The Graviton Propagator with a Non-Conserved External Generating Source

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Abstract
A novel general expression is obtained for the graviton propagator from Lagrangian field theory by taking into account the necessary fact that in the functional differential approach of quantum field theory, in order to generate non-linearities in gravitation and interactions with matter, the external source $T_{\mu\nu}$, coupled to the gravitational field, should \textit{a priori} not be conserved $\partial_{\mu}T_{\mu\nu} \neq 0$, so variations with respect to its ten components may be varied \textit{independently}. The resulting propagator is the one which arises in the functional approach and does \textit{not} coincide with the corresponding time-ordered product of two fields and it includes so-called Schwinger terms. The quantization is carried out in a gauge corresponding to physical states with two polarization states to ensure positivity in quantum applications.

KEY WORDS: Graviton propagator; quantum gravity; non-conserved external sources; Schwinger terms
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1 Introduction
A basic ingredient in quantum gravity computations is the graviton propagator ([cf.1-5]). The latter mediates the gravitational interaction between

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all particles to the leading order in the gravitational coupling constant. In the so-called functional differential treatment \([6–9, 11]\) of quantum field theory, referred as the quantum dynamical principle approach, based on functional derivative techniques with respect to external sources coupled to the underlying fields in a theory, functional derivatives are taken of the so-called vacuum-to-vacuum transition amplitude. The latter generates \(n\)-point functions by functional differentiations leading finally to transition amplitudes for various physical processes. For higher spin fields such as the electromagnetic vector potential \(A^\mu\), the gluon field \(A^a_\mu\), and certainly the gravitational field \(h^\mu\nu\), the respective external sources \(J^\mu\), \(J^a_\mu\), \(T^\mu\nu\), coupled to these fields, cannot \emph{a priori} taken to be conserved so that their respective components may be varied \emph{independently}. The consequences of relaxing the conservation of these external sources are highly non-trivial. For one thing the corresponding field propagators become modified. Also they have led to the rediscovery \([6,7]\) of Faddeev–Popov \([12]\) factors in non-abelian gauge theories and the discovery \([7]\) of even more generalized such factors, directly from the functional differential treatment, via the application of the quantum dynamical principle, in the presence of external sources, without using commutation rules, and without even going to the well known complicated structures of the underlying Hamiltonians. A brief account of this is given in the concluding section for the convenience of the reader.

For higher spin fields, the propagator and the time-ordered product of two fields do \emph{not} coincide as the former includes so-called Schwinger terms which, in general, lead to a simplification of the expression for the propagator over the time-ordered one. This is well known for spin 1, and, as shown below, is also true for the graviton propagator. Let \(h^{\mu\nu}\) denote the gravitational field (see Sect.2). We work in a gauge

\[
\partial_i h^{\mu\nu} = 0
\]

where \(i = 1, 2, 3; \; \nu = 0, 1, 2, 3\), which guarantees that only two states of polarization occur with the massless particle and ensures positivity in quantum applications avoiding non-physical states. Let \(T^{\mu\nu}\) denote an external source coupled to the gravitational field \(h^{\mu\nu}\) (see Sect.2), and let \(\langle 0_+ | 0_- \rangle^T\) denote the vacuum-to-vacuum transition amplitude in the presence of the external source. The propagator of the gravitational field is then defined by

\[
\Delta^{\mu\nu\sigma\lambda}_{+}(x, x') = i \left( \frac{\delta}{\delta T^{\mu\nu}(x)} \frac{\delta}{\delta T^{\sigma\lambda}(x')} \langle 0_+ | 0_- \rangle^T \right) \left/ \langle 0_+ | 0_- \rangle^T \right.,
\]

in the limit of the vanishing of the external source \(T^{\mu\nu}\). In more detail we
may rewrite (2) as

\[ \Delta^{\mu\nu;\sigma\lambda}(x, x') = i \frac{\langle 0_+ | (h^{\mu\nu}(x)h^{\sigma\lambda}(x')) + 0_- \rangle^T}{\langle 0_+ | 0_- \rangle^T} + \frac{\langle 0_+ | \delta T^{\mu\nu}(x)h^{\sigma\lambda}(x') \rangle 0_- \rangle^T}{\langle 0_+ | 0_- \rangle^T} \]  

(3)

in the limit of vanishing \( T_{\mu\nu} \), where the first term on the right-hand side, up to the \( i \) factor, denotes the time-ordered product. In the second term, the functional derivative with respect to the external source \( T_{\mu\nu}(x) \) is taken by keeping the independent field components of \( h^{\sigma\lambda}(x') \) fixed. The dependent field components depend on the external source and lead to extra terms on the right-hand side of (3) in addition to the time-ordered product and may be referred to as Schwinger terms. For a detailed derivation of the general identity in (3) see Ref. [10] (see also [11]). These additional terms lead to a simplification of the expression for the propagator over the time-ordered product. Accordingly, the propagator and the time-ordered product do not coincide and it is the propagator \( \Delta^{\mu\nu;\sigma\lambda} \) that appears in the functional approach and not the time-ordered product. The derivation of the explicit expression for \( \Delta^{\mu\nu;\sigma\lambda}(x, x') \) follows by relaxing the conservation of \( T_{\mu\nu} \) and it includes 30 terms in contrast to the well known case involving only 3 terms when a conservation law of \( T_{\mu\nu} \) is imposed. It is important to emphasize that our interest here is in the propagator, the basic component which appears in the theory, and not the time-ordered product. In the concluding section, some additional pertinent comments are made regarding our expression for the propagator. Our notation for the Minkowski meter is \( g^{\mu\nu} = \text{diag}[-1, 1, 1, 1] \), also quite generally we set \( i, j, k, l = 1, 2, 3 \), \( a, b = 1, 2 \), while \( \mu, \nu, \sigma, \lambda = 0, 1, 2, 3 \).

2 The Graviton Propagator

For the Lagrangian density of the gravitational field \( h^{\mu\nu} \) coupled to an external source \( T_{\mu\nu} \), we take

\[
\mathcal{L} = -\frac{1}{2} \partial^\alpha h^{\mu\nu} \partial_\alpha h_{\mu\nu} + \frac{1}{2} \partial^\alpha h^{\sigma\sigma} \partial_\alpha h_{\beta\beta} - \partial^\alpha h^{\alpha\beta} \partial^\beta h^{\sigma\sigma} \\
+ \frac{1}{2} \partial_\alpha h^{\alpha\nu} \partial^\beta h_{\beta\nu} + \frac{1}{2} \partial_\alpha h^{\mu\nu} \partial_\mu h_{\alpha\nu} + h^{\mu\nu} T_{\mu\nu},
\]

(4)

where \( h^{\mu\nu} = h^{\nu\mu} \), and as a result \( T_{\mu\nu} \) is chosen to be symmetric. We consider the ten components of \( T_{\mu\nu} \) to be independent by, \( a \text{ priori} \), not imposing a
conservation law for $T_{\mu\nu}$. The action corresponding to the Lagrangian density in (4), in the absence of the external source $T_{\mu\nu}$, is invariant under the gauge transformation $h^{\mu\nu} \rightarrow h^{\mu\nu} + \partial^{\mu}\xi^{\nu} + \partial^{\nu}\xi^{\mu} + \partial^{\mu}\partial^{\nu}\xi$. The gauge constraint in (1) allows us to solve, say, $h_{3\nu}$, in terms of other components:

$$h_{30} = -(\partial_3)^{-1}\partial_a h_{a0}, \quad h_{3a} = -(\partial_3)^{-1}\partial_b h_{ba},$$

$$h_{33} = -(\partial_3)^{-2}\partial_a \partial_b h_{ab}, \quad (5)$$

where $a, b = 1, 2$. Upon substituting the expressions for $h_{3\nu}$ in (4), and varying $h_{ab}$, we obtain

$$(\Box h_{ab} + T_{ab}) - \frac{\partial_b}{\partial_3}(\Box h_{a3} + T_{a3}) - \frac{\partial_a}{\partial_3}(\Box h_{3a} + T_{3a})$$

$$+ \frac{\partial_a \partial_b}{(\partial_3)^2}(\Box h_{33} + T_{33}) + \left[\delta_{ab} + \frac{\partial_a \partial_b}{(\partial_3)^2}\right](\partial^2 h_{00} - \Box h_{ii}) = 0, \quad (6)$$

$a, b = 1, 2$. Upon multiplying (6) by $(\delta_{ab} - \partial_a \partial_b/\partial^2)$, where $\partial^2 = \partial^i \partial_i$, $i = 1, 2, 3$, some tedious algebra leads to

$$-\partial^2 h_{00} = -\frac{1}{2}\Box h_{ii} + \frac{1}{2}\left(\delta_{ij} - \frac{\partial_i \partial_j}{\partial^2}\right) T_{ij}, \quad (7)$$

On the other hand, with the expressions for $h_{3\nu}$ in (5) replaced in (4), variations with respect to $h_{00}$, $h_{0a}$, $a = 1, 2$, give, respectively,

$$-\partial^2 h_{ii} = T_{00}, \quad (8)$$

$$-\partial^2 h_{0a} + \frac{\partial_a}{\partial_3} \partial^2 h_{03} = \left(T_{0a} - \frac{\partial_a}{\partial_3} T_{03}\right) \quad (9)$$

We note that (9) is valid if we formally replace $a$ by 3 since this simply gives $0 = 0$. Accordingly, we may rewrite (9) as

$$-\partial^2 h_{0i} + \frac{\partial_i}{\partial_3} \partial^2 h_{03} = T_{0i} - \frac{\partial_i}{\partial_3} T_{03} \quad (10)$$

where $i = 1, 2, 3$. Upon taking the divergence $\partial^i$ of (10) and using (1), we obtain

$$\frac{\partial_i}{\partial_3} \partial^2 h_{03} = \frac{\partial_i}{\partial^2} \left(\partial_j T_{0j} - \frac{\partial^2}{\partial_3 T_{03}}\right), \quad (11)$$
which upon substitution in (10) gives

$$-\nabla^2 h_{0i} = \left( \delta_{ij} - \frac{\partial_i \partial_j}{\partial^2} \right) T_{0j}. \quad (12)$$

Also upon substitution (8) in (7), and using the fact that \( \Box = \nabla^2 - \partial_0^2 \), we obtain for (7)

$$-\nabla^2 h_{00} = T_{00} + \frac{T}{2} - \frac{1}{2\partial^2} \left( \partial^0 \partial^0 T_{00} + \partial_i \partial_j T_{ij} \right), \quad (13)$$

where \( T = g^{\mu\nu} T_{\mu\nu} = T_{\nu \nu} \).

Equations (8), (12), (13) are not equations of motion as they involve no time derivatives of the corresponding fields and they yield to constraints which together the gauge condition in (1) give rise to two degrees of freedom corresponding to two polarization states for the graviton as it should be.

We now substitute the expression for \(-\nabla^2 h_{00}, \text{as given in (13), in (6)}\) and use (8) to obtain an equation involving \(h_{ij}, i,j = 1,2,3\). Upon multiplying the resulting equation from (6) by \(\partial_i \partial_j\) and using the expressions for \(h_{33}\) in (5) we obtain after some very tedious algebra

$$\left( \Box h_{33} + T_{33} \right) - \frac{1}{2} \left( 1 - \left( \frac{\partial 3}{\partial^2} \right)^2 \right) T + \frac{1}{2\partial^2} \left( -\partial^0 \partial^0 T_{00} + \partial^i \partial^j T_{ij} \right)
- \frac{2}{\partial^2} \partial^i \partial^j T_{ij}
+ \frac{1}{2\partial^2} \left( \frac{\partial 3}{\partial^2} \left( \partial^i \partial^j \partial^2 T_{ij} + \frac{\partial^0 \partial^0}{\partial^2} T_{00} \right) \right) = 0. \quad (14)$$

Similarly, upon multiplying (6) by \(\partial_i\) and using the expression for \(h_{b3}\) in (5), we obtain

$$\left( \Box h_{b3} + T_{b3} \right) - \frac{1}{\partial^2} \left[ \partial_3 \partial^i T_{ib} + \partial_b \partial^i T_{i3} - \frac{\partial_3}{2} \left( \frac{\partial^i \partial^j}{\partial^2} T_{ij} + \frac{\partial^0 \partial^0}{\partial^2} T_{00} \right) + T \right] = 0 \quad (15)$$

To obtain the equation for \(h_{ab}\), we substitute (14), (15) in (6), to obtain after some lengthy algebra

$$\left( \Box h_{ab} + T_{ab} \right) - \frac{1}{2} \left( \delta_{ab} - \frac{\partial_a \partial_b}{\partial^2} \right) T - \frac{1}{\partial^2} \left( \partial_a \partial^i T_{ib} + \partial_b \partial^i T_{ia} \right)
+ \frac{\delta_{ab}}{2\partial^2} \left( -\partial^0 \partial^0 T_{00} + \partial^i \partial^j T_{ij} \right)
+ \frac{\partial_a \partial_b}{2(\partial^2)^2} \left( \partial^i \partial^j T_{ij} + \partial^0 \partial^0 T_{00} \right) = 0. \quad (16)$$

Equations (14), (15), (16) may be now combined in the form

$$-\Box h_{ij} = T_{ij} - \frac{1}{2} \left( \delta_{ij} - \frac{\partial_i \partial_j}{\partial^2} \right) \left( T + \frac{\partial^0 \partial^0}{\partial^2} T_{00} \right)
- \frac{1}{\partial^2} \left[ \partial_i \partial^k T_{kj} + \partial_j \partial^k T_{ki} - \frac{1}{2} \left( \delta_{ij} + \frac{\partial_i \partial_j}{\partial^2} \right) \partial^k \partial^l T_{kl} \right], \quad (17)$$

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where \( i, j, k, l = 1, 2, 3 \).

Equations (17), (13), (12) give the equations for the various components of \( h_{\mu\nu} \). To obtain the unifying equation for \( h_{\mu\nu} \), we note that we may write

\[
    h_{\mu\nu} = g_{\mu i} h_{ij} g_{j\nu} + g_{\mu 0} h_{0j} g_{j\nu} + g_{\mu 0} g_{0\nu},
\]

with \( i, j = 1, 2, 3; \mu, \nu = 0, 1, 2, 3 \), and use in the process the identity

\[
    g_{\mu i} \partial_i = (\partial^\mu + N^\mu \partial_0),
\]

where \( N^\mu \) is the unit time-like vector \( (N^\mu N_\mu = -1) \)

\[
    (N^\mu) = (g^\mu_0) = (1, 0, 0, 0).
\]

Finally, we use the identity relating a tensor \( A_{\lambda\sigma} \), e.g., to the components \( A_{ij} \) as follows:

\[
    g^{\mu i} A_{ij} g^{\nu j} = \left[ g^{\mu \lambda} g^{\nu \sigma} + N^\mu N^\lambda g^{\nu \sigma} + N^\nu N^\sigma g^{\mu \lambda} + N^\mu N^\nu N^\lambda N^\sigma \right] A_{\lambda\sigma},
\]

and the fact that \( \Box = \partial^2 - \partial_0^2 \). A lengthy analysis from (12), (13), (17) then gives the following explicit expression for \( h_{\mu\nu} \):

\[
    h_{\mu\nu} = \frac{1}{(-\Box - i\epsilon)} \left\{ \frac{g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda} - g_{\mu\nu} g^{\lambda\sigma}}{2} \right. \\
    + \left. \frac{1}{2\partial^2} \left[ g^{\mu\nu} \partial^\nu \partial^\lambda + g^{\sigma\lambda} \partial^\nu \partial^\sigma - g^{\mu\sigma} \partial^\nu \partial^\lambda - g^{\mu\lambda} \partial^\nu \partial^\sigma \right. \\
    - g^{\mu\sigma} \partial^\nu \partial^\lambda - g^{\mu\lambda} \partial^\nu \partial^\sigma \left. + \frac{\partial^\nu \partial^\sigma \partial^\lambda \partial^\mu}{\partial^2} \right] \right. \\
    \left. + \frac{1}{2} \left( g^{\mu\nu} + \partial^\mu \partial^\nu \right) \left( \frac{N^\nu \partial^\lambda + N^\lambda \partial^\nu}{\partial^2} \right) \partial_0 + \frac{1}{2} \left( g^{\sigma\lambda} + \partial^\nu \partial^\lambda \right) \left( \frac{N^\nu \partial^\mu + N^\mu \partial^\nu}{\partial^2} \right) \partial_0 \\
    \right. \\
    \left. - \frac{1}{2} \left[ g^{\nu\sigma} (N^\mu \partial^\lambda + N^\lambda \partial^\nu) + g^{\mu\lambda} (N^\nu \partial^\sigma + N^\sigma \partial^\nu) \right] \frac{\partial_0}{\partial^2} \right. \\
    \left. + \frac{\partial^\mu \partial^\nu}{\partial^2} N^\sigma N^\lambda + \frac{\partial^\sigma \partial^\lambda}{\partial^2} N^\mu N^\nu \right\} T_{\sigma\lambda} \\
    \left. + \frac{1}{\partial^2} \left\{ \frac{\partial^\mu \partial^\nu}{\partial^2} N^\sigma N^\lambda + \frac{\partial^\sigma \partial^\lambda}{\partial^2} N^\mu N^\nu \right\} T_{\sigma\lambda}, \right.
\]

\( \epsilon \rightarrow +0 \).
From (22) the explicit expression for the graviton propagator $\Delta^{\mu\nu;\lambda\sigma}_{\mu\nu}(x,x')$ emerges as:

$$\Delta^{\mu\nu;\lambda\sigma}_{\mu\nu}(x,x') = \int \frac{dk}{(2\pi)^4} e^{ik(x-x')} \left[ \frac{\Delta^{\mu\nu;\lambda\sigma}_{\mu\nu}(k)}{k^2 - i\epsilon} + \frac{\Delta^{\mu\nu;\lambda\sigma}_{\mu\nu}(k)}{k^2} \right],$$  

(23)

$\epsilon \to +0$, where $(dk) = dk^0 dk^1 dk^2 dk^3$, $k^2 = k^2 - k^0$, and

$$\Delta^{\mu\nu;\lambda\sigma}_{1;\mu\nu}(k) = \left( g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\nu}g^{\lambda\sigma} \right) \frac{1}{2} \left[ g^{\mu\nu} + k^\mu k^\nu \right] k^0 + \left( \frac{N^\sigma k^\lambda + N^\lambda k^\sigma}{k^2} \right) \frac{k^0}{k^2} + \left( \frac{N^\nu k^\mu + N^\mu k^\nu}{k^2} \right) \frac{k^0}{k^2} + \frac{k^\mu k^\nu}{k^2} N^\sigma N^\lambda + \frac{k^\sigma k^\lambda}{k^2} N^\mu N^\nu,$$

(24)

$$\Delta^{\mu\nu;\lambda\sigma}_{2;\mu\nu}(k) = \frac{k^\mu k^\nu}{k^2} N^\sigma N^\lambda + \frac{k^\sigma k^\lambda}{k^2} N^\mu N^\nu.$$

(25)

The vacuum-to-vacuum transition amplitude for the gravitational field coupled to an external source is then given by

$$\langle 0_+ | 0_+ \rangle^T = \exp \left[ \frac{1}{2} \int (dx)(dx') T_{\mu\nu}(x) \Delta^{\mu\nu;\lambda\sigma}_{\mu\nu}(x,x') T_{\sigma\lambda}(x') \right],$$

(26)

with the graviton propagator given by the explicit expression in (23) - (25).

Now we are ready to make pertinent comments concerning the graviton propagator thus obtained.

3 Conclusion

We have derived a novel expression for the graviton propagator, from Lagrangian field theory, valid for the case when the external source $T_{\mu\nu}$ coupled to the gravitational field is not necessarily conserved, by working in a
gauge where only two polarization physical states of the graviton arise to ensure positivity in the quantum treatment thus avoiding non-physical states. That such a conservation should a priori not to be imposed is a necessary mathematical requirement so that all the ten components of the external source $T_{\mu\nu}$ may be varied independently in order to generate interactions of the gravitational field with matter and produce non-linearity of the gravitational field itself in the functional procedure. The latter requirement arises by noting that such interactions are generated by the application [cf.6, 7] of some functional $F[-i\delta/\delta T_{\mu\nu}]$ to $\langle 0_+ | 0_- \rangle^T$, where $\langle 0_+ | 0_- \rangle$ corresponding to other particles, as well as functional derivatives of their corresponding sources in $F$, have been suppressed to simplify the notation. Accordingly, to vary the ten components of $T_{\mu\nu}$ independently, no conservation may a priori be imposed. The $1/k^2$ terms in (23) - (25) are apparent singularities due to the sufficient powers in $k$ in the corresponding denominators and the three-dimensional character of space, in the same way that this happens for the photon propagator in the Coulomb gauge in quantum electrodynamics, and give rise to static $1/r$ type interactions complicated by the tensorial character of a spin two object. It is important to note that for a conserved $T_{\mu\nu}$, i.e., for $\partial^\mu T_{\mu\nu} = 0$, all the terms in the propagators in (23), with the exception of the terms $(g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\nu}g^{\sigma\lambda})/2$, do not contribute in (26) since all the other terms in (24), (25) involve derivatives of $T_{\mu\nu}$ and the graviton propagator $\Delta^{\mu\nu,\sigma\lambda}(x, x')$ effectively goes over to the well documented expression

$$\frac{1}{(-\Box - i\epsilon)} \frac{(g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\nu}g^{\sigma\lambda})}{2},$$

(27)

which has been known for years [cf.1, 2]. This is unlike the corresponding time-ordered product which does not go over to the result in (27) for $\partial^\mu T_{\mu\nu} = 0$. This may be shown by solving for the time-ordered product in (3) in terms of the propagator and carrying out explicitly, say, the functional derivatives $\delta h^{0i}/\delta T_{\mu\nu}$, $\delta h^{00}/\delta T_{\mu\nu}$, as arising on the right-hand side of (3), by using, in the process, Eqs. (12), (13). In any case, it is the propagator $\Delta^{\mu\nu,\sigma\lambda}_+(x, x')$, as given in (23), is the one that appears in the theory and not the time-ordered product as is often naively assumed. After all the functional derivatives with respect to $T_{\mu\nu}$ are carried out in the theory, one may impose a conservation law on $T_{\mu\nu}$ or even set $T_{\mu\nu}$ equal to zero if required on physical grounds. Such methods have led to the discovery [6,7], in the functional quantum dynamical principle differential approach, of Faddeev–Popov (FP) factors, and of their generalizations, in non-abelian gauge theories such as in QCD and in other theories.

Re-iterating the discussion above, the relevance of the analysis and the
explicit expression derived for the graviton propagator for, \textit{a priori}, not conserved external source $T_{\mu \nu} : \partial^\mu T_{\mu \nu} \neq 0$ is immediate. If, in contrast, a conservation law is \textit{a priori}, imposed then variations with respect to one of the components of $T_{\mu \nu}$ would automatically imply, via such a conservation law, variations with respect some of its \textit{other} components as well. A problem that may arise otherwise, may be readily seen from a simple example. The functional derivative of an expression like $a_{\mu \nu}(x) + b(x) \partial_\mu \partial_\nu |T^\mu \nu(x)|$, with respect to a component $T^\sigma \lambda(x')$ is $(1/2)[a_{\mu \nu}(x) + b(x)\partial_\mu \partial_\nu](\delta^\sigma \mu \delta^\lambda \nu + \delta^\lambda \mu \delta^\sigma \nu)\delta^4(x, x')$, where $a_{\mu \nu}(x), b(x)$, for example, depend on $x$, and not $(1/2)a_{\mu \nu}(x)(\delta^\sigma \mu \delta^\lambda \nu + \delta^\lambda \mu \delta^\sigma \nu)\delta^4(x, x')$ as one may na"ively assume by, \textit{a priori} imposing a conservation law. Also, as mentioned above, the present method, based on the functional differential treatment, as applied to non-abelian gauge theories such as QCD \cite{6,7} leads automatically to the presence of the FP determinant modifying na"ive Feynman rules. The physical relevance of such a factor is important as its omission would lead to a violation of unitarity. For the convenience of the reader we briefly review, before closing the concluding section, on how the FP determinant arises in the functional differential treatment \cite{6,7}.

Consider, for simplicity of the demonstration, the non-abelian gauge theory with Lagrangian density

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu \nu} G^a_{\mu \nu} + J_\mu^a A^a_{\mu}$$

(28)

where $J_\mu^a$ is an external source taken, \textit{a priori}, not to be conserved. Here

$$G^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_o f^{abc} A^b_\mu A^c_\nu$$

(29)

We work in the Coulomb gauge. The gauge field propagator, in analogy to the graviton one in \cite{24}, \cite{25}, is given by

$$D^\mu \nu_{ab} = \delta_{ab}[g^{\mu \nu} - \left(\frac{\partial^\mu \partial^\nu + N^\mu \partial^\nu \partial_0 + N^\nu \partial^\mu \partial_0}{\partial^2}\right)\frac{1}{-\Box - i\varepsilon}]$$

(30)

with $k = 1, 2, 3$.

The quantum dynamical principle states that

$$\frac{\partial}{\partial g_o} \langle 0_+ | 0_- \rangle = i \left\langle 0_+ \left| \int (dx) \frac{\partial}{\partial g_o} \mathcal{L}(x) \right| 0_- \right\rangle$$

(31)

where, with $k = 1, 2, 3$,

$$\frac{\partial}{\partial g_o} \mathcal{L}(x) = -f^{abc} A^b_k (A^c_\delta G^k_\alpha + \frac{1}{2} A^c_\delta G^k_\alpha)$$

(32)
and $G^{kl}_a$ may be expressed in terms of independent fields, that is, for which the canonical conjugate momenta do not vanish. On the other hand, $G^{k0}_a$ depends on the dependent field $A^0_a$. By using the identity

$$(-i)\frac{\delta}{\delta J^\mu_a(x')}\langle 0_+|\mathcal{O}|0_-\rangle = \langle 0_+|(A^\mu_a(x')\mathcal{O}(x))_+|0_-\rangle - i \left\langle 0_+ \left| \frac{\delta}{\delta J^\mu_a(x')}\mathcal{O}(x) \right| 0_- \right\rangle$$

(33)

for an operator $\mathcal{O}(x)$, where $(...)_+$ denotes the time-ordered product, and the functional derivative $\frac{\delta \mathcal{O}(x)}{\delta J^\mu_a(x')}$ in the second term on the right-hand side of (33) is taken by keeping the independent fields and their canonical conjugate kept fixed in $\mathcal{O}(x)$, after the latter is expressed in terms of these fields, together, possibly, in terms of the dependent fields and the external current [7, 10].

From the Lagrangian density in (28), the following relation follows

$$G^{k0}_a = \pi^k_a - \partial^k D_{ab} J^0_b$$

(34)

as a matrix equation, where $\pi^k_a$ denotes the canonical conjugate momentum of $A^k_a$, and $D_{ab}$ is the Green operator satisfying

$$[\delta^{ac} \partial^2 + g_o f^{abc} A^b_k \partial^k] D^{cd}(x, x'; g_o) = \delta^d(x, x') \delta^{ad}$$

(35)

Accordingly, with, a priori, non-conserved $J^\mu_a(x')$, we may vary each of its components independently to obtain from (34)

$$\frac{\delta}{\delta J^\mu_a(x')} G^{k0}_a(x) = -\delta^\mu_0 \partial^k D^{ac}(x, x'; g_o)$$

(36)

Hence from (32), (33), and (36), we may write

$$\left\langle 0_+ \left| \frac{\partial}{\partial g_o} \mathcal{L}(x) \right| 0_- \right\rangle = \left[ \left( \frac{\partial}{\partial g_o} \mathcal{L} \right)' + i f^{bca} A^b_k \partial^k D^{ac}(x, x'; g_o) \right] \langle 0_+ | 0_- \rangle$$

(37)

where the primes mean to replace $A^\mu_c(x)$ in the corresponding expressions by the functional differential operator $(-i)\delta/\delta J^\mu_c(x)$.

Clearly, upon an elementary integration over $g_o$ in (31) by using, in the process, (37) and the equation for $D^{ac}$ in (35), we obtain the FP determinant

$$\exp \text{Tr} \ln[1 - i g_o \frac{1}{\partial^2} A^\mu_k \partial^k]$$

(38)

as a multiplicative modifying differential operating factor in $\langle 0_+ | 0_- \rangle$. For additional related details see [6, 7] and also for further generalizations of the occurrence of such factors in field theory.
It is interesting to extend such analyses [6][7], as well as of gauge transformations [6], and covariance [13], to theories involving gravity. This would be exponentially much harder to do and will be attempted in further investigations. In this regard, our ultimate interest is in aspects of renormalizability [14] and rules for physical applications that would follow from our, a priori, systematic analysis carried out at the outset, in a quantum setting with the newly modified propagator, by a functional differential treatment, in the presence of external sources, to generate not-linearities in gravitation and interactions with matter.

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