Grand Canonical Partition Function for Unidimensional Systems: Application to Hubbard Model up to Order $\beta^3$

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Abstract

We exploit the Grassmannian nature of the variables involved in the path integral expression of the grand canonical partition function for self-interacting fermionic models to show, in one-space dimension, a general relation among the terms of its expansion in the high temperature limit and a combination of cofactors of a suitable matrix with commuting entries. As an application, we apply this framework to calculate the exact coefficients, up to order $\beta^3$, of the expansion of the grand canonical partition function for the Hubbard model in $d = (1 + 1)$ in the high temperature limit. The results are valid for any set of parameters that characterize the model.

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1. Introduction

A quantum system at thermal equilibrium can be completely described provided that one knows its grand canonical partition function, which can be expressed as a path integral. For bosonic systems, an advantageous feature of the path integral approach is that of employing commuting functions instead of non-commuting operators. For fermionic systems, however, such an advantage is not obvious to hold, as the integration variables are also non-commuting.

In Condensed Matter Physics we have self-interacting fermionic models that describe correlated electrons. Certainly one of the most studied models is the Hubbard model [1] in a variety of space dimensions.

Due to the non-commutative nature of the fermion fields, it is a common practice to bosonize the fermionic model. In the path integral formulation of the Hubbard model, this path integral can be re-written in terms of auxiliary bosonic fields using the Hubbard–Stratonovich transformation [2]. The Hubbard–Stratonovich transformation allows the introduction of different decompositions through different auxiliary bosonic fields [3]. Such decompositions are equivalent only when the path integral can be calculated exactly. However, due to the quartic interaction term in the Hubbard model, only perturbative results are attained, and we can no longer relate terms of the perturbative expansion obtained through different decompositions [3]. The ambiguity thus arose is handled by a suitable choice of decompositions, guided by the physical symmetries of the problem under consideration.

In the recent literature, we find many papers discussing the high temperature expansion of the Hubbard model in space dimensions larger than one [4]. Even though we have exact results for the Hubbard model in one space dimension at zero temperature [5], the study of the behavior of the unidimensional Hubbard model in the high temperature limit, we only find in the work by Takahashi [6] and Shiba–Pincus [7] in the seventies.

Takahashi [6] derives an integral equation for the thermodynamic potential. However, closed expressions for the physical quantities are only obtained in two limits: $U \to 0$ and $U \to \infty$. Shiba–Pincus [7] made a numerical analysis for the unidimensional half-filled-band Hubbard model as a function of temperature. They considered two kinds of boundary condition: i) finite chain with open ends; ii) finite ring with cyclic boundary conditions. In both cases, the model studied has at most six atoms. The results of the finite size model were
used to extrapolate to the case of an infinity chain at finite temperature.

We exploited in a previous paper the grassmannian nature of the fermionic fields to study the high temperature behavior of the grand canonical partition function for the anharmonic fermionic oscillator [8]. That model has zero (space) dimension, and a natural extension is the unidimensional Hubbard model [1].

In a recent paper [9], we calculated the exact first-order coefficient in $\beta$ $(\beta = \frac{1}{kT})$ of the grand canonical partition function for the unidimensional Hubbard model in the high temperature limit. Our result was free from any ambiguity, like the one introduced by the Hubbard–Stratonovich transformation [3]. Besides, it was valid for any set of parameters characterizing the model.

In this article, the method for evaluating the path integral of self–interacting fermionic systems presented in references [8,9] is improved, being thus extended to get the exact higher order coefficients of the grand canonical partition function for the unidimensional systems.

Exemplifying the application of our method, we consider the unidimensional periodic Hubbard model. The exact coefficients of order $\beta^2$ and $\beta^3$ of the grand canonical partition function expansion are calculated in the high temperature limit.

In section 2 we present the method of evaluating multivariable fermionic integrals, based on the grassmannian nature of the fermionic fields. The results derived in this section are valid for any unidimensional self–interacting fermionic model. We also write down the kernel of the Grassmann function associated to $\mathbf{K}$ in the Hubbard model, where $\mathbf{K} = \mathbf{H} - \mu \mathbf{N}$ (see eq. (2.2)). In section 3 we discuss the symmetries in one–space dimension that allow us to show that many multivariable integrals involved in the calculation are equal. Those symmetries guarantee that we have a small number of integrals to calculate. In section 4 we calculate the exact coefficients of order $\beta^2$ and $\beta^3$ in the expansion of the grand canonical partition function in the high temperature limit. In section 5 we obtain the Helmholtz free energy of the undimensional Hubbard model up to order $\beta^3$. The result is valid for any number $N$ of space sites. We derive the average energy by site, the average difference of number of spins up and down per site and finally the average of the square of magnetization per site. In Appendix A, we show that the moments of gaussian Grassmann multivariable integrals are equal to co–factors of an appropriate matrix.
2. Expansion in the High Temperature Limit and the Grassmann Multivariable Integrals

We have recently shown that the grand canonical partition function of the Hubbard model in \( d = (1 + 1) \), up to first order in \( \beta \), for a finite number \( N \) of space sites is [9],

\[
Z(\beta, \mu) = \int \prod_{l=1}^{N} \prod_{\sigma=\pm 1}^{M-1} d\psi_\sigma(x_l, \tau_l) d\bar{\psi}_\sigma(x_l, \tau_l) \\
\sum_{\epsilon_l=1}^{N} \sum_{\sigma=\pm 1}^{M-1} \psi_\sigma(x_l, \tau_l) (\psi_\sigma(x_l, \tau_l) - \psi_\sigma(x_l, \tau_{l+1})) e^{-\epsilon \sum_{i=0}^{M-1} K(\bar{\psi}_\sigma(x_l, \tau_l), \psi_\sigma(x_l, \tau_l)) + O(\beta^2)},
\]

(2.1)

where \( \beta = \frac{1}{kT} \), \( k \) is the Boltzmann constant, and \( T \) is the absolute temperature. The space lattice has \( N \) sites and \( M \) is the number of sites in the temperature lattice, such that \( \epsilon M = \beta \). \( K \) is given by

\[
K = H - \mu N,
\]

(2.2)

\( H \) is the hamiltonian of the system, \( \mu \) is the chemical potential and \( N \) is the total number of particles operator.

Using a previous result [10], we were able, for fixed \( N \) and \( M \), to re-write eq.(2.1) as a sum of determinants. We used the simbolic manipulation language MAPLE V.3 to calculate the determinants for fixed numerical values of \( N \) and \( M \), and get a recursive expression for the grand canonical partition function of the unidimensional Hubbard model up to order \( \beta \):

\[
Z(\beta, \mu; B) = 2^{2N} \left[ 1 - N \beta (E_0 - \mu) + \frac{U}{4} \right].
\]

(2.3)

It is important to point out that the evaluation of determinants of \( NM \times NM \) matrices were in order. Due to memory and speed limitations, such task got restricted to matrices for which \( NM \leq 30 \). It was also a matter of luck to obtain the expression (2.3).

Now, we present a new method to obtain the exact coefficients of the expansion of the grand canonical partition function for any self-interacting fermionic unidimensional model in the high temperature limit. The fermionic models have \( N \) space sites, where \( N \) is any integer value. From the results derived we can calculate the thermodynamic limit of \( N \to \infty \).
The grand canonical partition function in the limit of high temperature is,

\[ Z(\beta, \mu) = \text{Tr}(e^{-\beta K}) \]

\[ = \text{Tr}[\mathbb{1} - \beta K] + \frac{\beta^2}{2} \text{Tr}[K^2] - \frac{\beta^3}{3!} \text{Tr}[K^3] + \cdots, \tag{2.4} \]

where \( K \) is given by (2.2). Really, the second line on the r.h.s. of eq. (2.4) is the expansion of the exponential of an operator.

The trace of any fermionic operator, written in terms of creation (\( a_i^\dagger \)) and destruction (\( a_j \)) operators, can be mapped in terms of the generators of the Grassmann algebra \( \{\bar{\eta}_i, \eta_j\} \), provided we make the identification:

\[ a_i^\dagger \to \bar{\eta}_i \quad \text{and} \quad a_j \to \frac{\partial}{\partial \bar{\eta}_j}. \tag{2.5} \]

Let us consider a Grassmann algebra of dimension \( 2^{2N} \), whose generators are: \( \{\bar{\eta}_1, \cdots, \bar{\eta}_N; \eta_1, \cdots, \eta_N\} \). The generators \( \eta_i, \bar{\eta}_j \) satisfy anti-commutation relations:

\[ \{\eta_i, \eta_j\} = 0, \quad \{\bar{\eta}_i, \bar{\eta}_j\} = 0 \quad \text{and} \quad \{\bar{\eta}_i, \eta_j\} = 0, \]

where \( i, j = 1, 2, \cdots, N \).

The product of two fermionic operators \( A \) and \( B \), written in terms of the Grassmann generators, is

\[ (AB)(\bar{\eta}, \eta) = \int \prod_{I=1}^{N} d\eta_I^I d\bar{\eta}_I^I A(\bar{\eta}, \eta') B(\bar{\eta}', \eta) e^{-\sum_{j=1}^{J} \bar{\eta}_j \eta_j}, \tag{2.6} \]

where \( A(\bar{\eta}, \eta') \) and \( B(\bar{\eta}', \eta) \) are the kernel of the fermionic operators \( A \) and \( B \), respectively.

For fermionic operators in normal order,

\[ A = \sum_{n_1, \cdots, n_N = 0}^{1} A_{n_1, \cdots, n_N; m_1, \cdots, m_N}^\otimes (a_i^\dagger)^{n_1} \cdots (a_N^\dagger)^{n_N} (a_N)^{m_N} \cdots (a_1)^{m_1}, \tag{2.7a} \]

and

\[ B = \sum_{n_1, \cdots, n_N = 0}^{1} B_{n_1, \cdots, n_N; m_1, \cdots, m_N}^\otimes (a_i^\dagger)^{n_1} \cdots (a_N^\dagger)^{n_N} (a_N)^{m_N} \cdots (a_1)^{m_1}, \tag{2.7b} \]
with $A_{n_1,\ldots,n_N;m_1,\ldots,m_N}^{\oplus}$ and $B_{n_1,\ldots,n_N;m_1,\ldots,m_N}^{\oplus}$ being commuting constants. The kernel of these operators are given by,

$$A(\vec{\eta}, \vec{\eta}') = e^{\sum_{l=1}^{N} \bar{\eta}_l \eta'_l} A^{\oplus}(\vec{\eta}, \vec{\eta}')$$  \hfill (2.7c)

and

$$B(\vec{\eta}', \eta) = e^{\sum_{l=1}^{N} \bar{\eta}'_l \eta_l} B^{\oplus}(\vec{\eta}', \eta)$$  \hfill (2.7d)

where, in a naive way, we can say that we get $A^{\oplus}(\vec{\eta}, \vec{\eta}')$ from eq.(2.7a) by replacing $a_i^\dagger \rightarrow \bar{\eta}_i$ and $a_j \rightarrow \eta_j$, and similarly for $B^{\oplus}(\vec{\eta}', \eta)$.

All the operators considered in this paper are in normal order.

The trace of any fermionic operator $O$ is\cite{11}

$$\text{Tr}[O] = \int N \prod_{l=1}^{N} d\eta_l d\bar{\eta}_l \ O(\vec{\eta}, \vec{\eta}) \ e^{\sum_{j=1}^{N} \bar{\eta}_j \eta_j} ,$$  \hfill (2.8)

where we use the shorthand notation: $\vec{\eta} \equiv \{\bar{\eta}_1, \ldots, \bar{\eta}_N\}$ and $\eta \equiv \{\eta_1, \ldots, \eta_N\}$, and $O(\vec{\eta}, \vec{\eta})$ is the kernel of the fermionic operator $O$. For the case where the operator $O$ is a product of $n$ normal ordered fermionic operators $Q$, we use the relations (2.6–8) and algebraic manipulations to write the trace of such product of operators as

$$\text{Tr}[Q^n] = \int N \prod_{l=1}^{N} \prod_{\mu=0}^{n-1} d\eta_l(\tau_\mu) d\bar{\eta}_l(\tau_\mu) \ e^{\sum_{l=1}^{n-1} \bar{\eta}_l(\tau_\nu)\eta_l(\tau_{\nu+1})} \times \times O^{\oplus}(\bar{\eta}(\tau_0), \eta(\tau_0)) O^{\oplus}(\bar{\eta}(\tau_1), \eta(\tau_1)) \times \cdots \times O^{\oplus}(\bar{\eta}(\tau_{n-1}), \eta(\tau_{n-1})), \hfill (2.9)$$

where we use the convention: $\bar{\eta}(\tau_\nu) \equiv \{\bar{\eta}_1(\tau_\nu), \ldots, \bar{\eta}_N(\tau_\nu)\}$ and $\eta(\tau_\nu) \equiv \{\eta_1(\tau_\nu), \ldots, \eta_N(\tau_\nu)\}$ for $\nu = 0, 1, \ldots, n - 1$. The Grassmann variables in eq. (2.9) satisfy the boundary condition:

$$\eta_l(\tau_n) = -\eta_l(\tau_0) \quad \text{and} \quad \eta_l(\tau_\nu) = 0, \quad \text{for} \ \nu > n, \hfill (2.9a)$$

and $l = 1, 2, \ldots, N$. 

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Relation (2.9) is used to write the terms of the expansion of the grand canonical partition function in the high temperature limit as multivariable Grassmann integrals.

For a one–dimensional fermionic model, the index $l$ of the generators of the Grassmann algebra: $\{\eta(\tau_\mu), \bar{\eta}(\tau_\mu)\}$, conveys space and spin degrees of freedom. Explicitly separating the space index from the spin index in the generators of the non–commuting algebra, the coefficients of the expansion $Z(\beta, \mu)$ in eq.(2.4) become,

$$\text{Tr}[K^n] = \int \prod_{l=1}^{N} \prod_{\sigma=\pm 1} \prod_{\nu=0}^{n-1} d\eta_\sigma(x_l, \tau_\nu)d\bar{\eta}_\sigma(x_l, \tau_\nu) \sum_{\nu=0}^{n-1} \sum_{\nu=0}^{n-1} \bar{\eta}_\nu(x_l, \tau_\nu)[\eta_\nu(x_l, \tau_\nu)-\eta_\nu(x_l, \tau_{\nu+1})] \times$$

$$\times K^\oplus(\bar{\eta}_\nu(x, \tau_0), \eta_\nu(x, \tau_0)) K^\oplus(\bar{\eta}_\nu(x, \tau_1), \eta_\nu(x, \tau_1)) \times \cdots \times K^\oplus(\bar{\eta}_\nu(x, \tau_{n-1}), \eta_\nu(x, \tau_{n-1})),$$

where $N$ is the number of space site, $\sigma = \uparrow = +1, \sigma = \downarrow = -1$, and the boundary conditions (2.9a) become,

$$\eta_\sigma(x_l, \tau_n) = -\eta_\sigma(x_l, \tau_0) \quad \text{and} \quad \eta_\sigma(x_l, \tau_\nu) = 0, \quad \text{for} \ \nu > n,$$

$l = 1, 2, \ldots, N$.

To make use of previous results, where we relate the multivariable Grassmann integrals to a sum of determinants [10], it is interesting to map the generators $\eta_\sigma(x_l, \tau_\nu)$ and $\bar{\eta}_\sigma(x_l, \tau_\nu)$ into single–indexed anti–commuting variables. The particular mapping is a matter of choice; however, calculations are greatly simplified if we choose:

$$\eta^\uparrow(x_l, \tau_\nu) \equiv \eta_{\nu N+l} \quad (2.12a)$$

and

$$\eta^\downarrow(x_l, \tau_\nu) \equiv \eta_{(n+\nu)N+l}, \quad (2.12b)$$

where $l = 1, 2, \cdots, N$. $N$ is the number of space sites, and, $\nu = 0, 1, \cdots, n - 1$.

The mappings (2.12 a–b) can be summarized as:

$$\eta_\sigma(x_l, \tau_\nu) \equiv \eta_{\left[\frac{1-\sigma}{2}n+\nu\right]N+l} \quad (2.12c)$$
Using the redefined generators (2.12c), the sum in the exponential on the r.h.s. of eq.(2.10) is written as

\[
\sum_{l=1}^{N} \sum_{\sigma=\pm 1} \sum_{\nu=0}^{n-1} \bar{\eta}_{\sigma}(x_l, \tau_\nu)[\eta_{\sigma}(x_l, \tau_\nu) - \eta_{\sigma}(x_l, \tau_{\nu+1})] = \\
= \sum_{I,J=1}^{2nN} \bar{\eta}_I A_{IJ} \eta_J, \quad (2.13)
\]

where \(A_{IJ}\) are the entries of the block–matrix \(A\),

\[
A = \begin{pmatrix} E^{\uparrow\uparrow} & 0 \\ 0 & A^{\downarrow\downarrow} \end{pmatrix}, \quad (2.13a)
\]

whose entries are matrices of dimension \(nN \times nN\). The indices \(I, J\) are such that \(I, J = 1, 2, \cdots, nN\).

The matrices \(A^{\uparrow\uparrow}\) and \(A^{\downarrow\downarrow}\) are identical and have a block–structure. Taking into account the anti–periodic condition in temperature (2.11) in (2.13), the matrices \(A^{\sigma\sigma}, \sigma = \uparrow, \downarrow\), are

\[
A^{\uparrow\uparrow} = A^{\downarrow\downarrow} = \begin{pmatrix} \mathbb{1}_{N \times N} & -\mathbb{1}_{N \times N} & 0_{N \times N} & \cdots & 0_{N \times N} \\ 0_{N \times N} & \mathbb{1}_{N \times N} & -\mathbb{1}_{N \times N} & \cdots & 0_{N \times N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{1}_{N \times N} & 0_{N \times N} & 0_{N \times N} & \cdots & \mathbb{1}_{N \times N} \end{pmatrix}. \quad (2.14a)
\]

The non–null elements of \(A^{\sigma\sigma}, \sigma = \uparrow\) and \(\sigma = \downarrow\), are:

\[
A^{\sigma\sigma}_{IJ} = \begin{cases} 
  a_{II} = 1, & I = 1, 2, \cdots, nN \\
  a_{I,I+N} = -1, & I = 1, 2, \cdots, (n-2)N \\
  a_{(n-1)N+I,I} = 1, & I = 1, 2, \cdots, N 
\end{cases} \quad (2.14b)
\]

The matrices \(A^{\sigma\sigma}, \sigma = \uparrow, \downarrow\), have dimension \(nN \times nN\). The symbols \(\mathbb{1}_{N \times N}\) and \(0_{N \times N}\) stand for the identity and zero matrices of dimension \(N \times N\), respectively.

With the newly indexed generators, the expression of \(\text{Tr}[K^n]\) (eq.(2.10)) becomes,
\[ \text{Tr}[K^n] = \int \prod_{I=1}^{2nN} d\bar{\eta}_I d\eta_I \prod_{I,J=1}^{2nN} \sum_{\nu=0}^{e^{I,J=1}} \bar{\eta}_I A_{IJ} \eta_J \times \]
\[ \times K^{\oplus}(\bar{\eta}, \eta; \nu = 0) K^{\oplus}(\bar{\eta}, \eta; \nu = 1) \cdots K^{\oplus}(\bar{\eta}, \eta; \nu = n-1). \] (2.15)

Note that expression (2.15) is the exact coefficient in order \( \beta^n \) of the expansion in the high-temperature limit of the grand canonical partition function for any unidimensional self-interacting fermionic model. The specific model to be studied is represented by the function \( K^{\oplus} \). The matrix \( A \) is the same for all unidimensional fermionic models.

The Grassmann functions \( K^{\oplus} \) are polynomials in the generators of the algebra. Therefore, the r.h.s. of eq.(2.15) are moments of the multivariable Grassmann gaussian integrals. We show in Appendix A that these integrals can be written as co–factors of the matrix \( A \).

Once the sub–matrices \( A^{\uparrow\downarrow} \) and \( A^{\downarrow\uparrow} \) are null, the multivariable integral (2.15) is equal to the product of the contributions coming from the sectors: \( \sigma\sigma = \uparrow\uparrow \) and \( \sigma\sigma = \downarrow\downarrow \).

Even restricting ourselves to the calculation of the multivariable integrals of a fixed sector \( \sigma\sigma \), we have to obtain the determinant of non–diagonal matrices of dimension \( nN \times nN \). The evaluation of such determinants by means of some symbolic manipulation language obviously depends on hardware and software resources. For a fixed \( n \), for instance, we have an upper practical limit for \( N \) for doing the calculation of the determinants. One possibility for evaluating (2.15) is to fix different values for \( N \) and try to extrapolate the results for an arbitrary value of \( N \). If we are lucky, we may recognize some recursion expression for (2.15) for all \( N \).

The integrals in (2.15), for each sector \( \sigma\sigma \), have the form:

\[ M(L, K) = \int \prod_{i=1}^{nN} d\bar{\eta}_i d\eta_i \bar{\eta}_{l_1} \eta_{k_1} \cdots \bar{\eta}_{l_m} \eta_{k_m} e^{\sum_{i,j=1}^{nN} \bar{\eta}_i A_{ij} \eta_j}, \] (2.16)

with \( L = \{l_1, \cdots, l_m\} \) and \( K = \{k_1, \cdots, k_m\} \). The products \( \bar{\eta} \eta \) are ordered in such a way that \( l_1 < l_2 < \cdots < l_m \) and \( k_1 < k_2 < \cdots < k_m \). From eq.(A.10), the result of this type of integrals is equal to:

\[ M(L, K) = (-1)^{(l_1+l_2+\cdots+l_m)+(k_1+k_2+\cdots+k_m)} A(L, K), \] (2.17)

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where $A(L, K)$ is the determinant of the matrix obtained from matrix $A$ by deleting the lines \( \{l_1, \ldots, l_m\} \) and the columns: \( \{k_1, \ldots, k_m\} \).

Our approach to calculate the integral (2.16) is, for fixed $n$ and arbitrary $N$, to explore the block-structure of matrices $A^{\sigma\sigma}$, $\sigma = \uparrow$ and $\sigma = \downarrow$, diagonalizing it through a similarity transformation

$$P^{-1}A^{\sigma\sigma}P = D,$$

where the matrix $D$ is,

$$D = \begin{pmatrix}
\lambda_1 \mathbb{1}_{N \times N} & \Phi_{N \times N} & \cdots & \Phi_{N \times N} \\
\Phi_{N \times N} & \lambda_2 \mathbb{1}_{N \times N} & \cdots & \Phi_{N \times N} \\
\vdots & \vdots & \ddots & \vdots \\
\Phi_{N \times N} & \Phi_{N \times N} & \cdots & \lambda_n \mathbb{1}_{N \times N}
\end{pmatrix},$$

(2.18b)

\(\lambda_i, i = 1, 2, \ldots, n\), are the eigenvalues of matrices $A^{\sigma\sigma}$, $\sigma = \uparrow, \downarrow$, and calculate the co-factors of the matrix $D$.

The $j^{th}$-column of matrix $P$ is the eigenvector of $A^{\sigma\sigma}$ associated to the eigenvalue $\lambda_j$. Each eigenvalue of matrix $A^{\sigma\sigma}$ has degeneracy $N$. The matrices are not hermitian, thus some eigenvalues are complex. In the following, we will be working on the $\sigma\sigma = \uparrow\uparrow$ sector; however, the results for the $\sigma\sigma = \downarrow\downarrow$ sector are analogous since $A^{\uparrow\uparrow} = A^{\downarrow\downarrow}$.

We make a change of anti-commuting variables,

$$\eta' = P^{-1}\eta \quad \text{and} \quad \bar{\eta}' = \bar{\eta}P,$$

(2.19)

where $\eta' \equiv \{\eta'_1, \ldots, \eta'_{nN}\}$ and $\bar{\eta}' \equiv \{\bar{\eta}'_1, \ldots, \bar{\eta}'_{nN}\}$. The jacobian of the transformation (2.19) is equal to one.

In eqs. (2.19), we wrote the transformations of variables in a very simplified way; however, due to the fact that $A^{\uparrow\uparrow}$ is a block-matrix, the matrix $P$ also has a block structure. This fact implies that transformations (2.19) do not mix the index associated to the space site.

In a schematic way, the integrals $M(L, K)$ (eq.(2.16)) become:

$$M(L, K) = \int \prod_{i=1}^{nN} d\eta_i d\bar{\eta}_i (\bar{\eta}P^{-1})_{i_1}(P\eta)_{k_1} \cdots (\bar{\eta}P^{-1})_{i_m}(P\eta)_{k_m}$$
where $D_{ij}$ are the entries of matrix $D$.

From eq.(2.16), we have that $M(L,K)$ is a multivariable Grassmann integral, which in turn correspond to co–factors of the diagonalized matrix $D$ (eq.(2.17)). It is very simple to calculate these co–factors. Besides, taking into account that $\eta' = 0$ and $\bar{\eta}' = 0$ decreases the number of integrals to be calculated in (2.20).

This is a general approach, and it can be applied to any self–interacting unidimensional fermionic model. Thus we have reduced the calculation of fermionic path integral to the evaluation of co–factors of a diagonalized matrix.

2.1. Application to Unidimensional Periodic Hubbard Model

We will apply in section 4 the general approach presented in section 2 to the periodic Hubbard model [1] in $d = (1 + 1)$ with $N$ space sites in the presence of an external magnetic field in the $\hat{z}$ direction. However, we need to know the normal ordered Grassmann function $\mathcal{K}^\oplus$ for the specific model.

The hamiltonian that describes the Hubbard model in one space dimension is [1]:

$$H = \sum_{i=1}^{N} \sum_{\sigma=-1,1} t_{ij} a_{i\sigma}^\dagger a_{j\sigma} + U \sum_{i=1}^{N} a_{i\uparrow}^\dagger a_{i\uparrow} a_{i\downarrow}^\dagger a_{i\downarrow} + \lambda_B \sum_{i=1}^{N} \sum_{\sigma=-1,1} \sigma a_{i\sigma}^\dagger a_{i\sigma}$$

where $a_{i\sigma}^\dagger$ is the creation operator of an electron in site $i$ with spin $[12] \sigma$ and $a_{i\sigma}$ is the destruction operator of an electron in site $i$ with spin $\sigma$. All diagonal elements of $t_{ij}$ are equal, $t_{ii} = E_0$, where $E_0$ is the kinetic energy. The only non–null off–diagonal terms are $t_{i,i-1} = t_{i,i+1} = t$, where $i = 1, 2, \ldots, N$, and they contribute to the hopping term. $U$ is the strenght of the interaction between the electrons in the same site but with different spins. We have defined $\lambda_B = -\frac{1}{2}g\mu_B B$, where $g$ is the Landé’s factor and $\mu_B$ is the Bohr’s magneton.

The periodic boundary condition in space is implemented by imposing that $a_{0\sigma} \equiv a_{N\sigma}$ and $a_{N+1,\sigma} \equiv a_{1\sigma}$. Therefore, the hopping terms $t_{10} a_{1\sigma}^\dagger a_{0\sigma}$ and $t_{N,N+1} a_{N\sigma}^\dagger a_{N+1,\sigma}$ become
$t_1 N a_{1σ}^\dagger a_{Nσ}$ and $t_{N,1} a_{Nσ}^\dagger a_{1σ}$ respectively. We point out that the hamiltonian (2.1.1) is already in normal order.

From eq.(2.2) we have

$$K = H - \mu N.$$ 

The kernel of the operator $K$ of a one–dimensional Hubbard model in a lattice with $N$ space sites, written in terms of the generators $\eta_σ(x_l, τ_ν)$ and $\bar{η}_σ(x_l, τ_ν)$ is

$$K^{⊙}(\bar{η}_σ(x_l, τ_ν), η_σ(x_l, τ_ν)) = \sum_{l=1}^N \sum_{σ \pm 1} (E_0 + σλ_B - μ) \bar{η}_σ(x_l, τ_ν)η_σ(x_l, τ_ν) +
+ \sum_{l=1}^N \sum_{σ \pm 1} t[\bar{η}\sigma(x_l, τ_ν)η_σ(x_{l+1}, τ_ν) + \bar{η}_σ(x_l, τ_ν)η_σ(x_{l-1}, τ_ν)] +
+ \sum_{l=1}^N U \bar{η}↑(x_l, τ_ν)η↑(x_l, τ_ν)\bar{η}↓(x_l, τ_ν)η↓(x_l, τ_ν),$$

with the space periodic boundary condition imposed by:

$$t\bar{η}_σ(x_N, τ_ν)η_σ(x_{N+1}, τ_ν) \equiv t\bar{η}_σ(x_N, τ_ν)η_σ(x_1, τ_ν)$$

and

$$t\bar{η}_σ(x_1, τ_ν)η_σ(x_0, τ_ν) \equiv t\bar{η}_σ(x_1, τ_ν)η_σ(x_N, τ_ν),$$

and the anti–periodic boundary condition in $τ$:

$$η_σ(x_l, τ_n) = -η_σ(x_l, τ_0),$$

for $l = 1, 2, \cdots, N$, and $σ = ↑, ↓$.

Using the mapping (2.12c), the kernel of the operator $K$ becomes:

$$K^{⊙}(\bar{η}, η; ν)) = \sum_{l=1}^N \sum_{σ \pm 1} (E_0 + σλ_B - μ) \bar{η}_\frac{(i-σ)}{2}n+ν][N+l] \eta_\frac{(i-σ)}{2}n+ν][N+l]^+$$
$$\sum_{l=1}^{N} \sum_{\sigma=\pm} \left[ \tilde{\eta}_{\sigma}(1-\sigma) n+\nu | N+l \right] \eta_{\sigma}(1-\sigma) n+\nu | N+l+1 + \tilde{\eta}_{\sigma}(1-\sigma) n+\nu | N+l \right) +$$

$$+ \sum_{l=1}^{N} U \tilde{\eta}_{n+\nu N+l} \eta_{n+\nu N+l} \tilde{\eta}_{n+\nu N+l} \eta_{n+\nu N+l}.$$  \hspace{1cm} (2.1.3)

### 3. Useful Symmetries of the Multivariable Grassmann Integrals

In eq.(2.15), the number of terms to be calculated increases rapidly with $N$. To make this approach feasible, we have to explore symmetries, besides the trivial space translation, of the exponential in eq.(2.15). In section 2, we pointed out that in expression (2.15) the contributions from the sectors $\sigma \sigma = \uparrow \uparrow$ and $\sigma \sigma = \downarrow \downarrow$ are decoupled. In this section we discuss only one of the two sectors, e.g. $\sigma \sigma = \uparrow \uparrow$. Analogous results are valid for the sector $\sigma \sigma = \downarrow \downarrow$.

In the expression of $\text{Tr}[K^n]$ (eq.(2.15)), we deal with integrals of the following type:

$$I(m) = \int \prod_{l=1}^{nN} d\eta_l d\bar{\eta}_l \tilde{\eta}_{\nu_1 N+l_1} \eta_{\nu_1 N+k_1} \tilde{\eta}_{\nu_2 N+l_2} \eta_{\nu_2 N+k_2} \times \cdots \times$$

$$\times \tilde{\eta}_{\nu_m N+l_m} \eta_{\nu_m N+k_m} \times e^{t_{ij} \tau_{ij}},$$  \hspace{1cm} (3.1)

where $m \leq n$, and, $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_m$.

We want to study how the integral $I(m)$ transforms under a change of variables in the temperature parameter $\tau$. Consider the change of variables:

$$\eta'_{\nu N+l} = \eta(\nu-1) N+l, \quad \nu = 1, 2, \ldots, n-1, \hspace{1cm} (3.2a)$$

and

$$\eta'_l = -\eta(n-1) N+l; \hspace{1cm} (3.2b)$$

$$\tilde{\eta}'_{\nu N+l} = \tilde{\eta}(\nu-1) N+l, \quad \nu = 1, 2, \ldots, n-1, \hspace{1cm} (3.2c)$$

and
\[ \tilde{\eta}'_l = -\tilde{\eta}_{(n-1)N+l}. \] (3.2d)

In this relabelling of generators, we are shifting the temperature index by one unit, except for \( \eta_{n-1} \) and \( \tilde{\eta}_{n-1} \), identified to \(-\eta_0\) and \(-\tilde{\eta}_0\), respectively. The jacobian of the simultaneous transformations (3.2) is equal to one.

Before discussing how the integral \( I(m) \) transforms under (3.2), let us show that the sum in the exponentiml of eq.(3.1) is invarimnt under such transformations.

From eq.(2.14b), we have that the non–null terms of \( A^{\uparrow \uparrow} \) are:

\[
A^{\sigma\sigma}_{IJ} = \begin{cases} 
  a_{II} = 1, & I = 1, 2, \ldots, nN \\
  a_{I,I+N} = -1, & I = 1, 2, \ldots, (n-2)N \\
  a_{(n-1)N+l,I} = 1, & I = 1, 2, \ldots, N \\
\end{cases}
\] (3.3)

Therefore, the sum within the exponential in eq.(3.1) can be written as,

\[
\sum_{I,J=1}^{nN} \bar{\eta}_I A_{IJ}^{\uparrow \uparrow} \eta_J = \sum_{\nu=0}^{n-2} \sum_{l=1}^{N} \bar{\eta}_{\nu N+l} A_{\nu N+l,\nu N+l}^{\uparrow \uparrow} \eta_{\nu N+l} + \\
+ \sum_{l=1}^{N} \bar{\eta}_{(n-1)N+l} A_{(n-1)N+l,(n-1)N+l}^{\uparrow \uparrow} \eta_{(n-1)N+l} + \\
+ \sum_{\nu=0}^{n-3} \sum_{l=1}^{N} \bar{\eta}_{\nu N+l} A_{\nu N+l,\nu N+l}^{\uparrow \uparrow} \eta_{\nu N+l} + \\
+ \sum_{l=1}^{N} \bar{\eta}_{(n-2)N+l} A_{(n-2)N+l,(n-2)N+l}^{\uparrow \uparrow} \eta_{(n-2)N+l} + \\
+ \sum_{l=1}^{N} \bar{\eta}_{(n-1)N+l} A_{(n-1)N+l,N+l}^{\uparrow \uparrow} \eta_l. \] (3.4)

Under the change of variables (3.2), the terms on the r.h.s. of (3.4) transforms as:

1) \( \sum_{\nu=0}^{n-2} \sum_{l=1}^{N} \bar{\eta}_{\nu N+l} A_{\nu N+l,\nu N+l}^{\uparrow \uparrow} \eta_{\nu N+l} = \)

\[
= \sum_{\nu=0}^{n-2} \sum_{l=1}^{N} \bar{\eta}'_{\nu N+l} A_{(\nu-1)N+l,(\nu-1)N+l}^{\uparrow \uparrow} \eta'_{\nu N+l}
\]
\[
= \sum_{\nu=1}^{n-1} \sum_{l=1}^{N} \bar{\eta}_{\nu N+l} A_{\nu N+l,\nu N+l}^{\uparrow\uparrow} \eta_{\nu N+l} \cdot
\]  
(3.4a)

2) \[
\sum_{l=1}^{N} \bar{\eta}_{N(n-1)+l} A_{N(n-1)+l,N(n-1)+l}^{\uparrow\uparrow} \eta_{N(n-1)+l} =
\]
\[
= \sum_{l=1}^{N} \bar{\eta}_{l} A_{l,l}^{\uparrow\uparrow} \eta_{l}. 
\]  
(3.4b)

3) \[
\sum_{\nu=0}^{n-3} \sum_{l=1}^{N} \bar{\eta}_{\nu N+l} A_{\nu N+l,(\nu+1)N+l}^{\uparrow\uparrow} \eta_{(\nu+1)N+l} =
\]
\[
= \sum_{\nu=1}^{n-2} \sum_{l=1}^{N} \bar{\eta}_{\nu N+l} A_{(\nu-1)N+l,\nu N+l}^{\uparrow\uparrow} \eta_{(\nu+1)N+l}
\]
\[
= \sum_{l=1}^{n-2} \sum_{l=1}^{N} \bar{\eta}_{l} A_{\nu N+l,(\nu+1)N+l}^{\uparrow\uparrow} \eta_{(\nu+1)N+l}. 
\]  
(3.4c)

4) \[
\sum_{l=1}^{N} \bar{\eta}_{(n-2)N+l} A_{(n-2)N+l,(n-1)N+l}^{\uparrow\uparrow} \eta_{(n-1)N+l} =
\]
\[
= - \sum_{l=1}^{N} \bar{\eta}_{l} A_{(n-1)N+l,l}^{\uparrow\uparrow} \eta_{N+l}
\]
\[
= \sum_{l=1}^{N} \bar{\eta}_{l} A_{l,(n+1)N+l}^{\uparrow\uparrow} \eta_{N+l}. 
\]  
(3.4d)

5) \[
\sum_{l=1}^{N} \bar{\eta}_{(n-1)N+l} A_{(n-1)N+l,l}^{\uparrow\uparrow} \eta_{l} =
\]
\[
= - \sum_{l=1}^{N} \bar{\eta}_{l} A_{(n-2)N+l,(n-1)N+l}^{\uparrow\uparrow} \eta_{l}
\]
\[
= \sum_{l=1}^{N} \bar{\eta}_{(n-1)N+l} A_{(n-1)N+l,l}^{\uparrow\uparrow} \eta_{l}. 
\]  
(3.4e)
To obtain the results (3.4a–e), we have used eq.(3.3).

Summing the r.h.s. of (3.4a–e), we finally conclude that

$$
\sum_{I,J=1}^{nN} \bar{\eta}_I A_{IJ}^{\uparrow \uparrow} \eta_J = \sum_{I,J=1}^{nN} \bar{\eta'}_I A_{IJ}^{\uparrow \uparrow} \eta'_J.
$$

(3.5)

We point out that the invariance of the sum (3.5) is true for any value of $n$ and $N$. Moreover, since $A^{\uparrow \uparrow} = A^{\downarrow \downarrow}$, a similar invariance under (3.2) can be written for $A^{\downarrow \downarrow}$.

In (3.2) we translated the temperature index by one unit. However, the equality (3.5) is still valid if we translate the temperature index by any integer $\nu_0$, either positive or negative, i.e.,

$$
\eta_{\nu N+l} \rightarrow \eta_{(\nu+\nu_0)N+l}
$$

(3.6a)

and

$$
\eta_{\bar{\nu} N+l} \rightarrow -\eta_l, \quad \text{where} \quad \bar{\nu} + \nu_0 = n,
$$

(3.6b)

and perform an analogous transformation for $\bar{\eta}$. Transformation (3.6) is equivalent to applying $\nu_0$ distinct and consecutive transformations (3.2).

Under the transformation (3.6), the integral $I(m)$ (eq.(3.1)) becomes,

$$
I(m) = \int \prod_{I=1}^{nN} d\eta'_I d\bar{\eta'}_I \bar{\eta'}_{(\nu_1+\nu_0)N+l_1} \eta'_{(\nu_1+\nu_0)N+k_1} \times \cdots \times
\times \bar{\eta'}_{(\nu_m+\nu_0)N+l_m} \eta'_{(\nu_m+\nu_0)N+k_m} \times e^{\sum_{I,J=1}^{nN} \bar{\eta'}_I A_{IJ}^{\uparrow \uparrow} \eta'_J},
$$

(3.7)

where $\nu_0$ is a fixed integer, positive or negative; for $\bar{\nu}$ such that $\bar{\nu} + \nu_0 = 0$, we assume that

$$
\bar{\eta}_{(\bar{\nu}+\nu_0)N+l} \rightarrow -\bar{\eta}_l \quad \text{and} \quad \eta_{(\bar{\nu}+\nu_0)N+l} \rightarrow -\eta_l,
$$

and for $\nu + \nu_0 > n$

$$
\bar{\eta}_{(\nu+\nu_0)N+l} \rightarrow \bar{\eta}_{(\nu+\nu_0-n)N+l} \quad \text{and} \quad \eta_{(\nu+\nu_0)N+l} \rightarrow \eta_{(\nu+\nu_0-n)N+l}.
$$
An analogous result is valid for the sector $\sigma\sigma = \downarrow\downarrow$.

As the contributions to the integrals in eq.(2.15) of the sectors $\sigma\sigma = \uparrow\uparrow$ and $\sigma\sigma = \downarrow\downarrow$ factorize, we can choose different values for $\nu_0$ for each spin sector.

Another symmetry of the exponential in eq.(3.1) results from the arbitrariness of either clockwise or counterclockwise labelling of space sites in the periodic unidimensional chain. The result of the integrals is choice–independent. Let us label the space sites counterclockwise. For a fixed $\nu$, the Grassmann generators associated to the sites are: $\{\eta_{\nu N+1}, \eta_{\nu N+2}, \cdots, \eta_{\nu N+N}; \bar{\eta}_{\nu N+1}, \bar{\eta}_{\nu N+2}, \cdots, \bar{\eta}_{\nu N+N}\}$. If, on the other hand, we label the unidimensional chain clockwise, in this case the Grassmann generators will be: $\{\eta'_{\nu N+1}, \eta'_{\nu N+2}, \cdots, \eta'_{\nu N+N}; \bar{\eta}'_{\nu N+1}, \bar{\eta}'_{\nu N+2}, \cdots, \bar{\eta}'_{\nu N+N}\}$.

The correspondence among generators in the two labellings may be chosen to be

$$\eta_{\nu N+1} = \eta'_{\nu N+1} \quad \text{and} \quad \bar{\eta}_{\nu N+1} = \bar{\eta}'_{\nu N+1}, \quad (3.8a)$$

for $\nu = 0, 1, \cdots, n-1$, and

$$\eta_{\nu N+l} = \eta'_{\nu N+(N+2-l)} \quad \text{and} \quad \bar{\eta}_{\nu N+l} = \bar{\eta}'_{\nu N+(N+2-l)}. \quad (3.8b)$$

The jacobian associated to (3.8) is equal to one.

The sum within the exponential in the integrand of eq.(3.1), namely,

$$\sum_{l, j=1}^{nN} \bar{\eta}_l \ A_{l, j}^{\uparrow\uparrow} \eta_j = \sum_{\nu=0}^{n-1} \sum_{l=1}^{N} \bar{\eta}_{\nu N+l} A_{\nu N+l, \nu N+l}^{\uparrow\uparrow} \eta_{\nu N+l} + \sum_{\nu=0}^{n-2} \sum_{l=1}^{N} \bar{\eta}_{\nu N+l} A_{\nu N+l, (\nu+1)N+l}^{\uparrow\uparrow} \eta_{(\nu+1)N+l} + \sum_{l=1}^{N} \bar{\eta}_{(n-1)N+l} A_{(n-1)N+l, l}^{\uparrow\uparrow} \eta_l, \quad (3.9)$$

is invariant under the substitution (3.8). In order to show this, we use the relation among the non–zero elements of $A^{\uparrow\uparrow}$, given by eq.(3.3), and manipulate the expressions in a similar way as we have done before. In summary, we get that

$$\sum_{l, j=1}^{nN} \bar{\eta}_l \ A_{l, j}^{\uparrow\uparrow} \eta_j = \sum_{l, j=1}^{nN} \bar{\eta}'_l \ A_{l, j}^{\uparrow\uparrow} \eta'_j, \quad (3.10)$$
where the relation between \( \{ \eta, \bar{\eta} \} \) and \( \{ \eta', \bar{\eta}' \} \) is given by (3.8).

The integral \( I(m) \) (see eq.(3.1)) subjected to the change of variables (3.8) becomes,

\[
I(m) = \int \frac{d\eta'}{\nu_1 N + (N + 2 - l_1)} \eta_1^N + (N + 2 - k_1) \times \cdots \times \eta_m^N N + (N + 2 - k_m) \times e^{l \cdot J = 1} \sum_{I=1}^{n N} \eta_i A_{I, J} \eta_j \ldots \eta_j \cdot (3.11)
\]

The chiral space symmetry, represented by the transformations (3.8), is valid for any positive integers \( n \) and \( N \).

From eq.(2.15), we have that

\[
\text{Tr}[K^n] = \int 2^{n N} \prod_{L=1}^{2 n N} d\eta d\bar{\eta} \sum_{I=1}^{2 n N} \eta_i A_{I, J} \eta_j \times \ldots \times \eta_j \cdot (3.12)
\]

where the matrix \( A \) is given by (2.13a). For the one–dimensional Hubbard model with \( N \) space sites, the kernel \( K^\odot(\bar{\eta}, \eta; \nu) \) is given by eq.(2.1.3).

In order to simplify the algebraic manipulation for calculating eq.(3.12), we define:

\[
\mathcal{E}(\bar{\eta}, \eta; \nu; \sigma) \equiv \sum_{l=1}^{N} \bar{\eta}_l^{(1-\sigma)} n+\nu |N + l \eta_l^{(1-\sigma)} n+\nu N + l ; \quad (3.13a)
\]

\[
\mathcal{T}^-(\bar{\eta}, \eta; \nu; \sigma) \equiv \sum_{l=1}^{N} \bar{\eta}_l^{(1-\sigma)} n+\nu |N + l \eta_l^{(1-\sigma)} n+\nu N + l + 1 ; \quad (3.13b)
\]

and

\[
\mathcal{T}^+(\bar{\eta}, \eta; \nu; \sigma) \equiv \sum_{l=1}^{N} \bar{\eta}_l^{(1-\sigma)} n+\nu |N + l \eta_l^{(1-\sigma)} n+\nu N + l - 1 . \quad (3.13c)
\]

We call

\[
\mathcal{E}(\bar{\eta}, \eta; \nu) \equiv \sum_{\sigma = \pm 1} E(\sigma) \mathcal{E}(\bar{\eta}, \eta; \nu; \sigma), \quad (3.14a)
\]
\[ T^{-}(\bar{\eta}, \eta; \nu) \equiv \sum_{\sigma = \pm 1} t^{-}(\bar{\eta}, \eta; \nu; \sigma), \quad (3.14b) \]

\[ T^{+}(\bar{\eta}, \eta; \nu) \equiv \sum_{\sigma = \pm 1} t^{+}(\bar{\eta}, \eta; \nu; \sigma), \quad (3.14c) \]

and

\[ U(\bar{\eta}, \eta; \nu) \equiv \sum_{l=1}^{N} \bar{\eta}_{n+\nu \downarrow} N^{N+1} \eta_{n+\nu \uparrow} N^{l} \eta_{\nu N+l}. \quad (3.14d) \]

where \( E(\sigma) \equiv E_{0} - \sigma \lambda_{B} - \mu \). The term \( E(\bar{\eta}, \eta; \nu) \) represents the kinetic energy, \( T^{-}(\bar{\eta}, \eta; \nu) \) and \( T^{+}(\bar{\eta}, \eta; \nu) \) are the hopping terms and \( U(\bar{\eta}, \eta; \nu) \) is the fermionic interaction term.

Using the definitions (3.13) and (3.14), the Grassmann function \( K(\bar{\eta}, \eta; \nu) \) is written as,

\[ K(\bar{\eta}, \eta; \nu) = E(\bar{\eta}, \eta; \nu) + T^{-}(\bar{\eta}, \eta; \nu) + T^{+}(\bar{\eta}, \eta; \nu) + U(\bar{\eta}, \eta; \nu). \quad (3.15) \]

Since the terms in \( E(\bar{\eta}, \eta; \nu), T^{-}(\bar{\eta}, \eta; \nu), T^{+}(\bar{\eta}, \eta; \nu) \) and \( U(\bar{\eta}, \eta; \nu) \) are products of \( \bar{\eta} \) and \( \eta \) at the temperature index \( \nu \) these expressions still have the same form under the change of variables (3.6), except that they are defined at \( \nu + \nu_{0} \); that is,

\[ E(\eta, \eta; \nu) \rightarrow E(\eta, \eta; \nu + \nu_{0}) \quad (3.16a) \]
\[ T^{-}(\eta, \eta; \nu) \rightarrow T^{-}(\eta, \eta; \nu + \nu_{0}) \quad (3.16b) \]
\[ T^{+}(\eta, \eta; \nu) \rightarrow T^{+}(\eta, \eta; \nu + \nu_{0}) \quad (3.16c) \]
\[ U(\eta, \eta; \nu) \rightarrow U(\eta, \eta; \nu + \nu_{0}). \quad (3.16d) \]

Let us see the behavior of the expressions \( E(\bar{\eta}, \eta; \nu), T^{-}(\bar{\eta}, \eta; \nu), T^{+}(\bar{\eta}, \eta; \nu) \) and \( U(\bar{\eta}, \eta; \nu) \) under the chiral transformations (3.8). We begin by considering \( E(\bar{\eta}, \eta; \nu) \). From (3.13a) and (3.14a), we have that

\[ E(\bar{\eta}, \eta; \nu) = \sum_{\sigma = \pm 1} E(\sigma) \bar{\eta}_{n+\nu \downarrow} N^{N+1} \eta_{n+\nu \uparrow} N^{l} \eta_{\nu N+l}. \quad (3.17) \]

The transformation (3.8) for a sector \( \sigma \sigma \) is,
\[ \eta_{\frac{1}{2}n+\nu}N+1 = \eta'_{\frac{1}{2}n+\nu}N+1 \quad \text{and} \quad \bar{\eta}_{\frac{1}{2}n+\nu}N+1 = \bar{\eta}'_{\frac{1}{2}n+\nu}N+1, \]

(3.18a)

for \( l = 1 \) and \( \nu = 0, 1, \ldots, n - 1 \), and

\[ \eta_{\frac{1}{2}n+\nu}N+l = \eta'_{\frac{1}{2}n+\nu}N+N+2-l \quad \text{and} \quad \bar{\eta}_{\frac{1}{2}n+\nu}N+l = \bar{\eta}'_{\frac{1}{2}n+\nu}N+N+2-l, \]

(3.18b)

for \( l \neq 1 \) and \( \nu = 0, 1, \ldots, n - 1 \).

Substituting (3.18) in eq.(3.17), and defining the index \( k = N + 2 - l \) on the second term in the r.h.s. of eq.(3.17), we obtain

\[
E(\bar{\eta}, \eta; \nu) = \sum_{\sigma=\pm 1} E(\sigma)\eta'_{\frac{1}{2}n+\nu}N+1 \eta'_{\frac{1}{2}n+\nu}N+1 + \\
+ \sum_{\sigma=\pm 1} \sum_{k=2}^{N} E(\sigma)\bar{\eta}'_{\frac{1}{2}n+\nu}N+k \eta'_{\frac{1}{2}n+\nu}N+k - \\
= E(\bar{\eta}', \eta'; \nu) \quad (3.19)
\]

\( E(\bar{\eta}, \eta; \nu) \) is invariant under the chiral transformation (3.18).

Proceeding in a similar way, we also get that

\[
\mathcal{U}(\bar{\eta}, \eta; \nu) = \mathcal{U}(\bar{\eta}', \eta'; \nu), \quad (3.20)
\]

where the relation between the generators \( \{ \eta, \bar{\eta} \} \) and \( \{ \eta', \bar{\eta}' \} \) is given by (3.18).

Let us now take under consideration one of the hopping terms, namely,

\[
\mathcal{T}^{-}(\bar{\eta}, \eta; \nu) = \sum_{\sigma=\pm 1} t \bar{\eta}_{\frac{1}{2}n+\nu}N+1 \eta_{\frac{1}{2}n+\nu}N+2 + \\
+ \sum_{\sigma=\pm 1} \sum_{l=2}^{N-1} t \bar{\eta}_{\frac{1}{2}n+\nu}N+l \eta_{\frac{1}{2}n+\nu}N+l+1 + \\
+ \sum_{\sigma=\pm 1} t \bar{\eta}_{\frac{1}{2}n+\nu}N+N \eta_{\frac{1}{2}n+\nu}N+N+1, \quad (3.21)
\]
where we have already taken into account the periodic space boundary condition.

Applying the change of variables (3.18) in eq.(3.21) and defining the index \( k = N + 2 - l \), we have

\[
\mathcal{T}^-(\bar{\eta}, \eta; \nu) = \sum_{k=1}^{N} \sum_{\sigma=\pm1} \bar{\eta}_{ [(1-\sigma)n+\nu]N+k } \eta_{ [(1+\sigma)n+\nu]N+k-1 } = \mathcal{T}^+(\bar{\eta}, \eta; \nu)
\]

(3.22)

In a similar way, it may be shown that, under (3.18), \( \mathcal{T}^+(\bar{\eta}, \eta; \nu) \) transforms into \( \mathcal{T}^-(\bar{\eta}, \eta; \nu) \).

We point out that the results of this section are valid for any positive integer values of \( n \) and \( N \).

4. Calculation of the Coefficients of the Grand Canonical Partition Function for the Hubbard Model in the High Temperature Limit

Up to now, all that we have said about \( \text{Tr}[K^n] \) in sections 2 and 3, is valid for any value of \( n \) and \( N \).

The calculation of \( \text{Tr}[K^n] \), given by eq.(2.15), for arbitrary \( n \) can formally be done using the result (2.20). However, due to the fact that the number of terms that contribute to \( \text{Tr}[K^n] \) increases rapidly with \( n \), we are able to calculate the exact coefficients up to order \( \beta^3 \) of the grand canonical partition function of unidimensional Hubbard model in the high temperature limit.

When we say that our result is exact up to order \( \beta^3 \), we mean that it is valid for any value of the constant parameters \( E_0, t, U \) and \( \mu \), that characterize the model, and, for any value of the constant external magnetic field \( B \).

4.1. The Exact \( \beta^2 \) Coefficient in \( d=(1+1) \) Hubbard Model

From eq.(2.15), the expression of \( \text{Tr}[K^2] \) in terms of the Grassmann generators is
\[
\text{Tr}[K^2] = \int \prod_{l=1}^{4N} d\eta_l d\bar{\eta}_l \sum_{e^{I,J=1}}^{4N} \eta_l A_{IJ} \eta_J \times \mathcal{K}^\oplus(\bar{\eta}, \eta; \nu = 0) \mathcal{K}^\oplus(\bar{\eta}, \eta; \nu = 1),
\]

where the matrix \(A\) is given by (2.13a), and for the unidimensional periodic Hubbard model, \(\mathcal{K}^\oplus\) is given by eq.(2.1.3) or eq.(3.15).

For \(n = 2\), the matrices \(A^{\uparrow\uparrow}\) and \(A^{\downarrow\downarrow}\) (eq.(2.14a)) are:

\[
A^{\uparrow\uparrow} = A^{\downarrow\downarrow} = \begin{pmatrix}
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1
\end{pmatrix}.
\]

The eigenvalues \(\lambda_1\) and \(\lambda_2\) are \(N\)-fold degenerated.

\(\Psi\)From eq.(2.18a), \(P\) is the matrix that, through a similarity transformation, diagonalizes the matrices \(A^{\uparrow\uparrow}\) and \(A^{\downarrow\downarrow}\). For \(n = 2\), the matrix \(P\) is,

\[
P = \begin{pmatrix}
1_N \times N & 1_N \times N \\
i 1_N \times N & -i 1_N \times N
\end{pmatrix},
\]

and its inverse is:

\[
P^{-1} = \frac{1}{2} \begin{pmatrix}
1_N \times N & -i 1_N \times N \\
i 1_N \times N & i 1_N \times N
\end{pmatrix}.
\]

Since \(P\) and \(P^{-1}\) are block–matrices, their entries can be written as \(P_{\nu N+l, \nu' N+k} = \delta_{lk} p_{\nu \nu'}\) and \(P^{-1}_{\nu N+l, \nu' N+k} = \delta_{lk} q_{\nu \nu'}\). \(p_{\nu \nu'}\) and \(q_{\nu \nu'}\) are the non–null elements and for \(n = 2\) they are equal to

\[
p_{\nu \nu'} = \begin{pmatrix}
1 & 1 \\
i & -i
\end{pmatrix} \quad \text{and} \quad q_{\nu \nu'} = \frac{1}{2} \begin{pmatrix}
1 & -i \\
i & i
\end{pmatrix}.
\]

Therefore,
D ≡ P^{-1} A^{\sigma \sigma} P

= \begin{pmatrix}
(1 - e^{i \pi}) I_{N \times N} & Q_{N \times N} \\
Q_{N \times N} & (1 - e^{i \pi}) I_{N \times N}
\end{pmatrix}, \text{ for } \sigma = \uparrow, \downarrow. \quad (4.1.2e)

For \( n = 2 \), the change of variables (2.19): \( \eta = P \eta' \) and \( \bar{\eta} = \bar{\eta}' P^{-1} \), becomes,

\[
\eta_{\nu N+l} = \sum_{\nu' = 0}^{1} p_{\nu \nu'} \eta'_{\nu' N+l} \quad (4.1.3a)
\]

and

\[
\bar{\eta}_{\nu N+l} = \sum_{\nu' = 0}^{1} \bar{\eta}'_{\nu' N+l} q_{\nu' \nu}, \quad (4.1.3b)
\]

where \( \nu = 0, 1 \) and \( l = 1, 2, \cdots, N \).

We should note that due to the block structure of the matrices \( P \) and \( P^{-1} \), the space index \( l \) in the generators \( \eta \) and \( \bar{\eta} \) are not mixed up by the transformation (4.1.3).

From eq.(3.15), we write function \( K^@ (\bar{\eta}, \eta; \nu) \) as

\[
K^@ (\bar{\eta}, \eta; \nu) = \mathcal{E}(\bar{\eta}, \eta; \nu) + \mathcal{T}^- (\bar{\eta}, \eta; \nu) + \mathcal{T}^+ (\bar{\eta}, \eta; \nu) + \mathcal{U}(\bar{\eta}, \eta; \nu). \quad (4.1.4)
\]

In order to simplify the notation, we define

\[
< \mathcal{O}_1 (\nu_1) \cdots \mathcal{O}_m (\nu_m) > \equiv \int \prod_{I=1}^{2nN} d\eta_I d\bar{\eta}_I \sum_{l=1}^{2nN} A_{IJ} \eta_J \times \mathcal{O}_1 (\bar{\eta}, \eta; \nu_1) \cdots \mathcal{O}_m (\bar{\eta}, \eta; \nu_m) \quad (4.1.5a)
\]

and

\[
< \mathcal{O}_1 (\nu_1) \cdots \mathcal{O}_m (\nu_m) >_{\sigma} \equiv \int \prod_{I=(3-\sigma)nN+1}^{(3-\sigma)nN+2nN} d\eta_I d\bar{\eta}_I \sum_{l=1}^{2nN} A_{IJ} \eta_{(1-\sigma)nN+l} \times \mathcal{O}_1 (\bar{\eta}, \eta; \nu_1) \cdots \mathcal{O}_m (\bar{\eta}, \eta; \nu_m) \quad (4.1.5b)
\]

where \( \mathcal{O}_j (\bar{\eta}, \eta; \nu_j) \) are Grassmann functions.
When we substitute eq.(4.1.4) for \( \nu = 0 \) and \( \nu = 1 \) in eq.(4.1.1), we get sixteen terms. However, using the results (3.19), (3.20) and (3.22), and the fact that the terms:
\[<E(0)T^- (1)>,<E(0)T^+ (1)>,<T^-(0)T^- (1)>,<T^-(0)U(1)>,<T^+(0)T^+ (1)>, \text{ and }<T^+(0)U(1)> \] are null, we have that Tr\([K^2]\) reduces to the sum of four distinct terms:

\[
\text{Tr}[K^2] = <E(0)E(1)> + 2 <T^-(0)T^+ (1)> + 2 <E(0)U(1)> + <U(0)U(1)>. \tag{4.1.6}
\]

Before calculating these terms, lets us consider one of the null integrals and show the reasoning for its vanishing. For example, the term

\[
<E(0)T^- (1)> = 2t(E_0 - \mu)[<E(\uparrow,0)T^- (\uparrow,1)> + <E(\uparrow,0)T^- (\downarrow,1)>]. \tag{4.1.7}
\]

In expression (4.1.7) we identify the following types of terms:

i) Terms coming from \(<E(\uparrow,0)T^- (\uparrow,1)>\):

\[
\mathcal{I}_1 \equiv \int \prod_{I=1}^{2N} d\tilde{\eta}_I d\bar{\eta}_I \sum_{I,J=1}^{2N} \tilde{\eta}_I A_{IJ} \eta_J \tilde{\eta}_{l_1} \eta_{l_1} \tilde{\eta}_{N+l_2} \eta_{N+l_2} \times \\
\times \int \prod_{I=2N+1}^{4N} d\tilde{\eta}_I d\bar{\eta}_I \sum_{I,J=2N+1}^{4N} \tilde{\eta}_I A_{IJ} \eta_J , \tag{4.1.7a}
\]

where \( l_1 \) and \( l_2 \) are fixed and represent the space site index.

Making the change of variables (4.1.3) to diagonalize \( A^{\sigma \sigma} \), \( \sigma = \uparrow, \downarrow \), we get

\[
\mathcal{I}_1 = \det(A^{\uparrow \downarrow}) \sum_{\nu_1,\nu_1' = 0}^{1} q_{\nu_1,0} p_{0\nu_1'} q_{\nu_2,1} p_{1\nu_2'} \times \\
\times \int \prod_{I=1}^{2N} d\tilde{\eta}_I d\bar{\eta}_I \sum_{I,J=1}^{2N} \tilde{\eta}_I D_{IJ} \eta_J \tilde{\eta}_{\nu_1 N+l_1} \eta_{\nu_1' N+l_1} \tilde{\eta}_{\nu_2 N+l_2} \eta_{\nu_2' N+l_2} \times \\
\times \int \prod_{I=2N+1}^{4N} d\tilde{\eta}_I d\bar{\eta}_I \sum_{I,J=2N+1}^{4N} \tilde{\eta}_I A_{IJ} \eta_J , \tag{4.1.7b}
\]

The presence of \( \tilde{\eta}_i \) and \( \eta_j \) in the integrand of eq.(4.17b) implies that the integral is equal to the determinant of a matrix obtained from matrix \( D \) by cutting the line \( i \) and the column
As long as $D$ is diagonal, the only non–vanishing integrals are those for which $i = j$. In other words, if the $i$-th line of $D$ is cut, so must be its $i$-th column.

In eq.(4.1.7b) the space index $(l_2 + 1)$ never equals any other space index. Therefore, the minor obtained from matrix $D$ by cutting the lines: $(\nu_1 + l_1)$ and $(\nu_2 + l_2)$, and the columns: $(\nu'_1 + l_1)$ and $(\nu'_2 + l_2 + 1)$, will always have, at least, one line or row of zeros. Therefore, its determinant vanishes.

Then,

$$\mathcal{I}_1 = 0 \Rightarrow <\mathcal{E}(\uparrow, 0)\mathcal{T}^- (\uparrow, 1) > = 0.$$  \hspace{1cm} (4.1.8a)

ii) Terms coming from $<\mathcal{E}(\uparrow, 0)\mathcal{T}^- (\downarrow, 1)>$:

Using a similar reasoning as for the terms in $<\mathcal{E}(\uparrow, 0)\mathcal{T}^- (\uparrow, 1)>$, we get that

$$<\mathcal{T}^- (\downarrow, 1) >_1 = 0.$$  \hspace{1cm} (4.1.8b)

Therefore, using the results (4.1.8a–b), we finally obtain,

$$<\mathcal{E}(0)\mathcal{T}^- (1)> = 0.$$  \hspace{1cm} (4.1.9)

Proceeding in an analogous way, it is straightforward to show that the terms $<\mathcal{E}(0)\mathcal{T}^+ (1)>$, $<\mathcal{T}^- (0)\mathcal{T}^- (1)>$, $<\mathcal{T}^- (0)\mathcal{U}(1)>$, $<\mathcal{T}^+ (0)\mathcal{T}^+ (1)>$ and $<\mathcal{T}^+ (0)\mathcal{U}(1)>$ also vanish.

The results (4.1.8a–b) are a direct consequence to the fact that the change of variables (4.1.3) does not mix up space indices.

Taking into account the previous results and calculating the coefficients of the terms of eq.(4.1.6), $\text{Tr}[K^2]$ may be written as:

$$\text{Tr}[K^2] = 2((E_0 - \mu)^2 + \lambda_B^2) <\mathcal{E}(\uparrow, 0)\mathcal{E}(\uparrow, 1)> + 2((E_0 - \mu)^2 - \lambda_B^2) <\mathcal{E}(\downarrow, 0)><\mathcal{E}(\uparrow, 0)> +$$

$$+ 4t^2 <\mathcal{T}^- (\uparrow, 0)\mathcal{T}^+ (\uparrow, 1)> + 4(E_0 - \mu) <\mathcal{E}(\uparrow, 0)\mathcal{U}(1)> + <\mathcal{U}(0)\mathcal{U}(1)>.$$  \hspace{1cm} (4.1.10)
The terms on the r.h.s. of eq.(4.1.10) are sums of multivariable Grassmann integrals that reduce to the following types:

\[ \mathcal{H}_1(l) \equiv \int \prod_{I=1}^{2N} d\eta_I d\bar{\eta}_I e^{2N \sum_{I,J=1}^{2N} \bar{\eta}_I A^\uparrow_{IJ} \eta_J} \bar{\eta}_l \eta_l, \quad (4.1.11a) \]

\[ \mathcal{H}_2(l, k) \equiv \int \prod_{I=1}^{2N} d\eta_I d\bar{\eta}_I e^{2N \sum_{I,J=1}^{2N} \bar{\eta}_I A^\uparrow_{IJ} \eta_J} \bar{\eta}_l \eta_l \bar{\eta}_{N+k} \eta_{N+k}, \quad (4.1.11b) \]

and

\[ \mathcal{H}_3(l, k) \equiv \int \prod_{I=1}^{2N} d\eta_I d\bar{\eta}_I e^{2N \sum_{I,J=1}^{2N} \bar{\eta}_I A^\uparrow_{IJ} \eta_J} \bar{\eta}_l \eta_{l+1} \bar{\eta}_{N+k} \eta_{N+k-1}, \quad (4.1.11c) \]

in the sector \( \sigma \sigma = \uparrow \uparrow \), and equivalent integrals in the sector \( \downarrow \downarrow \).

Let us calculate in detail these integrals. Those integrals that contribute to order \( \beta^3 \) of the expansion of the grand canonical partition function in the high temperature limit are handled in a similar way. In the next section, we will restrict ourselves to give the results of such integrals.

Under the change of variables (4.1.3), the integral \( \mathcal{H}_1(l) \) becomes:

\[ \mathcal{H}_1(l) = \sum_{\nu, \nu' = 0}^{1} q_{\nu 0} p_{0 \nu'} \int \prod_{I=1}^{2N} d\eta'_I d\bar{\eta}'_I e^{2N \sum_{I,J=1}^{2N} \bar{\eta}'_I D_{IJ} \eta'_J} \bar{\eta}'_{l+1} \eta'_{N+l} \eta'_{N+l}. \quad (4.1.12) \]

We only get non–null results for eq.(4.1.12) when the \( i \)-th line and the \( i \)-th column of the matrix \( D \) are cut simultaneously. This only happens when \( \nu = \nu' \).

Therefore,

\[ \mathcal{H}_1(l) = \int \prod_{I=1}^{2N} d\eta'_I d\bar{\eta}'_I e^{2N \sum_{I,J=1}^{2N} \bar{\eta}'_I D_{IJ} \eta'_J} (q_{00} p_{00} \bar{\eta}'_l \eta'_l + q_{10} p_{01} \bar{\eta}'_{N+l} \eta'_{N+l}). \quad (4.1.13) \]

The first term in the integrand corresponds to cut the \( l \)-th line and \( l \)-th column of matrix \( D \). Since it is a diagonal block–matrix, the result of the integral is independent of \( l \) and it is equal to: \( \lambda_1^{N-1} \lambda_2^N \). The second term in the integrand corresponds to cutting its \( (N + l) \)-th line and
\((N + l)\)-th column. By the same reason as before, the result of this integral is independent of the value of \(l\), and it is equal to \(\lambda_1^N \lambda_2^{N-1}\).

Substituting the values of \(q_{ij}\) and \(p_{ij}\) (eq.(4.1.2d)) in eq.(4.1.13), we finally get

\[
H_1(l) \equiv \int \prod_{I=1}^{2N} d\eta_I d\bar{\eta}_I \, e^{\sum_{I,J=1}^{2N} \bar{\eta}_I A_{IJ}^T \eta_J} \bar{\eta}_l \eta_l, \\
= 2^{N-1},
\]  

(4.1.14)

for any value of \(l\).

Making the change of variables (4.1.3) in the integral \(H_2(l, k)\), it becomes:

\[
H_2(l, k) = \sum_{\nu_1, \nu_2 = 0}^1 q_{\nu_1 \nu_0} p_{0 \nu_1} q_{\nu_0 \nu_2} p_{1 \nu_2} \times \\
\times \int \prod_{I=1}^{2N} d\eta'_I d\bar{\eta}'_I \, e^{\sum_{I,J=1}^{2N} \bar{\eta}'_I D_{IJ} \eta'_J} \bar{\eta}'_{\nu_1 N + l} \eta'_{\nu_1 N + l} \bar{\eta}'_{\nu_2 N + k} \eta'_{\nu_2 N + k}. 
\]

(4.1.15a)

Once the \(i\)-th line and the \(i\)-th column need to be cut simultaneously if integral (4.1.15a) is to be non-vanishing, we have two situations to consider:

\( i) l \neq k. \)

In this case, the non-zero terms of eq.(4.1.15a) are:

\[
H_2(l, k) = \sum_{\nu_1, \nu_2 = 0}^1 q_{\nu_1 \nu_0} p_{0 \nu_1} q_{\nu_0 \nu_2} p_{1 \nu_2} \times \\
\times \int \prod_{I=1}^{2N} d\eta'_I d\bar{\eta}'_I \, e^{\sum_{I,J=1}^{2N} \bar{\eta}'_I D_{IJ} \eta'_J} \bar{\eta}'_{\nu_1 N + l} \eta'_{\nu_1 N + l} \bar{\eta}'_{\nu_2 N + k} \eta'_{\nu_2 N + k}. 
\]

(4.1.15b)

Since the result of the integrals in eq.(4.1.15b) i independent of the space indices \(l\) and \(k\) \((l \neq k)\), only the total number of cuts within each sector \(\nu\) is relevant to their evaluation. For example, for \(\nu_1 = 0\) and \(\nu_2 = 0\), there are two cuts in the sector \(\nu = 0\) of matrix \(D\). Thus the result of this integral is \(\lambda_1^{N-2} \lambda_2^N\). For \(\nu_1 = 0\) and \(\nu_2 = 1\), there is one cut in the sector \(\nu = 0\)
and another one in the sector $\nu = 1$ of matrix $D$; hence, the integral is equal to $\lambda_1^{N-1} \lambda_2^{N-1}$, the same result as for the case $\nu_1 = 1$ and $\nu_2 = 0$.

Using the previous reasoning and the values of $q_{ij}$ and $p_{ij}$ given by eq.(4.1.2d), we get,

$$H_2(l, k) = 2^{N-2}, \quad \text{for} \quad l, k = 1, 2, \ldots, N \quad \text{and} \quad l \neq k. \quad (4.1.15c)$$

$ii) l = k.$

In this case, we must take into account that $\eta_i^2 = \bar{\eta}_i^2 = 0$. For $l = k$ we have to compute all the possible permutations of the product $\bar{\eta}_l^i \eta_i^l \bar{\eta}_N^l \eta_N^l$. The only non–null terms in eq.(4.1.15a) are:

| $q_{\nu_10}$ | $p_{0\nu_1}$ | $q_{\nu_21}$ | $p_{1\nu_2}$ |
|--------------|--------------|--------------|--------------|
| $\bar{\eta}_l$ | $\eta_l$ | $\bar{\eta}_N^l$ | $\eta_N^l$ | $\Rightarrow$ | $q_{00} p_{00} q_{11} p_{11}$ |
| $\bar{\eta}_l$ | $\eta_{N+l}$ | $\bar{\eta}_N^l$ | $\eta_l$ | $\Rightarrow$ | $q_{00} p_{01} q_{11} p_{10}$ |
| $\bar{\eta}_{N+l}$ | $\eta_{N+l}$ | $\bar{\eta}_l$ | $\eta_l$ | $\Rightarrow$ | $q_{10} p_{01} q_{01} \vert_{10}$ |
| $\bar{\eta}_{N+l}$ | $\eta_l$ | $\bar{\eta}_l$ | $\eta_{N+l}$ | $\Rightarrow$ | $q_{10} p_{00} q_{01} p_{11}$ | (4.1.16) |

To write the multivariable Grassmann integrals as co–factors of matrix $D$, we have to rearrange the product to the pattern $\bar{\eta}_l^i \eta_i^l \bar{\eta}_N^l \eta_N^l$. Because of the permutations, the second and fourth term in (4.1.16) get a minus sign.

By the same reason as explained in the calculation of integral $H_1(l)$, the result of the integrals is independent of $l$, and it is equal to

$$H_2(l, l) = 2^{N-1}, \quad \text{for} \quad l = 1, 2, \ldots, N. \quad (4.1.17)$$

In summary, we have that

$$H_2(l, k) \equiv \int \prod_{l=1}^{2N} d\eta_l d\bar{\eta}_l \sum_{i,j=1}^{2N} \bar{\eta}_l A_i^{\uparrow \uparrow}_{jj} \eta_j \bar{\eta}_l \eta_l \bar{\eta}_N^k \eta_N^k, \quad 2^{N-2}, \quad \text{if} \ l \neq k$$

$$= \begin{cases} 2^{N-2}, & \text{if} \ l \neq k \\ 2^{N-1}, & \text{if} \ l = k \\ 0, & \text{otherwise} \end{cases}. \quad (4.1.18)$$
Finally, we calculate the integral $\mathcal{H}_3(l,k)$. Under the change of variables (4.1.3), it becomes:

$$\mathcal{H}_3(l,k) \equiv -\sum_{\nu_1,\nu_1'=0}^{1} q_{\nu_1} p_{0\nu_1'} q_{\nu_2} p_{1\nu_2'} \times$$

$$\times \int \prod_{l=1}^{2N} d\eta'_I d\bar{\eta}'_I e^{l'J=1} \bar{\eta}'_{\nu_1N+l} \eta'_{\nu_1N+k-1} \bar{\eta}'_{\nu_2N+k} \eta'_{\nu_2N+l+1}.$$  

(4.1.19)

The only non–null integrals in $\mathcal{H}_3(l,k)$ are the ones where $l = k - 1$. The terms on the r.h.s. of (4.1.19) that contribute to it are those where $\nu_1 = \nu_1'$ and $\nu_2 = \nu_2'$. Using the values of $q_{ij}$ and $p_{ij}$ given by eq.(4.1.2d), we get

$$\mathcal{H}_3(l,l+1) = 2^{N-2}, \quad l = 1, 2, \cdots, N. \quad (4.1.20)$$

Therefore, we have that,

$$\mathcal{H}_3(l,k) \equiv \int \prod_{l=1}^{2N} d\eta_I d\bar{\eta}_I e^{l'J=1} \bar{\eta}_l \eta_{l+1} \bar{\eta}_{N+k} \eta_{N+k-1}.$$  

(4.1.20a)

From the results (4.1.14), (4.1.18) and (4.1.20a) we evaluate the terms $<\mathcal{E}(\uparrow,0)\mathcal{E}(\uparrow,1)>$, $<\mathcal{E}(\sigma,0)>\sigma$, $<\mathcal{T}-(\uparrow,0)\mathcal{T}+(\uparrow,1)>$, $<\mathcal{E}(\uparrow,0)\mathcal{U}(1)>$ and $<\mathcal{U}(0)\mathcal{U}(1)>$. Substituting the results in (4.1.10), we finally get:

$$\text{Tr}[K^2] = 2^{2N} \left[ N^2 \left( (E_0-\mu)^2 + \frac{1}{2} (E_0-\mu)U + \frac{U^2}{16} \right) + \frac{N}{2} \left( (E_0-\mu)^2 + (E_0-\mu)U + \lambda_B^2 + 2t^2 + \frac{3}{8} U^2 \right) \right],$$  

(4.1.21)

where $\lambda_B = -\frac{1}{2} g\mu_B B$.  

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The result (4.1.21) is valid for any value of $N$ space sites, and any set of values: $(E_0, t, U, \mu)$ that characterize the fermionic system and its interaction with the external magnetic field $B$.

4.2. The Exact $\beta^3$ Coefficient in d=$(1+1)$ Hubbard Model

In section 4.1 we used the simpler case $\beta^2$ to exemplify, in great detail, how to use the results of Appendix A to calculate the coefficients of the expansion (2.4). In this section, we present the results of Tr$[K^3]$ for any number $N$ of space sites.

The expression of (3.12) for $n=3$ is

$$
\text{Tr}[K^3] = \int \prod_{l=1}^{6N} d\bar{\eta}_l d\eta_l \sum_{i,j=1}^{6N} \bar{\eta}_l A_{ij} \eta_j \times \\
\times K^\oplus(\bar{\eta}, \eta; \nu = 0) K^\oplus(\bar{\eta}, \eta; \nu = 1) K^\oplus(\bar{\eta}, \eta; \nu = 2),
$$

(4.2.1a)

where $A_{ij}$ are the entries of the block–matrix $A$ (eq.(2.13a)). For $n=3$, we have that

$$
A^{\uparrow\uparrow} = A^{\downarrow\downarrow} = 
\begin{pmatrix}
I_{N\times N} & -1_{N\times N} & 0_{N\times N} \\
0_{N\times N} & I_{N\times N} & -1_{N\times N} \\
1_{N\times N} & 0_{N\times N} & 1_{N\times N}
\end{pmatrix}.
$$

(4.2.1b)

The expression of the Grassmann function $K^\oplus(\bar{\eta}, \eta; \nu)$ for the case of the periodic unidimensional Hubbard model is given by eq.(2.1.3).

The r.h.s. of eq.(4.2.1a) is expanded into a sum of 64 terms, each of which contains a Grassmann multivariable integral. Their evaluation will be attained by the same technique used in section 4.1 for the case $n = 2$.

For $n = 3$, the matrices $A^{\sigma\sigma}$, $\sigma = \uparrow, \downarrow$, are diagonalized by the matrix $P$ (eq.(2.18a)),

$$
P = 
\begin{pmatrix}
I_{N\times N} & (1 + e^{2\pi i})I_{N\times N} & (1 + e^{2\pi i})I_{N\times N} \\
-I_{N\times N} & I_{N\times N} & I_{N\times N} \\
I_{N\times N} & (1 + e^{2\pi i})I_{N\times N} & (1 + e^{2\pi i})I_{N\times N}
\end{pmatrix},
$$

(4.2.2a)
the inverse of which is

\[
\mathbf{P}^{-1} = \frac{1}{3} \begin{pmatrix}
\mathbf{1}_{N \times N} & -\mathbf{1}_{N \times N} & \mathbf{1}_{N \times N} \\
(1 + e^{-\frac{2\pi i}{3}})\mathbf{1}_{N \times N} & \mathbf{1}_{N \times N} & (1 + e^{\frac{2\pi i}{3}})\mathbf{1}_{N \times N} \\
(1 + e^{\frac{2\pi i}{3}})\mathbf{1}_{N \times N} & \mathbf{1}_{N \times N} & (1 + e^{-\frac{2\pi i}{3}})\mathbf{1}_{N \times N}
\end{pmatrix}.
\] (4.2.2b)

For \( n = 3 \), the diagonalized matrix \( \mathbf{D} \) (eq.(2.18)) is

\[
\mathbf{D} = \begin{pmatrix}
2\mathbf{1}_{N \times N} & \mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} \\
\mathbf{0}_{N \times N} & (1 + e^{\frac{2\pi i}{3}})\mathbf{1}_{N \times N} & \mathbf{0}_{N \times N} \\
\mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} & (1 + e^{-\frac{2\pi i}{3}})\mathbf{1}_{N \times N}
\end{pmatrix}.
\] (4.2.3)

The eigenvalues of the matrix \( \mathbf{A}^{\sigma \sigma} \), \( \sigma = \uparrow, \downarrow \), are: 2, \((1 + e^{\frac{2\pi i}{3}})\) and \((1 + e^{-\frac{2\pi i}{3}})\). Each eigenvalue is \( N \)-fold degenerated.

The change of variables (2.19), for the case \( n = 3 \) becomes,

\[
\eta_{\nu N + l} = \sum_{\nu' = 0}^{2} p_{\nu \nu'} \eta'_{\nu' N + l} \tag{4.2.4a}
\]

and

\[
\tilde{\eta}_{\nu N + l} = \sum_{\nu' = 0}^{2} \tilde{\eta}'_{\nu' N + l} q_{\nu' \nu} \tag{4.2.4b}
\]

with \( l = 1, 2, \cdots, N \), and

\[
p_{\nu' \nu} = \begin{pmatrix}
1 & (1 + e^{\frac{2\pi i}{3}}) & (1 + e^{-\frac{2\pi i}{3}}) \\
-1 & 1 & 1 \\
1 & (1 + e^{-\frac{2\pi i}{3}}) & (1 + e^{\frac{2\pi i}{3}})
\end{pmatrix},
\]

and

\[
q_{\nu' \nu} = \frac{1}{3} \begin{pmatrix}
1 & 1 & (1 + e^{\frac{2\pi i}{3}}) \\
(1 + e^{-\frac{2\pi i}{3}}) & 1 & (1 + e^{\frac{2\pi i}{3}}) \\
(1 + e^{\frac{2\pi i}{3}}) & 1 & (1 + e^{-\frac{2\pi i}{3}})
\end{pmatrix}.
\] (4.2.4c)

Using the symmetries studied in section 3, it is simple to show which terms on the r.h.s. of eq.(4.2.1a) are equal. Besides, taking the same steps as in section 4.1, we get that the terms
Taking into account the symmetries of section 3 and the fact that the mentioned terms are zero, we get that $\text{Tr}[K^3]$ is equal to

$$
\text{Tr}[K^3] = < E(0)E(1)E(2) > + 3 < E(0)E(1)U(2) > + 6 < E(0)T^-(1)T^+(2) > + 3 < U(0)U(1)U(2) > + < U(0)U(1)U(2) >,
$$

(4.2.5)

All the terms on the r.h.s. of eq.(4.2.5) can be written as the sum of multivariable Grassmann integrals. These integrals are reduced to five different types of integrals. We will not detail the calculation of these integrals, however, for it follows the steps presented in section 4.1 to derive the results of the integrals $H_1(l), H_2(l,k)$ and $H_3(l,k)$.

The five types of integrals that appear in $\text{Tr}[K^3]$ and their respective results are [14]:

1)

$$
G_1(l) \equiv \int \prod_{I=1}^{3N} d\eta_I d\bar{\eta}_I \ e^{\sum_{I,J=1}^{3N} \bar{\eta}_I A_{IJ} \eta_J} \ \bar{\eta}_l \ \eta_l = 2^{N-1},
$$

(4.2.6a)

for $l = 1, 2, \ldots, N$.

2)

$$
G_2(l, k) \equiv \int \prod_{I=1}^{3N} d\eta_I d\bar{\eta}_I \ e^{\sum_{I,J=1}^{3N} \bar{\eta}_I A_{IJ} \eta_J} \ \bar{\eta}_l \ \eta_l \ \bar{\eta}_{N+k} \ \eta_{N+k}
$$

$$
= \begin{cases} 
2^{N-1}, & k = l, \quad l = 1, 2, \ldots, N \\
2^{N-2}, & k \neq l, \quad l, k = 1, 2, \ldots, N.
\end{cases}
$$

(4.2.6b)

3)

$$
J_1(l, k) \equiv \int \prod_{I=1}^{3N} d\eta_I d\bar{\eta}_I \ e^{\sum_{I,J=1}^{3N} \bar{\eta}_I A_{IJ} \eta_J} \ \bar{\eta}_l \ \eta_{l+1} \ \bar{\eta}_{N+k} \ \eta_{N+k-1}
$$

$$
= \begin{cases} 
2^{N-2}, & k = l + 1, \quad l = 1, 2, \ldots, N \\
0, & k \neq l + 1.
\end{cases}
$$

(4.2.6c)
4)

\[ J_2(l_1, l_2, l_3) \equiv \int \prod_{I=1}^{3N} d\eta_I d\bar{\eta}_I \sum_{I,J=1}^{3N} \bar{\eta}_I A_{I,J} \eta_J \eta_{N+l_2} \eta_{N+l_2+1} \eta_{2N+l_3} \eta_{2N+l_3-1} \]

\[ = \begin{cases} 2^{N-3}, & l_1 \neq l_2, l_1 \neq l_3 \text{ and } l_3 = l_2 + 1 \\ 2^{N-2}, & l_3 = l_1 \text{ and } l_2 = l_1 - 1 \\ 0, & \text{for any other case.} \end{cases} \] (4.2.6d)

5)

\[ G_3(l_1, l_2, l_3) \equiv \int \prod_{I=1}^{3N} d\eta_Id\bar{\eta}_I \sum_{I,J=1}^{3N} \bar{\eta}_I A_{I,J} \eta_J \eta_{N+l_2} \eta_{N+l_2} \eta_{2N+l_3} \eta_{2N+l_3} \]

\[ = \begin{cases} 2^{N-3}, & l_1 \neq l_2 \neq l_3 \\ 2^{N-2}, & l_1 = l_2 \neq l_3, l_1 = l_3 \neq l_2, l_2 = l_3 \neq l_1 \\ 2^{N-1}, & l_1 = l_2 = l_3. \end{cases} \] (4.2.6e)

Using the results (4.2.6) to calculate the terms on the r.h.s. of eq. (4.2.5), after some algebraic manipulation, we get

\[
\text{Tr}[K^3] = 2^{2N} \left[ N^3 \left( (E_0 - \mu)^3 + \frac{3}{4}(E_0 - \mu)^2U + \frac{3}{16}(E_0 - \mu)U^2 + \frac{1}{64}U^3 \right) + \right.
\]

\[
N^2 \left( \frac{3}{2}(E_0 - \mu)^3 + \frac{15}{8}(E_0 - \mu)^2U + \frac{3}{2}(E_0 - \mu)\lambda_B^2 + 3(E_0 - \mu)t^2 + \right.
\]

\[
\left. + \frac{15}{16}(E_0 - \mu)U^2 + \frac{3}{4}t^2U + \frac{3}{8}U\lambda_B^2 + \frac{9}{64}U^3 \right) + \]

\[
N \left( \frac{3}{8}(E_0 - \mu)^2U + \frac{3}{8}(E_0 - \mu)U^2 + \frac{3}{2}(E_0 - \mu)t^2 - \frac{3}{8}U\lambda_B^2 + \frac{3}{4}t^2U + \frac{3}{32}U^3 \right) \right].
\] (4.2.7)

5. The Grand Canonical Partition Function for the One–Dimensional Hubbard Model Up to Order $\beta^3$

In eq.(2.4), we write the expansion of the grand canonical partition function in the high temperature limit in terms of $\text{Tr}[K^n]$, namely
\[ Z(\beta, \mu) = \text{Tr}(e^{-\beta K}) \]
\[ = \text{Tr}[\mathbb{I} - \beta K] + \frac{\beta^2}{2} \text{Tr}[K^2] - \frac{\beta^3}{3!} \text{Tr}[K^3] + \cdots, \quad (5.1) \]

In reference [9], we calculated \( \text{Tr}[\mathbb{I} - K] \) for the unidimensional periodic Hubbard model for a chain with \( N \) space sites, where \( N \) can assume any arbitrary value,

\[ \text{Tr}[\mathbb{I} - K] = 2^{2N} \left[ 1 - N\beta ((E_0 - \mu) + \frac{U}{4}) \right]. \quad (5.2) \]

From expression (4.1.21), we have that

\[ \text{Tr}[K^2] = 2^{2N} \left[ N^2 ((E_0 - \mu)^2 + \frac{1}{2}(E_0 - \mu)U + \frac{U^2}{16}) + \frac{N}{2}((E_0 - \mu)^2 + (E_0 - \mu) + \lambda_B^2 + 2t^2 + \frac{3}{8}U^2) \right]. \quad (5.3) \]

In section 4.2, eq.(4.2.7), we got

\[ \text{Tr}[K^3] = 2^{2N} \left[ N^3 \left( (E_0 - \mu)^3 + \frac{3}{4}E_0 - \mu)^2 U + \frac{3}{16}(E_0 - \mu)U^2 + \frac{1}{64}U^3 \right) + \right. \]
\[ + N^2 \left( \frac{3}{2}E_0 - \mu)^3 + \frac{15}{8}(E_0 - \mu)^2 U + \frac{3}{2}(E_0 - \mu) + 3(E_0 - \mu)t^2 + \right. \]
\[ + \frac{15}{16}(E_0 - \mu)U^2 + \frac{3}{4}t^2 U + \frac{3}{8}U\lambda_B^2 + \frac{9}{64}U^3 \left) + \right. \]
\[ + N \left( \frac{3}{8}(E_0 - \mu)^2 U + \frac{3}{8}(E_0 - \mu)U^2 + \frac{3}{2}(E_0 - \mu)t^2 - \frac{3}{8}U\lambda_B^2 + \frac{3}{4}t^2 U + \frac{3}{32}U^3 \right) \]. \quad (5.4) \]

Substituting (5.2–4) in (5.1), we get the grand canonical partition function \( Z(\beta; \mu) \) of the unidimensional periodic Hubbard model. However, we will use the Helmholtz free energy \( W(\beta; \mu) \) to calculate physical quantities. The relation between \( Z(\beta; \mu) \) and \( W(\beta; \mu) \) is given by

\[ W(\beta; \mu) = \ln Z(\beta; \mu). \quad (5.5) \]

The Helmholtz free energy of the one–dimensional periodic Hubbard model, up to order \( \beta^3 \), is
\[ W(\beta; \mu) = N \left[ 2 \ln 2 + \right. \\
+ \left. \left( -\frac{U^3}{64} + \frac{1}{16} U \lambda_B^2 - \frac{1}{16} (E_0 - \mu)^2 U - \frac{1}{4} (E_0 - \mu)t^2 - \frac{1}{16} (E_0 - \mu)U^2 - \frac{1}{8} Ut^2 \right) \beta^3 + \\
+ \left. \frac{1}{4} (E_0 - \mu)^2 + \frac{1}{4} \lambda_B^2 + \frac{1}{4} (E_0 - \mu)U + \frac{t^2}{2} + \frac{3}{32} U^2 \right) \beta^2 - \left( (E_0 - \mu) + \frac{U}{4} \right) \beta \right] + O(\beta^4). \]

(5.6)

The coefficients of the terms \( \beta, \beta^2 \) and \( \beta^3 \) in the expansion of \( W(\beta; \mu) \) are exact.

Once we have the Helmholtz free energy in the high temperature limit, we can derive any physical quantity in this limit. Here, we calculate only three quantities:

\( i) \) average energy per site: \( \langle h \rangle \).

The simplest way to derive the average energy per site from the Helmholtz free energy, is to scale the constants: \( (E_0, t, U, \lambda B) \rightarrow (\alpha E_0, \alpha t, \alpha U, \alpha \lambda_B) \) and substitute in eq.(2.1.2) to obtain \( W(\beta; \mu; \alpha) \).

\[ \langle h \rangle (\beta) = - \frac{1}{\beta N} \frac{\partial W(\beta; \mu; \alpha)}{\partial \alpha} \bigg|_{\alpha=1}. \]

(5.7)

From eq.(5.6), we get that

\[ \langle h \rangle (\beta) = E_0 + \frac{U}{4} + \\
+ \left[ - \frac{t^2}{2} - \frac{3}{16} U^2 - \frac{1}{2} E_0 U + \frac{1}{4} U \mu - \frac{1}{2} E_0^2 + \frac{1}{2} E_0 \mu - \frac{1}{2} \lambda_B^2 \right] \beta^3 + \\
+ \left[ \frac{3}{8} t^2 U - \frac{1}{2} t^2 \mu + \frac{3}{16} E_0 U^2 - \frac{1}{8} U^2 \mu - \frac{3}{16} U \lambda_B^2 + \frac{3}{16} U E_0^2 - \frac{1}{4} U E_0 \mu + \\
+ \frac{1}{16} U \mu^2 + \frac{3}{64} U^3 + \frac{3}{4} t^2 E_0 \right] \beta^2 + O(\beta^3). \]

(5.8)

\( ii) \) difference between average numbers of spin up and spin down particles per site: \( \langle n_\uparrow \rangle - \langle n_\downarrow \rangle \).

From the definition of the Helmholtz free energy (eq.(5.5)), we have that
\[ < n_\uparrow > (\beta) - < n_\downarrow > (\beta) = -\frac{1}{\beta N} \frac{\partial W(\beta; \mu)}{\partial \lambda_B}. \] (5.9)

Up to order \( \beta^2 \), we get from eq.(5.6) that,

\[ < n_\uparrow > (\beta) - < n_\downarrow > (\beta) = \frac{\lambda_B}{8} (U \beta + 4) \beta + O(\beta^3). \] (5.10)

\( iii \) average of the square of the magnetization per site:

\[ < m_z^2 > (\beta) = \frac{\lambda_B}{B^2} < (n_\uparrow - n_\downarrow)^2 > = \left( \frac{1}{2} g \mu_B \right)^2 \frac{1}{\beta N} \left[ \frac{\partial W(\beta; \mu)}{\partial \mu} + 2 \frac{\partial W(\beta; \mu)}{\partial U} \right], \] (5.11)

where \( B \) is the external magnetic field.

From eq.(5.6), we obtain that

\[ < m_z^2 > (\beta) = \frac{1}{4} g^2 \mu_B^2 \left[ \frac{1}{2} + \frac{1}{8} U \beta + \left[ -\frac{1}{8} U E_0 + \frac{1}{8} \mu U - \frac{1}{32} U^2 + \frac{1}{8} \lambda_B^2 - \frac{1}{8} E_0^2 + \frac{1}{4} E_0 \mu - \frac{1}{8} \mu^2 \right] \beta^2 \right] + O(\beta^3), \] (5.12)

where \( g \) is the Landé’s factor and \( \mu_B \) is the Bohr’s magneton.

6. Conclusions

Using the grassmannian nature of the fermionic fields, we extend the results of our previous work [9] to higher order terms of the expansion of the grand canonical partition function for unidimensional self–interacting fermionic models in the high temperature limit.

We are able to write the path integral for such models as a sum of co–factors of a suitable matrix, the entries of which are commuting quantities. This approach avoids ambiguities like the ones yielded by Hubbard–Stratonovich transformation [3] for fermionic path integral.
We apply the developed method to the Hubbard model in one–space dimension with periodic boundary condition. We get the exact coefficients of the $\beta$–expansion ($\beta = \frac{1}{kT}$) of the Helmholtz free energy up to order $\beta^3$ for a ring with $N$ space sites, where $N$ is arbitrary.

The expansion (2.4) of the operator $e^{-\beta K}$ is valid for any value of $\beta$. However, a sound physical meaning can only be granted to its lower-order terms if it is assured that the expansion converges. For finite values of the constants, there exists such a range: the high temperature limit.

The calculation of the coefficients of $\beta^4$ and $\beta^5$ terms of the grand canonical partition function for the one–dimensional Hubbard is to appear soon.

The method presented here can be applied to any one–dimensional self–interacting fermionic model. We believe that this method can be extended to higher space dimensions.
Appendix A

Moments of Gaussian Grassmann Multivariable Integrals

It is a known result [11] for a Grassmann algebra of dimension $2^{2N}$, composed of the generators $\{\eta_1, \cdots, \eta_N; \overline{\eta}_1, \cdots, \overline{\eta}_N\}$, that

$$\int \prod_{i=1}^{N} d\eta_i d\overline{\eta}_i e^{\sum_{i,j=1}^{N} \overline{\eta}_i A_{ij} \eta_j} = \det(A), \quad (A.1)$$

where $A_{ij}$ are the entries of matrix $A$ and are commuting quantities.

We will show in this Appendix that the moments of integral (A.1) are co–factors of $A$.

We first consider the case where we have one product $\overline{\eta}_l \eta_k$ in the integrand of the gaussian integral (A.1), that is,

$$M(l,k) \equiv \int \prod_{i=1}^{N} d\eta_i d\overline{\eta}_i \overline{\eta}_l \eta_k e^{\sum_{i,j=1}^{N} \overline{\eta}_i A_{ij} \eta_j}, \quad (A.2)$$

where $l, k$ are fixed and $1 \leq l, k \leq N$.

Due to the fact that for all Grassmann generators we have: $\overline{\eta}_i^2 = \eta_i^2 = 0, i = 1, \cdots, N$, the only non–null terms in eq.(A.2) are the ones where the integrand has $N$ products of the form: $\overline{\eta}_i \eta_j$. Eq. (A.2) becomes:

$$M(l,k) = \int \prod_{i=1}^{N} d\eta_i d\overline{\eta}_i \overline{\eta}_l \eta_k \frac{1}{(N-1)!} \sum_{i_1, \cdots, i_{N-1}=1}^{N} A_{i_1 j_1} \cdots A_{i_{N-1} j_{N-1}} \times \overline{\eta}_{i_1} \eta_{j_1} \overline{\eta}_{i_2} \eta_{j_2} \cdots \overline{\eta}_{i_{N-1}} \eta_{j_{N-1}}, \quad (A.3)$$

and the indices are such that $i_n \neq l, n = 1, \cdots, N-1$, and $j_n \neq k, n = 1, \cdots, N-1$. Once the product $\overline{\eta}_{i_n} \eta_{i_n}$ is a commutative quantity, each term in the sum of (A.3) appears $(N-1)!$ times.

The $(N-1)!$ distinct terms in (A.3) can be generated by fixing one configuration for $\{i_1, i_2, \cdots, i_{N-1}\}$, for example, we choose: $\{i_1 = 1, \cdots, i_{l-1} = l-1, i_l = l+1, \cdots, i_{N-1} = N\}$,
and, taking all the terms coming from the sum over the indices $j_n, n = 1, \ldots, N - 1$. Therefore, $M(l, k)$ becomes

$$
M(l, k) = \int \prod_{i=1}^{N} d\bar{\eta}_i d\bar{\eta}_i \eta_i \eta_k \sum_{j_1, \ldots, j_{N-1} = 1 \atop j_n \neq k} A_{1j_1} \cdots A_{l-1,j_{l-1}} A_{l+1,j_l} \cdots A_{Nj_{N-1}} \times \\
\times \bar{\eta}_1 \eta_{j_1} \bar{\eta}_2 \eta_{j_2} \cdots \bar{\eta}_{l-1} \eta_{j_{l-1}} \bar{\eta}_{l+1} \eta_{j_l} \cdots \bar{\eta}_{N} \eta_{j_{N-1}}. \quad (A.4)
$$

Renaming the variables: $j_l \rightarrow j_{l+1}, j_{l+1} \rightarrow j_{l+2}, \ldots, j_{N-1} \rightarrow j_N$, we have that:

$$
M(l, k) = \int \prod_{i=1}^{N} d\bar{\eta}_i d\bar{\eta}_i \eta_i \eta_k \sum_{j_1, \ldots, j_{N} = 1} A_{1j_1} \cdots A_{l-1,j_{l-1}} A_{l+1,j_{l+1}} \cdots A_{Nj_{N}} \times \\
\times \bar{\eta}_1 \eta_{j_1} \bar{\eta}_2 \eta_{j_2} \cdots \bar{\eta}_{l-1} \eta_{j_{l-1}} \bar{\eta}_{l+1} \eta_{j_{l+1}} \cdots \bar{\eta}_{N} \eta_{j_{N}}. \quad (A.5)
$$

Defining the matrix $B(l, k)$ as:

$$
B_{ij}(l, k) = \begin{cases} 
A_{ij}, & \text{if } i \neq l \text{ and } j \neq k \\
\delta_{il}\delta_{jk}, & \text{if } i = l \text{ or } j = k 
\end{cases}, \quad (A.6)
$$

and $i, j = 1, 2, \ldots, N$.

Using the definition of matrix $B(l, k)$, the expression of $M(l, k)$ is re-written as:

$$
M(l, k) = \int \prod_{i=1}^{N} d\bar{\eta}_i d\bar{\eta}_i \eta_i \eta_k \sum_{j_1, \ldots, j_{N} = 1} B_{1j_1} \cdots B_{l-1,j_{l-1}} B_{l,j_l} \cdots B_{Nj_{N}} \times \\
\times \bar{\eta}_1 \eta_{j_1} \bar{\eta}_2 \eta_{j_2} \cdots \bar{\eta}_{l-1} \eta_{j_{l-1}} \bar{\eta}_{l+1} \eta_{j_{l+1}} \cdots \bar{\eta}_{N} \eta_{j_{N}}. \quad (A.7)
$$

Integrating over $\bar{\eta}_i$, and using the definition of determinant [10, 15], we finally have that

$$
M(l, k) = \det B = (-1)^{l+k} A(l, k), \quad (A.8)
$$

where $A(l, k)$ is the minor determinant of matrix $A$, when the line $l$ and the column $k$ are deleted. $M(l, k)$ is the cofactor of matrix $A$. 
Using an analogous procedure, we now consider the case of moments of the gaussian Grassmann multivariable integral when we have \( m \) products:

\[
\bar{\eta}_l_1 \eta_k_1 \bar{\eta}_l_2 \eta_k_2 \cdots \bar{\eta}_l_m \eta_k_m
\]

in the integrand of (A.2), where \( m \leq N \).

Consider the fixed sets: \( L = \{l_1, l_2, \ldots, l_m\} \) and \( K = \{k_1, k_2, \ldots, k_m\} \). We define \( M(L, K) \) as

\[
M(L, K) \equiv \int \prod_{i=1}^{N} d\eta_i d\bar{\eta}_i \; \bar{\eta}_{l_1} \eta_{k_1} \cdots \bar{\eta}_{l_m} \eta_{k_m} \; e^{\sum_{i,j=1}^{N} \bar{\eta}_i A_{ij} \eta_j}, \tag{A.9}
\]

and the products are ordered such that: \( l_1 < l_2 < \cdots < l_m \) and \( k_1 < k_2 < \cdots < k_m \).

Using an analogous reasoning, we obtain

\[
M(L, K) = \int \prod_{i=1}^{N} d\eta_i d\bar{\eta}_i \sum_{j_1, \ldots, j_N=1}^{N} B_{1j_1} \cdots B_{l-1,j_{i-1}} B_{l,j_i} \cdots B_{Nj_N} \times
\]

\[
\bar{\eta}_{l_1} \eta_{j_1} \cdots \bar{\eta}_{l_m} \eta_{j_N} \; = \; det B(L, K),
\]

\[
= (-1)^{(l_1+l_2+\cdots+l_m)+(k_1+k_2+\cdots+k_m)} A(L, K), \tag{A.10}
\]

where the matrix \( B(L, K) \) is defined as:

\[
B_{ij}(L, K) = \begin{cases} 
A_{ij}, & \text{if } i \neq l_1, \ldots, l_n \text{ and } j \neq k_1, \ldots, k_n \\
\delta_{i1} \delta_{j1}, & \text{if } i = l_1 \text{ or } j = k_1 \\
\vdots & \\
\delta_{il_m} \delta_{jk_m}, & \text{if } i = l_m \text{ or } j = k_m,
\end{cases} \tag{A.11}
\]

and \( i, j = 1, 2, \ldots, N \). \( A(K, L) \) is the determinant of the matrix obtained from matrix \( A \) by deleting the lines: \( \{l_1, l_2, \ldots, l_n\} \), and, the columns: \( \{k_1, k_2, \ldots, k_n\} \).

In summary, we can say that the effect of the presence of a product \( \bar{\eta}_l \eta_k \) within the integrand of the gaussian integral (A.2), is to replace the line \( l \) of matrix \( A \), \( A_{lj} \), by \( \delta_{jk} \), and its column \( k \), \( A_{ik} \), by \( \delta_{il} \). In their turn, the determinants of matrices \( B(L, K) \), eq.(A.10), are easily written in terms of determinants of matrices of smaller dimension. Hence products of
Grassmann generators cut down the dimension of the matrices the determinant of which we are to calculate.

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13. We continue to use the notation (4.1.5);
14. Due to the fact that \( A \) is a block–matrix, the contributions from the sectors \( \sigma \sigma = \uparrow \uparrow \) and \( \sigma \sigma = \downarrow \downarrow \) are decoupled. Because \( A^{\uparrow \uparrow} = A^{\downarrow \downarrow} \), the results of the integrals in the two sectors \( \sigma \sigma = \uparrow \uparrow \) and \( \sigma \sigma = \downarrow \downarrow \) are equal. That is why it is enough to calculate the integrals in only one sector;
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