Efficient Estimation and Control of Unknown Stochastic Differential Equations

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Abstract

Ito stochastic differential equations are ubiquitous models for dynamic environments. A canonical problem in this setting is that of decision-making policies for systems that evolve according to unknown diffusion processes. The goals consist of design and analysis of efficient policies for both minimizing quadratic cost functions of states and actions, as well as accurate estimation of underlying linear dynamics. Despite recent advances in statistical decision theory, little is known about estimation and control of diffusion processes, which is the subject of this work. A fundamental challenge is that the policy needs to continuously address the exploration-exploitation dilemma; estimation accuracy is necessary for optimal decision-making, while sub-optimal actions are required for obtaining accurate estimates.

We present an easy-to-implement reinforcement learning algorithm and establish theoretical performance guarantees showing that it efficiently addresses the above dilemma. In fact, the proposed algorithm learns the true diffusion process and optimal actions fast, such that the per-unit-time increase in cost decays with the square-root rate as time grows. Further, we present tight results for assuring system stability and for specifying fundamental limits of sub-optimalities caused by uncertainties. To obtain the results, multiple novel methods are developed for analysis of matrix perturbations, for studying comparative ratios of stochastic integrals and spectral properties of random matrices, and the new framework of policy differentiation is proposed.

1 Introduction

State-space models are ubiquitous for studying decision-making problems in dynamic environments. A popular setting consists of continuous-time stochastic multidimensional linear dynamical systems that are equipped with control actions and are disturbed by Brownian noise. In this setting, the control action at every time instant, together with the state at the time, determines the state evolution in continuous-time according to an Ito stochastic differential equation. The range of applications is extensive, including chemistry [1], biology [2], finance [3], insurance [4], and engineering [5].

In many applications, uncertainties about the true dynamical models necessitate statistical methods for estimating the unknown dynamics so that based on that, optimal decisions can be made. For applying such estimation procedures, adaptive control policies and reinforcement learning algorithms are studied for decision-making under uncertainty in discrete-time settings. The existing literature for discrete-time systems is fairly rich, including efficiency of algorithms based on optimism in the face of uncertainty [6, 7], posterior sampling [8, 9], and applying indigenous or exogenous randomizations [10, 11, 12]. Further, important limits are considered [13, 14, 15], and bounds for regret are available in the presence of domain knowledge [16, 17], partial observation [18], and multi-agent systems [19].

However, little is known about design and analysis of reinforcement learning algorithms for continuous-time stochastic systems, which is adopted by this work. Early works on consistency of estimating parameters [20, 21] provide technical results for adaptive control in a pure asymptotic sense that the time of interacting with the system is infinite [22, 23, 24]. However, implementation and computational tractability are not considered for the proposed methods, and the theoretical results rely on infinite-time system identification under significant technical assumptions. Indeed, efficiency of policies for addressing the exploration-exploitation trade-off as well as fast and effective parameter estimation are not studied.

\[1\text{the cumulative increase in cost, due to uncertainties}\]
Further existing works study special cases that accurate estimation is not needed, such as deterministic systems [25, 26]. Recently, sub-optimal policies that incur linear regret are proposed based on alternating control actions, and infinite-time consistency is shown under full-rankness of the true input matrix [27]. Ensuing papers focus on the problem of learning optimal policies in an offline fashion (i.e., from multiple trajectories), for univariate [28] or full-rank systems [29]. It is shown that if the time of interacting with the system is short and the estimation procedure can be repeated for arbitrarily many times, then, the average-case sub-optimality decays with the number of repetitions. Overall, the available frameworks are fairly restrictive and inapplicable to online or single-trajectory reinforcement learning.

This work provides the first comprehensive study on estimation and control of systems whose dynamics are governed by linear stochastic differential equations and their operating costs are quadratic. The main contributions of this paper can be summarized as follows. In Algorithm 1, we propose the first efficient policy for adaptive control of Ito stochastic differential equations and establish the rates for both learning the unknown dynamics matrices, as well as scaling of the regret. Importantly, Algorithm 1 has favorable time (i.e., computational) and space (i.e., memory) complexities, yet, it learns the optimal actions fast so that its regret at time $T$ is $O\left(\sqrt{T \log T}\right)$ (Theorem 5.3). In other words, the sub-optimality gap caused by uncertainties about the diffusion process is $O\left(\frac{1}{T^{1/2}} \log T\right)$, which indicates efficiency.

Furthermore, we study stability of dynamical systems for precluding explosions in the state trajectories due to inaccurate approximations of the true dynamics matrices. In this regard, we present the stabilizability margin of the system and establish the minimal information to ensure that the system operates stable (Theorem 3.2). Finally, a general analysis that fully captures sub-optimalities that a decision-maker needs to incur (due to uncertainty or inaccuracy in approximating the parameters) is discussed, and a reciprocal quantification for the regret is provided (Theorem 3.1). Technically, the presented result specifies the regret from the viewpoint of both average- and worst-case analyses. Note that the presented results are tight in the sense that the provided bounds can be achieved in some nontrivial settings. The technical assumptions adopted in this work are minimal, letting the presented results be applicable to an extensive class of continuous-time linear-quadratic systems.

For studying reinforcement learning algorithms in Ito stochastic differential equations, one needs to address several challenges. First, we need to study sensitivity of complex-valued eigenvalues of matrices to perturbations in the entries. Further requirements include showing anti-concentration bounds for singular values of slightly-random matrices, and characterizing effects of sub-optimal actions in terms of model uncertainties. Accordingly, to establish the presented results, we develop multiple novel frameworks for matrix-perturbation analysis, and for studying comparative ratios of stochastic integrals as well as spectral properties of random matrices. We also propose the new method of policy differentiation to precisely quantify the additional cost imposed by sub-optimal control actions. In addition, different tools from stochastic control, Ito calculus, and stochastic analysis [5, 30, 31] are employed, including Hamilton-Jacobi-Bellman equations, Ito isometry, as well as dominated and martingale convergence theorems.

This paper is organized as follows. In Section 2 we discuss the problem together with the preliminary materials. Then, in Section 3 we study system stability when the action is taken based on dynamics matrices other than the true ones, and establish stabilizability guarantees. Next, effects of sub-optimal actions and the regret they impose are examined in Section 4. Section 5 contains the estimation and control procedure we propose in Algorithm 1 as well as the theoretical analysis showing efficiency. The paper is concluded in Section 5 followed by the auxiliary results.

The following notation will be used in this work. The smallest (largest) eigenvalue of $A$, in magnitude, is denoted by $\lambda_{\min}(A)$ ($\lambda_{\max}(A)$). For $v \in \mathbb{C}^d$, the norm $\|v\|$ is defined as $\|v\|^2 = \sum_{i=1}^{d} |v_i|^2$. Moreover, we write $\|A\|$ for the operator norm of matrices; $\|A\| = \sup_{\|v\|=1} \|Av\|$, and $A^\dagger$ for Moore-Penrose generalized inverse. The sigma-field generated by the stochastic process $\{Y_s\}_{0 \leq s \leq t}$ is denoted by $\sigma(Y_{0:t})$. A multivariate normal distribution with mean $\mu$ and covariance matrix $\Sigma$ is shown by $\mathcal{N}((\mu, \Sigma))$. For $\lambda \in \mathbb{C}$, we use $\Re(\lambda), \Im(\lambda)$ to denote the real and imaginary parts of $\lambda$, respectively. The symbol $\lor$ (resp., $\land$) is used to show the maximum (resp., minimum) of two quantities. Finally, $O(\cdot)$ is used to refer to order of magnitudes.
2 Problem Statement

We study reinforcement learning algorithms for simultaneous estimation and control of a multidimensional Ito stochastic differential equation with unknown drift matrices. That is, the state vector at time $t$ is $X_t \in \mathbb{R}^{dx}$, which follows

$$dX_t = (A_sX_t + B_sU_t)dt + CdW_t.$$  

(1)

The vector $U_t \in \mathbb{R}^{du}$ is the control action at time $t$, and the exogenous disturbance $\{W_t\}_{t \geq 0}$ is a standard Brownian motion in a $dw$ dimensional Euclidean space. Technically, by fixing the probability space $(\Omega, \mathbb{F}, \mathbb{P})$, which is equipped with the null-sets of $\mathbb{P}$, let all stochastic objects belong to this probability space, and let $\mathbb{E} [\cdot]$ be expectation with respect to $\mathbb{P}$ (unless otherwise explicitly mentioned). The Brownian motion $\{W_t\}_{t \geq 0}$ start from the origin: $W_0 = 0$, and $W_t$ has independent normally distributed increments. That is, for all

$0 \leq t_1 < t_2 < t_3$, the vectors $W_{t_2} - W_{t_1}$ and $W_{t_3} - W_{t_2}$ are statistically independent, and for all non-negative reals $s < t$, it holds that $W_t - W_s \sim \mathcal{N}(0, (t-s)I_{dw})$. Furthermore, $C \in \mathbb{R}^{dx \times dw}$ reflects the effect of the Brownian noise $W_t$ on the state evolution.

We aim to design computationally tractable and provably efficient algorithms for learning to control the system in $\Pi$. That is, the transition matrix $A_s$, the input matrix $B_s$, and the noise-coefficient matrix $C$, all are unknown. The goal is to provide decision-making algorithms for minimizing the expected average cost

$$J(\pi) = \limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T c_\pi(X_t, U_t)dt \right],$$

where $c_\pi(X_t, U_t) = X_t^TQX_t + U_t^TRU_t$, is the quadratic instantaneous cost at time $t$. The value of $c_\pi(X_t, U_t)$ is determined by the positive definite cost matrices $Q, R$ of proper dimensions and the policy $\pi$ defined below.

Technically, $\pi$ is a non-anticipative closed-loop policy: At every time, $\pi$ decides about the control action $U_t \in \mathbb{R}^{du}$, according to the information available at the time. More precisely, $\pi$ maps the state observations up to time, $\{X_s\}_{0 \leq s \leq t}$, together with the previously taken actions, $\{U_s\}_{0 \leq s \leq t}$, to the current control action $U_t$. This mapping can be either stochastic or deterministic. Importantly, $\pi$ faces a fundamental challenge for minimizing the expected average cost because the dynamics matrices $A_s, B_s$ are unknown, and so need to be learned based on the state and action observations. The details of the exploration-exploitation dilemma for the above-mentioned continuous-time linear-quadratic systems will be discussed in Section [5]. We assume that $Q, R$ are known to the reinforcement learning algorithm, for which the rationale is that the decision-maker is aware of the objective he/she would like to achieve, but the uncertainty about the true dynamics impedes him/her from performing optimal decisions.

The benchmark for assessing the algorithms is the optimal policy $\pi^*$ which designs its optimal action $U_t$ using the true dynamical model $A_s, B_s$. In the sequel, we examine the effect of uncertainties about the true dynamics $A_s, B_s$ and the increase in cost compared to the cost of $\pi^*$. To that end, define the function $\Phi_{A_s, B_s}(\cdot) : \mathbb{R}^{dx \times dx} \to \mathbb{R}^{dx \times dx}$:

$$\Phi_{A_s, B_s}(M) = A_s^TM + MA_s - MB_sR^{-1}B_s^TM + Q.$$  

Then, let $K(A_s, B_s)$ solve the equation $\Phi_{A_s, B_s}(M) = 0$. In order to discuss existence and computation of $K(A_s, B_s)$, we need some the following notations.

Definition 2.1 (Notations $\overline{\lambda}(\cdot), \underline{\lambda}(\cdot), \mathcal{E}(\cdot)$). Let $\overline{\lambda}(M)$ be the largest real-part among those of the eigenvalues of the matrix $M$: $\overline{\lambda}(M) = \max\{\Re(\lambda) : \det(M - \lambda I) = 0\}$. Similarly, let $\underline{\lambda}(M)$ denote the smallest real-part of the eigenvalues. Finally, for arbitrary matrices $\hat{A} \in \mathbb{R}^{dx \times dx}, \hat{B} \in \mathbb{R}^{dx \times du}$, define $\mathcal{E}(\hat{A}, \hat{B}) = \|\hat{A} - A_s\| + \|\hat{B} - B_s\|$. So, $\mathcal{E}(\hat{A}, \hat{B})$ measures the deviation of $\hat{A}, \hat{B}$ from the true dynamics matrices $A_s, B_s$ in $\Pi$.

Note that unlike $\lambda_{\min}(\cdot), \lambda_{\max}(\cdot)$, which consider only magnitudes of the eigenvalues, $\underline{\lambda}(\cdot), \overline{\lambda}(\cdot)$ reflect the signs of the eigenvalues too, and so can be either positive, zero, or negative. However, they are related as follows: $\lambda_{\max}(e^M) = e^{\overline{\lambda}(M)}, \lambda_{\min}(e^M) = e^{\underline{\lambda}(M)}$. To proceed, assume that the true dynamics matrices $A_s, B_s$ are stabilizable:

Assumption 2.2 (stabilizability). There is $L \in \mathbb{R}^{du \times dx}$ such that $\overline{\lambda}(A_s + B_sL) < 0.$
Intuitively, Assumption 2.2 expresses that if one naively closes the loop by applying the linear feedback $U_t = LX_t$, the system can operate for a long period. That is because it gives

$$X_t = e^{(A_t + B_t) L} t \left( X_0 + \int_0^t e^{-(A_s + B_s) L} s C dW_s \right), \quad (2)$$

which by Assumption 2.2 does not grow unbounded. Importantly, existence of a stabilizer matrix $L$ is necessary for the problem to be well-defined; otherwise, the growing state vectors render the average cost infinite for all decision-making policies.

We show that Assumption 2.2 suffices for unique existence of $K(A_s, B_s)$, and discuss a method to explicitly find $K(A_s, B_s)$. In addition, we establish that an optimal policy is

$$\pi^*: \quad U_t = L(A_s, B_s)X_t, \quad \text{where} \quad L(A_s, B_s) = -R^{-1}B_s^T K(A_s, B_s). \quad (3)$$

These predicates are formalized in the following result.

**Theorem 2.3 (Optimal policy).** The matrix $K(A_s, B_s)$ uniquely exists, and $\pi^*$ in (3) gives $\inf \mathcal{J}(\pi) = \mathcal{J}(\pi^*) = \text{tr} \left( K(A_s, B_s)CC^T \right)$ and $\mathcal{X}(A_s + B_s L(A_s, B_s)) < 0$.

To compute $K(A_s, B_s)$, it suffices to solve the differential equation $\dot{M} = \Phi_{A_s, B_s}(M)$, starting from a positive semidefinite $M_0$. Note that it is equivalent to the integration $M_t = M_0 + \int_0^t \Phi_{A_s, B_s}(M_s)ds$. In the proof of Theorem 2.3 we show that $\lim_{t \to \infty} M_t = K(A_s, B_s)$.

Next, we formulate sub-optimality in the performance of decision-making policies and the increase in cost due to lack of access to the optimal actions $U_t = L(A_s, B_s)X_t$, caused by uncertainties about the truth $A_s, B_s$. For a general policy $\pi$, regret of $\pi$ at time $T$ is the cumulative increase in the cost up to time $T$. That is, the difference between the instantaneous cost of $\pi$ and that of the optimal policy $\pi^*$ in (3) is aggregated over the period $0$ to $T$:

$$\mathcal{R}_T(\pi) = \int_0^T \left( c_\pi(X_t, U_t) - c_{\pi^*}(X_t, U_t) \right) dt.$$ 

Clearly, randomness of state and action leads to that of regret. So, performance analyses of decision-making policies include worst-case analysis of establishing upper-bounds for $\mathcal{R}_T(\pi)$, as well as average-case one of showing growth rates for $\mathbb{E} [\mathcal{R}_T(\pi)]$. Further, for unknown $A_s, B_s$, we expect that the increasing observations of state and action over time will be effectively leveraged so that after a long time, the policy takes close-to-optimal actions. So, as $t$ grows, we expect $c_\pi(X_t, U_t) - c_{\pi^*}(X_t, U_t)$ to shrink. Accordingly, $\mathcal{R}_T(\pi)$ should scale sub-linearly with $T$ as $T$ grows. In the sequel, we establish results on $\mathcal{R}_T(\pi)$ and $\mathbb{E} [\mathcal{R}_T(\pi)]$, as well as their scaling with $T$.

Another quantity we study in this paper is the estimation accuracy of the unknown dynamics. So, letting $\hat{A}_t, \hat{B}_t$ be estimates of $A_s, B_s$ based on the state-action observations $\{X_s, U_s\}_{0 \leq s \leq t}$ by time $t$, we are interested in the decay rate of the estimation error $\mathcal{E} \left( \hat{A}_t, \hat{B}_t \right)$ in Definition 2.1. Similar to regret, $\mathcal{E} \left( \hat{A}_t, \hat{B}_t \right)$ can be stochastic as well.

### 3 Stability Analysis

Next, we study effects of uncertainties about the dynamical model, on stability of the system in (1). To that end, we specify the minimal information one needs to have access to, in order to ensure stabilization, and show that a coarse-grained approximation of the truth is sufficient for stabilizing the system. The results of this section will be used later in the design of the reinforcement learning algorithm of Section 5. Importantly, the analysis is general, captures effects of all the involved quantities, and provides to a tight result in the sense that the bound presented in Theorem 3.2 is required for guaranteeing stabilization. In addition, the results presented here are of independent interests, because stability is required for letting the system operate for a reasonable time period.
To proceed, first note that if hypothetically the optimal linear feedback in (3) is applied to the system in (1), then stability is guaranteed. Indeed, by applying $\mathcal{L}(A_*, B_*)$, the closed-loop transition matrix becomes $D* = A_* + B_*\mathcal{L}(A_*, B_*)$, which by Theorem 2.3 is a stable matrix. The issue is that when the true dynamics matrices $A_*, B_*$ are not known and need to be estimated based on the state-action trajectories. However, if matrices $\hat{A}, \hat{B}$ meet the technical conditions we shortly discuss, one can stabilize the system by applying the linear feedback $U_t = \mathcal{L}(\hat{A}, \hat{B})X_t$. Let $\rho > 0$ and $\zeta < \infty$ be such that

$$\mathcal{X}(\hat{D}) \leq -\rho, \quad \|\mathcal{K}(\hat{A}, \hat{B})\| \leq \zeta,$$  \tag{4}

where $\hat{D} = \hat{A} + \hat{B}\mathcal{L}(\hat{A}, \hat{B})$ is the closed-loop transition matrix of a system with dynamics matrices $\hat{A}, \hat{B}$. The conditions in (4) both are required for studying stability of the matrices $A_*, B_*\mathcal{L}(A_*, B_*)$, as elaborated in the sequel. The first condition quantifies the extent to which the linear feedback matrix $\mathcal{L}(\hat{A}, \hat{B})$ is able to stabilize a dynamical system of parameters $\hat{A}, \hat{B}$. Intuitively speaking, $\rho$ is the largest stability margin one can hope for, by applying $\mathcal{L}(\hat{A}, \hat{B})$ to the true system in (1), because $\mathcal{L}(\hat{A}, \hat{B})$ is purposefully designed for the dynamics matrices $\hat{A}, \hat{B}$ based on full certainty about them. Later on, we provide an explicit lower-bound for $\rho$. The second condition is somewhat guaranteed by the first one, since the following equation (which is proved in the proof of Theorem 2.3) holds true:

$$\mathcal{K}(\hat{A}, \hat{B}) = \int_0^\infty e^{\hat{D}^\top t} \left( Q + \mathcal{L}(\hat{A}, \hat{B})^\top R \mathcal{L}(\hat{A}, \hat{B}) \right) e^{\hat{D}t} dt.$$  \tag{5}

Thus, $\rho > 0$ implies $\zeta < \infty$, and the constant $\zeta$ is merely a notation for the sake of simplicity.

Towards the analysis of stability, we need further information about eigenvalues of $\hat{D} = \hat{A} + \hat{B}\mathcal{L}(\hat{A}, \hat{B})$, that Jordan form of this matrix provides. Suppose that eigenvalues of $\hat{D}$ are $\lambda_1, \ldots, \lambda_k$, and let the Jordan decomposition be $\hat{D} = P^{-1}\Lambda P$. So, $\Lambda = \text{diag}(\Lambda_1, \ldots, \Lambda_k)$ is a block-diagonal matrix, and all diagonal entries of $\Lambda_i$ are $\lambda_i$, the immediate off-diagonal entries above the diagonal of $\Lambda_i$ are 1, and all other entries of $\Lambda_i$ are 0, as shown in (1). Now, we define $\mu_{\hat{D}}$, which has a crucial effect for studying stability margins. It is based on the sizes of the above-mentioned blocks $\Lambda_i$, $i = 1, \ldots, k$, which we denote by $\mu_1, \ldots, \mu_k$.

**Definition 3.1** (Largest block-size $\mu_{\hat{D}}$). Letting $P$ and $\Lambda_i \in \mathbb{C}^{\mu_i \times \mu_i}$ be as in the Jordan decomposition $\hat{D} = P^{-1}\Lambda P$ explained above, define $\mu_{\hat{D}} = \max_{1 \leq i \leq k} \mu_i$.

The quantity $\mu_{\hat{D}}$, which corresponds to the largest size of the blocks $\Lambda_i$ in the Jordan decomposition, drastically determines the *order* of variations in eigenvalues. The following result establishes stability based on $\rho, \zeta, \mu_{\hat{D}}$.

**Theorem 3.2** (Stability margin). Using Definition 3.1 and $\rho, \zeta$ in (1), if

$$\mathcal{E}(\hat{A}, \hat{B}) < \left(1 \land \frac{\lambda_{\min}(R)}{\zeta \|\hat{B}\|} \right) \frac{(\rho - \delta) \wedge (\rho - \delta)^{\mu_{\hat{D}}}}{\mu_{\hat{D}}^{1/2}\|P\|\|P^{-1}\|},$$  \tag{6}

then, we have $\mathcal{X}(A_* + B_*\mathcal{L}(\hat{A}, \hat{B})) < -\delta$.

**Remark 3.3** (Sufficient condition). In the proof of Theorem 3.2 we provide $\epsilon_0, \rho, \zeta$, such that $\mathcal{E}(\hat{A}, \hat{B}) < \epsilon_0$ implies (4).

According to (2) if $\mathcal{E}(\hat{A}, \hat{B})$ is sufficiently small, then the matrix $A_* + B_*\mathcal{L}(\hat{A}, \hat{B})$ is stable and all of its eigenvalues lie in the open left half-plane of the complex plane. In addition, (3) reflects effects of different factors, as follows. First, $\lambda_{\min}(R)/\zeta\|\hat{B}\|$ is an upper-bound for $\|\mathcal{L}(\hat{A}, \hat{B})\|^{-1}$. The decrease of
\( \mathcal{E}(\hat{A}, \hat{B}) \) as \( \| \mathcal{L}(\hat{A}, \hat{B}) \| \) grows, is due to the fact that the difference \( B_* - \hat{B} \) is multiplied to \( \mathcal{L}(\hat{A}, \hat{B}) \) to become the difference between the corresponding closed-loop transition matrices. Moreover, as can be seen in Definition 3.1, \( \mu_D^{1/2} \| P \| \| P^{-1} \| \geq 1 \) quantifies non-diagonality of \( \hat{D} \), and it is 1 for diagonal matrices \( \hat{D} \).

The expression \( (\rho - \delta)\lambda(\rho - \delta)^{\mu} \) indicates that \( \mu \) determines the rates of bounding \( \lambda (A_* + B_* \mathcal{L}(\hat{A}, \hat{B})) \). The different rates for \( \rho - \delta < 1 \) and \( \rho - \delta > 1 \) are caused by a phenomena that appears in the sensitivity analysis of eigenvalues of a matrix to perturbations. In fact, in the proof of Theorem 3.2 [32], we present a generalized Bauer-Fike Theorem [32], which is of independent interests in the theory of matrix computations. We establish that larger blocks in the Jordan decomposition lead to remarkably larger fluctuations in eigenvalues of slightly different matrices. On the other hand, considering the dependence on the stability margins, according to (9), dynamical systems with smaller stability margins are more sensitive to model uncertainties and are harder to stabilize.

**Proof of Theorem 3.2.** First, we study eigenvalues of the sum of two matrices. Suppose that \( M, \Delta \) are arbitrary square matrices of the same size, and let \( M = P^{-1} \Delta P \) be the Jordan decomposition. That is, \( \Lambda \in \mathbb{C}^{d \times d} \) is a block diagonal matrix with blocks \( \Lambda_1, \cdots, \Lambda_k \), and

\[
\Lambda_i = \begin{bmatrix}
\lambda_i & 1 & 0 & \cdots & 0 & 0 \\
0 & \lambda_i & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \lambda_i & 1 \\
0 & 0 & \cdots & 0 & 0 & \lambda_i \\
\end{bmatrix} \in \mathbb{C}^{\mu_i \times \mu_i}.
\] (7)

Further, similar to Definition 3.1, let \( \mu_M = \max_{1 \leq i \leq k} \mu_i \). Then, it holds that

\[
\lambda_{\max}(M - \Delta) \leq \lambda_{\max}(M + \mu_M^{1/2} \| P \Delta P^{-1} \| \vee (\mu_M^{1/2} \| P \Delta P^{-1} \|)^{1/\mu_M}).
\] (8)

To show the above inequality, first let \( \lambda \) be an eigenvalue of \( M - \Delta \) that satisfies \( \Re(\lambda) > \lambda_{\max}(M) \). So, \( M - \Delta \) is an invertible matrix, and there exists at least one vector \( v \), such that \( v \neq 0 \) and \( (M - \Delta) P^{-1} v = 0 \). Then, \( (M - \Delta) P^{-1} = \Delta P^{-1} v \) implies that

\[
v = P (M - \Delta) P^{-1} \Delta P^{-1} v = (\Lambda - \mu I) P \Delta P^{-1} v.
\] (9)

Because \( \Lambda = \text{diag}(\Lambda_1, \cdots, \Lambda_k) \), the matrix \( \Lambda - \mu I \) is block diagonal as well, and we have \( (\Lambda - \mu I)^{-1} = \text{diag} \left( (\Lambda_1 - \mu_1 I)^{-1}, \cdots, (\Lambda_k - \mu_k I)^{-1} \right) \). On the other hand, it is straightforward to see that

\[
(\Lambda_i - \mu_1 I)^{-1} = \begin{bmatrix}
(\lambda - \lambda_1)^{-1} & (\lambda - \lambda_1)^{-2} & \cdots & (\lambda - \lambda_1)^{-\mu_1} \\
0 & (\lambda - \lambda_1)^{-1} & \cdots & (\lambda - \lambda_1)^{-\mu_1 + 1} \\
0 & \cdots & 0 & (\lambda - \lambda_1)^{-1}
\end{bmatrix}.
\]

Therefore, we have \( \| (\Lambda_i - \mu_1 I)^{-1} \| \leq \mu_1^{1/2} \| (\lambda - \lambda_1) \| \| (\lambda - \lambda_1)^{-\mu_1} \|^{-1} \). Using this bound for the operator norms of blocks of the block-diagonal matrix \( (\Lambda - \mu I)^{-1} \), since \( \mu_i \leq \mu_M \) and \( \Re(\lambda) > \lambda_{\max}(M) \), the equation in (9) implies that

\[
1 \leq \| (\Lambda - \mu I)^{-1} P \Delta P^{-1} \| \leq \| (\Lambda - \mu I)^{-1} \| \| P \Delta P^{-1} \|
\leq \mu_M^{1/2} \| P \Delta P^{-1} \| \left( (\Re(\lambda) - \lambda_{\max}(M)) \vee (\Re(\lambda) - \lambda_{\max}(M))^{\mu_M} \right)^{-1}.
\]

So, letting \( \lambda \) be an eigenvalue of \( M - \Delta \) that satisfies \( \Re(\lambda) = \lambda_{\max}(M - \Delta) \), we obtain (9).

Now, using (8), we compare \( A_* + B_* \mathcal{L}(\hat{A}, \hat{B}) \) and \( \tilde{D} = \hat{A} + \hat{B} \mathcal{L}(\hat{A}, \hat{B}) \). Since

\[
\Delta_* = A_* + B_* \mathcal{L}(\hat{A}, \hat{B}) - \tilde{D} = A_* - \hat{A} + (B_* - \hat{B}) R^{-1} \hat{B} K (\hat{A}, \hat{B}),
\] (10)
using (4), and letting \( M = \hat{D} \) in (8), we have

\[
\bar{X}(A_* + B_* \mathcal{L}(\hat{A}, \hat{B})) \leq -\rho + \mu^{1/2}_D \| P^{-1} \| P \| \Delta_* \| \vee \left( \mu^{1/2}_D \| P^{-1} \| P \| \Delta_* \| \right)^{1/\mu_D}.
\]

So, in order to have \( \bar{X}(A_* + B_* \mathcal{L}(\hat{A}, \hat{B})) < -\delta \), it suffices to show that

\[
\mu^{1/2}_D \| P^{-1} \| P \| \Delta_* \| < \rho - \delta, \quad \mu^{1/2}_D \| P^{-1} \| P \| \Delta_* \| < (\rho - \delta)^{\mu_D}.
\]  

(11)

However, since

\[
\| \Delta_* \| \leq \| A_* - \hat{A} \| + \| B_* - \hat{B} \| \| R^{-1} \| \| \hat{B} \| \mathcal{K}(\hat{A}, \hat{B}) \| \leq \mathcal{E}(\hat{A}, \hat{B}) \left( 1 \vee \| \hat{B} \| \zeta / \lambda_{\min}(R) \right),
\]

(6) provides the inequality in (11), which leads to the desired result.

Next, we specify \( \epsilon_0 \) such that \( \mathcal{E}(\hat{A}, \hat{B}) \leq \epsilon_0 \) is sufficient for (4). For this purpose, let \( D_* = A_* + B_* \mathcal{L}(A_* + B_*) = P_*^{-1} A_* P_* \) be the Jordan decomposition as defined in the beginning of the proof, and define the largest block size \( \mu_* = \mu_{D_*} \) similar to Definition 6.1. Further, suppose that the following constraints are satisfied:

\[
\epsilon_0 < \frac{1}{1 \vee \| \mathcal{L}(A_*, B_*) \|} \left( \frac{-\bar{X}(D_*) \wedge -\bar{X}(D_*)^{\mu_*}}{\mu^{1/2}_* \| P_*^{-1} \| P_* \|} \wedge \left[ 4 \int_0^\infty \| e^{D_* t} \|^2 \| \mathcal{L}(A_*, B_*) \| \right]^{-1} \right).
\]  

(12)

The inequality in (12) implies that if we write \( D_1 = \hat{A} + \hat{B} \mathcal{L}(A_*, B_*) = A_* + B_* \mathcal{L}(A_*, B_*) + \Delta_1 = D_* + \Delta_1 \), then, the matrix \( \Delta_1 = \hat{A} - A_* + (\hat{B} - B_*) \mathcal{L}(A_*, B_*) \) satisfies

\[
\| \Delta_1 \| \leq \frac{(-\bar{X}(D_*) \wedge -\bar{X}(D_*)^{\mu_*}}{\mu^{1/2}_* \| P_*^{-1} \| P_* \|}.
\]

So, taking \( M = D_* \), the bound in (6) implies that \( \bar{X}(D_1) < 0 \). Hence, we can employ Lemma 8.1 to study consequences of applying the linear feedback matrix \( \mathcal{L}(A_*, B_*) \) to a system of dynamics matrices \( \hat{A}, \hat{B} \), and get \( \mathcal{K}(\hat{A}, \hat{B}) \leq M \), where

\[
M = \mathcal{K}(\hat{A}, \hat{B}) + \int_0^\infty e^{D_1 t} \left[ \mathcal{L}(A_*, B_*) - \mathcal{L}(\hat{A}, \hat{B}) \right] R \left[ \mathcal{L}(A_*, B_*) - \mathcal{L}(\hat{A}, \hat{B}) \right]' e^{D_* t} dt,
\]

where we used the fact that the initial state \( X_0 = x \) in Lemma 8.1 is arbitrary, and so, the involved matrices are themselves equal. Further, similar to Lemma 8.1 it is straightforward to see that \( M = \int_0^\infty e^{D_1 t} \left[ Q + \mathcal{L}(A_*, B_*)' R \mathcal{L}(A_*, B_*) \right] e^{D_* t} dt \), which leads to

\[
Q + \mathcal{L}(A_*, B_*)' R \mathcal{L}(A_*, B_*) = -D_*^T M - M D_* = -D_*^T M - M D_* - \Delta_1^T M - M \Delta_1.
\]

Because \( \bar{X}(D_*) < 0 \), the latter equation provides

\[
M = \int_0^\infty e^{D_1 t} \left[ Q + \mathcal{L}(A_*, B_*)' R \mathcal{L}(A_*, B_*) + \Delta_1^T M + M \Delta_1 \right] e^{D_* t} dt \]

\[
= \int_0^\infty e^{D_1 t} \left[ Q + \mathcal{L}(A_*, B_*)' R \mathcal{L}(A_*, B_*) \right] e^{D_* t} dt + \int_0^\infty e^{D_1 t} \left[ \Delta_1^T M + M \Delta_1 \right] e^{D_* t} dt \]

\[
= \mathcal{K}(A_*, B_*) + \int_0^\infty e^{D_1 t} \left[ \Delta_1^T M + M \Delta_1 \right] e^{D_* t} dt,
\]

\[7\]
where in the last line above, we used (5). Therefore, it holds that \( \|M\| \leq \|K(A_*, B_*)\| + 2\|\Delta_1\|\|M\| \int_0^\infty \|e^{D_1 t}\|^2 dt \), which, according to (12) and \( K(\hat{A}, \hat{B}) \leq M \), yields to

\[
\|K(\hat{A}, \hat{B})\| \leq \|M\| \leq 2\|K(A_*, B_*)\|. \tag{13}
\]

To proceed, suppose that \( v \in C^{dx} \) satisfies \( \|v\| = 1 \) and \( \hat{D}v = \lambda v \). Now, (5) implies that

\[
v^*K(\hat{A}, \hat{B})v = \int_0^\infty \left\| Q + L(\hat{A}, \hat{B})^T RL(\hat{A}, \hat{B}) \right\|^{1/2} e^{\lambda t} v^2 dt,
\]

where \( \lambda^* \) is the transposed complex conjugate of \( \lambda \). Thus, maximizing the left-hand-side above while taking minimum on the right-hand-side, it holds that

\[
\|K(\hat{A}, \hat{B})\| \geq \lambda_{\text{min}}(Q) \int_0^\infty e^{2\Re(\lambda)t} dt \geq \lambda_{\text{min}}(Q) \frac{1}{2\Re(\lambda)}. \tag{14}
\]

Putting (13) and (14) together, we obtain \( \mathcal{X}(\hat{D}) \leq -\lambda_{\text{min}}(Q)(4\|K(A_*, B_*)\|)^{-1} \). Thus, the above result and (13) imply that \( \mathcal{E}(\hat{A}, \hat{B}) \leq \epsilon_0 \) is sufficient for satisfying (4), with \( \rho = \lambda_{\text{min}}(Q) 4^{-1}\|K(A_*, B_*)\|^{-1} \) and \( \zeta = 2\|K(A_*, B_*)\| \).

4 Sub-Optimality Analysis

In this section, we investigate sub-optimality and provide a tight expression for the regret a policy incurs due to taking actions other than the optimal one in (4). For reinforcement learning algorithms that need to estimate the unknown dynamics, sub-optimal actions originate from lack of knowledge about \( A_*, B_* \).

To proceed, let \( U_t \) be the control action of the policy \( \pi \) at time \( t \). In the following theorem, we quantify \( \mathcal{R}_T(\pi) \) in terms of deviations from the optimal feedback; \( U_t - \mathcal{L}(A_*, B_*) X_t \), and introduce \( \alpha_T \) that fully reflects the sub-optimality of policy \( \pi \). In fact, \( \alpha_T \) unifies the average- and worst-case analyses by capturing both \( \mathbb{E}[\mathcal{R}_T(\pi)] \) and \( \mathcal{R}_T(\pi) \). Further, Theorem 4.1 provides scalings with different parameters of the problem understudy, and show that the difference between \( \mathcal{R}_T(\pi) \) and \( \mathbb{E}[\mathcal{R}_T(\pi)] \) scales linearly with the dimension of the Brownian motion. After the statement of Theorem 4.1, we discuss its intuitions and applications, followed by its proof.

To establish Theorem 4.1, we utilize the theory of continuous-time martingales and develop new results on comparative ratios of stochastic integrals. More importantly, we introduce the novel framework of policy differentiation for accurately studying regrets of decision-making policies and finding tight bounds in this regard. Broadly speaking, policy differentiation consists of precise specification of infinitesimal sub-optimalities at all time instants. Then, we integrate these infinitesimal deviations and obtain \( \alpha_T \) so that the integrand \( \|R^{1/2}(L_t - \mathcal{L}(A_*, B_*)) X_t\|^2 \) plays a role similar to the derivative of regret. This framework is of independent interest in statistical decision theory and indicates new avenues for similar problems such as decision-making policies in non-linear (stochastic) differential equations.

**Theorem 4.1** (Regret analysis). Suppose that \( L_t \in \mathbb{R}^{d_U \times d_X} \) is a piecewise continuous function of \( t \), and \( \pi \) is the policy \( U_t = L_tX_t \). Then, we have \( \mathbb{E}[\mathcal{R}_T(\pi)] = \mathbb{E}[\alpha_T] \), and

\[
\mathcal{R}_T(\pi) = \alpha_T + O \left( \frac{\|B_*\|\|\mathbb{E}[K(A_*, B_*)]\|^{1/2}}{\|\lambda_{\text{min}}(Q)\|^{1/2}} d_W \right) \|D_* - A_* + B_* \mathcal{L}(A_*, B_*)\| e^{D_1 t},
\]

where \( \omega_W = \frac{\|B_*\|\|\mathbb{E}[K(A_*, B_*)]\|^{1/2}}{\|\lambda_{\text{min}}(Q)\|^{1/2}} d_W \), \( D_* = A_* + B_* \mathcal{L}(A_*, B_*) \), \( E_t = e^{D_1 t}, \mathcal{K}(A_*, B_*) e^{D_1 t}, \)

\[
\alpha_T = \int_0^T \left\| R^{1/2} (L_t - \mathcal{L}(A_*, B_*)) X_t \right\|^2 dt + 2 \int_0^T (X_t^T E_{T-t} B_* (L_t - \mathcal{L}(A_*, B_*)) X_t) dt.
\]

\(^2\)see Lemma 7.2
\(^3\)see Lemma 7.1
The piecewise continuity condition for $L_t$ is somewhat natural to ensure that the state evolution integral in (2) is well-defined. Note that because the optimal policy is a time-invariant feedback, violating piecewise continuity does not reduce the sub-optimality and regret. Furthermore, since by Theorem 2.3 we have $\bar{X}(D_\lambda) < 0$, the matrix $E_t$ exponentially decays as $t$ grows. Therefore, the second integral in the expression for $\alpha_T$ is always dominated by the first one. So, Theorem 4.1 shows that the sub-optimality $\pi$ incurs at time $t$ scales as square of the deviation $L_t - \mathcal{L}(A_\ast, B_\ast)$. The constant $\omega_\pi$ reflects effects of different problem parameters and indicates that the term $\mathcal{R}_T (\pi) - \alpha_T$ grows linearly with the dimension $d_W$.

The results of Theorem 4.1 are interesting along multiple directions. First, the exact equality $\mathbb{E} [\mathcal{R}_T (\pi)] = \mathbb{E} [\alpha_T]$ can be used for establishing lower-bounds for regrets of decision-making policies, by studying the fastest rates $\pi^\ast$ can be learned and the deviations $L_t - \mathcal{L}(A_\ast, B_\ast)$ shrinks. Further, since $\alpha_T^{1/2} \log \alpha_T = O (\alpha_T)$, not only the average-case criteria $\mathbb{E} [\mathcal{R}_T (\pi)]$, but also the worst-case sub-optimality $\mathcal{R}_T (\pi)$ are controlled by $\alpha_T$. Thus, in general, studying $\alpha_T$ is sufficient and necessary for regret analysis. In addition, Theorem 4.1 indicates that fluctuations of $\mathcal{R}_T (\pi)$ around its expectation $\mathbb{E} [\mathcal{R}_T (\pi)]$ are in magnitude smaller than the expected value itself. Therefore, it provides a general reciprocal relationship between the average-case and the worst-case performances, for all policies. Moreover, when $A_\ast, B_\ast$ are unknown, a reinforcement learning algorithm becomes progressively more capable for narrowing down the sub-optimality gap as time $t$ grows. More precisely, as soon as collecting enough observations to take actions satisfying $\| R^{1/2} (L_t - \mathcal{L}(A_\ast, B_\ast)) X_t \| < 1$, the regret grows much slower since $\alpha_T$ depends on squares of these deviations. For example, if the estimation accuracy satisfies the desired square-root rate $\mathbb{E} (\tilde{A}_t, \tilde{B}_t) = O (t^{1/2})$, then $\alpha_T = \log T$; i.e., the regret scales logarithmic with time. Unfortunately, due to the exploration-exploitation trade-off, this is not the case. Technically, to obtain the above error rate, $U_t$ needs to persistently deviate from $\mathcal{L}(A_\ast, B_\ast) X_t$, which causes a linearly growing regret. However, in the next section, we randomize the parameter estimates so that $U_t$ appropriately deviates from $\pi^\ast$, leading to $\mathbb{E} (\tilde{A}_t, \tilde{B}_t) = O (t^{-1/4})$.

Hence, we obtain $\alpha_T = O (T^{1/2})$, modulo a logarithmic factor, which is efficient. So, Theorem 4.1 provides both a general result for tightly analyzing reinforcement learning algorithms, as well as a useful insight on how to design them to minimize the regret. Later on, we leverage this framework to design Algorithm 1 and to establish Theorem 5.3.

Proof of Theorem 4.1. Throughout this proof, let $M = Q + \mathcal{L}(A_\ast, B_\ast)^\top R \mathcal{L}(A_\ast, B_\ast)$. For the policy $\pi$ that takes the action $U_t = L_t X_t$ at time $t$, and for a given $T$, suppose that $\epsilon > 0$ have a fixed small value. Then, let $N = \lceil T/\epsilon \rceil$, and define the sequence of policies $\{ \pi_i \}_{i=0}^N$ according to

$$\pi_i = \begin{cases} 
U_t = L_t X_t & t < i \epsilon \\
U_t = \mathcal{L}(A_\ast, B_\ast) X_t & t \geq i \epsilon.
\end{cases}$$

Note that as long as one concerns about times $t \leq T$, it holds that $\pi^\ast = \pi_0, \pi_N = \pi$. Clearly, since $\mathcal{R}_T (\pi_0) = 0$, it holds that $\mathcal{R}_T (\pi) = \sum_{i=0}^{N-1} (\mathcal{R}_T (\pi_{i+1}) - \mathcal{R}_T (\pi_i))$. Thus, using Lemma 7.1 we obtain $\mathcal{R}_T (\pi) = \sum_{i=0}^{N-1} (X_t^\top F_{ie} X_t \epsilon + 2X_t^\top g_{ie} \epsilon + \beta_{ie})$, where the matrix $F_{ie}$, the vector $g_{ie}$, and the scalar $\beta_{ie}$ are defined in (26), (27), and (28), respectively. Now, letting $\epsilon \to 0$, since $L_t$ is a piecewise continuous function of $t$, we have

$$\mathcal{R}_T (\pi) = \int_0^T \left( X_t^\top \bar{F}_t X_t + 2X_t^\top \bar{g}_t + \bar{\beta}_t \right) dt, \quad (15)$$

where $\bar{F}_t = \lim_{\epsilon \to 0, \epsilon \to t} \epsilon^{-1} F_{ie}$, $\bar{g}_t = \lim_{\epsilon \to 0, \epsilon \to t} \epsilon^{-1} g_{ie}$, and $\bar{\beta}_t = \lim_{\epsilon \to 0, \epsilon \to t} \epsilon^{-1} \beta_{ie}$. To calculate $\bar{F}_t, \bar{g}_t, \bar{\beta}_t$, using
Lemma 7.1 together with the piecewise continuity of \( L_t \), it is straightforward to see that \( \tilde{\beta}_t = 0 \), as well as

\[
\begin{align*}
\tilde{F}_t &= S_t + H_t^T \int_0^T e^{D_t^-(s-t)} Me^{D_s^-(s-t)} ds + \int_0^T e^{D_t^-(s-t)} M e^{D_s^-(s-t)} dH_t, \\
\tilde{g}_t &= \int_0^T \left( H_t^T e^{D_t^-(s-t)} M \int_t^s e^{D_s^-(s-u)} CdW_u \right) ds,
\end{align*}
\]

where \( D_* = A_* + B_* L_0, S_t = L_t^T R L_t - L_0 (A_* + B_*)^T R L_0 (A_* + B_*) \), and

\[
H_t = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ e^{(A_* + B_* L_t) \epsilon} - e^{(A_* + B_* L_0) \epsilon} \right] = B_* (L_t - L_0 (A_* + B_*)).
\]

by (5) and \( \int \int \int \int e^{D_t^-(s-t)} M e^{D_s^-(s-t)} ds \) is \( E_{T-t} \), the expression for \( \tilde{F}_t \) gives

\[
\tilde{F}_t = S_t + H_t^T K (A_* + B_*) + K (A_* + B_*) H_t - H_t^T E_{T-t} E_{T-t} H_t.
\]

So, after doing some algebra, we get

\[
S_t + H_t^T K (A_* + B_*) + K (A_* + B_*) H_t = (L_t - L_0 (A_* + B_*))^T R (L_t - L_0 (A_* + B_*)).
\]

Since \( W_u \) has independent increments, and in the expression for \( \tilde{g}_t \) we have \( u \geq t \), it gives

\[
E \left[ X_t^T \tilde{g}_t \right] = E \left[ E \left[ X_t^T \tilde{g}_t | \sigma (W_{0,t}) \right] \right] = E \left[ X_t^T E \left[ \tilde{g}_t | \sigma (W_{0,t}) \right] \right] = 0.
\]

Hence, (15), (16), (17), and Fubini’s Theorem imply that \( E [ R_T (\pi) ] = E [ \alpha_T ] \). To proceed towards establishing the second result, Stochastic Fubini Theorem [30, 31] leads to

\[
\int_0^T X_t^T \tilde{g}_t dt = \int_0^T \int_0^s X_t^T H_t^T e^{D_t^-(s-t)} Me^{D_s^-(s-u)} C dW_u ds dt = \int_0^T \int_0^u \left( X_t^T H_t^T e^{D_t^-(s-t)} Me^{D_s^-(s-u)} C \right) ds dt dW_u = \int_0^T Y_u^T dW_u,
\]

where, using the expression for \( H_t \), the vector \( Y_u \) can be written as

\[
Y_u^T = \int_0^u \left( X_t^T H_t^T e^{D_t^-(s-t)} Me^{D_s^-(s-u)} C \right) ds dt = \int_0^u \left( X_t^T (L_t - L_0 (A_* + B_*))^T P_{t,u}^T \right) dt,
\]

for \( P_{t,u}^T = \int_0^u e^{D_s^-(s-t)} Me^{D_s^-(s-u)} C dW_u \). Now, letting \( V_T = \int_0^T |Y_u|^2 du \), if \( V_T < 1 \), apply Ito Isometry [31], and if \( V_T \geq 1 \), then apply Lemma 7.2 to get

\[
\int_0^T Y_u^T dW_u = O \left( dW V_T^{1/2} \log^{1/2} V_T \right).
\]
However, by using the triangle inequality and Fubini’s Theorem, we obtain

\[ V_T \leq \int_0^T \int_0^T \| P_{t,u} (L_t - \mathcal{L}(A_s, B_s)) X_t \|^2 dt du \]

\[ = \int_0^T \left( X_t^T (L_t - \mathcal{L}(A_s, B_s))^T \left( \int_0^t P_{t,u} P_{t,u}^T du \right) (L_t - \mathcal{L}(A_s, B_s)) X_t \right) dt \]

\[ \leq \int_0^T \lambda_{\max} \left( \int_0^T R^{-1/2} P_{t,u} P_{t,u}^T R^{-1/2} du \right) \| R^{1/2} (L_t - \mathcal{L}(A_s, B_s)) X_t \|^2 dt. \]

The second part of the integrand above appears in \( \alpha_T \). So, we proceed by finding an upper-bound for the first part. For this purpose, we use the triangle inequality and (5) to obtain

\[ \lambda_{\max} \left( \int_0^T P_{t,u} P_{t,u}^T du \right) \leq \int_0^T \left\| B_s e^{D_t (s-u)} \right\|^2 \int_0^T \left\| e^{D_t (s-u)} Me^{D_t (s-u)} ds \right\|^2 \| C \|^2 du \]

\[ \leq \| B_s \|^2 \| K(A_s, B_s) \|^2 \| C \|^2 \int_0^\infty \left\| e^{D_t (u)} \right\|^2 du. \]

Therefore, by using (14), we get

\[ V_T \leq \frac{\| B_s \|^2 \| K(A_s, B_s) \|^2 \| C \|^2}{\lambda_{\min}(Q) \lambda_{\min}(R)} \int_0^T \| R^{1/2} (L_t - \mathcal{L}(A_s, B_s)) X_t \|^2 dt. \]

So, since \( E_t \) decays exponentially with \( t \), (19) gives the desired result. \( \square \)

## 5 Reinforcement Learning Algorithm

In this section, we discuss design and analysis of computationally tractable efficient algorithms for minimizing the average cost, subject to uncertainties about the dynamics matrices \( A_s, B_s \). First, we discuss the fundamental exploration-exploitation dilemma. Then, we investigate a procedure for estimating the unknown dynamics using the data of state-action trajectories. Based on that, a reinforcement learning algorithm that employs randomization of the parameter estimates for balancing exploration versus exploitation is presented in Algorithm 1. Then, a regret bound is established in Theorem 5.3 indicating that Algorithm efficiently minimizes the cost function so that the regret scales as square-root of the time of interacting with the system. We also specify the rates at which the estimation error decays.

First, according to Theorem 4.1, in order to incur a small regret, the policy needs to ensure that $U_t \approx \mathcal{L}(A_s, B_s) X_t$. Furthermore, since $A_s, B_s$ are unknown, at every time $t$ the policy needs to estimate them using the data \{\(X_s, U_s\)\}_{0 \leq s \leq t}, which is generated by (11). However, if $U_s \approx \mathcal{L}(A_s, B_s) X_s$, then some of the coordinates of the data point $X_s, U_s$ become (almost) uninformative as they are (approximately) a linear transformation of the rest of them. This renders accurate estimation of $A_s, B_s$ challenging, which defeats the purpose since (some) accurate approximations of $A_s, B_s$ are needed for designing an efficient control action according to (8). This, known as the exploration-exploitation trade-off, is the main obstacle for learning optimal actions, and depicts the fact that a low-regret policy needs to carefully diversify the actions \{\(U_s\)\}_{0 \leq s \leq t} by deviating from \{\(\mathcal{L}(A_s, B_s) X_s\)\}_{0 \leq s \leq t}.

Next, we discuss the learning procedure in Algorithm 1 based on the least-squares-estimator. Hypothetically, suppose that instead of the full state and action data \{\(X_s, U_s\)\}_{0 \leq s \leq t}, one has access to a sampled version at a discrete set of time points \{\(X_{ke}, U_{ke}\)\}_{k=0}^n. Then, as long as $\epsilon$ is small, we expect the data generation mechanism to be approximately

\[ X_{(k+1)e} - X_{ke} = (A_s X_{ke} + B_s U_{ke}) \epsilon + C (W_{(k+1)e} - W_{ke}). \]
So, a natural approach is to form a linear regression setting for estimating \( A_*, B_* \) by

\[
[\hat{A}_n, \hat{B}_n] \in \arg \min_{\hat{A}, \hat{B}} \sum_{k=0}^{n-1} \left\| X_{(k+1)}e - X_{ke} - \left( \hat{A}X_{ke} + \hat{B}U_{ke} \right) e \right\|^2.
\]

Solving this, we obtain

\[
[\hat{A}_n, \hat{B}_n] = \sum_{k=0}^{n-1} \left( X_{(k+1)}e - X_{ke} \right) Y_{ke}^\top e \left( \sum_{k=0}^{n-1} Y_{ke} Y_{ke}^\top e^2 \right)^{1/2},
\]

where \( Y_s = [X_s^\top, U_s^\top]^\top \).

Therefore, letting \( \epsilon \to 0 \), we get the continuous-time counterpart for the full trajectory \( \{X_s, U_s\}_{0 \leq s \leq t} \), and the summations in the above expression become integrations. The result, shown in (20) below, is used by Algorithm 1 as follows.

For some fixed \( \gamma > 1 \), the sequence of time instances \( \{\gamma^n\}_{n=0}^\infty \) are the time points the algorithm updates the parameter estimates. In fact, Algorithm 1 applies control actions \( U_t = L \left( \hat{A}_n, \hat{B}_n \right) X_t \) during the time period \( \gamma^n \leq t < \gamma^{n+1} \), where \( \hat{A}_n, \hat{B}_n \) are estimates of the unknown true dynamics matrices \( A_*, B_* \), based on the trajectory up to the time \( \gamma^n \).

Further, to ensure that the policy commits to sufficient exploration, at time \( \gamma^n \), a random matrix \( \Theta_n \) is added to the parameter estimates. Then, Algorithm 1 projects the resulting \( d_X \times (d_X + d_U) \) matrix onto the stabilization set \( S_0 \), which will be defined shortly (in Definition 5.1), to guarantee that the system evolves stable. Formally, letting \( \Pi_{S_0}(\cdot) \) denote the projection on \( S_0 \), define

\[
[\hat{A}_n, \hat{B}_n] = \Pi_{S_0} \left( \left[ \begin{array}{c} \gamma^n \\ \int_0^\gamma Y_s dX_s^\top \\ V_n^\top + \Theta_n \end{array} \right] \right), \tag{20}
\]

where \( Y_s = [X_s, U_s] \), \( V_n = \gamma^n \int_0^\gamma Y_s^\top ds \), and \( d_X \times (d_X + d_U) \) matrices \( \{\Theta_n\}_{n=0}^\infty \) are independent of everything else and of each others. Further, \( \Theta_n \) has independent Gaussian entries of mean zero and standard deviation \( \gamma^{-n}n^{1/4} \). This value of standard deviation is delicately adjusted for two purposes. First, \( \Theta_n \) is sufficiently large for randomizing the parameter estimates to ensure that effective exploration occurs and the current data is diverse enough so that we obtain accurate estimates in the future. At the same time, \( \Theta_n \) is sufficiently small to let the current estimates remain accurate and prevent large deviations from the optimal policy, which deteriorates efficient exploitation at the current time.

Next, to discuss the projection step, first note that for stability, it suffices to satisfy (6) for an arbitrary \( \delta > 0 \). Moreover, the condition in (6) is verifiable since \( \rho, \zeta \) depend on the known matrices \( \hat{A}, \hat{B} \). So, in lights of Theorem 5.2 we define the stabilization oracle \( S_0 \). By having access to \( S_0 \), the system is stabilized, despite uncertainties about the dynamics.

**Definition 5.1 (Stabilization oracle).** For a fixed \( \delta_0 > 0 \), let \( S_0 \) be a set containing matrices \( \hat{A}, \hat{B} \) for which (6) holds for \( \delta_0 \).

Note that by Remark 5.3 it is sufficient to have \( \mathcal{E} \left( \hat{A}, \hat{B} \right) \leq \epsilon_0 \), where \( \epsilon_0 \) satisfies (12). In general, \( S_0 \) can be significantly larger than the \( \epsilon_0 \)-neighborhood of \( A_*, B_* \). The following statement express that the step of projecting on \( S_0 \) is temporary, and can be removed after some time.

**Remark 5.2 (Transience of projection on \( S_0 \)).** Later on, (in Theorem 5.3) we show that \( \hat{A}_n, \hat{B}_n \) concentrate around \( A_*, B_* \) as the algorithm proceeds. So, by Remark 5.3 if \( t \geq O \left( \epsilon_0^{-1} \right) \), then we can remove the projection step \( \Pi_{S_0}(\cdot) \) in (20).

Note that if an initial stabilizing feedback \( L_0 \) is available at the beginning of interacting with the system, one can devote a finite time period to collect state-action observations and use them to obtain \( \epsilon_0 \)-accurate estimates of \( A_*, B_* \). Then, according to Theorem 5.2 those estimates are sufficient to stabilize the system [22] [20] [21]. The condition of having an initial stabilizing controller is commonly adopted in the literature [33] [34] [10] [12]. This condition is guaranteed, for example, in the dynamical systems that are in operation, prior to running Algorithm 1 or those that are open-loop-stable (i.e., \( X(A_*) < 0 \), and so \( L_0 = 0 \) is an initial stabilizer). On the other hand, if a RESET option is available in the algorithm that can immediately switch the policy or steer the state trajectory to small values, then, \( \mathcal{E} \left( \hat{A}, \hat{B} \right) \leq \epsilon_0 \) can be satisfied in
Algorithm 1: Reinforcement Learning Policy

Select \( \hat{A}_1, \hat{B}_1 \in S_0 \), arbitrarily, and formally let \( \gamma^{-1} = 0 \)

for \( n = 0, 1, 2, \ldots \) do

while \( \gamma^{-1} \leq t < \gamma^n \) do

Take action \( U_t = L(\hat{A}_{n-1}, \hat{B}_{n-1}) X_t \)

end while

Update learned dynamics \( \hat{A}_n, \hat{B}_n \) by (20)

end for

a relatively short time period by replicating the data collection step [27, 29]. In general, stabilization of unknown stochastic systems is an important problem in both continuous- [24, 33, 27], as well as discrete-time settings [21, 35, 30, 37, 38]. More details can be found in the aforementioned references.

The memory space occupied by running Algorithm 1 is remarkably small, as the algorithm can execute (20) using only the matrices \( V_n, \int_0^t Y_s dX^s \), which can be updated by continuously integrating new observations \( Y_t \) into the existing matrices. Furthermore, calculations are straightforward, making update of the parameter estimates at time \( \gamma^n \) immediately effective. The rationale for freezing the parameter estimates for exponentially growing time intervals \( \gamma^{-1} \leq t < \gamma^n \), is that Algorithm 1 can defer the learning step until collecting enough observation vectors \( Y_t \) so that a new update of parameter estimates is more effective than the previous one. The following result provides performance guarantees.

Theorem 5.3 (Analysis of Algorithm 1). Let the policy \( \pi \) and the estimates \( \hat{A}_n, \hat{B}_n \) be those in Algorithm 1. Then, we have

\[
\mathcal{R}_T(\pi) = O \left( \omega_n T^{1/2} \log T \right), \quad \mathcal{E} \left( \hat{A}_n, \hat{B}_n \right)^2 = O \left( \omega_{\hat{A}, \hat{B}} \gamma^{-n/2} \right),
\]

where \( d = d_X + d_U \), and

\[
\omega_{\hat{A}, \hat{B}} = \left( d_X + d_W \| C \|^2 \log \gamma \right) / \lambda_{\min} \left( CC^\top \right), \quad \omega_x = \left( \gamma - 1 \right) \| C \|^2 \| B_x \|^2 \| X \| \| K \| \| A_x \| \| B_x \|^2 \| R \| \log \gamma / \lambda_{\min} (Q) \lambda_{\min} (R) ^2 \omega_{\hat{A}, \hat{B}}.
\]

Theorem 5.3 indicates efficiency of Algorithm 1. At time \( T \), the sub-optimality gap is as small as \( O \left( T^{-1/2} \log T \right) \). It also provides \( \omega_{\pi}, \omega_{\hat{A}, \hat{B}} \) that reflect the dependence of estimation error and regret on different parameters in the problem. So, regret growth rate with the dimension is quadratic, while the estimation accuracy deteriorates linearly as the dimension grows. Further, the requirement \( \lambda_{\min} (CC^\top) > 0 \) is standard in estimation and control of diffusion processes [39, 40, 27]. Intuitively, it ensures that all coordinates of the state vectors are randomized by the Brownian motion \( \left\{ W_t \right\} \) at a short time period, and so have significant roles in the stochastic differential equation [39, 40]. From a modeling point of view, \( \lambda_{\min} (CC^\top) > 0 \) indicates that the stochastic differential equation is irreducible in the sense that a smaller subset of state variables is insufficient for capturing the dynamic behavior of the environment and the system under study is not over-parameterized.

Note that the estimation accuracy of Algorithm 1 is \( O \left( \gamma^{-n/4} \right) \), which is not as small as the classical square-root consistency. That is because the main priority of the algorithm is to efficiently minimize the regret. However, if the randomization matrices \( \Theta_n \) are persistent and do not diminish as \( n \) grows, then we obtain the square-root consistency. It is formalized in the following result. Of course, the price is that \( \mathcal{R}_T(\pi) \) grows linearly with \( T \) if \( \Theta_n \) does not dwindle.

Proposition 5.4 (Estimation Rates). If in Algorithm 1 the variance of entries of \( \Theta_n \) is \( \nu_n \), where \( \liminf_{n \to \infty} \nu_n > 0 \), then, for \( Y_s = [X_s^\top, U_s^\top]^\top \) and \( V_n = \int_0^\gamma Y_s Y_s^\top ds \), it holds that

\[
\left\| \left( \int_0^{\gamma^n} Y_s dX_s \right)^\top V_n^{-1} \right\|^2 = O \left( \omega_{\hat{A}, \hat{B}} \gamma^{-n/2} \right).
\]
Proof of Theorem \[5.3\] In order to establish Theorem \[5.3\], we study the estimation procedure in \[20\] and specify the accuracy at which the algorithm is able to estimate \(A_\ast, B_\ast\). To that end, Lemma \[8.2\] and Lemma \[8.3\] are utilized to study the Gram matrix \(V_n\) in \[20\], while Lemma \[7.2\] is used for bounding the estimation error. Then, by leveraging Lemma \[7.3\], we find the rates of deviating from the optimal policy in \[3\]. Finally, the resulting regret of Algorithm \[1\] is investigated in lights of Theorem \[4.1\].

First, observe that substituting for \(dX_t\) in \[20\] based on \(\{1\}\), we obtain

\[
\int_0^n Y_s dX_s^T \bigg|_{V_n}^T = \int_0^n Y_s Y_s^T [A_\ast, B_\ast]^T ds + \int_0^n Y_s dW_s^T C^T \bigg|_{V_n},
\]

So, as long as \(V_n\) is non-singular, (which we will show that is the case, see \[23\]), we have

\[
\int_0^n Y_s dX_s^T \bigg|_{V_n}^T = [A_\ast, B_\ast] + \int_0^n Y_s dW_s^T C^T \bigg|_{V_n},
\]

which, because \([A_\ast, B_\ast] \in S_0\), according to \[20\] leads to

\[
\mathcal{E} \left( \hat{A}_n, \hat{B}_n \right) \leq \left\| V_n^{-1} \int_0^n Y_s dW_s^T C^T \right\| + \| \Theta_n \|.
\]

Since the entries of \(\Theta_n\) are \(N \left( 0, \gamma^{-n/2} n^{1/2} \right)\), we have \(\mathbb{P} \left( \| \Theta_n \| \geq d_X^{1/2} (d_X + d_U)^{1/2} \gamma^{-n/4} n^{1/2} \right) = \mathcal{O} \left( e^{-n^{1/2}} \right)\),

which, by Borel-Cantelli Lemma, leads to \(\| \Theta_n \| = \mathcal{O} \left( d_X^{1/2} (d_X + d_U)^{1/2} \gamma^{-n/4} n^{1/2} \right)\). Thus, according to Lemma \[7.2\] it holds that

\[
\mathcal{E} \left( \hat{A}_n, \hat{B}_n \right) = (d_X + d_U)^{1/2} \mathcal{O} \left( d_X^{1/2} (d_X + d_U)^{1/2} \frac{\| C \| \log \lambda_{\max} \left( V_n \right)}{\lambda_{\min} \left( V_n \right)} \right)^{1/2} + d_X^{1/2} \gamma^{-n/4} n^{1/2} \right).
\]

(22)

Now, Lemma \[8.2\] provides \(\mathcal{O} \left( \log \lambda_{\max} \left( V_n \right) \right) = n \log \gamma\). Further, we will shortly show that

\[
\liminf_{n \to \infty} \gamma^{-n/2} \lambda_{\min} \left( V_n \right) \geq \lambda_{\min} \left( C C^T \right).
\]

Thus, \[22\] and \[20\] yield to

\[
\mathcal{E} \left( \hat{A}_n, \hat{B}_n \right) = \mathcal{O} \left( (d_X + d_U)^{1/2} \left( d_X^{1/2} + \frac{d_X^{1/2} \| C \| \log \gamma}{\lambda_{\min} \left( C C^T \right)^{1/2}} \right) \gamma^{-n/4} n^{1/2} \right).
\]

(23)

This gives the second result in Theorem \[5.3\]. To proceed towards proving the first result, using Lemma \[7.3\] we get

\[
\| \mathcal{L} \left( \hat{A}_n, \hat{B}_n \right) - \mathcal{L} \left( A_\ast, B_\ast \right) \|^2 = \mathcal{O} \left( (d_X + d_U) \beta^2_{\ast} \left( d_X + \frac{d_X \| C \| \log \gamma}{\lambda_{\min} \left( C C^T \right)} \right) \gamma^{-n/2} n \right),
\]

where \(\beta_{\ast}\) is defined in Lemma \[7.3\].

Now, since during the time period \(\gamma^{-1} \leq t < \gamma^n\), the feedback matrix is frozen to \(\mathcal{L} \left( \hat{A}_{n-1}, \hat{B}_{n-1} \right)\), according to Lemma \[8.2\] we have

\[
\int_0^n \left\| R^{1/2} \left( L_t - \mathcal{L} \left( A_\ast, B_\ast \right) \right) X_t \right\|^2 dt = \mathcal{O} \left( \sum_{k=1}^n \beta_{L} \gamma^{-k-1} \gamma^{-k/2} k \right),
\]

where

\[
\beta_{L} = (d_X + d_U) \beta^2_{\ast} \left( d_X + \frac{d_X \| C \| \log \gamma}{\lambda_{\min} \left( C C^T \right)} \right) \left( \gamma - 1 \right) \| R \| \| C \|^2.
\]
Moreover, since $\mathcal{X}(D_*) < \lambda$, the matrix $E_t$ in Theorem 4.1 decays exponentially with $T-t$. So, it holds that $\mathbb{E} \left[ \int_0^T (X_t^\top E_t B_* (L_t - L(A_*, B_*)) X_t) \, dt \right] = \mathcal{O} \left( \log^2 T \right)$. Therefore, according to Theorem 4.1, we have the following for the policy $\pi$ of Algorithm 1

$$
\mathcal{R}_T(\pi) = \mathcal{O} \left( \left( \frac{\log T}{\log \gamma} \right) \sum_{k=1}^{\gamma(k-1)/2} \gamma^{(k-1)/2} k \right) = \mathcal{O} \left( \frac{\beta_*}{\log \gamma} T^{1/2} \log T \right),
$$

which, according to $\beta_*$ in Lemma 7.3, completes the proof of the desired regret bound.

To prove (23), first by Lemma 8.2, letting $D_{k-1} = A_* + B_*(\hat{A}_{k-1}, \hat{B}_{k-1})$, we have

$$
\liminf_{k \to \infty} \gamma^{-k} \lambda_{\text{min}} \left( \int_{\gamma_{k-1}}^{\gamma_k} X_t X_t^\top \, dt \right) \geq \frac{1 - \gamma^{-1}}{1} \lambda_{\text{min}} \left( CC^\top \right) \geq \eta_k \lambda_{\text{min}} \left( CC^\top \right). \tag{24}
$$

Hence, (24) implies that to establish (23), it suffices to show that for some $0 \leq \ell < n - 1$,

$$
\liminf_{n \to \infty} \lambda_{\text{min}} \left( \sum_{k=\ell}^{n-1} \gamma^{-k} \right) \geq \max_{\ell \leq k \leq n - 1} \frac{1}{\eta_k}. \tag{25}
$$

For an arbitrary fixed $\epsilon > 0$, consider the event that the smallest eigenvalue above is less than $\epsilon$, and let $\mathcal{M}_n(\epsilon)$ be the set of matrices $[\hat{A}, \hat{B}]_{k=\ell}^{n-1}$ for which this event occurs. That is,

$$
\mathcal{M}_n(\epsilon) = \left\{ [\hat{A}, \hat{B}, \ldots, \hat{A}_{n-1}, \hat{B}_{n-1}] \in \mathbb{R}^{d_X \times (d_X + d_U) n} : \lambda_{\text{min}} (P_{\ell,n} P_{\ell,n}^\top) \leq \epsilon \right\},
$$

where the $(d_X + d_U) \times d_X (n - \ell)$ matrix $P_{\ell,n}$ is

$$
P_{\ell,n} = \begin{bmatrix} I_{d_X} & \cdots & I_{d_X} \\ \mathcal{L}(\hat{A}_{\ell}, \hat{B}_{\ell}) & \cdots & \mathcal{L}(\hat{A}_{n-1}, \hat{B}_{n-1}) \end{bmatrix}.
$$

Now, note that the set of all matrices

$$
F_n = \begin{bmatrix} I_{d_X} & \cdots & I_{d_X} \\ \mathcal{L}(L_{\ell} \cdots L_{\ell+1}) & \cdots & \mathcal{L}(L_{\ell+1} \cdots L_{n-1}) \end{bmatrix},
$$

for which there exists $v \in \mathbb{R}^{d_X + d_U}$ satisfying $\|v\| = 1$ and $F_n^\top v = 0$, is of dimension $d_X + d_U - 1 + (n - \ell) (d_U - 1)$. To show that, one one hand, the set of unit $d_X + d_U$ dimensional vectors is (a sphere) of dimension $d_X + d_U - 1$.

On the other hand, by writing $v = [v_1^\top, v_2^\top]^\top$, where $v_1 \in \mathbb{R}^{d_X}$ and $v_2 \in \mathbb{R}^{d_U}$, clearly, $F_n^\top v = 0$ is equivalent to $L_k^\top v_2 = -v_1$, for all $k = \ell, \cdots, n - 1$. The latter enforces every column of $L_k$ to be in a certain hyperplane in $\mathbb{R}^{d_U}$.

Thus, according to Lemma 5.3, the dimension of $\mathcal{M}_n(0)$ is at most $d_X + (d_U - 1)(n - \ell + 1) + (n - \ell) d_U$. Further, if $\ell$ is sufficiently large so that $\gamma^{-\ell + n/2} \epsilon < 1$, then for every $[\hat{A}_k, \hat{B}_k]_{k=\ell}^{n-1} \in \mathcal{M}_n(\epsilon)$, there exists some $[\hat{A}_k, \hat{B}_k]_{k=\ell}^{n-1} \in \mathcal{M}_n(0)$, such that for all $k = \ell, \cdots, n - 1$, it holds that

$$
\left\| [\hat{A}_k, \hat{B}_k] - [\hat{A}_k, \hat{B}_k] \right\| = \mathcal{O} \left( \gamma^{-k/2 + n/4} \epsilon^{1/2} \right).
$$

Remember that the random matrices $\{\Theta_k\}_{k=0}^{n-1}$ are independent, and entries of $\Theta_k$ are independent identically distributed $\mathcal{N}(0, \gamma^{-k/2} k^{1/2})$ random variables. Hence, we have

$$
P(\mathcal{M}_n(\epsilon)) = O \left( \gamma^{\ell/4} \epsilon^{-1/4} \gamma^{-\ell/2 + n/4} \epsilon^{1/2} \right) \wedge 1^{(d_X d_U - d_U + 1)(n - \ell) - d_X - d_U + 1},
$$

15
because, $\mathcal{M}_n(0)$ is a $d_X + (d_U - 1)(n-\ell + 1) + (n-\ell)d_X$ dimensional object in a $2d_X(d_X + d_U)(n-\ell)$ dimensional space. So, the exponent is at least their difference: $(d_X d_U - d_U + 1)(n-\ell) - d_X - d_U + 1$. Letting $\ell = n-5$, the above number is at least 5, and as $n$ grows, $O(\ell^{-1/4}(n-\ell)/\epsilon^{1/2}) < 1$ holds for $\epsilon = \max_{\ell \leq k \leq n-1} \eta_k^{-1}$. So, we have $\sum_{n=5}^{\infty} P(\mathcal{M}_n(\epsilon)) = \sum_{n=5}^{\infty} O(n^{-1/4})^5 < \infty$, which by Borel-Cantelli Lemma implies (25).

6 Conclusion

We studied estimation and control of unknown linear Ito differential equations and presented reinforcement learning algorithms that learn optimal control actions for minimizing quadratic costs. Three important problems are investigated, each of which received full treatments and the resulting comprehensive analyses are provided.

First, we addressed stabilization of diffusion processes based on coarse-grained approximations of the true dynamics matrices (Theorem 3.2). Then, proposing the novel framework of policy differentiation, we established a reciprocal result on the regret that decision-making policies incur due to deviations from the optimal policy (Theorem 5.1). Next, we presented the fast and efficient reinforcement learning Algorithm 4 and established that its estimation rate is $dI^{-1/4}\log^{1/2}T$, and has the regret bound $O(d^2T^{1/2} \log T)$ (Theorem 5.3).

As the first comprehensive study on design and analysis of statistical decision-making policies for stochastic differential equations, this work introduces several interesting problems for future directions. That includes, design and analysis of algorithms for finding the stabilization oracle $S_0$ in Definition 5.1 establishing anytime regret bounds that hold uniformly over time, extensions to high-dimensional diffusion processes for structured true dynamics matrices such as low-rank and/or sparse ones, studying efficient estimation and control under imperfect state-observations, and extensions to non-linear differential equations.

7 Technical Proofs

7.1 Proof of Proposition 5.4

By (21), it suffices to study $\Delta = V_n^\top \int_0^n Y_s dW_s C^\top$. We show that $\liminf_{n \to \infty} n^{-1} \nu_n \gamma_n \geq \lambda_{\min}(CC^\top)$. So, putting Lemma 7.2 and Lemma 7.3 together, we obtain the desired result, since $\|\Delta\|^2 = O(\|d_X + d_U\|d_W \|C\|^2 n^{-1/2} \log \gamma / \lambda_{\min}(CC^\top))$. Now, by (24), it is enough to show

$$\liminf_{n \to \infty} \lambda_{\min} \left( \sum_{k=\ell}^{n-1} \gamma_n^{k-n} \sum_{k=\ell}^{n-1} \begin{bmatrix} I_{d_X}^\top \mathcal{L} (\hat{A}_k, \hat{B}_k) \end{bmatrix} \left[ I_{d_X}^\top \mathcal{L} (\hat{A}_k, \hat{B}_k) \right]^\top \right) \geq \max_{\ell \leq k \leq n-1} \frac{1}{\eta_k},$$

for some $0 \leq \ell < n-1$. Next, for $\epsilon = \max_{\ell \leq k \leq n-1} \eta_k^{-1}$, let $\mathcal{M}_n(\epsilon)$ be the set of $[\hat{A}_k, \hat{B}_k]_{k=\ell}^{n-1}$ that the above does not hold: $\mathcal{M}_n(\epsilon) = \left\{ [\hat{A}_\ell, \hat{B}_\ell, \cdots, \hat{A}_{n-1}, \hat{B}_{n-1}] : \lambda_{\min}(P_{\ell,n}P_{\ell,n}^\top) \leq \epsilon \right\}$.

$$P_{\ell,n} = \gamma^{(\ell-n)/2} n^{1/2} \begin{bmatrix} I_{d_X}^\top \mathcal{L} (\hat{A}_\ell, \hat{B}_\ell) \end{bmatrix} \cdots \gamma^{-1/2} n^{1/2} \begin{bmatrix} I_{d_X}^\top \mathcal{L} (\hat{A}_{n-1}, \hat{B}_{n-1}) \end{bmatrix}.$$  

Similar to the proof of Theorem 5.3, for $[\hat{A}_k, \hat{B}_k]_{k=\ell}^{n-1} \in \mathcal{M}_n(\epsilon)$, there is $[\hat{A}_k, \hat{B}_k]_{k=\ell}^{n-1} \in \mathcal{M}_n(0)$, that $\| [\hat{A}_k, \hat{B}_k] - [\hat{A}_k, \hat{B}_k]_{k=\ell}^{n-1} \|^2 = O(\gamma^{n-k} n^{-1} \epsilon)$. Thus, $\liminf_{n \to \infty} v_n > 0$, together with the dimension of $\mathcal{M}_n(0)$ in the proof of Theorem 5.3 implies that

$$P(M_n(\epsilon)) = O\left( (\gamma^{(n-\ell)/2} n^{-1/2} \epsilon^{1/2}) \wedge 1 \right)^{\gamma (d_X d_U - d_U + 1)(n-\ell) - d_X - d_U + 1},$$
which for $\ell = n - 4$ gives $\sum_{n=4}^{\infty} \mathbb{P}(M_n(e)) < \infty$. Therefore, Borel-Cantelli Lemma implies the desired result.

### 7.2 Difference in regrets of two policies

**Lemma 7.1.** For fixed $0 \leq t_1 \leq t_2 \leq T$, define the policies $\pi_1, \pi_2$ according to

$$
\pi_i = \begin{cases} 
U_t = L X_t & t < t_i \\
U_t = \mathcal{L}(A_s, B_s) X_t & t \geq t_i
\end{cases}
$$

Then, we have $\mathcal{R}_T(\pi_2) - \mathcal{R}_T(\pi_1) = X_t^\top F_{t_1} X_{t_1} + 2X_{t_1}^\top g_{t_1} + \beta_{t_1}$, where

$$
F_{t_1} = \int_{t_1}^{t_2} \left( e^{D^\top_t (t-t_1)} S e^{D_t (t-t_1)} + 2 \Delta_t^\top M e^{D_t (t-t_1)} + \Delta_t^\top M \Delta_t \right) dt
$$

(26)

$$
g_{t_1} = \int_{t_1}^{t_2} \left( S \int_{t_1}^{t} e^{D_t (t-s)} C dW_s + \Delta_t^\top M \int_{t_1}^{t} e^{D_t (t-s)} C dW_s + e^{D^\top_t (t-t_1)} M Z_t + \Delta_t^\top M Z_t \right) dt
$$

(27)

$$
\beta_{t_1} = \int_{t_1}^{t_2} \left( \left\| S^{1/2} \int_{t_1}^{t} e^{D_t (t-s)} C dW_s \right\|^2 + 2 \int_{t_1}^{t} e^{D_t (t-s)} C dW_s + Z_t^\top M Z_t \right) dt
$$

(28)

**Proof.** Letting $X_{t_1}^{\pi_1}$ be the state of the system under the policy $\pi_1$, clearly, for $t_2 \leq t_1$, it holds that $X_{t_1}^{\pi_1} = X_{t_1}^{\pi_2}$. So, we use $X_{t_1}$ for both states at time $t_1$. Moreover, for $t_1 \leq t \leq t_2$, we have

$$
X_{t_1}^{\pi_1} = e^{D_{t_1}(t-t_1)} X_{t_1} + \int_{t_1}^{t} e^{D_t(t-s)} C dW_s, \quad X_{t_1}^{\pi_2} = e^{D(t-t_1)} X_{t_1} + \int_{t_1}^{t} e^{D(t-s)} C dW_s,
$$

where $D_s = A_s + B_s \mathcal{L}(A_s, B_s)$ and $D = A_s + B_s L$. Writing

$$
M_s = Q + \mathcal{L}(A_s, B_s)^\top R \mathcal{L}(A_s, B_s), \quad M = Q + LRL, \quad Y_t = X_{t_1}^{\pi_2} - X_{t_1}^{\pi_1},
$$

and denoting the instantaneous cost of policy $\pi_i$ at time $t$ by $c_{\pi_i}(t)$, we get $Y_t = \Delta_t X_{t_1} + Z_t$, as well as

$$
\int_{t_1}^{t_2} (c_{\pi_2}(t) - c_{\pi_1}(t)) dt = \int_{t_1}^{t_2} \left[ (X_{t_1}^{\pi_1} + Y_t)^\top M (X_{t_1}^{\pi_1} + Y_t) - X_{t_1}^{\pi_1}^\top M_s X_{t_1}^{\pi_1} \right] dt
$$

(29)
Thus, putting (29) and (30) together, we obtain the desired result for

\[ \det M_{t} = \det \left( e^{D_{t} - t_{1} - e^{D_{t}} - t_{1}} \right), \]

where \( \Delta_{t} = e^{D_{t} - t_{1} - e^{D_{t}} - t_{1}} \), \( Z_{t} = \int_{t_{1}}^{t} \left[ e^{D_{t} - s} - e^{D_{t}} - t_{1} \right] C_d W_s \), and \( S = M - M_{t} = L^{T} RL - \mathcal{L}(A_{*}, B_{*})^{T} R \mathcal{L}(A_{*}, B_{*}) \). On the other hand, for \( t \geq t_{2} \), it holds that

\[
X_{t}^{\pi_{i}} = e^{D_{t} - t_{2}} X_{t_{2}}^{\pi_{i}} + \int_{t_{2}}^{t} e^{D_{t} - s} C_d W_s,
\]

\[
Y_{t} = e^{D_{t} - t_{2}} \left[ X_{t_{2}}^{\pi_{i}} - X_{t_{2}}^{\pi_{i}} \right] Y_{t_{2}} = e^{D_{t} - t_{2}} \left[ \Delta_{t_{2}} X_{t_{1}} + Z_{t_{2}} \right],
\]

\[
\int_{t_{2}}^{T} (c_{\pi_{2}}(t) - c_{\pi_{1}}(t)) dt = \int_{t_{2}}^{T} \left[ (X_{t_{1}}^{\pi_{1}} + Y_{t})^{T} M_{t} (X_{t_{1}}^{\pi_{1}} + Y_{t}) - X_{t_{1}}^{\pi_{1}}^{T} M_{t} X_{t_{1}}^{\pi_{1}} \right] dt
\]

\[
= \int_{t_{2}}^{T} [2Y_{t}^{T} M_{t} Y_{t} + Y_{t}^{T} M_{t} Y_{t}] dt. \quad (30)
\]

Thus, putting (29) and (30) together, we obtain the desired result for

\[ \square \]

### 7.3 Scaling of self-normalized stochastic integrals

**Lemma 7.2.** Suppose that \( Y_{t} \in \mathbb{R}^{m} \) is a vector-valued stochastic process such that \( Y_{t} \) is \( \mathcal{F}_{t} \)-measurable for the natural filtration \( \mathcal{F}_{t} = \sigma \left( \{ W_{s} \}_{0 \leq s \leq t} \right) \). Then, we have

\[
\left\| \left( I + \int_{0}^{t} Y_{s}^{T} Y_{s} ds \right)^{-1/2} \int_{0}^{t} Y_{s} dW_{s}^{T} \right\|^{2} = \mathcal{O} \left( m d \log \lambda_{\text{max}}(Y_{t}) \right).
\]

**Proof.** First, fix \( t > 0 \), and for an arbitrary \( \epsilon > 0 \), let \( n = \lfloor t/\epsilon \rfloor \). Then, for \( k = 0, 1, \cdots, n \), consider the sequence of matrices \( M_{k} = \epsilon^{-1} I + \sum_{i=0}^{k} Y_{i} Y_{i}^{T} \). Then, for \( k = 1, \cdots, n \), consider the sequence of scalars \( \beta_{k} \) defined according to \( \beta_{k} = Y_{k}^{T} M_{k-1}^{-1} Y_{k} \). Using the formula for the determinant of a product of matrices, we have

\[
\det M_{k} = \det \left[ M_{k-1} \left( I + M_{k-1}^{-1} Y_{k} Y_{k}^{T} \right) \right] = \det \left( M_{k-1} \right) \det \left( I + M_{k-1}^{-1} Y_{k} Y_{k}^{T} \right).
\]

Since all eigenvalues of \( I + M_{k-1}^{-1} Y_{k} Y_{k}^{T} \) are unit, except one of them which is \( 1 + \beta_{k} \), we have \( (1 + \beta_{k}) \det M_{k-1} = \det M_{k} \). On the other hand, matrix inversion formula gives

\[
M_{k}^{-1} = (M_{k-1} + Y_{k} Y_{k}^{T})^{-1} = M_{k-1}^{-1} - \frac{1}{1 + Y_{k}^{T} M_{k-1}^{-1} Y_{k}} M_{k-1}^{-1} Y_{k} Y_{k}^{T} M_{k-1}^{-1},
\]

which leads to

\[
Y_{k}^{T} M_{k}^{-1} Y_{k} = Y_{k}^{T} M_{k-1}^{-1} Y_{k} Y_{k}^{T} - \frac{\beta_{k}^{2}}{1 + \beta_{k}} = 1 + \frac{1}{1 + \beta_{k}} = 1 - \frac{\det M_{k-1}}{\det M_{k}}.
\]

Further, by using the inequality \( 1 - \beta \leq - \log \beta \) for \( \beta > 0 \), the latter equality gives

\[
Y_{k}^{T} M_{k}^{-1} Y_{k} \leq \log \det M_{k} - \log \det M_{k-1}.
\quad (31)
\]
Thus, letting $F_k = \sum_{i=0}^{k} (W_{(i+1)e} - W_ie)^\top$, properties of conditional expectations give

$$
\mathbb{E} \left[ F_k^\top M_k^{-1} F_k \right] = \mathbb{E} \left[ \mathbb{E} \left[ F_k^\top M_k^{-1} F_k \mid \mathcal{F}_{k-1} \right] \right]
$$

$$
= \mathbb{E} \left[ \left( F_{k-1} + Y_{ke} (W_{(k+1)e} - W_{ke})^\top \right)^\top M_k^{-1} \left( F_{k-1} + Y_{ke} (W_{(k+1)e} - W_{ke})^\top \right) \right] \mid \mathcal{F}_{k-1} \right]
$$

$$
= \mathbb{E} \left[ F_{k-1}^\top M_k^{-1} F_{k-1} + \left( (W_{(k+1)e} - W_{ke}) Y_{ke} M_k^{-1} Y_{ke} (W_{(k+1)e} - W_{ke})^\top \right) \mid \mathcal{F}_{k-1} \right]
$$

where we used the facts that $Y_{ke}, F_{k-1}$, and $M_k$ all are $\mathcal{F}_{k-1}$-measurable, the Brownian motion $W_t$ has independent increments, and its covariance matrix is a multiple of identity. So, using (31) together with the fact that (as positive semidefinite matrices) the order $M_{k-1} \leq M_k$ holds, we get the telescopic relationships

$$
\lambda_{\max} (\mathbb{E} \left[ F_k^\top M_k^{-1} F_k \right]) - \lambda_{\max} (\mathbb{E} \left[ F_{k-1}^\top M_{k-1}^{-1} F_{k-1} \right]) \leq c \left( \log \frac{\det(\epsilon M_k)}{\det(\epsilon M_{k-1})} \right).
$$

Since $F_k^\top (M_k)^{-1} F_k$ is positive semidefinite, its trace is larger than its largest eigenvalue. Hence, adding up for $k = 0, 1, \cdots, n$, by interchanging trace and expectation, we obtain

$$
\mathbb{E} \left[ \lambda_{\max} \left( F_n^\top (M_n)^{-1} F_n \right) \right] \leq \mathbb{E} \left[ \text{tr} \left( F_n^\top (M_n)^{-1} F_n \right) \right] \leq dW \lambda_{\max} \left( \mathbb{E} \left[ F_n^\top (M_n)^{-1} F_n \right] \right),
$$

which, by $\epsilon M_0 \geq I$, leads to $\mathbb{E} \left[ \lambda_{\max} \left( F_n^\top (\epsilon M_n)^{-1} F_n \right) \right] \leq m dW \log \lambda_{\max} (\epsilon M_n)$. Thus, according to Doob’s Martingale Convergence Theorem [30, 31], we have

$$
\left\| (\epsilon M_n)^{-1/2} F_n \right\|^2 = O (m dW \log \lambda_{\max} (\epsilon M_n)).
$$

Finally, letting $\epsilon \to 0$, we obtain the desired result, because $\epsilon M_n, F_n$ are $\epsilon$-approximations of the corresponding integrals. \hfill \Box

### 7.4 Lipschitz continuity of optimal feedback matrix

**Lemma 7.3.** Using the Jordan decomposition $D_* = A_* + B_* L (A_*, B_*) = P_*^{-1} A_* P_*$, similar to Definition 7.4, and suppose that $E \left( \hat{A}, \hat{B} \right) \leq \kappa_*$, for

$$
\kappa_* = \frac{1}{1 \vee \| L (A_*, B_*) \|} \left( \frac{-\overline{X}(D_*) \wedge (-\overline{X}(D_*))^\mu_*}{\mu_*^{1/2} \| P_*^{-1} \| \| P_* \|} \right)^{-1} \left( 4 \int_0^\infty \| e^{D_* t} \|^2 dt \right)^{-1}.
$$

Then, we have

$$
\| L (\hat{A}, \hat{B}) - L (A_*, B_*) \| \leq \beta_* E \left( \hat{A}, \hat{B} \right),
$$

where

$$
\beta_* = \frac{2 \| K (A_*, B_*) \|}{\lambda_{\min} (R)} \left[ 1 + \frac{4 \| B_* \| \| K (A_*, B_*) \|}{\lambda_{\min} (Q) \| K (A_*, B_*) \|} \right] \left( 1 \vee \frac{2 \| B_* \| + \kappa_* \| K (A_*, B_*) \|}{\lambda_{\min} (R)} \right).
$$

In general, without the condition $E \left( \hat{A}, \hat{B} \right) \leq \kappa_*$, the constant $\beta_*$ is replaced with

$$
\beta = \frac{\| K (\hat{A}, \hat{B}) \|}{\lambda_{\min} (R)} + \frac{2 \| B_* \| ^2}{\lambda_{\min} (Q) \lambda_{\min} (R)} \left( 1 \vee \frac{\| B_* \| + E \left( \hat{A}, \hat{B} \right) \| K (\hat{A}_0, \hat{B}_0) \|}{\lambda_{\min} (R)} \right),
$$

for some convex combination $\hat{A}_0, \hat{B}_0 = \eta [\hat{A}, \hat{B}] + (1 - \eta) [A_*, B_*]$, and $0 \leq \eta \leq 1$. 19
Proof. Fix the matrices \( \hat{A}, \hat{B} \), and consider the matrix-valued curve
\[
\varphi = \left\{ \left( 1 - \eta \right) [A_*, B_*] + \eta \left[ \hat{A}, \hat{B} \right] \right\}_{0 \leq \eta \leq 1}.
\]
For an arbitrary \( \hat{A}_0, \hat{B}_0 \in \varphi \), we find the derivative of the matrix \( \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \) at \( \hat{A}_0, \hat{B}_0 \), assuming that the matrices \( \hat{A}_0, \hat{B}_0 \) vary along \( \varphi \). For this purpose, letting \( \Delta_A = \hat{A} - A_* \), \( \Delta_B = \hat{B} - B_* \), we first calculate \( \mathcal{K} \left( \hat{A}_1, \hat{B}_1 \right) \) for \( \hat{A}_1 = \hat{A}_0 + \eta \Delta_A, \hat{B}_1 = \hat{B}_0 + \eta \Delta_B \), and then let \( \eta \to 0 \). First, letting \( P = \mathcal{K} \left( \hat{A}_1, \hat{B}_1 \right) - \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \), we have
\[
\begin{align*}
\mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \hat{B}_1 R^{-1} \hat{B}_1^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) &= \eta \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \Delta_B R^{-1} \hat{B}_1^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) + \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \hat{B}_0 R^{-1} \hat{B}_1^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \\
&= \eta^2 \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \Delta_B R^{-1} \Delta_B^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) + \eta \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \Delta_B R^{-1} \hat{B}_1^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \\
&\quad + \eta \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \hat{B}_0 R^{-1} \Delta_B^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) + \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \hat{B}_0 R^{-1} \hat{B}_1^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right),
\end{align*}
\]
which because of
\[
\begin{align*}
\mathcal{K} \left( \hat{A}_1, \hat{B}_1 \right) \hat{B}_1 R^{-1} \hat{B}_1^\top \mathcal{K} \left( \hat{A}_1, \hat{B}_1 \right) &= \mathcal{K} \left( \hat{A}_1, \hat{B}_1 \right) \hat{B}_1 R^{-1} \hat{B}_1^\top P + \mathcal{K} \left( \hat{A}_1, \hat{B}_1 \right) \hat{B}_1 R^{-1} \hat{B}_1^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \\
&= P \hat{B}_1 R^{-1} \hat{B}_1^\top P + \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \hat{B}_1 R^{-1} \hat{B}_1^\top P + P \hat{B}_1 R^{-1} \hat{B}_1^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \\
&\quad + \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \hat{B}_1 R^{-1} \hat{B}_1^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right),
\end{align*}
\]
implies that
\[
\begin{align*}
\mathcal{K} \left( \hat{A}_1, \hat{B}_1 \right) \hat{B}_1 R^{-1} \hat{B}_1^\top \mathcal{K} \left( \hat{A}_1, \hat{B}_1 \right) &= \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \hat{B}_0 R^{-1} \hat{B}_0^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \\
&= P \hat{B}_1 R^{-1} \hat{B}_1^\top P + \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \hat{B}_1 R^{-1} \hat{B}_1^\top P + P \hat{B}_1 R^{-1} \hat{B}_1^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \\
&\quad + \eta \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \Delta_B R^{-1} \Delta_B^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \\
&\quad + \eta \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \Delta_B R^{-1} \hat{B}_0^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) + \eta \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \hat{B}_0 R^{-1} \Delta_B^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right).
\end{align*}
\]
So, plugging the above equality as well as
\[
\begin{align*}
\hat{A}_1^\top \mathcal{K} \left( \hat{A}_1, \hat{B}_1 \right) + \mathcal{K} \left( \hat{A}_1, \hat{B}_1 \right) \hat{A}_1 &= \hat{A}_0^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) + \hat{A}_1^\top P + \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \hat{A}_1 + P \hat{A}_1 \\
&= \hat{A}_0^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) + \eta \Delta_A^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) + \hat{A}_1^\top P + \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \hat{A}_0 + \eta \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \Delta_A + P \hat{A}_1,
\end{align*}
\]
in \( \Phi_{\hat{A}_0, \hat{B}_0} \left( \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \right) = 0 \) and \( \Phi_{\hat{A}_1, \hat{B}_1} \left( \mathcal{K} \left( \hat{A}_1, \hat{B}_1 \right) \right) = 0 \), we obtain
\[
0 = \Phi_{\hat{A}_1, \hat{B}_1} \left( \mathcal{K} \left( \hat{A}_1, \hat{B}_1 \right) \right) - \Phi_{\hat{A}_0, \hat{B}_0} \left( \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \right) \\
= \left[ \hat{A}_1^\top - \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \hat{B}_1 R^{-1} \hat{B}_1^\top \right] P + P \left[ \hat{A}_1 - \hat{B}_1 R^{-1} \hat{B}_1^\top \right] \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) - P \hat{B}_1 R^{-1} \hat{B}_1^\top P \\
+ \eta \Delta_A^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) + \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \Delta_A - \eta^2 \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \Delta_B R^{-1} \Delta_B^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \\
\quad - \eta \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \Delta_B R^{-1} \hat{B}_0^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) - \eta \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \hat{B}_0 R^{-1} \Delta_B^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right),
\]

20
or equivalently
\[ 0 = \tilde{A}^\top P + P\tilde{A} - P\tilde{B}_1R^{-1}\tilde{B}_1^\top P + \tilde{Q}, \]  
(32)
where \( \tilde{A} = \hat{A}_1 - \hat{B}_1R^{-1}\hat{B}_1^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \), and
\[
\tilde{Q} = \eta \Delta_A \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) + \eta \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \Delta_A - \eta^2 \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \Delta_B \left( R^{-1} \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \Delta_B \right) \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \\
- \eta \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \Delta_B R^{-1} \hat{B}_1^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \eta \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \hat{B}_0 \right) R^{-1} \Delta_B \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \\
- \eta^2 \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \Delta_B R^{-1} \Delta_B^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right).
\]

Suppose that \( \eta \) is sufficiently small so that \( \mathcal{X} (\hat{A}) < 0 \). Note that it is possible thanks to stabilizability of \( \hat{A}_0, \hat{B}_0 \), Theorem 2.73 and \( \eta \to 0 \hat{A} = \hat{A}_0 + \hat{B}_0 \mathcal{L} \left( \hat{A}_0, \hat{B}_0 \right) = \hat{D}_0 \). So, since \( P\tilde{B}_1R^{-1}\tilde{B}_1^\top P \) is a positive semidefinite matrix, (32) implies that
\[
P = \int_0^\infty e^{\tilde{A} t} \left( -P\tilde{B}_1R^{-1}\tilde{B}_1^\top P + \tilde{Q} \right) e^{\tilde{A} t} dt \leq \int_0^\infty e^{\tilde{A} t} \tilde{Q} e^{\tilde{A} t} dt \leq \left( \left\| \tilde{Q} \right\| \right) \left( \int_0^\infty \left\| e^{\tilde{A} t} \right\|^2 dt \right) I_{dx},
\]
which, because of \( \lim_{\eta \to 0} \tilde{Q} = 0 \), leads to \( \lim_{\eta \to 0} P = 0 \). Thus, letting \( \Delta_{\mathcal{K} (\hat{A}_0, \hat{B}_0)} = \lim_{\eta \to 0} \eta^{-1} P \), the expression in (32) provides
\[
\Delta_{\mathcal{K} (\hat{A}_0, \hat{B}_0)} = \int_0^\infty e^{\tilde{B}_0 t} \left( \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) M + M^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \right) e^{\tilde{B}_0 t} dt,
\]
(33)
where \( M = \Delta_A + \Delta_B \mathcal{L} \left( \hat{A}_0, \hat{B}_0 \right) \). Now, using

\[
\mathcal{K} \left( \tilde{A}, \tilde{B} \right) - \mathcal{K} \left( A, B \right) = \int \Delta_{\mathcal{K} (\hat{A}_0, \hat{B}_0)} d\varphi = \int_0^1 \Delta_\left( \left| A' \right|, \left| B' \right| \right) + \eta \left| A - B \right| d\varphi
\]
(43)

together with the Cauchy-Schwarz inequality provides
\[
\frac{\left\| \mathcal{K} \left( \tilde{A}, \tilde{B} \right) - \mathcal{K} \left( A, B \right) \right\|}{\mathcal{E} \left( \tilde{A}, \tilde{B} \right)} \leq \sup_{\hat{A}_0, \hat{B}_0} \frac{2 \left\| \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \right\| \left( 1 \left\| \mathcal{L} \left( \hat{A}_0, \hat{B}_0 \right) \right\| \right) \int_0^\infty \left\| B_0 \right\|^2 dt
\]
\[
\leq \frac{2 \lambda_{\text{min}} (Q)}{\lambda_{\text{min}} (R)} \sup_{\hat{A}_0, \hat{B}_0} \frac{\left\| \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \right\|^2 \left( 1 \left\| \mathcal{L} \left( \hat{A}_0, \hat{B}_0 \right) \right\| \right)}.
\]

where in the last line above, we used (44).

Next, note that \( \mathcal{E} \left( \tilde{A}, \tilde{B} \right) \leq \kappa_*, \) together with (12) and (13), implies that
\[
\left\| \mathcal{K} \left( \tilde{A}, \tilde{B} \right) - \mathcal{K} \left( A, B \right) \right\| \leq \mathcal{E} \left( \tilde{A}, \tilde{B} \right) (8 \left\| A, B \right\| ^2 \left( 1 \left\| B \right\| + \kappa_* \left\| \mathcal{K} \left( A, B \right) \right\| \right) / \lambda_{\text{min}} (Q) / \lambda_{\text{min}} (R)).
\]

Therefore using (13), putting the above inequality together with
\[
\left\| \mathcal{L} \left( \tilde{A}, \tilde{B} \right) - \mathcal{L} \left( A, B \right) \right\|
\[
= \left\| R^{-1} \left( \left| B - B \right| \mathcal{K} \left( \tilde{A}, \tilde{B} \right) + B \left( \mathcal{K} \left( A, B \right) - \mathcal{K} \left( \tilde{A}, \tilde{B} \right) \right) \right) \right\|
\[
\leq \left\| R^{-1} \left\| \left( \left| B - B \right| \right) \mathcal{K} \left( \tilde{A}, \tilde{B} \right) \left\| + \left\| B \right\| \left\| \mathcal{K} \left( A, B \right) - \mathcal{K} \left( \tilde{A}, \tilde{B} \right) \right\| \right. ,
\]

we get the first desired result. To establish the second result, it suffices to let \( \hat{A}_0, \hat{B}_0 \) be the one for which the above supremum over \( \varphi \) is achieved. \]
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8 Auxiliary Results

In this section, we state some auxiliary results that will be used for establishing the main results, and provide their proofs. First, in Lemma 8.1, we consider the total cumulative cost for the case of applying a sub-optimal time invariant feedback matrix to a deterministic system. Then, Lemma 8.2 focuses on explicit calculation of the empirical covariance matrix of the state vectors. Next, in Lemma 8.3 we specify the set of dynamics matrices that possess the same optimal linear feedback matrix.

8.1 Effects of sub-optimal linear feedback policies

**Lemma 8.1.** Consider a noiseless linear dynamical system with the stabilizable dynamics matrices \( \hat{A}, \hat{B} \). That is, \( dX_t = (\hat{A}X_t + \hat{B}U_t) dt \), starting from \( X_0 = x \). Then, if we apply the linear feedback \( U_t = LX_t \), as long as \( X(\hat{A} + \hat{B}L) < 0 \), it holds that

\[
\int_0^\infty c(X_t, U_t) dt = x^\top K (\hat{A}, \hat{B}) x + \int_0^\infty \| R^{1/2} (L - L (\hat{A}, \hat{B})) e^{(\hat{A}+\hat{B}L)t} x \|^2 dt.
\]

**Proof.** Denote \( \hat{D}_1 = \hat{A} + \hat{B}L (\hat{A}, \hat{B}) \) and \( D_2 = \hat{A} + \hat{B}L \). So, the dynamics equation \( dX_t = (\hat{A}X_t + \hat{B}LX_t) dt \) implies that \( X_t = e^{D_2t}x \), which leads to

\[
\int_0^\infty c(X_t, U_t) dt = \int_0^\infty x^\top (Q + L^\top RL) X_t dt = \int_0^\infty x^\top e^{D_2 t} (Q + L^\top RL) e^{D_2 t} x dt = x^\top P x,
\]

where

\[
P = \int_0^\infty e^{D_2^2 t} (Q + L^\top RL) e^{D_2 t} dt = \int_0^\varepsilon e^{D_2^2 t} (Q + L^\top RL) e^{D_2 t} dt + \int_0^\infty e^{D_2^2 t} (Q + L^\top RL) e^{D_2 t} dt = \int_0^\varepsilon e^{D_2^2 t} (Q + L^\top RL) e^{D_2 t} dt + e^{D_2^2 t} P e^{D_2^2 t},
\]

which yields to

\[
Q + L^\top RL = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon e^{D_2^2 t} (Q + L^\top RL) e^{D_2 t} dt
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ P - e^{D_2^2 t} P + e^{D_2^2 t} e^{D_2^2 t} P - e^{D_2^2 t} P e^{D_2^2 t} \right]
\]

\[
= -D_2^2 P - PD_2.
\]

Similar to (38), it holds that \( D_1^\top K (\hat{A}, \hat{B}) + K (\hat{A}, \hat{B}) \hat{D}_1 + Q + L (\hat{A}, \hat{B})^\top RL (\hat{A}, \hat{B}) \). So, subtracting the latter two equalities, we get

\[
\left( D_2 - \hat{D}_1 \right)^\top K (\hat{A}, \hat{B}) + K (\hat{A}, \hat{B}) \left( D_2 - \hat{D}_1 \right)
\]

\[
+ \ D_2^\top \left( P - K (\hat{A}, \hat{B}) \right) + \left( P - K (\hat{A}, \hat{B}) \right) D_2 + S = 0,
\]
which implies the desired result.

Proof. First, denote

\[ Y_t = dX_t X_t^\top + X_t dX_t^\top + dX_t dX_t^\top. \]

Plugging in for \( dX_t \) from (1), we obtain

\[ dY_t = (DX_t dt + C dW_t) X_t^\top + X_t (DX_t dt + C dW_t)^\top + C C^\top dt, \]

where we used the facts \( dt dt = 0 \), \( dW_t dW_t^\top = dt I_{dW} \), and Ito’s Formula (31) to find \( dY_t \).

Then, define the matrix \( Y_t = X_t X_t^\top \), and apply Ito’s Formula (31) to find \( dY_t \):

\[ dY_t = dX_t X_t^\top + X_t dX_t^\top + dX_t dX_t^\top. \]

Thus, \( P - \mathcal{K}(\hat{A}, \hat{B}) = \int_0^\infty e^{D^2 t} F e^{D^2 t} dt, \)

where

\[
F = S + \left[ \mathcal{L} (\hat{A}, \hat{B}) \right]^\top \left( \mathcal{B} \mathcal{K}(\hat{A}, \hat{B}) + \mathcal{K} (\hat{A}, \hat{B}) \mathcal{L} (\hat{A}, \hat{B}) \right].
\]

Then, using \( \mathcal{B} \mathcal{K}(\hat{A}, \hat{B}) = -\mathcal{L} (\hat{A}, \hat{B}) \)

after doing some algebra we obtain

\[ S = \left[ \mathcal{L} (\hat{A}, \hat{B}) \right]^\top \mathcal{R} \left[ \mathcal{L} (\hat{A}, \hat{B}) \right]. \]

Thus, \( P - \mathcal{K}(\hat{A}, \hat{B}) = \int_0^\infty e^{D^2 t} \left[ \mathcal{L} (\hat{A}, \hat{B}) \right]^\top \mathcal{R} \left[ \mathcal{L} (\hat{A}, \hat{B}) \right] e^{D^2 t} dt, \)

which implies the desired result.

8.2 Convergence of empirical covariance matrix of the state vectors

Lemma 8.2. Suppose that for \( t \geq \gamma \), the linear feedback \( L \) is applied to the system (1) such that \( \mathcal{X}(D) < 0 \), where \( D = A_\gamma + B_\gamma L \). Then, we have

\[
\lim_{T \to \infty} \frac{1}{T} \int_\gamma^{\gamma+T} X_t X_t^\top dt = \int_0^{\infty} e^{D s} C C^\top e^{D^\top s} ds.
\]

Proof. First, denote

\[ V_T = \frac{1}{T} \int_\gamma^{\gamma+T} X_t X_t^\top dt. \]

Then, define the matrix \( Y_t = X_t X_t^\top \), and apply Ito’s Formula (31) to find \( dY_t \):

\[ dY_t = dX_t X_t^\top + X_t dX_t^\top + dX_t dX_t^\top. \]

Plugging in for \( dX_t \) from (1), we obtain

\[ dY_t = (DX_t dt + C dW_t) X_t^\top + X_t (DX_t dt + C dW_t)^\top + C C^\top dt, \]

where we used the facts \( dt dt = 0 \), \( dW_t dW_t^\top = dt I_{dW} \), and Ito’s Isometry \( dW_t dW_t^\top = dt I_{dW} \). Thus, we have

\[ Y_{\gamma+T} - Y_\gamma = \int_\gamma^{\gamma+T} dY_t dt = \int_\gamma^{\gamma+T} (DX_t X_t^\top + X_t X_t^\top D^\top + C C^\top) dt + T M_{\gamma,T}, \]

where

\[ M_{\gamma,T} = \frac{1}{T} \int_\gamma^{\gamma+T} X_t dW_t^\top C^\top + \frac{1}{T} \left( \int_\gamma^{\gamma+T} X_t dW_t^\top C^\top \right)^\top. \]
This can equivalently be written as
\[
\frac{1}{T} (X_{\gamma+T}X_{\gamma+T}^\top - X_{\gamma}X_{\gamma}^\top) = DV_T + V_T D^\top + CC^\top + M_{\gamma,T}.
\]
Since \( \mathbf{X}(D) < 0 \), the latter equality implies that
\[
V_T = \int_0^\infty e^{Ds} \left( CC^\top + M_{\gamma,T} + \frac{1}{T} X_{\gamma}X_{\gamma}^\top - \frac{1}{T} X_{\gamma+T}X_{\gamma+T}^\top \right) e^{D^\top s} ds
\]
Now, according to the following statements, the above leads to the desired result, because the terms corresponding to \( M_{\gamma,T}, X_{\gamma}, X_{\gamma+T} \) vanish as \( T \) grows.

1. Clearly, it holds that \( \lim_{T \to \infty} T^{-1/2}\|X_{\gamma}\| = 0 \).
2. Since \( \mathbf{X}(D) < 0 \), the expression
\[
X_{\gamma+T} = e^{DT}X_{\gamma} + \int_{\gamma}^{\gamma+T} e^{D(T-s)}Cdw_w
\]
implies that \( \lim_{T \to \infty} T^{-1/2}\|X_{\gamma+T}\| = 0 \).
3. Putting \( \mathbf{X}(D) < 0 \) together with Doob’s Martingale Convergence Theorem [30, 31], we get \( \lim_{T \to \infty} M_{\gamma,T} = 0 \).

\[\Box\]

8.3 Manifolds of dynamical systems with equal optimal feedback matrices

**Lemma 8.3.** Consider the set of dynamics matrices \( \hat{A}, \hat{B} \) that share optimal feedback with \( \hat{A}_0, \hat{B}_0 \):
\[
\mathcal{M}_0 = \left\{ \left[ \hat{A}, \hat{B} \right] \in \mathbb{R}^{d_A \times (d_A + d_C)} : \mathcal{L} \left( \hat{A}, \hat{B} \right) = \mathcal{L} \left( \hat{A}_0, \hat{B}_0 \right) \right\}.
\]
Then, \( \mathcal{M}_0 \) is a manifold of dimension \( d_A^2 \).

**Proof.** Suppose that for the matrix \( \left[ \hat{A}, \hat{B} \right] = \left[ \hat{A}_0, \hat{B}_0 \right] + \epsilon [M, N] \), it holds that \( \mathcal{L} \left( \hat{A}, \hat{B} \right) = \mathcal{L} \left( \hat{A}_0, \hat{B}_0 \right) \). We find the derivative of \( \mathcal{L} \left( \hat{A}_0, \hat{B}_0 \right) \) along the direction \([M, N]\). First, using the expressions in [48] for \( \hat{A}, \hat{B} \) and for \( \hat{A}_0, \hat{B}_0 \), we get
\[
\left( D_0 + \epsilon M + \epsilon \mathcal{N} \mathcal{L} \left( \hat{A}_0, \hat{B}_0 \right) \right)^\top \mathcal{K} \left( \hat{A}, \hat{B} \right) + \mathcal{K} \left( \hat{A}, \hat{B} \right) \left( D_0 + \epsilon M + \epsilon \mathcal{N} \mathcal{L} \left( \hat{A}_0, \hat{B}_0 \right) \right)
\]
\[
= -Q - \mathcal{L} \left( \hat{A}_0, \hat{B}_0 \right)^\top R\mathcal{L} \left( \hat{A}_0, \hat{B}_0 \right) = D_0^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) + \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) D_0,
\]
where \( D_0 = \hat{A}_0 + \hat{B}_0 \mathcal{L} \left( \hat{A}_0, \hat{B}_0 \right) \). Simplifying the above expressions and letting \( \epsilon \to 0 \), for the matrix
\[
\Delta = \lim_{\epsilon \to 0} \epsilon^{-1} \left( \mathcal{K} \left( \hat{A}, \hat{B} \right) - \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \right)
\]
we have
\[
D_0^\top \Delta + \Delta D_0 + \left( M + \mathcal{N} \mathcal{L} \left( \hat{A}_0, \hat{B}_0 \right) \right)^\top \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) + \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \left( M + \mathcal{N} \mathcal{L} \left( \hat{A}_0, \hat{B}_0 \right) \right) = 0.
\]
Thus, since according to Theorem 2.3, $\bar{X}(D_0) < 0$, it yields to
\[
\Delta = \int_0^\infty e^{D_0^T t} F e^{D_0 t} dt,
\]
where
\[
F = \left( M + N \mathcal{L} \left( \hat{A}_0, \hat{B}_0 \right) \right)^T \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) + \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \left( M + N \mathcal{L} \left( \hat{A}_0, \hat{B}_0 \right) \right).
\]
On the other hand, $\mathcal{L} \left( \hat{A}, \hat{B} \right) = -R^{-1} \hat{B}^T \mathcal{K} \left( \hat{A}, \hat{B} \right)$ gives
\[
0 = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \hat{B}^T \mathcal{K} \left( \hat{A}, \hat{B} \right) - \hat{B}_0^T \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) \right)
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \left( \hat{B}^T - \hat{B}_0^T \right) \mathcal{K} \left( \hat{A}, \hat{B} \right) - \hat{B}_0^T \left( \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) - \mathcal{K} \left( \hat{A}, \hat{B} \right) \right) \right]
= N^T \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) + \hat{B}_0^T \Delta.
\]
So, $\mathcal{M}_0$ is a manifold, and its tangent space consists of matrices $M, N$ satisfying the above equation. To find the dimension, select a $d_X \times d_X$ matrix $P$ arbitrarily, and let
\[
N = -\mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right)^{-1} \int_0^\infty e^{D_0^T t} \left[ P^T \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) + \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) P \right] e^{D_0 t} \hat{B}_0 dt = 0.
\]
Note that since $\lambda_{\min} (Q) > 0$, the inverse $\mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right)^{-1}$ exists. Then, solve for $M$ according to $M + N \mathcal{L} \left( \hat{A}_0, \hat{B}_0 \right) = P$. Therefore, the matrices $M, N$ satisfy in $N^T \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) + \hat{B}_0^T \Delta = 0$, and so correspond to a member of $\mathcal{M}_0$. Conversely, every matrices $M, N$ in the tangent space of $\mathcal{M}_0$ provide a $d_X \times d_X$ matrix $P = M + N \mathcal{L} \left( \hat{A}_0, \hat{B}_0 \right)$ such that $N^T \mathcal{K} \left( \hat{A}_0, \hat{B}_0 \right) + \hat{B}_0^T \Delta = 0$. Thus, $\mathcal{M}_0$ is of dimension $d_X^2$, which is the desired result.

9 Proof of Theorem 2.3

Fixing $\epsilon > 0$, suppose that the control inputs $U_i$ are frozen in intervals of length $\epsilon$ and can change only at times $k \epsilon$, for $k = 0, 1, \cdots$. That is, for all times $t$ satisfying $k \epsilon \leq t < (k + 1) \epsilon$, the action vector is fixed: $U_i = U_{k \epsilon}$. Next, we proceed towards finding a decision-making policy for minimizing the expected average cost. Note that due to the above-mentioned freezing during $\epsilon$-length intervals, resulting decision-making policies can be sub-optimal, and indeed provide an upper bound for the optimal cost value.

Next, fix an arbitrary time horizon $T$, and denote the minimum cost-to-go at time $t$ by
\[
\mathcal{V}_t (X_t) = \inf \mathbb{E} \left[ \int_t^T c \left( X_s, U_s \right) ds \right| \mathcal{F}_t],
\]
where the infimum is taken over non-anticipating policies that freeze the control action in $\epsilon$-length intervals, as elaborated above, and the information at time $t$ is $\mathcal{F}_t = \sigma \left( X_{0:t}, U_{0:t} \right)$; the sigma-field generated by the state and action vectors up to the time. Now, finding an optimal policy is equivalent to applying dynamic programming principle and writing Bellman optimality equations. So, we have
\[
\mathcal{V}_{k \epsilon} (X_{k \epsilon}) = \min_{U_{k \epsilon}} \mathbb{E} \left[ \int_{k \epsilon}^{(k+1) \epsilon} c \left( X_t, U_{k \epsilon} \right) dt + \mathcal{V}_{(k+1) \epsilon} (X_{(k+1) \epsilon}) \right| \mathcal{F}_{k \epsilon}],
\]
subject to the dynamics equation in...
For the sake of simplicity, suppose that \( T/\epsilon \) is an integer. Solving (36) for \( k = T/\epsilon - 1 \), we get the optimal control action \( U^*_{T/\epsilon} = 0 \). Accordingly, this gives

\[
\mathcal{V}_{(k+1)\epsilon} (X_{(k+1)\epsilon}) = X_{(k+1)\epsilon}^T Q X_{(k+1)\epsilon} \epsilon,
\]

for \( k = T/\epsilon - 2 \), which, after substituting in (36), becomes

\[
\mathcal{V}_{k\epsilon} (X_{k\epsilon}) = \min_{U_{k\epsilon}} \int_{k\epsilon}^{(k+1)\epsilon} \mathbb{E} \left[ X_t^T Q X_t | F_{k\epsilon} \right] dt + U_{k\epsilon}^T R U_{k\epsilon} \epsilon + \mathbb{E} \left[ X_{(k+1)\epsilon}^T Q X_{(k+1)\epsilon} | F_{k\epsilon} \right] \epsilon, \tag{37}
\]

where we applied Fubini’s Theorem to derive

\[
\mathbb{E} \left[ \int_{k\epsilon}^{(k+1)\epsilon} c(X_t, U_{k\epsilon}) dt | F_{k\epsilon} \right] = \int_{k\epsilon}^{(k+1)\epsilon} \mathbb{E} \left[ X_t^T Q X_t | F_{k\epsilon} \right] dt + U_{k\epsilon}^T R U_{k\epsilon} \epsilon.
\]

However, solving the dynamics (11) for \( k\epsilon \leq t \leq (k+1)\epsilon \), we obtain

\[
X_t = e^{A_t (t-k\epsilon)} X_{k\epsilon} + \int_{k\epsilon}^{t} e^{A_s (t-s)} C dW_s + \int_{k\epsilon}^{t} e^{A_s (t-s)} ds B_s U_{k\epsilon},
\]

which together with Ito’s Lemma, \( dW_s dW^*_s = I_{dW} ds \) [30], yields to

\[
\mathbb{E} \left[ X_t^T Q X_t | F_{k\epsilon} \right] = \int_{k\epsilon}^{t} \text{tr} \left( e^{A_s^T (t-s) Q e^{A_s (t-s) C C^T}} \right) ds + \left( e^{A_{k\epsilon} (t-k\epsilon)} X_{k\epsilon} + \int_{k\epsilon}^{t} e^{A_s (t-s)} ds B_s U_{k\epsilon} \right)^T Q \left( e^{A_{k\epsilon} (t-k\epsilon)} X_{k\epsilon} + \int_{k\epsilon}^{t} e^{A_s (t-s)} ds B_s U_{k\epsilon} \right).
\]

Plugging these results in the dynamic programming equation in (37), the expression in front of the minimum becomes the following quadratic function of \( U_{k\epsilon} \):

\[
X_{k\epsilon}^T \tilde{Q} X_{k\epsilon} + 2X_{k\epsilon}^T \tilde{G} U_{k\epsilon} + U_{k\epsilon}^T \tilde{R} U_{k\epsilon} + \left( \tilde{A} X_{k\epsilon} + \tilde{B} U_{k\epsilon} \right)^T P_{k+1} \left( \tilde{A} X_{k\epsilon} + \tilde{B} U_{k\epsilon} \right) + \text{tr} \left( \tilde{P}_{k+1} CC^T \right),
\]

where \( P_{k+1} = Q \epsilon \), and

\[
\tilde{A} = e^{A_{k\epsilon} \epsilon}, \quad \tilde{B} = \int_{0}^{\epsilon} e^{A_s} ds B_s, \quad \tilde{Q} = \int_{0}^{\epsilon} e^{A_{k\epsilon}^T + t} Q e^{A_{k\epsilon} \epsilon} dt, \quad \tilde{G} = \int_{0}^{\epsilon} e^{A_{k\epsilon}^T + t} Q \left( \int_{0}^{\epsilon} e^{A_s \epsilon} ds \right) B_s dt, \quad \tilde{R} = R \epsilon + \int_{0}^{\epsilon} B_s^T \left( \int_{0}^{\epsilon} e^{A_{k\epsilon}^T + t} Q e^{A_s \epsilon} ds \right) B_s dt,
\]

\[
\tilde{P}_{k+1} = P_{k+1} \int_{0}^{\epsilon} e^{A_{k\epsilon} s} ds + \int_{0}^{\epsilon} \left( \int_{0}^{\epsilon} e^{A_{k\epsilon} s} Q e^{A_s} ds \right) dt.
\]
Note that in the last equation above, we used Ito Isometry \[31\] to find $\bar{P}_{k+1}$. Now, performing the minimization the optimal control action is

$$U_{k*} = - \left( \bar{B}^\top P_{k+1} \bar{B} + \bar{R} \right)^{-1} \left( \bar{B}^\top P_{k+1} \bar{A} + \bar{G}^\top \right) X_{k*},$$

and \[31\] leads to

$$\mathcal{V}_{k*}(X_{k*}) = X_{k*}^\top P_k X_{k*} + \text{tr} \left( CC^\top \left[ P_{k+1} \int_0^\epsilon e^{\bar{A}^* t} e^{A^* s} ds + \int_0^\epsilon \left( \int_0^t e^{\bar{A}^* s} Q e^{A^* s} ds \right) dt \right] \right), \quad (38)$$

where $P_k$ is calculated according to the discrete time Riccati equation

$$P_k = \bar{Q} + \bar{A}^\top P_{k+1} \bar{A} - \left( \bar{G} + \bar{A}^\top P_{k+1} \bar{B} \right) \left( \bar{B}^\top P_{k+1} \bar{B} + \bar{R} \right)^{-1} \left( \bar{B}^\top P_{k+1} \bar{A} + \bar{G}^\top \right). \quad (39)$$

It is shown that if there is some matrix $L$ such that $\lambda_{\text{max}} \left( \tilde{A} + \tilde{B} L \right) < 1$, then as $k \to -\infty$, the matrix $P_k$ in the above discrete time Riccati equation converges to a uniquely existing matrix $P$ that solves the algebraic Riccati equation

$$P = \bar{Q} + \bar{A}^\top P \bar{A} - \left( \bar{G} + \bar{A}^\top P \bar{B} \right) \left( \bar{B}^\top P \bar{B} + \bar{R} \right)^{-1} \left( \bar{B}^\top P \bar{A} + \bar{G}^\top \right), \quad (40)$$

regardless of the terminal matrix for the largest value $k+1$, which here corresponds to $P_{T/\epsilon}$ \[43\] \[44\] \[35\].

Next, we show that if $\epsilon$ is sufficiently small, then the matrix $L$ mentioned above exists. To that end, write

$$\tilde{A} = e^{A^* \epsilon} = \sum_{n=0}^\infty \frac{A^* \epsilon^n}{n!} = I_{dx} + \epsilon M(\epsilon) A^*,$$

$$\tilde{B} = \sum_{n=0}^\infty \int_0^\epsilon \frac{A^n \epsilon^n}{n!} ds B^* = \sum_{n=0}^\infty \frac{A^n \epsilon^{n+1}}{(n+1)!} B^* = \epsilon M(\epsilon) B^*,$$

where

$$M(\epsilon) = \sum_{n=1}^\infty \frac{A^n \epsilon^{n-1}}{n!} = I_{dx} + \epsilon \sum_{n=2}^\infty \frac{A^{n-1} \epsilon^{n-2}}{n!}.$$

Then, letting $L$ be as in Assumption \[22\] if $\epsilon$ is small enough, it holds that

$$\lambda_{\text{max}}(M(\epsilon) (A^* + B^* L)) < 0. \quad (41)$$

That is because the eigenvalues of the matrix $M(\epsilon) (A^* + B^* L)$ are continuous functions of $\epsilon$, and for $\epsilon = 0$ we have $\lambda(0) (A^* + B^* L) = \lambda((A^* + B^* L)) < 0$, according to Assumption \[22\]. Hence, $\tilde{A} + \tilde{B} L = I_{dx} + M(\epsilon) (A^* + B^* L) \epsilon$ implies that eigenvalues of $\tilde{A} + \tilde{B} L$ are exactly one plus the eigenvalues of $M(\epsilon) (A^* + B^* L)$. So, it holds that

$$\lambda_{\text{max}} \left( \tilde{A} + \tilde{B} L \right)^2 \leq 1 + 2 \lambda(0) (A^* + B^* L) + \lambda_{\text{max}}(M(\epsilon) (A^* + B^* L))^2 \epsilon^2. \quad (42)$$

Now, putting \[41\] and \[42\] together, if $\epsilon$ is small enough, then $\lambda_{\text{max}} \left( \tilde{A} + \tilde{B} L \right) < 1$. Henceforth, suppose that $\epsilon$ is sufficiently small so that the latter inequality holds true.

As long as $\epsilon > 0$ is small enough as described above, letting the time horizon $T$ tend to infinity, the $\epsilon$-length frozen optimal policy for minimizing the expected average cost is

$$U'_{k*} = - \left( \tilde{B}^\top P \tilde{B} + \tilde{R} \right)^{-1} \left( \tilde{B}^\top P \tilde{A} + \tilde{G}^\top \right) X_{k*}, \quad (43)$$
where $P$ is the unique solution of (40). On the other hand, for a fixed time horizon $T$, as $\varepsilon$ shrinks the discrete-time Riccati equation in (39) becomes a continuous-time Riccati equation as follows. First, we have

$$\lim_{\varepsilon \to 0} \frac{A - I_{dx}}{\varepsilon} = A_*, \quad \lim_{\varepsilon \to 0} \frac{B}{\varepsilon} = B_*, \quad \lim_{\varepsilon \to 0} \frac{Q}{\varepsilon} = Q, \quad \lim_{\varepsilon \to 0} \frac{G}{\varepsilon} = 0, \quad \lim_{\varepsilon \to 0} \frac{R}{\varepsilon} = R.$$  

Using these limits, letting $\varepsilon \to 0$ in (39) leads to

$$\lim_{\varepsilon \to 0} \frac{P_k - P_{k+1}}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\tilde{Q}}{\varepsilon} + \lim_{\varepsilon \to 0} \frac{\tilde{A}^T P_{k+1} \tilde{A} - P_{k+1}}{\varepsilon} - \lim_{\varepsilon \to 0} \frac{\tilde{G} + \tilde{A}^T P_{k+1} \tilde{B}}{\varepsilon} \left( \frac{\tilde{B}^T P_{k+1} \tilde{B} + \tilde{R}}{\varepsilon} \right)^{-1} \left( \frac{\tilde{B}^T P_{k+1} \tilde{A} + \tilde{G}^T}{\varepsilon} \right) = Q + A_*) P_{k+1} + P_{k+1} A_* - P_{k+1} B_* R^{-1} B_*^T P_{k+1},$$

that is, the backward differential equation

$$-\frac{dP_t}{dt} = \Phi_{A_*,B_*}(P),$$  \hspace{1cm} (44)

with the terminal condition $P_T = 0$. Thus, as $\varepsilon \to 0$, the optimal policy becomes

$$U_t^* = -R^{-1} B_*)^T P_t X_t,$$

where $P_t$ is the solution of (44). Similarly, letting $\varepsilon \to 0$ in (40), we get the optimal policy $U_t^* = \mathcal{L}(A_*,B_*) X_t$ for minimizing the infinite horizon expected average cost, where

$$\mathcal{L}(A_*,B_*) = -R^{-1} B_*)^T K(A_*,B_*),$$

and $K(A_*,B_*)$ solves $\Phi_{A_*,B_*}(P) = 0$. Equivalently, letting $P_{0,T}$ be the solution of (44) when the time horizon is $T$, it holds that $\lim_{T \to \infty} P_{0,T} = K(A_*,B_*)$, where $K(A_*,B_*)$ solves $\Phi_{A_*,B_*}(P) = 0$. Note that all these relationships rely on the convergence of discrete time Riccati equation (39) to the algebraic Riccati equation (40), as $T \to \infty$.

Next, subtracting $\mathcal{V}_{(k+1)\varepsilon}(X_{k\varepsilon})$ from both sides of (36), dividing by $\varepsilon$, and letting $\varepsilon \to 0$, Itô Isomery implies that

$$-\frac{\partial \mathcal{V}_{t}(X_{t})}{\partial t} dt = \min_{U_t} c(X_t,U_t) dt + E\left[dX_t^T \frac{\partial \mathcal{V}_{t}(X_{t})}{\partial X_t} + \frac{1}{2} dX_t^T \frac{\partial^2 \mathcal{V}_{t}(X_{t})}{\partial X_t \partial X_t^T} dX_t^T F_t\right],$$

where we used the limits of the matrices $\tilde{Q}, \tilde{G}, \tilde{R}$ as $\varepsilon \to 0$ to find the expression on the right-hand-side of the above equality. Note that the above partial derivatives exist according to (38) together with Dominated Convergence Theorem. Hence, substituting for $dX_t$ from the dynamics (1), and leveraging Itô’s Lemma, we obtain the Hamilton-Jacobi-Bellman [15] equation

$$-\frac{\partial \mathcal{V}_{t}(X_{t})}{\partial t} = \min_{U_t} c(X_t,U_t) + \frac{\partial \mathcal{V}_{t}(X_{t})}{\partial X_t}^T (A_* X_t + B_* U_t) + \frac{1}{2} \text{tr} \left( \frac{\partial^2 \mathcal{V}_{t}(X_{t})}{\partial X_t \partial X_t^T} CC^T \right).$$

(45)

Further, letting $\varepsilon \to 0$, the expression in (38) gives

$$\mathcal{V}_{t}(X_{t}) = X_t^T P_t X_t + \int_{t}^{T} \text{tr} \left( CC^T P_s \right) ds,$$  \hspace{1cm} (46)

where $P_t$ solve (44). This can be equivalently obtained using the fact that a quadratic function of the form $\mathcal{V}_{t}(X_{t}) = X_t^T F_t X_t + \varphi_t$ solves the partial differential equation (45), as long as

$$-\frac{d\varphi_t}{dt} - X_t^T \frac{dF_t}{dt} X_t = \min_{U_t} X_t^T Q X_t + U_t^T R U_t + 2X_t^T F_t (A_* X_t + B_* U_t) + \text{tr} \left( F_t CC^T \right),$$
which after solving for $U_t$ gives the optimal policy $U_t^* = -R^{-1}B_t^T F_t X_t$, as well as
\[-\frac{d\varphi_t}{dt} - X_t^T \frac{dF_t}{dt} X_t = X_t^T Q X_t + 2X_t^T F_t (A_* X_t) - X_t^T F_t B_* R^{-1} B_t^* F_t X_t + \text{tr} (F_t C C^T).\]
Because the equation above needs to hold for an arbitrary $X_t$, it splits to
\[-\frac{dF_t}{dt} = \Phi_{A_*,B_*}(F_t), \quad \frac{d\varphi_t}{dt} = -\text{tr} (F_t C C^T),\]
that is, $F_t$ solves (44). Further, note that cost-to-go at time $T$ is zero because time-to-go is zero, which provides the terminal condition $\mathcal{V}_T (X_T) = 0$, implying that $\varphi_T = \int_0^T \text{tr} (C C^T F_s)\,ds$. Therefore, the solutions $F_t, \varphi_t$ of (44) lead to the same expression as in (40).

Finally, the expected average cost of the policy $U_t = \mathcal{L} (A_*,B_*) X_t$ is the limit of the expected average cost of the policy $U_t = -R^{-1}B_t^T P_t,X_t X_t$, as $T \to \infty$;
\[
\limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T c(X_s, U_s)\,ds\right] = \limsup_{T \to \infty} \frac{1}{T} \int_0^T \text{tr} (C C^T P_{s,T})\,ds = \text{tr} (C C^T \lim_{T \to \infty} P_{s,T}) = \text{tr} (C C^T \mathcal{K}(A_*,B_*)).
\]

Moreover, suppose that $C = 0$, and apply the policy $U_t = \mathcal{L} (A_*,B_*) X_t$. Then, the state trajectory becomes $X_t = e^{D_* t} \sigma_0$, where $D_*= A_* + B_* \mathcal{L}(A_*,B_*)$. So, by (40), we have
\[
X_0^T \mathcal{K}(A_*,B_*) X_0 = \int_0^\infty X_t^T \left( Q + \mathcal{L}(A_*,B_*)^T \mathcal{R}(A_*,B_*) \right) X_t^T dt
\]
\[
= X_0^T \int_0^\infty e^{D_* t} \left( Q + \mathcal{L}(A_*,B_*)^T \mathcal{R}(A_*,B_*) \right) e^{D_* t} dt X_0,
\]
for an arbitrary initial state $X_0$. Thus, it holds that
\[
\mathcal{K}(A_*,B_*) = \int_0^\infty e^{D_* t} \left( Q + \mathcal{L}(A_*,B_*)^T \mathcal{R}(A_*,B_*) \right) e^{D_* t} dt.
\] (47)
Since $Q$ is positive definite, the above equality implies that $\mathcal{X}(D_*) < 0$, as well as
\[
D_*^T \mathcal{K}(A_*,B_*) + \mathcal{K}(A_*,B_*) D_* + Q + \mathcal{L}(A_*,B_*)^T \mathcal{R}(A_*,B_*) = 0.
\] (48)

So far, we have shown that by restricting our search for an optimal decision-making policy to the class of policies that the control action is frozen during intervals of length $\epsilon$, and then letting $\epsilon$ decay to vanish, we obtain optimal policies given by (44). Next, we shall show that these policies are optimal in the larger class of all control policies satisfying the information criteria at every time. That is, for all $t$, the control action $U_t$ can be determined using $F_t = \sigma (X_{0,t}, U_{0,t})$. For this purpose, first note that the decision-making policy $U_t = R^{-1}B_t^T \mathcal{K}(A_*,B_*) X_t$ provides an upper-bound for the optimal expected average cost. That is,
\[
\inf_{\pi} J(\pi) \leq \text{tr} (C C^T \mathcal{K}(A_*,B_*)).
\]

Now, suppose that there is another policy, denoted by $\bar{\pi}$, that satisfies $\mathcal{J}(\bar{\pi}) \leq \text{tr} (C C^T \mathcal{K}(A_*,B_*))$. Define cost-to-go of the policy $\bar{\pi}$ by
\[
\bar{V}_t (X_t) = \mathbb{E} \left[ \int_t^T c_{\bar{\pi}} (X_s, U_s)\,ds \middle| F_t \right],
\]
32
where $T$ is large enough to satisfy $\tilde{V}_t(X_t) \leq 2X_t^\top K(A_*, B_*) X_t + 2T \text{tr} \left( CC^\top K(A_*, B_*) \right)$, for all $0 \leq t \leq 1$. Note that such $T$ exists since $\tilde{\pi}$ provides a smaller expected average cost than the policy $U_t = R^{-1}B^\top K(A_*, B_*) X_t$, and the desired upper-bound for $\tilde{V}_t(X_t)$ is $2V_t(X_t)$; two times the cost-to-go of the policy $U_t = R^{-1}B^\top K(A_*, B_*) X_t$. Next, writing $\tilde{V}_t(X_t) = \mathbb{E} \left[ \int_0^{t+\epsilon} c_\pi(X_s, U_s) ds + \tilde{V}_{t+\epsilon}(X_{t+\epsilon}) \bigg| \mathcal{F}_t \right]$, subtract $\tilde{V}_{t+\epsilon}(X_t)$ from both sides, and divide by $\epsilon$. Letting $\epsilon$ decay to zero, the upper-bound for $\tilde{V}_t(X_t)$ in terms of $V_t(X_t)$ implies that according to Dominated Convergence Theorem, the following derivatives exist and it holds that

$$
-\frac{\partial \tilde{V}_t(X_t)}{\partial t} = c(X_t, \tilde{\pi}(\mathcal{F}_t)) + \frac{\partial \tilde{V}_t(X_t)^\top}{\partial X_t} (A_* X_t + B_* \tilde{\pi}(\mathcal{F}_t)) + \frac{1}{2} \text{tr} \left( \frac{\partial^2 \tilde{V}_t(X_t)}{\partial X_t \partial X_t^\top} CC^\top \right).
$$

Now, note that since $c(X_t, U_t)$ as well as $B_* U_t$ are continuous functions of $U_t$, the above partial differential equation for $\tilde{V}_t(X_t)$ indicates that $\tilde{\pi}(\mathcal{F}_t)$ is a continuous function of $X_t$. This, together with the fact that $W_t$ is an almost surely continuous function of time $t$, in lights of the dynamics equation in (1), leads to continuity of state trajectory $X_t$; i.e., $U_t = \tilde{\pi}(\mathcal{F}_t)$ is continuous as $t$ varies. Thus, decision-making policies that freeze for $\epsilon$-length intervals provide accurate approximations of $U_t = \tilde{\pi}(\mathcal{F}_t)$ in a sense that there exists a sequence $\left\{ U_t^{(n)} \right\}_{n=1}^\infty$ such that $U_t^{(n)}$ freezes during intervals of the length $1/n$, and it holds that

$$
\limsup_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E} \left[ \left\| U_t^{(n)} - \tilde{\pi}(\mathcal{F}_t) \right\| \right] dt = 0.
$$

Therefore, we have

$$
\mathcal{J}\left( \tilde{\pi} \right) \geq \inf_{\epsilon > 0} \inf \mathcal{J}(\pi) = \text{tr} \left( K(A_*, B_*) CC^\top \right),
$$

where the inner infimum is taken over all policies that freeze during $\epsilon$-length intervals. This shows that the policy $U_t = -R^{-1}B^\top K(A_*, B_*) X_t$ is an optimal one, which completes the proof.