1 Introduction

1.1

Let $A$ be an associative algebra with the unit over a commutative ring $k$. Its “categorical” analog is a monoidal category $\mathcal{A}$, i.e. a $k$-linear category equipped with a bilinear functor $\otimes : \mathcal{A}^2 \to \mathcal{A}$ which is associative. The category $\mathcal{A}$ has also a naturally defined unit object. Similarly, the categorical analog of a commutative associative algebra is a braided category (in this case we have functorial isomorphisms $X \otimes Y \to Y \otimes X$ satisfying natural properties, see [De1]).

Suppose that we have a $k$-module $A$ equipped with a $k$-linear product $m : A^\otimes 2 \to A$, $m(a \otimes b) = ab$, which is not necessarily associative. Assume that there is an algebraic group $G$ over $k$ acting on $A$, $a \mapsto a(g)$, which commutes with the product: $(ab)(g) = a(g)b(g)$. Then one can try to say what is a rational $G$-associativity. Naturally it should be an equality like $(a_1(g_1)a_2(g_2))a_3(g_3) = a_1(g_1)(a_2(g_2)a_3(g_3))$ valid for Zariski open subset of $G^3$. Here we can treat both sides as elements of the space of rational functions $\text{Funct}(G^3, A^\otimes 3)$. One can easily check that this “associativity for triples” does not imply the associativity for products of four elements, and so on. In other words one should consider all the spaces $\text{Funct}(G^n, A^\otimes n)$ and state the compatibility conditions using all of them. One can treat rational commutativity similarly.
This paper has arisen from the attempts to understand the categorical analogs of these and other examples. It does not contain deep results. It can be considered as an introduction to the axiomatics illustrated in a few interesting examples. It contains a part of the notes I have been taking on the topic for more than a year. I thank to Richard Borcherds who has convinced me to publish these notes. I would like to mention that he has developed independently the formalism of relaxed multi-linear categories and $G$-vertex algebras (see [Bo]) which is closely related to the topic of this paper.

1.2

Another motivation for this paper came from [BD] where the notion of pseudo-tensor category was introduced. As a special case it gives the notions of symmetric monoidal category and operad. The former corresponds to a representable pseudo-tensor structure. In this paper we advocate a similar point of view: if we want to speak about associativity and/or commutativity constraints depending on parameters we should do that in the language of operations rather than objects. In the case of a symmetric monoidal category these operations are given by $P_I(\{X_i\}, Y) = \text{Hom}(\otimes_I X_i, Y)$ (see [BD]). In our case we replace sets of operations by sheaves of operations, and the representability becomes a more sophisticated problem. The point is that we can work with operations without solving it.

Thus we naturally generalize pseudo-tensor categories in two directions: a) introducing an operad of spaces so that operations become sheaves (or rational or meromorphic sections of sheaves) over these spaces; b) introducing actions of braid groups on the operations instead of the actions of symmetric groups $\text{Aut}(I)$ in pseudo-tensor case.

1.3

Few typical examples include: finite-dimensional representations of quantum affine algebras, admissible representations of $GL(n,F)$, where $F$ is a local field, classical chiral algebras (see [BD]) as well as their q-deformations. In all these cases one can speak about analytic (rational, meromorphic) braided (or pseudo-braided) category. We discuss shortly these examples in the main body of the paper.
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2 Pseudo-braided categories

2.1

Suppose that for each $n \geq 1$ we are given a category $T(n)$. Suppose that:

a) the category $T(1)$ has a distinguished object $e$;

b) for any $n, k_1, \ldots, k_n$ we are given a functor $T(n) \times T(k_1) \times \cdots \times T(k_n) \to T(k_1 + \cdots + k_n)$ such that

$(T, T_1, \ldots, T_n) \mapsto T(T_1, \ldots, T_n)$.

It is called a composition functor;

c) the composition functors are strictly associative:

$T(T_1(X_{11}, \ldots, X_{1k_1}), \ldots, T_n(X_{n1}, \ldots, X_{nk_n})) = (T(T_1, \ldots, T_n))(X_{11}, \ldots, X_{nk_n})$;

d) $T(e, \ldots, e) = T, e(T) = T$ for any $T$.

We will call a collection $T = (T(n))_{n \geq 1}$ a strict monoidal 2-operad. It is strict, since we use equality of functors in c) (it can be replaced by an isomorphism of functors in non-strict case). It is monoidal since we do not require an action of the symmetric group $S_n$ on $T(n)$.

Example.

Let $n \geq 1$ be an integer. We denote by $T(n)$ the category of connected plane trees with $n$ external edges (= tails) equipped with the ordered labeling from 1 to $n$, and all the edges oriented down to the root vertex. Recall (see [GK] or [KM] for the details) that the morphisms in each category are either identities or compositions of contractions of edges. There are gluing functors between different categories: having a tree $T \in T(n)$, and a sequence of trees $T_i \in T(k_i) \ 1 \leq i \leq n$ one can construct a tree $T(T_1, T_2, \ldots, T_n)$ which is an object of $T(k_1 + \cdots + k_n)$.

Let $T$ be a strict monoidal 2-operad.
Definition 1.

A $\mathcal{T}$-pseudo-monoidal category is a class of objects $\mathcal{C}$ together with the following data:

a) A set $P_T(\{X_i\}, Y)$ given for any $T \in \mathcal{T}(n)$, a sequence $X_1, X_2, \ldots, X_n \in \mathcal{C}$ and $Y \in \mathcal{C}$.

b) A map $\phi_f: P_T(\{X_i\}, Y) \to P_{T'}(\{X_i\}; Y)$ given for any morphism $f: T' \to T$ in the category $\mathcal{T}(n)$. This map is functorial with respect to $f$.

c) A composition map

$$\Phi: P_T(\{X_i\}, Y) \times P_{T_1}(\{M_{ij}\}, X_1) \times \ldots \times P_{T_n}(\{M_{nj}\}, X_n) \to P_{T_1 \ldots T_n}(\{M_{ij}\}, Y)$$

given for any objects $T \in \mathcal{T}(n), T_i \in \mathcal{T}(k_i)$, and sequences of objects $\{X_1, \ldots, X_n\}, \{M_{ij}\}, \ldots, \{M_{nj}\}, 1 \leq j \leq k_i, 1 \leq i \leq n$.

The elements of the set $P_T$ are called operations, composition maps are called compositions of operations.

d) It is required that the composition maps are transitive with respect to the composition in $\mathcal{T}$, and that for any object $X \in \mathcal{C}$ there exists an element $id_X \in P_e(\{X\}, X)$ with the natural properties of the identity map (cf.[BD]).

If $\mathcal{T}$ is a strict monoidal 2-operad from the Example we will simply say that $\mathcal{C}$ is a pseudo-monoidal category. Unless we say otherwise we consider this case.

Remarks.

Suppose that we have a pseudo-monoidal category $\mathcal{C}$.

a) Each category $\mathcal{T}(n)$ contains the only tree $\delta_n$ without internal edges. If the morphisms in b) are isomorphisms, one can restate the definition above in terms of the sets $P_{\delta_n}$ only.

b) Similarly to [BD] we can treat a pseudo-monoidal structure as an extension of some categorical structure on $\mathcal{C}$. Indeed, considering the set of operations corresponding to the $\mathcal{T}(1)$ we get morphisms in $\mathcal{C}$.

Definition 2.

A pseudo-monoidal category $\mathcal{C}$ is called pseudo-braided if for any element $\sigma$ in the braid group $B_n$ we have (in the notations of the previous definition) a bijection

$$\mu_{\sigma}: P_T(\{X_i\}; Y) \to P_T(\{X_{\sigma(i)}\}; Y)$$

where $\sigma$ acts on the set $\{1, \ldots, n\}$ as the corresponding element of the symmetric group $S_n$. 
It is required that these bijections satisfy the following properties:

a) $\mu_{\sigma \tau} = \mu_\sigma \mu_\tau$,
   $\mu_1 = id$, where $1 \in B_n$ is the unit, $\sigma, \tau$ are elements of $B_n$.

b) Compatibility with the composition maps from 2.1c), which means that the following diagram is commutative:

$$
\begin{align*}
P_T(\{X_i\}_{i=1}^n, Y) \times \prod_{i=1}^n P_{T_i}(\{M_{ij}\}_{j=1}^{k_i}, X_i) & \longrightarrow P_{T(T_1, \ldots, T_n)}(\{M_{ij}\}, Y) \\
\downarrow & \\
P_T(\{X_{\sigma(i)}\}_{i=1}^n, Y) \times \prod_{i=1}^n P_{T_i}(\{M_{i\sigma(j)}\}_{j=1}^{k_i}, X_i) & \longrightarrow P_{T(T_1, \ldots, T_n)}(\{M_{\sigma(\sigma_1, \ldots, \sigma_n)(ij)}\}, Y)
\end{align*}
$$

Here $1 \leq i \leq n, 1 \leq j \leq k_i$, and $\sigma(\sigma_1, \ldots, \sigma_n)$ denotes the element of $B_{k_1 + \ldots + k_n}$ which is the image of the element $\sigma \times \sigma_1 \times \ldots \times \sigma_n$ under the natural map $B_n \times B_{k_1} \times \ldots \times B_{k_n} \to B_{k_1 + \ldots + k_n}$. Vertical arrows are actions of the braid groups, horizontal arrows are compositions of operations.

c) $\mu_\sigma$ commute with morphisms in $T(n)$.

d) $\mu_1$ preserves $id_X$ for any object $X$. Here $1$ is the unit in $B_1$.

The notion of a functor between pseudo-monoidal or pseudo-braided categories is defined in the natural way (see [BD] for the case of unordered sets). We will often denote pseudo-monoidal categories simply by $C$. Let $C$ be a pseudo-braided category such that all sets $P_T$ of operations depend on the sets $I$ of vertives of $T$ only. In this case the braid groups act by permutations, and we reproduce the notion of pseudo – tensor category from [BD]. We are going to use it later in the section devoted to chiral algebras. For a pseudo-tensor category we will denote the set of operations by $P_I$ instead of $P_T$.

Let $\mathcal{O}$ be a pseudo-braided category with one object. Then we will say that we are given a braided operad. In particular we can speak about functors from a braided operad to a pseudo-braided category.

2.2

We are going to generalize the notion of $T$-pseudo-monoidal category in the following way. We assume that for each object $T \in T(n)$ we are given a ringed space $(S_T, \mathcal{O}_{S_T})$ (topological space equipped with a sheaf of commutative rings).
rings, and this correspondence is functorial for each category $\mathcal{T}(n)$ and compatible with the composition functors. In particular we have a morphism of spaces $S_T \times \prod_i S_{T_i} \to S_{T(T_1,\ldots,T_n)}$ corresponding to the composition of objects $T_i$ and $T$. Thus we get a family of spaces $\mathcal{S} = (S_T)_{T \in \mathcal{T}(n)}, n \geq 1$. We say that $\mathcal{S}$ is a $\mathcal{T}$-operad.

Suppose that $\mathcal{T}$ is a strict 2-operad of plane trees, and $\mathcal{S}$ is a $\mathcal{T}$-operad.

**Definition 3.**

a) A pseudo-monoidal category over $\mathcal{S}$ consists of the following data:

(i) a class of objects $\mathcal{C}$;

(ii) for each plane tree $T \in \mathcal{T}(n)$, a sequence of objects $\{X_i\}_{i=1}^n$ and an object $Y$ we are given a sheaf (of sets, vector spaces, etc) $P_T(S_T; \{X_i\}, Y)$ such that the axioms of the Definition 1 are satisfied (with the obvious change “sets” to “sheaves”; here $\text{id}_X$ must be a section of the sheaf $P_e(S_e; \{X\}, X)$).

b) A pseudo-monoidal category over $\mathcal{S}$ is called pseudo-braided if for each $\mathcal{T}(n)$ we have an action of the braid group $B_n$ on the sheaves of operations such that the properties of the Definition 2 are satisfied ($B_n$ does not act on the base space $S_T$).

The notion of a functor between pseudo-monoidal or pseudo-braided categories over $\mathcal{S}$ is defined in the natural way. If we have two pseudo-monoidal categories defined over different operads of spaces, then a functor from one to another consists of a morphism of the $\mathcal{T}$-operads of spaces and morphisms of the sheaves of operations compatible with it. We leave to the interested reader to write down all the diagrams.

**Remarks.**

a) If all the spaces are points we recover the previous definitions. Sometimes we will omit $\mathcal{S}$ simply saying that we have a pseudo-monoidal or pseudo-braided category. We hope it will not be confusing.

b) If $\mathcal{F}$ is a sheaf over $X$, $\mathcal{G}$ is a sheaf over $Y$, and $f : X \to Y$ is a morphism of spaces then we say that a morphism of the sheaves is a morphism $\mathcal{F} \to f^*\mathcal{G}$. In particular, since our spaces are ringed, the composition maps are $\mathcal{O}_{S_{T(T_1,\ldots,T_n)}}$-linear (cf. [GK] for the notion of a sheaf on a topological operad).
2.3

Let $\mathcal{C}$ be a pseudo-monoidal category over $\mathcal{S}$. Assume that our spaces are reduced irreducible schemes or reduced irreducible complex analytic spaces. Then for each space $S_T$ the sheaf of operations is a sheaf of quasi-coherent $\mathcal{O}_{S_T}$-modules where $\mathcal{O}_X$ denotes the structure sheaf of $X$. We can extend the space of operations on $S_T$ to obtain a vector space over the field of rational functions on $S_T$ (for schemes) or over the field of meromorphic functions on $S_T$ (for complex analytic spaces). Let us denote either of these fields by $K(S_T)$. Clearly these vector spaces define a new pseudo-monoidal structure on $\mathcal{C}$. We will call it localization of the original one. The pseudo-braided case can be treated similarly.

Suppose now that we are given an operad of spaces as above, the vector spaces $P_T(S_T; \{X_i\}, Y)$ over $K(S_T)$ as well as maps $\phi_f$ and composition maps between them (see Definition 1).

Definition 4.

a) If the conditions of the Definition 1 are satisfied we say that $\mathcal{C}$ is:
   a rational pseudo-monoidal category if $\mathcal{S}$ is an operad of schemes;
   a meromorphic pseudo-monoidal category if $\mathcal{S}$ is an operad of complex analytic spaces.

b) If in addition we have actions of braid groups as in the Definition 2 we say that $\mathcal{C}$ is a rational (resp. meromorphic) pseudo-braided category.

Remark.

Intuitively one can imagine a pseudo-monoidal category over $\mathcal{S}$ as given by a class of objects and spaces of operations which are families parametrized by the spaces $S_T$. This parametrization is either complex analytic or algebraic regular depending on the category of spaces. Meromorphic or rational structures are given by similar families which might have singularities. In the pseudo-braided case the action of the braid group can be singular itself.

2.4 Example

Let $\mathcal{C}$ be a $\mathbb{C}$-linear category, $G$ be a complex analytic group acting on the objects from the left: $X \mapsto X(g)$ for any $X \in \mathcal{C}$ and $g \in G$.

We can define an operad of spaces in the following way. To each plane tree with $n$ tails we assign the group $G^n$, $n \geq 0$, $G^0$ is the trivial group. The
compositions are given by the maps \( G^n \times G^{k_1} \times ... \times G^{k_n} \to G^{k_1 + ... + k_n} \) such that \((g_1, ..., g_n; g_{11}, ..., g_{1k_1}; ..., g_{nk_n}) \mapsto (g_{1k_1}, ..., g_{nk_n})\). Suppose that for each \( G^n \) we have families of complex vector spaces \( P_T(G^n; \{X(g_i)\}, Y) \) parametrized by its points. Assume that in such a way we obtain analytic sheaves over \( G^n \) satisfying the condition a) (resp.b)) of the Definition 4. Thus we get a pseudo-monoidal (resp. pseudo-braided) category over \( G = (G^n)_{n \geq 0} \). If \( C \) is a pseudo-braided category we get a pseudo-braided category over \( G \). We call it pseudo-monoidal (resp. pseudo-braided) \( G - category \) if these sheaves are equivariant with respect to the left action of \( G \) on \( G^n \) (in pseudo-braided case the action of the braid group also has to be compatible with this equivariance). If the conditions of the Definition 5 are satisfied then we obtain a meromorphic pseudo-monoidal \( G - category \) (or its pseudo-braided version). This construction can be done in a pure algebraic setting as well (\( G \) has to be an algebraic group in this case). We will see that many examples which naturally appear in practice can be obtained in this way.

2.5

In this subsection we discuss the case of representable pseudo-monoidal and pseudo-braided structures. We restrict ourselves mostly to the case of pseudo-monoidal categories. The pseudo-braided case is similar.

Let us make a few comments. If we have a usual pseudo-monoidal category \( C \) then we say that its pseudo-monoidal structure is representable if for any plane tree \( T \in \mathcal{T}(n) \) there exists a functor \( F_T : C^n \to C \) such that we have a functorial isomorphism of sets (vector spaces, modules, etc.) \( P_T(\{X_i\}, Y) \to \text{Hom}(F_T(\{X_i\}), Y) \) valid for any sequence of objects \( X_1, X_2, ..., X_n, Y \) of \( C \). It is required to be compatible with the morphisms in \( \mathcal{T}(n) \) and with the gluing of trees and composition of functors in \( (\text{Funct}(C^n, C))_{n \geq 1} \).

A typical example is given by a monoidal category. Then each binary plane tree \( T \) gives rise to a functor (tensor product with the maximal bracketing prescribed by \( T \)). The Maclane coherence theorem allows us to extend this construction to get a functor \( F_T \) for each plane tree \( T \). Similarly a usual braided category gives an example of a representable pseudo-braided structure.

Suppose now that we have a pseudo-monoidal category \( C \) over \( S \). Then to say that it is representable we need to speak about families of objects of \( C \) parametrized by the spaces \( S_T \). We briefly recall an appropriate language...
of sheaves of categories.

Let \( X \) be a topological space (or a Grothendieck topology). We recall (see [Gr] for the details) that a sheaf (=stack) \( \mathcal{F} \) of categories on \( X \) is given by:

a) a presheaf of categories. It assigns a category to each open set from the category of open sets on \( X \) in such a way that the usual properties hold;

b) gluing (=descent) data for \( \mathcal{F} \) making this presheaf into a sheaf.

In particular for each open set \( U \) we have a fiber category \( \mathcal{F}_U \), and for each morphism of open sets \( f : V \to U \) there is a pull-back functor \( f^* : \mathcal{F}_U \to \mathcal{F}_V \). These structures are functorial in the natural way. If all fiber categories are subcategories of a category \( \mathcal{C} \) we say that we have a sheaf on \( X \) with values in \( \mathcal{C} \). If all fiber categories coincide with \( \mathcal{C} \) we have a constant sheaf with the fiber \( \mathcal{C} \).

A sheaf of categories is called \textit{locally constant} if it is locally isomorphic to a constant one. It is called also a \textit{local system} with values in \( \mathcal{C} \).

In the rest of this subsection we will assume that all \( S_T \) are complex analytic spaces unless we say otherwise. The case of schemes can be treated similarly.

Let \( \mathcal{C} \) be a \( \mathcal{C} \)-linear category. Then for any complex analytic space \( X \) we can construct a free sheaf of categories over \( X \), namely \( \mathcal{O}_X \otimes \mathcal{C} \). We say that a sheaf \( \mathcal{F} \) of categories over an analytic space \( \mathcal{M} \) is a \textit{bundle} with the fiber \( \mathcal{C} \) if for any open \( U \subset \mathcal{M} \) there is a morphism of the open subspaces \( f : V \to U \) in \( \mathcal{M} \) such that the pull-back category \( f^*(\mathcal{F}_U) \) is isomorphic to \( \mathcal{O}_V \otimes \mathcal{C} \).

In particular any object \( A \in \mathcal{C} \) can be viewed as a category with one object and hence defines a bundle with the fiber \( A \) : to each open \( U \subset \mathcal{M} \) it assigns \( \mathcal{O}(U) \otimes A \). There is a natural functor from the category of local systems with a \( \mathcal{C} \)-linear fiber \( \mathcal{C} \) to the category of bundles with the fiber \( \mathcal{C} \).

Let \( \mathcal{C} \) be as above, \( S \) be an operad of analytic spaces. Suppose that for each tree \( T \in \mathcal{T}(n) \) we are given a functor \( F_T \in \text{Funct}(\mathcal{C}^n, \mathcal{C}) \) such that the correspondence \( T \mapsto F_T \) is compatible with the morphisms and composition of trees (one can say that we are given a morphism of strict monoidal 2-operads \( T \mapsto \{\text{Funct}(\mathcal{C}^n, \mathcal{C})\}_{n \geq 1} \)). Then for each \( T \) and sequence of objects \( \{X_i\}_{1 \leq i \leq n} \) we have a trivial bundle over \( S_T \) isomorphic to \( \mathcal{O}_{S_T} \otimes \text{Hom}(F_T(\{X_i\}), Y) \) and these bundles form a pseudo-monoidal operad over \( S \). We denote each bundle by \( \mathcal{F}_T \). It is a bundle of categories with the fiber \( F_T \in \text{Funct}(\mathcal{C}^n, \mathcal{C}) \).

Assume now that we are given a pseudo-monoidal structure on \( \mathcal{C} \) extend-
ing its categorical structure (see Remark in 2.1).

Definition 5.

We say that the pseudo-monoidal structure of $\mathcal{C}$ is representable if for each
tree $T \in \mathcal{T}(n), n \geq 1$, it is locally isomorphic to the bundle of categories $\mathcal{F}_T$
on $S_T$ in such a way that we have a morphism of pseudo-monoidal categories
over $S$. This means that the following a) and b) hold:

a) $P_T(S_T; \{X_i\}, Y)$ is locally isomorphic (as an analytic sheaf) to $\mathcal{O}_{S_T} \otimes \text{Hom}(F_T(\{X_i\}), Y)$
(in self-explaining notations);

b) these isomorphisms are compatible with morphisms in $\mathcal{T}(n)$, with the
operadic structure on $\mathcal{T}$, composition maps between the sheaves of operations
and composition of functors $F_T$.

c) Suppose that all spaces $S_T$ are reduced and irreducible. We say that
we have a representable meromorphic pseudo-monoidal structure if a) and b)
hold for the corresponding spaces over the fields of meromorphic functions.
(In particular $P_T(S_T; \{X_i\}, Y)$ is isomorphic to $K(S_T) \otimes \text{Hom}(F_T(\{X_i\}), Y)$).

d) Suppose in addition to c) that all morphisms $\phi_f$ (see Definition 1)
and all composition maps are isomorphisms. Then we say that our category is
rational monoidal (in the case of schemes) or meromorphic monoidal (in the
case of complex analytic spaces). In this case the functor $F_{\delta_2}, \delta_2 \in \mathcal{T}(2)$ (see
Remark in 2.1 about $\delta_n$) is called a tensor product. In case if the representable
structure was pseudo-braided we would call $\mathcal{C}$ a rational (or meromorphic)
braided category.

Intuitively we can think about a meromorphic braided category as about
a class of objects $\mathcal{C}$ such that if $X$ and $Y$ are two of them then $\text{Hom}(X, Y)$
is a vector space over the field of meromorphic functions on $S_{\delta_1}$. The tensor
product of $X$ and $Y$ is a vector space $X \otimes Y$ over the field of meromorphic
functions on $S_{\delta_2}$. The generator of the braid group $B_2$ gives an isomorphism
of this vector space with the similar vector space $Y \otimes X$. The same can
be done for any tree $T \in \mathcal{T}(n)$ (in this case we have an action of $B_n$ of
course). Compatibility with the gluing of trees means that our tensor product
gives rise to an “associativity constraint with parameters” which might fail
to be an isomorphism on some closed subspaces of $S_T$. Similarly the action
of $B_n$ gives rise to a “commutativity constraint with parameters” which can
fail to be an isomorphism on some (other) closed subspaces of $S_T$. 

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Remarks.

a) This definition can be generalized to a quite general framework of Grothendieck topologies.

b) In the braided case we have an action of $B_n$ on $\text{Funct}(C^n, C)$ through the natural homomorphism $B_n \to S_n$ (the symmetric group $S_n$ acts on the functors via permutations of variables). This action is required to be compatible with the corresponding action on the operations of a given pseudo-braided category. It is clear how to generalize this picture to the meromorphic case. We are going to use all these generalizations without extra comments.

2.6 The case of braided categories

Let $\mathcal{C}$ be a braided monoidal category (see for ex. [De 1]). Then to any binary plane tree $T \in \mathcal{T}(n)$ we can assign a functor $F_T : C^n \to \mathcal{C}$ which maps a sequence of objects $\{X_i\}_{i=1}^n$ to their tensor product with the bracketing prescribed by $T$.

Let $\overline{M_{0,n+1}}$ be the moduli space of complex stable curves of genus 0 with $n + 1$ marked points and a non-zero tangent vector assigned to the last point. The real strata of this moduli space are parametrized by the elements of $\mathcal{T}(n) \times S_n$. Binary trees correspond to the zero-dimensional strata and the tree $\delta_n$ corresponds to the open strata. Following [De 2] one can construct a local system of categories on $\overline{M_{0,n+1}}$ with values in $\text{Funct}(C^n, \mathcal{C})$. To do this one uses the associativity constraint in $\mathcal{C}$. For example let $f : T \to \delta_3$ and $g : T' \to \delta_3$ are the only non-trivial morphisms in $\mathcal{T}(3)$. For the binary trees $T$ and $T'$ we have the corresponding functors $F_T$ and $F_{T'}$ (tensor products of three objects with two possible bracketings). Then we have an isomorphism of functors $F_T \to F_{T'}$ given by the associativity constraint.

From the geometric viewpoint we have two embeddings of the real strata corresponding to $(T, 1)$ and $(T', 1)$ to the boundary of $\overline{M_{0,4}}$ (here 1 is the unit of the symmetric group). One can construct a local system on the open stratum corresponding to $(\delta_3, 1)$ in such a way that it has a constant fiber $F_T$ and its specialization to the stratum corresponding to $(T', 1)$ is identified with $F_{T'}$ via the associativity constraint. In general, to get a local system on the moduli space $\overline{M_{0,n+1}}$ we use the action of the braid group (remark that we have $n!$ real components for $\overline{M_{0,n+1}}$ corresponding to a fixed tree). The braid group $B_n$ acts on $F_T$, $T \in \mathcal{T}(n)$ via the commutativity constraint. On the other hand it is a fundamental group of the corresponding moduli space. Note
that our local systems are compatible with the morphisms of trees. Namely if \( f : T' \to T \) is a morphism in \( \mathcal{T}(n) \) then we have a closed embedding of the component of the moduli space \( \overline{M}_{0,n+1} \) corresponding to \( T \) to the boundary of the component corresponding to \( T' \). Specialization of the local system to the boundary component is well-defined. This specialization is a local system isomorphic to the existing one (as above: the action of the braid group is interpreted as an action of the fundamental group). Gluing of trees is compatible with the embeddings of the products of smaller components to the larger ones. In the language of tensor products gluing of trees corresponds to the composition of functors.

According to [De 2] there is one-to-one correspondence between braided categories and local systems on the operad of spaces \( (\overline{M}_{0,n+1})_{n \geq 1} \) equipped with compatibilities described above. The corresponding vector bundles of \( \text{Hom}'s \) give an example of a representable pseudo-braided structure over \( (\overline{M}_{0,n+1})_{n \geq 1} \). Of course this pseudo-braided category is equivalent to a usual pseudo-braided category (over a point) which is also representable. The equivalence is given essentially by the restriction of the local systems to the zero-dimensional strata. This observation explains why braided categories often appear in the form of families of local systems. A famous example is the family of conformal blocks of the WZW model in Conformal Field Theory (see [MS]). They appear as the local systems of solutions of Knizhnik-Zamolodchikov equations and can be thought as local systems on \( \overline{M}_{0,n+1} \). These local systems give rise to a pseudo-braided category over \( (\overline{M}_{0,n+1})_{n \geq 1} \). On the other hand, according to [Dr] and [KL] this pseudo-braided category is representable, and moreover equivalent to a usual braided category (in [KL] it is a category of certain highest weight representations of a simply-laced affine Kac-Moody algebra with the fixed central charge. The tensor product is the so-called “fusion” tensor product. It corresponds to the operator product expansion in Conformal Field Theory).

2.7 Remark on meromorphic braided \( G \)-categories

Suppose that we are in the assumptions of the Example 2.4, and moreover we have a meromorphic monoidal category over \( G \). We say that we are given a meromorphic monoidal \( G \)–category if the action of \( G \) commutes with the tensor product \( : (X \otimes Y)(g) = X(g) \otimes Y(g) \). We define meromorphic \( G \)-braided categories in a similar fashion.
A typical situation when meromorphic monoidal $G$-categories can appear is the following. Suppose that $\mathcal{C}$ is a $\mathbf{C}$-linear category with the tensor product i.e. with a functor $\otimes : \mathcal{C}^2 \rightarrow \mathcal{C}$. It can happen that our tensor product is not necessarily associative or commutative. Assume that there is an analytic group $G$ acting on the category $\mathcal{C}$ in such a way that it commutes with the tensor product. It can happen that for any objects $X, Y, Z$ we have functorial isomorphisms $a_{X(g_1), Y(g_2), Z(g_3)} : (X(g_1) \otimes Y(g_2)) \otimes Z(g_3) \rightarrow X(g_1) \otimes (Y(g_2) \otimes Z(g_3))$ which is meromorphic on $G^3$ as well as similar isomorphisms for higher iterations of the tensor product. If all these isomorphisms are compatible with the compositions of the tensor product functors we say that we have a meromorphic monoidal $G$-category. We remark that the “higher” associativity constraints for the iterated tensor products should be given as a part of the data. This differs from the case of the usual monoidal categories.

Similarly, it can happen that we do not have an isomorphism $X \otimes Y$ and $Y \otimes X$ but we have a meromorphic isomorphism $c_{X(g_1), Y(g_2)} : X(g_1) \otimes Y(g_2) \rightarrow Y(g_2) \otimes X(g_1)$ which is compatible with the meromorphic associativity. Then we have a meromorphic $G$-braided structure on $\mathcal{C}$. We will consider an interesting example in the next section.

### 3 Quantum affine algebras

#### 3.1

Let $g$ be a complex finite-dimensional simple Lie algebra. We fix an invariant bilinear form on it. Then for a given non-zero complex number $q$ ($q$ is not a root of 1) one can define the Drinfeld-Jimbo quantized universal enveloping algebra $U_q(g)$ which is a complex Hopf algebra. Let $g'$ be the affine non-twisted Kac-Moody algebra corresponding to $g$. The corresponding quantized universal enveloping algebra $U_q(g')$ is a Hopf algebra containing $U_q(g)$. We are not going to reproduce its description here referring the reader to [L] and [KS]. Following [L] we will denote the generators of $U_{q}(g')$ by $E_i, F_i, K_\mu$. Here $\mu$ belongs to the co-weight lattice $\Lambda'$ of $g', i$ runs through the finite set $I$ corresponding to the affine irreducible root datum of $g'$. We denote by $i_0$ the special vertex of $I$. Then the Hopf subalgebra $U_q(g)$ is generated by the subset of the above-mentioned generators for which $i$ runs through $I - \{i_0\}$.
and $\mu$ belongs to the co-weight lattice of the Lie algebra $g$.

We recall that there is a natural group homomorphism $\mathbb{Z}[I] \to \Lambda$ where $\Lambda$ is the weight lattice of $g'$. The kernel is generated by the element $\sum_{i \in I} n_i i$ such that $n_{ia} = 1$. We denote by $K_i$ the generator $K_{\mu}$ with $\mu$ equals to $(i \cdot i)/2$. Then the element $Z = \prod_{i \in I} K_i^{n_i}$ is central in $U_q(g')$. For simplicity we will assume that $g$ is simply-laced.

3.2

We denote by $C$ the category of finite-dimensional unital $U_q(g')$-modules such that $Z$ acts on the objects as the identity morphism and all $K_{\mu}$ are diagonalizable with the eigen-values from $q^\mathbb{Z}$. The category $C$ is a monoidal category with the associativity constraint to be identity. There is an action of the group $\mathbb{C}^*$ on the category $C$. It comes from the corresponding action on $U_q(g')$ such that $E_i \to z E_i$, $F_i \to z^{-1} F_i$, and all $K_{\mu}$ are stable under the action.

This action makes an object $X$ into an object $X(z)$. It is clearly compatible with the tensor product in $C$. It was shown in [KS] that the universal quantum $R$-matrix of $U_q(g')$ defines a family of morphisms $c_{X(z_1),Y(z_2)} : X(z_1) \otimes Y(z_2) \to Y(z_2) \otimes X(z_1)$ which is meromorphic in $z_1/z_2$. The following result was proved in [KS].

**Theorem 1.**

The category $C$ carries a structure of a meromorphic braided $\mathbb{C}^*$-category. For any two objects $X$ and $Y$ the square of the commutativity constraint $c_{X(z_1),Y(z_2)} c_{Y(z_2),X(z_1)}$ is an elliptic function on the curve $E = \mathbb{C}^*/q^{2\mathbb{Z}}$ with values in the vector space $\text{End}_C(X \otimes Y)$.

3.3

In this subsection we are going to use the categories $\mathcal{O}^+_z$ of smooth $U_q(g')$-modules with the fixed central charge $z$ defined in [KS], Section 3. Weyl modules (induced from finite-dimensional simple $U_q(g)$-modules) are examples of the objects of $\mathcal{O}^+_z$. Let $\text{End}_p$ be the category of endomorphisms of $\mathcal{O}^+_z$ for $p = z q^h$ where $h$ is the dual Coxeter number of $g$. We assume that $|p|$ is greater than 1. Obviously $\text{End}_p$ is a monoidal category (tens or product
is given by the composition of functors). We can treat it as a meromorphic monoidal $G$-category for any analytic group $G$. We choose $G = \mathbb{C}^*$. Then one of the main results of [KS] can be reformulated as follows.

**Theorem 2.**

There is a functor $F : \mathcal{C} \to \text{End}_p$ of meromorphic monoidal $\mathbb{C}^*$-categories. Define $F(z)$ as $F(X(z))$ for any $X \in \mathcal{C}$, $z \in \mathbb{C}^*$. Then the functor $F(z)$ is isomorphic to $F(zp^2)$. Therefore the family $\{F(z)\}$ gives rise to a “line bundle” over the elliptic curve $\mathbb{C}^*/p^{2\mathbb{Z}}$.

### 3.4 Remarks on the Yangian case

For any simple complex Lie algebra $g$ V.Drinfeld has constructed an infinite-dimensional Hopf algebra $Y(g)$ called the Yangian of $g$ (see [Drinfeld, ICM-86]). The Hopf algebra $U_q(g')$ can be considered as a quantization of a centrally extend loop algebra $g[t, t^{-1}]$. Similarly the Yangian $Y(g)$ can be considered as a quantization of the regular loop algebra $g[t]$. Let us denote by $I_\mu$ and $J_\mu$ the standard generators of $Y(g)$ (see [D1]). Then there is an action of the additive group $\mathbb{C}$ on the Hopf algebra $Y(g)$ such that $I_\mu \mapsto I_\mu$ and $J_\mu \mapsto J_\mu + zI_\mu$ for any $z \in \mathbb{C}$. Therefore $\mathbb{C}$ acts on $Y(g)$-modules.

Theories of finite-dimensional modules of $U_q(g')$ and $Y(g)$ are completely parallel (see [D2]). In particular one has the following result.

**Theorem 3** The category of finite-dimensional $Y(g)$-modules is a meromorphic braided $\mathbb{C}$-category. The square of the commutativity constraint (cf. Theorem 1) is a $\mathbb{Z}$-periodic meromorphic function on $\mathbb{C}$.

One can also obtain an analog of the Theorem 2. For this one should consider a central extension of the double of the Yangian $D(Y(g))$ (see [S]).

### 4 Representations of $GL(F)$ and Eisenstein series

This section is based on the unpublished manuscript by M.Kapranov.
4.1

Let $F$ be a local field. We are going to consider admissible representations of the groups $GL(n, F), n \geq 0$ (see [BZ]).

We recall that if $V_i, i = 1, 2$ are admissible representations of $GL(n_i, F)$ then one can define a new admissible representation of $GL(n_1 + n_2, F)$ by the formula

$$V_1 \otimes V_2 = \text{Ind}_{P(n_1, n_2)}^{GL(n_1 + n_2, F)}(V_1 \otimes V_2)$$

where

$$P(n_1, n_2) = \begin{pmatrix} GL(n_1, F) & \ast \\ 0 & GL(n_2, F) \end{pmatrix}.$$

**Definition 6.**

$GL(F)$- module is a collection $V = (V_n)_{n \geq 0}$ of admissible representations such that $V_n$ is a representation of $GL(n, F)$.

Proof of the following proposition is straightforward.

**Theorem 4** The following operation makes the category $B$ of $GL(F)$-modules into a monoidal category

$$V \otimes W = \bigoplus_n (V \otimes W)_n$$

where

$$(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j$$

Let us remark that there is an action of the group $C$ on the $GL(n, F)$-modules, $V \mapsto V(z)$ where

$$V(z) = V \otimes |\det|^z$$

This action gives rise to the action of the group $C^\infty$ on the category $B$ where $C^\infty$ is the infinite product of the additive groups $C$. We recall (see loc.cit) that if $V_i$ are admissible representations of $GL(n_i, F), i = 1, 2$, then for generic complex numbers $z_i, i = 1, 2$ there is an intertwining operator:

$$A_{V_1(z_1), V_2(z_2)} : V_1(z_1) \otimes V_2(z_2) \rightarrow V_2(z_2 + n_1) \otimes V_1(z_1 - n_2)$$
Let us define a new tensor product of the above-mentioned representations:

\[ V_1 \cdot V_2 = V_1(\frac{n_2}{2}) \odot V_2(-\frac{n_1}{2}) \]

Then it is easy to see that the operator \( M_{V_1(z_1), V_2(z_2)} = A_{V_1(z_1 + \frac{n_2}{2}), V_2(z_2 - \frac{n_1}{2})} \) defines an intertwiner \( V_1(z_1) \cdot V_2(z_2) \to V_2(z_2) \cdot V_1(z_1) \) for generic \( z_1, z_2 \). Using the same definition as for \( \odot \) we extend the tensor product \( \cdot \) to the category \( \mathcal{B} \).

**Theorem 5** This makes \( \mathcal{B} \) into a meromorphic braided \( C^\infty \)-category.

Here we understand meromorphic function as being such when restricted to any finite product of analytic groups \( C \).

Now we need to recall some facts about Yangians and affine quantum algebras.

Let us fix \( l \) and \( n \) and consider all \( Y(sl(n)) \)-modules \( M \) whose \( sl(n) \)-irreducible components appear in \( (C^n)^{\otimes l} \). Suppose that \( l \) is the minimal number with this property. In this case we say that \( M \) has level \( l \). Every object in the category of \( Y(sl(n)) \)-modules has some level. Let us consider the category \( \mathcal{A}(n) \) formed by the sequences \( (M_i)_{i \geq 0} \) of \( Y(sl(n)) \)-modules, such that \( M_0 = 0 \), each \( M_i \) has level less or equal than \( i \).

Replacing \( Y(sl(n)) \) by \( U_q(sl(n)^\prime) \) where \( sl(n)^\prime \) is the non-twisted affine Lie algebra corresponding \( sl(n) \) we obtain the category \( \mathcal{E}(n) \) instead of \( \mathcal{A}(n) \). In this case we use the tensor power of the natural representation of \( U_q(sl(n)) \) to define the level. We refer the reader to [CP] for the details. It can be shown that \( \mathcal{A}(n) \) is a meromorphic braided \( C^\infty \)-category, and \( \mathcal{E}(n) \) is a meromorphic braided \( (C^*)^\infty \)-category.

Similarly to the category \( \mathcal{B} \) one can define the categories \( \mathcal{H} \) and \( \mathcal{L} \). The first one consists of the sequences \( (V_i)_{i \geq 0} \) such that \( V_0 = 0 \), and each \( V_i \) is a module over the affine Hecke algebra \( H_i \). In the second case we use sequences of modules over degenerate Hecke algebras \( \Lambda_i \) defined in [D3]. Both categories can be equipped with tensor products (each tensor product is similar to the parabolic induction in the \( GL(F) \)-case. It is called the Zelevinsky tensor product in [CP]).

The following result can be derived from [D3],[CP].

**Theorem 6** a) The category \( \mathcal{H} \) is a meromorphic braided \( (C^*)^\infty \)-category. The category \( \mathcal{L} \) is a meromorphic braided \( C^\infty \)-category.
b) There is an equivalence of $H$ to the subcategory of $E(n)$ consisting of those sequences $(V_i)$ for which $V_i$ has exactly level $i$ for $i < n$. This equivalence is compatible with the structures of meromorphic braided $(C^\infty)$-categories.

Similar result holds for the pair $L$ and $A(n)$.

4.2

The results of the previous subsection can be naturally extended to the global case. Namely let $k$ be a global field, $A$ be its ring of adeles. A representation of $\text{GL}(n, A)$ is called automorphic if it can be embedded into the regular representation in the space $C(\text{GL}(n, A)/\text{GL}(n, k))$. Let $M = \otimes_{x \in X} M_x$ be a $\text{GL}(n, A)$-module (here $X$ denotes the set of places of $k$). Using the adelic norm we can “twist” $M$ by $|\det|^s$. This gives an action of the additive group. Let $G$ be the the category consisting of sequences $(M_i)_{i \geq 0}$ such that $M_i$ is a $\text{GL}(i, A)$-module, $J$ be the subcategory of automorphic modules. Using the “local” definitions from the subsection 1 we define tensor products $\otimes$ and $\bullet$ on $G$. It is known that $J$ is closed under these tensor products. Using the twisting by $|\det|^s$ (here $|a|$ denotes the adelic norm of $a$) we make $G$ and $J$ into $C^\infty$-categories. Let us fix $\bullet$ as the tensor product on $G$.

**Theorem 7**  

a) The category $G$ becomes a meromorphic braided $C^\infty$-category.

b) The subcategory $J$ is a meromorphic tensor $C^\infty$-subcategory (i.e. the square of the commutativity constraint is 1).

The Eisenstein series construction defines a linear map $C(GL(n, A)/\text{GL}(n, k)) \bullet C(GL(m, A)/\text{GL}(m, k)) \rightarrow C(GL(n + m, A)/\text{GL}(n + m, k))$,

$$Eis(f) = \sum_{\gamma \in GL(n+m,k)/P(n,m)} f(g\gamma)$$

Let us consider $L = \{C(GL(n, A)/\text{GL}(n, k))\}_{n \geq 0}$ as an object of $J$. Then we have constructed a morphism $Eis: L \bullet L \rightarrow L$.

**Theorem 8**  

This makes $L$ into an associative algebra in the meromorphic tensor category $J$. 

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Remarks

a) We recall that we can consider a functor from any operad to a meromorphic pseudo-braided category. Taking for example the associative operad $Ass$ we arrive (cf. [BD]) to the notion of an associative algebra in a meromorphic tensor category.

b) We can restrict ourselves to the subgroup $C$ diagonally embedded into $C^\infty$. Then we obtain meromorphic functions in one variable. In particular part b) of Theorem 7 leads to functional equations for $L$-functions.

c) One can establish equivalences of the meromorphic tensor categories from Theorems 5 and 6 and a meromorphic tensor subcategory of $G$ corresponding to the weakly ramified representations. We leave to the reader this reformulation of the well-known results connecting representations of Hecke algebras and groups $GL$ over local fields.

5 Classical and quantum chiral algebras

5.1

In this subsection we follow [BD]. We are going to use a special case of the notion of pseudo-braided category, namely a pseudo-tensor category. We recall that the sets of operations for a pseudo-tensor category depend on the unordered sets of vertices of the trees. Let $X$ be a smooth complex curve. For any finite $I$ we denote by $\Delta^{(I)}$ the diagonal embedding $X \hookrightarrow X^I$. Let $j^{(I)} : U^{(I)} \hookrightarrow X^I$ be the embedding of the complement of the diagonal divisor. Let $\mathcal{M}(X)$ be the category of right $\mathcal{D}$-modules on $X$. Then for a sequence of objects $\{L_i\}_{i \in I}, N$ of this category one can define the following sets

\begin{align*}
a) P^*_I(\{L_i\}, N) &= \text{Hom}(\boxtimes L_i, \Delta^{(I)}_x N); \\
b) P^{ch}_I(\{L_i\}, N) &= \text{Hom}(j_*^{(I)} j^{(I)}(\boxtimes L_i), \Delta^{(I)}_x N); \end{align*}

where $\boxtimes$ denotes the external tensor product (it lives on $X^I$), all symbols like $f^*$ or $f_*$ denote the corresponding functors in the category of $\mathcal{D}_{X^I}$-modules, and Hom is taken in that category as well.

Theorem 9 ([BD]).
a) The sets from a) define a structure of a pseudo-tensor category on $\mathcal{M}(X)$. They are called $\ast I$-operations.

b) The sets from b) define a structure of a pseudo-tensor category on $\mathcal{M}(X)$. They are called chiral operations. The corresponding category is denoted by $\mathcal{M}^{ch}$.

Since the usual operads are just pseudo-tensor categories with one object, it is possible to speak about Lie algebras in pseudo-tensor categories (they are pseudo-tensor subcategories which are functorial images of the Lie-operad).

**Definition 7.**

Lie algebras in the pseudo-tensor category a) are called Lie$^{\ast}$-algebras. Lie algebras in the pseudo-tensor category b) are called weak chiral algebras.

If a weak chiral algebra contains a naturally defined unit (see [BD],1.6.3) it is called chiral algebra.

The notion of a chiral algebra can be considered as a generalization of the notion of a vertex algebra (see [FLM],[K],[Bo]) to the case of an arbitrary curve $X$. An extensive treatment of chiral algebras from the viewpoint of the theory of $\mathcal{D}$-modules can be found in [BD].

### 5.2 Remarks about q-deformed chiral algebras

One can ask about q-deformed version of the notion of chiral algebra. One cannot expect to have it for curves of the genuses higher than 1. Currently there exist few examples in genus zero case (see [FR]).

In the traditional approach a vertex algebra (=chiral algebra on the formal disk) is thought as a vector space $V$ equipped with a linear map $V \otimes V \to V[[z, z^{-1}]]$ satisfying certain properties. Equivalently, one can think of it as a linear map $V \to (\text{End}V)[[z, z^{-1}]], v \mapsto Y(v, z)$. One of the main properties is locality: $(z - w)^N([Y(a, z), Y(b, w)]) = 0$ for a sufficiently large integer $N$, and arbitrary $a, b \in V$. One can try to replace this condition by a more general one, like $f(z/w)Y(a, z)Y(b, w) = Y(b, w)Y(a, z)$ where the matrix-valued function $f(t)$ has singularities in the geometric series $q^n, n \in \mathbb{Z}$ or in a more general lattice. Thus the function $f$ becomes a new datum of the theory. This idea was used in [FR] where a preliminary definition of a q-vertex algebra was given.
One can try to interpret a q-deformed chiral algebra as a braided version of a Lie algebra in a certain meromorphic pseudo-braided $C^\ast$-category. We hope to return to this topic in the future.

5.3 About $G$-vertex algebras

R. Borcherds in [Bo] suggested a slightly different approach to vertex algebras. The motivation is more or less as follows. Let us interpret the vertex operator $Y(v, z)$ as $v^g$ where $g = exp(zL_{-1})$ and $L_{-1}$ is the generator of the Virasoro algebra. We can take as $g$ a more general element of the “local Virasoro group” $G$. Thus we can make products like $v^g_1 \cdots v^g_n$ corresponding to the products of vertex operators. Applying this expression to $1 \in V$ we obtain a space of operations $\mathcal{P}_\delta_n(G^n; V, \ldots, V)$ where $\delta_n$ is the only tree in $T(n)$ which has no internal edges. The space of operations can be informally interpreted as an extension of $V[[g_1, \ldots, g_n]]$ by $(g_i - g_j)^{-1}$. The same can be done for any bracketing in $v_1 \cdots v_n$. These spaces are related by the morphisms which roughly correspond to the restrictions to the complements of the sub-divisors (one of the smaller diagonals in this case). All this is compatible with the gluing of trees. Then one obtains a $G$-vertex algebra which can be interpreted as an associative algebra in the appropriate meromorphic pseudo-tensor $G$-category. We refer to [Bo] for the precise definition and more examples. In this way one can avoid $D$-modules and consider multidimensional generalizations of vertex algebras related to various groups $G$.

We remark that a group $G$ as a datum can be useful even in the case when a problem and the answer do no contain it. Let us return to the example of conformal blocks in WZW model (see the end of Section 3.6). Having a smooth curve $C$ of genus $0$ with $n$ marked points $z_1, \ldots, z_n$ and fixed (standard) local parameters at them one can consider the corresponding affine Kac-Moody algebras $g'_i$ “attached” to these points. Let $g'$ be an affine Kac-Moody algebra corresponding to $z = 0$ (the standard one), and $V_1, \ldots, V_n$ highest weight representations having the same fixed central charge $k$. Then under some mild conditions on $V_i$ and $k$, one can assign to each plane tree $T \in T(n)$ the “fusion” tensor product $\odot_i V_i$ with the bracketing prescribed by $T$. It is known that if $x$ belongs to the Virasoro algebra then $exp(x)$ acts on the highest weight representations. Thus we can make a tensor product $\odot_i V_i(g_i)$ where $g_i$ are of the form $exp(z_iL_{-1})$. In this way we get
an analytic braided $G$-category where $G$ is a 1-parametric group generated by $L_{-1}$. Of course this analytic braided category is equivalent to the usual braided category defined in [KL]. In fact many proofs in [KL] use either $G$ or the group $SL(2, \mathbb{C})$ with the Lie algebra generated by $(L_{-1}, L_0, L_1) \subset Vir$.

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