Set-valued uncertain process: definition and some properties

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Accepted: 7 January 2023 / Published online: 23 January 2023
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Abstract
Some phenomena develop over time, while they are uncertain sets at each moment. From an uncertain set, we mean an unsharp concept that is not clearly defined, even for an expert. The potential values of parameters describing this unsharp concept guide one to explain it quantitatively. For instance, in “recovering” from some disease, different levels of health might be defined, meaning that at each specific time, being healthy is measured by belonging some parameter values to a set with a specific belief degree. Clearly speaking, the set defining “recovery” at the beginning stage of a disease would be entirely or partially non-identical with it at other stages. These sets might be extracted using imprecisely observed data, or an expert opinion defines them by some belief degrees. An essential feature of these sets is their variation over time by considering a sequence of evolving sets over time. Such concepts would direct one to employ uncertainty theory as a solid axiomatic mathematical framework for modeling human reasoning. Analyzing the behavior of such a sequence motivated us to define the set-valued uncertain process. This concept combines uncertain set, uncertain process, and uncertain sequence to a new concept. Here, we introduce the main idea; some properties are extracted and clarified, along with some illustrative examples. We also put forward some potential state-of-the-art applications.

Keywords
Uncertainty theory · Uncertain set · Uncertain process · Set-valued uncertain process

1 Introduction
Uncertainty theory (Liu 2007), initiated in 2007 and later completed in 2009 (Liu 2009), is an axiomatic framework for modeling human reasoning in decision-making processes. Several concepts exist in this theory, such as uncertain sequence, uncertain set, and uncertain process. An uncertain sequence (Liu 2007) consists of infinite uncertain variables indexed by integers, mostly used when a variable evolves over another growing element, e.g., time. The concept of convergence is essential in studying such phenomena and is defined from different points of view in the literature. Some of these convergence notions dominate the other, meaning that convergence in one mode implies the other. However, they are not equivalent in general.

Uncertain set (Liu 2010) is another concept in the literature that is a set-valued function on an uncertainty space and aims to model “unsharp concepts” that are essentially set, while their boundaries are not clearly described. Two examples are “illness” and “recovery” since different degrees of illness and recovery would be defined. If some symptoms define an illness, the normal regions for these parameters are presented by some intervals. As these parameters tend to the extreme bounds of these sets, the degree of being healthy decreases. Dealing with such concepts and quantitatively deriving sensible results are the subject of uncertain set theory. These results would be later applied for more analysis and making practical decisions.

Uncertain process for modeling the evolution of uncertain phenomena over time has been also introduced (Liu 2008). For example, the exchange rate of two currencies, which varies as time progresses, could be considered as an uncertain process when the environment suggests using the uncertainty theory. Concepts of sample path, uncertainty distribution, independent increment process, extreme value, first hitting time, time integral, and stationary increment process have been introduced for an uncertain process. Recall that evolving factors in a phenomenon would be a collection of uncertain variables arrayed in a vector or a matrix. In such
a situation, the existing results and notions in the uncertain process literature must be generalized to a higher dimension with some potential complexities. For example, the absolute value must be replaced by a suitable norm, and other concepts, such as extreme value and first hitting time, must be redefined.

Generalization of the uncertain process may need more elaboration when the evolving object over time is an uncertain set. For example, “exacerbation” of and “recovery” from a disease are observed over time, while they are not defined sharply and would be expressed as evolving uncertain sets. Let us explain this problem with an example. Consider the disease “dementia”, according to WHO (https://www.who.int/news-room/fact-sheets/detail/dementia), it is a chronic symptom and commonly has progressive nature. Cognitive capability declines away from what might be anticipated from typical aging. It affects memory, reasoning, orientation, judgment, calculation and learning capacities, communication, and analysis. However, consciousness is not affected, and the impairment in cognitive function is commonly accompanied and occasionally preceded by a reduction in emotional control, social behavior, or motivation.

It is observed that dementia affects each person differently, depending upon the impact of the disease and the person’s personality before becoming ill. Signs and symptoms categorize the severity of dementia into three stages. The early stage is recognized with signs such as forgetfulness, losing track of time, and becoming lost in familiar places. As dementia progresses to the middle stage, the signs and symptoms become more apparent and more restricting. For example, some recent events and people’s names are forgotten. The most visible symptoms in this stage are getting lost at home, having increasing difficulty with communication, and needing help with personal care. In the severe stage, the patient reveals almost complete dependence and inactivity. Memory disturbances are serious, and physical signs become more evident. For instance, they become more unaware of the time and area, have difficulty remembering relatives and friends, and increasing demand for supported self-care. Moreover, other remarkable indicators may include difficulty walking and encountering behavior variations that may grow and be combined with aggression.

Several medical tests exist to diagnose the development of dementia. For instance, Montreal Cognitive Assessment (known as MoCA) test initiated in 2005 (Nasreddine et al. 2005) and then retrieved in 2017 (Montreal Cognitive Assessment 2017) is one the commonly used tests. It is a one-page 30-point test administered in approximately 10 minutes. The short-term memory recall task; visuospatial abilities, multiple aspects of executive functions; attention, concentration, and working memory; language, and abstract reasoning are assessed during this test. Its scores range between 0 and 30. A score of 26 or over is considered to be normal. Patients without cognitive impairment scored an average of 27.4, those with mild cognitive impairment scored an average of 22.1, and those with Alzheimer’s disease scored an average of 16.2. As mentioned above, while the score is compared with some criteria, a range of scores is used to diagnose the disease severity. Notably, the boundaries of these ranges are defined by experts. Moreover, the scoring process is carried out by an expert while it depends entirely on her expertise and reasoning. Further, these sets would not be naturally similar for patients of different ages, suggesting that they evolve over time. Developing a logical methodology would be of interest in finding an appropriate pattern for such advancing sets.

Set-valued processes have been studied using probability theory (see, e.g., Kisielewicz (2013); Schmelzer (2010)) and are formally referred to as set-valued stochastic processes. It has a rich theoretical background and a vast range of applications. Mainly it is used in set-valued differential equations along with a Brownian motion (see, e.g., Kisielewicz (2013); Zhu et al. (2020)) as well as addressing other problems in financial mathematics (see, e.g., Feinstein and Rudloff (2018); Zhang and Li (2016)). The primary assumption in these studies is that the probability space is known and the probability distribution is well-approximated. These models work well when frequency governs the environment and enough reliable data exist. However, many phenomena do not obey stochastic rules, and provided data are mainly extracted from experts’ opinions or measured imprecisely. Using probability theory in these situations might be misleading and produce unrealistic results.

One may be tempted to use the Fuzzy theory when data are extracted from an expert. The fuzzy process has been studied from different aspects (see for some recent applications in Luo et al. (2020); You and Bo (2021)). Observe that a fuzzy process is a function from \( T \times (\Theta, \mathcal{P}, \mathcal{C}) \) to the set of real numbers, where \( (\Theta, \mathcal{P}, \mathcal{C}) \) is a credibility space (Liu 2008). In fact, the object that evolves over a credibility space is a real value, not a fuzzy set. Fuzzy set theory has also been considered in combination with the set-valued stochastic process. Li and Guan (2007) considered fuzzy set-valued Gaussian processes and Brownian motions. The classical Gaussian stochastic process was extended to a case when the process elements are allowed to take values of fuzzy sets, and a new fuzzy Brownian motion was introduced. Further extensions have been proposed later (see, e.g., Li and Ren (2007); Wang and Zhang (2018)). While fuzzy set theory works well in some situations, some contradictions hinder being used everywhere (see Liu (2020) for a counter-intuitive example).

This paper introduces the set-valued uncertain process for analyzing an unsharp concept that evolves over time. We consider uncertainty theory (Liu 2020) as an underlying mathematical structure in our study. The central concept is defined formally, and the inverse set-valued uncertain pro-
cess is outlined. The membership function and convergence notions are proposed. Some illustrative examples are provided to clarify the concepts. Findings suggest a pattern for such developing sets to foster the decision-making process, say diagnostics in chronic diseases. The paper is organized in the following manner. Section 2 reviews some necessary facts from the uncertainty theory. In Section 3, the concept of set-valued uncertain process is introduced. The section continues with defining a set-valued sample path and presents some of its properties. These concepts are illustrated using some toy examples. The membership function of a set-valued uncertain process is defined and some properties are proposed. Some illustrative examples are provided to clarify the concepts. Findings suggest a pattern for such developing sets to foster the decision-making process, say diagnostics in chronic diseases.

2 Preliminary definitions

Consider a universal set $\Gamma$, with $\sigma$-algebra $\mathcal{L}$ and uncertain measure $\mathcal{M}$. Each $A \in \mathcal{L}$ is referred to as an event, and $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space where $\mathcal{M}$ satisfies the following four axioms (Liu 2007, 2009).

**Axiom 1** (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set $\Gamma$.

**Axiom 2** (Duality Axiom) $\mathcal{M}\{A\} + \mathcal{M}\{A^c\} = 1$ for any event $A$.

**Axiom 3** (Subadditivity Axiom) For every countable sequence of events $A_1, A_2, \ldots$, we have

$$\mathcal{M}\left\{ \bigcup_{i=1}^{\infty} A_i \right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{A_i\}. \quad (1)$$

**Axiom 4** (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{ \prod_{k=1}^{\infty} A_k \right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{A_k\}, \quad (2)$$

where $A_k$ is an arbitrarily chosen event from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

An uncertain variable is a function $\xi$ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\gamma \in \Gamma | \xi(\gamma) \in B\}$ is an event for any Borel set $B$ of real numbers (Liu 2007). An uncertain set is a function $\xi$ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to a collection of sets of real numbers such that both $\{\gamma \in \Gamma | B \subset \xi(\gamma)\}$ and $\{\gamma \in \Gamma | \xi(\gamma) \subset B\}$ are events for any Borel set $B$ of real numbers (Liu 2010). It has the membership function $\mu(x)$ if for any Borel set $B$ of real numbers, we have

$$\mathcal{M}\{\gamma | B \subset \xi(\gamma)\} = \inf_{x \in B} \mu(x), \quad (3)$$

$$\mathcal{M}\{\gamma | \xi(\gamma) \subset B\} = 1 - \sup_{x \in B^c} \mu(x). \quad (4)$$

Note that we use the notation $\xi(\gamma)$ for denoting both the uncertain variable and the uncertain set. The meaning can be derived from the context. We hope it raises no confusion.

Two uncertain sets $\xi_1$ and $\xi_2$ are independent (Liu 2013a) if for any Borel sets $B_1$ and $B_2$ of real numbers, we have

$$\mathcal{M}\{(\xi_1^+ \subset B_1) \cap (\xi_2^+ \subset B_2)\} = \mathcal{M}\{\xi_1^+ \subset B_1\} \wedge \mathcal{M}\{\xi_2^+ \subset B_2\}, \quad (5)$$

and

$$\mathcal{M}\{(\xi_1^+ \subset B_1) \cup (\xi_2^+ \subset B_2)\} = \mathcal{M}\{\xi_1^+ \subset B_1\} \vee \mathcal{M}\{\xi_2^+ \subset B_2\}, \quad (6)$$

where $\xi_i^+$ are arbitrarily chosen from $[\xi_i, \xi_i^+]$, $i = 1, 2$. Here, $\xi^c$ stands for the complement of the set $\xi$.

There are some standard defined uncertain sets in the literature. Here, we only remind the simplest intuitive one. An uncertain set $\xi$ is called triangular if it has the membership function

$$\mu(x) = \begin{cases} 
\frac{x-a}{b-a}, & \text{if } a \leq x \leq b \\
\frac{x-c}{b-c}, & \text{if } b \leq x \leq c,
\end{cases} \quad (7)$$

and denoted by $(a, b, c)$ where $a, b, c$ are real numbers with $a < b < c$.

3 Set-valued uncertain process

Here, we define the set-valued uncertain process that is an uncertain set at each moment of a time span. Let us define it formally.

**Definition 1** Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and $T$ be a totally ordered set (hereafter, we call it time). A set-valued uncertain process is a function $\xi_t(\gamma)$ from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to a collection of sets of real numbers such that both $\{\gamma | B \subset \xi_t(\gamma)\}$ and $\{\gamma | \xi_t(\gamma) \subset B\}$ are events for any Borel set $B$ of real numbers (Liu 2010).
\[ \xi_t(\gamma) \] and \( \{ \gamma \mid \xi_t(\gamma) \subset B \} \) are events for any Borel set \( B \) of real numbers at any time \( t \in T \).

**Example 1** Take an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \([\gamma_1, \gamma_2, \gamma_3]\) with power set and \( \mathcal{M}\{\gamma_1\} = 0.6, \mathcal{M}\{\gamma_2\} = 0.3, \mathcal{M}\{\gamma_3\} = 0.2 \). It is easy to verify that \( \xi_t(\gamma) \) defined by

\[
\xi_t(\gamma) = \begin{cases}
[t, 3 + t], & \text{if } \gamma = \gamma_1 \\
[2t - 1, 4 + t], & \text{if } \gamma = \gamma_2 \\
[3t, 5 + 2t], & \text{if } \gamma = \gamma_3,
\end{cases}
\]

(8)

defines a set-valued uncertain process since it is an uncertain set for every time \( t \geq 0 \).

**Definition 2** *(Set-valued sample path)* Let \( \xi_t \) be a set-valued uncertain process. For each \( \gamma \in \Gamma \), the set-valued function \( \xi_t(\gamma) \) is called a set-valued sample path of \( \xi_t \) and its graph is denoted by

\[ \mathcal{G}(\gamma) = \{(t, x) \in T \times \mathbb{R}, x \in \xi_t(\gamma)\} \]

A set-valued uncertain process is convex, closed, or polyhedral if its graph is likewise for every \( \gamma \in \Gamma \).

**Definition 3** *(Graph of a set-valued uncertain process)* For any time \( t \in T \) and any event \( \Lambda \in \mathcal{L} \), we write \( \xi_t(\Lambda) = \bigcup_{\gamma \in \Lambda} \mathcal{G}(\gamma) \). The graph of a set-valued uncertain process \( \xi_t \) is the set

\[ \mathcal{G}(\xi_t) = \bigcup_{\gamma \in \Gamma} \mathcal{G}(\gamma) = \bigcup_{\gamma \in \Gamma} \{(t, x) \in T \times \mathbb{R}, x \in \xi_t(\gamma)\} \]

A set-valued uncertain process is convex, closed, or polyhedral if its graph is likewise.

**Definition 4** *(Lower semi-continuity property)* A set-valued sample path \( \xi_t(\gamma) \) is Lower Semi-Continuous (LSC) at a point \((t, x)\) in its graph if for all neighborhood of \( x \) (open set \( U \subset \mathbb{R} \) containing \( x \)), \( \{t \mid \xi_t(\gamma) \cap U \neq \emptyset\} \) is an open set in \( T \) containing \( t \). In other words, for any sequence of points \((t_n)\) approaching \( t \) there is a sequence of points \( x_n \in \xi_{t_n}(\gamma) \) approaching \( x \). If, for a specific \( t \in T \), this property holds for all points \( x \in \xi_t \), we say \( \xi_t(\gamma) \) is LSC at \( t \). A set-valued sample path \( \xi_t(\gamma) \) is LSC if it is LSC for all \( t \in T \). A set-valued uncertain process is LSC if all its set-valued sample paths are LSC.

**Remark 1** Figure 1 depicts the set-valued sample path in Example 1 for \( \gamma = \gamma_1 \) (the region between two red lines). Figure 2 denotes a typical set-valued sample path; the region between red and black borders. Note that each set-valued sample path is a set-valued function of time \( t \). In addition, a set-valued uncertain process can be regarded as a function from an uncertainty space to a collection of set-valued sample paths. A set-valued uncertain process is nonempty if for all \( \gamma \) and \( t \in T \), we have \( \xi_t(\gamma) \neq \emptyset \). Furthermore, a set-valued sample path is convex, closed, or polyhedral if its graph is likewise. Observe that all set-valued sample paths in Example 1 are convex, closed, and polyhedral (see Figure 1 for \( \gamma = \gamma_1 \) and Figure 4a for all \( \gamma \)). The typical set-valued sample path depicted in Figure 2 is only closed when it includes the borders, but it is neither convex nor polyhedral. However, both set-valued sample paths in Figure 1, 2 are LSC.

**Remark 2** Observe that all set-valued uncertain processes are not LSC. For example, let \( \Gamma \) be the interval \([0, 1]\), with Borel algebra and Lebesgue measure. Define

\[
\xi_t(\gamma) = \begin{cases}
[0, 3\gamma] & \text{if } t \geq 1 \\
[0, \gamma/k] & \text{if } \frac{1}{k+1} \leq t < \frac{1}{k}, \quad k = 1, 2, 3, \ldots \\
[0, 1 + \gamma] & \text{if } t = 0.
\end{cases}
\]

(9)

For \( \gamma \geq \frac{1}{2} \), a typical set-valued sample path (red region) is denoted in Figure 3. It is not hard to verify that for any \( \gamma \in (0, 1) \), all set-valued sample paths are not LSC at \( t = 0 \) and \( x \in (0, 1 + \gamma] \subset \xi_0(\gamma) \).
4 Membership function

Analogous to the uncertain set, membership function can be defined for the set-valued uncertain process.

Definition 5 (Membership function) A set-valued uncertain process $\xi_t$ is said to have a membership function $\mu_t(x)$ for $t \in T$ if for any Borel set $B$ of real numbers, we have

$$M\{y | B \subset \xi_t(y)\} = \inf_{x \in B} \mu_t(x),$$
$$M\{y | \xi_t(y) \subset B\} = 1 - \sup_{x \in B^c} \mu_t(x). \quad (10)$$

Theorem 1 Let $\xi_t$ be a set-valued uncertain process whose membership function $\mu_t(x)$ exists. Then,

$$\mu_t(x) = M\{y | x \in \xi_t(y)\}, \quad (11)$$

for any $x \in \mathbb{R}$ and $t \in T$.

Proof: For any number $x \in \mathbb{R}$ and $t \in T$, it follows from (10) that

$$M\{y | x \in \xi_t(y)\} = M\{y | \{x\} \subset \xi_t(y)\} = \inf_{y \in \{x\}} \mu_t(y) = \mu_t(x).$$

The statement is proved. \qed

Example 2 Consider the set-valued uncertain process in Example 1. Different lines partition the domain of its membership function (see Figure 4a), while it is constant over each part. Table 1 denotes these regions and the membership function values over them. Observe that this membership function is not continuous. Figure 4b pictures this function by different colors.

Example 3 Let the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ be $[0, 1]$ with Borel algebra and Lebesgue measure. Define the set-valued uncertain process as

$$\xi_t(\gamma) = \left[ t - \frac{\gamma}{1+t}, t + \frac{\gamma}{1+t} \right], \forall \gamma \in [0, 1], \quad (12)$$

for all $t \geq 0$. Note that as $t$ increases, the uncertain set $\xi_t(\gamma)$ shrinks independently of the event $\gamma$, meaning that by proceeding in time, the uncertainty vanishes and the uncertain set reduces to the crisp value $t$. This behavior suggests a sense of convergence. We will discuss the convergence of a set-valued uncertain process later in detail.

One can verify that for every $t \geq 0$, membership function of (12) is

$$\mu_t(x) = \begin{cases} 1 - (1 + t)|x - t|, & \text{if } t - \frac{1}{1+t} \leq x \leq t + \frac{1}{1+t} \\ 0, & \text{Otherwise.} \end{cases} \quad (13)$$

Table 1 Domain of the membership function and its representation on each region in Example 1

| $t$    | $x$        | event $A$ | $M(A)$ |
|--------|------------|-----------|--------|
| $0 \leq t < 1$ | $-1 + 2t \leq x \leq t$ | $\{y_1, y_2\}$ | 0.3    |
| $t \leq x \leq 3t$ | $\{y_1, y_2\}$ | 0.8    |
| $3t \leq x \leq 4t$ | $\{y_1, y_2, y_3\}$ | 1      |
| $3 + t \leq x \leq 4 + t$ | $\{y_2, y_3\}$ | 0.4    |
| $4 + t \leq x \leq 5 + 2t$ | $\{y_3\}$ | 0.2    |
| $1 \leq t < \frac{3}{2}$ | $t \leq x \leq -1 + 2t$ | $\{y_1\}$ | 0.6    |
| $t \leq x \leq -1 + 2t$ | $\{y_1, y_2\}$ | 0.8    |
| $3 + t \leq x \leq 3 + t$ | $\{y_1, y_2, y_3\}$ | 1      |
| $3 + t \leq x \leq 4 + t$ | $\{y_2, y_3\}$ | 0.4    |
| $4 + t \leq x \leq 5 + 2t$ | $\{y_3\}$ | 0.2    |
| $\frac{3}{2} \leq t < 2$ | $t \leq x \leq -1 + 2t$ | $\{y_1\}$ | 0.6    |
| $t \leq x \leq -1 + 2t$ | $\{y_1, y_2\}$ | 0.8    |
| $3 + t \leq x \leq 3 + t$ | $\{y_2\}$ | 0.3    |
| $3 + t \leq x \leq 4 + t$ | $\{y_2, y_3\}$ | 0.4    |
| $4 + t \leq x \leq 5 + 2t$ | $\{y_3\}$ | 0.2    |
| $2 \leq t < 4$ | $t \leq x \leq -1 + 2t$ | $\{y_1\}$ | 0.6    |
| $t \leq x \leq -1 + 2t$ | $\{y_1, y_2\}$ | 0.8    |
| $3 + t \leq x \leq 3 + t$ | $\{y_2\}$ | 0.3    |
| $3 + t \leq x \leq 4 + t$ | $\{y_2, y_3\}$ | 0.4    |
| $4 + t \leq x \leq 5 + 2t$ | $\{y_3\}$ | 0.2    |
| $\{y_1\}$ | 0.6    |
| $t \leq x \leq 3 + t$ | $\{y_1\}$ | 0.6    |
| $3 + t \leq x \leq 4 + t$ | $\{y_1\}$ | 0.6    |
| $4 + t \leq x \leq 5 + 2t$ | $\{y_1\}$ | 0.6    |
| $-1 + 2t \leq x \leq 5 + 2t$ | $\{y_1\}$ | 0.6    |
| $5 + 2t \leq x \leq 3t$ | $\{y_1\}$ | 0.6    |
Fig. 4 Domain and membership function of the set-valued uncertain process defined by (8)

Fig. 5 Membership function of the set-valued uncertain process defined by (12)

**Definition 6** The inverse of a set-valued uncertain process $\xi_t$ is defined as $\xi_t^{-1} : \mathbb{R} \rightarrow (\Gamma, \mathcal{L}, \mathcal{M})$ by the relationship

$$\gamma \in \xi_t^{-1}(x) \iff x \in \xi_t(\gamma),$$

for $x \in \mathbb{R}$ and $t \in T$.

Observe that the inverse set-valued uncertain process defines a subset of the universal set $\Gamma$ for each real value $x$.

**Example 4** Consider the set-valued uncertain process defined by (12). According to Definition 6, $\gamma \in \xi_t^{-1}(x)$ if and only if $x \in \left[ t - \frac{\mu}{1 + \gamma}, t + \frac{\mu}{1 + \gamma} \right]$. This means that $\gamma$ must satisfy

$$(1 + t)|x - t| \leq \gamma,$$

since $\gamma \in [0, 1]$. The set of such $\gamma$’s is empty unless for the given $x, t \geq 0$ satisfies

$$(1 + t)|x - t| \leq 1.$$  \hspace{1cm} (14)

Thus, for each real value $x$, we have

$$\xi_t^{-1}(x) = \begin{cases} [(1 + t)|x - t|, 1] & \text{if } |x - t| \leq \frac{1}{1 + t} \\ \emptyset & \text{Otherwise.} \end{cases}$$  \hspace{1cm} (15)

Observe that (14) can be simplified for a given $x$ to find an explicit region of $t$. However, finding a closed form for
the domain of \( t \), and consequently, for \( \xi_t^{-1}(x) \) is out of our interest. Here, we only calculate its value for some specific values of \( x \). For instance,

\[
\xi_t^{-1}(1) = \begin{cases} 
(1 + t) |t - 1|, 1 & \text{if } 0 \leq t \leq \sqrt{2} \\
\emptyset & \text{Otherwise,}
\end{cases}
\]  

and

\[
\xi_t^{-1}(4) = \begin{cases} 
(1 + t) |t - 1|, 1 & \text{if } 3 + \sqrt{27} \leq t \leq 3 + \sqrt{29} \\
\emptyset & \text{Otherwise.}
\end{cases}
\]

**Remark 3** Let us elaborate the meaning of inverse set-valued uncertain process \( \xi_t^{-1}(x) \) within an example. Suppose we are interested in the values that define the unsharp concept of “recovery” in a disease. It would be defined by a set-valued uncertain process \( \xi_t(y) \). Let us fix the time \( t \) in the sequel. Here, each value of \( x \in \xi_t(y) \) denotes a level of recovery that is identified by the membership function \( \mu_t(x) \). On the other hand, for the fixed observed value \( x \) from a patient, we may be interested in knowing which uncertain events would result in observing \( x \) as a level of recovery. These events are members of the underlying \( \sigma \)-algebra over the universal set \( \Gamma \) with different belief degrees. As an intuitive case, let \( \xi_t(y) \) be defined by (8), and for \( t = 3 \), \( x = 5.5 \) be observed. Then, \( \xi_t^{-1}(5.5) = \{y_1, y_2\} \). This result is valid for all \( x \in [5, 6] \). Observe that \( \mathcal{M}(y_1, y_2) = 0.8 \) in this situation. If the set-valued uncertain process is defined by (12), and for \( t = 3.9 \), the observed value is \( x = 4 \), then \( \xi_t^{-1}(4) = [0.49, 1] \) and \( \mathcal{M}(\xi_t^{-1}(4)) = 0.51 \). This interpretation leads to the following concept.

**Definition 7 (Belief of observing)** Let \( \xi_t(y) \) be a set-valued uncertain process defined on the uncertainty space \( (\Gamma, \mathcal{L}, \mathcal{M}) \), and \( \xi_t^{-1}(x) \) be its inverse. Belief of observing \( x \) as an instance is defined as \( \mathcal{M}(\xi_t^{-1}(x)) \).

**Theorem 2** Let \( \xi_t(y) \) be a set-valued uncertain process defined on an uncertainty space \( (\Gamma, \mathcal{L}, \mathcal{M}) \) and \( \xi_t^{-1}(x) \) be its inverse. Belief of observing \( x \) is identical with the membership value of \( x \), that is,

\[
\mathcal{M}(\xi_t^{-1}(x)) = \mu_t(x).
\]

**Proof** Let \( x \) be a fixed real number. Then,

\[
\mathcal{M}(\xi_t^{-1}(x)) = \mathcal{M}(\gamma | \gamma \in \xi_t^{-1}(x)) = \mathcal{M}(\gamma | x \in \xi_t(y)) = \mu_t(x).
\]

The proof is complete. \( \square \)

**Definition 8 (Openness property)** We say a set-valued uncertain process \( \xi_t \) is open at a point \( (\hat{t}, \hat{x}) \) in its graph, if for all neighborhoods \( U \) of \( \hat{t} \), the point \( \hat{x} \) lies in \( \text{int}(U) \). Equivalently, for any sequence of points \( (x_n) \) approaching \( \hat{x} \), there is a sequence of points \( (t_n) \) approaching \( \hat{t} \) such that \( x_n \in \xi_{t_n} \) for all \( n \). If, for \( \hat{x} \) in the range, this property holds for all points \( t \in \xi_t^{-1}(\hat{x}) \), we say \( \xi_t \) is open at \( \hat{x} \).

**Example 5** We show that the set-valued uncertain process defined by (12) is open at any \( (\hat{t}, \hat{x}) \) in its graph. This means that \( \hat{x} \in \xi_t(\gamma) \) for all \( \gamma \in [0, 1] \). Let \( (x_n) \) be a sequence of real numbers converging to \( \hat{x} \). Thus, \( t_n \) must satisfy

\[
t_n - \frac{\gamma}{1 + t_n} \leq x_n \leq t_n + \frac{\gamma}{1 + t_n}.
\]

For such an appropriate \( t_n \), we must have

\[
t_n^2 + (1 - x_n)t_n - (x_n + \gamma) \leq 0
\]

\[
t_n^2 + (1 - x_n)t_n - (x_n - \gamma) \geq 0
\]

This means that

\[
x_n - \gamma \leq t_n^2 + (1 - x_n)t_n \leq x_n + \gamma.
\]

Without loss of generality, one may assume that (the midpoint of the interval)

\[
t_n^2 + (1 - x_n)t_n = \frac{(x_n - \gamma) + (x_n + \gamma)}{2} = x_n.
\]

By assuming \( t_n \rightarrow x_n \), a root of (24) as a sequence of time, we have \( \lim_{n \to \infty} t_n = \hat{t} = \hat{x} \). Considering (12), \( \hat{x} \in \xi_{t_n}(\gamma) \), and therefore, this set-valued uncertain process holds openness property.

There is an elegance relation between openness and LSC properties of a set-valued uncertain process and its inverse, respectively. The following theorem is an adjustment of Proposition 5.4.5 from Borwein and Lewis (2010). The proof is omitted.

**Theorem 3** (Openness and LSC property) Any set-valued uncertain process \( \xi_t(y) \) is LSC at a point \( (t, x) \) in its graph if and only if \( \xi_t^{-1}(x) \) is open at \( (x, t) \).

**Definition 9** A set-valued uncertain process \( \xi_t(y) \) defined on the uncertainty space \( (\Gamma, \mathcal{L}, \mathcal{M}) \) is called totally ordered for \( t \in T \) if \( \{\xi_t(y) | \gamma \in \Gamma\} \) is a totally ordered set. That is, for any given \( \gamma_1 \) and \( \gamma_2 \in \Gamma \), either \( \xi_t(\gamma_1) \subseteq \xi_t(\gamma_2) \) or \( \xi_t(\gamma_2) \subseteq \xi_t(\gamma_1) \) holds.

**Example 6** Consider Example 3. For a fixed \( t \) and \( 0 \leq \gamma_1 \leq \gamma_2 \leq 1 \), we have

\[
t - \frac{\gamma_2}{1 + t} < t - \frac{\gamma_1}{1 + t} < t + \frac{\gamma_1}{1 + t} < t + \frac{\gamma_2}{1 + t},
\]

that results in \( \xi_t(\gamma_1) \subseteq \xi_t(\gamma_2) \). Thus, the set-valued uncertain process defined by (12) is totally ordered.
A. Ghaffari-Hadigheh

5 Set and Arithmetic Operational Laws

This section will discuss the inclusion, union, intersection and complement of set-valued uncertain processes via membership functions. They are a generalization of set operations from uncertain sets to set-valued uncertain processes. For these laws, we need to have a notion of independence for two set-valued uncertain processes that in turn need the notion of complement of a set-valued uncertain process.

Definition 12 The complement \( \xi_t(\gamma) \) of the set-valued uncertain process \( \xi_t(\gamma) \) is

\[
\xi_t(\gamma) = \xi_t(\gamma)^c, \quad \forall \gamma \in \Gamma, \quad \forall t \in T. \tag{28}
\]

Definition 13 Let two set-valued uncertain processes \( \xi_t \) and \( \eta_t \) be defined on an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\). They are independent if for any Borel sets \(B_1\) and \(B_2\) of real numbers, we have

\[
\mathcal{M}\left\{ (\xi_t^* \subset B_1) \cap (\eta_t^* \subset B_2) \right\} = \mathcal{M}\left\{ \xi_t^* \subset B_1 \right\} \land \mathcal{M}\left\{ \eta_t^* \subset B_2 \right\}. \tag{29}
\]

and

\[
\mathcal{M}\left\{ (\xi_t^* \subset B_1) \cup (\eta_t^* \subset B_2) \right\} = \mathcal{M}\left\{ \xi_t^* \subset B_1 \right\} \lor \mathcal{M}\left\{ \eta_t^* \subset B_2 \right\}. \tag{30}
\]

for all \( t \in T \), where \( \xi_t^* \) and \( \eta_t^* \) are arbitrarily chosen from \( \{\xi_t, \xi_t^c\} \) and \( \{\eta_t, \eta_t^c\} \), respectively. Then, we have

First we present the concept of inclusion for two set-valued uncertain processes. It is a generalization of the results in Yao (2015).

Definition 14 Let \( \xi_t \) and \( \eta_t \) be independent uncertain sets defined on an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\), with membership functions \( \mu_t(x) \) and \( \nu_t(x) \), respectively. Then,

\[
\mathcal{M}\{\xi_t \subset \eta_t\} = \inf_{x \in \mathbb{R}} (1 - \mu_t(x)) \lor \nu_t(x). \tag{31}
\]

Definition 15 Let \( \xi_t \) and \( \eta_t \) be two set-valued uncertain processes on the uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\). Then,

(i) the union \( \xi_t \cup \eta_t \) of the set-valued uncertain processes \( \xi_t \) and \( \eta_t \) is

\[
(\xi_t \cup \eta_t)(\gamma) = \xi_t(\gamma) \cup \eta_t(\gamma), \quad \forall \gamma \in \Gamma, \quad \forall t \in T; \tag{32}
\]

(ii) the intersection \( \xi_t \cap \eta_t \) of the set-valued uncertain processes \( \xi_t \) and \( \eta_t \) is

\[
(\xi_t \cap \eta_t)(\gamma) = \xi_t(\gamma) \cap \eta_t(\gamma), \quad \forall \gamma \in \Gamma, \quad \forall t \in T; \tag{33}
\]
In the sequel, we present some theorems on the membership functions of produced set-valued uncertain processes via set operations.

**Theorem 5** Let $\xi_t$ and $\eta_t$ be independent set-valued uncertain processes defined on an uncertainty space $(\Gamma, \mathcal{E}, \mathcal{M})$, with membership functions $\mu_t$ and $\nu_t$, respectively. Then, their union $\xi_t \cup \eta_t$ has the membership function

$$\lambda_t(x) = \mu_t(x) \lor \nu_t(x).$$  \hspace{1cm} (34)

**Proof** Recall that for a fixed time moment $t \in T$, $\xi_t$ and $\eta_t$ are two independent uncertain sets with known membership functions. Using the results in Liu (2012) confirms the credibility of (34) for this fixed $t$. The claim is proved since $t$ is chosen arbitrarily in $T$. \hfill $\Box$

The proofs of the following two theorems are similar to the proof of Theorem 5 and omitted.

**Theorem 6** Let $\xi_t$ and $\eta_t$ be independent set-valued uncertain processes defined on an uncertainty space $(\Gamma, \mathcal{E}, \mathcal{M})$, with membership functions $\mu_t$ and $\nu_t$, respectively. Then, their intersection $\xi_t \cap \eta_t$ has the membership function

$$\lambda_t(x) = \mu_t(x) \land \nu_t(x).$$  \hspace{1cm} (35)

**Theorem 7** Let $\xi_t$ be set-valued uncertain process with the membership function $\mu_t$. Then, its complement $\xi_t^c$ has the membership function

$$\lambda_t(x) = 1 - \mu_t(x).$$  \hspace{1cm} (36)

In the rest of this subsection, we present an arithmetic operational law of set-valued uncertain processes, including addition, subtraction, multiplication, and division. The following theorem uses the inverse membership functions of the incorporating set-valued uncertain processes while the latter uses the membership functions themselves. Proofs are straightforward, since the incorporating set-valued uncertain processes are uncertain sets for a fixed value of $t$, and $t$ is an arbitrary time instance in $T$.

**Theorem 8** Let $\xi_t, 1, \xi_t, 2, \ldots, \xi_t, n$ be independent set-valued uncertain processes with inverse membership functions $\mu_{t, 1}^{-1}, \mu_{t, 2}^{-1}, \ldots, \mu_{t, n}^{-1}$, respectively, and $f$ be a measurable function. Then,

$$\xi_t = f(\xi_t, 1, \xi_t, 2, \ldots, \xi_t, n).$$  \hspace{1cm} (37)

is a set-valued uncertain process and has the inverse membership function

$$\lambda_t^{-1}(\alpha) = f(\mu_{t, 1}^{-1}(\alpha), \mu_{t, 2}^{-1}(\alpha), \ldots, \mu_{t, n}^{-1}(\alpha)).$$  \hspace{1cm} (38)

**Theorem 9** Let $\xi_t, 1, \xi_t, 2, \ldots, \xi_t, n$ be independent set-valued uncertain processes with membership functions $\mu_{t, 1}(x), \mu_{t, 2}(x), \ldots, \mu_{t, n}(x)$, respectively, and let $f$ be a measurable function. Then,

$$\xi_t = f(\xi_t, 1, \xi_t, 2, \ldots, \xi_t, n).$$  \hspace{1cm} (39)

has the membership function,

$$\lambda_t(x) = \sup_{f(x_1, x_2, \ldots, x_n) = x} \min_{1 \leq i \leq n} \mu_{t, i}(x_i).$$  \hspace{1cm} (40)

### 6 Expected sample path

The expected value of an uncertain set has been defined in Liu (2010). The expected value for a set-valued uncertain process $\xi_t$ for each $t$ is defined similarly, and denoted by $E[\xi_t]$ for each $t \geq 0$.

**Definition 16** *(Expected sample path)* Let $\xi_t$ be a nonempty set-valued uncertain process, and denote the expected value of $\xi_t$ for every $t \geq 0$ by $E[\xi_t]$. The produced sequence $E[\xi_t]$, indexed by $t$, is referred to as expected sample path.

It is important to note that a set-valued uncertain process is a set at any moment, and its expected value abstracts the whole set in a real number. Thus, the expected sample path defines an uncertain sequence independent of the existing uncertainty in the environment, but it is resulted from this situation.

Recall that if the membership function of an uncertain set is known, one can use the results in Liu (2010) to find its expected value, especially when the membership function is regular. We imitate these results and provide formulae for the expected sample path when the membership function $\mu_t(x)$ is given.

**Theorem 10** Let $\xi_t$ be a set-valued uncertain process with the membership function $\mu_t(x)$ for all real $x$. Then for each $t \in T$, the expected sample path $E[\xi_t]$ is calculated by

$$E[\xi_t] = x_0(t) + \frac{1}{2} \int_{-\infty}^{+\infty} \sup_{y \geq x} \mu_t(y)dy - \frac{1}{2} \int_{-\infty}^{x_0(t)} \sup_{y \leq x} \mu_t(y)dy,$$  \hspace{1cm} (41)

where $x_0(t)$ is a point such that $\mu_t(x_0(t)) = 1$.

**Proof** The proof is similar to the proof of (Liu 2020, Theorem 8.22) and omitted. \hfill $\Box$

**Theorem 11** Let $\xi_t$ be a set-valued uncertain process with a regular membership function $\mu_t$. Then,

$$E[\xi_t] = x_0(t) + \frac{1}{2} \int_{x_0(t)}^{+\infty} \mu_t(x)dx - \frac{1}{2} \int_{-\infty}^{x_0(t)} \mu_t(x)dx,$$  \hspace{1cm} (42)
where $x_0(t)$ is a point such that $\mu_1(x_0(t)) = 1$.

**Proof** The proof is similar to the proof of (Liu 2020, Theorem 8.23) and omitted.

**Remark 5** Let us highlight the differences between the set-valued sample path and the expected sample path of $\xi_t(\gamma)$. The former is a sequence of uncertain sets for each $\gamma$ in the universal set, while the latter is a sequence of real numbers depicting the expected behavior of the process over time, independent from what $\gamma$ is. For instance, consider Example 3 and let $\gamma = \frac{1}{2} \in (0, 1)$. Then, the set-valued sample path is the sequence of sets $\xi_t(\frac{1}{2}) = [t - \frac{1}{2(1+T)} , t + \frac{1}{2(1+T)}]$ while the expected sample path is $E[\xi_t] = t$.

### 7 Phase distance

The distance of two uncertain sets has been defined in Liu (2011). Here, we adjust this definition in a set-valued uncertain process for two different times $t_1$ and $t_2$.

**Definition 17** (Distance between two phases) Consider the set-valued uncertain set $\xi_t(\gamma)$, and $t_1, t_2 \in T$. The distance between $\xi_{t_1}(\gamma)$ and $\xi_{t_2}(\gamma)$ for each $\gamma \in \Gamma$ is defined as

$$d(\xi_{t_1}(\gamma), \xi_{t_2}(\gamma)) = E[|\xi_{t_1}(\gamma) - \xi_{t_2}(\gamma)|].$$

(43)

Obviously, $d(\xi_{t_1}(\gamma), \xi_{t_2}(\gamma))$ is a nonnegative real number. However, despite the distance in crisp sets, it is not zero if $t_1 = t_2$ (see Example 10).

In the sequel, we provide some tools for calculation of the distance between two phases of a set-valued uncertain process. First, we define the phase independence property. This property resembles the uncertain renewal process (Liu 2008) is an uncertain process in which events occur continuously and independently of one another.

**Definition 18** (Phase independence property) A set-valued uncertain process has phase independence property if for all $t \geq 0$, and $s > 0$, two uncertain sets $\xi_t(\gamma)$, and $\xi_{t+s}(\gamma)$ are independent uncertain sets for every $\gamma$.

Observe that phase independence property enables one to identify the membership function $\lambda$ of $\xi_{t_1} - \xi_{t_2}$ for every $t_1$ and $t_2$, or its inverse for the set-valued uncertain process $\xi_t$. The following theorem relates the inverse membership functions of $\mu_{t_1}^{-1}$ and $\mu_{t_2}^{-1}$ with the inverse membership function of $\xi_{t_1}(\gamma) - \xi_{t_2}(\gamma)$. The proof is similar to the proof of Theorem 8.18 in Liu (2020) and omitted.

**Theorem 12** Let $\xi_t$ be a set-valued phase independent uncertain process with the inverse membership function $\mu_t^{-1}(\alpha)$. Further, let $t_1, t_2 \in T$ be two arbitrary times. Then, $\xi_{t_1} - \xi_{t_2}$ has an inverse membership function,

$$\lambda^{-1}(\alpha) = \mu_{t_1}^{-1}(\alpha) - \mu_{t_2}^{-1}(\alpha).$$

(44)

The following theorem provides a formula for the membership function of $\xi_{t_1} - \xi_{t_2}$ in terms of the membership function $\mu_t(x)$ for the set-valued phase independent uncertain process $\xi_t$.

**Theorem 13** Let $\xi_t$ be a set-valued phase independent uncertain process with the membership function $\mu_t(x)$. Then, for $0 \leq t_1 \leq t_2$, $\xi_{t_1} - \xi_{t_2}$ has the membership function

$$\lambda(x) = \sup_{y \in \mathbb{R}} \mu_{t_1}(x + y) \cap \mu_{t_2}(y).$$

(45)

**Proof** Let $\lambda(x)$ be the membership function of $\xi_{t_1} - \xi_{t_2}$. According to (Liu 2020, Theorem 8.18), we define $f(\xi_{t_1}, \xi_{t_2}) = \xi_{t_1} - \xi_{t_2}$. Considering the phase independence property of $\xi_t$, two uncertain sets $\xi_{t_1}$ and $\xi_{t_2}$ are independent with inverse membership functions $\mu_{t_1}^{-1}$ and $\mu_{t_2}^{-1}$, respectively. Thus,

$$\lambda^{-1}(\alpha) = \mu_{t_1}^{-1}(\alpha) - \mu_{t_2}^{-1}(\alpha).$$

For a given real number $x$, let $\lambda(x) = \beta$. It holds

$$\lambda^{-1}(\beta) = \mu_{t_1}^{-1}(\beta) - \mu_{t_2}^{-1}(\beta).$$

Since $x \in \lambda^{-1}(\beta)$, there exist real numbers $x_1, x_2 \in \mu_{t_1}^{-1}(\beta)$ and $x_2 \in \mu_{t_2}^{-1}(\beta)$ such that $x_1 - x_2 = x$. Noting that $\mu_{t_1}(x_i) \geq \beta$ for $i = 1, 2$, we have

$$\lambda(x) = \beta \leq \mu_{t_1}(x_1) \cap \mu_{t_2}(x_2),$$

and then

$$\lambda(x) \leq \sup_{x_1 - x_2 = x} \mu_{t_1}(x_1) \cap \mu_{t_2}(x_2).$$

(46)

On the other hand, assume $x_1$ and $x_2$ are two real numbers with $x_1 - x_2 = x$. Let

$$\mu_{t_1}(x_1) \cap \mu_{t_2}(x_2) = \beta.$$ Again, using (Liu 2020, Theorem 8.18), we have

$$\lambda^{-1}(\beta) = \mu_{t_1}^{-1}(\beta) - \mu_{t_2}^{-1}(\beta).$$

Noting that $x_i \in \mu_{t_i}^{-1}(\beta)$ for $i = 1, 2$, we have

$$x = x_1 - x_2 \in \mu_{t_1}^{-1}(\beta) - \mu_{t_2}^{-1}(\beta) = \lambda^{-1}(\beta).$$

Hence,

$$\lambda(x) \geq \beta = \mu_{t_1}(x_1) - \mu_{t_2}(x_2),$$

and then,

$$\lambda(x) \geq \sup_{x_1 - x_2 = x} \mu_{t_1}(x_1) \cap \mu_{t_2}(x_2).$$

(47)
The statement follows from (46) and (47) with $x_2 = y$ and $x_1 = x + y$.

Observe that this theorem provides a formula for identifying the membership function of $\xi_t - \xi_{t_2}$; however, one may find it costly. In this case, one can use (26) when $\lambda^{-1}(\alpha)$ is available.

**Example 9** Consider Example 3 and let $0 \leq t_1 \leq t_2 \leq 1$. Further, let $\xi_t$ be defined by (12) and have phase independence property. Since both $\mu_{t_1}$ and $\mu_{t_2}$ are triangle membership functions, then $\xi_{t_1} - \xi_{t_2} = (a, b, c)$ is a triangular uncertain set (see Example 8.25 Liu (2020)), where $a = t_1 - t_2 - \left( \frac{1}{1 + t_1} + \frac{1}{1 + t_2} \right)$, $b = t_1 - t_2$ and $c = t_1 - t_2 + \left( \frac{1}{1 + t_1} + \frac{1}{1 + t_2} \right)$.

In other words, the membership function of $\xi_{t_1} - \xi_{t_2}$ is

$$
\lambda(x) = \frac{x - \left( (t_1 - t_2) - \left( \frac{1}{1 + t_1} + \frac{1}{1 + t_2} \right) \right)}{\frac{1}{1 + t_1} + \frac{1}{1 + t_2}},
$$

when $t_1 - t_2 - \left( \frac{1}{1 + t_1} + \frac{1}{1 + t_2} \right) \leq x \leq t_1 - t_2$;

$$
\lambda(x) = -\frac{x - \left( (t_1 - t_2) + \left( \frac{1}{1 + t_1} + \frac{1}{1 + t_2} \right) \right)}{\frac{1}{1 + t_1} + \frac{1}{1 + t_2}},
$$

when $t_1 - t_2 \leq x \leq t_1 - t_2 + \left( \frac{1}{1 + t_1} + \frac{1}{1 + t_2} \right)$ and $\lambda(x) = 0$ for other values of $x$. Moreover, $\lambda$ is regular since $\lambda(t_1 - t_2) = 1$. Observe that for $t_1 = t_2 = \bar{t}$, we have

$$
\xi_{\bar{t}} - \xi_{\bar{t}} = \left( -\frac{2}{1 + \bar{t}}, 0, \frac{2}{1 + \bar{t}} \right).
$$

The following theorem resembles (Liu 2015, Theorem 8.31); the proof is omitted.

**Theorem 14** Let $\xi_t$ be a set-valued phase independent uncertain process. For $0 \leq t_1 < t_2$, the distance between $\xi_{t_1}$ and $\xi_{t_2}$ is

$$
d(\xi_{t_1}, \xi_{t_2}) = \frac{1}{2} \int_{0}^{\infty} \left( \sup_{|y| \geq x} \lambda(y) + 1 - \sup_{|y| < x} \lambda(y) \right) dx,
$$

(50)

where $\lambda(x)$ is the membership function of $\xi_{t_1} - \xi_{t_2}$.

**Example 10** Let us determine the distance between $\xi_{\bar{t}}$ and $\xi_{2\bar{t}}$ for the set-valued uncertain process in Example 3, with the assumption that it holds phase independence property. Recall the membership of $\xi_{\bar{t}} - \xi_{2\bar{t}}$, calculated in Example 9. We only consider the case when $t_1 \geq t_2 + 1 \geq 0$. Other cases can be processed similarly with some considerations on the boundaries of integrals. With this consideration, $t_1 - t_2 - \left( \frac{1}{1 + t_1} + \frac{1}{1 + t_2} \right)$ is positive. Therefore, (50) reduces to

$$
d(\xi_{\bar{t}}, \xi_{2\bar{t}}) = \frac{1}{2} \left[ \int_{0}^{t_1 - t_2 - \left( \frac{1}{1 + t_1} + \frac{1}{1 + t_2} \right)} \sup_{|y| \geq x} \lambda(y) + 1 - \sup_{|y| < x} \lambda(y) dx + \int_{t_1 - t_2 + \left( \frac{1}{1 + t_1} + \frac{1}{1 + t_2} \right)} \sup_{|y| \geq x} \lambda(y) + 1 - \sup_{|y| < x} \lambda(y) dx \right].
$$

(51)

Observe that when $t_1 - t_2 - \left( \frac{1}{1 + t_1} + \frac{1}{1 + t_2} \right) \leq x \leq t_1 - t_2$,

$$
\sup_{|y| \geq x} \lambda(y) = 1, \quad \text{and} \quad \sup_{|y| < x} \lambda(y) = \lambda(x).
$$

Moreover, when $t_1 - t_2 \leq x \leq t_1 - t_2 + \left( \frac{1}{1 + t_1} + \frac{1}{1 + t_2} \right)$,

$$
\sup_{|y| \geq x} \lambda(y) = \lambda(x), \quad \text{and} \quad \sup_{|y| < x} \lambda(y) = 1.
$$

Thus, (51) reduces to

$$
d(\xi_{\bar{t}}, \xi_{2\bar{t}}) = \frac{1}{2} \left[ \int_{0}^{t_1 - t_2 - \left( \frac{1}{1 + t_1} + \frac{1}{1 + t_2} \right)} 2 - \lambda(x) dx + \int_{t_1 - t_2 + \left( \frac{1}{1 + t_1} + \frac{1}{1 + t_2} \right)} \lambda(x) dx \right].
$$

(52)

After some algebraic manipulation, we have

$$
d(\xi_{\bar{t}}, \xi_{2\bar{t}}) = \frac{t_1 + t_2 + 2}{(t_1 + 1)(t_2 + 1)}.
$$

(53)

Observe that when $t_1 = t_2 = \bar{t}$ we have

$$
d(\xi_{\bar{t}}, \xi_{2\bar{t}}) = \frac{2\bar{t} + 2}{(\bar{t} + 1)(\bar{t} + 1)} = \frac{2}{(\bar{t} + 1)},
$$

(54)

which is not zero.

**8 Convergence of a set-valued uncertain process**

Consider a set-valued uncertain process $\xi_t(y)$ and let $\xi(y)$ be a fixed uncertain set both are defined on a specific uncertain space. First observe that $d(\xi_t(y) - \xi(y))$ is an uncertain sequence indexed by time and each element of this sequence is an uncertain variable that depends on $y$. Therefore, different convergence notions can be defined for a set-valued uncertain process.
Definition 19 The set-valued uncertain process $\xi_t$ is said to be convergent a.s. to $\xi$ if there exists an event $\Lambda$ with $\mathcal{M}\{\Lambda\} = 1$ such that

$$\lim_{t \to \infty} d(\xi_t(\gamma), \xi(\gamma)) = 0,$$

(55)

for every $\gamma \in \Lambda$.

Example 11 Let the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ be $[0, 1]$ with Borel algebra and Lebesgue measure. Consider the set-valued uncertain process $\xi(x) = \{t\}$. For $t \geq 0$, observe that (56) is a special uncertain set with the following membership function

$$\mu_t(x) = \begin{cases} 1 & \text{if } x = t \\ 0 & \text{Otherwise.} \end{cases}$$

Using Theorem 13, the membership function of $\xi_t - \xi$ is

$$\lambda(x) = \sup_{y \in \mathbb{R}} \mu_t(x + y) \land \mu(y) = \mu_t(x + t) \land \mu(t) = \mu_t(x + t) \land 1 = \mu_t(x + t),$$

where $\mu_t(x)$ is calculated as (13). Thus,

$$\lambda(x) = \begin{cases} 1 - (1 + t)|x|, & \text{if } -\frac{1}{1+t} \leq x \leq \frac{1}{1+t} \\ 0, & \text{Otherwise.} \end{cases}$$

(57)

Observe that for every $x \geq 0$, we have

$$\sup_{|y| < x} \lambda(y) = \sup_{|y| < \min(x, \frac{1}{1+t})} \lambda(y) = \sup_{|y| < \min(x, \frac{1}{1+t})} 1 - (1 + t)|y| = 1.$$

Moreover, for $0 \leq x \leq \frac{1}{1+t}$, we have

$$\sup_{|y| \geq x} \lambda(y) = \lambda(x) = 1 - (1 + t)x,$$

and for $x \geq \frac{1}{1+t}$, it holds

$$\sup_{|y| \geq x} \lambda(y) = 0.$$

Therefore,

$$d(\xi_t, \xi) = \frac{1}{2} \int_0^{\frac{1}{1+t}} 1 - (1 + t)x \, dx = \frac{1}{4(1+t)}.$$

(59)

Obviously, when $t \to \infty$, $d(\xi_t, \xi)$ vanishes, proving the a.s. convergence of $\xi_t$ in Example 3 to the singleton $\{t\}$ (see Figures 5).

Sometimes, the limiting uncertain set of a set-valued uncertain process is unknown. In this case, we may have the following uniformly a.s. convergence concept.

Definition 20 The set-valued uncertain process $\xi_t$ is said to be uniformly convergent a.s. if there exists an event $\Lambda$ with $\mathcal{M}\{\Lambda\} = 1$ such that for any $\varepsilon > 0$, there exists a fix $t$ with $t_1 \geq t$ and $t_2 \geq t$,

$$d(\xi_t(\gamma), \xi_{t_2}(\gamma)) < \varepsilon,$$

(60)

for every $\gamma \in \Lambda$.

Example 12 Consider Example 11 and the special case $t_1 = t_2 + 1 \geq 0$. One can easily deduce from (53) and (54) that $d(\xi_t(\gamma), \xi_{t_2}(\gamma))$ reduces to zero for enough large $t_1$ and $t_2$. Thus, the set-valued uncertain process $\xi_t$ defined by (12) is uniformly convergent a.s.

Analogous to uncertain sequences (Liu 2020), other types of convergence can be defined.

Definition 21 The set-valued uncertain process $\xi_t$ is said to be a.s. convergent in measure to $\xi$ if

$$\lim_{t \to \infty} \mathcal{M}\{\xi_t \geq \varepsilon\} = 0,$$

(61)

for every $\varepsilon > 0$.

Definition 22 The set-valued uncertain process $\xi_t$ is said to be convergent in mean to $\xi$ if

$$\lim_{t \to \infty} \mathbb{E}[d(\xi_t, \xi)] = 0.$$

(62)

Definition 23 Let $\mu_t(x)$ and $\mu(x)$ be membership functions of the set-valued uncertain process $\xi_t$ and the uncertain set $\xi$, respectively. We say $\xi_t$ pointwise converges in membership function to $\xi$ if for all $x \in \mathbb{R}$ and for all $\varepsilon > 0$, there exists $\hat{t} \in T$ such that for $t \geq \hat{t}$,

$$|\mu_t(x) - \mu(x)| < \varepsilon.$$

(63)

Recall that the convergence in membership function defined as above is pointwise. Here, we define uniform convergence of a set-valued uncertain process in membership function.

Definition 24 Let $\mu_t(x)$ and $\mu(x)$ be membership functions of the set-valued uncertain process $\xi_t$ and the uncertain set $\xi$, respectively. We say $\xi_t$ uniformly converges in membership function to $\xi$ if for all $\varepsilon > 0$; there exists $\hat{t} \in T$ such that for $t \geq \hat{t}$ and for all $x \in \mathbb{R}$

$$|\mu_t(x) - \mu(x)| < \varepsilon.$$

(64)
One may define Cauchy criterion for uniform convergence in membership function. We leave details to the interested reader.

**Example 13** Consider the membership functions (13) and (57) of the set-valued uncertain process (12) and (56), respectively. One can verify promptly that for any \( \varepsilon > 0 \), and enough large \( t \), when \( x \notin [t - \frac{1}{1+t}, t + \frac{1}{1+t}] \) we have

\[
|\mu_t(x) - \mu(x)| = |0 - 0| = 0.
\]

On the other hand, when \( x \in [t - \frac{1}{1+t}, t + \frac{1}{1+t}] \) and \( x \notin t \), then \((1 + t)|x - t| \leq 1\), or \(- (1 + t)|x - t| \geq 0\), and thus,

\[
|\mu_t(x) - \mu(x)| = |1 - (1 + t)|x - t| - 0| \geq 0.
\]

Observe that the left-hand side of (65) would be larger than the given \( \varepsilon \) for some \( x \in [t - \frac{1}{1+t}, t + \frac{1}{1+t}] \). This means that the convergence in membership function in this instance is not uniform.

### 9 Potential applications

Here, we suggest three potential applications of the set-valued uncertain process. We believe it has more practical applications and will open new lines of research to interested scholars and practitioners.

#### 9.1 Uncertain differential inclusion

Differential inclusion

\[
\dot{x}(t) \in F(x(t)),
\]

is defined as an extension of the ordinary differential equation where for each \( x(t), F(x(t)) \) is a set. It reduces to ordinary differential equation when \( F(x(t)) \) is a singleton. With continuous right-hand side in an ordinary differential equation, it is natural to define solutions as continuously differentiable functions satisfying the equation in all points over some time interval. However, in many applied problems we have to consider differential inclusions with an upper semi-continuous right-hand side to guarantee the existence of solutions. Upper semi-continuous differential inclusions are mainly applied in the theory of differential equations with discontinuous right-hand side. Some well-known applications are systems with dry friction, electronic oscillators, and autopilot systems among many other applications. We refer to (Smirnov 2022, Chapter 4) for a rich discussion on the solution existence of a differential inclusion with different features. For the case of infinite-dimensional differential inclusion, one can refer to Tolstonogov (2012).

In the conjunction of stochastic events and set-valued evolving phenomena, *stochastic differential inclusions* are defined as a special form of stochastic differential equations for a theoretical description of stochastic control problems (Sosulski 2001).

Recall that one of the well-known stochastic differential equations is the classical Ito’s type equation,

\[
\begin{align*}
\dot{x}_t &= b(y, x_t) + \omega(t, x_t)dB_t, \\
\dot{x}_0 &= \eta,
\end{align*}
\]

with the stochastic integral form

\[
\begin{align*}
x_t &= \eta + \int_0^t b(s, x_s)ds + \int_0^t \sigma(s, x_s)dB_s, \\
\end{align*}
\]

where \( B_t, t \in [0, T] \) is a Brownian motion, \( b : [0, T] \times \Re \to \Re, \omega : [0, T] \times \Re \to \Re \) are Borel measurable functions with some conditions. It is widely used in the stochastic control and financial mathematics. It is based on the Wiener process and has an almost mature theory.

Observe that in complex systems, we can not assign an exact value \( x_t \) in (66) at time \( t \), but we may know it takes values in a set. In this case, we are faced with stochastic differential inclusion as

\[
dX_t = b(t, X_t)dt + \omega(t, X_t)dB_t,
\]

where \( b(t, X_t) \) and \( \omega(t, X_t) \) are closed subsets of \( \Re \). This equation can be written as the following set-valued stochastic integral form

\[
X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \omega(s, X_s)dB_s.
\]

Observe that the left- and right-hand sides of (68) are set, and therefore,

\[
x_t \in X_0 + \int_0^t b(s, X_s)ds + \int_0^t \omega(s, X_s)dB_s,
\]

refers to a sample (stochastic) path. We refer the interested reader to Kisielewicz (2013) for more details.

Here, we emphasize that all uncertain phenomena do not obey the probability theory principles, and one should employ other mathematical frameworks to deal with such situations. Liu process has been defined as a counterpart to the Wiener process when the indeterminacy of the environment is dealt with uncertainty theory. The Liu process is a stationary independent increment process whose increments are normal uncertain variables (Liu 2009). Therefore, counterpart uncertain differential equation to (67) is defined as

\[
dx_t = f(t, x_t)dt + g(t, x_t)dC_t,
\]
where $C_t$ is a Liu process and $f$ and $g$ are two functions. Observe that (69) has integral form as

$$x_t = x_0 + \int_0^t f(s, x_s)ds + \int_0^t g(s, x_s)dC_s. \quad (70)$$

Analogous to (68), one may be interested in defining uncertain differential inclusion that may have the following form

$$dX_t = F(t, X_t)dt + G(t, X_t)dC_t,$$ \quad (71)

where $F(t, X_t)$ and $G(t, X_t)$ are two set-valued functions from $[0, T] \times \mathbb{R}$ to $2^\mathbb{R}$. Considering its integral form,

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \omega(s, X_s)dC_s, \quad (72)$$

each

$$x_t \in X_0 + \int_0^t F(s, x_s)ds + \int_0^t G(s, x_s)dC_s,$$

denotes an uncertain sample path as defined in Definition 2. It means that the solution of (71) could be expressed as a set-valued uncertain process, as we introduced in this study.

Observe that integrands in (68) and (72) are set-valued functions. Their integrals are defined in the sense of Aumann integration (Aumann 1965). To keep this paper self-contained, we mention the integral of a set-valued function in a nutshell.

For each $s$ in interval $[0, t]$, let $F(s)$ be a nonempty subset of $\mathbb{R}$. Let $\mathcal{F}$ be the set of all point-valued functions $f$ from $[0, t]$ to $\mathbb{R}$ such that $f$ is integrable over $[0, t]$ and $f(s) \in F(s)$ for all $s \in [0, t]$. Define

$$\int_0^t F(s)ds = \left\{ \int_0^t f(s)ds : f(s) \in \mathcal{F} \right\},$$ \quad (73)

that is, the set of all integrals of members of $\mathcal{F}$. For the notational sake, let us denote the left-hand side of (73) by $\int_T F$. It is proved that $\int_T F$ is convex, and if $F$ is Borel measurable and integrably bounded, then $\int_F F$ is nonempty. From Borel measurable, we mean its graph $\{(s, x)|x \in F(s)\}$ is a Borel subset of $[0, t] \times \mathbb{R}$, and from integrably bounded, we mean there is a point-valued integrable function $h$ from $[0, t]$ to $\mathbb{R}$ such that $|x| \leq h(s)$ for all $x$ and $s$ such that $x \in F(s)$. For more details, we refer to Aumann (1965).

Similar to the uncertain differential equations, which have many applications in finance, the behavior of financial instruments and their pricing can be studied with the inclusion relation instead of equality. For an intuitive example, consider the European call option that gives the holder the right to buy a stock at an expiration time $s$ for a strike price $K$. This problem has been investigated based on uncertainty theory by Liu (2009). However, characterizing a single value for its price during a working day would rarely be possible in practice. It would be preferred to consider an uncertain set with a membership function depicting its price throughout the day. In this way, its price is a set-valued uncertain process over the exercising period, and equality must be replaced with inclusion in corresponding relations. This new paradigm needs more development to address such practical problems in financial mathematics.

### 9.2 Self-deriving vehicles

Managing self-driving cars is recently one of the most studied fields, developed by many universities, research centers, car companies, and companies of other industries all over the globe. The architecture of the autonomy system of these vehicles is typically systematized into the perception system and the decision-making system. Typically, the perception system is divided into several subsystems responsible for tasks such as localization, static obstacles mapping, moving obstacles detection and tracking, road mapping, and traffic signalization detection and recognition, among others. Detecting environmental conditions, such as slippery road situations and weather conditions, is essential in the perception system. In some situations, they must deal with freezing weather, heavy snowfalls, and slippery roads during execution.

Let us focus on slippery roads. Though the installed road signs could be detected via cameras, they have diverse meanings on different days of the year, suggesting that the concept of “slippery” could be expressed as an uncertain set evolving over time. To fortify the detection of slippery signs and their effect on control systems, the tools provided by the set-valued uncertain process would be effective.

The decision-making system is commonly partitioned into many subsystems devised to execute tasks such as route planning, path planning, behavior selection, motion planning, and control. This system commonly uses artificial intelligence via deep learning as an inference system. If this uncertain system is handled by the uncertainty theory framework (Liu 2010), one is faced with uncertain logic that is essentially defined based on uncertain sets. Let us consider a simple uncertain rule as

“if $X$ is $\xi$ then $Y$ is $\eta$’’

where $\xi$ and $\eta$ are two uncertain sets. Specifically, we may say

“if the weather is rainy then the road is slippery.’’

Here, $X = \text{weather}$, $Y = \text{road}$, $\xi = \text{rainy}$, and $\eta = \text{slippery}$. A control system may use sharp boundaries for considering the weather as rainy and the road as slippery, while identifying
these sharp boundaries is almost impossible and, in some situations, unreliable. The other option is to describe these concepts by uncertain sets. Thus, an introductory rule could be as

\[
\text{Rule: If } X \text{ is } \xi \text{ then } Y \text{ is } \eta \\
\text{From: } \xi \text{ is a constant } a \\
\text{Infer: } Y \text{ is } \eta^* = \eta|_{a \in \xi} \quad (74)
\]

It is proved that \( \eta^* \) has membership function (Liu 2020, Theorem 10.1). Recall that the concepts of rainy and slippery depend on time, respectively; they can be expressed as set-valued uncertain processes. In this case, one needs to develop uncertain inference rules to cover such assumptions and make the decision-making system more realistic. We hope this opens a new research horizon for the control system of self-driving vehicles.

### 9.3 Linear temporal logic

In logic, linear temporal logic (Huth and Ryan 2004) (LTL) is a modal temporal logic with modalities referring to time. It is a mathematical language describing linear-time propositions where evaluation occurs within a set of worlds. A proposition may be true in some worlds but not in others. Observe that the set of worlds corresponds to moments in time. In LTL, one can encode formulae about the future of paths, e.g., “a condition will eventually be true” and “a condition will be true until another fact becomes true.” It is widely used in program verification, motion planning in robotics, and process mining, among many other areas. One of the applications is vehicle trajectory prediction, which is mentioned in the previous subsection.

There are unitary and binary operators in LTL. Typical temporal unitary operators are \( \bigcirc \phi \), which means \( \phi \) is true in the next moment in time; \( \square \phi \) means \( \phi \) is true in all future moments, or a property is satisfied now and forever into the future; and \( \Diamond \phi \), \( \phi \) is true in some future moment; or “eventually”, a property is satisfied at some point in the future. An example of a binary operator is \( \phi U \psi \) among some others, which means that \( \phi \) is true until \( \psi \) is true, which must hold at the current or a future position.

The primary assumption of LTL is that each moment in time has a well-defined successor moment. However, this is not the case in practice. For example, consider the proposition “The patient is in good condition.” This proposition would be an inference from the parameters collected by some sensors and inferred by a specialist physician. Observe that this proposition would be true today but not the other days in the future because its value depends on some parameters that vary over time.

If these parameters are sensed in hospitals, they would be considered reliable by a high degree of belief. Remote patient monitoring devices are ubiquitous nowadays, and the data can automatically be transferred to a decision-making team. Many factors prevent the specialist from relying on these data with a higher degree of belief. For example, heart beating rates are collected regularly by a smartwatch, which are different values at different moments, depending on the patient’s conditions. If the specialist wants to use a proposition like “Heart is working normally” concluding from these data, the value of this proposition depends on the concept of “normality” which is not a sharp concept and may suggest denoting it as an uncertain set with approximated membership function. The final conclusion will also be an uncertain set with a membership function. Since the assumptions of the inference vary over time, the conclusion follows the same manner. Here is the situation we are faced with set-valued uncertain processes. It is expected that a combination of set-valued uncertain process and the rich theory of LTL would result in more realistic outcomes.

### 10 Concluding remarks

We introduced the notion of a set-valued uncertain process. Set-valued sample path and its LSC and openness properties were established. The membership function of a set-valued uncertain process was also defined. The inverse membership function was explained, and sufficient and necessary condition of being an inverse membership function was also constructed. Set and arithmetic operational laws were posed. The expected sample path was introduced, and its difference with the set-valued sample path was clarified. Additionally, a set-valued phase-independent uncertain process was defined that helped us to propose different convergence notions of a set-valued uncertain process. Some potential application of the paradigm was mentioned, too.

As a future work direction, we are interested in investigating more concepts in this paradigm, such as independent increment, extreme values theorem, first hitting time, time integral, and stationary increment for the set-valued uncertain process.

### Funding
The authors have not disclosed any funding.

### Data Availability
No data are used in this study.

### Declarations

**Conflict of interest** The authors declare that they have no conflicts of interest.

**Ethical approval** The research involves no human participants and animals and consequently no need for informed consent.
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