Blowup solutions of Grushin’s operator

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Abstract

In this note, we consider the blowup phenomenon of Grushin’s operator. By using the
knowledge of probability, we first get expression of heat kernel of Grushin’s operator. Then by
using the properties of heat kernel and suitable auxiliary function, we get that the solutions will
blow up in finite time.

Keywords: Grushin’s operator; Heat kernel; Blowup.

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1 Introduction

The finite time blowup phenomenon has been studied by many authors, see the book \cite{6}. There
are two cases to study this problem. One is bounded domain and the other is whole space. In this
paper, we only consider the problem in the whole space. For the whole space, the following ”Fujita
Phenomenon” has been attraction in the literature. Consider the following Cauchy problem

\begin{equation}
\begin{cases}
u_t = \Delta u + u^p, & x \in \mathbb{R}^d, \ t > 0, \ p > 0, \\
u(0, x) = u_0(x), & x \in \mathbb{R}^d.
\end{cases}
\end{equation}

It has been proved that:

(i) if \(0 < p < 1\), then every nonnegative solution is global, but not necessarily unique;

(ii) if \(1 < p \leq 1 + \frac{2}{d}\), then any nontrivial, nonnegative solution blows up in finite time;

(iii) if \(p > 1 + \frac{2}{d}\), then \(u_0 \in \mathcal{U}\) implies that \(u(t, x, u_0)\) exists globally;

(iv) if \(p > 1 + \frac{2}{d}\), then \(u_0 \in \mathcal{U}_\infty\) implies that \(u(t, x, u_0)\) blows up in finite time,

where \(\mathcal{U}\) and \(\mathcal{U}_\infty\) are defined as follows

\begin{equation*}
\mathcal{U} = \left\{ v(x) | v(x) \in BC(\mathbb{R}^d, \mathbb{R}_+) , v(x) \leq \delta e^{-k|x|^2}, \ k > 0, \delta = \delta(k) > 0 \right\},
\end{equation*}

\begin{equation*}
\mathcal{U}_\infty = \left\{ v(x) | v(x) \in BC(\mathbb{R}^d, \mathbb{R}_+) , v(x) \geq ce^{-k|x|^2}, \ k > 0, c \gg 1 \right\}.
\end{equation*}
Here $BC = \{\text{bounded and uniformly continuous functions}\}$, see Fujita [4, 5] and Hayakawa [7]. The proof of case (i)-(iii) relies on the properties of heat kernel and suitable auxiliary function. Comparison principle is the main tool to prove case (iv). In this note, we consider the degenerate parabolic operator–Grushin’s operator. We will consider the first three cases.

There are a lot of known results about the blowup phenomenon of parabolic equations. Blowup phenomenon of quasilinear parabolic equations with Robin boundary condition was considered by Enache [2], also see [3, 9]. Then the blowup phenomena of degenerate parabolic and nonlocal diffusion equations were considered by [8, 10, 11, 12, 13]. Seki [14] obtained the type II blowup mechanisms. Zhang-Wang [15] considered the blowup phenomenon of 3-D primitive equations of oceanic and atmospheric dynamics.

In this note, we consider a special degenerate parabolic operator–Grushin’s operator. Fortunately, we can obtain expression of Grushin’s operator. In next section, some preliminaries are given and the main results will be proved in section 3. Throughout this paper, we write $C$ as a general positive constant and $C_i, i = 1, 2, \cdots$ as a concrete positive constant.

2 Main results

Consider the Grushin’s operator

$$L = \frac{1}{2} \left( \partial_{x_1}^2 + x_2^2 \partial_{x_2}^2 \right),$$

which is the generator of the diffusion process $(X^1_t, X^2_t)$, where $(X^1_t, X^2_t)$ satisfies

$$\begin{cases}
    dX^1_t = dW^1_t,
    \\
    dX^2_t = X_t dW^2_t,
    \\
    X^1_0 = \mu_1, \quad X^2_0 = \mu_2.
\end{cases}$$

Here $W^i_t$ is a standard Brownian motion, $i = 1, 2$. It is easy to see that the process $(X^1_t, X^2_t)$ is a Gaussian stochastic process. Direct calculations show that

$$\begin{aligned}
    E \left( \begin{array}{c}
        X^1_t \\
        X^2_t
    \end{array} \right) &= \left( \begin{array}{c}
        \mu_1 \\
        \mu_2
    \end{array} \right), \\
    Cov(X^1_t, X^2_t) &= \left( \begin{array}{cc}
        t & \mu_1 t \\
        \mu_1 t & \frac{\mu_1^2 t}{2} + \frac{t^2}{2}
    \end{array} \right).
\end{aligned}$$

Therefore, we get the heat kernel of the operator $L$ is

$$K(t, x_1, t, x_2, \mu_2) = \frac{1}{2\pi t^{3/2}} \exp \left\{ -\frac{(x_1 - \mu_1)^2}{t} - \frac{|\mu_1(x_1 - \mu_1) - x_2 + \mu_2|^2}{t^2} \right\},$$

which yields that

$$\nabla_{x_1} K(t, x_1, x_2) = -\frac{2x_1}{t} K(t, x_1, x_2), \quad \nabla_{x_2} K(t, x_1, x_2) = -\frac{y}{t^2} K(t, x_1, x_2).$$

It is easy to see that for classical heat kernel, we have $x \sim \sqrt{t}$. But in our case, different axis has different scaling, that is,

$$x_1 \sim \sqrt{t}, \quad x_2 \sim t.$$

Now, we consider the following degenerate parabolic equation

$$\begin{cases}
    u_t = Lu + u^p, & x \in \mathbb{R}^2, \quad t > 0, \quad p > 0, \\
    u(0, x) = u_0(x) \geq 0, & x \in \mathbb{R}^2.
\end{cases} \quad (2.1)$$

The main results is as followings.
Theorem 2.1 Assume that $u_0$ is a bounded continuous non-negative function.

(i) If $0 < p < 1$, then the solution of (2.1) exists globally.

(ii) If $1 < p \leq 1 + \frac{2}{d}$, then all nontrivial solutions of (2.1) blow up in finite time. That is to say, there exists a positive $T^* > 0$ such that

$$u(t, x) = \infty, \quad t > T^*.$$

(iii) If $p > 1$, then the solution of (2.1) blows up in finite time provided the initial datum satisfies

$$\inf_{x \in \mathbb{R}^2} u_0(x) \geq \mu > 0,$$

where $\mu$ is a constant.

Remark 2.1 Comparing with the classical parabolic equation, that is to say, comparing [6, Theorem 5.5] with the above theorem 2.1, we find the value of $p$ is different. More precisely, it follows [6, Theorem 5.5] that when $1 < p \leq 2$ ($d$ is the dimension of space), the solutions of (2.1) with $L$ replaced by $\Delta$ will blow up in finite time under the condition that the initial data $u_0 \geq 0 (\neq 0)$ is bounded continuous function. However, in the case of (2.1), the index is $1 < p \leq 1 + \frac{2}{d}$.

The assumption of (iii) is too strict, one can weaken the assumption.

3 Proof of main results

Proof of Theorem 2.1 The solution of (2.1) can be expressed as

$$u(t, x) = K * u_0(x) + \int_0^t \int_{\mathbb{R}^2} K(t-s, x-y) u^p(s, y) dy.$$

Due to the positivity of heat kernel, it is easy to see that if the initial data is non-negative, then the solution will keep positive. The equality (3.1) yields that for any $T > 0$ and $t \in [0, T]$,

$$u(t, x) \leq \sup_{x \in \mathbb{R}^2} |u_0(x)| + \left[ \sup_{x \in \mathbb{R}^2} |u(t, x)| \right]^p,$$

where we used the properties of heat kernel. Hence we have for any $T > 0$ and $t \in [0, T]$,

$$\sup_{x, y \in \mathbb{R}} |u(t, x, y)| \leq C(T),$$

which implies the result of (i). Denote $u(t, x) = I_1(t, x) + I_2(t, x)$ and

$$I_1(t, x) = \int_{\mathbb{R}^2} K(t, x-y) u_0(y) dy, \quad I_2(t, x) = \int_0^t \int_{\mathbb{R}^2} K(t-s, x-y) u^p(s, y) dyds.$$

We may assume without loss of generality that $u_0(y) \geq C_1 > 0$ for $|y| < 1$ by the assumption. A direct computation shows that

$$I_1(t, x) \geq \frac{C_1}{t^{3/2}} \int_{B_1(0)} \exp \left( -\frac{2x_1^2 + 2y_1^2}{t} - \frac{2x_2^2 + 2y_2^2}{t^2} \right) dy$$

$$\geq \frac{C_1}{t^{3/2}} \exp \left( -\frac{2x_1^2}{t} - \frac{2x_2^2}{t^2} \right) \int_{|y| \leq \frac{1}{t}} \exp \left( -2y_1^2 - \frac{2y_2^2}{t} \right) dy$$

$$\geq \frac{C_1}{t^{3/2}} \exp \left( -\frac{2x_1^2}{t} - \frac{2x_2^2}{t^2} \right)$$

for $t > 1$ and $C_1 > 0$. 
It is easy to see that
\[
I_2(t, x) \geq \int_0^t \left( \int_{\mathbb{R}^2} K(t - s, x - y) u(s, y) dy \right)^p ds.
\]

Let
\[
G(t) = \int_{\mathbb{R}^2} K(t, x) u(t, x) dx.
\]

Then for \( t > 1 \),
\[
G(t) = \int_{\mathbb{R}^2} I_1(t, x) K(t, x) dx + \int_{\mathbb{R}^2} I_2(t, x) K(t, x) dx
\]
\[
\geq \frac{C_2}{t^{3/2}} + \int_0^t \int_{\mathbb{R}^2} K(t, x) \left( \int_{\mathbb{R}^2} K(t - s, x - y) u(s, y) dy \right)^p ds.
\]

It is clear that
\[
\int_{\mathbb{R}^2} K(t, x) K(t - s, x - y) dx
\]
\[
= \frac{1}{2\pi^{3/2}(t-s)^{3/2}} \int_{\mathbb{R}^2} \exp \left( -\frac{t|x_1|^2 + |x_2|^2}{t^2} - \frac{(t-s)|x_1 - y_1|^2 + |x_2 - y_2|^2}{(t-s)^2} \right) dx
\]
\[
= \frac{K(s, y)}{2\pi s^{3/2}} \times \frac{1}{4\pi^{3/2} s^{3/2}(t-s)^{3/2}} \int_{\mathbb{R}^2} \exp \left( \frac{s|x_1|^2 + |x_2|^2}{s^2} - \frac{t|x_1|^2 + |x_2|^2}{t^2} - \frac{(t-s)|x_1 - y_1|^2 + |x_2 - y_2|^2}{(t-s)^2} \right) dx.
\]

Since
\[
\frac{|y_1|^2}{s} - \frac{|x_1|^2}{t} - \frac{|x_1 - y_1|^2}{t-s} \geq \frac{|y_1|^2}{s} - \frac{|x_1 - y_1|^2 + |y_1|^2}{s} - \frac{(t-s)|y_1|^2}{t-s} - \frac{|x_1 - y_1|^2}{t-s}
\]
\[
= \frac{1}{t} \left( -2|x_1 - y_1||y_1| + \frac{t-s}{s} |y_1|^2 \right) - \frac{|x_1 - y_1|^2}{t-s}
\]
\[
\geq \frac{2|x_1 - y_1|^2}{t-s} \quad \text{for } 0 < s < t,
\]
and
\[
\frac{|y_2|^2}{s^2} - \frac{|x_2|^2}{t^2} - \frac{|x_2 - y_2|^2}{(t-s)^2} \geq \frac{|y_2|^2}{s^2} - \frac{|x_2 - y_2|^2 + |y_2|^2}{t^2} - \frac{2|x_2 - y_2||y_2|}{(t-s)^2}
\]
\[
= \frac{1}{t^2} \left( -2|x_2 - y_2||y_2| + \frac{t^2 - s^2}{s^2} |y_2|^2 \right) - \frac{|x_2 - y_2|^2}{(t-s)^2}
\]
\[
\geq -\frac{2|x_2 - y_2|^2}{(t-s)^2} \quad \text{for } 0 < s < t,
\]

for some constants \( C_1, C_2 \).
we get for $0 < s < t$
\[
\int_{\mathbb{R}^2} \exp \left( \frac{s|y_1|^2 + |y_2|^2}{s^2} - \frac{t|x_1|^2 + |x_2|^2}{t^2} - \frac{(t-s)|x_1 - y_1|^2 + |x_2 - y_2|^2}{(t-s)^2} \right) dx 
\geq \int_{\mathbb{R}^2} \exp \left( \frac{-2|x_1 - y_1|^2}{t-s} - \frac{2|x_2 - y_2|^2}{(t-s)^2} \right) dx 
= C_3(t-s)^{3/2}.
\]
Substituting the estimate into (3.3) and applying Jensen’s inequality, we obtain
\[
G(t) \geq \frac{C_2}{t^{3/2}} + C_3 \int_0^t \left( \frac{s^{3/2}}{t^{3/2}} \right)^p G^p(s) ds \quad \text{for } t > 1.
\]
We can rewrite the above inequality as
\[
t^{3p/2}G(t) \geq C_2 t^{3(p-1)/2} + C_3 \int_0^t s^{3p/2} G^p(s) ds =: g(t).
\] (3.4)
Then for $t > 1$, we have
\[
g(t) \geq C_2 e^{3(p-1)/2},
\]
\[
g'(t) \geq C_3 t^{3p/2} G(t) \geq C_3 t^{3p/2} \left( \frac{1}{t^{3p/2}} g(t) \right)^p = C_3 t^{3p(1-p)/2} g^p(t),
\]
which implies
\[
\frac{C_2^{1-p}}{p-1} t^{-3(p-1)/2} \geq \frac{1}{p-1} g^{1-p}(t) \geq C_3 \int_t^T s^{3p(1-p)/2} ds \quad \text{for } T > t \geq 1.
\]
If $p \leq 1 + \frac{2}{3p}$, the right-hand side of the above inequality is unbounded as $T \to \infty$, which gives a contradiction in this case. In the case $1 + \frac{2}{3p} < p < \frac{5}{3}$, we have $3(p-1)/2 > -1 + 3p(p-1)/2$, thus we get a contradiction by letting $T \to \infty$ and then taking $t \gg 1$.

In the case $p = \frac{5}{3}$, we derive from (3.2), for $t > 1$,
\[
u^p(t, x) \geq I_1^p(t, x) \geq \frac{C_1}{t^{3p/2}} \exp \left( -\frac{2px_1^2}{t} - \frac{2px_2^2}{t^2} \right).
\]
Substituting this estimate into the expression of $I_2$, we obtain, for $t > 2$,
\[
u(t, x) \geq I_2(t, x)
\]
\[
\geq \int_1^t \int_{\mathbb{R}^2} K(t-s, x-y) \frac{C_2}{s^{3p/2}} \exp \left( -\frac{2py_1^2}{s} - \frac{2py_2^2}{s^2} \right) dyds
\]
\[
\geq \frac{C_4}{t^{3/2}} \exp \left( -\frac{t|x_1|^2 + |x_2|^2}{t^2} \right) \int_1^{t/2} \frac{1}{s^{3/2}} \frac{s^{3/2}}{(t-s)^{3/2}} ds
\]
\[
\times \int_{\mathbb{R}^2} \exp \left( \frac{t|x_1|^2 + |x_2|^2}{t^2} - \frac{(t-s)(|x_1|^2 + |y_1|^2) + |x_2|^2 + |y_2|^2 - 2ps|y_1|^2 + 2p|y_2|^2}{s^2} \right) dy
\]
\[
\geq \frac{C_4}{t^{3/2}} \exp \left( -\frac{t|x_1|^2 + |x_2|^2}{t^2} \right) \int_1^{t/2} \frac{1}{s^{3/2}} \frac{t^{3/2}}{(t-s)^{3/2}} ds
\]
\[
\times \int_{\mathbb{R}^2} \exp \left( \frac{t|x_1|^2 + |x_2|^2}{t^2} - \frac{t(|x_1|^2 + |y_1|^2) + |x_2|^2 + |y_2|^2 - 2ps|y_1|^2 + 2p|y_2|^2}{s^2} \right) dy
\]
\[
\geq \frac{C_5}{t^{3/2}} \exp \left( -\frac{t|x_1|^2 + |x_2|^2}{t^2} \right) \int_1^{t/2} \frac{ds}{s}
\]
\[
= \frac{C_5}{t^{3/2}} \exp \left( -\frac{t|x_1|^2 + |x_2|^2}{t^2} \right) \log(t/2).
\]
Therefore, for \( t > 2 \), we have

\[
G(t) \geq \int_{\mathbb{R}} K(t, x) \frac{C_6}{t^{3/2}} \exp \left( -\frac{t|x_1|^2 + |x_2|^2}{t^2} \right) \log(t/2) dx \geq \frac{C_6}{t^{3/2}} \log(t).
\]

Using the above estimate, we obtain from (3.1), for \( t > 2 \),

\[
t^{3p/2}G(t) = \frac{1}{2} t^{3p/2}G(t) + \frac{1}{2} t^{3p/2}G(t) \geq C_7 t \log(t) + \frac{C_3}{2} \int_0^t s^{3p/2}G(s)^p ds.
\]

Denoting the right-hand side of the above inequality by \( g(t) \), we have

\[
g(t) \geq C_7 t \log(t)
\]

\[
g'(t) \geq C_7 \frac{3}{2} t^{3p/2}G(t)^p \geq C_7 \frac{3}{2} t^{3p/(1-p)}/2 g(t)^p,
\]

which implies that

\[
\frac{3}{2} C_7 \frac{2}{3} t^{3p} = \frac{3}{2} g^{-\frac{1}{p}}(t) \geq C_4 \int_T^t s^{-5/3} ds = C_4 t^{-\frac{2}{3}} - T^{-\frac{2}{3}}.
\]

Letting \( T \to \infty \) and \( t \gg 1 \), we get a contradiction. The proof of (ii) is complete.

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