $1/T^p$ with $p > 0$ do not arise in any order of perturbation theory for our contact interaction, provided that the bare filling factors are used. This is illustrated most easily by the example of the diagram shown in Fig. 4. The propagators in the cut A-B, giving rise to a term $T$ in the denominator, at the same time introduce the product of filling factors $\nu_1^4 \nu_2^4 (1 - \nu_3^1)(1 - \nu_4^4)$ in the numerator. It is apparent that for each choice of intermediate indices $N_i$, leading to a resonance, at least one of the four factors in the numerator must be zero. The case when all intermediate particles $i = 1, \ldots, 4$ are in the LL $N_0^1$ is not possible, because particles with the same spin do not interact.
It follows from the aforesaid that resonant processes, resulting in the maximally possible \((n - 1)\)th power of the singular denominator in Eq. (4.1) with \(p = 0; s_i = n - 1, k_i = 0; s_j = 0, k_j \neq 0\), determine the spectrum near the bare LL energy \(m \hbar \omega_c\) in the \(n\)th order of perturbation theory. With accuracy \((V/\hbar \omega_c)^2\), the summation over the LL’s in Eq. (2.6) is no longer important. In all the intermediate Green functions we can keep only the LL’s, leading to a \(\Sigma(\omega)\) which is resonant with the bare Green function \(G_0(\omega) = 1/(\omega - m \hbar \omega_c) = 1/x\). Here, the notation

\[ x = \omega - m \hbar \omega_c. \]  

(4.2)

has been used to denote the correction to the spectrum. Since the full Green function for the \(m\)th LL is \(G(x) = 1/[x - \Sigma(x)]\), the spectrum of interacting system is determined by the solution of the equation \(x = \Sigma(x)\) with resonant \(\Sigma\).

Tracing back the origin of the dimensionless factor \(\hbar \omega_c/\mu = 1/N_0\) in Eq. (3.5), one is led to the conclusion that it is due to the momentum-conserving \(\delta\)-function in Eq. (3.2). Analogously, each independent \(\delta\)-function in a vertex of any diagram will contribute a factor \(1/N_0\) to the estimate. Further we will mark each independent law of conservation of momentum by a circle around the vertex.

These rules will allow us to obtain quickly the dependence of any diagram on the values \(1/N_0, 1 - \nu_0, V, \) and \(x\).

V. HIGHER-ORDER PROCESSES

In the present section the essential higher-order processes are introduced and estimated. Depending on the index of the LL, different processes are important.

The classification of “large” and “small” processes, given in Section III, holds also for higher orders of perturbation theory. For LL’s \(N_0\) and \(N_0 + 1\), only “small” processes are possible, whereas for LL’s, situated further away from \(N_0\), “small” and “large” processes, as well as their combinations occur. Examples can be understood by looking at Fig. 3. The only exception is the LL \(N_0^\uparrow - 1\), for which only “large” processes exist (cf. discussion below).

Since “small” processes contain additional factor \(1 - \nu_0 \ll 1\), it is clear that the renormalization is governed by “large” processes, if any (the exact formulation is given in Section VII). Among all “large” processes, one should select those with the minimal power of the small parameter \(1/N_0\), i.e. with the minimal number of independent \(\delta\)-functions in a given order of perturbation theory. Such decay processes may be interpreted as a generalization of a second-order one, in which one of the intermediate particles or holes in turn undergoes a \(\Sigma(2)\) process, etc. This is shown in Fig. 5. An essential feature of this process is the necessity for a particle or hole to decay into particles and holes situated nearer to the LL \(N_0^\uparrow\) in each following process. This means that “large” processes up to \(k - 1\)th \((k\)th) order of perturbation theory are possible for the LL \(N_0 + k\) \((N_0 - k)\). The estimate of the decay process of the \(2n\)th order, carried out with the rules of Section IV, gives

\[ \Sigma_{\text{dec}, 1}(x) \propto \frac{V^{2n}}{N_0^\uparrow x^{2n-1}}. \]  

(5.1)
Any “large” decay processes result in the correction to the spectrum of the same order \( x \sim V/\sqrt{N_0} \) in each order of perturbation theory.

Analogously to the conventional Fermi-liquid theory\(^9\), it follows from energy conservation that all the intermediate particles and holes must lie between \( N_0 - k \) and \( N_0 + k \) in any diagram for the LL \( N_0 \). Hence, the effective energies of the LL’s further away from the LL \( N_0 \) cannot be found without knowledge of the energies of nearer LL’s. Let us select the essential diagrams for the LL’s near \( N_0 \).

1. For the LL \( N_0^{\uparrow} \), neither an electron- nor a hole-like excitation can interact with the electrons and holes of this LL, because the spin projection is the same.

2. For the LL’s \( N_0^{\downarrow} \) and \( N_0^{\downarrow} + 1 \), there are two series of diagrams, containing the minimal power of one of our two small parameters, \( 1/N_0 \) and \( 1 - \nu_0 \). The minimal power of \( 1/N_0 \) is contained in the “small” decay processes. In contrast to “large” decay processes for LL’s further away from \( N_0 \), the “small” ones exist in any order of perturbation theory. With including the next \( \Sigma^{(2)} \) in the self-energy, an additional particle is created in the unoccupied part of the LL \( N_0^{\uparrow} \). This contributes each time a factor \( 1 - \nu_0 \) to the estimate. For the “small” decay process of \( n \)th order one has

\[
\Sigma_{\text{dec, s}}^{(2n)}(x) \propto \frac{(1 - \nu_0)^n V^{2n}}{x^{2n} - 1 N_0^{2n}}. \tag{5.2}
\]

The correction to the energy is of the same order for all these processes \( x \sim V(1-\nu_0)^{1/2}/N_0^{1/2} \).

The ladder graphs represent the second essential series. An example of such a graph is given in Fig 6a. The estimate of a ladder graph, where the interaction of an initial hole in the LL \( N_0^{\downarrow} \) and a particle in the \( N_0^{\downarrow} + 1 \) with holes in the LL \( N_0^{\uparrow} \) is accounted for (Figs. 6b, 6c, respectively), contains only the first power of the parameter \( 1 - \nu_0 \) in any order of perturbation theory (because only one electron is created in the LL \( N_0^{\uparrow} \) in intermediate states), but higher powers of \( 1/N_0 \) in comparison with the previous series:

\[
\Sigma_{\text{lad}}^{(2n)}(x) \propto \frac{(1 - \nu_0)^n V^{2n}}{x^{2n} - 1 N_0^{2n}}. \tag{5.3}
\]

3. For the LL’s \( N_0^{\uparrow} + 1 \) and \( N_0^{\uparrow} - 1 \), only the processes of second order are essential, the “small” one for \( N_0^{\uparrow} + 1 \) and the “large” one for \( N_0^{\uparrow} - 1 \), presented in Figs. 7a, 7b. The ladder graphs are not important for the LL \( N_0^{\uparrow} + 1 \): Both possibilities – the interaction of the intermediate particle in the level \( N_0^{\downarrow} + 1 \) with electrons in the LL \( N_0^{\uparrow} \) and the interaction of the intermediate particle in the LL \( N_0^{\uparrow} \) with the holes in the LL \( N_0^{\downarrow} - 1 \) – give a power of the factor \( (1 - \nu_0)^k \) with \( k > 1 \) in any order of perturbation theory, beginning from the third. This is because more than one electron is created in the LL \( N_0^{\uparrow} \). As for decays, the possibilities for the particle in the LL \( N_0^{\downarrow} + 1 \) and the hole in the LL \( N_0^{\downarrow} - 1 \) to decay are already accounted for by their renormalized Green functions. The hole in the LL \( N_0^{\downarrow} - 1 \) does not feel the deviation of the filling factor \( \nu_0 \) from unity directly, because it does not interact with the electrons and holes in the LL \( N_0^{\uparrow} \), i.e. there exist only “large” processes for this LL.
VI. RENORMALIZATION OF THE LL’S NEAR THE LL $N_0$

In this section we carry out the program discussed above: for each LL we select the series $\Sigma^{(n)}(x)$ and sum them up, using the estimates of the previous section. Thus, we derive the Dyson equations (DE’s) and solve them.

1. In the LL $N_0^\uparrow$, neither a particle nor a hole can interact. This means that the full Green function coincides with the bare one and the energy does not change.

2. For the LL’s $N_0^\downarrow$ and $N_0^\downarrow + 1$, we have introduced two infinite series of relevant diagrams, which have minimal powers of one of our two small parameters, $1/N_0$ or $1 - \nu_0$.

If $\xi \ll 1$, it follows from the comparison of Eqs. (5.2) and (5.3) that the ladder graphs should be selected. The resulting DE is shown in Figs. 8a, 8b.

If $\xi \gg 1$, the decay processes become more important. However, it would be wrong to restrict oneself to the decay processes only. If this were done, then the DE would have the form of Figs. 9a, 9b with the bare vertex. After providing summation over frequencies $\omega_1$ and $\omega_2$ one would obtain

$$\Sigma(x) = \frac{(1 - \nu_0)V^2}{[x - \Sigma(x)]N_0}, \quad (6.2)$$

leading to a physically meaningless result, viz. that the full Green function, corresponding to Eq. (5.2), would have no poles. To select the correct series, and obtain the spectrum of the interacting system, let us notice that Eq. (5.2) is singular under the condition $x = \Sigma(x)$, determining the correction to the spectrum. Therefore, it is necessary to take into account terms with higher powers of the difference $x - \Sigma(x)$ in the denominator. One checks that all essential diagrams are included after the bare vertex is replaced with the vertex shown in Figs. 9a and 9b.

It is worth noticing that the Green function with the self-energy part Eq. (5.2) is mathematically well defined. Though it has no poles, its analytical properties are determined by the cut between two branching points, $x = \pm 2V[(1 - \nu_0)/N_0]^{1/2}$, on the complex plane $x$. However, a careful diagrammatic analysis reveals an effective dressing. (Its physical significance is discussed below.) As the result, the Green function will be shown to acquire poles.

Comparison of Figs. 8 and 9 shows that the bare Green functions are used in the DE of Fig. 8, instead of the full ones of Fig. 9. This implies that the solution of a DE for the case $\xi \ll 1$ should be recovered after omitting the self-energy part in a DE for the case $\xi \gg 1$. Further on, we treat DE for both cases simultaneously.

The dependencies on the momenta in the DE of Figs. 8a, 9a, are essentially simplified by using the Fourier transform. Accordingly, we search for the full vertex of the LL $N_0^\downarrow$ in the form

$$\Gamma(p_1 - p_3, p_1 - p_4) = \int dq_x \exp[iq_x(p_1 - p_3)] \Gamma(q_x, p_1 - p_4).$$

The vertex of the DE gives
\[
\Gamma(q, p_1 - p_3; \omega_1 + \omega_2) = W_{N_0, N_0}(q, p_1 - p_3) \sum_{m}^{2N_0} E_m \times \\
\frac{i(\omega_1 + \omega_2) - \epsilon_0 \uparrow - \epsilon_0 \downarrow - [\Sigma]}{i(\omega_1 + \omega_2) - \epsilon_0 \uparrow - \epsilon_0 \downarrow + E_m - [\Sigma]}.
\] 
(6.3)

Here, apart from the notations introduced in Fig. 8a, 9a,
\[
W_{1, n_2}(q) = U_0 \exp(-q^2/2) L_n(q^2/2),
\] 
(6.4)
where \(L_n\) is the Laguerre polynomial; \(\epsilon_0 \uparrow (\epsilon_0 \downarrow)\) is the bare energy of the LL \(N_0^\uparrow (N_0^\downarrow)\), \(m\) is even, and
\[
E_m = V \int dq^2 \exp(-q^2)[L_{N_0}(q^2/2)]^2 L_m(q^2).
\] 
(6.5)

The term \([\Sigma]\) in Eq. (6.3), as well as in the following equations, should be omitted for the case \(\xi \ll 1\) and kept for \(\xi \gg 1\). An estimate shows that \(E_m \sim V/N_0\) for \(1 \ll m \ll 2N_0\), which contain the range of \(m\) essential for summation. Using values \(E_m\), it is not difficult to rederive the estimates (2.7) and (5.3).

To calculate the self-energy, given by
\[
\Sigma(\omega) = T^2 \sum_{\omega_1 \omega_2} G(\omega_1)G_0(\omega_2)G_0(\omega + \omega_1 + \omega_2)\Gamma(\omega + \omega_2),
\] 
(6.6)
some assumptions about the full propagator \(G\) of the LL \(N_0^\downarrow\) have to be done. Later we will see that a propagator with several poles, one of them having the prevailing spectral weight, satisfy the requirement of selfconsistency. The following calculations will determine the values of this shift and explain the meaning of the poles.

Performing first the summation over the frequency \(\omega_1\), one obtains
\[
\Sigma(\omega) \propto T \sum_{\omega_2, m} \frac{E_m^2 G_0(\omega_2)}{i(\omega_2 + \omega) - \epsilon_0 \uparrow - \epsilon_0 \downarrow + E_m - [\Sigma]}. 
\] 
(6.7)
The final result of the summation over \(\omega_2\) and integration over the independent momenta reads:
\[
\Sigma(x) = \sum_{m} \frac{E_m^2 (1 - \nu_0)}{x + E_m - [\Sigma]}.
\] 
(6.8)
Equation (6.8) determines \(2N_0 + 1\) poles of the Green function, \(2N_0\) of them lie near \(x_m \sim [\Sigma] - E_m\). Considering the case \(\xi \ll 1\) and assuming \(x_m \ll E_m\), one obtains \(\Sigma \approx (1 - \nu_0)V\). The \((2N_0 + 1)\)th pole is obtained with use of \(x = \Sigma\),
\[
x \approx (1 - \nu_0)V.
\] 
(6.9)
One checks that \(x\) is indeed less than \(E_m\), \((1 - \nu_0)V \ll E_m\). For the case \(\xi \gg 1\) one obtains the same result after substituting the condition \(x = \Sigma(x)\), determining the correction to
the spectrum, into Eq. (6.8). Representing the full Green function as a sum of singular terms, one proves that the total weight of first \(2N_0\) poles is indeed smaller than that of the \((2N_0 + 1)\)th by factor of \(1 - \nu_0\), which justifies our initial assumption.

Appearance of \(E_m\) in the denominator of Eq. (6.8) is not accidental. These values are nothing but the eigenvalues of energy of a system of two interacting holes. They are discrete as they are classified by eigenvalues of total angular momentum.\(^{10}\) Thus, one interprets the considered effect as a dressing of the bare propagator by a “collective mode” — two interacting holes. As a result of this, the bare energy of the LL \(N_0^\uparrow\) is shifted and \(2N_0\) poles with small residues appear, reminiscent of the eigenvalues of the collective mode. Further we will consider only the shifted level.

For the analysis of the DE for the LL \(N_0^\uparrow + 1\), it is most natural to make use of the representation of the total transverse momentum,\(^{11}\) which simplifies the dependence of the vertex on the momenta:

\[
\Gamma(p_1 - p_4, q_y; \omega_1 - \omega_4) = F_{N_0,N_0+1}(p_1 - p_4, q_y) \times \frac{(i(\omega_1 - \omega_4) + \epsilon_0 - \epsilon_{N_0+1} - [\Sigma])}{i(\omega_1 - \omega_4) + \epsilon_0 - \epsilon_{N_0+1} - [\Sigma] - F_{N_0,N_0+1}(p_1 - p_4, q_y)}. \tag{6.10}
\]

In Eq. (6.10), \(F_{N_0,N_0+1}\) is the Fourier transform of the function \(W_{N_0,N_0+1}(x, y)\), performed on both of the coordinates \(x\) and \(y\), and \(\epsilon_0\) \((\epsilon_{N_0+1})\) denotes the energy of the LL \(N_0^\uparrow\) \((N_0^\uparrow+1)\).

As in the previous case, we have

\[
\Sigma = (1 - \nu_0) \int dp dq \frac{[F_{N_0,N_0+1}(p, q)]^2}{x - [\Sigma] + F_{N_0,N_0+1}(p, q)}, \tag{6.11}
\]

which leads to the same results as Eq. (6.8).

The bare propagator for the LL \(N_0^\uparrow + 1\) is dressed by another “collective mode”, interacting electron and hole. The eigenvalues of energy for such a system are classified by a continues parameter, the total momentum.\(^{12}\) That is way the integration is present in Eq. (6.11).

Thus, the bare energies of the LL’s \(N_0^\uparrow\) and \(N_0^\uparrow + 1\) are simply shifted in both regimes. What is more, the values and the signs of the shift are the same. In order to prove this, we multiply both sides of the expansion\(^{13}\)

\[
2V[\mbox{L}_n(r)]^2 = \sum_k L_{2k}(2r)E_k
\]

on \(e^{-\tau}\), then integrate over the variable \(r\), and obtain \(W_{N_0,N_0+1}(0) = \sum_k E_k\). This means that \(\Sigma\) from Eqs. (6.8) and (6.11) are identical on condition \(x = \Sigma\).

3. The LL’s \(N_0^\uparrow + 1\) and \(N_0^\uparrow - 1\) were shown in Section V to be renormalized by the second-order processes. Taking into account the identity of the shifts for LL’s \(N_0^\uparrow + 1\) and \(N_0^\uparrow\), this results in the splitting of the LL’s \(N_0^\uparrow - 1\) and \(N_0^\uparrow + 1\) in two sublevels for both \(\xi \gg 1\) and \(\xi \ll 1\) in the manner, similar to Section III. The splitting of the LL \(N_0^\uparrow + 1\) has the order of \(V(1 - \nu_0)^{1/2}N_0^{1/2}\), and that of the LL \(N_0^\uparrow - 1\) is of order \(VN_0^{1/2}\).

VII. REINORMALIZATION OF THE LL’S FURTHER AWAY FROM THE LL \(N_0\)

Once the energies of the LL’s near \(N_0\) have been established, one can proceed to LL’s, situated further. Here, the renormalization is determined by another dimensionless parameter than in Section VI because of the coexistence of “small” processes, \(x \sim x_1 = V(1 - \nu_0),\)
“large” processes, \( x \sim x_2 = V/\sqrt{N_0} \), as well as their combinations. The spectrum essentially differs in the following cases: (i) \( \sqrt{N_0}(1-\nu_0) \ll 1 \); (ii) \( \sqrt{N_0}(1-\nu_0) \approx 1 \); (iii) \( 1/\sqrt{N_0} \ll 1-\nu_0 \ll 1 \). In the present paper only the cases (i) and (iii) are analyzed. In case (ii) “large”, “small” processes, and their combinations are equally important and we were not able to select the essential series.

(i) For \( \sqrt{N_0}(1-\nu_0) \ll 1 \), the renormalization is governed by “large” decay processes, symmetric with respect to spin. Schematically one writes

\[
\Sigma(x) \propto \Sigma_{\text{dec}}(x) + \frac{V^3}{x^2N_0^2} + \frac{V^2(1-\nu_0)}{xN_0}, \tag{7.1}
\]

where the first term represents contributions from the “large” decays, the second and the third terms estimate the third-order “large” ladder-graph and the second-order “small” process, respectively. The second-order processes for the LL \( N_0^2 + 2 \) are shown in Fig. 3. The characteristic order of the renormalization \( x \sim V/\sqrt{N_0} \) is given by the first term in Eq. (7.1). One proves that any correction to \( \Sigma \) from “small” processes is smaller than one from “large” non-decay processes: The last two terms in Eq. (7.1) are of the same order only when \( 1-\nu_0 \approx 1/\sqrt{N_0} \), which contradicts to \( \sqrt{N_0}(1-\nu_0) \ll 1 \). Among all “large” processes, the decays should be selected.

According to Section V, there are \( k \) possible “large” decay processes for the LL’s \( N_0+k+1 \) and \( N_0-k \). Since the intermediate particles and holes in any “large” process, occurring for the LL’s \( N_0+k \) and \( N_0-k \), must lie in the LL’s, situated closer to the LL \( N_0, N_0 \pm |k-1| \), the Green function of each following LL depends on the propagators of all the previous LL’s. One determines the renormalized spectrum recursively, gradually increasing index \( k \). The LL’s \( N_0 \) and \( N_0+1 \) are not renormalized within the accuracy \( x \sim V/\sqrt{N_0} \). The next two levels, \( N_0-1 \) and \( N_0+2 \), are split into two sublevels by the second-order process in the manner described in Section III.

The fourth-order decay process for the LL’s \( N_0+3 \) and \( N_0-2 \) can be considered as a second-order process, in which the intermediate decay of the particle in the LL \( N_0+2 \) (or hole in the \( N_0-1 \)) is described by its Green function Eq. (3.7). Both processes give the same result:

\[
\Sigma_{\text{dec}}^{(4)}(x) = \frac{C_2^2 V^2}{2N_0(x-C_1 V/\sqrt{N_0})} + \frac{C_2^2 V^2}{2N_0(x+C_1 V/\sqrt{N_0})}, \tag{7.2}
\]

where the constants \( C_1, C_2 \) are of order of unity. The Green function corresponding to Eq. (7.2), has the form

\[
G^{(4)}(x) = \frac{1}{x[1-C_2^2 V^2/(N_0 x^2 - C_1^2 V^2)]} = \frac{\gamma_1}{x} + \frac{\gamma_2}{x-V a/\sqrt{N_0}} + \frac{\gamma_2}{x+V a/\sqrt{N_0}}
\]

with constants \( a = \sqrt{C_1^2 + C_2^2}, \gamma_1 = C_1^2/2a^2 \), and \( \gamma_2 = C_2^2/2a^2 \). We again observe a splitting of the bare LL’s, \( x = 0 \) and \( x = \pm V a/\sqrt{N_0} \) such that the total quasiparticle weight does not change, \( \gamma_1 + 2\gamma_2 = 1 \).

The energies of higher LL’s can be found analogously. Like the LL’s considered above, they are also split into sublevels with characteristic \( x \sim x_2 = V/\sqrt{N_0} \), and the total quasiparticle weight of each LL is equal to its bare value. It can be seen that the number of
sublevels growths rapidly when moving away from the LL \( N_0 \), however we were not able to derive any universal dependence. The resulting spectrum in this regime is presented in Fig. 10.

The typical value of the splitting \( x_k \) grows according to Eq. (3.6), \( x_k \sim kV/\sqrt{N_0} \), as the index of the LL \( N_0 \pm k \) deviates from \( N_0 \). Thus, one can formulate the analog of the Landau criterion: The Fermi-liquid description is applicable to our system up to the LL \( N_0 \pm k \), where \( k \) is such that two neighbouring LL’s do not overlap:

\[
kV/\sqrt{N_0} \ll \hbar \omega_c .
\]

(ii) Let us proceed to the next regime of renormalization, \( 1/\sqrt{N_0} \ll 1 - \nu_0 \ll 1 \). Now, the correction to the energy of any LL due to renormalization of the lower LL’s \( N_0^\downarrow \) and \( N_0^\downarrow + 1 \), found in the case \( \xi \gg 1 \) to be \( x \sim x_1 = V(1 - \nu_0) \), exceeds that of the “large” decay processes, \( x \sim x_2 = V/\sqrt{N_0} \).

The asymmetry we have found near the LL \( N_0 \), when the energy \( x^\uparrow \) of the LL’s \( N_0^\uparrow - 1 \) and \( N_0^\uparrow + 1 \) with spin up and the energy \( x^\downarrow \) of the LL’s \( N_0^\downarrow \) and \( N_0^\downarrow + 1 \) with spin down have different orders of magnitude, \( x^\uparrow \ll x^\downarrow \), is preserved also for higher numbers of the LL’s. Thus, the renormalization appears to depend on the spin of a LL.

For a particle placed in a LL with spin down, say \( N_0^\downarrow + 2 \) (Fig. 3), there exist “large” processes. However, their self-energy part is not resonant with the bare energy anymore due to different corrections to energies of the lower LL’s. The estimate of the “large” second-order process,

\[
\Sigma^{(2)}_\downarrow (x) \propto V^2/[N_0(x^\downarrow + x_{N_0}^\downarrow - x_{N_0+1}^\downarrow - x_{N_0+1}^\uparrow)] ,
\]

with \( x_{j}^{\downarrow,\uparrow} \) denoting the energy of the LL \( N_j^{\downarrow,\uparrow} \), gives the shift of the energy \( x^\downarrow \sim V(1 - \nu_0) \).

For a LL with spin up, this “detuning” does not take place. As an example, let us consider LL \( N_0^\uparrow + 2 \). The estimate of the second-order process, analogous to (7.4),

\[
\Sigma^{(2)}_\uparrow (x) \propto V^2/[N_0(x^\uparrow + x_{N_0}^\uparrow - x_{N_0+1}^\uparrow - x_{N_0+1}^\downarrow)] ,
\]

yields the correction of energy \( x^\uparrow \sim V/\sqrt{N_0} \ll V(1 - \nu_0) \).

It is not difficult to generalize Eqs. (7.4) and (7.5) to understand how the renormalization of the LL’s \( N_0^\downarrow \) and \( N_0^\uparrow + 1 \) affects the other LL’s. For a LL with spin down, the initial excitation jumps in a way that among the intermediate states only one state with spin down and two states with spin up are present. This leads to a shift of the level by an amount of \( x^\downarrow \sim V(1 - \nu_0) \). For a LL with spin up, one can always choose a hole and an electron in the intermediate states in the LL’s with spin down, so that the initial position of the LL does not change.

Thus, within accuracy \( x \sim V(1 - \nu_0) \), one effectively has a renormalization of the Zeeman splitting. The resulting spectrum is shown in Fig. 11. Let us notice that there are LL’s with opposite spin projections having the same spacing. As a result, the LL’s are split analogously to the first regime at the smaller scale of energy \( x \sim V\sqrt{N_0} \).

The Fermi-liquid description is applicable if

\[
V(1 - \nu_0) \ll g\hbar \omega_c ,
\]

so that the Zeeman-split LL’s do not overlap.
VIII. CONCLUSIONS

In the present paper, the spectrum of excitations of quasiparticle type is studied for the 2D electron gas with contact interactions within the conventional perturbative approach. The case of a weak magnetic field is considered, i.e. many Landau levels are filled. It is assumed that the upper Landau level $N_0 \gg 1$ is almost completely filled, $1 - \nu_0 \ll 1$.

For our model, an effect characteristic for a degenerate system, viz. a splitting of levels by virtue of interactions, is found and described. Due to absence of dispersion and due to the equal spacing between the Landau levels, resonances occur which split the bare spectrum. The resonances will appear for any kind of interactions, however, we were able to solve the problem only for contact interactions.

The spectrum of the interacting system has been found in two regimes. In the first regime, where $N_0^{1/2} (1 - \nu_0) \ll 1$, the bare LL’s are split. A scheme is suggested making it possible, in principle, to obtain the number of sublevels, their energies and filling factors. In the second regime, where $1/N_0^{1/2} \ll 1 - \nu_0 \ll 1$, the splitting is accompanied by an effective shift of the bare Zeeman-split LL’s. In the case intermediate between these two regimes, one is not able to select the essential processes. The spectrum found allows us to derive the criteria (7.3) and (7.6) determining the applicability of the Fermi-liquid description.

To our knowledge, the splitting of the LL’s has not been discussed in the literature yet; the Zeeman-splitting is known to be affected by long-range interactions for strong and intermediate magnetic fields. Apart from the strength of the field, another essential difference between Refs. 13, 14 and the second regime is that for long-range interactions the enhancement of the $g$-factor occurs already in the first order of perturbation, whereas for the present model higher-order corrections are also important. However, after accounting for all the essential processes, we obtain the similar final result: a shift of the spin-split LL on the value $x \sim V (1 - \nu_0)$. The electron gas in a weak magnetic field and interacting via Coulomb potential was studied in Ref. 27. There, the renormalization of the LL $N_0^\uparrow$ was governed by two characteristic energy scales: the first determined the energy gap in the tunneling density of states at a LL and the second described the characteristic shift of the Zeeman-split LL. In Refs. 13, 14, 27, a completely different interaction was used: the effective interaction between electrons in the LL with the same index and the spin projection was important, whereas in our model only electrons with different spin interact. Keeping in mind the similarities and differences in spectra obtain in Refs. 13, 14, 27, it would be interesting to find out by which way the spectrum changes with change of interactions and strength of the magnetic field.

In the present approach, the essential results arise due to energy denominators in the self-energy part $\Sigma$. They can be calculated exactly for any diagram. Although some approximations were used while estimating the dependence of any given diagram on the momenta, the character of renormalization is found exactly. For instance, to predict the same value of shift for the LL’s $N_0^\downarrow$ and $N_0^\downarrow + 1$ no approximation is needed, as soon as the essential series of diagrams is established.

We restricted ourselves to accounting for the physical effects arising in the minimal order of the small parameters involved. In this approach, only the bare energies of the LL’s are renormalized, but not their filling factors: The corrections to spectrum are of first order in the parameter $V/\hbar \omega_c$ and the bare filling factors are renormalized in second order of perturbation theory. One can hardly predict what kind of spectrum the system has in
the high orders of perturbation theory, f.e. whether it splits further or widens. In such an analysis, terms occur in $\Sigma$ proportional to $1/T$. The same happens when the system is additionally degenerated with respect to spin (i.e. for $g = 0, 2$). Having imposed the restriction on interactions between particles with the same projection, we thus exclude the FQHE. However, our model is reasonable and justified as it leads to no contradiction and predicts interesting results.

IX. ACKNOWLEDGMENTS

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Figure 1 a) Two interacting particles, changing their indices \(N_1, N_2\) and momenta \(p_1, p_2\) to \(N_3, N_4\) and \(p_3, p_4\), respectively. The point corresponds to the vertex \((2.7)\). b) The same process for a particular choice of the indices, \(N_2 = N_0\). The dimple in LL \(N_0^\uparrow\), corresponding to the chemical potential, depicts the incomplete filling of this level.

Figure 2 a) Second-order diagram for an electron b) Corresponding process for a particular choice of the indices, \(N_2 = N_0\). The dimple in LL \(N_0^\uparrow\), corresponding to the chemical potential, depicts the incomplete filling of this level.

Figure 3 Second-order processes for the LL \(N_0^\downarrow + 2\): a) a “large” one, no particle in the dimple in the LL \(N_0^\downarrow\). (b – c) “Small” ones, with a particle involved in the LL \(N_0^\downarrow\).

Figure 4 Third-order diagram representing interactions of quasiparticles with opposite spins. Dotted lines mark interactions, as in Ref.\(^21\). The process evolves with time \(t\). Each cut, corresponding to some time between interactions, contributes to the self-energy \(\Sigma\) according to Eq. \((4.1)\). Both the incoming and outgoing propagators contain external frequency \(\omega\) so that the denominator in Eq. \((4.1)\) due to cut A-B, \(\epsilon_1^\downarrow + \epsilon_2^\uparrow - \epsilon_3^\downarrow - \epsilon_4^\uparrow\), contains only energies of LL’s \(\epsilon_i\). This, however, does not lead to a term in \(\Sigma\), proportional to reversed power of temperature \(T\) even if \(\epsilon_1^\downarrow + \epsilon_2^\uparrow = \epsilon_3^\downarrow - \epsilon_4^\uparrow\), provided that the contact interactions are considered.

Figure 5 A diagram describing consecutive decay of either a particle or a hole. The circles around the vertex denote independent laws of momentum conservation.

Figure 6 a) A ladder graph representing the process of interaction of two intermediate states for the second-order diagram (Fig. 2) b) Essential interactions for the LL \(N_0^\downarrow\). c) The same for the LL \(N_0^\downarrow + 1\). In (b – c) the interactions are marked as in Ref.\(^21\).

Figure 7 Essential processes: a) for an electron in LL \(N_0^\downarrow + 1\); b) for a hole in LL \(N_0^\downarrow - 1\).

Figure 8 The Dyson equations in the limit \(N_0(1 - \nu_0) \ll 1\): a) for LL \(N_0^\downarrow\); b) for LL \(N_0^\downarrow + 1\). It is implied that each propagator depends both on \(\omega_i\) and \(p_i\). The projection of spins are explicitly shown in the equation for the self-energy but omitted in that of the vertex.

Figure 9 Same as Fig. 8 in the limit \(N_0(1 - \nu_0) \gg 1\).

Figure 10 Spectrum in the first regime of renormalization \(N_0^{1/2}(1 - \nu_0) \ll 1\). Left: bare spectrum. Right: renormalized spectrum. Solid lines mark the position of unchanged levels. Dashed and dotted lines correspond to the initial position of the LL’s and new levels arising due to the interaction, respectively.

Figure 11 Spectrum in the second regime of renormalization \(1/\sqrt{N_0} \ll 1 - \nu_0 \ll 1\). Only the relative shift of levels is shown. The meaning of the lines is the same as in Fig. 10.
Figure 1
Figure 2
Figure 3
Figure 4
Figure 5
Figure 6
Figure 7
\[ \begin{array}{c}
\omega \\
\downarrow \\
\end{array} =
\begin{array}{c}
\downarrow \\
\omega \\
\end{array} +
\begin{array}{c}
\omega_1 + \omega_2 + \omega_3 \\
\omega_2 + \omega_3 \\
\omega_4 + \omega_3 \\
\omega_4 + \omega_2 \\
\omega_1 \\
\omega_3 \\
\end{array}
\]

Figure 8
Figure 9
Figure 10
Figure 11