Scalar dark energy models mimicking ΛCDM with arbitrary future evolution

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Abstract

Dark energy models with various scenarios of evolution are considered from the viewpoint of the formalism for the equation of state. It is shown that these models are compatible with current astronomical data. Some of the models presented here evolve arbitrarily close to ΛCDM up to the present, but diverge in the future into a number of different possible asymptotic states, including asymptotic de-Sitter (pseudo-rip) evolution, little rips with disintegration of bound structures, and various forms of finite-time future singularities. Therefore it is impossible from observational data to determine whether the universe will end in a future singularity or not. We demonstrate that the models under consideration are stable for a long period of time (billions of years) before entering a Little Rip/Pseudo-Rip induced dissolution of bound structures or before entering a soft finite-time future singularity. Finally, the physical consequences of Little Rip, Type II, III and Big Crush singularities are briefly compared.

I. INTRODUCTION

The discovery of the accelerating expansion of the universe\textsuperscript{1,2} has raised a number of difficult problems in cosmology. The cosmic acceleration can be explained via the introduction of so-called dark energy (for recent review, see\textsuperscript{3,4}) with quite strange properties like negative pressure and/or negative entropy, invisibility in the early universe, etc. According to the latest supernovae observations, the dark energy currently accounts for 73\% of the total mass energy of the universe (see, for instance, Ref.\textsuperscript{3}).

The equation of state (EoS) parameter $w_D$ for dark energy is negative:

$$w_D = p_D / \rho_D < 0,$$

where $\rho_D$ is the dark energy density and $p_D$ is the pressure. Although astrophysical observations favor the standard ΛCDM cosmology, the uncertainties in the determination of the EoS dark energy parameter $w$ are still too large to define which of the three cases $w < -1$, $w = -1$, and $w > -1$ is realized in our universe: $w = -1.04^{+0.09}_{-0.10}$\textsuperscript{5,7}.

If $w < -1$ (phantom dark energy)\textsuperscript{8}, we are dealing with the most interesting and least understood theoretical case. For a phantom field, the violation of all of the four energy conditions occurs. Although this field is unstable from the quantum field theory viewpoint\textsuperscript{9}, it could be stable in classical cosmology. Some observations\textsuperscript{10} may be understood as the indication of the crossing of the phantom divide in the near past or in the near future. A very unpleasant property of phantom dark energy is the Big Rip future singularity\textsuperscript{8,11,14}, where the scale factor becomes infinite at a finite time in the future. A less dangerous future singularity caused by phantom or quintessence dark energy is the sudden (Type II) singularity\textsuperscript{15} where the scale factor is finite at Rip time. Nevertheless, the condition $w < -1$ is not sufficient for a singularity to occur. Mild phantom models where $w$ asymptotically tends to $-1$ and the energy density increases with time or remains constant but there is no finite-time future singularity are discussed in recent works\textsuperscript{16,17}. The key point is that if $w$ approaches $-1$ sufficiently rapidly, then it is possible to have a model in which the time required for the occurrence of the singularity is infinite, i.e., the singularity effectively does not occur. However, if the energy density grows, the disintegration of bound structures eventually occurs in a way similar to the case of the Big Rip singularity. Such phenomena, called the Little Rip and the Pseudo-Rip, destroy the bound structures in the universe at a finite time!

The most convenient formalism to construct the (scalar or fluid) dark energy is to use the EoS:

$$p = g(\rho),$$

where $g$ is a function of the energy density. The evolution of the universe then depends on the choice of the EoS.

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The aim of this article is to develop a general approach for the construction of dark energy models which are compatible with observational data and which provide various scenarios of evolution. Moreover, we are most interested in models which are stable for a long period of time before entering the Little rip/Pseudo-Rip induced dissolution of bound structures or before entering the mild finite-time future singularity (with a finite scale factor at the Rip). This question is analyzed from the viewpoint of the dark energy EoS and corresponding description in terms of scalar field theory. In Sec. II, the general approach to this problem is developed. The classification of cosmological models with different future evolutions is presented. The confrontation with observational data is also given. The non-singular Little Rip cosmology based on a massive scalar potential is considered in Sec. III. It is demonstrated that it may be indistinguishable from the $\Lambda$CDM cosmology, being stable for a long time before the disintegration of bound structures. Although the similar models are considered in [16], it is interesting to note that the simplest model with Klein-Gordon potential is described apparently for the first time. In Sec. IV the asymptotically de Sitter quintessence or phantom cosmology is proposed. The properties of such realistic cosmologies are similar to those of the previous section. In comparison, for example, with [17], our approach is based on equation-of-state formalism instead of setting of the Hubble parameter as function of time. The next two sections are devoted to the construction of quintessence dark energy with a Type III future singularity and phantom/quintessence dark energy with a Type II future singularity. These results based on EoS formalism are novel in cosmology. The estimation of time remaining before Type II or Type III singularity is made for first time. The comparison of the predictions of such models with observational data demonstrates that they may be indistinguishable the $\Lambda$CDM model up to the present. Moreover, they may be stable for billions of years before reaching a future singularity. In this sense, such models represent a quite viable alternative to $\Lambda$CDM. Sec. V is devoted to the construction of Big Crush scalar quintessence models and their comparison with the Little Rip or Type II future singularity cosmology. Specifically, some physical properties of the Big Crush versus the Rip are discussed. Phantom models are briefly reviewed in Sec. VIII. The main qualitative result of our study is that the current observations make it essentially impossible to determine whether or not the universe will end in a future singularity. We should note once more that the consistent application of equation-of-state formalism is the main feature of our work. Some summary and outlook are given in the Discussion section.

II. SCALAR DARK ENERGY MODELS

In this section, we consider the construction of one-scalar dark energy models and compare the models with observational data. We mainly concentrate on different phantom theories whose classical behavior is very similar to that of the $\Lambda$CDM model.

For the spatially-flat FRW universe with metric

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2),$$

(3)

the cosmological equations, that is, the FRW equations are given by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3}, \quad \dot{\rho} = -3 \left(\frac{\dot{a}}{a}\right)(\rho + p),$$

(4)

where $\rho$ and $p$ are the total energy density and pressure, $a$ is the scale factor, $' = d/dt$, and we use the natural system of units in which $8\pi G = c = 1$.

It is convenient to write the dark energy equation of state (EoS) in the following form:

$$p_D = -\rho_D - f(\rho_D),$$

(5)

where $f(\rho_D)$ is some function. The case $f(\rho_D) > 0$ corresponds to the EoS parameter $w < -1$ (phantom) while the case $f(\rho) < 0$ corresponds to the EoS parameter $w > -1$. If dark energy dominates, one can neglect the contribution of other components (matter, dark matter). Then from Eq. (4), one can get the following expression for time variable:

$$t = \frac{2}{\sqrt{3}} \int_{x_0}^{x} \frac{dx}{f(x)}, \quad x = \sqrt{\rho}.$$

(6)

Hereafter, it is convenient to omit the subscript D. The quintessence energy density decreases with time ($x < x_0$), while the phantom energy density increases ($x > x_0$). Following Ref. [14], one can find the following behavior for the expression (6):
1. The integral (6) converges at $\rho \rightarrow \infty$. Therefore a finite-time singularity occurs: the energy density becomes infinite at a finite time $t_f$. The expression for the scale factor

$$a = a_0 \exp \left( \frac{2}{3} \int_{x_0}^{x_f} \frac{x \, dx}{f(x)} \right),$$

indicates that there are two possibilities:

(a) The scale factor diverges at a finite time (Big Rip singularity [8, 11, 13]).
(b) The scale factor remains finite; however, a singularity ($\rho \rightarrow \infty$) occurs. This is a Type III singularity [19].

The key difference between (1a) and (1b) is that for (1b) the energy density grows so rapidly with time that the scale factor does not reach an infinite value.

2. The integral (10) diverges at $\rho \rightarrow \infty$. Such models are described in [16] (see also Refs. [18, 20]). The energy density grows with time but not rapidly enough for the emergence of the Big Rip singularity. According to Ref. [10], we have a so-called “Little Rip”; eventually a dissolution of bound structures occurs at a finite future time. Nevertheless, formally the future singularity does not occur (or, rather, it is shifted to the infinite future). Such scenarios are possible only in the case of phantom dark energy. The next two scenarios are possible for both phantom and quintessence dark energy.

3. The integral (11) diverges at $\rho \rightarrow \rho_c$. The dark energy density asymptotically tends to a constant value (“effective cosmological constant”). Such asymptotically de Sitter theories represent the natural alternative to the $\Lambda$CDM model, which also leads to non-singular cosmology. Nevertheless, even for a non-singular asymptotically de Sitter universe, the possibility of a dramatic rip which may lead to disappearance of bound structures in the universe remains possible [16, 18].

4. Another interesting case corresponds to $f(x) \rightarrow \pm \infty$ at $x = x_1$, i.e., the pressure of the dark energy becomes infinite at a finite energy density. The second derivative of the scale factor diverges while the first derivative remains finite. It is interesting to investigate the properties of dark energy with such a (sudden or Type II) finite-time future singularity [15, 19].

What does this last type of singularity mean from the physical viewpoint? As the universe expands, the relative acceleration between two points separated by a distance $l$ is given by $\ddot{l}/a$. If there is a particle with mass $m$ at each of the points, an observer at one of the masses will measure an inertial force on the other mass of

$$F_{\text{iner}} = ml\ddot{l}/a = ml \left( \dot{H} + H^2 \right).$$

Let us assume that two particles are bounded by a constant force $F_0$. If $F_{\text{iner}}$ is positive and greater than $F_0$, the two particles become unbounded. This is the rip produced by the accelerating expansion. Note that Eq. (8) shows that the rip always occurs when either $H$ diverges or $\dot{H}$ diverges (assuming $\dot{H} > 0$). The first case corresponds to the Big Rip singularity, while if $H$ is finite, but $\dot{H}$ diverges with $\dot{H} > 0$, we have a Type II or sudden future singularity, which also leads to a rip. Even if $H$ or $\dot{H}$ goes to infinity at the infinite future, the inertial force becomes larger and larger, and any bound object is ripped, i.e., the Little Rip cosmology emerges. If $H$ is finite and $\dot{H}$ is negative and diverges, then all the structures are crushed rather than ripped.

Eq. (8) can be rewritten as

$$F_{\text{iner}} = ml \left( \frac{x^2}{3} + \frac{f(x)}{2} \right).$$

The crush corresponds to $f(x) \rightarrow -\infty$ at $x \rightarrow x_f < \infty$ while the case $f(x) \rightarrow \infty$ at $x \rightarrow x_f$ describes the sudden future singularity. The equivalent description in terms of scalar theory can be derived using the equations:

$$\rho = \pm \dot{\phi}^2/2 + V(\phi), \quad p = \pm \dot{\phi}^2/2 - V(\phi),$$

where $\phi$ is the scalar field with potential $V(\phi)$. The sign “−” before kinetic term corresponds to the phantom energy. For the scalar field and its potential, one can derive the following expressions:

$$\phi(x) = \phi_0 \pm \frac{2}{\sqrt{3}} \int_{x_0}^{x} dx \frac{x}{\sqrt{|f(x)|}},$$

$$V(x) = x^2 + f(x)/2.$$
Combining Eqs. (11) and (12) gives the potential as function of the scalar field. For simplicity, we choose the sign “+” in Eq. (11) for phantom energy and “−” for quintessence. For the crush and sudden future singularity the potential of the scalar field has a pole, i.e. from the mathematical viewpoint, these singularities are equivalent. Note that singularities often correspond to the infinite value of the scalar field \( \phi \rightarrow \pm \infty \). For the sudden future singularity potential, we find \( V(\phi) \rightarrow +\infty \), and for the big crunch, \( V(\phi) \rightarrow -\infty \).

Confrontation of the theoretical models with observational data consists mainly of comparison with the distance modulus as a function of redshift from the Supernova Cosmology Project [28]. The distance modulus for a supernova with redshift \( z = a_0/a - 1 \) is

\[
\mu(z) = \text{const} + 5 \log D(z),
\]

where \( D(z) \) is the luminosity distance. As is well-known, the SNe data are well-fit by the \( \Lambda \)CDM cosmology. For such a model (which we call the “standard cosmology” (SC)), we obtain

\[
D_{\text{SC}} L = \frac{c}{H_0}(1 + z) \int_0^z (\Omega_m(1 + z)^3 + \Omega_\Lambda)^{-1/2} \, dz.
\]

Here, \( \Omega_m \) is the fraction of the total density contributed by matter, and \( \Omega_\Lambda \) is the fraction contributed by the vacuum energy density.

One can get also the deceleration parameter \( q_0 \) and jerk parameter \( j_0 \) [21]:

\[
q_0 = -\left. \frac{1}{aH^2} \frac{d^2 a}{dt^2} \right|_{t=t_0} = -\left. \frac{1}{H^2} \left\{ \frac{1}{2} \frac{d}{dN} (H^2) + H^2 \right\} \right|_{N=0},
\]

\[
j_0 = \left. \left\{ \frac{1}{aH^3} \frac{d^3 a}{dt^3} \right\} \right|_{t=0} = \left. \frac{1}{2H^2} \frac{d^2}{dN^2} (H^2) + \frac{3}{2H^2} \frac{d}{dN} (H^2) + 1 \right|_{N=0}.
\]

Here \( N \) is defined by

\[
N = -\ln (1 + z).
\]

For the current time \( t = t_0 \), we have \( N = 0 \). It is useful to note that

\[
\frac{d}{dN} = -(1 + z) \frac{d}{dz}, \quad \frac{d^2}{dN^2} = (1 + z) \frac{d}{dz} + (1 + z)^2 \frac{d^2}{dz^2}.
\]

Measuring the deceleration parameter and especially the jerk parameter is a much more difficult task than measuring \( H_0 \). In order to measure the Hubble constant, one needs to derive the distances to objects at \( \sim 100 \) Mpc; this corresponds to a redshift of \( z \gtrsim 0.02 \). To obtain \( q_0 \), one needs to observe objects to redshift \( z \gtrsim 1 \). Therefore, current observational results for the deceleration and jerk parameters are not totally reliable. For example, Ref. [22] gives for a flat model tight constraints on \( q_0 = -0.81 \pm 0.14 \) and \( j_0 = 2.16^{+0.81}_{-0.76} \) from type Ia supernovae and X-ray cluster gas mass fraction measurements. These results are consistent with \( \Lambda \)CDM at about the 1σ confidence level.

For the \( \Lambda \)CDM model, one gets

\[
q_0 = \frac{3}{2} \Omega_m - 1, \quad j_0 = 1.
\]

Therefore, measuring \( j_0 \) is important in the search for deviations from \( \Lambda \)CDM, since all \( \Lambda \)CDM models, regardless of the matter and cosmological constant energy densities, are characterized by \( j = 1 \).

This concludes our general discussion of scalar dark energy models. Below we present several examples of such models.

III. SCALAR LITTLE RIP COSMOLOGY

It is known that the Little Rip cosmology can be realized in the class of exponential or power-law scalar potentials [17, 18]. It is interesting that even for the case of the simplest Klein-Gordon potential, the Little Rip can occur.

Let us start from

\[
f(\rho) = \alpha^2 = \text{const}.
\]
From the conservation law for the dark energy fluid (4), it is easy to obtain
\[
\rho = \rho_0 - 3\alpha^2 \ln(1 + z).
\] (20)

The luminosity distance is given as
\[
D_L = \frac{c}{H_0}(1 + z) \int_0^z \left( \Omega_m(1 + z)^3 + (1 + 3(w_0 + 1))\Omega_D \right)^{-1/2} dz.
\] (21)

where the current EoS parameter \( w_0 \) is
\[
w_0 = -1 - \frac{\alpha^2}{\rho_0},
\] (22)

and \( \Omega_m \) and \( \Omega_D \) are the matter and dark energy contributions to the total energy budget. It is clear that if \( \alpha^2 \ll \rho_0 \) (and therefore \( w_0 \approx -1 \)) there is good agreement with the observational data: such a cosmology is indistinguishable from the \( \Lambda \)CDM model. The deceleration and jerk parameters are given by
\[
q_0 = \frac{3}{2}\Omega_m - 1 + \frac{3}{2}(w_0 + 1)\Omega_D, \quad j_0 = 1 - \frac{9}{2}(w_0 + 1)\Omega_D.
\] (23)

For \(-1.15 < w_0 < -1 \) and \( \Omega_D = 0.72 \), these parameters are in good agreement with observational data. The scalar field grows with time as
\[
\phi = \phi_0 + \alpha t,
\] (24)

and the potential describes the massive scalar field:
\[
V(\phi) = \frac{m^2}{2}(\phi - \phi^*)^2 + \frac{\alpha^2}{2}, \quad m^2 \equiv 3\alpha^2/2, \quad \phi^* \equiv \phi_0 - \frac{2\rho_0^{1/2}}{3^{1/2}\alpha}.
\] (25)

This type of potential is very popular in particle physics. For example, the light scalar particles like dilaton and moduli appear in superstring theories. The dark energy density increases with time. Hence, the universe accelerates. One can estimate the time required for disintegration of the Sun-Earth system, as an example. The dimensionless inertial force
\[
\bar{F}_{\text{iner}} = \frac{\ddot{a}}{aH_0^2},
\] (26)

can be expressed for \( t \gg t_0 \) as follows
\[
\bar{F}_{\text{iner}} \approx \Omega_D \frac{\rho}{\rho_0}.
\] (27)

Taking into account that the phantom energy density increases with time as
\[
\rho = \left( \rho_0^{1/2} - \frac{3^{1/2}}{2}(w_0 + 1)\rho_0 t \right)^2,
\] (28)

and that the Sun-Earth system disintegrates when \( \bar{F}_{\text{iner}} \sim 10^{23} \) (see [16]), one can find that the disintegration time is
\[
t_{\text{dis}} \approx \left( \frac{10^{12}}{|w_0 + 1|H_0} \right) \approx \left( \frac{10^{13}}{|w_0 + 1|} \right) \text{Gyr}.
\]

Hence, we have presented a realistic Little Rip cosmology caused by scalar dark energy with a standard particle physics massive potential. Note that if such cosmology occurs, one can speculate on visible reduction of galaxy clusters number in future cosmological surveys.
IV. ASYMPTOTICALLY DE SITTER EVOLUTION: PSEUDO-RIP

It is evident that some phantom/quintessence models may describe asymptotically de Sitter evolution. However, even in this case the disintegration of bound structures may take place. Such a scenario was dubbed the Pseudo-Rip cosmology \[16, 18\]. Of course, not all asymptotically de Sitter phantom scenarios lead to disintegration of bound structures.

Let us consider the example of a fluid which describes the phantom and/or quintessence field asymptotically approaching the de Sitter regime. Let

\[ f(\rho) = \alpha^2 \sin \left( \frac{\pi \rho}{\rho f} \right). \]  

(29)

If \( 2k\rho f < \rho < (2k+1)\rho f, \) \( k = 0, 1, \ldots, \) we have a phantom case while for \( (2k+1)\rho f < \rho < (2k+2)\rho f \) Eq. (29) describes quintessence. For \( \rho \ll \rho f, \) the parameter \( w \) is nearly constant:

\[ w \approx -1 - \frac{\alpha^2 \pi}{\rho f}, \]

\[ \rho = \rho f \left( \pm \frac{2}{\pi} \arctan \left( \tan \left( \frac{\pi \Delta}{2(\delta + 1)^2} \right) \right) + 2k \right), \quad \delta = 3\alpha^2/\pi \rho f, \quad \Delta = \rho_0/\rho f, \]  

(30)

where the “±” corresponds to phantom and quintessence theories, respectively. Depending on the parameter \( \Delta, \) the dark energy density tends to a single value from the set of “effective cosmological constants”

\[ \Lambda^{\text{eff}} = (2k + 1)\rho f. \]  

(31)

If the effective cosmological constant is sufficiently large \( (\ddot{a}/a \gg 0), \) then disintegration of bound structures can occur. It is obvious that this scenario occurs only for the phantom case (for quintessence with asymptotic de Sitter evolution, the acceleration of the universe can only decrease). The dimensionless inertial force

\[ F_{\text{inert}} = 3 \frac{\ddot{a}}{\rho_0 a}. \]  

(32)

tends to \( \rho_f/\rho_0 = \Delta^{-1} \) (for \( k = 0 \)) because \( a \sim \exp(\sqrt{\rho_f/3}t) \) at \( t \to \infty. \) Therefore if \( \Delta < 10^{-23} \) then \( F_{\text{inert}} > 10^{23} \) at \( t \to \infty \) and the disintegration of the Sun-Earth system eventually happens.

Such a scenario is compatible with observational data. For small \( \Delta, \) one can write with good accuracy the past dark energy density

\[ \rho \approx \rho_0(1 + z)^{-\delta}. \]  

(33)

For small \( \delta \) (\( w \to -1 \)), the dark energy density is nearly constant in the observable range \( 0 < z < 1.5. \) Therefore, the observable relation between modulus and redshift can be fulfilled in the model given by Eq. (29).

The deceleration and jerk parameters are within the range of the standard cosmology (as \( \delta \to 0 \)):

\[ q_0 = \frac{3}{2} \Omega_m - \frac{\delta}{2} \Omega_D - 1, \]  

(34)

\[ j_0 = \frac{1 + 3\delta + \delta^2}{2} \Omega_D. \]  

(35)

Thus, a realistic mild phantom scenario (Pseudo-Rip) may be easily realized.

V. DARK ENERGY MODELS WITH A TYPE III FUTURE SINGULARITY

We shall consider in this section a flat universe which ends up in a Type III future singularity \[19\]. Let us start from

\[ g(\rho) = -\beta^2 a^4 \rho^{1+\epsilon/3}, \]  

(36)

so that

\[ f(\rho) = \rho(-1 + \beta^2 a^4 \rho^{\epsilon/3}), \]
where $\beta$, $a_f$, and $\epsilon$ are positive constants. One can find the dependence of the dark energy density on the scale factor

$$\rho = \beta^{-6/\epsilon} (a_f^\epsilon - a^\epsilon)^{-3/\epsilon}.$$  \hfill (37)

Putting, for example, $\epsilon = 1$, we have

$$p = -\beta^2 a_f \rho^{2/3}.$$  

In this case, one can find the scale factor in parametric form

$$a = a_f \sin^2 \eta,$$

$$t = t_f + \frac{1}{\kappa} \left( \ln \left| \tan \frac{\eta}{2} \right| + \cos \eta + \frac{1}{3} \cos^3 \eta \right),$$  \hfill (38)

where

$$\kappa = \frac{1}{2 \sqrt{3 \beta^3 a_f^{-3/2}}}, \quad dt = \frac{\cos^4 \eta}{\kappa \sin \eta} d\eta.$$  

We now set $t_f = 0$. Therefore $\eta = 0$ corresponds to $t = -\infty$, $\eta = \pi/2$ to $t = 0$ (future singularity) and $\eta = \pi$ to $t = +\infty$. Hence, this solution describes two universes: the first one begins at $t = -\infty$ (Big Bang) and then expands to a future singularity which takes place at $t = 0$. The second solution begins at $t = 0$ (at a singularity) and then progressively contracts until a big crunch singularity at $t = \infty$. The asymptotic behavior of the scale factor near the future singularity in both cases is

$$a = a_f \left( 1 - \frac{5\kappa^2}{3|t|^{2/5}} \right), \quad t \sim 0.$$  

The addition of dark matter allows us to construct cosmological models in which the age of the universe is close to the conventional value of 10-20 Gyr. The dependence of the dark energy density on redshift is given by

$$\rho = \rho_0 (1 + z)^3 \left( \frac{N_0 - 1}{N_0 (1 + z)^\epsilon - 1} \right)^{3/\epsilon}.$$  \hfill (39)

where $N_0 = (a_f/a_0)^\epsilon$. For the current value of the EoS parameter $w_0$ we have

$$w_0 = -\beta^2 N_0 a_0^3 \rho_0^{\epsilon/3} = -\frac{N_0}{N_0 - 1}.$$  \hfill (40)

One can use the standard relation between redshift and time

$$H_0^{-1} \int \frac{dz}{(1 + z) \sqrt{h(z)}} = - \int dt,$$

$$h(z) = \Omega_m (1 + z)^3 + \Omega_D (1 + z)^3 \left( \frac{N_0 - 1}{N_0 (1 + z)^\epsilon - 1} \right)^{3/\epsilon},$$  \hfill (41)

for the calculation of the age of the universe and the estimation of the time of the future singularity. Integrating Eq. (41) from $z = \infty$ ($t = 0$, Big Bang) to $z = 0$ ($t = t_0$) gives the age of the universe:

$$t_0 = H_0^{-1} \int_0^\infty \frac{dz}{(1 + z) \sqrt{h(z)}},$$  \hfill (42)

For $N_0 \gg 1$ (i.e., for $w_0 \approx -1$) the function $h(z)$ can be approximated by

$$h(z) \approx \Omega_m (1 + z)^3 + \Omega_D.$$  \hfill (43)

Therefore, the age of the universe is eventually independent of $\epsilon$. This parameter, however, may change the remaining time before the future singularity $t_f - t_0$. Note that for the calculation of this time we can use Eq. (41), simply
Table I: A numerical calculation of the age of the universe, \( t_0 \), and the difference between the future singularity time, \( t_f \), and \( t_0 \) for various values of \( w_0 \) and \( \epsilon \). The time unit is 10\(^9\) years (Gyr) and we choose \( H_0^{-1} = 13.6 \) Gyr.

| \( w_0 \) | \( t_0 = -1.01 \) | \( t_0 = -1.05 \) | \( t_0 = -1.1 \) |
|---|---|---|---|
| \( \epsilon \) | \( t_f - t_0 \) | \( t_f - t_0 \) | \( t_f - t_0 \) | \( t_f - t_0 \) |
| 1 | 52.95 | 30.48 | 22.17 | 13.81 |
| 2 | 29.45 | 17.79 | 13.31 | 13.78 |
| 5 | 12.64 | 7.93 | 6.08 | 13.73 |
| 10 | 6.40 | 4.09 | 3.17 | 13.69 |
| 50 | 1.27 | - | 0.33 | - |
| 100 | 0.63 | - | 0.33 | - |
| 1000 | 0.06 | - | 0.03 | - |

assuming that the variable \( z \) can take negative values. The lower limit of integration corresponds to the scale factor \( a = a_f \), i.e., \( z_f = N_0^{-1/\epsilon} - 1 \). Therefore for \( t_f - t_0 \) one gets the following relation

\[
t_f - t_0 = H_0^{-1} \int_{z_f}^{0} \frac{dz}{(1 + z) \sqrt{H(z)}}.
\]

(44)

It is obvious that our model can fit the Supernova Cosmological Project data. For \( N \gg 1 \) the dark energy density is nearly constant in the interval \( 0 < t < t_0 \), i.e. the model mimics a cosmological constant in the past but leads to a finite-time future singularity. Moreover, the observational data do not impose any significant restrictions on the lifetime of the universe.

The numerical calculation of the age of the universe, \( t_0 \), and the difference between the future singularity time, \( t_f \), and \( t_0 \) for various values of \( w_0 \) and \( \epsilon \) are presented in Table I. The value of the Hubble parameter is chosen to be \( H_0^{-1} = 13.6 \) Gyr for this calculation.

The deceleration parameter is found to be

\[
g_0 = \frac{3}{2} \Omega_m - 1 + \frac{3}{2} \Omega_D(1 + w_0)
\]

(45)

and for \( w_0 \approx -1 \) \( g_0 \approx g_0^{SC} \). For the jerk parameter we have

\[
j_0 = \frac{3}{2}(3 + \epsilon)(w_0 + w_0^2)\Omega_D + 1.
\]

(46)

For sufficiently small \( \epsilon \) and \( w \approx -1 \), the jerk parameter is nearly equal to 1. Hence, a viable quintessence dark energy model which is compatible with observational data and leads to a Type III future singularity is constructed. Of course, it may be presented in terms of a scalar field with the field value \( \phi \) and potential \( V(\phi) \) written as functions of \( \eta \). Furthermore, the dependence of \( V(\phi) \) on \( \phi \) can be presented explicitly. We have

\[
\phi = \frac{\eta}{\alpha_0} + \phi_0,
\]

\[
V(\phi) = \alpha_1 (\cos(\alpha_0(\phi - \phi_0)))^{-6/\epsilon}(1 + \cos^{-2}(\alpha_0(\phi - \phi_0))),
\]

\[
\alpha_0 = \sqrt{\frac{1}{12} \epsilon}, \quad \alpha_1 = \frac{1}{2} \beta^{-6/\epsilon} a^{-3}_f.
\]

(47)

Other models with similar properties can be constructed.

VI. TYPE II FUTURE SINGULARITY DARK ENERGY

In this section we discuss realistic models of dark energy which contain a Type II future singularity \[19\] (or sudden future singularity \[13\]). It is known that such evolution may be also realized in \( f(R) \) modified gravity \[24\].

The simplest choice for an EoS producing a Type II future singularity is

\[
f(\rho) = \frac{\alpha^2}{1 - \rho/\rho_t},
\]

(48)
Figure 1: The scalar potential for the model (48). For a crush the scalar field rolls down from $-\infty$ to 0 and $V(\phi) \to \infty$; for a phantom sudden future singularity the scalar field rolls up and $V(\phi) \to \infty$ at some $\phi = \phi_s$.

where $\alpha$ and $\rho_f$ are positive constants. For $\rho_0 < \rho_f$, such a model describes phantom energy. Its energy density grows with time until the pressure tends to infinity and a phantom sudden future singularity occurs. If $\rho_0 > \rho_f$, the energy density decreases and a big crush occurs when $\rho$ is equal to $\rho_f$.

The time remaining before the future singularity is

$$t_f - t_0 = \frac{2}{\sqrt{3}} \int_{x_f}^{x_0} \frac{dx}{\alpha^2} \left(1 - \left(\frac{x}{x_f}\right)^2\right).$$

(49)

The corresponding description in terms of scalar field theory can be derived through Eqs. (12) and (11). The potential of the scalar field in parametric form is: (i) for phantom energy

$$\phi(y) = \frac{1}{\sqrt{3}y} \left(\arcsin y + y(1 - y^2)^{1/2}\right), \quad V(y) = \frac{\rho_f}{2} \left(2y^2 + \frac{\gamma}{1 - y^2}\right), \quad 0 \leq y \leq 1,$$

(50, 51)

(ii) for quintessence

$$\phi(y) = -\frac{1}{\sqrt{3}y} \left(\frac{\sqrt{1 - y^2}}{y^2} - \ln(1 - \sqrt{1 - y^2}) + \ln y\right),$$

$$V(y) = \frac{\rho_f}{2} \left(2y^{-2} + \gamma \frac{y^2}{y^2 - 1}\right), \quad 0 \leq y \leq 1,$$

(52, 53)

and $\gamma = \alpha^2/\rho_f$. This potential is depicted in Figure 1.

Such a model in principle can fit the latest supernova data from the Supernova Cosmology Project. The dependence of the dark energy density on the redshift $z$ can be derived from Eq. (7). After a simple algebraic calculation, one can obtain

$$\rho = \rho_f \left(1 \pm \left(1 - \Delta\right)^2 + 6\gamma \ln(1 + z)\right)^{1/2}, \quad \Delta = \rho_0/\rho_f, \quad \gamma = \alpha^2/x_i^2.$$

(54)

The sign “$+$” corresponds to the case of quintessence ($\Delta > 1$) while sign “$-$” to that of phantom energy ($\Delta < 1$). The current EoS parameter $w_0$ is

$$w_0 = -1 - \frac{\gamma}{\Delta(1 - \Delta)}.$$

(55)

Therefore, the dependence of the luminosity distance $D_L$ on the redshift $z$ is

$$D_L = \frac{c}{H_0} (1 + z) \int_0^z \left(\Omega_m(1 + z)^3 + \Omega_D h(z)\right)^{-1/2} dz,$$

$$h(z) = \Delta^{-1} \left(1 \pm \left(1 - \Delta\right)^2 + 6\gamma \ln(1 + z)\right)^{1/2}.$$

(56)
Table II: Numerical estimation of the difference $t_t - t_0$ in Gyr for various values of $\Delta$ and $\gamma$ in the case of a phantom model.

| $\Delta$ | $w_0 = -1.10$ | $w_0 = -1.08$ | $w_0 = -1.06$ | $w_0 = -1.04$ | $w_0 = -1.02$ |
|----------|----------------|----------------|----------------|----------------|----------------|
| 0.5      | 22.6           | 28.4           | 38.2           | 57.6           | 116.1          |
| 0.75     | 7.3            | 9.9            | 12.5           | 20.3           | 39.8           |
| 0.95     | 1.1            | 1.5            | 2.0            | 3.1            | 6.3            |

Table III: Numerical estimation of the difference $t_t - t_0$ in Gyr for various values of $\Delta$ and $\gamma$ in the case of quintessence.

| $w_0 = -0.98$ | $w_0 = -0.96$ | $w_0 = -0.94$ | $w_0 = -0.92$ | $w_0 = -0.90$ |
|----------------|----------------|----------------|----------------|----------------|
| $\Delta$ | $\Delta t_t$ | $\Delta t_d$ | $\Delta t_t$ | $\Delta t_d$ | $\Delta t_t$ | $\Delta t_d$ |
| 1.05 | 5.78           | 5.79           | 2.8           | 1.84          | 1.86          | 1.36          | 1.38          | 1.08          | 1.10          |
| 1.25 | 26.70          | 26.74          | 12.88         | 8.34          | 8.45          | 6.10          | 6.24          | 4.76          | 4.94          |
| 1.5  | 46.32          | 46.42          | 22.52         | 14.57         | 14.85         | 10.60         | 10.96         | 8.23          | 8.65          |

Eq. (59) coincides with (13) if $\gamma = 0$ ($f(x) = 0$). The SNe data are available in the range $0 < z < 1.5$. Therefore, if the parameters $\Delta$ and $\gamma$ are such that $(1 - \Delta)^2 \ll 6\gamma \ln(1 + z)$ in the observable range, the model under discussion is indistinguishable from $\Lambda$CDM cosmology.

The time remaining before a future singularity is

$$t_t - t_0 = \frac{1}{H_0} \int_0^0 du (1 + u)^{-1} \left( \Omega_m (1 + u)^3 + \Omega_D h(u) \right)^{-1/2}.$$  \tag{57}

The variable $u = a_0/a - 1$ varies from 0 (present time) to $\exp(- (1 - \Delta)^2/6\gamma) - 1$ ($\rho \to \infty$). The function $h(u)$ coincides with $h(z)$ in Eq. (58) (after changing $z \to u$). Numerical estimation of the difference $t_t - t_0$ for various values of $\Delta$ and $\gamma$ is given in Table II and Table III. For quintessence, the difference $t_d - t_0$ ($t_d$ is the moment of time when $\ddot{a} = 0$ and deceleration begins) is also calculated. We use the value of the Hubble parameter $H_0^{-1} = 13.6$ Gyr.

The difference $\delta \mu = 5 \log(D/D^{SC})$ ($\mu$ is the distance modulus) for $0 < z < 1.5$ lies in the interval $(-0.35$ to $0.35)$ for these parameter values. Taking into account that errors in the definition of the SNe modulus are $\sim 0.075 \div 0.5$, we conclude that our model fits these data with excellent precision. Note that further observational support for quintessence which leads to a Type II future singularity is given in Ref. [2].

The deceleration and jerk parameters are given by

$$q_0 = \frac{9\gamma}{2 \Delta (1 - \Delta)} \Omega_D + \frac{3}{2} \Omega_m - 1,$$

$$j_0 = -\frac{9\gamma}{2 \Delta (1 - \Delta)} \left( 1 + \frac{\gamma}{(1 - \Delta)^2} \right) \Omega_D + 1.$$  \tag{58}

It is convenient to present $q_0$ and $j_0$ through the parameters $w_0$ and $\Delta$:

$$q_0 = -\frac{9}{2} (w_0 + 1) \Omega_D + \frac{3}{2} \Omega_m - 1,$$

$$j_0 = \frac{9}{2} (w_0 + 1) \left( 1 - \frac{\Delta}{1 - \Delta} (w_0 + 1) \right) \Omega_D + 1.$$  \tag{59}

For $w_0 \approx -1$, the jerk parameter $j_0$ differs significantly from the standard value only for $\Delta \to 1$. The deceleration parameter $q_0$ for $-1.05 < w < -0.95$ lies in the interval $q_0^{SC} - 0.16 < q_0 < q_0^{SC} + 0.16$. Taking into account the errors in the definition of $q_0$ and $j_0$ one can conclude that dark energy models with a possible big crush or sudden future singularity fit well the current observational data.

The example considered above is a good theoretical illustration of dark energy models mimicking vacuum energy but leading to singularities of Type II. Dark energy with such behavior can be realized if the function $f(x)$ has a singularity at $x = x_t$.

An important remark is in order. For quintessence the disintegration of bound structures before a future singularity seems to be impossible. From Eqs. (5) and (6) it follows that

$$F_{iner} = -m \frac{1}{2} \left( w + \frac{1}{3} \right) \rho,$$  \tag{60}
Therefore, the maximal value of the inertial force for quintessence is

\[ F_{\text{iner}}^{\text{max}} = \frac{ml\rho}{3}. \]

The energy density of quintessence decreases when the universe expands and the inertial force also decreases with time. If we consider an EoS for which \( f(\rho) \) changes sign at \( x = x_{\text{ph}} \) then the energy density increases with time and tends to \( x_{\text{ph}} \). Hence, such an expanding universe is a de Sitter one.

### VII. BIG CRUSH DARK ENERGY MODELS

Another way to construct cosmological models with various types of evolution is to define the Hubble parameter as a function of time. Let us consider the Type II singularity, for example. This case occurs when \( H \) is finite and \( \dot{H} \) diverges but is negative. In this case, even though the universe is expanding, all structures are crushed rather than ripped. An example is given by

\[ H = H_0^{(0)} + H_1^{(0)} (t_c - t)^\alpha. \]  

(61)

Here \( H_0^{(0)} \) and \( H_1^{(0)} \) are positive constants and \( \alpha \) is a constant with \( 0 < \alpha < 1 \). If we choose \( \alpha \) to be given by the inverse of an odd number, \( \alpha = 1/(2n + 1) \), with positive integer \( n \), we can extend \( H \) beyond \( t = t_c \) by defining

\[ H = \begin{cases} 
H_0^{(0)} + H_1^{(0)} (t_c - t)^\alpha & \text{when } t < t_c \\
H_0^{(0)} - H_1^{(0)} (t - t_c)^\alpha & \text{when } t > t_c.
\end{cases} \]  

(62)

In a sense, the \( H \) obtained here is a smooth function of \( t \) although \( \dot{H} \) diverges at \( t = t_c \) since there is no sharp point, that is, a point where the line folds. At the point \( \dot{H} \) diverges but \( H \) is the continuous, single valued, and monotonically increasing or decreasing function of \( t \). One may consider the following model

\[ H = \begin{cases} 
H_0^{(0)} + H_1^{(0)} \left( \frac{\tanh \left( \frac{t_c - t}{t_0} \right)}{\tanh \left( \frac{t_0}{t_0} \right)} \right)^\alpha & \text{when } t < t_c \\
H_0^{(0)} - H_1^{(0)} \left( \frac{\tanh \left( \frac{t - t_c}{t_0} \right)}{\tanh \left( \frac{t_0}{t_0} \right)} \right)^\alpha & \text{when } t > t_c.
\end{cases} \]  

(63)

In the model (63), we find \( H \to \text{const} \) when \( t \to \pm \infty \), that is, the space-time is asymptotically de Sitter. Note that when \( H \) is finite, a rip occurs only for the phantom case, since \( \dot{H} > 0 \) before the singularity. However, a crush occurs for quintessence with \( \dot{H} < 0 \). In the models (62) and (63), a rip occurs when \( H_1^{(0)} < 0 \) and a crush occurs when \( H_1^{(0)} > 0 \).

From the FRW equations (1), the total energy density and pressure of dark energy as functions of time are:

\[ \rho(t) + \rho_m(t) = 3H^2, \quad p(t) = -3H^2 - 2\dot{H}. \]  

(64)

Neglecting the matter density, one can easily obtain the EoS of dark energy corresponding to model (62):

\[ p = \begin{cases} 
-\rho + 2\alpha 3^n H_1^{(0)}(\rho^{1/2} - \rho_1^{1/2})^{-2n}, & \rho_t = 3H_0^{(0)} \quad \text{when } t < t_c \\
-\rho - 2\alpha 3^n H_1^{(0)}(\rho^{1/2} - \rho_1^{1/2})^{-2n}, & \rho_t = 3H_0^{(0)} \quad \text{when } t > t_c.
\end{cases} \]  

(65)

When \( H_1^{(0)} < 0 \) or \( H_1^{(0)} > 0 \), \( p \to -\infty \) (phantom Type II singularity) or \( p \to +\infty \) (quintessence Type II singularity) at \( t \to t_c - 0 \). The scale factor is finite at \( t \to t - 0 \).

\[ a(t) = a_0 \exp \left\{ H_0^{(0)} t + H_1^{(0)} (t_c - t)^{(\alpha + 1)/(\alpha + 1)} - H_1^{(0)} t_c^{\alpha + 1}/(\alpha + 1) \right\}. \]  

(66)

In the following, we assume \( \alpha \) is given by \( \alpha = 1/(2n + 1) \) with positive integer \( n \). We now investigate if any object can be ripped or crushed at the singularity \( t = t_c \) although the inertial force \( F_{\text{iner}} \) diverges. For this purpose, we consider the work or the shift of the kinetic energy of the particle. For purposes of this estimation, we neglect all forces aside from the inertial force, and we neglect the first term in (5) since we assume only \( \dot{H} \) diverges. Then by solving the equation of motion

\[ m\ddot{x} = F_{\text{iner}} \sim ml\dot{H} = -mlH_1^{(0)} \alpha (t_c - t)^{\alpha - 1}, \]  

(67)
for the model \( \text{[32]} \), one finds
\[
x = x_0 + v_0 t - \frac{lH_1^{(0)}}{\alpha + 1} (t - t)\alpha + 1.
\] (68)

Then the shift of the kinetic energy can be estimated to be
\[
\Delta T = \int F\dot{x} \, dt \sim -mv_0 H_1^{(0)} \alpha \int (t - t)\alpha + 1 \, dt + ml^2 H_1^{(0)} 2 \alpha \int (t - t)\alpha + 1 \, dt.
\] (69)

Since \( \alpha > 0 \), the integration and therefore \( \Delta T \) is finite. Hence if the magnitude of the binding energy or the energy supporting the bound object is larger than the absolute value of \( \Delta T \), the object is not ripped or crushed although \( F_{\text{inert}} \) is infinite at \( t = t_c \).

In the case of a Type III singularity, \( H \) behaves as \( \text{[31]} \) but \( \alpha \) is negative and greater than unity, \( -1 < \alpha < 0 \). We also note that \( H_1^{(0)} > 0 \) since \( H > 0 \) when \( t < t_c \). Then in the inertial force \( \text{[9]} \), the first term behaves as \( -H (t - t)\alpha+1 \) and the second one as \( H^2 (t - t)\alpha+1 \). Since \( -1 < \alpha < 0 \), the first term dominates. Then in a way similar to \( \text{[69]} \), the shift of the kinetic energy can be estimated as
\[
\Delta T \sim -mv_0 H_1^{(0)} (t - t)\alpha + \frac{ml^2 H_1^{(0)} 2}{2} (t - t)\alpha + 1.
\] (70)

when \( t < t_c \) and it becomes positive and diverges when \( t \to t_c \). Therefore the Rip surely occurs even for a Type III singularity, which is different from the Type II singularity in \( \text{[69]} \), where all the objects are not always crushed or ripped.

By using the formulation in Ref. \( \text{[25]} \), we now consider what kind of scalar tensor model, whose action is given by
\[
S = \int d^4x\sqrt{-g}\left\{ \frac{1}{2\kappa^2} R - \frac{1}{2} \omega(\phi) \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) \right\},
\] (71)
can realize the evolution of \( H \) given by Eq. \( \text{[61]} \). Here, \( \omega(\phi) \) and \( V(\phi) \) are functions of the scalar field \( \phi \). If we consider the model where \( \omega(\phi) \) and \( V(\phi) \) are given by a single function \( f(\phi) \), as follows,
\[
\omega(\phi) = -\frac{2}{\kappa^2} f''(\phi), \quad V(\phi) = \frac{1}{\kappa^2} \left( 3f'(\phi)^2 + f''(\phi) \right),
\] (72)
the exact solution of the FRW equations has the following form:
\[
\phi = t, \quad H = f'(t).
\] (73)

Then for the model \( \text{[61]} \) with \( \alpha = 1/(2n + 1) \), we find
\[
\omega(\phi) = \frac{2H_1^{(0)} \alpha}{\kappa^2} (t - \phi)^{\frac{2n}{n+1}}, \quad V(\phi) = \frac{1}{\kappa^2} \left\{ \left( H_0^{(0)} + H_1^{(0)} (t - \phi)^{\frac{1}{n+1}} \right)^2 - 2H_1^{(0)} \alpha (t - \phi)^{\frac{2n}{n+1}} \right\}.
\] (74)

If we redefine the scalar field as
\[
\varphi = -\sqrt{\frac{(2n + 1)H_1^{(0)}}{\kappa(n + 1)}} (t - \phi)^{\frac{n+1}{n+1}},
\] (75)
the kinetic term in the action \( \text{[71]} \) becomes canonical
\[
-\frac{1}{2} \omega(\phi) \partial_{\mu} \phi \partial^{\mu} \phi = -\frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi,
\] (76)
and the potential is given by
\[
V(\phi) = \frac{1}{\kappa^2} \left\{ \left( H_0^{(0)} + H_1^{(0)} \left( \frac{\kappa(n + 1)}{\sqrt{2H_1^{(0)}(2n + 1)}} \right) \varphi \right)^{\frac{n+1}{n+1}} - H_1^{(0)} \alpha \left( \frac{\kappa(n + 1)}{\sqrt{2H_1^{(0)}(2n + 1)}} \varphi \right)^{\frac{2n}{n+1}} \right\}.
\] (77)
Note that $\varphi < 0$ when $\phi = t < t_c$ and $\varphi \to 0$ when $\phi = t \to t_c$. Near the singularity $\phi = t \to t_c$ ($\varphi \to 0$), only the last term in the potential (77) dominates:

$$V(\phi) \sim -\frac{H_1^{(0)}{\alpha}}{\kappa^2} \left( - \frac{\kappa(n + 1)}{\sqrt{2H_1^{(0)}(2n + 1)}} \right)^{-\frac{2n}{\kappa^2}}. \quad (78)$$

In particular, when $n = 1$, we find

$$V(\phi) \sim \frac{\alpha}{\kappa^2} \sqrt{\frac{3H_1^{(0)^3}}{2}} \varphi^{-1}. \quad (79)$$

Thus, the big crush occurs when the scalar field drops into the infinitely deep potential proportional to the inverse power of the scalar potential.

We have constructed models which generate the big crush and have given the explicit action in terms of the scalar field. After the big crush, the universe may evolve to asymptotic de Sitter space-time. Hence, big crush phenomenon looks much less dangerous than disintegration of bound structures.

**VIII. PHANTOM MODELS AND SINGULARITIES**

One way to realize many of the models proposed here is through a scalar field with a negative kinetic term (phantom models). The asymptotic future evolution of such models was examined systematically in Ref. [26], and we restate a number of those results here in order to show explicitly the relation between various types of future singularity.

The simplest phantom models are characterized by a field $\phi$ with a negative kinetic term. Such models evolve according to the equation

$$\ddot{\phi} + 3H\dot{\phi} - V'(\phi) = 0, \quad (80)$$

where the prime denotes the derivative with respect to $\phi$. A field evolving according to this equation rolls uphill in the potential. The density and pressure for the phantom field are given by

$$\rho_\phi = -\frac{1}{2} \dot{\phi}^2 + V(\phi), \quad (81)$$

and

$$p_\phi = -\frac{1}{2} \dot{\phi}^2 - V(\phi), \quad (82)$$

respectively, so the equation of state parameter is

$$w_\phi = \frac{(1/2)\dot{\phi}^2 + V(\phi)}{(1/2)\dot{\phi}^2 - V(\phi)}. \quad (83)$$

As noted in Ref. [26], the asymptotic behavior of the equation of state parameter depends on the corresponding asymptotic behavior of $V'/V$. If $V'/V \to 0$, then $w \to -1$. This set of models displays the most diverse behavior, since it can correspond to either a Big Rip, a Little Rip, or a Pseudo-Rip, depending on the exact functional form for $V(\phi)$. A Big Rip (Type I singularity) occurs when [26]

$$\int \sqrt{\frac{V(\phi)}{V'(\phi)}} d\phi \to \text{finite}. \quad (84)$$

If, instead, the integral in equation (84) diverges, we have either a Little Rip or a Pseudo-Rip. A Pseudo-Rip occurs if $V(\phi) \to \text{const}$, while a Little Rip occurs if $V(\phi) \to \infty$ (see also Section III).

The second set of models examined in Ref. [26] corresponds to $V'/V \to \text{constant}$. This gives a constant value for $w$ with $w < -1$, and produces a Big Rip (Type I) singularity.

Finally, if $V'/V \to \pm \infty$, we have $w \to -\infty$, which can result in a Type III singularity (see also Ref. [27]).
IX. CONCLUSION

In summary, dark energy models with various scenarios of evolution have been presented. Specifically, we constructed scalar dark energy models with Type II and Type III finite-time future singularities, Little Rip and Pseudo-Rip cosmologies with finite-time disintegration of bound structures and Big Crush cosmologies. It was shown that such models are consistent with observational data from the Supernova Cosmology Project and therefore may be viable alternatives to the ΛCDM cosmology. Moreover, they may be stable for billions of years before entering a soft future singularity (with a finite scale factor at the Rip) or before entering a finite-time dissolution of bound structures.

We have shown that the future evolution of the universe is determined by the selected EoS of dark energy. Unfortunately, current data for such important parameters as $q_0$ and $j_0$ are not very reliable, so the nature of the dark energy cannot yet be determined, and one can therefore only consider some typical models. In the future, more accurate measurements of the deceleration and jerk parameters as well as other cosmological parameters will help to define the exact nature of dark energy. Then we will acquire the information on the parameters of the fluid description for the EoS of dark energy in this paper and therefore also the information of the parameters in a reconstructed scalar field theory. In other words, more precise values of cosmological parameters may significantly constrain the dark energy models under discussion.

The key point is that the current observational data do not answer, even in principle, the question of whether or not the universe will end in a future singularity or Rip cosmology. One can construct (as we have here) models that mimic standard ΛCDM up the present, but evolve in the future into any number of possible future states, including Pseudo-Rip models, Little Rip models, and a variety of different future singularities. With a variety of ΛCDM-like cosmological models in hand, one can already start to think about future cosmological experiments to define the future of the universe more precisely.

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