A DECAY PROPERTY OF SOLUTIONS TO THE K-GENERALIZED KDV EQUATION

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Abstract. We use a Leibnitz rule type inequality for fractional derivatives to prove conditions under which a solution $u(x, t)$ of the k-generalized KdV equation is in the space $L^2(|x|^{2s} \, dx)$ for $s \in \mathbb{R}_+$. 

1. Introduction

The initial value problem for the modified Korteweg-de Vries equation (mKdV),

$$\partial_t u + \partial_x^2 u + \partial_x(u^3) = 0,$$  

$$u(x, 0) = u_0(x),$$

has applications to fluid dynamics (see [16], [20]), and plasmas (see [19]). It is also an example of an integrable system (see [5]). Ginibre and Y. Tsutsumi in [6] proved well-posedness in a weighted $L^2$ space. In [13], Kenig, Ponce, and Vega proved local well-posedness for $u_0$ in the Sobolev space $H^s$, when $s \geq \frac{1}{4}$ by a contraction mapping argument in mixed $L^p_x$ and $L^q_T$ spaces. Christ, Colliander, and Tao in [2] showed that (1.1) was locally well-posed for $u_0 \in H^s$, when $s \geq \frac{1}{4}$, by using a contraction mapping argument in the Bourgain spaces $X_{s,b}$. Colliander, Keel, Staffilani, Takaoka, and Tao proved global well-posedness for real initial data $u_0 \in H^s$, $s > \frac{1}{4}$ in [3]. Kishimoto in [15] and Guo in [7] proved global well-posedness for real data in the case $s = \frac{1}{4}$.

The focus of this work will be (1.1), but we will also consider the generalized Korteweg-de Vries equation,

$$\left\{ \begin{array}{l} 
\partial_t u + \partial_x^2 u + \partial_x(u^{k+1}) = 0, \\
u(x, 0) = u_0(x), \quad x \in \mathbb{R}. 
\end{array} \right.$$  

When $k \geq 4$, local well posedness was obtained for initial data $u_0 \in H^s$ with $s \geq \frac{k-4}{2k}$ in [13] using a contraction mapping argument in mixed $L^p_x$ and $L^q_T$ spaces. When $k = 3$, the optimal local well posedness result was proven by Tao in [22] for $u_0 \in H^s$ with $s \geq -\frac{1}{6}$ by using Bourgain spaces $X_{s,b}$.

Kato in [11] with energy estimates, and the fact that the operator

$$\Gamma_K \equiv x + 3t \partial_x^2$$

commutes with $\partial_t + \partial_x^2$, was able to prove the following: if $u_0 \in H^{2k}$ and $|x|^k u_0 \in L^2$ where $k \in \mathbb{Z}^+$, then for any other time $t$ when the solution
exists, \(|x|^k u(t) \in L^2_x\). Using slightly different techniques, we will prove the following theorem that extends this result slightly to \(k \in \mathbb{R}_+\).

**Theorem 1.1.** Suppose the initial data \(u_0\) satisfies \(|x|^s u_0 \in L^2\), and \(u_0 \in H^{2s+\varepsilon}\), for \(\varepsilon > 0\). Then for any other time \(t\), the solution \(u(x,t)\) to (1.2) satisfies \(|x|^s u(x,t) \in L^2\).

When \(s \geq \frac{1}{2}\), the result holds for \(\varepsilon = 0\). Namely, if \(|x|^s u_0 \in L^2\), and \(u_0 \in H^{2s}\), then for any other time \(t\), the solution \(u(x,t)\) to (1.2) satisfies \(|x|^s u(x,t) \in L^2\).

Analogous results for the NLS were first proved by Hayashi, Nakamitsu, and M. Tsutsumi in [8], [9], and [10]. They used the vector field (1.3) \(\Gamma_S = x + 2it\nabla\), which commutes with the operator \(\partial_t + i\Delta\), and a contraction mapping argument to show that if \(u_0 \in L^2(|x|^{2m} dx) \cap H^m\), where \(m \in \mathbb{N}\), then the solution \(u(x,t)\) at any other time is also in the space \(L^2(|x|^{2m} dx) \cap H^m\). These results were extended to the case when \(m \in \mathbb{R}_+\) by the author and G. Ponce in [18]. The corresponding results for the Benjamin-Ono equation were obtained in [1] by G. Ponce and G. Fonseca.

Inspired by these persistence results we prove the following as our main result.

**Theorem 1.2.** If \(u(x,t)\) is a solution of

\[
\begin{aligned}
\partial_t u + \partial_x^2 u + \partial_x (u^{k+1}) &= 0, \\
u(x,0) &= u_0(x), \quad x \in \mathbb{R},
\end{aligned}
\]

such that \(u_0 \in H^{s'} \cap L^2(|x|^s dx)\), where \(s \in (0,s']\). If \(k = 2\), and \(s' \geq \frac{1}{4}\), then \(u(\cdot,t) \in H^{s'} \cap L^2(|x|^s dx)\) for all \(t\) in the lifespan of \(u\).

If \(k \geq 4\), and \(s \geq \frac{k-1}{2k}\), then \(u(\cdot,t) \in H^{s'} \cap L^2(|x|^s dx)\) for all \(t\) in the lifespan of \(u\).

We only prove this property the most interesting case, (1.1). Note that the cases in (1.2) when \(k = 1\) or 4 are excluded from Theorem 1.2. We require our technique to be adapted to Bourgain spaces for these nonlinearities, which is an interesting open question.

The difficulty in the case of fractional decay lies in the lack of an operator \(\Gamma\) that sufficiently describes the relation between initial decay, and properties of the solution at another time (such as (1.3)). In order to solve this problem, we develop a Leibnitz rule type inequality for fractional derivatives.

We need some notation to illustrate this idea. If \(f\) is a complex valued function on \(\mathbb{R}\), we let \(\hat{f}\) (or \(\hat{f}\)) denote the Fourier transform of \(f\), and \(f^\wedge\) the inverse Fourier transform. For \(\alpha \in \mathbb{R}\), the operator \(D_x^\alpha\) is defined as \((D_x^2 f(x))^\wedge(\xi) \equiv |\xi|^\alpha \hat{f}(\xi)\). Let \(U(t)f\) denote the
solution $u(x, t)$ to the linear part of (1.1), with $u(x, 0) = f(x)$. Choose $\eta \in C^\infty_0(\mathbb{R})$ with $\text{supp}(\eta) \subset [\frac{1}{2}, 2]$ so that
\[
\sum_{N \in \mathbb{Z}} (\eta(\frac{x}{2^N}) + \eta(-\frac{x}{2^N})) = 1 \text{ for } x \neq 0.
\]

Define the operator $Q_N$ on a function $f$ as
\[
Q_N(f) \equiv ((\eta(\frac{\xi}{2^N}) + \eta(-\frac{\xi}{2^N})) \hat{f}(\xi))^\vee.
\]

If $\| \cdot \|_Y$ is a norm on some space of functions, we recall that
\[
\|Q_N(f)\|_{Y_{pN}} \equiv \|(\sum_{N \in \mathbb{Z}} |Q_N(f)|^p)^{\frac{1}{p}}\|_Y.
\]

Using Duhammel’s principle, we can formulate the problem (1.1) as an integral equation.
\[
u(x, t) = U(t)u_0 - \int_0^t U(t - t')\partial_x(u^3(x, t')) dt'.
\]

Using a Fourier transform, we can see how to commute an $x$ past $U(t)$,
\[
xU(t)f = (-i\partial_\xi(e^{it\xi^3}\hat{f}))^\vee
= (3t\xi^2e^{it\xi^3}\hat{f} - ie^{it\xi^3}\partial_\xi\hat{f})^\vee
= U(t)(3t\partial^2_x f + xf).
\]

We would like to use a similar argument with $|x|^{\frac{1}{8}}$ replacing $x$, but this would require that $D_\xi^{\frac{1}{8}}$ obey a product rule. We develop in inequality in Lemma 4.2 that is similar enough to the product rule that will allow this argument to work.

With Lemma 4.2 we will require that
\[
(1.4) \quad \left\| D_\xi^{\frac{1}{8}}Q_N(\frac{e^{it\xi^3}}{(1 + \xi^2)^{\frac{1}{8}}}) \right\|_{L^{1}_{t}L^{1}_{x}} < \infty.
\]

With less sophisticated techniques, we prove Theorem weak-decay in Section 2. We show (1.4) in Section 3, then prove our main result in Section 4. The proof of Lemma 4.2 is almost identical to the proof of a classical Leibnitz rule inequality. Because this proof requires different techniques than the rest of the paper, we present it in Appendix A.

We use the following notation throughout the paper. We let $A \lesssim B$ mean that the quantity $A$ is less than or equal to a fixed constant times the quantity $B$. Let $\langle x \rangle \equiv (1 + x^2)^{\frac{1}{2}}$, and similarly, $\langle D_\xi \rangle$. 
2. Weak Persistence Result

Using some standard estimates, we prove Theorem 1.1 which is a weaker persistence property for IVP for the gKdV equation for low regularity solutions, but holds for more values of \( k \) in (1.2) than our main result.

Following an argument by Kato, we multiply (1.2) by \( \phi(x)u(x,t) \) for some function \( \phi(x) \), and integrating over \( x \) and \( t \), we use integration by parts to obtain

\[
\int_{\mathbb{R}} \phi(x)u^2(x,T) \, dx - \int_{\mathbb{R}} \phi(x)u^2(x,0) \, dx - 3 \int_{[0,T]} \phi'(x)(\partial_x u)^2 \, dx \, dt \\
+ \int_{[0,T]} \int_{\mathbb{R}} \phi''(x)u^2 \, dx \, dt + \frac{k+1}{k+2} \int_{[0,T]} \int_{\mathbb{R}} \phi'(x)u^{k+2} \, dx \, dt = 0.
\]

(2.1)

Equation (2.1), along with the following two interpolation lemmas are the primary tools for the weak persistence result, Theorem 1.1.

**Lemma 2.1.** Let \( a,b > 0 \), and \( w(x) > \varepsilon > 0 \) a locally bounded function. Assume that \( \langle D_x \rangle^a f \in L^2(\mathbb{R}) \) and \( w^b(x)f \in L^2(\mathbb{R}) \). Then for any \( \theta \in (0,1) \)

\[
\| \langle D_x \rangle^a \left( w^{(1-\theta)b}(x)f \right) \|_2 \lesssim \| w^b(x)f \|_2^{1-\theta} \| \langle D_x \rangle^a f \|_2^\theta.
\]

**Proof.** This is an easy consequence of the Three Lines Lemma, and the fact that

\[
\| \langle D_x \rangle^a \left( w^{(1-z)b}(x)f \right) \|_2
\]

is an analytic function in \( z \) for \( \Re z \in (0,1) \), for a dense set of functions in the space \( H^a \cap L^2(w^{2b}(x) \, dx) \).

**Lemma 2.2.** For a solution \( u = u(x,t) \) of (1.2),

\[
\| \partial_x u \|_{L^{q/(q-1)}_x L^2_T} \leq c_T \| u_0 \|_{H^{2+\varepsilon}}.
\]

(2.2)

**Proof.** Consider the function

\[
F(z) = \int_{-\infty}^{\infty} \int_0^T D_z^{r(z)}(U(t)u_0) \psi(x,z) f(t) \, dt \, dx,
\]

where

\[
r(z) = (1-z)(1+2s+\varepsilon)+z(2s+\varepsilon), \quad \frac{1}{q(z)} = \frac{z}{2} + (1-z), \quad q = \frac{2}{2-2s-\varepsilon},
\]

\[
\psi(x,z) = \left| g(x) \right|^{q/q(z)} \frac{g(x)}{|g(x)|} \text{, with } \| g \|_{L^1_x(1-\delta-\frac{1}{2})} = \| f \|_{L^2([0,T])} = 1,
\]

which is analytic for \( \Re z \in (0,1) \). Using that

\[
\| \psi(\cdot, 0 + iy) \|_2 = \| \psi(\cdot, 1 + iy) \|_1 = 1,
\]
one gets from $H^{2s+\varepsilon}$ persistence and the Kato smoothing effect that
\[
\|\partial_x U(t)u_0\|_{L_x^{\frac{1}{s}+\varepsilon}L_T^2} \leq c \|D_x U(t)u_0\|_{L_x^{\frac{1}{s}+\varepsilon}L_T^2}
\]
\[
\leq c \sup_{y \in \mathbb{R}} \|D_x^{1+2s+\varepsilon+i\varepsilon}U(t)u_0\|_{L_x^{\frac{1}{s}+\varepsilon}L_T^2} \sup_{y \in \mathbb{R}} \|D_x^{2s+\varepsilon+i\varepsilon}U(t)u_0\|_{L_x^2L_T^2}
\]
\[
\leq c_T \|D_x^{2s+\varepsilon}U(t)u_0\|_2.
\]

Inserting the estimate (2.3) in the proof of the local well posedness for (1.2), the result follows.

Proof of Theorem 1.2. Let $\phi_N$ be a smooth function such that
\[
\phi_N(x) = \begin{cases} (x)^{2s} & \text{if } |x| \leq N, \\ (2N)^{2s} & \text{if } |x| > 3N. \end{cases}
\]
Then from (2.1),
\[
\int_\mathbb{R} \phi_N(x)u^2(x,T) \, dx - \int_\mathbb{R} \phi_N(x)u^2(x,0) \, dx =
\]
\[
3 \int_{[0,T]} \int_\mathbb{R} \phi_N'(x)(\partial_x u)^2 \, dx \, dt - \int_{[0,T]} \int_\mathbb{R} \phi_N''(x)u^2 \, dx \, dt
\]
\[
- \frac{k + 1}{k + 2} \int_{[0,T]} \int_\mathbb{R} \phi_N'(x)u^{k+2} \, dx \, dt.
\]
(2.3)

We only prove the result in the case where $s < 2$ of the KdV equation, when $k = 1$. Our main result, Theorem 1.2, is stronger when $k = 2$, and $k \geq 4$, and the proof for $s \geq 2$ or $k = 3$ is similar. We will use results from [12], which state that the smoothing effects and Strichartz estimates that hold for the linearized KdV and mKdV also hold for the KdV.

The $\phi_N''(x)u^2$ term in the right hand side of (2.3) can be bounded by the fact that $\phi_N''(x) \lesssim 1$ independently of $N$ for $s \leq \frac{1}{2}$, and $L^2$ persistence:

(2.4) \[ \left| \int_{[0,T]} \int_\mathbb{R} \phi_N''(x)u^2 \, dx \, dt \right| \lesssim T \|u\|_2^2. \]

The bounds on the other terms on the right hand side of (2.3) depend on whether $s < \frac{1}{2}$ or $s \geq \frac{1}{2}$. We first give the proof of the result in the case that $s < \frac{1}{2}$.

Since $|\phi_N'(x)| \lesssim (x)^{2s-1}$ independently of $N$, we can bound the first term on the right hand side of (2.3) by

(2.5) \[ \left| \int_{[0,T]} \int_\mathbb{R} \phi_N'(x)(\partial_x u)^2 \, dx \, dt \right| \lesssim \|\langle x\rangle^{s-\frac{1}{2}}\partial_x u\|_{L_x^2L_T^2}. \]
Using (2.5), Lemma 2.2, and the Hölder inequality,

\[
\left| \int_{[0,T]} \int_{\mathbb{R}} \phi_N'(x)(\partial_x u)^2 \, dx \, dt \right|
\]

\[
\lesssim \|\langle x \rangle^{s - \frac{1}{2}}\|_{L^2}^{\frac{2}{1 - 2(s + \frac{1}{4})}} \|D_x u(x,t)\|_{L^{s+\frac{1}{2}}_x L^2_t} < \infty.
(2.6)
\]

For the \(\phi_N'(x)u^{k+2}\) term in the right hand side of (2.3), we can bound this term with the Hölder inequality,

\[
\left| \int_{[0,T]} \int_{\mathbb{R}} \phi_N'(x)u^3 \, dx \, dt \right|
\]

\[
\lesssim \|\langle x \rangle^{2s-1}\|_{L^1_x L^2_t} \|u^3\|_{L^1_x L^2_t}
\]

\[
\leq \|u\|_{L^1_x L^\infty_t} \|\langle x \rangle^{s - \frac{1}{2}}\|_{L^2_x L^2_t}^2
\]

\[
\leq T^{\frac{3}{2}} \|u\|_{L_x^\infty L^2_t} \|u\|_{L^2_x L^2_t}^2.
(2.7)
\]

Since \(s - \frac{1}{2} < 0\), (2.7) is finite by the Strichartz estimates in [12], and \(L^2\) persistence.

It follows from (2.3) that

\[
\left| \int_{\mathbb{R}} (\phi_N(x)u^2(x,T)) \, dx \right| \leq \int_{\mathbb{R}} |\phi_N(x)u^2(x,0)| \, dx
\]

\[
+ 3 \int_{[0,T]} \int_{\mathbb{R}} |\phi_N'(x)(\partial_x u)^2| \, dx \, dt
\]

\[
+ \frac{2}{3} \int_{[0,T]} \int_{\mathbb{R}} |\phi_N'(x)u^3| \, dx \, dt
\]

\[
+ \int_{[0,T]} \int_{\mathbb{R}} |\phi_N'''(x)u^2| \, dx \, dt.
\]

By \(|x|^s u_0 \in L^2\), (2.6), (2.4), and (2.7), the result follows.

We now consider the case that \(s \in \left[\frac{1}{2}, 1\right)\). For the first term on the right hand side of (2.3), we use Lemma 2.1 and \(H^{2s}\) persistence to
The term in (2.9) is finite from the first part of the proof since

\[ \parallel T^\frac{1}{2} \parallel_{L^\infty_x} \parallel \phi_N(x) \parallel_{L^1_x} \]

Since \( \langle \phi_N(x) \rangle \overset{\text{max}}{\leq} \phi_N(x) \), it follows that

\[
\left| \int_{[0, T]} \int \phi_N(x) (\partial_x u)^2 \, dx \, dt \right| \leq \left( \parallel (D_x) \parallel_{L^\infty_x} \right)^2 \parallel \phi_N(x) \parallel_{L^1_x}
\]

(2.8)

For the \( \phi_N(x) u^{k+2} \) term,

\[
\left| \int_{[0, T]} \int \phi_N(x) u^3 \, dx \, dt \right| \leq \left( \parallel (D_x) \parallel_{L^\infty_x} \right)^2 \parallel \phi_N(x) \parallel_{L^1_x}
\]

(2.9)

The term in (2.9) is finite from the first part of the proof since \( s - \frac{1}{2} < \frac{1}{2} \).

From (2.3), (2.4), (2.9), (2.8), the fact that \( \phi_N(x) \leq \langle x \rangle^{2s} \) and our assumption on \( u(x, 0) \),

\[
\parallel \phi_N(x) u^2(x, T) \parallel_{L^2_x} \leq \parallel \langle x \rangle^{s} u^2(x, 0) \parallel_{L^2_x}^2 + \parallel u(x) \parallel_{L^2_x}^2 \]

(2.10)

The application of Bihari’s inequality (see [1]) to (2.10) yields a bound on \( \parallel \phi_N(x) u(x, T) \parallel_2 \) that is independent of \( N \). By taking \( N \) to infinity, the result follows.
3. Estimating a Derivative

We begin our computation of (1.4). We will show that by scaling out the fractional derivative, it will suffice to bound

\[ \left| Q_N \left( \frac{e^{it\xi^3}}{(1 + \xi^2)^\frac{3}{8}} \right) \right|. \]

Since the operator \( Q_N \) is convolution with a function whose Fourier transform is very localized, we require estimates on

\[ (3.1) \int_{\mathbb{R}} \varphi_\omega(\xi - z) \frac{e^{itz^3}}{(1 + z^2)^\frac{3}{8}} \, dz, \]

where \( \varphi_\omega \) is a function whose Fourier transform has support near \( \omega \).

We will use a contour integral argument. Because of this, we require estimates on the analytic continuation of \( \varphi_\omega \). These are contained in the following lemma.

**Lemma 3.1.** Let \( \xi \in \mathbb{R}, z = x + yi \) for \( x, y \in \mathbb{R} \), \( \varphi(\xi) \) be a function so that \( \hat{\varphi}(x) \) is a smooth function with support in \([\frac{1}{2}, 2]\), and for \( \omega \in \mathbb{R} \setminus \{0\} \), let \( \varphi_\omega(\xi) \) be the function with Fourier transform \( \hat{\varphi}(\frac{\xi}{\omega}) \). Then \( \varphi_\omega \) is an entire function that obeys the following estimates.

\[ |\varphi_\omega((\xi - z))] \lesssim \begin{cases} \frac{|e^{\frac{y^2}{\omega}} - e^{\frac{y^3}{\omega}}|}{\omega^2 y |\xi - z|^2} & \text{if } y \neq 0 \text{ and } x \neq \xi, \\ \frac{1}{|\omega|(|\xi - z|^3) & \text{if } y = 0 \text{ and } x \neq \xi. \end{cases} \]

**Proof.** That \( \varphi_\omega \) is entire follows from the Paley-Wiener theorem. Let \( y \neq 0 \). Since \( \hat{\varphi} \) is a smooth function with support in \([\frac{1}{2}, 2]\), we integrate by parts to obtain

\[ \varphi_\omega(\xi - z) = \int_{\mathbb{R}} \hat{\varphi}(\frac{\xi}{\omega}) \frac{1}{i(\xi - z)} \frac{d}{d\zeta} e^{i\zeta(\xi - z)} \, d\zeta 
= -\int_{[\frac{1}{2}, 2\omega]} \frac{1}{\omega} \hat{\varphi}'(\frac{\xi}{\omega}) \frac{1}{i(\xi - z)} e^{i\zeta(\xi - z)} \, d\zeta 
= \int_{[\frac{1}{2}, 2\omega]} \frac{1}{\omega} \hat{\varphi}'(\frac{\xi}{\omega}) \frac{1}{(\xi - z)^2} \frac{d}{d\zeta} e^{i\zeta(\xi - z)} \, d\zeta 
= -\int_{[\frac{1}{2}, 2\omega]} \frac{1}{\omega^2} \hat{\varphi}''(\frac{\xi}{\omega}) \frac{1}{(\xi - z)^3} e^{i\zeta(\xi - z)} \, d\zeta. \] (3.2)
From (3.2) we conclude that
\[ |\varphi_\omega(\xi - z)| \leq \int_{[\frac{\xi}{2}, \frac{\omega}{2}]} \frac{1}{\omega^2} \hat{\varphi}'' \left( \frac{\xi}{\omega} \right) \frac{1}{|\xi - z|^2} e^{i\zeta(\xi - z)} \, d\zeta \]
\[ \leq \int_{[\frac{\xi}{2}, \frac{\omega}{2}]} \frac{1}{\omega^2} |\hat{\varphi}''| \frac{1}{|\xi - z|^2} e^{\xi y} \, d\zeta \]
\[ \leq c^\omega \frac{|e^{2\omega y} - e^{\frac{1}{2}\omega y}|}{\omega^2 y |\xi - z|^2}. \]

The case \( y = 0 \) follows from taking the limit as \( y \to 0 \) of the first estimate.

From Lemma 3.1, we can infer the following about the analyticity of the integrand in (3.1).

**Corollary 3.1.** For \( \xi \in \mathbb{R} \), the function
\[ \varphi_\omega(\xi - z) e^{itz^3} \frac{1}{(1 + z^2)^{\frac{1}{8}}} \]
is analytic on \( \mathbb{C} \setminus \{ z : |\Im z| \geq 1, \Re z = 0 \} \).

The estimate in Lemma 3.1 has good \( x \) dependence away from \( \xi \). To estimate (3.1) near \( z = \xi \), we use an analytic continuation of the integrand and the Cauchy integral theorem, which we now describe.

The function \( \varphi_\omega \) oscillates with frequency near \( \omega \). For a fixed \( z_0 \in \mathbb{R} \), we think of the function \( \exp(itz^3) \) as oscillating with frequency \( tz_0^2 \) near the value \( z_0 \). For \( z = \xi \) where \( tz_0^2 \ll \omega \), the function \( \varphi_\omega \) oscillates much faster than \( \exp(itz^3) \), so Lemma 3.1 shows that analytic continuation of
\[ \varphi_\omega(\xi - z) e^{itz^3} \frac{1}{(1 + z^2)^{\frac{1}{8}}} \]
changes this rapid oscillation into decay, which yields good \( \omega \) dependence for (3.1). To formalize this, we make the following definition. Given \( t > 0 \), and \( \omega > 0 \), we say that \( \xi \in \mathbb{R} \) is **near** if
\[ |\xi| \leq \frac{1}{10} \frac{\omega}{t}. \]

Where the oscillation of \( \exp(itz^3) \) is much larger than \( \omega \), an analytic continuation of \( \exp(itz^3) \) has a similar property. We say that \( \xi \in \mathbb{R} \) is **far** if
\[ |\xi| > 10 \frac{\omega}{t}. \]

In the intermediate case where the oscillation of \( \exp(itz^3) \) is comparable to \( \omega \), analytic continuation does not help. This is where the worst
behavior of the estimate occurs. We say that \( \xi \in \mathbb{R} \) is \textbf{intermediate} if
\[
\frac{1}{10} \sqrt{\frac{\omega}{t}} < |\xi| \leq 10 \sqrt{\frac{\omega}{t}}.
\]
These heuristics are formalized in Lemma 3.3, then used to estimate \((1.4)\) in Lemma 3.4. We require an elementary integral estimate for Lemma 3.3.

One expects that since \( \sin t \approx t \), then
\[
\int_{[0,\pi]} e^{as \sin s} - e^{bs \sin s} \sin s ds \approx \int_{[0,\pi]} e^{as} - e^{bs} s ds = \int_{[0,\pi]} e^{as} - e^{bs} \cdot \frac{a}{s} ds = \int_{[0,\pi]} e^{t} e^{\frac{a}{t} s} - e^{t} e^{\frac{b}{t} s} \cdot \frac{a}{t} dt.
\]
This is what the next lemma proves.

**Lemma 3.2.** Let \( a < b < 0 \). Then
\[
\left| \int_{[0,\pi]} e^{as \sin s} - e^{bs \sin s} \sin s ds \right| \lesssim (\pi \frac{a}{b} - 1) + 1 + \frac{b}{\pi a} e^{-\frac{\pi a}{b}}.
\]

**Proof.** By making the change of variable \( r = -(s - \frac{\pi}{2}) \), we have for an arbitrary function \( f \),
\[
\int_{[\frac{\pi}{2},\pi]} f(\sin s) ds = \int_{[0,\frac{\pi}{2}]} f(\sin r) dr.
\]
Therefore,
\[
(3.4) \quad \left| \int_{[0,\pi]} e^{as \sin s} - e^{bs \sin s} \sin s ds \right| = 2 \int_{[0,\frac{\pi}{2}]} e^{bs \sin s} - e^{as \sin s} \sin s ds.
\]
Notice that for \( s \in [0, \frac{\pi}{2}] \), \( \frac{2s}{\pi} \leq \sin s \leq 2s \). We use this to bound \((3.4)\).
\[
\int_{[0,\frac{\pi}{2}]} e^{bs \sin s} - e^{as \sin s} \sin s ds \lesssim \int_{[0,\frac{\pi}{2}]} e^{\frac{2b}{\pi} s} - e^{2as} \frac{s}{s} ds = \int_{[0,\frac{\pi}{2}]} e^{\frac{2b}{\pi} s} - e^{2as} \cdot \frac{2b}{\pi} ds = \int_{[0,\frac{\pi}{2}]} e^{\frac{2b}{\pi} r} - e^{r} \cdot \frac{2b}{\pi} ds.
\]
(3.5)

Because \( a < b < 0 \), it follows that \( \frac{a}{b} > 1 \), and \( \frac{na}{b} > 1 \). For \( r < 0 \), \( \frac{2b}{\pi} r < r \), so that \( e^{\frac{2b}{\pi} r} - e^{r} < 0 \), and therefore
\[
(3.6) \quad \frac{e^{\frac{2b}{\pi} r} - e^{r}}{r} > 0.
\]
By the integrand in (3.5) is positive, so we can bound it with
\[
\int_{[b,0]} e^{\frac{\pi a r}{b}} - e^r \, dr \leq \int_{[-\infty,0]} e^{\frac{\pi a r}{b}} - e^r \, dr
\]
\[
= \int_{[-\infty,-1]} e^{\frac{\pi a r}{b}} - e^r \, dr + \int_{[-1,0]} e^{\frac{\pi a r}{b}} - e^r \, dr
\]
\[
\leq \frac{b}{\pi a} e^{-\frac{\pi a}{b}} + e^{-1} + \int_{[-1,0]} e^{\frac{\pi a r}{b}} - e^r \, dr.
\]
(3.7)

By Taylor expansion and an error estimate for alternating sums,
\[
\int_{[-1,0]} e^{\frac{\pi a r}{b}} - e^r \, dr = \int_{[-1,0]} \sum_{n=1}^{\infty} (\frac{\pi a}{b})^n - 1 \frac{1}{n!} r^{n-1} \, dr
\]
\[
= - \sum_{n=1}^{\infty} \frac{(\frac{\pi a}{b})^n - 1 (-1)^n}{n!} \frac{1}{n}
\]
\[
\leq (\frac{\pi a}{b}) - 1.
\]
(3.8)

Combining (3.7) and (3.8), the result follows.

Lemma 3.3. Let \( \varphi(\xi) \) be a function so that the Fourier transform \( \hat{\varphi}(x) \) is a smooth function with support in \([\frac{1}{2}, 2]\), and for \( \omega \in \mathbb{R} \setminus \{0\} \), let \( \varphi_\omega(\xi) \) be the function such that \( \hat{\varphi}_\omega = \varphi(\frac{\xi}{\omega}) \). Then

\[
\left| \int_{\mathbb{R}} \varphi_\omega(\xi - z) \frac{e^{it^3}}{(1 + z^2)^{\frac{3}{2}}} \, dz \right| \lesssim \begin{cases} 
(1 + t)\omega^{-\frac{1}{2}} & \text{if } \omega > 0, \\
(1 + t)|\xi| & \text{if } \omega \text{ intermediate}, \\
(1 + t)|\xi|^{-1} & \text{else}.
\end{cases}
\]

Proof. We consider separately the four different cases, \( \omega < 0, \omega > 0 \) and \( |\xi| \) near, \( \omega > 0 \) and \( |\xi| \) intermediate, and \( \omega > 0 \) and \( |\xi| \) far.

Case \( \omega < 0 \):

Instead of integrating over \( \mathbb{R} \) in (3.1), we will compute the integral over the contours \( \gamma_1 \) through \( \gamma_4 \) in Figure 1, taking the limit as \( R \)
approaches infinity. By Corollary 3.1 and the Cauchy integral theorem,
\[
\int_{\gamma_1} \varphi_\omega (\xi - z) \frac{e^{it z^3}}{(1 + z^2)^{\frac{3}{8}}} \, dz = - \int_{\gamma_2} \varphi_\omega (\xi - z) \frac{e^{it z^3}}{(1 + z^2)^{\frac{3}{8}}} \, dz - \int_{\gamma_3} \ldots
\]
\[
- \int_{\gamma_4} \ldots
\]
We will use estimates on the integrals over \(\gamma_2, \gamma_3,\) and \(\gamma_4\) to estimate (3.1). Along \(\gamma_2,\)
\[
\left| \int_{[0,\frac{1}{2}]} \varphi_\omega (\xi - R - yi) \frac{e^{it(\xi + yi)^3}}{(1 + (\xi + yi)^2)^{\frac{3}{8}}} i \, dy \right| \lesssim
\]
\[
\int_{[0,\frac{1}{2}]} \left| \varphi_\omega (\xi - R - yi) \frac{e^{it(\xi + yi)^3}}{(1 + (\xi + yi)^2)^{\frac{3}{8}}} \right| \, dy \lesssim
\]
(3.9)
\[
\int_{[0,\frac{1}{2}]} \frac{|e^{\omega y} - e^{\frac{1}{2} \omega y}|}{\omega^2 y |\xi - R - yi|^2} \frac{e^{-t(3R^2 - y^2)y}}{(1 + R^2)^{\frac{3}{8}}} \, dy.
\]
For fixed \(\omega,\) (3.9) approaches 0 as \(R \to \infty.\) A similar estimate applies for \(\gamma_4.\) We can estimate the integral along \(\gamma_3\) using Lemma 3.1,
\[
\left| \int_{[-R,R]} \varphi_\omega (\xi - x - \frac{i}{2}) \frac{e^{it(x + \frac{i}{2})^3}}{(1 + (x + \frac{i}{2})^2)^{\frac{3}{8}}} \, dx \right| \lesssim
\]
\[
\int_{[-R,R]} \left| \varphi_\omega (\xi - x - \frac{i}{2}) \frac{e^{it(x + \frac{i}{2})^3}}{(1 + (x + \frac{i}{2})^2)^{\frac{3}{8}}} \right| \, dx \lesssim
\]
\[
\int_{[-R,R]} \frac{|e^{\omega x} - e^{\frac{1}{2} \omega x}|}{\omega^2 (|\xi - x|^2 + 1)} \frac{e^{-t(\frac{x^2 - y}{1 + x^2})}}{(1 + x^2)^{\frac{3}{8}}} \, dx \lesssim
\]
\[
\frac{|e^{\omega} - e^{\frac{1}{2} \omega}|}{\omega^2} \int_{R} \frac{1}{(|\xi - x|^2 + 1)} \, dx \lesssim
\]
\[
\frac{|e^{\omega} - e^{\frac{1}{2} \omega}|}{\omega^2}.
\]
From (3.9) and (3.10) we estimate (3.1),
\[
\left| \int_{\mathbb{R}} \varphi_\omega (\xi - z) \frac{e^{it z^3}}{(1 + z^2)^{\frac{3}{8}}} \, dz \right| \lesssim \frac{|e^{\omega} - e^{\frac{1}{2} \omega}|}{\omega^2} \lesssim (1 + t)|\omega|^{-1}.
\]
End of Case \(\omega < 0.\)
Let \(\epsilon\) be some positive number that will be specified later. For the remaining three cases, we split up the integral (3.1) in the following
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\[ \xi - \frac{1}{10} \varepsilon \quad \xi \quad \xi + \frac{1}{10} \varepsilon \]

\[ \Gamma_1 \]

\[ \xi - \frac{1}{10} \varepsilon \]

**Figure 2.** The contour used when \( \omega > 0 \) and \( |\xi| \leq \frac{1}{10} \sqrt{\frac{\omega}{t}} \).

In the next three cases we estimate

\[ \int_{B_{\frac{1}{10} \varepsilon}^{\frac{1}{10} \varepsilon}} \varphi_{\omega}(\xi - z) \frac{e^{itz^3}}{(1 + z^2)^{\frac{3}{2}}} \, dz \]

We estimate the integral over \( \mathbb{R} \setminus B_{\frac{1}{10} \varepsilon}(\xi) \) using the decay of \( \varphi_{\omega} \), from Lemma 3.1

\[
\left| \int_{\mathbb{R} \setminus B_{\frac{1}{10} \varepsilon}(\xi)} \varphi_{\omega}(\xi - z) \frac{e^{itz^3}}{(1 + z^2)^{\frac{3}{2}}} \, dz \right| \leq \int_{\mathbb{R} \setminus B_{\frac{1}{10} \varepsilon}(\xi)} \left| \varphi_{\omega}(\xi - z) \frac{e^{itz^3}}{(1 + z^2)^{\frac{3}{2}}} \right| \, dz \\
\leq \int_{\mathbb{R} \setminus B_{\frac{1}{10} \varepsilon}(\xi)} |\varphi_{\omega}(\xi - z)| \, dz \\
\leq \int_{\mathbb{R} \setminus B_{\frac{1}{10} \varepsilon}(\xi)} \frac{1}{\omega(\xi - x)^2} \, dx \lesssim \frac{1}{\omega \varepsilon}.
\]

(3.11)

In the next three cases we estimate

\[ \int_{B_{\frac{1}{10} \varepsilon}^{\frac{1}{10} \varepsilon}} \varphi_{\omega}(\xi - z) \frac{e^{itz^3}}{(1 + z^2)^{\frac{3}{2}}} \, dz. \]

(3.12)

**Case \( \omega > 0 \), near:**

By Corollary 3.1 and the Cauchy integral theorem, we can estimate (3.12) by approximating the integral along the semicircle arc \( \Gamma_1 \) in Figure 2, as long as we avoid the rays where the integrand is not analytic.

If \( \frac{1}{10} \omega^{\frac{1}{2}} t^{-\frac{1}{2}} < 1 \), then let \( \varepsilon = \omega^{\frac{1}{2}} t^{-\frac{1}{2}} \). Otherwise, let \( \varepsilon = 1 \). We illustrate the estimate only for the case \( \varepsilon = 1 \), as the other case follows by
a similar argument.

\[\int_{[2\pi,\pi]} \varphi^\omega(\xi - z) \frac{e^{itz}}{(1 + z^2)^{\frac{1}{2}}} dz \lesssim \frac{1}{\omega} + \frac{t}{\omega} \lesssim (1 + t)\omega^{-1}.\]  

**End of Case $\omega > 0$, near.**

**Case $\omega > 0$, intermediate:** To estimate (3.12), we use the Young inequality, and the fact that $\|\varphi^\omega\|_1$ is uniformly bounded in $\omega$. Let $\varepsilon = \frac{1}{10} \sqrt{\frac{\tau}{\pi}}$.

\[\int_{B_{\frac{1}{10} \varepsilon}(\xi)} \varphi(\xi - z) \frac{e^{itz}}{(1 + z^2)^{\frac{1}{2}}} dz \lesssim \frac{1}{\omega} + \frac{t}{\omega} \lesssim (1 + t)\omega^{-1}.\]  

Since $|\xi| \leq \frac{1}{10} \sqrt{\frac{\tau}{\pi}}$ and $\varepsilon = 1 \leq \sqrt{\frac{\tau}{\pi}}$, it follows that

\[\left| \frac{1}{10} (3(\xi + \frac{1}{10} \varepsilon \cos s) - \frac{1}{100} \varepsilon^2) \right| \leq \frac{1}{10} (3(|\xi| + \frac{1}{10} \varepsilon^2) + \frac{1}{100} \varepsilon^2) \leq \frac{13}{1000} \omega.\]  

Using this and Lemma 3.2, we bound (3.13) with

\[\int_{[2\pi,\pi]} t \left| \frac{e^{isz} - e^{\frac{1}{20} \omega s}}{\omega^3 \sin s} \right| e^{-t \frac{3}{10} (\varepsilon + \frac{1}{10} \varepsilon^2 s^2) \varepsilon s} ds \lesssim \frac{t}{\omega}.\]  

From (3.14) and (3.11), we have the estimate

\[\int_{\mathbb{R}} \varphi^\omega(\xi - z) \frac{e^{itz^3}}{(1 + z^2)^{\frac{1}{2}}} dz \lesssim \frac{1}{\omega} + \frac{t}{\omega} \lesssim (1 + t)\omega^{-1}.\]  

**End of Case $\omega > 0$, near.**
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\[ \xi + \frac{t}{10} \varepsilon \]

\[ \xi - \frac{1}{10} \varepsilon \]

\[ \xi + \frac{1}{10} \varepsilon \]

\[ \Gamma_2 \]

**Figure 3.** The contour used when \( \omega > 0 \) and \( |\xi| > 10 \sqrt{\frac{\omega}{t}} \).

From (3.16) and (3.11), we have the estimate

\[
\int_{\mathbb{R}} \varphi_\omega(\xi - z) \frac{e^{itz^3}}{(1 + z^2)^{\frac{3}{4}}} \frac{e^{it\frac{1}{10} \varepsilon e^{is}}}{(1 + (\xi + \frac{1}{10} \varepsilon e^{is})^2)^{\frac{3}{4}}} ds \lesssim \sqrt{\frac{t}{\omega^2}} + t^{\frac{3}{4}} \omega^{-\frac{1}{4}} \lesssim (1 + t) \omega^{-\frac{1}{4}}.
\]

**End of Case \( \omega > 0 \), intermediate.**

**Case \( \omega > 0 \), far:** Let \( \varepsilon = \sqrt{\frac{\omega}{t}} \). We use an argument similar the near case, integrating along the the semicircle arc \( \Gamma_2 \) in Figure 3,

\[
\int_{[0, \pi]} \left| \varphi_\omega(\xi) \frac{e^{it\frac{1}{10} \varepsilon e^{is}}}{(1 + (\xi + \frac{1}{10} \varepsilon e^{is})^2)^{\frac{3}{4}}} ds \right| \lesssim \int_{[0, \pi]} \left| \varphi_\omega(\xi) \frac{1}{10} \varepsilon e^{is} ds \right| \lesssim \int_{[0, \pi]} t e^{\frac{1}{10} \varepsilon e^{is} s - \frac{1}{10} \sqrt{\omega \varepsilon} \sin s} e^{-\frac{1}{10} \varepsilon e^{is} (\xi + \frac{1}{10} \varepsilon \cos s)^2 - \frac{1}{100} \varepsilon e^{is} \sin s} s e^{-\frac{1}{10} \varepsilon e^{is} \sin s} ds.
\]

Since \( \xi > 10 \sqrt{\frac{\omega}{t}} \),

\[
-29.402 \omega \\
\leq -\frac{t}{10} \left( 3 \left( \sqrt{\frac{\omega}{t}} - \frac{1}{10} \sqrt{\frac{\omega}{t}} \right)^2 - \frac{1}{100} \sqrt{\frac{\omega}{t}} \right) \\
\leq -\frac{t}{10} \left( 3 \left( \xi + \frac{1}{10} \varepsilon \cos s \right)^2 - \frac{1}{100} \varepsilon e^{is} \sin s \right).
\]
We use this with Lemma 3.2 to bound (3.17) by
\[
\int_{[0,\pi]} t \left| e^{\frac{1}{2} \omega s \sin s} - e^{\frac{1}{5} \omega s \sin s} \right| e^{-29.402 \omega s \sin s} ds \approx t \frac{\omega^3}{\omega^3} \int_{[0,\pi]} \left| e^{-29.200 \omega s \sin s} - e^{-29.352 \omega s \sin s} \right| ds \lesssim \frac{t}{\omega^3}.
\]

From (3.18) and (3.11), we have the estimate
\[
\left| \int_{\mathbb{R}} \varphi_{\omega}(\xi - z) \frac{e^{it\xi^3}}{(1 + z^2)^{\frac{3}{4}}} dz \right| \lesssim \frac{\sqrt{t}}{\omega^2} + \frac{t}{\omega^3} \lesssim (1 + t)\omega^{-\frac{3}{2}} \lesssim (1 + t)\omega^{-1}.
\]
End of Case \( \omega > 0 \), far. \( \square \)

Lemma 3.4.
\[
\left\| D_{\xi}^{-\frac{1}{2}} \left( \frac{e^{it\xi^3}}{(1 + \xi^2)^{\frac{3}{4}}} \right) \right\|_{L^\infty_x L^1_N} \lesssim 1 + t.
\]

Proof. The operator \( Q_N^5 \) (see also Appendix A) is defined by
\[
Q_N^5 f \equiv (1 + 2^N \eta(x)) f(x).
\]
Since \( Q_N \) is just convolution against the Fourier transform of a scaled smooth function, by rescaling we obtain
\[
\left\| Q_N D_{\xi}^{-\frac{1}{2}} \frac{e^{it\xi^3}}{(1 + \xi^2)^{\frac{3}{4}}} \right\|_{L^\infty_x L^1_N} = \left\| 2^\frac{N}{2} Q_N^5 \left( \frac{e^{it\xi^3}}{(1 + \xi^2)^{\frac{3}{4}}} \right) \right\|_{L^\infty_x L^1_N}.
\]
We can estimate the low frequency part using the Young inequality in the following manner,
\[
\left\| 2^\frac{N}{2} Q_N^5 \left( \frac{e^{it\xi^3}}{(1 + \xi^2)^{\frac{3}{4}}} \right) \right\|_{L^\infty_x L^1_N} \leq \sum_{N \leq 0} 2^\frac{N}{2} \left\| Q_N^5 \left( \frac{e^{it\xi^3}}{(1 + \xi^2)^{\frac{3}{4}}} \right) \right\|_{L^\infty_x L^0_N} \lesssim \sum_{N \leq 0} 2^\frac{N}{2} \left\| \frac{e^{it\xi^3}}{(1 + \xi^2)^{\frac{3}{4}}} \right\|_{L^\infty_x} \lesssim 1.
\]
We use Lemma 3.3, noting that if \( t \) is fixed, for each \( |\xi| \), there is a unique dyadic \( 2^N \) so that \( \xi \) is intermediate. We use this to bound the
remaining frequencies.
\[
\left\| 2\pi \tilde{Q}_N (\frac{e^{it\xi^3}}{(1 + \xi^2)^{\frac{1}{2}}}) \right\|_{L_t^\infty L_x^2} = \sum_{N=1}^{\infty} 2\pi |Q_N (\frac{e^{it\xi^3}}{(1 + \xi^2)^{\frac{1}{2}}})|_{L_t^\infty} \\
\leq \sum_{2^N | \xi \text{ not intermediate}} 2^N |Q_N (\frac{e^{it\xi^3}}{(1 + \xi^2)^{\frac{1}{2}}})|_{L_t^\infty} \\
+ \sum_{2^N | \xi \text{ intermediate}} 2^N |Q_N (\frac{e^{it\xi^3}}{(1 + \xi^2)^{\frac{1}{2}}})|_{L_t^\infty} \\
\lesssim (\sum_{N=1}^{\infty} 2^N 2^{-N} + 1)(1 + t).
\]

4. Decay Estimates for mKdV Solutions

With our bound from Lemma 3.3, we will show that our main result follows. This will come from the fact that for \( \alpha \in (0, 1) \),
\[
(4.1) \quad \|D_x^\alpha (fg) - gD_x^\alpha f\|_2 \lesssim \|Q_N D_x^\alpha g\|_{L_t^\infty L_x^1} \|f\|_2.
\]

A classical Leibnitz type inequality for fractional derivatives is the following (see [13]).

**Lemma 4.1.** Let \( 0 < \alpha_1, \alpha_2 < 1, \alpha = \alpha_1 + \alpha_2, 1 < p, p_1, p_2 < \infty, \) and \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \). In addition, the \( \alpha_1 = \alpha, p = p_2, \) and \( p_1 = \infty \) is allowed. Then the following holds for functions \( f, g \) on \( \mathbb{R}^n \).
\[
\|D_x^\alpha (fg) - D_x^\alpha (f)g - fD_x^\alpha (g)\|_p \lesssim \|D_x^{\alpha_1} g\|_{p_1} \|D_x^{\alpha_2} f\|_{p_2}
\]

The proof uses the Littlewood-Paley Theorem (see [21]), which states that for any function \( f \), if \( 1 < p < \infty \), then
\[
(4.2) \quad \|Q_N (f)\|_{L_t^\infty L_x^2} \lesssim \|f\|_p \lesssim \|Q_N (f)\|_{L_t^\infty L_x^2}.
\]

Lemma 4.1 is not sufficient for our argument in the previous section, since we need to put the derivative term in the infinity norm. A product rule like this can be obtained by following the proof of Lemma 4.1 line for line. The only difference is that since (4.2) fails for \( p = \infty \), \( \|Q_N (D_x^\alpha g)\|_{L_t^\infty L_x^2} \) is not equivalent to \( \|D_x^\alpha g\|_\infty \). This idea was inspired by [14], where the authors use \( \|Q_N \cdot\|_{L_t^\infty L_x^2} \) in an estimate where the \( \| \cdot \|_{L_t^\infty L_x^2} \) norm may fail.

**Lemma 4.2.** Let \( 0 < \alpha < 1 \) and \( 1 < p < \infty \). For functions \( f \) and \( g \),
\[
\|D_x^\alpha (fg) - gD_x^\alpha f - fD_x^\alpha g\|_p \lesssim (\|Q_N D_x^\alpha g\|_{L_t^\infty L_x^2} + \|D_x^\alpha g\|_{L_t^\infty L_x^2}) \|f\|_p.
\]
In particular,

\[ \|D_x^\alpha (fg) - gD_x^\alpha f\|_2 \lesssim \|Q_N D_x^\alpha g\|_{L^2_t L_x^2} \|f\|_2. \]

The proof is in Appendix A.

For a number \(1 \leq p \leq \infty\), let \(p'\) denote the conjugate exponent. We recall the following properties of the operator \(U(t)\),

\[ \|\partial_x \int_0^t U(t-t') f(x, t') \, dt'\|_{L^2_x} \lesssim \|f\|_{L^p_t L_x^p}, \tag{4.3} \]

\[ \|\int_0^t U(t-t') f(t') \, dt'\|_{L^2_x} \lesssim \|f\|_{L^q_t L_x^q}, \tag{4.4} \]

where \(p \geq 2\), and \(q\) satisfy \(1 - p = \frac{1}{6} - \frac{1}{3p'}\). The proof of (4.3) can be found in [17], or [13]. Inequality (4.4) follows from the fact that \(U(t)\) is an \(L^2_x\) isometry, along with the dual of the homogenous Strichartz estimate for \(U(t)\) (see [6], page 1392).

The existence theorem for solutions to (1.1) is proved by a contraction mapping argument, which can also be found in [17].

**Theorem 4.1.** Let \(\| \cdot \|_{Y_T}\) denote the norm such that

\[ \|f\|_{Y_T} \equiv \|f\|_{L^4_t L^4_x} + \|D_x^\frac{1}{2} \partial_x f\|_{L^\infty_t L^2_x} + \|f\|_{L^\infty_t H^\frac{1}{4}_x} + \|\partial_x f\|_{L^2_t L^6_x} + \|D_x^\frac{1}{2} f\|_{L^4_t L^8_x}^2, \]

\[ Y_T \equiv \{ f \mid \|f\|_{Y_T} < \infty \}, \]

and let \(u_0 \in L^2\), and \(\Phi\) be the map from \(Y_T\) to \(Y_T\) such that

\[ \Phi(u) \equiv U(t)u_0 - \int_0^t U(t-t') \partial_x (u^3(t')) \, dt'. \]

Then

\[ \|\Phi(u)\|_{Y_T} \lesssim \|u_0\|_{H^\frac{1}{4}_x} + T^\frac{1}{2} \|u\|_{Y_T}^3. \tag{4.5} \]

This implies by contraction mapping that there exist \(T = c\|D_x^\frac{1}{2} u\|_{L^2_t}^{-4}\) and a unique strong solution \(u(t)\) of the IVP (1.1).

The proof requires a Leibnitz rule type inequality for \(L^p_t L^q_x\) norms, which we need as well.

**Lemma 4.3.** Let \(\alpha \in (0, 1), \alpha_1, \alpha_2 \in [0, \alpha]\) with \(\alpha = \alpha_1 + \alpha_2\). Let \(p, q, p_1, p_2, q_2 \in (1, \infty), q_1 \in (1, \infty)\) be such that

\[ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}. \]

Then

\[ \|D_x^\alpha (fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L^p_t L^q_x} \lesssim \|D_x^{\alpha_1} f\|_{L^p_1 L^{q_1}_x} \|D_x^{\alpha_2} f\|_{L^{p_2}_x L^{q_2}_x}. \]

Moreover, for \(\alpha_1 = 0\), the value \(q_1 = \infty\) is allowed.
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We will need an estimate on the Fourier transform $k(x)$ of $(1 + \xi^2)^{-\frac{1}{8}}$. We expect $k$ to have good decay properties since it is the inverse Fourier transform of a smooth function. Since

$$|\hat{\xi}|^{-\frac{1}{4}} = c_0 |x|^{-\frac{3}{4}},$$

we expect that $k(x) \approx |x|^{-\frac{3}{4}}$ for small $x$. This is formalized in the following lemma.

**Lemma 4.4.** Let $k(x)$ denote the Fourier transform of the function $(1 + \xi^2)^{-\frac{1}{8}}$. Then for any $n \in \mathbb{N}$,

$$|k(x)| \lesssim \frac{1}{|x|^\frac{3}{4}(1 + x^{2n})}.$$ 

In particular,

$$\int_{\mathbb{R}} |x|^\frac{1}{8} |k(x)| < \infty.$$

**Proof.** For $x > 1$, we can repeatedly integrate by parts as follows:

$$\int_{\mathbb{R}} (1 + \xi^2)^{-\frac{1}{8}} e^{-ix \xi} d\xi = \int_{\mathbb{R}} (1 + \xi^2)^{-\frac{1}{8}} \frac{d}{-ix} e^{-ix \xi} d\xi$$

$$= \frac{1}{ix} \int_{\mathbb{R}} \frac{1}{4} (1 + \xi^2)^{-\frac{3}{8}} e^{-ix \xi} d\xi$$

$$= \frac{1}{ix} \int_{\mathbb{R}} \frac{1}{4} (1 + \xi^2)^{-\frac{3}{8}} \frac{d}{-ix} e^{-ix \xi} d\xi$$

$$= \ldots$$

This argument gives us the decay in (4.7).

When $x < 1$, we split up the integral over the region $S = [\frac{1}{x}, 1]$. 

$$\int_{\mathbb{R}} (1 + \xi^2)^{-\frac{1}{8}} e^{-ix \xi} d\xi = \int_{S} (1 + \xi^2)^{-\frac{1}{8}} e^{-ix \xi} d\xi$$

$$+ \int_{\mathbb{R}\setminus S} (1 + \xi^2)^{-\frac{1}{8}} e^{-ix \xi} d\xi$$

$$= \mathcal{A} + \mathcal{B}.$$ 

Since $(1 + \xi^{-2})^{-\frac{1}{8}}$ is bounded,

$$|\mathcal{A}| \lesssim \int_{S} (1 + \xi^2)^{-\frac{1}{8}} d\xi$$

$$= \int_{S} |\xi|^{-\frac{1}{4}} (1 + \xi^{-2})^{-\frac{1}{8}} d\xi$$

$$\lesssim \int_{S} |\xi|^{-\frac{1}{4}} d\xi \lesssim |x|^{-\frac{3}{4}}.$$
By integration by parts,

\[
\mathcal{B} = \int_{\mathbb{R}\setminus S} (1 + \xi^2)^{-\frac{1}{8}} \frac{1}{-ix} \frac{d}{d\xi} e^{-ix\xi} \, d\xi
\]

\[
= (1 + x^{-2})^{-\frac{1}{8}} \frac{e^i}{-ix} + (1 + x^{-2})^{-\frac{1}{8}} \frac{e^{-i}}{ix}
\]

\[
+ \frac{1}{ix} \int_{\mathbb{R}\setminus S} \frac{1}{\xi^4} (1 + \xi^2)^{-\frac{9}{8}} e^{-ix\xi} \, d\xi.
\]

Therefore,

\[
|\mathcal{B}| \lesssim |x|^{-\frac{7}{8}} + \frac{1}{|x|} \int_{\mathbb{R}\setminus S} |\xi|(1 + \xi^2)^{-\frac{9}{8}} \, d\xi
\]

\[
= |x|^{-\frac{7}{8}} + \frac{1}{|x|} \int_{\mathbb{R}\setminus S} |\xi|^{-\frac{7}{8}}(1 + \xi^2)^{-\frac{9}{8}} \, d\xi
\]

\[
\lesssim |x|^{-\frac{7}{8}} + \frac{1}{|x|} \int_{\mathbb{R}\setminus S} |\xi|^{-\frac{3}{8}} \, d\xi \lesssim |x|^{-\frac{3}{4}}.
\]

Combining our estimates for \(|A|\) and \(|\mathcal{B}|\), the result follows. □

Before proving Theorem 1.2, we prove the corresponding decay result for solutions to the linear part of (1.1). This is necessary for the proof of Theorem 1.2, and it is also a simpler case that illustrates the main idea of our proof of Theorem 1.2. We note that it is also possible to prove this result using an argument like Lemma 2 in [18], but this proof does not generalize to solutions of (1.1) as discussed in the introduction.

**Lemma 4.5.** For \(u_0 \in C_0^\infty(\mathbb{R})\),

\[
|||x|^s U(t) u_0(x)||_2 \lesssim (1 + |t| + |t|^s) ||u_0||_{H^{2s}} + |||x|^s u_0||_2.
\]

**Proof.** For concreteness, it will suffice to prove the result in the case \(s = \frac{1}{8}\). By the definition of \(U(t)\) and the triangle inequality,

\[
|||x|^\frac{1}{8} U(t) u_0||_2 = ||D_{\xi}^{\frac{1}{8}} (e^{it\xi} \hat{u}_0) ||_{L^2_{\xi}}
\]

\[
= ||D_{\xi}^{\frac{1}{8}} \left( \frac{e^{it\xi}}{(1 + \xi^2)^{\frac{1}{8}}} (1 + \xi^2)^{\frac{1}{8}} \hat{u}_0 \right) ||_{L^2_{\xi}}
\]

\[
\lesssim ||D_{\xi}^{\frac{1}{8}} \left( \frac{e^{it\xi}}{(1 + \xi^2)^{\frac{1}{8}}} (1 + \xi^2)^{\frac{1}{8}} \hat{u}_0 \right) - \frac{e^{it\xi}}{(1 + \xi^2)^{\frac{1}{8}}} D_{\xi}^{\frac{1}{8}} ((1 + \xi^2)^{\frac{1}{8}} \hat{u}_0) ||_{L^2_{\xi}}
\]

\[
+ || \frac{e^{it\xi}}{(1 + \xi^2)^{\frac{1}{8}}} D_{\xi}^{\frac{1}{8}} ((1 + \xi^2)^{\frac{1}{8}} \hat{u}_0) ||_{L^2_{\xi}}
\]

\[
\equiv I + II.
\]
We can write term $\mathcal{I}$ as

$$
\mathcal{I} = \|(1 + \xi^2)^{-\frac{1}{8}} \tilde{D}_\xi \left( (1 + \xi^2)^{\frac{1}{8}} \hat{u}_0 \right) \|_2
$$

$$
\leq \|(1 + \xi^2)^{-\frac{1}{8}} \tilde{D}_\xi \left( (1 + \xi^2)^{\frac{1}{8}} \hat{u}_0 \right) \|_2 + \| \tilde{D}_\xi \hat{u}_0 \|_2
$$

(4.8)

We need to bound the commutator term in $\mathcal{I}$. For any function $h$, we use the Plancherel theorem, the Young inequality, and Lemma 4.4 to obtain

$$
\|(1 + \xi^2)^{-\frac{1}{8}} \tilde{D}_\xi h \|_{L^2_\xi} = \left\| \int_{\mathbb{R}} (|x|^{\frac{1}{8}} - |y|^{\frac{1}{8}}) k(x - y) \hat{h}(y) \, dy \right\|_{L^2_\xi}
$$

$$
\lesssim \left\| \int_{\mathbb{R}} |x - y|^{\frac{1}{8}} |k(x - y)| \hat{h}(y) \, dy \right\|_{L^3_\xi}
$$

$$
\lesssim c_k \| \hat{h} \|_{L^2_\xi} = c_k \| h \|_{L^2_\xi}.
$$

We apply this to (4.8),

$$
\| \mathcal{I} \| \lesssim c_k \| u_0 \|_{H^\frac{1}{2}} + \| |x|^\frac{1}{8} u_0 \|_2.
$$

For term $\mathcal{I}$, we use Lemma 4.2 with Lemma 3.4

$$
\| \mathcal{I} \| \lesssim \| Q_N \frac{e^{i \xi^3}}{(1 + \xi^2)^{\frac{1}{8}}} \|_{L^\infty_x L^2_t} \| u_0 \|_2.
$$

$$
\lesssim (1 + t) \| u_0 \|_2.
$$

Combining our estimates for $\mathcal{I}$ and $\mathcal{II}$, the result follows. \(\square\)

**Proof of Theorem 1.2.** For concreteness, we prove the result in the most interesting case when $k = 2$, $s = s' = \frac{1}{8}$, and $t > 0$. We use a contraction mapping argument to prove our decay estimate. The resolution space is

$$
\| f \|_{Z_T} \equiv \| |x|^{\frac{1}{8}} f \|_{L^\infty_t L^2_x} + \| f \|_{Y_T}.
$$

$$
Z_T \equiv \{ f | \| f \|_{Z_T} < \infty \}.
$$

Let $f(t) \equiv \partial_x (u^3(t))$ for convenience, and consider

$$
\Phi(u)(x, t) = U(t)u(x, 0) - \int_0^t U(t - t')f(t')dt'.
$$

(4.9)

Multiply (4.9) by $|x|^\frac{1}{8}$. The $|x|^\frac{1}{8} U(t)u(x, 0)$ term is bounded by Lemma 4.5 along with a density argument. We concentrate on the nonlinear
We bound term II in a similar fashion to term II in Lemma 4.5:

\[
\tag{4.10}
II \lesssim \left\| \int_0^t \frac{1}{(1 + \xi^2)\frac{3}{2}} D_{\xi}^{\frac{1}{2}} ((1 + \xi^2)^{\frac{3}{2}} f^\wedge(t')) \, dt' \right\|_{L_\infty^T L_2^\xi}
+ \left\| \int_0^t e^{i(t-t')\xi^3} \frac{1}{(1 + \xi^2)^{\frac{3}{2}}} D_{\xi}^{\frac{1}{2}} f^\wedge(t') \, dt' \right\|_{L_\infty^T L_2^\xi}
\lesssim \left\| \frac{1}{(1 + \xi^2)^{\frac{3}{2}}} D_{\xi}^{\frac{1}{2}} ((1 + \xi^2)^{\frac{3}{2}} f^\wedge(t')) \right\|_{L_\infty^T L_2^\xi}
\lesssim \left\| \frac{1}{(1 + \xi^2)^{\frac{3}{2}}} D_{\xi}^{\frac{1}{2}} ((1 + \xi^2)^{\frac{3}{2}} f^\wedge(t')) \right\|_{L_\infty^T L_2^\xi}
\approx II.1 + II.2.
\]
Specializing to the case of the mKdV, \( f(t') = \partial_x(u^3(t')) \), we bound II.1 using Theorem 4.1:

\[
\begin{align*}
II.1 \lesssim & \| \partial_x(u^3) \|_{L^2_t L^4_x} + \| D_x^\frac{3}{2} \partial_x(u^3) \|_{L^2_t L^2_x} \\
\lesssim & \frac{T^{\frac{1}{2}}}{2} \| \partial_x(u^3) \|_{L^2_t L^2_x} + \frac{T^{\frac{1}{2}}}{2} \| D_x^\frac{3}{2} \partial_x(u^3) \|_{L^2_t L^2_x} \\
\leq & \frac{T^{\frac{1}{2}}}{2} \| u \|_{L^2_t L^4_x}^2 \| \partial_x u \|_{L^\infty_t L^2_x} + \frac{T^{\frac{1}{2}}}{2} \| u \|_{L^2_t L^4_x} \| D_x^\frac{3}{2} \partial_x u \|_{L^\infty_t L^2_x} \\
& + \frac{T^{\frac{1}{2}}}{2} \| D_x^\frac{3}{2} (u^2) \|_{L^2_t L^4_x} \| \partial_x u \|_{L^\infty_t L^2_x} + \frac{T^{\frac{1}{2}}}{2} \| u \|_{L^2_t L^4_x} \| D_x^\frac{3}{2} u \|_{L^2_t L^4_x} \| \partial_x u \|_{L^\infty_t L^2_x} \\
\lesssim & \frac{T^{\frac{3}{2}}}{2} \| u \|_{L^3_t L^2_x}^3.
\end{align*}
\]

Let \( \phi(x) \in C^\infty_0(\mathbb{R}) \) have the property that \( \phi(x) = 1 \) for \( x \in (-1, 1) \). We handle II.2 with the following argument:

\[
\begin{align*}
\| \int_0^t U(t-t') |x|^\frac{3}{2} f(t') \, dt' \|_{L^\infty_t L^2_x} & \lesssim \| \int_0^t U(t-t') |x|^\frac{3}{2} \phi(x) f(t') \, dt' \|_{L^\infty_t L^2_x} \\
& + \| \int_0^t U(t-t') |x|^\frac{3}{2} (1-\phi(x)) f(t') \, dt' \|_{L^\infty_t L^2_x} \\
\lesssim & \| \int_0^t U(t-t') |x|^\frac{3}{2} (1-\phi(x)), \partial_x |u^3(t') \, dt' \|_{L^\infty_t L^2_x} \\
& + \| \int_0^t U(t-t') \partial_x ((|x|^\frac{3}{2} (1-\phi(x))) u^3(t')) \, dt' \|_{L^\infty_t L^2_x} \\
& + \| \int_0^t U(t-t') |x|^\frac{3}{2} \phi(x) \partial_x (u^3(t')) \, dt' \|_{L^\infty_t L^2_x} \\
\equiv & II.2.a + II.2.b + II.2.c.
\end{align*}
\]

For II.2.a, we use (4.4), and that for any function \( h \), and \( p \geq 1 \),

\[
\| |x|^\frac{3}{2} (1-\phi(x)), \partial_x |h \|_p \lesssim \| \frac{\partial}{\partial x} (|x|^\frac{3}{2} (1-\phi(x))) \|_\infty \| h \|_p,
\]

along with the Sobolev inequality to obtain the bound

\[
II.2.a \lesssim \| u^3 \|_{L^\infty_t L^\frac{12}{7}_x} = \| u \|_{L^\infty_t L^2_x}^3 \\
\lesssim \frac{T^{\frac{3}{2}}}{2} \| u \|_{L^3_t H^\frac{3}{2}_x}^3 \lesssim \frac{T^{\frac{3}{2}}}{2} \| u \|_{L^3_t H^\frac{3}{2}_x}^3.
\]
We use (4.3) to estimate II.2.b:

\[ II.2.b \lesssim \|x\|_2 \frac{1 - \phi(x)}{1 + \xi^2} \|u\|^3_{L^2_x L^\infty_t} \]

\[ \lesssim \|u\|^2_{L^2_x L^\infty_t} \|x\|_2 \frac{1 - \phi(x)}{1 + \xi^2} \|u\|_{L^2_x L^2_t} \]

\[ \lesssim \|u\|_2^2 L^\infty_t \|x\|_2 \frac{1 - \phi(x)}{1 + \xi^2} \|u\|_{L^2_x L^2_t} + \|u\|_2 L^2_t \]

\[ \lesssim T^{1/2} \|u\|_2^2 L^\infty_t \|x\|_2 \frac{1 - \phi(x)}{1 + \xi^2} \|u\|_{L^2_x L^2_t} + \|u\|_2 L^2_t \]

\[ \lesssim T^{1/2} \|u\|_2^2 \]

We use Theorem 4.1 and the fact that \( \phi \) has compact support to control II.2.c:

\[ II.2.c \lesssim \|x\|_2 \frac{1 - \phi(x)}{1 + \xi^2} \|u\|^3_{L^2_x L^2_t} \]

\[ \lesssim \|\partial_x (u^3)\|_{L^1_x L^2_t} \]

\[ \lesssim T^{1/2} \|u\|_2^2 L^\infty_t \|\partial_x u\|_{L^2_x L^2_t} \]

\[ \lesssim T^{1/2} \|u\|_2^3 \]

Term I from (4.10) can be controlled using Theorem 4.1 and the same argument as the bound for II.1:

\[ I \lesssim \|Q_N D^\frac{1}{2} e^{it\xi_3 (1 + \xi^2 \frac{1}{2})} (1 + D^2_x) \frac{1}{2} (u^2 \partial_x u)\|_{L^2_x L^\infty_t} \]

\[ \lesssim (1 + T^{1/2}) \|u^2 \partial_x u\|_{L^2_x L^2_t} + \|D^\frac{1}{2} (u^2 \partial_x u)\|_{L^2_x L^2_t} \]

\[ \lesssim T^{1/2} (1 + T) \|u^2\|_{L^2_x L^\infty_t} \|\partial_x u\|_{L^2_x L^2_t} + \|u\|_2 L^2_t \]

\[ + \|u\|_2 L^2_t \|D^\frac{1}{2} u\|_{L^2_x L^2_t} \|\partial_x u\|_{L^2_x L^2_t} \]

\[ \lesssim T^{1/2} (1 + T) \|u\|_2^3 \lesssim T^{1/2} (1 + T) \|u\|_2^3 \]

Putting these estimates together,

\[ \|x\|_2 \frac{1 - \phi(x)}{1 + \xi^2} \|u\|^3_{L^2_x L^\infty_t} \lesssim \|x\|_2 \frac{1 - \phi(x)}{1 + \xi^2} \|U(t)u_0\|_{L^2_x L^\infty_t} + I + II.1 + II.2.a + II.2.b + II.2.c \]

(4.11) \[ \lesssim \|x\|_2 \frac{1 - \phi(x)}{1 + \xi^2} \|u_0\|_{L^2_x L^\infty_t} + (1 + T) \|u_0\|_{H^\frac{1}{2}} + T^{\frac{1}{2}} (1 + T^{\frac{1}{2}} + T) \|u\|_2^3 \]

In order to get a contraction, we need to bound \( \|u\|_{Z_T} \) in terms of \( \|u\|_{Z_T} \). This follows from estimate (4.5) in Theorem 4.1. By combining this with (4.11), we obtain a contraction by taking \( T \) small enough,

\[ \|u\|_{Z_T} \lesssim \|x\|_2 \frac{1 - \phi(x)}{1 + \xi^2} \|u_0\|_{L^2_x L^\infty_t} + (1 + T) \|u_0\|_{H^\frac{1}{2}} + T^{\frac{1}{2}} (1 + T^{\frac{1}{2}} + T) \|u\|_2^3 \]

In order to show that \( \|x\|_2 \frac{1 - \phi(x)}{1 + \xi^2} \|u(t)\|_{L^2_x L^\infty_t} \) is finite, for \( t \in [0, T) \), apply \( \|x\|_2^{\frac{1}{2}} \cdot \|L^\infty_x L^2_t\| \) to (4.9) instead of \( \|x\|_2^{\frac{1}{2}} \cdot \|L^\infty_x L^2_t\| \), keeping in mind that \( \|x\|_2 \frac{1 - \phi(x)}{1 + \xi^2} \|u(t)\|_{L^2_x L^\infty_t} \) is finite. \( \square \)
5. Appendix A

For our proof of Lemma 4.1, we closely follow the proof of Theorem A.8 in [13]. This requires more notation. Let \( \alpha_1 = 0, \alpha_2 = \alpha \in [0, 1] \). For a function \( f \), let

\[
P_N f = \sum_{j \leq N-3} Q_j f.
\]

Define \( p(x) \) to be the function so that

\[
(P_N f)^\wedge = p(2^{-N} x) \hat{f}.
\]

Let \( \tilde{p} \in C_0^\infty(\mathbb{R}) \), with \( \tilde{p}(x) = 1 \) for \( x \in [-100, 100] \), and let

\[
(\tilde{P}_N f)^\wedge(x) = \tilde{p}(2^{-N} x) \hat{f}.
\]

Let \( \tilde{\eta} \in C_0^\infty(\mathbb{R}) \) with \( \tilde{\eta}(x) = 1 \) for \( x \in \left[\frac{1}{4}, 4\right] \), and \( \text{supp} \tilde{\eta} \in \left[\frac{1}{8}, 8\right] \). Then define \( (\tilde{Q}_k f)^\wedge(x) = \tilde{\eta}(2^{-k} x) \hat{f} \). Let

\[
\Psi^i(x) = |x|^{\alpha_i} p(x), \quad \eta^j(x) = \frac{\eta(x)}{|x|^{\alpha_j}},
\]

\[
(\Psi^j)^\wedge(x) = \Psi_j(2^{-k}) \hat{f}(x), \quad (Q^j_k f)^\wedge(x) = \eta^j(2^{-k} x) \hat{f}(x).
\]

Similarly, with \( \eta^3(x) = |x|^\alpha \tilde{p}(x), \eta^4(x) = |x|^{\alpha_1} \eta(x), \) and \( \eta^5(x) = |x|^{\alpha_2} \eta(x) \) we define \( Q^k_3, Q^k_4, Q^k_5 \). Let

\[
\eta^{\nu,j}(x) = \exp(i \nu x) \eta^j(x),
\]

\[
\eta^{\mu,j}(x) = \exp(i \mu x) x^{-\alpha_j} p(x),
\]

with \( j = 1, 2 \) and \( Q^\nu,j, Q^\mu,j \) the corresponding operators.

The following is Proposition A.2 from [13].
Lemma 5.1.

\[ D^a_\alpha(fg) - f D^a_\alpha g - g D^a_\alpha f \]

\[ = \sum_{|j| < 2} 2^j 2^{\alpha_2} \sum_k Q^3_k(Q^1_k(D^{\alpha_1}f)Q^2_k(D^{\alpha_2}g)) \]

\[ + \sum_k \tilde{Q}_k(\Psi^1_k(D^{\alpha_2}g)Q^1_k(D^{\alpha_1}f)) \]

\[ + \sum_k \tilde{Q}_k(Q^2_k(D^{\alpha_2}g)\Psi^2_k(D^{\alpha_1}f)) \]

\[ + \sum_{|j| \leq 2} 2^j 2^{\alpha_2} \sum_k Q^1_k(D^{\alpha_1}f)Q^4_k(D^{\alpha_2}g) \]

\[ + \sum_{|j| \leq 2} 2^j 2^{\alpha_2} \sum_k Q^2_k(D^{\alpha_2}g)Q^2_k(D^{\alpha_1}f) \]

\[ + \int_{\mathbb{R}} \int_{\mathbb{R}} \left[ \sum_k \tilde{Q}_k(Q^1_k(D^{\alpha_1}f)Q^2_k(D^{\alpha_2}g)) \right] r_1(\mu, \nu) \, d\nu \, d\mu \]

\[ + \int_{\mathbb{R}} \int_{\mathbb{R}} \left[ \sum_k \tilde{Q}_k(Q^2_k(D^{\alpha_2}g)Q^1_k(D^{\alpha_1}f)) \right] r_2(\mu, \nu) \, d\nu \, d\mu, \]

where \( r_1, r_2 \in \mathcal{S}(\mathbb{R}^2). \)

**Proof of Lemma 4.2.** From Lemma 5.1, we need to bound four types of terms:

1. \( \sum_{-\infty}^{\infty} Q^0_k(Q_k(f)Q_k(D^a_\alpha g)) \)
2. \( \sum_{-\infty}^{\infty} Q_k(\Psi_k(f)Q_k(D^a_\alpha g)) \)
3. \( \sum_{-\infty}^{\infty} Q^0_k(Q_k(f)\Psi_k(D^a_\alpha g)) \)
4. \( \sum_{-\infty}^{\infty} Q_k(f)Q_k(D^a_\alpha g) \)

Let \( \mathcal{M}h \) denote the Hardy Maximal operator applied to the function \( h. \) We control the first term using duality,

\[ | \int_{\mathbb{R}} \sum_{-\infty}^{\infty} Q_k(f)Q_k(D^a_\alpha g)h \, dx | = | \int_{\mathbb{R}} \sum_{-\infty}^{\infty} Q_k(f)Q_k(D^a_\alpha g)Q_k(h) \, dx | \]

\[ \lesssim \int_{\mathbb{R}} \left[ \sum_{-\infty}^{\infty} |Q_k(f)|^2 |Q_k(D^a_\alpha g)|^2 \right] \left[ \sum_{-\infty}^{\infty} |Q_k(h)|^2 \right] \, dx \]

\[ \lesssim \| Q_k(f)Q_k(D^a_\alpha g) \|_{L^2_{\alpha_k}} \| Q_k(h) \|_{L^\alpha_{\alpha_k}} \]

\[ \lesssim \| Q_k(f)Q_k(D^a_\alpha g) \|_{L^2_{\alpha_k}} \| Q_k(h) \|_{L^\alpha_{\alpha_k}} \]

\[ \lesssim \| \mathcal{M}(f) \|_{L^p_{\alpha_k}} \| Q_k(D^a_\alpha g) \|_{L^2_{\alpha_k}} \| Q_k(h) \|_{L^\alpha_{\alpha_k}} \]

\[ \lesssim \| f \|_{L^p_{\alpha_k}} \| Q_k(D^a_\alpha g) \|_{L^2_{\alpha_k}} \| Q_k(h) \|_{L^\alpha_{\alpha_k}} \]

The second item is treated as the first, with \( \Psi_k(f) \) replacing \( Q_k(f). \) A similar argument is used on the third term, with \( \Psi_k(D^a_\alpha g) \) replacing...
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Ψ_k(f), and the fact that

\[ \| M(D_\alpha^\alpha x g) \|_{L^\infty_x} \lesssim \| g \|_{L^\infty_x}, \]

because \( M \) is a bounded operator from \( L^\infty \) to \( L^\infty \).

The last term is treated with Cauchy-Schwartz,

\[ \| \sum_{-\infty}^\infty Q_k(f)Q_k(D_\alpha^\alpha x g) \|_p \lesssim \| Q_n(f) \|_{L^2_N} \| Q_k(D_\alpha^\alpha x g) \|_{L^2_N} \]

This proves the first part of the lemma.

The second part follows from

\[ \| D_\alpha^\alpha x (fg) - f D_\alpha^\alpha x g - g D_\alpha^\alpha x f \|_p \geq \| D_\alpha^\alpha x (fg) - g D_\alpha^\alpha x f \|_p - \| f D_\alpha^\alpha x g \|_p, \]

the observation that

\[ |D_\alpha^\alpha g| \leq \sum_{N} |Q_N(D_\alpha^\alpha g)|, \]

and for arbitrary functions \( \varphi_N \),

\[ \| \varphi_N \|_{L^2_N} \leq \| \varphi_N \|_{L^1_N} \| \varphi_N \|_{L^1_N} \leq \| \varphi_N \|_{L^1_N}. \]

\[ \Box \]

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