HOMOLOGY OF $\text{GL}_n$ OVER INFINITE FIELDS OUTSIDE THE STABILITY RANGE

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Abstract. For an infinite field $F$, we study the kernel of the map

$$H_n(\text{GL}_{n-1}(F), \mathbb{Z}\left[\frac{1}{(n-2)!}\right]) \to H_n(\text{GL}_n(F), \mathbb{Z}\left[\frac{1}{(n-2)!}\right]),$$

and the cokernel of

$$H_{n+1}(\text{GL}_{n-1}(F), \mathbb{Z}\left[\frac{1}{(n-2)!}\right]) \to H_{n+1}(\text{GL}_n(F), \mathbb{Z}\left[\frac{1}{(n-2)!}\right]).$$

We give conjectural estimates of these kernels and cokernels and prove our conjectures for $n \leq 4$.

Introduction

Let $F$ be a field. For any positive integer $r$, $\text{GL}_r(F)$ embeds naturally in $\text{GL}_{r+1}(F)$. The sequence of group embeddings $\text{GL}_1(F) \subseteq \text{GL}_2(F) \subseteq \text{GL}_3(F) \subseteq \cdots$ induces the sequence of homomorphisms of homology groups

$$H_n(\text{GL}_1(F), \mathbb{Z}) \to H_n(\text{GL}_2(F), \mathbb{Z}) \to H_n(\text{GL}_3(F), \mathbb{Z}) \to \cdots.$$ These homology group appear in many areas of Algebra and Geometry. Unfortunately it is hard to calculate them explicitly. Therefore all results allowing to compare them for different values of $n$ become quite important.

By an unpublished work of Quillen, if $F$ has more than two elements, then

$$(0.1) \quad H_n(\text{GL}_r(F), \mathbb{Z}) \to H_n(\text{GL}_{r+1}(F), \mathbb{Z})$$
is surjective for $r \geq n$ and is bijective for $r \geq n + 1$. (Quillen’s proof appears in his unpublished note [26, pp. 1–15]. Unfortunately the first and the second pages of this note are unreadable, which makes it hard to follow the proof. But nevertheless, see [28, Theorem A] for an exposition of Quillen’s argument. Quillen’s result also follows from [8, Theorem A].)

With a different method, Suslin showed that if the field is infinite, then

$$(0.1)$$
is an isomorphism for $r \geq n$. Moreover, he showed that the cokernel of

$$H_n(\text{GL}_{n-1}(F), \mathbb{Z}) \to H_n(\text{GL}_n(F), \mathbb{Z})$$
is isomorphic to $K^M_n(F)$, the $n$-th Milnor $K$-group of $F$ [30, Theorem 3.4] (see also Theorem 1.1 below).

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Not much is known about the map \((0.1)\) outside the stability range, i.e. when \(r < n\). The following conjecture is attributed to Suslin (see [27, Problem 4.13], [2, Remark 7.7], [5, Conjecture 2]).

**Suslin’s Injectivity Conjecture.** For any infinite field \(F\) and any \(n > r\), the natural map \(H_n(\text{GL}_r(F), \mathbb{Q}) \to H_n(\text{GL}_{r+1}(F), \mathbb{Q})\) is injective.

The conjecture is trivial for \(n = 1, 2\). It was known for number fields by the work of Borel and Yang [2, Corollary 7.6]. It was proved for \((n = 3, r = 2)\) by Sah [27, Remark 3.19] and Elbaz-Vincent [6, Theorem 1.22] and for \((n = 4, r = 3)\) by the author [19, Theorem 3].

Recently, the conjecture has been proved for \(r = n - 1\) by Galatius, Kupers and Randal-Williams in [9, Theorem 9.18] (see Theorem 1.2 below for a precise statement.)

Beside this, Galatius, Kupers and Randal-Williams have studied the cokernel of the map

\[
H_{n+1}(\text{GL}_{n-1}(F), \mathbb{Q}) \to H_{n+1}(\text{GL}_n(F), \mathbb{Q}).
\]

They showed that the group

\[
\bigoplus_{n \geq 1} H_{n+1}(\text{GL}_n(F), \text{GL}_{n-1}(F), \mathbb{Q})
\]

has a natural \(K^*_M(F)\mathbb{Q}\)-module structure and explained how to generate this module efficiently [9, Theorem D or Theorem 9.5]. This suggests that the above map has somewhat complicated cokernel in general.

In this article we study the kernel of

\[
H_n(\text{GL}_{n-1}(F), \mathbb{Z}[\frac{1}{(n-2)!}]) \to H_n(\text{GL}_n(F), \mathbb{Z}[\frac{1}{(n-2)!}]),
\]

and the cokernel of

\[
H_{n+1}(\text{GL}_{n-1}(F), \mathbb{Z}[\frac{1}{(n-2)!}]) \to H_{n+1}(\text{GL}_n(F), \mathbb{Z}[\frac{1}{(n-2)!}]).
\]

We show that both are related to the second homology of a complex, which we introduce now. The chain of maps

\[
(0.2) \quad F \times \otimes^n \mathbb{Z} K^M_0(F) \xrightarrow{\delta_0^{(n)}} F \times \otimes^{(n-1)} \mathbb{Z} K^M_1(F) \xrightarrow{\delta_1^{(n)}} \cdots \xrightarrow{\delta_{n-1}^{(n)}} F \times \otimes^2 \mathbb{Z} K^M_{n-2}(F) \xrightarrow{\delta_2^{(n)}} F \times \otimes \mathbb{Z} K^M_{n-1}(F) \xrightarrow{\delta_1^{(n)}} K^M_n(F) \to 0
\]

with differentials

\[
\delta_i^{(n)}(a_1 \otimes \cdots \otimes a_i \otimes \{b_1, \ldots, b_{n-i}\}) = \sum_{j=1}^{i} a_1 \otimes \cdots \hat{a}_j \otimes \cdots \otimes a_i \otimes \{a_j, b_1, \ldots, b_{n-i}\},
\]

is a complex. If \(n \geq 3\), it is easy to see that \(\ker(\delta_1^{(n)}) = \text{im}(\delta_2^{(n)})\) (see Remark 5.7).
For any $n \geq 1$, let
\[
\mathcal{B}_n(F) := \begin{cases} 
0 & \text{if } n = 1 \\
\tilde{B}(F) & \text{if } n = 2, \\
\ker(\delta_2^{(n)})/\im(\delta_3^{(n)}) & \text{if } n \geq 3
\end{cases}
\]
where $\tilde{B}(F)$ is an extension of the Bloch group $B(F)$ by $\mathbb{Z}/2$ if $\text{char}(F) \neq 2$ and $\mu_{2^\infty}(F)$ is finite and is $B(F)$ otherwise. More precisely, $\tilde{B}(F) \simeq K_3^{\text{ind}}(F)/\text{Tor}_1^F(\mu(F), \mu(F))$, where the map $\tilde{B}(F) \to B(F)$ comes from the Bloch-Wigner exact sequence ([31, Theorem 5.2], [23, Theorem 5.1]). Recall that $\mu(F)$ is the group of roots of unity and $\mu_{2^\infty}(F)$ is the group of 2-power roots of unity in $F$.

The groups $\mathcal{B}_n(F)$ seems to be interesting invariants of the field $F$. Our first main result (Theorem 2.1) is:

**Theorem A.** (i) For any positive integer $n$, there is a natural map
\[
\kappa_n : \mathcal{B}_n(F)[\frac{1}{(n-2)!}] \to \ker(H_n(\text{GL}_{n-1}(F), \mathbb{Z}[\frac{1}{(n-2)!}]) \to H_n(\text{GL}_n(F), \mathbb{Z}[\frac{1}{(n-2)!}]),
\]
where its image is $(n-1)$-torsion.

(ii) Let the sequence
\[
H_n(F^\times \times \text{GL}_{n-2}(F), \mathbb{Z}[\frac{1}{(n-2)!}]) \xrightarrow{\alpha_1-\alpha_2 \ast} H_n(F^\times \times \text{GL}_{n-1}(F), \mathbb{Z}[\frac{1}{(n-2)!}]) \\
\xrightarrow{\text{inc}} H_n(\text{GL}_n(F), \mathbb{Z}[\frac{1}{(n-2)!}]) \to 0
\]
be exact for a fixed $n \geq 3$. If the natural map
\[
H_n(\text{GL}_{m-1}(F), \mathbb{Z}[\frac{1}{(m-1)!}]) \to H_n(\text{GL}_m(F), \mathbb{Z}[\frac{1}{(m-1)!}])
\]
is injective for $m = n-1, n-2$, then $\kappa_n$ is surjective.

We conjecture that $\kappa_n$ is surjective for any $n$ (Conjecture 2.2) and prove it for $n \leq 4$ (Corollary 2.5). As an application we demonstrate that the natural homomorphisms
\[
H_3(\text{GL}_2(F), \mathbb{Z}[\frac{1}{2}]) \to H_3(\text{GL}_3(F), \mathbb{Z}[\frac{1}{2}]),
\]
\[
H_4(\text{GL}_3(F), \mathbb{Z}[\frac{1}{3}]) \to H_4(\text{GL}_4(F), \mathbb{Z}[\frac{1}{3}])
\]
are injective (Corollary 2.5). The first injectivity (the case $n = 3$) was already known by [18, Theorem 5.4]. The second injectivity seems to be new.

Our second main result (Theorem 5.1) concerns the quotient group
\[
H_{n+1}(\text{GL}_n(F), \mathbb{Z}[\frac{1}{(n-2)!}]) / H_{n+1}(\text{GL}_{n-1}(F), \mathbb{Z}[\frac{1}{(n-2)!}]).
\]
Inductively, the study of this group can be reduced to the study of the quotient group
\[
H_{n+1}(\text{GL}_n(F), \mathbb{Z}[\frac{1}{(n-2)!}]) / H_{n+1}(F^\times \times \text{GL}_{n-1}(F), \mathbb{Z}[\frac{1}{(n-2)!}]).
\]
Theorem B. Let
\[ H_m(\text{GL}_{m-1}(F), \mathbb{Z}\left[\frac{1}{(m-1)!}\right]) \to H_m(\text{GL}_m(F), \mathbb{Z}\left[\frac{1}{m!}\right]) \]
be injective for \( m = n - 1, n - 2 \). Then there is a natural map
\[ \chi_n: B_n(F)\left[\frac{1}{(n-2)!}\right] \to \frac{H_{n+1}(\text{GL}_n(F), \mathbb{Z}\left[\frac{1}{(n-2)!}\right])}{H_{n+1}(F^\times \times \text{GL}_{n-1}(F), \mathbb{Z}\left[\frac{1}{(n-2)!}\right])}. \]

We conjecture that \( \chi_n \) is always surjective (Conjecture 5.3) and prove it for \( n \leq 4 \) (Proposition 4.3, Corollary 5.4).

Remark. The above results hold over any commutative ring with many units in the sense of Guin [11, §1] (also see [19, §2]). The only exception is Proposition 4.3, where we need to assume that there is a field \( F \) and a ring homomorphism \( R \to F \) such that the restriction map \( \mu(R) \to \mu(F) \) is injective (see [23, Theorem 5.1]). Recall that \( R \) is a ring with many units if for any finite number of surjective linear forms \( f_i: R^2 \to R \), there exists \( v \in R^2 \) such that, for all \( i \), \( f_i(v) \in R^\times \). Important examples of rings with many units are semilocal rings with infinite residue fields.

In Section 6, we study \( B_n(F) \) over certain fields and explain how the above theorems can be improved over them. Using these improved results for \( n \leq 4 \) and some unpublished results of Galatius, Kupers and Randal-Williams (see Theorems 1.2, 6.1) we prove the following (Theorem 6.3):

Theorem C. Let \( F \) be a field such that \( F^\times \) is divisible. Then
(i) \( H_n(\text{GL}_{n-1}(F), \mathbb{Z}) \to H_n(\text{GL}_n(F), \mathbb{Z}) \) is injective for any \( n \),
(ii) \( H_{n+1}(\text{GL}_n(F), \mathbb{Z})/H_{n+1}(\text{GL}_{n-1}(F), \mathbb{Z}) \) is divisible for \( n \neq 2 \) and is uniquely divisible for \( n \geq 5 \).

A field \( F \) is called real if \(-1\) is not the sum of squares and is called real closed if it is real and has no real proper algebraic extension [13, Chap. XI].

Over these fields we get (Theorem 6.8):

Theorem D. Let \( F \) be a real closed field. Then
(i) \( H_3(\text{GL}_2(F), \mathbb{Z}) \to H_3(\text{GL}_3(F), \mathbb{Z}) \) is injective,
(ii) \( H_n(\text{GL}_{n-1}(F), \mathbb{Z}[\frac{1}{2}]) \to H_n(\text{GL}_n(F), \mathbb{Z}[\frac{1}{2}]) \) is injective for any \( n \),
(iii) \( H_{n+1}(\text{GL}_n(F), \mathbb{Z}[\frac{1}{2}])/H_{n+1}(\text{GL}_{n-1}(F), \mathbb{Z}[\frac{1}{2}]) \) is uniquely divisible for \( n \geq 5 \).

Notation. If \( A \to B \) is a homomorphism of abelian groups, by \( B/A \) and \( \text{im}(A) \) we mean \( \text{coker}(A \to B) \) and \( \text{im}(A \to B) \), respectively. We denote an element of \( B/A \) represented by \( b \in B \) again by \( b \). Any inclusion of groups \( H \subseteq G \) is denoted by \( \text{inc}: H \to G \). If \( R \) is a commutative ring and \( A \) an abelian group, by \( A_R \) we mean \( A \otimes R \). Moreover, by \( A[\frac{1}{n}] \) we mean \( A \otimes \mathbb{Z}[\frac{1}{n}] = A[1/n] \). We denote the \( i \)-th summand of \( F^\times k := F^\times \times \cdots \times F^\times \) by \( F_i^\times \).

Convention. In this article we assume that \( F \) always is an infinite field.
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1. The homology of general linear groups and Milnor K-groups

For an arbitrary group $G$, let $B_\bullet(G) \xrightarrow{\xi} \mathbb{Z}$ denote the right bar resolution of $G$. For any left $G$-module $N$, $H_n(G, N)$ coincides with the $n$-th homology of the complex $B_\bullet(G) \otimes_{\mathbb{Z}[G]} N$. In particular,

$$H_n(G, \mathbb{Z}) = H_n(B_\bullet(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}) = H_n(B_\bullet(G)G).$$

For any $n$-tuple $(g_1, g_2, \ldots, g_n)$ of pairwise commuting elements of $G$ and any commutative ring $\mathbb{k}$, let the homology class

$$c(g_1, g_2, \ldots, g_n) \in H_n(G, \mathbb{k})$$

be represented by the cycle

$$\sum_{\sigma \in \Sigma_n} \text{sign}(\sigma)[g_{\sigma(1)}] [g_{\sigma(2)}] \cdots [g_{\sigma(n)}] \otimes 1 \in B_n(G) \otimes_{\mathbb{Z}[G]} \mathbb{k},$$

where $\Sigma_n$ is the symmetric group of degree $n$. In fact $c(g_1, \ldots, g_n)$ is the image of $g_1 \wedge \cdots \wedge g_n$ under the composition

$$\wedge_{\mathbb{Z}}^G A_\mathbb{k} \to H_n(A, \mathbb{k}) \to H_n(G, \mathbb{k}),$$

where $A$ is the abelian subgroup of $G$ generated by $g_1, \ldots, g_n$ and the first map is the Pontryagin product [3, Chap. V, §5 and §6]. It follows immediately from the known properties of the Pontryagin product that:

(a) If $h_1 \in G$ commutes with all the elements $g_1, \ldots, g_n$, then

$$c(g_1 h_1, g_2, \ldots, g_n) = c(g_1, g_2, \ldots, g_n) + c(h_1, g_2, \ldots, g_n).$$

In particular if $h_1 = g_1^{-1}$, then we have

$$c(g_1^{-1}, g_2, \ldots, g_n) = -c(g_1, g_2, \ldots, g_n).$$

(b) For every $\sigma \in \Sigma_n$,

$$c(g_{\sigma(1)}, \ldots, g_{\sigma(n)}) = \text{sign}(\sigma)c(g_1, \ldots, g_n).$$

(c) The cup product of $c(g_1, \ldots, g_p) \in H_p(G, \mathbb{k})$ and $c(g_1', \ldots, g_q') \in H_q(G', \mathbb{k})$ is

$$c((g_1, 1), \ldots, (g_p, 1), (1, g_1'), \ldots, (1, g_q')) \in H_{p+q}(G \times G', \mathbb{k}).$$

Let $F$ be an infinite field. For any $a \in F^\times$ and any $1 \leq i \leq n$, let $D_{1,n}(a)$ be the diagonal matrix of size $n$ with $a$ in the $i$-th position of the diagonal and 1 everywhere else. It is well-known (see [30, Lemma 2.7.1] or [11, Proposition 3.1.2]) that the map

$$c_n : F^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} F^\times \to H_n(\mathrm{GL}_n(F), \mathbb{Z})/H_n(\mathrm{GL}_{n-1}(F), \mathbb{Z}),$$

$$a_1 \otimes \cdots \otimes a_n \mapsto c(D_{1,n}(a_1), \ldots, D_{n,n}(a_n)) \pmod{H_n(\mathrm{GL}_{n-1}(F), \mathbb{Z})}$$

is well-defined.
factors through the $n$-th Milnor $K$-group of $F$:
\[ \tilde{c}_n : K_n^M(F) \to H_n(\text{GL}_n(F), \mathbb{Z})/H_n(\text{GL}_{n-1}(F), \mathbb{Z}). \]

Observe that $c_n$ is nothing but the composite
\[ H_1(F^\times, \mathbb{Z})^{\otimes n} \cup H_n(T_n(F), \mathbb{Z}) \to H_n(\text{GL}_n(F), \mathbb{Z}) \]
\[ \to H_n(\text{GL}_{n}(F), \mathbb{Z})/H_n(\text{GL}_{n-1}(F), \mathbb{Z}), \]
with $T_n(F)$ the group of diagonal $(n \times n)$-matrices.

**Theorem 1.1** (Suslin, Nesterenko-Suslin). (i) The natural map
\[ \text{inc}_*: H_n(\text{GL}_r(F), \mathbb{Z}) \to H_n(\text{GL}_{r+1}(F), \mathbb{Z}) \]

is an isomorphism for any $r \geq n$.

(ii) There exists a natural map $s_n : H_n(\text{GL}_n(F), \mathbb{Z}) \to K_n^M(F)$ such that
\[ H_n(\text{GL}_{n-1}(F), \mathbb{Z}) \xrightarrow{\text{inc}} H_n(\text{GL}_n(F), \mathbb{Z}) \xrightarrow{s_n} K_n^M(F) \to 0 \]
is an exact sequence. Moreover, $s_n$ is the inverse of $\tilde{c}_n$.

(iii) The composite
\[ K_n^M(F) \to K_n(F) \xrightarrow{h_n} H_n(\text{GL}_n(F), \mathbb{Z}) \xrightarrow{\text{inc}^{-1}} H_n(\text{GL}_n(F), \mathbb{Z}) \xrightarrow{s_n} K_n^M(F) \]

coincides with multiplication by $(-1)^{n-1}(n-1)!$, where $h_n$ is the Hurewicz map.

*Proof.* See [30, Theorem 3.4, Corollary 4.4] or [24, Theorems 3.25, 4.1, Remark 3.27].

In [19, Proposition 4] the author showed that if $n \geq 3$ and if $k$ is a field such that $(n-1)! \in k^\times$, then the injectivity of
\[ H_n(\text{GL}_{n-1}(F), k) \to H_n(\text{GL}_n(F), k) \]
follows, by an induction process, from the exactness of the complex
\begin{equation}
H_n(F^\times \times \text{GL}_{n-2}(F), k) \xrightarrow{\alpha_1 - \alpha_2} H_n(F^\times \times \text{GL}_{n-1}(F), k) \xrightarrow{\text{inc}} H_n(\text{GL}_n(F), k) \to 0,
\end{equation}
where $\alpha_1(\text{diag}(a, b, A)) = \text{diag}(b, a, A)$ and $\alpha_2 = \text{inc}$.

The exactness of the above complex was known for $n = 3$ [18, Corollary 3.5] and $n = 4$ [19, Theorem 2]. Recently Galatius, Kupers and Randal-Williams have proved, by completely new and interesting methods, that this complex is indeed exact. Thus they proved Suslin’s injectivity conjecture for $r = n - 1$.

**Theorem 1.2** (Galatius, Kupers, Randal-Williams [9, Section 9.6]). Let $k$ be a field such that $(n-1)! \in k^\times$. Then

(i) For any $n \geq 3$, the complex (1.1) is exact,

(ii) For any $n$, the natural map $H_n(\text{GL}_{n-1}(F), k) \to H_n(\text{GL}_n(F), k)$ is injective.
Remark 1.3. We conjecture that for any $n$, the natural map 
\[ H_n(\text{GL}_{n-1}(F), \mathbb{Z}[\frac{1}{(n-1)!}]) \to H_n(\text{GL}_n(F), \mathbb{Z}[\frac{1}{(n-1)!}]) \]
is injective (see also Remark 2.6(ii) below). This follows from the exactness of the sequence 
\[ H_n(F^\times \times GL_{n-2}(F), \mathbb{Z}[\frac{1}{(n-1)!}]) \overset{\alpha_1 - \alpha_2}{\longrightarrow} H_n(F^\times \times GL_{n-1}(F), \mathbb{Z}[\frac{1}{(n-1)!}]) \overset{\text{inc}}{\longrightarrow} H_n(\text{GL}_n(F), \mathbb{Z}[\frac{1}{(n-1)!}]) \to 0 \]
for all $n \geq 3$ (see the proof of Theorem 2.1 below). The exactness of the above sequence does not follow directly from the result of Galatius, Kupers, Randal-Williams mentioned in Theorem 1.2 (at least we couldn’t prove this). But we are very hopeful that their methods developed in [9, Section 9] might be applied to this general setting.

2. On the kernel of $H_n(\text{GL}_{n-1}(F), \mathbb{Z}) \to H_n(\text{GL}_n(F), \mathbb{Z})$

In this section we show that the group $\mathcal{B}_n(F)$ is related to the kernel of 
\[ H_n(\text{GL}_{n-1}(F), \mathbb{Z}) \to H_n(\text{GL}_n(F), \mathbb{Z}). \]
This connection first was realised for $n = 3$ in [20, Remark 3.5]. The following is Theorem A of the introduction.

**Theorem 2.1.** (i) For any positive integer $n$, there is a natural map 
\[ \varphi_n: \mathcal{B}_n(F) \to \ker(H_n(\text{GL}_{n-1}(F), \mathbb{Z}) \to H_n(\text{GL}_n(F), \mathbb{Z})), \]
which factors through the multiplication by $(n-2)$-map $\mathcal{B}_n(F) \to (n-2)\mathcal{B}_n(F)$, $x \mapsto (n-2)x$, and its image is $(n-1)$-torsion. In particular, there is a natural map 
\[ \kappa_n: \mathcal{B}_n(F)[\frac{1}{(n-2)!}] \to \ker(H_n(\text{GL}_{n-1}(F), \mathbb{Z}[\frac{1}{(n-2)!}]) \to H_n(\text{GL}_n(F), \mathbb{Z}[\frac{1}{(n-2)!}]))), \]
where its image is $(n-1)$-torsion.

(ii) Let the sequence 
\[ H_n(F^\times \times GL_{n-2}(F), \mathbb{Z}[\frac{1}{(n-2)!}]) \overset{\alpha_1 - \alpha_2}{\longrightarrow} H_n(F^\times \times GL_{n-1}(F), \mathbb{Z}[\frac{1}{(n-2)!}]) \overset{\text{inc}}{\longrightarrow} H_n(\text{GL}_n(F), \mathbb{Z}[\frac{1}{(n-2)!}]) \to 0 \]
be exact for a fixed $n \geq 3$. If the natural map $H_m(\text{GL}_{m-1}(F), \mathbb{Z}[\frac{1}{(m-1)!}]) \to H_m(\text{GL}_m(F), \mathbb{Z}[\frac{1}{(m-1)!}])$ is injective for $m = n - 1, n - 2$, then $\kappa_n$ is surjective.

**Proof.** (i) We define $\varphi_1$ and $\varphi_2$ as trivial maps. So let $n \geq 3$. Let $x = \sum a \otimes b \otimes \{c_1, \ldots, c_{n-2}\} \in F^\times \otimes F^\times \otimes K^M_{n-2}(F)$ represents an element of $\mathcal{B}_n(F)$. Let $y$ be the image of $x$ by the map 
\[ \text{id}_{F^\times} \otimes \text{id}_{F^\times} \otimes \iota_{n-2}: F^\times \otimes F^\times \otimes K^M_{n-2}(F) \to F^\times \otimes F^\times \otimes H_{n-2}(\text{GL}_{n-2}(F), \mathbb{Z}). \]
Thus
\[ y = \sum a \otimes b \otimes [c_1, \ldots, c_{n-2}] = \sum a \otimes b \otimes c(C_{1,n-2},\ldots,C_{n-2,n-2}). \]

Consider the complex
\[ H_n(F^\times \times GL_{n-2}(F),\mathbb{Z}) \xrightarrow{\alpha_1^{n} \otimes \alpha_2^n} H_n(F^\times \times GL_{n-1}(F),\mathbb{Z}) \]
and set \( \alpha = \alpha_1^{n} - \alpha_2^n. \) The restriction of \( \alpha \) on
\[ F^n_1 \otimes F^n_2 \otimes H_{n-2}(GL_{n-2}(F),\mathbb{Z}) \subseteq H_n(F^\times \times GL_{n-2}(F),\mathbb{Z}) \]
factors through \( F^n_1 \otimes F^n_2 \otimes H_{n-1}(GL_{n-1}(F),\mathbb{Z}) \subseteq H_n(F^\times \times GL_{n-1}(F),\mathbb{Z}) \) and we have
\[
\alpha(y) = - \sum (b \otimes c(\text{diag}(a, I_{n-2}), \text{diag}(1, C_{1,n-2}), \ldots, \text{diag}(1, C_{n-2,n-2})))
\]
\[
\quad + a \otimes c(\text{diag}(I_{n-2}, a), C_{1,n-1}, \ldots, C_{n-2,n-1})
\]
\[
\quad + a \otimes c(\text{diag}(I_{n-2}, b), C_{1,n-1}, \ldots, C_{n-2,n-1})
\]
\[
\quad + a \otimes c(\text{diag}(I_{n-2}, b), C_{1,n-1}, \ldots, C_{n-2,n-1})
\]
(Here we denote the \( i \)-th summand of \( F^\times = F^\times \times F^\times \) by \( F^\times_i \) for \( i = 1, 2 \).)

On the other hand, the composite
\[ F^\times \otimes F^\times \otimes K^M_{n-2}(F) \xrightarrow{\delta^1_n} F^\times \otimes K^M_{n-1}(F) \xrightarrow{\text{id}_{F^\times} \otimes 1_{n-1}} F^\times \otimes H_{n-1}(GL_{n-1}(F),\mathbb{Z}) \]
takes \( x \) to \( z = \sum (b \otimes [a, c_1, \ldots, c_{n-2}] + a \otimes [b, c_1, \ldots, c_{n-2}]). \) We have
\[
\alpha((n-2)y) = -(n-2) \sum (b \otimes c(\text{diag}(I_{n-2}, a), C_{1,n-1}, \ldots, C_{n-2,n-1})
\]
\[
\quad + a \otimes c(\text{diag}(I_{n-2}, b), C_{1,n-1}, \ldots, C_{n-2,n-1})
\]
\[
\quad + a \otimes c(\text{diag}(I_{n-2}, b), C_{1,n-1}, \ldots, C_{n-2,n-1})
\]
\[
\quad + a \otimes c(\text{diag}(I_{n-2}, b), C_{1,n-1}, \ldots, C_{n-2,n-1})
\]
Let \( A_1, A_2 \) and \( B_1, B_2 \) be the following summands of \( H_n(F^\times \times GL_{n-1}(F),\mathbb{Z}) \) and \( H_n(F^\times \times GL_{n-2}(F),\mathbb{Z}) \), respectively:
\[
A_1 = H_n(GL_{n-1}(F),\mathbb{Z}), \quad B_1 = F^n_2 \otimes H_{n-1}(GL_{n-2}(F),\mathbb{Z}),
\]
\[
A_2 = F^\times \otimes H_{n-1}(GL_{n-1}(F),\mathbb{Z}), \quad B_2 = F^n_1 \otimes F^n_2 \otimes H_{n-2}(GL_{n-2}(F),\mathbb{Z}).
\]

Consider the following element of \( B_1: \)
\[
z = \sum (b \otimes c(a I_{n-2}, C_{1,n-2}, \ldots, C_{n-2,n-2})
\]
\[
\quad + a \otimes c(b I_{n-2}, C_{1,n-2}, \ldots, C_{n-2,n-2})
\]
The map \( \alpha|_{B_1 \otimes B_2} \) factors through \( A_1 \oplus A_2 \) and
\[
\alpha|_{B_1 \otimes B_2}(z, (n-2)y) = (-w, 0) \in A_1 \oplus A_2 \subseteq H_n(F^\times \times GL_{n-1}(F),\mathbb{Z}),
\]
where

\[ w = \sum (c(\text{diag}(I_{n-2}, b), \text{diag}(aI_{n-2}, 1), C_{1,n-1}, \ldots, C_{n-2,n-1}) + c(\text{diag}(I_{n-2}, a), \text{diag}(bI_{n-2}, 1), C_{1,n-1}, \ldots, C_{n-2,n-1})). \]

Clearly \( w \) is in the kernel of \( H_n(\text{GL}_{n-1}(F), \mathbb{Z}) \rightarrow H_n(\text{GL}_n(F), \mathbb{Z}) \). Now define

\[ \varphi_n : B_n(F) \rightarrow \ker(H_n(\text{GL}_{n-1}(F), \mathbb{Z}) \rightarrow H_n(\text{GL}_n(F), \mathbb{Z})), \quad \varphi(x) = w. \]

By a direct computation, using the definition of \( \varphi \), it is easy to see that

\[ \varphi_n(b \otimes c \otimes \{d_1, \ldots, d_{n-3}, a\} + a \otimes c \otimes \{d_1, \ldots, d_{n-3}, b\} + a \otimes b \otimes \{d_1, \ldots, d_{n-3}, c\} \]

is trivial. This shows that \( \varphi_n \) is well-defined.

To prove that \( w \) is \((n-1)\)-torsion, consider the composite

\[ F^\times \otimes K^M_{n-1}(F) \xrightarrow{\text{id}_F \otimes \iota_{n-1}} F^\times \otimes H_n(\text{GL}_{n-1}(F), \mathbb{Z}) \xrightarrow{\tau} H_n(\mathbb{F}^\times \times \text{GL}_{n-1}(F), \mathbb{Z}), \]

where \( \tau : \mathbb{F}^\times \times \text{GL}_{n-1}(F) \rightarrow \text{GL}_{n-1}(F) \) is the multiplication \((a, A) \mapsto aA = (aI_{n-1})A \). Under this composite,

\[
0 = (-1)^{n-2} \sum (b \otimes \{a, c_1, \ldots, c_{n-2}\} + a \otimes \{b, c_1, \ldots, c_{n-2}\})
= (n - 1)^{n-2} \sum (b \otimes \{c_1, \ldots, c_{n-2}, a\} + a \otimes \{c_1, \ldots, c_{n-2}, b\})
\]

maps to

\[
0 = (-1)^{n-2} \sum (c(bI_{n-1}, C_{1,n-1}, \ldots, C_{n-2,n-1}, A_{n-1,n-1}) + c(aI_{n-1}, C_{1,n-1}, \ldots, C_{n-2,n-1}, B_{n-1,n-1}))
= \sum (c(\text{diag}(bI_{n-2}, b), \text{diag}(aI_{n-2}, a^{-(n-2)}, C_{1,n-1}, \ldots, C_{n-2,n-1}) + c(\text{diag}(aI_{n-2}, a), \text{diag}(bI_{n-2}, b^{-(n-2)}, C_{1,n-1}, \ldots, C_{n-2,n-1})
= \sum (c(\text{diag}(bI_{n-2}, 1), \text{diag}(I_{n-2}, a^{-(n-2)}, C_{1,n-1}, \ldots, C_{n-2,n-1}) + c(\text{diag}(I_{n-2}, a), \text{diag}(bI_{n-2}, a^{-(n-2)}, C_{1,n-1}, \ldots, C_{n-2,n-1})
= (n - 1)w.
\]

(Observe that \( C_{i,n-1} = \text{diag}(C_{i,n-2}, 1) \) for \( 1 \leq i \leq n - 2 \).) Therefore \( w \) is \((n-1)\)-torsion.

The second part follows from tensoring \( \varphi_n \) with \( \mathbb{Z}[\frac{1}{(n-2)!}] \) and the fact that localization is exact. Following the above argument, it is easy to see that for any \( n \geq 3 \), \( \kappa_n \) is given by

\[
\sum a \otimes b \otimes \{c_1, \ldots, c_{n-2}\} \mapsto \frac{(-1)^{n-2}}{(n-2)!} w.
\]

(ii) The claim is trivial for \( n = 1, 2 \). So let \( n \geq 3 \). By Corollary ?? and the injectivity hypothesis for \( m = n - 1, n - 2 \), we have the decomposition

\[
H_m(\text{GL}_m(F), \mathbb{Z}[\frac{1}{(m-1)!}]) \simeq H_m(\text{GL}_{m-1}(F), \mathbb{Z}[\frac{1}{(m-1)!}]) \oplus K^M_m(F)[\frac{1}{(m-1)!}].
\]
For simplicity, we set $\mathbb{k} = \mathbb{Z}[\frac{1}{(n-2)!}]$. By the Künneth formula, and specifying a splitting (see [32, Theorem 2.7]), we have

$$H_n(F^\times \times \text{GL}_{n-1}(F), \mathbb{k}) \simeq \bigoplus_{i=1}^4 T_i,$$
$$H_n(F^\times^2 \times \text{GL}_{n-2}(F), \mathbb{k}) \simeq \bigoplus_{i=1}^4 U_i,$$

where

$$T_1 = H_n(\text{GL}_{n-1}(F), \mathbb{k}),$$
$$T_2 = F^\times \times H_{n-1}(\text{GL}_{n-1}(F), \mathbb{k}) \simeq T_2' \oplus T_2'',$$
$$T_2' = T_2'' \times H_{n-1}(\text{GL}_{n-2}(F), \mathbb{k}),$$
$$T_2'' = F^\times \times K^{n-1}_n(F) \mathbb{k},$$
$$T_3 = \bigoplus_{i=2}^n H_i(F^\times, \mathbb{k}) \otimes H_{n-i}(\text{GL}_{n-1}(F), \mathbb{k}),$$
$$T_4 = \bigoplus_{i=1}^n \text{Tor}_i^Z(H_i(F^\times, \mathbb{Z}), H_{n-i-1}(\text{GL}_{n-1}(F), \mathbb{Z})) \mathbb{k},$$

and

$$U_1 = H_n(\text{GL}_{n-2}(F), \mathbb{k}),$$
$$U_2 = \bigoplus_{i=1}^n H_i(F_1^\times, \mathbb{k}) \otimes H_{n-i}(\text{GL}_{n-2}(F), \mathbb{k}),$$
$$U_3 = \bigoplus_{i=1}^n H_i(F_2^\times, \mathbb{k}) \otimes H_{n-i}(\text{GL}_{n-2}(F), \mathbb{k}),$$
$$U_4 = F_1^\times \otimes F_2^\times \otimes H_{n-2}(\text{GL}_{n-3}(F), \mathbb{k}),$$
$$U_4' = F_1^\times \otimes F_2^\times \otimes K^{n-2}_n(F) \mathbb{k},$$
$$U_5 = \bigoplus_{i,j \geq 3} H_i(F_1^\times, \mathbb{k}) \otimes H_j(F_2^\times, \mathbb{k}) \otimes H_{n-i-j}(\text{GL}_{n-2}(F), \mathbb{k}),$$
$$U_6 = \bigoplus_{i,j \geq 2} \text{Tor}_i^Z(H_i(F_1^\times, \mathbb{Z}), H_{n-i-1}(\text{GL}_{n-2}(F), \mathbb{Z})) \mathbb{k},$$
$$U_7 = \bigoplus_{i,j \geq 2} \text{Tor}_i^Z(H_i(F_2^\times, \mathbb{Z}), H_{n-i-1}(\text{GL}_{n-2}(F), \mathbb{Z})) \mathbb{k},$$
$$U_8 = \bigoplus_{i,j \geq 2} \text{Tor}_i^Z(H_i(F_1^\times, \mathbb{Z}), H_{n-i-1}(F_2^\times, \mathbb{Z})) \mathbb{k},$$
$$U_9 = \bigoplus_{i,j \leq n-2} H_i(F_1^\times, \mathbb{Z}) \otimes \text{Tor}_i^Z(H_j(F_2^\times, \mathbb{Z}), H_{n-i-j-1}(\text{GL}_{n-2}(F), \mathbb{Z})) \mathbb{k},$$
$$U_{10} = \bigoplus_{i,j \leq n-2} \text{Tor}_i^Z(H_i(F_1^\times, \mathbb{Z}), \text{Tor}_i^Z(H_j(F_2^\times, \mathbb{Z}), H_{n-i-j}(\text{GL}_{n-2}(F), \mathbb{Z})) \mathbb{k},$$
$$U_{11} = \bigoplus_{i,j \leq n-2} \text{Tor}_i^Z(H_i(F_1^\times, \mathbb{Z}), \text{Tor}_i^Z(H_j(F_2^\times, \mathbb{Z}), H_{n-i-j-2}(\text{GL}_{n-2}(F), \mathbb{Z})) \mathbb{k}.$$

Let $t_1 \in \text{ker}(H_n(\text{GL}_{n-1}(F), \mathbb{k}) \rightarrow H_n(\text{GL}_{n}(F), \mathbb{k}))$. Then $t = (t_1, 0, 0, 0)$ is in the kernel of

$$\text{inc}_*: H_n(F^\times \times \text{GL}_{n-1}(F), \mathbb{k}) \rightarrow H_n(\text{GL}_{n}(F), \mathbb{k}).$$

Thus there exists $u = (u_1, \ldots, u_{11}) \in H_n(F^\times^2 \times \text{GL}_{n-2}(F), \mathbb{k})$ such that

$$\alpha(u) = t,$$

where $\alpha := \alpha_1 - \alpha_2$. Consider the complex

$$H_n(F^\times I_2 \times \text{GL}_{n-2}(F), \mathbb{k}) \oplus H_n(F^\times^3 \times \text{GL}_{n-3}(F), \mathbb{k}) \xrightarrow{\beta} H_n(F^\times^2 \times \text{GL}_{n-2}(F), \mathbb{k}) \xrightarrow{\alpha} H_n(F^\times \times \text{GL}_{n-1}(F), \mathbb{k}),$$

where $F^\times I_2$ is the group of the non-zero multiples of the $(2 \times 2)$ identity matrix and

$$\beta = (\text{inc}_*, \sigma_1, - \sigma_2, + \sigma_3).$$
with $\text{diag}(a,b,c,A) \xrightarrow{\sigma_1} \text{diag}(b,c,a,A)$, $\text{diag}(a,b,c,A) \xrightarrow{\sigma_2} \text{diag}(a,c,b,A)$ and $\text{diag}(a,b,c,A) \xrightarrow{\sigma_3=\text{inc}} \text{diag}(a,b,c,A)$.

Let $H_n(F^\times I_2 \times \text{GL}_{n-2}(F), k) \simeq V_1 \oplus V_2 \oplus V_3$, where

$$
V_1 = H_n(\text{GL}_{n-2}(F), k),
V_2 = \bigoplus_{i=1}^n H_i(F^\times I_2, k) \otimes H_{n-i}(\text{GL}_{n-2}(F), k),
V_3 = \bigoplus_{i=1}^{n-2} \text{Tor}_i^Z(H_i(F^\times I_2, \mathbb{Z}), H_{n-i-1}(\text{GL}_{n-2}(F), \mathbb{Z})).
$$

Since $\beta|_{H_n(F^\times I_2 \times \text{GL}_{n-2}(F), k)} = \text{inc}_*$, we have

$$
\beta((u_1, u_2, u_6), 0) = (u_1, u_2, u_6, 0, 0, 0, 0, 0).
$$

Observe that

$$
u - \beta((u_1, u_2, u_6), 0) = (0, 0, u_3 - u_2, u_4, u_5, 0, u_7 - u_6, u_8, u_9, u_{10}, u_{11})$$

and

$$
\alpha(u - \beta((u_1, u_2, u_6), 0)) = t.
$$

Thus from the beginning we may assume that $u_1 = u_2 = u_6 = 0$.

Let $W$ be the following summand of $H_n(F^\times 3 \times \text{GL}_{n-3}(F), k)$:

$$
W = \bigoplus_{i+j \geq 3 \atop i,j \geq 0} W_{i,j} = \bigoplus_{i+j \geq 3 \atop i,j \geq 0} H_i(F_2^\times, k) \otimes H_j(F_3^\times, k) \otimes H_{n-i-j}(\text{GL}_{n-3}(F), k).
$$

The restriction of $\beta$ on $W_{i,j}$ factors through $U_3 \oplus U_5$ and

$$
\beta|_{W_{i,j}} : W_{i,j} \rightarrow U_3 \oplus U_5 \subseteq H_n(F^\times 2 \times \text{GL}_{n-2}(F), k),
$$

$$
x \otimes y \otimes z \mapsto ((-\sigma_{2*} + \sigma_{3*})(x \otimes y \otimes z), x \otimes y \otimes \text{inc}_*(z)).
$$

By homological stability, $H_{n-i-j}(\text{GL}_{n-3}(F), k) \rightarrow H_{n-i-j}(\text{GL}_{n-2}(F), k)$ is an isomorphism for $i + j \geq 3$. Thus we may assume $u_5 = 0$ (similar to the elimination of $u_1, u_2, u_6$).

The restriction of $\beta$ on the summand $W' = F_2^\times \otimes F_3^\times \otimes H_{n-2}(\text{GL}_{n-3}(F), k)$ of $H_n(F^\times 3 \times \text{GL}_{n-3}(F), k)$ factors through $U_3 \oplus U_4'$ and

$$
\beta|_{W'} : W' \rightarrow U_3 \oplus U_4' \subseteq H_n(F^\times 2 \times \text{GL}_{n-2}(F), k),
$$

$$
a \otimes b \otimes z \mapsto ((-\sigma_{2*} + \sigma_{3*})(a \otimes b \otimes z), a \otimes b \otimes z).
$$

Thus we may assume that $u_4' = 0$. The restriction of $\beta$ on the summand

$$
W'' = \bigoplus_{i=1}^{n-2} \text{Tor}_i^Z(H_i(F_2^\times, \mathbb{Z}), H_{n-i-1}(F_3^\times, \mathbb{Z})), k
$$

of $H_n(F^\times 3 \times \text{GL}_{n-3}(F), k)$ factors through $U_7 \oplus U_8$ and

$$
\beta|_{W''} : W'' \rightarrow U_7 \oplus U_8 \subseteq H_n(F^\times 2 \times \text{GL}_{n-2}(F), k),
$$

$$
x \mapsto ((-\sigma_{2*} + \sigma_{3*})(x), x).
$$
Hence we may assume $u_8 = 0$. If $X, X'$ and $X''$ are the following summands

$$X = \bigoplus_{i+j \leq n-2} H_1(f_2^X, \mathbb{Z}) \otimes \text{Tor}_1^\mathbb{Z}(H_j(f_3^X, \mathbb{Z}), H_{n-i-j-1}(\text{GL}_{n-3}(F), \mathbb{Z}))_k,$$

$$X' = \bigoplus_{i+j \leq n-2} \text{Tor}_1^\mathbb{Z}(H_1(f_2^X, \mathbb{Z}), H_j(f_3^X, \mathbb{Z}) \otimes H_{n-i-j-1}(\text{GL}_{n-3}(F), \mathbb{Z}))_k,$$

$$X'' = \bigoplus_{i+j \leq n-2} \text{Tor}_1^\mathbb{Z}(H_1(f_2^X, \mathbb{Z}), \text{Tor}_1^\mathbb{Z}(H_j(f_3^X, \mathbb{Z}), H_{n-i-j-2}(\text{GL}_{n-3}(F), \mathbb{Z})))_k,$$

of $H_n(F^{\times 3} \times \text{GL}_{n-3}(F), k)$, then

$$\beta|_X : X \to U_3 \oplus U_7 \oplus U_9, \quad x \mapsto (\sigma_3(x), -\sigma_2(x), \text{inc}_s(x)),$$

$$\beta|_{X'} : X' \to U_7 \oplus U_{10}, \quad y \mapsto ((-\sigma_2 + \sigma_3)(y), \text{inc}_s(y),$$

$$\beta|_{X''} : X'' \to U_7 \oplus U_{11}, \quad z \mapsto ((-\sigma_2 + \sigma_3)(z), \text{inc}_s(z)).$$

Now by homological stability we may assume that $u_9 = u_{10} = u_{11} = 0$. Thus $u$ finds the following form

$$u = (u_3, u', u'' \in U_3 \oplus U'_4 \oplus U_7 \subseteq H_n(F^{\times 2} \times \text{GL}_{n-2}(F), k).$$

Let $U_3 = \bigoplus_{i=1}^n U_{3,i}$ and $u_3 = (u_{3,i})_{1 \leq i \leq n}$. If $T_3 = \bigoplus_{i=1}^n T_{3,i}$, then $\alpha$ factors as follow on the following summands:

$$\alpha|_{U_{3,1}} : U_{3,1} \to T_1 \oplus T'_2 \subseteq H_n(F^{\times} \times \text{GL}_{n-1}(F), k),$$

$$s \otimes z \mapsto (-s \cup z, s \otimes z),$$

$$\alpha|_{U_{3,i}} : U_{3,i} \to T_1 \oplus T_{3,i} \subseteq H_n(F^{\times} \times \text{GL}_{n-1}(F), k), \quad 2 \leq i \leq n,$$

$$r \otimes v \mapsto (r \cup v, r \otimes \text{inc}_s(v)),$$

$$\alpha|_{U_7} : U_7 \to T_1 \oplus T_4 \subseteq H_n(F^{\times} \times \text{GL}_{n-1}(F), k),$$

$$u_7 \mapsto (-\alpha_2(u_7), \alpha_1(u_7)).$$

Moreover, the restriction of $\alpha$ on $U'_4$ factors through $T_2 = T'_2 \oplus T''_2$. More precisely, we have

$$\alpha|_{U'_4} : U'_4 \to T_2 = T'_2 \oplus T''_2 \subseteq H_n(F^{\times} \times \text{GL}_{n-1}(F), k),$$

$$a \otimes b \otimes \{c_1, \ldots, c_{n-2}\} \mapsto t = (t', t''),$$

where

$$t = -\frac{(-1)^{n-3}}{(n-3)!} \left( b \otimes \text{diag}(a, I_{n-2}) + a \otimes \text{diag}(b, I_{n-2}) \right)$$

$$+ a \otimes \text{diag}(c_1, c_{n-2}),$$

$$= -\frac{(-1)^{n-2}}{(n-2)!} \left( b \otimes \text{diag}(a, I_{n-2}) \right)$$

$$+ a \otimes \text{diag}(c_1, c_{n-2}),$$

$$= -\frac{1}{(n-2)!} \left( b \otimes \text{diag}(a, I_{n-2}) \right)$$

$$+ a \otimes \text{diag}(c_1, c_{n-2}),$$

$$t' = -\frac{(-1)^{n-2}}{(n-2)!} \text{inc}_s(b \otimes \text{diag}(a I_{n-2}, C_{n-2, n-2})$$

$$+ a \otimes \text{diag}(b I_{n-2}, C_{n-2, n-2}),$$

$$t'' = -b \otimes \{c_1, \ldots, c_{n-2}\} - a \otimes \{b, c_1, \ldots, c_{n-2}\}.$$
These would imply that $v_{3,i} = 0$ for all $2 \leq i \leq n$ (here we use the homological stability $H_{n-1}(\text{GL}_{n-2}(F), \mathbb{k}) \simeq H_{n-1}(\text{GL}_{n-1}(F), \mathbb{k})$). Moreover, since the map $\alpha_1: U_7 \to H_4$ is induced by the homological stability $H_{n-1}(\text{GL}_{n-2}(F), \mathbb{Z}) \simeq H_{n-1}(\text{GL}_{n-1}(F), \mathbb{Z})$ for $1 \leq i \leq n - 2$, we have $u_7 = 0$. Thus we may assume that

$$u = (u_{3,1}, u_{4}^\prime, u_{4}^\prime, u_{4}^\prime, u_{4}^\prime) \in U_{3,1} \oplus U_4 \subseteq H_n(F^2 \times \text{GL}_{n-2}(F), \mathbb{k}).$$

Let $u_{3,1} = \sum s \otimes z$, $u_{4}^\prime = \sum a \otimes b \otimes \{c_1, \ldots, c_{n-2}\}$. Then

$$\alpha(u_{3,1}, u_{4}^\prime) = (t_1', (t_2', t_3'), t_3', t_4') = (t_1, 0, 0, 0),$$

where

$$t_1' = t_1 = -\sum s \cup z,$$

$$t_2' = 0 = \sum s \otimes z + \frac{(-1)^{n-2}}{(n-2)!} \sum (b \otimes c(a I_{n-2}, C_{1,n-2}, \ldots, C_{n-2,n-2}) + a \otimes c(b I_{n-2}, C_{1,n-2}, \ldots, C_{n-2,n-2})),

$$t_3' = 0 = -\sum (b \otimes \{a, c_1, \ldots, c_{n-2}\} + a \otimes \{b, c_1, \ldots, c_{n-2}\}).$$

Therefore

$$t_1 = \frac{(-1)^{n-2}}{(n-2)!} \sum (c(\text{diag}(I_{n-2}, b), \text{diag}(a I_{n-2}, 1), C_{1,n-1}, \ldots, C_{n-2,n-1}) + C(\text{diag}(I_{n-2}, a), \text{diag}(b I_{n-2}, 1), C_{1,n-1}, \ldots, C_{n-2,n-1})).$$

This shows that $\kappa_n$ is surjective. $\square$

Based on the above theorem we make the following conjectures.

**Conjecture 2.2.** (i) For any $n$, the natural map

$$\kappa_n: \mathcal{B}_n(F)[\frac{1}{(n-2)!}] \to \ker(H_n(\text{GL}_{n-1}(F), \mathbb{Z}[\frac{1}{(n-2)!}]) \to H_n(\text{GL}_n(F), \mathbb{Z}[\frac{1}{(n-2)!}]))$$

is surjective.

(ii) For any $n \geq 3$, the sequence

$$H_n(F^2 \times \text{GL}_{n-2}(F), \mathbb{Z}[\frac{1}{(n-2)!}]) \xrightarrow{\alpha_1, \alpha_2, \text{inc}} H_n(F^2 \times \text{GL}_{n-1}(F), \mathbb{Z}[\frac{1}{(n-2)!}]) \xrightarrow{\text{inc}} H_n(\text{GL}_n(F), \mathbb{Z}[\frac{1}{(n-2)!}]) \to 0$$

is exact.

**Corollary 2.3.** If Conjecture 2.2(ii) holds for all $n \geq 3$, then $\kappa_n$ is surjective for all $n \geq 1$.

**Proof.** The claim is trivial for $n = 1, 2$. So let $n \geq 3$. Let, by induction, the claim holds for any $m < n$. Then $\kappa_m$ is surjective and its image is $(m - 1)$-torsion. This implies that the map

$$H_m(\text{GL}_{m-1}(F), \mathbb{Z}[\frac{1}{(m-1)!}]) \to H_m(\text{GL}_m(F), \mathbb{Z}[\frac{1}{(m-1)!}])$$

is injective for any $m < n$. Now the claim follows from the previous theorem. $\square$
Remark 2.4. (i) Conjecture 2.2(ii) holds for \( n = 3, 4 \) [18, Corollary 3.5], [19, Theorem 2].
(ii) Conjecture 2.2(ii) follows from Conjecture 4.1 below, with \( \mathbb{Z}[\frac{1}{(n-2)!}] \)-coefficients. In fact by Conjecture 4.1, the groups \( E_{0,n}^2(n, \mathbb{Z}[\frac{1}{(n-2)!}]) \) and \( E_{1,n}^2(n, \mathbb{Z}[\frac{1}{(n-2)!}]) \) are trivial from which the exactness of the desired sequence follows.
(iii) Galatius, Kupers and Randal-Williams showed that Conjecture 2.2 holds if we replace \( \mathbb{Z}[\frac{1}{(n-2)!}] \) with a field \( \mathbb{k} \) such that \((n-1)! \in \mathbb{k}^\times \) (Theorem 1.2).

Corollary 2.5. For any \( n \leq 4 \), \( \kappa_n \) is surjective. In particular the natural maps \( H_3(\text{GL}_2(F), \mathbb{Z}[\frac{1}{n}]) \to H_3(\text{GL}_3(F), \mathbb{Z}[\frac{1}{n}]) \) and \( H_4(\text{GL}_3(F), \mathbb{Z}[\frac{1}{n}]) \to H_4(\text{GL}_4(F), \mathbb{Z}[\frac{1}{n}]) \) are injective.

Proof. Clearly \( \kappa_1 \) and \( \kappa_2 \) are surjective. Conjecture 2.2(ii) holds for \( n = 3 \) by [18, Corollary 3.5 (ii)] and for \( n = 4 \) by [19, Theorem 2]. Now the claim follows from Theorem 2.1.

Remark 2.6. (i) The case \( n = 3 \) of the above corollary already was known. In fact in [20, Remark 3.5], \( \kappa_3 \) is defined and is shown to be surjective.
(ii) It is an open problem whether the map \( H_3(\text{GL}_2(F), \mathbb{Z}) \to H_3(\text{GL}_3(F), \mathbb{Z}) \) is injective (see [21, Theorem 4.4]). Generalising this, one might ask if, in fact, the map
\[
H_n(\text{GL}_{n-1}(F), \mathbb{Z}[\frac{1}{(n-2)!}]) \to H_n(\text{GL}_n(F), \mathbb{Z}[\frac{1}{(n-2)!}])
\]
is injective? Up to Conjecture 2.2(i) this is equivalent to the triviality of \( \kappa_n \).
The answer to this question is positive when \( F^\times \) is divisible (Theorem 6.3) or when \( F \) is real closed (Theorem 6.8).

3. The cokernel of \( H_{n+1}(\text{GL}_{n-1}(F), \mathbb{Z}) \to H_{n+1}(\text{GL}_n(F), \mathbb{Z}) \)

For any positive integers \( n \) and \( r \), let
\[
H^n_{\text{GL}}(F, n) := H_n(\text{GL}_r(F), \mathbb{Z})/H_n(\text{GL}_{r-1}(F), \mathbb{Z}).
\]
By Theorem 1.1, \( H^n_{\text{GL}}(F, n) = 0 \) for \( r > n \) and \( H^n_{\text{GL}}(F, n) \simeq K_n^M(F) \). By Theorem 1.2,
\[
H^n_{\text{GL}}(F, n+1) \simeq H_{n+1}(\text{GL}_n(F), \mathbb{Z}),
\]
where \( \mathbb{k} \) is a field such that \((n-1)! \in \mathbb{k}^\times \).

From the inclusions of groups \( \text{GL}_{n-1}(F) \hookrightarrow F^\times \times \text{GL}_{n-1}(F) \hookrightarrow \text{GL}_n(F) \) we obtain the exact sequence
\[
\frac{H_{n+1}(F^\times \times \text{GL}_{n-1}(F), \mathbb{Z})}{H_{n+1}(\text{GL}_{n-1}(F), \mathbb{Z})} \to H^n_{\text{GL}}(F, n+1) \to \frac{H_{n+1}(\text{GL}_n(F), \mathbb{Z})}{H_{n+1}(F^\times \times \text{GL}_{n-1}(F), \mathbb{Z})} \to 0.
\]
Let $H_{n+1}(F^\times \times \text{GL}_{n-1}(F),\mathbb{Z}) = \bigoplus_{i=1}^6 S_i$, where

- $S_1 = H_{n+1}(\text{GL}_{n-1}(F),\mathbb{Z})$,
- $S_2 = F^\times \otimes H_n(\text{GL}_{n-1}(F),\mathbb{Z})$,
- $S_3 = H_2(F^\times,\mathbb{Z}) \otimes H_{n-1}(\text{GL}_{n-1}(F),\mathbb{Z})$,
- $S_4 = \bigoplus_{i\geq 3}^n H_i(F^\times,\mathbb{Z}) \otimes H_{n-i+1}(\text{GL}_{n-1}(F),\mathbb{Z})$,
- $S_5 = \text{Tor}_1^\mathbb{Z}(F^\times, H_{n-1}(\text{GL}_{n-1}(F),\mathbb{Z}))$,
- $S_6 = \bigoplus_{i=2}^n \text{Tor}_1^\mathbb{Z}(H_i(F^\times,\mathbb{Z}), H_{n-i}(\text{GL}_{n-1}(F),\mathbb{Z}))$.

Clearly $\Phi(S_1) = 0$. By homological stability

$$H_{n-i+1}(\text{GL}_{n-2}(F),\mathbb{Z}) \simeq H_{n-i+1}(\text{GL}_{n-1}(F),\mathbb{Z})$$

for $3 \leq i \leq n+1$ and thus $\Phi(S_4) = 0$. Furthermore, for any $2 \leq i \leq n$, the homological stability $H_{n-i}(\text{GL}_{n-2}(F),\mathbb{Z}) \simeq H_{n-i}(\text{GL}_{n-1}(F),\mathbb{Z})$ induces the isomorphism

$$\text{Tor}_1^\mathbb{Z}(H_i(F^\times,\mathbb{Z}), H_{n-i}(\text{GL}_{n-1}(F),\mathbb{Z})) \simeq \text{Tor}_1^\mathbb{Z}(H_i(F^\times,\mathbb{Z}), H_{n-i}(\text{GL}_{n-1}(F),\mathbb{Z}))$$

Thus we may assume that $\Phi(S_6) = 0$. Moreover,

$$\Phi(\text{im}(F^\times \otimes H_n(\text{GL}_{n-2}(F),\mathbb{Z}) \xrightarrow{\text{id} \otimes \text{id}_n} S_2)) = 0,$$

$$\Phi(\text{im}(H_2(F^\times,\mathbb{Z}) \otimes H_{n-1}(\text{GL}_{n-2}(F),\mathbb{Z}) \xrightarrow{\text{id} \otimes \text{id}_n} S_3)) = 0.$$

Thus the above exact sequence finds the following form

$$F^\times \otimes H^n_{\text{GL}}(F,n) + \bigoplus_{i=2}^n \mathbb{F} \otimes K^M_{i-1}(F) \oplus \text{Tor}_1^\mathbb{Z}(F^\times, H_{n-1}(\text{GL}_{n-1}(F),\mathbb{Z}))$$

$$\xrightarrow{\Phi} \text{im}(F^\times \otimes H_n(\text{GL}_{n-2}(F),\mathbb{Z}) \xrightarrow{\text{id} \otimes \text{id}_n} S_2) \text{Tor}_1^\mathbb{Z}(F^\times, H_{n-1}(\text{GL}_{n-1}(F),\mathbb{Z}) \xrightarrow{\text{id} \otimes \text{id}_n} S_3) = 0.$$
Moreover, we have the decomposition
\[ H^1_{GL}(F, 2) = H_2(GL_1(F), \mathbb{Z})/H_2(GL_0(F), \mathbb{Z}) \cong \bigwedge^2 F^\times. \]
But \( H^3_{GL}(F, 3) \) has much richer structure.

**Proposition 3.1.** For any infinite field \( F \), we have the exact sequence
\[ H_3(SL_2(F), \mathbb{Z})_{F^\times} \to H^2_{GL}(F, 3) \to F^\times \otimes K^M_2(F) \to K^M_3(F)/2 \to 0. \]
Moreover, we have the decomposition
\[ H^2_{GL}(F, 3)\left[\frac{1}{2}\right] \cong K^{\text{ind}}_3(F)\left[\frac{1}{2}\right] \oplus F^\times \otimes K^M_2(F)\left[\frac{1}{2}\right]. \]

**Proof.** By studying the relative Lyndon/Hochschild-Serre spectral sequence [7, Theorem 1.4] associated to the extension
\[ 1 \to SL_2(F) \to GL_2(F) \xrightarrow{\det} F^\times \to 1, \]
relative to the subgroup \( GL_1(F) \subseteq GL_2(F) \), we get the exact sequence
\[ H_3(SL_2(F), \mathbb{Z})_{F^\times} \to H^2_{GL}(F, 3) \to H_1(F^\times, H_2(SL_2(F), \mathbb{Z})) \to 0. \]
The inclusion \( SL_2 \to SL_3 \) induces the short exact sequence
\[ 0 \to H_1(F^\times, H_2(SL_2(F), \mathbb{Z})) \to H_1(F^\times, H_2(SL_3(F), \mathbb{Z})) \to K^M_3(F)/2 \to 0 \]
[12, Theorem 3.2]. Since \( H_2(SL_3(F), \mathbb{Z}) \cong K_2(F) \) [27, 2.1], \( F^\times \) acts trivially on \( H_2(SL_3(F), \mathbb{Z}) \). Thus
\[ H_1(F^\times, H_2(SL_3(F), \mathbb{Z})) \cong F^\times \otimes K^M_2(F). \]
Moreover, the map \( F^\times \otimes K^M_2(F) \cong H_1(F^\times, H_2(SL_3(F), \mathbb{Z})) \to K^M_3(F)/2 \) is induced by the natural map \( F^\times \otimes K^M_2(F) \to K^M_3(F) \) (see the proof of [12, Theorem 3.2]). Thus we have the exact sequence
\[ H_3(SL_2(F), \mathbb{Z})_{F^\times} \to H^2_{GL}(F, 3) \to F^\times \otimes K^M_2(F) \to K^M_3(F)/2 \to 0. \]
This proves the first claim.

By [18, Theorem 6.1], the map \( H_3(SL_2(F), \mathbb{Z}\left[\frac{1}{2}\right])_{F^\times} \to H^2_{GL}(F, 3)\left[\frac{1}{2}\right] \) is injective. Moreover, \( H_3(SL_2(F), \mathbb{Z}\left[\frac{1}{2}\right])_{F^\times} \cong K^{\text{ind}}_3(F)\left[\frac{1}{2}\right] \) by [18, Proposition 6.4] or [20, Theorem 3.7]. Thus we have the exact sequence
\[ 0 \to K^{\text{ind}}_3(F)\left[\frac{1}{2}\right] \to H^2_{GL}(F, 3)\left[\frac{1}{2}\right] \to F^\times \otimes K^M_2(F)\left[\frac{1}{2}\right] \to 0. \]

To finish the proof, we should show that the above short exact sequence splits. Let \( \phi \) be the composite
\[ F^\times \otimes K^M_2(F) \xrightarrow{\text{id}_{F^\times} \otimes \tau_2} F^\times \otimes H_2(GL_2(F), \mathbb{Z}) \xrightarrow{\cup} H_3(F^\times \times GL_2(F), \mathbb{Z}) \xrightarrow{\tau} H_3(GL_2(F), \mathbb{Z}) \to H^2_{GL}(F, 3), \]
where \( \tau: F^\times \times GL_2(F) \to GL_2(F) \) is given by \( (a, A) \mapsto aA \). By [21, Lemma 3.2 (ii)], the composite
\[ F^\times \otimes K^M_2(F) \xrightarrow{\phi} H^2_{GL}(F, 3) \to F^\times \otimes K^M_2(F) \]
coincides with multiplication by 2. Now it is easy to construct a splitting map.

In the next section we will describe $H^2_{\text{GL}}(F, 3)$ in a different way (Proposition 4.3).

Remark 3.2. For any field $F$ there is a natural map
\[ H_3(\text{SL}_2(F), \mathbb{Z})_{F^\times} \to K_3^{\text{ind}}(F) \]
(see [21, §1] for its construction). It is an open question, asked by Suslin, whether this map is an isomorphism [27, Question 4.4]. Hutchinson and Tao have proved that it is surjective [12, Lemma 5.1]. For more on this question see [21, Theorem 4.4].

4. The cokernel of $H_{n+1}(F^\times \times \text{GL}_{n-1}(F), \mathbb{Z}) \to H_{n+1}(\text{GL}_n(F), \mathbb{Z})$

To study the quotient group $H_{n+1}(\text{GL}_n(F), \mathbb{Z})/H_{n+1}(F^\times \times \text{GL}_{n-1}(F), \mathbb{Z})$, we look at a certain spectral sequence introduced and studied in [6, §2.2], [18, Section 3] and [19, Section 5].

Let $\mathbb{k}$ be a commutative ring. For any $l \geq 0$, let $D_l(F^n)$ be the free $\mathbb{k}$-module with a basis consisting of $(l + 1)$-tuples $\langle (w_0), \ldots, (w_l) \rangle$, where $0 \neq w_i \in F^n$, $\langle w_i \rangle = Fw_i$ and $\langle w_i \rangle \neq \langle w_j \rangle$ when $i \neq j$. Set $D_{-1}(F^n) := \mathbb{k}$. The group $\text{GL}_n(F)$ acts naturally on $D_l(F^n)$ on the left as follow:
\[ g.(\langle w_0), \ldots, (w_l) \rangle := (\langle gw_0), \ldots, (gw_l) \rangle. \]

We consider $D_{-1}(F^n) = \mathbb{k}$ as a trivial $\text{GL}_n(F)$-module. If it is necessary we convert these left actions to right actions by the definition $m.g := g^{-1}.m$.

Let define $\partial_0 : D_0(F^n) \to D_{-1}(F^n) = \mathbb{k}$ by $\sum_i n_i(\langle w_i \rangle) \mapsto \sum_i n_i$. For $l \geq 1$, we define the $l$-th differential operator $\partial_l : D_l(F^n) \to D_{l-1}(F^n)$, as an alternating sum of face operators which throw away the $i$-th component of generators.

For any integer $l \geq 0$, set $M_l = D_{l-1}(F^n)$. It is easy to see that the complex of $\text{GL}_n(F)$-modules
\[ M_* : 0 \leftarrow M_0 \leftarrow M_1 \leftarrow \cdots \leftarrow M_l \leftarrow \cdots \]

is exact (see the proof of [30, Lemma 2.2]).

Take a projective resolution $P_* \to \mathbb{Z}$ of $\mathbb{Z}$ over $\text{GL}_n(F)$. From the double complex $M_* \otimes_{\mathbb{Z}}[\text{GL}_n(F)] P_*$ we obtain the first quadrant spectral sequence converging to zero with $E^1_{\bullet, \bullet}$-terms
\[ E^1_{p, q}(n, \mathbb{k}) = \begin{cases} H_q(F^\times p \times \text{GL}_{n-p}(F), \mathbb{k}) & \text{if } 0 \leq p \leq 2 \\ H_q(\text{GL}_n(F), M_p) & \text{if } p \geq 3, \end{cases} \]
(see [18, Section 3], [19, Section 5]). It is easy to see that $d^1_{1, q}(n, \mathbb{k}) = \text{inc}_*$. In particular
\[ E^0_{0, q}(n, \mathbb{k}) = \frac{H_q(\text{GL}_n(F), \mathbb{k})}{H_q(F^\times \times \text{GL}_{n-1}(F), \mathbb{k})}. \]
Moreover, $d_{2,q}^1(n, k) = \alpha_1\ast - \alpha_2\ast$ which is discussed in the exact sequence (1.1). Now we would like to describe $E_{3,q}^1(n, k)$. The orbits of the action of $\text{GL}_n(F)$ on $M_3 = D_2(F^n)$ are represented by

$$w_1 = (\langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle) \text{ and } w_2 = (\langle e_1 \rangle, \langle e_2 \rangle, \langle e_1 + e_2 \rangle).$$

Thus

$$E_{3,q}^1(n, k) \simeq H_q(\text{Stab}_{\text{GL}_n(F)}(w_1), k) \oplus H_q(\text{Stab}_{\text{GL}_n(F)}(w_2), k)$$

$$\simeq H_q(F^{x, 3} \times \text{GL}_{n-3}(F), k) \oplus H_q(F^{x}I_2 \times \text{GL}_{n-2}(F), k),$$

where $F^{x}I_2 := \{aI_2 : a \in F^{x}\}$ (see [24, Theorem 1.11]). Moreover,

$$d_{3,q}^1(n, k)|_{H_q(F^{x, 3} \times \text{GL}_{n-3}(F), k)} = \sigma_1\ast - \sigma_2\ast + \sigma_3\ast,$$

$$d_{3,q}^1(n, k)|_{H_q(F^{x}I_2 \times \text{GL}_{n-2}(F), k)} = \text{inc}_\ast.$$

Note that $\text{diag}(a, b, c, A) \overset{\partial_3}{\rightarrow} \text{diag}(b, c, a, A)$, $\text{diag}(a, b, c, A) \overset{\partial_3}{\rightarrow} \text{diag}(a, c, b, A)$ and $\text{diag}(a, b, c, A) \overset{\sigma_3=\text{inc}}{\rightarrow} \text{diag}(a, b, c, A)$.

When $k$ is a field, with a similar method as in [16, Lemma 4.2], one can show that $E_{p,q}^2(n, k) = 0$ for $p = 0, 1$ and $q \leq n - 1$. Moreover, by Theorem 1.2, this is also true for $q = n$ if $(n - 1)! \in k^\times$.

**Conjecture 4.1.** Let $n \geq 3$ and let $k$ be either a field or a subring of $\mathbb{Q}$. Then for any $3 \leq i \leq n + 2$, $E_{i,n-i+2}^2(n, k) = 0$.

**Remark 4.2.**

(i) Since the spectral sequence converges to zero, it follows from Conjecture 4.1 that $E_{1,n}^1(n, k) = 0$. For $k = \mathbb{Z}\left[\frac{1}{n-2}\right]$ this is equivalent to Conjecture 2.2(ii). Thus Conjecture 4.1 implies Conjecture 2.2.

(ii) Moreover, it follows from Conjecture 4.1 that the differential

$$d_{2,n}^2(n, k) : E_{2,n}^2(n, k) \rightarrow E_{0,n+1}^2(n, k)$$

is surjective. For $k = \mathbb{Z}\left[\frac{1}{n-2}\right]$, this differential is used in the construction of the map $\chi_n$ of Theorem B. In fact, as will be clear from the definition of $\chi_n$, Conjecture 4.1 implies Conjecture 5.3 below.

(iii) Conjecture 4.1 is known for $n = 3, 4$ [18, §3], [19, §6].

Now we study the group $H_{n+1}(\text{GL}_n(F), \mathbb{Z})/H_{n+1}(F^x \times \text{GL}_{n-1}(F), \mathbb{Z})$ for $n = 1$ and $n = 2$. Clearly $H_2(\text{GL}_1(F), \mathbb{Z})/H_2(F^x \times \text{GL}_{0}(F), \mathbb{Z})$ is trivial. The group $H_3(\text{GL}_2(F), \mathbb{Z})/H_3(F^x \times \text{GL}_1(F), \mathbb{Z})$ is more interesting and is connected to the Bloch group of $F$.

**Proposition 4.3.** For any infinite field $F$,

$$H_3(\text{GL}_2(F), \mathbb{Z})/H_3(F^x \times \text{GL}_1(F), \mathbb{Z}) \simeq \mathcal{B}_2(F).$$

In particular, we have the exact sequence

$$\text{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F)) \oplus F^x \otimes H_1^\text{GL}(F, 2) \rightarrow H_2^\text{GL}(F, 3) \rightarrow \mathcal{B}_2(F) \rightarrow 0$$

where $\mu(F)$ is the group of roots of unity in $F$.
Proof. By studying the spectral sequence $E^1_{p,q}(2,\mathbb{Z})$, Suslin showed that there is a natural map $H_3(\text{GL}_2(F),\mathbb{Z}) \to B(F)$ such that the sequence

$$H_3(\text{GM}_2(F),\mathbb{Z}) \to H_3(\text{GL}_2(F),\mathbb{Z}) \to B(F) \to 0$$

is exact, where $\text{GM}_2(F)$ is the subgroup of monomial matrices in $\text{GL}_2(F)$ [31, Theorem 2.1]. Note that $\text{GM}_2(F) \simeq (F^\times \times F^\times) \rtimes \Sigma_2 = T_2(F) \rtimes \Sigma_2$, where $\Sigma_2 = \left\{ 1 := I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$.

The Lyndon/Hochschild-Serre spectral sequence associated to the extension $1 \to T_2(F) \to \text{GM}_2(F) \to \Sigma_2 \to 1$, i.e.

$$E^2_{p,q} = H_p(\Sigma_2, H_q(T_2(F),\mathbb{Z})), \to H_{p+q}(\text{GM}_2(F),\mathbb{Z}),$$

gives us a filtration of $H_3(\text{GM}_2(F),\mathbb{Z})$, $0 = F_{−1}H_3(\text{GM}_2(F),\mathbb{Z}) \subseteq \cdots \subseteq F_3H_3(\text{GM}_2(F),\mathbb{Z}) = H_3(\text{GM}_2(F),\mathbb{Z})$, such that

$$E^\infty_{0,3} \simeq F_0H_3(\text{GM}_2(F),\mathbb{Z}) = H_3(T_2(F),\mathbb{Z})\Sigma_2,$$

$$E^\infty_{1,2} \simeq F_1H_3(\text{GM}_2(F),\mathbb{Z})/F_0H_3(\text{GM}_2(F),\mathbb{Z}) \simeq E_{1,2}^2,$$

$$E^\infty_{2,1} \simeq F_2H_3(\text{GM}_2(F),\mathbb{Z})/F_1H_3(\text{GM}_2(F),\mathbb{Z}) = 0,$$

$$E^\infty_{3,0} \simeq H_3(\text{GM}_2(F),\mathbb{Z})/F_2H_3(\text{GM}_2(F),\mathbb{Z}) \simeq H_3(\Sigma_2,\mathbb{Z}),$$

(see [23, Section 4] for some details). Thus

$$E^2_{1,2} \simeq F_2H_3(\text{GM}_2(F),\mathbb{Z})/H_3(T_2(F),\mathbb{Z}).$$

Moreover, since the map $\text{GM}_2(F) \to \Sigma_2$ splits, we obtain the decomposition

$$H_3(\text{GM}_2(F),\mathbb{Z}) \simeq F_2H_3(\text{GM}_2(F),\mathbb{Z}) \oplus H_3(\Sigma_2,\mathbb{Z}).$$

The matrix $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is conjugate in $\text{GL}_2(F)$ to $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \in U_2(F)$, where

$$U_2(F) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in F^\times, b \in F \right\}.$$ 

Hence $\text{im}(H_3(\Sigma_2,\mathbb{Z})) \subseteq \text{im}(H_3(U_2(F),\mathbb{Z})) = \text{im}(H_3(T_2(F),\mathbb{Z}))$. Thus from the exact sequence (4.1) we obtain the exact sequence

$$E^2_{1,2} \to H_3(\text{GL}_2(F),\mathbb{Z})/H_3(F^\times \times \text{GL}_1(F),\mathbb{Z}) \to B(F) \to 0.$$ 

Observe that

$$E^2_{1,2} \simeq H_1(\Sigma_2, F^\times \otimes F^\times) \simeq H_1(\Sigma_2, \mu(F) \otimes \mu(F))$$

$$\simeq H_1(\Sigma_2, \mu_2(\mu(F)) \otimes \mu_2(\mu(F))) \simeq \frac{\langle \mu_2(\mu(F)) \otimes \mu_2(\mu(F)) \rangle}{\langle 1 + \sigma \rangle \langle \mu_2(\mu(F)) \otimes \mu_2(\mu(F)) \rangle}$$

$$= \langle \mu_2(\mu(F)) \otimes \mu_2(\mu(F)) \rangle = \langle \mu_2(\mu(F)) \otimes \mu_2(\mu(F)) \rangle$$

$$\simeq \begin{cases} 0 & \text{if } \mu_2(\mu(F)) \text{ is infinite or } \text{char}(F) = 2 \\ \mathbb{Z}/2 & \text{if } \mu_2(\mu(F)) \text{ is finite and } \text{char}(F) \neq 2 \end{cases},$$
(see [23, Lemma 4.1 and page 5088]). Recall that $\mu_{2^n}(F)$ is the group of 2-power roots of unity in $F$, i.e. $\mu_{2^n}(F) = \{a \in F : a^{2^n} = 1 \text{ for some } n \geq 0\}$ and for an abelian group $A$, $2A := \{a \in A : 2a = 0\}$.

Suslin also showed that there is a natural map $H_3(\text{GL}_3(F), \mathbb{Z}) \to B(F)$ such that the sequence

$$
H_3(\text{GL}_3(F), \mathbb{Z}) \oplus H_3(T_3(F), \mathbb{Z}) \to H_3(\text{GL}_3(F), \mathbb{Z}) \to B(F) \to 0
$$

is exact [31, Proposition 3.1] and the map $H_3(\text{GL}_2(F), \mathbb{Z}) \to B(F)$ in (4.1) factors through $H_3(\text{GL}_3(F), \mathbb{Z})$. Now from this one can obtain the exact sequence

$$
0 \to T_F \to H_3(\text{GL}_3(F), \mathbb{Z})/L \to B(F) \to 0,
$$

where $L = F^\times \otimes^3 F^\times \otimes H_2(F^\times, \mathbb{Z}) \oplus H_3(F^\times, \mathbb{Z})$ and $T_F$ sits in the following exact sequence

$$
0 \to \text{Tor}_1^\mathbb{Z}(\mu(F), \mu(F)) \to T_F \to H_1(\Sigma_2, \mu_{2^n}(F) \otimes \mu_{2^n}(F)) \to 0.
$$

Moreover,

$$
H_3(\text{GL}_3(F), \mathbb{Z})/L \simeq K_3^{\text{ind}}(F)
$$

(see the proof [23, Theorem 5.1]). From these results we obtain the exact sequence

$$
0 \to H_1(\Sigma_2, \mu_{2^n}(F) \otimes \mu_{2^n}(F)) \to K_3^{\text{ind}}(F)/\text{Tor}_1^\mathbb{Z}(\mu(F), \mu(F)) \to B(F) \to 0.
$$

Observe that $E^2_{1,2} \simeq H_1(\Sigma_2, \mu_{2^n}(F) \otimes \mu_{2^n}(F))$. Now from the commutative diagram with exact rows

$$
\begin{array}{ccc}
E^2_{1,2} & \to & H_3(\text{GL}_2(F), \mathbb{Z})/H_3(F^\times \times \text{GL}_1(F), \mathbb{Z}) \to B(F) \to 0 \\
\downarrow & & \downarrow \\
0 & \to & K_3^{\text{ind}}(F)/\text{Tor}_1^\mathbb{Z}(\mu(F), \mu(F)) \to B(F) \to 0
\end{array}
$$

it follows that

$$
\frac{H_3(\text{GL}_2(F), \mathbb{Z})}{H_3(F^\times \times \text{GL}_1(F), \mathbb{Z})} \simeq K_3^{\text{ind}}(F)/\text{Tor}_1^\mathbb{Z}(\mu(F), \mu(F)) \simeq B_2(F).
$$

This proves the first claim.

To prove the second claim, note that the image of the map

$$
F^\times \otimes H_2(\text{GL}_1(F), \mathbb{Z}) \to H^2_{\text{GL}}(F, 3)
$$

is generated by the elements $c(\text{diag}(a, 1), \text{diag}(1, b), \text{diag}(1, c))$, $a, b, c \in F^\times$, and the image of

$$
\bigwedge^2 \mathbb{Z} F^\times \otimes H_1(\text{GL}_1(F), \mathbb{Z}) \to H^2_{\text{GL}}(F, 3)
$$

is generated by the elements $c(\text{diag}(d, 1), \text{diag}(e, 1), \text{diag}(1, f))$, $d, e, f \in F^\times$.

The conjugation by $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $\text{GL}_2(F)$, i.e. the map $i_\sigma : \text{GL}_2(F) \to$
GL$_2(F)$, $A \mapsto \sigma A \sigma^{-1}$, induces identity on $H_3(GL_2(F), \mathbb{Z})$. This implies that
\[
c(\text{diag}(a, 1), \text{diag}(1, b), \text{diag}(1, c)) = c(\text{diag}(1, a), \text{diag}(b, 1), \text{diag}(c, 1))
= c(\text{diag}(b, 1), \text{diag}(c, 1), \text{diag}(1, a)).
\]

Thus the images of $F^\times \otimes H_2(GL_1(F), \mathbb{Z})$ and $\bigwedge^2 F^\times \otimes H_1(GL_1(F), \mathbb{Z})$ co-
incide in $H^3_{GL}(F, 3)$. □

**Remark 4.4.** (i) Proposition 4.3 holds for any field with more than nine
elements or more generally for any local domain where its residue field has
more than nine elements. The proof is the same, but for the exact sequences
(4.1) and (4.3) see [22, Section 5].

(ii) Putting the exact sequences of Propositions 3.1 and 4.3 in one diagram
we get the commutative diagram

\[
\begin{array}{cccccc}
\text{Tor}^2_1(\mu(F), \mu(F)) & \longrightarrow & H_3(SL_2(F), \mathbb{Z})_{F^\times} \\
\downarrow^{(\text{id}, 0)} & & \downarrow^\theta \\
\text{Tor}^2_1(\mu(F), \mu(F)) \oplus F^\times \otimes H^1_{GL}(F, 2) & \longrightarrow & H^2_{GL}(F, 3) & \longrightarrow & B_2(F) \to 0, \\
\downarrow & & \downarrow & & \\
F^\times \otimes K^M_2(F) & \longrightarrow & K^M_3(F)/2 & \to 0 \\
\end{array}
\]

where $\theta|_{F^\times \otimes H^1_{GL}(F, 2)}(c \otimes (a \wedge b)) = -a \otimes \{b, c\} + b \otimes \{a, c\}$ [21, Lemma 3.2],
$\theta|_{\text{Tor}^2_1(\mu(F), \mu(F))} = 0$ and $\text{Tor}^2_1(\mu(F), \mu(F)) \to H_3(SL_2(F), \mathbb{Z})_{F^\times}$ is induced
by the map $\mu(F) \to SL_2(F)$ given by $a \mapsto \text{diag}(a, a^{-1})$. Furthermore, the
surjective map $H_3(SL_2(F), \mathbb{Z})_{F^\times} \to B(F)$ factors through $K^\text{ind}_3(F)$.

5. The map $\chi_n$

The following is Theorem B of the introduction.

**Theorem 5.1.** Let the natural map
\[
H_m(GL_{m-1}(F), \mathbb{Z}\left[\frac{1}{(m-1)!}\right]) \to H_m(GL_m(F), \mathbb{Z}\left[\frac{1}{(m-1)!}\right])
\]
be injective for $m = n - 1, n - 2$. Then there is a natural map
\[
\chi_n : B_n(F)\left[\frac{1}{(n-2)!}\right] \to \frac{H_{n+1}(GL_n(F), \mathbb{Z}\left[\frac{1}{(n-2)!}\right])}{H_{n+1}(F^\times \times GL_{n-1}(F), \mathbb{Z}\left[\frac{1}{(n-2)!}\right])}.
\]

If Conjecture 4.1 holds with $\mathbb{Z}\left[\frac{1}{(n-2)!}\right]$ coefficients, then $\chi_n$ is surjective.
Proof. It is enough to construct $\chi_n$. The second part follows from the construction of $\chi_n$ below and an easy analysis of the spectral sequence $E_{p,q}^1(n, \mathbb{Z}[\frac{1}{(n-2)!}]).$

The map $\chi_1$ is the trivial map and $\chi_2$ is the isomorphism

$$B_2(F) \simeq H_3(\text{GL}_2(F), \mathbb{Z})/H_3(F^\times \times \text{GL}_1(F), \mathbb{Z})$$

of Proposition 4.3. So let $n \geq 3$.

For simplicity, set $k = \mathbb{Z}[\frac{1}{(n-2)!}]$. We construct a surjective map

$$\chi_n' : B_n(F)_k \rightarrow E^2_{2,n}(n, k).$$

Then $\chi_n$ is defined as the composite of $\chi_n'$ with the differential $d^2_{2,n}(n, k)$:

$$\chi_n := d^2_{2,n}(n, k) \circ \chi_n'.$$

The group $E^2_{2,n}(n, k)$ is the homology of the complex

$$H_n(F^x I_2 \times \text{GL}_{n-2}(F), \mathbb{Z}) \oplus H_n(F^x \times \text{GL}_{n-3}(F), \mathbb{Z}) \xrightarrow{d^1_{3,n}(n, k)} H_n(F^{x^2} \times \text{GL}_{n-2}(F), \mathbb{Z}) \xrightarrow{d^2_{2,n}(n, k)} H_n(F^x \times \text{GL}_{n-1}(F), \mathbb{Z}).$$

(Observe that this complex already appeared in the proof of Theorem 2.1(ii). In fact $d^1_{3,n}(n, k) = b$ and $d^2_{2,n}(n, k) = a$.) Consider the decompositions

$$H_n(F^x \times \text{GL}_{n-1}(F), \mathbb{Z}) = \bigoplus_{i=1}^4 T_i,$$

$$H_n(F^{x^2} \times \text{GL}_{n-2}(F), \mathbb{Z}) = \bigoplus_{i=1}^{11} U_i,$$

as in the proof of Theorem 2.1(ii) with $T_2 = T'_2 \oplus T''_2$, $U_4 = U'_4 \oplus U''_4$ and $U_3 = \bigoplus_{i=1}^{n-1} U_{3,i}$.

Let $u = (u_1, \ldots, u_{11})$ be in the kernel of $d^1_{3,n}(n, k) = \alpha_{1,*} - \alpha_{2,*}$, where $u_4 = (u'_4, u''_4)$ and $u_3 = (u_{3,i})_{1 \leq i \leq n}$. As in the proof of Theorem 2.1(ii), we can show that we may assume that $u_1, u_2, (u_{3,i})_{2 \leq i \leq n}, u'_4, u_5, u_6, u_7, u_8, u_9, u_{10}$ and $u_{11}$ are trivial. Hence

$$u = (u_{3,1}, u'_4) \in U_{3,1} \oplus U''_4 \subseteq H_n(F^{x^2} \times \text{GL}_{n-2}(F), \mathbb{Z}).$$

If $u_{3,1} = \sum s \otimes z$ and $u'_4 = \sum a \otimes b \otimes \{c_1, \ldots, c_{n-2}\}$, then as in the proof of Theorem 2.1(ii),

$$d^1_{2,n}(n, k)(u) = (t'_1, (t'_2, t''_2, t'_3, t'_4)) = 0 \in H_n(F^x \times \text{GL}_{n-1}(F), \mathbb{Z}) = \bigoplus_{i=1}^4 T_i,$$

where

$$t'_1 = - \sum s \cup z = 0,$$

$$t''_2 = 0 = \sum s \otimes z + \frac{(-1)^{n-2}}{(n-2)!} \sum \{b \times c(aI_{n-2}, C_{1,n-2}, \ldots, C_{n-2,n-2}) + a \otimes c(bI_{n-2}, C_{1,n-2}, \ldots, C_{n-2,n-2})\},$$

$$t''_2 = 0 = - \sum \{b \otimes \{a, c_1, \ldots, c_{n-2}\} + a \otimes \{b, c_1, \ldots, c_{n-2}\}\},$$

$$t'_3 = 0 = t'_4.$$

These calculations show that the map

$$\chi_n' : \ker(\delta_2^{(n)})_k \rightarrow E^2_{3,n}(n, k),$$
given by
\[ \sum a \otimes b \otimes \{c_1, \ldots, c_{n-2}\} \mapsto (u_{3,1}, \sum a \otimes b \otimes \{c_1, \ldots, c_{n-2}\}) \]
is surjective, where
\[ u_{3,1} = -\frac{(-1)^{n-2}}{(n-2)!} \sum (b \otimes c(aI_{n-2}, C_1, C_{n-2}, \ldots, C_{n-2, n-2})
+ a \otimes c(bI_{n-2}, C_1, C_{n-2, n-2})). \]

To finish the proof, we should check that this map factors through $\mathcal{B}_n(F)_k$ and for this we should show that
\[ \chi_n'(b \otimes c \otimes \{a, d_1, \ldots, d_{n-3}\} + a \otimes c \otimes \{b, d_1, \ldots, d_{n-3}\} + a \otimes b \otimes \{c, d_1, \ldots, d_{n-3}\}) \]
is trivial. Consider the following summands of $H_n(F^x \times \text{GL}_{n-3}(F), k)$:
\[ W''' = F_1^x \otimes F_2^x \otimes F_3^x \otimes H_{n-3}(\text{GL}_{n-3}(F), k) \]
\[ = W''_1 \otimes W''_2, \]
\[ W''_1 = F_1^x \otimes F_2^x \otimes F_3^x \otimes \text{im}(H_{n-3}(\text{GL}_{n-4}(F), k)), \]
\[ W''_2 = F_1^x \otimes F_2^x \otimes F_3^x \otimes K_{n-3}(F)_k. \]

Then $d_{3,n}(n, k)|_{W''_2}$ factors through $U_4$, and
\[ d_{3,n}(n, k)|_{W''_2} : W''_2 \to U_4 = U_3 \oplus U_4' \subseteq H_n(F^x \times \text{GL}_{n-2}(F), k), \]
\[ a \otimes b \otimes c \otimes \{d_1, \ldots, d_{n-3}\} \mapsto (u', u''), \]
where
\[ u' = \frac{(-1)^{n-3}}{(n-3)!} (b \otimes c \otimes c(aI_{n-3}, D_1, D_{n-3}, \ldots, D_{n-3, n-3})
+ a \otimes c \otimes c(bI_{n-3}, D_1, D_{n-3}, \ldots, D_{n-3, n-3})
+ a \otimes b \otimes c(cI_{n-3}, D_1, D_{n-3}, \ldots, D_{n-3, n-3})), \]
\[ u'' = b \otimes c \otimes \{a, d_1, \ldots, d_{n-3}\} + a \otimes c \otimes \{b, d_1, \ldots, d_{n-3}\}
+ a \otimes b \otimes \{c, d_1, \ldots, d_{n-3}\}. \]

On the other hand, for $W' = F_2^x \otimes F_3^x \otimes H_{n-2}(\text{GL}_{n-3}(F), k)$ the summand of $H_n(F^x \times \text{GL}_{n-3}(F), k)$, we have
\[ d_{3,n}(n, k)|_{W' \otimes W''_2} : W' \otimes W''_2 \to U_{3,1} \oplus U_4' \oplus U_4'' \subseteq H_n(F^x \times \text{GL}_{n-2}(F), k), \]
\[ (-u', a \otimes b \otimes c \otimes \{d_1, \ldots, d_{n-3}\}) \mapsto (\frac{(-1)^{n-3}}{(n-3)!} z, 0, u''), \]
where
\[ z = c \otimes c(\text{diag}(I_{n-3}, b), \text{diag}(aI_{n-3}, 1), D_1, D_{n-2}, \ldots, D_{n-3, n-2})
+ b \otimes c(\text{diag}(I_{n-3}, c), \text{diag}(aI_{n-3}, 1), D_1, D_{n-2}, \ldots, D_{n-3, n-2})
+ c \otimes c(\text{diag}(I_{n-3}, a), \text{diag}(bI_{n-3}, 1), D_1, D_{n-2}, \ldots, D_{n-3, n-2})
+ a \otimes c(\text{diag}(I_{n-3}, c), \text{diag}(bI_{n-3}, 1), D_1, D_{n-2}, \ldots, D_{n-3, n-2})
+ b \otimes c(\text{diag}(I_{n-3}, a), \text{diag}(cI_{n-3}, 1), D_1, D_{n-2}, \ldots, D_{n-3, n-2})
+ a \otimes c(\text{diag}(I_{n-3}, b), \text{diag}(cI_{n-3}, 1), D_1, D_{n-2}, \ldots, D_{n-3, n-2}). \]

(Note that $D_{n-2} = \text{diag}(D_{i,n-3}, 1) = \text{diag}(D_{i,n-3}(d_i), 1)$. Now through the composite $W''_2 \to U_4' \otimes U_4'' \to T_2' \otimes T_2''$ we have
\[ a \otimes b \otimes c \otimes \{d_1, \ldots, d_{n-3}\} \mapsto (u', u'') \mapsto (\frac{(-1)^{n-3}}{(n-3)!} z + u'''', u'''') = 0, \]
where
\[
 u''_1 = \frac{(-1)^{n-2}}{(n-2)!} \left(c \otimes c(bI_{n-2}, A_{1,n-2}, D_{2,n-2}, \ldots, D_{n-2,n-2})
 + b \otimes c(cI_{n-2}, A_{1,n-2}, D_{2,n-2}, \ldots, D_{n-2,n-2})
 + c \otimes c(aI_{n-2}, B_{1,n-2}, D_{2,n-2}, \ldots, D_{n-2,n-2})
 + a \otimes c(cI_{n-2}, B_{1,n-2}, D_{2,n-2}, \ldots, D_{n-2,n-2})
 + b \otimes c(aI_{n-2}, C_{1,n-2}, D_{2,n-2}, \ldots, D_{n-2,n-2})
 + a \otimes c(bI_{n-2}, C_{1,n-2}, D_{2,n-2}, \ldots, D_{n-2,n-2}) \right).
\]

Thus \( u''_1 = \frac{(-1)^{n-3}}{(n-3)!} z \) and we have \( \chi'_n(u'') = (-u''_1, u'') = \frac{(-1)^{n-3}}{(n-3)!} z, u'' \) which is trivial in \( E^2_{2,n}(n, k) \). This induces a well-defined surjective map \( B_n(F)_k \to E^2_{2,n}(n, k) \), which we denote again by \( \chi'_n \). This completes the proof of the theorem. \( \square \)

**Corollary 5.2.** For any positive integers \( n \) and any field \( k \) such that \( (n - 2)! \in k^{\times} \), there is a natural map
\[
 \chi_n : B_n(F)_k \to H_{n+1}(\text{GL}_n(F), k)/H_{n+1}(F^\times \times \text{GL}_{n-1}(F), k).
\]

If Conjecture 4.1 holds with \( k \) coefficients, then \( \chi_n \) is surjective. In particular, there is a surjective map
\[
 F^\times \otimes H^{n-1}_{\text{GL}}(F, n)_k \oplus \Lambda^2_F F^\times \otimes K^M_{n-1}(F)_k \oplus B_n(F)_k \to H^n_{\text{GL}}(F, n + 1)_k.
\]

**Proof.** These follow from Theorems 1.2 and 5.1 by replacing \( \mathbb{Z} \left[ \frac{1}{(n-2)!} \right] \) with the field \( k \). \( \square \)

Based on the above results we make the following conjecture.

**Conjecture 5.3.** For any positive integer \( n \), there is a natural surjective map
\[
 \chi_n : B_n(F) \left[ \frac{1}{(n-2)!} \right] \to \frac{H_{n+1}(\text{GL}_n(F), \mathbb{Z} \left[ \frac{1}{(n-2)!} \right])}{H_{n+1}(F^\times \times \text{GL}_{n-1}(F), \mathbb{Z} \left[ \frac{1}{(n-2)!} \right])}.
\]

**Corollary 5.4.** Conjecture 5.3 holds for \( 1 \leq n \leq 4 \). In particular, there are natural surjective maps
\[
 \chi_3 : B_3(F) \to H_4(\text{GL}_3(F), \mathbb{Z})/H_4(F^\times \times \text{GL}_2(F), \mathbb{Z}),
 \chi_4 : B_4(F) \left[ \frac{1}{2} \right] \to H_5(\text{GL}_4(F), \mathbb{Z} \left[ \frac{1}{2} \right])/H_5(F^\times \times \text{GL}_3(F), \mathbb{Z} \left[ \frac{1}{2} \right]).
\]

**Proof.** We know that Conjecture 4.1 holds for \( n = 3 \) [18, Section 3] and \( n = 4 \) [19, Section 6]. Therefore the desired results follow from Theorem 5.1. \( \square \)

**Corollary 5.5.** Let \( k \) be a field.

(i) There is a surjective map
\[
 F^\times \otimes H^2_{\text{GL}}(F, 3)_k \oplus \Lambda^2_F F^\times \otimes K^M_2(F)_k \oplus B_3(F)_k \to H^3_{\text{GL}}(F, 4)_k.
\]

(ii) If \( \text{char}(k) \neq 2 \), then there is a surjective map
\[
 F^\times \otimes H^2_{\text{GL}}(F, 4)_k \oplus \Lambda^2_F F^\times \otimes K^M_3(F)_k \oplus B_4(F)_k \to H^4_{\text{GL}}(F, 5)_k.
\]
Proof. These follow from Corollary 5.2 and Corollary 5.4 (by replacing the coefficients $\mathbb{Z} \left[ \frac{1}{(n-2)!} \right]$ with $k$). □

Remark 5.6. The groups $B_n(F)$ should be seen as a generalisation of the Bloch group $B(F)$ to higher dimensions, which is suitable for the study of the homology of $GL_n(F)$. There are other versions of $B_n(F)$ studied by Goncharov [10, p. 222], [4, §3] and Yagunov [35, §2]. Our version seems to be different. At the moment we do not know if there is any connection between our version and their version of $B_n(F)$ for $n \geq 3$.

Remark 5.7. Let $n \geq 3$ and $0 \leq k \leq n - 1$. We expect the $k$-th homology of Complex (0.2) to be related to the groups

$$K_n^M(F) \rightarrow F^\times \otimes K_{n-1}^M(F)/\text{im}(\delta_2^{(n)})$$

$$\{a_1, a_2, \ldots, a_n\} \mapsto a_1 \otimes \{a_2, \ldots, a_n\} \pmod{\delta_2^{(n)}}$$

is well-defined. To prove this, it is sufficient to show that the trivial element $\{1 - a, a, a_3, \ldots, a_n\}$ maps to zero. We have

$$0 = \{1 - a, a, a_3, \ldots, a_n\} \mapsto (1 - a) \otimes \{a, a_3, \ldots, a_n\} \pmod{\delta_2^{(n)}}$$

$$= -(1 - a) \otimes \{a_3, a, \ldots, a_n\} \pmod{\delta_2^{(n)}}$$

$$= a_3 \otimes \{1 - a, a, \ldots, a_n\} \pmod{\delta_2^{(n)}} = 0.$$ 

Now it can be checked directly that the map

$$\delta_1^{(n)}: F^\times \otimes K_{n-1}^M(F)/\text{im}(\delta_2^{(n)}) \rightarrow K_n^M(F)$$

is the inverse of the above map.

We do not know how to connect these groups for $2 \leq k \leq n - 1$.

6. Homology of $GL_n(F)$ over certain fields

In this section we study the homology of $GL_n$ over algebraically closed, real closed, global and local fields. As we will see many of the above results can be improved over these fields.
6.1. Fields with divisible multiplicative groups. Let $F$ be algebraically closed. Since $K^M_n(F)$ is uniquely divisible for $m \geq 2$ [1, Corollary 1.3], it follows that $B_n(F)$ is uniquely divisible for $n \geq 3$. Moreover, since the Bloch group $B(F)$ is uniquely divisible [31, Corollary 5.7] and $B_2(F) = B(F)$, $B_2(F)$ is uniquely divisible too. Therefore $B_n(F)$ is uniquely divisible for any $n$. Using this fact, we can improve almost all the results obtained in the previous sections. For example Theorem 2.1, Corollary 2.5, Theorem 5.1 and Corollary 5.4 hold with integral coefficients.

But using the next theorem and Theorem 1.2 of Galatius, Kupers and Randal-Williams, we can prove far better results.

**Theorem 6.1** (Galatius, Kupers, Randal-Williams). Let $p$ be a prime such that $F^\times \otimes \mathbb{Z}/p = 0$. Then $H_d(GL_n(F), GL_{n-1}(F), \mathbb{Z}/p) = 0$ in degrees $d < 3n/2$.

**Proof.** See [9, Theorem 9.11].

**Proposition 6.2.** Let $p$ be a prime such that $F^\times \otimes \mathbb{Z}/p = 0$. Then the relative homology group $H_{n+k}(GL_n(F), GL_{n-1}(F), \mathbb{Z})$ and the kernel of the natural map $H_{n+k-1}(GL_{n-1}(F), \mathbb{Z}) \to H_{n+k-1}(GL_n(F), \mathbb{Z})$ are uniquely $p$-divisible for $n > 2(k + 1)$.

**Proof.** By Theorem 6.1 we have $H_d(GL_n(F), GL_{n-1}(F), \mathbb{Z}/p) = 0$ in degrees $d < 3n/2$. This in particular implies that

$$H_{n+k}(GL_n(F), GL_{n-1}(F), \mathbb{Z}/p) = 0$$

for $n > 2k$ and

$$H_{n+k+1}(GL_n(F), GL_{n-1}(F), \mathbb{Z}/p) = 0$$

for $n > 2(k+1)$. From the long exact sequence induced by the exact sequence of coefficients

$$0 \to \mathbb{Z} \overset{p}{\to} \mathbb{Z} \to \mathbb{Z}/p \to 0$$

it follows that multiplication by $p$ on $H_{n+k}(GL_n(F), GL_{n-1}(F), \mathbb{Z})$ is a bijection for $n > 2(k + 1)$. Thus $H_{n+k}(GL_n(F), GL_{n-1}(F), \mathbb{Z})$ is uniquely $p$-divisible. This implies ker($H_{n+k-1}(GL_{n-1}(F), \mathbb{Z}) \to H_{n+k-1}(GL_n(F), \mathbb{Z})$) has the same property. □

The following is Theorem C of the introduction.

**Theorem 6.3.** Let $F$ be a field such that $F^\times$ is divisible. Then

(i) $H_n(GL_{n-1}(F), \mathbb{Z}) \to H_n(GL_n(F), \mathbb{Z})$ is injective for any $n \geq 1$,

(ii) $H^\delta_{GL}(F, n+1)$ is divisible for $n \neq 2$ and is uniquely divisible for $n \geq 5$.

**Proof.** Since $F^\times$ is divisible, $F^\times \otimes k$ is uniquely divisible for $k \geq 2$ (see the proof of [1, Proposition 1.2]). Thus $K^M_m(F)$ is divisible for $m \geq 1$ and $F^\times \otimes k \otimes K^M_m(F)$ is uniquely divisible for $k, m \geq 1$. This implies that $B_m(F)$ is uniquely divisible for any $m \geq 3$.

(i) The claim is trivial for $n = 1, 2$. By Corollary 2.5, $\kappa_3$ is surjective with 2-torsion image. Since $B_3(F)$ is uniquely divisible, $\kappa_3$ must be the trivial
map. Thus we get the injectivity for \( n = 3 \). Now let \( n = 4 \). By Theorem 2.1, there is a natural map

\[
\varphi_4 : B_4(F) \to \ker(H_4(GL_3(F), \mathbb{Z}) \to H_4(GL_4(F), \mathbb{Z})),
\]

with 3-torsion image. Since \( F^\times \) is divisible, for any \( a \in F^\times \), the polynomial \( X^2 - a \in F[X] \) splits into linear factors. Thus by [1, Proposition 1.2], \( K_3^M(F) \) is uniquely 2-divisible for any \( m \geq 2 \). Thus we have the decomposition

\[
H_3(GL_3(F), \mathbb{Z}) \simeq H_3(GL_2(F), \mathbb{Z}) \oplus K_3^M(F),
\]

where the splitting map \( K_3^M(F) \to H_3(GL_3(F), \mathbb{Z}) \) is given by \( \{a, b, c\} \mapsto \{a^{1/2}, b, c\} \). Now as the proof of Theorem 2.1(ii), using [19, Theorem 2], we can show that \( \varphi_4 \) is surjective. Since \( B_4(F) \) is uniquely divisible, \( \varphi_4 \) is the trivial map. This implies the injectivity for \( n = 4 \).

So let \( n \geq 5 \). Let \( p \) be a prime. Since \( F^\times \) is divisible, \( F^\times \otimes \mathbb{Z}/p = 0 \). By Proposition 6.2 (for \( k = 1 \)), the kernel of

\[
H_n(GL_{n-1}(F), \mathbb{Z}) \to H_n(GL_n(F), \mathbb{Z})
\]

is uniquely \( p \)-divisible. This holds for any prime and thus this kernel is uniquely divisible. On the other hand by Theorem 1.2 this kernel is torsion. Therefore it must be trivial.

(ii) If \( n \geq 5 \), then by Proposition 6.2, \( H_{n+1}(GL_n(F), GL_{n-1}(F), \mathbb{Z}) \) is uniquely \( p \)-divisible for any prime \( p \). This implies that it is uniquely divisible. By (i),

\[
H^n_{GL}(F, n + 1) \simeq H_{n+1}(GL_n(F), GL_{n-1}(F), \mathbb{Z}).
\]

Therefore \( H^n_{GL}(F, n + 1) \) is uniquely divisible.

If \( n = 1 \), then \( H^1_{GL}(F, 2) \simeq \bigwedge_2 F^\times \), so it is uniquely divisible. For \( n = 3 \), consider the exact sequence

\[
F^\times \otimes H^2_{GL}(F, 3) \oplus \bigwedge_2 F^\times \otimes K_2^M(F) \oplus \text{Tor}_1^Z(F^\times, K_2^M(F)) \to H_3^3(F, 4) \to H_4(GL_3(F), \mathbb{Z})/H_4(F^\times \otimes GL_2(F), \mathbb{Z}) \to 0.
\]

By Corollary 5.4, the map

\[
\chi_3 : B_3(F) \to \frac{H_4(GL_3(F), \mathbb{Z})}{H_4(F^\times \otimes GL_2(F), \mathbb{Z})}
\]

is surjective. Since \( B_3(F) \) is uniquely divisible, \( H_4(GL_3(F), \mathbb{Z})/H_4(F^\times \otimes GL_2(F), \mathbb{Z}) \) is divisible. Now it follows from the above exact sequence that \( H^3_{GL}(F, 4) \) is divisible. (Note that \( F^\times \otimes H^2_{GL}(F, 3), \bigwedge_2 F^\times \otimes K_2^M(F) \) and \( \text{Tor}_1^Z(F^\times, K_2^M(F)) \) are divisible.)

In a similar way we can show that \( H^4_{GL}(F, 5) \) is divisible. Here we need to show that we have a surjective map \( B_4(F) \to \frac{H_5(GL_4(F), \mathbb{Z})}{H_5(F^\times \otimes GL_3(F), \mathbb{Z})} \). But this can be proved as the proof of Theorem 5.1, using the validity of Conjecture 4.1 for \( n = 4 \) ([19, Section 6]) and the decomposition \( H_3(GL_3(F), \mathbb{Z}) \simeq H_3(GL_2(F), \mathbb{Z}) \oplus K_3^M(F) \). This completes the proof of the theorem.
Remark 6.4. The multiplicative group of an algebraically closed field is divisible. But the class of fields such that their multiplicative groups are divisible is much larger. For more on this see [25, Theorems 1, 2].

Corollary 6.5. Let $F$ be algebraically closed (or more generally a field such that $F^\times \otimes \mathbb{Z}/p = 0$ and the polynomial $X^p - 1 \in F[X]$ splits into linear factors, for any prime $p$). Then for any $n$,

(i) $H_n(\text{GL}_{n-1}(F), \mathbb{Z}) \to H_n(\text{GL}_n(F), \mathbb{Z})$ is injective,

(ii) $H_n(\text{GL}_n(F), \mathbb{Z}) \simeq H_n(\text{GL}_{n-1}(F), \mathbb{Z}) \oplus K^M_n(F)$, where the splitting map $K^M_n(F) \to H_n(\text{GL}_n(F), \mathbb{Z})$ is given by

\[
\{a_1, \ldots, a_n\} \mapsto [a_1^{-1}^{\frac{1}{n-1}}, \ldots, a_n].
\]

(iii) $H^\sigma_3(F, n+1)$ is divisible for $n \geq 1$ and is uniquely divisible for $n \neq 2$.

(iv) There is a natural map

\[
\chi_n: B_n(F) \to H_{n+1}(\text{GL}_n(F), \mathbb{Z})/H_{n+1}(F^\times \times \text{GL}_{n-1}(F), \mathbb{Z}).
\]

Moreover, if Conjecture 4.1 holds with $\mathbb{Z}$-coefficients over $F$, then $\chi_n$ is surjective.

Proof. (i) This follows from Theorem 6.3(i).

(ii) This follows from (i) and the fact that $K^M_n(F)$ is uniquely divisible for $n \geq 2$ [1, Proposition 1.2, Corollary 1.3].

(iii) The claim is trivial for $n = 1$. Let $n = 2$. Since $K^M_1(F)$ is uniquely divisible for $i \geq 2$, $F^\times \otimes K^M_2(F)$ is uniquely divisible and $K^M_2(F)/2 = 0$. Thus by Proposition 3.1, we have the exact sequence

\[
H_3(\text{SL}_2(F), \mathbb{Z}) \to H^\sigma_3(\text{GL}_2(F), 3) \to F^\times \otimes K^M_2(F) \to 0.
\]

By [18, Theorem 6.1(ii)], the natural map

\[
H_3(\text{SL}_2(F), \mathbb{Z}) = H_3(\text{SL}_2(F), \mathbb{Z})_{F^\times} \to H_3(\text{SL}(F), \mathbb{Z})
\]

is injective. Now it follows from the commutative diagram

\[
\begin{array}{ccc}
H_3(\text{SL}_2(F), \mathbb{Z}) & \longrightarrow & H_3(\text{SL}_3(F), \mathbb{Z}) \longrightarrow H_3(\text{SL}(F), \mathbb{Z}) \\
\downarrow & & \downarrow \\
H_3(\text{GL}_2(F), \mathbb{Z}) & \longrightarrow & H_3(\text{GL}_3(F), \mathbb{Z}) \cong H_3(\text{GL}(F), \mathbb{Z})
\end{array}
\]

that the map $H_3(\text{SL}_2(F), \mathbb{Z}) \to H_3(\text{GL}_2(F), \mathbb{Z})$ is injective. This implies that the map $H_3(\text{SL}_2(F), \mathbb{Z}) \to H^\sigma_3(\text{GL}_2(F), 3)$ is injective. Since

\[
H_3(\text{SL}_2(F), \mathbb{Z}) \simeq K^\text{ind}_3(F)
\]

[18, Proposition 6.4], similar to the proof of Proposition 3.1, we can show that

\[
H^\sigma_3(\text{GL}_2(F), 3) \simeq K^\text{ind}_3(F) \oplus F^\times \otimes K^M_2(F).
\]

But $K^\text{ind}_3(F)$ is divisible (see the proof of [9, Corollary 9.13]). Therefore $H^\sigma_3(\text{GL}_2(F), 3)$ is divisible. Now let $n = 3$. Then $F^\times \otimes H^\sigma_3(\text{GL}_2(F), 3), \wedge_2 F^\times \otimes K^M_2(F)$ are uniquely divisible (see the proof of [1, Proposition 1.2]) and
\[ \text{Tor}_2^F(F^\times, K^M_2(F)) = 0. \] Now as the proof of Theorem 6.3(ii) we can show that \( H^2_{\text{GL}}(F, 4) \) is uniquely divisible. In a similar way one can show that \( H^1_{\text{GL}}(F, 5) \) is uniquely divisible. This resolves the case \( n = 4 \). If \( n \geq 5 \), the claim follows from Theorem 6.3(ii).

(iv) We may assume \( n \geq 3 \). The proof is similar to the proof of Theorem 5.1. Here we should use (ii). In fact \( \chi_n = d^2_{2,n}(n, \mathbb{Z}) \circ \chi'_n \), where \( \chi'_n : \mathcal{B}_n(F) \to E^2_{2,n}(n, \mathbb{Z}) \) is surjective.

\[ \text{Remark 6.6.} \] (i) Let \( F \) be a field such that \( F^\times \) is divisible. Since \( F^\times = (F^\times)^2 \) and \( F^\times \otimes K^M_2(F) \) is uniquely divisible (see the proof of [1, Proposition 1.2]), as in the proof of Proposition 3.1, we can show that

\[ H^2_{\text{GL}}(F, 3) \simeq K^\text{ind}_3(F) \oplus F^\times \otimes K^M_2(F). \]

If \( \mu(F) \neq 1 \) (e.g. \( \text{char}(F) \neq 2 \)), Suslin’s Bloch-Wigner exact sequence

\[ 0 \to \text{Tor}_2^F(\mu(F), \mu(F)) \to K^\text{ind}_3(F) \to B(F) \to 0 \]

(see [31, Theorem 5.2]) shows that \( K^\text{ind}_3(F) \) has nontrivial torsion elements. Therefore \( H^2_{\text{GL}}(F, 3) \) is not uniquely divisible.

We do not know if \( H^2_{\text{GL}}(F, 3) \) is divisible, when \( F^\times \) is divisible. By Corollary 6.5 this is the case if \( F \) is algebraically closed.

(ii) The part (iii) of Corollary 6.5 gives a positive answer to a question asked by the author in [17, page 616].

6.2. \textbf{Real closed fields.} Let \( F \) be a real closed field [13, Chapter XI]. It is well-known that \( F \) has a unique order and \( F^\times \simeq \{ \pm 1 \} \times F^{>0} \). Any polynomial of odd degree has a root. Moreover, for any \( a \in F^{>0} \), \( X^2 - a \) has a root in \( F^{>0} \), i.e. \( \sqrt{a} \in F^{>0} \). Thus \( F^{>0} \) is a uniquely divisible group. The following result is well-known.

\[ \text{Lemma 6.7.} \] Let \( F \) be a real closed field. Then for any \( m \geq 1 \), \( K^M_m(F) \) is direct sum of a cyclic group of order 2 generated by \( \{-1, \ldots, -1\} \) and a divisible subgroup \( K^M_m(F) \circ \) generated by all symbols \( \{a_1, \ldots, a_m\}, a_1, \ldots, a_m \in F^{>0} \).

\[ \text{Proof.} \] For proof of the case \( F = \mathbb{R} \) see [14, Theorem 14.46, Corollary 14.47] or [15, Example 1.6, Theorem 1.4]. The proof of the general case is similar.

It follows from the above lemma that \( F^\times \otimes K^M_m(F) \) decomposes as a direct sum of the subgroup of order two, generated by \( (-1) \otimes \cdots \otimes (-1) \otimes \{-1, \ldots, -1\} \) and the uniquely divisible subgroup \( (F^{>0})^{\otimes i} \otimes K^M_m(F) \circ \) for \( i, m \geq 1 \) (see the proof of [1, Proposition 1.2]). Since

\[ \delta_3^{(m)}((-1) \otimes (-1) \otimes (-1) \otimes \{-1, \ldots, -1\}) = (-1) \otimes (-1) \otimes \{-1, \ldots, -1\}, \]

\( \mathcal{B}_n(F) \) is uniquely divisible for any \( n \geq 3 \).

Observe that \( \mathcal{B}_2(\mathbb{R}) \) is divisible. This follows from the fact that \( K_3(\mathbb{R}) \) is divisible [29, Theorem 4.9], since \( \mathcal{B}_2(\mathbb{R}) \) is a quotient of \( K^\text{ind}_3(\mathbb{R}) \).

The following is Theorem D of the introduction.
**Theorem 6.8.** Let $F$ be a real closed field. Then

(i) $H_3(\text{GL}_2(F), \mathbb{Z}) \to H_3(\text{GL}_3(F), \mathbb{Z})$ is injective,

(ii) $H_n(\text{GL}_{n-1}(F), \mathbb{Z}[\frac{1}{2}]) \to H_n(\text{GL}_n(F), \mathbb{Z}[\frac{1}{2}])$ is injective for any $n$,

(iii) $H_{3\ell}^{\text{GL}}(F, n + 1)[\frac{1}{2}]$ is uniquely divisible for $n \geq 5$.

**Proof.** (i) By Corollary 2.5, $\kappa_3$ is surjective with 2-torsion image. Since $\mathcal{B}_3(F)$ is divisible, $\kappa_3$ must be the trivial map. Thus the desired map is injective.

(ii) The claim is trivial for $n = 1, 2$. The case $n = 3$ follows from (i). The case $n = 4$ can be proved as the part (i). So we may assume that $n \geq 5$. Let $p$ be an odd prime. Then $F^x \otimes \mathbb{Z}/p = 0$. By Proposition 6.2, the kernel of $H_n(\text{GL}_{n-1}(F), \mathbb{Z}) \to H_n(\text{GL}_n(F), \mathbb{Z})$ is uniquely $p$-divisible. Since this holds for any odd prime, the kernel of the above map is 2-torsion. From this we obtain the desired injectivity.

(iii) The proof is similar to the proof of Theorem 6.3(ii). \(\square\)

**Corollary 6.9.** Let $n \geq 1$. Then

(i) $H_n(\text{GL}_{n-1}(\mathbb{R}), \mathbb{Z}[\frac{1}{2}]) \to H_n(\text{GL}_n(\mathbb{R}), \mathbb{Z}[\frac{1}{2}])$ is injective,

(ii) $H_n(\text{GL}_{n}(\mathbb{R}), \mathbb{Z}[\frac{1}{2}]) \simeq H_n(\text{GL}_{n-1}(\mathbb{R}), \mathbb{Z}[\frac{1}{2}]) \oplus K_n^M(\mathbb{R})^\circ$,

(iii) $H_{n+1}^{\text{GL}}(\mathbb{R}, n + 1)[\frac{1}{2}]$ is divisible and is uniquely divisible for any $n \neq 2$,

(iv) there is a natural map

$$\chi_n: B_n(\mathbb{R}) \to H_{n+1}(\text{GL}_n(\mathbb{R}), \mathbb{Z}[\frac{1}{2}])/H_{n+1}(\mathbb{R}^x \times \text{GL}_{n-1}(\mathbb{R}), \mathbb{Z}[\frac{1}{2}]).$$

Moreover, if Conjecture 4.1 holds with $\mathbb{Z}[\frac{1}{2}]$ coefficients for $\mathbb{R}$, then $\chi_n$ is surjective.

**Proof.** (i) This is an special case of Theorem 6.8(ii).

(ii) This follows from (i), Theorem 1.1, the fact that $K_n^M(\mathbb{R})^\circ$ is uniquely divisible [34, Example 7.2(c), Chap. III] and the isomorphism $K_n^M(\mathbb{R})^\circ \simeq K_n^M(\mathbb{R})[\frac{1}{2}]$.

(iii) Since $H_{\text{GL}}^1(\mathbb{R}, 2) \simeq \bigwedge_2^2 \mathbb{R}^x \otimes \mathbb{R}^+$, the claim is trivial for $n = 1$. By [29, Theorem 4.9], $K_3(\mathbb{R})$ is divisible. Thus by Lemma 6.7 and Proposition 3.1, $H_{2\ell}^{\text{GL}}(\mathbb{R}, 3)[\frac{1}{2}]$ is divisible. The cases $n = 3, 4$ can be proved similar to the cases $n = 3, 4$ done in the proof of Corollary 6.5(iii). If $n \geq 5$, the claim follows from Theorem 6.8(iii).

(iii) We may assume $n \geq 3$. The proof is similar to the proof of Theorem 5.1. In fact $\chi_n = d_{2,n}^2(n, \mathbb{Z}[\frac{1}{2}]) \circ \chi_n'$, where $\chi_n': \mathcal{B}_n(\mathbb{R}) \to E_{2,n}^2(n, \mathbb{Z}[\frac{1}{2}])$ is surjective. \(\square\)

6.3. **Global and local fields.** The homology of general linear groups over global fields is well studied. For example in [2, Corollary 7.6] Borel and Yang has shown that for a number field $F$, the natural map

$$H_d(\text{GL}_{n-1}(F), \mathbb{Q}) \to H_d(\text{GL}_n(F), \mathbb{Q})$$

is injective for any $d$ and is surjective if $d \leq 2n - 3$. Recently, Galatius, Kupers and Randal-Williams have proved the following result.
Theorem 6.10 (Galatius, Kupers and Randal-Williams). Let $F$ be a field with torsion $K_2(F)$. Then $H_d(\text{GL}_{n-1}(F), \mathbb{Q}) \to H_d(\text{GL}_n(F), \mathbb{Q})$ is surjective if $d < (4n - 1)/3$ and is injective if $d < (4n - 4)/3$.

Proof. See [9, Theorem E or Theorem 9.10].

It is known that the second $K$-group of a global field is torsion [33, p. 144, p. 158]. Thus the above theorem can be applied to global fields.

Let $F$ be a global field. Then by a theorem of Bass and Tate, for any $m \geq 3$,

$$K^M_m(F) \simeq (\mathbb{Z}/2)^{r_1},$$

where $r_1$ is the number of embeddings of $F$ in $\mathbb{R}$ [1, Theorem 2.1, Chap. II]. Thus $B_3(F)$ is torsion and for any $n \geq 5$, $B_n(F)$ is 2-torsion.

If $F$ is a local field, then $K^M_m(F)$ is uniquely divisible for any $m \geq 3$ [34, Proposition 7.1, Chap. VI]. In particular for any $n \geq 5$, $B_n(F)$ is uniquely divisible.

Proposition 6.11. Let $F$ be either a global field or a local field and let the sequence

$$H_n(F^x \times \text{GL}_{n-2}(F), \mathbb{Z}[\frac{1}{6}]) \xrightarrow{\alpha_{1,2}} H_n(F^x \times \text{GL}_{n-1}(F), \mathbb{Z}[\frac{1}{6}]) \xrightarrow{\text{inc}} H_n(\text{GL}_n(F), \mathbb{Z}[\frac{1}{6}]) \to 0$$

be exact for any $3 \leq n \leq s$. Then for any $1 \leq n \leq s$,

(i) $H_n(\text{GL}_{n-1}(F), \mathbb{Z}[\frac{1}{6}]) \to H_n(\text{GL}_n(F), \mathbb{Z}[\frac{1}{6}])$ is injective,

(ii) $H_n(\text{GL}_n(F), \mathbb{Z}[\frac{1}{6}]) \simeq H_n(\text{GL}_{n-1}(F), \mathbb{Z}[\frac{1}{6}]) \oplus K^M_n(F)[\frac{1}{6}]$. In case of local fields the splitting map $K^M_n(F)[\frac{1}{6}] \to H_n(\text{GL}_n(F), \mathbb{Z}[\frac{1}{6}])$ is given by

$$(-1)^{n-1}(n-1)!\{a_1, \ldots, a_n\} \mapsto [a_1, \ldots, a_n],$$

(iii) There is a natural map

$$\chi_n: B_n(F)[\frac{1}{6}] \to H_{n+1}(\text{GL}_n(F), \mathbb{Z}[\frac{1}{6}])/H_{n+1}(F^x \times \text{GL}_{n-1}(F), \mathbb{Z}[\frac{1}{6}]),$$

which is surjective if Conjecture 4.1 holds with $\mathbb{Z}[\frac{1}{6}]$ coefficients.

Proof. The claim is trivial for $n = 1, 2$. The cases $n = 3, 4$ follow from Corollaries 2.5 and 5.4. So let $n \geq 5$. The proof is by induction and is similar to the proofs of Theorems 2.1 and 5.1.

Remark 6.12. In the previous proposition, we believe it is enough to invert 2 (rather than 6). For this we need to prove the injectivity of

$$H_4(\text{GL}_3(F), \mathbb{Z}[\frac{1}{6}]) \to H_4(\text{GL}_3(F), \mathbb{Z}[\frac{1}{6}]),$$

(see Remark 2.6(ii)). So far we could prove the injectivity of the map $H_4(\text{GL}_3(F), \mathbb{Z}[\frac{1}{6}]) \to H_4(\text{GL}_3(F), \mathbb{Z}[\frac{1}{6}])$ (Corollary 2.5), which explains why in the previous proposition we invert 6. In the following we prove a special case, where we invert only 2 in the coefficients ring.
Proposition 6.13. Let $F$ be a local field such that $3 \nmid |\mu(F)|$. Then the natural map $H_4(\text{GL}_3(F), \mathbb{Z} \left[ \frac{1}{2} \right]) \to H_4(\text{GL}_4(F), \mathbb{Z} \left[ \frac{1}{2} \right])$ is injective. Moreover, $H_4(\text{GL}_4(F), \mathbb{Z} \left[ \frac{1}{2} \right]) \simeq H_4(\text{GL}_3(F), \mathbb{Z} \left[ \frac{1}{2} \right]) \oplus K^M_1(F)$.

Proof. Let $m = |\mu(F)|$. By a theorem of Moore, $K^M_2(F)$ is the direct sum of a uniquely divisible abelian group and a finite cyclic group, isomorphic to $\mu(F)$ [34, Theorem 6.2.4, Chap. III]. This would imply that $F^\times \otimes F^\times \otimes K^M_2(F)$ is the direct sum of a uniquely divisible abelian group and the $m$-torsion group $F^\times \otimes F^\times \otimes \mathbb{Z}/m \simeq F^\times / (F^\times m) \otimes F^\times / (F^\times m)$ (observe that $F^\times / (F^\times m)$ is finite). Since $K^M_3(F)$ is uniquely divisible, $F^\times \otimes K^M_3(F)$ also is uniquely divisible. Now from the definition of $\mathcal{B}_4(F)$ we have

$$\mathcal{B}_4(F) \simeq U \oplus A,$$

where $U$ is divisible and $A$ is $m$-torsion. Since $3 \nmid m$ and since the image of the surjective map

$$\kappa_4 : \mathcal{B}_4(F) \left[ \frac{1}{2} \right] \to \ker(H_4(\text{GL}_3(F), \mathbb{Z} \left[ \frac{1}{2} \right]) \to H_4(\text{GL}_3(F), \mathbb{Z} \left[ \frac{1}{2} \right])$$

(Corollary 2.5) of Theorem 2.1 is 3-torsion, we see that $\kappa_4$ is the trivial map. This proves the injectivity result. The other claim follows from Corollary ?? and the fact that $K^M_1(F)$ is uniquely divisible. \qed

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