Non–commutative Group Manifolds

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Abstract

We show that a chiral sector of a symplectic group manifold possesses a symmetry similar to, but somewhat weaker than the Lie–Poisson one.

1 Splitting of a cotangent bundle

For the dynamical systems with a high degree of symmetry it is natural to try to parametrize as much of the phase space, as possible by the global constants of motion.

In the case of geodesic motion on a semisimple group manifold (with the Hamiltonian being given by the quadratic Casimir invariant) all global constants of motion are described by the momentum mappings corresponding to the natural left and right actions of the group on its cotangent bundle. The equivariance of the momentum mappings allows one to split the phase space $T^*G$ into the sectors corresponding to the types of coadjoint orbits. In the case of compact groups this decomposition is quite simple as one has the unique type of the orbits of maximal dimension.

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It has been shown [8] that in such a case one can represent $T^*G$ by the Cartesian square of $G \times \mathcal{W}$ divided by suitable relations. ($\mathcal{W}$ stands for some chosen Weyl chamber, used to parametrize the space of coadjoint orbits).

We call $G \times \mathcal{W}$ a Chiral sector. The canonical symplectic form of the cotangent bundle pulls back onto the product of the two sectors as a difference $\Omega_L - \Omega_R$ of two components, each component living on one sector.

Each component is an exact two–form, giving each sector a structure of a symplectic manifold.

Thus the classical model can now be quantized in two different ways:

1. One can quantize the cotangent bundle. This is straightforward and yields known results. The operators corresponding to the matrix elements of any representation of $G$ commute.

2. One can quantize each of the sectors separately. It is much more complicated, but results in a very interesting class of non–commutative ($\mathcal{C}^*$ for compact groups) algebras describing a new class of quantum group manifolds. In each sector there is a non–commutative spectrum generating algebra (SGA).

The following diagram summarizes these ideas:

The detailed description of the above procedure called chiral splitting and fusion can be found in [8].
2 Symplectic structure of a sector

The symplectic structure of the chiral sector can be most transparently described in terms of Chevalley basis of $G$.

In case of compact groups the space of coadjoint orbits can be conveniently parametrized by choosing some Weyl chamber $W$ in $G$ (i.e. the dual of a Cartan subalgebra divided by the Weyl group of its discreet symmetries). $W$ intersects each regular orbit exactly once.

We shall use the following notation [4]. The set of simple roots dual to the chosen Weyl chamber is $\Delta$, the set of roots of the Lie algebra is $\Phi$, and the set of positive roots is $\Phi^+$. The element of the Cartan subalgebra $K$–dual to the root $\beta$ is $t_\beta = i[e_\beta, e_{-\beta}]$. In addition we introduce $\theta^{\alpha_i}$, the one–form dual to the simple root $t_{\alpha_i}$ and $\omega^\beta$, the left invariant one–form dual to the root vector $e_{\beta}$. Finally, $w_i$ is the coordinate in the Weyl chamber in the basis dual to the one formed by $t_{\alpha_i}$.

The left component (the symplectic form of the left sector) is given by

$$\Omega_L = \sum_{\alpha_i \in \Delta} dw_i \wedge \theta^{\alpha_i} + i \sum_{\beta \in \Phi^+} \langle w, t_{\beta} \rangle \omega^\beta \wedge \omega^{-\beta}. \quad (1)$$

It has a global symplectic potential:

$$\Omega_L = d \sum w_i \theta^{\alpha_i}. \quad (2)$$

For the 'right' component the expressions are analogous.

The symplectic structure gives Poisson brackets of matrix elements in arbitrary representations of $G$ as:

$$\{T_1 \otimes T_2\}_M(g) = (T_1 \otimes T_2)(g) r_{12}(w) \quad (3)$$

where

$$r_{12}(w) = \sum_{\beta \in \Phi^+} \frac{i}{\langle w, t_{\beta} \rangle} \left[ \tau_1(e_{-\beta}) \otimes \tau_2(e_{\beta}) - \tau_1(e_{\beta}) \otimes \tau_2(e_{-\beta}) \right]. \quad (4)$$

together with

$$\{w_i, T\}(g) = T(g) \tau(t_{\alpha_i}). \quad (5)$$

($T$ and $\tau$ label representations of $G$, and their differentials, respectively.)
3 Example: $G=SU(2)$

In the fundamental representation $T_f$:

$$T_f(g) = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} ; \quad w \in \mathbb{R}_+$$  \hfill (6)

$$a^* a + b^* b = 1. \quad \hfill (7)$$

$$\{a^*, a\}_M = \frac{i}{w} bb^* , \quad \{a, b\}_M = 0 \quad ,$$

$$\{a, b^*\}_M = \frac{i}{w} ab^* , \quad \{b, b^*\}_M = -\frac{i}{w} aa^* ,$$

$$\{a, w\}_M = -ia \quad , \quad \{a^*, w\}_M = ia^* ,$$

$$\{b, w\}_M = -ib \quad , \quad \{b^*, w\}_M = ib^* . \quad \hfill (8)$$

Geometric Quantization \cite{2} \cite{3} of this structure gives:

$$\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} = 1. \quad \hfill (10)$$

$$\Delta = 1 - \hat{a} \hat{a}^\dagger - \hat{b} \hat{b}^\dagger (= \hbar \hat{w}^{-1})$$

$$(1 + \Delta)\hat{a}^\dagger \hat{a} = \hat{a} \hat{a}^\dagger + \Delta$$

$$(1 + \Delta)\hat{b}^\dagger \hat{b} = \hat{b} \hat{b}^\dagger + \Delta$$

$$\quad (1 + \Delta)\hat{a}^\dagger \hat{b} = \hat{b} \hat{a}^\dagger$$

$$ab = ba \quad \hfill (12)$$

We can say that the above relations define the structure of non–commutative group manifold, namely “quantum” $S^3$. In the representation Hilbert space there is a unique ‘vacuum’ state $\varphi_o$ satisfying

$$\hat{a}^\dagger \varphi_o = 0 = \hat{b}^\dagger \varphi_o \quad \hfill (13)$$

We can expect that the procedure of geometric quantization will enable us to describe the corresponding non–commutative manifolds for all compact groups.
4 Trace of the Right Action Symmetry

By performing the chiral splitting we have broken the right symmetry of the dynamical system. This happened because we have chosen some fixed Weyl chamber in order to parametrize the coadjoint orbits.

The symmetry is restored after fusion of both sectors. One may ask however whether there is a trace of the broken symmetry in the chiral sector?

The natural right action of $G$ on the sector is given by:

$$(G \times W) \times G \ni (g, w, h) \mapsto (gh^{-1}, w) \in G \times W.$$ (14)

We assume that the acting group $G$ is equipped with the bracket $\{ , \}_G$, such that the above action preserves the quadratic Poisson brackets for the group elements in the chiral sector. We should stress that we do not demand the preservation of the brackets of functions on $W$ with the group elements.

The equation for the bracket $\{ , \}_G$ reads:

$$\{ T_1 \otimes T_2 \}_M (gh^{-1}) =$$

$$= \{ T_1 \otimes T_2 \}_M (g) (T_1 \otimes T_2) (h^{-1}) + (T_1 \otimes T_2) (g) \{ T_1 \otimes T_2 \}_G (h^{-1})$$

In terms of the Poisson tensor $calP_G$ corresponding to $\{ , \}_G$, the unique solution to (15) is:

$$P_G = L_g^* r - R_g^{s-1} r,$$ (16)

Where $r$ is that of (11).

Question: is $P_G$ Lie–Poisson? It has the familiar form of a 'Sklyanin bracket', but does the bivector $r$ satisfy YBE ?

For $SU(2)$ the answer is affirmative. In this case we obtain a family of Lie Poisson structures labeled by the Weyl chamber parameter $w$:

$$\{ \alpha, \beta \}_G = - \frac{i}{w} \alpha \beta ; \quad \{ \alpha^*, \beta^* \}_G = - \frac{i}{w} \alpha^* \beta^*;$$

$$\{ \alpha, \beta^* \}_G = - \frac{i}{w} \alpha \beta^* ; \quad \{ \alpha, \alpha^* \}_G = \frac{2i}{w} \beta^* \beta;$$

$$\{ \beta^*, \beta \}_G = 0 .$$ (17)

For the compact groups of higher rank the tensor $P_G$ is not a Poisson one as it breaks the Jacobi identity for the corresponding bracket of functions. It
is not clear to us at this moment how to realize the quantum version of the above symmetry as we don’t know which of the relations (12) are independent. We hope to get back to this problem in a forthcoming paper.

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