Covering 2-colored complete digraphs by monochromatic 
dominating digraphs

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Abstract

A digraph is $d$-dominating if every set of at most $d$ vertices has a common out-neighbor. For all integers $d \geq 2$, let $f(d)$ be the smallest integer such that the vertices of every 2-edge-colored (finite or infinite) complete digraph (including loops) can be covered by the vertices of at most $f(d)$ monochromatic $d$-dominating subgraphs. Note that the existence of $f(d)$ is not obvious – indeed, the question which motivated this paper was simply to determine whether $f(d)$ is bounded, even for $d = 2$. We answer this question affirmatively for all $d \geq 2$, proving $4 \leq f(2) \leq 8$ and $2d \leq f(d) \leq 2d \left(\frac{d+1}{d}\right)$ for all $d \geq 3$. We also give an example to show that there is no analogous bound for more than two colors.

Our result provides a positive answer to a question regarding an infinite analogue of the Burr-Erdős conjecture on the Ramsey numbers of $d$-degenerate graphs. Moreover, a special case of our result is related to properties of $d$-paradoxical tournaments.

1 Introduction

Throughout this note a directed graph (or digraph for short) is a pair $(V, E)$ where $V$ can be finite or infinite and $E \subseteq V \times V$ (so in particular, loops are allowed). A digraph is complete if $E = V \times V$. For $v \in V$, we write $N^+(v) = \{u : (v, u) \in E\}$ and $N^-(v) = \{u : (u, v) \in E\}$. For a positive integer $k$, we define $[k] := \{1, \ldots, k\}$. Note that regardless of whether $G = (V, E)$ is a graph or a digraph, if $H = (V', E')$ and $V' \subseteq V$ and $E' \subseteq E$, we will write $H \subseteq G$ and we will always refer to $H$ as a subgraph of $G$ rather than making a distinction between “subgraph” and “subdigraph.”

Let $G = (V, E)$ be a digraph. For $X, Y \subseteq V$ we say that $X$ dominates $Y$ if $(x, y) \in E$ for all $x \in X, y \in Y$. We say that $G$ is $d$-dominating if for all $S \subseteq V$ with $1 \leq |S| \leq d$, $S$ dominates some $w \in V$. Note that it is possible for $w \in S$, in which case we must have $(w, w) \in E$. Reversing all edges of a $d$-dominating digraph gives a $d$-dominated digraph. These notions are well studied for tournaments (see Section 3).

A cover of a digraph $G = (V, E)$ is a set of subgraphs $\{H_1, \ldots, H_t\}$ such that $V(G) = \bigcup_{i \in [t]} V(H_i)$. By a 2-coloring of $G = (V, E)$, we will always mean a 2-coloring of the edges of $G$; i.e. a function $c : E \rightarrow [2]$. Given a 2-coloring of $G$, we let $E_i$ be the set of edges receiving color $i$ (i.e. $E_i = c^{-1}(\{i\})$) and $G_i = (V, E_i)$ for $i \in [2]$. A cover of $G$ by monochromatic subgraphs is a cover $\{H_1, \ldots, H_t\}$ of $G$ such that for all $i \in [t]$ there exists $j \in [2]$ such that $H_i \subseteq G_j$.

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The following problem was raised in [4, Problem 6.6].

**Problem 1.1.** Given a 2-colored complete digraph $K$, is it possible to cover $K$ with at most four monochromatic 2-dominating subgraphs? (If not four, some other fixed number?)

Our main result is a positive answer for the qualitative part of Problem 1.1 in a more general form.

**Theorem 1.2.** Let $d$ be an integer with $d \geq 2$. In every 2-colored complete digraph $K$, there exists a cover of $K$ with at most $2 \times \sum_{i=1}^{d} d^{i} = 2d \left( \frac{d^{d+1}}{d-1} \right)$ monochromatic $d$-dominating subgraphs. In case of $d = 2$ there exists a cover of $K$ with at most eight monochromatic 2-dominating subgraphs.

For all integers $d \geq 1$, let $f(d)$ be the minimum number of monochromatic $d$-dominating subgraphs needed to cover an arbitrarily 2-colored complete digraph. Note that obviously $f(1) = 2$ since the two sets of monochromatic loops provide an optimal cover. For $d \geq 2$, Theorem 1.2 shows that $f(d)$ is well-defined. Example 1.3 below (adapted from [4, Proposition 6.3]) combined with Theorem 1.2 gives

$$4 \leq f(2) \leq 8 \quad \text{and} \quad 2d \leq f(d) \leq 2d \left( \frac{d^{d+1}}{d-1} \right) \quad \text{for all integers } d \geq 3. \quad (1)$$

**Example 1.3.** Let $K$ be a complete digraph on at least $2d$ vertices and partition $V(K)$ into non-empty sets $R_1, \ldots, R_d$ and $B_1, \ldots, B_d$, color all edges inside $R_i$ red, all edges inside $B_j$ blue, all edges from $R_i$ to $B_j$ red, all edges from $B_i$ to $R_j$ blue, all edges between $R_i$ and $R_j$ with $i \neq j$ blue, and all edges between $B_i$ and $B_j$ with $i \neq j$ red. One can check that every monochromatic $d$-dominating subgraph of $K$ is entirely contained inside one of the sets $R_1, \ldots, R_d, B_1, \ldots, B_d$.

Finally, the following example shows that for $d \geq 2$ there is no analogue of Theorem 1.2 for more than two colors (c.f. [4, Example 2.3]).

**Example 1.4.** Let $V$ be a totally ordered set and let $K$ be the complete digraph on $V$ where for all $i \in V$, $(i, i)$ is green and for all $i, j \in V$ with $i < j$, $(i, j)$ is red and $(j, i)$ is blue. Note that for $d \geq 2$ the only monochromatic $d$-dominating subgraphs are the green loops and thus no bound can be put on the number of monochromatic $d$-dominating subgraphs needed to cover $V$.

### 1.1 Motivation

A graph $G$ is $d$-degenerate if there is an ordering of the vertices $v_1, v_2, \ldots$ such that for all $i \geq 1$, $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| \leq d$ (equivalently, every subgraph has a vertex of degree at most $d$). Burr and Erdős conjectured [3] that for all positive integers $d$, there exists $c_d > 0$ such that every 2-coloring of $K_n$ contains a monochromatic copy of every $d$-degenerate graph on at most $c_d n$ vertices. This conjecture was recently confirmed by Lee [5].

The motivation for Problem 1.1 relates to the following conjecture also raised in [4, Problem 1.5, Conjecture 10.2] which can be thought of as an infinite analogue of the Burr-Erdős conjecture.

**Conjecture 1.5.** For all positive integers $d$, there exists a real number $c_d > 0$ such that if $G$ is a countably infinite $d$-degenerate graph with no finite dominating set, then in every 2-coloring of the edges of $K_n$, there exists a monochromatic copy of $G$ with vertex set $V \subseteq \mathbb{N}$ such that the upper density of $V$ is at least $c_d$. 


The case \( d = 1 \) was solved completely in \cite{3} (regardless of whether \( G \) has a finite dominating set or not). For certain 2-colorings of \( K_N \), described below, Theorem \ref{1.2} implies a positive solution to Conjecture \ref{1.5} for \( d \geq 2 \).

Suppose that for some finite subset \( F \subseteq \mathbb{N} \), we have a partition of \( \mathbb{N} \setminus F \) into (finitely or infinitely many) infinite sets \( X = \{X_1, \ldots, X_n, \ldots\} \). Also suppose that we have ultrafilters \( \mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n, \ldots \) on \( \mathbb{N} \) such that for all \( i \geq 1 \), \( X_i \in \mathcal{U}_i \). Finally, suppose that for all \( i, j \geq 1 \) there exists \( c_{i,j} \in [2] \) such that for all \( v \in X_i \), \( \{u, v\} \) has color \( c_{i,j} \) and \( X_j \in \mathcal{U}_j \). This last condition ensures that if there exists \( X_{i_1}, \ldots, X_{i_m} \) and \( X_j \) such that \( c_{i_{1,j}} = \cdots = c_{i_{m,j}} = c \), then every finite collection of vertices in \( X_{i_1} \cup \cdots \cup X_{i_m} \) has infinitely many common neighbors of color \( c \) in \( X_j \). Note that such a scenario can be realized as follows: For all \( i, j \), let \( c_{i,j} \in [2] \) and color the edges from \( X_i \) to \( X_j \) so that every vertex in \( X_i \) is incident with cofinitely many edges of color \( c_{i,j} \) (by using the half graph coloring\footnote{Given a totally ordered set \( Z \) and disjoint \( X, Y \subseteq Z \) the half graph coloring of the complete bipartite graph \( K_{X,Y} \) is a 2-coloring of the edges of \( K_{X,Y} \) where for all \( i \in X, j \in Y, \{i, j\} \) is red if and only if \( i \leq j \).}) when \( c_{i,j} \neq c_{j,i} \) for instance).

The above coloring of \( K_{\mathbb{N}} \) naturally corresponds to a 2-colored complete digraph in the following way: Let \( K \) be a 2-colored complete digraph on \( X \) where we color \((X_i, X_j)\) with color \( c \) if for all \( v \in X_i \), \( \{u, v\} \) has color \( c \). Now by Theorem \ref{1.2} \( K \) can be covered by \( t \leq f(d+1) \) monochromatic \((d+1)\)-dominating subgraphs \( G_1, \ldots, G_t \). Since \( \mathbb{N} \setminus F = \bigcup_{i \in [t]} \bigcup_{X \in V(G_i)} X \), there exists \( i \in [t] \) such that \( V_i := \bigcup_{X \in V(G_i)} X \) has upper density at least \( 1/f(d+1) \). Without loss of generality, suppose the edges of \( G_i \) are red. By the construction, \( V_i \) has the property that for all \( S \subseteq V_i \) with \( 1 \leq |S| \leq d+1 \), there is an infinite subset \( W \subseteq V_i \) such that every edge in \( E(S, W) \) is red. As shown in \cite[Proposition 6.1]{4}, if \( G \) is a graph satisfying the hypotheses of Conjecture \ref{1.5} then there exists a red copy of \( G \) which spans \( V_i \) and thus has upper density at least \( 1/f(d+1) \).

## 2 Covering digraphs, proof of Theorem \ref{1.2}

For a graph \( G \), we denote the order of a largest clique (pairwise adjacent vertices) in \( G \) by \( \omega(G) \). Given a 2-colored complete digraph \( K \) and a set \( U \subseteq V(K) \), define \( G[U]_{\text{blue}} \) to be the graph on \( U \) where \( \{u, v\} \in G[U]_{\text{blue}} \) if and only if \( (u, v) \) and \( (v, u) \) are blue in \( K \); define \( G[U]_{\text{red}} \) analogously.

Given positive integers \( \omega \) and \( d \), let \( f(\omega, d) \) be the smallest positive integer \( D \) such that if \( K \) is a 2-colored complete digraph on vertex set \( V \) where every loop has the same color, say red, and \( \omega(G[V]_{\text{blue}}) = \omega \), then \( V \) can be covered by at most \( D \) monochromatic \( d \)-dominating subgraphs. Also define \( f(0, d) = 0 \).

### Lemma 2.1.

1. \( f(1, 2) = 1 \)
2. \( f(\omega, d) \leq d(f(\omega-1, d) + 1) \) for all \( 1 \leq \omega \leq d \) (in particular, \( f(1, d) \leq d \)). In fact, all \( d \)-dominating subgraphs in the covering have the same color as the loops.

Note that the upper bound \( \omega \leq d \) is not strictly necessary, but we include it here for clarity since in the next lemma, we will prove a stronger result when \( \omega \geq d+1 \).

#### Proof.

Let \( K \) be a 2-colored complete digraph on vertex set \( V \) where all loops have the same color, say red.

1. is trivial since for all distinct \( u, v \in V \) both \( (u, u) \) and \( (v, v) \) are red and \( \omega(G[V]_{\text{blue}}) = 1 \) implies that either \( (u, v) \) or \( (v, u) \) is red.

To see (2), note first that we may assume that \( K \) itself is not spanned by a red \( d \)-dominating subgraph, otherwise we are done. This is witnessed by a set \( U = \{u_1, \ldots, u_d\} \subseteq V \), such that there is no \( w \in V \) with \( (u_i, w) \) red for all \( i \in [d] \).
For all $i \in [d]$ we define

$$W_i = \{v \in V : (v, u_i) \text{ is red}\}.$$ 

Note that $u_i \in W_i$ and $K[W_i]$ is spanned by a red $d$-dominating subgraph for all $i \in [d]$.

Set $V' = V \setminus \left( \bigcup_{i \in [d]} W_i \right)$ and define

$$T_i = \{v \in V' : (u_i, v) \text{ is blue}\}.$$ 

Note that by the definition of $V'$, $(v, u_i)$ is also blue for all $v \in T_i$ and $i \in [d]$. Moreover, from the selection of $U$, every vertex in $V'$ receives a blue edge from some vertex in $U$ and therefore $V' = \bigcup_{i=1}^{d} T_i$.

Note that if $\omega = 1$, then $T_i = \emptyset$ for all $i \in [d]$ and thus $\bigcup_{i \in [d]} W_i$ is a cover of $K$ with $d$ red $d$-dominating subgraphs; i.e. $f(1, d) \leq d = d(f(0, d) + 1)$.

Otherwise, we have that $\omega(K[T_i]\text{blue}) \leq \omega - 1$ and thus $K$ is covered by at most $d + d \cdot f(\omega - 1, d) = d(f(\omega - 1, d) + 1)$ red $d$-dominating subgraphs. \hfill \Box

**Lemma 2.2.** Let $K$ be a 2-colored complete digraph $K$ where $R$ is the set of red loops and $B$ is the set of blue loops. If $\omega(G[R]\text{blue}) \geq d + 1$, then $V(K)$ can be covered by at most $d$ red $d$-dominating subgraphs and at most one blue $d$-dominating subgraph. Likewise, if $\omega(G[B]\text{red}) \geq d + 1$. In particular, this implies $f(\omega, d) \leq d + 1$ for $\omega \geq d + 1$.

**Proof.** Suppose $\omega(G[R]\text{blue}) \geq d + 1$ and let $X = \{x_1, \ldots, x_d, x_{d+1}\} \subseteq R$ be a set of order $d + 1$ which witnesses this fact. For $i \in [d]$ we define

$$W_i = \{v \in V(K) : (v, x_i) \text{ is red}\}.$$ 

Note that $x_i \in W_i$ and $K[W_i]$ is spanned by a red $d$-dominating subgraph for all $i \in [d]$.

Set $V' = X \cup (V(K) \setminus (\bigcup_{i \in [d]} W_i))$ and note that for all $v \in V'$, $[v, X]$ is blue. Now let $S \subseteq V'$ such that $1 \leq |S| \leq d$. If $S \subseteq X$, then since $|S| < |X|$, there exists $x_i \in X \setminus S$ such that every edge in $[S, x_i]$ is blue; otherwise $|S \cap X| \leq d - 1$ and there exists $i \in [d]$ such that $x_i \notin S$ and every edge in $[S, x_i]$ is blue. So there is one blue $d$-dominating subgraph which covers $V'$, which together with the red $d$-dominating subgraphs $K[W_1], \ldots, K[W_d]$ gives the result.

When $\omega(G[B]\text{red}) \geq d + 1$, the proof is the same by switching the colors. \hfill \Box

Now we are ready to prove our main result.

**Proof of Theorem 2.2.** Let $V(K) = R \cup B$ where $R, B$ are the vertex sets of the red and blue loops, respectively. If $\omega(G[R]\text{blue}) \geq d + 1$ or $\omega(G[B]\text{red}) \geq d + 1$, then by Lemma 2.2 $R \cup B$ can be covered by at most $d + 1$ monochromatic $d$-dominating subgraphs. So suppose $\omega(G[R]\text{blue}) \leq d$ and $\omega(G[B]\text{red}) \leq d$. Now by Lemma 2.1 each of $K[R]$ and $K[B]$ can be covered by at most 4 monochromatic $d$-dominating subgraphs when $d = 2$, and by at most $\sum_{i=1}^{d} d^i \leq \sum_{i=1}^{d} d^i$ monochromatic $d$-dominating subgraphs when $d \geq 3$. \hfill \Box

### 3 Paradoxical tournaments

In the above section, we proved that $f(1, 2) = 1$ and $f(1, d) \leq d$ for all $d \geq 3$. Naturally, we wondered if the upper bound on $f(1, d)$ could be improved when $d \geq 3$ (since any improvement on $f(1, d)$ would improve the general upper bound on $f(d)$). In this section we show that it cannot; that is, $f(1, d) = d$ for all $d \geq 3$. 


A tournament is a digraph \((V,E)\) such that for all distinct \(x,y \in V\) exactly one of \((x,y),(y,x)\) is in \(E\) and \((x,x) \notin E\). Given a digraph \(G = (V,E)\), we say that \(S \subseteq V\) is an out-dominating set if for all \(v \in V \setminus S\), there exists \(u \in S\) such that \((u,v) \in E\), and we say that \(S\) is an in-dominating set if for all \(v \in V \setminus S\), there exists \(u \in S\) such that \((v,u) \in E\). Note that a tournament \(T\) is \(d\)-dominating (\(d\)-dominated) if and only if \(T\) has no in-dominating (out-dominating) set of order \(d\).

We call a \(d\)-dominating (\(d\)-dominated) tournament critical if its proper subtournaments are not \(d\)-dominating (\(d\)-dominated). For a tournament \(T\), let \(T^*\) be the digraph obtained from \(T\) by adding a loop at every vertex.

Our main result of this section is the following.

**Theorem 3.1.** For all integers \(d \geq 2\), if \(T\) is a critical \(d\)-dominated tournament with no \((d+1)\)-dominating subtournaments, then \(f(1,d+1) = d+1\).

However, before proving Theorem 3.1, we show that such a tournament exists for all \(d \geq 2\) from which we obtain the following corollary.

**Corollary 3.2.** For all \(d \geq 3\), \(f(1,d) = d\).

Note that the absence of loops and two-way oriented edges make the existence of \(d\)-dominated tournaments a nontrivial problem. This existence problem for \(d\)-dominated tournaments was proposed by Schütte (see [5]) and was first proved by Erdős [5] with the probabilistic method, then Graham and Spencer [6] gave an explicit construction using sufficiently large Paley tournaments.\(^2\)

Note that Babai [1] coined the term \(d\)-paradoxical tournament for what we refer to as \(d\)-dominated tournament. In this spirit, we say that a tournament is perfectly \(d\)-paradoxical if it is \(d\)-dominating, \(d\)-dominated, has no \((d+1)\)-dominating subtournaments, and has no \((d+1)\)-dominated subtournaments. A result of Esther and George Szekeres [7] combined with the fact that Paley tournaments are self-complementary implies that \(QT_{17}\) is perfectly 2-paradoxical and \(QT_{19}\) is perfectly 3-paradoxical. It is an open question (which to the best of our knowledge we are raising here for the first time) whether every Paley tournament is perfectly \(d\)-paradoxical for some \(d\). While we can’t settle that question, the following beautiful example of Bukh [2] shows that perfectly \(d\)-paradoxical tournaments exist for all \(d \geq 2\). We repeat his proof here (tailored to the terminology of this paper) for completeness.

**Example 3.3 (Bukh [2]).** For all integers \(d \geq 2\), there exists a perfectly \(d\)-paradoxical tournament. In particular, there exists a critical \(d\)-dominated tournament which has no \((d+1)\)-dominating subtournaments.

**Proof.** Let \(d\) be an integer with \(d \geq 2\) and let \(n = m(d+1)\) where \(m = 2^k\). Let \(V = \{0,1,\ldots,n-1\}\) and let \(G\) be the oriented graph on \(V\) such that \((i,j) \in E(G)\) if and only if \(1 \leq j-i \leq m-1\) (with addition modulo \(n\)). In other words \(G\) is the oriented \((m-1)\)st power of a cycle on \(n\) vertices. Now we define a tournament \(T\) by starting with the oriented graph \(G\) and for all distinct \(i,j \in V\), if \((i,j), (j,i) \notin E(G)\), then independently and uniformly at random let \((i,j) \in E(T)\) or \((j,i) \in E(T)\).

First note that every induced subgraph of \(G\) has an in-dominating set of order at most \(d+1\) and an out-dominating set of order at most \(d+1\) and thus the same is true of every subtournament of \(T\). This implies that \(T\) has no \((d+1)\)-dominating subtournaments and no \((d+1)\)-dominated subtournaments.

Now we claim that with positive probability, \(T\) has no out-dominating sets of order \(d\) and no in-dominating sets of order \(d\) and thus \(T\) is \(d\)-dominated and \(d\)-dominating. Let

\(^2\)For a prime power \(p\), \(p \equiv -1 \pmod{4}\), the Paley tournament \(QT_p\) is defined on vertex set \(V = [0,p-1]\) and \((a,b)\) is a directed edge if and only if \(a-b\) is a non-zero square in the finite field \(\mathbb{F}_p\).
\[ S \subseteq V \text{ with } |S| = d \text{ and let} \]

\[ N_G^+[S] = \{ v \in V : v \in S \text{ or there exists } u \in S \text{ such that } (u, v) \in E(G) \}. \]

Let \( V' := V \setminus N_G^+[S] \) and note that \(|N_G^+[S]| \leq dm\) and thus \(|V'| \geq m\). The probability that \( v \in V' \) is dominated by \( S \) in \( T \) is \( 1 - 2^{-d} \) and thus the probability that every vertex in \( V' \) is dominated by \( S \) is \((1 - 2^{-d})^{|V'|} \leq (1 - 2^{-d})^m \leq e^{-2^{-d}m} = e^{-4^d}\). Likewise for every vertex of \( V' \) dominating \( S \). So the expected number of out-dominating or in-dominating sets of order \( d \) is at most

\[ 2 \left( \frac{n}{d} \right) e^{-4^d} < 2(em)^d e^{-4^d} < 2(e^{3d+1})^d e^{-4^d} < 1 \]

(where the last inequality holds since \((3d + 1)d < 4^d\) for all \( d \geq 2 \)), which establishes the claim.

Starting with a perfectly \( d \)-paradoxical tournament \( T \), let \( T' \) be a minimal subtournament of \( T \) which is \( d \)-dominated. So \( T' \) is critical \( d \)-dominated and has no \((d + 1)\)-dominating subtournaments.

The proof of Theorem 3.1 will follow from two more general lemmas.

**Lemma 3.4.** Let \( T \) be a tournament and let \( d \geq 2 \). If \( T \) is 2-dominating and there exists a set \( W \subseteq V(T) \) with \(|W| = d \) such that \( W \) dominates exactly one vertex \( v \), then \( T^* \) is not \((d + 1)\)-dominating. In particular, if \( T \) is critical \( d \)-dominating, then \( T^* \) is not \((d + 1)\)-dominating.

**Proof.** Let \( W = \{w_1, \ldots, w_d\} \) and \( v \) be as in the statement. To see that \( T^* \) is not \((d + 1)\)-dominating, it is enough to prove that for some \( u \in N^+(v) \) the set \( W \cup \{u\} \) does not dominate any vertex in \( T^* \) (note that since \( T \) is 2-dominating, \( N^+(v) \neq \emptyset \)). Suppose for contradiction that this is not the case; that is, for all \( u \in N^+(v) \) the set \( W \cup \{u\} \) dominates some vertex \( x \) in \( T^* \). Note that by the definition of \( W \) and the fact that \( u \in N^+(v) \), it must be the case that \( x \in W \); without loss of generality, suppose \( x = w_1 \). This implies that for all \( i \in [d] \), \((w_i, w_1) \in E(T) \). But now this implies that for all \( u \in N^+(v) \), \( W \cup \{u\} \) dominates \( w_1 \). On the other hand since \( T \) is 2-dominating, it must be the case that there exists a vertex which is dominated by \( \{w_1, v\} \) in \( T \), but every outneighbor of \( v \) is an inneighbor of \( w_1 \) and thus we have a contradiction.

To get the second part of the lemma, first note that if \( T \) is critical \( d \)-dominating, then \( T \) is 2-dominating. Moreover, for all \( v \in V \), since \( T - v \) is not \( d \)-dominating there exists \( W = \{w_1, \ldots, w_d\} \subseteq V(T) \setminus \{v\} \) which does not dominate any vertex in \( V(T) \setminus \{v\} \), but since \( T \) is \( d \)-dominating, \( W \) must dominate \( v \).

If \( G = (V, E) \) is a digraph such that there exists \( w \in V \) such that \((v, w) \in E \) for all \( v \in V \) (including \( v = w \)), then note that \( G \) is \( d \)-dominating for all \( d \leq |V| \). In this case we call \( G \) trivially \( d \)-dominating.

**Lemma 3.5.** Let \( T \) be a tournament. If \( T \) is critical \( d \)-dominating, then \( T^* \) cannot be covered by less than \( d + 1 \) \((d + 1)\)-dominating subgraphs.

**Proof.** Suppose for contradiction that for some \( t \leq d \) there are \((d + 1)\)-dominating subgraphs \( H_1, \ldots, H_t \) which cover \( T^* \). Since \( T \) is critical \( d \)-dominating we have by Lemma 3.4 that \( T^* \) is not \((d + 1)\)-dominating, and thus all \( V(H_i) \) are proper subsets of \( V(T^*) \).

**Claim 3.6.** Each \( H_i \) is trivially \((d + 1)\)-dominating.
Proof.} The claim is obvious if $|V(H_i)| \leq d$; so suppose that $|V(H_i)| \geq d + 1$. Since $T$ is critical $d$-dominating, the subtournament $T_i$ of $T$ spanned by $V(H_i)$ is not $d$-dominating. This is witnessed by a set $W = \{w_1, \ldots, w_d\} \subseteq V(T_i)$ such that $W$ does not dominate any vertex in $U = V(T_i) \setminus W$. Let $u \in U$. Since $H_i$ is $(d+1)$-dominating, $W \cup u$ dominates some vertex $x \in V(H_i)$ which must be in $W$ from the definition of $W$. Without loss of generality, let $x = w_1$. This implies that $(u, x_1) \in E(T)$ and for all $i \in [d]$, $(w_i, w_1) \in E(T)$. But now this implies that for all $u \in U$, $W \cup \{u\}$ dominates $w_1$ and thus all vertices of $V(H_i)$ (including $w_1$) are oriented to $w_1$ proving the claim.

Claim 3.6 implies that for all $i \in [t]$ there is a vertex $v_i \in V(H_i)$ which is dominated by all vertices of $H_i$. But since $\bigcup_{i=1}^{t} V(H_i) = V(T)$, the set $\{v_1, \ldots, v_t\}$ does not dominate any vertex in $T$, contradicting the fact that $T$ is $d$-dominating.

**Proof of Theorem 3.1.** First note that $f(1, d+1) \leq d+1$ by Lemma 2.1.

Let $T_B$ be a tournament on vertex set $V$ such that $T_B$ is critical $d$-dominated and has no $(d+1)$-dominating subtournaments. Define the 2-colored complete digraph $K$ on $V$ by coloring all edges of $T_B$ blue, and all edges of $(V \times V) \setminus E(T_B)$ red. Let $T_R$ be the tournament with $E(T_R) = \{(y, x) : (x, y) \in E(T_B)\}$ and note that every edge of $T_R$ is red and $T_R$ has no loops. Since $T_B$ is critical $d$-dominated, this implies that $T_R$ is critical $d$-dominating (since $T_R$ is obtained by reversing all the edges of $T_B$).

Note that by the assumption on $T_B$, every monochromatic $(d+1)$-dominating subgraph in $K$ must be red. However, since $T_R$ is critical $d$-dominating, we get that $f(1, d+1) \geq d+1$ from Lemma 3.5.

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**References**

[1] L. Babai, Character sums, Weil’s estimate and paradoxical tournaments, [http://people.cs.uchicago.edu/~laci/reu02/paley.pdf](http://people.cs.uchicago.edu/~laci/reu02/paley.pdf) (2002).

[2] B. Bukh. Dominating sets in subtournaments of the Paley tournament, [https://mathoverflow.net/q/383245](https://mathoverflow.net/q/383245).

[3] S. A. Burr, P. Erdős. On the magnitude of generalized Ramsey numbers for graphs, Colloq. Math. Soc. János Bolyai 10 (1975), 215–240.

[4] J. Corsten, L. DeBiasio, P. McKenney. Density of monochromatic infinite subgraphs II, arXiv preprint [arXiv:2007.13277](https://arxiv.org/abs/2007.13277) (2020).

[5] P. Erdős. On a problem in Graph Theory, *Math. Gazette* 47 (1963) 220-223.

[6] R. L. Graham, J. H. Spencer, A constructive solution to a tournament problem, *Canadian Math. Bulletin* 14 (1971) 45-48.

[7] E. Szekeres, G. Szekeres, On a problem of Schütte and Erdős, *Math. Gazette* 49 (1965) 290-293.

[8] C. Lee. Ramsey numbers of degenerate graphs, *Ann. of Math.* 185 (2017), no. 3, 791-829.