Weighted Hardy-Sobolev Spaces and Complex Scaling of Differential Equations with Operator Coefficients

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Abstract

In this paper we study weighted Hardy-Sobolev spaces of vector valued functions analytic on double-napped cones of the complex plane. We introduce these spaces as a tool for complex scaling of linear ordinary differential equations with dilation analytic unbounded operator coefficients. As examples we consider boundary value problems in cylindrical domains and domains with quasicylindrical ends.

Key words: complex scaling, quasicylindrical ends, cylindrical ends, cusps, dilation analytic, Hardy classes, Hardy-Sobolev spaces, operator coefficients

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1 This work was funded by grant number 108898 awarded by the Academy of Finland.

Preprint submitted to Elsevier
1 Introduction

1.1 Motivation and structure of the paper

Complex scaling method has a long history in mathematical physics, going back to the original work of Combes [1]. The idea was developed by Aguilar, Balslev, Combes, and Simon in [2,3,4]. Van Winter independently studied the complex scaling method invoking a technique of spaces of analytic functions [5,6], however this approach did not get ensuing development. Further substantial steps were maid by Simon [7], Hunziker [8], Gérard [9], a microlocal approach to the complex scaling was devised by Helffer and Sjöstrand [10]; for a historical account see e.g. [11]. More recently, a version of the microlocal approach [10] appeared in the paper [12] by Wunsch and Zworski. Mazzeo and Vasy [39] implemented the complex scaling for studying the Laplacian on symmetric spaces.

The complex scaling method is widely used in mathematical physics as an effective tool for meromorphic continuation of the resolvent. Beyond that point, the method is finding increasing use in the construction of artificial perfectly matched absorbing layers for reliable numerical analysis of problems of mathematical physics, see e.g. [15,16,17,18,19]; here the original idea is due to Bérenger [13]. In spite of attracting a lot of attention, the method of complex scaling itself is poorly explored for boundary value problems in domains with quasicylindrical ends (quasicylindrical ends are unbounded domains that
can be transformed in a neighbourhood of infinity to a half-cylinder by suitable diffeomorphisms. To the author’s knowledge, in this setting there are only results obtained by applying the modal expansion method in cylindrical ends. The modal expansions immediately justify the complex scaling, see e.g. [14], but unfortunately the class of problems, where modal expansions can be employed, is quite narrow. On the other hand, a theory of elliptic boundary value problems in domains with quasicylindrical ends or with cusps was developed by Feigin [20], Bagirov and Feigin [21], Maz’ya and Plamenevskii [22,23], see also Kozlov et al [35,36] and references therein. Nevertheless analytic properties of solutions that may be used for the complex scaling are beyond the scope of this theory. We first need analyticity of solutions with respect to a local coordinate in a neighbourhood of infinity. In the case of a cylindrical end, this coordinate is the axial coordinate of the cylinder, for a quasicylindrical end it is a curvilinear coordinate along the quasicylinder. It is natural to assume that the coefficients of operators, the right hand side of the problem, and the boundary are analytic with respect to only that local coordinate. In particular, we will be interested in behavior of analytic continuations of solutions as the coordinate tends to infinity in a complex sector.

In order to study analytic properties of solutions near infinity, we implement a special localization procedure in every quasicylindrical end. This procedure preserves the analyticity of solutions in the neighborhoods of localization. The localized solutions satisfy some subsidiary problems in infinite cylinders, our task reduces to studying of complex scaling for the subsidiary problems. In many instances one can treat the subsidiary boundary value problems in infinite cylinders as linear ordinary differential equations with unbounded operator coefficients. We develop the complex scaling for these equations.

In many approaches to the complex scaling, original and scaled operators are studied at first separately, and then the relations between them are clarified with the help of a sufficiently large set of analytic functions. Our approach is principally different: we consider the original and scaled operators as one operator acting in spaces of analytic functions. Namely, we study differential operators with unbounded operator coefficients in Hardy-Sobolev spaces of analytic functions. Here we essentially rely on methods of the mentioned above theory of elliptic boundary value problems. In this paper we are mainly concerned with the Hardy-Sobolev spaces. Exploring the complex scaling, we restrict ourselves to a relatively simple case, assuming that the values of parameters are limited so that the operators are Fredholm. A more general case will be considered elsewhere.

The structure of the paper is as follows. We start with a detailed outline of the paper in Section 1.2, where we formulate our results in a simplified form and illustrate them by some examples. The main text consists of three parts (Sections 2–4) plus Appendix. Section 2 is devoted to a close examination
of weighted Hardy spaces of functions analytic on double-napped cones of the complex plane. In these spaces we consider Fredholm polynomial operator pencils, Section 2.4. In Section 3, by means of the Fourier-Laplace transformation, we pass from the Hardy spaces to the Hardy-Sobolev spaces and study properties of the latter. Finally, results of Sections 2 and 3 are applied in Section 4, where we investigate linear ordinary differential equations with unbounded operator coefficients. Every section is equipped with a short outline. Some technical proofs of Section 2 are located in Appendix.

1.2 Outline and examples of applications

Let \( \varphi \in (0, \pi) \) and let \( e^{i\varphi\mathbb{R}} \) stand for the line \( \{ \lambda \in \mathbb{C} : \Im \lambda = \Re \lambda \tan \varphi \} \) in the complex plane \( \mathbb{C} \). By \( \mathcal{K}^\varphi = \{ \lambda \in \mathbb{C} : \lambda = e^{i\psi \xi}, \xi \geq 0, \psi \in (0, \varphi) \} \) we denote the open double-napped cone. We define the Hardy class \( \mathcal{H}(\mathcal{K}^\varphi; X) \) of analytic functions \( \mathcal{K}^\varphi \ni \lambda \mapsto F(\lambda) \in X \) with values in a Hilbert space \( X \) as a class of all the functions \( F \) satisfying the estimate \( \| F; L_2(e^{i\varphi\mathbb{R}}; X) \| \leq \text{Const}(F) \) for \( \psi \in (0, \varphi) \). Here \( L_2(e^{i\varphi\mathbb{R}}; X) \) is the space of all square summable \( X \)-valued functions on \( e^{i\varphi\mathbb{R}} \). The functions \( F \in \mathcal{H}(\mathcal{K}^\varphi; X) \) can be extended to almost all points of the boundary \( \partial \mathcal{K}^\varphi = e^{i\varphi\mathbb{R}} \cup \mathbb{R} \) by the non-tangential limits (i.e. by the limits of \( F(\lambda) \) in \( X \) as \( \lambda \) goes to \( \mu \in \partial \mathcal{K}^\varphi \) in \( \mathcal{K}^\varphi \) by a non-tangential to \( \partial \mathcal{K}^\varphi \) path). The extended functions satisfy the inclusions \( F \in L_2(\mathbb{R}; X) \) and \( F \in L_2(e^{i\varphi\mathbb{R}}; X) \); in the case \( \varphi = \pi \) we distinguish the banks \( \lim_{\psi \to 0^+} e^{i\psi \mathbb{R}} \) and \( \lim_{\psi \to \varphi^+} e^{i\psi \mathbb{R}} \) of the boundary \( \partial \mathcal{K}^\varphi \). The Hardy class \( \mathcal{H}(\mathcal{K}^\varphi; X) \) endowed with the norm

\[
\| F; \mathcal{H}(\mathcal{K}^\varphi; X) \| = \| F; L_2(\mathbb{R}; X) \| + \| F; L_2(e^{i\varphi\mathbb{R}}; X) \|
\]

is a Banach space. For any \( \psi \in [0, \varphi] \) we can identify the elements \( F \in \mathcal{H}(\mathcal{K}^\varphi; X) \) with their traces \( F|_{e^{i\psi\mathbb{R}}} \in L_2(e^{i\psi\mathbb{R}}; X) \), then \( \mathcal{H}(\mathcal{K}^\varphi; X) \) is dense in \( L_2(e^{i\varphi\mathbb{R}}; X) \). As is known, the Fourier transformation yields an isometric isomorphism of the space \( L_2(\mathbb{R}; X) \) onto itself. It turns out that the Fourier transformation also yields an isometric isomorphism between \( \mathcal{H}(\mathcal{K}^\varphi; X) \) and a Hardy class \( H(K^\varphi; X) \) in the dual double-napped cone \( K^\varphi = \{ z \in \mathbb{C} : z = e^{-i\psi t}, t \leq 0, \psi \in (0, \varphi) \} \); here the elements of the Hardy classes are identified with their traces on \( \mathbb{R} \), the class \( H(K^\varphi; X) \) is defined in the same way as \( \mathcal{H}(\mathcal{K}^\varphi; X) \). Note that spaces similar to \( \mathcal{H}(\mathcal{K}^\varphi; X) \) and \( H(K^\varphi; X) \) were considered in context of the complex scaling method by Van Winter [5,6].

We introduce the weighted Hardy class \( \mathcal{H}^\ell(\mathcal{K}^\varphi; X) \), \( \ell \in \mathbb{R} \), that consists of all functions analytic in the cone \( \mathcal{K}^\varphi \) and satisfying the uniform in \( \psi \in (0, \varphi) \) estimate

\[
\|(1 + | \cdot |^2)^{\ell/2} F(\cdot); L_2(e^{i\varphi\mathbb{R}}; X) \| \leq \text{Const}(F).
\]  

Let \( \mathcal{H}^\ell(e^{i\varphi\mathbb{R}}; X) \) denote the weighted \( L_2 \)-space with the norm equal to the left hand side of the estimate (1.1). The class \( \mathcal{H}^\ell(\mathcal{K}^\varphi; X) \) supplied with the
norm
\[ \|F; \mathcal{H}^\ell(K^\varphi; X)\| = \|F; \mathcal{W}^\ell(\mathbb{R}; X)\| + \|F; \mathcal{W}^\ell(e^{i\varphi}\mathbb{R}; X)\| \]

is a Banach space. We extend the functions \( F \in \mathcal{H}^\ell(K^\varphi; X) \) to almost all points of the boundary \( \partial K^\varphi \) by the non-tangential limits. Here again we can identify the elements \( F \in \mathcal{H}^\ell(K^\varphi; X) \) with their traces \( F|_{e^{i\varphi}\mathbb{R}} \) for any \( \psi \in [0, \varphi] \), then \( \mathcal{H}^\ell(K^\varphi; X) \) is dense in \( \mathcal{W}^\ell(e^{i\varphi}\mathbb{R}; X) \). As is known, the Fourier transformation maps \( \mathcal{W}^\ell(\mathbb{R}; X) \) to the Sobolev space \( \mathcal{W}^\ell(\mathbb{R}; X) \) of tempered distributions with values in \( X \); see e.g. [28]. Clearly the Fourier transformation also maps \( \mathcal{H}^\ell(K^\varphi; X) \) to some subset \( \mathcal{H}^\ell(K^\varphi; X) \) of \( \mathcal{W}^\ell(\mathbb{R}; X) \), \( \mathcal{H}^\ell(K^\varphi; X) \) is dense in \( \mathcal{W}^\ell(\mathbb{R}; X) \). It turns out that \( \mathcal{H}^\ell(K^\varphi; X) \) is the Hardy-Sobolev space of order \( \ell \). In the main text we suppose that \( \ell \) is a real number, but in this introductory part we restrict ourselves to the case of nonnegative integer \( \ell \).

In this case the Hardy-Sobolev space \( \mathcal{H}^\ell(K^\varphi; X) \) can be introduced as a space of all functions \( F \in \mathcal{H}(K^\varphi; X) \) such that the complex derivatives \( D^j_F \), \( j = 1, \ldots, \ell \), are also in the space \( \mathcal{H}(K^\varphi; X) \). We define the norm in \( \mathcal{H}^\ell(K^\varphi; X) \) by the equality
\[ \|F; \mathcal{H}^\ell(K^\varphi; X)\| = \sum_{j=0}^\ell \|D^j_F; \mathcal{H}(K^\varphi; X)\|. \]

For any \( \psi \in [0, \varphi] \) the space \( \mathcal{H}^\ell(K^\varphi; X) \) can be viewed as a space of functions on the line \( e^{-i\psi}\mathbb{R} \), then \( \mathcal{H}^\ell(K^\varphi; X) \) is dense in the Sobolev space \( \mathcal{W}^\ell(e^{-i\psi}\mathbb{R}; X) \). Note that for \( \ell \geq 1 \) the functions from \( \mathcal{H}^\ell(K^\varphi; X) \) are in the class \( \mathcal{C}^{\ell-1}(K^\varphi; X) \) of \( \ell - 1 \) times continuously differentiable functions in the closure \( \overline{K^\varphi} \). For a fixed \( \psi \in [0, \varphi] \) the Fourier transformation
\[ F(z) = \frac{1}{\sqrt{2\pi}} \int_{e^{i\varphi}\mathbb{R}} e^{iz\lambda} F(\lambda) d\lambda, \quad z \in e^{-i\psi}\mathbb{R}, \]
defined first on the smooth compactly supported functions \( e^{i\psi}\mathbb{R} \ni \lambda \mapsto F(\lambda) \in X \), can be extended to an isomorphism between \( \mathcal{W}^\ell(e^{i\psi}\mathbb{R}; X) \) and \( \mathcal{W}^\ell(e^{-i\psi}\mathbb{R}; X) \), and also between \( \mathcal{H}^\ell(K^\varphi; X) \) and \( \mathcal{H}^\ell(K^\varphi; X) \).

Boundary value problems in infinite cylinders and differential equations with operator coefficients are traditionally studied in a scale of Sobolev spaces with exponential weights; see [35,36] and references therein. Therefore we introduce the exponential weight function \( e^\zeta : z \mapsto \exp(-i\zeta z) \), where \( \zeta \) is a complex weight number. We also want to shift the vertex of the cone \( K^\varphi \) to a point \( w \) of the complex plane, let \( K^\varphi_w = \{ z \in \mathbb{C} : z - w \in K^\varphi \} \). By \( \mathcal{H}^\ell(K^\varphi_w; X) \) we denote the weighted Hardy-Sobolev space that consists of all functions \( F \) such that \( (e^\zeta F)(\cdot - w) \in \mathcal{H}^\ell(K^\varphi; X) \). As the norm of \( F \) in \( \mathcal{H}^\ell(K^\varphi_w; X) \) we take the value \( \|(e^\zeta F)(\cdot - w)\|_{\mathcal{H}^\ell(K^\varphi; X)} \). Similarly we define the weighted Sobolev space \( \mathcal{W}^\ell(e^{-i\psi}\mathbb{R} + w; X) \) that consists of all functions \( F \) such that \( (e^\zeta F)(\cdot - w) \in \mathcal{W}^\ell(e^{-i\psi}\mathbb{R}; X) \); here \( e^{-i\psi}\mathbb{R} + w \) denotes the line \( \{ z \in \mathbb{C} : z - w \in e^{-i\psi}\mathbb{R} \} \). Let us stress that the behavior of the weight \( e^\zeta \) of the space \( \mathcal{W}^\ell(e^{-i\psi}\mathbb{R} + w; X) \) depends on the angle \( \psi \). For a fixed \( \psi \in [0, \varphi] \) the inverse
Fourier-Laplace transformation

\[ F(\lambda) = \frac{1}{2\pi} \int_{e^{-i\psi}R+w} e^{-i\lambda z} F(z) \, dz, \quad \lambda \in e^{i\psi}R + \zeta, \quad (1.2) \]

can be extended to an isomorphism between \( \mathcal{W}_z^\ell(e^{-i\psi}R+w; X) \) and \( \mathcal{W}_w^\ell(e^{i\psi}R+\zeta; X) \), where \( \mathcal{W}_w^\ell(e^{i\psi}R+\zeta; X) \) is the space of functions \( e^{i\psi}R+\zeta \ni \lambda \mapsto F(\lambda) \in X \) with the finite norm \( \| F; \mathcal{W}_w^\ell(e^{i\psi}R+\zeta; X) \| = \| \exp\{iw(\cdot - \zeta)\}F(\cdot - \zeta); \mathcal{W}_w^\ell(e^{i\psi}R; X) \| \).

Let \( K^\phi = \{ \lambda \in \mathbb{C}: \lambda - \zeta \in K^\phi \} \) be the cone \( K^\phi \) shifted by the weight number \( \zeta \). By \( \mathcal{H}_w^\ell(K^\phi; X) \) we denote the weighted Hardy space that consists of all functions such that \( \exp\{iw\cdot\} F(\cdot - \zeta) \in \mathcal{H}^\ell(K^\phi; X) \). The norm in \( \mathcal{H}_w^\ell(K^\phi; X) \) is given by

\[ \| F; \mathcal{H}_w^\ell(K^\phi; X) \| = \| \exp\{iw(\cdot - \zeta)\}F(\cdot - \zeta); \mathcal{H}^\ell(K^\phi; X) \|. \]

Then the extended Fourier-Laplace transformation (1.2) yields an isomorphism between the weighted Hardy-Sobolev space \( \mathcal{H}_w^\ell(K^\phi; X) \) and the weighted Hardy space \( \mathcal{H}_w^\ell(K^\phi; X) \). This enables us to adapt methods of the theory of ordinary differential equations with unbounded operator coefficients developed in the scale of Sobolev spaces \( \mathcal{W}_z^\ell(\mathbb{R}; X) \) (see e.g. [35]) to the case of the Hardy-Sobolev spaces \( \mathcal{H}_w^\ell(K^\phi; X) \).

The main purpose of this paper is to study the weighted spaces \( \mathcal{H}_w^\ell(K^\phi; X) \) and \( \mathcal{H}_w^\ell(K^\phi; X) \). Nevertheless, we demonstrate how to treat the complex scaling of differential equations with unbounded operator coefficients in terms of the weighted Hardy-Sobolev spaces. We also give examples of applications to boundary value problems in cylindrical domains and in domains with quasi-cylindrical end. Before presenting our results on the complex scaling, we give some preliminaries.

Let \( X_j \) denote a Hilbert space with the norm \( \| \cdot \|_j \). We introduce a set \( \{ X_j \}_{j=0}^m \) of Hilbert spaces such that \( \| u \|_j \leq \| u \|_{j+1} \) and \( X_{j+1} \) is dense in \( X_j \) for all \( j = 0, \ldots, m-1 \). Let \( \{ A_j \in \mathcal{B}(X_j, X_0) \}_{j=0}^m \) be a set of operators, where \( \mathcal{B}(X_j, X_0) \) stands for the set of all linear bounded operators \( A_j : X_j \to X_0 \). We start with the differential equation with constant operator coefficients

\[ \sum_{j=0}^m A_{m-j} D_j u(t) = F(t), \quad t \in \mathbb{R}, \quad (1.3) \]

where \( D_j = -i\partial_t \). Assume that the operator \( A(\lambda) = \sum_{j=0}^m A_{m-j} \lambda^j \) from \( \mathcal{B}(X_m, X_0) \) is Fredholm for all \( \lambda \in \mathbb{C} \) and is invertible for at least one value of \( \lambda \). Under these assumptions the operator \( A(\lambda) \) is invertible for all \( \lambda \in \mathbb{C} \) except for isolated eigenvalues of the operator pencil \( \mathbb{C} \ni \lambda \mapsto A(\lambda) \in \mathcal{B}(X_m, X_0) \).
These eigenvalues are of finite algebraic multiplicities and can accumulate only at infinity. We also assume that there exists $R > 0$ such that for all $f \in X_0$ the estimate

$$
\sum_{j=0}^{m} |\lambda|^j \|\mathcal{A}^{-1}(\lambda)f\|_{m-j} \leq c\|f\|_0, \quad \lambda \in \mathbb{R}, \ |\lambda| > R, \quad (1.4)
$$

is fulfilled. The assumptions we made are widely met in the theory of differential equations with operator coefficients, they are satisfied in many applications to boundary value problems for partial differential equations; see e.g. [35,36] and references therein. The assumption (1.4) in particular guaranties the existence of an angle $\vartheta \in (0, \pi/2)$ such that for any $\zeta \in \mathbb{C}$ and any $\varphi \in (0, \vartheta)$ the closed dual cone $K_{\zeta}^\varphi$ contains at most finitely many eigenvalues of the operator pencil $\mathcal{A}$, the estimate (1.4) remains valid for all $\lambda \in \overline{K_{\zeta}^\varphi}, \ |\lambda| > R$; see [35, Proposition 2.2.1]. One can find $\zeta \in \mathbb{C}$ and $\varphi \in (0, \vartheta)$ so that $K_{\zeta}^\varphi$ is free from the spectrum of $\mathcal{A}$. In this paper dealing with complex scaling we restrict ourselves by the assumption that $K_{\zeta}^\varphi$ is free from the spectrum of $\mathcal{A}$, a more general case will be considered elsewhere.

In the remaining part of this subsection we assume that $\varphi \in (0, \vartheta)$.

We introduce two scales of Banach spaces

$$
D^m_\zeta(\mathbb{R}) = \bigcap_{j=0}^{m} W^{m-j}(\mathbb{R}; X_j), \quad \|u; D^m_\zeta(\mathbb{R})\| = \sum_{j=0}^{m} \|u; W^{m-j}_\zeta(\mathbb{R}; X_j)\|; \quad (1.5)
$$

$$
D^m_\zeta(K_{\varphi}^\zeta) = \bigcap_{j=0}^{m} H^{m-j}_\zeta(K_{\varphi}^\zeta; X_j), \quad \|u; D^m_\zeta(K_{\varphi}^\zeta)\| = \sum_{j=0}^{m} \|u; H^{m-j}_\zeta(K_{\varphi}^\zeta; X_j)\|. \quad (1.6)
$$

We can identify the functions $u \in D^m_\zeta(K_{\varphi}^\zeta)$ with their non-tangential boundary limits $u|_\mathbb{R} \in D^m_\zeta(\mathbb{R})$. The space $D^m_\zeta(K_{\varphi}^\zeta)$ viewed as a space of functions on $\mathbb{R}$ is dense in $D^m_\zeta(\mathbb{R})$, the space $D^m_\zeta(\mathbb{R})$ does not depend on $\mathcal{R}_\zeta$. As is well-known [35, Theorem 2.4.1 and Remark 2.4.2], the operator $\mathcal{A}(D_t) : D^m_\zeta(\mathbb{R}) \to W^0_\zeta(\mathbb{R}; X_0)$ of the equation (1.3) yields an isomorphism if and only if the line $\mathbb{R} + \zeta$ is free from the eigenvalues of the pencil $\mathbb{C} \ni \lambda \mapsto \mathcal{A}(\lambda) \in \mathcal{B}(X_m, X_0)$; if there is an eigenvalue of the pencil on the line $\mathbb{R} + \zeta$ then the range of the operator $\mathcal{A}(D_t) : D^m_\zeta(\mathbb{R}) \to W^0_\zeta(\mathbb{R}; X_0)$ is not closed.

Note that if the line $\mathbb{R} + \zeta$ is free from the eigenvalues of the pencil $\mathcal{A}$ then for a sufficiently small angle $\varphi$ the closed dual cone $K_{\zeta}^\varphi$ is also free from the eigenvalues of the pencil $\mathcal{A}$. Now we are in position to formulate some results on the complex scaling.

**Theorem 1.1** Assume that the closed dual cone $K_{\zeta}^\varphi$ is free from the eigenvalues of the pencil $\mathbb{C} \ni \lambda \mapsto \mathcal{A}(\lambda) \in \mathcal{B}(X_m, X_0)$. Then the following assertions hold. (i) The operator $\mathcal{A}(D_t) : D^m_\zeta(K_{\varphi}^\zeta) \to H^0_\zeta(K_{\varphi}^\zeta; X_0)$ of the equation (1.3)
yields an isomorphism; here the analytic in $K_0^\varphi$ functions $u \in D^\omega_\varphi(K_0^\varphi)$, $F \in H^1_0(K_0^\varphi; X_0)$ are identified with their boundary limits $u \big|_{\mathbb{R}} \in D^\omega_\varphi(\mathbb{R})$, $F \big|_{\mathbb{R}} \in W^\omega_0(\mathbb{R}; X_0)$. (ii) Let $u \in D^m_\varphi(\mathbb{R})$ be a unique solution to the equation (1.3) with right hand side $F \in H^1_0(K_0^\varphi; X_0)$. Then $u \in D^m_\varphi(K_0^\varphi)$ and the function $\mathbb{R} \ni t \mapsto v(t) \equiv u(e^{-i\varphi}t) \in X_m$ is a unique solution $v \in D^m_{e^{-i\varphi}\varphi}(\mathbb{R})$ to the scaled equation $\mathcal{A}(e^{i\varphi}D_t)v(t) = F(e^{-i\varphi}t)$, $t \in \mathbb{R}$.

As an example of application of Theorem 1.1 we consider the Dirichlet problem in the cylinder $C = \{(y, t) : y \in \Omega, t \in \mathbb{R}\}$, where $\Omega$ is a domain in $\mathbb{R}^{n-1}$ with compact closure and a smooth boundary. We introduce the differential operator $\mathcal{L}(y, D_y, D_t) = \sum_{j=0}^m A_{m-j}(y, D_y)D_t^j$, where $A_{m-j}(y, D_y)$ are differential operators with smooth coefficients. Consider the Dirichlet problem

$$\begin{align*}
\mathcal{L}(y, D_y, D_t)u(y, t) &= F(y, t), \quad (y, t) \in C, \\
\partial_\nu u(y, t) &= 0, \quad (y, t) \in \partial C, \quad j = 0, \ldots, k - 1;
\end{align*}$$

(1.7)

here $\partial_\nu = \partial / \partial \nu$ and $\nu$ is the outward normal. Suppose that $2k/m$ is an integer and the order of $A_j$ does not exceed $2kj/m$ for $j = 0, \ldots, m$. To the problem (1.7) there corresponds the operator pencil

$$\mathbb{C} \ni \lambda \mapsto \mathcal{A}(\lambda) = \mathcal{L}(y, D_y, \lambda) \in \mathcal{B}(X_m, X_0),$$

(1.8)

where $X_0 = L^2(\Omega)$, $X_m = \{ U \in W^{2k}(\Omega) : \partial_\nu U = 0 \text{ on } \partial \Omega, j = 0, \ldots, k - 1\}$; here $W^{2k}(\Omega)$ stands for the Sobolev space of functions in $\Omega$. Assume that the operator $\mathcal{L}(y, D_y, D_t)$ is $(2k, m)$--elliptic, then the operator $\mathcal{A}(\lambda)$ is Fredholm for all $\lambda \in \mathbb{C}$, the estimate (1.4) is fulfilled for a sufficiently large $R > 0$ and for all $f \in X_0$; see [35, Section 2.5], where the unique solvability of the problem (1.7) is studied. Recall that $(2k, 2k)$--elliptic operators are standard elliptic operators of order $2k$, the class of $(2k, 1)$--elliptic operators includes parabolic operators.

We set $X_j = W^{2kj/m}(\Omega) \cap X_m$. The coefficients $A_j(y, D_y)$ satisfy the inclusions $A_j \in \mathcal{B}(X_j; X_0)$. In this case the space $W^\omega_0(\mathbb{R}; X_0)$ is the weighted space of square summable functions with the norm $\|e_\varphi F; L^2(\mathcal{C})\|$; here $e_\varphi$ is the same exponential weight function as before. The Hardy space $H^0_{\varphi}(K_0^\varphi; X_0)$ consists of analytic functions $K_0^\varphi \ni z \mapsto F(z) \equiv F(\cdot, z) \in L^2(\Omega)$, the elements of $H^0_{\varphi}(K_0^\varphi; X_0)$ are extended to almost all points of the boundary $\partial K_0^\varphi$. For any angle $\psi \in [0, \varphi]$ the set of functions $\{ \mathbb{R} \ni t \mapsto F(e^{-i\psi}t) : F \in H^0_{\varphi}(K_0^\varphi; X_0) \}$ is dense in the space $W^\omega_{-\psi\varphi}(\mathbb{R}; X_0)$. The space $D^\omega_{\varphi}(\mathbb{R})$ introduced in (1.5) consists of all functions $u$ such that $e_\varphi D_y^\alpha D_t^j u \in L^2(\mathcal{C})$ for $|\alpha| + 2kj/m \leq 2k$ and $\partial_\nu u = 0$ on $\partial C$ for $j = 0, \ldots, k - 1$. The elements of the space $D^m_{\varphi}(K_0^\varphi)$ defined in (1.6) are analytic functions $K_0^\varphi \ni z \mapsto u(z) \equiv u(\cdot, z) \in X_m$ extended to almost all points of the boundary $\partial K_0^\varphi$ by the non-tangential limits. For any $\psi \in [0, \varphi]$ the set of functions $\{ \mathbb{R} \ni t \mapsto u(e^{-i\psi}t) : u \in D^m_{\varphi}(K_0^\varphi) \}$ is dense in $D^m_{e^{-i\psi}\varphi}(\mathbb{R})$. 

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Suppose that the right hand side $F$ of the boundary value problem (1.7) is in the Hardy space $H^0_0(K_0^2; X_0)$. The Dirichlet problem (1.7) has a unique solution $u \in D^m_\zeta(\mathbb{R})$ for any $F \in W^0_\zeta(\mathbb{R}; X_0)$ if and only if the line $\mathbb{R} + \zeta$ is free from the spectrum of the operator pencil (1.8); recall that the spaces $D^m_\zeta(\mathbb{R})$ and $W^0_\zeta(\mathbb{R}; X_0)$ do not depend on $\Re \zeta$. If the line $\mathbb{R} + \zeta$ is free from the spectrum of $\mathcal{A}$ then the results of Theorem 1.1 are valid provided that $\varphi$ is a sufficiently small angle. By Theorem 1.1 a solution $u \in D^m_\zeta(\mathbb{R})$ belongs to the space $D^m_\zeta(K_0^2) \subset H^0_\zeta(K_0^2; X_0)$; moreover, the function $v(y, t) = u(y, e^{-i\varphi}t)$ is a unique solution $v \in D^m_{e^{-i\varphi}\zeta}(\mathbb{R})$ to the Dirichlet problem

$$\mathcal{L}(y, D_y, e^{i\varphi}D_t)v(y, t) = F(y, e^{-i\varphi}t), \ (y, t) \in \mathcal{C},$$
$$\partial_j^\alpha v(y, t) = 0, \ (y, t) \in \partial\mathcal{C}, \ j = 0, \ldots, k - 1. \tag{1.9}$$

The problem (1.9) is related to the original problem (1.7) by the complex scaling with the scaling coefficient $e^{-i\varphi}$. Similarly we can consider complex scaling of problems in the cylinder $\mathcal{C} = \{(y, t) : y \in \Omega, t \in \mathbb{R}\}$, where $\overline{\Omega}$ is a smooth compact manifold with or without boundary.

Now we consider the differential equation with variable operator coefficients

$$\sum_{j=0}^m \left(A_{m-j} + Q_{m-j}(t)\right)D_j^\alpha u(t) = F(t), \ t \in \mathbb{R}. \tag{1.10}$$

Here the coefficients $A_j$ are the same as in the equation (1.3), the coefficients $Q_j(t)$ are operator functions $\mathbb{R} \ni t \mapsto Q_j(t) \in \mathcal{B}(X_j, X_0)$ that are dilation analytic in the following sense:

i. for a large $T > 0$ and some $\alpha > 0$ the coefficients $Q_0, \ldots, Q_m$ can be extended to holomorphic operator functions

$$\{z \in \mathbb{C} : -\alpha \leq \arg(z - T) \leq 0\} \ni z \mapsto Q_j(z) \in \mathcal{B}(X_j, X_0);$$

ii. the values $\|Q_j(z); \mathcal{B}(X_j, X_0)\|$, $j = 0, \ldots, m$, uniformly tend to zero as $z$ goes to infinity inside the sector $\{z \in \mathbb{C} : -\alpha \leq \arg(z - T) \leq 0\}$.

As before we denote $\mathcal{A}(\lambda) = \sum_{j=0}^m A_{m-j} \lambda^j$. The next theorem presents the results of Section 4.2 in a simplified form.

**Theorem 1.2** Assume that the right hand side of the equation (1.10) is a compactly supported function $\mathbb{R} \ni t \mapsto F(t) \in X_0$, and there are no eigenvalues of the operator pencil $\mathbb{C} \ni \lambda \mapsto \mathcal{A}(\lambda) \in \mathcal{B}(X_m, X_0)$ in the closed dual cone $\overline{K_\zeta^\omega}$, where $\zeta \in \mathbb{C}$ and $\varphi$ does not exceed the angle $\alpha$ from the conditions i, ii. Let $T$ be a sufficiently large positive number. Then a solution $u \in D^m_\zeta(\mathbb{R})$ to
the equation (1.10) can be extended to a holomorphic function

\[ \{ z \in \mathbb{C} : -\varphi \leq \arg(z - T) \leq 0 \} \ni z \mapsto u(z) \in X_m \]
satisfying the uniform in \( \psi \in [-\varphi, 0] \) estimate

\[ \sum_{j=0}^{m} \int_{0}^{\infty} \| \exp(-i \zeta e^{i \psi} t) D_z^j u(e^{i \psi} t + T) \|^2_{m-j} dt \leq \text{Const}; \]

here \( D_z = -\frac{i}{2}(\partial_{\Re z} - i \partial_{\Im z}) \) is the complex derivative.

In order to illustrate Theorem 1.2 we consider an elliptic boundary value problem in a domain \( G \subset \mathbb{R}^n \) with quasicylindrical end. We assume that the boundary \( \partial G \) of \( G \) is smooth, the set \( \{ x \in G : x_n < 1 \} \) is bounded, and the set \( \{ x \in G : x_n > 1 \} \) coincides with the horn-like quasicylindrical end \( \{ x = (x', x_n) \in \mathbb{R}^n : x_n^{-a} x' \in \Omega, x_n > 1 \} \), where \( a < 0 \) and \( \Omega \) is a domain in \( \mathbb{R}^{n-1} \) with compact closure and smooth boundary. We introduce the Dirichlet problem

\[ \mathcal{L}(D_x)v(x) = f(x), \; x \in G; \quad \partial_j \nu v(x) = 0, \; x \in \partial G, \; j = 0, \ldots, k - 1, \quad (1.11) \]

for a \( 2k \) order elliptic differential operator \( \mathcal{L}(D_x) = \mathcal{L}(D_{x'}, D_{x_n}) \) with constant coefficients. The quasicylindrical end \( \{ x \in G : x_n > 1 \} \) can be transformed into the half-cylinder \( C_\alpha = \{(y, t) : y \in \Omega, t > (1 - a)^{-1}\} \) by the diffeomorphism \( (y, t) = \pi(x', x_n) = (x_n^{-a} x', (1 - a)^{-1} x_n^{1-a}) \). Let \( \chi \) be a smooth cutoff function such that \( \chi(x) = 0 \) for \( x_1 < 2 \) and \( \chi(x) = 1 \) for \( x_1 > 3 \). If \( v \) is a solution to the problem (1.11) then the function \( u(y, t) = (\chi v) \circ \pi^{-1}(y, t) \) extended from \( C_\alpha \) to the remaining part of the cylinder \( C = \{(y, t) : y \in \Omega, t \in \mathbb{R} \} \) by zero satisfies the Dirichlet problem

\[ (\mathcal{L}(D_y, D_t) + \mathcal{Q}(y, t, D_y, D_t)) u(y, t) = F(y, t), \quad (y, t) \in C, \]

\[ \partial_j u(y, t) = 0, \quad (y, t) \in \partial C, \; j = 0, \ldots, k - 1. \quad (1.12) \]

Here \( \mathcal{L}(D_y, D_t) \) denotes the homogeneous part (of order \( 2k \)) of \( \mathcal{L}(D_y, D_t) \), the operator \( \mathcal{L}(D_{x'}, D_{x_n}) \) written in the coordinates \( (y, t) \in C_\alpha \) is decomposed into \( \mathcal{L}(D_y, D_t) \) and \( \mathcal{Q}(y, t, D_y, D_t) \); without loss of generality we can assume that \( \mathcal{Q}(y, t, D_y, D_t) = 0 \) in \( C \setminus C_\alpha \). The subsidiary problem (1.12) can be represented in the form of the equation (1.10). The coefficients \( Q_j(y, z, D_y) \in \mathcal{B}(X_j, X_0) \) of the operator \( \mathcal{Q}(y, t, D_y, D_t) = \sum_{j=0}^{m} Q_{m-j}(y, t, D_y)D_t^j \) are dilation analytic in the sector \( \{ z \in \mathbb{C} : |\arg(z - T)| \leq \alpha \} \), where \( \alpha > 0 \) is an angle, the spaces \( X_0, \ldots, X_m \) are introduced in exactly the same way as for the problem (1.7). The operator \( \mathfrak{A}(\lambda) = \mathcal{L}(D_y, \lambda) : X_m \to X_0 \) is Fredholm for all \( \lambda \in \mathbb{C} \) and the estimate (1.4) is valid for a sufficiently large \( R > 0 \) because \( \mathcal{L} \) is elliptic. We suppose that the right hand side \( f \) of the problem (1.11) is a smooth compactly supported function. Then \( F \) in (1.12) is also smooth and compactly supported, Theorem 1.2 can be applied. As a result we have: if for a fixed \( \zeta \in \mathbb{C} \)
the line $\mathbb{R} + \zeta$ is free from the eigenvalues of the pencil $\mathbb{C} \ni \lambda \mapsto \mathfrak{A}(\lambda)$ and $v(x) = v(x', x_n)$ is a solution to the problem (1.11) such that

$$\sum_{j=0}^{m} \int_{(1-a)^{-1}}^{+\infty} \| \exp \left( -i\zeta t \right) D^j v \circ \kappa^{-1}(\cdot, t) \|_{m-j}^2 dt < \infty$$

(1.13)

then for a sufficiently large number $T > 0$ and an angle $\varphi > 0$ the solution $v$ can be continued to a holomorphic function

$$\{ z \in \mathbb{C} : |\arg(z - T)| \leq \varphi \} \ni z \mapsto v \circ \kappa^{-1}(\cdot, z) \in X_m$$

(1.14)

satisfying the uniform in $\psi \in [-\varphi, \varphi]$ estimate

$$\sum_{j=0}^{m} \int_{(1-a)^{-1}}^{+\infty} \| \exp \left( -i\zeta e^{i\psi} t \right) D^j v \circ \kappa^{-1}(\cdot, e^{i\psi} t + T) \|_{m-j}^2 dt \leq C$$

(1.15)

with a constant $C$. The condition (1.13) provides the inclusion $u \in D^m_{\zeta}(\mathbb{R})$ for the solution of (1.12). Since the function (1.14) is holomorphic we can perform the complex scaling of the problem (1.11) with respect to the coordinate $z$ in the conical neighbourhood $\{ z \in \mathbb{C} : |\arg(z - T)| \leq \varphi \}$ of infinity. The estimate (1.15) with the weight number $\zeta \in \mathbb{C}$ controls the behavior of the scaled solution at infinity. From properties of the holomorphic function (1.14) satisfying (1.15) it follows that for any $\phi > 0$ and $j \geq 0$ the value $|\exp(-i\zeta z)||z|^{1/2+j}\|D^j v \circ \kappa^{-1}(\cdot, z)\|_m$ uniformly tends to zero as $z \to \infty$, $|\arg(z - T)| \leq \varphi - \phi$.

Statements of elliptic problems in domains with quasicylindrical ends in scales of weighed Sobolev spaces and properties of solutions were studied in many works [20,21,22,23,35,36] and others. In particular, from the known results it follows that if the line $\mathbb{R} + \zeta$ is free from the eigenvalues of the pencil $\mathbb{C} \ni \lambda \mapsto \mathfrak{A}(\lambda)$ and the right hand side $f \in C^\infty_0(\mathcal{G})$ of the problem (1.11) is subjected to a finite number of orthogonality conditions then the problem has at least one solution (but not more than a finite number of solutions) $v \in W^m_{loc}(\mathcal{G})$ satisfying the conditions (1.13); here $W^m_{loc}(\mathcal{G})$ denotes the space of functions that are locally in the Sobolev space $W^m(\mathcal{G})$. The results on the analytic properties of the solutions to the problem (1.11) are new. Let us note that the assumption $f \in C^\infty_0(\mathcal{G})$ on the right hand side of the problem (1.11) is made for simplicity only, one can formulate a weaker assumption in terms of the Hardy-Sobolev spaces using Theorem 4.8 instead of Theorem 1.2, see also Remark 4.9. Moreover, quasicylindrical ends of many other geometric shapes (e.g. quasicylindrical ends that approach a cylinder at infinity) can be considered by choosing suitable diffeomorphisms that map quasicylinders onto the half-cylinder $\{(y, t) : y \in \Omega, t > 0\}$, c.f. [23] or [36, Chapter 9]. For other examples of applications of differential equations with operator coefficients to boundary value problems we refer to the books [35,36], some of these examples can also be considered in the scale of Hardy-Sobolev spaces.
2 Spaces of analytic vector valued functions

This section deals with weighted spaces of vector valued functions analytic in a dual complex cone or in a half-plane. In Subsection 2.1 we introduce the weighted Hardy classes $H^\ell_w(K_\varphi^\psi; X)$ and study their basic properties. In Subsection 2.2 we deal with some weighted Hardy classes in a half-plane, in particular we prove that these classes coincide with classical Hardy classes if the weight numbers are zeros. We also clarify relations between the Hardy classes in cones and in half-planes. In Subsection 2.3 we proceed to study the classes $H^\ell_w(K_\varphi^\psi; X)$; here we rely heavily on the material of Subsection 2.2. Finally, in Subsection 2.4 we consider Fredholm polynomial operator pencils in spaces of analytic vector valued functions.

2.1 Weighted Hardy classes in cones

Let $X$ be a Hilbert space and let $e^{i\psi} \mathbb{R} + \zeta = \{ \lambda \in \mathbb{C} : \lambda = e^{i\psi} \xi + \zeta, \xi \in \mathbb{R} \}$ be the line in the complex plane $\mathbb{C}$, where $\psi$ is an angle and $\zeta$ is a fixed complex parameter. For $\ell \in \mathbb{R}$ and $w \in \mathbb{C}$ we introduce the weighted $L_2$ space $\mathcal{W}_w^\ell(e^{i\psi} \mathbb{R} + \zeta; X)$ of (classes of) functions $e^{i\psi} \mathbb{R} + \zeta \ni \lambda \mapsto F(\lambda) \in X$ with the finite norm

$$\| F; \mathcal{W}_w^\ell(e^{i\psi} \mathbb{R} + \zeta; X) \|^2 = \int_{e^{i\psi} \mathbb{R} + \zeta} |\exp\{2iw\lambda\}|(1 + |\lambda|^2)^\ell \| F(\lambda) \|^2 |d\lambda|, \quad (2.1)$$

where $\| \cdot \|$ is the Hilbert norm in $X$. The space $\mathcal{W}_w^\ell(e^{i\psi} \mathbb{R} + \zeta; X)$ with the norm (2.1) is a Hilbert space [24]. It is clear that the space $\mathcal{W}_w^\ell(e^{i\psi} \mathbb{R} + \eta; X)$ and its norm does not change while $\eta$ travels along the line $e^{i\psi} \mathbb{R} + \zeta$. Let us also note that

$$\| F; \mathcal{W}_w^\ell(e^{i\psi} \mathbb{R} + \zeta; X) \| = e^{3(\zeta(w-v))} \| F; \mathcal{W}_w^\ell(e^{i\psi} \mathbb{R} + \zeta; X) \|, \quad v \in e^{-i\psi} \mathbb{R} + w. \quad (2.2)$$

The embedding $\mathcal{W}_w^\ell(e^{i\psi} \mathbb{R} + \zeta; X) \subset \mathcal{W}_w^s(e^{i\psi} \mathbb{R} + \zeta; X)$ is continuous for $\ell > s$.

By $K_\varphi^\psi$ we denote the open double-napped cone

$$K_\varphi^\psi = \{ \lambda \in \mathbb{C} : \lambda = e^{i\psi} \xi + \zeta, \xi \in \mathbb{R} \setminus \{0\}, 0 < \psi < \varphi \}$$

with the vertex $\zeta$ and the angle $\varphi \in (0, \pi]$. If $\varphi = \pi$, then $K_\varphi^\psi = \mathbb{C} \setminus (\mathbb{R} + \zeta)$.

**Definition 2.1** We introduce the weighted Hardy class $H^\ell_w(K_\varphi^\psi; X)$ as the set of all analytic functions $K_\varphi^\psi \ni \lambda \mapsto F(\lambda) \in X$ satisfying the uniform with respect to $\psi \in (0, \varphi)$ estimate

$$\| F; \mathcal{W}_w^\ell(e^{i\psi} \mathbb{R} + \zeta; X) \| \leq C(F). \quad (2.3)$$
Before we proceed further, we cite the following proposition, that contains some facts from the theory of Fourier transforms of analytic functions.

**Proposition 2.2** 1. Let \( \Phi \) be an analytic in the strip
\[ \Pi = \{ s \in \mathbb{C} : s = r + i\psi, r \in \mathbb{R}, 0 < \psi < \varphi \} \]
function taking the values in a Hilbert space \( X \). We set \( \Phi_\psi(r) \equiv \Phi(r + i\psi) \).
Suppose that \( \Phi \) satisfies the estimate
\[ ||\Phi_\psi; L_2(\mathbb{R}; X)|| \leq \text{Const}, \psi \in (0, \varphi); \quad (2.4) \]
here \( L_2(\mathbb{R}; X) (\equiv \mathcal{W}_0^0(\mathbb{R}; X)) \) denotes the space of square-summable functions \( \Phi_\psi : \mathbb{R} \rightarrow X \). The following assertions are true.

(i) The function \( \Phi \) has boundary limits \( \Phi_0, \Phi_\varphi \in L_2(\mathbb{R}; X) \) in the sense that for almost all \( r \in \mathbb{R} \) we have \( ||\Phi(s) - \Phi_0(r)|| \rightarrow 0 \) as \( s \) tends to \( r \) by a non-tangential to \( \mathbb{R} \) path, and \( ||\Phi(s) - \Phi_\varphi(r)|| \rightarrow 0 \) as \( s \) tends to \( r + i\varphi \) by a non-tangential to \( \mathbb{R} + i\varphi \) path; moreover,
\[ ||\Phi_0 - \Phi_\psi; L_2(\mathbb{R}; X)|| \rightarrow 0, \quad \psi \rightarrow 0^+; \quad ||\Phi_\varphi - \Phi_\psi; L_2(\mathbb{R}; X)|| \rightarrow 0, \quad \psi \rightarrow \varphi^- . \]

(ii) For all \( \psi \in [0, \varphi] \) the following estimate holds
\[ ||\Phi_\psi; L_2(\mathbb{R}; X)|| \leq ||\Phi_0; L_2(\mathbb{R}; X)|| + ||\Phi_\varphi; L_2(\mathbb{R}; X)|| . \]

(iii) For every compact set \( \mathfrak{K} \subset \Pi \) there is an independent of \( \Phi \) constant \( C(\mathfrak{K}) \) such that
\[ ||\Phi(s)|| \leq C(\mathfrak{K}) (||\Phi_0; L_2(\mathbb{R}; X)|| + ||\Phi_\varphi; L_2(\mathbb{R}; X)||), \quad s \in \mathfrak{K} . \]

(iv) The value \( ||\Phi(s)|| \) uniformly tends to zero as \( s \) goes to infinity in the strip
\{ \( s \in \Pi : \exists \psi \in [\phi, \varphi - \phi] \}, \phi > 0 \.

(v) The set \( \{ \Phi_{|\mathbb{R} + i\psi}: \Phi \in H(\Pi; X) \} \) is dense in the space \( \mathcal{W}_0^0(\mathbb{R} + i\psi; X) \) for any \( \psi \in [0, \varphi] \).

2. Let \( \Phi \) be an analytic function \( \Pi \ni s \mapsto \Phi(s) \in X \) and \( \Phi_\psi(r) \equiv \Phi(r + i\psi) \). Suppose that \( \Phi_0, \Phi_\varphi \in L_2(\mathbb{R}; X) \), and \( ||\Phi(s)|| \leq \text{const} \) for all \( s \in \Pi \). Then the estimate \((2.4)\) is valid.

3. A function \( \Pi \ni s \mapsto \Phi(s) \in X \) is an analytic function satisfying the estimate \((2.4)\) if and only if it can be represented in the form
\[ \Phi(s) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\Psi_0(r)}{r - s} dr - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\Psi_\varphi(r)}{r + i\varphi - s} dr, \quad s \in \Pi, \quad (2.5) \]
for some $\Psi_0, \Psi_\varphi \in L_2(\mathbb{R}; X)$.

**PROOF.** These results are well-known in the case of analytic functions taking their values in $\mathbb{C}$; see e.g. [25,26,27]. The generalization for the case of functions with values in a Hilbert space $X$ is straightforward due to the Fourier transformation $\mathcal{F}: L_2(\mathbb{R}; X) \to L_2(\mathbb{R}; X)$ that implements an isometric isomorphism; see e.g. [33,34]. The Parseval equality and other facts about the Fourier transformation of vector valued functions can be found in [28,33], see also Section 3.1 of this paper. \hfill \square

**Proposition 2.3** 1. Let $\mathcal{F} \in \mathcal{H}_w^\ell(\mathcal{K}_C^\varphi; X)$ with some $\ell \in \mathbb{R}$, $w, \zeta \in \mathbb{C}$, and $\varphi \in (0, \pi]$. Then the following assertions are true.

(i) $\mathcal{F}$ has boundary limits $\mathcal{F}_0 \in \mathcal{H}_w^\ell(\mathbb{R}; X)$ and $\mathcal{F}_\varphi \in \mathcal{H}_w^\ell(e^{i\psi}R + \zeta; X)$ in the sense that for almost all points $\mu$ of the boundary $\partial \mathcal{K}_C^\varphi = (\mathbb{R} + \zeta) \cup (e^{i\varphi}R + \zeta)$ we have $\|\mathcal{F}(\lambda) - \mathcal{F}_0(\mu)\| \to 0$ as $\lambda$ non-tangentially tends to $\mu \in \mathbb{R} + \zeta$, and $\|\mathcal{F}(\lambda) - \mathcal{F}_\varphi(\mu)\| \to 0$ as $\lambda$ non-tangentially tends to $\mu \in e^{i\varphi}R + \zeta$. Moreover,

$$
\|(e_w \mathcal{F}) \circ \varphi, \zeta - (e_w \mathcal{F}_0) \circ \varphi, \zeta; \mathcal{H}_w^\ell(\mathbb{R}; X)\| \to 0, \quad \psi \to 0^+,
$$

and

$$
\|(e_w \mathcal{F}) \circ \varphi, \zeta - (e_w \mathcal{F}_\varphi) \circ \varphi, \zeta; \mathcal{H}_w^\ell(\mathbb{R}; X)\| \to 0, \quad \psi \to \varphi^-,
$$

where $e_w$ denotes the weight function $\lambda \mapsto \exp(iw\lambda)$ and $\varphi, \zeta: \mathbb{R} \to e^{i\psi} \mathbb{R} + \zeta$ is the linear transformation $\varphi, \zeta(\lambda) = e^{i\psi}\lambda + \zeta$.

Here and elsewhere we shall suppose that every element $\mathcal{F} \in \mathcal{H}_w^\ell(\mathcal{K}_C^\varphi; X)$ is extended to the boundary $\partial \mathcal{K}_C^\varphi$ by the non-tangential limits. In the case $\varphi = \pi$ we distinguish the banks $\lim_{\psi \to 0^+} (e^{i\psi}R + \zeta)$ and $\lim_{\psi \to \pi^-} (e^{i\psi}R + \zeta)$ in $\partial \mathcal{K}_C^\pi$.

(ii) The estimate

$$
\|\mathcal{F}; \mathcal{H}_w^\ell(e^{i\psi}R + \zeta; X)\| \leq C \left( \|\mathcal{F}; \mathcal{H}_w^\ell(\mathbb{R}; X)\| + \|\mathcal{F}; \mathcal{H}_w^\ell(e^{i\psi}R + \zeta; X)\| \right) \quad (2.6)
$$

is valid, where the constant $C$ is independent of $\psi \in [0, \varphi]$, $\mathcal{F}$, and $w \in \mathbb{C}$.

(iii) For every compact set $\mathcal{K} \subset \mathcal{K}_C^\varphi$ there is an independent of $\mathcal{F}$ and $w \in \mathbb{C}$ constant $c(\mathcal{K})$ such that for all $\lambda \in \mathcal{K}$ we have

$$
|\exp{iw\lambda}|\|\mathcal{F}(\lambda)\| \leq c(\mathcal{K})(\|\mathcal{F}; \mathcal{H}_w^\ell(\mathbb{R} + \zeta; X)\| + \|\mathcal{F}; \mathcal{H}_w^\ell(e^{i\psi}R + \zeta; X)\|). \quad (2.7)
$$

(iv) The value

$$
|\exp{iw\lambda}|(1 + |\lambda|)^\delta|\lambda - \zeta|^{1/2}\|\mathcal{F}(\lambda)\|
$$

uniformly tends to zero as $\lambda$ goes to infinity (or $\lambda$ goes to $\zeta$) in the cone

$$
\{ \lambda \in \mathbb{C} : \lambda = e^{i\psi}t + \zeta, t \in \mathbb{R}, \psi \in [\phi, \varphi - \phi] \}, \phi > 0.
$$
(v) For any \( \psi \in [0, \varphi] \) the set \( \{ \mathcal{F} \mid e^{i\psi R + \zeta}; \mathcal{F} \in \mathcal{H}_w^\ell(K_\varphi; X) \} \) is dense in the space \( \mathcal{H}_w^\ell(e^{i\psi R + \zeta}; X) \).

2. Let \( \zeta \in \mathbb{C}, \varphi \in (0, \pi) \), and let \( \mathcal{F} \) be an analytic in \( K_\varphi \) function with values in \( X \). Suppose that for some \( \ell \in \mathbb{R} \) and \( w \in \mathbb{C} \) the estimate

\[
|\exp\{i\omega \zeta\}|(1 + |\omega|)^\ell|\lambda - \zeta|^{|1/2||\mathcal{F}(\lambda)|| \leq \text{Const}, \quad \lambda \in K_\varphi \setminus \{ \zeta \},
\]

holds and the inclusions \( \mathcal{F} \in \mathcal{H}_w^\ell(\mathbb{R} + \zeta; X), \mathcal{F} \in \mathcal{H}_w^\ell(e^{i\psi R} + \zeta; X) \) are valid. Then \( \mathcal{F} \in \mathcal{H}_w^\ell(K_\varphi; X) \).

**PROOF.** Let us define an equivalent norm in the space \( \mathcal{H}_w^\ell(e^{i\psi R} + \zeta; X) \) by the equality

\[
\|\mathcal{F}; \mathcal{H}_w^\ell(e^{i\psi R} + \zeta; X)\|^2 = \int_{e^{i\psi R} + \zeta} |\exp\{2i\omega \zeta\}|\|\lambda - \zeta - i|2^\ell\|\mathcal{F}(\lambda)\|^2 |d\lambda| + \int_{e^{i\psi R} + \zeta} |\exp\{2i\omega \zeta\}|\|\lambda - \zeta + i|2^\ell\|\mathcal{F}(\lambda)\|^2 |d\lambda|;
\]  

(2.8)

here \( e^{i\psi R} \pm \zeta = \{ \lambda \in \mathbb{C} : \lambda = \xi e^{i\psi} + \zeta, \xi \geq 0 \} \) and \( \varphi \in (0, \pi] \). The norm (2.8) is equivalent to the norm (2.1) because of the inequalities

\[
(1 + |\zeta - i|^2)^{-1}(1 + |\lambda|^2) \leq 3|\lambda - \zeta + i|^2 \leq 6(1 + |\zeta - i|^2)(1 + |\lambda|^2), \quad \Re(\lambda - \zeta) \geq 0,
\]

(2.9)

\[
(1 + |\zeta + i|^2)^{-1}(1 + |\lambda|^2) \leq 3|\lambda - \zeta - i|^2 \leq 6(1 + |\zeta + i|^2)(1 + |\lambda|^2), \quad \Re(\lambda - \zeta) \leq 0.
\]

Let \( \Pi \) be the same strip as in Proposition 2.2. For \( s \in \Pi \) we set

\[
\Psi(s) = \exp\{i\omega (\zeta - e^s) + s/2\}( -i - e^s)^\ell \mathcal{F}(\zeta - e^s), \\
\Phi(s) = \exp\{i\omega (\zeta + e^s) + s/2\}(i + e^s)^\ell \mathcal{F}(\zeta + e^s);
\]

(2.10)

here we use analytic in \( \Pi \) branches of the functions \((-i - e^s)^\ell\) and \((i + e^s)^\ell\).

The functions \( \Phi \) and \( \Psi \) are analytic in \( \Pi \). For all \( \psi \in (0, \varphi) \) we have

\[
\int_{-\infty}^{\infty} \|\Psi(r + i\psi)\|^2 dr + \int_{-\infty}^{\infty} \|\Phi(r + i\psi)\|^2 dr = \|\mathcal{F}; \mathcal{H}_w^\ell(e^{i\psi R} + \zeta; X)\|^2 \leq \text{Const}(\mathcal{F}).
\]

(2.11)

It remains to apply the items 1, 2 of Proposition 2.2 to the functions \( \Phi, \Psi \), and reformulate the results in terms of \( \mathcal{F} \). \( \square \)

From Proposition 2.3 it follows that we can take the right hand side of (2.6) as the constant in the inequality (2.3).
Proposition 2.4 The class $\mathcal{H}_w^K(\mathcal{K}_\xi^p; X)$ endowed with the norm

$$
\| F; \mathcal{H}_w^K(\mathcal{K}_\xi^p; X) \| = \| F; \mathcal{W}_w^K(\mathbb{R} + \xi; X) \| + \| F; \mathcal{W}_w^K(e^{i\psi} \mathbb{R} + \xi; X) \| 
$$

(2.12)
is a Banach space.

PROOF. With the help of (2.10) we can split the class $\mathcal{H}_w^K(\mathcal{K}_\xi^p; X)$ into two Hardy classes $H(\Pi; X)$ in the strip $\Pi$ such that the equality (2.11) holds. Since the Hardy class $H(\Pi; X)$ is complete (see e.g. [33,34]), the class $\mathcal{H}_w^K(\mathcal{K}_\xi^p; X)$ is also complete. \hfill \Box

The following proposition contains some elementary properties of the introduced classes $\mathcal{H}_w^K(\mathcal{K}_\xi^p; X)$, it is given without proof.

Proposition 2.5 Let $w, \xi \in \mathbb{C}, \ell \in \mathbb{R}$, and let $\varphi \in (0, \pi]$.

(i) If $\phi \in (0, \varphi]$ and $s \leq \ell$ then the embedding $\mathcal{H}_w^K(\mathcal{K}_\xi^p; X) \subseteq \mathcal{H}_w^K(\mathcal{K}_\xi^p; X)$ is continuous.

(ii) Let $F \in \mathcal{H}_w^K(\mathcal{K}_\xi^p; X)$ and $\mathcal{K}_\xi^{p, \pm} = \{ \lambda \in \mathcal{K}_\xi^p : \Im \lambda \gneq \Im \xi \}$. We set $F^+ = F$ on $\lambda \in \mathcal{K}_\xi^{p,+}$ and $F^+ = 0$ on $\mathcal{K}_\xi^{p,-}$; let also $F^- = F - F^+$. The mappings

$$
\mathcal{H}_w^K(\mathcal{K}_\xi^p; X) \ni F \mapsto F^+ \in \mathcal{H}_w^K(\mathcal{K}_\xi^p; X),
$$

$$
\mathcal{H}_w^K(\mathcal{K}_\xi^p; X) \ni F \mapsto F^- \in \mathcal{H}_w^K(\mathcal{K}_\xi^p; X)
$$

are continuous if $\Im(e^{i\psi} u) \geq \Im(e^{i\psi} w)$ and $\Im(e^{i\psi} v) \leq \Im(e^{i\psi} w)$ for all $\psi \in [0, \varphi]$.

(iii) Suppose that an analytic function $\overline{\mathcal{K}_\xi^p} \ni \lambda \mapsto p(\lambda) \in \mathbb{C}$ satisfies the estimate $|p(\lambda)| \leq C(1 + |\lambda|)^{s}$ for all $\lambda \in \overline{\mathcal{K}_\xi^p}$ and some $s \in \mathbb{R}$. Then the norm of the multiplication operator

$$
\mathcal{H}_w^K(\mathcal{K}_\xi^p; X) \ni F \mapsto pF \in \mathcal{H}_w^{\ell-s}(\mathcal{K}_\xi^p; X)
$$

is bounded uniformly in $w \in \mathbb{C}$.

(iv) The following mapping (2.13) is an isomorphism

$$
\mathcal{H}_w^K(\mathcal{K}_\xi^p; X) \ni F \mapsto (\cdot - \xi + i)^s F^+ (\cdot) + (\cdot - \xi - i)^s F^- (\cdot) \in \mathcal{H}_w^{\ell-s}(\mathcal{K}_\xi^p; X), \ s \in \mathbb{R};
$$

(2.13)

here $F^+$ and $F^-$ are the same as in (ii), we use an analytic in $\mathcal{K}_\xi^{p,+}$ branch of the function $(\cdot - \xi + i)^s$ and an analytic in $\mathcal{K}_\xi^{p,-}$ branch of $(\cdot - \xi - i)^s$. The norm of the mapping (2.13) and the norm of its inverse are uniformly bounded in $w \in \mathbb{C}$.  

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Proposition 2.6  (i) Let $\partial K_{\xi}^{\varphi, \pm}$ denote the boundary of $K_{\xi}^{\varphi, \pm} = \{ \lambda \in K_{\xi}^{\varphi} : \Im \lambda \geq \Im \zeta \}$, for a function $F \in \mathcal{H}_w^\ell(K_{\xi}^{\varphi}; X)$ the representations

$$
F(\lambda) = \int_{\partial K_{\xi}^{\varphi, +}} \frac{e^{iw(\mu-\lambda)}(\mu-\eta)^sF(\mu)}{2\pi i(\lambda-\eta)^s(\mu-\lambda)} d\mu, \quad \lambda \in K_{\xi}^{\varphi, +}, \eta \in K_{\xi}^{\varphi, -}, \ s \leq \ell, \ (2.14)
$$

$$
F(\lambda) = \int_{\partial K_{\xi}^{\varphi, -}} \frac{e^{iw(\mu-\lambda)}(\mu-\tau)^sF(\mu)}{2\pi i(\lambda-\tau)^s(\mu-\lambda)} d\mu, \quad \lambda \in K_{\xi}^{\varphi, -}, \tau \in K_{\xi}^{\varphi, +}, \ s \leq \ell, \ (2.15)
$$

are valid, where we use an analytic in $K_{\xi}^{\varphi, +}$ branch of the function $(\cdot - \eta)^s$ and an analytic in $K_{\xi}^{\varphi, -}$ branch of $(\cdot - \tau)^s$, the contours of integration are oriented such that $\lambda$ lies on the left side while $\mu$ travels along a contour. The integrals are absolutely convergent in the space $X$. The formulas (2.14), (2.15) recover a function $F \in \mathcal{H}_w^\ell(K_{\xi}^{\varphi}; X)$ from its boundary limits.

(ii) Let $F \in \mathcal{H}_w^\ell(K_{\xi}^{\varphi}; X)$. The value $\exp\{iw\lambda\}|\lambda|^{\ell}||F(\lambda)||$ uniformly tends to zero as $\lambda$ goes to infinity in the set $\{ \lambda \in K_{\xi}^{\varphi} : \text{dist}\{\lambda, \partial K_{\xi}^{\varphi}\} \geq \epsilon \}$, where $\epsilon > 0$.

The proof of Proposition 2.6 is displayed in Appendix, see page 66. The following proposition shows that in order to recover a function $F \in \mathcal{H}_w^\ell(K_{\xi}^{\varphi}; X)$ it is not necessary to know the boundary limits of $F$ on the whole boundary $\partial K_{\xi}^{\varphi}$.

Proposition 2.7 A function $F \in \mathcal{H}_w^\ell(K_{\xi}^{\varphi}; X)$ can be uniquely recovered from its boundary limits $F|_{\mathbb{R}+\zeta}$. Similarly, in order to recover $F$ it suffices to know the non-tangential boundary limits of $F$ on the part $e^{i\varphi}\mathbb{R} + \zeta$ of the boundary $\partial K_{\xi}^{\varphi}$ (or, equivalently, on the part $e^{i\varphi}\mathbb{R}^+ + \zeta \cup \mathbb{R}^- + \zeta \subset \partial K_{\xi}^{\varphi}$, or on the part $e^{i\varphi}\mathbb{R}^- + \zeta \cup \mathbb{R}^+ + \zeta \subset \partial K_{\xi}^{\varphi}$).

PROOF. Here again we split $F \in \mathcal{H}_w^\ell(K_{\xi}^{\varphi}; X)$ into the functions $\Psi$ and $\Phi$ in the Hardy class $H(\Pi; X)$, see (2.10) and (2.11). The non-tangential boundary limits of the functions $\Psi$, $\Phi$, and $F$ are related by the formulas (2.10). As is well-known (see e.g. [25] for the scalar case, and [33,34] for the case of $X$-valued functions), for all $\psi \in (0, \varphi)$ and $r \in \mathbb{R}$ we have

$$
\Phi(r + i\psi) = \mathcal{F}^{-1}_{t \rightarrow r}e^{t\psi}\mathcal{F}_{r \rightarrow t}^{-1}\Phi_0(r), \quad \Phi(r + i\psi) = \mathcal{F}^{-1}_{t \rightarrow r}e^{t(\psi - \varphi)}\mathcal{F}_{r \rightarrow t}\Phi_\varphi(r); \ (2.16)
$$

here $\mathcal{F} : L_2(\mathbb{R}; X) \rightarrow L_2(\mathbb{R}; X)$ is the Fourier transformation and $\mathcal{F}^{-1}$ is its inverse, as in Proposition 2.2 the boundary limits of $\Phi$ are denoted by $\Phi_0$ and $\Phi_\varphi$. Obviously, in the same way $\Psi$ can be recovered from $\Psi_0$ or $\Psi_\varphi$. Knowing $\Phi$ and $\Psi$ we get $F$ from the formulas (2.10).  \(\square\)
2.2 Weighted Hardy classes in a half-plane

Let $\mathbb{C}^+$ and $\mathbb{C}^-$ be the upper and the lower half-plane, $\mathbb{C}^\pm = \{ \lambda \in \mathbb{C} : \Im \lambda \geq 0 \}$. We shall use the notations

$$e^{i\varphi} \mathbb{R}^\pm + \zeta = \{ \lambda \in \mathbb{C} : \lambda = \xi e^{i\varphi} + \zeta, \pm \xi > 0 \},$$

$$e^{i\varphi} \mathbb{C}^\pm + \zeta = \{ \lambda \in \mathbb{C} : \lambda = e^{i\varphi} \mu + \zeta, \mu \in \mathbb{C}^\pm \},$$

where $\zeta \in \mathbb{C}$ and $\varphi$ is an angle. By $\mathcal{H}_w^\ell(e^{i\varphi} \mathbb{R}^\pm + \zeta; X)$ we denote by the Hilbert space of all (classes of) functions $e^{i\varphi} \mathbb{R}^\pm + \zeta \ni \lambda \mapsto F(\lambda) \in X$ with the finite norm

$$\| F; \mathcal{H}_w^\ell(e^{i\varphi} \mathbb{R}^\pm + \zeta; X) \|^2 = \int_{e^{i\varphi} \mathbb{R}^\pm + \zeta} \exp(2iw\lambda) |(1 + |\lambda|^2)^\ell F(\lambda)|^2 d\lambda.$$

**Definition 2.8** Let $\zeta, w \in \mathbb{C}$ and $\ell \in \mathbb{R}$. We introduce the Hardy class $\mathcal{H}_w^\ell(e^{i\varphi} \mathbb{C}^\pm + \zeta; X)$ as the set of all analytic functions $e^{i\varphi} \mathbb{C}^\pm + \zeta \ni \lambda \mapsto F(\lambda) \in X$ satisfying the uniform in $\psi \in (\varphi, \varphi + \pi)$ estimate

$$\| F; \mathcal{H}_w^\ell(e^{i\varphi} \mathbb{R}^\pm + \zeta; X) \| \leq C(F).$$

It is clear that $\mathcal{H}_w^\ell(e^{i\varphi} \mathbb{C}^\pm + \zeta; X) \equiv \mathcal{H}_w^\ell(e^{i(\varphi + \pi)} \mathbb{C}^- + \zeta; X)$. From Definition 2.8 and Definition 2.1 it is easily seen that a function $F$ is in the Hardy class $\mathcal{H}_w^\ell(\mathbb{K}_{\zeta}^\varphi; X)$ if and only if $F \in \mathcal{H}_w^\ell(\mathbb{C}^\pm + \zeta; X) \cap \mathcal{H}_w^\ell(\mathbb{C}^- + \zeta; X)$. Let us also note that if $F \in \mathcal{H}_w^\ell(\mathbb{C}^\pm + \zeta; X)$ then after extension of the function $F$ to the half-plane $\mathbb{C}^- + \zeta$ by zero we have the inclusion $F \in \mathcal{H}_w^\ell(\mathbb{K}_{\zeta}^\varphi; X)$ for all $\varphi \in (0, \pi)$. Moreover, a function $F$ is in the Hardy class $\mathcal{H}_w^\ell(\mathbb{C}^\pm + \zeta; X)$ if and only if the function $F$ extended to $\mathbb{C}^- + \zeta$ by zero is in the class $\mathcal{H}_w^\ell(\mathbb{K}_{\zeta}^\varphi; X)$. The next proposition contains some direct consequences of Proposition 2.3 and Proposition 2.6, it is presented without proof.

**Proposition 2.9** Let $\zeta, w \in \mathbb{C}$, $\ell \in \mathbb{R}$, and let $\varphi$ be an angle. For any function $F \in \mathcal{H}_w^\ell(e^{i\varphi} \mathbb{C}^\pm + \zeta; X)$ the following assertions are fulfilled.

(i) $F$ has boundary limit $F_\varphi \in \mathcal{H}_w^\ell(e^{i\varphi} \mathbb{R}^\pm + \zeta; X)$ in the sense that for almost all points $\mu$ of the boundary $e^{i\varphi} \mathbb{R}^\pm + \zeta$ we have $\| F(\lambda) - F_\varphi(\mu) \| \to 0$ as $\lambda$ tends to $\mu$ by a non-tangential to $e^{i\varphi} \mathbb{R}^\pm + \zeta$ path. Moreover

$$\|(e_w F) \circ \varphi_\psi, \zeta - (e_w F_\varphi) \circ \varphi_\psi, \zeta; \mathcal{H}_w^\ell(\mathbb{R}^\pm; X)\| \to 0, \ \psi \to \varphi^+, \$$

$$\|(e_w F) \circ \varphi_\psi, \zeta - (e_w F_\varphi) \circ \varphi_{\varphi + \pi}, \zeta; \mathcal{H}_w^\ell(\mathbb{R}^-; X)\| \to 0, \ \psi \to (\varphi + \pi)^-, \$$

where $e_w$ denotes the weight function $\lambda \mapsto \exp(iw\lambda)$ and $\varphi_\psi, \zeta : \mathbb{R} \to e^{i\psi} \mathbb{R} + \zeta$ is the linear transformation $\varphi_\psi, \zeta(\xi) = e^{i\psi} \xi + \zeta$.

Here and elsewhere we shall suppose that every element $F \in \mathcal{H}_w^\ell(e^{i\varphi} \mathbb{C}^\pm + \zeta; X)$ is extended to the boundary $e^{i\varphi} \mathbb{R} + \zeta$ by its non-tangential limits.
(ii) The estimate
\[ \|F; \mathcal{H}_w^\ell(e^{i\varphi} \mathbb{R}^+ + \zeta; X)\| \leq C\|F; \mathcal{H}_w^\ell(e^{i\varphi} \mathbb{R} + \zeta; X)\| \]
holds, where the constant \( C \) is independent of \( \psi \in [\varphi, \varphi + \pi] \), \( F \), and \( w \in \mathbb{C} \).

(iii) The value \( |\exp\{i\omega\lambda\}|\|\lambda\|^{\ell/2}\|F(\lambda)\| \) uniformly tends to zero as \( \lambda \) goes to infinity in the set \( \{ \lambda \in \mathbb{C} : \arg(\lambda - \zeta) \in [\varphi + \phi, \varphi + \pi - \phi] \} \), \( \phi > 0 \).

(iv) The value \( |\exp\{i\omega\lambda\}|\|\lambda\|\|F(\lambda)\| \) uniformly tends to zero as \( \lambda \) goes to infinity in the half-plane \( e^{i\varphi} \mathbb{C}^+ + \eta \), where \( \eta \in e^{i\varphi} \mathbb{C}^+ + \zeta \).

(v) The representation
\[ F(\lambda) = \int_{e^{i\varphi} \mathbb{R} + \zeta} \frac{e^{i\omega(\mu - \lambda)}(\mu - \eta)^s F'(\mu)}{2\pi i (\lambda - \eta)^s (\mu - \lambda)} d\mu, \quad \lambda \in e^{i\varphi} \mathbb{C}^+ + \zeta, \quad \eta \in e^{i\varphi} \mathbb{C}^- + \zeta, \quad s \leq \ell, \quad (2.17) \]
holds, where we use an analytic in \( e^{i\varphi} \mathbb{C}^+ + \zeta \) branch of the function \((\cdot - \eta)^s\), the contour of integration is oriented such that \( \lambda \) lies on the left side while \( \mu \) travels along the contour. The representation (2.17) recovers a function \( F \in \mathcal{H}_w^\ell(e^{i\varphi} \mathbb{C}^+ + \zeta; X) \) from its non-tangential boundary limits. The integral is absolutely convergent in \( X \).

As a consequence of Proposition 2.4 we get

**Proposition 2.10** The class \( \mathcal{H}_w^\ell(e^{i\varphi} \mathbb{C}^+ + \zeta; X) \) endowed with the norm
\[ \|F; \mathcal{H}_w^\ell(e^{i\varphi} \mathbb{C}^+ + \zeta; X)\| = \|F; \mathcal{H}_w^\ell(e^{i\varphi} \mathbb{R} + \zeta; X)\| \quad (2.18) \]
is a Banach space.

The next proposition in particular establishes the equivalence of Definition 2.8 and a universally accepted definition of the Hardy class in a half-plane; this fact is known for the Hardy class without weight [5].

**Proposition 2.11** (i) An analytic function \( e^{i\varphi} \mathbb{C}^+ + \zeta \ni \lambda \mapsto F(\lambda) \in X \) is in the Hardy class \( \mathcal{H}_w^\ell(e^{i\varphi} \mathbb{C}^+ + \zeta; X) \) if and only if it satisfies the uniform estimate
\[ \|F; \mathcal{H}_w^\ell(e^{i\varphi} \mathbb{R} + \eta; X)\| \leq C(F), \quad \eta \in e^{i\varphi} \mathbb{C}^+ + \zeta. \quad (2.19) \]

(ii) If \( F \in \mathcal{H}_w^\ell(e^{i\varphi} \mathbb{C}^+ + \zeta; X) \) then
\[ \|F(\cdot + \eta) - F(\cdot); \mathcal{H}_w^\ell(e^{i\varphi} \mathbb{R} + \zeta; X)\| \to 0, \quad \eta \to 0, \quad \eta \in e^{i\varphi} \mathbb{C}^+. \]

(iii) Let \( J \) be a function from the space \( \mathcal{H}_w^\ell(e^{i\varphi} \mathbb{R} + \zeta; X) \). In the right hand side of the equality (2.17) we replace \( F \) by \( J \). Then the equality (2.17) defines
a function \( F \in \mathcal{H}_w^k(e^{i\varphi}C^+ + \zeta; X) \), where \( k < s + 1/2 \) if \( \ell - s > 1/2 \), and \( k \leq s \) if \( \ell - s \leq 1/2 \). (Certainly this does not necessarily mean that the prescribed on \( e^{i\varphi}R + \zeta \) function \( J \) coincides with the boundary limit of the analytic in \( e^{i\varphi}C^+ + \zeta \) function \( F \).) The estimate
\[
\| F; \mathcal{H}_w^k(e^{i\varphi}C^+ + \zeta; X) \| \leq C \| J; \mathcal{W}_w^\ell(e^{i\varphi}R + \zeta; X) \| \tag{2.20}
\]
holds with an independent of \( w \in \mathbb{C} \) and \( J \in \mathcal{W}_w^\ell(e^{i\varphi}R + \zeta; X) \) constant \( C \).

**PROOF.** Without loss of generality we can suppose that the parameter \( \zeta, w, \ell, \) and the angle \( \varphi \) are equal to zero; if it is not the case then we identify \( F \in \mathcal{H}_w^\ell(e^{i\varphi}C^+ + \zeta; X) \) with \( G \in \mathcal{H}_0^0(C^+; X) \) by the rule
\[
G(\lambda) = \exp\{iwe^{i\varphi}\lambda\}(\lambda + i)^\ell F(e^{i\varphi}\lambda + \zeta), \quad \lambda \in \mathbb{C}^+ \tag{2.21}
\]
where we use an analytic in \( \mathbb{C}^+ \) branch of the function \((\cdot + i)^\ell\); cf. (2.9).

As it was already mentioned, the assertion (i) for the Hardy class \( \mathcal{H}_0^0(C^+; X) \) is proved in [5]. Since this result is not widely known we give an independent proof.

Due to Proposition 2.9, (v) we can recover a function \( F \in \mathcal{H}_0^0(C^+; X) \) from its non-tangential boundary limits by the Cauchy integral taken along the real axis. As is well-known [33,34], this immediately leads to the estimate
\[
\| F(\cdot + \eta); L_2(\mathbb{R}; X) \| \leq C(F), \quad \eta \in \mathbb{C}^+ \tag{2.22}
\]
that is the same as (2.19) in the case \( w = \zeta = 0, \ell = 0, \) and \( \varphi = 0 \). The necessity of the condition (2.19) is established.

Let \( F \in \mathcal{H}_0^0(C^+; X) \) and let \( \Pi \) denote the strip
\[
\Pi = \{ s \in \mathbb{C} : s = r + i\psi, r \in \mathbb{R}, 0 < \psi < \pi \}.
\]
We define the analytic function \( \Pi \ni s \mapsto \Phi(s) \in X \) by the formula \( \Phi(s) = F(e^s)e^{s/2} \). It is easy to see that
\[
\int_{-\infty}^{+\infty} \| \Phi(r + i\psi) \|^2 dr = \| F; \mathcal{W}_0^0(e^{i\varphi}R^+; X) \|^2 \leq Const, \quad \psi \in [0, \pi].
\]
From the assertion 3 of Proposition 2.2 we conclude that \( F \in \mathcal{H}_0^0(C^+; X) \) if and only if the function \( \Phi \) can be represented in the form (2.5) for some \( \Psi_0, \Psi_\varphi \in L_2(\mathbb{R}; X) \). We express the representation (2.5) in terms of \( F \) and get the equality
\[
\sqrt{\lambda} F(\lambda) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{G(\xi)}{(\log \xi - \log \lambda)^{1/2}} d\xi, \quad \lambda \in \mathbb{C}^+, \tag{2.23}
\]
with some $G \in L_2(\mathbb{R}; X)$; here we use analytic in $\mathbb{C}^+ \setminus \{0\}$ principal branches of logarithm and square root. Thus we can say that $F \in \mathcal{H}_0^0(\mathbb{C}^+; X)$ if and only if the equality (2.23) is valid for some $G \in L_2(\mathbb{R}; X)$. In order to prove the sufficiency of the condition (2.19), we shall show that for any analytic function $\mathbb{C}^+ \ni \lambda \mapsto F(\lambda) \in X$ satisfying the uniform estimate (2.22) there exists $G \in L_2(\mathbb{R}; X)$ such that the equality (2.23) is fulfilled. This part of the proof is arranged as Lemma 5.1 in Appendix.

Now we prove the assertion (ii). One of the universally accepted definitions of Hardy classes in $\mathbb{C}^+$ reads: the Hardy class $H(\mathbb{C}^+; X)$ is the set of all analytic functions $\mathbb{C}^+ \ni \lambda \mapsto F(\lambda) \in X$ satisfying the uniform estimate (2.22). Therefore the rule (2.21) allows us to identify the classes $H(\mathbb{C}^+; X)$ and $\mathcal{H}_{s}^0(e^{i\varphi}\mathbb{C}^+ + \zeta; X)$. If $w = \zeta = 0$, $\ell = 0$, and $\varphi = 0$ then the classes are coincident and the assertion (ii) is known. For nonzero parameters it is easily seen from (2.21).

Let us prove the assertion (iii). If the parameters $\ell$, $s$, $w$, are zeros then $F$ is the Cauchy integral of $J$, it is known that $F \in \mathcal{H}_0^0(e^{i\varphi}\mathbb{C}^+ + \zeta; X)$ and

$$\|F; \mathcal{H}_0^0(e^{i\varphi}\mathbb{C}^+ + \zeta; X)\| \leq c\|J; \mathcal{H}_0^0(e^{i\varphi}\mathbb{R} + \zeta; X)\|;$$

(2.24)

see e.g. [34]. If the parameters are not zeros then $\exp\{iw\}(-\eta)^sF(\cdot)$ is the Cauchy integral of $\exp\{iw\}(\cdot - \eta)^sJ(\cdot) \in \mathcal{H}_0^0(e^{i\varphi}\mathbb{R} + \zeta; X)$. We perform the corresponding changes of the functions in (2.24), after obvious estimates we arrive at the inequality

$$\|F; \mathcal{H}_u^s(e^{i\varphi}\mathbb{C}^+ + \zeta; X)\| \leq C\|J; \mathcal{H}_u^s(e^{i\varphi}\mathbb{R} + \zeta; X)\|.
$$

This immediately leads to the estimate (2.20), where $k \leq s \leq \ell$; see Proposition 2.5, (i). At this stage we can only guarantee the validity of the estimate (2.20) with $k \leq s$. This completely proves the assertion (iii) for the case $\ell - 1/2 \leq s \leq \ell$. In the case $s < \ell - 1/2$ the estimate (2.20) is to be improved so that $k \in (s, s + 1/2)$. We shall do this in Section 3.3, see Corollary 3.17 and Remark 3.18. Here we take this fact for granted. □

**Corollary 2.12** Let $\zeta, w \in \mathbb{C}$, $\ell \in \mathbb{R}$, and let $\varphi$ be an angle.

(i) The space $\mathcal{H}_u^s(e^{i\varphi}\mathbb{C}^+ + \eta; X)$ and its norm do not change while $\eta$ travels along the line $e^{i\varphi}\mathbb{R} + \zeta$.

(ii) For all $\eta \in e^{i\varphi}\mathbb{C}^+ + \zeta$ the embedding $\mathcal{H}_u^s(e^{i\varphi}\mathbb{C}^+ + \eta; X) \subset \mathcal{H}_u^s(e^{i\varphi}\mathbb{C}^+ + \zeta; X)$ and the estimate

$$\|F; \mathcal{H}_u^s(e^{i\varphi}\mathbb{C}^+ + \eta; X)\| \leq C\|F; \mathcal{H}_u^s(e^{i\varphi}\mathbb{C}^+ + \zeta; X)\|
$$

are valid. Here the constant $C$ is independent of $w$, $\eta$, and $F \in \mathcal{H}_u^s(e^{i\varphi}\mathbb{C}^+ + \zeta; X)$. 

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(iii) Every function $F \in \mathcal{H}_w^\ell(e^{i\varphi} \mathbb{R} + \zeta; X)$ can be represented as the sum $(F^+ + F^-)|_{e^{i\varphi} \mathbb{R} + \zeta}$ of the boundary limits of functions $F^\pm \in \mathcal{H}_w^s(e^{i\varphi} \mathbb{C}^\pm + \zeta; X)$, where

$$
s < 1/2 \quad \text{if} \quad \ell > 1/2; \quad s \leq [\ell] \quad \text{if} \quad \ell \leq 1/2 \quad \text{and} \quad [\ell] \leq \ell \leq [\ell] + 1/2; \quad (2.25)
s < [\ell] + 1/2 \quad \text{if} \quad \ell \leq 1/2 \quad \text{and} \quad [\ell] + 1/2 < \ell < [\ell].\nonumber
$$

In the case $s \geq -1/2$ the representation is unique. If $s < -1/2$ then the entire functions

$$
\mathbb{C} \ni \lambda \mapsto \lambda^j e^{-i\lambda w} f \in X, \quad j = 0, 1, \ldots, -[s + 3/2],
$$

are in the spaces $\mathcal{H}_w^s(e^{i\varphi} \mathbb{C}^\pm + \zeta; X)$ for any $f \in X$; if $F = (F^+ + F^-)|_{e^{i\varphi} \mathbb{R} + \zeta}$ with some functions $F^\pm \in \mathcal{H}_w^s(e^{i\varphi} \mathbb{C}^\pm + \zeta; X)$ then for coefficients $f_j \in X$ we have

$$
F^\pm(\lambda) = F(\lambda) \pm \sum_{j=0}^{-[s+3/2]} f_j \lambda^j e^{-i\lambda w}, \quad \lambda \in e^{i\varphi} \mathbb{C}^\pm + \zeta, \quad (2.26)
$$

**PROOF.** The assertions (i) and (ii) are readily apparent from the equalities (2.1), (2.18) together with Proposition 2.11, (i), (ii). Let us demonstrate (iii). Without loss of generality we can suppose that $\ell < 1$. We define $G \in \mathcal{H}_0^{\ell-[\ell]}(e^{i\varphi} \mathbb{R} + \zeta; X)$ by the formula

$$
G(\lambda) = \exp(iw\lambda)(\lambda - \eta)^{[\ell]} F(\lambda), \quad \lambda \in e^{i\varphi} \mathbb{R} + \zeta, \quad \eta \notin e^{i\varphi} \mathbb{R} + \zeta.
$$

As is well-known, any function $G \in \mathcal{H}_0^0(e^{i\varphi} \mathbb{R} + \zeta; X)$ can be uniquely represented in the form $G = (G^+ + G^-)|_{e^{i\varphi} \mathbb{R} + \zeta}$, where $G^\pm \in \mathcal{H}_0^0(e^{i\varphi} \mathbb{C}^\pm + \zeta; X)$, the functions $G^\pm$ are defined as the Cauchy integral of $G$, see e.g. [34]. In the case $\ell - [\ell] > 1/2$ the inclusion $G \in \mathcal{H}_0^{\ell-[\ell]}(e^{i\varphi} \mathbb{R} + \zeta; X)$ together with Proposition 2.11, (iii) allows us to see that $G^\pm \in \mathcal{H}_0^{k}(e^{i\varphi} \mathbb{C}^\pm + \zeta; X)$ for any $k < 1/2$. It remains to set

$$
F^\pm(\lambda) = \exp(-iw\lambda)(\lambda - \eta)^{-[\ell]} G^\pm(\lambda), \quad \lambda \in e^{i\varphi} \mathbb{C}^\pm + \zeta.
$$

Then the inclusions $F^\pm \in \mathcal{H}_w^s(e^{i\varphi} \mathbb{C}^\pm + \zeta; X)$ hold, where $s$ is the same as in (2.25). It is easy to see the validity of the remark concerning the uniqueness of the representation $F = (F^+ + F^-)|_{e^{i\varphi} \mathbb{R} + \zeta}$, we do not cite the proof here. \[\square\]

For $s \in \mathbb{R}$ and $v \in \mathbb{C}$ we introduce the operator $\mathcal{P}_{\eta,v}^s(\phi, \zeta)$ acting by the rule

$$
\left(\mathcal{P}_{\eta,v}^s G\right)(\lambda) = \int_{e^{i\varphi} \mathbb{R} + \zeta} \frac{e^{iv(\mu-\lambda)}(\mu - \eta)^s G(\mu)}{2\pi i(\lambda - \eta)^{s}(\mu - \lambda)} d\mu, \quad \lambda \in e^{i\varphi} \mathbb{C}^\pm + \zeta, \quad \eta \in e^{i\varphi} \mathbb{C}^\pm + \zeta, \quad (2.27)
$$
where we use an analytic in $e^{i\phi}\mathbb{C}^+ + \zeta$ branch of the function $(\cdot - \eta)^s$; here and elsewhere we shall omit the parameters $\phi$ and $\zeta$ in the notations of the operators $\mathcal{P}_{\eta,v}^s(\phi, \zeta)$ when it can be done without ambiguity.

Let $s \leq \ell$, $k = s$ if $\ell - s \leq 1/2$, and $k \in [s, s + 1/2]$ if $\ell - s > 1/2$. From the equality (2.2) and Proposition 2.11, (iii) we see that the operator

$$
\mathcal{P}_{\eta,v}^s : \mathcal{W}_w^{\ell}(e^{i\phi}\mathbb{R} + \zeta; X) \to \mathcal{H}_v^k(e^{i\phi}\mathbb{C}^- + \zeta; X), \quad v \in e^{-i\phi}\mathbb{R} + w,
$$

is continuous; moreover, the estimate

$$
\|\mathcal{P}_{\eta,v}^s G; \mathcal{H}_v^k(e^{i\phi}\mathbb{C}^- + \zeta; X)\| \leq Ce^\Im(\zeta(w - v))\|G; \mathcal{W}_w^{\ell}(e^{i\phi}\mathbb{R} + \zeta; X)\| \tag{2.29}
$$

holds with an independent of $w \in \mathbb{C}$ and $v \in e^{-i\phi}\mathbb{R} + w$ constant $C$. Proposition 2.9, (v) allows us to identify the elements of the space $\mathcal{H}_v^k(e^{i\phi}\mathbb{C}^- + \zeta; X)$ with their boundary limits in $\mathcal{W}_w^{\ell}(e^{i\phi}\mathbb{R} + \zeta; X)$. Then $\mathcal{P}_{\eta,v}^s \mathcal{F} = \mathcal{F}$ for any $\mathcal{F} \in \mathcal{H}_v^k(e^{i\phi}\mathbb{C}^- + \zeta; X)$ and $r \leq k$; cf. (2.27), (2.17). Therefore (2.28) is a projection operator with the property

$$
\mathcal{P}_{\eta,w}^r \mathcal{P}_{\eta,v}^s = \mathcal{P}_{\eta,v}^s, \quad r \leq s, \quad v \in e^{-i\phi}\mathbb{R}^- + w. \tag{2.30}
$$

It is easy to see that in the case of an integer nonpositive $s$ the operator

$$
(I - \mathcal{P}_{\eta,v}^s) : \mathcal{W}_w^{\ell}(e^{i\phi}\mathbb{R} + \zeta; X) \to \mathcal{H}_v^k(e^{i\phi}\mathbb{C}^+ + \zeta; X) \tag{2.31}
$$

is also continuous (see the proof of Corollary 2.12); here $I$ denotes the operator of the embedding $\mathcal{W}_w^{\ell}(e^{i\phi}\mathbb{R} + \zeta; X) \subseteq \mathcal{H}_v^k(e^{i\phi}\mathbb{R} + \zeta; X)$, we assume that the elements of the spaces $\mathcal{H}_v^k(e^{i\phi}\mathbb{C}^\pm + \zeta; X)$ are identified with their boundary limits in $\mathcal{W}_w^{\ell}(e^{i\phi}\mathbb{R} + \zeta; X)$.

In Subsection 3.3 we shall prove that the Fourier-Laplace transformation yields an isomorphism between $\mathcal{H}_v^k(e^{i\phi}\mathbb{C}^- + \zeta; X)$ and a Sobolev space of $X$-valued distributions supported on the half-line $e^{-i\phi}\mathbb{R}^+ + v$ (see Paley-Wiener Theorem 3.14), then we shall use the operator (2.28) to define a projection operator that maps distributions defined on the line $e^{-i\phi}\mathbb{R} + v$ to distributions supported on $e^{-i\phi}\mathbb{R}^+ + v$.

### 2.3 Further properties of Hardy classes in cones

In the next proposition we extend the result of Proposition 2.11, (iii) to the Hardy classes in cones.

**Proposition 2.13** Let $\mathcal{G}$ be a function defined on the boundary $\partial \mathcal{K}_\zeta^p = (\mathbb{R} + \zeta) \cup (e^{i\phi}\mathbb{R} + \zeta)$ of a cone $\mathcal{K}_\zeta^p$, and let $\mathcal{G} \in \mathcal{W}_w^{\ell}(\mathbb{R} + \zeta; X)$, $\mathcal{G} \in \mathcal{W}_w^{\ell}(e^{i\phi}\mathbb{R} + \zeta; X)$. In the right hand sides of the equalities (2.14) and (2.15) we replace $\mathcal{F}$ by
The equalities (2.14) and (2.15) define a function \( \mathcal{F} \in \mathcal{H}_w^k(\mathcal{K}_\zeta^\varphi; X) \), where \( k < s + 1/2 \) if \( \ell - s > 1/2 \), and \( k \leq s \) if \( \ell - s \leq 1/2 \). (This does not necessarily mean that the prescribed on the boundary \( \partial \mathcal{K}_\zeta^\varphi \) function \( \mathcal{G} \) coincides with boundary limits of the analytic in \( \mathcal{K}_\zeta^\varphi \) function \( \mathcal{F} \).) The estimate

\[
\| \mathcal{F}; \mathcal{H}_w^k(\mathcal{K}_\zeta^\varphi; X) \| \leq C \left( \| \mathcal{G}; \mathcal{W}_w^\ell(\mathbb{R} + \zeta; X) \| + \| \mathcal{G}; \mathcal{W}_w^\ell(e^{i\psi}\mathbb{R} + \zeta; X) \| \right) \tag{2.32}
\]

holds, where the constant \( C \) is independent of \( \mathcal{G} \) and \( w \in \mathbb{C} \).

**Proof.** We rewrite the representation (2.14) in the form

\[
\mathcal{F}(\lambda) = \int_{\mathbb{R}^+ + \zeta} \frac{e^{iw(\mu - \lambda)}(\mu - \eta)^s G_1(\mu)}{2\pi i(\lambda - \eta)^s(\mu - \lambda)} \, d\mu + \int_{e^{i\psi}\mathbb{R} + \zeta} \frac{e^{iw(\mu - \lambda)}(\mu - \eta)^s G_2(\mu)}{2\pi i(\lambda - \eta)^s(\mu - \lambda)} \, d\mu, \tag{2.33}
\]

where \( G_1(\mu) = G(\mu) \) for \( \mu \in \mathbb{R}^+ + \zeta \) and \( G_1(\mu) = 0 \) for \( \mu \in \mathbb{R}^- + \zeta \), \( G_2(\mu) = G(\mu) \) for \( \mu \in e^{i\psi}\mathbb{R}^+ + \zeta \) and \( G_2(\mu) = 0 \) for \( \mu \in e^{i\psi}\mathbb{R}^- + \zeta \). It is clear that \( G_1 \in \mathcal{W}_w^\ell(\mathbb{R} + \zeta; X) \) and \( G_2 \in \mathcal{W}_w^\ell(e^{i\psi}\mathbb{R} + \zeta; X) \). By Proposition 2.11, (iii) the first integral in (2.33) defines a function \( \mathcal{F}_1 \in \mathcal{H}_w^k(\mathbb{C}^+ + \zeta; X) \), the second integral defines a function \( \mathcal{F}_2 \in \mathcal{H}_w^k(e^{i\psi}\mathbb{C}^- + \zeta; X) \). The sum \( \mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 \) is analytic in \( \mathcal{K}_\zeta^\varphi \). We have the uniform in \( \psi \in (0, \varphi) \) estimates

\[
\| \mathcal{F}; \mathcal{W}_w^k(e^{i\psi}\mathbb{R}^+ + \zeta; X) \| \leq \| \mathcal{F}_1; \mathcal{W}_w^k(e^{i\psi}\mathbb{R}^+ + \zeta; X) \| + \| \mathcal{F}_2; \mathcal{W}_w^k(e^{i\psi}\mathbb{R}^- + \zeta; X) \| \leq C_0 \left( \| \mathcal{F}_1; \mathcal{H}_w^k(\mathbb{C}^+ + \zeta; X) \| + \| \mathcal{F}_2; \mathcal{H}_w^k(e^{i\psi}\mathbb{C}^- + \zeta; X) \| \right) \leq C_1 \left( \| \mathcal{G}_1; \mathcal{W}_w^\ell(\mathbb{R} + \zeta; X) \| + \| \mathcal{G}_2; \mathcal{W}_w^\ell(e^{i\psi}\mathbb{R} + \zeta; X) \| \right) \leq C_1 \left( \| \mathcal{G}; \mathcal{W}_w^\ell(\mathbb{R} + \zeta; X) \| + \| \mathcal{G}; \mathcal{W}_w^\ell(e^{i\psi}\mathbb{R} + \zeta; X) \| \right);
\]

see Proposition 2.9, (ii), the definition (2.18) of the norm in \( \mathcal{H}_w^\ell(e^{i\psi}\mathbb{C}^+ + \zeta; X) \), and Proposition 2.11, (iii). We apply a similar argument to the representation (2.15) and conclude that it defines an analytic in \( \mathcal{K}_\zeta^{\varphi^-} \) function \( \mathcal{F} \), the norm \( \| \mathcal{F}; \mathcal{W}_w^k(e^{i\psi}\mathbb{R}^- + \zeta; X) \| \) is bounded by the right hand side of the estimate (2.32) uniformly in \( \psi \in (0, \varphi) \). Hence \( \mathcal{F} \in \mathcal{H}_w^k(\mathcal{K}_\zeta^\varphi; X) \) and the estimate (2.32) holds. \( \square \)

**Proposition 2.14** Let \( \zeta, w \in \mathbb{C}, \ell \in \mathbb{R} \), and let \( \varphi \in (0, \pi] \). If \( \mathcal{F} \in \mathcal{H}_w^\ell(\mathcal{K}_\zeta^\varphi; X) \) then

\[
\| \mathcal{F}; \mathcal{W}_w^\ell(e^{i\psi}\mathbb{R}^- + \mu; X) \| \leq \| \mathcal{F}; \mathcal{H}_w^k(\mathcal{K}_\zeta^\varphi; X) \|, \quad \mu \in \mathcal{K}_\zeta^{\varphi^-}, \psi \in [0, \varphi], \tag{2.34}
\]

\[
\| \mathcal{F}; \mathcal{W}_w^\ell(e^{i\psi}\mathbb{R}^+ + \mu; X) \| \leq \| \mathcal{F}; \mathcal{H}_w^k(\mathcal{K}_\zeta^\varphi; X) \|, \quad \mu \in \mathcal{K}_\zeta^{\varphi^+}, \psi \in [0, \varphi], \tag{2.35}
\]

where \( \mathcal{K}_\zeta^{\varphi^\pm} = \{ \lambda \in \mathcal{K}_\zeta^\varphi : \exists \lambda \geq \exists \zeta \} \), the constant \( C \) is independent of \( w, \eta, \) and \( \mathcal{F} \).
PROOF. Let us prove the estimate (2.35). The estimate (2.34) can be proved in a similar way. We use the representation (2.14) with $s = \ell$ and the argument from the proof of Proposition 2.13, where we take the boundary limits of $\mathcal{F} \in \mathcal{H}_w^\ell(K_\zeta^\varphi; X)$ as $G$. This provides us with the representation $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ in $K_\zeta^\varphi$, where the functions $\mathcal{F}_1 \in \mathcal{H}_w^\ell(C^+ + \zeta; X)$ and $\mathcal{F}_2 \in \mathcal{H}_w^\ell(e^{i\varphi}C^+ + \zeta; X)$ satisfy the estimate

$$\|\mathcal{F}_1; \mathcal{H}_w^\ell(C^+ + \zeta; X)\| + \|\mathcal{F}_2; \mathcal{H}_w^\ell(e^{i\varphi}C^+ + \zeta; X)\| \leq C\|\mathcal{F}; \mathcal{H}_w^\ell(K_\zeta^\varphi; X)\|; \quad (2.36)$$

here $C$ is independent of $\mathcal{F}$ and $w$. Let $\mu \in K_\zeta^\varphi$. For all $\psi \in [0, \varphi]$ we have

$$\|\mathcal{F}_1; \mathcal{W}_w^\ell(e^{i\varphi}R^+ + \mu; X)\| \leq C_1\|\mathcal{F}_1; \mathcal{H}_w^\ell(C^+ + \mu; X)\| \leq C_2\|\mathcal{F}_1; \mathcal{H}_w^\ell(C^+ + \zeta; X)\|,$$

where $C_1$ and $C_2$ are independent of $\mu$, $\psi$, and $\mathcal{F}_1$; see Proposition 2.9, (ii) and Corollary 2.12, (i, ii). Similar estimates are valid for $\mathcal{F}_2$. This together with (2.36) finishes the proof. \(\square\)

Corollary 2.15 Let $\zeta, w \in \mathbb{C}$, $\ell \in \mathbb{R}$, and let $\varphi \in (0, \pi]$. The value

$$\sup_{\psi \in (0, \varphi)} \|\mathcal{F}; \mathcal{W}_w^\ell(e^{i\psi}R^- + \eta; X)\| + \sup_{\eta \in K_\zeta^\varphi} \|\mathcal{F}; \mathcal{W}_w^\ell(e^{i\psi}R^+ + \eta; X)\|$$

can be taken as an equivalent norm $\|\mathcal{F}; \mathcal{H}_w^\ell(K_\zeta^\varphi; X)\|$ in the Banach space $\mathcal{H}_w^\ell(K_\zeta^\varphi; X)$. Moreover, the inequalities

$$\|\mathcal{F}; \mathcal{H}_w^\ell(K_\zeta^\varphi; X)\| \leq c\|\mathcal{F}; \mathcal{H}_w^\ell(K_\zeta^\varphi; X)\| \leq 2c\|\mathcal{F}; \mathcal{H}_w^\ell(K_\zeta^\varphi; X)\|$$

hold with an independent of $w$ constant $c$.

PROOF. The inequality $\|\mathcal{F}; \mathcal{H}_w^\ell(K_\zeta^\varphi; X)\| \leq c\|\mathcal{F}; \mathcal{H}_w^\ell(K_\zeta^\varphi; X)\|$ readily apparent from Proposition 2.14. It is easy to see that the norm $\|\mathcal{F}; \mathcal{H}_w^\ell(K_\zeta^\varphi; X)\|$ does not exceed the value $\|\mathcal{F}; \mathcal{H}_w^\ell(K_\zeta^\varphi; X)\|$. Indeed, the value $\|\mathcal{F}; \mathcal{W}_w^\ell(e^{i\psi}R^- + \eta; X)\|$ comes arbitrarily close to $\|\mathcal{F}; \mathcal{W}_w^\ell(e^{i\varphi}C^- + \gamma; X)\|$ (or to $\|\mathcal{F}; \mathcal{W}_w^\ell(e^{i\varphi}C^+ + \zeta; X)\|$) as $\eta$ tends to $\zeta$ in $K_\zeta^\varphi$ and $\psi \to \varphi-$ (or $\psi \to 0+$). Similarly, the value $\|\mathcal{F}; \mathcal{W}_w^\ell(e^{i\psi}R^+ + \eta; X)\|$ comes arbitrarily close to $\|\mathcal{F}; \mathcal{W}_w^\ell(e^{i\psi}R^+ + \zeta; X)\|$ (or to $\|\mathcal{F}; \mathcal{W}_w^\ell(e^{i\psi}R^- + \zeta; X)\|$) as $\eta$ tends to $\zeta$ in $K_\zeta^\varphi$ and $\psi \to \varphi-$ (or $\psi \to 0+$); cf. Proposition 2.3, 1.(i). Due to the definition (2.12) of the norm in $\mathcal{H}_w^\ell(K_\zeta^\varphi; X)$ this completes the proof. \(\square\)

Corollary 2.16 Let $\zeta, w \in \mathbb{C}$, $\ell \in \mathbb{R}$, and $\varphi \in (0, \pi]$. Assume that $\mathcal{F} \in \mathcal{H}_w^\ell(K_\zeta^\varphi; X)$.

(i) Assume in addition that $\mathcal{F}$ is analytic in a neighbourhood $O_\zeta$ of the vertex $\zeta$ of the cone $K_\zeta^\varphi$. Then for any cone $K_{\eta}^\varphi$, $K_{\eta}^\varphi \subset O_\zeta \cup K_\zeta^\varphi$, we have $\mathcal{F} \in \mathcal{H}_w^\ell(K_{\eta}^\varphi; X)$. 

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(ii) Let $K^\phi_w$ be a cone such that $K^\phi_{\eta}^{\phi^+} \subseteq K^\phi_{\zeta}^{\phi^+}$ (or $K^\phi_{\eta}^{\phi^-} \subseteq K^\phi_{\zeta}^{\phi^-}$). We set $G(\lambda) = F(\lambda)$ for $\lambda \in K^\phi_{\eta}^{\phi^+}$ and $G(\lambda) = 0$ for $\lambda \in K^\phi_{\eta}^{\phi^-}$ (in the case $K^\phi_{\eta}^{\phi^-} \subseteq K^\phi_{\zeta}^{\phi^-}$ we set $G(\lambda) = F(\lambda)$ for $\lambda \in K^\phi_{\eta}^{\phi^-}$ and $G(\lambda) = 0$ for $\lambda \in K^\phi_{\eta}^{\phi^+}$). Then $G \in \mathcal{H}_w^{\ell}(K^\phi_{\eta}; X)$ and the inequality

$$\|G; \mathcal{H}_w^{\ell}(K^\phi_{\eta}; X)\| \leq C\|F; \mathcal{H}_w^{\ell}(K^\phi_{\zeta}; X)\| \quad (2.37)$$

holds, where the constant $C$ is independent of $w$, $F$, and $K^\phi_{\eta}$.

**PROOF.** (i) Let $\mathcal{U}$ be an open neighbourhood of the set $K^\phi_{\eta} \setminus K^\phi_{\zeta}$ and let $\overline{\mathcal{U}} \subseteq \partial \zeta$. We denote the characteristic function of $\mathcal{U}$ by $\chi$ (i.e. $\chi(\lambda) = 1$ if $\lambda \in \mathcal{U}$ and $\chi(\lambda) = 0$ if $\lambda \notin \mathcal{U}$). Since $F$ is analytic in $\partial \zeta$ the value $\|\chi F; \mathcal{H}_w^{\ell}(e^{i\psi} \mathbb{R} + \eta; X)\|$ is bounded uniformly in $\psi \in [0, \phi]$. The estimate

$$\|(1 - \chi) F; \mathcal{H}_w^{\ell}(e^{i\psi} \mathbb{R} + \eta; X)\| \leq C\|F; \mathcal{H}_w^{\ell}(K^\phi_{\zeta}; X)\|, \psi \in [0, \phi],$$

follows from Proposition 2.14. Thus $F \in \mathcal{H}_w^{\ell}(K^\phi_{\eta}; X)$ by Definition 2.1.

(ii) It is clear that $G$ is analytic in $K^\phi_{\eta}$. The estimate (2.37) follows from Proposition 2.14. Hence $G \in \mathcal{H}_w^{\ell}(K^\phi_{\eta}; X)$. $\square$

**Proposition 2.17** Let $\zeta, w \in \mathbb{C}$, $s \in \mathbb{R}$, and $0 < \varphi < \pi$. Assume that $v \in e^{i\varphi} \mathbb{R} + w$, $\phi \in [0, \varphi]$, and $G = \mathcal{P}_{\eta,v}^s(F |_{e^{i\varphi} \mathbb{R} + \zeta})$, where $\mathcal{P}_{\eta,v}^s$ is the projection operator (2.27), (2.28), and $F$ is a function in the space $\mathcal{H}_w^{\ell}(K^\phi_{\zeta}; X)$, $\ell \geq s$. We also suppose that the parameter $\eta$ in the definition (2.27) of the operator $\mathcal{P}_{\eta,v}^s$ does not belong to the union $e^{i\varphi} \mathbb{C}^- + \zeta \cup K^\phi_{\zeta}$. Then the function $G \in \mathcal{H}_w^{k}(e^{i\varphi} \mathbb{C}^- + \zeta; X)$ can be analytically extended to a function $G \in \mathcal{H}_w^{k}(K^\phi_{\zeta}; X)$; here $k = s$ if $\ell - s \leq 1/2$, and $k \in [s, s + 1/2)$ if $\ell - s > 1/2$. The estimate

$$\|G; \mathcal{H}_w^{k}(K^\phi_{\zeta}; X)\| \leq C e^{3\left(w-v\right)} \|F; \mathcal{H}_w^{\ell}(K^\phi_{\zeta}; X)\| \quad (2.38)$$

holds, where the constant $C$ is independent of $w$, $\phi$, and $v$.

**PROOF.** We introduce the function $\tilde{F}(\lambda) = (\lambda - \eta)^s F(\lambda)$, $\lambda \in K^\phi_{\zeta}$, where we use an analytic in $(e^{i\varphi} \mathbb{C}^- + \zeta) \cup K^\phi_{\zeta}$ branch of the function $(\cdot - \eta)^s$. The estimate

$$\|\tilde{F}; \mathcal{H}_w^{\ell-s}(K^\phi_{\zeta}; X)\| \leq C\|F; \mathcal{H}_w^{\ell}(K^\phi_{\zeta}; X)\| \quad (2.39)$$

holds. By Corollary 2.12,(iii) the function $\tilde{F} |_{e^{i\varphi} \mathbb{R} + \zeta} \in \mathcal{H}_w^{\ell-s}(e^{i\varphi} \mathbb{R} + \zeta; X)$ is uniquely representable as the sum $(F^+ + F^-)|_{e^{i\varphi} \mathbb{R} + \zeta}$ of the boundary limits of some functions $F^\pm \in \mathcal{H}_w^{k-s}(e^{i\varphi} \mathbb{C}^\pm + \zeta; X)$. The functions $F^\pm$ have analytic continuations to the cone $K^\phi_{\zeta}$ because $\tilde{F} = (F^+ + F^-)$ on the line $e^{i\varphi} \mathbb{R} + \zeta$, $\tilde{F}$ is analytic in $K^\phi_{\zeta}$, and $F^\pm$ are analytic in $e^{i\varphi} \mathbb{C}^\pm + \zeta$. It is clear that $F^- = \mathcal{P}_{\eta,v}^0(\tilde{F} |_{e^{i\varphi} \mathbb{R} + \zeta})$ and $G(\lambda) = (\lambda - \eta)^{-s} F^-(\lambda)$ for all $\lambda \in e^{i\varphi} \mathbb{C}^- + \zeta$. Thus $G$
has an analytic continuation to the cone $K^\varphi$. We intend to demonstrate the inclusion $F^{-} \in \mathcal{H}^{k-s}(K^\varphi; X)$ and the estimate

$$
\| F^{-}; \mathcal{H}^{k-s}(K^\varphi; X) \| \leq C e^{3(\zeta(w-v))} \| F; \mathcal{H}^\ell(K^\varphi; X) \|. 
$$

Taking into account the inequality (2.29) together with the estimate (2.6) and the norm (2.12) in $\mathcal{H}^\ell(K^\varphi; X)$ we get

$$
\| F^{-}; \mathcal{H}^{k-s}(e^{i\psi}R^+ + \zeta; X) \| \leq C e^{3(\zeta(w-v))} \| F; \mathcal{H}^\ell(K^\varphi; X) \|, \quad \psi \in [0, \phi],
$$

$$
\| F^{-}; \mathcal{H}^{k-s}(e^{i\psi}R^- + \zeta; X) \| \leq C e^{3(\zeta(w-v))} \| F; \mathcal{H}^\ell(K^\varphi; X) \|, \quad \psi \in [\phi, \varphi],
$$

where the constant $C$ is independent of $\psi$, $w$, and $v \in e^{i\phi}R + w$. Then we have

$$
\| F; \mathcal{H}^{k-s}(e^{i\psi}R^- + \zeta; X) \| \leq e^{3(\zeta(w-v))} \| F; \mathcal{H}^{k-s}(e^{i\psi}R^- + \zeta; X) \|
$$

$$
\| F; \mathcal{H}^{k-s}(e^{i\psi}R^+ + \zeta; X) \| \leq e^{3(\zeta(w-v))} \| F; \mathcal{H}^{k-s}(e^{i\psi}R^+ + \zeta; X) \|
$$

(2.42)

By analogy with (2.41) we derive the estimates

$$
\| F^{+}; \mathcal{H}^{k-s}(e^{i\psi}R^- + \zeta; X) \| \leq C e^{3(\zeta(w-v))} \| F; \mathcal{H}^{k-s}(K^\varphi; X) \|, \quad \psi \in [0, \phi],
$$

$$
\| F^{+}; \mathcal{H}^{k-s}(e^{i\psi}R^+ + \zeta; X) \| \leq C e^{3(\zeta(w-v))} \| F; \mathcal{H}^{k-s}(K^\varphi; X) \|, \quad \psi \in [\phi, \varphi].
$$

(2.43)

The equality $F^{-}(\lambda) = \tilde{F}(\lambda) - F^{+}(\lambda), \lambda \in K^\varphi$, and the estimates (2.41)—(2.43) establish the uniform in $\psi \in [0, \varphi]$ estimate

$$
\| F^{-}; \mathcal{H}^{k-s}(e^{i\psi}R^+ + \zeta; X) \| \leq C e^{3(\zeta(w-v))} \| F; \mathcal{H}^{k-s}(K^\varphi; X) \|.
$$

This together with (2.39) leads to (2.40). Since $G(\lambda) = (\lambda - \eta)^{-s}F^{-}(\lambda)$ we conclude that

$$
\| G; \mathcal{H}^{k}(K^\varphi; X) \| \leq C \| F^{-}; \mathcal{H}^{k-s}(K^\varphi; X) \|.
$$

Then the estimate (2.38) follows from (2.40). \qed

By analogy with $\mathcal{H}^\ell(K^\varphi; X)$ we introduce the Hardy class $\mathcal{H}^\ell(K^{-\varphi}; X)$ in the cone $K^{-\varphi} = \{ \lambda \in C : \lambda = e^{-i\varphi}\mu + \zeta, \mu \in K^\varphi \}$; here $\varphi \in (0, \pi]$. The class $\mathcal{H}^\ell(K^{-\varphi}; X)$ consists of all analytic functions $K^{-\varphi} \ni \lambda \mapsto F(\lambda) \in X$ satisfying the uniform in $\psi \in (-\varphi, 0)$ estimate (2.3). Let $\varphi : K^\varphi \to K^{-\varphi}$ denote the linear transformation $\varphi(\lambda) = e^{-i\varphi}(\lambda - \zeta) + \zeta$. It is easily seen that we can identify the classes $\mathcal{H}^\ell(K^{-\varphi}; X)$ and $\mathcal{H}^{k-s}(K^\varphi; X)$ by the rule $G = F \circ \varphi$, where $F \in \mathcal{H}^\ell(K^{-\varphi}; X)$, $G \in \mathcal{H}^{k-s}(K^\varphi; X)$, and $v = e^{-i\varphi}w$. Thus far, after obvious changes, everything said about the classes $\mathcal{H}^\ell(K^\varphi; X)$ is applicable.
These eigenvalues are of finite algebraic multiplicities and can accumulate only at infinity. We also assume that the pencil (2.44) satisfies the condition:

\[ \lambda \mapsto \mathcal{A}(\lambda) = \sum_{j=0}^{m} A_j \lambda^{m-j} \in \mathcal{B}(X_m, X_0). \]  

(2.44)

We assume that the operator \( \mathcal{A}(\lambda) \) is Fredholm for all \( \lambda \in \mathbb{C} \) and is invertible for at least one value of \( \lambda \). Under these assumptions the operator \( \mathcal{A}(\lambda) \) is invertible for all \( \lambda \in \mathbb{C} \) except for isolated eigenvalues of the pencil (2.44). These eigenvalues are of finite algebraic multiplicities and can accumulate only at infinity. We also assume that the pencil (2.44) satisfies the condition: there exist \( \vartheta \in (0, \pi/2) \) and \( R > 0 \) such that for all \( f \in X_0 \) the following
estimate is fulfilled
\[
\sum_{j=0}^{m} |\lambda|^j \|A^{-1}(\lambda) f\|_{m-j} \leq c\|f\|_0, \quad \lambda \in \overline{K_0^\|} \cup \overline{K_0^{-\|}}, |\lambda| > R.
\]

The condition (2.45) follows from the assumption (1.4), see [35, Proposition 2.2.1]. As it was already mentioned in the introductory part, the assumption (1.4) is widely met in the theory of operator pencils and is satisfied in many applications of the theory of boundary value problems for partial differential equations; see e.g. [29], [35,36] and references therein. It guarantees that for any \( \zeta \in \mathbb{C} \) and any \( \varphi \in (0, \pi) \) the closed cones \( \overline{K_\zeta^\varphi} \) and \( \overline{K_\zeta^{-\varphi}} \) contain at most finitely many eigenvalues of the pencil \( \mathfrak{A} \), one can find \( \zeta \) and \( \varphi \) so that \( \overline{K_\zeta^\varphi} \) and/or \( \overline{K_\zeta^{-\varphi}} \) are free from the spectrum of \( \mathfrak{A} \).

For \( \ell \in \mathbb{R}, \varphi \in (0, \pi) \), and \( \zeta, w \in \mathbb{C} \) we introduce the Banach space \( \mathcal{D}_w^\ell(K_\zeta^\varphi) \) of analytic functions \( K_\zeta^\varphi \ni \lambda \mapsto u(\lambda) \in X_m \) and the norm in \( \mathcal{D}_w^\ell(K_\zeta^\varphi) \) by the equalities
\[
\mathcal{D}_w^\ell(K_\zeta^\varphi) = \bigcap_{j=0}^{m} \mathcal{H}_w^{\ell-j}(K_\zeta^\varphi; X_j); \quad \|u; \mathcal{D}_w^\ell(K_\zeta^\varphi)\| = \sum_{j=0}^{m} \|u; \mathcal{H}_w^{\ell-j}(K_\zeta^\varphi; X_j)\|.
\]

It is convenient to introduce also the Banach space
\[
\mathcal{D}_w^\ell(e^{i\psi} \mathbb{R} + \zeta) = \bigcap_{j=0}^{m} \mathcal{H}_w^{\ell-j}(e^{i\psi} \mathbb{R} + \zeta; X_j); \quad \|u; \mathcal{D}_w^\ell(e^{i\psi} \mathbb{R} + \zeta)\| = \sum_{j=0}^{m} \|u; \mathcal{H}_w^{\ell-j}(e^{i\psi} \mathbb{R} + \zeta; X_j)\|.
\]

**Proposition 2.19** Let \( \zeta, w \in \mathbb{C}, \ell \in \mathbb{R}, \) and \( \varphi \in (0, \pi) \). The following assertions are valid.

(i) Every function \( u \in \mathcal{D}_w^\ell(K_\zeta^\varphi) \) has boundary limits \( u_0 \in \mathcal{D}_w^\ell(\mathbb{R} + \zeta) \) and \( u_\varphi \in \mathcal{D}_w^\ell(e^{i\varphi} \mathbb{R} + \zeta) \) in the sense that for almost all points \( \mu \) of the boundary \( \partial K_\zeta^\varphi = (\mathbb{R} + \zeta) \cup (e^{i\varphi} \mathbb{R} + \zeta) \) we have \( \|u(\lambda) - u_0(\mu)\|_m \to 0 \) as \( \lambda \) non-tangentially tends to \( \mu \in \mathbb{R} + \zeta \), and \( \|u(\lambda) - u_\varphi(\mu)\|_m \to 0 \) as \( \lambda \) non-tangentially tends to \( \mu \in e^{i\varphi} \mathbb{R} + \zeta \). Moreover,
\[
\|(e_w u) \circ \varphi_\psi, \zeta - (e_w u_0) \circ \varphi_\psi, \zeta; \mathcal{D}_0^\ell(\mathbb{R})\| \to 0, \quad \psi \to 0^+, \\
\|(e_w u) \circ \varphi_\psi, \zeta - (e_w u_\varphi) \circ \varphi_\psi, \zeta; \mathcal{D}_0^\ell(\mathbb{R})\| \to 0, \quad \psi \to \varphi^-;
\]

recall that \( e_w : \lambda \mapsto \exp(iw \lambda) \) and \( \varphi_\psi, \zeta : \mathbb{R} \to e^{i\psi} \mathbb{R} + \zeta \) is the linear transformation \( \varphi_\psi, \zeta(\xi) = e^{i\psi} \xi + \zeta \). From now on we suppose that every element \( u \in \mathcal{D}_w^\ell(K_\zeta^\varphi) \) is extended to the boundary \( \partial K_\zeta^\varphi \) by its non-tangential limits.
(ii) For all $\psi \in [0, \varphi]$ and $u \in \mathfrak{D}_w^\ell(\mathcal{K}_\xi^\varphi)$ the estimate

$$
\|u; \mathfrak{D}_w^\ell(e^{i\psi}\mathbb{R} + \zeta)\| \leq C \|u; \mathfrak{D}_w^\ell(\mathcal{K}_\xi^\varphi)\|
$$

(2.49)

holds, where the constant $C$ is independent of $u$, $\psi$, and $w$.

**Proof.** Due to the embedding $\mathfrak{D}_w^\ell(\mathcal{K}_\xi^\varphi) \subset \mathcal{H}_{w}^{\ell-m}(\mathcal{K}_\xi^\varphi; X_m)$ and Proposition 2.3.1.(i), the function $u \in \mathfrak{D}_w^\ell(\mathcal{K}_\xi^\varphi)$ has the boundary limit $u_0$ in the sense that for almost all points $\mu \in \mathbb{R} + \zeta$ we have $\|u(\lambda) - u_0(\mu)\|_m \to 0$ as $\lambda$ non-tangentially tends to $\mu$. Under the assumptions made on the spaces $X_j$, the convergence of $u(\lambda)$ to $u_0(\mu)$ in $X_m$ gives $u(\lambda) \to u_0(\mu)$ in $X_j$ for any $j = 0, \ldots, m$. This together with Proposition 2.3.1(i) and the embedding $\mathfrak{D}_w^\ell(\mathcal{K}_\xi^\varphi) \subset \mathcal{H}_{w}^{\ell-j}(\mathcal{K}_\xi^\varphi; X_j)$ provides us with the inclusion $u_0 \in \mathcal{W}_{\zeta}^{\ell-j}(\mathbb{R} + \zeta; X_j)$, $j = 0, \ldots, m$. Hence the boundary limit $u_0$ satisfies the inclusion $u_0 \in \mathfrak{D}_w^\ell(\mathbb{R} + \zeta)$. A similar argument demonstrates the inclusion $u_0 \in \mathfrak{D}_w^\ell(e^{i\psi}\mathbb{R} + \zeta)$. Now the assertion (ii) is readily apparent from Proposition 2.3.1.(ii) and the definition (2.46) of the norm in $\mathfrak{D}_w^\ell(\mathcal{K}_\xi^\varphi)$. The first relation in (2.48) is valid since the left hand side of the estimate (2.49) is bounded uniformly in $\psi \in [0, \varphi]$, and $\|u \circ \kappa_{\psi, \zeta}(\xi) - u_0 \circ \kappa_{0, \zeta}(\xi)\|_j \to 0$ as $\psi \to 0^+$ for $j = 0, \ldots, m$ and almost all $\xi \in \mathbb{R}$. In the same way one can see the second relation in (2.48). □

**Theorem 2.20** Let $\zeta, w \in \mathbb{C}$ and $\ell \in \mathbb{R}$. The following assertions are valid.

(i) Let $\varphi \in (0, \pi]$. The operator pencil (2.44) implements the continuous mapping

$$
\mathfrak{D}_w^\ell(\mathcal{K}_\xi^\varphi) \ni u \mapsto \mathfrak{A}u = \mathcal{F} \in \mathcal{H}_{w}^{\ell-m}(\mathcal{K}_\xi^\varphi; X_0);
$$

(2.50)

here and elsewhere $\mathfrak{A}u$ stands for the function $\mathcal{K}_\xi^\varphi \ni \lambda \mapsto \mathfrak{A}(\lambda)u(\lambda) \in X_0$. The estimate

$$
\|\mathfrak{A}u; \mathcal{H}_{w}^{\ell-m}(\mathcal{K}_\xi^\varphi; X_0)\| \leq C \|u; \mathfrak{D}_w^\ell(\mathcal{K}_\xi^\varphi)\|
$$

(2.51)

holds with an independent of $w \in \mathbb{C}$ and $u \in \mathfrak{D}_w^\ell(\mathcal{K}_\xi^\varphi)$ constant $C$.

(ii) Suppose that the pencil $\mathfrak{A}$ satisfies the condition (2.45) for some $\vartheta \in (0, \pi/2)$ and $R > 0$. Let the closed cone $\mathcal{K}_\xi^\varphi$, $\varphi < \vartheta$, be free from the eigenvalues of the pencil $\mathfrak{A}$. Then the mapping (2.50) is an isomorphism, the estimate

$$
\|\mathfrak{A}^{-1}\mathcal{F}; \mathfrak{D}_w^\ell(\mathcal{K}_\xi^\varphi)\| \leq C \|\mathcal{F}; \mathcal{H}_{w}^{\ell-m}(\mathcal{K}_\xi^\varphi; X_0)\|
$$

(2.52)

holds. Here the constants $C$ is independent of $w \in \mathbb{C}$ and $\mathcal{F} \in \mathcal{H}_{w}^{\ell-m}(\mathcal{K}_\xi^\varphi; X_0)$.

(iii) Suppose that the pencil $\mathfrak{A}$ satisfies the condition (2.45) for some $\vartheta \in (0, \pi/2)$ and $R > 0$. Let the line $e^{i\psi}\mathbb{R} + \zeta$, $|\psi| < \vartheta$, be free from the eigenvalues of the pencil $\mathfrak{A}$. Then the mapping

$$
\mathfrak{D}_w^\ell(e^{i\psi}\mathbb{R} + \zeta) \ni u \mapsto \mathfrak{A}u = \mathcal{F} \in \mathcal{H}_{w}^{\ell-m}(e^{i\psi}\mathbb{R} + \zeta; X_0)
$$
is an isomorphism, and the estimates
\[
\| A^{-1} F; D \left( e^{i\psi} R + \zeta \right) \| \leq c_1 \| F; H^\ell \left( e^{i\psi} R + \zeta; X_0 \right) \|
\]
\[
\leq c_2 \| A^{-1} F; D \left( e^{i\psi} R + \zeta \right) \|
\]
are valid with some independent of \( w \in \mathbb{C} \) and \( F \in \mathcal{H}^\ell \left( e^{i\psi} R + \zeta; X_0 \right) \)

constants \( c_1, c_2 \).

**PROOF.** (i) Recall that the set \( \{ A_j \in \mathcal{B}(X_j, X_0) \}_{j=0}^m \) consists of bounded, independent of \( w \in \mathbb{C} \) operators. For the pencil (2.44) we have
\[
\| A(\lambda)u(\lambda) \|_0 \leq \left( \sum_{j=0}^m |\lambda|^{m-j} \| A_j u(\lambda) \|_0 \right)^2 \leq C \sum_{j=0}^m (1+|\lambda|^2)^{(m-j)} \| u(\lambda) \|^2_j, \quad (2.53)
\]
where \( C \) is independent of \( w \in \mathbb{C} \) and \( u \in D \left( \mathcal{K}_\zeta^\varphi \right) \). Multiplying the inequalities (2.53) by the factor \( \exp\{2iw\lambda\}(1+|\lambda|^2)^{\ell-m} \) and integrating the result with respect to \( \lambda \in e^{i\psi} R + \zeta \), we arrive at the estimate
\[
\| A u; \mathcal{H}^\ell \left( e^{i\psi} R + \zeta; X_0 \right) \|^2 \leq C \sum_{j=0}^m \| u; \mathcal{H}_{w}^{\ell-m} \left( e^{i\psi} R + \zeta; X_j \right) \|^2, \quad \psi \in [0, \varphi].
\]
This together with (2.49) gives the estimate
\[
\| A u; \mathcal{H}_{w}^{\ell-m} \left( e^{i\psi} R + \zeta; X_0 \right) \| \leq C \| u; D \left( \mathcal{K}_\zeta^\varphi \right) \| \quad (2.54)
\]
with an independent of \( w \in \mathbb{C}, \psi \in [0, \varphi] \), and \( u \in D \left( \mathcal{K}_\zeta^\varphi \right) \) constant \( C \). Therefore the analytic function \( \mathcal{K}_\zeta^\varphi \ni \lambda \mapsto A(\lambda)u(\lambda) \in X_0 \) satisfies the uniform in \( \psi \in [0, \varphi] \) estimate (2.54). Thus \( A u \in \mathcal{H}_{w}^{\ell-m} \left( \mathcal{K}_\zeta^\varphi; X_0 \right) \) and the estimate (2.51) is valid.

(ii) The cone \( \overline{\mathcal{K}_\zeta^\varphi} \) is free from the eigenvalues of the pencil \( A \), consequently the operator function \( \overline{\mathcal{K}_\zeta^\varphi} \ni \lambda \mapsto A^{-1}(\lambda) \in \mathcal{B}(X_0, X_m) \) is holomorphic. For all \( f \in X_0 \) the estimate
\[
\| A^{-1}(\lambda)f \|_m \leq C(R)\| f \|_0, \quad \lambda \in \overline{\mathcal{K}_\zeta^\varphi}, |\lambda| < R + 1,
\]
is valid; here \( R \) is a sufficiently large positive number. Hence, for all \( f \in X_0 \) we have
\[
\sum_{j=0}^m |\lambda|^j \| A^{-1}(\lambda)f \|_{m-j} \leq c(R)\| f \|_0, \quad \lambda \in \overline{\mathcal{K}_\zeta^\varphi}, |\lambda| < R + 1.
\]
This together with the estimate (2.45) gives
\[
\sum_{j=0}^m |\lambda|^j \| A^{-1}(\lambda)F(\lambda) \|_{m-j} \leq c(R)\| F(\lambda) \|_0, \quad \lambda \in \overline{\mathcal{K}_\zeta^\varphi}, \quad (2.55)
\]
where \( F \in \mathcal{H}_w^\ell-m(\mathcal{K}_w^\varphi; X_0) \). We can rewrite the inequality (2.55) in the form
\[
\sum_{j=0}^{m} (1 + |\lambda|^2)^j \|\mathfrak{A}^{-1}(\lambda)F(\lambda)\|_{m-j}^2 \leq C\|F(\lambda)\|_{0}^2, \quad \lambda \in \mathcal{K}_w^\varphi; \tag{2.56}
\]
here the constant \( C \) is independent of \( w \in \mathbb{C} \) and \( F \in \mathcal{H}_w^\ell-m(\mathcal{K}_w^\varphi; X_0) \). We multiply the inequality (2.56) by the factor \(|\exp\{2iw\lambda\}|(1 + |\lambda|^2)^\ell-m\) and integrate the result with respect to \( \lambda \in e^{i\psi}\mathbb{R} + \zeta \). We get
\[
\sum_{j=0}^{m} \|\mathfrak{A}^{-1}F; \mathcal{W}_w^\ell-j(e^{i\psi}\mathbb{R} + \zeta; X_j)\|^2 \leq C\|F; \mathcal{W}_w^\ell-m(e^{i\psi}\mathbb{R} + \zeta; X_0)\|^2, \quad \psi \in [0, \varphi]. \tag{2.57}
\]
Clearly this leads to the estimate (2.52). The assertion (ii) is proved.

The proof of the assertion (iii) is similar. \( \square \)

3 Spaces of vector valued distributions

In this section we study weighted Hardy-Sobolev spaces \( H_\ell^w(\mathcal{K}_w^\varphi; X) \) of vector valued distributions. The space \( H_\ell^w(\mathcal{K}_w^\varphi; X) \) consists of Fourier-Laplace transforms of all functions from the Hardy space \( H_\ell^w(\mathcal{K}_w^\varphi; X) \). In the case \( \ell \geq 0 \) the elements of the space \( H_\ell^w(K_w^\varphi; X) \) are \( X \)-valued functions analytic in the cone \( K_w^\varphi \). In Subsection 3.1 we introduce the Fourier-Laplace transformation acting in weighted spaces of vector valued distributions. Then in Subsection 3.2 we introduce and study the spaces \( H_\ell^w(K_w^\varphi; X) \). Subsection 3.3 assembles different aspects related to the Sobolev, Hardy, and Hardy-Sobolev spaces. In particular we prove a variant of the Paley-Wiener theorem (Theorem 3.14) and a variant of the Paley-Wiener-Schwartz theorem (Proposition 3.20).

3.1 Fourier-Laplace transformation

Let \( C_0^\infty(\mathbb{R}; X) \) denote the space of smooth compactly supported functions with values in \( X \). For functions from \( C_0^\infty(\mathbb{R}; X) \) we can define the Fourier transform \( G = \mathcal{F}_{\xi}^{-1}G \) and the inverse Fourier transform \( G = \mathcal{F}_{t}^{-1}G \) by the formulas
\[
G(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{it\xi}G(\xi) \, d\xi; \quad \mathcal{G}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\xi t}G(t) \, dt. \tag{3.1}
\]

The integrals in (3.1) are absolutely convergent in \( X \), the Parseval equality
\[
\|G; L_2(\mathbb{R}; X)\| = \|\mathcal{G}; L_2(\mathbb{R}; X)\| \tag{3.2}
\]
holds. As is well-known [28,33], the Fourier transformation (3.1) can be continuously extended to an isometric isomorphism $\mathcal{F}_{\xi-t} : L_2(\mathbb{R}; X) \rightarrow L_2(\mathbb{R}; X)$.

**Lemma 3.1** Let $\zeta, w \in \mathbb{C}$ and let $\psi$ be an angle. For $\mathcal{F} \in C_0^\infty(e^{i\psi}\mathbb{R} + \zeta; X)$ we define the Fourier-Laplace transformation $\mathcal{T}_{\zeta,w} : \mathcal{F} \mapsto F$ by the equality

$$F(z) = \frac{e^{i\zeta w}}{\sqrt{2\pi}} \int_{e^{i\psi}\mathbb{R} + \zeta} e^{iz\lambda} \mathcal{F}(\lambda) d\lambda, \quad z \in e^{i\psi}\mathbb{R} + w, \quad (3.3)$$

where the integration runs from $e^{i\psi}(-\infty) + \zeta$ to $e^{i\psi}(+\infty) + \zeta$. Then the inverse transformation $(\mathcal{T}_{\zeta,w})^{-1}$ is given by the equality

$$\mathcal{F}(\lambda) = \frac{e^{-i\zeta w}}{\sqrt{2\pi}} \int_{e^{-i\psi}\mathbb{R} + w} e^{-i\lambda z} F(z) dz, \quad \lambda \in e^{i\psi}\mathbb{R} + \zeta, \quad (3.4)$$

where the integration runs from $e^{-i\psi}(-\infty) + \zeta$ to $e^{-i\psi}(+\infty) + \zeta$. The Parseval equality

$$\|\mathcal{F}; \mathcal{W}^0_\zeta(e^{i\psi}\mathbb{R} + \zeta; X)\| = \|F; \mathcal{W}^0_\zeta(e^{-i\psi}\mathbb{R} + w; X)\| \quad (3.5)$$

holds. The transformation $\mathcal{T}_{\zeta,w}$ can be continuously extended to an isometric isomorphism

$$\mathcal{T}_{\zeta,w} : \mathcal{W}^0_\zeta(e^{i\psi}\mathbb{R} + \zeta; X) \rightarrow \mathcal{W}^0_\zeta(e^{-i\psi}\mathbb{R} + w; X). \quad (3.6)$$

(The multipliers $e^{i\zeta w}$ and $e^{-i\zeta w}$ in the formulas (3.3) and (3.4) are added so that the transformation (3.6) is an isometry.)

**PROOF.** One can get the formulas (3.3), (3.4), and (3.5) from the formulas for the Fourier transformation (3.1) and (3.2). Indeed, it suffices to set

$$\mathcal{G}(\xi) = \exp\{iw(e^{i\psi}\xi + \zeta)\} \mathcal{F}(e^{i\psi}\xi + \zeta), \quad \xi \in \mathbb{R},$$

$$F(e^{-i\psi}t + w) = \exp\{i\psi + i\zeta(e^{-i\psi}t + w)\} G(t), \quad t \in \mathbb{R},$$

and $\lambda = e^{i\psi}\xi + \zeta$, $z = e^{-i\psi}t + w$. \[33\]
Every element $G \in \mathcal{S}'(\mathbb{R}; X)$ can be represented (in non-unique fashion) in the form $\sum_{m+n \leq \ell} D^m_{\xi}(1+|\xi|)D^n_{\eta}G_{mn}$, where $\ell$ is finite and $G_{mn} \in L_2(\mathbb{R}; X)$; i.e.

$$G(v) = \sum_{m+n \leq \ell} \int_{\mathbb{R}} G_{mn}(\xi)(1+|\xi|)^m D^n_{\eta}v(\xi) \, d\xi \quad \forall v \in \mathcal{S}(\mathbb{R}),$$

the integral is absolutely convergent in $X$. We have the embedding $L_2(\mathbb{R}; X) \subset \mathcal{S}'(\mathbb{R}; X)$, $\ell \in \mathbb{R}$, which means that to any $F \in L_2(\mathbb{R}; X)$ there corresponds a distribution $\mathcal{F} \in \mathcal{S}'(\mathbb{R}; X)$ given by the formula $\mathcal{F}(v) = \int \mathcal{F}(\xi)v(\xi) \, d\xi$, $v \in \mathcal{S}(\mathbb{R})$.

By $\mathcal{D}'(e^{-i\varphi}\mathbb{R} + w; X)$ we denote the space of all linear continuous mappings $C_0^\infty(e^{-i\varphi}\mathbb{R} + w) \to X$; here $w \in \mathbb{C}$ and $\varphi$ is an angle. Let $\tau_{\varphi,w}: \mathbb{R} \to e^{-i\varphi}\mathbb{R} + w$ be the linear transformation $\tau_{\varphi,w}(t) = e^{-i\varphi}t + w$. We say that $\mathcal{F} \in \mathcal{D}'(e^{-i\varphi}\mathbb{R} + w; X)$ if and only if $\mathcal{F} \circ \tau_{\varphi,w} \in \mathcal{D}'(\mathbb{R}; X)$, where $\mathcal{F} \circ \tau_{\varphi,w}(v) = \mathcal{F}(v \circ \tau_{\varphi,w}^{-1})$, $v \in C_0^\infty(\mathbb{R})$. In the same manner we introduce the space $\mathcal{D}'(e^{-i\varphi}\mathbb{R} + w; X)$ of tempered distributions on $e^{-i\varphi}\mathbb{R} + w$. The embedding $\mathcal{D}'(e^{-i\varphi}\mathbb{R} + w; X) \subset \mathcal{D}'(e^{-i\varphi}\mathbb{R} + w; X)$ holds.

We define the operator $D_{(\varphi)}: \mathcal{D}'(e^{-i\varphi}\mathbb{R} + w; X) \to \mathcal{D}'(e^{-i\varphi}\mathbb{R} + w; X)$ of differentiation along the line $e^{-i\varphi}\mathbb{R} + w$ by the equality

$$D_{(\varphi)}F = e^{i\varphi}(D_t(F \circ \tau_{\varphi,w})) \circ \tau_{\varphi,w}^{-1}, \quad F \in \mathcal{D}'(e^{-i\varphi}\mathbb{R} + w; X). \quad (3.8)$$

Let $\mathcal{O}_z$ be a neighbourhood in $\mathbb{C}$ of a point $x \in e^{-i\varphi}\mathbb{R} + w$ and let $\mathcal{O}_z \ni z \mapsto G(z) \in X$ be an analytic function. It can be easily seen that in the intersection $\mathcal{O}_z \cap (e^{-i\varphi}\mathbb{R} + w)$ we have $(D_zG)|_{e^{-i\varphi}\mathbb{R} + w} = D_{(\varphi)}(G|_{e^{-i\varphi}\mathbb{R} + w})$, where $D_z = -\frac{i}{2}(\partial_{\varphi z} - i\partial_{z \varphi})$ is the complex derivative. Alternatively, we can define at first the operator $D_{(\varphi)}$ in $C_0^\infty(e^{-i\varphi}\mathbb{R} + w)$ by the equality $D_{(\varphi)}v = (D_z\tilde{v})|_{e^{-i\varphi}\mathbb{R} + w}$, where $\tilde{v}$ is an almost analytic extension of $v$ ($\tilde{v}$ is a smooth extension of $v$ to a neighbourhood of the line $e^{-i\varphi}\mathbb{R} + w$ such that $D_z\tilde{v}$ vanishes to infinite order on $e^{-i\varphi}\mathbb{R} + w$, see e.g. [38]). Then we extend $D_{(\varphi)}$ to all $F \in \mathcal{D}'(e^{-i\varphi}\mathbb{R} + w; X)$.

**Lemma 3.2** Let $\zeta, w \in \mathbb{C}$ and let $\varphi$ be an angle. We introduce the sets

$$\mathcal{J}_w(e^{i\varphi}\mathbb{R} + \zeta; X) = \{ F \in \mathcal{D}'(e^{i\varphi}\mathbb{R} + \zeta; X) : e_wF \in \mathcal{D}'(e^{i\varphi}\mathbb{R} + \zeta; X) \}, \quad (3.9)$$

$$\mathcal{J}_\zeta(e^{-i\varphi}\mathbb{R} + w; X) = \{ F \in \mathcal{D}'(e^{-i\varphi}\mathbb{R} + w; X) : e_\zeta F \in \mathcal{D}'(e^{-i\varphi}\mathbb{R} + w; X) \}, \quad (3.10)$$

where $e_w$ and $e_\zeta$ denote the exponential weight functions

$$e_w : \lambda \mapsto \exp(iw\lambda), \quad \lambda \in e^{i\varphi}\mathbb{R} + \zeta; \quad e_\zeta : z \mapsto \exp(-i\zeta z), \quad z \in e^{-i\varphi}\mathbb{R} + w.$$ 

The transformation (3.6) can be continuously extended to all $F \in \mathcal{J}_w(e^{i\varphi}\mathbb{R} + \zeta; X)$ by the formula

$$T_{(\varphi,w)}F = e^{i\varphi}e^{-\zeta(\zeta \cdot \tau_{\varphi,w}^{-1}[D_{(\varphi)}F \circ \tau_{\varphi,w}] + \zeta \cdot \tau_{\varphi,w}^{-1}), \quad (3.11)$$
where \( \phi, \zeta : \mathbb{R} \to e^{i\psi} + \zeta \) and \( \tau^{-1}_{\psi, w} : e^{-i\psi} + w \to \mathbb{R} \) are the linear transformations \( \phi_{\psi, \zeta}(\xi) = e^{i\psi} + \zeta, \tau^{-1}_{\psi, w}(z) = e^{i\psi}(z - w) \). The Fourier-Laplace transformation (3.11) implements an isomorphism

\[
T_{\psi, w}^\zeta : \mathcal{S}_{W}(e^{i\psi} + \zeta; X) \to \mathcal{S}_{\zeta}^t(e^{-i\psi} + w; X).
\] (3.12)

We have the following differentiation rule

\[
D_{(\psi)}^j T_{\psi, w}^\zeta F = T_{\psi, w}^\zeta \lambda^j F, \quad F \in \mathcal{S}_{W}(e^{i\psi} + \zeta; X);
\] (3.13)

here the differential operator \( D_{(\psi)} \) is the same as in (3.8).

**PROOF.** The elements \( F \in \mathcal{S}_{W}(e^{i\psi} + \zeta; X) \) can be naturally identified with the elements \( G \in \mathcal{S}(\mathbb{R}; X) \) by the rule \( G = (e_{w}F) \circ \phi_{\psi, \zeta} \). We also identify the elements \( F \in \mathcal{S}_{\zeta}^t(e^{-i\psi} + w; X) \) and \( G \in \mathcal{S}(\mathbb{R}; X) \) by the rule \( F = e^{i\psi} e_{-\zeta}G \circ \tau^{-1}_{\psi, w} \). We set \( G = \mathcal{T}_{\xi} G \) and arrive at (3.11). The transformation (3.12) yields an isomorphism because \( \mathcal{T}_{\xi} : \mathcal{S}(\mathbb{R}; X) \to \mathcal{S}(\mathbb{R}; X) \) is an isomorphism. Since the Fourier transformation in \( \mathcal{S}(\mathbb{R}; X) \) is the continuous extension of the integral transformation (3.1) and \( \mathcal{W}_{w}^t(e^{i\psi} + \zeta; X) \subset \mathcal{S}(e^{i\psi} + \zeta; X) \) we conclude that (3.11) is the continuous extension of the Fourier-Laplace transformation (3.6); cf. the proof of Lemma 3.1. The differentiation rule (3.13) follows from (3.7), (3.8). \[ \square \]

### 3.2 Weighted Hardy-Sobolev spaces in cones

Let \( \zeta, w \in \mathbb{C}, \ell \in \mathbb{R} \), and let \( \psi \) be an angle. We introduce the space \( \mathcal{W}_{\ell}^t(e^{-i\psi} + w; X) \) of all distributions \( F \in \mathcal{S}_{\zeta}^t(e^{-i\psi} + w; X) \) that can be represented in the form

\[
F = T_{\psi, w}^\zeta F \quad \text{with some} \quad F \in \mathcal{W}_{w}^t(e^{i\psi} + \zeta; X); \quad \text{here} \quad T_{\psi, w}^\zeta \quad \text{is given in (3.11), and}
\]

\[
\mathcal{W}_{w}^t(e^{i\psi} + \zeta; X) \subset \mathcal{S}_{W}(e^{i\psi} + \zeta; X), \quad \text{see Lemma 3.2. Let the space} \quad \mathcal{W}_{\ell}^t(e^{-i\psi} + w; X) \quad \text{be endowed with the norm}
\]

\[
\| F; \mathcal{W}_{\ell}^t(e^{-i\psi} + w; X) \| = \| (T_{\psi, w}^\zeta)^{-1} F; \mathcal{W}_{w}^t(e^{i\psi} + \zeta; X) \|. \] (3.14)

If \( \ell = 0 \) then the equality (3.14) coincides with the Parseval equality (3.5). Using the formula (3.11) and the norm (2.1) of the space \( \mathcal{W}_{w}^t(e^{i\psi} + \zeta; X) \), one can see that the norm (3.14) is equivalent to the norm

\[
\| F; \mathcal{W}_{\ell}^t(e^{-i\psi} + w; X) \|^2 = \int_{\mathbb{R}} (1 + |\xi|^2)^\ell \| \mathcal{T}_{\psi, w}^\zeta \{ (e_{\zeta}F) \circ \tau^{-1}_{\psi, w} \} \|^2 d\xi;
\] (3.15)

moreover, the constants \( c_1 \) and \( c_2 \) in the inequalities

\[
\| F; \mathcal{W}_{\ell}^t(e^{-i\psi} + w; X) \| \leq c_1 \| F; \mathcal{W}_{\ell}^t(e^{-i\psi} + w; X) \| \leq c_2 \| F; \mathcal{W}_{\ell}^t(e^{-i\psi} + w; X) \|
\] (3.16)
are independent of $\psi$ and $w \in \mathbb{C}$. Therefore $W^\ell_\zeta(e^{-i\psi} \mathbb{R} + w; X)$ is the weighted Sobolev space of distributions with values in $X$, see [28]. Note that in (3.15) the exponential weight
\[ e_\zeta \circ \tau_{\psi,w}(t) = \exp\{-i\zeta(e^{-i\psi} t + w)\}, \quad t \in \mathbb{R}, \]
depends on the weight number $\zeta$ and the angle $\psi$. For all $v \in e^{-i\psi} \mathbb{R} + w$ we have
\[ \|F; W^\ell_\zeta(e^{-i\psi} \mathbb{R} + w; X)\| = \|F; W^\ell_\zeta(e^{-i\psi} \mathbb{R} + v; X)\|. \]
The space $W^\ell_\zeta(e^{-i\psi} \mathbb{R} + w; X)$ does not change while $\zeta$ travels along the line $e^{i\psi} \mathbb{R} + \eta$, $\eta \in \mathbb{C}$, the norm changes for the equivalent one
\[ \|F; W^\ell_\zeta(e^{-i\psi} \mathbb{R} + w; X)\| = e^{3\ell(\zeta-\eta)w}\|F; W^\ell_\eta(e^{-i\psi} \mathbb{R} + w; X)\|, \quad \zeta \in e^{i\psi} \mathbb{R} + \eta. \]
If $\ell \geq 0$ then the elements of the space $W^\ell_\zeta(e^{-i\psi} \mathbb{R} + w; X)$ are (classes of) functions. In this case we can define an equivalent norm in $W^\ell_\zeta(e^{-i\psi} \mathbb{R} + w; X)$ by the equality
\[ \|F; W^\ell_\zeta(e^{-i\psi} \mathbb{R} + w; X)\|^2 = \int \sum_{e^{-i\psi} \mathbb{R} + w} \|e^{-i\zeta z} D^\ell_{(\psi)} F(z)\|^2 |dz| \]
\[ + \int \int |z - u|^{-2(\ell-\lfloor \ell \rfloor)-1}\|e^{-i\zeta z} D^\lfloor \ell \rfloor_{(\psi)} F(z) - e^{-i\zeta u} D^\lfloor \ell \rfloor_{(\psi)} F(u)\|^2 |dz| |du|, \]
where $\lfloor \ell \rfloor$ is the integer part of $\ell$; see e.g. [31,32]. If $\ell$ is a nonnegative integer then from the rule (3.13) and the Parseval equality (3.5) it is easily seen that the norm (3.14) is equivalent to the norm
\[ \|F; W^\ell_\zeta(e^{-i\psi} \mathbb{R} + w; X)\|^2 = \int \sum_{e^{-i\psi} \mathbb{R} + w} \|e^{-i\zeta z} D^\ell_{(\psi)} F(z)\|^2 |dz|. \]
The differentiation in (3.17), (3.18) is understood in the sense of distributions. For the equivalent norms (3.17), (3.18) the inequalities (3.16) remain valid with some constants $c_1$ and $c_2$ independent of $w$, $\psi$ (and $\mathcal{F}$). We have the following corollary of Lemma 3.2.

**Corollary 3.3** The transformation (3.11) realizes an isometric isomorphism
\[ T^\psi_{\zeta,w} : W^\ell_w(e^{i\psi} \mathbb{R} + \zeta; X) \to W^\ell_\zeta(e^{-i\psi} \mathbb{R} + w; X). \]

Let $K^\varphi_w$ denote the open double-napped cone
\[ K^\varphi_w = \{z \in \mathbb{C} : z = e^{-i\psi} t + w, t \in \mathbb{R} \setminus \{0\}, 0 < \psi < \varphi\} \]
with the vertex $w$ and the angle $\varphi \in (0, \pi]$. 

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Definition 3.4 Let \( w, \zeta \in \mathbb{C}, \ell \geq 0, \) and \( \varphi \in (0, \pi]. \) We introduce the Hardy-Sobolev class \( H^\ell_\varphi(K_w^\varphi; X) \) of order \( \ell \) as the set of all analytic functions \( K_w^\varphi \ni z \mapsto F(z) \in X \) satisfying the uniform in \( \psi \in (0, \varphi) \) estimate
\[
\|F; \mathcal{W}^\ell_\psi(e^{-i\psi}R + w; X)\| \leq C(F). \tag{3.20}
\]

Theorem 3.5 Let \( \ell \geq 0, \varphi \in (0, \pi], \) and \( w, \zeta \in \mathbb{C}. \) The following assertions are fulfilled.

(i) Every function \( F \in H^\ell_\varphi(K_w^\varphi; X) \) has boundary limits \( F_0 \in \mathcal{W}^\ell_\psi(\mathbb{R} + w; X) \) and \( F_\varphi \in \mathcal{W}^\ell_\varphi(e^{-i\varphi}R + w; X) \) in the sense that for almost all points \( u \) of the boundary \( \partial K_w^\varphi \) we have \( \|F(z) - F_0(u)\| \to 0 \) as \( z \) non-tangentially tends to \( u \in \mathbb{R} + w, \) and \( \|F(z) - F_\varphi(u)\| \to 0 \) as \( z \) non-tangentially tends to \( u \in e^{-i\varphi}R + w. \) Moreover,
\[
\begin{align*}
\|(e^\zeta F) \circ \tau_{\psi, w} - (e^\zeta F_0) \circ \tau_{0, w}; \mathcal{W}^\ell_0(\mathbb{R}; X)\| \to 0, & \quad \psi \to 0+, \\
\|(e^\zeta F) \circ \tau_{\psi, w} - (e^\zeta F_\varphi) \circ \tau_{\varphi, w}; \mathcal{W}^\ell_0(\mathbb{R}; X)\| \to 0, & \quad \psi \to \varphi-;
\end{align*}
\]
here the linear transformation \( \tau_{\varphi, w} \) and the exponential weight function \( e^\zeta \) are the same as in Lemma 3.2. From now on we suppose that the elements \( F \in H^\ell_\varphi(K_w^\varphi; X) \) are extended to the boundary \( \partial K_w^\varphi \) by non-tangential limits. In the case \( \varphi = \pi \) we distinguish the banks \( \lim_{\psi \to 0+}(e^{-i\psi}R + w) \) and \( \lim_{\psi \to \pi-}(e^{-i\psi}R + w) \) in \( \partial K_w^\varphi. \)

(ii) For all \( F \in H^\ell_\varphi(K_w^\varphi; X) \) and \( \psi \in [0, \varphi] \) the estimate
\[
\|F; \mathcal{W}^\ell_\psi(e^{-i\psi}R + w; X)\| \leq C(\|F; \mathcal{W}^\ell_\psi(\mathbb{R} + w; X)\| + \|F; \mathcal{W}^\ell_\varphi(e^{-i\varphi}R + w; X)\|)
\]
holds, where the constant \( C \) is independent of \( F, \psi, \) and \( w. \)

(iii) The Hardy-Sobolev class \( H^\ell_\varphi(K_w^\varphi; X) \) endowed with the norm
\[
\|F; H^\ell_\varphi(K_w^\varphi; X)\| = \|F; \mathcal{W}^\ell_\psi(\mathbb{R} + w; X)\| + \|F; \mathcal{W}^\ell_\varphi(e^{-i\varphi}R + w; X)\|
\]
is a Banach space.

(iv) Let us identify \( F \in H^\ell_\varphi(K_w^\varphi; X) \) with the set \( \{F_\psi : \psi \in [0, \varphi]\} \) of functions \( F_\psi \in \mathcal{W}^\ell_\psi(e^{-i\psi}R + w; X) \), where \( F_\psi \) is the restriction of \( F \) to the line \( e^{-i\psi}R + w. \) In the same way we identify \( \mathcal{F} \in \mathcal{H}^\ell_\varphi(K_w^\varphi; X) \) with the set \( \{\mathcal{F}_\psi : \psi \in [0, \varphi]\} \), where \( \mathcal{F}_\psi = \mathcal{F}|_{e^{-i\psi}R + \zeta} \in \mathcal{H}^\ell_{\psi}(e^{i\psi}R + w; X). \) The Fourier-Laplace transformation
\[
\mathcal{H}^\ell_\varphi(K_w^\varphi; X) \ni \mathcal{F} \equiv \{F_\psi : \psi \in [0, \varphi]\}
\rightarrow \{T_{\psi, w}^\mathcal{F}_\psi : \psi \in [0, \varphi]\} \equiv F \in H^\ell_\varphi(K_w^\varphi; X) \tag{3.24}
\]
yields an isometric isomorphism.

(v) For any \( \psi \in [0, \varphi] \) the set of functions \( \{F|_{e^{-i\psi}R + w} : F \in H^\ell_\varphi(K_w^\varphi; X)\} \) is dense in the Sobolev space \( \mathcal{W}^\ell_\psi(e^{-i\psi}R + w; X). \)
The proof of Theorem 3.5 is preceded by the lemma.

**Lemma 3.6** Let \( F \in H_0^0(K_0^\sigma; X) , \phi \in (0, \pi) \). We denote \( K_0^{\sigma, \pm} = \{ z \in K_0^\sigma : \Im z \gtrless 0 \} \).

(i) Suppose that \( F(z) = 0 \) for \( z \in K_0^{\sigma, +} \). Then there exists an analytic function

\[
\bigcup_{\psi \in (0, \varphi)} e^{i\psi} C^- \ni \lambda \mapsto \mathcal{F}(\lambda) \in X
\]

such that for all \( \psi \in (0, \varphi) \) the equality \( \mathcal{F} = (T_{0,0}^\psi)^{-1}(F | e^{-i\psi} R) \) holds almost everywhere on the line \( e^{i\psi} R \).

(ii) Suppose that \( F(z) = 0 \) for \( z \in K_0^{\sigma, -} \). Then there exists an analytic function

\[
\bigcup_{\psi \in (0, \varphi)} e^{i\psi} C^+ \ni \lambda \mapsto \mathcal{F}(\lambda) \in X
\]

such that for all \( \psi \in (0, \varphi) \) the equality \( \mathcal{F} = (T_{0,0}^\psi)^{-1}(F | e^{-i\psi} R) \) holds almost everywhere on the line \( e^{i\psi} R \).

**PROOF.** We shall prove the assertion (i). The proof of the assertion (ii) is similar. Let us establish the equality

\[
\int_{e^{-i\psi} R^+} e^{-i\lambda z} F(z) \, dz = \int_{e^{-i\psi} R^+} e^{-i\lambda z} F(z) \, dz, \quad \lambda \in e^{i\psi} C^- \cap e^{i\phi} C^-, \tag{3.25}
\]

where \( 0 < \psi < \phi < \varphi \), the integrations run from 0 to \( e^{-i\psi}(+\infty) \) and from 0 to \( e^{-i\phi}(+\infty) \). Due to the inclusion \( F \in H_0^0(K_0^\sigma; X) \) we have \( F \in \mathcal{W}_0^0(e^{-i\psi} R; X) \) and \( F \in \mathcal{W}_0^0(e^{-i\phi} R; X) \). The integrals in (3.25) are absolutely convergent in \( X \). Since \( F \) is analytic in \( K_0^\sigma \), we get

\[
\int_{\mathcal{C}(a, \psi, \phi)} e^{-i\lambda z} F(z) \, dz = 0, \tag{3.26}
\]

where the integration runs along the closed contour

\[
\mathcal{C}(a, \psi, \phi) = \{ z : z = ae^{-i\theta}, \theta \in [\psi, \phi] \} \cup \{ z : z = te^{-i\phi}, 1/a \leq t \leq a \} \\
\cup \{ z : z = e^{-i\phi}/a, \theta \in [\psi, \phi] \} \cup \{ z : z = te^{-i\psi}, 1/a \leq t \leq a \}, \quad a > 0.
\]

Let us note that the value \( a^{-1/2} ||F(e^{-i\theta} a)|| \) tends to zero uniformly in \( \theta \in [\psi, \phi] \) as \( a \to 0 \) or \( a \to +\infty \). Indeed, this follows from the assertion 1.(iv) of Proposition 2.3 since the function \( K_0^\sigma \ni \lambda \mapsto F(e^{-i\phi} \lambda) \in X \) is in the class \( \mathcal{H}_0^0(K_0^\sigma; X) \). Let us also note that for any \( \lambda \in e^{i\psi} C^- \cap e^{i\phi} C^- \) the exponent \( \exp(-i\lambda ae^{-i\theta}) \) tends to zero uniformly in \( \theta \in [\psi, \varphi] \) as \( a \to +\infty \), and the value \( |\exp(-i\lambda e^{-i\theta}/a)| \) remains uniformly bounded as \( a \to +\infty \), \( \theta \in [\psi, \varphi] \).

Passing in (3.26) to the limit as \( a \to +\infty \) we arrive at (3.25).
It is clear that the integral
\[ F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{e^{-i\psi R}} e^{-i\lambda z} F(z) \, dz \] (3.27)
defines an analytic function \( e^{i\psi} C \ni \lambda \mapsto F(\lambda) \in X \). Due to (3.25), where \( \psi \) comes arbitrarily close to zero and \( \phi \) comes arbitrarily close to \( \varphi \), the function \( F \) has an analytic continuation to the set \( \cup_{\psi \in (0, \varphi)} e^{i\psi} C^+ \). It remains to show that for all \( \psi \in (0, \varphi) \) the equality \( F = (T_{0,0}^\psi)^{-1}(F|_{e^{-i\psi R}}) \) is valid almost everywhere on the line \( e^{i\psi} \mathbb{R} \).

Recall that \( F(z) = 0 \) for \( z \in K_0^{\varphi+} \). Let \( \zeta \in e^{i\psi} C^- \). Then we can rewrite the equality (3.27), where \( \lambda \in e^{i\psi} \mathbb{R} + \zeta \), in the form \( F = (T_{\zeta,0}^\psi)^{-1}(F|_{e^{-i\psi R}}) \) or, equivalently, in the form \( F(\cdot + \zeta) = (T_{\zeta,0}^\psi)^{-1}(e_\zeta F|_{e^{-i\psi R}}) \); cf. Lemma 3.1 and Lemma 3.2. From Corollary 3.3 we have
\[ ||F(\cdot + \zeta) - (T_{0,0}^\psi)^{-1}(F|_{e^{-i\psi R}}) ||_{\mathcal{W}_0^0(e^{i\psi} \mathbb{R}; X)} = ||e_\zeta F - F||_{\mathcal{W}_0^0(e^{-i\psi} \mathbb{R}; X)}. \] (3.28)
The right hand side of the equality (3.28) tends to zero as \( \zeta \to 0, \zeta \in e^{i\psi} C^- \). This is because \( F(z) = 0 \) for \( z \in e^{-i\psi} \mathbb{R}^- \), the exponential function \( e_\zeta \) is bounded on \( e^{-i\psi} \mathbb{R}^+ \) for \( \zeta \in e^{i\psi} C^- \), and \( e_\zeta \to 1 \) as \( \zeta \to 0 \). Thus \( F(\cdot + \zeta) \) tends to \( (T_{0,0}^\psi)^{-1}(F|_{e^{-i\psi R}}) \) in \( \mathcal{W}_0^0(e^{i\psi} \mathbb{R}; X) \) as \( \zeta \to 0, \zeta \in e^{i\psi} C^- \), and \( F = (T_{0,0}^\psi)^{-1}(F|_{e^{-i\psi R}}) \) almost everywhere on \( e^{i\psi} \mathbb{R} \). \( \square \)

**PROOF of Theorem 3.5.** We first prove the proposition for the case \( \zeta = w = 0 \). Let us prove that for every \( F \in \mathcal{H}_0^\ell(K_0^\varphi; X) \), \( \ell \geq 0 \), there exists \( F \in \mathcal{H}_0^\ell(K_0^\varphi; X) \) such that \( F|_{e^{-i\psi R}} = (T_{0,0}^\psi)^{-1}(F|_{e^{-i\psi R}}) \) for all \( \psi \in (0, \varphi) \). Since \( \ell \geq 0 \), the inclusion \( \mathcal{H}_0^\ell(K_0^\varphi; X) \subseteq \mathcal{H}_0^\ell(K_0^\varphi; X) \) holds and Lemma 3.6 can be applied. We set \( F^\pm(z) = F(z) \) for \( z \in K_0^{\varphi+} \) and \( F^\pm(z) = 0 \) for \( z \in K_0^{\varphi-} \). By Lemma 3.6 there exists an analytic function \( K_0^\varphi \ni \lambda \mapsto F(\lambda) \in X \) such that for all \( \psi \in (0, \varphi) \) the equality
\[ F = (T_{0,0}^\psi)^{-1}(F^+|_{e^{-i\psi R}} \pm F^-|_{e^{-i\psi R}}) = (T_{0,0}^\psi)^{-1}(F|_{e^{-i\psi R}}) \]
holds almost everywhere on the line \( e^{i\psi} \mathbb{R} \). By Definition 3.4 and Corollary 3.3 we have
\[ ||F; \mathcal{W}_0^\ell(e^{i\psi} \mathbb{R}; X)|| = ||F; \mathcal{W}_0^\ell(e^{-i\psi} \mathbb{R}; X)|| \leq Const, \quad \psi \in (0, \varphi). \]

This proves the inclusion \( F \in \mathcal{H}_0^\ell(K_0^\varphi; X) \), see Definition 2.1. In a similar way one can see that for every \( F \in \mathcal{H}_0^\ell(K_0^\varphi; X) \) there exists \( F \in \mathcal{H}_0^\ell(K_0^{\varphi+}; X) \) such that \( F|_{e^{-i\psi R}} = T_{0,0}^\psi(F|_{e^{i\psi R}}) \), we do not cite the proof of this implication.

We set \( F_0 = T_{0,0}^\psi(F|_\mathbb{R}) \) and \( F_\varphi = T_{0,0}^\psi(F|_{e^{i\psi R}}) \). The assertion 1. (i) of Proposition 2.3 together with Corollary 3.3 leads to the relations (3.21). It is clear that
$F_0$ and $F_\varphi$ coincide almost everywhere with non-tangential boundary limits of $F \in H^\ell_0(K_\varphi^\omega; X)$. Indeed, the embedding $W^\ell_0(\mathbb{R}; X) \subseteq W^\ell_0(\mathbb{R}; X)$ is continuous, and for $\ell = 0$ the assertion (i) can be established by the same argument as we used in the proof Proposition 2.3. The assertion (i) is proved.

The inequality (2.6) provides the estimate (3.22) for the assertion (ii). If we define the norm in $H^\ell_0(K_\varphi^\omega; X)$ by the equality (3.23) then we get

$$
||F; H^\ell_0(K_\varphi^\omega; X)|| = ||F; W^\ell_0(\mathbb{R}; X)|| + ||F; W^\ell_0(e^{i\varphi\mathbb{R}}; X)|| = ||F; H^\ell_0(K_\varphi^\omega; X)||.
$$

This equality finishes the proof of the assertion (iv). The space $H^\ell_0(K_\varphi^\omega; X)$ is complete because it is isomorphic to the complete space $H^\ell_0(K_\varphi^\omega; X)$. The assertion (iii) follows from (iv) and Proposition 2.4. Finally, the assertion (v) is a consequence of (iv) together with Corollary 3.3 and Proposition 2.3.1.(v).

The proposition is proved for the case $\zeta = w = 0$.

Let us consider the general case. We identify the classes $H^\ell_0(K_\varphi^\omega; X)$ and $H^\ell_0(K_\varphi^\omega; X)$ by setting $F(z - w) = \exp\{-i\zeta z\}G(z)$, $z \in K_\varphi^\omega$, where $G \in H^\ell_0(K_\varphi^\omega; X)$ and $F \in H^\ell_0(K_\varphi^\omega; X)$. Let us also identify the classes $H^\ell_0(K_\varphi^\omega; X)$ and $H^\ell_0(K_\varphi^\omega; X)$ according to the rule $G(\lambda - \zeta) = e^{-in\lambda}F(\lambda)$, $\lambda \in K_\varphi^\omega$. This allows us to reformulate the results proved for $F \in H^0_0(K_\varphi^\omega; X)$ and $F \in H^\ell_0(K_\varphi^\omega; X)$ in terms of $G \in H^\ell_0(K_\varphi^\omega; X)$ and $G \in H^\ell_0(K_\varphi^\omega; X)$.

If $\ell \in \mathbb{N}$ then the space $H^\ell_\zeta(K_\varphi^\omega; X)$ consists of all elements $F \in H^0_\zeta(K_\varphi^\omega; X)$ such that the complex derivatives $K_\varphi^\omega \ni z \mapsto D_z^jF(z) \in X$, $j = 1, \ldots, \ell$, are also in the class $H^0_\zeta(K_\varphi^\omega; X)$, the norm (3.23) in $H^\ell_\zeta(K_\varphi^\omega; X)$ is equivalent to the norm

$$
||F; H^\ell_\zeta(K_\varphi^\omega; X)|| = ||F; H^0_\zeta(K_\varphi^\omega; X)|| + \|D_z^jF; H^0_\zeta(K_\varphi^\omega; X)||.
$$

The next theorem presents an elementary embedding result.

**Theorem 3.7** If $m \in \mathbb{N}$ and $\ell > m - 1/2$ then $H^\ell_\zeta(K_\varphi^\omega; X) \subset C^{m-1}(\overline{K_\varphi^\omega; X})$.

**PROOF.** Let $G \in H^\ell_\zeta(K_\varphi^\omega; X)$. We set $F(z) = e^{-i\zeta z}G(z + w)$, $z \in K_\varphi^\omega$. Then $F \in H^0_\zeta(K_\varphi^\omega; X)$. It suffices to show that $F(z) \in C^{m-1}(\overline{K_\varphi^\omega; X})$. By the Sobolev embedding theorem we immediately get $F \in W^\ell_0(e^{-i\varphi\mathbb{R}}; X) \subset C^{m-1}(e^{-i\varphi\mathbb{R}}; X)$, $\psi \in [0, \varphi]$. Let $F \in H^\ell_0(K_\varphi^\omega; X)$ be the transform (3.24) of $F$. Since $\ell > m - 1/2 > 0$, the analytic functions $K_\varphi^\omega \ni \lambda \mapsto \lambda^j F(\lambda) \in X$, $j = 0, 1, \ldots, m - 1$, are in the class $H^\ell_0(K_\varphi^\omega; X)$. For $\psi \in [0, \varphi]$, we have

$$
D_\psi^j F(z) = \frac{1}{\sqrt{2\pi}} \int_{e^{i\varphi\mathbb{R}}} e^{i\lambda z} \lambda^j F(\lambda) d\lambda, \quad j = 0, 1, \ldots, m - 1,
$$

(3.29)
where the integral is absolutely convergent in $X$. Let us recall that $D^j_\psi \mathcal{F}(z) = D^j_\psi F(z)$ for $z \in e^{-i\psi} \mathbb{R} \setminus \{0\}$, $\psi \in (0, \varphi)$; here $D_\psi = -i\frac{d}{dz} - i\partial_\psi$. Using the same arguments as in the proof of Lemma 3.6 we arrive at the equality

$$\int_{e^{i\psi} \mathbb{R}^+} e^{i\lambda\psi} \lambda^j \mathcal{F}(\lambda) d\lambda = \int_{e^{i\phi} \mathbb{R}^+} e^{i\lambda\psi} \lambda^j \mathcal{F}(\lambda) d\lambda, \quad j = 0, 1, \ldots, m - 1, \quad (3.30)$$

where $0 < \psi < \phi < \varphi$ and $z \in e^{-i\psi} \mathbb{C}^+ \cap e^{-i\phi} \mathbb{C}^+$. The assertion 1.(i) of Proposition 2.3 and the inequality $\ell > j + 1/2$ allow us to pass in (3.30) to the limits as $\psi \to 0^+$ and $\phi \to \varphi^-$. As a result we get

$$\int_{e^{i\psi} \mathbb{R}^+} e^{i\lambda\psi} \lambda^j \mathcal{F}(\lambda) d\lambda = \int_{e^{i\varphi} \mathbb{R}^+} e^{i\lambda\psi} \lambda^j \mathcal{F}(\lambda) d\lambda, \quad \psi \in [0, \varphi], \quad z \in e^{-i\psi} \mathbb{R}^+. \quad (3.31)$$

In a similar way one can obtain the equality

$$\int_{e^{i\psi} \mathbb{R}^-} e^{i\lambda\psi} \lambda^j \mathcal{F}(\lambda) d\lambda = \int_{e^{i\varphi} \mathbb{R}^-} e^{i\lambda\psi} \lambda^j \mathcal{F}(\lambda) d\lambda, \quad \psi \in [0, \varphi], \quad z \in e^{-i\varphi} \mathbb{R}^- \quad (3.32)$$

the formulas (3.29), (3.31), and (3.32) give us the equality

$$D^j_\psi F(e^{-i\psi} t) = \frac{1}{\sqrt{2\pi}} \left( \int_{e^{i\varphi} \mathbb{R}^+} \exp\{i\lambda e^{-i\psi} t\} \lambda^j \mathcal{F}(\lambda) d\lambda \right)$$

$$+ \int_{e^{i\varphi} \mathbb{R}^+} \exp\{i\lambda e^{-i\psi} t\} \lambda^j \mathcal{F}(\lambda) d\lambda \quad (3.33)$$

for $j = 0, \ldots, m - 1$, $t \geq 0$ and $\psi \in [0, \varphi]$. Since $\mathcal{F} \in \mathcal{W}_0^\ell(\mathbb{R}; X)$, $\mathcal{F} \in \mathcal{W}_0^\ell(\mathbb{R}; X)$, and $\ell > j + 1/2$, the integrals in (3.33) are absolutely convergent in $X$ uniformly with respect to $t \geq 0$ and $\psi \in [0, \varphi]$. Thus $F \in C^{m-1}(\mathbb{R} \cap e^{-i\varphi} \mathbb{C}^+; X)$. Similarly we obtain the inclusion $F \in C^{m-1}(\mathbb{R} \cap e^{-i\psi} \mathbb{C}^+; X)$. Summing up we can say that $F \in C^{m-1}(\mathbb{R}^+; X)$. \square

Let $f \in \mathcal{D}'(\mathbb{R} \times (0, \varphi); X)$ and $v \in C_0^\infty(\mathbb{R})$. We define $f(v) \in \mathcal{D}'((0, \varphi); X)$ by the formula $f(v)(u) = f(uv)$, where $u \in C_0^\infty(0, \varphi)$. We say that a distribution $f \in \mathcal{D}'(\mathbb{R} \times (0, \varphi); X)$ is continuous with respect to $\psi$ and write $f \in C((0, \varphi); X)_\psi$ if for any $v \in C_0^\infty(\mathbb{R})$ there exists a function $f(v) \in C((0, \varphi); X)$ such that

$$f(v)(u) = \int_0^\varphi f(v)(\psi) u(\psi) d\psi, \quad \forall u \in C_0^\infty(0, \varphi).$$

If $f \in \mathcal{D}'((0, \varphi); X) \cap C((0, \varphi); X)_\psi$ then the restrictions $f(v) \in \mathcal{D}'(\mathbb{R}; X)$ of $f$ are well-defined for all $\psi \in (0, \varphi)$, and

$$f(v)(\psi) = f(v), \quad v \in C_0^\infty(\mathbb{R}), \quad \psi \in (0, \varphi);$$

an additional point to emphasize is that

$$f(\chi) = \int_0^\varphi f(v)(\chi(\cdot, \psi)) d\psi, \quad \forall \chi \in C_0^\infty(\mathbb{R} \times (0, \varphi)).$$
see e.g. [37]. For any $G \in H^0_\zeta(K^\varphi_w; X)$ we have $(D^j_{(\psi)} G) \circ \tau_{\psi,w} \in C((0, \varphi); X)_\psi$

because

$$\| (e^j D^j_{(\psi)} G) \circ \tau_{\psi,w} - (e^j D^j_{(\phi)} G) \circ \tau_{\phi,w} W_0^{-j}(\mathbb{R}; X) \| \to 0, \ \psi \to \phi, \ j = 0, 1, 2, \ldots,$$

where $\psi$ and $\phi$ are in the interval $[0, \varphi]$; cf. (3.21). Therefore if $G \in H^0_\zeta(K^\varphi_w; X)$ then to the set of distributions

$$\{ F_\psi \in W_0^{-j}(e^{-i\psi}\mathbb{R} + w; X) : F_\psi = D^j_{(\psi)}(G|_{e^{-i\psi}\mathbb{R} + w}), \psi \in (0, \varphi) \}$$

there corresponds a unique distribution $F \in \mathcal{D}'((0, \varphi) \times (0, \varphi) \times (\mathbb{R})^4)$ satisfying the conditions: for $\{ F_\psi : \psi \in (0, \varphi) \}$ there exist some constant $C(\mathcal{F})$, a finite $m \in \mathbb{N}$, and a set of functions $\{ G_j \in H^0_\zeta(K^\varphi_w; X) \}_{j=0}^m$ such that the estimate and the representation

$$\| F_\psi; W^j_\zeta(e^{-i\psi}\mathbb{R} + w; X) \| \leq C(F), \ \ F_\psi = \sum_{j=0}^m D^j_{(\psi)}(G_j|_{e^{-i\psi}\mathbb{R} + w}) \quad (3.34)$$

are valid for all $\psi \in (0, \varphi)$; here $D_{(\psi)}$ is the same as in (3.8).

Since the functions $G_j \in H^0_\zeta(K^\varphi_w; X)$ are analytic in $K^\varphi_w$ we have

$$(D^j_{(\psi)}(G_j|_{e^{-i\psi}\mathbb{R} + w}))(z) = D^j_{z}(G_j(z), z \in (e^{-i\psi}\mathbb{R} + w) \setminus \{ w \}, \psi \in (0, \varphi), \quad (3.35)$$

where $D^j_z = -i^j (\partial_{\zeta z} - i^j \partial_{\zeta z})$ is the complex derivative. From (3.34) and (3.35) it is easily seen that the distribution $F_{(\psi)}$ coincides with the analytic function $\sum D^j_{z} G_j$ on the set $(e^{-i\psi}\mathbb{R} + w) \setminus \{ w \}$. If $\{ F_\psi : \psi \in (0, \varphi) \} \in H^0_\zeta(K^\varphi_w; X)$ then the singular support of $F_\psi$ is empty or it consists of the only vertex $\{ w \}$ of the cone $K^\varphi_w$.

**Lemma 3.9** In the case $\ell \geq 0$ Definition 3.8 and Definition 3.4 are equivalent. If $F$ is a function from the class $H^\ell_\zeta(K^\varphi_w; X)$ by Definition 3.4 then the set

$$\{ F_\psi = F|_{e^{-i\psi}\mathbb{R} + w} \in W^j_\zeta(e^{-i\psi}\mathbb{R} + w; X) : \psi \in (0, \varphi) \} \quad (3.36)$$

is in the class $H^\ell_\zeta(K^\varphi_w; X)$ by Definition 3.8. If $\{ F_\psi : \psi \in (0, \varphi) \} \in H^\ell_\zeta(K^\varphi_w; X)$ by Definition 3.8 then in accordance with Definition 3.4 the class $H^\ell_\zeta(K^\varphi_w; X)$ contains the analytic in $K^\varphi_w$ function $F$ given by the equality

$$F(z) = F_\psi(z), \ z \in K^\varphi_w \cap (e^{-i\psi}\mathbb{R} + w), \ \psi \in (0, \varphi). \quad (3.37)$$
PROOF. If \( F \in \mathcal{H}_\zeta^f(K^\varphi_w;X) \) by Definition 3.4 then we can identify the analytic in \( K^\varphi_w \) function \( F \) with the set (3.36). Obviously, this set is in the class \( \mathcal{H}_\zeta^f(K^\varphi_w;X) \) according to Definition 3.8.

Let \( \ell \geq 0 \) and \( \{F_\psi : \psi \in (0, \varphi)\} \in \mathcal{H}_\zeta^f(K^\varphi_w;X) \) by Definition 3.8. Then from the representation in (3.34) it follows that the function \( F \) given by (3.37) is analytic in the cone \( K^\varphi_w \); see also (3.35). Taking into account the estimate from (3.34), we see that \( F \in \mathcal{H}_\zeta^f(K^\varphi_w;X) \) by Definition 3.4. \( \Box \)

Every set of distributions \( \{F_\psi : \psi \in (0, \varphi)\} \in \mathcal{H}_\zeta^f(K^\varphi_w;X) \) defines an analytic function \( K^\varphi_w \ni z \mapsto F(z) \in X \) by the rule (3.37). It is easy to see that in the general case elements of the space \( \mathcal{H}_\zeta^f(K^\varphi_w;X) \) cannot be reconstructed from the corresponding analytic functions. For instance, the set of distributions

\[
\{F_\psi = e^{\beta \psi}(D_t^{-1}\delta) \circ \tau_{\psi,w}^{-1} : \psi \in (0, \varphi)\} \in \mathcal{H}_\zeta^{-j}(K^\varphi_w;\mathbb{C}), \quad j \in \mathbb{N}, \tag{3.38}
\]

defines the analytic in \( K^\varphi_w \) function \( F \equiv 0 \); here \( \delta \) denotes the Dirac delta function on the real axis.

In what follows we remain to use the notations

\[
(e_\varsigma F) \circ \tau_{\psi,w}, \quad F|_{e^{-i\psi R+w}}, \quad \|F;\mathcal{W}_\varsigma^f(e^{-i\varphi R}+w;X)\|, \tag{3.39}
\]

which we employed studying the case \( \ell \geq 0 \). This can be done without ambiguity if by the notations (3.39) we shall mean

\[
(e_\varsigma F_\psi) \circ \tau_{\psi,w}, \quad F_\psi, \quad \|F_\psi;\mathcal{W}_\varsigma^f(e^{-i\varphi R}+w;X)\|.
\]

The next theorem in particular generalizes Theorem 3.5 to the case \( \ell \in \mathbb{R} \).

**Theorem 3.10** Let \( \ell \in \mathbb{R} \), \( \varphi \in (0, \pi) \), and \( w, \zeta \in \mathbb{C} \). The following assertions hold.

(i) Every set of distributions \( F \equiv \{F_\psi : \psi \in (0, \varphi)\} \in \mathcal{H}_\zeta^f(K^\varphi_w;X) \) has boundary limits \( F_0 \in \mathcal{W}_\zeta^f(\mathbb{R}+w;X) \) and \( F_\varphi \in \mathcal{W}_\zeta^f(e^{-i\varphi R}+w;X) \) such that the relations (3.21) hold. From now on we suppose that the elements \( F \in \mathcal{H}_\zeta^f(K^\varphi_w;X) \) are extended by continuity to the boundary \( \partial K^\varphi_w \), i.e. \( F \equiv \{F_\psi : \psi \in [0, \varphi]\} \).

(ii) For all \( F \in \mathcal{H}_\zeta^f(K^\varphi_w;X) \) and \( \psi \in [0, \varphi] \) the estimate (3.22) is valid.

(iii) The class \( \mathcal{H}_\zeta^f(K^\varphi_w;X) \) endowed with the norm (3.23) is a Banach space.

(iv) The transformation (3.24) implements an isometric isomorphism.

(v) Let \( \{F_\psi : \psi \in [0, \varphi]\} \in \mathcal{H}_\zeta^f(K^\varphi_w), \mathcal{F} \in \mathcal{M}_w^f(K^\varphi_\zeta) \), and let the equality

\[
F|_{e^{-i\psi R+w}} = \mathcal{T}_{\psi,w}^\zeta(F|_{e^{-i\psi R}})
\]
be valid for at least one value of \( \psi \in [0, \varphi] \). Then this equality is valid for all \( \psi \in [0, \varphi] \).

(vi) A set of distributions \( \{ F_\psi : \psi \in [0, \varphi] \} \in H^f_\xi(K_w^\varphi; X) \) can be uniquely recovered from any distribution \( F_\psi \) of this set.

(vii) For any \( \psi \in [0, \varphi] \) the set of distributions \( \{ F_\psi : F \in H^f_\xi(K_w^\varphi; X) \} \) is dense in the Sobolev space \( W^f_\xi(e^{-i\psi}R + w; X) \).

PROOF. It suffices to consider the case \( \ell < 0 \), the case \( \ell \geq 0 \) is covered by Theorem 3.5 and Lemma 3.9. Let us show that for every \( F_\psi \in \mathcal{F} \) there exists \( F \in H^f_\xi(K_w^\varphi; X) \) such that \( \mathcal{F} |_{e^{-i\psi}R + \zeta} = (T^\psi_{\zeta, w})^{-1}F_\psi \) for all \( \psi \in (0, \varphi) \). Here the operator \( (T^\psi_{\zeta, w})^{-1} \) is well-defined due to Corollary 3.3. Let \( \{ G^j \in \mathbb{H}_\zeta^0(K_w^\varphi; X) \}_{j=0}^m \) be a set of functions from Definition 3.8. By Theorem 3.5 there exists a unique set \( \{ G^j \in \mathcal{H}_w^0(K^\varphi_\zeta; X) \}_{j=0}^m \) such that \( \mathcal{G}^j \mid_{e^{-i\psi}R + w} \) for all \( \psi \in (0, \varphi) \) and \( j = 0, \ldots, m \). Denote \( \mathcal{F}^j(\lambda) = \sum_{j=0}^m \lambda^j G^j(\lambda) \), \( \lambda \in K^\varphi_\zeta \). It is clear that the function \( K^\varphi_\zeta \ni \lambda \mapsto \mathcal{F}^j(\lambda) \in X \) is analytic. Thus, the estimation (3.13) and the representation in (3.34) hold for \( F_\psi \) give

\[
\mathcal{F} |_{e^{-i\psi}R + \zeta} = (T^\psi_{\zeta, w})^{-1} \sum_{j=0}^m D^j(\psi) G^j \mid_{e^{-i\psi}R + w} = (T^\psi_{\zeta, w})^{-1} F_\psi, \quad \psi \in (0, \varphi).
\]

The estimate (3.34) together with Corollary 3.3 leads to the uniform in \( \psi \in (0, \varphi) \) estimate

\[
\| \mathcal{F} \mathcal{W}_\ell^f(e^{-i\psi}R + \zeta; X) \| \leq C.
\]

Thus, \( \mathcal{F} \in \mathcal{H}_\ell^f(K^\varphi_\zeta; X) \). We set \( F_0 = T^\psi_{\zeta, w}(\mathcal{F} \mid_{R + \zeta}) \) and \( F_\varphi = T^\psi_{\zeta, w}(\mathcal{F} \mid_{e^{-i\psi}R + \zeta}) \). Taking into account Corollary 3.3 and the items 1.(i), 1.(ii) of Proposition 2.3, we arrive at the relations (3.21) and the estimate (3.22). Moreover, if we define the norm in \( H^f_\xi(K_w^\varphi; X) \) as in (3.23) then we have

\[
\| F \|_{H^f_\xi(K_w^\varphi; X)} = \| \mathcal{F} \mathcal{W}_\ell^f(K^\varphi_\zeta; X) \|
\]

see (2.12). The assertions (i), (ii) are proved.

To complete the proof of the assertion (iv) it remains to show that for any \( \zeta \in \mathbb{R} \) the set \( \{ F_\psi : \psi \in (0, \varphi) \} \) is in the class \( H^f_\xi(K_w^\varphi; X) \) for every \( \mathcal{F} \in \mathcal{H}_\ell^f(K^\varphi_\zeta; X) \). Let \( \mathcal{F} \in \mathcal{H}_\ell^f(K^\varphi_\zeta; X) \) and let \( m \in \mathbb{N}, -m < \ell < 0 \). For all \( \lambda \in K^\varphi_\zeta \) we set

\[
G^{-}(\lambda) = (\lambda - \zeta - i)^{-m} \mathcal{F}^{-}(\lambda), \quad G^{+}(\lambda) = (\lambda - \zeta + i)^{-m} \mathcal{F}^{+}(\lambda);
\]

here the notations are the same as in Proposition 2.5. By this proposition we get \( \mathcal{G}^{\pm} \in \mathcal{H}_w^0(K^\varphi_\zeta; X) \). Due to Theorem 3.5 there exist unique functions \( G^+ \in H_0^0(K^\varphi_w; X) \) and \( G^- \in H_0^0(K^\varphi_w; X) \) such that \( G^\pm \mid_{e^{-i\psi}R + w} = T^\psi_{\zeta, w} \mathcal{G}^\pm \mid_{e^{-i\psi}R + \zeta}, \)

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ψ ∈ [0, φ]. Now by the rule (3.13) we have

\[ F_\psi = (D_\psi - \zeta + i)^m (G^+ \mid_{e^{-i\psi R + w}}) + (D_\psi - \zeta - i)^m (G^- \mid_{e^{-i\psi R + w}}). \] (3.40)

Obviously, the representation (3.40) can be rewritten in the same form as in (3.34). The estimate in (3.34) follows from the item 1.(ii) of Proposition 2.3, the definition (2.12) of the norm in \( \mathcal{H}_w^\ell(K_\zeta^\varphi; X) \), and Corollary 3.3. The assertion (iv) is proved. The assertion (iii) is readily apparent from (iv) and Proposition 2.4.

Let the assumptions of the assertion (v) be fulfilled. Then by (iv) there exists \( G \in \mathcal{H}_w^\ell(K_\zeta^\varphi; X) \) such that \( (T_{\psi}^\psi)^{-1} F_\psi = G \mid_{e^{i\psi R + \zeta}} \) for all \( \psi \in [0, \varphi] \). For some \( \psi \in [0, \varphi] \) we have \( G(\lambda) = F(\lambda), \lambda \in e^{i\psi R + \zeta} \). If \( \psi \in (0, \varphi) \) then the equality \( G(\lambda) = F(\lambda) \) can be extended by analyticity to all \( \lambda \in K_\zeta^\varphi \); if \( \psi = 0 \) or \( \psi = \varphi \) then the equality \( G(\lambda) = F(\lambda), \lambda \in K_\zeta^\varphi \), is a consequence of Proposition 2.7. Hence the elements \( G \) and \( F \) of the space \( \mathcal{H}_w^\ell(K_\zeta^\varphi; X) \) are coincident. The assertion (v) is proved. The proof of the assertion (vi) is similar. Finally, the assertion (vii) is a consequence of (iv) together with Corollary 3.3 and Proposition 2.3.1.(v). □

**Theorem 3.11** Let \( w, \zeta \in \mathbb{C}, \ell \in \mathbb{R} \), and \( \varphi \in (0, \pi] \). Suppose that \( \ell > r \) and the boundary limits of a set of distributions \( F \in \mathcal{H}_\zeta^r(K_w^\varphi; X) \) satisfy the inclusions \( F_0 \in \mathcal{W}_\zeta^r(\mathbb{R} + w; X) \) and \( F_\varphi \in \mathcal{W}_\zeta^r(e^{-i\varphi R} + w; X) \). Then \( F \in \mathcal{H}_\zeta^k(K_w^\varphi; X) \) for any \( k < \ell \).

**PROOF.** From Theorem 3.10, (iv) and Corollary 3.3 we see that \( F \) is the Fourier-Laplace transform of a function \( F \in \mathcal{H}_w^\ell(K_\zeta^\varphi; X) \) with the boundary limits \( F \mid_{R+\zeta} \in \mathcal{W}_w^\ell(\mathbb{R} + \zeta; X) \) and \( F \mid_{e^{i\psi R} + \zeta} \in \mathcal{W}_w^\ell(e^{i\varphi R} + \zeta; X) \). By Proposition 2.6, (i) the function \( F \) can be represented in the form (2.14), (2.15), where we take \( s \in (r - 1/2, r) \) such that \( \ell - s > 1/2 \). The boundary limits \( F \mid_{R+\zeta} \) and \( F \mid_{e^{i\psi R} + \zeta} \) play the role of the function \( G \) on \( \partial K_\zeta^\varphi \) in Proposition 2.13. This proposition gives \( F \in \mathcal{H}_w^k(K_\zeta^\varphi; X) \), where \( k \in (r, s + 1/2) \). Hence \( F \in \mathcal{H}_\zeta^k(K_w^\varphi; X) \) with the same \( k \). On the next step we can apply the same argument to \( F \in \mathcal{H}_\zeta^r(K_w^\varphi; X) \), where \( r = k \). Step by step the parameter \( k \) comes arbitrarily close to \( \ell \). □

### 3.3 Representation of distributions in terms of Hardy-Sobolev spaces

By analogy with \( \mathcal{H}_w^\ell(K_\zeta^\varphi; X) \) we can introduce the weighted Hardy-Sobolev space \( \mathcal{H}_\zeta^\ell(K_w^{-\varphi}; X) \) in the cone \( K_w^{-\varphi} = \{ z \in \mathbb{C} : z = e^{i\varphi} v + w, v \in K_0^\varphi \} \). We identify the spaces \( \mathcal{H}_\zeta^r(K_w^{-\varphi}; X) \) and \( \mathcal{H}_\eta^r(K_w^\varphi; X) \) by the rule: a set of distributions \( \{ F_\psi \mid \psi \in [0, \varphi] \} \) is in the space \( \mathcal{H}_\zeta^r(K_w^{-\varphi}; X) \) if and only if \( \{ F_\psi \circ \sigma_{\varphi, w} : \psi \in [0, \varphi] \} \) is in the space \( \mathcal{H}_\eta^r(K_w^\varphi; X) \), where \( \eta = e^{i\varphi} \zeta \) and
\(\sigma_{\varphi,w} : K^\varphi_w \rightarrow K^\varphi_w\) is the linear transformation \(\sigma_{\varphi,w}(z) = e^{i\varphi}(z - w) + w\). We equip the space \(H^\ell_\zeta(K^\varphi_w; X)\) with the norm
\[
\|F; H^\ell_\zeta(K^\varphi_w; X)\| = \|F_0; W^\ell_\zeta(\mathbb{R} + w; X)\| + \|F_{\varphi}; W^\ell_\zeta(e^{i\varphi}\mathbb{R} + w; X)\|.
\]

The following proposition is a direct consequence of Proposition 2.18.

**Proposition 3.12** Let \(\zeta, w \in \mathbb{C}, \ell \in \mathbb{R}\), and let \(0 < \varphi < \pi/2\). Then every distribution \(F \in W^\ell_\zeta(\mathbb{R} + w; X)\) can be represented as the sum \(F_0^+ + F_0^-\) of the boundary limits \(F_0^\pm \in W^\ell_\zeta(\mathbb{R} + w; X)\) of some sets of distributions \(F^+ \in H^\ell_\zeta(K^\varphi_w; X)\) and \(F^- \in H^\ell_\zeta(K^\varphi_w; X)\).

We introduce the space \(W^\ell_\zeta(e^{-i\phi}\mathbb{R}^\pm + w; X)\) as the set of all distributions \(F \in W^\ell_\zeta(e^{-i\phi}\mathbb{R} + w; X)\) supported on the set \((e^{-i\phi}\mathbb{R}^\pm + w) \cup \{w\}\); here \(\zeta, w \in \mathbb{C}, \ell \in \mathbb{R}\), and \(\phi\) is an angle. We equip the space \(W^\ell_\zeta(e^{-i\phi}\mathbb{R}^\pm + w; X)\) with the norm
\[
\|F; W^\ell_\zeta(e^{-i\phi}\mathbb{R}^\pm + w; X)\| = \|F; W^\ell_\zeta(e^{-i\phi}\mathbb{R} + w; X)\|;
\]
the norm in the Sobolev space \(W^\ell_\zeta(e^{-i\phi}\mathbb{R} + w; X)\) is defined in (3.15). It is clear that \(W^\ell_\zeta(e^{-i\phi}\mathbb{R} + w; X) = W^\ell_\zeta(e^{-i(\phi + \pi)}\mathbb{R}^- + w; X)\). As a direct consequence of the Sobolev embedding theorem we have \(D^j_{\phi} F(w) = 0, j = 0, \ldots, m - 1, \) for all \(F \in W^\ell_\zeta(e^{-i\phi}\mathbb{R}^\pm + w; X)\), where \(\ell > m - 1/2, m \in \mathbb{N}\). For the proof of the next proposition we refer to [29, Proposition 7.1].

**Proposition 3.13** Let \(\ell \geq 0\) and let \(W^\ell_\zeta(e^{-i\phi}\mathbb{R}^+ + w; X)\) denote the Sobolev space of functions \(e^{-i\phi}\mathbb{R}^+ + w \ni z \mapsto f(z) \in X\); we define the norm in \(W^\ell_\zeta(e^{-i\phi}\mathbb{R}^+ + w; X)\) by setting the value \(\|F; W^\ell_\zeta(e^{-i\phi}\mathbb{R}^+ + w; X)\|^2\) to be equal to the right hand side of the equality (3.17), where \(\mathbb{R}\) is replaced by \(\mathbb{R}^+\). Suppose that \(\ell \neq \lfloor \ell \rfloor + 1/2\), where \(\lfloor \ell \rfloor\) is the integer part of \(\ell\). Then a function \(F \in W^\ell_\zeta(e^{-i\phi}\mathbb{R}^+ + w; X)\) extended to the half-line \(e^{-i\phi}\mathbb{R}^- + w\) by zero falls into the space \(W^\ell_\zeta(e^{-i\phi}\mathbb{R}^+ + w; X)\) if and only if
\[
D^j_{\phi} F(w) = 0, \quad 0 \leq j \leq \lfloor \ell \rfloor - 1,
\]
\[
\int_{e^{-i\phi}\mathbb{R}^+ + w} |z - w|^{-2(\ell - \lfloor \ell \rfloor)} \|e^{-i\zeta} D^{|\ell|}_{\phi} F(z)\|^2 |dz| < +\infty, \quad \ell > \lfloor \ell \rfloor. \tag{3.42}
\]
(If \(\ell > \lfloor \ell \rfloor + 1/2\) then the condition (3.42) gives \(D^{|\ell|}_{\phi} F(w) = 0\).)

For all \(F \in W^\ell_\zeta(e^{-i\phi}\mathbb{R}^+ + w; X)\) the estimates
\[
\|F; W^\ell_\zeta(e^{-i\phi}\mathbb{R}^+ + w; X)\| \leq c_1 \|F; W^\ell_\zeta(e^{-i\phi}\mathbb{R} + w; X)\| \leq c_2 \|F; W^\ell_\zeta(e^{-i\phi}\mathbb{R}^+ + w; X)\|
\]
are valid, where the constants \(c_1\) and \(c_2\) are independent of \(F\).

In the sequel we shall need the following variant of the Paley-Wiener theorem.
Theorem 3.14 (Paley-Wiener) Let $\zeta, w \in \mathbb{C}$, $\ell \in \mathbb{R}$, and let $\phi$ be an angle. If we identify the functions $\mathcal{F}$ from the Hardy class $\mathcal{H}_w^\ell(e^{i\phi}\mathbb{C}^+ + \zeta; X)$ with their boundary limits $\mathcal{F}|_{e^{i\phi}\mathbb{R}+\zeta}$ then the Fourier-Laplace transformation (3.11) yields an isometric isomorphism

$$T_\phi^{\mathcal{F}} : \mathcal{H}_w^\ell(e^{i\phi}\mathbb{C}^+ + \zeta; X) \to \mathcal{W}_\zeta^\ell(e^{-i\phi}\mathbb{R}^- + w; X).$$

(3.43)

PROOF. An alternative proof of this theorem for the case $\ell \geq 0$ can be found in [29]. Proposition 2.9(v) allows us to identify the functions $\mathcal{F} \in \mathcal{H}_w^\ell(e^{i\phi}\mathbb{C}^+ + \zeta; X)$ with their boundary limits $\mathcal{F}_\phi \in \mathcal{W}_w^\ell(e^{i\phi}\mathbb{R} + \zeta; X)$. The mapping (3.43) is isometric due to Corollary 3.3 and the definitions (2.18), (3.41) of the norms in $\mathcal{H}_w^\ell(e^{i\phi}\mathbb{C}^+ + \zeta; X)$ and in $\mathcal{W}_\zeta^\ell(e^{i\phi}\mathbb{R}^- + \zeta; X)$. Let us prove that (3.43) is an isomorphism. Without loss of generality we can suppose that $\phi = 0$.

Epimorphism. Let us show that $T_{\zeta,w}^0 \mathcal{F} \in \mathcal{W}_\zeta^\ell(\mathbb{R}^- + w; X)$ if $\mathcal{F} \in \mathcal{H}_w^\ell(\mathbb{C}^+ + \zeta; X)$. Due to Corollary 3.3 the inclusion $T_{\zeta,w}^0 \mathcal{F} \in \mathcal{W}_\zeta^\ell(\mathbb{R} + w; X)$ holds. It remains to show that the distribution $T_{\zeta,w}^0 \mathcal{F}$ is supported on the set $(\mathbb{R}^- + w) \cup \{w\}$. We extend the function $\mathcal{F}$ to the half-plane $\mathbb{C}^- + \zeta$ by zero. Then the inclusion $\mathcal{F} \in \mathcal{H}_w^\ell(K_\zeta^-; X)$ is valid and $\mathcal{F}$ has boundary limits $\mathcal{F}_0, \mathcal{F}_\pi \in \mathcal{W}_w^\ell(\mathbb{R} + \zeta; X)$ on $\partial K_\zeta^\ell$; see Proposition 2.3. It is clear that $\mathcal{F}_0(\lambda) = 0$ for $\lambda \in \mathbb{R}^- + \zeta$ and $\mathcal{F}_\pi(\lambda) = 0$ for $\lambda \in \mathbb{R}^+ + \zeta$. Moreover, $\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_\pi$ on $\mathbb{R} + \zeta$. Therefore, $T_{\zeta,w}^0 = -T_{\zeta,w}^\pi$; this is because the transformation $T_{\zeta,w}^0$ is an extension of the transformation (3.3), where the integration runs in opposite directions for $\psi = 0$ and $\psi = \pi$, see Lemmas 3.1, 3.2. We have $T_{\zeta,w}^0 \mathcal{F} = T_{\zeta,w}^0 \mathcal{F}_0 - T_{\zeta,w}^\pi \mathcal{F}_\pi$.

Consider for example the transformation $T_{\zeta,w}^\pi/2 \mathcal{F}|_{i\mathbb{R}^- + \zeta} - 0$. Clearly, the function $F$ coincides with $T_{\zeta,w}^\pi \mathcal{F}|_{e^{i\psi}\mathbb{R} + \zeta}$ on the set $(i\mathbb{C}^- + w) \cap (e^{i\psi}\mathbb{R} + w)$ for all $\psi \in [0, \pi]$. Thus $T_{\zeta,w}^0 \mathcal{F}_0 = T_{\zeta,w}^\pi \mathcal{F}_\pi = F$ on $\mathbb{R}^+ + w$, the distribution $T_{\zeta,w}^0 \mathcal{F}_0 - T_{\zeta,w}^\pi \mathcal{F}_\pi$ is supported on the set $(\mathbb{R}^- + w) \cup \{w\}$, and $T_{\zeta,w}^0 \mathcal{F} \in \mathcal{W}_\zeta^\ell(\mathbb{R}^- + w; X)$.

Monomorphism. Here we prove that for any $F \in \mathcal{W}_\zeta^\ell(\mathbb{R}^- + w; X)$ there exists $\mathcal{F} \in \mathcal{H}_w^\ell(\mathbb{C}^+ + \zeta; X)$ such that $(T_{\zeta,w}^0)^{-1}F = \mathcal{F}|_{\mathbb{R}^+ + \zeta}$. We first consider the classical case $\ell = 0$. Since the function $F$ is supported on the set $\mathbb{R}^- + w$, the inverse Fourier-Laplace transform $(T_{\zeta,w}^0)^{-1}F$ defines an analytic function $\mathbb{C}^+ + \zeta \ni \lambda \mapsto \mathcal{F}(\lambda) \in X$ such that

$$\mathcal{F}|_{\mathbb{R}^+ + \zeta + t} = (T_{\zeta,w}^0)^{-1}e_\eta F, \quad \forall \eta \in \mathbb{C}^+, \quad (3.44)$$

where $e_\eta : z \mapsto \exp(-i\eta z)$ is the weight function. (The equality (3.44) can be easily seen from the integral representation (3.4) for $(T_{\zeta,w}^0)^{-1}F$.) The formula (3.44) together with Corollary 3.3 leads to the equalities

$$\|\mathcal{F}(\cdot + \eta); \mathcal{H}_w^0(\mathbb{R} + \zeta; X)\| = \|e_\eta F; \mathcal{W}_\zeta^0(\mathbb{R}^- + w; X)\|, \quad (3.45)$$

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It is easily seen that the uniform in $\eta \in \overline{C}$ estimate

$$\|e_\eta F; \mathcal{W}_\chi^{\theta}(\mathbb{R}^- + \omega; X)\| = C \|e_\eta F; \mathcal{W}_\chi^{\theta}(\mathbb{R}^- + \omega; X)\|. \quad (3.46)$$

is valid, where $C$ is independent of $F \in \mathcal{W}_\chi^{\theta}(\mathbb{R}^- + \omega; X)$; moreover,

$$\|e_\eta F - F; \mathcal{W}_\chi^{\theta}(\mathbb{R}^- + \omega; X)\| \to 0, \quad \eta \to \zeta, \eta \in \overline{C}. \quad (3.48)$$

The estimate (3.47) together with (3.45) proves the inclusion $\mathcal{F} \in \mathcal{H}_w^0(\mathbb{C}^+ + \zeta; X)$; see Proposition 2.11, (i). Due to (3.46) and (3.48) the inverse Fourier-Laplace transform $\mathcal{T}_\chi^{-\theta}_0 F$ and the boundary limits $\mathcal{F}|_{\mathbb{R}+\zeta}$ are coincident in $\mathcal{W}_w^0(\mathbb{R} + \omega; X)$.

Let us consider the case of a nonnegative integer $\ell$. In this case the norm (3.14) is equivalent to the norm (3.18) and $D^j_{(0, w)} F \in \mathcal{W}_\chi^{\theta}(\mathbb{R}^- + \omega; X)$, $j \leq \ell$. Thus far everything said in the previous case is applicable here. Taking into account the rule (3.13) we see that the functions $\lambda \mapsto \lambda^j F(\lambda)$, $j \leq \ell$, are in the space $\mathcal{H}_w^0(\mathbb{C}^+ + \zeta; X)$. Consequently $\mathcal{F} \in \mathcal{H}_w^0(\mathbb{C}^+ + \zeta; X)$ and $(\mathcal{T}_\chi^{-\theta}_0)^{-1} F = \mathcal{F}|_{\mathbb{R}+\zeta}$.

Let $\ell$ be a negative integer. If $F \in \mathcal{W}_\chi^{\theta}(\mathbb{R}^- + \omega; X)$ then $F$ is a distribution supported on $(\mathbb{R}^- + \omega) \cup \{w\}$ and representable in the form $F = \sum_{j \leq \ell} D^j_{(0, w)} G_j$, where $G_j \in \mathcal{W}_\chi^{\theta}(\mathbb{R}^- + \omega; X)$. Then $(\mathcal{T}_\chi^{-\theta}_0)^{-1} G_j = G_j|_{\mathbb{R}+\zeta}$ and $G_j \in \mathcal{H}_w^0(\mathbb{C}^+ + \zeta; X)$, $j \leq |\ell|$. The function $\mathcal{F}(\lambda) = \sum_{j \leq |\ell|} \lambda^j G_j(\lambda)$ is in the space $\mathcal{H}_w^0(\mathbb{C}^+ + \zeta; X)$. From the rule (3.13) it follows that $\mathcal{F}|_{\mathbb{R}+\zeta} = (\mathcal{T}_\chi^{-\theta}_0)^{-1} F$.

Now we are in position to consider the general case $\ell \in \mathbb{R}$. The embedding $\mathcal{W}_\chi^{\theta}(\mathbb{R}^- + \omega; X) \subseteq \mathcal{W}_\chi^{\theta}(\mathbb{R}^- + \omega; X)$ is fulfilled. Hence we have $(\mathcal{T}_\chi^{-\theta}_0)^{-1} F = \mathcal{F}|_{\mathbb{R}+\zeta}$, where $\mathcal{F} \in \mathcal{H}_w^0(\mathbb{C}^+ + \zeta; X)$ and $\mathcal{F} \in \mathcal{H}_w^0(\mathbb{R} + \omega; X)$. Define the function $\mathcal{G}(\lambda) = (\lambda - \zeta + i)^{-|\ell|} \mathcal{F}(\lambda)$. This function satisfies the inclusions $\mathcal{G} \in \mathcal{H}_w^{2|\ell|-\theta}(\mathbb{C}^+ + \zeta; X)$ and $\mathcal{G} \in \mathcal{W}_w^{2|\ell|}(\mathbb{R} + \omega; X)$; see the estimates (2.9). The last inclusion and Corollary 3.3 give $\mathcal{T}_\chi^{-\theta}_0(\mathcal{G}|_{\mathbb{R}+\zeta}) \in \mathcal{W}_\chi^{\theta}(\mathbb{R} + \omega; X)$. As we already know, the Fourier-Laplace transform $\mathcal{T}_\chi^{-\theta}_0(\mathcal{J}|_{\mathbb{R}+\zeta})$ of any function $\mathcal{J} \in \mathcal{H}_w^{2|\ell|-\theta}(\mathbb{C}^+ + \zeta; X)$ is supported on $(\mathbb{R}^- + \omega) \cup \{w\}$. Therefore the transform $\mathcal{G} = \mathcal{T}_\chi^{-\theta}_0(\mathcal{G}|_{\mathbb{R}+\zeta})$ is in the space $\mathcal{W}_\chi^{\theta}(\mathbb{R}^- + \omega; X)$. From the proved cases we conclude that the transformation $(\mathcal{T}_\chi^{-\theta}_0)^{-1} \mathcal{G}$ defines a function $\tilde{\mathcal{G}} \in \mathcal{H}_w^{\ell+1}(\mathbb{C}^+ + \zeta; X)$ such that $\mathcal{G}|_{\mathbb{R}+\zeta} = \tilde{\mathcal{G}}|_{\mathbb{R}+\zeta}$. By Proposition 2.9, (v) we have $\mathcal{G} \equiv \tilde{\mathcal{G}} \in \mathcal{H}_w^{\ell+1}(\mathbb{C}^+ + \zeta; X)$. Then the function $\mathcal{F}$ is in the space $\mathcal{H}_w^0(\mathbb{C}^+ + \zeta; X)$. □

**Corollary 3.15** Let $\zeta, w \in \mathbb{C}$, $\ell \in \mathbb{R}$, and let $\phi$ be an angle.

(i) The space $\mathcal{W}_\chi^{\theta}(e^{-i\phi}\mathbb{R}^+ + \omega; X)$ with the norm (3.41) is a Banach space.

(ii) For all $\eta \in e^{i\phi}\mathbb{C}^+ + \zeta$ we have $\mathcal{W}_\chi^{\theta}(e^{-i\phi}\mathbb{R}^- + \omega; X) \subseteq \mathcal{W}_\chi^{\theta}(e^{-i\phi}\mathbb{R}^- + \omega; X)$
and
\[
\| F; \mathcal{W}_\eta^\ell(e^{-i\phi} \mathbb{R}^- + w; X) \| \leq C e^{\delta \{ \eta - \zeta \} w} \| F; \mathcal{W}_\zeta^\ell(e^{-i\phi} \mathbb{R}^- + w; X) \|,
\]
where the constant \( C \) is independent of \( w, \eta \), and \( F \in \mathcal{W}_\zeta^\ell(e^{-i\phi} \mathbb{R}^- + w; X) \).

(iii) For all \( v \in e^{-i\phi} \mathbb{R}^- + w \) we have \( \mathcal{W}_\zeta^\ell(e^{-i\phi} \mathbb{R}^- + v; X) \subset \mathcal{W}_\zeta^\ell(e^{-i\phi} \mathbb{R}^- + w; X) \) and
\[
\| F; \mathcal{W}_\zeta^\ell(e^{-i\phi} \mathbb{R}^- + w; X) \| = \| F; \mathcal{W}_\zeta^\ell(e^{-i\phi} \mathbb{R}^- + v; X) \|, \quad F \in \mathcal{W}_\zeta^\ell(e^{-i\phi} \mathbb{R}^- + v; X).
\]

**PROOF.** (i) The space \( \mathcal{W}_\zeta^\ell(e^{-i\phi} \mathbb{R}^- + w; X) \) is complete because it is isometrically isomorphic to the Banach space \( \mathcal{H}_w^\ell(e^{i\phi} \mathbb{C}^+ + \zeta; X) \). (ii) For \( F \in \mathcal{W}_\zeta^\ell(e^{-i\phi} \mathbb{R}^- + w; X) \) we have the equality
\[
\mathcal{F}|_{e^{i\phi} \mathbb{R} + \eta} = e^{i(\eta - \zeta) w} (T_{\eta, w}^\phi)^{-1} F,
\]
where \( \mathcal{F} = (T_{\zeta, w}^\phi)^{-1} F \), \( \mathcal{F} \in \mathcal{H}_w^\ell(e^{i\phi} \mathbb{C}^+ + \zeta) \). For \( \ell \geq 0 \) the equality (3.49) can be easily seen from the integral representation (3.4). The rule (3.13) allows us to extend (3.49) to the case \( \ell < 0 \). Corollary 2.12,(ii) together with Corollary 3.3 and the equality (3.49) establishes the assertion. (iii) The embedding is obvious. The equality is valid because the norm of the space \( \mathcal{W}_\zeta^\ell(e^{i\phi} \mathbb{R} + v; X) \) does not change while \( v \) travels along the line \( e^{i\phi} \mathbb{R} + w \). \( \square \)

Let us note that the assertions (iii), (iv) of Proposition 2.9 furnish estimates for the transform \( \mathcal{F} = (T_{\zeta, w}^\phi)^{-1} F \) of a function \( F \in \mathcal{W}_\zeta^\ell(e^{i\phi} \mathbb{R}^- + w; X) \).

**Proposition 3.16** Let \( \ell > 1/2 \) and \( F \in \mathcal{W}_\zeta^\ell(e^{-i\phi} \mathbb{R} + w; X) \). We set \( G = F \) on \( e^{-i\phi} \mathbb{R}^- + w \) and \( G = 0 \) on \( e^{-i\phi} \mathbb{R}^- + w \). Then for any \( k < 1/2 \) the inclusion \( G \in \mathcal{W}_\zeta^k(e^{-i\phi} \mathbb{R}^- + w; X) \) and the estimate
\[
\| G; \mathcal{W}_\zeta^k(e^{-i\phi} \mathbb{R}^- + w; X) \| \leq C \| F; \mathcal{W}_\zeta^\ell(e^{-i\phi} \mathbb{R}^- + w; X) \|
\]
(3.50)
are valid. Here the constant \( C \) is independent of \( F \).

**PROOF.** For a function \( F \in \mathcal{W}_\zeta^\ell(e^{-i\phi} \mathbb{R}^- + w; X) \), \( \ell > 1/2 \), the Sobolev theorem gives \( F \in C(e^{-i\phi} \mathbb{R}^- + w; X) \). Hence the function \( e^{-i\phi} \mathbb{R}^- + w \ni z \mapsto \| F \| \) is bounded in a neighbourhood of the point \( w \), the condition (3.42) is satisfied with \( \ell \) replaced by \( k \), \( k < 1/2 \). Then from Proposition 3.13 it follows the inclusion \( G \in \mathcal{W}_\zeta^k(e^{-i\phi} \mathbb{R}^- + w; X) \) and the estimate (3.50). \( \square \)

Let us recall that the estimate (2.20) in Proposition 2.11, (iii) remains unproved in case \( s < \ell - 1/2 \), \( k \in (s, s + 1/2) \). The next corollary presents a result which enables us to finalize the proof of Proposition 2.11, (iii).

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Corollary 3.17 Let \( \ell > 1/2 \) and \( J \in \mathcal{W}_0^\ell(e^{i\phi} \mathbb{R} + \zeta; X) \). We set

\[
\mathcal{F}(\lambda) = \int_{e^{i\phi} \mathbb{R} + \zeta} (\lambda - \mu)^{-1} J(\mu) \, d\mu, \quad \lambda \in e^{i\phi} \mathbb{C} + \zeta.
\]

Then for any \( k < 1/2 \) the inclusion \( \mathcal{F} \in \mathcal{H}_0^k(e^{i\phi} \mathbb{C} + \zeta; X) \) and the estimate

\[
\| \mathcal{F}; \mathcal{H}_0^k(e^{i\phi} \mathbb{C} + \zeta; X) \| \leq c \| J; \mathcal{W}_0^\ell(e^{i\phi} \mathbb{R} + \zeta; X) \| \quad (3.51)
\]

are valid. The constant \( c \) is independent of \( J \).

**PROOF.** Let \( J \in \mathcal{W}_\zeta^\ell(e^{-i\phi} \mathbb{R}; X) \). Consider the Fourier-Laplace transform \( J = \mathbb{T}_{e^{i\phi} \zeta} J \). By Proposition 3.16 we can represent \( J \) as the sum \( F + G \), where \( F \in \mathcal{W}_\zeta^k(e^{-i\phi} \mathbb{R}; X) \) and \( G \in \mathcal{W}_\zeta^k(e^{-i\phi} \mathbb{R}^+; X) \), \( k < 1/2 \); moreover, the estimate

\[
\| F; \mathcal{W}_\zeta^k(e^{-i\phi} \mathbb{R}^-; X) \| \leq C \| J; \mathcal{W}_\zeta^k(e^{-i\phi} \mathbb{R}; X) \| \quad (3.52)
\]

holds. Then \( J = F + G \), where \( F = (\mathbb{T}_{e^{i\phi} \zeta})^{-1} F \) and \( G = (\mathbb{T}_{e^{i\phi} \zeta})^{-1} G \). As a consequence of Theorem 3.14 we have the estimate

\[
\| \mathcal{F}; \mathcal{H}_0^k(e^{i\phi} \mathbb{C} + \zeta; X) \| \leq C \| F; \mathcal{W}_\zeta^k(e^{-i\phi} \mathbb{R}^-; X) \|.
\]

This together with (3.52) establishes the estimate (3.51). We represented the function \( J \in \mathcal{H}_0^0(e^{i\phi} \mathbb{R} + \zeta; X) \) as the sum of boundary limits of functions \( \mathcal{F} \in \mathcal{H}_0^0(e^{i\phi} \mathbb{C} + \zeta; X) \) and \( \mathcal{G} \in \mathcal{H}_0^0(e^{i\phi} \mathbb{C}^- + \zeta; X) \). Hence \( \mathcal{F} \) is the Cauchy integral of \( J \). \( \Box \)

**Remark 3.18** To finalize the proof of Proposition 2.11, (iii) it suffices to apply Corollary 3.16 with the function \( J \) replaced by \( \exp\{iw\cdot\}(-\eta)^s J(\cdot) \) and \( \mathcal{F} \) replaced by \( \exp\{iw\cdot\}(-\eta)^s \mathcal{F}(\cdot) \); here \( \mathcal{F} \) and \( J \) are the same as in Proposition 2.11, (iii).

**Proposition 3.19** Let \( \zeta, w \in \mathbb{C}, \ell \in \mathbb{R} \), and let \( \phi \) be an angle.

(i) Every distribution \( F \in \mathcal{W}_\zeta^\ell(e^{-i\phi} \mathbb{R} + w; X) \) can be represented as \( F^+ + F^- \), where \( F^\pm \in \mathcal{W}_\zeta^\ell(e^{-i\phi} \mathbb{R}^\pm + w; X) \) for

\[
\begin{align*}
s < 1/2 & \quad \text{if } \ell > 1/2; \\
s \leq |\ell| & \quad \text{if } \ell \leq 1/2 \quad \text{and} \quad |\ell| \leq \ell \leq |\ell| + 1/2; \\
s < |\ell| + 1/2 & \quad \text{if } \ell \leq 1/2 \quad \text{and} \quad |\ell| + 1/2 < \ell < |\ell|.
\end{align*}
\]

If \( s < -1/2 \) then this representation is not unique due to the inclusions

\[
f \cdot (D^\ell_1 \delta) \circ \tau_{\phi,w}^{-1} \in \mathcal{W}_\zeta^s(e^{-i\phi} \mathbb{R}^+ + w; X) \cap \mathcal{W}_\zeta^s(e^{-i\phi} \mathbb{R}^- + w; X), \quad 0 \leq j \leq -[s+3/2];
\]

here \( f \) is a coefficient in \( X \), \( \delta \) denotes the Dirac delta function, and \( \tau_{\phi,w}^{-1} : e^{-i\phi} \mathbb{R} + w \to \mathbb{R} \) is the linear transformation \( \tau_{\phi,w}^{-1}(z) = e^{i\phi}(z - w) \).
(ii) For every \( F \in \mathcal{W}_\zeta^f(\mathbb{R} + w; X) \) there exists a set of distributions \( G \in H^s_%(\mathbb{K}^\pi_w; X) \) such that \( F = G_0 - G_\pi \), where \( s \) is the same as in (3.53), \( G_0 \) and \( G_\pi \) are boundary limits of \( G \) in the sense of Theorem 3.10. In the case \( s < -1/2 \) the set \( G \) is not unique. If \( G \in H^s_%(\mathbb{K}^\pi_w; X) \) and \( F = G_0 - G_\pi \) then the sets of distributions \( \tilde{G} \equiv \{ \tilde{G}_\psi : \psi \in [0, \pi] \} \) and \( G \equiv \{ G_\psi : \psi \in [0, \pi] \} \) satisfy the equality

\[
(e_\zeta \tilde{G}_\psi) \circ \tau_{\psi, w} - (e_\zeta G_\psi) \circ \tau_{\psi, w} = \sum_{j=0}^{\lfloor s+3/2 \rfloor} e^{i(j+1)\psi} f_j D^j_i \mathcal{P}^1_t \tag{3.54}
\]

for all \( \psi \in [0, \pi] \) and some coefficients \( f_j \in X \); here \( e_\zeta : z \mapsto \exp(-i\zeta z) \), the distribution \( \mathcal{P}^1_t(v), v \in \mathcal{F}(\mathbb{R}) \), is defined as the Cauchy principal value of the integral \( \int_\mathbb{R} t^{-1} v(t) \, dt \), and \( \tau_{\psi, w}(t) = e^{-i\psi t + w} \).

(iii) Let \( F \in \mathcal{W}_\zeta^f(\mathbb{R}^+ + w; X) \). Then there exists an analytic function

\[
\mathbb{C} \setminus \overline{\mathbb{R}^+ + w} \ni z \mapsto G(z) \in X
\]

such that for any \( v \in \mathbb{R}^+ + w \) the inclusion

\[
\{ G|_{e^{-i\psi \mathbb{R}^+ + w}} : \psi \in (0, \pi) \} \in H^s_%(\mathbb{K}^\pi_w; X) \tag{3.56}
\]

and the equality \( F = G_0 - G_\pi \) hold, where the distributions \( G_0 \) and \( G_\pi \) are boundary limits of the set (3.56) in the sense of Theorem 3.10. Moreover, there exists a set of distributions \( G^- \in H^s_%(\mathbb{K}^\pi_w; X) \) such that \( F = G^-_0 - G^-_\pi \) and

\[
G^-_\psi = G \text{ on the set } e^{-i\psi \mathbb{R} + w} \setminus \overline{\mathbb{R}^+ + w}, \quad \psi \in [0, \pi].
\]

A similar assertion is valid for \( F \in \mathcal{W}_\zeta^f(\mathbb{R}^+ + w; X) \).

**Proof.** (i) We represent \( \mathcal{F} = (\mathbb{T}_\zeta^\phi)^{-1} F \) in the form \( \mathcal{F} = (\mathcal{F}^+ + \mathcal{F}^-)|_{e^{i\varphi \mathbb{R}^+ + \zeta}} \), where \( \mathcal{F}^\pm \in \mathcal{H}^s_{\zeta}(e^{i\varphi \mathbb{C}^\pm + \zeta}; X) \); see Corollary 2.12, (iii). This together with Theorem 3.14 proves the assertion.

(ii) Here again we represent \( \mathcal{F} = (\mathbb{T}_\zeta^0 G)|_{\mathbb{R}^+ + \zeta} \), where \( \mathcal{F}^\pm \in \mathcal{H}^s_{\zeta}(\mathbb{C}^\pm + \zeta; X) \). Let us define \( \mathcal{G} \in \mathcal{H}^s_{\zeta}(\mathbb{K}^\pi_w; X) \) by setting \( \mathcal{G} = \mathcal{F}^+ \) on \( \mathbb{C}^+ + \zeta \) and \( \mathcal{G} = \mathcal{F}^- \) on \( \mathbb{C}^- + \zeta \). It is clear that \( (\mathcal{F}^+ + \mathcal{F}^-)|_{\mathbb{R}^+ + \zeta} = \mathcal{G}_0 + \mathcal{G}_\pi \), where \( \mathcal{G}_0 \) and \( \mathcal{G}_\pi \) are the boundary limits of \( \mathcal{G} \); see Propositions 2.3, 2.9. Let \( \mathcal{G} \in \mathcal{H}^s_{\zeta}(\mathbb{K}^\pi_w; X) \) be the set of distributions

\[
\mathcal{G} = \{ G_\psi = \mathbb{T}_\zeta^\phi(G|_{e^{i\varphi \mathbb{R}^+ + \zeta}}) : \psi \in [0, \varphi] \}. 
\]

It is clear that \( F = \mathbb{T}_\zeta^0 G^+ + \mathcal{F}^-|_{\mathbb{R}^+ + \zeta} = \mathbb{T}_\zeta^0 G_0 - \mathbb{T}_\zeta^0 G_\pi = G_0 - G_\pi. \) By Corollary 2.12, (iii) the representation \( \mathcal{F} = (\mathcal{F}^+ + \mathcal{F}^-)|_{\mathbb{R}^+ + \zeta} \) is not unique if \( \ell < -1/2 \). The relations (2.26) lead to the equality (3.54).
(iii) By Theorem 3.14, for $F = (T_{\phi}^w)^{-1} F$ we have the inclusion $F \in \mathcal{H}_w^{\ell}(\mathbb{C}^+ + \zeta; X)$. Let $G = F$ in $\mathbb{C}^+ + \zeta$ and $G = 0$ in $\mathbb{C}^- + \zeta$. It is clear that $G \in \mathcal{H}_v^{\ell}(K_\pi; X)$ for any $v \in \mathbb{R}^+ + w$ and $F|_{\mathbb{R}^+ + \zeta} = G_0 + G_\pi$, where $G_0$ and $G_\pi$ are boundary limits of $G$; see Proposition 2.3. We can define $G^- \in \mathcal{H}_\ell^0(K_{\pi}; X)$ by setting $G^- = T_{\zeta,w}^0(G|_{e^{i\varphi}\mathbb{R}^+ + w})$. Therefore $F = T_{\zeta,w}^0(G_0 + G_\pi) = G_0^- - G_\pi^-$. In the same way as in the item Epimorphism of the proof of Theorem 3.14 we see that the distributions $G_0$ and $G_\pi$ coincide on $\mathbb{R}^+ + w$ with an analytic function. Hence the set $G^- \in \mathcal{H}_\ell^0(K_{\pi}; X)$ defines an analytic function (3.55). Let us also consider the set of distributions

$$
\tilde{G} \equiv \{ e^{i\zeta(w-v)}T_{\zeta,v}^0(\mathcal{G}|_{e^{i\varphi}\mathbb{R}+\zeta}) : \psi \in [0, \pi] \} \in \mathcal{H}_\ell^0(K_{\pi}; X), \quad v \in \mathbb{R}^+ + w. \quad (3.57)
$$

By the same argument as above we conclude that the set (3.57) defines an analytic in $\mathbb{C} \setminus \mathbb{R}^+ + v$ function $\tilde{G}$. Note that $T_{\zeta,w}^0 = e^{i\zeta(w-v)}T_{\zeta,v}^0$ and $T_{\zeta,w}^\pi = e^{i\zeta(w-v)}T_{\zeta,v}^\pi$. Thus $\tilde{G}_0 = G_0$ and $\tilde{G}_\pi = G_\pi$. Consequently the set of distributions $\tilde{G}$ defines the same analytic function $\tilde{G}$ as the set $G^-$, we have the equality $G|_{e^{-i\varphi}\mathbb{R}+w} = \tilde{G}_\psi$, $\psi \in (0, \pi)$. □

The next proposition deals with compactly supported distributions, the first assertion is an integral form of the Paley-Wiener-Schwartz theorem.

**Proposition 3.20** Let $\zeta, w \in \mathbb{C}$, $\ell \in \mathbb{R}$, and let $\varphi$ be an angle. Assume that $v \in e^{i\varphi}\mathbb{R}^+ + w$.

(i) *(Paley-Wiener-Schwartz)* Let $F \in \mathcal{D}_\ell^0(e^{-i\varphi}\mathbb{R} + w; X)$. The inclusion

$$
F \in \mathcal{W}_0^0(e^{-i\varphi}\mathbb{R}^+ + w; X) \cap \mathcal{W}_0^0(e^{-i\varphi}\mathbb{R}^- + v; X) \quad (3.58)
$$

holds if and only if the inverse Fourier-Laplace transform $(T_{\varphi}^w)^{-1} F$ is an entire function $\mathbb{C} \ni \lambda \mapsto \mathcal{E}(\lambda) \in X$ satisfying the uniform in $\psi \in (0, \pi)$ estimates

$$
\| \mathcal{E}; \mathcal{W}_\ell^0(e^{i(\varphi+\psi)}\mathbb{R}^- + \zeta; X) \| \leq C; \quad \| \mathcal{E}; \mathcal{W}_\ell^0(e^{i(\varphi+\psi)}\mathbb{R}^+ + \zeta; X) \| \leq C. \quad (3.59)
$$

(ii) For any $F \in \mathcal{W}_0^0(\mathbb{R}^+ + w; X) \cap \mathcal{W}_0^0(\mathbb{R}^- + v; X)$ there exists a unique analytic function

$$
\mathbb{C} \setminus (\mathbb{R}^+ + w \cap \mathbb{R}^- + v) \ni z \mapsto G(z) \in X
$$

such that the following inclusion and equalities hold

$$
\{ G|_{e^{-i\varphi}\mathbb{R}+w} : \psi \in (0, \pi) \} \in \mathcal{H}_0^0(K_u^\pi; X), \quad u \in (\mathbb{R}^- + w) \cup (\mathbb{R}^+ + v); \quad (3.60)
$$

$$
F = G_\pi - G_0 \quad \text{if} \quad u \in \mathbb{R}^- + w; \quad F = G_0 - G_\pi \quad \text{if} \quad u \in \mathbb{R}^+ + v. \quad (3.61)
$$

Here the distributions $G_0$ and $G_\pi$ are the boundary limits of the set (3.60) in the sense of Theorem 3.10.
(iii) Let \( F \in \mathcal{W}_0^\ell(\mathbb{R}^+ + w; X) \cap \mathcal{W}_0^\ell(\mathbb{R}^- + v; X) \) and let \( \mathcal{E} \) be the corresponding entire function from the assertion (i). We set \( \mathcal{E}^\pm = \mathcal{E} \) on \( \mathbb{C}^\pm \) and \( \mathcal{E}^\pm = 0 \) on \( \mathbb{C}^\mp \). Then the sets of distributions

\[
\begin{align*}
G^- & \equiv \{ G^-_\psi = -\mathcal{T}_{0,w}^\psi(\mathcal{E}^-|_{e^{i\psi}\mathbb{R}}; \psi \in [0, \pi]) \} \in H_0^\ell(K_0^\pi; X), \\
G^+ & \equiv \{ G^+_\psi = \mathcal{T}_{0,v}^\psi(\mathcal{E}^+|_{e^{i\psi}\mathbb{R}}; \psi \in [0, \pi]) \} \in H_0^\ell(K_0^\pi; X)
\end{align*}
\]  

(3.62)

satisfy the conditions \( G^-_0 - G^-_\pi = -F \), \( G^+_0 - G^+_\pi = F \), and meet the relations

\[
\begin{align*}
G^-_\psi &= G \text{ on the set } e^{-i\psi}\mathbb{R} + w \setminus \overline{(\mathbb{R}^+ + w \cap \mathbb{R}^- + v)}, \psi \in [0, \pi], \\
G^+_\psi &= G \text{ on the set } e^{-i\psi}\mathbb{R} + v \setminus \overline{(\mathbb{R}^+ + w \cap \mathbb{R}^- + v)}, \psi \in [0, \pi],
\end{align*}
\]

where \( G \) denotes the analytic function from the assertion (ii).

**Proof.** (i) **Necessity.** If the inclusion (3.58) is valid then

\[
\text{supp } F \subseteq \mathbb{R}^+ + w \cap \mathbb{R}^- + v.
\]

Therefore we have \( F \in \mathcal{W}_0^\ell(e^{-i\varphi}\mathbb{R}^+ + w; X) \cap \mathcal{W}_0^\ell(e^{-i\varphi}\mathbb{R}^- + v; X) \) for any \( \varphi \in \mathbb{C} \). By Theorem 3.14 the transform \( (\mathcal{T}_{\varphi,w}^\psi)^{-1} F \) is the boundary limit of a function \( \mathcal{F}^+ \in \mathcal{H}_w^\ell(e^{i\varphi}\mathbb{C}^+ + \varphi; X) \), and \( (\mathcal{T}_{\varphi,v}^\psi)^{-1} F \) is the boundary limit of \( \mathcal{F}^- \in \mathcal{H}_w^\ell(e^{i\varphi}\mathbb{C}^- + \varphi; X) \). Since the equality \( (\mathcal{T}_{\varphi,w}^\psi)^{-1} F = e^{i\varphi(v-w)}(\mathcal{T}_{\varphi,w}^\psi)^{-1} 1 F \) holds, and \( \mathcal{F}^+ |_{\mathbb{R}^+ + \varphi} = e^{i(\eta - \varphi)\psi}(\mathcal{T}_{\eta,v}^\psi)^{-1} 1 F \) for \( \eta \in e^{i\varphi}\mathbb{C}^+ + \varphi \) (cf. (3.49)), we conclude that \( (\mathcal{T}_{\varphi,w}^\psi)^{-1} F \) defines an analytic in \( \mathbb{C} \) function \( \mathcal{E} \in \mathcal{H}_w^\ell(e^{i\varphi}\mathbb{C}^- + \varphi; X) \cap \mathcal{H}_w^\ell(e^{i\varphi}\mathbb{C}^+ + \varphi; X) \) such that \( \mathcal{E} = e^{i\varphi(v-w)}\mathcal{F}^+ \) on \( e^{i\varphi}\mathbb{C}^+ + \varphi \) and \( \mathcal{E} = \mathcal{F}^- \) on \( e^{i\varphi}\mathbb{C}^- + \varphi \). The estimates (3.59) are fulfilled due to Definition 2.8.

**Sufficiency.** If an analytic in \( \mathbb{C} \) function \( \mathcal{E} \) satisfies the estimates (3.59) then by Definition 2.8 we have \( \mathcal{E} \in \mathcal{H}_0^\ell(e^{i\varphi}\mathbb{C}^- + \varphi; X) \cap \mathcal{H}_w^\ell(e^{i\varphi}\mathbb{C}^+ + \varphi; X) \). For \( F = \mathcal{T}_{\varphi,w}^\psi \mathcal{E} \) Theorem 3.14 gives \( F \in \mathcal{W}_0^\ell(e^{-i\varphi}\mathbb{R}^+ + w; X) \cap \mathcal{W}_0^\ell(e^{-i\varphi}\mathbb{R}^- + v; X) \). Hence we have (3.63) and the inclusion (3.58) holds.

(ii, iii) Let \( \mathcal{E} \in \mathcal{H}_w^\ell(\mathbb{C}^-; X) \cap \mathcal{H}_w^\ell(\mathbb{C}^+; X) \) be the entire function from the assertion (i). We define the functions \( \mathcal{E}^+ \in \mathcal{H}_w^\ell(K_0^\pi; X) \) and \( \mathcal{E}^- \in \mathcal{H}_w^\ell(K_0^\pi; X) \) by the equalities \( \mathcal{E}^\pm = \mathcal{E} \) on \( \mathbb{C}^\pm \) and \( \mathcal{E}^\pm = 0 \) on \( \mathbb{C}^\mp \). It is clear that \( \mathcal{E} |_{\mathbb{R}^+} = \mathcal{E}_0^+ + \mathcal{E}_\pi^+ \) and \( \mathcal{E} |_{\mathbb{R}^-} = \mathcal{E}_0^- + \mathcal{E}_\pi^- \); here \( \mathcal{E}^\pm_0 \) and \( \mathcal{E}^\pm_\pi \) are boundary limits of \( \mathcal{E}^\pm \) in the sense of Proposition 2.3. Introduce the sets of distributions (3.62). Repeating the argument from the item *Epimorphism* in the proof of Theorem 3.14 we conclude that the distributions \( G^+_0 \) and \( G^+_\pi \) are equal to an analytic function on \( \mathbb{R}^+ + v \), the distributions \( G^-_0 \) and \( G^-_\pi \) are equal to an analytic function on \( \mathbb{R}^- + w \). Thus the set of distributions \( G^- \in H_0^\ell(K_0^\pi; X) \) defines an analytic in \( \mathbb{C} \setminus \mathbb{R}^+ + w \) function \( G^- \), and \( G^+ \in H_0^\ell(K_0^\pi; X) \) defines an analytic in \( \mathbb{C} \setminus \mathbb{R}^- + v \) function \( G^+ \). Then for any \( u \in \mathbb{R} + w \) we have \( F = \mathcal{T}_{0,u}^\psi(\mathcal{E} |_{\mathbb{R}}) = G^-_a - G^+_a = G^-_0 - G^+_0 \). It is clear that \( G^+_0 = G^+_\pi \). Therefore the analytic functions \( G^+ \) and
$G^-$ are coincident on $(\mathbb{R}^- + w) \cup (\mathbb{R}^+ + v)$. Hence we can define the needed analytic function $G$ as $G = G^+ = G^-$. Indeed, the inclusion $\mathcal{E}^- \in \mathcal{H}_u^f(\mathcal{K}_0^\pi; X)$, where $u \in \mathbb{R}^- + w$, and the inclusion $\mathcal{E}^+ \in \mathcal{H}_u^f(\mathcal{K}_0^\pi; X)$, where $u \in \mathbb{R}^+ + v$, allow us to see the relations

$$
\{ G |_{e^{-i\psi \mathbb{R}^+ + w}} = -\mathcal{T}_{0,u}^\psi(\mathcal{E}^- |_{e^{-i\psi \mathbb{R}^-}}) : \psi \in (0, \pi) \} \in \mathcal{H}_0^f(\mathcal{K}_0^\pi; X), \quad u \in \mathbb{R}^- + w;
$$

$$
\{ G |_{e^{-i\psi \mathbb{R}^- + w}} = \mathcal{T}_{0,u}^\psi(\mathcal{E}^+ |_{e^{-i\psi \mathbb{R}^+}}) : \psi \in (0, \pi) \} \in \mathcal{H}_0^f(\mathcal{K}_0^\pi; X), \quad u \in \mathbb{R}^+ + v.
$$

This proves the inclusions (3.60). The equalities $F = -\mathcal{T}_{0,u}^0(\mathcal{E}_0^- + \mathcal{E}_0^-)$ and $F = \mathcal{T}_{0,u}^0(\mathcal{E}_0^+ + \mathcal{E}_0^+)$ lead to (3.61). The function $G$ is unique due to Proposition 3.19, (i).

**Example 3.21** Consider the Dirac delta function $\delta \in \mathcal{W}_0^1(\mathbb{R}; \mathbb{C})$, $\ell < -1/2$, as an example of the distribution $F$ in Proposition 3.20, (ii) and (iii). Using the same notations as in Proposition 3.20 we have $\mathcal{E}(\lambda) = (2\pi)^{-1/2}$ for all $\lambda \in \mathbb{C}$. Then the analytic in $\mathbb{C} \setminus \{0\}$ function is $G(z) = -i/(2\pi z)$, the boundary limits $G_0$ and $G_\pi$ of the set (3.60) are given by the formulas

$$
G_0(t) = -\frac{1}{2} \delta(t) - \frac{i}{2\pi} \mathcal{P}_{\theta} \frac{1}{t} \quad \text{and} \quad G_\pi(t) = \frac{1}{2} \delta(t) - \frac{i}{2\pi} \mathcal{P}_{\theta} \frac{1}{t} \quad \text{if} \quad u \in \mathbb{R}^-;
$$

$$
G_0(t) = \frac{1}{2} \delta(t) - \frac{i}{2\pi} \mathcal{P}_{\theta} \frac{1}{t} \quad \text{and} \quad G_\pi(t) = -\frac{1}{2} \delta(t) - \frac{i}{2\pi} \mathcal{P}_{\theta} \frac{1}{t} \quad \text{if} \quad u \in \mathbb{R}^+.
$$

Here the distribution $\mathcal{P}_{\theta} / t$ is defined as the Cauchy principal value of the integral $\int_{\mathbb{R}} t^{-1} v(t) \, dt$. The distributions $G_0, G_\pi \in \mathcal{H}_0^{-1}(\mathcal{K}_0; \mathbb{C})$ are such that $G_0 \circ \tau_{\psi,0}(t) = e^{i\psi} \left( \frac{1}{2} \delta(t) - \frac{i}{2\pi} \mathcal{P}_{\theta} \frac{1}{t} \right)$ and $G_\pi \circ \tau_{\psi,0}(t) = e^{i\psi} \left( -\frac{1}{2} \delta(t) - \frac{i}{2\pi} \mathcal{P}_{\theta} \frac{1}{t} \right)$, where $\tau_{\psi,0}$ is the linear transformation $\tau_{\psi,0}(t) = e^{-i\psi t}$, cf. (3.11).

Let $\ell \in \mathbb{R}$ and $s \leq \ell$. For $v \in \mathbb{C}$ and $\eta \in e^{i\phi} \mathcal{C}^+ + \zeta$ we introduce the operator

$$
P_{\eta,v}(\phi, \zeta) = \mathcal{T}_{\phi}^\psi \mathcal{P}_{\eta,v}(\phi, \zeta)(\mathcal{T}_{\phi}^\psi)^{-1} : \mathcal{W}_0^f(e^{-i\phi \mathbb{R}^+} + v; X) \rightarrow \mathcal{W}_0^f(e^{-i\phi \mathbb{R}^+} + v; X)
$$

(3.64)

where $k = s$ if $\ell - s \leq 1/2$ and $k \in [s, s + 1/2]$ if $\ell - s > 1/2$. Recall that $\mathcal{P}_{\eta,v}(\phi, \zeta)$ denotes the projection operator (2.27), (2.28). We shall omit the parameters $\phi$ and $\zeta$ in the notations of the operators $P_{\eta,v}(\phi, \zeta)$ when it can be done without ambiguity. From (2.27) it is clearly seen that the operator $P_{\eta,v}$ does not depend on the parameter $\eta$. For any $F \in \mathcal{W}_0^f(e^{-i\phi \mathbb{R}^+} + v; X)$ we have $P_{\eta,v}F = F$ on $e^{-i\phi \mathbb{R}^+} + v$ and $P_{\eta,v}F = 0$ on $e^{-i\phi \mathbb{R}^-} + v$. In the case of an integer nonpositive $s$, any distribution $F \in \mathcal{W}_0^f(e^{-i\phi \mathbb{R}^+} + v; X)$, $\ell \geq s$, can be uniquely represented in the form $F = (D_0(\phi) - \eta)^{-s}G$ with some $G \in \mathcal{W}_0^f(e^{-i\phi \mathbb{R}^+} + v; X)$, then $P_{\eta,v}F = (D_0(\phi) - \eta)^{-s}P_{\eta,v}G$. Theorem 3.14 and Corollary 3.3 allows us to pass from (2.29) to the estimate

$$
\| P_{\eta,v}F; \mathcal{W}_0^f(e^{-i\phi \mathbb{R}^+} + v; X) \| \leq C \| F; \mathcal{W}_0^f(e^{-i\phi \mathbb{R}^+} + v; X) \|,
$$

(3.65)
where \( C \) is independent of \( v \in \mathbb{C} \) and \( F \in \mathcal{W}_\zeta(e^{-i\varphi}\mathbb{R} + v; X) \). In other words, the norm of the operator (3.64) is bounded uniformly in \( v \in \mathbb{C} \). Since the space \( \mathcal{W}_\zeta(e^{-i\varphi}\mathbb{R} + w; X) \) and its norm are independent of \( w \in e^{-i\varphi}\mathbb{R} + v \) the norm of the operator

\[
P_{s,v}^\ell : \mathcal{W}_\zeta(e^{-i\varphi}\mathbb{R} + w; X) \rightarrow \mathcal{W}_\zeta(e^{-i\varphi}\mathbb{R}^+ + v; X)
\]

is bounded uniformly in \( v \in \mathbb{C} \) and \( w \in e^{-i\varphi}\mathbb{R} + v \). As a consequence of the property (2.30) we get

\[
P_{s,v}^r P_{s,v}^s = P_{s,v}^s, \quad r \leq s, \quad v \in e^{-i\varphi}\mathbb{R}^+ + w.
\]

(3.66)

If \( s \) is an integer nonpositive number then also the norm of the operator

\[
(1 - P_{s,v}^s) : \mathcal{W}_\zeta(e^{-i\varphi}\mathbb{R} + v; X) \rightarrow \mathcal{W}_\zeta(e^{-i\varphi}\mathbb{R}^- + v; X), \quad \ell \geq s,
\]

(3.67)

is bounded uniformly in \( v \), cf. (2.31); here \( \mathcal{W}_\zeta(e^{-i\varphi}\mathbb{R} + v; X) \rightarrow \mathcal{W}_\zeta(e^{-i\varphi}\mathbb{R} + v; X) \) is the continuous embedding operator.

**Proposition 3.22** Let \( \zeta, w \in \mathbb{C}, \ell \in \mathbb{R}, \) and \( 0 < \varphi < \pi \). Assume that \( F \equiv \{F_\psi : \psi \in [0, \varphi]\} \) is a set of distributions in \( \mathcal{H}_\zeta^k(K_\psi^w; X) \). Let \( v \in e^{-i\varphi}\mathbb{R}^+ + w \) for some \( \phi \in [0, \varphi] \), and let \( G_\phi = P_{s,v}^s F_\phi \), where \( s \leq \ell, \eta \notin (e^{i\varphi}\mathbb{C}^- + \zeta) \cup K_\psi^w \), and \( P_{s,v}^s \) is the projection operator (3.64). Then the distribution \( G_\phi \in \mathcal{W}_\zeta^k(e^{-i\varphi}\mathbb{R} + w; X) \) can be uniquely extended to a set of distributions \( G \equiv \{G_\psi : \psi \in [0, \varphi]\} \in \mathcal{H}_\zeta^k(K_\psi^w; X) \); here \( k = s \) if \( \ell - s \leq 1/2 \) and \( k \in [s, s + 1/2] \) if \( \ell - s > 1/2 \). The set \( G \) satisfies the estimate

\[
\|G; \mathcal{H}_\zeta^k(K_\psi^w; X)\| \leq C \|F; \mathcal{H}_\zeta^k(K_\psi^w; X)\|,
\]

(3.68)

where the constant \( C \) is independent of \( w, \phi, \) and \( v \). Moreover, in the case of an integer nonpositive \( s \), the analytic in \( K_\psi^w \) function \( F \) defined by the set \( \{F_\psi : \psi \in [0, \varphi]\} \) and the analytic in \( K_\psi^w \) function \( G \) defined by the set \( \{G_\psi : \psi \in [0, \varphi]\} \) are coincident on \( K_\psi^{w,+} \).

**PROOF.** The existence of the set \( G \in \mathcal{H}_\zeta^k(K_\psi^w; X) \) and the estimate (3.68) are consequences of Proposition 2.17 and the definition (3.64) of the projection operator \( P_{s,v}^s \). The set \( G \) is unique by Theorem 3.10.(vi).

Let \( s = 0 \). Then the analytic in \( K_\psi^w \) function \( F \) defined by the set \( \{F_\psi : \psi \in [0, \varphi]\} \) and the analytic in \( K_\psi^w \) function \( G \) defined by the set \( \{G_\psi : \psi \in [0, \varphi]\} \) are coincident on \( K_\psi^{w,+} \). Indeed, this assertion is trivial if \( \phi \in (0, \varphi) \).

For the remaining cases we note that the assertion is nearly the same as the assertion (ii) of Corollary 2.16, it can be established in a similar way. In the case of a negative integer \( s \) every distribution from the set \( \{F_\psi : \psi \in (0, \varphi)\} \in \mathcal{H}_\zeta^k(K_\psi^w; X) \) can be represented in the form \( F_\psi = (D_\psi - \eta)^{-s} \tilde{F}_\psi \), where \( \{\tilde{F}_\psi : \psi \in (0, \varphi)\} \in \mathcal{H}_\zeta^k(K_\psi^w; X) \), and the parameter \( \eta \) is outside of the
Let $\phi \in [0, \varphi]$ be a fixed angle. Theorem 3.10, (vi) allows us to identify every element \{\(F_\psi : \psi \in [0, \varphi]\)\} of the space \(H_\xi^{k}(K_\nu^\varphi; X)\) with \(F_\phi \in W_\xi^{k}(e^{-i\varphi}X + w; X)\). Then \(H_\xi^{k}(K_\nu^\varphi; X)\) is dense in \(W_\xi^{k}(e^{-i\varphi}X + w; X)\) by Theorem 3.10, (v). All operators defined on \(W_\xi^{k}(e^{-i\varphi}X + w; X)\) can be restricted to \(H_\xi^{k}(K_\nu^\varphi; X)\). In particular, the projection operator \(P_{\eta, \nu}^* : H_\xi^{k}(K_\nu^\varphi; X) \rightarrow W_\xi^{k}(e^{-i\varphi}X + v; X)\) is well-defined for \(v \in e^{-i\varphi}X + w; \text{ cf. (3.64)}. If \(\eta \notin (e^{i\varphi}X + \zeta) \cup K_\xi^\varphi\) then by Proposition 3.22 the image of this operator is in the space \(H_\xi^{k}(K_\nu^\varphi; X) \subset W_\xi^{k}(e^{-i\varphi}X + v; X)\). Moreover, by the same proposition the projection operator

\[
P_{\eta, \nu}^* : H_\xi^{k}(K_\nu^\varphi; X) \rightarrow H_\xi^{k}(K_\nu^\varphi; X), \quad \eta \notin (e^{i\varphi}X + \zeta) \cup K_\xi^\varphi, \tag{3.69}
\]

satisfies the estimate

\[
\|P_{\eta, \nu}^* F; H_\xi^{k}(K_\nu^\varphi; X)\| \leq C \|F; H_\xi^{k}(K_\nu^\varphi; X)\|, \tag{3.70}
\]

where the constant \(C\) is independent of \(w, \phi,\) and \(v \in e^{-i\varphi}X + w\). In the same manner we can consider the differential operator

\[
D_{(\phi)}^j : W_\xi^{k}(e^{-i\varphi}X + w; X) \rightarrow W_\xi^{k-j}(e^{-i\varphi}X + w; X), \quad j \in \mathbb{N}, \tag{3.71}
\]

on the subspace \(H_\xi^{k}(K_\nu^\varphi; X)\) of \(W_\xi^{k}(e^{-i\varphi}X + w; X)\); here \(D_{(\phi)}^j\) is the same as in (3.8). Indeed, by Proposition 2.5, (iii) the operator of multiplication

\[
H_\nu^\phi(K_\nu^\varphi; X) \ni \phi \rightarrow (\cdot)^j \phi(\cdot) \in H_\nu^{k-j}(K_\nu^\varphi; X), \quad j \in \mathbb{N}
\]

is bounded uniformly in \(w \in \mathbb{C}\). Hence the norm of the mapping

\[
H_\nu^\phi(K_\nu^\varphi; X) \ni \{F_\psi : \psi \in [0, \varphi]\} \mapsto \{D_{(\psi)}^j F_\psi : \psi \in [0, \varphi]\} \in H_\nu^{k-j}(K_\nu^\varphi; X) \tag{3.72}
\]

is also bounded uniformly in \(w \in \mathbb{C}\); see the differentiation rule (3.13) and Theorem 3.10, (iv). Therefore the differential operator (3.71) maps the subspace \(H_\nu^\phi(K_\nu^\varphi; X)\) of \(W_\xi^{k}(e^{-i\varphi}X + w; X)\) to the subspace \(H_\nu^{k-j}(K_\nu^\varphi; X) \subset W_\xi^{k-j}(e^{-i\varphi}X + w; X)\), the operator

\[
D_{(\phi)}^j : H_\nu^\phi(K_\nu^\varphi; X) \rightarrow H_\nu^{k-j}(K_\nu^\varphi; X), \quad j \in \mathbb{N}, \quad \phi \in [0, \varphi], \tag{3.73}
\]

satisfies the uniform in \(w \in \mathbb{C}\) and \(\phi \in [0, \varphi]\) estimate

\[
\|D_{(\phi)}^j F; H_\nu^{k-j}(K_\nu^\varphi; X)\| \leq C \|F; H_\nu^\phi(K_\nu^\varphi; X)\|. \tag{3.74}
\]
4 Complex scaling of differential equations with unbounded operator coefficients

In this section we consider linear ordinary differential equations in spaces of analytic functions. The primary purpose here is to motivate the study made in the previous sections and to demonstrate the main ideas on the treatment of the complex scaling in terms of the Hardy-Sobolev spaces and the Fredholm polynomial operator pencils. For this reason we leave aside the important question of complex scaling in presence of operator pencil eigenvalues in the dual cone $K_{\zeta}^e$, this aspect will be detailed elsewhere. In Subsection 4.1 we introduce the complex scaling method for differential equations with constant operator coefficients. In Subsection 4.2 we consider equations with variable coefficients; here, for the sake of simplicity, we restrict ourselves by the case of spaces of positive integer orders. Two examples of applications to the complex scaling of boundary value problems were presented in Subsection 1.2. Examples of applications of differential equations with operator coefficients to boundary value problems can be found e.g. in [35,36], some of these examples can also be considered in context of the Hardy-Sobolev spaces.

4.1 Differential equations with constant coefficients

We shall use the same notations as in Section 2.4. Here again we suppose that the operator $\mathfrak{A}(\lambda)$ is Fredholm for all $\lambda \in \mathbb{C}$ and is invertible for at least one value of $\lambda$. We also assume that the condition (2.45) is satisfied for some $\vartheta \in (0, \pi/2)$ and $R > 0$. By $\mathfrak{A}(D_{(\phi)})$ we shall denote the differential operator $\sum_{j=0}^{m} A_j D_{(\phi)}^{m-j}$ on the line $e^{-i\phi} \mathbb{R} + w$; recall that for all $j = 0, \ldots, m - 1$ we have $A_j \in \mathcal{B}(X_j, X_0)$, $\|u\|_j \leq \|u\|_{j+1}$, and the Hilbert space $X_{j+1}$ is dense in $X_j$.

For $\ell \in \mathbb{R}$ and $w, \zeta \in \mathbb{C}$ we introduce the Banach space

$$ D_\zeta^\ell(e^{-i\phi} \mathbb{R} + w) = \bigcap_{j=0}^{m} W_\zeta^{\ell-j}(e^{-i\phi} \mathbb{R} + w; X_j); $$

$$ \|u; D_\zeta^\ell(e^{-i\phi} \mathbb{R} + w)\| = \sum_{j=0}^{m} \|u; W_\zeta^{\ell-j}(e^{-i\phi} \mathbb{R} + w; X_j)\|. $$

The operator

$$ \mathfrak{A}(D_{(\phi)}): D_\zeta^\ell(e^{-i\phi} \mathbb{R} + w) \to W_\zeta^{\ell-m}(e^{-i\phi} \mathbb{R} + w; X_0) \quad (4.1) $$

is bounded for any $\zeta, w \in \mathbb{C}$, $\ell \in \mathbb{R}$, and $\phi \in [-\pi, \pi)$. The next theorem is in essence a variant of Theorem 2.4.1 in [35].

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Theorem 4.1 Suppose that the operator pencil $\lambda \mapsto \mathfrak{A}(\lambda)$ satisfies the condition (2.45) for some $\vartheta \in (0, \pi/2)$ and $R > 0$. Let the line $e^{i\vartheta}R + \zeta$, $|\phi| < \vartheta$, be free from the spectrum of the operator pencil $\mathbb{C} \ni \lambda \mapsto \mathfrak{A}(\lambda)$. Then for all $\ell \in \mathbb{R}$ and $w \in \mathbb{C}$ the operator (4.1) yields an isomorphism and the estimates

$$
\|u; D_\xi^\ell(e^{-i\vartheta}R + w)\| \leq c_1 \|\mathfrak{A}(D_{(\phi)}u; H_\xi^{\ell-m}(e^{-i\vartheta}R + w; X_0)\| \\
\leq c_2 \|u; D_\xi^\ell(e^{-i\vartheta}R + w)\|
$$

hold. The constants $c_1, c_2$ does not depend on $w \in \mathbb{C}$ and $u \in D_\xi^\ell(e^{-i\vartheta}R + w)$.

PROOF. It is easily seen that the Fourier-Laplace transformation implements an isometric isomorphism $T_{\zeta,w}^\phi: D_\xi^\ell(e^{i\vartheta}R + \zeta) \to D_\xi^\ell(e^{-i\vartheta}R + w)$; see Corollary 3.3 and the definition (2.47) of the space $D_\xi^\ell(e^{i\vartheta}R + \zeta)$. Now the assertion follows from Theorem 2.20,(iii) and the differentiation rule (3.13). \(\square\)

Let us introduce the scale of Banach spaces

$$D_\xi^\ell(K_w^\varphi) = \bigcap_{j=0}^m H_\xi^{\ell-j}(K_w^\varphi; X_j); \|u; D_\xi^\ell(K_w^\varphi)\| = \sum_{j=0}^m \|u; H_\xi^{\ell-j}(K_w^\varphi; X_j)\|. \quad (4.2)$$

From Theorem 3.10 and the definitions (2.46), (4.2) of the spaces $D_{(\phi)}^\ell(K_w^\varphi)$ and $D_{\phi}^\ell(K_w^\varphi)$ we see that the Fourier-Laplace transformation $T_{\zeta,w}^\psi$ yields an isometric isomorphism between $D_{(\phi)}^\ell(K_w^\varphi)$ and $D_{\phi}^\ell(K_w^\varphi)$. The next proposition is a consequence of Proposition 2.19.

Proposition 4.2 Let $\zeta, w \in \mathbb{C}$, $\ell \in \mathbb{R}$, and $\varphi \in (0, \pi]$. The following assertions are valid.

(i) Every function $u \in D_\xi^\ell(K_w^\varphi)$ has boundary limits $u_0 \in D_\xi^\ell(R + w)$ and $u_\varphi \in D_\xi^\ell(e^{-i\varphi}R + w)$ in the sense that

$$\|(e_\zeta u) \circ \tau_{\psi,w} - (e_\zeta u_0) \circ \tau_{0,w}; D_0^\ell(\mathbb{R})\| \to 0, \quad \psi \to 0^+, \quad \|(e_\zeta u) \circ \tau_{\psi,w} - (e_\zeta u_\varphi) \circ \tau_{\psi,w}; D_0^\ell(\mathbb{R})\| \to 0, \quad \psi \to \varphi^-;$$

recall that $e_\zeta: z \mapsto \exp(-i\zeta z)$ and $\tau_{\psi,w}(t) = e^{-i\varphi}t + w$.

(ii) For all $\psi \in [0, \varphi]$ and $u \in D_\xi^\ell(K_w^\varphi)$ the estimate

$$\|u; D_\xi^\ell(e^{-i\psi}R + w)\| \leq C\|u; D_\xi^\ell(K_w^\varphi)\|$$

holds, where the constant $C$ is independent of $u$, $\psi$, and $w$.

By analogy with the case of the spaces $H_\xi^\ell(K_w^\varphi; X)$ and $W_\xi^\ell(e^{-i\varphi}R + w; X)$ we can identify a set of distributions $\{u_\psi: \psi \in [0, \varphi]\} \in D_\xi^\ell(K_w^\varphi)$ with the
correspondent element $u_\phi$ of the space $\mathcal{D}_\zeta^\ell(e^{-i\phi}\mathbb{R} + w)$, $\phi \in [0, \varphi]$; the argument is the same, see the explanation to the formulas (3.69), (3.73). Then we can interpret $\mathcal{D}_\zeta^\ell(K_w^\varphi)$ as a dense subspace of $\mathcal{D}_\zeta^\ell(e^{-i\phi}\mathbb{R} + w)$ and restrict the operator (4.1) to $\mathcal{D}_\zeta^\ell(K_w^\varphi)$. Since the operator $D_\zeta^\ell(\phi)$ yields the continuous mapping (3.73) the operator $\mathfrak{A}(D(\phi))$ continuously maps $\mathcal{D}_\zeta^\ell(K_w^\varphi)$ to the space $\mathcal{H}_\zeta^{\ell-m}(K_w^\varphi; X_0) \subset \mathcal{W}_\zeta^{\ell-m}(e^{-i\phi}\mathbb{R} + w; X_0)$.

Theorem 1.1 cited in the introductory part is a consequence of the following

**Theorem 4.3** Suppose that the operator pencil $\mathfrak{A}$ meets the condition (2.45) for some $\vartheta \in (0, \pi/2)$ and $R > 0$. Let $\varphi \in (0, \vartheta)$ and $\zeta \in \mathbb{C}$. If the closed cone $K_\zeta^\varphi$ is free from the spectrum of the operator pencil $\mathbb{C} \ni \lambda \mapsto \mathfrak{A}(\lambda)$ then the operator

$$
\mathfrak{A}(D(\phi)) : \mathcal{D}_\zeta^\ell(K_w^\varphi) \rightarrow \mathcal{H}_\zeta^{\ell-m}(K_w^\varphi; X_0), \quad \phi \in [0, \varphi],
$$

(4.3)

yields an isomorphism, and the estimates

$$
\|u; \mathcal{D}_\zeta^\ell(K_w^\varphi)\| \leq c_1\|\mathfrak{A}(D(\phi))u; \mathcal{H}_\zeta^{\ell-m}(K_w^\varphi; X_0)\| \leq c_2\|u; \mathcal{D}_\zeta^\ell(K_w^\varphi)\|
$$

are valid, where the constants $c_1$, $c_2$ are independent of $w \in \mathbb{C}$, $\phi \in [0, \varphi]$, and $u \in \mathcal{D}_\zeta^\ell(K_w^\varphi)$.

**PROOF.** The assertion directly follows from Theorem 2.20, (i) and (ii) because the transformation $\mathcal{T}_{\zeta,w}^\varphi : \mathcal{D}_\zeta^\ell(K_w^\varphi) \rightarrow \mathcal{D}_\zeta^\ell(K_w^\varphi)$ yields an isometric isomorphism. □

**Corollary 4.4** Let the assumptions of Theorem 4.3 be fulfilled and let $\phi \in [0, \varphi]$. If $u \in \mathcal{D}_\zeta^\ell(e^{-i\phi}\mathbb{R} + w)$ satisfies the equation $\mathfrak{A}(D(\phi))u = F$, and the right hand side $F$ is in the subspace $\mathcal{H}_\zeta^{\ell-m}(K_w^\varphi; X_0)$ of the space $\mathcal{W}_\zeta^{\ell-m}(e^{-i\phi}\mathbb{R} + w; X_0)$, then the solution $u$ is in the subspace $\mathcal{D}_\zeta^\ell(K_w^\varphi)$ of the space $\mathcal{D}_\zeta^\ell(e^{-i\phi}\mathbb{R} + w)$.

For the sake of simplicity we restrict ourselves in the next theorem to the case of an integer $\ell$, $\ell \geq m$. Moreover, we make an additional assumption on the regularity of the right hand side $F$ in a neighbourhood of the point $t = 0$. This allows us “to localize” the problem to the right half-line preserving the analyticity and without recourse to the spaces of negative orders.

**Theorem 4.5** Suppose that the operator pencil $\mathfrak{A}$ meets the condition (2.45) for some $\vartheta \in (0, \pi/2)$ and $R > 0$. Let $\ell$, $\ell \geq m$, be an integer number, and let the parameters $\varphi \in (0, \vartheta)$, $\zeta \in \mathbb{C}$ be such that the closed cone $K_\zeta^\varphi$ is free from the spectrum of the operator pencil $\mathbb{C} \ni \lambda \mapsto \mathfrak{A}(\lambda)$. Assume that $F$ is a function representable in the form $F = J + G|_\mathbb{R}$, where $J \in \mathcal{W}_\zeta^{\ell-m}(\mathbb{R}^-; X_0)$ and $G \in \mathcal{H}_\zeta^{\ell-m}(K_0^\varphi; X_0)$. In addition we assume that $\chi F \in \mathcal{W}_0^\ell(\mathbb{R}; X_0)$, where $\chi \in \mathcal{C}^\infty(\mathbb{R})$ is a compactly supported cutoff function, $\chi = 1$ in a neighbourhood of the point $t = 0$. Then a solution $u \in \mathcal{D}_\zeta^\ell(\mathbb{R})$ of the equation $\mathfrak{A}(D_t)u = F$
can be extended to an analytic function $K_0^\varphi\ni z \mapsto u(z) \in X_m$ such that: (i) the extension meets the inclusions $u \in C^{\ell-j-1}(K_0^\varphi \cup \mathbb{R}; X_j)$, where $0 \leq j \leq \max\{m, \ell - 1\}$; (ii) for almost all $t > 0$ the value $\|D_\ell^{t-m}u(z) - D_\ell^{t-m}u(t)\|_m$ tends to zero as $z$ goes to $t$ by a non-tangential to $\mathbb{R}$ path in $K_0^\varphi$. Moreover, $\mathfrak{A}(D_z)u(z) = G(z)$ for $z \in K_0^\varphi$, and the estimate
\[
\sum_{j=0}^{\ell} \int_{e^{-i\psi} \mathbb{R}^+} |e^{-iz}D_\ell^j u(z)|^2 dz \leq C \left( \| J; \mathcal{W}_\xi^{\ell-m}(\mathbb{R}^-, X_0) \|_2^2 + \| \Psi J + \mathcal{G}; \mathcal{W}_\xi^{\ell-m}(\mathbb{R}; X_0) \|_2^2 \right)
\]
holds, where the constant $C$ is independent of $\psi \in (0, \varphi)$, $J$, $G$, and $\chi$.

**PROOF.** Let $\rho \in C^\infty(\mathbb{R})$ be a cutoff function such that $\rho = 1$ in a neighborhood of the point $t = 0$, and $\rho x = \rho$. Then $\mathfrak{A}(D_t)\rho u = \rho F + [\mathfrak{A}(D_t), \rho]u$, where $\rho F = \rho F \in \mathcal{W}_\xi^{\ell}(\mathbb{R}; X_0)$ and $\text{ord}[\mathfrak{A}(D_t), \rho] \leq m - 1$. The operator $\mathfrak{A}(D_t) : D_s^\xi(\mathbb{R}) \to \mathcal{W}_s^{\ell-m}(\mathbb{R}; X_0)$ yields an isomorphism for all $s \in \mathbb{R}$ (Theorem 4.1). Hence we have $\rho u \in D_s^{\ell-m}(\mathbb{R})$ and
\[
\| \rho u; D_s^{\ell+m}(\mathbb{R}) \| \leq C \left( \| \rho F; \mathcal{W}_\xi^0(\mathbb{R}; X_0) \| + \| F; \mathcal{W}_\xi^{\ell-m}(\mathbb{R}; X_0) \| \right).
\]
By the Sobolev theorem the function $\rho u \in D_s^{\ell+m}(\mathbb{R}) = \bigcap_{j=0}^m \mathcal{W}_s^{\ell+m-j}(\mathbb{R}; X_j)$ has traces $(D_t^j \rho u)(0) = (D_t^j u)(0) \in X_m$, $j \leq \ell - 1$, the estimates
\[
\| (D_t^j u)(0); X_m \| \leq \| \rho u; D_s^{\ell+m}(\mathbb{R}) \|, \quad j = 0, \ldots, \ell - 1,
\]
hold. Then there exists an analytic function $K_0^\varphi \ni z \mapsto \Phi(z) \in X_m$ such that
\[
(D_t^j \Phi)(0) = (D_t^j u)(0) \in X_m, \quad j = 0, \ldots, \ell - 1,
\]
\[
\sum_{j=0}^{\ell} \int_{e^{-i\psi} \mathbb{R}^+} |e^{-iz} (D_t^j \Phi)(z)|^2 dz \leq C \left( \| \rho F; \mathcal{W}_\xi^0(\mathbb{R}; X_0) \|^2 + \| F; \mathcal{W}_\xi^{\ell-m}(\mathbb{R}; X_0) \|^2 \right),
\]
where $\psi \in [0, \varphi]$.

One can find a function satisfying the conditions (4.7), (4.8) in the form $e^{\gamma z} \sum_{j=0}^{\ell-1} a_j z^j$, where $\gamma \in \mathbb{C}$ and $a_j \in X_m$; cf. (4.5) and (4.6). From Proposition 3.13 together with (4.7) we get $\theta(u - \Phi) \in \bigcap_{j=0}^m \mathcal{W}_s^{\ell-j}(\mathbb{R}^+; X_j)$ and, in particular, $\theta(u - \Phi) \in D_s^\ell(\mathbb{R})$; here $\theta$ denotes the Heaviside unit step function. Due to the relations (4.7) the equality $[\mathfrak{A}(D_t), \theta](u - \Phi) = 0$ holds, where $[\cdot, \cdot]$ stands for the commutator $[a, b] = ab - ba$. Therefore $\mathfrak{A}(D_t) \theta(u - \Phi) = \theta(F - \mathfrak{A}(D_t) \Phi) = \theta(G - \mathfrak{A}(D_z) \Phi)$. Moreover, it is easy to see that $D_t^j (G - \mathfrak{A}(D_z) \Phi)(0) = 0$ for $j = 0, \ldots, \ell - m - 1$. The embedding result of Theorem 3.7 states that $G \in C^{\ell-m-1}(K_0^\varphi; X_0)$. Thus we have $D_t^j (G - \mathfrak{A}(D_z) \Phi)(0) = 0$ for
It is clear that the estimates (4.9), (4.10), and (4.8) gives
\[
\|\theta(G - \mathfrak{A}(D_z)\Phi); H^{\ell-m}(K_0^{\varphi}; X_0)\|^2 \\
\leq C \left(\|J; W^{\ell-m}(\mathbb{R}^-; X_0)\|^2 \\
+ \|G; H^{\ell-m}(K_0^{\varphi}; X_0)\|^2 + \|\chi F; W_0(\mathbb{R}; X_0)\|^2 \right). 
\] (4.9)

As a consequence of Theorem 4.3 (see also Corollary 4.4) we get the estimate
\[
\|\theta(u - \Phi); D_\xi(K_0^{\varphi})\|^2 \leq c\|\theta(G - \mathfrak{A}(D_z)\Phi); H^{\ell-m}(K_0^{\varphi}; X_0)\|^2. 
\] (4.10)

Recall that \(D_\xi(K_0^{\varphi}) = \cap_{j=0}^m H^{\ell-j}(K_0^{\varphi}; X_j)\). Hence by Theorem 3.7 we have the inclusions \(\theta(u - \Phi) \in C^{\ell-j}(K_0^{\varphi}; X_j)\), where \(0 \leq j \leq \max\{m, \ell - 1\}\); by the Sobolev embedding theorem with the same restrictions on \(j\) we have \(u \in C^{\ell-j}(\mathbb{R}; X_j)\). This together with the relations (4.7) proves the property (i). We also note that \(D^{\ell-m}_\xi(\theta u - \theta \Phi)\) is in the subspace \(H_0^0(K_0^{\varphi}; X_m)\) of the space \(W_0^0(\mathbb{R}; X_m)\); see (3.72). The property (ii) follows from Theorem 3.5, (i). It is clear that \(\mathfrak{A}(D_z)u(z) = G(z)\) for all \(z \in K_0^{\varphi+}\). The estimate (4.4) is readily apparent from the estimates (4.9), (4.10) and (4.8). \(\square\)

### 4.2 Differential equations with variable coefficients

The aim of this section is to give an analog of Theorem 4.5 for the case of equations with variable operator coefficients. We shall consider the equation
\[
\mathfrak{A}(D_t)u(t) - \sum_{j=0}^m Q_j(t)D_t^{m-j}u(t) = F(t), \quad t \in \mathbb{R}. 
\] (4.11)

Here \(\mathfrak{A}(D_t)\) is the same as before, the coefficients \(Q_0, \ldots, Q_m\) are operator functions \(\mathbb{R} \ni t \mapsto Q_j(t) \in \mathcal{B}(X_j, X_0)\) satisfying the following conditions:

i. for a large \(T > 0\) and some \(\alpha > 0\) the coefficients \(Q_0, \ldots, Q_m\) can be extended to holomorphic operator functions \(K_T^{\varphi+} \ni z \mapsto Q_j(z) \in \mathcal{B}(X_j, X_0)\);

ii. for all \(n = 0, 1, \ldots\) the values \(\|D_{\zeta}^n Q_j(z); \mathcal{B}(X_j, X_0)\|, j = 0, \ldots, m\), uniformly tend to zero as \(z \to \infty, z \in K_T^{\varphi+}\).

If the operator functions \(\mathbb{R} \ni t \mapsto D_{\zeta}^n Q_j(t) \in \mathcal{B}(X_j, X_0), j \leq m\), are bounded for all \(n = 0, 1, \ldots\) then the operator \((\mathfrak{A}(D_t) - \sum_{j=0}^m Q_j(t)D_t^{m-j}) : D_\zeta^{\ell}(\mathbb{R}) \to W^{\ell-m}(\mathbb{R}; X_0)\) of the equation (4.11) is continuous for any \(\ell \in \mathbb{R}\) and \(\zeta \in \mathbb{C}\). For simplicity we restrict ourselves to the case of an integer \(\ell \geq m\).
We start with the subsidiary equation

$$\mathcal{A}(D_{(\phi)})U_\phi(z) - \sum_{j=0}^m Q_j(z)D_{(\phi)}^{m-j}P_{\eta,w}^{m-j}U_\phi(z) = G_\phi(z), \quad z \in e^{-i\phi}\mathbb{R} + w, \quad (4.12)$$

where $w \in \overline{K_T^{\alpha,+}}$ and $\phi \in [0, \alpha]$. Here $P_{\eta,w}^{m-j}$ is the projection operator (3.64) with the parameter $\eta \in \bigcap_{\phi \in [0, \alpha]} e^{i\phi}\mathbb{C}^+ + \zeta$. The coefficients $Q_0, \ldots, Q_m$ are defined on the set $e^{-i\phi}\mathbb{R} + w \cap \overline{K_T^{\alpha,+}}$ as the analytic extensions of the correspondent coefficients of the equation (4.11). Without loss of generality we can assume that $Q_j(z) = 0$ for $z \notin \overline{K_T^{\alpha,+}}$ because $\text{supp}\{P_{\eta,w}^{-j}U_\phi\} \subset \overline{K_T^{\alpha,+}}$. Let us note that the norms of the operators

$$D_{(\phi)}^{m-j}P_{\eta,w}^{m-j}: W_\zeta^{\ell-j}(e^{-i\phi}\mathbb{R} + w; X_j) \to W_\zeta^{\ell-m}(e^{-i\phi}\mathbb{R}^+ + w; X_j), \quad j \leq m,$$

are bounded uniformly in $w \in \mathbb{C}$; see (3.65). Since $Q_0(z), \ldots, Q_m(z)$ satisfy the condition ii the mappings

$$W_\zeta^{\ell-m}(e^{-i\phi}\mathbb{R}^+ + w; X_j) \ni F \mapsto Q_jF \in W_\zeta^{\ell-m}(e^{-i\phi}\mathbb{R}^+ + w; X_0), \quad j \leq m,$$

tend to zero in the operator norms as $w \to \infty$, $w \in \overline{K_T^{\alpha,+}}$. Hence the operator

$$\mathbf{Q}^+_w(z, D_{(\phi)}) = \sum_{j=0}^m Q_j(z)D_{(\phi)}^{m-j}P_{\eta,w}^{m-j}, \quad (4.13)$$

$$\mathbf{Q}^+_w(z, D_{(\phi)}): D_\zeta^{\ell}(e^{-i\phi}\mathbb{R} + w) \to W_\zeta^{\ell-m}(e^{-i\phi}\mathbb{R}^+ + w; X_0) \quad (4.14)$$

tends to zero in the operator norm as $w \to \infty$, $w \in \overline{K_T^{\alpha,+}}$.

**Theorem 4.6** Let the coefficients $Q_0, \ldots, Q_m$ of the equation (4.11) meet the conditions i, ii in some cone $K_T^{\alpha,+}$, and let $w \in \overline{K_T^{\alpha,+}}$ be a complex number with a sufficiently large modulus. Suppose that $\zeta \in \mathbb{C}$, $\phi \in [0, \alpha]$ and $\phi < \vartheta$; here $\vartheta$ is the angle from the condition (2.45). If the line $e^{i\phi}\mathbb{R} + \zeta$ is free from the spectrum of the pencil $\mathbb{C} \ni \lambda \mapsto \mathcal{A}(\lambda)$ then the operator

$$\mathcal{A}(D_{(\phi)}) - \mathbf{Q}^+_w(z, D_{(\phi)}): D_\zeta^{\ell}(e^{-i\phi}\mathbb{R} + w) \to W_\zeta^{\ell-m}(e^{-i\phi}\mathbb{R} + w; X_0) \quad (4.15)$$

of the subsidiary equation (4.12) yields an isomorphism for all $\ell \geq m$, $\ell \in \mathbb{Z}$.

**Proof.** By Theorem 4.1 the operator (4.1) with constant coefficients is invertible, and the norm of the inverse operator is bounded uniformly in $w \in \mathbb{C}$. Let us denote the inverse operator by $\mathfrak{R}$. Since the norm of the operator (4.14) tends to zero and the norm of $\mathfrak{R}$ remains bounded as $w \to \infty$, $w \in \overline{K_T^{\alpha,+}}$, the norm of the composition

$$\mathbf{Q}^+_w(z, D_{(\phi)})\mathfrak{R}: W_\zeta^{\ell-m}(e^{-i\phi}\mathbb{R} + w; X_0) \to W_\zeta^{\ell-m}(e^{-i\phi}\mathbb{R}^+ + w; X_0)$$

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is strictly less than 1 as far as \( w \in \overline{K_T^{\alpha,\varphi}^+} \) and \( |w| \) is a sufficiently large positive number. Then the operator (4.15) with variable coefficients is invertible, its inverse operator is defined as the Neumann operator series

\[
\mathfrak{A} \sum_{n=0}^{+\infty} \left( \Omega_w^+(z, D(\phi)) R \right)^n : W_{\xi}^{\ell-m}(e^{-i\phi}R + w; X_0) \rightarrow D_\xi(e^{-i\phi}R + w).
\]

\[\square\]

Now we consider the operator (4.15) on the subspace \( D_\xi(K_w^\varphi) \) of the subsidiary equation (4.12) of the subsidiary equation (4.12). Let the coefficients of the equation (4.11) tend to zero in the operator norm as \( w \rightarrow \infty \) from the condition (2.45). If the cone \( K_w^\varphi \) is free from the spectrum of the operator pencil \( z \mapsto \mathfrak{A}(\lambda) \) then the operator

\[
\mathfrak{A}(D(\phi)) - \Omega_w^+(z, D(\phi)) : D_\xi(K_w^\varphi) \rightarrow H_\xi^{\ell-m}(K_w^\varphi; X_0)
\]

tends to zero in the operator norm as \( w \rightarrow \infty \), \( w \in \overline{K_T^{\alpha,\varphi}^+} \).

**Theorem 4.7** Let the coefficients of the equation (4.11) meet the conditions i, ii in some cone \( K_T^{\alpha,\varphi} \), and let \( w \in \overline{K_T^{\alpha,\varphi}^+} \) be a complex number with a sufficiently large module. Suppose that \( \zeta \in \mathbb{C} \), \( \varphi \in (0, \alpha] \) and \( \varphi < \vartheta \); here \( \vartheta \) is the angle from the condition (2.45). If the cone \( K_w^\varphi \) is free from the spectrum of the operator pencil \( \mathbb{C} \ni \lambda \mapsto \mathfrak{A}(\lambda) \) then the operator

\[
\mathfrak{A}(D(\phi)) - \Omega_w^+(z, D(\phi)) : D_\xi(K_w^\varphi) \rightarrow H_\xi^{\ell-m}(K_w^\varphi; X_0)
\]

tends to zero in the operator norm as \( w \rightarrow \infty \).

**Proof.** The proof is similar to the proof of Theorem 4.6. Indeed, by Theorem 4.3 the operator \( \mathfrak{A}(D(\phi)) \) implements an isomorphism (4.3) and its inverse
operator is bounded uniformly in \( w \in \mathbb{C} \). It remains to note that under the assumptions of the theorem the norm of the operator (4.17) is sufficiently small. \( \square \)

Now we are in position to extend the results of Theorem 4.5 to the case of equations with variable coefficients. The following theorem presents a more general result than Theorem 1.2 cited in the introductory part.

**Theorem 4.8** Let the conditions i, ii on the coefficients of the equation (4.11) be fulfilled in some cone \( K_T^{\varphi^+} \). Suppose that the polynomial operator pencil \( \mathbb{C} \ni \lambda \mapsto \mathfrak{A}(\lambda) \) satisfies the condition (2.45) for some \( \vartheta \in (0, \pi/2) \) and \( R > 0 \). Let the angle \( \varphi \in (0, \alpha] \), \( \varphi < \vartheta \), and the vertex \( \zeta \in \mathbb{C} \) of the cone \( K_{\zeta}^{\varphi} \) be such that the closed cone \( \overline{K}_{\zeta}^{\varphi} \) is free from the spectrum of the pencil \( \mathbb{C} \ni \lambda \mapsto \mathfrak{A}(\lambda) \).

Assume that \( \ell \geq m \), \( \ell \in \mathbb{Z} \), and the right hand side \( F \in \mathcal{W}_{\zeta}^{\ell-m}(\mathbb{R}; X_0) \) of the equation (4.11) is representable in the form \( F = J + G\chi_{[\mathbb{R}]} \), where \( J \in \mathcal{W}_{\zeta}^{\ell-m}(\mathbb{R}^{-} + T; X_0) \), \( G \in \mathcal{H}_{\zeta}^{\ell-m}(K_{\zeta}^{\varphi}; X_0) \), and \( T \) is a sufficiently large positive number. In addition we assume that \( \chi F \in \mathcal{W}_{0}^{\ell}(\mathbb{R}; X_0) \), where \( \chi \in C^{\infty}(\mathbb{R}) \) is a compactly supported cutoff function, \( \chi = 1 \) in a neighbourhood of the point \( t = R \). Then a solution \( u \in \mathcal{D}_{\ell}^{0}(\mathbb{R}) \) of the equation (4.11) can be extended to an analytic function \( K_{R}^{\varphi^+} \ni z \mapsto u(z) \in X_m \) such that: (i) the extension meets the inclusions \( u \in C^{\ell-j-1}(K_{R}^{\varphi^+} \cup \mathbb{R}; X_j) \), \( j = 0, \ldots, \max\{m, \ell - 1\} \); (ii) for almost all \( t > R \) the value \( \|D_{z}^{j}u(z) - D_{z}^{j}u(t)\|_{m} \) tends to zero as \( z \) goes to \( t \) by a non-tangential to \( \mathbb{R} \) path in \( K_{R}^{\varphi^+} \). Moreover, the extension satisfies the following equation and estimate

\[
\mathfrak{A}(D_{z})u(z) - \sum_{j=0}^{m} Q_{j}(z)D_{z}^{m-j}u(z) = G(z), \quad z \in K_{R}^{\varphi^+},
\]

\[
\sum_{j=0}^{\ell} \int_{0}^{1} e^{-i\psi z} D_{z}^{j}u(z + T)\|z\|_{\ell-j} dz \leq C\left(\|\chi u; D_{0}^{\ell}(\mathbb{R})\|^{2}
\right.
\]

\[
+ \|G; H_{\zeta}^{\ell-m}(K_{\zeta}^{\varphi}; X_0)\|^{2} + \|\chi(J + G); \mathcal{W}_{0}^{\ell}(\mathbb{R}; X_0)\|^{2}\),
\]

where the constant \( C \) is independent of \( \psi \in (0, \varphi) \), \( J \), \( G \), and \( \chi \).

**PROOF.** Let \( \rho \in C^{\infty}(\mathbb{R}) \) be a cutoff function such that \( \rho = 1 \) in a neighbourhood of the point \( t = T \), \( \rho(t) = 0 \) for \( t < T - 1/2 \), and \( \rho \chi = \rho \). Then

\[
\left(\mathfrak{A}(D_{t}) - \mathcal{Q}_{T-1}^{+}(t, D_{t})\right)\rho u = \rho \chi F + [\mathfrak{A}(D_{t}) - \mathcal{Q}_{T-1}^{+}(t, D_{t})]\rho \chi u,
\]

where \( \rho \chi F = \rho \chi F \in \mathcal{W}_{\zeta}^{\ell}(\mathbb{R}; X_0) \) and \( \text{ord}[\mathfrak{A}(D_{t}) - \mathcal{Q}_{T-1}^{+}(t, D_{t})] \leq m - 1 \). The operator \( \mathfrak{A}(D_{t}) - \mathcal{Q}_{T-1}^{+}(t, D_{t}) : \mathcal{D}_{\ell}(\mathbb{R}) \rightarrow \mathcal{W}_{\zeta}^{\ell-m}(\mathbb{R}; X_0) \) yields an isomorphism.
for all $s \geq m$, $s \in \mathbb{Z}$ (Theorem 4.6). Hence we have $\rho u \in D^{\ell+m}_\xi(\mathbb{R})$ and
\[
\|\rho u; D^{\ell+m}_\xi(\mathbb{R})\| \leq C\left(\|\chi F; W^0_0(\mathbb{R}; X_0)\| + \|\chi u; D^\ell_0(\mathbb{R})\|\right).
\]
(4.20)
The function $\rho u \in \cap_{j=0}^m W^{\ell+m-j}_\xi(\mathbb{R}; X_j)$ has traces satisfying the estimates
\[
\|(D^j_{\psi}(u); X_m)\| \leq \|\rho u; D^{\ell+m}_\xi(\mathbb{R})\|, \quad j = 0, \ldots, \ell - 1.
\]
(4.21)
Let $\Phi$ denote an analytic function $\overline{K^\ell_T} \ni z \mapsto \Phi(z) \in X_m$ such that
\[
(D^j_{\psi}(\Phi))(T) = (D^j_{\psi}(u))(T) \in X_m, \quad j = 0, \ldots, \ell - 1,
\]
(4.22)
\[
\sum_{j=0}^{\ell+m} \int_{e^{-i\psi} \mathbb{R}} e^{-i\xi z} \|D^j_{\psi}(\Phi)(z + T)\|^2_m |dz| \leq C\left(\|\chi F; W^0_0(\mathbb{R}; X_0)\|^2 + \|\chi u; D^\ell_0(\mathbb{R})\|^2\right), \quad \psi \in [0, \varphi].
\]
(4.23)
By Proposition 3.13 we have $\theta(\cdot - T)(u - \Phi) \in \cap_{j=0}^m W^{\ell-j} \xi(\mathbb{R}^+ + T; X_j)$. The equalities
\[
[\mathcal{A}(D_t) - \Omega^+_T(t, D_t), \theta(\cdot - T)](u - \Phi) = 0,
\]
\[
(\mathcal{A}(D_t) - \Omega^+_T(t, D_t))\theta(\cdot - T)(u - \Phi) = \theta(\cdot - T)(G - \mathcal{A}(D_t)\Phi + \Omega^+_T(t, D_t)\Phi)
\]
hold. Moreover, one can easily see that $D^j_T(G - \mathcal{A}(D_t)\Phi + \Omega^+_T(t, D_t)\Phi)(T) = 0$ for $j = 0, \ldots, \ell - m - 1$. By Theorem 3.7 we have $G \in C^{\ell-m-1}(\overline{K^\ell_T}; X_0)$, thus for $\psi \in [0, \varphi]$ and $j = 0, \ldots, \ell - m - 1$ we get
\[
D^j_{(\psi)}(G - \mathcal{A}(D_t)\Phi + \Omega^+_T(t, D_t)\Phi)|_{t=T} = 0.
\]
This together with Proposition 3.13 and the estimate (4.23) gives
\[
\|\theta(\cdot - T)(G - \mathcal{A}(D_t)\Phi + \Omega^+_T(t, D_t)\Phi); H^{\ell-m}_\xi(\mathbb{R}^+; X_0)\| \leq C\left(\|\chi u; D^\ell_0(\mathbb{R})\|^2 + \|G; H^{\ell-m}_\xi(\mathbb{R}^+; X_0)\|^2 + \|\chi F; W^\ell_0(\mathbb{R}; X_0)\|^2\right).
\]
(4.24)
As a consequence of Theorem 4.6 we get the estimate
\[
\|\theta(\cdot - T)(u - \Phi); D^{\ell}_\xi(\mathbb{R})\|^2 \leq c\|\theta(\cdot - T)(G - \mathcal{A}(D_t)\Phi + \Omega^+_T(t, D_t)\Phi); H^{\ell-m}_\xi(\mathbb{R}^+; X_0)\|^2.
\]
(4.25)
By Theorem 3.7 the inclusion $\theta(\cdot - T)(u - \Phi) \in C^{\ell-j-1}(\overline{K^\ell_T}; X_j)$ is fulfilled for $j = 0, \ldots, \max\{m, \ell - 1\}$; by the Sobolev theorem for $u \in D^{\ell}_\xi(\mathbb{R})$ we have $u \in C^{\ell-j-1}(\mathbb{R}; X_j)$ with the same restrictions on $j$. This together with the relations (4.22) proves the property (i). The property (ii) follows from Theorem 3.5, (i). It is clear that $u$ satisfies the equation (4.18). The estimate (4.19) is a consequence of the estimates (4.24), (4.25) and (4.23). □
Remark 4.9 In Theorem 4.8 the assumptions on the right hand side $F \in \mathcal{W}_\ell^{\ell-m}(\mathbb{R}; X_0)$ of the equation (4.11) are a priori fulfilled for all $T > T_0$ if the function $F$ can be extended to an analytic function $K_{T_0}^{\varphi, +} \ni z \mapsto F(z) \in X_0$ satisfying the uniform in $\psi \in [0, \varphi]$ estimate

$$\sum_{j=0}^{\ell-m} \int e^{-i\zeta z} D_z^j F(z + T_0) ||_0^2 |dz| \leq \text{Const.} \quad (4.26)$$

Indeed, from the estimate (4.26) it follows that the functions $K_{T_0}^{\varphi, +} \ni z \mapsto D_z^j F(z) \in X_0$, $j = 0, \ldots, \ell - m$, extended to $K_{T_0}^{\varphi, -}$ by zero are in the class $\mathcal{H}_w^0(K_{T_0}^{\varphi, -}; X_0)$; here $K_{T_0}^{\varphi, -} = K_{T_0}^{\varphi, +}$. Then by Proposition 2.14 we have

$$\sum_{j=0}^{\ell-m} \int e^{-i\zeta z} D_z^j F(z + T) ||_0^2 |dz| \leq C, \quad T > T_0, \ \psi \in [0, \varphi].$$

We find an entire function $C \ni z \mapsto \Phi(z) \in X_0$ such that $D_z^j \Phi(T) = D_z^j F(T)$ for all $j = 0, \ldots, \ell - m - 1$, and

$$\sum_{j=0}^{\ell-m} \int e^{-i\zeta z} D_z^j \Phi(z + T) ||_0^2 |dz| \leq C, \quad \psi \in [0, \varphi].$$

By setting $G = \Phi$ on $K_{T_0}^{\varphi, -}$ and $G = F$ on $K_{T_0}^{\varphi, +}$ we define $G \in H_\ell^{\ell-m}(K_{T_0}^{\varphi, -}; X_0)$. It is clear that $J = F - G \in \mathcal{W}_\ell^{\ell-m}(\mathbb{R}^+ + T; X_0)$.

5 Appendix. Proof of Proposition 2.6, Lemma 5.1

PROOF of Proposition 2.6. (i) Here we prove the representation

$$\mathcal{F}(\lambda) = \int_{\partial K_{\zeta}^{\varphi, +}} \frac{e^{i w (\mu - \lambda)(\mu - \eta)^s \mathcal{F}(\mu)}}{2 \pi i (\lambda - \eta)^s (\mu - \lambda)} \ d\mu, \quad \lambda \in K_{\zeta}^{\varphi, +}, \ \eta \notin K_{\zeta}^{\varphi, +}, \ s \leq \ell, \quad (5.1)$$

for a function $\mathcal{F} \in \mathcal{H}_w^\ell(K_{\zeta}^{\varphi}; X)$, where $K_{\zeta}^{\varphi, +} = \{ \lambda \in K_{\zeta}^{\varphi}; \Im \lambda > \Im \zeta \}$ and $\partial K_{\zeta}^{\varphi, +}$ is the boundary of $K_{\zeta}^{\varphi, +}$, the function $(\cdot - \eta)^s$ is analytic in $K_{\zeta}^{\varphi, +}$; cf. (2.14). The proof of the second representation (2.15) in Proposition 2.6(i) is similar.

Let us define the function $G \in \mathcal{H}_0^0(K_{\zeta}^{\varphi}; X)$ by the equality

$$G(\lambda - \zeta) = \begin{cases} \exp\{i w \lambda\} (\lambda - \eta)^s \mathcal{F}(\lambda), & \lambda \in K_{\zeta}^{\varphi, +}; \\ \exp\{i w \lambda\} (\lambda - \tau)^s \mathcal{F}(\lambda), & \lambda \in K_{\zeta}^{\varphi, -}. \end{cases} \quad (5.2)$$
To prove the representation (5.1) it suffices to establish that

$$G(\lambda) = \frac{1}{2\pi i} \left( \int_0^{+\infty} \frac{G(\xi)}{\xi - \lambda} d\xi - \int_0^{+\infty} \frac{G(e^{i\varphi} \xi)}{e^{i\varphi} \xi - \lambda} e^{i\varphi} d\xi \right) \quad \lambda \in \mathcal{K}_0. \quad (5.3)$$

By the Cauchy integral theorem we have

$$G(\lambda) = \frac{1}{2\pi i} \int_R^{R+1} da \oint_{C(a,\psi,\phi)} \frac{G(\mu)}{\mu - \lambda} d\mu, \quad (5.4)$$

where $R > 0$, the contour integration runs anticlockwise along the closed path

$$\mathcal{C}(a,\psi,\phi) = \{ \mu : \mu = ae^{i\vartheta}, \vartheta \in [\psi, \phi] \} \cup \{ \mu : \mu = xe^{i\phi}, 1/a \leq x \leq a \} \cup \{ \mu : \mu = xe^{i\psi}, 1/a \leq x \leq a \}, \quad 0 < \psi < \phi < \varphi,$$

and $\lambda$ is inside of the contour $\mathcal{C}(R,\psi,\phi)(\subset \mathcal{K}_0^\varphi)$. We will show that for all $\lambda \in \mathcal{K}_0^\varphi$ the equality

$$\int_0^{+\infty} \frac{G(\xi)}{\xi - \lambda} d\xi - \int_0^{+\infty} \frac{G(e^{i\varphi} \xi)}{e^{i\varphi} \xi - \lambda} e^{i\varphi} d\xi = \lim_{\psi \to 0^+} \lim_{\phi \to \varphi^-} \int_R^{R+1} da \oint_{C(a,\psi,\phi)} \frac{G(\mu)}{\mu - \lambda} d\mu$$

is fulfilled, where the limits are taken in the space $X$. The equalities (5.4), (5.5) prove the representation (5.3) (and consequently the representation (5.1)).

Let us demonstrate (5.5). At first we estimate the norm in $X$ of the integral

$$\int_R^{R+1} \int_{\{\mu : \mu = ae^{i\vartheta}, \vartheta \in [\psi, \phi] \}} \frac{G(\mu)}{\mu - \lambda} d\mu \, da = \int_R^{R+1} \int_{\psi}^{\phi} \frac{G(ae^{i\vartheta})}{ae^{i\vartheta} - \lambda} da d\vartheta.$$

Interchanging the order of integration and applying the Cauchy-Schwarz inequality, we get

$$\left\| \int_R^{R+1} \int_{\psi}^{\phi} \frac{G(ae^{i\vartheta})}{ae^{i\vartheta} - \lambda} a \, d\vartheta \, da \right\|^2 \leq (\phi - \psi) \int_{\psi}^{\phi} \left( \int_R^{R+1} \left\| \frac{G(ae^{i\vartheta})}{ae^{i\vartheta} - \lambda} \right\| \, da \right)^2 d\vartheta$$

$$\leq (\phi - \psi) \int_{\psi}^{\phi} \left( \int_R^{R+1} \left\| \frac{a}{ae^{i\vartheta} - \lambda} \right\|^2 \, da \right) \left( \int_R^{R+1} \left\| G(ae^{i\vartheta}) \right\|^2 \, da \right) d\vartheta$$

$$\leq C \int_{\psi}^{\phi} \int_R^{R+1} \left\| G(ae^{i\vartheta}) \right\|^2 \, da \, d\vartheta. \quad (5.6)$$
In the same way we derive

\[
\begin{align*}
\left\| \int_{R}^{R+1} \int_{\{\mu; \mu = e^{i\phi}/a, \phi \in [\psi, \phi]\}} \frac{G(\mu)}{\mu - \lambda} d\mu \right\| & \leq \left( \int_{R}^{R+1} \int_{\psi}^{\phi} \left\| \frac{G(e^{i\phi}/a)}{e^{i\phi}/a - \lambda} \right\| d\phi \right) d\theta \\
& \leq (\phi - \psi) \int_{\psi}^{\phi} \left( \int_{R}^{R+1} \left| e^{i\phi}/a - \lambda \right|^{-2} d\theta \right) \left( \int_{R}^{R+1} \left\| G(e^{i\phi}/a) \right\|^2 a^{-2} d\theta \right) d\phi \\
& \leq C \int_{\psi}^{\phi} \int_{1/(R+1)}^{1/R} \left\| G(\alpha e^{i\phi}) \right\|^2 d\alpha d\phi.
\end{align*}
\]

(5.7)

Since \( G \in \mathcal{H}_0^0(K^\phi; X) \), the integrals \( \int_{R}^{R+1} \left\| G(\alpha e^{i\phi}) \right\|^2 d\alpha \), \( \int_{1/(R+1)}^{1/R} \left\| G(\alpha e^{i\phi}) \right\|^2 d\alpha \) are uniformly bounded and tend to zero as \( R \to +\infty \). Consequently, the right hand sides of the inequalities (5.6) and (5.7) tend to zero as \( R \to +\infty \).

Let us consider the integral

\[
I(\psi, R) = \int_{R}^{R+1} da \int_{\{\mu; \mu = e^{i\psi}/a, 1/a \leq \xi \leq a\}} \frac{G(\mu)}{\mu - \lambda} d\mu.
\]

For brevity, in the next formula we write \([\ldots]\) instead of the integrand

\[
G(\xi e^{i\phi})/(\xi e^{i\phi} - \lambda).
\]

Interchanging the order of integration, we arrive at the formula

\[
I(\psi, R) = \int_{R}^{R+1} da \int_{1/a}^{\infty} \{\ldots\} d\xi
\]

\[
= \int_{1/R}^{R} d\xi \int_{R}^{R+1} da [\ldots] + \int_{1/(R+1)}^{1/R} d\xi \int_{1/\xi}^{R} da [\ldots] + \int_{R}^{R+1} d\xi \int_{1}^{\infty} da [\ldots]
\]

\[
= \int_{1/R}^{R} d\xi [\ldots] + \int_{1/(R+1)}^{1/R} d\xi (R + 1 - 1/\xi)[\ldots] + \int_{R}^{R+1} dt(t - R)[\ldots].
\]

Note that the last two integrals tend in \( X \) to zero as \( R \to +\infty \). Indeed, the estimates

\[
\left\| \int_{1/(R+1)}^{1/R} (R + 1 - 1/\xi)G(\xi e^{i\phi})(\xi e^{i\phi} - \lambda)^{-1} d\xi \right\|^2 \leq C \int_{1/(R+1)}^{1/R} \left\| G(\xi e^{i\phi}) \right\|^2 d\xi,
\]

\[
\left\| \int_{R}^{R+1} (\xi - R)G(\xi e^{i\phi})(\xi e^{i\phi} - \lambda)^{-1} d\xi \right\|^2 \leq C \int_{R}^{R+1} \left\| G(\xi e^{i\phi}) \right\|^2 d\xi,
\]

are valid, where the right hand sides tend to zero as \( R \to +\infty \) whenever \( G \in \mathcal{H}_0^0(K^\phi; X) \) and \( \psi \in (0, \varphi) \). We have proved the equality

\[
G(\lambda) = \frac{1}{2\pi i} \left( \int_{0}^{+\infty} \frac{G(e^{i\phi} \xi)}{e^{i\phi} \xi - \lambda} e^{i\phi} d\xi - \int_{0}^{+\infty} \frac{G(e^{i\phi} \xi)}{e^{i\phi} \xi - \lambda} e^{i\phi} d\xi \right), \quad (5.8)
\]

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where \( \arg \lambda \in (\psi, \phi) \), \( \lambda \neq 0 \), and \( 0 < \psi < \phi < \varphi \). Due to the assertion (i) of Proposition 2.3, we can pass in (5.8) to the limit as \( \psi \to 0^+ \) and \( \phi \to \varphi^- \). As a result we obtain the equality (5.5). The representation (5.3) is proved.

(ii) To prove the assertion (ii) of Proposition 2.6 it suffices to show that the value \( \|G(\lambda)\| \) uniformly tends to zero as \( \lambda \) goes to infinity in the set \( \{ \lambda \in K_0^\varphi : \text{dist}\{\lambda, \partial K_0^\varphi\} \geq \varepsilon > 0\} \); here \( G \in \mathcal{H}_0^\varphi (K_0^\varphi ; X) \) is identified with \( F \in \mathcal{H}_w^\varphi (K_0^\varphi ; X) \) by the rule (5.2). For all \( 0 < \varrho < \varepsilon \) we have

\[
G(\lambda) = \frac{1}{2\pi i} \int_{|\mu - \lambda| = \varrho} \frac{G(\mu)}{\mu - \lambda} \, d\mu = \frac{1}{2\pi} \int_0^{2\pi} G(\lambda + \varrho e^{i\alpha}) \, d\alpha,
\]

\[
\pi \varrho^2 G(\lambda) = \int_0^\varepsilon \int_0^{2\pi} G(\lambda + \varrho e^{i\alpha}) \, d\alpha \, d\varrho. \tag{5.9}
\]

Using the Cauchy-Schwarz inequality, we deduce from (5.9) that

\[
\pi \varrho^2 \|G(\lambda)\|^2 \leq 2\pi \int_0^\varepsilon \varrho \, d\varrho \cdot \int_0^{2\pi} \|G(\lambda + \varrho e^{i\alpha})\|^2 \, d\alpha \, d\varrho. \tag{5.10}
\]

Since the circle \( \{ \mu \in \mathbb{C} : |\mu - \lambda| = \varepsilon \} \) is contained in the set

\[
\{ \mu \in \mathbb{C} : \arg \lambda - 2\varepsilon / |\lambda|, \arg \lambda + 2\varepsilon / |\lambda|, |\lambda| - \varepsilon \leq |\mu| \leq |\lambda| + \varepsilon \},
\]

we have

\[
\int_0^\varepsilon \int_0^{2\pi} \|G(\lambda + \varrho e^{i\alpha})\|^2 \, d\alpha \, d\varrho \leq \int_{-2\varepsilon / |\lambda|}^{2\varepsilon / |\lambda|} \, d\psi \int_{|\lambda| - \varepsilon}^{\varepsilon} \left\| G(\varepsilon e^{i(\arg \lambda + \psi)}) \right\|^2 \, d\xi \leq C \int_{-2\varepsilon / |\lambda|}^{2\varepsilon / |\lambda|} \, d\psi \int_{|\lambda| - \varepsilon}^{\varepsilon} \left\| G(\varepsilon e^{i(\arg \lambda + \psi)}) \right\|^2 \, d\xi \tag{5.11}
\]

Due to the inclusion \( G \in \mathcal{H}_0^\varphi (K_0^\varphi ; X) \) the integral \( \int_{|\lambda| - \varepsilon}^{\varepsilon} \left\| G(\varepsilon e^{i(\arg \lambda + \psi)}) \right\|^2 \, d\xi \) is uniformly bounded in \( \psi \in [-2\varepsilon, 2\varepsilon] \) for all sufficiently large \( |\lambda| \) and tends to zero as \( |\lambda| \to +\infty \). This together with (5.10) and (5.11) finishes the proof. \( \square \)

**Lemma 5.1** For any analytic function \( \mathbb{C}^+ \ni \lambda \mapsto F(\lambda) \in X \) satisfying the uniform in \( \eta \in \mathbb{C}^+ \) estimate

\[
\|F(\cdot + \eta); L_2(\mathbb{R}; X)\| \leq C(F) \tag{5.12}
\]

there exists a function \( G \in L_2(\mathbb{R}; X) \) such that for all \( \lambda \in \mathbb{C}^+ \) the equality

\[
\sqrt{\lambda} F(\lambda) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{G(\xi)}{(\log \xi - \log \lambda)^{1/2}} \, d\xi \tag{5.13}
\]

is valid; here we use the analytic in \( \mathbb{C}^+ \setminus \{0\} \) principal branches of logarithm and square root, the integral is absolutely convergent in \( X \).
PROOF. Let $R > 0$ and $0 < \eta_1 < \eta_2$. From the Cauchy integral theorem it follows the equality

$$\sqrt{\lambda} \mathcal{F}(\lambda) = \frac{1}{2\pi i} \int_{R}^{R+1} d\lambda \int_{C(a,\eta_1,\eta_2)} \mathcal{F}(\mu) \left(\frac{\log\mu - \log\lambda}{\sqrt{\mu}}\right) d\mu,$$

(5.14)

where $|\Re\lambda| < R$, $\Re\lambda \in (\eta_1, \eta_2)$, and the closed contour $C(a,\eta_1,\eta_2)$ is given by the formula

$$C(a,\eta_1,\eta_2) = \{\mu \in \mathbb{C} : \mu = \xi + i\eta, \xi \in [-a, a]\} \cup \{\mu \in \mathbb{C} : \mu = -\xi + i\eta, \xi \in [-a, a]\}$$

$$\cup \{\mu \in \mathbb{C} : \mu = -a - i\eta, \eta \in [-\eta_2, -\eta_1]\}.$$ 

Let us check that passing to the limit in (5.14) as $R \to +\infty$ leads to the equality

$$\sqrt{\lambda} \mathcal{F}(\lambda) = \frac{1}{2\pi i} \left(\int_{R+i\eta_1}^{R+1} d\lambda \int_{C(a,\eta_1,\eta_2)} \mathcal{F}(\mu) \left(\frac{\log\mu - \log\lambda}{\sqrt{\mu}}\right) d\mu - \int_{R+i\eta_2}^{R+1} d\lambda \int_{C(a,\eta_1,\eta_2)} \mathcal{F}(\mu) \left(\frac{\log\mu - \log\lambda}{\sqrt{\mu}}\right) d\mu\right).$$

(5.15)

Indeed, by Cauchy-Schwarz inequality we get

$$\left(\int_{R}^{R+1} d\lambda \int_{\eta_1}^{\eta_2} \left(\frac{\mathcal{F}(\pm a + i\eta)}{|(\log(\pm a + i\eta)) - \log\lambda|\sqrt{\pm a + i\eta}}\right)^2 d\eta\right)^{1/2} \leq \left(\int_{R}^{R+1} \mathcal{F}(\pm a + i\eta) \left(\frac{2|\log(\pm a + i\eta) - \log\lambda|^{-2} \pm a + i\eta|^{-1}}{d\eta}\right)\right)^{1/2} \times \left(\int_{R}^{R+1} \mathcal{F}(\pm a + i\eta)|^2 d\eta\right).$$

(5.16)

Here the integral $\int_{R}^{R+1} \mathcal{F}(\pm a + i\eta)|^2 d\eta$ is uniformly bounded in $\eta \in \mathbb{R}^+$ and tends to zero as $R \to +\infty$ due to the estimate (5.12). It is easy to see that

$$\int_{R}^{R+1} |\log(\pm a + i\eta) - \log\lambda|^{-2} \pm a + i\eta|^{-1} d\eta \leq \int_{R}^{+\infty} (\log a - \log |\lambda|)^{-2} a^{-1} d\eta = 1/ \log(R/|\lambda|) \to 0, \quad R \to +\infty.$$ 

(5.17)

Thus the right hand side of the inequality (5.16) tends to zero as $R \to +\infty$.

We now consider the integral

$$\int_{R}^{R+1} \int_{-a}^{a} \mathcal{F}(\xi + i\eta) \left(\frac{\log(\xi + i\eta) - \log\lambda}{\sqrt{\xi + i\eta}}\right) d\xi d\eta = \int_{-R}^{R} \mathcal{F}[\ldots] d\xi$$

$$+ \int_{-R}^{R} (R + 1 + \xi)[\ldots] d\xi + \int_{R}^{R+1} (R + 1 - \xi)[\ldots] d\xi,$$

(5.18)

where, for brevity, $[\ldots]$ denotes the same integrand as in the left hand side of the equality (5.18), and $\eta = \eta_1$ or $\eta = \eta_2$. The interchange of integrations in
(5.18) is legal because of (5.12) and the estimate
\[ \int_{-\infty}^{+\infty} |\log(\xi + i\eta) - \log \lambda|^{-2} |\xi + i\eta|^{-1} d\xi \leq \text{Const}(\eta, \lambda), \quad \lambda \notin \mathbb{R} + i\eta. \]

Note that the last two integrals in (5.18) go to zero in \( X \) as \( R \to +\infty \). Indeed, for the first of these integrals it follows from (5.17), (5.12), and the estimate
\[ \int_{-\infty}^{+\infty} \left| \int_{-R}^{-1} \frac{\mathcal{F}(\xi + i\eta)}{(\log(\xi + i\eta) - \log \lambda)\sqrt{\xi + i\eta}} d\xi \right|^2 \]
\[ \leq \left( \int_{-R}^{-1} |\log(\xi + i\eta) - \log \lambda|^{-2} |\xi + i\eta|^{-1} d\xi \right) \left( \int_{-R}^{-1} |\mathcal{F}(\xi + i\eta)|^2 d\xi \right) ; \]
a similar estimate is valid for the last integral in (5.18).

It is easily seen that the first integral in the right hand side of the equality (5.18) tends to the first integral in the formula (5.15) (as \( R \to +\infty \)) if \( \eta = \eta_1 \) and to the second integral if \( \eta = \eta_2 \). We have shown that the right hand side of the equality (5.14) tends to the right hand side of (5.15) in the space \( X \) as \( R \to +\infty \). The equality (5.15) is proved.

On the next step we pass in (5.15) to the limit as \( \eta_1 \to 0^+ \) and then as \( \eta_2 \to +\infty \). As a result we get the equality (5.13) and complete the proof.

Let us establish the uniform in \( \eta \in [0, \epsilon] \) estimate
\[ \int_{-\infty}^{+\infty} |\log(\xi + i\eta) - \log \lambda|^{-2} |\xi + i\eta|^{-1} d\xi \leq \text{Const}(\epsilon, \lambda), \quad \epsilon > 0, \]
where \( \lambda \in \mathbb{C}^+ \) and \( \epsilon \) is a sufficiently small positive number, \( \epsilon << \Im \lambda \). If \( |\xi| < \epsilon, \eta \in [0, \epsilon], \) and \( \epsilon \) is sufficiently small then \( |\xi + i\eta| < \exp(-2 \max\{1, |\log \lambda|\}) \) and the inequalities
\[ |\log(\xi + i\eta) - \log \lambda| \geq |\log |\xi + i\eta|| - |\log \lambda| \geq |\log |\xi + i\eta||/2, \]
\[ |\log(\xi + i\eta) - \log \lambda|^2 |\xi + i\eta| \geq (\log |\xi + i\eta|)^2 |\xi + i\eta|/4 \geq (\log |\xi|)^2 |\xi|/4 \]
hold. Let \( R \) be a fixed number, \( R > \max\{1, |\lambda|\} \). For \( \xi \in [-R, -\epsilon] \cup [\epsilon, R] \) we have
\[ |\log(\xi + i\eta) - \log \lambda|^{-2} |\xi + i\eta|^{-1} \leq C(\epsilon), \quad \eta \in [0, \epsilon]. \]

Now the estimate (5.19) follows from the inequalities
\[ \int_{-\infty}^{+\infty} |\log(\xi + i\eta) - \log \lambda|^{-2} |\xi + i\eta|^{-1} d\xi \leq 8 \int_{0}^{\epsilon} (\log |\xi|)^{-2} |\xi|^{-1} d\xi \]
\[ + 2 \int_{\epsilon}^{R} C(\epsilon) d\xi + 2 \int_{R}^{+\infty} (\log \xi - \log |\lambda|)^{-2} \xi^{-1} d\xi \leq \text{Const}(\epsilon, \lambda). \]
Since \((\log(\xi + i\eta) - \log \lambda)^{-1}(\xi + i\eta)^{-1/2}\) tends to \((\log \xi - \log \lambda)^{-1}\xi^{-1/2}\) as \(\eta \to 0^+\) for \(\xi \in \mathbb{R} \setminus \{0\}\), and the estimate (5.19) holds, we conclude that

\[
\| (\log(\cdot + i\eta) - \log \lambda)^{-1}(\cdot + i\eta)^{-1/2} - (\log \cdot - \log \lambda)^{-1}(\cdot)^{-1/2}; L_2(\mathbb{R}) \| \to 0 \tag{5.20}
\]
as \(\eta \to 0^+\). Proposition 2.2 asserts that every analytic function \(C^+ \ni \lambda \mapsto \mathcal{F}(\lambda) \in X\) satisfying the uniform estimate (5.12) has non-tangential boundary limits \(\mathcal{F}_0 \in L_2(\mathbb{R}; X)\) and

\[
\| \mathcal{F}(\cdot + i\eta) - \mathcal{F}_0(\cdot); L_2(\mathbb{R}; X) \| \to 0, \quad \eta \to 0^+. \tag{5.21}
\]

Having (5.20) and (5.21) we can pass in (5.15) to the limit as \(\eta_1 \to 0^+\). We have

\[
\sqrt{\lambda} \mathcal{F}(\lambda) = \frac{1}{2\pi i} \left( \int_{\mathbb{R}} \frac{\mathcal{F}_0(\mu)}{(\log \mu - \log \lambda)\sqrt{\mu}} d\mu - \int_{\mathbb{R} + i\eta_2} \frac{\mathcal{F}(\mu)}{(\log \mu - \log \lambda)\sqrt{\mu}} d\mu \right). \tag{5.22}
\]

It remains to show that the second integral in (5.22) goes to zero in \(X\) as \(\eta_2 \to +\infty\). Let \(R > \max\{1, |\lambda|\}\). For all \(\eta > R\) we have

\[
\left| \int_{-\infty}^{+\infty} \log(\xi + i\eta) - \log \lambda \right|^{-2} d\xi \left| \int_{-\infty}^{+\infty} \frac{1}{\log(\xi + i\eta) - \log \lambda} d\xi \right|^{-1} \leq \int_{-R}^{R} \left( \log |\eta| - \log |\lambda| \right)^{-2} d\xi \\
+ 2 \int_{R}^{+\infty} \left( \log \xi - \log |\lambda| \right)^{-2} \xi^{-1} d\xi \leq \text{Const}(\lambda, R).
\]

As a consequence we get

\[
\| (\log(\cdot + i\eta) - \log \lambda)^{-1}(\cdot + i\eta)^{-1/2}; L_2(\mathbb{R}) \| \to 0, \quad \eta \to +\infty. \tag{5.23}
\]

This together with (5.12) allows us to pass in (5.22) to the limit as \(\eta_2 \to +\infty\). The equality (5.13) is valid for \(\mathcal{G} \equiv \mathcal{F}_0\). \(\Box\)

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