On weak (measure valued)–strong uniqueness for Navier–Stokes–Fourier system with Dirichlet boundary condition

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Abstract. In this article, our goal is to define a measure valued solution of compressible Navier–Stokes–Fourier system for a heat conducting fluid with Dirichlet boundary condition for temperature in a bounded domain. The definition will be based on the weak formulation of entropy inequality and ballistic energy inequality. Moreover, we obtain the weak (measure valued)–strong uniqueness property of this solution with the help of relative energy.

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1. Introduction

We consider the Navier–Stokes–Fourier system in the time-space cylinder $Q_T = (0, T) \times \Omega$, where $T > 0$ and $\Omega$ is a bounded domain in $\mathbb{R}^d$ with $d = 2$ or 3. The time evolution of the density $\varrho = \varrho(t, x)$, velocity $u = u(t, x)$ and the absolute temperature $\vartheta = \vartheta(t, x)$ of a compressible, viscous and heat conducting fluid is given by the following system of field equations describing conservation of mass, momentum and internal energy:

\begin{align}
\frac{\partial}{\partial t} \varrho + \text{div}_x(\varrho u) &= 0, \\
\frac{\partial}{\partial t}(\varrho u) + \text{div}_x(\varrho u \otimes u) + \nabla_x p(\varrho, \vartheta) &= \text{div}_x S + \varrho g, \\
\frac{\partial}{\partial t}(\varrho e(\varrho, \vartheta)) + \text{div}_x(\varrho e(\varrho, \vartheta)u) + \text{div}_x q &= S: D_x u - p(\varrho, \vartheta) \text{div}_x u,
\end{align}

where $D_x u = \frac{1}{2}(\nabla_x u + \nabla_x^t u)$, $p = p(\varrho, \vartheta)$ is the pressure and $e = e(\varrho, \vartheta)$ is the internal energy and $g$ is an external force. The pressure and the internal energy are interrelated by means of Gibbs’ equation.
\[
\vartheta Ds = De + pD \left( \frac{1}{\varrho} \right),
\]
where \( s = s(\varrho, \vartheta) \) is the entropy and \( D = \left( \frac{\partial}{\partial \varrho} \right) \). We assume the fluid is Newtonian, hence the viscous stress tensor \( S \) is given by
\[
S(\nabla_x u) = \mu(\varrho, \vartheta) \left( \frac{\nabla_x u + \nabla_T u}{2} - \frac{1}{d} (\text{div}_x u \mathbb{I}) \right) + \lambda(\varrho, \vartheta) (\text{div}_x u \mathbb{I}),
\]
where \( \mu \) is the shear viscosity coefficient and \( \lambda \) is the bulk viscosity coefficient with \( \mu > 0 \) and \( \lambda \geq 0 \). The heat flux \( q \) is given by the Fourier law,
\[
q(\varrho, \vartheta, \nabla_x \vartheta) = -\kappa(\varrho, \vartheta) \nabla_x \vartheta,
\]
where \( \kappa \) is the heat conductivity coefficient. Since we consider the bounded domain \( \Omega \), our goal is to discuss the solvability of the initial-boundary value problem for the system (1.1)–(1.3) endowed with the constitutive relations (1.4), (1.5) and (1.6) where we consider homogeneous Dirichlet boundary condition for the velocity and inhomogeneous Dirichlet boundary condition in the temperature
\[
\begin{align*}
\mathbf{u} |_{\partial \Omega} &= 0, \\
\vartheta |_{\partial \Omega} &= \vartheta_B > 0,
\end{align*}
\]
where \( \vartheta_B \in C^2([0, T] \times \partial \Omega) \).

The consideration of (1.8) is closely related to the celebrated Rayleigh-Bénard problem, which, in general, describes the behavior of a fluid confined between two horizontal, parallel plates with a temperature profile given on the horizontal boundaries. In other words, if we consider \( \Omega = \mathbb{T}^{d-1} \times (0, 1) \) which is a \((d-1)\)-dimensional flat torus, the condition (1.8) can be interpreted as
\[
\vartheta |_{x_d=0} = \vartheta_{\text{bottom}} \quad \text{and} \quad \vartheta |_{x_d=1} = \vartheta_{\text{top}} \quad \text{with} \quad \vartheta_{\text{bottom}} > \vartheta_{\text{top}} > 0.
\]
The Rayleigh-Bénard problem is significant due to its investigation of fluid convection and pattern formation, offering fundamental insights into phase transitions, chaotic behavior, and pattern emergence, see Davidson [13].

For the above system (1.1)–(1.3) with the constitutive relations (1.4)–(1.8) and the pressure following Boyle’s law, the existence of a local in time strong solution was proved by Valli and Zajaczkowski [33]. For the global in time weak solutions there are many articles and monographs by Lions [28], Bresch and Desardirin [4], Feireisl and Novotný [22], Bresch and Jabin [5] which focus mainly on space-periodic domains or energy-conserving boundary conditions for the temperature, i.e., either \( \Omega = \mathbb{T}^d \) or
\[
q \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial \Omega.
\]
Also for thermodynamically open systems with in/outflow boundary conditions, Feireisl and Novotný [24] prove the existence of weak solutions. It also requires the control of the internal (heat) energy flux \( q \) in \( \partial \Omega \).

Recently, the concept of weak solution was introduced for the inhomogeneous Dirichlet boundary condition for temperature in [11]. Although this
analysis considers a more general case of inflow-outflow boundary data, the assumption (1.7)–(1.8) can be viewed as a particular instance of it. This weak solution approach is slightly different from the earlier considerations that we will discuss in Sect. 2. This concept of weak solution is adapted by Pokorný [32] for steady compressible viscous flows.

The concept of measure valued solutions in the context of inviscid and incompressible fluids has been extensively explored by several authors, starting with the seminal work of DiPerna [14] and DiPerna and Majda [15]. Related results have also been contributed by Kröner and Zajaczkowski [26], Nečasová and Novotný [30], and Neustupa [31]. Recently, Březina et al. [8] introduced measure-valued solutions for compressible viscous heat conducting fluids, considering the no-flux boundary condition (1.9) for temperature. Their definition of measure-valued solution aligns with the approach formulated by Brenier et al. [3] by describing the measure valued solution as a general object, while an alternative approach is proposed by Málek et al. [29], identifying measure-valued solutions as weak limits of weak solutions. In this study, we adopt the approach presented in [8].

For a generalized (weak or measure valued) solution, it is quite important to prove the weak (measure valued)–strong uniqueness principle. The principle asserts that given the same initial data a weak solution will coincide with the strong or classical solution if the latter exists. Considering the boundary condition (1.9), Feireisl and Novotný [23] (see also Feireisl [16]) have shown that the weak-strong uniqueness principle holds. A similar result for measure valued solution is achieved by in Březina et al. [8]. The main idea here is to utilize the relative energy inequality, which was first introduced by Dafermos [12] for scalar conservation law and later adapted for the compressible fluid by Feireisl and Novotný [22].

An important application of measure valued solutions is their identification as limits of numerical schemes. Interesting results in numerical analysis have been obtained by Fjordholm et al. [25], Feireisl and Lukáčová-Medvid’ová [18] and Feireisl et al. [20]. Together with the existing weak (measure valued)–strong uniqueness principle in the class of measure valued solutions, one can show that numerical solutions converge strongly to a strong solution of the system as long as the latter exists. In the context of compressible Navier–Stokes–Fourier system with no-flux boundary condition see Feireisl et al. [21] and a detailed discussion is available in the monograph by Feireisl et al. [19]. The result obtained in this paper are expected to have an analogous applications in the context of physically relevant models (Rayleigh–Bénard model) concerning compressible, viscous, and heat-conducting fluids with inhomogeneous boundary condition for temperature.

The plan for the article is as follows:

- At first in Sect. 2, we devote ourselves to the weak formulation of the problem along with a proper definition of the measure valued solutions.
- In Sect. 3, we derive the relative energy inequality for the measure valued solutions.
We present our main results in Sect. 4. We conclude the weak (measure valued)—strong uniqueness property of the system. Our first two theorems, Theorems 4.1 and 4.2, requires some additional hypothesis on measure valued solutions, while Theorem 4.3 is does not need any of such extra assumption on solutions, although we impose some physically relevant structural assumptions on transport coefficients $\mu, \lambda$ and $\kappa$.

Finally, in Sect. 5, we briefly discuss the limitation of our results, some comments on existence of solution and validation of the definition.

2. Measure valued solution

2.1. Weak formulation: revisited

We assume that $p, e \in C^2((0, \infty) \times (0, \infty)) \cap C([0, \infty) \times [0, \infty))$ and they satisfy Gibbs’ relation (1.4). As a direct consequence of (1.2) and (1.3), we have the total energy equation

$$\partial_t \left[ \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right] + \text{div}_x \left( \left[ \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e + p(\varrho, \vartheta) \right] \mathbf{u} \right) + \text{div}_x \mathbf{q} = \text{div}_x (S \mathbf{u}) + \varrho \mathbf{u} \cdot \mathbf{g}. \quad (2.1)$$

Also, using the Gibb’s relation (1.4), we deduce that internal energy equation (1.3) is equivalent to the entropy equation which is given by

$$\partial_t (\varrho s(\varrho, \vartheta)) + \text{div}_x (\varrho s(\varrho, \vartheta) \mathbf{u}) + \text{div}_x \left( \frac{\mathbf{q}}{\vartheta} \right) = \frac{1}{\vartheta} \left( S : \mathbf{D}_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right). \quad (2.2)$$

No flux boundary condition for temperature At first we quickly recall the weak formulation described in Feireisl and Novotný [22, Chapter 2] which deals with a no-flux boundary condition for temperature (1.9). The weak formulation of the problem depends on the weak formulation of continuity equation, momentum equation and replaces the internal energy equation (1.3) by the entropy inequality

$$\partial_t (\varrho s(\varrho, \vartheta)) + \text{div}_x (\varrho s(\varrho, \vartheta) \mathbf{u}) + \text{div}_x \left( \frac{\mathbf{q}(\varrho, \vartheta, \nabla_x \vartheta)}{\vartheta} \right) \geq \frac{1}{\vartheta} \left( S(\varrho, \vartheta, \mathbf{D}_x \mathbf{u}) : \mathbf{D}_x \mathbf{u} - \frac{\mathbf{q}(\varrho, \vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right),$$

along with the weak form of total energy balance:

$$\frac{d}{dt} \int_{\Omega} \left[ \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right] dx = \int_{\Omega} \varrho \mathbf{u} \cdot \mathbf{g} dx. \quad (2.3)$$

The last equality (2.3) follows from integrating the total energy equation over space and using the boundary condition (1.9), which gives, in particular,

$$\int_{\partial \Omega} \mathbf{q} \cdot \mathbf{n} \, d\sigma_x = 0,$$

where $\sigma_x$ is the surface measure on the boundary $\partial \Omega$. 
Inhomogeneous Dirichlet boundary condition temperature

Instead of boundary condition (1.9), if we consider inhomogeneous Dirichlet boundary condition for the temperature (1.8), we need to proceed as described in [11, Section 2.4]. In this case, assuming all quantities in consideration are smooth, integrating the (2.1) we obtain
\[
\frac{d}{dt} \int_{\Omega} \left[ \frac{1}{2} \varrho |u|^2 + \varrho e \right] \, dx + \int_{\partial\Omega} q \cdot n \, d\sigma_x = \int_{\Omega} \varrho u \cdot g \, dx. \tag{2.4}
\]

Due to the presence of the term \( \int_{\partial\Omega} q \cdot n \, d\sigma_x \) in the left hand side of (2.4), the total energy balance is unavailable. Next, considering a smooth function \( \tilde{\vartheta} \) such that \( \tilde{\vartheta} > 0 \) in \( (0, T) \times \Omega \) with \( \tilde{\vartheta} = \vartheta_B \) on \( \partial\Omega \), and multiplying the entropy inequality (2.3) by \( \tilde{\vartheta} \) along with using the integrating by parts, we obtain
\[
-\frac{d}{dt} \int_{\Omega} \tilde{\vartheta} \varrho s \, dx - \int_{\partial\Omega} q \cdot n \, d\sigma_x \\
\leq - \int_{\Omega} \frac{\tilde{\vartheta}}{\vartheta} \left( S : D_x u - \frac{q \cdot \nabla \tilde{\vartheta}}{\vartheta} \right) \, dx - \int_{\Omega} \varrho s \left( \partial_t \tilde{\vartheta} + u \cdot \nabla \tilde{\vartheta} \right) + \frac{q}{\vartheta} \cdot \nabla \tilde{\vartheta} \right) \, dx. \tag{2.5}
\]

In the above inequality we have the term \( \int_{\partial\Omega} q \cdot n \, d\sigma_x \) in the left hand side with exact opposite sign of (2.4). Therefore, adding (2.4) and (2.5) we obtain the ballistic energy inequality
\[
\frac{d}{dt} \int_{\Omega} \left[ \frac{1}{2} \varrho |u|^2 + \varrho e - \tilde{\vartheta} \varrho s \right] \, dx + \int_{\Omega} \frac{\tilde{\vartheta}}{\vartheta} \left( S : D_x u - \frac{q \cdot \nabla \tilde{\vartheta}}{\vartheta} \right) \, dx \\
\leq \int_{\Omega} \varrho u \cdot g \, dx - \int_{\Omega} \left[ \varrho s \left( \partial_t \tilde{\vartheta} + u \cdot \nabla \tilde{\vartheta} \right) + \frac{q}{\vartheta} \cdot \nabla \tilde{\vartheta} \right] \, dx.
\]

We recall that for a smooth function \( \tilde{\Theta} \), the ballistic energy (denoted by \( H_{\tilde{\Theta}}(\varrho, \vartheta) \)) is defined as
\[
H_{\tilde{\Theta}}(\varrho, \vartheta) = \varrho [e(\varrho, \vartheta) - \tilde{\Theta}s(\varrho, \vartheta)].
\]

Hence, we consider the weak formulation with ballistic energy inequality instead of energy balance. In this paper our goal is to provide a suitable definition of a measure valued solution based on the above discussion for boundary condition (1.8). To define it, we take motivation from Březina, Feireisl and Novotný [8] that covers the boundary condition (1.9). This definition will be given in terms of Young measures and suitable defect measures. To simplify our analysis, we will assume \( g = 0 \) in (1.2), although this can be treated in a similar manner as shown in [11].

**2.1.1. Phase space and Young measure.** We first introduce some notation:

- \( C_0(\mathbb{R}^d) \) denotes the closure under the supremum norm of compactly supported, continuous functions on \( \mathbb{R}^d \), that is the set of continuous functions on \( \mathbb{R}^d \) vanishing at infinity;
\( \mathcal{M}(\mathbb{R}^d) \) denotes the dual space of \( C_0(\mathbb{R}^d) \) consisting of signed Radon measures with finite mass, equipped with the dual norm of total variation; 
\( \mathcal{M}^+(\mathbb{R}^d) \) denotes the cone of non-negative Radon measures on \( \mathbb{R}^d \) and \( \mathcal{P}(\mathbb{R}^d) \) indicates the space of probability measures, i.e. for \( \nu \in \mathcal{P}(\mathbb{R}^d) \subset \mathcal{M}^+(\mathbb{R}^d) \) we have \( \nu[\mathbb{R}^d] = 1 \).

- For \( \mathbb{A} = (a_{ij})_{i,j=1}^d \in \mathbb{R}^{d \times d} \), we consider the symmetric part and the traceless part of \( \mathbb{A} \) as 
\[ \mathbb{D}(\mathbb{A}) = \frac{\mathbb{A} + \mathbb{A}^T}{2} \quad \text{and} \quad \mathbb{D}_0(\mathbb{A}) = \frac{\mathbb{A} + \mathbb{A}^T}{2} - \frac{1}{d} \text{Tr}(\mathbb{A}) \mathbb{I}, \]
respectively, where \( \text{Tr}(\mathbb{A}) = \sum_{i=1}^d a_{ii} \).

A natural candidate for the phase space is given by the state variables \([\varrho, \mathbf{u}, \vartheta]\). Since we are looking for a more general class of solutions and \( \nabla_x \mathbf{u} \) and \( \nabla_x \vartheta \) are present in the system (1.1)–(1.3), thus gradient of the velocity and the temperature have been included along with the natural choices. Hence a proper phase space is
\[ \mathcal{F} = \{ (\varrho, \mathbf{u}, \vartheta, \mathbb{D}_\mathbf{u}, \mathbb{D}_\vartheta) \mid \varrho \in [0, \infty), \mathbf{u} \in \mathbb{R}^d, \vartheta \in [0, \infty), \mathbb{D}_\mathbf{u} \in \mathbb{R}^{d \times d}_{\mathrm{sym}}, \mathbb{D}_\vartheta \in \mathbb{R}^d \}. \]

We consider a Young measure \( \mathcal{V} \) such that \( \mathcal{V} = \{ \mathcal{V}_{t,x} \}_{(t,x) \in (0,T) \times \Omega} \) and
\[ \mathcal{V} \in L_{\text{weak-}(*)}((0,T) \times \Omega; \mathcal{P}(\mathcal{F})). \]

**2.1.2. Initial data.** Let \( \varrho_0 \in L^\infty(\Omega; \mathcal{P}(\mathcal{F}_0)) \) such that \( \mathcal{F}_0 = \{ (\varrho_0, \mathbf{u}_0, \vartheta_0) \mid \varrho_0 \in [0, \infty), \mathbf{u}_0 \in \mathbb{R}^d, \vartheta_0 \in [0, \infty) \} \) with finite ballistic energy, i.e.,
\[ \int_\Omega \left( \mathcal{V}_{0,x} \left( \frac{1}{2} |\mathbf{u}|^2 + \varrho \mathcal{E}(\varrho, \vartheta) - \tilde{\Theta}(0, x) \mathcal{G}(\varrho, \vartheta) \right) \right) \, dx < \infty \quad \text{for every} \quad 0 < \tilde{\Theta} \in C^1([0, T] \times \bar{\Omega}) \quad \text{with} \quad \tilde{\Theta} = \vartheta_B \quad \text{on} \quad \partial \Omega. \]

**2.1.3. Compatibility relations for the Young measure.** We consider a very general phase space for the Young measure. We expect the following compatibility conditions.

- **Velocity compatibility** The identity
\[ - \int_0^T \int_\Omega \langle \mathcal{V}_{t,x}; \mathbf{u} \rangle \cdot \nabla_x \mathbb{T} \, dx \, dt = \int_0^T \int_\Omega \langle \mathcal{V}_{t,x}; \mathbb{D}_\mathbf{u} \rangle : \mathbb{T} \, dx \, dt \]
holds for any \( \mathbb{T} \in C^1(\overline{Q_T}; \mathbb{R}^{d \times d}_{\mathrm{sym}}) \).

- **Temperature compatibility** The identity
\[ - \int_0^T \int_\Omega \langle \mathcal{V}_{t,x}; \vartheta - \tilde{\Theta} \rangle \, \nabla_x \mathbf{\psi} \, dx \, dt = - \int_0^T \int_\Omega \langle \mathcal{V}_{t,x}; \mathbb{D}_\vartheta - \nabla_x \tilde{\Theta} \rangle \cdot \mathbf{\psi} \, dx \, dt \]
holds for any \( \mathbf{\psi} \in C^1(\overline{Q_T}; \mathbb{R}^d) \) and \( \tilde{\Theta} \in C^1([0, T] \times \bar{\Omega}) \) with \( \tilde{\Theta} = \vartheta_B \) on \( \partial \Omega \).
2.1.4. Field equations with defect.

- **Equation of continuity** For a.e. $\tau \in (0, T)$ and $\psi \in C^1([0, T] \times \Omega)$, the following equation holds:

$$\int_{\Omega} \langle \mathcal{V}_{\tau,x}; \varrho \rangle \psi(\tau, x) \, dx - \int_{\Omega} \langle \mathcal{V}_{0,x}; \varrho \rangle \psi(0, x) \, dx$$

$$= \int_0^\tau \int_{\Omega} \left[ \langle \mathcal{V}_{t,x}; \varrho \rangle \partial_t \psi + \langle \mathcal{V}_{t,x}; \varrho \mathbf{u} \rangle \cdot \nabla_x \psi \right] \, dx \, dt. \quad (2.11)$$

- **Momentum equation** There exists a measure $\tau^M \in L^\infty_{\mathrm{weak-*}}(0; \mathcal{M}(\bar{\Omega}; \mathbb{R}^{d \times d}))$ such that for a.e. $\tau \in (0, T)$ and every $\varphi \in C^1([0, T] \times \bar{\Omega}; \mathbb{R}^d)$, $\varphi|_{\partial \Omega} = 0$ the following equation holds:

$$\int_{\Omega} \langle \mathcal{V}_{\tau,x}; \varrho \mathbf{u} \rangle \cdot \varphi(\tau, x) \, dx - \int_{\Omega} \langle \mathcal{V}_{0,x}; \varrho \mathbf{u} \rangle \cdot \varphi(0, x) \, dx$$

$$= \int_0^\tau \int_{\Omega} \left[ \langle \mathcal{V}_{t,x}; \varrho \mathbf{u} \rangle \cdot \partial_t \varphi + \langle \mathcal{V}_{t,x}; \varrho \mathbf{u} \otimes \mathbf{u} \rangle : \nabla_x \varphi \right] \, dx \, dt$$

$$+ \langle \mathcal{V}_{t,x}; p(\varrho, \vartheta) \rangle \mathrm{div}_x \varphi \right] \, dx \, dt$$

$$- \int_0^\tau \int_{\Omega} \langle \mathcal{V}_{t,x}; \mathcal{S}(\varrho, \vartheta, \mathcal{D} \mathbf{u}) \rangle : \nabla_x \varphi \, dx \, dt$$

$$+ \int_0^\tau \langle \tau^M; \nabla_x \varphi \rangle \{ \mathcal{M}(\bar{\Omega}; \mathbb{R}^{d \times d}), C(\bar{\Omega}; \mathbb{R}^{d \times d}) \} \, dt. \quad (2.12)$$

- **Entropy inequality** For a.e. $\tau \in (0, T)$ and $0 \leq \phi \in C^1_c([0, T] \times \Omega)$, we have the following inequality:

$$\int_{\Omega} \langle \mathcal{V}_{\tau,x}; \varrho \mathcal{S}(\varrho, \vartheta) \rangle \phi(\tau, x) \, dx - \int_{\Omega} \langle \mathcal{V}_{0,x}; \varrho \mathcal{S}(\varrho, \vartheta) \rangle \phi(0, x) \, dx$$

$$\geq \int_0^\tau \int_{\Omega} \langle \mathcal{V}_{t,x}; \varrho \mathcal{S}(\varrho, \vartheta) \rangle \partial_t \phi(t, x) \, dx \, dt$$

$$+ \int_0^\tau \int_{\Omega} \left[ \langle \mathcal{V}_{t,x}; \varrho \mathcal{S}(\varrho, \vartheta) \rangle \mathbf{u} - \frac{\kappa(\varrho, \vartheta)}{\varrho} \mathcal{D} \vartheta \right] \cdot \nabla_x \phi(t, x) \, dx \, dt$$

$$+ \int_0^\tau \int_{\Omega} \left[ \frac{1}{\vartheta} \left( \mathcal{S}(\varrho, \vartheta, \mathcal{D} \mathbf{u}) : \mathcal{D} \mathbf{u} + \frac{\kappa(\varrho, \vartheta)}{\varrho} |\mathcal{D} \vartheta|^2 \right) \phi(t, x) \right] \, dx \, dt. \quad (2.13)$$

- **Ballistic energy inequality** For any $\mathbf{\Theta} \in C^1([0, T] \times \bar{\Omega})$, $\mathbf{\Theta} > 0$, $\mathbf{\Theta}|_{\partial \Omega} = \vartheta_B$, there exists a dissipation defect $\mathcal{D}_{\mathbf{\Theta}}$ such that

$$\mathcal{D}_{\mathbf{\Theta}} \in L^\infty(0, T), \mathcal{D}_{\mathbf{\Theta}} \geq 0,$$
and the following inequality holds:

\[
\int_{\Omega} \left\langle V_{t,x}; \left( \frac{1}{2} \varrho |u|^2 + \varphi \epsilon(\varrho, \vartheta) - \tilde{\Theta}(t, x) \varphi s(\varrho, \vartheta) \right) \right\rangle \, dx \\
+ \int_0^\tau \int_{\Omega} \left\langle V_{t,x}; \frac{1}{\varrho} \left( S(\varrho, \vartheta, D_u) : D_u + \frac{\kappa(\varrho, \vartheta)}{\varrho} |D\varphi| \right) \right\rangle \tilde{\Theta}(t, x) \, dx \, dt + \mathcal{D}_{\tilde{\Theta}}(\tau) \\
\leq \int_{\Omega} \left\langle V_{0,x}; \left( \frac{1}{2} \varrho |u|^2 + \varphi \epsilon(\varrho, \vartheta) - \tilde{\Theta}(0, x) \varphi s(\varrho, \vartheta) \right) \right\rangle \, dx \\
- \int_0^\tau \int_{\Omega} \left\langle V_{t,x}; \varphi s(\varrho, \vartheta) \right\rangle \partial_t \tilde{\Theta} \, dx \, dt - \int_0^\tau \int_{\Omega} \left\langle V_{t,x}; \varphi s(\varrho, \vartheta) u \right\rangle \cdot \nabla_x \tilde{\Theta} \, dx \, dt \\
+ \int_0^\tau \int_{\Omega} \left\langle V_{t,x}; \frac{\kappa(\varrho, \vartheta)}{\varrho} D\varphi \right\rangle \cdot \nabla_x \tilde{\Theta} \, dx \, dt.
\]

(2.14)

2.1.5. Compatibility of defect measures. For any \( \tilde{\Theta} \in C^1([0, T] \times \overline{\Omega}) \), \( \tilde{\Theta} > 0 \), \( \tilde{\Theta}|_{\partial\Omega} = \vartheta_B \) and \( \varphi \in C^1(\overline{Q}_T) \), we have

\[
|\langle r^M(\tau); \nabla_x \varphi \rangle_{\mathcal{M}(\overline{\Omega}; \mathbb{R}^{d \times d}), C(\overline{\Omega}; \mathbb{R}^{d \times d})}| \leq \xi(\tau) \mathcal{D}_{\tilde{\Theta}}(\tau) \| \varphi \|_{C^1(\overline{\Omega})}
\]

(2.15)

where \( \xi \in L^1(0, T) \) and \( \varphi \in C^1(\overline{\Omega}) \).

2.1.6. Generalized Korn–Poincaré inequality. The following version of Korn–Poincaré inequality is true:

\[
\int_0^\tau \int_{\Omega} \left\langle V_{t,x}; |u - \tilde{U}|^2 \right\rangle \, dx \, dt \leq C_p \int_0^\tau \int_{\Omega} \left\langle V_{t,x}; |D_0(D_u) - D_0(\nabla_x \tilde{U})|^2 \right\rangle \, dx \, dt
\]

(2.16)

for any \( \tilde{U} \in L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^d)) \).

Remark 2.1. We must note that the ballistic energy inequality (2.14) is given for a large class of functions \( \tilde{\Theta} \) and in general the dissipation defect \( \mathcal{D}_{\tilde{\Theta}} \) depends of \( \tilde{\Theta} \). On the other hand the defect measure \( r^M \) in (2.12) is independent of \( \tilde{\Theta} \). Therefore, we assume a stronger compatibility criterion (2.15). However for some physical equation state we will able to conclude that \( \mathcal{D}_{\tilde{\Theta}} \) is independent of \( \tilde{\Theta} \), we will discuss that in Sect. 5.

2.2. Definition of a measure valued solution

Here we provide the definition of the measure valued solutions.

Definition 2.2. Let \( 0 < \vartheta_B \in C^2((0, T] \times \partial\Omega) \) with and \( \tilde{\Theta} \) belongs to the class \{ \( \tilde{\Theta} \in C^1([0, T] \times \overline{\Omega}) \mid \tilde{\Theta} > 0 \), \( \tilde{\Theta}|_{\partial\Omega} = \vartheta_B \) \}. Moreover, we assume that the the initial condition \( \mathcal{V}_0 \) satisfies (2.8). Then \{\( \mathcal{V}, \mathcal{D}_{\tilde{\Theta}} \)\} is called a measure valued solution for the system (1.1)–(1.8) if it satisfies (2.11)–(2.14) with the compatibility conditions (2.9),(2.10), (2.15) and (2.16) and \( \mathcal{V}|_{t=0} = \mathcal{V}_0 \).
3. Relative energy inequality

Following the approach of Feireisl and Novotný [22, Chapter 9], we express the relative energy with the help of ballistic energy and it is given by

\[
E(\varrho, \vartheta, \mathbf{u} | \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}) = \frac{1}{2} \varrho |\mathbf{u} - \tilde{\mathbf{u}}|^2 + \left( H_{\tilde{\vartheta}}(\varrho, \vartheta) - \frac{\partial H_{\tilde{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta})}{\partial \varrho} (\varrho - \tilde{\varrho}) - H_{\tilde{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta}) \right),
\]

where \((\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}})\) are smooth functions such that \(\tilde{\varrho} > 0, \tilde{\vartheta} > 0\) with \(\tilde{\vartheta} |_{\partial \Omega} = \vartheta_B\).

As observed in [24, Section 1.2], if the relative energy is interpreted in terms of the conservative entropy variables \((\varrho, S = \varrho s, \mathbf{m} = \varrho \mathbf{u})\), it represents a Bregman divergence associated with the energy functional

\[
E(\varrho, S, \mathbf{m}) = \frac{1}{2} |\mathbf{m}|^2 + \varrho e(\varrho, S).
\]

To obtain that the relative energy functional is non-negative, we need the hypothesis of thermodynamic stability, i.e.

\[
\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0 \quad \text{and} \quad \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0.
\]

This hypothesis leads to the convexity of the internal energy \(\varrho e(\varrho, S)\) with respect to the variables \((\varrho, S)\). Additionally, from (1.4), we have.

\[
\frac{\partial (\varrho e(\varrho, S))}{\partial \varrho} = e - \varrho s + \frac{p}{\varrho} \quad \text{and} \quad \frac{\partial (\varrho e(\varrho, S))}{\partial S} = \vartheta.
\]

The time evolution of the relative energy is given by

\[
\mathcal{E}(\varrho, \vartheta, \mathbf{u} | \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}) = \int_{\Omega} E(\varrho, \vartheta, \mathbf{u} | \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}) \, dx.
\]

Given a measure valued solution \(\{\mathcal{V}_{t,x}\}_{(t,x) \in Q_T}\) of the Navier–Stokes–Fourier system, we adapt the relative energy as

\[
\mathcal{E}_{mv}(\tau) := \int_{\Omega} \langle \mathcal{V}_{\tau,x}; E(\varrho, \vartheta, \mathbf{u} | \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}) \rangle \, dx.
\]

At first, using the standard expansion, we have

\[
\mathcal{E}_{mv}(\tau) = \int_{\Omega} \langle \mathcal{V}_{\tau,x}; \frac{1}{2} \varrho |\mathbf{u}|^2 + H_{\tilde{\vartheta}}(\varrho, \vartheta) \rangle \, dx - \int_{\Omega} \langle \mathcal{V}_{\tau,x}; \varrho \mathbf{u} \rangle \cdot \tilde{\mathbf{u}} \, dx
+ \int_{\Omega} \langle \mathcal{V}_{\tau,x}; \varrho \rangle \left( \frac{1}{2} |\tilde{\mathbf{u}}|^2 - \frac{\partial H_{\tilde{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta})}{\partial \varrho} \right) \, dx
+ \int_{\Omega} p(\tilde{\varrho}, \tilde{\vartheta})(\tau, \cdot) \, dx = \Sigma_{i=1}^4 L_i.
\]
and the continuity equation (2.11), respectively. For the term \( L_4 \) we use the
identity
\[
\begin{align*}
\int_0^\tau \int_\Omega \partial_t p(\tilde{\rho}, \tilde{\vartheta}) \, dx \, dt &= \left[ \int_\Omega p(\tilde{\rho}, \tilde{\vartheta})(t, \cdot) \, dx \right]_{t=0}^{t=\tau}.
\end{align*}
\]
Thus, using the Gibbs’ relation (1.4) suitably, (3.5) yields
\[
\begin{align*}
\mathcal{E}_{mv}(\tau) + \int_0^\tau \int_\Omega \left[ \langle \mathcal{V}_{t,x}; \frac{\partial}{\partial \rho} \mathcal{S}(\rho, \vartheta, \mathbb{D} u) : \mathbb{D} u \rangle + \tilde{\vartheta} \left\langle \mathcal{V}_{t,x}; \frac{\kappa(\rho, \vartheta)}{\vartheta^2} |\mathbb{D} \vartheta|^2 \right\rangle \right] \, dx \, dt
\end{align*}
\]
\[
- \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; \mathcal{S}(\rho, \vartheta, \mathbb{D} u) \rangle : \nabla_x \tilde{u} \, dx \, dt + \mathcal{D}_{\tilde{\vartheta}}(\tau)
\]
\[
\leq \mathcal{E}_{mv}(0) - \int_0^\tau \int_\Omega \left[ \langle \mathcal{V}_{t,x}; \rho \cdot \mathcal{S}(\rho, \vartheta) \rangle \partial_t \tilde{\vartheta}
\right.
\]
\[
+ \left. \langle \mathcal{V}_{t,x}; \rho \cdot (\mathcal{S}(\rho, \vartheta) \mathcal{D} u - \kappa(\rho, \vartheta) \mathbb{D} \vartheta) \rangle : \nabla_x \tilde{u} \right] \, dx \, dt
\]
\[
- \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; p(\rho, \vartheta) \rangle \, div_x \tilde{u} \, dx \, dt
\]
\[
+ \int_0^\tau \int_\Omega \left[ \langle \mathcal{V}_{t,x}; \rho \rangle \partial_t \tilde{\vartheta} \, s(\tilde{\rho}, \tilde{\vartheta}) + \langle \mathcal{V}_{t,x}; \rho \mathcal{D} u \rangle \cdot \nabla_x \tilde{\vartheta} \, s(\tilde{\rho}, \tilde{\vartheta}) \right] \, dx \, dt
\]
\[
+ \int_0^\tau \int_\Omega \left[ \langle \mathcal{V}_{t,x}; \rho \rangle \frac{1}{\tilde{\rho}} \partial_t p(\tilde{\rho}, \tilde{\vartheta}) - \langle \mathcal{V}_{t,x}; \rho \mathcal{D} u \rangle \cdot \frac{1}{\tilde{\rho}} \nabla_x \partial_t p(\tilde{\rho}, \tilde{\vartheta}) \right] \, dx \, dt
\]
\[
+ \int_0^\tau \left\langle r^M; \nabla_x \tilde{u} \right\rangle_{\{ \mathcal{M}(\bar{\Omega}; \mathbb{R}^{d \times d}), \mathcal{C}(\bar{\Omega}; \mathbb{R}^{d \times d}) \}} \, dt.
\]
(3.6)

Our main goal is to establish the weak (measure valued)– strong uniqueness
property. To obtain this result we choose \((\tilde{\rho}, \tilde{u}, \tilde{\vartheta})\) as a strong solution of the
problem (1.1)–(1.8) emanating from the initial data \((\tilde{\rho}_0, \tilde{u}_0, \tilde{\vartheta}_0)\). We consider
that the initial data satisfies
\[
0 < \tilde{\rho}_0 \in C^2(\bar{\Omega}), \quad \tilde{u}_0 \in C^2(\bar{\Omega}; \mathbb{R}^d) \text{ and } 0 < \tilde{\vartheta}_0 \in C^2(\bar{\Omega})
\]
(3.7)

Also, we assume that the strong solution belongs the following class
\[
0 < \tilde{\rho} \in C^2([0, T]; C^2(\Omega)), \quad \tilde{u} \in C^2([0, T]; C^2(\bar{\Omega}; \mathbb{R}^d)) \text{ and }
0 < \tilde{\vartheta} \in C^2([0, T]; C^2(\bar{\Omega})).
\]
(3.8)
Therefore, assuming that \((\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{\vartheta})\) is a strong solution in the class \((3.8)\) with initial data satisfying \((3.7)\), the inequality \((3.6)\) becomes

\[
\mathcal{E}_{mv}(\tau) + \int_0^\tau \int_\Omega \left[ \left\langle \mathcal{V}_{t,x}; \frac{\tilde{\vartheta}}{\vartheta} \mathcal{S}(\varrho, \vartheta, \mathbb{D}\mathbf{u}) : \mathbb{D}\mathbf{u} \right\rangle \right] \, dx \, dt

+ \int_0^\tau \int_\Omega \tilde{\varrho} \left\langle \mathcal{V}_{t,x}; \frac{\kappa(\varrho, \vartheta)}{\varrho} \mathbb{D}\vartheta \left( \frac{D\vartheta}{\varrho} - \frac{\nabla_x \tilde{\vartheta}}{\varrho} \right) \right\rangle \, dx \, dt

- \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \mathcal{S}(\varrho, \vartheta, \mathbb{D}\mathbf{u}) \right\rangle : \nabla_x \mathbf{u} \, dx \, dt

- \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; (\mathbb{D}\mathbf{u} - \mathbb{D}(\tilde{\mathbf{u}})) \right\rangle : \mathcal{S}(\tilde{\varrho}, \tilde{\vartheta}, \nabla_x \tilde{\mathbf{u}}) \, dx \, dt \quad + \mathcal{D}(\tau)

\leq \mathcal{E}_{mv}(0) - \int_0^\tau \int_\Omega \tilde{\varrho} \left\langle \mathcal{V}_{t,x}; \nabla_x \varrho \right\rangle \cdot \nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) \, dx \, dt

- \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \varrho - \varrho \right\rangle \cdot \nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) \, dx \, dt

+ \int_0^\tau \int_\Omega \left[ \left\langle \mathcal{V}_{t,x}; \varrho - \varrho \right\rangle \frac{1}{\varrho} \partial_t p(\tilde{\varrho}, \tilde{\vartheta}) - \left\langle \mathcal{V}_{t,x}; \varrho \mathbf{u} \right\rangle \frac{1}{\varrho} \nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) \right] \, dx \, dt

+ \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \mathcal{S}(\tilde{\varrho}, \tilde{\vartheta}, \nabla_x \tilde{\mathbf{u}}) \right\rangle \, dx \, dt

+ \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \mathbb{D}\mathbf{u} \right\rangle \, dx \, dt

\]

where

\[
R_1 = \left\langle \mathcal{V}_{t,x}; \varrho(\mathbf{u} - \mathbf{u}) \otimes (\mathbf{u} - \tilde{\mathbf{u}}) \right\rangle : \mathbb{D}(\nabla_x \tilde{\mathbf{u}})

+ \left\langle \mathcal{V}_{t,x}; \left( \frac{\varrho}{\vartheta} - 1 \right) (\tilde{\mathbf{u}} - \mathbf{u}) \right\rangle \cdot \nabla_x \mathcal{S}(\tilde{\varrho}, \tilde{\vartheta}, \mathbb{D}(\nabla_x \tilde{\mathbf{u}}))

+ \left\langle \mathcal{V}_{t,x}; \left( \frac{\varrho}{\vartheta} - 1 \right) (\tilde{\mathbf{u}} - \mathbf{u}) \right\rangle \cdot \nabla_x p(\tilde{\varrho}, \tilde{\vartheta})

- \left\langle \mathcal{V}_{t,x}; \varrho(s(\varrho, \vartheta) - s(\tilde{\varrho}, \tilde{\vartheta}))(\tilde{\mathbf{u}} - \mathbf{u}) \right\rangle \cdot \nabla_x \tilde{\vartheta}

- \left\langle \mathcal{V}_{t,x}; (\varrho - \bar{\varrho})(s(\varrho, \vartheta) - s(\tilde{\varrho}, \tilde{\vartheta})) \right\rangle \left( \partial_t \tilde{\vartheta} + \tilde{\mathbf{u}} \cdot \nabla_x \tilde{\vartheta} \right).

Moreover, adjusting a few terms suitably, the above inequality becomes

\[
\mathcal{E}_{mv}(\tau) + \int_0^\tau \int_\Omega \left[ \left\langle \mathcal{V}_{t,x}; \frac{\tilde{\vartheta}}{\vartheta} \mathcal{S}(\varrho, \vartheta, \mathbb{D}\mathbf{u}) : \mathbb{D}\mathbf{u} \right\rangle \right] \, dx \, dt

- \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \mathcal{S}(\varrho, \vartheta, \mathbb{D}\mathbf{u}) \right\rangle : \nabla_x \mathbf{u} \, dx \, dt

+ \int_0^\tau \int_\Omega \tilde{\varrho} \left\langle \mathcal{V}_{t,x}; \frac{\kappa(\varrho, \vartheta)}{\varrho} \mathbb{D}\vartheta \cdot \left( \frac{D\vartheta}{\varrho} - \frac{\nabla_x \tilde{\vartheta}}{\varrho} \right) \right\rangle \, dx \, dt

- \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; (\mathbb{D}\mathbf{u} - \mathbb{D}(\nabla_x \tilde{\mathbf{u}})) \right\rangle : \mathcal{S}(\tilde{\varrho}, \tilde{\vartheta}, \nabla_x \tilde{\mathbf{u}}) \, dx \, dt \quad + \mathcal{D}(\tau)

\leq \mathcal{E}_{mv}(0) + \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \varrho - \varrho \right\rangle \, \text{div}_x \left( \frac{\kappa(\tilde{\varrho}, \tilde{\vartheta}) \nabla_x \tilde{\vartheta}}{\tilde{\vartheta}} \right) \, dx \, dt

+ \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \left( 1 - \frac{\varrho}{\vartheta} \right) \right\rangle \left( \mathcal{S}(\tilde{\varrho}, \tilde{\vartheta}, \nabla_x \tilde{\mathbf{u}}) : \mathbb{D}(\nabla_x \tilde{\mathbf{u}}) + \frac{\kappa(\tilde{\varrho}, \tilde{\vartheta}) \nabla_x \tilde{\vartheta}}{\tilde{\vartheta}} \cdot \nabla_x \tilde{\vartheta} \right) \, dx \, dt

\]
\[+ \int_0^\tau \langle r^M; \nabla_x \tilde{u} \rangle_{\{ M(\Omega; R^{d \times d}), C(\Omega; R^{d \times d}) \}} \, dt + \int_0^\tau \int_\Omega R_2 \, dx \, dt,\]

where

\[
R_2 = R_1 + \left\langle \mathcal{V}_{t,x}; \left( 1 - \frac{\rho}{\bar{\rho}} \right) (\tilde{u} - u) \right\rangle \cdot \nabla_x p(\bar{\rho}, \bar{\vartheta})
+ \left\langle \mathcal{V}_{t,x}; p(\bar{\rho}, \bar{\vartheta}) - \frac{\partial p(\bar{\rho}, \bar{\vartheta})}{\partial \rho}(\bar{\rho} - \rho) - \frac{\partial p(\bar{\rho}, \bar{\vartheta})}{\partial \theta}(\bar{\vartheta} - \vartheta) - p(\rho, \vartheta) \right\rangle \text{div}_x \tilde{u}
+ \left\langle \mathcal{V}_{t,x}; s(\bar{\rho}, \bar{\vartheta}) - \frac{\partial s(\bar{\rho}, \bar{\vartheta})}{\partial \rho}(\bar{\rho} - \rho) - \frac{\partial s(\bar{\rho}, \bar{\vartheta})}{\partial \theta}(\bar{\vartheta} - \vartheta) \right\rangle \bar{\rho} \left( \partial_t \bar{\vartheta} + \tilde{u} \cdot \nabla_x \bar{\vartheta} \right). \tag{3.9}\]

Now we use the compatibility of the Young measure (2.9) and (2.10) to deduce

\[
\mathcal{E}_{mv}(\tau) + \int_0^\tau \int_\Omega \left[ \left\langle \mathcal{V}_{t,x}; \frac{\bar{\rho}}{\partial} S(\rho, \vartheta, \mathbb{D}_u) : \mathbb{D} \right\rangle \right] \, dx \, dt
+ \left\langle \mathcal{V}_{t,x}; \frac{\bar{\rho}}{\partial} S(\bar{\rho}, \bar{\vartheta}, \nabla_x \tilde{u}) : \mathbb{D}(\nabla_x \tilde{u}) \right\rangle \, dx \, dt
- \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; S(\rho, \vartheta, \mathbb{D}_u) \rangle : \nabla_x \tilde{u} \, dx \, dt
- \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; \mathbb{D}_u \rangle : S(\bar{\rho}, \bar{\vartheta}, \nabla_x \tilde{u}) \, dx \, dt
+ \int_0^\tau \int_\Omega \bar{\rho} \left\langle \mathcal{V}_{t,x}; \frac{\kappa(\rho, \vartheta)}{\theta} \mathbb{D}_\vartheta \left( \frac{\mathbb{D}_\vartheta}{\partial} - \frac{\nabla_x \bar{\vartheta}}{\partial} \right) \right\rangle \, dx \, dt
+ \int_0^\tau \int_\Omega \bar{\rho} \left\langle \mathcal{V}_{t,x}; \vartheta \left( \frac{\nabla_x \bar{\vartheta}}{\partial} - \frac{\mathbb{D}_\vartheta}{\partial} \right) \right\rangle \, dx \, dt + \mathcal{D}(\tilde{\vartheta} (\tau)
\leq \mathcal{E}_{mv}(0) + \int_0^\tau \langle r^M; \nabla_x \tilde{u} \rangle_{\{ M(\Omega; R^{d \times d}), C(\Omega; R^{d \times d}) \}} \, dt + \int_0^\tau \int_\Omega R_2 \, dx \, dt, \tag{3.10}\]

where the term \( R_2 \) is given by (3.9) and it consists of quadratic error terms. Using the notation introduced for symmetric and traceless part of a matrix in Sect. 2.1.1, we write the Newtonian stress tensor as

\[
S(\nabla_x u) = \mu(\rho, \vartheta) \left( \frac{\nabla_x u + \nabla_x^T u}{2} - \frac{1}{\rho} (\text{div}_x u) \mathbb{I} \right) + \lambda(\rho, \vartheta) (\text{div}_x u) \mathbb{I},
= \mu(\rho, \vartheta) \mathbb{D}_0(\nabla_x u) + \lambda(\rho, \vartheta) \text{div}_x u \mathbb{I}.\]
It helps us to rewrite the inequality (3.10) as
\[
\mathcal{E}_{mv}(\tau) + \int_0^\tau \int_\Omega \left( \mathcal{V}_{t,x}; \mu(\varrho, \vartheta) \frac{\tilde{\vartheta}}{\vartheta} \right) \left| \frac{D_0(\mathbb{D}u)}{\vartheta} - \frac{\varrho}{\vartheta} \mathbb{D}0(\nabla_x \tilde{u}) \right|^2 dx \, dt \\
+ \int_0^\tau \int_\Omega \mathbb{D}0(\nabla_x \tilde{u}); \left( \mathcal{V}_{t,x}(\mu(\varrho, \vartheta) - \mu(\tilde{\varrho}, \tilde{\vartheta})) \left( \mathbb{D}_0(\mathbb{D}u) - \frac{\varrho}{\vartheta} \mathbb{D}_0(\nabla_x \tilde{u}) \right) \right) dx \, dt \\
+ \int_0^\tau \int_\Omega \left( \mathcal{V}_{t,x}; \lambda(\varrho, \vartheta) \frac{\tilde{\vartheta}}{\vartheta} \right) \left| \text{Tr}(\mathbb{D}u) - \frac{\varrho}{\vartheta} \text{div}_x \tilde{u} \right|^2 dx \, dt \\
+ \int_0^\tau \int_\Omega \text{div}_x \tilde{u}; \left( \mathcal{V}_{t,x}(\lambda(\varrho, \vartheta) - \lambda(\tilde{\varrho}, \tilde{\vartheta})) \left( \text{Tr}(\mathbb{D}u) - \frac{\varrho}{\vartheta} \text{div}_x \tilde{u} \right) \right) dx \, dt \\
+ \int_0^\tau \int_\Omega \left( \mathcal{V}_{t,x}; \kappa(\varrho, \vartheta) \right) \left| \frac{D_\vartheta}{\vartheta} - \frac{\nabla_x \tilde{\vartheta}}{\vartheta} \right|^2 dx \, dt \\
+ \int_0^\tau \int_\Omega \kappa(\tilde{\varrho}, \tilde{\vartheta}) \frac{\nabla_x \tilde{\vartheta}}{\vartheta} \cdot \left( \mathcal{V}_{t,x}(\vartheta - \tilde{\vartheta}) \right) \left( \frac{\nabla_x \tilde{\vartheta}}{\vartheta} - \frac{D_\vartheta}{\vartheta} \right) dx \, dt \\
- \int_0^\tau \int_\Omega \nabla_x \tilde{\vartheta} \cdot \left( \mathcal{V}_{t,x}(\kappa(\varrho, \vartheta) - \kappa(\tilde{\varrho}, \tilde{\vartheta})) \right) \left( \frac{\nabla_x \tilde{\vartheta}}{\vartheta} - \frac{D_\vartheta}{\vartheta} \right) dx \, dt \\
\leq \mathcal{E}_{mv}(0) + \int_0^\tau \langle r^M; \nabla_x \tilde{u} \rangle_{\mathcal{M}(\Omega; \mathbb{R}^{d \times d}), C(\Omega; \mathbb{R}^{d \times d})} dt + \int_0^\tau \int_\Omega R_2 dx \, dt.
\] (3.11)

The inequality (3.11) is called the **relative energy inequality** associated with the problem. Therefore we summarize the above discussion in the following lemma:

**Lemma 3.1.** Let the transport coefficients \( \kappa(\varrho, \vartheta), \mu(\varrho, \vartheta) \) and \( \lambda(\varrho, \vartheta) \) be continuously differentiable and positive for \( \varrho > 0, \vartheta > 0 \) and the thermodynamic functions satisfy Gibbs equation (1.4) and the thermodynamic stability assumption (3.2). We consider a a measure valued solution \( \{ \mathcal{V}, D_\vartheta \} \) of the system (1.1)–(1.3) with the initial data \( V_0 \) and a strong solution \( \{ \tilde{\varrho}, \tilde{u}, \tilde{\vartheta} \} \) in the class (3.8) and initial data in the class and (3.7). Under these conditions, we can establish the inequality (3.11), where the remainder term \( R_2 \) is given by (3.9).

### 3.1. A suitable reduction of relative energy inequality

In this subsection, our objective is to simplify the expression (3.11). We have observed that the relative energy is a non-negative functional, and the remainder term \( R_2 \) contains certain quadratic terms.

#### 3.1.1. Essential and residual parts of a function.

At first, we introduce a cutoff function \( \chi_\delta \) such that
\[
\chi_\delta \in C^\infty_c (\mathbb{R}^2), 0 \leq \chi_\delta \leq 1, \chi_\delta(\varrho, \vartheta) = 1 \text{ if } \delta \leq \varrho \leq \frac{1}{\delta} \text{ and } \delta \leq \vartheta \leq \frac{1}{\delta} \text{ for some } \delta > 0.
\]

For a function \( H = H(\varrho, \vartheta, u, D_\varrho, D_\vartheta) \), we set
\[
[H]_{\text{ess}} = \chi_\delta(\varrho, \vartheta) H(\varrho, \vartheta, u, D_\varrho, D_\vartheta), \quad [H]_{\text{res}} = (1 - \chi_\delta(\varrho, \vartheta)) H(\varrho, \vartheta, u, D_\varrho, D_\vartheta).
\]
If \( \tilde{\rho} \) and \( \tilde{\vartheta} \) are strictly positive and bounded above and below, and \(|\tilde{u}|\) is also bounded, then with the help of the above notation, for any \( \delta > 0 \), we have

\[
E(\rho, \vartheta, u \mid \tilde{\rho}, \tilde{\vartheta}, \tilde{u}) \geq c(\delta, \tilde{\rho}, \tilde{\vartheta}) \left( \| \vartheta - \tilde{\vartheta} \|^2 + \| \tilde{\rho} \|^2 + |u - \tilde{u}|^2 \right)_{\text{ess}} \\
+ \left[ 1 + \vartheta + \vartheta|s(\vartheta, \vartheta)| + \varrho e(\vartheta, \varrho) + \varrho |u|^2 \right]_{\text{res}}.
\]

**(Remark 3.2.** The inequality (3.12) implies that the relative energy functional is coercive, see [6, Section 3.2.2].

At this point, our goal is either to control the remainder term \( R_2 \) of (3.11), by the integral of the relative energy or to absorb it into a non-negative term on the left-hand side of (3.11).

**Lemma 3.3.** Let the hypothesis in Lemma 3.1 remains true. Then with the help of (3.12), the inequality (3.11) reduces to

\[
\mathcal{E}_{mv}(\tau) + \int_{0}^{\tau} \int_{\Omega} \left( \mathcal{V}_{t,x}; \mu(\vartheta, \varrho) \frac{\partial \tilde{\vartheta}}{\partial \vartheta} \right) \left( \mathbb{D}_0(\mathbb{D} u) - \frac{\partial}{\partial \vartheta} \mathbb{D}_0(\nabla_x \tilde{u}) \right)^2 \, dx \, dt \\
+ \int_{0}^{\tau} \int_{\Omega} \mathcal{D}_0(\nabla_x \tilde{u}) \cdot \left( \mathcal{V}_{t,x}; (\mu(\vartheta, \varrho) - \mu(\tilde{\vartheta}, \tilde{\rho})) \left( \mathbb{D}_0(\mathbb{D} u) - \frac{\partial}{\partial \vartheta} \mathbb{D}_0(\nabla_x \tilde{u}) \right) \right) \, dx \, dt \\
+ \int_{0}^{\tau} \int_{\Omega} \left( \mathcal{V}_{t,x}; \kappa(\vartheta, \varrho) \frac{\partial \tilde{\rho}}{\partial \vartheta} + \frac{\partial}{\partial \vartheta} \left( \mathbb{D}_\vartheta \tilde{u} \right) \right)^2 \, dx \, dt \\
+ \int_{0}^{\tau} \int_{\Omega} \kappa(\tilde{\vartheta}, \tilde{\rho}) \frac{\partial}{\partial \vartheta} \left( \mathcal{V}_{t,x}; (\vartheta - \tilde{\vartheta}) \left( \frac{\partial \tilde{\rho}}{\partial \vartheta} - \frac{\partial}{\partial \vartheta} \mathbb{D}_\vartheta \tilde{u} \right) \right) \, dx \, dt \\
- \int_{0}^{\tau} \int_{\Omega} \frac{\partial}{\partial \vartheta} \cdot \left( \mathcal{V}_{t,x}; (\kappa(\vartheta, \varrho) - \kappa(\tilde{\vartheta}, \tilde{\rho})) \left( \frac{\partial \tilde{\rho}}{\partial \vartheta} - \frac{\partial}{\partial \vartheta} \mathbb{D}_\vartheta \tilde{u} \right) \right) \, dx \, dt \\
+ \mathcal{D}_\rho(\tau) \\
\leq \mathcal{E}_{mv}(0) + C(\delta, \tilde{\rho}, \tilde{\vartheta}) \int_{0}^{\tau} \mathcal{E}_{mv}(t) \, dt + \int_{0}^{\tau} \mathcal{D}_\vartheta(t) \, dt \\
+ \int_{0}^{\tau} \int_{\Omega} \left( \mathcal{V}_{t,x}; |\vartheta + |p(\vartheta, \varrho)| + |u - \tilde{u}| + \varrho |s(\vartheta, \varrho)| \right) |u|^2 \, dx \, dt. \tag{3.13}
\]

**Proof.** The proof is quite straightforward, since \( R_2 \) contains quadratic terms, therefore the essential parts of \( R_2 \) will be controlled by \( C(\delta, \tilde{\rho}, \tilde{\vartheta}) \int_{0}^{T} \mathcal{E}_{mv}(t) \, dt \). For the residual parts of \( R_2 \), we control some of the terms that are compatible with the right hand side of (3.12) and we keep the other terms on the RHS of (3.13). \( \square \)
4. Main results: weak (measure valued)–strong uniqueness

4.1. Conditional weak (measure valued)–strong uniqueness

By the term ‘conditional’, we mean that, along with the measure valued solution if we assume some additional hypothesis, then we will be able to achieve the desired weak (measure valued)–strong uniqueness property.

4.1.1. First conditional result. First conditional result when density and temperature are uniformly bounded.

Theorem 4.1. Let the transport coefficients $\kappa(\varrho, \vartheta)$, $\mu(\varrho, \vartheta)$ and $\lambda(\varrho, \vartheta)$ be continuously differentiable and positive for $\varrho > 0$, $\vartheta > 0$. Let the thermodynamic functions satisfy Gibbs equation (1.4) and the thermodynamic stability (3.2). Additionally, let us assume that $(\varrho, \tilde{u}, \vartheta)$ represents a strong solution of the system (1.1)–(1.8) in the class (3.8), over the time-space cylinder $[0, T] \times \Omega$ emanating from the initial data $(\varrho_0, u_0, \vartheta_0)$, which belongs to the class (3.7). Also, we consider that $\mathcal{V}$ be a measure valued solution of the same problem following the Definition 2.2 and it satisfies

$$\mathcal{V}_{t,x} \left\{ 0 < \varrho < \varrho_0, 0 < \vartheta < \vartheta_0 \right\} = 1 \text{ for a.e. } (t, x) \in QT,$$

(4.1)

for some constants $\varrho_0, \vartheta_0 > 0$ and the initial data coincide, i.e.

$$\mathcal{V}_{0,x} = \delta_{[\varrho_0, u_0, \vartheta_0]} \text{ for a.e. } x \in \Omega.$$

Then

$$\mathcal{V}_{t,x} = \delta_{[\tilde{\varrho}(t, x), \tilde{u}(t, x), \tilde{\vartheta}(t, x), \nabla_x \tilde{u}(t, x), \nabla_x \tilde{\vartheta}(t, x)]} \text{ for a.e. } (t, x) \in QT.$$

Proof. At first we note that there exists a $\delta > 0$, such that $\delta \leq \min \{ \varrho, \vartheta \}$ and $\max \{ \tilde{\varrho}, \tilde{\vartheta} \} \leq \frac{1}{\delta} \varrho$. Therefore, we consider a cut-off function $\chi_\delta$ as mentioned in Sect. 3.1.1. The hypothesis (4.1) implies

$$\int_0^T \int_\Omega \left( \mathcal{V}_{t,x} : [\vartheta + |p(\varrho, \vartheta)| + |u - \tilde{u}| + \varrho |s(\varrho, \vartheta)|] |u| \right) dx \, dt = 0.$$

Let $\epsilon > 0$. Based on the assumption (4.1), we observe that

$$\int_0^T \int_\Omega \left( \mathcal{V}_{t,x} ; \mu(\varrho, \vartheta) \frac{\vartheta}{\vartheta} \right) \left\| \mathcal{D}_0(D_u) - \frac{\vartheta}{\vartheta} \mathcal{D}_0(\nabla_x \tilde{u}) \right\|^2 dx \, dt$$

$$+ \int_0^T \int_\Omega \left( \mathcal{V}_{t,x} ; \lambda(\varrho, \vartheta) \frac{\vartheta}{\vartheta} \right) \left\| \text{tr}(D_u) - \frac{\vartheta}{\vartheta} \text{div}_x \tilde{u} \right\|^2 dx \, dt$$

$$+ \int_0^T \int_\Omega \left( \mathcal{V}_{t,x} ; \kappa(\varrho, \vartheta) \right) \left\| \frac{D_\varrho}{\vartheta} - \frac{\vartheta}{\vartheta} \right\|^2 dx \, dt$$

$$\geq C(\delta, \tilde{\varrho}, \tilde{\vartheta}, \tilde{u}) \int_0^T \int_\Omega \left( \mathcal{V}_{t,x} ; \left\| \mathcal{D}_0(D_u) - \frac{\vartheta}{\vartheta} \mathcal{D}_0(\nabla_x \tilde{u}) \right\|^2 \right) dx \, dt$$

$$+ \left\| \text{tr}(D_u) - \frac{\vartheta}{\vartheta} \text{div}_x \tilde{u} \right\|^2 + \left\| \frac{D_\varrho}{\vartheta} - \frac{\vartheta}{\vartheta} \right\|^2 dx \, dt.$$
and
\[
\int_0^\tau \int_\Omega \mathbb{D}_0(\nabla_x \tilde{\mathbf{u}}) \left\langle \mathcal{V}_{t,x}; (\mu(\rho, \vartheta) - \mu(\bar{\rho}, \bar{\vartheta})) \left( \mathbb{D}_0(\mathbb{D}_u) - \frac{\partial}{\partial \vartheta} \mathbb{D}_0(\nabla_x \tilde{\mathbf{u}}) \right) \right\rangle \, dx \, dt \\
+ \int_0^\tau \int_\Omega \text{div}_x \tilde{\mathbf{u}} \left\langle \mathcal{V}_{t,x}; (\lambda(\rho, \vartheta) - \lambda(\bar{\rho}, \bar{\vartheta})) \left( \text{Tr}(\mathbb{D}_u) - \frac{\partial}{\partial \vartheta} (\text{div}_x \tilde{\mathbf{u}}) \right) \right\rangle \, dx \, dt \\
- \int_0^\tau \int_\Omega \nabla_x \tilde{\vartheta} \cdot \left\langle \mathcal{V}_{t,x}; (\kappa(\rho, \vartheta) - \kappa(\bar{\rho}, \bar{\vartheta})) \left( \frac{\nabla_x \tilde{\vartheta}}{\vartheta} - \frac{\text{D}_\vartheta}{\vartheta} \right) \right\rangle \, dx \, dt \\
+ \int_0^\tau \int_\Omega (\bar{\kappa}(\bar{\rho}, \bar{\vartheta}) \nabla_x \tilde{\vartheta}) \cdot \left\langle \mathcal{V}_{t,x}; (\vartheta - \bar{\vartheta}) \left( \frac{\nabla_x \tilde{\vartheta}}{\vartheta} - \frac{\text{D}_\vartheta}{\vartheta} \right) \right\rangle \, dx \, dt \\
\leq C(\epsilon, \bar{\rho}, \bar{\vartheta}, \tilde{\mathbf{u}}) \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; |\vartheta - \bar{\vartheta}|^2 + |\vartheta - \bar{\vartheta}|^2 \right\rangle \, dx \, dt \\
+ \epsilon \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \left| \mathbb{D}_0(\mathbb{D}_u) - \frac{\partial}{\partial \vartheta} \mathbb{D}_0(\nabla_x \tilde{\mathbf{u}}) \right|^2 \right\rangle \\
+ \left| \text{Tr}(\mathbb{D}_u) - \frac{\partial}{\partial \vartheta} \text{div}_x \tilde{\mathbf{u}} \right|^2 + \left| \frac{\text{D}_\vartheta}{\vartheta} - \frac{\nabla_x \tilde{\vartheta}}{\vartheta} \right|^2 \right\rangle \, dx \, dt ,
\]
where $C(\delta, \bar{\rho}, \bar{\vartheta}, \tilde{\mathbf{u}}) > 0$. Therefore, choosing $\epsilon$ suitably, we deduce
\[
\mathcal{E}_{mv}(\tau) + \frac{1}{2} C(\delta, \bar{\rho}, \bar{\vartheta}, \tilde{\mathbf{u}}) \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \left| \mathbb{D}_0(\mathbb{D}_u) - \frac{\partial}{\partial \vartheta} \mathbb{D}_0(\nabla_x \tilde{\mathbf{u}}) \right|^2 \right\rangle \\
+ \left| \text{Tr}(\mathbb{D}_u) - \frac{\partial}{\partial \vartheta} \text{div}_x \tilde{\mathbf{u}} \right|^2 + \left| \frac{\text{D}_\vartheta}{\vartheta} - \frac{\nabla_x \tilde{\vartheta}}{\vartheta} \right|^2 \right\rangle \, dx \, dt + \mathcal{D}_\vartheta(\tau) \quad (4.2)
\]
\[
\leq \mathcal{E}_{mv}(0) + C(\delta, \bar{\rho}, \bar{\vartheta}, \tilde{\mathbf{u}}) \int_0^\tau \mathcal{E}_{mv}(t) \, dt + \int_0^\tau \mathcal{D}_\vartheta(t) \, dt .
\]
Employing Grönwall’s lemma we obtain the desired result. \(\square\)

**Discussion of the result** One important observation is that in this analysis, no specific assumptions on the transport coefficients or the pressure are required; they can be quite general. The main takeaway from this result is that if we can establish boundedness of the solutions, particularly for density and temperature, then the desired weak (measure-valued)–strong principle holds. This is highly beneficial for numerical solutions of the system, as numerical approximations are typically constructed to be bounded. Therefore, we can infer that the numerical approximate solutions are converging toward measure-valued solutions, with some additional assumptions given by (4.1).

### 4.1.2. Second conditional result

Our next result pertains to a more physically relevant situation where the pressure law follows Boyle’s law. The weak-strong uniqueness we prove is imposing certain condition on entropy and additional hypothesis on transport coefficients.

- The bulk and shear viscosity coefficient are given by
\[
\mu(\rho, \vartheta) = C_\mu (1 + \vartheta) \text{ and } \lambda(\rho, \vartheta) = C_\lambda (1 + \vartheta) \text{ with } C_\mu > 0, C_\lambda \geq 0. \quad (4.3)
\]
The heat conductivity coefficient is given by
\[ \kappa(\varrho, \vartheta) = \kappa(1 + \vartheta) \] with \( \kappa > 0 \). (4.4)
Indeed, conditions (4.3) and (4.4) imply that the transport coefficients may be unbounded.

**Theorem 4.2.** Let the transport coefficients \( \kappa(\varrho, \vartheta) \), \( \mu(\varrho, \vartheta) \) and \( \lambda(\varrho, \vartheta) \) be given by (4.3) and (4.4). Let the thermodynamic functions satisfy Gibbs equation and the thermodynamic stability assumption with
\[ p(\varrho, \vartheta) = \varrho \vartheta, \quad c(\varrho, \vartheta) = c_v \vartheta, \quad s(\varrho, \vartheta) = \log\left(\frac{\varrho^{c_v}}{\vartheta}\right), \quad c_v > 1. \] (4.5)
Additionally, let us assume that \((\tilde{\varrho}, \tilde{\vartheta}, \tilde{\vartheta})\) represents a strong solution of the system (1.1)–(1.8) in the class (3.8), over the time-space cylinder \([0, T] \times \Omega\) emanating from the initial data \((\varrho_0, \mathbf{u}_0, \vartheta_0)\), which belongs to the class (3.7). Also, we consider that \( \mathcal{V} \) be a measure valued solution of the same problem following the Definition 2.2 and it satisfies
\[ \mathcal{V}_{t,x} \left\{ |s(\varrho, \vartheta)| \leq \bar{s} \right\} = 1 \text{ for a.e. } (t, x) \in Q_T, \] (4.6)
for some \( \bar{s} > 0 \) and
\[ \mathcal{V}_{0,x} = \delta_{(\varrho_0, \mathbf{u}_0, \vartheta_0)} \text{ for a.e. } x \in \Omega. \]

Then
\[ \mathcal{V}_{t,x} = \delta_{(\tilde{\varrho}(t,x), \tilde{\mathbf{u}}(t,x), \tilde{\vartheta}(t,x), \mathbb{D}_x \tilde{\mathbf{u}}(t,x), \nabla_x \tilde{\vartheta}(t,x))] \text{ for a.e. } (t, x) \in Q_T. \]

**Proof.** For the sake of simplicity we assume \( C_{\lambda} = 0 \). Using the hypothesis (4.3), we get
\[ \int_0^T \int_\Omega \left\langle \mathcal{V}_{t,x}; \mu(\varrho, \vartheta) \frac{\partial \tilde{\vartheta}}{\partial \vartheta} \right| \mathbb{D}_0(\mathbb{D}_\mathbf{u}) - \frac{\vartheta}{\varrho} \mathbb{D}_0(\nabla_x \tilde{\mathbf{u}}) \right| \right\rangle \, dx \, dt \]
\[ \geq \int_0^T \int_\Omega \left\langle \mathcal{V}_{t,x}; \frac{1}{2} C_\mu (1 + \vartheta) \frac{\partial \tilde{\vartheta}}{\partial \vartheta} \right| \mathbb{D}_0(\mathbb{D}_\mathbf{u}) - \frac{\vartheta}{\varrho} \mathbb{D}_0(\nabla_x \tilde{\mathbf{u}}) \right| \right\rangle \, dx \, dt \]
\[ + \int_0^T \int_\Omega \left\langle \mathcal{V}_{t,x}; C_\mu \frac{1}{4} \nabla_x \tilde{\mathbf{u}} \right| \mathbb{D}_0(\mathbb{D}_\mathbf{u}) - \mathbb{D}_0(\nabla_x \tilde{\mathbf{u}}) \right| \right\rangle \, dx \, dt \]
\[ - \int_0^T \int_\Omega \left\langle \mathcal{V}_{t,x}; \tilde{\vartheta} C_\mu \left(1 - \frac{\partial}{\partial \vartheta}\right) \mathbb{D}_0(\nabla_x \tilde{\mathbf{u}}) \right| \right\rangle \, dx \, dt \]
and
\[ \left| \int_0^T \int_\Omega \mathbb{D}(\nabla_x \tilde{\mathbf{u}}) \left\langle \mathcal{V}_{t,x}; (\mu(\varrho, \vartheta) - \mu(\tilde{\varrho}, \tilde{\vartheta})) \left( \mathbb{D}_0(\mathbb{D}_\mathbf{u}) - \frac{\vartheta}{\varrho} \mathbb{D}_0(\nabla_x \tilde{\mathbf{u}}) \right) \right\rangle \, dx \, dt \right| \]
\[ \leq \epsilon \int_0^T \int_\Omega \left\langle \mathcal{V}_{t,x}; \left| \mathbb{D}_0(\mathbb{D}_\mathbf{u}) - \frac{\vartheta}{\varrho} \mathbb{D}_0(\nabla_x \tilde{\mathbf{u}}) \right|^2 \right\rangle \, dx \, dt \]
\[ + C(\epsilon, \tilde{\vartheta}, \tilde{\mathbf{u}}) \int_0^T \int_\Omega \left\langle \mathcal{V}_{t,x}; \vartheta - \tilde{\vartheta} \right| \right\rangle \, dx \, dt. \]
Also, (4.4) implies
\[
\int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \kappa(1 + \vartheta) \left| \frac{D\vartheta}{\vartheta} - \frac{\nabla_x \vartheta}{\vartheta} \right|^2 \right\rangle dx \, dt \geq \kappa \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \left| \frac{D\vartheta}{\vartheta} - \frac{\nabla_x \vartheta}{\vartheta} \right|^2 \right\rangle dx \, dt.
\]

Similarly, For any \( \epsilon > 0 \), using Young’s inequality, we infer the following estimates
\[
\left| \int_0^\tau \int_\Omega \kappa(1 + \tilde{\vartheta}) \nabla_x \tilde{\vartheta} \cdot \left\langle \mathcal{V}_{t,x}; (\vartheta - \tilde{\vartheta}) \left( \frac{\nabla_x \tilde{\vartheta}}{\tilde{\vartheta}} - \frac{D\vartheta}{\vartheta} \right) \right\rangle dx \, dt \right|
\leq \epsilon \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \left| \frac{D\vartheta}{\vartheta} - \frac{\nabla_x \vartheta}{\vartheta} \right|^2 \right\rangle dx \, dt
+ C(\epsilon, \tilde{\vartheta}) \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \left| \vartheta - \tilde{\vartheta} \right|^2 \right\rangle dx \, dt
\]
and
\[
\left| \int_0^\tau \int_\Omega \nabla_x \tilde{\vartheta} \cdot \left\langle \mathcal{V}_{t,x}; (\kappa(\varrho, \vartheta) - \kappa(\tilde{\varrho}, \tilde{\vartheta}) \left( \frac{\nabla_x \tilde{\vartheta}}{\tilde{\vartheta}} - \frac{D\vartheta}{\vartheta} \right) \right\rangle dx \, dt \right|
\leq \epsilon \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \left| \frac{D\vartheta}{\vartheta} - \frac{\nabla_x \vartheta}{\vartheta} \right|^2 \right\rangle dx \, dt
+ C(\epsilon, \tilde{\vartheta}) \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \left| \vartheta - \tilde{\vartheta} \right|^2 \right\rangle dx \, dt.
\]

Now, for pressure law (4.5), we note that
\[
|p(\varrho, \vartheta)| \leq 1 + \varrho e(\varrho, \vartheta) + \varrho |s(\varrho, \vartheta)|.
\]

Then using compatibility condition (2.16) and (3.12) we obtain
\[
\mathcal{E}_{mv}(\tau) + \frac{1}{2} \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; C_\mu(1 + \vartheta) \frac{\tilde{\vartheta}}{\vartheta} \left| \mathcal{D}_0(\nabla_x \mathcal{U}) - \frac{\vartheta}{\tilde{\vartheta}} \mathcal{D}_0(\nabla_x \tilde{\mathcal{U}}) \right|^2 \right\rangle dx \, dt
+ \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \frac{1}{4} \left( \frac{\tilde{\vartheta}}{\vartheta} \right)^2 \left| \mathcal{D}_0(\nabla_x \mathcal{U}) - \mathcal{D}_0(\nabla_x \tilde{\mathcal{U}}) \right|^2 \right\rangle dx \, dt
+ \int_0^\tau \int_\Omega \tilde{\vartheta} \left\langle \mathcal{V}_{t,x}; \kappa(1 + \vartheta) \left| \frac{D\vartheta}{\vartheta} - \frac{\nabla_x \vartheta}{\vartheta} \right|^2 \right\rangle dx \, dt + \mathcal{D}_\vartheta(\tau)
\leq \mathcal{E}_{mv}(0) + C(\delta) \int_0^\tau \mathcal{E}_{mv}(t) \, dt + \int_0^\tau \mathcal{D}_\vartheta(t) \, dt
+ \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; [\vartheta^2 + \varrho |s(\varrho, \vartheta)|] |\mathcal{U}|_{res} \right\rangle dx \, dt
+ \epsilon \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \left| \frac{D\vartheta}{\vartheta} - \frac{\nabla_x \vartheta}{\vartheta} \right|^2 \right\rangle dx \, dt.
\[ + \epsilon \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \left| \mathbb{D}_0(D_u) - \frac{\partial}{\partial \vartheta} \mathbb{D}_0(\nabla_x \tilde{u}) \right|^2 \right\rangle \, dx \, dt \]
\[ + \epsilon \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; |\mathbb{D}_0(D_u) - \mathbb{D}_0(\nabla_x \tilde{u})|^2 \right\rangle \, dx \, dt. \quad (4.7) \]

Now using (4.6) we have \( \vartheta^{\vartheta} \leq C(\vartheta) \varrho \) and \( \varrho s u \leq C(s) (\varrho + \varrho |u|^2) \). This implies
\[ \vartheta^{\vartheta} + 1 \leq C(s) \varrho \vartheta = C(s) \varrho \vartheta. \]

By proper choice of \( \epsilon \), inequality (4.7) reduces to
\[
\mathcal{E}_{mv}(\tau) + \frac{1}{4} \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; C\mu (1 + \vartheta) \frac{\partial}{\partial \vartheta} \left| \mathbb{D}_0(D_u) - \frac{\partial}{\partial \vartheta} \mathbb{D}_0(\nabla_x \tilde{u}) \right|^2 \right\rangle \, dx \, dt \\
+ \frac{1}{8} \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \tilde{\vartheta}^2 \left| \mathbb{D}_0(D_u) - \mathbb{D}_0(\nabla_x \tilde{u}) \right|^2 \right\rangle \, dx \, dt \\
+ \frac{1}{2} \int_0^\tau \int_\Omega \tilde{\vartheta} \left\langle \mathcal{V}_{t,x}; \kappa (1 + \vartheta) \left| \mathbb{D}_\vartheta - \frac{\nabla_x \tilde{\vartheta}}{\vartheta} \right|^2 \right\rangle \, dx \, dt \\
+ \mathcal{D}_\vartheta(\tau) \\
\leq \mathcal{E}_{mv}(0) + C(\delta) \int_0^\tau \mathcal{E}_{mv}(t) \, dt + \int_0^\tau \mathcal{D}_\vartheta(t) \, dt.
\]

Again, invoking Grönwall’s lemma, we complete the proof. The case \( C_\lambda \neq 0 \) can be computed similarly using Korn-type inequalities. \[ \square \]

**Discussion of the result** Here, we consider, certain structural hypothesis on transport coefficient and a particular form of pressure. The condition (4.6) somehow suggests that the entropy is bounded and this is a much weaker assumption than the previous result as (4.1) implies (4.6). A recent study by Basarić et al. [1] presents a conditional regularity result for the system with the pressure law (4.5). The combination of conditional regularity results and the weak (measure valued)–strong uniqueness principle within the class of measure-valued solutions serves as a powerful tool for establishing the convergence of numerical schemes.

### 4.2. Unconditional weak (measure valued)–strong uniqueness

Here we consider a general pressure law and some structural assumption on the transport coefficients. We call it unconditional result because, here we will not assume any further restriction on measure valued solution. Let us first consider the pressure law in the following way:
\[
p(\varrho, \vartheta) = p_M(\varrho, \vartheta) + p_R(\varrho, \vartheta), \quad (4.8)
\]
where \( p_M \) stands for the molecular pressure and \( p_R \) is the radiation pressure. The relation between the molecular pressure \( p_M \) and the associated internal
energy $e_M$ is given by
\[ p_M(\varrho, \vartheta) = \frac{2}{3} \varrho e_M(\varrho, \vartheta). \] (4.9)

From the Gibb’s relation (1.4), we have
\[ p_M(\varrho, \vartheta) = \vartheta \frac{5}{2} P \left( \frac{\varrho}{\vartheta^{\frac{2}{3}}} \right), \]
for some function $P$. Moreover, following [22], we assume that $P \in C^1[0, \infty) \cap C^5(0, \infty)$,
\[ P(0) = 0, \quad P'(q) > 0, \quad \text{for all } q > 0, \quad 0 < \frac{5}{3} P(q) - P'(q)q < c \text{ for all } q < 0, \quad \lim_{q \to -\infty} \frac{P(q)}{q^{\frac{5}{3}}} = \overline{p} > 0. \] (4.10)

We rewrite the internal energy $e_M$ associated with molecular pressure as
\[ e_M(\varrho) = \frac{3}{2} \varrho^{\frac{5}{2}} P \left( \frac{\varrho}{\vartheta^{\frac{2}{3}}} \right), \]
and, again using Gibbs relation, we have
\[ s_M = S \left( \frac{\varrho}{\vartheta^{\frac{2}{3}}} \right), \] (4.11)
for some function $S$ with the property
\[ S'(q) = -\frac{3}{2} \frac{5}{3} P(q) - P'(q)q < 0. \] (4.12)

Finally, we impose the third law of thermodynamics in the form
\[ \lim_{q \to -\infty} S(q) = 0. \] (4.13)

The structural assumptions on transport coefficient read as
\[ \kappa(\varrho, \vartheta) = \kappa_1 + \kappa_2 \vartheta^2, \text{ with } \kappa_1 > 0, \quad \kappa_2 \geq 0 \quad \text{and} \quad \beta \leq 2 \]
\[ \mu(\varrho, \vartheta) = \mu_0 + \mu_1 \vartheta, \quad \mu_0, \mu_1 > 0, \] (4.14)
\[ \lambda(\varrho, \vartheta) = \lambda_0 + \lambda_1 \vartheta, \quad \lambda_0, \lambda_1 \geq 0. \]

The hypothesis (4.14) encompasses important properties of gases, as they exhibit the characteristic behavior of both $\mu$ and $\kappa$ being unbounded as $\vartheta \to \infty$, as noted by Becker [2].

**Theorem 4.3.** Let the pressure follows (4.8) with $p_M, e_M$ and $s_M$ given by (4.9), (4.11) and (4.13). Moreover, $\kappa, \mu, \lambda$ satisfy (4.14). The radiation pressure along with associated internal energy and entropy are given by
\[ p_R = a \vartheta^2, \quad e_R = a \frac{\vartheta^2}{\varrho} \quad \text{and} \quad s_R = 2a \frac{\vartheta}{\varrho} \text{ with } a > 0. \] (4.15)

Additionally, let us assume that $(\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{\vartheta})$ represents a strong solution of the system (1.1)–(1.8) in the class (3.8), over the time-space cylinder $[0, T] \times \Omega$ emanating from the initial data $(\varrho_0, \mathbf{u}_0, \vartheta_0)$, which belongs to the class (3.7).
Also, we consider that \( \{ \mathcal{V}_{t,x} \}_{(t,x) \in Q_T} \) is a measure valued solution of the same problem following the Definition 2.2 such that

\[
\mathcal{V}_{0,x} = \delta_{[\varrho_0, u_0, \varrho_0]} \text{ for a.e. } x \in \Omega.
\]

Then

\[
\mathcal{V}_{t,x} = \delta_{[\varrho(t,x), u(t,x), \varrho(t,x), D_x u(t,x), \nabla_x \varrho(t,x)]} \text{ for a.e. } (t, x) \in Q_T.
\] (4.16)

**Proof.** At first, using the structural assumptions (4.14), we have

\[
\begin{align*}
\int_0^T \int_\Omega & \left\langle \mathcal{V}_{t,x}; \mu(\varrho, \varrho) \frac{\partial \varrho}{\partial x} - \frac{\varrho}{\varrho} D_0(D u) - \frac{\varrho}{\varrho} D_0(\nabla_x \tilde{u}) \right\rangle \, dx \, dt \\
+ \int_0^T \int_\Omega & \left\langle D_0(\nabla_x \tilde{u}); \left( \mathcal{V}_{t,x}; (\mu(\varrho, \varrho) - \mu(\tilde{\varrho}, \tilde{\varrho})) \left( D_0(D u) - \frac{\varrho}{\varrho} D_0(\nabla_x \tilde{u}) \right) \right) \right\rangle \, dx \, dt \\
= & \int_0^T \int_\Omega \left\langle \mathcal{V}_{t,x}; \mu_0 \frac{\partial \varrho}{\partial x} \left| D_0(D u) - \frac{\varrho}{\varrho} D_0(\nabla_x \tilde{u}) \right|^2 \right\rangle \, dx \, dt \\
+ & \int_0^T \int_\Omega \left\langle \mathcal{V}_{t,x}; \mu_1 \frac{\partial \varrho}{\partial x} \left| D_0(D u) - \frac{\varrho}{\varrho} D_0(\nabla_x \tilde{u}) \right|^2 \right\rangle \, dx \, dt \\
+ & \int_0^T \int_\Omega \left\langle \mathcal{V}_{t,x}; \mu_1 (\varrho - \tilde{\varrho}) \left( D_0(D u) - \frac{\varrho}{\varrho} D_0(\nabla_x \tilde{u}) \right) \right\rangle \, dx \, dt \\
+ & \int_0^T \int_\Omega \left\langle \mathcal{V}_{t,x}; \frac{1}{\varrho} \mu_1 (\varrho - \tilde{\varrho})^2 D_0(\nabla_x \tilde{u}) \right\rangle \, dx \, dt.
\end{align*}
\]

Using Young’s inequality, for any \( \epsilon > 0 \), we obtain

\[
\begin{align*}
\int_0^T \int_\Omega & D_0(\nabla_x \tilde{u}); \left\langle \mathcal{V}_{t,x}; \mu_1 (\varrho - \tilde{\varrho}) (D_0(D u) - D_0(\nabla_x \tilde{u})) \right\rangle \, dx \, dt \\
\leq & \epsilon \int_0^T \int_\Omega \left\langle \mathcal{V}_{t,x}; \mu_1 \frac{\partial \varrho}{\partial x} \left| D_0(D u) - D_0(\nabla_x \tilde{u}) \right|^2 \right\rangle \, dx \, dt \\
& + C(\nabla_x \tilde{u}) \frac{1}{4 \epsilon} \int_0^T \int_\Omega \left\langle \mathcal{V}_{t,x}; \left| \varrho - \tilde{\varrho} \right|^2 \right\rangle_{\text{ess}} \, dx \, dt. + \left[1 + \varrho e_R(\varrho, \varrho)\right]_{\text{res}}
\end{align*}
\]

Similarly, for any \( \epsilon > 0 \), we derive the following inequalities:

\[
\begin{align*}
\left| \int_0^T \int_\Omega \kappa(\tilde{\varrho}) \frac{\nabla_x \tilde{\varrho}}{\varrho} \cdot \left\langle \mathcal{V}_{t,x}; (\varrho - \tilde{\varrho}) \left( \frac{\nabla_x \tilde{\varrho}}{\varrho} - \frac{D \varrho}{\varrho} \right) \right\rangle \, dx \, dt \right| \\
\leq & \epsilon \int_0^T \int_\Omega \left\langle \mathcal{V}_{t,x}; \left| \frac{D \varrho}{\varrho} - \frac{\nabla_x \tilde{\varrho}}{\varrho} \right|^2 \right\rangle \, dx \, dt + C(\epsilon, \tilde{\varrho}) \int_0^T \int_\Omega \left\langle \mathcal{V}_{t,x}; \left| \varrho - \tilde{\varrho} \right|^2 \right\rangle \, dx \, dt.
\end{align*}
\]
Remark 4.4. We proved the weak strong uniqueness result for $p$ our analysis quite general and physically relevant. Specifically, our choice of $\kappa$ and $\beta > 0$. The consideration of unbounded coefficients and a general pressure law makes stability of strong solutions within a large class of measure-valued solutions.

\[ \int_0^\tau \int_\Omega \nabla_x \tilde{\vartheta} \cdot \left( \nu_{t,x}; \left( \kappa(\varrho, \vartheta) - \kappa(\tilde{\varrho}, \tilde{\vartheta}) \right) \left( \frac{\nabla_x \tilde{\vartheta}}{\tilde{\vartheta}} - \frac{D\vartheta}{\varrho} \right) \right) \, dx \, dt \]

\[ \leq \epsilon \int_0^\tau \int_\Omega \nu_{t,x} \left( 1 + \tilde{\vartheta}^2 \right) \left| \frac{D\vartheta}{\varrho} - \frac{\nabla_x \tilde{\vartheta}}{\tilde{\vartheta}} \right|^2 \, dx \, dt \]

\[ + C(\epsilon, \tilde{\vartheta}) \left( \int_0^\tau \int_\Omega \left[ \nu_{t,x} \left( |\tilde{\vartheta}|^2 \right) \right] \, dx \, dt + \int_0^\tau \int_\Omega \nu_{t,x} \left[ |1 + \tilde{\vartheta}|^2 \right] \, dx \, dt \right). \]

Since $\beta \leq 2$, the second term is controlled by the radiation pressure $p_R$. Therefore, our choice of $p$ yields

\[ \int_0^\tau \int_\Omega \nu_{t,x} \left[ \vartheta^2 + |p(\varrho, \vartheta)| + |u - \tilde{u}| + \varrho |s_M(\varrho, \vartheta)| \right] \, dx \, dt \]

\[ \leq C(\epsilon) \int_0^\tau \mathcal{E}_{mv}(t) \, dt + \epsilon \int_0^\tau \int_\Omega \nu_{t,x} \left[ |u - \tilde{u}|^2 \right] \, dx \, dt \]

\[ + \int_0^\tau \int_\Omega \nu_{t,x} \left[ \varrho |s_M(\varrho, \vartheta)|^2 + \varrho s_R |u| \right] \, dx \, dt. \]

Furthermore, for any $\epsilon > 0$, we get the following inequality

\[ \varrho s_R |u| \leq C(\tilde{u})(\vartheta |u - \tilde{u}| + \varrho s_R) \leq C(\tilde{u}) \left( \frac{1}{4\epsilon} \vartheta^2 + \epsilon |u - \tilde{u}|^2 + \varrho s_R \right). \] (4.17)

On the other hand, the structural assumption on $p_M$ and $S_M$ gives us

\[ \varrho |s_M(\varrho, \vartheta)|^2 \leq C(1 + \varrho + \varrho e_M). \] (4.18)

Finally, we conclude that

\[ \int_0^\tau \int_\Omega \nu_{t,x} \left[ \vartheta^2 + |p(\varrho, \vartheta)| + |u - \tilde{u}| + \varrho |s_M(\varrho, \vartheta)| \right] \, dx \, dt \]

\[ \leq C(\epsilon) \int_0^\tau \mathcal{E}_{mv}(t) \, dt + 2\epsilon \int_0^\tau \int_\Omega \nu_{t,x} \left[ |u - \tilde{u}|^2 \right] \, dx \, dt. \]

Now proper choice of $\epsilon$, generalized Korn–Poincaré inequality (2.16) and Grönwall’s argument gives us the desired result. \( \square \)

**Remark 4.4.** We proved the weak strong uniqueness result for $\kappa(\vartheta) = \kappa_1 + \kappa_2 \vartheta^2$, $0 \leq \beta \leq 2$. This method does not allow us to prove the weak (measure valued)–strong uniqueness when $\beta > 2$.

**Discussion of the result** Indeed, our approach enables us to identify the stability of strong solutions within a large class of measure-valued solutions. The consideration of unbounded coefficients and a general pressure law makes our analysis quite general and physically relevant. Specifically, our choice of $p_R$ in (4.15) is motivated by models of Neutron stars, as discussed in Lattimer et al. [27].
5. Comments on measure valued solution

In this section, we discuss the existence of a measure-valued solution following Definition 2.2. Although we do not provide an exact construction for a measure valued solution, we present the following two important concepts:

- First, for a certain pressure law, with some assumptions on transport coefficients, we derive a priori estimates for the system.
- Next, we demonstrate that if a suitable and consistent approximation of the system satisfies certain uniform bounds, then we get compatibility conditions (2.9) and (2.10) along with a better characterization of the defect measure $D_{\tilde{\Theta}}$ in (2.14).

We assume that the pressure law follows (4.8) with (4.9) and (4.15) and the transport coefficients $\mu$, $\lambda$ and $\kappa$ follow

\begin{align}
0 < \mu (1 + \vartheta) &\leq \mu(\vartheta) \leq \bar{\mu} (1 + \vartheta), \quad |\mu'(\vartheta)| \leq c \text{ for all } \vartheta \geq 0, \\
0 &\leq \lambda(\vartheta) \leq \bar{\lambda} (1 + \vartheta), \\
0 < \kappa (1 + \vartheta^\beta) &\leq \kappa(\vartheta) \leq \bar{\kappa} (1 + \vartheta^\beta), \quad \beta \geq 2.
\end{align}

We also invoke the third law of thermodynamics (4.13).

5.1. A priori estimate

Let us first recall the ballistic energy inequality

\begin{align}
\frac{d}{dt} \int_\Omega \left[ \frac{1}{2} \vartheta |u|^2 + \varrho e - \tilde{\varrho} \varrho_s \right] dx + \int_\Omega \tilde{\varrho} \left( \mathbb{S} : \mathbb{D} u - \frac{q \cdot \nabla \vartheta}{\vartheta} \right) dx \\
\leq \int_\Omega \varrho u \cdot g dx - \int_\Omega \left[ \varrho_s \left( \partial_t \tilde{\varrho} + u \cdot \nabla \tilde{\varrho} \right) + \frac{q}{\vartheta} \cdot \nabla \tilde{\varrho} \right] dx,
\end{align}

where $\tilde{\varrho}$ is a smooth function such that $\tilde{\varrho} > 0$ in $(0, T) \times \Omega$ with $\tilde{\varrho} = \varrho_B$ on $\partial \Omega$.

In order to control the last integral in (5.2), we proceed analogously as in [11, Section 4.1]. Hence, we consider the extension $\hat{\varrho}$ to be the unique solution of the Laplace equation

$$
\Delta \hat{\varrho}(\tau, \cdot) = 0 \text{ in } \Omega, \quad \hat{\varrho}(\tau, \cdot)|_{\partial \Omega} = \varrho_B \text{ for any } \tau \in [0, T].
$$

The maximum principle for the Laplace equation gives

$$
\min_{[0,T] \times \partial \Omega} \varrho_B \leq \hat{\varrho}(t, x) \leq \max_{[0,T] \times \partial \Omega} \varrho_B \text{ for any } (t, x) \in (0, T) \times \Omega.
$$

Let us denote this particular extension by $\hat{\varrho_B}$. At first we note that

\begin{align*}
- \int_\Omega \frac{q}{\vartheta} \cdot \nabla_x \hat{\varrho_B} dx &= \int_\Omega \frac{\kappa(\tilde{\varrho})}{\vartheta} \nabla_x \tilde{\varrho} \cdot \nabla_x \hat{\varrho_B} dx = \int_\Omega \nabla_x K(\tilde{\varrho}) \cdot \nabla_x \hat{\varrho_B} dx \\
&= \int_{\partial \Omega} K(\varrho_B) \nabla_x \varrho_B \cdot n,
\end{align*}
where \( K'(\vartheta) = \frac{\kappa(\vartheta)}{\vartheta} \). Next, since \( \bar{\vartheta}B \) belongs to \( C^2([0, T] \times \overline{\Omega}) \) (due to \( \partial_B \in C^2([0, T] \times \partial\Omega) \)), we obtain
\[
- \int_{\Omega} \varrho s \partial_t \bar{\vartheta}B \, dx \leq \left[ 1 + \int_{\Omega} \left( \frac{1}{2} \varrho |u|^2 + \varrho e - \bar{\vartheta}gs \right) \, dx \right].
\]
We note that
\[
- \int_{\Omega} \varrho s u \cdot \nabla_x \bar{\vartheta}B \, dx = - \int_{\Omega} \varrho S \left( \frac{\varrho}{\vartheta} \right) u \cdot \nabla_x \bar{\vartheta}B \, dx - 2a \int_{\Omega} \varrho u \cdot \nabla_x \bar{\vartheta}B \, dx.
\]
Using the inequality in [23, Section 4, formula (4.6)], we have
\[
\varrho \left| S \left( \frac{\varrho}{\vartheta} \right) \right| \leq (\varrho + \varrho |\log(\vartheta)| + \varrho [\log(\vartheta)]^+) \quad \text{for any} \ \varrho \geq 0, \ \vartheta \geq 0.
\]
Consequently, it yields
\[
\left| \int_{\Omega} \varrho S \left( \frac{\varrho}{\vartheta} \right) u \cdot \nabla_x \bar{\vartheta}B \, dx \right| \leq \epsilon \left\| u \right\|_{W^{1,2}(\Omega; R^4)}^2 + c(\epsilon) \int_{\Omega} \varrho^2 \, dx.
\]
For the other term, we notice that
\[
\left| \int_{\Omega} \varrho u \cdot \nabla \bar{\vartheta}B \, dx \right| \leq \epsilon \left\| u \right\|_{W^{1,2}(\Omega; R^4)}^2 + c(\epsilon) \int_{\Omega} \varrho^2 \, dx.
\]
From the definition of the radiation pressure, we conclude
\[
\int_{\Omega} \varrho^2 \, dx \leq \int_{\Omega} (\varrho e - \bar{\vartheta}gs) \, dx + c(\vartheta_B).
\]
Therefore, choosing \( \epsilon \) properly, we are able to get
\[
\int_{\Omega} \left( \frac{1}{2} \varrho |u|^2 + \varrho e - \bar{\vartheta}gs \right) \, dx
+ \inf_{[0, T] \times \partial\Omega} \{ \vartheta_B \} \int_0^\tau \int_{\Omega} \left( \left\| u \right\|_{W^{1,2}(\Omega; R^4)}^2 + \frac{\kappa(\vartheta) |\nabla_x \vartheta|^2}{\vartheta^2} \right) \, dx \, dt
\leq \int_{\Omega} \left( \frac{1}{2} \varrho_0 |u_0|^2 + \varrho_0 e(\vartheta_0, \vartheta_0) - \bar{\vartheta}_0 s(\vartheta_0, \vartheta_0) \right) \, dx
+ c(\vartheta_B) \left[ 1 + \int_0^\tau \int_{\Omega} \left( \frac{1}{2} \varrho |u|^2 + \varrho e - \bar{\vartheta}gs \right) \, dx \, dt \right].
\]
Now we can use Grönwall’s argument to deduce a priori estimate for \( \frac{1}{2} \varrho |u|^2 + \varrho e - \bar{\vartheta}gs \).
Moreover for this particular pressure law (4.8) with (4.9) and (4.15) along with the third law of thermodynamics (4.13), following Feireisl and Březina [7], we are able to conclude that
\[
\esssup_{(0, T)} \| gs \|_{L^q(\Omega)} + \esssup_{(0, T)} \| gsu \|_{L^q(\Omega)} \leq C
\]
for some $q > 1$. Although they consider only the molecular pressure, the radiation pressure case is quite straightforward. Therefore, we obtain the following a priori estimate:

$$
\text{ess sup}_{(0, T)} \int_{\Omega} \left[ \varrho + \varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + |\varrho| s(\varrho, \vartheta)|^q \right] \, dx \\
+ \int_0^T \int_{\Omega} \left[ \mu \left( 1 + \frac{1}{\vartheta} \right) \left| \nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{d} \text{div}_x \mathbf{u} \right|^2 + \frac{\lambda(\vartheta)}{2\vartheta} |\text{div}_x \mathbf{u}|^2 \right] \, dx \, dt \\
+ \int_0^T \int_{\Omega} \kappa \left( \frac{1}{\vartheta^2} + \vartheta^{\beta-2} \right) |\nabla_x \vartheta|^2 \, dx \, dt \leq C(\vartheta_B),
$$

for some $q > 1$.

**Remark 5.1.** One can notice that the function $\varrho e - \tilde{\vartheta} \varrho s$ is not necessarily non-negative in general. However, as discussed in Sect. 3, the map $(\varrho, S) \mapsto \varrho e(\varrho, S) - \tilde{\vartheta} S$ is convex, where $S = \varrho s$. Consequently, if $\tilde{\vartheta}$ is bounded both above and below, we can deduce that $\varrho e - \tilde{\vartheta} \varrho s + A(\tilde{\vartheta})(\varrho) + B(\tilde{\vartheta}) > 0$, where $A(\tilde{\vartheta})$ and $B(\tilde{\vartheta})$ are constants dependent on $\tilde{\vartheta}$. Since we are dealing with a bounded domain $\Omega$ and have the mass balance, we can justify the application of Grönwall’s lemma.

### 5.2. Comments on compatibility conditions

Let us assume a Young measure $\mathcal{V}$ is generated by a family of sequences

$$
\{X_\varepsilon = (\varrho_\varepsilon, \mathbf{u}_\varepsilon, \mathcal{D}_x \mathbf{u}_\varepsilon, \vartheta_\varepsilon, \nabla_x \vartheta_\varepsilon)\}_{\varepsilon > 0}
$$

that satisfies the bound

$$
\text{ess sup}_{(0, T)} \int_{\Omega} \left[ \varrho_\varepsilon + \varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) + |\varrho_\varepsilon| s(\varrho_\varepsilon, \vartheta_\varepsilon)|^q \right] \, dx \\
+ \int_0^T \int_{\Omega} \left[ \frac{\mu(\varrho_\varepsilon, \vartheta_\varepsilon)}{2\vartheta_\varepsilon} \left| \nabla_x \mathbf{u}_\varepsilon + \nabla_x^T \mathbf{u}_\varepsilon - \frac{2}{d} \text{div}_x \mathbf{u}_\varepsilon \right|^2 + \frac{\mu(\varrho_\varepsilon, \vartheta_\varepsilon)}{2\vartheta_\varepsilon} |\text{div}_x \mathbf{u}_\varepsilon|^2 \right] \, dx \, dt \\
+ \int_0^T \int_{\Omega} \kappa(\varrho_\varepsilon, \vartheta_\varepsilon) \log \vartheta_\varepsilon \, dx \, dt \leq C,
$$

(5.3)

uniformly with respect to $\varepsilon$ and $\mu, \lambda$ and $\kappa$ satisfies (5.1) and $q > 1$. The bound (5.3) is motivated by the discussion in Sect. 5.1. One can consider $\{X_\varepsilon\}_{\varepsilon > 0}$ is a sequence of approximate solutions of the Navier–Stokes–Fourier system. Then following [8, Section 2.4], we are able to deduce the velocity and temperature compatibility. Also, Generalized Korn–Poincaré inequality can be derived in the similar way.

**Defect measures and its compatibility.** The bound (5.3), gives a uniform estimate of $\left( \frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) - \tilde{\vartheta} \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \right)_{\varepsilon > 0}$ with respect to $\varepsilon$ in $L^\infty(0, T; L^1(\Omega))$. We denote the weak* limit in $L_{\text{weak-}^*}^\infty(0, T; \mathcal{M}(\Omega))$ by

$$
\left( \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \tilde{\vartheta} \varrho s(\varrho, \vartheta) \right).
$$
Next, we define the corresponding defect measure as
\[
\mathcal{D}_{\tilde{\Theta}} = \left( \frac{1}{2} \rho |u|^2 + \rho e(\rho, \vartheta) - \tilde{\Theta} \rho s(\rho, \vartheta) \right)
- \left\langle \mathcal{V}_{t,x}; \left( \frac{1}{2} \rho |u|^2 + \rho e(\rho, \vartheta) - \tilde{\Theta} \rho s(\rho, \vartheta) \right) \right\rangle.
\]

Now from the estimate (5.3) for entropy, we are able to conclude
\[
\left\langle \mathcal{V}_{t,x}; \rho s(\rho, \vartheta) \right\rangle = \rho s(\rho, \vartheta) \text{ for a.e. } (t, x) \in (0, T) \times \Omega,
\]
where \( \rho s(\rho, \vartheta) \) is a weak limit of \( \rho \epsilon s_{\epsilon} \). This implies \( \mathcal{D}_{\tilde{\Theta}} \) is independent of \( \tilde{\Theta} \) and eventually we have
\[
\mathcal{D} = \left( \frac{1}{2} \rho |u|^2 + \rho e(\rho, \vartheta) \right) - \left\langle \mathcal{V}_{t,x}; \left( \frac{1}{2} \rho |u|^2 + \rho e(\rho, \vartheta) \right) \right\rangle.
\]

From the comparison lemma for defect measures in Feireisl et. al [17, Lemma 2.1], we can claim the compatibility of defects (2.15), since \( r^M \) is only related to the term \( \rho u \otimes u + p(\rho, \vartheta) I \). Therefore, this particular pressure law (4.8) allows us to have weak convergence for the terms \( \rho s \) and \( \rho s u \) in some \( L^q(0, T; L^q(\Omega)) \), for some \( q > 1 \). Hence, we avoid a defect measure coming from the terms \( \rho s \) and \( \rho s u \) in ballistic energy inequality.

Concluding remarks

Here we briefly discuss the possible extensions and limitations of the problem:

- We have given no proof of the existence of a measure valued solution. One way to prove existence is to use consistent approximations. By consistent approximation, we mean an approximated system of the original system with error terms that vanish in the limiting case. The approximate solutions with some uniform bounds of state variables generate a Young measure, which is eventually a measure valued solution of the system. For a detailed discussion, see [19, Chapter 5, Section 5.3.1] and [9].

- Another way to construct measure valued solution is to consider it as a weak limit of weak solutions. From [11], we know that there exists a weak solution for
\[
0 < \kappa (1 + \vartheta^\beta) \leq \kappa(\vartheta) \leq \overline{\kappa}(1 + \vartheta^\beta), \quad \beta > 6, \quad p_R = a \vartheta^4,
\]
Therefore, at least for system (1.1)–(1.8) with (4.14) and pressure following (4.9), (4.15), we expect to obtain a measure valued solution as a suitable limit of these weak solutions. If we consider a modified radiation pressure and the heat conductivity coefficient as
\[
p_{\epsilon,R} = a \vartheta^2 + \epsilon \vartheta^4
\]
and
\[
\kappa_{\epsilon}(\vartheta) = 1 + \vartheta^\beta + \epsilon \vartheta^\gamma, \quad \text{with } \beta \leq 2 \text{ and } \gamma > 6.
\]
Then one can expect to generate a measure value solution by sequence of weak solutions \( \{\rho_\epsilon, u_\epsilon, \vartheta_\epsilon\}_{\epsilon > 0} \).
• Navier-slip boundary condition for velocity: Instead of boundary condition (1.7), we could consider the Navier-Slip boundary condition

\[ u \cdot n |_{\partial \Omega} = [S(\nabla_x u) \cdot n]_{\text{tan}} |_{\partial \Omega} = 0. \]

We can provide a similar definition by modifying the Korn–Poincaré inequality suitably, see [10, Section 1.3.6]. A similar weak (measure valued)–strong uniqueness result is expected to be true.

• Limitation with radiation pressure following Stefan–Boltzman law: Unfortunately, we are not able to prove our results for the radiation pressure following Stefan–Boltzman law \( p_R(\vartheta) = a\vartheta^4 \) with \( a > 0 \). The main difficulty is to deal with the term \( [\varrho s_R | u]_{\text{res}} \). Following Feireisl [16], we notice that, the estimation of the term needs certain Sobolev embedding which is missing in our definition.

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Declarations

Conflict of interest The work is original to the best knowledge of the author and he has no other conflict of interest to declare.

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