On the MISO Channel with Feedback: Can Infinitely Massive Antennas Achieve Infinite Capacity?

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Abstract

We consider communication over a multiple-input single-output (MISO) block fading channel in the presence of an independent noiseless feedback link. We assume that the transmitter and receiver have no prior knowledge of the channel state realizations, but the transmitter and receiver can acquire the channel state information (CSIT/CSIR) via downlink training and feedback. For this channel, we show that increasing the number of transmit antennas to infinity will not achieve an infinite capacity, for a finite channel coherence and a finite input constraint on the second or fourth moment. This insight follows from our new capacity bounds that hold for any linear and nonlinear coding strategies, and any channel training schemes. In addition to the channel capacity bounds, we also provide a characterization on the beamforming gain that is also known as array gain or power gain, at the regime with large number of antennas.

I. INTRODUCTION

Motivated by the increasing demand for higher data rates in wireless communication systems, a significant effort is being made to study the use of massive multiple-input multiple-output (massive MIMO) systems [1]–[3]. As equipped with a large number of antennas, the massive MIMO system has a potential to boost the channel’s beamforming gain that is also known as array gain or power gain (cf. [4], [5]). In the massive MIMO channels, for example in a massive multiple-input single-output (MISO) channel, the capacity may increase logarithmically with the number of antennas (cf. [4]–[7]), which implies that infinitely massive antennas may allow us to achieve an infinite capacity, even with a finite power constraint at the transmitter.

However, the above exciting result is based on the key assumption that the instantaneous fading coefficients are perfectly known to the receiver/transmitter (perfect CSIR/CSIT). In general, CSIT and CSIR entail channel training and feedback. In a typical system with frequency-division-duplex (FDD) mode, CSIT comes from channel training and feedback operating over the downlink channel and feedback channel respectively. The overhead of the training and feedback may in turn affect the channel capacity. Therefore, it remains open if a massive MIMO system could still provide a significant capacity benefit as we expected. Specifically, we might ask the following question: Can infinitely massive antennas always achieve an infinite capacity in a massive MIMO channel?

In this work, we study this question by focusing on a massive MISO block fading channel with output feedback. We assume that the transmitter and receiver have no prior knowledge of the channel state realizations, but the transmitter and receiver can acquire the channel state information via downlink training and feedback. We begin with a simple case where the coherence of the channel block is $T_c = 2$ and the input signals are limited by a finite second-moment constraint that is also known as long-term average power constraint. Since the coherence is $T_c = 2$, the transmitter could use the first and the second channel uses of each channel block for channel training and data transmission, respectively. Based on this scheme, one might tentatively expect an infinite rate for the case with infinite number of transmit antennas, because a little channel state information might be very useful for this case. However, we show that in this setting increasing the transmit-antenna number to infinity will not yield an infinite capacity. The result holds for any linear and nonlinear coding strategies, and any channel training schemes. This result is in sharp
contrast to the result of the setting with perfect CSIT/CSIR (e.g., through a genie-aided training and feedback), in which the capacity will go to infinity as the antenna number grows to infinity (cf. [7]).

As a main contribution of this work, we derive both capacity upper bound and lower bound for the MISO channel with feedback under the second moment and the fourth moment input constraints, respectively. For the case with a finite channel coherence and a finite input constraint on the fourth moment, the result reveals again that increasing the transmit-antenna number to infinity will not yield an infinite capacity. In addition to the capacity bounds, this work also provides a characterization on the channel’s beamforming gain at the regime with large number of antennas. Similarly to the degrees-of-freedom metric (cf. [8]) that usually captures the prelog factor of capacity at the high power regime, beamforming gain is used in this work to capture the prelog factor of capacity at the high antenna-number regime.

The capacity of the channels with feedback, or with imperfect CSIT/CSIR, has been studied extensively in the literature for varying settings, e.g., the point-to-point channels [9]–[21] and the broadcast channel [22]–[26]. However, a common assumption in those works above is that imperfect CSIT and CSIR were acquired without considering the overhead in channel training. The channel training overhead can not be negligible when the number of channel parameters to be estimated is large and the channel coherence is relatively small. This work is categorized in the line of works studying the multiple-antenna networks where CSIT and CSIR were acquired via channel training with overhead, such as [27]–[32]. To the best of our knowledge, the previous works in this direction usually considered some specific assumptions, e.g., linear coding strategy, dedicated channel training (a certain time is dedicated specifically for channel training) and short-term input constraint. In this work, we study a broad setting that considers any linear and nonlinear coding strategies, any channel training schemes, any short-term and long-term input constraints.

The remainder of this work is organized as follows. Section II describes the system model. Section III provides the main results of this work. The converse and achievability proofs are described in Sections IV, V, VI and the appendices. The work is concluded in Section VII. Throughout this work, nonlinear coding strategies, any channel training schemes, any short-term and long-term input constraints.

II. System model

We consider a MISO channel where a transmitter with \( M \) \( (M \geq 2) \) antennas sends information to a single-antenna user, as illustrated in Fig. 1. The signal received by the user at time \( t \) is given as

\[
y_t = h_t^\top x_t + z_t, \tag{1}
\]

\( t = 1, 2, \ldots, n \), where \( x_t \) denotes the transmitted signal vector at time \( t \), \( z_t \sim \mathcal{CN}(0, 1) \) denotes the additive white Gaussian noise (AWGN), \( h_t \defeq [h_{t,1}, h_{t,2}, \ldots, h_{t,M}]^\top \sim \mathcal{CN}(0, I_M) \) denotes the \( M \times 1 \)
channel vector at time $t$, and $h_{t,m}$ denotes a channel coefficient of the $m$th transmit antenna at time $t$. We assume a block fading model (cf. [28], [33]), in which the channel coefficients remain constant during a coherence block of $T_c$ channel uses and change independently from one block to the next, i.e.,

$$h_{t,T_c + 1} = h_{t,T_c + 2} = \cdots = h_{t,T_c + T_c} \neq h_{(t+1)T_c + 1}$$

for $\ell = 0, 1, \cdots, L - 1$ and $L = n/T_c$, where $n, L, T_c$ are assumed to be integers. We assume that the channel coefficients in each block are initially unknown to the transmitter and the user. At the end of each time $t$, the user can feed back the channel outputs to the transmitter over an independent feedback link. For simplicity we assume that the feedback link is noiseless (error-free) and with a unit time delay, i.e., at the beginning of time $t + 1$, the transmitter knows $y_t \equiv (y_1, y_2, \cdots, y_t)$.

For this feedback communication of total $n$ channel uses, the transmitter wishes to send the user a message index $w$ that is uniformly distributed over $\{1, 2, \cdots, 2^{nR}\}$. We specify a $(2^{nR}, n)$ feedback code with encoding maps $x_t : \{1, 2, \cdots, 2^{nR}\} \times C^{t-1} \rightarrow C^M$, $t = 1, 2, \cdots, n$ (2)

that result in codewords (or code functions, more precisely)

$$x^n(w, y^{n-1}) = (x_1(w), x_2(w, y_1), \cdots, x_n(w, y^{n-1}))$$. (3)

Then the user decodes the message with decoding maps $\hat{w}_n : C^n \rightarrow \{1, 2, \cdots, 2^{nR}\}$. (4)

We consider two cases of constraints on the input signals. At first we consider the second moment input constraint such that

$$\frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left[ \|x_t(w, y^{t-1})\|^2 \right] \leq P$$ (5)

where the expectation is over all possible noise and fading sequences as well as the message $w$. This second moment constraint is also known as the average power constraint. We then consider the fourth moment input constraint such that

$$\frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left[ \|x_t(w, y^{t-1})\|^4 \right] \leq \kappa^2 P^2$$ (6)

where $\kappa$ is a positive constant. The fourth moment input constraint has been introduced in several communication scenarios (cf. [34]–[38]). For some certain cases, imposing the fourth moment constraint is identical to imposing a limitation on the kurtosis that is a measure of peakedness of the signal (cf. [35]–[37]). The probability of error $P_{e}^{(n)}$ is defined as

$$P_{e}^{(n)} \triangleq \frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} \mathbb{P} \{ \hat{w}_n(y^n) \neq w | w = w \}.$$
A rate $R$ (bits per channel use) is said to be achievable if there exists a sequence of $(2^{nR}, n)$ codes with $P_e(n) \to 0$ as $n \to \infty$. The capacity of this channel $C$ is defined as the supremum of all achievable rates.

In this work, we specifically focus on the capacity effect of the channel with large number of antennas, which may be captured by the metric of beamforming gain. For the capacity effect of the channel with high power, one might consider the metric of degrees-of-freedom that is beyond the scope of this work. For a finite input constraint $P$ (or $\kappa P$) and for a fixed

$$\alpha \triangleq \frac{\log T_c}{\log M}, \quad \alpha \geq 0$$

that is the ratio between coherence and antenna number in a logarithmic scale, or equivalently $T_c \triangleq M^\alpha$, the beamforming gain of the channel is defined as

$$b(\alpha) \triangleq \limsup_{M \to \infty} \frac{C(\alpha, P, M)}{\log M}.$$ 

Similarly to the definition of generalized degrees-of-freedom (GDoF, see [39]), the beamforming gain $b(\alpha)$ captures the capacity prelog factor for a class of channels with a fixed $\alpha$, at the regime with a large number of antennas. In this setting $b = 0$ means zero beamforming gain, while $b = 1$ denotes a full beamforming gain. For the ideal case with perfect CSIT and CSIR (e.g., through a genie-aided method) one might achieve a full beamforming gain. However, for this setting where CSIR and CSIT are acquired via downlink training and feedback, the beamforming gain is generally unknown so far. In the following we seek to characterize the beamforming gain of this setting.

III. MAIN RESULTS

This section provides the main results for a MISO channel with feedback defined in Section II. The proofs are shown in Sections IV, V, VI and the appendices. Before showing the main results of this work, let us first revisit the ideal case of MISO channel with perfect CSIT and CSIR, and with a second moment input constraint. For this ideal case, the work in [7] characterized the channel capacity in a closed form

$$C_{\text{ideal}} = \max_{\gamma: f_\gamma} \log(1 + \bar{P}(\gamma) \cdot \gamma) f_\gamma(\gamma) d\gamma$$

where $\gamma \triangleq \|h_t\|^2$, $f_\gamma(\gamma)$ is the probability density function of $\gamma$, and the optimal power allocation of $\bar{P}(\gamma)$ is based on a water-filling algorithm (cf. [7]). When the antenna-number $M$ is large, the capacity in (7) tends to $\log(1 + PM)$, which is summarized in the following proposition.

**Proposition 1** (Ideal case). For the MISO channel with perfect CSIT and CSIR, and with a finite second-moment input constraint $P$, the channel capacity tends to $\log(1 + PM)$ for a large $M$.

Proposition 1 follows from the capacity expression in (7) and the asymptotic analysis that is provided in Appendix A. Proposition 1 reveals that the capacity of a MISO channel with perfect CSIT and CSIR will go to infinity as the antenna-number $M$ grows to infinity, even with a finite input constraint $P$.

Let us now go back to the MISO channel with feedback defined in Section II where the transmitter and receiver have no prior knowledge of the channel state realizations, but the transmitter and receiver can acquire the CSIT/CSIR via downlink training and feedback. We begin with a specific case of coherence, $T_c = 2$, and provide a capacity result in the following theorem.

**Theorem 1** (Capacity upper bound). For the MISO channel with feedback defined in Section II and given a coherence $T_c = 2$, the capacity is upper bounded as

$$C \leq 1.5 \log(1 + P) + 0.5$$

under the second moment input constraint in (5).
The proof of Theorem 1 is provided in Section IV. Theorem 1 reveals that under a finite second-moment input constraint \( P \) and \( T_c = 2 \), the capacity of this MISO channel with feedback will not go to infinity as the antenna-number \( M \) grows to infinity. This result in sharp contrast to the result of the perfect CSIT/CSIR case, in which the capacity will be infinite as \( M \) grows to infinity (see Proposition 1).

In Theorem 1 we just consider the specific case of \( T_c = 2 \). When \( T_c = 1 \) it is reduced to the fast fading case without CSIT — in this case increasing the antenna number \( M \) to infinity will not improve too much on the channel capacity even with perfect CSIR (cf. [6]). Note that for the case without CSIT but with perfect CSIR, the channel capacity tends to \( \log(1 + P) \) for large \( M \), under the second moment input constraint (cf. [6]). When \( T_c \geq 3 \), we conjecture that the channel capacity will still not go to infinity as \( M \) grows to infinity, for a finite \( P \) and a finite \( T_c \). The following result summarizes the capacity upper bound for the case with a fourth moment input constraint.

**Theorem 2 (Capacity upper bound).** For the MISO channel with feedback defined in Section II, the capacity is upper bounded by

\[
C \leq \log\left(1 + \min\{M + 2, \sqrt{2}(T_c + 1)\} \cdot \kappa P\right)
\]

under the fourth moment input constraint in (6).

The proof of Theorem 2 is provided in Section V and the appendices. Theorem 2 reveals that given a finite fourth-moment input constraint \( \kappa P \) and a finite \( T_c \), again, the capacity will not go to infinity as the antenna number \( M \) grows to infinity. The following result summarizes the capacity lower bound.

**Theorem 3 (Capacity lower bound).** For the MISO channel with feedback defined in Section II, the capacity is lower bounded by

\[
C \geq \frac{T_c - T_\tau}{T_c} \cdot \log\left(1 + \frac{P \cdot \max\{(T_\tau - 1)/2, 1/2\}}{2 + \frac{1}{P}}\right) - \frac{1}{\max\{T_\tau, 2\}}
\]

under the second moment input constraint; while under the fourth moment input constraint the capacity is lower bounded by

\[
C \geq \frac{T_c - T_\tau}{T_c} \cdot \log\left(1 + \frac{P_o \cdot \max\{(T_\tau - 1)/2, 1/2\}}{2 + \frac{1}{P_o}}\right) - \frac{1}{\max\{T_\tau, 2\}}
\]

where \( P_o \triangleq \frac{\kappa P}{\sqrt{3}} \) and \( T_\tau \triangleq \left\lceil \log\max\{4, \min\{M,T_c\}\}\right\rceil \).

The capacity lower bound in Theorem 3 is a lower bound on the achievable rate of a proposed scheme (see Section VI). The proposed scheme is a simple scheme that uses only \( T_\tau \) number of transmit antennas (\( T_\tau \leq M \)). The scheme consists of a downlink training phase and a data transmission phase for each coherence block of the channel. Specifically, for each coherence block with \( T_c \) channel uses, the duration of downlink training phase is \( T_\tau \) number of channel uses, while the duration of data transmission phase is \( T_c - T_\tau \) number of channel uses. The choice of \( T_\tau \) is critical to the scheme performance, because with too small \( T_\tau \) there is not enough time for the channel training, while with too large \( T_\tau \) there is not enough time for the data transmission. In our scheme we set \( T_\tau = \left\lceil \log\max\{4, \min\{M,T_c\}\}\right\rceil \) which yields the achievable rate in Theorem 3. Note that the rate in Theorem 3 can be further improved since we just focus on the simple scheme.

In this work, we specifically focus on the channel capacity effect of the system with large number of antennas, which may be captured by the metric of beamforming gain. The following results summarize the beamforming gain of the channel under two input constraints respectively.
Proposition 2 (Beamforming gain). For the MISO channel with feedback defined in Section II and given a coherence $T_c = 2$, the beamforming gain is

$$b = 0$$

under the second moment input constraint.

Proposition 2 follows directly from Theorem I. Proposition 2 reveals that there is no beamforming gain at a specific case of $T_c = 2$. In this case, increasing the antenna number $M$ to infinity will not improve too much on the capacity.

Theorem 4 (Beamforming gain). For the MISO channel with feedback defined in Section II, the beamforming gain is characterized as

$$b(\alpha) = \min\{\alpha, 1\}$$

under the fourth moment input constraint.

Theorem 4 follows from the capacity bounds in Theorems 2 and 3 under the fourth moment input constraint (see Appendix D for the proof). As illustrated in Fig. 2, a full beamforming gain, $b = 1$, is achievable when $\alpha \geq 1$. Intuitively, for the case with large $\alpha$, $\alpha \gg 1$, the coherence is very large and consequently the channel can be considered as a static channel, in which a full beamforming gain could be achieved easily via sufficiently long downlink training, as the time overhead of downlink training can be negligible in this static case. Theorem 4 reveals an interesting insight that, instead of $\alpha \gg 1$, $\alpha = 1$ is sufficient for achieving a full beamforming gain. When $0 \leq \alpha \leq 1$ (or, equivalently, $T_c \leq M$) the beamforming gain grows linearly with $\alpha$, which implies that in this case the capacity grows logarithmically with the coherence $T_c$ in an asymptotic sense.

IV. CONVERSE: THE CASE WITH A SECOND MOMENT INPUT CONSTRAINT

For the channel model defined in Section II we will show that the rate is upper bounded by

$$R \leq 1.5 \log(1 + P) + 0.5$$

(bits/channel use) under the second moment input constraint in (5) and given a fixed coherence $T_c = 2$.

The result reveals that, given a fixed coherence $T_c = 2$ and a finite $P$, increasing the antenna number $M$ to infinity will not yield an infinite capacity. The result holds for any linear and nonlinear schemes, and holds for any channel training schemes. Our proof is mainly based on minimum mean square error (MMSE) estimation techniques and MIMO techniques.
In the proof we will use Lemmas 1 - 3 that are shown at the end of this section. At first we bound the rate as follows:

\[ nR = \mathbb{H}(w) = \mathbb{I}(w; y^n) + \mathbb{H}(w|y^n) \leq \mathbb{H}(w; y^n) + n\epsilon_n \]

(8)

\[ = \sum_{t=1}^{n} \left( h(y_t|y^{t-1}) - h(y_t|w, y^{t-1}) \right) + n\epsilon_n \]

(9)

\[ \leq \sum_{t=1}^{n} \left( h(y_t|y^{t-1}) - h(y_t|w, y^{t-1}, h_t, \tilde{x}_t) \right) + n\epsilon_n \]

(10)

\[ = \sum_{t=1}^{n} \left( h(y_t|y^{t-1}) - \log(\pi e) \right) + n\epsilon_n \]

(11)

\[ = \sum_{t=1}^{n} \mathbb{E}_{y^{t-1}} \left[ h(y_t|y^{t-1} = y^{t-1}) \right] - n \log(\pi e) + n\epsilon_n \]

(12)

where (8) follows from Fano’s inequality and \( \epsilon_n \rightarrow 0 \) as \( n \rightarrow \infty \); (9) results from chain rule; (10) uses the fact that conditioning reduces differential entropy; (11) is from that \( h(y_t|w, y^{t-1}, h_t, \tilde{x}_t) = h(y_t - h_t^\top \tilde{x}_t|w, y^{t-1}, h_t, \tilde{x}_t) = h(z_t) = \log(\pi e) \).

We proceed to upper bound the differential entropy \( h(y_t|y^{t-1} = y^{t-1}) \) in (12). Note that the average power of \( y_t \) given \( (y^{t-1} = y^{t-1}) \) is

\[ \mathbb{E}[|y_t|^2|y^{t-1} = y^{t-1}] = \mathbb{E}[(h_t^\top \tilde{x}_t + z_t)^2|y^{t-1} = y^{t-1}] = 1 + \mathbb{E}[|h_t^\top \tilde{x}_t|^2|y^{t-1} = y^{t-1}] \]

Since differential entropy is maximized by a circularly symmetric complex Gaussian distribution with the same average power, we have

\[ h(y_t|y^{t-1} = y^{t-1}) \leq \log(\pi e (1 + \mathbb{E}[|h_t^\top \tilde{x}_t|^2|y^{t-1} = y^{t-1}])) \]

(13)

Then, by plugging (13) into (12) it gives the following bound on the rate:

\[ nR - n\epsilon_n \leq \sum_{t=1}^{n} \mathbb{E} \left[ \log(\pi e (1 + \mathbb{E}[|h_t^\top \tilde{x}_t|^2|y^{t-1}])) \right] - n \log(\pi e) \]

\[ = \sum_{t=1}^{n} \mathbb{E} \left[ \log \left( 1 + \mathbb{E}[|h_t^\top \tilde{x}_t|^2|y^{t-1}] \right) \right] \]

(14)

Let us now focus on the expectation term \( \mathbb{E}[|h_t^\top \tilde{x}_t|^2|y^{t-1}] \) in (14). Note that \( \tilde{x}_t \) is a function of \( (y^{t-1}, w) \), which implies that \( \tilde{x}_t \) and \( h_t \) can be correlated. Based on this fact, computing the value of \( \mathbb{E}[|h_t^\top \tilde{x}_t|^2|y^{t-1}] \) could be challenging in general. We also note that the value of \( \mathbb{E}[|h_t^\top \tilde{x}_t|^2|y^{t-1}] \) could be a function of \( M \), depending on the correlation of \( \tilde{x}_t \) and \( h_t \). In what follows we seek to bound the value of \( \mathbb{E}[|h_t^\top \tilde{x}_t|^2|y^{t-1}] \) for the specific case of \( T_c = 2 \). We will use the notations of \( \hat{h}_t, \tilde{h}_t \) and \( \Omega_t \) given as

\[ \hat{h}_t \equiv \begin{cases} 0 & \text{if } t \in \{2\ell + 1\}_{\ell=0}^{L-1} \\ \frac{\tilde{x}_{t-1}^\top y_{t-1}}{\|\tilde{x}_{t-1}\|^2 + 1} & \text{if } t \in \{2\ell + 2\}_{\ell=0}^{L-1} \end{cases} \]

(15)

\[ \Omega_t \equiv \begin{cases} I_M & \text{if } t \in \{2\ell + 1\}_{\ell=0}^{L-1} \\ I_M - \frac{\tilde{x}_{t-1}^\top \tilde{x}_{t-1}}{\|\tilde{x}_{t-1}\|^2 + 1} & \text{if } t \in \{2\ell + 2\}_{\ell=0}^{L-1} \end{cases} \]

(16)
\begin{equation}
\hat{h}_t = \hat{h}_t - \hat{h}_t \quad \forall t \in \{1, 2, \ldots, n\}.
\end{equation}

When \( T_c = 2 \), the channel changes every two channel uses. From Lemma 1 (see below) it implies that \( \hat{h}_t \) is the MMSE estimate of \( h_t \) given \( (y^{-1}, w) \). Now we bound the value of \( \mathbb{E}[\|h_t x_t\|^2 | y^{-1}] \) as follows

\begin{align}
\mathbb{E}[\|h_t x_t\|^2 | y^{-1}] &= \mathbb{E}[\mathbb{E}[\|h_t x_t\|^2 | y^{-1}, w] | y^{-1}] \\
&= \mathbb{E}[\mathbb{E}[\text{tr}(h_t^* h_t x_t x_t^*) | y^{-1}, w] | y^{-1}] \\
&= \mathbb{E}[\text{tr}( \mathbb{E}[h_t^* h_t | y^{-1}, w] x_t x_t^*) | y^{-1}] \\
&= \text{tr}( \mathbb{E}[(\hat{h}_t + \hat{h}_t)^* (\hat{h}_t + \hat{h}_t)] | y^{-1}, w] x_t x_t^*) | y^{-1}] \\
&= \text{tr}( \mathbb{E}[\hat{h}_t^* \hat{h}_t^T + \hat{h}_t^* \hat{h}_t^T + \hat{h}_t^* \hat{h}_t^T + \hat{h}_t^* \hat{h}_t^T | y^{-1}, w] x_t x_t^*) | y^{-1}] \\
&= \text{tr}( (\hat{h}_t^* \hat{h}_t^T + \Omega_t) x_t x_t^*) | y^{-1}] \\
&\leq \lambda_{\max}(\hat{h}_t^* \hat{h}_t^T) \cdot \text{tr}(x_t x_t^*) + \lambda_{\max}(\Omega_t) \cdot \text{tr}(x_t x_t^*) | y^{-1}] \\
&\leq \mathbb{E}[\|\hat{h}_t\|^2 \cdot \|x_t\|^2 + \|x_t\|^2 | y^{-1}] \\
\end{align}

where (18) follows from the identity that \( \mathbb{E}[a] = \mathbb{E}[\mathbb{E}[a | b]] \) for any two random variables \( a \) and \( b \); (19) results from the fact that \( \|h_t x_t\|^2 = \text{tr}(h_t^* x_t x_t^* h_t^*) = \text{tr}(h_t^* x_t x_t^* h_t^*) \) by using the identity of \( \text{tr}(AB) = \text{tr}(BA) \) for any matrices \( A \in \mathbb{C}^{m \times q}, B \in \mathbb{C}^{q \times m} \); (20) stems from the fact that \( x_t \) is a deterministic function of \( (y^{-1}, w) \) given the encoding maps as in (2); in (21) we just replace \( h_t \) with \( \hat{h}_t + \hat{h}_t \), where \( \hat{h}_t \) and \( \hat{h}_t \) are defined in (15)-(17); (22) results from the fact that \( h_t \) is a deterministic function of \( (y^{-1}, w) \) given the encoding maps as in (2), and the fact that the conditional density of \( h_t \) given \( (y^{-1}, w) \) is

\begin{equation}
\mathbb{E}[\|h_t\|^2 | y^{-1}, w] \sim \mathcal{CN}(0, \Omega_t)
\end{equation}

(see Lemma 1 below); (23) follows from the identity that \( \text{tr}(AB) \leq \lambda_{\max}(A) \text{tr}(B) \), where \( \lambda_{\max}(A) \) corresponds to the maximum eigenvalue of matrix \( A \), for positive semidefinite \( m \times m \) Hermitian matrices \( A, B \); (24) results from the facts that \( \lambda_{\max}(\hat{h}_t^* \hat{h}_t^T) = \|\hat{h}_t\|^2, \text{tr}((x_t x_t^*)) = \|x_t\|^2, \) and \( \lambda_{\max}(\Omega_t) \leq 1 \) (see Lemma 2 below). At this point, by plugging (24) into (14) we can bound the rate as

\begin{equation}
nR - n\epsilon_n \leq \sum_{t=1}^{n} \mathbb{E} \left[ \log \left( 1 + \mathbb{E}[\|\hat{h}_t\|^2 \cdot \|x_t\|^2 | y^{-1}] + \mathbb{E}[\|x_t\|^2 | y^{-1}] \right) \right]
\end{equation}

Note that both \( \hat{h}_t \) and \( x_t \) are the functions of \( (y^{-1}, w) \), which implies that computing \( \mathbb{E}[\|\hat{h}_t\|^2 \cdot \|x_t\|^2 | y^{-1}] \) in (25) would be challenging. In the following we will bound the value of \( \|\hat{h}_t\|^2 \) given \( y^{-1} \) for the specific case of \( T_c = 2 \). When \( T_c = 2 \), the channel changes every two channel uses. In this case, \( h_t \) can not be estimated from the knowledge of \( (y^{-1}, w) \) if \( t \) is an odd number. If \( t \) is an even number, \( h_t \) can be estimated from the knowledge of \( (y^{-1}, w) \) at a certain degree of precision. From (15) we note that \( h_t \) is the MMSE estimate of \( h_t \) given \( (y^{-1}, w) \). Given the expression of \( h_t \) in (15), now we compute \( \|\hat{h}_t\|^2 \) as

\begin{equation}
\|\hat{h}_t\|^2 \begin{cases} 0 & \text{if } t \in \{2\ell + 1\}_{\ell=0}^{\frac{L-1}{2}} \\
\|x_t\|^2 | y^{-1} \|^2 & \text{if } t \in \{2\ell + 2\}_{\ell=0}^{\frac{L-1}{2}} \end{cases}
\end{equation}
When $t$ is an even number, $\|\hat{h}_t\|^2$ can be bounded as

$$\|\hat{h}_t\|^2 = \frac{\|x_{t-1}\|^2 \cdot |y_{t-1}|^2}{(\|x_{t-1}\|^2 + 1)^2} \leq \frac{|y_{t-1}|^2}{\|x_{t-1}\|^2 + 1} \leq |y_{t-1}|^2, \quad \text{if } t \in \{2\ell + 2\}_{\ell=0}^{L-1} \quad (27)$$

By plugging (26) and (27) into (25) it gives

$$nR - n\epsilon_n$$

$$\leq \sum_{\ell=0}^{L-1} \mathbb{E} \left[ \log \left( 1 + \mathbb{E}[\|\hat{h}_{2\ell+1}\|^2 \cdot \|x_{2\ell+1}\|^2 | y_{2\ell}^2] + \mathbb{E}[\|x_{2\ell+1}\|^2 | y_{2\ell}^2] \right) \right]$$

$$+ \sum_{\ell=0}^{L-1} \mathbb{E} \left[ \log \left( 1 + \mathbb{E}[\|x_{2\ell+1}\|^2 | y_{2\ell}^2] + \mathbb{E}[\|x_{2\ell+2}\|^2 | y_{2\ell+1}^2] \right) \right] \quad (28)$$

$$\leq \sum_{\ell=0}^{L-1} \mathbb{E} \left[ \log \left( 1 + \mathbb{E}[\|x_{2\ell+1}\|^2 | y_{2\ell}^2] \right) \right]$$

$$+ \sum_{\ell=0}^{L-1} \mathbb{E} \left[ \log \left( 1 + \mathbb{E}[\|y_{2\ell+1}\|^2 + 1] \cdot \mathbb{E}[\|x_{2\ell+2}\|^2 | y_{2\ell+1}^2] \right) \right] \quad (29)$$

$$= \sum_{\ell=0}^{L-1} \mathbb{E} \left[ \log \left( 1 + \mathbb{E}[\|x_{2\ell+1}\|^2 | y_{2\ell}^2] \right) \right] + \sum_{\ell=0}^{L-1} \mathbb{E} \left[ \log \left( 1 + (|y_{2\ell+1}|^2 + 1) \cdot \mathbb{E}[\|x_{2\ell+2}\|^2 | y_{2\ell+1}^2] \right) \right] \quad (30)$$

$$\leq \sum_{\ell=0}^{L-1} \log \left( 1 + \mathbb{E}[\|x_{2\ell+1}\|^2] \right) + \sum_{\ell=0}^{L-1} \mathbb{E} \left[ \log \left( (|y_{2\ell+1}|^2 + 1) \cdot (1 + \mathbb{E}[\|x_{2\ell+2}\|^2 | y_{2\ell+1}^2] \right) \right]$$

$$= \sum_{\ell=0}^{L-1} \log \left( 1 + \mathbb{E}[\|x_{2\ell+1}\|^2] \right) + \sum_{\ell=0}^{L-1} \mathbb{E} \left[ \log \left( 1 + \mathbb{E}[\|x_{2\ell+2}\|^2 | y_{2\ell+1}^2] \right) \right] + \sum_{\ell=0}^{L-1} \mathbb{E} \left[ \log \left( 1 + |y_{2\ell+1}|^2 \right) \right] \quad (31)$$

where $n$ is assumed to be an even number and $L \triangleq n/2$ for this case of $T_c = 2$; (28) results from splitting one summation term in (25) into two summation terms; (29) is from (26) and (27). In the first summation term in (30) the outer expectation is moved inside the logarithmic function, which will not reduce the value. (30) also follows from the fact that $1 + a_1(1 + a_2) \leq (1 + a_1)(1 + a_2)$ for $a_1, a_2 \geq 0$. In the second and third summation terms in (31), the outer expectations are moved inside the logarithmic functions, respectively, which again will not reduce the values. In the next step we will bound the expectation term $\mathbb{E}[|y_{2\ell+1}|^2]$ in (31). In this case of $T_c = 2$, the channel changes independently at time $t = 2\ell + 1$, which implies that the input signal $x_{2\ell+1}$ should be independent of the channel $h_{2\ell+1}$. Therefore,

$$\mathbb{E}[|y_{2\ell+1}|^2] = 1 + \mathbb{E}[h_{2\ell+1}^* x_{2\ell+1}^1 | x_{2\ell+1}^1]$$

$$= 1 + \mathbb{E}[\text{tr}(h_{2\ell+1}^* x_{2\ell+1}^1 x_{2\ell+1}^1 h_{2\ell+1}^* h_{2\ell+1}^1)]$$

$$= 1 + \mathbb{E}[\text{tr}(h_{2\ell+1}^* h_{2\ell+1}^* x_{2\ell+1}^1 x_{2\ell+1}^1 h_{2\ell+1}^1)]$$

$$= 1 + \mathbb{E}[\text{tr}(h_{2\ell+1}^* h_{2\ell+1}^* x_{2\ell+1}^1 x_{2\ell+1}^1)]$$

$$= 1 + \mathbb{E}[h_{2\ell+1}^* x_{2\ell+1}^1] \cdot \mathbb{E}[x_{2\ell+1}^1 x_{2\ell+1}^1]$$

$$= 1 + \mathbb{E}[h_{2\ell+1}^* x_{2\ell+1}^1]$$

$$= 1 + \mathbb{E}[\|x_{2\ell+1}\|^2] \quad (32)$$

$$= 1 + \mathbb{E}[\|x_{2\ell+1}\|^2] \quad (33)$$

$$= 1 + \mathbb{E}[\|x_{2\ell+1}\|^2] \quad (34)$$
where (32) follows from the independence between $x_{2\ell+1}$ and $h_{2\ell+1}$; (33) is from that $h_{2\ell+1} \sim \mathcal{CN}(0, I_M)$. By plugging (34) into (31) it then yields
\[
\begin{align*}
2R - n\epsilon_n & \leq \sum_{\ell=0}^{L-1} \log \left(1 + \mathbb{E} \left[\|x_{2\ell+1}\|^2\right]\right) + \sum_{\ell=0}^{L-1} \log \left(1 + \mathbb{E} \left[\|x_{2\ell+2}\|^2\right]\right) + \sum_{\ell=0}^{L-1} \log \left(2 + \mathbb{E} \left[\|x_{2\ell+1}\|^2\right]\right) \\
& = \sum_{t=1}^{n} \log \left(1 + \mathbb{E} \left[\|x_t\|^2\right]\right) + \sum_{\ell=0}^{L-1} \log \left(2 + \mathbb{E} \left[\|x_{2\ell+1}\|^2\right]\right) \\
& \leq \max_{t=1}^{n} \sum_{\ell=0}^{L-1} \log \left(1 + \mathbb{E} \left[\|x_t\|^2\right]\right) + \sum_{\ell=0}^{L-1} \log \left(2 + \mathbb{E} \left[\|x_{2\ell+1}\|^2\right]\right) \\
& \leq \max_{t=1}^{n} \sum_{\ell=0}^{L-1} \log \left(1 + \mathbb{E} \left[\|x_t\|^2\right]\right) + \max_{\ell=0}^{L-1} \sum_{\ell=0}^{L-1} \log \left(2 + \mathbb{E} \left[\|x_{2\ell+1}\|^2\right]\right) \\
& = n \log (1 + P) + \frac{n}{2} \log (2 + 2P) \\
& = 1.5n \log (1 + P) + 0.5n
\end{align*}
\] (35)

where (36) uses the definition that $n = 2L$ for this case of $T_c = 2$; (37) follows from that the right-hand-side (RHS) of (36) is maximized by the optimal power allocation subject to the power constraint in (5); in (38) the maximization is moved into two summation terms, which will not reduce the value; (39) follows from Lemma 3 (see below). Finally, by taking $n \to \infty$ it gives the bound $R \leq 1.5 \log (1 + P) + 0.5$ which completes the proof.

Now we provide some lemmas used in our proofs. The following lemma describes the well-known MMSE estimate result (see, for example, [40, Chapter 15.8]).

**Lemma 1.** [40, Chapter 15.8] Consider two independent random vectors $u \in \mathcal{C}^{M \times 1}$, $z \in \mathcal{C}^{N \times 1}$, with two vectors being complex Gaussian, that is, $z \sim \mathcal{CN}(0, I_N)$ and $u \sim \mathcal{CN}(\hat{u}, \Omega)$ for some fixed $\hat{u}$ and fixed Hermitian positive semidefinite $\Omega$. Let $A \in \mathcal{C}^{N \times M}$ be a fixed matrix and let
\[
y = Au + z
\]
then the conditional density of $u$ given $y$ is
\[
u | y \sim \mathcal{CN}(\hat{u}, \Omega)
\]
where
\[
\begin{align*}
\hat{u} & = \hat{u} + \Omega_1 A^H (A \Omega_1 A^H + I_N)^{-1} (y - A \hat{u}), \\
\Omega & = \Omega - \Omega_1 A^H (A \Omega_1 A^H + I_N)^{-1} A \Omega_1.
\end{align*}
\]

Furthermore, the two random vectors $\hat{u}$ and $v \triangleq u - \hat{u}$ are independent, and the conditional density of $v$ given $y$ is
\[
v | y \sim \mathcal{CN}(0, \Omega).
\]

Note that in Lemma 1, $\hat{u}$ and $v$ are two jointly proper complex Gaussian vectors and the covariance matrix of those two vectors vanishes, which implies that $\hat{u}$ and $v$ are independent — the lack of correlation implies independence for two jointly proper Gaussian vectors [41].

In our setting, we consider the case of $y_t = x_t^H h + z_t$, where $x_t$ a deterministic function of $(y^{t-1}, w)$ given the encoding maps as in (2). When $T_c = 2$ and $t$ is an even number, Lemma 1 reveals that $h_t = \frac{x_t^H y_{t-1}}{\|x_{t-1}\|^2 + 1}$ is the MMSE estimate of $h_t$ given $(y^{t-1}, w)$ and $h_t | (y^{t-1}, w) \sim \mathcal{CN}(\hat{h}_t, \Omega_t)$, where $\hat{h}_t$ and $\Omega_t$ are defined in (15) and (16).


Lemma 2. Consider any vectors $\mathbf{e}_i \in \mathbb{C}^{M \times 1}$, $i \in \mathbb{Z}$, and let

$$
K_t \triangleq I_M - \sum_{i=1}^{t-1} K_i \mathbf{e}_i^* \mathbf{e}_i^T K_i \mathbf{e}_i^* + 1, \quad t = 2, 3, 4, \ldots
$$

and $K_1 \triangleq I_M$, then we have

$$
0 \preceq K_t \preceq I_M, \quad \forall t \in \{1, 2, 3, \ldots\}.
$$

Proof. The proof is shown in Appendix B-A.

Lemma 3. The solution for the following maximization problem

maximize $\sum_{t=1}^{n} \log(1 + s_t)$

subject to $\sum_{t=1}^{n} s_t \leq m$

$s_t \geq 0, \quad t = 1, 2, \ldots, n$

is $s_1^* = s_2^* = \cdots = s_n^* = m/n$, for a positive constant $m > 0$.

Proof. The proof is shown in Appendix B-B.

V. CONVERSE: THE CASE WITH A FOURTH MOMENT INPUT CONSTRAINT

For the channel model defined as in Section II, we will show that the rate is upper bounded by

$$
R \leq \log \left( 1 + \min \{ M + 2, \sqrt{2(T_c + 1)} \} \cdot \kappa \rho \right)
$$

under the fourth moment input constraint in (6). The result reveals that, given a finite coherence $T_c$ and a finite $\kappa \rho$ on the fourth moment input constraint, then increasing the antenna-number $M$ to infinity will not yield an infinite capacity. The result holds for any linear and nonlinear schemes, and holds for any channel training schemes. Similarly to the proof in Section IV for the case with a second moment input constraint, the proof for this case with a fourth moment input constraint is based on MMSE estimation techniques, MIMO techniques, as well as Cauchy-Schwarz inequality.

In the proof we will use Lemmas 1-6 shown in Section IV and in this section (see below). Following the steps (8)-(11), we bound the rate as follows:

$$
nR \leq \sum_{t=1}^{n} \left( h(y_t|y^{t-1}) - \log(\pi e) \right) + n\epsilon_n \quad (42)
$$

$$
\leq \sum_{t=1}^{n} h(y_t) - n \log(\pi e) + n\epsilon_n \quad (43)
$$

where (42) is from (8)-(11), (43) results from the fact that conditioning reduces differential entropy. We proceed to upper bound the differential entropy $h(y_t)$ in (43). Note that the average power of $y_t$ is

$$
\mathbb{E}[|y_t|^2] = \mathbb{E}[|h_t^T \mathbf{x}_t + z_t|^2] = 1 + \mathbb{E}[|h_t^T \mathbf{x}_t|^2].
$$

Again, by using the fact that differential entropy is maximized by a circularly symmetric complex Gaussian distribution with the same average power, we have

$$
h(y_t) \leq \log \left( \pi e (1 + \mathbb{E}[|h_t^T \mathbf{x}_t|^2]) \right). \quad (44)$$
Then, by combining (44) and (43) it yields the following bound on the rate:

\[
R - n \epsilon_n \leq \sum_{t=1}^{n} \log \left( \pi e \left( 1 + \mathbb{E} \left[ |h_t^T x_t|^2 \right] \right) \right) - n \log(\pi e)
\]

\[
= \sum_{t=1}^{n} \log \left( 1 + \mathbb{E} \left[ |h_t^T x_t|^2 \right] \right).
\]

(45)

Let us now focus on the expectation term \( \mathbb{E} \left[ |h_t^T x_t|^2 \right] \) in (45). Similarly to the case with a second moment input constraint, \( x_t \) is a function of \((y^{t-1}, w)\), which implies that \( x_t \) and \( h_t \) can be correlated and consequently computing the value of \( \mathbb{E} \left[ |h_t^T x_t|^2 \right] \) could be challenging in general. In what follows we seek to bound the value of \( \mathbb{E} \left[ |h_t^T x_t|^2 \right] \). We will use the notations of \( \tilde{h}_t \), \( \hat{h}_t \) and \( \Omega_t \) given as

\[
\tilde{h}_t \triangleq \sum_{i=0}^{t-1} \Omega_i x_i^T (y_i - x_i^T h_i) x_i^T \Omega_i x_i^T + 1
\]

for \( t \neq T_c + 1, \)

(46)

\[
\Omega_t \triangleq I_M - \sum_{i=0}^{t-1} \Omega_i x_i^T x_i^T \Omega_i x_i^T + 1
\]

for \( t \neq T_c + 1, \)

(47)

\[
\hat{h}_t \triangleq h_t - \tilde{h}_t \quad \forall t \in \{1, 2, \cdots, n\}
\]

(48)

and \( \hat{h}_{T_c+1} = 0, \) \( \Omega_{T_c+1} = I_M, \) \( \forall t \in \{0, 1, \cdots, L - 1\} \). From Lemma 4 (see below) we note that \( \hat{h}_t \) is the MMSE estimate of \( h_t \) given \((y^{t-1}, w)\). By following the similar steps in (18)-(24), now we bound the value of \( \mathbb{E} \left[ |h_t^T x_t|^2 \right] \) as

\[
\mathbb{E} \left[ |h_t^T x_t|^2 \right] = \mathbb{E} \left[ |h_t^T x_t|^2 \mid y^{t-1}, w \right]
\]

\[
= \mathbb{E} \left[ \mathbb{E} \left[ |h_t^T x_t|^2 \mid y^{t-1}, w, \tilde{h}_t, \hat{h}_t, \Omega_t \right] \right]
\]

\[
= \mathbb{E} \left[ \mathbb{E} \left[ |h_t^T x_t|^2 \mid y^{t-1}, w, \tilde{h}_t, \hat{h}_t, \Omega_t \right] \cdot \tilde{h}_t^T \hat{h}_t + \Omega_t \right]
\]

\[
= \mathbb{E} \left[ \mathbb{E} \left[ |h_t^T x_t|^2 \mid y^{t-1}, w, \tilde{h}_t, \hat{h}_t, \Omega_t \right] \right]
\]

\[
= \mathbb{E} \left[ \lambda_{\max}(\tilde{h}_t^T \hat{h}_t^T) \cdot \mathbb{E} \left[ |x_t^T x_t^H|^2 \right] + \lambda_{\max}(\Omega_t) \cdot \mathbb{E} \left[ |x_t^T x_t^H|^2 \right] \right]
\]

\[
\leq \mathbb{E} \left[ \lambda_{\max}(\tilde{h}_t^T \hat{h}_t^T) \cdot \left\| x_t \right\|^2 + \lambda_{\max}(\Omega_t) \cdot \left\| x_t \right\|^2 \right]
\]

(52)

where (49) stems from the fact that \( x_t \) is a deterministic function of \((y^{t-1}, w)\) given the encoding maps as in (2); in (50) we just replace \( \hat{h}_t \) with \( \tilde{h}_t + \hat{h}_t \), where \( \tilde{h}_t \) and \( \hat{h}_t \) are defined in (46)-(48); (51) results from the fact that \( \tilde{h}_t \) is a deterministic function of \((y^{t-1}, w)\) given the encoding maps as in (2), and the fact that \( \tilde{h}_t \mid (y^{t-1}, w) \sim \mathcal{CN}(0, \Omega_t) \) (cf. Lemma 4); (52) follows from the identity that \( \text{tr}(AB) \leq \lambda_{\max}(A)\text{tr}(B) \) for positive semidefinite \( m \times m \) Hermitian matrices \( A, B \); (53) results from the facts that \( \lambda_{\max}(\tilde{h}_t^T \hat{h}_t^T) = \left\| \tilde{h}_t \right\|^2 \), \( \mathbb{E} \left[ \left\| x_t \right\|^2 \right] = \left\| x_t \right\|^2 \), and \( \lambda_{\max}(\Omega_t) \leq 1 \) (see Lemma 2 in Section IV).
At this point, by combining (53) and (45) we bound the rate as
\[
nR - n\epsilon_n \leq \sum_{t=1}^{n} \log \left( 1 + \mathbb{E}[\|\hat{h}_t\|^2 \cdot \|x_t\|^2] + \mathbb{E}[\|x_t\|^2] \right)
= \sum_{t=1}^{n} \log \left( 1 + \mathbb{E}[\|\hat{h}_t\|^2 + 1] \cdot \|x_t\|^2] \right). \tag{54}
\]

Since \(\hat{h}_t\) and \(x_t\) are two functions of \((y^{t-1}, w)\), it implies that computing \(\mathbb{E}[\|\hat{h}_t\|^2 \cdot \|x_t\|^2] \) or \(\mathbb{E}[\|\hat{h}_t\|^2 + 1] \cdot \|x_t\|^2\) in (54) is also challenging. In order to bound \(\mathbb{E}[\|\hat{h}_t\|^2 + 1] \cdot \|x_t\|^2\) in (54), we use Cauchy-Schwarz inequality, that is, \(\mathbb{E}[ab] \leq \sqrt{\mathbb{E}[a^2]} \cdot \sqrt{\mathbb{E}[b^2]}\) for any two random variables \(a\) and \(b\). Then, it yields
\[
\mathbb{E}[\|\hat{h}_t\|^2 + 1] \cdot \|x_t\|^2] \leq \sqrt{\mathbb{E}[\|\hat{h}_t\|^2 + 1^2]} \cdot \sqrt{\mathbb{E}[\|x_t\|^4]}
\]
which, together with (54), gives the following bound on the rate
\[
nR - n\epsilon_n \leq \sum_{t=1}^{n} \log \left( 1 + \sqrt{\mathbb{E}[\|\hat{h}_t\|^2 + 1]^2]} \cdot \sqrt{\mathbb{E}[\|x_t\|^4]} \right) \tag{55}
\]
\[
\leq \sum_{t=1}^{n} \log \left( 1 + \sqrt{\min\{M^2+4M+1, 2[(t-1) \text{ mod } T_c]^2 + 7[(t-1) \text{ mod } T_c] + 1\}} \cdot \sqrt{\mathbb{E}[\|x_t\|^4]} \right) \tag{56}
\]
\[
\leq \sum_{t=1}^{n} \log \left( 1 + \min\{M + 2, \sqrt{2(T_c + 1)}\} \cdot \sqrt{\mathbb{E}[\|x_t\|^4]} \right) \tag{57}
\]
\[
\leq \max_{\mathbb{E}[\|x_t\|^4] \leq n\epsilon^2 P^2} \sum_{t=1}^{n} \log \left( 1 + \min\{M + 2, \sqrt{2(T_c + 1)}\} \cdot \sqrt{\mathbb{E}[\|x_t\|^4]} \right) \tag{58}
\]
\[
= n \log(1 + \min\{M + 2, \sqrt{2(T_c + 1)}\} \cdot \kappa P) \tag{59}
\]
where (55) results from (54) and Cauchy-Schwarz inequality; (56) follows from the (66) in Lemma 5 (see below); \([t \text{ mod } T_c]\) denotes a modulo operation; (57) stems from that \(M^2+4M+1 < (M + 2)^2\) and that \(2[(t-1) \text{ mod } T_c]^2 + 7[(t-1) \text{ mod } T_c] + 1 \leq 2(Tc - 1)^2 + 7(Tc - 1) + 1 < 2(Tc + 1)^2\); (58) results from maximizing the RHS of (57) under a fourth moment constraint (cf. (6)); where (59) follows from Lemma 6. At this point, as \(n \to \infty\), we have the bound \(R \leq \log(1 + \min\{M + 2, \sqrt{2(T_c + 1)}\} \cdot \kappa P)\) and complete the proof.

Now we provide some lemmas used in our proofs. The following lemma is the extension of Lemma 1

**Lemma 4.** Consider independent random vectors \(\vec{u} \in \mathcal{C}^{M \times 1} \) and \(\vec{z}_t \in \mathcal{C}^{N \times 1}, t = 1, 2, \cdots, T\), with each vector being complex Gaussian, that is, \(\vec{z}_t \sim \mathcal{CN}(0, I_N)\) and \(\vec{u} \sim \mathcal{CN}(\hat{\vec{u}}_1, \Omega_1)\) for some fixed \(\hat{\vec{u}}_1\) and some fixed Hermitian positive semidefinite \(\Omega_1\). Let
\[
\vec{y}_t = \vec{A}_t \vec{u} + \vec{z}_t, t = 1, 2, \cdots, T,
\]
where \(\vec{A}_t \in \mathcal{C}^{N \times M}\) is a deterministic function of \((\vec{y}^{t-1}, w)\); \(w\) is a fixed parameter (or a set of fixed parameters). Then, the conditional density of \(\vec{u}\) given \((\vec{y}^{t-1}, w)\) is
\[
\vec{u} \big| (\vec{y}^{t-1}, w) \sim \mathcal{CN}(\hat{\vec{u}}_t, \Omega_t)
\]
where
\[
\hat{u}_t \triangleq \hat{u}_1 + \sum_{i=1}^{t-1} \Omega_i A_i^h (A_i \Omega_i A_i^h + I_N)^{-1} (y_i - A_i \hat{u}_i) \tag{60}
\]
\[
\Omega_t \triangleq \Omega_1 - \sum_{i=1}^{t-1} \Omega_i A_i^h (A_i \Omega_i A_i^h + I_N)^{-1} A_i \tag{61}
\]
for \(t = 2, 3, \cdots, T\). Furthermore, \(\hat{u}_t\) and \(\nu_t \triangleq u - \hat{u}_t\) are conditionally independent given \((y^{t-2}, w)\) for \(t = 2, 3, \cdots, T\), and the conditional density of \(\nu_t\) given \((y^{t-1}, w)\) is
\[
\nu_t | (y^{t-1}, w) \sim \mathcal{CN}(0, \Omega_t).
\]

Proof. The proof is shown in Appendix B-D.

**Lemma 5.** For \(\hat{h}_t\) defined in \(46\), we have
\[
\mathbb{E}[\|\hat{h}_t\|^2] \leq [(t - 1) \mod T_c]
\]
\[
\mathbb{E}[\|\hat{h}_t\|^2] \leq M
\]
\[
\mathbb{E}[\|\hat{h}_t\|^4] \leq 2[(t - 1) \mod T_c]^2 + 5[(t - 1) \mod T_c]
\]
\[
\mathbb{E}[\|\hat{h}_t\|^4] \leq M^2 + 2M
\]
\[
\mathbb{E}[\|\hat{h}_t\|^2 + 1] \leq \min\{M^2 + 4M + 1, 2[(t - 1) \mod T_c]^2 + 7[t \mod T_c] + 1\}
\]
for \(t = 1, 2, \cdots, n\). \([t \mod T_c]\) denotes a modulo operation.

Proof. The proof is shown in Appendix B-E.

**Lemma 6.** The solution for the following maximization problem
\[
\max \quad \sum_{t=1}^{n} \log(1 + c\sqrt{s_t})
\]
subject to
\[
\sum_{t=1}^{n} s_t \leq m
\]
\[
s_t \geq 0, \quad t = 1, 2, \cdots, n
\]
is \(s^*_1 = s^*_2 = \cdots = s^*_n = m/n\), for positive constants \(m > 0\) and \(c > 0\).

Proof. The proof is shown in Appendix B-C.

**VI. Achievability**

In this section we provide an achievability scheme for the MISO channel with feedback. To this end, the proposed scheme can achieve a rate \(R\) (bits/channel use) that is lower bounded by
\[
R \geq \frac{T_c - T_r}{T_c} \cdot \log \left(1 + \frac{P \cdot \max\{(T_r - 1)/2, 1/2\}}{2 + \frac{1}{P}} - \frac{1}{\max\{T_r, 2\}}\right)
\tag{67}
\]
under the second moment input constraint (cf. \(5\)), where \(T_r = \lceil \frac{\log \max\{\min\{M, T_c\}, 1\}}{\log \max\{4, \min\{M, T_c\}\}} \rceil\). For the case with a fourth moment input constraint (cf. \(6\)), the proposed scheme achieves the similar rate \(R\) with difference being that in the latter case \(P\) is replaced with \(P_0 \triangleq \frac{P}{\sqrt{3}}\). Note that by replacing the input power \(P\) with
$P_o$, the proposed scheme will satisfy the fourth moment input constraint and achieve the declared rate. In the following we will just describe the scheme for the case with a second moment input constraint.

The proposed scheme is a simple scheme that uses no more than $T_c$ number of transmit-antennas. The scheme consists of a downlink training phase and a data transmission phase for each coherence block of the channel. The choice of phase duration is critical to the scheme performance, because with too small duration for training phase there is not enough time for the channel training, while with too large duration for training phase there is not enough time for the data transmission. In this scheme we set the durations of the training phase and data transmission phase as

$$T_\tau = \left\lceil \frac{\min\{M, T_c\}}{\log \max\{4, \min\{M, T_c\}\}} \right\rceil, \quad T_d = T_c - T_\tau$$

respectively, as illustrated in Fig 3. Without loss of generality we focus on the scheme description for the first channel block, corresponding to the time index $t \in \{1, 2, \ldots, T_c\}$. Note that $h_1 = h_2 = \cdots = h_{T_c}$ and $h_\tau = [h_{1,1}, h_{1,2}, \cdots, h_{1,M}]^T$.

### A. Downlink training

The goal of the downlink training phase with feedback is to allow both user and transmitter to learn the channel state information. At time $t$, $t = 1, 2, \ldots, T_\tau$, the downlink training is operated over the $t$th transmit-antenna in order to estimate the channel $h_{1,t}$, where $h_{1,t}$ denotes the channel coefficient between the $t$th transmit antenna and the user during the first channel block. By setting the pilot signal as $x_t = \sqrt{P} [0, 0, \cdots, 0, 1, 0, \cdots, 0]^T$, where the nonzero value is placed at the $t$th element, then the received signal of user at time $t$ is given as

$$y_t = \sqrt{P} h_{1,t} + z_t, \quad t = 1, 2, \ldots, T_\tau.$$  

As a result, the user observes $T_\tau$ channel training outputs that can be written in a vector form:

$$y_\tau = \sqrt{P} h_\tau + z_\tau,$$

where $y_\tau \triangleq [y_1, y_2, \cdots, y_{T_\tau}]^T$, $h_\tau \triangleq [h_{1,1}, h_{1,2}, \cdots, h_{1,T_c}]^T$ and $z_\tau \triangleq [z_1, z_2, \cdots, z_{T_\tau}]^T$.

After receiving the channel training outputs, the user can estimate channel $h_\tau$ with MMSE estimator:

$$\hat{h}_\tau = \frac{\sqrt{P}}{P + 1} y_\tau.$$  

The MMSE estimate $\hat{h}_\tau$ and estimation error $\tilde{h}_\tau \triangleq h_\tau - \hat{h}_\tau$ are two independent complex Gaussian vectors, where $\tilde{h}_\tau \sim \mathcal{C}\mathcal{N}\left(0, \frac{P}{P+1} I\right)$ and $\hat{h}_\tau \sim \mathcal{C}\mathcal{N}\left(0, \frac{1}{P+1} I\right)$.

After MMSE estimation, the user feeds back the value of $\hat{h}_\tau$ to the transmitter over an independent feedback link (the transmitter can also obtain the MMSE estimate $\hat{h}_\tau$ if the user feeds back the channel outputs to the transmitter).
B. Data transmission

After obtaining the channel state information of $\hat{h}_\tau$ (CSIT) the transmitter sends the data information with linear precoding:

$$x_t = \sqrt{P} \frac{\hat{h}_\tau^*}{\|\hat{h}_\tau\|} s_t, \quad t = T_\tau + 1, T_\tau + 2, \cdots, T_c$$

(focusing on the first channel block), where $s_t$ denotes the information symbol with unit average power. The corresponding signal received at the user is given as:

$$y_t = \hat{h}_\tau^* x_t + z_t$$

$$= \sqrt{P} (\hat{h}_\tau + \hat{h}_\tau) \frac{\hat{h}_\tau}{\|\hat{h}_\tau\|} s_t + z_t$$

$$= \sqrt{P} \|\hat{h}_\tau\| s_t + \sqrt{P} \frac{\hat{h}_\tau^*}{\|\hat{h}_\tau\|} \frac{\hat{h}_\tau}{\|\hat{h}_\tau\|} s_t + z_t, \quad t = T_\tau + 1, T_\tau + 2, \cdots, T_c$$  \hspace{1cm} (72)

(again, focusing on the first channel block). The channel input-output relationship in (72) can be further expressed in a vector form:

$$y_d = \sqrt{P} \|\hat{h}_\tau\| s_d + \sqrt{P} \frac{\hat{h}_\tau^*}{\|\hat{h}_\tau\|} \frac{\hat{h}_\tau}{\|\hat{h}_\tau\|} s_d + z_d$$ \hspace{1cm} (73)

where $y_d \triangleq [y_{T_\tau+1}, y_{T_\tau+2}, \cdots, y_{T_c}]^T$, $s_d \triangleq [s_{T_\tau+1}, s_{T_\tau+2}, \cdots, s_{T_c}]^T$ and $z_d \triangleq [z_{T_\tau+1}, z_{T_\tau+2}, \cdots, z_{T_c}]^T$. Note that the conditional distribution of $\frac{\hat{h}_\tau^* \hat{h}_\tau}{\|\hat{h}_\tau\| \|\hat{h}_\tau\|}$ given $\hat{h}_\tau$ is a Gaussian distribution, that is, $\frac{\hat{h}_\tau^* \hat{h}_\tau}{\|\hat{h}_\tau\| \|\hat{h}_\tau\|} \sim \mathcal{CN}(0, \frac{1}{P+1})$.

Rate analysis: We now analyze the achievable rate of the proposed scheme. At first we assume that the input symbol $s_t, \forall t$, is circularly symmetric complex Gaussian distributed, i.e., $s_t \sim \mathcal{CN}(0, 1)$, and is independent of $\hat{h}_\tau$ and $\hat{h}_\tau$. The following proposition provides a lower bound on the achievable ergodic rate.

**Proposition 3.** The achievable ergodic rate for the scheme with Gaussian input, training and feedback, and data transmission as described in Sections VI-A, VI-B is bounded as

$$R \geq \frac{T_c - T_\tau}{T_c} \log \left(1 + \frac{P \cdot \max\{(T_\tau - 1), 1/2\}}{2 + \frac{1}{P}} \cdot \frac{1}{\max\{T_\tau, 2\}}\right)$$

under the second moment input constraint (cf. (5)), where $T_\tau = \lceil \frac{\min\{M, T_c\}}{\log \max\{4, \min\{M, T_c\}\}} \rceil$.

**Proof.** The proof is shown in Appendix C \hfill \Box

VII. Conclusion

In this work we provided capacity bounds for the MISO block fading channel with a noiseless feedback link, under the second and fourth moment input constraints respectively. The result holds for any linear and nonlinear coding strategies, any channel training schemes, any long-term and short-term input constraints. The result reveals that, increasing the transmit-antenna number $M$ to infinity will not yield an infinite capacity, for the case with a finite second-moment input constraint and $T_c = 2$, and for the case with a finite fourth-moment input constraint and a finite coherence $T_c$. In addition to the capacity bounds, this work also provided a characterization on the channel’s beamforming gain for some cases. Specifically, for the case with a finite fourth-moment input constraint and $T_c = M^\alpha$, the result reveals that $\alpha = 1$ is sufficient for achieving a full beamforming gain. When $0 \leq \alpha \leq 1$, the beamforming gain increases linearly with $\alpha$. The result has provided some practical insights for the massive MIMO system operating with FDD mode.
Appendix A
Proofs of Proposition 1

In this section we provide the proof of Proposition 1 for the ideal case of MISO channel with perfect CSIT and CSIR, and with a second moment input constraint. For this ideal case, the channel capacity is characterized in [7] in a closed form

\[
C_{\text{ideal}} = \max_{\bar{P}(\gamma); \gamma \in \mathbb{R}^+} \mathbb{E}_\gamma \left[ \log \left( 1 + \bar{P}(\gamma) \cdot \gamma \right) \right] \tag{74}
\]

where \( \bar{P}(\gamma) \) is the probability density function of \( \gamma \), and the optimal power allocation of \( \bar{P}(\gamma) \) is based on a water-filling algorithm (cf. [7]). We here focus on the asymptotic analysis when the antenna-number \( M \) is large.

For the capacity \( C_{\text{ideal}} \) expressed in (74), it can be upper bounded as:

\[
C_{\text{ideal}} = \max_{\bar{P}(\gamma); \gamma \in \mathbb{R}^+} \mathbb{E}_\gamma \left[ \log \left( 1 + \bar{P}(\gamma) \cdot \gamma \right) \right] \\
\leq \max_{\bar{P}(\gamma); \gamma \in \mathbb{R}^+} \mathbb{E}_\gamma \left[ \log \left( 1 + \bar{P}(\gamma) \right) \right] + \mathbb{E}_\gamma \left[ \log \left( 1 + \gamma \right) \right] \tag{75}
\]

\[
\leq \max_{\bar{P}(\gamma); \gamma \in \mathbb{R}^+} \log \left( 1 + \mathbb{E}_\gamma [\bar{P}(\gamma)] \right) + \log \left( 1 + \mathbb{E}_\gamma [\gamma] \right) \tag{76}
\]

\[
= \log \left( 1 + \bar{P} \right) + \log \left( 1 + M \right) \tag{77}
\]

\[
= \log \left( 1 + PM + P + M \right) \tag{78}
\]

where (75) results from the identity that \( \log(1 + a_1a_2) \leq \log(1 + a_1) + \log(1 + a_2) \) for any \( a_1 \geq 0 \) and \( a_2 \geq 0 \); (76) stems from Jensen’s inequality; (77) follows from the fact that \( \mathbb{E}_\gamma [\gamma] = \mathbb{E}[\| \mathbf{h} \|^2] = M \).

Let us now focus on the lower bound on \( C_{\text{ideal}} \) expressed in (74). Since \( C_{\text{ideal}} \) is determined by the optimal power allocation of \( \bar{P}(\gamma) \) over all possible power allocation strategies. Clearly, setting \( \bar{P}(\gamma) = P \), \( \forall \gamma \) (equal power allocation) gives a lower bound on \( C_{\text{ideal}} \). Therefore,

\[
C_{\text{ideal}} = \max_{\bar{P}(\gamma); \gamma \in \mathbb{R}^+} \mathbb{E}_\gamma \left[ \log \left( 1 + \bar{P}(\gamma) \cdot \gamma \right) \right] \\
\geq \mathbb{E}_\gamma \left[ \log \left( 1 + P \cdot \gamma \right) \right] \tag{79}
\]

\[
\geq \left( \mathbb{E}_\gamma \left[ \log (P \cdot \gamma) \right] \right)^+ \tag{80}
\]

\[
= \left( \mathbb{E}_\gamma \left[ \log (2\gamma) \right] + \log \left( \frac{P}{2} \right) \right)^+ \tag{81}
\]

\[
\geq \left( \log \max \{2M - 2, 1\} + \log \left( \frac{P}{2} \right) \right)^+ \tag{82}
\]

\[
\geq \log \left( 1 + (M - 1)P \right) - 1
\]

where (79) uses a suboptimal power allocation, i.e., \( \bar{P}(\gamma) = P \), \( \forall \gamma \), which will not increase the value of \( C_{\text{ideal}} \); (80) uses the notation of \( (\bullet)^+ = \max\{\bullet, 0\} \); (81) stems from Lemma 7 (see below), that is, \( \mathbb{E}_\gamma [\log (2\gamma)] \geq \log \max \{2M - 2, 1\} \); note that \( 2\gamma = 2\| \mathbf{h} \|^2 \sim \mathcal{X}^2(2M) \); (82) follows from the identity that \( (\log x)^+ \geq \log(1 + x) - 1 \) for a positive \( x \). Therefore, combing the upper bound and lower bound in (78) and (82) leads to the following conclusion:

\[
\log(1 + (M - 1)P) - 1 \leq C_{\text{ideal}} \leq \log \left( 1 + PM + P + M \right)
\]

For a finite \( P \), we have

\[
\lim_{M \to \infty} \frac{\log(1 + (M - 1)P) - 1}{\log(1 + PM)} = 1 \quad \text{and} \quad \lim_{M \to \infty} \frac{\log(1 + PM + P + M)}{\log(1 + PM)} = 1 \tag{83}
\]
which implies that \( \lim_{M \to \infty} \frac{C_{\text{ideal}}}{\log(1 + PM)} = 1 \), i.e., channel capacity \( C_{\text{ideal}} \) tends to \( \log(1 + PM) \) for a large \( M \). At this point, we complete the proof.

**Lemma 7.** If \( u \sim \chi^2(k) \) is a chi-square random variable with \( k \geq 2 \) degrees of freedom, \( k \) is an even number, then

\[
\mathbb{E}[\log u] \geq \log \max\{k - 2, 1\}.
\]

**Proof.** If \( u \) is a chi-square random variable with \( k \geq 2 \) degrees of freedom, its probability density function is given by

\[
f_{\chi^2}(u) = \begin{cases} 
\frac{u^{k/2-1}e^{-u/2}}{2^{k/2}\Gamma(k/2)} & u > 0 \\
0 & \text{else}
\end{cases}
\] (84)

where \( \Gamma(\cdot) \) is a Gamma function (cf. [42]). When \( k \geq 2 \) and \( k \) is an even number, we have

\[
\mathbb{E}[\ln u] = \psi(k/2) + \ln 2
\]

(see 4.352-1 in [43]), where \( \psi(x) \) is the digamma function. Note that \( \psi(1) = -\gamma_o \), where \( \gamma_o \approx 0.57721566 \) is Euler’s constant, and for any integer \( x > 1 \) the digamma function \( \psi(x) \) can be expressed as

\[
\psi(x) = -\gamma_o + \sum_{p=1}^{x-1} \frac{1}{p}
\]

(cf. [44], [45]). Therefore, when \( k > 2 \) and \( k \) is an even number, we have

\[
\mathbb{E}[\ln u] = \psi(k/2) + \ln 2
\]

\[
= -\gamma_o + \sum_{p=1}^{k/2-1} \frac{1}{p} + \ln 2
\]

\[
\geq \ln(k/2 - 1) + \ln 2
\]

\[
= \ln(k - 2)
\]

(85)

where (85) uses the identity of Harmonic series \( \sum_{p=1}^{m} \frac{1}{p} \geq \ln m + \gamma_o \) for any positive natural number \( m \) (cf. [46]). When \( k = 2 \), then

\[
\mathbb{E}[\ln u] = \psi(1) + \ln 2
\]

\[
= -\gamma_o + \ln 2
\]

\[
\geq 0.
\]

(87)

Finally, by combing (86) and (87), we have

\[
\mathbb{E}[\log u] = \frac{1}{\ln 2} \mathbb{E}[\ln u] \geq \frac{1}{\ln 2} \ln(\max\{k - 2, 1\}) = \log(\max\{k - 2, 1\}).
\]

\[
\square
\]

**Appendix B**

**Proofs of Lemmas 2 - 6**

In this section we provide the proofs of Lemmas 2 - 6.
A. Proof of Lemma 2

We will prove that, for any vectors $\mathbf{e}_i \in \mathbb{C}^{M \times 1}$ for $i \in \mathcal{Z}$, and for
\[
K_t \triangleq I_M - \sum_{i=1}^{t-1} \frac{K_i \mathbf{e}_i^* \mathbf{e}_i^\top K_i}{\mathbf{e}_i^\top K_i \mathbf{e}_i^* + 1}, \quad t = 2, 3, 4, \ldots
\]
(88)

and $K_1 \triangleq I_M$, then
\[
0 \leq K_t \leq I_M, \quad \forall t \in \{2, 3, \cdots\}.
\]

From the definition in (88), we have
\[
K_{t+1} = K_t - \frac{K_t \mathbf{e}_t^* \mathbf{e}_t^\top K_t}{\mathbf{e}_t^\top K_t \mathbf{e}_t^* + 1}, \quad t \in \{1, 2, 3, \cdots\}. \tag{89}
\]

One can easily check from (89) that, if $K_t$ is a Hermitian matrix, then $K_{t+1}$ is also a Hermitian matrix for $t \in \{1, 2, 3, \cdots\}$. Since $K_1 \triangleq I_M$ is a Hermitian matrix, then from the above recursive argument it is true that $K_t$ is a Hermitian matrix for $t \in \{1, 2, 3, \cdots\}$.

In the second step, we will prove that if the Hermitian matrix $K_t$ is positive semidefinite, then the Hermitian matrix $K_{t+1}$ is also positive semidefinite for $t \in \{1, 2, 3, \cdots\}$. Specifically, if the Hermitian matrix $K_t$ is positive semidefinite, $t \in \{1, 2, 3, \cdots\}$, then for any vector $\mathbf{x} \in \mathbb{C}^{M \times 1}$ we have
\[
\mathbf{x}^* K_{t+1} \mathbf{x} = \mathbf{x}^* (K_t - \frac{K_t \mathbf{e}_t^* \mathbf{e}_t^\top K_t}{\mathbf{e}_t^\top K_t \mathbf{e}_t^* + 1}) \mathbf{x}
\]
\[
= \mathbf{x}^* K_t \mathbf{x} - \frac{\mathbf{x}^* K_t \mathbf{e}_t^* \mathbf{e}_t^\top K_t \mathbf{x}}{\mathbf{e}_t^\top K_t \mathbf{e}_t^* + 1}
\]
\[
= b^* b - \frac{|b^* c|^2}{c^* c + 1}
\]
\[
= \frac{||b||^2 + ||b||^2 ||c||^2 - |b^* c|^2}{||c||^2 + 1}
\]
\[
\geq \frac{||b||^2 + ||b||^2 ||c||^2 - ||b||^2 ||c||^2}{||c||^2 + 1}
\]
\[
\geq 0 \tag{90}
\]
\[
\geq 0 \tag{91}
\]
(91)

where (90) is from the definition in (89); the Hermitian positive semidefinite matrix $K_t$ is decomposed as $K_t \triangleq U \Lambda U^\dagger = U \Lambda^{1/2} U^\dagger U \Lambda^{1/2} U^\dagger$ using singular value decomposition method, where $U$ and $\Lambda$ are the unitary matrix and diagonal matrix respectively, $b \triangleq U \Lambda^{1/2} U^\dagger \mathbf{x}$ and $c \triangleq U \Lambda^{1/2} U^\dagger \mathbf{e}^*$; where (91) results from Cauchy-Schwarz inequality, i.e., $|b^* c|^2 \leq ||b||^2 ||c||^2$. Since the Hermitian matrix $K_1$ is positive semidefinite, then from the above recursive argument it is true that the Hermitian matrix $K_t$ is positive semidefinite, $t \in \{1, 2, 3, \cdots\}$.

From the above steps we have proved that the matrix $K_t$ is Hermitian positive semidefinite, $t \in \{1, 2, 3, \cdots\}$, which means that
\[
0 \leq K_t, \quad \forall t \in \{1, 2, 3, \cdots\}
\]
(92)

In the next step we will prove that
\[
K_t \leq I_M, \quad \forall t \in \{1, 2, 3, \cdots\}
\]
(93)

From the definition in (88), we have
\[
I_M - K_t \triangleq \sum_{i=1}^{t-1} \frac{K_i \mathbf{e}_i^* \mathbf{e}_i^\top K_i}{\mathbf{e}_i^\top K_i \mathbf{e}_i^* + 1}, \quad t = 2, 3, 4, \cdots
\]
Since matrix $K_t$ is Hermitian positive semidefinite, $t \in \{1, 2, 3, \cdots \}$, it holds true that
\[
\frac{K_t e_t^* e_t^T K_t}{e_t^T K_t e_t^* + 1} \succeq 0, \quad t = 1, 2, 3, \cdots
\]
(94)
because for any vector $x \in C^{M \times 1}$ we have $x^H \left( \frac{K_t e_t^* e_t^T K_t}{e_t^T K_t e_t^* + 1} \right) x = \frac{|x^H K_t e_t^*|^2}{e_t^T K_t e_t^* + 1} \succeq 0$. Then combing (94) and (93) it gives
\[
I_M - K_t \triangleq \sum_{i=1}^{t-1} K_i e_i^* e_i^T K_i e_i^T e_i^* + 1 \succeq 0, \quad t = 2, 3, 4, \cdots
\]
which implies that
\[
I_M \succeq K_t, \quad t = 2, 3, 4, \cdots
\]
At this point, we completes the proof.

B. Proof of Lemma 3

This subsection shows that the solution for the following minimization problem
\[
\min_{s_1, s_2, \cdots, s_n} - \sum_{t=1}^{n} \log(1 + s_t)
\]
subject to
\[
\sum_{t=1}^{n} s_t \leq m
\]
\[
-s_t \leq 0, \quad t = 1, 2, \cdots, n
\]
is $s_1^* = s_2^* = \cdots = s_n^* = m/n$, for a positive constant $m > 0$.

Note that the Lagrangian of this convex optimization problem is given as
\[
L(s_1, s_2, \cdots, s_n, \beta, \mu_1, \mu_2, \cdots, \mu_n) = - \sum_{t=1}^{n} \log(1 + s_t) + \beta \left( \sum_{t=1}^{n} s_t - m \right) + \sum_{t=1}^{n} \mu_t \cdot (-s_t),
\]
where $\beta, \mu_1, \mu_2, \cdots, \mu_n$ are Lagrangian parameters. By solving the following Karush-Kuhn-Tucker (KKT) conditions:
\[
\frac{\partial L(s_1, s_2, \cdots, s_n, \beta, \mu_1, \mu_2, \cdots, \mu_n)}{\partial s_t} = - \frac{1}{(1 + s_t) \cdot \ln 2} + \beta - \mu_t = 0, \quad t = 1, 2, \cdots, n,
\]
(95)
\[
\beta \left( \sum_{t=1}^{n} s_t - m \right) = 0,
\]
(96)
\[
-\mu_t s_t = 0, \quad t = 1, 2, \cdots, n
\]
(97)
\[
\sum_{t=1}^{n} s_t - m \leq 0,
\]
\[
-s_t \leq 0, \quad t = 1, 2, \cdots, n
\]
(99)
\[
\beta \geq 0, \quad \mu_t \geq 0, \quad t = 1, 2, \cdots, n
\]
(100)
then it gives the optimal solution of $s_1^* = s_2^* = \cdots = s_n^* = m/n$. Note that $s_1^* = s_2^* = \cdots = s_n^* = m/n$, $\beta^* = \frac{n}{(n+m) \cdot \ln 2}$, and $\mu_1^* = \mu_2^* = \cdots = \mu_n^* = 0$ satisfy the above KKT conditions. At this point we completes the proof. Note that one could also use the symmetric optimization method to prove the result of this lemma.
C. Proof of Lemma 6

This subsection shows that the solution for the following minimization problem

\[
\begin{align}
\text{minimize} & \quad -\sum_{t=1}^{n} \log(1 + c\sqrt{s_t}) \\
\text{subject to} & \quad \sum_{t=1}^{n} s_t \leq m \\
& \quad -s_t \leq 0, \quad t = 1, 2, \cdots, n
\end{align}
\]

is \( s_1^* = s_2^* = \cdots = s_n^* = m/n \), for positive constants \( m > 0 \) and \( c > 0 \).

The proof is similar to that of Lemma 3. Note that the Lagrangian of this convex optimization problem is given as

\[
L(s_1, s_2, \cdots, s_n, \beta, \mu_1, \mu_2, \cdots, \mu_n) = -\sum_{t=1}^{n} \log(1 + c\sqrt{s_t}) + \beta \left( \sum_{t=1}^{n} s_t - m \right) + \sum_{t=1}^{n} \mu_t \cdot (-s_t),
\]

where \( \beta, \mu_1, \mu_2, \cdots, \mu_n \) are Lagrangian parameters. By solving the following KKT conditions:

\[
\frac{\partial L(s_1, s_2, \cdots, s_n, \beta, \mu_1, \mu_2, \cdots, \mu_n)}{\partial s_t} = -\frac{c}{2} \cdot \frac{s_t^{-1/2}}{(1 + c\sqrt{s_t}) \cdot \ln 2} + \beta - \mu_t = 0, \quad t = 1, 2, \cdots, n, \quad (101)
\]

\[
\beta \left( \sum_{t=1}^{n} s_t - m \right) = 0, \quad (102)
\]

\[
-\mu_t s_t = 0, \quad t = 1, 2, \cdots, n \quad (103)
\]

\[
\sum_{t=1}^{n} s_t - m \leq 0, \quad (104)
\]

\[
-\mu_t s_t = 0, \quad t = 1, 2, \cdots, n \quad (105)
\]

\[
\beta \geq 0, \quad \mu_t \geq 0, \quad t = 1, 2, \cdots, n \quad (106)
\]

then it gives the optimal solution of \( s_1^* = s_2^* = \cdots = s_n^* = m/n \). Note that \( s_1^* = s_2^* = \cdots = s_n^* = m/n \), \( \beta^* = \left( \frac{m^2 + \frac{1}{2} \sqrt{m^2}}{2m^2} \right)^2 \ln 2 \), and \( \mu_1^* = \mu_2^* = \cdots = \mu_n^* = 0 \) satisfy the above KKT conditions.

D. Proof of Lemma 7

We will prove the following statement. Consider independent random vectors \( \hat{u} \in \mathbb{C}^{M \times 1} \) and \( z_t \in \mathbb{C}^{N \times 1}, \) \( t = 1, 2, \cdots, T, \) with each vector being complex Gaussian, that is, \( z_t \sim \mathcal{CN}(0, I_N) \) and \( u \sim \mathcal{CN}(\hat{u}, \Omega) \) for some fixed \( \hat{u} \) and Hermitian positive semidefinite \( \Omega. \) Let

\[
y_t = A_t \hat{u} + z_t, \quad t = 1, 2, \cdots, T,
\]

where \( A_t \in \mathbb{C}^{N \times M} \) is a deterministic function of \( (y^{t-1}, w); \) \( w \) is a fixed parameter (or a set of fixed parameters). \( y^t \) denotes an empty term if \( \ell \leq 0. \) Then, the conditional density of \( u \) given \( (\hat{y}^{t-1}, w) \) is

\[
\begin{align}
\Pr\{ u \mid (\hat{y}^{t-1}, w) \} & \sim \mathcal{CN}(\hat{u}, \Omega_t) \\
\end{align}
\]

where

\[
\begin{align}
\hat{u}_t & \triangleq \hat{u}_{t-1} + \hat{u}_{t-1,t}, \quad \Omega_t \triangleq \Omega_{t-1} - \Omega_{t-1,t}
\end{align}
\]

and

\[
\begin{align}
\hat{u}_{t-1,t} & \triangleq \Omega_{t-1} A^h_{t-1} (A_{t-1} \Omega_{t-1} A^h_{t-1} + I_N)^{-1}(y_{t-1} - A_{t-1} \hat{u}_{t-1}) \\
\Omega_{t-1,t} & \triangleq \Omega_{t-1} A^h_{t-1} (A_{t-1} \Omega_{t-1} A^h_{t-1} + I_N)^{-1} A_{t-1} \Omega_{t-1}
\end{align}
\]
for \( t = 2, 3, \ldots, T \). Let
\[
\mathbf{v}_t \triangleq \mathbf{u} - \hat{\mathbf{u}}_t
\]
then \( \hat{\mathbf{u}}_t \) and \( \mathbf{v}_t \) are conditionally independent given \((\mathbf{y}^{t-2}, \mathbf{w})\), and the conditional density of \( \mathbf{v}_t \) given \((\mathbf{y}^{t-1}, \mathbf{w})\) is
\[
\mathbf{v}_t | (\mathbf{y}^{t-1}, \mathbf{w}) \sim \mathcal{CN}(0, \Omega_t)
\]
for \( t = 2, 3, \ldots, T \). This lemma is the extension of the well-known MMSE estimate result that is expressed in Lemma 1.

Now we proceed with the proof. We first consider the simple case of \( t = 2 \). From Lemma 1 we conclude that the conditional density of \( \mathbf{u} \) given \((\mathbf{y}_1, \mathbf{w})\) is
\[
\mathbf{u} | (\mathbf{y}_1, \mathbf{w}) \sim \mathcal{CN}(\hat{\mathbf{u}}_2, \Omega_2)
\]
(108)
where
\[
\hat{\mathbf{u}}_2 = \hat{\mathbf{u}}_1 + \Omega_1 \mathbf{A}_1^h (\mathbf{A}_1 \Omega_1 \mathbf{A}_1^h + I_N)^{-1} (\mathbf{y}_1 - \mathbf{A}_1 \hat{\mathbf{u}}_1)
\]
(109)
and
\[
\Omega_2 = \Omega_1 - \Omega_1 \mathbf{A}_1^h (\mathbf{A}_1 \Omega_1 \mathbf{A}_1^h + I_N)^{-1} \mathbf{A}_1 \Omega_1
\]
(110)
where \( \mathbf{A}_1 \) is a deterministic function of \( \mathbf{w} \) by definition. It follows from Lemma 1 that \( \hat{\mathbf{u}}_2 \) and \( \mathbf{v}_2 \triangleq \mathbf{u} - \hat{\mathbf{u}}_2 \) are independent; the conditional density of \( \mathbf{v}_2 \) given \((\mathbf{y}_1, \mathbf{w})\) is
\[
\mathbf{v}_2 | (\mathbf{y}_1, \mathbf{w}) \sim \mathcal{CN}(0, \Omega_2)
\]
We then consider the case of \( t = 3 \) \((T \geq 3)\). By using the result in (108), that is, \( \mathbf{u} | (\mathbf{y}_1, \mathbf{w}) \sim \mathcal{CN}(\hat{\mathbf{u}}_2, \Omega_2) \), it yields the following conclusion:
\[
\begin{bmatrix}
\hat{\mathbf{z}}_2 \\
\hat{\mathbf{u}}_2
\end{bmatrix} | (\mathbf{y}_1, \mathbf{w}) \sim \mathcal{CN}
\left(
\begin{bmatrix}
0 \\
\hat{\mathbf{u}}_2
\end{bmatrix},
\begin{bmatrix}
I_N & 0_{N \times M} \\
0_{M \times N} & \Omega_2
\end{bmatrix}
\right)
\]
(111)
Let us now look at the following vector
\[
\begin{bmatrix}
\mathbf{y}_2 \\
\mathbf{u}
\end{bmatrix} = \begin{bmatrix}
I_N & \mathbf{A}_2 \\
0_{M \times N} & I_M
\end{bmatrix}
\begin{bmatrix}
\hat{\mathbf{z}}_2 \\
\hat{\mathbf{u}}_2
\end{bmatrix}
\]
(112)
where \( \mathbf{A}_2 \) is a deterministic function of \((\mathbf{y}_1, \mathbf{w})\). It is well known that the affine transformation of a complex proper Gaussian vector also yields a complex proper Gaussian vector, that is, if \( \mathbf{v} \in \mathcal{C}^{q \times 1} \sim \mathcal{CN}(\mu, \mathbf{Q}) \), then it holds true that \( \mathbf{B} \mathbf{v} \sim \mathcal{CN}(\mathbf{B} \mu, \mathbf{B} \mathbf{Q} \mathbf{B}^H) \) for fixed \( \mu \in \mathcal{C}^{q \times 1} \), \( \mathbf{B} \in \mathcal{C}^{p \times q} \) and \( \mathbf{Q} \in \mathcal{C}^{q \times q} \) (see, e.g., [6], [41]). Therefore, by combining (111) and (112) it gives
\[
\begin{bmatrix}
\mathbf{y}_2 \\
\mathbf{u}
\end{bmatrix} | (\mathbf{y}_1, \mathbf{w}) \sim \mathcal{CN}
\left(
\begin{bmatrix}
\mathbf{A}_2 \hat{\mathbf{u}}_2 \\
\hat{\mathbf{u}}_2
\end{bmatrix},
\begin{bmatrix}
\mathbf{K}_{1,1} & \mathbf{K}_{1,2} \\
\mathbf{K}_{2,1} & \mathbf{K}_{2,2}
\end{bmatrix}
\right)
\]
(113)
where
\[
\begin{align*}
\mathbf{K}_{2,2} &= \Omega_2, \\
\mathbf{K}_{2,1} &= \Omega_2 \mathbf{A}_2^H, \\
\mathbf{K}_{1,2} &= \mathbf{A}_2 \Omega_2, \\
\mathbf{K}_{1,1} &= \mathbf{A}_2 \Omega_2 \mathbf{A}_2^H + I_N.
\end{align*}
\]
(114)
Let us consider a new vector obtained from the following affine transformation:
\[
\begin{bmatrix}
\mathbf{u} - \mathbf{K}_{2,1} \mathbf{K}_{1,1}^{-1} \mathbf{y}_2
\end{bmatrix} = \mathbf{B} \begin{bmatrix}
\mathbf{y}_2 \\
\mathbf{u}
\end{bmatrix}
\]
where
\[
\mathbf{B} = \begin{bmatrix}
I_N & 0_{N \times M} \\
-K_{2,1} \mathbf{K}_{1,1}^{-1} & I_M
\end{bmatrix}
\]
As mentioned, affine transformation of a complex proper Gaussian vector also yields a complex proper Gaussian vector. Therefore,

\[
\begin{bmatrix}
  \frac{y_2}{u - K_{2,1}K_{1,1}^{-1}y_2}
\end{bmatrix} \sim \mathcal{CN}\left(\begin{bmatrix}
  A_2 \hat{u}_2 \\
  K_{1,1}^{-1}
\end{bmatrix}, \begin{bmatrix}
  K_{1,1} & 0_{N \times M} \\
  0_{M \times N} & K_{2,2} - K_{2,1}K_{1,1}^{-1}K_{1,2}
\end{bmatrix}\right).
\]

(115)

The result in (115) implies that

\[
u - K_{2,1}K_{1,1}^{-1}y_2 \mid (y_1, w) \sim \mathcal{CN}\left(\hat{u}_2 - K_{2,1}K_{1,1}^{-1}A_2 \hat{u}_2, K_{2,2} - K_{2,1}K_{1,1}^{-1}K_{1,2}\right).
\]

(116)

The result in (115) also implies that the two vectors \(y_2\) and \(\nu - K_{2,1}K_{1,1}^{-1}y_2\) are conditionally independent given \((y_1, w)\) because their conditional cross-covariance vanishes. Based on this independence and (116), it gives

\[
u - K_{2,1}K_{1,1}^{-1}y_2 \mid (y_2, y_1, w) \sim \mathcal{CN}\left(\hat{u}_2 - K_{2,1}K_{1,1}^{-1}A_2 \hat{u}_2, K_{2,2} - K_{2,1}K_{1,1}^{-1}K_{1,2}\right)
\]

(117)

and

\[
u - K_{2,1}K_{1,1}^{-1}y_2 + K_{2,1}K_{1,1}^{-1}y_2 \mid (y_2, y_1, w) \sim \mathcal{CN}\left(\hat{u}_2 + K_{2,1}K_{1,1}^{-1}(y_2 - A_2 \hat{u}_2), K_{2,2} - K_{2,1}K_{1,1}^{-1}K_{1,2}\right)
\]

(118)

Finally, plugging (114) into (118) leads to the following conclusion:

\[
u \mid (y_2, y_1, w) \sim \mathcal{CN}(\hat{u}_3, \Omega_3)
\]

where \(\hat{u}_3 = \hat{u}_2 + \Omega_2 A_2^\dagger (A_2 \Omega_2 A_2^\dagger + I_N)^{-1} (y_2 - A_2 \hat{u}_2)\) and \(\Omega_3 = \Omega_2 - \Omega_2 A_2^\dagger (A_2 \Omega_2 A_2^\dagger + I_N)^{-1} A_2 \Omega_2\), as defined in (107). Let \(v_3 \doteq \nu - \hat{u}_3\). Then, the conditional density of \(v_3\) given \((y_2, y_1)\) is

\[
v_3 \mid (y_2, y_1, w) \sim \mathcal{CN}(0, \Omega_3).
\]

Note that \(\hat{u}_3\) is conditionally independent of \(v_3\) given \((y_1, w)\), since

\[
\text{Cov}(\hat{u}_3, v_3 \mid y_1, w) \doteq \mathbb{E}\left[(\hat{u}_3 - \mathbb{E}[\hat{u}_3 \mid y_1, w])(v_3 - \mathbb{E}[v_3 \mid y_1, w])^H \mid y_1, w\right] = 0
\]

and the vectors \(\hat{u}_3\) and \(v_3\) are two jointly proper Gaussian vectors given \((y_1, w)\). The lack of correlation implies independence for two jointly proper Gaussian vectors.

For the general case when \(t = 4, 5, \cdots, T\), the proof is similar to the previous case. At this point we complete the proof.

E. Proof of Lemma 5

For \(\hat{h}_t\) defined as in (46) and (47), we will prove the following bounds

\[
\mathbb{E}\left[\|\hat{h}_t\|^2\right] \leq [(t - 1) \mod T_c]
\]

(119)

\[
\mathbb{E}\left[\|\hat{h}_t\|^2\right] \leq M
\]

(120)

\[
\mathbb{E}\left[\|\hat{h}_t\|^4\right] \leq 2[(t - 1) \mod T_c]^2 + 5[(t - 1) \mod T_c]
\]

(121)

\[
\mathbb{E}\left[\|\hat{h}_t\|^4\right] \leq M^2 + 2M
\]

(122)

\[
\mathbb{E}\left[\|\hat{h}_t\|^2 + 1\right]^2 \leq \min\{M^2 + 4M + 1, 2[(t - 1) \mod T_c]^2 + 7[(t - 1) \mod T_c] + 1\}
\]

(123)

for \(t = 1, 2, \cdots, n\).
Let us provide some lemmas that will be used in our proof. At first we rewrite the definitions of $\hat{h}_t$ and $\Omega_t$ in (46) and (47) as

$$\hat{h}_{t+1} \triangleq \hat{h}_t + \hat{h}_{t,t+1}, \quad \hat{h}_{t,t+1} \triangleq \Omega_t \hat{x}_t^* (y_t - \hat{x}_t^* \hat{h}_t) / \hat{x}_t^* \Omega_t \hat{x}_t^* + 1$$

for $t + 1 \neq \ell T_c + 1$ (124)

$$\Omega_{t+1} \triangleq \Omega_t - \Omega_t \hat{x}_t^* \hat{x}_t^* \Omega_t / \hat{x}_t^* \Omega_t \hat{x}_t^* + 1$$

for $t + 1 \neq \ell T_c + 1$ (125)

and $\hat{h}_{\ell T_c + 1} = 0$, $\Omega_{\ell T_c + 1} = I_M, \forall \ell \in \{0, 1, \ldots, L - 1\}$.

**Lemma 8.** For $\hat{h}_{t+1}$ and $\hat{h}_{t,t+1}$ defined as in (124) and (125), $t \in \{1, 2, \ldots, T_c - 1\}$, we have

$$\mathbb{E}[||\hat{h}_1 + \hat{h}_{1,2} + \hat{h}_{2,3} + \cdots + \hat{h}_{t,t+1}||^2] = \mathbb{E}[||\hat{h}_1||^2 + ||\hat{h}_{1,2}||^2 + ||\hat{h}_{2,3}||^2 + \cdots + ||\hat{h}_{t,t+1}||^2].$$

**Proof.** See Appendix B-F.

**Lemma 9.** For $\hat{h}_{t,t+1}$ defined as in (124) and (125), $t \in \{1, 2, \ldots, T_c - 1\}$, the following bounds hold

$$\mathbb{E}[||\hat{h}_{t,t+1}||^2 | w, y^{t-1}] \leq 1, \quad (126)$$

$$\mathbb{E}[||\hat{h}_{t,t+1}||^2] \leq 1. \quad (127)$$

**Proof.** See Appendix B-G.

**Lemma 10.** For $\hat{h}_{t,t+1}$ defined as in (124) and (125), $t \in \{1, 2, \ldots, T_c - 1\}$, the following inequalities hold

$$\mathbb{E}[||\hat{h}_{t,t+1}||^4 | w, y^{t-1}] \leq 3, \quad (128)$$

$$\mathbb{E}[||\hat{h}_{t,t+1}||^4] \leq 3. \quad (129)$$

**Proof.** See Appendix B-H.

**Lemma 11.** [47, Theorem 6] Let $u \in C^{M \times 1} \sim C\mathcal{N}(\emptyset, \Omega)$. For a fixed Hermitian matrix $A \in C^{M \times M}$, then

$$\mathbb{E}[(u^H A u)^2] = 2 \text{tr}(A \Omega A \Omega) + (\text{tr}(A \Omega))^2.$$  

**Lemma 12.** For $\hat{h}_{t+1} = \hat{h}_t + \hat{h}_{t,t+1}$ defined in (124) and (125), $t \in \{1, 2, \ldots, T_c - 1\}$, we have

$$\mathbb{E}[||\hat{h}_{t+1}||^4] \leq \mathbb{E}[||\hat{h}_t||^4] + 4 \cdot \mathbb{E}[||\hat{h}_t||^2] + 3.$$  

**Proof.** See Appendix B-I.

Now we are ready to prove Lemma 5. At first we focus on the case of $t \in \{1, 2, \ldots, T_c\}$ and prove (62) in Lemma 5 (or equivalently, (119)):

$$\mathbb{E}[||\hat{h}_t||^2] = \mathbb{E}[||\hat{h}_{1,2}||^2 + ||\hat{h}_{2,3}||^2 + \cdots + ||\hat{h}_{t-1,t}||^2] \leq 1 + \cdots + 1$$

$$= t - 1.$$  

where (130) results from Lemma 8, (131) follows from Lemma 9. For the general case of $t \in \{1, 2, \ldots, n\}$, we note that $\hat{h}_t$ is a function of $(\hat{x}_t^{\ell-1}, \hat{y}_t^{\ell-1})^{\ell-1} \in \{T_c \mid \hat{c}_t \in \{T_c \mid \hat{c}_t \}}$ (cf. (46), (47)), where $y^{\ell-1}_t$ corresponds to
the channel outputs (up to time \(t - 1\)) within the current channel block associated with time \(t\). We also note that the previous results in (132) only depends on the number of channel outputs within the current channel block. Therefore, one can easily follow the previous steps and show that

\[
\mathbb{E}[\|\hat{h}_t\|^2] = \sum_{i=T_c, \frac{t-1}{T_C}+1}^{t-1} \mathbb{E}[\|\hat{h}_{i,i+1}\|^2]
\leq 1 + 1 + \cdots + 1
= [(t-1) \mod T_c], \quad t \in \{1, 2, \cdots, n\}
\]

(133)

where \(\hat{h}_{i,i+1} \triangleq \frac{\Omega_i g^T(y_i - g^T \bar{h}_i)\hat{h}_i}{g^T \Omega_i g^T \hat{h}_i \bar{h}_i} \) for \(T_c \leq \frac{t-1}{T_C} + 1 \leq t - 1\); where \(\hat{h}_i\) and \(\Omega_i\) are defined in (46) and (47); (133) is again from Lemma 9.

We now prove (63) in Lemma 5 (or (120)):

\[
\mathbb{E}[\|\hat{h}_t\|^2] \leq \mathbb{E}[\|\hat{h}_t\|^2] + \mathbb{E}[\|\hat{h}_t\|^2]
= \mathbb{E}[\|\hat{h}_t + \hat{h}_t\|^2] - \mathbb{E}[\hat{h}_t^H \hat{h}_t] - \mathbb{E}[\hat{h}_t^H \hat{h}_t]
= \mathbb{E}[\|\hat{h}_t\|^2]
= M
\]

(135)

(136)

(137)

where \(\hat{h}_t \triangleq \hat{h}_t - \hat{h}_t\); (135) is from the identity that \(\|a + b\|^2 = \|a\|^2 + \|b\|^2 + a^H b + b^H a\) for any two vectors \(a, b \in \mathbb{C}^{M \times 1}\); (136) follows from the fact that \(\mathbb{E}[\hat{h}_t^H \hat{h}_t] = \mathbb{E}[\mathbb{E}[\hat{h}_t^H \hat{h}_t|w, y^{t-1}]] = \mathbb{E}[0] = 0\) by using the result that \(\hat{h}_t|\{w, y^{t-1}\} \sim \mathcal{CN}(0, \Omega_i)\) and that \(\hat{h}_t\) is deterministic given \(\{w, y^{t-1}\}\); similarly, \(\mathbb{E}[\hat{h}_t^H \hat{h}_t] = 0\); (137) is from the assumption that \(\hat{h}_t \sim \mathcal{CN}(0, I_M)\).

We focus on the case of \(t \in \{1, 2, \cdots, T_c\}\) and prove (64) in Lemma 5 (or (121)):

\[
\mathbb{E}[\|\hat{h}_t\|^4] \leq \mathbb{E}[\|\hat{h}_{t-1}\|^4] + 4 \cdot \mathbb{E}[\|\hat{h}_{t-1}\|^2] + 3
\leq \mathbb{E}[\|\hat{h}_{t-1}\|^4] + 4(t-1) + 3
\leq \mathbb{E}[\|\hat{h}_1\|^4] + 4(1-1) + 4(2-1) + \cdots + 4(t-1) + 3(t-1)
= 2(t-1)^2 + 5(t-1)
\]

(138)

(139)

(140)

(141)

where (138) follows from Lemma 12; (139) is from the result in (132); (140) follows by repeating the steps of (138) and (139); (141) uses the definition that \(\hat{h}_t = 0\). For the general case of \(t \in \{1, 2, \cdots, n\}\), we again note that \(\hat{h}_t\) is a function of \(\{x^{t-1}_{T_c, \frac{t-1}{T_c}+1}, y^{t-1}_{T_c, \frac{t-1}{T_c}+1}\}\). Therefore, one can easily follow the previous steps and show that

\[
\mathbb{E}[\|\hat{h}_t\|^4] \leq \mathbb{E}[\|\hat{h}_{t, \frac{t-1}{T_C}+1}\|^4] + \sum_{k=0}^{(t-1) \mod T_c} 4k + 3([t \mod T_c] - 1)
= 2([t-1] \mod T_c)^2 + 5([t-1] \mod T_c)
\]

(142)

(143)

where (143) uses the definition of \(\hat{h}_{T_c, \frac{t-1}{T_C}+1} = 0\).

We now prove (65) in Lemma 5 (or (122)):

\[
\mathbb{E}[\|\hat{h}_t\|^4] \leq \mathbb{E}[\|\hat{h}_t\|^4] + \mathbb{E}[\|\hat{h}_t\|^4] + \mathbb{E}[2\|\hat{h}_t\|^2\|\hat{h}_t\|^2] + \mathbb{E}[4\Re^2(\hat{h}_t^H \hat{h}_t)] + \mathbb{E}[4(\|\hat{h}_t\|^2 + \|\hat{h}_t\|^2) \cdot \Re(\hat{h}_t^H \hat{h}_t)]
= \mathbb{E}[\|\hat{h}_t + \hat{h}_t\|^4]
= \mathbb{E}[\|\hat{h}_t\|^4]
= M^2 + 2M
\]

(144)

(145)

(146)
where $\hat{h}_t \triangleq h_t - \hat{h}_t$; (145) stems from the identity that
\[ \|a + b\|^4 = \|a\|^4 + \|b\|^4 + 2\|a\|^2\|b\|^2 + 4\text{Re}(a^*b) + 4(\|a\|^2 + \|b\|^2) \cdot \text{Re}(a^*b) \]
for any two vectors $a, b \in \mathbb{C}^M$, where $\text{Re}(\cdot)$ denotes the real part of the argument; (146) follows from Lemma 11; (144) results from the fact that $\hat{h}_t$ is deterministic given $(w, y^{t-1})$; (147) results from the fact that $\mathbb{E}[\text{Re}(a^*b)] = \text{Re}(\mathbb{E}[a^*b]) = 0$ and $\mathbb{E}[\|b\|^2 \cdot \text{Re}(a^*b)] = \text{Re}(\mathbb{E}[a^*b \cdot \|b\|^2]) = 0$ for a fixed vector $a$ and a Gaussian vector $b \sim \mathcal{CN}(0, K)$; note that the odd-order moments of a complex proper Gaussian vector are zeros (see, e.g., [48]).

Finally, (66) in Lemma 5 follows from (62), (63), (64), and (65). Specifically, combing (63) and (65) gives $\mathbb{E}[\|\hat{h}_t\|^2 + 1]^2 = \mathbb{E}[\|\hat{h}_t\|^4 + 2\|\hat{h}_t\|^2 + 1 \leq M^2 + 4M + 1$, while combing (62) and (64) gives $\mathbb{E}[\|\hat{h}_{t+1}\|^2 + 1]^2 = \mathbb{E}[\|\hat{h}_{t+1}\|^4 + 2\|\hat{h}_{t+1}\|^2 + 1 \leq 2[(t-1) \mod T_c]^2 + 7[(t-1) \mod T_c] + 1. At this point we complete the proof of Lemma 5.

**F. Proof of Lemma 5**

We here prove that, for $\hat{h}_{t+1}$ and $\hat{h}_{t,t+1}$ defined as in (124) and (125), $t \in \{1, 2, \ldots, T_c - 1\}$, we have
\[ \mathbb{E}[\|\hat{h}_1 + \hat{h}_{1,2} + \hat{h}_{2,3} + \cdots + \hat{h}_{t,t+1}\|^2] = \mathbb{E}[\|\hat{h}_1\|^2 + \|\hat{h}_{1,2}\|^2 + \|\hat{h}_{2,3}\|^2 + \cdots + \|\hat{h}_{t,t+1}\|^2]. \]
For the case of $t \in \{1, 2, \ldots, T_c - 1\}$, we have
\[ \mathbb{E}[\|\hat{h}_1 + \hat{h}_{1,2} + \hat{h}_{2,3} + \cdots + \hat{h}_{t,t+1}\|^2] \]
\[ = \mathbb{E}[\|\hat{h}_t + \hat{h}_{t,t+1}\|^2] \]
\[ = \mathbb{E}[\|\hat{h}_t\|^2 + 2\|\hat{h}_{t,t+1}\|^2 + \|\hat{h}_t\|^2 + \hat{h}_t^H \hat{h}_{t,t+1} \hat{h}_t | w, y^{t-1}] \]
\[ = \mathbb{E}[\|\hat{h}_t\|^2 + \|\hat{h}_{t,t+1}\|^2 | w, y^{t-1}] + 0 + 0 \]
\[ = \mathbb{E}[\|\hat{h}_t\|^2] + \mathbb{E}[\|\hat{h}_{t,t+1}\|^2 | w, y^{t-1}] \]
\[ = \mathbb{E}[\|\hat{h}_t\|^2] + \mathbb{E}[\|\hat{h}_{t,t+1}\|^2] \]
\[ = \mathbb{E}[\|\hat{h}_{t-1} + \hat{h}_{t-1,t}\|^2] + \mathbb{E}[\|\hat{h}_{t,t+1}\|^2] \]
\[ = \mathbb{E}[\|\hat{h}_{t-1}\|^2] + \mathbb{E}[\|\hat{h}_{t-1,t}\|^2] + \mathbb{E}[\|\hat{h}_{t,t+1}\|^2] \]
\[ \vdots \]
\[ = \mathbb{E}[\|\hat{h}_1\|^2] + \mathbb{E}[\|\hat{h}_{1,2}\|^2] + \mathbb{E}[\|\hat{h}_{2,3}\|^2] + \cdots + \mathbb{E}[\|\hat{h}_{t,t+1}\|^2] \]
where (149) uses the definitions of $\hat{h}_{t+1}$ and $\hat{h}_{t,t+1}$ as in (124) and (125); (150) is from the identity that $\mathbb{E}[a] = \mathbb{E}[\mathbb{E}[a|b]]$ for random $a$ and $b$; (151) follows from the fact that $\hat{h}_{t,t+1}$ is a complex Gaussian
vector with zero mean given \((\mathbf{w}, \mathbf{y}^{-1})\) (cf. Lemma \[4\]) and the fact that \(\hat{\mathbf{h}}_t\) is a deterministic function of \((\mathbf{w}, \mathbf{y}^{-1})\) given the encoding maps as in \((2)\); \((152)\) use the definitions of \(\hat{\mathbf{h}}_{t-1}\) and \(\hat{\mathbf{h}}_{t-1,t}\) as in \((124)\) and \((125)\); \((153)\) follows from the previous steps in \((149)-(152)\); \((154)\) follows from the same step in \((153)\). Note that \(\hat{\mathbf{h}}_1 = 0\).

**G. Proof of Lemma \[9\]**

We will prove that, for \(\hat{\mathbf{h}}_{t,t+1}\) defined as in \((124)\) and \((125)\), \(t \in \{1, 2, \cdots, T_c - 1\}\), the following bounds hold

\[
\mathbb{E}[||\hat{\mathbf{h}}_{t,t+1}||^2 | \mathbf{w}, \mathbf{y}^{-1}] \leq 1,
\]

\[
\mathbb{E}[||\hat{\mathbf{h}}_{t,t+1}||^2] \leq 1.
\]

We will just prove the first inequality, as the second inequality follows immediately from the first inequality and the identity that \(\mathbb{E}[||\hat{\mathbf{h}}_{t,t+1}||^2] = \mathbb{E}[\mathbb{E}[||\hat{\mathbf{h}}_{t,t+1}||^2 | \mathbf{w}, \mathbf{y}^{-1}]]\).

Given that \(\hat{\mathbf{h}}_{t,t+1} = \frac{\Omega_t \mathbf{w}^T y_t - \mathbf{z}_t}{\mathbf{x}_t^T \mathbf{y}_t} \), for \(t \in \{1, 2, \cdots, T_c - 1\}\), we have

\[
\mathbb{E}[||\hat{\mathbf{h}}_{t,t+1}||^2 | \mathbf{w}, \mathbf{y}^{-1}] = \mathbb{E}\left[\left(\Omega_t \mathbf{w}^T (y_t - \mathbf{z}_t)\right)^H \left(\Omega_t \mathbf{w}^T (y_t - \mathbf{z}_t)\right) | \mathbf{w}, \mathbf{y}^{-1}\right] = \mathbb{E}\left[\frac{\left(\Omega_t \mathbf{w}^T (x^*_t \mathbf{h}_t + z_t)\right)^H \left(\Omega_t \mathbf{w}^T (x^*_t \mathbf{h}_t + z_t)\right)}{\left(x^*_t \mathbf{h}_t + 1\right)^2} | \mathbf{w}, \mathbf{y}^{-1}\right] = \frac{\mathbf{x}_t^H \Omega_t \mathbf{x}_t \cdot \mathbb{E}\left[(x^*_t \mathbf{h}_t + z_t) (x^*_t \mathbf{h}_t + z_t) | \mathbf{w}, \mathbf{y}^{-1}\right]}{\left(x^*_t \mathbf{h}_t + 1\right)^2} = \frac{\left(x_t^H \Omega_t \mathbf{x}_t \right)^2 + 1 + 0 + 0}{\left(x^*_t \mathbf{h}_t + 1\right)^2} = 1
\]

where \(\hat{\mathbf{h}}_{t,t+1}\) is defined in \((124)\) and \((125)\); where \((155)\) use the definition of \(\hat{\mathbf{h}}_t \triangleq \mathbf{h}_t - \hat{\mathbf{h}}_t\) and the fact that \(y_t - \mathbf{z}_t\) is defined as in \((124)\) and \((125)\); \((156)\) results from the facts that \(\mathbf{x}_t\) is a deterministic function of \((\mathbf{y}^{-1}, \mathbf{w})\) and \(\Omega_t\) is a deterministic function of \((\mathbf{y}^{-2}, \mathbf{w})\) given the encoding maps as in \((2)\); \((157)\) follows from the facts that \(\hat{\mathbf{h}}_t | (\mathbf{y}^{-1}, \mathbf{w}) \sim \mathcal{CN}(0, \Omega_t)\) and that \(z_t\) is independent of \(\hat{\mathbf{h}}_t\); \((158)\) follows from that

\[
\mathbf{x}_t^H \Omega_t \mathbf{x}_t \leq \mathbf{x}_t^H \Omega_t \mathbf{x}^*_t \leq \mathbf{x}_t^H \Omega_t \mathbf{x}_t^* + 1,
\]

where the first inequality follows from that \(0 \preceq \Omega_t \preceq I_M\) (cf. Lemma \[2\]) and that \(\mathbf{x}_t^H \Omega_t\mathbf{x}_t - \mathbf{x}_t^H \Omega_t \mathbf{x}_t^* \triangleq \mathbf{x}_t^H \Omega_t \mathbf{U} \mathbf{U}^H \mathbf{x}_t^* - \mathbf{x}_t^H \mathbf{U} \mathbf{U}^H \mathbf{U} \mathbf{U}^H \mathbf{x}_t^* = \mathbf{x}_t^H \mathbf{U} (\Lambda - \Lambda^2) \mathbf{U}^H \mathbf{x}_t^* \geq 0\) by using the singular value decomposition of \(\Omega_t \triangleq \mathbf{U} \mathbf{U}^H\), where \(\mathbf{U}\) and \(\Lambda\) are the unitary matrix and diagonal matrix respectively. Note that if \(0 \preceq \Omega_t \preceq I_M\), then \(\mathbf{U} (\Lambda - \Lambda^2) \mathbf{U}^H \succeq 0\). At this point we completes the proof.
H. Proof of Lemma 10

We will prove that, for $\tilde{h}_{t,t+1}$ defined as in (124) and (125), $t \in \{1, 2, \cdots, T_c - 1\}$, the following inequalities hold

$$\mathbb{E} \left[ ||\tilde{h}_{t,t+1}||^4 | w, y^{t-1} \right] \leq 3, \quad (160)$$

$$\mathbb{E} \left[ ||\tilde{h}_{t,t+1}||^4 \right] \leq 3. \quad (161)$$

We will just prove the first inequality in (160), as the second inequality in (161) follows immediately from (160) and the identity that $\mathbb{E} \left[ ||\tilde{h}_{t,t+1}||^4 \right] = \mathbb{E} \left[ \mathbb{E} \left[ ||\tilde{h}_{t,t+1}||^4 | w, y^{t-1} \right] \right]$.

The proof of (160) follows from the proof steps of Lemma 9. For $\tilde{h}_{t,t+1}$ defined as in (124) and (125), $t \in \{1, 2, \cdots, T_c - 1\}$, we have

$$\mathbb{E} \left[ ||\tilde{h}_{t,t+1}||^4 | w, y^{t-1} \right] = \mathbb{E} \left[ \left( \frac{\text{Re}(\tilde{h}_{t+1}^T(y_t - \tilde{x}_t^T\tilde{h}_t))}{\tilde{x}_t^T\tilde{x}_t + 1} \right)^4 \right] | w, y^{t-1} \right]$$

$$= \mathbb{E} \left[ \left( \frac{\text{Re}(\tilde{h}_{t+1}^T(y_t - \tilde{x}_t^T\tilde{h}_t))}{\tilde{x}_t^T\tilde{x}_t + 1} \right)^4 \right] \mathbb{E} \left[ \left| \tilde{h}_t \right|^2 \mathbb{E} \left[ \left| \tilde{h}_t \right|^2 \right] \right]$$

$$\leq \mathbb{E} \left[ \left( \frac{\text{Re}(\tilde{h}_{t+1}^T(y_t - \tilde{x}_t^T\tilde{h}_t))}{\tilde{x}_t^T\tilde{x}_t + 1} \right)^4 \right] \mathbb{E} \left[ \left| \tilde{h}_t \right|^2 \right]$$

$$\leq 3 \mathbb{E} \left[ \left( \frac{\text{Re}(\tilde{h}_{t+1}^T(y_t - \tilde{x}_t^T\tilde{h}_t))}{\tilde{x}_t^T\tilde{x}_t + 1} \right)^4 \right] \mathbb{E} \left[ \left| \tilde{h}_t \right|^2 \right] \mathbb{E} \left[ \left| \tilde{h}_t \right|^2 \right]$$

where (162) use the definition of $\tilde{h}_t \triangleq h_t - \hat{h}_t$; (163) follows from the fact that $x_t$ is a deterministic function of $(y^{t-1}, w)$ and $\Omega_t$ is a deterministic function of $(y^{t-2}, w)$ given the encoding maps as in (2). Let us focus on the inner expectation term in (163). Note that, for two complex numbers $a$ and $b$, we have

$$|(a + b)|^4 = |a|^4 + |b|^4 + 2|a|^2|b|^2 + 4Re^2(ab^*) + 4(|a|^2 + |b|^2) \cdot Re(ab^*). \quad (164)$$

In the following, we will replace $a$ and $b$ with $x_t^T\tilde{h}_t$ and $z_t$ respectively and compute $\mathbb{E} \left[ \left( x_t^T\tilde{h}_t + z_t \right)^4 | w, y^{t-1} \right]$. At first we note that given $z_t \sim \mathcal{CN}(0, 1)$ and $\tilde{h}_t|(y^{t-1}, w) \sim \mathcal{CN}(\Omega_t, \Omega_t)$, the following equalities holds true:

$$\mathbb{E}[z_t] = 0 \quad (165)$$

$$\mathbb{E}[|z_t|^2] = 1 \quad (166)$$

$$\mathbb{E}[z_tz_t^*] = 0 \quad (167)$$

$$\mathbb{E}[z_t \cdot |z_t|^2] = 0 \quad (168)$$

$$\mathbb{E}[|z_t|^4] = 3 \quad (169)$$

$$\mathbb{E}[[x_t^T\tilde{h}_t] | w, y^{t-1}] = 0 \quad (170)$$

$$\mathbb{E}[[x_t^T\tilde{h}_t]^2 | w, y^{t-1}] = x_t^T\Omega_t x_t^* \quad (171)$$

$$\mathbb{E}[[x_t^T\tilde{h}_t]^4 | w, y^{t-1}] = 3(x_t^T\Omega_t x_t^*)^2 \quad (172)$$
where (172) follows from Lemma \[11\] in Appendix B-E, i.e., $\mathbb{E}[(x_t^\top \hat{h}_t | y, y^{-1})] = \mathbb{E}[(\tilde{h}_t | x_t^\top \hat{h}_t)^2 | w, y^{-1}] = 2\text{tr}(x_t^\top \Omega_t x_t^\top \Omega_t) + (\text{tr}(x_t^\top \Omega_t))^2 = 3(x_t^\top \Omega_t x_t^\top)^2$; (169) also follows from Lemma \[11\]. By using (164)-(172), we have

$$
\mathbb{E}\left[ |(x_t^\top \tilde{h}_t + z_t)|^4 \bigg| w, y'y^{-1}\right] = \mathbb{E}\left[ |x_t^\top \tilde{h}_t|^4 + |z_t|^4 + 2|x_t^\top \tilde{h}_t|^2 |z_t|^2 + 4|z_t|^4 \right] \cdot \text{Re}(x_t^\top \tilde{h}_t z_t^*) \mathbb{E}[ |w, y'y^{-1}|] \right] = 3(x_t^\top \Omega_t x_t^\top)^2 + 3 + 2x_t^\top \Omega_t x_t^\top + \mathbb{E}\left[ 4\text{Re}(x_t^\top \tilde{h}_t z_t^*) |w, y'y^{-1}| + 0 \right]
$$

where (173) is from (164); (174) follows from (165)-(172) as well as the fact that $z_t$ is independent of $x_t$ and $\tilde{h}_t$; (175) stems from the following conclusion for two independent complex random variables $a$ and $b$, $a \sim \mathcal{CN}(0, 1)$, that is, $\mathbb{E}[4\text{Re}(ab^*)] = \mathbb{E}[(ab^* + a^*b)(ab^* + a^*b)] = \mathbb{E}[2|a|^2|b|^2 + aa^*(bb^*) + (aa^*)bb^*] = 2\mathbb{E}[|a|^2|b|^2]$ (cf. (167)). In the above we replace $a$ and $b$ with $x_t^\top \tilde{h}_t$ and $z_t$ respectively. The last step in (176) follows from (171).

By plugging (176) into (163), we have

$$
\mathbb{E}[\|\hat{h}_{t+1} - \hat{h}_{t}\|^4 | w, y'y^{-1}] = \frac{(x_t^\top \Omega_t x_t^\top)^4 + 2(x_t^\top \Omega_t x_t^\top)^2 + 3}{(x_t^\top \Omega_t x_t^\top + 1)^4} \cdot \mathbb{E}[\|\hat{h}_t\|^4 | w, y'y^{-1}] \leq \frac{(x_t^\top \Omega_t x_t^\top)^2 + 3}{(x_t^\top \Omega_t x_t^\top + 1)^2} \cdot \mathbb{E}[\|\hat{h}_t\|^4 | w, y'y^{-1}] \leq 3 \cdot \frac{(x_t^\top \Omega_t x_t^\top + 1)^2}{(x_t^\top \Omega_t x_t^\top + 1)^2} = 3
$$

where (178) uses the fact that $x_t^\top \Omega_t x_t^\top \geq 0$ since $\Omega_t \succeq 0$ (cf. Lemma \[2\]); (179) follows from the same step in (180), i.e., $x_t^\top \Omega_t x_t^\top \leq x_t^\top x_t + 1$. At this point we completes the proof.

I. Proof of Lemma \[12\]

We will prove that, for $\hat{h}_{t+1} = \hat{h}_t + \hat{h}_{t+1}$ defined in (124) and (125), $t \in \{1, 2, \ldots, T_e - 1\}$, we have

$$
\mathbb{E}[\|\hat{h}_{t+1} - \hat{h}_t\|^4] \leq \mathbb{E}[\|\hat{h}_t\|^4] + 4 \cdot \mathbb{E}[\|\hat{h}_t\|^2] + 3.
$$

We will at first focus on the upper bound of $\mathbb{E}[\|\hat{h}_t + \hat{h}_{t+1} - \hat{h}_t\|^4 | w, y'y^{-1}]$. Remind that $\hat{h}_{t+1} = \frac{\Omega_t x_t^\top (y^{-1} - \hat{h}_t)}{x_t^\top x_t + 1} = \frac{\Omega_t x_t^\top (y^{-1} + \hat{h}_t + \hat{h}_{t+1})}{x_t^\top x_t + 1}$ and that $\hat{h}_t | (y^{-1}, w) \sim \mathcal{CN}(0, \Omega_t)$ (cf. Lemma \[4\]), where $\hat{h}_t \triangleq \hat{h}_t - \hat{h}_t$. Thus, one can easily conclude that

$$
\hat{h}_{t+1} | (y^{-1}, w) \sim \mathcal{CN}(0, \frac{\Omega_t x_t^\top x_t^\top + 1}{x_t^\top x_t + 1}).
$$

(181)
Note that, for any two vectors \( \mathbf{a}, \mathbf{b} \in \mathbb{C}^{M \times 1} \), \( \| \mathbf{a} + \mathbf{b} \|^4 \) can be expanded as in (164). Then, by replacing \( \mathbf{a} \) and \( \mathbf{b} \) with \( \hat{\mathbf{h}}_t \) and \( \hat{\mathbf{h}}_{t,t+1} \) respectively, we have

\[
\mathbb{E}
\left[
\| \hat{\mathbf{h}}_t + \hat{\mathbf{h}}_{t,t+1} \|^4 | \mathbf{w}, \mathbf{y}^{-1}
\right]
\]

\[
= \mathbb{E}
\left[
\| \hat{\mathbf{h}}_t \|^4 + 4\| \hat{\mathbf{h}}_{t,t+1} \|^2\| \hat{\mathbf{h}}_t \|^2 + 4 \Re^2(\hat{\mathbf{h}}_t^H \hat{\mathbf{h}}_{t,t+1}) + 4(\| \hat{\mathbf{h}}_t \|^2 + \| \hat{\mathbf{h}}_{t,t+1} \|^2) \cdot \Re(\hat{\mathbf{h}}_t^H \hat{\mathbf{h}}_{t,t+1})
\right] | \mathbf{w}, \mathbf{y}^{-1}
\]

(182)

\[
= \| \hat{\mathbf{h}}_t \|^4 + \mathbb{E}\left[\| \hat{\mathbf{h}}_{t,t+1} \|^4 | \mathbf{w}, \mathbf{y}^{-1}\right] + 4\| \hat{\mathbf{h}}_t \|^2\| \hat{\mathbf{h}}_{t,t+1} \|^2 + 4 \Re^2(\hat{\mathbf{h}}_t^H \hat{\mathbf{h}}_{t,t+1}) + 4(\| \hat{\mathbf{h}}_t \|^2 + \| \hat{\mathbf{h}}_{t,t+1} \|^2) \cdot \Re(\hat{\mathbf{h}}_t^H \hat{\mathbf{h}}_{t,t+1})
\]

(183)

\[
\leq \| \hat{\mathbf{h}}_t \|^4 + 3 + 2\| \hat{\mathbf{h}}_t \|^2 + 4 \Re^2(\hat{\mathbf{h}}_t^H \hat{\mathbf{h}}_{t,t+1}) | \mathbf{w}, \mathbf{y}^{-1}
\]

(184)

\[
\leq \| \hat{\mathbf{h}}_t \|^4 + 3 + 2\| \hat{\mathbf{h}}_t \|^2 + 2\| \hat{\mathbf{h}}_t \|^2
\]

(185)

where (182) results from (164); (183) follows from the fact that \( \hat{\mathbf{h}}_t \) is deterministic given \( \mathbf{w}, \mathbf{y}^{-1} \), and the identities that \( \mathbb{E}[\Re(\mathbf{a}^H \mathbf{b})] = \Re(\mathbb{E}[\mathbf{a}^H \mathbf{b}]) = 0 \) and \( \mathbb{E}[\| \mathbf{b} \|^2 \cdot \Re(\mathbf{a}^H \mathbf{b})] = \Re(\mathbb{E}[\mathbf{a}^H \mathbf{b}] \cdot \| \mathbf{b} \|^2 \) = 0 for a fixed vector \( \mathbf{a} \) and a Gaussian vector \( \mathbf{b} \sim \mathcal{CN}(0, \mathbf{K}) \); note that the odd-order moments of a complex proper Gaussian vector are zeros (see, e.g., [48]); (184) results from Lemma 9 and Lemma 10, i.e., \( \mathbb{E} \left[ \| \hat{\mathbf{h}}_{t,t+1} \|^4 | \mathbf{w}, \mathbf{y}^{-1} \right] \leq 3 \) and \( \mathbb{E} \left[ \| \hat{\mathbf{h}}_{t,t+1} \|^2 | \mathbf{w}, \mathbf{y}^{-1} \right] \leq 1 \); (185) follows from that

\[
4 \mathbb{E}[\Re^2(\hat{\mathbf{h}}_t^H \hat{\mathbf{h}}_{t,t+1}) | \mathbf{w}, \mathbf{y}^{-1}]
\]

(186)

\[
= \mathbb{E}[\| \hat{\mathbf{h}}_t \|^4 \cdot \| \hat{\mathbf{h}}_{t,t+1} \|^2 | \mathbf{w}, \mathbf{y}^{-1}]
\]

(187)

\[
= 2 \cdot \| \hat{\mathbf{h}}_t \|^2 \cdot \| \hat{\mathbf{h}}_{t,t+1} \|^2 + 2 \Re^2(\hat{\mathbf{h}}_t^H \hat{\mathbf{h}}_{t,t+1})
\]

(188)

\[
= 2 \cdot \| \hat{\mathbf{h}}_t \|^2 \cdot \| \hat{\mathbf{h}}_{t,t+1} \|^2 + 2 \Re^2(\hat{\mathbf{h}}_t^H \hat{\mathbf{h}}_{t,t+1})
\]

(189)

\[
= 2 \cdot \| \hat{\mathbf{h}}_t \|^2 \cdot \| \hat{\mathbf{h}}_{t,t+1} \|^2 + 2 \Re^2(\hat{\mathbf{h}}_t^H \hat{\mathbf{h}}_{t,t+1})
\]

(190)

\[
= 2\| \hat{\mathbf{h}}_t \|^2 \cdot \| \hat{\mathbf{h}}_{t,t+1} \|^2
\]

(191)

where (186) follows from the identity that \( 4 \Re^2(\mathbf{a}^H \mathbf{b}) = (\mathbf{a}^H \mathbf{b} + \mathbf{b}^H \mathbf{a}) \cdot (\mathbf{a}^H \mathbf{b} + \mathbf{b}^H \mathbf{a}) = 2 \mathbf{a}^H \mathbf{b} \cdot \mathbf{a} + 2 \Re(\mathbf{a}^H \mathbf{b} \cdot \mathbf{b}) \) for two vectors \( \mathbf{a} \) and \( \mathbf{b} \) with the same dimension; (187) stems from (181), i.e., \( \hat{\mathbf{h}}_{t,t+1}(\mathbf{y}^{-1}, \mathbf{w}) \sim \mathcal{CN}(0, \frac{\Omega^H \mathbf{x}^H \mathbf{x} \Omega}{\| \mathbf{x} \|^2 + 1}) \); (188) follows from the identity that \( \mathbb{E}[\mathbf{a}^H \mathbf{b}] = 0 \) for a fixed vector \( \mathbf{a} \) and a complex Gaussian vector \( \mathbf{b} \sim \mathcal{CN}(0, \mathbf{K}) \); note that if \( \mathbf{b} \sim \mathcal{CN}(0, \mathbf{K}) \), then \( \mathbb{E}[\mathbf{a}^H \mathbf{b}] = \mathcal{CN}(0, \mathbf{a}^H \mathbf{K} \mathbf{a}) \) and \( \mathbb{E}[\mathbf{c} \cdot \mathbf{c}] = 0; \) (189) follows from the identities that \( \text{tr}(\mathbf{A} \mathbf{B}) \leq \lambda_{\text{max}}(\mathbf{A}) \text{tr}(\mathbf{B}) \) for positive semidefinite \( m \times m \) Hermitian matrices \( \mathbf{A}, \mathbf{B} \), and that \( \lambda_{\text{max}}(\hat{\mathbf{h}}_t \hat{\mathbf{h}}_t^H) = \| \hat{\mathbf{h}}_t \|^2 \); (190) follows from the same step in (185), i.e., \( \| \mathbf{x}^H \Omega \mathbf{x} \| \| \mathbf{x}^H \Omega \mathbf{x} \| \leq \| \mathbf{x}^H \Omega \mathbf{x} \|^2 + 1 \). Finally, from the step in (185), we have the following inequality

\[
\mathbb{E}
\left[
\| \hat{\mathbf{h}}_t + \hat{\mathbf{h}}_{t,t+1} \|^4 | \mathbf{w}, \mathbf{y}^{-1}\right] \leq \| \hat{\mathbf{h}}_t \|^4 + 4\| \hat{\mathbf{h}}_t \|^2 + 3.
\]

By taking the expectation on both sides of the above inequality, and using the identity that \( \mathbb{E}[\| \hat{\mathbf{h}}_t + \hat{\mathbf{h}}_{t,t+1} \|^4] = \mathbb{E}[\mathbb{E}[\| \hat{\mathbf{h}}_t + \hat{\mathbf{h}}_{t,t+1} \|^4 | \mathbf{w}, \mathbf{y}^{-1}]] \), it yields

\[
\mathbb{E}[\| \hat{\mathbf{h}}_t + \hat{\mathbf{h}}_{t,t+1} \|^4] \leq \mathbb{E}[\| \hat{\mathbf{h}}_t \|^4] + 4 \cdot \mathbb{E}[\| \hat{\mathbf{h}}_t \|^2] + 3
\]

(192)

which completes the proof.
In this section we provide the proof of Proposition 3. Note that our rate analysis is closely inspired by [9], [28]. For the proposed scheme with Gaussian input, training and feedback described in Sections VI-A, VI-B, the scheme achieves the following ergodic rate

\[ R = \frac{1}{T_c} \| (s_d; y_T, y_d) \]

by encoding the message over sufficiently large number of channel blocks, where the relationship between \( s_d, y_T \) and \( y_d \) are given in \( (70) \) and \( (73) \). The achievable rate can be lower bounded as:

\[ T_c R = \| (s_d; y_T, y_d) \]

\[ = \| (s_d; \hat{h}_T, y_T, y_d) \]

\[ \geq \| (s_d; \hat{h}_T, y_d) \]

\[ = \| (s_d; \hat{h}_T) + \| (s_d; y_d | \hat{h}_T) \]

\[ = \| (s_d; y_d | \hat{h}_T) \]

\[ \geq \| (\sqrt{P} \| \hat{h}_T \| s_d; y_d | \hat{h}_T) \]

\[ = h(\sqrt{P} \| \hat{h}_T \| s_d) - h(\sqrt{P} \| \hat{h}_T \| s_d | y_d, \hat{h}_T) \]

\[ = T_d \cdot \mathbb{E} \left[ \log(\pi e P \| \hat{h}_T \|^2) \right] - h(\sqrt{P} \| \hat{h}_T \| s_d | y_d, \hat{h}_T) \]

(198)

where (193) results from the fact that \( \| \hat{h}_T \| \) is a function of \( y_T \); where (194) and (197) are from the fact that adding more information will not reduce the mutual information; (195) is from our input assumption that \( s_d \) and \( \| \hat{h}_T \| \) are independent; (196) uses the fact that \( \sqrt{P} \| \hat{h}_T \| s_d \) is a function of \( s_d \) and \( \hat{h}_T \); (198) follows from the fact that \( s_d \sim \mathcal{CN}(0, 1 T_d) \), where \( T_d = T_c - T_r \) (cf. (68)). Let us focus on the second term in (198):

\[ h(\sqrt{P} \| \hat{h}_T \| s_d | y_d, \hat{h}_T) \leq \sum_{t=0}^{T_c-1} h(\sqrt{P} \| \hat{h}_T \| s_t | y_t, \hat{h}_T) \]

(199)

\[ = \sum_{t=0}^{T_c-1} h(\sqrt{P} \| \hat{h}_T \| s_t - \beta_t y_t | y_t, \hat{h}_T) \]

(200)

\[ \leq \sum_{t=0}^{T_c-1} h(\sqrt{P} \| \hat{h}_T \| s_t - \beta_t y_t | \hat{h}_T) \]

(201)

\[ \leq \sum_{t=0}^{T_c-1} \mathbb{E} \left[ \log(\pi e \mathbb{E} \left[ |\sqrt{P} \| \hat{h}_T \| s_t - \beta_t y_t | \hat{h}_T|^2 \right]) \right] \]

(202)

where (199) is from chain rule and the fact that conditioning reduces differential entropy, where \( y_d \triangleq [y_{T_r+1}, y_{T_r+2}, \ldots, y_{T_c}]^\top, s_d \triangleq [s_{T_r+1}, s_{T_r+2} \cdots, s_{T_c}]^\top \) and

\[ y_t = \sqrt{P} \| \hat{h}_T \| s_t + \sqrt{P} h_T^\top \hat{h}_T s_t + z_t \]

(203)

(cf. (72); (200) results from that \( h(\sqrt{P} \| \hat{h}_T \| s_t | y_t, \hat{h}_T) = h(\sqrt{P} \| \hat{h}_T \| s_t - \beta_t y_t | y_t, \hat{h}_T) \) for any deterministic function \( \beta_t \) of \( y_t \) and \( \hat{h}_T \); (201) is from the fact that conditioning reduces differential entropy; (202) uses
the fact that Gaussian distribution is the differential entropy maximizer given the same second moment of \( \mathbb{E} \left[ \sqrt{P} \| \hat{h}_\tau \| s_t - \beta_t y_t \|^2 \right] \).

In the next step we will focus on a single term inside the summation in (202). Specifically, we will choose a proper \( \beta_t \) to minimizes \( \mathbb{E} \left[ \sqrt{P} \| \hat{h}_\tau \| s_t - \beta_t y_t \|^2 \right] \), which will in turn tighten the bound in (202), where \( y_t \) is expressed in (203). This is equivalent to the MMSE estimation problem. The optimal \( c_t \) to minimize \( \mathbb{E} \left[ |u - c v|^2 \right] \) is \( c_t = \frac{\mathbb{E}[uv^*]}{\mathbb{E}[v^2]} \) and in this case \( \mathbb{E} \left[ |u - c^* v| \right] = \mathbb{E} \left[ |u|^2 \right] - \frac{\mathbb{E}[uv^*]^2}{\mathbb{E}[v^2]} \), for two random variables \( u \) and \( v \) with zero mean. Therefore, the optimal \( \beta_t \) can be chosen as

\[
\beta_t = \frac{\mathbb{E} \left[ \sqrt{P} \| \hat{h}_\tau \| s_t y_t^* \right] \hat{h}_\tau}{\mathbb{E} \left[ |y_t|^2 \right] \hat{h}_\tau} = \frac{P \| \hat{h}_\tau \|^2}{P \| \hat{h}_\tau \|^2 + P \sigma^2 + 1}
\]

where

\[
\sigma^2 \triangleq \frac{1}{P + 1}
\]

corresponding to the variance of \( \hat{h}_\tau^T \hat{h}_\tau \) given \( \hat{h}_\tau \). Remind that \( \hat{h}_\tau \) and \( \tilde{h}_\tau \) are independent with each other, \( \hat{h}_\tau \sim \mathcal{CN}(0, \frac{P}{P + 1} I) \) and \( \tilde{h}_\tau \sim \mathcal{CN}(0, 1 \frac{1}{P + 1} I) \). By setting \( \beta_t \) as in (204), we have

\[
\mathbb{E} \left[ \sqrt{P} \| \hat{h}_\tau \| s_t - \beta_t y_t \|^2 \right] = \mathbb{E} \left[ \sqrt{P} \| \hat{h}_\tau \| s_t \|^2 \right] \hat{h}_\tau = \frac{\mathbb{E} \left[ \sqrt{P} \| \hat{h}_\tau \| s_t y_t^* \right] \hat{h}_\tau}{\mathbb{E} \left[ |y_t|^2 \right] \hat{h}_\tau} = \frac{P \| \hat{h}_\tau \|^2 - (P \| \hat{h}_\tau \|^2 + P \sigma^2 + 1)^2 \| \hat{h}_\tau \|^2 \| \hat{h}_\tau \|^2 + P \sigma^2 + 1}{P \| \hat{h}_\tau \|^2 \| \hat{h}_\tau \|^2 + P \sigma^2 + 1}
\]

By plugging (205) and (202) into (198), we have:

\[
T_c R \geq T_d \cdot \mathbb{E} \left[ \log(\pi e P \| \hat{h}_\tau \|^2) \right] - T_d \cdot \mathbb{E} \left[ \log \left( \pi e \cdot \frac{P \| \hat{h}_\tau \|^2 \| \hat{h}_\tau \|^2 + P \sigma^2 + 1}{\| \hat{h}_\tau \|^2 + P \sigma^2 + 1} \right) \right]
\]

\[
= T_d \cdot \mathbb{E} \left[ \log \left( 1 + \frac{P \| \hat{h}_\tau \|^2}{P \sigma^2 + 1} \right) \right]
\]

Note that \( \hat{h}_\tau \sim \mathcal{CN}(0, \frac{P}{P + 1} I) \) and \( \delta \hat{h}_\tau \sim \mathcal{CN}(0, 2I) \), for

\[
\delta \triangleq \sqrt{\frac{2(P + 1)}{P}}
\]

It then implies that \( \| \delta \hat{h}_\tau \|^2 \) is chi-squared distributed with \( 2T_c \) degrees of freedom, that is, \( \| \delta \hat{h}_\tau \|^2 \sim \chi^2(2T_c) \). If \( u \) is a chi-square random variable with \( k \geq 2 \) degrees of freedom, its probability density function is given by (84) and its probability density function is zero when \( u \leq 0 \). Therefore, without loss
of generality we consider \( \|\hat{h}_s\|^2 \) as a positive chi-squared random variable with \( 2T_r \) degrees of freedom. Then, from (206) we further have

\[
T_c R \geq T_d \cdot \mathbb{E} \left[ \log \left( 1 + \frac{P}{P\sigma^2 + 1} \right) \right]
\]

\[
= T_d \cdot \mathbb{E} \left[ \log(\|\hat{h}_s\|^2) \right] + T_d \cdot \mathbb{E} \left[ \log \left( \frac{1}{\|\hat{h}_s\|^2} + \frac{P/\delta^2}{P\sigma^2 + 1} \right) \right]
\]

\[
\geq T_d \cdot \mathbb{E} \left[ \log(\|\hat{h}_s\|^2) \right] + T_d \cdot \log \left( \frac{1}{\mathbb{E} \|\hat{h}_s\|^2} + \frac{P/\delta^2}{P\sigma^2 + 1} \right)
\]

(207)

\[
= T_d \cdot \mathbb{E} \left[ \log(\|\hat{h}_s\|^2) \right] + T_d \cdot \log \left( \frac{1}{2T_r} + \frac{P/\delta^2}{P\sigma^2 + 1} \right)
\]

(208)

where (207) follows from the fact that \( g(x) = \log(1 + x) \) is a convex function since \( \frac{\partial^2 g(x)}{\partial x^2} \geq 0 \) for any \( x > 0 \), where \( c > 0 \) is a constant; (208) results from that \( \mathbb{E} \|\hat{h}_s\|^2 = 2T_r \), since \( \|\hat{h}_s\|^2 \sim \chi^2(2T_r) \). Let us now we focus on the first term in (208). From Lemma 7 described in Appendix A we note that if \( u \sim \chi^2(k) \) is a chi-square random variable with \( k \geq 2 \) degrees of freedom, \( k \) is an even number, then

\[
\mathbb{E}[\log u] \geq \log \max \{k - 2, 1\}
\]

which, together with the fact that \( \|\hat{h}_s\|^2 \sim \chi^2(2T_r) \), implies that

\[
\mathbb{E}[\log(\|\hat{h}_s\|^2)] \geq \log(\max(2(T_r - 1), 1)).
\]

(209)

Finally, by plugging (209) into (208) we have:

\[
T_c R \geq T_d \cdot \log(\max\{2(T_r - 1), 1\}) + T_d \cdot \log \left( \frac{1}{2T_r} + \frac{P/\delta^2}{P\sigma^2 + 1} \right)
\]

\[
= T_d \cdot \log \left( \max\{2(T_r - 1), 1\} \right) + T_d \cdot \log \left( \frac{1}{2T_r} + \frac{P/\delta^2}{P\sigma^2 + 1} \right)
\]

\[
= T_d \cdot \log \left( 1 - \frac{1}{\max\{T_r, 2\}} \right) + T_d \cdot \log \left( 1 - \frac{1}{\max\{T_r, 2\}} \right)
\]

(210)

where \( \delta^2 \triangleq \frac{2(P+1)}{P} \), \( \sigma^2 \triangleq \frac{1}{P+1} \) and \( T_d \triangleq (T_c - T_r) \). By dividing both two sides of (210) with \( T_c \), it gives the final lower bound on the achievable rate of the proposed scheme. At this point we complete the proof.

**Appendix D**

**Proof of Theorem 4**

Theorem 4 is based on the capacity bounds in Theorems 2 and 3. Theorem 2 reveals that the capacity of a MISO channel with feedback is upper bounded as

\[
C \leq \log \left( 1 + \min\{M + 2, \sqrt{2}(T_c + 1)\} \cdot \kappa P \right)
\]

under the fourth moment input constraint. By replacing \( T_c \) with \( M^\alpha \), then the beamforming gain is upper bounded as

\[
b(\alpha) \leq \lim_{M \to \infty} \frac{\log \left( 1 + \min\{M + 2, \sqrt{2}(M^\alpha + 1)\} \cdot \kappa P \right)}{\log M}
\]

\[
= \lim_{M \to \infty} \frac{\log \left( 1 + \min\{M, M^\alpha\} \right)}{\log M}
\]

\[
= \min\{1, \alpha\}
\]
Regarding the lower bound, Theorem 3 reveals that the capacity is also bounded as

\[ C \geq \frac{T_c - T \tau}{T_c} \cdot \log \left( 1 + \frac{P_o \cdot \max \{ (T \tau - 1) / 2, 1/2 \}}{2 + \frac{P_o}{\tau}} \right) \]

where \( P_o \triangleq \frac{\alpha P}{\sqrt{3}} \) and \( T \tau \triangleq \left\lfloor \frac{\min \{ M, T \}}{\log \max \{ 4, \min \{ M, T \} \}} \right\rfloor \). Thus, when \( \alpha > 0 \), the beamforming gain is lower bounded as

\[ b(\alpha) \geq \lim_{M \to \infty} \frac{T_c - T \tau}{T_c} \cdot \log \left( 1 + \frac{P_o \cdot \max \{ (T \tau - 1) / 2, 1/2 \}}{2 + \frac{P_o}{\tau}} \right) \cdot \log M \]

\[ = \lim_{M \to \infty} \frac{T_c - T \tau}{T_c} \cdot \log \left( 1 + \frac{P_o \cdot \max \{ (T \tau - 1) / 2, 1/2 \}}{2 + \frac{P_o}{\tau}} \right) \cdot \log M \]

\[ = \lim_{M \to \infty} \log \left( 1 + \min \{ M, \alpha \} \right) \cdot \log M \]

\[ = \lim_{M \to \infty} \log \left( 1 + \min \{ M, \alpha \} \right) \cdot \log M \]

which matches the upper bound. Note that when \( \alpha = 0 \), the upper and lower bounds on the beamforming gain are matched immediately. At this point we complete the proof.

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