A Unified Algebraic Approach to Few and Many-Body Correlated Systems

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Abstract

The present article is an extended version of the paper Phys. Rev. B 59, R2490 (1999), where, we have established the equivalence of the Calogero-Sutherland model to decoupled oscillators. Here, we first employ the same approach for finding the eigenstates of a large class of Hamiltonians, dealing with correlated systems. A number of few and many-body interacting models are studied and the relationship between their respective Hilbert spaces, with that of oscillators, is found. This connection is then used to obtain the spectrum generating algebras for these systems and make an algebraic statement about correlated systems. The procedure to generate new solvable interacting models is outlined. We then point out the inadequacies of the present technique and make use of a novel method for solving linear differential equations to diagonalize the Sutherland model and establish a precise connection between this correlated system’s wave functions, with those of the free particles on a circle. In the process, we obtain a new expression for the Jack polynomials. In two dimensions, we analyze the Hamiltonian having Laughlin wave function as the ground-state and point out the natural emergence of the underlying linear $W_{1+\infty}$ symmetry in this approach.

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I. INTRODUCTION

The Calogero-Sutherland model (CSM),1–3 Sutherland model (SM)4 and their generalizations have found physical applications in various fields such as the universal conductance fluctuations in mesoscopic systems,5–7 quantum Hall effect,8–10 wave propagation in stratified fields,11 random matrix theory,12,13–16 chaos,17,18 fractional statistics,19–23 gravity,24–26 anyons27–29 and gauge theories.30,31 This class of many-body systems with long-range interactions have been studied quite extensively in the current literature.32 Ref.[ 33] is a classic review; newer developments have been covered in Refs.[ 32, 34–36]. Interestingly, CSM and SM type long-range interactions have also manifested in models dealing with pairing interactions37 and phase transitions.38

The increasing appearance of correlated systems in diverse physical phenomena makes these exactly solvable models, with in-built correlation due to the presence of the Jastrow factor in the eigenfunctions, an ideal testing ground, for ideas ranging from Haldane statistics20 to black hole physics.26,31,39 In recent times, several new mathematical structures like $S_N$-extended Heisenberg algebra,40,41 Yangians42,43 and deformed algebras44–46 have been employed for unravelling the precise structure of the Hilbert space of these systems. The linearity of the eigen spectra of the CSM and the fact that interaction only modifies the ground-state has led Calogero to conjecture the existence of an exact mapping between the CSM and decoupled oscillators.47 The $S_N$-extended Heisenberg algebra, with its oscillator like creation and annihilation operators, partially achieved the above objective, although the oscillators were not decoupled and consequently, the members of the Hilbert space, although linearly independent, were not orthogonal. Recently, we have been able to show the precise equivalence of the CSM to decoupled oscillators, without involving any additional mathematical construct.48 The formal appearance of a new inner product in Ref. [ 48] has been explicated recently,49 as also the successful application of our procedure50 to supersymmetric CSM,51 the scattering Hamiltonian52 and other related models.53–59 The method also applies to Hamiltonians, containing other competing long-range interactions.60 However, the subtleties involved in extending the present procedure to models based on root systems other than $A_{N-1}$,33 as also to higher dimensions, still remains to be explored. Furthermore, it can be seen that the above mentioned method to diagonalize the CSM is inadequate for the diagonalization of SM and its generalizations. Unlike the CSM, the eigen spectra of SM is quadratic in nature, where the interaction modifies the ground state as well as the excited states. The present article intends to deal with the various generalizations of the CSM and makes use of a novel method of solving linear differential equations61,62 for diagonalizing the SM and finding its connection with free particles on a circle.

The paper is organized as follows. Sec. II is devoted to a brief summary of the results obtained in Ref. [ 48]. The $B_N$-type CSM and a recently proposed four particle model in one-dimension63 are then studied in detail. In the case of $B_N$ model, it is found that the underlying spectrum generating algebra is $SU(1, 1)$, unlike the $A_{N-1}$ CSM, where it was the Heisenberg algebra. This also leads to the straightforward construction of the linear $W_\infty$ algebra. The eigenfunctions and commuting constants of motion, behind the quantum integrability, are presented. We point out in Sec. III, how this method can be used to construct more general interacting models of the CSM type. Sec. IV deals with the inadequacies of the procedure used for diagonalization of the CSM and illustrates a new
method of solving linear differential equations, which can be employed for the diagonalization of both CSM and SM. Through this approach we establish the precise connection between free particles on a circle and interacting particles in SM. In the process, we obtain a novel expression for the Jack polynomials in terms of the monomial symmetric functions. In Sec. V, we analyze a two-dimensional interacting Hamiltonian for electrons under the influence of both electromagnetic and Chern-Simons gauge fields. This model has Laughlin wave function as the ground-state and is also relevant for implementing the Jain picture of the fractional quantum Hall effect. We explicitly show the connection of this interacting model with non-interacting harmonic oscillators, and point out the intricacies involved in extending the present procedure to higher dimensions. In this process the underlying linear $W_{1+\infty}$ symmetry algebra with the Laughlin wave function as the highest weight vector of this algebra is brought out in a transparent manner. We conclude in the last section after pointing out several areas, where the present procedure can be profitably employed.

II. DIAGONALIZING CALOGERO-SUTHERLAND TYPE INTERACTING MODEL

In this section, we outline the procedure employed in \textit{Phys. Rev.} B 59, R2490 (1999) to diagonalize the CSM and map it to a set of free harmonic oscillators. This equivalence provides an elegant and straightforward method to construct the complete set of eigenstates of the CSM, starting from the symmetrized form of the eigenstates of the harmonic oscillators we first employ the same procedure to diagonalize the $B_N$-type Calogero model and establish the precise connection between the Hilbert spaces of CSM and the present model. A large class of CSM type models in one and higher dimensions can also be solved in an analogous manner.

The mapping of the CSM to decoupled oscillators involves a series of similarity transformations (ST). The CSM Hamiltonian, in the units, $\hbar = \omega = m = 1$ (we will use these units throughout the text, unless specified otherwise) is given by

$$H_{\text{CSM}} = -\frac{1}{2} \sum_i^N \partial_{x_i}^2 + \frac{1}{2} \sum_i^N x_i^2 + \frac{\eta^2}{2} \sum_{i \neq j}^N \frac{1}{(x_i - x_j)^2} ,$$

where, $\partial_{x_i} \equiv \partial/\partial x_i$ and $\eta^2 > -1/4$.

The above N-body quantum problem is first solved in one sector of the configuration space such that $x_1 < x_2 < \ldots < x_N$. The solutions are then analytically continued to other sectors of the Hilbert space. This is possible in one dimension, because the repulsive potential in CSM does not allow the particles to overtake each other. The many-body system can also be quantized either as bosons or fermions, without loss of generality. We list the following definitions for the sake of later convenience:

$$Z \equiv \Pi_{i<j} (x_i - x_j)^{\beta} ,$$
$$\hat{A}(\beta) \equiv \frac{1}{2} \sum_i \partial_{x_i}^2 + \beta \sum_{i \neq j} \frac{1}{(x_i - x_j)} \partial_{x_i} ,$$
$$\hat{E}(\beta) \equiv \exp\{-\hat{A}(\beta)/2\} .$$
\[ G \equiv \exp\left\{-\frac{1}{2} \sum_i x_i^2\right\}. \]

Combining the above three transformations, we find that the operator,
\[ \hat{T} \equiv ZG\hat{E}(\beta) \] (2)
diagonalizes the original \( H_{CSM} \) (with \( \beta(\beta - 1) = g^2 \)):
\[ \hat{T}^{-1}H_{CSM}\hat{T} = \sum_i x_i\partial x_i + E_0 \equiv H'_{CSM}. \] (3)

Here \( E_0 = N/2 + N(N - 1)/2\beta \) is the ground state energy. The above equation has also been obtained by Sogo in an entirely different context.66

Eq. (3) reveals that, starting from a symmetric polynomial basis on \( e \) can construct the eigenfunctions of the CSM by an inverse similarity transformation. The CSM can also be mapped to the decoupled oscillators using another inverse similarity transformation:
\[ G\hat{E}(0)H'_{CSM}\hat{E}^{-1}(0)G^{-1} = -\frac{1}{2} \sum_i \partial^2 x_i + \frac{1}{2} \sum_i x_i^2 + (E_0 - \frac{1}{2} N) = H_{\text{free}}. \] (4)

These results can be succinctly written as
\[ \hat{O}^{-1}H_{CSM}\hat{O} = H_{\text{oscillator}} + E_0 - \frac{N}{2}, \] (5)

where, \( \hat{O} \equiv \hat{TE}^{-1}(0)G^{-1} \). As anticipated by Calogero and indicated by the structure of the eigenspectrum, the CSM Hamiltonian can be mapped to \( N \) free oscillators, apart from the coupling dependent overall shift of the energy eigenvalues. Hence, it follows that, the excited energy levels and the degeneracy structure of both these systems are identical, a fact known since the original solution of the interacting model.1

We now proceed to the \( B_N \) model and first point out the fact that, while applying the various similarity transformations one needs to ensure that the resulting eigenfunctions are members of the Hilbert space. This is the origin of differences in the spectrum generating algebras of various CSM type models. It will be shown that for the \( B_N \) case the underlying algebra is not the Heisenberg one as in the CSM but the \( SU(1,1) \) algebra. The \( B_N \)-type CSM (BnCSM) Hamiltonian is given by,33
\[ H_{B_N} = -\frac{1}{2} \sum_i^N \partial^2 x_i + \frac{1}{2} \sum_i^N x_i^2 + \frac{1}{2} g^2 \sum_{i \neq j}^N \left\{ \frac{1}{(x_i - x_j)^2} + \frac{1}{(x_i + x_j)^2} \right\} + \frac{1}{2} g_1^2 \sum_i^N x_i^2, \] (6)

where, \( g^2 \) and \( g_1^2 \) are two independent coupling constants. Since, the ground-state wave function of \( H_{B_N} \), when the system is quantized as bosons, is given by
\[ \psi_0 = \prod_{1 \leq j < k \leq N} |x_i - x_j|^\lambda |x_i + x_j|^\lambda |x_k|^\lambda \exp\{-\frac{1}{2} \sum_i x_i^2\} \] (7)

one can make the following ST, after confining oneself in one sector of the configuration space:
\[
H \equiv \psi_0^{-1} H \psi_0 = \sum_i x_i \partial x_i + E_0' + \hat{F} \quad .
\] (8)

Here,
\[
\hat{F} \equiv -\left( \frac{1}{2} \sum_i \partial^2 x_i + \lambda \sum_{i<j} \frac{1}{(x_i^2 - x_j^2)} (x_i \partial x_i - x_j \partial x_j) + \lambda \sum_i \frac{1}{x_i} \partial x_i \right) \quad ,
\]
\[
g^2 = \lambda(\lambda - 1) \quad ,
\]
\[
g^2_1 = \lambda_1(\lambda_1 - 1) \quad ,
\]
and the ground-state energy \(E_0'\) is given by
\[
E_0' = N\left( \frac{1}{2} + (N - 1)\lambda + \lambda_1 \right) \quad .
\]

As in the case of the CSM, the eigenfunctions of \(\tilde{H}\) must be totally symmetric with respect to the exchange of any two particle coordinates. One can easily establish the following commutation relation:
\[
\left[ \sum_i x_i \partial x_i \ , \ \exp\{\hat{F}/2\} \right] = -\hat{F} \exp\{\hat{F}/2\} \quad .
\] (9)

Making use of Eq. (9) in Eq. (8), one gets
\[
\exp\{-\hat{F}/2\} \tilde{H}_{B_N} \exp\{\hat{F}/2\} = \sum_i x_i \partial x_i + E_0' \quad .
\] (10)

The above two results are identical to the CSM case; this is due the fact that the Euler operator is only sensitive to the degree of the operator \(\hat{F}\). Hence, the ST by the operator \(\tilde{S} \equiv \psi_0 \exp\{\hat{F}/2\}\) diagonalizes \(H_{B_N}\) i.e.,
\[
\tilde{S}^{-1} H_{B_N} \tilde{S} = \sum_i x_i \partial x_i + E_0' \quad .
\] (11)

Furthermore, the following similarity transformation on Eq. (10) makes the connection of the BnCSM with the decoupled oscillators explicit:
\[
G \hat{E}(0) \tilde{S}^{-1} H_{B_N} \tilde{S} \hat{E}^{-1}(0) G^{-1} = -\frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \sum_i x_i^2 + (E_0' - \frac{1}{2}N) \quad ,
\] (12)

where, \(\hat{E}(0)\) and \(G\) are as defined earlier.

**A. Mapping between the CSM and BnCSM**

It is interesting to note that, \(H_{B_N}\) can also be made equivalent to the CSM as follows;
\[
\hat{T} \tilde{S}^{-1} H_{B_N} \tilde{S} \hat{T}^{-1} = -\frac{1}{2} \sum_i \partial^2 x_i + \frac{1}{2} \sum_i x_i^2 + \sum_{i<j} \frac{\beta(\beta - 1)}{(x_i - x_j)^2} + (E_0' - E_0) \quad ,
\] (13)

where, \(E_0\) is the ground-state energy of the CSM. One needs to check the spaces of functions on which all the above ST’s are well defined. This point will be elaborated more after finding out the Hilbert space of the \(B_N\) model. This will also lead to the precise relationship between the respective eigenspaces of \(H_{CSM}\) and \(H_{B_N}\).
B. Hilbert space of BnCSM

We can make use of Eq. (10) for the explicit construction of the eigenfunctions of Eq. (6). It is easy to see that an arbitrary homogeneous symmetric function of $x_i$'s is an eigenfunction of $\sum_i x_i \partial_{x_i}$; however, unlike the case of CSM, only the homogeneous symmetric functions of the square of the particle co-ordinates are the ones on which the action of $\exp\{\hat{F}/2\}$ gives a polynomial solution. At this moment, it is also worth noting that if $\lambda = 1$, then Eq. (8) reduces to the differential equation, in $y_i = x_i^2$ variables for the multivariate Lagurre polynomials. This is the basic reason why, the action of the operator $\exp\{\hat{F}/2\}$ on the homogeneous symmetric functions of the square of the particle co-ordinates gives a polynomial solution. For the sake of illustration, we choose here the power sum basis $P_l \equiv \sum_i (x_i^2)^l$. The eigenfunctions and the energy eigenvalues are respectively given by

$$\psi_n = \psi_0 \left[ \exp\{\hat{F}/2\} \prod_{l=1}^{N} P^n_l \right], \quad \text{(14)}$$

and

$$E_n = 2 \sum_{l} l n_l + E_0; \quad n = \sum_{l} l n_l. \quad \text{(15)}$$

Note that, $\hat{T}\hat{S}^{-1}$ maps all eigenfunctions of BnCSM to the even sector of CSM and hence the connection mentioned earlier between the CSM and BnCSM is valid only in the even sector.

C. Underlying algebraic structure of the BnCSM

Akin to the CSM case, one can obtain the creation and annihilation operators for the BnCSM from Eq. (10);

$$b_i^+ = \hat{S} x_i \hat{S}^{-1},$$

and

$$b_i^- = \hat{\partial} x_i \hat{S}^{-1},$$

such that, the symmetrized form of the operators,

$$K_i^+ = \frac{1}{2} b_i^{+2}, \quad \text{(16)}$$

acts on the ground-state obtained from

$$b_i^- |0> = 0; \quad i = 1, 2, \cdots, N$$

and creates the eigenstates of the BnCSM. In terms of the creation and annihilation operators, the BNCSM Hamiltonian can be written as

$$H = \sum_i H_i + (E_0 - N/2), \quad \text{(17)}$$
where,
\[ H_i \equiv b_i^+ b_i^- + \frac{1}{2} \, . \]

It can be easily checked that, the commutation relations
\[ [K_i^-, K_j^+] = \delta_{ij} H_i \quad , \quad (18) \]
and
\[ [H_i, K_j^\pm] = \pm 2 \delta_{ij} K_i^\pm \quad , \quad (19) \]
generate \( N \) copies of \( SU(1, 1) \) algebra; here,
\[ K_i^- = \frac{1}{2} b_i^2 \quad . \]

It is worth noticing that these, \( N \), commuting \( SU(1, 1) \) generators act on the even sector of the harmonic oscillator basis and hence generate a complete set of eigenfunctions. Thus, we conclude that, the underlying algebraic structure of the BnCSM is \( SU(1, 1) \). In comparison, the eigenfunctions of the CSM contain both the even and odd sectors of the oscillator basis.\(^{48}\)

With the generators of the above mentioned \( SU(1, 1) \) algebras of BnCSM, one can define a linear \( W_\infty \) algebra for which there exist several basis sets.\(^{69}\)

### D. New innerproduct

In parallel to the case of CSM, one can also define \( << 0 | S_n(\{ \frac{1}{2} b_i^2 \}) = << n | \) and \( S_n(\{ \frac{1}{2} b_i^2 \}) | 0 >= | n > \) as the bra and ket vectors; \( S_n \) is a symmetric homogeneous function of degree \( n \) and \( << 0 | \frac{1}{2} b_i^2 = \frac{1}{2} b_i^2 | 0 >= 0 \). Since all the \( N \) \( SU(1, 1) \) algebras are decoupled, the inner product between these bra and ket vectors proves that any ket \( | n > \), with a given partition of \( n \), is orthogonal to all the bra vectors, with different \( n \) and also to those with different partitions of the same \( n \). The normalization for any state \( | n > \) can also be found out from the ground state normalization, which is known.\(^{70}\)

### E. Quantum integrability

The quantum integrability of the BnCSM and the identification of the constants of motion become transparent after establishing its equivalence to free oscillators. It is easy to verify that \([H, H_k] = [H_i, H_j] = 0 \) for \( i, j, k = 1, 2, \cdots, N \). Therefore, the set \( \{ H_1, H_2, \cdots, H_N \} \) provides the \( N \) conserved quantities. One can construct linearly independent symmetric conserved quantities analogous to that of CSM. Here, we would like to point out that, the present proof is entirely different from earlier works.\(^{71-74}\)
F. The Haschke-Rühl model

Now, we analyse a one-dimensional model of four identical particles, with both two-body and four-body inverse-square interactions given by the Hamiltonian

\[
H = -\frac{1}{2} \sum_{i=1}^{4} \partial_{x_i}^2 + \frac{1}{2} \sum_{i=1}^{4} x_i^2 + g_1 \sum_{i,j, i \neq j} (x_i - x_j)^{-2} + g_2 \sum_{3 \text{independent terms}} (x_i + x_j - x_k - x_l)^{-2}. \tag{20}
\]

The correlated bosonic ground-state of \(H\) is given by

\[
\psi_0 = \prod_{i<j} |x_i - x_j|^\alpha \prod_{3 \text{indep.terms}} (x_i + x_j - x_k - x_l)^\beta \exp\{-\frac{1}{2} \sum_i x_i^2\},
\]

where, \(\alpha = \frac{1}{2} (1 + \sqrt{1 + 4g_1})\) and \(\beta = \frac{1}{2} (-1 + \sqrt{1 + 2g_2})\). By performing a ST on \(H\) with respect to \(\psi_0\), one gets

\[
\psi_0^{-1} H \psi_0 \equiv \tilde{H} = \sum_i x_i \partial_{x_i} - \hat{D} + E_0 \quad ,
\]

where,

\[
\hat{D} = \frac{1}{2} \sum_i \partial_{x_i}^2 + \alpha \sum_{i \neq j} \frac{1}{x_i - x_j} \partial_{x_i} + \beta \sum_{3 \text{indep.terms}} \frac{1}{(x_i + x_j - x_k - x_l)} (\partial_{x_i} + \partial_{x_j} - \partial_{x_k} - \partial_{x_l})
\]

and \(E_0 = 2 + 6\alpha + 3\beta\). Another ST by \(\hat{I} = \exp\{-\hat{D}/2\}\) on \(\tilde{H}\) diagonalizes it completely:

\[
\hat{I}^{-1} \tilde{H} \hat{I} \equiv \hat{H} = \sum_i x_i \partial_{x_i} + E_0 \quad .
\]

The explicit connection of \(H\) with the decoupled oscillators can be obtained by one more ST on \(\hat{H}\)

\[
\hat{T}^{-1} \hat{H} \hat{T} = -\frac{1}{2} \sum_{i=1}^{4} \partial_{x_i}^2 + \frac{1}{2} \sum_{i=1}^{4} x_i^2 + E_0 - 2 \quad ,
\]

where, \(\hat{T} \equiv \exp(-\frac{1}{2} \sum_{i=1}^{4} x_i^2) \exp(-\frac{1}{2} \sum_{i=1}^{4} \partial_{x_i}^2)\).

Eigenfunctions for this model can be constructed straightforwardly following the previous examples.

III. METHOD FOR CONSTRUCTION OF NEW SOLVABLE MODELS

More general interacting models of the CSM type, which can be mapped to decoupled oscillators, can be constructed in the following manner.

One starts with a general Hamiltonian of the type

\[
H = -\frac{1}{2} \sum_{i=1}^{N} \partial_{x_i}^2 + V(x_1, x_2, x_3, \cdots, x_N) \quad ,
\]

where, \(V(x_1, x_2, x_3, \cdots, x_N)\) is a suitable function of the particle coordinates.

The approach involves finding a suitable transformation \(\hat{T}\) that maps the Hamiltonian to a decoupled form, i.e.,

\[
\hat{T}^{-1} H \hat{T} = \sum_i \partial_{x_i}^2 + E_0 \quad .
\]

The detailed procedure involves the following steps:

1. **Identification of Terms**: Identify the terms in the Hamiltonian that are responsible for the interactions.
2. **Transformation Idea**: Construct a transformation \(\hat{T}\) that maps the Hamiltonian to its decoupled form.
3. **Diagonalization**: Ensure that the transformed Hamiltonian is diagonal or quasi-diagonal.
4. **Eigenfunction Construction**: Once the Hamiltonian is diagonalized, construct the eigenfunctions.

This approach, when applied systematically, can lead to the construction of new solvable models in quantum mechanics.
having \( \Phi_0 \) as the ground-state wave function and

\[
V(x) = \epsilon_0 + \frac{1}{2\Phi_0} \sum_{i=1}^{N} \partial_x^2 \Phi_0 ,
\]

where, \( \epsilon_0 \) is the ground-state energy. In order to bring \( H \) to the following form

\[
\Phi_0^{-1} H \Phi_0 \equiv \tilde{H} = \sum_{i} x_i \partial_{x_i} - \hat{A} + \epsilon_0 \quad , \tag{25}
\]

the ground-state wave function must be of the form \( \Phi_0 = GJ \); where,

\[
\hat{A} \equiv \frac{1}{2} \sum_{i=1}^{N} \partial_x^2 + \sum_{i=1}^{N} \partial_{x_i} (\ln J) \partial_{x_i} ,
\]

and \( J \) is so far an arbitrary function.

\( \tilde{H} \) can be mapped to the Euler operator by another ST

\[
\hat{S}^{-1} \tilde{H} \hat{S} \equiv \bar{H} = \sum_{i=1}^{N} x_i \partial_{x_i} + \epsilon_0 \quad , \tag{26}
\]

provided, the following equation holds,

\[
\left[ \bar{H}, \exp\{-\hat{A}/2\} \right] = \left[ \sum_{i} x_i \partial_{x_i}, \exp\{-\hat{A}/2\} \right] = \hat{A} \exp\{-\hat{A}/2\} \quad . \tag{27}
\]

The above condition restricts \( J \) to be a homogeneous function of the particle coordinates.

Now, it is easy to see that, the Hamiltonian in Eq.(24) can be mapped to free oscillators by a series of STs,

\[
G\hat{E}(0) \exp\{\hat{A}/2\} \Phi_0^{-1} H \Phi_0 \exp\{-\hat{A}/2\} \hat{E}^{-1}(0) G^{-1} = -\frac{1}{2} \sum_{i} \partial_x^2 + \frac{1}{2} \sum_{i} x_i^2 + (\epsilon_0 - \frac{1}{2} N) . \tag{28}
\]

However, as mentioned earlier, it is important to check that, the action of \( \exp\{-\hat{A}/2\} \) on an appropriate linear combination of the eigenstates of \( \sum_{i=1}^{N} x_i \partial_{x_i} \), yields solutions which are normalizable with respect to \( \Phi_0^2 \) as the weight function. Appropriate choices of \( J \) will yield new solvable models having linear spectra. This can also be generalized to the higher-dimensional interacting models,\(^{75-77} \) provided one is careful about the intricacies arising in higher dimensions.

**IV. A NEW TECHNIQUE TO SOLVE LINEAR DIFFERENTIAL EQUATIONS AND DIAGONALIZATION OF THE SUTHERLAND MODEL**

In the previous section, we have dealt with a class of many-body Hamiltonians, which, after a similarity transformation separate into a part containing the Euler operator and a constant and another operator \( \hat{A} \) having a definite degree. These Hamiltonians were then
diagonalized, taking advantage of the fact that \[ \sum_i x_i \partial x_i, e^{\hat{A}/d} = \hat{A} e^{\hat{A}/d}, \]
where \( d \) is the degree of the operator \( \hat{A} \), i.e., \[ \sum_i x_i \partial x_i, \hat{A} = d \hat{A}. \]
We notice that the above procedure fails in the cases, where the similarity transformed Hamiltonian \( \hat{H} \) contains an operator \( \hat{F} \) possessing the following properties.

**Case i:** When \( \hat{F} = \sum_{k=1}^{s} \hat{F}_k \), where, \( s \) is an arbitrary integer and \( \hat{F}_k \)'s are operators with different degrees and \[ [\hat{F}_k, F] \neq 0; \text{ for } k, l = 1, 2, \ldots s. \]

**Case ii:** When the degree, \( d \), of the operator \( \hat{F} \) is zero.

**Case iii:** When \( \hat{H} \) contains operators like \( \sum_i (x_i \partial x_i)^n \), with \( n \geq 2 \).

As will be shown below, one needs a method which works for the second and third cases mentioned above, in order to diagonalize the Sutherland model and to show its equivalence to free particles on a circle.

A recently proposed method\(^{61}\) is illustrated below which overcomes all the difficulties mentioned above and achieves the goal of connecting the solutions of differential equations to monomials.

Consider first the general, single variable linear differential equation,

\[
\left( F(D) + \hat{P} \right) y(x) = 0 ,
\]

where, \( D \equiv x \frac{d}{dx} \) and \( F(D) = \sum_{n=-\infty}^{\infty} a_n D^n \), is a diagonal operator in the space of monomials. \( \hat{P} \) can be an arbitrary operator, having a well-defined action in the space spanned by \( x^n \). Here, \( a_n \)'s are some parameters.

The solution of Eq. (29) is given by

\[
y(x) = C_\lambda \left\{ \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{F(D)} \hat{P} \right]^m \right\} x^\lambda
\equiv C_\lambda \hat{G}_\lambda x^\lambda ,
\]

provided, \( F(D)x^\lambda = 0 \) and the coefficient of \( x^\lambda \) in \( y(x) - C_\lambda x^\lambda \) is zero (no summation over \( \lambda \)); here, \( C_\lambda \) is a constant. The latter condition guarantees that, the solutions, \( y(x) \)'s, are non-singular. The proof is straightforward and follows by direct substitution.\(^{61,62}\) Note that, the detailed properties of \( \hat{P} \) are not needed to prove that \( y(x) \) in Eq. (30) is a solution of Eq. (29). However, naturally, these are required while constructing the explicit solutions of any given linear differential equation. This procedure gives novel expression for the solutions of known differential equations and brings out the properties of the orthogonal polynomials and functions in a natural way.\(^{61,62}\)

Eq. (30), which connects the solutions of a differential equation to the monomials, can be generalized to many-variables as follows.

Consider,

\[
\left( \sum_{n=-\infty}^{\infty} b_n \left( \sum_i D_i^n \right) + \hat{A} \right) Q_\lambda(x) = B_\lambda(x) ,
\]

where, \( b_n \)'s are some parameters, \( D_i \equiv x_i \partial x_i \); \( \hat{A} \) can be a function of \( x_i, \partial x_i \) and also some other well-defined composite operators and \( B_\lambda(x) \) is a source term. Solutions of Eq. (31) can be obtained for different cases.
Case (i): When $B_{\lambda}(x) = 0$ and $\hat{A}m_{\lambda} = \epsilon_{\lambda}m_{\lambda} + \sum_{\mu<\lambda} C_{\mu\lambda}m_{\mu}$; where, $m_{\lambda}$'s are the monomial symmetric functions and $\epsilon_{\lambda}$ and $C_{\lambda\mu}$ are some coefficients.

Using Eq. (30), the solution can be obtained as,

$$Q_{\lambda}(x) = \sum_{r=0}^{\infty} (-1)^{r} \left[ \frac{1}{\left( \sum_{n=-\infty}^{\infty} b_{n}(\sum_{i} D_{i}^{n}) - \sum_{n=-\infty}^{\infty} b_{n}(\sum_{i} \lambda_{i}^{n}) \right)} \left( \hat{A} - \epsilon_{\lambda} \right) \right]^{r} m_{\lambda}(x) \quad (32)$$

with, $\sum_{n=-\infty}^{\infty} b_{n}(\sum_{i} \lambda_{i}^{n}) + \epsilon_{\lambda} = 0$.

Case (ii): When $B_{\lambda}(x) \neq 0$.

$$Q_{\lambda}(x) = \sum_{r=0}^{\infty} (-1)^{r} \left[ \frac{1}{\left( \sum_{n=-\infty}^{\infty} b_{n}(\sum_{i} D_{i}^{n}) - \sum_{n=-\infty}^{\infty} b_{n}(\sum_{i} \lambda_{i}^{n}) \right)} \right]^{r} \times \left( \sum_{n=-\infty}^{\infty} b_{n}(\sum_{i} \lambda_{i}^{n}) \right) B_{\lambda}(x), \quad (33)$$

provided, the coefficient of the divergent part in the right hand side of the above equation is zero. As will be soon seen in the context of the SM, this requirement yields the eigenvalues for the Hamiltonians, with which the above differential equations are associated.

We now proceed to use the above mentioned method to diagonalize the SM and establish a precise connection between the Hilbert space of SM and that of free particles on a circle. The Schrödinger equation is,

$$\left( -\sum_{i=1}^{N} \frac{\partial^{2}}{\partial x_{i}^{2}} + 2\beta(\beta - 1) \frac{\pi}{L^{2}} \sum_{i<j} \frac{1}{\sin^{2}[\pi(x_{i} - x_{j})/L]} - E_{\lambda} \right) \psi_{\lambda}({\{x_{i}\}}) = 0 \quad . \quad (34)$$

Choosing, $z_{j} = e^{2\pi i x_{j}/L}$ and writing $\psi_{\lambda}({\{z_{i}\}}) = \prod_{i} z_{i}^{-(N-1)/2} \prod_{i<j} (z_{i} - z_{j})^{\beta} J_{\lambda}({\{z_{i}\}})$, the above equation becomes,

$$\left( \sum_{i} D_{i}^{2} + \beta \sum_{i<j} \frac{z_{i} + z_{j}}{z_{i} - z_{j}} (D_{i} - D_{j}) + \tilde{E}_{0} - \tilde{E}_{\lambda} \right) J_{\lambda}({\{z_{i}\}}) = 0 \quad , \quad (35)$$

where, $D_{i} \equiv z_{i} \frac{\partial}{\partial z_{i}}$, $\tilde{E}_{\lambda} \equiv (\frac{\beta}{2\pi})^{2} E_{\lambda}$, $\tilde{E}_{0} \equiv (\frac{\beta}{2\pi})^{2} E_{0}$ and $E_{0} = \frac{1}{3} (\frac{\beta}{L})^{2} N(N^{2}-1)$, is the ground-state energy. Here, $J_{\lambda}({\{z_{i}\}})$'s are known as the Jack polynomials.\(^{57,78-80}\) $\sum_{i} D_{i}^{2}$ is a diagonal operator in the space spanned by the monomial symmetric functions, $m_{\{\lambda\}}$, with eigenvalues $\sum_{i=1}^{N} \lambda_{i}^{2}$. Rewriting Eq. (35) in the form,

$$\left( \sum_{i} (D_{i}^{2} - \lambda_{i}^{2}) + \beta \sum_{i<j} \frac{z_{i} + z_{j}}{z_{i} - z_{j}} (D_{i} - D_{j}) + \tilde{E}_{0} + \sum_{i} \lambda_{i}^{2} - \tilde{E}_{\lambda} \right) J_{\lambda}({\{z_{i}\}}) = 0 \quad , \quad (36)$$

and using Eq. (32), one can immediately show that,

$$J_{\lambda}({\{z_{i}\}}) = C_{\lambda} \left\{ \sum_{n=0}^{\infty} (-1)^{n} \left[ \frac{1}{\sum_{i<j} \frac{z_{i} + z_{j}}{z_{i} - z_{j}} (D_{i} - D_{j}) + \tilde{E}_{0} + \sum_{i} \lambda_{i}^{2} - \tilde{E}_{\lambda}} \right]^{n} \right\} \equiv C_{\lambda} \hat{G}_{\lambda} m_{\lambda}({\{z_{i}\}}) \quad . \quad (37)$$
It can be checked that, $\hat{G}_\lambda$ maps the SM to free particles on a circle, i.e.,

$$(\psi_0 \hat{G}_\lambda)^{-1} H_S(\psi_0 \hat{G}_\lambda) = \left(\frac{2\pi}{L}\right)^2 \left(\sum_i D_i^2 - \sum_i \lambda_i^2 + \tilde{E}_\lambda\right)$$

$$= -\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} - \left(\frac{2\pi}{L}\right)^2 \sum_i \lambda_i^2 + E_\lambda$$

(38)

where, $H_S$ is the SM Hamiltonian and $\psi_0$ is its ground-state wave function. For the sake of convenience, we define

$$\hat{S} \equiv \frac{1}{\sum_i (D_i^2 - \lambda_i^2)} \hat{Z},$$

and

$$\hat{Z} \equiv \beta \sum_{i<j} \frac{z_i + z_j}{z_i - z_j} (D_i - D_j) + \tilde{E}_0 + \sum_i \lambda_i^2 - \tilde{E}_\lambda.$$

(39)

The action of $\hat{S}$ on $m_\lambda(\{z_i\})$ yields singularities, unless one chooses the coefficient of $m_\lambda$ in $\hat{Z} m_\lambda(\{z_i\})$ to be zero; this condition yields the well-known eigenspectrum of the Sutherland model:

$$\tilde{E}_\lambda = \tilde{E}_0 + \sum_i (\lambda_i^2 + \beta [N + 1 - 2i] \lambda_i)$$

The action of $\hat{S}$ on $m_\lambda(\{z_i\})$ yields singularities, unless one chooses the coefficient of $m_\lambda$ in $\hat{Z} m_\lambda(\{z_i\})$ to be zero; this condition yields the well-known eigenspectrum of the Sutherland model:

$$\tilde{E}_\lambda = \tilde{E}_0 + \sum_i (\lambda_i^2 + \beta [N + 1 - 2i] \lambda_i)$$

Using the above, one can write down a novel expression for the Jack polynomials as,

$$J_\lambda(\{z_i\}) = \sum_{n=0}^{\infty} (-\beta)^n \left[\frac{1}{\sum_i (D_i^2 - \lambda_i^2)} \left(\sum_{i<j} \frac{z_i + z_j}{z_i - z_j} (D_i - D_j) - \sum_i (N + 1 - 2i) \lambda_i\right)\right]^n$$

$$\times m_\lambda(\{x_i\}).$$

(40)

This procedure for constructing the Jack polynomials does not involve additional mathematical tools like Dunkl derivatives and $S_N$ extended Heisenberg algebra. The implications of the just established connection between SM wave functions and free particles are currently under investigation. In particular, its relevance to generalized statistics is of deep interest.

The connection of Jack polynomials with the monomial symmetric functions can be used for constructing ladder operators, since at the level of the monomials, the construction of the ladder operators is straightforward. Currently, these operators have been constructed only for a few particle system. We are also studying the symmetry properties of the Jack and other multivariate polynomials, using the above technique. It is worth pointing out that Jack polynomial and its supersymmetric generalizations are being studied extensively, for their relevance to multi-variate statistics and mathematical physics.

V. LAUGHLIN WAVE FUNCTION AND DECOUPLED HARMONIC OSCILLATORS

In this section, we study a planar many-body Hamiltonian relevant for the description of the quantum Hall effect. This Hamiltonian describes electrons in a magnetic field, with
two-body and three-body inverse-square interactions arising due to the Chern-Simons gauge field and have Laughlin wave function as the ground-state. We explicitly prove that, these models can be exactly mapped to a set of free harmonic oscillators on the plane. As a consequence, the existence of linear $W_{1+\infty}$ algebra with Laughlin wave function as its highest weight vector, is pointed out, in a rather elegant and straightforward manner. This study also brings out the potential difficulties one can face in more than one dimension.

We start with the Hamiltonian:

$$ H = \frac{1}{2} N \sum_{i=1}^{N} (-4\partial_{z_i} \partial_{\bar{z}_i} + z_i \partial_{z_i} - \bar{z}_i \partial_{\bar{z}_i} + \frac{1}{4} \bar{z}_i z_i) + 2\eta N \sum_{i \neq j} \frac{1}{(z_i - z_j)} (\partial_{z_i} - \frac{1}{4} z_i)$$

$$ - 2\eta N \sum_{i \neq j} \frac{1}{(z_i - z_j)} (\partial_{\bar{z}_i} - \frac{1}{4} \bar{z}_i) + 2\eta^2 N \sum_{i,j,k} \frac{1}{(z_i - z_j)(\bar{z}_i - \bar{z}_k)} .$$

The ground-state of this model was found to be of Laughlin form,

$$ \psi_0 = \prod_{i<j} (z_i - z_j)^n \exp\left\{ -\frac{1}{4} \sum_i \bar{z}_i z_i \right\} .$$

By performing a ST, one gets

$$ \psi_0^{-1} H \psi_0 \equiv \tilde{H} = \sum_i z_i \partial_{z_i} - \hat{A} + \frac{1}{2} N ,$$

where,

$$ \hat{A} \equiv 2 \sum_i \partial_{z_i} \partial_{\bar{z}_i} + 2\eta \sum_{i \neq j} \left( \frac{1}{z_i - z_j} \partial_{z_i} - \frac{1}{z_i - z_j} \partial_{\bar{z}_i} \right) .$$

It is easy to check that

$$ [\sum_i z_i \partial_{z_i} , \hat{A}] = -\hat{A} + 4\pi \eta \sum_{i \neq j} (z_i - z_j) \delta^2(z_i - z_j) \partial_{z_i} .$$

In view of the identity i.e., $x\delta(x) = 0$, the above equation reduces to

$$ [\sum_i z_i \partial_{z_i} , \hat{A}] = -\hat{A} .$$

Performing a ST by $\exp\{-\hat{A}\}$, Eq. (42) becomes

$$ \exp\{\hat{A}\} \tilde{H} \exp\{-\hat{A}\} \equiv \bar{H} = \sum_i z_i \partial_{z_i} + \frac{1}{2} N .$$

Finally, the following ST by $\hat{W} \equiv \exp\{2 \sum_i \partial_{z_i} \partial_{\bar{z}_i}\} \exp\{\frac{1}{4} \sum_i \bar{z}_i z_i\}$ brings the above Hamiltonian to a Hamiltonian of $N$ free harmonic oscillators,

$$ \hat{W}^{-1} H \hat{W} = \frac{1}{2} N \sum_{i=1}^{N} (-4\partial_{z_i} \partial_{\bar{z}_i} + z_i \partial_{z_i} - \bar{z}_i \partial_{\bar{z}_i} + \frac{1}{4} \bar{z}_i z_i) .$$

13
By defining \( a_i^+ = \hat{S}^{-1}z_i \hat{S} \) and \( a_i^- = \hat{S}^{-1}\partial_{z_i} \hat{S} \); where \( \hat{S} = \psi_0 \exp\{\hat{A}\} \), and making use of Eq. (45), one can rewrite Eq. (41) as

\[
H = \sum_i H_i + \frac{1}{2} N = \sum_i a_i^+ a_i^- + \frac{1}{2} N
\]

(47)

where, \( H_i \equiv a_i^+ a_i^- \), such that

\[
[a_i^-, a_j^+] = \delta_{ij}
\]

and

\[
[H_i, a_j^\pm] = \pm a_j^\pm \delta_{ij}
\]

These \( N \) quantities \( H_i \) serve as the conserved quantities and are in involution, \textit{i.e.}, \([H_i, H_j] = 0\).

Since \( a_i^- \) and \( a_i^+ \) obey non-interacting oscillator algebra, one can make use of this fact to define a \textit{linear} \( W_{1+\infty} \) algebra, making use of the basis given in Ref. [87]. The highest weight vector obtained from \( L_{m,n} \psi_0 = 0 \) for \( n > m \geq -1 \) is nothing but the Laughlin wave function.

**VI. CONCLUSIONS AND DISCUSSIONS**

In conclusion, we have extended the procedure for the exact diagonalization of the N-particle Calogero-Sutherland model to a host of related models and have demonstrated the precise correspondence between these models and free oscillators. The underlying spectrum generating algebras, the construction of the excited state eigenfunctions and the conserved quantities responsible for the quantum integrability were also explicated. Although, we have worked out only the \( A_{N-1} \) and \( B_N \) type models, for the sake of brevity, it is evident that our method generalizes to other root systems. In light of the fact that, the eigenvectors of the CSM have correspondence with the singular vectors of the \( W_N \) algebras,\(^{88}\) it is worth studying in greater detail, the properties of our exact expressions for the excited state eigenvectors. This will throw more light on the emergence of conformal field theory results\(^{89-91}\) from the CSM in various limiting conditions. The construction of the coherent states for these correlated systems is now straightforward, starting from the known coherent states of the oscillator systems. In the two particle case, it is known that, the interaction brings in interesting behaviour in the coherent states.\(^{92}\) Hence, the N-particle case needs further investigation. The method elaborated here can also be applied to other models involving nearest and next-to-nearest neighbour interactions.\(^{56-59}\) These models are related to pseudo-integrable systems, which are currently attracting considerable attention in connection with physical systems like metal-insulator transitions.\(^{93}\) The procedure for diagonalizing the CSM has been found inadequate for dealing with SM and its generalizations. A recently developed method was used for mapping the SM to free particles on a circle, which also yielded a novel connection between the Jack polynomial and monomial symmetric functions. The precise relationship between the wave functions of SM and the free particle eigenfunctions is currently under study, in light of its relevance to generalized statistics as also the problem of finding ladder operators for the SM.
In two dimensions, the consequences of the linear $W_{1+\infty}$ symmetry algebra, the realization of the Laughlin wave function as the highest weight vector and the oscillator connection of the Chern-Simons Hamiltonians need further exhaustive study. Apart from the deep interest of the lowest Landau level physics, the aforementioned Hamiltonians involving Chern-Simons gauge fields, also provide realizations of the Jain picture.\textsuperscript{65} It is worth mentioning that, the lowest Landau level physics can be mapped to a one dimensional quantum mechanical system of the Calogero-Sutherland type.\textsuperscript{8–10} Hence, the results of this paper for one dimensional models may find relevance for electrons in a strong magnetic field. Further investigations, along the above lines are currently under progress and will be reported elsewhere. Recently, Calogero-Sutherland type models are also being extended to non-commutative spaces,\textsuperscript{9} which has relevance to lowest Landau level physics.\textsuperscript{95,96} Hence, extending the present approach to the noncommutative space is also an interesting area to explore.
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