Preserving Diversity when Partitioning: A Geometric Approach

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Abstract

Diversity plays a crucial role in multiple contexts such as team formation, representation of minority groups and generally when allocating resources fairly. Given a community composed by individuals of different types, we study the problem of partitioning this community such that the global diversity is preserved as much as possible in each subgroup. We consider the diversity metric introduced by Simpson in his influential work that, roughly speaking, corresponds to the inverse probability that two individuals are from the same type when taken uniformly at random, with replacement, from the community of interest. We provide a novel perspective by reinterpreting this quantity in geometric terms. We characterize the instances in which the optimal partition exactly preserves the global diversity in each subgroup. When this is not possible, we provide an efficient polynomial-time algorithm that outputs an optimal partition for the problem with two types. Finally, we discuss further challenges and open questions for the problem that considers more than two types.

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1 Introduction

Diversity is a complex and multidimensional concept that is regularly used as a way to summarize the structure of a community. Addressing diversity concerns in decision-making tasks and socio-technical systems at large has become an imperative goal to achieve fairness and equity. Societal issues around emerging technologies and technical artifacts, such as datasets, have recently motivated the machine learning and artificial intelligence communities to study the notion of diversity and the related concepts of inclusion and representation [39, 17, 18, 23]. However, towards this goal, we face an immediate challenge: how to measure diversity. The chosen metric highly depends on the problem and the decision-maker’s objectives. As Baumgärtner [9] observes: the choice of the diversity metric is conditioned to some extent on the aspects of diversity that the decision-maker considers more important. Numerous technical notions of diversity have been proposed in the literature such as those based in abundance and similarity—common in biology and ecology—e.g. [43, 37, 8, 31, 35], geometry or distance-based, e.g. [22, 26, 14, 42] and more recent ones that incorporate the concept of inclusion [39].

Diversity indices based on abundance aim to gauge the variety or heterogeneity of a community, without focusing on the specific attributes of each individual. Particularly, there is a long-standing consensus in ecology on recommending the usage of the Hill numbers [28, 27, 16, 21], which satisfy key mathematical axioms and possess other desired properties [21]. This class of indices includes the well-known Simpson dominance index [43]. Roughly speaking, this index represents the inverse of the probability that two individuals taken uniformly at random, with replacement, from the community of interest, are from the same type. Formally, consider a community with \( r \) types of individuals where each type \( i \in [r] := \{1, \ldots, r\} \) has a relative abundance \( p_i \in (0, 1) \) with \( \sum_{i \in [r]} p_i = 1 \). Then, the Simpson dominance index corresponds to \( \frac{1}{\sum_{i \in [r]} p_i^2} \).

The Simpson dominance index, as other Hill numbers, weighs more on common types than rarer ones. The index reaches its maximum value \( r \) when all types have equal relative abundance (evenness), i.e., \( p_i = 1/r \) for all \( i \in [r] \). On the other hand, the index attains its minimum value of 1 when a single type has a relative abundance close to 1. Namely, there is some \( i^* \in [r] \) with \( p_{i^*} \approx 0 \) for every \( i \neq i^* \) and \( p_{i^*} \approx 1 \). The index’s value always lies in the interval \([1, r]\) and uniquely depends on the abundance profile. Baumgärtner [9] provides the example in Table 1 to compare the effects of the abundance profile in the value of the index. Note that subgroup \( S_1 \) has 4 types where each type is equally abundant. Similarly, subgroup \( S_2 \) has an index value of 5, however, it is richer than \( S_1 \) since it has more types. Observe in subgroups \( S_3 \) and \( S_4 \) how the index decreases as the abundance of type 5 decreases. Subgroup \( S_5 \) shows that it is equally richer than \( S_4 \) but much less even. Finally, subgroup \( S_6 \) is richer than \( S_5 \) but almost equally even.

| Type \( i \) | Relative abundance \( p_i \) in subgroup \( S \) |
| --- | --- | --- | --- | --- | --- | --- |
| \( S_1 \) | \( S_2 \) | \( S_3 \) | \( S_4 \) | \( S_5 \) | \( S_6 \) |
| 1 | 0.25 | 0.2 | 0.24 | 0.249 | 0.50 | 0.50 |
| 2 | 0.25 | 0.2 | 0.24 | 0.249 | 0.30 | 0.30 |
| 3 | 0.25 | 0.2 | 0.24 | 0.249 | 0.10 | 0.10 |
| 4 | 0.25 | 0.2 | 0.24 | 0.249 | 0.07 | 0.07 |
| 5 | - | 0.2 | 0.04 | 0.004 | 0.03 | 0.01 |
| 6 | - | - | - | - | - | 0.01 |
| 7 | - | - | - | - | - | 0.01 |

| Simpson index | 4.00 | 5.00 | 4.48 | 4.08 | 3.42 | 3.53 |

Table 1: Comparison for different Simpson dominance index values.
In this work, we study the effects on diversity when partitioning a finite community of individuals into subgroups and we consider the Simpson dominance index as a diversity metric. Broadly speaking, the main question that we aim to address is the following:

*Given a positive integer value \( k \), how do we divide a community into \( k \) subgroups such that each subgroup’s diversity is approximately as good as the global diversity?*

Our main contributions in this work are the following: (1) to the best of our knowledge, we introduce the first model that incorporates diversity requirements in a partition problem; (2) we give a geometric interpretation of each subgroup’s diversity and its relationship with the global diversity; (3) we characterize the instances in which the diversity of each subgroup is guaranteed to be at least as good as the global diversity; (4) we provide a polynomial-time algorithm that outputs a partition of \( k \) subgroups that preserve the global diversity as much as possible when the community is composed by 2 types; (5) we discuss further challenges and open questions for the problem that considers more than 2 types.

**Motivating example.** There are countless benefits and practical applications of finding diverse partitions. We offer this motivating example for concreteness [31]. Here we use loosely the term diversity to refer to Simpson dominance index. Other metrics of diversity yield similar conclusions (see [21] for other metrics). We consider a population of 48 entities with 3 types: 12 blues, 16 greens and 20 pinks. The population is divided into three islands. In Figure 1, we show two possible configurations of these divisions. The islands are represented by dashed circles. In configuration 1 (on the top), we have a homogeneous division of the population; in configuration 2 (on the bottom), we have a more heterogeneous division of the population. The global diversity of the population is 

\[
\gamma = \frac{1}{((12/48)^2 + (16/48)^2 + (20/48)^2)} = 2.88.
\]

Suppose that some instantaneous natural catastrophe wipes out Island 1, leaving no survivors in that island. In configuration 1, after all blue types disappear, the new global diversity is 

\[
\gamma' = \frac{1}{((16/36)^2 + (20/36)^2)} \approx 1.975;
\]

hence, the diversity of the population decreases by \( \approx 31.4\% \). In configuration 2, the new diversity, after Island 1 disappears, is 

\[
\gamma'' = \frac{1}{((9/36)^2 + (12/36)^2 + (15/36)^2)} = 2.88 = \gamma;
\]

hence, there is no loss in diversity under the Simpson dominance index. We can see from this example that ensuring diversity on each part (island) provides a more resilient configuration.
1.1 Our Model

We consider the problem of partitioning a community with \( r \in \mathbb{Z}_+ \) types into \( k \in \mathbb{Z}_+ \) groups or parts that preserve as much as possible the global diversity. Formally, the input of the problem corresponds to a vector \( b = (b_1, \ldots, b_r) \in \mathbb{Z}_+^r \), where the number \( b_i \) denotes the amount of entities of type \( i \in [r] \) in the community. The output of the problem is a collection of vectors \( x_1, \ldots, x_k \in \mathbb{Z}_+^r \) such that \( \sum_{i=1}^k x_i = b \). We call such a collection a \( k \)-partition of \( b \). The size of the community—the overall number of entities—corresponds to \( n = \sum_{i=1}^r b_i \). Note that without loss of generality we can assume that \( b_1 \leq b_2 \leq \ldots \leq b_r \). In this work, we consider the case when \( k \leq b_1 \), i.e., there exists a sufficient number of entities of each type for each part. Our goal is to form partitions \( x_1, \ldots, x_k \) of \( b \) that are as diverse as possible. We measure diversity using the Simpson dominance index [43]. First, note that the relative abundance of type \( i \in [r] \) is \( p_i = b_i / \sum_{j=1}^r b_j = b_i / n \), so the global diversity corresponds to \( n^2 / \sum_{j=1}^r b_j^2 \). We now formalize the definition of this diversity measure for any vector \( x \in \mathbb{Z}_+^r \).

**Definition 1.** Given a vector \( x \in \mathbb{Z}_+^r \), the Simpson dominance index of \( x \) is given by

\[
D(x) = \frac{\|x\|_1^2}{\|x\|_2^2},
\]

where \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) denote the \( \ell_1 \) norm and \( \ell_2 \) norm, respectively.

In what follows, we refer to the Simpson dominance index of \( x \), \( D(x) \), just by diversity of \( x \), unless specified. The global diversity of a community determined by \( b \in \mathbb{Z}_+^r \) is \( D(b) \). The first natural question that arises is the following: Can we form a \( k \)-partition such that each part completely preserves the diversity of the entire community. Formally, we define the perfect partition problem as follows:

**Problem 1 (Perfect Partition).** Given \( b = (b_1, \ldots, b_r) \in \mathbb{Z}_+^r \) and \( k \in \mathbb{Z}_+ \), does there exist a partition \( x_1, \ldots, x_k \in \mathbb{Z}_+^r \) of \( b \) that satisfies \( D(x_i) \geq D(b) \) for all \( i \in [k] \)?

When the previous problem has a negative answer, we address the following question: how close can we get to the global diversity? In other words, our goal is to compute a partition such that the diversity of each part is as close as possible to the global diversity. Formally, we define the problem of maximin diversity as follows,

**Problem 2 (Partition of Maximin Diversity (PMD)).** Given \( b = (b_1, \ldots, b_r) \in \mathbb{Z}_+^r \) and \( k \in \mathbb{Z}_+ \), what is the minimum \( \varepsilon \geq 0 \) such that there exists a partition \( x_1, \ldots, x_k \in \mathbb{Z}_+^r \) of \( b \) that satisfies \( D(x_i) \geq (1 - \varepsilon) \cdot D(b) \) for every \( i \in [k] \)? We denote this value by \( \varepsilon(b,k) \).

Note that Problem 2 can be equivalently stated as

\[
\max \left\{ \min \{D(x_1), \ldots, D(x_k)\} : x \in \mathbb{Z}_+^k \text{ and } x_1 + \cdots + x_k = b \right\},
\]

which can be seen as an analogous of the fair division problem [13, 36, 11].

**Tentative approaches.** To give some insights behind the difficulty of finding these optimal partitions, consider the natural approach of distributing items in a proportional manner by the so-called balanced solutions or round-robin procedures. The idea is to balance types over different parts in an iterative manner. We show an example for which some balanced solutions are not always optimal. Consider \( b = (6, 14, 21) \) and \( k = 2 \). We have \( D(b) = 1681/673 \approx 2.497 \). A balanced
solution is \( x_1 = (3, 7, 10) \) and \( x_2 = (3, 7, 11) \). This solution ensures that the diversity of each part is \( \varepsilon \approx 0.013 \) away from the global diversity. However, the solution: \( x'_1 = (3, 6, 10) \) and \( x'_2 = (3, 8, 11) \) ensures that the diversity of each part is \( \varepsilon \approx 0.003 \) away from the global, almost four times smaller than the balanced solution.

In other example, consider \( b = (6, 15, 21) \) and \( k = 2 \). We have \( D(b) = 98/39 \approx 2.512 \). One possible partition of \( b \) into 2 parts is: \( x_1 = (3, 7, 10) \) and \( x_2 = (3, 8, 11) \) obtained by setting the entries of \( x_1 \) to be the floor of \( b_i/2 \) with \( i \in \{1, 2, 3\} \); the rest is given to \( x_2 \). It is easy to check that \( D(x_2)/D(b) \approx 0.992 \). On the other hand, the partition: \( x'_1 = (2, 5, 7) \) and \( x'_2 = (4, 10, 14) \) holds \( D(x'_1) = D(x'_2) = D(b) \), that is, we can match exactly the global diversity with this optimal partition \( (x'_1, x'_2) \).

**Notation.** In the remainder of the manuscript, we follow the following notation. We denote by 1 the all-1 vector in \( \mathbb{R}^r \), \( \langle \cdot, \cdot \rangle \) the standard euclidean inner product, \( \theta_x \) the angle formed between vectors \( x \in \mathbb{R}^r \) and 1. Also, denote by \( \gcd(b) = \gcd(b_1, \ldots, b_r) \) the greatest common divisor of the elements \( b_1, \ldots, b_r \).

### 1.2 Our Results

Our first result fully answer Problem 1. We show that a \( k \)-partition with each part as diverse as the the total population can be achieved if and only if \( k \) is at most the greatest common divisor of \( b = (b_1, \ldots, b_r) \). An immediate implications of this result is that diversity deteriorates when \( k \) is larger than the greatest common divisor. The analysis and proof of this result can be found in Section 2.

Our next result answers the PMD problem (Problem 2) for the case of two types, \( r = 2 \). We present an algorithm that outputs a partition \( (x_1, \ldots, x_k) \) achieving \( \varepsilon(b, k) \). Moreover, the implementation of this algorithm is \( \mathcal{O}(k \log^2 \max\{b_1, b_2\}) \), which is polynomial in the input \( (b_1, b_2) \) and polynomial in the output of size \( k \). More specifically, we show that Problem 2 has the following geometric interpretation: Find a piecewise linear curve that joins 0 and \( b \) with \( k-1 \) breakpoints and such that each part has a diversity as good as \( b \) (see Figure 2). A simple calculation shows that the diversity of a vector \( x \) is proportional to \( (\cos \theta_x)^2 \), where \( \theta_x \) is the angle formed between \( x \) and the vector 1 = \( (1, \ldots, 1) \). Thus, we aim to find a piecewise approximation of the line \( L = \{t \cdot b : t \in [0, 1]\} \) where each segment forms angles as close as possible with the vector 1.

As a warm up, we first present the analysis for \( k = 2 \) in Section 3. For the case \( k \geq 3 \), we show a structural result that characterizes the optimal solutions with \( k \) parts under mild assumptions. Then, by decreasing appropriately \( b_1 \) and \( b_2 \), we can find an instance where we can solve the \( k \) partition problem and we implement this procedure recursively. The analysis can be found in Section 3. Finally, in Section 4, we discuss the geometric challenges that the PMD problem poses in higher dimension and we propose two tentative approaches to find 2-partitions. We also discuss the main open questions related to the complexity of the problem and the usage of other diversity metrics.

### 1.3 Related Work

One of the objectives of diversity in ecology is to gauge rare species in a population. There is a spectrum of viewpoints; on one side, rare species are the main focus; while on the other side, communities are important and only measuring common species matters. In his Nature’s influential work [43], Simpson introduced a sample-driven metric of diversity based on abundance and richness of a population. The Simpson dominance index weighs heavily on rare species. Also, it
has been generalized to the Hill numbers [28], and more generally, it has been derived as special cases of entropy indices [32]. These more general numbers allow practitioners to weigh on more common species. More modern metrics include similarities between different types or species [35]. For additional indices and metrics of diversity, we refer the interested reader to [21]. In this work, we digress from the Simpson’s index probabilistic viewpoint and we interpret this index as a geometric object on a high dimensional space. Closely related to our geometric approach is the cosine similarity [45].

Diversity has an essential role in many areas outside ecology. For instance, fairness in data summarization [14, 15, 29], where the goal is to select a small group of representative data that exhibits diversity in the feature space and is fair among sensitive features. A closely related approach is fair clustering [4, 24, 19, 30]. The majority of this literature has focused on producing clusters where no protected class is underrepresented. Our techniques could help provide new geometric insights in the design of fair and diverse outputs. There has also been a growing interest in building algorithms that are diverse in the sense of membership-aware [3, 6, 2].

Another area where diversity has been extensively studied is recommendation systems [1, 12, 44]. In this context, the main goal is to create better content-based recommendations by diversifying and not just rely on similar contents [38, 46]. As we mentioned in Section 1.1, Problem 2 can be equivalently formulated as the fair division problem (1). This problem can be interpreted as the maximin guarantee [13] used in fair allocation of indivisible goods, which has been extensively studied [36, 11, 41, 33, 7, 25, 5]. The problem of splitting attributes in the construction of decision trees (see e.g. [34] and the references therein) is closely related to the problem we introduce in this work. The goal is to design splitting procedures that minimize the impurity of the partitions, where impurity is measured with the Entropy or Gini metric. For example, given \( x \in \mathbb{Z}_+^r \), the Gini impurity measure corresponds to \( 1 - 1/D(x) \). In particular, Laber et al. [34] design splitting procedures with constant approximation guarantees. Finally, other notions of diversity and inclusion for subset selection tasks has been recently proposed in [39].

2 The Perfect Partition Problem

In this section, we characterize those instances in which there exists a partition that does not deteriorate the diversity of the whole population. First, we show the following properties of the Simpson diversity index.

Proposition 1. The Simpson diversity index \( D \) satisfies the following:

(a) \( D(ax) = D(x) \) for any \( a \neq 0 \).

(b) For \( x \geq 0, D(x) = r(\cos \theta_x)^2 \) where \( \theta_x \) is the angle formed between \( x \) and \( 1 = (1, \ldots, 1) \in \mathbb{Z}^r \).

Proof. Recall that the Simpson diversity is defined as \( D(x) = (\|x\|_1/\|x\|_2)^2 \), from where we get directly that \( D \) is invariant under scaling, which corresponds to Property (a). For Property (b), observe that since \( x \) is non-negative we have \( \|x\|_1 = \langle x, 1 \rangle = \|x\|_2 \sqrt{r} \cos \theta_x \).

For other mathematical properties of this index, we refer the interested reader to [21]. Let us recall Problem 1: Given a vector of types \( b \), does there exists a \( k \)-partition \( x_1, \ldots, x_k \) such that \( D(x_i) \geq D(b) \) for all \( i \in [k] \)? Observe, that this question is equivalent to characterize \( \varepsilon(b,k) = 0 \) versus \( \varepsilon(b,k) > 0 \) in Problem 2. In the main result of this section, we show that is possible to solve Problem 1 if, and only if, the number of parts do not exceed the greatest common divisor of \( b \).
**Theorem 1.** For every \( b = (b_1, \ldots, b_r) \in \mathbb{Z}_+^r \) and \( k \in \mathbb{Z}_+ \), there exists a \( k \)-partition \( x_1, \ldots, x_k \in \mathbb{Z}_+^r \) of \( b \) that satisfies \( D(x_i) \geq D(b) \) for all \( i \in [k] \), if and only if, \( k \leq \gcd(b) \).

**Proof.** We prove both implications separately. Throughout the proof we denote \( d = \gcd(b) \). Let \( k \) be the maximum integer such that there exists a partition \( x_1, \ldots, x_k \in \mathbb{Z}_+^r \) of \( b \) that satisfies \( D(x_i) \geq D(b) \) for all \( i \in [k] \). We aim to show that \( k = d = \gcd(b) \). First, observe that since \( d \) divides \( b \) for every \( i \in [r] \), then \( b/d \) is an integral vector. Moreover, \( D(b/d) = D(b) \) by Property (a) in Proposition 1. Thus, \( b/d \) is a feasible solution of the PMD problem with \( \epsilon = 0 \) and \( \epsilon(b, d) = 0 \). Therefore, \( k \geq d \). Now, suppose by contradiction that \( k \geq d + 1 \). Take any solution \( x_1, \ldots, x_k \) of the PMD problem with \( k \) parts and \( \epsilon = 0 \), that exists by the choice of \( k \). Then we have \( D(x_i) \geq D(b) \) for all \( i \in [k] \). We claim that the vectors \( x_1, \ldots, x_k \) are aligned, that is, for every \( j \in \{2, 3, \ldots, k\} \) there exists a positive \( a_j \) such that \( x_j = a_j x_1 \). Indeed, if the vectors are not aligned, we have that

\[
\| b \|_1 = \sum_{i=1}^{k} \| x_i \|_1 \geq \sum_{i=1}^{k} D(b) \| x_i \|_2 > \sqrt{D(b)} \cdot \left\| \sum_{i=1}^{k} x_i \right\|_2 = \sqrt{D(b)} \| b \|_2,
\]

where the first inequality is a consequence of \( D(x_i) \geq D(b) \) for all \( i \in [k] \) and the strict inequality holds by the triangle inequality, which is strict since the vectors \( x_1, \ldots, x_k \) are not aligned. The above chain of inequalities implies that \( D(b) < \| b \|_1 \| / \| b \|_2 \), which is a contradiction since this is an equality. We conclude that the vectors \( x_1, \ldots, x_k \) are aligned, and consequently, we have for each \( j \in \{2, 3, \ldots, k\} \) that \( x_j = \tilde{a}_j b \) where \( \tilde{a}_j = a_j / (\sum_{j=2}^{k} a_j + 1) \) and let \( \beta_j, \delta_j \) coprime such that \( \tilde{a}_j = \beta_j / \delta_j \). From this, we note that for any \( i \in [r] \), \( \tilde{b}_j b_i = \delta_j x_{ij} \) and so \( \delta_j \) divides \( b_i \) for each \( j \in \{1, \ldots, k\} \), since \( \beta_j \) and \( \delta_j \) are coprime. This shows that, for each \( j \in \{1, \ldots, k\} \), \( \delta_j \) is a common divisor of \( b_1, \ldots, b_r \), and therefore \( \delta_j \leq d \). We deduce from here that \( \tilde{a}_j = \beta_j / \delta_j \geq 1/d \). Then,

\[
b = \sum_{j=1}^{k} x_j = \sum_{j=1}^{k} \tilde{a}_j b \geq \frac{k}{d} b,
\]

which implies \( k \leq d \). Let us prove the opposite implication. Consider an integer value \( k \leq d \) and let \( x_j = b/d \) for all \( j \in \{1, \ldots, d\} \). As in the previous part, we have \( D(x_i) = D(b) \). Now define \( x'_j = x_j = b/d \) for every \( j \in \{1, \ldots, k-1\} \) and

\[
x'_{k} = b - \sum_{j=1}^{k-1} x_j = (d - k + 1) \frac{b}{d},
\]

Then, by Proposition 1 (a) we have that \( D(x'_j) = D(b) \) for every \( j \leq k \). This finishes the proof. \( \square \)

### 3 The Partition of Maximin Diversity for Two Types

In this section, we present optimal algorithms for the PMD problem when the number of types equals two. To illustrate the challenges of our problem, consider a geometric interpretation of the resources in the two dimensional plane. Represent each part \( x_1, \ldots, x_k \) of the budget \( b \) as points in the two dimensional integer lattice. Let \( L = \{tb : t \in [0,1]\} \) be the segment that joins the origin and \( b \). A \( k \)-partition can be visualized as a linear piecewise approximation of the line segment \( L \) with \( k - 1 \) breakpoints given by \( \sum_{i=1}^{j} x_i \) for \( j \in \{1, \ldots, k-1\} \) (Figure 2). Since we are assuming \( b_1 \leq b_2 \), intuitively, points lying above the line \( L \) will define the diversity of the partition. This is because the vector \( 1 \) lies below the line \( L \) and \( D(x) = 2(\cos \theta_x)^2 \) (Proposition 1), so if \( \theta_x \) increases, then \( D(x) \) decreases.
We show that the optimal solution is characterized by the $k$ closest points to the line $L$ that lie above the line. It is worth remarking that this is a non-trivial fact. For instance, a close point to the line that is also close to the origin could exhibit a larger angle to $1$ than a point that is further from the origin and that is also further from $L$. In fact, we show that the distance of points of the form $(i, \lceil b_2 i / b_1 \rceil)$ with $i \in \{1, \ldots, b_1 - 1\}$ to the line are unique integral multiples of $1 / b_1$. This allows us to show that the closest points to the line are also minimizing in terms of the angles formed with the vector $1$. A naive implementation computes all the $b_1 - 1$ values and selects the best $k$ ones. We show how to improve this polylogarithmically in $b_1 + b_2$ by using some basics of remainders.

First, and for the ease of explanation, we present formally our ideas for the case $k = 2$. We present an algorithm based on computing the closest point above the line $L$. In the second part, we analyze the general case of $k$-partition for two types. We build upon the results for 2-partition and we show its correctness.

### 3.1 Warm-up: Analysis for 2-Partitions

In this section, we present a deterministic algorithm with complexity $O(\log^2(\max\{b_1, b_2\}))$. Let $\mathcal{A} = \{x \in \mathbb{Z}_+^2 : 0 \leq x \leq b, 1 \leq \|x\|_1 \leq \|b\|_1 - 1\}$ be the set of all non-trivial feasible allocations for the first part in the division. Note that if $x \in \mathcal{A}$ is the allocation for the first part, then the allocation for the second part corresponds to $b - x$, which also belongs to $\mathcal{A}$. Hence, any 2-partition of the PMD problem can be represented as $(x, b - x)$ where $x \in \mathcal{A}$.

Observe that the lattice representation of any partition for $b$ is given by a linear piecewise approximation of the line segment $L$ with one breakpoint $x \in \mathcal{A}$. The key idea behind our algorithm is to be as close as possible to $b$, while being integral at the same time. This idea is inspired by our result for the separation problem in Theorem 1. Our approach is formally presented in Algorithm 1. Intuitively, our algorithm is searching over the values $i \in \{1, \ldots, b_1 - 1\}$ and selecting the closest integer point above the point $b_2 i / b_1$ in the line segment $L$. Algorithm 1 can be implemented using the well-known Extended Euclidean algorithm (Theorem 2) in time $O(\log^2(\max\{b_1, b_2\}))$; a proof of this result can be found in [20].

![Figure 2: Lattice representation of a partition of $b = (5, 7)$ into three parts. The partition is given by $x_1 = (1, 2), x_2 = (2, 3)$ and $x_3 = (2, 2)$. The breakpoints correspond to $x_1 = (1, 2)$ and $x_1 + x_2 = (3, 5)$.](image)
Theorem 2 ([20]). Given \( b_1, \ldots, b_r \) non-negative integers and \( d = \gcd(b) \), there is an algorithm that obtains integers \( k_1, \ldots, k_r \) such that \( k_1b_1 + \cdots + k_rb_r = d \) in \( O(\log^2(b_1 + \cdots + b_r)) \) time.

Algorithm 1 Two Types and Two Parts

**Input:** Budget \( b = (b_1, b_2) \) with \( b_1 \leq b_2 \).

**Output:** A partition \((x, b - x)\) of maximin diversity.

1. If \( d = \gcd(b_1, b_2) \geq 2 \) return the partition \((x, b - x) = (b/d, b - b/d)\)
2. If \( \gcd(b_1, b_2) = 1 \), compute \( \tau = b_2 \mod b_1 \) and let \( i^* \in \{1, \ldots, b_1 - 1\} \) be such that \( \tau \cdot i^* = b_1 - 1 \) (mod \( b_1 \)).
3. Return the partition \((x, b - x)\) given by \( x = (i^*, \lfloor b_2i^*/b_1 \rfloor) \).

In the following, we show the main result of this section: the correctness of Algorithm 1. Note that we only need to analyze the case when \( \gcd(b_1, b_2) = 1 \), since the other one was shown in Theorem 1.

Theorem 3. For every budget \( b = (b_1, b_2) \) such that \( \gcd(b_1, b_2) = 1 \) and \( b_1 \leq b_2 \), the 2-partition \((x, b - x)\) computed by Algorithm 1 solves the PMD problem with two parts. The algorithm runs in time \( O(\log^2 \max\{b_1, b_2\}) \).

To prove this theorem, we follow the next steps. Note that the line segment \( \mathcal{L} \) divides the region \( \mathcal{A} \) into two symmetric parts, \( \mathcal{A}_+ = \{y \in \mathcal{A} : y_2 \geq (b_2/b_1)y_1\} \) and \( \mathcal{A}_- = \{y \in \mathcal{A} : y_2 \leq (b_2/b_1)y_1\} \); above and below the line segment, respectively. In particular, we can assume that the upper part of the region \( \mathcal{A} \) contains a solution of the PMD problem. Therefore, solving the PMD problem for two parts is equivalent in this case to the problem \( \min_{y \in \mathcal{A}_+} (1 - D(y)/D(b)) \). We then characterize the value \( \epsilon(b, 2) \) as an equivalent optimization problem over \( \mathcal{A}_+ \). Finally, we prove that the solution computed by Algorithm 1 solves this optimization problem; hence it solves the PMD problem.

Proposition 2. Let \((x, b - x)\) be an optimal solution of the PMD problem with two parts, such that \( x \in \mathcal{A}_+ \). Then, we have \( \epsilon(b, 2) = \min_{y \in \mathcal{A}_+} (1 - D(y)/D(b)) = 1 - D(x)/D(b) \).

**Proof.** Since \((x, b - x)\) is an optimal solution for the PMD problem, we have that \( \min\{D(x), D(b - x)\} = (1 - \epsilon(b, 2))/D(b) \) and therefore \( \epsilon(b, 2) = 1 - \min\{D(x), D(b - x)\}/D(b) \). On the other hand, by Proposition 1 (b) we have that \( D(x) = 2(\cos \theta_x)^2 \) where \( \theta_x \) is the angle formed between \( x \) and the vector \( 1 \). Similarly, we get \( D(b - x) = 2(\cos \theta_{b - x})^2 \) and \( D(b) = 2(\cos \theta_b)^2 \). Then we have

\[
\epsilon(b, 2) = 1 - \min \left\{ (\cos \theta_x / \cos \theta_b)^2, (\cos \theta_{b - x} / \cos \theta_b)^2 \right\}.
\]

Since \( x \in \mathcal{A}_+ \) while \( b - x \in \mathcal{A}_- \), we have that \( \theta_x > \theta_{b - x} \). Cosine is a decreasing function in \([0, \pi/2]\) and therefore we conclude that \( \epsilon(b, 2) = 1 - (\cos \theta_x / \cos \theta_b)^2 = 1 - D(x)/D(b) \). \( \square \)

We refer by the vertical distance from the point \((i, j) \in \mathbb{Z}_+^2\) to the line segment \( \mathcal{L} \) to the quantity \( |j - b_2i/b_1| \). By abusing notation, we will refer to this quantity by simply vertical distance. The following proposition states that all vertical distances of points \((i, \lfloor b_2i/b_1 \rfloor), i = 1, \ldots, b_1 - 1\), to the segment \( \mathcal{L} \) are in one-to-one correspondence to the set \( \{1/b_1, 2/b_1, \ldots, (b_1 - 1)/b_1\} \). We will use this fact to argue that no two different point \((i, \lfloor b_2i/b_1 \rfloor)\) and \((j, \lfloor b_1j/b_1 \rfloor)\) have the same vertical distance.
Proposition 3. For every \( i \in \{1, \ldots, b_1 - 1\} \), the vertical distance from the point \((i, \lceil b_2 i / b_1 \rceil)\) to the line segment \(L\) belongs to the set \(\{1/b_1, 2/b_1, \ldots, (b_1 - 1)/b_1\}\). Furthermore, for any \( i \neq j \) in \(\{1, \ldots, b_1 - 1\}\), the vertical distances from \((i, \lceil b_2 i / b_1 \rceil)\) and \((j, \lceil b_2 j / b_1 \rceil)\) to the line segment \(L\) are different.

Proof. Recall that we are assuming \(\gcd(b_1, b_2) = 1\). For every \( i \in \{1, \ldots, b_1 - 1\} \) we have that \(b_2 i = \lceil b_2 i / b_1 \rceil b_1 + \tau_i\) where \(\tau_i \in \{1, 2, \ldots, b_1 - 1\}\) is the remainder in the division. We remark that the remainders are non-zero since \(b_1\) and \(b_2\) are coprime. Therefore, the vertical distance from the point \((i, \lceil b_2 i / b_1 \rceil)\) to \(L\) is such that \([b_2 i / b_1] - b_2 i / b_1 = (b_1 - \tau_i) / b_1 \in \{1/b_1, 2/b_1, \ldots, (b_1 - 1)/b_1\}\). One-to-one correspondence follows since the remainders are uniquely defined for the values \(i \in \{1, 2, \ldots, b_1 - 1\}\). □

Proposition 4. Let \((x, b - x)\) be the partition computed by Algorithm 1. Then, \(x = (i^*, \lfloor b_2 i^* / b_1 \rfloor)\) is the point in \(A_+\) that minimizes the vertical distance to the line segment \(L\). In particular, the vertical distance from \(x\) to the line segment \(L\) is \(1/b_1\).

Proof. Recall that we are assuming \(\gcd(b_1, b_2) = 1\). For each \(i \in \{1, \ldots, b_1 - 1\}\), the closest point in \(A_+ \cap \{(i, t) : t \in \mathbb{Z}_+\}\) to the line segment \(L\) is the point \((i, \lceil b_2 i / b_1 \rceil)\) and the vertical distance from this point to \(L\) corresponds to \([b_2 i / b_1] - b_2 i / b_1 = (b_1 - \tau_i) / b_1\), where \(\tau_i \in \{1, \ldots, b_1 - 1\}\) is the remainder \(\tau_i = b_2 \cdot i \mod b_1 = \tau \cdot i \mod b_1\) and \(\tau\) is the remainder \(\tau = b_2 \mod b_1\). The minimum of \((b_1 - \tau_i) / b_1\) is attained when \(\tau_i = b_1 - 1\), which is exactly what Algorithm 1 computes. In particular, the vertical distance from this point \(x\) to the segment is \(1/b_1\). □

We are now ready to prove Theorem 3.

![Figure 3: Every point \((i, \lceil b_2 i / b_1 \rceil)\) with \(i \in (i^*, 2i^*)\) has vertical distance to the segment \(L\) of at least \(3/b_1\).](image)

Proof of Theorem 3. Observe that solving the PMD problem corresponds \(\min_{y \in A_+} (1 - D(y) / D(b))\), and by Proposition 1 (b), this can be written as

\[
\min_{x \in A_+} \left(1 - \left(\cos \theta_x / \cos \theta_b\right)^2\right).
\]

This last problem is equivalent to \(\max_{x \in A_+} \cos \theta_x\), which consists in finding \(x \in A_+\) of minimum slope. In other words, we look for \(i \in \{1, 2, \ldots, b_1 - 1\}\) that minimizes \([b_2 i / b_1] / i\). We show in what follows that the part \(x\) constructed by Algorithm 1 solves this problem of minimum slope.
Observe that \( \left( \left\lfloor \frac{b_i}{b_1} \right\rfloor - \frac{b_i}{b_1} \right) / i = \left\lfloor \frac{b_i}{b_1} \right\rfloor / i - \frac{b_i}{b_1} \) for every \( i \in \{1, 2, \ldots, b_1 - 1\} \). Therefore, the problem of minimizing the slope is equivalent to minimize the vertical distance from a point \( (i, \left\lfloor \frac{b_i}{b_1} \right\rfloor) \) to the line segment \( L \) normalized by \( i \), with \( i \in \{1, \ldots, b_1 - 1\} \). Let \( i^* \) be the solution found by Algorithm 1 to the problem \( i \cdot \tau = b_1 - 1 \pmod{b_1} \). We study the area between the line that passes through the origin and \( (i^*, \left\lfloor \frac{b_i}{b_1} \right\rfloor) \) and the line \( L \). Both of these lines can be explicitly parameterized by

\[
y(t) = \frac{b_2}{b_1} t \quad \text{and} \quad z(t) = \left( \frac{1}{i^*} \left\lfloor \frac{b_i}{b_1} \right\rfloor \right) t,
\]

with \( t \in [0, b_1] \). We define \( L_{i^*} = \{(t, z(t)) : t \in [0, b_1]\} \) and observe that \( L = \{(t, y(t)) : t \in [0, b_1]\} \). Note that for every positive integer \( \ell \) we have that

\[
z(\ell i^*) - y(\ell i^*) = \left( \frac{1}{i^*} \left\lfloor \frac{b_i}{b_1} \right\rfloor \right) \ell \cdot i^* = \ell \left( \left\lfloor \frac{b_i}{b_1} \right\rfloor - \frac{b_i}{b_1} \right) = \frac{\ell}{b_1},
\]

where the last equality holds due to Proposition 4. That is, the vertical distance between two points in the two lines, sharing the same first coordinate, is increasing as multiples of \( 1/b_1 \) when the first coordinate is an integer multiple of \( i^* \). We claim that in the interior of \( \{(t, w) : 0 \leq t \leq b_1, y(t) \leq w \leq z(t)\} \) there are no integral points. Observe that for any positive integer \( \ell \) such that \( \ell \cdot i^* \leq b_1 \) we have that the interior of the area between the line segments \( L_{i^*} \) and \( L \) when the first coordinate lives in \( (\ell i^*, (\ell + 1)i^*) \) does not contain any integral point (see Figure 3). This is because the vertical distance between \( (i, \left\lfloor \frac{b_i}{b_1} \right\rfloor) \) to \( L \) is at least \( (\ell + 1)/b_1 \) for \( i \notin \{i^*, 2i^*, \ldots, \ell i^*\} \) and \( i \in \{1, \ldots, b_1 - 1\} \), which is true by Proposition 3. This concludes that among all possible points \( (i, \left\lfloor \frac{b_i}{b_1} \right\rfloor) \), the point \( (i^*, z(i^*)) \) achieves minimum slope which finishes the proof. \( \square \)

### 3.2 Analysis for \( k \)-Partitions

In this section, we present an algorithm for the general PMD problem with any \( k \geq 3 \) and two types. Given a budget \( (b_1, b_2) \), Algorithm 2 recursively reduces the instance size until reaching an initial condition where a pattern-like solution of \( k \) parts can be easily computed (Lemma 2). In what follows, we assume that \( b_1 \) does not divide \( b_2 \) since otherwise we can handle it by Theorem 1. We denote by \( \text{slope}(x, y) = y/x \) the slope of the point \( (x, y) \). Let \( k = \left\lfloor \frac{b_2}{b_1} \right\rfloor \) and define \( b'_1 = b_1 \) and \( b'_2 = b_2 - kb_1 \). Let \( m = \left\lfloor \frac{b'_1}{b'_2} \right\rfloor \). We consider two functions \( \phi_b \) and \( \eta_{b'} \) defined as follows. Function \( \phi_b \) maps a point \( (x, y) \in A_+ = \{(u, v) \in A : v \geq (b_2/b_1)u\} \) to the point \( (x, y - \kappa x) \). Consider

\[
B^b_+ = \phi_b(A_+) \cap \left\{(u, v) : v \leq b'_2\right\} \cap \left\{(u, v) : v \leq (1 - mb'_2/b'_1)u\right\}.
\]

Function \( \eta_{b'} \) maps \( (x, y) \in B^b_+ \) to the point \( (x - my, y) \). A depiction of these two functions appear in Figure 4. The following proposition summarizes the main properties of the functions \( \phi_b \) and \( \eta_{b'} \).

**Proposition 5.** Consider a budget vector \( (b_1, b_2) \). Then, the following holds:

(a) Functions \( \phi_b : A_+ \to \phi_b(A_+) \) and \( \eta_{b'} : B^b_+ \to \eta_{b'}(B^b_+) \) are one-to-one.

(b) For any two points \( (x, y), (\overline{x}, \overline{y}) \in A_+ \), the slope between \( \phi(x, y) \) and \( \phi(\overline{x}, \overline{y}) \) is equal to

\[
\frac{y - \overline{y}}{x - \overline{x}} - \kappa.
\]

Moreover, \( D(x, y) \geq D(\overline{x}, \overline{y}) \) if and only if slope \( (\phi_b(x, y)) \leq \text{slope} (\phi_b(\overline{x}, \overline{y})) \).
(c) For any two points \((x, y), (\bar{x}, \bar{y})\) \(\in \mathcal{B}_+^b\), the slope between \(\eta^b(x, y)\) and \(\eta^b(\bar{x}, \bar{y})\) is equal to
\[
\frac{y - \bar{y}}{x - \bar{x} - m(y - \bar{y})}.
\]
Moreover, \(\text{slope}(x, y) \leq \text{slope}(\bar{x}, \bar{y})\) if and only if \(\text{slope}(\eta^b(x, y)) \leq \text{slope}(\eta^b(\bar{x}, \bar{y}))\).

Proof. We have that (a) holds directly since \(\phi^b\) and \(\eta^b\) are linear transformation defined by nonsingular matrices. The first part of (b) holds by directly computing the slope between the two points \((x, y - \kappa x)\) and \((\bar{x}, \bar{y} - \kappa \bar{x})\). Note that the slope of the vector \(\phi^b(x, y) \in \phi^b(A)\) equals \(y/x - \kappa\), which is a translation of the slope of \((x, y)\). Therefore, \(\text{slope}(x, y) \leq \text{slope}(\bar{x}, \bar{y})\) if and only if \(\text{slope}(\phi^b(x, y)) \leq \text{slope}(\phi^b(\bar{x}, \bar{y}))\). The angle \(\theta_{(x,y)}\) formed between \((x, y)\) and 1 also holds \(\theta_{(x,y)} \leq \theta_{(\bar{x},\bar{y})}\) if and only if \(\text{slope}(x, y) \leq \text{slope}(\bar{x}, \bar{y})\). Since \(D(x, y) = 2(\cos \theta_{(x,y)})^2\), and cosine is decreasing for \(\theta_{(x,y)} \in [0, \pi/2]\), we conclude (b). The proof of part (c) is analogous to the previous point and the conclusion follows by the monotonicity of the function \(f(s) = s/(1 - sm)\), where \(s = (y - \bar{y})/(x - \bar{x})\).

Algorithm 2 Two types and \(k\) parts

Input: Budget \(b = (b_1, b_2)\) with \(b_1 \leq b_2\) and \(k \geq 3\) parts.

Output: A set of \(k - 1\) breakpoints \(w_1, \ldots, w_{k-1}\).

1: Compute \(\kappa, b' = (b'_1, b'_2), m\) and functions \(\phi^b\) and \(\eta^b\).
2: if \(b'_1 - \ell b'_2 + 1 \leq k \leq b'_1 - (\ell - 1)b'_2\) for some \(\ell \in \{1, \ldots, m\}\) then
3: Construct points \((w'_1, \ldots, w'_{k-1})\) as follows:
4: Select the \(b'_2\) points given by \(w'_i = (i\ell, i)\) for \(i \in \{1, \ldots, b'_2\}\) and complete the rest with any \(k - b'_2 - 1\) points on the line joining \((\ell b'_2, b'_2)\) and \(b'\).
5: else \(k \leq b'_1 - mb'_2\)
6: Recur on the input \((\eta^b(b'), k)\) and obtain points \((w''_1, \ldots, w''_{k-1})\).
7: Define \((w'_1, \ldots, w'_{k-1}) = (\eta^{-1}_b(w''_1), \ldots, \eta^{-1}_b(w''_{k-1}))\).
8: Return \((w_1, \ldots, w_{k-1}) = (\phi^{-1}_b(w'_1), \ldots, \phi^{-1}_b(w'_{k-1}))\).
We give a brief explanation of the role of the functions \( \phi_k \) and \( \eta_{\nu'} \). Any optimal \( k \) segments joined by \( k-1 \) breakpoints in \( A_+ \) can be reordered by decreasing slope. Thus the diversity of the first segment corresponds to the diversity of the \( k \) parts (Lemma 1). By mapping the \( k-1 \) points using transformation \( \phi_k \) and using Proposition 5, we see that the \( k \) segments joining these new points are ordered by decreasing slope. The converse is also true by the same argument. If the number of segments required is larger than \( b'_1 - mb'_2 \), then the slope of any \( k \) segments joining 0 with \( b' \) must have a slope of at least \( 1/\ell \) for an appropriate \( \ell \). We are able to explicitly construct a solution with slope \( 1/\ell \) which can be mapped back to a solution of the \( k \)-partition problem using \( \phi_k^{-1} \). If the number of parts required is at most \( b'_1 - mb'_2 \), then, intuitively, pairs of points with large slope can be discarded, namely points in \( \phi_k(A_+) \setminus B^b_+ \). Since the slopes are preserved under \( \eta_{\nu'} \) due to \( \eta \)-order invariance, we can recur over the instance \( \eta_{\nu'}(B^b_+) \).

The points found in the recursion can be brought back to the initial instance using \( \eta_{\nu'}^{-1} \) and \( \phi_k \) and using their corresponding order invariance. This is formally presented in Algorithm 2. We describe the algorithm for \( k \geq 3 \) since \( k = 2 \) is already solved by Algorithm 1. From the breakpoints \( w_1, \ldots, w_{k-1} \) computed by the algorithm, we recover a \( k \)-partition by defining \( x_1 = w_1, \)

\[
x_k = b - w_{k-1}
\]

(2) for every \( j \in \{2, \ldots, k-1\} \). Note that the algorithm is well-defined. Indeed, by the choice of \( \kappa \) and \( m \), in each recursive call we always have \( b_1 \leq b_2 \). In each iteration \( b_1 \) and \( b_2 \) decrease, so given that \( k \geq 3 \), there must be a recursive call where the corresponding \( b'_1 \) and \( b'_2 \) hold \( b'_1 - \ell b'_2 + 1 \leq k \leq b'_1 - (\ell - 1)b'_2 \) for some \( \ell \in \{1, \ldots, m\} \).

Note that the maximum number of calls is bounded by the number of times that takes to reach \( (b_1, b_2) = (1, 1) \). Due to the implementation of the algorithm, we can see that this is at most \( O(\log \max\{b_1, b_2\}) \) recursive calls. Thus, the overall number of operations is \( O(k \log \max\{b_1, b_2\}) \). Including the time of arithmetic operations, we see that the time complexity is increased by at most \( O(\log \max\{b_1, b_2\}) \) factor, which gives us an algorithm with overall time complexity

\[
O(k \log^2 \max\{b_1, b_2\}),
\]

that is polynomial in the input \( (b_1, b_2), k \), and the output length, a vector of length \( k \). The following theorem summarizes our main result in this section.

**Theorem 4.** For every budget \( b = (b_1, b_2) \) with gcd\( (b) = 1 \) and \( b_1 \leq b_2 \), the \( k \)-partition \( x_1, \ldots, x_k \) in (2) obtained from Algorithm 2 solves the PMD problem with \( k \geq 3 \) parts. The algorithm runs in time \( O(k \log^2 \max\{b_1, b_2\}) \).

The proof of the theorem is a consequence of the following two structural results.

**Lemma 1.** For \( b_1 \leq b_2 \) and \( k \geq 2 \), there is a \( k \)-partition solution \( x_1, \ldots, x_k \) of PMD described by points \( w_0 = 0, w_1, \ldots, w_{k-1}, w_k = b \) as \( x_i = w_i - w_{i-1} \) for \( i \in [k] \) and such that the slopes of the segments \( w_{i-1} - w_i \) are decreasing and \( \min_{i \in [k]} D(x_i) = D(w_1) \).

**Lemma 2.** Let \( (b_1, b_2) \) with \( b_1 \leq b_2 \) and let \( k \geq 3 \). Suppose that for some \( \ell \in [m] \) we have \( b'_1 - \ell b'_2 + 1 \leq k \leq b'_1 - (\ell - 1)b'_2 \), where \( b'_1 = b_1 \) and \( b'_2 = b_2 - [b_2/b_1]b_1 \). Then, any \( k-1 \) points in \( \phi_k(A_+) \) will have a segment joining two points with a slope at least \( 1/\ell \).

Before we provide the proofs of the lemmata, we conclude Theorem 4.

**Proof of Theorem 4.** Since optimality is preserved under \( \phi_k \) and \( \eta_{\nu'} \), it is enough to show the result for one level of the recursion. Assume that for some \( \ell \in [m] \) we have \( b'_1 - \ell b'_2 + 1 \leq k \leq b'_1 -
Adding up the points (but this contradicts the assumption that the diversity of slopes are sorted in decreasing order by Proposition 5. Now consider the solution constructed by Algorithm 2, namely \((w'_1, \ldots, w'_{k-1})\). Since the largest slope of this solution is exactly \(1/\ell\), we have slope\((w'_i) = 1/\ell \leq \text{slope}(\bar{w}_1)\). Using Proposition 5 again we obtain \(D(\phi^{-1}_b(w_1)) \geq D(w_1)\), which concludes the optimality of the solution provided by Algorithm 2.

**Proof of Lemma 1.** Consider an optimal solution \(x_1, \ldots, x_k \in \mathbb{Z}_+^2\). Since these points are vectors in \(\mathbb{R}^2\) we can compute their slopes \(\text{slope}(x_i)\). Without loss of generality, we assume that the points \(x_1, \ldots, x_k\) are sorted by decreasing slopes: \(\text{slope}(x_1) \geq \cdots \geq \text{slope}(x_k)\). Define \(w_0 = 0\) and \(w_i = w_{i-1} + x_i\) for \(i \in [k]\). Thus \(w_k = b\).

We first claim that \(\min_{i \in [k]} D(x_i) = \min\{D(x_1), D(x_k)\}\). Indeed, let \(a_i\) be the angle formed by \(x_i\) and \((1, 0)\). Then, by the order of \(x_i\) we have \(a_1 \geq \cdots \geq a_k\). Define \(\tilde{\theta}_i\) for \(i \in [k]\) as follows: \(\tilde{\theta}_i = \theta_{x_i}\) if \(x_i\) is above the line \(\{(i, t) : t \in \mathbb{R}\}\) and \(\tilde{\theta}_i = -\theta_{x_i}\) otherwise. Thus \(\tilde{\theta}_i + \pi/4 = a_i\) for all \(i\). By the monotonicity of \(a_i\), we have \(\pi/4 \geq \tilde{\theta}_1 \geq \cdots \geq \tilde{\theta}_k \geq -\pi/4\). Since \(\cos(x)\) is concave for \(x \in [-\pi/2, \pi/2]\) and even\(^1\), we have that the minimum of \(\cos(x)\) for \(x \in \{\theta_{x_1}, \ldots, \theta_{x_k}\}\) must be attained at \(x \in \{\theta_{x_1}, \theta_{x_k}\}\). Using \(D(x_i) = 2(\cos \theta_{x_i})^2\), the result follows.

We now show that the diversity is defined just by \(x_1\), and if not, then we can modify slightly the solution \(x_1, \ldots, x_k\) to achieve this. Note that if \(\theta_k \geq 0\), with \(\theta_k\) defined as before, then the result follows by the monotonicity of the cosine function. Suppose then that \(\theta_k < 0\). Let \(L' = \{(i, \kappa) : 0 \leq t \leq b_1\}\) be the continuous line joining \((0, \kappa)\) and \(b = (b_1, b_2)\), where \(\tau = b_2 \mod b_1\) and \(\kappa = [b_2/b_1]\). Thus the point \(w_{k-1} \in \tilde{L'}\) lies above the line \(L'\). Let \(i^*\) be the first index where \(w_{i^*}\) is below \(L'\) and \(w_{i^*+1}\) is above the line \(L'\). Without loss of generality \(w_{i^*} = (p, q)\), thus, the first component of \(w_{i^*+1}\) is at least \(p + 1\). This implies that \(k - 1 - i^*\), the number of points \(w_{i^*+1}, \ldots, w_{k-1}\), is at most \(b_1 - 1 - p\). Consider the following solution: \(w'_i = w_i\) for \(i \leq i^*\) and \(w'_{i^*+1} = (p+1, \kappa(p+1) + \tau)\), \(\ldots, w'_{k-1} = (p+\ell, \kappa(p+\ell) + \tau)\), where \(\ell = k - 1 - i^*\), and \(w'_{k} = b\). Then, we observe that the parts \(x'_i = w'_i - w'_{i-1}\) for \(i \in [k]\) exhibit a diversity as good as the diversity of \(x_1, \ldots, x_k\), the slopes of \(x'_i\) are in decreasing order, and all \(x'_i\) lie above or in the line \(\{(i, t) : t \in \mathbb{R}\}\), which ensures that their corresponding angle \(\tilde{\theta}'_i \geq 0\) for all \(i\).

**Proof of Lemma 2.** By contradiction, assume there is a solution of \(k - 1\) points where the \(k\) segments have slope smaller than \(1/\ell\). Take any of these segments and suppose that \((x, y)\) and \((\pi, \gamma)\) are its endpoints, then the ratio between \(\Delta y = y - \gamma\) and \(\Delta x = x - \pi\) is strictly smaller than \(1/\ell\). Let us assume without loss of generality that \(x \geq \pi\). Since the points \(x, \pi, y\) and \(\gamma\) are integer and \(\ell\) is also an integer, we deduce

\[
\Delta x \geq 1 + \ell \cdot \Delta y.
\]

Adding up the \(\Delta x's\) of all segments, we obtain

\[
b'_1 = \sum_{\text{segments}} \Delta x \geq \sum_{\text{segments}} (1 + \ell \cdot \Delta y) = k + \ell b'_2,
\]

but this contradicts the assumption \(b'_1 \leq \ell b'_2 + k - 1\). Thus, in any set of \(k - 1\) points in \(\phi_b(A_+)\) there must be a pair of points where the segment joining them has slope of at least \(1/\ell\). \(\square\)

\(^1\)A function \(f\) over \(\mathbb{R}\) is even if \(f(x) = f(-x)\) for every \(x \in \mathbb{R}\).
4 Challenges in Higher Dimension and Open Questions

In this section, we provide a brief insight on the challenges that our problem poses when dealing with more than two types and we discuss some of the remaining open questions. First, for \( r = 2 \), the two dimensional geometry makes the problem more approachable in that we are able to narrow the search and focus on the nearest points around the line segment \( L \) determined by \((b_1, b_2)\). This approach works for both the case \( k = 2 \) and the recursive procedure for \( k \geq 3 \). This is possible since the diversity \( D(x) \) of a vector \( x \), the angle \( \theta_x \) between \( x \) and \( 1 \), and the slope of \( x \) are directly connected. Moreover, during the analysis of \( k = 2 \), we uncovered that the distances of the closest points are in one-to-one correspondence with the set \( \{1, \ldots, \tau\} \), where \( \tau = b_2 \mod b_1 \). Hence the closest point to the line \( L \) defines a 2-partition of maximin diversity.

One of the main remaining open questions relates to the complexity of the general problem. It might not be solvable in polynomial time. This question aligns with the open question stated by Laber et al. \([34]\), where the complexity of the problem remains unanswered for the Gini impurity measure. Also, the approach taken in \([34]\) might be useful to design algorithms with constant approximation guarantees for our problem.
Another open question corresponds to the scarce resource setting where $k > b_j$ for some $j \in [r]$. In this case, the diversity of the resulting partition might be considerably worse than the global diversity, since some subgroups do not get individuals of type $j$. Finally, it would be interesting to study our framework under other diversity indices, such as the general class of Hill numbers or similarity-based indices [28, 21]. Formally, for $x \in \mathbb{Z}_+^r$, the class of Hill numbers is defined as

$$D_q(x) = \left( \sum_{i \in [r]} \frac{x_i^q}{\|x\|_1^q} \right)^{\frac{1}{1-q}},$$

where $q \in [0, \infty]$ is known as the order of the diversity. For $q = 2$, we recover the Simpson dominance index. For $q = 1$, the limit exists and corresponds to the exponential of the Shannon entropy. For $q = \infty$, the metric measures only the maximum of the entries. The Hill numbers are the only family of ecological diversity metrics that are known to satisfy key mathematical axioms [21]. For $q \neq 2$, our proof techniques are not directly applicable since in this case the metric may not have a geometric interpretation, in particular Property (b) in Proposition 1 does not apply.

5 Conclusions

This work presented a novel framework for the partition problem under diversity requirements. We provide a geometric interpretation of the relationship between the global diversity of the community and each subgroup’s diversity. We show that a perfect partition exists only when the number of parts $k$ divides the gcd of $b$. We also design a polynomial time algorithm for the case of $r = 2$ types. Finally, we discuss the technical challenges that we face in higher dimensions and some open questions.

Addressing diversity concerns has posed numerous challenges. Long-term interdisciplinary efforts and multiple views on the matter are needed to appropriately progress towards fairness and equity [18, 10]. We hope that, from a technical perspective, our work helps in understanding the effects on diversity when dividing a community into subgroups. We think that our framework and results could provide deeper insights in other resource constrained settings that look to incorporate diversity requirements such as clustering, classification and scheduling.

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