THE POSITIVE PART OF THE QUANTIZED UNIVERSAL ENVELOPING ALGEBRA OF TYPE $A_n$ AS A BRAIDED QUANTUM GROUP

CÉSAR BAUTISTA

Abstract. A generalized Hopf algebra structure for the positive (negative) part of the Drinfeld-Jimbo quantum group of type $A_n$ is established without make any use of the usual deformation of the abelian part of $sl_{n+1}$.

1. Introduction

The aim of this paper is to explain the bialgebra structure of the positive part of the quantized universal enveloping algebra (Drinfeld-Jimbo quantum group) of type $A_n$ using the Lie algebra theory concepts.

Recently has been introduced a generalization of Lie algebras, the basic $T$-Lie algebras [1]. Using the $T$-Lie algebra concept some new (we think) quantum groups of type $A_n$ can be constructed. Such quantum groups arise as universal enveloping algebras of certain deformations as generalized Lie algebras of the Lie algebras form by upper triangular matrices $sl_{n+1}$.

Let us explain, embedded in the positive (negative) parts $U_q(sl_{n+1})$ of the Drinfeld-Jimbo quantum groups of type $A_n$ there are some generalized Lie algebras $(sl_{n+1}^+)_q$ called $T$-Lie algebras, [1]. Such $T$-Lie algebras satisfy not only a generalized antisymmetry and a generalized Jacobi identity, but an additional condition called multiplicativity. Through these $T$-Lie algebras the Poincaré-Birkhoff-Witt theorem for $U_q^\pm(sl_{n+1})$ can be explained.

The Poincaré-Birkhoff-Witt theorem is a general property for the universal enveloping algebras of adequate $T$-Lie algebras. In order to keep the proof of such theorem closest to the classical one [2] a Gurevich’s condition of multiplicativity is needed, [3], [4].

On the other hand, the next natural step in the study of $(sl_{n+1}^+)_q$ as a $T$-Lie algebra is to give to its universal enveloping algebra a structure of Hopf algebra.

Now a structure as a generalized Hopf algebra (braided quantum group) of the universal enveloping algebra $U_q^+(sl_{n+1})$ of $(sl_{n+1}^+)_q$ is presented. As a matter of fact, $U_q^+(sl_{n+1})$ has the usual algebra structure: generators $x_1, \ldots, x_n$ and relations

\[x_ix_j - x_jx_i = 0, \text{ if } |i - j| > 1\]  \hspace{1cm} (1.1)
\[x_i^2 - (q + q^{-1})x_ix_i + x_i = 0, \text{ if } |i - j| = 1\]  \hspace{1cm} (1.2)

but now the tensorial product $U_q^+(sl_{n+1}) \otimes U_q^+(sl_{n+1})$ has a non-standard algebra structure: the multiplication is given by

\[(a \otimes b)(c \otimes d) = aa(\sigma b \otimes c)d\]
where \( \sigma : U_q^+ (sl_{n+1}) \otimes U_q^+ (sl_{n+1}) \to U_q^+ (sl_{n+1}) \otimes U_q^+ (sl_{n+1}) \) is not the usual switch. This is so, because \( U_q^+ (sl_{n+1}) \) has a coproduct given by
\[
\phi(x_i) = x_i \otimes 1 + 1 \otimes x_i, \quad i = 1, \ldots, n.
\]
We obtain a non-trivial deformation of the classical universal enveloping algebra \( U(sl_{n+1}) \) without make any use of the usuals \( K^\pm_1, \ldots, K^\pm_2 \) (which they form a deformation of the abelian part of \( sl_{n+1} \)). Such developments leads to the following question: are there any non-trivial deformation of \( U_q(sl_{n+1}) \) constructed without the usuals \( K^\pm_1, \ldots, K^\pm_2 \)? We do not have an answer yet.

2. Braids and Coproduct

Let \( k \) be a unitary commutative ring and \( q \in k^* \). Denote with \( e_{ij}, 1 \leq i, j \leq n \) canonical basis of \( gl_n \), the matrices \( n \times n \), with \( B \) the canonical basis of \( sl_n^+ \), and with \([,]\) the usual bracket in \( gl_n \). Define \( m = n(n+1)/2 \) and \( c_{ij,ab} \in \mathbb{Z}, 1 \leq i < j \leq m, 1 \leq a < b \leq m \) such that \([e_{ij}, e_{ji}], e_{ab}] = c_{ij,ab}e_{ab}. \) Put
\[
e_{ij} < e_{ab} \text{ if } i+j < a+b \text{ or } (i+j = a+b \text{ and } j < b).
\]
Let \( (sl_{n+1})^+_q \) be \( sl_n^+ \) with structure of \( T \)-Lie algebra. This means: \( (sl_{n+1})^+_q \) is \( sl_n^+ \) in its structure of \( k \)-module, besides a bracket \([,]\)_q such that
\[
[e_{ij}, e_{ab}]_q = [e_{ij}, e_{ab}], \quad \text{if } e_{ij} \leq e_{ab},
\]
a \( k \)-linear morphism \( S : (sl_{n+1})^+_q \otimes_k (sl_{n+1})^+_q \to (sl_{n+1})^+_q \otimes_k (sl_{n+1})^+_q \) called presymmetry such that
\[
[\cdot,\cdot]_q S = -[\cdot,\cdot]_q
\]
defined by
\[
S(e_{ij} \otimes e_{ab}) = q^{c_{ij,ab}}e_{ab} \otimes e_{ij}, \quad \text{if } e_{ij} < e_{ab}
\]
\[
S(e_{ij} \otimes e_{ij}) = e_{ij} \otimes e_{ij}, \quad S^2 = 1
\]
and a \( k \)-linear morphism \( \langle,\rangle : (sl_{n+1})^+_q \otimes_k (sl_{n+1})^+_q \to (sl_{n+1})^+_q \otimes_k (sl_{n+1})^+_q \) called pseudobracket such that
\[
[\cdot,\cdot]_q \langle,\rangle = 0, \quad \langle,\rangle S = -\langle,\rangle,
\]
in the case \( e_{ij} < e_{ab} \) defined by
\[
\langle e_{ij}, e_{ab} \rangle = \begin{cases} (q - q^{-1})e_{aj} \otimes e_{ab}, & \text{if } a < j < b \text{ and } i < a < j, \\ 0, & \text{otherwise.} \end{cases}
\]
The algebra \( (sl_{n+1})^+_q \) satisfies generalized Lie algebra axioms, see [1].

On the other hand, since \([e_{ij}, e_{ab}], e_{ab}] = \delta_{jk}e_{al} - \delta_{jl}e_{kj} \) and \( \delta \) is the Kronecker delta, we get
\[
e_{ij,ab} = -\delta_{b,i} + \delta_{b,j} + \delta_{i,a} - \delta_{j,a}, \quad \forall e_{ij}, e_{ab} \in B
\]
(2.1)
We shall define a new symmetry \( \sigma \). Let \( \sigma : (sl_{n+1})^+_q \otimes_k (sl_{n+1})^+_q \to (sl_{n+1})^+_q \otimes_k (sl_{n+1})^+_q \) such that
\[
\sigma(e_{ij} \otimes e_{ab}) = q^{c_{ij,ab}}e_{ab} \otimes e_{ij}, \quad \forall e_{ij}, e_{ab} \in B.
\]
Proposition 2.1. Let $[,]_q$ be the bracket of $(sl_{n+1}^*)_q$. The linear morphism $\sigma$ satisfies the multiplicativity conditions:

$$\sigma(1 \otimes [,]_q) = ([,]_q \otimes 1)\sigma_{23}\sigma_{12},$$  \hspace{1cm} (2.2)

$$\sigma([,]_q \otimes 1) = (1 \otimes [,]_q)\sigma_{12}\sigma_{23}.$$  \hspace{1cm} (2.3)

**Proof.** The Jacobi identity in $sl_{n+1}^*$ ensures that

$$[[e_{ij}, e_{jq}], [e_{ab}, e_{uv}]] = (c_{ij,ab} + c_{ij,uv})[e_{ab}, e_{uv}]$$

this implies

$$\sigma(e_{ij}\otimes[e_{ab},e_{uv}]_q) = q^{c_{ij,ab}+c_{ij,uv}}[e_{ab}, e_{uv}]_q \otimes e_{ij}$$

$$= ([,]_q \otimes 1)\sigma_{23}\sigma_{12}(e_{ij} \otimes e_{ab} \otimes e_{uv}).$$ \hspace{1cm} (2.4)

In a similar way we obtain

$$\sigma([e_{ij}, e_{ab}]_q \otimes e_{uv}) = (1 \otimes [,]_q)\sigma_{23}\sigma_{12}(e_{ij} \otimes e_{ab} \otimes e_{uv})$$

\hfill \Box

Now we may extend $\sigma$ to $U_q^+(sl_{n+1})$. To do so, let us consider $T$ as the $k$-tensorial algebra of $(sl_{n+1}^*)_q$. Define $\hat{\sigma} : T \otimes T \rightarrow T \otimes_k T$ by

$$\hat{\sigma}(z_1 \ldots z_n \otimes y_1 \ldots y_m) = \prod_{i,j} p_{zi,yj} y_1 \ldots y_m \otimes z_1 \ldots z_m, \forall z_i, y_j \in B,$$

where $\sigma(z_i \otimes y_j) = p_{zi,yj} y_j \otimes z_i, p_{zi,yj} \in k$.

In particular, the objects made by finite tensor products of one dimensional spaces $(x), x \in B$ form a braided tensor category with braiding $\hat{\sigma}$, [1, Chapter 11] and the proposition 2.1 says that the bracket $[,]_q$ is compatible with the braiding $\hat{\sigma}$.

**Lemma 2.2.** Let $\langle , \rangle$ be the pseudobracket of $(sl_{n+1}^*)_q$ and $m$ the product of the tensor algebra $T$. The linear morphism $\sigma$ satisfies the multiplicativity conditions

$$\hat{\sigma}(1 \otimes m(\langle , \rangle)) = (m(\langle , \rangle) \otimes 1)\hat{\sigma}_{23}\hat{\sigma}_{12},$$ \hspace{1cm} (2.5)

$$\hat{\sigma}(m(\langle , \rangle) \otimes 1) = (1 \otimes m(\langle , \rangle))\hat{\sigma}_{12}\hat{\sigma}_{23}. \hspace{1cm} (2.6)$$

**Proof.** Let $e_{ab}, e_{uv}, e_{\alpha\beta}$ be elements of $B$. We have to prove

$$\hat{\sigma}(e_{ab} \otimes m(e_{uv}, e_{\alpha\beta})) = (m(\langle , \rangle) \otimes 1)\hat{\sigma}_{23}\hat{\sigma}_{12}, \hspace{1cm} (2.7)$$

$$\hat{\sigma}(m(e_{uv} \otimes e_{\alpha\beta}) \otimes e_{ab}) = (1 \otimes m(\langle , \rangle))\hat{\sigma}_{12}\hat{\sigma}_{23}(e_{uv} \otimes e_{\alpha\beta} \otimes e_{ab}). \hspace{1cm} (2.8)$$

The left side of (2.7) is

$$q^{c_{ab,uv}+c_{ab,u\beta}+c_{ab,\alpha\beta}}(q - q^{-1})e_{uv} \otimes e_{uv} \otimes e_{ab}$$

while the right side is

$$q^{c_{ab,uv}+c_{ab,u\beta}+c_{ab,\alpha\beta}}(e_{uv} \otimes e_{\alpha\beta} \otimes e_{ab})$$

Since (2.1) we obtain $c_{ab,uv} + c_{ab,u\beta} = c_{ab,uv} + c_{ab,\alpha\beta}$. We conclude that (2.7) holds. And by similar calculations (2.8) holds too. \hfill \Box
Proposition 2.3. The linear morphism $\tilde{\sigma}$ can be extended to a linear morphism
\[ \sigma : U^+_q(sl_{n+1}) \otimes_k U^+_q(sl_{n+1}) \to U^+_q(sl_{n+1}) \otimes_k U^+_q(sl_{n+1}) \]
where $U^+_q(sl_{n+1})$ is the universal enveloping algebra of $(sl_{n+1}^+)_q$, such that the following diagram commutes,
\[ \begin{array}{ccc}
U^+_q(sl_{n+1}) \otimes_k U^+_q(sl_{n+1}) & \xrightarrow{\sigma} & U^+_q(sl_{n+1}) \otimes_k U^+_q(sl_{n+1}) \\
\uparrow & & \uparrow \\
(sl^+_{n+1})_q \otimes_k (sl^+_{n+1})_q & \xrightarrow{\sigma} & (sl^+_{n+1})_q \otimes_k (sl^+_{n+1})_q
\end{array} \]
besides the multiplicativity conditions
\[ \sigma(1 \otimes m) = (m \otimes 1)(1 \otimes \sigma)(\sigma \otimes 1) \quad (2.9) \]
\[ \sigma(m \otimes 1) = (1 \otimes m)(\sigma \otimes 1)(1 \otimes \sigma) \quad (2.10) \]
holds, where $m$ denotes the multiplication on $U^+_q(sl_{n+1})$.

Now we may apply some remarks from Durdević [3]. Suppose that we are in the conditions of the proposition 2.3. The tensorial product $U(L) \otimes U(L)$ is a natural $U(L)$-bimodule and we can define a multiplication over $U(L) \otimes U(L)$ by
\[ (a \otimes b)(c \otimes d) = a\sigma(b \otimes c)d \quad (2.11) \]

Corollary 2.4. $U^+_q(sl_{n+1}) \otimes_k U^+_q(sl_{n+1})$ is an associative algebra with multiplication defined by equation (2.11).

Proof. The multiplicativity conditions (2.9), (2.10) imply the associativity of (2.11), [3] \qed

In a similar way to the classical case define $\phi : (sl^+_{n+1})_q \to U^+_q(sl_{n+1}) \otimes_k U^+_q(sl_{n+1})$ by
\[ \phi(e_{i(i+1)}) = e_{i(i+1)} \otimes 1 + 1 \otimes e_{i(i+1)}, \quad 1 \leq i \leq n. \quad (2.12) \]

Theorem 2.5. The morphism defined by (2.12) can be extended to a morphism of $k$-algebras
\[ \phi : U^+_q(sl_{n+1}) \to U^+_q(sl_{n+1}) \otimes_k U^+_q(sl_{n+1}) \]

Proof. The $k$-algebra $U^+_q(sl_{n+1})$ is generated by $x_i = e_{i(i+1)}$, $i = 1, \ldots, n$ modulo the relations
\[ x_i^2 - (q + q^{-1})x_i = 0 \quad (2.13) \]
\[ x_i x_j - x_j x_i = 0, \quad 1 \leq i \neq j \leq n. \quad (2.14) \]
We have to prove that $\phi$ preserves these relations. Let us put $\sigma(x_i \otimes x_j) = q^{c_{ij}} x_j \otimes x_i$, $1 \leq i, j \leq n$.
\[ \phi(x_i)^2 \phi(x_j) = x_i^2 \otimes 1 + x_i^2 \otimes x_j + q^{c_{ij}} x_i x_j \otimes x_i + x_i \otimes x_i x_j + q^{2c_{ij}} x_i x_j \otimes x_i + q^2 x_i \otimes x_i x_j + q^{2c_{ij}} x_j \otimes x_i + x_i^2 x_j \]
\[ + q^{2c_{ij}} x_j \otimes x_i + x_i^2 x_j \]
while
\[ \phi(x_j)\phi(x_i)^2 = x_jx_i^2 \otimes 1 + x_jx_i \otimes x_i + q^2 x_j x_i \otimes x_i + x_j \otimes x_i^2 + q^{2c_{ij}} x_i^2 \otimes x_j + q^{2+\epsilon_{ij}} x_j \otimes x_j x_i \]

and
\[ \phi(x_i)\phi(x_j)\phi(x_i) = x_i x_j x_i \otimes 1 + x_i x_j \otimes x_i + q^{c_{ij}} x_i^2 \otimes x_j + x_i \otimes x_j x_i \]
\[ + q^{c_{ij}+2} x_j x_i \otimes x_i + q^{c_{ij}} x_j \otimes x_i^2 \]
\[ + q^{c_{ij}+2} x_i x_j + 1 \otimes x_i x_j x_i \]
(2.15)

It follows
\[ \phi(x_i)^2 \phi(x_j) + \phi(x_j) \phi(x_i)^2 = q^{-1}(q + q^{-1})(x_i^2 \otimes x_j) + (q^{-1} + q)x_i \otimes x_i x_j + (q + q^{-1})x_j \otimes x_i^2 \]
\[ + (q + q^{-1})x_i x_j x_i \otimes 1 + 1 \otimes (q + q^{-1})x_i x_j x_i \]
\[ = (q + q^{-1})\phi(x_i)\phi(x_j)\phi(x_i) \]
(2.16)

We shall define the counit. This can be done following the classical case.

The commutative ring \( k \) is a basic \( T \)-Lie algebra in the obvious way and the zero morphism \( 0 : (sl_{n+1}^+) \rightarrow k \) is a morphism \( \epsilon = U(0) : U_q^+(sl_{n+1}) \rightarrow U(k) \simeq k \) of \( k \)-algebras.

**Definition 2.6.**
\[ x_i = e_{i(n+1)}, \quad i = 1, \ldots, n. \]

**Proposition 2.7.** Let \( C = \{x_1, \ldots, x_n\} \). Then
\[ \phi(x_{i_1}) \cdots \phi(x_{i_m}) = x_{i_m} \cdots x_{i_1} \otimes 1 + \sum_j u_j \otimes v_j = 1 \otimes x_{i_1} \cdots x_{i_m} + \sum_l a_l \otimes b_l \]
where each \( u_j, v_j, a_l, b_l \) is a non-empty product of basic elements in \( C \). It follows
\[ (1 \otimes \epsilon)\phi(x_{i_1}) \cdots \phi(x_{i_m}) = x_{i_1} \cdots x_{i_m} = (\epsilon \otimes 1)\phi(x_{i_1}) \cdots \phi(x_{i_m}). \]

Since \( C \) is a generator set of \( U_q^+(sl_{n+1}) \) we get that \( \epsilon \) is the counit for \( \phi \).

3. **Antipode**

Let \( L \) be \( (sl_{n+1}^+) \) as a \( T \)-Lie algebra.

**Definition 3.1.**
1. The **opposite** \( T \)-Lie algebra \( L^{op} \) is defined as
\[ [\cdot, \cdot]^{op} = [\cdot, \cdot]_q, \quad S^{op} = S, \quad (\cdot)^{op} = \sigma^{-1}(\cdot) \]
2. Let \( m \) be the product of \( U(L) \). The opposite algebra \( U(L)^{op} \) is defined as \( U(L) \) itself in its \( k \)-module structure and product given by
\[ m^{op} = m\sigma. \]
Proposition 3.2. Let $L$ be $(st^+_{n+1})_q$.
1. The $k$-algebra $U(L)^{op}$ is associative and $L^{op}$ is a basic $T$-Lie algebra.
2. The map $\eta : L \to L$, $x \mapsto -x$ is a $T$-Lie algebra morphism.
3. There exist an isomorphism
$$U(L^{op}) \simeq U(L)^{op}$$
of $k$-algebras.

Proposition 3.3. There exist a morphism
$$\eta : U(L) \to U(L)^{op}$$
of $k$-algebras such that $\eta(x) = -x$, $\forall x \in L$.

Now, let us consider the quantum plane $A_{q^2|0}$ defined by the ring
$$A_{q^2|0} = k\langle x, y \rangle/yx - qxy$$
where $k\langle x, y \rangle$ means an associative algebra freely generated by $x, y$.

For positive integers $i \leq n$, we define the numbers
$$(m_i)_q$$
by the equation in $A_{q^2|0} = k\langle x, y \rangle/yx - qxy$,
$$(x + y)^m = \sum_{i=0}^{m} \left(\begin{array}{c} m \\ i \end{array}\right)_q x^{m-i} y^i$$

Lemma 3.4.
$$\sum_{i=0}^{m} \left(\begin{array}{c} m \\ i \end{array}\right)_q (-1)^i q^{i(i-1)} = 0$$
(3.1)

Proof: If $m = 1$ the equation (3.1) holds. Now, suppose (3.1). Then,
$$\sum_{i=0}^{m+1} \left(\begin{array}{c} m+1 \\ i \end{array}\right)_q (-1)^i q^{i(i-1)} = \left(\begin{array}{c} m+1 \\ 0 \end{array}\right)_q + \sum_{i=1}^{m} \left(\begin{array}{c} m \\ i \end{array}\right)_q (-1)^i q^{i(i-1)} + \left(\begin{array}{c} m+1 \\ m+1 \end{array}\right)_q (-1)^{m+1} q^{(m+1)m}$$
(3.2)
and because
$$\left(\begin{array}{c} m+1 \\ i \end{array}\right)_q = \left(\begin{array}{c} m \\ i-1 \end{array}\right)_q + q^i \left(\begin{array}{c} m \\ i \end{array}\right)_q$$
for $1 \leq i \leq m$,
(see [3, p. 74]) then (3.2) is equal to
$$1 + \sum_{i=1}^{m} \left(\begin{array}{c} m \\ i-1 \end{array}\right)_q (-1)^i q^{i(i-1)} + \sum_{i=1}^{m} \left(\begin{array}{c} m \\ i \end{array}\right)_q (-1)^i q^{i^2+i} + (-1)^{n+1} q^{n(n+1)n}$$
$$= 1 + \sum_{j=0}^{m} \left(\begin{array}{c} m \\ j \end{array}\right)_q (-1)^{j+1} q^{(j+1)j} + \sum_{i=1}^{m} \left(\begin{array}{c} m \\ i \end{array}\right)_q (-1)^i q^{i(i+1)} + (-1)^{n+1} q^{n(n+1)}$$
$$= 0$$
\qed
Lemma 3.5. If $\iota : U(L)^{op} \to U(L)$ is the natural $k$-module morphism then

1. $\phi(x_j)^m = \sum_{i=0}^{m} \binom{m}{i} q^{i} x_j^{m-i} \otimes x_j^i$, $1 \leq j \leq n$

2. $(x_{j_1}^{n_1-1} \otimes x_{j_2}^{n_2-1}) (x_{j_2}^{n_2-2} \otimes x_{j_2}^{n_2-2}) \ldots (x_{j_u}^{n_u-1} \otimes x_{j_u}^{n_u-1}) = q \sum_{a < b} c_{ja_jb}^{i_u} x_{j_1}^{n_1-1} x_{j_2}^{n_2-2} \ldots x_{j_u}^{n_u-1}$

3. $(1 \otimes \eta)(x_{j_1}^{n_1-1} \otimes x_{j_2}^{n_2-2} \ldots x_{j_u}^{n_u-1} \otimes x_{j_u}^{n_u-1}) = q \sum_{a < b} c_{ja_jb}^{i_u} x_{j_1}^{n_1-1} x_{j_2}^{n_2-2} \ldots x_{j_u}^{n_u-1} \otimes (1 \otimes \eta(x_{j_1}^{i_1}) \ldots \eta(x_{j_1}^{i_1}))$

4. For $j = 1, \ldots, n$, $\eta(x_j) = (-1)^{j} q^{l(i-1)(\ell x_j)^{i}}$

5. Denote with $m$ the product of $U_q^+(sl_{n+1})$. Then $m(1 \otimes \eta)(x_{j_1}^{i_1} \ldots x_{j_u}^{i_u}) = 0$, $1 \leq j_1, \ldots, j_u \leq n$.

   if $n_1, \ldots, n_u$ are all positive integers.

Proof. By straightforward computations. By example, if $c = \sum_{a < b} c_{ja_jb}^{i_u} x_{j_u}^{n_u-1}$ then

$$m(1 \otimes \eta)(x_{j_1}^{i_1} \ldots x_{j_u}^{i_u}) = m(1 \otimes \eta)(x_{j_1}^{n_1} \ldots x_{j_u}^{n_u}) = \sum_{i_1, \ldots, i_u=0}^{n_1, \ldots, n_u} \binom{n_1}{i_1} \ldots \binom{n_u}{i_u} q^{i_1} m(x_{j_1}^{n_1-1} \ldots x_{j_u}^{n_u-1} \otimes \eta(x_{j_1}^{i_1} \ldots x_{j_1}^{i_1}))$$

$$= \sum_{i_1, \ldots, i_u=0}^{n_1, \ldots, n_u-1} \binom{n_1}{i_1} \ldots \binom{n_u}{i_u} q^{i_1} \sum_{a < b} c_{ja_jb}^{i_u} x_{j_1}^{n_1-1} \ldots x_{j_u}^{n_u-1} \eta(x_{j_1}^{i_1}) \ldots \eta(x_{j_1}^{i_1})$$

$$= \sum_{i_1, \ldots, i_u=0}^{n_1, \ldots, n_u-1} \binom{n_1}{i_1} \ldots \binom{n_u}{i_u} q^{i_1} \sum_{a < b} c_{ja_jb}^{i_u} x_{j_1}^{n_1-1} \ldots x_{j_u}^{n_u-1} \eta(x_{j_1}^{i_1}) \ldots \eta(x_{j_1}^{i_1}) = 0 \quad (3.3)$$

\[ \square \]

Proposition 3.6. Let $\iota : U(L)^{op} \to U(L)$ the natural $k$-morphism. Then, the $k$-module morphism

$$\kappa = \iota \eta : U(L) \to U(L)$$

is the antipode for the coproduct $\phi$.

Proof.
4. The additional condition of the braided quantum group definition

Lemma 4.1. If
\[
(\sigma \otimes 1)(1 \otimes \sigma)(\phi \otimes 1) = (1 \otimes \phi)\sigma; \\
(1 \otimes \sigma)(\sigma \otimes 1)(1 \otimes \phi) = (\phi \otimes 1)\sigma.
\]
then
\[
(\sigma \otimes 1^2)(1 \otimes \phi \otimes 1)(\sigma^{-1} \otimes 1)(1 \otimes \phi) = (1^2 \otimes \sigma)(1 \otimes \phi \otimes 1)(1 \otimes \sigma^{-1})(\phi \otimes 1) \tag{4.3}
\]
Proof. 
\[
(\sigma \otimes 1^2)(1 \otimes \phi \otimes 1)(\sigma^{-1} \otimes 1)(1 \otimes \phi) = ((\sigma \otimes 1)(1 \otimes \phi) \otimes 1)(\sigma^{-1} \otimes 1)(1 \otimes \phi)
\]
\[
= (1 \otimes \sigma^{-1})(1 \otimes \phi)\sigma \otimes 1)(\sigma^{-1} \otimes 1)(1 \otimes \phi)
\]
on the other hand,
\[
(1^2 \otimes \sigma)(1 \otimes \phi \otimes 1)(1 \otimes \sigma^{-1})(\phi \otimes 1) = (1 \otimes (1 \otimes \sigma)(\phi \otimes 1))(1 \otimes \sigma^{-1})(\phi \otimes 1)
\]
\[
= (1 \otimes (\sigma^{-1} \otimes 1)(\sigma \phi)(1) \otimes \sigma^{-1})(\phi \otimes 1)
\]
We conclude that \((4.3)\) holds.

Proposition 4.2. The following equations holds, 
\[
(\sigma \otimes 1)(1 \otimes \sigma)(\phi \otimes 1) = (1 \otimes \phi)\sigma \tag{4.4}
\]
\[
(1 \otimes \sigma)(\sigma \otimes 1)(1 \otimes \phi) = (\phi \otimes 1)\sigma. \tag{4.5}
\]
Proof. Suppose that \(x_{j_1}, \ldots, x_{j_n}\) and \(x_{k_1}, \ldots, x_{k_\nu}\) are elements of \(\{x_1, \ldots, x_n\}\). Put \(x_J = x_{j_1}^{a_1} \ldots x_{j_n}^{a_n}, x_K = x_{k_1}^{a_1} \ldots x_{k_\nu}^{a_\nu}\). Then
\[
(\phi \otimes 1)(x_J \otimes x_K) = \sum_{i_1=0, \ldots, i_n=0}^n \frac{n_1}{i_1} \frac{n_2}{n_2} \cdots \frac{n_\nu}{i_\nu} \frac{q^2}{q^2} q^a_{\sum a < b} c_{a_k b_k} a u(n_{j_b} - i_b)
\]
\[
\sum_{i_1=0, \ldots, i_n=0}^n x_{j_1}^{a_1 - i_1} \ldots x_{j_n}^{a_n - i_n} \otimes x_{j_1}^{i_1} \cdots x_{j_n}^{i_n} \otimes x_{k_1}^{a_1} \cdots x_{k_\nu}^{a_\nu}
\]
it follows,
\[
(1 \otimes \sigma)(\phi \otimes 1)(x_J \otimes x_K) = \sum_{i_1=0, \ldots, i_n=0}^n \frac{n_1}{i_1} \frac{n_2}{n_2} \cdots \frac{n_\nu}{i_\nu} \frac{q^2}{q^2} q^a_{\sum a < b} c_{a_k b_k} a u(n_{j_b} - i_b)
\]
and
\[
(\sigma \otimes 1)(1 \otimes \sigma)(\phi \otimes 1)(x_J \otimes x_K) = \sum_{i_1=0, \ldots, i_n=0}^n \frac{n_1}{i_1} \frac{n_2}{n_2} \cdots \frac{n_\nu}{i_\nu} \frac{q^2}{q^2} q^a_{\sum a < b} c_{a_k b_k} a u(n_{j_b} - i_b)
\]
\[
\sum_{i_1=0, \ldots, i_n=0}^n x_{j_1}^{a_1} \cdots x_{j_n}^{a_n} \otimes x_{j_1}^{i_1} \cdots x_{j_n}^{i_n} \otimes x_{j_1}^{n_1} \cdots x_{j_n}^{n_n} \otimes \phi(x_{k_1} \cdots x_{k_\nu})
\]
By similar calculations \((4.7)\) hold too.
Suppose that $A, B$ are $k$-algebras (non-associative, perhaps), and $\beta : A \otimes_k B \to B \otimes_k A$ a $k$-morphism. Moreover, denote with $A \otimes B$ to $A \otimes_k B$ in its $k$-module structure and product given by

$$(a \otimes b)(c \otimes d) = a\beta(b \otimes c)d$$

Theorem 4.3. The algebra $U_q^+(sl_{n+1})$ is a braided quantum group with
1. coproduct:
$$\phi : U_q^+(sl_{n+1}) \to U_q^+(sl_{n+1}) \otimes U_q^+(sl_{n+1})$$
induced by $\phi(x_i) = x_i \otimes 1 + 1 \otimes x_i$, $i = 1, \ldots, n$;
2. counit:
$$\epsilon : U_q^+(sl_{n+1}) \to k$$
induced by $\epsilon(x) = 0$, $\forall x \in (sl_{n+1}^+)_q$;
3. antipode:
$$\eta : U_q^+(sl_{n+1}) \to U_q^+(sl_{n+1})$$
induced by $\eta(x) = -x$, $\forall x \in (sl_{n+1}^+)_q$.

Acknowledgment

I would like to thank M. Durdević. The existence of a braided product (2.11) in the case $A_n$ was suggested by Durdević, besides the lemma 4.1 belongs to him.

References

[1] C. Bautista, A Poincaré-Birkhoff-Witt theorem for generalized color Lie algebras, preprint q-alg/9706016. Submitted to the J. of Math. Phys.
[2] N. Bourbaki, “Lie Groups and Lie Algebras,” Elements of Mathematics. Chap. 1-3. Springer-Verlag, Great Britain, 1989.
[3] M. Durdević, On braided Quantum Groups, preprint q-alg/9412003.
[4] D. Gurevich, The Yang-Baxter Equation and a Generalization of Formal Lie Theory, Soviet Math. Dokl. 33 (1986), No. 758-762.
[5] C. Kassel, “Quantum Groups,” GTM 155. Springer-Verlag, New York, 1995.
[6] R. Street. “Quantum Groups: an entrée to modern algebra, Preliminary version. Macquarie University. 1993-1994.