DEFORMATIONS OF LEGENDRIAN CURVES

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Abstract. We construct versal and equimultiple versal deformations of the parametrization of a Legendrian curve.

1. Contact Geometry

Let $(X, \mathcal{O}_X)$ be a complex manifold of dimension 3. A differential form $\omega$ of degree 1 is said to be a contact form if $\omega \wedge d\omega$ never vanishes. Let $\omega$ be a contact form. By Darboux’s theorem for contact forms there is locally a system of coordinates $(x, y, p)$ such that $\omega = dy - pdx$. If $\omega$ is a contact form and $f$ is a holomorphic function that never vanishes, $f\omega$ is also a contact form. We say that a locally free subsheaf $L$ of $\Omega^1_X$ is a contact structure on $X$ if $L$ is locally generated by a contact form. If $L$ is a contact structure on $X$ the pair $(X, L)$ is called a contact manifold. Let $(X_1, L_1)$ and $(X_2, L_2)$ be contact manifolds. Let $\chi : X_1 \to X_2$ be a holomorphic map. We say that $\chi$ is a contact transformation if $\chi^*\omega$ is a local generator of $L_1$ whenever $\omega$ is a local generator of $L_2$.

Let $\theta = \xi dx + \eta dy$ denote the canonical 1-form of $T^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{C}^2$. Let $\pi : \mathbb{P}^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{P}^1 \to \mathbb{C}^2$ be the projective cotangent bundle of $\mathbb{C}^2$, where $\pi(x, y; \xi : \eta) = (x, y)$. Let $U [V]$ be the open subset of $\mathbb{P}^*\mathbb{C}^2$ defined by $\eta \neq 0 [\xi \neq 0]$. Then $\theta/\eta [\theta/\xi]$ defines a contact form $dy - pdx$ on $U$, where $p = -\xi/\eta [q = -\eta/\xi]$. Moreover, $dy - pdx$ and $dx - qdy$ define a structure of contact manifold on $\mathbb{P}^*\mathbb{C}^2$.

If $\Phi(x, y) = (a(x, y), b(x, y))$ with $a, b \in \mathbb{C}\{x, y}\}$ is an automorphism of $(\mathbb{C}^2, (0, 0))$, we associate to $\Phi$ the germ of contact transformation

$$\chi : (\mathbb{P}^*\mathbb{C}^2, (0, 0; 0 : 1)) \to (\mathbb{P}^*\mathbb{C}^2, (0, 0; -\partial_x b(0, 0) : \partial_x a(0, 0)))$$

defined by

$$\chi(x, y; \xi : \eta) = (a(x, y), b(x, y); \partial_y b\xi - \partial_x b\eta : -\partial_y a\xi + \partial_x a\eta).$$

If $D\Phi_{(0, 0)}$ leaves invariant $\{y = 0\}$, then $\partial_x b(0, 0) = 0$, $\partial_x a(0, 0) \neq 0$ and $\chi(0, 0; 0 : 1) = (0, 0; 0 : 1)$. Moreover,

$$\chi(x, y, p) = (a(x, y), b(x, y), (\partial_y bp + \partial_x b)/(\partial_y ap + \partial_x a)).$$

Let $(X, L)$ be a contact manifold. A curve $L$ in $X$ is called Legendrian if $\omega|_L = 0$ for each section $\omega$ of $L$.

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Let \( Z \) be the germ at \((0, 0)\) of an irreducible plane curve parametrized by
\begin{equation}
\varphi(t) = (x(t), y(t)).
\end{equation}
We define the conormal of \( Z \) as the curve parametrized by
\begin{equation}
\psi(t) = (x(t), y(t) ; -y'(t) : x'(t)).
\end{equation}

The conormal of \( Z \) is the germ of a Legendrian curve of \( \mathbb{P}^* \mathbb{C}^2 \).

We will denote the conormal of \( Z \) by \( \mathbb{P}^* _{} Z \mathbb{C}^2 \) and the parametrization (1.3) by \( \text{Con} \varphi \).

Assume that the tangent cone \( C(Z) \) is defined by the equation \( ax + by = 0 \), with \( (a, b) \neq (0, 0) \). Then \( \mathbb{P}^* _{} Z \mathbb{C}^2 \) is a germ of a Legendrian curve at \((0, 0; a : b)\).

Let \( f \in \mathbb{C} \{ t \} \). We say the \( f \) has order \( k \) and write \( \text{ord} f = k \) or \( \text{ord}_t f = k \) if \( f/t^k \) is a unit of \( \mathbb{C} \{ t \} \).

Remark 1.1. Let \( Z \) be the plane curve parametrized by (1.2). Let \( L = \mathbb{P}^* _{} Z \mathbb{C}^2 \).

Then:
(i) \( C(Z) = \{ y = 0 \} \) if and only if \( \text{ord} y > \text{ord} x \). If \( C(Z) = \{ y = 0 \} \), \( L \) admits the parametrization
\[
\psi(t) = (x(t), y(t), y'(t)/x'(t))
\]
on the chart \((x, y, p)\).
(ii) \( C(Z) = \{ y = 0 \} \) and \( C(L) = \{ x = y = 0 \} \) if and only if \( \text{ord} x < \text{ord} y < 2\text{ord} x \).
(iii) \( C(Z) = \{ y = 0 \} \) and \( \{ x = y = 0 \} \nsubseteq C(L) \subseteq \{ y = 0 \} \) if and only if \( \text{ord} y \geq 2\text{ord} x \).
(iv) \( C(L) = \{ y = p = 0 \} \) if and only if \( \text{ord} y > 2\text{ord} x \).
(v) \( \text{mult} L \leq \text{mult} Z \). Moreover, \( \text{mult} L = \text{mult} Z \) if and only if \( \text{ord} y \geq 2\text{ord} x \).

If \( L \) is the germ of a Legendrian curve at \((0, 0; a : b)\), \( \pi(L) \) is a germ of a plane curve of \((\mathbb{C}^2, (0, 0))\). Notice that all branches of \( \pi(L) \) have the same tangent cone.

If \( Z \) is the germ of a plane curve with irreducible tangent cone, the union \( L \) of the conormals of the branches of \( Z \) is a germ of a Legendrian curve. We call \( L \) the conormal of \( Z \).

If \( C(Z) \) has several components, the union of the conormals of the branches of \( Z \) is a union of several germs of Legendrian curves.

If \( L \) is a germ of Legendrian curve, \( L \) is the conormal of \( \pi(L) \).

Consider in the vector space \( \mathbb{C}^2 \), with coordinates \( x, p \), the symplectic form \( dp \land dx \). We associate to each symplectic linear automorphism
\[
(p, x) \mapsto (\alpha p + \beta x, \gamma p + \delta x)
\]
of \( \mathbb{C}^2 \) the contact transformation
\begin{equation}
(x, y) = (\gamma p + \delta x, y + \frac{1}{2} \alpha \gamma p^2 + \beta \gamma x p + \frac{1}{2} \beta \delta x^2, \alpha p + \beta x).
\end{equation}
We call (1.4) a paraboloidal contact transformation.
In the case \( \alpha = \delta = 0 \) and \( \gamma = -\beta = 1 \) we get the so called Legendre transformation
\[
\Psi(x, y, p) = (p, y - px, -x).
\]

We say that a germ of a Legendrian curve \( L \) of \( (\mathbb{P}^*\mathbb{C}^2, (0, 0; a : b)) \) is in generic position if \( C(L) \not\supset \pi^{-1}(0, 0) \).

**Remark 1.2.** Let \( L \) be the germ of a Legendrian curve on a contact manifold \((X, \mathcal{L})\) at a point \( o \). By the Darboux’s theorem for contact forms there is a germ of a contact transformation \( \chi : (X, o) \to (U, (0, 0, 0)) \), where \( U = \{ \eta \neq 0 \} \) is the open subset of \( \mathbb{P}^*\mathbb{C}^2 \) considered above. Hence \( C(\pi(\chi(L))) = \{ y = 0 \} \). Applying a paraboloidal transformation to \( \chi(L) \) we can assume that \( \chi(L) \) is in generic position. If \( C(L) \) is irreducible, we can assume \( C(\chi(L)) = \{ y = p = 0 \} \).

Following the above remark, from now on we will always assume that every Legendrian curve germ is embedded in \((\mathbb{C}^3(x,y,p), \omega)\), where \( \omega = dy - pdx \).

**Example 1.3.**

(1) The plane curve \( Z = \{ y^2 - x^3 = 0 \} \) admits a parametrization \( \varphi(t) = (t^2, t^3) \). The conormal \( L \) of \( Z \) admits the parametrization \( \psi(t) = (t^2, t^3, \frac{3}{2}t) \). Hence \( C(L) = \pi^{-1}(0, 0) \) and \( L \) is not in generic position. If \( \chi \) is the Legendre transformation, \( C(\chi(L)) = \{ y = p = 0 \} \) and \( L \) is in generic position. Moreover, \( \pi(\chi(L)) \) is a smooth curve.

(2) The plane curve \( Z = \{(y^2 - x^3)(y^2 - x^5) = 0\} \) admits a parametrization given by
\[
\varphi_1(t_1) = (t_1^2, t_1^3), \quad \varphi_2(t_2) = (t_2^2, t_2^5).
\]
The conormal \( L \) of \( Z \) admits the parametrization given by
\[
\psi_1(t_1) = (t_1^2, t_1^3, \frac{3}{2}t_1), \quad \psi_2(t_2) = (t_2^2, t_2^5, \frac{5}{2}t_2^3).
\]
Hence \( C(L_1) = \pi^{-1}(0, 0) \) and \( L \) is not in generic position. If \( \chi \) is the paraboloidal contact transformation
\[
\chi : (x, y, p) \mapsto (x + p, y + \frac{1}{2}p^2, p),
\]
then \( \chi(L) \) has branches with parametrization given by
\[
\chi(\psi_1)(t_1) = (t_1^2 + \frac{3}{2}t_1, t_1^3 + \frac{9}{8}t_1^2, \frac{3}{2}t_1),
\]
\[
\chi(\psi_2)(t_2) = (t_2^2 + \frac{5}{2}t_2^3, t_2^5 + \frac{25}{8}t_2^6, \frac{5}{2}t_2^3).
\]
Then
\[
C(\chi(L_1)) = \{ y = p - x = 0 \}, \quad C(\chi(L_2)) = \{ y = p = 0 \}
\]
and \( L \) is in generic position.

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Set \( x = (x_1, \ldots, x_n) \) and \( z = (z_1, \ldots, z_m) \). Let \( I \) be an ideal of the ring \( \mathbb{C}\{z\} \). Let \( \tilde{I} \) be the ideal of \( \mathbb{C}\{x, z\} \) generated by \( I \).

**Lemma 2.1.** (a) Let \( f \in \mathbb{C}\{x, z\} \). Then \( f \in \tilde{I} \) if and only if \( a_\alpha \in I \) for each \( \alpha \).
(b) If \( f \in \tilde{I} \), then \( \partial_x f \in \tilde{I} \) for \( 1 \leq i \leq n \).
(c) Let \( a_1, \ldots, a_{n-1} \in \mathbb{C}\{x, z\} \). Let \( b, \beta_0 \in \tilde{I} \). Assume that \( \partial_x \beta_0 = 0 \). If \( \beta \) is the solution of the Cauchy problem

\[
\partial_x \beta - \sum_{i=1}^{n-1} a_i \partial_{x_i} \beta = b, \quad \beta - \beta_0 \in \mathbb{C}\{x, z\} x_n, \tag{2.1}
\]

then \( \beta \in \tilde{I} \).

**Proof.** There are \( g_1, \ldots, g_\ell \in \mathbb{C}\{z\} \) such that \( I = (g_1, \ldots, g_\ell) \). If \( a_\alpha \in I \) for each \( \alpha \), then \( h_{i,\alpha} \in \mathbb{C}\{z\} \) such that \( a_\alpha = \sum_{i=1}^\ell h_{i,\alpha} g_i \). Hence \( f = \sum_{i=1}^\ell (\sum_{\alpha} h_{i,\alpha} x^\alpha) g_i \in \tilde{I} \).

If \( f \in \tilde{I} \), then \( H_i \in \mathbb{C}\{x, z\} \) such that \( f = \sum_{i=1}^\ell H_i g_i \). There are \( b_{i,\alpha} \in \mathbb{C}\{z\} \) such that \( H_i = \sum_{\alpha} b_{i,\alpha} x^\alpha \). Therefore \( a_\alpha = \sum_{i=1}^\ell b_{i,\alpha} g_i \in I \).

We can perform a change of variables that rectifies the vector field \( \partial_x - \sum_{i=1}^{n-1} a_i \partial_{x_i} \), reducing the Cauchy problem (2.1) to the Cauchy problem

\[
\partial_x \beta = b, \quad \beta - \beta_0 \in \mathbb{C}\{x, z\} x_n, \tag{2.2}
\]

Hence statements (b) and (c) follow from (a). \( \square \)

Let \( J \) be an ideal of \( \mathbb{C}\{z\} \) contained in \( I \). Let \( X, S \) and \( T \) be analytic spaces with local rings \( \mathbb{C}\{x\}, \mathbb{C}\{z\}/I \) and \( \mathbb{C}\{z\}/J \). Hence \( X \times S \) and \( X \times T \) have local rings \( \mathcal{O} := \mathbb{C}\{x, z\}/I \) and \( \mathcal{O} := \mathbb{C}\{x, z\}/J \). Let \( a_1, \ldots, a_{n-1}, b \in \mathcal{O} \) and \( g \in \mathcal{O}/x_n \mathcal{O} \). Let \( a_i, b \in \mathcal{O} \) and \( g \in \mathcal{O}/x_n \mathcal{O} \) be representatives of \( a_i, b \) and \( g \). Consider the Cauchy problems

\[
\partial_x f + \sum_{i=1}^{n-1} a_i \partial_{x_i} f = b, \quad f + x_n \mathcal{O} = g \tag{2.2}
\]

and

\[
\partial_x f + \sum_{i=1}^{n-1} a_i \partial_{x_i} f = b, \quad f + x_n \mathcal{O} = g \tag{2.3}
\]

**Theorem 2.2.** (a) There is one and only one solution of the Cauchy problem (2.2).
(b) If \( f \) is a solution of (2.2), \( f = f + \tilde{I} \) is a solution of (2.3).
(c) If \( f \) is a solution of (2.3) there is a representative \( \tilde{f} \) of \( f \) that is a solution of (2.2).
Proof. By Lemma 2.1, \( \partial_{x_i} \tilde{I} = \tilde{I} \). Hence (b) holds.

Assume \( J = (0) \). The existence and uniqueness of the solution of (2.2) is a special case of the classical Cauchy-Kowalevski Theorem. There is one and only one formal solution of (2.2). Its convergence follows from the majorant method.

The existence of a solution of (2.3) follows from (b).

Let \( f_1, f_2 \) be two solutions of (2.3). Let \( f_j \) be a representative of \( f_j \) for \( j = 1, 2 \). Then \( \partial_{x_n} (f_2 - f_1) + \sum_{i=1}^{n-1} a_i \partial_{x_i} (f_2 - f_1) \in \tilde{I} \) and \( f_2 - f_1 + x_n \tilde{O} \in \tilde{I} + x_n \tilde{O} \). By Lemma 2.1, \( f_2 - f_1 \in \tilde{I} \). Therefore \( f_1 = f_2 \). This ends the proof of statement (a). Statement (c) follows from statements (a) and (b).

\[ \square \]

Set \( \Omega^1_{X|S} = \bigoplus_{i=1}^n \mathcal{O}dx_i \). We call the elements of \( \Omega^1_{X|S} \) germs of relative differential forms on \( X \times S \). The map \( d : \mathcal{O} \to \Omega^1_{X|S} \) given by \( df = \sum_{i=1}^n \partial_{x_i} f dx_i \) is called the relative differential of \( f \).

Assume that \( \dim X = 3 \) and let \( \mathcal{L} \) be a contact structure on \( X \). Let \( \rho : X \times S \to X \) be the first projection. Let \( \omega \) be a generator of \( \mathcal{L} \). We will denote by \( \mathcal{L}_S \) the sub \( \mathcal{O} \)-module of \( \Omega^1_{X|S} \) generated by \( \rho^* \omega \). We call \( \mathcal{L}_S \) a relative contact structure of \( X \times S \). We call \( (X \times S, \mathcal{L}_S) \) a relative contact manifold. We say that an isomorphism of analytic spaces

\[ (2.4) \]

\[ \chi : X \times S \to X \times S \]

is a relative contact transformation if \( \chi(0, s) = (0, s) \), \( \chi^* \omega \in \mathcal{L}_S \) for each \( \omega \in \mathcal{L}_S \) and the diagram

\[ (2.5) \]

\[
\begin{array}{ccc}
X & \xrightarrow{id_X} & X \\
\downarrow & & \downarrow \\
X \times S & \xrightarrow{\chi} & X \times S \\
\downarrow & & \downarrow \\
S & \xrightarrow{id_S} & S
\end{array}
\]

commutes.

The demand of the commutativeness of diagram (2.5) is a very restrictive condition but these are the only relative contact transformations we will need. We can and will assume that the local ring of \( X \) equals \( \mathbb{C}\{x,y,p\} \) and that \( \mathcal{L} \) is generated by \( dy - pdx \).

Set \( \mathcal{O} = \mathbb{C}\{x,y,p,z\}/I \) and \( \mathcal{O} = \mathbb{C}\{x,y,p,z\}/J \). Let \( m_X \) be the maximal ideal of \( \mathbb{C}\{x,y,p\} \). Let \( m_{\mathcal{O}} \) be the maximal ideal of \( \mathbb{C}\{z\}/I[\mathbb{C}\{z\}/J] \). Let \( n_\mathcal{O} \) be the ideal of \( \mathcal{O}[\mathcal{O}] \) generated by \( m_X m_{\mathcal{O}} [m_X m_{\mathcal{O}}] \).
Remark 2.3. If (2.4) is a relative contact transformation, there are \( \alpha, \beta, \gamma \in \mathfrak{n} \) such that \( \partial_x \beta \in \mathfrak{n} \) and
\[
(2.6) \quad \chi(x, y, p, z) = (x + \alpha, y + \beta, p + \gamma, z).
\]

**Theorem 2.4.** (a) Let \( \chi : X \times S \to X \times S \) be a relative contact transformation. There is \( \beta_0 \in \mathfrak{n} \) such that \( \partial_p \beta_0 = 0 \), \( \partial_x \beta_0 \in \mathfrak{n} \), \( \beta \) is the solution of the Cauchy problem
\[
(2.7) \quad \left(1 + \frac{\partial \alpha}{\partial x} + p \frac{\partial \alpha}{\partial y}\right) \frac{\partial \beta}{\partial p} - \left(p \frac{\partial \alpha}{\partial y} - \frac{\partial \alpha}{\partial x} \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial p} \frac{\partial \beta}{\partial y}\right) = \frac{\partial \alpha}{\partial p}, \quad \beta - \beta_0 \in p\mathcal{O}
\]
and
\[
(2.8) \quad \gamma = \left(1 + \frac{\partial \alpha}{\partial x} + p \frac{\partial \alpha}{\partial y}\right)^{-1} \left(\frac{\partial \beta}{\partial x} + p \left(\frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial x} \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial p} \frac{\partial \beta}{\partial y}\right)\right).
\]
(b) Given \( \alpha, \beta_0 \in \mathfrak{n} \) such that \( \partial_p \beta_0 = 0 \) and \( \partial_x \beta_0 \in \mathfrak{n} \), there is a unique contact transformation \( \chi \) verifying the conditions of statement (a). We will denote \( \chi \) by \( \chi_{\alpha, \beta_0} \).
(c) Given a relative contact transformation \( \tilde{\chi} : X \times T \to X \times T \) there is one and only one contact transformation \( \chi : X \times S \to X \times S \) such that the diagram
\[
\begin{array}{ccc}
X \times S & \xrightarrow{\chi} & X \times S \\
\uparrow & & \uparrow \\
X \times T & \xrightarrow{\tilde{\chi}} & X \times T
\end{array}
\]
commutes.
(d) Given \( \alpha, \beta_0 \in \mathfrak{n} \) and \( \tilde{\alpha}, \tilde{\beta}_0 \in \tilde{\mathfrak{n}} \) such that \( \partial_p \beta_0 = 0 \), \( \partial_p \tilde{\beta}_0 = 0 \), \( \partial_x \beta_0 \in \mathfrak{n} \), \( \partial_x \tilde{\beta}_0 \in \tilde{\mathfrak{n}} \) and \( \tilde{\alpha}, \tilde{\beta}_0 \) are representatives of \( \alpha, \beta_0 \), set \( \chi = \chi_{\alpha, \beta_0}, \tilde{\chi} = \chi_{\tilde{\alpha}, \tilde{\beta}_0} \).
Then diagram (2.9) commutes.

**Proof.** Statements (a) and (b) are a relative version of Theorem 3.2 of [1]. In [1] we assume \( S = \{0\} \). The proof works as long \( S \) is smooth. The proof in the singular case depends on the singular variant of the Cauchy-Kowalevski Theorem introduced in [2]. Statement (c) follows from statement (b) of Theorem 2.2. Statement (d) follows from statement (c) of Theorem 2.2. \( \square \)

**Remark 2.5.** (1) The inclusion \( S \hookrightarrow T \) is said to be a small extension if the surjective map \( \mathcal{O}_T \to \mathcal{O}_S \) has one dimensional kernel. If the kernel is generated by \( \varepsilon \), we have that, as complex vector spaces, \( \mathcal{O}_T = \mathcal{O}_S \oplus \varepsilon \mathbb{C} \).
Every extension of Artinian local rings factors through small extensions.

**Theorem 2.6.** Let \( S \hookrightarrow T \) be a small extension such that
\[
\mathcal{O}_S \cong \mathbb{C}\{z\},
\]
\[
\mathcal{O}_T \cong \mathbb{C}\{z, \varepsilon\}/(\varepsilon^2, \varepsilon z_1, \ldots, \varepsilon z_m) = \mathbb{C}\{z\} \oplus \mathbb{C}\varepsilon.
\]
Assume $\chi : X \times S \rightarrow X \times S$ is a relative contact transformation given at the ring level by

$$(x, y, p) \mapsto (H_1, H_2, H_3),$$

$\alpha, \beta_0 \in m_X$, such that $\partial_p \beta_0 = 0$ and $\beta_0 \in (x^2, y)$. Then, there are uniquely determined $\beta, \gamma \in m_X$ such that $\beta - \beta_0 \in pO_X$ and $\tilde{\chi} : X \times T \rightarrow X \times T$, given by

$$\tilde{\chi}(x, y, p, z, \varepsilon) = (H_1 + \varepsilon \alpha, H_2 + \varepsilon \beta, H_3 + \varepsilon \gamma, z, \varepsilon),$$

is a relative contact transformation extending $\chi$ (diagram (2.9) commutes). Moreover, the Cauchy problem (2.7) for $\tilde{\chi}$ takes the simplified form

$$(2.10) \quad \frac{\partial \beta}{\partial p} = p \frac{\partial \alpha}{\partial p}, \quad \beta - \beta_0 \in \mathbb{C}\{x, y, p\}p$$

and

$$(2.11) \quad \gamma = \frac{\partial \beta}{\partial x} + p \left( \frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial x} \right) - p^2 \frac{\partial \alpha}{\partial y}.$$ 

Proof. We have that $\tilde{\chi}$ is a relative contact transformation if and only if there is $f = f' + \varepsilon f'' \in O_T\{x, y, p\}$ with $f \notin (x, y, p)O_T\{x, y, p\}$, $f' \in O_S\{x, y, p\}$, $f'' \in \mathbb{C}\{x, y, p\} = O_X$ such that

$$(2.12) \quad d(H_2 + \varepsilon \beta) - (H_3 + \varepsilon \gamma)d(H_1 + \varepsilon \alpha) = f(dy - pdx).$$

Since $\chi$ is a relative contact transformation we can suppose that

$$dH_2 - H_3dH_1 = f'(dy - pdx).$$

Using the fact that $\varepsilon \mathfrak{m}_{O_T}$, we see that (2.12) is equivalent to

$$\begin{cases}
\frac{\partial \beta}{\partial p} = p \frac{\partial \alpha}{\partial p}, \\
\gamma = \frac{\partial \beta}{\partial x} + p \left( \frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial x} \right) - p^2 \frac{\partial \alpha}{\partial y}, \\
f'' = \frac{\partial \beta}{\partial y} - p \frac{\partial \alpha}{\partial y}.
\end{cases}$$

As $\beta - \beta_0 \in (p)\mathbb{C}\{x, y, p\}$ we have that $\beta$, and consequently $\gamma$, are completely determined by $\alpha$ and $\beta_0$. \qed

Remark 2.7. Set $\alpha = \sum_k \alpha_k p^k$, $\beta = \sum_k \beta_k p^k$, $\gamma = \sum_k \gamma_k p^k$, where $\alpha_k, \beta_k, \gamma_k \in \mathbb{C}\{x, y\}$ for each $k \geq 0$ and $\beta_0 \in (x^2, y)$. Under the assumptions of Theorem 2.6,

(i) $\beta_k = \frac{k-1}{k} \alpha_{k-1}, \quad k \geq 1$.

(ii) Moreover,

$$\gamma_0 = \frac{\partial \beta_0}{\partial x}, \quad \gamma_1 = \frac{\partial \beta_0}{\partial y} - \frac{\partial \alpha_0}{\partial x}, \quad \gamma_k = -\frac{1}{k} \frac{\partial \alpha_{k-1}}{\partial x} - \frac{1}{k-1} \frac{\partial \alpha_{k-2}}{\partial y}, \quad k \geq 2.$$ 

Since,

$$\frac{\partial}{\partial y} \gamma_0 = \frac{\partial}{\partial x} \left( \frac{\partial \alpha_0}{\partial x} + \gamma_1 \right),$$

$\beta_0$ is the solution of the Cauchy problem

$$\frac{\partial \beta_0}{\partial x} = \gamma_0, \quad \frac{\partial \beta_0}{\partial y} = \frac{\partial \alpha_0}{\partial x} + \gamma_1, \quad \beta_0 \in (x^2, y).$$
A category \( C \) is called a groupoid if all morphisms of \( C \) are isomorphisms.

Let \( p : F \rightarrow C \) be a functor.

Let \( S \) be an object of \( C \). We will denote by \( F(S) \) the subcategory of \( F \) given by the following conditions:

- \( \Psi \) is an object of \( F(S) \) if \( p(\Psi) = S \).
- \( \chi \) is a morphism of \( F(S) \) if \( p(\chi) = \text{id}_S \).

Let \( \chi : \Psi \rightarrow \Psi \) be a morphism [an object] of \( F \) over \( f : S' \rightarrow S \). We say that \( \chi \) is a morphism [an object] of \( F \) over \( f \) if \( p(\chi) = f [p(\Psi) = S] \).

A morphism \( \chi' : \Psi' \rightarrow \Psi \) of \( F \) over \( f \) is cartesian if for each morphism \( \chi'' : \Psi'' \rightarrow \Psi \) of \( F \) over \( f \) there is exactly one morphism \( \chi : \Psi'' \rightarrow \Psi' \) over \( \text{id}_S \) such that \( \chi'' \circ \chi = \chi' \).

If the morphism \( \chi' : \Psi' \rightarrow \Psi \) is cartesian, \( \Psi' \) is well defined up to a unique isomorphism. We will denote \( \Psi' \) by \( f^* \Psi \) or \( \Psi \times S S' \).

A fibered groupoid is a fibered category such that \( F(S) \) is a groupoid for each \( S \in C \).

Lemma 3.1. If \( p : F \rightarrow C \) satisfies (1) and \( F(S) \) is a groupoid for each object \( S \) of \( C \), then \( F \) is a fibered groupoid over \( C \).

Proof. Let \( \chi : \Phi \rightarrow \Psi \) be an arbitrary morphism of \( F \). It is enough to show that \( \chi \) is cartesian. Set \( f = p(\chi) \). Let \( \chi' : \Phi' \rightarrow \Psi \) be another morphism over \( f \). Let \( f^* \Psi \rightarrow \Psi \) be a cartesian morphism over \( f \). There are morphisms \( \alpha : \Phi' \rightarrow f^* \Psi, \beta : \Phi \rightarrow f^* \Psi \) such that the solid diagram

\[
\begin{array}{ccc}
\Phi' & \xrightarrow{\alpha} & f^* \Psi \\
\downarrow{\chi'} & & \downarrow{\chi} \\
\Phi & \xrightarrow{\beta} & \Psi
\end{array}
\]  

commutes. Hence \( \beta^{-1} \circ \alpha \) is the only morphism over \( f \) such that diagram (3.1) commutes.

Let \( \mathbb{A} \) be the category of analytic complex space germs. Let 0 denote the complex vector space of dimension 0.

Definition 3.2. Let \( T \) be an analytic complex space germ. Let \( \psi \) be an object of \( \mathbb{A}(0) \) [\( \mathbb{A}(T) \)]. We say that \( \Psi \) is a versal deformation of \( \psi \) if given

- a closed embedding \( f : T'' \rightarrow T' \),
• a morphism of complex analytic space germs $g : T'' \to T$,
• an object $\Psi'$ of $\mathcal{F}(T')$ such that $f^*\Psi' \cong g^*\Psi$,

there is a morphism of complex analytic space germs $h : T' \to T$ such that

$$h \circ f = g \quad \text{and} \quad h^*\Psi \cong \Psi'.$$

If $\Psi$ is versal and for each $\Psi'$ the tangent map $T(h) : T' \to T_T$ is determined by $\Psi'$, $\Psi$ is called a \textit{semuniversal deformation} of $\psi$.

Let $T$ be a germ of a complex analytic space. Let $A$ be the local ring of $T$ and let $\mathfrak{m}$ be the maximal ideal of $A$. Let $T_n$ be the complex analytic space with local ring $A/\mathfrak{m}^n$ for each positive integer $n$. The canonical morphisms

$$A \to A/\mathfrak{m}^n \quad \text{and} \quad A/\mathfrak{m}^n \to A/\mathfrak{m}^{n+1}$$

induce morphisms $\alpha_n : T_n \to T$ and $\beta_n : T_{n+1} \to T_n$.

A morphism $f : T'' \to T'$ induces morphisms $f_n : T''_n \to T'_n$ such that the diagram

$$
\begin{array}{ccc}
T'' & \xrightarrow{f} & T' \\
\downarrow{\alpha_n''} & & \downarrow{\alpha'_n} \\
T''_n & \xrightarrow{f_n} & T'_n \\
\downarrow{\beta_n''} & & \downarrow{\beta'_n} \\
T''_{n+1} & \xrightarrow{f_{n+1}} & T'_{n+1}
\end{array}
$$

commutes.

\textbf{Definition 3.3.} We will follow the terminology of Definition 3.2. Let $g_n = g \circ \alpha_n''$. We say that $\Psi$ is a \textit{formally versal deformation} of $\psi$ if there are morphisms $h_n : T'_n \to T$ such that

$$h_n \circ f_n = g_n, \quad h_n \circ \beta_n' = h_{n+1} \quad \text{and} \quad h_n^*\Psi \cong \alpha'_n^*\Psi'.$$

If $\Psi$ is formally versal and for each $\Psi'$ the tangent maps $T(h_n) : T'_{T_n} \to T_T$ are determined by $\alpha'_n^*\Psi'$, $\Psi$ is called a \textit{formally semiuniversal deformation} of $\psi$.

\textbf{Theorem 3.4.} (\cite{4}, Theorem 5.2). Let $\mathfrak{F} \to \mathcal{C}$ be a fibered groupoid. Let $\psi \in \mathfrak{F}(0)$. If there is a versal deformation of $\psi$, every formally versal \textit{[semiuniversal deformation of $\psi$]} is versal \textit{[semiuniversal]}.

Let $Z$ be a curve of $\mathbb{C}^n$ with irreducible components $Z_1, \ldots, Z_r$. Set $\bar{C} = \bigsqcup_{i=1}^r C_i$ where each $C_i$ is a copy of $\mathbb{C}$. Let $\varphi_i$ be a parametrization of $Z_i$, $1 \leq i \leq r$. Let $\varphi : \bar{C} \to \mathbb{C}^n$ be the map such that $\varphi |_{C_i} = \varphi_i$, $1 \leq i \leq r$. We call $\varphi$ the \textit{parametrization} of $Z$. 

9
Let $T$ be an analytic space. A morphism of analytic spaces $\Phi : \bar{C} \times T \rightarrow \mathbb{C}^n \times T$ is called a deformation of $\varphi$ over $T$ if the diagram

\[
\begin{array}{ccc}
\bar{C} & \xrightarrow{\varphi} & \mathbb{C}^n \\
\downarrow & & \downarrow \\
\bar{C} \times T & \xrightarrow{\Phi} & \mathbb{C}^n \times T \\
\downarrow & & \downarrow \\
T & \xrightarrow{id_T} & T
\end{array}
\]

commutes. The analytic space $T$ is called the base space of the deformation.

We will denote by $\Phi_i$ the composition

\[
\bar{C}_i \times T \hookrightarrow \bar{C} \times T \xrightarrow{\Phi} \mathbb{C}^n \times T \rightarrow \mathbb{n}, \quad 1 \leq i \leq r.
\]

The maps $\Phi_i$, $1 \leq i \leq r$, determine $\Phi$.

Let $\Phi$ be a deformation of $\varphi$ over $T$. Let $f : T' \rightarrow T$ be a morphism of analytic spaces. We will denote by $f^*\Phi$ the deformation of $\varphi$ over $T'$ given by

\[
(f^*\Phi)_i = \Phi_i \circ (id_{\bar{C}_i} \times f).
\]

We call $f^*\Phi$ the pullback of $\Phi$ by $f$.

Let $\Phi' : \bar{C} \times T \rightarrow \mathbb{C}^n \times T$ be another deformation of $\varphi$ over $T$. A morphism from $\Phi'$ into $\Phi$ is a pair $(\chi, \xi)$ where $\chi : \mathbb{C}^n \times T \rightarrow \mathbb{C}^n \times T$ and $\xi : \bar{C} \times T \rightarrow \bar{C} \times T$ are isomorphisms of analytic spaces such that the diagram

\[
\begin{array}{ccc}
T & \xleftarrow{\text{id}_T} & \bar{C} \times T & \xrightarrow{\Phi} & \mathbb{C}^n \times T & \xrightarrow{\text{id}_T} & T \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
T & \xleftarrow{\text{id}_T} & \bar{C} \times T & \xrightarrow{\Phi'} & \mathbb{C}^n \times T & \xrightarrow{\text{id}_T} & T
\end{array}
\]

commutes.

Let $\Phi'$ be a deformation of $\varphi$ over $S$ and $f : S \rightarrow T$ a morphism of analytic spaces. A morphism of $\Phi'$ into $\Phi$ over $f$ is a morphism from $\Phi'$ into $f^*\Phi$. There is a functor $p$ that associates $T$ to a deformation $\Psi$ over $T$ and $f$ to a morphism of deformations over $f$.

Given $t \in T$ let $Z_t$ be the curve parametrized by the composition

\[
\bar{C} \times \{t\} \hookrightarrow \bar{C} \times T \xrightarrow{\Phi} \mathbb{C}^n \times T \rightarrow \mathbb{C}^n.
\]

We call $Z_t$ the fiber of the deformation $\Phi$ at the point $t$. 

10
Let $\varphi : \mathbb{C} \to \mathbb{C}^2$ be the parametrization of a plane curve $Z$. We will denote by $\text{Def}_\varphi[\text{Def}_\varphi^{\text{em}}]$ the category of deformations [equimultiple deformations] $\Phi$ of (the parametrization $\varphi$ of) the plane curve $Z$.

Consider in $\mathbb{C}^3$ the contact structure given by the differential form $dy - p\,dx$. Let $\psi : \mathbb{C} \to \mathbb{C}^3$ be the parametrization of a Legendrian curve $L$. We say that a deformation $\Psi$ of $\psi$ is a Legendrian deformation of $\psi$ if all of its fibers are Legendrian. We say that $(\chi, \xi)$ is an isomorphism of Legendrian deformations if $\chi : X \times T \to X \times T$ is a relative contact transformation.

We will denote by $\tilde{\text{Def}}_\psi[\text{Def}_\psi^{\text{em}}]$ the category of Legendrian [equimultiple Legendrian] deformations of $\psi$. All deformations are assumed to have trivial sections (see [3]).

Assume that $\psi = \text{Con}\varphi$ parametrizes a germ of a Legendrian curve $L$, in generic position, in $(\mathbb{C}^3(x,y,p),\omega)$. If $\Phi \in \text{Def}_\varphi$ is given by

$$\Phi_i(t_i,s) = (X_i(t_i,s), Y_i(t_i,s)), \quad 1 \leq i \leq r,$$

such that $P_i(t_i,s) := \partial_t Y_i(t_i,s)/\partial_t X_i(t_i,s) \in \mathbb{C}\{t_i,s\}$ for $1 \leq i \leq r$, then

$$\Psi_i(t_i,s) = (X_i(t_i,s), Y_i(t_i,s), P_i(t_i,s)).$$

defines a deformation $\Psi$ of $\psi$ which we call conormal of $\Phi$. Notice that in this case all fibers of $\Phi$ have the same tangent space $\{y = 0\}$. We will denote $\Psi$ by $\text{Con}\Phi$. If $\Psi \in \tilde{\text{Def}}_\psi$ is given by (3.3), we call plane projection of $\Psi$ to the deformation $\Phi$ of $\varphi$ given by (3.2). We will denote $\Phi$ by $\Psi^\pi$.

Let us consider the full subcategory $\rightarrow \text{Def}_\varphi$ of the deformations $\Phi \in \text{Def}_\varphi^{\text{em}}$ such that all fibers of $\Phi$ have the same tangent space $\{y = 0\}$.

**Remark 3.5.** We see immediately that if $\Phi \in \tilde{\text{Def}}_\varphi$ then $\text{Con}\Phi$ exists. However, it should be noted that there are more deformations for which the conormal is defined:

Let $\Phi$ be the deformation of $\varphi = (t^3, t^{10})$ given by

$$X(t,s) = st + t^3; \quad Y(t,s) = \frac{5}{12}st^8 + t^{10}.$$ 

Then $\text{Con}\Phi$ exists, but $\Phi$ is not equimultiple.

We define in this way the functors

$$\text{Con} : \tilde{\text{Def}}_\varphi \to \tilde{\text{Def}}_\psi, \quad \pi : \tilde{\text{Def}}_\psi \to \text{Def}_\varphi.$$ 

Notice that the conormal of the plane projection of a Legendrian deformation always exists and we have that $\text{Con}(\Psi^\pi) = \Psi$ for each $\Psi \in \tilde{\text{Def}}_\psi$ and $(\text{Con}\Phi)^\pi = \Phi$ where $\Phi \in \tilde{\text{Def}}_\varphi$.

Let us denote by $\text{Def}_\varphi$ the subcategory of equimultiple deformations $\Phi$ of $\varphi$ such that all fibers of $\Phi$ have fixed tangent space $\{y = 0\}$ with conormal in generic position. Then $\text{Def}_\varphi \subset \text{Def}_\varphi$ and if $\Phi \in \text{Def}_\varphi$ is given by 3.2
then $\Phi \in \overline{\text{Def}}_{\varphi}$ iff

$$\text{ord}_i Y_i \geq 2 \text{ord}_i X_i, \quad 1 \leq i \leq r.$$  

Because we demand that $\Phi$ is equimultiple and all branches have tangent space $\{y = 0\}$, (3.4) is equivalent to

$$\text{ord}_i Y_i \geq 2 m_i, \quad 1 \leq i \leq r,$$

where $m_i$ is the multiplicity of the component $Z_i$ of $Z$.

Lemma 3.6. Under the assumptions above,

$$\text{Con}(\overline{\text{Def}}_{\varphi}) \subset \overline{\text{Def}}_{\varphi}^\text{em} \quad \text{and} \quad (\overline{\text{Def}}_{\varphi}^\text{em})^\pi \subset \overline{\text{Def}}_{\varphi}.$$

Proof. Let $m_i$ be the multiplicity of the component $Z_i$ of $Z$. Let $Z_{i,s}[L_{i,s}]$ be the fiber of $\Phi[\Psi]$ (given by 3.2). If $\Phi \in \overline{\text{Def}}_{\varphi}$, $C(L_{i,s}) \not\supset \pi^{-1}(0,0)$ for each $s$, so $\text{ord}_i Y_i \geq 2 \text{ord}_i X_i = 2 m_i$. Hence $\text{ord}_i P_i \geq m_i$ and $\Psi$ is equimultiple.

If $\Psi \in \overline{\text{Def}}_{\varphi}^\text{em}$, $\text{ord}_i P_i \geq \text{ord}_i X_i$ and we get that $C(L_{i,s}) \not\supset \pi^{-1}(0,0)$ for each $s$. Each component $L_{i,s}$ has multiplicity $m_i$ for each $s$. Hence $\text{mult} Z_{i,s} \geq m_i$ for each $s$. Since multiplicity is semicontinuous, $\text{mult} Z_{i,s} = m_i$ for each $s$ and $\Phi$ is equimultiple.

Lemma 3.7. If $\mathcal{E}$ is one of the categories $\overline{\text{Def}}_{\varphi}$, $\overline{\text{Def}}_{\varphi}^\text{em}$, $p : \mathcal{E} \to \mathfrak{An}$ is a fibered groupoid.

Proof. Let $f : S \to T$ be a morphism of $\mathfrak{An}$. Let $\Psi$ be a deformation over $T$. Then, $(\overline{\chi}, \overline{\xi}) : f^*\Psi \to \Psi$ is cartesian, with

$$\overline{\xi}(t_i, s) = (t_i, s), \quad \overline{\chi}(x, y, p, s) = (x, y, p, s).$$

This is because if $(\chi, \xi) : \Psi' \to \Psi$ is a morphism over $f$, then by definition of morphism of deformations over different base spaces, $(\chi, \xi)$ is a morphism from $\Psi'$ into $f^*\Psi$ over $id_S$.  

4. Equimultiple Versal Deformations

For Sophus Lie a contact transformation was a transformation that takes curves into curves, instead of points into points. We can recover the initial point of view. Given a plane curve $Z$ at the origin, with tangent cone $\{y = 0\}$, and a contact transformation $\chi$ from a neighbourhood of $(0; dy)$ into itself, $\chi$ acts on $Z$ in the following way: $\chi \cdot Z$ is the plane projection of the image by $\chi$ of the conormal of $Z$. We can define in a similar way the action of a relative contact transformation on a deformation of a plane curve $Z$, obtaining another deformation of $Z$.

We say that $\Phi \in \overline{\text{Def}}_{\varphi}(T)$ is trivial (relative to the action of the group of relative contact transformations over $T$) if there is $\chi$ such that $\chi \cdot \Phi := \pi \circ \chi \circ \text{Con} \Phi$ is the constant deformation of $\phi$ over $T$, given by

$$(t_i, s) \mapsto \varphi_i(t_i), \quad i = 1, \ldots, r.$$
Let \( Z \) be the germ of a plane curve parametrized by \( \varphi : \mathbb{C} \to \mathbb{C}^2 \). In the following we will identify each ideal of \( O_Z \) with its image by \( \varphi^* : O_Z \to O_{\mathbb{C}} \).

Hence

\[
O_Z = \mathbb{C} \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_r \\ y_1 \\ \vdots \\ y_r \end{bmatrix} \right\} \subset \bigoplus_{i=1}^r \mathbb{C} \{ t_i \} = O_{\mathbb{C}}.
\]

Set \( \dot{x} = [\dot{x}_1, \ldots, \dot{x}_r]^t \), where \( \dot{x}_i \) is the derivative of \( x_i \) in order to \( t_i \), \( 1 \leq i \leq r \).

Let

\[
\dot{\varphi} := \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y}
\]

be an element of the free \( O_{\mathbb{C}} \)-module

\[
O_{\mathbb{C}} \frac{\partial}{\partial x} \oplus O_{\mathbb{C}} \frac{\partial}{\partial y}.
\]

Notice that (4.1) has a structure of \( O_Z \)-module induced by \( \varphi^* \).

Let \( m_i \) be the multiplicity of \( Z_i \), \( 1 \leq i \leq r \). Consider the \( O_{\mathbb{C}} \)-module

\[
\bigoplus_{i=1}^r t_i^{m_i} \mathbb{C}\{ t_i \} \frac{\partial}{\partial x} \oplus \bigoplus_{i=1}^r t_i^{2m_i} \mathbb{C}\{ t_i \} \frac{\partial}{\partial y}.
\]

Let \( m_{\mathbb{C}\dot{\varphi}} \) be the sub \( O_{\mathbb{C}} \)-module of (4.2) generated by

\[
(a_1, \ldots, a_r) \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right),
\]

where \( a_i \in t_i \mathbb{C}\{ t_i \}, 1 \leq i \leq r \). For \( i = 1, \ldots, r \) set \( p_i = \dot{y}_i / \dot{x}_i \). For each \( k \geq 0 \) set

\[
P^k = \begin{bmatrix} p_1^k & \ldots & p_r^k \end{bmatrix}^t.
\]

Let \( \tilde{\mathcal{I}} \) be the sub \( O_Z \)-module of (4.2) generated by

\[
P^k \frac{\partial}{\partial x} + k + 1 \frac{\partial}{\partial y}, \quad k \geq 1.
\]

Set

\[
\hat{M}_\varphi = \left( \bigoplus_{i=1}^r t_i^{m_i} \mathbb{C}\{ t_i \} \frac{\partial}{\partial x} \right) \oplus \left( \bigoplus_{i=1}^r t_i^{2m_i} \mathbb{C}\{ t_i \} \frac{\partial}{\partial y} \right) + m_{\mathbb{C}\dot{\varphi}} + (x, y) \frac{\partial}{\partial x} \oplus (x^2, y) \frac{\partial}{\partial y} + \tilde{\mathcal{I}}.
\]

Given a category \( \mathcal{C} \) we will denote by \( \mathcal{C} \) the set of isomorphism classes of elements of \( \mathcal{C} \).

**Theorem 4.1.** Let \( \psi \) be the parametrization of a germ of a Legendrian curve \( L \) of a contact manifold \( X \). Let \( \chi : X \to \mathbb{C}^3 \) be a contact transformation such that \( \chi(L) \) is in generic position. Let \( \varphi \) be the plane projection of \( \chi \circ \psi \).

Then there is a canonical isomorphism

\[
\hat{\text{Def}}_{\psi} (T_e) \sim \hat{M}_\varphi.
\]
Proof. Let $\Psi \in \widehat{Def}_{\phi}^{\em} (T_\varepsilon)$. By Lemma 3.6, $\Psi$ is the conormal of its projection $\Phi \in \widehat{Def}_{\phi}(T_\varepsilon)$. Moreover, $\Psi$ is given by

$$\Psi_i(t_i, \varepsilon) = (x_i + \varepsilon a_i, y_i + \varepsilon b_i, p_i + \varepsilon c_i),$$

where $a_i, b_i, c_i \in \mathbb{C}\{t_i\}$, odd $a_i \geq m_i$, odd $b_i \geq 2m_i$, $i = 1, \ldots, r$. The deformation $\Psi$ is trivial if and only if $\Phi$ is trivial for the action of the relative contact transformations. $\Phi$ is trivial if and only if there are

$$\xi_i(t_i) = \tilde{t}_i = t_i + \varepsilon h_i,$$

such that $\chi$ is a relative contact transformation, $\xi_i$ is an isomorphism, $\alpha, \beta, \gamma \in (x, y, p)\mathbb{C}\{x, y, p\}$, $h_i \in t_i\mathbb{C}\{t_i\}$, $1 \leq i \leq r$, and

$$x_i(t_i) + \varepsilon a_i(t_i) = x_i(\tilde{t}_i) + \varepsilon \alpha(x_i(\tilde{t}_i), y_i(\tilde{t}_i), p_i(\tilde{t}_i)),$$

$$y_i(t_i) + \varepsilon b_i(t_i) = y_i(\tilde{t}_i) + \varepsilon \beta(x_i(\tilde{t}_i), y_i(\tilde{t}_i), p_i(\tilde{t}_i)),$$

for $i = 1, \ldots, r$. By Taylor’s formula $x_i(\tilde{t}_i) = x_i(t_i) + \varepsilon \dot{x}_i(t_i) h_i(t_i)$, $y_i(\tilde{t}_i) = y_i(t_i) + \varepsilon \dot{y}_i(t_i) h_i(t_i)$ and

$$\varepsilon \alpha(x_i(\tilde{t}_i), y_i(\tilde{t}_i), p_i(\tilde{t}_i)) = \varepsilon \alpha(x_i(t_i), y_i(t_i), p_i(t_i)),$$

$$\varepsilon \beta(x_i(\tilde{t}_i), y_i(\tilde{t}_i), p_i(\tilde{t}_i)) = \varepsilon \beta(x_i(t_i), y_i(t_i), p_i(t_i)),$$

for $i = 1, \ldots, r$. Hence $\Phi$ is trivialized by $\chi$ if and only if

$$(4.3) \quad a_i(t_i) = \dot{x}_i(t_i) h_i(t_i) + \alpha(x_i(t_i), y_i(t_i), p_i(t_i)),$$

$$(4.4) \quad b_i(t_i) = \dot{y}_i(t_i) h_i(t_i) + \beta(x_i(t_i), y_i(t_i), p_i(t_i)),$$

for $i = 1, \ldots, r$. By Remark 2.7 (i), (4.3) and (4.4) are equivalent to the condition

$$a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} \in m_{\mathbb{C}} \hat{\varphi} + (x, y) \frac{\partial}{\partial x} \oplus (x^2, y) \frac{\partial}{\partial y} + \hat{I}.$$ 

Set

$$M_{\varphi} = \left( \bigoplus_{i=1}^{r} t_i^{m_i} \mathbb{C}\{t_i\} \frac{\partial}{\partial x} \oplus \left( \bigoplus_{i=1}^{r} t_i^{m_i} \mathbb{C}\{t_i\} \frac{\partial}{\partial y} \right) \right),$$

$$M_{\varphi}^{-} = \left( \bigoplus_{i=1}^{r} t_i^{m_i} \mathbb{C}\{t_i\} \frac{\partial}{\partial x} \oplus \left( \bigoplus_{i=1}^{r} t_i^{2m_i} \mathbb{C}\{t_i\} \frac{\partial}{\partial y} \right) \right).$$

By Proposition 2.27 of [3],

$$\widehat{Def}_{\phi}^{\em}(T_\varepsilon) \cong M_{\varphi}^{-}. $$

A similar argument shows that

$$\widehat{Def}_{\phi}(T_\varepsilon) \cong M_{\varphi}.$$


We have linear maps

\[(4.5) \quad M_\varphi \xhookrightarrow{\iota} \tilde{M}_\varphi \to \hat{M}_\varphi.\]

**Theorem 4.2** ([3], II Theorem 2.38 (3)). Set \(k = \dim M_\varphi\). Let \(a^j, b^i \in \bigoplus_{i=1}^r t_i^m \mathbb{C}\{t_i\}, 1 \leq j \leq k\). If

\[(4.6) \quad a^j \frac{\partial}{\partial x} + b^j \frac{\partial}{\partial y} = \begin{bmatrix} a^j_1 \\ \vdots \\ a^j_r \end{bmatrix} \frac{\partial}{\partial x} + \begin{bmatrix} b^j_1 \\ \vdots \\ b^j_r \end{bmatrix} \frac{\partial}{\partial y},\]

\(1 \leq j \leq k\), represents a basis of \(M_\varphi\), the deformation \(\Phi : \tilde{\mathbb{C}} \times \mathbb{C}^k \to \mathbb{C}^2 \times \mathbb{C}^k\) given by

\[(4.7) \quad X_i(t_i, s) = x_i(t_i) + \sum_{j=1}^k a^j_i(t_i)s_j, \quad Y_i(t_i, s) = y_i(t_i) + \sum_{j=1}^k b^j_i(t_i)s_j, \quad i = 1, \ldots, r,\]

is a semiuniversal deformation of \(\varphi\) in \(\text{Def}^{\text{em}} \varphi\).

**Lemma 4.3.** Set \(k = \dim \hat{M}_\varphi\). Let \(a^j \in \bigoplus_{i=1}^r t_i^m \mathbb{C}\{t_i\}, b^j \in \bigoplus_{i=1}^r t_i^{2m} \mathbb{C}\{t_i\}, 1 \leq j \leq k\). If \((4.6)\) represents a basis of \(\hat{M}_\varphi\), the deformation \(\Phi_\ast\) given by

\[(4.7) \quad X_i(t_i, s) = x_i(t_i) + \sum_{j=1}^k a^j_i(t_i)s_j, \quad Y_i(t_i, s) = y_i(t_i) + \sum_{j=1}^k b^j_i(t_i)s_j, \quad i = 1, \ldots, r,\]

is a semiuniversal deformation of \(\varphi\) in \(\text{Def}^{\text{em}} \varphi\). Moreover, \(\text{Con} \Phi_\ast\) is a versal deformation of \(\psi\) in \(\text{Def}^{\text{em}} \psi\).

**Proof.** We will only show the completeness of \(\Phi\) and \(\text{Con} \Phi\). Since the linear inclusion map \(\iota\) referred in \((4.5)\) is injective, the deformation \(\Phi\) is the restriction to \(\hat{M}_\varphi\) of the deformation \(\Phi\) introduced in Theorem 4.2. Let \(\Phi_0 \in \text{Def}^{\text{em}} (T)\). Since \(\Phi_0 \in \text{Def}^{\text{em}} \varphi(T)\), there is a morphism of analytic spaces \(f : T \to M_\varphi\) such that \(\Phi_0 \cong f^\ast \Phi\). Since \(\Phi_0 \in \text{Def}^{\text{em}} \varphi(T)\), \(f(T) \subset \hat{M}_\varphi\). Hence \(f^\ast \Phi = f^\ast \Phi\).

If \(\Psi \in \text{Def}^{\text{em}} \psi(T)\), \(\Psi^\ast \in \text{Def}^{\text{em}} \varphi(T)\). Hence there is \(f : T \to \hat{M}_\varphi\) such that \(\Psi^\ast \cong f^\ast \Phi\). Therefore \(\Psi = \text{Con} \Psi^\ast \cong \text{Con} f^\ast \Phi = f^\ast \text{Con} \Phi\).

**Theorem 4.4.** Let \(a^j \in \bigoplus_{i=1}^r t_i^m \mathbb{C}\{t_i\}, b^j \in \bigoplus_{i=1}^r t_i^{2m} \mathbb{C}\{t_i\}, 1 \leq j \leq \ell\). Assume that \((4.6)\) represents a basis \([\text{a system of generators}]\) of \(\hat{M}_\varphi\). Let \(\Phi\) be the deformation given by \((4.7)\), \(1 \leq i \leq r\). Then \(\text{Con} \Phi\) is a semiuniversal \([\text{versal}]\) deformation of \(\psi\) in \(\text{Def}^{\text{em}} \psi\).

**Proof.** By Theorem 3.4 and Lemma 4.3 it is enough to show that \(\text{Con} \Phi\) is formally semiuniversal \([\text{versal}]\).

Let \(i : T' \hookrightarrow T\) be a small extension. Let \(\Psi \in \text{Def}^{\text{em}} \psi(T)\). Set \(\Psi' = i^\ast \Psi\). Let \(\eta' : T' \to \mathbb{C}^\ell\) be a morphism of complex analytic spaces. Assume that
\((\chi', \xi')\) define an isomorphism
\[
\eta'^*\text{Con} \Phi \cong \Psi'.
\]
We need to find \(\eta : T \to \mathbb{C}^l\) and \(\chi, \xi\) such that \(\eta' = \eta \circ \iota\) and \(\chi, \xi\) define an isomorphism
\[
\eta^*\text{Con} \Phi \cong \Psi
\]
that extends \((\chi', \xi')\). Let \(A'[A']\) be the local ring of \(T[T']\). Let \(\delta\) be the generator of \(\text{Ker}(A \to A')\). We can assume \(A' \cong \mathbb{C}\{z\}/I\), where \(z = (z_1, \ldots, z_m)\). Set
\[
\widetilde{A}' = \mathbb{C}\{z\} \quad \text{and} \quad \widetilde{A} = \mathbb{C}\{z, \varepsilon\}/(\varepsilon^2, \varepsilon z_1, \ldots, \varepsilon z_m).
\]
Let \(m_A\) be the maximal ideal of \(A\). Since \(m_A\delta = 0\) and \(\delta \in m_A\), there is a morphism of local analytic algebras from \(\widetilde{A}\) onto \(A\) that takes \(\varepsilon\) into \(\delta\) such that the diagram
\[
(4.8) \quad \begin{array}{ccc}
\widetilde{A} & \longrightarrow & \widetilde{A}' \\
\downarrow & & \downarrow \\
A & \longrightarrow & A'
\end{array}
\]
commutes. Assume \(\widetilde{T} \supset \widetilde{T}'\) has local ring \(\widetilde{A}[\widetilde{A}']\). We also denote by \(\iota\) the morphism \(\widetilde{T}' \hookrightarrow \widetilde{T}\). We denote by \(\kappa\) the morphisms \(T \hookrightarrow \widetilde{T}\) and \(T' \hookrightarrow \widetilde{T}'\). Let \(\widetilde{\Psi} \in \hat{D}^{\text{con}}_{\Psi}(\widetilde{T})\) be a lifting of \(\Psi\).

We fix a linear map \(\sigma : A' \hookrightarrow \widetilde{A}'\) such that \(\kappa^*\sigma = \text{id}_{A'}\). Set \(\widetilde{\chi}' = \chi_{\sigma(\alpha), \sigma(\beta_0)}\), where \(\chi' = \chi_{\alpha, \beta_0}\). Define \(\widetilde{\eta}'\) by \(\widetilde{\eta}'s_i = \sigma(\eta^*s_i), i = 1, \ldots, l\). Let \(\widetilde{\xi}'\) be the lifting of \(\xi'\) determined by \(\sigma\). Then
\[
\widetilde{\Psi}' := \widetilde{\chi}'^{-1} \circ \widetilde{\eta}' \circ \text{Con} \Phi \circ \widetilde{\xi}'^{-1}
\]
is a lifting of \(\Psi'\) and
\[
(4.9) \quad \widetilde{\chi}' \circ \widetilde{\Psi}' \circ \widetilde{\xi}' = \widetilde{\eta}' \circ \text{Con} \Phi.
\]
By Theorem 2.4 it is enough to find liftings \(\widetilde{\chi}, \widetilde{\xi}, \widetilde{\eta}\) of \(\chi', \xi', \eta'\) such that
\[
\widetilde{\chi} \cdot \widetilde{\Psi} \circ \widetilde{\xi} = \widetilde{\eta} \circ \Phi
\]
in order to prove the theorem.
Consider the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{C} \times \tilde{T}' & \xrightarrow{\psi'} & \mathcal{C} \times \tilde{T} & \xrightarrow{\Phi} & \mathcal{C} \times \mathcal{C}' \\
\mathcal{C}^3 \times \tilde{T}' & \xrightarrow{pr} & \mathcal{C}^3 \times \tilde{T} & \xrightarrow{\tilde{\Psi}} & \mathcal{C} \times \mathcal{C}' \\
\tilde{T}' & \xrightarrow{\tilde{\eta}'} & \tilde{T} & \xrightarrow{\tilde{\eta}} & \mathcal{C}'.
\end{array}
\]

If \( \text{Con} \Phi \) is given by

\[
X_i(t_i, s), \ Y_i(t_i, s), \ P_i(t_i, s) \in \mathbb{C}\{s, t_i\},
\]
then \( \tilde{\eta}^* \text{Con} \Phi \) is given by

\[
X_i(t_i, \tilde{\eta}^*(z)), \ Y_i(t_i, \tilde{\eta}^*(z)), \ P_i(t_i, \tilde{\eta}^*(z)) \in \tilde{A}'\{t_i\} = \mathbb{C}\{z, t_i\}
\]
for \( i = 1, \ldots, r \). Suppose that \( \tilde{\Psi}' \) is given by

\[
U_i'(t_i, z), \ V_i'(t_i, z), \ W_i'(t_i, z) \in \mathbb{C}\{z, t_i\}.
\]
Then, \( \tilde{\Psi} \) must be given by

\[
U_i = U_i' + \varepsilon u_i, \ V_i = V_i' + \varepsilon v_i, \ W_i = W_i' + \varepsilon w_i \in \tilde{A}\{t_i\} = \mathbb{C}\{z, t_i\} \oplus \varepsilon \mathbb{C}\{t_i\}
\]
with \( u_i, v_i, w_i \in \mathbb{C}\{t_i\} \) and \( i = 1, \ldots, r \). By definition of deformation we have that, for each \( i \),

\[
(U_i, V_i, W_i) = (x_i(t_i), y_i(t_i), p_i(t_i)) \mod \mathfrak{m}_{\tilde{A}}.
\]
Suppose \( \tilde{\eta} : \tilde{T}' \to \mathbb{C}' \) is given by \( (\tilde{\eta}_1', \ldots, \tilde{\eta}_r') \), with \( \tilde{\eta}_i' \in \mathbb{C}\{z\} \). Then \( \tilde{\eta} \) must be given by \( \tilde{\eta} = \tilde{\eta}' + \varepsilon \tilde{\eta}^0 \) for some \( \tilde{\eta}^0 = (\tilde{\eta}_1^0, \ldots, \tilde{\eta}_r^0) \in \mathbb{C}' \). Suppose that \( \tilde{\chi}' : \mathbb{C}^3 \times \tilde{T}' \to \mathbb{C}^3 \times \tilde{T}' \) is given at the ring level by

\[
(x, y, p) \mapsto (H_1', H_2', H_3'),
\]
such that \( H' = id \mod \mathfrak{m}_{\tilde{A}'} \) with \( H_i' = (x, y, p)A'\{x, y, p\} \). Let the automorphism \( \tilde{\zeta} : \mathbb{C} \times \tilde{T}' \to \mathbb{C} \times \tilde{T}' \) be given at the ring level by

\[
t_i \mapsto h_i'
\]
such that \( h' = id \mod \mathfrak{m}_{\tilde{A}'} \) with \( h_i' \in (t_i)\mathbb{C}\{z, t_i\} \).

Then, from (4.9) follows that

\[
X_i(t_i, \tilde{\eta}') = H_1'(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')),
\]
(4.10)

\[
Y_i(t_i, \tilde{\eta}') = H_2'(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')),
\]

\[
P_i(t_i, \tilde{\eta}') = H_3'(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')).
\]
Now, \( \eta' \) must be extended to \( \eta \) such that the first two previous equations extend as well. That is, we must have

\[
X_i(t_i, \eta) = (H'_{1i} + \epsilon \alpha)(U_i(h'_i + \epsilon h^0_i), V_i(h'_i + \epsilon h^0_i), W_i(h'_i + \epsilon h^0_i)),
\]

\[
Y_i(t_i, \eta) = (H'_{2i} + \epsilon \beta)(U_i(h'_i + \epsilon h^0_i), V_i(h'_i + \epsilon h^0_i), W_i(h'_i + \epsilon h^0_i)).
\]

with \( \alpha, \beta \in (x, y, p) \mathbb{C}\{x, y, p\} \), \( h^0_i \in (t_i) \mathbb{C}\{t_i\} \) such that

\[
(x, y, p) \mapsto (H'_1 + \epsilon \alpha, H'_2 + \epsilon \beta, H'_3 + \epsilon \gamma)
\]
gives a relative contact transformation over \( \tilde{T} \) for some \( \gamma \in (x, y, p) \mathbb{C}\{x, y, p\} \).

The existence of this extended relative contact transformation is guaranteed by Theorem 2.6. Moreover, again by Theorem 2.6 this extension depends only on the choices of \( \alpha \) and \( \beta_0 \). So, we need only to find \( \alpha, \beta_0, \eta^0 \) and \( h^0_i \) such that (4.11) holds. Using Taylor’s formula and \( \varepsilon^2 = 0 \) we see that

\[
X_i(t_i, \eta' + \varepsilon \eta^0) = X_i(t_i, \eta') + \varepsilon \sum_{j=1}^{\ell} \frac{\partial X_i}{\partial s_j}(t_i, \eta') \eta^0_j
\]

(4.12)

\[
(\varepsilon m_{\tilde{A}} = 0) \quad \Rightarrow \quad X_i(t_i, \eta') + \varepsilon \sum_{j=1}^{\ell} \frac{\partial X_i}{\partial s_j}(t_i, 0) \eta^0_j,
\]

\[
Y_i(t_i, \eta' + \varepsilon \eta^0) = Y_i(t_i, \eta') + \varepsilon \sum_{j=1}^{\ell} \frac{\partial Y_i}{\partial s_j}(t_i, 0) \eta^0_j.
\]

Again by Taylor’s formula and noticing that \( \varepsilon m_{\tilde{A}} = 0, \varepsilon m_{\tilde{A}'} = 0 \) in \( \tilde{A}, h' = id \mod m_{\tilde{A}'}, \) and \((U_i, V_i) = (x_i(t_i), y_i(t_i)) \mod m_{\tilde{A}} \) we see that

\[
U_i(h'_i + \epsilon h^0_i) = U_i(h'_i) + \epsilon \tilde{U}_i(h'_i) h^0_i
\]

(4.13)

\[
= U'_i(h'_i) + \epsilon (\tilde{x}_i h^0_i + u_i),
\]

\[
V_i(h'_i + \epsilon h^0_i) = V'_i(h'_i) + \epsilon (\tilde{y}_i h^0_i + v_i).
\]

Now, \( H' = id \mod m_{\tilde{A}'}, \) so

\[
\frac{\partial H'_1}{\partial x} = 1 \mod m_{\tilde{A}'}, \quad \frac{\partial H'_1}{\partial y}, \frac{\partial H'_1}{\partial p} \in m_{\tilde{A}'} \mathbb{A}'(x, y, p).
\]

In particular,

\[
\varepsilon \frac{\partial H'_1}{\partial y} = \varepsilon \frac{\partial H'_1}{\partial p} = 0.
\]

By this and arguing as in (4.12) and (4.13) we see that

\[
(H'_1 + \epsilon \alpha)(U'_i(h'_i) + \epsilon (\tilde{x}_i h^0_i + u_i), V'_i(h'_i) + \epsilon (\tilde{y}_i h^0_i + v_i), W'_i(h'_i) + \epsilon (\tilde{p}_i h^0_i + w_i))
\]

\[
= H'_1(U'_i(h'_i), V'_i(h'_i), W'_i(h'_i)) + \epsilon (\alpha(U'_i(h'_i), V'_i(h'_i), W'_i(h'_i)) + 1(\tilde{x}_i h^0_i + u_i))
\]

\[
= H'_1(U'_i(h'_i), V'_i(h'_i), W'_i(h'_i)) + \epsilon (\alpha(x_i, y_i, p_i) + \tilde{x}_i h^0_i + u_i),
\]

\[
(H'_2 + \epsilon \beta)(U'_i(h'_i) + \epsilon (\tilde{x}_i h^0_i + u_i), V'_i(h'_i) + \epsilon (\tilde{y}_i h^0_i + v_i), W'_i(h'_i) + \epsilon (\tilde{p}_i h^0_i + w_i))
\]

\[
= H'_2(U'_i(h'_i), V'_i(h'_i), W'_i(h'_i)) + \epsilon (\beta(x_i, y_i, p_i) + \tilde{y}_i h^0_i + v_i)
\]

\[
= H'_3(U'_i(h'_i), V'_i(h'_i), W'_i(h'_i)) + \epsilon (\gamma(x_i, y_i, p_i) + \tilde{w}_i h^0_i + w_i).
\]
Substituting this in (4.11) and using (4.10) and (4.12) we see that we have to find $\eta^0 = (\eta^0_1, \ldots, \eta^0_\ell) \in \mathbb{C}^\ell$, $h^0_i$ such that

$$
(4.14) \quad (u_i(t_i), v_i(t_i)) = \sum_{j=1}^\ell \eta^0_j \left( \frac{\partial X_i}{\partial s_j}(t_i, 0), \frac{\partial Y_i}{\partial s_j}(t_i, 0) \right) - h^0_i (\dot{x}_i(t_i), \dot{y}_i(t_i)) - (\alpha(x_i(t_i), y_i(t_i), p_i(t_i)), \beta(x_i(t_i), y_i(t_i), p_i(t_i))).
$$

Note that, because of Remark 2.7 (i), $\alpha(x_i(t_i), y_i(t_i), p_i(t_i))$, $\beta(x_i(t_i), y_i(t_i), p_i(t_i)) \in \tilde{I}$ for each $i$. Also note that $\Psi \in \text{Def}_\psi (\tilde{T})$ means that $u_i \in t^m_i \mathbb{C}\{t_i\}, v_i \in t^m_i \mathbb{C}\{t_i\}$. Then, if the vectors

$$
\left( \frac{\partial X_1}{\partial s_j}(t_1, 0), \ldots, \frac{\partial X_r}{\partial s_j}(t_r, 0) \right) \frac{\partial}{\partial x} + \left( \frac{\partial Y_1}{\partial s_j}(t_1, 0), \ldots, \frac{\partial Y_r}{\partial s_j}(t_r, 0) \right) \frac{\partial}{\partial y} = (a^1_j(t_1), \ldots, a^r_j(t_r)) \frac{\partial}{\partial x} + (b^1_j(t_1), \ldots, b^r_j(t_r)) \frac{\partial}{\partial y}, \quad j = 1, \ldots, \ell
$$

form a basis of [generate] $\widehat{M}_\rho$, we can solve (4.14) with unique $\eta^0_1, \ldots, \eta^0_\ell$ [respectively, solve] for all $i = 1, \ldots, r$. This implies that the conormal of $\Phi$ is a formally semiuniversal [respectively, versal] equimultiple deformation of $\psi$ over $\mathbb{C}^\ell$. 

\[5. \text{ Versal Deformations}\]

Let $f \in \mathbb{C}\{x_1, \ldots, x_n\}$. We will denote by $\int f dx_i$ the solution of the Cauchy problem

$$
\frac{\partial g}{\partial x_i} = f, \quad g \in (x_i).
$$

Let $\psi$ be a Legendrian curve with parametrization given by

$$
(5.1) \quad t_i \mapsto (x_i(t_i), y_i(t_i), p_i(t_i)) \quad i = 1, \ldots, r.
$$

We will call fake plane projection of (5.1) to the plane curve $\sigma$ with parametrization given by

$$
(5.2) \quad t_i \mapsto (x_i(t_i), p_i(t_i)) \quad i = 1, \ldots, r.
$$

We will denote $\sigma$ by $\psi^{\sigma f}$.

Given a plane curve $\sigma$ with parametrization (5.2), we will call fake conormal of $\sigma$ to the Legendrian curve $\psi$ with parametrization (5.1), where

$$
y_i(t_i) = \int p_i(t_i) \dot{x}_i(t_i) dt_i.
$$

We will denote $\psi$ by $\text{Con}_f \sigma$. Applying the construction above to each fibre of a deformation we obtain functors

$$
\pi_f : \widehat{\text{Def}}_\psi \rightarrow \widehat{\text{Def}}_\sigma, \quad \text{Con}_f : \text{Def}_\sigma \rightarrow \widehat{\text{Def}}_\psi.
$$

Notice that

$$
(5.3) \quad \text{Con}_f (\Psi^{\sigma f}) = \Psi, \quad (\text{Con}_f (\Sigma))^{\sigma f} = \Sigma
$$
for each \( \Psi \in \mathcal{D}ef_{f, \psi} \) and each \( \Sigma \in \mathcal{D}ef_{f, \sigma} \).

Let \( \psi \) be the parametrization of a Legendrian curve given by (5.1). Let \( \sigma \) be the fake plane projection of \( \psi \). Set \( \hat{\sigma} := \dot{x} \frac{\partial}{\partial x} + \dot{p} \frac{\partial}{\partial p} \). Let \( I^f \) be the linear subspace of

\[
\mathfrak{m} \frac{\partial}{\partial x} \oplus \mathfrak{m} \frac{\partial}{\partial p} = \left( \bigoplus_{i=1}^{r} t_i \mathbb{C} \{ t_i \} \frac{\partial}{\partial x} \right) \oplus \left( \bigoplus_{i=1}^{r} t_i \mathbb{C} \{ t_i \} \frac{\partial}{\partial p} \right)
\]

generated by

\[
\alpha \frac{\partial}{\partial x} - \left( \frac{\partial \alpha}{\partial x} + \frac{\partial \alpha}{\partial y} \right) p \frac{\partial}{\partial p}, \quad \left( \frac{\partial \beta}{\partial x} + \frac{\partial \beta}{\partial y} \right) \frac{\partial}{\partial p},
\]

and

\[
\alpha_k p^k \frac{\partial}{\partial x} - \frac{1}{k+1} \left( \frac{\partial \alpha_k}{\partial x} p^{k+1} + \frac{\partial \alpha_k}{\partial y} p^{k+2} \right) \frac{\partial}{\partial p}, \quad k \geq 1,
\]

where \( \alpha_k \in (x, y), \beta_0 \in (x^2, y) \) for each \( k \geq 0 \). Set

\[
M^f = \frac{\mathfrak{m} \frac{\partial}{\partial x} \oplus \mathfrak{m} \frac{\partial}{\partial p}}{\mathfrak{m} \hat{\sigma} + I^f}.
\]

**Theorem 5.1.** Assuming the notations above, \( \mathcal{D}ef_{f, \psi}(T_\varepsilon) \cong M^f \).

**Proof.** Let \( \Psi \in \mathcal{D}ef_{f, \psi}(T_\varepsilon) \) be given by

\[
\Psi_i(t_i, \varepsilon) = (X_i, Y_i, P_i) = (x_i + \varepsilon a_i, y_i + \varepsilon b_i, p_i + \varepsilon c_i),
\]

where \( a_i, b_i, c_i \in \mathbb{C} \{ t_i \} t_i \) and \( Y_i = \int P_i \partial t_i X_i dt_i, \ i = 1, \ldots, r \). Hence

\[
b_i = \int (\dot{x}_i c_i + \dot{a}_i p_i) dt_i, \quad i = 1, \ldots, r.
\]

By (5.3) \( \Psi \) is trivial if and only if there an isomorphism \( \xi : \mathbb{C} \times T_\varepsilon \to \mathbb{C} \times T_\varepsilon \) given by

\[
t_i \rightarrow \tilde{t}_i = t_i + \varepsilon h_i, \quad h_i \in \mathbb{C} \{ t_i \} t_i, \ i = 1, \ldots, r,
\]

and a relative contact transformation \( \chi : \mathbb{C}^3 \times T_\varepsilon \to \mathbb{C}^3 \times T_\varepsilon \) given by

\[
(x, y, p, \varepsilon) \mapsto (x + \varepsilon\alpha, y + \varepsilon\beta, p + \varepsilon\gamma, \varepsilon)
\]

such that

\[
X_i = x_i(\tilde{t}_i) + \varepsilon \alpha(x_i(\tilde{t}_i), y_i(\tilde{t}_i), p_i(\tilde{t}_i)),
\]

\[
P_i = p_i(\tilde{t}_i) + \varepsilon \beta(x_i(\tilde{t}_i), y_i(\tilde{t}_i), p_i(\tilde{t}_i)),
\]

\( i = 1, \ldots, r \). Following the argument of the proof of Theorem 4.1, \( \Psi^f \) is trivial if and only if

\[
a_i(t_i) = \dot{x}_i(t_i) h_i(t_i) + \alpha(x_i(t_i), y_i(t_i), p_i(t_i)),
\]

\[
c_i(t_i) = \dot{p}_i(t_i) h_i(t_i) + \gamma(x_i(t_i), y_i(t_i), p_i(t_i)),
\]

\( i = 1, \ldots, r \). The result follows from Remark 2.7 (ii). \( \square \)
Lemma 5.2. Let \( \psi \) be the parametrization of a Legendrian curve. Let \( \Phi \) be the semiuniversal deformation in \( \text{Def}_f \) of the fake plane projection \( \sigma \) of \( \psi \). Then \( \text{Con}_f \Phi \) is a versal deformation of \( \psi \) in \( \text{Def}_f \).

Proof. It follows the argument of Lemma 4.3.

Theorem 5.3. Let \( a^j, c^j \in m_{\mathbb{C}} \) such that

\[
(5.4) \quad a^i \frac{\partial}{\partial x} + c^j \frac{\partial \ell}{\partial p} = \begin{bmatrix} a^i_1 \\ \vdots \\ a^i_\ell \\ c^j_1 \\ \vdots \\ c^j_\ell \end{bmatrix} \frac{\partial}{\partial x} + \begin{bmatrix} c^j_1 \\ \vdots \\ c^j_\ell \end{bmatrix} \frac{\partial \ell}{\partial p},
\]

\( i = 1, \ldots, \ell \), represents a basis [a system of generators] of \( M^j_f \). Let \( \Phi \in \text{Def}_f \) be given by

\[
(5.5) \quad X_i(t, s) = x_i(t_i) + \sum_{j=1}^{\ell} a^j_i(t_i)s_j, \quad P_i(t, s) = p_i(t_i) + \sum_{j=1}^{\ell} c^j_i(t_i)s_j,
\]

\( i = 1, \ldots, r \). Then \( \text{Con}_f \Phi \) is a semiuniversal [versal] deformation of \( \psi \) in \( \text{Def}_f \).

Proof. It follows the argument of Theorem 4.4 using Remark 2.7(ii). □

6. Examples

Example 6.1. Let \( \varphi(t) = (t^3, t^{10}) \), \( \psi(t) = (t^3, t^{10}, t^7) \), \( \sigma(t) = (t^3, \frac{10}{3}t^7) \). The deformations given by

- \( X(t, s) = t^3, \quad Y(t, s) = s_1 t^4 + s_2 t^5 + s_3 t^7 + s_4 t^8 + t^{10} + s_5 t^{11} + s_6 t^{14}; \)
- \( X(t, s) = s_1 t + s_2 t^2 + t^3, \quad Y(t, s) = s_3 t + s_4 t^2 + s_5 t^4 + s_6 t^5 + s_7 t^7 + s_8 t^8 + t^{10} + s_9 t^{11} + s_{10} t^{14}; \)

are respectively

- an equimultiple semiuniversal deformation;
- a semiuniversal deformation

of \( \varphi \). The conormal of the deformation given by

\[
X(t, s) = t^3, \quad Y(t, s) = s_1 t^7 + s_2 t^8 + t^{10} + s_3 t^{11};
\]

is an equimultiple semiuniversal deformation of \( \psi \). The fake conormal of the deformation given by

\[
X(t, s) = s_1 t + s_2 t^2 + t^3, \quad P(t, s) = s_3 t + s_4 t^2 + s_5 t^4 + s_6 t^5 + \frac{10}{3} t^7 + s_7 t^8;
\]

is a semiuniversal deformation of the fake conormal of \( \sigma \). The conormal of the deformation given by

\[
X(t, s) = s_1 t + s_2 t^2 + t^3, \quad Y(t, s) = \alpha_2 t^2 + \alpha_3 t^3 + \alpha_4 t^4 + \alpha_5 t^5 + \alpha_6 t^6 + \alpha_7 t^7 + \alpha_8 t^8 + \alpha_9 t^9 + \alpha_{10} t^{10} + \alpha_{11} t^{11};
\]

\( \alpha_2, \ldots, \alpha_{11} \) being arbitrary constants.
is a semiuniversal deformation of $\psi$.

**Example 6.2.** Let $Z = \{(x, y) \in \mathbb{C}^2 : (y^3 - x^5)(y^3 - x^7) = 0\}$. Consider the parametrization $\varphi$ of $Z$ given by
\[
x_1(t_1) = t_1^2, \quad y_1(t_1) = t_1^5, \quad x_2(t_2) = t_2^2, \quad y_2(t_2) = t_2^7.
\]

Let $\sigma$ be the fake projection of the conormal of $\varphi$ given by
\[
x_1(t_1) = \frac{t_1^2}{2}, \quad p_1(t_1) = \frac{5}{2}t_1^3, \quad x_2(t_2) = t_2^2, \quad p_1(t_2) = \frac{7}{2}t_2^5.
\]

The deformations given by
- $X_1(t_1, s) = t_1^2$, $Y_1(t_1, s) = s_1 t_1^4 + t_1^5$,
- $X_2(t_2, s) = t_2^2$, $Y_2(t_2, s) = s_2 t_2^2 + s_3 t_2^3 + s_4 t_2^4 + s_5 t_2^5 + s_6 t_2^6 + t_2^7 + t_2^8 + s_8 t_2^{10} + s_9 t_2^{12}$,
- $X_1(t_1, s) = s_1 t_1 + t_1^2$, $Y_1(t_1, s) = s_3 t_1 + s_4 t_1^2 + t_1^5$,
- $X_2(t_2, s) = s_2 t_2 + t_2^2$, $Y_2(t_2, s) = s_5 t_2 + s_6 t_2^2 + s_7 t_2^3 + s_8 t_2^4 + s_9 t_2^5 + s_{10} t_2^6 + t_2^7 + s_{11} t_2^8 + s_{12} t_2^{10} + s_{13} t_2^{12}$,

are respectively
- an equimultiple semiuniversal deformation;
- a semiuniversal deformation

of $\varphi$. The conormal of the deformation given by
\[
x_1(t_1, s) = t_1^2, \quad Y_1(t_1, s) = t_1^5,
\]
\[
x_2(t_2, s) = t_2^2, \quad Y_2(t_2, s) = s_1 t_2^4 + s_2 t_2^5 + s_3 t_2^6 + t_2^7 + s_4 t_2^8;
\]

is an equimultiple semiuniversal deformation of the conormal of $\varphi$. The fake conormal of the deformation given by
\[
x_1(t_1, s) = s_1 t_1 + t_1^2, \quad P_1(t_1, s) = s_3 t_1 + \frac{5}{2} t_1^3,
\]
\[
x_2(t_2, s) = s_2 t_2 + t_2^2, \quad P_2(t_2, s) = s_4 t_2 + s_5 t_2^2 + s_6 t_2^3 + s_7 t_2^4 + \frac{7}{2} t_2^5 + s_8 t_2^6;
\]
is a semiuniversal deformation of the fake conormal of $\sigma$. The conormal of the deformation given by

$$
X_1(t_1, s) = s_1 t_1 + t_1^2, \quad Y_1(t_1, s) = \alpha_2 t_1^2 + \alpha_3 t_1^3 + \alpha_4 t_1^4 + t_1^5,
$$

$$
X_2(t_2, s) = s_2 t_2 + t_2^2, \quad Y_2(t_2, s) = \beta_2 t_2^2 + \beta_3 t_2^3 + \beta_4 t_2^4 + \beta_5 t_2^5 + \beta_6 t_2^6 + \beta_7 t_2^7 + \beta_8 t_2^8;
$$

with

$$
\alpha_2 = \frac{s_1 s_3}{2}, \quad \alpha_3 = \frac{2 s_3}{3}, \quad \alpha_4 = \frac{5 s_1}{8},
$$

$$
\beta_2 = \frac{s_2 s_4}{2}, \quad \beta_3 = \frac{2 s_4 + s_2 s_5}{3}, \quad \beta_4 = \frac{2 s_5 + s_2 s_6}{4},
$$

$$
\beta_5 = \frac{2 s_6 + s_2 s_7}{5}, \quad \beta_6 = \frac{4 s_7 + 7 s_2}{12}, \quad \beta_7 = 1 + \frac{s_2 s_8}{7}, \quad \beta_8 = \frac{2 s_8}{8},
$$

is a semiuniversal deformation of the conormal of $\varphi$.

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