Anomalous diffusion: fractional Brownian motion vs fractional Ito motion

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Abstract

Generalizing Brownian motion (BM), fractional Brownian motion (FBM) is a paradigmatic self-similar model for anomalous diffusion. Specifically, varying its Hurst exponent, FBM spans: sub-diffusion, regular diffusion, and super-diffusion. As BM, also FBM is a symmetric and Gaussian process, with a continuous trajectory, and with a stationary velocity. In contrast to BM, FBM is neither a Markov process nor a martingale, and its velocity is correlated. Based on a recent study of self-similar Ito diffusions, we explore an alternative self-similar model for anomalous diffusion: fractional Ito motion (FIM). The FIM model exhibits the same Hurst-exponent behavior as FBM, and it is also a symmetric process with a continuous trajectory. In sharp contrast to FBM, we show that FIM: is not a Gaussian process; is a Markov process; is a martingale; and its velocity is not stationary and is not correlated. On the one hand, FBM is hard to simulate, its analytic tractability is limited, and it generates only a Gaussian dissipation pattern. On the other hand, FIM is easy to simulate, it is analytically tractable, and it generates non-Gaussian dissipation patterns. Moreover, we show that FIM has an intimate linkage to diffusion in a logarithmic potential. With its compelling properties, FIM offers researchers and practitioners a highly workable analytic model for anomalous diffusion.

Keywords: sub-diffusion, super-diffusion, self-similarity, Hurst exponent, non-Gaussian diffusion, diffusion in a logarithmic potential

(Some figures may appear in colour only in the online journal)
1. Introduction

A recent study [1] characterized the intersection of two principal classes of one-dimensional random motions: the class of selfsimilar processes, and the class of Ito diffusion processes. The former class manifests random motions whose trajectories are, statistically, fractal objects [2]. The latter class manifests random motions whose dynamics are governed by Ito’s stochastic differential equation (SDE) [3–5]. Both these classes are of major importance, theoretical and practical alike. The intersection of these classes is also of major importance, due to the following fact [6]: selfsimilar Ito diffusions emerge universally when transcending from microscopic to macroscopic spatio-temporal scales. In this paper, progressing from the theoretical studies [1, 6] to statistical-physics applications, we explore a practical model for anomalous diffusion.

Regular and anomalous diffusion assume central roles in science and engineering [7–13]. The paradigmatic model for regular diffusion is Brownian motion (BM) [14, 15], which is a mathematical object of exquisite beauty and riches [16]. Maintaining key properties of BM—namely, being a selfsimilar process with finite variance and with a continuous trajectory—the paradigmatic model for anomalous diffusion is fractional Brownian motion (FBM) [17–29]. On the flip side, FBM fails to maintain other key properties of BM; specifically, FBM is neither a Markov process, nor a martingale [2]. Consequently, both the numerical simulation and the analytic tracking of FBM are challenging tasks.

In analogy with FBM, we term this paper’s anomalous-diffusion model fractional Ito motion (FIM). As FBM, FIM also maintains the aforementioned key properties of BM: it is a selfsimilar process with finite variance and with a continuous trajectory. In sharp contrast to FBM, FIM is a Markov process, as well as a martingale. Consequently, FIM is a highly useful and a highly workable model for anomalous diffusion.

A Stratonovich counterpart of FIM—in which the underpinning Ito integration is replaced by Stratonovich integration—was explored in [30]. A special case of FIM was investigated in [31], and, most recently, FIM was investigated in the context of stochastic resetting [32]. In this paper, taking on a selfsimilarity approach, we conduct a detailed comparison between two anomalous diffusion models: FBM vs FIM.

The paper is organized as follows. We begin with an overview of regular and anomalous diffusion, and with a concise description of the three aforementioned models (section 2): BM, FBM, and FIM. We then present the detailed comparison between FBM and FIM (section 3). Thereafter, we present a ‘diffusion-in-a-logarithmic-potential’ representation of FIM (section 4), and conclude with a summary (section 5).

A note about notation. Along this paper E[·] denotes the operation of statistical expectation (in the context of random variables), and ≈ denotes asymptotic equality (in the context of real-valued functions).

2. Regular and anomalous diffusion

Diffusion is a generic name for random motions that diffuse with time. Arguably, the most common method to measure diffusivity is mean square displacement (MSD). To describe this method, consider a one-dimensional random motion whose trajectory is $X(t)$ ($t \geq 0$). Namely, the motion initiates at time 0, and its position at time $t$ is $X(t)$ (a point on the real line). Hence, at time $t$, the motion’s displacement—relative to its initial position—is $|X(t) - X(0)|$. In turn, the motion’s MSD at time $t$ is $\phi(t) = E[(X(t) - X(0))^2]$. Rather generally, the random motion under consideration can be said to be a diffusion if: its MSD function $\phi(t)$ increases with time, and diverges in the temporal limit $t \to \infty$. 

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Often, diffusions take place at microscopic spatio-temporal scales, but yet they are observed at macroscopic spatio-temporal scales. As explained in the methods, on the macroscopic level the only admissible form of the MSD function is a power-law:

$$\phi(t) = c \cdot t^\epsilon,$$

where $c$ is a positive coefficient, and where $\epsilon$ is a positive exponent. A broad and rich framework for random motions that yield such a power-law MSD form are self-similar processes with finite variance [2]. To describe the notion of self-similarity consider, as above, a one-dimensional random motion whose trajectory is $X(t)$ ($t \geq 0$).

The random motion under consideration is selfsimilar if—for any given positive scale $s$—its two following versions are statistically equal [2]: the time-scaled version $X(st)$ ($t \geq 0$); and the space-scaled version $s^H X(t)$ ($t \geq 0$), where $H$ is a positive Hurst exponent. Namely, with regard to the random motion under consideration, selfsimilarity means that: changing the underlying temporal scale from 1 to $s$ is statistically equivalent to changing the underlying spatial scale from 1 to $s^H$. Selfsimilarity implies that the motion initiates at the spatial origin $X(0) = 0$, and that the random variable $X(t)$ is equal in law to the random variable $t^H \cdot X(1)$. Consequently—in the finite-variance case—selfsimilarity yields the power-law MSD form of equation (1) with: coefficient $c = \mathbb{E}[|X(1)|^2]$, and exponent $\epsilon = 2H$.

The above discussion is mathematically neat, yet it is merely theoretical. Does the real world corroborate the theory? The answer is a resounding yes: real-world observations happen to be in perfect accord with the theory. Indeed, pioneered by the experimental observations of Jan Ingen-Housz [33], of Brown [34], and of Perrin [35], a vast body of empirical evidence established the ubiquity of diffusions with linear MSD functions, $\phi(t) = c \cdot t$. Such diffusions are so prevalent that they were assumed to manifest diffusive motions at large. In turn, the convention was to quantify a given diffusive motion by its ‘diffusion coefficient’—the slope of its linear MSD function.

In the 1970s new experimental observations began to challenge the then commonplace assumption that diffusive motions have linear MSD functions [36–40]. As the new body of empirical evidence grew larger and larger, it became clear that in addition to the well-known diffusive motions with linear MSD functions, there are also diffusive motions with power-law MSD functions. Consequently, diffusions were distinguished by three different categories: regular diffusion, characterized by linear MSD functions; sub-diffusion, characterized by power-law MSD functions with sub-linear exponents $\epsilon < 1$; and super-diffusion, characterized by power-law MSD functions with super-linear exponents $\epsilon > 1$.

The experimental discovery of sub-diffusive and super-diffusive random motions ushered in the multidisciplinary scientific field of anomalous diffusion [7–13]. This field attracted substantial scientific interest [41–51], and the scientific exploration of the field led to the conclusions that ‘anomalous is normal’ [52] and that ‘anomalous is ubiquitous’ [53].

Over forty years after its inception, anomalous diffusion continues to be a vibrant multidisciplinary scientific field. Examples of recent anomalous-diffusion research include: anomalous diffusion in complex media [54]; anomalous diffusion in the evolution of inhomogeneous systems [55]; anomalous diffusion in random dynamical systems [56]; anomalous diffusion and recurrent neural networks [57]; heterogeneous diffusion processes [58]; anomalous diffusion in heterogeneous binary media [59]; anomalous diffusion and time-dependent diffusivity [60]; anomalous diffusion in the noisy voter model [61, 62]; anomalous diffusion under stochastic resetting [63, 64]; anomalous diffusion in comb structures [64–67]; and random diffusivity [26, 68–70].
We now turn to describe the three random-motion models that were noted in the introduction. We begin with the paradigmatic models of BM and FBM. Thereafter, we present the model of FIM.

2.1. Brownian motion and fractional Brownian motion

Following the trailblazing theoretical works of Bachelier in finance [71, 72], of Einstein and Smoluchowski in physics [73, 74], and of Wiener in mathematics [75], a particular random motion emerged as the paradigmatic model for regular diffusion in science and engineering [14, 15]: BM, $B(t)$ ($t \geq 0$). Named in honor of Sir Robert Brown, and also termed Wiener process in honor of Norbert Wiener, BM is a profound object that exhibits a host of amicable mathematical, statistical, and geometric properties [16].

In this paper we shall address the following key BM properties. (I) BM has finite variance, i.e. its positions have finite first-order and second-order moments; hence BM has a well-defined MSD function. (II) BM is a symmetric process, i.e. its trajectory is statistically identical to its mirror trajectory, $-B(t)$ ($t \geq 0$). (III) The trajectory of BM is continuous. (IV) BM is a selfsimilar process with Hurst exponent $H = \frac{1}{2}$; hence the MSD function of BM is linear, $\epsilon = 1$, and hence it is a regular diffusion indeed. (V) BM is a Gaussian process, i.e. its finite-dimensional distributions are multivariate normal. (VI) BM is a Levy process, i.e. its increments are stationary and its non-overlapping increments are independent. (VII) BM is a Markov process, i.e.—at any given time point $t$—its future trajectory depends only at its present position, $B(t)$, and it does not depend on its past trajectory. (VIII) BM is a martingale.

The properties of BM imply that its positions, as well as its increments, are normal random variables with zero means. The inherent scale of BM is set so that the random variable $B(1)$—the position of BM at the time point 1—is ‘standard normal’, i.e.: $B(1)$ is a normal random variable with zero mean and with unit variance.

In order to model anomalous diffusion one needs to go beyond BM. To do so—within the realm of symmetric and selfsimilar processes with finite variance and continuous trajectories—leads to the following model: FBM, $B_H(t)$ ($t \geq 0$), where the subscript $H$ manifests the underlying Hurst exponent. Pioneered by Kolmogorov [76], by Yaglom [77], and by Mandelbrot and Van Ness [78], FBM is a well established generalization of BM [79–82]. The Hurst exponent of FBM takes values in the range $0 < H < 1$, and hence the diffusivity of FBM is as follows: sub-diffusion in the exponent range $0 < H < \frac{1}{2}$; super-diffusion in the exponent range $\frac{1}{2} < H < 1$; and regular diffusion at the exponent value $H = \frac{1}{2}$—in which case FBM is BM.

One the one hand—as BM—FBM is a Gaussian process, its trajectory is continuous, and its increments are stationary [2]. On the other hand—in sharp contrast to BM—the non-overlapping increments of FBM are dependent, and FBM is neither a Markov process nor is it a martingale [2]. Moreover, FBM is not even a semi-martingale, and this fact has major implications in the context of stochastic integration [2].

To gain insight regarding the difference between BM and FBM, we turn to their velocities. The velocity of BM, $\dot{B}(t)$ ($t \geq 0$), is commonly known as white noise. The velocity of FBM, $\dot{B}_H(t)$ ($t \geq 0$), is a moving average—with a power-law kernel—of white noise. Specifically, the velocity of FBM with Hurst exponent $H \neq \frac{1}{2}$ admits the following moving-average representation [83]:

$$
\dot{B}_H(t) = \left( H - \frac{1}{2} \right) \int_{-\infty}^t (t-u)^{H-\frac{1}{2}} \dot{B}(u) \, du.
$$

(2)
The moving-average representation of equation (2) actually uses two independent white noises: one defined over the negative time axis, \( \dot{B}(u) \) \((u < 0)\); and one defined over the non-negative time axis, \( \dot{B}(u) \) \((u \geq 0)\).

As FBM is a self-similar process, it initiates at the spatial origin \( BH(0) = 0 \). Hence, integrating equation (2) yields

\[
BH(t) = \int_{-\infty}^{0} \left[ (t-u)^{H-\frac{1}{2}} - (0-u)^{H-\frac{1}{2}} \right] \dot{B}(u) du + \int_{0}^{t} (t-u)^{H-\frac{1}{2}} \dot{B}(u) du. \tag{3}
\]

The integral representation of equation (3) comprises two parts: an integral of the white noise \( \dot{B}(u) \) over the negative temporal axis, \(-\infty < u < 0\); and an integral of the white noise \( \dot{B}(u) \) over the temporal interval \( 0 \leq u \leq t \). The integral representation of equation (3) covers also the Hurst exponent \( H = \frac{1}{2} \); and setting this Hurst exponent in equation (3) yields BM: \( BH(1/2) = B(t) \). Hence, FBM is indeed a generalization of BM. Schematic illustrations of the integration kernel of equation (3)—whose shape is determined by the value of the Hurst exponent \( H \)—are depicted in figure 1.

As noted above, FBM is not a Markov process. There are one-dimensional random motions that are non-Markov, but yet they can be transformed to a Markov-process representation by elevating from a single dimension to several dimensions. FBM is a highly non-Markov process in the following sense: in order to transform FBM to a Markov-process representation one has to elevate from a single dimension to infinitely many dimensions [83]. Consequently, the numerical simulation of FBM is a challenging computational task [84–93].

So, on the one hand, FBM is a random motion model—which is self-similar and Gaussian, whose positions have a finite variance, and whose trajectory is continuous—that produces both sub-diffusion and super-diffusion. On the other hand, working with FBM is quite challenging from the perspectives of stochastic integration and numerical simulation, as well as from the perspective of analytic tractability. The intricacy of FBM gives rise to the following question: is there an alternative anomalous-diffusion model that shares the ‘upside’ features of FBM, but not the ‘downside’ features of FBM? The answer, as we shall now argue, is affirmative indeed.

### 2.2. Fractional Ito motion

Consider a one-dimensional deterministic motion \( X(t) \) \((t \geq 0)\), whose dynamics are governed by the ordinary differential equation (ODE) \( \dot{X}(t) = \mu [X(t)] \). Namely, when the deterministic motion is at the position \( x \) (a point on the real line) then its velocity is \( \mu (x) \) (a real number).

Adding white noise to the dynamics changes the motion from deterministic to random, and changes the ODE to a Langevin SDE [94–96]: \( \dot{X}(t) = \mu [X(t)] + \nu \dot{B}(t) \), where \( \nu \) is a positive parameter that manifest the white-noise magnitude. In turn, replacing the constant white-noise magnitude \( \nu \) by a position-dependent magnitude changes the Langevin SDE to an Ito SDE [3–5, 97, 98]: \( \dot{X}(t) = \mu [X(t)] + \sigma [X(t)] \dot{B}(t) \). In the jargon of mathematical finance [99]: when the random motion is at the position \( x \) (a point on the real line) then its ‘drift’ is \( \mu (x) \) (a real number), and its ‘volatility’ is \( \sigma (x) \) (a positive number).

With the notion of Ito SDEs recalled, we introduce the following model: FIM, \( IH(t) \) \((t \geq 0)\), where the subscript \( H \) manifests the underlying Hurst exponent. In analogy with FBM, we name FIM in honor of Kioshi Ito—the mathematician that invented the white-noise stochastic calculus. The dynamics of FIM are governed by the Ito SDE

\[
\dot{IH}(t) = |IH(t)|^{1+\frac{H}{2}} \dot{B}(t). \tag{4}
\]
Figure 1. Schematic illustrations of the shapes of the FBM integration kernel, and of the FIM volatility landscape. Depicted in the left panels, the FBM integration kernel (for a fixed positive time point $t$) is: $\kappa(t; u) = (t - u)^{H - \frac{1}{2}} - (0 - u)^{H - \frac{1}{2}}$ over the negative temporal axis, $-\infty < u < 0$; and $\kappa(t; u) = (t - u)^{H - \frac{1}{2}}$ over the temporal interval $0 \leq u \leq t$. Depicted in the right panels, the FIM volatility landscape is: $\sigma(x) = |x|^{1 - \frac{1}{2H}}$ over the spatial axis $-\infty < x < \infty$. The temporal kernel function $\kappa(t; u)$ and the spatial volatility function $\sigma(x)$ display markedly different shapes in the three different diffusivity categories: sub-diffusion $0 < H < \frac{1}{2}$ (top panels); super-diffusion $\frac{1}{2} < H < 1$ (bottom panels); and regular diffusion $H = \frac{1}{2}$ (middle panels).
Namely, FIM has a zero drift, \( \mu(x) = 0 \), and a power-law volatility, \( \sigma(x) = |x|^{1-H} \). In general, a random motion whose dynamics are governed by an Ito SDE is a Markov process, and its trajectory is continuous [16]; hence, in particular, FIM exhibits these properties.

We initiate FIM from the spatial origin \( I_H(0) = 0 \). Hence, integrating equation (4) yields

\[
I_H(t) = \int_0^t |I_H(u)|^{1-H} B(u) du.
\]

The right-hand side of equation (5) is a running Ito integral. A general running Ito integral—with an integrand that does not ‘look into the future’—is a symmetric process and a martingale [16]; hence, in particular, FIM exhibits these properties.

According to general results regarding selfsimilar Ito diffusions [1], FIM is a selfsimilar process, and the Hurst exponent of FIM takes values in the range \( 0 < H < 1 \). Hence, the diffusivity of FIM is identical to the aforementioned diffusivity of FBM: sub-diffusion in the exponent range \( 0 < H < \frac{1}{2} \); super-diffusion in the exponent range \( \frac{1}{2} < H < 1 \); and regular diffusion at the exponent value \( H = \frac{1}{2} \)—in which case FIM is BM. Indeed, setting \( H = \frac{1}{2} \) in equation (5) yields BM: \( I_{1/2}(t) = B(t) \). Schematic illustrations of the FIM ‘volatility landscape’—whose shape is determined by the value of the Hurst exponent \( H \)—are depicted in figure 1.

So, on the one hand, FIM shares the ‘upside’ features of FBM: it is a random-motion model—which is symmetric and selfsimilar, and whose trajectory is continuous—that generalizes BM, and that produces both sub-diffusion and super-diffusion. On the other hand, as FIM is a Markov process and a martingale, it circumvents the ‘downside’ features of FBM: it well applies in the context of stochastic integration, its numerical simulation is easy and straightforward, and it is analytically tractable. Moreover, according to scaling-limit results regarding general SDEs [6]: selfsimilar Ito diffusions—hence FIM in particular—emerge universally when transcending from microscopic to macroscopic spatio-temporal scales.

Replacing the Ito SDE (4) by an identical Stratonovich SDE results in a ‘fractional Stratonovich motion’ counterpart of FIM; this counterpart of the FIM model was explored in [30]. A special case of the FIM model was investigated in [31]; in this special case the motion runs over the positive half-line \( 0 < x < \infty \), and the Hurst exponent is in the range \( \frac{1}{2} \leq H < 1 \). Most recently, FIM was also investigated in the context of stochastic resetting [32]. To the best of our knowledge, the comprehensive and in-depth comparison between the FBM model and the FIM model—which is the core this paper, and which will be presented in the next section—is entirely new.

3. Fractional Brownian motion vs fractional Ito motion

Describing the FBM model and the FIM model, the previous section revealed some of the marked differences between these two anomalous-diffusion models. In this section we carry on with a series of analytic comparisons that examine and pinpoint additional profound differences between FBM and FIM. Visual comparisons between simulated trajectories of FBM and FIM are offered by figure 2 (for sub-diffusion) and by figure 3 (for super-diffusion).

3.1. Dissipation comparison

As FBM and FIM are selfsimilar processes, they both initiate from the spatial origin. Hence, from a probabilistic perspective, at time 0 both these random motions manifest a unit mass that is placed at the origin. In turn, at a positive time \( t \), this unit mass dissipates, and the ‘shape of the dissipation’ is quantified by a probability density function: the density of the random
Figure 2. Simulated trajectories of sub-diffusive FBM (left panels) and of sub-diffusive FIM (right panels).

The properties of FBM imply that the random variable $B_H(t)$ is normal with mean zero and with variance $\text{Var}[B_H(t)] = \text{Var}[B_H(1)] \cdot t^{2H}$. Hence, setting $b = \text{Var}[B_H(1)]$, the density of the random variable $B_H(t)$ is

$$
\frac{1}{\sqrt{2\pi b}} \cdot \frac{1}{t^H} \exp \left( -\frac{x^2}{2bt^{2H}} \right)
$$

($-\infty < x < \infty$). This density is a symmetric ‘bell curve’ (see figure 4): it vanishes at $x \to \pm \infty$, and it has a unimodal shape at the spatial origin $x = 0$. We emphasize that the shape of this density is the same for all the values of the Hurst exponent $H$.

It follows from a general result regarding selfsimilar Ito diffusions [1] that the density of the random variable $I_H(t)$ is

$$
\frac{1}{2H\Gamma(1-H)} \cdot \left( \frac{2H^2}{t} \right)^{1-H} \exp \left( -\frac{2H^2}{t} |x|^H \right) |x|^H t^{-2}
$$

($-\infty < x < \infty$). This density is symmetric, and it vanishes at $x \to \pm \infty$. The shape of this density is determined by the value of the Hurst exponent $H$, as follows (see figure 4).

- In the sub-diffusion range, $0 < H < \frac{1}{2}$, the density has a bimodal shape: it vanishes at the spatial origin $x = 0$, and it peaks at the spatial points $x = \pm \left( \frac{1-2H}{2H^2} \right)^{1/H}$; at these points the density’s peak height is $c_H/t^H$, where $c_H$ is a constant that depends on the Hurst exponent $H$. 
Figure 3. Simulated trajectories of super-diffusive FBM (left panels) and of super-diffusive FIM (right panels).

- At the regular-diffusion value, $H = \frac{1}{2}$, the density is a ‘bell curve’: it has a unimodal shape that peaks at the spatial origin $x = 0$.
- In the super-diffusion range, $\frac{1}{2} < H < 1$, the density has a unimodal shape that explodes at the spatial origin $x = 0$.

Evidently, the differences between the shape of the Gaussian FBM density of equation (6) and the shape of the non-Gaussian (for $H \neq \frac{1}{2}$) FIM density of equation (7) are dramatic (figure 4, as well as figures 5 and 6). On the one hand, changing the Hurst exponent $H$ in the FBM model has no qualitative effect on the shape of the dissipation pattern. On the other hand, changing the Hurst exponent $H$ in the FIM model has a profound qualitative effect on the shape of the dissipation pattern.

The dissipation pattern of ‘fractional Stratonovich motion’—the Stratonovich counterpart of FIM—displays a behavior which is similar to that of the FIM density of equation (7) [30]. Non-Gaussian diffusions (i.e. diffusions with non-Gaussian dissipation patterns), such as FIM and its Stratonovich counterpart, attracted substantial scientific interest recently [100–108].

3.2. Tail-behavior comparison

Consider a symmetric one-dimensional random motion whose trajectory is $X(t) (t \geq 0)$. The tail behavior of the random motion under consideration is the asymptotic behavior of the probability $\Pr [|X(t)| > l]$ in the limit $l \to \infty$. Namely, the tail behavior quantifies the asymptotic likelihood of the following rare event: the motion’s displacement, at time $t$, relative to its initial position, is greater than the level $l \gg 1$.

In this subsection we conduct three tail-behavior comparisons: FBM vs BM; FIM vs BM; and FIM vs FBM. These tail-behavior comparisons are presented in table 1. The calculations of
Figure 4. Schematic illustrations of the shapes of the FBM and FIM dissipation patterns (for a fixed positive time point \( t \)). Depicted in the left panels, the FBM dissipation pattern is the probability density function of the random variable \( B_H(t) \) (equation (6)). Depicted in the right panels, the FIM dissipation pattern is the probability density function of the random variable \( I_H(t) \) (equation (7)). While the FBM dissipation pattern displays the same unimodal ‘bell-curve’ shape, the FIM dissipation pattern displays markedly different shapes in the three different diffusivity categories: bimodal for sub-diffusion \( 0 < H < \frac{1}{2} \) (top panels); unimodal and explosive for super-diffusion \( \frac{1}{2} < H < 1 \) (bottom panels); and unimodal ‘bell-curve’ for regular diffusion \( H = \frac{1}{2} \) (middle panels).
Figure 5. ‘Heat maps’ for three examples of the FBM dissipation pattern—the probability density function of the random variable $B_H(t)$ (equation (6)). The examples correspond to the three different diffusivity categories of FBM: sub-diffusion (top panel); regular diffusion (middle panel); super diffusion (bottom panel).

The limits appearing in table 1 are detailed in the methods. These calculations use L’Hospital’s rule and the probability density functions of equations (6) and (7), and they reveal the asymptotics that lead to the limits. In turn, the limits yield the following ‘asymptotic orderings’. In the case of sub-diffusion ($0 < H < \frac{1}{2}$): the tail behavior of FIM is infinitely ‘lighter’ than the
Figure 6. ‘Heat maps’ for three examples of the FIM dissipation pattern—the probability density function of the random variable $I_H(t)$ (equation (7)). The examples correspond to the three different diffusivity categories of FIM: sub-diffusion (top panel); regular diffusion (middle panel); super diffusion (bottom panel).

tail behavior of FBM—which, in turn, is infinitely ‘lighter’ than the tail behavior of BM. In the case of super-diffusion ($\frac{1}{2} < H < 1$): the tail behavior of FIM is infinitely ‘heavier’ than the tail behavior of FBM—which, in turn, is infinitely ‘heavier’ than the tail behavior of BM.
Table 1. Tail-behavior comparisons. The comparisons are specified with regard to the two anomalous-diffusion categories: sub-diffusion, \( (0 < H < 1/2) \); and super-diffusion \( (1/2 < H < 1) \). The result regarding the FBM vs BM comparison holds in the temporal ray \( t > b^{1/(1-2H)} \). For sub-diffusion: the tail behavior of FIM is infinitely ‘lighter’ than the tail behavior of FBM—which, in turn, is infinitely ‘lighter’ than the tail behavior of BM. Conversely, for super-diffusion: the tail behavior of FIM is infinitely ‘heavier’ than the tail behavior of FBM—which, in turn, is infinitely ‘heavier’ than the tail behavior of BM.

| Comparison | Limit | Sub diffusion \( (0 < H < 1/2) \) | Super diffusion \( (1/2 < H < 1) \) |
|------------|-------|-----------------------------------|-----------------------------------|
| FBM vs BM  | \( \lim_{t \to \infty} \frac{\Pr [|B_H(t)| > l]}{\Pr [|B(t)| > l]} \) | 0 \( \infty \) | |
| FIM vs BM  | \( \lim_{t \to \infty} \frac{\Pr [|J_H(t)| > l]}{\Pr [|B(t)| > l]} \) | 0 \( \infty \) | |
| FIM vs FBM | \( \lim_{t \to \infty} \frac{\Pr [|I_H(t)| > l]}{\Pr [|B_H(t)| > l]} \) | 0 \( \infty \) | |

3.3. Inequality comparison

Inequality indices are quantitative gauges that—using a socioeconomic-inequality perspective—score the statistical heterogeneity of non-negative random variables with positive means [109–111]. We shall now use a particular inequality index in order to compare the inherent statistical heterogeneity of FBM to the inherent statistical heterogeneity of FIM.

Inequality indices are widely applied in economics and in the social sciences to measure wealth inequalities in human societies [112–115]. An inequality index takes values in the unit interval, and with regard to a human society under consideration: the lower the inequality-index score—the more egalitarian the distribution of wealth among the society members; and the higher the inequality-index score—the less egalitarian the distribution of wealth among the society members. In particular, the inequality index yields a zero score only when the society under consideration is communist—in which case all the society members share a common (positive) wealth value. Inequality indices are invariant with respect to the specific currency via which wealth is measured (e.g. Dollar or Euro).

Given an inequality index \( \mathcal{I} \), and given a non-negative random variable \( W \) with a positive mean, the statistical heterogeneity of the random variable can be measured by the inequality index as follows [111]: deem the random variable \( W \) to manifest the personal wealth value of a member that is sampled at random from a virtual human society; then, set the statistical-heterogeneity score of the random variable \( W \) to be the inequality-index score of the virtual human society. In what follows we denote by \( \mathcal{I}(W) \) the statistical-heterogeneity score that the inequality index \( \mathcal{I} \) assigns to the random variable \( W \).

As \( \mathcal{I} \) is an inequality index, the score \( \mathcal{I}(W) \) exhibits the three following properties. (I) It takes values in the unit interval, \( 0 \leq \mathcal{I}(W) \leq 1 \). (II) It vanishes if and only if the random variable is deterministic: \( \mathcal{I}(W) = 0 \iff W = \text{const} \) (with probability one). (III) It is invariant with respect to changes of scale of the random variable: \( \mathcal{I}(s \cdot W) = \mathcal{I}(W) \), where \( s \) is any positive scale.

The particular inequality index \( \mathcal{I} \) that we shall use is based on the first-order and second-order moments of the random variable \( W \), and its statistical-heterogeneity score is [111, 116]: \( \mathcal{I}(W) = 1 - \mathbb{E}[W]^2 / \mathbb{E}[W^2] \). This statistical-heterogeneity score is unique in the following...
sense: it is the only score that has a one-to-one correspondence with the coefficient of variation (CV) of the random variable $W$, i.e. the ratio of the random-variable’s standard deviation to the random-variable’s mean. In short, we henceforth term this statistical-heterogeneity score ‘CV score’.

Consider a selfsimilar one-dimensional random motion whose trajectory is $X(t) (t \geq 0)$, and whose Hurst exponent is $H$. We set the focus on the random variable $|X(t)|$—the motion’s displacement, at time $t$, relative to its initial position. The motion’s selfsimilarity implies that the random variable $|X(t)|$ is equal in law to the random variable $t^H \cdot |X(1)|$. In turn, assuming that the random variable $|X(1)|$ has a positive mean, the scaling property of inequality indices implies that: $\mathcal{I}(|X(t)|) = \mathcal{I}(|X(1)|)$. Hence, the quantity $\mathcal{I}(|X(1)|)$ is, in effect, a statistical-heterogeneity score of the selfsimilar random motion under consideration.

In the case of FBM the random variable $B_H(1)$ is a normal random variable with zero mean. In turn, a probabilistic calculation that is detailed in the methods asserts that the CV score of FBM is

$$\mathcal{I}[|B_H(1)|] = 1 - \frac{2}{\pi}. \text{ (8)}$$

This CV score does not depend on the value of the Hurst exponent $H$.

In the case of FIM the statistical distribution of the random variable $I_H(1)$ is governed by the density function of equation (7). In turn, a probabilistic calculation that is detailed in the methods asserts that the CV score of FIM is

$$\mathcal{I}[|I_H(1)|] = 1 - \frac{\sin(\pi H)}{\pi H}. \text{ (9)}$$

This CV score depends on the value of the Hurst exponent $H$, and as a function of the Hurst exponent $0 < H < 1$ (see figure 7): it increases from the zero lower-bound score $\lim_{H \to 0} \mathcal{I}[|I_H(1)|] = 0$ to the unit upper-bounds score $\lim_{H \to 1} \mathcal{I}[|I_H(1)|] = 1$.

Evidently, the difference between the CV score of equation (8) and the CV score of equation (9) is dramatic. On the one hand, changing the Hurst exponent $H$ in the FBM model has no effect on the inherent statistical heterogeneity. On the other hand, changing the Hurst exponent $H$ in the FIM model has a profound effect on the inherent statistical heterogeneity—which spans the full unit-interval range of statistical-heterogeneity scores.

### 3.4. Divergence comparison

Consider a selfsimilar one-dimensional random motion whose trajectory is $X(t) (t \geq 0)$, and whose Hurst exponent is $H$. As noted above, the motion’s selfsimilarity implies that the random variable $X(t)$ is equal in law to the random variable $t^H \cdot X(1)$. Hence, the random variable $X(1)$—the motion’s position at the time point 1—is a ‘benchmark’ for the motion’s positions at all times.

In the previous subsection we focused on quantifying the inherent statistical heterogeneity of the random variable $|X(t)|$—the motion’s displacement, at time $t$, relative to its initial position. In this subsection we shall focus on quantifying the statistical divergence of the random variable $X(t)$ from its benchmark—the random variable $X(1)$. To that end we use the Kulback–Leibler (KL) divergence [117, 118], which is arguably the most widely applied measure of statistical divergence. Specifically, the KL divergence of the random variable $X(t)$ from its benchmark $X(1)$ is: $D[X(t) \| X(1)] = \int_{-\infty}^{\infty} \ln \left[ \frac{f_t(x)}{f_1(x)} \right] f_t(x) \, dx$, where $f_t(x)$ is the density of the random variable $X(t)$ (and, in particular, $f_1(x)$ is the density of the benchmark $X(1)$).
Figure 7. The FIM CV score of equation (9), as a function of the Hurst exponent $0 < H < 1$. Varying the value of the Hurst exponent, the full unit-interval range of the CV score is attained.

With regard to FBM, a probabilistic calculation that is detailed in the methods asserts that the KL divergence of the position $B_H(t)$ from the benchmark position $B_H(1)$ is

$$D[B_H(t) \| B_H(1)] = \frac{1}{2} (2^H - 1) - H \ln(t).$$  (10)

The asymptotic behavior of this KL divergence, in the temporal limit $t \to \infty$, is $D[B_H(t) \| B_H(1)] \approx \frac{1}{2} 2^H$.

With regard to FIM, a probabilistic calculation that is detailed in the methods asserts that the KL divergence of the position $I_H(t)$ from the benchmark position $I_H(1)$ is

$$D[I_H(t) \| I_H(1)] = (1 - H)(t - 1 - \ln(t)).$$  (11)

The asymptotic behavior of this KL divergence, in the temporal limit $t \to \infty$, is $D[I_H(t) \| I_H(1)] \approx (1 - H) t$.

As a function of the temporal variable $t$, the KL divergences of equations (10) and (11) share, qualitatively, a common U shape with: a unique global minimum, whose value is zero, that is attained at the time point 1. However, quantitatively, these KL divergences yield dramatically different asymptotic behaviors. Using these asymptotic behaviors, we conduct three divergence comparisons: FBM vs BM; FIM vs BM; and FIM vs FBM. These divergence comparisons are presented in table 2, and they yield the following 'asymptotic orderings'. In the case of sub-diffusion ($0 < H < \frac{1}{2}$): the FIM asymptotics are equivalent to the BM asymptotics—which, in turn, are infinitely 'lighter' than the FBM asymptotics. In the case of super-diffusion ($\frac{1}{2} < H < 1$):
Consider a symmetric one-dimensional random motion whose trajectory is \( X(t) \), where \( t \geq 0 \), and whose positions have a finite variance. The motion’s displacement over the temporal interval \([t, t + \Delta]\)—where \( \Delta \) is the interval’s positive length—is \( |X(t + \Delta) - X(t)| \). In turn, the motion’s MSD over the temporal interval \([t, t + \Delta]\) is \( E[|X(t + \Delta) - X(t)|^2] \). And, as the motion is symmetric, this MSD is equal to \( \text{Var} [X(t + \Delta) - X(t)] \)—the variance of the increment \( X(t + \Delta) - X(t) \) in \([t, t + \Delta] \).

In this subsection we shall compare the variance of the FBM increment \( B_H(t + \Delta) - B_H(t) \) to the variance of the FIM increment \( I_H(t + \Delta) - I_H(t) \).

As FBM initiates at the spatial origin, and as the increments of FBM are stationary, the FBM increment \( B_H(t + \Delta) - B_H(t) \) is equal in law to the random variable \( B_H(\Delta) \). In turn, as FBM is a selfsimilar process with Hurst exponent \( H \), the random variable \( B_H(\Delta) \) is equal in law to the random variable \( \Delta^H \cdot B_H(1) \). Consequently, the FBM increment \( B_H(t + \Delta) - B_H(t) \) is a normal random variable with mean zero and with variance

\[
\text{Var} [B_H(t + \Delta) - B_H(t)] = \text{Var} [B_H(1)] \cdot \Delta^{2H}.
\]  

Due to the fact that the increments of FBM are stationary, the variance of equation (12) depends only on the length \( \Delta \) of the temporal interval \([t, t + \Delta]\).

Using the fact that FIM is a selfsimilar process with Hurst exponent \( H \), as well as the fact that FIM is a martingale, it is shown in the methods that the FIM increment \( I_H(t + \Delta) - I_H(t) \) is a random variable with mean zero and with variance

\[
\text{Var} [I_H(t + \Delta) - I_H(t)] = \text{Var} [I_H(1)] \cdot [(t + \Delta)^{2H} - t^{2H}].
\]  

The variance of equation (13) depends on the starting point \( t \), as well as on the length \( \Delta \), of the temporal interval \([t, t + \Delta]\). Hence, equation (13) implies that the increments of FIM are not stationary.

The asymptotic behavior of the variance of equation (13), in the temporal limit \( t \to \infty \), is \( \text{Var} [I_H(t + \Delta) - I_H(t)] \approx v \cdot t^{2H-1} \), where \( v = 2\Delta H \text{Var} [I_H(1)] \) (see the methods for the details of this asymptotic equality). Consequently, for sub-diffusion \( 0 < H < \frac{1}{2} \), equation (13) implies that: \( \lim_{t \to \infty} \text{Var} [I_H(t + \Delta) - I_H(t)] = 0 \); namely, the variance of the FIM increment

\[
\text{Var} [I_H(t + \Delta) - I_H(t)] = 0.
\]

### Table 2. Asymptotic KL-divergence comparisons. The comparisons are specified with regard to the two anomalous-diffusion categories: sub-diffusion \((0 < H < \frac{1}{2})\); and super-diffusion \((\frac{1}{2} < H < 1)\). For sub-diffusion: the FIM asymptotics—which are equivalent to the BM asymptotics—are infinitely ‘lighter’ than the FBM asymptotics. Conversely, for super-diffusion: the FIM asymptotics—which are equivalent to the BM asymptotics—are infinitely ‘heavier’ than the FBM asymptotics.

| Comparison | Limit | Sub diffusion \((0 < H < \frac{1}{2})\) | Super diffusion \((\frac{1}{2} < H < 1)\) |
|------------|-------|----------------------------------|----------------------------------|
| FBM vs BM  | \( t\to\infty \) \( \mathcal{D} [B_H(t) || B_H(1)] \) | 0 | \( \infty \) |
| FIM vs BM  | \( t\to\infty \) \( \mathcal{D} [I_H(t) || I_H(1)] \) | 2(1 - H) | 2(1 - H) |
| FIM vs FBM | \( t\to\infty \) \( \mathcal{D} [B_H(t) || B_H(1)] \) | 0 | \( \infty \) |

1): the FIM asymptotics are equivalent to the BM asymptotics—which, in turn, are infinitely ‘heavier’ than the FBM asymptotics.

### 3.5. Aging comparison

Consider a symmetric one-dimensional random motion whose trajectory is \( X(t) \), and whose positions have a finite variance. The motion’s displacement over the temporal interval \([t, t + \Delta]\)—where \( \Delta \) is the interval’s positive length—is \( |X(t + \Delta) - X(t)| \). In turn, the motion’s MSD over the temporal interval \([t, t + \Delta]\) is \( E[|X(t + \Delta) - X(t)|^2] \). And, as the motion is symmetric, this MSD is equal to \( \text{Var} [X(t + \Delta) - X(t)] \)—the variance of the increment \( X(t + \Delta) - X(t) \) in \([t, t + \Delta] \).

As FBM initiates at the spatial origin, and as the increments of FBM are stationary, the FBM increment \( B_H(t + \Delta) - B_H(t) \) is equal in law to the random variable \( B_H(\Delta) \). In turn, as FBM is a selfsimilar process with Hurst exponent \( H \), the random variable \( B_H(\Delta) \) is equal in law to the random variable \( \Delta^H \cdot B_H(1) \). Consequently, the FBM increment \( B_H(t + \Delta) - B_H(t) \) is a normal random variable with mean zero and with variance

\[
\text{Var} [B_H(t + \Delta) - B_H(t)] = \text{Var} [B_H(1)] \cdot \Delta^{2H}.
\]  

Due to the fact that the increments of FBM are stationary, the variance of equation (12) depends only on the length \( \Delta \) of the temporal interval \([t, t + \Delta]\).

Using the fact that FIM is a selfsimilar process with Hurst exponent \( H \), as well as the fact that FIM is a martingale, it is shown in the methods that the FIM increment \( I_H(t + \Delta) - I_H(t) \) is a random variable with mean zero and with variance

\[
\text{Var} [I_H(t + \Delta) - I_H(t)] = \text{Var} [I_H(1)] \cdot [(t + \Delta)^{2H} - t^{2H}].
\]  

The variance of equation (13) depends on the starting point \( t \), as well as on the length \( \Delta \), of the temporal interval \([t, t + \Delta]\). Hence, equation (13) implies that the increments of FIM are not stationary.

The asymptotic behavior of the variance of equation (13), in the temporal limit \( t \to \infty \), is \( \text{Var} [I_H(t + \Delta) - I_H(t)] \approx v \cdot t^{2H-1} \), where \( v = 2\Delta H \text{Var} [I_H(1)] \) (see the methods for the details of this asymptotic equality). Consequently, for sub-diffusion \((0 < H < \frac{1}{2})\), equation (13) implies that: \( \lim_{t \to \infty} \text{Var} [I_H(t + \Delta) - I_H(t)] = 0 \); namely, the variance of the FIM increment is

\[
\text{Var} [I_H(t + \Delta) - I_H(t)] = 0.
\]
FIM velocities are uncorrelated: that end with two positive time points, as evident from equations (2) and (4), FBM and FIM have markedly different velocities. In 3.6. Correlation comparison

As evident from equations (2) and (4), FBM and FIM have markedly different velocities. In this subsection we shall compare the correlation structures of the FBM and FIM velocities. To that end we set two positive time points, \( t_1 \) and \( t_2 \), and address the covariance of the velocities at these points.

As FBM is a self-similar process with Hurst exponent \( H \), and as its increments are stationary, it follows that the covariance of the FBM positions \( B_H(t_1) \) and \( B_H(t_2) \) is [2]:

\[
\text{Cov} [B_H(t_1), B_H(t_2)] = \frac{1}{2} \text{Var} [B_H(1)] \cdot \left( 2^H - |t_1 - t_2|^{2H} + t_2^{2H} \right). \tag{14}
\]

Differentiating the covariance of equation (14) with respect to the temporal variable \( t_1 \), and then with respect to the temporal variable \( t_2 \), implies that the covariance of the FBM velocities \( \dot{B}_H(t_1) \) and \( \dot{B}_H(t_2) \) is:

\[
\text{Cov} [\dot{B}_H(t_1), \dot{B}_H(t_2)] = \text{Var} [B_H(1)] \cdot \frac{H(2H - 1)}{|t_1 - t_2|^{2H} - t_2^{2H}}. \tag{15}
\]

With regard to distinct time points, \( t_1 \neq t_2 \), the covariance of equation (15) has the following implications. For sub-diffusion (0 < \( H < \frac{1}{2} \)) the FBM velocities are negatively correlated: \( \text{Cov}[\dot{B}_H(t_1), \dot{B}_H(t_2)] < 0 \). For super-diffusion (\( \frac{1}{2} < H < 1 \)) the FBM velocities are positively correlated: \( \text{Cov} [\dot{B}_H(t_1), \dot{B}_H(t_2)] > 0 \). Consequently, the non-overlapping increments of FBM are negatively/positively correlated respectively.

As FIM is a self-similar process with Hurst exponent \( H \), and as it is a martingale, it is shown in the methods (see equation (65)) that the covariance of the FIM positions \( I_H(t_1) \) and \( I_H(t_2) \) is:

\[
\text{Cov} [I_H(t_1), I_H(t_2)] = \text{Var} [I_H(1)] \cdot \left( \min \{t_1, t_2\} \right)^{2H}. \tag{16}
\]

Differentiating the covariance of equation (16) with respect to the temporal variable \( t_1 \), and then with respect to the temporal variable \( t_2 \), implies that the covariance of the FIM velocities \( \dot{I}_H(t_1) \) and \( \dot{I}_H(t_2) \) is:

\[
\text{Cov} [\dot{I}_H(t_1), \dot{I}_H(t_2)] = 2H \text{Var} [I_H(1)] \cdot \left( \min \{t_1, t_2\} \right)^{2H - 1} \delta(t_1 - t_2), \tag{17}
\]

where \( \delta(\cdot) \) denotes Dirac’s ‘delta function’.

With regard to distinct time points, \( t_1 \neq t_2 \), the covariance of equation (17) implies that the FIM velocities are uncorrelated: \( \text{Cov} [\dot{I}_H(t_1), \dot{I}_H(t_2)] = 0 \). Consequently, the non-overlapping increments of FBM are also uncorrelated.
Evidently, the difference between the covariance of equation (15) and the covariance of equation (17) is dramatic. On the one hand, changing the Hurst exponent $H$ in the FBM model has a profound effect, both qualitatively and quantitatively, on the correlations of the FBM velocities. On the other hand, changing the Hurst exponent $H$ in the FIM model has no effect on the correlations of the FIM velocities—as these velocities are always uncorrelated.

To further elucidate the dramatic difference between the FBM velocity covariance and the FIM velocity covariance, consider a one-dimensional random motion whose trajectory is $X(t)$ ($t \geq 0$). Assume that the motion initiates at the spatial origin $X(0) = 0$, and that its positions have finite variance. Then, as shown in the methods, the variance of the motion’s positions admits the following representation in terms of motion’s velocity covariance:

$$\text{Var}[X(t)] = \int_0^t \int_0^t \text{Cov}[X(t_1), X(t_2)] \, dt_1 \, dt_2.$$  

Two antithetical observations follow from this variance–covariance formula. On the one hand, for FBM the diagonal of the velocity covariance (equation (17)) constitutes 0% of the ‘mass’ of the position variance ($\text{Var}[B_H(t)]$). On the other hand, for FIM the diagonal of the velocity covariance (equation (17)) constitutes 100% of the ‘mass’ of the position variance ($\text{Var}[I_H(t)]$).

### 3.7 Fourier comparison

Consider a one-dimensional random motion whose trajectory is $X(t)$ ($t \geq 0$). Assume that the motion initiates at the spatial origin $X(0) = 0$, and that its positions have zero mean and finite variance. Consequently, the motion’s square displacement at time $t$ is $|X(t)|^2$, and its MSD is $E[|X(t)|^2] = \text{Var}[X(t)]$. In turn, the variance–covariance formula noted in the previous subsection manifests the MSD in terms of motion’s velocity covariance: $E[|X(t)|^2] = \int_0^t \int_0^t \text{Cov}[\dot{X}(t_1), \dot{X}(t_2)] \, dt_1 \, dt_2$.

The motion under consideration takes place over the real line. We now shift the motion from the real line to the complex plane via the following mapping of positions: $X(t) \mapsto X(t; \omega) = \int_0^\infty \exp(i\omega u) \dot{X}(u) \, du$, where $\omega$ is a real ‘Fourier parameter’. Namely, the position $X(t; \omega)$ of the newly constructed ‘complex motion’ at time $t$ is the Fourier transform of the velocity of the ‘original motion’ over the temporal interval $[0, t]$. Note that when the Fourier parameter is zero, $\omega = 0$, then the complex motion coincides with the original motion: $X(t; 0) = X(t)$ ($t \geq 0$). Thus, in effect, the complex motion is a generalization of the original motion.

As the original motion, the complex motion initiates at the spatial origin $X(0; \omega) = 0$ (now the origin it that of the complex plane), and its positions have zero mean and finite variance. The square displacement of the complex motion at time $t$ is $|X(t; \omega)|^2 = X(t; \omega) \cdot X(t; -\omega)$, and its MSD is $E[|X(t; \omega)|^2]$. As shown in the methods, the counterpart of the above variance–covariance formula is the following MSD-covariance formula: $E[|X(t; \omega)|^2] = \int_0^t \int_0^t \exp[i\omega(t_1 - t_2)] \text{Cov}[X(t_1), \dot{X}(t_2)] \, dt_1 \, dt_2$. Namely, the MSD-covariance formula manifests the MSD of the complex motion in terms of the velocity covariance of the original motion.

Now, further consider the original motion to be self-similar with Hurst exponent $H$. Then, as noted in section 2, the MSD of the original motion is a power-law: $E[|X(t)|^2] = \text{Var}[X(1)] \cdot t^{2H}$. Shifting from the original motion to the complex motion, the natural question that arises is: what behavior does the MSD of the complex motion exhibit? We shall answer this question with regard to FBM and FIM.

Substituting the FBM velocity covariance of equation (15) into the MSD-covariance formula, a calculation that is detailed in the methods yields

$$E[|B_H(t; \omega)|^2] \approx \text{Var}[B_H(1)] \Gamma(1 + 2H) \sin((\pi H)\omega)^{1-2H} \cdot t,$$  

(18)
where: the asymptotic equality is in the temporal limit \( t \to \infty \), and it holds for all non-zero values of the Fourier parameter \( \omega \). So, FBM displays a markedly different MSD behavior for \( \omega = 0 \) and for \( \omega \neq 0 \). On the one hand, for \( \omega = 0 \) the MSD behavior manifests: sub-diffusion when \( 0 < H < \frac{1}{2} \); super-diffusion when \( \frac{1}{2} < H < 1 \); and regular diffusion when \( H = \frac{1}{2} \). On the other hand, for \( \omega \neq 0 \) the MSD behavior manifests, asymptotically, regular diffusion for all Hurst exponents \( 0 < H < 1 \).

Substituting the FIM velocity covariance of equation (17) into the MSD-covariance formula, a calculation that is detailed in the methods yields

\[
E \left[ |I_H (t; \omega)|^2 \right] = \text{Var} [I_H (1)] \cdot t^{2H},
\]  

(19)

where the equality holds for all real values of the Fourier parameter \( \omega \). So, FIM displays the following MSD behavior: the MSD of the complex motion does not depend on the value of the Fourier parameter \( \omega \)—and hence, in particular, it is identical to the MSD of the original motion. In other words, the MSD of FIM does not change when shifting from the original motion to the complex motion.

The FBM model and the FIM model are indistinguishable via the MSD perspective. However, shifting from FBM and FIM to their corresponding ‘complex versions’, the two models become highly distinguishable via the MSD perspective. Indeed, the ‘original’ power-law MSD of FBM switches to the asymptotically linear MSD of equation (18), whereas the ‘original’ power-law MSD of FIM is maintained by the MSD of equation (19).

We conclude this subsection with a ‘spectral remark’. Consider the above random motion, whose trajectory is \( X(t) (t \geq 0) \). The motion’s spectral density is defined as the limit \( \lim_{t \to \infty} \frac{1}{t} E[|X(t; \omega)|^2] \), provided that this limit exists. In particular, if the motion has a stationary velocity then, as shown in the methods: the motion’s spectral density coincides with the Fourier transform of the velocity’s auto-covariance function. The spectral density manifests, asymptotically, the diffusion coefficient of the motion’s ‘complex version’. The FIM model offers an example of a motion whose complex version does not have a diffusion coefficient—as the motion’s complex version is an anomalous diffusion. A general Poisson model of motions whose complex versions are anomalous diffusions was established in [119].

4. Diffusion in a logarithmic potential

As described above, the Itô stochastic dynamics of FIM have a zero drift and a position-dependent volatility. Namely, the evolution of FIM is governed by an Itô SDE of the form \( \dot{X}(t) = \sigma [X(t)] \dot{B}(t) \). Many researchers in the physical sciences and in engineering are used to work with Langevin stochastic dynamics [94–96], which have a position-dependent drift and a constant volatility. Up to an inherent time scale, the latter evolution is governed by a Langevin SDE of the form \( \dot{X}(t) = \mu [X(t)] + \dot{B}(t) \).

As noted above, a special case of the FIM model was investigated in [31]: in this special case the motion runs over the positive half-line \( 0 < x < \infty \), and the Hurst exponent is in the range \( \frac{1}{2} \leq H < 1 \). It was shown in [31] that the special case of FIM admits a Langevin representation. In this section we shall present a Langevin representation for the (general) FIM model.

There is a one-to-one mapping between FIM and a particular random motion that is generated by specific Langevin stochastic dynamics. Indeed, map the FIM trajectory \( I_H (t) (t \geq 0) \) to the trajectory \( \xi_H (t) (t \geq 0) \) via the following transformation: \( \xi_H (t) = \varphi [I_H (t)] \), where
\( \varphi(x) = 2H|x|^{1/2H} \) \( \text{sign}(x) \), and where \( \text{sign}(x) \) is the sign of \( x \).\(^1\) Then, using Ito’s formula and equation (4), it is shown in the methods that the stochastic dynamics of the random motion \( \xi_H(t) \) is governed by the Langevin SDE

\[
\dot{\xi}_H(t) = \left( \frac{1}{2} - H \right) \frac{1}{\xi_H(t)} + \dot{B}(t).
\] (20)

Shifting back from the trajectory \( \xi_H(t) \) \( (t \geq 0) \) to the FIM trajectory \( I_H(t) \) \( (t \geq 0) \) is via the corresponding inverse transformation: \( I_H(t) = \varphi^{-1}[\xi_H(t)] \), where \( \varphi^{-1}(y) = (\frac{1}{2H}|y|^{2H} \text{sign}(y) \).

The drift of the Langevin SDE (20) is harmonic, \( \mu(x) = \left( \frac{1}{2} - H \right) / x \). This drift underscores and highlights the three different diffusivity categories of FIM—which are determined by the value of its Hurst exponent \( H \). When FIM is sub-diffusive \( (0 < H < \frac{1}{2}) \) then the Langevin drift is ‘repulsive’; it pushes away from the spatial origin. When FIM is super-diffusive \( (\frac{1}{2} < H < 1) \) then the Langevin drift is ‘attractive’: it pushes towards the spatial origin. And when FIM is a regular diffusion \( (H = \frac{1}{2}) \) then the Langevin drift is zero.

A Langevin SDE of the form \( \dot{X}(t) = \mu[X(t)] + \dot{B}(t) \) is commonly characterized by its ‘potential function’: a function \( U(x) \) whose negative gradient is the drift, \( -U'(x) = \mu(x) \). The potential function \( U(x) \) manifests a ‘potential landscape’ that underpins and governs the Langevin stochastic dynamics. The random motions that are generated by a Langevin SDE with a logarithmic potential \( U(x) = r \cdot \ln(|x|) \), where \( r \) is a real parameter, are termed ‘diffusion in a logarithmic potential’ (DLP). The topic of DLP attracted significant scientific interest recently [120–129].

Arguably, the best known example of DLP is the Bessel process [130–134]: the Euclidean distance of a BM—which takes place in the \( d \)-dimensional Euclidean space—from the \( d \)-dimensional spatial origin. In this example DLP runs over the positive half-line \( (0 < x < \infty) \), and its parameter is \( r = \frac{1}{2}(d - 1) \). So, for dimensions \( d = 2, 3, 4, 5, 6, \ldots \), the Bessel process yields the DLP parameter values \( r = \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{7}{16}, \frac{9}{32}, \ldots \).

FIM yields, via the Langevin SDE (20), another example of DLP. Indeed, the underpinning potential function of the Langevin SDE (20) is logarithmic with parameter \( r = H - \frac{1}{2} \). As the Hurst exponent of FIM takes values in the range \( 0 < H < 1 \), the corresponding DLP parameter takes values in the range \( -\frac{1}{2} < r < \frac{1}{2} \). Negative values of the DLP parameter correspond to sub-diffusive FIM, and positive values of the DLP parameter correspond to super-diffusive FIM.

With the one-to-one mapping between FIM and its corresponding DLP at hand, one can shift between properties of these different motions. For example, one can shift from the density of the FIM position \( I_H(t) \) to the density of the corresponding DLP position \( \xi_H(t) \), and vice versa. In subsection 3.1 we obtained the FIM density of equation (7) from a general result regarding selfsimilar Ito diffusions [1]. And indeed, an alternative way to obtain the FIM density of equation (7) is via known results regarding the DLP density.

As noted above, the potential function \( U(x) \) manifests the ‘potential landscape’ that underpins and governs the Langevin SDE \( \dot{X}(t) = \mu[X(t)] + \dot{B}(t) \). Analogously, the volatility function \( \sigma(x) \) manifests the ‘volatility landscape’ that underpins and governs the drift-less Ito SDE \( \dot{X}(t) = \sigma|X(t)|\dot{B}(t) \). The one-to-one mapping between FIM and its corresponding DLP induces a one-to-one mapping between their respective underpinning landscapes: the power-law volatility landscape \( \sigma(x) = |x|^{\frac{1}{2d}} \), and the logarithmic potential landscape \( U(x) = \)

\(^1\) Namely, \( \text{sign}(x) = 1 \) if \( x \) is positive, and \( \text{sign}(x) = -1 \) if \( x \) is negative.
Figure 8. Schematic illustrations of the shapes of the FIM volatility landscape, and of the corresponding DLP potential landscape. Depicted in the right panels, the FIM volatility landscape is: $\sigma(x) = |x|^{1 - \frac{1}{2H}}$ over the spatial axis $-\infty < x < \infty$. Depicted in the left panels, the DLP potential landscape is: $U(x) = r \cdot \ln(|x|)$ over the spatial axis $-\infty < x < \infty$. The one-to-one correspondence between the two landscapes is via the relation $r = H - \frac{1}{2}$, where $0 < H < \frac{1}{2}$ and $-\frac{1}{2} < r < \frac{1}{2}$. The two landscapes display markedly different shapes in the three different diffusivity categories: sub-diffusion $0 < H < \frac{1}{2}$ and $-\frac{1}{2} < r < 0$ (top panels); super-diffusion $\frac{1}{2} < H < 1$ and $0 < r < \frac{1}{2}$ (bottom panels); and regular diffusion $H = \frac{1}{2}$ and $r = 0$ (middle panels).
Table 3. A ‘bird’s-eye view’ comparison between the FBM and FIM models with Hurst exponents \( H \neq \frac{1}{2} \). The comparison is with respect to the ‘benchmark’ BM model—which the FBM and FIM models yield at the Hurst exponent \( H = \frac{1}{2} \). Each row of the table specifies a different key property of BM. The top four rows of the table list the properties that FBM and FIM share in common with BM. The bottom six rows of the table list key differences between the three models. The last row of the table refers to non-overlapping increments.

| Property                   | BM   | FBM | FIM |
|----------------------------|------|-----|-----|
| Finite variance            | YES  | YES |     |
| Symmetric process          | YES  | YES |     |
| Continuous trajectory      | YES  | YES |     |
| Selfsimilar process        | YES  | YES |     |
| Hurst exponent             | \( H = \frac{1}{2} \) | \( 0 < H < 1 \) | \( 0 < H < 1 \) |
| Gaussian process           | YES  | YES | NO  |
| Markov process             | YES  | NO  | YES |
| Martingale                 | YES  | NO  | YES |
| Stationary increments      | YES  | YES | NO  |
| Uncorrelated increments    | YES  | NO  | YES |

\((H - \frac{1}{2}) \ln |x|\). Schematic illustrations of these corresponding landscapes are depicted in figure 8.

5. Summary

Setting off from the paradigmatic model for regular diffusion, BM, in this paper we examined and compared two anomalous-diffusion models that generalize BM: the popular and well-known FBM model, and the FIM model. A ‘bird’s-eye view’ of the FBM model vs the FIM model is summarized in table 3. Going over a list of key BM properties, table 3: compares FBM and FIM to BM; and compares FBM to FIM.

From the perspective of mean-square-displacement measurement, both FBM and FIM exhibit the very same modeling ‘efficacy’. Indeed, in both these models the range of the Hurst exponent is \( 0 < H < 1 \). Consequently, both these models yield: sub-diffusion in the Hurst range \( 0 < H < \frac{1}{2} \); super-diffusion in the Hurst range \( \frac{1}{2} < H < 1 \); and regular diffusion at the Hurst value \( H = \frac{1}{2} \)—in which case both FBM and FIM are BM. Thus, the FBM and FIM models are indistinguishable from the mean-square-displacement perspective.

However, the stochastic propagation mechanisms of FBM and FIM are dramatically different. On the one hand, the FBM velocity (equation (2)) is a linear moving average of the underlying white noise—where the averaging is temporal, and it goes from the present (the time point \( t \)) all the way back to the infinitely distant past (the time point \( -\infty \)). We emphasize that the FBM velocity does not depend on the FBM positions. On the other hand, the FIM velocity (equation (4)) is a non-linear mapping of the following present (the time point \( t \)) quantities: the present FIM position, and the present value of the underlying white noise.

The dramatic differences between the stochastic propagation mechanisms of FBM and FIM yield profound differences between these anomalous-diffusion models. We explored various FBM vs FIM differences via a set of comprehensive and in-depth comparisons; highlights of these comparisons are summarized in table 4. In addition to these comparisons, we showed that there is a one-to-one mapping between FIM and DLP—with the coefficient of the logarithmic potential being \( H - \frac{1}{2} \).
Table 4. Highlights of the in-depth comparisons between FBM and FIM. See the relevant sections (noted in the right column of the table) for the full details. In a nutshell: $D [X(t) \| X(1)]$ is the KL divergence of the position $X(t)$ from the ‘benchmark’ position $X(1)$; and $X(t, \omega) = \int_0^t \exp(i \omega u) X(u) du$ is the Fourier transform of the velocity over the temporal interval $[0, t]$. In the cells of the table $\approx$ manifests asymptotic equality in the temporal limit $t \to \infty$.

| Statistic                                      | FBM $X(t) = B_H(t)$                                                                 | FIM $X(t) = I_H(t)$                                                                 | Section |
|-----------------------------------------------|------------------------------------------------------------------------------------|------------------------------------------------------------------------------------|---------|
| $\dot{X}(t)$                                  | $(H - \frac{1}{2}) \int_{-\infty}^t (t - u)^H \frac{1}{2} B(u) du$                | $|X(t)|^{1-\frac{1}{2H}} B(t)$                                                     | 2.1 and 2.2 |
| Law of $X(t)$                                 | Gaussian                                                                           | Non-Gaussian                                                                      | 3.1 and 3.2 |
| $E \left[|X(t)|^2\right]$                    | $\frac{2}{\pi}$                                                                   | $\frac{\sin(\pi H)}{\pi H}$                                                     | 3.3     |
| $D [X(t) \| X(1)]$                            | $\approx \frac{1}{2} t^{2H}$                                                      | $\approx (1 - H) t$                                                              | 3.4     |
| $\text{Var} [X(t + \Delta) - X(t)]$          | 1                                                                                 | $2H (\frac{1}{\Delta})^{2H-1}$                                                  | 3.5     |
| $\text{Var} [X(t)]$                           | $H (2H - 1) |t_1 - t_2|^{2H-2}$                                                              | $2H (\min \{t_1, t_2\})^{2H-1} \delta (t_1 - t_2)$                             | 3.6     |
| $\text{Cov} [\dot{X}(t_1), \dot{X}(t_2)]$   | $H (2H - 1) |t_1 - t_2|^{2H-2}$                                                              | $2H (\min \{t_1, t_2\})^{2H-1} \delta (t_1 - t_2)$                             | 3.6     |
| $\text{Var} [X(1)]$                           | $H (2H - 1) |t_1 - t_2|^{2H-2}$                                                              | $2H (\min \{t_1, t_2\})^{2H-1} \delta (t_1 - t_2)$                             | 3.6     |
| $E \left[|X(t; \omega)|^2\right]$            | $\approx \Gamma(1 + 2H) \sin(\pi H) (|\omega| t)^{1-2H}$                         | $= 1$                                                                             | 3.7     |
| $E \left[|X(t)|^2\right]$                    |                                                                     |                                                                     |         |
Due to its properties, FBM is a rather challenging model from the perspectives of numerical simulation, of stochastic integration, and of analytic tractability. Conversely, from these very perspectives, FIM is an amicable working model. With its compelling properties, its easy numerical simulation, its analytic tractability, and its linkage to DLP—we believe that FIM can serve as an instrumental working model for anomalous diffusion in various fields of science and engineering.

6. Methods

6.1. The macroscopic power-law structure of the MSD function

Consider a one-dimensional random motion whose trajectory is \( X(t) \) (\( t \geq 0 \)). Given a positive number \( n \), speed up the temporal scale by the factor \( n \). To compensate for speeding up time, shrink the spatial scale by the positive factor \( \phi(n) \) (where \( \phi(n) \) is a continuous function of the positive number \( n \)). This spatio-temporal scaling shifts the trajectory \( X(t) \) (\( t \geq 0 \)) to the trajectory \( X_n(t) \) (\( t \geq 0 \)), where

\[
X_n(t) = \frac{1}{\phi(n)} X(nt)
\]  

(21)

The MSD function of the random motion is \( \phi(t) = E[|X(t) - X(0)|^2] \). In turn, the MSD function of the scaled random motion is

\[
\phi_n(t) = E\left[\left|X_n(t) - X_n(0)\right|^2\right]
\]

\[
= \frac{1}{\phi(n)^2} E\left[\left|X(nt) - X(n0)\right|^2\right]
\]

\[
= \frac{\phi(nt)}{\phi(n)^2} \phi(nt) \phi(n)^{-2}
\]

(22)

This MSD function can be written as follows

\[
\phi_n(t) = \frac{\phi(n)}{\phi(n)^2} \cdot \frac{\phi(nt)}{\phi(n)^2}
\]  

(23)

Now, consider the MSD function \( \phi_n(t) \) in the scaling limit \( n \to \infty \). In general, in order to obtain a non-trivial point-wise limit \( \phi_\infty(t) = \lim_{n \to \infty} \phi_n(t) \) (for any positive time \( t \)), two conditions must be met. Firstly, the spatial scaling function \( \phi(n) \) should be asymptotically equivalent (in the limit \( n \to \infty \)) to the function \( \sqrt{\phi(n)} \). Secondly, the MSD function \( \phi(t) \) should be regularly varying at infinity [135]. If these two conditions hold then the non-trivial point-wise limit is

\[
\phi_\infty(t) = c \cdot \epsilon',
\]  

(24)

where: \( c \) is a positive constant that stems from the first condition; and \( \epsilon \) is a positive regular-variation exponent that stems from the second condition.
6.2. The limits of Table 1

In this subsection we use the following shorthand notation: \( f_{\text{FBM}}(t; x) \) is the FBM density of equation (6); \( f_{\text{FIM}}(t; x) \) is the FIM density of equation (7); and \( f_{\text{BM}}(t; x) = (1/\sqrt{2\pi t}) \exp \left(-x^2/(2t)\right) \) \((\infty < x < \infty)\) is the density of the random variable \( B(t) \) (the position of BM at time \( t \)). Also, in this subsection \( \approx \) denotes asymptotic equivalence—rather than asymptotic equality (as in the rest of this paper)—in the limit \( x \to \infty \). Namely, for two real-valued functions \( \varphi_1(x) \) and \( \varphi_2(x) \), the notation \( \varphi_1(x) \approx \varphi_2(x) \) means that:

\[
\lim_{x \to \infty} \frac{\varphi_1(x)}{\varphi_2(x)} = v, \]

where \( v \) is a positive limit value.

6.2.1. FBM vs BM. With regard to the limit \( l \to \infty \), L’Hospital’s rule implies that

\[
\frac{\Pr \left[ B_H(t) > I \right]}{\Pr \left[ B(t) > I \right]} \approx \frac{2f_{\text{FBM}}(t; I)}{2f_{\text{BM}}(t; I)} = \frac{1}{\sqrt{2\pi t}} \cdot \frac{1}{\sqrt{2\pi t}} \exp \left(-\frac{\kappa^2}{\pi} \right) \approx \exp \left(\frac{1}{2} \kappa^2\right),
\]

where

\[
\kappa = \frac{1}{l} - \frac{1}{bt^2H}.
\]

For \( 0 < H < \frac{1}{2} \) and \( t > l^{1/(1-2H)} \) note that: \( \kappa < 0 \) and hence

\[
\lim_{l \to \infty} \frac{\Pr \left[ B_H(t) > I \right]}{\Pr \left[ B(t) > I \right]} = 0.
\]

For \( \frac{1}{2} < H < 1 \) and \( t > l^{1/(1-2H)} \) note that: \( \kappa > 0 \) and hence

\[
\lim_{l \to \infty} \frac{\Pr \left[ B_H(t) > I \right]}{\Pr \left[ B(t) > I \right]} = \infty.
\]

6.2.2. FIM vs BM. With regard to the limit \( l \to \infty \), L’Hospital’s rule implies that

\[
\frac{\Pr \left[ I_H(t) > I \right]}{\Pr \left[ B(t) > I \right]} \approx \frac{2f_{\text{FIM}}(t; I)}{2f_{\text{BM}}(t; I)} = \frac{1}{\pi^{1/2-H}} \left[ \left( \frac{2H}{\pi} \right)^{1-H} \exp \left( -\frac{2H^2}{\pi} \right) \right]^{1/2-H} \exp \left( -\frac{\psi(I)}{\pi} \right)
\]

\[\approx \exp \left(\psi(I)\right),\]

where

\[
\psi(I) = \frac{1}{2t} - \frac{2H^2}{\pi} + \left( \frac{1}{H} - 2 \right) \ln(t).
\]

For \( 0 < H < \frac{1}{2} \) and any fixed \( t \) note that: \( \lim_{l \to \infty} \psi(I) = -\infty \) and hence

\[
\lim_{l \to \infty} \frac{\Pr \left[ I_H(t) > I \right]}{\Pr \left[ B(t) > I \right]} = 0.
\]
For $\frac{1}{2} < H < 1$ and any fixed $t$ note that: $\lim_{l \to \infty} \psi_l(l) = \infty$ and hence

$$\lim_{l \to \infty} \frac{\Pr [\{I_H(0) > l\}]}{\Pr [\{B_H(0) > l\}]} = \infty. \quad (32)$$

### 6.2.3. FIM vs FBM

With regard to the limit $l \to \infty$, L’Hospital’s rule implies that

$$\frac{\Pr [\{|I_H(t)| > l\}]}{\Pr [\{|B_H(t)| > l\}]} \approx \frac{2 f_{\text{FIM}} (t; l)}{2 f_{\text{FBM}} (t; l)} \approx \frac{\frac{1}{2} \left( \frac{2H^2}{t} \right)^{1-H} \exp \left( -\frac{2H^2}{t} \right) t^{\frac{1}{2}}}{\frac{1}{2} \left( \frac{2H^2}{t} \right)^{1-H} \exp \left( -\frac{2H^2}{t} \right) t^{\frac{1}{2}} - \frac{1}{2}}$$

$$\approx \exp [\psi_l(l)], \quad (33)$$

where

$$\psi_l(l) = \frac{1}{2bt^2H^2} - \frac{2H^2}{t} t^H + \left( \frac{1}{H} - 2 \right) \ln(t). \quad (34)$$

For $0 < H < \frac{1}{2}$ and any fixed $t$ note that: $\lim_{l \to \infty} \psi_l(l) = -\infty$ and hence

$$\lim_{l \to \infty} \frac{\Pr [\{I_H(0) > l\}]}{\Pr [\{B_H(0) > l\}]} = 0. \quad (35)$$

For $\frac{1}{2} < H < 1$ and any fixed $t$ note that: $\lim_{l \to \infty} \psi_l(l) = \infty$ and hence

$$\lim_{l \to \infty} \frac{\Pr [\{I_H(0) > l\}]}{\Pr [\{B_H(0) > l\}]} = \infty. \quad (36)$$

### 6.3. The CV scores of FBM and FIM

We start with the random variable

$$Z = 2H^2 \cdot |I_H(1)|^{1/H}. \quad (37)$$

The cumulative distribution function of the random variable $Z$ is

$$\Pr (Z \leq z) = \Pr \left( 2H^2 \cdot |I_H(1)|^{1/H} \leq z \right)$$

$$= \Pr \left[ |I_H(1)| \leq \left( \frac{z}{2H^2} \right)^H \right]$$

$$= 2 \Pr \left[ I_H(1) \leq \left( \frac{z}{2H^2} \right)^H \right] - 1 \quad (38)$$

($z \geq 0$). In the transition from the second line of equation (38) to the third line we used the fact that the random variable $I_H(1)$ is symmetric (i.e. the random variable $I_H(1)$ is equal in law to the random variable $-I_H(1)$).
Denote by \( g(z) (z \geq 0) \) the density function of the random variable \( Z \), and denote by \( f(x) (-\infty < x < \infty) \) the density function of the random variable \( I_H(1) \). Differentiating equation (38), and then using the density function of equation (7), implies that

\[
g(z) = 2f\left[\left(\frac{z}{2H^2}\right)^H\right] \frac{1}{(2H^2)^H} z^{H-1}
\]

\[
= \frac{2H}{(2H^2)^H} f\left[\left(\frac{z}{2H^2}\right)^H\right] z^{H-1}
\]

\[
= \frac{2H}{(2H^2)^H} \left\{ \frac{H ! - 2H}{2H ! (1 - H)} \right\} \exp\left[ -2H^2 \left(\frac{z}{2H^2}\right)^H \right] \left[\left(\frac{z}{2H^2}\right)^H\right]^{-2} \}
\]

\[
= \left\{ \frac{2H}{(2H^2)^H} \frac{H ! - 2H}{2H ! (1 - H)} \right\} \exp(-z) \left\{ z^{1 - 2H} z^{H-1} \right\}
\]

\[
= \frac{1}{\Gamma(1 - H)} \exp(-z) z^{-H}.
\] (39)

In turn, equation (39) implies that

\[
g(z) = \frac{1}{\Gamma(1 - H)} \exp(-z) z^{(1 - H) - 1}.
\] (40)

The density function of equation (40) characterizes a Gamma distribution with Gamma exponent \( 1 - H \). Consequently, the mean of the random variable \( Z^p \)—where \( p \) is a positive power—is

\[
\mathbb{E}[Z^p] = \frac{\Gamma(1 - H + p)}{\Gamma(1 - H)}.
\] (41)

Equation (37) implies that

\[
|I_H(1)| = \left(\frac{1}{2H^2}\right)^H \cdot Z^H.
\] (42)

In turn, for any inequality index \( \mathcal{I} \) we have

\[
\mathcal{I}(|I_H(1)|) = \mathcal{I}(Z^H).
\] (43)

In particular, for the inequality index \( \mathcal{I} \) that yields the CV score we have

\[
\mathcal{I}(|I_H(1)|) = 1 - \frac{\mathbb{E}[Z^H]^2}{\mathbb{E}[Z^{2H}]}
\]

\[
= 1 - \frac{\left[\frac{\Gamma(1 - H + 2H)}{\Gamma(1 - H)}\right]^2}{\Gamma(1 - H + 2H)} = 1 - \frac{\Gamma(1)^2}{\Gamma(1 + H) \Gamma(1 - H)}
\]

\[
= 1 - \frac{1}{H \Gamma(H) \cdot \Gamma(1 - H)} = 1 - \frac{1}{H \Gamma(H) \Gamma(1 - H)}
\]

\[
= 1 - \frac{\sin(\pi H)}{\pi} = 1 - \frac{\sin(\pi H)}{\pi H}.
\] (44)
In equation (44) we used the following (well known) properties of the Gamma function: \( \Gamma (1) = 1; \ \Gamma (1 + H) = H \Gamma (H); \) and \( \Gamma (H) \Gamma (1 - H) = \pi / \sin (\pi H). \) Equation (44) proves equation (9).

As noted above, at the Hurst exponent \( H = \frac{1}{2} \) the random variable \( I_{1/2} (1) \) is ‘standard normal’, i.e. it is a normal random variable with mean zero and with variance one. Also, as noted above, the random variable \( B_H (1) \) is normal with mean zero and with variance \( b = \text{Var} [B_H (1)]. \) Hence, the random variables \( I_{1/2} (1) \) and \( B_H (1) \) differ only by a scale factor:

\[
B_H (1) = \sqrt{b} \cdot I_{1/2} (1),
\]

where the equality is in law. In turn, any inequality index \( \mathcal{I} \) will assign the random variables \( |I_{1/2} (1)| \) and \( |B_H (1)| \) the same statistical-heterogeneity score. In particular, for the CV score equation (44) implies that

\[
\mathcal{I} (|B_H (1)|) = \mathcal{I} (|I_{1/2} (1)|) = 1 - \frac{\sin \left(\frac{\pi}{2}\right)}{\pi^{1/2}} = 1 - \frac{2}{\pi}.
\]

Equation (46) proves equation (8).

### 6.4. The Kulback–Leibler divergence of FBM and FIM

Consider a one-dimensional random motion whose trajectory is \( X(t) \) \( (t \geq 0) \), and denote by \( f_t (x) \) the density of the random variable \( X(t). \) As noted above, the KL divergence of the random variable \( X(t) \) from the random variable \( X(1) \) is:

\[
\mathcal{D} [X(t) \| X(1)] = \int_{-\infty}^{\infty} \ln \left[ \frac{f_t (x)}{f_1 (x)} \right] f_t (x) \, dx.
\]

#### 6.4.1. FBM case

For FBM, equation (6) implies that

\[
\frac{f_t (x)}{f_1 (x)} = \frac{1}{\sqrt{2\pi b} t} \exp \left( -\frac{x^2}{2bt} \right)
\]

\[
= \frac{1}{\sqrt{2\pi b} t} \exp \left[ -\frac{x^2}{2b} \left( 1 - \frac{1}{t^{2H}} \right) \right].
\]

and hence

\[
\ln \left[ \frac{f_t (x)}{f_1 (x)} \right] = -H \ln (t) + \frac{1}{2b} \left( 1 - \frac{1}{t^{2H}} \right) \cdot x^2.
\]

Setting the random motion \( X(t) \) \( (t \geq 0) \) to be FBM, and substituting equation (49) into equation (47) yields
\[ \mathcal{D} [B_H(t) \| B_H(1)] = \int_{-\infty}^{\infty} \left[ -H \ln (t) + \frac{1}{2b} \left( 1 - \frac{1}{t^\beta} \right) \cdot x^2 \right] f_i(x) \, dx \]
\[ = -H \ln (t) \int_{-\infty}^{\infty} f_i(x) \, dx + \frac{1}{2b} \left( 1 - \frac{1}{t^\beta} \right) \int_{-\infty}^{\infty} x^2 f_i(x) \, dx \]
\[ = -H \ln (t) + \frac{1}{2b} \left( 1 - \frac{1}{t^\beta} \right) \mathbb{E} \left[ B_H(t)^2 \right]. \tag{50} \]

As noted above, the properties of FBM imply that the random variable \( B_H(t) \) is normal with mean zero and variance \( \text{Var} [B_H(t)] = b \cdot t^{2H} \)—and hence
\[ \mathbb{E} \left[ B_H(t)^2 \right] = b \cdot t^{2H}. \tag{51} \]

Combining equations (50) and (51) together, we obtain that
\[ \mathcal{D} [B_H(t) \| B_H(1)] = -H \ln (t) + \frac{1}{2b} \left( 1 - \frac{1}{t^\beta} \right) b t^{2H} \]
\[ = \frac{1}{2} (t^{2H} - 1) - H \ln (t). \tag{52} \]

Equation (52) proves equation (10).

6.4.2. FIM case. For FIM, equation (7) implies that
\[ f_i(x) = \frac{1}{2bH t^{1-H}} \cdot \left( \frac{2H^2}{t} \right)^{\frac{1-H}{2}} \exp \left( -\frac{2H^2}{t} |x|^\beta \right) \frac{|x|^\beta \cdot 2 t^{2H} |x|^\beta \cdot 2 t^{2H}}{2H^2 |x|^\beta \cdot 2 t^{2H} |x|^\beta} \]
\[ = \frac{1}{t^{1-H}} \exp \left[ 2H^2 |x|^\beta \left( 1 - \frac{1}{t} \right) \right]. \tag{53} \]

and hence
\[ \ln \left[ \frac{f_i(x)}{f_1(x)} \right] = -(1 - H) \ln (t) + 2H^2 \left( 1 - \frac{1}{t} \right) \cdot |x|^\beta. \tag{54} \]

Setting the random motion \( X(t) (t \geq 0) \) to be FIM, and substituting equation (54) into equation (47) yields
\[ \mathcal{D} [I_H(t) \| I_H(1)] = \int_{-\infty}^{\infty} \left[ -(1 - H) \ln (t) + 2H^2 \left( 1 - \frac{1}{t} \right) \cdot |x|^\beta \right] f_i(x) \, dx \]
\[ = -(1 - H) \ln (t) \int_{-\infty}^{\infty} f_i(x) \, dx + 2H^2 \left( 1 - \frac{1}{t} \right) \int_{-\infty}^{\infty} |x|^\beta f_i(x) \, dx \]
\[ = -(1 - H) \ln (t) + 2H^2 \left( 1 - \frac{1}{t} \right) \mathbb{E} \left[ I_H(t)^\beta \right]. \tag{55} \]
The selfsimilarity of FIM implies that \( I_H (t) = t^H \cdot I_H (1) \) (the equality being in law), and hence
\[
\mathbb{E} \left[ |I_H (t)|^\beta \right] = \mathbb{E} \left[ |t^H \cdot I_H (1)|^\beta \right] \\
= t \cdot \mathbb{E} \left[ |I_H (1)|^\beta \right]. \tag{56}
\]
Using equation (7), and setting \( \lambda = 2H^2 \), note that the density of the random variable \( |I_H (1)| \) is
\[
\frac{\lambda^{1-H}}{H^\Gamma (1-H)} \exp \left( -\lambda x \frac{1}{2} \right) x^{\frac{1}{2} - 2} \tag{57}
\]
\((0 < x < \infty)\), and hence (using the change-of-variables \( y = \lambda x \frac{1}{2} \)):
\[
\mathbb{E} \left[ |I_H (1)|^\beta \right] = \int_0^\infty \frac{\lambda^{1-H}}{H^\Gamma (1-H)} \exp \left( -\lambda x \frac{1}{2} \right) x^{\frac{1}{2} - 2} \, dx \\
= \frac{\lambda^{1-H}}{H^\Gamma (1-H)} \int_0^\infty \exp \left( -\lambda x \frac{1}{2} \right) x^{\frac{1}{2} - 2} \, dx \\
= \frac{1}{\lambda \Gamma (1-H)} \int_0^\infty \exp (-y)^{2-H} \, dy \\
= \frac{1}{\lambda \Gamma (1-H)} \Gamma (2-H) \\
= \frac{1-H}{\lambda} = \frac{1-H}{2H^2}. \tag{58}
\]
Combining equations (55) and (56) together with equation (58), we obtain that
\[
D \left[ I_H (t) \mid I_H (1) \right] = - (1 - H) \ln (t) + 2H^2 \left( 1 - \frac{1}{t} \right) \mathbb{E} \left[ |I_H (t)|^\beta \right] \\
- (1 - H) \ln (t) + 2H^2 \left( 1 - \frac{1}{t} \right) \cdot \frac{1-H}{2H^2} \\
= (1 - H) \left[ t - 1 - \ln (t) \right]. \tag{59}
\]
Equation (59) proves equation (11).

6.5. The mean and the variance of the FIM increment

Consider the FIM increment \( I_H (t + \Delta) - I_H (t) \). As FIM is a symmetric process, its positions have zero means, and hence so do its increments; thus, in particular, the increment \( I_H (t + \Delta) - I_H (t) \) has a zero mean. The variance of the increment \( I_H (t + \Delta) - I_H (t) \) satisfies
\[
\text{Var} [I_H (t + \Delta) - I_H (t)] = \text{Var} [I_H (t + \Delta)] - 2 \text{Cov} [I_H (t + \Delta), I_H (t)] \\
+ \text{Var} [I_H (t)]. \tag{60}
\]
As FIM is a selfsimilar process with Hurst exponent \( H \), the random variable \( I_H (t) \) is equal in law to the random variable \( t^H \cdot I_H (1) \), and the random variable \( I_H (t + \Delta) \) is equal in law to
the random variable \((t + \Delta)^H \cdot I_H(1)\). Hence, setting \(v_1 = \text{Var}[I_H(1)]\), we have

\[
\text{Var}[I_H(t)] = v_1 \cdot t^{2H}, \tag{61}
\]

and

\[
\text{Var}[I_H(t + \Delta)] = v_1 \cdot (t + \Delta)^{2H}. \tag{62}
\]

The fact that the positions of FIM have zero means implies that

\[
\text{Cov}[I_H(t + \Delta), I_H(t)] = E[I_H(t + \Delta) \cdot I_H(t)]. \tag{63}
\]

Denote by \(\mathcal{F}_t\) the sigma-field generated by the trajectory of FIM over the temporal interval \([0, t]\). The martingale property of FIM implies that—given the information \(\mathcal{F}_t\)—the conditional mean of the random variable \(I_H(t + \Delta)\) is:

\[
E[I_H(t + \Delta) \mid \mathcal{F}_t] = I_H(t).
\]

Hence, applying conditioning to the right-hand side of equation (63), and using the fact that the random variable \(I_H(t)\) has a zero mean, we have

\[
E[I_H(t + \Delta) \cdot I_H(t)] = \left(\frac{1}{t^2} \right) \cdot \text{Var}[I_H(t)]. \tag{64}
\]

Combining together equations (61), (63), and (64) yields

\[
\text{Cov}[I_H(t + \Delta), I_H(t)] = \text{Var}[I_H(t)] = v_1 \cdot t^{2H}. \tag{65}
\]

In turn, substituting equations (61) and (62) and equation (65) into equation (60) yields

\[
\text{Var}[I_H(t + \Delta) - I_H(t)] = v_1 \cdot (t + \Delta)^{2H} - 2v_1 \cdot t^{2H} + v_1 \cdot t^{2H} = v_1 \cdot [(t + \Delta)^{2H} - t^{2H}]. \tag{66}
\]

Equation (66) proves equation (13).

In the limit \(t \to \infty\) note that

\[
\lim_{t \to \infty} \frac{(t + \Delta)^{2H} - t^{2H}}{t^{2H-1}} = \Delta \cdot \lim_{t \to \infty} \left(\frac{1 + \Delta/t}{t} \right)^{2H} - 1 = \Delta \cdot 2H. \tag{67}
\]

Hence, in the limit \(t \to \infty\), equations (66) and (67) yield the following asymptotic equality:

\[
\text{Var}[I_H(t + \Delta) - I_H(t)] \approx (v_1 \Delta 2H) \cdot t^{2H-1}. \tag{68}
\]

6.6. Covariance formulae and their FBM and FIM applications

In this subsection consider a one-dimensional random motion whose trajectory is \(X(t) \quad (t \geq 0)\), and whose positions have finite variance.

6.6.1. The variance–covariance formula. Assume that the motion initiates at the spatial origin \(X(0) = 0\). Then, the motion’s position at time \(t\) is \(X(t) = \int_0^t X(u) \, du\). In turn, the variance of
the random variable $X(t)$ is
\[
\text{Var}[X(t)] = \text{Cov}[X(t), X(t)] \\
= \text{Cov}\left[\int_0^t \dot{X}(t_1) \, dt_1, \int_0^t \dot{X}(t_2) \, dt_2\right] \\
= \int_0^t \int_0^t \text{Cov}[\dot{X}(t_1), \dot{X}(t_2)] \, dt_1 \, dt_2. \quad (69)
\]

6.6.2. MSD-covariance formulae. Shift from the real position $X(t)$ to the complex position

$X(t; \omega) = \int_0^t \exp(i\omega u) \dot{X}(u) \, du$, where $\omega$ is a real ‘Fourier parameter’. Assuming that the '
original motion' $X(t) (t \geq 0)$ has zero mean, so does the 'complex motion' $X(t; \omega) (t \geq 0)$. In

turn, the MSD of the random variable $X(t; \omega)$ is
\[
E\left[|X(t; \omega)|^2\right] = E[X(t; \omega) \cdot X(t; -\omega)] \\
= \text{Cov}[X(t; \omega), X(t; -\omega)] \\
= \text{Cov}\left[\int_0^t \exp(i\omega t_1) \dot{X}(t_1) \, dt_1, \int_0^t \exp(-i\omega t_2) \dot{X}(t_2) \, dt_2\right] \\
= \int_0^t \int_0^t \exp[i\omega(t_1 - t_2)] \text{Cov}[\dot{X}(t_1), \dot{X}(t_2)] \, dt_1 \, dt_2. \quad (70)
\]

Focus on the integral appearing in the bottom line of equation (70). Assuming that the diagonal of the covariance $\text{Cov}[\dot{X}(t_1), \dot{X}(t_2)]$ contributes 'zero mass' to this integral, this integral can be split as follows
\[
\int_0^t \int_0^t \exp[i\omega(t_1 - t_2)] \text{Cov}[\dot{X}(t_1), \dot{X}(t_2)] \, dt_1 \, dt_2 \\
= \int_0^t \int_0^t \exp[i\omega(t_1 - t_2)] \text{Cov}[\dot{X}(t_1), \dot{X}(t_2)] \, dt_1 \, dt_2 \\
+ \int_0^t \int_0^t \exp[i\omega(t_1 - t_2)] \text{Cov}[\dot{X}(t_1), \dot{X}(t_2)] \, dt_1 \, dt_2. \quad (71)
\]

Note that the integral appearing in the bottom line of equation (71) can be re-written as follows
\[
\int_0^t \int_0^t \exp[i\omega(t_1 - t_2)] \text{Cov}[\dot{X}(t_1), \dot{X}(t_2)] \, dt_1 \, dt_2 \\
= \int_0^t \int_0^t \exp[\omega(t_2 - t_1)] \text{Cov}[\dot{X}(t_2), \dot{X}(t_1)] \, dt_2 \, dt_1 \\
= \int_0^t \int_0^t \exp[-\omega(t_1 - t_2)] \text{Cov}[\dot{X}(t_1), \dot{X}(t_2)] \, dt_1 \, dt_2. \quad (72)
\]
Combined together, equations (71) and (72) imply that
\[\int_{0}^{t} \int_{0}^{t} \exp \left[ i \omega (t_1 - t_2) \right] \text{Cov}[\dot{X}(t_1), \dot{X}(t_2)] \, dt_1 \, dt_2 \]
\[= \int \int_{0 < t_1 < t_2 < t} \exp \left[ i \omega (t_1 - t_2) \right] + \exp \left[ -i \omega (t_1 - t_2) \right] \text{Cov}[\dot{X}(t_1), \dot{X}(t_2)] \, dt_1 \, dt_2 \]
\[= \int \int_{0 < t_1 < t_2 < t} 2 \cos \left[ \omega (t_1 - t_2) \right] \text{Cov}[\dot{X}(t_1), \dot{X}(t_2)] \, dt_1 \, dt_2. \tag{73}\]

In turn, combined together, equations (70) and (73) imply that
\[E[|X(t; \omega)|^2] = \int \int_{0 < t_1 < t_2 < t} \cos \left[ \omega (t_1 - t_2) \right] \text{Cov}[\dot{X}(t_1), \dot{X}(t_2)] \, dt_1 \, dt_2. \tag{74}\]

6.6.3. The case of a stationary velocity. Assuming that the motion’s velocity is stationary, the velocity covariance admits the form
\[\text{Cov}[\dot{X}(t_1), \dot{X}(t_2)] = \rho(t_1 - t_2), \tag{75}\]
where \(\rho(u) (-\infty < u < \infty)\) is the velocity’s auto-covariance function. Note that the auto-covariance function is symmetric: \(\rho(-u) = \rho(u)\). Substituting equations (75) into equation (74) yields
\[E[|X(t; \omega)|^2] = \int \int_{0 < t_1 < t_2 < t} \cos \left[ \omega (t_1 - t_2) \right] \rho(t_1 - t_2) \, dt_1 \, dt_2 \tag{76}\]
(using the symmetry of the auto-covariance function)
\[= 2 \int \int_{0 < t_1 < t_2 < t} \cos \left[ \omega (t_2 - t_1) \right] \rho(t_2 - t_1) \, dt_1 \, dt_2 \]
\[= 2 \int_{0}^{t} \left\{ \int_{0}^{t_2} \cos \left[ \omega (t_2 - t_1) \right] \rho(t_2 - t_1) \, dt_1 \right\} \, dt_2 \tag{77}\]
(using the change of variables \(t_1 \rightarrow u = t_2 - t_1\) in the inner integral)
\[= 2 \int_{0}^{t} \left\{ \int_{0}^{t_0} \cos (\omega u) \rho(u) \, du \right\} \, dt_2. \tag{78}\]

Equations (76)–(78) imply that
\[E[|X(t; \omega)|^2] = \int_{0}^{t} \left\{ 2 \int_{0}^{t} \cos (\omega u) \rho(u) \, du \right\} \, dt. \tag{79}\]
and hence
\[ \frac{\partial}{\partial t} E \left[ |X(t)\omega|^2 \right] = 2 \int_0^t \cos(\omega u) \rho(u) \, du. \] (80)

The Fourier transform of the auto-covariance function \( \rho(u) \) is
\[ \hat{\rho}(\omega) := \int_{-\infty}^{\infty} \exp(i\omega u) \rho(u) \, du = 2 \int_0^\infty \cos(\omega u) \rho(u) \, du \] (81)
\((-\infty < \omega < \infty)\); equation (81) used the symmetry of the auto-covariance function \( \rho(u) \).

L’Hospital’s rule, equations (80), and (81) imply that
\[ \lim_{t \to \infty} \frac{E \left[ |X(t; \omega)|^2 \right]}{t} = \lim_{t \to \infty} \frac{\partial}{\partial t} E \left[ |X(t; \omega)|^2 \right] = \hat{\rho}(\omega). \] (82)

Thus, in the temporal limit \( t \to \infty \), the following asymptotic MSD behavior is attained:
\[ E \left[ |X(t; \omega)|^2 \right] \approx \hat{\rho}(\omega) \cdot t. \] (83)

Hence, asymptotically, the diffusivity behavior of the complex motion \( X(t; \omega) (t \geq 0) \) is that of a regular diffusion with ‘diffusion coefficient’ \( \hat{\rho}(\omega) \)—the Fourier transform of the auto-covariance function \( \rho(u) \).

6.6.4. FBM and FIM. The velocity of FBM is stationary, and equation (15) implies that the auto-covariance function of the FBM velocity is
\[ \rho_{\text{FBM}}(u) = \text{Var}[B_H(1)] H (2H - 1) \cdot |u|^{2H - 2}. \] (84)

A calculation using the Gamma function implies that the Fourier transform of the auto-covariance function \( \rho_{\text{FBM}}(u) \) is
\[ \hat{\rho}_{\text{FBM}}(\omega) = \text{Var}[B_H(1)] \Gamma(1 + 2H) \sin(\pi H) \cdot |\omega|^{1-2H}. \] (85)

Substituting the Fourier transform \( \hat{\rho}_{\text{FBM}}(\omega) \) into equation (83) yields
\[ E \left[ |B_H(t; \omega)|^2 \right] \approx \text{Var}[B_H(1)] \Gamma(1 + 2H) \sin(\pi H) \cdot |\omega|^{1-2H} \cdot t, \] (86)
where the asymptotic equality is in the temporal limit \( t \to \infty \). Equation (86) proves equation (18).

Substituting the FIM velocity covariance of equation (17) into equation (70) yields
\[ E \left[ |I_H(t; \omega)|^2 \right] = \int_0^t \int_0^t \exp[i\omega(t_1 - t_2)] \text{Cov} [I_H(t_1), I_H(t_2)] \, dt_1 \, dt_2 \]
\[ = \int_0^t \int_0^t \exp[i\omega(t_1 - t_2)] \left\{ \text{Var}[I_H(1)] 2H \times (\min \{t_1, t_2\})^{2H-1} \delta(t_1 - t_2) \right\} \, dt_1 \, dt_2 \]
\[
\begin{align*}
\text{Var} [I_H (1)] \int_0^t \left\{ \int_0^t \exp \left[ i \omega (t_1 - t_2) \right] 2H \\
\times \left( \min \{ t_1, t_2 \} \right)^{2H - 1} \delta (t_1 - t_2) \, dt_1 \right\} \, dt_2 \\
= \text{Var} [I_H (1)] \int_0^t 2H t_2^{2H - 1} \, dt_2 = \text{Var} [I_H (1)] \cdot t^{2H}.
\end{align*}
\]

Equation (86) proves equation (19).

6.7 The linkage between FIM and Langevin stochastic dynamics

The stochastic dynamics of FIM are governed by the Ito SDE
\[
\dot{I}_H (t) = \sigma [I_H (t)] \dot{B} (t),
\]
with the power-law ‘volatility’ \( \sigma (x) = |x|^{1 - \frac{1}{2H}} \). Consider a monotone increasing function \( \varphi (x) \) that maps the real line to the real line, and introduce the following map of FIM: the random motion \( \xi_H (t) = \varphi [I_H (t)] (t \geq 0) \). Ito’s formula [16] and the stochastic dynamics of FIM imply that
\[
\dot{\xi}_H (t) = \frac{1}{2} \varphi'' [I_H (t)] \sigma [I_H (t)]^2 + \{ \varphi' [I_H (t)] \sigma [I_H (t)] \} \dot{B} (t).
\]  

Our goal is that the stochastic dynamics of random motion \( \xi_H (t) (t \geq 0) \) be governed by a Langevin SDE of the form \( \dot{\xi}_H (t) = \mu [\xi_H (t)] + \dot{B} (t) \). To that end, the function \( \varphi (x) \) must satisfy the condition \( \varphi' (x) \sigma (x) = 1 \). As \( \sigma (x) = |x|^{1 - \frac{1}{2H}} \), the only function \( \varphi (x) \) that satisfies this condition is
\[
\varphi (x) = 2H |x|^{\frac{1}{2H}} \cdot \text{sign} (x), \tag{89}
\]
where \( \text{sign} (x) \) is the sign of \( x \). Indeed, the first derivative of the function appearing in equation (89) is
\[
\varphi' (x) = |x|^{\frac{1}{2H} - 1} = \frac{1}{\sigma (x)}. \tag{90}
\]
And, the second derivative of the function appearing in equation (89) is
\[
\varphi'' (x) = \left( \frac{1}{2H} - 1 \right) |x|^{\frac{1}{2H} - 2} \cdot \text{sign} (x). \tag{91}
\]

Equations (89)–(91) imply that
\[
\frac{1}{2} \varphi'' (x) \sigma (x) = \frac{1}{2} \left( \frac{1}{2H} - 1 \right) |x|^{\frac{1}{2H} - 2} \cdot \text{sign} (x) \cdot |x|^{2 - \frac{1}{2H}} = \frac{1}{2} \left( \frac{1}{2H} - 1 \right) |x|^{-\frac{1}{2H}} \cdot \text{sign} (x) = \frac{1}{2} \left( 1 - 2H \right) \frac{1}{2H |x|^{\frac{1}{2H}} \cdot \text{sign} (x)} = \left( \frac{1}{2} - H \right) \frac{1}{\varphi (x)}. \tag{92}
\]
In turn, substituting equations (90) and (92) into equation (88) implies that

\[ \dot{\xi}_H(t) = \left( \frac{1}{2} - H \right) \frac{1}{\varphi[U_H(t)]} + \dot{B}(t) \]

\[ = \left( \frac{1}{2} - H \right) \frac{1}{\xi_H(t)} + \dot{B}(t). \]  \hspace{1cm} (93)

Equation (93) proves equation (20).

Data availability statement

No new data were created or analysed in this study.

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