Variational Analysis of Mass Spectra and Decay Constants for Ground State Pseudoscalar and Vector Mesons in Light-Front Quark Model

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Abstract Using the variational principle, we compute mass spectra and decay constants of ground state pseudoscalar and vector mesons in the light-front quark model (LFQM) with the QCD-motivated effective Hamiltonian including the hyperfine interaction. To avoid the issue of negative infinity in applying the variational principle to the computation of meson mass spectra, we smear out the Dirac delta function in the hyperfine interaction. We compare the results from this variational analysis with the previous computation handling the hyperfine interaction as perturbation. We also take a couple of different trial functions for the variation to examine the sensitivity of the variational results with respect to the choice of the trial wave functions. Comparing the results with the experimental data of mass spectra and decay constants for the ground state pseudoscalar and vector mesons from $\pi$ to $\Upsilon$, we note that the analysis of the light meson sectors may need to be separately handled from the analysis of the heavy meson sectors as the degree of improvement in the LFQM results appears dependent on the meson sectors.

1 Introduction

Effective degrees of freedom to describe a strongly interacting system of hadrons have been one of the key issues in understanding the non-perturbative nature of QCD in the low energy regime. Within an impressive array of effective theories available nowadays, the constituent quark model has been quite useful in providing a good physical picture of hadrons just like the atomic model for the system of atoms. Absorbing the complicated effect of quark, antiquark and gluon interactions into the effective constituent degrees of freedom, one may make the problem more tractable yet still keep some key features of the underlying QCD to provide useful predictions [1]. The effective potentials used in constituent quark models are typically described by the flux tube configurations generated by the gluon fields as well as the effective “one-gluon-exchange” calculation in QCD [2, 3]. In the QCD-motivated effective Hamiltonian, a proper way of dealing with the relativistic effects in the hadron system is quite essential due to the nature of strong interactions. In particular, proper care and handling of relativistic effects has been emphasized in describing the hadrons made of $u$, $d$, and $s$ quarks and antiquarks.

As a proper way of handling relativistic effects, the light-front quark model (LFQM) [4–8] appears to be one of the most efficient and effective tools in hadron physics as it takes advantage of the distinguished features of the light-front dynamics (LFD) [9, 10]. In particular, the LFD carries the maximum number (seven) of the kinetic (or interaction independent) generators and thus the less effort in dynamics is necessary in order to get the QCD solutions that reflect the full Poincaré symmetries. Moreover, the rational energy-momentum dispersion relation of LFD, namely $p^- = (p_+^2 + m^2)/p^+$, yields the sign correlation between the light-front (LF) energy $p^- (= p_0 - p^3)$ and the LF longitudinal momentum $p^+ (= p_0 + p^3)$ and leads to the suppression of quantum fluctuations of the vacuum, sweeping the complicated vacuum fluctuations into the zero-modes in the limit of $p^+ \to 0$ [11–13]. This simplification is a remarkable advantage in LFD and facilitates the partonic interpretation of the amplitudes. Based on the advantages of the LFD, the LFQM has been developed [14] and subsequently applied for various meson phenomenologies such as the mass spectra of both heavy and light mesons [15], the decay constants, distribution amplitudes, form factors and generalized parton distributions [10, 14–23].

Despite these successes in reproducing the general features of the data, however, it has proved very difficult to obtain direct connection between the LFQM and QCD. Typically, rigorous derivations of the connection between the
effective constituent degrees of freedom and the fundamental QCD quark, antiquark and gluon degrees of freedom have been explored by solving momentum-dependent mass gap equations as discussed in many-body Hamiltonian approach [24], Dyson-Schwinger approach [25], etc. Although one has not yet explored solving the momentum-dependent mass gap equation in LFD, there has been some attempt to derive an effective LF Hamiltonian starting from QCD using the discrete light-cone quantization (DLCQ) and solve the corresponding equation of motion approximately for the quark and antiquark bound-states to provide semianalytical expressions for the masses of pseudoscalar and vector mesons [26]. The attempt to link between QCD and LFQM is also supported by our recent analyses of quark-antiquark distribution amplitudes for pseudoscalar and vector mesons in LFQM [27], where we presented a self-consistent covariant description of twist 2 and twist 3 quark-antiquark distribution amplitudes for these mesons to discuss the link between the chiral symmetry of QCD and the LFQM. Our results for the pseudoscalar and vector mesons [27] effectively indicated that the constituent quark and antiquark in the LFQM could be considered as the dressed constituents including the zero-mode quantum fluctuations from the QCD vacuum. These developments motivate our present work for the more-in-depth analysis of the mass spectra and decay constants for the ground state pseudoscalar and vector mesons in LFQM.

In LFQM, the LF wave function is independent of all reference frames related by the front-form boosts because the longitudinal boost operator as well as the LF transverse boost operators are all kinematical. This is clearly an advantageous feature unique to LFQM, which makes the calculation of observables such as mass spectra, decay constants, form factors, etc. much more effective. Computing the meson mass spectra, however, we have previously [14, 15] treated the hyperfine interaction as a perturbation rather than including it in the variation procedure to avoid the negative infinity from the Dirac delta function contained in the hyperfine interaction. In the present work, we smear out the Dirac delta function by a Gaussian distribution and resolve the infinity problem when variational principle is applied to the hyperfine interaction. We obtain optimal model parameters in our variational analysis including the hyperfine interaction and examine if it improves phenomenologically our numerical results compared to the ones obtained by the perturbative treatment of the hyperfine interaction. For our trial wave function, we also take a larger harmonic oscillator (HO) basis to see if it provides any phenomenological improvement in our predictions of mass spectra and decay constants for ground state pseudoscalar and vector mesons.

The paper is organized as follows: In Sec. 2, we describe our QCD-motivated effective Hamiltonian with the smeared-out hyperfine interaction. Using two different radial wave functions, i.e. the ground state HO wave function and the mixture of the two lowest order HO states, as trial wave functions of the variational principle, we find the analytic formulae of the mass eigenvalues for the ground state pseudoscalar and vector mesons. The optimum values of model parameters are also presented in this section. In Sec. 3, we present our numerical results of the mass spectra obtained from both trial wave functions and compare them with the experimental data as well as our previous calculations [14, 15]. To test our trial wave functions with the parameters obtained from the variational principle, we also calculate the meson decay constants and compare them with the experimental data as well as other available theoretical predictions. We discuss an improvement of our numerical results by taking a larger HO basis in the trial wave function in comparison with the case of taking just the ground state HO wave function as the trial wave function. Summary and conclusion follow in Sec. 4. The detailed procedure of fixing our parameters through variational principle is presented in Appendix A.

2 Model Description

As mentioned in the introduction, there has been an attempt to derive an effective LF Hamiltonian starting from QCD using DLCQ [26]. Transforming the LFD variables to the ordinary variables in the instant form dynamics (IFD), one may see the equivalence between the resulting effective LF Hamiltonian for the quark and antiquark bound-states and the usual relativistic constituent quark model Hamiltonian for mesons typically given in the rest frame of the meson, i.e. the center of mass (C.M.) frame for the constituent quark and antiquark system. It may be more intuitive to express the effective LF Hamiltonian describing the relativistic constituent quark model system for mesons in terms of the ordinary IFD variables. Effectively, the meson system at rest is then described as an interacting bound system of effectively dressed valence quark and antiquark typically given by the following QCD-motivated effective Hamiltonian in the quark and antiquark C.M. frame [14, 15]:

$$H_{\text{C.M.}} = \sqrt{m_q^2 + \mathbf{k}^2} + \sqrt{m_{\bar{q}}^2 + \mathbf{k}^2} + V,$$

(1)

where $\mathbf{k} = (k_\perp, k_z)$ is the relativistic three-momentum of the constituent quarks and $V$ is the effective potential between quark and antiquark in the rest frame of the meson. The effective potential $V$ is typically given by the linear confining potential $V_{\text{conf}}$ plus the effective one-gluon-exchange potential $V_{\text{oge}}$. For $S$-wave pseudoscalar and vector mesons, the effective one-gluon-exchange potential reduces to the coulomb potential $V_{\text{conf}}$ plus the hyperfine interaction $V_{\text{hyp}}$. Thus, one may summarize $V$ as

$$V = V_{\text{conf}} + V_{\text{oge}}$$
\[
\begin{align*}
\text{conf} & = a + b r + \frac{4\alpha_s}{3r} + \frac{2S_\text{hyp}}{3m_q m_i} V_{\text{conf}}, \\
\text{hyp} & = \text{conf} + \frac{2S_\text{hyp}}{3m_q m_i} V_{\text{hyp}},
\end{align*}
\]
where \(\alpha_s\) is the strong interaction coupling constant \(^1\), \(\langle S_q \rangle\) is \(1/4 \sim (-3/4)\) for the vector (pseudoscalar) meson and \(\nabla^2 V_{\text{conf}} = (16\pi/3)\delta^3(r)\). Reduction of the LF Hamiltonian in QCD to a similar effective Hamiltonian in the C.M. frame of the quark and antiquark system given by Eqs. (1) and (2) was discussed in Ref. [26]. For the hyperfine interaction in this work smearing out the Dirac delta function to resolve the infinity problem, we naturally introduce a smearing parameter which may effect the Dirac delta function. Such relativization is important for the \(\delta^3(r)\)-type potential without any smearing in computing particularly the light meson sector. Since we apply the variational principle even for the hyperfine interaction in this work, we compute the expectation value using the variation principle. Alternatively, one may consider computing the expectation value from the free energies of the constituents is given by Eq. (1), i.e. \(H_{\text{C.M.}}\). Since the eigenvalues and the expectation values are same for the eigenstates, we compute the expectation value \(\langle H_{\text{C.M.}} \rangle\) using the variational principle. Alternatively, one may consider computing the expectation value \(\langle H_{\text{C.M.}}^{2/3} \rangle\) in view of the LFQD mass square operator \(P^+ P^- - P_\perp^2\) (that provides the eigenvalues \(P^+ P^- - P_\perp^2\)) as the square of the effective Hamiltonian given by Eq. (1), i.e. \(H_{\text{C.M.}}^{2/3}\). The wave function \(\langle \chi_{\text{meson}} \rangle\) can be suggested in variational analysis. In this work, we examine the \(\chi^2\) values of our computational results in comparison with experimental data to get optimal parameter values in the \(\langle H_{\text{C.M.}} \rangle\) computation. This will provide useful ground information for any alternative and/or further works beyond the present analysis.

As discussed earlier, the longitudinal boost operator as well as the LF transverse boost operators are all kinematical and thus the LF wave function does not depend on the external momentum, i.e. \(P^+\) and \(P_\perp\). In effect, the determination of the LF wave function in the meson rest frame such as \(P^+ = M_{\bar q q}\) and \(P_\perp = 0\) won’t hinder its use for any other values of \(P^+\) and \(P_\perp\). This provides the applicability of LFQM for the computation of observables beyond the meson mass spectra.

The wave function is thus represented by the Lorentz invariant internal variables \(x_i = P_i^0/P^+\), \(\bar{\mathbf{k}}_{\perp i} = \mathbf{p}_{\perp i} - x_i \mathbf{P}_\perp\), and helicity \(\chi_i\), where \(P_i^0\) is the momenta of constituent quarks. Explicitly, the LF wave function of the ground state mesons is given by

\[
\Psi_{00}(x_i, \mathbf{k}_{\perp i}, \chi_i) = \mathcal{R}_{\chi_i, \mathbf{k}_{\perp i}}(x_i, \mathbf{k}_{\perp i}) \phi(x_i, \mathbf{k}_{\perp i}),
\]

where \(\phi\) is the radial wave function and \(\mathcal{R}_{\chi_i, \mathbf{k}_{\perp i}}\) is the interaction-independent spin-orbit wave function. The spin-orbit wave functions for pseudoscalar and vector mesons are given by [14, 28]

\[
\begin{align*}
\mathcal{R}_{\chi_i, \mathbf{k}_{\perp i}}^{00} &= -\frac{\tilde{u}_{\chi_i}(p_q)}{\sqrt{2} \sqrt{M_0^2 - (m_q - m_{\bar q})^2}}, \\
\mathcal{R}_{\chi_i, \mathbf{k}_{\perp i}}^{11} &= -\frac{\tilde{u}_{\chi_i}(p_q)}{\sqrt{2} \sqrt{M_0^2 - (m_q - m_{\bar q})^2}} \epsilon^\mu(J_\mu),
\end{align*}
\]

where \(\epsilon^\mu(J_\mu)\) is the polarization vector of the vector meson and the boost invariant meson mass squared \(M_0^2\) obtained from the free energies of the constituents is given by

\[
M_0^2 = \frac{\boldsymbol{k}_{\perp i}^2 + m_{\bar q}^2}{x} + \frac{\boldsymbol{k}_{\perp i}^2 + m_q^2}{1-x}.
\]

The spin-orbit wave functions satisfy the relation \(\mathcal{R}_{\chi_i, \mathbf{k}_{\perp i}}^{00} \mathcal{R}_{\chi_i, \mathbf{k}_{\perp i}}^{11} = 1\) for both pseudoscalar and vector mesons.

To use a variational principle, we take our trial wave function as an expansion of the true wave function in the HO basis. We use the same trial wave function \(\phi\) for both pseudoscalar and vector mesons, but we try two different forms: one simply takes the 1S-state HO wave function \(\phi_{\text{1S}}\) and the other one is expanded with the two lowest order HO wave functions \(\phi_0 = \sum_{\chi_{\text{meson}}} c_{\chi_{\text{meson}}} \phi_{\text{1S}}\), where

\[
\phi_{\text{1S}}(x_i, \mathbf{k}_{\perp i}) = 4\pi^{3/4} \beta^{3/2} \left| x^2 - \beta \right| e^{-\frac{x^2 + \beta^2}{2\beta^2}},
\]

\[
\phi_{2x}(x_i, \mathbf{k}_{\perp i}) = 4\pi^{3/4} \beta^{3/2} \left(2\mathbf{k}^2 - 3\beta^2\right) \left| x^2 - \beta \right| e^{-\frac{x^2 + \beta^2}{2\beta^2}}.
\]

and \(\beta\) is the variational parameter. The longitudinal momentum \(x_i\) is defined by \(k_i = (x - 1/2)M_0 + (m_q - m_{\bar q})/2M_0\) and thus the Jacobian of the variable transformation \((x, \mathbf{k}_{\perp i}) \rightarrow \mathbf{k} = (\mathbf{k}_{\perp i}, k_z)\) is given by \(\partial k_i / \partial x = M_0 [1 - (m_q^2 - m_{\bar q}^2)/M_0^2] / 4x(1-x)\). The wave function \(\phi_{\text{AS}}\) satisfies the following normalization:

\[
\int_0^1 dx \int d^2 \mathbf{k}_{\perp i} \left| \phi_{\text{AS}}(x_i, \mathbf{k}_{\perp i}) \right|^2 = 1.
\]
With $\phi_{A(B)}$, we evaluate the expectation value of the Hamiltonian in Eq. (1), i.e., $\langle \phi_{A(B)} | H_{C.M.} | \phi_{A(B)} \rangle$ which depends on the variational parameter $\beta$. According to the variational principle, we can set the upper limit of the ground state’s energy by calculating the expectation value of the system’s Hamiltonian with a trial wave function. In our previous calculations [14, 15], which we call “CJ model”, we first evaluate the expectation value of the central Hamiltonian $T + V_{\text{conf}} + V_{\text{coul}}$ with the trial function $\phi_A$, where $T$ is the kinetic energy part of the Hamiltonian. Once the model parameters are fixed by minimizing the expectation value $\langle \phi_A | (T + V_{\text{conf}} + V_{\text{coul}}) | \phi_A \rangle$, then the mass eigenvalue of each meson is obtained as $M_M = \langle \phi_A | H_{C.M.} | \phi_A \rangle$. The hyperfine interaction $V_{\text{hypo}}$ in CJ model, which contains a Dirac delta function, was treated as perturbation to the Hamiltonian and was left out in the variational process that optimizes the model parameters. The main reason for doing this was to avoid the negative infinity generated by the delta function as was pointed out in [29]. Specifically, $\langle \phi_A | V_{\text{hypo}} | \phi_A \rangle$ for pseudoscalar mesons decreases faster than other terms that increase as $\beta$ increases and the expectation value of the Hamiltonian is unbounded from below.

To avoid the negative infinity, we use a Gaussian smearing function to weaken the singularity of $\delta^4(\mathbf{r})$ in hyperfine interaction, viz. [29, 30], $\delta^4(\mathbf{r}) \rightarrow (\sigma^3/\pi^{3/2}) e^{-\sigma^2 r^2}$. Once the delta function is smeared out like this, a true minimum for the mass occurs at a finite value of $\beta$. The analytic formulae of mass eigenvalues for our modified Hamiltonian with the smeared-out hyperfine interaction, i.e. $M_{M(q)}^{A(B)} = \langle \phi_{A(B)} | H_{C.M.} | \phi_{A(B)} \rangle$, are found as follows

$$M_{M(q)}^{A} = \alpha + \frac{b}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}} \sum_{i=q, \bar{q}} m_i^2 e^{z_i^2/2} K_1 \left( z_i \sqrt{\frac{2}{3}} \right), \quad (\text{9})$$

$$M_{M(q)}^{B} = a + \frac{b}{\sqrt{2\pi}} \left[ 3 - c_i^2 - 2 \sqrt{\frac{2}{3}} c_1 c_2 \right]$$

$$+ \beta \sqrt{\frac{2}{3}} \sum_{i=q, \bar{q}} \left\{ \sqrt{\pi} \left( 6c_1 c_2 - 3c_i^2 \right) \right\} \cdot \left\{ U \left( \frac{1}{2}, -2, z_i \right) \right\}$$

$$+ \beta \sum_{i=q, \bar{q}} \left\{ \sqrt{\pi} \left( 6c_1 c_2 - 3c_i^2 \right) \right\} \cdot \left\{ U \left( \frac{1}{2}, -2, z_i \right) \right\},$$

$$- \frac{4\beta}{\sqrt{2\pi}} \left[ 3 + c_i^2 + 6 \sqrt{2/3} c_1 c_2 \right],$$

$$- \frac{4\beta}{\sqrt{2\pi}} \left[ (S_i - S_{\bar{q}}) \right] \left\{ \left[ 2 \sqrt{6} c_1 c_2 + 3 - c_i^2 \right] \right\} \cdot \left\{ \left[ 2 \sqrt{6} c_1 c_2 + 3 - c_i^2 \right] \right\} \cdot \left\{ \left[ 2 \sqrt{6} c_1 c_2 + 3 - c_i^2 \right] \right\},$$

$$+ 2\beta \left[ 2c_i^2 + \sqrt{6} c_1 c_2 \right] \cdot \left\{ \left[ 2c_i^2 + \sqrt{6} c_1 c_2 \right] \right\} \cdot \left\{ \left[ 2c_i^2 + \sqrt{6} c_1 c_2 \right] \right\},$$

$$\text{where } z_i = m_i^2/\beta^2 \text{ and } K_1 \text{ is the modified Bessel function of the second kind and } U(a, b, z) \text{ is Tricomi’s (confluent hypergeometric) function. We then apply the variational principle, i.e. } \partial M_{M(q)}^{A(B)}/\partial \beta = 0, \text{ to find the optimal model parameters in order to get a best fit for the mass spectra of ground state pseudoscalar and vector mesons (a more detailed description of this procedure can be found in Appendix A).}$$

Our optimized potential parameters are obtained as $\{a = -0.5575 \text{ GeV}, b = 0.18 \text{ GeV}^2, \alpha_4 = 0.5174\}$ for $M_{M(q)}^{A}$ and $\{a = -0.6664 \text{ GeV}, b = 0.18 \text{ GeV}^2, \alpha_4 = 0.5348\}$ for $M_{M(q)}^{B}$, respectively. We should note that the two sets of potential parameters are quite comparable with the ones suggested by Scora and Isgur [31], where they obtained $a = -0.81 \text{ GeV}, b = 0.18 \text{ GeV}^2,$ and $\alpha_4 = 0.3 \sim 0.6$. For a comparison, the coupling constant we found in our previous model [14, 15] was $\alpha_4 = 0.31$. For the best fit of the ground state mass spectra, we obtain $c_1 = +\sqrt{0.7}$ and $c_2 = +\sqrt{0.3}$ for $\phi = \phi_0$ case.

Since we included the hyperfine interaction with smearing function entirely in our variational process, we now obtain the two different sets of $\beta$ values, one for pseudoscalar and the other for vector mesons, respectively. Our optimal constituent quark masses and the smearing parameters $\sigma$ are listed in Table 1. The optimal Gaussian parameters $\beta_{q\bar{q}}$ for pseudoscalar and vector mesons are also listed in Table 2 and 3, respectively.

Table 1 Constituent quark masses [GeV] and the smearing parameter $\sigma$ [GeV] obtained by the variational principle for the Hamiltonian with a smeared-out hyperfine interaction. Here $q = u$ and $d$.

| Model | $m_q$ | $m_s$ | $m_c$ | $m_b$ | $\sigma$ |
|-------|-------|-------|-------|-------|-------|
| $\phi_A$ | 0.220 | 0.432 | 1.77 | 5.2 | 0.405 |
| $\phi_B$ | 0.221 | 0.456 | 1.77 | 5.2 | 0.423 |

Our updated model with the smeared-out hyperfine interaction appears to improve the result of mass spectrum, which is presented in the next section. This may suggest that when using constituent quark models, the contact interactions has to be smeared out. In fact, we think this smeared-out interaction seems to be more consistent with the physical picture for a system of finite-sized constituent quarks.

For practical application of our model, we also compute the decay constants for the ground state pseudoscalar and vector mesons. The decay constants are defined by

$$\langle 0 | \bar{q} \gamma^\mu p q | P \rangle = i f_P p^\mu,$$

$$\langle 0 | \bar{q} \gamma^\mu q | V(P, h) \rangle = f_V V e^{i h} (h),$$

for pseudoscalar and vector mesons, respectively. The experimental values of the pion and rho meson decay constants are $f_\pi \approx 131 \text{ MeV from } \pi \rightarrow \mu \nu$ and $f_\rho \approx 220 \text{ MeV from } \rho \rightarrow e^+ e^-$. 


Table 2 The Gaussian parameter $\beta$ [GeV] for ground state pseudoscalar mesons obtained by the variational principle. \( q = u \) and \( d \).

| Model | $\beta_{\ell q}$ | $\beta_{\ell p}$ | $\beta_{bc}$ | $\beta_{cs}$ | $\beta_{cc}$ | $\beta_{qb}$ | $\beta_{bs}$ | $\beta_{bc}$ | $\beta_{bb}$ |
|-------|-----------------|-----------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $\phi_A$ | 0.6376 | 0.5513 | 0.5810 | 0.5994 | 0.7916 | 0.6686 | 0.7132 | 1.0577 | 1.6455 |
| $\phi_B$ | 0.4520 | 0.3799 | 0.3960 | 0.4078 | 0.5286 | 0.4461 | 0.4757 | 0.6891 | 1.0549 |

Table 3 The Gaussian parameter $\beta$ [GeV] for ground state vector mesons obtained by the variational principle. \( q = u \) and \( d \).

| Model | $\beta_{\ell q}$ | $\beta_{\ell p}$ | $\beta_{bc}$ | $\beta_{cs}$ | $\beta_{cc}$ | $\beta_{qb}$ | $\beta_{bs}$ | $\beta_{bc}$ | $\beta_{bb}$ |
|-------|-----------------|-----------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $\phi_A$ | 0.3480 | 0.3952 | 0.5283 | 0.5727 | 0.7849 | 0.6436 | 0.7010 | 1.0554 | 1.6450 |
| $\phi_B$ | 0.2416 | 0.2742 | 0.3579 | 0.3892 | 0.5233 | 0.4278 | 0.4671 | 0.6871 | 1.0544 |

Using the plus component \((\mu = +)\) of the currents, one can calculate the decay constants. The explicit formulae of pseudoscalar and vector meson decay constants are given by \[14, 28\]

\[
f_P = \sqrt{6} \int_0^1 dx \int \frac{d^2k}{8\pi^3} \frac{\phi(x, k)}{\sqrt{\Delta^2 + k^2}},
\]

\[
f_V = \sqrt{6} \int_0^1 dx \int \frac{d^2k}{8\pi^3} \frac{\phi(x, k)}{\sqrt{\Delta^2 + k^2}} \left[ \sqrt{\Delta^2 + \Delta_{LF}^2} \right],
\]

where $\Delta = (1 - x)m_q + xm_{\bar{q}}$ and $\Delta_{LF} = M_0 + m_q + m_{\bar{q}}$. We perform the decay constant calculations for both trial wave functions $\phi_A$ and $\phi_B$ using the corresponding set of parameters fixed for each trial wavefunction, respectively.

3 Results and Discussion

We show in Fig. 1 our prediction of the meson mass spectra obtained from the variational principle to the effective Hamiltonian with the smeared-out hyperfine interaction using two different trial functions $\phi_A$ (blue lines) and $\phi_B$ (purple lines) and compare them with the experimental data (green lines) \[32\]. We also include the results (black lines) obtained from the CJ model with the linear confining potential \[14\]. We should note that the masses of $\pi$ and $\rho$ mesons are used as inputs in our calculation. As one can see, the single 1S state HO wave function $\phi_A$ already generates good results for the spectrum, and a more complicated trial wave function $\phi_B$ does not change the 1S results too much. In fact, the $\chi^2$ value for this modified model is 0.014 [0.018] for $\phi_A$ [$\phi_B$], which is more than half reduced from $\chi^2 = 0.039$ for the CJ model \[15\]. Except for the mass of $K$, our predictions for the masses of 1S-state pseudoscalar and vector mesons are within 4% error. Especially, our effective Hamiltonian with the smeared hyperfine interaction using both trial functions $\phi_A$ and $\phi_B$ clearly improves the predictions of heavy-light and heavy quarkonia systems such as $(\eta_c, J/\psi, B_c, \eta_b, \Upsilon)$ compared to the CJ model adopting the contact hyperfine interaction. Although the experimental data for $B_c^*$ is not yet available, our predictions of $B_c^*$, i.e. 6343 (6324) MeV for $\phi_{A(B)}$, are quite comparable with the lattice prediction 6331(9) MeV \[33\] as well as other quark model predictions such as 6340 MeV \[30\] and 6345.8 MeV \[34\].

In Table 4, we list our predictions for the decay constants of light mesons $(\pi, K, \rho, K^*)$ obtained by using $\phi_{A(B)}$
and compare them with CJ model [35] and the experimental data [32]. As one can see, our updated model calculation including the hyperfine interaction in the variation procedure doesn’t seem to improve the results of CJ model. In particular, the trial wave function \( \phi_A \) generates decay constants that are quite high for light mesons \((\pi, \rho, K, K^*)\) indicating that just 1S-state HO wave function alone cannot be a good trial wave function for the entire Hamiltonian including the smeared hyperfine interaction. However, using the mixed wave function \( \phi_B \) of 1S and 2S states, we can see a dramatic decrease in the numerical results consistent with the variational principle. Indeed, the results from \( \phi_B \) are much closer to the experimental data than those from \( \phi_A \). Especially for \( \pi \), the decay constant changes from 155 MeV to 139 MeV, which is much closer to the experimental value. Although the CJ model yields the experimental value of \( f_\pi \) much better than the updated model results from \( \phi_B \), an overall improvement due to the change of trial wave functions from \( \phi_A \) to \( \phi_B \) seems quite clear. Since the experimental values are very well known for light mesons, this improvement is very encouraging.

In Table 5, we list our predictions for the charmed meson decay constants \((f_{D^0}, f_{D^+}, f_{D^+}, f_{D_s^0}, f_{D_s^+})\) together with CJ model [36], lattice QCD [37–40], QCD sum rules [41], relativistic Bethe-Salpeter (BS) model [42], relativized quark model [43], and other relativistic quark model (RQM) [44] predictions as well as the available experimental data [32, 36]. We extract the experimental value \((f_{J/\Psi}/\exp) = (407 \pm 5)\) MeV from the data \(\Gamma_{\exp}(J/\Psi \rightarrow e^+e^-) = (5.55 \pm 0.14)\) keV [32] and the formula
\[
\Gamma(V \rightarrow e^+e^-) = \frac{4\pi}{3} \alpha_{\text{QED}}^2 e^2 \frac{f_V^2}{M_V},
\]
where \(\alpha_{\text{QED}}\) is the fine structure constant of the electromagnetic interaction.

4 Summary and Conclusion

In this work, we updated our LFQM by smearing out the Dirac delta function in the hyperfine interaction and including the smeared hyperfine interaction in our calculation based on the variational principle rather than using the perturbation method (CJ model) to handle the delta function in the contact hyperfine interaction. Using the two trial wave functions, i.e. the 1S state HO wave function \(\phi_A\) and the mixed wave function \(\phi_B \) of 1S and 2S HO states, we calculated both the mass spectra of the ground state pseudoscalar and vector mesons and the decay constants of the corresponding mesons.

In comparison with the CJ model, the variational analysis including the hyperfine interaction appears to provide the better agreement to the experimental data in particular for the heavy meson mass spectra regardless whether we use \(\phi_A\) or \(\phi_B\). We notice in general the meson mass spectra are irrespective of whether we use \(\phi_A\) or \(\phi_B\). However, a rather sizable difference between \(\phi_A\) and \(\phi_B\) is found in the decay constants. It is noticeable that \(\phi_B\) provides a rather significantly better agreement with the data of decay constants than \(\phi_A\) does. The variational analysis with \(\phi_B\) seems to improve the agreement with the data of heavy meson decay constants over the results of the CJ model. For the light mesons, however, the variation analysis including the hyperfine interac-
tion seem to improve the agreement with the data neither of mass spectra nor decay constants over the CJ model. Although the results from $\phi_A$ appear phenomenologically better than those from $\phi_A$ for the light meson decay constants, they don’t seem to be any better than the results from the CJ model. These results seem to suggest that the analysis of the light meson sectors better be separately handled from the analysis of the heavy meson sectors. This distinction between the heavy meson sector and the light meson sector appears rather natural from the commonality of the chiral symmetry between QCD and LFQM [27]. To get more definitive conclusion in this respect, further analysis of other wave function related observables such as various meson elastic and transition form factors may be useful. It may be also interesting to analyze the results of $\langle H_{C.M.} \rangle$ vs. $\langle H_{C.M.}^2 \rangle$ as discussed in our model description, Sec. 2, as well as radially excited meson states using the larger HO basis.

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Appendix A: Fixation of the model parameters using variational principle

In our model, we assumed SU(2) symmetry and have the following parameters that need to be fixed: constituent quark masses $(m_u, m_d, m_s, m_c)$, potential parameters $(a, b, \alpha_s)$, gaussian parameter $\beta$, and the smearing parameter $\sigma$. For our second trial wave function $\phi_B = \sum_{n=1}^{2} c_n \phi_{nS}$, we also

| Table 5 | Charmed meson decay constants (in unit of MeV) obtained from our updated LFQM. |
|---------|-----------------------------------------------|
| Model   | $f_D$  | $f_{D'}$ | $f_{D_s}$ | $f_{D'_s}$ | $f_{B_s}$ | $f_{J/\Psi}$ |
| $\phi_A$ | 244    | 279     | 276      | 322      | 406      | 460        |
| $\phi_B$ | 218    | 241     | 249      | 282      | 354      | 390        |
| CJ [36]  | 197    | 239     | 232      | 273      | 326      | 360        |
| Lattice [37] | $211 \pm 3 \pm 17$ | $245 \pm 2 \pm 13$ | $231 \pm 2 \pm 18$ | $272 \pm 2 \pm 16$ | –          | –          |
| QCD [38, 39] | $208 \pm 7 \pm 38$ | –       | $250 \pm 7 \pm 38$ | –          | $387 \pm 7 \pm 39$ | $418 \pm 9 \pm 39$ |
| Sum-rules [41] | $201 \pm 12 \pm 23$ | $242 \pm 20 \pm 23$ | $238 \pm 13 \pm 23$ | $293 \pm 19 \pm 15$ | –          | –          |
| BS [42]    | $230 \pm 25$ | $340 \pm 23$ | $248 \pm 27$ | $375 \pm 24$ | $292 \pm 25$ | $459 \pm 28$ |
| QM [43]    | $240 \pm 20$ | –       | $290 \pm 20$ | –          | –          | –          |
| RQM [44]   | $234$    | $310$ | $268$ | $315$ | –          | –          |
| Exp.       | $206.7 \pm 8.9 \pm 32$ | –       | $257.5 \pm 6.1 \pm 32$ | –          | $335 \pm 75 \pm 45$ | $407 \pm 5 \pm 32$ |

| Table 6 | Bottomed meson decay constants (in unit of MeV) obtained from our updated LFQM. |
|---------|-----------------------------------------------|
| Model   | $f_B$  | $f_{B'}$ | $f_{B_s}$ | $f_{B'_s}$ | $f_{B_{s}}$ | $f_R$ |
| $\phi_A$ | 229    | 243     | 267      | 288      | 805      | 871        |
| $\phi_B$ | 195    | 202     | 229      | 242      | 654      | 692        |
| CJ [36]  | 171    | 185     | 205      | 220      | 507      | 529        |
| Lattice [37] | $179 \pm 18 \pm 34$ | $196 \pm 24 \pm 29$ | $204 \pm 2 \pm 0$ | $229 \pm 2 \pm 1$ | –          | –          |
| QCD [38, 46] | $189 \pm 8 \pm 38$ | –       | $228 \pm 8 \pm 38$ | –          | –          | $649 \pm 3 \pm 46$ |
| Sum-rules [41] | $207 \pm 17 \pm 0$ | $210 \pm 10 \pm 0$ | $242 \pm 12 \pm 0$ | $251 \pm 14 \pm 16$ | –          | –          |
| BS [42]    | $196 \pm 29$ | $238 \pm 18$ | $216 \pm 32$ | $272 \pm 20$ | –          | $498 \pm 20$ |
| QM [43]    | $155 \pm 15$ | –       | $210 \pm 20$ | –          | –          | –          |
| RQM [44]   | $189$    | $219$ | $218$ | $251$ | –          | –          |
| Exp.       | $229 \pm 36 \pm 34 \pm 47$ | –       | –       | –          | $689 \pm 5 \pm 32$ | –          |

| Table 7 | Bottom-charmed meson decay constants (in unit of MeV) obtained from our updated LFQM. |
|---------|-----------------------------------------------|
| Model   | $f_{B_c}$ | $f_{B'_c}$ | $f_{B_{s}}$ | $f_{B'_{s}}$ | $f_{B_{s}}$ | $f_{J/\Psi}$ |
| $\phi_A$ | 488    | 406     | 349      | 360      | 433      | 500 460 60 | 517 410 40 |
| $\phi_B$ | 350    | 432     | 369      | –        | 503      | 500 460 60 | 517 – – – |
| CJ [15]  | 349    | 360     | 433      | 500      | 460 60   | 517 410 40 |
| | 300    | 350     | 433      | 500      | 460 60   | 517 410 40 |
have the mixing factor \( c_n(n = 1, 2) \) that we have to adjust. Notice that the \( \beta \) values here are not only different for different quark combinations, but also different for pseudoscalar and vector mesons of the same quark combination. The reason for this is that the hyperfine interaction we included in our parameterization process gives different contributions to the masses of pseudoscalar and vector mesons and thus induces different parameterizations under variational principle.

We now illustrate our procedure for fixing these parameters. The variational principle gives us one constraint:

\[
\frac{\partial \langle \Psi'|H|\Psi \rangle}{\partial \beta} = \frac{\partial M_{\bar{q}q}}{\partial \beta} = 0. \tag{A.1}
\]

We can use this equation to rewrite the coupling constant \( \alpha \), \( c \) in terms of other parameters and plug it back into Eqs. (9) and (10) and thus eliminate \( \alpha \). The string tension \( b \) is fixed to be 0.18 GeV, a well known value from other quark model analysis [30, 31, 55]. We will leave the quark masses and smearing parameter \( \sigma \) (and the mixing factor \( c_1 \) for \( \phi_b \)) as externally adjustable variables. We picked a set of values for \( (m_{u(d)}, m_s, m_c, m_b, \sigma) \) when using \( \phi_s \) and \( (m_{u(d)}, m_s, m_c, m_b, \sigma, c_1) \) when using \( \phi_b \), and proceed with the following procedure to solve for the rest of parameters.

We are left with 3 more parameters \( (\alpha, \beta_p, \beta^e_{\bar{q}q}) \) for mesons of a certain quark combination \( (q\bar{q}) \), where \( \beta_p, \beta^e_{\bar{q}q} \) are the gaussian parameters for pseudoscalar \((p)\) and vector \((v)\) mesons, respectively. Using the masses of \( \pi \) and \( \rho \) as our input values for \( M^\pi_{\bar{q}q} \) in Eq. (9) \([M^\rho_{\bar{q}q} \text{ in Eq. (10)}\], and the condition that our coupling constants \( \alpha \) are the same for all these ground state pseudoscalar and vector mesons, we can fix the three model parameters \( (\alpha, \beta_p, \beta^e_{\bar{q}q}) \) for \( q = u \) or \( d \) from the following three equations:

\[
M_\pi(\beta^e_{\bar{q}q}, \alpha) = 0.140, \tag{A.2a}
\]

\[
M^\rho_{\bar{q}q}(\beta^e_{\bar{q}q}, \alpha) = 0.77, \tag{A.2b}
\]

\[
\alpha_s(\beta^e_{\bar{q}q}, \alpha) = \alpha_s(\beta^e_{\bar{q}q}, \alpha). \tag{A.2c}
\]

Solving these equations not only gives us the remaining parameters \( \alpha_p, \beta_p, \beta^e_{\bar{q}q} \) but also the coupling constant \( \alpha \), which we assumed to be the same for all the mesons we consider here. We can then solve for the \( \beta \) values of all the other mesons using the known \( \alpha \) value, by equating the \( \alpha \) expressions for different mesons that we got from Eq. (A.1). We thus fixed all parameters for the ground state pseudoscalar and vector mesons we consider here.

We then assign a different set of values to the externally adjustable variables, i.e. \( (m_{u(d)}, m_s, m_c, m_b, \sigma) \) when using \( \phi_s \) and \( (m_{u(d)}, m_s, m_c, m_b, \sigma, c_1) \) when using \( \phi_b \), and repeat the above procedure until we find a set of values that give best fit for the meson mass spectra.

Through our trial and error type of analysis, we found \( m_q = 0.220 \text{ GeV}, m_s = 0.432 \text{ GeV}, m_c = 1.77 \text{ GeV}, m_b = 5.2 \text{ GeV}, \sigma = 0.405 \text{ GeV} \) gives best fit of the meson mass spectrum when using trial wave function \( \phi_s \), while \( m_q = 0.221 \text{ GeV}, m_s = 0.456 \text{ GeV}, m_c = 1.77 \text{ GeV}, m_b = 5.2 \text{ GeV}, \sigma = 0.423 \text{ GeV}, c_1 = \sqrt{0.7} \) gives best fit when using trial wave function \( \phi_b \). For these values, our obtained \( \beta \) values are listed in Tables 2 and 3.

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