A New Ternary Interpolatory Subdivision Scheme for Polyhedral Meshes with Arbitrary Topology

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Abstract Interpolation using ternary subdivision is an attractive characteristic in geometric modeling due to its advantage over commonly used binary subdivision. In this paper, a new ternary interpolatory subdivision scheme with one parameter based on arbitrarily triangular meshes is proposed. It can be used to any polyhedral meshes with arbitrary topology. The sufficient conditions of the convergence and the tangent plane continuity of the presented subdivision scheme are obtained using geometric methods and algebraic manipulations. It is simple, fast and has a better local property in comparison with the existing ternary interpolatory surface subdivision schemes.

1. Introduction

Subdivision schemes have become important for their efficiency, convenience, flexibility and arbitrary topological properties. They can be classified in approximating and interpolating ones. There have been some results in the area of binary interpolatory ones [1-4]. Recently ternary subdivision has attracted much attention [5-12]. Generally speaking, ternary subdivision scheme performs quicker topological refinement than the usual dyadic split operation, so we can get a quicker generation of a subdivision curve or a subdivision surface by using a ternary one than by using a binary one. Ternary interpolatory subdivision scheme unites the advantages of interpolatory subdivision and ternary subdivision. For example, the 4-point ternary interpolatory subdivision scheme [6] can generate higher smoothness than the 4-point binary one [1] by using the same number of control points.

Due to the good properties of the 4-point ternary one, much attention has been given to extend its ability in modeling interpolatory surfaces. Li and Wang et al. generalized it to the ternary interpolatory subdivision schemes based on quadrilateral meshes with arbitrary topology [9-10], respectively. Dodgson et al. generalized it to the ternary interpolatory subdivision scheme based on regular triangular mesh [11], but the actual continuity analysis need to be performed. Zheng and Ye discussed the constructions and analyses of smooth ternary interpolatory subdivision schemes based on regular triangular meshes [12], but the local properties of the subdivision schemes need to be improved.

In this paper, a new ternary interpolatory subdivision scheme with one parameter for triangular meshes with arbitrary topology is proposed and analyzed. It can be used freely to refine any polyhedral meshes with arbitrary topology. The parameter is introduced to control and modify the shape of the subdivision surface. It is shown that the presented subdivision scheme can generate a tangent plane continuous interpolatory subdivision surface.

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2. A new ternary interpolatory subdivision scheme for triangular meshes with arbitrary topology

In this section we propose a new ternary interpolatory subdivision scheme for triangular control meshes with arbitrary topology. Since the subdivision scheme is interpolatory, we leave all the old control points (V-points) unchanged, introduce one new face-point (F-point) for a triangular face, and two new edge-points (E-points) for an old edge in the old control mesh. We use a 1-to-9 triangle connecting rules for every triangular face and the connecting result is illustrated in Fig. 1.

Since we want to develop simple rules for a smooth ternary interpolatory subdivision scheme, the geometric smoothing rules to create new points need to be simple. They are as follows.

(1) Keep the old V-points unchanged.
(2) The new F-point $F$ corresponding to each inner triangular face (see the shadowed region in Fig. 2(a)) is computed by using a 6-neighborhood stencil according to the following expression

$$F = \eta(P_1 + P_2 + P_3) + \kappa(P_4 + P_5 + P_6),$$

where $\eta = \frac{2}{3} - \mu, \kappa = \mu - \frac{1}{3}$, and $\mu$ is the parameter. Its stencil is depicted in Fig. 2(b). While a 12-neighborhood stencil is used in [11] and thus the face-point rule is more complicated than (1).

(3) In the case of open triangular mesh, for triangular face containing one or two boundary edges (see the shadowed region in Fig. 2(c), here boundary points of the mesh are marked by circular hollow dots), where some stencil points for the new F-point rule may not exist, virtual points are introduced to use the normal new F-point rule (2) (see Fig. 2(c), virtual point is marked by square).

$$E_1 = \alpha P_1 + \beta P_2 + \gamma(P_3 + P_4) + \delta(P_5 + P_6),$$
$$E_2 = \beta P_1 + \alpha P_2 + \gamma(P_3 + P_4) + \delta(P_5 + P_6),$$

where $\alpha = \frac{4}{3} - 2\mu, \beta = -\frac{1}{3} + 2\mu, \gamma = \frac{1}{3} - \mu$ and $\delta = -\frac{1}{3} + \mu$. The stencil of $E_1$ is depicted in Fig. 3(b), and that of $E_2$ is depicted in Fig. 3(c). While the new edge-points are computed by using a 10-neighborhood stencil in [11] and a 8-neighborhood stencil in [12] respectively, so both of the edge-point rules are more complicated and can only be used in the case of regular triangular mesh.
Fig. 3 The generation and the stencils of new E-points of an inner edge \((n_i \geq 5 \text{ and } n_j \geq 5)\).

The two new edge-points corresponding to an extraordinary edge (at least one of the two end points \(P_i\) and \(P_j\) are extraordinary points with vertex orders of \(n_i \neq 6\) or \(n_j \neq 6\)) of each inner triangle are generated as follows.

1. If \(n_i \geq 5\), \(E_i\) is still computed by (2); If \(n_j \geq 5\), \(E_j\) is still computed by (3).
2. If \(n_i = 4\), suppose the direct neighbors of \(P_i\) are \(P_2, P_3, P_4\) and \(P_5\) (see Fig. 4(a)), we have
\[
E_i = \alpha P_1 + \beta P_2 + \gamma (P_3 + P_4) + \lambda P_5 ,
\]
where \(\alpha, \beta\) and \(\gamma\) are as before, \(\lambda = \frac{2}{3} + 2\mu\). The stencil of \(E_i\) is depicted in Fig. 4(b).

If \(n_j = 4\), \(E_2\) is computed by using a similar 5-neighborhood stencil (see Fig. 4(c) and (a)) as follows
\[
E_2 = \beta P_1 + \alpha P_2 + \gamma (P_3 + P_4) + \lambda P_5 .
\]

Fig. 4 The generation and the stencils of new E-points of an inner edge \((n_i = n_j = 4)\).

3. If \(n_i = 3\), suppose the direct neighbors of \(P_i\) are \(P_2, P_3\) and \(P_4\) (see Fig. 4(a)), we have
\[
E_i = \alpha_2 P_1 + \beta_2 P_2 + \gamma_3 (P_3 + P_4) ,
\]
where \(\alpha_2 = \frac{26}{27}, \beta_3 = \frac{19}{81}, \gamma_3 = -\frac{8}{81}\). The stencil of \(E_i\) is depicted in Fig. 5(a). If \(n_j = 3\), \(E_2\) is computed by using a similar 4-neighborhood stencil (see Fig. 5(b) and Fig. 4(a)) as follows
\[
E_2 = \beta_1 P_1 + \alpha_3 P_2 + \gamma_3 (P_3 + P_4) .
\]

In the case of open triangular mesh, to get two new E-points for boundary edges, virtual points can be introduced to use the new E-points rules (4) or (5).

Thus, in all the cases we can get new E-points for an edge according to the values of vertex order \(n_i\) and \(n_j\) of the two end points \(P_i\) and \(P_j\) of the edge.
The ternary interpolatory subdivision scheme proposed in this section can be used to refine any polyhedral meshes with arbitrary topology. If the initial control mesh is not triangulated and is a polyhedral mesh with an arbitrary topology involving non-triangular faces (extraordinary faces) and extraordinary vertices in it, we need a pretreatment process to get a triangular mesh. We can attain our goal by introducing a barycenter for each extraordinary face and connecting it to every old points of the extraordinary face. Thus we can generate a triangulation. So the presented subdivision scheme can be used to deal with any polyhedral meshes of arbitrary topology.

3. Analysis of convergence

The analysis of the convergence and the continuity property of the presented subdivision scheme in Section 2 may be done locally by the observation that after one subdivision step, all points sharing neighbors of an extraordinary point are ordinary points (with vertex order of 6). The local region to be studied consist of an extraordinary point \( v \) with vertex order of \( n \) and all the points of the triangular mesh sharing an edge with \( v \) (see Fig. 6). These points shall be contained in the set \( P^0 = \{P^0_0, P^0_1, \ldots, P^0_{n-1}\} \), and \( P^0_{(i+1) \mod n} \) shares an edge with \( P^0_i \). In this way, \( P^0 \) is cyclically ordered. It is always assumed that all subscripts are to be taken \( \text{mod } n \).

**Theorem 1.** If \( n \geq 5 \), when \(-\frac{1}{3} < \mu < \frac{2}{3}\) and \( \max_{j=0,\ldots,\sigma-1} \left| \beta + 2\gamma \cos \frac{2\pi}{n} j + 2\delta \cos \frac{4\pi}{n} j \right| < 1 \), the surfaces generated by the presented subdivision scheme converge at \( v \). If \( n < 5 \), when \( \frac{1}{9} < \mu < \frac{2}{3} \), they converge at \( v \).

Proof. Let \( P^i = \{P^i_0, P^i_1, \ldots, P^i_{n-1}\} \) be the set consisting of the points sharing an edge with \( v \) after \( k \) subdivision steps which depend on only \( v \) and the points in \( P^0 \). \( P^i \) is also cyclically ordered.

**Case 1.** \( n \geq 5 \). In this case the points \( P^i \)'s in \( P^i \) are computed according to (2). Let \( Q^i = \frac{1}{n} \sum_{i=0}^{n} P^i \) be an average of the \( P^i \)'s. The proof of convergence of the subdivision scheme is given in two parts. In the first part, it is shown that when \(-\frac{1}{3} < \mu < \frac{2}{3}\), as \( k \to \infty \), \( Q^i \to v \). While this average may be approaching \( v \) as \( k \to \infty \), the individual \( P^i \)'s might be diverging. So in the second part it is shown...
that when \( \max_{\beta, \gamma, \delta} \left\{ \beta + 2\gamma \cos \frac{2\pi}{n} j + 2\delta \cos \frac{4\pi}{n} j \right\} < 1 \), \( P_i^k \rightarrow Q_i^k \) (\( i = 0,1,\cdots,n-1 \)) as \( k \rightarrow \infty \) by using the concepts of discrete convolution and discrete Fourier transform. Here we will not give the details of the proof owing to the limitation of space.

**Case 2.** \( n = 4 \). In this case the points in \( P_i^k \) are computed according to (4). The proof is done by analyzing the eigenstructure of the subdivision matrix.

By computation we find that the eigenvalues of subdivision matrix are \( 1, \frac{1}{3}, \frac{1}{3}, 2\mu - \frac{1}{3}, 6\mu - \frac{5}{3} \). It is easy to know that in this case, when \( \frac{1}{9} < \mu < \frac{2}{3} \), the surfaces generated by the ternary subdivision scheme converge at the extraordinary point \( v \) with vertex order of \( n = 4 \).

**Case 3.** \( n = 3 \). In this case the points in \( P_i^k \) are computed according to (6). Since in this case the eigenvalues of subdivision matrix are \( 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{9} \), it is obvious that the surfaces generated by the ternary subdivision scheme converge at the extraordinary point \( v \) with vertex order of \( n = 3 \).

### 4. Analysis of tangent plane continuity

About the tangent plane continuity of the subdivision scheme we have the following theorems.

**Theorem 2.** If \( n \geq 5 \), \( s = \max_{\beta, \gamma, \delta} \left\{ \beta + 2\gamma \cos \frac{2\pi}{n} j + 2\delta \cos \frac{4\pi}{n} j \right\} \in (0,1) \), when \( 1 - s < \alpha < 1 + s \), the surfaces generated by the ternary subdivision scheme are tangent plane continuous at \( v \).

Proof. Define \( t_i^k = P_i^k - v \) \( (i = 0,1,\cdots,n-1) \) be the edge vectors from \( v \) to \( P_i^k \) after \( k \) iterations of the subdivision algorithm. To cancel the contracti on of edge vectors between iterations of the algorithm, we introduce a series of scalars \( \left( \frac{1}{s} \right)^j \) (an exact value for \( s \) can be given as \( s = \max_{\beta, \gamma, \delta} \left\{ \beta + 2\gamma \cos \frac{2\pi}{n} j + 2\delta \cos \frac{4\pi}{n} j \right\} \in (0,1) \)), and denote \( T_i^k = \left( \frac{1}{s} \right)^j t_i^k \), \( T_i = \{ T_i^0, T_i^1, \cdots, T_i^k \} \).

Let \( G = \{ G_0, G_1, \cdots, G_n \} \), where \( G_j = \frac{2}{n} \cos \left( \frac{2\pi}{n} j \right) \), \( j = 0,1,\cdots,n-1 \). By using the concepts of discrete convolution and discrete inverse Fourier transform, we can find that as \( k \rightarrow \infty \), \( T_i^k \rightarrow G \ast P_i^k \) (here \( \ast \) denotes discrete convolution), namely

\[
T_i^k \rightarrow \frac{2}{n} \sum_{j=0}^{n} G_j P_j^0 = \frac{2}{n} \sum_{j=0}^{n} \cos \frac{2\pi(i-j)}{n} P_j^0 = t_i \cos \frac{2\pi i}{n} + t_2 \sin \frac{2\pi i}{n}, \quad i = 0,1,\cdots,n-1,
\]

where \( t_i = \frac{2}{n} \sum_{j=0}^{n} \cos \frac{2\pi j}{n} P_j^0 \) and \( t_2 = \frac{2}{n} \sum_{j=0}^{n} \sin \frac{2\pi j}{n} P_j^0 \) are two constant vectors.

If \( t_i \) and \( t_j \) are not parallel, from (7) we know that as \( k \rightarrow \infty \), the vectors \( T_i^k \)’s will lie in a plane determined by \( t_i \) and \( t_j \), which will be the tangent plane of the subdivision surface at the extraordinary point \( v \) and whose normal vector will be \( n = t_i \times t_j \).

If \( t_i \) and \( t_j \) are parallel, we may prove it by disturbing the local region around \( v \) slightly.

Since the subdivision surfaces have well defined tangent planes at \( v \) with vertex order of \( n \geq 5 \) for almost all initial control meshes, so they are tangent plane continuous.

**Theorem 3.** For \( \mu \in \left( \frac{2}{9}, \frac{1}{3} \right) \), the subdivision surface is tangent plane continuous at the extraordinary point \( v \) with vertex order of \( n = 4 \) and \( n = 3 \).
Proof. It is easy to know that the subdivision scheme satisfies the necessary condition of tangent plane continuity for \( \frac{2}{9} < \mu \leq \frac{1}{3} \).

**Case 1.** \( n = 4 \). Let \( V^i = [v^i, P^0_1, P^1_1, P^0_2, P^1_2] \) denote the vertex vector corresponding to the new refinement local meshes consisting of \( v \) and its 1-ring neighbor points after \( k \) subdivision steps. We can find that as \( k \to \infty \), \( v^i, P^0_1, P^1_1, P^0_2, P^1_2 \) will all converge to a constant vector \( c_i = v \).

Furthermore, we find that as \( k \to \infty \), the unit edge vectors \( \frac{t^i}{|t^i|} (i = 0, \ldots, 3) \) will converge to some constant vectors which lie in a plane passing through \( c_i \) determined by \( c_2 \) and \( c_6 \) (suppose they are not parallel, otherwise the similar disturbing method may be used), where

\[
\begin{align*}
\mathbf{n} &= \frac{1}{2} (P_1^0 - P_2^0), \quad \mathbf{c}_i = \frac{1}{2} (P_i^0 - P_i^1).
\end{align*}
\]

This plane will exactly be the tangent plane of the subdivision surface at \( v \), whose normal vector will be \( \mathbf{n} = \mathbf{c}_i \times \mathbf{c}_j \).

**Case 2.** \( n = 3 \). The proof is similar to case 1.

In conclusion, Theorem 3 holds.

Fig. 7 show the results of applying the presented algorithm after four subdivision steps for the same initial triangular control mesh (see Fig. 7(a)). In Fig. 7(b) and (c) the surfaces are obtained with \( \mu = \frac{5}{18} \) and \( \mu = \frac{1}{4} \) respectively. According to the theorems presented in this paper both of the limit surfaces will be tangent plane continuous.

![Fig. 7](image_url)

**Fig. 7** Initial control mesh and the subdivision surfaces generated by applying the presented scheme

5. **Conclusions**

In this paper, we present a new interpolatory subdivision algorithm for triangular control meshes. It can be used to deal with not only extraordinary faces but also extraordinary vertices in polyhedral meshes of arbitrary topologies. Implementation of the presented algorithm is easy, efficient and fast because of its local, ternary and simple properties.

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