INJECTIONS OF ARTIN GROUPS

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Abstract. We study those Artin groups which, modulo their centers, are fi-
nite index subgroups of the mapping class group of a sphere with at least 5
punctures. In particular, we show that any injective homomorphism between
these groups is parameterized by a homeomorphism of a punctured sphere
together with a map to the integers. We also give a generating set for the au-
tomorphism group of the pure braid group on at least 4 strands. The technique,
following Ivanov, is to prove that every superinjective map of the complex of
curves of a sphere with at least 5 punctures is induced by a homeomorphism.

1. Introduction

We investigate injective homomorphisms between Artin groups which, modulo their
center, embed as finite index subgroups in the mapping class group of an m-times
punctured sphere $S_m$, where $m \geq 5$.

The extended mapping class group of a surface $F$ is the group of isotopy classes of
homeomorphisms of $F$:

$$\text{Mod}(F) = \pi_0(\text{Homeo}(F)).$$

Theorem 1. Let $m \geq 5$. If $G$ is a finite index subgroup of $\text{Mod}(S_m)$ and $\rho : G \to \text{Mod}(S_m)$ is an injective homomorphism, then there is a unique $f \in \text{Mod}(S_m)$ so that $\rho(g) = fgf ^{-1}$ for all $g \in G$.

In particular, the Theorem 1 applies to four infinite families of Artin groups modulo
their centers: $A(A_n)/Z$, $A(B_n)/Z$, $A(\tilde{C}_{n-1})$, and $A(\tilde{A}_{n-1})$ where $n = m - 2$ (see below for definitions). Throughout, $Z$ denotes the center of the ambient group; the
groups $A(\tilde{C}_{n-1})$ and $A(\tilde{A}_{n-1})$ have trivial center. Thus, Theorem 1 is a general-
ization of work of Charney–Crisp, who computed the automorphism groups of the
aforementioned Artin groups using similar techniques [9].

Following Ivanov, we prove Theorem 1 by translating the problem into one about
the curve complex $C(S_m)$. This is the abstract simplicial flag complex with vertices
corresponding to isotopy classes of essential curves in $S_m$ and edges corresponding
to disjoint pairs of curves. To this end, we focus on particular elements of $G$: powers of Dehn twists; each such element is associated to a unique isotopy class of curves in $S_m$ (see Section 2). We show that the injection $\rho$ must take powers of Dehn twists to powers of Dehn twists, thus giving an action $\rho_*$ on the vertices of $C(S_m)$. Since $\rho_*$ is easily seen to be superinjective in the sense of Irmak (i.e. $\rho_*$ preserves disjointness and nondisjointness), we will be able to derive Theorem 1 from the following theorem.

**Theorem 2.** Let $m \geq 5$. Every superinjective map of $C(S_m)$ is induced by a unique element of $\text{Mod}(S_m)$.

The proofs of both theorems are modeled on previous work of Ivanov, who showed that every isomorphism between finite index subgroups of $\text{Mod}(F)$ is the restriction of an inner automorphism of $\text{Mod}(F)$, when the genus of $F$ is at least 2 [20]. To do this, he applied his theorem that every automorphism of $C(F)$ is induced by an element of $\text{Mod}(F)$. His method has been used to prove similar theorems by various other authors [24, 27, 22, 18, 17, 16, 28, 19, 14, 30, 8]. In particular, Korkmaz proved that every automorphism of $C(S_m)$ is induced by an element of $\text{Mod}(S_m)$ [24], and Irmak showed that every superinjective map of $C(F)$, for higher genus $F$, is induced by an element of $\text{Mod}(F)$, thus obtaining the analog of Theorem 1 for surfaces of genus at least 2 [18, 17, 16].

After the completion of the work presented in this paper, the final cases of Irmak’s theorem were completed by Behrstock–Margalit [2], Shackleton [32], and a subsequent paper by the authors [4].

**Artin groups.** Before we explain the applications of Theorem 1 to Artin groups, we recall the basic definitions. An **Artin group** is any group with a finite set of generators $\{s_1, \ldots, s_n\}$ and, for each $i \neq j$, a defining relation of the form

$$s_is_j \cdots = s_js_i \cdots$$

where $s_is_j \cdots$ denotes an alternating string of $m_{ij} = m_{ji}$ letters. The value of $m_{ij}$ must lie in the set $\{2, 3, \ldots, \infty\}$ with $m_{ij} = \infty$ signifying that there is no defining relation between $s_i$ and $s_j$.

It is convenient to define an Artin group by a **Coxeter graph**, which has a vertex for each generator $s_i$ and an edge labelled $m_{ij}$ connecting the vertices corresponding to $s_i$ and $s_j$ if $m_{ij} > 2$. The label 3 is suppressed. The Coxeter graphs $A_n$, $B_n$, $C_{n-1}$, and $A_{n-1}$ for the Artin groups $A(A_n)$, $A(B_n)$, $A(C_{n-1})$, and $A(A_{n-1})$ are displayed in Figure 1.

**Artin groups and mapping class groups.** The Artin group $A(A_n)$ is better known as the braid group on $n + 1$ strands. If $D_{n+1}$ is the disk with $n + 1$ punctures, and $\text{Homeo}^+(D_{n+1}, \partial D_{n+1})$ is the space of (orientation preserving) homeomorphisms of $D_{n+1}$ which are the identity on the boundary, then $A(A_n)$ is isomorphic to

$$\text{Mod}(D_{n+1}, \partial D_{n+1}) = \pi_0(\text{Homeo}^+(D_{n+1}, \partial D_{n+1}))$$

(see, e.g., [6]). The **pure braid group** $P(A_n)$ is the (finite index) subgroup of $A(A_n)$ consisting of elements which fix each puncture of $D_{n+1}$. The group $A(B_n)$ is isomorphic to a subgroup of $A(A_n)$ fixing one given puncture (see [1] or [9]).
The center of $A(A_n)$ is generated by the Dehn twist about a curve isotopic to $\partial D_{n+1}$; we denote this element by $z$. Both $A(B_n)$ and $P(A_n)$ inherit the same center.

We can also identify $A(A_n)/Z$, $A(B_n)/Z$, and $A(\tilde{C}_{n-1})$ with the subgroups of $\text{Mod}(S_{n+2})$ consisting of orientation preserving elements which fix one, two, and three particular punctures, respectively; further $P(A_n)/Z$ is isomorphic to the pure mapping class group $\text{PMod}(S_{n+2})$, which is the finite index subgroup of $\text{Mod}(S_{n+2})$ consisting of orientation preserving elements which fix every puncture. The group $A(\tilde{A}_{n-1})$ is also isomorphic to a finite index subgroup of $\text{Mod}(S_{n+2})$. A complete description of these isomorphisms appears in the paper of Charney–Crisp [9]. The proofs are due to Allcock, Kent–Peifer, Charney–Peifer, Crisp, and Charney–Crisp [1, 9, 10, 11, 23].

**Applications.** We now give some consequences of Theorem 1. A group is co-Hopfian if each of its injective endomorphisms is an isomorphism.

**Corollary 3.** For $n \geq 3$, all finite index subgroups of $\text{Mod}(S_{n+2})$ are co-Hopfian; in particular, the groups $A(A_n)/Z$, $A(B_n)/Z$, $A(\tilde{C}_{n-1})$, $A(A_{n-1})$, and $P(A_n)/Z$ are co-Hopfian.

For each $0 \leq k \leq m = n + 2$, let $G_k$ be a subgroup of $\text{Mod}(S_m)$ consisting of orientation preserving elements which fix $k$ given punctures. Note that $G_0$ is the index 2 subgroup of $\text{Mod}(S_m)$ consisting of orientation preserving elements, $G_1 \cong A(A_n)/Z$, and $G_{m-1} = G_m = \text{PMod}(S_m)$. Also, $G_2 \cong A(B_n)/Z$ and $G_3 \cong A(\tilde{C}_{n-1})$.

**Corollary 4.** Suppose $n \geq 3$ and let $G$ and $H$ be any of the groups in Figure 2. Then there exists an injection $\rho : G \to H$ if and only if there is a directed path from $G$ to $H$ in Figure 2.

A concrete way to verify this corollary is to compare the indices of the groups $G$ and $H$ in $\text{Mod}(S_m)$.

One might also ask whether or not any injection from Corollary 4 is unique up to automorphisms of $H$. The answer is no. For instance, since $G_m$ is normal in $G_0$, we may conjugate $G_m$ by any element of $\text{Mod}(S_m)$ to get an injective homomorphism.
\[ G_m \to G_k \text{ for any } k. \] However, if \( k > 0 \), then \( f \) might not fix the \( k \) punctures fixed by \( G_k \), and so there is no automorphism of \( G_k \) which achieves the injection.

We are also able to characterize injections between the groups \( A(A_n) \), \( A(B_n) \), and \( P(A_n) \) (with their centers). There are inclusions: \( P(A_n) \to A(B_n) \to A(A_n) \) (see Section 5); all other injections between these groups are described by the following corollary to the Theorem 1.

**Theorem 5.** Suppose \( n \geq 3 \). Let \( G \) be a finite index subgroup of \( A(A_n) \). If \( \rho : G \to A(A_n) \) is an injective homomorphism, then there is an induced injection \( G/Z \to A(A_n)/Z \). Moreover, there is a unique \( f \in \text{Mod}(S_{n+2}) \) so that, after identifying \( A(A_n)/Z \) with the group \( G_{1} \), we have

\[
\rho(g)Z = f(gZ)f^{-1}
\]

for all \( g \in G \).

In Section 5, we explain how this theorem may be applied to give an explicit list of all injections of \( A(A_n) \), \( A(B_n) \), and \( P(A_n) \) into \( A(A_n) \). The case of \( A(A_n) \) was already handled in a previous paper of the authors [3].

Combining Theorem 5 with Corollary 4, we immediately obtain an analogue of Corollary 4 for \( A(A_n) \). Precisely, if \( L_k \) is a subgroup of \( A(A_n) \) corresponding to elements which fix \( k \) particular punctures, then there is an injective homomorphism \( L_j \to L_k \) if and only if \( j > k \). In particular, there is an injective homomorphism between two of the groups \( A(A_n) \), \( A(B_n) \), and \( P(A_n) \) if and only if there is an obvious one.

Finally, combining a theorem of Korkmaz with our understanding of injections of \( P(A_n) \) into \( A(A_n) \), we will prove the following two results. The first is a theorem of Charney–Crisp.

**Theorem 6.** For \( n \geq 3 \), we have

\[
\text{Aut}(A(B_n)) \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes (G_2 \times \mathbb{Z})
\]

where \( G_2 \) is the group defined above.

Whereas the Charney–Crisp proof of Theorem 6 relies on a semidirect product decomposition of \( A(B_n) \) due to Kent–Peifer, we work directly from the isomorphism \( A(B_n) \cong L_1 \).

**Theorem 7.** Suppose \( n \geq 3 \) and let \( N = \binom{n+1}{2} \). We have a short exact sequence

\[
1 \to \mathbb{Z}_2 \ltimes \mathbb{Z}^{N-1} \to \text{Aut}(P(A_n)) \to \text{Aut}(P(A_n)/Z) \to 1
\]

where the third map is the natural one.
A consequence of Theorem 7 is that we get a generating set for Aut($P(A_n)$) from the standard generating sets for $\mathbb{Z}_2 \rtimes \mathbb{Z}^{N-1}$ and $\text{Aut}(P(A_n)/\mathbb{Z}) \cong \text{Mod}(S_{n+2})$ (the last isomorphism is a theorem of Korkmaz).

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2. Background

Curves. By a curve in a surface $F$, we mean the isotopy class of a simple closed curve in $F$ which is not isotopic to a point, a puncture, or a boundary component of $F$. We will often not make the distinction between a representative curve and its isotopy class.

We denote by $i(a, b)$ the geometric intersection number between two curves $a$ and $b$.

A maximal collection of pairwise disjoint curves in $S_m$ is called a pants decomposition. Any pants decomposition of $S_m$ or $D_{m-1}$ has $m - 3$ curves.

A curve in $D_{n+1}$ with $k$ punctures in its interior is called a $k$-curve.

Curve complex. The curve complex $C(F)$ for a surface $F$, defined by Harvey, is the abstract simplicial flag complex with a vertex for each curve in $F$ and edges corresponding to geometric intersection zero [15].

A map $\phi : C(F) \to C(F)$ is called superinjective if for any two vertices $v$ and $w$ of $C(F)$, thought of as curves in $F$, we have $i(v, w) = 0$ if and only if $i(\phi(v), \phi(w)) = 0$. Superinjective maps of $C(S_m)$ are injective for $m \geq 5$ since, given two distinct curves, there is a curve which is disjoint from one but not the other.

Twists. A Dehn twist about a curve $a$, denoted $T_a$ is the element of the mapping class group which has support on an annular neighborhood of $a$, and is described on that annulus by Figure 3.

If $a$ is a 2-curve, we define the half twist about $a$, denoted $H_a$, to be the element of the mapping class group which has support the interior of $a$, and is described inside this twice-punctured disk by Figure 4.

For each $f \in \text{Mod}(S_m)$, let $\epsilon(f) = 1$ if $f$ preserves orientation and $\epsilon(f) = -1$ if not. We will use the following connection between the topology and algebra of Dehn twists in $\text{Mod}(S_m)$.

Fact 8. Suppose $f \in \text{Mod}(S_m)$. Then $fT_a f^{-1} = T_{f(a)}^{\epsilon(f)}$. In particular, $[f, T_a] = 1$ implies $f(a) = a$, and powers of Dehn twists commute if and only if the curves have geometric intersection zero.
For a group \( \Gamma \), we define its \textit{rank}, \( \text{rk}(\Gamma) \), to be the maximal rank of a free abelian subgroup of \( \Gamma \). It follows from work of Birman–Lubotzky–McCarthy that for any surface \( F \), \( \text{rk}\text{Mod}(F) \) is realized by any subgroup generated by powers of Dehn twists about curves forming a pants decomposition for \( F \) \cite{7}; thus, \( \text{rk}\text{Mod}(S_m) = m - 3 \). The following theorem of Ivanov gives another connection between the algebra and topology of \( \text{Mod}(S_m) \) \cite{20}. We restrict our attention here to the genus 0 case, which has a particularly simple statement.

**Theorem 9.** Let \( m \geq 5 \) and let \( P \) be a finite index subgroup of \( \text{PMod}(S_m) \). An element \( g \) of \( P \) is power of Dehn twist if and only if \( \mathbb{Z}(\mathcal{C}_P(g)) \cong \mathbb{Z} \) and \( \text{rk}\mathcal{C}_P(g) = m - 3 \).

We now state a group theoretical lemma, due to Ivanov–McCarthy \cite{22}, which will be used in Proposition 11.

**Lemma 10.** Let \( \rho : \Gamma \to \Gamma' \) be any injective homomorphism of groups, where \( \text{rk}\Gamma' = \text{rk}\Gamma < \infty \). If \( G < \Gamma \) is a free abelian subgroup of maximal rank, and \( g \in G \). Then

\[
\text{rk}\mathbb{Z}(\mathcal{C}_{\Gamma'}(\rho(g))) \leq \text{rk}\mathbb{Z}(\mathcal{C}_{\Gamma}(g)).
\]

Note that Lemma 10 applies whenever \( g \) is a power of a Dehn twist and both \( \Gamma \) and \( \Gamma' \) are finite index subgroups of \( \text{PMod}(S_m) \).

### 3. Subgroups of \( \text{Mod}(S_m) \)

Let \( \rho : G \to \text{Mod}(S_m) \) be an injective homomorphism, where \( G \) is a finite index subgroup of \( \text{Mod}(S_m) \) with \( m \geq 5 \).
Proposition 11. For each curve \(a\) in \(S_m\), there are nonzero integers \(k\) and \(k'\) and a curve \(a'\) such that \(\rho(T^k_a) = T^{k'}_{a'}\).

**Proof.** Let \(Q = \text{PMod}(S_m)\), and let \(P = Q \cap \rho^{-1}(Q)\). Since \(P\) is a finite index subgroup of \(\text{Mod}(S_m)\), we can choose a \(k\) so that \(g = T^k_a\) belongs to \(P\). By Theorem 9, \(Z(C_P(g)) \cong \mathbb{Z}\). Lemma 10 and the fact that \(\rho\) is injective imply that \(Z(C_Q(\rho(g))) \cong \mathbb{Z}\). Since \(\text{rk} \rho(C_P(g)) = \text{rk Mod}(S_m)\), Theorem 9 says that \(\rho(g)\) must be a power of a Dehn twist. \(\square\)

By Proposition 11, \(\rho\) induces a well-defined action \(\rho_*\) on curves given by

\[\rho(T^k_a) = T^{k'}_{\rho_*(a)}\]

Applying Fact 8, we have:

**Proposition 12.** The map \(\rho_*\) is a superinjective map of \(C(S_m)\).

We are now ready to complete the proof of Theorem 1, assuming Theorem 2.

**Proof.** By Propositions 11 and 12, the injection \(\rho\) gives rise to a superinjective map \(\rho_*\) of \(C(S)\), which by Theorem 2 is induced by a unique \(f \in \text{Mod}(S_m)\); that is to say, \(\rho_*(c) = f(c)\) for every curve \(c\). Since \(f\) is unique, we can check that \(\rho(g) = fgf^{-1}\) by checking that \(fg(c) = \rho(g)f(c)\) for any curve \(c\).

\[T_{fg(c)}^{k'} = \rho(T^k_{g(c)}) = \rho(gT^k_c g^{-1}) = \rho(g)\rho(T^{\pm k}_c)\rho(g)^{-1} = \rho(g)T^{k''}_{f(c)}\rho(g)^{-1} = T^{\pm k''}_{\rho(g)f(c)}\]

Thus, \(T_{fg(c)}^{k'} = T^{\pm k''}_{\rho(g)f(c)}\), which implies that \(fg(c) = \rho(g)f(c)\). \(\square\)

4. SUBGROUPS OF \(A(A_n)\)

Let \(G < A(A_n)\) be a finite index subgroup and \(\rho : G \to A(A_n)\) an injective homomorphism. To prove Theorem 5, we need to show that \(\rho\) induces an injective homomorphism \(G/Z \to A(A_n)/Z\) and apply Theorem 1.

As with Theorem 1, we shall require the existence of a superinjective map \(\rho_*\) of \(C(D_{n+1})\) which is induced by \(\rho\) in the sense that for any curve \(a\) we have

\[\rho(T^k_a) = T^{k'}_{\rho_*(a)}\]

for some integers \(k\), \(k'\), and \(k''\) (\(k\) and \(k'\) nonzero); as usual \(z\) is the generator of the center of \(A(A_n)\). The argument is exactly the same as in Proposition 11, with Theorem 9 replaced by the following corollary of Theorem 9.

**Corollary 13.** Let \(P\) be a finite index subgroup of \(P(A_n)\). An element \(g\) of \(P\) is the product of a central element and a nontrivial power of a noncentral Dehn twist if and only if \(Z(C_P(g)) \cong \mathbb{Z}^2\) and \(\text{rk} C_P(g) = n\).

We now prove the theorem.
Proof of Theorem 5. Let $G$ be a finite index subgroup of $A(A_n)$ and $\rho : G \to A(A_n)$ an injective homomorphism. We know that $G$ has nontrivial center $Z(G)$ since it is finite index in $A(A_n)$. Further we have $Z(G) \cong \mathbb{Z}$. Indeed, if $z$ is an element of $Z(G)$, then $z$ must fix every curve in $D_{n+1}$ by Fact 8 and the fact that $G$ is finite index; hence $z$ is a power of $z$.

Let $\zeta$ denote a generator of $Z(G)$. We now show that $\rho(Z(G)) < Z$ by showing $\rho(\zeta) \in Z$. Since $rk \ G = rk \ A(A_n)$, we have that $\rho(\zeta^k) \in Z$ for some nonzero $k$.

Choose a pants decomposition $P$ of $D_{n+1}$. As in Section 3, we know that $\rho_*(P)$ is also a pants decomposition ($\rho_*$ exists by the discussion at the start of this section). Further, because $\zeta$ is central and $\rho$ is injective, it follows that $\rho(\zeta)$ fixes each element of $\rho_*(P)$. Since an orientation preserving element of $Mod(S)$ is determined by its action on the punctures, it follows that $\rho(\zeta)$ lies in the free abelian subgroup generated by half twists and Dehn twists in the curves of $\rho_*(P)$. Since $\rho(\zeta)^k \in Z$, it now follows that $\rho(\zeta) \in Z$.

Moreover, we have that $\rho^{-1}(Z) < Z(G)$, by the injectivity of $\rho$. Thus, $\rho$ induces a well-defined injection $G/Z(G) \to A(A_n)/Z$. Since $G/Z(G)$ is finite index in $A(A_n)/Z$, we may apply Theorem 1. Thus, fixing an identification $A(A_n)/Z < \text{Mod}(S_{n+2})$, there is a unique $f \in \text{Mod}(S_{n+2})$ so that

$$\rho(g)Z = f(gZ)f^{-1}$$

for all $g \in G$. This proves the theorem. $\square$

We now take a moment to interpret Theorem 5 in a way that will be useful to us in the next section. The element $f \in \text{Mod}(S_{n+2})$ from the theorem does not necessarily correspond to an element of $\text{Mod}(D_{n+1})$, for it may switch the puncture in $S_{n+2}$ corresponding to the boundary of $D_{n+1}$ with another puncture. However, even in this case, since $C(D_{n+1}) \cong C(S_{n+2})$, the element $f$ induces an automorphism $f_*$ of $C(D_{n+1})$. Since $f$ is an element of $\text{Mod}(S_{n+2})$ as opposed to $\text{Mod}(D_{n+1})$, the map $f_*$ may take a $k$-curve to an $(n-k+2)$-curve.

Now, let $g$ be a power of a noncentral Dehn twist or half twist in $G$; for concreteness, $g = T^k_a$. The coset $gZ$ is a power of a Dehn twist, also denoted $T^k_a$, thought of as an element of $\text{Mod}(S_{n+2})$. The conjugate $f(gZ)f^{-1}$ is equal to $T^k_{f_*(a)}$. It follows from Theorem 5 that $\rho(g)$ is a product of $T^k_{f_*(a)}$ with a central element.

We again emphasize that $f$ is an element of the mapping class group of $S_{n+2}$, and not $D_{n+1}$, and so $f_*$ can take a curve to one which is not topologically equivalent in $D_{n+1}$. In fact, we will see examples of this “nongeometric” phenomenon in the next section, where we classify injective homomorphisms of $A(A_n)$, $A(B_n)$, and $P(A_n)$ into $A(A_n)$.

As each of these groups is generated by half twists and Dehn twists, we will be able to understand these injections via the following corollary to Theorem 5, which summarizes the above discussion.

Corollary 14. Let $G$ be a finite index subgroup of $A(A_n)$ and $\rho : G \to A(A_n)$ an injective endomorphism. There is an $f \in \text{Mod}(S_{n+2})$ so that for any power of a Dehn twist $T^k_a \in G$, we have

$$\rho(T^k_a) = T^k_{f_*(a)}.$$
for some integer \( t = t(T_k) \).

The analogous statement for half twists also holds. We remark that the reason we focus on Dehn twists and half twists here is that in \( A(A_n) \) there is a natural representative of a Dehn twist coset of \( A(A_n)/Z \), and so, combined with the action of \( f_* \) on curves, there is a relatively simple form for the image under \( \rho \) of a power of a twist.

**Moving punctures criterion.** Another fact which will be useful in the next section is that \( f \) must send moving punctures to moving punctures; that is, the set of punctures of \( S_{n+2} \) which are not fixed by every element of \( G/Z(G) \) must be sent by \( f \) to into the \( n+1 \) punctures which are not fixed by \( A(A_n)/Z \). This is because conjugation by \( f \) sends fixed punctures to fixed punctures and moving punctures to moving punctures. Below, we call this the **moving punctures criterion**. We remark that this criterion can be used to derive Corollary 4 from Theorem 1.

## 5. Catalogue of injections

We now use Theorem 5 to list all injections of the groups \( P(A_n), A(B_n), \) and \( A(A_n) \) into \( A(A_n) \). As usual, we denote the generator of \( Z \) by \( z \).

Instead of applying Theorem 4 directly, we will instead use Corollary 14 and the moving punctures criterion. We use the notation of Corollary 14: given an element \( f \in \text{Mod}(S_{n+2}) \), the symbol \( f_* \) denotes the induced automorphism of \( C(S_{n+2}) \); the identification \( C(D_{n+1}) \cong C(S_{n+2}) \) comes from the identification \( A(A_n)/Z < \text{Mod}(S_{n+2}) \).

**Injections of \( A(A_n) \).** The Artin group \( A(A_n) \) is defined via the presentation given by Figure 1. We denote the generators by \( \sigma_1, \ldots, \sigma_n \). Under the identification with \( \text{Mod}(D_{n+1}, \partial D_{n+1}) \), each generator \( \sigma_i \) corresponds to a half twist \( H_{a_i} \) about a curve \( a_i \) in \( D_{n+1} \) (see [6]).

Let \( \rho : A(A_n) \to A(A_n) \) be an injective homomorphism. Applying the moving punctures criterion, we see that the element \( f \in \text{Mod}(S_{n+2}) \) given by Theorem 5 must send the puncture of \( S_{n+2} \) fixed by \( A(A_n)/Z \) to itself. Therefore, we may think of \( f \) as an element of \( \text{Mod}(D_{n+1}) \). Corollary 14 then implies that \( \rho \) is described on generators by the formula

\[
\rho(H_{a_i}) = H_{f_*(a_i)}^{t_i} \rho(z) = Z^{t_i} Z^{n(n+1)} = Z^{1+t(n(n+1))}
\]

As \( n \geq 3 \), we see \( t(n(n+1)) \) cannot be \( -1 \), so \( \rho(z) \) is not trivial, and the kernel of \( \rho \) is trivial. Thus, we have an injection for any \( t \); moreover, the map is not surjective.
when $t \neq 0$: the preimage of $Z$ is contained in $Z$, but $z \mapsto z^{1+t(n(n+1))}$, so nothing maps to $z$.

It follows that $\text{Aut}(A(A_n)) \cong \text{Mod}(D_{n+1})$. This was first proven by Dyer–Grossman [13]. Ivanov was the first to compute $\text{Aut}(A(A_n))$ from the perspective of mapping class groups [21].

**Injections of $A(B_n)$.** Again, this group has a presentation given by Figure 1. We denote the generators for $A(B_n)$, from left to right, by $s_1, \ldots, s_n$. The usual inclusion $A(B_n) \to A(A_n)$ is given by $s_1 \to \sigma_1^2$ and $s_i \to \sigma_i$ for $i > 1$.

Let $\rho : A(B_n) \to A(A_n)$ be an injective homomorphism. There are two punctures fixed by $A(B_n)/Z \lt \text{Mod}(S_{n+2})$. By the moving punctures criterion, the element $f$ given by Theorem 5 must send one of these two punctures to the puncture fixed by $A(A_n)/Z$. Identifying $\sigma_i$ with $H_{a_i}$ as above, this means that $f_*$ takes $a_i$ to a 2-curve when $i > 1$, and $f_*(a_1)$ is either a 2-curve or an $n$-curve. By Theorem 6, $\text{Aut}(A(B_n)) \to \text{Aut}(A(B_n)/Z)$ is surjective, so we are forced to consider these nongeometric maps $f_*$.

As above, the homomorphism $\rho$ is given on generators by

$$\rho(H_{a_i}) = H_{f_*(a_i)}^{t_i} z^{t_i}.$$  

Since the $s_i$ are conjugate for $i > 1$, we have that $t_i$ is the same for these $i$; set $u = t_1$ and $v = t_2$. Conversely, we have a well-defined homomorphism for any $u$ and $v$. If $g \mapsto 1$, we again have that $g \in Z$. Since $z = (s_1 \cdots s_n)^n$, we have $z \mapsto z^{1+nu+n(n-1)v}$. But there are no $u$ and $v$ which make the latter trivial (as $n$ and $n(n-1)$ are not relatively prime), so every choice of $u$ and $v$ leads to an injection.

**Injections of $P(A_n)$.** We identify $P(A_n)$ with the elements of $A(A_n)$ which fix each puncture of $D_{n+1}$. There is a standard generating set for $P(A_n)$, due to Artin, consisting of one Dehn twist $T_{a_i,j}$ for each pair of punctures of $D_{n+1}$ (see [6]). If the punctures of $D_{n+1}$ lie in a horizontal line, then each $a_{i,j}$ can be realized as the boundary of a regular neighborhood of an arc which lies below this horizontal and connects the $i$th and $j$th punctures; note that $a_i = a_{i,i+1}$.

Let $\rho : P(A_n) \to A(A_n)$ be an injective homomorphism. As in the previous cases, we apply Corollary 14 and deduce that $\rho$ is described on generators by

$$\rho(T_{a_i,j}) = T_{f_*(a_i,j)}^{t_{i,j}} z^{t_{i,j}}$$

for some $f \in \text{Mod}(S_{n+2})$. In the case of $P(A_n)/Z$, there are no moving punctures in $S_{n+2}$, and so the moving punctures criterion gives no restriction for the action of $f$ on the punctures of $S_{n+2}$. We will see in Section 6 that in fact every $f \in \text{Mod}(S_{n+2})$ gives rise to an automorphism of $P(A_n)$, and so the $f$ associated to $\rho$ is arbitrary.

Conversely, since all of Artin’s defining relations of $P(A_n)$ are commutation relations (see [6]), it follows that even if the $t_{i,j}$ are all different, $\rho$ is a well-defined homomorphism. Again, the kernel of $\rho$ must be contained in $Z$. In the generators of $P(A_n)$, $z$ can be written as

$$(T_{a_1,2} T_{a_{1,3}} \cdots T_{a_{1,n+1}}) \cdots (T_{a_{n-1,n}} T_{a_{n-1,n+1}})(T_{a_{n,n+1}})$$
and so we see that
\[ z \mapsto z^{1+\sum t_{i,j}}. \]
Hence, there is an affine hyperplane in \( \mathbb{Z}^N \), where \( N = \binom{n+1}{2} \), corresponding to noninjective homomorphisms of \( P(A_n) \) into \( A(A_n) \).

**Remark.** The abstract commensurator \( \text{Comm}(G) \) of a group \( G \) is the collection of isomorphisms of finite index subgroups of \( G \), where two such isomorphisms are equivalent if they agree on some common finite index subgroup. For a given group \( A(A_n) \), \( A(B_n) \), or \( P(A_n) \), different choices of the function \( t \) give rise to distinct elements of \( \text{Comm}(A(A_n)) \). After the first version of this paper was written, Leininger–Margalit proved that \( \text{Comm}(A(B_n)) \cong \text{Mod}(S_{n+2}) \ltimes (\mathbb{Q}^\times \ltimes \mathbb{Q}^\infty) \) [26].

### 6. Automorphisms

In this section we construct a lift \( \xi_k \) of the natural map \( \text{Aut}(L_k) \to \text{Aut}(L_k/Z) \), where (as in the introduction) \( L_k \) is the subgroup of \( A(A_n) \) consisting of elements which fix \( k \) particular punctures. Using the facts that \( L_1 \cong A(B_n) \) and \( L_{n+1} \cong P(A_n) \), we will compute \( \text{Aut}(A(B_n)) \) explicitly (reproving a result of Charney–Crisp), and we will give a generating set for \( \text{Aut}(P(A_n)) \).

Before we begin in earnest, we note that any \( L_k \) can be generated by Artin’s generators for \( P(A_n) \) plus a collection of half twists which are lifts of the elements of the symmetric group on \( n+1 \) letters which are in the image of \( L_k \).

Charney–Crisp define a **transvection** of a group \( G \) with infinite cyclic center \( Z = \langle z \rangle \) to be a homomorphism \( G \to G \) of the form \( x \mapsto x z^{t(x)} \), where \( t : G \to Z \) is a homomorphism. They observe that such a map is an automorphism if and only if its restriction to \( Z \) is surjective; this holds if and only if \( t(z) = \pm 1 \), i.e., \( z \mapsto z^{\pm 1} \).

We denote by \( \text{Tv}(G) \) the **transvection subgroup** of \( \text{Aut}(G) \).

We consider the following sequence:

\[ 1 \to \text{Tv}(L_k) \to \text{Aut}(L_k) \to \text{Aut}(L_k/Z) \to 1 \]

To find a generating set for \( \text{Aut}(L_k) \), it suffices to construct the lift \( \xi_k : \text{Aut}(L_k/Z) \to \text{Aut}(L_k) \) (so the sequence is exact) and to find generating sets for \( \text{Tv}(L_k) \) and \( \text{Aut}(L_k/Z) \). The group \( \text{Tv}(L_k) \) can often be computed directly from a presentation of \( L_k \), and (by Theorem 1, say) \( \text{Aut}(L_k/Z) \) is isomorphic to the subgroup \( \tilde{G}_{k+1} \) of \( \text{Mod}(S_{n+2}) \) consisting of elements which preserve a set of \( k+1 \) punctures (the group \( \tilde{G}_{k+1} \) is generated by Dehn twists, half twists, and a reflection). In the case of \( L_1 \cong A(B_n) \), we will show that the above exact sequence is split.

#### 6.1. Auxiliary groups

Intuitively, we would like to “blow up” the punctures fixed by \( L_k \) (the ones that are fixed by \( L_k \)) into boundary components so that the group \( \tilde{G}_{k+1} \cong \text{Aut}(L_k/Z) \) cannot distinguish between the original boundary of \( D_{n+1} \) and the fixed punctures. In particular, we want \( \tilde{G}_{k+1} \) to be able to interchange \( \partial D_{n+1} \) with the fixed punctures. We now make this precise.

If one is only interested in Theorems 6 and 7, this subsection can be skipped; see the remarks at the end of Sections 6.5 and 6.6.
Let $\bar{S}$ be a sphere with $n + 2$ boundary components. We choose a set $P$ of distinguished points in $\bar{S}$, one in each boundary component. Then, we define $\text{Mod}(\bar{S})$ to be the group of homeomorphisms of $\bar{S}$ fixing $P$ as a set, modulo isotopies which fix $P$.

We fix an embedding $\bar{S} \to D_{n+1}$ which induces an isomorphism on the level of curve complexes (send each boundary component to a circle around a puncture or a circle parallel to $\partial D_{n+1}$). We will use the same names for the curves which are equivalent under this isomorphism (and the other isomorphisms below).

![Figure 5. Generalized half twist.](image)

We get the embedding $\iota : A(A_n) \to \overline{\text{Mod}(\bar{S})}$ as follows. If the generators $\sigma_i$ correspond to half twists $H_{a_i}$ about the 2-curves $a_i$, then we define $\iota(H_{a_i})$ to be the generalized half twist about $a_i$, as indicated in Figure 5. The generalized half twist about a curve $a$ is denoted $\tilde{H}_a$.

For our definition of $\iota$ to be precise, we must specify the points of $P$. If $\{d_i\}$ are the boundary components of $\bar{S}$, we choose the unique such labelling consistent with the isomorphism $\mathcal{C}(D_{n+1}) \cong \mathcal{C}(\bar{S})$ and the choice of the $\{a_i\}$. We draw $\bar{S}$ in the plane so that $d_{n+2}$ is the outer boundary component and the other $d_i$ are Euclidean circles which lie in a horizontal line. Then, the points of $P$ are chosen to be the leftmost point of each circle (this choice is consistent with Figure 5).

To see that $\iota$ is a homomorphism, one only needs to check the two braid relations. The commuting relation obviously holds. In Figure 6, we show the effect of $\iota(H_{a_i} H_{a_{i+1}} H_{a_i}) = \iota(H_{a_{i+1}} H_{a_i} H_{a_{i+1}})$. We can also see that $\iota$ is injective; indeed, the map $\bar{S} \to D_{n+1}$ induces a left inverse $\pi : \overline{\text{Mod}(\bar{S})} \to A(A_n)$. Of course, $\iota$ restricts to an injection $L_k \to \overline{\text{Mod}(\bar{S})}$, also called $\iota$, for any $k$.

![Figure 6. The braid relation in $\overline{\text{Mod}(\bar{S})}$.](image)

We introduce another surface $\tilde{S}_k$, obtained by gluing punctured disks to the $d_i$ corresponding to the punctures in $D_{n+1}$ not fixed by $L_k$ (the surface $\tilde{S}_k$ is a sphere with $k + 1$ boundary components and $n - k + 1$ punctures). The inclusion $\tilde{S} \to \tilde{S}_k$ identifies $\mathcal{C}(\tilde{S})$ with $\mathcal{C}(\tilde{S}_k)$ and induces a map $\eta : \iota(L_k) \to \overline{\text{PMod}(\tilde{S}_k)}$, where by $\overline{\text{PMod}(\tilde{S}_k)}$ we mean the isotopy classes of homeomorphisms of $\tilde{S}_k$ which are the identity on the boundary. Fixing a set of points $P \subset \partial \tilde{S}_k$ (one for each boundary),
we can alternatively think of $\text{PMod}(\bar{S}_k)$ as a normal subgroup of the group $\text{Mod}(\bar{S}_k)$, which consists of homeomorphisms of $\bar{S}_k$ fixing $P$ as a set (modulo isotopies fixing $P$).

The map $\iota_k = \eta \circ \iota$ is again injective since there is an inverse $\pi_k$ induced by gluing punctured disks to $k$ of the components of $\partial \bar{S}_k$.

We also want a map from $\text{Mod}(\bar{S}_k)$ to $\bar{G}_{k+1}$. We glue punctured disks to the $k + 1$ boundary components of $\bar{S}_k$ in order to obtain the surface $S_{n+2}$. The inclusion of surfaces induces a surjective map $\text{Mod}(\bar{S}_k) \to \bar{G}_{k+1}$.

We encode the key relationships between all of our groups in Figure 7.

6.2. Generalized lantern relation. In order to define our lift $\xi_k : \text{Aut}(L_k/Z) \to \text{Aut}(L_k)$, we will need a relation in $\text{PMod}(\bar{S}_k)$ called the generalized lantern relation. Let $\{T_{a_{i,j}}\}$ be the set of Artin generators for $P(A_n)$, and let $\{d_i\}$ be the set of boundary components of $\bar{S}$. In the language we have developed, the relation is

$$\iota(z) = \iota((T_{a_{1,2}}T_{a_{1,3}}\cdots T_{a_{1,n+1}})\cdots (T_{a_{n-1,n}}T_{a_{n-1,n+1}})(T_{a_{n,n+1}}))$$

$$= T_{d_1}^{-1}T_{d_2}^{-1}\cdots T_{d_{n+1}}^{-1}T_{d_{n+2}}$$

(the first equality is the well-known relation in $P(A_n)$, and the second equality is the generalized lantern relation). This relation appears in the work of Wajnryb [33], who writes that this relation can be checked “by induction (by drawing many pictures)”. In Section 7, we give a straightforward proof of the relation.

Without reference to $\iota$, the generalized lantern relation is simply:

$$(T_{a_{1,2}}T_{a_{1,3}}\cdots T_{a_{1,n+1}})\cdots (T_{a_{n-1,n}}T_{a_{n-1,n+1}})(T_{a_{n,n+1}}) = T_{d_1}^{-1}T_{d_2}^{-1}\cdots T_{d_{n+1}}^{-1}T_{d_{n+2}}$$

In the case of $n = 2$, this relation is precisely the famous lantern relation, known to Dehn [12].

Stated in this alternate way, the relation exhibits an obvious asymmetry in $P(A_n)$ between the punctures of $D_{n+1}$ and $\partial D_{n+1}$ when $n \neq 2$. In our first description of the relation, there still is an asymmetry (in the signs), and we will see that this is what prevents us from finding a homomorphism $\text{Aut}(P(A_n)/Z) \to \text{Aut}(P(A_n))$.

6.3. The lift. We now define lift $\xi_k$ from $\bar{G}_{k+1} \cong \text{Aut}(L_k/Z)$ to $\text{Aut}(L_k)$. Given an element $f \in \bar{G}_{k+1}$, we choose a lift $\bar{f}$ in $\text{Mod}(\bar{S}_k)$. Since $\text{PMod}(\bar{S}_k)$ is normal in

![Figure 7. Groups used in the definition of $\xi_k : \text{Aut}(L_k/Z) \to \text{Aut}(L_k)$.](image)
\(\phi_*\) induces an automorphism \(\psi_f\) of \(\text{PMod}(\hat{S}_k)\); this automorphism is well-defined since any two lifts differ by a central element of \(\text{PMod}(\hat{S}_k)\). We can now define an endomorphism of \(L_k\) via the composition \(\pi_k \circ \psi_f \circ \iota_k\). To see that this composition of homomorphisms is actually an automorphism of \(L_k\), we will show that it is surjective. This suffices since \(L_k\) is \(\text{Hopfian}\), that is, every surjective endomorphism is an automorphism: braid groups are linear by a result of Krammer and Bigelow [5, 25] and finitely generated linear groups are Hopfian by results of Mal’cev.

The homomorphism \(\pi_k \circ \psi_f \circ \iota_k\) clearly induces a surjection from \(L_k\) to \(L_k/Z\), and by the generalized lantern relation, it also induces a surjection \(Z \to Z \,(z\,\text{maps to either} \,z\,\text{or} \,z^{-1})\). It follows that \(\pi_k \circ \psi_f \circ \iota_k\) is a surjection. We are now justified in calling the composition \(\xi_k(f)\), and this defines our lifting (it is clear that \(\xi_k\) is a lift).

### 6.4. Generalized Artin generators

We will see below that \(\text{Aut}(L_k)\) does not preserve the conjugacy classes (in \(A(A_n)\)) of the generators for \(L_k\). Thus, in order to get a nice statement for how \(\text{Aut}(L_k)\) acts on \(L_k\), we expand the generating set for \(L_k\).

Let \(a\) be either a 2-curve or an \(n\)-curve in \(D_{n+1}\). We denote by \(g(a, \alpha)\) the following element of \(L_k\):

\[
g(a, \alpha) = \begin{cases} H_a & \text{a is a 2-curve, } H_a \in L_k \\ T_a & \text{a is a 2-curve, } H_a \not\in L_k \\ T_a z^{-1} & \text{a is an \(n\)-curve with moveable puncture in exterior} \\ T_a z^\alpha & \text{a is an \(n\)-curve with fixed puncture in exterior} \end{cases}
\]

Each \(g(a, \alpha)\) is called a generalized Artin generator for \(L_k\). We will see that the above classification of these generators completely describes the symmetry of \(L_1 \cong A(A_n)\) and suggests an asymmetry in \(L_{n+1} = P(A_n)\).

We will need to know the image under \(\iota\) of each generalized Artin generators. We first figure this out for the usual standard generators, and then use conjugation to get the rest.

By definition \(\iota(H_a)\) is equal to \(\hat{H}_{a_1}\). Since \(T_{a_1,2} = H_{a_1}^2\), it is straightforward to check that \(\iota(T_{a_1,2}) = \hat{H}_{a_1}^2\) is equal to \(T_{a_1,2} T_{d_1}^{-1} T_{d_2}^{-1}\) (refer to Figure 5). If \(h \in A(A_n)\) and \(h_*(a_{1,2}) = a\), then we see that

\[
i\iota(g(a, \alpha)) = i(h H_{a_1}^q h^{-1}) = i(h) i((H_{a_1})^q h^{-1} = \hat{H}_{a_1}^q h_*(a_{1,2}) = \hat{H}_a^q
\]

where \(q \in \{1, 2\}\). The third equality holds because \(i(h)\) and \(h\) induce the same maps of \(\mathcal{C}(D_{n+1})\) (which is identified with \(\mathcal{C}(\hat{S})\)). As part of our proof of the generalized lantern relation in Section 7, we will use our understanding of the action of \(\iota\) on each \(H_a\) and \(H_a^2\) to show that if \(a\) is a curve surrounding each puncture but the \(i^{th}\) and \(\alpha = \pm 1\), then \(\iota(T_a z^\alpha)\) is equal to \(T_a (T_{d_1}^{-1} z)^\alpha\).

### 6.5. Automorphisms of \(A(B_n)\)

Recall that \(A(B_n)\) is isomorphic to \(L_1\), and that \(A(B_n)\) is generated by elements \(s_i\) where \(s_1 = T_{a_1}\) and \(s_i = H_{a_i}\) for \(i > 1\). We
now compute the transvection subgroup of $\text{Aut}(A(B_n))$ and show that our lifting $\xi_1$ from $\tilde{G}_2 \cong \text{Aut}(A(B_n)/Z)$ to $\text{Aut}(L_1) \cong \text{Aut}(A(B_n))$ is a homomorphism.

In Section 5, we classified all transvections of $A(B_n)$ in terms of two integers, $u$ and $v$. These were defined by $s_1 \mapsto s_1z^u$ and $s_2 \mapsto s_2z^v$. We also found that $z \mapsto z^{1+nu+n(n-1)v}$. Again, in order for a transvection to be an automorphism, we need $z \mapsto z^{\pm 1}$. We see that $z \mapsto z$ if and only if $nu + n(n-1)v = 0$ and $z \mapsto z^{-1}$ if and only if $nu + n(n-1)v = -2$. The latter case actually cannot happen, since $nu + n(n-1)v$ is divisible by $n \geq 3$ while $-2$ is not. Thus, $\text{Tv}(A(B_n)) \cong \mathbb{Z}$.

We now want to show that $\xi_1$ is a splitting of the sequence (1). There is a homomorphism $\delta : \tilde{G}_2 \to \mathbb{Z}_2$ given by the action on the two punctures of $S_{n+2}$ which are fixed setwise by $G_2$. Recall that $\epsilon : \tilde{G}_2 \to \mathbb{Z}_2$ is the homomorphism which records whether or not elements are orientation preserving.

Given $f \in \tilde{G}_2 \cong \text{Aut}(L_1/Z)$, we have the following simple formula for the action of $\xi_1(f)$ on the generalized Artin generators of $L_1$:

$$\xi_1(f)(g(a, \alpha)) = g(f_\ast(a), \delta(f)\alpha)^{\xi(f)}.$$  

It follows easily that $\xi_1$ is a homomorphism. Checking the above formula is straightforward for each of the 4 types of generators; the work was done in their careful classification into the 4 types. To give the idea, we check the formula for the fourth type of generator (this is the only case where $\delta$ is important). Let $a$ be a curve which surrounds all punctures but the fixed one.

$$\iota_1(g(a, \alpha)) = \iota_1(T_az^\alpha) = T_a(T_{d_1}^{-1}T_{d_2})^\alpha$$

$$\psi_f \circ \iota_1(g_a) = (T_{f_\ast(a)}T_{f_\ast(d_1)}T_{f_\ast(d_2)})^\alpha\xi(f)$$

$$\xi_1(f)(T_az^\alpha) = \pi_1 \circ \psi_f \circ \iota_1(T_az^\alpha) = \begin{cases} (T_{f_\ast(a)}z^\alpha)^\xi(f) & \text{if } \delta(f) = +1 \\ (T_{f_\ast(a)}z^{-\alpha})^\xi(f) & \text{if } \delta(f) = -1 \end{cases}$$

$$= g(f_\ast(a), \delta(f)\alpha)^{\xi(f)}$$

We now have that $\text{Aut}(A(B_n)) \cong \tilde{G}_2 \ltimes \mathbb{Z}$. The group $\tilde{G}_2$ is isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes G_2$, where, as in the introduction, the group $G_2$ is the group of orientation preserving elements of $\tilde{G}_2$ which fix two particular punctures. Thus, we can write $\text{Aut}(A(B_n))$ as $(\mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes G_2 \ltimes \mathbb{Z}$. As noted by Charney–Crisp, the elements of $G_2$ commute with the transvections of $\text{Aut}(A(B_n))$, and so, finally, we obtain Theorem 6: $\text{Aut}(A(B_n)) \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes (G_2 \times \mathbb{Z})$.

**Remark.** In the special case of $k = 1$, we can give a more straightforward definition of the lift $\xi_k$. Given an $f \in \tilde{G}_2$, we define $\xi_1(f)$ directly by the formula $\xi_1(f)(g(a, \alpha)) = g(f_\ast(a), \delta(f)\alpha)^{\xi(f)}$. Using the presentation of $A(B_n)$, and the generalized lantern relation, one can directly check that this defines a homomorphism $\tilde{G}_2 \to \text{Aut}(A(B_n))$.

6.6. **Automorphisms of $P(A_n)$**. Having constructed $\xi_{n+1}$ we have completed our proof of Theorem 7. We now address the question of whether or not there is a splitting $\text{Aut}(P(A_n)/Z) \to \text{Aut}(P(A_n))$. More specifically, we will explain why the map $\xi_{n+1}$ is not a splitting. Since $\xi_{n+1}$ seems like the most natural candidate for a splitting, we conjecture that there is no splitting.
Let \( a \) be a curve in \( D_{n+1} \) which surrounds all punctures except the first. Let \( g \in \text{Mod}(S_{n+2}) \cong \text{Aut}(P(A_n)/Z) \) be an element whose lift \( \bar{g} \in \text{Mod}(\bar{S}) \) satisfies \( \bar{g}(d_1) = d_1 \) and \( \bar{g}(d_{n+2}) = d_2 \). Similarly to our calculations for \( A_n(B_n) \), we can check that \( \xi_{n+1}(g) \) takes \( T_{a_{z-1}} \) to \( T\bar{g}(a_{z-1}) \).

Let \( f \in \text{Mod}(S_{n+2}) \) be such that \( f \circ g(a) = a \), but \( f \circ \bar{g}(d_1) = d_{n+2} \) and \( f \circ \bar{g}(d_{n+2}) = d_1 \). Then \( \xi_{n+1}(f) \) takes \( T\bar{g}(a_{z-1}) \) to \( T_{a_{z-1}} \). However, \( \xi_{n+1}(fg) \) takes \( T_{a_{z-1}} \) to \( T_{a_{z-1}} \). Thus, \( \xi_{n+1} \) is not a homomorphism.

**Remark.** There is a simpler proof that \( \text{Aut}(P(A_n)) \) surjects onto \( \text{Aut}(P(A_n)/Z) \). Given any \( f \in \text{Mod}(S_{n+2}) \cong \text{Aut}(P(A_n)/Z) \), one can use the presentation of \( P(A_n) \) to directly show that for any choice of \( \{ t_{i,j} \} \), the map given by \( T_{a_{i,j}} \mapsto T_{f(a_{i,j}) \cdot z^{t_{i,j}}} \) is a homomorphism (a convenient presentation to use for this is the “modified” Artin presentation in [29]). Thus, in order to obtain a lift of \( f \), one only needs to choose \( \{ t_{i,j} \} \) so that \( z \mapsto z \pm 1 \); that is, \( \sum t_{i,j} \) needs to be \( 0 \) or \( -2 \). Since we don’t have explicit presentations for the other \( L_k \), this method does not work in general.

We also note here that an earlier version of this paper contained an incorrect computation of \( \text{Aut}(P(A_n)) \).

### 7. Generalized lantern relation

We now prove the “generalized lantern relation”, used in the proof of Theorem 7. We freely use the notation of Section 6.

Our goal is to understand \( \iota(z) \). We think of \( z \) as a product of elements

\[
g_i = T_{a_{i+1}} T_{a_{i+2}} \cdots T_{a_{n+1}}
\]

and we will first understand each \( \iota(g_i) \) individually. We draw \( \bar{S} \) in the plane as in Section 6. This allows us to see the \( a_{i,j} \) in \( \bar{S} \) exactly as they appear in \( D_{n+1} \).

![Figure 8](image-url)

**Figure 8**

We can think of each \( T_{a_{i,j}} \) as a “push map”, where the \( i^{th} \) boundary component moves around the \( i^{th} \) boundary component, while travelling clockwise inside \( a_{i,j} \) in such a way that it never turns (Figure 5 represents the halfway point of this push map).

We can thus think of \( g_i \) as a product of these push maps (see Figure 8). We now see the following intuitive relation: \( g_i \) can also be obtained by pushing the \( i^{th} \) boundary component around the \( n - i + 1 \) boundary components to its right all at once (see Figure 9). We then observe that this latter push map is equivalent to

\[
T_{c_{i,n+1}} T_{c_{i+1,n+1}} T_{d_i}^{-1}
\]
where the curves are as shown in Figure 10. This intuitive relation (which is already an interesting relation in the mapping class group) is explained more formally in the remark below (see also [29]).

We can now compute \( \iota(z) \) as the product of the \( \iota(g_i) \):

\[
(T_{c_1,n} T_{c_2,n}^{-1} T_{d_1}^{-1})(T_{c_2,n} T_{c_3,n}^{-1} T_{d_2}^{-1}) \cdots (T_{c_n,n+1} T_{c_{n,n}}^{-1} T_{d_{n+1}}^{-1})
\]

All of the \( T_{c_{i,j}} \) elements cancel except the first, which is equal to \( z \), and the last, which is equal to \( T_{d_n}^{-1} \). Thus, \( \iota(z) \) is equal to the product of \( z \) with \( T_{d_1}^{-1} \cdots T_{d_{n+1}}^{-1} \), and this is exactly the generalized lantern relation.

We notice that, applying the map \( \pi \) to \( \iota(z) \), we see that we have proven that the product of the \( g_i \) is indeed equal to \( z \) in \( P(A_n) \).

Also, since \( \iota \) takes conjugates of the \( T_{a_{i,j}} \) to the corresponding conjugates of the \( \iota(T_{a_{i,j}}) \), the same holds for the conjugates of the \( g_i \). This fact was used in Section 6.

**Remark.** We now explain a more formal framework for proving the intuitive relation that pushing a disk around two loops is the same as pushing it around a composite loop. Let \( X \) be the subset of the configuration space of \( n + 1 \) ordered points in the unit tangent bundle of the disk, where each point lies in a different fiber. There is a natural map from \( \pi_1(X) \to \text{Mod}(S) \) (the projection to the disk of each point in \( X \) specifies the location of a particular boundary component and the vector specifies the rotation; if we like, we can replace boundary components with rigid disks in the sphere). The relation described above simply follows from the fact that this map is a homomorphism. Putman has observed that relations in \( \pi_1(X) \) thus give rise to many different “generalized lantern relations”.
8. Superinjective maps of curve complexes

Let \( S_m \) be a sphere with \( m \geq 5 \) punctures, and let \( \phi \) be a superinjective map of \( \mathcal{C}(S_m) \). We will prove Theorem 2, i.e., that \( \phi \) is induced by a unique element of \( \text{Mod}(S_m) \). The basic strategy is to show that \( \phi \) preserves the topological types of curves (Lemma 18), that \( \phi \) preserves the simplest type of nontrivial intersection between curves (Lemma 20), and then to use induction to show that \( \phi \) is surjective (Proposition 22). Theorem 2 is then a consequence of the following theorem of Korkmaz [24]:

**Theorem 15.** Let \( m \geq 5 \). Every automorphism of \( \mathcal{C}(S_m) \) is induced by a unique element of \( \text{Mod}(S_m) \).

A side of a curve \( a \) in \( S_m \) is one of the two connected components of \( S_m - a \). The curve \( a \) is called a \( k \)-curve if the minimum of the numbers of punctures on each side is \( k \). Two 2-curves \( a \) and \( b \) in \( S_m \) are said to be adjacent if \( i(a, b) = 2 \).

**Lemma 16 (Sides).** If \( a \) and \( b \) are two distinct curves which lie on the same side of a curve \( w \), then \( \phi(a) \) and \( \phi(b) \) lie on the same side of \( \phi(w) \).

**Proof.** Choose a curve \( d \) which intersects \( a \) and \( b \), but not \( w \). Since \( \phi \) is superinjective, \( \phi(d) \) intersects \( \phi(a) \) and \( \phi(b) \), but not \( \phi(w) \), and so the lemma follows. \( \square \)

**Lemma 17 (2-curves).** If \( a \) is a 2-curve, then \( \phi(a) \) is a 2-curve.

**Proof.** Choose a pants decomposition \( \{a = c_1, c_2, \ldots, c_{m-3}\} \). Applying Lemma 16, we see that \( \phi(c_2), \ldots, \phi(c_{m-3}) \) must all lie on the same side of \( \phi(a) \). It follows that \( \phi(a) \) is a 2-curve. \( \square \)

**Lemma 18 (k-curves).** If \( w \) is a \( k \)-curve, then \( \phi(w) \) is a \( k \)-curve.

**Proof.** By Lemma 17, we may assume that \( w \) is not a 2-curve. Any pants decomposition \( \mathcal{P} \) containing \( w \) has \( k-2 \) curves on one side and \( m-k-2 \) curves on the other side. By Lemma 16 and the injectivity of \( \phi \), the curve \( \phi(w) \) must either be a \( k \)-curve or a 2-curve. Thus, it suffices to rule out the latter possibility.

First, suppose that \( w \) has an even number of punctures on one of its sides. Choose maximal collections \( C \) and \( C' \) of disjoint 2-curves on each side of \( w \). The set \( C \cup C' \), and hence \( \phi(C) \cup \phi(C') \), realizes the maximal number of disjoint 2-curves in \( S_m \). Since \( \phi \) is injective, \( \phi(w) \) cannot be a 2-curve.

Now suppose that \( w \) has an odd number of punctures on both of its sides. There are two cases. In the first case, one of the sides of \( w \) has at least five punctures. Choose maximal collections \( C_0 \) and \( C_1 \) of 2-curves on each side of \( w \), where \( C_0 \) has at least two 2-curves. If \( \phi(w) \) is a 2-curve, then the collection \( C = C_0 \cup C_1 \cup \{w\} \) maps to a maximal collection of disjoint 2-curves. Choose \( c \in C_0 \) and an adjacent 2-curve \( x \) with \( i(x, y) = 0 \) for all \( y \in C - \{c\} \). Since \( \phi(c) \) and \( \phi(x) \) are distinct 2-curves and \( \phi(C) \) is maximal, \( \phi(c) \cup \phi(x) \) must separate two other curves of \( \phi(C) \). But for any curves \( c_1, c_2 \in C \) not equal to \( c \), we can find a curve \( y \) which intersects \( c_1 \) and \( c_2 \) but does not intersect \( c \) or \( x \) (such a curve will not exist in the case that...
Proof. We claim that 2-curves \( w \) curve \( x \) intersects \( y \) and \( b \). Let \( F \) of containing \( a \) and \( c \). Note that on a thrice-punctured disk, there can be at most three families of disjoint parallel arcs. However, since \( a \) \( w \) pants decomposition containing \( a \) \( y \) and \( x \) in one of these families. The same is true for \( b \). By Lemma 18, we have that \( F_i \) is homeomorphic to \( F_i' \) and \( \phi \) induces maps from the vertices of \( \mathcal{C}(F_i) \) to those of \( \mathcal{C}(F_i') \) for \( i = 1, 2 \).

**Lemma 19** (Sides II). Let \( w \) be a curve in \( S_m \), let \( F_1 \) and \( F_2 \) be the sides of \( w \), and let \( F_1' \) and \( F_2' \) be the sides of \( \phi(w) \). Up to renumbering, \( F_i \) is homeomorphic to \( F_i' \) and \( \phi \) induces maps from the vertices of \( \mathcal{C}(F_i) \) to those of \( \mathcal{C}(F_i') \) for \( i = 1, 2 \).

**Proof.** By Lemma 18, we have that \( F_i \) is homeomorphic to \( F_i' \) for \( i = 1, 2 \). If \( \mathcal{P} \) is a pants decomposition containing \( w \), then \( \mathcal{P} - \{ w \} \) restricts to pants decompositions of \( F_1 \) and \( F_2 \), and an application of Lemma 16 completes the proof.

**Lemma 20** (Adjacency). If \( a \) and \( b \) are adjacent, then \( \phi(a) \) and \( \phi(b) \) also adjacent.

**Proof.** We claim that 2-curves \( a \) and \( b \) are adjacent if and only if there exists a curve \( w \) and curves \( x \) and \( y \) so that: \( a \) and \( b \) lie on a thrice-punctured side of \( w \), \( x \) intersects \( a \) and \( w \) but not \( b \) and not \( y \), and \( y \) intersects \( b \) and \( w \) but not \( a \) and not \( x \). By Lemma 18, Lemma 19 and the definition of superinjectivity, all of these properties are preserved by \( \phi \), and thus the lemma will follow.

One direction is easy: if \( a \) and \( b \) are adjacent, then we can find curves \( w \), \( x \), and \( y \) which satisfy the given properties. Now suppose that there exist curves \( w \), \( x \), and \( y \) which satisfy the given properties. We restrict our attention to the side of \( w \) containing \( a \) and \( b \). On this subsurface \( S' \), \( x \) and \( y \) are collections of disjoint arcs. Note that on a thrice-punctured disk, there can be at most three families of disjoint parallel arcs. However, since \( a \) is a curve disjoint from \( y \), arcs of \( y \) can only appear in one of these families. The same is true for \( x \), and we see that the arcs of \( x \) are not parallel to the arcs of \( y \). Thus, \( a \) must lie in the component of \( S' - y \) which is a twice punctured disk. There is only one such curve. Likewise, there is only one choice for \( b \), and we see that \( a \) and \( b \) are adjacent.

We alter the definition of the curve complex for \( S_4 \) so it is the flag complex with vertices for curves in \( S_4 \) and edges corresponding to adjacency. It is well-known that \( \mathcal{C}(S_4) \) is isomorphic to the classical Farey graph (see [31]). Further it is not hard to see that an injective simplicial map of the Farey graph to itself is determined by what it does to a single triangle. We thus have the following fact.

**Lemma 21.** Any injective simplicial map \( \mathcal{C}(S_4) \to \mathcal{C}(S_4) \) is surjective.

As discussed at the start of this section, the following proposition completes the proof of Theorem 2.

**Proposition 22.** \( \phi \) is surjective.
Proof. We proceed by induction on \( m \), starting with base case \( m = 4 \), which is covered by Lemma 21. Now assume that \( m \geq 5 \), and that the proposition is true for all spheres with fewer punctures.

Let \( c \) be any curve in \( S_m \). By Lemma 19, \( \phi \) induces superinjective maps from the curve complexes of the sides of \( c \) to the curve complexes of the sides of \( S_m - \phi(c) \); if any of the components are homeomorphic to \( S_4 \), then we apply Lemma 21, and if any of the components are homeomorphic to \( S_3 \), then we simply throw it out.

By induction, we conclude that these induced maps are surjective. In other words, for any curve in the image of \( \phi \), the entire set of points joined to that curve by an edge in \( C(S_m) \) is in the image of \( \phi \). Since \( C(S_m) \) is connected, it follows that \( \phi \) is surjective.

\[ \square \]

Remark. An earlier version of this paper used a different argument between Lemma 20 and the conclusion of the proof. The approach, already used in several papers [20, 24, 18, 17, 16, 2], was to show that \( \phi \) induces a map of the arc complex and to find an element of \( \text{Mod}(S_m) \) which agrees with \( \phi \) on some maximal simplex of the arc complex (injective simplicial maps of the arc complex are determined by their action on a single maximal simplex). We did this by quoting parts of Korkmaz’s proof that automorphisms of \( C(S_m) \) induce injective simplicial maps of the arc complex.

The idea of showing the surjectivity of \( \phi \) and directly applying the theorem of Korkmaz came much later, and gives a much simpler way of completing the proof. Concurrently with the revision of this paper, the induction argument was used by the authors to give a very short argument that all superinjective maps are surjective [4].

We also remark that the proof can also be simplified using the idea of the adjacency graph, introduced by Behrstock–Margalit and Shackleton [2] [32]. However, so as not to overly confuse the chronological relationships between these papers, we refrain from employing this useful tool here.

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