Analysis of virtual meson production in solvable (1+1) dimensional scalar field theory

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Light-front time-ordered amplitudes are investigated in the virtual scalar meson production process in (1 + 1) dimensions using the solvable scalar field theory extended from the conventional Wick-Cutkosky model. There is only one Compton form factor (CFF) in the (1 + 1) dimensional computation of the virtual meson production process, and we compute both the real and imaginary parts of the CFF for the entire kinematic regions of $Q^2 > 0$ and $t < 0$. We then analyze the contribution of each and every light-front time-ordered amplitude to the CFF as a function of $Q^2$ and $t$. In particular, we discuss the significance of the “cat’s ears” contributions for gauge invariance and the validity of the “handbag dominance” in the formulation of the generalized parton distribution (GPD) function used typically in the analysis of deeply virtual meson production processes. We explicitly derive the GPD from the “handbag” light-front time-ordered amplitudes in the $-t/Q^2 \ll 1$ limit and verify that the integrations of the GPD over the light-front longitudinal momentum fraction for the DGLAP and ERBL regions correspond to the valence and nonvalence contributions of the electromagnetic form factor that we have recently reported [Phys. Rev. D 103, 076002 (2021)]. We also discuss the correspondence of the GPD to the parton distribution function for the analysis of the deep inelastic lepton-hadron scattering process and the utility of the new light-front longitudinal spatial variable $z$.

I. INTRODUCTION

One of the main goals in hadron physics is to understand the properties and structure of hadrons in terms of quarks and gluons. Since the elastic electron scattering from the nucleon unveiled the non-pointlike structure of the nucleon [1], the information on the spatial distributions of charge and current inside the nucleon has been obtained from the electromagnetic (EM) form factors. The information on the momentum distributions of quarks and gluons inside nucleons can also be accessed by the parton distribution functions (PDFs) measured through deep inelastic scattering (DIS) processes [2].

Since the DIS with the longitudinally polarized beam and the polarized target proton yielded the surprising result that the quarks and antiquarks inside the proton carry about only 30% of the total spin of the proton [3], one of the key observables to explore the so-called ‘proton spin puzzle’ has been the set of generalized parton distributions (GPDs) [4–12]. GPDs describe internal parton structures of the hadron, unifying the investigation of form factors and PDFs. In particular, the parton’s orbital angular momentum contribution to the nucleon spin may be estimated with the formulation of GPDs. The extraction of GPDs from the experimental data can be accessed mainly by hard exclusive electroproduction processes such as the deeply virtual Compton scattering (DVCS) and the deeply virtual meson production (DVMP) processes. In these processes, an electron interacts with a parton from the hadron, e.g., the nucleon, by the exchange of a virtual photon, and the struck parton radiates a real photon (DVCS process) or hadronizes into a meson (DVMP process) [13–18]. Both in DVCS and DVMP, the GPD formalism relies on the “handbag dominance” representing the factorization of the hard and soft parts in the respective scattering amplitudes. Here, the light-front dynamics (LFD) plays an important role in providing both the skewness $\xi$ and the light-front (LF) longitudinal momentum fraction $x$ of the parton struck by the probing virtual photon off the target. It is well known that the integrals of the leading-twist GPDs in the $s$ - and $u$-channel handbag amplitudes of both DVCS and DVMP processes carry the factorized denominator factors such as $1/(x - \xi)$ and $1/x$, respectively.

While the virtual Compton scattering (VCS) process is coherent with the Bethe-Heitler (BH) process [19], the virtual meson production (VMP) process does not possess complications from the involvement of the BH process and offers a unique way for experimental exploration of the hadronic structure for the study of Quantum Chromodynamics and strong interactions. In particular, the coherent electroproduction of pseudoscalar ($J^{PC} = 0^{-+}$) or scalar ($0^{++}$) mesons off a scalar target (for example, the $^4$He nucleus [20]) provides an excellent experimental terrain to discuss the fundamental nature of hadrons without involving much complication from the spin degrees of freedom. In Ref. [21], two of us discussed the most general formulation of the differential cross sections for the meson ($0^{-+}$ or $0^{++}$) production processes which involve only one or two hadronic form factors, respectively, when the target is a scalar particle. In particular, the beam spin asymmetry was discussed and our findings from the general formulation were contrasted with respect to the GPD formulation.

In the present article, we investigate the electroproduction process of a scalar meson off a scalar target, simulating for example, $\gamma^* + {}^4\text{He} \rightarrow f_0(980) + {}^4\text{He}$, in the one-loop level of an exactly solvable (1+1) dimensional scalar field theory extended from the conventional Wick-Cutkosky model [22]. The same scalar field model theory was previously applied to the analysis of the longitudinal charge density [23]. As the transverse rotations are absent in (1 + 1) dimensions, the advantage of the

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LFD with the LF time \( x^+ = x^0 + x^3 \) as the evolution parameter is maximized in contrast to the usual instant form dynamics (IFD) with the ordinary time \( x^0 \) as the evolution parameter. In LFD, the individual \( x^+ \)-ordered amplitudes contributing to the hadronic form factor are invariant under the boost, i.e., frame-independent, while the individual \( x^0 \)-ordered amplitudes in IFD are not invariant under the boost but dependent on the reference frame. As only one hadronic form factor is involved in \((1 + 1)\) dimensions for the electroproduction process of a scalar meson off a scalar target, the analysis may be regarded relatively simple without involving the beam spin asymmetry. In this work, we focus on analyzing the essential features of the LFD by benchmarking the \((1 + 1)\) dimensional characteristics of the VMP process.

The extraction of the hadronic form factor, the so-called Compton form factor (CFF), is made by utilizing the general formulation of the hadronic currents presented in Ref. [21]. The real and imaginary parts of the CFF are extracted explicitly. In order to explore the applicability of the handbag dominance adopted in the GPD formulation, we extract the GPD in the DVMP limit and verify that the integrations of the GPD over \( x \) for the DGLAP (\( 1 > x > \xi \)) and ERBL (\( \xi > x > 0 \)) regions correspond to the respective valence and nonvalence contributions of the electromagnetic form factor that we have recently presented in Ref. [23]. The correspondence of the GPD to the PDF is also discussed with the new LF longitudinal spatial variable \( \xi = x^2 - p^+ \) recently introduced in Ref. [24]. We then contrast the CFF obtained from the GPD formulation with the CFF result from the general formulation of the VMP process in the present exactly solvable scalar field model.

This article is organized as follows. In Section II, we present the kinematics of the virtual meson production process off the scalar target. Section III is devoted to the derivation of the exact form of the CFF in the VMP process within the one-loop level of the scalar field model in \((1 + 1)\) dimensions. Complete analyses for various LF time-ordered diagrams involved in the VMP process are presented as well. In Section IV, we extract the GPD, PDF, longitudinal probability density (LPD) in the LF coordinate space, and the EM form factor in the DVMP limit. Section V presents our numerical results for the CFF, GPD, PDF, LPD, and EM form factor of the scalar target simulating the mass arrangement of the \( \gamma^+ + ^4\text{He} \to f_0(980) + ^4\text{He} \) process. We summarize and conclude in Section VI.

## II. KINEMATICS

We begin with the kinematics involved in the virtual-photon scattering off the scalar target (\( M \)) for the production of the scalar meson (\( S \)),

\[
\gamma^+(q) + M(p) \to S(q') + M(p'),
\]

where the initial (final) scalar target state is characterized by the momentum \( p \) (\( p' \)) and the incoming virtual-photon and the outgoing meson by \( q \) and \( q' \), respectively. We shall use the component notation \( a = (a^+, a^-) \) in \((1 + 1)\) dimensions and the metric is specified by \( a^+ = a^\pm a^\mp \) and \( a \cdot b = (a^+ b^- + a^- b^+)/2 \).

Defining the four momentum transfer \( \Delta = p - p' \), we have

\[
p = \left( p^+, \frac{M_T^2}{p^+} \right),
\]

\[
p' = \left( (1 - \zeta) p^+, \frac{1}{p^+} \left( M_T^2 - \frac{t}{\zeta} \right) \right),
\]

and

\[
\Delta = \left( \zeta p^+, \frac{1}{\zeta p^+} \right),
\]

where \( M_T \) is the target mass and \( \zeta = \Delta^+ / p^+ \) is the skewness parameter describing the asymmetry in plus momentum. The squared momentum transfer then reads

\[
t = \Delta^2 = 2p \cdot \Delta = -\zeta^2 M_T^2 / (1 - \zeta) \leq 0,
\]

which defines \( \zeta \) in terms of \( t \) as

\[
\zeta = \frac{1}{2M_T^2} \left( t + \sqrt{t^2 - 4t M_T^2} \right),
\]

so that \( 0 \leq \zeta \leq 1 \) is taken. Considering the fact that \( q^+ \neq 0 \) in \((1 + 1)\) dimensions, we choose the momenta \( q \) and \( q' \) as

\[
q = \left( (\mu_s \zeta' - \zeta) p^+, \frac{Q^2}{p^+} \left( \frac{1}{\zeta'} + \tau \zeta \right) \right),
\]

\[
q' = \left( \mu_s \zeta' p^+, \frac{Q^2}{\zeta' p^+} \right),
\]

where \( \mu_s = M_S^2 / Q^2 \) and \( \tau = -t / Q^2 \) with \( q^2 = -Q^2 \) and \( M_S \) being the mass of the produced scalar meson. It also gives the definition of \( \zeta' \) through the relation with \( \zeta \) as

\[
\frac{\zeta'}{\zeta} = \frac{2}{1 + \mu_s - \tau + \sqrt{(1 + \mu_s + \tau)^2 - 4\tau}},
\]

where \( 0 \leq \mu_s \zeta' < \zeta \leq 1 \) is taken in deriving Eq. (7) so that \( q^+ < 0 \). The Bjorken variable \( x_{ Bj } = Q^2 / (2p \cdot q) \) is then given by

\[
x_{ Bj } = \frac{2t}{t(1 + \mu_s + \tau) - \sqrt{t(t - 4M_T^2)[(1 + \mu_s + \tau)^2 - 4\tau]}},
\]

The maximum value of \( x_{ Bj } \) for a given value of \( Q^2 \) is obtained by the condition that

\[
\frac{dx_{ Bj }}{dt} \bigg|_{t=t_{ th}} = 0,
\]

which determines the threshold momentum transfer squared \( t_{ th}(Q^2) \) as

\[
t_{ th}(Q^2) = \frac{M_T^2 (M_S^2 + Q^2)}{M_T + M_S}.
\]

This corresponds to the threshold point where momentum directions of the outgoing target and the produced meson are swapped in the center-of-momentum frame.
In deeply virtual limit where $Q^2$ is very large compared to other scales, $M^2_T$, $M^2_S$, and $|t|$, one can easily find that $\zeta \approx x_{Bj}$, i.e., $\zeta$ plays the role of $x_{Bj}$. We also note that $\zeta$ and $t$ are not independent in $(1 + 1)$ dimensions while they are in general independent of each other in $(3 + 1)$ dimensions because of the nonzero transverse component of $\Delta$. Explicitly, we have $t = - (\zeta^2 M^2_T + \Delta^2) / (1 - \zeta)$ in $(3 + 1)$ dimensions [12] where $\Delta^2$ is the transverse momentum transfer squared. Defining the skewness parameters $\zeta = \zeta_{1+1}$ and $\zeta = \zeta_{3+1}$ in $(1 + 1)$ and $(3 + 1)$ dimensions, respectively, one can obtain the allowed range of $\zeta_{3+1}$ as $0 \leq \zeta_{3+1} \leq \zeta_{1+1}$ for a fixed value of $t$. Here, we note that the limit $t \rightarrow 0$ implies $\zeta_{1+1} \rightarrow 0$, while there is no such correlation between $t \rightarrow 0$ and $\zeta_{3+1} \rightarrow 0$ unless $\Delta^2 \rightarrow 0$ is imposed as well. This indicates that the $(1 + 1)$ dimensional computation simulates only the forward production of the meson in the $(3+1)$ dimensional computations as expected intuitively. As shown in Eq. (4), the value of $-t$ is also not independent of the target mass $M_T$ in the $(1 + 1)$ dimensional computations. Thus, for a given $-t$ value, the skewness parameters $\zeta_{1+1}$ gets smaller as $M_T$ increases. The consequence of such constraint in $(1 + 1)$ dimensions will be revealed in the comparison of the CFF between the predictions from the GPD formulation deduced in the DVMP limit and our exact VMP computations obtained in the present work. In particular, the condition that $Q^2 > M^2_T$ may not be required for the DVMP limit in the $(1 + 1)$ dimensional computations because of the correlation among $\zeta_{1+1}$, $t$, and $M^2_T$. This would then imply that the condition $Q^2 > M^2_T$ may also not be required for the forward production of mesons in the $(3+1)$ dimensional computations.

### III. MODEL CALCULATIONS FOR VIRTUAL MESON PRODUCTION

In general, the total scattering amplitude $\mathcal{M}_{tot}^{\mu}$ for scalar meson production off the scalar target of Eq. (1) is expressed in terms of two independent CFFs [21] as

$$\mathcal{M}_{tot}^{\mu} = [ (\Delta \cdot q) q^\mu - q^2 \Delta^\mu ] \mathcal{F}_1 + [ (\Delta \cdot q) \mathcal{P}^\mu - (\mathcal{P} \cdot q) \Delta^\mu ] \mathcal{F}_2$$

$$\equiv A^{\mu} \mathcal{F}_1 + B^{\mu} \mathcal{F}_2,$$  

(11)

which defines $A^{\mu}$ and $B^{\mu}$, where $\mathcal{P} = p + p'$. The EM current conservation in Eq. (11) is assured by the condition $q \cdot \mathcal{F}_{tot} = 0$. The CFFs are measurable physical quantities and are related to the GPDs in the deeply virtual kinematic region, e.g., $Q^2 \gg -t$. For the VCS process, it is not possible to distinguish whether the emitted real photon comes from the loop process in the hadronic sector or from the scattered electron, i.e., the BH process [19]. However, the VMP does not have a BH-type process since the scalar meson cannot be emitted from the electron.

Furthermore, the two CFFs $\mathcal{F}_1$ and $\mathcal{F}_2$ in Eq. (11) are not linearly independent in $(1 + 1)$ dimensions since the two covariant

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1 The two CFFs given by Eq. (13) in Ref. [21] are obtained from the replacement of $\mathcal{F}_1 \rightarrow -(\mathcal{F}_1 + \mathcal{F}_2)$ and $\mathcal{F}_2 \rightarrow -\mathcal{F}_2$ in Eq. (11).

2 From now on, we drop the superscript $(1 + 1)$ in $\mathcal{M}_{tot}^{\mu(1+1)}$. 

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FIG. 1. Relevant Feynman diagrams for the reaction of $\gamma^* (q) + M(\mu) \rightarrow S(q') + M(p')$. (a) The $s$-box diagram, (b) the $u$-box diagram, and (c) the cat’s ears diagram.
‘c-channel’ amplitude. The inclusion of the c-channel amplitude is crucial to satisfy the gauge invariance.

In the solvable covariant BS model of (1 + 1)-dimensional scalar field theory, the scattering amplitudes of s-, u-, and c-channels in the one-loop expression are written as

\[ M_{s}^{u} = i e Q_{s} N \int \frac{d^{2}k}{(2\pi)^{2}} \frac{(2 + q)^{u}}{N_{k}N_{k+q}N_{k+\Delta k_{f}}N_{k-p}}, \]

\[ M_{s}^{u} = i e Q_{s} N \int \frac{d^{2}k}{(2\pi)^{2}} \frac{(2 - 2q + q)^{u}}{N_{k}N_{k+q}N_{k-\Delta k_{f}}N_{k-p}}, \]

\[ M_{c}^{u} = -i e Q_{c} N \int \frac{d^{2}k}{(2\pi)^{2}} \frac{(2 - 2p - q)^{u}}{N_{k}N_{k+q}D_{k-p}D_{k-p}}, \]  

where the denominators are coming from the intermediate scalar propagators shown in Fig. 1. Here, \( N_{1} = p_{1}^{2} - m_{Q_{s}}^{2} + i\epsilon \) and \( D_{p_{2}} = p_{2}^{2} - m_{Q_{c}}^{2} + i\epsilon \). The normalization constant \( N \) includes the coupling constants involved in this reaction. The electric charges satisfy the charge conservation, \( e_{Q_{s}} + e_{Q_{c}} = e_{M} \), where \( e_{M} \) is the charge of the scalar target and \( e_{Q_{j}} = -e_{Q_{j}} \).

While one may perform the manifestly covariant calculations of Eq. (15) using the Feynman parametrization, it has technical difficulties in analyzing the pole structures associated with the multi-dimensional integral of Feynman parameters. On the other hand, the LF calculations in (1 + 1) dimensions avoid such difficulties since it involves only the one-dimensional integral, as we shall show below.

In terms of the LF variables, the s-channel amplitude \( M_{s}^{u} \) in Eq. (15) can be rewritten as

\[ M_{s}^{u} = \frac{i e Q_{s} N}{(2\pi)^{2}} \int \frac{dk^{+}dk^{-}}{C_{s}} \frac{2k^{u} + q^{u}}{(k^{-} - k_{-})(k^{+} - k_{+})(k^{+} - k_{+})}, \]  

where \( C_{s} = k^{+}(k^{+} + q^{+})(k^{+} - \Delta^{+})(k^{+} - p^{+}) \) and

\begin{align*}
    k_{+} &= \frac{m_{Q_{s}}^{2}}{k^{+}} - i \frac{\epsilon}{k^{+}}, \\
    k_{-} &= \Delta^{+} - \frac{m_{Q_{s}}^{2}}{k^{+} - \Delta^{+}} - i \frac{\epsilon}{k^{+} - \Delta^{+}}, \\
    k_{+} &= -q^{-} + \frac{m_{Q_{s}}^{2}}{k^{+} + q^{+}} - i \frac{\epsilon}{k^{+} + q^{+}}, \\
    k_{b} &= p^{+} + \frac{m_{Q_{s}}^{2}}{k^{+} - p^{+}} - i \frac{\epsilon}{k^{+} - p^{+}}.
\end{align*}

Similar expressions can be obtained for \( M_{s}^{u} \) and \( M_{c}^{u} \). The obtained twelve LF time-ordered diagrams for the scattering amplitudes \( (M_{s}^{u}, M_{u}^{u}, M_{c}^{u}) \) corresponding to \( (s, u, c) \)-channels are depicted in Fig. 2.

For the s-channel amplitude \( M_{s}^{u} \) in Eq. (16), the Cauchy integration over \( k^{-} \) gives the three LF time \((\bar{c})\)-ordered contributions to the residue calculations, i.e., those coming from regions S1 \( (\Delta^{+} < k^{+} < p^{+}) \), S2 \( (q^{+} < k^{+} < \Delta^{+}) \), and S3 \( (0 < k^{+} < -q^{+}) \), respectively. There are several comments in order. We note that \( q^{+} \) in Eq. (6) is chosen to be \( q^{+} < 0 \) and the region S2 is absent for \( \mu_{z} = 0 \) limit as in the case of DVCS. The kinematic region S1 corresponds to the so-called “Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP)” region \([25–27]\), and the other two regions, S2 and S3, correspond to the so-called “Efremov-Radyushkin-Brodsky-Lepage (ERBL)” region \([28–30]\). The DGLAP and ERBL regions correspond to the valence contribution representing the particle-number-conserving process and the nonvalence one representing the particle-number-changing process, respectively.

In the DGLAP region of S1, where \( \Delta^{+} < k^{+} < p^{+} \), the residue is at the pole of \( k^{-} = k^{-}_{+} \) which is placed in the upper half of the complex \( k^{-} \) plane. Therefore, the Cauchy integration of \( M_{s}^{u} \) in Eq. (16) over \( k^{-} \) in this region leads to

\[ M_{s,\text{hand}} = -\frac{e_{Q_{s}} N}{4\pi} \int \frac{dk^{u}}{C_{s}(\Delta k_{f})}\frac{2k^{u}_{+} + q^{u}_{+}}{(\Delta k_{f})(\Delta k_{f})}, \]  

where \( k^{u}_{+} = (k^{+} + q^{+}) \) and \( \Delta k_{f} = k^{+} - k_{-}^{-} \). This amplitude corresponds to the “handbag” diagram shown in Fig. 2(d) in the s-channel.

In the ERBL region of S2, where \( -q^{+} < k^{+} < \Delta^{+} \), while two poles \( (k^{-}_{+}, k^{-}_{-}) \) are placed in the lower half of the complex \( k^{-} \) plane, the two other poles \( (k^{-}_{-}, k^{-}_{-}) \) lie on the upper half of the complex \( k^{-} \) plane. Taking the two poles \( k^{-} = (k^{-}_{+}, k^{-}_{-}) \) and using some mathematical manipulations for the denominators, e.g., \( 1/(\Delta k_{f}^{(i)}\Delta k_{f}^{(j)}) = (1/\Delta k_{f}^{(i)}) + 1/\Delta k_{f}^{(j)} \), we obtain two different types of LF time-ordered amplitudes in the S2 region as

\[ M_{s,\text{str}}^{u} = \frac{e_{Q_{s}} N}{4\pi} \int \frac{dk^{u}}{C_{s}(\Delta k_{f})}\frac{2k^{u}_{+} + q^{u}_{+}}{(\Delta k_{f})(\Delta k_{f})}, \]

\[ M_{s,\text{open}}^{u} = \frac{e_{Q_{s}} N}{4\pi} \int \frac{dk^{u}}{C_{s}(\Delta k_{f})}\frac{2k^{u}_{+} + q^{u}_{+}}{(\Delta k_{f})(\Delta k_{f})}, \]  

where \( M_{s,\text{str}}^{u} \) and \( M_{s,\text{open}}^{u} \) correspond to the so-called “stretched box” and “open diamond” diagrams shown in Figs. 2(b) and 2(c) in the s-channel, respectively.

In the other ERBL region of S3, where \( 0 < k^{+} < -q^{+} \), the residue is at the pole of \( k^{+} = k^{-}_{-} \), which is placed in the lower half of the complex \( k^{-} \) plane. The Cauchy integration of \( M_{s}^{u} \) in Eq. (16) over \( k^{-} \) in the S3 region leads to

\[ M_{s,\text{twist}}^{u} = \frac{e_{Q_{s}} N}{4\pi} \int \frac{dk^{u}}{C_{s}(\Delta k_{f})}\frac{2k^{u}_{+} + q^{u}_{+}}{(\Delta k_{f})(\Delta k_{f})}, \]  

which corresponds to what we call “twisted stretched box” diagram as shown in Fig. 2(a) in the s-channel.

Similarly, we can obtain the LF time-ordered amplitudes for \( M_{u}^{u} \) of Figs. 2(e)-(h) in the u-channel and \( M_{c}^{u} \) of Figs. 2(i)-(l) in the c-channel. Their explicit expressions read

\[ M_{c,\text{str}}^{u} = \frac{e_{Q_{c}} N}{4\pi} \int \frac{dk^{u}}{C_{c}(\Delta k_{f})}\frac{2k^{u}_{+} + q^{u}_{+}}{(\Delta k_{f})(\Delta k_{f})}, \]

\[ M_{c,\text{out}}^{u} = \frac{e_{Q_{c}} N}{4\pi} \int \frac{dk^{u}}{C_{c}(\Delta k_{f})}\frac{2k^{u}_{+} + q^{u}_{+}}{(\Delta k_{f})(\Delta k_{f})}. \]
FIG. 2. The LF time-ordered diagrams for the scattering amplitudes ($\mathcal{M}_s^\mu$, $\mathcal{M}_u^\mu$, $\mathcal{M}_c^\mu$) corresponding to the ($s, u, c$)-channels.

\begin{align}
\mathcal{M}_u^\mu &\approx -\frac{e_Q u}{4\pi} \int_{q^+}^{\Delta^+} dk^+ \frac{-2k_b^\mu + 2q^\mu - q^\mu}{C_u(\Delta k_{ab}^-)(\Delta k_{ba}^-)(\Delta k_{ib}^-)}, \\
\mathcal{M}_u^\mu_{\text{hand}} &\approx -\frac{e_Q u}{4\pi} \int_{p^+}^{\Delta^+} dk^+ \frac{2k_b^\mu - 2q^\mu + q^\mu}{C_u(\Delta k_{bf}^-)(\Delta k_{bu}^-)(\Delta k_{ib}^-)}, \\
\text{and} &\approx \frac{e_Q u}{4\pi} \int_{q^+}^{\Delta^+} dk^+ \frac{2k_b^\mu + 2p^\mu - q^\mu}{C_c(\Delta k_{cb}^-)(\Delta k_{bc}^-)(\Delta k_{cu}^-)}, \quad \mathcal{M}_c^\mu_{iji} = \mathcal{M}_c^\mu_{ijji} = \mathcal{M}_c^\mu_{ijij},
\end{align}

where $C_u = k^+(k^+ - q^+)(k^+ - \Delta^+)(k^+ - p^+)$, $C_c = k^+(k^+ - q^+)(k^+ - p^+ - q^+)(k^+ - p^+)$, and

\begin{align}
k_u^+ &= q^+ + \frac{m_{Q_1}^2}{k^+ - q^+} - i\frac{\epsilon}{k^+ - q^+}, \\
k_c^- &= p^+ + q^+ + \frac{m_{Q_2}^2}{k^+ - p^+ - q^+} - i\frac{\epsilon}{k^+ - p^+ - q^+}.
\end{align}

### B. Amplitudes from Effective Tree Diagrams

For the neural target, where $e_M = e_Q_1 + e_Q_2 = 0$, the gauge invariance condition $q \cdot \mathcal{M}_u^\mu = 0$ is guaranteed when $\mathcal{M}_u^\mu_{\text{loop}} = \mathcal{M}_u^\mu_{\text{open}}$. However, for the case of a charged target such as the “helium” nucleus, additional diagrams called “effective” tree diagrams, where the photon line is attached to the charged target, are required to ensure the gauge invariance. The effective tree contribution to the scattering amplitude is decomposed as

\begin{align}
\mathcal{M}_u^\mu_{\text{ET}} = \mathcal{M}_u^\mu_{s,\text{ET}} + \mathcal{M}_u^\mu_{u,\text{ET}},
\end{align}

where the corresponding LF time-ordered diagrams are presented in Fig. 3. The covariant scattering amplitudes $\mathcal{M}_u^\mu_{s,\text{ET}}$ and $\mathcal{M}_u^\mu_{u,\text{ET}}$ are obtained as

\begin{align}
\mathcal{M}_u^\mu_{s,\text{ET}} &= \frac{ieM_N}{(p + q)^2 - M_T^2} \int \frac{d^2 k}{2\pi} \frac{2p^\mu + q^\mu}{N_k N_{k - p - q}}, \\
\mathcal{M}_u^\mu_{u,\text{ET}} &= \frac{ieM_N}{(p - q)^2 - M_T^2} \int \frac{d^2 k}{2\pi} \frac{2p^\mu + q^\mu - 2q^\mu}{N_k N_{k - q - p - q}}.
\end{align}

In the LF calculations, the Cauchy integration over $k^-$ in Eq. (25) gives two LF time-ordered contributions to the residue calculations. For $\mathcal{M}_u^\mu_{s,\text{ET}}$, one comes from the valence region ($q^+ < k^+ < p^+ + q^+$) as shown in Fig. 3(a) and the other from the nonvalence region ($0 < k^+ < q^+$) as shown in Fig. 3(b). In the case of $\mathcal{M}_u^\mu_{u,\text{ET}}$, they come from the valence
FIG. 3. The LF time-ordered effective tree diagrams in s- and u-channels for a charged target case. (a,b) The valence and nonvalence contributions to $\mathcal{M}^{\mu}_{s,ET}$, respectively, and (c,d) the valence and nonvalence contributions to $\mathcal{M}^{\mu}_{u,ET}$, respectively.

region ($q^+ < k^* < p^*$) as shown in Fig. 3(c) and from the nonvalence region ($0 < k^* < q^*$) as shown in Fig. 3(d). In the valence (nonvalence) region for $\mathcal{M}^{\mu}_{s,ET}$ the residue is at the pole of $k^* = k^*_b(k^*_c)$, which is placed in the upper (lower) half of the complex $k^*$ plane. Similarly, in the valence (nonvalence) region for $\mathcal{M}^{\mu}_{u,ET}$ the residue is at the pole of $k^* = k^*_b(k^*_c)$, which is placed in the upper (lower) half of the complex $k^*$ plane. Thus, the Cauchy integrations of $\mathcal{M}^{\mu}_{s,ET}$ over $k^*$ lead to

\[
\mathcal{M}^{\mu}_{s,ET} = \frac{e_{MN}}{4\pi[(p + q)^2 - M_T^2]} \left[ \int_0^{q^+} dk^+ \frac{2p^\mu + q^{\mu}}{C_s'((k^*_b)^(-)(k^*_c)^-)} + \int_{q^+}^{p^+ + q^+} \frac{2p^\mu + q^{\mu}}{C_s'((k^*_b)^+)(k^*_c)^-)} \right],
\]

\[
\mathcal{M}^{\mu}_{u,ET} = \frac{e_{MN}}{4\pi[(p - q)^2 - M_T^2]} \left[ \int_0^{q^+} dk^+ \frac{2p^\mu + q^{\mu} - 2q^{\mu}}{C_u'((k^*_b)^+)(k^*_c)^-)} + \int_{q^+}^{p^+ - q^+} \frac{2p^\mu + q^{\mu} - 2q^{\mu}}{C_u'((k^*_b)^+)(k^*_c)^-)} \right],
\]

(26)

where $C_s' = k^+(k^* - q^*)(k^* - p^* - q^*)$ and $C_u' = k^+(k^* - q^*)(k^* - p^* - q^*)$.

It should be noted that the full amplitudes are obtained by including the exchanged diagrams ($Q_1 \leftrightarrow Q_2$) in Figs. 1-3. Although we do not give the corresponding amplitudes explicitly, their expressions can straightforwardly be obtained from the formulæ given above with the exchange of $Q_1 \leftrightarrow Q_2$. It should be understood that the contributions from the exchange of $Q_1 \leftrightarrow Q_2$ are included in our numerical computation of the full amplitudes even if they are not explicitly mentioned. Thus, the total scattering amplitudes for the neutral and charged tar-

gets may be summarized as

\[
\mathcal{M}^{\mu}_{\text{neutral}} = \mathcal{M}^{\mu}_{\text{loop}}(Q_1, Q_2) + (1 \leftrightarrow 2),
\]

\[
\mathcal{M}^{\mu}_{\text{charged}} = \mathcal{M}^{\mu}_{\text{loop}}(Q_1, Q_2) + \mathcal{M}^{\mu}_{\text{ET}}(Q_1, Q_2) + (1 \leftrightarrow 2),
\]

(27)

respectively. The CFF $\mathcal{F}_c$ for the charged target such as the “helium” nucleus is then computed by

\[
\mathcal{F}_c^{\text{VMP}}(Q^2, t) = \frac{\mathcal{M}^{\mu}_{\text{charged}}((\Delta \cdot q)q^{\mu} - q^2s^{\mu})}{\mathcal{M}^{\mu}_{\text{loop}}(Q_1, Q_2)},
\]

(28)

which is valid for each component ($\mu = +, -$) of the current in $(1 + 1)$ dimensions.

IV. DEEPLY VIRTUAL MESON PRODUCTION LIMIT

In this section, we analyze the amplitude in the DVMP limit, where $Q^2$ is larger than the other scales, namely, $Q^2 \gg (M_T^2, M_s^2, - i)$. From $Q^2 \gg (M_T^2, - i)$, we have ($\mu_s, \tau \rightarrow 0$, which leads to $\zeta = \zeta', q^- = q'' = Q^2/\zeta p^s$, $q^+ = -\Delta^s$, and $q^{*0} = 0$ from Eqs. (6) and (7). Furthermore, we also have $\Delta k^i_b = \Delta k_f^i = \Delta k^i_s = q^-$ in the energy denominators for the scattering amplitudes given by Eqs. (18)-(20). However, it should be noted that the condition $Q^2 \gg M_T^2$ is not used here in taking the DVMP limit.

A. Generalized Parton Distribution

In the DVMP limit, the time-ordered amplitudes for the s-channel with $\mu = +$ given by Eqs. (18)-(20) are now reduced to

\[
\mathcal{M}^{\mu}_{s,\text{hand}}^{\text{DVMP}} = \frac{e_{MN}}{4\pi q} \int_{s^*}^{\Delta^2} dk^+ \frac{-2k^+ + \Delta^s}{C_s'(k^*_b - k^*_i)(k^*_b - k^*_f)},
\]

\[
\mathcal{M}^{\mu}_{s,\text{stret}}^{\text{DVMP}} = \frac{e_{MN}}{4\pi q} \int_{0}^{\Delta^2} dk^+ \frac{2k^+ - \Delta^s}{C_s'(k^*_b - k^*_i)(k^*_b - k^*_f)},
\]

(29)

at the leading order in $Q^2$, where we have used $q^+ = -\Delta^s$. The kinematic region for both the “open diamond” and “stretched box” diagrams given by Eq. (19) vanishes in the limit of $q^+ = -\Delta^s$. Also, the “effective tree” amplitude $\mathcal{M}^{\mu}_{s,ET}^{\text{DVMP}}$ in Eq. (26) does not contribute in the DVMP limit.

Similarly, the reduced amplitudes for the $u$-channel in the DVMP limit are given by

\[
\mathcal{M}^{\mu}_{u,\text{hand}}^{\text{DVMP}} = \frac{e_{MN}}{4\pi q} \int_{s^*}^{\Delta^2} dk^+ \frac{2k^+ - \Delta^s}{C_u'(k^*_b - k^*_i)(k^*_b - k^*_f)},
\]

\[
\mathcal{M}^{\mu}_{u,\text{stret}}^{\text{DVMP}} = \frac{e_{MN}}{4\pi q} \int_{0}^{\Delta^2} dk^+ \frac{-2k^+ + \Delta^s}{C_u'(k^*_b - k^*_i)(k^*_b - k^*_f)}
\]

(30)

from Eq. (21). Neither the “effective tree” amplitude $\mathcal{M}^{\mu}_{u,ET}^{\text{DVMP}}$ in Eq. (26) nor the amplitudes of the “cat’s ears” in Eq. (22) contribute in the DVMP limit.
Combining both $s$- and $u$-channel amplitudes given by Eqs. (29) and (30) and using the longitudinal momentum fraction $x = k^+/p^+$ ($0 \leq x \leq 1$), we obtain the DVMP amplitude in the leading order of $Q^2$ as the factorized form of the hard and soft parts given by

$$M^{\text{DVMP}}_{s+u} = \frac{\epsilon_0}{4\pi Q^2} \int_0^1 dx \left( \frac{1}{x-\zeta} - \frac{1}{x} \right) H(\xi, x, t),$$

(31)

where

$$H(\xi, x, t) = \begin{cases} H_{\text{ERBL}}(\xi, x, t), & \text{for } 0 \leq \xi \leq \zeta, \\ H_{\text{DGLAP}}(\xi, x, t), & \text{for } \zeta < \xi \leq 1 \end{cases}$$

(32)

is identified as the GPD [4–11]. The GPD function $H(\xi, x, t)$ is naturally represented by the sum of the LF nonvalence contribution to the ERBL ($0 \leq \xi < \zeta$) and the valence contribution to the DGLAP ($\xi \leq x \leq 1$) region as shown in Fig. 4.

Respectively, $H_{\text{ERBL}}(\xi, x, t)$ and $H_{\text{DGLAP}}(\xi, x, t)$ are given by

$$H_{\text{ERBL}} = \frac{N}{x(\xi-x)(1-x)} \left( \frac{2x-\xi}{M_f^2 - M_0^2} \right) \left( t + \frac{\zeta^2m_1Q_1}{x(\xi-\zeta)} \right)$$

$$H_{\text{DGLAP}} = \frac{N}{x(x-\xi)(1-x)} \left( \frac{2x-\xi}{1-\xi} \right) \left( \frac{t}{M_f^2 - M_0^2} \right)$$

(33)

where

$$M_f^2 = \frac{m_1^2}{x'} + \frac{m_2^2}{1-x}, \quad M_0^2 = \frac{m_1^2}{x'} + \frac{m_2^2}{1-x}$$

(34)

with $x' = (x-\xi)/(1-\xi)$. It can be checked that $H_{\text{ERBL}}(\xi, x, t)$ and $H_{\text{DGLAP}}(\xi, x, t)$ obtained in this model are continuous and finite at the boundary $x = \xi$, namely, $H_{\text{ERBL}}(\xi, \xi, t) = H_{\text{DGLAP}}(\xi, \xi, t) = H(\xi, \xi, t)$, which is written explicitly as

$$H(\xi, \xi, t) = \frac{N\xi}{(1-\xi)m_1^4 + \zeta^2m_1^2Q_1m_2^2 + (\xi-1)M_f^2}.$$  

(35)

It is related with the imaginary part of the DVMP amplitude $M^{\text{DVMP}}_{s+u}$ in Eq. (31). As we have mentioned in Sec. II, $\zeta$ and $t$ are not independent of each other in $(1+1)$ dimensions unlike the $(3+1)$ dimensional case and the explicit expression in Eq. (35) is given by a function of $\zeta$ only.

In the DVMP limit and at the leading order of $Q^2$, the contributions from “effective tree” amplitudes are suppressed and only the $s$- and $u$-channel loop amplitudes contribute. Effectively, the DVMP results given by $M^{\text{DVMP}}_{s+u}$ are independent of the electric charge of the target, whether it is charged or neutral. Taking into account the corresponding prefactor in Eq. (13) relating the scattering amplitude to the CFF given by

$$\langle dz \rangle = \frac{1}{2} Q^2 \zeta p^+ \left[ 1 + \frac{t}{Q^2} + \cdots \right],$$

(36)

we obtain the CFF in the DVMP limit at the leading order of $Q^2$ denoted by $F^{\text{DVMP}}$ as

$$F^{\text{DVMP}}(Q^2, t) = \frac{M^{\text{DVMP}}_{s+u}}{2Q^2 \zeta p^+}.$$  

(37)

In view of the QCD collinear factorization theorem at the leading twist for the DVMP process [31], the comparison of $F^{\text{DVMP}}(Q^2, t)$ of Eq. (37) with the exact results $T_r(Q^2, t)$ of Eq. (28) would be very interesting as it allows us to explore the valid kinematic region for the GPD formulation based on the handbag dominance in the leading order of $Q^2$. We compare the numerical results of $F_r$ and the leading twist $F^{\text{DVMP}}(Q^2, t)$ in Sec. V.

B. Parton Distribution Functions

In the forward limit $(\zeta, t) \rightarrow 0$, we have $H(0, x, 0) = H_{\text{DGLAP}}(0, x, 0) \equiv q_v(x)$, where

$$q_v(x) = \frac{2N}{x(1-x)(M^2_f - M^2_0)^2}.$$  

(38)

In this limit, $H_{\text{ERBL}}(0, x, 0) = 0$ and $q_v(x)$ corresponds to the ordinary PDF representing the probability to find the constituent inside the hadron as a function of the momentum fraction $x$ carried by the constituent in the valence sector. The corresponding LF wave function $\psi(x)$ of the target hadron in the momentum space may be written as

$$\psi(x) = \sqrt{q_v(x)},$$

(39)

which satisfies $\int_0^1 dx |\psi(x)|^2 = \frac{1}{2} \int_1^0 dy q_v(y) = 1$. We then obtain the $n$-th moment of $q_v(x)$ defined by [33]

$$\langle y_n \rangle = \int_1^0 dy y^n q_v(y),$$

(40)

where $y = 2x - 1$.

Introducing the longitudinally boost-invariant dimensionless LF spatial variable $\tilde{z} = p^+ x'$, which is canonically conjugate to $x$ [24, 34, 35], the LF wave function $\psi(\tilde{z})$ in the LF coordinate space $\tilde{z}$ evaluated at $x' = 0$ can be defined by

$$\psi(\tilde{z}) = \frac{1}{\sqrt{2\pi}} \int dx \psi(x) e^{ikx} = \frac{1}{\sqrt{2\pi}} \int_0^1 dx \psi(x)e^{i\tilde{z} x/2}.$$  

(41)
as the Fourier transform of $\psi(x)$ in $(1 + 1)$ dimensions. Then, the longitudinal probability density $g(\vec{z})$ in the LF coordinate space $\vec{z}$ is given by

$$g(\vec{z}) = |\psi(\vec{z})|^2,$$

(42)

which satisfies $\int_0^\infty g(\vec{z}) d\vec{z} = 1$. Detailed discussions on the three dimensional version of Eq. (41), $\psi(\vec{z}, \vec{b})$, which includes the transverse distance $\vec{b}$ of the struck constituent from the transverse center of momentum were provided in Refs. [24, 34, 35].

C. Moments of GPD

In general, the $n$-th moment of the GPD is defined by

$$F_n(\zeta, t) = \int_0^1 \frac{dx}{1 - \zeta/2} x^{n-1} H(\zeta, x, t).$$

(43)

It is well known that the polynomiality conditions [36, 37] for the $n$-th moment of the GPD require that the highest power of $\zeta$ in the polynomial expression of $F_n(\zeta, t)$ should not be larger than $n$. These polynomiality conditions are fundamental properties of the GPD, which follow from the Lorentz invariance.

The first moment of $H(\zeta, x, t)$ is related to the EM form factor $F_M(t)$ of the target $M$ by the following sum rule [4–7]:

$$F_M(t) = \int_0^1 \frac{dx}{1 - \zeta/2} H(\zeta, x, t).$$

(44)

In the $(3 + 1)$ dimensional analysis, the full result of the EM form factor $F_M(t)$ is independent of $\zeta$ so that $F_1(\zeta, t) = F_M(t)$ since the two variables $\zeta$ and $t$ are independent of each other. However, in $(1 + 1)$ dimensions, the moment $F_n(\zeta, t)$ should be a function of a single variable, $F_n(\zeta)$ or $F_n(t)$, since the two variables are related to each other. For example, $\zeta = 0$ and $t = 0$ correspond to $\zeta = \infty$ and $t = \infty$, respectively. In other words, the interval of $\zeta = [0, 1]$ covers the entire range of the momentum transfer squared $-t = [0, \infty]$ in the $n$-th moment of the GPD in $(1 + 1)$ dimensions. Furthermore, all the moments vanish at $\zeta = 1$, i.e., $F_n(\zeta = 1) = 0$, which hinders to check the polynomiality conditions. To circumvent this problem in checking the polynomiality conditions due to $F_n(\zeta = 1) = 0$, we redefine the moments as

$$\tilde{F}_n(\zeta) = F_n(\zeta)/F_1(\zeta),$$

(45)

so that $\tilde{F}_1(\zeta)$ is independent of $\zeta$. In Sec. V, we will discuss how our model calculations for $\tilde{F}_n(\zeta)$ satisfy the polynomiality conditions.

The normalization factor $N$ is fixed by the condition $F_M(0) = 1$ and given by

$$N = \frac{m^2_{Q_1} m^2_{Q_2} (1 - \omega^2)^2}{1 - \omega^2 + \omega \sqrt{1 - \omega^2} T_\omega},$$

(46)

where

$$\omega = \frac{M^2_T - m^2_{Q_2} - m^2_{Q_1}}{2m_{Q_1}m_{Q_2}},$$

(47)

and

$$T_\omega = \tan^{-1} \left( \frac{m_{Q_1} + m_{Q_2} \omega}{m_{Q_1} \sqrt{1 - \omega^2}} \right) + \tan^{-1} \left( \frac{m_{Q_1} + m_{Q_2} \omega}{m_{Q_2} \sqrt{1 - \omega^2}} \right).$$

(48)

We note that the EM form factor obtained by using Eqs. (33) and (44) is identical to the form factor obtained in our previous publication [23] within the same solvable $\phi^3$ scalar field model in $(1 + 1)$ dimensions [38–40].

V. NUMERICAL RESULTS

For the numerical computation, we simulate the electroproduction of a scalar meson off the scalar target $^4$He with the electric charge $e_M = +2e$ using the $(1 + 1)$ dimensional scalar field theory discussed in the present work. For our numerical calculations and analyses, we thus take the target and scalar meson masses as $M_T = 3.7$ GeV and $M_S = 0.98$ GeV. In this case, the threshold momentum transfer at $Q^2 = 0$ is given by

$$t_{th}(Q^2 = 0) = -M_T M_S^2 / (M_T + M_S) \approx -0.76 \text{ GeV}^2.$$

For the constituent mass, we use the equal mass for both constituents, $m_{Q_1} = m_{Q_2} = 2 \text{ GeV}$, so that the $^4$He target is a weakly bound state, as $M_T < m_{Q_1} + m_{Q_2}$ but $M_T^2 > m^2_{Q_1} + m^2_{Q_2}$. However, we will discuss the cases with some variations of constituent masses as needed for comments. For the consistency of our numerical analysis, we use the same normalization constant $N$ given by Eq. (46) for all physical observables such as the CFF $\tilde{F}(Q^2, t)$, the GPD $H(\zeta, x, t)$, and the EM form factor $F_M(t)$ throughout the present work.

A. CFF in VMP

As we have discussed before, the total scattering amplitude for a charged target $\mathcal{M}^\mu = \mathcal{M}^\mu_{\text{ch}}$ given by Eq. (27), i.e., the full results without any approximation, should satisfy the condition that $q \cdot \mathcal{M} = 0$. Furthermore, since $\mathcal{M}^\mu$ is in general a complex-valued function, even for the space-like region $Q^2 > 0$, the real and imaginary part of $\mathcal{M}^\mu$ can be shown to satisfy the gauge invariance condition separately, i.e., $\text{Re}[q \cdot \mathcal{M}] = 0$ and $\text{Im}[q \cdot \mathcal{M}] = 0$. We first check numerically whether these conditions are met by our amplitudes. The solid line of Fig. 5 shows that the gauge invariance is observed by the exact amplitude $\mathcal{M}^\mu_{\text{ch}} = \mathcal{M}^\mu_{\text{ch}}(\mathcal{S} \pm \mathcal{C} + \mathcal{ET})$ for the range of $0 \leq Q^2 \leq 40 \text{ GeV}^2$ at $t = t_{th}(Q^2 = 0)$. To estimate the c-channel contribution, we turn off the c-channel contributions, and the results for the real and imaginary parts of $q \cdot \mathcal{M}_{\text{ch}}(\mathcal{S} \pm \mathcal{C} + \mathcal{ET})$ are plotted by the dashed and dotted lines, respectively, in Fig. 5. This evidently shows that the omission of the “cat’s ears” diagrams violates the Ward identity. The violation is

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3 Therefore, we mimic the reaction of $f_0(980)$ production off the $^4$He targets in $(1 + 1)$ dimensions assuming that the $^4$He nucleus is a bound state of two (scalar) deuterons.
more serious at the smaller $Q^2$ region, although the degree of deviation weakens at the larger $Q^2$ region as anticipated.

We then compute the CFF in VMP, $\mathcal{F}_{c,\text{VMP}}(Q^2, t)$. Shown in Fig. 6 are the real part, the imaginary part, and the modulus of the amplitude $\mathcal{F}_{c,\text{VMP}}$ obtained from $\mathcal{M}_{\text{charged}}$ of Eq. (28) in the range of $0 \leq Q^2 \leq 10 \text{ GeV}^2$ at $t = t_0(Q^2 = 0) \approx -0.7593 \text{ GeV}^2$. In order to explore the sensitivity of $\mathcal{F}_{c,\text{VMP}}(Q^2, t)$ on the constituent mass, we vary the constituent mass, and repeat the computations for $m_{Q_1} = 1.9, 2.0$, and 2.1 GeV, while keeping $m_{Q_2} = m_{Q_1}$.\(^4\) The results for $m_{Q_1} = 1.9, 2.0$, and 2.1 GeV are presented by the dot-dashed, solid, and dashed lines, respectively. The close inspection of Fig. 6 leads to the following comments. (i) The real part has a hump structure, and the peak locates at the higher values of $Q^2$ with the lesser pronounced hump structures as the binding energy increases, as shown in Fig. 6(a). (ii) The magnitude of the imaginary part gets larger as the binding energy increases as shown in Fig. 6(b). (iii) As a result, the hump behavior of $|\mathcal{F}_{c,\text{VMP}}(Q^2, t)|$ shown in Fig. 6(c) appears strong near $Q^2 = t_0(Q^2 = 0)$ for weak binding energies, but it goes away as the binding energy increases. Also, there is no hump structure in $|\mathcal{F}_{c,\text{VMP}}(Q^2, t)|$ for $Q^2 > t_0(Q^2 = 0)$ region and the binding energy effect is getting smaller as $Q^2$ increases.

The left and right panels of Fig. 7 respectively show the three-dimensional and contour plots of $|\mathcal{F}_{c,\text{VMP}}|^2$, $\text{Im}[\mathcal{F}_{c,\text{VMP}}]^2$, and $|\mathcal{F}_{c,\text{VMP}}|^2$ for the range of $0 \leq Q^2 \leq 20 \text{ GeV}^2$ and $-2 \text{ GeV}^2 \leq t \leq 0$. Both the real (top panel) and imaginary (middle panel) parts are going to zero as $t \to 0$ regardless of the value of $Q^2$. For $|t| \leq Q^2$, the real part of $\mathcal{F}_{c,\text{VMP}}$ shows a gradual crest along the straight line of $|t| \approx Q^2$, and the imaginary part has a trough located at $|t| \approx Q^2$. Both the real and imaginary parts rapidly approach to zero as $|t|$ decreases to zero for the small $Q^2$ region from the crest and trough, respectively. The modulus of the CFF also has a crest around $|t| \approx Q^2$ and gradually decreases as $Q^2$ increases and $|t|$ decreases.

B. CFF and GPD in DVMP limit

In the DVMP limit, where $Q^2 \gg (M_T^2, |t|)$ but not explicitly involving $Q^2 \gg M_T^2$ due to the correlation among $\zeta_{1+1}, t$ and $M_T^2$ as discussed in Sec. II, the scattering amplitude for the scalar meson production from either neutral or charged scalar target is reduced to the DVMP amplitude $\mathcal{M}_{s+u}^{\alpha+\text{DVMP}}$. Regardless of neutral or charged target, the $s$- and $u$-channel amplitudes are factorized in the DVMP limit as discussed in Sec. IV A and $\mathcal{M}_{s+u}^{\alpha+\text{DVMP}}$ is given by the factorized form of the hard scattering part and the soft GPD $H(\zeta, x, t)$ as shown in Eq. (31). In order to find the region where the DVMP limit is valid, we compare the CFF $\mathcal{F}_{\text{DVMP}}$ in Eq. (37) obtained from $\mathcal{M}_{s+u}^{\alpha+\text{DVMP}}$ with the exact solution $\mathcal{F}_{c,\text{VMP}}$ presented in Fig. 6.

In Fig. 8, we compare $\mathcal{F}_{\text{DVMP}}$ (solid lines) and the leading twist $\mathcal{F}_{\text{DVMP}}$ (dashed lines) for the range of $0 \leq Q^2 \leq 10 \text{ GeV}^2$. Figures 8 (a,c) and (b,d) show $\text{Re}[\mathcal{F}], \text{Im}[\mathcal{F}]$, at $t = t_0(Q^2 = 0) \approx -0.7593 \text{ GeV}^2$ and $t = -2 \text{ GeV}^2$, respectively. There are several points to comment on the results shown in Fig. 8. (i) While the exact solution $\mathcal{F}_{c,\text{VMP}}$ is finite at $Q^2 = 0$ with a hump behavior near $Q^2 = -t$, the results of $\mathcal{F}_{\text{DVMP}}$ obtained at the leading order of $Q^2$ in the DVMP limit do not have any hump structure but blows up in the vicinity of $Q^2 = 0$. (ii) The agreement between $\mathcal{F}_{c,\text{VMP}}$ and $\mathcal{F}_{\text{DVMP}}$ can be seen at large $Q^2$ region, but it reaches faster as the smaller $-t$ value is used. For instance, the VMP and DVMP results agree for $Q^2 \geq 6 \text{ GeV}^2$ when $-t \approx 0.7593 \text{ GeV}^2$ is fixed. This indicates that the validity of the GPD handbag approximation is governed not just either by $-t$ or $Q^2$, but by $-t/Q^2$. Figure 8 shows that it is valid in the region of $-t/Q^2 \leq 0.125$ with $-t \approx 0.7593 \text{ GeV}^2$. On the other hand, for $-t \approx 2 \text{ GeV}^2$, the agreement of the VMP and DVMP CFFs can be seen at higher $Q^2$, i.e., $Q^2 > 10 \text{ GeV}^2$, which corresponds to $-t/Q^2 = 0.2$. To have better agreement both for the real and imaginary parts of the CFF, $Q^2$ should be even larger to get $-t/Q^2 \lesssim 0.1$. Therefore, we find that the GPD handbag approximation can be valid only for a small value of $-t/Q^2$, although the critical value of $-t/Q^2$ appears somewhat larger as $-t$ increases in our $(1+1)$ dimensional results. This indicates that, for realistic VMP measurements in $(3+1)$ dimensions, very forward scattering region should be required to invoke the GPD handbag approximation as the forward scattering region would allow a very small value of $-t$. Since $-t$ is not independent of the target mass $M_T$ as shown in Eq. (4) in $(1+1)$ dimensions, the skewness parameter $\zeta$ gets smaller as $M_T$ increases for given $-t$ value. This indicates that the GPD handbag approximation agrees with the exact VMP result faster with a larger $M_T$ than a smaller $M_T$, which appears the characteristic of the $(1+1)$

\(^4\) These masses give the binding energy $B = m_{Q_1} + m_{Q_2} - M_T$ in the range of (0.1–0.5) GeV.
FIG. 6. The Compton form factor $F_{\text{VMP}}(Q^2, t)$. (a) its real part, (b) its imaginary part, and (c) its modulus for three constituent masses, $m_{Q_1} = m_{Q_2} = 1.9, 2.0$, and $2.1$ GeV at the momentum transfer squared $t = t_0 = -0.7593$ GeV$^2$.

FIG. 7. (Left) Three-dimensional and (right) contour plots for the real part, imaginary part, and modulus of the Compton form factor $F_{\text{VMP}}$ at $t = -0.7593$ GeV$^2$.

FIG. 8. The real and imaginary parts of the Compton form factor determined from $\mathcal{M}^{+}_{s+e^-+e^+}$ and $\mathcal{M}^{+}_{\text{VMP}}$ (left panel) for $t = -0.7593$ GeV$^2$ and (right panel) for $t = -2$ GeV$^2$.

by Eq. (5). Figure 9 shows the three-dimensional plots of $H(\zeta, x, t)$ for two parameter sets, $(m_{Q_1}, m_{Q_2}) = (2, 2)$ GeV and $(m_{Q_1}, m_{Q_2}) = (1, 3)$ GeV, of which results are presented in the upper and lower panels of Fig. 9, respectively, in the range of $0 \leq x \leq 1$ and $-4$ GeV$^2 \leq t \leq 0$. The red and green regions correspond to GPDs in the ERBL ($0 \leq x \leq \zeta$) and DGLAP ($\zeta \leq x \leq 1$) regions, respectively. The crossover boundaries (black lines) between the two regions correspond to the lines $x = \zeta$, i.e., $H_{\text{ERBL}}(\zeta, \zeta, t) = H_{\text{DGLAP}}(\zeta, \zeta, t)$. Again $t$ and $\zeta$ are not independent variables in $(1 + 1)$ dimensions. In the crossover boundary for a given parameter set, the longitudinal momentum fraction $x$ carried by the struck constituent $Q_1$ gradually increases as $|t|$ increases. Also, the peak position of GPD always exists in the DGLAP region. Comparing

dimensional analysis.

As we have mentioned before, in $(1 + 1)$ dimensions, the GPD $H(\zeta, x, t)$ given by Eq. (33) is essentially a function of $x$ and $t$ since $\zeta$ and $t$ are related to each other
the two parameter sets, the value of \( x \) at the peak is found to decrease as the mass ratio \( m_{Q_1}/m_{Q_2} \) decreases.

### C. PDF and EM form factor

Figure 10 shows the ordinary valence PDF for the “helium” target multiplied by \( x \), i.e., \( xq_v(x) \) for three values of \( m_{Q_1} = m_{Q_2} \) with \( M_T = 3.7 \text{ GeV} \). The dotted, solid, and dashed lines in this figure represent the results obtained with \( m_{Q_1(2)} = 1.9 \), 2.0, and 2.1 GeV, respectively. Since \( q_v(x) \) in Eq. (38) is symmetric under the exchange of \( x \rightarrow 1 - x \) for the case of equal constituent mass, \( xq_v(x) \) is somehow asymmetric. In Fig. 10, this asymmetric behavior is getting noticeable as the binding gets stronger.

The \( n \)-th moments of \( q_v(x) \) for the scalar target are summarized in Table I. Since \( q_v(y) \) in Eq. (40) is an even function of \( y \), the odd-numbered moments vanish. Our results in Table I show that the heavier the constituent mass (or equivalently, the larger the binding energy) is, the greater the values of even-numbered moments are. This implies that the shape of the PDF, \( q_v(y) \), is more narrowly peaked at \( y = 0 \) and more suppressed at the endpoints \( (y = \pm 1) \) as the binding energy of the scalar target decreases.

Shown in Fig. 11 is the longitudinal probability density \( \varrho(\bar{z}) = |\psi(\bar{z})|^2 \) (See Eqs. (41) and (42)) for the scalar target with \( M_T = 3.7 \text{ GeV} \) in the LF coordinate space of \( \bar{z} = x^\ast p^\ast \) which is completely Lorentz-invariant in \( (1 + 1) \) dimensions. The dot-dashed, solid, and dashed lines represent the results for \( m_{Q_1(2)} = 1.9 \), 2.0, and 2.1 GeV, respectively. In order to clearly show the behavior of the longitudinal probability density, we plot \( \varrho(\bar{z}) \) in two ways. In Fig. 11(a), \( \varrho(\bar{z}) \) is shown in the range of \( 0 \leq \bar{z} \leq 50 \) in linear scale. This shows that the stronger bound state (\( m_{Q_1(2)} = 2.1 \text{ GeV} \)) has a more concentrated distribution near \( \bar{z} = 0 \) than weakly bound states have. The long range behavior of \( \varrho(\bar{z}) \) is shown in Fig. 11(b) that plots the same function in logarithmic scale for a wider range of \( \bar{z} \). One can verify that \( \varrho(\bar{z}) \) shows the oscillating behavior for large \( \bar{z} \). Furthermore, the onset of the oscillation appears earlier, and the amplitude of the oscillation is larger for the strongly bound states than the weakly bound states. Our observation is consistent with those for the pion case reported in Refs. [24, 34, 35].

We also investigate the dependence of our results on the value of the target mass \( M_T \). In Fig. 12, we show the first three moments \( F_n(\zeta,t) = F_n(\zeta) = F_n(\zeta)/F_1(\zeta) \) (\( n = 1, 2, 3 \)) of GPD \( H(\zeta,x,t) \) (See Eqs. (43)-(45)). We consider a weekly
bound state and a strongly bound state. Presented in Fig. 12(a) are the results for the weakly bound scalar target with \( m_{Q_1(2)} = 2 \) GeV and \( M_T = 3.7 \) GeV, which is dubbed “helium” target. For comparison, we also show in Fig. 12(b) the results for the strongly bound scalar target with \( m_{Q_1(2)} = 0.25 \) GeV and \( M_T = 0.14 \) GeV, which is dubbed “pion” target [23]. The black, blue, and red lines represent \( F_1(ζ) \), \( F_2(ζ) \), and \( F_3(ζ) \), respectively. The dashed, dotted, and solid lines represent the valence contributions, the nonvalence contributions, and their sum, respectively. The valence and nonvalence contributions are obtained by replacing \( H(ζ, x, t) \) with \( H_{DGLAP}(ζ, x, t) \) and \( H_{ERBL}(ζ, x, t) \) in Eq. (33), respectively. The moments shown in Fig. 12 are indeed for the entire spacelike momentum transfer region since \( 0 \leq ζ \leq 1 \) corresponds to \( 0 \leq -t \leq \infty \). In other words, the skewness parameter \( ζ \) is zero only at \( t = 0 \) and the nonvalence contributions always exist for nonzero skewness (\( ζ > 0 \)).

Our results presented in Fig. 12 give the following observations. (i) The first moments, \( F_n(ζ) \), given by solid black lines are defined to be \( ζ \)-independent while the sum rule for \( n = 1 \) yields the physical EM form factor. (ii) The redefined higher moments, \( F_2(ζ) \) and \( F_3(ζ) \), satisfy the polynomiality condition. In the figures, we plot the fitted \( F_2(ζ) \) (orange diamonds) and \( F_3(ζ) \) (green diamonds) by finding the corresponding polynomials up to the second order of \( ζ \). (iii) The nonvalence contribution for the weakly bound helium target does not exceed the valence contribution for the entire mo-

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*As we showed before, in \((1+1)\) dimensions, \( ζ \) and \( t \) are related to each other. But the relation depends on the mass. For example, \( ζ = 0.5 \) corresponds to \(-t = 6.85 \) GeV\(^2\) for the helium target but it corresponds to \(-t = 0.01 \) GeV\(^2\) for the pion target.*
momentum transfer region as shown in Fig. 12(a). However, the nonvalence contribution for the strongly bound “pion” is not negligible and indeed takes over the valence contribution at some points of \( \zeta \) (or equivalently \(-t\)) as shown in Fig. 12(b). For example, the nonvalence contribution to the “pion” EM form factor \( n = 1 \) case is greater than the valence one for \( \zeta \geq 0.8 \) values.

Finally, we compute the EM form factor \( F_M(t) \) of the “helium” target as the first moment of \( H(\zeta, x, t) \) for the spacelike \( 0 \leq -t < 40 \text{ GeV}^2 \) region, and the results are given in Fig. 13.

The dashed, dotted, and solid lines represent the valence contributions, the nonvalence contributions, and their sums, respectively. In particular, we obtain the form factors by taking the plus (+) and minus (−) components of the current to examine the valence and nonvalence contributions in taking different components of the current while confirming that the sum of the valence and nonvalence contributions coincide whichever component is taken. Our results for \( F_M(t) \) shown in Fig. 13(a) are obtained by using Eqs. (31), (32), and (44) together with the “+” component of the current in the DVMP limit, which is exactly the same as our recent result reported in Ref. [23] based on the direct calculations of the triangle diagrams. Although one typically uses the “+” current to compute the form factor \( F_M(t) \), we compute here \( F_M(t) \) using the “−” component of the current as well in our direct triangle diagram calculation [23]. For comparison, we show in Fig. 13(b) the EM form factor obtained from the “−” component of the current using Eqs. (1) and (2) in Ref. [23]. All these results confirm that the total EM form factors are completely the same independent of the adopted component of the current and either “+” or “−” component of the current can be used to obtain the EM form factor. However, the decomposition of the form factor depends on which component of the current is used for the calculation. This is apparent as shown in Figs. 13(a,b), where one can see that the nonvalence contribution is quite suppressed for the entire spacelike region when the “+” current is used, while its contribution is not negligible but even exceeds the valence contribution for \(|t| \geq 8.5 \text{ GeV}^2\) when the “−” current is used. Furthermore, the nonvalence contribution for the “−” current case does not vanish even at \( t = 0 \) while it is zero for the “+” current case. Therefore, the decomposition of the form factor as the valence and nonvalence contributions depends on which component of the current is used. It should be interpreted with great care noting which component of the current is used.

VI. SUMMARY AND CONCLUSION

In the present work, we investigated the light-front amplitudes of the virtual meson production process off the scalar target in \((1 + 1)\) dimensions using the solvable scalar field theory. Noting that there is only one CFF in \((1 + 1)\) dimensions for this process, we obtained the analytic expressions for all possible LF time-ordered amplitudes shown in Fig. 2. The obtained LF time-ordered amplitudes are individually boost-invariant, and the sum of all LF time-ordered amplitudes turns out gauge invariant as they must be.

With the analytic solutions of the amplitudes at hand, we investigated various quantities, including CFF, GPD, PDF, and EM form factor. We first tested the “handbag dominance” that has been adopted in the GPD formulation for the large \( Q^2 \) region. In particular, we explored the role of the “cat’s ears” contributions, which have been typically ignored.

To quantify the individual contribution of the LF time-ordered amplitudes, we simulated the typical mass arrangement of the \( \gamma^* + ^4\text{He} \rightarrow J/\psi(980)+^4\text{He} \) process. Our numerical results showed that the gauge invariance is largely violated if one neglects the “cat’s ears” contribution. In particular, the
addition of the “cat’s ears” contribution is crucial for the low $Q^2$ region. Although the violation appears smaller at $Q^2 > 10\text{-}20 \text{ GeV}^2$ for the imaginary part of the total amplitude, it is still noticeable for the real part even at large $Q^2$ as shown in Fig. 5. This appears to limit the validity of the “handbag dominance” to the region of small $-t/Q^2$ [4, 32]. Our numerical calculations in $(1 + 1)$ dimensions presented in Fig. 8 show that the “handbag dominance” appears limited to the kinematic region $-t/Q^2 \lesssim 0.1$ for its applicability both to real and imaginary parts of the CFF. The relaxation of the condition $Q^2 \gg M_F^2$ in reaching the DVMP limit would also apply only for the forward production of the meson in $(3 + 1)$ dimensions. Therefore, the direct use of the “handbag dominance” in the analyses of the proposed experiments at JLab [41], where such small values of $-t/Q^2$ are not reached, may be treacherous necessitating great care taking into account the higher order $-t/Q^2$ corrections not only from the kinematic higher twist contributions but also from the dynamic higher twist GPDs [16, 42, 43].

In this respect, the future Electron-Ion Collider project [44] is strongly called for the proper extraction of GPDs from the experimental data off the nucleon and nuclei targets focusing on the forward angle.

In our simple $(1 + 1)$ dimensional model computations, the forward limit $(\zeta,t) \to 0$ of the GPD, $H(0,x,0) = H_{DGLAP}(0,x,0)$, provides the PDF, $q_\nu(x)$, which could be interpreted as the probability to find the constituent inside the hadron as a function of the momentum fraction $x$ carried by the constituent. It is equivalent to the square of the LF wave function, i.e., $q_\nu(x) = \mid \phi(x) \mid^2$. In $(1+1)$ dimensions, the LF spatial variable $\tilde{z} = x^+p^-$ is completely Lorentz-invariant providing an intrinsic longitudinal probability density $\rho(\tilde{z})$. Our numerical results showed that the stronger bound state concentrates the density more at $\tilde{z} = 0$ than the weaker bound states do, which appears to be consistent with the intuitive understanding of the bound-state system. The polynomiality condition for the moments of GPD appears also well satisfied. The GPD sum rule provided the EM form factor $F_1M(t)$ confirming the valence and non-valence contributions that we obtained previously [23], which corresponded to the GPD contributions from the DGLAP and ERBL regions, respectively.

In the calculation of the electromagnetic properties of hadrons in the LF formulation, one may use not only the plus (+) component but also any other component of the current as they are supposed to give the identical results. As shown in Fig. 13, we indeed confirmed that the two components (+ or −) led to the identical form factor. However, the decomposition of the form factor into valence and nonvalence contributions appears quite different depending on the component of the current used in the extraction. This indicates that it requires great care in interpreting the valence and nonvalence contributions to the form factor.

Our $(1 + 1)$-dimensional analyses performed in the present work would extend to the more realistic $(3 + 1)$-dimensional analyses, where the contributions from the transverse component of the current would be important. In particular, the two CFFs $F_1$ and $F_2$ involved in the scalar meson production off the scalar target are independent of each other in $(3 + 1)$-dimensions. Thus, the investigation of the beam spin asymmetry proportional to $F_1F_2^* - F_2F_1^*$ for this process would provide a unique opportunity not only to explore the imaginary part of the hadronic amplitude in our general formulation but also to examine the significance of the chiral-odd GPD contribution in the leading-twist GPD formulation as discussed in Ref. [21]. The work along this line of thought is currently underway.

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