HYPERBOLIC SETS THAT ARE NOT CONTAINED
IN A LOCALLY MAXIMAL ONE

ADRIANA DA LUZ
Iguá 4225 Esq. Mataojo
C.P. 11400 Montevideo, Uruguay

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Abstract. In this paper we study two properties related to the structure of hyperbolic sets. First we construct new examples answering in the negative the following question posed by Katok and Hasselblatt in [12], p. 272

Question. Let Λ be a hyperbolic set, and let V be an open neighborhood of Λ. Does there exist a locally maximal hyperbolic set ˆΛ such that Λ ⊂ ˆΛ ⊂ V?

We show that such examples are present in linear Anosov diffeomorphisms of T^3, and are therefore robust.
Also we construct new examples of sets that are not contained in any locally maximal hyperbolic set. The examples known until now were constructed by Crovisier in [7] and by Fisher in [9], and these were either in dimension equal or bigger than 4 or they were not transitive. We give a transitive and robust example in T^3. And show that such examples cannot be build in dimension 2.

1. Introduction. In the ’60s, Anosov ([3]) and Smale ([23]) began the study of some compact invariant sets, whose tangent space splits into invariant, uniformly contracting and uniformly expanding directions. More precisely, a hyperbolic set is defined to be a compact invariant subset of a compact manifold Λ ⊂ M of a diffeomorphism f such that the tangent space at every x ∈ Λ admits an invariant splitting that satisfies:

- T_xM = E^s(x) ⊕ E^u(x)
- Df_x(E^s(x)) = E^s(f(x)) and Df_x(E^u(x)) = E^u(f(x))
- There are constants C > 0 and λ ∈ (0, 1) such that for every n ∈ N one has ∥Df^n(v)∥ ≤ Cλ^n ∥v∥ for v ∈ E^s(x) and ∥Df^{-n}(v)∥ ≤ Cλ^{-n} ∥v∥ for v ∈ E^u(x).

A specially interesting case is when the hyperbolic set Λ is the non-wandering set of f. Particularly when we also have that the set of periodic points of f is dense in the non-wandering set Ω(f), we say that f is Axiom A. These systems are very important because, If f satisfies Axiom A, Ω(f) = Λ_1 ∪ ... ∪ Λ_n, where each Λ_n is compact, invariant, hyperbolic, and transitive. We call these components, basic sets.

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On the other hand when there are no cycles between the basic pieces, these systems have some kind of stability in the following sense: the non wandering set of all nearby systems have an equivalent dynamic. The no cycle condition means that there is no closed chain of basic sets (where the chain is formed by one basic’s set unstable manifold intersecting the stable manifold of the next basic set).

Given the relevance of these diffeomorphisms in the study of hyperbolic dynamics is natural to ask what kind of sets may or may not be a basic piece of some spectral decomposition.

All basic pieces have the following property:

**Definition 1.** A hyperbolic set $\Lambda$ is called locally maximal (or isolated) if there exists a neighborhood $V$ of $\Lambda$ in $M$ such that $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(V)$.

We will focus now on whether or not a set is locally maximal. If it is not, we will be interested in studying whether or not the set is contained in another set that is locally maximal.

This kind of sets are interesting on themselves since they can be thought of, locally, in coordinates of the stable and unstable manifold of a point. Also this property is equivalent to having the shadowing property or having local product structure, which gives us information about the dynamics of a neighborhood of the set.

Many of the best known examples of hyperbolic sets are locally maximal. Some examples could be the solenoid or a horseshoe. Also, there are examples of simple sets that do not verify this, for instance, the closure of the orbit of a homoclinic point. In any neighborhood of this set there is a bigger set containing it that does have local product structure.

In the 1960’s Alexseyev asked the following question (that was later posed by Katok and Hasselblatt in [12, p. 272])

**Question 1.** Let $\Lambda$ be a hyperbolic set, and let $V$ be any open neighborhood of $\Lambda$. Does there exist a locally maximal hyperbolic set $\tilde{\Lambda}$ such that $\Lambda \subset \tilde{\Lambda} \subset V$?

Also the following related question was unanswered:

**Question 2.** Given a hyperbolic set $\Lambda$ does there exist a hyperbolic locally maximal set such that $\Lambda \subset \tilde{\Lambda}$?

Both questions remained open until 2001, when Crovisier [7] constructed an example (based on an example by Shub in [14]) that answers question 2 in the negative. This example is on the $4$-torus.

Later, Fisher [9] constructed several other examples of this sort. He constructed robust examples in any dimension, and transitive examples in dimension $4$.

This examples motivate the following definitions as in [1]

**Definition 2.** We say that a hyperbolic set $\Lambda \subset M$ is premaximal, if there exists a hyperbolic locally maximal set $\Delta \subset M$ such that $\Lambda \subset \Delta$.

**Definition 3.** We say that a hyperbolic set $\Lambda$ is locally premaximal, if for every neighborhood $U$ of $\Lambda$, there is a hyperbolic locally maximal set $\Delta$ such that $\Lambda \subset \Delta \subset U$.

In spite of this there are still some natural questions left to answer

- For any manifold $M$, does there exist an open set $U \in Diff^1(M)$ such that every $f \in U$ possesses an invariant transitive hyperbolic set that is not pre-maximal on any manifolds?
Does there exist robust and transitive examples of non locally premaximal sets on manifolds with dimension lower than 4?

Does there exist an example of non locally premaximal sets that are locally maximal?

In section (3) we will show

**Theorem A.** Let \( f_A : T^3 \to T^3 \) be an Anosov diffeomorphism. There is a connected, compact proper invariant subset of \( T^3 \), such that the only locally maximal hyperbolic set containing it is \( T^3 \).

This answers our last question.

Note that the same will be true for any \( g \) conjugate to \( f_A \), therefore, the property is robust.

In section (4) we describe a well known example by Mañe in [15] that we will use on section (5) to construct a new example of a set that is not included in any locally maximal set. This example gives a partial answer to our second question. It is the first example in dimension 3 that is robust and transitive (here we do not refer to the fact that the diffeomorphism is transitive on \( T^3 \) but to the fact that the set from the example is transitive),

**Theorem B.** There exists \( U \subset \text{Diff}(T^3) \) such that for every \( g \in U \) there is a transitive, hyperbolic subset of \( T^3 \), such that there is no locally maximal hyperbolic set containing it.

In the case of surfaces our first 2 questions can be combined in the following

**Question.** If \( \dim(M) = 2 \), and \( \Lambda \subset M \) is a transitive hyperbolic set, Is \( \Lambda \) locally premaximal?

We will give a positive answer to this question. In section (6) we will show:

**Theorem C.** Let \( M \) a compact surface. \( f : M \to M \) be a diffeomorphism, and \( \Lambda \subset M \) a compact hyperbolic invariant set. If we also have that \( \Omega(f|_{\Lambda}) = \Lambda \) then for any neighborhood \( V \) of \( \Lambda \), there exist \( \bar{\Lambda} \) such that \( \bar{\Lambda} \) is compact hyperbolic invariant and with local product structure and,

\[
\Lambda \subset \bar{\Lambda} \subset V.
\]

This result was already known by Fisher, and can be found in an unpublished note in [10]. However our proof is different and in some sense simpler since we do not use the fact that every hyperbolic set is included in a set with Markov partition.

2. **Preliminaries.** Let \( M \) be a compact manifold, \( f \) a \( C^r \) diffeomorphism, and \( \Lambda \) a hyperbolic set.

For \( \varepsilon > 0 \) sufficiently small and \( x \in \Lambda \) the local stable and unstable manifolds are respectively:

\[
W^s_\varepsilon(x, f) = \{ y \in M | \text{for all } n \in \mathbb{N}, d(f^n(x), f^n(y)) < \varepsilon \},
\]

and

\[
W^u_\varepsilon(x, f) = \{ y \in M | \text{for all } n \in \mathbb{N}, d(f^{-n}(x), f^{-n}(y)) < \varepsilon \}.
\]

The stable and unstable manifolds are respectively:

\[
W^s(x, f) = \bigcup_{n \geq 0} f^{-n}(W^s_\varepsilon(f^n(x), f)),
\]

and

\[
W^u(x, f) = \bigcup_{n \geq 0} f^n(W^u_\varepsilon(f^n(x), f)).
\]
and

\[ W^u(x, f) = \bigcup_{n \geq 0} f^n(W^u_\varepsilon(f^{-n}(x), f)) . \]

The stable and unstable manifolds are \( C^r \) injectivity immersed submanifolds. If two points of \( \Lambda \) are sufficiently close, the local stable and unstable manifolds intersect transversely at a single point.

Since the hyperbolic systems are very sensitive with respect to the initial point, it is necessary to know what kind of meaningful information can be extracted from an only approximated knowledge of the behavior of an orbit. A notion of “approximated knowledge of the behavior of an orbit” could be the following: We say that a sequence of points is an \( \alpha \)-pseudo orbit if given a term, its image by the diffeomorphism is “\( \alpha \) close” to the next term of the sequence. A hyperbolic set will be such that any \( \alpha \)-pseudo is “shadowed” by an actual orbit of the system (an orbit such that its iterates are close to the terms of the sequence).

Some interesting consequences of this are related to the following property (we refer the reader to [12] or [20] to for proofs and further discussion)

**Definition 4.** Let \( f : M \to M \) be a diffeomorphism and \( \Delta \) a compact invariant hyperbolic set. We say that \( \Delta \) has local product structure if there exists \( \delta > 0 \) such that if \( x, y \in \Delta \), and \( d(x, y) < \delta \) then \( W^s_\varepsilon(x) \cap W^u_\varepsilon(y) \in \Delta \) where \( \varepsilon \) is as in the stable manifold theorem.

When this property occurs, not only do we have the shadowing property, but also the orbit that “shadows” remains inside the hyperbolic set. As a consequence of the shadowing theorem we have:

**Corollary 1.** If in addition to the other hypothesis we have that \( \Lambda \) has local product structure, if and only if every \( \alpha \)-pseudo orbit in \( \Lambda \) is \( \beta \) shadowed by an orbits in \( \Lambda \).

With this we can show an other equivalence that will be particularly useful for us.

**Corollary 2.** A hyperbolic set \( \Lambda \) is locally maximal if and only if \( \Lambda \) has local product structure.

3. Proof of Theorem A: A set that is not locally premaximal. In this section we prove that there is a subset of the \( T^3 \), invariant under a linear Anosov diffeomorphism \( f_A \), that is not locally premaximal.

Let \( f_A \) be a Anosov diffeomorphism in \( T^3 \) that is induced by \( A \in GL(3, \mathbb{Z}) \) which is a hyperbolic toral automorphism with only one eigenvalue greater than one, and all eigenvalues real, positive, simple, and irrational. Let \( \pi : \mathbb{R}^3 \to T^3 \) be such that \( \pi \circ A = f \circ \pi \). \( f_A \) has a fixed points \( p \), and \( \pi(0, 0, 0) = p \). As a consequence of the results in [11] we have:

**Theorem 1.** Let \( f_A : T^3 \to T^3 \) be a hyperbolic automorphism, there exists a path \( \gamma \) in \( T^3 \), such that it contains \( p \) and the set \( \overline{O(\gamma)} \) is connected proper and non trivial.

This curve can also be constructed so that its image avoids a neighborhood of another fixed point \( q \). For this \( \gamma \) we note \( \Lambda = \overline{O(\gamma)} \).

We will prove now that in this conditions the only set with local product structure containing \( \Lambda \) is the whole \( T^3 \), following mainly the ideas in [10]. Here Mañe proves that every compact, connected, locally maximal subset of \( T^n \) under a linear hyperbolic automorphism must be of the form \( \Delta = x + G \), where \( x \) is a fixed point
and $G$ is an invariant compact subgroup. In particular in dimension 3 this implies that $\Delta = T^3$ or $\Delta = x$. We will adapt the proof to the case where $\Delta$ is not connected but contains non trivial compact, connected, invariant set that contains a fixed point.

**Definition 5.** Let $\Lambda \subset T^3$ be a compact, connected and invariant set, such that $p \in \Lambda$. We say that a curve $\gamma : [0,1] \to T^3$ is $\delta$-adapted to $\Lambda$, if there are $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $\gamma(t) \in \Lambda$, and $d(\gamma(t), \gamma(t_j)) < \delta$ for all $t_j \leq t \leq t_{j+1}$ and $0 \leq j \leq m$.

We define $\Gamma_\delta$ the set of classes of arcs $\delta$-adapted to $\Lambda$ which is a subgroup of $\pi_1(T^3, p) = \mathbb{Z}^3$. Note that if $\delta_1 < \delta_2$ then $\Gamma_{\delta_1} \subset \Gamma_{\delta_2}$.

Using the continuity of $A$ we have that, given $\delta$ there is a $\delta_1$, such that $A(\Gamma_{\delta'}) \subset \Gamma_{\delta}$ for all $0 < \delta' < \delta_1$.

The idea now is to define a $\Gamma_0$ which we would naively define as the subgroup limit of $\Gamma_\delta$ with $\delta$ going to zero. A first attempt to define it would be to consider $\bigcap_{\delta > 0} \Gamma_\delta$ but that set might be empty and not represent what we want it to. Instead we define $N_\delta$ as the subspace of $\mathbb{R}^3$ generated by $\Gamma_\delta$. We define $N_0 = \bigcap_{\delta > 0} N_\delta$ and $\Gamma_0 = N_0 \cap Z^3$. Note that $A(N_0) = N_0$.

**Lemma 1.** In the above mentioned conditions, $(N_0/\Gamma_0)$ is $T^3$ or $p$.

**Proof.** First we note that $(N_0/\Gamma_0)$ is $f$-invariant since $A(N_0) = N_0$, and $A(\Gamma_0) = \Gamma_0$. So $(N_0/\Gamma_0)$ is invariant and since $\Gamma_0 = N_0 \cap Z^3$ then $(N_0/\Gamma_0)$ is a compact subtorus. A result from [13] tells us that if the stable or unstable manifold are 1 dimensional then the only connected, locally connected, compact, invariant subsets of a hyperbolic toral automorphisms are fixed points and the whole torus.

Note that since $\Gamma_0 = Z^3 \cap N_0$ then the previous lemma implies that $\Gamma_0 = Z^3$ or $\Gamma_0 = 0$. If $\Gamma_0 = Z^3$ then $N_0 = R^3$ and $R^3 = N_0 = E^s \oplus E^u$ the splitting in stable and unstable eigenspaces of $A$.

**Lemma 2.** If $\Gamma_0 = Z^3$ and $\pi^s : N_0 \to E^u$ is the projection associated with the splitting $N_0 = E^s \oplus E^u$, then there exists $\delta_0$ such that $\pi^s(\Gamma_\delta)$ is dense in $E^u$.

**Proof.** To see this, note that $N_{\delta_1} \subset N_{\delta_2}$ if $\delta_1 \leq \delta_2$. This implies that $N_\delta = N_0$ since $N_0 = R^3$. It follows that $\Gamma_\delta \subset Z^3 \cap N_\delta = Z^3 \cap N_0$.

On the other hand $\dim(N_0) = \dim(N_\delta) = \text{ran}(\Gamma_\delta)$, so $\text{ran}(\Gamma_\delta) = 3$ and there is an isomorphism $\phi : \Gamma_\delta \to Z_3$. $\pi^s : \Gamma_\delta \to E^u$ in injective. If there where $a'$ and $a''$ such that $\pi^s(a') = \pi^s(a'') = a$, then $a'' = E^s + a'$. This is impossible since $a', a'' \in Z^3$ and $E^s$ is totally irrational.

We define now $\varphi : \pi^s(\Gamma_\delta) \to \pi^s(Z^3)$ as $\varphi(a) = \pi^s(\phi(a'))$ which is an isomorphism. Therefore since $\pi^s(Z^3)$ is dense in $E^u$, $\pi^s(\Gamma_\delta)$ is dense in $E^u$.

**Lemma 3.** Let $\Lambda \subset \mathbb{T}^n$ be a compact, connected, invariant set such that $p \in \Lambda$ and let $\Gamma_0$ be the group defined before. Let $\rho : (R^3/\Gamma_0, p) \to (R^3/Z^3, p)$ be the canonical covering map and $f_A : (R^3/\Gamma_0, p) \to (R^3/\Gamma_0, p)$ be the lift of $f_A$. Then there exists a continuous function $\varphi : (\Lambda, p) \to (R^3/\Gamma_0, p)$ such that the following diagram commutes:

**Proof.** Since $N_{\delta_1} \subset N_{\delta_2}$ if $\delta_1 < \delta_2$, then for some $\delta_0$ the chain stops. That is, $\delta_0$ such that $N_\delta = N_0$ for all $\delta < \delta_0$. 
We also have that for \( \delta < \delta_0 \),
\[
\Gamma_\delta \subset \mathbb{Z}^3 \cap N_\delta = \mathbb{Z}^3 \cap N_0 = \Gamma_0,
\]
that is, \( \Gamma_\delta \subset \Gamma_0 \). To define \( \varphi(x) \) we take \( \gamma : [0, 1] \to \mathbb{T}^3 \), \( \delta \)-adapted to \( \Lambda \) such that \( \Gamma_\delta \subset \Gamma_0 \) and \( \gamma(0) = p \), \( \gamma(1) = x \).

Let \( \tilde{\gamma} : [0, 1] \to \mathbb{R}^3/\Gamma_0 \) be such that \( \tilde{\gamma}(0) = p \) and \( \rho \circ \tilde{\gamma} = \gamma \). Since \( \Gamma_\delta \subset \Gamma_0 \) we have that \( \tilde{\gamma}(1) \) does not depend on the choice of the \( \delta \)-adapted \( \gamma \) joining \( x \) with \( p \).

Definimos entonces \( \varphi(x) = \tilde{\gamma}(1) \)

As a consequence of the commutation of the diagram and the fact that \( \varphi \) is continuous we have that if \( \Gamma_0 \) is trivial then \( \varphi(\Lambda) \) is a compact \( \tilde{\varphi} \) invariant set of \( (\mathbb{R}^3/\Gamma_0, \rho) = \mathbb{R}^3 \) and then it must be a fixed point. Now we consider \( \Lambda \) to be the set described by Hancock (1). Then \( \Lambda \) is compact, connected, invariant, it contains a fixed point \( p \) and is not trivial. Since \( \Lambda \) is not trivial then \( \Gamma_0 \) is not trivial. Let us suppose there exists a set \( \Delta \) with local product structure containing \( \Lambda \), and let us call it’s lift \( \hat{\Delta} \).

The strategy now is to see that such a \( \Delta \), must contain a dense set in the unstable manifold of \( p \) (which is of dimension 1). Since \( \Delta \) is compact then \( \Delta = \mathbb{T}^3 \).

**Definition 6.** We say that \( x \) and \( y \) are \( n\varepsilon \)-related in \( \hat{\Delta} \) if there exists sequence of point \( x = x_0, x_1, \ldots, x_n = y \) such that:

- \( x_i \in \hat{\Delta} \) for \( i = 1, \ldots, n \)
- \( \pi^r(x_{i+1} - x_i) \leq \varepsilon \) for \( 1 \leq i \leq n \)
- \( \pi^u(x_{i+1} - x_i) \leq \varepsilon \) for \( 1 \leq i \leq n \)

**Lemma 4.** If \( x, y \in \hat{\Delta} \) are \( n\varepsilon \)-related, with \( \varepsilon \) sufficiently small, then \( (x + E^s) \cap (y + E^u) \in \hat{\Delta} \).

**Proof.** We take \( \varepsilon < \delta \) with \( \delta \) from the local product structure. We prove this lemma by induction. For \( n = 1 \) the property is verified by the local product structure.

Suppose now that \( x, y \in \hat{\Delta} \) are \( n\varepsilon \)-related. We have \( x = x_0, x_1, \ldots, x_n = y \) as in the definition. We define \( x_j = (x_j + E^s) \cap (x_{j+1} + E^u) \) for \( 0 \leq j \leq n - 1 \). Note that \( x_0 \) and \( x_{n-1} \) are \( (n-1)\varepsilon \)-related because:

- \( x_j \in \hat{\Delta} \) for \( j = 1, \ldots, n - 1 \) by induction hypothesis,
- \( \pi^r(x_{j+1} - x_j) = \pi^r(x_{j+1} - x_j) \leq \varepsilon \) for \( 1 \leq j \leq n - 1 \),
- \( \pi^u(x_{j+1} - x_j) = \pi^u(x_{j+1} - x_j) \leq \varepsilon \) for \( 1 \leq j \leq n - 1 \).

Then we then have, \( (x_0 + E^s) \cap (x_{n-1} + E^u) = z \in \hat{\Delta} \).

Since we also have \( (x_0 + E^s) = (x_0 + E^s) \) and \( (x_{n-1} + E^u) = (x_n + E^u) \), we conclude that \( (x_0 + E^s) \cap (x_n + E^u) = z \in \hat{\Delta} \). □

The following theorem implies theorem [\( \boxed{\Lambda} \)]
**Theorem 2.** Let $\Lambda$ be a compact, connected, invariant set, such that $p \in \Lambda$, and $p \neq \Lambda$. Suppose there is $\Delta$ such that $\Lambda \subset \Delta$ and $\Delta$ is compact invariant and with local product structure. Then $\Delta = \mathbb{T}^3$.

**Proof.** Let $\hat{\Delta}$ and $\hat{\Lambda}$ be the lifts of $\Delta$ and $\Lambda$ respectively.

If $\Delta$ is compact invariant and has local product structure, then by Lemma 3 if we have two points $x, y \in \hat{\Delta}$ which are $n \varepsilon$-related, we have $(x + E^u) \cap (y + E^u) \in \hat{\Delta}$.

We observe that $p$ and any point in $\Gamma_\delta$ are $n \delta$-related, therefore $\pi_s(\Gamma_\delta) \subset \hat{\Delta}$. Since $\pi_s(\Gamma_\delta)$ by 2 is dense in $E^u$, then $\pi_s(\Gamma_\delta) = E^u \subset \hat{\Delta}$ and $\mathbb{T}^3 = \pi(E^u) \subset \Delta$,

obtaining the desired result. \qed

4. **Mañ’s robustly transitive diffeomorphisms that is not Anosov.** In this section we will describe an example constructed by Mañ in [15]. This example is very well described in numerous references (see for instance [5], or [19]), but we will include a description for the convenience of the reader, and because we will emphasize some properties of the example that will be useful later on. However we will not include the proofs, which can be found in any of the given references.

As in the previous section, let us starts with a linear Anosov diffeomorphism $f_A$ in $\mathbb{T}^3$ that is induced form $A \in GL(3, \mathbb{Z})$ which is a hyperbolic toral automorphism with only one eigenvalue grater than one, and all eigenvalues real, positive, simple, and irrational. Let $0 < \lambda^s < \lambda^c < 1 < \lambda^u$ be the eigenvalues. Let $\mathcal{F}^c$ be the foliation corresponding to the eigenvalue $\lambda^c$, similarly with $\mathcal{F}^s$ and $\mathcal{F}^u$. We remind you that all of these leaves are dense. We may also assume that $f_A$ has at least two fixed points, $p$ and $q$, and the unstable eigenvalue $\lambda^u$ have modulus greater than 3 (if not, replace $A$ by some power).

Following the construction in [15] we define $f$ by modifying $f_A$ in $C$, a sufficiently small domain contained in $B_{\rho}(q)$, keeping invariant the foliation $\mathcal{F}^c$. Where $\rho > 0$ is a small enough number to be determined in what follows. Let us observe that

![Figure 1. A $n \varepsilon$-relation between $x$ and $y$.](image-url)
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\[ f_A|_{B_{\varphi}(q)^c} = f|_{B_{\varphi}(q)^c}. \] In particular

\[ \Gamma = \bigcap_{n \in \mathbb{Z}} f^n(B_{\varphi}(q)^c) = \bigcap_{n \in \mathbb{Z}} f_A^n(B_{\varphi}(q)^c). \] \hspace{1cm} (1)

We can take \( \rho \) sufficiently small so that \( p \in \Gamma \). Inside \( C \) the point \( q \) undergoes a bifurcation as shown in the figure 2 in the direction of \( \mathcal{F}^c \), which changes the unstable index (the dimension of the unstable manifold) of \( q \) increasing it in 1. Also two other fixed points, \( x_2 \) and \( x_3 \) are created, with the same index \( q \) had under \( f_A \).

As a result, we get a difeomorphism \( f \) which is strongly partially hyperbolic. That is

\[ T\mathbb{T}^3 = E^s_f \oplus E^c_f \oplus E^u_f, \]

where \( E^s_f \) is uniformly contracting and \( E^u_f \) is uniformly expanding. In fact, \( E^s_f \) and \( E^u_f \) are contained in some small cones around \( E^s \) and \( E^u \) respectively. Then by a well known results (see [14]) we get that the bundles \( E^c_f \) and \( E^u_f \) are uniquely integrable to foliations \( \mathcal{F}^c_f \) and \( \mathcal{F}^u_f \) called the (strong) stable and unstable foliations. Moreover, they are quasi-isometric Since we preserved the central foliation we have \( \mathcal{F}^c_f = \mathcal{F}^c, \mathcal{F}^s_f \oplus \mathcal{F}^c_f \) and \( \mathcal{F}^u_f \oplus \mathcal{F}^c_f \) are also integrable, by what we call the center-stable and center-unstable foliations respectively. In [15] it is shown that the leaves of \( \mathcal{F}^c_f \) are dense in \( \mathbb{T}^3 \) in a robust fashion(see also [5]).

It is particularly relevant for us that, not only is the central foliation robustly minimal, but also the unstable foliation is robustly minimal as well. This is shown for instance in the following theorem (see for example [19] or [14]).

**Theorem 3.** (2.0.1 in [19]). There exists a neighborhood \( \mathcal{U} \) of \( f \), in the \( C^1 \) topology such that for every \( g \in \mathcal{U} \) the bundles \( E^s_g, E^c_g \) and \( E^u_g \), uniquely integrates to invariant foliations \( \mathcal{F}^s_g, \mathcal{F}^c_g \) and \( \mathcal{F}^u_g \), respectively). Furthermore, the central and unstable foliations \( g \in \mathcal{U} \) are minimal, i.e., all leaves are dense.

The following lemma is a consequence of the shadowing theorem (see [23]).

**Lemma 5.** Let \( A \in GL(3, \mathbb{Z}) \) which is a hyperbolic toral automorphism and let \( G : \mathbb{R}^3 \to \mathbb{R}^3 \) be a homeomorphism such that \( \|A(x) - G(x)\| \leq r \) for all \( x \in \mathbb{R}^3 \). Then there exists \( H : \mathbb{R}^3 \to \mathbb{R}^3 \) continuous and onto such that \( A \circ H = H \circ G \). Moreover \( H : \mathbb{R}^3 \to \mathbb{R}^3 \) is uniformly continuous.

**Figure 2.** Perturbing a neighborhood of \( q \)
Note that $H(x) = H(y)$ if and only if $\| G^n(x) - G^n(y) \| \leq 2C r \ \forall n \in \mathbb{Z}$. This is a consequence of the uniqueness in the shadowing theorem.

Let us now consider a $g$ in a $C^1$ neighborhood $\mathcal{U}$ of $f$. Since $f$ is partially hyperbolic and this is a $C^1$ condition we can take $\mathcal{U}$ so that $g$ is also partially hyperbolic In this conditions we have that the lift $G$ is in the conditions of our previous lema and $H$ induces an $h : \mathbb{T}^3 \to \mathbb{T}^3$ continuous and onto such that $f_A \circ h = h \circ g$.

Denote by $\hat{F}^\sigma$, $\sigma = s, c, u, cs, cu$ the lift of the stable, central, unstable, central stable and central unstable foliation respectively.

Now let us see how $H$ behaves with respect of the invariant foliations.

**Lemma 6.** There exists a neighborhood $\mathcal{U}$ of $f$ such that if $g \in \mathcal{U}$ and $G$ is the lift of $g$ then, for $H$, $A$ as above we have that

1. $H\hat{F}^\sigma_G(x) = \hat{F}^\sigma_A(H(x))$ and $H\hat{F}^u_G(x) = \hat{F}^u_A(H(x))$.
2. $H\hat{F}^c_G(x) = \hat{F}^c_A(H(x))$.
3. $H\hat{F}^u_G(x) = \hat{F}^u_A(H(x)) = H(x) + E^u_A$ and $H|_{\hat{F}(x)}$ is a homeomorphism for every $x$.
4. For any $x, y \in \mathbb{R}^3$,
   \[ \# \left\{ \hat{F}^s_G(x) \cap \hat{F}^u_G(y) = 1 \right\} \text{ and } \# \left\{ \hat{F}^c_G(x) \cap \hat{F}^c_G(y) \right\} = 1. \]
5. If $H(x) = H(y)$, Then $x$ and $y$ belong to the same central leaf.

These results follow mainly from the expansivity of $A$. For a proof see [19]. It can also be shown that $h : \mathbb{T}^3 \to \mathbb{T}^3$ inherits similar properties.

### 4.1. Transitivity of the set $\Gamma$.

In this section we prove that the set:

$$
\Gamma = \bigcap_{n \in \mathbb{Z}} f^n(B_{x_1}(x))^c = \bigcap_{n \in \mathbb{Z}} f^n_A(B_{x_1}(x))^c,
$$

is transitive. For this it will be necessary to introduce some definitions and results on Markov partitions and subshifts of finite type. For a proof of the results that we mention below there are numerous references, one can see for example: [12] .

**Definition 7.** For a diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ and a hyperbolic set $\Lambda$, we say that $R \subset \Lambda$ is a rectangle if the diameter of the set is less than the expansivity constant of $f$ in $\Lambda$, and if $x, y \in R$ and $\hat{W}^s(x) \cap \hat{W}^u(y) = z$ then $z \in R$.

we note that

$$
R \cong \hat{W}^s(x, R) \times \hat{W}^u(x, R),
$$

where

$$
\hat{W}^s(x, R) = \hat{W}^s(x, R) \cap R \text{ and } \hat{W}^u(x, R) = \hat{W}^u(x, R) \cap R.
$$

**Definition 8.** A Markov partition of $\Lambda$ is a finite collection of rectangles $\{ R_i \}_{i=1}^m$ that have the following properties:

- $\Lambda = \bigcup_{i=1}^m R_i$
- $R_i = (int(R_i))$ for all $i$
- If $i \neq j$ then $int(R_i) \cap int(R_j) = \emptyset$.
- If $z \in int(R_i) \cap f^{-1}(int(R_j))$ then:
  - $f(W^u(x, R_i)) \supset W^u(f(x), R_j)$ and
  - $f(W^s(x, R_i)) \subset W^s(f(x), R_j)$
For a Markov partition we define the transition matrix in the following way:

**Definition 9.** We call the transition matrix $B$ of the Markov partition $\mathcal{R} = \{ R_i \}_{i=1}^{m}$ to a matrix $m \times m$ such that

- $B_{i,j} = 1$ if $\text{int}(R_i) \cap f^{-1}(\text{int}(R_j))$ and
- $B_{i,j} = 0$ in the other case.

**Definition 10.** We say that the transition matrix $B$ is irreducible if there exists $k$ such that all the entries of the matrix $B^k$ are positive.

**Lemma 7.** If the transition matrix $B$ is irreducible, then $\Lambda$ is transitive for $f$.

Now we only need to have some control over the size of the rectangles so that if we take a sufficiently small partition so that taking out the ball $B_\rho^2(x_1)$ is equivalent to take out a rectangle or some integer quantity of rectangles. For this we have the following theorem:

**Theorem 4.** Let $f : T^3 \to T^3$ be an Anosov diffeomorphism, then, for $\beta > 0$ there exists a Markov partition $M$ for $f$ with diameter less than $\beta$ and such that the transition Matrix is irreducible.

**Lemma 8.** Choosing $A$ and $B_\rho^2(x_1)$ appropriate, $\Gamma$ is transitive since $f_A|_{B_\rho^2(x_1)^c} = f|_{B_\rho^2(x_1)^c}$.

**Proof.** Since $f_A$ is Anosov it admits a Markov partition with arbitrarily small rectangles. We can take the ball $B_\rho^2(x_1)$ as a rectangle $R_i$ of this partition. Also because $f_A$ is Anosov, it is topologically transitive so there exists an integer $n$ such that $f^n_A$ is such the image of every rectangle intersects every other rectangle. If we eliminate a rectangle $R_i$ we obtain a Markov partition for the complement of the union of the iterates of $R_i$, that is, a partition of $\Gamma$. It is also true that the iterates of the partition of $\Gamma$ must still intersect every other element of the partition. This means that the transition matrix of some powder $f^{n+1}_A$ is such that all entries are positive and so $\Gamma$ is transitive for $f_A$. \qed

**5. Proof of Theorem**

A set that is robustly not premaximal in $T^3$.

Let $f : T^3 \to T^3$ be as in the previous section, the diffeomorphism form Mañé’s example, and let us consider a $C^1$ neighborhood $\mathcal{U}$ around $f$. In this section we will prove that for any $g \in \mathcal{U}$, there is a set on $T^3$ that cannot be included on any set with local product structure.

For this we will show that the set $\Lambda$ from section 4 does not intersect some ball around $q$. So possibly taking a smaller $\rho$ we can construct a diffeomorphism $f$ as the one from the previous section and such that $\Lambda \subset \Gamma = \bigcap_{n \in \mathbb{Z}} f^n(B_\rho^2(x_1)^c)$.

Note that the set $\Gamma = \bigcap_{n \in \mathbb{Z}} f^n(B_\rho^2(x_1)^c)$ can be made to be transitive as in (4.1).

So $\Lambda$ is a compact, hyperbolic set, invariant under $f$ since it is invariant under $f_A$ and

$$\Gamma = \bigcap_{n \in \mathbb{Z}} f^n(B_\rho^2(q)^c) = \bigcap_{n \in \mathbb{Z}} f^n_A(B_\rho^2(q)^c).$$  \hspace{1cm} (2)

For any $g$ sufficiently close to $f$, there is a hyperbolic set $\Lambda_g$ which is the hyperbolic continuation of $\Lambda$, and that has essentially the same properties in all that concerns us. We will call both sets $\Lambda$, for simplicity.

The aim of this section is to show that if there is a set $\Delta$ containing $\Lambda$ with local product structure, then $\Delta = T^3$ which in this case is not possible since $g$ is not
Anosov. The first thing one may try is to use the semiconjugacy $h$ with the linear Anosov and the results in section (3) to show that $h(\Delta) = \mathbb{T}^3$. If this was possible then one can conclude that $\Delta = \mathbb{T}^3$ by observing that the only invariant set $h$ can send to the hole torus is $\mathbb{T}^3$.

However one can not deduce the local product structure of $h(\Delta)$ from the local product structure of $\Delta$ since to points that are close on $h(\Delta)$ might be separated in $\Delta$ and therefore their local stable and unstable manifolds need not intersect.

We will prove that if there is a set $\Delta$ containing $\Lambda$ with local product structure, then $\Delta \cap \mathcal{F}^u_g(p)$ is dense in some small interval of $\mathcal{F}^u_g(p)$, and then $\Delta$ is dense in $\mathbb{T}^3$ in virtue of the minimality of $\mathcal{F}^u_g$. Since in this context the unstable leaves are not straight lines it would be convenient to redefine the $n\cdot \varepsilon$-relation.

Let $p^c_{G} : \mathbb{R}^3 \to \hat{\mathcal{F}}^c_G(x)$ and $p^s_{G} : \mathbb{R}^3 \to \hat{\mathcal{F}}^s_G(x)$ be the projections along the center stable and unstable foliation respectively. We note as $\hat{\Delta}$ and $\hat{\Lambda}$ the lifts of $\Delta$ and $\Lambda$ respectively that lift $p$ to $(0,0,0)$.

**Definition 11.** We say that $x$ and $y$ are $n\cdot \varepsilon$-related in $\hat{\Delta}$ if there exists a sequence of point $x = x_0, x_1, \ldots, x_n = y$ such that:

- $x_i \in \hat{\Delta}$ for $i = 1, \ldots, n$
- $d(\hat{p}^u(x_{i+1}), x_{i+1}) \leq \varepsilon$ for $1 \leq i \leq n$
- $d(\hat{p}^c(x_i), x_i) \leq \varepsilon$ for $1 \leq i \leq n$

The main problem which we are dealing with now, is that lemma 4 relies heavily on the linearity of $A$. We will fix this problem by finding a tube $V$ around $(0,0,0)$ so that both the distance between the center-sable foliations of $x$ and $y$ and the distance between the unstable foliations in $V$ are small when $x$ and $y$ are close enough. The interval of the unstable foliation in which $\hat{\Delta} \cap \hat{\mathcal{F}}^c_G((0,0,0))$ will be dense, will be contained in this $V$.

Another important difference is that 4 also makes a strong use of the linearity therefore we will not try to prove that the projection of all $\Gamma_\delta$ is in $\hat{\Delta}$. It will be enough to find a point of $\Gamma_\delta$ outside $V$ and project the points of the $\delta$-chain joining $(0,0,0)$ with that point.

For two points $x$ and $y$ in the same leaf of the unstable foliation, we define $l^u(x, y)$ to be the length of the arc joining $x$ with $y$. For a fixed $\varepsilon$, we will prove first that for any two points $x$, $y$ in $\mathbb{R}^3$, there exist a $\delta$ such that if $d(x, y) < \delta_0$. Then, if we choose any $z$ in $\hat{\mathcal{F}}^c(x)$, then $l^u(z, p^u_\delta(z)) < \varepsilon$.

**Lemma 9.** For any $\varepsilon > 0$ there exists a $\delta$ such that for every $x$ and $y \in \hat{\mathcal{F}}^c_G(x)$ such that $l^u(x, y) \leq \delta$ then $l^u(z, p^u_\delta(z)) < \varepsilon$, for any $z$ in $\hat{\mathcal{F}}^c_G(x)$.

**Proof.** Suppose that this is not the case. Then there must exist an $\varepsilon_0$ such that there exist there sequences $\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}} \subset \hat{\mathcal{F}}^c_G(x)$ and $\{z_n\}_{n \in \mathbb{N}}$ such that $l^u(x_n, y_n) \leq 1/n$, and $l^u(z_n, p^u_\delta(z_n)) \geq \varepsilon_0$. Now let us recall that from 4 we have that $H|_{\hat{\mathcal{F}}^c_G(x)}$ is a homeomorphism, for simplicity we note $H|_{\hat{\mathcal{F}}^c_G(x)} = H_{ux}$. Let $\delta_0$ be the one given by de uniform continuity of $H$ (??). Since $H_{ux}$ is a homeomorphism, for $\delta'$ we can find a $\delta_0$ (independent of $x$) such that if $x$ and $y$ are such that $y \in \hat{\mathcal{F}}^c_G(x)$ and $d(H_{ux}(x), H_{ux}(y)) \leq \delta_0$, then $l^u(x, y) \leq \delta'$.

Let us consider $n_0$ such that $1/n_0 < \delta'$ and $z' = H(z_{n_0})$. Note that $z' \in \hat{\mathcal{F}}^c_A(H(x_{n_0}))$. 
For perhaps a bigger $n$, we have that $l^u(x_n, y_n) \leq 1/n$, and $d(H(x_n), H(y_n)) \leq \delta_0$, from the continuity of $H$. But for $A$, $\hat{F}_A^u$ are parallel planes so since the length of the unstable segment between $z'$ and $p^u_{H(y_n)}(z')$ is less than $\delta_0$ (see figure 3) and therefore
\[ \varepsilon_0 > \delta' > l^u(z_n, H^{-1}_{uX}(p^u_{H(y_n)}(z'))) = l^u(z_n, p^u_{y}(z_n)) \geq \varepsilon_0. \]

If two points are sufficiently close their unstable manifolds remain close in some neighborhood. This is a consequence of the continuity of the foliation.

For every $\varepsilon > 0$, there is $\beta > 0$ and $\eta > 0$ such that if $y \in \hat{F}_G^u(x)$ and $d(x, y) < \eta$, then for any $z \in \hat{F}_G^u(x)$ such that $l^u(x, z) < \beta$, we have that $d(z, p^u_y(z)) < \varepsilon$. We can also take $\beta$ to be uniform since the foliations are lifts of foliations in a compact set (see figure 4).

Now we will put everything together. Let $\varepsilon = \delta_p$ be the one from the local product structure of $\Delta$. For this $\varepsilon$ we find $\eta > 0$ and $\beta$ from our previous observation. This will ensure us that if $d(x, y) < \eta$ their unstable leaves will remain closer than $\delta_p$ in a ball of radius $\beta$ from $x$.

We can choose the $\varepsilon_0$ from the lemma 9 smaller $\delta_p$ and $\beta$, so the lemma ensures us that there exists a $\delta_0$ such that if $x$ and $y \in \hat{F}_G^u(x)$ and $l^u(x, y) \leq \delta_0$ then the center stable foliations of $x$ and $y$ will not separate more than $\varepsilon_0$.

For this last $\delta_0$ we will take a compact neighborhood of $(0, 0, 0)$ in $\hat{F}_u^u((0, 0, 0))$ that we call $U^u$ with $diam(U^u) < \delta_0$. In this conditions we define
\[ V = \bigcup_{x \in U^u} \hat{F}_G^u(x). \]

We have proved the following for $V$.

**Lemma 10.** There exist a compact neighborhood of $(0, 0, 0)$, $U^u$, in $\hat{F}_u^u((0, 0, 0))$, such that the set $V$ define as $V = \bigcup_{x \in U^u} \hat{F}_G^u(x)$, satisfies:
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Figure 4. Separation between the unstable leaves

Figure 5. The tube $V$

- If $x \in V$ and $y \in \mathcal{F}_G(x) \cap V$, then $d(x, y) < \delta$ and $d(x, y) < \beta$.
- For every $\varepsilon$ there is $\delta$ such that if $x \in V$ and $d(x, y) < \delta$ then $d(z, p^u_y(z)) < \varepsilon$, for any $z$ in $\mathcal{F}_G(x)$.
- For every $\varepsilon$ there is $\delta$ such that if $x \in V$ and $d(x, y) < \delta$ then $d(z, p^e_y(z)) < \varepsilon$, for any $z$ in $\mathcal{F}_G(x) \cap V$.

The following implies Theorem B:

Theorem 5. Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be as in Mañe’s example. There exists a neighborhood $U$ of $f$ such that for $g \in U$ there exists a transitive hyperbolic compact set which is not included in any hyperbolic set with local product structure.

Proof. We first recall that for every $g \in \mathcal{U}(f)$ the existence of a set $\Lambda$ which is a connected, non trivial, $g$-invariant, hyperbolic set, has already been stated.
As before we start by supposing that there is a set $\Delta$ with local product structure containing $\Lambda$. Let $\hat{\Lambda}$ be the lifts of $\Lambda$, lifting $p$ to $(0,0,0)$ and $\Delta$ the lift of $\Delta$. Recall that the strategy to see that such a $\Delta$ can not exist, is to find an interval of the unstable foliation in which $\Delta \cap \hat{\cal F}_G^u(x_0)$ will dense.

Let $V \subset \mathbb{R}^3$ be an open tube as defined in [10].

**Assertion.** Suppose that $x$ and $y \in \Delta \cap V$ are $n$-related, with $x_0, \ldots, x_n \in \Delta \cap V$ and:

- For any $z \in \hat{\cal F}_G^u(x_{i+1})$, $d(z, p_{x_i}^u(z)) < \varepsilon$, for every $0 \leq i \leq n - 1$.
- For any $z \in \hat{\cal F}_G^u(x_i) \cap V$, $d(z, p_{x_{i+1}}^u(z)) < \varepsilon$, for every $0 \leq i \leq n - 1$.

Then $p_{x_0}^u(x_i) \in \hat{\Delta}$ for every $0 \leq i \leq n - 1$.

**Proof.** We will prove this by induction. The base case is given by the fact that we chose $\varepsilon > 0$ smaller than $\delta_p/2$, where $\delta_p$ is from the local product structure of $\hat{\Delta}$. Suppose now that $x, y \in \hat{\Delta} \cap V$ are $n$-related. We have $x = x_0, x_1, \ldots, x_n = y$ as in the definition. We define $\hat{\cal F}_j^u = \hat{\cal F}_G^u(x_j) \cap \hat{\cal F}_G^u(x_{j+1})$ for $0 \leq j \leq n - 1$. Note that $\hat{\cal F}_0$ and $\hat{\cal F}_{n-1}$ are $(n-1)$-related because:

- $\hat{x}_j \in \hat{\Delta}$ for $j = 1, \ldots, n - 1$ by induction hypothesis,
- Since $\hat{x}_j$ is in $\hat{\cal F}_G^u(x_{j+1})$, by our hypothesis we have that $d(\hat{x}_j, p_{x_j}^u(\hat{x}_j)) < \varepsilon$
  
  
  On the other hand since $\hat{x}_{j-1} \in \hat{\cal F}_G^u(x_j)$ then, $d(\hat{x}_j, p_{x_{j-1}}^u(\hat{x}_j)) < \varepsilon$.
- Since $\hat{x}_j = \hat{\cal F}_G^u(x_j) \cap \hat{\cal F}_G^u(x_{j+1})$, we have that, $\hat{x}_j \in \hat{\cal F}_G^u(x_j) \cap V$. Again by our assertion’s hypothesis we have that $d(\hat{x}_j, p_{x_{j+1}}^u(\hat{x}_j)) < \varepsilon$ On the other hand, $\hat{x}_{j+1} = \hat{\cal F}_G^u(x_{j+1})$, so $d(\hat{x}_j, p_{x_{j+1}}^u(\hat{x}_j)) < \varepsilon$.

This allows us to conclude that $p_{x_{i+1}}^u(x_i) \in \hat{\Delta}$. This proves our assertion. 

Returning to the proof of the theorem, for any $\varepsilon < \delta_p/2$ we take the $\delta < \varepsilon$ from the definition of $V$ as in lemma [10].

Let us take any point $r$ of $V$ which is not in $V$. $(0,0,0)$ is $n$-related to $r$. We call $x_{j+1}$ the first element of the sequence relating $(0,0,0)$ to $r$ that is not in $V$. As in the proof of the assertion, we can construct a new sequence from the sequence that $j$-relates $(0,0,0)$ to $x_j$ as follows.

We define $\hat{x}_i = \hat{\cal F}_i^u(x_i) \cap \hat{\cal F}_{i+1}^u(x_{i+1})$ for $0 \leq i \leq j - 1$. Note that $\hat{x}_0$ and $\hat{x}_{j-1}$ are $(j-1)$-related because:

- $\hat{x}_i \in \hat{\Delta}$ for $i = 1, \ldots, j - 1$ since $\delta/2 < \varepsilon < \delta_p/2$ from the local product structure.
- $\hat{x}_i \in \hat{\cal F}_i^u(x_{i+1})$, and $x_i, x_{i+1} \in V$ with $d(x_i, x_{i+1}) < \delta$. From [10] we have that $d(\hat{x}_i, p_{x_i}^u(\hat{x}_i)) < \varepsilon$. As in the assertion this implies $d(\hat{x}_i, p_{x_{i+1}}^u(\hat{x}_i)) < \varepsilon$ for $i = 1, \ldots, j - 1$.
- Since $\hat{x}_i = \hat{\cal F}_i^u(x_i) \cap \hat{\cal F}_{i+1}^u(x_{i+1})$, we have that, $\hat{x}_i \in \hat{\cal F}_i^u(x_i) \cap V$. Since $d(x_i, x_{i+1}) < \delta$. As before from [10] we have that $d(\hat{x}_i, p_{x_{i+1}}^u(\hat{x}_i)) < \varepsilon$ On the other hand, $\hat{x}_{i+1} = \hat{\cal F}_i^u(x_{i+1})$, so $d(\hat{x}_i, p_{x_{i+1}}^u(\hat{x}_i)) < \varepsilon$ for $i = 1, \ldots, j - 1$.

In addition to this, $\hat{\cal F}_j^u(\hat{x}_j) \cap V = \hat{\cal F}_j^u(x_j) \cap V$ for $1 \leq i \leq j - 1$ and $\hat{\cal F}_j^u(x_j) = \hat{\cal F}_j^u(x_{j+1})$. Recall that since $x_i \in V$ for $i = 1, \ldots, j$ and $d(x_i, x_{i+1}) < \delta$, so by [10] we have:

- For any $z$ in $\hat{\cal F}_j^u(\hat{x}_j) \cap V$, $d(z, p_{x_j}^u(z)) = d(z, p_{x_{i+1}}^u(z)) < \varepsilon$, for every $0 \leq i \leq j - 1$. 


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For any $z$ in $\hat{F}^u(\pi) \cap V$, $d(z, p_{x_{i+1}}^u(z)) = d(z, p_{x_{i+1}}^u(z)) < \varepsilon$, for every $0 \leq i \leq j - 1$.

This proves that the new sequence we have defined is in the hypothesis of our assertion so, $p_{x_{i+1}}^u(x_i) \in \hat{\Delta}$ for $i = 0, \ldots, j - 1$. But then $p_{x_0}^u(x_i) \in \hat{\Delta}$ for $i = 0, \ldots, j$.

Figure 6. A $\varepsilon$-relation between $(0, 0, 0)$ and $x_j$.

6. In dimension 2 transitive sets are locally premaximal. In this section we prove Theorem C. For this, we rely on a result by Anosov [2] (for a proof in an other context see also [6] proposition 4.3) and one by Fisher in [9], which we state below.

Theorem 6. (Anosov) Let $f : M \to M$ be a diffeomorphism of a compact manifold and let $F \subset M$ be a compact hyperbolic invariant set with zero topological dimension. For every neighborhood $U$ of $F$, there exists a set $\Lambda$ which is compact invariant hyperbolic and has local product structure, such that $F \subset \Lambda \subset U$.

Theorem 7. (Fisher) Let $f : M \to M$ be a diffeomorphism of a compact surface $M$. If $\Lambda$ has local product structure and has non empty interior, then $\Lambda = M = T^2$. 


\[ \delta - \text{adapted curve} \]

\[ d(p_{x_{j+1}}^u(x_j), x_j) \leq \varepsilon \]

\[ d(p_{x_j}^s(x_{j-1}), x_{j-1}) \leq \varepsilon \]
Remark 1. Since $M$ is a compact surface (and therefor Hausdorff locally compact connected space), then zero dimensional subsets are exactly the totally disconnected subsets.

We will prove some lemmas that will imply Theorem

In what follows $f : M \to M$ will be a diffeomorphism of $M$ a compact surface and $\Lambda \subset M$ will be a compact hyperbolic invariant set such that $\Omega(f |_\Lambda) = \Lambda$

Definition 12. We denote $\Lambda_0$ as the union of all points $p$ in $\Lambda$ such that the connected component of $p$ in $\Lambda$ is $p$ itself. We define $\Lambda_1$ as $\Lambda_1 = \Lambda \setminus \Lambda_0$.

Note that $\Lambda_0$ is totally disconnected and therefore 0 dimensional.

In what follows $W^s_i(x)$ with $i = u, s$ will denote the local stable and unstable manifolds for a suitable $\varepsilon$ (for instance as in the theorem of the stable manifold). As before, for a sufficiently small neighborhood of $x \in \Lambda_1$ we define $p^s_x : U(x) \cap \Lambda_1 \to W^s_i(x)$ to be the local projection along the stable manifolds. We define analogously $p^u_x : U(x) \cap \Lambda_1 \to W^u_i(x)$. Recall that both stable and unstable manifolds are one dimensional. Unless in the case where there are sinks or sources in $\Lambda$ but then $\Lambda$ is just the point (since $\Omega(f |_\Lambda) = \Lambda$), so in what follows we will not deal with this case.

Lemma 11. Periodic points are dense in $\Lambda_1$. Moreover for any $x \in \Lambda_1$ we have that $W^u(x) \subset \Lambda_1$ or $W^s(x) \subset \Lambda_1$ (or both).

Proof. We define $lcc(x)$ as the connected component of $x$ in $\Lambda_1 \cap B(\varepsilon, x)$ and from the definition of $\Lambda_1$ we have that $lcc(x)$ is not trivial. Then for any $x \in \Lambda_1$ we have that either $p^s_x(lcc(x))$ contains a nontrivial connected set (an arc) or $p^u_x(lcc(x))$ contains a nontrivial connected set. This will also be true for any smaller $U(x)$.

Since $\Omega(f |_\Lambda) = \Lambda$, using the shadowing lemma we have that if $x \in \Lambda$ then it is approximated by periodic points (which a priori would not be in $\Lambda$).

Let us suppose that $p^s_x(lcc(x))$ contains an arc. Then we define

$$V = \bigcup_{y \in p^s_x(lcc(x))} W^s_i(y)$$

and $\hat{V}$ is not empty. We take $z \in \hat{V} \cap lcc(x)$ If $\{ p_n \}_{n \in \mathbb{N}}$ is such that $p_n \to z$ (where $p_n$ are periodic points), then $p_n \in \hat{V}$ for all $n$ greater than some $n_0$.

Since $\Lambda$ is invariant, and $f$ a diffeomorphism it follows that $\Lambda_1$ is invariant too (otherwise it would take a nontrivial connected component into a point). This implies that $\omega(p_n) \in \Lambda_1$ but then $p_n \in \Lambda_1$ for all $n > n_0$.

Now we take $p_n$ sufficiently close to $z$. Then $W^s_i(p_n) \cap lcc(x) \neq \emptyset$.

Iterating for the future $f^n(lcc(x))$ accumulates on both branches of the unstable manifold containing $p_n$ in its interior and since $diam(lcc(x)) > 0$, $f^n(lcc(x))$ accumulates on an arc of $W^u_i(p_n)$ containing $p_n$. Therefore this arc of $W^u_i(p_n)$ is contained in $\Lambda$, since $\Lambda$ is compact. Such an arc must be contained in $\Lambda_1$ because it clearly does not belong to $\Lambda_0$. The invariance of $\Lambda_1$ implies that $W^u_i(p_n) \subset \Lambda_1$ and as $p_n \to z$ implies that $W^u_i(z) \subset \Lambda_1$. We can take now a sequence of $z_n$ as before that are contained in neighborhoods $U_n(x)$ which are each time smaller, and $z_n \to x$ so $W^u_i(x) \subset \Lambda_1$.

The situation is analogous if $p^s_x(lcc(x))$ is a stable arc. \qed
Figure 7. the local connected component of $x$ and $p^u_x(lcc(x))$

**Lemma 12.** Let $\Lambda^u_1 = \{ x \in \Lambda_1 \mid W^u(x) \subset \Lambda_1 \}$ and $\Lambda^s_1 = \{ x \in \Lambda_1 \mid W^s(x) \subset \Lambda_1 \}$. Then $\Lambda^u_1$ and $\Lambda^s_1$ are compact sets with local product structure.

**Proof.** We will prove first that $\Lambda^u_1$ is closed and therefore compact. Let $\{ x_n \}_{n \in \mathbb{N}} \subset \Lambda^u_1$ and $x_n \to y$ then the unstable manifold of $y$ is in $\Lambda$ since $\Lambda$ is compact. Hence $y \in \Lambda^u_1$. Let $x, y \in \Lambda^u_1$ such that $d(x, y) < \delta$ for some $\delta$ appropriate such that $W^u_{loc}(x) \cap W^s_{loc}(y) \neq \emptyset$. Since $W^u(x) \subset \Lambda_1$, and $W^s(y) \subset \Lambda_1$,

$$W^u_{loc}(x) \cap W^s_{loc}(y) \subset W^u(x) \subset \Lambda_1.$$ observe that logically if $x \in \Lambda^u_1$ then $z \in \Lambda^u_1$ for all $z \in W^u(x)$ ($= W^u(z)$).

The situation is analogous for $\Lambda^s_1$. 

**Corollary 3.** The set $\Lambda_1$ is compact, invariant and has local product structure.

**Proof.** The sets $\Lambda^u_1$ and $\Lambda^s_1$ defined in lemma 12 are such that

$$\Lambda_1 = \Lambda^u_1 \cup \Lambda^s_1,$$

as a consequence of lemma 11.

If $\Lambda^u_1$ and $\Lambda^s_1$ are not disjoint and if one takes $p \in \Lambda^u_1$ and $q \in \Lambda^s_1$ periodic points and are close, then $W^u_{loc}(p) \cap W^s_{loc}(q) = z \in \Lambda^u_1 \cap \Lambda^s_1$. Then we can see that actually $q \in \Lambda^u_1$ since the iterates of $z$ accumulates on $q$ and are in $\Lambda^u_1$ which is compact. Now if we take $x \in \Lambda^u_1$ and $y \in \Lambda^s_1$ close, the fact that periodic points are dense in $\Lambda^u_1$ and what we just observed yields that $x$ and $y$ are in $\Lambda^u_1$ which has local product structure.

Therefore $\Lambda_1$ is compact and has local product structure.
Corollary 4. Either $\Lambda_1$ is the disjoint union of an attractor $\Lambda_1^u$ and a repeller $\Lambda_1^s$, or $\Lambda = M = \mathbb{T}^2$ and $f$ is Anosov.

Proof. $\Lambda_1^u$ is an attractor since

$$V = \bigcup_{x \in \Lambda_1^u} W^u_x(x)$$

is a neighborhood of $\Lambda_1^u$, that satisfies $f(V) \subset V$ and $\bigcap f^n(V) = \Lambda_1^u$ because of the theorem of the unstable manifold and the fact that $\Lambda_1^u$ is invariant. Suppose that there is $x \in \Lambda_1^u \cap \Lambda_1^s$. Then $W^u(x)$ and $W^s(x) \in \Lambda_1$. Since $\Lambda_1$ has local product structure this implies that

$$\bigcup_{y \in W^s_{\text{loc}}(x)} W^u_{\text{loc}}(y) \subset \Lambda_1,$$

and so $\Lambda_1$ has non empty interior. It follows from Theorem 7 that $\Lambda = \Lambda_1 = M = \mathbb{T}^2$ and $f$ is Anosov.

Lemma 13. The set $\Lambda_0$ is compact and disjoint from $\Lambda_1$. Moreover for any neighborhood $V$ of $\Lambda_0$, we can find a hyperbolic set $\Lambda_0'$ with local product structure such that $\Lambda_0' \cap \Lambda_1 = \emptyset$ and $\Lambda_0 \subset \Lambda_0' \subset \bar{V}$.

Proof. From its definition, $\Lambda_0$ is disjoint from $\Lambda_1$. Suppose that $\Lambda_0$ is not empty (the lemma holds trivially if it is empty). Suppose that there is a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \Lambda_0$ such that $x_n \to y \in \Lambda_1$. By corollary 4 we have that $\Lambda_1$ is the disjoint union of attractors and a repellers so suppose that $y \in \Lambda_1^s$. For a sufficiently big $n$, the sequence $x_n$ must be in the (open) basin of attraction of $\Lambda_1^s$ and therefore a neighborhood of $x_n$ never intersects with itself again, which is impossible since $\Omega(f|_\Lambda) = \Lambda$. For the second part, let us consider $\bar{V} = V \setminus \Lambda_1$. Since $\Lambda_0$ is disjoint from $\Lambda_1$ this is still a neighborhood of $\Lambda_0$. Since $\Lambda_0$ is 0-dimensional, by 6 there is a set $\Lambda_0'$ such that $\Lambda_0 \subset \Lambda_0' \subset \bar{V} \subset V$ and $\Lambda$ disjoint from $\Lambda_1$.

Now we are in the right conditions to prove:

Theorem 8. Let $f : M \to M$ be a diffeomorphism, $M$ a compact surface and $\Lambda \subset M$ a compact hyperbolic invariant set. If we also have that $\Omega(f|_\Lambda) = \Lambda$ then for any neighborhood $V$ of $\Lambda$, there exist $\bar{\Lambda}$ a compact hyperbolic invariant set with local product structure such that,

$$\Lambda \subset \bar{\Lambda} \subset V.$$

Proof. Let $V$ be any open set containing $\Lambda$ and $V' = V \setminus \Lambda_1$. Note that $V'$ is open and contains $\Lambda_0$.

From lemma 13 we have that there exists a hyperbolic set $\Lambda_0'$ with local product structure such that $\Lambda_0' \cap \Lambda_1 = \emptyset$ and $\Lambda_0 \subset \Lambda_0' \subset V'$. On the other hand $\Lambda_1$ has local product structure from corollary 3.

We conclude that $\bar{\Lambda} = \Lambda_0' \cup \Lambda_1$ has local product structure, is contained in $V$ and contains $\Lambda$.

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E-mail address: adaluz@cmat.edu.uy