Quantized Consensus by the ADMM: Probabilistic versus Deterministic Quantizers

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Abstract—This paper develops efficient algorithms for distributed average consensus with quantized communication using the alternating direction method of multipliers (ADMM). We first study the effects of probabilistic and deterministic quantizations on a distributed version of the ADMM. With probabilistic quantization, this approach yields linear convergence to the desired average in the mean sense with bounded variance. When deterministic quantization is employed, the distributed ADMM converges to a consensus within $3 + \lceil \log_2(1+\delta) \rceil$ iterations where $\delta > 0$ depends on the network topology and $\Omega$ is a polynomial of quantization resolution, agents’ data, and the network topology. A tight upper bound on the consensus error is also obtained, which depends only on the quantization resolution and the average degree of the graph. This bound is much preferred in large scale networks over existing algorithms whose consensus errors are increasing in the range of agents’ data, quantization resolution and the number of agents. We finally propose our algorithm which combines the probabilistic and deterministic quantizations. Simulations show that the consensus error of our algorithm is typically less than one quantization resolution for all connected networks with agents’ data of arbitrary magnitudes. This is so far the best known result for quantized consensus.

Index Terms—Quantized consensus, dither, probabilistic quantization, deterministic quantization, alternating direction method of multipliers, linear convergence.

I. INTRODUCTION

In recent years there has been considerable interest in distributed average consensus where a group of agents aim to reach a consensus on the average of their measurements [1]–[19]. This is largely motivated by numerous applications in control, signal processing, and computer science. For example, the distributed averaging is a fundamental problem in ad hoc network applications, such as distributed agreement and synchronization [3], distributed coordination of mobile autonomous agents [4], and distributed data fusion in sensor networks [5]. It has also found applications in load balancing for parallel computers [6].

We consider in this paper distributed averaging algorithms where nodes only exchange information with their immediate neighbors. These algorithms are extremely attractive for large scale networks characterized by the lack of centralized access to information. They are also energy efficient and enhance the survivability of the networks, compared with fusion center based processing. However, a number of factors such as limited bandwidth, sensor battery power and computing resources place tight constraints on the rate and form of information exchange amongst neighboring nodes, resulting in quantized consensus problems [11, 7]. This paper is specifically devoted to developing efficient algorithms for quantized consensus in connected networks with static topologies.

A. Related work

There have been three widely used methods for solving distributed averaging problems. A classical approach is to update the state of each node with a weighted average of values from neighboring nodes [8]–[10]. The matrix, consisting of the weights associated with the edges, is chosen to be doubly stochastic to ensure convergence to the average. Another method is a gossip based algorithm which was initially introduced in [20] for consensus problems and further studied by many researchers, e.g., [7], [11], [12]. The third approach is to employ the ADMM which is an iterative algorithm for solving convex problems and has received much attention recently (see [21] and references therein for details). The idea is to formulate the data average as the solution to a least-squares minimization problem and manipulate the ADMM updates to derive a decentralized algorithm [13–15].

In the most ideal case where agents are able to send and receive real values with infinite precision, the three methods all lead to the desired consensus at the average. When quantization is imposed, however, these methods do not directly apply. A well studied approach for quantized consensus is to use dithered quantizers which add noises to agents’ variables before quantization [22]. By imposing certain conditions, the quantization error sequence becomes independent and identically distributed (i.i.d.) and is also independent of the input sequence. The classical approach and the gossip based algorithm then yield the almost sure consensus at a common but random quantization level with the expectation of the consensus value equal to the desired average [16, 17, 19]. To the best of our knowledge, there have been no existing results on the ADMM based method for quantized consensus. Nevertheless, since the quantization error of dithered quantizer is zero-mean and has bounded variance, we can immediately extend the results in [14, 15] to quantized consensus (see Section IV). That is, the ADMM based method using dithered quantization leads to the consensus at the data average in the mean sense whose variance converges to a finite value.

Meanwhile, studies on distributed average consensus with deterministic quantizers have been scarcely reported. Deterministic quantization makes the problems much harder to deal with as the error terms caused by quantization no longer possess tractable statistical characteristics [16, 17]. The authors in [10] show that the classical approach, where a quantization...
rule that rounds the values down is adopted, converges to a consensus with an error from the average depending on the quantization resolution, number of agents, agents’ data and the updated weights of each agent. A recent result indicates that this approach, with appropriate choices of the weights, reaches a quantized consensus close to the average in finite time or leads all agents’ variables to cycle in a small neighborhood around the average \[18\]. The gossip based algorithms in \[19\] have similar results to those of the classical approach. The ADMM based algorithms for deterministically quantized consensus, however, have not yet been explored.

B. Our contributions

One shall note that the consensus error for deterministically quantized consensus in \[10, 18\] is much undesired when the number of agents or the range of agents’ data is very large. Unfortunately, this is typically the case in large scale networks or big data settings. The ADMM has been known to be an efficient algorithm for large scale optimizations and used in various applications such as regression and classification \[21\]. Moreover, \[23, 26\] validate the fast convergence of the ADMM and \[14, 15\] demonstrate the resilience of the ADMM to noise, link failures, etc. We therefore expect ADMM based methods to work well for quantized consensus problems, with regards to both consensus error and convergence time.

We first study the effect of the probabilistic quantization \[25\], which is equivalent to a dithering method as shown by \[16\] Lemma 2, on the ADMM based method. Utilizing the first and second order moments of the probabilistic quantizer output, we establish the convergence to the average in the mean sense based on existing convergence results of the ADMM. A simple method is proposed such that the consensus is reached in probability, which is slightly weaker than the almost sure consensus. We remark that even though our result is weaker than existing results in \[16, 17, 19\], they are established directly from the convergence of the ADMM. Furthermore, recent work of \[24\] enables us to immediately characterize the convergence rate of the distributed ADMM with probabilistic quantization.

The main contribution of this paper is to establish convergence results for deterministic quantization. Our results show that a distributed deterministically quantized ADMM algorithm converges to a consensus within finite time as long as an initialization condition is satisfied. The initialization condition is rather mild; indeed, simply setting the variables to be 0 suffices. We derive a tight upper bound for the consensus error, which only depends on the quantization resolution and the average degree of the undirected graph (two times the ratio of the number of edges to the number of nodes). This is much preferred over existing methods for large scale networks as it does not rely on the number of agents nor the agent’s data. We finally capture the convergence speed, that is, it converges to a consensus within approximately \(\log_{\delta+1+\delta} \Omega\) iterations where \(\delta > 0\) depends on the network topology and \(\Omega\) is a polynomial of quantization resolution, agents’ data and the network topology.

We notice that our deterministically quantized ADMM method does not converge to global optima. For algorithms that converge to local optima, it is well known that a good starting point usually helps. This inspires our approach for quantized consensus which first uses the probabilistic method to obtain a good starting point and then employs the deterministic algorithm to reach a consensus. Simulations show that our algorithm performs the best among all existing methods in terms of the consensus error.

C. Paper organization

The rest of this paper is organized as follows. Section II applies the ADMM to the distributed averaging problem without quantization, which leads to a distributed ADMM algorithm. We then develop several convergence results of this algorithm; they will be used later in establishing our main results. Section III defines the probabilistic and deterministic quantization schemes. Their effects on the distributed ADMM are studied respectively in Sections IV and V. Section VI describes the proposed algorithm for quantized consensus which combines the two quantized ADMM methods, followed by simulation results in Section VII. Section VIII concludes the paper.

D. Notations

Denote by \(\|x\|_2\) the Euclidean norm of a vector \(x\) and \((x, y)\) the inner product of two vectors \(x\) and \(y\). Given a semidefinite matrix \(G\) with proper dimensions, the \(G\)-norm of \(x\) is \(\|x\|_G = \sqrt{x^T G x}\). Also denote \(\sigma_{\max}(M)\) as the largest singular value of a square matrix \(M\) and \(\sigma_{\min}(M)\) as the smallest nonzero singular value of \(M\).

We use two definitions of rate of convergence for an iterative algorithm. A sequence \(x^k\), where the superscript \(k\) stands for time index, is said to converge \(Q\)-linearly to a point \(x^*\) if there exists a number \(v \in (0, 1)\) such that \(\lim_{k \to \infty} \frac{\|x^{k+1} - x^*\|}{\|x^k - x^*\|} = v\) with \(\| \cdot \|\) being a vector norm. A sequence \(y^k\) is said to converge \(R\)-linearly to \(y^*\) if for all \(k\), \(\|y^k - y^*\| \leq \|x^k - x^*\|\) where \(x^k\) converges \(Q\)-linearly to \(x^*\).

II. DISTRIBUTED AVERAGE CONSENSUS BY THE ADMM

This section introduces the distributed consensus ADMM (DC-ADMM) for average consensus without quantization. This ideal case provides a good understanding of how the ADMM works for distributed average consensus. We start with the setting of the distributed averaging problem.

A. Problem Setting

Consider a connected network of \(N\) agents which are bidirectionally connected by \(E\) edges (and thus \(2E\) arcs). We describe this network as a symmetric directed graph \(G_d = \{V, A\}\) or an undirected graph \(G_u = \{V, E\}\), where \(V\) is the set of vertices with cardinality \(|V| = N\), \(A\) is the set of arcs with \(|A| = 2E\) and \(E\) is the set of edges with \(|E| = E\). Assume that the topology of the network is fixed throughout this paper. Let \(x_i \in \mathbb{R}\) be the data available at node \(i\), \(i = 1, 2, \ldots, N\), and \(r \in \mathbb{R}^N\) the vector concatenating all
The goal of distributed average consensus is to compute the data average
\[ x_{\text{avg}} = \frac{1}{N} \sum_{i=1}^{N} r_i \]  
by local data exchanges among neighboring nodes.

### B. Application of the ADMM to distributed average consensus: DC-ADMM

The ADMM applies in general to the convex optimization problem in the form of
\[
\begin{align*}
\text{minimize} & \quad g_1(y_1) + g_2(y_2) \\
\text{subject to} & \quad C_1 y_1 + C_2 y_2 = c,
\end{align*}
\]
where \( y_1 \) and \( y_2 \) are optimization variables, \( g_1 \) and \( g_2 \) are convex functions, and \( C_1 y_1 + C_2 y_2 = c \) is a linear constraint on \( y_1 \) and \( y_2 \). The ADMM solves a sequence of subproblems involving \( g_1 \) and \( g_2 \) one at a time and iterate to converge when, e.g., \( g_1 \) and \( g_2 \) are proper closed convex functions and the Laplacian of \( \mathcal{G} \) has a saddle point [21].

To apply the ADMM, we first formulate (11) as a convex optimization problem
\[ x_{\text{avg}} = \arg \min_{\bar{x}} \sum_{i=1}^{N} \frac{1}{2}(\bar{x} - r_i)^2, \]
that is, the data average is the solution to a least-squares minimization problem. We continue to reformulate (3) in the form of (2) as
\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{N} \frac{1}{2}(x_i - r_i)^2 \\
\text{subject to} & \quad x_i = z_{ij}, \quad x_j = z_{ij}, \forall (i, j) \in \mathcal{A},
\end{align*}
\]
where \( x_i \) is the local copy of the common optimization variable \( \bar{x} \) at node \( i \) and \( z_{ij} \) is an auxiliary variable imposing the consensus constraint on neighboring nodes \( i \) and \( j \). Since the network is connected, this constraint ensures the consensus to be achieved over the entire network, i.e., \( x_i = x_j, \forall i, j \in \mathcal{A} \), which in turn guarantees the solution to (4) is the data average \( x_{\text{avg}} \). Further define \( x \in \mathbb{R}^N \) as a vector concatenating all \( x_i \), \( z \in \mathbb{R}^{2E} \) as a vector concatenating all \( z_{ij} \), and
\[
f(x) = \frac{1}{2} \| x - r \|^2.
\]
Then (4) can be written in a matrix form as
\[
\begin{align*}
\text{minimize} & \quad f(x) + g(z) \\
\text{subject to} & \quad Ax + Bz = 0,
\end{align*}
\]
where \( g(z) = 0 \), and \( 0 \) is a column vector with proper dimensions and all entries being 0. Here \( B = [-I_{2E}; -I_{2E}] \) with \( I_{2E} \) being a \( 2E \times 2E \) identity matrix and \( A = [A_1; A_2] \) with \( A_1, A_2 \in \mathbb{R}^{2E \times N} \). If \( (i, j) \in \mathcal{E} \) and \( z_{ij} \) is the \( q \)th entry of \( z \), then the \((q, i)\)th entry of \( A_1 \) and the \((q, j)\)th entry of \( A_2 \) are 1; otherwise the corresponding entries are 0.

We are now ready to apply the ADMM to solve the consensus problem. The augmented Lagrangian of [5] is
\[
L_\rho(x, z, \lambda) = f(x) + \langle \lambda, Ax + Bz \rangle + \frac{\rho}{2} \| Ax + Bz \|^2,
\]
where \( \lambda = [\beta; \gamma] \) with \( \beta, \gamma \in \mathbb{R}^{2E} \) is the Lagrange multiplier and \( \rho \) is a positive algorithm parameter. At iteration \( k + 1 \), the ADMM first obtains \( x^{k+1} \) by minimizing \( L_\rho(x, z^k, \lambda^k) \), then calculates \( z^{k+1} \) by minimizing \( L_\rho(x^{k+1}, z, \lambda^k) \) and finally updates \( \lambda^{k+1} \) using \( x^{k+1} \) and \( z^{k+1} \). The updates are
\[
\begin{align*}
\text{x-update:} & \quad \nabla f(x^{k+1}) + A^T \lambda^k + \rho A^T (Ax^{k+1} + Bz^k) = 0 \\
\text{z-update:} & \quad B^T \lambda^k + \rho B^T (Ax^{k+1} + Bz^{k+1}) = 0 \\
\text{\lambda-update:} & \quad \lambda^{k+1} - \lambda^k - \rho (Ax^{k+1} + Bz^{k+1}) = 0,
\end{align*}
\]
where \( \nabla f(x^{k+1}) = x^{k+1} - r \) is the gradient of \( f \) at \( x^{k+1} \).

A nice property of the ADMM, known as global convergence, states that the sequence \( (x^k, z^k, \lambda^k) \) generated by (7) has a single limit point \((x^*, z^*, \lambda^*)\) which is a primal-dual solution to (6). Proofs can be found in [21], [26]. Noting that our objective function \( f(x) \) is strongly convex in \( x \), we obtain \( x^* = 1 x_{\text{avg}} \) as our unique primal solution where \( I \) denotes the \( N \)-dimensional column vector with all entries being 1. To summarize, we have

\textit{Lemma 1 (Global convergence of the ADMM [27], [26])}:

For any initial values \( x^0 \in \mathbb{R}^N \), \( z^0 \in \mathbb{R}^{2E} \) and \( \lambda^0 \in \mathbb{R}^{2E} \), the updates in [7] yield that as \( k \to \infty \),
\[
x^k \to x^*, \quad z^k \to z^*, \quad \lambda^k \to \lambda^*,
\]
where \((x^*, z^*, \lambda^*)\) is a primal-dual solution to (6) and \( x^* = 1 x_{\text{avg}} \) is unique for the distributed average consensus problem [5].

While [7] provides an efficient algorithm to solve (3), it is not clear whether [7] can be carried out in a distributed manner, i.e., data exchanges only occur within neighboring nodes. Interestingly, as established in Lemma 1, the ADMM allows any initial values \( x^0 \in \mathbb{R}^N \), \( z^0 \in \mathbb{R}^{2E} \) and \( \lambda^0 \) for global convergence; there indeed exist initial values that decentralize [7]. Define \( M_x = A_1^T + A_2^T \) and \( M_z = A_1^T - A_2^T \) which are respectively the unoriented and oriented incidence matrices with respect to the undirected graph \( G_\omega \). Initialize \( \beta^0 = -\gamma^0 \) and \( z^0 = \frac{1}{2} M_x^T x^0 \). As shown in [26], the updates in [7] lead to
\[
\begin{align*}
x_i^{k+1} &= \frac{1}{1 + 2\rho |N_i|} \left( \rho |N_i| x_i^k + \rho \sum_{j \in N_i} x_j^k - \alpha_i^k + r_i \right), \\
\alpha_i^{k+1} &= \alpha_i^k + \rho \left( |N_i| x_i^{k+1} - \sum_{j \in N_i} x_j^{k+1} \right),
\end{align*}
\]
at node \( i \), where \( N_i \) denotes the set of neighbors of node \( i \) and \( \alpha_i^k \) is the \( i \)th entry of \( \alpha^k = M - \beta^k \in \mathbb{R}^N \). Obviously, (8) is fully decentralized as the updates of \( x_i^{k+1} \) and \( \alpha_i^{k+1} \) only rely on local and neighboring information. Therefore (8) can be used for distributed average consensus. With a little abuse of terminology, we refer to [8] as the DC-ADMM for distributed average consensus [1].

1 Some authors refer to this algorithm as distributed ADMM or consensus ADMM. DC-ADMM also represents dual consensus ADMM in some papers.
If we further initialize $\beta^0$ in the column space of $M^T$ (e.g., $\beta^0 = 0$), then $\beta^k$ lies in the column space of $M^T$ and converges to a unique $\beta^*$. We will use this result immediately but postpone its proof to Lemma 4. Note that this implies $M \cdot M^T$ is rank deficient.

Lemma 2: For $\beta$ and $\beta'$ in the column space of $M^T$, define $\alpha = M \cdot \beta$ and $\alpha' = M \cdot \beta'$. Then $\alpha$ and $\beta$ are one-to-one correspondence; i.e., if $\beta = \beta'$ then $\alpha = \alpha'$, and vice versa.

Proof: That $\beta = \beta'$ implying $\alpha = \alpha'$ is straightforward. Consider $\alpha = M \cdot \beta$ and write $\beta = M^T b$ for some $b \in \mathbb{R}^N$. $\alpha'$, $\beta'$ and $b'$ are similarly defined. Then

$$\|\alpha - \alpha'\|_2 = \|M \cdot M^T (b - b')\|_2$$

$$\geq \alpha_{\min} (M \cdot M^T) \|b - b'\|_2$$

$$= \alpha_{\min} (M \cdot M^T) \|\beta - \beta'\|_2,$$

where $\alpha_{\min}(M \cdot M^T)$ is the smallest nonzero singular value of $M \cdot M^T$, whose existence is guaranteed for a connected graph. Therefore, we have $\beta = \beta'$ if $\alpha = \alpha'$.

It is therefore meaningful to define an initialization condition for the DC-ADMM. A similar global convergence property for the DC-ADMM is given in Lemma 3.

**Initialization condition for the DC-ADMM:** $x^0$ can be any vector in $\mathbb{R}^N$ and $\alpha^0$ lies in the column space of $M \cdot M^T$.

Lemma 3 (Global convergence of the DC-ADMM): For any $x^0$ and $\alpha^0$ satisfying the initialization condition, the DC-ADMM leads to

$$x^k \to x^*$$

and $\alpha^k \to \alpha^*$ as $k \to \infty$,

where $x^* = 1_x \text{avg}$ and $\alpha^* = r - 1_x \text{avg}$ which lies in the column space of $M \cdot M^T$ are both unique.

Proof: Global convergence follows from Lemmas 1 and 2, together with the fact that $\beta^k$ converges to a unique $\beta^*$ which lies in the column space of $M^T$.

Now taking $k \to \infty$ in (8) and using the fact that $x^*_i = x_{\text{avg}}$ for $i = 1, 2, \ldots, N$, we have

$$\alpha^*_i = r_i - x_{\text{avg}}.$$

Throughout the rest of this paper, we assume that the DC-ADMM, wherever used, is initialized to satisfy the initialization condition.

C. Linear convergence of the DC-ADMM

We investigate two properties of the DC-ADMM: the first property is built on global convergence while the second considers the rate of convergence.

Define $L_+ = \frac{1}{2} M \cdot M^T$ and $L_- = \frac{1}{2} M \cdot M^T$ which are respectively the signless and signed Laplacian matrices with respect to $G_0$. Let $W \in \mathbb{R}^{N \times N}$ be the degree matrix related to the underlying network, i.e., a diagonal matrix with its $(i, i)$th entry being the degree of node $i$ and other entries being 0. Then Lemma 2 is an immediate result from the property of $L_-$ and $W = \frac{1}{2} (L_+ + L_-)$ [27]. Rewrite in the matrix form as

$$x^{k + 1} = (I_N + 2\rho W)^{-1} (\rho L_+ x^k - \alpha^k + r),$$

or equivalently,

$$s^{k + 1} = D s^k,$$

with

$$s^k = \begin{bmatrix} x^k \\ \alpha^k \\ r^k \end{bmatrix},$$

and

$$D = \begin{bmatrix} \rho D_0 L_+ & -D_0 & D_0 \\ \rho^2 L_- D_0 L_+ & I_N - \rho L_- D_0 & \rho L_- D_0 \\ 0_N & 0_N & I_N \end{bmatrix},$$

for fixed $\alpha_1, \alpha_2 \in \mathbb{R}^N$ and $C \in \mathbb{R}^{N \times N}$.

Proof: By Lemma 3 we have for any $s^0$ that satisfies the initialization condition,

$$s^\infty = \begin{bmatrix} x^\infty \\ \alpha^\infty \\ r^\infty \end{bmatrix} = \begin{bmatrix} x^* \\ \alpha^* \\ r^* \end{bmatrix} = \begin{bmatrix} 1_x \text{avg} \\ r - 1_x \text{avg} \end{bmatrix}.$$
theorem. The remaining blocks, $D_{22}$ and $D_{33}$, follow directly from the multiplication operation of matrices.

Given global convergence, we now turn our attention to the rate of convergence of the DC-ADMM. Recent work of [23], [20] has established the linear convergence of the ADMM. Unfortunately, their results do not apply to the DC-ADMM as their conditions are not satisfied here. In [23], the step size of the dual variable update, i.e., $\rho$ in the $\lambda$-update of (7), needs to be sufficiently small while our DC-ADMM has a fixed step size $\rho$ that can be any positive number (see Remark 2 for further discussion on the choice of $\rho$). The linear convergence in [20] is established provided that either $g(z)$ is strongly convex or $B$ is full row-rank in $G$. In our formulation, however, $g(z) = 0$ is not strongly convex and $B = [-I_{2}; -I_{2}]$ is row-rank deficient. Nevertheless, we first give Lemma 4 with regards to the convergence rate of a vector concatenating $z$ and $\beta$. A more general result can be found in [24, Theorem 1]. Our proof is similar to that of [24] but much simpler.

**Lemma 4 ([24 Theorem 1]):** Consider the ADMM iteration (7) that solves (5). Define

$$ u = \begin{bmatrix} z \\ \beta \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} \rho I_{2} & 0_{2} \\ 0_{2} & \frac{1}{\rho} I_{2} \end{bmatrix}, $$

where $\beta$ is the dual variable. If we initialize $z^0 = \frac{1}{2} M_{T}^{T} x^0$, $\beta^0 = -\gamma^0$ where $\gamma$ is the other dual variable and $\beta^0$ is in the column space of $M_{T}$, then for $k = 0, 1, \ldots$, $z^{k} = \frac{1}{2} M_{T}^{T} x^{k}$, $\beta^{k}$ lies in the column space of $M_{T}$, and $(x^{k}, z^{k}, \beta^{k})$ converges uniquely to $(x^{*}, z^{*}, \beta^{*})$ with $x^{*} = 1_{x\text{avg}}$, $z^{*} = \frac{1}{2} M_{T}^{T} 1_{x\text{avg}}$ and $\beta^{*}$ being a vector in the column space of $M_{T}$. Furthermore, $u^{k} = [z^{k}; \beta^{k}]$ converges $Q$-linearly to its optimal $u^{*} = [z^{*}; \beta^{*}]$ with respect to the $G$-norm

$$ \|u^{k+1} - u^{*}\|_{G}^{2} \leq \frac{1}{1 + \delta} \|u^{k} - u^{*}\|_{G}^{2}, $$

where

$$ \delta = \min \left\{ \frac{\sigma_{\min}^{2}(M_{-})}{2 \sigma_{\max}^{2}(M_{+})}, \frac{4 \rho \sigma_{\min}^{2}(M_{-})}{\rho^{2} \sigma_{\min}^{2}(M_{+}) \sigma_{\max}^{2}(M_{-}) + 8} \right\}, $$

$\sigma_{\max}(M_{-})$ denotes the spectral norm or the largest singular value of $M_{+}$, and $\sigma_{\min}(M_{-})$ denotes the smallest positive singular value of $M_{-}$.\[\]

**Proof:** See Appendix.

With this lemma, we can now establish the linear convergence rate of the DC-ADMM.

**Theorem 2 (Linear convergence of the DC-ADMM):** For any $s^{0}$ satisfying the initialization condition,

$$ \|s^{k+1} - s^{*}\|_{2} = \|(D^{k+1} - D^{*}) s^{0}\|_{2} \leq \left(1 + \sqrt{\frac{\rho}{1 + \delta} \sigma_{\max}(M_{-})} \right) \|u^{k} - u^{*}\|_{G}, $$

where $u^{k}$ is defined in Lemma 4. Therefore, $s^{k}$ is $R$-linearly convergent to $s^{*}$.

**Proof:** Notice that the initializations in Lemma 4 decentralize the ADMM iteration (7) into the DC-ADMM. Thus $x^{k}$ is the same in the ADMM iteration (7) and the DC-ADMM iteration (8) while $\alpha^{k} = M_{-} \beta^{k}$. Then (15) implies

$$ \|x^{k+1} - x^{*}\|_{2} \leq \|u^{k} - u^{*}\|_{G}. $$

We also have

$$ \| \alpha^{k+1} - \alpha^{*} \|_{2} = \| M_{-} (\beta^{k+1} - \beta^{*}) \|_{2} \leq \sigma_{\max}(M_{-}) \| \beta^{k+1} - \beta^{*} \|_{2} \leq \sqrt{\rho} \sigma_{\max}(M_{-}) \| u^{k+1} - u^{*} \|_{G} \leq \sqrt{\frac{\rho}{1 + \delta}} \sigma_{\max}(M_{-}) \| u^{k} - u^{*} \|_{G}, $$

where the last two inequalities are from the definitions of $u$ and $G$, and (11), respectively. Thus,

$$ \|s^{k+1} - s^{*}\|_{2} \leq \|x^{k+1} - x^{*}\|_{2} + \| \alpha^{k+1} - \alpha^{*} \|_{2} \leq \left(1 + \sqrt{\frac{\rho}{1 + \delta}} \sigma_{\max}(M_{-}) \right) \| u^{k} - u^{*} \|_{G}. $$

**III. QUANTIZED CONSENSUS**

To model the effect of quantized communication, we assume that each agent can store and compute real values with infinite precision; however, an agent can only transmit quantized data through the channel which are received by its neighbors without any error. The quantization operation is defined as follows. Let $\Delta > 0$ be a given quantization resolution and define the quantization lattice in $\mathbb{R}$ by

$$ \Lambda = \{ t \Delta : t \in \mathbb{Z} \}. $$

A quantizer is a function $Q : \mathbb{R} \rightarrow \Lambda$ that maps a real value to some point in $\Lambda$. Among all quantizers there are two that are widely used:

1) **Probabilistic quantizer** $Q_p$ defined as follows: for $y \in [t \Delta, (t+1) \Delta)$,

$$ Q_p(y) = \begin{cases} t \Delta, & \text{with probability } \frac{y}{\Delta} - t, \\ (t + 1) \Delta, & \text{with probability } t + 1 - \frac{y}{\Delta}. \end{cases} $$

2) **Rounding quantizer** $Q_d$ which projects $y \in \mathbb{R}$ to its nearest point in $\Lambda$:

$$ Q_d(y) = t \Delta, \quad \text{if } \left( t - \frac{1}{2} \right) \Delta \leq y < \left( t + \frac{1}{2} \right) \Delta. $$

We point out that probabilistic quantization is equivalent to a dithered quantization method (see [16, Lemma 2]) while rounding quantization is one of the deterministic quantization schemes. Throughout the rest of this paper, we use $Q(y)$ or $y|Q)$ to denote the quantized value of $y \in \mathbb{R}$ regardless of its quantization scheme; we use $Q_p(y)$ (or $y|Q_p)$ and $Q_d(y)$ (or $y|Q_d)$ when it is necessary to specify the quantization scheme. Quantizing a vector means quantizing each of its entries. Define $e|Q) = y|Q) - y$ as the quantization error. It is clear that

$$ |e|_Q| \leq \Delta \quad \text{and} \quad |e|_{Q_d}| \leq \frac{1}{2} \Delta, \quad \text{for any } y \in \mathbb{R}. $$

As seen from Section III the DC-ADMM has the advantage of global and linear convergence for solving the average consensus problem as long as the initialization condition is met. The authors in [14], [15] have also shown the good behavior of the ADMM in distributed settings when noise or random link failures are imposed. The rest of this paper
is devoted to investigating the effects of the two quantization schemes defined in (13) and (14) on the performance of the DC-ADMM. We remark that the results of probabilistic and rounding quantizations hold respectively for other dithered and deterministic cases, which will be elaborated in Sections IV and V.

IV. PROBABILISTIC QUANTIZATION

For ease of presentation, we only study the probabilistic quantization defined in (13). The results can be easily extended to any other dithered quantization as the only information used here is the first and second order moments of the probabilistic quantizer output which are stated in the following lemma. See [25] for a proof.

Lemma 5 (Lemma 2)): For every $y \in \mathbb{R}$, it holds that

$$
\mathbb{E}[Q_p(y)] = y \quad \text{and} \quad \mathbb{E}[(y - Q_p(y))^2] \leq \frac{\Delta^2}{4}.
$$

The iterations in (8) now take the form of

$$
x_i^{k+1} = \frac{1}{1 + 2\rho|N_i|} \left( \rho|N_i|x_i^k + \rho \sum_{j \in N_i} x_{j(Q_a)}^k - \alpha_i^k + r_i \right),
$$

$$
\alpha_i^{k+1} = \alpha_i^k + \rho \left( |N_i|x_i^{k+1} - \sum_{j \in N_i} x_{j(Q_a)}^{k+1} \right).
$$

(16)

As illustrated in (14), iterations (16) can be interpreted as stochastic gradient updates. Viewed from this point, the quantization error $x_i^k$ fluctuates around the quantization-free updates (8). Our convergence claims are

Theorem 3: Let $x^0$ and $\alpha^0$ satisfy the initialization condition. The probabilistically quantized DC-ADMM (PQ-DC-ADMM) iteration (16) generates $x_i^k, i = 1, 2, \cdots, N$, which converges linearly to the data average $x_{avg}$ in the mean sense as $k \to \infty$. In addition, the variance of $x_i^k$ converges to a finite value which depends on $\Delta$ and the network topology.

Proof: Taking expectation of both sides of (16), we have

$$
\mathbb{E}[x_i^{k+1}] = \frac{1}{1 + 2\rho|N_i|} \left( \rho|N_i|\mathbb{E}[x_i^k] + \rho \sum_{j \in N_i} \mathbb{E}[x_{j(Q_a)}^k] \right)
$$

$$
- \mathbb{E}[\alpha_i^k] + r_i,
$$

$$
\mathbb{E}[\alpha_i^{k+1}] = \mathbb{E}[\alpha_i^k] + \rho \left( |N_i|\mathbb{E}[x_i^{k+1}] - \sum_{j \in N_i} \mathbb{E}[x_{j(Q_a)}^{k+1}] \right).
$$

(17)

Noting that Lemma 5 implies $\mathbb{E}[x_{j(Q_a)}^k] = \mathbb{E}[x_i^k]$, we see that (17) takes exactly the same iterations in the mean sense as the DC-ADMM. By initializing $\alpha^0$ in the column space of $L_a$, $\mathbb{E}[\alpha^0] = \alpha_{avg}$ satisfies the initialization condition. The linear convergence of $\mathbb{E}[x_i^k]$ to $x_{avg}$ is thus ensured due to Theorem 2.

Since Lemma 5 also indicates the bounded variance of quantization error, the second claim follows directly from (14) Proposition 3.

We notice a disadvantage with our algorithm (16): the convergence of $\mathbb{E}[x_i^k]$ to $x_{avg}$ does not imply that $x_i^k$ reaches a consensus when $k \to \infty$. Nevertheless, a simple method fixes this problem using the law of large numbers. The idea is to calculate the running average $\bar{x}_i^k = \frac{1}{k} \sum_{j=1}^k x_i^j, k \geq 1$ at node $i$ and then deterministically quantize it, which yields $Q_d(\bar{x}_i^k)$. By the strong law of large numbers, we have $\bar{x}_i^k \to x_{avg}$ almost surely as $k \to \infty$. Unless $x_{avg} = (t + \frac{1}{2})\Delta$ where $t$ is an integer, $Q_d(\bar{x}_i^k) \to Q_d(x_{avg})$ almost surely as $k \to \infty$ for $i = 1, 2, \cdots, N$, i.e., the consensus is reached. The result is summarized as follows.

Corollary 1: Let $\bar{x}_i^k = \frac{1}{k} \sum_{j=1}^k x_i^j$ denote the running average of $x_i$ at node $i$. Then

$$
\lim_{k \to \infty} Q_d(\bar{x}_i^k) = Q_d(x_{avg})\quad \text{in probability},
$$

for $i = 1, 2, \cdots, N$.

V. DETERMINISTIC QUANTIZATION

Deterministic quantization is usually much harder to handle as the quantization error is not stochastic. Unlike probabilistic quantization, the accumulated error term is very likely to blow up, and only a few methods could relieve such difficulties (see (10), (18), (19)); yet their algorithms either do not guarantee a consensus or reach a consensus with an error from the desired average that depends on the number of agents, quantization resolution and agents’ data. Utilizing the linear convergence rate of the DC-ADMM, our approach leads to a finite upper bound on the sum of the absolute value of each error term and hence establishes the convergence of the accumulated error term.

Let the local data $x_i^k$ be also quantized for the $(k+1)$th update at node $i$. The updates become

$$
x_i^{k+1} = \frac{1}{1 + 2\rho|N_i|} \left( \rho|N_i|x_i^{k(Q_a)} + \rho \sum_{j \in N_i} x_{j(Q_a)}^k - \alpha_i^k + r_i \right),
$$

$$
\alpha_i^{k+1} = \alpha_i^k + \rho \left( |N_i|x_i^{k+1} - \sum_{j \in N_i} x_{j(Q_a)}^{k+1} \right).
$$

(18)

Rewrite $x_i^{k(Q_a)} = x_i^k + e_i^{k(Q_a)}$ with $e_i^{k(Q_a)} \in [-\Delta/2, \Delta/2]$ according to (14). Then the $\alpha_i$-update, $i = 1, 2, \cdots, N$, is equivalent to

$$
\alpha_i^{k+1} = \alpha_i^k + \rho \left( |N_i|x_i^{k+1} - \sum_{j \in N_i} x_{j(Q_a)}^{k+1} \right)
$$

or written in the matrix form,

$$
\alpha^{k+1} = \alpha^k + \rho L_x x^{k+1} + \rho L_e e^{k+1},
$$

(19)

where $e^{k(Q_a)}$ denotes the vector concatenating all $e_i^{k(Q_a)}$. Recalling the ideal DC-ADMM update (2), we have the matrix form of (18) as

$$
s^{k+1} = D(s^k + s_x^k) + s_e^k
$$

(20)

where $s_x^k = [e_i^{k(Q_a)}; 0; 0]$ and $s_e^k = [0; \rho L_e^{k(Q_a)}; 0]$. Our main results are stated in the following theorem.
Theorem 4: Consider the deterministically quantized DC-ADMM (DQ-DC-ADMM) iteration \(13\). Let \(\alpha^0\) and \(\alpha^0\) satisfy the initialization condition for the DC-ADMM. Then we have

1) **Convergence:** the sequence \((x^k_{[Q_d]}, \alpha^k)\) generated by \(13\) converges to a finite value \((13)\), where \(x^k_{[Q_d]} \in \Lambda\) and \(\alpha^k\) (hence \(\alpha^0\)) lies in the column space of \(L_-\).

2) **Consensus error:** an upper bound for the consensus error is given by

\[
|x^k_{[Q_d]} - x_{\text{avg}}| \leq \left( \frac{1}{2} + \rho \frac{2E}{N} \right) \Delta.
\]

3) **Number of iterations:** \(13\) converges in finite iterations and the number of iterations is upper bounded by \(\lfloor \log_{1+\rho} \Omega \rfloor + k_2\), where \(\delta > 0\) is given in Lemma 4

\[
k_2 = \left\lfloor \frac{1}{2N} \sqrt{\rho \sigma_{\text{max}}(M_-) + 1} \right\rfloor + 2,
\]

\[
\Omega = \max \left\{ \left( \frac{2T_0(1+\delta)}{\Delta(1+\delta-1)} \right)^2, \frac{\rho^2 \sigma_{\text{max}}^2(M_-) \sigma_{\text{max}}^2(M_+)}{\Delta^2} \right\}
\]

\[
+ 4\sigma_{\text{max}}^2(M_-) \frac{\alpha^0 - (r - x_{\text{avg}})^2}{\Delta^2 \sigma_{\text{min}}^2(M_-)} \}
\]

Here \(\lfloor y \rfloor, y \in \mathbb{R}\) is the ceiling function, i.e., \(\lfloor y \rfloor\) is the smallest integer that is greater than or equal to \(y\), and

\[
T_0 = \frac{1}{4} \Delta \sqrt{\rho N} \left( 1 + \sqrt{1 + \frac{\rho}{1 + \delta} \sigma_{\text{max}}(M_-)} \right) \times (\sigma_{\text{max}}(M_-) + \sigma_{\text{max}}(M_-)).
\]

If we further pick \(\rho = 1\), then \(k_2 = 3\).

Proof: We will prove the three claims one by one.

**Convergence:** Since \(\alpha^0\) is initialized in the column space of \(L_-\), that \(\alpha^0\) also lies in the column space of \(L_-\) follows directly from \(19\).

We next show the convergence of \(s^k\). Following \(20\), we have

\[
s^k = D(s^{k-1} + s^{k-1}) + s^{k-1}
\]

\[
= D (D(s^{k-2} + s^{k-2}) + s^{k-2}) + Ds^{k-1} + s^{k-1}
\]

\[
= \ldots
\]

\[
= D^k s^0 + \left( \sum_{i=1}^k D_i s^k_{\alpha} + \sum_{j=0}^{k-1} D_j s^{k-1-j} \right).
\]

We only need to show the convergence of the accumulated error term \(\sum_{i=1}^k D_i s^k_{\alpha} + \sum_{j=0}^{k-1} D_j s^{k-1-j}\) in \(22\) as the first term is simply the ideal DC-ADMM update. Notice that \(D_i s^k_{\alpha}\) is the \(i\)th update of the DC-ADMM with the initial value \(s^0\). Let \(u^k_{k-1} = [z^j_{k-1}; \beta^j_{k-1}]\) be the vector that concatenates the primal and dual variables in the ADMM iteration \(11\), with initial values \(z^j_{k-1} = \frac{1}{2} M^T e^{k-1}_{[Q_d]}\) and \(\beta^j_{k-1} = 0\) corresponding to \(s^k_{\alpha} = [e^{k-1}_{[Q_d]}; 0; 0]\). We then have, for \(G\) defined in Lemma 4

\[
\|u^k_{k-1}\|_G^2 = \| \frac{1}{2} M^T e^{k-1}_{[Q_d]} \|_2^2
\]

\[
\leq \frac{1}{4} \rho \sigma_{\text{max}}^2(M_+) \| e^{k-1}_{[Q_d]} \|_2^2
\]

\[
\leq \frac{1}{16} \rho N \Delta^2 \sigma_{\text{max}}^2(M_+),
\]

where the last inequality is from \(15\). Since Theorem 1 indicates the form of \(\alpha^0\), we get \(D^{\ast} s^{k-1}_{\alpha} = 0\), i.e., \(x^{k-1}_{\alpha} = 0\) and \(\sigma_{\text{max}} = 0\). Therefore, \(u^k_{k-1} = [z^j_{k-1}; \beta^j_{k-1}] = 0\) from Lemma 4 and the fact that \(z^j_{k-1} = \frac{1}{2} M^T x^{k-1}_{\alpha}\). Noting also that the initializations \(s^{k-1}_{\alpha} = 0\) and \(\beta_{k-1}^j = \beta_{k-1} \) meet the condition of Lemma 4 we thus have

\[
\|D^k s^k_{\alpha} - (D^k - D^\ast) s^{k-1}_{\alpha} \|_2 \leq \left( 1 + \frac{\rho}{1 + \delta} \sigma_{\text{max}}(M_-) \right) \| u^{k-1}_{k-1} - u^{k-1}_{k-1} \|_G
\]

\[
\leq \left( 1 + \frac{\rho}{1 + \delta} \sigma_{\text{max}}(M_-) \right) \left( \frac{1}{1 + \delta} \right)^i \times \frac{1}{4} \Delta \sigma_{\text{max}}(M_-) \sqrt{\rho N},
\]

where \((a)\) is from Theorem 2 and \((b)\) is due to Lemma 4 together with the fact that \(u^k_{k-1} = 0\). Similarly, we have for \(j \geq 1\)

\[
\| D_j s^{k-1-j}_{\alpha} \|_2 \leq \left( 1 + \frac{\rho}{1 + \delta} \sigma_{\text{max}}(M_-) \right) \left( \frac{1}{1 + \delta} \right)^j \times \frac{1}{4} \Delta \sigma_{\text{max}}(M_-) \sqrt{\rho N},
\]

and when \(j = 0\)

\[
\| D_j s^{k-1-j}_{\alpha} \|_2 \leq \| s^{k-1}_{\alpha} \|_2 \leq \frac{1}{4} \Delta \sigma_{\text{max}}(M_-) \rho \sqrt{N}.
\]

Therefore,

\[
\left\| \sum_{i=1}^k D^i s^k_{\alpha} + \sum_{j=0}^{k-1} D_j s^{k-1-j}_{\alpha} \right\|_2 \leq \sum_{i=1}^k \| D^i s^k_{\alpha} \|_2 + \sum_{j=0}^{k-1} \| D_j s^{k-1-j}_{\alpha} \|_2
\]

\[
\leq \| s^{k-1}_{\alpha} \|_2 + \sum_{i=1}^k \left( \| D^i s^k_{\alpha} \|_2 + \| D_j s^{k-1-j}_{\alpha} \|_2 \right)
\]

\[
\leq \left( \frac{1}{4} \Delta \sigma_{\text{max}}(M_-) \rho \sqrt{N} \right) + \left( 1 + \frac{\rho}{1 + \delta} \sigma_{\text{max}}(M_-) \right) \sum_{i=1}^k \left( \frac{1}{1 + \delta} \right)^i \times \frac{1}{4} \Delta \sigma_{\text{max}}(M_-) \rho \sqrt{N},
\]

\[
\left(\frac{1}{4} \Delta \sigma_{\text{max}}(M_-) \rho \sqrt{N} \right) + \left( 1 + \frac{\rho}{1 + \delta} \sigma_{\text{max}}(M_-) \right) \sum_{i=1}^k \left( \frac{1}{1 + \delta} \right)^i \times \frac{1}{4} \Delta \sigma_{\text{max}}(M_-) \rho \sqrt{N},
\]

\[
= \frac{1}{4} \Delta \sigma_{\text{max}}(M_-) \rho \sqrt{N} + T_0 \sum_{i=1}^k \left( \frac{1}{1 + \delta} \right)^i,
\]

where \((a)\) is from \(23\), \(25\), and \(T_0\) is defined in \(21\). Then \(26\) converges to a finite value as \(k \to \infty\) since \(\delta > 0\). Thus, we have \(\sum_{i=1}^k D^i s^k_{\alpha} + \sum_{j=0}^{k-1} D_j s^{k-1-j}_{\alpha}\) converges to a finite value by the comparison theorem and the fact that
absolute convergence implies convergence \(^{(28)}\). Therefore, \(s^k\), the sum of two convergent sequences, converges. Now that \(\alpha^{k+1} = \alpha^k + \rho L - x_{[Q_d]}^k\) converges, \(x_{[Q_d]}^k\) must reach a consensus by the property of \(L_{-}\). \(^{(22)}\)

**Consensus error**: The consensus error may be studied directly by calculating the accumulated error term in \(^{(22)}\). However, the bound in \(^{(20)}\) is quite loose in general as the bounds in \(^{(11)}\) and \(^{(21)}\) are themselves loose for the respective quantities. We alternatively study the update \(^{(18)}\) based on the fact that \(x_{[Q_d]}^k\) converges to a consensus as \(k \to \infty\).

Let \(x_{[Q_d]}^i = \Lambda\) denote the convergent quantized value. Then \(x_{[Q_d]}^i = x_{[Q_d]}^i\) for \(i = 1, 2, \ldots, N\), and \(x_{[Q_d]}^i = x_{[Q_d]}^i - e_{i[Q_d]}^i\). Summing up both sides of \(^{(18)}\) from \(i = 1\) to \(N\), we have

\[
\sum_{i=1}^{N} (1 + 2\rho|N_i|) \left( x_{[Q_d]}^i - e_{i[Q_d]}^i \right) = \sum_{i=1}^{N} \left( |N_i| x_{[Q_d]}^i + \rho \sum_{j \in N_i} x_{[Q_d]}^j + r_k \right),
\]

which is equivalent to

\[
x_{[Q_d]}^* = \frac{1}{N} \sum_{i=1}^{N} r_i + \frac{1}{N} \sum_{i=1}^{N} (1 + 2\rho|N_i|) e_{i[Q_d]}^i.
\]

Here we use the fact that \(\alpha^*\) lies in the column space of \(L_{-}\), i.e., \(\alpha^* = L_{-} b^*\) where \(b^* \in \mathbb{R}^N\). Then \(\sum_{i=1}^{N} \alpha_{i}^* = (L_{-} b^*)^T 1 = (b^*)^T (L^T 1) = 0\). Recalling that \(x_{avg} = \frac{1}{N} \sum_{i=1}^{N} r_i\) and \(|e_{i[Q_d]}| \leq \frac{\Delta}{2}\), we finally obtain

\[
\left| x_{[Q_d]}^* - x_{avg} \right| \leq \left( \frac{1}{2} + \frac{2\rho}{N} \right) \Delta. \quad (27)
\]

We next use a simple example to show the tightness of this bound. Consider a simple two-node network with \(r_1 = -\frac{\Delta}{2}\) and \(r_2 = -\frac{\Delta}{2}\). Set both \(\Delta\) and \(\rho\) to be 1. In this case, we have \(E = 1, N = 2\) and

\[
L_{-} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.
\]

We start with \(x_{[Q_d]}^0 = x_{[Q_d]}^0 = 1\) and \(\alpha_{i}^0 = -\alpha_{i}^0 = 1\). One can easily check that our initialization condition is met, and \(x_{[Q_d]}^1 = x_{[Q_d]}^1 = 1\) and \(\alpha_{i}^1 = -\alpha_{i}^1 = 1, k = 1, 2, \ldots\), in the updates of \(^{(18)}\). Hence \(x_{[Q_d]}^* = 1\) and the consensus error is

\[
\left| x_{[Q_d]}^* - x_{avg} \right| = \frac{3}{2} \leq \left( \frac{1}{2} + \frac{2\rho}{N} \right) \Delta.
\]

This coincides with the error bound in \(^{(27)}\).

**Number of iterations**: The convergence of \(^{(22)}\) implies that there exists a finite \(k_0 \geq 1\) such that \(\|\alpha^k - \alpha^*\|_2 < \Delta\) and \(\|x_{[Q_d]}^k\|_2 < \Delta\) for \(k \geq k_0\). Then \(\alpha^k = \alpha^*\) and \(x_{[Q_d]}^k = 1\) for \(k \geq k_0\) due to quantization. Therefore, \(^{(18)}\) converges in finite steps.

One may notice that the two terms in \(^{(22)}\) converge relatively fast: \(D^k s^0\) is the \(k\)th update of the DC-ADMM thus linearly convergent and \(\sum_{i=1}^{k} D^i s_{\alpha}^{k-1-i} + \sum_{j=0}^{\infty} D^i s_{\alpha}^{k-1-j}\) is absolutely bounded by the sum of a constant term plus the sum of a geometric series whose common ratio is positive and strictly less than 1. As such, we expect an upper bound for the number of iterations that guarantees the convergence of \(^{(22)}\). Our purpose here is to give a rough idea of how fast the convergence can be; we will not make any effort to tighten the bound.

We first consider the number of iterations, denoted by \(k_1\), that guarantees the convergence of \(\alpha^k\). Write \(\alpha^k = \alpha_i^k + \alpha_E^k\) where \(\alpha_i^k\) and \(\alpha_E^k\) are the corresponding vectors in the ideal update \(D^k s^0\) and the accumulated error term \(\sum_{i=1}^{k} D^i s_{\alpha}^{k-1-i} + \sum_{j=0}^{\infty} D^i s_{\alpha}^{k-1-j}\), respectively. Denote \(x^k = x_i^k + x_E^k\), \(\alpha^* = \alpha_i^* + \alpha_E^*\) and \(x^* = x_i^* + x_E^*\) in the same fashion, so that \(x_i^* = x_{avg}\) and \(\alpha_E^* = r - x_{avg}\) by \(^{(3)}\). Then we have

\[
\|\alpha^k - \alpha^*\|_2 = \|\alpha_i^k - \alpha_i^* + \alpha_E^k - \alpha_E^*\|_2 \leq \|\alpha_i^k - \alpha_i^*\|_2 + \|\alpha_E^k - \alpha_E^*\|_2 \leq \sqrt{\rho \sigma_{\max}(M_{-})} \|u_i^k - u_i^*\|_G
\]

\[
+ \lim_{k \to \infty} \left\| \sum_{i=k+1}^{K_1} D^i s_{\alpha}^{k-1-i} + \sum_{j=k}^{K_1} D^j s_{\alpha}^{K_1-1-j} \right\|_2,
\]

where \(u_i^k = [x_i^k; \beta_i^k]\) is the vector in the ideal DC-ADMM update with initial values \(u_i^0 = [x_i^0; \beta_i^0]\) corresponding to \(x^0\) and \(\alpha^0\), and \(^{(28)}\) is due to \(^{(12)}\) and \(^{(22)}\). We then choose \(k_1\) such that the two terms in \(^{(28)}\) become small enough, i.e.,

\[
\sqrt{\rho \sigma_{\max}(M_{-})} \|u_i^k - u_i^*\|_G < \frac{\Delta}{2}, \quad (29)
\]

\[
\lim_{k \to \infty} \left\| \sum_{i=k+1}^{K_1} D^i s_{\alpha}^{k-1-i} + \sum_{j=k}^{K_1} D^j s_{\alpha}^{K_1-1-j} \right\|_2 < \frac{\Delta}{2}. \quad (30)
\]

Using \(^{(4)}\) and \(^{(26)}\), we can pick \(k_1\) such that

\[
\sqrt{\rho \sigma_{\max}(M_{-})} \|k_1 - u_i^*\|_G \leq \sqrt{\rho \sigma_{\max}(M_{-})} \|u_i^0 - u_i^*\|_G \left( \frac{1}{1 + \delta} \right) ^{k_1} < \frac{\Delta}{2},
\]

and

\[
\lim_{k_1 \to \infty} \left\| \sum_{i=k_1+1}^{K_1} D^i s_{\alpha}^{k_1-1-i} + \sum_{j=k_1}^{K_1} D^j s_{\alpha}^{K_1-1-j} \right\|_2 < \frac{\Delta}{2}. \quad (31)
\]

Since we also have

\[
\|u_i^0 - u_i^*\|_G^2 = \rho \|x_i^0 - x_i^*\|_2^2 + \frac{1}{\rho} \|\beta_i^0 - \beta_i^*\|_2^2
\]

\[
= \rho \left( \frac{1}{2} M_i^T (x^0 - x_i^*)^2 + \frac{1}{\rho} \|\beta_i^0 - \beta_i^*\|_2^2 \right)
\]

\[
\leq \frac{1}{4} \rho \sigma_{\min}(M_{+}) \|x^0 - x_i^*\|_2^2
\]

\[
+ \frac{1}{\rho \sigma_{\min}(M_{+})} \|x^0 - \alpha_i^*\|_2^2,
\]
where the last inequality is from (45), it suffices to pick
\[ k_1 = \lceil \log_{1+\delta} \Omega \rceil. \]

Though we have obtained \( \alpha^k = \alpha^* \) for \( k \geq k_1 \), it is not enough to conclude \( x_{i[Q]}^k = x_i^* \). We still need more steps, denoted by \( k_2 \), to ensure the convergence \( x_{i[Q]}^k \). Since \( \alpha^k \) has converged, we have \( x_{i[Q]}^k \) reaching a consensus according to the property of \( \mathcal{L}_- \). We can write \( x_{i[Q]}^k = 1\eta^k \) with \( \eta^k \in \Lambda \) for \( k \geq k_1 + 1 \). By (29) and (45), we obtain
\[ \|x_{i}^{k+1} - x_i^*\|_2 \leq \frac{\Delta}{\sqrt{2\rho\sigma_{\max}(\mathcal{L}_-)}}. \]

Using (40) and (41), we get
\[ \|x_{i}^{k+1} - x_i^*\|_2 \leq \frac{\Delta}{2}. \]

Therefore,
\[ \|x_{i}^{k+1} - x_i^*\|_2 \leq \frac{1}{2\sqrt{\rho\sigma_{\max}(\mathcal{L}_-)}} + \frac{1}{2} \Delta, \]
and therefore,
\[ |\eta^{k+1} - x_{i[Q]}^*| \leq \frac{1}{2N} \left( \frac{1}{\sqrt{\rho\sigma_{\max}(\mathcal{L}_-)}} + 1 \right) \Delta + \Delta. \]

Now for \( i = 1, \cdots, N \) and \( k \geq k_1 + 1 \), (18) implies
\[ x_{i}^{k+2} = \frac{2\rho[i][N_i]}{1 + 2\rho[i][N_i]} \eta^{k+1} - \frac{1}{1 + 2\rho[i][N_i]}(\alpha_i^* - r_i), \]
\[ \eta^{k+1} = Q_d(x_{i}^{k+1}). \]

Therefore, \( x_{i}^{k+2} \) remains the same as \( x_{i}^{k+1} \) once \( \eta^{k+1} > \eta^k \) if \( \eta^{k+1} > \eta^k \) then \( \eta^{k+1} \geq \eta^k + 1 \); if \( \eta^{k+1} < \eta^k \) then \( \eta^{k+1} < \eta^k + 1 \). Together with the fact \( \eta^k \rightarrow x_{i[Q]}^k \) as \( k \rightarrow \infty \), we can choose \( k_2 \) to be the smallest integer that is greater than or equal to \( \frac{\Delta}{\sqrt{\rho\sigma_{\max}(\mathcal{L}_-)}} + 1 \), i.e., \( k_2 = \lceil \frac{\Delta}{\sqrt{\rho\sigma_{\max}(\mathcal{L}_-)}} + 1 \rceil \). Since the network is connected and has at least two nodes, \( k_2 = 3 \) if we pick \( \rho \) is large.

In summary, the DQ-DC-ADMM converges within \( k_1 + k_2 \) iterations.

**Remark 1:** We shall mention that the limit \( (1x_{i[Q]}^*, \alpha^*) \) need not be unique. Note also that the given example illustrating the tightness of the consensus error bound is poorly initialized and we usually have smaller errors than (27) in practice (see Figure 1). We hence expect better consensuses when \( (\alpha^0, \alpha^0) \) is initialized closer to the ideal optimas, which leads to our algorithm for quantized consensus in Section VI.

**Remark 2:** An interesting observation of our main results is the ADMM parameter \( \rho \). While a small \( \rho \) indicates a small consensus error bound, it is not easy to quantify how it affects the convergence rate. Here we do not study the optimal selection of \( \rho \) but simply set \( \rho = 1 \). Therefore we do not regard \( \rho \) as a factor affecting our algorithm’s performance. We refer readers to (21), (24), (29) for detailed discussions on how \( \rho \) affects the ADMM’s performance.

**Remark 3:** The number of iterations that guarantees convergence is bounded by \( 3 + \lceil \log_{1+\delta} \Omega \rceil \). This bound is much desired for large scale networks compared with that characterized by a polynomial of the number of nodes, quantization resolution and agents’ data in (19). However, \( \delta \) can be very small when the graph is dense, which results in slow convergence. See simulations in Figure 2.

**Remark 4:** Our main results for rounding quantizers also apply to other deterministic quantizers as the only information used in our proof is the bounded quantization error. In contrast with (7), (10) where the algorithms may fail to converge for some deterministic quantization schemes, e.g., the rounding quantization, our results work for all deterministic quantization schemes as long as a finite quantization error bound is provided.

**Remark 5:** A major difference between the DQ-DC-ADMM and PQ-DC-ADMM updates is that \( x_{i}^k \) is quantized for the \( (k+1) \)th update at node \( i \) but is not in (18). Indeed, we can use \( Q_d(x_{i}^k) \) for the \( (k+1) \)th update in the probabilistic case as \( E[Q_d(x_{i}^k)] = E[x_{i}^k] \), and the results still hold with an increased variance. However, if we do not quantize \( x_{i}^k \) at node \( i \) in the deterministic case, the property of \( \alpha^k \) lying in the column space of \( \mathcal{L}_- \) or \( \sum_{i=1}^N \alpha_i^k = 0 \) is not preserved. This can lead to very large consensus errors.

**VI. DC-ADMM Based Algorithm for Quantized Consensus**

Let us summarize the two quantized versions of the DC-ADMM: the PQ-DC-ADMM converges linearly to the data average in the mean sense, but it does not guarantee a consensus or needs infinite time for reaching a consensus in probability; the DQ-DC-ADMM, on the other hand, converges to a consensus within finite iterations but results in an error from the average.

As discussed in Remark 1, we can first run the PQ-DC-ADMM \( 2K \) times to obtain
\[ Q_d \left( x_i = \frac{1}{K} \sum_{k=K+1}^{2K} x_i^k \right), \]
which is a reasonable estimate of \( x_{\text{avg}} \) at node \( i \). Here \( K \) can be chosen such that \( E[x_{i[Q_d]}^K] \) is close enough to \( x_{\text{avg}} \) when we have the knowledge of agents’ data and the network topology. Otherwise, we can simply set \( K = 10N \) or as large as permitted. We then run the PQ-DC-ADMM with \( Q_d(x_i) \) as initial values. Even though \( Q_d(x_{i}) \) in the probabilistic case is also a good estimate, unfortunately we cannot use \( Q_d(x_{i}) \) for subsequent iterations since it is not guaranteed to lie in the column space of \( \mathcal{L}_- \). The probabilistically quantized DC-ADMM followed by deterministically quantized DC-ADMM (PQDQ-DC-ADMM) is presented in Algorithm VI.

**VII. Simulations**

This section compares the performance of the DQ-DC-ADMM, the PQDQ-DC-ADMM, the gossip based method in (27) and the classical method in (10) via numerical examples. To construct a connected graph with \( N \) nodes and \( E \) edges, we first generate a complete graph consisting of \( N \) nodes, and then uniformly randomly remove \( \frac{N(N-1)}{2} - E \) edges while ensuring that the network stays connected. Set \( \Delta = 1 \) and
Algorithm 1 PQDQ-DC-ADMM for quantized consensus

Require: Initialize \( x^0 = 0 \), \( \alpha^0 = 0 \) and \( \rho > 0 \). Set \( K = 10N \).

1. for \( k = 0, 1, \ldots, 2K - 1 \), every node \( i \) do
   2. \( x_{i}^{k+1} \leftarrow \frac{1}{1 + 2\rho|N_i|} \left( \rho|N_i| x_i^k + \rho \sum_{j \in N_i} x_j^k x_j[Q_i] \right) \)
   \[ - \alpha_i^k + r_i \]
   \( \alpha_i^{k+1} \leftarrow \alpha_i^k + \rho \left( |N_i| x_i^{k+1} - \sum_{j \in N_i} x_j^{k+1} x_j[Q_i] \right) \).
3. end for
4. set \( x^0 = Q_d \left( \frac{1}{K} \sum_{k=K+1}^{2K} x^k \right) \), \( \alpha^0 = 0 \) and \( k = 0 \).
5. repeat
   6. For \( i = 1, 2, \ldots, N \),
   \[ x_{i}^{k+1} \leftarrow \frac{1}{1 + 2\rho|N_i|} \left( \rho|N_i| x_i^k + \rho \sum_{j \in N_i} x_j^k x_j[Q_i] \right) \]
   \[ - \alpha_i^k + r_i \]
   \( \alpha_i^{k+1} \leftarrow \alpha_i^k + \rho \left( |N_i| x_i^{k+1} - \sum_{j \in N_i} x_j^{k+1} x_j[Q_i] \right) \).
7. set \( k = k + 1 \).
8. until a predefined stopping criterion (e.g., a maximum iteration number) is satisfied.

Assume that the range of \( r \) is increasing with the size of the network, e.g., let \( r_i \) be uniformly distributed in \([-N, N]\). Our settings are:

- DQ-DC-ADMM: Set \( \rho = 1 \), \( x^0 = 0 \) and \( \alpha^0 = 0 \).
- PQDQ-DC-ADMM: Set \( \rho = 1 \).
- Classical method: Let \( W \) denote the weight matrix of the graph \( G_d = \{ V, A \} \). The updating rule is given by \( x^{k+1} = W x_{[Q]}^k \) where the weights are
  \[ W_{ij} = \begin{cases} \frac{1}{N}, & i \neq j \text{ and } (i, j) \in A, \\ 1 - \sum_{k \in N_i} W_{ik}, & i = j, \\ 0, & \text{otherwise}. \end{cases} \]
- Gossip based method: We uniformly randomly pick one edge in \( A \) and perform the updating, i.e., if \((i, j) \in A \) is chosen, then \( x_{i}^{k+1} = x_{j}^{k+1} = \frac{1}{2}(x_i[k] + x_j[k]). \)

Consensus error: In Figure 1(a) we fix the average degree \( \bar{E} = 20 \) while letting the node number \( N \) vary. We can see that the classical method and gossip based method both have increasing consensus errors as \( N \) increases. The consensus error of the DQ-DC-ADMM is relatively small compared with the upper bound \( (1 + \frac{2E}{N}) \Delta = 21 \) and decreases when \( N \) becomes larger. Our proposed algorithm, the PQDQ-DC-ADMM, has the smallest consensus error whose average of 100 runs is less than 0.3 for all \( N \). We next fix \( N = 50 \) and vary the number \( E \) of edges until the network is complete in Figure 1(b). As the graph becomes denser, the consensus errors of the classical method and gossip based method decrease. The consensus error of the DQ-DC-ADMM, however, increases as the average degree and thus the error bound increase. Our DPDQ-DC-ADMM still performs the best with average consensus errors less than 0.33 for all \( E \).

One should note from Figure 1 that the classical method is highly dependent on the graph density; its consensus error can be extremely large for a sparsely connected graph.

Convergence time: We study the convergence time of the four algorithms in Figure 2. When the average degree is fixed and the number of nodes is increasing, all the algorithms have increasing convergence time. When the number of nodes is fixed and the average degree (or graph density) becomes larger, the classical method and gossip based method converge faster while the DQ-DC-ADMM requires more time to converge. The convergence time of the PQDQ-DC-ADMM, on the other hand, does not depend much on the average degree (or graph density) when the number of nodes is fixed.

We observe that the gossip based method and the PQDQ-DC-ADMM need much more time than the classical method and the DQ-DC-ADMM. This is because each update of the gossip based method only involves two nodes while the classical method and the DQ-DC-ADMM update the states of all the nodes at one iteration. In terms of the communication energy, however, one may argue that the gossip based method consumes the least amount of energy. The significant portion of the convergence time for the PQDQ-DC-ADMM is spent on achieving an approximate estimate of \( x_{avg} \), i.e., running the PQ-DC-ADMM. With good starting points, the DQ-DC-ADMM converges almost immediately. Besides, the gossip based method and the classical method need much longer time to reach convergence when the graph is sparse.

We summarize our simulation results as follows: our PQDQ-DC-ADMM provides the best consensus error but need a longer time to converge to a consensus; the DQ-DC-ADMM converges faster but results in higher consensus errors; the gossip based method needs the least communication energy to reach a consensus yet with an adequate consensus error; the classical method converges fast with a small consensus error when the graph is dense but performs very poorly when the network is sparsely connected. Therefore, our algorithm, the PQDQ-DC-ADMM, provides the best performance where the priority is the precision of the consensus reached at convergence.

VIII. CONCLUSION

In this paper we have proposed an efficient algorithm, the PQDQ-DC-ADMM, for quantized consensus problems. We first study the effects of both probabilistic and deterministic quantizations on the DC-ADMM. With probabilistic quantization, the PQ-DC-ADMM converges linearly to the data average in the mean sense. In the deterministic case, we can bound the sum of the absolute value of each error term caused by quantization using the global and linear convergence of the DC-ADMM and thus prove the convergence of the DC-ADMM. We finally combine the two quantized versions of
the DC-ADMM to obtain the PQDQ-DC-ADMM algorithm, which uses the PQ-DC-ADMM to get an estimate of the data average and then runs the DQ-DC-ADMM to reach a consensus. Simulations show that our PQDQ-DC-ADMM provides the best result than all existing methods in terms of the consensus error.

Our approach also motivates a number of further research directions:

1) We assume the quantized data communication between agents to be perfect in this paper. In practice, channel impairment may lead to imperfect transmissions. Moreover, the links between agents may fail and the network topology may vary randomly, as studied in [10], [14], [17]. It is thus meaningful to investigate how our algorithm performs in these settings.

2) The algorithm parameter $\rho$ is another interesting topic in the DQ-DC-ADMM. Its choice should be guided depending on whether a small consensus error or fast consensus speed is desired.

3) We only consider the unbounded quantization scheme in this paper. It is also interesting to consider bounded quantization that is used in many applications.
Proof of Lemma 2] We first manipulate (37) to derive equivalent updates
\[
\nabla f(x^{k+1}) + A^T \lambda^k + \rho A^T B(z^k - z^{k+1}) = 0, \tag{33}
\]
\[
B^T \lambda^{k+1} = 0, \tag{34}
\]
\[
\lambda^{k+1} - \lambda^k - \rho(A x^{k+1} + B z^{k+1}) = 0, \tag{35}
\]
where (33) and (34) are from multiplying the two sides of the \( \lambda \)-update by \( A^T \) and \( B^T \) and adding them to the \( x \)-update and \( z \)-update, respectively. Recalling \( \lambda = [\beta; \gamma] \) with \( \beta, \gamma \in \mathbb{R}^2 \) and \( B = [-I_2: -I_2] \), we know that \( \beta^{k+1} = -\gamma^k \) from (33). Since we initialize \( \beta^0 = -\gamma^0 \), we have \( B^T \lambda^k = 0 \) for \( k = 0, 1, \ldots \). Equation (33) then reduces to \( \nabla f(x^{k+1}) + M_\beta \beta^{k+1} - \rho M_\beta^T(z^k - z^{k+1}) = 0 \), and (35) splits into \( \beta^{k+1} - \beta^k - \rho A_1 x^{k+1} + \rho z^{k+1} = 0 \) and \( \gamma^{k+1} - \gamma^k - \rho A_2 z^{k+1} + \rho z^{k+1} = 0 \). Summing and subtracting these two equations we have \( \frac{1}{2} M_T^T x^{k+1} - z^{k+1} = 0 \) and \( \beta^{k+1} - \beta^k - \frac{1}{2} M_T^T x^{k+1} = 0 \). With the initialization \( z^0 = \frac{1}{2} M_T^T x^0 \), \( \frac{1}{2} M_T^T x^k - z^k = 0 \) holds true for \( k = 0, 1, \ldots \). Since \( x^* \) is unique and equal to 1\( x_{avg} \) according to Lemma 1, \( z^* = \frac{1}{2} M_T^T x^* \) is also unique. To summarize, with the initialization \( \beta^0 = -\gamma^0 \) and \( z^0 = \frac{1}{2} M_T^T x^0 \), (33)-(35) reduce to
\[
\nabla f(x^{k+1}) + M_\beta \beta^{k+1} - \rho M_\beta^T(z^k - z^{k+1}) = 0, \tag{36}
\]
\[
\beta^{k+1} - \beta^k - \rho M_T^T x^{k+1} = 0, \tag{37}
\]
\[
\frac{1}{2} M_T^T x^{k+1} - z^{k+1} = 0, \tag{38}
\]
which further lead to \( x^k \to x^* = 1 x_{avg} \) and \( z^k \to \frac{1}{2} M_T^T x^* = \frac{1}{2} M_T^T x_{avg} \) uniquely as \( k \to \infty \). Taking \( k \to \infty \) in (36)-(38) and using global convergence, we get
\[
\nabla f(x^*) + M_\beta \beta^* = 0, \tag{39}
\]
\[
M_T^T x^* = 0, \tag{40}
\]
\[
\frac{1}{2} M_T^T x^* - z^* = 0. \tag{41}
\]
We can now use (39) to demonstrate the uniqueness of \( \beta^* \) if we also initialize \( \beta_0 \) in the column space of \( M_T^T \). Note that if \( \beta^0 \) lies in the column space of \( M_T^T \) then (37) indicates that \( \beta^k \) also lies in the column space of \( M_T^T \), \( k = 0, 1, \ldots \). The uniqueness of \( \beta^* \) then follows from the uniqueness of \( x^* \) and Lemma 2.

Next we show the linear convergence of \( u^k \). Subtracting (36)-(38) from (39)-(41), respectively, and using \( \nabla f(x) = x - r \), we have
\[
x^{k+1} - x^* = \rho M_T^T(z^k - z^{k+1}) - M_\beta(\beta^{k+1} - \beta^*), \tag{42}
\]
\[
\frac{\rho}{2} M_T^T(x^{k+1} - x^*) = \beta^{k+1} - \beta^*, \tag{43}
\]
\[
\frac{1}{2} M_T^T(x^{k+1} - x^*) = z^{k+1} - z^*. \tag{44}
\]
We therefore obtain
\[
\|x^{k+1} - x^*\|^2 \leq \delta \|u^{k+1} - u^*\|^2 + \frac{\delta}{\rho} \|\beta^{k+1} - \beta^*\|^2, \tag{45}
\]
where \( \delta \) is from (42), \( \rho \) is from (45) and (44), and \( \delta \) is from the definitions of \( u \) and \( G \).

To prove (11) we only need to show
\[
\|u^k - u^{k-1}\|^2_G + \|x^{k+1} - x^*\|^2 \geq \delta \|u^k - u^*\|^2_G, \tag{46}
\]
which is equivalent to
\[
\rho \|z^{k+1} - z^*\|^2 + \frac{1}{\rho} \|\beta^{k+1} - \beta^*\|^2 + \|x^{k+1} - x^*\|^2 \geq \delta \|u^{k+1} - u^*\|^2_G. \tag{47}
\]
It then suffices to show
\[
\rho \|z^{k+1} - z^*\|^2 + \|x^{k+1} - x^*\|^2 \geq \delta \|u^k - u^*\|^2_G + \frac{\delta}{\rho} \|\beta^{k+1} - \beta^*\|^2. \tag{48}
\]
The rest of this proof is to establish that \( \delta \rho \|z^{k+1} - z^*\|^2 \) and \( \delta \rho \|\beta^{k+1} - \beta^*\|^2 \) are upper bounded by two non-overlapping parts of the left side of (47), respectively.

We first have from (48) that
\[
\|z^{k+1} - z^*\|^2 = \frac{1}{4} \|M_T^T(x^{k+1} - x^*)\|^2 \leq \frac{1}{4} \sigma^2_{\min}(M_\gamma)\|x^{k+1} - x^*\|^2. \tag{49}
\]
To upper bound \( \|\beta^{k+1} - \beta^*\|^2 \), we first notice that \( \beta^{k+1} - \beta^* \) lies in the column space of \( M_T^T \). Therefore,
\[
\|M_\beta(\beta^{k+1} - \beta^*)\|^2 \geq \sigma^2_{\min}(M_\gamma)\|\beta^{k+1} - \beta^*\|^2. \tag{50}
\]
Now using (48) and (42) we get
\[
\|\beta^{k+1} - \beta^*\|^2 \leq \frac{\rho}{\sigma^2_{\min}(M_\gamma)}\|x^{k+1} - z^\gamma\|^2 \leq \frac{1}{\sigma^2_{\min}(M_\gamma)}\|x^{k+1} - z^\gamma\|^2 + \rho \sigma^2_{\min}(M_\gamma)\|z^{k+1} - z^\gamma\|^2 \tag{51}
\]
where \( \gamma \) is from the Cauchy-Schwarz inequality together with the fact \( 2\rho p_1 p_2 \leq p_1^2 + p_2^2 \) for any \( p_1, p_2 \in \mathbb{R} \). Combining (47) and (49), we have
\[
\rho \sigma^2_{\max}(M_\gamma)\|z^{k+1} - z^\gamma\|^2 \leq \rho \sigma^2_{\min}(M_\gamma)\|z^{k+1} - z^\gamma\|^2 + \frac{\rho}{\sigma^2_{\max}(M_\gamma)}\|x^{k+1} - x^*\|^2 \leq \rho \sigma^2_{\min}(M_\gamma)\|z^{k+1} - z^\gamma\|^2 \leq \rho \sigma^2_{\max}(M_\gamma)\|z^{k+1} - z^\gamma\|^2, \tag{52}
\]
The proof is thus complete by picking

$$\delta = \min \left\{ \frac{\sigma_{\min}^2(M_\pi)}{2\sigma_{\max}^2(M_\pi)}, \frac{4\sigma_{\min}^2(M_\pi)}{\rho^2\sigma_{\max}^2(M_\pi)}\delta_{\min}^2(M_\pi) + 8 \right\}.$$  

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