Asymptotics of Solution Curves of Kirchhoff Type Elliptic Equations with Logarithmic Kirchhoff Function

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Received: 28 October 2022 / Accepted: 15 February 2023 / Published online: 8 March 2023
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Abstract
We study the one-dimensional nonlocal elliptic equation of Kirchhoff type with logarithmic Kirchhoff function. We establish the precise asymptotic formulas for the solution $u_\lambda(x)$ as $\lambda \to \infty$. Here, $\lambda > 0$ is the bifurcation parameter.

Keywords Nonlocal elliptic equations · Bifurcation curves · Asymptotic formulas

Mathematics Subject Classification 34C23 · 34F10

1 Introduction
In this paper, we consider the typical example of the one-dimensional nonlocal elliptic equation

\[
\begin{cases}
- \log(a\|u\|_2^2 + b\|u\|_2^2 + 1)u''(x) = \lambda u(x)^p, \quad x \in I := (0, 1), \\
u(x) > 0, \quad x \in I, \\
u(0) = u(1) = 0,
\end{cases}
\]

(1.1)

where $a \geq 0$, $b > 0$, $p > 1$ are given constants, $\lambda > 0$ is a bifurcation parameter and $\| \cdot \|_2$ denotes the usual $L^2$-norm. Equation (1.1) is the nonlocal problem of Kirchhoff type, which is motivated by the following problem (1.2) in [7]:

\[
\begin{cases}
- A \left( \int_0^1 |u(x)|^q dx \right) u''(x) = \lambda f(x, u(x)), \quad x \in I, \\
u(x) > 0, \quad x \in I, \\
u(0) = u(1) = 0,
\end{cases}
\]

(1.2)
where \( A = A(w) \geq 0 \), which is called Kirchhoff function, is a continuous function of \( w \geq 0 \).

Nonlocal problems have been studied by many investigators, since many problems come from the phenomena of, for instance, biological problems such as population dynamics. Moreover, nonlocal problems have been also derived from numerous physical models and the other area of science, and have been studied intensively. We refer to [1–4, 6–11, 13–19], and the references therein. It seems that the main interests in this area are existence, nonexistence and the number of positive solutions. On the other hand, as far as the author knows, there are a few works which treat (1.1) as the bifurcation problems. We refer to [14, 19] and the references therein. In [19], the case where \( A(\|u\|_2^2) = a \|u\|_2^2 + b \) and \( f(x, u) = u^p \) in (1.2) has been considered and the existence of a branch of positive solutions bifurcating from infinity at \( \lambda = 0 \) was studied.

The purpose of this study is to investigate the global shape of the solution curves of nonlocal elliptic equations from the viewpoint of bifurcation problems. More precisely, we would like to clarify how the properties of Kirchhoff functions have effect on the global shape of the solution curves of nonlocal elliptic equations. To do this, we try to characterize the global behavior of solution curves by the growth order of \( A \).

In the typical elementary functions, it is clear that the most slowly increasing function is the logarithmic function. As an important step to accomplish our purpose, we study the problem (1.1), which has never before, to the author’s knowledge, been studied. By the results obtained here, we are able to compare the global shape of solution curves of (1.1) with that which was obtained in the case \( A(\|u\|_2^2) = \|u\|_2^2 + 1 \) in [16]. Then we find that the global shape of solution curves of these problems are completely different from each other.

The final goal of our study is to treat the equations with more general nonlocal Kirchhoff functions \( A \) in a unified fashion. However, as mentioned above, once \( A \) varies, the global shape of the solutions are completely changed. Moreover, the mathematical technique of calculation is different from each other. To overcome these difficulties, it is quite important to investigate as many typical examples of \( A \) including (1.1) as possible, and obtain many examples of solution curves. Then we extract commonalities of them to characterize the Kirchhoff functions for which the similar shape of global solution curves appear as the next step. By these observations, we will be able to characterize the structures of the bifurcation diagrams in the field of nonlocal elliptic problems. These investigations will give us the appropriate approach to the final goal of our problems.

We also emphasize that, as far as the author knows, Kirchhoff function \( A \) contains only one of \( \|u\|_2 \) or \( \|u\|_2 \). However, the Kirchhoff function in (1.1) contains both of them simultaneously. It seems that there are few papers which treat such problems as (1.1). Therefore, little is known about the properties of the solutions of (1.1), and our results here seem to be novel. To state our results, we prepare the following notation. For \( p > 1 \), let

\[
\begin{align*}
-W''(x) &= W(x)^p, \quad x \in I, \\
W(x) &> 0, \quad x \in I, \\
W(0) &= W(1) = 0.
\end{align*}
\]
We know from [5, p.224] that there exists a unique solution $W_p(x)$ of (1.3). For $m \geq 1$ and $q \geq 0$, we put

$$L_{p,q} := \int_0^1 \frac{s^q}{\sqrt{1 - s^{p+1}}} ds, \quad M_{p,m} := \int_0^1 (1 - s^{p+1})^{(m-1)/2} ds. \quad (1.4)$$

Then we have

$$\|W_p\|_m^m = \frac{2mp^{p-1}(p+1)^{m/(p-1)}L_{p,0}^{(mp+m-1)/(p-1)}M_{p,m}}{p-1} \quad (1.5)$$

$$\|W_p\|_\infty = (2(p+1))^{1/(p-1)}L_{p,0}^{2/(p-1)} \quad (1.6)$$

(1.5) and (1.6) have been given in [16]. For completeness, the proof of (1.5) and (1.6) will be given in Appendix.

In the following Theorems 1.1 – 1.3, we consider the case $a = 0$ and $b$, which is rewritten by $d$ in (1.1), since it is convenient to distinguish between the constant in front of $\|u\|_2$ in the case $a = 0$ and $a \neq 0$. Namely, we first consider the equation

$$\begin{cases}
- \log(d\|u\|_2^2 + 1)u''(x) = \lambda u(x)^p, & x \in I := (0, 1), \\
u(x) > 0, & x \in I, \\
u(0) = u(1) = 0,
\end{cases} \quad (1.7)$$

where $d > 0$ is a given constant. Now we state our main results.

**Theorem 1.1** Consider (1.7). Assume that $p > 3$. Then for any $\lambda > 0$, there exists a unique solution $u_\lambda$ of (1.7). Further, as $\lambda \to \infty$,

$$u_\lambda(x) = \left(\frac{\lambda\|W_p\|_2^{1-p}}{d}\right)^{1/(3-p)} \left(1 + \frac{1}{2(3-p)}d^{(p-1)/(p-3)}(\lambda\|W_p\|_2^{1-p})^{2/(3-p)}(1 + o(1))\right)$$

$$\times \|W_p\|_2^{-1}W_p(x). \quad (1.8)$$

Moreover, as $\lambda \to 0$,

$$u_\lambda(x) = \left(\frac{2}{p-1}\right)^{1/(p-1)} \lambda^{-1/(p-1)} \left(\log \frac{1}{\lambda}\right)^{1/(p-1)}$$

$$\times \left(1 + \frac{1}{p-1} \frac{\log \log 1}{\log 1} (1 + o(1))\right) W_p(x). \quad (1.9)$$

**Theorem 1.2** Consider (1.7). Assume that $1 < p < 3$. Furthermore, put

$$\nu := \frac{2}{p-1} \frac{dt_2^{(3-p)/2}}{dt_2 + 1} \|W_p\|_2^{p-1}, \quad (1.10)$$

where $t_2 > 0$ is a constant determined later. Then the following three cases occur:
(i) If $0 < \lambda < \nu$, then there exist exactly two solutions $u_{\lambda,1}$ and $u_{\lambda,3}$ of (1.7) satisfying $u_{\lambda,1} < u_{\lambda,3}$ in I. Furthermore, as $\lambda \to 0$,

$$u_{\lambda,1}(x) = \left( \frac{\lambda \|W_p\|_2^{1-p}}{d} \right)^{1/(3-p)} \left( 1 + \frac{1}{2(3-p)} \right) \left( \frac{1}{2} \right)^{2/(3-p)} d^{(1-p)/(3-p)} (1 + o(1)) \times \|W_p\|_2^{-1} W_p(x),$$

(1.11)

$$u_{\lambda,3}(x) = \left( \frac{2}{p-1} \right)^{1/(p-1)} \lambda^{-1/(p-1)} \left( \log \frac{1}{\lambda} \right)^{1/(p-1)} \times \left\{ 1 + \frac{1}{p-1} \frac{\log(\log \frac{1}{\lambda})}{\log \frac{1}{\lambda}} (1 + o(1)) \right\} W_p(x).$$

(1.12)

(ii) If $\lambda = \nu$, then there exists exactly one solution $u_\lambda(x)$ of (1.7).

(iii) If $\lambda > \nu$, then there exists no solution $u_\lambda(x)$ of (1.7).

**Theorem 1.3** Consider (1.7). Let $p = 3$.

(i) Let $\lambda \geq d \|W_3\|_2^2$. Then there exists no solution of (1.7).

(ii) Let $0 < \lambda < d \|W_3\|_2^2$. Then there exists exactly one solution $u_\lambda$ of (1.7).

(iii) Let $0 < \lambda < d \|W_3\|_2^2$. Then as $\lambda \to d \|W_3\|_2^2$,

$$u_\lambda(x) = \frac{\sqrt{2}}{d} \sqrt{d - \lambda \|W_3\|_2^{-2}} \left\{ 1 + \frac{2}{3} (d - \lambda \|W_3\|_2^{-2})(1 + o(1)) \right\} \|W_3\|_2^{-1} W_3(x).$$

(1.13)

Furthermore, as $\lambda \to 0$,

$$u_\lambda(x) = \lambda^{-1/2} \left( \log \frac{1}{\lambda} \right)^{1/2} \left\{ 1 + \frac{\log(\log \frac{1}{\lambda})}{2 \log \frac{1}{\lambda}} (1 + o(1)) \right\} W_p(x).$$

(1.14)

Now we consider the Eq. (1.1). We show that (1.1) is reduced to (1.7).

**Theorem 1.4** Consider (1.1). Then (1.1) is reduced to (1.7) with $d = d_0$, where

$$d_0 := \frac{4a L_{p,0}^2 M_{p,2}}{L_{p,2}} + b.$$

(1.15)

Namely, the solution $u_\lambda$ of (1.1) with $a > 0$ and $b > 0$ satisfies (1.7) with $d = d_0$. Therefore, the solution $u_\lambda$ of (1.1) satisfies all the results in Theorems 1.1–1.3 with $d_0$.

The rest of this paper is organized as follows. In Sec. 2, we prove Theorems 1.1–1.3 by using the argument in [1] and time map method (cf. [12]). In Sect. 3, we prove Theorem 1.4. The final section is the Appendix, in which the proofs of (1.5) and (1.6) will be given for the reader’s convenience.
2 Proofs of Theorems 1.1—1.3

In this section, we consider (1.7). By [5], we know that if \( u_\lambda \) is a solution of (1.7), then \( u_\lambda \) satisfies
\[
\begin{align*}
  u_\lambda(x) &= u_\lambda(1 - x), \quad 0 \leq x \leq \frac{1}{2}, \quad (2.1) \\
  \alpha &:= \|u_\lambda\|_\infty = u_\lambda \left( \frac{1}{2} \right), \quad (2.2) \\
  u'_\lambda(x) &> 0, \quad 0 \leq x < \frac{1}{2}. \quad (2.3)
\end{align*}
\]

For a given \( \lambda > 0 \), let \( w_\lambda(x) \) be a unique solution of
\[
\begin{align*}
  \begin{cases}
    -w''(x) = \lambda w(x)^p, & x \in I, \\
    w(x) > 0, & x \in I, \\
    w(0) = w(1) = 0.
  \end{cases} \quad (2.4)
\end{align*}
\]

It is clear that \( w_\lambda = \lambda^{-1/(p-1)} W_p \). We explain the existence of the solutions \( u_\lambda \) of (1.7) by using the idea in [1]. We put \( M(t) := \log(dt + 1) \) and consider the equation for \( t > 0 \):
\[
M(t) = \log(dt + 1) = \|w_\lambda\|_2^{1-p} t^{(p-1)/2}. \quad (2.5)
\]

Assume that \( t_\lambda > 0 \) satisfies (2.5). We put \( \gamma := t_\lambda^{1/2} \|w_\lambda\|_2^{-1} \) and
\[
\begin{align*}
  u_\lambda := \gamma w_\lambda &= t_\lambda^{1/2} \|w_\lambda\|_2^{-1} w_\lambda = t_\lambda^{1/2} \|W_\lambda\|_2^{-1} W_\lambda. \quad (2.6)
\end{align*}
\]

Then by (2.5), we have
\[
M \left( \|\gamma w_\lambda\|_2^2 \right) = M(t_\lambda) = \gamma^{p-1}. \quad \text{Then we have}
\]
\[
\begin{align*}
  -M \left( \|u_\lambda\|_2^2 \right) u''_\lambda(x) &= -M \left( \|\gamma w_\lambda\|_2^2 \right) \gamma w''_\lambda(x) \\
  &= \gamma^p \lambda w^p_\lambda = \lambda (\gamma w_\lambda(x))^p \\
  &= \lambda u_\lambda(x)^p. \quad (2.7)
\end{align*}
\]

Let
\[
f(t) := \frac{\log(dt + 1)}{t^{(p-1)/2}}. \quad (2.8)
\]

By (2.5) and (2.7), to find the solutions of (1.7), we look for solutions \( t_\lambda > 0 \) of the following equation of \( t > 0 \):
\[
f(t) = \lambda \|W_p\|_2^{1-p}. \quad (2.9)
\]
Assume that $u_\lambda$ is a solution of (1.7). Then $u_\lambda$ is a solution of (2.4) with $\lambda / M (\|u_\lambda\|_2^2)$. Therefore, by the uniqueness of $W_p$ in (1.3), there exists a unique constant $\Lambda > 0$ such that $u_\lambda = \Lambda w_\lambda$. Then we see that $\Lambda = \|u_\lambda\|_2 \|w_\lambda\|_2^{-1}$. Then we put $u_\lambda = \|u_\lambda\|_2 \|w_\lambda\|_2^{-1} w_\lambda$ and $t_\lambda := \|u_\lambda\|_2^2$. Since $u_\lambda$ satisfies (1.7), we have

$$- M \left( \|u_\lambda\|_2^2 \right) \|u_\lambda\|_2 \|w_\lambda\|_2^{-1} w_\lambda''(x) = \lambda \|u_\lambda\|_2^p \|w_\lambda\|_2^{-p} w_\lambda(x)^p. \quad (2.10)$$

This implies that

$$M(t_\lambda) = M \left( \|u_\lambda\|_2^2 \right) = \|u_\lambda\|_2^{-1} \|w_\lambda\|_2^{1-p} = \lambda \|W_p\|_2^{1-p} \|u_\lambda\|_2^{1-p} = \lambda \|W_p\|_2^{1-p} t_\lambda^{(p-1)/2}. \quad (2.11)$$

This implies (2.5). Therefore, the solution of (1.7) coincides with the solution $t$ of (2.5).

**Lemma 2.1** Assume that $p > 3$. Then for any given $\lambda > 0$, there exists a unique $t_\lambda > 0$ such that $f(t_\lambda) = \lambda \|W_p\|_2^{1-p}$. Furthermore, as $\lambda \to \infty$,

$$u_\lambda(x) = \left( \frac{\lambda \|W_p\|_2^{1-p}}{d} \right)^{1/(3-p)} \left( 1 + \frac{1}{2(3-p)} d^{(p-1)/(p-3)} \lambda \|W_p\|_2^{1-p} \right)^{2/(3-p)} (1 + o(1))$$

$$\times \|W_p\|_2^{1-p} W_p(x). \quad (2.12)$$

Moreover, as $\lambda \to 0$,

$$u_\lambda(x) = \left( \frac{2}{p-1} \right)^{1/(p-1)} \left( \frac{\lambda}{\lambda_0} \right)^{-1/(p-1)} \left( \log \frac{1}{\lambda} \right)^{1/(p-1)}$$

$$\times \left\{ 1 + \frac{1}{p-1} \frac{\log \left( \log \frac{1}{\lambda} \right)}{\log \left( \frac{1}{\lambda_0} \right)} (1 + o(1)) \right\} W_p(x). \quad (2.13)$$

**Proof** By (2.8), we have

$$f'(t) = -\frac{p-1}{2} t^{-(1+p)/2} \log(dt + 1) + t^{(1-p)/2} \frac{d}{dt + 1}$$

$$= t^{-(1+p)/2} g(t), \quad (2.14)$$

where

$$g(t) := -\frac{p-1}{2} \log(dt + 1) + \frac{dt}{dt + 1}. \quad (2.15)$$

By direct calculation, we have

$$g'(t) = \frac{d}{(dt + 1)^2} \left\{ -\frac{p-1}{2} (dt + 1) + 1 \right\}. \quad (2.16)$$
By this, we see that $g'(t) < 0$ for $t > 0$. We know that $g(0) = 0$. Therefore, $g(t) < 0$ for $t > 0$. By this and (2.13), we see that $f'(t) < 0$ for $t > 0$. Namely, $f(t)$ is strictly decreasing for $t > 0$. Further, $\lim_{t \to 0} f(t) = \infty$ and $\lim_{t \to \infty} f(t) = 0$. Therefore, there exists a unique $t_\lambda > 0$ such that the Eq. (2.9) holds.

We first assume that $\lambda \to \infty$. We easily see that $t_\lambda \to 0$ as $\lambda \to \infty$. By this and (2.17), we have

$$f(t_\lambda) = \frac{\log(dt_\lambda + 1)}{t_\lambda^{(p-1)/2}} = dt_\lambda^{(3-p)/2} - \frac{1}{2} d t_\lambda^{(5-p)/2} (1 + o(1)) = \lambda \|W_p\|_2^{1-p}.$$ (2.17)

We put

$$t_\lambda = \left( \frac{\lambda \|W_p\|_2^{1-p} d}{3-p} \right)^{2/(3-p)} (1 + R_\lambda),$$ (2.18)

where $R_\lambda$ is the remainder term, which satisfies $R_\lambda \to 0$ as $\lambda \to \infty$. By this and (2.17), we have

$$\lambda \|W_p\|_2^{1-p} (1 + R_\lambda)^{(3-p)/2} - \frac{1}{2} d^{(1-p)/(3-p)} \left( \lambda \|W_p\|_2^{1-p} \right)^{(5-p)/(3-p)} (1 + o(1)).$$ (2.19)

Then by (2.19) and Taylor expansion, we have

$$\frac{3-p}{2} \lambda \|W_p\|_2^{1-p} R_\lambda = \frac{1}{2} d^{(1-p)/(3-p)} \left( \lambda \|W_p\|_2^{1-p} \right)^{(5-p)/(3-p)} (1 + o(1)).$$ (2.20)

This implies

$$R_\lambda = \frac{1}{3-p} d^{(p-1)/(p-3)} \left( \lambda \|W_p\|_2^{1-p} \right)^{2/(3-p)} (1 + o(1)).$$ (2.21)

By (2.6), (2.18), (2.21) and Taylor expansion, as $\lambda \to \infty$, we obtain

$$u_\lambda(x) = t_\lambda^{1/2} \|W_p\|_2^{-1} W_p(x) \left( \frac{\lambda \|W_p\|_2^{1-p} d}{3-p} \right)^{1/(3-p)} \left( 1 + \frac{1}{2(3-p)} d^{(p-1)/(p-3)} \left( \lambda \|W_p\|_2^{1-p} \right)^{2/(3-p)} (1 + o(1)) \right)$$
This implies (2.11). Next, we assume that \( \lambda \to 0 \) and show (2.12). It is easy to see that \( t_\lambda \to \infty \) as \( \lambda \to 0 \). We look for \( t_\lambda \) of the form
\[
t_\lambda = C_\lambda \lambda^{-2/(p-1)} \left( \log \frac{1}{\lambda} \right)^q (1 + R_\lambda),
\]
where \( C \) and \( q \) are constants and \( R_\lambda \) is the remainder term satisfying \( R_\lambda \to 0 \) as \( \lambda \to 0 \). By Taylor expansion, we have
\[
\log(dt_\lambda + 1) = \log t_\lambda + O(1) = \frac{2}{p-1} \log \frac{1}{\lambda} + q \log \left( \log \frac{1}{\lambda} \right) + R_\lambda + O(1),
\]
(2.23)
\[
t_\lambda^{(p-1)/2} \lambda \|W_p\|_2^{1-p} = \lambda \left\{ C \lambda^{-2/(p-1)} \left( \log \frac{1}{\lambda} \right)^q (1 + R_\lambda) \right\}^{(p-1)/2} \|W_p\|_2^{1-p}
\]
\[
= \|W_p\|_2^{1-p} C^{(p-1)/2} \left( \log \frac{1}{\lambda} \right)^{q(p-1)/2} \times \left\{ 1 + \frac{p-1}{2} R_\lambda + o(R_\lambda) \right\}.
\]
(2.24)
This implies \( C = (2/(p-1))^{2/(p-1)} \|W_p\|_2^2 \) and \( q = 2/(p-1) \). By this, (2.23) and (2.24), we have
\[
\frac{2}{p-1} \log \left( \log \frac{1}{\lambda} \right) + R_\lambda + O(1) = R_\lambda (1 + o(1)) \log \frac{1}{\lambda}.
\]
(2.25)
This implies that
\[
R_\lambda = \frac{2}{p-1} \frac{\log \left( \log \frac{1}{\lambda} \right)}{\log \left( \frac{1}{\lambda} \right)} (1 + o(1)).
\]
(2.26)
By this and (2.6), as \( \lambda \to 0 \),
\[
u_\lambda(x) = t_\lambda^{1/2} \|W_p\|^{-1}_2 W_p(x)
\]
\[
= C^{1/2} \lambda^{-1/(p-1)} \left( \log \frac{1}{\lambda} \right)^{q/2} (1 + R_\lambda + o(R_\lambda))^{1/2} \|W_p\|_2^{-1} W_p(x)
\]
\[
= \left( \frac{2}{p-1} \right)^{(p-1)/2} \lambda^{-1/(p-1)} \left( \log \frac{1}{\lambda} \right)^{1/(p-1)} \times \left( 1 + \frac{1}{p-1} \frac{\log \left( \log \frac{1}{\lambda} \right)}{\log \left( \frac{1}{\lambda} \right)} (1 + o(1)) \right) W_p(x).
\]
(2.27)
This implies (2.12). Thus the proof is complete. \(\square\)

We obtain Theorem 1.1 by Lemma 2.1. We next prove Theorem 1.2.
Lemma 2.2 Assume that $1 < p < 3$. Then there exists a constant $v$ and the following three cases occur:

(i) If $0 < \lambda < v$, then there exist two solutions $u_{\lambda,1}$ and $u_{\lambda,3}$ of (1.7).

(ii) If $\lambda = v$, then there exists exactly one solution $u_{\lambda}(x)$ of (1.7).

(iii) If $\lambda > v$, then there exists no solution $u_{\lambda}(x)$ of (1.7).

Proof By (2.15), we see that if $t_0 = (3 - p)/(d(p - 1))$, then $g'(t_0) = 0$. This implies that $g(t)$ is increasing in $0 < t < t_0$ and attains the maximum at $t = t_0$ and decreasing in $t > t_0$. Since $g(0) = 0$, $g(t_0) > 0$ and $g(t) \to -\infty$ as $t \to \infty$, we see that there exists $t = t_2$ such that $g(t_2) = 0$. Namely, $f'(t_2) = 0$. We know that $\lim_{x \to 0} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 0$. Furthermore, $f(t)$ attains the maximum that $t = t_2$. By (2.13) and (2.14), we see that $t_2$ satisfies

$$\log(dt_2 + 1) = \frac{2}{p - 1} \frac{dt_2}{dt_2 + 1}. \quad (2.28)$$

We note that there exists exactly one $t_2$ satisfying (2.28). The reason is simple. We consider the graph of $h(x) := \log(x + 1) - \frac{2}{(x+1)(x+2)}$. Then $h'(x) = \frac{1}{(x+1)^2} (x - \frac{3}{p-1})$. Therefore, $h(x)$ is strictly decreasing in $0 < x < (3 - p)/(p - 1)$ and strictly increasing in $x > (3 - p)/(p - 1)$. Further, $h(0) = 0$ and $\lim_{x \to \infty} h(x) = \infty$. Therefore, there exists a unique $t_2$ such that $(3 - p)/(p - 1) < t_2 < C$ satisfying (2.28), since (2.28) does not hold for $t_2 \gg 1$. By this, (2.8) and (2.28), we have

$$f(t_2) = \frac{2}{p - 1} \frac{dt_2^{(3-p)/2}}{dt_2 + 1} = v\|W_p\|_2^{1-p}, \quad (2.29)$$

where $v := \frac{2}{p - 1} \frac{dt_2^{(3-p)/2}}{dt_2 + 1} \|W_p\|_2^{p-1}$. If $0 < \lambda < v$, then there exist exactly two $t_1 < t_3$ such that $f(t_j) = \lambda\|W_p\|_2^{p-1} (j = 1, 3)$. This implies that the exists exactly two solutions of (1.7) $u_{\lambda,1}$ and $u_{\lambda,3}$ corresponding to $t_1$ and $t_3$. Similarly, if $\lambda = v$, then there exists one solution of (1.7) and if $\lambda > v$, then there exist no solutions of (1.7). Thus the proof is complete. \qed

Now we consider the asymptotic behavior of $u_{\lambda,1}$ and $u_{\lambda,3}$ as $\lambda \to 0$.

Lemma 2.3 Assume that $1 < p < 3$. Then as $\lambda \to 0$,

$$u_{\lambda,1}(x) = \left(\frac{\lambda \|W_p\|_2}{d}\right)^{1/(3-p)} \left(1 + \frac{1}{2(3-p)} \left(\frac{\lambda \|W_p\|_2}{1-p}\right)^{2/(3-p)} d^{(1-p)/(3-p)} (1 + o(1))\right)^{1/(3-p-1)}$$

$$\times \|W_p\|_2^{-1} W_p(x), \quad (2.30)$$

$$u_{\lambda,3}(x) = \left(\frac{2}{p - 1}\right)^{1/(p-1)} \left(\frac{\lambda}{1-p}\right)^{-1/(p-1)}$$

$$\times \left\{1 + \frac{1}{p - 1} \log\left(\frac{1}{\lambda}\right) \log\left(\frac{1}{\lambda}\right) (1 + o(1))\right\} W_p(x). \quad (2.31)$$
**Proof** Since \(1 < p < 3\), and \(t_1 < t_2 < t_3\), we easily see that \(t_1 \to 0\) and \(t_3 \to \infty\) as \(\lambda \to 0\). We first prove (2.30). By (2.4) and Taylor expansion, we have

\[
\frac{dt_1 - (dt_1)^2/2 + o(t_1^2)}{t_1^{(p-1)/2}} = dt_1^{(3-p)/2} - \frac{1}{2} \lambda^2 t_1^{(5-p)/2} (1 + o(1)) = \lambda \|W_p\|_2^{1-p}. 
\]

(2.32)

By this, we have

\[
t_1 = \left(\frac{\lambda \|W_p\|_2^{1-p}}{d} \right)^{2/(3-p)} (1 + \eta),
\]

(2.33)

where \(\eta \to 0\) as \(\lambda \to 0\). By (2.32), (2.33) and Taylor expansion, we have

\[
\lambda \|W_p\|_2^{1-p} (1 + \eta)^{(3-p)/2} - \frac{1}{2} \lambda^2 \left(\frac{\lambda \|W_p\|_2^{1-p}}{d} \right)^{(5-p)/(3-p)} (1 + o(1)) - \frac{1}{2} \lambda^2 \left(\frac{\lambda \|W_p\|_2^{1-p}}{d} \right)^{(5-p)/(3-p)} (1 + o(1)) = \lambda \|W_p\|_2^{1-p}.
\]

(2.34)

This implies that

\[
\eta = \frac{1}{3-p} \left(\frac{\lambda \|W_p\|_2^{1-p}}{d} \right)^{2/(3-p)} d^{(1-p)/(3-p)} (1 + o(1)).
\]

(2.35)

By (2.6), (2.33), (2.35) and Taylor expansion, we have

\[
\begin{align*}
\log(bt_3 + 1) &= \log t_3 + O(1) = t_3^{(p-1)/2} \lambda \|W_p\|_2^{1-p}. 
\end{align*}
\]

(2.37)
Then the situation is the same as that of (2.23), (2.24) and (2.26). Therefore, by the same argument as that to obtain (2.12), we obtain (2.31). So we omit the proof. Thus the proof is complete. □

Lemma 2.4 Assume that $p = 3$.

(i) Let $\lambda \geq d\|W_3\|_2^2$. Then there exists no solution of (1.7).

(ii) Let $0 < \lambda < d\|W_3\|_2^2$. Then there exists exactly one solution $u_\lambda$ of (1.7).

(iii) Let $0 < \lambda < d\|W_3\|_2^2$. Then as $\lambda \to d\|W_3\|_2^2$,

$$u_\lambda(x) = \frac{\sqrt{2}}{d} \sqrt{d - \lambda\|W_3\|_2^{-2}} \times \left\{1 + \frac{2}{3}(d - \lambda\|W_3\|_2^{-2})(1 + o(1))\right\}\|W_3\|_2^{-1}W_3(x). \quad (2.38)$$

Furthermore, as $\lambda \to 0$,

$$u_\lambda(x) = \lambda^{-1/2} \left\{\log \left(\frac{1}{\lambda}\right)\right\}^{1/2} \left\{1 + \frac{\log(\log \frac{1}{\lambda})}{2 \log \frac{1}{\lambda}} (1 + o(1))\right\} W_p(x). \quad (2.39)$$

Proof We first prove (i) and (ii). By (2.8), we have

$$f(t) = \frac{\log(dt + 1)}{t}. \quad (2.40)$$

Then by (2.14) and (2.15), for $t > 0$, we have

$$g(t) = -\log(dt + 1) + \frac{dt}{dt + 1}, \quad (2.41)$$

$$g'(t) = -\frac{d^2t}{(dt + 1)^2} < 0. \quad (2.42)$$

By this, $g(t)$ is strictly decreasing for $t \geq 0$ and $g(0) = 0$. This implies that $g(t) < 0$ for $t > 0$. By this and (2.13), $f'(t) < 0$ for $t > 0$ and $f(0) = \lim_{t \to 0} f(t) = d > 0$. Further, $f(t) \to 0$ as $t \to \infty$. By this and (2.9), we see that if $\lambda\|W_3\|_2^{-2} \geq d$, then there exists no solution $t_\lambda > 0$ of (2.9). Further, if $0 < \lambda\|W_p\|_2^{-2} < d$, then there exists exactly one solution $t_\lambda > 0$ of (2.9). Therefore, we obtain (i) and (ii).

We next prove (iii). In this case, it is clear that $t_\lambda \to 0$ as $\lambda \to d\|W_3\|_2^{-2}$. By this, (2.8), (2.9) and Taylor expansion, we have

$$f(t_\lambda) = \frac{\log(dt_\lambda + 1)}{t_\lambda} = \frac{dt_\lambda - (1/2)d^2t_\lambda^2 + (1/3)d^3t_\lambda^3 + O(t_\lambda^4)}{t_\lambda}$$

$$= d - \frac{1}{2}d^2t_\lambda + \frac{1}{3}d^3t_\lambda^2 + O(t_\lambda^3) = \lambda\|W_3\|_2^{-2}. \quad (2.43)$$
This implies that

$$t_\lambda = \frac{2}{d^2} \left( d - \lambda \|W_3\|_2^{-2} \right) + R, \quad (2.44)$$

where $R$ is a remainder term. By this, (2.43) and direct calculation, we have

$$R = \frac{8}{3d^2} \left( d - \lambda \|W_3\|_2^{-2} \right)^2 (1 + o(1)). \quad (2.45)$$

By (2.6), (2.44) and (2.45), we obtain

$$u_\lambda(x) = \frac{t_1}{2} \lambda \|W_3\|_2^{-1} W_3(x)
= \frac{\sqrt{2}}{d} \sqrt{d - \lambda \|W_3\|_2^{-2}} \times \left\{ 1 + \frac{2}{3} (d - \lambda \|W_3\|_2^{-2}) (1 + o(1)) \right\} \|W_3\|_2^{-1} W_3(x). \quad (2.46)$$

This implies (2.39). Finally, we see that the argument as that to obtain (1.12) is available in the case $p = 3$. Therefore, (1.14) follows from (1.12) by putting $p = 3$. Thus the proof is complete. \(\square\)

3 Proof of Theorem 1.4

In this section, we consider (1.1). Let $a > 0$ and $b > 0$ in (1.1). We show that (1.1) is equivalent to (1.7) if we put

$$d = d_0 := \frac{4aL^2_{p,0}M_{p,2}}{L_{p,2}} + b. \quad (3.1)$$

To do this, we need two lemmas.

**Lemma 3.1** Assume that $u_\lambda$ satisfies (1.1). Then

$$\|u'_\lambda\|_2^2 = \frac{4L^2_{p,0}M_{p,2}}{L_{p,2}} \|u_\lambda\|_2^2. \quad (3.2)$$

**Proof** We put $H := \log \left( a \|u'_\lambda\|_2 + b \|u_\lambda\|_2^2 + 1 \right)$. By (1.1), we have

$$Hu'_\lambda(x) + \lambda u_\lambda(x)^p = 0. \quad (3.3)$$

This implies

$$\left\{ Hu''_\lambda(x) + \lambda u_\lambda(x)^{p+1} \right\} u'_\lambda(x) = 0. \quad (3.4)$$
We recall that $\alpha = \|u_\lambda\|_\infty$, which is defined in (2.2). By this, (2.2) and putting $x = 1/2$, for $x \in \bar{I}$, we have
\[
\frac{1}{2} Hu'_\lambda(x)^2 + \frac{1}{p+1} \lambda u_\lambda(x)^p = \text{constant} = \frac{1}{p+1} \lambda \alpha^{p+1}.
\] (3.5)
By this and (2.3), for $0 \leq x \leq 1/2$, we have
\[
u'_\lambda(x) = \sqrt{k(\alpha^{p+1} - u_\lambda(x)^{p+1})},
\] (3.6)
where $k := 2\lambda/(H(p+1))$. By this, (1.4), (2.1), (2.3) and putting $u_\lambda = \theta = \alpha s$, we have
\[
\|u'_\lambda\|_2^2 = 2 \int_0^{1/2} \sqrt{k(\alpha^{p+1} - u_\lambda(x)^{p+1})} u'_\lambda(x) dx
\]
\[
= 2 \int_0^\alpha \sqrt{k(\alpha^{p+1} - \theta^{p+1})} d\theta
\]
\[
= 2\sqrt{k} \alpha^{(p+3)/2} \int_0^1 \sqrt{1-s^{p+1}} ds = 2\sqrt{k} M_{p,2} \alpha^{(p+3)/2},
\] (3.7)
\[
\|u_\lambda\|_2^2 = 2 \int_0^{1/2} u_\lambda(x)^2 \frac{u'_\lambda(x)}{\sqrt{k(\alpha^{p+1} - u_\lambda(x)^{p+1})}} dx
\]
\[
= 2 \int_0^\alpha \frac{\theta^2}{\sqrt{k(\alpha^{p+1} - \theta^{p+1})}} d\theta
\]
\[
= \frac{2}{\sqrt{k}} \alpha^{(5-p)/2} \int_0^1 \frac{s^2}{\sqrt{1-s^{p+1}}} ds = \frac{2}{\sqrt{k}} L_{p,2} \alpha^{(5-p)/2}.
\] (3.8)
\[
\frac{1}{2} = \int_0^{1/2} \frac{u'_\lambda(x)}{\sqrt{k(\alpha^{p+1} - u_\lambda(x)^{p+1})}} dx
\]
\[
= \frac{1}{\sqrt{k}} \int_0^\alpha \frac{1}{\sqrt{\alpha^{p+1} - \theta^{p+1}}} d\theta
\]
\[
= \frac{1}{\sqrt{k}} \alpha^{(1-p)/2} \int_0^1 \frac{1}{\sqrt{1-s^{p+1}}} ds
\]
\[
= \frac{1}{\sqrt{k}} L_{p,0,2} \alpha^{(1-p)/2}.
\] (3.9)
By this, we have
\[
\sqrt{k} = 2L_{p,0,2} \alpha^{(1-p)/2}.
\] (3.10)
By this, (3.7) and (3.8), we have
\[
\|u'_\lambda\|_2^2 = 4L_{p,0,2} M_{p,2} \alpha^2, \|u_\lambda\|_2^2 = \frac{L_{p,0,2}}{L_{p,2}} \alpha^2, \|u'_\lambda\|_2^2 = \frac{4L_{p,0,2}^2 M_{p,2}}{L_{p,2}} \|u_\lambda\|_2^2.
\] (3.11)
Thus the proof is complete. \(\square\)
Lemma 3.2 Assume that \( u_\lambda \) satisfies (1.7). Then
\[
\|u'_\lambda\|_2^2 = \frac{4L_{p,0}^2 M_{p,2}}{L_{p,2}} \|u_\lambda\|_2^2.
\] (3.12)

We obtain Lemma 3.2 by the same argument as that to prove Lemma 3.1. Therefore, we omit the proof.

Proof of Theorem 1.4 Let \( u_\lambda \) be a solution to (1.7). We put
\[
d_0 := \frac{4aL_{p,0}^2 M_{p,2}}{L_{p,2}} + b.
\] (3.13)

Then by Lemmas 3.1 and 3.2, we have
\[
d_0\|u_\lambda\|_2^2 + 1 = a\|u'_\lambda\|_2^2 + b\|u_\lambda\|_2^2 + 1.
\] (3.14)

Therefore, the solution \( u_\lambda \) of (1.7) is also the solution of (1.1), since (3.14) holds. Therefore, we are able to apply the argument in Sect. 2 and obtain Theorem 1.4. Thus the proof is complete. \( \square \)

4 Appendix

Let \( p > 1 \). We show (1.5) and (1.6), which was proved in [16], for completeness. We apply the time map argument to (1.3) (cf. [12]). Since (1.3) is autonomous, as (2.1)–(2.3), we have
\[
W_p(x) = W_p(1-x), \quad 0 \leq x \leq \frac{1}{2},
\] (4.1)
\[
\xi := \|W_p\|_{\infty} = \max_{0 \leq x \leq 1} W_p(x) = W_p\left(\frac{1}{2}\right),
\] (4.2)
\[
W'_p(x) > 0, \quad 0 \leq x < \frac{1}{2}.
\] (4.3)

By (1.3), for \( 0 \leq x \leq 1 \), we have
\[
\left\{ W''_p(x) + W_p(x)^p \right\} W'_p(x) = 0.
\] (4.4)

By this and (4.2), we have
\[
\frac{1}{2} W'_p(x)^2 + \frac{1}{p+1} W_p(x)^{p+1} = \text{constant} = \frac{1}{p+1} W_p\left(\frac{1}{2}\right)^{p+1} = \frac{1}{p+1} \xi^{p+1}.
\] (4.5)
By this and (4.3), for $0 \leq x \leq 1/2$, we have, using $\theta = \xi s$,

$$\frac{W_p'(x)}{2} = \sqrt{\frac{2}{p+1}} (\xi^{p+1} - W_p(x)^{p+1}).$$  \hspace{1cm} (4.6)

By (4.1) and (4.6), we have

$$\frac{1}{2} = \frac{1}{2} \int_0^{1/2} W_p'(x) dx = \frac{1}{2} \int_0^{1/2} \frac{W_p'(x)}{\sqrt{2(p+1)(\xi^{p+1} - W_p(x)^{p+1})}} dx$$

$$= \sqrt{\frac{p+1}{2}} \int_0^{\xi} \frac{1}{\sqrt{\xi^{p+1} - \theta^{p+1}}} d\theta$$

$$= \sqrt{\frac{p+1}{2}} \xi^{(1-p)/2} \int_0^1 \frac{1}{\sqrt{1 - \xi^{p+1}-s^{p+1}}} ds$$

$$= \sqrt{\frac{p+1}{2}} \xi^{(1-p)/2} L_{p,0}.$$  \hspace{1cm} (4.7)

By this, we have

$$\xi = (2(p+1))^{1/(p-1)} L_{p,0}^{2/(p-1)}. $$  \hspace{1cm} (4.8)

This implies (1.6). We next show (1.5). By (4.1), (4.2), (4.6) and (4.8), we have

$$\|W_p'\|_m^m = 2 \int_0^{1/2} W_p'(x)^{m-1} W_p'(x) dx$$

$$= 2 \left( \frac{2}{p+1} \right)^{(m-1)/2} \int_0^{1/2} \left( \xi^{p+1} - W_p(x)^{p+1} \right)^{(m-1)/2} W_p'(x) dx$$

$$= 2^{(m+1)/2} (p+1)^{-(m-1)/2} \int_0^{\xi} (\xi^{p+1} - \theta^{p+1})^{(m-1)/2} d\theta$$

$$= 2^{(m+1)/2} (p+1)^{-(m-1)/2} \xi^{(m-1)(p+1)/2+1} \int_0^1 (1 - s^{p+1})^{(m-1)/2} ds$$

$$= 2^{mp/(p-1)} (p+1)^{m/(p-1)} L_{p,0}^{(mp+m-p+1)/(p-1)} M_{p,m}. $$  \hspace{1cm} (4.9)

This implies (1.5). Thus the proof is complete.

Author Contributions  All the theorems in this manuscript are proved by Tetsutaro Shibata.

Funding  This work was supported by JSPS KAKENHI Grant no. JP21K03310.

Declarations

Conflict of interest  The author declares that he has no conflict of interest.
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