Unions of 1-factors in $r$-graphs and overfull graphs

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Abstract

We prove lower bounds for the fraction of edges of an $r$-graph which can be covered by the union of $k$ 1-factors. The special case $r = 3$ yields some known results for cubic graphs. Furthermore, we introduce the concept of $k$-overfull-free $r$-graphs and achieve better bounds for these graphs.

Keywords: $r$-graphs; 1-factors; overfull graphs

1 Introduction

We consider finite graphs $G$ with vertex set $V(G)$ and edge set $E(G)$. Graphs do not contain loops in this paper. For $v, w \in V(G)$, the number of edges between $v$ and $w$ is denoted by $\mu(v, w)$ and $\mu(G) = \max\{\mu(v, w) : v, w \in V(G)\}$. $\mu(v, w)$ is called the multiplicity of $vw$ and $\mu(G)$ the multiplicity of $G$. A graph is simple if $\mu(v, w) \leq 1$ for any two vertices $v, w$. The number of edges which are incident to vertex $v$ is the vertex degree of $v$ which is denoted by $d_G(v)$. The maximum vertex degree of $G$ is $\max\{d_G(v) : v \in V(G)\}$ and it is denoted by $\Delta(G)$. Further $\delta(G)$ denotes the minimum degree of a vertex of $G$.

1.1 1-factor covering

The following celebrated conjecture, often referred to as the Berge-Fulkerson conjecture, is due to Fulkerson and appears first in [5]:

**Conjecture 1.1** (Berge-Fulkerson conjecture [5]). Every bridgeless cubic graph $G$ has six 1-factors such that each edge of $G$ is contained in precisely two of them.

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A set of such six 1-factors in the conjecture is called a Fulkerson cover of $G$. It is straightforward that Berge-Fulkerson Conjecture implies the existence of five 1-factors whose union is the edge-set of the graph $G$. This naturally raises a seemingly weaker conjecture, attributed to Berge (unpublished, see e.g. [27]).

**Conjecture 1.2** (Berge conjecture). *Every bridgeless cubic graph $G$ has five 1-factors such that each edge of $G$ is contained in at least one of them.*

A set of the five 1-factors in Berge Conjecture is called a Berge cover of $G$. Recently, Mazzuoccolo [11] proved that the previous two conjectures are equivalent. It is unclear whether the same equivalence holds for every single bridgeless cubic graph, in other words, does a graph having a Berge cover always have a Fulkerson cover?

Let $r$ be a positive integer. A graph $G$ is $r$-regular, if $d_G(v) = r$ for all $v \in V(G)$. Let $X \subseteq V(G)$ be a set of vertices. The subgraph of $G$ induced by $X$ is denoted by $G[X]$, and the set of edges with precisely one end in $X$ by $\partial_G(X)$. An $r$-regular graph $G$ is an $r$-graph if $|\partial_G(X)| \geq r$ for every odd set $X \subseteq V(G)$.

A cubic graph is a 3-graph if and only if it is bridgeless. Moreover, it was proved in [21] that every $r$-graph has a 1-factor. Hence, it is natural to consider similar questions on perfect matching covering for $r$-graphs as for bridgeless cubic graphs. In particular, aforementioned two conjectures were generalized to $r$-graphs. In 1979, Seymour [21] proposed the generalized Berge-Fulkerson conjecture:

**Conjecture 1.3** (Generalized Berge-Fulkerson conjecture [21]). *Every $r$-graph has $2r$ 1-factors such that each edge is contained in precisely two of them.*

Trivially, this conjecture implies the following generalized form of Conjecture 1.2 first proposed by Mazzuoccolo [12].

**Conjecture 1.4** (Generalized Berge conjecture [12]). *Every $r$-graph $G$ has $2r - 1$ 1-factors such that each edge is contained in at least one of them.*

The value $2r - 1$ in the conjecture is best possible, that is, it can not be smaller, as shown in [12]. In the same paper, Mazzuoccolo proved the equivalence between the generalized Berge-Fulkerson conjecture and the generalized Berge conjecture, in a similar way as he did for cubic case.

The excessive index $\chi'_e(G)$ of a graph $G$ is the minimum number of 1-factors needed to cover $E(G)$. This parameter, also called the perfect matching index in [4], was widely studied in the literature, e.g., [1] [2] [12] [14] [15] [18]. It is reasonable to consider the excessive index for $r$-graphs in the context that it can be arbitrary large for some family of bridgeless $r$-regular graphs, constructed in [15]. However, it is an open question whether there exists a constant $k$ such that $\chi'_e(G) \leq k$ for all $r$-graphs $G$ for any fixed $r \geq 3$. The result of Mazzuoccolo [12] shows that if such $k$ exists then it is at least $2r - 1$. The generalized Berge conjecture asserts that such $k$ exists and $k = 2r - 1$.  

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Partial covers of $r$-graphs with 1-factors are of great interest, see e.g. [8, 22]. In this paper, we consider the following relaxed form of the generalized Berge conjecture: Over all $r$-graphs $G$ for any fixed $r$, what is the maximum constant $c$ ($c \leq 1$), such that $G$ has $2r - 1$ 1-factors whose union contains at least $c|E(G)|$ edges? Note that the generalized Berge conjecture asserts that $c = 1$. We will show that $c \geq 1 - e^{-2} \approx 0.8647$. We will also show a second lower bound for $c$ which depends on $r$, but which is always greater than $1 - e^{-2}$. In fact, this second lower bound is an approximation to the following more general problem.

Given an $r$-graph $G$, let $\mathcal{M}$ be the set of distinct 1-factors in $G$. Fix a positive integer $k$. Define

$$m(r, k, G) = \max_{M_1, \ldots, M_k \in \mathcal{M}} \frac{|\bigcup_{i=1}^{k} M_i|}{|E(G)|},$$

and

$$m(r, k) = \inf_G m(r, k, G),$$

where the infimum is taken over all $r$-graphs. Clearly, $m(r, k) \leq m(r, k + 1) \leq 1$. With this notation, the generalized Berge conjecture can be reformulated as follows:

**Conjecture 1.5.** $m(r, 2r - 1) = 1$ for every integer $r$ with $r \geq 3$.

The parameter $m(r, k)$ has primarily been studied in cubic case, i.e. $r = 3$. Berge’s conjecture states that $m(3, 5) = 1$. Kaiser, Král and Norine [9] proposed a lower bound for $m(3, k)$ as

$$m(3, k) \geq 1 - \prod_{i=1}^{k} \left( \frac{r^2 - 3r + 1}{r^2 - 2r - 1} \right),$$

and verified it for the case $k \in \{2, 3\}$. Meanwhile, Patel [16] conjectured that $m(3, 2) = \frac{4}{5}$, $m(3, 3) = \frac{4}{5}$ and $m(3, 4) = \frac{14}{15}$. Since the example of Petersen graph, the result of Kaiser, Král and Norine confirms that $m(3, 2) = \frac{4}{5}$. But the exact values for $m(3, 3)$ and $m(3, 4)$ are still unknown. A complete proof for the lower bound in (1) was later given by Mazzuoccolo [13].

In Section 3, we obtain the following lower bound for $m(r, k)$:

$$m(r, k) \geq 1 - \prod_{i=1}^{k} \left( \frac{r^2 - 3r + 1}{r^2 - 2r - 1} \right),$$

for any even $r \geq 4$ and any $k \geq 1$, and

$$m(r, k) \geq 1 - \prod_{i=1}^{k} \left( \frac{r^2 - 2r - 1}{r^2 - r - 2} \right),$$

for any odd $r \geq 3$ and any $k \geq 1$.

For instance of small $r$ and $k$, the values of this lower bound are listed in Table 1.1. In particular, if we take $r = 3$, this lower bound coincides with the established bound in (1); if we take $k = 2r - 1$, it gives a partial result to the generalized Berge conjecture, and the approximate value of $m(r, 2r - 1)$ is shown in bold in Table 1.1.
Let $f(x) = \frac{r^2 - 3r + 1}{r^2 - 2r - 1}$, and the derivative

$$f'(x) = \frac{-2}{[r^2 - 2r - 1]x - (r^2 - 4r - 1)]^2} < 0.$$  

Hence, $f(i) \leq \frac{r-1}{r}$ for each $i \geq 1$. So when we take $k = 2r - 1$, the lower bound in (2) is greater than

$$1 - \left(\frac{r-1}{r}\right)^{2r-1} > 1 - e^{-2},$$

where the last inequality is given by Corollary 3.2. Similarly, when $k = 2r - 1$, we can deduce that the lower bound in (3) is greater than $1 - e^{-2}$ as well.

## 1.2 Edge-colorings and overfull graphs

A graph $G$ is $k$-overfull if $|V(G)|$ is odd, $\Delta(G) \leq k$ and $\frac{|E(G)|}{\frac{1}{2}|V(G)|} > k$. It is easy to see that $G$ is $k$-overfull if and only if $2|E(G)| > k(|V(G)| - 1)$. Furthermore, the $k$-deficiency of $G$ is $k|V(G)| - 2|E(G)|$, and it is denoted by $s_k(G)$.

A $k$-edge-coloring of $G$ is a mapping $c : E(G) \rightarrow \{1, \ldots, k\}$ such that adjacent edges are colored differently. The chromatic index $\chi'(G)$ is the minimum number $k$ such that $G$ has a $k$-edge-coloring. Vizing [25] proved that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + \mu(G)$, in particular if $G$ is simple, then $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$. We say that $G$ is class 1 if $\chi'(G) = \Delta(G)$, and it is class 2 otherwise.

Clearly, $\Delta(G)$ is a lower bound for the chromatic index of $G$. Overfull graphs are class 2 graphs for the trivial reason that they contain too many edges. In general we have that $\chi'(G) \geq$
\[ \max_{H \subseteq G} \frac{|E(H)|}{\lfloor \frac{1}{2} |V(H)| \rfloor}. \]

A graph \( G \) is critical with respect to \( \chi'(G) \), if \( \chi'(G - e) < \chi'(G) \) for every \( e \in E(G) \). For simple graphs we have the definition of a \( k \)-critical graph which says that a critical graph \( H \) is \( k \)-critical, if \( \Delta(H) = k \) and \( \chi'(H) = k + 1 \). Vizing [20] proved the classical result that a simple class 2 graph with maximum degree \( k \) contains a \( t \)-critical subgraph for every \( t \in \{2, \ldots, k\} \). These results are the motivation for the result of Section 4 which proves that a \( k \)-overfull graph contains a \( t \)-overfull subgraph for every \( t \in \{2, \ldots, k\} \).

A graph \( G \) is \( k \)-overfull-free, if it does not contain a \( k \)-overfull subgraph. Clearly, there are no 1-critical graphs and the 2-critical graphs are the odd circuits which are also the connected 2-overfull graphs. Hence we have: A graph is 2-overfull-free if and only if it is bipartite. We study \( k \)-overfull-free graphs in Section 4. If an \( r \)-regular graph \( G \) is class 1, then surely \( G \) is an \( r \)-graph and \( G \) is \( r \)-overfull-free. For \( i \in \{1, 2\} \) let \( G_i \) be an \( r_i \)-regular graph. We say that an \( r \)-graph \( G \) is decomposable into \( G_1 \) and \( G_2 \) if \( r = r_1 + r_2 \), \( V(G_i) = V(G) \) and \( E(G) = E(G_1) \cup E(G_2) \). We will characterize some decomposable \( r \)-graphs in terms of excluded overfull subgraphs.

2 The perfect matching polytope

Let \( G \) be a graph and \( w \) be a vector of \( \mathbb{R}^{E(G)} \). The entry of \( w \) corresponding to an edge \( e \) is denoted by \( w(e) \), and for \( A \subseteq E \), we define \( w(A) = \sum_{e \in A} w(e) \). The vector \( w \) is a fractional 1-factor if it satisfies

(i) \( 0 \leq w(e) \leq 1 \) for every \( e \in E(G) \), and

(ii) \( w(\partial\{v\}) = 1 \) for every \( v \in V(G) \), and

(iii) \( w(\partial(S)) \geq 1 \) for every \( S \subseteq V(G) \) with odd cardinality.

Let \( F(G) \) denote the set of all fractional 1-factors of a graph \( G \). If \( M \) is a 1-factor, then its characteristic vector \( \chi^M \) is contained in \( F(G) \). Furthermore, if \( w_1, \ldots, w_n \in F(G) \), then any convex combination \( \sum_{i=1}^{n} \alpha_i w_i \) (where \( \alpha_1, \ldots, \alpha_n \) are nonnegative real numbers summing up to 1) also belongs to \( F(G) \). It follows that \( F(G) \) contains the convex hull of all the vectors \( \chi^M \) where \( M \) is a 1-factor of \( G \). The following theorem by Edmonds asserts that the converse inclusion also holds:

**Theorem 2.1 (Perfect Matching Polytope Theorem [3])**. For any graph \( G \), the set \( F(G) \) coincides with the convex hull of the characteristic vectors of all 1-factors of \( G \).

Towards the generalized Berge-Fulkerson conjecture, Seymour [21] gave an alternative proof of the following theorem, which is a corollary of Edmonds’s matching polytope theorem (see [21] for the details between these two theorems).
Theorem 2.2 ([21]). For any r-graph \( G \), there is a positive integer \( p \) such that \( G \) has \( rp \) 1-factors and each edge is contained in precisely \( p \) of them.

We will use this theorem to deduce our first lower bound in the next section. Moreover, the following property on fractional 1-factors will play a crucial role in the proof for our second lower bound.

Lemma 2.3 ([9]). Let \( w \) be a fractional 1-factor of a graph \( G \) and \( c \in \mathbb{R}^{E(G)} \). Then \( G \) has a 1-factor \( M \) such that \( c \cdot \chi^M \geq c \cdot w \), where \( \cdot \) denotes the scalar product, and \( |M \cap C| = 1 \) for each edge-cut \( C \) of odd cardinality and with \( w(C) = 1 \).

The proof of this lemma was given in [9], where Theorem 2.1 is the main tool for the proof.

3 Lower bounds for \( m(r, k) \)

We are going to deduce a lower bound for the parameter \( m(r, k) \) by using Theorem 2.2 only.

Theorem 3.1. \( m(r, k) \geq 1 - \left( \frac{r-1}{r} \right)^k \) for every positive integers \( r \) and \( k \) with \( r \geq 3 \).

Proof: (induction on \( k \).) Since every \( r \)-graph has a 1-factor, which covers a fraction \( \frac{1}{r} \) of the edges, the proof is trivial for \( k = 1 \). We proceed to the induction step. Let \( G \) be any \( r \)-graph and \( E = E(G) \). By the induction hypothesis, \( G \) has \( k-1 \) many 1-factors \( M_1, \ldots, M_{k-1} \) such that

\[
\frac{|\bigcup_{i=1}^{k-1} M_i|}{|E|} \geq 1 - \left( \frac{r-1}{r} \right)^{k-1}.
\]  

Moreover, by Theorem 2.2, there exists a positive integer \( p \) such that \( G \) has \( rp \) 1-factors \( F_1, \ldots, F_{rp} \) and each edge is contained in precisely \( p \) of them. It follows that for every \( X \subseteq E \), graph \( G \) has a 1-factor \( F \) among \( F_1, \ldots, F_{rp} \) such that \( |F \cap X| \geq \frac{|X|}{r} \). In particular, let \( X = E \setminus \bigcup_{i=1}^{k-1} M_i \) and consequently, take \( M_k = F \). Thus,

\[
|M_k \cap (E \setminus \bigcup_{i=1}^{k-1} M_i)| \geq \frac{|E \setminus \bigcup_{i=1}^{k-1} M_i|}{r},
\]  

that is,

\[
\frac{|\bigcup_{i=1}^{k-1} M_i| - |\bigcup_{i=1}^{k-1} M_i|}{|E|} \geq \frac{1}{r} (1 - \frac{|\bigcup_{i=1}^{k-1} M_i|}{|E|}).
\]  

It follows that

\[
\frac{|\bigcup_{i=1}^{k} M_i|}{|E|} \geq (1 - \frac{1}{r})|\bigcup_{i=1}^{k-1} M_i| + \frac{1}{r} \geq 1 - \left( \frac{r-1}{r} \right)^k
\]  

where the last inequality follows by using the inequality [4]. Therefore, \( m(r, k, G) \geq 1 - \left( \frac{r-1}{r} \right)^k \) and by the choice of \( G \), we have \( m(r, k) \geq 1 - \left( \frac{r-1}{r} \right)^k \).

In particular, if we take \( k = 2r - 1 \), we can further deduce from this theorem a constant lower bound for \( m(r, 2r - 1) \).

Corollary 3.2. For every integer \( r \geq 3 \), we have \( m(r, 2r - 1) \geq 1 - e^{-2} \approx 0.8647 \).
Proof: Let $f(r)$ denote the function $1 - (\frac{r-1}{r})^{2r-1}$. It is easy to see that $f(r)$ is strictly monotonic decreasing with respect to $r$. Moreover, $\lim_{r \to +\infty} f(r) = 1 - e^{-2}$. It follows with Theorem 3.1 that $m(r, 2r - 1) \geq f(r) \geq 1 - e^{-2}$.

We now prove the following theorem, which will be used to deduce a second lower bound for $m(r, k)$. An $i$-cut of a graph $G$ is an edge cut of $G$ of cardinality $i$. The proof of the theorem is conducted by induction. In the induction step, we apply Lemma 2.3 to a well-chosen fractional 1-factor, whose existence can be guaranteed by both inclusions of the induction hypothesis, one on the union of 1-factors and the other on $i$-cuts. The resulting 1-factor with its properties described in Lemma 2.3 and the 1-factors given by the induction hypothesis together complete the proof.

**Theorem 3.3.** Let $G$ be an $r$-graph, and $V = V(G)$ and $E = E(G)$.

(a) If $r$ is even and $r \geq 4$, then for any positive integer $k$, graph $G$ has $k$ 1-factors $M_1, \ldots, M_k$ such that

$$\frac{|\bigcup_{i=1}^k M_i|}{|E|} \geq 1 - \prod_{i=1}^k \left(\frac{r^2 - 3r + 1}{r^2 - 2r - 1} - \frac{r^2 - 5r + 3}{r^2 - 4r + 1}\right),$$

and $\sum_{i=1}^k \chi^{M_i}(C) \leq (r - 1)k + 2$ for each $(r + 1)$-cut $C$.

(b) If $r$ is odd and $r \geq 3$, then for any positive integer $k$, graph $G$ has $k$ 1-factors $M_1, \ldots, M_k$ such that

$$\frac{|\bigcup_{i=1}^k M_i|}{|E|} \geq 1 - \prod_{i=1}^k \left(\frac{r^2 - 3r + 1}{r^2 - 2r - 1} - \frac{r^2 - 5r + 3}{r^2 - 4r + 1}\right),$$

and

$$\sum_{i=1}^k \chi^{M_i}(C) = k \text{ for each } r\text{-cut } C \text{ and } \sum_{i=1}^k \chi^{M_i}(D) \leq rk + 2 \text{ for each } (r + 2)\text{-cut } D.$$

**Proof:** (induction on $k$).

**Statement (a).** The statement holds for $k = 1$, since the required $M_1$ can be an arbitrary 1-factor of $G$. Assume that $k \geq 2$. By the induction hypothesis, $G$ has $k - 1$ many 1-factors $M_1, \ldots, M_{k-1}$ such that

$$\frac{|\bigcup_{i=1}^{k-1} M_i|}{|E|} \geq 1 - \prod_{i=1}^{k-1} \left(\frac{r^2 - 3r + 1}{r^2 - 2r - 1} - \frac{r^2 - 5r + 3}{r^2 - 4r + 1}\right),$$

and

$$\sum_{i=1}^{k-1} \chi^{M_i}(C) \leq (r - 1)(k - 1) + 2$$

for each $(r + 1)$-cut $C$.

For $e \in E$, let $n(e)$ denote the number of 1-factors among $M_1, \ldots, M_{k-1}$ which contain $e$, and define

$$w_k(e) = \frac{(r - 2)k - (r - 4) - n(e)}{(r^2 - 2r - 1)k - (r^2 - 4r - 1)}.$$

We claim that $w_k$ is a fractional 1-factor of $G$, that is, $w_k \in F(G)$. Since $k \geq 2, r \geq 4$ and $0 \leq n(e) \leq k - 1$, we can deduce that $\frac{1}{r + 4} < w_k(e) < 1$. Moreover, note that for every $X \subseteq E$,
the equality \( \sum_{e \in X} a(e) = \sum_{i=1}^{k-1} \chi^M_i(X) \) always holds and so

\[
w_k(X) = \sum_{e \in X} w_k(e) = \frac{|(r-2)k-(r-4)||X| - \sum_{i=1}^{k-1} \chi^M_i(X)}{(r^2-2r-1)k-(r^2-4r-1)}.
\] (9)

Thus for \( v \in V \), since \( \sum_{i=1}^{k-1} \chi^M_i(\partial(\{v\})) = k-1 \), we have \( w_k(\partial(\{v\})) = \frac{(r-2)k-(r-4)|\partial(\{v\})|-(k-1)}{(r^2-2r-1)k-(r^2-4r-1)} = 1 \).

Finally, let \( S \subseteq V \) with odd cardinality. Since \( G \) is an \( r \)-graph, we have \( |\partial(S)| \geq r \). On the other hand, by recalling that \( w_k(e) > \frac{1}{r-3} \) for each edge \( e \), we have \( w_k(\partial(S)) > 1 \) provided by \( |\partial(S)| \geq r+3 \). Hence, we may next assume that \( |\partial(S)| = r+1 \) by parity. Since in this case \( S \) is a \((r+1)\)-cut, the formula (8) implies \( \sum_{i=1}^{k-1} \chi^M_i(\partial(S)) \leq (r-1)(k-1) + 2 \), and thus with the help of the formula (9), we deduce \( w_k(\partial(S)) \geq \frac{(r-2)k-(r-4)(r+1)-(r-1)(k-1)+2}{(r^2-2r-1)k-(r^2-4r-1)} = 1 \). This completes the proof of the claim.

By Lemma 2.3 the graph \( G \) has a 1-factor \( M_k \) such that

\[(1 - \chi^{U_{[1]}^{k-1}M_i}) \cdot \chi^M_k \geq (1 - \chi^{U_{[1]}^{k-1}M_i}) \cdot w_k.
\]

Since the left side is just \( |\bigcup_{i=1}^{k-1} M_i| - |\bigcup_{i=1}^{k-1} M_i| \) and the right side equals \( (r-2)k-(r-4)(r+1)-(r-1)(k-1)+2 \), it follows that

\[
|\bigcup_{i=1}^{k-1} M_i| \geq \frac{(r^2-3r+1)k-(r^2-5r+3)}{(r^2-2r-1)k-(r^2-4r-1)} |\bigcup_{i=1}^{k-1} M_i| + \frac{(r-2)k-(r-4)}{(r^2-2r-1)k-(r^2-4r-1)} |E|,
\]

which leads to

\[
|\bigcup_{i=1}^{k-1} M_i| \geq 1 - \prod_{i=1}^{k} \frac{(r^2-3r+1)i-(r^2-5r+3)}{(r^2-2r-1)i-(r^2-4r-1)}.
\]

as desired.

Moreover, let \( C \) be an edge cut with cardinality \( r+1 \). Clearly, \( \chi^M_k(C) \leq r+1 \). Thus, if \( \sum_{i=1}^{k-1} \chi^M_i(C) \leq (r-1)(k-1) \) then \( \sum_{i=1}^{k} \chi^M_i(C) \leq (r-1)k+2 \), as desired. By the formula (8) and by parity, we may next assume that \( \sum_{i=1}^{k-1} \chi^M_i(C) = (r-1)(k-1) + 2 \). In this case, we calculate from the formula (9) that \( w_k(C) = 1 \). Thus \( \chi^M_k(C) = 1 \) by Lemma 2.3 which yields \( \sum_{i=1}^{k} \chi^M_i(C) = (r-1)k+r+4 < (r-1)k+2 \), as desired. This completes the proof of Statement (a).

**Statement (b).** We follow a similar way to prove this statement as we did for Statement (a). Let \( w_1 \) be a vector of \( \mathbb{R}^E \) defined by \( w_1(e) = \frac{1}{i} \) for \( e \in E \). Clearly, \( w_1 \in F(G) \). By Lemma 2.3 \( G \) has a 1-factor \( M_1 \) such that \( \chi^M_1(C) = 1 \) for each edge cut \( C \) with odd cardinality and with \( w_1(C) = 1 \), that is, for each \( r \)-cut \( C \). Therefore, the statement is true for \( k = 1 \).

Assume \( k \geq 2 \). By the induction hypothesis, \( G \) has \( k-1 \) many 1-factors \( M_1, \ldots, M_{k-1} \) such that

\[
|\bigcup_{i=1}^{k-1} M_i| \geq 1 - \prod_{i=1}^{k-1} \frac{(r^2-2r-1)i-(r^2-4r+1)}{(r^2-r-2)i-(r^2-3r-2)}.
\]

and for each \( r \)-cut \( C \)

\[
\sum_{i=1}^{k-1} \chi^M_i(C) = k-1,
\] (10)
and for each \((r + 2)\)-cut \(D\)
\[
\sum_{i=1}^{k-1} \chi^M_i(D) \leq r(k - 1) + 2. 
\]  
(11)

For \(e \in E\), let \(n(e)\) denote the number of 1-factors among \(M_1, \ldots, M_{k-1}\) that contains \(e\), and define
\[
w_k(e) = \frac{(r - 1)k - (r - 3) - 2n(e)}{(r^2 - r - 2)k - (r^2 - 3r - 2)}.
\]

We claim that \(w_k \in F(G)\). Since \(k \geq 2, r \geq 3\) and \(0 \leq n(e) \leq k - 1\), we can deduce that \(0 < \frac{1}{r+2} < w_k(e) < 1\). Moreover, note that for every \(X \subseteq E\), the equality \(\sum_{e \in X} n(e) = \sum_{i=1}^{k-1} \chi^M_i(X)\) always holds and so
\[
w_k(X) = \frac{[(r - 1)k - (r - 3)]|X| - 2 \sum_{i=1}^{k-1} \chi^M_i(X)}{(r^2 - r - 2)k - (r^2 - 3r - 2)}. 
\]  
(12)

Thus for \(v \in V\), since \(\sum_{i=1}^{k-1} \chi^M_i(\partial(\{v\})) = k-1\), we have \(w_k(\partial(\{v\})) = \frac{[(r - 1)k - (r - 3)](r - 2)(k - 1)}{(r^2 - r - 2)k - (r^2 - 3r - 2)} = 1\). Finally, let \(S \subseteq V\) with odd cardinality. Since \(G\) is an \(r\)-graph, \(|\partial(S)| \geq r\). On the other hand, by recalling that \(w_k(e) > \frac{1}{r+2}\) for each edge \(e\), we have \(w_k(\partial(S)) > 1\) provided by \(|\partial(S)| \geq r + 4\). Hence, we may next assume that either \(|\partial(S)| = r\) or \(|\partial(S)| = r + 2\) by parity. In the former case, the formula \((11)\) implies \(\sum_{i=1}^{k-1} \chi^M_i(\partial(S)) = k - 1\), and thus we can calculate from the formula \((12)\) that \(w_k(\partial(S)) = 1\). In the latter case, the formula \((11)\) implies \(\sum_{i=1}^{k-1} \chi^M_i(\partial(S)) \leq r(k - 1) + 2\) and similarly, we get \(w_k(\partial(S)) \geq \frac{[(r - 1)k - (r - 3)](r - 2)(r - 1) + 2}{(r^2 - r - 2)k - (r^2 - 3r - 2)} = 1\). This proves the claim.

By Lemma 2.3 the graph \(G\) has a 1-factor \(M_k\) such that
\[
(1 - \chi^M_{k+1}(M)) \cdot \chi^{M_k} \geq (1 - \chi^M_{k+1}(M)) \cdot w_k.
\]
Since the left side is just \(|\bigcup_{i=1}^{k} M_i| - |\bigcup_{i=1}^{k-1} M_i|\) and the right side equals to \(\frac{(r - 1)k - (r - 3)}{(r^2 - r - 2)k - (r^2 - 3r - 2)}(|E| - |\bigcup_{i=1}^{k-1} M_i|)\), it follows that
\[
|\bigcup_{i=1}^{k} M_i| \geq \frac{(r - 1)k - (r - 3)}{(r^2 - r - 2)k - (r^2 - 3r - 2)}|E| + \frac{(r^2 - 2r - 1)k - (r^2 - 4r + 1)}{(r^2 - r - 2)k - (r^2 - 3r - 2)} \sum_{i=1}^{k-1} M_i,
\]
which leads to
\[
\frac{|\bigcup_{i=1}^{k} M_i|}{|E|} \geq 1 - \prod_{i=1}^{k} \frac{(r^2 - 2r - 1)i - (r^2 - 4r + 1)}{(r^2 - r - 2)i - (r^2 - 3r - 2)}
\]
as desired.

Moreover, let \(C\) be an edge cut of cardinality \(r\). The formula \((11)\) implies \(\sum_{i=1}^{k-1} \chi^M_i(C) = k - 1\). On the other hand, We can calculate from the formula \((12)\) that \(w_k(C) = 1\), and thus \(\chi^M_k(C) = 1\) by Lemma 2.3. Therefore, \(\sum_{i=1}^{k} \chi^M_i(C) = k\), as desired.

We next let \(D\) be an edge cut of cardinality \(r + 2\). Clearly, \(\chi^M_k(D) \leq r + 2\). Thus if \(\sum_{i=1}^{k-1} \chi^M_i(D) \leq r(k - 1)\), then \(\sum_{i=1}^{k} \chi^M_i(D) \leq rk + 2\), as desired. By the formula \((11)\) and by parity, we may next assume that \(\sum_{i=1}^{k-1} \chi^M_i(D) = r(k - 1) + 2\). By calculation we can get \(w_k(D) = 1\), and thus \(\chi^M_k(D) = 1\) by Lemma 2.3 which also yields \(\sum_{i=1}^{k} \chi^M_i(D) \leq rk + 2\). This completes the proof of this theorem.
The following corollary is a direct consequence of this theorem.

**Corollary 3.4.** Let \( r \) and \( k \) be two positive integers with \( r \geq 3 \). If \( r \) is even then

\[
m(r, k) \geq 1 - \frac{k}{k} \left( \frac{r^2 - 3r + 1}{r^2 - 2r - 1} \right) - \frac{(r^2 - 5r + 3)}{(r^2 - 4r - 1)}.
\]

and if \( r \) is odd then

\[
m(r, k) \geq 1 - \frac{k}{k} \left( \frac{r^2 - 2r - 1}{r^2 - r - 2} \right) - \frac{(r^2 - 4r + 1)}{(r^2 - 3r - 2)}.
\]

### 4 Overfull graphs

We start with the following observations.

**Observation 4.1.** Let \( r \geq 2 \) be an integer. Every \( r \)-overfull-free \( r \)-regular graph is an \( r \)-graph.

**Observation 4.2.** A graph \( G \) is \( 2 \)-overfull-free if and only if \( G \) is bipartite.

If \( G \) is a graph, then \( \omega(G) \) denotes the number of odd components of \( G \). We will use the following theorem of Tutte.

**Theorem 4.3** ([24]). A graph \( G \) has a 1-factor if and only if \( \omega(G - S) \leq |S| \) for all \( S \subseteq V(G) \).

**Proposition 4.4.** Let \( k \geq 2 \). If \( G \) is a \( k \)-overfull graph, then \( 0 \leq s_k(G) \leq k - 2 \) and \( k < \frac{|V(G)|}{|V(G)| - \Delta(G)} \).

**Proof:** Since \( G \) is \( k \)-overfull, \( 2|E(G)| > k(|V(G)| - 1) \) and \( |V(G)| \) is odd. Notice that the two sides of this inequality has the same parity. So, \( 2|E(G)| \geq k(|V(G)| - 1) + 2 \), that is, \( s_k(G) \leq k - 2 \). Moreover, by Handshaking Lemma, \( 2|E(G)| = \sum_{v \in V(G)} d_G(v) \leq \Delta(G)|V(G)| \). Combining it with the fact that \( 2|E(G)| > k(|V(G)| - 1) \), we deduce that \( k < \frac{|V(G)|}{|V(G)| - \Delta(G)} \).

**Theorem 4.5.** Let \( k \geq 3 \) be an integer. Every \( k \)-overfull graph contains a \((k - 1)\)-overfull subgraph.

**Proof:** Suppose to the contrary that the statement is not true. Then there is a \( k \)-overfull graph \( G \) which does not contain a \((k - 1)\)-overfull subgraph. We may assume that \( |V(G)| \) is minimum and according to this property \( |E(G)| \) is minimum as well.

It holds \( \Delta(G) = k \), since for otherwise \( G \) is \((k - 1)\)-overfull as well, a contradiction.

**Claim 4.5.1.** Let \( H \) be a proper subgraph of \( G \). If \( H \) is of odd order, then \( s_k(H) \geq k \).

By the minimality of \( G \), the subgraph \( H \) is not \( k \)-overfull. Note that \( H \) is of odd order and has maximum degree at most \( k \). Thus, \( \frac{2|E(H)|}{|V(H)| - 1} \leq k \) and therefore, \( s_k(H) = k|V(H)| - 2|E(H)| \geq k|V(H)| - k|V(H)| + k = k \).

**Claim 4.5.2.** \( s_k(G) = k - 2 \), that is, \( 2|E(G)| = k(|V(G)| - 1) + 2 \).
Choose any edge $e$ of $G$. By Claim 4.5.1, $s_k(G-e) \geq k$. It follows that $s_k(G) = s_k(G-e) - 2 \geq k - 2$. On the other hand, $s_k(G) \leq k - 2$ by Proposition 4.1. Therefore, $s_k(G) = k - 2$.

Claim 4.5.3. For every $z \in V(G)$, the graph $G - z$ has a 1-factor.

Let $G' = G - z$. Then $s_k(G') = s_k(G) + d_G(z) - (k-d_G(z)) = k - 2 + 2d_G(z) - k = 2d_G(z) - 2 \leq 2k - 2$.

Suppose to the contrary that $G'$ does not have a 1-factor. By Theorem 4.3, there is $S \subseteq V(G')$ such that $o(G' - S) > |S|$. Let $O_1, \ldots, O_n$ be the odd components of $G' - S$. Since $G - z$ has even order, $n$ and $|S|$ have the same parity. Thus, $n \geq |S| + 2$.

With Claim 4.5.1 it follows that $s_k(O_i) \geq k$. Hence, $|\partial_G(S)| \leq \sum_{i=1}^n s_k(O_i) - s_k(G') \geq nk - 2k + 2 = k(n - 2) + 2 \geq k|S| + 2$, a contradiction.

We now deduce the statement. If $G$ is regular, then $s_k(G) = 0 = k - 2$. So $k = 2$, a contradiction. Hence, there is $z \in V(G)$ such that $d_G(z) < k$. By Claim 4.5.3, $G - z$ has a 1-factor $F$. Let $G'' = G - F$. Then $\Delta(G'') = k - 1$, $|E(G'')| = |E(G)| - \frac{1}{2}(|V(G)| - 1)$, and $|V(G'')| = |V(G)|$. Hence, $\frac{2|E(G'')|}{|V(G'')|-1} = \frac{2|E(G)| - (|V(G)| - 1)}{|V(G)| - 1} = \frac{2|E(G)|}{|V(G)|} - 1 > k - 1$. This contradicts our assumption that $G$ does not contain a $(k-1)$-overfull subgraph and the statement is proved.

The following corollaries are immediate consequences of Theorem 4.5. The first one has the same flavor as a result of Vizing 28 that a class 2 graph with chromatic index $k$ contains critical subgraphs with chromatic index $t$ for every $t \in \{2, \ldots, k\}$.

Corollary 4.6. Let $k \geq 2$ be an integer and $G$ be a graph. If $G$ is $k$-overfull, then $G$ contains a $t$-overfull subgraph for every $t \in \{2, \ldots, k\}$.

Corollary 4.7. Let $k \geq 2$ be an integer and $G$ be a graph. If $G$ is $k$-overfull-free, then $G$ is $t$-overfull-free for every $t \geq k$.

Corollary 4.8. Let $2 \leq k \leq r$ be integers and $G$ be an $r$-regular graph. If $G$ is $k$-overfull-free, then $G$ is an $r$-graph and $G$ can be decomposed into a $(r-k)$-graph that is class 1 and a $k$-graph.

Proof: By Corollary 4.7, $G$ is $r$-overfull-free and further, by Observation 4.1, $G$ is an $r$-graph. Let $F_1$ be a 1-factor of $G$. Consider $G - F_1$. If $k = r$, then we are done. Hence, we may assume $k \leq r - 1$. Similarly, we can deduce that $G - F_1$ is an $(r-1)$-graph having a 1-factor $F_2$. Continue as above till $G' = G - \bigcup_{i=1}^{r-k} F_i$. Then $G''$ and $G''' = (V(G), \bigcup_{i=1}^{r-k} F_i)$ is the desired decomposition.

Corollary 4.8 gives a sufficient condition for an $r$-graph decomposable into a $r_1$-graph and a $r_2$-graph for some $r_1$ and $r_2$. It also shows that for any $t$-overfull-free $r$-graph with $2 \leq t \leq r$, we can obtain a better lower bound of $m(r, k, G)$ than the one of $m(r, k)$. More precisely, for $k \leq r - t$, take $k$ pairwise disjoint 1-factors of the class 1 graph from the decomposition by Corollary 4.8, which gives $m(r, k, G) = \frac{r}{t}$. For $k > r - t$, applying Theorem 3.3 to the $t$-graph from the decomposition.
by Corollary 4.8 gives $k$ 1-factors, which together with any $r - t$ many pairwise disjoint 1-factors of the class 1 graph from the decomposition leads to a better lower bound for $(m, k, G)$.

Moreover, Corollary 4.8 confirms the following classical result.

**Theorem 4.9.** Let $k \geq 0$ be an integer. Every $k$-regular bipartite graph is class 1.

**Proof:** For $k \in \{0, 1\}$, the proof is trivial. For $k \geq 2$, let $G$ be a $k$-regular bipartite graph. By Observation 12, $G$ is 2-overfull-free. By Corollary 4.8, $G$ is decomposable into a class 1 subgraph and a 1-factor. Thus, $G$ is class 1.

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