Asymptotic distribution of capital in a model of an investment market with competition

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30 November 2018

Abstract

We consider a stochastic game-theoretic model of an investment market in continuous time where investors compete for dividend income from several assets. Asset prices are determined endogenously from the equality of supply and demand. The main results are related to the question in what proportions the total capital will be distributed among the investors, and what will be the prices of the assets asymptotically on the infinite time horizon depending on the strategies of the investors. We prove that there exists a strategy with the following properties: the proportion of its capital on the entire time horizon is separated from zero with probability 1 regardless of the strategies of competitors; the relative asset prices are asymptotically determined by it; the proportion of capital of investors who follow essentially different strategies tends to zero. We also show that investors who follow this strategy will accumulate capital faster than their competitors under several definitions of the speed of capital growth.

Keywords: market competition, endogenous prices, capital distribution, growth optimal strategies, martingale convergence.

1 Introduction

We consider a stochastic game-theoretic model with continuous time that can be used in finance to describe competition in investment markets. Mainly, we study questions about optimality of investment strategies on the infinite time horizon. Unlike the classical theory, which assumes that investors follow some rational rules, for example, maximize their utility functions, this paper is based on another approach. We will be interested in questions of evolutionary nature: what strategies survive in the competition for the distribution of profits, what strategies dominate, what get extinct, and how

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they affect the market structure and the asset prices. This circle of questions has been studied in a number of papers in the literature, but mostly in discrete time (see the review [6]). The model considered here is one of the few in continuous time.

A market in our model consists of several investors who invest their capital in several assets which periodically pay dividends. The dividends are distributed proportionally to the number of shares of the assets purchased by the investors. Amounts of the dividends are specified by some exogenous stochastic processes, while the asset prices are determined endogenously by the equality of supply and demand. In a natural situation, the higher the expected profit from holding an asset, the greater the demand for it, and, accordingly, its price. This causes competition for possession of shares of the assets and the dividends.

We consider the following two main questions: what will be the prices of the assets and in what proportions the capital will be distributed among the investors on the infinite time horizon depending on the strategies they follow. It is not assumed that the investors are rational in the sense that their strategies are solutions to some optimization problems. For example, they can use strategies that mimic other market participants, follow some empirical rules, etc. It is also not assumed that the investors know the strategies of their competitors. Despite such a general setting, it turns out possible to get concrete answers to the questions we are interested in, of both qualitative and quantitative nature. The key circumstance is that the model is studied in the limit as the length of the time horizon tends to infinity.

Generally, the main results are as follows. There exists a class of strategies such that if an investor follows one of them, then she survives in the market – with probability one her share in the total capital does not become infinitely small. Any investor who follows an essentially different strategy gets extinct – her share of capital tends to zero. Asymptotically, all the surviving strategies are, in a certain sense, close to each other and they determine the evolution of the asset prices on a long time horizon.

Description of the class of survival strategies will be given in an explicit form: we will construct one particular survival strategy, and then show that the property of survival of any strategy can be characterized as being asymptotically close to that strategy. The key idea will be to find a strategy such that the process of its capital proportion in the total capital of all the investors is a submartingale. As it will be shown, its existence follows from Gibbs’ inequality applied to a suitable representation of the capital process. Then, using results on convergence of submartingales, we will establish the survival property of that strategy, and this will also allow to find asymptotics of the asset prices.

This approach was used for the first time in the paper [2], which studied a fairly general discrete-time model with short-lived assets. Such assets pay dividends in the next period of time after they are purchased and then dis-
appear, i.e. cannot be traded again. For particular instances of this model, similar results had been known before (see the review [6]), but they mainly used ideas based on the Law of Large Numbers, which limited them only to dividend sequences with independent increments. One can also mention the paper [1], where an approach similar to [2] was used in a model with long-lived assets, which represent a usual model of shares of stock. There, in order to prove analogous results, more subtle arguments were required. In this sense, our work is closer to [2], we also consider a model with only short-lived assets, but in continuous time.

It is worth mentioning that survival strategies bear some similarity with optimal growth strategies in models of financial mathematics without competition between investors and with exogenous asset prices (such strategies are also called numeraires or benchmark portfolios). For example, we will show in Section 5 that a survival strategy achieves the highest asymptotic rate of capital growth compared to the strategies competing with it in the market, and also has other related optimality properties. However, the major difference between survival strategies and optimal growth strategies is that the former generally cannot be constructed as solutions of some optimization problems for the amount of capital. Moreover, optimal growth strategies are, in some sense, both locally and asymptotically optimal, while survival strategies in our model are in general only asymptotically optimal, and it is not possible to find a strategy that would be locally optimal.

The paper is organized as follows. Section 2 describes the model and constructs a particular survival strategy. In Section 3 we formulate the main results about its optimality: the properties of survival and dominance, and its relation to the prices of the assets. Proofs of the results are provided in Section 4. In Section 5, we establish additional results related to growth optimality of survival strategies.

2 The market model

Before we formulate a general model in continuous time, let us briefly look at a model in discrete time, in which the main objects and formulas have a clear interpretation. Passage to continuous time will be done, as usual, by “replacing” sums with integrals and differences with differentials. The discrete time model will remain a particular case of the continuous time model.

2.1 Preliminary consideration: a model in discrete time

All the random variables introduced below are assumed to be defined on a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a filtration \(\mathcal{F} = (\mathcal{F}_t)_{t=0}^\infty\). In this section the time is discrete: \(t = 0, 1, 2, \ldots\)
A market in the model consists of \( M \geq 2 \) investors who put their capital into \( N \geq 2 \) assets paying dividends. At each moment of time \( t \geq 1 \) asset \( n \) pays a dividend \( A^t_n \geq 0 \), which is distributed among the investors proportionally to their investments into this asset. Namely, if investor \( m \) at a moment of time \( t - 1 \) invests into asset \( n \) an amount of money \( h^{m,n}_{t-1} \geq 0 \), then her profit from this asset at the next moment of time will be

\[
v^m_{t} = \frac{h^{m,n}_{t}}{\sum_k h^{k,n}_{t}} A^t_n. \tag{1}
\]

The denominators in the right-hand side, \( p^t_n := \sum_k h^{k,n}_{t} \), can be interpreted as the asset prices at time \( t - 1 \), provided that the total supply of each asset is equal to one (it can be always normalized in this way, without loss of generality), and the prices are determined by the equality of supply and demand. The investors choose the amounts \( h^{k,n}_{t} \), which they are going to invest, simultaneously and independently of each other, so this model can be considered as a simultaneous-move game.

It is assumed that the random sequences \( A^n \), which are defined endogenously, are non-negative and adapted to the filtration \( \mathcal{F} \), i.e. \( A^n_t \) are \( \mathcal{F}_t \)-measurable for all \( n,t \); the sequences \( h^{m,n} \) are non-negative and predictable, i.e. \( h^{m,n}_{t-1} \) are \( \mathcal{F}_{t-1} \)-measurable. When there is the indeterminacy \( 0/0 \) in formula (1), which happens when no one invests in asset \( n \), we set \( v^{m,n}_{t} = A^t_n / M \), so the dividends from that asset are divided equally.

At a moment of time \( t \) the capital of investor \( m \) becomes equal to \( \sum (\delta_t h^{m,n}_{t} + v^{m,n}_{t}) \), where \( \delta_t \in (0, 1] \) is a given (exogenous) \( \mathcal{F}_t \)-measurable random variable. The proportion \( 1 - \delta_t \) of the invested capital is lost (for example, due to amortization of capital goods, or taxation), while the amount of money \( \sum \delta_t h^{m,n}_{t} \) returns to the investor together with the dividends (1). It is assumed that \( \delta_t \) is the same for all the assets. It would be interesting to consider different \( \delta_t \)'s, but at the moment it is not clear whether the main results of the paper will remain true in that case. This is left for future work.

Let the initial capital of the investors be given by \( \mathcal{F}_0 \)-measurable random variables \( x^m_0 > 0 \). If investor \( m \) follows an investment strategy \( h^m \), then her capital \( X^m_t \) at moments of time \( t \geq 0 \) is defined by the equation

\[
X^m_0 = x^m_0, \quad X^m_t = \sum_n (\delta_t h^{m,n}_{t} + v^{m,n}_{t}), \quad t \geq 1. \tag{2}
\]

We will call a strategy \( h^m \) self-financing if \( \sum_n h^{m,n}_{t} = X^m_{t-1} \) for all \( t \), i.e. each time the whole current capital is reinvested. Everywhere below the word strategy will always mean a self-financing strategy. Using the self-financing condition, it is more convenient to rewrite (2) in terms of the proportions of capital of investor \( m \) put into each of the assets. By these proportions we
call the quantities \( \lambda^{m,n}_t = h^{m,n}_t / X^{m}_{t-1} \). Then the capital \( X^m_t \) and the prices \( p^n_t \) will satisfy the equations

\[
X^m_t = X^m_{t-1} \left( \delta_t + \sum_n \frac{\lambda^{m,n}_t A^n_t}{\sum_k \lambda^{k,n}_t X^k_{t-1}} \right),
\]

(3)

\[
p^n_t = \sum_k \lambda^{k,n}_t X^k_{t-1}.
\]

(4)

Note that the components of the strategies \( \lambda^{m,n}_t \) depend on a random outcome \( \omega \in \Omega \), but do not depend on the investors’ capital or their strategies. This means that the investors, when making decisions how to invest, take into consideration only dividend payments. In the paper [2], such strategies were called basic. One could consider a more general setting, where, for example, \( \lambda^m_t = \lambda^m_t(\omega, X_0, \ldots, X_{t-1}, \lambda_0, \ldots, \lambda_{t-1}) \), but this will not essentially increase the generality of the main results of our paper, see Remark 2 below.

The characteristic of investment strategies, which will present the main interest to us, is the proportion of the capital of one investor in the total capital of all the investors. The total capital is defined by the random sequence \( W_t = \sum_m X^m_t \), and, assuming that the strategies of the investors are self-financing, \( W_t \) can be defined by the equation

\[
W_t = \delta_t W_{t-1} + \sum_n A^n_t.
\]

The proportion of capital of investor \( m \) is defined by the ratio \( r^m_t = X^m_t / W_t \). Note that the sequence \( r^m_t \) depends not only on a random outcome \( \omega \) and the strategy \( \lambda^m \) (or \( h^m \)) of investor \( m \), but also on the initial conditions \( x_0 \) and the strategies of the other investors.

This paper concerns questions about the behavior of the proportion of capital \( r^m_t \) and the asset prices \( p^n_t \) on a long time horizon, i.e. as \( t \to \infty \). Mainly, we are interested in finding strategies that survive in the market.

By definition (see the next section), a strategy \( \lambda \) of investor \( m \) will be called survival, if for any initial condition \( x^m_0 > 0 \) and any strategies \( \lambda^k \) of the other investors \( k \neq m \) the inequality \( \inf_{t \geq 0} r^m_t > 0 \) holds with probability 1. It means that, regardless of the strategies of the competitors, the proportion of capital of investor \( m \) doesn’t become infinitely small. It turns out that the presence of investors who follow such strategies determines the asymptotics of the prices \( p^n_t \) as \( t \to \infty \). Moreover, all the survival strategies are close to each other in some sense, which will allow to find that asymptotics explicitly.

2.2 The general model

From now on assume given a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) which satisfies the usual assumptions, i.e. the filtration \( \mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+} \) is right-continuous (\( \mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s \)), and \( \mathcal{F}_0 \) contains all the \( \mathbb{P} \)-null sets of \( \mathcal{F} \).
We will use the following notation. For vectors $x, y \in \mathbb{R}^N$, by $|x|$ will be denoted the $l_1$-norm of a vector, and by $xy$ the scalar product; for a scalar function $f: \mathbb{R} \to \mathbb{R}$ the notation $f(x)$ means the coordinate-wise application of the function:

$$
|x| := \sum_n |x^n|, \quad xy := \sum_n x^n y^n, \quad f(x) := (f(x^1), \ldots, f(x^N)).
$$

If $G_t = G(t)$ is a non-decreasing function, then for a measurable function $f_t$

$$
f \cdot G_t := \int_0^t f_s dG_s,
$$

provided that the integral is well-defined (as a Lebesgue-Stieltjes integral). Functions $f, G$ may be random, then $f \cdot G_t(\omega)$ is defined pathwise for each $\omega$. The upper limit of integration $t$ may be equal to $+\infty$. If $f$ is a vector-valued function and $G$ is scalar-valued, then $f \cdot G_t = (f^1 \cdot G_t, \ldots, f^N \cdot G_t)$; if both are vector-valued, then $f \cdot G_t = \sum_n f^n \cdot G^n_t$.

For random processes $X_t(\omega), Y_t(\omega)$ the equality $X = Y$ is understood as the equality for all $t, \omega$; in the same way we treat inequalities. We will also use the notion of equality up to an evanescent set. A set $A \subset \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$ is called evanescent if $P\{\omega : \exists t \text{ such that } (\omega, t) \in A\} = 0$. Where it doesn’t lead to ambiguity, the words “up to an evanescent set” will be omitted. Similarly, properties of trajectories (continuity, monotonicity, etc.) are assumed to hold for all $\omega$, unless else is specified.

By the predictable $\sigma$-algebra $\mathcal{P}$ on $\Omega \times \mathbb{R}_+$ we call, as usual, the $\sigma$-algebra generated by all the left-continuous adapted processes. A process is predictable if it is measurable with respect to $\mathcal{P}$ as a map from $\Omega \times \mathbb{R}_+$ to $\mathbb{R}$ or to $\mathbb{R} \cup \{\pm \infty\}$.

As in the discrete-time model, there are $M \geq 2$ investors and $N \geq 2$ assets. Instead of the sequences of dividend payments $A^n_t$, it will be more convenient to assume that processes of cumulative dividends $S^n_t$ are given (in the discrete-time model $A^n_t = S^n_t - S^n_{t-1}$). It is assumed that they are adapted to the filtration $\mathcal{F}$ and have non-decreasing càdlàg paths (right-continuous with left limits) and $S^n_0 = 0$. Similarly, instead of the coefficients $\delta_t$, we will use an adapted non-decreasing càdlàg cumulative discounting process $D_t$ such that its jumps $\Delta D_t \in [0, 1)$ (as usual, $\Delta D_t = D_t - D_{t-}$, where $D_{t-} = \lim_{s \to t} D_s$ and $\Delta D_0 = 0$); in discrete time, $\Delta D_t = 1 - \delta_t$. To avoid problems with non-integrability (see Section 2.3), we will assume that the jumps of $D_t$ are uniformly bounded away from 1, i.e. there exists a constant $\varepsilon_D \in (0, 1]$ such that

$$
\Delta D_t \leq 1 - \varepsilon_D \quad \text{for all } t \geq 0.
$$

A strategy of investor $m$ is identified with a predictable process of proportions of invested capital $\lambda^m_t$, which assumes values in the standard simplex.
in $\mathbb{R}^N$, i.e. $\lambda_{t}^{m,n} \geq 0$ and $\sum_{n} \lambda_{t}^{m,n} = 1$. As it was noted above, we consider only basic strategies, in the sense that $\lambda_{t}^{m}$ doesn’t depend on the “past history” of the processes $\lambda_{t}^{k}$ up to time $t$; this doesn’t reduce the generality of the main results.

The capitals of the investors are described by strictly positive càdlàg processes $X^{m}$ that satisfy the equation (a continuous-time analogue of (3))

$$dX_{t}^{m} = X_{t}^{m} \left( \sum_{n} \lambda_{t}^{m,n} \sum_{k} \lambda_{t}^{k,n} X_{t}^{k} - dS_{t}^{n} - dD_{t} \right), \quad X_{0}^{m} = x_{0}^{m}$$

(6)

with $\mathcal{F}_{0}$-measurable initial conditions $x_{0}^{m} > 0$. If $\lambda_{t}^{k,n} = 0$ for all $k$, then we assume that the value of the ratio in the right-hand side is $1/(MX_{t}^{m})$ for the corresponding $n$. As usual, this equation should be understood in the integral form:

$$X_{t}^{m} = x_{0}^{m} + \sum_{n} \int_{0}^{t} \lambda_{u}^{m,n} X_{u}^{m} - \int_{0}^{t} \sum_{k} \lambda_{u}^{k,n} X_{u}^{k} - dS_{u}^{n} - \int_{0}^{t} X_{u}^{m} dD_{u}, \quad t \geq 0.$$  

(7)

The integrals here are pathwise Lebesgue-Stieltjes integrals (the processes $S_{t}, D_{t}$ don’t decrease). It is not difficult to see that if $X_{t}^{m}$ satisfies (7), then it has finite variation on any interval $[0, t]$. The next proposition shows that this equation has a unique solution, hence the capital processes are well-defined.

**Proposition 1.** For any $\mathcal{F}_{0}$-measurable random variables $x_{0}^{m} > 0$ and strategies $\lambda^{m}$, $m = 1, \ldots, M$, there exists a unique adapted strictly positive càdlàg process $X = (X^{1}, \ldots, X^{M})$ which satisfies (7).

For given initial capital amounts $x_{0}^{m}$, strategies $\lambda^{m}$ and the corresponding capital processes $X^{m}$, define the process of the total capital of all the investors $W$ and the proportion $r^{m}$ of investor $m$’s capital in the total capital:

$$W_{t} = |X_{t}|, \quad r_{t}^{m} = \frac{X_{t}^{m}}{W_{t}}.$$  

Equation (6) implies that the total capital satisfies the equation $dW_{t} = d|S_{t}| - W_{t} \cdot dD_{t}$ with the initial condition $W_{0} = |x_{0}|$. In the case when it is necessary to emphasize that the introduced processes depend on the initial capitals and the strategies, we will use the notation $r_{t}^{m}(x_{0}, \Lambda)$, where $\Lambda = (\lambda^{1}, \ldots, \lambda^{M})$ denotes the aggregate of the strategies of all the investors.

The prices and the relative prices of the assets are defined (like in (4)) as the predictable processes

$$p_{t}^{n} = \sum_{m} \lambda_{t}^{m,n} X_{t}^{m}, \quad \pi_{t}^{n} = \frac{p_{t}^{n}}{p_{t}} = \frac{p_{t}^{n}}{W_{t}} = \sum_{m} \lambda_{t}^{m,n} r_{t}^{m}.$$  

7
Since we assume that the supply of each asset is normalized to 1, the relative prices are just the proportions in which the total capital is divided between the assets.

The central definition of the present paper is the notion of a **survival strategy**. We call a strategy \( \lambda \) survival, if for any initial conditions \( x_0^m > 0, m = 1, \ldots, M \), and an aggregate of strategies \( \Lambda = (\lambda^1, \ldots, \lambda^M) \) with \( \lambda^1 = \lambda \) and arbitrary strategies \( \lambda^m, m = 2, \ldots, M \), holds the inequality

\[
\inf_{t \geq 0} r_t^1(x_0, \Lambda) > 0 \text{ a.s.}
\]

In the next section we construct one such strategy.

It is worth emphasizing that the property of survival is much stronger than just the possibility to keep a fraction of capital bounded away from zero. In fact, we will show that survival strategies determine the market structure and the asset prices. For that reason, in the seminal paper [3] survival strategies are termed **dominating** strategies. However, we prefer a slightly different terminology, which is from [1, 2].

### 2.3 Construction of a survival strategy

This section relies on several known facts from stochastic calculus, which can be found, for example, in [8].

Define the \( N \)-dimensional process of cumulative relative dividends \( R_t \),

\[
dR_t^n = \frac{dS_t^n}{W_t}, \quad R_0^n = 0.
\]

The process \( R_t \) is càdlàg and non-decreasing in each coordinate. Split it into the continuous part and the sum of jumps. Let \( B_t \) denote its continuous part, which is the non-decreasing process

\[
B_t^n = R_t^n - \sum_{s \leq t} \Delta R_s^n,
\]

where \( \Delta R_s^n = R_s^n - R_{s-}^n \), and for \( s = 0 \) we set \( \Delta R_0^n = 0 \). It will be convenient to work with jumps \( \Delta R_t^n \) and \( \Delta D_t \) using the measure of jumps of the \((N + 1)\)-dimensional process \((R_t, D_t)\). It is defined as the integer-valued random measure on \((S, \mathcal{B}(S))\), where \( S = \mathbb{R}_+ \times \mathbb{R}_+^{N+1} \) and \( \mathcal{B} \) is the Borel \( \sigma \)-algebra, by the formula

\[
\mu(\omega, A) = \sum_{t \geq 0} \mathbb{1}(\Delta(R_t, D_t)(\omega) \neq 0, (t, \Delta(R_t, D_t)(\omega)) \in A), \quad A \in \mathcal{B}(S)
\]

(actually, we can even assume \( S = \mathbb{R}_+ \times \mathbb{R}_+^N \times [0, 1 - \varepsilon_D] \), where \( \varepsilon_D \) is the constant from the bound for jumps of \( D_t \), see [5]). For the integral of an
\(\mathcal{F} \otimes \mathcal{B}(\mathcal{S})\)-measurable function with respect to a random measure we will use the notation

\[
f * \mu_t(\omega) = \int_{(0,t] \times \mathbb{R}_+^{N+1}} f(\omega, s, x, y) \mu(\omega, ds, dx, dy),
\]

assuming that the integral is well-defined (as a Lebesgue integral), possibly being \(+\infty\) or \(-\infty\). Henceforth, the variable \(x \in \mathbb{R}_+^{N+1}\) corresponds to the jumps \(\Delta R_t\), and \(y \in \mathbb{R}_+\) to the jumps \(\Delta D_t\). The integral (8) can be defined for a general random measure; in the particular case when \(\mu\) is our measure of jumps, it can be simply written as the sum

\[
f * \mu_t(\omega) = \sum_{s \leq t} f(\omega, s, \Delta R_s(\omega), \Delta D_s(\omega)) \mathbf{1}(\Delta(R_s, D_s)(\omega) \neq 0).
\]

In the case when \(f\) is a vector-valued measurable function, we treat the integral (8) as vector-valued and compute it coordinatewisely. In particular, the process \(R_t\) can be represented as

\[
R_t = B_t + x * \mu_t,
\]

where \(x = (x^1, \ldots, x^N)\) and \(x^n\) here stands for the function \((x, y) \mapsto x^n\).

Let \(\overline{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+^{N+1})\) be the predictable \(\sigma\)-algebra on \(\Omega \times \mathbb{R}_+ \times \mathbb{R}_+^{N+1}\). A random measure \(\nu\) is called predictable, if for any \(\overline{\mathcal{P}}\)-measurable non-negative function \(f(\omega, t, x, y)\) the process \(f * \nu_t\) is predictable (\(P\)-measurable).

By a compensator of the measure of jumps \(\mu\) we call a predictable random measure \(\nu\), such that for any \(\overline{\mathcal{P}}\)-measurable non-negative function \(f\) holds the equality

\[
E(f * \mu_\infty) = E(f * \nu_\infty),
\]

or, equivalently, \(f * (\mu - \nu)_t\) is a local martingale, provided that the process \(|f| * \mu_t\) is locally integrable. The measure of jumps of an adapted càdlàg process always has a compensator, which is unique up to indistinguishability with respect to the probability measure \(P\) \(\S\ II.1\). Since the processes \(R\) and \(D\) don’t decrease, the inequality \(|x| \wedge 1 + y) * \nu_t(\omega) < \infty\) holds a.s. for all \(t\), see \([12, \S\ 4.1]\).

From the general theory, it is known that there exists a predictable non-decreasing locally integrable scalar process \(G\) (a process of operational time) such that up to an evanescent set

\[
B_t = b \cdot G_t, \quad \nu(\omega, dt, dx, dy) = K_{\omega,t}(dx, dy)dG_t(\omega),
\]

where \(b_t\) is a non-negative predictable \(N\)-dimensional process, and \(K_{\omega,t}(dx, dy)\) is a transition kernel from \((\Omega \times \mathbb{R}_+, \mathcal{P})\) to \((\mathbb{R}_+^{N+1}, \mathcal{B}(\mathbb{R}_+^{N+1}))\) which for all \(\omega, t\) satisfies the properties

\[
K_{\omega,t}(\{0\}) = 0, \quad \int_{\mathbb{R}_+^{N+1}} (|x| \wedge 1 + y) K_{\omega,t}(dx, dy) < \infty.
\]
Often, it will be convenient to use the process
\[ G_t = |B_t| + (|x| \wedge 1 + y) \ast \nu_t. \] (11)
The possibility of representation (10) for this process can be proved similarly to Proposition II.2.9 in [8].

For \( b, K, G \) satisfying (10), define the non-negative predictable process \( a_t \) with values in \( \mathbb{R}^N_{+} \) by the formula
\[ a^n_t = \int_{\mathbb{R}^{N+1}_+} \frac{x^n}{1 - y + |x|} K_t(dx, dy), \]
and define the strategy \( \hat{\lambda} \) by
\[ \hat{\lambda}_t = \frac{a_t + b_t}{|a_t + b_t|}, \] (12)
where we set \( \hat{\lambda}_n^t = 1/N \) for all \( n \) whenever \( |a_t + b_t| = 0 \). This strategy will be a candidate for a survival strategy. Note that the continuous part of the process \( D_t \) is not involved in its definition.

The strategy \( \hat{\lambda} \) doesn’t essentially depend on the choice of an operational time process in the following sense. Let \( G_t \) be defined by (11), and \( a_t, b_t, \hat{\lambda} \) be constructed from it as described above. Suppose \( G'_t \) is another predictable process satisfying (10) with some process \( b'_t \) and transition kernel \( K'_t \). Then the random measure generated by \( G_t \) on \( \mathbb{R}^+ \) is a.s. absolutely continuous with respect to the measure generated by \( G'_t \), and according to [8, Proposition I.3.13] there exists a non-negative predictable process \( \rho_t \) such that \( G = \rho \cdot G' \) up to an evanescent set. Define the measure \( Q = P \otimes G \) on \((\Omega \times \mathbb{R}^+, \mathcal{P})\), i.e. \( Q(A) = E(\int_0^\infty 1((\omega, t) \in A)dG_t(\omega)) \) for \( A \in \mathcal{P} \). Since \( b \cdot G = (\rho b) \cdot G' \) up to an evanescent set, we have \( \rho b = b' \) (\( P \otimes G'\)-a.s. and, hence, \( Q\)-a.s.). In a similar way, \( \rho K = K' \) (\( Q\)-a.s.), which implies \( \rho a = a' \) (\( Q\)-a.s.). Then, from (12), we have \( \hat{\lambda} = \hat{\lambda}' \) (\( Q\)-a.s.).

**Remark 1.** Obviously, the discrete-time model of Section 2.1 is a particular case of the general model. In that model, \( \Delta R^n_t = A^n_t/W_{t-1} \), where \( A^n_t = \Delta S^n_t \), and \( \Delta D_t = 1 - \delta_t \) for \( t \in \mathbb{N} \). One can take \( G_t = [t] \) (the integer part), \( B_t = 0 \), and \( \nu(\cdot, \{t\} \times dx \times dy) \) being the regular conditional distribution of the jump \((\Delta R_t, \Delta D_t)\) with respect to the \( \sigma \)-algebra \( \mathcal{F}_{t-1} \). Then
\[ a^n_t = E\left(\frac{A^n_t}{W_t} \mid \mathcal{F}_{t-1}\right), \quad b_t = 0, \quad \hat{\lambda}_t = \frac{a_t}{|a_t|}, \] (13)

3 The main results

3.1 Statements

In this section we assume given and fixed an operational time process \( G \) for which the representation (10) holds, and \( a_t, b_t, K_t, \hat{\lambda}_t \) constructed from \( G_t \).
as in the previous section. We also define the scalar process

\[ H_t = |a + b| \cdot G_t. \]

For an adapted scalar process \( V \), we will denote by \( \mathcal{M}(V) = \{ \tau_a(V), a \in \mathbb{R}_+ \} \) the class of Markov times when \( V \) exceeds a level \( a \) for the first time:

\[ \tau_a(V) = \inf\{ t \geq 0 : V_t \geq a \}, \]

where \( \inf\emptyset = +\infty. \)

**Theorem 1.** Suppose a strategy \( \lambda \) satisfies the following conditions:

(a) the set \( \bigcup_n \{ (\omega, t) : \lambda^n(\omega) = 0, \hat{\lambda}^n(\omega) \neq 0 \} \) is evanescent,

(b) the process \( U_t := \hat{\lambda}_t (\ln \hat{\lambda}_t - \ln \lambda_t) \) satisfies \( U \cdot H_\infty < \infty \) a.s.,

(c) \( \mathbb{E}(U_t \Delta H_t I(\tau < \infty)) < \infty \) for any \( \tau \in \mathcal{M}(U \cdot H) \).

Then, if investor \( m \) follows the strategy \( \lambda \),

(A) for any strategies \( \lambda^k \) of the other investors the limit \( \lim_{t \to \infty} r^m_t > 0 \) exists with probability one, and the strategy \( \lambda \) is survival;

(B) the relative prices \( \pi \) satisfy the inequality \( |\hat{\lambda} - \pi|^2 \cdot H_\infty < \infty \) a.s.

This theorem states that if a strategy \( \lambda \) is close to \( \hat{\lambda} \), then it survives (in particular, \( \hat{\lambda} \) survives). The closeness is essentially determined by the condition (b), while (a) and (c) are technical assumptions. Moreover, if such a strategy is present in the market, then it asymptotically determines the relative prices in view of (B), and, hence, the prices \( p_t = \pi_t W_t \).

Let us clarify that in the conditions (b), (c) on the sets \( \{ \lambda^n = 0 \} \) and \( \{ \hat{\lambda}^n = 0 \} \) the corresponding term in the scalar product in the definition of \( U_t \) is assumed to be zero (which agrees with the condition (a)). Also observe that the process \( U_t \) is non-negative as follows from Gibbs' inequality (see (20) below). Therefore, the integral \( U \cdot H_\infty \) and the expectation in the condition (c) are always well-defined, though they may take on the value \( +\infty \).

The next simple proposition can be useful for verification of the conditions (a), (b) of Theorem 1 in particular models (a simple sufficient condition for the validity of (c) is the continuity of the process \( G \), and hence \( H \)).

**Proposition 2.** Suppose the strategy \( \hat{\lambda} \) satisfies the condition \( \inf_{t \geq 0} \hat{\lambda}^n_t > 0 \) a.s. for all \( n \). In that case, if a strategy \( \lambda \) satisfies the inequalities \( \inf_{t \geq 0} \lambda^n_t > 0 \) a.s. for all \( n \) and \( |\hat{\lambda} - \lambda|^2 \cdot H_\infty < \infty \) a.s., then it satisfies the conditions (a), (b) of Theorem 1.

The next result, which follows from Theorem 1, shows that all the survival strategies are, in some sense, asymptotically close to the strategy \( \hat{\lambda} \).

**Corollary 1.** If a strategy \( \lambda \) is survival, then \( |\hat{\lambda} - \lambda|^2 \cdot H_\infty < \infty \) a.s.
The second corollary provided below answers the question under what conditions an investor who follows a survival strategy asymptotically dominates in the market, i.e. her proportion of the total capital tends to 1 as \( t \to \infty \). It will be shown that if the conditions of Theorem 1 are satisfied, then this happens if the representative strategy of the other investors is in some sense essentially different from \( \hat{\lambda} \).

The representative strategy is defined in the following way. Consider the model with \( N \) assets, which have cumulative dividend processes \( S^m \) and a discounting process \( D \), and \( M \) investors who follow strategies \( \lambda^m \) and their capital processes \( X^m \) are defined by (7). For the same assets with the processes \( S^m, D \), on the same filtered probability space consider the model with only two investors: the first one has the starting capital \( \tilde{x}^1_0 = x_0 \) and uses the strategy \( \tilde{\lambda}^1 = \lambda^1 \), while the second one has the starting capital \( \tilde{x}^2_0 = \sum_{m=2}^M x^m_0 \) and uses the strategy \( \tilde{\lambda}^2 \) (which we call the representative strategy of the original investors \( m = 2, \ldots, M \)) with the components

\[
\tilde{\lambda}^2, n_t = \sum_{m=2}^M \lambda^m, n_t r^m_t(\Lambda) / (1 - r^1_t(\Lambda)).
\]

Then equation (7) implies that

\[
\tilde{X}^1_t = X^1_t, \quad \tilde{X}^2_t = \sum_{m=2}^M X^m_t, \quad r^1_t(\Lambda) = r^1_t(\tilde{\Lambda}).
\]

It is clear from here that in order to analyze the proportion of capital \( r^1_t \) in the case when \( M > 2 \) investors are present, it is enough to consider the above model with two investors, where the second investor follows the representative strategy. Observe that \( \tilde{\lambda}^2, n_t \) in formula (14) are evaluated for given and fixed \( \Lambda \) and \( r_t(\Lambda) \), hence, they are predictable processes. This allows to treat \( \tilde{\lambda}^2 \) as a basic strategy.

**Corollary 2.** Suppose \( M = 2 \) and investor 1 follows a strategy \( \lambda \) which satisfies the conditions of Theorem 1. Then for any strategy \( \tilde{\lambda} \) of the second investor, \( \lim_{t \to \infty} r^1_t = 1 \) a.s. on the set \( \{ |\tilde{\lambda} - \tilde{\lambda}|^2 \cdot H_\infty = \infty \} \).

**Remark 2.** As it has been noted, all strategies considered in the present paper are basic in the sense that their components are functions of \( t \) and \( \omega \) only. It is possible to consider general strategies, where \( \lambda_t \) also depends on paths of the processes \( X, \lambda \) up to a time \( t \) in an appropriate non-anticipatory way. As one can see from the proofs of the main results, Theorem 1, Proposition 2 and Corollaries 1, 2 remain valid in this case, if one additionally assumes that the strategy \( \lambda \) is basic (as for any aggregate \( \Lambda \) of general strategies it is possible to find an aggregate of basic strategies which generate the same process of capital proportions \( r(\Lambda) \)). In particular, Theorem 1 implies the
remarkable result that in a model with general strategies a survival strategy exists and can be found among basic strategies ($\lambda$ is such a strategy). However, survival strategies will be asymptotically close only in the class of basic strategies, i.e. Corollary 1 doesn’t hold if one allows $\lambda$ to be a general survival strategy. A counterexample is provided in the paper [2] for a different model, but it can be easily carried to our setting as well.

3.2 Other results in the literature

A model very close to ours in discrete time was studied in the paper [2]. Its main difference is that the invested capital is not returned, i.e. the equation defining the capital sequences of investors, instead of (2), is the following one:

$$X_{t}^{m} = \sum_{n} v_{t}^{m,n} = \sum_{n} h_{t}^{m,n} A_{t}^{n} \sum_{n} h_{t}^{m,n} A_{t}^{n}.$$  (15)

Note that this equation can be formally obtained from (2) by setting $\delta_{t} = 0$. The main results of the paper [2] also consist in finding a survival strategy in an explicit form and proving that all the survival strategies are asymptotically close to it. In that model, a survival strategy is defined by the same formula as (13) with $\delta_{t} = 0$.

Similar results were also obtained in the paper [1] for a more difficult model in discrete time, which assumes that investors can resell their assets at subsequent moments of time for the price determined by the equality of supply and demand – such an assumption is natural for a model of a stock market. Quite remarkably, a survival strategy in that model also exists and can be found in the class of basic strategies which depend only on the structure of dividend sequences, but not on the actions of investors.

Let us also mention the paper [3] – one of the first in this direction – where a result similar to the statement (B) of Theorem 1 was obtained. The model considered in that paper is a simple particular case of (15), where at each moment of time only one asset pays a dividend and its value, if paid, is known in advance.

A review of other results (mostly in discrete time) can be found in the paper [6]. Note that the model (15) cannot be straightforwardly generalized to the case of continuous time, since it should allow that during an “infinitely short” period of time the amount of dividends $A_{t}$ can be “infinitely small” – but then equation (15) makes no sense. There is no such a problem in our model since $\delta_{t} > 0$.

Among (few) other results in this direction for continuous time, let us mention the paper [13], where convergence of a discrete-time model to a continuous-time model was studied, and the paper [14], where questions of survival and dominance of investment strategies were investigated in the case when dividends are represented by absolutely continuous non-decreasing processes.
4 Proofs

Proof of Proposition 1. Introduce the function $F : \mathbb{R}_+^{M+N+M} \rightarrow \mathbb{R}_+^{M+N}$ which specifies the distribution of dividends in equation (7):

$$[F(\lambda, x)]^{m,n} = \frac{\lambda^{m,n} x^m}{\sum_k \lambda^{k,n} x^k}.$$ 

If $\lambda^{k,n} = 0$ for some $n$ and all $k$, we define $[F(\lambda, x)]^{m,n} = 1/M$. It is straightforward to check that $|\partial[F(\lambda, x)]^{m,n}/\partial x^k| \leq 1/x^k$. Hence, $F$ is Lipschitz continuous in $x$ on any set $\{x : x \geq a\}$, where $a \in \mathbb{R}_+^M$ is a vector with strictly positive coordinates. Let $l(a)$ be a constant such that $|F(\lambda, x) - F(\lambda, y)| \leq l(a)|x - y|$ for any $x, y \geq a$ and $\lambda \in \mathbb{R}_+^{M,N}$.

Define the sequence of stopping times $\tau_i, i \geq 0$,

$$\tau_0 = 0, \quad \tau_i = \inf\{t \geq \tau_{i-1} : |S_t| \geq |S_{\tau_{i-1}}| + (4l(x_0))^{-1} \text{ or } D_t \geq D_{\tau_{i-1}} + 1/4\} \wedge (\tau_{i-1} + 1), \quad i \geq 1,$$

where $x_0$ is the initial condition in equation (7), and $\inf \emptyset = \infty$. Clearly, $\tau_i \leq i$ for all $i$ and $\tau_i \to \infty$ as $i \to \infty$.

We will construct the solution of (7) by induction on the intervals $[0, \tau_i]$, $i \geq 0$. Namely, we will define a sequence of strictly positive adapted càdlàg processes $X^{(i)}$ with finite variation on any interval $[0, t]$ that satisfy (7) on $[0, \tau_i]$ and also $X^{(i)}_t = X^{(i-1)}_t$ for $t \leq \tau_{i-1}$. For $i = 0$, let $X^{(0),m}_t = x_0^m$ for all $t \geq 0$.

Suppose the process $X^{(i-1)}$ is constructed, let us show how to find $X^{(i)}$. Denote by $\mathcal{D}$ the Banach space of bounded càdlàg functions $f : \mathbb{R}_+ \rightarrow \mathbb{R}^M$ with the norm $\|f\| = \sup_{t \geq 0} |f_t|$ and by $\mathbb{D}_x$, $x \in \mathbb{R}_+^M$, its closed subspace consisting of $f$ such that $f_0^m \geq x^m$ for all $m$.

For each $\omega$ consider the operator $H$ which maps $f \in \mathcal{D}_{x_0(\omega)}$ to

$$H(\omega, f)_t = X^{(i-1)}_t(1(t \leq \tau_{i-1}) + \int_{(\tau_{i-1}, t]} \text{I}(u < \tau_i)(F(\lambda_u, f_u-)dS_u - f_u_-dD_u))\text{I}(t > \tau_{i-1}),$$

where the random variables in the right-hand side are evaluated for a given $\omega$. It’s not difficult to see that $H$ maps $\mathcal{D}_{x_0(\omega)}$ to itself. Indeed, the function $H(f)$ is clearly càdlàg, and its boundedness follows from the estimate $|H(f)_t| \leq |X^{(i-1)}_{\tau_{i-1}}| + |S_t| + \|f\|D_t$ for all $t$, where we use that $\tau_i \leq i$ and $|F(\lambda, x)| \leq 1$. Moreover, $H$ preserves adaptedness of processes in the sense that if $X$ is an adapted process, then so is $H(X)$. 

14
By the choice of the stopping times \( \tau_i \), for any \( f, \tilde{f} \in D_{x_0(\omega)} \) we have
\[
\sup_{t \geq 0} |H(f) - H(\tilde{f})| \\
\leq \int_{(\tau_{i-1}, \tau_i]} (|F(\lambda_t, f_{t^-}) - F(\lambda_t, \tilde{f}_{t^-})|dS_t + |f_{t^-} - \tilde{f}_{t^-}|dD_t) \\
\leq \frac{1}{2} \sup_{t \geq 0} |f_t - \tilde{f}_t|,
\]
which implies that \( H \) is a contraction mapping on \( D_{x_0(\omega)} \). Therefore it has a fixed point \( \tilde{X} \), which satisfies equation (7) on the half-interval \([0, \tau_i)\). Also, the process \( \tilde{X} \) is adapted, since it can be obtained as the limit (for each \( \omega \) and \( t \)) of the adapted processes \( H^n(X^{(i-1)}) \) as \( n \to \infty \), where \( n \) stands for \( n \)-times application of \( H \). Define
\[
X^{(i)}_t = \tilde{X}_t I(t < \tau_i) + \left[ \tilde{X}_{\tau_i} + F(\lambda_{\tau_i}, \tilde{X}_{\tau_i}) \Delta S_{\tau_i} - \tilde{X}_{\tau_i} \Delta D_{\tau_i} \right] I(t \geq \tau_i).
\]
Then \( X^{(i)} \) is the sought-for process which satisfies (7) on the whole interval \([0, \tau_i]\). Using that \( \Delta D_t \leq 1 - \varepsilon_D \), from (7) one can see that \( X^{(i)} \) is a strictly positive process.

The uniqueness of the solution of (7) follows from the uniqueness of the fixed point of the operator \( H \) on each step of the induction. \( \square \)

**Proof of Theorem 1.** Suppose investor \( m \) uses a strategy \( \lambda \) which satisfies the conditions (a)–(c). First we are going to prove that the process \( Y_t := \ln r^m_t + U \cdot H_t - \ln r^m_0 \) is a local submartingale.

It will be convenient to represent the capital process \( X^m \) and the process of total capital \( W \) as stochastic exponents. Recall that the stochastic exponent of a scalar semimartingale \( Z \) is defined as the semimartingale \( \mathcal{E}(Z) \) that solves the stochastic differential equation (which always has a unique strong solution, see [8, §I.4f])
\[
d\mathcal{E}(Z)_t = \mathcal{E}(Z)_{t^-}dZ_t, \quad \mathcal{E}(Z)_0 = 1.
\]
In all the cases we are going to consider, only adapted càdlàg processes with finite variation on any interval \([0, t] \) will be used as \( Z \), and hence this equation should be understood in the sense of pathwise Lebesgue–Stieltjes integration. The Doolean–Dade formula implies that
\[
\mathcal{E}(Z)_t = e^{Z_t} \prod_{s \leq t} (1 + \Delta Z_s)e^{-\Delta Z_s}.
\]
Let us associate with the strategy \( \lambda^m = \lambda \) of investor \( m \) the \( N \)-dimensional predictable process \( L_t \) with the components
\[
L^n_t = \lambda^n_t / \pi^n_t,
\]
where \( \pi_{nt} = \sum_k \lambda_{t,k}^{n,n} r_{t}^{k} \) is the relative price of asset \( n \), and in the case of the indeterminacy \( 0/0 \) we assume \( L_{t}^{n} = (M r_{t}^{m})^{-1} \). Then

\[
dX_{t}^{m} = X_{t}^{m} \left( \sum_{n} L_{t}^{n} dR_{t}^{n} - dD_{t} \right), \quad dW_{t} = W_{t} \left( \sum_{n} dR_{t}^{n} - dD_{t} \right).
\]

Therefore,

\[
X_{t}^{m} = X_{0}^{m} e^{\mathcal{E}(L \cdot R - D)_{t}}, \quad W_{t} = W_{0} e^{\mathcal{E}(|R| - D)_{t}}. \tag{17}
\]

Introduce the process \( Z_{t} = \ln \left( r_{m}^{m}/r_{0}^{m} \right) = \ln \left( e^{\mathcal{E}(L \cdot R - D)_{t}} - e^{\mathcal{E}(|R| - D)_{t}} \right) \).

As follows from (16),

\[
Z_{t} = (L - 1) \cdot B_{t} + \sum_{s \leq t} \ln \left( \frac{1 - \Delta D_{s} + L_{s} \Delta R_{s}}{1 - \Delta D_{s} + |\Delta R_{s}|} \right)
\]

(the unit is subtracted from each coordinate of \( L \)). Define the predictable function \( f: \Omega \times \mathbb{R}_{+} \times \mathbb{R}_{+}^{N+1} \rightarrow \mathbb{R} \) by

\[
f(\omega, t, x, y) = \ln \left( \frac{1 - y + L_{t}(\omega)x}{1 - y + |x|} \right).
\]

Using this function, it is possible to write

\[
Z_{t} = (L - 1) \cdot B_{t} + f \ast \mu_{t}.
\]

Let us prove the representation (up to an evanescent set)

\[
Z_{t} = g \cdot G_{t} + f \ast (\mu - \nu)_{t} \tag{18}
\]

with the function

\[
g_{t} = (L_{t} - 1)b_{t} + \int_{\mathbb{R}_{+}^{N+1}} f_{t}(x, y)K_{t}(dx, dy).
\]

To prove (18), it is sufficient to show that \( f \ast \nu_{t} < +\infty \) and \( g \cdot G_{t} > -\infty \) a.s. for all \( t \) (and then we’ll also have \( |f| \ast \nu_{t} < +\infty \) a.s.).

Consider the Markov times \( \tau_{i} = \inf\{t \geq 0 : r_{t}^{n} \leq 1/i\} \) with \( \inf \emptyset = +\infty \).

It’s not difficult to see that \( L_{t}^{n} \leq (r_{t}^{m})^{-1} \) for all \( n, t \). Then \( f \leq i \int_{1 - \epsilon D_{i} + |x|}^{x} \frac{d\nu_{t}}{i} \) on the set \( \{t < \tau_{i}(\omega)\} \). Since \( (|x| \wedge 1) \ast \nu_{t} < +\infty \) a.s. for all \( t \), then also \( f \ast \nu_{\tau_{i}} < +\infty \) a.s. Because \( \tau_{i} \rightarrow \infty \) as \( i \rightarrow \infty \) (due to the strict positivity of \( r_{t}^{m} \)), passing to the limit \( i \rightarrow \infty \), we obtain that \( f \ast \nu_{t} < +\infty \) for all \( t \).

Let us prove that \( g \cdot G_{t} > -\infty \) a.s. for all \( t \). For convenience of notation, introduce the function \( \overline{x} = \ln x \) if \( x > 0 \) and \( \overline{x} = -1 \) if \( x \leq 0 \), and the set \( \mathcal{X}(\omega, t) = \{(x, y) \in \mathbb{R}_{+}^{N+1} \setminus \{0\} : x^{n} = 0 \text{ if } L_{t}^{n}(\omega) = 0, \ n = 1, \ldots, N\} \).
Using Jensen’s inequality and the concavity of the logarithm, we find that for any \( x \in \mathcal{X}(\omega, t) \)

\[
f_t(x, y) = \ln \left( \frac{1 - y}{1 - y + |x|} + \frac{|x|}{1 - y + |x|} \right) \geq \frac{|x|}{1 - y + |x|} \ln \left( \frac{L_t x}{|x|} \right)
\]

\[
\geq \frac{x}{1 - y + |x|} \ln L_t
\]

This implies that a.s. for each \( t \)

\[
\int_{\mathbb{R}^{N+1}} f_t(x, y) K_t(dx, dy) = \int_{\mathcal{X}_t} f_t(x, y) K_t(dx, dy) \geq a_t \ln L_t,
\]

where we use that \( K_t(\mathbb{R}^{N+1} \setminus \mathcal{X}_t) = 0 \). Indeed, the set \( \mathbb{R}^{N+1} \setminus \mathcal{X}_t \) consists of \((x, y)\) such that \( L_t^n(\omega) = 0 \) but \( x^n > 0 \) for some \( n \). On the set \( \{L^n_t = 0\} \) we have \( \lambda^n_t = 0 \), so by the condition (a) of the theorem, \( \hat{\lambda}^n_t = 0 \) a.s. on this set, and therefore \( K_{\omega, t}(\{x^n > 0\}) = 0 \).

Then we can write

\[
g_t \geq (L_t - 1)b_t + a_t \ln L_t \geq (a_t + b_t) \ln L_t \geq \hat{\lambda}_t(\ln \lambda_t - \ln \pi_t)|a_t + b_t|,
\]

where in the second inequality we use that \( L^n_t - 1 \geq \ln L^n_t \).

Next use well-known Gibbs’ inequality. Suppose vectors \( \alpha, \beta \in \mathbb{R}^N_+ \) are such that \(|\alpha| \geq |\beta|\) and for any \( n \) the equality \( \beta^n = 0 \) implies \( \alpha^n = 0 \). Then

\[
\alpha \ln \alpha \geq \alpha \ln \beta.
\]

This inequality is well-known for vectors with strictly positive coordinates, but it is easy to extend it to our case as well. Applying (20) to the vectors \( \alpha = \hat{\lambda}_t \) and \( \beta = \pi_t \), from formula (19) we find that

\[
g_t \geq \hat{\lambda}_t(\ln \lambda_t - \ln \hat{\lambda}_t)|a_t + b_t| = -U_t|a_t + b_t|,
\]

where the equality holds up to an evanescent set according to the condition (a) of the theorem. Then, by the condition (b), \( g \cdot G_t \geq -U \cdot H_t > -\infty \) a.s., which proves the representation (18). In particular, \(|f| \ast \nu_t < \infty \) a.s. for any \( t \). Since a predictable non-decreasing finite-valued process is locally integrable \([12, \text{Lemma 1.6.1}]\), the process \(|f| \ast \nu_t\) is locally integrable, and, hence, \( f \ast (\mu - \nu)_t\) is a local martingale. Therefore, \( Y_t = Z_t + U \cdot H_t \) is a local submartingale. Following a standard technique, let us show that this fact together with the condition (c) imply that \( Z_t \) has an a.s. finite limit as \( t \to \infty \).

Consider the sequence of Markov times

\[
\tau_i = \inf \{t \geq 0 : U \cdot H_t \geq i\}, \quad i \in \mathbb{N},
\]

17
where \( \inf \emptyset = +\infty \). By the condition (b), for a.a. \( \omega \) we have \( \tau_i(\omega) = \infty \) starting from some \( i \). For each \( i \), the process \( Y^{(i)}_t := Y_{t \wedge \tau_i} I(\tau_i^m \geq i^{-1}), \ t \geq 0, \) is a local submartingale and, moreover, for all \( t \geq 0 \)

\[
Y^{(i)}_t \leq (-\ln(\tau_i^m) + U \cdot H_{\tau_i})I(\tau_i^m \geq i^{-1}) \leq \ln i + i + U \tau_i \Delta H_{\tau_i} I(\tau_i < \infty),
\]

where in the first inequality we use that \( Z_t = \ln(\tau_i^m / \tau_i^0) \leq -\ln(\tau_i^0) \). From the condition (c), it follows that the random variable in the right-hand side of the above inequality is integrable. Consequently, \( Y^{(i)}_t \) is a usual submartingale and there exists the finite limit \( \lim_{t \to \infty} Y^{(i)}_t = Y_{\tau_i} I(\tau_i^m \geq i^{-1}) \) a.s. (by Doob’s martingale convergence theorem, see, e.g., Theorem I.1.39 in [2]). Letting \( i \to \infty \) we obtain the existence of the a.s. finite limit \( Y_\infty = \lim_{t \to \infty} Y_t \) and \( Z_\infty = Y_\infty - U \cdot H_\infty \). This implies \( \lim_{t \to \infty} r_t^m = r_0^m \exp(\Delta Z_\infty) > 0 \) a.s., which proves the statement (A).

In order to prove (B), we’ll need the following strengthening of Gibbs’ inequality (20). Suppose vectors \( \alpha, \beta \in \mathbb{R}_+^N \) are such that \( |\alpha| = |\beta| = 1 \) and for each \( n \) the equality \( \beta^n = 0 \) implies \( \alpha^n = 0 \). Then the following inequality is true:

\[
|\alpha - \beta|^2 \leq 4N\alpha(\overline{\alpha} - \overline{\beta}).
\]  

(22)

For vectors with strictly positive coordinates it follows from the inequality for the Kullback–Leibler and Hellinger–Kakutani distances (a direct proof can be found, for example, in [3], Lemma 2): it is sufficient to consider \( \alpha, \beta \) as probability distributions on a set of \( N \) elements. For vectors which may have null coordinates, instead of \( \alpha, \beta \) one should take \( c\alpha + (1-c)u, c\beta + (1-c)u \), where \( c \in (0,1) \) and \( u \) is a vector with strictly positive coordinates and \( |u| = 1 \). Then let \( c \to 1 \) and use the continuity of the function \( x \overline{\alpha} x \) to obtain (22).

Inequalities (19) and (22) imply

\[
|\hat{\lambda} - \pi|^2 \cdot H_\infty \leq 4N(\overline{\hat{\lambda}} - \overline{\pi}) \cdot H_\infty \leq 4N(g \cdot G_\infty + U \cdot H_\infty).
\]

It remains to show that \( g \cdot G_\infty + U \cdot H_\infty < \infty \) a.s. Consider the Markov times \( \tau_i \) defined in (21). Then for any \( t \)

\[
E((g \cdot G_{t \wedge \tau_i} + U \cdot H_{t \wedge \tau_i})I(r_0^m \geq i^{-1})) = EY_{t}^{(i)}(i) \leq \ln i + i + E(U_{\tau_i} \Delta H_{\tau_i} I(\tau_i < \infty)),
\]

where the equality holds because \( (g \cdot G_{t \wedge \tau_i} + U \cdot H_{t \wedge \tau_i})I(r_0^m \geq i^{-1}) \) is the compensator of the submartingale \( Y^{(i)}_t \). Passing to the limit \( t \to \infty \), by the monotone convergence theorem \( E((g \cdot G_{\tau_i} + U \cdot H_{\tau_i})I(r_0^m \geq i^{-1})) < \infty \), and hence \( (g \cdot G_{\tau_i} + U \cdot H_{\tau_i})I(r_0^m \geq i^{-1}) < \infty \) a.s. Passing to the limit \( i \to \infty \), we see that \( g \cdot G_\infty + U \cdot H_\infty < \infty \) a.s., which proves (B).

**Proof of Proposition 2.** It is clear that if the conditions of the proposition are satisfied then the strategy \( \lambda \) satisfies the condition (a). Denote
\( \hat{\delta} = \inf_{t,n} \hat{\lambda}_t^n \) and \( \delta = \inf_{t,n} \lambda_t^n \). Then we have the inequalities
\[
(\lambda_t^n - \hat{\lambda}_t^n)(\ln \lambda_t^n - \ln \hat{\lambda}_t^n) \leq (\lambda_t^n - \hat{\lambda}_t^n)^2 / \hat{\lambda}_t^n \quad \text{if} \quad \lambda_t^n \geq \hat{\lambda}_t^n,
\]
\[
(\lambda_t^n - \hat{\lambda}_t^n)(\ln \lambda_t^n - \ln \hat{\lambda}_t^n) \leq \ln(\delta)(\lambda_t^n - \hat{\lambda}_t^n)^2 / ((\delta - 1)\hat{\lambda}_t^n) \quad \text{if} \quad \lambda_t^n < \hat{\lambda}_t^n,
\]
where we use the inequalities \( \ln(1 + x) \leq x \) if \( x \geq 0 \) and \( \ln(1 + x) \geq x^{-1}\ln(1 + \varepsilon) \) if \( x \in [\varepsilon, 0] \), \( \varepsilon > -1 \), applied to \( x = (\lambda_t^n - \hat{\lambda}_t^n) / \hat{\lambda}_t^n \) and \( \varepsilon = \delta - 1 \). By Gibbs' inequality, \( U_t \leq \ln(\delta)|\lambda_t - \hat{\lambda}_t|^2 / ((\delta - 1)\hat{\lambda}_t^n) \). Consequently, \( U \cdot H_\infty < \infty \) a.s., so \( \lambda \) satisfies the condition (b).

**Proof of Corollary 1** Consider the model where investor 1 uses the strategy \( \lambda^1 = \lambda \), and all the other investors use the strategy \( \tilde{\lambda} \), i.e. \( \lambda^m = \tilde{\lambda} \), \( m = 2, \ldots, M \). In this case \( \pi_t = \lambda_t r^1_t + \lambda_t (1 - r^1_t) \) and \( |\tilde{\lambda} - \pi_t| = r^1_t |\tilde{\lambda} - \lambda_t| \). Then from the statement (B) of Theorem 4 we obtain \( (r^1_t |\tilde{\lambda} - \lambda|^2) \cdot H_\infty < \infty \) a.s. Since the strategy \( \lambda \) is survival, \( \inf_t r^1_t > 0 \) a.s. Therefore, \( |\lambda - \lambda|^2 \cdot H_\infty < \infty \) a.s.

**Proof of Corollary 2** From the statement (B) of Theorem 4 we obtain \( |\lambda - \lambda + (1 - r^1_t)(\lambda - \tilde{\lambda})|^2 \cdot H_\infty = |\tilde{\lambda} - \lambda|^2 \cdot H_\infty < \infty \) a.s. Since the strategy \( \lambda \) is survival, \( |\tilde{\lambda} - \lambda|^2 \cdot H_\infty < \infty \) a.s. by Corollary 1. From these inequalities, it follows that \( (1 - r^1_t)^2|\lambda - \tilde{\lambda}|^2 \cdot H_\infty < \infty \) a.s. According to the statement (A) of Theorem 4 there exists the finite limit \( r^1_\infty = \lim_{t \to \infty} r^1_t \) a.s. Then necessarily \( r^1_\infty = 1 \) a.s. on the set \( \{ |\lambda - \tilde{\lambda}|^2 \cdot H_\infty = \infty \} \), which a.s. coincides with the set \( \{ |\lambda - \tilde{\lambda}|^2 \cdot H_\infty = \infty \} \) as follows from Corollary 1.

### 5 Growth optimality of survival strategies

A large body of literature in financial mathematics is devoted to **optimal growth strategies** in market models with exogenous prices and without competition (they are also called **benchmark portfolios** or **numeraires**). This theory concerns the problem of finding a strategy that grows faster than any other strategy on the infinite time horizon; there are several (almost) equivalent definitions of what **faster** means, see the subsequent references. First results in this direction appeared in the 1950-60s (beginning with the papers [4, 11]); the modern review of the theory in discrete time can be found in [7] and [5, Ch. 16]. A complete result for continuous time was obtained in the paper [1], where, in particular, a necessary and sufficient condition for the existence of a numeraire in terms of the no-arbitrage property of a market was proved. Various applications of optimal growth strategies in the context of stock and derivatives markets are described in the book [13].

It is well-known that optimal growth strategies have a number of optimality properties for the rate of capital growth. In this section we’ll show
that in our model with endogenous prices and competition, survival strategies and, in particular, the strategy $\hat{\lambda}$, have similar properties if they are compared with strategies of competitors.

Our results will exhibit a certain similarity between optimal growth and survival strategies. However, there is an essential distinction between them: an optimal growth strategy is obtained as a solution of some optimization problem for the capital process, while a survival strategy cannot be obtained in such a way (because competitors’ strategies are unknown to an investor). In particular, optimal growth strategies turn out to be, in some sense, both asymptotically optimal and locally optimal. For example, they maximize the asymptotic capital growth rate and minimize the expected time to reach a large capital level (asymptotic properties), and also solve the logarithmic utility maximization problem and have the highest local rate of capital growth (local properties); the details can be found in [9, 10]. On the other hand, survival strategies in our model have similar properties of asymptotic optimality, but the analogous local properties hold only if there are two investors in a market (or the strategy of one investor is compared to the representative strategy of the competitors).

### 5.1 Maximization of the asymptotic capital growth rate

Let $X_t$ denote the capital process of some strategy. The asymptotic capital growth rate is the quantity $\limsup_{t \to \infty} t^{-1} \ln X_t$ (which may be infinite).

If investor 1 follows a survival strategy, then for any strategies of the other investors the inequality $\inf_t r_1 > 0$ holds with probability one. Then $\sup_t W_t/X_t < \infty$ and therefore $\sup_t X_t^m / X_t < \infty$ for any $m$. Hence for any process $T_t$ such that $\lim_{t \to \infty} T_t = +\infty$ a.s. (for example, simply $T_t = t$) we have the inequality

$$\limsup_{t \to \infty} \frac{1}{T_t} \ln \frac{X_t^m}{X_t} \leq 0 \text{ a.s.}$$

It implies that any survival strategy with probability 1 maximizes the asymptotic capital growth rate: $\limsup_t t^{-1} \ln X_t \geq \limsup_t t^{-1} \ln X_t^m$ for any $m$.

### 5.2 Maximization of the logarithmic utility

Suppose $M = 2$ (there are only two investors in a market, or the strategy of one investor is compared to the representative strategy of the competitors), investor 1 follows the strategy $\hat{\lambda}$ defined in Section 2.3, and investor 2 follows an arbitrary strategy $\lambda$, the starting capitals are equal $x_0^1 = x_0^2 = 1$. Let $\tau$ be a stopping time such that

$$E \sup_{s \leq \tau} |\ln W_s| < \infty. \quad (23)$$

We will now show that

$$E \ln X_\tau^1 \geq E \ln X_\tau^2, \quad (24)$$
i.e. the strategy \( \hat{\lambda} \) leads to a not smaller value of the expected logarithmic utility compared to any competing strategy.

From the proof of Theorem 1 it follows that \( Z_t = \ln(r_t^1/r_0^1) \) is a submartingale. Indeed, we have shown that \( Y_t = Z_t + U \cdot H_t \) is a local submartingale, but now \( U = 0 \), so the claim follows from that a local submartingale bounded from above \((Z_t \leq \ln 2)\) is a submartingale.

Then by Jensen’s inequality \( r_t^1 \) is also a submartingale, and hence \( r_t^2 = 1 - r_t^1 \) is a supermartingale, and, again by Jensen’s inequality, \( \ln(r_t^2 + \varepsilon) \) is a supermartingale for any \( \varepsilon > 0 \). This implies that \( \ln(X_t^3 + \varepsilon W_t) - \ln X_t^1 \) is a supermartingale as well. Applying Doob’s stopping theorem to the bounded stopping time \( \tau \wedge t \) with arbitrary \( t \geq 0 \), we obtain the inequality 
\[
E \ln(X_{\tau \wedge t}^3 + \varepsilon W_{\tau \wedge t}) \leq E \ln X_{\tau \wedge t}^1 + \ln(1 + 2\varepsilon)
\]
(here we need the condition (23) to ensure that the expectations exist). Passing to the limit \( t \to \infty \) by Fatou’s lemma, and using that \( \varepsilon \) can be arbitrarily small, we get (24).

Note that for \( M > 2 \) a similar result is no longer true. It can happen that investor 1 uses the strategy \( \hat{\lambda} \), investor 2 acts in an unoptimal way, and investor 3 manages to find a strategy which deprives investor 2 of a larger fraction of profit than the strategy \( \lambda^2 \) to ensure that the expectations exist). Passing to the limit \( t \to \infty \) by Fatou’s lemma, and using that \( \varepsilon \) can be arbitrarily small, we get (24).

5.3 Maximization of the local capital growth rate

By the local growth rate of investor \( m \)’s capital we call the drift coefficient of the process \( \tilde{Z}_t^m = \ln X_t^m - \ln x_0^m \) (the initial value is subtracted for convenience in the subsequent formulas). In order to have it well-defined, let us assume from now on that the process \( \ln W_t \) is locally integrable, i.e. there exists a sequence of stopping times \( \tau_i \to \infty \) a.s. for which \( E \ln W_{\tau_i} < \infty \) (for example, this holds if \( \ln W \) has uniformly bounded jumps). Let \( D_t^x = D_t - y \ast \mu_t \) denote the continuous part of the process \( D_t \). The logarithm of the total capital can be represented in the form (see (16) and (17))
\[
\ln W_t = \ln W_0 + |B_t| - D_t^x + \ln(1 - y + |x|) \ast \mu_t.
\]
(25)

This implies that the process \( \ln(1 - y + |x|) \ast \mu_t \) is locally integrable, and therefore \( \ln(1 - y + |x|) \ast \nu_t < \infty \) a.s. for all \( t \). Then we have (omitting \( m \) for brevity)
\[
\tilde{Z}_t = L \cdot B_t - D_t^x + \tilde{f} \ast \mu_t = \tilde{g} \cdot G_t - D_t^x + \tilde{f} \ast (\mu - \nu)_t
\]
(26)
with the $N$-dimensional process $L^m_t = \lambda^m_t / \pi^m_t$ (in the same way as in the proof of Theorem 1), and the functions
\[
\tilde{f}_t(x, y) = \ln(1 - y + L_t x), \quad \tilde{g}_t = L_t b_t + \int_{\mathbb{R}^{N+1}_+} \tilde{f}_t(x, y) K_t(dx, dy).
\]
In the second equality in (26), we used the inequality $|f| * \nu_t < \infty$ a.s. for all $t$. It can be proved similarly to formula (18), using $\ln(1 - y + |x|) * \nu_t < \infty$.

In the representation (26), the process $f * (\mu - \nu) t$ is a local martingale, while $\tilde{g} \cdot G_t$, $D_t^\nu$ are predictable processes. Therefore, it is natural to call $\tilde{g} \cdot G_t - D_t^\nu$ the drift process of $Z^m_t$. Assume additionally that $D_t^\nu$ is absolutely continuous with respect to $G_t$, i.e. $D_t^\nu = \delta \cdot G_t$ with some non-negative process $\delta_t$ (this can be always achieved by passing to the process $\tilde{G} = G_t + D_t^\nu$). Then we call $\theta^m_t = \tilde{g}_t - \delta_t$ the drift coefficient with respect to $G_t$ or the (local) capital growth rate of investor $m$.

For simplicity, let’s assume that the initial capitals $x^m_0 > 0$ of all the investors are non-random, and $G$ is a strictly increasing process, though all the following arguments can be easily carried over to the general case. Let $\theta^m_t = |b_t| - \delta_t + \int_{\mathbb{R}^{N+1}_+} \ln(1 - y + |x|) K_t(dx, dy)$ denote the local growth rate of the total capital $W$. We’re going to show that if $M = 2$ and investor 1 uses the strategy $\hat{\lambda}$, then a.s. for all $t$
\[
\theta^1_t \geq \theta^*_t \geq \theta^2_t.
\]
Indeed, this inequality is equivalent to $g^1_t \geq 0 \geq g^2_t$, where $g^m_t$ is the drift coefficient of the process $Z^m_t = \ln(r^m_t / r^m_0)$, which is defined as in the proof of Theorem 1 i.e. $g^m_t = (L^m_t - 1) b_t + \int_{\mathbb{R}^{N+1}_+} f^m_t(x) K_t(dx, dy)$ with the function $f^m_t(x) = \ln((1 - y + L_t x)/(1 - y + |x|))$. As it was noted above, if investor 1 uses the strategy $\hat{\lambda}$, then the process $\ln r^1_t$ is a submartingale and $g^1_t \geq 0$. Moreover, $\ln r^2_t$ is a local supermartingale and, hence, $g^2_t \leq 0$ (here we use that $G_t$ is strictly increasing).

5.4 Minimization of the mean time to reach a given capital level when $R$ and $D$ are Lévy processes

Recall that a Lévy process is an adapted càdlàg process with stationary independent increments. Let us assume that the process of cumulative relative payments $R$ and the discounting process $D$ are non-decreasing Lévy process (subordinators), and the jumps of $D$ are bounded, $\Delta D_t \leq 1 - \varepsilon D$, as above. In this case, the continuous parts of these processes are non-random, $B_t = bt$, $D^\nu_t = \delta t$, where $b \in \mathbb{R}_+^N$, $\delta \in \mathbb{R}_+$, and the compensator of the measure of jumps is non-random as well and admits the representation $\nu(dt, dx, dy) = K(dx, dy)dt$, where $K$ is a measure on $\mathbb{R}_{+}^{N+1}$ with the properties $\int_{\mathbb{R}_{+}^{N+1}} (|x| \wedge 1 + y) K(dx, dy) < \infty$ and $K(\{0\}) = 0$. We will use $G_t = t$ as a process of operational time.
For simplicity, let us make the additional assumption that the jumps \( \Delta |R_t| \) are bounded, which results in that for some constant \( c \)

\[
K(\{(x,y) : |x| \geq c\}) = 0. \tag{29}
\]

Then the processes \( R_t, D_t \) admit the representations \( R_t = bt + x (\mu_t - \nu_t), D_t = \delta t + y (\mu_t - \nu_t) \) (cf. \((9))\). Also, in order to avoid uninteresting complications, let us assume that each component of \( R_t \) is not identically constant, which is equivalent to the inequality

\[
b^n + K(\{(x,y) : x^n > 0\}) > 0 \text{ for all } n. \tag{30}
\]

Finally, since we are going to investigate the expected time when the capital of an investor reaches some (large) level \( l \), it is natural to require that at least the total capital \( W_t \) can reach any level with probability 1. For this, it is enough to assume

\[
\theta^* = |b| - \delta + \int_{\mathbb{R}^{N+1}_+} \ln(1 - y + |x|)K(dx,dy) > 0, \tag{31}
\]

since from the Law of Large Numbers and \((25)\) we have \((\ln W_t)/t \to \theta^*\) a.s. In the rest of this section, we assume that the conditions \((29) - (31)\) are satisfied.

Consider the model with \( M \) investors who have non-random starting capitals \( x_{m0} > 0 \). For an arbitrary constant \( l > 0 \) define the first moment of time when the capital of investor \( m \) exceeds \( l \):

\[
\tau_{l}^{m} := \tau_l(X^m) = \inf\{t \geq 0 : X_t^m \geq l\},
\]

where, as usual, \( \inf \emptyset = +\infty \). We are going to show that if investor 1 follows the strategy \( \hat{\lambda} \), then for any constant \( \gamma > 0 \)

\[
\limsup_{l \to \infty} \frac{E \tau_{l1}^{1}}{E \tau_{l}^{m}} \leq 1, \tag{32}
\]

i.e. the mean time required to reach a large capital level will be asymptotically minimal for an investor who follows \( \hat{\lambda} \) (one is typically interested in the case \( \gamma = 1 \)). Note that the strategy \( \hat{\lambda} \) will be non-random and not depending on \( t \): \( \hat{\lambda} = (a+b)/(a+b) \), where \( a = \int_{\mathbb{R}^{N+1}_+} x/(1 - y + |x|)K(dx,dy) \).

Also, the condition \((30)\) implies that \( \hat{\lambda}^n > 0 \) for any \( n \).

Clearly, it is sufficient to prove this statement for \( M = 2 \), as the capital of any investor \( m \neq 1 \) doesn’t exceed the capital of the representative strategy of all the investors except the first one. First let us show that the proof of \((32)\) can be reduced to the case when there exists a constant \( \varepsilon > 0 \) such that \( \lambda^{2,n}_t \geq \varepsilon \) for all \( t, n \). Indeed, consider the model with the same processes \( R \) and \( D \), starting capitals \( \tilde{x}_0^1 = x_0^1/2, \tilde{x}_0^2 = x_0^2 + x_0^1/2 \), and suppose
that investor 1 follows the strategy $\tilde{\lambda}$, while investor 2 follows the strategy $\tilde{\lambda}_2^1 = \alpha\tilde{\lambda}_2 + (1 - \alpha)\tilde{\lambda}$, where $\alpha = x_0^1/(x_0^1 + x_0^2/2)$. Then $\tilde{\lambda}_2^1 \geq \varepsilon > 0$ with $\varepsilon = (1 - \alpha)\min_n \tilde{\lambda}_n$. The capital processes satisfy the relations $\tilde{X}_t^1 = X_t^1/2$, $\tilde{X}_t^2 = X_t^2 + X_t^1/2$. Therefore, $\tau_{\tilde{\lambda}}(X) = \tau_{\tilde{\lambda}/2}(X^1)$, $\tau(X^1) \geq \tau(X^2)$. Thus, if we prove (32) for the processes $\tilde{X}_1$, $\tilde{X}_2$ and the constant $\tilde{\gamma} = \gamma/2$, then the same statement will be true for $X^1$, $X^2$, $\gamma$.

So, let us prove (32) assuming that $\tilde{\lambda}_2^1 \geq \varepsilon$. For the logarithm of the capital of each investor $Z_t^m = \ln(X_t^m/x_0^m)$ we have the representation (26) with $G_t = t$. It is not difficult to see that $L_t^{n,m} := \lambda_t^{n,m}/\pi_t^n$, $m = 1, 2$, satisfies the inequality $L_t^{m,n} \leq \beta := 1/(\varepsilon \min_n \tilde{\lambda}_n)$. Then the condition (29) implies that the process $\tilde{f}_m = (\mu - \nu)_t$ is integrable and, hence, a martingale (here, $\tilde{f}_m = \ln(1 - y + L_t^m x)$, as in (24)).

Suppose for the second investor $E\tau_1^2 < \infty$, so $\tau_1^2 < \infty$ a.s. Then we have
\[
\ln((1/x_0^2)) \leq E\tilde{Z}_1^2, \leq E(\theta^2 \cdot G_{\tilde{\lambda}_2}), \leq \theta^* E\tau_1^2,
\]
where in the second inequality we use Doob’s stopping theorem, and in the last one we use (23). The possibility of application of Doob’s theorem can be justified as follows. First apply it to the bounded stopping time $\tau_1^2 \wedge t$ with arbitrary $t \geq 0$, which yields $E\tilde{Z}_1^2, t \leq \tau_1^2 < \infty \wedge t, m = 1, 2$, satisfies the condition (31) and that the strategy $\tilde{\lambda}$ is survival.

Consider now the capital of the first investor. Observe that $\tau_1^2 < \infty$, which follows from the condition (31) and that the strategy $\tilde{\lambda}$ is survival.

From the inequality $L_t^{1,n} \leq \beta$ and the condition (29), it follows that the jumps of the process $\tilde{Z}_1^1$ can be bounded from above by the constant $\beta c$ (because the function $\tilde{f}_1^m$ can be bounded by $L_t^m x$). Then we have
\[
\ln(\gamma l/x_0^1) \geq E(\tilde{Z}_1^1, \tau_1^1) - \beta c \geq \theta^* E\tau_1^1 - \beta c,
\]
where the second inequality can be obtained similarly to (33), but now using that $\tilde{Z}_1^1, \tau_1^1 < \infty \liminf \tau_1^1 < \infty$.

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