On the singular value decomposition of (skew-)involutory and (skew-)coninvolutory matrices

Abstract: The singular values \( \sigma > 1 \) of an \( n \times n \) involutory matrix \( A \) appear in pairs \((\sigma, \frac{1}{\sigma})\). Their left and right singular vectors are closely connected. The case of singular values \( \sigma = 1 \) is discussed in detail. These singular values may appear in pairs \((1, 1)\) with closely connected left and right singular vectors or by themselves. The link between the left and right singular vectors is used to reformulate the singular value decomposition (SVD) of an involutory matrix as an eigendecomposition. This displays an interesting relation between the singular values of an involutory matrix and its eigenvalues. Similar observations hold for the SVD, the singular values and the cone-eigenvalues of (skew-)coninvolutory matrices.

Keywords: singular value decomposition, (skew-)involutory matrix, (skew-)coninvolutory, consimilarity

MSC: 15A23, 65F99

1 Introduction

Inspired by the work [7] on the singular values of involutory matrices some more insight into the singular value decomposition (SVD) of involutory matrices is derived. For any matrix \( A \in \mathbb{C}^{n \times n} \) there exists a singular value decomposition (SVD), that is, a decomposition of the form

\[
A = U\Sigma V^H,
\]

where \( U, V \in \mathbb{C}^{n \times n} \) are unitary matrices and \( \Sigma \in \mathbb{R}^{n \times n} \) is a diagonal matrix with non-negative real numbers on the diagonal. The diagonal entries \( \sigma_j \) of \( \Sigma \) are the singular values of \( A \). Usually, they are ordered such that \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0 \). The number of nonzero singular values of \( A \) is the same as the rank of \( A \). Thus, a nonsingular matrix \( A \in \mathbb{C}^{n \times n} \) has \( n \) positive singular values. The columns \( u_j, j = 1, \ldots, n \) of \( U \) and the columns \( v_j, j = 1, \ldots, n \) of \( V \) are the left singular vectors and right singular vectors of \( A \), respectively. From (1.1) we have \( Av_j = \sigma_j u_j, j = 1, \ldots, n \). Any triplet \((u, v, \sigma)\) with \( Av = \sigma u \) is called a singular triplet of \( A \). In case \( A \in \mathbb{R}^{n \times n} \), \( U \) and \( V \) can be chosen to be real and orthogonal.

While the singular values are unique, in general, the singular vectors are not. The nonuniqueness of the singular vectors mainly depends on the multiplicities of the singular values. For simplicity, assume that \( A \in \mathbb{C}^{n \times n} \) is nonsingular. Let \( s_1 > s_2 > \cdots > s_k > 0 \) denote the distinct singular values of \( A \) with respective multiplicities \( \theta_1, \theta_2, \ldots, \theta_k \geq 1 \), \( \sum_{j=1}^k \theta_j = n \). Let \( A = U\Sigma V^H \) be a given singular value decomposition with \( \Sigma = \text{diag}(s_1 I_{\theta_1}, s_2 I_{\theta_2}, \ldots, s_k I_{\theta_k}) \). Here \( I_j \) denotes as usual the \( j \times j \) identity matrix. Then

\[
\hat{U} = U [W_1 \oplus W_2 \oplus \cdots \oplus W_k] \quad \text{and} \quad \hat{V} = V [W_1 \oplus W_2 \oplus \cdots \oplus W_k]
\]
with unitary matrices $W_j \in \mathbb{C}^{n \times n}$ yield another SVD of $A$, $A = \tilde{U}\tilde{\Sigma}\tilde{V}^H$. This describes all possible SVDs of $A$, see, e.g., [4, Theorem 3.1.1'] or [8, Theorem 4.28]. In case $\theta_j = 1$, the corresponding left and right singular vector are unique up to multiplication with some $e^{i\theta}$, $\alpha_j \in \mathbb{R}$, where $i = \sqrt{-1}$. For more information on the SVD see, e.g., [2, 4, 8].

A matrix $A \in \mathbb{C}^{n \times n}$ with $A^2 = I_n$, or equivalently, $A = A^{-1}$ is called an involutory matrix. Thus, for any involutory matrix $A$, $A$ and its inverse $A^{-1}$ have the same SVD and hence, the same (positive) singular values. Let $A = U\Sigma V^H$ be the usual SVD of $A$ with the diagonal of $\Sigma$ ordered by magnitude, $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$. Noting that an SVD of $A^{-1}$ is given by $A^{-1} = V\Sigma^{-1}U^H$ and that the diagonal elements of $\Sigma^{-1}$ are ordered by magnitude, $\frac{1}{\sigma_1} \geq \frac{1}{\sigma_2} \geq \ldots \geq \frac{1}{\sigma_n}$, we conclude that $\sigma_1 = \frac{1}{\sigma_n}, \sigma_2 = \frac{1}{\sigma_{n-1}}, \ldots, \sigma_n = \frac{1}{\sigma_1}$. That is, the singular values of an involutory matrix $A$ are either 1 or pairs $(\sigma, \frac{1}{\sigma})$, where $\sigma > 1$. This has already been observed in [7] (see Theorem 1 for a quote of the findings). Here we will describe the SVD $A = UEV^H$ for involutory matrices in detail. In particular, we will note a close relation between the left and right singular vectors of pairs $(\sigma, \frac{1}{\sigma})$ of singular values: if $(u, v, \sigma)$ is a singular triplet, then so is $(v, u, \frac{1}{\sigma})$. We will observe that some of the singular values $\sigma = 1$ may also appear in such pairs. That is, if $(u, v, 1)$ is a singular triplet, then so is $(v, u, 1)$. Other singular values $\sigma = 1$ may appear as a single singular triplet, that is, $(u, \pm u, 1)$. These observations allow to express $U$ as $U = VT$ for a real elementary orthogonal matrix $T$. With this the SVD of $A$ reads $A = VTSV^H$. Taking a closer look at the real matrix $T\Sigma$ we will see that $T\Sigma$ is an involutory matrix just as $A$. All relevant information concerning the singular values and the eigenvalues of $A$ can be read off of $T\Sigma$. Some of these findings also follow from [6, Theorem 7.2], see Section 2.

As any skew-involutory matrix $B \in \mathbb{C}^{n \times n}, B^2 = -I_n$ can be expressed as $B = iA$ with an involutory matrix $A \in \mathbb{C}^{n \times n}$, the results for involutory matrices can be transferred easily to skew-involutory ones.

We will also consider the SVD of coninvolutory matrices, that is of matrices $A \in \mathbb{C}^{n \times n}$ which satisfy $A\overline{A} = I_n$, see, e.g., [5]. As involutory matrices, coninvolutory are nonsingular and have $n$ positive singular values. Moreover, the singular values appear in pairs $(\sigma, \frac{1}{\sigma})$ or are 1, see [4]. Similar to the case of involutory matrices, we can give a relation between the matrices $U$ and $V$ in the SVD $U\Sigma V^H$ of $A$ in the form $U = TVT$. $T\Sigma$ is a complex coninvolutory matrix consimilar to $A$ and consimilar to the identity. Similar observations have been given in [5]. Some of our findings also follow from [6, Theorem 7.1], see Section 4.

Skew-coninvolutory matrices $A \in \mathbb{C}^{2n \times 2n}$, that is, $A\overline{A} = -I_{2n}$, have been studied in [1]. Here we will briefly state how with our approach findings on the SVD of skew-coninvolutory matrices given in [1] can easily be rediscovered.

Our goal is to round off the picture drawn in the literature about the singular value decomposition of the four classes of matrices considered here. In particular we would like to make visible the relation between the singular vectors belonging to reciprocal pairs of singular values in the form of the matrix $T$.

The SVD of involutory matrices is treated at length in Section 2. In Section 3 we make immediate use of the results in Section 2 in order to discuss the SVD of skew-involutory matrices. Section 4 deals with coninvolutory matrices. Finally, the SVD of skew-coninvolutory matrices is discussed in Section 5.

## 2 Involutory matrices

Let $A \in \mathbb{C}^{n \times n}$ be involutory, that is, $A^2 = I_n$ holds. Thus, $A = A^{-1}$. Symmetric permutation matrices and Hermitian unitary matrices are simple examples of involutory matrices. Nontrivial examples of involutory matrices can be found in [3, Page 165, 166, 170]. The spectrum of any involutory matrix can have at most two elements, $\lambda(A) \subset \{1, -1\}$ as from $Ax = \lambda x$ for $x \in \mathbb{C}^n \setminus \{0\}$ it follows that $A^2 x = \lambda x$ which gives $x = \lambda^2 x$.

The SVD of involutory matrices has already been considered in [7]. In particular, the following theorem is given.

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1 Two matrices $A$ and $B \in \mathbb{C}^{n \times n}$ are said to be consimilar if there exists a nonsingular matrix $S \in \mathbb{C}^{n \times n}$ such that $A = SB\overline{S}^{-1}$, [5].
Theorem 1. Let $A \in \mathbb{C}^{n \times n}$ be an involutory matrix. Then $B_1 = \frac{1}{2}(I_n - A)$ and $B_2 = \frac{1}{2}(I_n + A)$ are idempotent, that is, $B_i^2 = B_i$, $i = 1, 2$. Assume that $A$ has $r \leq \frac{n}{2}$ eigenvalues $-1$ and $n - r$ eigenvalues $+1$ (or $r \leq \frac{n}{2}$ eigenvalues $+1$ and $n - r$ eigenvalues $-1$). Then $B_1$ ($B_2$) is of rank $r$ and can be decomposed as $B_1 = Q W^H (B_2 = Q W^H)$ where $Q, W \in \mathbb{C}^{n \times n}$ have $r$ linear independent columns. Moreover, $W^H W - I$ is at least positive semidefinite. There are $n - 2r$ singular values of $A$ equal to 1, $r$ singular values which are equal to the eigenvalues of the matrix $(W^H W)^\frac{1}{2} + (W^H W - I_n)^\frac{1}{2}$ and $r$ singular values which are the reciprocal of these.

Thus, in [7] it was noted that the singular values of an involutory matrix $A$ may be 1 or may appear in pairs $(\sigma, \frac{1}{\sigma})$. Moreover, the minimal number of singular values 1 is given by $2n - r$ where $r \leq \frac{n}{2}$ denotes the number of eigenvalues $-1$ of $A$ or the number of eigenvalues $+1$ of $A$, whichever is smallest. Further, in [7] it is said "that the singular values $\neq 1$ of the involutory matrix $A$ are the roots of $(W^H W)^{\frac{1}{2}} + (W^H W - I_n)^{\frac{1}{2}}$, and $(W^H W)^{\frac{1}{2}} - (W^H W - I_n)^{\frac{1}{2}}$. This does not imply that the matrix $(W^H W)^{\frac{1}{2}} + (W^H W - I_n)^{\frac{1}{2}}$ does not have eigenvalues equal to +1. This can be illustrated by $A = \text{diag}(1, -1, -1)$ with one eigenvalue +1 and two eigenvalues $-1$. Hence, in Theorem 1, $r = 1$ and there is (at least) one singular value 1, as $n - 2r = 1$. For the other two singular values, the matrix $B_2 = \frac{1}{2}(I_n + A) = Q W^T$ with $Q = [1, 0, 0]^T$ and $W^T = [1, 0, 0]$ and the $1 \times 1$ matrix $(W^T W)^{\frac{1}{2}} + (W^T W - I_n)^{\frac{1}{2}} = 1$ needs to be considered. Obviously, $W$ has the eigenvalue +1 which, according to Theorem 1 is a singular value of $A$. Moreover, its reciprocal has to be a singular value. Hence, $A$ has three singular values 1, two appearing as a pair $(\sigma, \frac{1}{\sigma}) = (1, 1)$ and a 'single' one.

In the following discussion we will see that apart from the pairing of the singular values, there is even more structure in the singular value decomposition of an involutory matrix by taking a closer look at $U$ and $V$. This will highlight the difference between the two types of singular values 1 (pairs of singular values $(1, 1)$ and single singular values 1).

Let $(u, v, \sigma)$ be a singular triplet of $A$, that is, let $A v = \sigma u$ hold. This is equivalent to $v = \sigma^{-1} u = \sigma A u$ and further to $A u = \frac{1}{\sigma} v$. Thus, the singular triplet $(u, v, \sigma)$ of $A$ is always accompanied by the singular triplet $(v, u, \frac{1}{\sigma})$ of $A$. In case $\sigma = 1$, this may collapse into a single triplet if $u = v$ or $u = -v$. This allows for three cases:

1. In case $u = v$, we have a single singular value 1 with the triplet $(u, u, 1)$. In this case, it follows immediately that $A$ has an eigenvalue 1, as we have $A u = u$.
2. In case $u = -v$, we have a single singular value 1 with the triplet $(u, -u, 1)$ (or the triplet $(-u, u, 1)$). In this case, it follows immediately that $A$ has an eigenvalue $-1$, as we have $A u = -u$.
3. In case $u \neq \pm v$, we have a pair $(1, 1)$ of singular values associated with the two triplets $(u, v, 1)$ and $(v, u, 1)$.

Therefore, singular values $\sigma > 1$ will appear in pairs $(\sigma, \frac{1}{\sigma})$, while a singular value $\sigma = 1$ may appear in a pair in the sense that there are two triplets $(u, v, 1)$ and $(v, u, 1)$ or it may appear by itself as a triplet $(u, \pm u, 1)$. It follows immediately, that an involutory matrix $A$ of odd size $n$ must have at least one singular value 1.

Clearly, due to the nonuniqueness of the SVD, no every SVD of $A$ has to display the pairing of the singular values identified above. But every SVD of $A$ can be easily modified so that this can be read off. Let us consider a small example first.

Example 2. Consider the involutory, Hermitian and unitary matrix

$$A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \in \mathbb{R}^{4 \times 4}.
$$

One possible SVD of $A$ is given by $A = \tilde{U} \Sigma \tilde{V}^T$ with

$$\tilde{U} = \begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2} & 0 \\
0 & \frac{-1}{\sqrt{2}} & \frac{\sqrt{2}}{2} \\
-\frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \Sigma = I_4, \quad \tilde{V} = \begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2} & 0 \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\
\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\
0 & 0 & 1
\end{bmatrix}.$$
We have \( \tilde{A}v_1 = \tilde{u}_1, \tilde{A}v_2 = \tilde{u}_2, \tilde{A}v_3 = \tilde{u}_3, \) and \( \tilde{A}v_4 = \tilde{e}_4 = \tilde{u}_4. \) It can be seen immediately that there is the singular triplet \( (1, e_4, e_4) \). Thus, there needs to be at least one other singular triplet of this type, but it can not be seen straightaway whether the other two singular values \( 1 \) correspond to a pair of singular values with connected singular vectors or not. Please note that \( \tilde{U} = UW, \tilde{V} = VW \) holds for

\[
U = [e_1, e_3, e_2, e_4], \quad V = [e_1, e_2, e_3, e_4], \quad W = \tilde{V}.
\]

Thus \( A = U\Sigma V^H \) is another SVD of \( A \) with

\[ Ae_1 = e_1, \quad Ae_2 = e_3, \quad Ae_3 = e_2, \quad Ae_4 = e_4. \]

Hence, there is one singular value which is part of the two related triplets \( (e_2, e_3, 1) \) and \( (e_3, e_2, 1) \) and two singular values which have no partner as their triplets are \( (e_1, 1, 1) \) and \( (e_4, 1, 1) \). In a similar way, any other SVD of \( A \) can be modified in order to display the structure of the singular values and vectors.

Assume that an SVD \( A = U\Sigma V^H \) is given where the singular values are ordered such that \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0 \).

Let us first assume that \( \sigma_1 > \sigma_2 \).

Then \( \sigma_{n-1} = \frac{1}{\sigma_1} > \sigma_n = \frac{1}{\sigma_2} \).

Moreover, \( \tilde{A}v_1 = \sigma_1 u_1 \) and \( \tilde{A}v_n = \sigma_n u_n = \frac{1}{\sigma_1} u_n \),

or, equivalently, \( \tilde{A}v_1 = \sigma_1 u_1 \) and \( \tilde{A}u_n = \sigma_1 v_n \).

Due to the essential uniqueness of singular vectors of singular values with multiplicity \( 1 \), it follows that \( u_n = v_1 e^{i\alpha} \) and \( v_n = u_1 e^{i\alpha} \) for some \( \alpha \).

Thus,

\[
U = [u_1 | u_2 \cdots u_{n-1} | v_1 e^{i\alpha}] = \tilde{U} \left[ I_{n-1} | e^{i\alpha} \right],
\]

\[
V = [v_1 | v_2 \cdots v_{n-1} | u_1 e^{i\alpha}] = \tilde{V} \left[ I_{n-1} | e^{i\alpha} \right]
\]

with the unitary matrices

\[
\tilde{U} := [u_1 | u_2 \cdots u_{n-1} | v_1], \quad \tilde{V} := [v_1 | v_2 \cdots v_{n-1} | u_1].
\]

We have \( A = U\Sigma V^H = \tilde{U}\Sigma \tilde{V}^H \) and \( \tilde{U}\Sigma \tilde{V}^H \) is a valid SVD of \( A \) displaying the pairing for the singular value \( \sigma_1 \).

In this fashion all singular values \( > 1 \) of multiplicity \( 1 \) can be treated.

Next let us assume that \( \sigma_1 = \sigma_2 = \cdots = \sigma_{\ell} > \sigma_{\ell+1} \).

Then \( \sigma_{n-\ell} = \frac{1}{\sigma_1} > \sigma_{n-\ell-1} = \cdots = \sigma_n = \frac{1}{\sigma_{\ell}} \).

Due to our observation concerning the singular vectors of pairs of singular values \( (\sigma_1, \frac{1}{\sigma_1}) \) and the essential uniqueness of the SVD (1.2), it follows that

\[
[u_{n-\ell+1} \cdots u_{n-1} u_n] = [v_1 v_2 \cdots v_{\ell}] W, \\
[v_{n-\ell+1} \cdots v_{n-1} v_n] = [u_1 u_2 \cdots u_{\ell}] W
\]

for some unitary \( \ell \times \ell \) matrix \( W \).

Thus,

\[
U = \tilde{U} \left[ I_{n-\ell} \right], \\
V = \tilde{V} \left[ I_{n-\ell} \right]
\]

with the unitary matrices

\[
\tilde{U} := [u_1 u_2 \cdots u_{\ell} | u_{\ell+1} \cdots u_{n-\ell} | v_1 v_2 \cdots v_{\ell}], \\
\tilde{V} := [v_1 v_2 \cdots v_{\ell} | v_{\ell+1} \cdots v_{n-\ell} | u_1 u_2 \cdots u_{\ell}].
\]

We have \( A = U\Sigma V^H = \tilde{U}\Sigma \tilde{V}^H \) and \( \tilde{U}\Sigma \tilde{V}^H \) is a valid SVD displaying the pairing for the singular value \( \sigma_1 = \sigma_2 = \cdots = \sigma_{\ell} \).

In this fashion all singular values \( > 1 \) of multiplicity \( > 1 \) can be treated.
Let us assume that $A$ has $v$ singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_v > 1$ and $n - 2v$ singular values equal to one such that $A = U \Sigma V^H$ with

$$
\Sigma = S \oplus I_{n-2v} \oplus S^{-1}, \quad S = \text{diag}(\sigma_1, \ldots, \sigma_v).
$$

Please note that non-standard ordering of the singular values on the diagonal of $\Sigma$. We have $S^{-1} = \text{diag}(\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_v}) = \text{diag}(\sigma_{n-v+1}, \ldots, \sigma_n)$. Partition $U$ and $V$ conformal to $S$,

$$
U = [U_1 \ U_2 \ U_3], \quad V = [V_1 \ V_2 \ V_3], \quad U_1, U_3, V_1, V_3 \in \mathbb{C}^{n \times v}.
$$

Let us further assume that $S = \text{diag}(t_1 I_{\mu_1}, t_2 I_{\mu_2}, \ldots, t_\ell I_{\mu_\ell})$, where $t_1 > t_2 > \cdots > t_\ell > 1$ denote the distinct singular values larger than one with respective multiplicities $\mu_1, \mu_2, \ldots, \mu_\ell \geq 1$, $\sum_{j=1}^\ell \mu_j = v$. Then we can summarize our findings so far as

$$
U_3 = V_1 \begin{bmatrix} W_1 & W_2 & \cdots & W_\ell \end{bmatrix} \quad \text{and} \quad V_3 = U_1 \begin{bmatrix} W_1 & W_2 & \cdots & W_\ell \end{bmatrix}
$$

with unitary matrices $W_j \in \mathbb{C}^{\mu_j \times \mu_j}$. Thus,

$$
U = [U_1 \ U_2 \ U_3] = [U_1 \ U_2 \ V_1] \begin{bmatrix} I_\nu & I_{n-2v} \end{bmatrix}_W = \tilde{U} \begin{bmatrix} I_{n-\nu} \end{bmatrix}_W,
$$

$$
V = [V_1 \ V_2 \ V_3] = [V_1 \ V_2 \ U_1] \begin{bmatrix} I_\nu & I_{n-2v} \end{bmatrix}_W = \tilde{V} \begin{bmatrix} I_{n-\nu} \end{bmatrix}_W
$$

with $W = \text{diag}(W_1, W_2, \ldots, W_\ell) \in \mathbb{C}^{\nu \times \nu}$. We have $A = U \Sigma V^H = \tilde{U} \tilde{\Sigma} \tilde{V}^H$ and $\tilde{U} \tilde{\Sigma} \tilde{V}^H$ is a valid SVD displaying the pairing of the singular values unequal to one.

Finally we need to consider the singular values 1. Similar as before, we can give a relation between the columns of $U_2$ and $V_2$, so that the relation between the singular vectors becomes apparent.

This gives rise to the following theorem.

**Theorem 3 (SVD of an involutory matrix).** Let $A \in \mathbb{C}^{n \times n}$ be involutory. Assume that $A$ has $v$ singular values $\sigma_1 \geq \cdots \geq \sigma_v > 1$. These singular values appear in pairs $(\sigma_j, \frac{1}{\sigma_j})$ associated with the singular triplets $(u_j, v_j, \sigma_j)$ and $(v_j, u_j, \frac{1}{\sigma_j})$, $j = 1, \ldots, v$. Assume further that $A$ has $\mu$ singular values 1 which appear in pairs $(1, 1)$ associated with the singular triplets $(\hat{u}_j, \hat{v}_j, 1)$ and $(\hat{v}_j, \hat{u}_j, 1)$, $j = 1, \ldots, \mu$. Finally, assume that $A$ has $k = n - 2v - 2\mu$ single singular values 1 associated with the singular triplet $(\hat{u}_j, \pm \hat{u}_j, 1)$, $j = 1, \ldots, k$.

Thus the SVD of $A$ is given by

$$
\Sigma = \begin{bmatrix} S & I_\delta & I_\mu \\ I_\delta & S^{-1} & I_\mu \\ I_\mu & I_\mu & I_\eta \end{bmatrix}
$$

with $S = \text{diag}(\sigma_1, \ldots, \sigma_v)$ and

$$
U = [u_1, \ldots, u_v, \hat{u}_1, \ldots, \hat{u}_\mu, \hat{u}_{\delta+1}, \ldots, \hat{u}_{\delta+\mu}],
$$

$$
V = [v_1, \ldots, v_v, \hat{v}_1, \ldots, \hat{v}_\mu, \pm \hat{u}_1, \ldots, \pm \hat{u}_\delta, \hat{u}_{\delta+1}, \ldots, \hat{u}_{\delta+\mu}, \hat{u}_{\mu+\delta}, \ldots, \hat{u}_{\mu+\eta}],
$$

where $\nu + \mu + \eta = m$ and

$$
\delta = \begin{cases} 
\eta & \text{if } n = 2m \\
\eta + 1 & \text{if } n = 2m + 1.
\end{cases}
$$

In particular, the signs do not need to be equal for all $\hat{u}_j, j = 1, \ldots, \delta + \eta$. 
For $U$ and $V$ from Theorem 3 we have

$$
U = V \begin{bmatrix}
0 & 0 & 0 & I_\nu & 0 & 0 \\
0 & 0 & 0 & 0 & I_\mu & 0 \\
0 & 0 & D_\delta & 0 & 0 & 0 \\
I_\nu & 0 & 0 & 0 & 0 & 0 \\
0 & I_\mu & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & E_\eta
\end{bmatrix} = V \begin{bmatrix}
0 & 0 & I_{\nu+\mu} & 0 \\
0 & D_\delta & 0 & 0 \\
I_{\nu+\mu} & 0 & 0 & 0 \\
0 & 0 & 0 & E_\eta
\end{bmatrix} = VT \tag{2.1}
$$

where $D_\delta \in \mathbb{R}^{6\times 6}$, $E_\eta \in \mathbb{R}^{6\times 6}$ denote diagonal matrices with $\pm 1$ on the diagonal. The particular choice depends on the sign choice in the sequence $\pm \hat{u}_j$ in $V$ in Theorem 3. Clearly, $D_\delta$ and $E_\eta$ as well as $T$ are involutory.

Hence, $A$ is unitarily similar to the real involutory matrix $T \Sigma$. This canonical form is the most condensed involutory matrix unitarily similar to $A$. All relevant information concerning the singular values and the eigenvalues of $A$ can be read off of $T \Sigma$.

Making use of the fact that all diagonal elements of $S$ are positive, we can rewrite $S$ as $S = S_+^\frac{1}{2} S_-^\frac{1}{2}$ with $S_+^\frac{1}{2} = \text{diag}(\sqrt{\sigma_1}, \sqrt{\sigma_2}, \ldots, \sqrt{\sigma_n})$. Thus

$$
A = V \begin{bmatrix}
S_+^\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & I_\mu & 0 & 0 & 0 & 0 \\
0 & 0 & I_\delta & 0 & 0 & 0 \\
0 & 0 & 0 & S_-^\frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & I_\mu & 0 \\
0 & 0 & 0 & 0 & 0 & I_\eta
\end{bmatrix} T \begin{bmatrix}
S_-^\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & I_\mu & 0 & 0 & 0 & 0 \\
0 & 0 & I_\delta & 0 & 0 & 0 \\
0 & 0 & 0 & S_+^\frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & I_\mu & 0 \\
0 & 0 & 0 & 0 & 0 & I_\eta
\end{bmatrix} V^H
$$

= $Z T Z^{-1}$.

Hence, $A$ and $T$ are similar matrices and their eigenvalues are identical. Taking a closer look at $T$, we immediately see that $T$ is similar to

$$
P T P^T = \begin{bmatrix}
-I_{\nu+\mu} & 0 & 0 & 0 \\
0 & I_{\nu+\mu} & 0 & 0 \\
0 & 0 & D_\delta & 0 \\
0 & 0 & 0 & E_\eta
\end{bmatrix} \tag{2.3}
$$

with the orthogonal matrix

$$
P = \begin{bmatrix}
\frac{1}{\sqrt{2}} I_{\nu+\mu} & 0 & \frac{1}{\sqrt{2}} I_{\nu+\mu} & 0 \\
\frac{1}{\sqrt{2}} I_{\nu+\mu} & 0 & \frac{1}{\sqrt{2}} I_{\nu+\mu} & 0 \\
0 & I_\delta & 0 & 0 \\
0 & 0 & 0 & I_\eta
\end{bmatrix}.
$$

Assume that there are $\eta_1$ positive and $\eta_2$ negative signs in the sequence $\pm \hat{u}_j$ in $V$, $j = 1, \ldots, \delta + \eta = \eta_1 + \eta_2$. Then $\text{diag}(D_\delta, E_\eta)$ is similar to $\text{diag}(I_{\eta_1}, -I_{\eta_2})$. It is straightforward to see that

$$
\det(T - \lambda I_6) = (\lambda + 1)^{\nu+\mu}(\lambda - 1)^{\nu+\mu}(\lambda - 1)^{\nu+\mu}(\lambda + 1)^{\eta_2}.
$$

Each pair of singular triplets $(u, v, \sigma)$ and $(v, u, \frac{1}{\sigma})$ (including those with $\sigma = 1$) corresponds to a pair of eigenvalues $(+1, -1)$. A single singular triplet $(u, z u, 1)$ corresponds to an eigenvalue $+1$ or $-1$ depending on the sign in $(u, z u, 1)$. Our findings are summarized in the following corollary.
Corollary 4 (Canonical Form, Eigendecomposition). Let \( A \in \mathbb{C}^{n \times n} \) be involutory. Let \( A = U \Sigma V^H \) be the SVD of \( A \). Assume that \( A \) has \( v \) singular values > 1, \( \mu \) singular values 1 which appear in pairs \((1,1)\), \( \eta_1 \) single singular values 1 associated with the singular triplet \((u_j, \tilde{u}_j, 1)\) and \( \eta_2 \) single singular values 1 associated with the singular triplet \((\tilde{u}_j, -\tilde{u}_j, 1)\). Here \( \eta_1 + \eta_2 = \delta + \eta \) for \( \delta, \eta \) as in Theorem 3. Then \( A \) is unitarily similar to the real involutory matrix \( T \Sigma \) as in \(2.2\) and diagonalizable to \( \text{diag}(-I_{v+\mu+\eta_2}, I_{v+\mu+\eta}), \) see \(2.3\).

Remark 5. If \( A \) is involutory, then \( B = \frac{1}{2}(I \pm A) \) is idempotent, \( B^2 = B \). This has been used in [7], see Theorem 1. Using \(2.2\) we obtain for \( B = \frac{1}{2}(I \pm V \Sigma V^H) = \frac{1}{2} V(I \pm \Sigma) V^H = \frac{1}{2} V T(I \pm \Sigma) V^H \) holds as \( T \) is involutory. We can easily construct an SVD of \( T \Sigma \),

\[
T \pm \Sigma = \begin{bmatrix}
\pm S & 0 & 0 & I_{\nu} & 0 & 0 \\
0 & \pm I_{\mu} & 0 & 0 & I_{\mu} & 0 \\
0 & 0 & D_{\delta} \pm I_{\delta} & 0 & 0 & 0 \\
I_{\nu} & 0 & 0 & \pm S^{-1} & 0 & 0 \\
0 & I_{\mu} & 0 & 0 & \pm I_{\mu} & 0 \\
0 & 0 & 0 & 0 & 0 & E_{\eta} \pm I_{\eta}
\end{bmatrix}.
\]

First permute \( T \pm \Sigma \) with

\[
P_1 = \begin{bmatrix}
I_{\nu} & 0 & 0 & 0 & 0 \\
0 & 0 & I_{\mu} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & I_{\nu} & 0 & 0 & 0 \\
0 & 0 & I_{\mu} & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

such that

\[
P_1^T (T \pm \Sigma) P_1 = \begin{bmatrix}
\pm S & I_{\nu} & 0 & 0 & 0 & 0 \\
I_{\nu} & \pm S^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & \pm I_{\mu} & I_{\mu} & 0 & 0 \\
0 & 0 & I_{\mu} & \pm I_{\mu} & 0 & 0 \\
0 & 0 & 0 & 0 & D_{\delta} \pm I_{\delta} & 0 \\
0 & 0 & 0 & 0 & 0 & E_{\eta} \pm I_{\eta}
\end{bmatrix}.
\]

Next permute the blocks \( \begin{bmatrix} \pm S & I_{\nu} \\ I_{\nu} & \pm S^{-1} \end{bmatrix} \) and \( \begin{bmatrix} \pm I_{\mu} & I_{\mu} \\ I_{\mu} & \pm I_{\mu} \end{bmatrix} \) to block diagonal form \( \tilde{S} = \text{diag} \left( \begin{bmatrix} \pm \sigma_1 & 1 \\ 1 & \pm \sigma_1^{-1} \end{bmatrix}, \ldots, \begin{bmatrix} \pm \sigma_v & 1 \\ 1 & \pm \sigma_v^{-1} \end{bmatrix} \right) \) and \( \tilde{I} = \text{diag} \left( \begin{bmatrix} \pm 1 & \pm 1 \\ 1 & \pm 1 \end{bmatrix}, \ldots, \begin{bmatrix} \pm 1 & \pm 1 \\ 1 & \pm 1 \end{bmatrix} \right) \). Let \( P_2 \) be the corresponding permutation matrix such that

\[
P_2^T P_1^T (T \pm \Sigma) P_1 P_2 = \begin{bmatrix}
\tilde{S} & 0 & 0 & 0 \\
0 & \tilde{I} & 0 & 0 \\
0 & 0 & D_{\delta} \pm I_{\delta} & 0 \\
0 & 0 & 0 & E_{\eta} \pm I_{\eta}
\end{bmatrix}
\]

is block diagonal with 1×1 and 2×2 diagonal blocks. The 2×2 blocks are real symmetric and can be diagonalized by an orthogonal similarity transformation

\[
\begin{bmatrix} \pm \sigma & 1 \\ 1 & \pm \sigma^{-1} \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} \pm (\sigma + \sigma^{-1}) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix},
\]

with \( c = \sqrt{\frac{\sigma}{\sigma + \sigma^{-1}}} \) and \( s = \frac{\sigma}{\sigma + \sigma^{-1}} \). Let \( X \) be the orthogonal matrix which diagonalizes \( P_2^T P_1^T (T \pm \Sigma) P_1 P_2 \),

\[
X^T P_2^T P_1^T (T \pm \Sigma) P_1 P_2 X = \begin{bmatrix}
\tilde{S} & 0 & 0 & 0 \\
0 & \tilde{I} & 0 & 0 \\
0 & 0 & D_{\delta} \pm I_{\delta} & 0 \\
0 & 0 & 0 & E_{\eta} \pm I_{\eta}
\end{bmatrix}
\]
with $\tilde{S} = \text{diag}(\sigma_1 + \sigma_1^{-1}, 0, \ldots, \sigma_v + \sigma_v^{-1}, 0)$ and $\tilde{T} = \text{diag}(2, 0, \ldots, 2, 0)$. This gives an SVD of $T + \Sigma$ and thus of $B$. In case $B = \frac{1}{2}(I + A)$ has been chosen, we have

$$T + \Sigma = P_1 P_2 X \text{diag}(\tilde{S}, \tilde{T}, D_\delta + I_\delta, E_\eta + I_\eta)X^T P_1^T P_2^T,$$

and

$$B = VTP_1 P_2 X \text{diag}(1/2 \tilde{S}, \frac{1}{2} \tilde{T}, \frac{1}{2} (D_\delta + I_\delta), \frac{1}{2} (E_\eta + I_\eta))X^T P_1^T P_2^T V^H.$$

In case $B = \frac{1}{2}(I - A)$ has been chosen, we need to take care of the minus sign in front of $\tilde{S}$ and $\tilde{T}$

$$T - \Sigma = P_1 P_2 X \text{diag}(-\tilde{S}, -\tilde{T}, D_\delta - I_\delta, E_\eta - I_\eta)X^T P_1^T P_2^T$$

$$= P_1 P_2 X \text{diag}(\tilde{S}, \tilde{T}, D_\delta - I_\delta, E_\eta - I_\eta)Y^T P_1^T P_2^T$$

with $Y = -X$. The SVD of $B$ is immediate.

Hence, if $A$ has pairs of singular values $(\sigma, \sigma^{-1})$ and $(1, 1)$, then $B$ has pairs of singular values $(\sigma + \sigma^{-1}, 0)$ and $(1, 0)$. Moreover, $D_\delta \pm I_\delta$ and $E_\eta \pm I_\eta$ give singular values 1 or 0.

Before we turn our attention to the skew-involutory case, we would like to point out that most of our observations given in this section also follow from [6, Theorem 7.2]. For the ease of the reader, this theorem is stated next.

**Theorem 6.** Let $A \in \mathbb{C}^{n \times n}$. If $A^2$ is normal, then $A$ is unitarily $H$-congruent\(^2\) to a direct sum of blocks, each of which is

$$[\lambda] \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ \mu & 0 \end{bmatrix}, \quad \tau \in \mathbb{R}, \lambda, \mu \in \mathbb{C}, \tau > 0, \text{ and } |\mu| < 1. \tag{2.4}$$

This direct sum is uniquely determined by $A$, up to permutation of its blocks. Conversely, if $A$ is unitarily $H$-congruent to a direct sum of blocks of the form (2.4), then $A^2$ is normal.

In case $A^2 = I$, Theorem 6 gives for the $1 \times 1$ blocks that $A = \pm 1$ has to hold. For the $2 \times 2$ blocks it follows $\tau^2 \mu = 1$. Thus, $\mu \in (0, 1)$ and $\tau \mu = \frac{1}{\tau}$. Let $U \in \mathbb{C}^{n \times n}$ be the unitary matrix which transforms the involutory matrix $A$ as described in Theorem 6;

$$U^H AU = \left( \bigoplus_{i=1}^{n_1} [\lambda_i] \right) \oplus \left( \bigoplus_{j=1}^{n_2} \tau_j \begin{bmatrix} 0 & 1 \\ \mu_j & 0 \end{bmatrix} \right), \tag{2.5}$$

with $\tau_j, \mu_j \in \mathbb{R}, \lambda_i \in \{+1, -1\}, \tau_j > 0, \mu_j \in (0, 1)$ and $\tau_j \mu_j = \frac{1}{\tau_j}$. Clearly, $\tau_j \neq 1$ as $\mu_j \neq 1$.

The unitary $H$-congruence of $A$ as in (2.5) can be modified into an SVD. For any $1 \times 1$ block $[\lambda_i]$, the corresponding singular value is 1. At the same time, any $1 \times 1$ block $[\lambda_i]$ represent an eigenvalue of $A$. An eigenvector $u_i$ corresponding to $\lambda_i$ can be read off of $U;AU_i = \lambda_i u_i$. This eigenvector $u_i$ will serve as the corresponding right singular vector. The left singular vector will be chosen as $u_i$ in case the $\lambda_i = 1$ and as $-u_i$ in case $\lambda_i = -1$. The SVD of a $2 \times 2$ block is given by

$$\tau \begin{bmatrix} 0 & 1 \\ \mu & 0 \end{bmatrix} = I_2 \begin{bmatrix} \tau & 0 \\ 0 & \tau \mu \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = W \Sigma Z^H.$$

Thus, as $\tau \mu = \frac{1}{\tau}$ and $\mu \in (0, 1)$ holds, the singular values $\tau > 1$ appear in pairs $(\tau, \frac{1}{\tau})$. There are columns $v, w$ from $U$ such that

$$A[v \ w] = [v \ w] \tau \begin{bmatrix} 0 & 1 \\ \mu & 0 \end{bmatrix} = [v \ w] \Sigma Z^H.$$

\(^2\) Two matrices $A$ and $B \in \mathbb{C}^{n \times n}$ are said to be $H$-congruent if there exists a nonsingular matrix $S \in \mathbb{C}^{n \times n}$ such that $A = SBS^H$. Thus, two unitarily $H$-congruent matrices are unitarily similar.
This gives
\[ A[w \ v] = [v \ w] \begin{bmatrix} \tau & 0 \\ 0 & \frac{1}{\tau} \end{bmatrix}. \]

Hence, singular values \( \tau > 1 \) appear in pairs \((\tau, \frac{1}{\tau})\) and are associated with the singular triplets \((v, w, \tau)\) and \((w, v, \frac{1}{\tau})\). The fact that singular values 1 can also appear in pairs \((1, 1)\) with singular triplets in the form \((u, v, 1)\) and \((v, u, 1)\) does not follow from Theorem 6.

The unitary \( H \)-congruence of \( A \) can also be modified into an eigendecomposition. The \( 1 \times 1 \) blocks represent eigenvalues of \( A \), a corresponding eigenvector can be read off of \( U \). The eigenvalues of the \( 2 \times 2 \) blocks are \(+1\) and \(-1\). Each \( 2 \times 2 \) block can be diagonalized by a unitary matrix. Thus, Theorem 6 yields that any involutory matrix is unitarily diagonalizable to \( \text{diag}(-I_{n_1}, n_2, +I_{n_1}n_2) \). Here, it is assumed that there are \( n_2 \) \( 2 \times 2 \) blocks and \( n_1 = n - 2n_2 \). \( 1 \times 1 \) blocks with \( n_{11} \) blocks \([-1]\) and \( n_{12} \) blocks \([+1]\). Comparing this result to our one in Corollary 4 we see that \( v = n_2, \mu + \eta_2 = n_{11} \) and \( \mu + \eta_1 = n_{12} \).

### 3 Skew-involutory matrices

Any skew-involutory matrix \( A \in \mathbb{C}^{n \times n} \) can be expressed as \( A = iC \) with an involutory matrix \( C \in \mathbb{C}^{n \times n} \). Thus we can immediately make use of the results from Section 2. As for the spectrum of an involutory matrix \( C \) we have \( \lambda(C) \subset \{1, -1\} \), it follows that \( \lambda(A) \subset \{i, -i\} \) holds. Moreover, if the singular value decomposition of \( C \) is given by \( C = U\Sigma V^H \) with \( U, \Sigma, V \) as in Theorem 3, then \( A = U\Sigma(iV^H) \) is an SVD of \( A \).

In particular, it holds that any singular value \( \sigma > 1 \) of \( A \) appears as a pair \((\sigma, \frac{1}{\sigma})\). The singular triplet \((u, v, \sigma)\) of \( A \) is always accompanied by the singular triplet \((-v, u, \frac{1}{\sigma})\) of \( A \). In case \( \sigma = 1 \), this may collapse into a single triplet if \( u = iv \) or \( u = -iv \). Thus, the SVD of \( A \) can be given as in Theorem 3 where \( V \) is modified such that
\[ V = [v_1, \ldots, v_v, \ldots, \bar{v}_\mu, \pm \bar{v}_\delta, \ldots, -u_1, \ldots, -u_v, -\bar{u}_1, \ldots, -\bar{u}_\mu, \pm \bar{u}_\delta, \ldots, \mp \bar{u}_{\eta, \delta}]. \]

Similar to before, \( U \) and \( V \) are closely connected
\[ U = V \begin{bmatrix} 0 & 0 & -I_{v+\mu} & 0 \\ 0 & iD_\delta & 0 & 0 \\ I_{v+\mu} & 0 & 0 & 0 \\ 0 & 0 & 0 & iE_\eta \end{bmatrix} = VT. \]

Hence, we have
\[ A = U\Sigma V^H = VT\Sigma V^H = V \begin{bmatrix} 0 & 0 & -S^{-1} & 0 & 0 \\ 0 & 0 & 0 & -I_\mu & 0 \\ 0 & 0 & iD_\delta & 0 & 0 \\ S & 0 & 0 & 0 & 0 \\ 0 & I_\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & iE_\eta \end{bmatrix} V^H. \]

In other words, \( A \) is unitarily similar to the complex skew-involutory matrix \( T\Sigma \) which reveals all relevant information about the singular values and the eigenvalues of \( A \). Moreover, \( A \) is diagonalizable to \( \text{diag}(-i_{v+\mu+\eta_1}, i_{v+\mu+\eta_2}) \) with \( \eta_1, \eta_2 \) as in Corollary 4.

Please note, that Theorem 6 holds for skew-involutory matrices. Similar comments as those given at the end of Section 2 hold here.

### 4 Coninvolutary matrices

For any coninvolutary matrix \( A \in \mathbb{C}^{n \times n} \) we have \( A^{-1} = \overline{A} \) as \( A\overline{A} = I_n \). Any coninvolutary matrix can be expressed as \( A = e^{i\theta} \) for \( R \in \mathbb{R}^{n \times n} \), see, e.g., [4]. Any real coninvolutary matrix is also involutory. Since \( A^{-1} = \overline{A} \),
the singular values of \( A \) are either 1 or pairs \( \sigma, \frac{1}{\sigma} \). Moreover, any coninvolutory matrix is condiaogonalisable\(^3\), see, e.g., [5, Chapter 4.6].

Let \((u, v, \sigma)\) be singular triplet of a coninvolutory matrix \( A \), that is, let \( Av = \sigma u \) hold. This is equivalent to \( v = \sigma A^{-1} u \) and further to \( A^2 = \frac{1}{\sigma} v \). Thus, the singular triplet \((u, v, \sigma)\) of \( A \) is always accompanied by the singular triplet \((\overline{v}, \overline{u}, \frac{1}{\sigma})\) of \( A \). In case \( \sigma = 1 \), this may collapse into a single triplet if \( v = e^{i\alpha} \overline{u} \) for a real scalar \( \alpha \in [0, 2\pi) \) (as \( e^{i\alpha} = e^{i(\alpha + \pi)} \), there is no need to consider \( v = -e^{i\alpha} \overline{u} \)). The case \( \sigma = 1 \), \( v = e^{i\alpha} \overline{u} \) implies that \( A \) has a coneigenvalue \( e^{-i\alpha} \) as \( A \overline{u} = e^{-i\alpha} u \) holds. It follows immediately, that coninvolutory matrix \( A \) of odd size \( n \) must have a singular value 1.

This gives rise the following theorem.

**Theorem 7.** Let \( A \in \mathbb{C}^{n \times n} \) be coninvolutory. Assume that \( A \) has \( \nu \) singular values \( \sigma_1 \geq \cdots \geq \sigma_\nu > 1 \). These singular values appear in pairs \((\sigma_j, \frac{1}{\sigma_j})\) associated with the singular triplets \((u_j, v_j, \sigma_j)\) and \((\overline{v}_j, \overline{u}_j, \frac{1}{\sigma_j})\), \( j = 1, \ldots, \nu \). Assume further that \( A \) has \( \mu \) singular values 1 which appear in pairs \((1, 1)\) associated with the singular triplets \((u_{\nu+j}, v_{\nu+j}, 1)\) and \((\overline{v}_{\nu+j}, \overline{u}_{\nu+j}, 1)\), \( j = 1, \ldots, \mu \). Then \( A \) has \( k = n - 2\nu - 2\mu = \delta + \eta \) single singular values 1 associated with the singular triplet \((u_{\nu+j}, e^{i\delta} \overline{v}_{\nu+j}, 1)\), \( j = 1, \ldots, k \).

Thus the SVD of \( A \) is given by

\[
\Sigma = \begin{bmatrix}
S & I_\mu & I_\delta \\
I_\mu & S^{-1} & I_\eta \\
I_\mu & I_\eta & S^{-1}
\end{bmatrix}
\]

with \( S = \text{diag}(\sigma_1, \ldots, \sigma_\nu) \) and

\[
U = \begin{bmatrix}
u_1, & \ldots, & \nu_\nu, & u_{\nu+1}, & \ldots, & u_{\nu+\mu}, & u_{\nu+\mu+1}, & \ldots, & u_{\nu+\mu+\delta}, & \overline{v}_1, & \ldots, & \overline{v}_\nu, & \overline{v}_{\nu+1}, & \ldots, & \overline{v}_{\nu+\mu}, & \overline{u}_{\nu+\mu}, & \overline{u}_{\nu+\mu+1}, & \ldots, & \overline{u}_{\nu+\mu+\delta+\eta}
\end{bmatrix},
\]

\[
V = \begin{bmatrix}
u_1, & \ldots, & \nu_\nu, & v_{\nu+1}, & \ldots, & v_{\nu+\mu}, & e^{i\delta} \overline{v}_{\nu+1}, & \ldots, & e^{i\delta} \overline{v}_{\nu+\mu}, & \overline{u}_1, & \ldots, & \overline{u}_\nu, & \overline{u}_{\nu+1}, & \ldots, & \overline{u}_{\nu+\mu}, & e^{i\delta+1} \overline{u}_{\nu+1}, & \ldots, & e^{i\delta+1} \overline{u}_{\nu+\mu}, & e^{i\delta+\eta} \overline{u}_{\nu+\mu+1}, & \ldots, & e^{i\delta+\eta} \overline{u}_{\nu+\mu+\delta+\eta}
\end{bmatrix}
\]

where \( \nu + \mu + \eta = m \)

\[
\delta = \begin{cases}
\eta & \text{if } n = 2m \\
\eta + 1 & \text{if } n = 2m + 1
\end{cases}
\]

For \( U \) and \( V \) from Theorem 7 we have \( U = \overline{V}^T \)

\[
U = \overline{V} \begin{bmatrix}
0 & 0 & I_{\nu+\mu} \\
0 & D_\delta & 0 \\
I_{\nu+\mu} & 0 & 0 \\
0 & 0 & E_\eta
\end{bmatrix} = \overline{V}^T \tag{4.1}
\]

with the unitary and coninvolutory diagonal matrices

\[
D_\delta = \text{diag}(e^{i\delta_1}, \ldots, e^{i\delta_\delta}),
\]

\[
E_\eta = \text{diag}(e^{i\delta_{1+1}}, \ldots, e^{i\delta_{\delta+\eta}}).
\]

---

\(^3\) \( A \in \mathbb{C}^{n \times n} \) is said to be condiaogonalizable if there exists a nonsingular matrix \( S \in \mathbb{C}^{n \times n} \) such that \( D = S^{-1} A S \) is diagonal [5].

\(^4\) A nonzero vector \( x \in \mathbb{C}^n \) such that \( Ax = \lambda x \) for some \( \lambda \in \mathbb{C} \) is said to be a coneigenvector of \( A \); the scalar \( \lambda \) is a coneigenvalue of \( A \), [5].
Thus we have $A = UΣV^H = \overline{V}TΣV^H$ with
\[
TΣ = \begin{bmatrix}
0 & 0 & 0 & S^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & I_\mu & 0 \\
0 & 0 & D_\delta & 0 & 0 & 0 \\
S & 0 & 0 & 0 & 0 & 0 \\
0 & I_\mu & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & E_\eta
\end{bmatrix}.
\] (4.2)

In other words, $A$ is unitarily consimilar to the complex coninvolutory matrix $TΣ$. A similar statement is given in [5, Exercise 4.6P27] (just write $D_\delta$ and $E_\eta$ as a product of their square roots and move the square roots into $V$ and $V^H$).

As in Section 2 we obtain
\[
A = V \begin{bmatrix}
S^{-\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & I_\mu & 0 & 0 & 0 & 0 \\
0 & 0 & I_\delta & 0 & 0 & 0 \\
0 & 0 & 0 & S^{-\frac{1}{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & I_\mu & 0 \\
0 & 0 & 0 & 0 & 0 & I_\eta
\end{bmatrix} T \begin{bmatrix}
S^{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & I_\mu & 0 & 0 & 0 & 0 \\
0 & 0 & I_\delta & 0 & 0 & 0 \\
0 & 0 & 0 & S^{\frac{1}{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & I_\mu & 0 \\
0 & 0 & 0 & 0 & 0 & I_\eta
\end{bmatrix} V^H,
\]

Hence, $A$ and $T$ are consimilar matrices and their coneigenvalues are identical. Taking a closer look at $T$, we immediately see that $T$ is unitarily consimilar to the identity

\[
P^\tau P = \begin{bmatrix}
I_{v+\mu} & 0 & 0 & 0 \\
0 & I_\delta & 0 & 0 \\
0 & 0 & I_{v+\mu} & 0 \\
0 & 0 & 0 & I_\eta
\end{bmatrix}
\]

with the unitary matrix
\[
P = \begin{bmatrix}
\frac{1}{\sqrt{2}} I_{v+\mu} & 0 & \frac{1}{\sqrt{2}} I_{v+\mu} & 0 \\
0 & D_\delta^{\frac{1}{2}} & 0 & 0 \\
-\frac{1}{\sqrt{2}} I_{v+\mu} & 0 & \frac{1}{\sqrt{2}} I_{v+\mu} & 0 \\
0 & 0 & 0 & E_\eta^{-\frac{1}{2}}
\end{bmatrix}.
\]

Thus, all coneigenvalues of a coninvolutory matrix are +1. This has already been observed in [5, Theorem 4.6.9].

The following corollary summarizes our findings.

**Corollary 8** (Canonical Form, Coneigen decomposition). Let $A \in \mathbb{C}^{n\times n}$ be coninvolutory. Then $A$ is unitarily consimilar to the coninvolutory matrix $TΣ$ as in (4.2) and consimilar to the identity.

Before we turn our attention to the skew-coninvolutory case, we would like to point out that most of our observations given in this section also follow easily from [6, Theorem 7.1]. For the ease of the reader, this theorem is stated next.

**Theorem 9.** Let $A \in \mathbb{C}^{n\times n}$. If $\overline{A}A$ is normal, then $A$ is unitarily congruent to a direct sum of blocks, each of which is
\[
\begin{bmatrix}
\lambda & 0 \\
0 & 1
\end{bmatrix}, \quad \lambda, \tau \in \mathbb{R}, \lambda \geq 0, \tau > 0, \mu \in \mathbb{C}, \text{ and } \mu \neq 1.
\] (4.3)

This direct sum is uniquely determined by $A$ up to permutation of its blocks and replacement of any nonzero parameter $\mu$ by $\mu^{-1}$ with a corresponding replacement of $\tau$ by $\tau|\mu|$. Conversely, if $A$ is unitarily congruent to a direct sum of blocks of the form (4.3), then $\overline{A}A$ is normal.
In case $\overline{AA} = I$, Theorem 9 gives for the $1 \times 1$ blocks that $\lambda = \pm 1$ has to hold. For the $2 \times 2$ blocks it follows $\tau^2 \mu = 1$ and $\tau^2 \overline{\nu} = 1$. Thus, as $\mu \neq 1$ we have $\tau \neq 1$. Analogous to the discussion at the end of Section 2 part of our findings, in particular the pairing $(\tau, \frac{1}{\tau})$ of the singular values $\tau > 1$ and the relation of the corresponding singular vectors, follows from this; see also [6, Corollary 8A]. The fact, that singular values $1$ may also appear in pairs $(1, 1)$ with related singular triplets does not follow from Theorem 9.

5 Skew-coninvolutory matrices

For any skew-coninvolutory matrix $A \in \mathbb{C}^{m \times m}$ we have $A^{-1} = -\overline{A}$ as $AA = -I_m$. Skew-coninvolutory matrices exist only for even $m$, as det $AA$ is nonnegative for any $A \in \mathbb{C}^{m \times m}$. Properties and canonical forms of skew-coninvolutory matrices have been analyzed in detail in [1].

From $A = U\Sigma V^H$ we see that $A^{-1} = V\Sigma^{-1}U^H = -U\Sigma V^T$. Thus, the singular values appear in pairs $\sigma, \frac{1}{\sigma}$ (see [1, Proposition 5]) and the singular triplet $(u, v, \sigma)$ of $A$ is always accompanied by the singular triplet $(-\overline{v}, \overline{u}, \frac{1}{\sigma})$ of $A$. There is no need to consider singular values $\sigma = 1$ separately, as their singular vectors do not satisfy any additional condition. This gives rise to the following theorem.

**Theorem 10.** Let $A \in \mathbb{C}^{2n \times 2n}$ be skew-coninvolutory. $A$ has $n$ singular values $\sigma_1 \geq \ldots \geq \sigma_n \geq 1$. These singular values appear in pairs $(\sigma_j, \frac{1}{\sigma_j})$ associated with the singular triplets $(u_j, v_j, \sigma_j)$ and $(-\overline{v}_j, \overline{u}_j, \frac{1}{\sigma_j}), j = 1, \ldots, n$.

Thus the SVD of $A$ is given by

$$\Sigma = \begin{bmatrix} S & 0 \\ 0 & S^{-1} \end{bmatrix}$$

with $S = \text{diag}(\sigma_1, \ldots, \sigma_n)$ and

$$U = \begin{bmatrix} u_1, \ldots, u_n, -\overline{v}_1, \ldots, -\overline{v}_n \\ v_1, \ldots, v_n, \overline{u}_1, \ldots, \overline{u}_n \end{bmatrix},$$

$$V = \begin{bmatrix} u_1, \ldots, u_n, -\overline{v}_1, \ldots, -\overline{v}_n \end{bmatrix}.$$

For $U$ and $V$ from Theorem 10 we have $U = -\overline{V}J_n$ with $J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. Thus we have $A = U\Sigma V^H = -\overline{V}J_n\Sigma V^H$.

Any skew-coninvolutory matrix is unitarily consimilar to the elementary skew-coninvolutory matrix $-J_n\Sigma$. Observe

$$J_n\Sigma = \begin{bmatrix} 0 & S^{-1} \\ -S & 0 \end{bmatrix} = \begin{bmatrix} S^{-\frac{1}{2}} & 0 \\ 0 & S^{\frac{1}{2}} \end{bmatrix} J_n \begin{bmatrix} S^{\frac{1}{2}} & 0 \\ 0 & S^{-\frac{1}{2}} \end{bmatrix},$$

(see [1, Theorem 12]). Let $Z = V \begin{bmatrix} S^{-\frac{1}{2}} & 0 \\ 0 & S^{\frac{1}{2}} \end{bmatrix}$. Thus $A = -ZJ_nZ^{-1}$ and $-J_n$ are consimilar matrices (see [1, Theorem 13]).

In summary, we have the following corollary.

**Corollary 11 (Canonical Form, Consimilarity).** Let $A \in \mathbb{C}^{2n \times 2n}$ be skew-coninvolutory. Then $A$ is unitarily consimilar to the skew-coninvolutory matrix $-J_n\Sigma$ as in (5.1) and consimilar to $-J_n$.

Please note, that Theorem 9 holds for skew-coninvolutory matrices. Similar comments as those given at the end of Section 4 hold here, see also [6, Theorem 8.3].

6 Concluding remarks

We have described the SVD of (skew-)involutory and (skew-)coninvolutory matrices in detail. In order to do so we made use of the fact that for any matrix in one of the four classes of matrices the singular values $\sigma > 1$ appear in pairs $(\sigma, \frac{1}{\sigma})$, while singular values $\sigma = 1$ may appear in pairs or by themselves. As the singular
vectors of pairs of singular values are closely related, the SVD reveals all relevant information also about the eigenvalues and eigenvectors. Some of our findings are new, some are rediscoveries of known results.

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