Statistical inference for heavy tailed series with extremal independence

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Abstract

We consider stationary time series \(\{X_j, j \in \mathbb{Z}\}\) whose finite dimensional distributions are regularly varying with extremal independence. We assume that for each \(h \geq 1\), conditionally on \(X_0\) to exceed a threshold tending to infinity, the conditional distribution of \(X_h\) suitably normalized converges weakly to a non degenerate distribution. We consider in this paper the estimation of the normalization and of the limiting distribution.

1 Introduction

Let \(\{X_j, j \in \mathbb{Z}\}\) be a strictly stationary univariate time series. We say that the time series \(\{X_j\}\) is regularly varying if all its finite dimensional distributions are regularly varying, i.e. for each \(h \geq 0\), there exists a nonzero boundedly finite measure \(\nu_h\) on \(\mathbb{R}^{h+1} \setminus \{0\}\) infinity, such that

\[
\frac{\mathbb{P}(x^{-1}(X_0, \ldots, X_h) \in \cdot)}{\mathbb{P}(|X_0| > x)} \xrightarrow{\nu} \nu_h, \tag{1.1}
\]
on \(\mathbb{R}^{h+1} \setminus \{0\}\), as \(x \to \infty\), where \(\xrightarrow{\nu}\) means vague convergence. Following [Kal17], we say that a measure \(\nu\) defined on a complete separable metric space \(E\) (endowed with its Borel \(\sigma\)-field) is boundedly finite if \(\nu(B)\) for all Borel bounded sets and a sequence of boundedly finite measures \(\{\nu_n\}\) is said to converge vaguely to a measure \(\nu\) if \(\nu_n(f) \to \nu(f)\) for all continuous functions with bounded support. See also [HL06] who use the terminology of \(\mathcal{M}_0\)-convergence. Here the metric space considered is \(\mathbb{R}^{h+1} \setminus \{0\}\) endowed with the metric

\[
d_h(x, y) = |x - y| \vee |x|^{-1} - |y|^{-1},
\]
where \(|\cdot|\) is an arbitrary norm on \(\mathbb{R}^{h+1}\). This metric induces the usual topology and makes \(\mathbb{R}^{h+1} \setminus \{0\}\) a complete separable space and bounded sets are sets separated from zero.

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Moreover, \( \mathbb{R}^{h+1} \setminus \{0\} \) is still locally compact so this definition essentially yields the same notion as the classical vague convergence without the need for compactification at infinity.

This assumption implies that there exists \( \alpha > 0 \) such that the measure \( \nu_h \) is homogeneous of degree \(-\alpha\) and the marginal distribution of \( X_0 \) is regularly varying and satisfies the balanced tail condition: there exists \( p \in [0, 1] \) such that
\[
p = \lim_{x \to \infty} \frac{P(X_0 > x)}{P(|X_0| > x)} = 1 - \lim_{x \to \infty} \frac{P(X_0 < -x)}{P(|X_0| > x)}.
\]

Without loss of generality, we assume that \( p > 0 \).

If \( h \geq 1 \), there exist two fundamentally different cases: either the exponent measure is concentrated on the axes or it is not. The former case is referred to as extremal independence and the latter as extremal dependence. In other words, extremal independence means that no two components can be extremely large at the same time, and extremal dependence means that some pairs components can be simultaneously extremely large.

In a time series context, we may want to assess the influence of an extreme event at time zero on future observations. If the finite dimensional distributions of the time series model under consideration are extremely independent or more generally if the vector \((X_0, X_m, \ldots, X_h)\) is extremally independent for some \( h \geq m \geq 1 \), then, for any Borel set \( A \) which is bounded away from zero in \( \mathbb{R}^{h-m+1} \) and \( y_0 > 0 \),
\[
\lim_{x \to \infty} \frac{P(X_0 > xy_0, (X_m, \ldots, X_h) \in xA)}{P(|X_0| > x)} = 0.
\]
(1.2)

Thus in case of extremal independence the exponent measure \( \nu_h \) provides no information on (most) extreme events occurring after an extreme event at time 0.

In order to obtain a non degenerate limit in (1.2) and a finer analysis of the sequence of extreme values, it is necessary to change the normalization in (1.1), and possibly the space on which we will assume that vague convergence holds. One idea is to find a sequence of normalizations \( b_j(x), j \geq 1 \) such that for each \( h \geq 1 \), the conditional distribution of \((X_0/x, X_1/b_1(x), \ldots, X_h(x)/b_h(x))\) given \(|X_0| > x\) has a non degenerate limit. Pursuing in the direction opened by [HR07] and [DR11], [LRR14] and [KS15] we will consider vague convergence on the set \( (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^h \) endowed with the metric \( d^0_h \) on defined by
\[
d^0_h(x, y) = |x - y| \vee |x_0^{-1} - y_0^{-1}|.
\]
The bounded sets for this metric are those sets \( A \) such that \( x = (x_0, \ldots, x_h) \in A \) implies \( |x_0| > \epsilon \) for some \( \epsilon > 0 \). Note that under the present definition of vague convergence, we avoid the pitfalls described in [DJJe17].

**Assumption 1.1.** There exist scaling functions \( b_j, j \geq 1 \) and nonzero measures \( \mu_{0,h}, h \geq 1 \), boundedly finite on \( \mathbb{R} \setminus \{0\} \times \mathbb{R}^h \), \( h \geq 1 \), such that
\[
\frac{1}{P(|X_0| > x)} P\left( \left( \frac{X_0}{x}, \frac{X_1}{b_1(x)}, \ldots, \frac{X_h}{b_h(x)} \right) \in \cdot \right) \Rightarrow \mu_{0,h},
\]
(1.3)
on \( \mathbb{R} \setminus \{0\} \times \mathbb{R}^h \) and for every \( y_0 > 0 \), the measures \( \mu_{0,h}([y_0, \infty) \times \cdot \cdot \cdot) \) and \( \mu_{0,h}((\infty, -y_0) \times \cdot \cdot \cdot) \) on \( \mathbb{R}^h \) are not concentrated on a hyperplane.
This assumption does not exclude regularly varying time series with extremal dependence for which \( b_j(x) = x \) for all \( j \geq 0 \). But our interest will be in extremally independent time series for which \( b_j(x) = o(x) \) for all \( j \geq 0 \). This assumption is fulfilled by many time series, like stochastic volatility models with heavy tailed noise or heavy tailed volatility, exponential moving averages and certain Markov chains with regularly varying initial distribution and appropriate conditions on the transition kernel. See [KS15], [MR13] and [JD16].

An important consequence of Assumption 1.1 is that the functions \( b_j, j \geq 1 \) are regularly varying (see [HR07, Proposition 1] and [KS15].) To put emphasis on the regular variation of the functions \( b_j \), we recall the following definition of [KS15].

**Definition 1.2** (Conditional scaling exponent). Under Assumption 1.1, for \( h \geq 1 \), we call the index \( \kappa_h \) of regular variation of the functions \( b_h \) the (lag \( h \)) conditional scaling exponent.

The exponents \( \kappa_h, h \geq 1 \) reflect the influence of an extreme event at time zero on future lags. Even though we expect this influence to decrease with the lag in the case of extremal independence, these exponents are not necessarily monotone decreasing. The measures \( \mu_{0,h} \) also have some important homogeneity properties: For all Borel sets \( A_0 \subset \mathbb{R} \setminus \{0\} \), \( A_1, \ldots, A_h \subset \mathbb{R} \),

\[
\mu_{0,h} \left( tA_0 \times \prod_{i=1}^h t^{\kappa_i} A_i \right) = t^{-\alpha} \mu_{0,h} \left( \prod_{i=0}^h A_i \right). \tag{1.4}
\]

Equivalently, for all bounded measurable functions \( f \),

\[
\int_{(\mathbb{R} \setminus \{0\}) \times \mathbb{R}^h} f(t^{-1}x_0, t^{-\kappa_1}x_1, \ldots, t^{-\kappa_h}x_h) \mu_{0,h}(dx) = t^{-\alpha} \int_{(\mathbb{R} \setminus \{0\}) \times \mathbb{R}^h} f(x) \mu_{0,h}(dx). \tag{1.5}
\]

Cf. [HR07, Proposition 1] and [KS15, Lemma 2.1]. Define the probability measure \( \sigma_h \) on \( \{-1, 1\} \times \mathbb{R}^h \) by

\[
\sigma_h(\{\epsilon\} \times A) = \int_{c_{u_0} > 1} \int_A \mu_{0,h}(du_0, u_0^{\kappa_1}du_1, \ldots, u_0^{\kappa_h}du_h),
\]

for \( \epsilon \in \{-1, 1\} \) and \( A \) a Borel subset of \( \mathbb{R}^h \). Let \( W = (W_0, W_1, \ldots, W_h) \) be an \( \mathbb{R}^{h+1} \) valued random vector with distribution \( \sigma_h \). Then, for every Borel subsets \( A \subset (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^h \), we have

\[
\mu_{0,h}(A) = \int_0^\infty \mathbb{P}((sW_0, s^{\kappa_1}W_1, \ldots, s^{\kappa_h}W_h) \in A) \alpha s^{-\alpha - 1}ds. \tag{1.6}
\]

See [KS15, Section 2.4]. Let \( Y_0 \) be a Pareto random variable with tail index \( \alpha \), independent of \( W_h \). Then, as \( x \to \infty \),

\[
\mathcal{L} \left( \left( \frac{X_0}{x}, \frac{X_1}{b_1(x)}, \ldots, \frac{X_h}{b_h(x)} \right) \mid |X_0| > x \right) \xrightarrow{d} Y_0W_h. \tag{1.7}
\]
In particular, we define for $h > 0$ the distribution function $\Psi_h$ on $\mathbb{R}$:

$$\Psi_h(y) = \mathbb{P}(Y_0 W_h \leq y) = \lim_{x \to \infty} \mathbb{P}(X_h \leq b_h(x)y \mid |X_0| > x),$$

for all $y \in \mathbb{R} \setminus \{0\}$ since the distribution of $Y_0 W_h$ is continuous at all points except possibly $0$.

The goal of this paper is to complement the investigation of this assumption started in [KS15] by providing valid statistical procedures to estimate the conditional scaling functions $b_h$, the conditional limiting distributions $\Psi_h$ and scaling exponents $\kappa_h$.

2 Statistical inference

Let $F_0$ be a distribution of $|X_0|$. All our results we be proved under the following $\beta$-mixing assumptions.

**Assumption 2.1.**

(A1) The sequence $\{X_j, j \in \mathbb{Z}\}$ is $\beta$-mixing with rate $\{\beta_n, n \geq 1\}$.

(A2) There exist a non decreasing sequence $u_n$, non decreasing sequences of integers $r_n$ and $l_n$ such that

$$\lim_{n \to \infty} l_n = \lim_{n \to \infty} r_n = \lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{r_n}{l_n} = \infty,$$

$$\lim_{n \to \infty} \frac{n}{r_n} \beta_{l_n} = 0,$$

$$\lim_{n \to \infty} u_n = n F_0(u_n) = \infty, \quad \lim_{n \to \infty} r_n F_0(u_n) = 0.$$ (2.1, 2.2, 2.3)

2.1 Non parametric estimation of the limiting conditional distribution

In order to define an estimator of $\Psi_h$, we must first consider the infeasible statistic

$$\tilde{I}_{h,n}(s, y) = \frac{1}{n F_0(u_n)} \sum_{j=1}^{n-h} \mathbb{1}\{|X_j| > u_n s, X_{j+h} \leq b_h(u_n)y\}.$$ (2.4)

Then, Assumption 1.1 and the homogeneity property (1.5) imply that for all $s > 0$ and $y \in \mathbb{R}$,

$$\lim_{n \to \infty} \mathbb{E}[\tilde{I}_{h,n}(s, y)] = \lim_{n \to \infty} \frac{n \mathbb{P}(|X_0| > u_n s, X_h \leq b(u_n)y)}{n F_0(u_n)} = \mu_{0,h}((s, \infty) \times \mathbb{R}^{h-1} \times (-\infty, y]) = s^{-\alpha} \Psi_h(s^{-\kappa_h}y).$$

We consider weak convergence of the processes $\tilde{I}_{h,n}$ and $\tilde{I}_{h,n}$ defined on $(0, \infty) \times \mathbb{R}$ by

$$\tilde{I}_{h,n}(s, y) = \sqrt{n F_0(u_n)} \{\tilde{I}_{h,n}(s, y) - \mathbb{E}[\tilde{I}_{h,n}(s, y)]\},$$

$$\tilde{I}_{h,n}(s, y) = \sqrt{n F_0(u_n)} \{\tilde{I}_{h,n}(s, y) - s^{-\alpha} \Psi_h(s^{-\kappa_h}y)\}.$$
**Assumption 2.2.** For all \( s, t > 0 \),

\[
\lim_{\ell \to \infty} \limsup_{n \to \infty} \frac{1}{F_0(u_n)} \sum_{\ell < |j| \leq r_n} \mathbb{P}(|X_0| > u_n s, |X_j| > u_n t) = 0 .
\] (2.5)

**Assumption 2.3.** There exists \( s_0 \in (0, 1) \) such that

\[
\lim_{n \to \infty} \sqrt{n F_0(u_n)} \sup_{s \geq s_0, y \in \mathbb{R}} \left| \mathbb{P}(|X_0| > u_n s, X_h \leq b(u_n)y) - s^{-\alpha} \Psi_h(s^{-\kappa_h}y) \right| = 0 .
\] (2.6)

**Remark 2.4.** An assumptions similar to (2.5) is unavoidable. Its purpose is to prove the convergence of the intrablock variance in the blocking method and tightness. The present one is taken from [KSW15]. Similar ones have been checked directly for extremally dependent time series like GARCH(1,1) or ARMA models (see e.g. [Dre02]), or for Markov chains that satisfy a drift condition (cf. [KSW15]). This assumption will be checked in Section 3 for some specific models. Assumption 2.3 is unavoidable if one wants to remove bias. This will not be discussed in the paper. The condition holds for some sequences \( u_n \).

Let \( \mathbb{I}_h \) be the Gaussian process on \((0, \infty) \times \mathbb{R} \) with covariance \( \text{cov}(\mathbb{I}_h(s, y), \mathbb{I}_h(t, z)) = (s \vee t)^{-\alpha} \Psi_h((s \vee t)^{-\kappa_h}(y \wedge z)) \), \( s, t > 0, y, z \in \mathbb{R} \). We note that

\[
\mathbb{W}(u) = \mathbb{I}_h(1, \Psi^{-1}_h(u)) , \quad u \in (0, 1) ,
\]

is a standard Brownian motion on \((0, 1) \). The following theorem establishes weak convergence of the tail empirical process and forms the basis for statistical inference on \( \Psi_h \). Its proof is given in Section 6.2.

**Theorem 2.5.** Let \( \{X_j, j \in \mathbb{Z}\} \) be a strictly stationary regularly varying sequence such that Assumption 1.1 with extremal independence at all lags. Assume moreover that Assumptions 2.1 and 2.2 hold and that the function \( \Psi_h \) is continuous on \( \mathbb{R} \). Then the process \( \mathbb{I}_{h,n} \) converges weakly in \( L^\infty([s_0, \infty) \times \mathbb{R}) \) to \( \mathbb{I}_h \). If moreover Assumption 2.3 holds, then \( \mathbb{I}_{h,n} \) converges weakly in \( L^\infty([s_0, \infty) \times \mathbb{R}) \) to \( \mathbb{I}_h \).

We now need proxies to replace \( u_n \) and \( b(u_n) \) which are unknown in order to obtain a feasible statistical procedure. As usual, \( u_n \) will be replaced by an order statistic. To estimate the scaling functions \( b_h \) we will exploit their representations in terms of conditional mean. Therefore, we need additional conditions.

**Assumption 2.6.** There exists \( \delta > 0 \) and \( s_0 > 0 \) such that

\[
\lim_{n \to \infty} r_n \{n F_0(u_n)\}^{-\delta/2} = 0 ,
\] (2.7)
2.9 may seem to be too restrictive.

and Assumption 2.6

\[ \lim \sup_{n \to \infty} \frac{1}{\sqrt{F_0(u_n)}} \sum_{\ell < |j| \leq r_n} \mathbb{E} \left[ \frac{|X_h|}{b_h(u_n)} \frac{|X_{j+h}|}{b_h(u_n)} \mathbb{1}\{|X_0 > s_0 u_n \}} \mathbb{1}\{|X_j > s_0 u_n \}} \right] = 0 , \]  

(2.9)

\[ \lim \sup_{n \to \infty} \sqrt{n} \frac{1}{\sqrt{F_0(u_n)}} \mathbb{E} \left[ \frac{\mathbb{1}\{|X_0 > u_n s \}}{b_h(u_n)} \right] - \mathbb{E}[|W_h|] s^{\kappa_h - \alpha} = 0 . \]  

(2.10)

Condition (2.8) requires \( \alpha > 2 \) and implies that the sequence \( (b_h^{-1}(u_n) X_h)^2 \) is uniformly integrable conditionally on \( |X_0| > u_n \) and therefore,

\[ \lim_{x \to \infty} \mathbb{E} \left[ b_h^{-1}(x) \right] |X_h| | X_0 > x \right] = \int_{-\infty}^{\infty} |y|^i \Psi_h(dy) = \mathbb{E}[Y_0^i] \mathbb{E}[|W_h|] < \infty , \quad i = 1, 2 . \]  

(2.11)

Since the function \( b_h \) and the limiting distribution \( \Psi_h \) are defined up to a scaling constant, we can and will assume without loss of generality that

\[ \mathbb{E}[|W_h|] = \int_{-\infty}^{\infty} |y| \Psi_h(dy) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x_h| \mu_{0,h}(x) . \]  

Condition (2.9) is again unavoidable and must be checked for specific models. Condition (2.10) is a bias condition which will not be further discussed.

Set \( k = n F_0(u_n) \) and let \( \{X_{(i:1)} \leq |X_{(i:2)} \leq \cdots \leq |X_{(i:n)} \} \) be the order statistics of \( |X_1|, \ldots, |X_n| \). Define an estimator of \( b_h(u_n) \) by

\[ \hat{b}_{h,n} = \frac{1}{k} \sum_{j=1}^{n-h} |X_{j+h}| \mathbb{1}\{|X_j > |X_{(n-k)} \}} . \]  

(2.12)

**Corollary 2.7.** Let the assumptions of Theorem 2.5 and Assumption 2.6 hold with extremal independence at all lags. Then

\[ \left( \Pi_{h,n}, \sqrt{k} \left( \frac{|X_{(n-k)}}{u_n} - 1 \right), \sqrt{k} \left( \frac{\hat{b}_{h,n}}{b_h(u_n)} - 1 \right) \right) \]

\[ \xrightarrow{w} \left( \Pi_h, \alpha^{-1} \mathbb{W}(1), \int_{0}^{1} |\Psi_h^{-1}(u)| d\mathbb{B}(u) + \alpha^{-1} \kappa_h \mathbb{W}(1) \right) , \]

where \( \mathbb{W}(u) = \Pi_h(1, \Psi_h^{-1}(u)) \) is a standard Brownian motion and \( \mathbb{B}(u) = \mathbb{W}(u) - u \mathbb{W}(1) \) is a standard Brownian bridge on \( [0, 1] \).

**Remark 2.8.** The moment conditions in Assumption 2.6 may seem to be too restrictive. In fact, we can consider a family of estimators \( \hat{b}_{h,n}(\zeta) \), where \( |X_{j+h}| \) in (2.12) is replaced with \( |X_{j+h}|^{\zeta} \) with some \( \zeta > 0 \). However, we do not pursue it in this paper.
Define now the following estimator of $\Psi_h$:  
\[
\hat{\Psi}_{h,n}(y) = \frac{1}{k} \sum_{j=1}^{n-h} \mathbb{1}\{X_j > X_{(n,n-k)}\} \mathbb{1}\{X_{j+h} \leq \hat{b}_{h,n} y\} = \tilde{I}_{h,n} \left( \frac{X_{(n,n-k)}}{u_n}, \frac{\hat{b}_{h,n} y}{b_h(u_n)} \right).
\]

(2.13)

The theory for this estimator is easily obtained by applying Corollary 2.7 and the $\delta$-method.

**Corollary 2.9.** Under the assumptions of Corollary 2.7 and if the function $\Psi_h$ is differentiable,  
\[
\sqrt{k} \left\{ \hat{\Psi}_{h,n} - \Psi_h \right\} \xrightarrow{w} \Lambda_h \text{ in } \ell^\infty([s_0, \infty)), \text{ where the process } \Lambda_h \text{ is defined by}
\]
\[
\Lambda_h(y) = \mathbb{B}(\Psi_h(y)) + y \Psi_h'(y) \int_0^1 |\Psi_h^{-1}(u)| d\mathbb{B}(u),
\]

(2.14)

where $\mathbb{B}$ is the standard Brownian bridge.

**Remark 2.10.** The additional term in the limiting distribution is due to the method of estimation of the conditional scaling function. Note that the limiting distribution depends only on $\Psi_h$ and therefore can be used for a Kolmogorov-Smirnov type goodness of fit test of the conditional distribution.

### 2.2 Estimation of the conditional scaling exponent

We now consider the estimation of the scaling exponent $\kappa_h$. We will use the following result.

**Lemma 2.11.** Let Assumption 1.1 hold and assume moreover that  
\[
\lim_{\epsilon \to 0} \limsup_{x \to \infty} \frac{\mathbb{P}(|X_0 X_h| > x b_h(x), |X_0| \leq \epsilon x)}{\mathbb{P}(|X_0| > x)} = 0.
\]

(2.15)

Then $\mathbb{E}[|W_h|^{\alpha/(1+\kappa_h)}] < \infty$ and

\[
\lim_{x \to \infty} \frac{\mathbb{P}(|X_0 X_h| > x b_h(x) y)}{\mathbb{P}(|X_0| > x)} = \mathbb{E}[|W_h|^{\alpha/(1+\kappa_h)}] y^{-\alpha/(1+\kappa_h)}.
\]

(2.16)

This is [KS15, Proposition 2], where the finiteness of $\mathbb{E}[|W_h|^{\alpha/(1+\kappa_h)}]$ is assumed, but it is easily seen that this is actually a consequence of (2.15). At this moment this is all we need to state our results but we will need to prove in Section 6.1 a generalized version of Lemma 2.11; see Lemma 6.4. It must be noted that Condition (2.15) does not hold for an i.i.d. sequence. See also Section 3.1.

If (2.15) holds, then the product $X_0 X_h$ has tail index $\alpha/(1+\kappa_h)$. Hence, we can suggest the following estimation procedure of the scaling exponent $\kappa_h$.  

[7]
• Let $\gamma = 1/\alpha$, where $\alpha$ is the tail index of the sequence $\{|X_j|\}$. Estimate $\gamma$ using the Hill estimator $\hat{\gamma}$ based on an intermediate sequence $k$, i.e.

$$\hat{\gamma} = \frac{1}{k} \sum_{j=1}^{n} \log(|X_{(n-j+1)}|/|X_{(n-k)}|).$$

• Let $\gamma_h = (1+\kappa_h)\gamma = (1+\kappa_h)/\alpha$ be estimated by $\hat{\gamma}_h$, the Hill estimator of the tail index of $|X_{0}X_{h}|$, based on the sequence $V_j = |X_jX_{j+h}|$, $j = 1, \ldots, n$ (assuming without loss of generality that we have $n+h$ observations) and on the same intermediate sequence:

$$\hat{\gamma}_h = \frac{1}{k} \sum_{j=1}^{n} \log(V_{(n-j+1)}/V_{(n-k)}).$$

• Estimate $\kappa_h = \gamma_h/\gamma - 1$ by

$$\hat{\kappa}_h = \frac{\hat{\gamma}_h}{\hat{\gamma}} - 1. \quad (2.17)$$

Asymptotic normality of the Hill estimator for beta-mixing sequences is well known. See e.g. [Dre00, Dre02]. The asymptotic normality of $\hat{\kappa}_h$ will follow from the delta method. To state the result, we need additional anti-clustering and second-order conditions.

**Assumption 2.12.** For all $s,t > 0$,

$$\lim_{\ell \to \infty} \limsup_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \text{Pr}(|X_{0}X_{h}| > u_nb(u_n)\alpha, |X_{j}X_{j+h}| > u_nb(u_n)t) = 0. \quad (2.18)$$

Furthermore,

$$\lim_{\ell \to \infty} \limsup_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \text{E} \left[ \log_+ (|X_{0}|/u_n) \log_+ (|X_{j}|/u_n) \right] = 0, \quad (2.19)$$

$$\lim_{\ell \to \infty} \limsup_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \text{E} \left[ \log_+ (|X_{0}X_{h}|/(u_nb(u_n))) \log_+ (|X_{j}X_{j+h}|/(u_nb(u_n))) \right] = 0. \quad (2.20)$$

**Theorem 2.13.** Let $\{X_j, j \in \mathbb{Z}\}$ be a strictly stationary regularly varying sequence such that Assumption 1.1 holds with independence at all lags. Assume moreover that Assumptions 2.1 to 2.3 and 2.12 and the bound (2.15) hold and that $k = n\text{F}_0(u_n)$ is chosen in such a way that

$$\lim_{n \to \infty} \sqrt{k} \sup_{s \geq s_0} \left| \text{Pr}(|X_{0}X_{h}| > u_nb(u_n)\alpha)/\text{F}_0(u_n) - s^{-\alpha/(1+\kappa_h)} \right| = 0 \quad (2.21)$$

for some $s_0 \in (0,1)$. Then

$$\sqrt{k}(\hat{\kappa}_h - \kappa_h) \xrightarrow{d} \mathcal{N} \left(0, (1+\kappa_h)\text{E}||W_h||^{1+\kappa_h} - 1\right)\right).$$
3 Examples

3.1 Stochastic volatility process

Consider the sequence \( X_j = \varepsilon_j \exp(Y_j), \; j \in \mathbb{Z} \), where \( \{Y_j, j \in \mathbb{Z}\} \) is a Gaussian process independent of the i.i.d. sequence \( \{\varepsilon_j, j \in \mathbb{Z}\} \), regularly varying with index \( \alpha \). For simplicity we assume that the random variables \( \varepsilon_j \) are nonnegative. We list the properties of \( X_j \) (see [DM01], [KS11], [KS15]).

(i) The sequence \( \{X_j, j \in \mathbb{Z}\} \) is regularly varying with extremal independence. It satisfies Assumption 1.1 with \( b_h \equiv 1 \) for all \( h \geq 1 \).

(ii) By Breiman’s lemma, \( \mathbb{P}(X_0 > x) \sim \mathbb{E}[(\exp(\alpha Y_0))] \mathbb{P}(\varepsilon_0 > x) \) as \( x \to \infty \).

(iii) By [Bra05, Theorem 5.2a),(c)], if the spectral density of the Gaussian sequence \( \{Y_j, j \in \mathbb{Z}\} \) is bounded away from zero and if \( \text{cov}(Y_0, Y_n) = O(n^{-\delta}) \) with \( \delta > 2 \) then \( \beta(n) = O(n^{2-\delta}) \);

(iv) Conditioning on the sequence \( Y = \{Y_j, j \in \mathbb{Z}\} \), the equivalence between the tails of \( \varepsilon_0 \) and \( X_0 \) and Potter’s bounds yield for \( \delta > 0 \),

\[
\frac{1}{F_0(u_n)} \sum_{\ell < |j| \leq r_n} \mathbb{P}(X_0 > u_n s, X_j > u_n s) = \frac{F^2_\varepsilon(u_n)}{F_0(u_n)} \sum_{\ell < |j| \leq r_n} \mathbb{E} \left[ \frac{\mathbb{P}(\varepsilon_0 > u_n s \exp(-Y_0) | Y) \mathbb{P}(\varepsilon_0 > u_n s \exp(-Y_j) | Y)}{\mathbb{P}(\varepsilon_0 > u_n)} \right] = O(F^2_\varepsilon(u_n)) \sum_{\ell < |j| \leq r_n} \mathbb{E} [\exp((\alpha + \delta)(Y_0 + Y_j)) \vee 1] = O(r_n F_\varepsilon(u_n)) = o(1),
\]

as \( n \to \infty \) if (2.3) holds.

(v) Fix \( \delta > 0 \). We again condition on the sequence \( Y \) and apply Potter’s bounds:

\[
\frac{1}{F_0(u_n)} \sum_{\ell < |j| \leq r_n} \mathbb{E}[[X_h X_{j+h}] | \mathbb{1}\{X_0 > u_n s\} \mathbb{1}\{X_j > u_n s\}]] = \frac{F^2_\varepsilon(u_n)}{F_0(u_n)} (\mathbb{E}[||\varepsilon_0||])^2 \times \sum_{\ell < |j| \leq r_n} \mathbb{E} \left[ e^{Y_h} \frac{\mathbb{P}(\varepsilon_0 > u_n s \exp(-Y_0) | Y)}{\mathbb{P}(\varepsilon_0 > u_n)} \frac{\mathbb{P}(\varepsilon_0 > u_n s \exp(-Y_j) | Y)}{\mathbb{P}(\varepsilon_0 > u_n)} \right] = O(F^2_\varepsilon(u_n)) \sum_{\ell < |j| \leq r_n} \mathbb{E} [\exp((Y_h + Y_{j+h}) \{\exp((\alpha + \delta)(Y_0 + Y_j)) \vee 1\}] = O(r_n F_\varepsilon(u_n)) = o(1),
\]

whenever (2.3) holds and \( \mathbb{E}[||\varepsilon_0||] < \infty \).
In summary, the results in Section 2.1 are applicable to the stochastic volatility model.

On the other hand, condition (2.15) does not hold and hence the method of estimating the conditional scaling exponent is not applicable here (note however that the exponent itself is zero).

### 3.2 Markov chains

As in [KSW15], assume that \( \{X_j, j \in \mathbb{Z}\} \) is a function of a stationary Markov chain \( \{Y_j, j \in \mathbb{Z}\} \), defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), with values in a measurable space \((E, \mathcal{E})\). That is, there exists a measurable real valued function \( g : E \to \mathbb{R} \) such that \( X_j = g(Y_j) \). Assume moreover that:

**Assumption 3.1.** (i) The Markov chain \( \{Y_j, j \in \mathbb{Z}\} \) is strictly stationary under \( \mathbb{P} \).

(ii) The sequence \( \{X_j = g(Y_j), j \in \mathbb{Z}\} \) is regularly varying with tail index \( \alpha > 0 \).

(iii) The sequence \( \{X_j = g(Y_j), j \in \mathbb{Z}\} \) satisfies Assumption 1.1.

(iv) There exist a measurable function \( V : E \to [1, \infty) \), \( \gamma \in (0, 1) \), \( x_0 \geq 1 \) and \( b > 0 \) such that for all \( y \in E \),

\[
\mathbb{E}[V(Y_1) \mid Y_0 = y] \leq \gamma V(y) + b .
\]  

(v) There exist an integer \( m \geq 1 \) and for all \( x \geq x_0 \), there exists a probability measure \( \nu \) on \((E, \mathcal{E})\) and \( \epsilon > 0 \) such that, for all \( y \in \{V \leq x\} \) and all measurable sets \( B \in \mathcal{E} \),

\[
\mathbb{P}(Y_m \in B \mid Y_0 = y) \geq \epsilon \nu(B) .
\]

(vi) There exist \( q_0 \in (0, \alpha) \) and a constant \( c > 0 \) such that

\[
|g|^{q_0} \leq cV .
\]

(vii) For every \( s > 0 \),

\[
\limsup_{n \to \infty} \frac{1}{b^{q_0}(u_n) F(u_n)} \mathbb{E}[V(Y_0) \mathbb{1}\{g(Y_0) > u_n s\}] < \infty ,
\]  

where \( b(x) = b_1(x) \).

In [KSW15] we showed that the above assumptions (without (iii) and with \( b(u_n) = u_n \) in (3.2)) imply that \( \{X_j, j \in \mathbb{Z}\} \) is \( \beta \)-mixing with geometric rates and the conditions (2.2), (2.5) and (2.8)-(2.9) are satisfied. Following the calculations in [KSW15] we can argue that (2.8)-(2.9) hold with \( b_h(u_n) = o(u_n) \). Therefore, we conclude the following result.
Corollary 3.2. Assume that Assumption 3.1 holds. Assume moreover that the conditions (2.1), (2.3), (2.6) are satisfied. Then the conclusion of Theorem 2.5 holds. If also (2.10) is satisfied, then the of Corollary 2.7 holds. If moreover $\Psi_h$ is differentiable, then the conclusion of Corollary 2.9 holds.

Example 3.3 (Exponential AR(1)). Consider $X_j = e^{\xi_j}$, $\xi_j = \phi \xi_{j-1} + \epsilon_j$, where $\phi \in (0, 1)$ and $P(e^{\xi_0} > x) = x^{-\alpha} L(x)$. Then the stationary solution has a regularly varying right tail and is tail equivalent to $e^{\xi_0}$, cf. [MR13], [KS15]. If $\alpha > 1$, then $E[X_1 | X_0 = y] = y^\phi E[e^{\xi_0}]$. Hence, the drift function is $V(y) = y^\phi$. Condition (3.2) holds with $q_0 = \phi < \alpha$.

4 Simulations

We simulated from Exponential AR(1) model $X_j = e^{\xi_j}$, $j = 1, \ldots, 500$, where $\xi_j = \phi \xi_{j-1} + \epsilon_j$, and $\epsilon_j$ are i.i.d. with exponential distribution and the parameter $\alpha$. Hence, $\kappa_1 = \phi$, $\kappa_2 = \phi^2$, $\kappa_3 = \phi^3$.

On Figure 1 we plot estimates of the tail index of $X$ using the Hill estimator along with the confidence intervals:

$$\hat{\alpha}_k \pm 1.96 \frac{1}{\sqrt{k}} \hat{\alpha}_k, \quad k = 10, \ldots, 500,$$

where $\hat{\alpha}_k$ is the reciprocal of the Hill estimator based on $k$ order statistics. On the same graph we plot the estimates of the tail index for products, along with the confidence intervals (left panel). On the right panels we display estimates of the scaling exponent $\kappa_1$ along with the confidence interval:

$$\hat{\kappa}_1(k) \pm 1.96 \frac{1}{\sqrt{k}} (1 + \hat{\kappa}_1(k)) \sqrt{E[W_1^{\alpha/(1+\kappa) - 1}]},$$

where $\hat{\kappa}_1(k)$ indicates that the estimator of the scaling exponent is based on $k$ order statistics. The factor $\sqrt{E[W_1^{\alpha/(1+\kappa) - 1}]}$ is computed in two ways. First, note that for our EXPAR(1) we have $W_1 = e^{\xi_0}$. Thus, $W_1$ is exponential with rate $\alpha$. In the first case we plug in known values of $\alpha$ and $\kappa_1 = \phi$ into the expectation and evaluate the multiplicative factor by Monte Carlo simulation. In the second case, we make the factor depending on $k$, plug-in the estimates $\hat{\alpha}_k$ and $\hat{\kappa}_1(k)$ into the expectation and performing Monte Carlo for each $k$. The first set of confidence intervals is marked in blue, while the second one is plotted in red.

Figure 2 displays boxplots for the estimates of the scaling exponent obtained from 1000 Monte Carlo simulations, for selected choices of the number of order statistics.

5 Data Analysis

In this section we apply our theory to the volumes of sales of Microsoft stock prices from January 1, 2010. The data has been detrended by applying simple linear regression. There
Figure 1: Exponential AR(1) model with $\phi = 0.5$ and $\alpha = 2$. Left panel - tail index estimation for the original data and products. Right panel: estimation of $\kappa_1$ along with two types of confidence intervals. The horizontal lines indicate the true values and 1, the latter indicates extremal dependence.

Figure 2: Exponential AR(1) model with $\phi = 0.5$, $\alpha = 4$ and sample size $n = 500$. Estimation of $\kappa_1$ (left panel), $\kappa_2$ (middle panel) and $\kappa_3$ (right panel) for $k = 0.05n$ (left box), $k = 0.1n$ (middle box), $k = 0.2n$ (right box), based on 1000 repetitions.
is some correlation in data and the absolute values of residuals. The estimated tail index for residuals is around 2, while for the products at lag 1, around 1.3-1.4. This indicates extremal independence, since under extremal dependence we would expect the tail of the product to be 1. The estimate of the scaling exponent returns 0.6, with the upper confidence interval clearly separated from 1. The confidence intervals were calculated under the assumption that the underlying process is EXPAR(1), described in Section 4.

6 Proofs

In this section we prove our results. In Section 6.1 we prove general results on weak convergence of tail array sums; see Theorems 6.2 and 6.3. Many details are skipped, since the arguments follow basically the lines of the proofs in [KSW15], appropriately modified to incorporate the CEV assumption. These results will be applied to prove all the results.
Figure 5: Microsoft volumes data: estimation of the tail index (left panel), tail index for products (mid-panel) and the conditional scaling exponent for lag 1 of Section 2.

6.1 Convergence of tail arrays sums

For $0 \leq i_1 \leq i_2 \in \mathbb{N}$, $0 \leq j_1 \leq j_2 \in \mathbb{N}$ such that $i_2 - i_1 = j_2 - j_1$ denote

$$X_{i_1,i_2} = (X_{i_1}, \ldots, X_{i_2})$$

$$X_{i_1,i_2}/b_{j_1,j_2}(u_n) = (X_{i_1}/b_{j_1}(u_n), \ldots, X_{i_2}/b_{j_2}(u_n)).$$

Let $\psi : \mathbb{R}^{h+1} \to \mathbb{R}_+$ be a measurable function such that

$$\lim_{\epsilon \to 0} \lim_{x \to \infty} \sup_{\ell < |j| \leq r_n} \frac{\mathbb{E} \left| \psi \left( \frac{X_0}{u_n} \frac{X_1}{b_{1}(u_n)}, \ldots, \frac{X_h}{b_{h}(u_n)} \right) \right| 1 \{|X_0| \leq \epsilon x\}}{F_0(x)} = 0, \quad (6.1)$$

and either $\psi$ is bounded or there exists $\delta > 0$ such that

$$\sup_{n \geq 1} \frac{\mathbb{E} \left[ \psi^{2+\delta} \left( \frac{X_0}{u_n} \frac{X_1}{b_{1}(u_n)}, \ldots, \frac{X_h}{b_{h}(u_n)} \right) \right]}{F_0(u_n)} < \infty. \quad (6.2)$$

Furthermore, we need a version of the anticlustering condition:

$$\lim_{\ell \to \infty} \lim_{n \to \infty} \frac{1}{F_0(u_n)} \sum_{\ell < |j| \leq r_n} \mathbb{E} \left[ \psi \left( \frac{X_{0,h}}{b_{0,h}(u_n)} \right) \psi \left( \frac{X_{j,j+h}}{b_{j,h}(u_n)} \right) \right] = 0, \quad (6.3)$$

where $r_n$ is the sequence from (A2) of Assumption 2.1.

**Definition 6.1.** By $\mathcal{M}_\psi$ we denote the linear space of bounded functions $\phi : \mathbb{R}^{h+1} \to \mathbb{R}$ such that:
• $|\phi| \leq \text{cst } \psi$ where \text{cst} depends on $\phi$;

• for all $j \geq 0$, the function $x_{0,j+h} \mapsto \phi(x_{j,j+h})$ is almost surely continuous with respect to $\mu_{0,j+h}$.

In this section we are interested in convergence of the tail array sums of the form

$$\tilde{M}_{h,n}(\phi) = \frac{1}{\sqrt{nF_0(u_n)}} \sum_{j=1}^{n-h} \phi \left( \frac{X_j}{u_n}, \frac{X_{j+1}}{b_1(u_n)}, \ldots, \frac{X_{j+h}}{b_h(u_n)} \right).$$

We consider finite dimensional convergence of the process

$$\tilde{M}_{h,n}(\phi) = \sqrt{nF_0(u_n)} \left\{ \tilde{M}_{h,n}(\phi) - \mathbb{E}[\tilde{M}_{h,n}(\phi)] \right\}$$

indexed by the set $\mathcal{M}_\psi$.

**Theorem 6.2.** Let $\{X_j, j \in \mathbb{Z}\}$ be a strictly stationary regularly varying sequence such that Assumption 1.1 holds with extremal independence at all lags and that Assumption 2.1 is satisfied. Let $\psi$ be a measurable function such that (6.1), (6.3) hold and either $\psi$ is bounded or there exists $\delta \in (0,1]$ such that (6.2) and

$$\lim_{n \to \infty} \frac{r_n}{(nF_0(u_n))^{\delta/2}} = 0$$

hold. Then

$$\left\{ \tilde{M}_{h,n}(\phi) = \sqrt{nF_0(u_n)} \left\{ \tilde{M}_{h,n}(\phi) - \mathbb{E}[\tilde{M}_{h,n}(\phi)] \right\}, \phi \in \mathcal{M}_\psi \right\} \overset{d}{\to} \{ \mathcal{M}_h(\phi), \phi \in \mathcal{M}_\psi \},$$

where $\mathcal{M}_h$ is a Gaussian process indexed by $\mathcal{M}_\psi$ with covariance function $(\phi, \varphi) \mapsto \mu_{0,h}(\phi \varphi)$.

In Lemma 6.4 we will justify that under (6.1) the limit

$$\mu_{0,h}(\phi) = \lim_{n \to \infty} \mathbb{E}[\tilde{M}_{h,n}(\phi)]$$

is finite for $\phi \in \mathcal{M}_\psi$. This allows us to consider weak convergence of the process

$$\mathcal{M}_{h,n}(\phi) = \sqrt{nF_0(u_n)} \left\{ \tilde{M}_{h,n}(\phi) - \mu_{0,h}(\phi) \right\}$$

indexed by a subset of $\mathcal{M}_\psi$. Let $\mathcal{G} \subseteq \mathcal{M}_\psi$ be equipped with a semi-metric $\rho_h$. The following result mimics Theorem 2.4 in [KSW15] which in turn is an adaptation of [vdVW96, Theorem 2.11.1]. Hence, it is stated without a proof.

**Theorem 6.3.** Let $\{X_j, j \in \mathbb{Z}\}$ be a strictly stationary regularly varying sequence such that Assumption 1.1 holds with extremal independence at all lags. Suppose that assumptions of Theorem 6.2 are satisfied. If moreover

(i) \( \mathcal{G} \) is pointwise separable;
(ii) the envelope function \( \Phi_{\mathcal{G}} = \sup_{\phi \in \mathcal{G}} |\phi| \) is in \( \mathcal{M}_\psi \);
(iii) \( \mathcal{G} \) is a VC subgraph class or a finite union of such classes;
(iv) \((\mathcal{G}, \rho_h)\) is totally bounded;
(v) for every sequence \( \{\delta_n\} \) which decreases to zero,

\[
\limsup_{n \to \infty} \sup_{\phi, \psi \in \mathcal{G}, \rho_h(\phi, \psi) \leq \delta_n} \frac{\mathbb{E} \left[ \phi \left( \frac{X_{0,h}}{b_0(h,u_n)} \right) - \psi \left( \frac{X_{0,h}}{b_0(h,u_n)} \right) \right]^2}{F_0(u_n)} = 0 ,
\]

then

\[
\widehat{M}_{h,n}(\phi) = \sqrt{n F_0(u_n)} \left\{ \widehat{M}_{h,n}(\phi) - \mathbb{E}[\widehat{M}_{h,n}(\phi)] \right\} \Rightarrow M_h(\phi)
\]
in \( \ell^\infty(\mathcal{G}) \). If moreover

\[
\limsup_{n \to \infty} \sup_{\phi \in \mathcal{G}} \sqrt{n F_0(u_n)} |\mathbb{E}[\widehat{M}_{h,n}(\phi)] - \mu_{0,h}(\phi)| = 0 ,
\]

then

\[
\mathbb{M}_{h,n}(\phi) = \sqrt{n F_0(u_n)} \left\{ \mathbb{M}_{h,n}(\phi) - \mu_{0,h}(\phi) \right\} \Rightarrow \mathbb{M}_h(\phi)
\]
in \( \ell^\infty(\mathcal{G}) \).

The proof of Theorem 6.2 will be prefaced by several lemmas. For brevity, write

\[
V_{j,n}(\phi) = \phi \left( \frac{X_j}{u_n}, \frac{X_{j+1}}{b_1(u_n)}, \ldots, \frac{X_{j+h}}{b_h(u_n)} \right) , \quad S_n(\phi) = \sum_{j=1}^{r_n} V_{j,n}(\phi).
\]

Lemma 6.4. Let Assumption 1.1 hold with extremal independence at all lags. Let \( \psi \) be a measurable function such that (6.1) holds and either \( \psi \) is bounded or (6.2) holds. Then, for all \( \phi \in \mathcal{M}_\psi \) we have \( \mu_{0,h}(\phi^2) < \infty \) and

\[
\lim_{n \to \infty} \frac{\mathbb{E}[V_{0,n}^2(\phi)]}{\mathbb{P}(\{|X_0| > u_n\})} = \mu_{0,h}(\phi^2) = \int_0^\infty \mathbb{E}[\phi^2(sW_0, s^{\kappa_1}W_1, \ldots, s^{\kappa_h}W_h)]s^{\alpha-1}ds .
\]

Moreover, for all \( j \neq 0 \) and \( \phi, \varphi \in \mathcal{M}_\psi \),

\[
\lim_{n \to \infty} \frac{1}{F_0(u_n)} \mathbb{E}[V_{0,n}(\phi)V_{j,n}(\varphi)] = 0 .
\]
Proof. The proof of the first part is similar to [KS15, Proposition 2].

Assume that \( \psi \) is bounded. For \( \epsilon > 0 \), we write

\[
\frac{1}{F_0(u_n)} \mathbb{E}[|V_{0,n}(\phi)|^2] = \frac{1}{F_0(u_n)} \mathbb{E}[|V_{0,n}(\phi)| \mathbb{I}\{|X_0| > \epsilon u_n\}] + \frac{1}{F_0(u_n)} \mathbb{E}[|V_{0,n}(\phi)| \mathbb{I}\{|X_0| \leq \epsilon u_n\}].
\]

By vague convergence and boundedness of \( \phi \) the first expression on the right hand side converges to \( \mu_{0,h}(\phi^2 \mathbb{I}\{|y_0| > \epsilon\}) < \infty \). Application of (6.1) implies that \( \mu_{0,h}(\phi^2) \) is finite.

If \( \psi \) is unbounded, then then for all \( A > 0 \), applying Markov and Hölder inequalities, we obtain

\[
\lim_{A \to \infty} \limsup_{n \to \infty} \frac{\mathbb{E}[V_{0,n}(\phi)V_{j,n}(\varphi)]}{F_0(u_n)} = \lim_{A \to \infty} \text{cst} \cdot A^{-2/\delta} \sup_{n \geq 1} \frac{\mathbb{E}[|V_{0,n}(\phi)|^{2+\delta}]}{F_0(u_n)} = 0.
\]

Thus, we can split \( V_{0,n}(\phi) \) as \( V_{0,n}(\phi) \mathbb{I}\{|V_{0,n}(\phi) \leq A\} + V_{0,n}(\phi) \mathbb{I}\{|V_{0,n}(\phi) > A\} \) and apply the truncation argument.

As for the second part, thanks to the truncation argument, we can consider bounded functions \( \phi, \varphi \in \mathcal{M}_\psi \). We have

\[
\lim_{n \to \infty} \frac{\mathbb{E}[V_{0,n}(\phi)V_{j,n}(\varphi)]}{F_0(u_n)} = \lim_{n \to \infty} \frac{\mathbb{E}[V_{0,n}(\phi)V_{j,n}(\varphi) \mathbb{I}\{|X_0| \wedge \vert X_j \vert > \epsilon u_n\}]}{F_0(u_n)} + \lim_{n \to \infty} \frac{\mathbb{E}[V_{0,n}(\phi)V_{j,n}(\varphi) \mathbb{I}\{|X_0| \wedge \vert X_j \vert \leq \epsilon u_n\}]}{F_0(u_n)}. \tag{6.8}
\]

The term in (6.8) vanishes, for each \( \epsilon > 0 \), by boundedness of \( \varphi, \phi \) and extremal independence. The term in (6.9) is bounded by

\[
\lim_{n \to \infty} \frac{1}{F_0(u_n)} \{\|\varphi\|_{\infty}\mathbb{E}[V_{0,n}(\phi) \mathbb{I}\{|X_0| \leq \epsilon u_n\}] + \|\phi\|_{\infty}\mathbb{E}[V_{0,n}(\varphi) \mathbb{I}\{|X_0| \leq \epsilon u_n\}]\}
\]

and hence vanishes as \( \epsilon \to 0 \) by (6.1).

\[\square\]

Lemma 6.5. Let Assumption 1.1 hold with extremal independence at all lags and Assumption 2.1 hold. Let \( \psi \) be a measurable function such that (6.1), (6.3) hold and either \( \psi \) is bounded or (6.2) holds. Then, for all \( \phi, \varphi \in \mathcal{M}_\psi \),

\[
\lim_{n \to \infty} \frac{\mathbb{E}[S_n(\phi)S_n(\varphi)]}{r_n F_0(u_n)} = \mu_{0,h}(\phi \varphi),
\]

and

\[
\lim_{n \to \infty} \frac{\text{cov}(S_n(\phi), S_n(\varphi))}{r_n \mathbb{P}(|X_0| > u_n)} = \mu_{0,h}(\phi \varphi).
\]

Proof. By stationarity we can write, for \( \ell \geq 1 \),

\[
\frac{\mathbb{E}[S_n(\phi)S_n(\varphi)]}{r_n F_0(u_n)} = \sum_{j=-\ell}^{\ell} \left( 1 - \frac{|j|}{r_n} \right) \frac{\mathbb{E}[V_{0,n}(\phi)V_{j,n}(\varphi)]}{F_0(u_n)} + O \left( \sum_{j=\ell+1}^{r_n} \frac{\mathbb{E}[V_{0,n}(\phi)V_{j,n}(\varphi)]}{F_0(u_n)} \right).
\]
Since $V_{0,n}(\phi) V_{0,n}(\varphi) = V_{0,n}(\phi \varphi)$, Lemma 6.4 shows that the term on the right hand side converges to $\mu_{0,h}(\phi \varphi)$. The second term vanishes by assumption (6.3), upon letting $n \to \infty$ and then $\ell \to \infty$.

Finally, by Assumption 2.1,

$$
\frac{1}{F_0(u_n)} \sum_{j=1}^{r_n} \mathbb{E}[V_{0,n}(\phi)] \mathbb{E}[V_{j,n}(\varphi)] = \left( \frac{\mathbb{E}[V_{0,n}(\phi)]}{F_0(u_n)} \right)^2 r_n F_0(u_n) \to 0
$$

and hence the result for the covariances follows.

The next result can be proven along the same lines as of [KSW15, Lemmas 3.6-3.7]. In the case of unbounded functions, we need additionally (6.4).

**Lemma 6.6.** Let Assumption 1.1 hold with extremal independence at all lags and Assumption 2.1 hold. Let $\psi$ be a measurable function such that (6.1), (6.3) hold and either $\psi$ is bounded or there exists $\delta \in (0,1]$ such that (6.2) and (6.4) hold. Then

$$
\lim_{n \to \infty} \frac{\mathbb{E} \left[ S_n^2(\phi) \mathbb{1} \left\{ |S_n(\phi)| > \delta \sqrt{n F_0(u_n)} \right\} \right]}{r_n F_0(u_n)} = 0. \quad (6.10)
$$

**Proof of Theorem 6.2.** Define $m_n = \lceil n/r_n \rceil$ and let $\{X_{n,i}^+, 1 \leq i \leq m_n, n \geq 1\}$ be a triangular array of random variables such that the blocks $\{X_{(i-1)r_n+1}^+, \ldots, X_{ir_n}^+\}$ are independent and each have the same distribution as the original stationary blocks, i.e. the same distribution as $(X_1, \ldots, X_n)$ by stationarity of the original sequence. For $i = 1, \ldots, m_n$, define

$$
V_{j,n}^+(\phi) = \phi \left( \frac{X_{j,n}^+}{u_n}, \frac{X_{j+1,n}^+}{b_1(u_n)}, \ldots, \frac{X_{j+h,n}^+}{b_h(u_n)} \right), \quad S_{n,i}^+(\phi) = \sum_{j=(i-1)r_n+1}^{ir_n} V_{j,n}^+(\phi),
$$

Arguing as in the proof of [DR10, Theorem 2.8] or [KSW15], the $\beta$-mixing property and the rate condition (2.2) implies that it suffices to prove weak convergence of the process

$$
\tilde{M}_{h,n}^+(\phi) = \left\{ n F_0(u_n) \right\}^{-1/2} \sum_{i=1}^{m_n} \left\{ S_{n,i}^+(\phi) - \mathbb{E}[S_{n,i}^+(\phi)] \right\},
$$

along with the appropriate bias condition. In the first step we show existence of the limiting covariance. For $\phi, \varphi$ we have

$$
\frac{1}{F_0(u_n)} \sum_{j=1}^{r_n} \mathbb{E}[V_{0,n}(\phi)] \mathbb{E}[V_{j,n}(\varphi)] = \left( \frac{\mathbb{E}[V_{0,n}(\phi)]}{F_0(u_n)} \right) \left( \frac{\mathbb{E}[V_{0,n}(\varphi)]}{F_0(u_n)} \right) r_n F_0(u_n) \to 0,
$$

by (6.7) and (2.3). Application of Lemma 6.5 yields the limiting covariance:

$$
\lim_{n \to \infty} \text{cov} \left( \tilde{M}_{h,n}^+(\phi), \tilde{M}_{h,n}^+(\varphi) \right) = \mu_{0,h}(\phi \varphi).
$$

Lemma 6.6 finishes the proof. \hfill \Box
6.2 Proof of Theorem 2.5

We only consider the case of extremal independence, since the extremally dependent case can be concluded directly from [KSW15]. Let $s_0 \in (0, 1)$. We apply the results of Section 6.1 to the function $\psi(x_0, \ldots, x_h) = \mathbb{1}\{|x_0| > s, x_h \leq y\}$ and the class $\mathcal{G}_0 = \{\phi_{s,y} : (x_0, \ldots, x_h) \rightarrow \mathbb{1}\{|x_0| > s, x_h \leq y\}, s \geq s_0, y \in \mathbb{R}\}$. Then
\[ \tilde{I}_{h,n}(s,y) = \tilde{M}_{h,n}(\phi_{s,y}) , \quad \bar{I}_{h,n}(s,y) = \bar{M}_{h,n}(\phi_{s,y}) . \tag{6.11} \]

We need to check assumptions (6.1)-(6.3):

- Condition (6.1) trivially holds since its right hand side vanishes whenever $\epsilon \in (0, s_0)$;
- Condition (6.3) for the function $\psi$ is implied by the anticlustering condition (2.5) of Assumption 2.2;

Since Assumptions 1.1 and 2.1 are already assumed in Theorem 2.5, by Theorem 6.2, the finite dimensional distributions of
\[ \sqrt{n \mathbb{F}_0(u_n)}\{\tilde{I}_{h,n}(s,y) - \mathbb{E}[\tilde{I}_{h,n}(s,y)]\} \]
converge to those of $\mathbb{I}_h$.

To prove tightness, we apply Theorem 6.3. Condition (6.6) reduces to (2.6). It remains to verify (i)-(v). Define the semi-metric $\rho_h$ on $(0, \infty) \times \mathbb{R}$ by
\[ \rho_h((s,y),(t,z)) = |s - t| + |\Psi_h(y) - \Psi_h(z)| . \tag{6.12} \]
This also defines the semi-metric on $\mathcal{G}_0$. The class is clearly pointwise separable and the envelope function is $\mathbb{1}\{(s_0, \infty) \times \mathbb{R}^h\} \in \mathcal{M}_\psi$. Also, the class of indicators of the sets $(s, \infty) \times (-\infty, y]$ is a VC class of index 2. The proof of (iv)-(v) follows the same lines as that of [KSW15, Theorem 2.11]. It remains to identify the limiting Gaussian process. For $0 < s < t$ and $y, z \in \mathbb{R}$, we have
\[ \text{cov}(\mathbb{M}(\phi_{s,y}), \mathbb{M}(\phi_{t,z})) = \mu_{0,h}(\phi_{s,y} \phi_{t,z}) = t^{-\alpha} \Psi_h(t^{-\kappa_h}(y \wedge z)) . \]
This finishes the proof.

6.3 Central limit theorem for the conditional mean

We state another corollary to Theorem 6.3. For $s > 0$, set $\psi_s(x_0, \ldots, x_h) = |x_h| \mathbb{1}\{|x_0| > s\}$ and define the class
\[ \mathcal{G}_1 = \{\psi_s, s \geq s_0\} . \]

Then,
\[ \mathbb{M}_{h,n}(\psi_s) = \sqrt{n \mathbb{F}_0(u_n)} \left\{ \frac{1}{n \mathbb{F}_0(u_n)} \sum_{j=1}^{n-h} \frac{|X_{j+h}|}{\bar{b}_h(u_n)} \mathbb{1}\{|X_j| > u_n s\} - \mu_{0,h}(\psi_s) \right\} . \tag{6.13} \]
By the homogeneity property (1.5) and the assumption \( \int_{-\infty}^{\infty} |y| \Psi_h(dy) = 1 \), we have
\[
\mu_{0,h}(\psi_s) = s^{\kappa_h - \alpha}.
\] (6.14)

**Corollary 6.7.** Let \( \{X_j, j \in \mathbb{Z}\} \) be a strictly stationary regularly varying sequence such that Assumption 1.1 holds with extremal independence at all lags and that Assumption 2.1 is satisfied. Assume that Assumptions 2.2, 2.3 and 2.6 and (6.4) hold. Then
\[
M_{h,n} \xrightarrow{w} M_h, \text{ in } \ell^\infty(G_0 \cup G_1).
\]

**Proof.** Let \( s_0 \in (0,1) \). We apply the results of Section 6.1. We need to check the assumptions (6.1)-(6.3):

- Condition (6.1) holds trivially for \( \psi(x_0, \ldots, x_h) = |x_h| 1\{|x_0| > s_0\} \).
- (2.8) of Assumption 2.6 implies (6.2).
- (2.9) of Assumption 2.6 implies (6.3).

For tightness, we apply Theorem 6.3. Condition (6.6) reduces to (2.10). It remains to verify (i)-(v). The class \( G_1 \) is separable and linearly ordered, hence VC-subgraph class. The envelope function \( |x_h| 1\{|x_0| > s_0\} \) belongs to \( G_1 \).

Define the semi-metric \( \rho_h \) on \( G_1 \) by
\[
\rho_h(\psi_s, \psi_t) = E[Y_0^2 W_h^2 1\{Y_0 \in (s,t]\}].
\] (6.15)

Since Assumption 2.6 implies \( \alpha > 2 \) we have for \( s < t \),
\[
\rho_h(\psi_s, \psi_t) = \frac{\alpha}{\alpha - 2} E[W_h^2(s^{-\alpha+2} - t^{-\alpha+2})] \leq \text{cst}(t - s).
\]

Hence, \( (G_1, \rho_h) \) is totally bounded. Moreover, by regular variation and the uniform convergence theorem, the convergence
\[
\lim_{n \to \infty} \frac{1}{b_h^2(u_n)F_0(u_n)} E[X_h^2 1\{X_h > u_n s\} - 1\{X_h > u_n t\}] = \frac{\alpha}{\alpha - 2} E[W_h^2(s^{-\alpha+2} - t^{-\alpha+2})]
\]

is uniform on compact sets of \((0, \infty)\). Hence, (v) holds. The joint convergence holds by applying Theorem 2.5 and considering the class \( G = G_0 \cup G_1 \).

### 6.4 Proof of Corollaries 2.7 and 2.9

**Proof of Corollary 2.7.** Set \( T_n(s) = k^{-1} \sum_{j=1}^{n} 1\{|X_j| > u_n s\} \). By Corollary 6.7 and Vervaat’s lemma [Ver72], we obtain the joint convergence
\[
\left\{ M_{h,n}(\psi_s), 1_{h,n}(s,y), \sqrt{k} (T_n(s) - s^{-\alpha}), \sqrt{k} (u_n^{-1} |X_{(n,k)}| - 1) \right\} \xrightarrow{w} \left\{ M_h(\psi_s), M_h(\phi_{s,y}), M_h(\phi_{s,\infty}), \alpha^{-1} M_h(\phi_{1,\infty}) \right\}.
\] (6.16)
Write $\xi_n = u_n^{-1}|X|_{(n,k)}$ and note that the above weak convergence implies that $\xi_n \to 1$ in probability. Note that by (6.14), we have
$$\mu_{0,h}(\psi_{\xi_n}) = \xi_n^{\kappa_h - \alpha}.$$  
In view of this identity and (6.13), we have
$$\sqrt{k} \left( \frac{\hat{b}_{h,n}}{b_h(u_n)} - 1 \right) = \mathcal{M}_{h,n}(\psi_{\xi_n}) + \sqrt{k} \left\{ \xi_n^{\kappa_h - \alpha} - 1 \right\}. \quad (6.17)$$  
Thus,
$$\sqrt{k} \left( \frac{\hat{b}_{h,n}}{b_h(u_n)} - 1 \right) \overset{d}{\to} \mathcal{M}_h(\psi_1) + \alpha^{-1}(\kappa_h - \alpha)\mathcal{M}_h(\phi_{1,\infty}).$$  
The process $\tilde{W}$ defined by $\tilde{W}(u) = \mathcal{M}_h(\phi_{1,\infty}(u))$ is a standard Brownian motion and $\mathbb{B}(u) = \tilde{W}(u) - u\mathbb{W}(1)$ is a standard Brownian bridge. Therefore,
$$\mathcal{M}_h(\psi_1) = \int_0^1 |\Psi_1^{-1}(u)|\tilde{W}(du).$$  
By assumption, we have
$$\int_0^1 |\Psi_1^{-1}(u)|\,du = \int_{-\infty}^\infty |y|\Psi_h(dy) = 1,$$  
thus $\mathcal{M}_h(\psi_1) - \mathcal{M}_h(\phi_{1,\infty}) = \int_0^1 |\Psi_1^{-1}(u)|\mathbb{B}(du).$  

**Proof of Corollary 2.9.** Writing $\theta_n = b_h^{-1}(u_n)b_{h,n}$, we have
$$\sqrt{k} \left\{ \hat{\psi}_{h,n}(y) - \psi_h(y) \right\} = \sqrt{k} \left\{ \tilde{I}_{h,n} \left( \frac{X_{(n,n-k)}}{u_n}, \frac{\hat{b}_{h,n}}{b_h(u_n)}y \right) - \psi_h(y) \right\}$$  
$$= \mathbb{I}_{h,n}(\xi_n, \theta_n y) + \sqrt{k} \left( \xi_n^{-\alpha}\psi_h(\xi_n^{-\kappa_h}\theta_n y) - \psi_h(y) \right). \quad (6.18)$$  
By Corollary 2.7, $\xi_n \overset{p}{\to} 1$ and $\theta_n \overset{p}{\to} 1$. Hence, the first term in (6.18) converges to $\mathcal{M}_h(\phi_1,y)$. The delta method implies that the limiting behaviour of the second term in (6.18) is the same as that of
$$\sqrt{k} \psi_h(y) \left\{ \xi_n^{-\alpha} - 1 \right\} + \xi_n^{-\alpha}\sqrt{k} y \psi'_h(y) \left\{ \xi_n^{-\kappa_h} - 1 + \xi_n^{-\kappa_h}(\theta_n - 1) \right\},$$  
which by Corollary 2.7 converges weakly to
$$-\psi_h(y)\mathcal{M}_h(\phi_{1,\infty}) + y\psi'_h(y)\left\{ \mathcal{M}_h(\psi_1) - \mathcal{M}_h(\phi_{1,\infty}) \right\}.$$  
Moreover, with $\tilde{W}$ and $\mathbb{B}$ as above, we have
$$\left\{ \mathcal{M}_h(\phi_{1,y}) - \psi_h(y)\mathcal{M}_h(\phi_{1,\infty}) : y \in \mathbb{R} \right\} = \mathbb{B} \circ \psi.$$  
This concludes the proof.  

\[\square\]
6.5 Proof of Theorem 2.13

Set \( \beta_h = \alpha / (1 + \kappa_h) \) and \( \gamma_h = \beta_h^{-1} \). At the first step we justify functional convergence of the tail empirical process based on products:

\[
\sqrt{n} F_0(u_n) \left\{ \frac{1}{n F_0(u_n)} \sum_{j=1}^{n-h} 1 \{ |X_j X_{j+h}| > \|W_h\| b_h(u_n) s \} - s^{-\beta_h} \right\}.
\]

(6.19)

Define \( \psi(x_0, \ldots, x_h) = 1\{|x_0| > s_0\} + 1\{|\|W_n\| b_h x_0 x_h| > s_0\} \) and the class \( \mathcal{G} = \mathcal{G}_0' \cup \mathcal{G}_1' \) with

\[
\mathcal{G}_0' = \{ I_s : (x_0, \ldots, x_h) \rightarrow 1\{|x_0| > s\}, s \geq s_0 \},
\]

\[
\mathcal{G}_1' = \{ \varphi_s : (x_0, \ldots, x_h) \rightarrow 1\{|x_0 x_h| > s \|W_n\| b_h\}, s \geq s_0 \}.
\]

With this notation the process defined in (6.19) can be written as \( M_{h,n}(\varphi_s) \). Similarly, the tail empirical process of \( X_j \)'s can be written as \( M_{h,n}(I_s) \):

\[
M_{h,n}(I_s) = \sqrt{n} F_0(u_n) \left\{ \frac{1}{n F_0(u_n)} \sum_{j=1}^{n} 1 \{ |X_j| > u_n s \} - s^{-\alpha} \right\}.
\]

We note that \( \mathcal{G}_0' \subseteq \mathcal{G}_0 \), where \( \mathcal{G}_0 \) was defined in the proof of Theorem 2.5 and \( I_s = \phi_{s,\infty} \). Assumptions (2.15) and (2.18) imply (6.1) and (6.3) for the class \( \mathcal{G}_1' \). The bias condition (6.6) is implied by (2.21). Therefore, by Theorem 6.2, the finite dimensional distributions of \( M_{h,n} \) converge to those of \( M_h \) of \( \mathcal{G}_1' \). Moreover, the envelope function of \( \mathcal{G} \) is \( \psi \). Furthermore,

- The class \( \mathcal{G} \) is the union of two linearly ordered classes, hence is a VC subgraph class.
- We consider the metric \( \rho_h \) on \( \mathcal{G} \) induced by the covariance, that is for \( \phi, \varphi \in \mathcal{G} \),

\[
\rho_h(\phi, \varphi) = \sqrt{\text{var}(M_h(\phi) - M_h(\varphi))}.
\]

The metric \( \rho_h \) restricted to \( \mathcal{G}_0' \) and \( \mathcal{G}_1' \) respectively becomes

\[
\rho_h(I_s, I_t) = \sqrt{|s^{-\alpha} - t^{-\alpha}|}, \quad \rho_h(\varphi_s, \varphi_t) = \sqrt{|s^{-\beta_h} - t^{-\beta_h}|}.
\]

Thus it easily seen that both \( \mathcal{G}_0' \) and \( \mathcal{G}_1' \) are totally bounded for the metric \( \rho_h \). This means that Condition (iv) holds.

- By regular variation, the convergence

\[
\lim_{n \to \infty} \frac{1}{F_0(u_n)} \mathbb{E} [1\{|X_0 X_h| \in u_n b_h(u_n)(s,t)\}] = \rho_h^2(\varphi_s, \varphi_t)
\]

is uniform with respect to \( s, t \in [s_0, \infty) \) (see [BGT89, Theorem 1.5.2]). Therefore, (v) holds on \( \mathcal{G}_1' \).
Thus we have proved that
\[ \mathbb{M}_{h,n} \Rightarrow \mathbb{M}_h, \quad \text{in } \ell^\infty(\mathcal{G}). \]
Recall that \( \xi_n = |X|/(n:n-k)/u_n. \) Set \( V_j = |X_j X_{j+h}| \) and \( v_n = u_n b(u_n) \| W_h \| \beta_h. \) Define
\[ \zeta_n = V_{(n:n-k)}/v_n. \]
As a consequence of the functional convergence and Vervaat’s Lemma, we obtain
\[ \left( \mathbb{M}_{h,n}, \sqrt{k}(\xi_n - 1), \sqrt{k}(\zeta_n - 1) \right) \Rightarrow \left( \mathbb{M}_h, \gamma \mathbb{M}_h(\mathcal{I}_1), \gamma_h \mathbb{M}_h(\varphi_1) \right); \quad (6.20) \]
By elementary algebra and Fubini’s theorem, we have
\[ \sqrt{k}(\hat{\gamma} - \gamma) = \int_1^\infty \mathbb{M}_{h,n}(\mathcal{I}_s) \frac{ds}{s} + \int_1^1 \mathbb{M}_{h,n}(\mathcal{I}_s) \frac{ds}{s} + \gamma \sqrt{k}(\xi_n^{-1/\gamma} - 1), \]
\[ \sqrt{k}(\hat{\gamma}_h - \gamma_h) = \int_1^\infty \mathbb{M}_{h,n}(\varphi_s) \frac{ds}{s} + \int_1^1 \mathbb{M}_{h,n}(\varphi_s) \frac{ds}{s} + \gamma_h \sqrt{k}(\zeta_n^{-1/\gamma_h} - 1). \]
The middle term of both expansions is \( o_P(1). \) The last terms are dealt with by Slutsky’s theorem and (6.20):
\[ \left( \sqrt{k}(\xi_n^{-1/\gamma} - 1), \sqrt{k}(\zeta_n^{-1/\gamma_h} - 1) \right) \Rightarrow \left( -\mathbb{M}_h(\mathcal{I}_1), -\mathbb{M}_h(\varphi_1) \right). \quad (6.21) \]
Furthermore, jointly with the previous convergences,
\[ \left( \int_1^\infty \mathbb{M}_{h,n}(\mathcal{I}_s) \frac{ds}{s}, \int_1^\infty \mathbb{M}_{h,n}(\varphi_s) \frac{ds}{s} \right) \Rightarrow \left( \int_1^\infty \mathbb{M}_h(\mathcal{I}_s) \frac{ds}{s}, \int_1^\infty \mathbb{M}_h(\varphi_s) \frac{ds}{s} \right). \quad (6.22) \]
Indeed, for a fixed \( A > 1, \) we can write
\[ \int_1^\infty \mathbb{M}_{h,n}(\mathcal{I}_s) \frac{ds}{s} = \int_1^A \mathbb{M}_{h,n}(\mathcal{I}_s) \frac{ds}{s} + \int_A^\infty \mathbb{M}_{h,n}(\mathcal{I}_s) \frac{ds}{s}, \]
\[ \int_1^\infty \mathbb{M}_{h,n}(\varphi_s) \frac{ds}{s} = \int_1^A \mathbb{M}_{h,n}(\varphi_s) \frac{ds}{s} + \int_A^\infty \mathbb{M}_{h,n}(\varphi_s) \frac{ds}{s}. \]
By Theorem 6.3, we have
\[ \left( \int_1^A \mathbb{M}_{h,n}(\mathcal{I}_s) \frac{ds}{s}, \int_1^A \mathbb{M}_{h,n}(\varphi_s) \frac{ds}{s} \right) \Rightarrow \left( \int_1^A \mathbb{M}_h(\mathcal{I}_s) \frac{ds}{s}, \int_1^A \mathbb{M}_h(\varphi_s) \frac{ds}{s} \right). \]
Moreover, as \( A \to \infty, \)
\[ \left( \int_1^A \mathbb{M}_h(\mathcal{I}_s) \frac{ds}{s}, \int_1^A \mathbb{M}_h(\varphi_s) \frac{ds}{s} \right) \Rightarrow \left( \int_1^\infty \mathbb{M}_h(\mathcal{I}_s) \frac{ds}{s}, \int_1^\infty \mathbb{M}_h(\varphi_s) \frac{ds}{s} \right). \]
Therefore, we must prove that for all \( \epsilon > 0 \),

\[
\lim_{A \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \left| \int_A^\infty M_{h,n}(I_s) \frac{ds}{s} \right| > \epsilon \right) = 0 ,
\]

\[
\lim_{A \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \left| \int_A^\infty M_{h,n}(\varphi_s) \frac{ds}{s} \right| > \epsilon \right) = 0 .
\]

The proof of these bounds rely on the \( \beta \) mixing property and Conditions (2.19) and (2.20). We will only prove the first one, the second being exactly similar. First note that

\[
\int_\infty^\infty T_n(I_s) \frac{ds}{s} = \frac{1}{k} \sum_{j=1}^n \log_+ (|X_j|/Au_n)
\]

In view of Assumption 2.1, it suffices to consider independent blocks and thus to compute the variance of one block, that is we must prove that

\[
\lim_{A \to \infty} \limsup_{n \to \infty} m_n \text{var} \left( k^{-1/2} \sum_{i=1}^{r_n} \log_+ (|X_j|/Au_n) \right) = 0
\]

Let the variance term in the previous display be denoted by \( \mathcal{V}_n(A) \). Then,

\[
\mathcal{V}_n(A) \leq \frac{\mathbb{E}[\log_+^2 (|X_0|/u_n)]}{F_0(u_n)} + 2 \sum_{j=m+1}^{r_n} \frac{\mathbb{E}[\log_+ (|X_0|/u_n) \log_+ (|X_j|/u_n)]}{F_0(u_n)}
\]

By regular variation, extremal independence and condition (2.19), for every \( \epsilon > 0 \), we can choose an integer \( m \) such that

\[
\lim_{n \to \infty} \sum_{j=m+1}^{r_n} \mathbb{E}[\log_+ (|X_0|/u_n) \log_+ (|X_j|/u_n)] \leq A^{-a} \int_1^\infty \log^2(y) \alpha y^{-\alpha-1} dy + \epsilon .
\]

Thus \( \lim_{A \to \infty} \limsup_{n \to \infty} \mathcal{V}_n(A) \leq \epsilon \). Since \( \epsilon \) is arbitrary, this proves the requested bound. Combining (6.21) and (6.22), we have proved that

\[
\sqrt{k} (\hat{\gamma} - \gamma, \hat{\gamma}_h - \gamma_h) \xrightarrow{d} \left( \int_1^\infty M_h(I_s) \frac{ds}{s} - M_h(I_1), \int_1^\infty M_h(\psi_s) \frac{ds}{s} - M_h(\psi_1) \right) .
\]

(6.23)

Define the Gaussian process \( G \) on \((0, \infty) \times \mathbb{R} \) by \( G(s, y) = M_h(\phi_s, y) \). Then,

\[
\int_1^\infty M_h(I_s) \frac{ds}{s} - M_h(I_1) = \int_0^\infty \int_{-\infty}^\infty \{ \log_+ (u) - \gamma \mathbb{1}\{u > 1\} \} G(dudv) ,
\]

\[
\int_1^\infty M_h(\varphi_s) \frac{ds}{s} - G(\varphi_1) = \int_0^\infty \int_{-\infty}^\infty \{ \log_+ (u|v|) - \gamma_h \mathbb{1}\{|u|v| > 1\} \} G(dudv) .
\]
We note that
\[
\sqrt{h}(\hat{k}_h - \kappa) = \sqrt{h} \{\hat{\gamma}/\gamma - \gamma_h/\gamma\} = \sqrt{h} \frac{1}{\gamma} \{\hat{\gamma} - \gamma_h\} - \sqrt{h} \frac{\gamma_h}{\gamma} \{\hat{\gamma}^{-1} - \gamma^{-1}\}.
\]

Applying the joint convergence (6.23), Slutsky lemma and the delta method, we obtain,
\[
\sqrt{h}(\hat{k}_h - \kappa) \xrightarrow{d} (1 + \kappa) \int_0^\infty \int_{-\infty}^\infty (\beta_h \log_+(u|v|) - \mathbb{1}\{u|v| > 1\} - \alpha \log_+(u) + \mathbb{1}\{u > 1\}) \mathbb{G}(du, dv).
\]

There remains to compute the variance \(\sigma^2\) of the limiting distribution which is Gaussian with zero mean. Setting
\[
h(u, v) = \beta_h \log_+(u|v|) - \mathbb{1}\{u|v| > 1\} - \alpha \log_+(u) + \mathbb{1}\{u > 1\},
\]
and combining with (1.6) we obtain
\[
\sigma^2 = (1 + \kappa)^2 \mathbb{E} \left[ \int_0^\infty h^2(s, s^{\kappa_h}|W_h|)s^{-\alpha-1}ds \right].
\]

Since \(h^2(s, s^{\kappa_h}|W_h|) = 0\) if \(s < 1\) and \(s^{1+\kappa_h}|W_h| < 1\), we have
\[
\sigma^2 = (1 + \kappa)^2 \mathbb{E} \left[ \int_{1\wedge W_h^{-1/(1+\kappa_h)}}^\infty h^2(s, s^{\kappa_h}|W_h|)s^{-\alpha-1}ds \right].
\]

Setting \(A = |W_h|^\beta_h\) and substituting \(t = s^{-\alpha}\) we obtain
\[
\int_{1\wedge W_h^{-1/(1+\kappa_h)}}^\infty h^2(s, s^{\kappa_h}|W_h|)s^{-\alpha-1}ds = \int_0^{1\wedge A} h^2(t^{-1/\alpha}, t^{-\kappa_h/\alpha}|W_h|)dt.
\]

We also note that
\[
h(t^{-1/\alpha}, t^{-\kappa_h/\alpha}|W_h|) = \{\log_+(t^{-1}A) - \mathbb{1}\{A > t\}\} - \{\log_+(t^{-1}) - \mathbb{1}\{t < 1\}\}.
\]

If \(A < 1\), then we evaluate
\[
\int_0^1 h^2(t^{-1/\alpha}, t^{-\kappa_h/\alpha}|W_h|)dt = A \log^2(A) + \int_A^1 \{\log(t) + 1\}^2 dt.
\]

Since the primitive of \(\{\log(t) + 1\}^2\) is \(t \log^2(t) + t\), the right hand side becomes \(1 - A\). If \(A > 1\), then
\[
\int_0^1 h^2(t^{-1/\alpha}, t^{-\kappa_h/\alpha}|W_h|)dt = \log^2(A) + \int_1^A \{\log(A) - \log(t) - 1\}^2 dt = A - 1.
\]

Summarizing,
\[
\sigma^2 = (1 + \kappa)^2 \mathbb{E}[|1 - |W_h|^\beta_h|].
\]
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