The Axioms of Team Logic

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Abstract. A framework is developed that extends calculi for propositional, modal and predicate logics to calculi for team-based logics. This method is applied to classical and quantified propositional logic, first-order logic and the modal logic K. Complete axiomatizations for propositional team logic PTL, quantified propositional team logic QPTL, modal team logic MTL and the dependence-atom-free fragment of first-order team logic TL are presented.

1. Introduction

Propositional and modal logics, while their history goes back to ancient philosophers, have assumed an outstanding role in the age of modern computer science, with plentiful applications in software verification, modeling, artificial intelligence, or protocol design. An important property of a logical framework is completeness, i.e., that the act of mechanical reasoning can actually effectively be done by a computer. The actual question of completeness of first-order logic, which is the foundation of today’s mathematics, was not settled until the famous result of Gödel in 1929. Today the area of proof theory achieved tremendous progress and is still a growing field, especially with respect to many variants of propositional and modal logics as well as non-classical logics (see e.g. Fitting [Fit83]).

A recent extension to logics is the generalization to teams, i.e., sets of valuations. So-called team based logic allows a more sophisticated expression of facts that regard multiple states of a system simultaneously as well as the internal relationship towards each other. The concepts of team logic originated from the concept of quantifier dependence and independence. The question was simple and long-known in linguistics: How can the statement

For every x there is y, and for every u there is v such that P(x,y,u,v)

be formalized? Suggestions were Henkin’s branching quantifiers [Hen61], the independence-friendly logic IF by Hintikka and Sandu [HS89] or the dependence logic D by Väänänen [Vä07]. Hodges found that a compositional semantics of IF can be formulated with the concept of teams [Hod97], which was adapted by Väänänen together with an atom of dependence, written =((x, y) or dep(x, y) [Vä07, Vää08].
Beside Väänänen’s dependence atom a variety of atomic formulas solely for the reasoning in teams were introduced. Galliani and others found a connection to database theory; they defined common constraints like independence ⊥, inclusion ⊆ and exclusion | in the framework of team semantics [Gal12, GV13]. Beside first-order logic, all these atoms were also adapted for modal logic \( \mathcal{ML} \) [Vää08] and recently propositional logic \( \mathcal{PL} \) [Yan14].

As for any logic system, the question of axiomatizability arose. After all, team logics enable reasoning about sets of valuations, and predicate logic with set quantifiers (\( \mathcal{SO} \)) is not axiomatizable. An important connection between these two was found by Väänänen when he showed that dependence logic \( \mathcal{D} \) is as powerful as existential second-order logic \( \mathcal{SO}(\exists) \) [Vä07], and its extension \( \mathcal{TL} \) (where a semantical negation \( \sim \) is provided) is even equivalent to full second-order logic \( \mathcal{SO} \) [KN09]; therefore both are non-axiomatizable. Later Kontinen and Väänänen showed that there is a partial axiomatization in the sense that \( \mathcal{FO} \) consequences of \( \mathcal{D} \) formulas are derivable [KV13].

For many weaker team logics the question of complete axiomatizability is open. Exceptions are certain fragments of propositional and modal team logic. They were axiomatized by Sano and Virtema [SV15] and Yang [Yan14], but however these solutions rely on the absence of Boolean negation. For dependence atoms itself there are Armstrong’s axioms, Kontinen gives a nice overview for axioms of dependence and independence atoms [Kon13].

**Contribution**

In this paper the dependence atom \( =() \) is identified as the source of incompleteness of \( \mathcal{D} \) and \( \mathcal{TL} \), by giving a complete set of axioms for the \( =() \)-free fragment of \( \mathcal{TL} \), or to be more precise, for the variant with lax semantics as defined by Galliani [Gal12]. This means that reasoning about teams can in fact be axiomatized; but only if we cannot talk about the internal dependencies between the team members.

A crucial step in the completeness proof is the surprising fact that \( \mathcal{TL} \) without \( =() \) collapses to \( \mathcal{B}(\mathcal{FO}) \), the team-semantical Boolean closure of classical first-order logic \( \mathcal{FO} \). Whether logics strictly stronger than \( \mathcal{B}(\mathcal{FO}) \) have axiomatizations is beyond the scope of this paper, but the result itself is still useful: The propositional and modal team logics \( \mathcal{PTL}, \mathcal{QPTL} \) and \( \mathcal{MTL} \) can define their dependence atoms via other operators (which is not possible in \( \mathcal{FO} \)). Therefore they collapse to \( \mathcal{B}(\mathcal{PL}) \) and \( \mathcal{B}(\mathcal{ML}) \) in a similar fashion and we obtain complete axiomatizations as a byproduct.

The paper is built as follows: After reminding the reader of several foundational definitions (Section 2), a complete axiomatization for the team-semantical Boolean closure \( \mathcal{B}(\mathcal{L}) \) of classical logics is presented (Section 3), where \( \mathcal{L} \in \{ \mathcal{PL}, \mathcal{QBFL}, \mathcal{ML}, \mathcal{FO} \} \). The collapses to \( \mathcal{B}(\mathcal{L}) \) are then proven step-wise by axiomatizing the elimination of splitting (Section 4), modalities (Section 5) and finally team-semantical quantifiers (Section 7). Section 6 points out some differences between first-order logic and propositional or modal logic that are relevant for the axiomatization.
2. Preliminaries

2.1. Classical logics

We define a logic as a triple $\mathcal{L} = (\Phi, \mathfrak{A}, \models)$. The set $\Phi$ is a countable set consisting of finite words over some alphabet $\Sigma$, the so-called formulas of $\mathcal{L}$. The set $\mathfrak{A}$ contains possible valuations of formulas in $\Phi$, and the binary relation $\models$ is the truth or satisfaction relation between $\mathfrak{A}$ and $\Phi$. To distinguish between different satisfaction relations we sometimes write $\models_{\mathcal{L}}$. The letter $\mathcal{F}$ instead of $\mathcal{L}$ will often be used to emphasize a flat logic as opposed to non-flat team logics, where flat means that the usual team semantics and classical semantics coincide in a certain way made explicit in this paper.

We write $\varphi \models \psi$ if $A \models \varphi$ implies $A \models \psi$ for all $A \in \mathfrak{A}$. If $\Psi \subseteq \Phi$, then $A \models \Psi$ means $A \models \psi$ for all $\psi \in \Psi$. $\Psi \models \varphi$ means that $A \models \varphi$ for all valuations $A \in \mathfrak{A}$. Finally, the shortcuts $\varphi \equiv \psi$ and $\Psi \equiv \Theta$ mean that $\varphi \models \psi$ and $\psi \models \varphi$, respectively $\Psi \models \Theta$ and $\Theta \models \varphi$.

For brevity we will also write $\varphi \in \mathcal{L}$ instead of $\varphi \in \Phi$.

Define the satisfying valuations, or models, of $\varphi$ as $\text{Mod}(\varphi) := \{ A \in \mathfrak{A} \mid A \models \varphi \}$. If two logics $\mathcal{L}, \mathcal{L}'$ share the same set of valuations, then $\mathcal{L} \leq \mathcal{L}'$ means that for every $\varphi \in \mathcal{L}$ there is a $\varphi' \in \mathcal{L}'$ such that $\text{Mod}(\varphi) = \text{Mod}(\varphi')$. $\mathcal{L} \equiv \mathcal{L}'$ means $\mathcal{L} \leq \mathcal{L}'$ and $\mathcal{L}' \leq \mathcal{L}$.

In contrast $\mathcal{L} \subseteq \mathcal{L}'$ will mean that $\mathcal{L}$ has a subset of the formulas of $\mathcal{L}'$, but the valuations and truth on the common formulas are identical. Then $\mathcal{L}$ is a fragment of $\mathcal{L}'$.

Propositional logic

A foundational part of logic is the propositional logic $\mathcal{P}\mathcal{L}$. It is built on a collection $\mathcal{P}\mathcal{S}$ of propositional statements, or atomic propositions. The propositions in $\mathcal{P}\mathcal{S}$ are denoted with latin letters $\{a, b, c, \ldots\}$. $\mathcal{P}\mathcal{S}$ shall be countable infinite, so we can also write $\{x_1, x_2, \ldots\}$.

The set of $\mathcal{P}\mathcal{L}$ formulas is obtained by closing $\mathcal{P}\mathcal{S}$ under Boolean connectives: Every $x \in \mathcal{P}\mathcal{S}$ is a valid formula, and if $\alpha$ and $\beta$ are valid formulas, then so are $(\alpha \rightarrow \beta)$ and $(\neg \alpha)$. Define $\top := (\alpha \rightarrow \alpha)$ for some fixed formula $\alpha$, and $\bot := \neg \top$.

The set of quantified Boolean formulas, $\mathcal{Q}\mathcal{B}\mathcal{F}$, is defined as the closure of $\mathcal{P}\mathcal{S}$ under $\neg, \rightarrow$ and propositional quantification: If $\alpha \in \mathcal{Q}\mathcal{B}\mathcal{F}$ and $x \in \mathcal{P}\mathcal{S}$, then $\forall x \alpha \in \mathcal{Q}\mathcal{B}\mathcal{F}$.

The valuations of $\mathcal{P}\mathcal{L}$ and $\mathcal{Q}\mathcal{B}\mathcal{F}$ consist of Boolean functions $f : \mathcal{P}\mathcal{S} \rightarrow \{0, 1\}$. The truth relation is defined inductively. For $x$ in $\mathcal{P}\mathcal{S}$ we define $f \models x$ as $f(x) = 1$, further it holds $f \models \neg \alpha$ if $f \not\models \alpha$ and $f \models \alpha \rightarrow \beta$ if $f \models \beta$ or $f \not\models \alpha$. For $\mathcal{Q}\mathcal{B}\mathcal{F}$ also it holds $f \models \forall x \alpha$ if $f \models \alpha[x/\top]$ and $f \models \alpha[x/\bot]$, where $\alpha[\beta/\gamma]$ is the formula that is obtained from $\alpha$ if all occurrences of the subformula $\beta$ are replaced with $\gamma$.

Modal logic

The static nature of propositional logic was early expanded to comprise modalities like “permitted”, “possible”, “obligatory”, “necessarily” and so on. The standard modal logic
First-order logic

First-order logic $\mathcal{FO}$ consists of terms and formulas over some agreed countable vocabulary $\tau = (R_1, R_2, \ldots; f_1, f_2, \ldots; c_1, c_2, \ldots)$. The $R_i$ are relation symbols and the $f_i$ are function symbols, each with their own respective arity $\text{ar}(R_i) \in \mathbb{N}$, $\text{ar}(f_i) \in \mathbb{N}$. The $c_i$ are constants. Also there is a countable infinite set $\text{Var} := \{x_1, x_2, \ldots\}$ of first-order variables. Every variable $x \in \text{Var}$ and every constant $c \in \tau$ is a $\tau$-term. If $t_1, \ldots, t_n$ are $\tau$-terms and $f \in \tau$ is an $n$-place function symbol, then $f(t_1, \ldots, t_n)$ is a $\tau$-term.

An atomic $\tau$-formula is either one of $t_1 = t_2$ or $R(t_1, \ldots, t_n)$ for $\tau$-terms $t_1, \ldots, t_n$, and $n \in \mathbb{N}$, where $R \in \tau$ is an $n$-place relation symbol. Lastly, if $\alpha, \beta$ are $\tau$-formulas, then so are $(\alpha \rightarrow \beta)$, $(\neg \alpha)$ and $(\forall x \alpha)$.

A first-order $\tau$-structure $\mathcal{A} = (A, \tau^A)$ is a non-empty domain $A$ with interpretation for each $\tau$-symbol. The shortcut $\tau^A$ stands for $R_1^A, R_2^A, \ldots; f_1^A, f_2^A, \ldots; c_1^A, c_2^A, \ldots$ and so on. It holds $c^A \in A$ for each constant $c \in \tau$, $f^A: \text{ar}(f) \to A$ for each function $f \in \tau$, and $R^A \subseteq A^{\text{ar}(R)}$ for each relation $R \in \tau$. A first-order $\tau$-interpretation is a pair $(A, s)$ of a $\tau$-structure $\mathcal{A} = (A, \tau^A)$ and an assignment $s: \text{Var} \to A$. For arbitrary $\tau$-terms $t$ then inductively define $t(A, s) \in A$ as follows. If $t \in \text{Var}$, then $t(A, s) := s(t)$. If $t = c$ for a constant $c$, then $t(A, s) := c^A$. If $t = f(t_1, \ldots, t_n)$, then $t(A, s) := f^A(t_1(A, s), \ldots, t_n(A, s))$.

The logic $\mathcal{FO}[\tau]$ is then defined as the logic of $\tau$-formulas with all $\tau$-interpretations as valuations. The satisfaction relation $\models$ is defined as follows. $(A, s) \models t_1 = t_2$ if $t_1(A, s) = t_2(A, s)$, and $(A, s) \models R(t_1, \ldots, t_n)$ if $t(A, s) \in R^A$. The Boolean connectives work as expected. $(A, s) \models \forall x \varphi$ if for all $a \in A$ it holds that $(A, s^a) \models \varphi$, where $s^a$ maps $a$ to $a$ but otherwise maps $y \in \text{Var} \setminus \{x\}$ to $s(y)$.

If the exact vocabulary $\tau$ does not matter we usually drop it and write only $\mathcal{FO}$.

2.2. Team logics

Let $\mathcal{L}$ be a logic. We introduce two new operators to $\mathcal{L}$: The strong negation $\sim$ and the material implication $\rightarrow$. W.l.o.g. they do not occur in formulas of $\mathcal{L}$. The logic $\mathcal{B}(\mathcal{L})$ is the closure of $\mathcal{L}$ under $\sim$ and $\rightarrow$. We say $\mathcal{B}(\mathcal{L})$ is the Boolean closure of $\mathcal{L}$. We further use the symbol $\bot$ (strong falsum), $\bot := \sim(\psi \rightarrow \psi)$. We abbreviate $(\varphi \otimes \psi) := (\sim \varphi \rightarrow \psi)$,
\((\varphi \otimes \psi) := \sim (\varphi \rightarrow \sim \psi)\) and \((\varphi \otimes \psi) := (\varphi \rightarrow \psi) \otimes (\psi \rightarrow \varphi)\). The semantics of \(B(L)\) extend \(L\) by \(A \models \sim \varphi \leftrightarrow A \models \varphi\) and \(A \models \varphi \rightarrow \psi \leftrightarrow A \models \varphi \lor A \models \psi\). The operators are truth functional or extensional, their truth depends only on the truth of subformulas on the same valuation.

If \(L = (\Phi, \mathfrak{A}, \models)\) and \(\sigma: \mathfrak{A} \rightarrow \mathcal{P}(\mathfrak{A} \times \mathfrak{A})\), then call \(\sigma\) a splitting function of \(L\). The splitting team logic \(S_{\sigma}(L)\) is the closure of \(L\) under \(\sim\), \(\rightarrow\) and the binary operator \(\sim_{\sigma}\), which we call linear implication. At this point we follow Väänänen \[Vä07\] to avoid confusion with the propositional operators. Abbreviate \(\varphi \otimes_{\sigma} \psi := \sim (\varphi \rightarrow \sim_{\sigma} \psi)\). These operators are not truth-functional: For a valuation \(A \in \mathfrak{A}\) it is \(A \models \varphi \rightarrow \sim_{\sigma} \psi\) if for all \((A_1, A_2) \in \sigma(A)\) it is the case that \(A_1 \models \varphi\) implies \(A_2 \models \psi\). \((A_1, A_2)\) is called splitting or division of \(A\). Hence for any logic \(L\), given a splitting function \(\sigma\), we can extend its syntax and semantics to \(S_{\sigma}(L)\) as above. If the function \(\sigma\) is known from the context then we drop the index. The fragment \(S^+(L) \subseteq S(L)\) is the closure of \(L\) under \(\otimes, \varnothing\) and \(\otimes\) only.

The quantified team logic \(Q(L)\) is additionally closed under team quantification. We use the quantifiers \(\forall x\) and \(!x\) (shriek). If \(x\) is a variable and \(\varphi, \psi \in Q(L)\), then the formulas \(\sim \varphi, \varphi \rightarrow \psi, \psi \rightarrow \varphi, !x \varphi, \forall x \varphi\) are all in \(Q(L)\). Abbreviate \(\exists x \varphi := \sim !x \sim \varphi\).

Finally define modal team logic \(M(L)\) as the closure of \(L\) under \(\sim, \rightarrow, \otimes\) and the unary modalities \(\Box\) and \(\Delta\). Abbreviate \(\Diamond := \sim \Delta \sim\).

We drop parentheses according to the usual precedence rules; further we assume \(\rightarrow, \sim\) and \(\otimes\) as right-associative and the shortcuts \(\land, \lor, \varnothing, \otimes\) as left-associative.

Propositional team logic \(PTL\) is the logic of \(S(PL)\)-formulas. A valuation of \(PTL\) is a team \(T\) which is a (possibly empty) set of propositional assignments \(s: PS \rightarrow \{0, 1\}\). If \(\varphi \in PL\) then \(T \models \varphi\) if \(s \models \varphi\) for all \(s \in T\) in \(PL\) semantics. A split of a team \(T\) is simply a pair \((S, U)\) such that \(S \cup U = T\).

Modal team logic \(ML\) is the logic of \(M(ML)\) formulas, which is interpreted on Kripke structures. However valuations are not pointed Kripke structures \((K, w)\), but instead have the form \((K, T)\), where \(T \subseteq W\) is called team (of worlds). For \(\varphi \in ML\) it holds \((K, T) \models \varphi\) if \((K, w) \models \varphi\) in \(ML\) semantics for all \(w \in T\). A split of \((K, T)\) is a pair \(((K, S), (K, U))\) such that \(S \cup U = T\).

A successor team \(T'\) of a team \(T\) in \(K\) is a team such that every \(w \in T\) has a successor \(v \in T'\), but also every \(v \in T'\) has a predecessor \(w \in T\). A team has a successor team if and only if every of its worlds has at least one successor world, but every team \(T\) has a unique global successor \(R[T]\) which just consists of all successors of all worlds \(w \in T\). Using these terms we say \((K, T) \models \Box \varphi\) if \((K, R[T]) \models \varphi\) and \((K, T) \models \Delta \varphi\) if for all successor teams \(T'\) of \(T\) it holds \((K, T') \models \varphi\).

Next we define the semantics of \(Q(FO)\). In first-order team logic, an interpretation \((A, T)\) consists of a first-order structure \(A = (A, \tau^A)\) with a team \(T\) of assignments \(s: \text{Var} \rightarrow A\). As for the other logics define \((A, T) \models \varphi\) for \(\varphi \in FO\) if \((A, s) \models \varphi \forall s \in T\) in \(FO\) semantics. Similarly a split of \((A, T)\) is a pair \(((A, S), (A, U))\) with \(S \cup U = T\).

\footnote{Obviously this adds nothing to logics which already have Boolean conjunction and negation. Team logics however do not have negation a priori.}
A function $f: T \rightarrow \mathcal{P}(A) \setminus \emptyset$ is a supplementing function, and the corresponding supplementing team $T[f/x]$ of $T$ is $T[f/x] := \{ s_a^x | s \in T, a \in f(s) \}$. If $\forall s f(s) = A$ then write short $T[A/x]$ and call $T[A/x]$ the duplicating team.

Then $(A,T) \models \forall x\varphi$ if $(A,T[A/x]) \models \varphi$ and $(A,T) \models ! x\varphi$ if $(A,T[f/x]) \models \varphi$ for all supplementing functions $f: T \rightarrow \mathcal{P}(A)$.

Finally, the logic $\mathcal{QPTL}$ is defined as the logic of $\mathcal{Q}(\mathcal{QBF})$ formulas with similar semantics of team quantifiers as $\mathcal{Q}(\mathcal{FO})$, but the supplementing functions have the domain $\mathcal{P}\{(0,1)\} \setminus \emptyset$, and the duplicating function is $f(x) := \{0,1\}$. The valuations of $\mathcal{QPTL}$ and their splitting function are the same as for $\mathcal{PTRL}$.

In the following we reserve the letters $\alpha, \beta, \gamma, \ldots$ for classical $\mathcal{PL}, \mathcal{QBF}, \mathcal{ML}, \mathcal{FO}$ formulas; and we use $\varphi, \psi, \vartheta, \ldots$ for general, team-logical formulas.

### 2.3. Proof systems

Proof systems or calculi are connected to the so-called Entscheidungsproblem, the problem of algorithmically deciding if a given formula $\varphi$ of a logic $\mathcal{L}$ is valid. Formally a proof system is a tuple $\Omega = (\Xi, \Psi, I)$ where $\Xi$ is a set of formulas, $\Psi \subseteq \Xi$ is a set of axioms, and $I \subseteq \mathcal{P}(\Xi) \times \Xi$ is a set of inference rules. $\Xi, \Psi$ and $I$ are all countable and decidable.

An $\Omega$-proof $\mathcal{P}$ from a given set of premises $\Phi \subseteq \Xi$ is a finite sequence $\mathcal{P} = (P_0, \ldots, P_n)$ of finite sets $P_i \subseteq \Xi$ such that $\zeta \in P_i$ implies $\zeta \in P_{i-1} \cup \Psi \cup \Phi$ or $I(P_{i-1}, \zeta)$ for some $P'_{i-1} \subseteq P_{i-1}$.

We say that $\mathcal{P}$ proves or derives a formula $\varphi$ from $\Phi$ if $\varphi \in P_n$ and $\mathcal{P}$ is an $\Omega$-proof from $\Phi$. We write $\Phi \vdash_{\Omega} \varphi$ if there is some $\Omega$-proof that proves $\varphi$ from $\Phi$. If $\Omega$ is clear then we just write $\Phi \vdash \varphi$. If two formulas $\varphi$ and $\varphi'$ prove each other, i.e., $\{\varphi\} \vdash \varphi'$ and $\{\varphi'\} \vdash \varphi$, then we write $\varphi \vdash \varphi'$. For sets write $\Phi \vdash \Phi'$ if for every $\varphi \in \Phi$ it holds $\varphi' \vdash \varphi$, and for every $\varphi' \in \Phi'$ it holds $\Phi \vdash \varphi'$.

A calculus $\Omega$ is sound for a logic $\mathcal{L}$ if for $\Phi \subseteq \mathcal{L}, \varphi \in \mathcal{L}$ it holds that $\Phi \vdash_{\Omega} \varphi$ implies $\Phi \models_{\mathcal{L}} \varphi$, and it is complete if conversely $\Phi \models_{\mathcal{L}} \varphi$ implies $\Phi \vdash_{\Omega} \varphi$. We say $\Omega'$ is stronger than $\Omega$, in symbols $\Omega' \succeq \Omega$, if $\Phi \vdash_{\Omega} \varphi$ implies $\Phi \vdash_{\Omega'} \varphi$. Clearly if $\Omega'$ is sound, then $\Omega$ is sound, and if $\Omega$ is complete, then $\Omega'$ is complete. The union of two systems $\Omega, \Omega'$ is defined as component-wise union and just written $\Omega \Omega'$.

One well known family of sound and complete calculi are the manifold flavors of Hilbert-style deductive systems. They have in general a minimum of deductive rules, the most prominent being the modus ponens $\{\alpha, \alpha \rightarrow \beta\} \vdash \beta$, but often multiple axioms. Different systems with no axioms but many deductive rules are Gentzen’s natural deduction $\mathcal{NK}$ and his sequent calculus $\mathcal{LK}$.

The proof systems in this paper are based on Hilbert-style textbook axiomatizations of propositional and first-order logic, see Figure 1. We call $H^0$ the system of $\mathcal{PL}$ which has as axioms all substitution instances of $(A1)$–$(A3)$ with the rule $(E\rightarrow)$. A substitution...
\begin{align*}
\text{(A1)} & \quad \alpha \to (\beta \to \alpha) \\
\text{(A2)} & \quad (\alpha \to (\beta \to \gamma)) \to (\alpha \to \beta) \to (\alpha \to \gamma) \\
\text{(A3)} & \quad (\neg \alpha \to \neg \beta) \to (\beta \to \alpha) \\
\text{(K)} & \quad \Box (\alpha \to \beta) \to (\Box \alpha \to \Box \beta) \\
\text{(A4)} & \quad \forall x \alpha \to \alpha[x/t], \text{ } t \text{ term} \\
\text{(A5)} & \quad \forall x(\alpha \to \beta) \to (\alpha \to \forall x \beta), \text{ } x \text{ not free in } \alpha \\
\text{(A6)} & \quad x = x \\
\text{(A7)} & \quad x = y \to (\alpha \to \alpha[x/y]) \\
\text{(E→)} & \quad \frac{\alpha \alpha \to \beta}{\beta} \\
\text{(Nec)} & \quad \frac{\alpha}{\Box \alpha} \text{ (} \alpha \text{ theorem) } \\
\text{(UG∀)} & \quad \frac{\alpha}{\forall x \alpha(t/x)} \text{ (} \alpha \text{ theorem, } t \text{ term)}
\end{align*}

Figure 1: Hilbert-style axiomatizations of \( \mathcal{PL} \), \( \mathcal{ML} \) and \( \mathcal{FO} \)

instance here means a formula resulting from a given axiom by replacing the letters \( \alpha, \beta, \gamma, \ldots \) with arbitrary formulas (but all occurrences of \( \alpha \) with the same formula, and likewise for \( \beta, \gamma, \ldots \)).

Write \( H^\Box \) for the system of \( \mathcal{ML} \) with the axioms (A1)–(A3), (K) and the rules (E→) and (Nec). Write \( H \) for the system of \( \mathcal{FO} \) with axioms (A1)–(A7) and the rules (E→) and (UG∀).

The inference rules (Nec) and (UG∀) require explanation. In contrast to the modus ponens (E→) they cannot be applied to arbitrary derived formulas, but only to theorems \( \alpha \). That \( \alpha \) is a theorem means that it was derived from only the axioms of the proof system and not from the given premises \( \Phi \). Consider the necessitation rule (Nec). Surely if \( \alpha \) is a tautology, then \( \Box \alpha \) is, but \( \alpha \) alone does not entail \( \Box \alpha \). This is the “failure of the deduction theorem” which is discussed e.g. by Hakli and Negri [HN12], Fitting [Fit07]. There are several ways to remedy it, one is exactly the restriction of \( \alpha \) to be a theorem, i.e., no local assumptions from \( \Phi \) were used for deriving the given formula.\(^3\) This restores the deduction theorem for \( \mathcal{H}^\Box \) and \( \mathcal{H} \), for a detailed proof see Hakli and Negri [HN12].

For the classical logics \( \mathcal{PL}, \mathcal{ML}, \mathcal{FO} \) it holds:

**Theorem 2.1.** \( \mathcal{H}^0 \) is sound and complete for \( \mathcal{PL} \). \( \mathcal{H}^\Box \) is sound and complete for \( \mathcal{ML} \). \( \mathcal{H} \) is sound and complete for \( \mathcal{FO} \).

We defined classical logics to have the flatness property, i.e., a formula is satisfied by a team \( T \) in team semantics exactly when all of \( T \)'s members satisfy it in classical semantics. The team semantics of the respective logics are the semantics defined earlier

\(^3\)To encode this formally, we can for instance use as judgement set \( \mathcal{L} \cup \mathcal{L} \times \{0,1\} \), where \((\alpha,1)\) counts as axiom or theorem and \((\alpha,0)\) does not, and the 1-bit is maintained until a “0-formula” is used.
in this section, i.e., with valuations generalized to teams of assignments resp. worlds. The flatness property itself has an important consequence regarding axiomatizability.

**Theorem 2.2.** Let $\mathcal{F} \in \{\mathcal{PL}, \mathcal{QBF}, \mathcal{ML}, \mathcal{FO}\}$, $\Gamma \subseteq \mathcal{F}, \alpha \in \mathcal{F}$. Then $\Gamma \models \alpha$ holds in team semantics if and only if it holds in classical semantics.

**Proof.** The equivalence follows from the flatness property, which again follows directly from the defined semantics. We prove only the $\mathcal{FO}$ case, the other cases are proven similar. Assume $\Gamma \models \alpha$ holds in classical semantics. Then for every interpretation $(A, s)$ we have that $(A, s) \models \Gamma$ implies $(A, s) \models \alpha$. Let now $(B, T)$ be a valuation which satisfies $\Gamma$ in team semantics. This means that $(B, s) \models \Gamma$ for all $s \in T$, so $(B, s) \models \alpha$ in for all $s \in T$. Thus $(B, T) \models \alpha$ in team semantics.

The other direction is proven by contraposition. If $\Gamma \not\models_{\mathcal{FO}} \varphi$, then there is a valuation $(A, s)$ such that $(A, s) \not\models_{\mathcal{FO}} \Gamma$, $(A, s) \not\models_{\mathcal{FO}} \varphi$. But then also $(A, \{s\}) \not\models_{\mathcal{TL}} \Gamma$ and $(A, \{s\}) \not\models_{\mathcal{TL}} \varphi$, so $\Gamma \not\models_{\mathcal{TL}} \varphi$. ☐

**Corollary 2.3.** In team semantics the calculi $H^0$, $H^\Box$ and $H$ are sound and complete for $\mathcal{PL}, \mathcal{ML}$ and $\mathcal{FO}$.

Other immediate consequences of flatness are the following, where $\mathcal{F} \in \{\mathcal{PL}, \mathcal{ML}, \mathcal{FO}\}$:

**Theorem 2.4** (Downward closure). If $A \models \alpha$ for a formula $\alpha \in \mathcal{F}$, then $A_1 \models \alpha$ and $A_2 \models \alpha$ for all divisions $(A_1, A_2)$ of $A$.

**Theorem 2.5** (Union closure). If $A_1 \models \alpha$ and $A_2 \models \alpha$ for a formula $\alpha \in \mathcal{F}$, then $A \models \alpha$ for all $A$ which have a division into $(A_1, A_2)$.

**Theorem 2.6** (Flatness of $\otimes$). For all $\alpha, \beta \in \mathcal{F}$ it holds $A \models \alpha \otimes \beta$ if and only if $A \models \alpha \otimes \beta$.

3. Axioms of the Boolean closure

We begin the development of a proof system for team logic with the operators $\rightarrow$ and $\sim$. They are purely truth-functional; hence we can only reason about Boolean combinations of classical formulas. The presented calculus for this is the system Lift or just $L$ of lifted propositional axioms, shown in Figure 2. The axioms of $L$ just reflect their propositional counterparts in $H^0$, with exception of (L4) which relates the propositional and the material implication. In this section it is shown how complete proof systems for flat logics $\mathcal{F}$ can be augmented with $L$ to obtain a complete system for $\mathcal{B}(\mathcal{F})$.

For the team-logical material implication $\rightarrow$ a new *modus ponens* inference rule is introduced. While the systems $H^0$, $H^\Box$ and $H$ can only be applied to classical formulas $\alpha, \beta, \gamma, \ldots$, i.e., where no team logical operators occur, the axioms and rules in $L$ are permitted for general team-logical formulas $\varphi, \psi, \vartheta, \ldots$. 

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3.1. The deduction theorem for team logics

We begin the proof of completeness of $L$ with a team-logical variant of the deduction theorem. The thought behind this strategy is that the deduction theorem implies Lindenbaum’s lemma which allows the construction of a maximal consistent set, the usual method for completeness proofs of propositional axioms. We begin with identifying a family of proof systems which guarantee a deduction theorem, improving the ideas of Hakli and Negri [HN12].

Definition 3.1. Let $\mathcal{L} = (\Xi, \Psi, I)$ be a calculus. Say that a rule $(\{\xi_1, \ldots, \xi_k\}, \psi) \in I$ has weakening if $\{\varphi \rightarrow \xi_i \mid i \in [n]\} \vdash \varphi \rightarrow \psi$ for all $\varphi \in \Xi$.

In other words, every derivation using an inference rule can also be done under arbitrary assumptions. Say that a calculus $\Omega$ has weakening if all inference rules have weakening.

Lemma 3.2. If $\Omega \succeq L$ and $\Omega$ has weakening then it has the deduction theorem: $\Phi \vdash (\varphi \rightarrow \psi)$ if and only if $\Phi \cup \{\varphi\} \vdash \psi$.

Proof. The direction from left to right is clear as $L$ has $(E\rightarrow)$. For right to left we do an induction over the length of a shortest proof of $\psi$. If $\psi \in \Phi$, $\psi = \varphi$, or if $\psi$ is an axiom, then by $(L1)$ and $(E\rightarrow) \Phi \vdash (\varphi \rightarrow \psi)$. For $n = 1$ these are the only cases. Let $n > 1$. Then $\psi$ could be obtained by application of some inference rule $(\{\xi_1, \ldots, \xi_k\}, \psi)$. $\xi_1, \ldots, \xi_k$ all have a proof of length $\leq n - 1$ from $\Phi \cup \{\varphi\}$, so by induction hypothesis $\Phi \vdash \varphi \rightarrow \xi_i$ for $i \in [k]$. By weakening $\Phi \vdash \varphi \rightarrow \psi$. 

Lemma 3.3 (Deduction theorem of $L$). If $\Omega \succeq L$ and all inference rules of $\Omega$ except $(E\rightarrow)$ and $(E_\psi)$ produce only theorems, then $\Omega$ has the deduction theorem.

Proof. By Lemma 3.2 we can just show that $\Omega$ has weakening. If an inference rule produces only theorems $\psi$ then $\psi$ is provable without assumptions, and so by $(L1)$ we can prove $\xi \rightarrow \psi$ for any $\xi$. The rule $(E\rightarrow)$ is $(\{\varphi, \varphi \rightarrow \psi\}, \psi)$. From the assumptions $\xi \rightarrow (\varphi \rightarrow \psi)$ and $\xi \rightarrow \varphi$ derive $\xi \rightarrow \psi$ by $(L2)$. For $(E\rightarrow)$ weakening is shown in the following derivation.

Figure 2: Hilbert-style axiomatization $L$ of $B(\mathcal{F})$
3.2. Completeness of the Boolean closure

The typical textbook proof of completeness of propositional or first-order logic uses Lindenbaum’s lemma to construct a maximal consistent set. For this we need the notion of inconsistency first.

Definition 3.4. Let \( \Omega = (\Xi, \Psi, I) \) be a proof system. A set \( \Phi \) is \( \Omega \)-inconsistent (or just inconsistent) if \( \Phi \vdash \Xi \). \( \Phi \) is \( \Omega \)-consistent (or just consistent) if it is not \( \Omega \)-inconsistent.

Lemma 3.5. Let \( \Omega \succeq L \). The following statements are equivalent:

1. \( \Phi \vdash \varphi \) and \( \Phi \vdash \sim \varphi \) for some \( \varphi \),
2. \( \Phi \) is inconsistent,
3. \( \Phi \vdash \bot \), \( \bot \).

Proof. For 1. \( \Rightarrow \) 2. we have to show \( \Phi \vdash \xi \) for all judgements \( \xi \). First \( \Phi \vdash (\sim \xi \rightarrow \sim \varphi) \) follows from \( \Phi \vdash \sim \varphi \), (L1) and (E→); by (L3) and (E→) then follows \( \Phi \vdash (\varphi \rightarrow \xi) \), and again by (E→) then \( \Phi \vdash \xi \). 3. is a special case of 2. For 3. \( \Rightarrow \) 1. it suffices to derive \( (\psi \rightarrow \psi) \) by a textbook proof, since \( \bot := \sim (\psi \rightarrow \psi) \).

Lemma 3.6. Let \( \Omega \succeq L \) have weakening and let \( \Phi \) be consistent. Then \( \Phi \not\vdash \varphi \) implies that \( \Phi \cup \{\sim \varphi\} \) is consistent, and \( \Phi \vdash \varphi \) implies that \( \Phi \cup \{\varphi\} \) is consistent.

Proof. If \( \Phi \not\vdash \varphi \) and \( \Phi \cup \{\sim \varphi\} \) was inconsistent, then \( \Phi \vdash \varphi \) and \( \Phi \cup \{\sim \varphi\} \vdash \sim \psi \) for any axiom \( \psi \) and thus by Lemma 3.2 \( \Phi \vdash (\sim \varphi \rightarrow \sim \psi) \). By (L3) then \( \Phi \vdash \psi, \psi \rightarrow \varphi \), so by (E→) \( \Phi \vdash \varphi \), contradiction.

If \( \Phi \vdash \varphi \) and \( \Phi \cup \{\varphi\} \) was inconsistent, then again \( \Phi \vdash \varphi, \varphi \rightarrow \bot \): contradiction to consistency of \( \Phi \) and Lemma 3.5.

Definition 3.7. If \( \Omega = (\Xi, \Psi, I) \) then \( \Phi \subseteq \Xi \) is maximal consistent if it is consistent and contains \( \xi \) or \( \sim \xi \) for every \( \xi \in \Xi \).

Lemma 3.8 (Lindenbaum’s Lemma). If \( \Omega \succeq L \) has the deduction theorem, then every consistent set has a maximal consistent superset.
Proof. Let \( \Phi \) be consistent, \( \Omega = (\Xi, \Psi, I) \). We can write the countable set \( \Xi \) as \\
\( \Xi := \{\xi_1, \xi_2, \ldots\} \).

Let \( \Phi_0 := \Phi \), and for each \( i \geq 1 \) define \( \Phi_i \) as

\[
\Phi_i := \begin{cases} 
\Phi_{i-1} \cup \{\xi_i\} & \text{if } \Phi_{i-1} \vdash \{\xi_i\}, \\
\Phi_{i-1} \cup \{\sim \xi_i\} & \text{otherwise}.
\end{cases}
\]

By Lemma 3.6 the consistency of \( \Phi_{i-1} \) implies that of \( \Phi_i \). Hence by induction all \( \Phi_n \) for \( n \geq 0 \) are consistent. Let \( \Phi^* := \bigcup_{n \geq 0} \Phi_n \). \( \Phi^* \) is again consistent, otherwise it could prove \( \bot \) already from a finite set of assumptions. \( \Phi^* \) is maximal by construction.

The next step in usual completeness proofs is to give an explicit model for maximal consistent sets. The application of Lindenbaum’s lemma is usually as follows: If a set \( \Phi \) is maximal consistent, then there is a model fulfilling all its atomic formulas. By the maximality of \( \Phi \) then one can claim that also all Boolean combinations of atomic formulas, if they are in \( \Phi \), are automatically fulfilled by the model as well. The latter part goes through for the Boolean connectives \( \land, \sim \) as well, but some more work is required for the induction basis — the model satisfying the atomic formulas. In our context, an “atom” is in fact any formula of the underlying classical, flat logic, in this case \( \mathcal{PL}, \mathcal{ML} \) or \( \mathcal{FOL} \). This additional complexity requires the next property as an additional step to completeness.

Definition 3.9. Let \( \mathcal{F} \) be a logic. \( \mathcal{F} \) admits counter-model merging if it has the following property for arbitrary sets \( \Gamma, \Delta \subseteq \mathcal{F} \): If for every \( \delta \in \Delta \) there is a valuation falsifying \( \delta \) and satisfying \( \Gamma \), then there is a valuation falsifying all formulas in \( \Delta \) and satisfying \( \Gamma \).

Lemma 3.10. \( \mathcal{PL} \) and \( \mathcal{ML} \) admit counter-model merging.

Proof. We prove only the \( \mathcal{ML} \) case as \( \mathcal{PL} \) works similar. Let \( \Gamma, \Delta \subseteq \mathcal{ML} \). Assume for each \( \delta \in \Delta \) a Kripke structure \( (\mathcal{K}_\delta, T_\delta) \) that falsifies \( \delta \) and satisfies \( \Gamma \). Define \( \mathcal{K} \) as the disjoint union (see Goranko and Otto [GO07]) of all Kripke structures \( \mathcal{K}_\delta \). Then \( (\mathcal{K}, T_\delta) \models \Gamma \), \( (\mathcal{K}, T_\delta) \not\models \delta \) as \( \mathcal{ML} \) is invariant under disjoint union of structures [GO07] and due to flatness of \( \mathcal{ML} \). Define the team \( \mathcal{T} := \bigcup_{\delta \in \Delta} T_\delta \). As \( \mathcal{ML} \) is union closed (Theorem 2.5), \( (\mathcal{K}, \mathcal{T}) \) satisfies \( \Gamma \), and as it is downwards closed (Theorem 2.4), it falsifies each \( \delta \in \Delta \).

Definition 3.11. A calculus \( \Omega \) is refutation complete for \( \mathcal{L} \) if for every unsatisfiable \( \Phi \subseteq \mathcal{L} \) there is a \( \varphi \) s.t. \( \Phi \vdash \varphi, \sim \varphi \).

Write \( \sim \mathcal{F} \) for the fragment of \( \mathcal{B}(\mathcal{F}) \) restricted to the formulas \( \{ \sim \varphi \mid \varphi \in \mathcal{F} \} \).

Lemma 3.12. If \( \mathcal{F} \) has counter-model merging and \( \Omega \) is complete for \( \mathcal{F} \), then \( \Omega \) is refutation complete for \( \mathcal{F} \cup \sim \mathcal{F} \).

Proof. Let a set \( \Phi \subseteq \mathcal{F} \cup \sim \mathcal{F} \) be unsatisfiable. Abbreviate \( \Gamma := \Phi \cap \mathcal{F} \) and \( \Delta := \Phi \cap \sim \mathcal{F} \).

It is not the case that \( \Gamma \cup \{\sim \delta\} \) is satisfiable for every \( \sim \delta \in \Delta \), because then \( \Phi \) would be satisfiable by counter-model merging. Hence for some \( \sim \delta \in \Delta \) the set \( \Gamma \cup \{\sim \delta\} \) is unsatisfiable, i.e., \( \Gamma \models \delta \). But then \( \Gamma \models \delta \) due to the completeness of \( \Omega \) for \( \mathcal{F} \), so \( \Phi \models \delta, \sim \delta \).
Let us emphasize again the difference to classical logics, say, $\mathcal{PL}$: $\mathcal{PL}$ has $\mathcal{PS}$ as its atoms, and the analogously defined fragment $\mathcal{PS} \cup \lnot \mathcal{PS}$ is trivially “refutation complete”: A set $\Gamma \subseteq \mathcal{PS} \cup \lnot \mathcal{PS}$ is contradictory if and only if contains $p, \lnot p$ for some proposition $p$, so there is nothing to do for a proof system. This is different for team logics.

Not every logic has counter-model merging: The completeness of $\mathcal{B}(\mathcal{FO})$ is shown via a different approach in Section 6.

After the atoms are handled correctly by the proof system (by refutation completeness of $\mathcal{F} \cup \lnot \mathcal{F}$), the induction step goes just for classical logic, and then results in completeness of $\mathcal{B}(\mathcal{F})$.

**Lemma 3.13.** If $\Omega \succeq \mathcal{L}$ is refutation complete for $\mathcal{F} \cup \lnot \mathcal{F}$ and $\Omega$ has the deduction theorem, then $\Omega$ is refutation complete for $\mathcal{B}(\mathcal{F})$.

**Proof.** We must show that every unsatisfiable $\Phi \subseteq \mathcal{B}(\mathcal{F})$ derives $\varphi$ and $\lnot \varphi$ for some $\varphi$, or, equivalently due to Lemma 3.5, that it is inconsistent. We prove for contraposition that every consistent $\Phi \subseteq \mathcal{B}(\mathcal{F})$ has a model.

If $\Phi$ is consistent, then it has a maximal consistent superset $\Phi^*$ by Lemma 3.8. Certainly $\Phi^* \cap (\mathcal{F} \cup \lnot \mathcal{F})$ is consistent as well. By refutation completeness it has a model $\mathcal{A}$. We show that $\psi \in \Phi^* \iff \mathcal{A} \vDash \psi$ for all $\psi \in \mathcal{B}(\mathcal{F})$, then in particular $\Phi$ is satisfiable.

The rest of the proof will be an induction over $|\psi|$. Let $\psi \in \mathcal{F}$. If $\psi \in \Phi^*$, then $\mathcal{A} \vDash \psi$ by definition of $\mathcal{A}$. If $\psi \notin \Phi^*$, then $\lnot \psi \in \Phi^*$ due to the maximality of $\Phi^*$, so $\lnot \psi \in \Phi^* \cap \lnot \mathcal{F}$, and again $\mathcal{A} \vDash \lnot \psi$ by the definition of $\mathcal{A}$, hence $\mathcal{A} \nvDash \psi$ by definition of $\lnot$.

The induction step $\psi = \lnot \theta$ is clear due to the consistency and maximality of $\Phi^*$. So let $\psi = \psi_1 \rightarrow \psi_2$. Assume $\psi \in \Phi^*$. Then either $\psi_1 \notin \Phi^*$ or $\lnot \psi_2 \notin \Phi^*$, otherwise by modus ponens $\Phi^*$ is inconsistent. But then either $\mathcal{A} \nvDash \psi_1$ or $\mathcal{A} \nvDash \psi_2$ by induction hypothesis, hence $\mathcal{A} \nvDash \psi_1 \rightarrow \psi_2$.

If $\psi \notin \Phi^*$, then $\lnot \psi \in \Phi^*$. If now $\mathcal{A} \vDash \psi_2$, then $\psi_2 \in \Phi^*$ by induction hypothesis. From $\psi_2$ we can derive $\psi$ via (L1). If $\mathcal{A} \nvDash \psi_1$, then $\lnot \psi_1 \in \Phi^*$. From $\lnot \psi_1$ we can infer $\lnot \psi_2 \rightarrow \lnot \psi_1$ again with (L1) and by contraposition (L3) we obtain the conditional $\psi$. But in both cases $\Phi$ would then be inconsistent, so $\mathcal{A} \nvDash \psi_1$ and $\mathcal{A} \nvDash \psi_2$, hence $\mathcal{A} \nvDash \psi_1 \rightarrow \psi_2$. 

**Theorem 3.14** (Completeness of $\mathcal{L}$, I). If $\Omega \mathcal{L}$ is refutation complete for $\mathcal{F} \cup \lnot \mathcal{F}$ and has weakening, then it is complete for $\mathcal{B}(\mathcal{F})$.

**Proof.** Let $\Phi \subseteq \mathcal{B}(\mathcal{F})$ and $\varphi \in \mathcal{B}(\mathcal{F})$. We have to show that from $\Phi \vDash \varphi$ it follows $\Phi \vdash \varphi$. Assume for contraposition that $\Phi \nvDash \varphi$. Then $\Phi$ is consistent by definition and due to Lemma 3.6 so is $\Phi \cup \{\lnot \varphi\}$. $\Omega \mathcal{L}$ has the deduction theorem due to Lemma 3.2, so by Lemma 3.13 $\Omega \mathcal{L}$ is refutation complete for $\mathcal{B}(\mathcal{F})$. Hence the consistent set $\Phi \cup \{\lnot \varphi\}$ must be satisfiable which implies $\Phi \nvDash \varphi$. 

**Theorem 3.15.** If $\Omega$ is sound for $\mathcal{F}$ and $\Omega$ has $(E\rightarrow)$, then $\Omega \mathcal{L}$ is sound for $\mathcal{B}(\mathcal{F})$.

**Proof.** The proof is done by showing that all axioms and inference rules are sound. By induction on the length of proofs then the soundness of $\Omega \mathcal{L}$ follows. The axioms and
rules of $\Omega$ apply only to $\mathcal{F}$ and are hence sound by assumption. As $(E \rightarrow)$ is sound, it holds $\{\alpha, \alpha \rightarrow \beta\} \models \beta$, but then $\{\alpha \rightarrow \beta\} \models (\alpha \rightarrow \beta)$. The other axioms of $L$ are sound by definition of the strong negation $\sim$ and material implication $\rightarrow$, and so is the $\rightarrow$-modus ponens. Hence $\Omega L$ is sound for $\mathcal{B}(\mathcal{F})$.

**Corollary 3.16.** $H^0L$ is sound and complete for $\mathcal{B}(\mathcal{P}L)$. $H^\ominus L$ is sound and complete for $\mathcal{B}(\mathcal{M}L)$.

**Proof.** Follows from the previous theorem, Corollary 2.3, Lemma 3.3, 3.10 and 3.12 and Theorem 3.14.

The next theorems show that all Boolean tautologies over $\sim, \rightarrow$ are provable in $L$. We can thus for instance prove distributive laws, De Morgan’s laws, double negation elimination and so on without having to know what the respective subformulas exactly mean.

**Definition 3.17.** If $\mathcal{F}$ is a logic, call a set $\mathcal{P} \subseteq \mathcal{B}(\mathcal{F})$ independent if $\sim \mathcal{P} \cup (\mathcal{P} \setminus \mathcal{P}')$ is satisfiable for all $\mathcal{P}' \subseteq \mathcal{P}$.

An example of an independent set is $\mathcal{P}S$.

**Lemma 3.18.** Let $\mathcal{P} \subseteq \mathcal{F}$ be independent and $\varphi \in \mathcal{B}(\mathcal{P})$, i.e., $\varphi$ is a Boolean combination of independent formulas. Then $\models \varphi$ implies $\vdash L \varphi$.

**Proof.** By Lemma 3.3 $L$ has the deduction theorem. $L$ is trivially refutation complete for $\mathcal{P} \cup \sim \mathcal{P}$. If a set $\Gamma \subseteq \mathcal{P} \cup \sim \mathcal{P}$ is unsatisfiable, then $\psi, \sim \psi \in \Gamma$ for some $\psi \in \mathcal{P}$ due to the independence of $\mathcal{P}$. By Lemma 3.13 $L$ is refutation complete for $\mathcal{B}(\mathcal{P})$.

Assume now for the sake of contradiction that $\not\models L \varphi$. Then $\{\sim \varphi\}$ is consistent by Lemma 3.6 and thus satisfiable. Hence $\not\models \varphi$.

A substitution instance of a formula $\varphi \in \mathcal{B}(\mathcal{P}S)$ is a formula $\sigma \varphi \in \mathcal{B}(\mathcal{L})$ where $\sigma : \mathcal{P}S \rightarrow \mathcal{L}$, and $\sigma \varphi$ is obtained from $\varphi$ by replacing in parallel the occurrences of any $p \in \mathcal{P}S$ with $\sigma(p)$.

**Theorem 3.19 (Completeness of $L$, II).** Let $\varphi \in \mathcal{B}(\mathcal{P}S)$. If $\models \varphi$ in team semantics, then $\vdash L \sigma \varphi$ for any substitution instance $\sigma \varphi$ of $\varphi$.

**Proof.** $\mathcal{P}S$ is independent in team semantics. Hence $\models L \varphi$ by Lemma 3.18. The proof of $\vdash L \sigma \varphi$ is by induction over the length of the proof of $\models L \varphi$. If $\varphi$ is an axiom of $L$, then also $\sigma \varphi$ is. If $\varphi$ is obtained by $\psi \rightarrow \varphi, \psi$, then by induction hypothesis $\vdash L \sigma(\psi \rightarrow \varphi), \sigma \psi$. As $\sigma(\psi \rightarrow \varphi) = \sigma \psi \rightarrow \sigma \varphi$, apply $(E \rightarrow)$ to derive $\sigma \varphi$. Note that $\varphi$ cannot be an instance of $(L4)$, as then it would not be a $\mathcal{B}(\mathcal{P}S)$ formula.

Some comments follow to conclude this section. If a set $\mathcal{P} \subseteq \mathcal{B}(\mathcal{F})$ is said to be inconsistent in a system $\Omega$, then $\Omega$ can derive all $\mathcal{B}(\mathcal{F})$ formulas including $\bot$. The strong falsum $\bot$ does not necessarily have to be a valid formula in $\mathcal{F}$, but it is introduced in its Boolean closure $\mathcal{B}(\mathcal{F})$ as the negation of truth.
The ability to derive all formulas is called *absolute inconsistency* by Hilbert; recently Mossakowski and Schröder [MS15] discussed the difference between absolute inconsistency and so-called ⊥-inconsistency, where ⊥-inconsistency just means that ⊥ can be derived. This notions are equivalent if ⊥ is a *proof-theoretic falsum*, i.e., any formula can be proven from it. Mossakowski and Schröder further call a set *Aristotle inconsistent* for a given negation symbol ¬ if α and ¬α can be derived, and ¬ is a *proof-theoretic negation* if ¬α is derivable from α ⊢ ⊥ [MS15]. They distinguish this from *semantic falsity* and negation which behave in the sense of Tarski, defined via satisfaction in a model. In our context ¬ and ⊥ are both semantic and proof-theoretic.

A proof system may have a symbol like ⊥, but it does not necessarily have proof-theoretic or semantic falsity or negation, as it is the case for many team logics. Instead ⊥ is then the absence of valuations in the team. If a logic ℱ is flat, then every set Φ ⊆ ℱ is satisfiable, even if Φ is inconsistent in a proof system which is sound and complete for ℱ. This is the case for the team semantical variants of ℙℒ, ℳℒ and Kℒ, but also for their closures under ⊙, ⊕, ⊗, or to be more general, for any logic with the empty team property in the sense of Väänänen [Va07]. The terms regarding consistency require careful handling: In classical logic the mentioned notions of inconsistency clearly coincide, and so also do inconsistency and unsatisfiability. If the same logics are interpreted in team semantics then we can still derive anything from ⊥, or from ¬α and α, so the three notions of inconsistency still coincide. But every set is satisfiable, so the logics have falsum and negation in the proof-theoretic sense, but not in the semantical sense, otherwise inconsistency would imply unsatisfiability [MS15].

If we take a further step and introduce ⊥ and ∼, then we recover semantic falsity and negation which are at the same time proof-theoretic. However then we cannot infer everything from ⊥ or α, ¬α anymore. Mossakowski and Schröder [MS15] call such falsum resp. negation *paraconsistent*, and they claim that if ¬ is paraconsistent, then ⊥ and ¬ cannot both be proof-theoretic anymore — implying that ¬ and ∼ are not equivalent.

The term “paraconsistent” is a little inappropriate for team logics, as the whole flat fragment ℱ ⊆ ℬ(ℱ) is still derivable from α, ¬α, and paraconsistent logics mean to avoid the explosion principle at all. Here it would be interesting to develop a “stratified” concept of consistency which matches the stratified approach to syntax and semantics for team logics.

4. The axioms of splitting

In the previous sections we considered classical logics in the setting of team semantics, and their closure under the Boolean operators of team logic. With this operators we can express in essence three facts: The existence of certain members in the team, the absence of other members in the team, and further Boolean combinations thereof. The evaluation of such a formula on a team is nothing more than a truth function of the truth values of the classical subformulas.

An essential addition to team semantics is the splitting disjunction ⊗, or sometimes *splitjunction* or tensor. The expression φ ⊗ ψ can be seen as a per-member decision for
either $\varphi$, $\psi$, or, as in the classical disjunction, both. This is called lax semantics. In the strict semantics the two subteams of the division may not overlap; so the strict $\otimes$ is better seen as a member-wise “exclusive or”. In this work we will only consider the (easier) lax semantics as defined in Section 2.

Due to flatness, the splitting disjunction of two classical formulas is equivalent to the their usual disjunction (Theorem 2.6). This however should not distract from the fact that the splitting operator is a set quantification in disguise. Consider propositional team logic with splitting $\otimes$ and negation $\sim$; in terms of computational complexity its validity problem is complete for the class $\text{AEXP}(\text{poly})$: The class of sets decidable by Turing machines with exponential runtime and polynomial number of alternations between existential und universal quantification $[\text{Vir15}]$.

The non-truth-functional nature of splitting disjunction is an obstacle to axiomatizability; the key however is to consider it as a special type of modality. Necessity and possibility as the classical non-truth-functional operators have long been successfully axiomatized.

This section aims to prove the following theorem:

**Theorem 4.1.** Every $\mathcal{PTL}$ formula is provably equivalent to a $\mathcal{B}(\mathcal{P\mathcal{L}})$ formula.

In other words, formulas applying the splitting operator on non-classical formulas can always be reduced to just a Boolean combination of classical formulas. Of course already by complexity theoretic reasons the equivalent formulas may be very long.

It was shown by Yang $[\text{Yan14}]$ that $\mathcal{PTL}$ formulas are equivalent to $\sim$-free formulas except the atom of non-emptiness ($\text{NE} := \sim \bot$). However her argumentation is model-theoretic and not syntactical. Indeed her axiomatization works only on this special case, i.e., $S^+(\mathcal{PL} \cup \text{NE})$. She argues that the axiomatization of $S^+(\mathcal{PL} \cup \text{NE})$ should be easier for the reason that non-emptiness is easier to handle than semantical negation. An oversight of her is however that this restriction demands a rather complicated set of rules for many special cases, whereas the permission of Boolean operators allows a very natural axiomatization of $\mathcal{B}(\mathcal{PL})$ itself which is similar to the axioms of classical propositional logic. The axiom system $S$ shown in Figure 3 then suffices to reduce splitting disjunctions to Boolean combinations of classical formulas.

Note that the matter is not so easy for strict splitting semantics. For instance the formula $\text{NE} \otimes \sim(\text{NE} \otimes \text{NE})$ is true in strict semantics if and only if the team contains exactly one element.

**Theorem 4.2.** There is no finite set $\Phi \subseteq \mathcal{B}(\mathcal{PL})$ that is equivalent to $\text{NE} \otimes \sim(\text{NE} \otimes \text{NE})$ in strict semantics.

**Proof.** Assume for the sake of contradiction that there was some finite $\Phi \subseteq \mathcal{B}(\mathcal{PL})$ as above. W.l.o.g. the variable $x$ does not occur in $\Phi$. Let $T \models \Phi$, then $T = \{s\}$ for some assignment $s$. It can be easily shown by induction that $\{s_0^x, s_1^x\} \models \Phi$, where $s_c^x(x) = c$ and $s_c^x(y) = s(y)$ for $x \neq y$. \qed
One remark about the naming of the "necessitation" rule in Figure 3. This rule is similar to the deductive rule in $H^\Box$. In the context of teams, a subteam can as well be seen as a type of "other world". We cannot, as typical for modal logics, derive knowledge about $\psi$ in a subteam from knowledge about $\psi$ in the current team. Instead we can see team logics as logics with countable many modalities of the form "$\varphi \rightarrow$" and introduce corresponding necessitation and distribution rules.

Theorem 4.3. The proof system $H^0LS$ is sound for $\mathcal{PTL}$.

Proof. We show that all axioms are valid and the inference rule preserves truth. By induction then the soundness of the calculus follows. All instances of axioms of $H^0$ are classical $\mathcal{PL}$ tautologies and hence true by Theorem 2.1. The rule ($E \rightarrow$) is sound for the same reason. By definition of $\rightarrow$, $\sim$ and $\rightarrow$ also the axioms of $L$ and ($E \rightarrow$) are sound for general $\mathcal{PTL}$ formulas.

The rule (I$\rightarrow$) of $S$ states that classical formulas are downward closed (Theorem 2.4). Similar ($E \otimes$) states that $\otimes$ is flat, i.e., there is a split into $\alpha \otimes \beta$ if and only if every member satisfies $\alpha$ or $\beta$ (Theorem 2.6). The lax substitution (Lax) says that whenever $T \models \varphi, \varphi \rightarrow \psi$, every subteam $S \subseteq T$ satisfies $\psi$, since $(T,S)$ is always a split of $T$ in lax semantics. This in turn implies $T \models \psi \rightarrow \varphi$ for arbitrary $\psi$. For ($E \sim$) assume by contradiction that $T$ has a split into $S, U, U'$ s.t. $S \models \psi$, $U \models \varphi$, $U' \not\models \psi$. Then $S \cup U' \models \varphi \otimes \sim \psi$, hence $S \cup U' \models \sim (\psi \rightarrow \varphi)$. But $U \not\models \varphi$, hence the division $(U, S \cup U')$ falsifies $\varphi \rightarrow (\psi \rightarrow \varphi)$. For ($C \rightarrow$) assume for contradiction that $T$ has a split into $S, U$ s.t. $S \models \psi$ but $U \models \varphi$. Then $(U, S)$ is a witness that $(\varphi \rightarrow \sim \psi)$ is false as well. The necessitation rule (Nec$\rightarrow$) and the distribution axiom (Dis$\rightarrow$) are known similar from in modal logic, here just applied with the pseudo-modality "$\psi \rightarrow$". ($Nec \rightarrow$) is applied to $\varphi$ only if $\varphi$ is a theorem, i.e., $\vdash \varphi$. A simple induction shows the soundness of (Nec$\rightarrow$).

4.1. Completeness of propositional team logic

In this section we first prove that the given calculus suffices to show Theorem 4.1. The completeness is then just a consequence of the completeness of $B(\mathcal{PL})$, i.e., Corollary 3.16. The proof will be built on several lemmas. Besides the deduction theorem we need a number of meta-rules in our proofs.
Lemma 4.4. If a proof system $\Omega$ has the deduction theorem, then it admits the following meta-rules:

- If $\Omega \succeq L$: Reductio ad absurdum (RAA): $\Phi \cup \{\varphi\} \vdash \psi, \sim \psi \Rightarrow \Phi \vdash \sim \varphi$ and $\Phi \cup \{\sim \varphi\} \vdash \psi, \sim \psi \Rightarrow \Phi \vdash \varphi$.
- If $\Omega \succeq LS$: Modus ponens in $\rightarrow$ ($MP\rightarrow$): $\vdash \varphi \rightarrow \psi$, $\Phi \vdash \vartheta \rightarrow \varphi \Rightarrow \Phi \vdash \vartheta \rightarrow \psi$.
- If $\Omega \succeq LS$: Modus ponens in $\otimes$ ($MP\otimes$): $\vdash \varphi \rightarrow \psi$, $\Phi \vdash \vartheta \otimes \varphi \Rightarrow \Phi \vdash \vartheta \otimes \psi$.

Proof. In the first case of (RAA) we have $\Phi \vdash \varphi \rightarrow \sim \psi, \varphi \rightarrow \psi$ by the deduction theorem. $L$ proves the Boolean tautologies $(\varphi \rightarrow \sim \psi) \rightarrow (\psi \rightarrow \sim \varphi)$ and $(\varphi \rightarrow \sim \varphi) \rightarrow \sim \varphi$, hence $\Phi \vdash \sim \varphi$ by (E$_\rightarrow$). The second case is proven with the theorem $\sim \sim \varphi \rightarrow \varphi$ of $L$. ($MP\otimes$) is just a shortcut for (Nec$_\rightarrow$), (Dis$_\rightarrow$) and (E$_\rightarrow$). ($MP\otimes$) is proven as follows.

|   |   |   |
|---|---|---|
| A | $\vdash \varphi \rightarrow \psi$ | MP$_\otimes$ |
| B | $\{\vartheta \otimes \varphi\}$ |   |
| 1 | $\vdash \sim \psi \rightarrow \sim \varphi$ | L (A) |
| 2 | $\vdash (\vartheta \rightarrow \sim \psi) \rightarrow (\vartheta \rightarrow \sim \varphi)$ | Nec$_\rightarrow$, Dis$_\rightarrow$ (1) |
| 3 | $\sim (\vartheta \rightarrow \sim \varphi)$ | Def. (B) |
| 4 | $\sim (\vartheta \rightarrow \sim \varphi)$ | L (2 + 3) |
| 5 | $\vartheta \otimes \psi$ | Def. (4) |

$\vartheta \otimes \psi$

Definition 4.5. Let $\mathfrak{f}$ be an $n$-ary connective. Say that a proof system $\Omega$ has substitution in $\mathfrak{f}$ if $\varphi_1, \ldots, \varphi_n \vdash \psi_n$ implies $\mathfrak{f}(\varphi_1, \ldots, \varphi_n) \vdash \mathfrak{f}(\psi_1, \ldots, \psi_n)$.

Note that due to symmetry it suffices to prove only $\mathfrak{f}(\varphi_1, \ldots, \varphi_n) \vdash \mathfrak{f}(\psi_1, \ldots, \psi_n)$ to show the substitution in $\mathfrak{f}$.

Lemma 4.6. If $\Omega \succeq L$ has the deduction theorem, then $\Omega$ has substitution in $\sim$ and $\rightarrow$.

Proof. Let $\varphi = \xi_1 \rightarrow \xi_2$, $\xi_1 \vdash \psi_1$ and $\xi_2 \vdash \psi_2$. Then $\{\psi_1, \varphi\} \vdash \xi_2$ in $L$ and hence $\{\psi_1, \varphi\} \vdash \psi_2$. By the deduction theorem $\varphi \vdash \psi_1 \rightarrow \psi_2$.

Let $\varphi = \sim \xi$ and $\xi \vdash \psi$. Obviously $\{\varphi, \psi\} \vdash \xi, \sim \xi$ in $L$. By Lemma 4.4 (RAA) is derivable, so we obtain $\varphi \vdash \sim \psi$.

Lemma 4.7. If $\Omega \succeq LS$ has the deduction theorem, then $\Omega$ has substitution in $\rightarrow$.

Proof. Let $\varphi = \xi_1 \rightarrow \xi_2$, $\xi_1 \vdash \psi_1$ and $\xi_2 \vdash \psi_2$. Due to Lemma 4.4 the following derivation is possible.
If $\alpha$ is a classical formula, in the following write $E\alpha$ as an abbreviation for $\neg \neg \alpha$.

Lemma 4.8. Let $\Omega \geq H^0LS$ have the deduction theorem. Then $\Omega$ proves the axioms $S'$ (see Figure 4).

Proof. For the derivations see Appendix A.1.

We formally describe the translation from $\mathcal{PTL}$ to $\mathcal{B}(\mathcal{PL})$ as $\neg \neg$-elimination, a special case of a more general translation, as later we will also eliminate other operators from stronger logics.

Definition 4.9. Let $\mathcal{F}$ be a logic. Let $\Omega$ be a proof system. Let $f$ be a connective of arity $n$. We say that $\mathcal{B}(\mathcal{F})$ has $f$-elimination in $\Omega$ if $\xi_1 \vdash \xi'_1, \ldots, \xi_n \vdash \xi''_n$ for $\xi_1, \ldots, \xi_n \in \mathcal{B}(\mathcal{F})$ implies $f(\xi_1, \ldots, \xi_n) \vdash \varphi$ for some $\varphi \in \mathcal{B}(\mathcal{F})$.

In other words: If $\xi_1, \ldots, \xi_n$ have provable equivalent $\mathcal{B}(\mathcal{F})$ formulas, then so has $f(\xi_1, \ldots, \xi_n)$. To actually prove $\neg \neg$-elimination we do a stepwise translation with the help of the following lemmas. We implicitly use Lemma 4.6 and 4.7 which permit derivations applied
to subformulas inside $\neg$, $\sim$ and $\odot$, as well as the meta-rules in Lemma 4.4 and the theorems in Lemma 4.8.

**Lemma 4.10** (And/Or swapping). If $\Omega$ has the deduction theorem and $\Omega \succeq H^0 LS$, then $\bigotimes_{i=1}^n E \beta_i \leftrightarrow \bigotimes_{i=1}^{n-1} E \beta_i$ is a theorem of $\Omega$.

**Proof.** The proof of the implication from left to right is by induction over $n$. $\otimes$ is left-associative, so assume by induction hypothesis $\bigotimes_{i=1}^{n-1} E \beta_i$ and $E \beta_n$ as premises. Certainly $\top \lor \top$ is a theorem of $H^0$ and hence $\top \otimes \top$ of $H^0 S$ by $F \otimes$. Assume $\bigotimes_{i=1}^{n-1} E \beta_i \rightarrow \sim E \beta_n$, then by (Lax) first derive $(\top \rightarrow \sim E \beta_n)$, by (C$\leftrightarrow$) $(E \beta_n \rightarrow \sim \top)$ and by (Lax) $(\top \rightarrow \sim \top)$, contradicting $\top \otimes \top$. (RAA) yields $(\bigotimes_{i=1}^{n-1} E \beta_i \rightarrow \sim E \beta_n)$, hence $\bigotimes_{i=1}^{n-1} E \beta_i \otimes E \beta_n = \bigotimes_{i=1}^{n-1} E \beta_i$.

The theorems $(\text{Abs} \otimes)$, $(\text{Ass} \otimes)$ and $(\text{Com} \otimes)$ are used to derive each conjunct for the implication from right to left. \hfill $\Box$

**Lemma 4.11** (Generalized distributive law). If $\Omega$ has the deduction theorem and $\Omega \succeq H^0 LS$, then $\alpha \otimes (\bigotimes_{i=1}^n E \beta_i) \leftrightarrow \bigotimes_{i=1}^n (\alpha \otimes E \beta_i)$ is a theorem of $\Omega$.

**Proof.** First apply the previous lemma to substitute the right-hand side of the conjunction, then distribute $\alpha$ with repeated application of $(D \otimes)$, $(\text{Ass} \otimes)$ and $(\text{Com} \otimes)$. \hfill $\Box$

**Lemma 4.12** (E isolation). If $\Omega$ has the deduction theorem and $\Omega \succeq H^0 LS$, then $\bigotimes_{i=1}^n (\alpha_i \otimes E \beta_i) \leftrightarrow \bigotimes_{i=1}^n (\alpha_i) \otimes \bigotimes_{i=1}^n E (\alpha_i \land \beta_i)$ is a theorem of $\Omega$.

**Proof.** From $\bigotimes_{i=1}^n (\alpha_i \otimes E \beta_i)$ we obtain $\bigotimes_{i=1}^n \alpha_i$ by $(\text{Ass} \otimes)$, $(\text{Com} \otimes)$ and $(\text{MP} \otimes)$ as $(\alpha_i \otimes E \beta_i) \rightarrow \alpha_i$ for all $i \in \{1, \ldots, n\}$. Apply $(\text{JoinE})$ to also derive $\bigotimes_{i=1}^n E (\alpha_i \land \beta_i)$ the same way, and by Lemma 4.10 then $\bigotimes_{i=1}^n E (\alpha_i \land \beta_i)$.

For the other implication we repeatedly apply the theorem $(\text{IsolateE})$, i.e., $(\varphi \otimes \alpha) \otimes E (\alpha \land \beta) \rightarrow \varphi \otimes (\alpha \otimes E \beta)$ as follows: Assume that the formula has the following form after $k$ applications.

$$\left( \bigotimes_{i=1}^k (\alpha_i \otimes E \beta_i) \otimes \bigotimes_{i=k+1}^n \alpha_i \right) \otimes \bigotimes_{i=k+1}^n E (\alpha_i \land \beta_i).$$

For $k = 0$ this is indeed the case. With commutative and associative laws we isolate a single subformula on each side:

$$\left( \bigotimes_{i=1}^k (\alpha_i \otimes E \beta_i) \otimes \bigotimes_{i=k+2}^n \alpha_i \right) \otimes \bigotimes_{i=k+2}^n E (\alpha_{k+1} \land \beta_{k+1}) \otimes \bigotimes_{i=k+2}^n E (\alpha_i \land \beta_i)$$

Then we apply the theorem on the left two conjuncts, resulting in

$$\left( \bigotimes_{i=1}^k (\alpha_i \otimes E \beta_i) \otimes \bigotimes_{i=k+2}^n \alpha_i \right) \otimes (\alpha_{k+1} \otimes E \beta_{k+1}) \otimes \bigotimes_{i=k+2}^n E (\alpha_i \land \beta_i)$$
and again with commutative and associative laws in
\[
\left(\bigotimes_{i=1}^{k+1} (\alpha_i \otimes E\beta_i) \otimes \bigotimes_{i=k+2}^{n} E\alpha_i\right) \otimes \bigotimes_{i=k+2}^{n} E(\alpha_i \land \beta_i)
\]
so that we arrive at the same form again and repeat the steps. \(\square\)

**Lemma 4.13** (Flatness 1). \(\bigotimes_{i=1}^{n} \alpha_i \iff \bigvee_{i=1}^{n} \alpha_i\) is a theorem of \(\Omega \geq H^0_{\text{LS}}\) if \(\Omega\) has the deduction theorem.

**Proof.** By induction over \(n\). The case \(n = 1\) is trivial. For \(n > 1\) let \(\varphi := \bigotimes_{i=1}^{n-1} \alpha_i\) and \(\gamma := \bigvee_{i=1}^{n-1} \alpha_i\) be given. By induction hypothesis \(\varphi \vdash_{\alpha_n} \gamma\). Then \(\varphi \otimes \alpha_n \vdash_{\alpha_n} \gamma \otimes \alpha_n\) by (Com\(\otimes\)) and (MP\(\otimes\)), and by (F\(\otimes\)) we obtain \(\varphi \otimes \alpha_n \vdash -\gamma \otimes \alpha_n \vdash \gamma \lor \alpha_n\). \(\square\)

**Lemma 4.14** (Flatness 2). \(\bigotimes_{i=1}^{n} \alpha_i \iff \bigwedge_{i=1}^{n} \alpha_i\) is a theorem of \(\Omega \geq H^0_{\text{LS}}\) if \(\Omega\) has the deduction theorem.

**Proof.** We do an induction as above, so let \(\varphi := \bigotimes_{i=1}^{n-1} \alpha_i\) and \(\gamma := \bigwedge_{i=1}^{n-1} \alpha_i\) be given such that \(\varphi \vdash_{\alpha_n} \gamma\). Then \(\varphi \otimes \alpha_n \vdash -\gamma \otimes \alpha_n\) in \(L\). To obtain \(\varphi \otimes \alpha_n \vdash \gamma \otimes \alpha_n \vdash -\gamma \land \alpha_n\) we prove the general theorem \(\alpha \land \beta \vdash -\alpha \otimes \beta\) for classical \(\alpha, \beta\).

Clearly \(\alpha \land \beta\) proves \(\alpha\) and \(\beta\) in \(H^0\) and thus \(\alpha \otimes \beta\) in \(H^0_{\text{LS}}\).

To prove \(\alpha \land \beta\) from \(\alpha \otimes \beta\) we use (RAA), i.e., assume \(\alpha \otimes \beta\) and \(\neg(\alpha \land \beta)\). It holds \((\alpha \land \beta) \vdash \neg(\alpha \rightarrow \neg \beta)\) in \(H^0\), so we derive \(E(\alpha \rightarrow \neg \beta)\) from \(\neg(\alpha \land \beta)\). Via \(L\) and the theorem (JoinE) we obtain in two steps first \(E(\alpha \land \alpha \rightarrow \neg \beta)\) and then \(E(\beta \land \alpha \land (\alpha \rightarrow \neg \beta))\) which turns into \(E\bot\) in \(H^0_{\text{LS}}\). But \(E\bot = \neg
\neg\bot\) is equivalent to \(\bot\), contradiction. \(\square\)

With the above lemmas we are ready to state the full translation from \(\text{PTL}\) to \(\mathcal{B}(\mathcal{P}L)\).

**Lemma 4.15** (\(\neg\)-elimination). Let \(\mathcal{F}\) be a logic closed under \(\neg, \lor, \land\). Let \(\Omega\) be a proof system with the deduction theorem s. t. \(\Omega \geq H^0_{\text{LS}}\). Then \(\mathcal{B}(\mathcal{F})\) has \(\neg\)-elimination in \(\Omega\).

**Proof.** Let \(\varphi = \psi \rightarrow \vartheta\). Let \(\psi', \vartheta' \in \mathcal{B}(\mathcal{F})\) such that \(\psi \vdash_{\alpha} \psi'\) and \(\vartheta \vdash_{\alpha} \vartheta'\). It holds \(\vartheta' \vdash \neg \neg \vartheta'\) in \(L\). Lemma 4.7 applies to \(\Omega\), so by substitution in \(\neg\) we can translate \(\varphi := \psi \rightarrow \vartheta\) to \(\psi' \rightarrow \neg \neg \vartheta'\), and therefore in \(L\) to \(\neg(\psi' \rightarrow \neg \neg \vartheta') = \neg(\psi' \otimes \neg \vartheta')\).

By Theorem 3.19 we can apply De Morgan’s laws and distributive laws on both \(\psi'\) and \(\neg \neg \vartheta'\). We can derive two formulas \(\psi''\), \(\vartheta''\) in disjunctive normal form (DNF) over \(\otimes, \oplus\); and obtain with substitution in \(\neg\) (Lemma 4.6) and \(\neg\) the following equivalent form of \(\varphi\), where all \(\alpha, \beta, \gamma, \delta, \ldots \in \mathcal{F}\):

\[
\sim \left[\bigotimes_{i=1}^{n} \left(\bigotimes_{j=1}^{m_i} \alpha_{i,j} \oplus \bigotimes_{j=1}^{m_i} E\beta_{i,j}\right)\right] \otimes \left[\bigotimes_{i=1}^{n} \left(\bigotimes_{j=1}^{m'_i} \alpha'_{i,j} \oplus \bigotimes_{j=1}^{m'_i} E\beta'_{i,j}\right)\right]
\]

That the negative literals can be represented with \(E\) prefix is due to \(H^0\) having \(\neg\neg\) introduction. Apply the following derivation in \(H^0_{\text{LS}}\):

\[
\begin{align*}
\text{Lemma 4.14} & \quad \neg \rightarrow \sim \left[\bigotimes_{i=1}^{n} \left(\bigotimes_{j=1}^{m_i} E\beta_{i,j}\right)\right] \otimes \left[\bigotimes_{i=1}^{n'} \left(\bigotimes_{j=1}^{m'_i} E\beta'_{i,j}\right)\right]
\end{align*}
\]
Theorem 4.18. The soundness was proven in Theorem 4.3.

Proof. This to:

Corollary 4.19. Yang has proven $\mathcal{PTL} \equiv S^+(\mathcal{PL} \cup \mathcal{NE})$ [Yan14]. Soundness and completeness improve this to:
Corollary 4.20. Every $\mathcal{PTL}$ formula has a provable equivalent $S^+(\mathcal{PL} \cup \text{NE})$ formula in $\mathcal{H}^0\mathcal{LS}$.

Observe that the perhaps unintuitive rule (Lax) is still independent of the remaining axioms of $\mathcal{H}^0\mathcal{LS}$. (Lax) is false in strict semantics of $\otimes$, but the system $\mathcal{H}^0\mathcal{LS}\setminus\text{(Lax)}$ is sound for the strict semantics, so it cannot derive (Lax).

5. Modal team logic

In modal team logic we have additionally team-wide modalities. As for the splitting operator, we axiomatize the modalities just enough so that we can eliminate them. Kontinen, Müller, Schnoor and Vollmer [Kon+15] proved that $\mathcal{MTL} \equiv \mathcal{B}(\mathcal{ML})$, i.e., every $\mathcal{MTL}$ formula is equivalent to an $\mathcal{B}(\mathcal{ML})$ formula. The idea is as follows: Every $\mathcal{MTL}$ formula $\varphi$ contains a finite number of propositional atoms. Therefore it can be written as a finite Boolean combination of the Hintikka formulas of the bisimulation types of the models satisfying $\varphi$ (generalized to team bisimulation in a suitable way). A satisfiable modal formula has infinitely many models, but they share only finitely many different bisimulation types, thus $\varphi$ has an equivalent $\mathcal{B}(\mathcal{ML})$ formula. But as with Yang’s argument we improve this result by giving a purely syntactical derivation procedure which does not rely on model-theoretic aspects.

Together with Corollary 3.16 this yields a complete axiomatization for $\mathcal{MTL}$, settling an open question of Kontinen et al. [Kon+15].

![Figure 5: The modal team logic axioms M](image)

Lemma 5.1. $D\Diamond \otimes$ is sound for $\mathcal{MTL}$.

Proof. “$\Rightarrow$”: Assume $\mathcal{K} = (W, R, V)$ and $(\mathcal{K}, T) \models \Diamond (\varphi \otimes \psi)$. Then $T$ has a successor team $T'$ which satisfies $\varphi \otimes \psi$, i.e., there are $S'$ and $U'$ such that $T' = S' \cup U'$, $(\mathcal{K}, S') \models \varphi$ and $(\mathcal{K}, U') \models \psi$. We have to find a split $(S, U)$ of $T$ such that $(\mathcal{K}, S) \models \Diamond \varphi$ and $(\mathcal{K}, U) \models \Diamond \psi$. Define $S := \{ v \in T \mid vRu, u \in S' \}$ and $U := \{ v \in T \mid vRu, u \in U' \}$. Now every world
Theorem 5.2. The proof system $\mathcal{H}^2\text{LSM}$ is sound for $\mathcal{MTL}$.

Proof. The system $\mathcal{H}^2$ applies only to $\mathcal{ML}$ formulas and is hence sound by Corollary 2.3. The system $\mathcal{L}$ is easily confirmed sound, and the soundness of $\mathcal{S}$ is proven as in Theorem 4.3. So we prove only the axioms $\mathcal{M}$.

(\text{Lin}\Box): Assume $(\mathcal{K}, T) \models \Box \varphi$. Then the unique successor team $R[T]$ does not satisfy $\varphi$. So it is not the case that $(\mathcal{K}, R[T]) \models \varphi$, hence $(\mathcal{K}, T) \not\models \Box \varphi$ by definition of $\Box$. The other direction is similar. The flatness axiom (F\Diamond) follows from the definition of a successor team. For (D\Diamond) see Lemma 5.1. (\text{E}\Box), (\Box), (\text{Dis}\Box), (\text{Dis}\Delta) are clear, and as well are (\text{Nec}\Box) and (\text{Nec}\Delta): If a formula $\varphi$ is a theorem and hence holds in all teams, then it certainly holds for the global successor team of any team and all successor teams in general.

5.1. Completeness of modal team logic

Lemma 5.3. The proof system $\mathcal{H}^2\text{LSM}$ admits the following meta-rules:

- \text{Modus ponens in } \Box (\text{MP}\Box): \vdash \varphi \to \psi, \Phi \vdash \Box \varphi \Rightarrow \Phi \vdash \Box \psi.
- \text{Modus ponens in } \Delta (\text{MP}\Delta): \vdash \varphi \to \psi, \Phi \vdash \Delta \varphi \Rightarrow \Phi \vdash \Delta \psi.
- \text{Modus ponens in } \Diamond (\text{MP}\Diamond): \vdash \varphi \to \psi, \Phi \vdash \Diamond \varphi \Rightarrow \Phi \vdash \Diamond \psi.

Proof. (\text{MP}\Box) and (\text{MP}\Delta) are just shortcuts for (\text{Nec}\Box), (\text{Dis}\Box) resp. (\text{Nec}\Delta), (\text{Dis}\Delta) and (E\to). By Lemma 4.4 we have (\text{RAA}), so the proof for (\text{MP}\Diamond) can be done as follows:
Lemma 5.4. $H^{\Box}LSM$ has substitution in $\rightarrow, \sim, \rightarrow, \Box$ and $\Delta$.

Proof. $H^{\Box}LSM$ has the deduction theorem due to Lemma 3.3, so the first three cases follow from Lemma 4.6 and 4.7. The cases $\varphi = \Box \xi$ and $\varphi = \Delta \xi$ are easily shown with (MP$\Box$) and (MP$\Delta$).

Lemma 5.5. $H^{\Box}LSM$ proves the system $M'$ (see Figure 6).

Proof. The $\rightarrow$ part of (D$\Box\rightarrow$) is (Dis$\Box$). See Appendix A.2 for the other derivations.

\[ \begin{array}{ll}
\Box(\varphi \rightarrow \psi) \leftrightarrow (\Box \varphi \rightarrow \Box \psi) & (D\Box\rightarrow) \\
\Diamond(\varphi \otimes \psi) \leftrightarrow (\Diamond \varphi \otimes \Diamond \psi) & (D\Diamond\otimes) \\
\Diamond(\alpha \otimes E\beta) \leftrightarrow \Diamond \alpha \otimes E \Box \neg(\alpha \land \beta) & (\Diamond \text{IsolateE})
\end{array} \]

Figure 6: Alternative axioms $M'$ for modalities

Lemma 5.6. $B(\mathcal{ML})$ has $\Box$-elimination in $H^{\Box}LSM$.

Proof. We have to show: If $\varphi$ has an equivalent $B(\mathcal{ML})$ formula, then so has $\Box \varphi$. $H^{\Box}LSM$ has the deduction theorem by Lemma 3.3. Hence $\varphi \leftrightarrow \varphi'$ is a theorem of $H^{\Box}LSM$ for some $\varphi' \in B(\mathcal{ML})$. Then via $M$ also $\Box(\varphi \leftrightarrow \varphi')$ and therefore $\Box \varphi \leftrightarrow \Box \varphi'$ are theorems.

The formula $\Box \varphi'$ can be transformed to an equivalent $B(\mathcal{ML})$ formula with repeated application of the theorem (D$\Box\rightarrow$) and the axiom (Lin$\Box$); other non-classical operators do not occur since $\varphi' \in B(\mathcal{ML})$. The $\Box$ is pushed inside until it precedes only classical $\mathcal{ML}$ formulas, resulting in another $B(\mathcal{ML})$ formula. This is possible as Lemma 5.4 allows the required substitution in subformulas.

Lemma 5.7. $B(\mathcal{ML})$ has $\Delta$-elimination in $H^{\Box}LSM$.

Proof. We can as in the previous lemma assume that the deduction theorem and substitution are available. Let $\Delta \varphi$ be given s.t. $\varphi \vdash \varphi'$ for $\varphi' \in B(\mathcal{ML})$. With $L$ and (MP$\Delta$) we can prove $\Delta \varphi$ equivalent to $\sim \sim \Delta \sim \varphi' = \sim \Diamond \sim \varphi'$. By Theorem 3.19 we can apply
De Morgan’s laws and distributive laws on $\sim \varphi'$ such that it is provable equivalent to a formula in DNF:

$$
\bigvee_{i=1}^{n} \left( \bigwedge_{j=1}^{o_i} \alpha_{i,j} \otimes \bigwedge_{j=1}^{k_i} E_{i,j} \right)
$$

Then $\Delta \varphi$ itself is provable equivalent to:

$$
\sim \bigdiamond \bigvee_{i=1}^{n} \left( \bigwedge_{j=1}^{o_i} \alpha_{i,j} \otimes \bigwedge_{j=1}^{k_i} E_{i,j} \right)
$$

Lemma 4.14 $\iff$ $\sim \bigdiamond \bigvee_{i=1}^{n} k_i \left( \alpha_i \otimes E_{i,j} \right)

Lemma 4.11 $\iff$ $\sim \bigdiamond \bigwedge_{i=1}^{n} k_i \left( \alpha_i \otimes E_{i,j} \right)

(D \otimes) $ $\iff$ $\bigvee_{i=1}^{n} k_i \left( \alpha_i \otimes E_{i,j} \right)

(Lemma 5.5 $\iff$ $\bigvee_{i=1}^{n} k_i \left( \alpha_i \otimes E_{i,j} \right)$}

(F) $\iff$ $\bigvee_{i=1}^{n} k_i \left( \neg \Box \neg \alpha_i \otimes E_{i,j} \right)

(Renaming) $\iff$ $\bigvee_{i=1}^{n} k_i \left( \mu_{i,j} \otimes \nu_{i,j} \right)$

where $\mu_{i,j}, \nu_{i,j} \in \mathcal{M}$

Lemma 4.12 $\iff$ $\bigvee_{i=1}^{k_i} \left( \bigwedge_{j=1}^{k_i} \mu_{i,j} \otimes \bigwedge_{j=1}^{k_i} E_{i,j} \right)$

Lemma 4.13 $\iff$ $\bigvee_{i=1}^{k_i} \left( \bigwedge_{j=1}^{k_i} \mu_{i,j} \otimes \bigwedge_{j=1}^{k_i} E_{i,j} \right)$

$\in \mathcal{B}(\mathcal{M})$. $\square$

**Theorem 5.8.** Every $\mathcal{MTL}$ formula is provable equivalent to a $\mathcal{B}(\mathcal{M})$ formula in $\mathcal{H}^{\square}$. stems.

**Proof.** Let $\varphi \in \mathcal{MTL}$. We do again an induction over $|\varphi|$ as in the proof of Theorem 4.16. If $\varphi \in \mathcal{B}(\mathcal{M})$, then $\varphi' := \varphi$. If $\varphi = \psi \rightarrow \vartheta$ or $\varphi = \sim \psi$ then by induction hypothesis there are $\psi', \vartheta' \in \mathcal{B}(\mathcal{M})$ s.t. $\psi \vdash \psi'$ and $\vartheta \vdash \vartheta'$. By substitution $\varphi$ is provable equivalent to an $\mathcal{B}(\mathcal{M})$ formula. The case $\varphi = \psi \rightarrow \vartheta$ follows from Lemma 4.15. The cases $\varphi = \Box \psi$ and $\varphi = \Delta \psi$ follow from Lemma 5.6 and Lemma 5.7. $\square
Theorem 5.9. The proof system $H^2\text{LSM}$ is sound and complete for $\mathcal{MTL}$.

Proof. The soundness was proven in Theorem 5.2. The completeness follows from soundness, Theorem 5.8, Corollary 3.16 and Lemma 4.17.

Corollary 5.10. The compactness theorem holds for $\mathcal{MTL}$.

Theorem 5.11. Every $\mathcal{MTL}$ formula has a provable equivalent $\mathcal{S}^+(\mathcal{ML} \cup \text{NE})$ formula in $H^2\text{LSM}$.

Proof. By Theorem 5.8 and Theorem 5.9 it is sufficient that for every $\varphi \in \mathcal{B}(\mathcal{ML})$ an equivalent $\mathcal{S}^+(\mathcal{ML} \cup \text{NE})$ formula exists. As seen in previous proofs we assume $\varphi$ to be in DNF, i.e., of the form

$$\bigvee_{i=1}^n \left( \bigwedge_{j=1}^{o_i} \alpha_{i,j} \otimes \bigwedge_{j=1}^{k_i} E\beta_{i,j} \right)$$

for $\mathcal{ML}$ formulas $\alpha_{i,j}, \beta_{i,j}$. As the expression $E\beta_{i,j}$ is equivalent to $(p \lor \neg p) \otimes (\text{NE} \otimes \beta_{i,j})$, by the definition of $\mathcal{S}^+$ the formula $\varphi$ has an equivalent $\mathcal{S}^+(\mathcal{ML} \cup \text{NE})$ formula.

6. First-order logic

First-order logic $\mathcal{FO}$ does not have the counter-model merging property. Consider the first-order sentences $R(c)$ and $\neg R(c)$. Clearly either of them can be falsified by an appropriate interpretation in team semantics, but to falsify both in the same structure is impossible, even with teams consisting of multiple assignments.

The point is of course that $R(c)$ and $\neg R(c)$ are contradicting sentences. It is shown that on the other hand this case is the only obstacle. The problem can be remedied by introduction of an additional deduction rule; we can then prove a contradiction from formulas whose models are already distinguishable by sentences. Note that this is not a contradiction to $\mathcal{FO}$ on sentences not having a counter-model merge property: Some satisfiable formulas just have no common model (in contrast to $\mathcal{PL}$ and $\mathcal{ML}$), but at least it is provable that there is none.

The new axiom is the unanimity axiom:

$$\sim \alpha \rightarrow \sim \alpha \quad (\alpha \in \mathcal{FO}^0) \quad (U)$$

Here $\mathcal{FO}^0$ is the set of sentences of $\mathcal{FO}$, i.e., the formulas which have no free variables. The axiom is crucial for the following lemma, which is a first step towards completeness.

If in the following $\Gamma$ is a set of first-order sentences, then we write $\mathcal{A} \models \Gamma$ instead of $(\mathcal{A}, s) \models \Gamma$ as the assignment is irrelevant.

Lemma 6.1. Every unsatisfiable $\Delta \subseteq \sim \mathcal{FO}$ is inconsistent in $HUL$. 

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Proof. To be unsatisfiable, $\Delta$ must be non-empty. We show $\Delta \models \bot$. As $\bot \rightarrow \delta$ holds for any $\sim \delta \in \Delta$ and this implies $\models \sim \delta \rightarrow \sim \bot$, $\Delta$ is then inconsistent as $\Delta \models \bot, \sim \bot$.

Define $\Gamma := \{ \exists x_1 \ldots \exists x_n \delta(x_1, \ldots, x_n) \mid \sim \delta(x_1, \ldots, x_n) \in \Delta \}$, it is the “$\exists$-closure” of $\Delta$. Here, $\delta(x_1, \ldots, x_n)$ means that $\delta$ has the free variables $x_1, \ldots, x_n$.

To complete the proof we show that on the one hand $\Delta \models \Gamma$, and on the other hand that $\Gamma$ has no first-order model, as then it proves $\bot$ by the completeness of $H$ for $\mathcal{FO}$.

For the first part we use the unanimity axiom: It holds $\forall x_1 \ldots \forall x_n \delta(x_1, \ldots, x_n) \models H \delta(x_1, \ldots, x_n)$, hence $\sim \delta(x_1, \ldots, x_n)$

$\Gamma : \Gamma_L \sim \forall x_1 \ldots \forall x_n \delta(x_1, \ldots, x_n)$

$\Gamma_U \sim \forall x_1 \ldots \forall x_n \delta(x_1, \ldots, x_n)$

$\Gamma_H \exists x_1 \ldots \exists x_n \sim \delta(x_1, \ldots, x_n)$.

For the second part assume for contradiction that $\Gamma$ has a first-order model $\mathcal{A}$. Then for every $\exists x_1 \ldots \exists x_n \sim \delta(x_1, \ldots, x_n)$ there is some assignment $s_\delta$ s.t. $(\mathcal{A}, s_\delta) \models \sim \delta(x_1, \ldots, x_n)$. In team semantics this implies $(\mathcal{A}, \{s_\delta\}) \models \sim \delta(x_1, \ldots, x_n)$. But $\mathcal{FO}$ has downward closure (Theorem 2.4), so $\sim \mathcal{FO}$ has “upward closure”: If a team satisfies a formula, then all teams containing it do as well. Thus $\Delta$ has a model $(\mathcal{A}, \{s \mid \sim \delta \in \Delta, (\mathcal{M}, s) \not\models \delta\})$, contradiction. □

To lift the partial completeness result one step higher—from $\sim \mathcal{FO}$ to ($\mathcal{FO} \cup \sim \mathcal{FO}$)—we vastly simplify the problem by first proving compactness of the $\mathcal{FO} \cup \sim \mathcal{FO}$ fragment.

Let $\Phi \subseteq \mathcal{FO} \cup \sim \mathcal{FO}$. W.l.o.g. assume that no bound variable in a formula of $\Phi$ has the same name as a free variable of the same formula. Further let $C$ be a countable infinite set of constant symbols not used in $\Phi$. Then the first-order translation $\Phi_f$ of $\Phi$ is the set

$\Phi_f : = \{ \gamma(c_\delta^1, \ldots, c_\delta^n) \mid \gamma(x_1, \ldots, x_n) \in \Phi \cap \mathcal{FO}, \sim \delta \in \Phi \cap \sim \mathcal{FO} \} \cup$

$\{ \sim \delta(c_\delta^{x_1}, \ldots, c_\delta^{x_n}) \mid \sim \delta(x_1, \ldots, x_n) \in \Phi \cap \sim \mathcal{FO} \}$,

where the symbols $c_\delta^x \in C$ are constant symbols and $\gamma(t_1, \ldots, t_n)$ is the formula obtained from $\gamma$ by replacing the term $x_i$ with $t_i$, similar for $\sim \delta$.

Lemma 6.2. Let $\Phi \subseteq \mathcal{FO} \cup \sim \mathcal{FO}$. Then $\Phi$ is satisfiable in team semantics if and only if $\Phi_f$ is satisfiable in classical semantics.

Proof. The idea is to encode the assignments of a team directly into the model with the help of the new constant symbols $c_\delta^x$. Let again $\Gamma : = \Phi \cap \mathcal{FO}$ and $\Delta : = \Phi \cap \sim \mathcal{FO}$.

For “$\Rightarrow$” assume $(\mathcal{A}, T) \models \Phi$ for some first-order structure $\mathcal{A}$. Enrich the interpretation of $\mathcal{A}$ by assigning the new constants as follows, resulting in a structure $\mathcal{A}'$. If $\sim \delta \in \Delta$, then there is a non-empty set $\{s \in T \mid (\mathcal{A}, s) \not\models \delta\}$. By the axiom of choice we can extract some witness $s_\delta \in T$ for each $\sim \delta$ and interpret $(c_\delta^x)^{\mathcal{A}'} := s_\delta(x)$. It holds that $\mathcal{A}' \models \Phi_f$:

$\gamma(c_\delta^{x_1}, \ldots, c_\delta^{x_n}) \in \Phi_f$

4If the set $D$ of symbols occurs, replace them by $\{0\} \times D$ and define $C := \{1\} \times D$. 27
\[ \Rightarrow \gamma(x_1, \ldots, x_n) \in \Gamma \]
\[ \Rightarrow \forall s \in T : (\mathcal{A}, s) \models \gamma(x_1, \ldots, x_n) \]
\[ \Rightarrow (\mathcal{A}, s_\delta) \models \gamma(x_1, \ldots, x_n) \]
\[ \Rightarrow \mathcal{A}' \models \gamma(c_1^x, \ldots, c_n^x) \]

\[ \neg \delta(e_1^x, \ldots, e_n^x) \in \Phi_f \]
\[ \Rightarrow \sim \delta(x_1, \ldots, x_n) \in \Delta \]
\[ \Rightarrow (\mathcal{A}, s_\delta) \not\models \delta(x_1, \ldots, x_n) \]
\[ \Rightarrow \mathcal{A}' \models \neg \delta(e_1^x, \ldots, e_n^x) \]

For “\( \Leftarrow \)” assume \( \mathcal{A} \models \Phi_f \) for some model \( \mathcal{A} = (\mathcal{A}, \tau^A) \). For each \( \sim \delta \in \Delta \) define now \( s_\delta : \text{Var} \to \mathcal{A} \) as \( s_\delta(x) := (c_1^x)^A \). Then

\[ \gamma(x_1, \ldots, x_n) \in \Gamma \]
\[ \Rightarrow \forall \sim \delta \in \Delta : \gamma(c_1^x, \ldots, c_n^x) \in \Phi_f \]
\[ \Rightarrow \forall \sim \delta \in \Delta : \mathcal{A} \models \gamma(c_1^x, \ldots, c_n^x) \]
\[ \Rightarrow \forall \sim \delta \in \Delta : (\mathcal{A}, s_\delta) \models \gamma(x_1, \ldots, x_n) \]
\[ \Rightarrow \forall \sim \delta \in \Delta : (\mathcal{A}, \{s_\delta\}) \vDash \gamma(x_1, \ldots, x_n) \]

\[ \sim \delta(x_1, \ldots, x_n) \in \Delta \]
\[ \Rightarrow \neg \delta(e_1^x, \ldots, e_n^x) \in \Phi_f \]
\[ \Rightarrow \mathcal{A} \models \neg \delta(e_1^x, \ldots, e_n^x) \]
\[ \Rightarrow (\mathcal{A}, s_\delta) \models \neg \delta(x_1, \ldots, x_n) \]
\[ \Rightarrow (\mathcal{A}, s_\delta) \not\models \delta(x_1, \ldots, x_n) \]
\[ \Rightarrow (\mathcal{A}, \{s_\delta\}) \vDash \sim \delta(x_1, \ldots, x_n) \]

Define \( T := \{ s_\delta \mid s_\delta \text{ is the witness of } \sim \delta, \sim \delta \in \Delta \} \). By downwards closure of \( \mathcal{F} \mathcal{O} \) (see Theorem 2.4) and hence upwards closure of \( \sim \mathcal{F} \mathcal{O} \) it holds \( (\mathcal{A}, T) \models \sim \delta(x_1, \ldots, x_n) \) for all \( \sim \delta \in \Delta \). By union closure of \( \mathcal{F} \mathcal{O} \) (see Theorem 2.5) it holds \( (\mathcal{A}, T) \models \gamma(x_1, \ldots, x_n) \) for all \( \gamma \in \Gamma \). So \( \Phi = \Gamma \cup \Delta \) is satisfiable.

From the above construction we also obtain a generalization of the empty team property (which itself states that every \( \Phi \subseteq \mathcal{F} \mathcal{O} \) is satisfied by the empty team).

**Corollary 6.3.** Every satisfiable set \( \Phi \subseteq \mathcal{F} \mathcal{O} \cup \sim \mathcal{F} \mathcal{O} \) is satisfied by a team of cardinality \( |\Phi \cap \sim \mathcal{F} \mathcal{O}| \).

In particular the team can always be chosen countable.

**Lemma 6.4.** If a set \( \Phi \subseteq \mathcal{F} \mathcal{O} \cup \sim \mathcal{F} \mathcal{O} \) is unsatisfiable, then it has a finite unsatisfiable subset.
Proof. Let $\Phi$ be unsatisfiable. By the previous lemma $\Phi_f$ has no first-order model. First-order logic has the compactness theorem, so a finite subset $\Phi' \subseteq \Phi_f$ is already unsatisfiable. Extend $\Phi'$ to a finite set of the form $\Phi^* \subseteq \Phi$, then $\Phi^*$ is finite and unsatisfiable as well. □

Lemma 6.5. The calculus HUL is refutation complete for $\mathcal{FO} \cup \neg \mathcal{FO}$.

Proof. Let $\Phi \subseteq \mathcal{FO} \cup \neg \mathcal{FO}$ be unsatisfiable. By the previous lemma a finite $\Phi' \subseteq \Phi$ is unsatisfiable. Let $\Gamma := \Phi' \cap \mathcal{FO}$, $\Delta := \Phi' \cap \neg \mathcal{FO}$. Let $\gamma \in \mathcal{FO}$ be a conjunction of all formulas in $\Gamma$. Then $\Gamma \cup \Delta \vdash \{ \gamma \} \cup \Delta$, so we need only to prove that $\{ \gamma \} \cup \Delta$ is inconsistent (it is unsatisfiable).

For this we “adjoin” $\gamma$ to all $\Delta$ formulas: Let $\Delta^\gamma := \{ \neg (\delta \lor \neg \gamma) \mid \neg \delta \in \Delta \}$. It is $\{ \gamma \} \cup \Delta \vdash \Delta^\gamma$: It holds $\{ \gamma, \delta \lor \neg \gamma \} \vdash \delta$, so by contraposition $\{ \gamma, \neg \delta \} \vdash \neg (\delta \lor \neg \gamma)$ in HL. So it suffices to prove a contradiction from $\Delta^\gamma$. Assume that the set $\Delta^\gamma$ is satisfied by some $(A, T)$. Then for each $\neg \delta \in \Delta$ there is at least one $s_\delta \in T$ s. t. $(A, s_\delta) \not\models \delta \lor \neg \gamma$, so $(A, s_\delta) \models \neg \delta, \gamma$. Then the model $(A, \{ s \mid (A, s) \not\models \delta \lor \neg \gamma, \neg \delta \in \Delta \})$ satisfies $\{ \gamma \} \cup \Delta$ which is a contradiction. So $\Delta^\gamma$ is unsatisfiable as well.

But by Lemma 6.1, and since $\Delta^\gamma \subseteq \neg \mathcal{FO}$, the set $\Delta^\gamma$ must be inconsistent in HUL, so we can prove a contradiction from it and therefore from $\Phi$. □

Theorem 6.6. HUL is sound and complete for $B(\mathcal{FO})$.

Proof. The soundness of the unanimity axiom (U) is proven as follows: If $(M, T) \models \neg \alpha$, then $(M, s) \models \neg \alpha$ for some $s \in T$. $\neg \alpha$ is a sentence, so $(M, s') \models \neg \alpha$ for any $s'$, hence $(M, s) \models \neg \alpha$ for all $s \in T$. The systems $H, L$ are easily checked sound for $B(\mathcal{FO})$.

HUL has weakening by Lemma 3.3. By Theorem 3.14 we then obtain completeness. □

The fragment $\neg \mathcal{FO}$ was investigated by Galliani [Gal14]. One of his results is that every $\neg \mathcal{FO}$ formula without free variables is equivalent to a $\mathcal{FO}$ sentence.\(^5\) Therefore the fragment of $\neg \mathcal{FO}$ sentences is as weak as $\mathcal{FO}^0$. We give a similar result with respect to formulas containing free variables.

Theorem 6.7 (Sentence Interpolation). Let $\Delta \subseteq \neg \mathcal{FO}$ and $\alpha \in \mathcal{FO}$. If $\Delta \models \alpha$ then there is a sentence $\varepsilon \in \mathcal{FO}^0$ such that $\Delta \models \varepsilon$ and $\varepsilon \models \alpha$.

Proof. If $\alpha$ has free variables $x_1, \ldots, x_n$, then define $\varepsilon := \forall x_1 \ldots \forall x_n \alpha$.

Proof of $\Delta \models \varepsilon$: Let $(A, T) \models \Delta$. Let $\mathcal{V}$ be the team of all assignments to $x_1, \ldots, x_n$.

Then certainly $T \subseteq \mathcal{V}$.p and by downward closure of $\mathcal{FO}$, and hence upward closure of $\neg \mathcal{FO}$, it holds $(A, \mathcal{V}) \models \Delta$. By assumption it holds $(A, \mathcal{V}) \models \alpha$. By definition of $\mathcal{V}$ it holds $(A, \{ \emptyset \}) \models \varepsilon$ where $\{ \emptyset \}$ is the team which contains only the empty assignment. As $\varepsilon$ is a sentence this again implies $(A, T) \models \varepsilon$.

For the proof of $\varepsilon \models \alpha$ we can equivalently consider classical semantics due to Theorem 2.2. So let $(A, s) \models \varepsilon$. Since $\varepsilon = \forall x_1 \ldots \forall x_n \alpha$, $(A, v) \models \alpha$ for every assignment $v$ to $x_1, \ldots, x_n$, including $s$; therefore $(A, s) \models \alpha$. But $(A, s)$ was arbitrary, hence $\varepsilon \models \alpha$. □

\(^5\)Equivalence here means that the $\mathcal{FO}$ sentence is satisfied in classical first-order sense; as $\neg \mathcal{FO}$ has no empty team property but $\mathcal{FO}^0$ has, such an equivalence cannot exist in team semantics.

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7. Quantifier elimination

The first-order team quantifiers for $Q(\cdot)$ and modalities of $M(\cdot)$ can be axiomatized in a similar fashion, as seen in Figure 7. $\forall$ behaves like $\Box$, and $\exists$ behaves like $\Diamond$. Note that the precondition of the (Iv) axiom of $Q$ is dropped in contrast to the corresponding axiom (I$\Box$) of $M$, since structures always have a non-empty domain, and so there is always some supplementing function. In Kripke structures there is not always a successor team, namely if some world in the team has no successor.

| ∀$x$¬$\varphi$ ↔ ¬∀$x$.$\varphi$ | (Lin$\forall$) | $\forall$ is linear. |
|∃$x$$\alpha$ ↔ ¬∀¬$\alpha$ | (F$\exists$) | Flatness of $\exists$. |
|∃($\varphi \otimes \psi$) ↔ ¬∃$x$.$\varphi$ $\otimes$ $\exists$$x$$\psi$ | (D$\exists\otimes$) | $\exists$ distributes over splitting (for lax $\exists$). |
|∀$x$$\alpha$ → !(∀$x$.$\alpha$) | (Ev) | Suppl. teams are subteams of dupl. team. |
|!$x$$\psi$ → ∀$x$$\psi$ | (Iv) | The dupl. team is a suppl. team (for lax $\exists$). |
|∀$x$($\varphi \rightarrow \psi$) → (∀$x$.$\varphi$ $\rightarrow$ ∀$x$$\psi$) | (Dis$\forall$) | Distribution axiom |
|!(∀$x$.$\varphi$ $\rightarrow$ !(∀$x$$\psi$)) | (Dis$!$) | Distribution axiom |

\[\frac{\varphi}{!(∀x\varphi)} \quad (\varphi \text{ theorem}) \quad \text{(UG)$!$} \quad \text{Universal generalization} \]

Figure 7: The team quantifier axioms $Q$

**Lemma 7.1.** $D\exists\otimes$ is sound for $Q(FO)$ and $Q(QBF)$.

**Proof.** We show only the $Q(FO)$ case, the $Q(QBF)$ case is proven similar.

“$\rightarrow$” Assume $(A,T) \models \exists x(\varphi \otimes \psi)$ where $A = (A,\tau^A)$. There is a supplementing function $f : T \rightarrow 3(A) \setminus \emptyset$ s.t. $(A,T[f/x]) \models \varphi \otimes \psi$, hence $(A,S) \models \varphi$ and $(A,U) \models \psi$ for some $S,U$ with $T[f/x] = S \cup U$.

Let $S' := \{ t \in T \mid \exists s \in S : s(y) = t(y) \forall y \neq x \}$. $S'$ is the subteam of all assignments $t \in T$ such that for some $a$ the supplemented assignment $t[a/x]$ is in $S$. Similar define $U'$. It holds that $S$ is a supplemented team of $S'$ and $U$ of $U'$. Therefore $(A, S') \models \exists x\varphi$ and $(A, U') \models \exists x\psi$. It remains to show that $T = S' \cup U'$. Clearly $S', U' \subseteq T$ by definition. Let $t \in T$. Then $t[a/x] \in T[f/x]$ for some $a$ and hence $t[a/x] \in S' \cup U'$. But then $t \in S'$ or $t \in U'$ by definition of $S'$ and $U'$. Thus $(A,T) \models \exists x\varphi \otimes \exists x\psi$.

“$\leftarrow$” Let $(A,S) \models \exists x\varphi$ and $(A,U) \models \exists x\psi$ such that $T = S \cup U$. Let $S', U'$ be the supplemented teams such that $(A,S') \models \varphi$ and $(A,U') \models \psi$. Then $S' \cup U'$ is a supplemented team of $T$ and fulfills $\varphi \otimes \psi$, hence $(A,T) \models \exists x(\varphi \otimes \psi)$.

**Lemma 7.2.** The system $Q$ is sound for $Q(FO)$ and $Q(QBF)$.

**Proof.** For the soundness of $D\exists\otimes$ see the preceding lemma. (Lin$\forall$) is proven similar to (Lin$\Box$). (F$\exists$) follows from the definition of supplementing functions and flatness. The axiom (Ev) is clear as the supplementing teams always are subteams of the duplicating teams, and classical formulas are downward closed (see Theorem 2.4). (Iv) follows from the fact that the duplicating team is just the team that supplements with the whole
domain. The generalization (UG!) is clear. The axiom (Dis\forall) is clear as well: If the duplicating team satisfies \( \varphi \rightarrow \psi \) and \( \varphi \) at the same time, then it satisfies \( \psi \). The axiom (Dis!) works similar, but quantifies all supplementing teams.

**Theorem 7.3.** The proof system HULSQ is sound for \( \mathcal{Q}(\mathcal{F}\mathcal{O}) \).

*Proof.* See the preceding lemma and the proofs of Theorem 4.3 and Theorem 6.6. \( \square \)

**Lemma 7.4.** If a proof system \( \Omega \succeq \text{LQ} \) has the deduction theorem, then it admits the following meta-rules:

- **Modus ponens in \( \forall \)(MP\forall):** \( \Gamma \vdash \varphi \rightarrow \psi \), \( \Gamma \vdash \forall x \varphi \Rightarrow \Gamma \vdash \forall x \psi \).
- **Modus ponens in \( ! \)(MP!):** \( \Gamma \vdash \varphi \rightarrow \psi \), \( \Gamma \vdash ! x \varphi \Rightarrow \Gamma \vdash ! x \psi \).
- **Modus ponens in \( \exists \)(MP\exists):** \( \Gamma \vdash \varphi \rightarrow \psi \), \( \Gamma \vdash \exists x \varphi \Rightarrow \Gamma \vdash \exists x \psi \).

*Proof.* The rules are proven as (MP\Box), (MP\Delta) and (MP\Diamond) in Lemma 5.3 with the corresponding axioms of Q. The required rule \( \vdash \varphi \Rightarrow \vdash \forall x \varphi \) is simulated by (I\forall) and (Nec!). \( \square \)

### 7.1. Completeness of first-order team logic

In the following we regard quantifiers of the form \( \forall x \) and \( ! x \) as unary connective. For countable many variables \( x_1, x_2, \ldots \) then countable many connectives are introduced. Analogously a logic is *closed under universal quantification* if for any syntactically valid formula \( \varphi \) and every \( x_i \) there is a syntactically valid formula \( \forall x_i \varphi \). These connectives can then be eliminated in the system Q the same way as the modal operators \( \Box \) and \( \Diamond \).

In our context of syntactical elimination inside proof systems we do not distinguish between variables as first-order placeholders and propositional statements, in both cases we consider them just as a countable set of quantifiable symbols.

As for the other logics we require the notion of substitution in our logical connectives.

**Lemma 7.5.** If \( \Omega \succeq \text{LSQ} \) has the deduction theorem, then \( \Omega \) has substitution in \( \rightarrow, \sim, \neg, \forall x \) and \( ! x \).

*Proof.* The cases for the Boolean connectives and splitting are covered by Lemma 4.6 and 4.7. The cases \( \forall x \varphi \) and \( ! x \varphi \) are proven by (MP\forall) and (MP!). \( \square \)

**Lemma 7.6** (Elimination of \( \forall \)). Let \( \mathcal{F} \) be a logic closed under universal quantification. Let \( \Omega \) be a proof system with the deduction theorem s.t. \( \Omega \succeq \text{H}^0\text{LSQ} \). Then \( \mathcal{B}(\mathcal{F}) \) has \( \forall x \)-elimination in \( \Omega \).

*Proof.* Proven as for \( \Box \) in Lemma 5.6 with the axioms Q instead of M. The closure under universal quantification causes that \( \forall x \alpha \) for classical formulas \( \alpha \) is again a classical formula, so we can “push inside” \( \forall x \) similar to \( \Box \). \( \square \)
The other lemma, $!x$-elimination, essentially is proven as the elimination of $\Delta$ in Lemma 5.7, but here with the axioms $Q$ instead of $M$. In almost all derivations found in Appendix A.2 the symbols $\square$ and $\Delta$ can just be interchanged with $\forall x$ and $!x$, as the analogous axioms just can be used. There is only one proof in Appendix A.2 which must be further adapted, namely that of $(\text{Join} \diamond)$, as it is the only derivation which explicitly makes use of $H \square$ for the provability of $\neg \square \neg \bot \vdash \bot$ in one step. If the analogous theorem $\neg \forall x \neg \bot \vdash \bot$ is assumed provable in $\mathcal{U}$, then the step can be replaced, the proof for $\Delta$-elimination similarly goes through for $!x$ and we obtain:

**Lemma 7.7** (Elimination of $!$). Let $\mathcal{F}$ be a logic closed under $\neg, \lor, \land$ and universal quantification. Let $\Omega$ be a proof system with the deduction theorem s. t. $\Omega \models H^0 \mathsf{LSQ}$. If $\Omega$ proves $\neg \forall x \neg \bot \vdash \bot$ for all variables $x$, then $\mathcal{B}(\mathcal{F})$ has $!x$-elimination in $\Omega$.

**Theorem 7.8.** Every $\mathcal{Q}(\mathcal{FO})$ formula is provable equivalent in $\mathsf{HLSQ}$ to a $\mathcal{B}(\mathcal{FO})$ formula.

**Proof.** Let $\varphi \in \mathcal{Q}(\mathcal{FO})$. We do again an induction over $|\varphi|$ as in Theorem 5.8. The new cases are $\varphi = \forall x \psi$ and $\varphi = !x \psi$.

The system $\mathsf{HLSQ}$ certainly has the deduction theorem as of Lemma 3.3. If here $\psi$ is provable equivalent to a $\mathcal{B}(\mathcal{FO})$ formula we therefore can apply Lemma 7.6 in the $\forall$ case. In the $!$ case we have to apply Lemma 7.7, but here it is required to show that $\neg \forall x \neg \bot \vdash \bot$ is provable. This is however possible already in $H$, as $H$ is complete for $\mathcal{FO}$.

**Theorem 7.9.** The proof system $\mathsf{HULSQ}$ is sound and complete for $\mathcal{Q}(\mathcal{FO})$.

**Proof.** For the soundness see Theorem 7.3. The completeness follows from soundness, Lemma 4.17, Theorem 6.6 and Theorem 7.8.

**Corollary 7.10.** The compactness theorem holds for $\mathcal{Q}(\mathcal{FO})$. The set of valid $\mathcal{Q}(\mathcal{FO})$ formulas is recursively enumerable.

A similar proof as in Theorem 5.11 yields:

**Corollary 7.11.** Every $\mathcal{Q}(\mathcal{FO})$ formula is provable equivalent to a $S^+(\mathcal{FO} \cup \text{NE})$ formula in $\mathsf{HULSQ}$.

### 7.2. Completeness of Quantified Boolean Team Logic

The elimination of team-semantical quantifiers $\forall$ and $!$ may not only be applied to first-order logic, but also to the simpler quantified Boolean logic in its team variant $\mathcal{QPTL}$. For completeness of the underlying classical logic, $\mathcal{QBetr}$, we consider the following straightforward axiom for the expansion of quantifiers and reduce this problem to the completeness of propositional calculus.

\[
\forall x \alpha \leftrightarrow (\alpha[x/\top] \land \alpha[x/\bot]) \quad (X) \quad \text{Expansion of quantifier}
\]
Propositional calculus is known to have the deduction theorem for propositional implication $\rightarrow$, i.e., $\Gamma \cup \{\alpha\} \vdash \beta$ implies $\Gamma \vdash \alpha \rightarrow \beta$. This property is unchanged for stronger systems as long as no new inference rules are added (cf. Hakli and Negri [HN12]).

Lemma 7.12. $\Gamma \cup \{\alpha\} \vdash \beta$ implies $\Gamma \vdash \alpha \rightarrow \beta$ in $\sf{H}^{\emptyset}\mathcal{X}$.

Lemma 7.13. Every $\sf{QBF}$ formula has a provable equivalent $\sf{PL}$ formula in $\sf{H}^{\emptyset}\mathcal{X}$.

Proof. By induction over the number of occurring quantifiers. Assume $\forall x\alpha \in \sf{QBF}$. It holds $\forall x\alpha \vdash \alpha[x/\top] \land \alpha[x/\bot]$. By induction hypothesis and Lemma 7.12 $\alpha[x/\top] \leftrightarrow \beta_1$ and $\alpha[x/\bot] \leftrightarrow \beta_2$ are theorems of $\sf{H}^{\emptyset}\mathcal{X}$ for some $\beta_1,\beta_2 \in \sf{PL}$.

By completeness of propositional calculus however the formula $(\alpha \rightarrow \beta) \rightarrow (\gamma \rightarrow \delta) \rightarrow (\alpha \land \gamma \rightarrow \beta \land \delta)$ is a theorem of $\sf{H}^{\emptyset}$ for any $\alpha, \beta, \gamma, \delta$, therefore $\forall x\alpha \vdash \alpha[x/\top] \land \alpha[x/\bot] \vdash \beta_1 \land \beta_2$. \hfill $\square$

Lemma 7.14. Every $\sf{B}(\sf{QBF})$ formula has a provable equivalent $\sf{B}(\sf{PL})$ formula in $\sf{H}^{\emptyset}\mathcal{X}$.

Proof. $\sf{H}^{\emptyset}\mathcal{X}$ has substitution in $\sim$ and $\rightarrow$ by Lemma 3.3 and 4.6. In $\sf{H}^{\emptyset}\mathcal{X}$ therefore we can prove the equivalence to a $\sf{B}(\sf{PL})$ formula by substituting all classical $\sf{QBF}$ subformulas via Lemma 7.13. \hfill $\square$

Theorem 7.15. Every $\sf{Q}(\sf{QBF})$ formula is provable equivalent to a $\sf{B}(\sf{PL})$ formula in $\sf{H}^{\emptyset}\mathcal{X}$.

Proof. We apply the preceding lemma and thus give only the translation to $\sf{B}(\sf{QBF})$. This in turn is proven by induction similar to the $\sf{FO}$ case (Theorem 7.8), as elimination is available due to Lemma 7.6 and 7.7. The required theorem $\neg \forall x \neg \bot \rightarrow \bot$ is provable in $\sf{H}^{\emptyset}\mathcal{X}$. \hfill $\square$

It follows the completeness of $\sf{QPTL}$ itself, again by provable equivalence to $\sf{B}(\sf{PL})$. All the required steps have already been introduced in earlier sections.

Theorem 7.16. The proof system $\sf{H}^{\emptyset}\mathcal{X}$ is sound and complete for $\sf{QPTL}$.

Proof. The soundness is proven as in the $\sf{Q}(\sf{FO})$ case, see Theorem 7.3. The completeness again follows from soundness, the provable equivalence to $\sf{B}(\sf{PL})$ (Theorem 7.15) and the completeness of $\sf{B}(\sf{PL})$ (Corollary 3.16). \hfill $\square$

Corollary 7.17. The compactness theorem holds for $\sf{QPTL}$.

Again similar to Theorem 5.11 it holds:

Corollary 7.18. Every $\sf{QPTL}$ formula is provable equivalent to a $S^+(\sf{QBF} \cup \sf{NE})$ formula in $\sf{H}^{\emptyset}\mathcal{X}$.
8. Conclusion

Figure 8 visualizes the landscape of fragments of Väänänen’s team logic $\mathcal{TL}$ and dependence logic $\mathcal{D}$ [Vä07]. It was shown that without its dependence atom, $\mathcal{TL}$ with lax semantics collapses down to $\mathcal{B}(\mathcal{FO})$ and is then axiomatizable, hence it is then also recursively enumerable and compact.

The team-semantical extensions of propositional logic $\mathcal{PL}$ and modal logic $\mathcal{ML}$, i.e., $\mathcal{PTL}$ and $\mathcal{MTL}$, even the quantified variant $\mathcal{QPTL}$, have been shown axiomatizable, in fact because they also collapse to the Boolean closures of their flat base logics, i.e., $\mathcal{B}(\mathcal{PL})$ and $\mathcal{B}(\mathcal{ML})$.

An important detail there is the use of lax semantics for the operators $\otimes$, $\diamondsuit$ and $\exists$. It is possible to define the semantics in a strict way, i.e., to define team divisions via partitions; to choose exactly one successor per world for $\diamondsuit$; and to require supplementation functions to have range $A$ instead of $\mathcal{P}(A) \setminus \emptyset$. The semantics of $\otimes$ would then allow to count certain elements in the team. The strictness of $\diamondsuit$ can be chosen accordingly: It distributes over $\otimes$ if and only if both are strict or both are lax. For $\exists$ and $\otimes$ the same is true.

But even if we recover the distributive laws at this point — counting cannot be finitely expressed in the $\mathcal{B}(\cdot)$ closure (see Theorem 4.2), so there can be no completeness proof based on operator elimination as in the style of this paper. It is open how complete axiomatizations can be obtained for team logics strictly stronger than $\mathcal{B}(\cdot)$.

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\[
\mathcal{T \mathcal{L}} \equiv Q(\mathcal{FO} \cup \text{dep}) \\
B(\mathcal{FO}) \equiv Q(\mathcal{FO}) \\
D \quad \text{non-arithmetic} \quad \text{not complete} \quad \text{not compact} \\
\mathcal{FO} \quad \text{r.e.} \quad \text{complete} \quad \text{compact} \\
\text{negation closed} \\
\text{downward closed} \\
\text{r.e.} \quad \text{complete} \quad \text{compact}
\]

Figure 8: The logics below Väänänen’s team logic \( \mathcal{T \mathcal{L}} \).

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A. Appendix

In the following several auxiliary theorems are proven that are used for the completeness proofs of the presented logics. The right column of each proof shows the applied rules with the line numbers of the arguments. If the line numbers are omitted then just the preceding lines are the arguments. The meta-rule “L” means that the axioms and rules of the system L were used, but the exact steps were omitted, as L proves all Boolean tautologies (see Theorem 3.19). The $^aH^0_\pi$ and $^aH^0_\land$ meta-rules are similar. Applications of modus ponens (E→) are mostly omitted for readability.

A.1. Derivations of the splitting elimination

A.1.1. Basic laws

| A | {φ ⊗ ψ} | Com⊗ | A | {φ ⊗ ψ} | Aug⊗ |
|---|---|---|---|---|---|
| 1 | {¬(φ ⊗ φ)} | Def. | 1 | {φ ⊗ ψ} | |
| 2 | (φ → ¬φ) | | 2 | (φ ⊗ ψ) → (φ → ¬ψ) | L |
| 3 | φ → ¬φ | L | 3 | | |
| 4 | φ → ¬ψ | C→ | 4 | φ → ¬(ψ ⊗ ψ) | L |
| 5 | φ → ¬ψ | L | 5 | φ → ¬(ψ ⊗ ψ) | MP→ (1+4) |
| 6 | φ → ¬ψ | Def. | 6 | φ → ¬ψ | Dis→ |
| 7 | φ → ¬ψ | RAA (A+6) | 7 | φ → ¬ψ | E→ (B+6) |
| 8 | φ → ¬ψ | Def. (A) | 8 | φ → ¬ψ | |
| 9 | φ → ¬ψ | RAA | 9 | φ → ¬ψ | |
| 10 | (φ ⊗ ψ) ⊗ ψ | |

A | {φ ⊗ φ} | Ass⊗ |
|---|---|
| 1 | {φ ⊗ (φ → ψ)} | |
| 2 | φ → ¬ψ | |
| 3 | φ → ¬ψ | L |
| 4 | φ → ¬ψ | |
| 5 | φ → ¬ψ | |
| 6 | φ → ¬ψ | |
| 7 | φ → ¬ψ | |
| 8 | φ → ¬ψ | |
| 9 | φ → ¬ψ | |
| 10 | φ → ¬ψ | |

A | {φ ⊗ ψ} | E→ |
|---|---|
| 1 | T → (¬α → α) | |
| 2 | T → (¬α → α) | |
| 3 | α → ¬α | |
| 4 | α → ¬α | |
| 5 | α → ¬α | |
| 6 | α → ¬α | |
| 7 | α → ¬α | |
| 8 | α → ¬α | |
| 9 | α → ¬α | |
| 10 | α → (α ⊗ ¬α) | |
| 11 | α → (α ⊗ ¬α) | |
| 12 | α → (α ⊗ ¬α) | |
| 13 | α → (α ⊗ ¬α) | |
| 14 | α → (α ⊗ ¬α) | |
| 15 | α → (α ⊗ ¬α) | |

α
### A.1.2. Distributive laws

| A  | \{α ⊗ (ϕ ⊗ ψ)\} | D⊗⊗ |
|----|-------------------|-----|
| 1 α | L                 |     |
| 2 (α ⊗ ϕ) → α | F→   |     |
| 3 ⊥ α → (¬(α ⊗ ψ) → ¬ψ) | L     |     |
| 4 (α ⊗ ϕ) → (¬(α ⊗ ψ) → ¬ψ) | MP→ (2+3) |     |
| 5 ψ → α | F→ (1) |     |
| 6 ⊥ α → (¬(α ⊗ ψ) → ¬ψ) | L     |     |
| 7 ψ → (¬(α ⊗ ψ) → ¬ψ) | MP→ (5+6) |     |
| 8 \{¬(α ⊗ ϕ) ⊗ (α ⊗ ψ)\} | Def., E¬ |     |
| 9 (α ⊗ ϕ) → ¬ψ | Dis¬ (4+9) |     |
| 10 (α ⊗ ϕ) → ¬ψ | C¬   |     |
| 11 ψ → ¬ψ | L (A) |     |
| 12 ψ → ¬ψ | Com⊗ |     |
| 13 ψ ⊗ ϕ | ⊸   |     |
| 14 ψ ⊗ ϕ | Def. |     |
| 15 ¬(ψ → ¬ψ) | RAA (12+15) |     |
| 16 (α ⊗ ϕ) ⊗ (α ⊗ ψ) | RAA (12+15) |     |

### A.1.3. Laws for E

| A  | \{α ⊗ (ϕ ⊗ ϕ)\} | D⊗⊗ |
|----|-------------------|-----|
| 1 α | L                 |     |
| 2 (α ⊗ ϕ) → α | MP⊗ |     |
| 3 α ⊗ (α ⊗ ϕ) | Com⊗ |     |
| 4 ⊥ (α ⊗ ϕ) → α | L     |     |
| 5 α ⊗ α | MP⊗ |     |
| 6 α ∨ α | F⊗  |     |
| 7 α | H₀  |     |
| 8 ⊥ (α ⊗ ϕ) → ϕ | L     |     |
| 9 (α ⊗ ϕ) ⊗ ϕ | MP⊗ (A+8) |     |
| 10 ϕ ⊗ (α ⊗ ϕ) | Com⊗ |     |
| 11 ⊥ (α ⊗ ϕ) → ϕ | L     |     |
| 12 ϕ ⊗ ϕ | MP⊗ |     |
| 13 ϕ ⊗ ϕ | Com⊗ |     |
| 14 α ⊗ (ϕ ⊗ ϕ) | L (7+13) |     |

| A  | \{α ⊗ (ϕ ⊗ ϕ)\} | D⊗⊗ |
|----|-------------------|-----|
| 1 α | L                 |     |
| 2 (ϕ ⊗ (ψ ⊗ ϕ)) | Def. |     |
| 3 ϕ → (ϕ ⊗ (ψ ⊗ ϕ)) | L     |     |
| 4 (ϕ ⊗ (ψ ⊗ ϕ)) | MP⊗ |     |
| 5 ϕ → (ϕ ⊗ (ψ ⊗ ϕ)) | L     |     |
| 6 ϕ → ~ψ | MP→ (3+4) |     |
| 7 ϕ → ~ψ | MP→ (3+5) |     |
| 8 ~ϕ → (ϕ → ~ψ) | L (6) |     |
| 9 ~ϕ → (ϕ → ~ψ) | L (7) |     |
| 10 ~ϕ → (ϕ → ~ψ) | Def. (8) |     |
| 11 ~ϕ → (ϕ ⊗ ϕ) | Def. (9) |     |
| 12 ~ϕ → (ϕ ⊗ ϕ) | L     |     |
| 13 ϕ ⊗ (ψ ⊗ ϕ) | RAA (A+12) |     |

| A  | \{α ⊗ Eβ\} | JoinE |
|----|------------|-------|
| 1 α | L (A) |     |
| 2 α | E→ (1+2) |     |
| 3 ~α | E (A) |     |
| 4 Eβ | L (A) |     |
| 5 ~α → ~β | RAA |     |
| 6 ~α → ~β | H₀, L₄ |     |
| 7 ~α → ~β | ~L |     |
| 8 E(α ∧ β) | E→, Def. (5+7) |     |

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A.2. Derivations of the modality elimination

### A.2.1. Distributive laws

| A | \{ψ ⊗ ϕ\} | D ⊗ |
|---|---|---|
| 1 | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} | L |
| 2 | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} | L |
| 3 | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} | Def. |
| 4 | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} | L (3) |
| 5 | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} | L (3) |
| 6 | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} |
| 7 | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} |
| 8 | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} |
| 9 | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} |
| 10 | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} |
| 11 | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} |
| 12 | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} |
| 13 | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} |
| 14 | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} |
| 15 | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} | \{ψ \rightarrow (ψ \rightarrow (ψ ⊗ ψ))\} |

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\[\lozenge (\psi \otimes \phi)\]
A.2.2. Laws of $E$ and $\Diamond$

| A                  | Com$\neg E$ | B                  | Com$\neg E$ |
|--------------------|--------------|--------------------|-------------|
| $\{\phi \rightarrow \psi\}$ | Def.         | $\{\phi \}$       | Join$\phi$  |
| $\{\phi \}$       | Aug$\phi$    | $\{E \rightarrow \phi\}$ | Join$\phi$  |

| 1                  | $\neg \phi \rightarrow \psi$ | $\neg \phi \rightarrow \psi$ | $\neg \phi \rightarrow \psi$ |
| 2                  | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ |
| 3                  | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ |
| 4                  | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ |
| 5                  | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ |
| 6                  | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ |
| 7                  | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ |
| 8                  | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ |
| 9                  | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ |
| 10                 | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ |
| 11                 | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ |
| 12                 | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ | $\Delta \psi \rightarrow \Delta \phi$ |

$\phi \rightarrow \psi$
\[ \text{\textcircled{A}} \{ \diamond (\alpha \otimes E\beta) \} \quad \text{\textcircled{IsolateE}} \]

| Step | Rule | Description |
|------|------|-------------|
| 1    | ⊢ 1  | \((\alpha \otimes E\beta) \rightarrow E(\alpha \land \beta)\) JoinE |
| 2    | ⊢ 1  | \((\alpha \otimes E\beta) \rightarrow \alpha\) L |
| 3    | ⊢ 1  | \(\alpha\) MP\(\diamond\) (A+11) |
| 4    | ⊢ 1  | \(E(\alpha \land \beta)\) MP\(\diamond\) (A+10) |
| 5    | ⊢ 1  | \(E \neg \neg (\alpha \land \beta)\) Com\(\diamond\)E |
| 6    | ⊢ 1  | \(\alpha \otimes E \neg \neg (\alpha \land \beta)\) L (12+14) |

\[ \text{\textcircled{A}} \{ \diamond \alpha \otimes E \neg \neg (\alpha \land \beta) \} \quad \text{\textcircled{IsolateE}} \]

| Step | Rule | Description |
|------|------|-------------|
| 1    | ⊢ 1  | \(\neg \neg (\alpha \land \beta)\) L |
| 2    | ⊢ 1  | \(\diamond (\alpha \land \beta)\) F\(\diamond\) |
| 3    | ⊢ 1  | \(E \neg \neg (\alpha \land \beta)\) L (1) |
| 4    | ⊢ 1  | \(\diamond E(\alpha \land \beta)\) Com\(\diamond\)E |
| 5    | ⊢ 1  | \(\diamond (\alpha \land \beta) \otimes E(\alpha \land \beta)\) Join\(\diamond\) (3+5) |
| 6    | ⊢ 1  | \(\neg \neg (\alpha \land \beta) \otimes E \neg \neg (\alpha \land \beta)\) Ded. Thm. |
| 7    | ⊢ 1  | \(\diamond (\alpha \otimes E\beta)\) L, SubE |
| 8    | ⊢ 1  | \(\diamond (\alpha \otimes E\beta)\) MP\(\diamond\) |
| 9    | ⊢ 1  | \(\neg \neg (\alpha \land \beta) \otimes E \neg \neg (\alpha \land \beta)\) L (A) |
| 10   | ⊢ 1  | \(\alpha \rightarrow (\alpha \otimes \alpha)\) L |
| 11   | ⊢ 1  | \(\diamond (\alpha \otimes \alpha)\) MP\(\diamond\) |
| 12   | ⊢ 1  | \(\diamond (\alpha \otimes E\beta) \otimes \diamond (\alpha \otimes \alpha)\) Lax\(\diamond\) |
| 13   | ⊢ 1  | \(\diamond (\alpha \otimes E\beta) \otimes (\alpha \otimes \alpha)\) D\(\diamond\) |
| 14   | ⊢ 1  | \(\diamond (\alpha \otimes E\beta) \otimes (\alpha \otimes E\beta)\) Com\(\diamond\), MP\(\diamond\) |
| 15   | ⊢ 1  | \(\alpha \otimes E\beta \rightarrow E\beta\) D\(\diamond\) |
| 16   | ⊢ 1  | \(\alpha \otimes E\beta \rightarrow E\beta\) Com\(\diamond\), Abs\(\diamond\) |
| 17   | ⊢ 1  | \(\alpha \otimes E\beta \rightarrow (\alpha \otimes E\beta)\) L |
| 18   | ⊢ 1  | \(\alpha \otimes (\alpha \otimes E\beta)\) MP\(\diamond\) (20+22) |

\(\diamond (\alpha \otimes E\beta)\)