Borel and Prime Quantization Schemes

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Abstract.

We briefly review the two main quantization schemes to which Prof. Doebner made contributions and point out their relationships to some of the other schemes existing in the current literature. Unfortunately, it will not be possible to do more than just enumerate the topics touched by Prof. Doebner’s work. Extensive reviews of his works are now available in various places.

1. A word of appreciation

Prof. Doebner’s work on quantization, carried out over several decades, in collaboration with a large number of students and coworkers, centered mainly around Borel and prime quantizations. A description of much of that work and the most important results may be found in the two references [1, 2] and additional references to the original literature cited therein.

But before attempting a resume of that work, I would like to say a few words, in Prof. Doebner’s own language.

Lieber Herr Professor Doebner,

heute stehen Sie im Zenit einer reichhaltigen wissenschaftlichen und professionellen Karriere, mit der Sie sich den Beifall und die Hochschätzung von Freunden, Kollegen und Studenten gleichermaßen verdient haben.

Mein eigenes Leben haben Sie auf unzählige Weisen berührt. Erlauben Sie mir daher bitte, Ihnen die wärmsten Gratulationen und herzlichsten guten Wünsche zu diesem besonderen Geburtstag zu überbringen.

Persönlich hatte ich das Glück, Sie während fast der Hälfte Ihrer achtzig Lebensjahre zu kennen. Während dieser Zeit habe ich - und unzähligen anderen ging es genauso wie mir - sowohl wissenschaftlich als auch persönlich von Ihrer Bekanntheit profitiert. Besonders hoch schätze ich meine Jahre in Clausthal, als ich unter Ihrer Anleitung arbeiten und mich von Ihrem offenen, internationalen Geist inspirieren lassen durfte. Die heutige wissenschaftliche Welt ist ein globales Kaffeehaus, und viele von uns haben durch Sie den Zugang zu diesem Club gefunden.

Wir alle wünschen Ihnen von ganzem Herzen, dass dieser Festtag viele glückliche Wiederholungen haben möge.

2. What is quantization?

Quantization is the process of lifting a classical mechanical theory to its quantum counterpart. Classical mechanics lives on a phase space, \( \Gamma \), and is defined by the symplectic geometry of this
space. This is an even dimensional differential manifold and comes equipped with a two-form, \( \Omega \), which defines a Poisson bracket on the functions on \( \Gamma \). If one starts with the configuration space \( \mathcal{Q} \) of the classical system, the phase space is the cotangent bundle of this space, \( \Gamma = T^* \mathcal{Q} \).

The states of classical mechanics are normalized probability measures \( \mu \) on \( \Gamma \),

\[
\int_{\Gamma} d\mu = 1. 
\]

Pure states are the extremal points of this set, i.e., \( \delta \)-measures, concentrated at individual points \( \gamma \in \Gamma \). Observables in classical mechanics are real-valued functions on \( \Gamma \). The dynamics of a classical system is governed by the Hamiltonian equations.

Quantum mechanics lives on a separable, complex Hilbert space, \( \mathcal{H} \), and its structure is defined mainly by the functional analytic properties of \( \mathcal{H} \). Observables are self-adjoint operators and states are normalized, positive, trace class operators, \( \rho \), on the Hilbert space \( \mathcal{H} \), also called density matrices:

\[
\text{Tr}[\rho] = 1. 
\]

Pure states are again the extremal points of this set, i.e., they are one-dimensional projection operators,

\[
\rho = \frac{|\phi\rangle\langle\phi|}{\|\phi\|^2}, \quad \phi \in \mathcal{H}. 
\]

The dynamics of a quantum system is governed by the Schrödinger equation.

Naturally associated to the Hilbert space \( \mathcal{H} \) is a symplectic manifold, the projective space \( \mathbb{CP}(\mathcal{H}) \), which is infinite dimensional if \( \mathcal{H} \) is infinite dimensional. This manifold comes equipped with its own two form, \( \Omega_{\text{FS}} \), the Fubini-Study two-form.

Quantization may now be looked upon as one of two processes.

1. Starting with a classical system \((\Gamma, \Omega)\), find a Hilbert space \( \mathcal{H} \) and a mapping \( Q \) from the smooth functions on \( \Gamma \) to the self-adjoint operators on \( \mathcal{H} \), which for some appropriate subset of functions \( f, g, \ldots \), maps the Poisson bracket to the commutator bracket:

\[
[Q(f), Q(g)] = i\hbar Q(\{f, g\}),
\]

where \( [A, B] = AB - BA \) is the commutator bracket of the operators \( A, B \) on \( \mathcal{H} \) and \( \{f, g\} \) is the Poisson bracket of the functions \( f, g \) on \( \Gamma \).

It is well-known that this relation cannot be expected to hold exactly for all functions. However, in an asymptotic sense, it can be made to hold for all smooth functions on \( \Gamma \). In geometric quantization, or Borel quantization, studied by Doebner, et al, one only looks for the above condition to hold on a subset of functions. In coherent state quantization, Berezin-Töplitz quantization or prime quantization, studied by Doebner, et al, the focus is more on the asymptotic validity of the above relation.

2. Starting with the classical system \((\Gamma, \Omega)\), find a Hilbert space \( \mathcal{H} \) and a symplectomorphism

\[
s : (\Gamma, \Omega) \longrightarrow (\mathbb{CP}(\mathcal{H}), \Omega_{\text{FS}}),
\]

i.e., a map which preserves the symplectic structures of the spaces. This is usually the focus of coherent state quantization.

It is also possible to attempt a quantization by starting with the configuration space \( \mathcal{Q} \) rather than on the phase space \( \Gamma \). However, in such cases as, for example with Borel or Segal quantization, the process can later be also lifted to the phase space by passing to the cotangent bundle \( T^* \mathcal{Q} \).

We move on now to look at the two quantization schemes to which Prof. Doebner, along with his students and co-workers, have made extensive contributions over several decades.
3. Borel quantization

The theme of Borel quantization dominated research at Prof. Doebner’s Clausthal institute for over two decades, from the mid-seventies to the mid-nineties. Several generations of students, visitors and co-workers participated in this development. At the end it was a tour de force, which vastly generalized and in fact completed a technique that was independently proposed by Segal.

In many ways, Borel quantization is a much more constructive scheme than many other quantization techniques that are fashionable to study these days. It is aesthetically pleasing in that it shares the simplicity of canonical quantization, yet greatly generalizes it. Borel quantization starts out with the configuration space $Q$ of the system. The question it addresses is how to take into account any non-trivial geometry that $Q$ may have and also any internal degrees of freedom that may be present.

Consider a flat configuration space $Q = \mathbb{R}^N$, for a system with $N$ degrees of freedom. The phase space is $\Gamma = T^*Q \simeq \mathbb{R}^{2N}$. Canonical quantization of the system gives us the rule:

$$H = L^2(\mathbb{R}^N, dx)$$

$q_i \mapsto Q(q_i) = \text{operator of multiplication by } x_i \text{ on } H$

$p_i \mapsto Q(p_i) = -i\hbar \frac{\partial}{\partial x_i}$

the constant function $1 \mapsto I_H \text{ (identity operator on } H)\$,

The resulting commutations relations are well-known:

$$[Q(q_i), Q(p_j)] = i\hbar Q(\{q_i, p_j\}) = i\hbar \delta_{ij}$$

Note that according to this rule, a real function $f(q) = f(x)$ on $Q$ would be quantized as:

$$Q(f) = \text{operator of multiplication by } f(x)$$

and a general vector field $X = \sum_{i=1}^N \alpha_i(x) \frac{\partial}{\partial x_i}$, $\alpha_i = \text{ real functions}$, would (up to questions about self-adjointness) be quantized as:

$$Q(X) = -i\hbar \times (\text{the corresponding symmetrized differential operator})$$

The resulting commutation rules are then,

$$[Q(f), Q(g)] = 0$$

$$[Q(X), Q(Y)] = -i\hbar \times (\text{differential operator of the commutator } [X, Y])$$

$$[Q(f), Q(X)] = i\hbar Q(X(f))$$

The question now is how to extend these rules if the geometry of $Q$ is non-trivial.

The method proposed by Segal was to take the Hilbert space $H = L^2(Q, \mu)$, where $\mu$ is a measure on $Q$ which is locally equivalent to the Lebesgue measure,

$$d\mu(x) = \rho(x) dx_1 dx_2 \ldots dx_N,$$

in local coordinates and $\rho$ is a positive, non-vanishing function. Such a measure can always be found if $Q$ is orientable. One then defines $Q(f)$ as the operator of multiplication by $f$ and for a vector field $X = \sum_{i=1}^N \alpha_i(x) \frac{\partial}{\partial x_i}$,

$$Q(X) = -i\hbar \left( X + \frac{1}{2} \left[ X(\log \rho) + \sum_{i=1}^N \frac{\partial \alpha_i}{\partial x_i} \right] \right).$$
The commutation rules

\[
\begin{align*}
\{Q(f), Q(g)\} &= 0, \\
\{Q(X), Q(Y)\} &= -i\hbar \times \text{(differential operator of the commutator \([X, Y]\))}, \\
\{Q(f), Q(X)\} &= i\hbar Q(X(f))
\end{align*}
\]

are then preserved. This is clearly a generalization of the canonical quantization method. A group theoretical method was suggested by Mackey, within the context of the theory of induced representations of finite groups. The method of Borel quantization developed by Doebner, \textit{et al}, is a much more general method, which combines the Segal and Mackey techniques. In a group theoretical sense, Borel quantization is somewhat akin to finding suitable unitary irreducible representations of \(\text{Diff}(\mathcal{Q})\), the diffeomorphism group of the manifold \(\mathcal{Q}\). These representations are required to admit systems of imprimitivity based on \(\mathcal{Q}\).

Borel quantization starts out, basically the same way as Segal quantization, with the configuration space manifold, \(\mathcal{Q}\) and the Hilbert space

\[
\mathcal{H} = \mathbb{C}^N \otimes L^2(\mathcal{Q}, \mu)
\]

The \(\mathbb{C}^N\) accomodates internal degrees of freedom. Once again functions \(f : \mathcal{Q} \to \mathbb{R}\) define operators of multiplication \(Q(f)\) on \(\mathcal{H}\), which constitute the set of position operators of the theory. In particular, the characteristic functions \(\chi_E\) of Borel sets of \(\mathcal{Q}\) define projection operators \(P(E) = Q(\chi_E)\) on \(\mathcal{Q}\). The momentum operators of the quantized theory are obtained from the set of complete vector fields \(X\) of \(\mathcal{Q}\). Such vector fields define shifts\(^1\) or one-parameter abelian groups of diffeomorphisms

\[
\phi_s^X, \quad s \in \mathbb{R}
\]

of \(\mathcal{Q}\). The quantization of these shifts give rise to one-parameter unitary groups

\[
V(\phi_s^X), \quad s \in \mathbb{R}.
\]

Moreover, \(\{V(\phi_s^X), P\}\) is a system of imprimitivity with respect to the group of real numbers \(\mathbb{R}\) and the Borel \(\mathbb{R}\)-space \(\mathcal{Q}\) with group action \(\phi_s^X\), i.e.,

\[
V(\phi_s^X)P(E)V(\phi_{s}^X) = P(\phi_{s}^X(E)).
\]

The triple \(\{\mathcal{H}, P, V\}\) is called a localized quantum system with shifts. The generators of these unitary groups are then the quantized momentum operators \(Q(X)\), whose action on a vector \(\psi \in \mathcal{H}\) is given by

\[
Q(X)\psi = -i\hbar \mathcal{L}_X \psi - \frac{i\hbar}{2} \text{div}_\mu(X)\psi + \omega(X)\psi,
\]

where \(\mathcal{L}_X\psi\) is the Lie derivative of \(\psi\) along \(X\).

It is now possible to show that the following commutation relations hold:

\[
\begin{align*}
\{Q(f), Q(g)\} &= 0, \\
\{Q(X), Q(f)\} &= -i\hbar Q(\mathcal{L}_X f), \\
\{Q(X), Q(Y)\} &= -i\hbar Q([X, Y]) - i\hbar \Omega(X, Y),
\end{align*}
\]

for all \(f, g \in \mathcal{C}^\infty(\mathcal{Q})\), \(X, Y \in \mathfrak{X}_\mathcal{Q}(\mathcal{Q})\), and where,

\[
\Omega(X, Y) = -i\hbar [\omega(X), \omega(Y)] + \mathcal{L}_X \omega(Y) - \mathcal{L}_Y \omega(X) - \omega([X, Y]).
\]

\(^1\) Also called flows in the literature.
The two-form $\Omega$ and the one-form $\omega$ on $\mathcal{Q}$ are related in the same way as the curvature two-form $\frac{i}{\hbar}\Omega$ of a $C^1$-bundle and its connection form $\frac{i}{\hbar}\omega(X)$. Indeed, one can show that if $D$ is the covariant derivative defined by the connection, then $D\Omega = 0$, which is the Bianchi identity.

Finally, there is the question of classification or enumeration of the number of possible such quantizations, starting from a given configuration space manifold $\mathcal{Q}$. For $N = 1$, i.e., $\mathfrak{h} = L^2(\mathcal{Q}, \mu)$ one has the result

**Theorem 3.1** The equivalence classes of Borel quantizations in the above sense are in one-to-one correspondence with $I^2(\mathcal{Q}) \times \pi_1(\mathcal{Q})^* \times \mathbb{R}$, where $I^2(\mathcal{Q})$ denotes the set of closed real two-forms $B$ on $\mathcal{Q}$, satisfying the integrality condition

$$\frac{1}{2\pi\hbar} \int_{\Sigma} B \in \mathbb{Z},$$

for all closed two-surfaces $\Sigma$ in $\mathcal{Q}$ and where $\pi_1(\mathcal{Q})^*$ denotes the group of characters of the first fundamental group of $\mathcal{Q}$.

For arbitrary $k$ one has a somewhat weaker result.

### 4. Prime quantization

Let us now say a few words about the other quantization technique that Prof. Doebner worked on. This technique, to which the name prime quantization was given, has connections to coherent state quantization and Berezin-Töplitz quantization. The method starts out on the phase space $\Gamma$ of the classical system and exploits the functional analytical rather than geometric structures of the system. This quantization also sheds light on the time honoured ordering problem of quantum mechanics.

Very simply, we start with the symplectic manifold $(\Gamma, \Omega)$ and the Hilbert space $\mathfrak{h} = L^2(\Gamma, \Omega)$. The next step is to look for reproducing kernel subspaces of $\mathfrak{h}$. This means finding subspace $\mathfrak{h}_K$ of $\mathfrak{h}$ on which the evaluation maps

$$E_{\gamma} : \mathfrak{h}_K \rightarrow \mathbb{C}, \quad E_{\gamma}(\psi) = \psi(\gamma), \quad \gamma \in \Gamma, \; \psi \in \mathfrak{h}_K$$

are continuous. Then,

$$K(\gamma, \gamma') = E_{\gamma}E_{\gamma'}^*$$

is a reproducing kernel.

For any Borel set $\Delta \subset \Gamma$, the operator

$$a_K(\Delta) = \int_{\Delta} E_{\gamma}^*E_{\gamma} \, d\mu(\gamma)$$

on $\mathfrak{h}_K$ is positive. A classical observable $f : \Gamma \rightarrow \mathbb{R}$ is then quantized by the prescription,

$$Q(f) = \int_{\Gamma} f(\gamma)E_{\gamma}^*E_{\gamma} \, d\mu(\gamma).$$

One then has to prove the asymptotic validity of

$$[Q(f), Q(g)] \simeq i\hbar Q(\{f, g\}).$$

Clearly, the challenge in this method is to find the appropriate reproducing kernel Hilbert space $\mathfrak{h}_K$, which is reflective of the physical problem. Examples are many.

As mentioned earlier, the method is similar to
1. Berezin-Töplitz quantization when $\Gamma$ is a complex Kähler manifold and $\mathcal{H}_K$ is the subspace of holomorphic functions in $L^2(\Gamma, \Omega)$.

2. Coherent state quantization, with the coherent states being given by the vectors

$$\psi_\gamma = K(\gamma, \cdot), \quad \gamma \in \Gamma.$$

**Acknowledgements**

The author would like to thank the organizers of the QTS7 meeting for having given him this opportunity to speak at this special session in honour of Prof. Doebner.

**References**

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