High-order asymptotics for the Spin-Weighted Spheroidal Equation at large real frequency

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The spin-weighted spheroidal eigenvalues and eigenfunctions arise in the separation by variables of spin-field perturbations of Kerr black holes. We derive a large, real-frequency asymptotic expansion of the spin-weighted spheroidal eigenvalues and eigenfunctions to high order. This expansion corrects and extends existing results in the literature and we validate it via a numerical calculation.

I. INTRODUCTION

Teukolsky derived a single “master” equation for spin-field perturbations of rotating (Kerr) black holes [1, 2]. This (3 + 1)-dimensional master equation separates by variables, with the polar-angular factor in the solution being the so-called spin-weighted spheroidal eigenfunction. The corresponding eigenvalue also appears in the equation satisfied by the radial factor of the Teukolsky master solution. Thus, both the eigenvalues and the eigenfunctions are important for studying perturbations of astrophysical black holes.

Neither the spin-weighted spheroidal eigenfunctions nor eigenvalues are known in closed form but they can be calculated using numerical and analytical techniques (see [3, 4] for a review). As for the analytical techniques, for example, expansions have been obtained for small frequency [5–8] and asymptotic analyses have been carried out for large, purely-imaginary frequency [3, 4, 9–11].

In this paper we are instead interested in the asymptotics for large, real frequency. These asymptotics are interesting for various reasons, such as for synchrotron radiation [9, 12], for the study of divergences in either the quantum or classical field theories (e.g., [13, 14] for WKB in the case of spherically-symmetric space-times and [15, 16] for expressions for expectation values involving (spin-weighted) spheroidal harmonics in Kerr) and gravitational waves from rapidly rotating black holes [17, 18]. Analytic approximations are also extremely valuable as checks on numerical calculation schemes. The large, real frequency behaviour of the scalar spheroidal eigenfunctions and eigenvalues was studied in [19–21]. The first large, real frequency study in the non-zero spin case was carried out in [9]. However, this work contained an error which was corrected by Breuer, Ryan and Waller [22] (BRW). BRW provided an asymptotic expansion for the eigenvalue up to six leading orders, which depended crucially on a parameter $s_{q\ell m}$ that was left undetermined for the case of non-zero spin. Furthermore, their analysis for non-zero spin had an error in the asymptotic behaviour of the eigenfunctions which was later corrected in [23]. This correction further allowed [23] to analytically obtain the parameter $s_{q\ell m}$ as well as the correct first term in a large real-frequency series expansion for the eigenfunctions.

As it turns out, however, the last three orders in the asymptotic expansion for the eigenvalue provided in BRW formally in terms of $s_{q\ell m}$ were also incorrect. In this paper, we correct these 3rd-to-6th leading orders and extend the expansion up to four higher orders, thus providing the correct ten leading orders of the eigenvalue for large real frequency. We also provide the first few coefficients in the large, real-frequency expansion of the eigenfunctions, thus going, for the first time, beyond leading order. We compare our asymptotic expansions for both the eigenvalues and eigenfunctions with high-precision numerical calculations and find excellent agreement. The results of this paper together with those in [23] thus provide a correct, high-order asymptotic expansion of the eigenvalues and eigenfunctions for large, real frequency.

The layout of the rest of the paper is as follows. In Sec. [1] we introduce the spin-weighted spheroidal equation and its symmetries. In Sec. [11] we perform the large frequency asymptotic analysis of the spin-weighted spheroidal eigenfunctions and eigenvalues. We compare our asymptotic analysis and our numerical results in Sec. [IV]. In Appendix [A] we give explicit expressions for the coefficients in the series for the eigenfunctions and in Appendix [B] we describe the implementation of the asymptotics for the eigenvalue in a Mathematica toolkit.

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II. SPIN-WEIGHTED SPHEROIDAL EQUATION

Teukolsky [1, 2] managed to decouple and separate by variables the linear spin-field perturbations of Kerr black holes. He achieved this for the radiative components of the massless fields of spin $s = 0$ (scalar), $\pm 1/2$ (neutrino), $\pm 1$ (electromagnetic) and $\pm 2$ (gravitational). The polar-angular factor of the perturbations are the so-called spin-weighted spheroidal harmonics $sS_{lmc}$. These functions satisfy the following linear, second-order ordinary differential equation (ODE):

$$\left( \frac{d}{dx} \left( (1 - x^2) \frac{d}{dx} \right) - \frac{c^2 x^2 - 2scx}{1 - x^2} - \frac{(m + sx)^2}{1 - x^2} + sE_{lmc} - s^2 \right)sS_{lmc}(x) = 0,$$

(2.1)

where $x \equiv \cos \theta \in [-1, +1]$ is the physical region of interest and $\theta \in [0, \pi]$ is the (Boyer-Lindquist) polar angle. Here, $c \equiv a\omega$, where $a \in \mathbb{R}$ is the angular momentum per unit mass of the black hole and $\omega \in \mathbb{C}$ is the frequency of the field mode. The multipole number $\ell = |s|, |s| + 1, |s| + 2, \ldots$ serves to label the eigenvalue $sE_{lmc}$ and $m = -\ell, -\ell + 1, \ell + 2, \ldots, +\ell$ is the azimuthal number. The ODE (2.1) has two regular singular points at $x = \pm 1$ and an irregular singular point at $x = \infty$. The eigenvalue $sE_{lmc}$ is chosen so that the corresponding solution $sS_{lmc}$ is regular over $x \in [-1, +1]$. The case $s = 0$ yields the (scalar) spheroidal equation [19, 21], whereas the case $c = 0$ yields the spin-weighted spherical equation [24] (in which case $sS_{lmc} = \ell(\ell + 1)$).

Other common parameterizations of the eigenvalue are

$$s\lambda_{lmc} \equiv sE_{lmc} - s(s + 1) - c^2 - 2mc,$$

$$sA_{lmc} \equiv sE_{lmc} - s(s + 1),$$

(2.2a, b)

The manifest symmetries of the spin-weighted spheroidal equation imply that

$$-sS_{lmc}(x) = (-1)^{\ell+m}sS_{lmc}(-x), \quad sS_{l(-m)}(-x) = (-1)^{\ell+s}sS_{lmc}(-x),$$

(2.3)

where the choice of signs ensures consistency with the so-called Teukolsky-Starobinsky identities [25, 26], and

$$-sE_{lmc} = sE_{lmc}, \quad sE_{l(-m)c} = sE_{l(-m)c}.$$  

(2.4)

While $s\lambda_{lmc}$ is most common in the current literature, we shall use $sA_{lmc}$ in the following sections to ease comparison with BRW.

III. LARGE REAL-FREQUENCY ASYMPTOTICS

In this paper we are interested in the large, real "frequency" (by which we really mean $c \to \pm \infty$) behaviour of the eigenvalues and eigenfunctions. In addition, by Eqs. (2.3) and (2.4) we may assume $c$ positive and deduce the $c$ negative behaviour from changing $m$ to $-m$. We therefore restrict ourselves to $c > 0$ from now on.

We here generalize to arbitrary spin Flammer’s [20] approach in the scalar case – this is essentially BRW’s path, although they obtained some incorrect results which we specify and correct below. We start by writing solutions of the spin-weighted spheroidal equation (2.1) as

$$sS_{lmc}^\pm(x) = (1 - x)^{|m|+|s|/2}(1 + x)^{|m-s|/2}e^{-c(1\mp x)}g_\pm(x),$$

(3.1)

where $g_\pm(x)$ are regular functions. The powers of $(1 - x)$ and $(1 + x)$ are dictated by the Frobenius method, so that the solution is regular at both boundary points $x = \pm 1$ and $-1$. The exponential factor $e^{-c(1\mp x)}$ is included for convenience when looking for an asymptotic solution “near” $x = \pm 1$.

From Eq. (2.3), it follows that the solution $g_{-}(x)$ is obtained from $g_{+}(x)$ under the transformation $\{ s \to -s, x \to -x \}$, modulo an overall sign of the solution. Hence, from now on we focus on $g_{+}$.

Looking for the asymptotic solution valid near $x = +1$, we introduce $u \equiv 2c(1-x)$. (For a full discussion see [23], where this procedure defines an asymptotic solution $S^{inn,+1}(x)$ for $0 \leq 1 - x \leq \mathcal{O}(c^\delta)$ with $-1 < \delta < 0$.) Inserting

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1 The symbol $s$ really corresponds to the helicity of the spin-field, although in keeping with general convention, we refer to it as the spin.
2 This corrects a typographical error in Eq. (1.3) [19].
3 The function $S^{inn,+1}(x)$ in [23] corresponds to the leading-order term in the expansion for $g_{+}$ which we provide in this paper.
the expression (3.1) into the ODE (2.1), we find that $g_+(u)$ satisfies the following equation,

$$u g_+'' + (m + s + 1 - u) g_+' + \frac{1}{2} (s q_{\ell m} - |m + s| - 1 - s) g_+ +$$

$$s \tilde{A}_{\ell mc} g_+ = \frac{1}{4c} \left( u (u g_+'' + (m + s + |m - s| + 2 - u) g_') + \right.$$

$$\left. \frac{1}{2} ((m + s + 1)(m - s + 1) - (m + 1)^2 + s^2 - (|m + s| + |m - s| + 2 + 2s)u) g_+ \right) = 0, \quad (3.2)$$

where primes denote derivatives with respect to $u$. Here we have defined

$$s \tilde{A}_{\ell mc} = \frac{1}{4c} \left( s(s + 1) - m(m + 1) + c^2 - 2s q_{\ell m} c + s A_{\ell mc} \right), \quad (3.3)$$

introducing the parameter $s q_{\ell m}$, discussed in the Introduction, which is chosen so that

$$s \tilde{A}_{\ell mc} = o(1), \quad \text{as } c \to \infty. \quad (3.4)$$

BRW left the parameter $s q_{\ell m}$ undetermined for non-zero spin. Its value was determined in [23] to be $\ell + 1 - z_0$,

$$s q_{\ell m} = \ell + 1 - z_0, \quad \text{if } \ell \geq |s| \ell_m, \quad (3.5)$$

$$s q_{\ell m} = 2 \ell + 1 - |s| \ell_m, \quad \text{if } \ell < |s| \ell_m, \quad (3.6)$$

where $\ell_m \equiv |m + s| + s$, and

$$z_0 \equiv \begin{cases} 0 & \text{if } \ell + m \text{ even}, \\ 1 & \text{if } \ell + m \text{ odd}. \end{cases} \quad (3.7)$$

The value of $z_0$ indicates whether $s S_{\ell mc}(x)$ has a zero “near” $x = 0$ for large-c: it is $z_0 = 1$ if $s S_{\ell mc}(x)$ has a zero “near” $x = 0$ and it is $z_0 = 0$ if it does not. Regarding the values of $s q_{\ell m}$, we note, in particular, that: (i) $s q_{\ell m}$ is an integer if $s$ is an integer, whereas $s q_{\ell m}$ is a half-integer if $s$ is a half-integer; (ii) $s q_{\ell m} = -s q_{\ell m}$.

In the limit of infinitely large $c$, only the first line in Eq. (3.2) survives and the solution of the resulting ODE which is regular at $u = 0$ ($x = 1$) is

$$1 F_1(-s p_{\ell m+}, |m + s| + 1, u),$$

where we have introduced $s p_{\ell m+} \equiv \frac{1}{2} (s q_{\ell m} - |m + s| - s - 1)$ and $1 F_1$ is the regular confluent hypergeometric function [19]. We note that $s p_{\ell m+} \in \mathbb{Z}$ for $2s \in \mathbb{Z}$.

Eq. (3.2) then suggests that we express the function $g_+$ as

$$g_+(u) = \sum_{n=-\infty}^{\infty} a_n 1 F_1(-s p_{\ell m+} - n, |m + s| + 1, u), \quad (3.8)$$

where without loss of generality we assume that $a_0 = 1$. The series coefficients $a_n$ satisfy a three-term recurrence relation

$$(2n + s q_{\ell m} - |m - s| + s + 1)(2n + s q_{\ell m} - |m + s| - s + 1)a_{n+1} +$$

$$2(8cn - (2n + s q_{\ell m})^2 + 2s^2 - (m + 1)^2 - 8c s \tilde{A}_{\ell mc})a_n +$$

$$(2n + s q_{\ell m} + |m - s| - s + 1)(2n + s q_{\ell m} + |m + s| - s - 1)a_{n-1} = 0. \quad (3.9)$$

These recurrence relations are obtained by inserting the series representation (3.8) into (3.2) and using the following recurrence relations satisfied by the hypergeometric functions [19]:

$$u_1 1 F_1''(\alpha, \beta, u + (\gamma - u)) 1 F_1'(\alpha, \beta, u) = (\gamma - \beta)_1 1 F_1'(\alpha, \beta, u) + \alpha_1 1 F_1(\alpha, \beta, u), \quad (3.10a)$$

$$u_1 1 F_1'(\alpha, \beta, u) = \alpha_1 1 F_1(\alpha + 1, \beta, u - 1 1 F_1(\alpha, \beta, u)), \quad (3.10b)$$

$$u_1 1 F_1(\alpha, \beta, u) = \alpha_1 1 F_1(\alpha + 1, \beta, u) - (2\alpha - \beta)_1 1 F_1(\alpha, \beta, u) + (\alpha - \beta)_1 1 F_1(\alpha - 1, \beta, u), \quad (3.10c)$$

$^4$ It can be checked that our expressions for $s q_{\ell m}$ and $z_0$ here are equivalent to –but simpler than– those given in Eqs. (4.5)–(4.6) in [23].
for constant $\alpha$, $\beta$ and $\gamma$. Because of the analyticity of the coefficients of the spin-weighted spheroidal differential equation in the parameter $c$, we can assume a series expansion in powers of $1/c$ for $sA_{\ell mc}$ \cite{27}:

$$sA_{\ell mc} \sim \sum_{k=1}^{\infty} A_k c^{-k} \quad \text{as } c \to \infty. \quad (3.11)$$

Correspondingly we now expand the coefficients for large-$c$:

$$a_n \sim \sum_{k=|n|}^{\infty} a_{n,k} c^{-k} \quad \text{as } c \to \infty, \quad (3.12)$$

where the structure of this coefficient expansion follows from the dominant first term in the coefficient of $a_n$ in the recurrence relation. We give explicit expressions for the series coefficients $a_{n,k}$ for $n : -3 \to 3$ and $k : |n| \to 3$ in Appendix \[A\].

Inserting the expansions Eqs. \(3.11\) and \(3.12\) into the recurrence relation \(3.9\) and requiring it to be satisfied order-by-order determines the expansion coefficients. Specifically, we find

$$sE_{\ell mc} = -c^2 + 2s_{\ell mc} c - \frac{1}{2} \left( s_{\ell mc}^2 - m^2 - 2s^2 + 1 \right) + \sum_{k=1}^{7} A_k \frac{1}{c^k} + O \left( \frac{1}{c^8} \right), \quad (3.13)$$

where, dropping the subscripts on $s_{\ell mc}$ for compactness,

$$A_1 = -\frac{1}{8} \left( q^3 - m^2 q + q - 2s^2 (q + m) \right), \quad (3.14)$$

$$A_2 = \frac{1}{64} \left( -m^4 + 6m^2 q^2 + 2m^2 - 5q^4 - 10q^2 - 1 + 4s^2 (m^2 + 4mq + 3q^2 + 1) \right), \quad (3.15)$$

$$A_3 = \frac{1}{512} \left( -q(37 + 13m^4 + 114q^2 + 33q^4 - 2m^2(25 + 23q^2)) + 4(13m - m^3 + 25q + 9m^2q + 33mq^2 + 23q^3)s^2 - 8(m + q)s^4 \right), \quad (3.16)$$

$$A_4 = \frac{1}{1024} \left( -14 + 2m^6 - 239q^2 - 340q^4 - 63q^6 - 3m^4(6 + 13q^2) + 10n^2(3 + 23q^2 + 10q^4) + 4(-m^4 - 9m^3 q + 5m^2(2 + 3q^2) + mq(93 + 73q^2) + 5(3 + 23q^2 + 10q^4))s^2 - 8(2 + 3m^2 + 9mq + 6q^2)s^4 \right), \quad (3.17)$$

$$A_5 = \frac{1}{8192} \left( -q(1009 - 53m^6 + 5221q^2 + 4139q^4 + 527q^6 + 5m^4(127 + 93q^2) - m^2(1591 + 3750q^2 + 939q^4)) + 2(14m^5 - 45m^4q + 130m^2q^3 + q^5) - 20m^3(7 + 18q^2) + 2m(303 + 1820q^2 + 685q^4) + q(1591 + 3750q^2 + 939q^4))s^2 - 80(7m + m^3 + 11q + 9m^2q + 18mq^2 + 10q^3)s^4 + 16(m + q)s^6 \right), \quad (3.18)$$

$$A_6 = \frac{1}{131072} \left( -3747 - 51m^8 - 86940q^2 - 205898q^4 - 101836q^6 - 9387q^8 + 12m^6(85 + 167q^2) - 6m^4(939 + 5078q^2 + 1855q^4) + 12m^2(701 + 8657q^2 + 9575q^4 + 1547q^6) + 8(19m^6 + 270m^5q - m^4(191 + 309q^2) - 4m^3q(919 + 725q^2) + m^2(949 + 1482q^2 - 255q^4) + 2mq(8135 + 15310q^2 + 3363q^4) + 3(701 + 8657q^2 + 9575q^4 + 1547q^6))s^2 + 16(-467 + 17m^4 - 236m^3q - 3438q^2 - 1455q^4 - 4mq(919 + 725q^2) - 2m^2(407 + 849q^2))s^4 + 128(4 + 7m^2 + 19mq + 12q^2)s^6 \right). \quad (3.19)$$
A_2 = \frac{1}{2097152} \left( q(822221 + 4093m^8 + 5771940q^2 + 7568470q^4 + 2520820q^6 + 175045q^8 - 1540m^{b}(65 + 43q^2) + 42m^4(16371 + 29350q^2 + 6375q^4) - 4m^2(353449 + 1345421q^2 + 847819q^4 + 95167q^6)) - 8(257m^7 - 1253m^6q + 35m^4q(379 + 169q^2) - 35m^5(181 + 381q^2) + 7m^2q(-6821 + 6070q^2 + 3567q^4) + 7m^3(5389 + 32190q^2 + 12045q^4) - q(353449 + 1345421q^2 + 847819q^4 + 95167q^6)) - m(112285 + 1057707q^2 + 953715q^4 + 136773q^6)s^2 + 112(31m^5 + 363m^4q - 6m^3(107 + 131q^2) - 10m^2q(1135 + 749q^2) - q(10573 + 23530q^2 + 5673q^4) - m(5389 + 32190q^2 + 12045q^4))s^4 + 896(73m + 19m^3 + 105q + 113m^2q + 189mq^2 + 95q^3)s^6 - 640(m + q)s^8 \right). \quad (3.20)

While for compactness we have given just the first ten orders (to order $1/c^7$) for $sE_{\ell mc}$ in Eq. (3.13), the process is easy to automate as it is for the $a_n$’s. We have implemented code into the SpinWeightedSpheroidalHarmonics package of the Black Hole Perturbation Toolkit to compute the high-frequency expansion of the eigenvalue – see Appendix B. We also provide additional code to compute the $a_{n,k}$’s and $A_0$’s to arbitrary order.

We note that BRW gave an expansion for $sE_{\ell mc}$ to the first six orders (i.e., to order $1/c^3$) but, while their first three orders were as in Eq. (3.13), our values of $A_1$, $A_2$ and $A_3$ correct the corresponding last three orders in Eq. (4.12) in BRW.

IV. COMPARISON WITH NUMERICAL CALCULATION

We validate our high-frequency asymptotic expansions by comparing them against a numerical calculation. For the numerical results we use the SpinWeightedSpheroidalHarmonics Mathematica package which is part of the Black Hole Perturbation Toolkit [28]. This package employs both a spectral method [20] and Leaver’s method [30][31], combining them in a similar fashion to the method used by Ref. [32] for the $s=0$ case, to rapidly compute high precision values for the spin-weighted spheroidal-harmonics and their eigenvalues.

For the eigenvalue calculation we use the SpinWeightedSpheroidalEigenvalue to compare against Eq. (3.13). Note that Eq. (4.13) gives the expansion for $sE_{\ell mc}$, whereas the SpinWeightedSpheroidalEigenvalue command returns $\lambda_{\ell mc}$, so we use Eq. (2.2a) to convert between them. The results of the comparison are shown in Fig. 1 which shows that our high-frequency expansion agrees extremely well with the numerical results for large $c$. The comparison is further discussed in the figure’s caption.

For the numerical calculation of the eigenfunctions we use the SpinWeightedSpheroidalHarmonicS command. These harmonics are normalized such that

$$2\pi \int_0^\pi s_{\ell,m,c}(\theta)s_{\ell',m',c}(\theta) \sin \theta d\theta = \delta_{\ell_1,\ell_2} \delta_{m_1,m_2}, \quad (4.1)$$

where $\delta$ is the Kronecker delta function. On the other hand, we do not know the normalization of the $sS_{\ell mc}^\pm$, in Eq. (3.11) with the $q_{\pm}$ given by (3.8) and its ($s \rightarrow -s, x \rightarrow -x$) counterpart. To make a meaningful comparison with the numerical calculation of the harmonics we numerically integrate $sS_{\ell mc}^+$ over $x \in [1,0]$, and $sS_{\ell mc}^-$ over $x \in [0,-1]$ to obtain their normalization. With this information we can ensure that the numerical and asymptotic approximate solutions are normalized the same. Figure 2 presents an example of the excellent agreement we find between the numerical calculation and the high-frequency approximation of the eigenfunctions.

For the eigenvalues and the eigenfunctions, the excellent agreement we observe between the high-frequency asymptotics and the numerical results gives us confidence in both.

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$^5$ Eq. (4.12) in BRW was merely reproduced in [23] and in [31] without previously checking it, and so containing the last three erroneous terms of the original BRW version.
FIG. 1. Difference between the numerically computed eigenvalue, $\lambda_{\text{num}}$, (computed to 40-digits of accuracy) and its high-frequency expansion, $\lambda_{\text{HF}}$, for the case $\{s, \ell, m\} = \{2, 2, 2\}$. We present the same results on both a log-log scale (left panel) and a log scale (right panel). The different curves are computed using successively higher orders in the high frequency expansion. On the right of the graph, the top curve plots the numerical value of the eigenvalue. The subsequent lower curves are computed by subtracting the high-frequency series truncated at $O(c^1), O(c^0), O(c^{-1}), \ldots O(c^{-27})$, respectively. For $c \gtrsim 1$ including additional terms in the high frequency series improves the comparison with the numerical results up to a point. After this, adding more terms does not improve the agreement. The shape of the curve beyond which adding terms not does improve the agreement is clearest on a log scale (right panel). This suggests that in addition to admitting a series expansion in $c^{-1}$ there is an exponential term which the power law expansion cannot capture. Finally, as one would expect of a high frequency expansion, for $c \lesssim 1$ adding terms acts to worsen the agreement with the numerical results. The eigenvalue is better approximated by small frequency expansions around $c = 0$ in this region.

FIG. 2. Example of the high-frequency approximation to the spheroidal-harmonic eigenfunction for parameters $\{s, \ell, m, c\} = \{2, 7, 3, 20\}$. (Left panel) The (red) solid curve shows the numerically computed value of $2S_{7,3,20}$. The leading-order approximation is shown with the (blue) dotted curve. (Right panel) Including higher-order terms in the expansion improves the agreement with the numerical results. In this figure the top curve is the difference between the leading-order expansion and the numerical data. Successive lower curves are the difference between the numerical expansion and successively higher-order expansions.

Appendix A: Eigenfunction coefficients

For completeness we here give the first three orders for the coefficients in the eigenfunction asymptotic expansion – see Eqs. (3.8) and (3.12). The coefficients $a_{n,k}$ may conveniently be expressed in terms of $s p_{\ell m} = \frac{1}{2} (s q_{\ell m} - |m \pm s| \mp s - 1)$, again dropping the subscripts on $s p_{\ell m}$ and $s q_{\ell m}$ for compactness:

$$a_{-1,1} = \frac{1}{2} p_- - p_+,$$

$$a_{1,1} = -\frac{1}{4} (p_- - q - s)(p_+ - q + s),$$

(A1a) (A1b)
\[ a_{-2,2} = \frac{1}{32} p_-(p_- - 1)p_+(p_+ - 1) \]
\[ a_{-1,2} = \frac{1}{2} p_-(q - 1), \]
\[ a_{1,2} = -\frac{1}{8} (p_- - q - s)(p_+ - q + s)(q + 1), \]
\[ a_{2,2} = \frac{1}{32} (p_- - q - s)(p_- - q - s - 1)(p_+ - q + s)(p_+ - q + s - 1), \]
\[ a_{-3,3} = \frac{1}{384} p_-(p_- - 1)(p_- - 2)p_+(p_+ - 1)(p_+ - 2), \]
\[ a_{-2,3} = \frac{1}{128} p_-(p_- - 1)p_+(p_+ - 1)(2q - 3), \]
\[ a_{-1,3} = \frac{1}{128} p_-(p_- - q - s + 1)(q - s - 2)(p_+(p_+ - q + s + 1) - q + s - 2) + 2(q - 2)(5q - 2) - 2s^2), \]
\[ a_{1,3} = \frac{1}{128} (p_- - q - s)(p_+ - q + s)(p_+(p_+ - q + s + 1) - 2)(p_+(p_+ - q + s + 1) - 2) + 2(q + 2)(5q + 2) - 2s^2), \]
\[ a_{2,3} = \frac{1}{64} (p_- - q - s)(p_- - q - s - 1)(p_+ - q + s)(p_+ - q + s - 1)(2q + 3), \]
\[ a_{3,3} = -\frac{1}{384} (p_- - q - s)(p_- - q - s - 1)(p_- - q - s - 2)(p_+ - q + s)(p_+ - q + s - 1)(p_+ - q + s - 1). \]

These series coefficients were not given in BRW or, to the best of our knowledge, anywhere else in the literature. As noted in the body of the paper, it is \( s_{\ell m} - c \in \mathbb{Z} \) for \( 2a \in \mathbb{Z} \), and it is straightforward to show that \( s_{\ell m} - c \geq 0 \) for \( s \geq 0 \) and \( s_{\ell m} - c \geq 0 \) for \( s \leq 0 \). The structure of the expanded recursion relations then shows that the functional expansion Eq. (3.3) terminates with finite lower limit “\(-s_{\ell m} - c\)” for \( s \geq 0 \), reflected in the vanishing of the coefficients \( a_{n, k} \) for \( n < -s_{\ell m} - c \). Corresponding comments hold for \( s \leq 0 \) with \( s_{\ell m} - c \) replaced by \( s_{\ell m} + c \). For \( s = 0 \), \( a_{n, k} = 0 \) and our observation agrees with Eq. (8.2.9) of Flammer [20].

Appendix B: Implementation in the Black Hole Perturbation Toolkit

We have implemented the calculation of the high frequency expansion of the spin-weighted spheroidal eigenvalue and eigenfunction into the Mathematica SpinWeightedSpheroidalHarmonics package which is part of the open-source Black Hole Perturbation Toolkit. This package allows for the numerical and (where possible) analytic calculation of the eigenvalue and eigenfunction of the spin-weighted spheroidal equation. It also allows the user to compute small frequency expansions of these functions using the standard Mathematica Series[ ] function. Following this work, we have implemented the high (real) frequency expansion of the eigenfunction as well.

As an example, the high-frequency expansion of the eigenvalue, \( s_{\lambda_{\ell m c}} \), for \( \{s, \ell, m\} = \{2, 7, 3\} \) about \( c = \infty \) can be computed via

\[
\text{Series[SpinWeightedSpheroidalEigenvalue}[2,7,3,c], \{c,\infty,5\}] = \]
\[10 c - 30 - \frac{45}{c} + \frac{405}{2 c^2} - \frac{9855}{8 c^3} - \frac{17685}{2 c^4} - \frac{2261115}{32 c^5} + O[c^{-6}]\]

The expansion can also be computed around \( c = -\infty \), for example:

\[
\text{Series[SpinWeightedSpheroidalEigenvalue}[2,7,3,-c], \{c,\infty,5\}] = \]
\[22 c - 30 - \frac{51}{c} - \frac{501}{2 c^2} - \frac{13017}{8 c^3} - \frac{24603}{2 c^4} - \frac{3283149}{32 c^5} + O[c^{-6}]\]

We have also included an example notebook in the Toolkit which demonstrates the use of this function and provides code to calculate the \( a_{n,k} \) and \( A_k \) coefficients that appear in Eq. (3.12) and Eq. (3.13), respectively.

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