Euler characteristics, aspherical manifolds and
the Zimmer program

Shengkui Ye

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Abstract

Let $\text{SL}_n(\mathbb{Z})$ ($n \geq 3$) be the special linear group and $M^r$ be a closed manifold. When the Euler characteristic $\chi(M) \not\equiv 0 \mod 3$ (mod 6, resp.), it is proved that any group action of $\text{SL}_n(\mathbb{Z})$ ($n \geq r + 2$) on $M^r$ by diffeomorphisms is trivial (finite, resp.). When the manifold $M$ is aspherical, it is proved that when $r < n$, a group action of $\text{SL}_n(\mathbb{Z})$ on $M^r$ by homeomorphisms is trivial if and only if the induced group homomorphism $\text{SL}_n(\mathbb{Z}) \to \text{Out}(\pi_1(M))$ is trivial. For (almost) flat manifolds, we prove a similar result in terms of holonomy groups. Especially, when $\pi_1(M)$ is nilpotent, the group $\text{SL}_n(\mathbb{Z})$ cannot act nontrivially on $M$ when $r < n$. This confirms a conjecture related to Zimmer’s program for these manifolds.

1 Introduction

Let $\text{SL}_n(\mathbb{Z})$ be the special linear group over integers. There is an action of $\text{SL}_n(\mathbb{Z})$ on the sphere $S^{n-1}$ induced by the linear action on the Euclidean space $\mathbb{R}^n$. It is believed that this action is minimal in the following sense.

Conjecture 1.1 Any group action of $\text{SL}_n(\mathbb{Z})$ ($n \geq 3$) on a compact smooth connected $r$-manifold by diffeomorphisms factors through a finite group action if $r < n - 1$.

This conjecture was formulated by Farb and Shalen [12], which is related to the Zimmer program concerning group actions of lattices in Lie groups on manifolds (see the survey articles [13, 36] for more details). When $r = 1$, Conjecture 1.1 is already proved by Witte [31]. Weinberger [32] confirms the conjecture when $M = T^r$ is a torus. Bridson and Vogtmann [4] confirm the conjecture when $M = S^r$ is a sphere. It seems that very few other cases have been confirmed (for group actions preserving extra structures, many results have been obtained, cf. [13, 36]).

In this article, we obtain a general result as following.
Theorem 1.2 Let \( M^r \) be a closed smooth connected manifold with Euler characteristic \( \chi(M) \not\equiv 0 \mod 3 \). Then any group action of \( \text{SL}_n(\mathbb{Z}) \) \((n > r + 1)\) on \( M^r \) by diffeomorphisms is trivial.

Remark 1.3 (i) The bound of \( n \) couldn’t be improved, since \( \text{SL}_n(\mathbb{Z}) \) acts non-trivially on \( S^{n-1} \).

(ii) Belolipetsky and Lubotzky [2] prove that for any finite group \( G \) and any dimension \( r \geq 2 \), there exists a hyperbolic manifold \( M \) such that the isometric group \( \text{Isom}(M) \cong G \). Therefore, the group \( \text{SL}_n(\mathbb{Z}) \) could act non-trivially through a finite quotient group on such a hyperbolic manifold. This implies that the condition of Euler characteristic couldn’t be dropped.

(iii) The dimension \( r \) of a manifold \( M \) with nonvanishing Euler characteristic is necessarily even. Let \( \{g_i\} \) be a sequence of nonnegative integers with \( g_i \not\equiv 1 \mod 3 \) and \( \Sigma_{g_i} \), a closed orientable surface of genus \( g_i \). For any even number \( r \),

\[
M^r = \Sigma_{g_1} \times \Sigma_{g_2} \times \cdots \times \Sigma_{g_r}
\]

has nonzero (modulo 3) Euler characteristic and thus satisfies the condition of Theorem 1.2.

Our next result confirms Conjecture 1.1 for manifolds with nonvanishing Euler characteristic modulo 6.

Theorem 1.4 Let \( M^r \) be a closed smooth connected manifold with Euler characteristic \( \chi(M) \not\equiv 0 \mod 6 \). Then any group action of \( \text{SL}_n(\mathbb{Z}) \) \((n > r + 1)\) on \( M^r \) by diffeomorphisms factors through a finite group action.

When \( r = 2 \), there are many progress on Conjecture 1.1. For example, when the group action is smooth real-analytic and \( M^r \) is a compact surface other than the torus or Klein bottle, Farb and Shalen [12] prove that Conjecture 1.1 is true for \( n \geq 5 \) (more generally for 2-big lattices). When the group action is smooth real-analytic and volume-preserving, they also show that this result could be extended to all compact surfaces. Polterovich (see Corollary 1.1.D of [20]) proves that if \( n \geq 3 \), then any action of \( \text{SL}_n(\mathbb{Z}) \) on a closed surface other than the sphere \( S^2 \) and the torus \( T^2 \) by area preserving diffeomorphisms factors through a finite group action. When \( r = 2 \) and the group action is by area preserving diffeomorphisms, Franks and Handel [14] prove that Conjecture 1.1 is true for an almost simple group containing a subgroup isomorphic to the three-dimensional integer Heisenberg group (eg. any finite-index subgroup of \( \text{SL}_n(\mathbb{Z}) \) for \( n \geq 3 \)). The conjecture is still open for general smooth actions on 2-dimensional manifolds. In the special case of \( r = 2 \), we improve Theorem 1.4 as the following.

Corollary 1.5 Let \( \Sigma_h \) be a closed orientable surface with genus \( h \not\equiv 1 \mod 6 \). Then any group action of \( \text{SL}_n(\mathbb{Z}) \) \((n \geq 4)\) on \( \Sigma_h \) by homeomorphisms factors through a finite group action.
Recall that a manifold $M$ is aspherical if the universal cover $\tilde{M}$ is contractible. For a group $G$, denote by $\text{Out}(G)$ the outer automorphism group. Our next result is the following.

**Theorem 1.6** Let $M^r$ be an aspherical manifold. A group action of $\text{SL}_n(\mathbb{Z})$ ($n \geq 3$) on $M^r$ ($r \leq n - 1$) by homeomorphisms is trivial if and only if the induced group homomorphism $\text{SL}_n(\mathbb{Z}) \to \text{Out}(\pi_1(M))$ is trivial. In particular, Conjecture 1.1 holds if the set of group homomorphisms

$$\text{Hom}(\text{SL}_n(\mathbb{Z}), \text{Out}(\pi_1(M))) = 1.$$ 

An obvious application is the following.

**Corollary 1.7** Any group action of $\text{SL}_n(\mathbb{Z})$ ($n \geq 3$) on an aspherical manifold $M^r$ ($r \leq n - 1$) by homotopic-identity homeomorphisms is trivial.

For aspherical manifolds with finitely generated nilpotent fundamental groups (e.g., Nil-manifolds), we confirm Conjecture 1.1 as follows.

**Theorem 1.8** Let $M^r$ be an aspherical manifold. If the fundamental group $\pi_1(M)$ is finitely generated nilpotent, any group action of $\text{SL}_n(\mathbb{Z})$ ($n \geq 3$) on $M^r$ ($r \leq n - 1$) by homeomorphisms is trivial.

When $M^r = T^r$ is a torus, Theorem 1.8 recovers the results obtained by Witte [31] and Weinberger [32], since $\text{Hom}(\text{SL}_n(\mathbb{Z}), \text{Out}(\pi_1(T^r))) = 1$ (cf. Corollary 3.6).

We now study group actions on (almost) flat manifolds. Recall that a closed manifold $M$ is almost flat if for any $\varepsilon > 0$ there is a Riemannian metric $g_\varepsilon$ on $M$ such that $\text{diam}(M, g_\varepsilon) < 1$ and $g_\varepsilon$ is $\varepsilon$-flat.

**Theorem 1.9** Let $M^r$ be a closed almost flat manifold with holonomy group $\Phi$. A group action of $\text{SL}_n(\mathbb{Z})$ ($n \geq 3$) on $M^r$ ($r \leq n - 1$) by homeomorphisms is trivial if and only if the induced group homomorphism $\text{SL}_n(\mathbb{Z}) \to \text{Out}(\Phi)$ is trivial. In particular, Conjecture 1.1 holds if the set of group homomorphisms

$$\text{Hom}(\text{SL}_n(\mathbb{Z}), \text{Out}(\Phi)) = 1.$$ 

Surprisingly, the proof of Theorem 1.9 will use knowledge on algebraic $K$-theory (Steinberg groups and $K_2(\mathbb{Z})$, especially).

In order to confirm Conjecture 1.1 it’s enough to show that every group homomorphism from $\text{SL}_n(\mathbb{Z})$ to the outer automorphism group of the fundamental (or holonomy) group is trivial, by Theorem 1.6 and Theorem 1.9. Actually, Conjecture 1.1 could be confirmed in this way for many other manifolds in addition to manifolds with nilpotent fundamental groups proved in Theorem 1.8. These includes the following:

- Flat manifolds with abelian holonomy group (see Corollary 7.4 for a more general result):
Almost flat manifolds with dihedral, symmetric or alternating holonomy group (cf. Lemma 7.5);

- Flat manifold of dimension $r \leq 5$ (cf. Corollary 7.6).

The article is organized as follows. In Section 2, we study Euler characteristics of manifolds with effective finite group actions and give proofs of Theorems 1.2, 1.4 and 1.5. In Section 3, we study the group action of Steinberg group on spheres and acyclic manifolds. Theorem 1.6 is proved in Section 4. In Section 5 and Section 6, we study group action on flat manifolds and Theorem 1.9 is proved. In the last section, we give some applications to flat manifolds with special holonomy groups.

2 Finite group actions and Euler characteristic

2.1 Homology manifolds

The generalized manifolds studied in this article are following Bredon’s book [6]. Let $L = \mathbb{Z}$ or $\mathbb{Z}/p$. All homology groups in this section are Borel-Moore homology with compact supports and coefficients in a sheaf $\mathcal{A}$ of modules over a principle ideal domain $L$. The homology groups of $X$ are denoted by $H^c_\ast(X; \mathcal{A})$ and the Alexander-Spanier cohomology groups (with coefficients in $\mathbb{Z}/p$ and compact supports) are denoted by $H^c_\ast(X; \mathbb{Z}/p)$.

We define $\dim_L X = \min\{ n \mid H^{n+1}_c(U; \mathbb{Z}/p) = 0 \text{ for all open } U \subset X \}$. If $L = \mathbb{Z}/p$, we write $\dim_p X$ for $\dim_L X$. For integer $k \geq 0$, let $\mathcal{O}_k$ denote the sheaf associated to the presheaf $U \mapsto H^c_k(X, X \setminus U; \mathbb{Z}/p)$.

Definition 2.1 An $n$-dimensional homology manifold over $L$ (denoted $n$-hm$_L$) is a locally compact Hausdorff space $X$ with $\dim_L X < +\infty$, and $\mathcal{O}_n(X; \mathbb{Z}/p) = 0$ for $p \neq n$ and $\mathcal{O}_n(X; \mathbb{Z}/p)$ is locally constant with stalks isomorphic to $\mathbb{Z}/p$. The sheaf $\mathcal{O}_n$ is called the orientation sheaf.

There is a similar notion of cohomology manifold over $L$, denoted $n$-cm$_L$ (cf. [6], p.373). For a prime $p$, denote by $\dim_p X$ the cohomological dimension of $X$.

Definition 2.2 If $X$ is a compact $m$-hm$_L$ and $H^c_\ast(X; \mathbb{Z}/p) \cong H^c_\ast(S^m; \mathbb{Z}/p)$, then $X$ is called a generalized $m$-sphere over $L$. Similarly, if $H^c_0(X; \mathbb{Z}/p) = L$ and $H^c_k(X; L) = 0$ for $k > 0$, then $X$ is said to be $L$-acyclic.

We will need the following lemmas. The first is a combination of Corollary 19.8 and Corollary 19.9 (page 144) in [6] (see also Theorem 4.5 in [4]).

Lemma 2.3 Let $p$ be a prime and $X$ be a locally compact Hausdorff space of finite dimension over $\mathbb{Z}/p$. Suppose that $\mathbb{Z}/p$ acts on $X$ with fixed-point set $F$.

(i) If $H^c_\ast(X; \mathbb{Z}/p) \cong H^c_\ast(S^m; \mathbb{Z}/p)$, then $H^c_\ast(F; \mathbb{Z}/p) \cong H^c_\ast(S^r; \mathbb{Z}/p)$ for some $r$ with $-1 \leq r \leq m$. If $p$ is odd, then $r - m$ is even.
(ii) If $X$ is $\mathbb{Z}_p$-acyclic, then $F$ is $\mathbb{Z}_p$-acyclic (in particular nonempty and connected).

The following is a relation between dimensions of fixed point set and the whole space (cf. Borel \[8\], Theorem 4.3, p.182).

**Lemma 2.4** Let $G$ be an elementary $p$-group operating on a first countable cohomology $n$-manifold $X$ mod $p$. Let $x \in X$ be a fixed point of $G$ on $X$ and let $n(H)$ be the cohomology dimension mod $p$ of the component of $x$ in the fixed point set of a subgroup $H$ of $G$. If $r = n(G)$, we have

$$n - r = \sum_H (n(H) - r)$$

where $H$ runs through the subgroups of $G$ of index $p$.

The following lemma is proved by Bredon [5] (Theorem 7.1).

**Lemma 2.5** Let $G$ be a group of order 2 operating effectively on an $n$-cm over $\mathbb{Z}$, with nonempty fixed points. Let $F_0$ be a connected component of the fixed point set of $G$, and $r = \dim_2 F_0$. Then $n - r$ is even (respectively odd) if and only if $G$ preserves (respectively reverses) the local orientation around some point of $F_0$.

### 2.2 Proofs of Theorem [1.2] and Theorem [1.4]

**Lemma 2.6** (Mann and Su [23], Theorem 2.2) Let $G$ be an elementary $p$-group of rank $k$ operating effectively on a first countable connected cohomology $r$-manifold $X$ mod $p$. Suppose $\dim_p F(G) = r_0 \geq 0$ where $F(G)$ is the fixed point set of $G$ on $X$. Then $k \leq \frac{r - r_0}{2}$ if $p \neq 2$ and $k \leq r - r_0$ if $p = 2$.

The following lemma generalizes Lemma 2.6 to the case of orientation-preserving actions.

**Lemma 2.7** Let $G$ be an elementary 2-group of rank $k$ operating effectively on a first countable connected cohomology $r$-manifold $X$ over $\mathbb{Z}$ by orientation-preserving homeomorphisms. Suppose $\dim_p F(G) = r_0 \geq 0$ where $F(G)$ is the fixed point set of $G$ on $X$. Then $k \leq r - 1 - r_0$.

**Proof.** If there is an element $g \in G$ such that the dimension of the fixed point set $\text{Fix}(g)$ is $r$, the element $g$ acts trivially by invariance of domain (cf. \[4\], Cor. 16.19). This is a contradiction to the assumption that $G$ acts effectively. Therefore, we could assume that $\text{Fix}(g)$ is of non-trivial even codimension (cf. \[8\], Theorem 2.5, p.79.). The lemma is obvious if $r = 1$. When $r = 2$, the dimension of $\text{Fix}(H)$ is zero for any nontrivial subgroup $H < G$. This is impossible by Borel’s formula in Lemma 2.4. Choose a nontrivial element $g \in G$ and fix a connected component $M$ of $\text{Fix}(g)$ containing a connected component of $F(G)$ with largest dimension. The complement $G_0$ of $\langle g \rangle$ in $G$ acts invariantly on $M$. 5
Since \( \text{Fix}(g) \) is dimension at most \( r - 2 \), the rank of \( G_0 \) is at most \( r - 2 - r_0 \) by Lemma 2.6. Therefore,

\[
k = \text{rank}(G_0) + 1 \leq r - 1 - r_0.
\]

The inequality in Lemma 2.7 is sharp, by considering the linear action of the diagonal subgroup \((\mathbb{Z}/2)^{n-1} \ltimes \text{SL}_n(\mathbb{Z})\) on \( \mathbb{R}^n \).

Let \( G \) be a finite group acting on a closed smooth manifold \( M \) by diffeomorphisms. The manifold \( M \) has an equivariant smooth triangulation (cf. [18]) and \( p : M \rightarrow M/G \) is a covering of orbifolds (cf. [29]). Denote by \( \chi(M) \) the Euler characteristic class.

**Lemma 2.8 (Thurston [29], Prop. 13.3.4)** Let \( G \) be a finite group acting effectively on a closed smooth connected manifold \( M \) by diffeomorphisms. We have

\[
\chi(M) = |G| \sum_{s_i} (-1)^{\dim s_i} \frac{1}{|G_{s_i}|}.
\]

Here \( s_i \) ranges over cells of \( X/G \) and \( G_{s_i} \) is the stabilizer of a point in \( p^{-1}(s_i) \).

For a finite group \( G \) and prime \( p \), denote by \( G_p \) the Sylow \( p \)-subgroup and \( p^n_p = |G_p| \) the cardinality, and \( p^e_p \) the exponent of elements in \( G_p \).

**Proposition 2.9** Let \( G \) be a finite abelian group acting on a closed smooth manifold \( M \) effectively. Then we have

\[
p^{n_p-\lfloor \frac{r}{2} \rfloor e_p} | \chi(M),
\]

when \( p \neq 2 \) (where \( \lfloor \frac{r}{2} \rfloor \) is the integer part) and

\[
2^{n_2-r e_2} | \chi(M)
\]

when \( p = 2 \). If the group action is by orientation-preserving, we have

\[
2^{n_2-r e_2+e_2} | \chi(M).
\]

**Proof.** Without loss of generality, we suppose \( G = G_p \). By Lemma 2.8, we have

\[
\chi(X) = |G| \sum_{s_i} (-1)^{\dim s_i} \frac{1}{|G_{s_i}|} = \sum_{s_i} (-1)^{\dim s_i} |G/G_{s_i}|.
\]

By Lemma 2.6

\[
|G_{s_i}| \leq p^{n_p e_p} \text{ if } p \neq 2
\]

and

\[
|G_{s_i}| \leq p^{e_p} \text{ if } p = 2.
\]
Thus
\[ |G/G_s| \geq p^{n_p - \lfloor \frac{r}{2} \rfloor} r_p \text{ if } p \neq 2 \]
and
\[ |G/G_s| \geq p^{n_p - r_p} \text{ if } p = 2. \]
Therefore, we have
\[ p^{n_p - \lfloor \frac{r}{2} \rfloor} \mid \chi(X) \text{ if } p \neq 2 \]
and
\[ 2^{n_2 - r_2} \mid \chi(M). \]
If the group action is orientation-preserving, a similar argument using Lemma 2.7 shows that
\[ 2^{n_2 - r_2 + e_2} \mid \chi(M). \]

\[ \blacksquare \]

**Remark 2.10** When \( M \) is a surface, Proposition 1.3 is already known to Kulakarni [20].

Theorem 1.2 is a special case of the following result when \( k = 2 \).

**Theorem 2.11** Assume that \( r \) and \( k \geq 2 \) are positive integers. Let \( M^r \) be a closed smooth connected manifold with Euler characteristic \( \chi(M) \not\equiv 0 \mod 3^{\lfloor \frac{k}{2} \rfloor} \). Then any group action of \( SL_n(\mathbb{Z}) \) \((n \geq r + k)\) on \( M^r \) by diffeomorphisms is trivial.

**Proof.** Let \( f : SL_n(\mathbb{Z}) \to Diff(M) \) be a group homomorphism. Set
\[ A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}. \]
Since the subgroup \( \langle A \rangle \in SL_n(\mathbb{Z}) \) is isomorphic to \( \mathbb{Z}/3 \), the group \( SL_n(\mathbb{Z}) \) contains \( \lfloor \frac{r}{2} \rfloor \) copies of \( \mathbb{Z}/3 \).

Case (i) Suppose that the subgroup \( (\mathbb{Z}/3)^{\lfloor \frac{r}{2} \rfloor} \) acts on \( M \) effectively. By Proposition 2.9, we get
\[ 3^{\lfloor \frac{r}{2} \rfloor - \lfloor \frac{k}{2} \rfloor} \mid \chi(M). \]
By assumption, the integer \( r \) is necessarily even. Since \( n \geq r + k \), we have that
\[ \chi(M) \equiv 0 \mod 3^{\lfloor \frac{k}{2} \rfloor}. \]
This is a contradiction.
Case (ii) If the subgroup \((\mathbb{Z}/3)^{[n]})\) can’t act effectively on \(M^r\), choose a nontrivial element \(g \in \ker f\). Without loss of generality, suppose \(n = 4\). Note that \(g = \text{diag}(A^{i_1}, A^{i_2})\) is a diagonal sum of powers of \(A\), at least one component is nontrivial. After conjugating by a permutation matrix, we may assume \(i_1 \not\equiv 0 \mod 3\). Denote by \(e_{ij}(a)\) the matrix with 1s along the diagonal, integer \(a\) in the \((i,j)\)-th position and zeros elsewhere. Let \(x\) be a \(2 \times 2\) matrix and

\[ X = \begin{pmatrix} I_2 & 0 \\ x & I_2 \end{pmatrix}. \]

Direct computation shows that the commutator

\[ [X, g] = XgX^{-1}g^{-1} = \begin{pmatrix} I_2 & 0 \\ x - A^{i_2}xA^{-i_1} & I_2 \end{pmatrix} \in \ker f. \]

We could always choose \(x\) such that one entry of \(x - A^{i_2}xA^{-i_1}\) is 1. For example, when \(g = \text{diag}(A^2, A)\), choose

\[ x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and then } x - A^{i_2}xA^{-i_1} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}. \]

Then the commutator

\[ [e_{34}(1), [X, g]] = e_{31}(1) \in \ker f. \]

Note that \(\text{SL}_n(\mathbb{Z})\) is normally generated by \(e_{31}(1)\) (cf. [24]). Therefore, the group action of \(\text{SL}_n(\mathbb{Z})\) is trivial. All other cases of \(g\) could be proved similarly.

Theorem 1.4 is a special case of the following theorem when \(k = 2\).

**Theorem 2.12** Assume that \(r\) and \(k \geq 2\) are positive integers. Let \(M^r\) be a closed smooth connected manifold with Euler characteristic \(\chi(M) \not\equiv 0 \mod 2^{k-1} \cdot 3^{\lfloor \frac{k}{2} \rfloor}\). Then any group action of \(\text{SL}_n(\mathbb{Z})(n \geq r + k)\) on \(M^r\) by diffeomorphisms is trivial.

**Proof.** Let \(f : \text{SL}_n(\mathbb{Z}) \to \text{Diff}(M)\) be a group homomorphism as before.

Since \(\text{SL}_n(\mathbb{Z})\) is perfect, i.e. \(\text{SL}_n(\mathbb{Z}) = [\text{SL}_n(\mathbb{Z}), \text{SL}_n(\mathbb{Z})]\), the group action is orientation-preserving. If \(\chi(M) \not\equiv 0 \mod 2^{k-1} \cdot 3^{\lfloor \frac{k}{2} \rfloor}\), the group action is trivial by Theorem 1.12. Suppose that \(\chi(M) \equiv 0 \mod 3\). For each integer \(2 \leq i \leq n\), denote by \(A_i\) the diagonal matrix \(\text{diag}(-1, \cdots, -1, \cdots, 1)\), where the second \(-1\) is the \(i\)-th entry. The subgroup \(G := \langle A_2, \ldots, A_n \rangle\) is isomorphic to \((\mathbb{Z}/2)^{n-1}\).

Case (i) Suppose that the subgroup \((\mathbb{Z}/2)^{n-1}\) acts on \(M\) effectively. By Proposition 2.9, we have that

\[ 2^{n-r} | \chi(M). \]

Since \(n \geq r + k\), we get that \(2^k | \chi(M)\) and thus \(2^k \cdot 3^{\lfloor \frac{k}{2} \rfloor} | \chi(M)\). This is a contradiction.
Case (ii) If the action of $G$ on $M^r$ is not effective, the kernel $\ker f$ contains a nontrivial element $g$. If $g$ is not central, i.e. $g \neq \text{diag}(-1, \cdots, -1)$, the normal subgroup generated by $g$ is of finite index (cf. [3], Prop. 3.4). Therefore, the image $\text{Im} f$ is a finite group. If $g = \text{diag}(-1, \cdots, -1)$ (note that this can only happen when $n$ is even), the group action factors through $\text{PSL}_n(\mathbb{Z})$. Since $n \geq r + k$, the image of $\langle A_3, \ldots, A_n \rangle \cong (\mathbb{Z}/2)^{n-2} \in \text{PSL}_n(\mathbb{Z})$ cannot act on $M^r$ effectively by Proposition 2.9 (otherwise, we get $2^{k-1} \mid \chi(M)$ and thus $2^{k-1} \cdot 3^{k-1} \mid \chi(M)$, which is a contradiction). Therefore, the subgroup $\ker f$ contains a noncentral element and thus of finite index. This proves that the group action of $\text{SL}_n(\mathbb{Z})$ is finite.

\[ A_0 = \begin{pmatrix} -1 & & & \\ -1 & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \in \text{SL}_4(\mathbb{Z}). \]

Note that the image of $A_0$ in $\text{PSL}_n(\mathbb{Z})$ has order two and the subgroup generated by the image of $\langle A_0, A_2, A_4 \rangle$ is isomorphic to $(\mathbb{Z}/2)^3$. If the action of $(\mathbb{Z}/2)^3$ is not effective, we finish the proof as before. If the action of this $(\mathbb{Z}/2)^3$ is effective, we get that $2^2 \mid \chi(M)$ and thus $12 \mid \chi(M)$, which is a contradiction. 

\section{The action of Steinberg groups on spheres and acyclic manifolds}

\subsection{Steinberg group}

For a unitary associative ring $R$, the Steinberg group $\text{St}_n(R)$ ($n \geq 3$), is generated by $x_{ij}(r)$ for $1 \leq i, j \leq n$ and $r \in R$ subject to the relations:

1. $x_{ij}(r_1) \cdot x_{ij}(r_2) = x_{ij}(r_1 + r_2)$;
2. $[x_{ij}(r_1), x_{jk}(r_2)] = x_{ik}(r_1 r_2)$;
3. $[x_{ij}(r_1), x_{pq}(r_2)] = 1$ if $i \neq p$ and $j \neq q$.

Let $R = \mathbb{Z}$, the integers. There is a natural group homomorphism $f : \text{St}_n(\mathbb{Z}) \to \text{SL}_n(\mathbb{Z})$ mapping $x_{ij}(r)$ to the matrix $e_{ij}(r)$, which is a matrix with 1s along the diagonal, $r$ in the $(i, j)$-th position and zeros elsewhere. Denote by
\[ \omega_{ij}(-1) = x_{ij}(-1)x_{ji}(1)x_{ij}(-1), \quad h_{ij} = \omega_{ij}(-1)\omega_{ij}(-1) \text{ and } a = h_{12}^2. \] We call a Steinberg symbol, denoted by \{-1, -1\} usually.

**Lemma 3.1** (Milnor [24], Theorem 10.1) For \( n \geq 3 \), the group \( \text{St}_n(\mathbb{Z}) \) is a central extension
\[ 1 \to K \to \text{St}_n(\mathbb{Z}) \to \text{SL}_n(\mathbb{Z}) \to 1, \]
where \( K \) is the cyclic group of order \( 2 \) generated by \( a = (x_{12}(-1)x_{21}(1)x_{12}(-1))^4 \).

**Lemma 3.2** For distinct integers \( i, j, s, t \), we have the following.

(i) \([h_{ij}, h_{st}] = 1;\)

(ii) \([h_{ij}, h_{is}] = a;\)

(iii) The subgroup \( \langle h_{ij}, h_{is} \rangle \) is isomorphic to the quaternion group \( Q_8 \).

**Proof.** (i) follows the third Steinberg relation easily. (ii) is Lemma 9.7 of Milnor [24] (p. 74). A direct computation shows that \( h_{ij}, h_{is} \) and \( h_{ij}h_{is} \) are all elements of order \( 4 \). Considering (ii), \( \langle h_{ij}, h_{is} \rangle \) is isomorphic to the quaternion group \( Q_8 \).

Denote by \( q : \text{SL}_n(\mathbb{Z}) \to \text{SL}_n(\mathbb{Z}/n) \) or \( \text{St}_n(\mathbb{Z}) \to \text{St}_n(\mathbb{Z}/n) \) the group homomorphism induced by the ring homomorphism \( \mathbb{Z} \to \mathbb{Z}/n \), for some integer \( n \).

Let \( \text{SL}_n(\mathbb{Z}, n\mathbb{Z}) = \ker(\text{SL}_n(\mathbb{Z}) \to \text{SL}_n(\mathbb{Z}/n)) \) and \( \text{St}_n(\mathbb{Z}, n\mathbb{Z}) = \ker(\text{St}_n(\mathbb{Z}) \to \text{St}_n(\mathbb{Z}/n)) \) be the congruence subgroups.

**Lemma 3.3** Let \( N \) be a normal subgroup of \( \text{St}_n(\mathbb{Z}) \). If \( f(N) \) contains \( \text{SL}_n(\mathbb{Z}, 2\mathbb{Z}) \), then \( B \) contains \( a \). In particular, the normal subgroup generated by \( h_{ij} \) \((n \geq 3) \) or \( h_{ij}h_{st} \) \((n \geq 5) \) contains \( a \) for distinct integers \( i, j, s, t \).

**Proof.** Recall that \( e_{ij}(r) \) is a matrix with 1s along the diagonal, \( r \) in the \((i, j)\)-th position and zeros elsewhere. Since \( f(N) \) contains \( \text{SL}_n(\mathbb{Z}, 2\mathbb{Z}) \), we have \( x_{pq}(2) \) or \( x_{pq}(2) \in N \) for any integers \( p \neq q \in [1, n] \). Note that \( \text{St}_n(\mathbb{Z}, 2\mathbb{Z}) \) is normally generated by \( x_{pq}(2) \) (cf. 13.18 of Magurn [22], p.448). Therefore, \( \text{St}_n(\mathbb{Z}, 2\mathbb{Z}) \) or \( \text{St}_n(\mathbb{Z}, 2\mathbb{Z}) \subset N \). However, it is known that the Steinberg symbol \( q(a) \in \text{St}_n(\mathbb{Z}/2) \) is trivial (cf. Corollary 9.9 of Milnor [24], p.75). Thus, \( a \in a \cdot \text{St}_n(\mathbb{Z}, 2\mathbb{Z}) = \text{St}_n(\mathbb{Z}, 2\mathbb{Z}) \subset N \). The image \( f(h_{ij}) = \text{diag}(1, \ldots, -1, \ldots, -1, \ldots, 1) \) (or \( f(h_{ij}h_{st}) \)) normally generates the congruence subgroup \( \text{SL}_n(\mathbb{Z}, 2\mathbb{Z}) \) (cf. Ye [35]). The proof is finished.

### 3.2 Steinberg group acting on \( R^n \) and \( S^n \)

**Lemma 3.4** Let \( X \) be a generalized \( m \)-sphere over \( \mathbb{Z}/2 \) (or a \( \mathbb{Z}/2 \)-acyclic \( m \)-hm\( \mathbb{Z}/2 \), resp.). Suppose that \( \tau \) is an involution of \( X \) and \( F \) is a closed \( \tau \)-invariant submanifold. If \( F \) containing \( \text{Fix}(\tau) \) is an \((m-1)\)-sphere (or \( \mathbb{Z}/2 \)-acyclic \((m-1)\)-hm\( \mathbb{Z}/2 \), resp.), then \( X \setminus F \) has two \( \mathbb{Z}/2 \)-acyclic components and \( \tau \) interchanges them.
Proof. The proof is exactly the same as that of Lemma 4.11 of Bridson and Vogtmann \[4\], where \( F = \text{Fix}(\tau) \).

We now study the group action of Steinberg groups \( \text{St}_n(\mathbb{Z}) \) on spheres and acyclic manifolds. Compared with the proof of actions of \( \text{SL}_n(\mathbb{Z}) \), there is no enough involutions in \( \text{St}_n(\mathbb{Z}) \). Note that the element \( h_{ij} \in \text{St}_n(\mathbb{Z}) \) corresponding to the involution \( \text{diag}(1, \cdots, -1, \cdots, -1, \cdots, 1) \) is of order 4. Moreover, \( h_{ij} \) and \( h_{is} \) do not commute with each other. All these facts make the study of \( \text{St}_n(\mathbb{Z}) \)'s actions difficult and the proof for the action of \( \text{SL}_n(\mathbb{Z}) \) presented in \[4\] could not be carried to study that of \( \text{St}_n(\mathbb{Z}) \) easily.

Theorem 3.5 We have the following.

1. Any group action of \( \text{St}_n(\mathbb{Z}) \) \((n \geq 4)\) on a generalized \( k \)-sphere \( M^k \) over \( \mathbb{Z}/2 \) \((k \leq n-2)\) by homeomorphisms is trivial.

2. Any group action of \( \text{St}_n(\mathbb{Z}) \) \((n \geq 4)\) on a \( \mathbb{Z}/2 \)-acyclic \( k \)-hm \( \mathbb{Z}/2M^k \) \((k \leq n-1)\) by homeomorphisms is trivial.

Proof. Let \( a = (x_{12}(1)x_{21}(-1)x_{12}(1))^4 \) as in Lemma 3.1.

Case 1 If \( a \) acts trivially on \( M^k \), the group action of \( \text{St}_n(\mathbb{Z}) \) factors through an action of \( \text{SL}_n(\mathbb{Z}) \). However, it is proved by Bridson and Vogtmann \[4\] that the group action of \( \text{SL}_n(\mathbb{Z}) \) is trivial. Suppose now that \( a \) acts non-trivially.

Case 2 \( \text{Fix}(a) \neq \emptyset \) and \( \text{Fix}(a) \neq M^k \). Since \( \text{St}_n(\mathbb{Z}) \) \((n \geq 3)\) is perfect, every element acts by orientation-preserving homeomorphism. Bredon’s result (cf. Lemma 2.5) shows that \( \text{Fix}(a) \) is of even dimension. If \( \dim_2 \text{Fix}(a) = k \), \( \text{Fix}(a) = M^k \) by the invariance of domain. This is a contradiction to the fact that \( a \) acts non-trivially. Therefore,

\[
\dim_2 \text{Fix}(a) \leq k - 2.
\]

Note that \( h_{12}h_{34} \) is of order 2 and \( A := (a, h_{12}h_{34}) \) is isomorphic to \( (\mathbb{Z}/2)^2 \). Write \( r = \dim_2(\text{Fix}(A)) \) and \( n(H) = \dim_2(\text{Fix}(H)) \) for each non-trivial cyclic subgroup \( H < A \). By Borel’s formula (cf. Lemma 2.4),

\[
k - r = \sum n(H) - r, \tag{1}
\]

where \( H \) ranges over the nontrivial subgroup of index 2. Since \( a \) is in the center of \( \text{St}_n(\mathbb{Z}) \), there is an group action of \( \text{SL}_n(\mathbb{Z}) \) acts on the acyclic \( \mathbb{Z}/2 \)-manifold or generalized sphere \( \text{Fix}(a) \) induced by that of \( \text{St}_n(\mathbb{Z}) \). This group action is trivial by Bridson and Vogtmann \[4\]. Thus,

\[
n((h_{12}h_{34})) \geq r = \dim_2 \text{Fix}(a).
\]

Case 2.1 If \( n((h_{12}h_{34})) = r \), we have \( n((a \cdot h_{12}h_{34})) = k \) by (1). By invariance of domain, \( a \cdot h_{12}h_{34} \) acts trivially on \( M^k \). Take \( \omega = h_{12}\omega_{12}(-1)\omega_{34}(-1) \) and
$C = \langle a, \omega \rangle$. Note that $\omega^2 = a \cdot h_{12}h_{34}$ and $f(\omega) = f(a \cdot \omega)$ has the form

$$
\begin{pmatrix}
-1 & 1 \\
1 & -1
\end{pmatrix}
\in \text{SL}_2(\mathbb{Z}).
$$

Therefore, $C$ acts on $M^k$ as a group isomorphic to $(\mathbb{Z}/2)^2$. If $\dim_2 \text{Fix}(\omega) = k$, or $\dim_2 \text{Fix}(a \cdot \omega) = k$, i.e. $\omega$ or $a \cdot \omega$ acts trivially on $M^k$, the normal subgroup in $St_n(\mathbb{Z})$ generated by $\omega$ or $a \cdot \omega$ contains $a$ by Lemma 3.3. This is a contradiction to the fact that $\text{Fix}(a) \neq M^k$. Therefore, we may assume that $\dim_2 \text{Fix}(\omega) \leq k - 2$ and $\dim_2 \text{Fix}(a \cdot \omega) \leq k - 2$, considering Lemma 2.5. According to the Borel’s formula (1), we have invariance of domain, $\text{Fix}(a)$ is impossible. Similarly, $n(\langle a \cdot h_{12}h_{34} \rangle) \neq r$.

**Case 2.2** If $n(\langle h_{12}h_{34} \rangle) - r \geq 2$ and $n(\langle a \cdot h_{12}h_{34} \rangle) - r \geq 2$ (noting that $n(\langle h_{12}h_{34} \rangle) - r$ is even), then

$$
k - r = 2(n(\langle h_{12}h_{34} \rangle) - r) \geq 4. \tag{2}
$$

(Note that when $n \geq 5$, $h_{15}h_{13}h_{34}h_{15}^{-1} = a \cdot h_{12}h_{34}$ by Lemma 3.2 and thus $n(\langle h_{12}h_{34} \rangle) = n(\langle a \cdot h_{12}h_{34} \rangle)$). When $k = 4$, we have $r = 0$ and $n(\langle h_{12}h_{34} \rangle) = 2$. Therefore, $\text{Fix}(h_{12}h_{34})$ is $S^2$ or $\mathbb{R}^2$ (cf. [6], 16.32, p.388). If $\text{Fix}(h_{12}h_{34}) = \mathbb{R}^2$, the quaternion group $\langle h_{12}, h_{13} \rangle$ acts on $\text{Fix}(h_{12}h_{34})$ with a global fixed point in $\text{Fix}(a)$. Since any finite group of orientation-preserving homeomorphisms of the plane that fix the origin is cyclic, $\langle h_{12}, h_{13} \rangle$ cannot act effectively. A nontrivial element in $\langle h_{12}, h_{13} \rangle$ will normally generate a group containing $a$ by Lemma 3.3 which is impossible.

If $\text{Fix}(h_{12}h_{34}) = S^2$, we have $n \geq 6$. Denote by $B = \langle a \cdot h_{12}h_{34}, a \cdot h_{34}h_{56} \rangle \cong (\mathbb{Z}/2)^2$. Note that $\text{Fix}(a) \subset \text{Fix}(B)$. Write $r' = \dim_2(\text{Fix}(B))$ and $n(H) = \dim_2(\text{Fix}(H))$ for each non-trivial cyclic subgroup $H \subset B$. By Borel’s formula (cf. Lemma 2.4),

$$
k - r' = \sum n(H) - r', \tag{3}
$$

where $H$ ranges over the nontrivial subgroup in $B$ of index 2. Since any two nontrivial elements in $B$ are conjugate (cf. [22], 12.20, p.418), we get

$$
k - r' = 3n(\langle h_{12}h_{34} \rangle) - r' \tag{4}
$$

and $r' = 1$. Therefore, $\text{Fix}(a)$ is a submanifold of $\text{Fix}(B)$ of codimension 1. By Lemma 3.3, $a$ permutes the two components of $\text{Fix}(B) \setminus \text{Fix}(a)$. However, $h_{12}^2 = a$ and $h_{12}\text{Fix}(B) = \text{Fix}(B)$. This is impossible.

Similar arguments using (2) and (4) prove the following. When $k = 5$, $r = 1, n(\langle h_{12}h_{34} \rangle) = 3$ and $r' = 2$. When $k = 6, r = 2, n(\langle h_{12}h_{34} \rangle) = 4$.
and \( r' = 3 \). When \( k = 7, r = 3, n((h_{12h_{34}})) = 5 \) and \( r' = 4 \). In all these cases, \( \text{Fix}(a) \) is still a submanifold in \( \text{Fix}(B) \) of codimension 1. By Lemma 3.4, this is impossible. When \( k = 8, r = 4, n((h_{12h_{34}})) = 6, r' = 5 \) or \( r = 0, n((h_{12h_{34}})) = 4, r' = 2 \). The former is impossible for the same reason as \( k = 7 \), while the latter is impossible for the same reason as \( k = 4 \).

When \( k \geq 9 \), we have \( n \geq 10 \). If \( k - r' \geq 6 \), then \( \text{St}_{n-6}(\mathbb{Z}) \) generating by all \( x_{ij}(r) (7 \leq i \neq j \leq n) \) acts on \( \text{Fix}(B) \). By Smith theory (cf. Lemma 2.3), \( \text{Fix}(B) \) is still a generalized sphere over \( \mathbb{Z}/2 \) or \( \mathbb{Z}/2 \)-acyclic \( h_{34} \). An inductive argument shows that \( \text{St}_{n-6}(\mathbb{Z}) \) acts trivially on \( \text{Fix}(B) \). Therefore, \( \text{Fix}(B) = \text{Fix}(a) \). However, this is impossible considering formulas (2) and (4). If \( k - r' \leq 5 \), we have

\[
3(n((h_{12h_{34}})) - r') \leq 5.
\]

When \( n((h_{12h_{34}})) - r' = 0, \text{Fix}(B) = M^k \). Then \( h_{12h_{34}} \) acts trivially on \( M \). The normal subgroup generated by \( h_{12h_{13}} \) contains \( a \) (cf. Lemma 3.3), which means \( a \) acts trivially. This is a contradiction. When \( n((h_{12h_{34}})) - r' = 1, \) we have \( k - r' = 3, k - n((h_{12h_{34}})) = 2 \) and \( k - r = 4 \). Therefore, \( \text{Fix}(a) \) is a submanifold in \( \text{Fix}(B) \) of codimension 1, which is impossible as above by Lemma 3.3.

Case 3 \( \text{Fix}(a) = \emptyset \). According to the Lefschetz fixed-point theorem, this can only happen when \( M \) is a generalized sphere of odd dimension.

When \( n = 1, M = S^1 \) (cf. 6, 16.32, p.388). The group \( (h_{12}, h_{13}) \), which is isomorphic to quaternion group \( Q_8 \) (cf. Lemma 3.2), acts freely on \( S^1 \). However, this is impossible since any finite subgroup of \( \text{Homeo}(S^1) \) is isomorphic to a subgroup of \( \text{SO}(2; \mathbb{R}) \) (cf. Navas 24, Prop. 1.1.1).

Assume \( k = 3 \). Recall that \( A := \langle a, h_{12h_{34}} \rangle \) is isomorphic to \((\mathbb{Z}/2)^2 \). By Smith theory, \((\mathbb{Z}/2)^2 \) cannot act freely and thus \( \text{Fix}(h_{12h_{34}}) \) is not empty. Bredon’s result (cf. Lemma 2.3) shows that \( \text{Fix}(h_{12h_{34}}) \) is of even dimension. If \( \dim_2 \text{Fix}(h_{12h_{34}}) = 3, h_{12h_{34}} \) acts trivially on \( M \). The normal subgroup \( \text{St}_n(\mathbb{Z}) \) generated by \( h_{12h_{34}} \) contains \( a \), which is a contradiction to the fact that \( \text{Fix}(a) = \emptyset \). Therefore, \( \text{Fix}(h_{12h_{34}}) = S^1 \). Note that the quaternion group \( \langle h_{12}, h_{13} \rangle \) commutes with \( h_{12h_{34}} \). Since \( a \) acts freely on \( \text{Fix}(h_{12h_{34}}) \), so does \( \langle h_{12}, h_{13} \rangle \), which is impossible as well.

When \( k = 5 \), take \( B = \langle a \cdot h_{12h_{34}}, a \cdot h_{34}h_{56} \rangle \cong (\mathbb{Z}/2)^2 \). Since \( \text{Fix}(h_{12h_{34}}) \) is a generalized sphere over \( \mathbb{Z}/2 \), \( \langle a, ah_{34}h_{56} \rangle \) can not act freely on it. Therefore, \( \text{Fix}(B) \neq \emptyset \). By (4), we have

\[
k - r' = 3(n((ah_{12h_{34}})) - r').
\]

If \( n((ah_{12h_{34}})) = r' \), we have \( k = r' \). Thus \( ah_{12h_{34}} \) acts trivially on \( M \). The normal subgroup in \( \text{St}_n(\mathbb{Z}) \) generated by \( ah_{12h_{34}} \) contains \( a \), a contradiction. Therefore, \( r' = 2 \) and \( n((ah_{12h_{34}})) - r' = 1 \). By Lemma
3.4. a permutes the two components of $\text{Fix}(ah_{12}h_{34}) \setminus \text{Fix}(B)$, which is impossible by noting that $h_{12}^2 = a$.

When $k = 7$, we may assume $n(\langle ah_{12}h_{34} \rangle) \neq r'$ as in the proof of the case when $k = 5$. Considering formula (2), we have either $r' = 4, n(\langle ah_{12}h_{34} \rangle) = 5$ or $r' = 1, n(\langle ah_{12}h_{34} \rangle) = 3$. For the former, apply Lemma 3.4 to get a contradiction. For the latter, the quaternion group $\langle h_{12}, h_{13} \rangle$ acts on $\text{Fix}(B) = S^1$ freely, which is impossible as the case of $k = 3$.

Suppose that $k \geq 9$. If $k - r' \geq 6$, the subgroup $\text{St}_{n-6}(\mathbb{Z})$ generating by all $x_{ij}(r)$ ($7 \leq i \neq j \leq n$) acts trivially on $\text{Fix}(B)$ by an inductive argument. This is a contradiction to the fact that $\text{Fix}(a) = \emptyset$. If $k - r' \leq 5$, $n(\langle h_{12}h_{34} \rangle) - r' = 0$ or 1. If $n(\langle h_{12}h_{34} \rangle) = r'$, $k = r'$, and thus $\text{Fix}(B) = M^k$. Then $h_{12}h_{34}$ acts trivially on $M$. The normal subgroup generated by $h_{12}h_{13}$ contains a (cf. Lemma 3.3), which is a contradiction to the fact $\text{Fix}(a) = \emptyset$. If $n(\langle h_{12}h_{34} \rangle) - r' = 1$, the element $a$ permutes $\text{Fix}(h_{12}h_{34}) \setminus \text{Fix}(B)$ by Lemma 3.4. This is impossible by noting that $h_{12}^2 = a$. The whole proof is finished.

**Corollary 3.6** Any group homomorphism $f : \text{St}_n(\mathbb{Z}) \to \text{GL}_k(\mathbb{Z})$ ($n \geq 3, k \leq n - 1$) is trivial.

**Proof.** When $k = 1$, $\text{GL}_k(\mathbb{Z})$ is abelian. Since $\text{St}_n(\mathbb{Z})$ is perfect, $f$ is trivial. When $k = 2$, $f$ has its image in $\text{SL}_2(\mathbb{Z})$. Note that the projective linear group $\text{PSL}_2(\mathbb{Z}) = \mathbb{Z}/2 \ast \mathbb{Z}/3$, a free product. Thus $\text{SL}_2(\mathbb{Z})$ does not have nontrivial perfect subgroup (cf. [3], 5.8, p.48). This means that $f$ is trivial. The group $\text{GL}_k(\mathbb{Z})$ acts naturally on the Euclidean space $\mathbb{R}^k$ by linear transformations. When $k \geq 3$, Theorem 3.5 implies that the image $\text{Im} f$ acts trivially on $\mathbb{R}^k$. Therefore, $\text{Im} f = 1$. ■

4 Proof of Theorem 1.6

We need the following lemma.

**Lemma 4.1** Denote by $Q$ a quotient group of $\text{SL}_n(\mathbb{Z})$. Let $\pi$ be a torsion-free abelian group. For any $n \geq 3$, the second cohomology group

$$H^2(Q; \pi) = 0.$$  

**Proof.** By van der Kallen [30], the second homology group $H_2(\text{SL}_n(\mathbb{Z}); \mathbb{Z}) = \mathbb{Z}/2$ when $n \geq 5$ and

$$H_2(\text{SL}_3(\mathbb{Z}); \mathbb{Z}) = H_2(\text{SL}_4(\mathbb{Z}); \mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$
Since $\text{SL}_n(\mathbb{Z})$ is perfect, $H_1(\text{SL}_n(\mathbb{Z}); \mathbb{Z}) = 0$ for any $n \geq 3$. By the universal coefficient theorem, $H^2(\text{SL}_n(\mathbb{Z}); \pi) = 0$ for any $n \geq 3$. Dennis and Stein proved that
\[
H_2(\text{SL}_n(\mathbb{Z}/k); \mathbb{Z}) = \mathbb{Z}/2, \text{ for } k \equiv 0(\text{mod}4),
\]
while $H_2(\text{SL}_n(\mathbb{Z}/k); \mathbb{Z}) = 0$, otherwise (cf. [11] corollary 10.2). By the universal coefficient theorem again, $H^2(\text{SL}_n(\mathbb{Z}/k); \pi) = 0$ for any $k$. Let $f : \text{SL}_n(\mathbb{Z}) \to Q$ be a surjective homomorphism. If $f$ is trivial, $Q = \text{SL}_n(\mathbb{Z})$ and thus $H^2(Q; \pi) = 0$. If $f$ is nontrivial, the congruence subgroup property \[\text{[1]}\] implies that $Q$ is a quotient of $\text{SL}_n(\mathbb{Z}/k)$ by a central subgroup $K$ for some non-zero integer $k$. From the Serre spectral sequence
\[
H^p(Q; H^q(K; \pi)) \implies H^{p+q}(\text{SL}_n(\mathbb{Z}/k); \pi),
\]
we have the exact sequence
\[
0 \to H^1(Q; \pi) \to H^1(\text{SL}_n(\mathbb{Z}/k); \pi) \to H^0(Q; H^1(K; \pi)) \to H^2(Q; \pi) \to H^2(\text{SL}_n(\mathbb{Z}/k); \pi).
\]
This implies $H^2(Q; \pi) = 0$. □

**Proof of Theorem 1.6.** If $\text{SL}_n(\mathbb{Z})$ acts trivially on $M^r$, it is obvious that the induced homomorphism $\text{SL}_n(\mathbb{Z}) \to \text{Out}(\pi_1(M))$ is trivial. It is enough to prove the other direction. Denote by $\text{Homeo}(M^r)$ the group of homeomorphisms of $M^r$. Suppose that $f : \text{SL}_n(\mathbb{Z}) \to \text{Homeo}(M^r)$ is the group homomorphism. We have a group lifting
\[
1 \to \pi_1(M) \to G' \to \text{Im} f \to 1,
\]
where $\tilde{M}$ is the universal cover of $M$ and $G'$ is a subgroup of $\text{Homeo}(\tilde{M})$. By the assumption that the group homomorphism $\text{SL}_n(\mathbb{Z}) \to \text{Out}(\pi_1(M))$ is trivial, $\text{Im} f$ acts trivially on the center $C(\pi)$. Since $M$ is aspherical, $\pi = \pi_1(M)$ is torsion-free and the center $C(\pi)$ is torsion-free as well. By Lemma 4.1, $H^2(\text{Im} f; C(\pi)) = 0$, which implies that the exact sequence is split. Therefore, $\text{Im} f$ is isomorphic to a subgroup of $G'$, which implies that the group $\text{SL}_n(\mathbb{Z})$ and thus $\text{St}_n(\mathbb{Z})$ could act on the acyclic manifold $\tilde{M}$ through $\text{Im} f$. However, Theorem 5.7 implies that any group action of $\text{St}_n(\mathbb{Z})$ ($n \geq 4$) on the acyclic manifold $\tilde{M}$ is trivial. When $n = 3$, for each integer $2 \leq i \leq n$, denote by $A_i$ the diagonal matrix $\text{diag}(-1, \ldots, -1, \cdots, 1)$, where the second $-1$ is the $i$-th entry. The subgroup $G := \langle A_2, A_3 \rangle < \text{SL}_3(\mathbb{Z})$ is isomorphic to $(\mathbb{Z}/2)^2$. By Smith theory (cf. Lemma 2.3), the group action of $G$ on $\tilde{M}$ has a global fixed point. The Borel formula (cf. Lemma 2.4) implies that the action of $G$ on $\tilde{M}$ is trivial. Therefore, the group action of $\text{SL}_3(\mathbb{Z})$ on $M$ factors through the projective linear group $\text{PSL}_3(\mathbb{Z}/2)$. Using Smith theory and the Borel formula once again for the subgroup $(\mathbb{Z}/2)^2 \cong \langle e_{12}(1), e_{13}(1) \rangle < \text{PSL}_3(\mathbb{Z}/2)$, we see that the group action of $\text{PSL}_3(\mathbb{Z}/2)$ and thus $\text{SL}_3(\mathbb{Z})$ on $M$ is trivial. This implies $\text{Im} f$ is trivial, i.e. the group action of $\text{SL}_n(\mathbb{Z})$ on $M$ is trivial. □
5 Aspherical manifolds with nilpotent fundamental groups

Let $G$ be a group. Denote by $Z_1 = Z(G)$, the center. Inductively, define $Z_{i+1}(G) = p_i^{-1}Z(G/Z_{i-1}(G))$, where $p_i : G \to G/Z_i(G)$ is the quotient group homomorphism. We have the upper central sequence

$$1 \subset Z_1 \subset Z_2 \subset ... \subset Z_i \subset ....$$

If $Z_i = G$ for some $i$, we call $G$ a nilpotent group. For two groups $G$ and $H$, denote by $\text{Hom}(G, H)$ the set of group homomorphisms from $G$ to $H$.

Lemma 5.1 Let

$$1 \to N \to \pi \xrightarrow{\rho} Q \to 1$$

be a central extension, i.e. an exact sequence with $N < Z(\pi)$. Suppose that $G$ is a group with the second cohomology group $H^2(G; N) = 0$, where $G$ acts on $N$ trivially. Then

$$\text{Hom}(G, \pi) \xrightarrow{\rho} \text{Hom}(G, Q)$$

is surjective.

Proof. This is an easy exercise in homological algebra. For completeness, we give a proof here. The central extension gives a principal fibration $BN \to B\pi \to BQ$ and thus a fibration

$$B\pi \to BQ \xrightarrow{h} K(N, 2),$$

where $B(-)$ is a classifying space and $K(N, 2)$ is a simply connected CW complex with the second homotopy group $N$ and all other homotopy groups trivial (cf. [3], 8.2, p.64). Let $\alpha : G \to Q$ be any group homomorphism. The composite

$$BG \xrightarrow{Bo} BQ \xrightarrow{h} K(N, 2)$$

is null-homotopic, by the assumption that $H^2(G; N) = 0$. Therefore, $\alpha$ could be lifted to be a group homomorphism $\alpha' : G \to \pi$. □

Lemma 5.2 Let $\pi$ be a group with center $Z = Z(\pi)$. Suppose that one of the following holds:

(i) $G$ is a perfect group with $H_2(G; \mathbb{Z})$ finite, $\pi$ and $\pi/Z$ are torsion-free; or

(ii) $G$ is a perfect group with $H_2(G; \mathbb{Z}) = 0$.

If the set of group homomorphisms

$$\text{Hom}(G, \text{Aut}(Z)) = 1 \text{ and } \text{Hom}(G, \text{Out}(\pi/Z)) = 1,$$

then

$$\text{Hom}(G, \text{Out}(\pi)) = 1.$$

Here 1 denote the trivial group homomorphism.
Proof. Considering the quotient group homomorphism \( \pi \to \pi/Z \), we have the following commutative diagram

\[
\begin{array}{ccccccccc}
1 & \to & \text{Inn}(\pi) & \to & \text{Aut}(\pi) & \to & \text{Out}(\pi) & \to & 1 \\
\downarrow & & \downarrow f & & \downarrow g & & \\
1 & \to & \text{Inn}(\pi/Z) & \to & \text{Aut}(\pi/Z) & \to & \text{Out}(\pi/Z) & \to & 1.
\end{array}
\]

Note that \( \text{Inn}(\pi) = \pi/Z \). By the snake lemma for groups (cf. [22], 11.8, p.363), the following sequence is exact

\[
1 \to Z(\pi/Z) \to \ker f \to \ker g \to 1.
\] (5)

By diagram chase, the action of \( \ker g \) on the center \( Z(\pi/Z) \) is by inner automorphisms of \( \pi/Z \) and thus trivial. This proves that the previous exact sequence (5) is a central extension. Since

\[
\text{Hom}(G, \text{Out}(\pi/Z)) = 1
\]

by assumption, it suffices to prove \( \text{Hom}(G, \ker g) = 1 \). Let \( \alpha : G \to \ker g \) be any group homomorphism. In the case of (i), when \( \pi/Z \) is torsion-free, the center \( Z(G/Z) \) is torsion-free. Since \( G \) is perfect and \( H_2(G;Z) \) is finite, we have \( H^2(G; Z(\pi/Z)) = 0 \) by the universal coefficient theorem. In the case of (ii), we also have \( H^2(G; Z(\pi/Z)) = 0 \) using the universal coefficient theorem. Lemma 5.1 implies that \( \alpha \) could be lifted to be a group homomorphism \( \alpha' : G \to \ker f \).

Let \( F : \text{Aut}(\pi) \to \text{Aut}(Z) \) be the restriction of automorphisms of \( \pi \) to that of the center \( Z \). Since the image \( F(\ker f) \) is a subgroup of \( \text{Aut}(Z) \) and

\[
\text{Hom}(G, \text{Aut}(Z)) = 1
\]

by assumption, the map \( \alpha' \) has image in \( \ker F \cap \ker f \). It is well-known that \( (\ker F \cap \ker f) \) is isomorphic to \( H^1(\pi/Z;Z) \) (cf. [16], Prop. 5, p.45). Since \( G \) is perfect, \( \alpha' \) has trivial image. This proves that \( \alpha \) is trivial and thus \( \text{Hom}(G, \text{Out}(\pi)) = 1 \).

Recall that a group \( G \) has cohomological dimension \( k \) (denoted as \( \text{cd}(G) = k \)) if the cohomological group \( H^i(G;A) = 0 \) for any \( i > k \) and \( \mathbb{Z}G \)-module \( A \), but \( H^n(G; M) \neq 0 \) for some \( \mathbb{Z}G \)-module \( M \). The Hirsch number or Hirsch length of a polycyclic group \( G \) is the number of infinite factors in its subnormal series. The following lemma is well-known (see Gruenberg [16], p.152).

**Lemma 5.3** If \( G \) is finitely generated, torsion-free nilpotent group, then \( \text{cd}(G) = h(G) \), where \( \text{cd}(G) \) is the cohomological dimension and \( h(G) \) is the Hirsch number.

**Lemma 5.4** Let \( 1 \to Z \to G \to Q \to 1 \) be a central extension with \( Z = Z(G) \) the center and \( G \) a torsion-free nilpotent group. Then \( Q \) is torsion-free.

17
Proof. It’s known that all the quotient $Z_i/Z_{i-1}$ is torsion-free. Suppose that the nilpotency class of $G$ is $n$, i.e. $Z_n = G$. Then $G/Z_{n-1}$ is (torsion-free) abelian and we have exact sequence

$$1 \to Z_n/Z_{n-1} \to G/Z_{n-2} \to G/Z_{n-1} \to 1.$$ 

Since both $G/Z_{n-1}$ and $Z_n/Z_{n-2}$ are both finitely generated (Note that every subgroup of a finitely generated nilpotent group is finitely generated) and of finite cohomology dimensions, $G/Z_{n-2}$ is of finite cohomological dimension and thus torsion-free. Inductively, we prove the lemma. ■

Lemma 5.5 Let $G$ be a finitely generated torsion-free nilpotent group of cohomological dimension $k$. When $k < n$, the set of group homomorphism

$$\text{Hom}(\text{St}_n(Z), \text{Out}(G)) = 1,$$

and thus $\text{Hom}(\text{SL}_n(Z), \text{Out}(G)) = 1$.

Proof. Let $Z$ be the center of $G$. Note that cohomological dimension $\text{cd}(G) = h(G)$ and $\text{cd}(G/Z) = h(G/Z)$ and $h(G/Z) \leq h(G) - 1$. When $G/Z$ is abelian, we have $h(G/Z) \leq k$ and $\text{Out}(G/Z) = \text{GL}(h(G/Z))(Z)$. Noting that $\text{Hom}(\text{St}_n(Z), \text{GL}_k(Z)) = 1$ for any $k \leq n-1$ by Lemma 5.5, we have $\text{Hom}(\text{St}_n(Z), \text{Out}(G/Z)) = 1$. Similarly, we get $\text{Hom}(\text{SL}_n(Z), \text{Aut}(Z)) = 1$ by noting that $Z$ is torsion-free abelian and $h(Z) \leq k$. Using Lemma 5.2 repeatedly, we have $\text{Hom}(\text{St}_n(Z), \text{Out}(G)) = 1$. ■

Let $M$ be aspherical manifolds with a finitely generated nilpotent fundamental group $\pi_1(M)$. Any group action of $\text{SL}_n(Z)$ ($n \geq 3$) on $M$ ($k < n$) is trivial, as following.

Proof of Theorem 1.8 Since $M$ is aspherical, $M$ is a classifying space for $\pi_1(M)$ and thus the cohomological dimension $\text{cd}(\pi_1(M)) \leq r$. By Lemma 5.5 any group homomorphism $\text{SL}_n(Z) \to \text{Out}(\pi_1(M))$ is trivial. By Theorem 1.6 any group action of $\text{SL}_n(Z)$ on $M$ is trivial. ■

6 Flat and almost flat manifolds

A closed manifold $M$ is almost flat if for any $\varepsilon > 0$ there is a Riemannian metric $g_\varepsilon$ on $M$ such that $\text{diam}(M, g_\varepsilon) < 1$ and $g_\varepsilon$ is $\varepsilon$-flat. Let $G$ be simply connected nilpotent Lie group. Choose a maximal compact subgroup $C$ of $\text{Aut}(G)$. If $\pi$ is a torsion-free uniform discrete subgroup of the semi-direct product $G \rtimes C$, the orbit space $M = \pi \backslash G$ is called an infra-nilmanifold and $\pi$ is called a generalized Bieberbach group. The group $F := \pi / (\pi \cap G)$ is called holonomy group of $M$. When $G = \mathbb{R}^n$, the abelian Lie group, $M$ is called a flat manifold. By Gromov and Ruh [15, 27], every almost flat is diffeomorphic to an infra-nilmanifold. Note that $N := \pi \cap G$ is the unique maximal nilpotent normal subgroup of $\pi$. 

18
The automorphism of $\pi$ is studied by Igodt and Malfait [17], generalizing the corresponding result for flat manifolds obtained by Charlap and Vasquez [9]. Let’s recall the relevant results as follows. Let $\psi : F \to \text{Out}(N)$ be a injective group homomorphism and

$$1 \to N \to \pi \to F \to 1$$

be a group extension compatible with $\psi$. The extension determines a cohomology class $a \in H^2(F; N)$. Let $p : \text{Aut}(N) \to \text{Out}(N)$ be the natural quotient map and $\mathcal{M}_\psi = p^{-1}(N_{\text{Out}(N)}(F))$, the preimage of the normalizer of $\psi(F)$ in $\text{Out}(N)$. Denote by $\mathcal{M}_\psi,a$ the stabilizer of $a$ under the action of $\mathcal{M}_\psi$ on $H^2(F; N)$ by conjugations. Let $A : \text{Aut}(\pi) \to \text{Aut}(N)$ be the group homomorphisms of restrictions. For an element $n \in N$, write $\mu(n) \in \text{Aut}(N)$ the inner automorphism determined by $n$, i.e. $\mu(n)(x) = nxn^{-1}$. Denote by $* : \mathcal{M}_\psi \to \text{Aut}(F)$ the group homomorphism given by $*\psi = \psi \circ \mu(p(v)) \circ \psi^{-1}$ (cf. [17], Lemma 3.3).

We will need the following result (cf. [17], Theorem 4.8).

**Lemma 6.1** The following sequences are exact:

$$1 \to Z^1(F; Z(N)) \to \text{Aut}(\pi) \xrightarrow{A} \mathcal{M}_\psi,a \to 1$$

and

$$1 \to H^1(F; Z(N)) \to \text{Out}(\pi) \xrightarrow{A} Q \to 1.$$  

The quotient $Q = (\mathcal{M}_\psi,a/\text{Inn}(N))/F$ and fits into an exact sequence

$$1 \to Q_2 = ((\ker(*)) \cap \mathcal{M}_\psi,a)/\text{Inn}(N)\to Z(F) \to Q$$

$$\to Q_1 = \text{Im}(\ast|_{\mathcal{M}_\psi,a})/\text{Inn}(F) \to 1,$$  

(6)

where $(\ker(*)) \cap \mathcal{M}_\psi,a)/\text{Inn}(N)$ is contained in $(\ker(*)) \cap \mathcal{M}_\psi)/\text{Inn}(N) = N_{\text{Out}(N)}(F)$.

**Proof of Theorem 1.9.** If $\text{SL}_n(Z)$ acts trivially on $M'$, it is obvious that the induced homomorphism $\text{SL}_n(Z) \to \text{Out}(F)$ is trivial. In order to prove the converse, it’s enough to prove $\text{SL}_n(Z) \to \text{Out}(\pi)$ is trivial for a given group action of $\text{SL}_n(Z)$ on $M$, considering Theorem 1.6. We will use the exact sequence (6) in Lemma 6.1. Note that $Q_1$ is a subgroup of $\text{Out}(F)$. By the assumption that the group homomorphism $\text{SL}_n(Z) \to \text{Out}(F)$ is trivial, the composite

$$\text{SL}_n(Z) \to \text{Out}(\pi) \to Q \to Q_1$$

has to be trivial. Therefore, the map $\text{SL}_n(Z) \to \text{Out}(\pi) \to Q$ has image in $Q_2$. Denote by $K = (\ker(*)) \cap \mathcal{M}_\psi,a)/\text{Inn}(N)$ and $Z = Z(F)$ to fit into an exact sequence

$$1 \to Z \to K \to Q_2 \to 1.$$  

(7)

Since $K$ is a subgroup of $N_{\text{Out}(N)}(F)$, $Z(F)$ lies in the center of $K$. Therefore, the exact sequence (7) is a central extension. Let $\text{St}_n(Z)$ be the Steinberg group and denote by the composite

$$\alpha : \text{St}_n(Z) \to \text{SL}_n(Z) \to \text{Out}(\pi) \to Q_2$$

19
Since $H_2(\text{St}_n(\mathbb{Z}); \mathbb{Z}) = 0$, we have $H^2(\text{St}_n(\mathbb{Z}); \mathbb{Z}) = 0$ by the universal coefficient theorem. Therefore, $\alpha$ could be lifted to be a group homomorphism $\alpha' : \text{St}_n(\mathbb{Z}) \to K$ by Lemma 5.1. Note that $K$ is a subgroup of $\text{Out}(N)$ and the cohomological dimension of $N$ is at most $r$. By Lemma 5.3, $\alpha'$ is trivial and thus $\alpha$ is trivial. This implies the group homomorphism $\text{SL}_n(\mathbb{Z}) \to \text{Out}(\pi)$ has image in $H^1(F; Z(N))$. Since $H^1(F; Z(N))$ is abelian and $\text{SL}_n(\mathbb{Z})$ is perfect, the map $\text{SL}_n(\mathbb{Z}) \to \text{Out}(\pi)$ has to be trivial. The proof is finished.

7 Examples

In this section, we give further applications of Theorem 1.6 and Theorem 1.9.

The proof of the following lemma is similar to that of the corresponding result for $\text{SL}_n(\mathbb{Z})$, proved by Kielak [19] (Theorem 2.3.4).

**Lemma 7.1** Let $p$ be a prime. Then $\text{Hom}(\text{St}_n(\mathbb{Z}), \text{GL}_k(\mathbb{Z}/p)) = 1$, i.e. any group homomorphism $g : \text{St}_n(\mathbb{Z}) \to \text{GL}_k(\mathbb{Z}/p)$ is trivial, for $k < n - 1$.

**Proof.** Let $N = \ker g$. Since the image of $N$ in $\text{SL}_n(\mathbb{Z})$ is of finite index, the map $g$ factors through $g' : \text{St}_n(\mathbb{Z}/N) \to \text{GL}_k(\mathbb{Z}/p)$ for some integer $N$. Note that $\text{St}_n(R_1 \times R_2) = \text{St}_n(R_1) \times \text{St}_n(R_2)$ for rings $R_1$ and $R_2$. Without loss of generality, we assume that $N$ is a power of a prime number. Let $Z$ be the center of $\text{St}_n(\mathbb{Z}/N)$. Suppose that $\text{GL}_k(\mathbb{Z}/p)$ acts on $(\mathbb{Z}/p)^k$ naturally. We could assume that the action of $\text{Im} g'$ on $(\mathbb{Z}/p)^k$ is irreducible. Note that $(\mathbb{Z}/p)^k$ is the intersection of eigenspaces of $g'(v), v \in Z$ (if necessary, we may consider the algebraic closure of $\mathbb{Z}/p$). After change of basis in $(\mathbb{Z}/p)^k$, we get that $g'(N)$ lies in the center of $\text{GL}_k(\mathbb{Z}/p)$. Therefore, $g'$ induces a map $g'' : \text{PSL}_n(\mathbb{Z}/N) \to \text{PGL}_k(\mathbb{Z}/p)$. However, it’s known that $g''$ has to be trivial by Landazuri–Seitz [21].

Let $A$ be a finite abelian group. For a prime $p$, define the $p$-rank $\text{rank}_p(A)$ as the dimension of $A \otimes_{\mathbb{Z}/p} \mathbb{Z}/p$, as a vector space over $\mathbb{Z}/p$.

**Lemma 7.2** Let $A$ be a finite abelian group with $\text{rank}_p(A) < n - 1$ for every prime $p$. Then every group homomorphism $\text{St}_n(\mathbb{Z}) \to \text{Aut}(A)$ is trivial for $n \geq 3$.

**Proof.** Since a group homomorphism preserves $p$-Sylow subgroup of $A$, it’s enough to prove the theorem for a $p$-group $A$. If $A$ is an elementary $p$-group, $\text{Aut}(A) = \text{GL}_k(\mathbb{Z}/p), k = \text{rank}_p(A)$. Thus $\text{Hom}(\text{St}_n(\mathbb{Z}), \text{Aut}(A))$ is trivial by Lemma 7.1. If $A$ is not elementary, the subgroup $A_1$ consisting of elements of order $p$ is a characteristic subgroup $A$. Inductively, we assume that $\text{Hom}(\text{St}_n(\mathbb{Z}), \text{Aut}(A/A_1))$ and $\text{Hom}(\text{St}_n(\mathbb{Z}), \text{Aut}(A_1))$ are both trivial. Let

$$B = \{ f \in \text{Aut}(A) : f|_{A_1} = \text{id}_{A_1} \}.$$ 

The group $\text{ker}(\text{Aut}(A) \to \text{Aut}(A/A_1)) \cap B = H^1(A/A_1; A_1)$ is abelian (cf. [16], Prop. 5, p. 45). Since $\text{St}_n(\mathbb{Z})$ $(n \geq 3)$ is perfect, we have $\text{Hom}(\text{St}_n(\mathbb{Z}), \text{Aut}(A))$ is trivial.
Lemma 7.3  Let $\pi$ be a finite nilpotent group with the upper central series $1 = Z_0 < Z_1 < \cdots < Z_k = \pi$. Suppose that $\text{rank}_p(Z_i/Z_{i-1}) < n - 1$ for each prime $p$ and each $i = 1, \cdots, k$. Then the set of group homomorphisms $\text{Hom}(\text{St}_n(\mathbb{Z}), \text{Out}(\pi)) = 1$.

Proof. When $k = 1$, this is proved in Lemma 7.2. Considering the central extension 
$$1 \to Z_i/Z_{i-1} \to \pi/Z_{i-1} \to \pi/Z_i \to 1$$
the statement could be proved inductively using Lemma 7.2 and Lemma 5.2. □

Corollary 7.4  Let $M^r$ be a closed almost flat manifold with the holonomy group $\Phi$ nilpotent satisfying the condition in Lemma 7.3. Then Conjecture 1.1 holds for $M$. In particular, when $M$ is a closed flat manifold with abelian holonomy group, Conjecture 1.1 holds.

Proof. By Lemma 7.3, $\text{Hom}(\text{St}_n(\mathbb{Z}), \text{Out}(\Phi)) = 1$ and thus $\text{Hom}(\text{SL}_n(\mathbb{Z}), \text{Out}(\Phi)) = 1$. Theorem 1.9 implies that Conjecture 1.1 is true. When $M$ is flat, $\Phi$ is a subgroup of $\text{GL}_r(\mathbb{Z})$. If $\Phi$ is abelian, elements in $\Phi$ could be simultaneously diagonalizable in $\text{GL}_r(\mathbb{C})$. Therefore, $\text{rank}_p(\Phi) \leq r - 1 < n - 1$ for each prime $p$. □

Lemma 7.5  Let $\Phi$ be a dihedral group $D_{2k}$, symmetric group $S_k$ or alternating group $A_k$. Then $\text{Hom}(\text{SL}_n(\mathbb{Z}), \text{Out}(\Phi)) = 1$ for $n \geq 3$.

Proof. It’s well-known that $\text{Aut}(D_{2k}) = \mathbb{Z}/k \rtimes \text{Aut}(\mathbb{Z}/k)$, a solvable group. Therefore, $\text{Out}(D_{2k})$ is solvable. When $\Phi$ is $S_k$ or $A_k$, we have $\text{Out}(\Phi)$ is abelian. However, $\text{SL}_n(\mathbb{Z})$ is perfect and thus $\text{Hom}(\text{SL}_n(\mathbb{Z}), \text{Out}(\Phi)) = 1$. □

For flat manifolds of low dimensions, we obtain the following.

Corollary 7.6  Conjecture 1.1 holds for closed flat manifolds $M^r$ of dimension $r \leq 5$.

Proof. The proof depends on the classifications of low-dimensional holonomy groups $\Phi$. When $r \leq 3$, $\Phi = \{1\}, \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4, \mathbb{Z}/6$ or $\mathbb{Z}/2 \rtimes \mathbb{Z}/2$ (cf. [31] Corollary 3.5.6, p.118). They are all abelian. By Lemma 7.3, Conjecture 1.1 is true. When $r = 4$, the nonabelian $\Phi = D_6, D_8, D_{12}$ or $A_4$ (cf. [7]). By Lemma 7.5 and Theorem 1.9, Conjecture 1.1 is true. When $r = 5$, the nonabelian $\Phi = D_6, D_8, D_{12}, D_6 \rtimes \mathbb{Z}/2, D_8 \rtimes \mathbb{Z}/3, D_{12} \rtimes \mathbb{Z}/2, A_4, A_4 \rtimes \mathbb{Z}/2, A_4 \rtimes \mathbb{Z}/2 \rtimes \mathbb{Z}/2, S_4$ or $(\mathbb{Z}/2 \rtimes \mathbb{Z}/2) \rtimes \mathbb{Z}/4$ (cf. [28], Theorem 1, or [16], Theorem 4.2). By Lemma 7.5, it’s enough to consider the $\Phi$ with two factors. By Lemma 7.3 and Lemma 5.2, $\text{Hom}(\text{SL}_n(\mathbb{Z}), \text{Out}(\Phi)) = 1$ and thus Conjecture 1.1 holds by Theorem 1.9. □

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References

[1] H. Bass, J. Milnor, and J.P. Serre, The congruence subgroup property for $SL_n$ ($n > 3$) and $Sp_{2n}$, ($n > 2$), Publ. Math. d’IHES. 33 (1967) 59-137.

[2] M. Belolipetsky and A. Lubotzky, Finite groups and hyperbolic manifolds, Invent. Math. 162 (2005), 459-472.

[3] A. J. Berrick, An approach to algebraic K-Theory, Pitman Research Notes in Math 56, London, 1982.

[4] M. Bridson, K. Vogtmann, Actions of automorphism groups of free groups on homology spheres and acyclic manifolds, Commentarii Mathematici Helvetici 86 (2011), 73-90.

[5] G. Bredon, Orientation in generalized manifolds and applications to the theory of transformation group, Michigan Math. Journal 7 (1960), 35-64.

[6] G. Bredon, Sheaf Theory, second edition. Graduate Texts in Mathematics, 170. Springer-Verlag, New York, 1997.

[7] H. Brown, R. Bulow, J. Neubuser, H. Wondratschek, and H. Zassenhaus, Crystallography groups of four-dimensional space, Wiley Monographs in Crystallography, Wiley-Interscience, New York, 1978.

[8] A. Borel, Seminar on transformation group, Annals of Mathematics Studies, No. 46, Princeton University Press, Princeton, N.J. 1960.

[9] L. Charlap and A. Vasquez, Compact Flat Riemannian Manifolds III: The Group of Affinities, American Journal of Mathematics, 95 (1973), 471-494.

[10] C. Cid and T. Schulz, Computation of Five- and Six-Dimensional Bieberbach Groups, Exp. Math. 10(2001), 109-115.

[11] R. Dennis and M. Stein, The functor K: a survey of computations and problems, Algebraic K-Theory II, Lecture Notes in Math., vol. 342, Springer-Verlag, Berlin and New York, 1973, pp. 243-280.

[12] B. Farb and P. Shalen, Real-analytic action of lattices, Invent. math. 135 (1999), 273-296.

[13] D. Fisher, Groups acting on manifolds: Around the Zimmer program, In Geometry, Rigidity, and Group Actions 72-157. Univ. Chicago Press, Chicago, 2011.

[14] J. Franks, M. Handel: Area preserving group actions on surfaces, Geom. Topol. 7, 757-771 (2003)

[15] M. Gromov, Almost flat manifolds, Journal of Differential Geometry 13 (1978), 231–241.
[16] K.W. Gruenberg, *Cohomological topics in group theory*, Lecture Notes in Mathematics, 143, Springer, Berlin (1970).

[17] P. Igodt and W. Malfait, *Extensions realising a faithful abstract kernel and their automorphisms*, Manuscripta mathematica 84(1994), 135-162.

[18] S. Illman, *Smooth equivariant triangulations of G-manifolds for G a finite group*, Math. Ann., 233(1978), 199-220.

[19] D. Kielak, *Free and linear representations of outer automorphism groups of free groups*, PhD thesis, Oxford University. https://www.math.uni-bielefeld.de/~dkielak/DPhil.pdf.

[20] R. Kulkarni, *Symmetries of surfaces*, Topology, 26(1987), 195-203.

[21] V. Landazuri and G. Seitz, *On the minimal degrees of projective representations of the finite Chevalley groups*, J. Algebra, 32(1974):418-443.

[22] B. Magurn. *An algebraic introduction to K-theory*, Cambridge University Press, 2002.

[23] L.N. Mann and J.C. Su, *Actions of elementary p-groups on manifolds*, Trans. Amer. Math. Soc., 106 (1963), 115-126.

[24] J. Milnor, *Introduction to algebraic K-theory*, Princeton University Press, Princeton, N.J., 1971. Annals of Mathematics Studies, No. 72.

[25] A. Navas, *Groups of Circle Diffeomorphisms*, Univ. Chicago Press, Chicago, 2011.

[26] L. Polterovich, *Growth of maps, distortion in groups and symplectic geometry*, Invent. math. 150, 655-686 (2002)

[27] E. Ruh, *Almost flat manifolds*, Journal of Differential Geometry 17(1982), 1-14.

[28] A. Szczepański, *Holonomy groups of five-dimensional Bieberbach groups*, Manuscripta Math. 90(1996), 383-389.

[29] W. Thurston, Chapter 13: Orbifolds, part of the Geometry and Topology of Three Manifolds, unpublished manuscript, 2002, available at http://www.msri.org/publications/books/gt3m/

[30] W. van der Kallen, *The Schur multipliers of SL(3, Z) and SL(4, Z)*, Math. Ann. 212 (1974/75), 47-49.

[31] D. Witte, *Arithmetic groups of higher Q-rank cannot act on 1-manifolds*, Proc. AMS, 122(1994), 333-340.

[32] S. Weinberger, *SL(n, Z) cannot act on small tori*, Geometric topology (Athens, GA, 1993), 406-408, AMS/IP Stud. Adv. Math., Amer. Math. Soc., Providence, RI, 1997.
[33] S. Weinberger, *Some remarks inspired by the C₀ Zimmer program*, in Rigidity and Group actions, Chicago lecture notes, 2011.

[34] J. Wolf, *Spaces of Constant Curvature*, Sixth Edition, AMS Chelsea Publishing, 2011.

[35] S. Ye, *Low-dimensional representations of matrix groups and group actions on CAT(0) spaces and manifolds*, J. Algebra 409(2014), 219-243.

[36] R. Zimmer and D.W. Morris, *Ergodic Theory, Groups, and Geometry*, American Mathematical Society, Providence, 2008.

Department of Mathematical Sciences, Xi’an Jiaotong-Liverpool University, 111 Ren Ai Road, Suzhou, Jiangsu 215123, China.  
E-mail: Shengkui.Ye@xjtlu.edu.cn