ON A CONTINUED FRACTION EXPANSION FOR EULER’S CONSTANT.

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Abstract. Recently, A. I. Aptekarev and his collaborators found a sequence of rational approximations to Euler’s constant $\gamma$ defined by a third-order homogeneous linear recurrence. In this paper, we give a new interpretation of Aptekarev’s approximations to $\gamma$ in terms of Meijer $G$-functions and hypergeometric-type series. This approach allows us to describe a very general construction giving linear forms in 1 and $\gamma$ with rational coefficients. Using this construction we find new rational approximations to $\gamma$ generated by a second-order inhomogeneous linear recurrence with polynomial coefficients. This leads to a continued fraction (though not a simple continued fraction) for Euler’s constant. It seems to be the first non-trivial continued fraction expansion convergent to Euler’s constant sub-exponentially, the elements of which can be expressed as a general pattern. It is interesting to note that the same homogeneous recurrence generates a continued fraction for the Euler-Gompertz constant found by Stieltjes in 1895.

1. Introduction

In 1978, R. Apéry [3, 31] stunned the mathematical world with a proof that $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational. Since then many different proofs of this fact have appeared in the literature (see [15] and the references given there). The main idea of all the known proofs is essentially the same and consists in constructing a sequence of linear forms $I_n = u_n\zeta(3) - v_n$, $n = 0, 1, 2, \ldots$, which satisfy the following conditions:

$$\limsup_{n \to \infty} |I_n|^{1/n} \leq (\sqrt{2} - 1)^4 = 0.0294372\ldots,$$

$I_n \neq 0$ for infinitely many $n$, and $u_n \in \mathbb{Z}$, $2D_n^3v_n \in \mathbb{Z}$, where $D_n$ is the least common multiple of the numbers $1, 2, \ldots, n$. If we suppose that $\zeta(3)$ is a rational number $a/b$, then $2bD_n^3I_n$ is a non-zero integer for infinitely many $n$ and, on the other hand, it tends to zero as $n \to \infty$ (since $D_n^{1/2} \to e$ and $e^3(\sqrt{2} - 1)^4 = 0.591\ldots < 1$), which is a contradiction.

The diversity of all the proposed proofs of the irrationality of $\zeta(3)$ is presented by different interpretations and ways of obtaining the linear forms $I_n$ and its various representations. Apéry showed that the sequences $u_n$ and $v_n$ are given by the formulas

$$u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad v_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left( \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{m^3(nm)(n+m)} \right)$$

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and satisfy the second-order linear recurrence relation
\[(n + 1)^3y_{n+1} - (34n^3 + 51n^2 + 27n + 5)y_n + n^3y_{n-1} = 0\]
with the initial conditions \(u_0 = 1, u_1 = 5, v_0 = 0, v_1 = 6.\) This implies immediately that \(\frac{w_n}{u_n}\) is the \(n\)-th convergent of the following continued fraction:
\[\zeta(3) = \frac{6}{5} - \frac{1}{117} - \frac{64}{535} - \cdots - \frac{n^6}{34n^3 + 51n^2 + 27n + 5} - \cdots.\]
The shorter proof of Apéry’s theorem has been found in 1979 by F. Beukers [8], who used multiple Euler-type integrals and Legendre polynomials. Beukers discovered the following elegant representation for the linear forms \(I_n:\)
\[2I_n = \int\!\int_{[0,1]^2} \frac{P_n(x)P_n(y)\log(xy)}{xy - 1} dxdy = \int\!\int\!\int_{[0,1]^3} \frac{x^n(1-x)^n(1-y)^nw^n(1-w)^n}{(1-(1-xy)w)^{n+1}} dxdydw,\]
where \(P_n(z) = \frac{(-1)^n}{n!}(z^n(1-z)^n)^{(n)}\) is the Legendre polynomial.
In 1996 inspired by the works of F. Beukers [9] and L. Gutnik [16], Yu. V. Nesterenko [28] proposed another proof of the irrationality of \(\zeta(3)\) and a new expansion of this number into continued fraction. His proof was based on the hypergeometric type series
\[\frac{-1}{2} \sum_{k=1}^{\infty} R_n'(k) = \frac{-1}{2} \sum_{k=1}^{\infty} \frac{d}{dt} \frac{\Gamma^4(t)}{\Gamma^2(t-n)\Gamma^2(t+n+1)} \bigg|_{t=k} = u_n\zeta(3) - v_n\]
that can be written (by the residue theorem) as a complex integral or a Meijer G-function (see [27, §5.2], for definition)
\[-\sum_{k=1}^{\infty} R_n'(k) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} R_n(s) \left(\frac{\pi}{\sin \pi s}\right)^2 ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma^2(n+1-s)\Gamma^4(s)}{\Gamma^2(n+1+s)} ds\]
\[= G_{4,4}^{4,2}\left(\begin{array}{c} -n, -n, n+1, n+1 \\ 0, 0, 0, 0 \end{array} \left| 1 \right)\),
here \(c\) is an arbitrary real number satisfying \(0 < c < n + 1\) and \(R_n(t)\) is a rational function defined by
\[R_n(t) = \frac{(t-1)^2 \cdots (t-n)^2}{t^2(t+1)^2 \cdots (t+n)^2}.
Using an identity of Bailey (see [7, Section 6.3, formula (2)]) it can be shown (see [45]) that the complex integral in (3) is equal to the very-well-poised hypergeometric series (see [15, §2], for definition)
\[\frac{n!(3n+2)!}{(2n+1)!^5} \, _7F_6\left(\begin{array}{c} 3n+2, 3n+2, n+1, \ldots, n+1 \\ \frac{3}{2}n+1, 2n+2, \ldots, 2n+2 \end{array} \left| 1 \right)\).
This series was first introduced by K. Ball in attempts to give an elementary proof of Apéry’s theorem similar to the proof of the irrationality of the number \(\zeta\) by Fourier. As we can see now, the hypergeometric construction (2)–(4) appeared to be more transparent for generalizations. A generalization of the series (4) allowed T. Rivoal [33] to
prove his remarkable result on infiniteness of irrational numbers in the set of odd zeta values \( \zeta(2n+1), n \geq 1 \). The best finite version of it belongs to W. Zudilin [44] and states that at least one of the four numbers \( \zeta(5), \zeta(7), \zeta(9), \zeta(11) \) must be irrational. Another generalization of (2), (3) to values of the polylogarithmic function

\[
L_k(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^k}, \quad k \geq 2, |z| \leq 1,
\]

enabled the second author [24, 25] to prove some results on arithmetical nature of values of the polylogarithm at rational points lying inside the disk of convergence far enough from zero. Namely, for any integer \( q \neq 0, 2 \) at least one of the two numbers \( L_2(1/q), L_3(1/q) \) is irrational and for any integer \( q \in (-\infty, -4] \cup [7, +\infty) \) at least one of the three numbers \( L_3(1/q), L_4(1/q), L_5(1/q) \) is irrational. For small \( |1/q| \) we have more precise results due to M. Hata [18, 17] and G. Rhin and C. Viola [32]: \( L_2(1/q) \) is irrational for any \( q \in (-\infty, -5] \cup [6, +\infty) \), and \( L_3(1/q) \) is irrational for any integer \( q \) with \( |q| \geq 1038 \).

Further extension of the hypergeometric approach led to the discoveries of second-order linear recursions and Apéry-like continued fractions for other remarkable number-theoretic constants including Catalan’s constant [46], \( \pi^4 \), [11, 47, 39], \( \pi \coth \pi \) [20]. For other related examples we refer the reader to [2, 43, 48].

Euler’s constant was first introduced by Leonhard Euler in 1734 as

\[
\gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right) = 0.57721566490153286\ldots
\]

It can be considered as an analogue of the value "\( \zeta(1) \)" of Riemann’s zeta function if we compensate the partial sums of the divergent harmonic series by the natural logarithm. It is not known whether \( \gamma \) is an irrational or transcendental number. The question of its irrationality remains a famous unresolved problem in the theory of numbers. Even obtaining good rational approximations to it was unknown until recently. First such approximations defined by a third-order linear recurrence were proposed by A. I. Aptekarev and his collaborators [4] in 2007. More precisely, the numerators \( \tilde{p}_n \) and denominators \( \tilde{q}_n \) of the approximations are positive integers generated by the following recurrence relation:

\[
(16n-15)y_{n+1} = (128n^3 + 40n^2 - 82n - 45)y_n - n^2(256n^3 - 240n^2 + 64n - 7)y_{n-1} + n^2(n-1)^2(16n+1)y_{n-2}
\]

with the initial conditions

\[
\tilde{p}_0 = 0, \quad \tilde{p}_1 = 2, \quad \tilde{p}_2 = 31,
\]
\[
\tilde{q}_0 = 1, \quad \tilde{q}_1 = 3, \quad \tilde{q}_2 = 50
\]

and having the following asymptotics:

\[
\tilde{q}_n = (2n)! e^{\sqrt{2n}} \left( \frac{1}{\sqrt{n}(4e)^{3/8}} + O(n^{-1/2}) \right),
\]

\[
\tilde{p}_n - \gamma \tilde{q}_n = (2n)! e^{-\sqrt{2n}} \left( \frac{2\sqrt{\pi}}{(4e)^{3/8}} + O(n^{-1/2}) \right).
\]
The remainder of the above approximations is given by the integral [5]

\[ \int_0^\infty Q_n(x)e^{-x}\log(x)\,dx = \tilde{p}_n - \gamma \tilde{q}_n, \]

where

\[ Q_n(x) = \frac{1}{n!} \frac{e^x}{1-x} (x^n x^n (1-x)^{2n+1} e^{-x})^{(n/2)} \]

is a multiple Jacobi-Laguerre orthogonal polynomial on \([0,1]\) and \([1, +\infty)\) with respect to the two weight functions \(w_1(x) = (1-x)e^{-x}\), \(w_2(x) = (1-x)\log(x)e^{-x}\). The integral (7) can also be written as a multiple integral (see [22, Lemma 4])

\[ \int_0^\infty Q_n(x)e^{-x}\log(x)\,dx = \int_0^\infty \int_0^\infty \frac{x^n y^n (x-1)^{2n+1} e^{-x}}{(xy+1)^{n+1}(y+1)^{n+1}} \,dxdy. \]

The integrality of the sequences \(\tilde{p}_n\) and \(\tilde{q}_n\) is not evident and cannot be deduced directly from the recurrence equation (5). Tulyakov [41] proved independently that \(\tilde{p}_n\) and \(\tilde{q}_n\) are integers, by considering a more "dense" sequence of rational approximations to \(\gamma\).

The authors [23] found explicit representations for \(\tilde{p}_n\) and \(\tilde{q}_n\):

\[ \tilde{q}_n = \sum_{k=0}^n \binom{n}{k}^2 (n+k)!, \quad \tilde{p}_n = \sum_{k=0}^n \binom{n}{k}^2 (n+k)! (H_{n+k} + 2H_{n-k} - 2H_k), \]

here \(H_n = \sum_{k=1}^n \frac{1}{k}\) is the \(n\)-th harmonic number, \(H_0 := 0\). Formulas (9) imply that \(\tilde{q}_n\) and \(p_n\) are integers divisible by \(n!\) and \(n!\frac{D_n}{D_0}\), respectively. Although the coefficients of the linear forms (6) can be canceled out by the big common factor \(\frac{n!}{D_0}\), it is still not enough to prove the irrationality of \(\gamma\), since the linear \(\gamma\)-forms with integer coefficients:

\[ \frac{\tilde{p}_n D_n}{n!} - \frac{\tilde{q}_n D_n}{n!} \gamma \in \mathbb{Z} + \mathbb{Z}\gamma \]

do not tend to zero as \(n\) tends to infinity:

\[ \frac{\tilde{p}_n D_n}{n!} - \frac{\tilde{q}_n D_n}{n!} = O(4^n n^{n-1} e^{-\sqrt{2n}}). \]

Nevertheless, the sequence \(\frac{\tilde{p}_n}{\tilde{q}_n}\) provides good rational approximations to Euler’s constant \(\gamma\), by using multiple Laguerre polynomials

\[ A_n(x) = \frac{1}{n!} e^x (x^n e^{-x})^{(n)}(n). \]

His construction is based on the following third-order recurrence:

\[ (n+3)^2 (8n+11)(8n+19)y_{n+3} = (n+3)(8n+11)(24n^2 + 145n + 215)y_{n+2} - (8n+27)(24n^3 + 105n^2 + 124n + 25)y_{n+1} + (n+2)^2 (8n+19)(8n+27)y_n, \]

In 2009, T. Rivoal [35] found another example of rational approximations to the Euler constant \(\gamma\), by using multiple Laguerre polynomials

\[ A_n(x) = \frac{1}{n!} e^x (x^n e^{-x})^{(n)}(n). \]
which provides two sequences of rational numbers $P_n$ and $Q_n$, $n \geq 0$, with the initial values

$$
P_0 = -1, \quad P_1 = 4, \quad P_2 = 77/4, \\
Q_0 = 1, \quad Q_1 = 7, \quad Q_2 = 65/2
$$

such that $\frac{P_n}{Q_n}$ converges to $\gamma$. The sequences $P_n$, $Q_n$ satisfy the inclusions (see [22, Corollary 5])

$$
n!Q_n, \quad n!D_nP_n \in \mathbb{Z}
$$

and provide better approximations to $\gamma$

$$
\left| \frac{P_n}{Q_n} - \gamma \right| \leq e_0 e^{-n/2n^{2/3} + 3/2n^{1/3}} \quad \text{as } n \to \infty.
$$

Unfortunately, this convergence is not fast enough to imply the irrationality of $\gamma$.

In this paper, we give a new interpretation of Aptekarev’s approximations to $\gamma$ in terms of Meijer $G$-functions and hypergeometric-type series, which can be considered as an analog of the complex integral (3) and series (2) for Euler’s constant. This approach allows us to describe a very general construction giving linear forms in 1 and $\gamma$ with rational coefficients. Using this construction we find new rational approximations to $\gamma$ generated by a second-order inhomogeneous linear recurrence with polynomial coefficients. This leads to a continued fraction (though not a simple continued fraction) for Euler’s constant. It seems to be the first non-trivial continued fraction expansion convergent to Euler’s constant sub-exponentially, the elements of which can be expressed as a general pattern. It is interesting to note that the same homogeneous second-order linear recurrence generates a continued fraction for the Euler-Gompertz constant found by Stieltjes in 1895.

2. ANALOGS OF HYPERGEOEMTRIC TYPE SERIES AND COMPLEX INTEGRALS FOR EULER’S CONSTANT

Note that the first attempt to generalize series (2) to find suitable approximations for Euler’s constant $\gamma$ was made by Sondow [38]. He introduced the following series:

$$
I_n := \sum_{\nu=n+1}^{\infty} \int_{\nu}^{\infty} \left( \frac{n!}{x(x+1)\cdots(x+n)} \right)^2 \, dx
$$

and proved that

$$
I_n = \left( \frac{2n}{n} \right) \gamma + L_n - \sum_{i=0}^{n} \binom{n}{i}^2 H_{n+i} = O(2^{-4n}n^{-1}),
$$

where

$$
L_n = 2 \sum_{k=1}^{n} \sum_{i=0}^{k-1} \binom{n}{i}^2 (H_{n-i} - H_i) \log(n + k).
$$

Prévost [30] showed that the construction (10) is a consequence of the Padé approximation to the function $(\log u)/(1 - u)$ and proposed simpler approximations to Euler’s
constant given by
\[ J_{n,m} := \int_0^1 u^{n-m} P_n(u) \left( \frac{1}{\log u} + \frac{1}{1-u} \right) \, du = \gamma - 2H_n + L_{n,m}, \]
where
\[ L_{n,m} = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{n+k}{k} \log(n-m+k+1) \]
is a linear form in logarithms and \( P_n(u) \) is the Legendre polynomial. Rivoal [34] mentioned on the following connection
\[ \gamma - S_n = \int_0^1 u^n \left( \frac{1}{\log u} + \frac{1}{1-u} \right) \, du, \]
where \( S_n = \sum_{k=1}^{n} \frac{1}{k} - \log(n+1) \), and obviously, \( \gamma = \lim_{n \to \infty} S_n \). In particular, it was shown in [19] that
\[ J_{n,m} = \gamma - \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{n+k}{k} S_{k+n-m}. \]
This gave a new view on the approximating formulas for Euler’s constant \( \gamma \) found much earlier by different methods by C. Elsner [12]:
\[ \left| \gamma - \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{k+n+\tau-1}{k+\tau-1} S_{k+n-m} \right| \leq \frac{1}{2n\tau(n+\tau)}, \quad \tau \in \mathbb{N}, \]
and led to proving many similar formulas for \( \gamma \) and some other constants (see [13, 19, 21, 34]). Unfortunately, this type of approximations involving \( S_n \) contains linear forms in logarithms and this enables only to obtain conditional irrationality criteria for \( \gamma \), accelerate convergence of the sequence \( S_n \) and provide ways for fast calculation of Euler’s constant.

In this section, we obtain new representations for Aptekarev’s linear form \( \tilde{p}_n - \gamma \tilde{q}_n \) distinct from (7) and (8) which can be considered as analogs of the hypergeometric-type series (2) and complex integral (3). It can be easily done by using explicit formulas (9).

**Proposition 1.** For each \( n = 0, 1, 2, \ldots \), the following equality holds:
\[ \tilde{f}_n := \tilde{p}_n - \gamma \tilde{q}_n = n!^2 \sum_{k=0}^{n} \frac{d}{dt} \left( \frac{\Gamma(n+t+1)}{\Gamma^2(t+1)\Gamma^2(n-t+1)} \right) \bigg|_{t=k}. \]

**Proof.** The straightforward verification shows that
\[ \frac{d}{dt} \left( \frac{\Gamma(n+t+1)}{\Gamma^2(t+1)\Gamma^2(n-t+1)} \right) = \frac{\Gamma(n+t+1)}{\Gamma^2(t+1)\Gamma^2(n-t+1)} \times \left( \psi(n+t+1) - 2\psi(t+1) + 2\psi(n-t+1) \right), \]
where \( \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \) is the logarithmic derivative of the gamma function, also known as the digamma function. Summing (11) for \( t = 0, 1, \ldots, n \) and using the well-known properties of the digamma function
\[ \psi(1) = -\gamma, \quad \psi(n+1) = H_n - \gamma, \quad n \geq 1, \]
we have
\[ n!^2 \sum_{k=0}^{n} \frac{d}{dt} \left( \frac{\Gamma(n + t + 1)}{\Gamma^2(t + 1)} \right) \bigg|_{t=k} = n!^2 \sum_{k=0}^{n} \frac{\Gamma(n + k + 1)}{\Gamma^2(k + 1)\Gamma^2(n - k + 1)} (\psi(n + k + 1) - 2\psi(k + 1) + 2\psi(n - k + 1)) \]
\[ = \sum_{k=0}^{n} \binom{n}{k}^2 (n + k)! (H_{n+k} - 2H_k + 2H_{n-k} - \gamma) = \tilde{p}_n - \gamma \tilde{q}_n, \]
as required.

Proposition 2. For each \( n = 0, 1, 2, \ldots \), we have
\[ \tilde{f}_n = \frac{n!^2}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(n + t + 1)}{\Gamma^2(t + 1)\Gamma^2(n - t + 1)} \left( \frac{\pi}{\sin \pi t} \right)^2 dt = \frac{n!^2}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(n + t + 1)\Gamma^2(t - n)}{\Gamma^2(t + 1)} dt = n!^2 G_{3,2}^{0,3} \begin{pmatrix} -n, n + 1, n + 1 \end{pmatrix}_{0, 0}, \]
\[ \tilde{q}_n = \frac{n!^2}{2\pi i} \int_{L} \frac{\Gamma(n + t + 1) e^{i\pi t}}{\Gamma^2(t + 1)\Gamma^2(n - t + 1)} \cdot \frac{\pi}{\sin \pi t} dt \]
\[ = (-1)^n n!^2 G_{3,2}^{0,2} \begin{pmatrix} -n, n + 1, n + 1 \end{pmatrix}_{0, 0}, \]
where \( c \) is an arbitrary real number satisfying \( c > n \) and \( L \) is a loop beginning and ending at \(-\infty\) and encircling the points \( n, n - 1, n - 2, \ldots \) exactly once in the positive direction.

Proof. We first note that the equality of two complex integrals in (12) and (13) follows easily by the reflection formula for the gamma function:
\[ \Gamma(t - n)\Gamma(n - t + 1) = \left( \frac{-1}{\sin \pi t} \right)^n \pi. \]
The third equality in both formulas (12) and (13) follows by the definition of the Meijer G-function. To prove the first equality in (12), we consider the integrands of the complex integrals (12) on the rectangle contour with vertices \( c - iN, c + iN, -N - 1/2 \pm iN \), where \( N \) is a sufficiently large integer, \( N > c \). Then, by the residue theorem, we have that the integral
\[ \frac{1}{2\pi i} \left( \int_{c-iN}^{c+iN} + \int_{c-iN}^{-N-\frac{1}{2}+iN} + \int_{-N-\frac{1}{2}+iN}^{-N-\frac{1}{2}-iN} + \int_{-N-\frac{1}{2}-iN}^{c-iN} \right) \frac{\Gamma(n + t + 1)\Gamma^2(t - n)}{\Gamma^2(t + 1)} dt \]
is equal to the sum of the residues of the integrand at integer points \( t = k, \) \( 0 \leq k \leq n. \)

Using the expansion

\[
\left( \frac{\pi}{\sin \pi t} \right)^2 = \frac{1}{(t-k)^2} + O(1)
\]

in a neighborhood of the integer point \( t = k \) we obtain that the integral (14) is equal to (15)

\[
\sum_{k=0}^{n} \left. \text{res} \left( \frac{\Gamma(n+t+1)}{\Gamma^2(t+1)\Gamma^2(n-t+1)} \left( \frac{\pi}{\sin \pi t} \right)^2 \right) \right|_{t=k} = \sum_{k=0}^{n} \frac{d}{dt} \left( \frac{\Gamma(n+t+1)}{\Gamma^2(t+1)\Gamma^2(n-t+1)} \right) \bigg|_{t=k}.
\]

Since on the sides \([c+iN, -N-1/2+iN], [-N-1/2+iN, -N-1/2-iN], [-N-1/2-iN, c-iN] \) of the rectangle we have \(|t| = O(N)\), it follows that

\[
\left| \frac{\Gamma^2(t-n)}{\Gamma^2(t+1)} \right| = \frac{1}{|t^2(t-1)^2 \cdots (t-n)^2|} = O \left( \frac{1}{N^{2n+2}} \right).
\]

For large \(|z|\) the asymptotic expansion of the gamma function is [27, §2.11]

\[
\log \Gamma(z) = \left( z - \frac{1}{2} \right) \log z - z + \frac{1}{2} \log(2\pi) + O(|z|^{-1}),
\]

where \(|\arg z| \leq \pi - \varepsilon, \varepsilon > 0\) and the constant in \( O \) is independent of \( z \). Then for \( t = x + iN, -N \leq x \leq c, \) we have

\[
|\Gamma(n+t+1)| = |\Gamma(x+n+1 \pm iN)| = O \left( e^{(x+n+\frac{1}{2}) \log N \mp N \arg(x+n+1 \pm iN)} \right) \leq O \left( N^{c+n+\frac{1}{2}} e^{-\frac{\pi}{2} N} \right).
\]

On the segment \([-N-1/2 + iN, -N-1/2 - iN] \) we use the trivial estimate

\[
|\Gamma(n+t+1)| = |\Gamma(\Re(n+t+1))| = |\Gamma(n+1/2-N)| = \frac{\pi}{\Gamma(N+1/2 - n)} = O(e^{-N \log N}.N).
\]

Summarizing (16), (18), (19) and letting \( N \) tend to infinity in (14), by (15) and Proposition 1, we get the first equality in (12).

Now prove the first equality in (13). For this purpose, suppose that \( C_1 \) and \( C_2 \) are points of intersections of the loop \( L \) with the vertical line \( \Re(t) = -N - 1/2 \) and consider a closed contour \( L^* \) oriented in the positive direction and consisting of the segment \( C_1C_2 \) and the right part of the loop \( L \) connecting the points \( C_1 \) and \( C_2 \), which we denote by \( \widetilde{C_1C_2} \). Then, by the residue theorem, we obtain

\[
n!^2 \int_{L^*} \frac{\Gamma(n+t+1) e^{i\pi t}}{\Gamma^2(t+1)\Gamma^2(n-t+1) \sin \pi t} \, dt = n!^2 \sum_{k=0}^{n} \left. \text{res} \left( \frac{\Gamma(n+t+1) e^{i\pi t}}{\Gamma^2(t+1)\Gamma^2(n-t+1) \sin \pi t} \right) \right|_{t=k} = \tilde{q}_n.
\]

On the other hand, we have

\[
\int_{L^*} = \int_{C_1C_2} + \int_{C_2C_1}.
\]

Since

\[
\frac{\Gamma(n+t+1) e^{i\pi t}}{\Gamma^2(t+1)\Gamma^2(n-t+1) \sin \pi t} = \frac{1}{2\pi i} \left( \frac{\Gamma^2(t-n)}{\Gamma^2(t+1)\Gamma(n+t+1)(e^{2i\pi t} - 1)} \right)
\]

we have
and the function $e^{2i\pi t} - 1$ is bounded on the vertical segment $C_1C_2$, by (16), (19) and (21) we get

$$\int_{L^*} = \int_{C_1C_2} + O(e^{-N\log N+N}).$$

Now letting $N$ tend to infinity and taking into account (20), we get the required formula. □

**Remark.** Note that the asymptotics of linear forms $\tilde{\rho}_n - \gamma \tilde{q}_n$ and denominators $\tilde{q}_n$ can be also obtained by application of the saddle-point method to the complex integrals (12) and (13).

More generally, let us consider an arbitrary function $F(n,t)$ of the form

(22)

$$F(n,t) = \frac{\prod_{j=1}^{s} \Gamma(a_j n + b_j t + 1)}{\prod_{j=1}^{u} \Gamma(c_j n + d_j t + 1)},$$

where $a_j, b_j, c_j, d_j \in \mathbb{Z}$, $\sum_{j=1}^{s} b_j \neq \sum_{j=1}^{u} d_j$, and all gamma values in the numerator are well defined for $0 \leq t \leq M(n)$. We say that $\Gamma(an + bt + 1)$ is well defined if $an + bt + 1$ is not a negative integer or zero.

**Proposition 3.** Let $F(n,t)$ be defined as above, $M(n)$ be a non-negative integer, and

$$F_n = \sum_{t=0}^{M(n)} \frac{d}{dt} F(n,t).$$

Then for each $n = 0, 1, 2, \ldots$, we have $F_n = p_n - \gamma q_n$ with

$$q_n = \left(\sum_{j=1}^{s} b_j - \sum_{j=1}^{u} d_j\right) \cdot \sum_{k=0}^{M(n)} F(n,k)$$

and

$$p_n = \sum_{k=0}^{M(n)} F(n,k) \left(\sum_{j=1}^{s} b_j H_{a_j n + b_j k} - \sum_{j=1}^{u} d_j H_{c_j n + d_j k}\right).$$

**Proof.** Differentiating $F(n,t)$ with respect to $t$ and summing over $t = 0, 1, \ldots, M(n)$, we have

$$\sum_{t=0}^{M(n)} \frac{d}{dt} F(n,t) = \sum_{k=0}^{M(n)} \frac{\prod_{j=1}^{s} \Gamma(a_j n + b_j k + 1)}{\prod_{j=1}^{u} \Gamma(c_j n + d_j k + 1)}$$

$$\times \left(\sum_{j=1}^{s} b_j \psi(a_j n + b_j k + 1) - \sum_{j=1}^{u} d_j \psi(c_j n + d_j k + 1)\right)$$

$$= \sum_{k=0}^{M(n)} \frac{\prod_{j=1}^{s} (a_j n + b_j k)！}{\prod_{j=1}^{u} (c_j n + d_j k)！} \left(\sum_{j=1}^{s} b_j (H_{a_j n + b_j k} - \gamma) - \sum_{j=1}^{u} d_j (H_{c_j n + d_j k} - \gamma)\right) = p_n - \gamma q_n.$$

□
3. A second-order inhomogeneous linear recurrence for Euler’s constant

In this section, we consider application of Proposition 3 to the function

\[ F(n, t) = \frac{\Gamma^2(n + 1)}{\Gamma(t + 1)\Gamma^2(n - t + 1)}, \quad n \in \mathbb{Z}, \quad n \geq 0. \]

Then for

\[ F_n := \sum_{t=0}^{n} \frac{d}{dt} F(n, t), \]

we have \( F_n = p_n - \gamma q_n \) with

\[ q_n = \sum_{k=0}^{n} \binom{n}{k}^2 k!, \quad p_n = \sum_{k=0}^{n} \binom{n}{k}^2 k!(2H_{n-k} - H_k), \quad n = 0, 1, 2, \ldots. \]

**Lemma 1.** The sequence \( \{q_n\}_{n=0}^\infty \) is a solution of the second-order homogeneous linear recurrence

\[ q_{n+2} - 2(n + 2)q_{n+1} + (n + 1)^2 q_n = 0 \]

with the initial values \( q_0 = 1, \quad q_1 = 2 \), and the sequence \( \{p_n\}_{n=0}^\infty \) is a solution of the second-order inhomogeneous linear recurrence

\[ p_{n+2} - 2(n + 2)p_{n+1} + (n + 1)^2 p_n = -\frac{n}{n + 2} \]

with the initial values \( p_0 = 0, \quad p_1 = 1 \).

**Proof.** Applying Zeilberger’s algorithm of creative telescoping [29, Ch.6] to the function \( F(n, t) \) we get for each \( n = 0, 1, 2, \ldots \) the identity

\[ F(n + 2, t) - 2(n + 2)F(n + 1, t) + (n + 1)^2 F(n, t) = G(n, t + 1) - G(n, t), \]

where

\[ G(n, t) = \frac{\Gamma^2(n + 2) \cdot r(n, t)}{\Gamma(t + 1)\Gamma^2(n - t + 3)} \quad \text{and} \quad r(n, t) = t(t^2 - (2n + 3)t + n(n + 2)). \]

To prove (26) it is sufficient to multiply both sides of (26) by \( \Gamma^2(n-t+3)\Gamma(t+1)/\Gamma^2(n+2) \) and after cancelation of gamma factors to verify the identity

\[ (n + 2)^2 - 2(n + 2)(n - t + 2)^2 + (n - t + 2)^2(n - t + 1)^2 = \frac{(n - t + 2)^2}{t + 1}r(n, t + 1) - r(n, t). \]

Summing equality (26) over \( t = 0, 1, 2, \ldots \) and taking into account that

\[ \lim_{t \to k} G(n, t) = 0, \quad k = n + 3, n + 4, \ldots, \]

we get the difference equation for \( q_n : \)

\[ q_{n+2} - 2(n + 2)q_{n+1} + (n + 1)^2 q_n = -G(n, 0) = 0. \]

In order to get the recurrence relation for the sequence \( p_n \), it is convenient to rewrite \( F_n \) as an infinite sum

\[ F_n := \sum_{t=0}^{\infty} \frac{d}{dt} F(n, t), \]
taking into account that
\[
\lim_{t \to k} \frac{d}{dt} F(n,t) = 0 \quad \text{for} \quad k = n + 1, n + 2, \ldots.
\]
Then differentiating (26) with respect to \(t\) and summing over \(t = 0, 1, 2, \ldots\) we get for each \(n = 0, 1, 2, \ldots\),
\[
(27) \quad F_{n+2} - 2(n+2)F_{n+1} + (n+1)^2 F_n = \lim_{k \to \infty} G'(n, k + 1) - G'(n, 0).
\]
Since
\[
G'(n, t) = (n+1)!^2 \left( \frac{t^3 - (2n+3)t^2 + tn(n+2)}{\Gamma(t+1)\Gamma^2(n-t+3)}(2\psi(n-t+3) - \psi(t+1)) 
+ \frac{3t^2 - 2t(2n+3) + n(n+2)}{\Gamma(t+1)\Gamma^2(n-t+3)} \right)
\]
we see that
\[
\lim_{k \to \infty} G'(n, k + 1) = 0 \quad \text{and} \quad G'(n, 0) = \frac{n}{n+2},
\]
and consequently, (27) becomes
\[
F_{n+2} - 2(n+2)F_{n+1} + (n+1)^2 F_n = -\frac{n}{n+2}, \quad n = 0, 1, 2, \ldots.
\]
This implies that the sequence \(p_n = F_n + \gamma q_n\) satisfies the same inhomogeneous recurrence, and the lemma is proved.

Now let us define a complex integral \(I_n\) by means of the Meijer \(G\)-function:
\[
(28) \quad I_n := n!^2 G^0,2_{2,0} \left( \begin{array}{c} n+1, n+1 \\ 0 \end{array} \right) \left| 1 \right) = \frac{n!^2}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma^2(t-n)}{\Gamma(t+1)} dt,
\]
where \(c > n\) is an arbitrary constant.

**Lemma 2.** The following formula holds:
\[
F_n = I_n + O \left( \frac{1}{n^2} \right) \quad \text{as} \quad n \to \infty,
\]
where the constant in \(O\) is absolute.

**Proof.** Since
\[
(29) \quad \frac{\Gamma^2(t-n)}{\Gamma(t+1)} = \Gamma(t-n) \left( \frac{\Gamma(t-n)}{\Gamma(t+1)} \right),
\]
by similar arguments as in the proof of Proposition 2, considering the integrand (29) on the rectangle contour with vertices \(c \pm iN, -N - 1/2 \pm iN\), where \(N\) is a sufficiently large integer, we conclude that the integral (28) can be evaluated as a sum of residues.
at the points $n, n - 1, \ldots$. It is easily seen that the function (29) has double poles at the points $0, 1, 2, \ldots, n$ and simple poles at $-1, -2, \ldots$. Therefore, we have

\[ I_n = n!^2 \sum_{k=-\infty}^{n} \text{res}_{t=k} \left( \frac{\Gamma^2(t-n)}{\Gamma(t+1)} \right) \]

\[ = n!^2 \sum_{k=-\infty}^{-1} \text{res}_{t=k} \left( -\frac{\pi}{\sin \pi t} \cdot \frac{1}{\Gamma(n-t+1) \cdot t(1-t) \cdots (n-t)} \right) \]

\[ + n!^2 \sum_{k=0}^{n} \text{res}_{t=k} \left( \frac{1}{\Gamma(t+1) \Gamma^2(n-t+1)} \right) \]

\[ = n!^2 \sum_{k=-\infty}^{-1} \frac{(-1)^k}{(n-k)! \cdot k(1-k) \cdots (n-k)} \]

\[ + n!^2 \sum_{k=0}^{n} \frac{d}{dt} \left( \frac{1}{\Gamma(t+1) \Gamma^2(n-t+1)} \right) \bigg|_{t=k} = \frac{1}{(n+1)^2} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(n+2)_k} + F_n. \]

Since

\[ \left| \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(n+2)_k} \right| \leq \sum_{k=0}^{\infty} \frac{k!}{(n+2)_k} \leq \sum_{k=0}^{\infty} \frac{1}{k!} = e, \]

we get the desired assertion. \qed

**Lemma 3.** The following asymptotic formulas hold:

\[ F_n = n! \frac{e^{-2\sqrt{n}}}{n^{1/4}} \left( \sqrt[4]{\frac{\pi}{e}} + O(n^{-1/2}) \right) \quad \text{as} \quad n \to \infty, \]

\[ q_n = n! \frac{e^{2\sqrt{n}}}{n^{1/4}} \left( \frac{1}{2\sqrt{n}e} + O(n^{-1/2}) \right) \quad \text{as} \quad n \to \infty. \]

**Proof.** To prove the asymptotic formula for $F_n$, we apply the saddle point method (see [14, Ch.6, Th.3.1]) to the complex integral (28). We begin by considering the quadratic polynomial

\[ p(\tau) = n(\tau - 1)^2 - \tau, \]

which has two real roots $1 + \frac{1}{2n} \pm \sqrt{\frac{1}{n} + \frac{1}{4n^2}}$, and denote the largest of them by

\[ \tau_0 = 1 + \frac{1}{2n} + \sqrt{\frac{1}{n} + \frac{1}{4n^2}} = 1 + \frac{1}{n^{1/2}} + \frac{1}{2n} + \frac{1}{8n^{3/2}} + O(n^{-2}). \]

Then put $c = n\tau_0$ and suppose that the contour of integration in (28) passes through the point $n\tau_0$. Then the asymptotic expansion of the gamma function (17) implies that
the following formula holds on the contour of integration:

$$\frac{\Gamma^2(t - n)}{\Gamma(t + 1)} = \exp\{2 \log \Gamma(t - n) - \log \Gamma(t + 1)\}$$

$$= \exp\left\{2 \left(t - n - \frac{1}{2}\right) \log(t - n) - \left(t + \frac{1}{2}\right) \log(t + 1)
+ 2n - t + 1 + \log \sqrt{2\pi} + O(n^{-1/2})\right\}.$$  

Making the change of variable $t = n\tau$ and fixing the branches of logarithms that take real values on the interval $(1, +\infty)$ of the real axis, we get

$$\frac{\Gamma^2(t - n)}{\Gamma(t + 1)} = \sqrt{2\pi} e^{2n-(2n+3/2) \log n} g(\tau) e^{nf(\tau)} (1 + O(n^{-1/2})).$$

where

$$f(\tau) = 2(\tau - 1) \log(\tau - 1) - \tau \log \tau - \tau + \log n, \quad g(\tau) = \frac{1}{(\tau - 1)1/2}.$$  

This gives

$$I_n = \sqrt{2\pi} n \int_{\tau_0 = 1}^{\tau_0 + i\infty} e^{nf(\tau)} g(\tau) d\tau (1 + O(n^{-1/2})).$$

Now we determine the saddle points of the integrand (31), i.e., zeros of the derivative of the function $f(\tau)$. It is easily seen that the zeros of the derivative

$$f'(\tau) = 2 \log(\tau - 1) - \log \tau + \log n$$

are simultaneously the roots of the polynomial $p(\tau)$ defined in (30). It is clear that $\tau_0$ is the unique saddle point of the function $f(\tau)$ and the contour of integration in (31) passes through it. Now for $\tau = \tau_0 + iy, -\infty < y < +\infty$ by the Cauchy-Riemann conditions, we have

$$\frac{d}{dy} \text{Re} f(\tau_0 + iy) = -\text{Im} \frac{d}{d\tau} f(\tau_0 + iy) = -2 \arg(\tau - 1) + \arg \tau.$$  

Since Re $\tau_0 > 1$, for $y < 0$ we get

$$-\frac{\pi}{2} < \arg(\tau - 1) < \arg \tau < 0$$

and therefore,

$$\frac{d}{dy} \text{Re} f(\tau_0 + iy) = (\arg \tau - \arg(\tau - 1)) - \arg(\tau - 1) > 0.$$  

This implies that Re $f(\tau_0 + iy)$ strictly increases as $y$ increases from $-\infty$ to 0. If $y > 0$ then we have

$$0 < \arg \tau < \arg(\tau - 1) < \frac{\pi}{2}$$

and hence

$$\frac{d}{dy} \text{Re} f(\tau_0 + iy) = (\arg \tau - \arg(\tau - 1)) - \arg(\tau - 1) < 0.$$
Therefore, the function $Re f(\tau_0 + iy)$ strictly decreases as $y$ increases from 0 to $+\infty$. This proves that $Re f(\tau_0 + iy)$ attains its maximum on the vertical line $Re \tau = \tau_0$ at the unique point $\tau_0$ and we can apply the saddle-point method to calculate the asymptotics of $I_n$,

$$I_n = \frac{\sqrt{2\pi n}}{i} \frac{(2\pi)^{1/2}}{n^{1/2}} e^{\frac{\tau_0}{2} - \frac{i}{2} \arg f''(\tau_0)} |f''(\tau_0)|^{-\frac{1}{2}} g(\tau_0) e^{n f(\tau_0)} (1 + O(n^{-1/2})).$$

Since

$$f''(\tau_0) = \frac{\tau_0 + 1}{\tau_0 (\tau_0 - 1)} = 2n^{1/2} (1 + O(n^{-1/2})), \quad g(\tau_0) = n^{1/2} (1 + O(n^{-1/2}))$$

and

$$f(\tau_0) = -2 \log(\tau_0 - 1) - \tau_0 = \log n - 1 - \frac{2}{n^{1/2}} - \frac{1}{2n} + O(n^{-3/2}),$$

we get

$$I_n = \frac{\pi \sqrt{2n^{1/4}}}{\sqrt{e}} e^{n \log n - 2\sqrt{n}} (1 + O(n^{-1/2})) = \sqrt{\pi} \frac{n!}{e^{1/4}} e^{-2\sqrt{n}} (1 + O(n^{-1/2})).$$

To compute the asymptotics of $q_n$, we note that $q_n = n! L_n(-1)$, where $L_n(x) = \frac{1}{n!} (x^n e^{-x}) (n) = \sum_{k=0}^{n} \binom{n}{k} (\frac{x}{k!})^k$ is the Laguerre polynomial. Then the Perron asymptotics for the Laguerre polynomials (see [40, p. 199]) yields

$$q_n = n! e^{2\sqrt{n}} \left( \frac{1}{2\sqrt{\pi e}} + O(n^{-1/2}) \right),$$

and the lemma is proved.

\[\square\]

**Theorem 1.** Let \( \{q_n\}_{n \geq 0}, \{p_n\}_{n \geq 0} \) be defined by (23). Then $q_n \in \mathbb{Z}$, $D_n p_n \in \mathbb{Z}$ for each $n = 0, 1, 2, \ldots$, and

$$p_n - \gamma q_n = n! \frac{e^{-2\sqrt{n}}}{\sqrt{n}} \left( \sqrt{\frac{\pi}{e}} + O(n^{-1/2}) \right), \quad q_n = n! \frac{e^{2\sqrt{n}}}{\sqrt{n}} \left( \frac{1}{2\sqrt{\pi e}} + O(n^{-1/2}) \right)$$

as $n \to \infty$.

**Corollary 1.** The sequence $p_n/q_n$ converges to Euler’s constant sub-exponentially:

$$\frac{p_n}{q_n} - \gamma = e^{-4\sqrt{n}} (2\pi + O(n^{-1/2})) \quad \text{as} \quad n \to \infty.$$

4. A CONTINUED FRACTION FOR EULER’S CONSTANT

It is interesting to mention on connection of our construction with a continued fraction of the Euler-Gompertz constant

$$\delta := \int_0^\infty \frac{e^{-x}}{x + 1} \, dx = \int_0^\infty \log(x + 1) e^{-x} \, dx = 0.5963473623 \ldots$$

found by Stieltjes in 1895

$$\delta = \frac{1}{2} + \sum_{m=1}^{\infty} \left( \frac{-m^2}{2(m + 1)} \right) = \frac{1}{2} - \frac{1^2}{4} - \frac{2^2}{6} - \frac{3^2}{8} - \ldots.$$
The $n$-th convergent of this continued fraction has the form $\frac{s_n}{q_n}$, where $q_n$ is defined in (23) and the sequence $s_n$ is generated by the homogeneous linear recurrence (24) with the initial conditions $s_0 = 0$, $s_1 = 1$. Stieltjes derived (34) from the continued fraction of Gauss

$$
F(a, b + 1; c + 1; z) = \frac{1}{1 - \frac{a(z-1)}{1 + \frac{b+1}{1 + \frac{(c-a+1)z}{1 + \frac{(c+1)(c+2)z}{1 + \ldots}}}}},
$$

where $F(a, b; c; z)$ is the Gauss hypergeometric series

$$
F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k.
$$

We now recall how to deduce (34) following [42, Ch. 18]. Setting $b = 0$ and replacing $z$ by $-cz$ in (35), and then letting $c$ tend to infinity we get the expansion

$$
\sum_{k=0}^{\infty} (a)_k k! (-z)^k = \frac{a z}{1 + \frac{a+1}{1 + \frac{2z}{1 + \frac{a+2}{1 + \ldots}}}}.
$$

Here the equality sign can be considered as purely formal since the series on the left-hand side diverges excepting for $z = 0$. Although the power series is divergent, nevertheless the continued fraction converges to an analytic function of $z$, namely to

$$
g(z) = \frac{1}{\Gamma(a)} \int_0^\infty \frac{e^{-u} u^{a-1}}{1 + zu} du.
$$

More exactly, one has [42, p. 353]

$$
\frac{1}{\Gamma(a)} \int_0^\infty \frac{e^{-u} u^{a-1}}{1 + zu} du = \frac{a z}{1 + \frac{a+1}{1 + \frac{2z}{1 + \frac{a+2}{1 + \ldots}}}},
$$

which is valid for $a \in \mathbb{C}$, Re $a > 0$, and $z \in \mathbb{C} \setminus (-\infty, 0]$. Replacing $z$ by $1/z$ and then dividing by $z$ in (36) yields

$$
\frac{1}{\Gamma(a)} \int_0^\infty \frac{e^{-u} u^{a-1}}{z + u} du = \frac{a}{z + 1} + \frac{a+1}{z + 1} + \frac{2}{z + 1} + \frac{a+2}{z + 1} + \ldots,
$$

valid in the range of (36). Taking the even part of the continued fraction (corresponding to the $2n$-th convergents) on the right-hand side of (37) gives

$$
\frac{1}{\Gamma(a)} \int_0^\infty \frac{e^{-u} u^{a-1}}{z + u} du = \frac{1}{z + a} + \frac{2(1+a)}{z + a + 2} + \frac{3(2+a)}{z + a + 4} + \ldots,
$$

valid in the range of (36). Now taking $a = z = 1$ in (38) leads to the expansion (34). If we take $a = z = 1$ in (36) we get the following continued fraction expansion:

$$
\delta = \frac{1}{1 + 1 + 1 + 2 + 3 + 4 + \ldots}.
$$

Note that continued fraction expansion (36) can be found independently if we define

$$
h(a, b) := \frac{1}{\Gamma(a)} \int_0^\infty \frac{e^{-u} u^{a-1}}{(1 + xu)^b} du, \quad a > 0,
$$
and indicate the following relations, which can be easily deduced by integration by parts:

\[ h(a, b) = h(a, b + 1) + axh(a + 1, b + 1), \]
\[ h(a, b + 1) = h(a + 1, b + 1) + bxh(a + 1, b + 2). \]

Then we can easily see that

\[ f(a, b + 1) = \frac{1}{1 + \frac{ax}{1 + \frac{(b + 1)x}{1 + \frac{(a + 1)x}{1 + \frac{(b + 2)x}{1 + \cdots}}}}}, \]

which for \( b = 0 \) implies (36).

On the other hand, we note that the convergence of the continued fraction (34) is slow enough to imply certain results on arithmetical nature of \( \delta \). As it was mentioned by Aptekarev in [6]

\[ s_n - \delta q_n = O \left( \frac{n!}{\sqrt{n}} \right) e^{-2\sqrt{n}} \]

and therefore, one has only rational approximations to \( \delta \):

\[ \frac{s_n}{q_n} - \delta = O(e^{-4\sqrt{n}}) \quad \text{as} \quad n \to \infty. \]

Notice that the irrationality of the Euler-Gompertz constant \( \delta \) is still an open problem. All we know is that at least one of the two numbers \( \gamma, \delta \) must be irrational [6]. Indeed, from (33) we have \( \delta = eE_1(1) \), where

\[ E_1(z) = \int_z^\infty \frac{e^{-t}}{t} \, dt, \quad |\arg z| < \pi, \]

is the exponential integral. Using the formula (see [1, p. 228])

\[ E_1(z) = -\gamma - \log z - \sum_{k=1}^\infty \frac{(-1)^k z^k}{k!}, \quad |\arg z| < \pi, \]

which implies that

\[ -\delta = e \gamma + e \sum_{k=1}^\infty \frac{(-1)^k}{k!}, \]

and the fact that the numbers \( e \) and \( \sum_{k=1}^\infty \frac{(-1)^k}{k!} \) are algebraically independent (see [37, Ch. 7, Th. 1]), we get that the numbers \( e \) and \( \delta/e + \gamma \) are algebraically independent and therefore at least one of the numbers \( \gamma, \delta \) must be irrational. Recently, Rivoal [36] gave a quantitative version of this statement by proving that for any \( \varepsilon > 0 \) there exists a positive constant \( d(\varepsilon) \) such that for any integers \( p, q, r, q \neq 0 \), one has

\[ \left| \gamma - \frac{p}{q} \right| + \left| \delta - \frac{r}{q} \right| \geq \frac{d(\varepsilon)}{H^{3+\varepsilon}}, \]

where \( H = \max(|p|, |q|, |r|) \).

Using the following theorem from the general theory of continued fractions we will be able to find a continued fraction expansion (though not a simple one) for Euler’s constant \( \gamma \) whose \( n \)-th numerator and denominator coincide with the \( p_n \) and \( q_n \), respectively. It seems to be the first non-trivial continued fraction expansion convergent to
Euler’s constant sub-exponentially, the elements of which can be expressed as a general pattern.

**Theorem A.** ([26, Th. 2.2]) Let \( \{A_n\}, \{B_n\} \) be sequences of complex numbers such that
\[
A_{-1} = 1, \quad A_0 = b_0, \quad B_{-1} = 0, \quad B_0 = 1,
\]
and
\[
A_nB_{n-1} - A_{n-1}B_n \neq 0, \quad n = 0, 1, 2, \ldots.
\]
Then there exists a uniquely determined continued fraction \( b_0 + K(a_n/b_n) \) with \( n \)th numerator \( A_n \) and denominator \( B_n \) for all \( n \). Moreover,
\[
b_0 = A_0, \quad a_1 = A_1 - A_0B_1, \quad b_1 = B_1,
\]
\[
a_n = \frac{A_{n-1}B_n - A_nB_{n-1}}{A_{n-1}B_{n-2} - A_{n-2}B_{n-1}}, \quad b_n = \frac{A_nB_{n-2} - A_{n-1}B_{n-1}}{A_{n-1}B_{n-2} - A_{n-2}B_{n-1}}, \quad n = 2, 3, 4, \ldots.
\]

Applying the above theorem to the sequences \( \{p_n\}_{n \geq 0} \) and \( \{q_n\}_{n \geq 0} \) we get the following.

**Theorem 2.** Euler’s constant \( \gamma \) has the following continued-fraction expansion:
\[
\gamma = \frac{\infty}{K} (a_n/b_n) = \frac{1}{2} - \frac{1}{4 - 16 + 59 - 404 + \cdots + a_n/b_n + \cdots},
\]
where
\[
a_1 = 1, \quad a_2 = -1, \quad a_3 = -5, \quad a_4 = 36, \quad a_5 = -15740,
\]
\[
b_1 = 2, \quad b_2 = 4, \quad b_3 = 16, \quad b_4 = 59, \quad b_5 = 404,
\]
and
\[
a_n = -\frac{(n - 1)^2}{4} \Delta_n \Delta_{n-2}, \quad b_n = n^2 \Delta_{n-1} + \frac{(n - 1)(n - 2)}{2} q_{n-2}, \quad n \geq 6.
\]

Here \( q_n \) is a sequence of positive integers defined by (23) and \( \Delta_n \) is a sequence of integers generated by the third-order linear recurrence:
\[
(n - 1)(n - 2) \Delta_{n+2} = (n - 2)(n + 1)(n^2 + 3n - 2) \Delta_{n+1}
\]
\[
- n^2(2n^3 + n^2 - 7n - 4) \Delta_n + (n - 1)^2 n^4 \Delta_{n-1}, \quad n \geq 3,
\]
with the initial values \( \Delta_1 = -1, \Delta_2 = -2, \Delta_3 = -5, \Delta_4 = 8 \). Moreover, \( \Delta_n \) is positive for any \( n \geq 4 \), and \( \Delta_{2n} \) is even for any \( n \geq 1 \).

**Proof.** Consider sequences \( \{p_n\}_{n \geq 0} \) and \( \{q_n\}_{n \geq 0} \) from (23) and put \( p_{-1} = 1, q_{-1} = 0 \). For \( n \geq 0 \), define \( \vartheta_n := p_{n-1}q_n - p_nq_{n-1} \). Then we have
\[
\vartheta_0 = 1, \quad \vartheta_1 = \vartheta_2 = -1, \quad \vartheta_3 = -\frac{5}{3}, \quad \vartheta_4 = 2, \quad \vartheta_5 = \frac{787}{5}.
\]

From the recurrent equations (24), (25) we get the relation
\[
\vartheta_n = (n - 1)^2 \vartheta_{n-1} + \frac{n - 2}{n} q_{n-1}, \quad n \geq 1.
\]

Since \( q_n \) is positive for any \( n \geq 0 \) and \( \vartheta_4 > 0 \), it follows easily by induction that \( \vartheta_n \) is positive for any \( n \geq 4 \). Taking into account (42), we get \( \vartheta_n \neq 0 \) for \( n = 0, 1, 2, \ldots \).

Moreover, (43) implies that \( n\vartheta_n \in \mathbb{Z} \) for any \( n \) and \( n\vartheta_n/2 \in \mathbb{Z} \) if \( n \) is even.
Now by Theorem A, we get that there exists a uniquely determined continued fraction
\[ b_0 + K(a_n/b_n) \] with \( n \)-th numerator \( p_n \) and denominator \( q_n \) for all \( n \), where \( b_0 = p_0 = 0 \), \( a_1 = p_1 - p_0q_1 = 1 \), \( b_1 = q_2 = 1 \), and
\[
\begin{align*}
  a_n &= -\frac{\Delta_n}{\Delta_{n-1}}, \\
  b_n &= \frac{p_{n-2}q_n - p_nq_{n-2}}{\Delta_{n-1}}, \quad n \geq 2,
\end{align*}
\]
such that
\[
\gamma = K\left(\frac{a_n}{b_n}\right) = \frac{1}{2} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \ldots.
\]
From (43), (44) we have
\[
a_n = -(n-1)^2 - \frac{n-2}{n} \cdot \frac{q_{n-1}}{\Delta_{n-1}}, \quad n \geq 1.
\]
From the recurrent equations (24), (25) we obtain
\[
p_{n-2}q_n - p_nq_{n-2} = 2n\Delta_{n-1} + \frac{n-2}{n} \cdot q_{n-2}
\]
and hence
\[
b_n = 2n + \frac{n-2}{n} \cdot \frac{q_{n-2}}{\Delta_{n-1}}, \quad n \geq 1.
\]
Now let \( \Delta_n := n\Delta_n \). Then we have \( \Delta_n \in \mathbb{Z} \), \( \Delta_n \) is positive for any \( n \geq 4 \) and \( \Delta_{2n} \) is even for any \( n \geq 1 \). Define \( \rho_0 = \rho_1 = \rho_2 = 1 \), \( \rho_3 = 3 \), \( \rho_4 = 10 \),
\[
\rho_n = \frac{n(n-1)\Delta_{n-1}}{2} = \frac{n\Delta_{n-1}}{2}, \quad n \geq 5,
\]
and make the equivalence transformation of the fraction (45) by the rule (see [26, Th. 2.6])
\[
a^*_n = \rho_n\rho_{n-1}a_n, \quad b^*_n = \rho_nb_n, \quad n = 1, 2, 3, \ldots,
\]
then using first several values of the sequence \( q_n \),
\[
q_0 = 1, \quad q_1 = 2, \quad q_2 = 7, \quad q_3 = 34, \quad q_4 = 209
\]
and formulas (44), (46) we get (39), (40). What is left is to show that the sequence \( \Delta_n \) satisfies the recurrence (41). From (43) we have
\[
q_{n-1} = \frac{\Delta_n - n(n-1)\Delta_{n-1}}{n-2}, \quad n \geq 3.
\]
Substituting this expression in (24) we get the four-term recurrence relation (41), which completes the proof. □
5. Concluding remarks

In section 3 we considered the function $F(n,t)$, which is a special case of a more general function (22), such that the sequences $q_n$ and $F_n$ defined by

\[(47) \quad q_n = \sum_{t=0}^{n} F(n,t) = \sum_{t=0}^{\infty} F(n,t), \quad F_n = \sum_{t=0}^{n} \frac{d}{dt} F(n,t) = \sum_{t=0}^{\infty} \frac{d}{dt} F(n,t)\]

satisfy the second-order homogeneous and inhomogeneous linear recurrences (24) and (25), respectively. In this connection, it would be interesting to investigate the following question: does there exist a function $F(n,t)$ of the form (22) such that both sequences (47) satisfy the same second-order homogeneous linear recurrence? If the answer is positive it would be therefore of interest to examine the convergence of $\frac{F_n}{q_n}$ that could lead to a simpler continued fraction for Euler’s constant. If this is not so, it would be curious to have a proof of this fact. Note that if we remove the condition

$$\sum_{j=1}^{s} b_j \neq \sum_{j=1}^{u} d_j$$

in (22), the existence of such functions $F(n,t)$ is known. For example, the rational function

$$F(n,t) = \frac{\Gamma(2t+3n+1)\Gamma^6(t+n)\Gamma^2(t+3n+1)}{\Gamma(2t+3n)\Gamma^6(t+2n+1)\Gamma^2(t)}$$

and the sequences

$$q_n = \sum_{t=1}^{\infty} F(n,t), \quad F_n = \sum_{t=1}^{\infty} \frac{d}{dt} F(n,t)$$

that satisfy the second-order homogeneous linear recurrence

$$(n+1)^5 F_{n+1} + 3(2n+1)(3n^2 + 3n + 1)(15n^2 + 15n + 4)F_n - 3n^3(3n-1)(3n+1)F_{n-1} = 0$$

were used in [47] for constructing an Apéry-like continued fraction for $\zeta(4)$.

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