Abstract. We consider the facility location problem in two dimensions. In particular, we consider a setting where agents have Euclidean preferences, defined by their ideal points, for a facility to be located in $\mathbb{R}^2$. For the utilitarian objective and an odd number of agents, we show that the coordinate-wise median mechanism (CM) has a worst-case approximation ratio (WAR) of $\sqrt{2} \frac{\sqrt{n^2+1}}{n+1}$. Further, we show that CM has the lowest WAR for this objective in the class of strategyproof, anonymous, continuous mechanism. For the $p$-norm social welfare objective, we find that the WAR for CM is bounded above by $2^\frac{1}{p}$ for $p \geq 2$. Since it follows from previous results in one-dimension that any deterministic strategyproof mechanism must have WAR at least $2^{\frac{1}{p}}$, our upper bound guarantees that the CM mechanism is very close to being the best deterministic strategyproof mechanism for $p \geq 2$.

1 Introduction

We consider the problem of locating a single facility in $\mathbb{R}^2$, given a finite set of agents who have Euclidean costs defined by their private ideal locations. A central authority wishes to choose a facility location that optimizes some measure of social welfare. Since the ideal points are private information, the mechanism choosing the facility location based on reported ideal points must be strategyproof. Hence, the problem is to check if the mechanism choosing a socially optimal location is strategyproof, and if not, to find a strategyproof mechanism that best approximates the optimal social cost.

This problem has been extensively studied in the literature known as Approximate Mechanism Design without money. It was first introduced by Procaccia and Tennenholtz [26] who studied the setting of locating a single facility on a

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real line under the utilitarian (sum of individual costs) and egalitarian (maximum of individual costs) objectives. Since then, the problem has received much attention, with extensions to alternative objective functions ([11], [12], [13], [6]), multiple facilities ([19], [10], [27]), obnoxious facilities [7], different networks ([1], [9], [21]), and other variations. In the class of deterministic strategyproof mechanisms for locating a facility, the median mechanism has been shown to be optimal under various objectives and domains. [20] provides a good survey of results on approximation ratios achieved in several of these settings.

There is also a large literature in social choice theory on characterising the set of strategyproof mechanisms under different assumptions on preference domains ([23], [25]). In multiple dimensions, the characterizations typically include or are completely described by the coordinate-wise median mechanism ([17], [5], [24], [3]). The strong axiomatic foundations of the coordinate-wise median mechanism, together with the nice properties of the median in one dimension, motivate our study of the coordinate-wise median mechanism against the optimal mechanism in two dimensions under a variety of social cost functions.

There has been some recent work in extending the facility location problem to multiple dimensions. [18] characterises regions in the Euclidean plane that would lead to the optimal facility location being strategyproof when agents may want the facility to be close or far from their location on a real segment. [31] and [30] show that the generalized median mechanisms are not group-strategyproof in Euclidean space and the incentive of group misreport is unbounded. The most closely related work is Meir ([22]), who uses techniques different from ours to find the approximation ratio for the coordinate-wise median mechanism for the case of three agents.

1.1 Our contribution

Our main contribution is to show that for \( n \) odd, the coordinate-wise median mechanism has a worst-case approximation ratio of \( \sqrt{2 \frac{n^2 + 1}{n^2 + 2}} \) under the minimisation objective. Using the worst-case profile and the characterisation from [17], we establish that there is no deterministic, strategyproof, anonymous and continuous mechanism that does better. Using the one-dimensional analysis of [11], we show that for the \( p - \text{norm} \) objective, the worst-case approximation ratio for the coordinate-wise median mechanism is bounded above by \( 2^{\frac{1}{p - \frac{1}{p}}} \) for \( p \geq 2 \). Since it follows from [11] that any deterministic strategyproof mechanism must have
approximation ratio at least $2^{1-\frac{1}{p}}$, our bounds suggest that the coordinate-wise median mechanism is (at worst) very nearly optimal.

2 Preliminaries

We study the problem in which a facility is to be located in $\mathbb{R}^2$. Let $N = \{1, 2, \ldots, n\}$ be the set of agents. Each agent $i$ has a Euclidean cost; that is, there is a point $x_i = (a_i, b_i)$, called agent $i$’s ideal point, such that her cost for locating the facility at $y$ is $C(y, x_i) = d(y, x_i)$. We denote by $x$ the profile of ideal points: $x = (x_1, \ldots, x_n)$.

We assume there is a welfare objective summarized by a social cost function $sc$, in the sense that locating the facility at $y$ is (weakly) better than locating the facility at $z$ if and only if $sc(y, x) \leq sc(z, x)$.

2.1 Mechanisms

A mechanism is a function $f : (\mathbb{R}^2)^n \rightarrow \mathbb{R}^2$. A mechanism is said to be strategyproof if no agent can benefit by misreporting her ideal point, regardless of the reports of the other agents. Formally:

Definition 1. A mechanism $f$ is strategyproof if for all $i \in N$, $x_i, x_i' \in \mathbb{R}^2$, $x_{-i} \in (\mathbb{R}^2)^{n-1}$,

$$C(f(x_i, x_{-i}), x_i) \leq C(f(x_i', x_{-i}), x_i).$$

Definition 2. A mechanism $f$ is anonymous if for any permutation $\pi : [n] \rightarrow [n]$,

$$f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$$

A mechanism $f$ is continuous if it is a continuous function. A social cost function is a function $sc : \mathbb{R}^2 \times (\mathbb{R}^2)^n \rightarrow \mathbb{R}$. We denote by $OPT(sc, x)$ the set of minimizers for $sc$ given $x$:

$$OPT(sc, x) = \arg\min_y sc(y, x).$$

When $OPT(sc, \cdot)$ is singleton-valued, we will abuse notation and use $OPT(sc, x)$ to refer to the unique element contained therein. When $sc$ is clear from context, we will suppress the first argument and write $OPT(sc, x)$ simply as $OPT(x)$.

To measure how closely a strategyproof mechanism approximates the optimal social cost for a given preference profile, we use the notion of approximation ratio.
Definition 3. The approximation ratio of a mechanism $f$ at a profile $x \in (\mathbb{R}^2)^n$ is

$$AR(f, x) = \frac{sc(f(x), x)}{sc(OPT(x), x)}.$$ 

To measure how closely a strategyproof mechanism approximates the optimal social cost in the worst case, we use the worst-case approximation ratio.

Definition 4. The worst case approximation ratio of a mechanism $f$ is

$$WAR(f) = \sup_{x \in (\mathbb{R}^2)^n} AR(f, x).$$

The coordinate-wise median mechanism

Definition 5. The coordinate-wise median mechanism is given by

$$c(x) = (\text{median}(a_1, a_2, \ldots, a_n), \text{median}(b_1, b_2, \ldots, b_n)).$$

The coordinate-wise median has strong axiomatic foundations in the literature (see, e.g., [5], [24], [17]). In particular, the strategyproofness of CM follows from the following result of [17]:

Lemma 1 (Kim and Roush, 1984). A mechanism $f$ is strategyproof, anonymous and continuous if and only if there exist points $p_1, p_2, \ldots, p_{n+1} \in (\{-\infty, \infty\} \cup \mathbb{R})^2$ such that $f(x) = c(x_1, \ldots, x_n, p_1, \ldots, p_{n+1})$.

Note that for $n$ odd, the coordinate-wise median mechanism is obtained by taking, e.g., $p_1 = \cdots = p_{n+1} = (-\infty, -\infty)$ and $p_{n+1+1} = \cdots = p_{n+1} = (\infty, \infty)$.

Geometric median A point minimizing the sum of distances from a finite set of points in a Euclidean space is known as a geometric median for that set of points. The geometric median is characterised by the following result:

Lemma 2. Let $X$ be a Euclidean space. Given $x \in X^n$, a point $y \in X$ is a geometric median for $x$ if and only if there are vectors $u_1, \ldots, u_n$ such that

$$\sum_{i=1}^{n} u_i = 0$$

\[1\text{ Note that this is only well-defined in the case that } n \text{ is odd. We will also discuss mechanisms which are adaptations of the coordinate-wise median mechanism to even } n.\]

\[2\text{ See, e.g., } 33\]
where for \( x_i \neq y \), \( u_i = \frac{x_i - y}{|x_i - y|} \) and for \( x_i = y \), \( |u_i| \leq 1 \).

This characterisation yields conditions under which changing a profile of points does not change the geometric median, as summarized in the following corollary:

**Corollary 1.** Let \( x \in X^n \), and denote by \( y \) the geometric median of \( x \). For any \( i \), if \( x_i \neq y \) and if \( x'_i \in \{ y + t(x_i - y) \mid t \in \mathbb{R}_{\geq 0} \} \), then the geometric median for the profile \( (x'_i, x_{-i}) \) is also \( y \).

Informally, moving a point directly away from or directly towards (but not past) the geometric median leaves the geometric median unchanged. We will use this observation repeatedly in the sequel and note here that in fact it will be the only characteristic of the geometric median that we use for much of the paper.

For the special case of \( n = 3 \), a more explicit characterisation is easily obtained from Lemma 23 if any angle of the triangle formed by the three points is at least \( 120^\circ \), \( g(x) \) lies on the vertex of that angle; otherwise, it is the unique point inside the triangle that subtends an angle of \( 120^\circ \) to all three pairs of vertices.

### 2.2 Notation

We refer to the coordinates of points in \( \mathbb{R}^2 \) by \( a \) and \( b \). We refer to the sets \( \mathbb{R} \times \{0\} \) and \( \{0\} \times \mathbb{R} \) as the \( a \)-axis and the \( b \)-axis, respectively. We refer to the sets \( \pm \mathbb{R}_{\geq 0} \times \{0\} \) and \( \{0\} \times \pm \mathbb{R}_{\geq 0} \) as the \( \pm a \)-axes and \( \pm b \)-axes, respectively. We refer to the geometric median by \( g(x) \). We use the notation \( a_g(x) \) and \( b_g(x) \) to denote the first and second coordinates of \( g(x) \), respectively. Similarly, we denote by \( c(x) \) the coordinate-wise median of \( x \). We denote by \( a_c(\cdot) \) and \( b_c(\cdot) \) the coordinates of \( c(\cdot) \), so that \( c(x) = (a_c(x), b_c(x)) \).

We use the notation \( [yz] \) to denote the line segment joining \( y \) and \( z \): \( \{ ty + (1 - t)z : t \in [0, 1] \} \). Similarly, we denote by \( (y, z) \) the set \( [yz] \setminus \{ y, z \} \).

### 3 Coordinate-wise Median mechanism’s WAR for the minisum objective is not bad

In this section, we consider the objective \( sc(y, x) = (\sum_{i=1}^{n} |y - x_i|) \). Assume that \( n \) is odd and so the geometric median is unique. Consider the geometric median

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3 This characterisation was obtained by Torricelli (1646) and is referred to as the **Torricelli point**.

4 The geometric median is unique whenever \( n \) is odd or points are not collinear.
mechanism, which chooses the geometric median \(g(x)\) at any profile \(x\). Since this mechanism is anonymous, continuous, and not a (generalized) coordinate-wise median mechanism, it follows from Lemma 1 that it is not strategyproof.

Note that by definition, the geometric median mechanism has constant (and hence worst-case) approximation ratio of 1. However, the question remains what the best possible strategyproof approximation to the geometric median mechanism is, in the sense of having the minimum possible \(WAR\). Due to its strong axiomatic foundations ([24], [17], [5], [25], [29] etc.), we consider the coordinate-wise median mechanism as a candidate and investigate the problem of finding its \(WAR\).

We begin by stating our results. We then provide a full proof for the case that \(n = 3\), as we find the approach taken in its proof to be simple enough to be digestible yet sufficiently similar to the more nuanced approach required for arbitrary odd \(n\) as to be illuminating. We then provide a sketch of the proof for all odd \(n\), relegating the formal proof for this case to the appendix.

3.1 Results

Theorem 1. For \(n\) odd, the worst-case approximation ratio for the coordinate-wise median mechanism is given by:

\[
WAR(CM) = \sqrt{\frac{2}{n+1}} \sqrt{n^2 + 1}.
\]

The argument for obtaining the exact value of \(WAR(CM)\) is rather involved. Interestingly, establishing an (asymptotically tight) upper bound of \(\sqrt{2}\) for any odd \(n\) is fairly straightforward:

Lemma 3. For any profile \(x \in (\mathbb{R}^2)^n\),

\[
sc(c(x), x) \leq \sqrt{2} \cdot sc(OPT(x), x).
\]

The proof follows from observing that the median is optimal in one dimension and that for any right triangle, the sum of the lengths of the legs is at most \(\sqrt{2}\) times the length of the hypotenuse.

Remark 1. When \(n = 2m\) is even, the version of the coordinate-wise median mechanism given by \(c(x) = (\text{median}(\infty, a), \text{median}(\infty, b))\) has worst-case approximation ratio

\[
\text{WAR} \text{ monotonically increasing and converges to } \sqrt{2} \text{ as } n \to \infty.
\]
approximation ratio equal to $\sqrt{2}$. This follows from the bound in the previous lemma and the worst-case profile $\mathbf{x}$ where $x_1 = x_2 = \ldots x_m = (1,0)$ and $x_{m+1} = x_{m+2} = \ldots x_{2m} = (0,1)$.

In both the $n = 3$ case and the general case, the key to the proof is to reduce the search space for the worst-case profile from $\mathbb{R}^2$ to a much smaller space of profiles that have a simple structure. In many cases, this involves “transforming” one profile into another profile that has a higher approximation ratio and a simpler structure. One important transformation that helps in significantly reducing the search space involves moving a point $x_i$ directly towards $g(\mathbf{x})$, getting as close as possible to $g(\mathbf{x})$ without changing $c(\mathbf{x})$. Because this transformation will be used repeatedly throughout this section, we provide here a proof that this transformation leads to a profile $(x'_i, x_{-i})$ with a weakly higher approximation ratio.

**Lemma 4 (Towards geometric median).** Let $\mathbf{x}$ be a profile and $i \in N$, and let $\mathbf{x}'$ be any profile such that

1. $x'_i \in [x_i, g(\mathbf{x})]$,
2. for all $j \neq i$, $x'_j = x_j$, and
3. $c(\mathbf{x}') = c(\mathbf{x})$.

Then $AR(\mathbf{x}') \geq AR(\mathbf{x})$.

**Proof.** By definition, $g(\mathbf{x}') = g(\mathbf{x})$ and $c(\mathbf{x}') = c(\mathbf{x})$. The change in optimal social cost is given by $|x_i - x'_i|$ while the change in social cost with respect to coordinate-wise median is $|c(\mathbf{x}) - x'_i| - |c(\mathbf{x}) - x_i|$. By triangle inequality, $|x_i - x'_i| \geq |c(\mathbf{x}) - x'_i| - |c(\mathbf{x}) - x_i|$. Thus, the $sc(OPT(\cdot, \cdot))$ reduces by a greater amount than $sc(CM(\cdot, \cdot))$ as we move $x_i$ to $x'_i$. Since the ratio is always at least 1, it follows that $AR(\mathbf{x}') \geq AR(\mathbf{x})$. \qed

### 3.2 Proof for $n=3$

Define the set of Centered perpendicular (CP) profiles as follows:

$$CP = \{ \mathbf{x} \in (\mathbb{R}^2)^3 : c(\mathbf{x}) = (0,0) \text{ and } \forall i, \text{ either } a_i = 0 \text{ or } b_i = 0 \}.$$

In words, a profile is in $CP$ if the coordinate-wise median is at the origin and all points in $\mathbf{x}$ are on the axes.
Define the set of *Isosceles-centered perpendicular (I-CP)* profiles as follows:

\[ I-CP = \{ x \in CP : \exists t \text{ such that } x = ((t, 0), (-t, 0), (0, 1)) \text{ and } g(x) = (0, 1) \} \]

In words, a profile is in \( I-CP \) if there are two points on the \( a \)-axis equidistant from the origin and the third point is at \((0, 1)\), which is also the geometric median.

We first show that we can reduce the search space for the worst-case profile from \((\mathbb{R}^2)^3\) to \(CP\).

**Lemma 5.** For any profile \( x \in (\mathbb{R}^2)^3 \), there is a profile \( \chi \in CP \) such that \( AR(\chi) \geq AR(x) \).

**Proof.** Let \( x \in (\mathbb{R}^2)^3 \) be a profile. Let \( x' \) be the profile where \( x'_i = x_i - c(x) \). Then \( x' \) has the same approximation ratio as \( x \) and \( c(x') = (0, 0) \). Denote \( A = \{ i : a_i = 0 \} \) and \( B = \{ i : b_i = 0 \} \). Note that since \( c(x') = (0, 0) \), it follows from the definition of \( c(x') \) that \( A \neq \emptyset \) and \( B \neq \emptyset \). For each \( i \), define \( x''_i \) as follows. Let \( \Gamma = \{ (a, b) \in \mathbb{R}^2 : a = 0 \text{ or } b = 0 \} \). If \( i \in A \cup B \), let \( x''_i = x'_i \); otherwise, let \( x''_i \) be the point in \([x'_i, g(x')] \cap \Gamma \) that is closest to \( x'_i \). Then \( x''_i \in \Gamma \) for all \( i \) and \( c(x'') = (0, 0) \), so \( x'' \in CP \). Further, it follows from Lemma 4 that \( AR(x'') \geq AR(x') = AR(x) \); hence, taking \( \chi = x'' \) completes the proof.

\[ \square \]

\[ \text{Figure 3.2: Towards geometric median} \]

Now we show that we can further reduce the search space from \( CP \) to \( I-CP \).

\[ \text{The set } [x'_i, g(x')] \cap \Gamma \text{ is non-empty because } g(x') \text{ cannot be in the same quadrant as } x'_i. \text{ Any point in the same quadrant as } x'_i \text{ subtends an angle of less than } 90^\circ \text{ with the other two points and hence it cannot be the Torricelli point.} \]
Lemma 6. For any profile $x \in CP$, there exists a profile $\chi \in I - CP$ such that $AR(\chi) \geq AR(x)$.

Proof. Let $x$ be a profile in $CP$.

Without loss of generality, we may assume that all $x_i$ are weakly above the $a$-axis and there are at least two $x_i$ on the $a$-axis, since reflecting a profile in $CP$ across the $a$-axis, the $b$-axis, or the line $a = b$ gives a profile in $CP$ with the same approximation ratio. Hence, we can label the points such that $x_1 = (-a, 0)$, $x_2 = (b, 0)$, and $x_3 = (0, c)$, for some $a, b, c \geq 0$.

If $c = 0$, then $AR(x) = 1$, and so every profile has approximation ratio weakly greater than $x$. Hence, we may further assume that $c > 0$.

Since $x_1$ and $x_2$ are on the $a$-axis, it follows from the characterization of the geometric median for three points given in Section 2.1 that $-a \leq a_g(x) \leq b$ and $0 < b_g(x) \leq c$. Hence, moving $x_3$ to $g(x)$ then (if necessary) translating all points by the same vector so that the coordinate-wise median is at the origin yields a profile in $CP$ which has higher approximation ratio. Hence, we may further assume that $g(x) = x_3$.

Let $x'$ be the profile where $x'_1 = (-a + b)/2, 0, x'_2 = ((a + b)/2, 0)$, and $x'_3 = (0, c)$. By definition, $sc(g(x'), x') \leq sc(g(x), x')$ and by an argument that exploits the convexity of the distance function, $sc(g(x), x') \leq sc(g(x), x)$. Combining these inequalities gives $sc(g(x'), x') \leq sc(g(x), x)$, and a simple calculation shows that $sc(c(x'), x') = sc(c(x), x)$. Thus, $AR(x') \geq AR(x)$.

Note that under $x'$, $g(x') = (0, k)$ for some $k \leq c$. Define $x''$ to be the profile with $x''_1 = x'_1$, $x''_2 = x'_2$, and $x''_3 = g(x')$. Then, by Lemma 4, $AR(x'') \geq AR(x')$.

Finally, define $x'''$ such that $x'''_i = \frac{1}{t} x''_i$ for each $i$. Then since $AR(\cdot)$ is homogeneous of degree 0, $AR(x''') = AR(x'')$, and so $AR(x''') \geq AR(x)$. Further, $c(x''') = (0, 0)$, $x'''_1 = (-t, 0)$, $x'''_2 = (t, 0)$, and $x'''_3 = (0, 1)$ for some $t \geq 0$; in fact, it follows from the characterisation of the geometric median that $t \geq \sqrt{3}$. Hence, $x''' \in I - CP$, and so taking $\chi = x'''$ completes the proof.

\[ \Box \]

Denote by $\eta_t = ((t, 0), (-t, 0), (0, 1))$. It follows from the arguments in the proof of Lemma 4 that $I - CP = \{ \eta_t : t \geq \sqrt{3} \}$. Let $\alpha(t) = \frac{2t + 1}{2\sqrt{3}}$. A simple calculation shows that for $t \geq \sqrt{3}$, $AR(\eta_t) = \alpha(t)$. In particular, it follows that $WAR(CM)$ is equal to by sup$_{t \geq \sqrt{3}} \alpha(t)$. Since $\alpha(t)$ achieves its global maximum at $t^* = 2 > \sqrt{3}$, $WAR(CM) = AR(\eta_2) = \alpha(2)$. Since $\alpha(2) = \frac{\sqrt{3}^2 + 1}{3 + 1}$, the result follows.
3.3 Outline for general (odd) $n$

We now consider the case of $n = 2m + 1$ agents. We begin by defining classes of profiles analogous to those used in the proof for $n = 3$.

We define the class of Centered Perpendicular (CP) profiles as all profiles $\mathbf{x} \in (\mathbb{R}^2)^n$ such that

- $c(\mathbf{x}) = (0, 0)$
- for all $i$, either $a_i = 0$ or $b_i = 0$ or $x_i = g(\mathbf{x})$
- if $x_i' \in (x_i, g(\mathbf{x}))$, then $c(x_i', x-i) \neq (0, 0)$

Since the last condition is slightly more subtle than the others and will be important in the sequel, we describe it now in words. This condition says that any (nonzero) movement of any $x_i$ towards the geometric median would result in a change in the coordinate-wise median.

We define the class of Isosceles-Centered Perpendicular (I-CP) profiles as all $\mathbf{x} \in \text{CP}$ for which there exists $t \geq 0$ such that

- $x_1 = \cdots = x_m = (t, 0)$
- $x_{m+1} = (-t, 0)$
- $x_{m+2} = \cdots = x_{2m+1} = (0, 1)$
- $g(\mathbf{x}) = (0, 1)$.

The proof proceeds much as in the proof for $n = 3$. We first show that for every profile, there is some profile in CP with weakly higher approximation ratio. The approach used in the $n = 3$ case extends naturally here: first, translate the profile $\mathbf{x} \in (\mathbb{R}^2)^n$ so that coordinate-wise median moves to the origin; then, starting from $i = 1$ and going to $i = n$, move $x_i$ directly towards the geometric median until either it reaches the geometric median or moving it further would move the coordinate-wise median. The resulting profile is in CP and has an approximation ratio that is weakly greater than $x$'s.

![CP profile](image3.3.png)
Next, we show that for any profile in $CP$, there is some profile in $I - CP$ with weakly higher approximation ratio. The approach used in the $n = 3$ case for this step does not extend in a straightforward manner to the general case—the main obstruction arises from the fact that for a profile $x$ in $CP$, there may be $i \in N$ such that $x_i = g(x)$, which may not be on either axis. The next subsection is devoted to giving an overview of the procedure used to transform a profile in $CP$ to one in $I - CP$ with weakly higher approximation ratio.

Finally, the approach used to calculate the worst-case approximation ratio for profiles in $I - CP$ has much the same structure as in the $n = 3$ case. We define $\eta_t = (x^1_t, \ldots, x^{2m+1}_t)$, where

$$
x^i_t = \begin{cases} 
(t, 0), & i = 1, \ldots, m \\
(-t, 0), & i = m + 1 \\
(0, 1), & i = m + 2, \ldots, 2m + 1 
\end{cases}
$$

and we show that $I - CP = \{ \eta_t : t \geq \sqrt{\frac{2m+1}{2m-1}} \}$. Defining $\alpha(t) = \frac{(m+1)t + m}{(m+1)\sqrt{t^2+1}}$, we show that for $t \geq \sqrt{\frac{2m+1}{2m-1}}$, $AR(\eta_t) = \alpha(t)$, and that $\alpha(t)$ has a global maximum at $t^* = \frac{m+1}{m} > \sqrt{\frac{2m+1}{2m-1}}$, from which it follows that

$$WAR(CM) = \alpha \left( \frac{m+1}{m} \right) = \sqrt{2} \frac{\sqrt{(2m + 1)^2 + 1}}{(2m + 1) + 1} = \sqrt{2} \frac{\sqrt{n^2 + 1}}{n + 1}.$$

![Figure 3.3: Worst case profile](image-url)
3.4 Reduction from \( CP \) to \( I - CP \)

Next, we discuss informally some transformations that allow us to deal with the profiles in \( CP \). Without loss of generality (using reflections if necessary as in the \( n = 3 \) case), we may restrict consideration to profiles \( x \in CP \) with \( g(x) = (a_g, b_g) \) such that \( a_g \geq 0, b_g \geq 0, \) and \( b_g \geq a_g \).

1. **Reducing axes**: In this step, we move all points on \( -b \)-axis to \( -a \)-axis while keeping them equidistant from \( c(x) = (0, 0) \). This works because the \( sc(c(\cdot), \cdot) \) remains the same while \( sc(g(\cdot), \cdot) \) reduces, as the points move closer to the old geometric median. Thus, we get a profile in which all points are either on one of the \( +a \)-, \( +b \)-, or \( -a \)-axes or at \( g(x) \).

2. **Convexity**: Consider a profile obtained after applying step 1. Transform the profile so that all points on the \( +a \)-, \( +b \)-, and \( -a \)-axes are at their mean coordinates on the \( +a \)-, \( +b \)-, and \( -a \)-axes respectively. Again, \( sc(c(\cdot), \cdot) \) remains the same while \( sc(g(\cdot), \cdot) \) falls because of convexity of the distance function. Thus, we get a profile with weakly higher approximation ratio which has \( k \) points at \( (-b, 0) \), \( m + 1 - k \) points at \( (0, c) \), \( m + 1 - k \) points at \( (a, 0) \) and \( k - 1 \) points at \( g(x) \). Note that we are able to pin down the exact cardinalities of these sets because of the third condition in the definition of \( CP \), which requires that if any of the points were to move towards \( g(x) \), then \( c(x) \) would change.

3. **Double Rotation**: Consider a profile obtained after applying step 2. Transform the profile by moving the \( k - 1 \) points at \( g(x) \) to \( (0, \alpha) \), where \( \alpha = d(c(x), g(x)) \), and moving \( k - 1 \) of the \( k \) points at \( (-b, 0) \) to \( (\beta, 0) \), where \( \beta \) is the unique positive number such that \( d(g(x), (\beta, 0)) = d(g(x), (-b, 0)) \). In this case, one can show that the increase in \( sc(c(\cdot), \cdot) \) is at least \( \sqrt{2} \) times the increase in \( sc(g(\cdot), \cdot) \) and therefore, by Lemma 3, it follows that the approximation ratio weakly increases. Applying convexity again, we get a profile such that there is one point at \( (-b, 0) \), \( m \) points at \( (0, c) \) and \( m \) points at \( (a, 0) \). Note that \( g(x) \) may still not be on the axes.

4. **Geometric to axis**: Consider a profile obtained after applying step 3. Transform the profile so that the \( m \) points at \( (0, c) \) are at \( g(x) \), then translate all points by the same amount so that the coordinate-wise median is back to the origin. Doing so weakly increases the approximation ratio and yields a profile where one point is at \( (-b, 0) \), \( m \) points are at \( (0, c) \), \( m \) points are at \( (a, 0) \) and \( g(x) = (0, c) \).
4 Coordinate-wise Median mechanism’s WAR for the $p$-norm objective is not bad

In this section, we consider the objective $sc(y, x) = (\sum_{i=1}^{n} |y - x_i|^p)^{\frac{1}{p}}$ for $p \geq 2$. We refer to this objective as the $p$-norm objective. Feigenbaum et al. [11] consider this objective for the one-dimensional problem and obtain the following result:

**Lemma 7** (Feigenbaum et al, 2017). *Suppose there are $n$ agents with ideal points $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. Then the social cost incurred by the median mechanism under the $p$-norm is at most $2^{1-\frac{1}{p}}$ times the optimal social cost. Further, there are no deterministic strategyproof mechanisms with a lower worst-case approximation ratio.*

In particular, if $a_c$ is the median of $(a_1, a_2, \ldots, a_n)$ and $\text{OPT}(a)$ is the optimal location, then
$$\sum_{i=1}^{n} |a_c - a_i|^p \leq 2^{p-1} \sum_{i=1}^{n} |\text{OPT}(a) - a_i|^p$$

We consider the problem of quantifying the WAR for the coordinate-wise median mechanism under the $p$-norm objective when agents have ideal points in $\mathbb{R}^2$.

**Theorem 2.** *For the $p$-norm objective with $p \geq 2$, $2^{1-\frac{1}{p}} \leq \text{WAR}(CM) \leq 2^{\frac{3}{2} - \frac{1}{p}}$.*

The lower bound follows directly from Lemma 7. The upper bounds are obtained by using the following inequalities, together with Lemma 7:

$$\alpha^2 + \beta^2 \geq (\alpha^p + \beta^p) \quad \alpha^p + \beta^p \geq 2^{1-\frac{1}{p}}(\alpha^2 + \beta^2)^{\frac{1}{p}}.$$

5 Coordinate-wise Median mechanism’s WAR is pretty good

In this section, we aim to compare the performance of the coordinate-wise median mechanism against other strategyproof mechanisms. We begin by stating the conjecture that motivates this work on the coordinate-wise median:
Conjecture 1. For $n$ odd and $p \geq 1$, CM has the lowest $WAR$ for the $p$-norm among all deterministic strategyproof mechanisms.

For the case $p = 1$, which corresponds to the minisum objective, we have the following result, which we view as progress towards the conjecture.

**Theorem 3.** For $n$ odd and the minisum objective, CM has the lowest $WAR$ among all deterministic, strategyproof, anonymous, and continuous mechanisms.

**Remark 2.** We note that our proof of Theorem 3 makes heavy use of the worst case profile $w$ derived in Theorem 1. Specifically, the proof uses the characterisation of Lemma 1 and shows that for any generalized median mechanism $f$, we can find a profile $w'$, where $w'_i = w_i + \theta$ for some fixed $\theta \in \mathbb{R}^2$, such that $f(w') \in P + \theta$, where $P = \left\{ \frac{-(m+1)}{m}, 0, \frac{-(m+1)}{m} \right\} \times \{0,1\}$. Then, we show that for any of these six locations, we can find a profile which is the same as $w$ up to translation and reflection, and whose AR for the mechanism under consideration is at least as high as $WAR(CM)$. The formal proof is in appendix.

For general $p$, we are able to show quantitatively that CM can not be much worse than the optimal deterministic strategyproof mechanism. As we show in Theorem 2, $WAR(CM)$ for the $p$-norm is bounded above by $\frac{2^{\frac{1}{p}} - \frac{1}{p}}{1 - 2^{\frac{1}{2p}}}$ for $p \geq 2$. In addition, Lemma 7 gives a lower bound on $WAR$ for any deterministic strategyproof mechanism, as summarized in the following corollary:

**Corollary 2.** Any deterministic strategyproof mechanism for facility location in $\mathbb{R}^2$ has $WAR$ at least $2^{1 - \frac{1}{p}}$ for the $p$-norm objective.

Since the ratio of $WAR(CM)$ and this lower bound is at most $\sqrt{2}$ for $p \geq 2$, it follows that for such $p$ no deterministic strategyproof mechanism has a worst-case approximation ratio that is better than $CM$ by more than a factor of $\sqrt{2}$. This implies that the coordinate-wise median mechanism is already very close to being optimal. While Theorem 2 gives bounds on $WAR(CM)$ for arbitrary $p$, more precise results follow directly for $p = 2$ and $p = \infty$ from previous work in the one-dimensional setting. Even though these results extend in a straightforward way, we are not aware of anyone stating them explicitly, and so we mention them here for completeness. For $p = 2$ (also referred to in the literature as the minisOS objective), both the bounds in Theorem 2 are equal to $\sqrt{2}$ and thus $WAR(CM) = \sqrt{2}$. Thus, it follows from the second part of Lemma 7 that there
is no deterministic strategyproof mechanism with a better $WAR$ for $p = 2$. For $p = \infty$ (also referred to in the literature as the minimax objective), any deterministic strategyproof mechanism has $WAR \geq 2$ [26]. Also, any pareto optimal mechanism has $WAR \leq 2$. Together, we get that $WAR(CM) = 2$ for the minimax objective, and hence again there is no deterministic strategyproof mechanism that does better.

6 Conclusion

We show that the social cost of the coordinate-wise median is always within $\sqrt{7n^2 + 1} \frac{1}{n + 1}$ of the social cost obtained under the optimal mechanism. Using the worst case profile, we further show that there is no deterministic, strategyproof, anonymous, and continuous mechanism that does better. For the $p$-norm objectives, we find that the worst-case approximation ratio for the coordinate-wise median mechanism is bounded above by $2^{\frac{1}{p} - \frac{1}{2}} \cdot 2^{1 - \frac{1}{p}}$ for $p \geq 2$, and we observe that, by [11], the worst-case approximation ratio for any deterministic strategyproof mechanism is bounded below by $2^{1 - \frac{1}{p}}$. Thus, our bounds show that in these cases, at worst the coordinate-wise median mechanism is within a factor of $\sqrt{2}$ of being optimal.

Work on higher-dimensional facility location problems remains hitherto relatively limited, but we are optimistic that the results and methods used in this paper will encourage further research in this fundamental domain. We wish to stress, in particular, that the simple structure of the worst-case profile can provide a ready tool for researchers looking to advance progress on the conjectural optimality of the coordinate-wise median mechanism. We also hope that the techniques used in the proof of Theorem [1] can be adapted to the problem for other $p$-norms or more general families of social welfare objectives.
References

1. Alon, N., Feldman, M., Procaccia, A.D., Tennenholtz, M.: Strategyproof approximation of the minimax on networks. Mathematics of Operations Research 35(3), 513–526 (2010)
2. Barbera, S.: An introduction to strategy-proof social choice functions. Social Choice and Welfare pp. 619–653 (2001)
3. Barberà, S., Gul, F., Stacchetti, E.: Generalized median voter schemes and committees. Journal of Economic Theory 61(2), 262–289 (1993)
4. Black, D.: On the rationale of group decision-making. Journal of political economy 56(1), 23–34 (1948)
5. Border, K.C., Jordan, J.S.: Straightforward elections, unanimity and phantom voters. The Review of Economic Studies 50(1), 153–170 (1983)
6. Cai, Q., Filos-Ratsikas, A., Tang, P.: Facility location with minimax envy. AAAI Press/International Joint Conferences on Artificial Intelligence (2016)
7. Cheng, Y., Yu, W., Zhang, G.: Strategy-proof approximation mechanisms for an obnoxious facility game on networks. Theoretical Computer Science 497, 154–163 (2013)
8. De Keijzer, B., Wojtczak, D.: Facility reallocation on the line. In: Proceedings of the Twenty-Seventh International Joint Conference on Artificial Intelligence. International Joint Conferences on Artificial Intelligence Organization (2018)
9. Dokow, E., Feldman, M., Meir, R., Nehama, I.: Mechanism design on discrete lines and cycles. In: Proceedings of the 13th ACM Conference on Electronic Commerce. pp. 423–440 (2012)
10. Escoffier, B., Gourves, L., Thang, N.K., Pascual, F., Spanjaard, O.: Strategy-proof mechanisms for facility location games with many facilities. In: International conference on algorithmic decisiontheory. pp. 67–81. Springer (2011)
11. Feigenbaum, I., Sethuraman, J., Ye, C.: Approximately optimal mechanisms for strategyproof facility location: Minimizing lp norm of costs. Mathematics of Operations Research 42(2), 434–447 (2017)
12. Feldman, M., Wilf, Y.: Strategyproof facility location and the least squares objective. In: Proceedings of the fourteenth ACM conference on Electronic commerce. pp. 873–890 (2013)
13. Filos-Ratsikas, A., Li, M., Zhang, J., Zhang, Q.: Facility location with double-peaked preferences. Autonomous Agents and Multi-Agent Systems 31(6), 1209–1235 (2017)
14. Fotakis, D., Tzamos, C.: Strategyproof facility location for concave cost functions. In: Proceedings of the fourteenth ACM conference on Electronic commerce. pp. 435–452 (2013)
15. Fotakis, D., Tzamos, C.: On the power of deterministic mechanisms for facility location games. ACM Transactions on Economics and Computation (TEAC) 2(4), 1–37 (2014)
16. Gershkov, A., Moldovanu, B., Shi, X.: Voting on multiple issues: What to put on the ballot? Theoretical Economics 14(2), 555–596 (2019)
17. Kim, K.H., Roush, F.W.: Nonmanipulability in two dimensions. Mathematical Social Sciences 8(1), 29–43 (1984)
18. Kyropoulou, M., Ventre, C., Zhang, X.: Mechanism design for constrained heterogeneous facility location. In: International Symposium on Algorithmic Game Theory. pp. 63–76. Springer (2019)
19. Lu, P., Sun, X., Wang, Y., Zhu, Z.A.: Asymptotically optimal strategy-proof mechanisms for two-facility games. In: Proceedings of the 11th ACM conference on Electronic commerce. pp. 315–324 (2010)
20. Meir, R.: Strategic voting. Synthesis Lectures on Artificial Intelligence and Machine Learning 13(1), 1–167 (2018)
21. Meir, R.: Strategyproof facility location for three agents on a circle. In: International Symposium on Algorithmic Game Theory. pp. 18–33. Springer (2019)
22. Meir, R.: Strategyproof facility location for three agents on a circle (2019)
23. Moulin, H.: On strategy-proofness and single peakedness. Public Choice 35(4), 437–455 (1980)
24. Peters, H., van der Stel, H., Storcken, T.: Pareto optimality, anonymity, and strategy-proofness in location problems. International Journal of Game Theory 21(3), 221–235 (1992)
25. Plott, C.R.: A notion of equilibrium and its possibility under majority rule. The American Economic Review 57(4), 787–806 (1967)
26. Procaccia, A.D., Tennenholtz, M.: Approximate mechanism design without money. In: Proceedings of the 10th ACM conference on Electronic commerce. pp. 177–186 (2009)
27. Procaccia, A.D., Tennenholtz, M.: Approximate mechanism design without money. ACM Transactions on Economics and Computation (TEAC) 1(4), 1–26 (2013)
28. Schummer, J., Vohra, R.V.: Strategy-proof location on a network. Journal of Economic Theory 104(2), 405–428 (2002)
29. Shepsle, K.A., Weingast, B.R.: Structure-induced equilibrium and legislative choice. Public choice 37(3), 503–519 (1981)
30. Sui, X., Boutilier, C.: Approximately strategy-proof mechanisms for (constrained) facility location. In: Proceedings of the 2015 international conference on autonomous agents and multiagent systems. pp. 605–613 (2015)
31. Sui, X., Boutilier, C., Sandholm, T.: Analysis and optimization of multi-dimensional percentile mechanisms. In: Twenty-Third International Joint Conference on Artificial Intelligence (2013)
32. Tang, P., Yu, D., Zhao, S.: Group-strategyproof mechanisms for facility location with euclidean distance. arXiv preprint arXiv:1808.06320 (2018)

33. Wikipedia contributors: Geometric median — Wikipedia, the free encyclopedia (2020), https://en.wikipedia.org/w/index.php?title=Geometric_median&oldid=951011519, [Online; accessed 16-May-2020]

34. Zou, S., Li, M.: Facility location games with dual preference. In: Proceedings of the 2015 International Conference on Autonomous Agents and Multiagent Systems. pp. 615–623 (2015)
7 Appendix

Proof (Lemma 3). Let \( x_i = (a_i, b_i), c(x) = (a_c(x), b_c(x)) \) and \( g(x) = (a_g(x), b_g(x)) \).

For any right triangle, the sum of the lengths of the legs is at most \( \sqrt{2} \) times the length of the hypotenuse. Thus, together with the optimality of the median mechanism in one dimension we get

\[
\sqrt{2} \sum_{i=1}^{n} |x_i - g(x)| \geq \sum_{i=1}^{n} |a_i - a_g| + \sum_{i=1}^{n} |b_i - b_g|
\]

\[
\geq \sum_{i=1}^{n} |a_i - a_c| + \sum_{i=1}^{n} |b_i - b_c|
\]

\[
\geq \sum_{i=1}^{n} |x_i - c(x)|
\]

Hence, the worst-case approximation ratio for the coordinate-wise median mechanism is at most \( \sqrt{2} \).

Proof (Theorem 1).

Define Centered Perpendicular (CP) profiles as all profiles \( x \in (\mathbb{R}^2)^n \) such that

- \( c(x) = (0, 0) \)
- for all \( i \), either \( a_i = 0 \) or \( b_i = 0 \) or \( x_i = g(x) \)
- if \( x'_i \in (x_i, g(x)) \), then \( c(x'_i, x_{-i}) \neq (0, 0) \)

Lemma 8 (CP). For any profile \( x \in (\mathbb{R}^2)^n \), there exists a profile \( \chi \in CP \) such that \( AR(\chi) \geq AR(x) \).

Proof. Let \( x \in (\mathbb{R}^2)^n \) be a profile. Let \( x' \) be the profile where \( x'_i = x_i - c(x) \). Then \( x' \) has the same approximation ratio and \( c(x') = (0, 0) \). Denote \( A = \{ i : a_i = 0 \} \) and \( B = \{ i : b_i = 0 \} \). Note that since \( c(x') = (0, 0) \), it follows from the definition of \( c(x') \) that \( A \neq \emptyset \) and \( B \neq \emptyset \). Let \( \Gamma = \{(a, b) : a = 0 \text{ or } b = 0\} \cup g(x') \). Starting from \( i = 1 \) and going till \( n \), define \( x''_i \) to be the point in \([x'_i, g(x')] \cap \Gamma \) that is closest to \( g(x') \) under the constraint that \( c(x''_1, x''_2, \ldots, x''_i, x_n) = (0, 0) \). Then \( x'' \in CP \). Further, by lemma 8 \( AR(x'') \geq AR(x') = AR(x) \); hence, taking \( \chi = x'' \) completes the proof.

Define Isosceles-Centered Perpendicular (I-CP) profiles as all \( x \in CP \) for which there exists \( t \geq 0 \) such that
Next, we prove some lemmas that will be useful in reducing the search space for the worst-case profile from $CP$ to $I - CP$.

First, we show that we can reduce the number of half-axes that the points lie on from (at most) four to (at most) three.

**Lemma 9 (Reduce axes).** Suppose $x$ and $x'$ are profiles which differ only at $i$ where for some $a > 0$, $x_i = (0, -a)$ and $x'_i = (-a, 0)$, and for which $c(x) = c(x') = (0, 0)$ and $b_g(x) \geq a_g(x) \geq 0$. Then $AR(x') \geq AR(x)$.

**Proof.** Again $c(x') = c(x)$ and $sc(c(x'), x') = sc(c(x), x)$. Thus, it is sufficient to show that $sc(g(x'), x') \leq sc(g(x), x)$. For this, we just need to show that $d(x'_i, g(x)) \leq d(x_i, g(x))$. This follows from the following simple calculation:

$$d(x'_i, g(x))^2 = (a_g(x) + a)^2 + b_g(x)^2$$
$$\leq a_g(x)^2 + b_g(x)^2 + 2ab_g(x) + a^2$$
$$= a_g(x)^2 + (b_g(x) + a)^2$$
$$= d(x_i, g(x))^2.$$

□

Next, we show that we can combine points on each of the three half-axes while weakly increasing the approximation ratio.

**Lemma 10 (Convexity).** Let $x \in CP$ and let $S \subseteq N$ be such that for all $i \in S$, $a_i > 0$ and $b_i = 0$. Let $x_S$ be the mean of the $x_i$ across $i \in S$. Let $x'$ be the profile where

1. $x'_j = x_j$ for $j \notin S$ and
2. $x'_j = x_S$ for $j \in S$.

Then $AR(x') \geq AR(x)$.

**Proof.** It is immediate that $c(x') = c(x)$. Hence, it will be sufficient to show that $AR$ for $x'$ with $c(x)$ and $g(x)$ instead of $c(x')$ and $g(x')$ is at least as big
as \( AR(x) \). Indeed, \( sc(c(x), x') = sc(c(x), x) \) and \( sc(g(x'), x') < sc(g(x), x') < sc(g(x), x) \) where the last inequality follows from convexity of the distance function.

The same argument applies for any of the other strict half axes.

Next, we show that we can move all the points that are on the geometric median to the axis in a way that weakly increases the approximation ratio.

**Lemma 11 (Double Rotation).** Let \( x \) and \( x' \) be profiles that differ only at \( i_1 \) and \( i_2 \), such that for some \( a \geq 0 \)

\[- c(x) = (0, 0), \]
\[- b_g(x) \geq a_g(x) > 0, \]
\[- x_{i_1} = (-a, 0), \]
\[- x'_{i_1} = (a + 2a_g(x), 0), \]
\[- x_{i_2} = g(x), \] and
\[- x'_{i_2} = (0, d(g(x), (0, 0))). \]

Then \( c(x') = (0, 0) \) and \( AR(x') \geq AR(x) \).

**Proof.** The first claim is immediate.

For the second claim, let

\[ A = \sum_{i \neq i_1} d(x_i, c(x)) \]
\[ B = \sum_{i \neq i_2} d(x_i, g(x)). \]

By a previous result,

\[ A + d(x_{i_1}, c(x)) \leq \sqrt{2}B. \]

Hence, it follows that

\[ [A + d(x_{i_1}, c(x))]d(x'_{i_2}, g(x)) \leq \sqrt{2}Bd(x'_{i_2}, g(x)). \]

But since \( b_g(x) \geq a_g(x) \), it follows that \( d(x'_{i_2}, g(x)) \leq \sqrt{2}a_g(x) \). Hence,

\[ [A + d(x_{i_1}, c(x))]d(x'_{i_2}, g(x)) \leq 2Ba_g(x) \]
\[ = B(d(x'_{i_1}, c(x)) - d(x'_{i_2}, c(x))). \]
From this it follows that
\[
(A + d(x_{i_1}, c(x)))(B + d(x'_{i_2}, g(x))) = AB + Bd(x_{i_1}, c(x)) + [A + d(x_{i_1}, c(x))]d(x'_{i_2}, g(x)) \\
\leq AB + Bd(x_{i_1}, c(x)) \\
= (A + d(x'_{i_1}, c(x)))B
\]
and hence
\[
AR(x) = \frac{A + d(x_{i_1}, c(x))}{B} \\
\leq \frac{A + d(x'_{i_1}, c(x))}{B + d(x'_{i_2}, g(x))} \\
= \frac{A + d(x'_{i_1}, c(x'))}{B + d(x'_{i_2}, g(x))} \\
\leq AR(x').
\]

Once we have all the points on the three half-axes, we now show that we can move the geometric median to the axis as well.

**Lemma 12 (Geometric to axis).** Suppose that \( x \) is a profile such that there are \( a \geq 0 \) and \( b, c > 0 \) and subsets \( L, R, U \subseteq N \) with \( L \cap R = L \cap U = R \cap U = \emptyset, L \cup R \cup U = N, |L| = 1, |U| = |R| = m, \) and

- \( x_i = (0, -a) \) for \( i \in L \)
- \( x_i = (0, b) \) for \( i \in U \)
- \( x_i = (c, 0) \) for \( i \in R \)

and so that \( b_g(x) \geq a_g(x) > 0 \).

Let \( x' \) be the profile which is the same as \( x \) for \( i \notin U \) and which has \( x'_i = g(x) \) for \( i \in U \). Then \( AR(x') \geq AR(x) \).

**Proof.** Define
\[
h(t) = \frac{(a + (1-t)a_g(x)) + m(c - (1-t)a_g(x)) + mb}{d((-a, 0), g(x)) + m((c, 0), g(x)) + mtd((0, b), g(x))}.
\]
Then \( AR(x) = h(1) \) and \( AR(x') = h(0) \). Hence, it will be sufficient to show that \( h(1) \leq h(0) \).

To see this, note that since the denominator of \( h(t) \) is strictly positive for \( t \geq 0 \) and since both the numerator and the denominator are linear in \( t \), \( h(t) \) is
monotonic on $[0, \infty)$. Now, note that since the approximation ratio is always at least 1, $h(0) = AR(x') \geq 1$. Further,

$$\lim_{t \to \infty} h(t) = \frac{(m - 1)a_g(x)}{md((0, b), g(x))} < \frac{a_g(x)}{d((0, b), g(x))} < 1. $$

Hence, there is some $t > 0$ such that $h(t) < 1 \leq h(0)$, and so since $h(t)$ is monotonic on $[0, \infty)$, it follows that $h(t)$ is decreasing on $[0, \infty)$. Thus, $AR(x') = h(0) \geq h(1) = AR(x)$.  

Finally, the following lemma shows that we can use convexity to make the triangle formed by the three groups of points isosceles.

**Lemma 13 (Isosceles).** Let $x$ be a profile such for which are $m$ points at $(a, 0)$, 1 point at $(-b, 0)$ and $m$ points at $(0, c)$, and for which $g(x) = (0, c)$ and $c(x) = (0, 0)$. Let $x'$ be the profile where there are $m$ points at $\left(\frac{ma + b}{m + 1}, 0\right)$, 1 point at $\left(-\frac{ma + b}{m + 1}, 0\right)$, and $m$ points at $(0, c)$. Then, $AR(x') \geq AR(x)$.

**Proof.** Note that $c(x) = c(x') = (0, 0)$. Since $ma + b = m \cdot \frac{ma + b}{m + 1} + \frac{ma + b}{m + 1}$, we get that the numerator in $AR(x)$ and $AR(x')$ remains the same. Thus, we only need to argue that the denominator goes down as we go from $AR(x)$ to $AR(x')$.

Even though $g(x')$ may not be equal to $g(x)$ we have that $sc(g(x), x') \leq sc(g(x), x)$ by the convexity of the distance function which would imply $sc(g(x'), x') \leq sc(g(x), x)$ by definition of $g(x)$. Thus, we have that $AR(x') \geq AR(x)$.

Now, we use above lemmas to reduce the search space to I-CP.

**Lemma 14 (ICP).** For every $x \in CP$, there exists $\chi \in I - CP$ such that $AR(\chi) \geq AR(x)$.

**Proof.** Without loss of generality, consider any profile $x \in CP$ such that $b_g(x) \geq a_g(x) \geq 0$. Applying Lemma 9 to all points on the negative b axis gives a profile $x'$ with a weakly higher approximation ratio. In $x'$, we have all points on positive a, negative a, positive b and the geometric median. Using lemma 10 we can combine the points on positive a, negative a, positive b to some
(a, 0), (0, b), (−c, 0) while weakly increasing AR. Let this profile be \( x'' \). Now, we use lemma 11 to move points on the geometric median to +b-axis. Using 10 again, we get a profile \( x''' \) with \( m \) points on some \((a, 0)\), 1 point on \((−c, 0)\) and \( m \) points on \((0, b)\). Now we use lemma 12 to move the geometric median to the axis. Then, we use lemma 13 which gives a profile \( x'''' \) such that \( x'''' \in I−CP \) and \( AR(x''') \geq AR(x) \). Setting \( \chi = x'''' \) completes the proof.

\[ \Box \]

Using Lemma 14 we can now restrict attention to profiles in \( I−CP \). Define

\[ \eta_t = (x^t_1, \ldots, x^t_{2m+1}) \]

where

\[ x^t_i = \begin{cases} 
(t, 0) & i = 1, \ldots, m \\
(-t, 0) & i = m + 1 \\
(0, 1) & i = m + 2, \ldots, 2m + 1 
\end{cases} \]

Then, \( I−CP = \{ \eta_t : t \geq \sqrt{\frac{2m+1}{2m-1}} \} \). Defining \( \alpha(t) = \frac{(m+1)t+m}{(m+1)\sqrt{t^2+1}} \), we get that for \( t \geq \sqrt{\frac{2m+1}{2m-1}} \), \( AR(\eta_t) = \alpha(t) \), and that \( \alpha(t) \) is maximized at \( t^* = \frac{m+1}{m} > \sqrt{\frac{2m+1}{2m-1}} \), from which it follows that

\[ WAR(CM) = \alpha \left( \frac{m+1}{m} \right) = \sqrt{\frac{\sqrt{(2m+1)^2+1}}{2m+1}} = \sqrt{\frac{\sqrt{n^2+1}}{n+1}}. \]

Thus, we get that \( WAR(CM) = \sqrt{2} \frac{\sqrt{n^2+1}}{n+1} \) as required. \( \Box \)

**Proof (Theorem 2).**

The lower bound follows directly from lemma 7.

Consider any profile \( x = (a_i, b_i) \in (\mathbb{R}^2)^n \). Let \( g(x) = (a_g(x), b_g(x)) \) and \( c(x) = (a_c(x), b_c(x)) \). Then, we have that
\[ sc(g(x), x)^p = \sum_{i=1}^{n} |g(x) - x_i|^p \]
\[ \geq \left( \sum_{i=1}^{n} |a_g(x) - a_i|^p + \sum_{i=1}^{n} |b_g(x) - b_i|^p \right) \]
\[ \geq \left( \sum_{i=1}^{n} |OPT(a) - a_i|^p + \sum_{i=1}^{n} |OPT(b) - b_i|^p \right) \]
\[ \geq \frac{1}{2^{p-1}} \left( \sum_{i=1}^{n} |c_a - a_i|^p + \sum_{i=1}^{n} |c_b - b_i|^p \right) \]
\[ \geq \frac{2^{1-\frac{p}{2}}}{2^{p-1}} \sum_{i=1}^{n} |c(x) - x_i|^p \]
\[ = 2^{2-\frac{p}{2}} sc(c(x), x)^p \]

Thus, we get \( WAR(CM) \leq 2^{2-\frac{p}{2}} \) for \( p \geq 2 \) as required.

\[ \square \]

**Proof. (Theorem 3)** Using Lemma 1, we know that every deterministic, strategyproof, anonymous, and continuous mechanism \( f \) is defined by points \( p_1, p_2, \ldots, p_{n+1} \) such that \( f(x) = c(x, p) \). Consider any arbitrary such mechanism \( f \). The worst case profile \( w \) defines six important points which are \( P = \{(a, b) \in \mathbb{R}^2 : a \in \{-\frac{(m+1)}{m}, 0, \frac{(m+1)}{m}\}, b \in \{0, 1\}\} \) as illustrated in figure 7.

Observe that if \( f(w) \notin P \), there exists some \( \theta \in \mathbb{R}^2 \) such that \( f(w+\theta) \in P+\theta \). Thus, without loss of generality, we restrict attention to the case where \( f(w) \in P \) and show that no matter which point \( f \) chooses under \( w \) in \( P \), we can find a profile \( x' \) such that \( AR(f, x') \geq WAR(CM) = AR(w) \).

If \( f(w) \in \{(-\frac{(m+1)}{m}, 0), (0, 0)\} \), then we set \( x' = w \) and we are done. If \( f(w) = (\frac{m+1}{m}, 0) \), consider the \( w' \) obtained by reflecting \( w \) around the \( b \)-axis. It follows that \( f(w') \in \{(\frac{(m+1)}{m}, 0), (0, 0)\} \) where \( (\frac{(m+1)}{m}, 0) \) only has 1 agent on it in \( w' \). Thus, setting \( x' = w' \), we are done.

Now, if \( f(w) \in \{(-\frac{(m+1)}{m}, 1), (0, 1), (\frac{(m+1)}{m}, 1)\} \), consider the \( w' \) obtained by reflecting \( w \) around \( a \)-axis. It follows by definition of \( f \) that \( f(w') \in \{(-\frac{(m+1)}{m}, 0), (0, 0), (\frac{(m+1)}{m}, 0)\} \). This is same as the previous case and hence, we get that there is no deterministic, strategyproof, anonymous, and continuous mechanism with a better \( WAR \) than the coordinate-wise median mechanism.

\[ \square \]
Figure 7: The six points