Critical behavior of noise-induced phase synchronization

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Abstract – We present for the first time in detail the set of the main critical exponents associated with the phase transition of the Kuramoto model under multiplicative noise action. This was done considering the equilibrium thermodynamics for the states of synchronization as well as the subsequent analysis of the critical behavior of the free energy and entropy of the model. We reinforce the concept of the synchronization field for a system of oscillators with multiplicative noise where an expression for the susceptibility is analytically obtained at the critical limit. These results complete the gap that was lacking in obtaining all the critical exponents associated with the phase transition of a Kuramoto-type model.

Introduction. – One of the essential properties of many physical quantities in the vicinity of phase transitions is that they are well described through their critical exponents [1]. For systems with a specific spatial dimensionality and symmetries, these exponents are universal, which allows us to group a large class of phenomena within the same general behavior. Landau proposed a theory for continuous phase transition [2] wherein the definition of an order parameter that characterizes the transition symmetry plays a fundamental role. The Landau theory, associated with the Ginzburg criterion [3] establishing its applicability limit, allows us to find the main critical exponents analytically for some systems. It is well known that very close to the critical temperature $T \rightarrow T_c$, the specific heat $C_H$ at constant field $H$, the magnetization $m$, and the magnetic susceptibility $\chi$ behave, respectively, as $C_H \propto \tau^{-\alpha}$, $m \propto \tau^{\beta}$, and $\chi \propto \tau^{-\gamma}$, where $\tau = 1 - T/T_c$. These equations are valid for $H = 0$ and $\tau \rightarrow 0$. Moreover, for $\tau = 0$ and $H \rightarrow 0$ we have $m \propto H^{1/\delta}$. The set of exponents $\alpha$, $\beta$, $\gamma$, $\delta$ are the main critical exponents in the study of phase transitions. Starting from the Landau theory it is possible to construct more easily a mean-field theory whose exponents are $\alpha = 0$, $\beta = 1/2$, $\gamma = 1$, and $\delta = 3$.

The study of the synchronization of coupled oscillators has traditionally been part of dynamical systems theory. The most prominent applications of the theory were in biological systems [4]. In fact, the first systematic mathematical approach using coupled phase oscillators was proposed by Winfree [5], who was a pioneer in elucidating the fundamental mechanism of the synchronization process, directing the quantitative study of the synchronization phenomenon.

It is recognized that one of the major conceptual advances in the understanding of synchronization phenomena in physical systems was its interpretation as a phase transition phenomenon by physicist Kuramoto. In fact, by making analogies with the mean-field theories for magnetic systems, Kuramoto developed an order parameter for the coupled oscillators model that allowed the characterization under various conditions of the transition of the oscillator system to the synchronized state [6]. It was subsequently found that the distribution of natural frequencies influences the type of phase transition model, which can exhibit continuous or discontinuous transitions and different critical exponents [7,8]. Moreover, a phase transition has also been characterized by a symmetry breaking and a discontinuity in the first derivative of the order parameter in coupled oscillators [9]. Other studies have shown that in addition to the frequency distribution, the form of the coupling function can also change the critical exponent of the order parameter in the transition [10,11].

Although much has been discussed about the critical exponent of the order parameter, there is still a lack of studies that systematically address other critical exponents that characterize the complete theory of phase transitions.
of these coupled phase oscillators systems, i.e., the state of a particular phase transition theory in Kuramoto’s approach has not yet reached the stage of other models such as the Ising model. In part, this is due to the lack of proposals that deal with the thermodynamic extension of these models, especially the absence in the literature of the field concept associated with synchronization, as well as an analytical expression for susceptibility. Recently, several studies have appeared [12–14] that address the susceptibility to phase oscillators but give priority to its numerical analysis.

We present a broad analytical approach to the critical behavior of phase oscillators, where we calculate the main critical exponents for the generalized Kuramoto model systematically developed from a thermodynamic equilibrium approach [15]. We analyze the critical behavior of the various thermodynamic quantities involved, such as entropy, free energy, specific heat, synchronization field and susceptibility. We derive a fluctuation-dissipation relation connecting the order parameter fluctuation with susceptibility.

**The model.** – We introduced the Itô stochastic differential equation for phase oscillators [15] in the form

$$\dot{\theta}_i = \omega_i + f_i(\{\theta\}) + g_i(\{\theta\})\xi_i(t),$$  
(1)

for $i = 1, 2, \ldots, N$ oscillators, where $\omega_i$ are the natural frequencies of the oscillators, and the drift force $f_i(\{\theta\})$ and the noise strength $g_i(\{\theta\})$ are general functions of the phases $\{\theta\} = \theta_1, \ldots, \theta_N$ and $\xi$ is a Gaussian white noise which obeys the relations

$$\langle \xi_i(t) \xi_j(t') \rangle = 2D \delta_{ij} \delta(t - t') \quad \text{with} \quad \langle \xi_i(t) \rangle = 0,$$

(2)

in which $D$ is the diffusion constant. We employ now Kuramoto’s drift force [6] for phase oscillators, given by

$$f_i(\{\theta\}) = \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i),$$

(3)

which is controlled by the coupling constant $K$ that can take positive values (attractive interaction) or negative (repulsive interaction). Now we adopt an identical natural frequency $\omega_i = \omega$ and considering the rotating frame, we can set $\omega = 0$. This allows us to write eq. (1) in the mean-field approach as

$$\dot{\theta}_i = f(\theta_i) + g(\theta_i)\xi_i(t),$$

(4)

where the drift force eq. (3) can be reduced to

$$f(\theta_i) = rK \sin(\psi - \theta_i),$$

(5)

for which the order parameter $r$ and the average phase $\psi$ are defined by

$$r e^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j}.$$  
(6)

Here, $g(\theta_i)$ can also be written in terms of mean-field parameters $r$ and $\psi$. The order parameter $r$ measures the phase coherence, i.e., for $r = 1$, the system is fully synchronized, whereas for $r = 0$, the system is fully incoherent. A partially synchronized state is obtained when $0 < r < 1$. Note that when the interactions of oscillator $i$ with other oscillators of the system are no longer considered individually but in terms of the mean-field effects of the system acting on the oscillator $i$, we can omit the index $i$ of the individual oscillator $\theta_i = \theta$, such that eq. (4) is given by

$$\dot{\theta} = f(\theta) + \sqrt{g(\theta)}\xi(t).$$

(7)

To determine the noise strength $g(\theta)$, we consider that eq. (7) establishes a general Itô phase equation to an oscillator system with strong limit cycle attractor [16–18] in the form

$$\dot{\theta} = \tilde{D} \dot{Z}(\theta) Z'(\theta) + Z(\theta) \xi(t),$$

(8)

where $Z'(\theta) = \partial Z(\theta)/\partial \theta$ and $\tilde{D}$ refers to the diffusion constant related to a small perturbation of the trajectory in the limit cycle for the oscillator, which must satisfy $|\tilde{D}| \ll 1$. In this case, the function $g(\theta)$ is derived from $f(\theta)$ as

$$f(\theta) = \tilde{D} Z(\theta) Z'(\theta) = rK \sin(\psi - \theta),$$

(9)

$$g(\theta) = Z(\theta)^2 = 1 + r \sigma \cos(\psi - \theta),$$

(10)

where we define $\sigma = 2K/\tilde{D}$, with $-1 \leq \sigma \leq 1$. The parameter $\sigma$ is the noise coupling that determines the intensity of $\sqrt{g(\theta)}$, i.e., the global modulation of the multiplicative noise. Here we point out that the parameters $\tilde{D}$ in eq. (2) and $\tilde{D}$ in eq. (8) are not necessarily the same. Note that the diffusive parameter $\tilde{D}$ is associated with the magnitude of the random perturbations in the orbits of the limit cycle while the diffusion constant $D$ is exclusively defined by the noise white correlation eq. (2) and has no action in the drift force term of the model. Ensuring that parameter $\tilde{D}$ is small and the coupling $K$ is weak, it is possible to absorb them in the definition of $\sigma$.

The corresponding Fokker-Planck equation from eq. (7), in Itô prescription, is written as

$$\frac{\partial P}{\partial t} = D \frac{\partial^2}{\partial \theta^2} [g(\theta)P] - \frac{\partial}{\partial \theta} [f(\theta)P].$$

(11)

This equation satisfies the thermodynamic equilibrium condition. This can be verified by making a transformation in the Fokker-Planck eq. (11) using the following change of variables:

$$\phi(\theta) = \frac{1}{\sqrt{|D|}} \int_0^\theta \frac{d\theta'}{\sqrt{g(\theta')}},$$

(12)

which results in the temporal evolution of distribution $P(\phi,t)$, in the form

$$\frac{\partial P(\phi,t)}{\partial t} = \frac{\partial^2 P}{\partial \phi^2} - \frac{\partial}{\partial \phi} [\mathcal{F}(\phi)P].$$

(13)
The function $\mathcal{F}(\phi)$ is the drift term associated to the Langevin equation $\dot{\phi} = \mathcal{F}(\phi) + \xi(t)$, see [15] for details. The drift term $\mathcal{F}(\phi)$ is conservative
\begin{equation}
\int \mathcal{F}(\phi) d\phi = 0,
\end{equation}
and the system reaches thermal equilibrium. Using eqs. (9) and (10), the Fokker-Planck eq. (11) can be written as
\begin{equation}
\frac{\partial \rho}{\partial t} = D \frac{\partial^2}{\partial \theta^2} [(1 + r \sigma \cos(\psi - \theta)) \rho] - \frac{\partial}{\partial \theta} [r K \sin(\psi - \theta) \rho].
\end{equation}
This equation has an exact analytical expression for the stationary equilibrium distribution $\rho_s(\theta)$, given by
\begin{equation}
\rho_s(\theta) = N^{-1} [z + \text{sgn}(\sigma) \sqrt{z^2 - 1} \cos(\psi - \theta)]^\nu,
\end{equation}
where $\text{sgn}(\sigma)$ is the sign function. The normalization constant $N$ and parameters $z$ and $\nu$ are
\begin{align}
N &= 2 \pi P_0^0(z), \\
z &= (1 - \sigma^2 r^2)^{-1/2}, \\
\nu &= \frac{K}{D \sigma} - 1,
\end{align}
where $P_0^0(z)$ is the associated Legendre function of zero order.

**Critical behavior.** – To study the critical behavior of the system, we introduce the first law of thermodynamics to phase oscillators already established in ref. [15], given by
\begin{equation}
dF = -SdT + H_s dr,
\end{equation}
where $F = F(T, r)$ is the Helmholtz free energy, and $S$, $T$, and $r$ are entropy, temperature and order parameter, respectively. Moreover, we have defined a new quantity $H_s$ that plays the role of a synchronization field on the oscillator system.

**Order parameter and temperature.** At this point it is useful to obtain the order parameter in the critical region. From stationary density $\rho_s$ the order parameter $r$ is given by
\begin{equation}
r = \int_0^{2\pi} e^{i(\theta - \nu)} \rho_s (\theta) d\theta = \frac{\text{sgn}(\sigma) P_0^1(z)}{1 + \nu P_0^0(z)}.
\end{equation}
In the critical region $r \approx 0$, we can take the first terms of expanded eq. (21) for $z \approx 1$, which leads to
\begin{equation}
r = \frac{\sqrt{2} \text{sgn}(\sigma) \nu}{2} (z - 1)^{1/2} - \frac{\sqrt{2} \text{sgn}(\sigma)(\nu^2 + \nu + 1) \nu}{8} (z - 1)^{3/2}.
\end{equation}
For $r \approx 0$, the $z$ parameter can be expanded as
\begin{equation}
z = (1 - r^2 \sigma^2)^{-1/2} \approx 1 + \frac{r^2 \sigma^2}{2} + \frac{3r^4 \sigma^4}{8}.
\end{equation}
Note that from eq. (23) the order parameter eq. (22) is a highly transcendental function in $r$. Starting from the equations above, we obtain the critical coupling $K_c$ of the system. Retaining the first terms in $r$ we get $r \approx \nu \sigma / 2$ and using $\nu = K_c / D \sigma - 1$, $K_c$ is given by
\begin{equation}
K_c = D(\sigma + 2).
\end{equation}
Note that for $\sigma = 0$, we retrieve the Kuramoto-Sakaguchi model with additive noise and equal natural frequency $K_c = 2D$.

In order to analyze the effects of multiplicative noise in the phase transition of the system, we define the temperature in terms of the critical coupling as
\begin{equation}
T \equiv \frac{K_c}{K} = \frac{2D}{K},
\end{equation}
such that $T_c = 1$ is the critical temperature for $\sigma = 0$.

An important limit associated with order parameter eq. (21) is the asymptotic condition $\nu \to \infty$ when $\sigma \approx 0$, which results in
\begin{align}
\varepsilon &\equiv \nu \cosh^{-1}(z), \\
z &\equiv (1 - \sigma^2 r^2)^{-1/2}, \\
r(\varepsilon) &= \lim_{\nu \to \infty} r \approx \frac{I_1(\varepsilon)}{I_0(\varepsilon)},
\end{align}
where $I_0(x)$ and $I_1(x)$ are the modified Bessel functions of first kind of order 0 and 1, respectively. Furthermore, in the limit $\sigma \to 0$, the order parameter eq. (21) is given by
\begin{align}
k &= \lim_{\sigma \to 0} \frac{2\pi(k)}{T}, \\
r(k) &= \lim_{\sigma \to 0} r(\varepsilon) = \frac{I_1(k)}{I_0(k)}
\end{align}
which is the case for which we do not have a multiplicative noise, but only a Gaussian white noise.

**Entropy.** As we already have shown, the free energy for a phase oscillators system with multiplicative noise is given by
\begin{equation}
F = -T \ln[2\pi P_0^0(z)],
\end{equation}
which allows us to obtain entropy $S$ by the definition eq. (20) as
\begin{equation}
S = - \left( \frac{\partial F}{\partial T} \right) = \ln[2\pi P_0^0(z)] - (\nu + 1) \frac{\partial}{\partial \nu} \ln[2\pi P_0^0(z)].
\end{equation}
Here we use $\nu = 2 / T \sigma - 1$. We can rewrite the above equation by doing the following transformation of variables $S(\nu) \to S(-\nu - 1)$:
\begin{equation}
S = \ln[2\pi P_{-\nu-1}^0(z)] - \nu \frac{\partial}{\partial \nu} \ln[2\pi P_0^0(z)],
\end{equation}
and using the property of the functions of Legendre $P_{-\nu-1}^0(z) = P_{\nu}^0(z)$, we find
\begin{equation}
S = \left( 1 - \nu \frac{\partial}{\partial \nu} \right) \ln[2\pi P_0^0(z)].
\end{equation}
Since we are interested only in the variation of entropy \( dS(\nu) = dS(-\nu -1) \), this transformation can be performed without loss of generality. We now analyze the entropy eq. (32) in the critical region by expanding the Legendre function for \( z \approx 1 \), given by

\[
P^0_\nu(z) \approx 1 + \frac{\nu(\nu + 1)(z - 1)}{2},
\]

which results in the critical entropy, given by

\[
S = S_{\text{max}} - \frac{\nu^2 \sigma^2}{4} r^2. \quad (36)
\]

Here, \( S_{\text{max}} = \ln(2\pi) \) is the maximum entropy of the system.

**Synchronization field.** The synchronization field is properly defined by the first law of thermodynamics eq. (20) as

\[
H_s = -\left( \frac{\partial F}{\partial r} \right)_T = \frac{T(1 + \nu)z^3 \sigma^2 \sigma^2 \text{sgn}(\sigma)}{\sqrt{z^2 - 1}}. \quad (37)
\]

We have already shown that the field \( H_s \) is associated with the noise effect of the system, i.e., the Gaussian white noise and multiplicative noise. Hence, \( H_s \) can be decomposed as

\[
H_s = H_0 + H_\sigma,
\]

where \( H_0 \) and \( H_\sigma \) are the internal and external synchronization fields, respectively, defined as

\[
H_0 = \lim_{\sigma \to 0} H_s = 2r = 2L_1(k), \quad (39)
\]

\[
H_\sigma = H_s(\sigma, r, T) - H_0. \quad (40)
\]

Here we have used the results of eqs. (29) and (30). Note that \( H_0 \) is associated to part of field \( H_s \), which does not depend explicitly on \( \sigma \), i.e., \( H_0 \) is the field related to the Gaussian white noise behavior. On the other hand, \( H_\sigma \) corresponds to the part of field \( H_s \) which explicitly depends on \( \sigma \). This is directly associated to the non-null multiplicative noise.

Now we are interested in obtaining the field \( H_s \) in the critical region. Thus expanding eq. (37) for \( r \approx 0 \) (with \( z \approx 1 + r^2 \sigma^2 / 2 \), eq. (23)), it results in

\[
H_s \approx 2\text{sgn}(\sigma) r = \frac{1}{2} T \nu^2 \sigma^2 r + \frac{1}{2} T \nu \sigma^2 r. \quad (41)
\]

On the left-hand side of the above equation, we employ the order parameter eq. (22). Furthermore, by inserting \( \nu = 2/T - 1 \) in eq. (41) and retaining the terms in \( r \), we obtain the critical synchronization field

\[
H_s = \frac{2r}{T} - \sigma r = H_0 + H_\sigma. \quad (42)
\]

Finally, in the critical region \( T \approx T_c = 1 \), the critical internal field is \( H_0 = \lim_{\nu \to 0} H_s = 2r/T \) and

\[
H_\sigma = -\sigma r \quad (43)
\]

is the critical external field. Indeed, the field \( H_\sigma \) depends explicitly on the noise coupling \( \sigma \), as expected. The negative sign also shows that synchronization \( r \) should decrease when \( -\sigma \) increases in the critical region.

**Specific heat.** We can now define the critical specific heat at constant external field \( H_\sigma \). First, we express the critical entropy eq. (36) in terms of \( H_\sigma \), eq. (43), which gives

\[
S = S_{\text{max}} - \frac{r^2}{T^2} - \frac{H_\sigma}{T} r - \frac{H_\sigma^2}{4}. \quad (44)
\]

It follows that the critical specific heat at constant external field is given by

\[
C_{H_\sigma} = T \left( \frac{\partial S}{\partial T} \right)_{H_\sigma} = \frac{2r^2}{T^3} - \frac{2r}{T} \frac{\partial r}{\partial T} + H_\sigma \left( \frac{r}{T} \frac{\partial r}{\partial T} \right). \quad (45)
\]

It is important to emphasize that we have achieved the main thermodynamic functions in the critical region for the phase oscillator system. Furthermore, the concept of critical synchronization field shown above is of crucial importance for determining the complete thermodynamics. Indeed, this reflects the precise determination of the main critical exponents for the model, as we will see below.

**Fluctuation-dissipation relation.** We can also examine the relation between order parameter fluctuation \( \langle r^2 \rangle \) and its response function in the vicinity of the phase transition. We begin by considering the probability of fluctuation \( w(r) \) of the order parameter that can be directly obtained as

\[
w(r) \propto \exp(\Delta S), \quad (46)
\]

where \( \Delta S \) is the change in entropy in the fluctuation that can be expressed as \( \Delta S = -W_{\text{min}}/T \) [19], such that \( W_{\text{min}} \) is the minimum external work that must be performed on the system in order to reversibly produce this fluctuation. At the critical region, the minimum external work is then related to the external field \( H_\sigma \), eq. (43), according to

\[
W_{\text{min}} = \int_0^r H_\sigma dr = -\frac{\sigma r^2}{2}. \quad (47)
\]

On the other hand, from the definition of susceptibility

\[
\chi^{-1} = \left( \frac{\partial H_\sigma}{\partial r} \right)_T = -\sigma, \quad (48)
\]

which allows to rewrite eq. (46) in the form

\[
w(r) \propto \exp \left( \frac{r^2}{2\chi T} \right), \quad (49)
\]

which is the usual Gaussian distribution, whose mean square fluctuation \( \langle r^2 \rangle \) takes the form

\[
\langle r^2 \rangle = T_c \chi. \quad (50)
\]

This is the fluctuation-dissipation relation for the oscillators system, which relates its linear response \( \chi \) due to the action of the external field \( H_\sigma \) with the fluctuations of the order parameter around the equilibrium region \( \langle r \rangle = 0 \).

Indeed, a similar relationship holds in magnetic systems for the magnetization fluctuation [20].
Critical exponents. – We are now able to determine the main critical exponents for the phase oscillator system. First we must define, in the critical region, the quantity

$$\tau = \frac{T_c - T}{T_c} = 1 - T,$$  \hspace{1cm} (51)

with $T \approx T_c$, where $T_c = 1$ is the critical temperature.

Order parameter. We start by determining the critical exponents of the order parameter $r$. Note that eq. (22) can be written in terms of external field eq. (43) as

$$r^2 \Lambda = 1 - \frac{1}{T} + \frac{\sigma}{2} = 1 - \frac{1}{T} - \frac{H_\sigma}{2T},$$  \hspace{1cm} (52)

where

$$\Lambda = \frac{\nu \sigma^3}{16} (3 - \nu^2 - \nu - 1) = \frac{H_\sigma^2}{8r^2} + \frac{H_\sigma^2}{2rT^2} = \frac{1}{2T^4}. \hspace{1cm} (53)$$

We should now impose that $T \to T_c = 1$ for null field $H_\sigma = 0$, such that eq. (53) goes to $\Lambda = -1/2$ and eq. (52) converges to

$$r = \lim_{\tau \to 0} \left[ \Lambda^{-1} (1 - \frac{1}{T}) \right]^{1/2} \propto (1 - T)^{1/2} \propto \tau^\beta,$$  \hspace{1cm} (54)

where $\beta = 1/2$ is the critical exponent for the order parameter $r$, as has already been obtained for the Kuramoto model [7,11]. Here however, we have presented a rigorous demonstration through the synchronization field concept. Furthermore, note also that along the isotherm $T = T_c = 1$, eq. (52) leads to

$$r^3 \Lambda = -\frac{1}{2} H_\sigma,$$  \hspace{1cm} (55)

and taking $\tau = 0$ and $H_\sigma \to 0$ (with $\Lambda = -1/2$), we get the relation between $r$ and external field $H_\sigma$ as

$$r = -\lim_{H_\sigma \to 0} \frac{1}{2} \left( 2\Lambda \right)^{1/3} H_\sigma^{1/3} \propto H_\sigma^{1/3},$$  \hspace{1cm} (56)

where we obtain the critical exponent $\delta = 3$ for the system.

Specific heat. In order to obtain the critical exponent for the specific heat, we start from eq. (45) for null field $H_\sigma = 0$ and $\tau \to 0$, which leads to

$$C_{H_\sigma=0} = \lim_{\tau \to 0} \left[ \frac{g \sigma^2}{T^2} - \frac{2g \partial r}{T \partial T} \right]$$

$$= \left(-2r^2 \frac{\partial r}{\partial T} \right)_{r \to 0} \propto \tau^{\gamma} \tau^{\frac{\nu}{2}} \propto \tau^\alpha,$$  \hspace{1cm} (57)

where $\alpha = 0$ is the critical exponent for the specific heat.

Susceptibility. Now, from the definition of inverse susceptibility $\chi^{-1}$ and taking eq. (55) for the null field $H_\sigma = 0$, $\Lambda = -1/2$, and $\tau \to 0$, we obtain

$$\chi^{-1} = \left( \frac{\partial H_\sigma}{\partial r} \right)_{H_\sigma=0} \propto r^2 \propto \tau,$$  \hspace{1cm} (58)

where we use $r \propto \tau^{1/2}$. This strictly shows that the susceptibility is divergent, i.e.,

$$\chi \propto \tau^{-\gamma},$$  \hspace{1cm} (59)

with critical exponent $\gamma = 1$, in accordance with the mean-field theory. According to eq. (59), the mean square fluctuation of the order parameter eq. (50) increases as $1/\tau$ when $T \to T_c$.

These results conclude the study of the four main critical exponents of the generalized Kuramoto model for which we have also shown the fluctuation-dissipation relation.

Conclusions. – In this work, we have systematically investigated the critical behavior of phase oscillators with multiplicative noise from a thermodynamic equilibrium approach. We derived the set of the four main mean-field critical exponents $\alpha = 0$, $\beta = 1/2$, $\gamma = 1$ and $\delta = 3$. Indeed, this is the first time that all of these exponents have been presented for phase oscillator systems. The critical behavior associated with phase oscillators may appear in many physical systems, such as biomolecular networks and neural systems [21–23], particularly in the neuronal avalanches phenomenon [24] where synchronization transition is present. Furthermore, susceptibility and specific heat as presented in this study can play important roles in the description of the thermodynamics of these physical systems.

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