STRONG INSTABILITY OF STANDING WAVES FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH A PARTIAL CONFINEMENT

Dedicated to Professor Vladimir Georgiev on the occasion of his sixtieth birthday

MASAHITO OHTA
Department of Mathematics, Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

ABSTRACT. We study the instability of standing wave solutions for nonlinear Schrödinger equations with a one-dimensional harmonic potential in dimension $N \geq 2$. We prove that if the nonlinearity is $L^2$-critical or supercritical in dimension $N-1$, then any ground states are strongly unstable by blowup.

1. Introduction. In this paper, we study the instability of standing wave solutions $e^{i\omega t}\phi_{\omega}(x)$ for the nonlinear Schrödinger equation with a one-dimensional harmonic potential

$$i\partial_t u = -\Delta u + x_N^2 u - |u|^{p-1} u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

where $N \geq 2$, $x_N$ is the $N$-th component of $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $\Delta$ is the Laplacian in $x$, and $1 < p < 1 + 4/(N - 2)$. Here, $1 + 4/(N - 2)$ stands for $\infty$ if $N = 2$.

The Cauchy problem for (1) is locally well-posed in the energy space $X$ (see [6, Theorem 9.2.6]). Here, the energy space $X$ for (1) is defined by

$$X = \left\{ v \in H^1(\mathbb{R}^N) : x_N v \in L^2(\mathbb{R}^N) \right\}$$

with the norm

$$\|v\|_X = \left( \|\nabla v\|_{L^2}^2 + \|v\|_{L^2}^2 + \|x_N v\|_{L^2}^2 \right)^{1/2}.$$

**Proposition 1.** Let $1 < p < 1 + 4/(N - 2)$. For any $u_0 \in X$ there exist $T_{\text{max}} = T_{\text{max}}(u_0) \in (0, \infty]$ and a unique maximal solution $u \in C([0, T_{\text{max}}), X) \cap C^1([0, T_{\text{max}}), X^*)$ of (1) with initial condition $u(0) = u_0$. The solution $u(t)$ is maximal in the sense that if $T_{\text{max}} < \infty$, then $\|u(t)\|_X \to \infty$ as $t \nearrow T_{\text{max}}$.

Moreover, the solution $u(t)$ satisfies the conservation laws

$$\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2, \quad E(u(t)) = E(u_0)$$

for all $t \in [0, T_{\text{max}})$, where the energy $E$ is defined by

$$E(v) = \frac{1}{2}\|\nabla v\|_{L^2}^2 + \frac{1}{2}\|x_N v\|_{L^2}^2 - \frac{1}{p+1}\|v\|_{L^{p+1}}^{p+1}.$$
Next, we consider the stationary problem
\[-\Delta \phi + x_N^2 \phi + \omega \phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^N,\] (3)
where \(\omega \in \mathbb{R}\). Note that if \(\phi(x)\) solves (3), then \(e^{i\omega t}\phi(x)\) is a solution of (1).

Moreover, (3) can be written as \(S_\omega(v) = 0\), where
\[S_\omega(v) = E(v) + \frac{\omega}{2} \|v\|_{L^2}^2 - \frac{1}{p+1} \|v\|_{L^{p+1}}^{p+1},\]
is the action. The set of all ground states for (3) is defined by
\[G_\omega = \{ \phi \in \mathcal{A}_\omega : S_\omega(\phi) \leq S_\omega(v) \text{ for all } v \in \mathcal{A}_\omega \},\] (4)
where
\[\mathcal{A}_\omega = \{ v \in X : S_\omega'(v) = 0, \ v \neq 0 \}\]
is the set of all nontrivial solutions for (3).

Then, we have the following result on the existence of ground states for (3).

**Proposition 2.** Let \(1 < p < 1 + 4/(N-2)\) and \(\omega \in (-1, \infty)\). Then, the set \(G_\omega\) is not empty, and it is characterized by
\[G_\omega = \{ v \in X : S_\omega(v) = d(\omega), K_\omega(v) = 0, \ v \neq 0 \},\] (5)
where
\[K_\omega(v) = \partial_x S_\omega(\lambda v) |_{\lambda = 1} = \|\nabla v\|_{L^2}^2 + \|x_N v\|_{L^2}^2 + \omega \|v\|_{L^2}^2 - \|v\|_{L^{p+1}}^{p+1},\]
is the Nehari functional, and
\[d(\omega) = \inf \{ S_\omega(v) : v \in X, \ K_\omega(v) = 0, \ v \neq 0 \}.\] (6)

Although Proposition 2 can be proved by the standard concentration compactness argument, for the sake of completeness, we give the proof of Proposition 2 in Section 3.

Here, we remark that by Heisenberg’s inequality
\[\|v\|_{L^2}^2 \leq 2\|\nabla v\|_{L^2} \|x_N v\|_{L^2},\]
for any \(\omega \in (-1, \infty)\) there exist positive constants \(C_1(\omega)\) and \(C_2(\omega)\) such that
\[C_1(\omega) \|v\|_{L^2}^2 \leq \|\nabla v\|_{L^2}^2 + \|x_N v\|_{L^2}^2 + \omega \|v\|_{L^2}^2 \leq C_2(\omega) \|v\|_{L^2}^4\]
for all \(v \in X\). Indeed, we can choose
\[C_1(\omega) = \min \left\{ 1, \frac{1 + \omega}{2} \right\}, \quad C_2(\omega) = \max \{1, \omega\}.\]

Now we state our main result in this paper.

**Theorem 1.1.** Assume that \(N \geq 2, 1 + 4/(N-1) \leq p < 1 + 4/(N-2)\), and let \(\phi_\omega \in G_\omega\) for \(\omega \in (-1, \infty)\). Then, for any \(\omega \in (-1, \infty)\), the standing wave solutions \(e^{i\omega t}\phi_\omega\) of (1) are strongly unstable in the following sense. For any \(\varepsilon > 0\) there exists \(u_0 \in X\) such that \(\|u_0 - \phi_\omega\|_{L^2} < \varepsilon\) and the solution \(u(t)\) of (1) with \(u(0) = u_0\) blows up in finite time.
Notice that Theorem 1.1 covers the physically relevant case $N = 3$ and $p = 3$ as a borderline case.

Here, we recall some known results related to Theorem 1.1.

First, we consider the nonlinear Schrödinger equations without potential

$$i\partial_t u = -\Delta u - |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

where $1 < p < 1 + 4/(N-2)$. For any $\omega \in (0, \infty)$, there exists a unique positive radial solution $\phi_\omega(x)$ of the stationary problem

$$-\Delta \phi + \omega \phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^N$$

(see [13] for the uniqueness).

When $1 < p < 1 + 4/N$, the standing wave solution $e^{i\omega t}\phi_\omega$ of (9) is orbitally stable for all $\omega > 0$ (see [7]). While, if $1 + 4/N \leq p < 1 + 4/(N-2)$, then the standing wave solution $e^{i\omega t}\phi_\omega$ of (9) is strongly unstable for all $\omega > 0$ (see [3] and also [6, Theorem 8.2.2]).

Next, we consider the nonlinear Schrödinger equations with a harmonic potential

$$i\partial_t u = -\Delta u + |x|^2 u - |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

where $1 < p < 1 + 4/(N-2)$. For any $\omega \in (-N, \infty)$, there exists a unique positive radial solution $\phi_\omega(x)$ of the stationary problem

$$-\Delta \phi + |x|^2 \phi + \omega \phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^N$$

(see [11, 12] for the uniqueness).

When $\omega$ is sufficiently close to $-N$, the standing wave solution $e^{i\omega t}\phi_\omega$ of (10) is orbitally stable for any $1 < p < 1 + 4/(N-2)$ (see [9]). We remark that $N$ is the first eigenvalue of $-\Delta + |x|^2$.

On the other hand, when $\omega$ is sufficiently large, the standing wave solution $e^{i\omega t}\phi_\omega$ of (10) is orbitally stable for the case $1 < p \leq 1 + 4/N$ (see [8, 9]), and it is strongly unstable for the case $1 + 4/N < p < 1 + 4/(N-2)$ (see [17] and also [10] for an earlier result on the orbital instability).

Finally, we consider the nonlinear Schrödinger equations with a partial confinement of the form

$$i\partial_t u = -\Delta u + (x_1^2 + \cdots + x_d^2)u - |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

where $N \geq 2$, $1 \leq d \leq N-1$, $x = (x_1, \ldots, x_d, x_{d+1}, \ldots, x_N)$. The typical case is that $N = 3$ and $d = 2$.

Recently, Bellazzini, Bousaïd, Jeanjean and Visciglia [2] constructed orbitally stable standing wave solutions of (11) for the case

$$1 + 4/N < p < \min\{1 + 4/(N-d), 1 + 4/(N-2)\}$$

(see Theorem 1 and Remark 1.9 of [2]). It should be remarked that the bottom of the spectrum of $-\Delta + (x_1^2 + \cdots + x_d^2)$ is not an eigenvalue, so that unlike (10) with a complete confinement, the existence of stable standing wave solutions for (11) is highly nontrivial in the $L^2$-supercritical case $p > 1 + 4/N$.

We also remark that for the case $d \geq 2$, the assumption (12) becomes $1 + 4/N < p < 1 + 4/(N-2)$. On the other hand, for the case $d = 1$, the assumption (12) becomes $1 + 4/N < p < 1 + 4/(N-1)$, and there is a chance to consider the case $1 + 4/(N-1) \leq p < 1 + 4/(N-2)$. This is our main motivation for Theorem 1.1 in the present paper (see also [1, 5, 20] for related results).

Although it is not clear whether the standing wave solutions constructed by [2] are ground states in the sense of (4) (see Definition 1.1 and Remark 1.10 of [2]), it
would be safe to conclude from our Theorem 1.1 that the upper bound on $p$ in (12) is optimal for the existence of stable standing wave solutions of (11).

The rest of the paper is organized as follows. In Section 2, we give the proof of Theorem 1.1. The proof is based on a virial type identity (13) associated with the scaling (14), the characterization of ground states (5) by the minimization problem on the Nehari manifold, and Lemma 2.1 below. We remark that the classical method by Berestycki and Cazenave [3] is not applicable to (1) directly. Instead, we use and modify the ideas of Zhang [21] and Le Coz [14], which give an alternative approach to the strong instability (see also [17, 18, 19] for recent developments).

In Section 3, we give the proof of Proposition 2. The proof is based on the standard concentration compactness argument.

2. Proof of Theorem 1.1. We define

$$\Sigma = \{ v \in H^1(\mathbb{R}^N) : |x|v \in L^2(\mathbb{R}^N) \}.$$

First, we derive a virial type identity.

**Proposition 3.** Let $1 < p < 1 + 4/(N - 2)$ if $u_0 \in \Sigma$, then the solution $u(t)$ of (1) with $u(0) = u_0$ satisfies $u \in C([0, T_{\text{max}}), \Sigma)$. Moreover, the function

$$t \mapsto F(t) = \sum_{j=1}^{N-1} \int_{\mathbb{R}^N} x_j^2 |u(t,x)|^2 \, dx$$

is in $C^2([0, T_{\text{max}}), \Sigma)$, and satisfies

$$F''(t) = 16 P(u(t))$$

for all $t \in [0, T_{\text{max}})$, where

$$P(v) = \frac{1}{2} \sum_{j=1}^{N-1} ||\partial_j v||^2_{L^2} - \frac{\alpha}{2(p+1)} ||v||^p_{L^{p+1}}, \quad \alpha = \frac{(N-1)(p-1)}{2}.$$  

**Proof.** We state formal calculations for the identity (13) only. These formal calculations can be justified by the classical regularization argument as in [6, Proposition 6.5.1] (see also [16]).

Let $u(t, x)$ be a smooth solution of (1). Then, we have

$$F'(t) = 2 \sum_{j=1}^{N-1} \text{Im} \int_{\mathbb{R}^N} x_j^2 \pi (-\Delta u + x_N^2 u - |u|^{p-1} u) \, dx$$

$$= -2 \sum_{j=1}^{N-1} \text{Im} \int_{\mathbb{R}^N} x_j^2 \pi N u \, dx = 4 \sum_{j=1}^{N-1} \text{Im} \int_{\mathbb{R}^N} \pi x_j \partial_j u \, dx.$$  

Moreover, we have

$$F''(t) = -4 \sum_{j=1}^{N-1} \text{Im} \int_{\mathbb{R}^N} \partial_t u (2x_j \partial_j + \pi) \, dx.$$  

Here, we consider the scaling

$$v^{\lambda}(x) = \lambda^{(N-1)/2} v(\lambda x_1, ..., \lambda x_{N-1}, x_N)$$  (14)
for $\lambda > 0$ and $x = (x_1, ..., x_{N-1}, x_N) \in \mathbb{R}^N$. Then, we have

$$\partial_\lambda v^\lambda(x)|_{\lambda=1} = \sum_{j=1}^{N-1} x_j \partial_j v(x) + \frac{N-1}{2} v(x),$$

$$E(v^\lambda) = \frac{\lambda^2}{2} \sum_{j=1}^{N-1} \|\partial_j v\|_{L^2}^2 - \frac{\lambda^\alpha}{p+1} \|v\|_{L^{p+1}}^{p+1} + \frac{1}{2} \|\partial_N v\|_{L^2}^2 + \frac{1}{2} \|x_N v\|_{L^2}^2,$$

and

$$P(v) = \frac{1}{2} \partial_\lambda E(v^\lambda)|_{\lambda=1}.$$ 

Thus, we have

$$F''(t) = 8 \text{Re} \int_{\mathbb{R}^N} (-\Delta u + x_N^2 u - |u|^{p-1} u) \overline{\partial_\lambda u^\lambda}|_{\lambda=1} \, dx$$

$$= 8 \partial_\lambda E(u^\lambda)|_{\lambda=1} = 16P(u(t)).$$

As stated above, these formal calculations can be justified by the regularization argument. \qed

Notice that

$$\alpha = \frac{(N-1)(p-1)}{2} \geq 2$$

for the case $1 + 4/(N - 1) \leq p < 1 + 4/(N - 2)$.

The following lemma is a modification of the ideas of Zhang [21] and Le Coz [14] (see also [17, 18, 19]).

**Lemma 2.1.** Assume that $1 + 4/(N - 1) \leq p < 1 + 4/(N - 2)$ and $\omega \in (-1, \infty)$. If $v \in X$ satisfies $P(v) \leq 0$ and $v \neq 0$, then $d(\omega) \leq S_\omega(v) - P(v)$.

**Proof.** Since $\omega > -1$ and $v \neq 0$, by Heisenberg's inequality (7), we have

$$C_0 := \|\partial_N v\|_{L^2}^2 + \|x_N v\|_{L^2}^2 + \omega \|v\|_{L^2}^2 \geq 2\|\partial_N v\|_{L^2}^2 \|x_N v\|_{L^2} + \omega \|v\|_{L^2}^2 \geq (\omega + 1) \|v\|_{L^2}^2 > 0.$$

Then, it follows from $P(v) \leq 0$ that

$$K_\omega(v^\lambda) = \lambda^2 \sum_{j=1}^{N-1} \|\partial_j v\|_{L^2}^2 - \lambda^\alpha \|v\|_{L^{p+1}}^{p+1} + C_0$$

$$\leq \left( \frac{\alpha \lambda^2}{p+1} - \lambda^\alpha \right) \|v\|_{L^{p+1}}^{p+1} + C_0$$

for $\lambda > 0$. Since $\alpha \geq 2$ and $v \neq 0$, there exists $\lambda_0 \in (0, \infty)$ such that $K_\omega(v^{\lambda_0}) = 0$.

Here, we remark that

$$\frac{\alpha \lambda^2}{p+1} - \lambda^\alpha = \frac{p-1}{p+1} \lambda^2$$

for the case $\alpha = 2$.

Then, by the definition (6) of $d(\omega)$, we have $d(\omega) \leq S_\omega(v^{\lambda_0})$.

Moreover, since $\alpha \geq 2$, the function

$$(0, \infty) \ni \lambda \mapsto S_\omega(v^\lambda) - \lambda^2 P(v) = \frac{\alpha \lambda^2 - 2 \lambda^\alpha}{2(p+1)} \|v\|_{L^{p+1}}^{p+1} + \frac{C_0}{2}$$

attains its maximum at $\lambda = 1$. 
Thus, since $P(v) \leq 0$ again, we have
\[ d(\omega) \leq S_\omega(u^{\lambda_0}) \leq S_\omega(u^{\lambda_0}) - \lambda_0^2 P(v) \leq S_\omega(v) - P(v). \]
This completes the proof.

Once we have obtained Lemma 2.1, the rest of the proof is the same as in the classical argument of Berestycki and Cazenave [3].

**Lemma 2.2.** Assume that $1 + 4/(N - 1) \leq p < 1 + 4/(N - 2)$ and $\omega \in (-1, \infty)$.
The set
\[ B_\omega = \{ v \in X : S_\omega(v) < d(\omega), \ P(v) < 0 \} \]
is invariant under the flow of (1). That is, if $u_0 \in B_\omega$, then the solution $u(t)$ of (1) with $u(0) = u_0$ satisfies $u(t) \in B_\omega$ for all $t \in [0, T_{\max})$.

**Proof.** This follows from the conservation laws (2), Lemma 2.1, and the continuity of the function $t \mapsto P(u(t))$.

**Theorem 2.3.** Assume that $1 + 4/(N - 1) \leq p < 1 + 4/(N - 2)$ and $\omega \in (-1, \infty)$.
If $u_0 \in B_\omega \cap \Sigma$, then the solution $u(t)$ of (1) with $u(0) = u_0$ blows up in finite time.

**Proof.** Let $u_0 \in B_\omega \cap \Sigma$ and let $u(t)$ be the solution of (1) with $u(0) = u_0$. Then, it follows from Lemma 2.2 and Proposition 3 that $u(t) \in B_\omega \cap \Sigma$ for all $t \in [0, T_{\max})$.
Moreover, by the virial identity (13), the conservation laws (2) and Lemma 2.1, we have
\[
\frac{1}{16} \frac{d^2}{dt^2} \sum_{j=1}^{N-1} \int_{\mathbb{R}^N} x_j^2 |u(t,x)|^2 \, dx = P(u(t)) \leq S_\omega(u(t)) - d(\omega) = S_\omega(u_0) - d(\omega) < 0
\]
each $t \in [0, T_{\max})$. This implies $T_{\max} < \infty$.

Finally, we give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** First, by the elliptic regularity theory, we see that $\phi_\omega \in \Sigma$ (see, e.g., [6, Theorem 8.1.1]).

Next, since $S'_\omega(\phi_\omega) = 0$, the function
\[
(0, \infty) \ni \lambda \mapsto S_\omega(\lambda \phi_\omega) = \lambda^2 \left\{ \frac{1}{2} \| \nabla \phi_\omega \|_{L^2}^2 + \frac{1}{2} \| x_N \phi_\omega \|_{L^2}^2 + \omega \frac{1}{2} \| \phi_\omega \|_{L^2}^2 \right\} - \frac{\lambda^{p+1}}{p+1} \| \phi_\omega \|_{L^{p+1}}^{p+1}
\]
attains its maximum at $\lambda = 1$. Thus, we have
\[
S_\omega(\lambda \phi_\omega) < S_\omega(\phi_\omega) = d(\omega)
\]
for all $\lambda > 1$.
Moreover, since $\phi_\omega \in \Sigma$ and $e^{i \omega t} \phi_\omega$ is a solution of (1), it follows from Proposition 3 that $P(\phi_\omega) = 0$. Thus, we have
\[
P(\lambda \phi_\omega) = \frac{\lambda^2}{2} \sum_{j=1}^{N-1} \| \partial_j \phi_\omega \|_{L^2}^2 - \frac{\alpha \lambda^{p+1}}{2(p+1)} \| \phi_\omega \|_{L^{p+1}}^{p+1} < 0
\]
for all $\lambda > 1$.
Therefore, we see that $\lambda \phi_\omega \in B_\omega \cap \Sigma$ for all $\lambda > 1$, and it follows from Theorem 2.3 that the solution $u(t)$ of (1) with $u(0) = \lambda \phi_\omega$ blows up in finite time.
Hence, the result follows, since $\lambda \phi_\omega \to \phi_\omega$ in $X$ as $\lambda \to 1$. 

3. Proof of Proposition 2. In this section, we prove Proposition 2 by using the standard concentration compactness argument. Throughout this section, we assume that $1 < p < 1 + 4/(N - 2)$ and $\omega \in (-1, \infty)$.

We define
\[ J_\omega(v) = S_\omega(v) - \frac{1}{p+1} K_\omega(v) \]
\[ = \frac{p-1}{2(p+1)} (\|\nabla v\|_{L^2}^2 + \|x_N v\|_{L^2}^2 + \omega \|v\|_{L^2}^2). \]

Note that by (8), there exists a positive constant $C_0$ depending only on $\omega$ and $p$ such that
\[ J_\omega(v) \geq C_0 \|v\|_{X}^2, \quad v \in X. \]  

We also remark that by (15) and (6), we have
\[ d(\omega) = \inf \{ J_\omega(v) : v \in X, \ K_\omega(v) = 0, \ v \neq 0 \}. \]  

Lemma 3.1. $d(\omega) > 0$.

Proof. Let $v \in X$ satisfy $K_\omega(v) = 0$ and $v \neq 0$. Then, by $K_\omega(v) = 0$, the Sobolev inequality and (16), there exist positive constants $C_1$ and $C_2$ depending only on $N$, $p$ and $\omega$ such that
\[ J_\omega(v) = \frac{p-1}{2(p+1)} \|v\|_{L^{p+1}}^{p+1} \leq C_1 \|v\|_{H^1}^{p+1} \leq C_1 \|v\|_{X}^{p+1} \leq C_2 J_\omega(v)^{(p+1)/2}. \]

Since $v \neq 0$, we have $J_\omega(v) > 0$ and $J_\omega(v)^{(p+1)/2} \geq 1/C_2$.

Thus, by (17), we have
\[ d(\omega) \geq \frac{1}{C_2^{2/(p+1)}} > 0. \]

This completes the proof. \qed

Lemma 3.2. If $v \in X$ satisfies $K_\omega(v) < 0$, then $d(\omega) < J_\omega(v)$.

Proof. Since $K_\omega(v) < 0$ and
\[ K_\omega(\lambda v) = \lambda^2 (\|\nabla v\|_{L^2}^2 + \|x_N v\|_{L^2}^2 + \omega \|v\|_{L^2}^2) - \lambda^{p+1} \|v\|_{L^{p+1}}^{p+1} \]
for $\lambda > 0$, there exists $\lambda_0 \in (0, 1)$ such that $K_\omega(\lambda_0 v) = 0$.

Thus, by (17) and (15), we have
\[ d(\omega) \leq J_\omega(\lambda_0 v) = \lambda_0^2 J_\omega(v) < J_\omega(v). \]

This completes the proof. \qed

The following lemma is a variant of the classical result of Lieb [15] (see also [2, Lemma 3.4]).

Lemma 3.3. Assume that a sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in $X$, and satisfies
\[ \limsup_{n \to \infty} \|u_n\|_{L^{p+1}}^{p+1} > 0. \]

Then, there exist a sequence $(y^n)_{n \in \mathbb{N}}$ in $\mathbb{R}^{N-1}$ and $u \in X \setminus \{0\}$ such that $(\tau_{y^n} u_n)_{n \in \mathbb{N}}$ has a subsequence which converges to $u$ weakly in $X$. Here we define
\[ \tau_y v(x) = v(x_1 - y_1, \ldots, x_{N-1} - y_{N-1}, x_N) \]
for $x = (x_1, \ldots, x_{N-1}, x_N) \in \mathbb{R}^N$ and $y = (y_1, \ldots, y_{N-1}) \in \mathbb{R}^{N-1}$.
Proof. Without loss of generality, we may assume that
\[ C_1 := \inf_{n \in \mathbb{N}}\|u_n\|_{L^{p+1}}^{p+1} > 0. \]

Moreover, we put
\[ C_2 := \sup_{n \in \mathbb{N}}\|u_n\|_X^2, \quad C_3 := \frac{C_2 + 1}{C_1}, \]
and for \( y = (y_1, ..., y_{N-1}) \in \mathbb{Z}^{N-1} \), we define
\[ Q_y = (y_1, y_1 + 1) \times \cdots \times (y_{N-1}, y_{N-1} + 1) \times \mathbb{R} \]
\[ = \{(x_1, ..., x_{N-1}, x_N) \in \mathbb{R}^N : y_j < x_j < y_j + 1 (j = 1, ..., N - 1)\}. \]

Then, by the definition of \( C_3 \), we see that for any \( n \in \mathbb{N} \), there exists \( y^n \in \mathbb{Z}^{N-1} \) such that
\[ \|u_n\|_{X(Q_y^n)}^2 < C_4 \|u_n\|_{L^{p+1}(Q_{y^n})}^{p+1}. \]

where we put
\[ \|v\|_{X(Q_y)}^2 = \|v\|_{H^1(Q_y)}^2 + \|x_N v\|_{L^2(Q_y)}^2. \]

Here, we define \( v_n = \tau_{-y^n} u_n \). Then, we have
\[ \|v_n\|_{X(Q_0)}^2 < C_3 \|v_n\|_{L^{p+1}(Q_0)}^{p+1}. \]

for all \( n \in \mathbb{N} \). In particular, \( \|v_n\|_{L^{p+1}(Q_0)} > 0 \) for all \( n \in \mathbb{N} \).

Moreover, by the Sobolev inequality, we have
\[ C_4 \|v_n\|_{L^{p+1}(Q_0)}^{p+1} \leq \|v_n\|_{H^1(Q_0)}^2 \leq \|v_n\|_{X(Q_0)}^2 \]
for all \( n \in \mathbb{N} \), where \( C_4 \) is a positive constant depending only on \( N \) and \( p \).

Thus, we have
\[ \frac{C_4}{C_3} < \|v_n\|_{L^{p+1}(Q_0)}^{p-1}, \quad n \in \mathbb{N}. \quad (19) \]

Since \((v_n)_{n \in \mathbb{N}}\) is bounded in \( X \), there exist a subsequence \((v_{n'})\) of \((v_n)\) and \( u \in X \) such that \((v_{n'})\) converges to \( u \) weakly in \( X \).

Finally, since the embedding \( X(Q_0) \hookrightarrow L^{p+1}(Q_0) \) is compact, it follows from (19) that
\[ 0 < \frac{C_4}{C_3} \leq \|u\|_{L^{p+1}(Q_0)}^{p-1}, \]
which implies \( u \neq 0 \). This completes the proof. \( \square \)

We define the set of all minimizers for (6) by
\[ \mathcal{M}_\omega = \{ v \in X : S_\omega (v) = d(\omega), \ K_\omega (v) = 0, \ v \neq 0 \}. \]

Lemma 3.4. The set \( \mathcal{M}_\omega \) is not empty.

Proof. Let \((u_n)\) be a sequence in \( X \) such that \( K_\omega (u_n) = 0 \), \( u_n \neq 0 \) for all \( n \in \mathbb{N} \), and \( S_\omega (u_n) \to d(\omega) \).

Then, by (16) and \( J_\omega (u_n) = S_\omega (u_n) \to d(\omega) \), we see that the sequence \((u_n)_{n \in \mathbb{N}}\) is bounded in \( X \).

Moreover, it follows from \( K_\omega (u_n) = 0 \) and Lemma 3.1 that
\[ \|u_n\|_{L^{p+1}}^{p+1} = \frac{2(p+1)}{p-1} J_\omega (u_n) \to \frac{2(p+1)}{p-1} d(\omega) > 0. \]
Thus, by Lemma 3.3, there exist a sequence \((y^n)\) in \(\mathbb{R}^{N-1}\), a subsequence of \((\tau_n u_n)\), which is denoted by \((v_n)\), and \(v \in X \setminus \{0\}\) such that \((v_n)\) converges to \(v\) weakly in \(X\). By the weakly lower semicontinuity of \(J_\omega\), we have
\[
J_\omega(v) \leq \liminf_{n \to \infty} J_\omega(v_n) = d(\omega). \tag{20}
\]

Moreover, by the Brezis-Lieb Lemma (see [4]), we have
\[
K_\omega(v_n) - K_\omega(v_n - v) \to K_\omega(v),
\]
which implies \(K_\omega(v) \leq 0\).

Indeed, suppose that \(K_\omega(v) > 0\). Since \(K_\omega(v_n) = 0\), we have \(K_\omega(v_n - v) < 0\) for large \(n\). Then, by Lemma 3.2, we have \(d(\omega) < J_\omega(v_n - v)\), and
\[
J_\omega(v) = \lim_{n \to \infty} \{J_\omega(v_n) - J_\omega(v_n - v)\} \leq 0.
\]

On the other hand, by \(v \neq 0\) and (16), we have \(J_\omega(v) > 0\). This is a contradiction. Thus, we obtain \(K_\omega(v) \leq 0\).

Furthermore, by Lemma 3.2 and (20), we have \(K_\omega(v) = 0\). Since \(v \neq 0\) again, it follows from (6) and (20) that
\[
d(\omega) \leq S_\omega(v) = J_\omega(v) \leq d(\omega).
\]

Hence, we have \(S_\omega(v) = d(\omega)\) and \(v \in M_\omega\). This completes the proof. \(\square\)

**Lemma 3.5.** \(M_\omega \subset G_\omega\).

**Proof.** Let \(\phi \in M_\omega\). Then, there exists a Lagrange multiplier \(\mu \in \mathbb{R}\) such that \(S'_\omega(\phi) = \mu K'_\omega(\phi)\). Thus, we have
\[
0 = K_\omega(\phi) = \langle S'_\omega(\phi), \phi \rangle = \mu \langle K'_\omega(\phi), \phi \rangle.
\]

Here, by (18), \(K_\omega(\phi) = 0\) and \(\phi \neq 0\), we have
\[
\langle K'_\omega(\phi), \phi \rangle = \partial_\lambda K_\omega(\lambda \phi)|_{\lambda=1}
= 2 \left( \| \nabla \phi \|_{L^2}^2 + \| x_N \phi \|_{L^2}^2 + \omega \| \phi \|_{L^2}^2 \right) - (p + 1) \| \phi \|_{L^{p+1}}^{p+1}
= -(p - 1) \| \phi \|_{L^{p+1}}^{p+1} < 0.
\]

Thus, we have \(\mu = 0\) and \(S'_\omega(\phi) = 0\), which shows that \(\phi \in A_\omega\).

Moreover, for any \(v \in A_\omega\), we have \(K_\omega(v) = \langle S'_\omega(v), v \rangle = 0\) and \(v \neq 0\), so it follows from the definition (6) of \(d(\omega)\) that \(S_\omega(\phi) = d(\omega) \leq S_\omega(v)\).

Therefore, we have \(\phi \in G_\omega\), and we conclude that \(M_\omega \subset G_\omega\). \(\square\)

Finally, we give the proof of Proposition 2.

**Proof of Proposition 2.** By Lemma 3.5, it is enough to show that \(G_\omega \subset M_\omega\).

Let \(\phi \in G_\omega\). By Lemma 3.4, we can take an element \(v \in M_\omega\). Then, by Lemma 3.5, we have \(v \in G_\omega \subset A_\omega\).

Moreover, since \(\phi \in G_\omega\), by the definition (4) of \(G_\omega\), we have
\[
S_\omega(\phi) \leq S_\omega(v) = d(\omega).
\]

On the other hand, since \(\phi\) satisfies \(K_\omega(\phi) = 0\) and \(\phi \neq 0\), by the definition (6) of \(d(\omega)\), we have \(d(\omega) \leq S_\omega(\phi)\).

Hence, we conclude that \(S_\omega(\phi) = d(\omega)\) and \(\phi \in M_\omega\). This completes the proof. \(\square\)
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E-mail address: mohta@rs.tus.ac.jp