Denseness conditions, morphisms and equivalences of toposes

Olivia Caramello

August 6, 2019

Abstract

We establish a general theorem providing necessary and sufficient explicit conditions for a morphism of sites to induce an equivalence of toposes. This stems from a detailed analysis of arrows in Grothendieck toposes and denseness conditions, which yields results of independent interest. We also derive site characterizations of the property of a geometric morphism to be an inclusion (resp. a surjection, hyper-connected, localic), as well as site-level descriptions of the surjection-inclusion and hyperconnected-localic factorizations.

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1 Introduction

In this paper we establish a number of results around the theme of equivalences of toposes and their characterizations in terms of sites. Its contents can be summarized as follows.

In section 2 we establish a basic result allowing us to describe the subobjects of the sheafification of a certain object $A$ in terms of subobjects of $A$ which are closed with respect to the associated closure operation, and recall the notion of a morphism of sites, which will play a key role in the following parts of the paper. We shall work in the setting of small-generated sites (in the sense of [7]) (also called essentially small sites in [4]) rather than in the usual, but restricted, context of small sites since all the fundamental results for morphisms of sites and flat functors actually hold at this higher level of generality and their formulations at this level provide a much greater flexibility both at the theoretical level and in connection with applications.

In section 3 we show that arrows in a topos $\text{Sh}(\mathcal{C}, J)$ between objects of the form $l^f_J(c)$ for $c \in \mathcal{C}$ (where $l^f_J$ is the functor $\mathcal{C} \to \text{Sh}(\mathcal{C}, J)$ given by the composite of the Yoneda embedding $y_C : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \text{Set}]$ with the associated
sheaf functor $a_J : [C^{op}, \text{Set}] \to \text{Sh}(C, J)$) can all be locally represented in terms of arrows coming from the site. Then we investigate, more generally, arrows in $\text{Sh}(C, J)$ between objects of the form $a_J(P)$, and obtain an explicit characterization for them in terms of $J$-functional relations, also describing the operation of composition of such relations which corresponds to composition of the corresponding arrows. Further, we discuss the specialization of this characterization in the case of arrows between objects of the form $f^*_J(c)$ for $c \in C$ and its relationship with the alternative description previously obtained.

In section 4 we recall the notion of a dense morphism of sites and show that, if the target site is subcanonical, it corresponds precisely to the property of the associated geometric morphism to be an equivalence. In order to generalize this result to the setting of arbitrary sites, we introduce the notion of a weakly dense morphism of sites, giving an explicit characterization of it, and discuss an example of a weakly dense morphism which is not dense. The resulting general theorem providing necessary and sufficient conditions for a morphism of sites to induce an equivalence of toposes represents a vast extension of Grothendieck’s Comparison Lemma [11]. Next we deduce from this result a criterion for a $J$-continuous flat functor $C \to E$ to induce, via Diaconescu’s equivalence, an equivalence of toposes $E \simeq \text{Sh}(C, J)$, and a criterion for the toposes of sheaves on two small-generated sites to be equivalent. Lastly, we introduce some notions of local faithfulness, local fullness and local surjectivity and show that they are naturally related to our denseness conditions.

In section 5, by applying results obtained in the previous section, we explicitly characterize the morphisms of sites whose corresponding geometric morphism is a surjection (resp. an inclusion, hyperconnected, localic); this applies in particular to continuous flat functors, giving necessary and sufficient conditions for the geometric morphisms corresponding to them to satisfy such properties. This analysis notably leads to a characterization of the property of a geometric morphism to be an inclusion entirely in terms of its inverse image functor. We then give site-level descriptions of the surjection-inclusion and hyperconnected-localic factorizations, and derive alternative criteria for a morphism of sites (resp. a continuous flat functor) to induce an equivalence of toposes. In this section, we also identify a most general framework for defining induced Grothendieck topologies (which subsumes the classical notion of Grothendieck topology induced on a dense subcategory), and investigate a notion of image of a Grothendieck topology under a functor.

In section 6 we investigate the same properties of geometric morphisms in the context of morphisms induced by comorphisms of sites, obtaining site-theoretic characterizations for them which generalize, in a natural but
non-trivial way, a number of known results.

2 Preliminaries

Recall that a small-generated site (in the sense of [7]) is a site \((C, J)\) such that \(C\) is locally small and has a small \(J\)-dense subcategory. Notice that, for any Grothendieck topos \(E\), the site \((E, J_E^{\text{can}})\), where \(J_E^{\text{can}}\) is the canonical topology on \(E\), is small-generated.

Even though results about morphisms of sites are usually formulated only for small sites, they admit a straightforward extension to small-generated sites (see, for instance, [7]).

For any small-generated site \((C, J)\), we shall denote by \(l^J_C\) (or simply by \(l\), when there is no risk of ambiguity) the functor \(C \to \text{Sh}(C, J)\) given by the composite of the Yoneda embedding \(y_C : C \to [\mathcal{C}^{\text{op}}, \text{Set}]\) with the associated sheaf functor \(a_J : [\mathcal{C}^{\text{op}}, \text{Set}] \to \text{Sh}(C, J)\). We shall say that a family of arrows in \(C\) with common codomain is \(J\)-covering if the sieve generated by it is. The unit of the adjunction between the inclusion functor \(i\) of \(\text{Sh}(C, J)\) into \([\mathcal{C}^{\text{op}}, \text{Set}]\) and the associated sheaf functor \(a_J\) will be denoted by \(\eta_J\) (or simply by \(\eta\)).

Notice that we have a closure operation \(c_J\) on subobjects in the presheaf category \([\mathcal{C}^{\text{op}}, \text{Set}]\), which admits the following explicit description: for any subobject \(A \to E\) in \([\mathcal{C}^{\text{op}}, \text{Set}]\), we have

\[
c_J(A)(c) = \{ x \in E(c) \mid \{ f : d \to c \mid E(f)(x) \in A(d) \} \in J(c) \}
\]

for any \(c \in C\). Note also that, for any small-generated site \((\mathcal{C}, J)\), the canonical inclusion functor \(\text{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \text{Set}]\) admits a left adjoint \(a_J\), which we call, by analogy with the classical case, the associated sheaf functor, and which is (colimit and) finite-limit-preserving. Indeed, if \(\mathcal{D}\) is a \(J\)-dense small subcategory of \(\mathcal{C}\) then, by the Comparison Lemma, we have an equivalence \(c : \text{Sh}(\mathcal{D}, J|_\mathcal{D}) \to \text{Sh}(\mathcal{C}, J)\) which is the restriction, along the inclusions \(\text{Sh}(\mathcal{D}, J|_\mathcal{D}) \hookrightarrow [\mathcal{D}^{\text{op}}, \text{Set}]\) and \(\text{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \text{Set}]\), of the functor \([\mathcal{D}^{\text{op}}, \text{Set}] \to [\mathcal{C}^{\text{op}}, \text{Set}]\) given by the right Kan extension along the embedding \(i\) of \(\mathcal{D}^{\text{op}}\) into \(\mathcal{C}^{\text{op}}\), whence the composite functor \(c \circ a_{\mathcal{D}|_\mathcal{D}} \circ (\circ i) : [\mathcal{C}^{\text{op}}, \text{Set}] \to \text{Sh}(\mathcal{C}, J)\) yields a left adjoint to the inclusion \(\text{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \text{Set}]\).

The closure operation \(c_J\) can be notably used for obtaining site-theoretic descriptions of many properties and constructions on the topos \(\text{Sh}(\mathcal{C}, J)\) which can be expressed in terms of subobjects in it; for instance, we have the following result:

**Lemma 2.1.** Let \((\mathcal{C}, J)\) be a small-generated site and \(\alpha : F \to G\) an arrow in the presheaf category \([\mathcal{C}^{\text{op}}, \text{Set}]\). Then
if and only if the diagonal monomorphism

We can thus conclude that

\( F \) if the diagonal arrow

**Proof** (i) An arrow \( A \to B \) in a category with pullbacks is a monomorphism if and only if the diagonal monomorphism \( A \to A \times_B A \) is an isomorphism. We can thus conclude that \( a_J(\alpha) \) is a monomorphism in \( \text{Sh}(C, J) \) if and only if the diagonal arrow \( F \to F \times_G F \) is \( c_J \)-dense, that is if and only if for every \( c \in C \) and any elements \( x, x' \in F(c) \) such that \( \alpha(c)(x) = \alpha(c)(x') \), the sieve \( \{ f : d \to c \mid F(f)(x) = F(f)(x') \} \) is \( J \)-covering.

(ii) \( a_J(\alpha) \) is an epimorphism in \( \text{Sh}(C, J) \) if and only if for every \( c \in C \) and \( x \in G(c) \), the sieve \( \{ f : d \to c \mid G(f)(x) \in \text{Im}(\alpha(d)) \} \) is \( J \)-covering.

(iii) \( a_J(\alpha) \) is an isomorphism in \( \text{Sh}(C, J) \) if and only if both conditions (ii) and (iii) are satisfied.

**Remark 2.2.** Let \( A_{c,x} := \{ f : d \to c \mid G(f)(x) \in \text{Im}(\alpha(d)) \} \) (for each \( c \in C \) and \( x \in G(c) \)). Then we have that for any arrow \( \xi : c \to c' \) and any elements \( x \in G(c) \) and \( x' \in G(c') \) such that \( G(\xi)(x') = x \), \( A_{c,x} = \xi^*(A_{c',x'}) \). So, if the codomain of \( \alpha \) is a representable \( y_C(c) \), condition (ii) becomes equivalent to the requirement that the sieve \( \{ f : d \to c \mid f \in \text{Im}(\alpha(d)) \} \) be \( J \)-covering.

### 2.1 Subsheaves and closed subobjects

The following result is probably well-known, but we did not find a proof of it in the literature, so we give it here.

**Proposition 2.3.** Let \( (C, J) \) be a small-generated site. Then for any object \( F \) of \([C^{\text{op}}, \text{Set}]\), denoting by \( \text{ClSub}_{[C^{\text{op}}, \text{Set}]}(F) \) the sub-lattice of \( \text{Sub}_{[C^{\text{op}}, \text{Set}]}(F) \) consisting of the \( c_J \)-closed subobjects, we have a lattice isomorphism

\[
\text{Sub}_{\text{Sh}(C, J)}(a_J(F)) \cong \text{ClSub}_{[C^{\text{op}}, \text{Set}]}(F)
\]

which sends a subobject in \( \text{ClSub}_{[C^{\text{op}}, \text{Set}]}(F) \) to its image under \( a_J \) and a subobject of \( a_J(F) \) to the pullback of it under the unit arrow \( F \to a_J(F) \).
Proof. Given a subobject \( m : A \to a_J(F) \) in \( \mathbf{Sh}(C, J) \), let us associate with it the subobject \( n : A' \hookrightarrow F \) of \( F \) defined by the following pullback diagram in \([C^{\text{op}}, \mathbf{Set}]\):

\[
\begin{array}{ccc}
A' & \to & A \\
\downarrow^{n} & & \downarrow^{m} \\
F & \to & a_J(F)
\end{array}
\]

Recall that the \( c_J \)-closure \( c_J(k) : A'' \hookrightarrow F \) of a subobject \( k : A' \hookrightarrow F \) in \([C^{\text{op}}, \mathbf{Set}]\) is characterized by the following pullback diagram:

\[
\begin{array}{ccc}
A'' & \to & a_J(A') \\
\downarrow^{c_J(k)} & & \downarrow^{s_J(k)} \\
F & \to & a_J(F)
\end{array}
\]

It thus follows that \( n \) is \( c_J \)-closed, since by applying the pullback-preserving functor \( a_J \) to the former pullback square we obtain that \( m \cong a_J(n) \). Conversely, we associate with any \( c_J \)-closed subobject \( n \) the subobject \( a_J(n) \) of \( \mathbf{Sh}(C, J) \). It is immediate to see that these two operations are inverse to each other, and that they are order-preserving; so they are lattice isomorphisms. □

Remarks 2.4. (a) Given an arrow \( \alpha : P \to Q \) in \([C^{\text{op}}, \mathbf{Set}]\), the \( c_J \)-closed subobject \( Q_\alpha \) of \( Q \) corresponding via Proposition 2.3 to the image of the arrow \( a_J(\alpha) \) is given by

\[
Q_\alpha(c) = \{ x \in Q(c) \mid \{ f : d \to c \mid Q(f)(x) \in \text{Im}(\alpha(d)) \} \in J(c) \}
\]

for any \( c \in C \).

(b) Proposition 2.3 clearly generalizes to the setting of elementary toposes, replacing \([C^{\text{op}}, \mathbf{Set}]\) by an arbitrary elementary topos \( \mathcal{E} \), \( J \) by a local operator on it and \( c_J \) by the associate closure operation on subobjects in \( \mathcal{E} \).

2.2 Morphisms of sites

Let us recall the notion of a morphism of sites:

Definition 2.5. A functor \( F : C \to D \) is said to be a morphism of sites \((C, J) \to (D, K)\), where \( J \) is a collection of sieves in \( C \) and \( K \) is a Grothendieck topology on \( D \), if it satisfies the following conditions:
(i) $F$ sends every $J$-covering family in $C$ into a $K$-covering family in $D$.

(ii) Every object $d$ of $D$ admits a $K$-covering family

$$d_i \to d, \quad i \in I,$$

by objects $d_i$ of $D$ which have morphisms

$$d_i \to F(c'_i)$$

to the images under $F$ of objects $c'_i$ of $C$.

(iii) For any objects $c_1, c_2$ of $C$ and any pair of morphisms of $D$

$$g_1 : d \to F(c_1), \quad g_2 : d \to F(c_2),$$

there exists a $K$-covering family

$$g'_i : d_i \to d, \quad i \in I,$$

and a family of pairs of morphisms of $C$

$$f_1^i : c'_i \to c_1, \quad f_2^i : c'_i \to c_2, \quad i \in I,$$

and of morphisms of $D$

$$h_i : d_i \to F(c'_i), \quad i \in I,$$

making the following squares commutative:

(iv) For any pair of arrows $f_1, f_2 : c_1 \Rightarrow c_2$ of $C$ and any arrow of $D$

$$g : d \to F(c_1)$$

satisfying

$$F(f_1) \circ g = F(f_2) \circ g,$$

there exist a $K$-covering family

$$g_i : d_i \to d, \quad i \in I,$$
and a family of morphisms of $C$

$$k_i : c'_i \rightarrow c_1, \; i \in I,$$

satisfying

$$f_1 \circ k_i = f_2 \circ k_i, \; \forall i \in I,$$

and of morphisms of $D$

$$h_i : d_i \rightarrow F(c'_i), \; i \in I,$$

making the following squares commutative:

\[
\begin{array}{ccc}
F(c'_i) & \xrightarrow{F(k_i)} & F(c_1) \\
\downarrow g & & \downarrow g \\
F(c'_i) & \xrightarrow{g_i} & d
\end{array}
\]

**Remark 2.6.** Definition 2.5 is obtained by expressing entirely in terms of $F$ the property of the functor $l' \circ F : C \rightarrow \text{Sh}(\mathcal{D}, K)$ to be flat (equivalently, filtering, in the sense of Definition VII8.1 [6]) and $J$-continuous, where $l'$ is the canonical functor $\mathcal{D} \rightarrow \text{Sh}(\mathcal{D}, K)$ given by the composite of the Yoneda embedding with the associated sheaf functor (by using the possibility of $K$-locally representing the arrows in the image of $l'$ in terms of arrows in $\mathcal{D}$, cf. Proposition 3.1(i)); so we have that a functor $F : C \rightarrow \mathcal{D}$ is a morphism of sites $(C, J) \rightarrow (\mathcal{D}, K)$ if and only if $l' \circ F$ is a $J$-continuous flat functor $C \rightarrow \text{Sh}(\mathcal{D}, K)$, and, conversely, for any small-generated site $(C, J)$ and any Grothendieck topos $\mathcal{E}$, a $J$-continuous flat functor $C \rightarrow \mathcal{E}$ is precisely a morphism of sites $(C, J) \rightarrow (\mathcal{E}, J_{\text{can}}^{\mathcal{E}})$. It thus follows at once from Diaconescu’s equivalence between geometric morphisms and flat functors that, if $J$ is a Grothendieck topology, then morphisms of sites $(C, J) \rightarrow (\mathcal{D}, K)$ are precisely the functors $F : C \rightarrow \mathcal{D}$ which induce a geometric morphism $\text{Sh}(F) : \text{Sh}(\mathcal{D}, K) \rightarrow \text{Sh}(C, J)$ making the following square commutative:

\[
\begin{array}{ccc}
C & \xrightarrow{F} & \mathcal{D} \\
\downarrow l & & \downarrow l' \\
\text{Sh}(C, J) & \xrightarrow{\text{Sh}(F)^*} & \text{Sh}(\mathcal{D}, K)
\end{array}
\]

This result actually holds for any notion $J$ of covering family in $C$ (by replacing $J$ with the Grothendieck topology generated by it in the definition of sheaves on $(C, J)$ – cf. Proposition 2.7 below).
Definition 2.5 is equivalent to the one given in [7], which specifies that a functor is a morphism of sites when it is cover-preserving and covering-flat (in the sense that for any finite diagram \( D \) in \( C \) every cone over an object of the form \( F(c) \) factors locally through the \( F \)-image of a cone over \( D \)), and also proves the above result by using this latter definition.

**Proposition 2.7.** Let \( J' \) a collection of sieves on a category \( C \), and \( J \) the Grothendieck topology on \( C \) generated by it. Then any morphism of sites \((C, J') \to (D, K)\) is a morphism of sites \((C, J) \to (D, K)\).

**Proof** We have to show that a morphism of sites \( F : (C, J') \to (D, K) \) is cover-preserving as a morphism \((C, J) \to (D, K)\) (the other conditions in the notion of morphism of sites being independent from the Grothendieck topology in the source site). For this, it clearly suffices to prove that the collection of sieves \( S \) in \( C \) such that \( F(S) \) generates a \( K \)-covering sieve is a Grothendieck topology containing \( J' \).

The maximality and transitivity axioms are clearly satisfied, so it remains to prove the pullback-stability axiom. We shall deduce this from the fact that \( F : (C, J') \to (D, K) \) satisfies conditions (ii) and (iii) in the definition of morphism of sites. We have to prove that if \( S \) is a sieve on \( c \) such that the sieve \( < F(S) > \) generated by \( F(S) \) is \( K \)-covering then for any arrow \( f : c' \to c \), the sieve \( < F(f^*(S)) > \) generated by \( F(f^*(S)) \) is also \( K \)-covering. For this, using the transitivity axiom for \( K \), we are reduced to verifying that for each \( \xi : d \to F(c') \) in \( F(f)^*(< F(S) >) \), \( \xi^*(< F(f^*(S)) >) \) is \( K \)-covering. Since \( \xi \in F(f)^*(< F(S) >) \), there exist \( s : c'' \to c \) in \( S \) and \( \chi : d \to F(c'') \) in \( D \) such that \( F(f) \circ \xi = F(s) \circ \chi \). By condition (ii) of Definition 2.5 there exist a \( K \)-covering family

\[
g'_i : d_i \to d, \quad i \in I,
\]

and a family of pairs of morphisms of \( C \)

\[
f'_1 : c'_i \to c' , \quad f'_2 : c'_i \to c' , \quad i \in I,
\]

and of morphisms of \( D \)

\[
h_i : d_i \to F(c'_i), \quad i \in I,
\]

such that \( F(f'_i) \circ h_i = \chi \circ g'_i \) and \( F(f'_2) \circ h_i = \xi \circ g'_i \). Now, for each \( i \in I \), consider the arrows \( s \circ f'_1 \circ f'_2 : c'_i \to c \), which satisfy \( F(s \circ f'_1) \circ h_i = F(f \circ f'_2) \circ h_i \). By condition (iii) of Definition 2.5 there exist a \( K \)-covering family

\[
g'_j : d'_j \to d_j, \quad j \in J_i,
\]

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and a family of morphisms of $C$

$$k_j^i : c^i_j \rightarrow c_i^j, \quad j \in J_i,$$

satisfying

$$s \circ f_i^j \circ k_j^i = f \circ f_i^j \circ k_j^i, \quad \forall j \in J_i,$$

and of morphisms of $D$

$$h_j^i : d_j^i \rightarrow F(c_i^j), \quad j \in J_i,$$

such that $h_i \circ g_j^i = F(k_j^i) \circ h_j^i$.

Therefore, the family of arrows $\{g_i^j \circ g_j^i \mid i \in I, j \in J_i\}$ is $K$-covering and is contained in the sieve $\xi^*(< F(f^*(S)) >)$. Indeed, $\xi \circ g_i^j \circ g_j^i = F(f_i^j \circ h_i \circ g_j^i = F(f_i^j \circ k_j^i) \circ h_j^i$, which belongs to $< F(f^*(S)) >$ since the equality $s \circ f_i^j \circ k_j^i = f \circ f_i^j \circ k_j^i$ implies that $f_i^j \circ k_j^i \in f^*(S)$ and hence that $F(f_i^j \circ k_j^i) \in F(S)$. So $\xi^*(< F(f^*(S)) >)$ is $K$-covering, as desired. □

3 Arrows in a Grothendieck topos

In this section we make a detailed site-theoretic analysis of arrows in a Grothendieck topos, with the purpose of preparing the ground for the results in the following sections of the paper.

3.1 Site-theoretic description of arrows between objects coming from the site

Given a site $(C, J)$, for two arrows $h, k : c \rightarrow d$ in $C$ we shall write $h \equiv_J k$ for $J$-local equality, that is, to mean that there exists a $J$-covering sieve $S$ on $c$ such that $h \circ f = k \circ f$ for every $f \in S$. Notice that $l(h) = l(k)$ if and only if $h \equiv_J k$.

**Proposition 3.1.** Let $(C, J)$ be a small-generated site.

(i) Then for any arrow $\xi : l(c) \rightarrow l(d)$ in $\text{Sh}(C, J)$ there exists a family of arrows $\{f_u : c_u \rightarrow c, g_u : c_u \rightarrow d \mid u \in U\}$ such that $\{f_u : c_u \rightarrow c \mid u \in U\}$ generates a $J$-covering sieve, for any object $e$ and arrows $h : e \rightarrow c_u$ and $k : e \rightarrow c_{u'}$ such that $f_u \circ h = f_{u'} \circ k$ we have $g_u \circ h \equiv_J g_{u'} \circ k$, and $\xi \circ l(f_u) = l(g_u)$ for every $u \in U$. 

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(ii) Conversely, any family of arrows $\mathcal{F} = \{f_u : c_u \to c \mid u \in U\}$ such that $\{f_u : c_u \to c \mid u \in U\}$ generates a $J$-covering sieve and for any object $e$ and arrows $h : e \to c_u$ and $k : e \to c_{u'}$ such that $f_u \circ h = f_{u'} \circ k$ we have $g_u \circ h \equiv_f g_{u'} \circ k$, determines a unique arrow $\xi_{\mathcal{F}} : l(c) \to l(d)$ in $\text{Sh}(\mathcal{C}, J)$ such that $\xi_{\mathcal{F}} \circ l(f_u) = l(g_u)$ for every $u \in U$.

(iii) Two families $\mathcal{F} = \{f_u : c_u \to c, g_u : c_u \to d \mid u \in U\}$ and $\mathcal{F}' = \{f'_u : e_v \to c, g'_u : e_v \to d \mid v \in V\}$ as in (ii) determine the same arrow $l(c) \to l(d)$ (i.e. $\xi_{\mathcal{F}} = \xi_{\mathcal{F}'}$) if and only if there exist a $J$-covering family $\{a_k : b_k \to c \mid k \in K\}$ and factorizations of it through both of them by arrows $x_k : b_k \to c_{u(k)}$ and $y_k : b_k \to e_v(k)$ (i.e. $f_{u(k)} \circ x_k = a_k = f'_{v(k)} \circ y_k$ for every $k \in K$) such that $g_{u(k)} \circ x_k \equiv_f g'_{v(k)} \circ y_k$ for every $k \in K$.

(iv) Given two families $\mathcal{F} = \{f_u : c_u \to c, g_u : c_u \to d \mid u \in U\}$ and $\mathcal{G} = \{h_v : d_v \to d, k_v : d_v \to e \mid v \in V\}$ in $\text{Sh}(\mathcal{C}, J)$, the composite arrow $\xi_{\mathcal{G}} \circ \xi_{\mathcal{F}} : l(c) \to l(e)$ is induced as in (ii) by the family $\{f_u \circ x : \text{dom}(x) \to c, k_v \circ y : \text{dom}(y) \to e \mid (u,v,x,y) \in Z\}$, where $Z = \{(u,v,x,y) \mid u \in U, v \in V, \text{dom}(x) = \text{dom}(y), \text{cod}(x) = c_u, \text{cod}(y) = d_v, h_v \circ y = g_u \circ x\}$.

**Proof** (i) Consider the following pullback square in $[\mathcal{C}^{\text{op}}, \text{Set}]$:

\[
\begin{array}{ccc}
R & \xrightarrow{f} & y_C(d) \\
\downarrow{\xi_{\eta_{\mathcal{C}(c)}}} & & \downarrow{h_{\mathcal{C}(d)}} \\
y_C(c) & \xrightarrow{\eta_{\mathcal{C}(c)}} & l(d)
\end{array}
\]

and the sieve $S = \{f : e \to c \mid \exists g : e \to d \text{ with } (f, g) \in R\}$ on $c$ given by the image in $[\mathcal{C}^{\text{op}}, \text{Set}]$ of the arrow $r : R \to y_C(c)$. This sieve is $J$-covering since it is sent by $a_J$ to an isomorphism. Let us show that for any $(f, g) \in R$, $\xi \circ l(f) = l(g)$. Consider the adjunction between $i$ and $a_J$; the arrow $\eta_{\mathcal{C}(d)} \circ y_C(g)$ is the transpose of $l(g)$ and the arrow $\xi \circ \eta_{\mathcal{C}(c)} \circ y_C(f)$ is the transpose of $\xi \circ l(f)$, so it is equivalent to verify that $\eta_{\mathcal{C}(d)} \circ y_C(g) = \xi \circ \eta_{\mathcal{C}(c)} \circ y_C(f)$. But this follows by the commutativity of the above square, since $(f, g) \in R$. To check that the family of arrows $\{(f, g) \mid (f, g) \in R\}$ satisfies the required condition, we have to check that for any $(f, g), (f', g') \in R$ and arrows $h, h'$ such that $f \circ h = f' \circ h'$, $g \circ h \equiv_f g' \circ h'$, equivalently $l(g \circ h) = l(g' \circ h')$. Since $(f \circ h, g \circ h) \in R$ as $R$ is functorial, we have that $\xi \circ l(f \circ h) = l(g \circ h)$; similarly, we have that $\xi \circ l(f' \circ h') = l(g' \circ h')$. So $l(g \circ h) = l(g' \circ h')$, as required.

(ii) Any sieve $S$ on an object $c$ of $\mathcal{C}$, regarded as a presheaf on $\mathcal{C}$, is the colimit of the canonical diagram defined on the full subcategory $\int S$ of $\mathcal{C}/c$ on the objects which are elements of $S$. If $S$ is $J$-covering then
the canonical monomorphism $S \rightarrow y_C(c)$ is sent by the associated sheaf functor $a_J$ to an isomorphism, so, since $a_J$ is colimit-preserving, $l(c)$ is the colimit of the functor $a_S : \int S \rightarrow \text{Sh}(C, J)$ sending any object $f : d \rightarrow c$ of $\int S$ to $l(d)$ and any arrow $h : (f : d \rightarrow c) \rightarrow (f' : d' \rightarrow c)$ in $\int S$ to $l(h) : l(d') \rightarrow l(d)$. Therefore, giving an arrow $l(c) \rightarrow l(d)$ in $\text{Sh}(C, J)$ amounts to giving a cocone on $a_S$ with vertex $l(d)$. Any family of arrows $\{f_u : c_u \rightarrow c, g_u : c_u \rightarrow d \mid u \in U\}$ such that $\{f_u : c_u \rightarrow c \mid u \in U\}$ generates a $J$-covering sieve and for any object $e$ and arrows $h : e \rightarrow c_u$ and $k : e \rightarrow c_u'$ such that $f_u \circ h = f_{u'} \circ k$ we have $g_u \circ h \equiv_J g_{u'} \circ k$, defines such a cocone. Indeed, by taking $S$ equal to the sieve generated by the family $\{f_u : c_u \rightarrow c\}$ we obtain, thanks to the property that for any object $e$ and arrows $h : e \rightarrow c_u$ and $k : e \rightarrow c_u'$ such that $f_u \circ h = f_{u'} \circ k$ we have $g_u \circ h \equiv_J g_{u'} \circ k$, a well-defined family of arrows $l(\text{dom}(f)) \rightarrow l(d)$ (for $f \in S$) satisfying the commutativity conditions for a cocone. It is clear that the arrow $l(c) \rightarrow l(d)$ thus defined satisfies the required property.

(iii) Let us first prove the ‘if’ direction. Since the sieve generated by the family $\{a_k : b_k \rightarrow c \mid k \in K\}$ is $J$-covering, the arrows $l(a_k)$ to $l(c)$ are jointly epimorphic and hence $\xi_F = \xi_{F'}$ if and only if $\xi_F \circ l(a_k) = \xi_{F'} \circ l(a_k)$ for every $k \in K$. But $\xi_F \circ l(a_k) = \xi_{F'} \circ l(f_u(k)) \circ l(x_k) = l(g_u(k) \circ x_k) = l(g'_u(k) \circ y_k) = \xi_{F'} \circ l(f'_{v}(k)) \circ l(y_k) = \xi_{F'} \circ l(a_k)$, where the equality $l(g_u(k) \circ x_k) = l(g'_v(k) \circ y_k)$ follows from the fact that $g_u(k) \circ x_k \equiv_J g'_v(k) \circ y_k$. It thus remains to prove the ‘only if’ direction. Suppose that $\xi_F = \xi_{F'}$. Let us define the family $\{a_k : b_k \rightarrow c \mid k \in K\}$ to be the intersection of the sieve generated by the family $\{f_u : c_u \rightarrow c \mid u \in U\}$ with that generated by the family $\{f'_v : e_v \rightarrow c \mid v \in V\}$; since these two sieves are $J$-covering by our hypotheses, their intersection is $J$-covering as well. Let us choose factorizations $x_k : b_k \rightarrow c_{u(k)}$ and $y_k : b_k \rightarrow e_{v(k)}$ of it through the arrows of these families, so that $f_u(k) \circ x_k = a_k = f'_{v(k)} \circ y_k$ for every $k \in K$. We want to prove that, for every $k \in K$, $g_u(k) \circ x_k \equiv_J g'_v(k) \circ y_k$, equivalently that $l(g_u(k) \circ x_k) = l(g'_v(k) \circ y_k)$. But $l(g_u(k) \circ x_k) = l(g_u(k) \circ l(x_k)) = \xi_F \circ l(f_u(k)) \circ l(x_k) = \xi_{F'} \circ l(f'_{v(k)} \circ y_k) = l(g'_v(k) \circ y_k)$, as required.

(iv) First of all, we have to show that the family of arrows $\{f_u \circ x \mid (u, v, x, y) \in Z\}$ is $J$-covering. To this end, consider the pullback

$$
\begin{array}{ccc}
P & \rightarrow & H \\
\downarrow & & \downarrow \\
G & \rightarrow & y_C(d)
\end{array}
$$

where $H$ is the sieve on $d$ generated by the arrows $\{h_v : d_v \rightarrow d \mid v \in I\}$, $G$ is the sieve on $d$ generated by the arrows $\{g_u : c_u \rightarrow d \mid u \in U\}$ and the arrows are the canonical inclusion monomorphisms.
Since $H$ is $J$-covering, the canonical monomorphism $H \to y_C(d)$ is $c_J$-dense and hence the monomorphism $P \to G$ in the above square is $c_J$-dense too; this means that for any $u \in U$, the sieve \( \{ \xi : \text{dom}(\xi) \to c_u \mid \exists v \exists y (u, v, x, y) \in Z \} \) is $J$-covering. But $P = H \cap G$, so this means that for any $u \in U$, the sieve \( \{ x : \text{dom}(x) \to c_u \mid \exists v \exists y (u, v, x, y) \in Z \} \) is $J$-covering. Since this sieve is contained in the pullback along $f_u$ of the sieve \( \{ f_u \circ x \mid (u, v, x, y) \in Z \} \), it follows from the transitivity axiom for Grothendieck topologies that this latter sieve is $J$-covering, as desired.

To conclude the proof of (iv), we have to verify that for every $(u, v, x, y) \in Z$, $\xi_G \circ \xi_F \circ l(f_u) \circ l(x) = l(k_u) \circ l(y)$. Recalling the definitions of $\xi_F$ and $\xi_G$, we obtain that $\xi_G \circ \xi_F \circ l(f_u) \circ l(x) = \xi_G \circ l(g_u) \circ l(x) = \xi_G \circ l(h_v) \circ l(y) = l(k_v) \circ l(y) = l(k_v \circ y)$, as required. \qed

**Remarks 3.2.** (a) Proposition 3.1 could alternatively have been derived from the description of the topos $\text{Sh}(C, J)$ as a completion of the site $(C, J)$ provided by Theorem 8.22 [7].

(b) If we assume the axiom of choice, given a family of arrows \( \{ f_u : c_u \to c, g_u : c \to d \mid u \in U \} \) such that \( \{ f_u : c_u \to c \mid u \in U \} \) generates a $J$-covering sieve and for any object $e$ and arrows $h : e \to c_u$ and $k : e \to c_{u'}$ such that $f_u \circ h = f_{u'} \circ k$ we have $g_u \circ h = g_{u'} \circ k$, as in points (i) and (ii) of Proposition 3.1 we can clearly suppose without loss of generality $U$ to be the $J$-covering sieve on $c$ generated by the family \( \{ f_u : c_u \to c \mid u \in U \} \).

So such a family can be equivalently identified with a family of arrows

\[ g_f : \text{dom}(f) \to d \]

indexed by the arrows $f$ of a $J$-covering sieve $S$ on $c$ with the property that for any arrow $h$ composable with $f$, $g_{f \circ h} \equiv_{j} g_f \circ h$.

(c) Under the identification of Remark 3.2(b), the family \( \{ a_k : b_k \to c \mid k \in K \} \) in point (iii) of Proposition 3.1 can be assumed to be the intersection of the sieves $U$ and $V$.

**Proposition 3.3.** Let $(C, J)$ be a small-generated site and $h : c \to d$, $k : c \to e$ arrows in $C$. Then there exists an arrow $\xi : l(e) \to l(d)$ such that $\xi \circ l(h) = l(k)$ if and only if there exists a family of arrows $g_f : \text{dom}(f) \to d$ indexed by the arrows $f$ of a $J$-covering sieve $S$ on $e$ such that $g_{f \circ z} = g_f \circ z$ whenever $z$ is composable with $f$, and such that for any $\chi \in h^*(S)$, $k \circ \chi \equiv_{j} g_{h \circ \chi}$.

**Proof** By Proposition 3.1 giving an arrow $\xi : l(e) \to l(d)$ amounts to specifying a family of arrows $g_f : \text{dom}(f) \to d$ indexed by the arrows $f$ of a $J$-covering sieve $S$ on $e$ such that $g_{f \circ z} = g_f \circ z$ whenever $z$ is composable with $f$, which satisfies $\xi \circ l(f) = l(g_f)$ for each $f \in S$. Now, $\xi \circ l(h) = l(k)$ if and only if $\xi \circ l(h) \circ l(\chi) = l(k) \circ l(\chi)$ for every $\chi \in h^*(S)$ (since,
Proposition 3.4. Let \((C, J)\) be a small-generated site and \(\xi : A \to l(c)\) an arrow of \(\mathbf{Sh}(C, J)\). Then the family of arrows \(\chi : l(d) \to A\), where \(d \in C\), such that \(\xi \circ \chi = l(f)\) for some arrow \(f : d \to c\) in \(C\) is epimorphic on \(A\).

Proof The family \(R\) of arrows to \(A\) whose domain is of the form \(l(e)\) for some \(e \in C\) is clearly epimorphic on \(A\). Let us denote by \(T\) the family of arrows \(\chi : l(d) \to A\), where \(d \in C\), such that \(\xi \circ \chi = l(f)\) for some arrow \(f : d \to c\) in \(C\). To show that \(T\) is epimorphic, it clearly suffices, by transitivity, to show that for any \(\alpha \in R\), \(\alpha^*(T)\) is epimorphic. But this follows from Proposition 3.1(i) applied to the arrow \(\xi \circ \alpha\), since that implies that there exists a \(J\)-covering family \(\{f_i \mid i \in I\}\) of arrows on \(\text{dom}(\alpha)\) and for each \(i \in I\) an arrow \(g_i : \text{dom}(f_i) \to c\) such that \(\xi \circ \alpha \circ l(f_i) = l(g_i)\); indeed, the family of arrows \(\{l(f_i) \mid i \in I\}\) is epimorphic and contained in \(\alpha^*(T)\).

3.2 Locally functional relations

Let us now consider the problem of explicitly describing arrows in a sheaf topos \(\mathbf{Sh}(C, J)\). Given two objects \(P\) and \(Q\) of \([C^{\text{op}}, \text{Set}]\), we shall describe the arrows \(a_J(P) \to a_J(Q)\) in \(\mathbf{Sh}(C, J)\) by explicitly characterizing their graphs. Notice that a relation \(r : R \to A \times B\) in a topos is the graph of an arrow \(A \to B\) if and only if the composite of the canonical projection \(A \times B \to A\) with \(r\) is an isomorphism. By Proposition 2.3, the relations \(r : R \to a_J(P) \times a_J(Q) \cong a_J(P \times Q)\) in \(\mathbf{Sh}(C, J)\) can be identified with the \(c_J\)-closed subobjects of \(P \times Q\), via the correspondence sending \(r\) to its pullback \(r' : R' \to P \times Q\) along the arrow \(\eta_{P \times Q} : P \times Q \to a_J(P \times Q)\); so \(R \cong a_J(R')\). Notice that a subobject \(R' \to P \times Q\) is \(c_J\)-closed if and only if for any \(c \in C\) and any \((x, y) \in P(c) \times Q(c)\), if the sieve \(\{f : d \to c \mid (P(f)(x), Q(f)(y)) \in R(d)\}\) is \(J\)-covering then \((x, y) \in R(c)\). The condition for \(R\) to be the graph of an arrow \(a_J(P) \to a_J(Q)\) in \(\mathbf{Sh}(C, J)\) can thus be reformulated as the requirement that the arrow \(\pi_P \circ r'\) should be sent by \(a_J\) to an isomorphism. In light of Lemma 2.1, this motivates the following definition.

Definition 3.5. In a presheaf topos \([C^{\text{op}}, \text{Set}]\), a relation \(R : P \to Q\) (that is, an assignment \(c \to R(c)\) to each object \(c\) of \(C\)) is said to be \(J\)-functional from \(P\) to \(Q\) if it satisfies the following properties:
(b) Any relation

By the above discussion, the $J$-functional relations $R \rightarrow P \times Q$ from $P$ to $Q$ are precisely the $c_J$-closed relations $R$ on $\langle \mathcal{C}^{op}, \textbf{Set} \rangle$ whose image under $a_J$ is the graph of an arrow $a_J(P) \rightarrow a_J(Q)$ in $\textbf{Sh}(\mathcal{C}, J)$, that is the subobjects corresponding to the functional relations from $a_J(P)$ to $a_J(Q)$ under the bijection of Proposition 2.3.

(b) Any relation $R \rightarrow P \times Q$ on $\langle \mathcal{C}^{op}, \textbf{Set} \rangle$ satisfying conditions (ii) and (iii) of Definition 3.5 admits a $J$-closure $\overline{R}$, consisting of all the pairs $(x, y) \in P(c) \times Q(c)$ such that $\{ f : d \rightarrow c \mid (P(f)(x), Q(f)(y)) \in R(d) \} \in J(c)$, which satisfies all the conditions of the definition.

(c) The $J$-functional relation $R_{\xi}$ corresponding to an arrow $\xi : a_J(P) \rightarrow a_J(Q)$ can be concretely described as follows: for any $(x, y) \in P(c) \times Q(c)$, $(x, y) \in R_{\xi}$ if and only if $(\xi \circ \eta_P)(c)(x) = (\eta_Q(c))(y)$, where $\eta_P : P \rightarrow a_J(P)$ and $\eta_Q : Q \rightarrow a_J(Q)$ are the unit arrows.

The following result shows that $J$-functional relations from $P$ to $Q$ can be identified with certain ways of assigning (possibly empty) $\equiv_J$-equivalence classes of elements of $Q$ to elements of $P$.

**Proposition 3.7.** We have a natural bijection between the $J$-functional relations $R$ from $P$ to $Q$ and the functions $f$ assigning to each element $x \in P(c)$ a $\equiv_J$-equivalence class of elements of $Q(c)$ (that is, a subset $A$ of $Q(c)$ such that for any $y \in A$ and $y' \in Q(c)$, $y \equiv_J y'$ if and only if $y' \in A$) which is functorial in the sense that for any arrow $g : d \rightarrow c$ in $\mathcal{C}$, if $y \in f(x)$ then $Q(g)(y) \in f(P(g)(x))$, $J$-closed in the sense that if $Q(g)(y) \in f(P(g)(x))$ for all the arrows $g$ of a $J$-covering sieve then $y \in f(x)$ and $J$-locally pointed in the sense that for any $x \in P(c)$, $\{ g : d \rightarrow c \mid f(P(g)(x)) \neq \emptyset \} \in J(c)$. This correspondence sends

- a function $f$ satisfying the above properties to the relation $R_f \rightarrow P \times Q$ consisting of the pairs $(x, y)$ such that $y \in f(x)$;
- a $J$-functional relation $R$ from $P$ to $Q$ to the function $f_R$ given by:

$$f_R(x) = \{ y \in Q(c) \mid (x, y) \in R(c) \}$$
for any $x \in P(c)$.

**Proof** First, let us show that the correspondence is well-defined. Given $f$, let us verify that $R_f$ is a $J$-functional relation. The functoriality of $R_f$ follows from that of $f$. The relation $R_f$ satisfies condition (i) by the $J$-closedness of $f$, condition (ii) since $f$ takes values in $\equiv_J$-equivalence classes and condition (iii) since $f$ is $J$-locally pointed. Conversely, let us show that, given a $J$-functional relation $R$ from $P$ to $Q$, the function $f_R$ takes values in $\equiv_J$-equivalence classes, is functorial, $J$-closed and $J$-locally pointed. Given $x \in P(c)$ and $y, y' \in Q(c)$ such that $y \in f_R(x)$, let us prove that $y \equiv_J y$ if and only if $y' \in f_R(x)$). The ‘if’ direction follows at once from the fact that $R$ satisfies condition (ii), so it remains to prove the ‘only if’ one. If $y \equiv_J y$ then there is a $J$-covering sieve $T$ on $c$ such that for any $t \in T$, $Q(t)(y) = Q(t)(y')$. Now, since $(x, y) \in R(c)$, by the functoriality of $R$ we have that $(P(t)(x), Q(t)(y)) \in R(\text{dom}(t))$ for each $t \in T$; condition (i) thus implies that $(x, y') \in R(c)$, as required. The fact that $f_R$ is functorial (resp. $J$-closed, $J$-locally pointed) follows from the fact that $R$ is functorial (resp. satisfies condition (i), condition (iii)). Now that we have proved that the assignments $R \mapsto f_R$ and $f \mapsto R_f$ are well-defined, it remains to show that they are inverse to each other. The equality $f = f_{R_f}$ follows from the fact that for any $x$, $f_{R_f} = \{y \mid (x, y) \in R_f\} = \{y \mid y \in f(x)\} = f(x)$, while the equality $R = R_{f_R}$ follows from the fact that $R_{f_R} = \{(x, y) \mid y \in f_R(x)\} = \{(x, y) \mid (x, y) \in R\}$.

Now that we have seen that we can naturally represent arrows $a_J(P) \to a_J(Q)$ in $\text{Sh}(C, J)$ in terms of $J$-functional relations from $P$ to $Q$, it is natural to consider how composition of arrows in $\text{Sh}(C, J)$ can be described in terms of an operation of composition of such relations in $[\text{C}^{\text{op}}, \text{Set}]$. The $J$-functional relation corresponding to the composite of two arrows in $\text{Sh}(C, J)$ induced by $J$-functional relations $R$ and $S$ is the $c_J$-closure of the relation given by the (relational) composite of $R$ and $S$ in $[\text{C}^{\text{op}}, \text{Set}]$. Indeed, the operation of composition of relations is clearly preserved by geometric functors; but $a_J$ is such a functor and, by Proposition 2.3, there is just one $c_J$-closed subobject whose image under $a_J$ is a given subobject. Notice that the composition of relations in a presheaf topos is computed pointwise.

Summarizing, we have the following result:

**Theorem 3.8.** Let $(C, J)$ be a small-generated site. Then, for any presheaves $P, Q \in [\text{C}^{\text{op}}, \text{Set}]$, the arrows $a_J(P) \to a_J(Q)$ in $\text{Sh}(C, J)$ are in natural bijection with the $J$-functional relations from $P$ to $Q$ in $[\text{C}^{\text{op}}, \text{Set}]$.

Moreover, under this bijection, the composition of arrows in $\text{Sh}(C, J)$ corresponds to the $c_J$-closure of the composition in $[\text{C}^{\text{op}}, \text{Set}]$ of the associated
J\text{-}functional relations. This operation, which we shall denote by the symbol \(*\), admits the following explicit description: given a J\text{-}functional relation \(R\) from \(P\) to \(Q\) and a J\text{-}functional relation \(S\) from \(Q\) to \(Z\), \(S \ast R\) is given by the formula

\[(S \ast R)(c) = \{(x, z) \in P(c) \times Z(c) \mid \{f : d \to c \mid \exists y \in Q(d) \quad (P(f)(x), y) \in R(d) \text{ and } (y, Z(f)(z)) \in S(d)\} \in J(c)\}\]

for any \(c \in C\).

Theorem 3.8 can be notably applied to the description of the full subcategory \(a_J(C)\) of \(\text{Sh}(C, J)\) on the objects of the form \(l(c)\) for \(c \in C\).

**Corollary 3.9.** Let \((C, J)\) be a small-generated site. Then the full subcategory \(a_J(C)\) of \(\text{Sh}(C, J)\) on the objects of the form \(l(c)\) for \(c \in C\) is equivalent to the category \(C_J\) defined as follows:

- the objects of \(C_J\) are the objects of \(C\);
- the arrows \(c \to d\) are the collections \(R\) of pairs of arrows \((f : e \to c, g : e \to d)\) where \(e\) varies among the objects of \(C\) (we write \((f, g) \in R(e)\) to mean that \((f, g) \in R\) and \(\text{dom}(f) = \text{dom}(g) = e\) satisfying the following properties:
  
  (i) for any arrow \(k : e' \to e\), if \((f, g) \in R\) then \((f \circ k, g \circ k) \in R\);
  
  (ii) for any \(e \in C\) and any arrows \(x : e \to c, y : e \to d\), if \(\{f : e' \to e \mid (x \circ f, y \circ f) \in R\}\) \(\in J(e)\) then \((x, y) \in R\);
  
  (iii) for any \(e \in C\) and any \((x, y), (x', y') \in R(e)\), if \(x = x'\) then \(\{f : e' \to e \mid y \circ f = y' \circ f\} \in J(e)\);
  
  (iv) for any \(e \in C\) and any \(x : e \to c\), \(\{f : e' \to e \mid \exists y : e' \to d\} \text{ such that } (x \circ f, y) \in R(e)\} \in J(e)\);
- the composite \(S \ast R\) of two arrows \(R : c \to d\) and \(S : d \to a\) is given by the following formula:

  \[(S \ast R)(e) = \{(x : e \to c, z : e \to a) \mid \{f : e' \to e \mid \exists y : e' \to d \quad (x \circ f, y) \in R(e') \text{ and } (y, z \circ f) \in S(e')\} \in J(e)\}\]

for any \(e \in C\).

One half of the equivalence \(C_J \simeq a_J(C)\) is given by the functor \(C_J \to a_J(C)\) acting on objects by sending any \(c \in C_J\) to \(l(c)\) and any arrow \(R : c \to c'\) in \(C_J\) to the arrow \(l(\text{dom}(R)) \to l(\text{cod}(R))\) in \(a_J(C)\) whose graph is \(a_J(R)\).
Remark 3.10. Let us clarify the relationship between the description of arrows in $a_J(C)$ given by Proposition 3.1 with that provided by Corollary 3.9 (modulo the equivalence $a_J(C) \simeq C_J$). We stress that, unlike the former description (as equivalence classes of ways of representing them), the latter is canonical; in particular, the composition operation, not being defined by using representatives, does not require any choices. Indeed, $J$-functional relations are equivalence classes themselves.

Given an arrow $R : c \to d$ in $C_J$, by property (iv) we have (by taking $e = c$ and $x$ equal to the identity arrow on $c$) that $S := \{ f : e \to c \mid \exists y : e \to d \text{ such that } (f, y) \in R \} \in J(c)$. By using the axiom of choice, we can therefore choose, for each $f \in S$, an arrow $y_f : e \to d$ such that $(f, y_f) \in R$, and hence the family $\{ f : e \to c, y_f : e \to d \mid f \in S \}$ presents the arrow $l(c) \to l(d)$ corresponding to $R$ in the sense of Proposition 3.1. In the converse direction, we claim that, given a family $\mathcal{F} = \{ f : \text{dom}(f) \to c, g_f : \text{dom}(f) \to d \mid f \in S \}$ inducing an arrow $\xi : l(c) \to l(d)$ in the sense of Proposition 3.1 (we present $\mathcal{F}$ in the form of Remark 3.2(b) for simplicity), the $J$-functional relation $R$ corresponding to the arrow $\xi$ admits the following explicit description:

$$(h, k) \in R \iff \text{for all } \chi \in h^*(S), k \circ \chi \equiv_J g_{h^*\chi}$$

By definition of $R$ as the pullback of the graph of the arrow $\xi : l(c) \to l(d)$ along the arrow $\eta_{bc(c)} \times \eta_{bc(d)}$, $R$ is characterized by the following universal property: for any $e \in C$, $(h, k) \in R(e)$ if and only if $\xi \circ l(h) = l(k)$. Now, since the sieve $S$ is $J$-covering, for any $h$, the sieve $h^*(S)$ is $J$-covering as well and hence the family of arrows $\{ l(\chi) \mid \chi \in h^*(S) \}$ is epimorphic on $l(e)$. So $\xi \circ l(h) = l(k)$ if and only if $\xi \circ l(h) \circ l(\chi) = l(k) \circ l(\chi)$ for every $\chi \in h^*(S)$. But $\xi \circ l(h) \circ l(\chi) = \xi \circ l(h \circ \chi) = l(g_{h^*\chi})$, while $l(k) \circ l(\chi) = l(k \circ \chi)$, so $\xi \circ l(h) \circ l(\chi) = (l(k) \circ l(\chi))$ if and only if $l(g_{h^*\chi}) = l(k \circ \chi)$, i.e. if and only if $k \circ \chi \equiv_J g_{h^*\chi}$, as required.

The following corollary of Theorem 3.8 will be instrumental in section 5.5.4 for obtaining a site-level description of the hyperconnected-localic factorization of a geometric morphism.

Corollary 3.11. Let $(C, J)$ be a small-generated site. Then the full subcategory of $\text{Sh}(C, J)$ on the objects of the form $a_J(S)$ for $c \in C$ and $S$ a (J-closed) sieve on $c$ is equivalent to the category $C_J^*$ defined as follows:

- the objects of $C_J^*$ are the pairs $(c, S)$ consisting of an object $c$ of $C$ and a (J-closed) sieve $S$ on $c$;
• the arrows \((c, S) \to (d, T)\) are the collections \(R\) of pairs of arrows \((x : e \to c, y : e \to d)\) where \(e\) varies among the objects of \(C\) (we write \((x, y) \in R(e)\) to mean that \((x, y) \in R\) and \(\text{dom}(x) = \text{dom}(y) = e\)) satisfying the following properties:

  (i) for any \((x, y) \in R, x \in S\) and \(y \in T;\)
  
  (ii) for any arrow \(f : e' \to e\), if \((x, y) \in R\) then \((x \circ f, y \circ f) \in R;\)
  
  (iii) for any \(e \in C\) and any \((x : e \to c, y : e \to d) \in S \times T\), if \(\{f : e' \to e \mid (x \circ f, y \circ f) \in R\} \in J(e)\) then \((x, y) \in R;\)
  
  (iv) for any \(e \in C\) and any \((x, y), (x', y') \in R(e),\) if \(x = x'\) then \(\{f : e' \to e \mid y \circ f = y' \circ f\} \in J(e);\)
  
  (v) for any \(e \in C\) and any \(x : e \to c\) in \(S,\) \(\{f : e' \to e \mid \exists y : e' \to d \text{ such that } (x \circ f, y) \in R(e')\} \in J(e);\)

• the composite \(R' \ast R : (c, S) \to (a, Z)\) of two arrows \(R : (c, S) \to (d, T)\) and \(R' : (d, T) \to (a, Z)\) is given by the following formula:

\[
(R' \ast R)(e) = \{(x : e \to c, z : e \to a) \in S \times Z \mid \{f : e' \to e \mid \exists y : e' \to d (x \circ f, y) \in R(e') \text{ and } (y, z \circ f) \in R'(e')\} \in J(e)\}
\]

for any \(e \in C.\)

\[
\square
\]

**Remark 3.12.** If \(C\) is a geometric category (i.e. a well-powered category which has finite limits, images of arrows which are stable under pullback and arbitrary unions of subobjects that are stable under pullback) and \(J\) is the geometric topology on it (whose covering sieves are those which contain small families of arrows the union of whose images is the given object) then \(C\) is closed under subobjects in \(\text{Sh}(C, J),\) so the category \(C^J\) is equivalent to \(C.\)

The following proposition characterizes the \(J\)-functional relations \(R\) from \(P\) to \(Q\) which induce a monomorphism (resp. an epimorphism) \(a_J(P) \to a_J(Q):\)

**Proposition 3.13.** Let \(R\) be a \(J\)-functional relation from \(P\) to \(Q\) in \([C^{op}, \text{Set}]\) and \(\alpha_R : a_J(P) \to a_J(Q)\) the arrow in \(\text{Sh}(C, J)\) induced by \(R.\) Then

(i) \(\alpha_R\) is a monomorphism if and only if for any \(c \in C\) and any elements \((x, y), (x', y') \in R(e)\) such that \(y = y',\) the sieve \(\{f : d \to c \mid P(f)(x) = P(f)(x')\}\) is \(J\)-covering.
(ii) \( \alpha_R \) is an epimorphism if and only if for any \( c \in C \) and \( y \in Q(c) \), the sieve \( \{ f : d \to c \mid (\exists x \in P(d))(x, Q(f)(y)) \in R(d) \} \) is \( J \)-covering.

**Proof** Given an arrow \( \alpha : A \to B \) in a topos and its graph \( R_\alpha : A \times B \), we clearly have that \( \alpha \) is a monomorphism (resp. an epimorphism) if and only if \( \pi_B : R_\alpha \to B \) is a monomorphism (resp. an epimorphism). But by Lemma 2.1, \( \alpha_R \) is a monomorphism if and only if for every \( c \in C \) and any elements \( (x,y), (x',y') \in R(c) \) such that \( y = y' \), the sieve \( \{ f : d \to c \mid P(f)(x) = P(f)(x') \} \) is \( J \)-covering, while \( \alpha_R \) is an epimorphism if and only if for any \( c \in C \) and \( y \in Q(c) \), the sieve \( \{ f : d \to c \mid (\exists x \in P(d))(x, Q(f)(y)) \in R(d) \} \) is \( J \)-covering. \( \square \)

### 4 Denseness conditions and equivalences of toposes

**Definition 4.1.** A morphism of sites \( F : (C, J) \to (D, K) \) is said to be dense if it satisfies the following properties:

(i) \( P \) is a \( J \)-covering family in \( C \) if and only if \( F(P) \) is a \( K \)-covering family in \( D \);

(ii) \( F \) is \( K \)-dense in the sense that for any object \( d \) of \( D \) there exists a \( K \)-covering family of arrows \( d_i \to d \) whose domains \( d_i \) are in the image of \( F \);

(iii) for every \( c_1, c_2 \in C \) and any arrow \( g : F(c_1) \to F(c_2) \) in \( D \), there exist a \( J \)-covering family of arrows \( f_i : c'_i \to c_1 \) and a family of arrows \( k_i : c'_i \to c_2 \) such that \( g \circ F(f_i) = F(k_i) \) for all \( i \).

**Remark 4.2.** This definition is adapted from section 11 of [7], where a functor \( F : C \to D \) is defined to be a dense morphism of sites if it satisfies the three conditions above plus the following one (which we shall call \( J \)-faithfulness in section 4.3):

(iv) for any arrows \( f_1, f_2 : c_1 \to c_2 \) in \( C \) such that \( F(f_1) = F(f_2) \) there exists a \( J \)-covering family of arrows \( k_i : c'_i \to c_1 \) such that \( f_1 \circ k_i = f_2 \circ k_i \) for all \( i \).

In fact, Theorem 11.2 [7](b) proves that every such functor \( F \) is a morphism of sites \( (C, J) \to (D, K) \). On the other hand, if \( F \) is already a morphism of sites, condition (iv) is unnecessary as it is implied by condition (i) (cf. condition (iv) in Definition 2.5).
By Theorem 11.8 [7], if $F$ is a dense morphism of sites then the associated geometric morphism $\text{Sh}(F) : \text{Sh}(\mathcal{D}, K) \to \text{Sh}(\mathcal{C}, J)$ is an equivalence. The following proposition shows that, conversely, if $F$ is a morphism of sites such that $(\mathcal{D}, K)$ is subcanonical and $\text{Sh}(F)$ is an equivalence then $F$ is a dense morphism of sites.

**Proposition 4.3.** Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a morphism of sites. Suppose that $K$ is subcanonical. Then, if the geometric morphism $\text{Sh}(F) : \text{Sh}(\mathcal{D}, K) \to \text{Sh}(\mathcal{C}, J)$ is an equivalence, $F$ is a dense morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$.

**Proof** Let us check that all the conditions in the definition of a dense morphism of sites are satisfied.

(i) This condition follows from the commutativity of the diagram

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow F & & \downarrow F' \\
\text{Sh}(\mathcal{C}, J) & \xrightarrow{\text{Sh}(F)^*} & \text{Sh}(\mathcal{D}, K)
\end{array}
$$

by exploiting the fact that a sieve $P$ (resp. $Q$) in $\mathcal{C}$ (resp. in $\mathcal{D}$) is $J$-covering (resp. $K$-covering) if and only if it is sent by $l$ (resp. by $l'$) to an epimorphic family, since for any arrow $f$ in $\mathcal{C}$, $l'(F(f)) \cong \text{Sh}(F)^*(l(f))$ and $\text{Sh}(F)^*$ is an equivalence.

(ii) By the commutativity of the above square, the objects of the form $l'(F(c)) \cong \text{Sh}(F)^*(l(c))$ form a separating set for the topos $\text{Sh}(\mathcal{D}, K)$. Therefore, for any $d \in \mathcal{D}$, the family of arrows from objects of the form $l'(F(c))$ (for $c \in \mathcal{C}$) to $l'(d)$ is epimorphic. Since, by the subcanonicity of $K$, all these arrows are of the form $l'(f)$ for $f$ an arrow in $\mathcal{D}$, this means that the family of arrows from objects of the form $F(c)$ to $d$ is $K$-covering in $\mathcal{D}$, as required.

(iii) By the commutativity of the above square, we have that $l'(g) : l'(F(x)) \to l'(F(y))$ is the image under $\text{Sh}(F)^*$ of an arrow $\xi : l(x) \to l(y)$ in $\text{Sh}(\mathcal{C}, J)$. By Proposition 3.2(i), there exists a $J$-covering family of arrows $f_i : x_i \to x$ and a family of arrows $g_i : x_i \to y$ such that $\xi \circ l(f_i) = l(g_i)$ for all $i$. Applying $\text{Sh}(F)^*$ yields $l'(g) \circ l'(F(f_i)) = l'(F(g_i))$ for all $i$, whence, since $l'$ is faithful ($K$ being subcanonical), $g \circ F(f_i) = F(g_i)$ for all $i \in I$, as required.

Recall that if $\mathcal{C}$ is the full subcategory of a Grothendieck topos $\mathcal{E}$ on a family of objects which is separating for it then $\mathcal{E}$ is equivalent to the topos $\text{Sh}(\mathcal{C}, J_{\mathcal{E}}^\text{can}|\mathcal{C})$, where $J_{\mathcal{E}}^\text{can}|\mathcal{C}$ is the Grothendieck topology on $\mathcal{C}$ induced by the canonical topology $J_{\mathcal{E}}^\text{can}$ on $\mathcal{E}$: the $J_{\mathcal{E}}^\text{can}|\mathcal{C}$-covering sieves are those which
generate $J^\text{can}$-covering sieves in $\mathcal{E}$. Given a small-generated site $(\mathcal{C}, J)$, we shall denote by $C^J_0$ the Grothendieck topology $J^\text{can}_{\mathbf{Sh}(\mathcal{C}, J)}|_{a_J(\mathcal{C})}$; so we have an equivalence

$$\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(a_J(\mathcal{C}), C^J_0).$$

This equivalence is clearly induced by the morphism of sites

$$l : (\mathcal{C}, J) \to (a_J(\mathcal{C}), C^J_0).$$

4.1 Weakly dense morphisms of sites

In light of Proposition 4.3, it is natural to give the following definition.

**Definition 4.4.** A morphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ is weakly dense if the morphism of sites $l' \circ F : (\mathcal{C}, J) \to (a_K(\mathcal{D}), C^K_0)$ is dense.

Notice that, if $K$ is subcanonical then any weakly dense morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$ is dense, but the converse does not hold in general (see Example 4.9 below).

The following proposition gives an explicit characterization of weakly dense morphisms of sites (for conciseness we give the constructively stronger formulation relying on the axiom of choice, but the reader who prefers the constructive phrasing in terms of more general families satisfying the conditions in Proposition 3.1(i)-(ii) can obtain it by replacing all the families of arrows indexed by a covering sieve occurring in the statement with these more general families).

**Proposition 4.5.** Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a morphism of sites. Then $F$ is a weakly dense morphism of sites if and only if it satisfies the following conditions:

(i) $P$ is a $J$-covering family in $\mathcal{C}$ if and only if $F(P)$ is a $K$-covering family in $\mathcal{D}$;

(ii) for any object $d$ of $\mathcal{D}$ there exist a family $\{S_i \mid i \in I\}$ of $K$-covering sieves on objects of the form $F(c_i)$ (where $c_i$ is an object of $\mathcal{C}$) and for each $f \in S_i$ an arrow $g_f : \text{dom}(f) \to d$ such that $g_{f \circ z} \equiv_K g_f \circ z$ whenever $z$ is composable with $f$, such that the family of arrows $g_f$ (for $f \in S_i$ for some $i$) is $K$-covering;

(iii) for any objects $x, y$ of $\mathcal{C}$ and any family of arrows $g_h : \text{dom}(h) \to F(y)$ indexed by the arrows of a $K$-covering sieve $U$ on $F(x)$ such that $g_{h \circ k} \equiv_K g_h \circ k$ for every arrow $k$ composable with $h$, there exist a $J$-covering family of arrows $\{f_i : x_i \to x \mid i \in I\}$ and arrows $k_i : x_i \to y$.

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(for each \(i \in I\)) such that for every arrows \(w\) and \(z\) such that \(F(f_i) \circ w = h \circ z\), we have \(g_h \circ z \equiv_K F(k_i) \circ w\) (for every \(h \in U\) and \(i \in I\)).

**Proof** Conditions (i) for \(F\) is clearly equivalent to condition (i) in the definition of a dense morphism of sites for the functor \(l' \circ F\).

Let us now reformulate condition (ii) in the definition of a dense morphism of sites for the functor \(l' \circ F\), namely the condition that for every object \(d\) of \(\mathcal{D}\) there exists an epimorphic family of arrows \(\{l'(F(c_i)) \to l'(d) \mid i \in I\}\) whose domains are objects in the image of the functor \(l' \circ F\). Our thesis thus immediately follows from the description of such arrows provided by Proposition 3.1.

Let us show that we can reformulate condition (iii) in the definition of a dense morphism of sites for the functor \(l' \circ F\) as property (iii) of the proposition. For any arrow \(\xi : l'(F(x)) \to l'(F(y))\) in \(\text{Sh}(\mathcal{D}, K)\), by Proposition 3.1 there exists a \(K\)-covering sieve \(\{h : \text{dom}(h) \to F(x) \mid h \in U\}\) for each \(h \in U\) an arrow \(g_h : \text{dom}(h) \to F(y)\) such that \(\xi \circ l'(h) = l'(g_h)\) for each \(h\) and \(g_h \equiv_K g_k \circ k\) for any arrow \(k\) composable with \(h\). The existence of a \(J\)-covering family of arrows \(f_i : x_i \to x\) and a family of arrows \(k_i : x_i \to y\) such that \(\xi \circ l'(F(f_i)) = l'(k_i)\) for each \(i\) can be reformulated in terms of the arrows \(h\) and \(g_h\), as follows. Let us consider, for each \(i \in I\) and \(h \in U\), the following pullback square:

\[
\begin{array}{ccc}
P_{i,h} & \xrightarrow{\pi_1^{i,h}} & l'(F(x_i)) \\
\pi_2^{i,h} & \searrow & \downarrow_{l'(\pi)} \\
l'(\text{dom}(h)) & \xrightarrow{l'(h)} & l'(F(x))
\end{array}
\]

It is easy to see (cf. Proposition 3.1) that the collection \(\mathcal{F}_{i,h}\) of arrows \(\chi : l'(c) \to P_{i,h}\) such that both \(\pi_1^{i,h} \circ \chi\) and \(\pi_2^{i,h} \circ \chi\) are images \(l'(w) : l'(c) \to l'(F(x_i))\) and \(l'(z) : l'(c) \to l'(\text{dom}(h))\) under \(l'\) of arrows \(w\) and \(z\) in \(\mathcal{D}\) (where \(c\) varies among the objects of \(\mathcal{C}\)) is epimorphic. At the cost of composing with another covering family on \(l'(c)\), we can suppose that \(F(f_i) \circ w = h \circ z\) without loss of generality. Therefore, since for any \(i \in I\) the collection of arrows \(\{\pi_1^{i,h} \circ \chi \mid h \in U, \chi \in \mathcal{F}_{i,h}\}\) is epimorphic, \(\xi \circ l'(F(f_i)) = l'(k_i)\) if and only if for any \(h \in U\) and arrows \(w\) and \(z\) such that \(F(f_i) \circ w = h \circ z\), \(\xi \circ e_{i,h} \circ \chi = l'(k_i) \circ \pi_1^{i,h} \circ \chi\). But \(\pi_1^{i,h} \circ \chi = l'(w)\) and \(\xi \circ e_{i,h} \circ \chi = \xi \circ l'(h) \circ \pi_2^{i,h} \circ \chi = l'(g_h) \circ l'(z)\). So the condition \(\xi \circ l'(F(f_i)) = l'(k_i)\) is equivalent to the condition \(g_h \circ z \equiv_K k_i \circ w\) for every arrow \(h \in U\) and arrows \(w\) and \(z\) such that \(F(f_i) \circ w = h \circ z\).

\[\square\]
Remark 4.6. Condition (iv) in the definition of dense morphism of sites (cf. Remark 4.2) for the functor \( l' \circ F \) admits the following reformulation in terms of \( F \):

(iv) For any arrows \( f_1, f_2 : c_1 \to c_2 \) in \( C \) such that \( F(f_1) \equiv_K F(f_2) \) there exists a \( J \)-covering family of arrows \( k_i : c'_i \to c_1 \) such that \( f_1 \circ k_i = f_2 \circ k_i \) for all \( i \).

If a functor \( F \) satisfies the conditions of Proposition 4.5 plus condition (iv) then \( l' \circ F \) is a dense morphism of sites and hence, by Remark 2.6, \( F : (C, J) \to (D, K) \) is a weakly dense morphism of sites.

Summarizing, we have the following result:

**Theorem 4.7.** Let \( F : (C, J) \to (D, K) \) be a morphism of sites. Then the following conditions are equivalent:

(i) The geometric morphism \( \text{Sh}(F) : \text{Sh}(D, K) \to \text{Sh}(C, J) \) is an equivalence.

(ii) \( l' \circ F : (C, J) \to (a_K(D), C^D_K) \) is a dense morphism of sites.

(iii) \( F \) is a weakly dense morphism of sites \( (C, J) \to (D, K) \) (i.e. it satisfies the conditions of Proposition 4.5).

**Proof** Since \( \text{Sh}(l' \circ F) \) is the composite of \( \text{Sh}(F) \) with the canonical equivalence \( \text{Sh}(a_K(D), C^D_K) \simeq \text{Sh}(D, K) \) and the site \( (a_K(D), C^D_K) \) is subcanonical, the direction (ii) \( \Rightarrow \) (i) follows from Theorem 11.8 [7]. The equivalence between (ii) and (iii) follows by definition of a weakly dense morphism of sites (in light of Proposition 4.5). Lastly, the implication (i) \( \Rightarrow \) (ii) follows from Proposition 4.3. \( \square \)

**Remark 4.8.** Theorem 4.7 constitutes a vast generalization of Grothendieck’s Comparison Lemma (Theorem 4.1 from Chapter III of [1]).

**Example 4.9.** Let us discuss an example of a weakly dense morphism of sites which is not dense. Let \( 2 \) be the preorder category with two distinct objects 0 and 1 and just one arrow \( 0 \to 1 \) apart from the identities. Notice that \( 2 \) is cartesian, since it is a meet-semilattice, and the functor \( F : 2 \to 2 \) sending 0 to 1, 1 to 1 and the arrow \( 0 \to 1 \) to the identity arrow on 1 is cartesian as it is a meet-semilattice homomorphism. Let us equip \( 2 \) with the atomic topology \( J_1 \), whose covering sieves are the maximal ones plus the sieve on 1 consisting of the arrow \( 0 \to 1 \). Since it is cartesian and cover-preserving, the functor \( F \)
is a morphism of sites \((2, J_{at}) \to (2, J_{at})\). Now, by the Comparison Lemma, the topos \(\text{Sh}(2, J_{at})\) is equivalent to the topos of sheaves on the one-object full subcategory \(\{0\}\) with respect to the induced topology on it, namely the maximal one; so we have \(\text{Sh}(2, J_{at}) \simeq \text{Set}\). Therefore the geometric morphism \(\text{Sh}(F) : \text{Sh}(2, J_{at}) \to \text{Sh}(2, J_{at})\) is necessarily (isomorphic) to the identity morphism (since the only geometric morphism \(\text{Set} \to \text{Set}\) is the identity one); in particular, it is an equivalence. Therefore, by Theorem 4.7, \(F\) is a weakly dense morphism of sites. However, \(F\) is not dense; for instance, it does not satisfy condition (ii) in the definition of a dense morphism of sites. Indeed, there is no \(J_{at}\)-covering family of arrows to the object 0 whose domain is in the image of the functor \(F\). On the other hand, it satisfies condition (ii) in the characterization of weakly dense morphisms of sites provided by Proposition 4.5. Indeed, if \(d = 1\) then the arrow \(k = 1_1\) satisfies the condition (by taking \(h = 1_1, x = 0 = 1, S = \{0 \to 1\}\) and \(g_{0 \to 1} = 0 \to 1\)), while if \(d = 0\) then the arrow \(k = 1_0\) satisfies the condition (by taking \(h = 0 \to 1, x = 0 = 1, S = \{0 \to 1\}\) and \(g_{0 \to 1} = 1_0\)).

**Remark 4.10.** In section 5.6 we shall obtain an alternative criterion, given by Corollary 5.40, for a morphism of sites to induce an equivalence of toposes.

### 4.2 Two criteria for equivalence

Given a flat functor \(F : \mathcal{C} \to \mathcal{E}\) defined on an essentially small category \(\mathcal{C}\) with values in a Grothendieck topos \(\mathcal{E}\), if \(F\) is \(J\)-continuous for a Grothendieck topology \(J\) on \(\mathcal{C}\) then we know by Diaconescu’s equivalence that it induces a geometric morphism \(f : \mathcal{E} \to \text{Sh}(\mathcal{C}, J)\). The following result provides a necessary and sufficient condition, phrased entirely in terms of \(F\), for \(f\) to be an equivalence:

**Corollary 4.11.** Let \((\mathcal{C}, J)\) be a small-generated site, \(\mathcal{E}\) a Grothendieck topos and \(F : \mathcal{C} \to \mathcal{E}\) a \(J\)-continuous flat functor. Then the geometric morphism \(f : \mathcal{E} \to \text{Sh}(\mathcal{C}, J)\) induced by \(F\) is an equivalence if and only if \(F\) satisfies the following conditions:

(i) If the image under \(F\) of a sieve \(S\) in \(\mathcal{C}\) is epimorphic in \(\mathcal{E}\) then \(S\) is \(J\)-covering;

(ii) the family of objects of the form \(F(c)\) for \(c \in \mathcal{C}\) is separating for \(\mathcal{E}\);

(iii) for every \(x, y \in \mathcal{C}\) and any arrow \(g : F(x) \to F(y)\) in \(\mathcal{E}\), there exist a \(J\)-covering family of arrows \(f_i : x_i \to x\) and a family of arrows \(g_i : x_i \to y\) such that \(g \circ F(f_i) = F(g_i)\) for all \(i\).
Proof  A $J$-continuous flat functor $F : C \to \mathcal{E}$ is clearly the same thing as a morphism of sites $F : (C, J) \to (\mathcal{E}, J_\mathcal{E}^{\text{can}})$, and the geometric morphism $f : \mathcal{E} \to \text{Sh}(C, J)$ corresponding to $F$ via Diaconescu’s equivalence is precisely $\text{Sh}(F)$. Our thesis thus follows from Theorem 4.7, since the conditions of the corollary are precisely those for $F : (C, J) \to (\mathcal{E}, J_\mathcal{E}^{\text{can}})$ to be dense (notice that the site $(\mathcal{E}, J_\mathcal{E}^{\text{can}})$ is subcanonical). □

Remark 4.12. In section 5.6 we shall encounter an alternative criterion, provided by Corollary 5.36, for a continuous flat functor to induce an equivalence of toposes.

We can deduce from Theorem 4.7 a criterion for two (essentially small) sites to give rise to equivalent toposes of sheaves on them.

Theorem 4.13. Let $(C, J)$ and $(D, K)$ be two small-generated sites. Then the following conditions are equivalent:

(i) The toposes $\text{Sh}(C, J)$ and $\text{Sh}(D, K)$ are equivalent.

(ii) There exist a category (resp. an essentially small category, if $C$ and $D$ are essentially small) $A$, a Grothendieck topology $Z$ on $A$ (which can be supposed subcanonical) and two functors $H : C \to A$ and $K : D \to A$ satisfying the following conditions:

(i) $P$ is a $J$-covering family in $C$ if and only if $H(P)$ is a $Z$-covering family in $A$;

(ii) $Q$ is a $K$-covering family in $D$ if and only if $K(Q)$ is a $Z$-covering family in $A$;

(iii) for any object $a$ of $A$ there exists a $Z$-covering sieve whose arrows factor both through an arrow whose domain is in the image of $H$ and through an arrow whose domain is in the image of $K$;

(iv) for every $x, y \in C$ (resp. $x', y' \in D$) and any arrow $g : H(x) \to H(y)$ (resp. $g' : K(x') \to K(y')$) in $A$, there exist a $J$-covering family of arrows $f_i : x_i \to x$ (resp. a $K$-covering family of arrows $f'_j : x'_j \to x'$) and a family of arrows $g_i : x_i \to y$ (resp. a family of arrows $g'_j : x'_j \to y'$) such that $g \circ H(f_i) = H(g_i)$ for all $i$ (resp. $g' \circ K(f'_j) = K(g'_j)$ for all $j$);

(v) for any arrows $h, k : x \to y$ (resp. $h', k' : x' \to y'$) in $C$ (resp. in $D$) such that $H(h) = H(k)$ (resp. $K(h') = K'(k')$) there exists a $J$-covering (resp. $K$-covering) family of arrows $f_i : x_i \to x$ (resp. $f'_j : x'_j \to x'$) such that $h \circ f_i = k \circ f_i$ for all $i$ (resp. $h' \circ f'_j = k' \circ f'_j$ for all $j$).
Proof  The given conditions are precisely those for the functor $H$ and $K$ to respectively define dense morphisms of sites $(\mathcal{C}, J) \to (\mathcal{A}, Z)$ and $(\mathcal{D}, K) \to (\mathcal{A}, Z)$. Such morphisms induce by Theorem 11.8 [7] (cf. also Theorem 4.7) equivalences of toposes $\text{Sh}(\mathcal{C}, J) \simeq \text{Sh}(\mathcal{A}, Z)$ and $\text{Sh}(\mathcal{D}, K) \to \text{Sh}(\mathcal{A}, Z)$, whence an equivalence $\text{Sh}(\mathcal{C}, J) \simeq \text{Sh}(\mathcal{D}, K)$. Conversely, if $\text{Sh}(\mathcal{C}, J) \simeq \text{Sh}(\mathcal{D}, K)$ then, by taking $\mathcal{A}$ to be the full subcategory of this topos on the objects that are either coming from the site $(\mathcal{C}, J)$ or from the site $(\mathcal{D}, K)$ with the Grothendieck topology $Z$ induced on it by the canonical topology on the topos, we obtain by the Comparison Lemma equivalences $\text{Sh}(\mathcal{C}, J) \simeq \text{Sh}(\mathcal{A}, Z)$ and $\text{Sh}(\mathcal{D}, K) \to \text{Sh}(\mathcal{A}, Z)$, and hence by Theorem 4.7 the canonical functors $\mathcal{C} \to \mathcal{A}$ and $\mathcal{D} \to \mathcal{A}$ are respectively dense morphisms of sites $(\mathcal{C}, J) \to (\mathcal{A}, Z)$ and $(\mathcal{D}, K) \to (\mathcal{A}, Z)$. □

4.3 Local faithfulness, local fullness and local surjectivity

In this section we shall introduce some notions which are naturally related to the denseness conditions considered above.

Definition 4.14. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $J$ (resp. $K$) a Grothendieck topology on $\mathcal{C}$ (resp. on $\mathcal{D}$). Then $F$ is said to be

(a) $(J, K)$-faithful (resp. $J$-faithful) if whenever $F(h) \equiv_K F(k)$ (resp. $F(h) = F(k)$), $h \equiv_J k$;

(b) $(J, K)$-full (resp. $J$-full) if for every $x, y \in \mathcal{C}$ and any arrow $g : F(x) \to F(y)$ in $\mathcal{D}$, there exist a $J$-covering family of arrows $f_i : x_i \to x$ and arrows $g_i : x_i \to y$ (for each $i \in I$) such that $g \circ F(f_i) \equiv_K F(g_i)$ (resp. $g \circ F(f_i) = F(g_i)$) for all $i$;

(c) $K$-dense if for every $d \in \mathcal{D}$, there exists a $K$-covering family of arrows whose domains are in the image of $F$.

Remarks 4.15. (a) If $K$ is the canonical topology on $\mathcal{D}$ then the relation $\equiv_K$ reduces to equality and $(J, K)$-faithfulness (resp. $(J, K)$-fullness) reduces to $J$-faithfulness (resp. $J$-fullness).

(b) If $F$ satisfies the covering-lifting property (that is, the property that for any $c \in \mathcal{C}$ and any $K$-covering sieve $S$ on $F(c)$ there is a $J$-covering sieve $R$ on $c$ such that $F(R) \subseteq S$) then the local equality $\equiv_K$ in conditions (a) and (b) of Definition 4.14 can be replaced by strict equality, whence $(J, K)$-faithfulness (resp. $(J, K)$-fullness) coincides with $J$-faithfulness (resp. $J$-fullness) for $F$. 27
(c) Any dense morphism of sites \((\mathcal{C}, J) \to (\mathcal{D}, K)\) is \(J\)-faithful, \(J\)-full and \(K\)-dense.

(d) Inverse images functors of geometric inclusions of toposes (in particular, associated sheaf functors) satisfy a form of local fullness (cf. Corollary 5.16 below).

**Proposition 4.16.** Let \(F : \mathcal{C} \to \mathcal{D}\) be a functor and \(J\) a Grothendieck topology on \(\mathcal{C}\). Then \(F\) is \(J\)-full and \(J\)-faithful if and only if for any \(c \in \mathcal{C}\), the canonical arrow \(l(c) \to a_J(\text{Hom}_D(F(-), F(c)))\) is an isomorphism in \(\text{Sh}(\mathcal{C}, J)\).

**Proof** The canonical arrow \(l(c) \to l(\text{Hom}_D(F(-), F(c)))\) is the image under the associated sheaf functor \(a_J : [\mathcal{C}^{\text{op}}, \text{Set}] \to \text{Sh}(\mathcal{C}, J)\) of the canonical arrow \(y_C(c) \to (\text{Hom}_D(F(-), F(c)))\). Our thesis thus immediately follows from Lemma 2.1.

**Proposition 4.17.** Let \(F : \mathcal{C} \to \mathcal{D}\) be a \(J\)-full, \(K\)-dense, cover-reflecting and cover-preserving functor (for instance, a dense morphism of sites \(F : (\mathcal{C}, J) \to (\mathcal{D}, K)\)). Then \(F : (\mathcal{C}, J) \to (\mathcal{D}, K)\) has the covering-lifting property.

**Proof** Let \(c\) be an object of \(\mathcal{C}\) and \(S\) be a \(K\)-covering sieve on \(F(c)\). Then, for each \(g \in S\), by \(K\)-denseness there exists a \(K\)-covering sieve \(U_g\) on \(\text{dom}(g)\) generated by a family of arrows \(\{\xi^g_i \mid i \in I\}\) whose domains are of the form \(F(c^g_i)\), where \(c^g_i\) are objects of \(\mathcal{C}\). Now, by \((J,K)\)-fullness, there is a \(J\)-covering family of arrows \(\xi^g_i\) to \(c^g_i\) and for each \(h^g_{(g,i)}\) an arrow \(\chi^{(g,i)}\) from \(\text{dom}(h^g_{(g,i)})\) to \(c\) such that \(g \circ \xi^g_i \circ F(h^g_{(g,i)}) = F(\chi^{(g,i)})\). So the arrows \(\chi^{(g,i)}\) generate a sieve whose image under \(F\) factors through \(g\) and hence belongs to \(S\). This sieve is \(J\)-covering since \(F\) is both cover-preserving and cover-reflecting; indeed, the sieve generated by the arrows \(g \circ \xi^g_i \circ F(h^g_{(g,i)})\) is \(K\)-covering since \(S\) is \(K\)-covering, the family of arrows \(\{\xi^g_i \mid i \in I\}\) is \(K\)-covering and the family of arrows \(h^g_{(g,i)}\) is \(J\)-covering (whence its image under \(F\) is \(K\)-covering). So \(F\) satisfies the covering-lifting property. 

Given a morphism of sites \(F : (\mathcal{C}, J) \to (\mathcal{D}, K)\), we have an induced functor \(a_F : a_J(\mathcal{C}) \to a_K(\mathcal{D})\) given by the restriction of the inverse image of the induced geometric morphism \(\text{Sh}(F)\), which is a morphism of sites if \(a_J(\mathcal{C})\) and \(a_K(\mathcal{D})\) are endowed with the Grothendieck topologies \(C_J^F\) and \(C_K^D\) respectively induced by the canonical ones on \(\text{Sh}(\mathcal{C}, J)\) and \(\text{Sh}(\mathcal{D}, K)\).
Proposition 4.18. Let \( F : (C, J) \to (D, K) \) be a morphism of sites. Then

(i) \( F \) is \((J, K)\)-faithful if and only if \( a_F \) is faithful;

(ii) If \( F \) is \((J, K)\)-faithful and satisfies the covering-lifting property then \( F \) is \((J, K)\)-full if and only if \( a_F \) is full;

(iii) If \( F \) satisfies the covering-lifting property then \( F \) is \( K \)-dense if and only if \( a_F \) is \( C_K^D \)-dense.

Proof (i) If \( a_F \) is faithful then for any parallel arrows \( h, k \) in \( C \), the condition \( l(h) = l(k) \) (that is, \( h \equiv_J k \)) is equivalent to the condition \( l'(F(h)) = a_F(l(h)) = a_F(l(k)) = l'(F(k)) \) (that is, \( F(h) \equiv_K F(k) \)).

Conversely, let us now suppose that \( F \) is \((J, K)\)-faithful, and prove that \( a_F \) is faithful. If \( u, v : l(c) \to l(c') \) are arrows in \( \text{Sh}(C, J) \) then there are \( J \)-covering sieves \( S \) and \( T \) on \( c \) such that for each \( f \in S \) there is an arrow \( \xi_f : \text{dom}(f) \to c' \) such that \( u \circ l(f) = l(\xi_f) \) for each \( f \), and for each \( g \in T \) there is an arrow \( \chi_g : \text{dom}(g) \to c' \) such that \( v \circ l(g) = l(\chi_g) \). Notice that the sieve \( S \cap T \) is \( J \)-covering as it is the intersection of two \( J \)-covering sieves. Suppose that \( a_F(u) = a_F(v) \). Then \( a_F(u) \circ a_F(l(h)) = a_F(v) \circ a_F(l(h)) \) for each \( h \in S \cap T \). Therefore \( l'(F(\xi_h)) = a_F(l(\xi_h)) = a_F(l(\chi_h)) = l'(F(\chi_h)) \) for each \( h \in S \cap T \), that is \( F(\xi_h) \equiv_K F(\chi_h) \). Since \( F \) is \((J, K)\)-faithful, it follows that \( \xi_h \equiv_J \chi_h \) for each \( h \in S \cap T \). But this means that \( u \circ l(h) = l(\xi_h) = l(\chi_h) = v \circ l(h) \) for each \( h \in S \cap T \). Since \( S \cap T \) is \( J \)-covering, this implies that \( u = v \), as required.

(ii) Suppose that \( a_F \) is full. Then for every \( x, y \in C \) and any arrow \( g : F(x) \to F(y) \) in \( D \), there is an arrow \( \xi : l(x) \to l(y) \) such that \( a_F(\xi) = l'(g) \). By Proposition 3.1 there are a \( J \)-covering family of arrows \( f_i : x_i \to x \) and arrows \( g_i : x_i \to y \) (for each \( i \in I \)) such that \( \xi \circ l(f_i) = l(g_i) \). Applying \( a_F \), we thus obtain that \( l'(g) \circ l'(F(f_i)) = l'(F(g_i)) \), equivalently \( g \circ F(f_i) \equiv_K F(g_i) \). So \( F \) is \((J, K)\)-full. In fact, this argument shows that the fullness condition applied to arrows of the form \( l'(g) \) for some arrow \( g \) in \( D \) is precisely equivalent to the \((J, K)\)-fullness condition. Conversely, let us suppose that \( F \) is \((J, K)\)-full and satisfies the covering-lifting property. Given an arrow \( \xi : l'(F(c)) \to l'(F(c')) \), by Proposition 3.1 there exist a \( K \)-covering sieve \( S \) on \( F(c) \) and for each arrow \( g \) in \( S \) an arrow \( t_g : \text{dom}(g) \to F(c') \) such that \( \xi \circ l'(g) = l'(t_g) \). Since \( F \) satisfies the covering-lifting property, there is a \( J \)-covering sieve \( R \) on \( c \) such that \( F(R) \subseteq S \). For any \( r \in R \), consider the arrow \( t_{F(r)} : F(\text{dom}(r)) \to F(c') \). Since \( F \) is \((J, K)\)-full, there exist a \( J \)-covering family of arrows \( \{f_j^r : y_j \to \text{dom}(r) \mid j \in H_r\} \) and, for each \( j \in H_r \), an arrow \( k_j^r : y_j \to c' \) such that \( t_{F(r)} \circ F(f_j^r) \equiv_K F(k_j^r) \). We want to show that the arrows \( k_j^r : y_j \to c' \) indexed by the \( J \)-covering family of
arrows $r \circ f_j^r$ to $c$ define an arrow $\chi : l(c) \to l(c')$ such that $a_F(\chi) = \xi$. For this, by Proposition 3.1 we have to verify that if $m$ and $n$ are arrows such that $r \circ f_j^r \circ m = r' \circ f_j^{r'} \circ n$, then $k_j^r \circ m \equiv_j k_j^{r'} \circ n$. Now, since $F$ is $(J, K)$-faithful, the latter condition is equivalent to $F(k_j^r \circ m) \equiv_K F(k_j^{r'} \circ n)$. But $l'(F(k_j^r \circ m)) = l'(F(k_j^{r'})) \circ l'(F(m)) = l'(t_{F(r)}) \circ l'(F(f_j^r)) \circ l'(F(m)) = l'(F(r) \circ f_j^r \circ m) = \xi \circ l'(F(r)) \circ l'(F(f_j^r)) \circ l'(F(m)) = \xi \circ l'(F(r') \circ f_j^{r'} \circ n)) = \xi \circ l'(F(r')) \circ l'(F(f_j^{r'})) \circ l'(F(n)) = l'(t_{F(r')}) \circ l'(F(f_j^{r'})) \circ l'(F(n)) = l'(F(f_j^{r'})) \circ l'(F(n)) = l'(F(k_j^{r'} \circ n))$, as required. The fact that $a_F(\chi) = \xi$ follows immediately from the definition of $\chi$, by using the fact that $F$ is cover-preserving.

(iii) If $F$ is $K$-dense then, clearly, $a_F$ is $C_K^D$-dense. Suppose instead that $a_F$ is $C_K^D$-dense. Given an object $d$ of $\mathcal{D}$, we want to show that there is a $K$-covering family of arrows whose domains are in the image of $F$. Consider the object $l'(d)$. Then there is an epimorphic family $F \in \text{Sh}(\mathcal{D}, K)$ of arrows $\xi$ to $l'(d)$ whose domains are of the form $l'(F(c))$ for some $c \in \mathcal{C}$. For any such $\xi : l'(F(c_\xi)) \to l'(d)$, by Proposition 3.1 there exists a $K$-covering sieve $S_\xi$ on $F(c_\xi)$ and for each arrow $g$ in $S_\xi$ an arrow $t_g : \text{dom}(g) \to d$ such that $\xi \circ l'(g) = l'(t_g)$. Since $F$ satisfies the covering lifting property, there is a $J$-covering sieve $R_\xi$ on $c$ such that $F(R_\xi) \subseteq S_\xi$. Now, the collection of arrows $\{t_{F(r)} : F(\text{dom}(r)) \to d \mid r \in R_\xi, \xi \in F\}$ is $K$-covering. Indeed, by using the fact that $F$ is cover-preserving, one immediately sees that the image of this family under $l'$ is epimorphic in $\text{Sh}(\mathcal{D}, K)$. This completes our proof.  

\begin{remark}
Proposition 4.18 implies that every morphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ whose underlying functor has the covering-lifting property, is $J$-faithful, $J$-full and $K$-dense induces an equivalence of toposes. In fact, as shown by the following result, these morphisms are precisely the dense morphisms of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$.
\end{remark}

\begin{corollary}
Let $F$ be a morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$. Then the following conditions are equivalent:

(i) $F$ is a weakly dense morphism of sites which has the covering-lifting property;

(ii) $F$ is dense (that is, $K$-dense, $J$-full and cover-reflecting).
\end{corollary}

\begin{proof}
(i) $\Rightarrow$ (ii) By definition, any weakly dense morphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ is cover-reflecting and satisfies the property that $a_F$ is $C_K^D$-dense; so by Proposition 4.18(iii) $F$ is $K$-dense. The fact that $F$ is $J$-full follows from the proof of Proposition 4.18(ii) in light of Theorem 4.7 and Remark 4.15(b).

(ii) $\Rightarrow$ (i) This follows from Propositions 4.17 and 4.18.
\end{proof}
Remarks 4.21. (a) The morphism of sites $F : (\mathbb{2}, J_{at}) \to (\mathbb{2}, J_{at})$ of Example 4.9 is weakly dense but does not satisfy the covering-lifting property (the Sieve on $1 = F(0) = F(1)$ generated by the arrow $0 \to 1$ cannot be lifted neither to a $J_{at}$-covering sieve on $0$ nor on $1$ since the image under $F$ of any sieve is the maximal one).

(b) Every weakly dense morphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, T)$ is $(J, T)$-faithful if $T$ is a subcanonical topology on $\mathcal{D}$ (in which cases the local equality $\equiv_T$ reduces to strict equality). On the other hand, every weakly dense morphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ satisfying the covering lifting property is $(J, K)$-faithful (see Proposition 4.5 and Remark 4.2).

5 Site characterizations of some properties of geometric morphisms

Corollary 4.11 gives a characterization of the $J$-continuous flat functors $\mathcal{C} \to \mathcal{E}$ whose corresponding geometric morphism $f : \mathcal{E} \to \text{Sh}(\mathcal{C}, J)$ is an equivalence. In this section we shall characterize the $J$-continuous flat functors whose associated geometric morphism $f$ is a geometric inclusion (resp. a surjection, localic, hyperconnected).

5.1 Surjections and inclusions

Proposition 5.1. Let $(\mathcal{C}, J)$ be a small-generated site, $\mathcal{E}$ a Grothendieck topos and $F : \mathcal{C} \to \mathcal{E}$ a $J$-continuous flat functor. Then the geometric morphism $f : \mathcal{E} \to \text{Sh}(\mathcal{C}, J)$ induced by $F$ is a surjection (that is, $f^*$ is faithful) if and only if $F$ is cover-reflecting (in the sense that the $J$-covering families in $\mathcal{C}$ are precisely those which are sent by $F$ to epimorphic families in $\mathcal{E}$).

Proof Let us preliminarily notice that $f^*$ is faithful if and only if for every monomorphism $m$, if $f^*(m)$ is an isomorphism then $m$ is an isomorphism. Indeed, one direction follows by considering the equalizer of the given pair of arrows, while the other follows from the balancedness of $\text{Sh}(\mathcal{C}, J)$ (which allows to check that $m$ is an isomorphism by showing that it is both a monomorphism and an epimorphism).

Suppose that $f^*$ is faithful. Given a sieve $S$ in $\mathcal{C}$ on an object $c$, consider the corresponding monomorphism $m_S : S \to \text{Hom}_{\mathcal{C}}(\cdot, c)$. We have that $S$ is $J$-covering if and only if $a_J(m_S)$ is an isomorphism. But this is equivalent, since $f^*$ is faithful, to $f^*(a_J(m_S))$ being an isomorphism, that is, to the family $\{F(f) \mid f \in S\}$ being epimorphic.
Conversely, let us suppose that $F$ is cover-reflecting. Given a monomorphism $m : P \to Q$ in $\text{Sh}(\mathcal{C}, \mathcal{J})$, we have to prove that if $f^*(m)$ is an isomorphism then $m$ is an isomorphism. By considering the pullbacks $S_\alpha \to l(c)$ of $m$ along all the arrows $\alpha : l(c) \to Q$ where $c$ is an object of $\mathcal{C}$, we obtain that, since $F$ is cover-reflecting, the monomorphisms $S_\alpha \to l(c)$ are all isomorphisms, whence $m$ is an isomorphism as well (the family of arrows $\alpha$ being epimorphic and $\mathcal{E}$ being balanced).

Let us apply this result in the context of interpretations between geometric theories. Recall from section 2.1.3 of [2] that an interpretation of a geometric theory $T$ into a geometric theory $S$ is a geometric functor $I : \mathcal{C}_T \to \mathcal{C}_S$ between their geometric syntactic categories. Notice that $I$ acts on contexts $\vec{x}$ for geometric formulae over the signature of $T$ sending a context $\vec{x}$ to the context $I(\vec{x})$ appearing in the formula $I(\{\vec{x} \cdot \top\}) = \{I(\vec{x}) \cdot \xi\}$; since $I$ preserves monomorphisms, we have that $I(\{\vec{x} \cdot \phi\}) = \{I(\vec{x}) \cdot \chi\}$ for some geometric formula $\chi$ in the context $I(\vec{x})$; we shall write $I(\phi)$ for $\chi$ (when the context for $\phi$ can be unambiguously inferred). Therefore, with any sequent $\sigma \equiv (\phi \vdash_S \psi)$ over the signature of $T$ can be associated a sequent $I(\sigma) \equiv (I(\phi) \vdash_{I(\vec{x})} I(\psi))$ over the signature of $S$ which is provable in $S$ if the former sequent is provable in $T$.

**Corollary 5.2.** (i) An interpretation $I : \mathcal{C}_T \to \mathcal{C}_S$ between geometric theories $T$ and $S$ induces a geometric surjection $\text{Set}[S] \to \text{Set}[T]$ between their classifying toposes if and only if, with the above notation, for every sequent $\sigma$ over the signature of $T$, if the sequent $I(\sigma)$ is provable in $S$ then $\sigma$ is provable in $T$.

(ii) In particular, if $I$ is the canonical interpretation given by an expansion $T'$ of $T$ (in the sense of section 7.1 of [2]) then $I$ induces a geometric surjection if and only if $T'$ is a conservative expansion of $T$, that is, every geometric sequent over the signature of $T$ which is provable in $T'$ is provable in $T$.

**Proof** It suffices to apply Proposition 5.1 observing that the $J_\mathcal{T}$-continuous flat functor given by the composite $y_S \circ I$ is cover-reflecting if and only if for every subobject $[\psi] : \{\vec{x} \cdot \psi\} \to \{\vec{x} \cdot \phi\}$ in $\mathcal{T}$, if $I([\psi])$ is the identity subobject (equivalently, the sequent $(I(\phi) \vdash_{I(\vec{x})} I(\psi))$ is provable in $S$ then $[\psi]$ is the identity subobject (equivalently, the sequent $(\phi \vdash_{\vec{x}} \psi)$ is provable in $T$).
Let us recall from Theorem A4.2.10 [4] that every geometric morphism can be factored, uniquely up to commuting equivalence, as a surjection followed by an inclusion. Let us describe this factorization in terms of sites.

In the proof of the following theorem, we exploit the possibility of regarding the inverse image functor $f^* : E \to F$ of a geometric morphism $f : F \to E$ as a morphism of small-generated sites $(E, J^E_F) \to (F, J^F_K)$.

**Theorem 5.3.** Let $F : (C, J) \to (D, K)$ be a morphism of small-generated sites. Then:

(i) The geometric morphism $\text{Sh}(F) : \text{Sh}(D, K) \to \text{Sh}(C, J)$ induced by $F$ is a surjection if and only if $F$ is cover-reflecting (that is, if the image of a family of arrows with a fixed codomain is $K$-covering then the family is $J$-covering).

(ii) The surjection-inclusion factorization of the geometric morphism $\text{Sh}(F) : \text{Sh}(D, K) \to \text{Sh}(C, J)$ induced by $F$ can be identified with the factorization $\text{Sh}(i_{J^F_F}) \circ \text{Sh}(F_r)$, where $J^F_F$ is the Grothendieck topology on $C$ whose covering sieves are exactly those whose image under $F$ are $K$-covering families, $i_{J^F_F} : (C, J) \to (C, J^F_F)$ is the morphism of sites given by the canonical inclusion functor and $F_r : (C, J^F_F) \to (D, K)$ is the morphism of sites given by $F$.

(iii) The geometric morphism $\text{Sh}(F) : \text{Sh}(D, K) \to \text{Sh}(C, J)$ induced by $F$ is an inclusion if and only if $F_r : (C, J^F_F) \to (D, K)$ is a weakly dense morphism of sites (equivalently, if $K$ is subcanonical, a dense morphism of sites); in particular, if $K$ is subcanonical then $\text{Sh}(F)$ is an inclusion if and only if the following conditions are satisfied:

(i) for any object $d$ of $D$ there exists a $K$-covering family of arrows $d_i \to d$ whose domains $d_i$ are in the image of $F$;

(ii) for every $x, y \in C$ and any arrow $g : F(x) \to F(y)$ in $D$, there exist a $J^F_F$-covering family of arrows $f_i : x_i \to x$ and a family of arrows $g_i : x_i \to y$ such that $g \circ F(f_i) = F(g_i)$ for all $i$.

**Proof** (i) This follows as an immediate consequence of Proposition 5.1 by observing that a family of arrows in $D$ with common codomain is sent by the canonical functor $D \to \text{Sh}(D, K)$ to an epimorphic family if and only if it is $K$-covering.

(ii) The morphism $\text{Sh}(i_{J^F_F})$ is the canonical inclusion of $\text{Sh}(C, J^F_F)$ into $\text{Sh}(C, J)$. On the other hand, by definition of $J^F_F$, the morphism of sites $F_r$ is
cover-reflecting and hence, by point (i), induces a surjective geometric morphism. So \( \text{Sh}(F_{r}) \) and \( \text{Sh}(i_{J_F}) \) give a factorization of \( \text{Sh}(F) \) as a surjection followed by an inclusion.

(iii) A geometric morphism is an inclusion if and only if the surjection part of its surjection-inclusion factorization is an equivalence; our thesis thus follows from Theorem 4.7.

□

Theorem 5.3 can be notably applied to a \( J \)-continuous flat functor \( F : C \to E \), regarded as a morphism of sites \( (C, J) \to (E, J_{E}^{\text{can}}) \), giving the following result:

Corollary 5.4. A \( J \)-continuous flat functor \( F : C \to E \) induces a geometric inclusion if and only if it satisfies the following conditions:

(i) every object of \( E \) can be covered by objects which are in the image of \( F \);

(ii) for every \( x, y \in C \) and any arrow \( g : F(x) \to F(y) \) in \( E \), there exist a family of arrows \( f_i : x_i \to x \) which is sent by \( F \) to an epimorphic family and a family of arrows \( g_i : x_i \to y \) such that \( g \circ F(f_i) = F(g_i) \) for all \( i \).

5.2 Induced topologies

The following result, which is a corollary of Theorem 5.3(ii), shows that the natural context for defining ‘induced Grothendieck topologies’ is that provided by flat functors (equivalently, geometric morphisms) or more generally morphisms of sites.

Proposition 5.5. Let \( f : E \to [C^{\text{op}}, \text{Set}] \) be a geometric morphism (equivalently, a flat functor \( F : C \to E \)). Then there exists a Grothendieck topology \( J_f \) (resp. \( J_F \)) on \( C \), called the Grothendieck topology induced by \( f \) (resp. \( F \)) whose covering sieves are precisely the sieves which are sent by \( f^* \) (resp. by \( F \)) to epimorphic families in \( E \). This applies in particular to a morphism of sites \( G : (C, J) \to (D, K) \), yielding a Grothendieck topology \( J_G \) on \( C \) whose covering sieves are exactly those whose image under \( G \) are \( K \)-covering families (cf. Theorem 5.3(ii)).

□

Remarks 5.6. (a) If \( F : (C, J) \to (D, K) \) is a cover-reflecting morphism of sites (for instance, a weakly dense morphism of sites) then \( J \) is a Grothendieck topology (since it coincides with \( J_F \)).
The classical definition of Grothendieck topology induced on a full $K$-dense subcategory $C$ of a category $D$ is subsumed by this general definition of induced topology; indeed, it is readily seen that the embedding of $C$ into $D$ defines a morphism of sites $(C, T) \rightarrow (D, K)$, where $T$ is the trivial Grothendieck topology on $C$.

The following result shows in particular that if $F : (C, J) \rightarrow (D, K)$ is a dense morphism of sites then the Grothendieck topology $K$ can be recovered from $J$. Notice that, by Corollary 4.20, any flat functor $F : C \rightarrow E$ on a small-generated category $C$ with values in a Grothendieck topos $E$ which induces an equivalence $E \simeq \mathbf{Sh}(C, J_F)$ (equivalently, satisfies the conditions of Corollary 4.11 cf. also Corollary 5.9 below) is a dense morphism of sites $(C, J_F) \rightarrow (E, J_E)$.

**Proposition 5.7.** Let $F : (C, J) \rightarrow (D, K)$ be a $K$-dense functor with the covering-lifting property (for instance, a dense morphism of sites $(C, J) \rightarrow (D, K)$ – cf. Corollary 4.20). Then the Grothendieck topology $K$ can be recovered from $J$, as follows: for any sieve $T$ on an object $d$ of $D$, $T \in K(d)$ if and only if for any object $c \in C$ and any arrow $\xi : F(c) \rightarrow d$ in $D$, there exists a $J$-covering sieve $R$ on $c$ such that $F(R) \subseteq \xi^*(T)$.

**Proof** By the transitivity and stability axioms for Grothendieck topologies, $T$ is $K$-covering if and only if for any object $c \in C$ and any arrow $\xi : F(c) \rightarrow d$ in $D$, $\xi^*(T) \in K(\text{dom}(\xi))$. But, $F$ having the covering-lifting property, $\xi^*(T) \in K(\text{dom}(\xi))$ if and only if there exists a $J$-covering sieve $R$ on $c$ such that $F(R) \subseteq \xi^*(T)$. □

**Remark 5.8.** Every dense morphism of sites $F : (C, J) \rightarrow (D, K)$ is cover-reflecting, so $J$ can be recovered from $K$ as the collection of sieves which are sent by $F$ to $K$-covering families.

The notion of induced topology can be very profitably applied for establishing equivalences of toposes. Indeed, if we have a flat functor $F : C \rightarrow E$, we dispose of easily applicable criteria for $F$ to induce an equivalence of toposes

$$E \simeq \mathbf{Sh}(C, J_F),$$

as shown by the following result (which is an immediate consequence of Corollary 4.11): 

**Corollary 5.9.** Let $C$ be an essentially small category, $E$ a Grothendieck topos and $F : C \rightarrow E$ a flat functor. Then $F$ induces an equivalence

$$E \simeq \mathbf{Sh}(C, J_F),$$

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if and only if \( F \) is \( J_F \)-full and the objects of the form \( F(c) \) for \( c \in \mathcal{C} \) form a separating set for the topos \( \mathcal{E} \).

The following proposition describes a general setup for building equivalences of toposes.

**Proposition 5.10.** Let \( \mathcal{E} \) be a Grothendieck topos, represented as the category \( \text{Sh}(\mathcal{D}, K) \) of sheaves on a small-generated site \((\mathcal{D}, K)\). Let \( F : \mathcal{C} \to \mathcal{D} \) be a functor and \( J_F \) the collection of sieves in \( \mathcal{C} \) whose image under \( F \) is \( K \)-covering. If \( F \) is a weakly dense morphism of sites \((\mathcal{C}, J_F) \to (\mathcal{D}, K)\) then \( J_F \) is a Grothendieck topology on \( \mathcal{C} \), and we have an equivalence of toposes

\[
\text{Sh}(\mathcal{D}, K) \simeq \text{Sh}(\mathcal{C}, J_F).
\]

The following conditions are equivalent to \( F \) being a dense (or weakly dense, if \( K \) is subcanonical) morphism of sites \((\mathcal{C}, J_F) \to (\mathcal{D}, K)\):

(i)’ For any object \( d \) of \( \mathcal{D} \) there exists a \( K \)-covering family of arrows \( d_i \to d \) whose domains \( d_i \) are in the image of \( F \).

(ii)’ For every \( c_1, c_2 \in \mathcal{C} \) and any arrow \( g : F(c_1) \to F(c_2) \) in \( \mathcal{D} \), there exist a family of arrows \( f_i : c'_i \to c_1 \) which is sent by \( F \) to a \( K \)-covering family and a family of arrows \( k_i : c'_i \to c_2 \) such that \( g \circ F(f_i) = F(k_i) \) for all \( i \).

(iii)’ For any arrows \( f_1, f_2 : c_1 \to c_2 \) in \( \mathcal{C} \) such that \( F(f_1) = F(f_2) \) there exists a family of arrows \( k_i : c'_i \to c_1 \) which is sent by \( F \) to a \( K \)-covering family such that \( f_1 \circ k_i = f_2 \circ k_i \) for all \( i \).

**Proof** The first part of the proposition follows from Theorem 4.7 and Remark 5.6(a).

The second part of the proposition follows from the fact that every dense morphism is weakly dense and that the two notions are equivalent when \( K \) is subcanonical.

\[\square\]

### 5.3 Coinduced topologies

The following result, established in the proof of Lemma C2.3.19(ii) [4], represents a kind of converse to Proposition 5.7. It involves a notion of *image* of a Grothendieck topology under a functor.
Proposition 5.11 (cf. Lemma C2.3.19(ii) [4]). Let \( F : \mathcal{C} \to \mathcal{D} \) be a functor and \( J \) a Grothendieck topology on \( \mathcal{C} \). Then there is a Grothendieck topology \( J^F \) on \( \mathcal{D} \), called the image of \( J \) along \( F \) (or the Grothendieck topology coinduced by \( J \) along \( F \)), such that for any sieve \( T \) on an object \( d \) of \( \mathcal{D} \), \( T \in J^F(d) \) if and only if for any object \( c \in \mathcal{C} \) and any arrow \( \xi : F(c) \to d \) in \( \mathcal{D} \), there exists a \( J \)-covering sieve \( R \) on \( c \) such that \( F(R) \subseteq \xi^*(T) \). Then the functor \( F : (\mathcal{C}, J) \to (\mathcal{D}, J^F) \) is \( J^F \)-dense and has the covering-lifting property.

The Grothendieck topology \( J^F \) enjoys the following universal property: the subtopos \( \text{Sh}(\mathcal{D}, J^F) \hookrightarrow [\mathcal{D}^{\text{op}}, \text{Set}] \) is the image (in the sense of surjection-inclusion factorization) of the composite of the induced geometric morphism \( [F^{\text{op}}, \text{Set}] : [\mathcal{C}^{\text{op}}, \text{Set}] \to [\mathcal{D}^{\text{op}}, \text{Set}] \) with the subtopos inclusion \( \text{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \text{Set}] \).

**Proof** First, let us show that \( J^F \) is indeed a Grothendieck topology on \( \mathcal{C} \). The maximality and pullback-stability axiom trivially hold, so it remains to prove the transitivity one. It is clear that any sieve containing a \( J^F \)-covering one is also \( J^F \)-covering. Let us denote, given a sieve \( S \) on an object \( d \) and, for each arrow \( f \in S \), a sieve \( S_f \) on \( \text{dom}(f) \), by \( S \times \{S_f \ | \ f \in S \} \) the sieve on \( d \) consisting of the arrows of the form \( f \circ h \) where \( f \in S \) and \( h \in S_f \).

We have to prove that, if \( S \in J^F(d) \) and, for each \( f \in S \), \( S_f \in J^F(\text{dom}(f)) \), then \( S \times \{S_f \ | \ f \in S \} \in J^F(d) \). Given an arrow \( \xi : F(c) \to d \), we have that \( F(R) \subseteq \xi^*(S) \) for some \( R \in J(c) \). For any \( r \in R \), \( \xi \circ F(r) \in S \), so, since \( S_{\xi \circ F(r)} \in J^F(F(\text{dom}(r))) \), there is a sieve \( R_r \in J(F(\text{dom}(r))) \) such that \( F(R_r) \subseteq S_{\xi \circ F(r)} \). So the \( J \)-covering sieve \( R \times \{R_r \ | \ r \in R \} \) satisfies the condition \( F(R \times \{R_r \ | \ r \in R \}) \subseteq \xi^*(S \times \{S_f \ | \ f \in S \}) \).

The fact that \( F : (\mathcal{C}, J) \to (\mathcal{D}, J^F) \) is \( J^F \)-dense and has the covering-lifting property is obvious.

The last part of the proposition follows at once from Proposition 5.11 below. Indeed, the construction of the geometric morphism induced by a comorphism of sites is functorial, the geometric morphism \( [G^{\text{op}}, \text{Set}] : [\mathcal{A}^{\text{op}}, \text{Set}] \to [\mathcal{B}^{\text{op}}, \text{Set}] \) induced by a functor \( G : \mathcal{A} \to \mathcal{B} \) is precisely \( L_G \), regarding \( G \) as a comorphism of sites \( (\mathcal{A}, T_A) \to (\mathcal{B}, T_B) \) where \( T_A \) and \( T_B \) are the trivial Grothendieck topologies respectively on \( \mathcal{A} \) and \( \mathcal{B} \), and any canonical geometric inclusion \( \text{Sh}(\mathcal{A}, Z) \hookrightarrow [\mathcal{A}^{\text{op}}, \text{Set}] \) is induced by the comorphism of sites \( 1_A : (\mathcal{A}, Z) \to (\mathcal{A}, T) \) given by the identity functor \( 1_A \) on \( \mathcal{A} \).

**Remark 5.12.** If \( F : (\mathcal{C}, J) \to (\mathcal{D}, K) \) satisfies the covering-lifting property then \( K \subseteq J^F \). This follows immediately from the fact that \( K \) satisfies the pullback-stability property. In fact, in Lemma C2.3.19(ii) [4], \( J^F \) is characterized as the largest Grothendieck topology on \( \mathcal{C} \) for which \( F \) satisfies the covering-lifting property.
Lemma 5.13. Let $F : \mathcal{C} \to \mathcal{D}$ be a $J$-full and $J$-faithful functor, for a Grothendieck topology $J$ on $\mathcal{C}$. Then:

(i) for any sieve $R$ on an object $c$ of $\mathcal{C}$ and any arrow $t$ with codomain $c$, if $F(t)$ belongs to the sieve generated by $F(R)$ then there exists a $J$-covering sieve $U$ on $\text{dom}(t)$ such that $t \circ u \in R$ for each $u \in U$;

(ii) the functor $F : (\mathcal{C}, J) \to (\mathcal{D}, J^F)$ is cover-reflecting.

Proof (i) Suppose that $F(t) = F(r) \circ \xi$ for some $r \in R$, where $t : a \to c$, $r : b \to c$ and $\xi : F(a) \to F(b)$. Since $F$ is $J$-full, there exist a $J$-covering sieve $S$ on $a$ and for each $s \in S$ an arrow $t_s : \text{dom}(s) \to b$ such that $\xi \circ F(s) = F(t_s)$. Since $F(t) = F(r) \circ \xi$, we have that $F(t \circ s) = F(t) \circ F(s) = F(r) \circ \xi \circ F(s) = F(r) \circ F(t_s) = F(r \circ t_s)$ for each $s \in S$. The fact that $F$ is $J$-faithful then implies that $t \circ s \equiv_J r \circ t_s$, that is, there is a $J$-covering sieve $V_s$ on $\text{dom}(s)$ such that for each $v \in V_s$, $t \circ s \circ v = r \circ t_s \circ v$. Therefore the $J$-covering sieve $U := S \ast \{V_s \mid s \in S\}$ on $a$ satisfies the property that for each $u \in U$, $t \circ u \in R$.

(ii) Let $R$ be a sieve on an object $c$ of $\mathcal{C}$. If $F(R)$ generates a $J^F$-covering sieve then there is a $J$-covering sieve $T$ on $c$ such that the sieve generated by $F(T)$ is contained in the sieve generated by $F(R)$. So, for each $t \in T$, by point (i) of the Lemma, there is a $J$-covering sieve $U_t$ on $\text{dom}(t)$ such that $t \circ u \in R$ for each $u \in U_t$. So the sieve $R$ is $J$-covering, since it contains the $J$-covering sieve $T \ast \{U_t \mid t \in T\}$. □

Corollary 5.14. Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a $K$-dense, cover-preserving and cover-reflecting functor with the covering-lifting property (for instance, a dense morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$ - cf. Corollary 4.20). Suppose that $K$ is generated by a family $K'$ of sieves each of which contains the image $F(R)$ under $F$ of some $J$-covering sieve $R$, and that $F$ is $J'$-full and $J'$-faithful, where $J'$ is the Grothendieck topology generated by the sieves $R$ such that $F(R) \in K'$. Then $J = J'$.

Proof Let $J'$ be the Grothendieck topology on $\mathcal{C}$ generated by the sieves $R$ such that $F(R)$ generates a $K$-covering sieve. We have to prove that $J' = J$. Consider the topology $J^F$. We clearly have that $K' \subseteq J^F$, whence $K \subseteq J^F$. On the other hand, we have that $K = J^F$ by Proposition 5.7. Now, the fact that $F$ is cover-reflecting implies that $J' \subseteq J$; so $J^F \subseteq J^F = K$, and hence $J^F = K$. Now, since both $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ and $F : (\mathcal{C}, J') \to (\mathcal{D}, J^F)$ are cover-reflecting (the former by our hypothesis and the latter by Lemma 5.13), $J^F = K$ implies that $J = J'$. □
Remark 5.15. If $F$ is a weakly dense morphism of sites $(C, J) \to (D, K)$, the condition that $F$ be $J'$-full and $J$-faithful is also necessary for $J$ to be equal to $J'$ (see Proposition 6.17).

5.4 Intrinsic characterization of geometric inclusions

The first part of the following result characterizes the property of a geometric morphism being an inclusion entirely in terms of its inverse image. It can be useful, for instance, in cases where one disposes of a description of the inverse image of the morphism which is simpler or more tractable than that of its direct image.

Corollary 5.16. Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. Then

(i) $f$ is an inclusion if and only if $f^*$ satisfies the following conditions:

(i) $f^*$ is locally surjective, that is every object of $\mathcal{F}$ can be covered by objects in the image of $f^*$;

(ii) $f^*$ is locally full, that is for every $x, y \in \mathcal{E}$ and any arrow $g : f^*(x) \to f^*(y)$ in $\mathcal{F}$, there exists a family of arrows $s_i : x_i \to x$ in $\mathcal{E}$ which is sent by $f^*$ to an epimorphic family and a family of arrows $g_i : x_i \to y$ such that $g \circ f^*(s_i) = f^*(g_i)$ for all $i$.

(ii) $f$ is an equivalence if and only if $f^*$ is faithful, locally full and locally surjective.

Proof (i) The above conditions amount precisely to the property of the morphism of sites $(\mathcal{E}, J^\text{can}_{\mathcal{E} \leftarrow f}) \to (\mathcal{F}, J^\text{can}_{\mathcal{F}})$ to be dense, so the thesis follows from Theorem 5.3.

(ii) This follows from (i), given that a geometric morphism is an equivalence if and only if it is a surjection and an inclusion. □

Remark 5.17. Since a Grothendieck topos has all coproducts, all the covering families arising in the formulations of the properties in part (i) of Corollary 5.16 can be supposed, without loss of generality, to consist of a single element. More precisely, the condition for $f$ to be an inclusion is equivalent to the conjunction of the following ones:

(i) every object of $\mathcal{F}$ is a quotient of an object in the image of $f^*$;

(ii) for every $x, y \in \mathcal{E}$ and any arrow $g : f^*(x) \to f^*(y)$ in $\mathcal{F}$, there exist an arrow $s : x' \to x$ in $\mathcal{E}$ which is sent by $f^*$ to an epimorphism and an arrow $g' : x' \to y$ such that $g \circ f^*(s) = f^*(g')$. 
So, $f$ is an equivalence if and only if $f^*$ is faithful and satisfies the above conditions.

**Example 5.18.** Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism between Grothendieck toposes and $A$ an object of $\mathcal{E}$. Then we have a geometric morphism $f_A : \mathcal{F}/f^*(A) \to \mathcal{E}/A$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{F}/f^*(A) & \xrightarrow{f_A} & \mathcal{F} \\
\downarrow & & \downarrow f \\
\mathcal{E}/A & \xrightarrow{f} & \mathcal{E}
\end{array}
$$

where the horizontal morphisms are the canonical ones, commutes. The inverse image of $f_A$ is the functor $\mathcal{E}/A \to \mathcal{F}/f^*(A)$ sending an object $u : B \to A$ of $\mathcal{E}/A$ to $f^*(u) : f^*(B) \to f^*(A)$ (and acting accordingly on arrows).

Let us show, by applying Corollary 5.16(i), that if $f$ is an inclusion then $f_A$ is also an inclusion. First, let us show that $f_A^*$ is locally surjective. Given an object $u : C \to f^*(A)$ of $\mathcal{F}/f^*(A)$, since $f^*$ is locally surjective there is an object $B$ of $\mathcal{E}$ and an epimorphism $q : f^*(B) \to f^*(A)$ (and acting accordingly on arrows). Consider the composite arrow $u \circ q : f^*(B) \to f^*(A)$. Since $f^*$ satisfies condition (ii) of Corollary 5.16(i), there is an arrow $s : B' \to B$ in $\mathcal{E}$ and an arrow $g : B' \to A$ in $\mathcal{E}$ such that $f^*(s)$ is an epimorphism and $u \circ q \circ f^*(s) = f^*(g)$ (cf. Remark 5.17). Therefore the arrow $q \circ f^*(s)$ is an epimorphism from $f_A^*(g)$ to $u$ in $\mathcal{F}/f^*(A)$. The fact that $f_A^*$ inherits local fullness from $f^*$ is obvious.

### 5.5 Hyperconnected and localic morphisms

Recall that a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be **hyperconnected** if $f^*$ is full and faithful and its image is closed under subobjects in $\mathcal{F}$, and is said to be **localic** if every object of $\mathcal{F}$ is a subquotient (that is, a quotient of a subobject) of an object of the form $f^*(A)$ for $A \in \mathcal{E}$. By Theorem A4.6.5 [4], every geometric morphism can be factored, uniquely up to commuting equivalence, as the composite of a hyperconnected morphism followed by a localic one.

In this section we shall characterize the property of a geometric morphism to be hyperconnected (or localic) in terms of sites, and also provide a natural site-level description of the hyperconnected-localic factorization.

#### 5.5.1 Hyperconnected morphisms

The following lemma will be instrumental in the sequel.
Lemma 5.19. Let $q : X \to X'$ be the coequalizer of a pair of arrows $r_1, r_2 : R \to X$ in a Grothendieck topos $\mathcal{E}$ and $f : Y \to X'$ a monomorphism. Then, the arrows $t_1, t_2 : R \times_{X'} Y \to Z$ which make the diagram

\[
\begin{array}{ccc}
R \times_{X'} Y & \xrightarrow{t_1} & R \\
\downarrow{t_2} & & \downarrow{r_1} \\
Z & \xrightarrow{h} & X \\
\downarrow{k} & & \downarrow{q} \\
Y & \xrightarrow{f} & X'
\end{array}
\]

commutative, where the bottom square is a pullback, satisfy the property that $< t_1, t_2 > : R \times_{X'} Y \to Z \times Z$ is (isomorphic to) the pullback of $< r_1, r_2 >$ along $h \times h$.

Moreover, $k$ is the coequalizer of the arrows $t_1$ and $t_2$.

Proof. By using the internal language, we shall argue as if $\mathcal{E}$ were the topos $\text{Set}$ of sets.

Let $U \to Z \times Z$ be the pullback of $< r_1, r_2 >$ along $h \times h$. We have that

\[ U = \{ ((z_1, z_2), \xi) \mid (z_1, z_2) \in Z, \xi \in R, h(z_1) = r_1(\xi), h(z_2) = r_2(\xi) \} \].

We want to prove that $U \to Z \times Z$ is isomorphic to $< t_1, t_2 > : R \times_{X'} Y \to Z \times Z$ as arrows over $Z \times Z$.

We have that

\[ R \times_{X'} Y = \{ (\xi, y) \mid \xi \in R, y \in Y, q(r_1(\xi)) = q(r_2(\xi)) = f(y) \} \].

Now, the arrow $t_1 : R \times_{X'} Y \to Z$ (resp. $t_2 : R \times_{X'} Y \to Z$) sends an element $(\xi, y) \in R \times_{X'} Y$ to the unique element $z_1 \in Z$ such that $k(z_1) = y$ and $h(z_1) = r_1(\xi)$ (resp. to the unique element $z_2 \in Z$ such that $k(z_2) = y$ and $h(z_2) = r_2(\xi)$). Let us define arrows $R \times_{X'} Y \to U$ and $U \to R \times_{X'} Y$ over $Z \times Z$ that are inverse to each other. In one direction, we send an element $(\xi, y) \in R \times_{X'} Y$ to the element $(t_1(\xi, y), t_2(\xi, y), \xi)$. This is well-defined by definition of the arrows $t_1$ and $t_2$. In the converse direction, we observe that, given an element $((z_1, z_2), \xi)$ of $U$, we have that $k(z_1) = k(z_2)$. Indeed, since $f$ is monic, this condition is equivalent to $f(k(z_1)) = f(k(z_2))$; but $f(k(z_1)) = q(h(z_1)) = q(r_1(\xi)) = q(r_2(\xi)) = q(h(z_2)) = f(k(z_2))$. We can therefore define an arrow $U \to R \times_{X'} Y$ which sends an element $((z_1, z_2), \xi)$ of $U$ to the element $(\xi, k(z_1))$. This is well-defined since $q(r_1(\xi)) = q(h(z_1)) = f(k(z_1))$ and $q(r_2(\xi)) = q(h(z_2)) = f(k(z_2))$ (as $k(z_1) = k(z_2)$). It is now straightforward to check that the two arrows just defined are inverse to each other.
other and compatible with the structure arrows to $Z \times Z$. This completes the proof of the first part of the lemma. The fact that $k$ is the coequalizer of the arrows $t_1$ and $t_2$ follows from the fact that in a Grothendieck topos colimits are stable under pullback.

**Remark 5.20.** The proof of the lemma shows that the general version of it (for $f$ not necessarily monic) reads as follows: $\langle t_1, t_2 \rangle$ is isomorphic over $Z \times Z$ to the composite with the canonical embedding $\zeta_Y : Z \times_Y Z \hookrightarrow Z \times Z$ of the pullback of $\langle r_1, r_2 \rangle$ along $(h \times h) \circ \zeta_Y$.

**Proposition 5.21.** Let $C$ be a separating set for a Grothendieck topos $\mathcal{E}$ and $f : \mathcal{F} \to \mathcal{E}$ a geometric morphism. Then the image of $f^*$ is closed under subobjects if and only if for any $c \in C$, every subobject of $f^*(c)$ is in the image of $f^*$.

**Proof** Let $m : B \to f^*(A)$ be a monomorphism. Let us represent $A$ as a quotient $q : \coprod_{c \in C_A} c \to A$ of a coproduct of objects coming from $C$. Then, since the image of $f^*|_C$ is closed under subobjects and coproducts are stable under pullbacks, the pullback of $m$ along $f^*(q)$ is of the form $f^*(z)$ for some subobject $z : Z \hookrightarrow \coprod_{c \in C_A} c$ in $\mathcal{E}$. Now, if $q$ is the coequalizer of its kernel pair $\langle r_1, r_2 \rangle$ in $\mathcal{E}$ then $f^*(q)$ is the coequalizer of the pair $f^*(r_1), f^*(r_2)$. Let $\langle t_1, t_2 \rangle : T \to Z$ be the pullback in $\mathcal{E}$ of $\langle r_1, r_2 \rangle$ along $z$. Then, since $f^*$ preserves pullbacks, $\langle f^*(t_1), f^*(t_2) \rangle : f^*(T) \to f^*(Z)$ is the pullback of $\langle f^*(r_1), f^*(r_2) \rangle$ along $f^*(z)$.

By Lemma 5.19, it follows that $m$ is isomorphic to the canonical arrow $U \to f^*(A)$ from the codomain $U$ of the coequalizer $f^*(Z) \to U$ of the arrows $f^*(t_1), f^*(t_2)$. But, since $f^*$ preserves coequalizers, this arrow is the image under $f^*$ of the canonical arrow $\xi : Z_T \to A$ from the codomain $Z_T$ of the coequalizer $Z \to Z_T$ of the two arrows $t_1, t_2$ to $A$, that is, $m \cong \mathrm{Im}(f^*(\xi)) \cong f^*(\mathrm{Im}(\xi))$. 

**Proposition 5.22.** Let $(C, J)$ be a small-generated site, $\mathcal{E}$ a Grothendieck topos and $F : C \to \mathcal{E}$ a $J$-continuous flat functor. Then the geometric morphism $f : \mathcal{E} \to \mathbf{Sh}(C, J)$ induced by $F$ is hyperconnected if and only if $F$ is cover-reflecting and for every subobject $A \hookrightarrow F(c)$ in $\mathcal{E}$ there exists a ($J$-closed) sieve $R$ on $c$ such that $A$ is the union of the images of the arrows $F(f)$ for $f \in R$.

**Proof** It is clear that if $f^*$ is faithful and its image is closed under subobjects then $f^*$ is also full (see the proof of the implication $(vi) \Rightarrow (i)$ at p. 229 of [4]). So it follows from Propositions 5.1 and 5.21 that $f$ is hyperconnected if and
only if \( F \) is cover-reflecting and, for every object \( c \) of \( C \), all the subobjects of \( f^*(l(c)) \) are images under \( f^* \) of subobjects of \( l(c) \) in \( \text{Sh}(C, J) \). Now, since the subobjects of \( l(c) \) in \( \text{Sh}(C, J) \) are all images under the associated sheaf functor \( a_J : [C^{\text{op}}, \text{Set}] \to \text{Sh}(C, J) \) of subobjects of the form \( R \mapsto \text{Hom}_C(\cdot, c) \) (which can be supposed \( J \)-closed without loss of generality), and this subobject is the union of the images of the arrows of the form \( y_C(f) \) for \( f \in R \) (where \( y_C \) is the Yoneda embedding), we can reformulate the latter condition as the requirement that for any object \( c \) of \( C \) and any subobject \( A \) of \( F(c) \) in \( E \) there should exist a \((J \)-closed\) sieve \( R \) on \( c \) such that \( A \) is the union of the images of the arrows \( F(f) \) for \( f \in R \). \( \square \)

Let us apply this result in the context of interpretations of geometric theories (see section 5.1 for the notation).

**Corollary 5.23.** An interpretation \( I : C_T \to C_S \) between geometric theories \( T \) and \( S \) induces an hyperconnected geometric morphism \( \text{Set}[S] \to \text{Set}[T] \) if and only if for every sequent \( \sigma \) over the signature of \( T \), if the sequent \( I(\sigma) \) is provable in \( S \) then \( \sigma \) is provable in \( T \), and for any sorts \( A_1, \ldots, A_n \) of the signature of \( T \), every geometric formula \( \psi \) over the signature of \( S \) in the context \( I((x_1^{A_1}, \ldots, x_n^{A_n})) \) is \( T \)-provably equivalent (in its context) to a formula \( I(\phi) \), where \( \phi \) is a geometric formula over the signature of \( T \) in the context \((x_1^{A_1}, \ldots, x_n^{A_n})\).

In particular, if \( I \) is the canonical interpretation given by an expansion \( T' \) of \( T \) (in the sense of section 7.1 of [2]) then the geometric morphism induced by \( I \) is hyperconnected if and only if every sequent over the signature of \( T \) which is provable in \( T' \) is provable in \( T \), and for any sorts \( A_1, \ldots, A_n \) of the signature of \( T \), every geometric formula \( \psi \) in the context \((x_1^{A_1}, \ldots, x_n^{A_n})\) over the signature of \( T' \) is \( T \)-provably equivalent (in its context) to a geometric formula over the signature of \( T \).

**Proof** Our thesis follows immediately from Proposition 5.22 in light of Corollary 5.2. \( \square \)

Let us now apply Proposition 5.22 in the context of morphisms of sites.

**Proposition 5.24.** Let \( F : (C, J) \to (D, K) \) be a morphism of small-generated sites. Then the geometric morphism \( \text{Sh}(F) : \text{Sh}(D, K) \to \text{Sh}(C, J) \) induced by \( F \) is hyperconnected if and only if \( F \) is cover-reflecting and closed-sieve-lifting, in the sense that for every object \( c \) of \( C \) and any \( K \)-closed sieve \( S \) on \( F(c) \) there exists a \((J \)-closed\) sieve \( R \) on \( c \) such that \( S \) coincides with the \((K \)-closed\) sieve on \( F(c) \) generated by the arrows \( F(f) \) for \( f \in R \).
Proof The thesis follows as an immediate consequence of Proposition 5.22 in light of Proposition 2.24. Indeed, the condition of Proposition 5.22 amounts precisely to the requirement that for every subobject \( A \rightarrow l'(F(c)) \), where \( l' \) is the canonical functor \( D \rightarrow \text{Sh}(D, K) \), there should be a \((J\text{-closed})\) sieve \( R \) on \( c \) such that \( A \) is the union of the images of the arrows \( l'(F(f)) \) for \( f \in R \). Now, the subobjects of \( l'(F(c)) \) in \( \text{Sh}(D, K) \) are the images under the associated sheaf functor \( a_K : [D^{\text{op}}, \text{Set}] \rightarrow \text{Sh}(D, K) \) of \( K\)-closed sieves \( S \) on \( F(c) \), and the union of the images of the arrows \( l'(F(f)) \) for \( f \in R \) is the image under \( a_K \) of the union of the images of the arrows \( y_D(F(f)) \) for \( f \in R \); our claim thus follows from the fact that, by Proposition 2.24, two subobjects of a given object in \([D^{\text{op}}, \text{Set}]\) have isomorphic images under \( a_K \) if and only if their \( K\)-closures coincide. \( \square \)

5.5.2 Essential surjectivity

Proposition 5.25. Let \((C, J)\) be a small-generated site, \( \mathcal{E} \) a Grothendieck topos and \( F : C \rightarrow \mathcal{E} \) a \( J\)-continuous flat functor inducing a geometric morphism \( f : \mathcal{E} \rightarrow \text{Sh}(C, J) \). If the image of \( f^* \) is closed under subobjects then \( f^* \) is essentially surjective if and only if the objects of the form \( F(c) \) for \( c \in C \) form a separating set for the topos \( \mathcal{E} \).

Proof If the objects of the form \( F(c) \) for \( c \in C \) form a separating set of objects for \( \mathcal{E} \) then for every \( A \in \mathcal{E} \), we have an epimorphism \( q : f^*(\coprod_{i \in I} l(c_i)) \cong \coprod_{i \in I} f^*(l(c_i)) \rightarrow A \). Since the image of \( f^* \) is closed under subobjects, the kernel pair of \( q \) is of the form \( f^*(R) \), where \( R \) is a relation on \( \coprod_{i \in I} l(c_i) \). Then \( q \) is isomorphic to the image under \( f^* \) of the coequalizer of \( R \); in particular, \( A \) is isomorphic to an object in the image of \( f^* \), as desired.

Conversely, if \( f^* \) is essentially surjective then for every object \( A \) of \( \mathcal{E} \), there exists an object \( B \) of \( \text{Sh}(C, J) \) such that \( A \cong f^*(B) \). But \( B \) can be covered in \( \text{Sh}(C, J) \) by objects of the form \( l(c) \) for \( C \), whence \( A \) can be covered in \( \mathcal{E} \) by objects of the form \( F(c) \) for \( c \in C \), as required. \( \square \)

Let us apply Proposition 5.25 in the context of interpretations of theories.

Corollary 5.26. Let \( I : C_\Sigma \rightarrow C_\Sigma \) be an interpretation of a geometric theory \( \mathcal{T} \) into a geometric theory \( \mathcal{S} \). Then the geometric morphism \( \text{Set}[\mathcal{S}] \rightarrow \text{Set}[\mathcal{T}] \) induced by \( I \) satisfies the property that its inverse image is essentially surjective and its image is closed under subobjects if and only if for every sort \( B \) of the signature of \( \mathcal{S} \), the object \( \{x^B \in \mathcal{T}\} \) is \( J_\Sigma\)-covered by objects in the image of \( I \) and the image of \( I \) is closed under subobjects (that is, for any sorts \( A_1, \ldots, A_n \) of the signature of \( \mathcal{T} \), every geometric formula \( \psi \) over the
signature of $S$ in the context $I((x^{A_1}_1, \ldots, x^{A_n}_n))$ is $T$-provably equivalent (in its context) to a formula $I(\phi)$, where $\phi$ is a geometric formula over the signature of $T$ in the context $(x^{A_1}_1, \ldots, x^{A_n}_n)$.

**Proof** By Propositions 5.21 and 5.25, since the geometric syntactic category $C_S$ is closed under subobjects in the topos $\text{Sh}(C_S, J_S)$, it suffices to show, assuming that the image of $I$ is closed under subobjects, that the objects in the image of $I$ form a separating set for the topos $\text{Set}[S]$ if and only if every object of the form $\{x^B \cdot \top\}$ (for $B$ a sort of the signature of $S$) is $J_S$-covered by objects in the image of $I$. But this follows from the fact that every object of $C_S$ is a subobject of a finite product of objects of the form $\{x^B \cdot \top\}$, using the fact that $I$ preserves finite products and its image is closed under subobjects. □

Let us now apply Proposition 5.25 in the context of morphisms of sites.

**Proposition 5.27.** Let $F : (C, J) \to (D, K)$ be a morphism of small-generated sites. Then the geometric morphism $\text{Sh}(F) : \text{Sh}(D, K) \to \text{Sh}(C, J)$ induced by $F$ satisfies the property that $\text{Sh}(F)^*$ is essentially surjective and its image is closed under subobjects if and only if $F$ is closed-sieve-lifting (in the sense of Proposition 5.24) and satisfies condition (ii) of Proposition 4.5 (equivalently, if $K$ is subcanonical, every object of $D$ admits a $K$-covering sieve generated by arrows whose domain lies in the image of $F$).

**Proof** The equivalence between the property of the image of $\text{Sh}(F)^*$ to be closed under subobjects and the property of $F$ to be closed-sieve-lifting follows from the proof of Proposition 5.24 while the equivalence between the property of $\text{Sh}(F)^*$ to be essentially surjective and condition (ii) of Proposition 4.5 follows from the proof of Proposition 4.5. □

### 5.5.3 Localic morphisms

The following proposition shows the natural behavior of the property of a geometric morphism being localic in terms of separating sets for toposes.

**Proposition 5.28.** Let $C$ (resp. $D$) be a separating set for a Grothendieck topos $\mathcal{E}$ (resp. $\mathcal{F}$) and $f : \mathcal{F} \to \mathcal{E}$ a geometric morphism. Then:

(i) The full subcategory $\mathcal{G}$ of $\mathcal{F}$ on the objects which are subquotients of objects of the form $f^*(A)$ for $A \in \mathcal{E}$ coincides with the full subcategory of $\mathcal{F}$ on the objects which can be covered by objects which are domains of subobjects of objects of the form $f^*(A)$ for $A$ in $\mathcal{C}$.
(ii) $f$ is localic if and only if every object of $\mathcal{D}$ can be covered by objects which are domains of subobjects of objects of the form $f^*(A)$ for $A$ in $\mathcal{C}$.

**Proof** (i) Let $B \in \mathcal{G}$. Then $B$ is a quotient $d \to B$ of a subobject $d \to f^*(A)$. But $A$ is a quotient $q : \coprod_{i \in I} c_i \to A$ of a coproduct of objects $c_i$ in $\mathcal{C}$, so, by considering the pullback

$$
\begin{array}{ccc}
d' & \xrightarrow{f^*(\coprod_{i \in I} c_i)} & \coprod_{i \in I} f^*(c_i) \\
\downarrow & & \downarrow q \\
d & \xrightarrow{f^*(A)} &
\end{array}
$$

we see that $B$ is a quotient of $d'$, which is a coproduct of subobjects of the objects $f^*(c_i)$. Therefore $B$ can be covered by the domains of these subobjects, as required.

Conversely, let us suppose that $B$ is an object of $\mathcal{F}$ that can be covered by objects $u_i$ (for $i \in I$) which are domains of subobjects $u_i \hookrightarrow f^*(c_i)$ of objects of the form $f^*(c_i)$ for $c_i$ in $\mathcal{C}$. Then $B$ is a quotient of the coproduct of the $u_i$ (for $i \in I$), which is a subobject of the coproduct of the $f^*(c_i)$; but this coproduct is in the image of $f^*$ as $f^*$ preserves coproducts.

(ii) By definition of a localic morphism, $f$ is localic if and only if $\mathcal{G} = \mathcal{F}$. So, by point (i), if $f$ is localic then every object of $\mathcal{D}$ can be covered by objects which are domains of subobjects of objects of the form $f^*(A)$ for $A$ in $\mathcal{C}$. Conversely, again by (i), we have to prove that if every object of $\mathcal{D}$ can be covered by objects which are domains of subobjects of objects of the form $f^*(A)$ for $A$ in $\mathcal{C}$ then every object of $\mathcal{F}$ can be covered by objects which are domains of subobjects of objects of the form $f^*(A)$ for $A$ in $\mathcal{C}$; but this immediately follows from the fact that $\mathcal{D}$ is separating for $\mathcal{F}$. □

**Remark 5.29.** We know from section A4.6 of [1] that if $\mathcal{E}$ and $\mathcal{F}$ are Grothendieck toposes then the category $\mathcal{G}$ defined in point (i) of Proposition 5.28 is a Grothendieck topos; so this part of the proposition actually asserts that the collection of objects which are subobjects of an object of the form $f^*(A)$ for $A \in \mathcal{C}$ is a separating set for $\mathcal{G}$.

The following result is an immediate corollary of Proposition 5.28.

**Proposition 5.30.** Let $(\mathcal{C}, J)$ be a small-generated site, $\mathcal{E}$ a Grothendieck topos and $F : \mathcal{C} \to \mathcal{E}$ a $J$-continuous flat functor inducing a geometric morphism $f : \mathcal{E} \to \text{Sh}(\mathcal{C}, J)$. Then $f$ is localic if and only if the subobjects of objects of the form $F(c)$ for $c \in \mathcal{C}$ form a separating set for the topos $\mathcal{E}$.

□

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Let us now apply Proposition 5.28 in the general context of morphisms of sites.

**Proposition 5.31.** Let \( F : (\mathcal{C}, J) \to (\mathcal{D}, K) \) be a morphism of small-generated sites. Then the geometric morphism \( \mathbf{Sh}(F) : \mathbf{Sh}\mathcal{D} \to \mathbf{Sh}\mathcal{C} \) induced by \( F \) is localic if and only if for any object \( d \) of \( \mathcal{D} \) there exist a family \( \{ S_i \mid i \in I \} \) of sieves on objects of the form \( F(c_i) \) (where \( c_i \) is an object of \( \mathcal{C} \)) and for each \( f \in S_i \) an arrow \( g_f : \text{dom}(f) \to d \) such that \( g_f \equiv_K g_f \circ z \) whenever \( z \) is composable with \( f \), such that the family of arrows \( g_f \) (for \( f \in S_i \) for some \( i \)) is \( K \)-covering.

**Proof** Let us suppose that \( \mathbf{Sh}(F) \) is localic. Let \( l' \) be the canonical functor \( \mathcal{D} \to \mathbf{Sh}\mathcal{D} \). By Proposition 5.28, for every \( d \in \mathcal{D} \) there exist objects \( c_i \) of \( \mathcal{C} \) (for \( i \in I \)), subobjects \( A_i \to l'(F(c_i)) \) and a jointly epimorphic family of arrows \( A_i \to l'(d) \) (for \( i \in I \)). Now, we can write \( A_i = a_K(S_i) \), where \( S_i \) is a sieve on \( F(c_i) \). For each \( f \in S_i \), composing the arrow \( a_K(S_i) \to l'(d) \) with the canonical arrow \( l'(\text{dom}(f)) \to a_K(S_i) \) thus yields an arrow \( l'(\text{dom}(f)) \to l'(d) \) which we can suppose without loss of generality of the form \( l'(g_f) \) for an arrow \( g_f : \text{dom}(f) \to d \) (at the cost of replacing \( S_i \), by using Proposition 3.1, which a smaller sieve with the same image under \( a_K \)); clearly, we have that, for each \( z \) composable with \( f \), \( l'(g_f \circ z) = l'(g_f \circ z) \), equivalently \( g_f \equiv_K g_f \circ z \). The property of our arrows \( A_i \to l'(d) \) to be jointly epimorphic can thus be reformulated as the property of the arrows \( g_f \) to be sent by \( l' \) to an epimorphic family, equivalently to be \( K \)-covering.

Conversely, any family of arrows \( g_f : \text{dom}(f) \to d \) indexed by a sieve \( S_i \) such that \( g_f \equiv_K g_f \circ z \) whenever \( z \) is composable with \( f \) induces an arrow \( a_K(S_i) \to l'(d) \). So if we have a family of sieves \( S_i \) on objects of the form \( F(c_i) \) and one such family of arrows for each sieve, the resulting arrows \( a_K(S_i) \to l'(d) \) will be jointly epimorphic if (and only if) the family of arrows \( g_f \) (for \( f \in S_i \) for some \( i \)) is \( K \)-covering. \( \square \)

The following result characterizes the interpretations between geometric theories which induce a localic geometric morphism.

**Proposition 5.32.** Let \( I : \mathcal{C}_T \to \mathcal{C}_S \) be an interpretation of a geometric theory \( T \) into a geometric theory \( S \). Then the geometric morphism \( \text{Set}[S] \to \text{Set}[T] \) induced by \( I \) is localic if and only if every object of \( \mathcal{C}_S \) can be \( J_S \)-covered by formulae over the signature of \( S \) in a context of the form \( I(x) \) which \( T \)-provably entail a formula of the form \( I(\phi) \), where \( \phi \) is a geometric formula in the context \( \vec{x} \) over the signature of \( T \).

In particular, the canonical interpretation of \( T \) into an expansion of it over a signature which does not contain any new sorts with respect to the signature of \( T \) induces a localic geometric morphism.
Proof This is an immediate consequence of Proposition 5.28 by using the fact that the subobjects in $\text{Set}[S] \simeq \text{Sh} (C_s, J_s)$ of an object of the form $I(\{ \vec{x} : \phi \})$ for a geometric formula-in-context $\phi(\vec{x})$ over the signature of $T$ can be identified with the $(S$-provable equivalence classes of) geometric formulae in the context $I(\vec{x})$ over the signature of $S$ which $S$-provably entail $I(\phi)$. 

\[ \square \]

\section{The hyperconnected-localic factorization}

In this section we shall provide a site-level description of the hyperconnected-localic factorization (in the sense of \[ \text{[5]} \] cf. also section A4.6 of \[ \text{[4]} \]) of a geometric morphism between Grothendieck toposes.

Let us refer to Corollary 5.11 for the notation. Let $C_j$ be the Grothendieck topology induced on $C_j$ by the canonical topology on the topos $\text{Sh}(C, J)$.

We have a canonical functor $C \to C^j$, sending an object $c$ of $C$ to the pair $(c, M_c)$, where $M_c$ is the maximal sieve on $c$, which yields a morphism of sites $i^j : (C, J) \to (C^j, C^j_j)$. This morphism induces an equivalence $\text{Sh}(i^j) : \text{Sh}(C^j, C^j_j) \simeq \text{Sh}(C, J)$.

\[ \text{Theorem 5.33. Let } F : (C, J) \to (D, K) \text{ be a morphism of small-generated sites. Then the hyperconnected-localic factorization of the geometric morphism } \text{Sh}(F) \circ \text{Sh}(i^j_K) : \text{Sh}(D^j_K, C^j_K) \simeq \text{Sh}(D, K) \to \text{Sh}(C, J) \text{ induced by } F \text{ can be identified with the factorization } \text{Sh}(F^s) \circ \text{Sh}(i^s_K) \text{, where } D^s_K \text{ is the full subcategory of } D^j_K \text{ on the objects of the form } (F(c), S) \text{ for an object } c \text{ of } C \text{ and a } K \text{-closed sieve } S \text{ on } F(c), C^s_K|_{D^s_K} \text{ is the Grothendieck topology induced by } C^j_K \text{ on } D^s_K, i^s_K : (D^s_K, C^j_K|_{D^s_K}) \to (D^j_K, C^j_K) \text{ is the morphism of sites given by the canonical inclusion functor } D^K_F \to D^K_F, \text{ and } F_s \text{ is the morphism of sites } (C, J) \to (D^K_F, C^j_K|_{D^K_F}) \text{ given by the functor } F. \]

Proof By Remark 5.29 the topos $G$ arising in the hyperconnected-localic factorization $p' \circ p$ of the geometric morphism $\text{Sh}(F)$ admits as separating set of objects the family of domains of subobjects of objects of the form $\text{Sh}(D, K)$ on such objects is equivalent to $D^K_F$ by Corollary 5.11 so we have an equivalence $\xi : G \to \text{Sh}(D^K_F, C^K_K|_{D^K_F})$. The inverse image of the geometric morphism $p : \text{Sh}(D, K) \to G$ is precisely the inclusion functor of $G$ into $\text{Sh}(D, K)$, and the morphism of sites $i^s_K : (D, K) \to (D^s_K, C^s_K)$ yields an equivalence $\text{Sh}(D^s_K, C^s_K) \simeq \text{Sh}(D, K)$. So the canonical inclusion functor $i^s_K$ of $D^s_K$ into $D^j_K$ defines a morphism of sites $D^K_F, C^K_K|_{D^K_F} \to (D^K_F, C^K_K)$, since it induces the geometric morphism $\xi \circ p \circ \text{Sh}(i^s_K)$. On the other hand, the inverse image of the geometric morphism $p' : G \to \text{Sh}(C, J)$ given by the localic part of the
The hyperconnected-localic factorization of \( \text{Sh}(F) \) is simply \( \text{Sh}(F)^* \), regarded as taking values in the subcategory \( G \) of \( \text{Sh}(D, K) \). Therefore the functor \( F_s \) is a morphism of sites \((C, J) \to (D^*_F, C^*_K | D^*_F)\) as it induces the geometric morphism \( p' \circ \xi^{-1} \). This completes our proof. □

Let us now apply Theorem 5.33 in the context of interpretations between geometric theories.

**Corollary 5.34.** Let \( I : C_T \to C_S \) be an interpretation of a geometric theory \( T \) into a geometric theory \( S \). Let \( T_I \) be the expansion of \( T \) whose signature is obtained by adding a predicate symbol \( R_\psi \) of sorts \( I(\vec{x}) \) for any context \( \vec{x} \) over the signature of \( T \) and any geometric formula \( \psi \) in the context \( I(\vec{x}) \) over the signature of \( S \), and whose axioms are all the sequents whose image under the extension of \( I \) sending each predicate symbol \( R_\psi \) to the corresponding formula \( \{ I(\vec{x}) \cdot \psi \} \) is provable in \( S \), and let \( I_s \) be the obvious interpretation of \( T \) into \( T_I \). Then the hyperconnected-localic factorization of the geometric morphism \( \text{Set}[S] \to \text{Set}[T] \) induced by \( I \) can be identified with the composite \( \text{Sh}(I') \circ \text{Sh}(I_s) \).

**Proof** It is immediate to see that the factorization of \( I \) as \( I_s \) followed by \( I' \) can be identified with the factorization of \( I \), regarded as a morphism of sites \((C_T, J_T) \to (C_S, J_S)\), provided by Theorem 5.33 by using Proposition 5.1, Corollary 5.2 and Remark 3.12. □

**Remark 5.35.** Corollary 5.34 generalizes the corresponding result for expansions of theories proved as Theorem 7.1.3 in [2].

### 5.6 Equivalence of toposes

From the above results we can obtain a criterion (alternative to Corollary 4.11) for a \( J \)-continuous flat functor to induce an equivalence of toposes:

**Corollary 5.36.** Let \( (C, J) \) be a small-generated site, \( E \) a Grothendieck topos and \( F : C \to E \) a \( J \)-continuous flat functor inducing a geometric morphism \( f : E \to \text{Sh}(C, J) \). Then \( f \) is an equivalence if and only if the following conditions are satisfied:

(i) \( F \) is cover-reflecting;

(ii) for every subobject \( A \to F(c) \) in \( E \) there exists a \( (J \text{-closed}) \) sieve \( R \) on \( c \) such that \( A \) is the union of the images of the arrows \( F(f) \) for \( f \in R \);
(iii) the objects of the form $F(c)$ for $c \in C$ form a separating set for the topos $\mathcal{E}$.

**Proof** We have that $f^*$ is one half of an equivalence of categories if and only if it is full and faithful and essentially surjective. Since faithfulness and closure of its image under subobjects implies fullness for the functor $f^*$, we conclude that $f^*$ is one half of an equivalence if and only if it is faithful, essentially surjective and its image is closed under subobjects. Our thesis thus follows from Propositions 5.1, 5.21 and 5.25. □

It is interesting to compare this characterization with the different, but equivalent, criterion provided by Corollary 4.11: the difference lies in condition (ii) of Corollary 5.36 which is replaced by the “local fullness” condition of Corollary 4.11.

This corollary can be notably applied in the context of interpretations between geometric theories to characterize those which induce Morita equivalences.

**Corollary 5.37.** Let $I : C_T \to C_S$ be an interpretation of a geometric theory $T$ into a geometric theory $S$. Then $I$ induces a Morita-equivalence between $T$ and $S$ if and only if the following conditions are satisfied:

(i) for every sequent $\sigma$ over the signature of $T$, if the sequent $I(\sigma)$ is provable in $S$ then $\sigma$ is provable in $T$;

(ii) the image of $I$ is closed under subobjects (that is, for any sorts $A_1, \ldots, A_n$ of the signature of $T$, every geometric formula $\psi$ over the signature of $S$ in the context $I((x_1^{A_1}, \ldots, x_n^{A_n}))$ is $T$-provably equivalent (in its context) to a formula of the form $I(\phi)$ for a geometric formula $\phi$ over the signature of $T$ in the context $(x_1^{A_1}, \ldots, x_n^{A_n})$);

(iii) for every sort $B$ of the signature of $S$, the object $\{x^B . \top\}$ is $J_S$-covered by objects in the image of $I$.

**Proof** Our thesis immediately follows from Corollaries 5.2 and 5.23. □

**Remark 5.38.** Conditions (i) and (ii) of Corollary 5.37 can be reformulated as the requirement that the functor $I$ should be an equivalence onto its image.

**Example 5.39.** Corollary 5.37 can for instance be applied to the interpretation of the algebraic theory $MV$ of MV-algebras into the theory $L_u$ of lattice-ordered abelian groups with strong unit established in [3], which, as proved in [3], induces a Morita equivalence. Condition (iii) is satisfied
since the formula \( \{ x \cdot \top \} \) is covered in \( C_L \) by the objects in the image of \( I \). Indeed, by the axiom expressing the property of strong unit, the arrows \( \{ x' \cdot 0 \leq x' \leq u \} \rightarrow \{ x \cdot \top \} \) given by \( x = nx' \) (for \( n \in \mathbb{Z} \)) generate a \( J_{nu} \)-covering sieve. One can also understand concretely why condition (ii) is satisfied; indeed, by using the construction of the \( \ell \)-group associated with an MV-algebra in terms of good sequences of elements of the latter, one can prove that for every atomic formula \( \chi(\bar{x}) \) over the signature of \( L_u \), there exists a Horn formula \( \chi^{MV}(\bar{x}) \) in the same context over the signature of \( MV \) such that the formula \( I(\chi^{MV}(\bar{x})) \) is provably equivalent in \( L_u \) (in the context \( \bar{x} \)) to the formula \( \chi \land (0 \leq \bar{x} \leq u) \).

From Proposition 5.27 and Theorem 5.3(a), we deduce the following criterion, alternative to Theorem 4.7, for a morphism of sites to induce an equivalence of toposes:

**Corollary 5.40.** Let \( F : (C, J) \rightarrow (D, K) \) be a morphism of small-generated sites. Then the geometric morphism \( Sh(F) : Sh(D, K) \rightarrow Sh(C, J) \) induced by \( F \) is an equivalence if and only if \( F \) is cover-reflecting, closed-sieve-lifting and satisfies condition (ii) of Proposition 4.5 (recall that, if \( K \) is subcanonical, the latter condition is equivalent to \( F \) being \( K \)-dense).

**Proof** Since a geometric morphism \( f \) is an equivalence if and only if it is hyperconnected and its inverse image is essentially surjective, the thesis follows from Proposition 5.27 and Theorem 5.3(a). \( \square \)

The following criterion, alternative to Corollary 5.16, for a geometric morphism to be an inclusion (resp. an equivalence) follows as an immediate consequence of Corollary 5.40.

**Corollary 5.41.** Let \( f : \mathcal{F} \rightarrow \mathcal{E} \) be a geometric morphism. Then

(i) \( f \) is an inclusion if and only if \( f^* \) is locally surjective and its image is closed under subobjects.

(ii) \( f \) is an equivalence if and only if \( f^* \) is faithful, locally surjective and its image is closed under subobjects.

**Proof** As observed in the proof of Corollary 5.16, \( f \) is an inclusion if and only if the morphism of sites \( f^* : (\mathcal{E}, (J^\text{can})_{f^*}) \rightarrow (\mathcal{F}, J^\text{can}) \) is dense (equivalently, weakly dense). Now, this morphism is \( J^\text{can} \)-dense if and only if \( f^* \) is locally surjective (in the sense of Corollary 5.16(i)), while it is closed-sieve-lifting if and only if the image of \( f^* \) is closed under subobjects; indeed, for any Grothendieck topology \( K \) on a Grothendieck topos \( \mathcal{G} \) containing the canonical
one, any $K$-closed sieve is equal to the principal sieve on a subobject. This proves the first part of the corollary; the second part follows from the first by using that a geometric morphism is an equivalence if and only if it is both a surjection and an inclusion.

5.7 An example

Given a small-generated site $(C, J)$ and a presheaf $P : C^{\text{op}} \to \text{Set}$, let us prove that we have an equivalence of toposes

$$\text{Sh}(C, J)/a_{J}(P) \simeq \int P, J_{P},$$

where $\int P$ is the category of elements of $P$ and $J_{P}$ is the Grothendieck topology on $\int P$ whose covering sieves are the sieves which are sent by the canonical projection $\pi_{P} : \int P \to C$ to $J$-covering families. This equivalence is induced by the (flat) $J_{P}$-continuous functor $F_{P} : \int P \to \text{Sh}(C, J)/a_{J}(P)$ sending any object $(c, s) \in \int P$ to the object $y_{C}(c) \to P$ of $[C^{\text{op}}, \text{Set}]$ induced by $(c, x)$ via the Yoneda lemma. First, we observe that, since we have an equivalence

$$[C^{\text{op}}, \text{Set}]/P \simeq [(\int P)^{\text{op}}, \text{Set}]$$

(see, for instance, Proposition A1.1.7 [1]) induced (by composition with the Yoneda embedding $\int P \to [(\int P)^{\text{op}}, \text{Set}]$) by the canonical functor $G_{P} : \int P \to [C^{\text{op}}, \text{Set}]/P$ (sending any object $(c, x) \in \int P$ to the object $y_{C}(c) \to P$ of $[C^{\text{op}}, \text{Set}]/P$ given by the arrow corresponding to $x$ via the Yoneda Lemma), $G_{P}$ is flat. Now, $F_{P}$ is the composite of $G_{P}$ with the inverse image of the geometric morphism $\text{Sh}(C, J)/a_{J}(P) \to [C^{\text{op}}, \text{Set}]/P$ induced by the canonical geometric inclusion $\text{Sh}(C, J) \hookrightarrow [C^{\text{op}}, \text{Set}]$ as in Example 5.18, so it is flat as $G_{P}$ is. Notice that the Grothendieck topology $J_{P}$ is precisely the topology induced by $F_{P}$ (in the sense of Proposition 5.5). Let us apply Corollary 5.30 to prove that $F_{P}$ induces an equivalence. First, we have to show that the objects in the image of the functor $F_{P}$ define a separating set of objects for the topos $\text{Sh}(C, J)/a_{J}(P)$. Given two distinct arrows $\alpha, \beta : (\gamma : Q \to a_{J}(P)) \to (\xi : Q' \to a_{J}(P))$ in $\text{Sh}(C, J)/a_{J}(P)$, we have to show that there are an object $c$ of $C$, an element $x \in P(c)$, corresponding to an arrow $P_{x} : l(c) \to a_{J}(P)$ in $\text{Sh}(C, J)$, and an element $y \in Q(c)$, corresponding to an arrow $\gamma : l(c) \to Q$ in $\text{Sh}(C, J)$, such that $P_{x} = \gamma \circ \gamma$ and $\alpha \circ \gamma \neq \beta \circ \gamma$. Consider the pullback $\tilde{\gamma} : Q \to P$ of $\gamma : Q \to a_{J}(P)$ along the unit $P \to a_{J}(P)$. Since the canonical arrow $z : Q \to Q$ is sent by $a_{J}$ to an isomorphism and $a_{J}$ is left adjoint to the inclusion functor, $\alpha \circ z \neq \beta \circ z$. So there exists an object $c$ of
\( C \) and an element \( y' \in \tilde{Q}(c) \) such that \( \alpha(z(c)(y')) \neq \beta(z(c)(y')) \). So, posing \( x = \gamma(c)(y') \) and \( y = z(c)(y') \), we obtain that \( F_x = \gamma \circ y \) and \( \alpha \circ \gamma \neq \beta \circ y \), as required.

It now remains to show that condition (ii) is satisfied by \( F_P \); but this is clear, since, for any object \((c, x)\) of \( \int P \), any subobject \( A \) of the object \( P_x : l(c) \to a_J(P) \) in \( \text{Sh}(C, J)/a_J(P) \) corresponds to a subobject of \( l(c) \) in \( \text{Sh}(C, J) \) and hence to a sieve \( S \) on \( c \), which clearly lifts to a sieve \( S_\pi \) on the object \((c, x)\) in \( \int P \) such that \( A \) is isomorphic to the union of the images of the arrows \( F_P(f) \) where \( f \in S_\pi \).

### 5.8 A characterization of quotient interpretations

Let us now apply Corollaries 4.11 and 5.36 in the context of interpretations of geometric theories, in order to characterize the interpretations of a geometric theory \( T \) which are isomorphic to the canonical interpretation of \( T \) into a quotient of it.

**Corollary 5.42.** Let \( I : C_T \to C_S \) be an interpretation of a geometric theory \( T \) into a geometric theory \( S \). Then \( I \) is isomorphic (over \( C_T \)) to the canonical interpretation \( I^\pi_T : C_T \to C_T' \) of \( T \) into a quotient \( T' \) of \( T \) if and only if it satisfies the following conditions:

(i) The image of \( I \) is closed under subobjects (that is, for any sorts \( A_1, \ldots, A_n \) of the signature of \( S \) in the context \( I((x_1^{A_1}, \ldots, x_n^{A_n})) \) is \( T \)-provably equivalent (in its context) to a formula of the form \( I(\phi) \) for a geometric formula \( \phi \) over the signature of \( T \) in the context \((x_1^{A_1}, \ldots, x_n^{A_n})) \).

(ii) For every sort \( B \) of the signature of \( S \), the object \( \{x^B . T\} \) is \( J_S \)-covered by objects in the image of \( I \).

Condition (i) can be equivalently replaced by the following condition:

(i)' For every objects \( \{\bar{x} . \phi\} \) and \( \{\bar{y} . \psi\} \) of \( C_T \) and any \( S \)-provably functional formula \( \theta(I(\bar{x}), I(\bar{y})) \) from \( I(\{\bar{x} . \phi\}) \) to \( I(\{\bar{y} . \psi\}) \) there exist a set of \( T \)-provably functional formulae \( \{\chi_i(\bar{x}, \bar{z})\} : \{\bar{x} . \phi_i\} \to \{\bar{y} . \phi\} \) such that the sequent \( (I(\phi) \vdash_{I(\bar{x})} \bigvee_i (\exists I(\bar{x}) I(\chi_i(\bar{x}, \bar{z})))) \) is provable in \( S \) and \( T \)-provably functional formulae \( \xi_i(\bar{x}, \bar{y}) : \{\bar{x} . \phi_i\} \to \{\bar{y} . \psi\} \) (for each \( i \)) such that \( [\theta] \circ I(|\chi_i|) = I(|\xi_i|) \) in \( C_S \) (i.e. the bi-sequent \((\exists I(\bar{x}))(\theta \land I(|\chi_i|)) \vdash_{I(\bar{x})} I(|\xi_i|)) \) is provable in \( S \).

**Proof** By the duality theorem between quotients and subtoposes established in Chapter 3 of [2], we have to characterize the interpretations \( I \) which induce
a geometric inclusion to the classifying topos of \( T \). The formulation of this condition as the conjunction of conditions (i) and (ii) follows from Corollary 5.36 by observing that the geometric morphism induced by \( I \) is induced by the \((J_T)_I\)-continuous flat functor \( C_T \to \text{Sh}(C_S, J_S) \) given by the composite of the Yoneda embedding \( C_S \to \text{Sh}(C_S, J_S) \) with \( I \).

On the other hand, conditions (i)' and (ii) are equivalent to the property of the interpretation \( I \) to induce a dense morphism of sites \((C_T, (J_T)_I) \to (C_S, J_S)\), equivalently, by Theorem 5.3(c), a geometric inclusion from the classifying topos of \( S \) to the classifying topos of \( T \). □

Remarks 5.43. (a) The quotient \( T' \) of \( T \) corresponding to an interpretation \( I \) satisfying the conditions of Corollary 5.42 is axiomatized by the sequents \( \sigma \) over the signature of \( T \) such that \( I(\sigma) \) is provable in \( S \) (cf. 5.37).

(b) It is interesting to understand why property (i)' is true in the case of the canonical interpretation of \( T \) into a quotient \( T' \) of it. Given a \( T' \)-provably functional formula \( \theta(I(\vec{x}), I(\vec{y})) \) from \( I(\{\vec{x} . \phi\}) \) to \( I(\{\vec{y} . \psi\}) \), we can take \( \chi \) to be the formula \((\theta \wedge \phi \wedge \psi)(\vec{x}, \vec{y}) \wedge \vec{x} = \vec{x}'\), which is \( T \)-provably functional from \( \{\vec{x}, \vec{y} . \theta \wedge \phi \wedge \psi\} \) to \( \{\vec{x}'. \phi[\vec{x}'/\vec{x}]\} \), and \( \xi \) to be the formula \((\theta \wedge \phi \wedge \psi)(\vec{x}, \vec{y}) \wedge \vec{y} = \vec{y}'\), which is \( T \)-provably functional from \( \{\vec{x}, \vec{y} . \theta \wedge \phi \wedge \psi\} \) to \( \{\vec{y}'. \psi[\vec{y}'/\vec{y}]\} \).

6 Geometric morphisms induced by comorphisms of sites

Recall that a \textit{comorphism of sites} \((D, K) \to (C, J)\) (where \( J \) and \( K \) are Grothendieck topologies respectively on \( C \) and \( D \)) is a functor \( F : D \to C \) which has the covering lifting property, that is the property that for every \( d \in D \) and any \( J \)-covering sieve \( S \) on \( F(d) \) there is a \( K \)-covering sieve \( R \) on \( c \) such that \( F(R) \subseteq S \).

Comorphisms of sites notably arise in the context of flat functors inducing equivalences of toposes. Let \( F : C \to E \) be a flat functor on an essentially small category \( C \) with values in a Grothendieck topos \( E \), and let \( J_F \) be the induced Grothendieck topology on \( C \) (in the sense of Proposition 5.5). Suppose that \( F \) induces an equivalence \( E \simeq \text{Sh}(C, J_F) \) (equivalently, satisfies the conditions of Corollary 4.11) that is, is \( J_F \)-full and the objects of the form \( F(c) \) for \( c \in C \) form a separating family of objects of \( E \). Then \( F \) is both a morphism and a comorphism of sites \((C, J_F) \to (E, J_E^{\text{for}})\) (cf. Corollary
4.20), and the geometric morphisms induced by them are quasi-inverse to each other.

As it is well-known (cf. Theorem VII.5 [6]), every comorphism of sites $F : (\mathcal{D}, K) \to (\mathcal{C}, J)$ induces a $J$-continuous flat functor

$$A_F : \mathcal{C} \to \text{Sh}(\mathcal{D}, K),$$

which assigns to each object $c$ of $\mathcal{C}$ the $K$-sheaf given by $a_K(\text{Hom}_\mathcal{C}(F(-), c))$ (and acts on the arrows in the obvious way), and hence, by Diaconescu’s equivalence, a geometric morphism $L_F : \text{Sh}(\mathcal{D}, K) \to \text{Sh}(\mathcal{C}, J)$ (whose inverse image $L_F^* : \text{Sh}(\mathcal{C}, J) \to \text{Sh}(\mathcal{D}, K)$ sends a $J$-sheaf $P$ to $a_K(P \circ F^{\text{op}})$).

Notice that the canonical geometric inclusion $\text{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \text{Set}]$ can be seen either as induced by the morphism of sites $(\mathcal{C}, T) \to (\mathcal{C}, J)$ given by the identity functor $1_\mathcal{C}$ on $\mathcal{C}$, where $T$ is the trivial Grothendieck topology on $\mathcal{C}$, or as induced in this way by the comorphism of sites $1_\mathcal{C} : (\mathcal{C}, J) \to (\mathcal{C}, T)$.

In this section we shall investigate the conditions on a comorphism of sites which correspond to the property of the corresponding geometric morphism to be a surjection (resp. an inclusion, hyperconnected, localic), and also describe the surjection-inclusion and hyperconnected-localic factorizations at the level of comorphisms of sites.

### 6.1 Surjections

**Proposition 6.1.** The geometric morphism $L_F : \text{Sh}(\mathcal{D}, K) \to \text{Sh}(\mathcal{C}, J)$ induced by a comorphism of sites $F : (\mathcal{D}, K) \to (\mathcal{C}, J)$ is a surjection if and only if $J = K^F$ (in the sense of Proposition 5.11), that is, if a sieve $S$ on an object $c \in \mathcal{C}$ satisfies the property that for every object $d$ of $\mathcal{D}$ and arrow $x : F(d) \to c$ in $\mathcal{C}$, there exists a $K$-covering sieve $T$ on $d$ such that $F(T) \subseteq x^*(S)$ then $S$ is $J$-covering. This condition implies that $F$ is $J$-dense and is equivalent to it if $F$ is cover-preserving.

**Proof** By Proposition 5.1 $L_F$ is a surjection if and only the functor $A_F$ is cover-reflecting, that is, for any sieve $S$ on an object $c$, if the family of arrows $A_F(f)$ for $f \in S$ is epimorphic then $S$ is $J$-covering. Now, since a family of morphisms in $[\mathcal{D}^{\text{op}}, \text{Set}]$ is sent by the associated sheaf functor $a_K : [\mathcal{D}^{\text{op}}, \text{Set}] \to \text{Sh}(\mathcal{D}, K)$ to an epimorphic family if and only if it is $K$-locally surjective (cf. Corollary III.6 [6]), we obtain the following criterion: the family $\{A_F(f) \mid f \in S\}$ is epimorphic if and only if for every object $d$ of $\mathcal{D}$ and arrow $x : F(d) \to c$ there exists a $K$-covering sieve $T$ on $d$ such that for every $g \in T$, $x \circ F(g) \in S$, that is, if and only if $K^F \subseteq J$. From this, in light of Remark 5.12 the first part of our thesis follows immediately.
It remains to show that the condition $J = K^F$ implies that $F$ is $J$-dense and is equivalent to it if $F$ is cover-preserving. The fact that if $J = K^F$ then $F$ is $J$-dense follows from the fact that, for any $c \in \mathcal{C}$, the sieve $S_c$ on $c$ generated by the arrows from objects of the form $F(d)$ (for $d \in \mathcal{D}$) to $c$ is $K^F$-covering, by definition of $K^F$, and therefore $J$-covering. Conversely, let us show that if $F$ is cover-preserving and $J$-dense then $K^F \subseteq J$ (equivalently, $K^F = J$). Given a sieve $S$ on an object $c \in \mathcal{C}$ satisfying the property that for every object $d$ of $\mathcal{D}$ and arrow $x : F(d) \to c$ in $\mathcal{C}$, there exists a $K$-covering sieve $T$ on $d$ such that $F(T) \subseteq x^*(S)$, we want to prove that $S$ is $J$-covering. Since $F$ is cover-preserving, for any arrow $x : F(d) \to c$ in $\mathcal{C}$, $x^*(S) \in J(\text{dom}(x))$; the fact that $S \in J(c)$ thus follows from the fact that $F$ is $J$-dense by the transitivity axiom for Grothendieck topologies. \hfill \Box

**Remarks 6.2.** (a) If $F$ is $K$-full then the condition $J = K^F$ implies that $F$ is cover-preserving. Indeed, for any $R \in K(d')$, the sieve $\langle F(R) \rangle$ is $J$-covering, since, by the $K$-fullness of $F$, for any arrow $x : F(d) \to F(d')$ in $\mathcal{C}$ there are a set $I$ and arrows $f_i : d_i \to d, g_i : d_i \to d'$ (for $i \in I$) such that the family $\{f_i : d_i \to d \mid i \in I\}$ is $K$-covering and $x \circ F(f_i) = F(g_i)$, and hence the sieve $T := \{f_i \circ h \mid i \in I, h \in g_i(R)\}$ is $K$-covering and satisfies the property $F(T) \subseteq x^*(S)$.

(b) If $J$ and $K$ are the trivial topologies then $F$ is clearly cover-preserving and the condition that $F$ be $J$-dense is equivalent to the requirement that every object of $\mathcal{D}$ should be a retract of an object in the image of $F$. In fact, this was also proved with other means in Example A4.2.7(b) \[4\].

### 6.2 Inclusions

To characterize the comorphisms of sites which give rise to a geometric inclusion, we shall use the criterion provided by Corollary 5.2.4.

The following lemma expresses certain properties of arrows in the image of the functor $A_F$ introduced above in terms of functional relations associated with them.

**Lemma 6.3.** Let $c, c' \in \mathcal{C}$, $\xi : A_F(c) = a_K(\text{Hom}_\mathcal{C}(F(-), c)) \to A_F(c') = a_K(\text{Hom}_\mathcal{C}(F(-), c'))$ an arrow in $\text{Sh}(\mathcal{D}, K)$ and $R_\xi$ the functional relation from $\text{Hom}_\mathcal{C}(F(-), c)$ to $\text{Hom}_\mathcal{C}(F(-), c')$ associated with it as in Remark 6.2(c). Then

(i) For any arrows $f : c' \to c$ and $g : c'' \to c'$, $\xi \circ A_F(f) = A_F(g)$ if and only if for every $x : F(c) \to c''$, $(f \circ x, g \circ x) \in R_\xi$.

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(ii) There exist a family of arrows $f_i : c_i \to c$ in $C$ which is sent by $A_F$ to an epimorphic family and a family of arrows $g_i : c_i \to c'$ in $C$ such that $\xi \circ A_F(f_i) = A_F(g_i)$ for all $i$ if and only if there exists a family $(u_j, v_j)$ of elements of $R_\xi$ such that the family $\{A_F(u_j)\}$ is epimorphic and for any arrow $s : F(e) \to \text{dom}(u_j) = \text{dom}(v_j)$ in $C$, $(u_j \circ s, v_j \circ s) \in R_\xi$.

**Proof** (i) By Remark 3.6(c), we have that $(f \circ x, g \circ x) \in R_\xi$ if and only if $\xi \circ \eta^e(f \circ x) = \eta^e(g \circ x)$, where $\eta^e : \text{Hom}_C(F(-), c) \to a_K(\text{Hom}_C(F(-), c))$ and $\eta'' : \text{Hom}_C(F(-), c') \to a_K(\text{Hom}_C(F(-), c'))$ are the canonical unit arrows. But $\xi \circ \eta^e(f \circ x) = (\xi \circ A_F(f) \circ \eta''(x))$ and $\eta^e(g \circ x) = (A_F(g) \circ \eta''(x))$, where $\eta'' : \text{Hom}_C(F(-), c') \to a_K(\text{Hom}_C(F(-), c'))$ is the canonical unit arrow.

(ii) Let us suppose that there exist a family of arrows $f_i : c_i \to c$ in $C$ which is sent by $A_F$ to an epimorphic family and a family of arrows $g_i : c_i \to c'$ in $C$ such that $\xi \circ A_F(f_i) = A_F(g_i)$ for all $i$. Then we may take as arrows $(u_j, v_j)$ those of the form $(f_i \circ x, g_i \circ x)$ for $x$ an arrow $F(e) \to c_i$ for some $i \in I$. We have that, by (i), every pair of the form $(f_i \circ x, g_i \circ x)$ is in $R_\xi$, so it remains to show that the family of arrows $\{A_F(u_j)\}$ is epimorphic (equivalently, for any arrow $u : F(e) \to e$ there exists a $K$-covering sieve $T$ on $e$ such that $u \circ F(t)$ factors through some $u_i$ for every $t \in T$). But this is clear since, the family $\{A_F(f_i) | i \in I\}$ being epimorphic, there is a $K$-covering sieve $T_u$ on $e$ such that, for every $t \in T_u$, $u \circ F(t)$ factors through some $f_i$, that is, of the form $f_i \circ x$ for some arrow $x$.

Conversely, if there exists a family $(u_j, v_j)$ of elements of $R_\xi$ such that the family $\{A_F(u_j)\}$ is epimorphic and for any arrow $s : F(e) \to \text{dom}(u_j) = \text{dom}(v_j)$ in $C$, $(u_j \circ s, v_j \circ s) \in R_\xi$ then by (i) we have that $\xi \circ A_F(u_j) = A_F(v_j)$ for each $j$. So we may take as family of arrows $(f_i, g_i)$ precisely the family of arrows $(u_j, v_j)$.

**Proposition 6.4.** Let $F : (\mathcal{D}, K) \to (\mathcal{C}, J)$ be a comorphism of sites. Then the flat functor $A_F : \mathcal{C} \to \text{Sh}(\mathcal{D}, K)$ satisfies condition (ii) of Corollary 5.4 if and only if for every $K$-functional relation $R$ from $\text{Hom}_C(F(-), c)$ to $\text{Hom}_C(F(-), c')$ in $[\mathcal{D}^{\text{op}}, \text{Set}]$ there exists a family $(f_i, g_i)$ of arrows $f_i : \text{dom}(f_i) \to c$ and $g_i : \text{dom}(g_i) \to c'$ with common domain $\text{dom}(f_i) = \text{dom}(g_i)$ such that

- for any arrow $x : F(e) \to c$ there exists a $K$-covering sieve $T$ on $e$ such that, for every $t \in T$, $x \circ F(t)$ factors through some $f_i$;

- for any arrow $x : F(e) \to \text{dom}(f_i) = \text{dom}(g_i)$ in $C$, $(f_i \circ x, g_i \circ x) \in R$.

In particular, this condition is satisfied if $F$ is $K$-full.
If $F$ is cover-preserving then the conjunction of the two conditions above is equivalent to the requirement that for any $(f, g) \in R$ and any arrow $x : F(e) \to \text{dom}(f) = \text{dom}(g)$ in $C$, $(f \circ x, g \circ x) \in R$.

**Proof** Recall that condition (ii) of Corollary 5.4 states that for every $c, c' \in C$ and any arrow $\xi : A_F(c) \to A_F(c')$ in $\text{Sh}(D, K)$, there exist a family of arrows $f_i : c_i \to c$ in $C$ which is sent by $A_F$ to an epimorphic family and a family of arrows $g_i : c_i \to c'$ in $C$ such that $\xi \circ A_F(f_i) = A_F(g_i)$ for all $i$.

By Theorem 6.3 by observing that a family of arrows $\{A_F(u_i)\}$ is epimorphic if and only if for any arrow $x : F(e) \to e$ there exists a $K$-covering sieve $T$ on $e$ such that, for every $t \in T$, $x \circ F(t)$ factors through some $u_j$.

If $F$ is $K$-full then, for any arrow $\xi : A_F(c) \to A_F(c')$, the family of pairs of arrows in $R_\xi$ satisfies both conditions in the statement of the Proposition. The first condition is satisfied since, $R_\xi$ being a $K$-functorial relation, the family of arrows $\{A_F(x) \mid x \in \pi_1(R_\xi)\}$, where $\pi_1$ is the canonical projection $\text{Hom}_C(F(-), c) \times \text{Hom}_C(F(-), c') \to \text{Hom}_C(F(-), c)$, is epimorphic in $\text{Sh}(D, K)$, while the second holds by the following argument. Given $(f, g) \in R_\xi$ and $x : F(e) \to \text{dom}(f) = \text{dom}(g)$, the $K$-fullness of $F$ implies that there exist a $K$-covering family $T$ on $e$ and for each $t \in T$ an arrow $s_t : \text{dom}(t) \to e$ such that $x \circ F(t) = F(s_t)$. Now, by the $K$-closure property of $R_\xi$, the condition $(f \circ x, g \circ x) \in R_\xi$ is equivalent to $(f \circ x \circ F(t), g \circ x \circ F(t)) \in R_\xi$ for every $t \in T$. But $(f \circ x \circ F(t), g \circ x \circ F(t)) = (f \circ F(s_t), g \circ F(s_t))$, and $(f \circ F(s_t), g \circ F(s_t)) \in R_\xi$ by the functoriality of $R_\xi$.

It remains to prove the last part of the proposition, namely that, provided that $F$ is cover-preserving, if $R$ satisfies the two conditions above then $R$ is closed under composition on the right with arbitrary arrows in $C$. Note that the converse direction follows from the fact that if $R$ satisfies this property then, by taking the family $\{(f_i, g_i)\}$ to consist of the pairs in $R$, the first condition in the proposition is automatically satisfied by the $K$-functionality of $R$. So, let us suppose that $(f : F(e) \to c, g : F(e) \to c') \in R$ and that $x : F(a) \to F(e)$ is an arrow in $C$; we want to show that $(f \circ x, g \circ x) \in R$.

By the first condition, there is a $K$-covering sieve $T$ on $e$ such that, for each $t \in T$, $f \circ F(t)$ factors through some $f_{i_t}$, say, $f \circ F(t) = f_{i_t} \circ u_{i_t}$. Now, by the functoriality of $R$, $(f, g) \in R$ implies that $(f \circ F(t), g \circ F(t)) \in R$. On the other hand, the second condition in the proposition implies that
\((f_t \circ u_t^l, g_t \circ u_t^l) \in R\). So, the \(K\)-functionality of \(R\) implies that \(g \circ F(t) \equiv_K g_t \circ u_t^l\). Since \(F\) is cover-preserving, the sieve \(\langle F(T) \rangle\) generated by the arrows \(F(t)\) for \(t \in T\) is \(J\)-covering; therefore its pullback along \(x\) is also \(J\)-covering and hence by the covering-lifting property there is a \(K\)-covering sieve \(H\) on \(a\) such that for each \(h \in H\), \(x \circ F(h) = F(t_h) \circ z_h\) for some arrow \(t_h \in T\) and some arrow \(z_h : F(\text{dom}(h)) \to F(\text{dom}(t_h))\). Now, by the \(K\)-closure of \(R\), \((f \circ x, g \circ x) \in R\) if and only if for every \(h \in H\), \((f \circ x \circ F(h), g \circ x \circ F(h)) \in R\). But \((f \circ x \circ F(h), g \circ x \circ F(h)) = (f_{t_h} \circ u_{t_h}^l \circ z_h, g \circ F(t_h) \circ z_h)\). Now, we have that \(g \circ F(t_h) \equiv_K g_{t_h} \circ u_{t_h}^l\), whence \(g \circ F(t_h) \circ z_h \equiv_K g_{t_h} \circ u_{t_h}^l \circ z_h\). Since \(R\) is a \(K\)-functional relation, the condition \((f_{t_h} \circ u_{t_h}^l \circ z_h, g \circ F(t_h) \circ z_h) \in R\) is equivalent to \((f_{t_h} \circ u_{t_h}^l \circ z_h, g_{t_h} \circ u_{t_h}^l \circ z_h) \in R\) (cf. Proposition \ref{prop:5.7}), and this is satisfied by the second condition in the proposition. \(\square\)

For investigating the satisfaction of condition (i) of Corollary \ref{cor:5.4}, we need the following lemma. Notice that if \(F : (D, K) \to (C, J)\) is a comorphism of sites then for any \(d \in D\) we have an arrow \(\chi_d : l'(d) \to A_F(F(d))\) corresponding, via the associated sheaf adjunction and the Yoneda embedding, to the element of \(A_F(F(d))(d)\) given by the identity arrow on \(F(d)\).

**Lemma 6.5.** Let \(F : (D, K) \to (C, J)\) be a comorphism of sites. Then for any object \(d \in D\) and any arrow \(g : d' \to d\) in \(D\), the arrows \(\xi : A_F(F(d')) \to l'(d)\) such that \(\xi \circ \chi_{d'} = l'(g)\) can be identified with the relations \(R\) from \(\text{Hom}_C(F(-), F(d'))\) to \(\text{Hom}_D(-, d)\) which assign to each object \(e \in C\) a collection \(R(e)\) of pairs of arrows \((x : F(e) \to F(d'), y : e \to d)\) in such a way that the following properties are satisfied:

(i) for any \(\xi : e' \to e\) in \(D\), if \((x, y) \in R\) then \((x \circ F(\xi), y \circ \xi) \in R;\)

(ii) for any \(e \in D\) and any \((x, y) \in \text{Hom}_C(F(e), F(d')) \times \text{Hom}_C(e, d)\), if \(\{\xi : e' \to e \mid (x \circ F(\xi), y \circ \xi) \in R(e')\}\) is \(K(e)\) then \((x, y) \in R(e);\)

(iii) for any \(e \in D\), if \((x, y), (x', y') \in R\) and \(x = x'\) then \(y \equiv_K y';\)

(iv) for any \(e \in D\) and any arrow \(x : F(e) \to F(d')\) in \(C\), \(\{t : e' \to e \mid \exists y : e' \to d, (x \circ F(t), y) \in R(e')\}\) is \(K(e);\)

(v) for any arrow \(f : e \to d'\) in \(D\), \((F(f), g \circ f) \in R(e).\)

In particular, if \(F\) is \(K\)-full and \(K\)-faithful then for every \(d \in D\) the arrow \(\chi_{d'} : l'(d) \to A_F(F(d))\) is an isomorphism.

**Proof** The first part of the lemma follows from the proof of Theorem \ref{thm:3.8}. It thus remains to show that if \(F\) is \(K\)-full and \(K\)-faithful then there exists
a relation which verifies the conditions in the statement of the lemma and is inverse to $\chi_d$. We can define a relation $R_F$ on pairs of arrows $(x : F(e) \to F(d), y : e \to d)$ by stipulating that $(x, y) \in R_F$ if and only if $F(y) = x$. If $F$ is $K$-faithful and $K$-full then the relation $R_F$ clearly satisfies all the conditions of the lemma for the family of objects $\{c_i \mid i \in I\}$ given by the singleton $\{F(d)\}$ except for the closure condition (ii), which, by Remark 3.6.b, can be made to hold (without affecting the satisfaction of the other conditions) by taking its $K$-closure. The corresponding arrow $A_F(F(d)) \to l'(d)$ (via the bijection of Theorem 3.8) is actually inverse to the arrow $\chi_d$ (cf. Proposition 4.16), as required.

**Proposition 6.6.** The geometric morphism $L_F : \text{Sh}(\mathcal{D}, K) \to \text{Sh}(\mathcal{C}, J)$ induced by a comorphism of sites $F : (\mathcal{D}, K) \to (\mathcal{C}, J)$ is an inclusion if and only if $F$ satisfies the condition of Proposition 6.4 and for every object $d$ of $\mathcal{D}$ there exist a $K$-covering family $\{g_i : d_i \to d \mid i \in I\}$ and arrows $\xi_i : A_F(F(d_i)) \to l'(d)$ in $\text{Sh}(\mathcal{D}, K)$ such that $\xi_i \circ \chi_{d_i} = l'(g_i)$ (equivalently, relations $R_i$ from $\text{Hom}_C(F(-), F(d_i))$ to $\text{Hom}_D(-, d)$ satisfying the conditions of Lemma 6.5).

In particular, if $F$ is $K$-full and $K$-faithful then the geometric morphism $L_F$ is an inclusion. Conversely, if $F$ is cover-preserving and $L_F$ is an inclusion then $F$ is $K$-full.

**Proof** By Corollary 5.4, $L_F$ is an inclusion if and only if conditions (i) and (ii) of the Corollary are satisfied by the flat functor $A_F$. Condition (ii) is equivalent to the condition of Proposition 6.4. On the other hand, condition (i) is satisfied by $A_F$ if and only if for each $d \in \mathcal{D}$, there exists a set $\{c_i\}$ of objects of $\mathcal{C}$ and an epimorphic family of arrows $\xi'_j : A_F(c_j) \to l'(d)$. But, by using Proposition 3.13, one can easily see that this amounts to the existence of a $K$-covering family $\{g_i : d_i \to d \mid i \in I\}$ on $d$ and for each $i \in I$ an arrow $u_i : F(d_i) \to c_{j_i}$ in $\mathcal{C}$ (for some $j_i$) such that $l'(g_i) = \xi'_{j_i} \circ A_F(u_i) \circ \chi_{d_i}$. So the composite of the arrow $\xi_i := \xi'_{j_i} \circ A_F(u_i) : A_F(F(d_i)) \to l'(d)$ with $\chi_{d_i}$ is equal to $l'(g_i)$.

If $F$ is $K$-full and $K$-faithful then by Proposition 6.4 $F$ satisfies the condition in it and by Lemma 5.5 for every $d \in \mathcal{D}$ the arrow $\chi : l'(d) \to A_F(F(d))$ is an isomorphism whence we can take the trivial covering family on each $d$ to make the condition of the proposition satisfied.

Let us now show that if $L_F$ is an inclusion and $F$ is cover-preserving then $F$ is $K$-full. Let $f, f' : e \to d$ be arrows in $\mathcal{D}$ such that $F(f) = F(f')$. Let $\{g_i : d_i \to d \mid i \in I\}$ be a $K$-covering family and $\xi_i : A_F(F(d_i)) \to l'(d)$ be arrows in $\text{Sh}(\mathcal{D}, K)$, corresponding to $K$-functional relations $R_i$ from $\text{Hom}_C(F(-), F(d_i))$ to $\text{Hom}_D(-, d)$, such that $\xi_i \circ \chi_{d_i} = l'(g_i)$. Let us first
prove that, for any \(i \in I\), we have \(\chi_d \circ \xi_i = A_F(F(g_i))\). The \(K\)-functional relation corresponding to this arrow is the composite \(R'_i\) of \(R_i\) with the relation \(R_{\chi_d}\) associated with \(\chi_d\). Since \((1_{F(d_i)}, g_i) \in R_i\) (this follows from the fact that \(R_i\) satisfies property (v) of Lemma 6.5 with respect to the arrow \(g_i\)) and \((g_i, F(g_i)) \in R_{\chi_d}\), we have that \((1_{F(d_i)}, F(g_i)) \in R'_i\). So, by the last part of Proposition 6.4, for any arrow \(x : F(e) \to F(d_i)\), \((x, F(g_i) \circ x) \in R'_i\); in other words, \(\chi_d \circ \xi_i = A_F(F(g_i))\), as required.

The sieve \(\langle \{ F(g_i) \mid i \in I \} \rangle\) generated by the arrows \(F(g_i)\) for \(i \in I\) is \(J\)-covering; so, for any arrow \(y : F(e) \to F(d)\) in \(C\), its pullback along \(y\) is also \(J\)-covering, whence by the covering-lifting property there is a \(K\)-covering sieve \(H\) on \(e\) such that for each \(h \in H\), \(y \circ F(h) = F(g_{i_h}) \circ z_h\) for some \(i_h \in I\) and some arrow \(z_h : F(\text{dom}(h)) \to F(d_{i_h})\). Now, since \(R_i\) is \(K\)-functional, there is a \(K\)-covering sieve \(T_h\) on \(\text{dom}(h)\) and for each \(t \in T_h\) an arrow \(a_t : \text{dom}(t) \to d\) such that \((z_h \circ F(t), a_t) \in R_i\). Then, from the fact (proved above) that \(\chi_d \circ \xi_i = A_F(F(g_i))\) it follows that \(F(a_t)\) is \(K\)-locally equal to \(F(g_{i_h}) \circ z_h \circ F(t)\), that is, there is a \(K\)-covering sieve \(W_t\) on \(\text{dom}(t)\) such that \(F(a_t) \circ F(w) = F(g_{i_h}) \circ z_h \circ F(t) \circ F(w) = y \circ F(h) \circ F(t) \circ F(w)\) for each \(w \in W_t\). Therefore the \(K\)-covering sieve \(U\) given by the multicomposite of \(H\), the \(T_h\)'s (for \(h \in H\)) and the \(W_t\)'s (for \(t \in T_h\)) satisfies the property that for any \(u \in U\) there is an arrow \(a'_u : \text{dom}(u) \to d\) such that \(y \circ F(u) = F(a'_u)\) for each \(u \in U\) (indeed, if \(u = h \circ t \circ w\) then we can take \(a'_u = a_t \circ w\)). So \(F\) is \(K\)-full, as desired. \(\qed\)

**Remark 6.7.** If \(J\) and \(K\) are the trivial topologies then one may deduce from Proposition 6.6 the following criterion for the essential geometric morphism \(L_F : [\mathcal{D}^{\text{op}}, \text{Set}] \to [\mathcal{C}^{\text{op}}, \text{Set}]\) induced by \(F\) to be an inclusion: \(L_F\) is an inclusion if and only if \(F\) is full and faithful.

Indeed, the ‘if’ direction follows immediately from the proposition, while the converse one can be proved as follows. The fullness of \(F\) follows from the last part of the proposition, while its faithfulness follows from the fact that the condition in the proposition implies that, for every \(d \in D\), there is an arrow \(\xi : A_F(F(d)) \to l'(d)\) such that \(\xi \circ \chi_d = 1_{l'(d)}\).

An alternative way of proving that if \(L_F\) is an inclusion then \(F\) is full and faithful is to observe that, the inverse image functor of the geometric morphism \(L_F\) having adjoints on both sides, the right adjoint is full and faithful if and only if the left adjoint is; but this latter condition implies that \(F\) is full and faithful, since \(F\) can be recovered from this functor as its restriction to the representables.

The surjection-inclusion factorization of the geometric morphism induced by a comorphism of sites admits a particularly simple description when the latter is cover-preserving, as shown by the following proposition.
Proposition 6.8. Let \( F : (\mathcal{D}, K) \to (\mathcal{C}, J) \) be a comorphism of sites which is cover-preserving. Then the surjection-inclusion factorization of the geometric morphism \( L_F : \text{Sh}(\mathcal{D}, K) \to \text{Sh}(\mathcal{C}, J) \) induced by \( F \) can be identified with \( L_i \circ L_{F'} \), where \( F' \) is the functor \( F \) regarded as a comorphism of sites from \( (\mathcal{D}, K) \) to the site \( (\mathcal{C}', J') \) whose underlying category \( \mathcal{C}' \) is the full subcategory of \( \mathcal{C} \) on the objects in the image of \( F \) and whose Grothendieck topology \( J' \) is the restriction of \( J \) to \( \mathcal{C}' \), and \( i \) is the inclusion functor of \( \mathcal{C}' \) into \( \mathcal{C} \), regarded as a comorphism of sites \( (\mathcal{C}', J') \to (\mathcal{C}, J) \).

Proof Since \( F \) is cover-preserving, by definition of the Grothendieck topology \( L \), \( F' \) is also cover-preserving. It is also surjective on objects, so, by Proposition 6.1, \( L_{F'} \) is a surjection. On the other hand, \( i \) is full and faithful, so, by Proposition 6.6, \( L_i \) is an inclusion. By the uniqueness (up to equivalence) of the surjection-inclusion factorization, it follows that \( L_i \circ L_{F'} \) is the surjection-inclusion factorization of \( L_{F} \), as required. \( \square \)

Remark 6.9. Proposition 6.8 generalizes the result (cf. Example A4.2.12(b) [4]) describing the surjection-inclusion factorization of the essential geometric morphism induced by an arbitrary functor between small categories.

6.3 Localic morphisms

We shall now characterize the comorphisms of sites which induce localic geometric morphisms.

Lemma 6.10. Let \( F : (\mathcal{D}, K) \to (\mathcal{C}, J) \) be a comorphism of sites. Then for any object \( d \) of \( \mathcal{D} \) and any arrow \( g : d' \to d \) in \( \mathcal{D} \), the arrows \( \xi \) to \( l'(d) \) whose domain is a subobject \( s : S \to A_F(F(d')) \) through which \( \chi_d \) factors as \( \chi_d = s \circ \overline{\chi_d} \), and such that \( \xi \circ \overline{\chi_d} = l'(g) \), can be identified with the relations \( R \) from \( \text{Hom}_C(F(-), F(d')) \) to \( \text{Hom}_D(-, d) \) satisfying all the conditions of Lemma 6.3 but condition (iv).

The functor \( F \) is \( K \)-faithful if and only if for every object \( d \) of \( \mathcal{D} \), the arrow \( \chi_d : l'(d) \to A_F(F(d)) \) is a monomorphism.

Proof The first part of the lemma follows from the proof of Theorem 3.8 so it remains to prove the second part. We notice that \( \chi_d = a_K(\gamma_d) \), where \( \gamma_d \) is the arrow \( y_D(d) \to \text{Hom}_C(F(-), F(d)) \) corresponding to the element \( 1_{F(d)} \) of \( \text{Hom}_C(F(d), F(d)) \) via the Yoneda embedding. So by Lemma 2.1(i) \( \chi_d \) is a monomorphism if and only if for every \( x, x' : e \to d \) such that \( F(x) = F(x') \), \( x \equiv_K x' \), that is, if and only if \( F \) is \( K \)-faithful. \( \square \)
Proposition 6.11. The geometric morphism $L_F : \text{Sh}(D, K) \to \text{Sh}(C, J)$ induced by a comorphism of sites $F : (D, K) \to (C, J)$ is localic if and only if for every object $d$ in $D$ there exist a $K$-covering sieve $\{g_i : d_i \to d \mid i \in I\}$ on $d$ and relations $R_i$ from $\text{Hom}_C(F(-), F(d_i))$ to $\text{Hom}_D(-, d)$ satisfying the conditions of Lemma 6.10.

In particular, if $F$ is $K$-faithful then the geometric morphism $L_F$ is localic.

Proof The proof uses Lemma 6.10 in a way analogous to that in which Lemma 6.5 is used in the proof of Proposition 6.6. In fact, stating that, for each $d \in D$, there is a set $\{c_j\}$ of objects of $C$ and an epimorphic family of arrows $\{\xi_j : a_K(S_j) \to l'(d)\}$, where each $S_j$ is a $K$-closed sieve on $c_j$ yielding a canonical subobject $s_j : a_K(S_j) \hookrightarrow A_F(c_j)$, amounts to saying that there are a $K$-covering family $\{g_i : d_i \to d \mid i \in I\}$ and for each $i$ an element $j_i$, and an arrow $u_{j_i} : F(d_i) \to c_{j_i}$ in $S_{j_i}$ such that $(u_{j_i}, g_i) \in R_{j_i}$ (where $R_{j_i}$ is the functional relation corresponding to $\xi_{j_i}$). For each $i \in I$, by considering the pullback

$$
\begin{array}{ccc}
a_K(S_{j_i}) & \xrightarrow{s_{j_i}} & A_F(F(d_i)) \\
\downarrow & & \downarrow A_F(u_{j_i}) \\
a_K(S_{j_i}) & \xrightarrow{s_{j_i}} & A_F(c_{j_i})
\end{array}
$$

of each subobject $s_{j_i} : a_K(S_{j_i}) \hookrightarrow A_F(c_{j_i})$ along $A_F(u_{j_i})$, we obtain a subobject $s'_{j_i} : a_K(S'_{j_i}) \hookrightarrow A_F(F(d_i))$, where $S'_{j_i}$ is $K$-closed sieve on $F(d_i)$, through which $\chi_{d_i}$ factors (this follows from the universal property of the above pullback, since $s_{j_i} \circ \tilde{u}_{j_i} = A_F(u_{j_i}) \circ \chi_{d_i}$, where $\tilde{u}_{j_i}$ is the arrow $l'(d_i) \to a_K(S_{j_i})$ corresponding to the element $u_{j_i}$ of $S_{j_i}$) and for which the condition $(u_{j_i}, g_i) \in R_{j_i}$ can be reformulated as the condition $\xi_{j_i} \circ s'_{j_i} \circ \chi_{d_i} = l'(g_i)$, where $\chi_{d_i}$ denotes the factorization of $\chi_{d_i}$ through $s'_{j_i} : a_K(S'_{j_i}) \hookrightarrow A_F(F(d_i))$. $\square$

6.4 Hyperconnected morphisms

Proposition 6.12. The geometric morphism $L_F : \text{Sh}(D, K) \to \text{Sh}(C, J)$ induced by a comorphism of sites $F : (D, K) \to (C, J)$ is hyperconnected if and only if $F$ satisfies the property of Proposition 6.1 and for every object $c$ of $C$ and any set $A$ of arrows of the form $x : F(d) \to c$ (for an object $d$ of $D$) which is functorial (in the sense that if $x \in A$ then $x \circ F(g) \in A$ for any arrow $g : d' \to d$ in $D$) and $K$-closed (in the sense that for any $K$-covering sieve $T$ on $d$, if $x \circ F(t) \in A$ for every $t \in T$ then $x \in A$) there exists a (J-closed) sieve $S$ on $c$ such that

$$A = \{x : F(d) \to c \mid d \in D, \{t : \text{dom}(t) \to d \mid x \circ F(t) \in S\} \in K(d)\}.$$
If particular, if $F$ is $K$-full this latter condition is satisfied.

**Proof** The first part of this proposition follows immediately from Proposition 5.22 in light of Remark 2.4(a), so it remains to prove the second part. Given a set $A$ of arrows $F(d) \to c$ (for $d \in D$) satisfying the property in Proposition 6.12 let us define $S$ to be the sieve on $c$ generated by it. We want to prove that $A = \{ x : F(d) \to c \mid d \in D \text{ and } \{ t : \text{dom}(t) \to d \mid x \circ F(t) \in S \} \in K(d) \}$. The inclusion $\subseteq$ is clear since $A \subseteq S$. To prove the converse one, since $A$ is functorial and $K$-closed, it clearly suffices to show that for any arrow $x : F(d) \to c$, if $x \in S$ then $x \in A$. If $x \in S$ then, by definition of $S$, there exists an arrow $y : F(d') \to c$ in $A$ and an arrow $\xi : F(d) \to F(d')$ such that $x = y \circ \xi$. Now, since $F$ is $K$-full, there exist a $K$-covering family $\{ h : \text{dom}(h) \to d \mid h \in R \}$ on $d$ and for each $h \in R$ an arrow $s_h : \text{dom}(h) \to d'$ such that $\xi \circ F(h) = F(s_h)$ for each $h \in R$. Since $A$ is $K$-closed, to show that $x \in A$ it suffices to show that $x \circ F(h) \in A$ for each $h \in R$. But $x \circ F(h) = y \circ \xi \circ F(h) = y \circ F(s_h)$, which belongs to $A$ since $A$ is functorial. □

**Corollary 6.13.** Let $f : D \to C$ be a functor between essentially small categories. Then the geometric morphism $L_F : [D^{op}, \text{Set}] \to [C^{op}, \text{Set}]$ is hyperconnected if and only if $F$ is full and every object of $D$ is a retract of an object in the image of $F$.

**Proof** By Proposition 6.12 and Remark 6.2(b), $L_F$ is hyperconnected if and only if every object of $D$ is a retract of an object in the image of $F$ and for every functorial set $A$ of arrows $x : F(d) \to c$ there exists a sieve $S$ on $c$ such that for any arrow $x : F(d) \to c$ in $D$, $x \in A$ if and only if $x \in S$. Let us prove that this latter condition is satisfied if and only if $F$ is full. If $F$ is full, then we can get the condition satisfied by defining $S$ to be the sieve on $c$ generated by the arrows in $A$. Indeed, if $x \in A$ then $x \in S$ since $A \subseteq S$. Conversely, suppose that $x : F(d) \to c$ is in $S$; then we can write $x = y \circ g$, where $y : F(d') \to c$ is in $A$. But by the fullness of $F$, $g = F(\xi)$ for some arrow $\xi : d \to d'$ in $D$ and hence $x = y \circ F(\xi)$ in $A$ by the functoriality of $A$. Conversely, supposing that $F$ satisfies the condition, we want to prove that it is full. Given $d \in D$, let $A_d$ be the collection of all the arrows of the form $F(\xi)$ for an arrow $\xi$ of $D$ with codomain $d$. If $A_d$ satisfies the condition then there is a sieve $S_d$ on $F(d)$ such that an arrow $F(d') \to F(d)$ lies in $S_d$ if and only if it lies in $A_d$. It follows that the identity $1_{F(d)}$ lies in $S_d$, so every arrow $F(d') \to F(d)$ in $C$ (where $d'$ is an object of $D$) lies in $S_d$ and hence in $A_d$; that is, it is of the form $F(\xi)$ for some $\xi$. □
Remark 6.14. Corollary 6.13 generalizes and strengthens Example A4.6.9 [4], where it is observed that if $F$ is bijective on objects and full then the geometric morphism $L_F$ is hyperconnected.

The following result describes the hyperconnected-localic factorization of the geometric morphism induced by a cover-preserving comorphism of sites; it generalizes the well-known result for presheaf toposes (Example A4.6.9 [4]).

Proposition 6.15. Let $F : (D, K) \to (C, J)$ be a comorphism of sites which is cover-preserving. Then the hyperconnected-localic factorization of the geometric morphism $L_F : \text{Sh}(D, K) \to \text{Sh}(C, J)$ induced by $F$ can be identified with $L_{\tilde{F}} \circ L_{\pi}$, where $\tilde{F}$ is the functor $F$ regarded as a comorphism of sites from the site $(E, L)$ whose underlying category $E$ is the quotient of the category $D$ by the congruence induced by $F$ and whose Grothendieck topology $L$ has as covering sieves the sieves whose inverse image under the canonical projection functor $\pi : D \to E$ is $K$-covering.

Proof Since the functor $\pi$ is cover-preserving and full on objects and arrows, by Propositions 6.12 and 6.1, $L_{\pi}$ is hyperconnected. On the other hand, since $\tilde{F}$ is faithful then $L_{\tilde{F}}$ is localic by Proposition 6.11. Since we clearly have $F = \tilde{F} \circ \pi$, our thesis follows. □

6.5 Equivalences of toposes

Finally, we can deduce from Propositions 6.11 and 6.12 a criterion for a comorphism of sites to induce an equivalence of toposes (by observing that a geometric morphism is an equivalence if and only if it is both localic and hyperconnected):

Proposition 6.16. The geometric morphism $L_F : \text{Sh}(D, K) \to \text{Sh}(C, J)$ induced by a comorphism of sites $F : (D, K) \to (C, J)$ is an equivalence if and only if the following conditions are satisfied:

(i) for every object $d$ of $D$ there exist a $K$-covering sieve $\{g_i : d_i \to d \mid i \in I\}$ on $d$ and relations $R_i$ from $\text{Hom}_C(F(-), F(d_i))$ to $\text{Hom}_D(-, d)$ satisfying all the conditions of Lemma 6.5 but condition (iv);

(ii) $J = K^F$, that is, if a sieve $S$ on an object $c \in C$ satisfies the property that for every object $d$ of $D$ and arrow $x : F(d) \to c$ there exists a $K$-covering sieve $T$ on $d$ such that for every $g \in T$, $x \circ F(g) \in S$ then $S$ is $J$-covering;
(iii) for every object \(c\) of \(\mathcal{C}\) and any set \(A\) of arrows of the form \(x : F(d) \to c\) for an object \(d\) of \(\mathcal{D}\) which is functorial (in the sense that if \(x \in A\) then \(x \circ F(g) \in A\) for any arrow \(g : d' \to d\) in \(\mathcal{D}\)) and \(K\)-closed (in the sense that for any \(K\)-covering sieve \(T\) on \(d\), if \(x \circ F(t) \in A\) for every \(t \in T\) then \(x \in A\)) there exists a (\(J\)-closed) sieve \(S\) on \(c\) such that

\[
A = \{x : F(d) \to c \mid d \in \mathcal{D}, \{t : \text{dom}(t) \to d \mid x \circ F(t) \in S\} \in K(d)\}.
\]

In particular, if \(F\) is \(K\)-full and \(K\)-faithful and \(J\)-dense then \(L_F\) is an equivalence.

\[\square\]

The following result provides a criterion for a functor which is both a morphism and a comorphism of sites to induce an equivalence.

**Proposition 6.17.** Let \(F : \mathcal{D} \to \mathcal{C}\) be a functor which is both a comorphism of sites and a morphism of sites \((\mathcal{D}, K) \to (\mathcal{C}, J)\), where \(J\) and \(K\) are Grothendieck topologies respectively on \(\mathcal{C}\) and \(\mathcal{D}\). Then the following conditions are equivalent:

(i) The geometric morphism \(L_F\) is an equivalence with quasi-inverse \(\text{Sh}(F)\).

(ii) \(F\) is \(K\)-full and \(J\)-dense.

**Proof** (i) \(\Rightarrow\) (ii) If \(F\), as a morphism of sites \((\mathcal{D}, K) \to (\mathcal{C}, J)\), induces an equivalence then, since \(F\) has the covering-lifting property, by Corollary 4.20 \(F\) is \(K\)-full and \(J\)-dense.

(ii) \(\Rightarrow\) (i) Since the construction of the geometric morphism induced by a comorphism of sites is functorial, and any canonical geometric inclusion \(\text{Sh}(\mathcal{A}, Z) \hookrightarrow [\mathcal{A}^{\text{op}}, \text{Set}]\) is induced by the identity functor on \(\mathcal{A}\), regarded as comorphism of sites \((\mathcal{A}, Z) \to (\mathcal{A}, T)\) where \(T\) is the trivial topology on \(\mathcal{A}\), we have a commutative square

\[
\begin{array}{ccc}
\text{Sh}(\mathcal{D}, K) & \xrightarrow{L_F} & \text{Sh}(\mathcal{C}, J) \\
[\mathcal{D}^{\text{op}}, \text{Set}] & \xrightarrow{[F^{\text{op}}, \text{Set}]} & [\mathcal{C}^{\text{op}}, \text{Set}]
\end{array}
\]

where the vertical arrows are the canonical geometric inclusions. Let \(\text{Sh}(F) : \text{Sh}(\mathcal{C}, J) \to \text{Sh}(\mathcal{D}, K)\) be the geometric morphism induced by \(F\) as a morphism of sites \((\mathcal{D}, K) \to (\mathcal{C}, J)\). Since \(F\) is \(J\)-dense and has the covering-lifting property, by Proposition 5.7 \(J = K_F\) and hence, by Lemma 5.13 \(F\)
is cover-reflecting. So $F$ is a dense morphism of sites (cf. Remark 4.2) and hence $\text{Sh}(F)$ is an equivalence. To prove that $g$ is a quasi-inverse to $L_F$, it is clearly enough to show that $L^*_F \circ \text{Sh}(F)^* \circ l_D \cong l_D$. Now, $\text{Sh}(F)^* \circ l_D \cong l_C \circ F$, so $L^*_F \circ \text{Sh}(F)^* \circ l_D \cong L^*_F \circ \text{Sh}(F^\text{op}) \circ \text{Set}^* \circ l_D \circ F$. But $L^*_F \circ l_C \cong L^*_F \circ a_J \circ y_C \cong a_K \circ [\text{F}^\text{op}, \text{Set}]^* \circ y_C$, where the second isomorphism follows from the commutativity of the above square. Now, By Proposition 4.16 since $F$ is $K$-full and $K$-faithful, the canonical arrow $l_D \cong a_K \circ y_D \to a_K \circ E(F)^* \circ y_C \circ F$ is an isomorphism, so $L^*_F \circ g^* \circ l_D \cong l_D$, as required. 

Acknowledgements: We gratefully acknowledge MIUR for the support in the form of a “Rita Levi Montalcini” position, and IHÉS for a visiting position during which part of this work has been written. We also thank Laurent Lafforgue for his careful reading of a preliminary version of this text.

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Olivia Caramello
Dipartimento di Scienza e Alta Tecnologia, Università degli Studi dell’Insubria, via Valleggio 11, 22100 Como, Italy.
E-mail address: olivia.caramello@uninsubria.it