INTEGRALITY STRUCTURES IN TOPOLOGICAL STRINGS I: FRAMED UNKNOT

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Abstract. We study the open string integrality invariants (LMOV invariants) for toric Calabi-Yau 3-folds with Aganagic-Vafa brane (AV-brane). In this paper, we focus on the case of the resolved conifold with one out AV-brane in any integer framing $\tau$, which is the large $N$ duality of the Chern-Simons theory for a framed unknot with integer framing $\tau$ in $S^3$. We compute the explicit formulas for the LMOV invariants in genus $g=0$ with any number of holes, and prove their integrality. For the higher genus LMOV invariants with one hole, they are reformulated into a generating function $g_m(q,a)$, and we prove that $g_m(q,a) \in \left(q^{1/2} - q^{-1/2}\right)^{-2}\mathbb{Z}[\left(q^{1/2} - q^{-1/2}\right)^2, a^{\pm 1/2}]$ for any integer $m \geq 1$. As a by product, we compute the reduced open string partition function of $\mathbb{C}^3$ with one AV-brane in framing $\tau$. We find that, for $\tau \leq -1$, this open string partition function is equivalent to the Hilbert-Poincaré series of the Cohomological Hall algebra of the $|\tau|$-loop quiver. It gives an open string GW/DT correspondence.

Contents

1. Introduction 2
   2. Topological strings 5
      2.1. Closed strings and Gromov-Witten invariants 5
      2.2. Open strings 6
      2.3. Integrality 7
      2.4. Lower genus cases 8
   3. Chern-Simons theory and large N duality 9
      3.1. Quantum invariants 9
      3.2. Large N duality 10
      3.3. Integrality of the quantum invariants 12
   4. LMOV invariants for framed unknot $U_\tau$ 13
      4.1. $a$-deformed A-polynomial as the mirror curve 13
      4.2. Disc countings 15
      4.3. Annulus counting 18
      4.4. Genus $g=0$ with more holes 21
      4.5. Genus $g \geq 1$, with one hole 21
   5. An open string GW/DT correspondence 26
      5.1. Reduced open string partition function of $(\mathbb{C}^3, D_\tau)$ 26
      5.2. Hilbert-Poincaré series of the Cohomological Hall algebra of the $m$-loop quiver 28
      5.3. The correspondence 29
   6. Appendix 29
References 30
1. Introduction

We study the integrality structures in topological string theory. Let $X$ be a Calabi-Yau 3-fold, by the work of Gopakumar and Vafa [22], the closed string free energy $F^X$, which is the generating function of Gromov-Witten invariants $K_{g,Q}$, has the following structure:

$$F^X = \sum_{g \geq 0} g_s^{2g-2} \sum_{Q \neq 0} K_{g,Q} e^{-Q \omega} = \sum_{g \geq 0, d \geq 1} \sum_{Q \neq 0} \frac{1}{d} N_{g,Q} \left( 2 \sin \frac{dg_s}{2} \right)^{2g-2} e^{-dQ \omega}$$

where $N_{g,Q}$ are integers and vanish for large $g$ or $Q$. When $X$ is a toric Calabi-Yau 3-fold, the above Gopakumar-Vafa conjecture was proved in [53, 26].

In the open string case, let us consider a Calabi-Yau 3-fold $X$ with a Lagrange submanifold $D$ in it. According to the work of Ooguri and Vafa [52], the generating function of the open Gromov-Witten invariants can also be expressed in terms of a series of new integers which were refined by Labastida, Mariño and Vafa [37, 38, 39]:

$$\sum_{g \geq 0} \sum_{Q \neq 0} g_s^{2g-2+l(\mu)} K_{\mu,g,Q} e^{-Q \omega} = \sum_{g \geq 0} \sum_{Q \neq 0} \sum_{d|\mu} (-1)^{l(\mu)+g} \prod_{i=1}^{l(\mu)} n_{\mu/d,g,Q} \prod_{j=1}^{l(\mu)} \left( 2 \sin \frac{dg_s}{2} \right)^{2g-2} e^{-dQ \omega}.$$  \hspace{1cm} (1)

These new integer $n_{\mu,g,Q}$ (here $\mu$ denote a partition of a positive integer) are called the LMOV invariants in this paper.

Although for any toric Calabi-Yau 3-fold with Agangica-Vafa brane (AV-brane for short) [1], we have the method of topological vertex [4, 33] to compute the open string partition function and furthermore the open Gromov-Witten invariants $K_{\mu,g,Q}$, it is difficult to compute the corresponding LMOV invariants $n_{\mu,g,Q}$ at the righthand side of the formula (1).

We will study these LMOV invariants $n_{\mu,g,Q}$. In this paper, we only focus on a special toric Calabi-Yau 3-fold, i.e. the resolved conifold $\hat{X}$ with one special Lagrangian submanifold (AV-brane $D_\tau$ in integer framing $\tau$). More general toric Calabi-Yau 3-fold will be discussed in a separated paper.

According to the large $N$ duality, the open string theory of $(\hat{X}, D_\tau)$ is the large $N$ duality of the Chern-Simons theory of $(S^3, U_\tau)$, where $U_\tau$ denotes a framed unknot (trivial knot) with integer framing $\tau$. The large $N$ duality of Chern-Simons and topological string theory was proposed by Witten [61], and developed further by [23, 52, 39]. Later, Mariño and Vafa [49] generalized it to the case of the knot including integer framing. The large $N$ duality of $(\hat{X}, D_\tau)$ and $(S^3, U_\tau)$ is expressed in terms of the following identity:

$$Z_{CS}^{(S^3, U_\tau)}(q, a; \chi) = Z_{str}^{(\hat{X}, D_\tau)}(g_s, a; \chi), \quad q = e^{-1/g_s}$$  \hspace{1cm} (2)

where the explicit expressions of the above two partitions in identity (2) are given by the formulas (13) and (14) respectively. The identity (2) implies the Mariño-Vafa formula [49, 34, 51], a very powerful Hodge integral identity, which implies various important results in intersection theory of moduli spaces of curves, see [63] for a review of the applications of Mariño-Vafa formula. The identity (2) was proved by J. Zhou [62] based on his previous joint works with C.-C. Liu and K. Liu [34, 38].

On the other hand side, through mirror symmetry, the partition function $Z_{str}^{(\hat{X}, D_\tau)}(g_s, a; \chi)$ can also be computed by B-model. The mirror geometry information of $(\hat{X}, D_\tau)$ is encoded in the mirror curve $\mathcal{C}_X$. The disc counting invariants of $(\hat{X}, D_\tau)$ were given by the coefficients of the superpotential related to the mirror curve [13], and this fact was proved in [16]. Furthermore,
the open Gromov-Witten invariants of higher genus with more holes can be computed by the Eynard-Orantin topological recursions [12]. This approach named the BKMP conjecture, was proposed by Bouchard, Klemm, Mariño and Pasquetti [7], and was fully proved in [13, 17] for any toric Calabi-Yau 3-fold with AV-brane, so we can also use the BKMP method to compute the LMOV invariants for \((X, D_r)\).

In conclusion, now we have three different approaches to compute the open string partition function \(Z_{str}^{(X, D_r)}(g_s, a; \mathbf{x})\) and their LMOV invariants \(n_{\mu, g, Q}(\tau)\): topological vertex [4, 33], Chern-Simons partition function [13] and the BKMP method [7].

In this paper, we first compute the genus 0 LMOV invariants by BKMP method. At first step, we illustrate the computations of the mirror curve of \((X, D_r)\) by using the new approach of [2]. It turns out that the mirror curve is given by:

\[
y - 1 - a^{-\frac{1}{2}}(-1)^\tau xy^\tau (ay - 1) = 0.\tag{3}
\]

By using mirror curve (3), we obtain the genus 0 with one-hole LMOV invariants \(n_{m, 0, l - \frac{m}{2}}(\tau)\) which is denoted by \(n_{m, l}(\tau)\) for brevity:

\[
n_{m, l}(\tau) = \sum_{d|m, d | l} \frac{\mu(d)}{d^2} c_{\frac{m}{d}, \frac{l}{d}}(\tau),
\]

where

\[
c_{m, l}(\tau) = \frac{(-1)^{m\tau + m + l}}{m^2} \binom{m\tau + l - 1}{l} \binom{l}{m - 1}.
\]

We prove the integrality of the number \(n_{m, l}(\tau)\).

**Theorem 1.1.** For any \(\tau \in \mathbb{Z}, m \geq 1, l \geq 0\), we have \(n_{m, l}(\tau) \in \mathbb{Z}\).

For LMOV invariants of genus 0 with two holes, we study the Bergmann kernel expansion in the BKMP construction, and find an explicit formula for the LMOV invariants \(n_{(m_1, m_2), 0, \frac{m_1 + m_2}{2}}(\tau)\) which is denoted by \(n_{(m_1, m_2)}(\tau)\) for short,

\[
n_{(m_1, m_2)}(\tau) = \frac{1}{m_1 + m_2} \sum_{d|m_1, d|m_2} \frac{\mu(d)}{d} (-1)^{(m_1 + m_2)(\tau + 1)/d} \binom{(m_1\tau + m_1)/d - 1}{m_1/d} \binom{(m_2\tau + m_2)/d}{m_2/d}.
\]

Then, we prove that

**Theorem 1.2.** For \(m_1, m_2 \geq 1\), and \(\tau \in \mathbb{Z}\), \(n_{(m_1, m_2)}(\tau) \in \mathbb{Z}\).

For the genus 0 LMOV invariants with more than two holes, we can compute the LMOV invariant \(n_{\mu, g, Q}(\tau)\) for general \(Q\) by using the BKMP construction. But it is hard to give an explicit formula for general \(Q\), except the case \(Q = \frac{|\mu|}{2}\), in which the Mariño-Vafa formula holds,

\[
n_{\mu, 0, \frac{|\mu|}{2}}(\tau) = (-1)^{|\mu|} \sum_{d|\mu} \frac{\mu(d)}{d^{l(\mu) - 1}} K_{\mu, 0, \frac{|\mu|}{2}}^\tau(\tau, \tau + 1)
\]

where

\[
K_{\mu, 0, \frac{|\mu|}{2}}^\tau(\tau, \tau + 1) = (-1)^{|\mu|} \tau(\tau + 1)^{l(\mu) - 1} \prod_{i=1}^{l(\mu)} \left( \frac{\mu_i (\tau + 1) - 1}{\mu_i - 1} \right) ^{l(\mu) - 3}.
\]

It is clear that \(K_{\mu, 0, \frac{|\mu|}{2}}^\tau(\tau) \in \mathbb{Z}\), for any \(\tau \in \mathbb{Z}\) and since \(l(\mu) \geq 3\), it immediately implies that:
Theorem 1.3. For a partition $\mu$ with $l(\mu) \geq 3$ and $\tau \in \mathbb{Z}$, $n_{\mu,0}[\mu](\tau) \in \mathbb{Z}$.

Next, we study the Chern-Simons partition function $Z_{CS}^{(S^3, U_\tau)}(q, a; x)$ whose explicit formula is given in (13). Following the works of [19] [22], we formulate the LMOV conjecture for a general framed knots, see Conjectures [33] and [1.14]. The mathematical structures of the LMOV conjecture for general link were first studied by K. Liu and P. Peng [40]. Then we formulate the higher genus and one hole LMOV invariants $n_{m,g,Q}(\tau)$ into a unified generating function $g_m(q, a)$. The integrality of the LMOV invariants $n_{m,g,Q}(\tau)$ is equivalent to the following theorem which will be proved in Section 4.5.

Theorem 1.4. Let $g_m(q, a) = \sum_{d|m} \mu(d) Z_{m/d}(q^d, a^d)$, where

$$Z_m(q, a) = (-1)^{m\tau} \sum_{|\nu|=m} \frac{1}{\nu \cdot \nu} \{ m \nu \}_{\nu} \{ \nu \}_{\nu},$$

see formula (37) for the definitions of the above quantum integers. For any integer $m \geq 1$ and any $\tau \in \mathbb{Z}$, there exist integers $n_{m,g,Q}(\tau)$, such that

$$g_m(q, a) = \sum_{g \geq 0} \sum_{Q} n_{m,g,Q}(\tau) z^{2g-2} a^Q \in z^{-2} \mathbb{Z}[z^2, a^{-\frac{1}{2}}],$$

where $z = q^\frac{1}{2} - q^{-\frac{1}{2}} = \{ 1 \}$.

In Section 5, we introduce the definition of the reduced open string partition function motivated by the work [2]. And we compute the reduced open string partition function $Z_{str}^{(C^3, D_\tau)}(g_s, x)$ for the trivial Calabi-Yau 3-fold $(C^3, D_\tau)$ (see the formula (17)). For brevity, we let $Z_\tau(q, x) = Z_{str}^{(C^3, D_\tau)}(g_s, x)$, it turns out that

$$Z_{\tau}(q, x) = \sum_{n \geq 0} \frac{(-1)^n(\tau-1) q^n \sum_{\nu=0}^{n-1} q^{\nu} x^{n/2}}{(1-q)(1-q^2)\cdots(1-q^n)} x^n.$$

By comparing with the expression of the Hilbert-Poincaré series $P_m(q, t)$ (see formula (??) of the Cohomological Hall algebra [30] of the $m$-loop quiver [54], we obtain the following open string GW/DT correspondence:

Theorem 1.5. For $\tau \leq -1$ (i.e. $-\tau \geq 1$), we have

$$Z_{\tau}(q, x) = P_{-\tau}(q, (-1)^{\tau-1} x q^{\frac{1}{2}}).$$

The main property of the Hilbert-Poincaré series $P_m(q, t)$ is the following factorization formula:

Theorem 1.6 (Conjecture 3.3 [54] or Theorem 2.3 [30]). There exists a product expansion

$$P_m(q, (-1)^{m-1} t) = \prod_{n \geq 1} \prod_{k \geq 0} \left( 1 - q^{l-k} t \right)^{-(1)^{(m-1)n} c_{n,k}}$$

for nonnegative integers $c_{n,k}$, such that only finitely many $c_{n,k}$ are nonzero for any fixed $n$.

The series $DT_n^{(m)}(q) = \sum_{k \geq 0} c_{n,k} q^k$ is called the quantum Donaldson-Thomas invariant in [54].

Besides, we also formulate the reduced LMOV conjecture (see conjecture [52]), which can be viewed as a weak form of the LMOV conjecture due to the original work of [54]. In particular, the reduced LMOV conjecture in the case of $(C^3, D_\tau)$ says:
Conjecture 1.7. There exist nonnegative integers \( N_{m,k}(\tau) \), and only finitely many \( N_{m,k}(\tau) \) are nonzero for any fixed \( m \geq 1 \). Such that

\[
Z_{\tau}(q, x) = \prod_{m \geq 1} \prod_{k \in \mathbb{Z}} \prod_{l \geq 0} \left( 1 - q^{k/2+l}x^m \right)^{N_{m,k}(\tau)}.
\]

Theorem 1.5 and Theorem 1.6 imply that, for \( \tau \leq -1 \), the reduced open string partition function \( Z_{\tau}(q, x) \) on \((\mathbb{C}^3, D_\tau)\) carries the product factorization:

\[
Z_{\tau}(q, x) = \prod_{n \geq 1} \prod_{k \geq 0} \prod_{l \geq 0} (1 - q^{n/2+l-k}x^n)^{(-1)^{n-1}c_{n,k}}.
\]

It provides the correspondence of the Ooguri-Vafa invariants (or weak LMOV invariants) \( N_{m,k}(\tau) \) and the Donaldson-Thomas invariants \( c_{n,k} \) for \( \tau \leq -1 \).

The toric diagram of the trivial Calabi-Yau 3-fold \((\mathbb{C}^3, D_\tau)\) is a topological vertex with one framed leg. The above open string GW/DT correspondence shows that there is a corresponding quiver with self-loops. Now we can ask the following questions:

**Questions:** What is the DT correspondence for the reduced open string partition \( \tilde{Z}_{\text{str}}^{(\hat{X}, D_\tau)} \) of the resolved conifold \((\hat{X}, D_\tau)\)? More general, we can ask, for a toric Calabi-Yau 3-fold with one out AV-brane \((X, D)\), if there exists a corresponding quiver with self-loops, such that the reduced open partition function \((X, D)\) is equal to the Hilbert-Poincaré series of the Cohomological Hall algebra attached to this quiver? And also for a framed knot \( K_\tau \), if there exits a corresponding quiver?

We will study these questions in our further work.

The rest of this paper is organized as follow: In section 2, we review the definitions of topological string partition functions, free energies, and the integrality structures appearing in topological strings. We introduce the definitions of Gopakumar-Vafa invariants in closed strings, and LMOV invariants in open strings. In section 3, we first review the Witten’s Chern-Simons theory for 3 manifolds and links, and the large N duality between the Chern-Simons theory and the topological strings. Then, a basic example of for the case of framed unknot was illustrated. We formulate the LMOV conjecture for the framed knot. In section 4, we study the LMOV invariants for framed unknot in detail. We first illustrate the computations of the mirror curve of \((\hat{X}, D_\tau)\) by using the new approach of \([2]\). Then, we compute the explicit formulas for genus 0 LMOV invariants by using mirror curve, and prove the integrality of them. Next, we formulate the higher genus with one hole LMOV invariants into a unified generating function by using LMOV conjecture for framed knot. We prove the integrality of these invariants, i.e. this generating function lies in a certain integral ring. In section 5, we introduce the definitions of the reduced partition functions and establish a correspondence of the open string on \((\mathbb{C}^3, D_\tau)\) and the Cohomological Hall algebra of a quiver with self-loops. In section 6, the appendix provides a proof of the integrality of the other BPS invariants obtained in \([18]\) by our method used in this paper.

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2. Topological strings

2.1. Closed strings and Gromov-Witten invariants. Topological strings on a Calabi-Yau 3-fold \( X \) have two types, the A-models and the B-models. The mathematical theory for A-model is Gromov-Witten theory. Let \( \overline{M}_{g,n}(X, Q) \) be the moduli space of stable maps \((f, \Sigma_g, p_1, ..., p_n)\), where \( f : \Sigma_g \to X \) is a holomorphic map from the nodal curve \( \Sigma_g \) to the Kähler manifold.
with \( f_\ast([\Sigma_g]) = \beta \in H_2(X, \mathbb{Z}) \). In general, \( \overline{\mathcal{M}}_{g,n}(X, Q) \) carries a virtual fundamental class \([\overline{\mathcal{M}}_{g,n}(X, Q)]^{vir}\) in the sense of [14]. The virtual dimension is given by:

\[
\text{vdim}[\overline{\mathcal{M}}_{g,n}(X, Q)]^{vir} = \int_Q c_1(X) + (\dim X - 3)(1 - g) + n.
\]

When \( X \) is a Calabi-Yau 3-fold, i.e. \( c_1(X) = 0 \), then \( \text{vdim}[\overline{\mathcal{M}}_g(X, Q)]^{vir} = 0 \). The genus \( g \), degree \( Q \) Gromov-Witten invariants of \( X \) is defined by

\[
K^X_{g,Q} = \int_{[\overline{\mathcal{M}}_{g,0}(X, Q)]^{vir}} 1
\]

which is usually denoted by \( K_{g,Q} \) for brevity without any confusions. In the A-model, the genus \( g \) closed free energy \( F^X_g \) of \( X \) is the generating function of Gromov-Witten invariants \( K_{g,Q} \).

\[
F^X_g = \sum_{Q \neq 0} K_{g,Q} e^{-Q \cdot \omega},
\]

where \( \omega \) is the Kähler class for \( X \). We define the total free energy \( F^X \) and partition function \( Z^X \) as

\[
F^X = \sum_{g \geq 0} g_s^{2g - 2} F^X_g, \quad Z^X = \exp(F^X).
\]

where \( g_s \) is the string coupling constant. The mathematical computations of the free energy \( F^X \) is mainly by the method of localizations [25] [21]. Especially, when \( X \) is a toric Calabi-Yau 3-fold, we have a more effective approach to obtain the partition function \( Z^X \) by the method of topological vertex [33].

Usually, the Gromov-Witten invariants \( K_{g,Q} \) are rational numbers, from M-theory, Gopakumar and Vafa [22] expressed the total free energy \( F^X \) in terms of the generating function of the new integer number \( N_{g,Q} \) obtained by counting BPS states:

\[
F^X = \sum_{g \geq 0} \sum_{Q \neq 0} g_s^{2g - 2} K_{g,Q} e^{-Q \cdot \omega} = \sum_{g \geq 0} \sum_{d \geq 1} \sum_{Q \neq 0} 1 \cdot N_{g,Q} \left( 2 \sin \frac{dg_s}{2} \right)^{2g - 2} e^{-dQ \cdot \omega}
\]

The integrality of the Gopakumar-Vafa invariants \( N_{g,Q} \) was first proved by P. Peng in the case of toric Del Pezzo surfaces [53]. The proof for general toric Calabi-Yau threefolds was given by Konishi in [26].

2.2. Open strings. Let us now consider the open sector of topological A-model of a Calabi-Yau 3-fold \( X \) with a submanifold \( D \) with dim \( H_1(D, \mathbb{Z}) = L \). The open sector topological A-model can be described by holomorphic maps \( \phi \) from open Riemann surface of genus \( g \) with \( l \)-holes \( \Sigma_{g,l} \) to \( X \), with Dirichlet condition specified by \( D \). These holomorphic maps are called open string instantons. More precisely, an open string instanton is a holomorphic map \( \phi : \Sigma_{g,h} \to X \) such that \( \partial \Sigma_{g,l} = \bigcup_{i=1}^l C_i \to D \subset X \) where the boundary \( \partial \Sigma_{g,l} \) of \( \Sigma_{g,l} \) consists of \( l \) connected components \( C_i \) mapped to Lagrangian manifold \( D \) of \( X \). Therefore, the open string instanton \( \phi \) is described by the following two different kinds of data: the first is the "bulk part" which is given by \( \phi_*[\Sigma_{g,l}] = Q \in H_2(X, L) \), and the second is the "boundary part" which is given by \( \phi_*[C_i] = w_i^a \gamma_\alpha \), for \( i = 1, ..., l \), where \( \gamma_\alpha, \alpha = 1, ..., L \) is a basis of \( H_1(D, \mathbb{Z}) \) and \( w_i^a \in \mathbb{Z} \). Let \( \vec{w} = (w^1, ..., w^L) \), and where \( w^a = (w_1^a, ..., w_L^a) \in \mathbb{Z} \), for \( \alpha = 1, ..., L \). We expect there exist the corresponding open Gromov-Witten invariants \( K_{\vec{w}; g, Q} \) determined by the data \( \vec{w}, Q \) in the genus
Now, the total free energy \( F_{\text{str}}^{(X, D)} \) is defined as

\[
F_{\text{str}}^{(X, D)}(g) = \sum_{g \geq 0} \sum_{l \geq 1} \sum_{\vec{w}} g_{\vec{w}}^{2g-2+l} F_{\vec{w}, g}(\omega) \frac{1}{l!} \prod_{i=1}^{L} \prod_{\alpha=1}^{l} \text{Tr} V^{\omega_{\alpha}}
\]

where \( \omega \) is also the Kähler class of \( X \), and \( V \) is a holonomy matrix of gauge group \( U(\infty) \) on the source A-brane [61].

Usually, we write the total free energy \( F_{\text{str}}^{(X, D)} \) in the form of the summation over all partitions [49].

\[
F_{\text{str}}^{(X, D)}(g) = \sum_{g \geq 0} \sum_{\vec{w}} \left( \prod_{i=1}^{L} \prod_{\alpha=1}^{l} \frac{1}{\text{Aut}(\vec{\mu})} g_{\vec{w}}^{2g-2+l(\vec{\mu})} F_{\vec{w}, g}(\vec{\mu}) \right) p_{\vec{\mu}}(\vec{x}),
\]

where \( p_{\vec{\mu}}(\vec{x}) = \prod_{\alpha=1}^{L} p_{\mu_{\alpha}}(x_{\alpha}), \) and for a partition \( \mu \in \mathcal{P}^{+} \), \( p_{\mu}(x) = \prod_{\alpha=1}^{h} p_{\mu_{\alpha}}(x) \). \( p_{n}(x) \) is the power sum symmetric function [16] given by \( p_{n}(x) = x_{1}^{n} + x_{2}^{n} + \cdots \). Where \( \mathcal{P}^{+} \) denotes the set of all the partitions of positive integers. Moreover, let \( \mathcal{P} = \mathcal{P}^{+} \cup \{0\} \). The notations \( \mathcal{P}^{+}, \mathcal{P} \) will be used frequently throughout this paper.

In the following, we only need to consider the case of \( L=1 \). It is useful to write the A-model generating function of \( F_{w, g}^{(X, D)} \) in the fixed genus \( g \) as follow:

\[
F_{(g, l)}^{(X, D)} = \sum_{w \in (\mathbb{Z}^{+})^{l}} F_{w, g}^{(X, D)} x_{1}^{w_{1}} \cdots x_{l}^{w_{l}}.
\]

The central problem in topological string theory is how to calculate \( F_{(g, l)}^{(X, D)} \). In particular, when \( X \) is a toric Calabi-Yau 3-fold, and \( D \) is a special Lagrangian submanifold named as Aganagic-Vafa A-brane in the sense of [1]. The open string partition function \( Z_{\text{str}}^{(X, D)} = \exp(F_{\text{str}}^{(X, D)}) \) can be computed by the method of topological vertex [4] [33]. However, in this case, there exists another more effective method to compute \( F_{(g, l)}^{(X, D)} \) by using the topological recursion of Eynard and Orantin [12]. This approach was first proposed by M. Mariño [47], and developed further by Bouchard, Klemm, Mariño and Pasquetti [7], so the conjectural equivalence of these two different approaches was called the BKMP conjecture. Finally, the BKMP conjecture was proved in [13] [17].

2.3. Integrality. Let \( q = e^{\sqrt{-1}g_{s}}, a = e^{-\omega} \), for the open string free energy \( F_{\text{str}}^{(X, D)}(q, a) \), we define the generating functions \( f_{\lambda}(q, a) \) by the following expansion formula,

\[
f_{\lambda}^{(X, D)}(q, a) = \sum_{d=1}^{\infty} \sum_{\lambda \in \mathcal{P}^{+}} f_{\lambda}(q^{d}, a^{d}) s_{\lambda}(x^{d}),
\]

where \( s_{\lambda}(x) \) is the Schur symmetric functions.

Just as in the closed string case [22], the open topological strings compute the partition function of BPS domain walls in a related superstring theory [52]. It follows that \( F^{(X, D)}(q, a) \) also has an integral expansion. This integrality structure was further refined in [37] [38] [39] which showed that \( f_{\lambda}(q, a) \) has the following integral expansion

\[
f_{\lambda}(q, a) = \sum_{g=0}^{\infty} \sum_{Q \neq 0} \sum_{|\mu|=|\lambda|} M_{\lambda}(q) N_{\mu, g, Q}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{2g-2} a^{Q},
\]
where \( N_{\mu,g,Q} \) are integers which compute the net number of BPS domain walls and \( M_{\lambda\mu}(q) \) is defined by

\[
M_{\lambda\mu}(q) = \sum_{\mu} \frac{\chi_{\lambda}(C_{\mu})\chi_{\mu}(C_{\nu})}{3\nu} \prod_{j=1}^{l(\nu)} (q^{-\nu_j/2} - q^{\nu_j/2})
\]

For convenience, we usually introduce the new integers

\[
n_{\mu,g,Q} = \sum_{\nu} \chi_{\nu}(C_{\mu})N_{\nu,g,Q}.
\]

**Definition 2.1.** These integers \( N_{\mu,g,Q} \) and \( n_{\mu,g,Q} \) are both called LMOV invariants.

Therefore,

\[
f_\lambda(q,a) = \sum_{g \geq 0} \sum_{Q \neq 0} \sum_{\mu \in \mathcal{P}} \frac{\chi_{\lambda}(C_{\mu})}{3\mu} n_{\mu,g,Q} \prod_{j=1}^{l(\mu)} (q^{-\mu_j/2} - q^{\mu_j/2}) (q^{-1/2} - q^{1/2})^{2g-2} a^Q
\]

By using the orthogonal relation \( \sum_{\lambda} \frac{\chi_{\lambda}(C_{\mu})\chi_{\lambda}(C_{\nu})}{3\nu} \chi_{\mu}(C_{\nu}) = \delta_{\mu,\nu} \), we obtain the following multiple covering formula for open string illustrated in [49]:

\[
\sum_{g \geq 0} \sum_{Q \neq 0} g_s^{2g-2+l(\mu)} K_{\mu,g,Q} a^Q = \sum_{g \geq 0} \sum_{Q \neq 0} \sum_{d|\mu} \frac{(-1)^{-l(\mu)+g}}{d^{l(\mu)-1}} n_{\mu/d,g,Q} \prod_{j=1}^{l(\mu)} (2 \sin \frac{\mu_j g_s}{2}) (2 \sin \frac{dg_s}{2})^{2g-2} a^Q.
\]

Hence we have the following integrality structure conjecture which is called the Labastida-Mariño-Ooguri-Vafa (LMOV) conjecture for open string.

**Conjecture 2.2** (LMOV conjecture for open string). Let \( F_{\mu}^{(X,D)} \) be the generating function function defined by

\[
F_{str}^{(X,D)} = \sum_{\mu} F_{\mu}^{(X,D)} p_{\mu}(x),
\]

then \( F_{\mu}^{(X,D)} \) has the integral expansion as in the righthand side of the formula [3].

There is no general definition for the open Gromov-Witten invariants \( K_{\mu,g,Q} \). However, just as mentioned in the previous subsection, when \( X \) is a toric Calabi-Yau \( 3 \)-fold, and \( D \) is the Aganagic-Vafa A-brane [1], the open string partition function \( Z_{str}^{(X,D)} \) can be fully computed by using the method of topological vertex [4, 33]. The open Gromov-Witten invariants \( K_{\mu,g,Q} \) can also be computed by the topological recursion formula [7]. It is natural to ask how to prove the Conjecture 2.2 in the case of toric Calabi-Yau 3-fold? In this paper, we study this conjecture for the resolved conifold with one AV-brane in integer framing \( \tau \). We first compute some LMOV integers as predicted by the formula (6), and then prove that they are really integers.

**2.4. Lower genus cases.** We illustrate some lower genus cases for the above multiple covering formula (6). By using the expansion \( \sin x = \sum_{k \geq 1} \frac{x^{2k-1}}{(2k-1)!} \), and taking the coefficients of \( g_s^{2g-2+l(\mu)} a^Q \) in formula (6), we obtain

\[
K_{\mu,0,Q} = \sum_{d|\mu} \frac{(-1)^{-l(\mu)+d(\mu)-3}}{d^{l(\mu)-3}} n_{\mu/d,0,Q}.
\]
\[ K_{\mu,1,Q} = \sum_{d|\mu} (-1)^{l(\mu)+1} \left( d^{l(\mu)-1} n^Q_{\frac{\mu}{d}, \frac{\mu}{d}} + \left( \frac{\sum_{j=1}^{l(\mu)} \mu_j^2}{24} d^{l(\mu)-3} - \frac{1}{12} d^{l(\mu)-1} \right) n^Q_{0,0,0} \right) \]

\[ K_{\mu,2,Q} = \sum_{d|\mu} (-1)^{l(\mu)} \left( d^{l(\mu)+1} n^Q_{\frac{\mu}{d}, \frac{\mu}{d}} + \frac{\sum_{j=1}^{l(\mu)} \mu_j^2}{24} d^{l(\mu)-1} n^Q_{\frac{\mu}{d}, \frac{\mu}{d}} \right) \]

\[ + \left( \frac{\sum_{j=1}^{l(\mu)} \mu_j^4}{1920} d^{l(\mu)-3} + \frac{\sum_{i<j} \mu_i^2 \mu_j^2}{576} d^{l(\mu)-3} - \frac{\sum_{j=1}^{l(\mu)} \mu_j^2}{288} d^{l(\mu)-1} + \frac{1}{240} d^{l(\mu)+1} \right) n^Q_{0,0,0} \]

for \( g = 0, \ g = 1 \) and \( g = 2 \) respectively. In fact, these formulas were firstly computed in [49].

Therefore

\[ F_{(0,1)} = \sum_{|\mu| = l} \sum_{Q} K_{\mu,0,Q} x_1^{\mu_1} \cdots x_l^{\mu_l} \]

\[ = \sum_{|\mu| = l} \sum_{Q} \sum_{d|\mu} (-1)^{l(\mu)} d^{l(\mu)-3} n^Q_{\frac{\mu}{d}, \frac{\mu}{d}} a^{Q} x_1^{\mu_1} \cdots x_l^{\mu_l} \]

\[ = (-1)^l \sum_{|\mu| = l} \sum_{Q} \sum_{d \geq 1} d^{-3} n_{\mu,0,Q} a^{Q} x_1^{\mu_1} \cdots x_l^{\mu_l}. \]

In particular

\[ F_{(0,1)} = - \sum_{m \geq 1} \sum_{d \geq 1} \sum_{Q} \frac{n_{m,0,Q}}{d^2} a^{Q} x_1^{dm}, \]

and for \( g = 1, l = 1, \)

\[ F_{(1,1)} = \sum_{m \geq 0} \sum_{Q} K_{(m),1,Q} a^{Q} x^m \]

\[ = \sum_{m \geq 0} \sum_{Q} \left( \sum_{d|m} n_{m/d,1,Q/d} + \left( \frac{m^2}{24} d^{-2} - \frac{1}{12} \right) n_{m/d,0,Q/d} \right) a^{Q} x^m \]

\[ = \sum_{m \geq 0} \sum_{Q} \sum_{d \geq 1} \frac{1}{d} \left( n_{m,1,Q} + \left( \frac{m^2}{24} - \frac{1}{12} \right) \right) a^{Q} x^{dm}. \]

3. Chern-Simons Theory and Large N Duality

3.1. Quantum Invariants. In the seminal paper [59], E. Witten defined a topological invariant of a 3-manifold \( M \) as a partition function of quantum Chern-Simons theory. Let \( G \) be a compact gauge group which is a Lie group, and \( M \) be an oriented three-dimensional manifold. Let \( \mathcal{A} \) be a \( g \)-valued connection on \( M \) where \( g \) is the Lie algebra of \( G \). The Chern-Simons action is given by

\[ S(\mathcal{A}) = \frac{k}{4\pi} \int_M Tr \left( \mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right) \]

where \( k \) is an integer called the level.

Chern-Simons partition function is defined as the path integral in quantum field theory

\[ Z^G(M; k) = \int e^{iS(\mathcal{A})} D\mathcal{A} \]

where the integral is over the space of all \( g \)-valued connections \( \mathcal{A} \) on \( M \). Although it is not rigorous, Witten developed some techniques to calculate such invariants.
If the 3-manifold $M$ contains a link $\mathcal{L}$, we let $\mathcal{L}$ be an $L$-component link with $\mathcal{L} = \bigsqcup_{j=1}^{L} \mathcal{K}_j$. Define $W_R_j(\mathcal{K}_j) = \text{Tr}_{R_j} \exp \oint_{\mathcal{K}_j} A$ to be the trace of holonomy along $\mathcal{K}_j$ taken in representation $R_j$. Then Witten’s invariant of the pair $(M, \mathcal{L})$ is given by

$$Z^G(M, \mathcal{L}; \{R_j\}; k) = \int e^{iS(A)} \prod_{j=1}^{L} W_{R_j}(\mathcal{K}_j) D A.$$ 

We often use the following normalization form

$$P_R^G(M, \mathcal{K}; k) = \frac{Z^G(M, \mathcal{K}; R; k)}{Z^G(M; k)}.$$ 

When $M = S^3$ and the lie algebra of $G$ is the semisimple lie algebra, Reshetikhin and Turaev [55, 56] developed a systematic way to constructed the above invariants by using the representation theory of quantum groups. Their construction led to the definition of the colored HOMFLY-PT invariants [38, 45], which can be viewed as the large $N$ limit of the quantum $U_q(sl_N)$ invariants. Usually, we use the notation $W_{\lambda_1, \ldots, \lambda_t}(L; q, a)$ to denote the (framing-independent) colored HOMFLY-PT invariants for a (oriented) link $\mathcal{L} = \bigsqcup_{j=1}^{L} \mathcal{K}_j$, where each component $\mathcal{K}_j$ is colored by an irreducible representation $V_{\lambda_j}$ of $U_q(sl_N)$. Some basic structures for $W_{\lambda_1, \ldots, \lambda_t}(L; q, a)$ were proved in [40, 41, 64]. It is difficult to obtain an explicit formula of a given link for any irreducible representations $\lambda$. We refer to [45] for an explicit formula for torus links, and a series of works due to Morozov et al (see for example [48]) proposed many conjectural formulas for the twist knots. However, in this paper, we only need the following explicit formula for a trivial knot (unknot) $U$ (for example, see formula (4.6) in [40]).

$$W_{\lambda}(U; q, a) = \sum_{\mu} \chi_{\lambda}(\mu) \prod_{i=1}^{l(\mu)} \frac{a^{\mu_i} - a^{-\mu_i}}{q^{\mu_i} - q^{-\mu_i}}.$$

3.2. Large N duality. In another fundamental work of Witten [61], the $SU(N)$ Chern-Simons gauge theory on a three-manifold $M$ was interpreted as an open topological string theory on $T^*M$ with $N$ topological branes wrapping $M$ inside $T^*M$. Furthermore, Gopakumar and Vafa [23] conjectured that the large $N$ limit of $SU(N)$ Chern-Simons gauge theory on $S^3$ is equivalent to the closed topological string theory on the resolved conifold. Furthermore, Ooguri and Vafa [52] generalized the above construction to the case of a knot $K$ in $S^3$. They introduced the Chern-Simons partition function $Z_{CS}^{(S^3, K)}$ for $(S^3, K)$ which is the generating function of the colored HOMFLY-PT invariants in all irreducible representations.

$$Z_{CS}^{(S^3, K)}(q, a, x) = \sum_{\lambda \in P} W_{\lambda}(\mathcal{L}, q, a) s_{\lambda}(x).$$

Ooguri and Vafa conjectured that for any knot $K$ in $S^3$, there exists a corresponding Lagrangian submanifold $\mathcal{D}_K$, such that the Chern-Simons partition function $Z_{CS}^{(S^3, D)}$ is equal to the open topological string partition function $Z_{str}^{(X, D_K)}$ on $(X, \mathcal{D}_K)$. They have established this duality in the case of a trivial knot $U$ in $S^3$, and the link case was further discussed in [39].

In general, we first should find a way to construct the Lagrangian submanifold $\mathcal{D}_L$ corresponding to the link $\mathcal{L}$ in geometry. See [39, 27, 58, 10] for the constructions for some special links. Furthermore, if we have found the Lagrangian submanifold, we need to compute the open string partition function under this geometry. For the trivial knot in $S^3$, the dual open string partition function was computed by J. Li and Y. Song [43] and S. Katz and C.-C.M. Liu [28].
On the other hand side, Aganagic and Vafa [11] introduced the special Lagrangian submanifold in toric Calabi-Yau 3-fold which we call Aganagi-Vafa A-brane (AV-brane) and studied its mirror geometry, then they computed the counting of the holomorphic disc end on AV-brane by using the idea of mirror symmetry. Moreover, Aganagic and Vafa surprisingly found the computation by using mirror symmetry and the result from Chern-Simons knot invariants [52] are matched. Furthermore, in [3], Aganagic, Klemm and Vafa investigated the integer ambiguity appearing in the disc counting and discovered that the corresponding ambiguity in Chern-Simons theory was described by the framing of the knot. They checked that the two ambiguities match for the case of the unknot, by comparing the disk amplitudes on both sides.

Then, Mariño and Vafa [19] generalized the large N duality to the case of knots with arbitrary framing. They studied carefully and established the large N duality between a framed unknot in $S^3$ and the open string theory on resolved conifold with AV-brane by using the mathematical approach in [28]. By comparing the coefficient of the highest degree of the Kähler parameter in this duality, they derived a remarkable Hodge integral identity which now is called the Mariño-Vafa formula. Two mathematical proofs for the Mariño-Vafa formula were given in [34] and [51] respectively. We describe this duality in more details. For a framed knot $K$ with framing $\tau \in \mathbb{Z}$, we define the framed colored HOMFLYPT invariants $\mathcal{K}_\tau$ as follow,

$$H_\lambda(K_\tau, q, a) = (-1)^{|\lambda|} q^{\frac{\kappa_\lambda}{2}} W_\lambda(K, q, a),$$

where $\kappa_\lambda = \sum_{i=1}^{l(\lambda)} \lambda_i (\lambda_i - 2i + 1)$.

The Chern-Simon partition function for $(S^3, K_\tau)$ is given by

$$Z_{CS}^{(S^3, K_\tau)}(q, a; x) = \sum_{\lambda \in \mathbb{P}} H_\lambda(K_\tau, q, a) s_\lambda(x),$$

where $\mathcal{X} := \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ be the resolved conifold, and $D_\tau$ be the corresponding AV-brane. The open string partition function for $(\mathcal{X}, D_\tau)$ has the structure

$$Z_{str}^{(\mathcal{X}, D_\tau)}(g, a; x) = \exp \left( - \sum_{g \geq 0, \mu} \frac{\sqrt{-1} \mathcal{A}_{\mu}(\mu)}{A_{\mu}(\mu)} g^2 q^{2g-2+l(\mu)} F_{\mu,g}(a)p_{\mu}(x) \right),$$

where $F_{\mu,g}(a) = \sum_{Q \in \mathbb{Z}/2} K_{\mu,g,Q} a^Q$ and $K_{\mu,g,Q}$ is the open Gromov-Witten invariants defined by

$$K_{\mu,g,Q} = \int_{[\mathcal{M}_{g,|\mu|}(D^2, S^3; 2Q, \mu_1, \ldots, \mu_l)]} e(\mathcal{V}),$$

defined in S. Katz and C.-C. Liu [28]. In particular, when $Q = \frac{|\mu|}{2}$, the computations in [28] gives

$$K_{\mu,g,|\mu|/2} = (-1)^{|\mu|} (\tau + 1)^{l(\mu) - 1} \prod_{i=1}^{l(\mu)} \frac{\Pi_{j=1}^{\mu_i - 1} (\mu_i j + j)}{(\mu_i - 1)!} \left\{ \int_{[\mathcal{M}_{g,|\mu|}(D^2, S^3)]} \frac{\Lambda_{g}^{\prime}(1) \Lambda_{g}^{\prime}(\tau - 1) \Lambda_{g}^{\prime}(\tau)}{\prod_{j=1}^{l(\mu)} (1 - \mu_j \psi_j)} \right\},$$

where $\Lambda_{g}^{\prime}(\tau) = \tau^g - \lambda_1 \tau^{g-1} + \cdots + (-1)^g \lambda_g$. Therefore, the large N duality in this case is given the following identity:

$$Z_{CS}^{(S^3, U_\tau)}(q, a; x) = Z_{str}^{(\mathcal{X}, D_\tau)}(g, a; x)$$

where $q = e^{ig}$. By taking the coefficients of $a^{\frac{|\mu|}{2}}$ of the following equality:

$$[p_{\mu}(x) g_{s}^{2g-2+l(\mu)}] \log Z_{CS}^{(S^3, U_\tau)}(q, a; x) = [p_{\mu}(x) g_{s}^{2g-2+l(\mu)}] \log Z_{str}^{(\mathcal{X}, D_\tau)}(g, a; x),$$

the following identity:
we get the Mariño-Vafa formula which is a Hodge integral identity with triple $\lambda$ classes. It provides a very powerful tool in studying the intersection theory of moduli space of curves. From it, we can derive the Witten conjecture [60] [24], the ELSV formula [11], and various Hodge integral identities, see [35] [32] [63].

Combining the duality ideas above, together with several new technical ingredients, Aganagic, Klemm, Mariño and Vafa finally developed a systematic method, gluing the topological vertex, to compute all loop topological string amplitudes on toric Calabi-Yau manifolds [5, 4]. The mathematical theory for topological vertex was finally established in [33]. This method give an effective way to compute both the closed and open string partition function for a toric Calabi-Yau 3-fold with AV-brane. Therefore, we have an explicit formula for the partition function of resolved conifold $Z_{\text{str}}(X,D_C)(g_s,a)$, by comparing the explicit formula $Z_{CS}^{(S^3,U^3)}(g,a)$ of Chern-Simons partition function describe above, J. Zhou proved the identity [16] in [62] based on the results of their previous works [34, 36, 33].

3.3. Integrality of the quantum invariants. Now, let us collect the above discussions together. Let $\mathcal{L}$ be a link in $S^3$, the large N duality predicts there exists a Lagrangian submanifold $\mathcal{D}_\mathcal{L}$ in the resolved conifold $\hat{X}$, and provides us the identity [16]. Since $Z_{\text{str}}^{(X,D_C)}(g_s,a,x)$ has the integrality structures by the discussions in section 2.3, it implies that $Z_{CS}^{(S^3,U^3)}(g,a,x)$ also inherits the integrality structure. Usually, this integrality structure is called the LMOV conjecture for link in [40]. Furthermore, as mentioned previously, the large N duality was generalized to the case of framed knot $K_\tau$ with framing $\tau \in \mathbb{Z}$ in [49], with the Chern-Simons partition $Z_{CS}^{(S^3,K_\tau)}$ for framed knot $K_\tau$ given in formula [13]. For convenience, we only formulate the LMOV conjecture for framed knot $K_\tau$ in the following, although the conjecture should also holds for any framed link, see [32].

**Conjecture 3.1** (LMOV conjecture for framed knot or framed LMOV conjecture). Let

$$F_{CS}^{(S^3,K_\tau)}(q,a,x) = \log Z_{CS}^{(S^3,K_\tau)}(q,a,x)$$

be the Chern-Simons free energy for a framed knot $K_\tau$ in $S^3$. Then there exist functions $f_\lambda(K_\tau;q,a)$ such that

$$F_{CS}^{(S^3,K_\tau)}(q,a,x) = \sum_{d=1}^{\infty} \frac{1}{d} \sum_{\lambda \in \mathcal{P}} f_\lambda(K_\tau;q^d,a^d) s_\lambda(x^d).$$

Let $\hat{f}_\mu(K_\tau;q,a) = \sum_\lambda f_\lambda(K_\tau;q,a) M_\lambda(q)^{-1}$, where $M_\lambda(q)$ is defined in the formula [2]. Denote $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, then for any $\mu \in \mathcal{P}^+$, there are integers $N_{\mu,g,Q}(\tau)$ such that

$$\hat{f}_\mu(K_\tau;q,a) = \sum_{g \geq 0} \sum_Q N_{\mu,g,Q}(\tau) z^{2g-2} a^Q \in z^{-2} \mathbb{Z}[z^2,a^{\pm \frac{1}{2}}].$$

Therefore,

$$\hat{g}_\mu(K_\tau;q,a) = \sum_\nu \chi_\nu(C_\mu) \hat{f}_\nu(K_\tau;q,a)$$

$$= \sum_{g \geq 0} \sum_Q n_{\mu,g,Q}(\tau) z^{2g-2} a^Q \in z^{-2} \mathbb{Z}[z^2,a^{\pm \frac{1}{2}}].$$

where $n_{\mu,g,Q}(\tau) = \sum_\nu \chi_\nu(C_\mu) N_{\nu,g,Q}(\tau)$.

K. Liu and P. Peng [40] first studied the mathematical structures of LMOV conjecture for general links (as to the Chern-Simons partition [11]), which is equivalent to the framed LMOV
conjecture for any links in framing zero. They provided a proof for this case by using cut-and-join analysis and the cabling technique [45]. Motivated by the work [49], K. Liu and P. Peng [42] formulated the framed LMOV conjecture for any links in framing zero. They provided a proof for this case by using cut-and-join analysis and the cabling technique [45]. Motivated by the work [49], K. Liu and P. Peng [42] formulated the framed LMOV conjecture for any links in framing zero. They provided a proof for this case by using cut-and-join analysis and the cabling technique [45].

4. LMOV invariants for framed unknot $U_\tau$

In Section 3.2, we have showed that, for a framed unknot $U_\tau$ in $S^3$, the large N duality holds [62]:

$$Z_{CS}^{(S^3,U_\tau)}(q, a; x) = Z_{str}^{(\hat{X},D_\tau)}(g_s, a; x), \quad q = e^{\sqrt{-1}g_s}. $$

So one can compute LMOV invariants completely by using the colored HOMFLY-PT invariants of the framed unknot $U_\tau$. On the other hand, by using mirror symmetry, one can also compute the partition function $Z_{str}^{(\hat{X},D_\tau)}(g_s, a; x)$ from B-model. The mirror geometry information of $(\hat{X},D_\tau)$ is encoded in the mirror curve $C_{\hat{X}}$. The disc counting information of $(\hat{X},D_\tau)$ was given by the superpotential related to the mirror curve [1, 3], and this fact was proved in [16].

Furthermore, the open Gromov-Witten invariants with higher genus with more holes can be computed by the Eynard-Orantin topological recursions [12]. This approach named the BKMP conjecture, was proposed by Bouchard, Klemm, Mariño and Pasquetti [7], and was fully proved in [13, 17] for any toric Calabi-Yau 3-fold with AV-brane, so we can also use the BKMP method to compute the LMOV invariants for $(\hat{X},D_\tau)$. To determine the mirror curve of $(\hat{X},D_\tau)$, there is standard method in toric geometry. However, in [2], Aganagic and Vafa proposed another effective method to compute the mirror curve, their method can be applied to the more general large N geometry of any knot in $S^3$. The rest contents of this section will be organized as follows, we first illustrate the computations of the mirror curve of $(\hat{X},D_\tau)$ by using the new approach of [1]. Then, we compute the explicit formulas for genus 0 LMOV invariants, and prove the integrality of them. Next, we formulate the higher genus with one hole LMOV invariants into a unified generating function, and we prove this generating function lies in a certain integral ring.

4.1. $a$-deformed A-polynomial as the mirror curve. The method used in [2] to compute the mirror curve is based on the fact that, colored HOMFLY-PT invariants colored by a partition with a single row is a $q$-holonomic function, this fact was conjectured and used in many literatures, such as [14, 15], and was finally proved in [20]. In fact, such idea can go back to [19].

Now, we illustrate the computation for the framed unknot $U_\tau$. We first compute the noncommutative $a$-deformed $A$-polynomial (it is called the $Q$-deformed $A$-polynomial in [2], the variable $Q$ in [2] is the variable $a$ here) for $U_\tau$.

By formula (11), the colored HOMFLY-PT invariants colored by partition $(n)$ for the unknot $U$ is given by

$$W_n(U; q, a) = a^{\frac{n}{2}} - a^{-\frac{n}{2}} + a^{\frac{n-1}{2}}q^{\frac{n+1}{2}} - a^{-\frac{n-1}{2}}q^{-\frac{n+1}{2}}.$$

It gives the recursion

$$(q^{n+1} - 1)W_{n+1}(U; q, a) - (a^{\frac{1}{2}} q^{n+\frac{1}{2}} - a^{-\frac{1}{2}} q^{-\frac{1}{2}})W_n(U; q, a) = 0.$$ 

By formula (12), the framed colored HOMFLY-PT invariants for the framed unknot with framing $\tau \in \mathbb{Z}$ is

$$\mathcal{H}_n(U_\tau; q, a) = (-1)^{n\tau} q^{\frac{n(n-1)}{2}} W_n(U; q, a).$$
So we get the recursion for $\mathcal{H}_n(U_\tau; q, a)$ as follow

$$(17) \quad (-1)^n (q^{n+1} - 1) \mathcal{H}_{n+1}(U_\tau; q, a) - (a^\frac{1}{2} q^{n+\frac{1}{2}} - a^{-\frac{1}{2}} q^{\frac{1}{2}}) q^{n \tau} \mathcal{H}_n(U_\tau; q, a) = 0.$$  

For a general series $\{\mathcal{H}_n(q, a)\}_{n \geq 0}$, we introduce two operators $M$ and $L$ act on $\{\mathcal{H}_n(q, a)\}_{n \geq 0}$ as follow:

$$M \mathcal{H}_n = q^n \mathcal{H}_n, \quad L \mathcal{H}_n = \mathcal{H}_{n+1}.$$  

then $LM = qML$.

**Definition 4.1.** The noncommutative $a$-deformed A-polynomial for series $\{\mathcal{H}_n(q, a)\}_{n \geq 0}$ is a polynomial $\hat{A}(M, L; q, a)$ of operators $M, L$, such that

$$\hat{A}(M, L; q, a) \mathcal{H}_n(q, a) = 0, \text{ for } n \geq 0.$$  

and $A(M, L; a) = \lim_{q \to 1} \hat{A}(M, L; q, a)$ is called the a-deformed A-polynomial.

Therefore, from the recursion (17), we obtain the noncommutative a-deformed A-polynomial for $U_\tau$ as follow:

$$\hat{A}_{U_\tau}(M, L; q; a) = (-1)^\tau (qM - 1)L - M^\tau (a^\frac{1}{2} q^\frac{1}{2} M - a^{-\frac{1}{2}} q^{\frac{1}{2}}).$$  

and the a-deformed A-polynomial is

$$A_{U_\tau}(M, L; a) = \lim_{q \to 1} \hat{A}_{U_\tau}(M, L; q; a) = (-1)^\tau (M - 1)L - M^\tau (a^\frac{1}{2} M - a^{-\frac{1}{2}}).$$

In order to get the mirror curve of $U_\tau$, we need the following general result which is written in the following lemma. Let $Z(x) = \sum_{k \geq 0} \mathcal{H}_k(q, a)x^k$ be a generating function of the series $\{\mathcal{H}_k(q, a)|k \geq 0\}$. We also introduce two operators $\hat{x}, \hat{y}$ act on $Z(x)$ as follow:

$$\hat{x}Z(x) = xZ(x), \quad \hat{y}Z(x) = Z(qx).$$

then $\hat{y}\hat{x} = q\hat{x}\hat{y}$. It is easy to obtain the following result (see Lemma 2.1 in [18] for the similar statement).

**Proposition 4.2.** Given a noncommutative A-polynomial $\hat{A}(M, L, q, a) = \sum_{i, j} c_{i, j} M^i L^j$ for the series $\{\mathcal{H}_k(q, a)|k \geq 0\}$, then we have

$$(18) \quad \hat{A}(\hat{y}, \hat{x}^{-1}, q, a)Z(x) = \sum_{i, j} \sum_{-j \leq k \leq -1} \mathcal{H}_{k+j} q^{ki} x^k.$$  

**Proof.** Since

$$\hat{A}(\hat{y}, \hat{x}^{-1}, q, a)Z(x) = \sum_{i, j} c_{i, j} \hat{y}^i \hat{x}^{-j} Z(x)$$

$$= \sum_{i, j} c_{i, j} q^{-ij} x^{-j} Z(q^i x)$$

$$= \sum_{i, j} c_{i, j} \sum_{n \geq 0} \mathcal{H}_n q^{(n-j)i} x^{n-j}$$

$$= \sum_{i, j} c_{i, j} \sum_{k \geq 0} \mathcal{H}_{k+j} q^{ki} x^k + \sum_{i, j} \sum_{-j \leq k \leq -1} a_{k+j} q^{ki} x^k.$$  

and by the definitions of the operators $M, L, \hat{A}(M, L, q, a)\mathcal{H}_k = 0$ gives

$$\sum_{i, j} c_{i, j} q^{ki} \mathcal{H}_{k+j} = 0, \text{ for } k \geq 0.$$  

We obtain the formula (18).
Finally, the mirror curve is given by
\[ A(y, x^{-1}; a) = \lim_{q \to 1} \hat{A}(\hat{y}, \hat{x}^{-1}; q, a) = 0 \]
In our case, the mirror curve is:
(19) \[ A_{U_{\tau}}(y, x^{-1}; a) = y - 1 - a^{-\frac{1}{\tau}}(-1)^{\tau}xy^\tau(ay - 1) = 0. \]

4.2. Disc countings. For convenience, we let \( X = a^{-\frac{1}{\tau}}(-1)^{\tau}x, \) and \( Y = 1 - y, \) then the mirror curve (19) becomes the functional equation
(20) \[ Y = X(1 - Y)\tau(1 - a(1 - Y)). \]
In order to solve the above equation, we introduce the following Lagrangian inversion formula [57].

**Lemma 4.3.** Let \( \phi(\lambda) \) be an invertible formal power series in the indeterminate \( \lambda. \) Then the functional equation \( Y = X\phi(Y) \) has a unique formal power series solution \( Y = Y(X). \) Moreover, if \( f \) is a formal power series, then
(21) \[ f(Y(X)) = f(0) + \sum_{n \geq 1} \frac{X^n}{n} \left[ \frac{df(\lambda)}{d\lambda} \phi(\lambda)^n \right]_{\lambda = 0} \]

**Remark 4.4.** In the following, we will frequently use the binomial coefficient \( \binom{n}{k} \) for all \( n \in \mathbb{Z}. \) That means for \( n < 0, \) we define \( \binom{n}{k} = \frac{(-1)^k(-n+k-1)}{k!}. \)

In our case, we take \( \phi(Y) = (1 - Y)^\tau(1 - a(1 - Y)). \) Let \( f(Y) = 1 - Y, \) by formula (21), we obtain
\[ y(X) = 1 - Y(X) = 1 + \sum_{n \geq 1} \frac{X^n}{n} \sum_{i \geq 0} (-1)^{n+i} \binom{n}{i} \left( \frac{n\tau + i}{n-1} \right) a^i \]
since \( \phi(\lambda)^n \) has the expansion
\[ \phi(\lambda)^n = (1 - \lambda)^{n\tau}(1 - a(1 - \lambda))^n = \sum_{i \geq 0} \binom{n}{i} (-a)^i(1 - \lambda)^{n\tau+i} = \sum_{i,j \geq 0} \binom{n}{i} (-1)^{i+j}(\frac{n\tau + i}{j}) a^i \lambda^j. \]
Moreover, if we let \( f(Y(X)) = \log(1 - Y(X)), \) then
\[ \left[ \frac{df(\lambda)}{d\lambda} \phi(\lambda)^n \right]_{\lambda = 0} = \sum_{i \geq 0} (-1)^i \binom{n}{i} \sum_{j = 0}^{n-1} (-1)^{j+1} \binom{n\tau + i}{j} a^i = \sum_{i \geq 0} (-1)^i \binom{n}{i} (-1)^n \binom{n\tau + i - 1}{n - 1} a^i \]
where we have used the combinatoric identity:
\[ \sum_{j = 0}^{n-1} (-1)^{j+1} \binom{m}{j} = (-1)^n \binom{m-1}{n-1}. \]
Formula (21) gives
\[ \log(y(X)) = \log(1 - Y(X)) = \sum_{n \geq 1} \frac{X^n}{n} \sum_{i \geq 0} (-1)^{n+i} \binom{n}{i} \left( \frac{n\tau + i - 1}{n - 1} \right) a^i. \]
i.e.
\[ \log(y(x)) = \sum_{n \geq 1} \frac{x^n}{n^2} \sum_{i \geq 0} (-1)^{n\tau + i} \binom{n}{i} \left( \frac{n\tau + i - 1}{n - 1} \right) a^{i - \frac{n}{2}}. \]

By BKMP’s construction in genus 0 with one hole, one obtains
\[ F_{(0,1)} = \int \log(y(x)) \frac{dx}{x} \]
\[ = \sum_{n \geq 1} \frac{x^n}{n^2} \sum_{i \geq 0} (-1)^{n\tau + i} \binom{n}{i} \left( \frac{n\tau + i - 1}{n - 1} \right) a^{i - \frac{n}{2}}. \]

By formula (23), and if we let \( n_{m,l}(\tau) = n_{m,0,l-m/2}(\tau) \), then
\[ F_{(0,1)} = - \sum_{m \geq 1} \sum_{d | m,d | l} d^{-2} n_{m,l-m/2}(\tau) x^m a^{l-m/2}. \]

If we let
\[ c_{m,l}(\tau) = - \left( \frac{-1}{m\tau + m + l} \right) \left( \frac{l}{m} \right) \left( \frac{l}{m - 1} \right), \]
by comparing the coefficients of \( x^m a^{l-m/2} \) in (23) and (22), we have
\[ c_{m,l}(\tau) = \sum_{d | m,d | l} \frac{n_{m,d,l/d}(\tau)}{d^2}. \]

By Möbius inversion formula,
\[ n_{m,l}(\tau) = \sum_{d | m,d | l} \frac{\mu(d)}{d^2} c_{m,l}(\tau). \]

**Theorem 4.5.** For any \( \tau \in \mathbb{Z} \), \( m \geq 1, l \geq 0 \), we have \( n_{m,l}(\tau) \in \mathbb{Z} \).

Before proving Theorem 4.5 we define the following function, for nonnegative integer \( n \) and prime number \( p \),
\[ f_p(n) = \prod_{i=1, p \nmid i}^{n} i = \frac{n!}{p^{[n/p]} [n/p]!}. \]

**Lemma 4.6.** For odd prime numbers \( p \) and \( \alpha \geq 1 \) or for \( p = 2, \alpha \geq 2 \), we have \( p^{2\alpha} \mid f_p(p^{\alpha} n) - f_p(p^{\alpha})^n \). For \( p = 2, \alpha = 1 \), \( f_2(2n) \equiv (-1)^{[n/2]} \) (mod 4)

**Proof.** With \( \alpha \geq 2 \) or \( p > 2 \), \( p^{\alpha-1}(p - 1) \) is even,
\[ f_p(p^{\alpha} n) - f_p(p^{\alpha}(n - 1)) f_p(p^{\alpha}) \]
\[ = f_p(p^{\alpha}(n - 1)) \left( \prod_{i=1, p \nmid i}^{p^{\alpha}} (p^{\alpha}(n - 1) + i) - f_p(p^{\alpha}) \right) \]
\[ \equiv p^{\alpha}(n - 1) f_p(p^{\alpha}(n - 1)) f_p(p^{\alpha}) \left( \sum_{i=1, p \nmid i}^{p^{\alpha}} \frac{1}{i} \right) \pmod{p^{2\alpha}} \]
\[ \equiv p^\alpha (n - 1) f_p(p^\alpha (n - 1)) f_p(p^\alpha) \left( \sum_{i=1,p|i} \left( \frac{1}{i} + \frac{1}{p^\alpha - i} \right) \right) \pmod{p^{2 \alpha}} \]

Thus the first part of the Lemma is proved by induction. For \( p = 2, \alpha = 1 \), the formula is straightforward.

**Lemma 4.7.** For odd prime number \( p \) and \( m = p^\alpha a, l = p^\beta b, p \nmid a, p \nmid b, \alpha \geq 1, \beta \geq 0 \), we have

\[ p^{2 \alpha} \mid \left( \frac{m}{l} \right) \left( \frac{m \tau + l - 1}{m - 1} \right) - \left( \frac{m}{l} \right) \left( \frac{m \tau + l - 1}{m - 1} \right) \]

where for \( \beta = 0 \), the second term is defined to be zero.

**Proof.**

\[ \left( \frac{m}{l} \right) \left( \frac{m \tau + l - 1}{m - 1} \right) - \left( \frac{m}{l} \right) \left( \frac{m \tau + l - 1}{m - 1} \right) \]

\[ = \left( \frac{m}{l} \right) \left( \frac{m \tau + l - 1}{m - 1} \right) \left( \frac{f_p(m)}{f_p(l)f_p(m - l)} \cdot \frac{f_p(m \tau + l)}{f_p(m \tau + l)} \right) - \left( \frac{m}{l} \right) \left( \frac{m \tau + l - 1}{m - 1} \right) \]

Write \( \left( \frac{m}{l} \right) = \frac{m}{l} \left( \frac{m - 1}{l - 1} \right) \) and \( \left( \frac{m \tau + l - 1}{m - 1} \right) = \frac{m}{m \tau + l} \left( \frac{m \tau + l}{l} \right) \), both are divisible by \( p^{\max(\alpha - \beta, 0)} \). Each element of \( \{m, l, m - l, m \tau + l, m(\tau - 1) + l\} \) is divisible by \( p^{\min(\alpha, \beta)} \), so by Lemma 4.6.

\[ \frac{f_p(m)}{f_p(l)f_p(m - l)} \cdot \frac{f_p(m \tau + l)}{f_p(m \tau + l)} - 1 \]

is divisible by \( p^{2 \min(\alpha, \beta)} \) (including the case \( \beta = 0 \)) in \( p \)-adic number field. Thus (25) is divisible by \( p^{2 \max(\alpha - \beta, 0) + 2 \min(\alpha, \beta)} = p^{2 \alpha} \).

**Lemma 4.8.** For \( m = 2^\alpha a, l = 2^\beta b, \alpha \geq 1, \beta \geq 0 \),

\[ 2^{2 \alpha} \mid (-1)^{m \tau + m + l} \left( \frac{m}{l} \right) \left( \frac{m \tau + l - 1}{m - 1} \right) - (-1)^{m \tau + m + l} \left( \frac{m}{l} \right) \left( \frac{m \tau + l - 1}{m - 1} \right), \]

where the second term is set to zero for \( \beta = 0 \).

**Proof.** For the case \( \alpha \geq 2, \beta \geq 2 \), both \( m \tau + m + l \) and \( (m \tau + m + l)/2 \) are even, the Lemma is proved as in Lemma 4.7. For the case \( \beta = 0 \), both \( \left( \frac{m}{l} \right) \) and \( \left( \frac{m \tau + l - 1}{m - 1} \right) \) are divisible by \( 2^\alpha \), and the Lemma is also proved. For remaining cases \( \alpha > \beta = 1 \) or \( \beta \geq \alpha = 1 \), we compute similarly as (25).

\[ (-1)^{m \tau + m + l} \left( \frac{m}{l} \right) \left( \frac{m \tau + l - 1}{m - 1} \right) - (-1)^{m \tau + m + l} \left( \frac{m}{l} \right) \left( \frac{m \tau + l - 1}{m - 1} \right) \]

\[ = \left( \frac{m}{l} \right) \left( \frac{m \tau + l - 1}{m - 1} \right) \left( \frac{f_2(m)}{f_2(l)f_2(m - l)} \cdot \frac{f_2(m \tau + l)}{f_2(m \tau + l)} \right) - (-1)^{m \tau + m + l} \]

Both \( \left( \frac{m}{l} \right) \) and \( \left( \frac{m \tau + l - 1}{m - 1} \right) \) are divisible by \( 2^{\alpha - 1} \), it suffice to prove that the third factor is divisible by 4, which is, by Lemma 4.6.

\[ (-1)^{\frac{m \tau + l - 1}{2}} \left( \frac{m \tau + l - 1}{m - 1} \right) - (-1)^{m \tau + m + l} \]

\[ \equiv 0 \pmod{4}. \]
4.3. Annulus counting. The Bergmann kernel of the curve \((20)\) is

\[
B(X_1, X_2) = \frac{dY_1dY_2}{(Y_1 - Y_2)^2}.
\]

By the construction of BKMP \([7]\), the annulus amplitude is calculated by the integral

\[
\int \left( B(X_1, X_2) - \frac{dX_1dX_2}{(X_1 - X_2)^2} \right) = \ln \left( \frac{Y_2(X_2) - Y_1(X_1)}{X_2 - X_1} \right)
\]

More precisely, for \(m_1, m_2 \geq 1\), the coefficients \(\left[ \ln \left( \frac{Y_2(X_2) - Y_1(X_1)}{X_2 - X_1} \right) \right]_{x_1^{m_1}x_2^{m_2}d^l} \) gives the annulus Gromov-Witten invariants \(K_{(m_1, m_2), 0, l}\).

Let \(b_{n,i} = \frac{(-1)^{n+i}}{n+1} \binom{n+i}{n} \left( \binom{n+1}{2} \right)^{i} \) and \(b_n = \sum_{i=0}^{n} b_{n,i}a^i\). In particular \(b_0 = 1 - a\). Then

\[
Y(X) = \sum_{n \geq 1} b_n X^n.
\]

and

\[
\frac{Y_2(X_2) - Y_1(X_1)}{X_2 - X_1} = (1 - a) + \sum_{n \geq 1} b_n \left( \sum_{i=0}^{n} X_1^{n}X_2^{n-i} \right).
\]

Let \(\tilde{b}_{m,l} = \sum_{i=0}^{l} b_{m,i}\) and \(\tilde{b}_{n} = \sum_{l=0}^{n} \tilde{b}_{m,l}a^l\). For \(m_1 \geq 1, m_2 \geq 1\), the coefficients \(c_{(m_1, m_2)}\) of \([X_1^{m_1}X_2^{m_2}]\) in the expansion

\[
\ln \left( 1 + \sum_{n \geq 1} \tilde{b}_{n} \left( \sum_{i=0}^{n} X_1^{n}X_2^{n-i} \right) \right)
\]
we have

\[ \text{Theorem 4.10.} \]

Then we have the following integrality result:

\[ S_\mu(m_1) \]

In particular, when

\[ l \]

is given by

\[ \text{Lemma 4.9} \]

We introduce the following Lemma first.

\[ \text{For nonnegative integer } n \text{ and prime number } q, \text{ define} \]

\[ f_q(n) = \prod_{i=1}^{n} i = \frac{n!}{q^{[n/q]}[n/q]!} \]

It is obvious that

\[ f_q(q^n k) \equiv f_q(q^n) k \equiv (-1)^k \pmod{p^n} \]

We introduce the following Lemma first.
Lemma 4.11. If \( p^\beta \| (a,b), p^\alpha \| a + b \), then \( p^{\alpha-\beta} \) divides
\[
\binom{a\tau + a - 1}{a} \binom{b\tau + b}{b}.
\]

Proof. Power of prime \( p \) in \( n! \) is \( \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor \). Apply this to the binomial coefficients to find that the power of \( p \) in \( \binom{a\tau + a - 1}{a} \binom{b\tau + b}{b} \) is
\[
\sum_{i=1}^{\infty} \left( \frac{a\tau + a - 1}{p^i} \right) + \left( \frac{b\tau + b}{p^i} \right) - \left( \frac{a\tau - 1}{p^i} \right) + \left( \frac{b\tau}{p^i} \right) - \left( \frac{a}{p^i} \right) - \left( \frac{b}{p^i} \right)
\]
\[
= \sum_{i=1}^{\infty} \left( \frac{(a + b)(\tau + 1)}{p^i} - 1 \right) - \left( \frac{(a + b)\tau}{p^i} - 1 \right) - \sum_{i=1}^{\infty} \frac{a + b}{p^i} - \sum_{i=\beta+1}^{\infty} \frac{a + b}{p^i}
\]
\[
= \alpha - \beta + 1
\]
where we use the fact that for \( k \mid m + n + 1, k > 1 \), \( [m/k] + [n/k] = (m + n)/k - 1 \) and for \( k \mid m + n, k \not\mid m, [m/k] + [n/k] = (m + n)/k - 1 \).

Now, we can finish the proof of Theorem 4.10

Proof. By definition,
\[
n_{m_1,m_2}(\tau) = \frac{1}{m_1 + m_2} \sum_{d|m_1,d|m_2} \mu(d)(-1)^{(m_1+m_2)(\tau+1)/d}
\]
\[
\times \binom{m_1\tau + m_1}{m_1/d} \binom{m_2\tau + m_2}{m_2/d}
\]
(33)
Let \( p \) be any prime divisor of \( m_1 + m_2, p^\alpha \| m_1 + m_2 \). We will prove \( p^\alpha \) divides the summation in (33), thus \( m_1 + m_2 \) also divides and \( n_{m_1,m_2} \) are integers.

If \( p \nmid m_1 \), each summand in (33) corresponds to \( p \nmid d \), so \( p^\alpha \mid (m_1 + m_2)/d \) and \( p \mid m_1/d \). By Lemma 4.11 applies to \( a = m_1/d, b = m_2/d, p^\alpha \) divides each summand and thus the summation.

If \( p^\beta \mid m_1, \beta \geq 1 \), consider two summands in (33) corresponding to \( d \) and \( pd \) such that \( pd \mid (m_1 + m_2), \mu(pd) \neq 0 \). When \( p \) is an odd prime or \( \alpha \geq 2 \), the sign \((-1)^{(m_1+m_2)(\tau+1)/d} \) and \((-1)^{(m_1+m_2)(\tau+1)/(pd)} \) are equal. When \( p = 2, \alpha = 1 \), modulo 2 the sign is irrelevant. Write \( a = m_1/d, b = m_2/d \), then \( p^\alpha \mid a + b, p^\beta \| a \).
\[
\binom{a\tau + a - 1}{a} \binom{b\tau + b}{b} - \binom{(a\tau + a)/p - 1}{a/p} \binom{(b\tau + b)/p}{b/p}
\]
\[
= \binom{(a\tau + a)/p - 1}{a/p} \binom{f_p(a\tau + a)f_p(b\tau + b)}{f_p(a\tau)f_p(a)f_p(b\tau)f_p(b) - 1}
\]
\[
= \binom{(a\tau + a)/p - 1}{a/p} \binom{f_p(a\tau + a)f_p(b\tau + b) - f_p(a\tau)f_p(a)f_p(b\tau)f_p(b)}{f_p(a\tau)f_p(a)f_p(b\tau)f_p(b)}
\]
(34)
The term \( \binom{(a\tau + a)/p - 1}{a/p} \binom{f_p(a\tau + a)f_p(b\tau + b)}{f_p(a\tau)f_p(a)f_p(b\tau)f_p(b)} \) is divisible by \( p^{\alpha-\beta} \) by Lemma 4.11. The numerator of the fraction term in (34) is divisible by \( p^\beta \) by (32), and the denominator is not divisible by \( p \). We proved that \( p^\alpha \) divides (34), take summation over \( d \), we get that \( p^\alpha \) divides the summation in (33). This is true for any \( p \mid m_1 + m_2 \), thus \( n_{m_1,m_2}(\tau) \) is an integer.
4.4. Genus $g=0$ with more holes. By formula (15), we have

$$K_{\mu,0,\frac{\omega}{\rho}}^\tau = (-1)^{|\mu|} [\tau(\tau + 1)]^{l(\mu)-1} \prod_{i=1}^{l(\mu)} \left( \frac{\mu_i (\tau + 1) - 1}{\mu_i - 1} \right) \prod_{b_i \geq 0} \sum_{i=1}^{l(\mu)} \mu_i \left( \int \mathcal{M}_{g,l(\mu)} \frac{\Gamma_0(\tau)}{\prod_{i=1}^{l(\mu)} (1 - \mu_i \psi_i)} \right)$$

When $g = 0$ and $l \geq 3$, then $\Gamma_0(\tau) = 1$ and the Hodge integrals

$$\langle \tau_{b_1} \ldots \tau_{b_l} \rangle_{0,l} = \left( \begin{array}{c} l - 3 \\ b_1, \ldots, b_l \end{array} \right).$$

Hence, we have

(35) $$K_{\mu,0,\frac{\omega}{\rho}}^\tau = (-1)^{|\mu|} [\tau(\tau + 1)]^{l(\mu)-1} \prod_{i=1}^{l(\mu)} \left( \frac{\mu_i (\tau + 1) - 1}{\mu_i - 1} \right) \left( \sum_{i=1}^{l(\mu)} \mu_i \right)^{l(\mu)-3}$$

And by formula (7), we obtain

(36) $$n_{\mu,0,\frac{\omega}{\rho}}(\tau) = (-1)^{|\mu|} \sum_{d \mid |\mu|} \mu(d) d^{l(\mu)-1} K_{d,0,\frac{\omega}{\rho}}^\tau$$

By formula (35), it is clear that $K_{\mu,0,\frac{\omega}{\rho}}^\tau \in \mathbb{Z}$, and since $l(\mu) \geq 3$, we obtain

**Theorem 4.12.** For a partition $\mu$ with $l(\mu) \geq 3$,

$$n_{\mu,0,\frac{\omega}{\rho}}(\tau) \in \mathbb{Z}.$$
and we introduce the notation

\[ \hat{F}_\mu(K_\tau) = \frac{F_\mu(K_\tau)}{\{\mu\}}. \]

**Remark 4.13.** For two partitions \( \nu^1 \) and \( \nu^2 \), the notation \( \nu^1 \cup \nu^2 \) denotes the new partition by combing all the parts in \( \nu^1, \nu^2 \). For example \( \mu = (2, 2, 1) \), then the set of pairs \( (\nu^1, \nu^2) \) such that \( \nu^1 \cup \nu^2 = (2, 2, 1) \) is

\[
(\nu^1 = (2), \nu^2 = (2, 1)), \ (\nu^1 = (2, 1), \nu^2 = (2)), \\
(\nu^1 = (1), \nu^2 = (2, 2)), \ (\nu^1 = (2, 2), \nu^2 = (1)),
\]

For a rational function \( f(q, a) \in \mathbb{Q}(q^{\pm}, a^{\pm}) \), we define the adams operator \( \Psi_d(f(q, a)) = f(q^d, a^d) \).

Then, we have

\[ \hat{g}_\mu(K_\tau) = \sum_{d|\mu} \frac{\mu(d)}{d} \Psi_d(\hat{F}_{\mu/d}(K_\tau)) \quad (38) \]

The LMOV conjecture for framed knot \( K_\tau \) says:

**Conjecture 4.14.** For any partition \( \mu \), there exist integers \( n_{\mu, g, Q}(\tau) \), such that

\[ \hat{g}_\mu(K_\tau) = \sum_{g \geq 0} \sum_{Q} n_{\mu, g, Q}(\tau) z^{\mu g - 2} a^Q \in z^{-2} \mathbb{Z}[z^2, a^{\pm \frac{1}{2}}], \]

where \( z = q^{\frac{1}{2}} - q^{-\frac{1}{2}} = \{1\} \).

**4.5.2. Framed unknot \( U_\tau \).** For convenience, we define the function

\[ \phi_{\mu, \nu}(x) = \sum_{\lambda} \chi_\lambda(C_\mu) \chi_\lambda(C_\nu) x^{\kappa_\lambda}. \]

By Lemma 5.1 in [8], for \( d \in \mathbb{Z}_+ \), we have

\[ \phi(d, \nu)(x) = \left\{ \frac{dx}{d} \right\} x^2. \]

By the formula of colored HOMFLYPT invariant for unknot [10], we obtain

\[ Z_\mu(U_\tau) = \sum_{\lambda} \chi_\lambda(C_\mu) H_\lambda(U_\tau) \]

\[ = (-1)^{|\mu|_\tau} \sum_{\lambda} \chi_\lambda(C_\mu) q^{\frac{\mu}{2} \tau} \sum_{\nu} \chi_\lambda(C_\nu) \frac{\{\nu\}_a}{\nu} \]

\[ = (-1)^{|\mu|_\tau} \sum_{\nu} \frac{1}{3\nu} \phi_{\mu, \nu}(q^\tau) \frac{\{\nu\}_a}{\nu}. \]

In particular, for \( \mu = (m) \), \( m \in \mathbb{Z} \), we have

\[ Z_m(U_\tau) = (-1)^{m_\tau} \sum_{|\nu| = m} \frac{1}{3\nu} \frac{\{mn\} \{\nu\}_a}{\nu}. \]

For brevity, we let \( Z_m(q, a) = \frac{1}{m} Z_m(U_\tau) = (-1)^{m_\tau} \sum_{|\nu| = m} \frac{1}{3\nu} \frac{\{mn\} \{\nu\}_a}{\nu} \) and \( g_m(q, a) = \hat{g}_m(U_\tau) \). Then, by formula (38), we have

\[ g_m(q, a) = \sum_{d|m} \frac{\mu(d)}{d} Z_{m/d}(q^d, a^d). \quad (39) \]
Then the integrality of the higher genus with one hole LMOV invariants is encoded in the following theorem.

**Theorem 4.15.** For any integer \( m \geq 1 \), there exist integers \( n_{m,g,Q}(\tau) \), such that

\[
g_m(q, a) = \sum_{g \geq 0} \sum_{Q} n_{m,g,Q}(\tau) z^{2g-2} a^Q \in z^{-2} \mathbb{Z}[z^2, a^{\pm \frac{1}{2}}],
\]

where \( z = q^{\frac{1}{2}} - q^{-\frac{1}{2}} = \{1\} \).

The proof of Theorem 4.15 is divided into several steps. First, we need the following lemmas.

**Lemma 4.16.** Suppose \( k \) is a positive integer, then the number

\[
c_m(k, y) = \sum_{|\lambda| = m} \frac{1}{k^{\ell(\lambda)}} \{\lambda\} y^2
\]

is equal to the coefficient of \( t^m \) in \((\frac{1-t/y}{1-t y})^k\).

**Proof.** Suppose the number of \( i \)'s in the partition \( \lambda \) is \( a_i, i = 1, \ldots \). Then

\[
c_m(k, y) = \sum_{\sum a_i = m} \prod_{i=1}^{\infty} \frac{1}{a_i!} k^{a_i} (y^i - y^{-i})^{a_i}
\]

\[
= \left[ \prod_{i=1}^{\infty} \left( \sum_{j=0}^{\infty} t^j k^{a_i} (y^i - y^{-i})^j \right) \right]_{t^m}
\]

\[
= \left[ \prod_{i=1}^{\infty} \exp(t^i k (y^i - y^{-i})^i) \right]_{t^m}
\]

\[
= \left[ \left( \frac{1-t/y}{1-t y} \right)^k \right]_{t^m}
\]

\[
\square
\]

**Lemma 4.17.** Let \( R = \mathbb{Q}[q^{\pm 1/2}, a^{\pm 1/2}] \). Then

\[
\{m\} \{m \tau\} g_m(q, a) = \sum_{d | m} \sum_{|\mu| = m/d} \frac{\mu(d)(-1)^{m/\tau,d} \{m \mu \tau\}}{\{d \mu\}} \{d \mu\} a
\]

is divisible by \( \{m \tau\} \{m\} \{1\}^2 \) in \( R \).

**Proof.** By the definition \[39\] of \( g_m(q, a) \), we have the formula \[40\]. It is clear that

\[
\{m\} \{m \tau\} g_m(q, a) \in R.
\]

Denote \( \Phi_n(q) = \prod_{d | n} (q^d - 1)^{\mu(n/d)} \) to be the \( n \)-th cyclotomic polynomial, which is irreducible over \( R \). Then \( q^n - 1 = \prod_{d | n} \Phi_d(q) \), and

\[
\{m\} \{m \tau\} = q^{-\frac{m+m \tau}{2}} \prod_{m_1 | m} \Phi_{m_1}(q) \prod_{m_1 | m \tau} \Phi_{m_1}(q)
\]

\[
= q^{-\frac{m+m \tau}{2}} \prod_{m_1 | m} \Phi_{m_1}(q)^2 \prod_{m_1 | \tau, m_1 | m} \Phi_{m_1}(q)
\]

(i) For \( m_1 \mid m \tau, m_1 \nmid m \), and any \( |\mu| = m/d \), at least one of \( d \mu_i \)'s are not divisible by \( m_1 \), thus \( \{m_\mu \tau\} / \{d \mu_i\} \) is divisible by \( \Phi_{m_1}(q) \). So \( \Phi_{m_1}(q) \) divides \( \{m\} \{m \tau\} g_m(q, a) \).
(ii) For $m_1 \mid m$ and any $|\mu| = m/d$, if not all $d\mu_i$ are divisible by $m_1$, then at least two of them are not divisible. Then two of corresponding $\{m_\mu \tau\}/\{d\mu\}$ are divisible by $\Phi_{m_1}(q)$.

We consider modulo $\{m_1\}^2$ in the ring $R$. It is easy to see, for $a, b \geq 1$,

$$\frac{\{abm_1\}}{\{bm_1\}} \equiv a \left( \frac{q^{m_1/2} + q^{-m_1/2}}{2} \right)^{(a-1)b} \pmod{\{m_1\}^2}$$

We write $x = (q^{m_1/2} + q^{-m_1/2})/2$, then $x^2 \equiv 1 \pmod{\{m_1\}^2}$.

Then modulo $\Phi_{m_1}(q)^2$, we have

$$\{m\}/\{m\} \Phi_{m_1}(q,a)$$

\begin{align*}
\equiv & \sum_{d|m/|\mu|=m/d, m_1/d} \frac{\mu(d)(-1)^{m_\mu/d}}{3\mu} \left\{m_\mu \tau\}/\{d\mu\} \right\}_{d\mu}
\equiv & \sum_{d|m/|\mu|=m/d, m_1/d} \frac{\mu(d)(-1)^{m_\mu/d}}{3\mu} \left( \frac{m_\tau}{d} \right) x^{(m_\mu|\tau - d|\mu)/m_1} \left\{d\mu\right\}_{d\mu}
\equiv & \sum_{d|m/|\lambda|=m/lcm(d,m_1)} \mu(d)(-1)^{m_\mu/d} x^{m_\mu/\lambda} \left( \frac{m_\tau}{lcm(d, m_1)} \right) ^{\{\lambda\}_{d\lambda}}
\end{align*}

(42)

- For the cases $m_1$ with an odd prime factor $p$, or $p = 2$ divides $m_1$ and 4 $| m$, or $p = 2$ divides $m_1$ and 2 $| \tau$: Consider those $d$ with $\mu(d) \neq 0$ and $p \mid d$, we have $lcm(d, m_1) = lcm(pd, m_1)$ and parity of $m_\tau/d$ equals parity of $m_\tau/(pd)$, but $\mu(d) = -\mu(pd)$. Thus two terms in (42) corresponding to $d$ and $pd$ cancelled.

- For the remaining case $2 \mid m, m_1 = 2, 2 \mid \tau$: $\Phi_{m_1}(q)^2 = (q^{1/2} + q^{-1/2})^2 = 2x + 2$. Coefficients of $x$ in (42) equals sum of terms corresponds to odd $d \mid m, \mu(d) \neq 0$, while constant term coefficients equals to sum of terms corresponds to $2d \mid m, \mu(2d) \neq 0$. The coefficients of term for $d$ and $2d$ match, so (42) is divisible by $x + 1$.

In summary, we have proved that for $m_1 \mid m\tau, m_1 \mid m$, $\Phi_{m_1}(q)$ divides $\{m\}/\{m\} \Phi_{m_1}(q)$; for $m_1 \mid m, m_1 \neq 1$, $\Phi_{m_1}(q)^2$ divides $\{m\}/\{m\} \Phi_{m_1}(q,a)$. By (41), the lemma is proved. \hfill \Box

**Lemma 4.18.** For any integer $m \geq 1$, we have

$$g_m(q,a) \in z^{-2}Q[z^2, a^{\pm 2}]$$

**Proof.** By Lemma 4.17 we have

$$f(q,a) := z^2 g_m(q,a) = \{1\}/\{m\} \sum_{d|m/|\mu|=m/d} \frac{\mu(d)(-1)^{m_\mu/d}}{3\mu} \left\{m_\mu \tau\}/\{d\mu\} \right\}_{d\mu} \in Q[z^{\pm 2}, a^{\pm 1}]$$

As a function of $q$, it is clear $f(q,a)$ admits $f(q,a) = f(q^{-1}, a)$. Furthermore, for any $d|m$ and $|\mu| = m/d$, we have

$$m_\mu|\tau - d|\mu| - m\tau - m \equiv m^2\tau/d - m\tau = m\tau(m/d - 1) \equiv 0 \pmod 2,$$

which implies $f(q,a) = f(-q,a)$. Therefore, $f(q,a) = z^2 g_m(q,a) \in Q[z^2, a^{\pm 1}]$. The lemma is proved. \hfill \Box
Lemma 4.19. For any \( \tau \in \mathbb{Z} \), we have

\[
(43) \quad \{m\{m\tau\} \mathcal{Z}_m(q, a) \in \mathbb{Z}[q^{\frac{1}{2}}, a^{\frac{1}{2}}].
\]

Proof. Since

\[
(-1)^{m\tau} \{m\{m\tau\} \mathcal{Z}_m(q, a) = \sum_{|\mu| = m} \frac{m\tau \mu}{\delta \mu \{\mu\}} \{\mu\}_a
\]

we construct a generating function

\[
(44) \quad f(x) = \sum_{n \geq 0} x^n \sum_{\sum_{j \geq 1} jk_j = n} \frac{\prod_{j \geq 1} (\{m\tau j\} \{j\}_a x^j)^{k_j}}{\prod_{j \geq 1} j^{k_j} k_j!}
\]

Then \((-1)^{m\tau} \{m\{m\tau\} \mathcal{Z}_m(q, a) = [f(x)]_{x^m}.

For \( \tau = 0 \), it is the trivial case.

For \( \tau \geq 1 \), we use the expansion \( \{m\tau j\} \{j\}_a = \sum_{k=0}^{m\tau - 1} q^{\frac{j(m\tau - 2k - 1)}{2}} \)

\[
f(x) = \exp \left( \sum_{k=0}^{m\tau - 1} \sum_{j \geq 1} \left( \frac{(q^{\frac{m\tau - 1 - 2k}{2}} a^{\frac{1}{2}} x)^j}{j} - \frac{(q^{\frac{m\tau - 1 - 2k}{2}} a^{\frac{1}{2}} x)^j}{j} \right) \right)
\]

\[
= \exp \left( \sum_{k=0}^{m\tau - 1} \log \frac{1 + q^{\frac{m\tau - 1 - 2k}{2}} a^{\frac{1}{2}} x}{1 + q^{\frac{m\tau - 1 - 2k}{2}} a^{\frac{1}{2}} x} \right)
\]

\[
= \prod_{k=0}^{m\tau - 1} \frac{1 + q^{\frac{m\tau - 1 - 2k}{2}} a^{\frac{1}{2}} x}{1 + q^{\frac{m\tau - 1 - 2k}{2}} a^{\frac{1}{2}} x}
\]

We introduce the Gaussian binomial coefficients defined by

\[
\binom{m}{r}_q = \frac{(1 - q^m)(1 - q^{m-1}) \cdots (1 - q^{m-r+1})}{(1 - q)(1 - q^2) \cdots (1 - q^r)}
\]

for \( r \leq m \), and in particular \( \binom{m}{0}_q = 1 \). The Gaussian binomial coefficients \( \binom{m}{r}_q \in \mathbb{Z}[q] \) (see Chapter 2 of [29] for \( q \)-calculus). There are analogs of the binomial formula, and of Newton’s generalized version of it for negative integer exponents,

\[
\prod_{k=0}^{n-1} (1 + q^k t) = \sum_{k=0}^{n} q^{k(k-1)/2} \binom{n}{k}_q t^k
\]

\[
\prod_{k=0}^{n-1} \frac{1}{(1 - q^k t)} = \sum_{k=0}^{n} (n + k - 1) \binom{k}{k}_q t^k.
\]
Therefore, the coefficient \([f(x)]_{x^m}\) of \(x^m\) in \(f(x)\) is given by
\[
\sum_{j+k=m} (-1)^k q^{j(j-1)-(m-r-1)m} a^{k-j} \binom{m\tau}{j} q^{m\tau + k - 1},
\]
which lies in the ring \(\mathbb{Z}[q^{\pm \frac{1}{2}}, a^{\pm \frac{1}{2}}]\) by the integrality of Gaussian binomial.

For the case \(\tau \leq -1\), we write \(\{m\tau j\} = -\{m\tau j\}\) in the formula (44), then the similar computations give the formula (43).

\[\text{Remark 4.20.}\] In fact, by using the Theorem 3.2 in [5], we have the following refined integrality structure:
\[\{m\}^2 Z_m(q,a) \in \mathbb{Z}[z^2, a^{\pm \frac{1}{2}}].\]
Together with Lemma 4.18 it can also be used to complete the proof of Theorem 4.15.

Now, we can finish the proof of Theorem 4.15 as follow:

\[\text{Proof.}\] Lemma 4.18 implies that there exist rational numbers \(n_{m,g,Q}(\tau)\), such that
\[z^2 g_m(q,a) = \sum_{g \geq 0} \sum_{Q} n_{m,g,Q}(\tau) z^{2g} a^Q \in \mathbb{Q}[z^2, a^{\pm \frac{1}{2}}].\]
So we only need to show \(n_{m,g,Q}(\tau)\) are integers. By lemma 4.19 and the formula (39) for \(g_m(q,a)\), we have
\[\{m\} \{m\tau\} z^2 g_m(q,a) \in \mathbb{Z}[q^{\pm \frac{1}{2}}, a^{\pm \frac{1}{2}}],\]
which is equivalent to
\[(q^{\frac{3}{2}} - q^{-\frac{3}{2}})(q^{\frac{m\tau}{2}} - q^{-\frac{m\tau}{2}}) \sum_{g \geq 0} \sum_{Q} n_{m,g,Q}(\tau) (q^{1/2} - q^{-1/2})^{2g} a^Q \in \mathbb{Z}[q^{\pm 1}, a^{\pm \frac{1}{2}}].\]
So it is easy to get the contradiction if we assume there exists \(n_{m,g,Q}(\tau)\) which is not integer. \(\square\)

5. AN OPEN STRING GW/DT CORRESPONDENCE

5.1. Reduced open string partition function of \((C^3, D_\tau)\).

\[\text{Definition 5.1.}\] We define the reduced open string partition function of \((X, D)\) as
\[(45) \quad \tilde{Z}_{\text{str}}^{(X,D)} (g_s, a, x) = Z_{\text{str}}^{(X,D)} (g_s, a, x = (x, 0, 0, ...)).\]
similarly, the reduced Chern-Simons partition function of \((S^3, K)\)
\[(46) \quad \tilde{Z}_{CS}^{(S^3,K)} (q, a, x) = Z_{CS}^{(S^3,K)} (q, a, x = (x, 0, 0, ...)) = \sum_{n \geq 0} \mathcal{H}_n (K; q, a) x^n.\]
since by the definition of Schur function \(s_\lambda(x = (x, 0, 0, ...)) = x^{|\lambda|}\) which is nozero only when \(\lambda\) is an one row partition.

Now we consider the trivial Calabi-Yau 3-fold \(C^3\) with one AV-brane in framing \(\tau\) which can be viewed as the limit case of the resolved conifold geometry \((\tilde{X}, D_\tau)\) by study their toric diagrams. In fact, the open string free energy on \((C^3, D_\tau)\) is given by
\[F_{\text{str}}^{(C^3,D_\tau)} (g_s, x) = - \sum_{g \geq 0, \mu} \sqrt{\frac{1 - (\mu)}{|\text{Aut}(\mu)|}} g_s^{2g - 2 + l(\mu)} K_{\mu, g, |\mu|/2}^\tau p_\mu(x),\]
where \(K_{\mu, g, |\mu|/2}^\tau\) is the triple Hodge integral given by formula (15).
We define
\[
H_\lambda(q) = [H_\lambda(U_\tau, q, a)]_{\lambda \tau} = (-1)^{|\lambda|\tau} q^{\frac{n|\lambda^{\tau}}{2}} \sum_{\mu} \frac{\chi_\lambda(C_\mu)}{3\mu} \frac{1}{\{\mu\}}.
\]

In particular
\[
H_n(q) = (-1)^n \tau (1 - q)(1 - q^2) \cdots (1 - q^n).
\]

By large N duality (2), we have
\[
Z^{(C^3, D_\tau)}_{str}(g_s, x) = \exp \left( \int_{str} (C^3, D_\tau) (g_s, x) \right) = \sum_{\lambda \in \mathcal{P}} H_\lambda(q) s_\lambda(x).
\]

Then, the reduced open string partition function of \((C^3, D_\tau)\) is given by
\[
\tilde{Z}^{(C^3, D_\tau)}_{str}(g_s, x) = \sum_{n \geq 0} H_n(q) x^n.
\]

For brevity, we let
\[
Z_\tau(q, x) = \tilde{Z}^{(C^3, D_\tau)}_{str}(g_s, x) = \sum_{n \geq 0} H_n(q) x^n
\]
\[
= \sum_{n \geq 0} (-1)^n \tau q^{\frac{n^2}{2}} (1 - q)(1 - q^2) \cdots (1 - q^n) x^n.
\]

In fact, the LMOV conjecture [31] provides a factorization for the partition function \(Z_\tau(q, a)\), which will be showed as follow. We first formulate the LMOV conjecture for the general reduced partition function \(\tilde{Z}\) as in formulas (45) and (46). We take the reduced Chern-Simon partition function for example.
\[
\tilde{Z}^{(S^3, K_\tau)}_{CS}(q, a, x) = \sum_{n \geq 0} H_n(K_\tau; q, a) x^n.
\]

By LMOV conjecture for \(K_\tau\) (Conjecture 3.1), there exist functions \(f_m(K_\tau; q, a)\) such that
\[
\tilde{Z}^{(S^3, K_\tau)}_{CS}(q, a, x) = \sum_{m \geq 1} \sum_{d \geq 1} \frac{1}{d} f_m(K_\tau; q^d, a^d) x^{dm}.
\]

One can compute \(f_m(K_\tau; q, a)\) explicitly, for example
\[
f_1(K_\tau; q, a) = H_1(K_\tau).
\]
\[
f_2(K_\tau; q, a) = H_2(K_\tau) - \frac{1}{2} H_1(K_\tau)^2 - \frac{1}{2} \Psi_2(H_1(K_\tau)).
\]

Then reduced LMOV conjecture asserts the following weak form of the integrality which was first proposed in [52].

**Conjecture 5.2** (Reduced LMOV conjecture for \(K_\tau\)). There exist integers \(N_{m, i, k}(\tau)\), and only finitely many \(N_{m, i, k}(\tau)\) are nonzero for any fixed \(m \geq 1\). Such that
\[
f_m(K_\tau; q, a) = - \sum_{i, k \in \mathbb{Z}} \frac{N_{m, i, k}(\tau) a^i q^{\frac{k+1}{2}}}{1 - q}.
\]
The integers \( N_{m,i,k}(\tau) \) are called the Ooguri-Vafa invariants which were first studied by Ooguri and Vafa in [52].

Therefore, by Conjecture 5.2, we have

\[
\tilde{Z}_{C_S^{(3, D_\tau)}}(q, a, x) = \exp \left( - \sum_{m \geq 1} \sum_{i,k} N_{m,i,k}(\tau) \sum_{l \geq 0} \frac{1}{d} (a^\frac{1}{2} q^\frac{k+1}{2} + l x^m)^d \right)
\]

\[
= \exp \left( \sum_{m \geq 1} \sum_{i,k} N_{m,i,k}(\tau) \sum_{l \geq 0} \log \left( 1 - a^\frac{1}{2} q^\frac{k+1}{2} + l x^m \right) \right)
\]

\[
= \prod_{m \geq 1} \prod_{i,k} \prod_{l \geq 0} \left( 1 - a^\frac{1}{2} q^\frac{k+1}{2} + l x^m \right)^{N_{m,i,k}(\tau)}
\]

Now, we consider the reduced open string partition function \( Z_\tau(q, x) = \tilde{Z}_{str}^{(C^3, D_\tau)}(g_a, x) \), then the corresponding reduced LMOV conjecture assert that:

**Conjecture 5.3** (Reduced LMOV conjecture for \((C^3, D_\tau)\)). There exist integers \( N_{m,k}(\tau) \), and only finitely many \( N_{m,k}(\tau) \) are nonzero for any fixed \( m \geq 1 \). Such that

\[
Z_\tau(q, x) = \prod_{m \geq 1} \prod_{k} \prod_{l \geq 0} \left( 1 - q^\frac{k+1}{2} + l x^m \right)^{N_{m,k}(\tau)}
\]

### 5.2. Hilbert-Poincaré series of the Cohomological Hall algebra of the \( m \)-loop quiver

We first review the definition and the main results of Cohomological Hall algebra [30] for the \( m \)-loop quiver, \( m \in \mathbb{N} \). Here we mainly following the expositions in [54] (i.e. Section 4 in [54]).

Fix a nonnegative integer \( m \geq 1 \). For a complex vector space \( V \), we denote by \( E_V = \text{End}(V)^m \) the space of \( m \)-tuples of endomorphisms of \( V \). Then the group \( G_V = \text{GL}(V) \) acts on \( E_V \) by simultaneous conjugation. We study the equivariant cohomology with rational coefficient \( H^*_G(V) \). For two complex vector spaces \( V \) and \( W \), Kontsevich and Soibelman [30] constructed a map:

\[
H^*_G(V) \otimes H^*_G(W) \to H^*_{G(V \oplus W)}(V \oplus W).
\]

They proved such maps induce an associative unital \( \mathbb{Q} \)-algebra structure on \( \mathcal{H} = \oplus_{n \geq 0} H^*_G(E_{C^n}) \), which is \( \mathbb{N} \times \mathbb{Z} \)-graded if \( H^k_{G(E_{C^n})} \) is placed in bidegree \((n, (m-1)n - \frac{k}{2})\). This algebra \( \mathcal{H} \) is called the Cohomological Hall algebra of the \( m \)-loop quiver in [30]. We define the Hilbert-Poincaré series of \( \mathcal{H} \) as following:

\[
P_m(q, t) = \sum_{n \geq 0} \sum_{k \in \mathbb{Z}} \dim_{\mathbb{Q}} \mathcal{H}_{n,k} q^{-k} t^n.
\]

Please note that we use the parameter \( q^{-1} \) instead of \( q \) in Section 4 of [54].

**Proposition 5.4** (Lemma 4.2 [54]). The series

\[
P_m(q, t) = \sum_{n \geq 0} \frac{q^{-(m-1)\frac{n(n-1)}{2}}}{(1-q)(1-q^2)\ldots(1-q^n)} t^n \in \mathbb{Q}(q)[[t]].
\]

The main property of \( P_m(q, t) \) is the following factorization formula.

**Theorem 5.5** (Conjecture 3.3 [54] or Theorem 2.3 [30]). There exists a product expansion

\[
P_m(q, (-1)^{m-1}t) = \prod_{n \geq 1} \prod_{k \geq 0} \prod_{l \geq 0} (1 - q^{l-k} t^n)^{(-1)^{m-1} c_{n,k}}
\]

for nonnegative integers \( c_{n,k} \), such that only finitely many \( c_{n,k} \) are nonzero for any fixed \( n \).
Let $DT_n^{(m)}(q) = \sum_{k \geq 0} c_{n,k} q^k$ which is called the quantum Donaldson-Thomas invariant in [54]. The above theorem implies that $DT_n^{(m)}(q)$ is a polynomial with nonnegative coefficients. The explicit formula for $DT_n^{(m)}(q)$ was given in [54].

5.3. The correspondence. One can write the partition function $Z_\tau(q,a)$ in the following form:

$$Z_{-\tau}(q,x) = \sum_{n \geq 0} \frac{q^{-(\tau-1)n(n-1)/2}}{(1-q)(1-q^2)\cdots(1-q^n)}(-1)^{\tau-1}xq^{\frac{1}{2}n}.$$  

By comparing with the Hilbert-Poincaré series $P_m(q,t)$ (??), we obtain

**Theorem 5.6.** For $\tau \leq -1$ (i.e. $-\tau \geq 1$), we have

$$Z_\tau(q,x) = P_{-\tau}(q,(-1)^{\tau-1}xq^{\frac{1}{2}}).$$

Theorem 5.6 can be viewed as an open string GW/DT correspondence, we refer to [50] for a discussion of the GW/DT correspondence for toric 3-folds.

Theorem 5.5 implies that, for $\tau \leq -1$, the reduced open string partition function $Z_\tau(q,x)$ on $(\mathbb{C}^3,D_\tau)$ carries the product factorization:

$$Z_\tau(q,x) = \prod_{n \geq 1} \prod_{k \geq 0} \prod_{l \geq 0} (1-q^{n+l-k}x^n)^{-(-1)^{\tau-1}n}c_{n,k}.$$

Comparing with the Conjecture 5.3, it provides the correspondence of the Ooguri-Vafa invariants $N_{m,k}(\tau)$ and the Donaldson-Thomas invariants $c_{n,k}$ for $\tau \leq -1$.

**Remark 5.7.** For the simplicity of the discussion of the LMOV invariants, Garoufalidis, Kucharski and Sulkowski [18] introduced the notion of extremal LMOV invariant. In fact, the Ooguri-Vafa invariants (or weak LMOV invariants) $N_{m,k}(\tau)$ for $(\mathbb{C}^3,D_\tau)$ in Conjecture 5.3 are the extremal LMOV invariants in the sense of [18] for framed unknot $U_\tau$. The relationship of the extremal LMOV invariants and the work of Reineke [54] was extensively studied in the recent paper [31].

6. Appendix

In [18], Garoufalidis, Kucharski and Sulkowski obtained the following extremal BPS invariants of twist knots:

$$b_{K_p,r}^- = -\frac{1}{r^2} \sum_{d|r} \mu(r) \left(\frac{3d-1}{d-1}\right), \quad b_{K_p,r}^+ = \frac{1}{r^2} \sum_{d|r} \mu(r) \left(\frac{2|p|+1}{d-1}\right)$$

for $p \leq -1$ and

$$b_{K_p,r}^- = -\frac{1}{r^2} \sum_{d|r} \mu(r) (-1)^{d+1} \left(\frac{2d-1}{d-1}\right), \quad b_{K_p,r}^+ = \frac{1}{r^2} \sum_{d|r} \mu(r) (-1)^d \left(\frac{2p+2d-1}{d-1}\right)$$

for $p \geq 2$. See the formulas (1.4) and (1.5) in [18].

In fact, in their later work [31], Kucharski and Sulkowski found the work of Reineke [54] can be used to interpret the integrality of above BPS invariants $b_{K_p,r}^-$ and $b_{K_p,r}^+$. But in this appendix, we provide a direct proof of the integrality of the BPS invariants $b_{K_p,r}^-$ and $b_{K_p,r}^+$ by the same method used in the proofs of Theorems 4.5, 4.10.

**Theorem 6.1.** $b_{K_p,r}^-$ and $b_{K_p,r}^+$ given in formulas (48) and (49) are integers.
For nonnegative integer $n$ and prime number $q$, define

$$f_q(n) = \prod_{i=1, q \nmid i}^{n} i = \frac{n!}{q^{[n/q]}[n/q]!}$$

**Lemma 6.2** (=Lemma 4.6). For odd prime numbers $q$ and $\alpha \geq 1$ or for $q = 2$, $\alpha \geq 2$, we have $q^{2\alpha} \mid f_q(q^\alpha n) - f_q(q^\alpha n)$. For $q = 2$, $\alpha = 1$, $f_2(2n) \equiv (-1)^{[n/2]} \pmod{4}$

**Lemma 6.3.** For prime number $q$ and $m = q^\alpha a, q \nmid a, \alpha \geq 1, k \geq 1$, $q^{2\alpha}$ divides

$$( -1)^{(k+1)m} \binom{km - 1}{m - 1} - (-1)^{(k+1)m/q} \frac{f_q(km)}{f_q((k-1)m)f_q(m)} - (-1)^{(k+1)(m-m/q)}.$$

Proof.

$$= (-1)^{(k+1)m} \binom{km - 1}{m - 1} - (-1)^{(k+1)m/q} \frac{f_q(km)}{f_q((k-1)m)f_q(m)} - (-1)^{(k+1)(m-m/q)}$$

For $q > 2$ or $q = 2, \alpha > 1$, then $m - m/q$ is even, thus (50) is divisible by $q^{2\alpha}$ by Lemma 6.2. For $q = 2, \alpha = 1$, (50) is divisible by 4 if

$$\left[\frac{km}{4}\right] + \left[\frac{(k-1)m}{4}\right] + \left[\frac{m}{4}\right] - (k + 1)(m - \frac{m}{2}) \equiv 0 \pmod{2}$$

which depends only on $k \pmod{2}$, verify for $k \in \{1, 2\}$ to get the results. \hfill \Box

Now, we can finish the proof of Theorem 6.1.

**Proof.** For each prime divisor $q$ of $r$, in the summation in (48) over $d \mid r$, pairing the terms with nonzero $\mu(d/q)$ and $\mu(d'/q)$. Sum of two terms of each pair is divisible by $q^{2\alpha}$ by Lemma 6.3. This is true for all prime divisors of $r$, thus $b_{K_p,r}^-$ and $b_{K_p,r}^+$ are integers. \hfill \Box

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