Moments of the Virtual Photon Structure Function

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Abstract

The photon structure function is a useful testing ground for QCD. It is perturbatively computable apart from a contribution from what is usually called the hadronic component of the photon. There have been many proposals for this nonperturbative part of the real photon structure function. By studying moments of the virtual photon structure function, we explore the extent to which these proposed nonperturbative contributions can be identified experimentally.

1 Introduction

With the discovery of asymptotic freedom, QCD has become the accepted theory of hadrons. However, at the energy scale of the formation of hadrons (about a few GeV or less), perturbative QCD (PQCD) can no longer be applied alone, and nonperturbative QCD (NP) becomes important and may even rule. In this regime, we have very limited tools we can use to understand what is going on.

There is a fundamental dimensional constant giving a qualitative criterion of separating these two energy regions, $\Lambda_{\overline{MS}} \equiv \Lambda$. We use the modified minimal subtraction renormalization scheme, $\overline{MS}$, throughout this paper. We can hope for some progress in understanding
the interplay of nonperturbative and perturbative QCD in those processes where both play a role. It may even be important to understand the relation of perturbative and nonperturbative contributions in order to obtain $\Lambda$ from experiment.

The photon structure function in $e^+e^- \to e^+e^- + \text{hadrons}$ has been a useful tool to study QCD. It has been calculated perturbatively to nonleading order, which is necessary to obtain a meaningful comparison with QCD predictions for other processes [1]. While valid asymptotically, the PQCD prediction is modified at attainable $Q^2$ by the presence of a NP part, usually called the “hadronic” component of the photon. (This is an unfortunate usage and we will refer in the following to the PQCD and NP contributions to the photon structure function.) Our point of view in this paper is that the photon structure function is valuable precisely because it contains both PQCD and NP contributions at not too large $Q^2$. This may make it a unique process in which we can study both of these as functions of the kinematic variables.

The most information can be extracted from this process by studying the photon structure function for a virtual photon target whose invariant mass, $-P^2$, can vary over a range from zero to some appreciable $P^2 \ll Q^2$.

Here we exploit the moments of the photon structure function from the calculation of Uematsu and Walsh for the perturbative part of the structure function [2]. (In particular, we use the normalization and conventions of that paper.) We use the suggestions of various authors for the nonperturbative part [3], [4], [5]. This “hadronic” part to the function has been obtained by these authors by fitting experimental data from other processes. Typically, one finds the pion structure function from a semihadronic process such as $\pi + p \to \mu^+\mu^- + \text{anything}$. The static quark model and vector dominance are then used to obtain a conjectured NP component to the photon structure function.

The basic idea of this paper is to examine the behavior of the total photon structure function (including both PQCD and NP parts) as a function of $Q^2$ and $P^2$, with the aim mentioned above of finding out how to distinguish the two experimentally. First we describe models for the NP contribution—including the $Q^2$ dependence often left out of consideration.
We then discuss the real and virtual photon structure functions. We then present our results for the $P^2$ behavior of the two components of the structure function.

We are interested in the kinematic region $2 - 4 \text{ GeV}^2 \leq Q^2 \leq 50 \text{ GeV}^2$ and $0 \leq P^2 \leq 1 \text{ GeV}^2$. We would like to stimulate experimental studies of this range of variables.

Throughout this paper, we set the quark mass to zero and the number of quark flavors to four. While interesting and relevant to a comparison with experiment at high $Q^2$, a consideration of quark mass effects here would distract from our main point.

2 Models for the Nonperturbative Component

We think that it is very much an open question how the nonperturbative component of the photon structure function is to be treated. For the present, we describe the conventional view. We return to the question later in the paper. According to this conventional view, the overall real photon structure function is a sum of two parts,

$$F^\gamma_2(x, Q^2) = F^\gamma_{2,PQCD}(x, Q^2) + F^\gamma_{2,NP}(x, Q^2). \tag{1}$$

It is customary to write the nonperturbative contribution to the photon structure function $F^\gamma_{2,NP}$ as a coherent sum of light vector meson states. From this vector meson dominance model we can then identify the NP part of the photon structure function $[7]$, approximated by the $\rho$ meson contribution,

$$F^\gamma_{2,NP}(x, Q^2) \simeq \frac{\alpha \pi}{\gamma^2_{\rho}} F^\rho_2(x, Q^2). \tag{2}$$

Where $\gamma^2_{\rho} = f^2_{\rho}/\pi \simeq 2.2$ and $F^\rho_2$ can be obtained from the pion structure function using the quark model $[8]$,

$$F^\rho_2(x, Q^2) = F^\pi\hspace{-1pt}\not{\gamma}(x, Q^2) = x(\frac{4}{9} f_{\bar{u}/\pi^-} + \frac{1}{9} f_{\bar{d}/\pi^-}) = \frac{5}{9} x f_{\bar{u}/\pi^-}. \tag{3}$$

Here we denote $x f_{\bar{u}/\pi^-}$ as $x \bar{u}(x, Q^2)$ for convenience and similarly for the others. We also put $\bar{u}(x, Q^2) = d(x, Q^2)$ in the pion. This assumes that we are at low enough $Q^2$ so that we
can neglect sea quarks. Then equation (2) becomes

\[ F_{\gamma,NP}^2(x, Q^2) = \left( \frac{\alpha \pi}{\gamma^2} \right)^{\frac{5}{9}} x \bar{u}(x, Q^2). \]  

(4)

From now on we confine ourselves to \( Q^2 \leq 50 \text{ GeV}^2 \) where we expect that our neglect of sea quarks will be adequate. From definitions of the non-singlet and the singlet quark distribution functions we can write a quark structure function in terms of the non-singlet and singlet parts of the structure function \[ 6 \].

\[ xq_i(x, Q^2) = xq_{NS}^i(x, Q^2) + \frac{1}{2f} xq^S(x, Q^2), \]

(5)

where \( q^S \equiv \sum_i (q_i + \bar{q}_i) \) and \( i \) runs over all available flavors, here we choose the number of flavors to four, \( f = 4 \). The final form of \( F_{\gamma,NP}^2 \) of the photon structure function becomes

\[ F_{\gamma,NP}^2 = \left( \frac{\alpha \pi}{\gamma^2} \right)^{\frac{5}{9}} [xq_{NS}^i(x, Q^2) + \frac{1}{2f} xq^S(x, Q^2)]. \]

(6)

We have \( xq_{NS}^i \equiv x\bar{u}^{NS} \) for the pion in this example. The moments of the nonperturbative part of the structure function are then

\[ F_{\gamma,NP}^2(n, Q^2) = \int_0^1 dx x^{n-2} F_{\gamma,NP}^2(x, Q^2). \]

(7)

Our procedure will be to obtain \( xq^{NS} \) and \( xq^S \) at all \( Q^2 \) through numerical integration of the Altarelli-Parisi equations. So we can calculate \( M_{\gamma,NP}^r(Q^2) \) at any \( Q^2 \), given its starting value. We now need to determine the starting value of the \( \rho \) meson or pion structure function at some low \( Q_0^2 \).

We use three different fits which were obtained by three groups Newman et. al. \[ 3 \], NA3 collaboration at CERN \[ 4 \], and Castorina et al. \[ 5 \]. All these fits have been obtained at \( Q^2 \simeq 2 - 4 \text{ GeV}^2 \). We will show all the solutions to the Altarelli-Parisi equations for the corresponding NP contributions to the structure function. We emphasize that we have no convictions as to which of the resulting NP parts is the correct one. Our aim later in this paper will be to see to what extent they can be distinguished experimentally for \( P^2 > 0 \).

We will include three examples of Non-Singlet (NS), Singlet (S), Gluon (G) parts of the structure function at \( Q_0^2 \).
The general form of the parametrizations of the hadronic functions can be written as 
\[ x\bar{u} = Cx^a(1 - x)^b \] since the three fits are given as 
\[
\begin{align*}
\text{(A) Newman et al.} & : x\bar{u}(x, Q_0^2) = 0.52(1 - x) \\
\text{(B) NA3 collaboration} & : x\bar{u}(x, Q_0^2) = 0.57x^{0.4}(1 - x) \\
\text{(C) Castorina et al.} & : x\bar{u}(x, Q_0^2) = 0.766x^{0.5}(1 - x)^{0.6}
\end{align*}
\]
which yield the nonperturbative structure functions considering only the dominant \( \rho \) meson 
\[
F_{2,\text{NP}}^\gamma(x, Q_0^2) = \begin{cases} 
0.13 \alpha (1 - x) & \text{(A)} \\
0.14 \alpha x^{0.4}(1 - x) & \text{(B)} \\
0.19 \alpha x^{0.5}(1 - x)^{0.6} & \text{(C)}
\end{cases}
\]
(9)
The first of these is slightly at odds with our neglect of sea quarks, since it goes to a finite limit as \( x \to 0 \). Since our aim is only to compare different assumptions about the \( \rho \) meson structure function for high moments, we will use it anyway.

Let us consider the ratios of the momenta carried by quarks and gluons. Since \( \bar{u} = d \) inside the \( \pi^- \) at low \( Q^2 \), we can only consider the \( \bar{u} \) for quarks. We define the ratios for the quarks and gluons as 
\[
\begin{align*}
r_{\bar{u}} &= \int_0^1 dx x f_{\bar{u}/\pi^-} = \int_0^1 dx x\bar{u}(x, Q^2) \\
r_g &= \int_0^1 dx x f_{g/\pi^-} = \int_0^1 dx xg(x, Q^2).
\end{align*}
\]
(10)
We can calculate \( r_{\bar{u}} \) using the three fits 
\[
r_{\bar{u}} = C \int_0^1 dx x^a(1 - x)^b = B(a + 1, b + 1),
\]
(11)
where \( B \) is the Euler Beta function with \( \text{Re } (a + 1) > 0, \text{Re } (b + 1) > 0 \). Therefore the values of \( r_{\bar{u}} \) for each fit are 
\[
r_{\bar{u}} = \begin{cases} 
0.52 B(1, 2) & = 0.26 \quad \text{(A)} \\
0.57 B(1.4, 2) & = 0.17 \quad \text{(B)} \\
0.766 B(1.5, 1.6) & = 0.28 \quad \text{(C)}
\end{cases}
\]
(12)
From the QCD momentum sum rule we have \( r_g = 1 - 2r_{\bar{u}} \) so that
\[
r_g = 1 - 2 C B(a + 1, b + 1) = \begin{cases} 
0.48 \quad \text{(A)} \\
0.66 \quad \text{(B)} \\
0.44 \quad \text{(C)}
\end{cases}
\]
(13)
We can determine \( r_g \) directly from \( r_{\bar{u}} \) with the assumption of no sea quark contribution to the total momentum at the low initial value of \( Q^2 \) regardless of the form of the gluon structure function. The parametrization of \( xg(x, Q_0^2) \) is up to us to determine. The simplest form of the gluon structure function at \( Q_0^2 \) is

\[
xg(x, Q_0^2) = C_1(1 - x)^{C_2}.
\] (14)

We can find the value of \( C_1 \) from the QCD momentum sum rule after making an assumption of the value of \( C_2 \) (\( C_2 \geq 0 \)) which could in principle be determined from experiment. We choose \( C_2 = 2 \) here. Using the momentum sum rule again we obtain

\[
r_g = \frac{C_1}{(1 + C_2)} = 1 - 2C B(a + 1, b + 1).
\] (15)

This equation yields

\[
C_2 = 2, \quad C_1 = \begin{cases} 
0.624 & \text{(A)} \\
1.584 & \text{(B)} \\
1.344 & \text{(C)}. 
\end{cases}
\] (16)

Let us look at the values of \( r_{\bar{u}} \) for the three examples. In example (A) 26% of the total momentum is carried by the \( \bar{u} \) quark and thus 48% of it is carried by gluons in \( \rho \) (or \( \pi \))-meson. Similar argument applies to other examples although, with the model that we suggest, the example (B) seems to have too big gluon momentum (66%) compared to quarks (34%). The authors of reference (C) chose \( xg(x, Q_0^2) = 1.408(1 - x)^3 \). Then \( r_g = 0.35 \) which seems small compared to our result 0.44. However, one needs to keep in mind that the model for the gluon structure function here is only the simplest one and all three fits to the experimental data are perfectly acceptable.

This analysis is independent of the number of vector mesons. However, in the nonperturbative part of the structure function we need to sum over all the vector mesons coherently. For example, we will have about 60% enhancement in the nonperturbative part when we include the incoherent sum of all vector mesons \( \omega \) and \( \phi \) in addition to the dominant \( \rho \) [7]. In this paper we confine ourselves only to the \( \rho \)-term for simplicity.

In the numerical calculation, we evolve the equations from \( Q_0^2 = 3 \) GeV\(^2 \) to \( Q^2 = 45 \) GeV\(^2 \) with \( \Lambda = 0.2 \) GeV. The evolution of the nonsinglet function can be obtained in a
straight forward manner using the above hadronic functions. In coupled equations of singlet and gluonic functions we assume the initial gluon distribution in the form of equation (14) with no initial sea quark distribution. We use the normalization factor $\alpha \ln(Q^2/\Lambda^2)$ and we cut at very low $x$ ($x < 0.025$) and high $x$ ($x > 0.975$) to avoid numerical problems at the end points.

We show the moments of the NP part of the structure function, $F_{2,NP}(n, Q^2)$, in Figure 1 with the normalization factor, $\alpha \ln(Q^2/\Lambda^2)$. The qualitative behavior of NP of three fits are same. They decrease fast at low $Q^2$ while the rate of the decrement becomes small at high $Q^2$. This means that we do not expect very small NP contribution at a given moment to the structure function even at high $Q^2$. Therefore we expect the NP contribution to the total structure function persists even at high $Q^2$ as we will see in the next section. We will come back to this point in the real and the virtual cases.

3 The Real Photon Structure Function

The PQCD moments of the real photon structure function can be written as

$$F_{2,PQCD}^\gamma(n, Q^2) = \int_0^1 dxx^{n-2}F_2^\gamma(x, Q^2) = \frac{\alpha^2}{e^2}(\frac{16\pi^2}{\beta_0 g^2})a_n + b_n)$$

$$\simeq \frac{\alpha^2}{e^2}(a_n \ln\frac{Q^2}{\Lambda^2} + \tilde{a}_n \ln \ln\frac{Q^2}{\Lambda^2} + b_n).$$ (17)

Where the expressions and the values of $a_n$, $b_n$, and $\tilde{a}_n$ are given in reference [1]. The factor $1/e^2$ in the right hand side of the equation is from the difference in normalizations of structure functions between reference [2] on which our calculation is based and reference [1]. The approximation comes from omitting the higher orders in

$$\beta_0 \ln\frac{Q^2}{\Lambda} O\left[\left(\frac{\beta_1}{\beta_0} \frac{\ln \ln(Q^2/\Lambda^2)}{\ln(Q^2/\Lambda^2)}\right)^2\right].$$ (18)

$\Lambda$ is given in the $\overline{MS}$ scheme and $\tilde{g}^2(Q^2) \equiv \tilde{g}_{\overline{MS}}^2(Q^2)$ is the coupling constant with two-loop order term,

$$\frac{\tilde{g}^2(Q^2)}{16\pi^2} = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)} - \frac{\beta_3^3 \ln \ln(Q^2/\Lambda^2)}{\beta_0^3 \ln^2(Q^2/\Lambda^2)}.$$ (19)
Let us go back to the equation (17); then we have
\[ \tilde{a}_n = \beta_1 \beta_0 a_n. \]  
(20)

We split the total structure function into two parts as before. We use the results in [2] for the perturbative QCD part of the structure function and the previous three examples for the NP part. We can recover Bardeen and Buras’s moments from Uematsu and Walsh’s ones formally by putting \( P^2 = \Lambda^2 \).

We write the moments of the real photon structure function in equation (1) as
\[ F_{\gamma, \text{real}}^2(n, Q^2) = F_{\gamma, \text{PQCD}}^2(n, Q^2) + F_{\gamma, \text{NP}}^2(n, Q^2). \]  
(21)

We obtain the total real photon structure function in each moment by summing the nonperturbative and the perturbative contribution for the corresponding expressions. As we saw in Figure [1], the NP parts of the structure functions decrease relatively fast at low \( Q^2 \) while the rate slows down as \( Q^2 \) increases. On the other hand PQCD increases fast at low \( Q^2 \) and slowly at high \( Q^2 \) as we see in Figure [2]. This is a reasonable result since we expect that NP contribution becomes larger than that of PQCD at a low \( Q^2 \). If we add the two contributions, the total structure functions, \( F_{\gamma, \text{tot}}^2(n, Q^2) \), for \( n = 4 \) increase very slowly at high \( Q^2 \) after normalizing by the factor \( \alpha \ln(Q^2/\Lambda^2) \). Here the dashed curve indicates the PQCD part and A, B, and C indicate the total structure functions for the three examples. All the moments (for \( n > 4 \)) increase rather slowly after \( Q^2 > 15 \text{ GeV}^2 \). We choose different scales for Figure [2](a) and Figure [2](b) to distinguish explicitly the curves for those examples. We also notice the faster decrease of the NP part compared to PQCD at higher moments. The total real structure functions for all three fits are more or less the same. We found that the NP contribution to the total structure function is about 10% around \( Q^2 = 3 \text{ GeV}^2 \) when \( n = 4 \) and slightly less when the moments become higher and \( Q^2 \) gets larger.

4 Virtual Photon Structure Function

Let us now move on to a virtual target photon. Consider a deep inelastic scattering involving two photons with momenta \( q \) for the probe photon and \( p \) for the target photon. We have
\( q^2 = -Q^2 \) and \( p^2 = -P^2 \) as the probe photon mass and the target photon mass respectively.

We take the kinematic region of the momenta of the photons as follows.

\[
\bar{\Lambda}^2 \ll P^2 \ll Q^2. \tag{22}
\]

The moments of the structure function can be calculated by solving the renormalization group equation for the Wilson coefficient in the operator product expansion of photon-photon scattering to leading order in \( \alpha \) and to next to leading order in \( \alpha_s \).

\[
F_\gamma^{\gamma}(x, Q^2, P^2) \equiv \int_0^1 dx x^{n-2} F_2^{\gamma}(x, Q^2, P^2) = \frac{1}{16\pi^2 \beta_0} \left[ \sum_i \bar{P}_i^n \frac{1}{1 + \lambda_i^n / 2\beta_0} \frac{16\pi^2}{\bar{g}^2(Q^2)} \left\{ 1 - \left( \frac{\bar{g}^2(Q^2)}{\bar{g}^2(P^2)} \right)^{\lambda_i^n/2\beta_0 + 1} \right\} 
+ \sum_i A_i^n \left\{ 1 - \left( \frac{\bar{g}^2(Q^2)}{\bar{g}^2(P^2)} \right)^{\lambda_i^n/2\beta_0} \right\} 
+ \sum_i B_i^n \left\{ 1 - \left( \frac{\bar{g}^2(Q^2)}{\bar{g}^2(P^2)} \right)^{\lambda_i^n/2\beta_0 + 1} \right\} + C_i^n \right], \tag{23}
\]

where \( \bar{P}_i^n, A_i^n, B_i^n, C_i^n \), and \( \lambda_i^n \) are given in references [2], [8] and the index \( i \) runs over \(+, -, NS\). Notations, \(+, -, NS\) are from the eigenvalues, \( \lambda_{\pm}^n, \lambda_{NS}^n \), of the one-loop hadronic anomalous dimension matrix. \( \bar{g}^2(Q^2) \) is the coupling constant to two loops. Now, equation (23) is valid for \( n = 2 \) since the singular behavior of the coefficient \( A_i^n \), which causes a problem in the real photon structure function, is cancelled by the factor, \( 1 - (\bar{g}^2(Q^2)/\bar{g}^2(P^2))^{\lambda_i^n/2\beta_0} \) when \( \lambda_i^n \) goes to zero. Since the expression (23) is independent of any renormalization scheme, to change the equation from one scheme to another all we have to do is to write \( \bar{g}, A_i^n, B_i^n, \) and \( \Lambda \) in the new scheme.

We now write

\[
F_2^{\gamma}(x, Q^2, P^2) = F_{2,PQCD}^{\gamma}(x, Q^2, P^2) + F_{2,NP}^{\gamma}(x, Q^2, P^2) \tag{24}
\]

with the corresponding moment functions

\[
F_2^{\gamma}(n, Q^2, P^2) = F_{2,PQCD}^{\gamma}(n, Q^2, P^2) + F_{2,NP}^{\gamma}(n, Q^2, P^2). \tag{25}
\]

We continue to follow the convention and the simple vector meson dominance model for the
nonperturbative part of the structure function by writing its dependence on $P^2$ as

$$F_{2,NP}^\gamma(x, Q^2, P^2) = \frac{F_{2,NP}^\gamma(x, Q^2)}{(1 + \frac{P^2}{M^2})^2}.$$  \hspace{1cm} (26)$$

Where $F_{2,NP}^\gamma$ is the real photon nonperturbative part which we have been discussing up to this point. In the previous section we simply had $P^2 = 0$. This is the only consistent way of including the $P^2$ dependence once we adopt the notion that the NP part of the photon structure function can be approximated by the simple vector dominance model.

As $n$ increases, the moments are sharply suppressed in exactly the same way as for the real photon case. Note that the $n = 2$ case is possible for both the nonperturbative and perturbative parts in the virtual photon structure function while only the nonperturbative part appears in the real structure function case.

Since the PQCD structure function of the real photon can be obtained from the virtual photon structure function formally by putting $P^2 = \Lambda^2$, it makes sense to replace $P^2$ in our expressions by $P^2 + \Lambda^2$. Then we return to the real photon structure function by setting $P^2 = 0$ rather than $\Lambda^2$. This modification does not affect our calculation much since we need $P^2 = 0$ and $P^2 \gg \Lambda^2$. Henceforth we write the PQCD piece with this change.

In Figure 3 we show the total virtual photon structure function, $F_{2,tot}^{\gamma,vir}(n, Q^2, P^2)$, versus the probe photon mass $Q^2$ from PQCD and NP after normalizing the structure function by $\alpha \ln(Q^2/(P^2 + \Lambda^2))$. These display the strong nonperturbative dependence of the total structure function at low $Q^2$ ($Q^2 < 10$ GeV$^2$) and at low moments. Perhaps the most interesting behavior of the function occurs in the regions $0.4$ GeV$^2 \leq P^2 \leq 0.6$ GeV$^2$ and $Q \geq 20$ GeV$^2$. As the moments increase we observe similar behavior to that in the real photon case.

It is useful to look at the $n = 2$ moment in the virtual structure function since there is no such moment in the real case. In fact we found interesting behavior of the structure functions in our three examples. When $P^2 = 0.2$ GeV$^2$, we have qualitatively same behavior for all three fits while the fit (A) behaves very differently when $P^2 = 0.4$ GeV$^2$. However, one should keep in mind that we have used only $\rho$ meson in the analysis.
From these plots we can clearly see how the NP contribution behaves, given different assumptions about its form at $P^2 = 0$.

We close with a comment on the PQCD transition to the real photon case. The virtual photon structure function can be reduced to the real structure function by taking the target photon mass equal to zero (i.e., $P^2 = \bar{\Lambda}^2$ in equation (23)). Then $\bar{g}^2(P^2 = \bar{\Lambda}^2)$ goes to the infinity where quark confinement start occurring. The equation (23) becomes

$$\int_0^1 dx x^{n-2} F_2^\gamma(x, Q^2, P^2 = \bar{\Lambda}^2) = \frac{1}{16\pi^2 \alpha_s} \left[ \sum_i \tilde{P}_i^n \frac{1}{1 + \lambda_i^n / 2\beta_0} \frac{16\pi^2}{\bar{g}^2(Q^2)} + \sum_i A_i^n + \sum_i B_i^n + C_i^n \right].$$

Now this expression is valid only for $n > 2$ since we cannot calculate the moment for $n = 2$ case in the real structure function due to the existence of unknown hadronic matrix elements. If we compare two equations (17) and (27) in the \text{MS} scheme (\bar{\Lambda} and $\bar{g}^2_{\text{MS}}$ are defined in the equation (19)),

$$\frac{\alpha_s^2}{e^2} \left[ \sum_i \tilde{P}_i^n \frac{1}{1 + \lambda_i^n / 2\beta_0} \frac{16\pi^2}{\bar{g}^2(Q^2)} + \sum_i A_i^n + \sum_i B_i^n + C_i^n \right].$$

From the above equation we can easily identify

$$a_n = \frac{1}{2\beta_0} \left( \sum_i \tilde{P}_i^n \frac{1}{1 + \lambda_i^n / 2\beta_0} \frac{16\pi^2}{\bar{g}^2(Q^2)} \right)$$

$$b_n = \frac{1}{2\beta_0} \left( \sum_i A_i^n + \sum_i B_i^n + C_i^n \right).$$

Therefore we obtain all coefficients in Bardeen and Buras’ form from those in Uematsu and Walsh.

5 Discussion

The nonperturbative part of the photon structure function has been of interest for a long time. At low $Q^2$ it complicates efforts to check QCD experimentally and to extract the scale factor $\bar{\Lambda}$. These problems can be dealt with in part by studying the photon structure function for higher moments where the prediction of QCD is stable against the higher order
corrections \[1\] and where the nonperturbative part is small. (Of course this depends at least in part on the common assumption that the NP part vanishes quickly as the scaling variable \(x \to 1\).) The agreement of PQCD with experiment and a value \(\Lambda \simeq 200\ \text{MeV}\) is striking \[7\]. Nevertheless, it is still of interest to know to what extent the NP part of the structure function is really under control at low \(Q^2\).

We have argued here that one should explore the dependence of the virtual photon structure function on target \(P^2\). Then one can determine to an adequate extent just how important the NP part is. We think that this is clearly possible, given our results for the several NP models which we have studied.

This has a number of consequences. First, by learning the actual size of \(F_{NP}\), we can improve checks on the PQCD prediction and extract a more reliable value for \(\Lambda\). Secondly, the NP part and its relationship to the conventional PQCD prediction is of interest by itself. We know that checks of PQCD for high moments are satisfactory. However, we also know that we have no prediction for the \(n = 2\) moment at \(P^2 = 0\). It is a surprising and interesting fact that the virtual photon structure function has a calculable \(n = 2\) moment. If we want to understand the relationship of perturbative and nonperturbative QCD, this problem needs to be explored experimentally. Equivalently, we need to understand the behavior of the structure function as \(x \to 0\) for both real and virtual photon targets. The existence of a finite or logarithmically rising cross section for the center of mass energy \(W \to \infty\) at fixed \(P^2 \ll Q^2\) is a presumably nonperturbative phenomenon. This is related in some yet unknown way to the relation of perturbative and nonperturbative contributions to the structure function for small \(x\). We think that this is an important issue. It is as yet unexplored by experiment.

We can illustrate one unresolved issue by returning to our bland assumption that the NP part of the structure function can be obtained from the vector meson dominance model. While we have used this assumption to show that the NP part can be studied, we do question it. The \(P^2\) dependence can be studied by means of a dispersion relation in the target mass squared \(m^2 = \mu^2\), with threshold at \(\mu^2 = 4m^2_\pi\) \[9\], \[10\]. The NP part of the structure function
does have a contribution from small values of the mass parameter in this dispersion relation. In particular, there is a double pole at \( \mu^2 = m_\rho^2 \) corresponding to the simple \( \rho \) dominance model which we have studied. However, the NP terms in the photon structure function which are inverse powers of \( P^2 \ll \Lambda^2 \) are not in fact so simple. We expect that the dominant terms behave as [10],

\[
F_{2,NP}(x, Q^2, P^2) \simeq \frac{f_1(x, Q^2)}{1 + \frac{P^2}{m_1^2}} + \frac{f_2(x, Q^2)}{1 + \frac{P^2}{m_2^2}}. \tag{30}
\]

In writing this, we have chosen to turn an expansion in powers of \( 1/P^2 \) into one which is well behaved as \( P^2 \to 0 \). This represents an assumption about how higher order terms behave which we think is reasonable (but not proven). Only one of these terms can be thought of as corresponding to the vector meson double pole term, if \( m_1 \to m_\rho \). The other might have physically an interference term between a vector meson pole (for \( m_2 \to m_\rho \)) and another term with no pole in \( \mu^2 \). We can see no reason why the second term cannot be important. We do not know of theoretical arguments which could establish its physical significance or size.

We believe that these considerations make it clear why the \( P^2 \) dependence of the photon structure function is worth careful experimental study. We hope that these experiments will be carried out. As yet, little has been done in this promising area at the interface of perturbative and nonperturbative quantum chromodynamics.

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Figures

1. This figure set shows moments of the NP part of the photon structure function at $P^2 = 0$, with $Q^2$ ranges from $Q_0^2 = 3$ to 45 GeV$^2$ The moments for $n = 2, 3, 4$ correspond to plots 1(a), 1(b), 1(c). Each curve contains NP behavior of the photon structure function for the three fits A, B, C mentioned in the text.

2. These plots show the total real photon structure functions for the three fits. 2(a) and 2(b) correspond to $n = 4, 6$ moments respectively. Each plot shows the corresponding total moment for the PQCD and NP structure functions superimposed. Dashed curves are PQCD parts and solid curves are total moments for the examples A, B, C.

3. The figures 3(a), 3(b), and 3(c) correspond to $n = 2, n = 4, n = 6$ moments of the virtual photon structure function respectively. There are also $P^2$ dependence in the figures. $n = 2$ case is the most interesting one since we can not see this moment in the real photon structure function. Dashed curves correspond to PQCD parts of the structure function and solid curves to total moments for the examples, A, B, C.
$F_{2,NP}^{\gamma}(n=2, Q^2)/\alpha_s(Q^2)\log(Q^2/\Lambda^2)$
\[ F_{2,\gamma}^{NP}(n=4,Q^2)/\alpha_s(Q^2)\log(Q^2/\Lambda^2) \]
Fig. 1 (c)

\[ F_{2,\text{NP}}(n=6,Q^2)/\alpha_s(Q^2)\text{Log}(Q^2/\Lambda^2) \]
Fig. 3 (a1)

\[ F_2^{\gamma_{\text{tot}}(n=2,Q^2,P^2,\Lambda^2)/\alpha_s(Q^2)} \] 

\[ \text{Log}(Q^2/(P^2+\Lambda^2)) \]

\[ P^2=0.2 \]

Graph showing \( F_2^{\gamma_{\text{tot}}(n=2,Q^2,P^2,\Lambda^2)/\alpha_s(Q^2)} \) as a function of \( Q^2 \) for different curves labeled A, B, and C, and a PQCD curve.

### Notes
- The graph plots the function \( F_2^{\gamma_{\text{tot}}(n=2,Q^2,P^2,\Lambda^2)/\alpha_s(Q^2)} \text{Log}(Q^2/(P^2+\Lambda^2)) \) as a function of \( Q^2 \) for different values of \( P^2 = 0.2 \).
- The curves A, B, and C represent different theoretical or experimental scenarios.
- The PQCD curve provides a reference for comparison with the theoretical predictions.

### Mathematical Expression
\[ F_2^{\gamma_{\text{tot}}(n=2,Q^2,P^2,\Lambda^2)/\alpha_s(Q^2)} \text{Log}(Q^2/(P^2+\Lambda^2)) \]
Fig. 3 (b1)

\[ F_{\gamma,\text{vir}}^{\gamma,\text{tot}}(n=4, Q^2, P^2, P_s^2, L^2, \alpha_s(Q^2)) = \frac{\log(Q^2)}{(P^2 + \Lambda^2)} \]

- A
- B
- C
- PQCD

\( P^2 = 0.2 \)
\[ F_{2}^{\gamma_{\text{vir}}, \text{tot}}(n=6, Q^2, P^2, P^2/\alpha_s(Q^2) \log(Q^2/(P^2+\Lambda^2))) \]

\( P^2 = 0.2 \)

Fig. 3 (c1)
Fig. 2 (a)

\[ F_{2}^{\gamma_{\text{tot}}} (n=4, Q^{2}) / \alpha_{s} (Q^{2}) \log (Q^{2}/\Lambda^{2}) \]
Fig. 2 (b)

\[ F_2^{\gamma, \text{real}}(n=6, Q^2)/\alpha_s(Q^2) \log(Q^2/\Lambda^2) \]

The graph shows the function \( F_2^{\gamma, \text{real}}(n=6, Q^2)/\alpha_s(Q^2) \log(Q^2/\Lambda^2) \) as a function of \( Q^2 \). The graph includes curves labeled A, B, C, and PQCD, each representing different theoretical models. The x-axis represents \( Q^2 \) ranging from 3 to 43, and the y-axis represents the function values ranging from 0.0200 to 0.0250.
Fig. 3 (a2)

\[ F_2^{\gamma,\text{vir}}_{\text{tot}}(n=2, Q^2, P^2, \alpha_s, \log(Q^2/(P^2+\Lambda^2))) \]

- \( P^2 = 0.4 \)

Curves A, B, C, and PQCD are plotted against \( Q^2 \) from 3 to 43.
\[ F_{2, \text{tot}}^{\gamma,\text{vir}} \left( n=4, Q^2, \frac{P^2}{P^2 + \Lambda^2} \right) \alpha_s \left( Q^2 \right) \log(\frac{Q^2}{P^2 + \Lambda^2}) \]

Fig. 3 (b2)
\[ F_{\gamma \text{vir}}(n=6, Q^2, P^2, \alpha_s(Q^2)) = \frac{Q^2}{(P^2 + \Lambda^2)} \log(Q^2/(P^2 + \Lambda^2)) \]

\[ P^2 = 0.4 \]
$F_2^{\gamma_{\text{vir}}} = \frac{n=2, Q^2, P^2, \alpha_s(Q^2)}{\alpha_s(Q^2) \log(Q^2/(P^2+\Lambda^2))}$

$P^2 = 0.6$

Fig. 3 (a3)
\[
F_2^{\gamma, \text{vir}}_{\text{tot}}(n=4, Q^2, P^2, \alpha_s(Q^2)/(P^2+\Lambda^2))
\]

\[
F_2^{\gamma, \text{vir}}_{\text{tot}}(n=4, Q^2, P^2, \alpha_s(Q^2)/(P^2+\Lambda^2))
\]

\[
P^2 = 0.6
\]

Fig. 3 (b3)
Fig. 3 (c3)

$F_{\gamma}^{\nu}_{\text{tot}}(n=6, Q^2, P^2, Q_s^2, \alpha_s(Q^2) \log(Q^2/(P^2+\Lambda^2)))$

$P^2=0.6$

$Q^2$

$F_{\gamma}^{\nu}_{\text{tot}}$

$PQCD$

A

B

C

PQCD