Periods, cycles, and $L$-functions: a relative trace formula approach

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1. Two classical examples
2. Automorphic period and L-values
3. Special cycles and L-derivatives
4. Higher Gross–Zagier formula
5. Relative trace formula and arithmetic fundamental lemma
Two classical examples

The central theme of this talk

Period integral  Algebraic cycle  \( \rightarrow \)  L-functions

with an emphasis on the *relative trace formula* approach. We first discuss two examples

- Dirichlet’s solution to *Pell’s equation*, and two formulas of Dirichlet.
- Heegner’s solution to *elliptic curve*, and the formula of Gross–Zagier and of Birch–Swinnerton-Dyer.
Pell’s equation

\[ x^2 - dy^2 = \pm 1. \]

For simplicity, assume that \( d = p \equiv 1 \mod 4 \) is a prime. Dirichlet constructed an “explicit" triangular solution

\[
x + y \sqrt{p} = \theta_p \]

\[
= \frac{\prod_{a \not\equiv \square \mod p} \sin \frac{a\pi}{p}}{\prod_{b \equiv \square \mod p} \sin \frac{b\pi}{p}} \\
0 < a, b < p/2.
\]
Two formulas of Dirichlet

Let \( \left( \frac{\cdot}{p} \right) \) denote the Legendre symbol for quadratic residues. Let

\[
L \left( s, \left( \frac{\cdot}{p} \right) \right) = \sum_{n \geq 1, \ p \nmid n} \left( \frac{n}{p} \right) n^{-s}.
\]

Dirichlet’s first formula,

\[
L' \left( 0, \left( \frac{\cdot}{p} \right) \right) = \log \theta_p,
\]

and the second formula

\[
L' \left( 0, \left( \frac{\cdot}{p} \right) \right) = h_p \log \epsilon_p,
\]

where \( h_p \) is the class number and \( \epsilon_p > 1 \) is the fundamental unit of \( K = \mathbb{Q}[\sqrt{p}] \),
Modular parameterization of elliptic curves over $\mathbb{Q}$

- $E$: an elliptic curve over $\mathbb{Q}$.
- $\mathcal{H} = \{\tau \in \mathbb{C} : \text{Im}(\tau) > 0\}$ the upper half plane.
- $\exists$ a modular parameterization

$$\varphi : \mathcal{H} \longrightarrow E_{\mathbb{C}} .$$

modular functions
The elliptic curve

\[ E : y^2 = x^3 - 1728 \]

is parameterized by \((\gamma_2, \gamma_3)\):

\[
\gamma_2(\tau) = \frac{E_4}{\eta^8} = \frac{1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n}{q^{1/3} \prod_{n=1}^{\infty} (1 - q^n)^8},
\]

\[
\gamma_3(\tau) = \frac{E_6}{\eta^{12}} = \frac{1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n}{q^{1/2} \prod_{n=1}^{\infty} (1 - q^n)^{12}},
\]

where \(q = e^{2\pi i \tau}, \tau \in \mathcal{H}\).
Modular solution: Heegner point

- $K = \mathbb{Q}[\sqrt{-d}] \subset \mathbb{C}$: a (suitable) imaginary quadratic number field.
- Heegner point: some of $\varphi(K \cap \mathcal{H})$ produces $\mathcal{P}_K \in E(K)$.

- $L(s, E/K)$: the Hasse–Weil L-function of $E$ over $K$ (centered at $s = 1$).

**Theorem (Gross–Zagier formula (1980s))**

*There is an explicit $c > 0$ such that*

$$L'(1, E/K) = c \cdot \langle \mathcal{P}_K, \mathcal{P}_K \rangle_{NT}$$

*where the RHS is the Nerón–Tate height pairing.*
Conjecture of Birch and Swinnerton-Dyer (1960s)

- The order $r = \text{ord}_{s=1} L(s, E/Q)$ equals to $\text{rank } E(\mathbb{Q})$.
- the leading term of the Taylor expansion

\[
\frac{L^{(r)}(1, E/Q)}{r! \cdot c_E} = \#\Sha \cdot \text{Reg}(E)
\]

where
- $\Sha$: Tate–Shafarevich group.
- Reg($E$) is the regulator ($\sim$ the “volume” of the abelian group $E(\mathbb{Q})$ in $E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{R}$ w.r.t. the Néron–Tate metric).
- $c_E = \Omega_E \prod_{\ell \text{ prime}} c_\ell$, $\Omega_E$ is the real period, $c_\ell$ the number of connected components of the special fiber of Néron model at $\ell$. 
Theorem (Skinner, Z., ∼ ’14)

If \( \text{ord}_{s=1} L(s, E/\mathbb{Q}) = 3 \) (or any odd integer \( \geq 3 \)), then either

- \( \#\text{III} = \infty \), or
- \( \text{rank } E(\mathbb{Q}) \geq 3 \).
Two classical examples

Automorphic period and L-values

Special cycles and L-derivatives

Higher Gross–Zagier formula

Relative trace formula and arithmetic fundamental lemma
Automorphic period integral

1. $G$ reductive group over a global field $F$, and (spherical) $H \subset G$.
2. The automorphic quotients $[H] := H(F) \backslash H(\mathbb{A}) \rightarrow [G]$.
3. $\pi$: a (tempered) cuspidal automorphic repn. of $G$.

Automorphic period integral

$$\mathcal{P}_H(\phi) := \int_{[H]} \phi(h) dh, \quad \phi \in \pi.$$ 

Automorphic periods are often related to (special) values of $L$-functions, e.g. the Rankin–Selberg pair $(\text{GL}_{n-1}, \text{GL}_{n-1} \times \text{GL}_n)$. 
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Gan–Gross–Prasad pairs \((H, G)\)

- \(F'/F\): quadratic extension of number fields.
- \(W\): \(F'/F\)-Hermitian space, \(\dim_F W = n\).
- \(W^b \subset W\), codimension one, \(U(W^b) \subset U(W)\).
- Diagonal embedding

\[
H = U(W^b) \subset G = U(W^b) \times U(W).
\]

The pair \((H, G)\) is called the unitary Gan–Gross–Prasad pair. Similar construction applies to orthogonal groups.
Global Gan–Gross–Prasad conjecture

- \((H, G)\): the Gan–Gross–Prasad pair (unitary/orthogonal).
- \(\pi\): a tempered cusp. automorphic repn. of \(G\).
- \(L(s, \pi, R)\): the Rankin–Selberg L-function for the endoscopic functoriality transfer of \(\pi\).

**Conjecture (Gan–Gross–Prasad)**

The following are equivalent

1. The automorphic \(H\)-period integral does not vanish on \(\pi\), i.e., \(\mathcal{P}_H(\phi) \neq 0\) for some \(\phi \in \pi\).
2. \(L\left(\frac{1}{2}, \pi, R\right) \neq 0\) (and \(\text{Hom}_{H(\mathbb{A})}(\pi, \mathbb{C}) \neq 0\)).
The unitary Gan–Gross–Prasad pair

**Theorem**

Let \((H, G)\) be the unitary GGP pair. The conjecture holds if

\[
\text{there exists a place } v \text{ of } F \text{ split in } F' \text{ where } \pi_v \text{ is supercuspidal.}
\]

**Remark**

- The same holds for a refined GGP conjecture of Ichino–Ikeda.
- \(n = 2\) (i.e., \(G \cong U(1) \times U(2)\)): Waldspurger (1980s).
- \(n > 2\) : due to a series of work on Jacquet–Rallis relative trace formula by several authors: Yun, Beuzart-Plessis, Xue, and the author.
- Work in progress by Zydor and Chaudouard on the spectral side will remove the above local condition.
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Shimura datum: \((G, X_G)\)

- \(G\): (connected) reductive group over \(\mathbb{Q}\),
- \(X_G = \{h_G\}\): a \(G(\mathbb{R})\)-conjugacy class of \(\mathbb{R}\)-group homomorphisms \(h_G : \mathbb{C}^\times \rightarrow G_{\mathbb{R}}\), satisfying Deligne’s list of axioms (in particular, \(X_G\) is a *Hermitian symmetric domain*).
Examples of \((G_{\mathbb{R}}, X_G)\)

1. **Type A** \(G_{\mathbb{R}} = U(r, s)\) (for \(r + s = n\)) and \(X_G = \frac{U(r,s)}{U(r) \times U(s)}\). When \(r = 1\), \(X_G = D_{n-1} = \{ z \in \mathbb{C}^{n-1} : z \cdot \overline{z} < 1 \}\) is the unit ball.

2. **Type B, D** Tube domains: \(G_{\mathbb{R}} = SO(n, 2)\), \(X_G = \frac{SO(n,2)}{SO(n) \times SO(2)}\).

3. **Type C** Siegel upper half space \(\{ z \in \text{Symm}_{g \times g}(\mathbb{C}) : \text{Im}(z) > 0 \}\).
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A special pair of Shimura data is a homomorphism

\[(H, X_H) \rightarrow (G, X_G)\]

such that

1. the pair \((H, G)\) is spherical, and
2. the dimensions (as complex manifolds) satisfy

\[\dim \mathbb{C} X_H = \frac{\dim \mathbb{C} X_G - 1}{2}.\]

Example (Gross–Zagier pair)

Let \(F = \mathbb{Q}[(\sqrt{-d})]\) be an imaginary quadratic field. Let

\[H = \mathbb{R}_{F/\mathbb{Q}} \mathbb{G}_m \subset G = \text{GL}_2, \mathbb{Q}.

Then \(\dim X_G = 1, \dim X_H = 0.\)
Some more examples (over $\mathbb{R}$)

1. Gan–Gross–Prasad pairs

|                      | $G_\mathbb{R}$                          | $H_\mathbb{R}$                       |
|----------------------|-----------------------------------------|--------------------------------------|
| unitary groups       | $U(1, n - 2) \times U(1, n - 1)$        | $U(1, n - 2)$                        |
| orthogonal groups    | $SO(2, n - 2) \times SO(2, n - 1)$      | $SO(2, n - 2)$                       |

2. Symmetric pairs

|                      | $G_\mathbb{R}$                          | $H_\mathbb{R}$                       |
|----------------------|-----------------------------------------|--------------------------------------|
| unitary groups       | $U(1, 2n - 1)$                          | $U(1, n - 1) \times U(0, n)$        |
| orthogonal groups    | $SO(2, 2n - 1)$                         | $SO(2, n - 1) \times SO(0, n)$      |
We now focus on the unitary GGP pair \((H, G)\) that can be enhanced to a special pair of Shimura data.

- The *arithmetic diagonal cycle*

\[
\text{Sh}_H \longrightarrow \text{Sh}_G ,
\]

(for certain level subgroups \(K^o_H, K^o_G\)).

- \(\exists\) a *PEL* type variant of the GGP Shimura varieties, with *smooth* integral models \(\text{Sh}_H\) and \(\text{Sh}_G\) [Rapoport–Smithling–Z. ’17].

Define

\[
\text{Int}(f) = \left(f \ast [\text{Sh}_H], \ [\text{Sh}_H]\right)_{\text{Sh}_G} , \quad f \in \mathcal{H} (G, K^o_G) ,
\]

where the action is through the Hecke correspondence associated to certain \(f\) in the Hecke algebra \(\mathcal{H} (G, K^o_G)\).
One version of the arithmetic GGP conjecture

**Conjecture**

There is a decomposition

\[ \text{Int}(f) = \sum_{\pi} \text{Int}_{\pi}(f), \quad \text{for all } f \in \mathcal{H}(G, K_G^o), \]

- \( \pi \): cohomological automorphic repn. of \( G(\mathbb{A}) \),
- \( \text{Int}_{\pi} \): eigen-distribution for the spherical Hecke algebra \( \mathcal{H}^S(\tilde{G}) \) away from the set \( S \) of bad places, with eigen-character given by \( \pi \).

Moreover, if \( \pi \) is tempered, the following are equivalent

1. \( \text{Int}_{\pi} \neq 0. \)
2. \( L'(\frac{1}{2}, \pi, R) \neq 0 \) (and \( \text{Hom}_{\mathcal{H}_\infty}(\pi_\infty, \mathbb{C}) \neq 0 \)).
**Theorem (Gross–Zagier ’86, Yuan–S. Zhang–Z. ’12)**

When \( n = 2 \) (i.e., \( G = U(1) \times U(2) \)), the conjecture holds.

**Corollary**

Let \( F \) be a totally real number field, and \( \pi \) a cusp. automorphic repn. of \( \text{PGL}_2(\mathbb{A}_F) \) with \( \pi_\infty \) parallel weight two. Then

\[
\mathcal{L}'(1/2, \pi) \geq 0.
\]

**Question:** What about \( n \geq 3 \), i.e., when the Shimura variety is of dimension higher than one?
GGP, and Arithmetic GGP

Central value  \( 1^{st} \) central derivative

Waldspurger

\[ \downarrow \]

GGP

\[ \downarrow \]

Ichino–Ikeda

Gross–Zagier

\[ \downarrow \]

Arithmetic GGP
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Higher Gross–Zagier formula
(in positive equal char. case)

\[
\begin{array}{c}
\{ \text{Number fields} \} \\
\{ [F : \mathbb{Q}] < \infty \} \\
\{ \text{Function fields} \} \\
\{ [F : \mathbb{C}(t)] < \infty \}
\end{array}
\rightarrow
\begin{array}{c}
\{ \text{Function fields} \} \\
\{ [F : \mathbb{F}_q(t)] < \infty \}
\end{array}
\]
Higher Gross–Zagier formula
(in positive equal char. case)
Drinfeld Shtukas

- \( k = \mathbb{F}_q \), and \( X/k \) a curve.
- Shtukas of rank \( n \) with \( r \)-legs: for \( S \) over \( \text{Spec} k \)

\[
\text{Sht}_{\text{GL}_n, X}^r(S) = \left\{ \begin{array}{l}
\text{vector bundles } \mathcal{E} \text{ of rank } n \text{ on } X \times S \\
+ \text{simple modification } \mathcal{E} \rightarrow (\text{id} \times \text{Frob}_S)^* \mathcal{E} \\
\text{at } r\text{-marked points } x_i : S \rightarrow X, 1 \leq i \leq r
\end{array} \right\}
\]

\[
\text{Sht}_{\text{GL}_n, X}^r \\
\downarrow
\]

\[
X^r = X \times_{\text{Spec} k} \cdots \times_{\text{Spec} k} X \\
\underbrace{\text{r times}}_{\text{r times}}
\]
The special case $r = 0$, $G = \text{GL}_n$

$$\text{Sht}^r_G = \text{Bun}_G(k)$$

{ rank $n$ vector bundles over $X$}

Weil

$[G] = G(F) \backslash G(\mathbb{A})$

$H^0_c(\text{Sht}^r_G)$

$C_c^\infty([G])$
Fix an étale double covering $X' \to X$. We have a natural morphism

$$ \text{Sht}_{GL_{n/2}, X'} \to \text{Sht}_{GL_{n}, X}. $$

They have dimensions

$$ \frac{nr}{2}, \quad nr. $$

A technical simplification: we pass to $\text{PGL}_{n}$, then take base change to $(X')^r$:

$$ \theta^r : \text{Sht}_H \to \text{Sht}_{G} := \text{Sht}_G \times^r (X')^r $$

where

$$ H = R_{X'/X}(\text{GL}_{n/2})/\mathbb{G}_{m,X} \subset G = \text{PGL}_{n,X}. $$
Fix an étale double covering $X' \to X$. We have a natural morphism

$$
\xymatrix{ 
\text{Sht}_{GL_{n/2}, X'}^r \ar[r] & \text{Sht}_{GL_n, X}^r.
}
$$

They have dimensions

$$
\frac{nr}{2}, \quad nr.
$$

A technical simplification: we pass to $\text{PGL}_n$, then take base change to $(X')^r$:

$$
\theta^r : \text{Sht}_H^r \longrightarrow \text{Sht}_{G}^r := \text{Sht}_{G}^r \times_{X^r} (X')^r
$$

where

$$
H = R_{X'/X} (\text{GL}_{n/2})/ \mathbb{G}_{m, X} \subset G = \text{PGL}_{n, X}.
$$
Higher Gross–Zagier formula, $n = 2$

- Now $G = \text{PGL}_2$, and $\text{Sht}_G'$, for even integer $r \geq 0$.
- $V_r = H_c^{2r} \left( \text{Sht}_{\text{PGL}_2} \otimes_k k, \overline{Q}_\ell \right)$ has a spectral decomposition

$$V_r = \left( \bigoplus_{\pi} V_{r,\pi} \right) \oplus \text{"Eisenstein part"},$$

$\pi$: unramified cusp. automorphic repn. of $\text{PGL}_2(\mathbb{A})$.
- $L(s, \pi \chi')$ : the (normalized) base change $L$-function.

**Theorem (Yun–Z.)**

Let $Z_r \in V_r$ be the cycle class of Heegner–Drinfeld cycle, and $Z_{r,\pi} \in V_{r,\pi}$. Then

$$L^{(r)}(1/2, \pi \chi') = c \cdot \left( Z_{r,\pi}, Z_{r,\pi} \right),$$

where $(\cdot, \cdot)$ is the intersection pairing, and $c > 0$ is explicit.
A comparison with the number field case

1. When \( r = 0 \), the automorphic quotient space (versus \( \text{Bun}_n(\mathbb{F}_q) \))
   
   \[ [G] = G(F) \backslash G(\mathbb{A}). \]

2. When \( r = 1 \), Shimura variety (versus moduli of Shtukas)

\[ 
\begin{align*}
\text{Sh}_G & \quad \downarrow \\
\text{Spec}\mathbb{Z} & \quad \downarrow \\
\text{Sht}^r_{GL_n} & \quad \downarrow \\
X^r & = X \times_{\text{Spec} k} \cdots \times_{\text{Spec} k} X
\end{align*}
\]

\( r \) times
An indirect example: Faltings heights of CM abelian varieties

**Kronecker limit formula** for an imaginary quadratic field $K = \mathbb{Q}[\sqrt{-d}]$:

$$h_{\text{Fal}}(E_d) = -\frac{L'(0, \chi_{-d})}{L(0, \chi_{-d})} - \frac{1}{2} \log |d|,$$

where $E_d$ is an elliptic curve with complex multiplication by $O_K$.

**Colmez conjecture** generalizes the identity to CM abelian varieties.

An averaged version is recently proved by Yuan–S. Zhang and by Andreatta–Goren–Howard–Madapusi-Pera.
A summary

Central value    1\textsuperscript{st} derivative    \( r^n \) derivative

\textcolor{blue}{\textit{Waldspurger}} \quad \textcolor{red}{\textit{Gross–Zagier}} \quad \textcolor{green}{\textit{Higher G-Z}}

\textcolor{blue}{\textit{GGP}} \quad \textcolor{red}{\textit{Arithmetic GGP}} \quad \textcolor{green}{\textit{\ast\ast\ast}}

\textcolor{blue}{\textit{Ichino–Ikeda}}
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The basic strategy is to compare two relative trace formulas:

- one for the “geometric" side (intersection numbers of algebraic cycles),
- the other for the “analytic" side (L-values).

Below we consider the two cases

- Higher Gross–Zagier formula.
- GGP and its arithmetic version.
**Geometric side:** Let $f$ be an element in the spherical Hecke algebra $\mathcal{H}$. Set

$$\text{Int}_r(f) := \left( f \ast [\text{Sht}^r_{H}], \quad [\text{Sht}^r_{H}] \right)_{\text{Sht}_G^r}.$$

**Analytic side:** consider the triple $(G', H'_1, H'_2)$ where $G' = G = \text{PGL}_2$ and $H'_1 = H'_2$ are the diagonal torus $A$ of $\text{PGL}_2$.

$$\mathbb{J}(f, s) := \int_{[H'_1]} \int_{[H'_2]} K_f(h_1, h_2) |h_1 h_2|^s \eta(h_2) \, dh_1 \, dh_2, \quad s \in \mathbb{C}$$

where $\eta_{F'/F}$ is a quadratic character, and

$$K_f(x, y) := \sum_{\gamma \in G'(F)} f(x^{-1} \gamma y), \quad x, y \in G'(\mathbb{A}), f \in C_c^\infty(G'(\mathbb{A})).$$

Note that this is a weighted version of

$$\left( f \ast [\text{Sht}^0_{H'_1}], [\text{Sht}^0_{H'_2}] \right)_{\text{Sht}_G^0} = \left( f \ast [\text{Bun}_A(k)], [\text{Bun}_A(k)] \right)_{\text{Bun}_G(k)}.$$
Geometric RTF (over function fields)

Let

\[ J_r(f) = \left. \frac{d^r}{ds^r} \right|_{s=0} J(f, s). \]

The following key identity, which we may call a geometric RTF (in contrast to the arithmetic intersection numbers in the number field case (AGGP) below).

**Theorem (Yun–Z.)**

Let \( f \in \mathcal{H} \). Then

\[ I_r(f) = (-\log q)^{-r} J_r(f). \]

Informally we may state the identity as

\[
\left( f \ast [\text{Sh}_H^r], [\text{Sh}_H^r] \right)_{\text{Sh}_G^r} = \left. \frac{d^r}{ds^r} \right|_{s=0} \left( f_{s, \eta} \ast [\text{Sh}_A^0], [\text{Sh}_A^0] \right)_{\text{Sh}_G^0}.
\]
We now move to the number field case. Similarly, we define linear functionals on Hecke algebras:

- $\mathbb{I}(f)$ for the unitary GGP triple $(G, H, H)$, and
- $\mathbb{J}(f', s)$ for the Jacquet–Rallis triple $(G', H'_1, H'_2)$ where

\[
G' = \mathbb{R}_{F'/F}(GL_{n-1} \times GL_n)
\]
\[
H'_1 = \mathbb{R}_{F'/F}GL_{n-1}, \quad H'_2 = GL_{n-1} \times GL_n.
\]

Then we have an analogous RTF identity

**Theorem**

There is a natural correspondence (for nice test functions) $f \leftrightarrow f'$ such that

\[
\mathbb{I}(f) = \mathbb{J}(f', 0).
\]
An arithmetic intersection conjecture

Let

$$\partial \mathbb{J}(f') = \frac{d}{ds}\bigg|_{s=0} \mathbb{J}(f', s).$$

Recall we have defined an arithmetic intersection number

$$\text{Int}(f) = \left( f \ast [\text{Sh}_H], [\text{Sh}_H] \right)_{\text{Sh}_G}, \quad f \in \mathcal{H}(G, K_G^\circ).$$

Conjecture (Z. '12, Rapoport–Smithling–Z. ’17)

There is a natural correspondence (for nice test functions) $f \leftrightarrow f'$ such that

$$\text{Int}(f) = -\partial \mathbb{J}(f').$$
For nice $f'$, we have a decomposition as a sum of *relative characters* for the triple $(G', H_1', H_2')$

$$\mathcal{J}(f', s) = \sum_{\Pi} \mathcal{J}_{\Pi}(f', s),$$

and, for cuspidal $\Pi$, a factorization into certain *local relative characters*

$$\mathcal{J}_{\Pi}(f', s) = 2^{-2} \mathcal{L}(s + 1/2, \pi) \prod_{\nu} \mathcal{J}_{\Pi_{\nu}}(f'_{\nu}, s).$$
Passing to the local situation

\[
\begin{align*}
\{ \text{Number fields} & \} & \{ \text{Function fields} & \} \\
[F : \mathbb{Q}] < \infty & \quad & [F : \mathbb{F}_q(t)] < \infty \\
\downarrow & & \downarrow \\
\{ \text{\(p\)-adic fields} & \} & \{ \text{local function fields} & \} \\
[F : \mathbb{Q}_p] < \infty & & [F : \mathbb{F}_q((t))] < \infty
\end{align*}
\]
$F'/F$: an unramified quadratic extension of $p$-adic fields.

$X_n$: $n$-dim’l *Hermitian supersingular formal* $O_{F'}$-*modules of signature* $(1, n−1)$ (unique up to isogeny).

$\mathcal{N}_n$: the unitary Rapoport–Zink formal moduli space over $\text{Spf}(O_{\mathbb{F}})$ (parameterizing “deformations" of $X_n$).

The group $\text{Aut}^0(X_n)$ is a unitary group in $n$-variable and acts on $\mathcal{N}_n$.

The $\mathcal{N}_n$’s are non-archimedean analogs of Hermitian symmetric domains. They have a “skeleton" given by a union of Deligne–Lusztig varieties for unitary groups over finite fields.
A natural closed embedding $\delta : \mathcal{N}_{n-1} \to \mathcal{N}_n$, and its graph

$$\Delta : \mathcal{N}_{n-1} \longrightarrow \mathcal{N}_{n-1,n} = \mathcal{N}_{n-1} \times_{\text{Spf} \mathcal{O}_F} \mathcal{N}_n.$$ 

Denote by $\Delta_{\mathcal{N}_{n-1}}$ the image of $\Delta$.

The group $G(F) := \text{Aut}^0(\mathbb{X}_{n-1}) \times \text{Aut}^0(\mathbb{X}_n)$ acts on $\mathcal{N}_{n-1,n}$. For (nice) $g \in G(F)$, we define the intersection number

$$\text{Int}(g) = \left( \Delta_{\mathcal{N}_{n-1}}, g \cdot \Delta_{\mathcal{N}_{n-1}} \right)_{\mathcal{N}_{n-1,n}}$$

$$: = \chi \left( \mathcal{N}_{n-1,n}, \mathcal{O}_{\Delta_{\mathcal{N}_{n-1}}} \otimes^L \mathcal{O}_{g \cdot \Delta_{\mathcal{N}_{n-1}}} \right).$$
The arithmetic fundamental lemma (AFL) conjecture

Define a family of (weighted) orbital integrals:

\[ \text{Orb} \left( \gamma, 1_{\text{gl}_n(O_F)}, s \right) = \int_{\text{GL}_{n-1}(F)} 1_{\text{gl}_n(O_F)}(h^{-1}\gamma h) \left| \det(h) \right|^{s} (-1)^{\text{val}(\det(h))} dh. \]

This serves as the local version of the analytic RTF. Then the local version of the global “arithmetic intersection conjecture” is

**Conjecture (Z. '12)**

Let \( \gamma \in \text{gl}_n(F) \) match an element \( g \in G(F) \). Then

\[ \pm \frac{d}{ds} \bigg|_{s=0} \text{Orb} \left( \gamma, 1_{\text{gl}_n(O_F)}, s \right) = -\text{Int}(g) \cdot \log q. \]
The status

Theorem (Z. ’12)

*The AFL conjecture holds when* \( n \leq 3 \).

A simplified proof when \( p \geq 5 \) is given by Mihatsch. For \( n > 3 \), we only have some partial results.

Theorem (Rapoport–Terstiege–Z. ’13)

*When* \( p \geq \frac{n}{2} + 1 \), *the AFL conjecture holds for minuscule elements* \( g \in G(F) \).

A simplified proof is given by Chao Li and Yihang Zhu.
Non-archimedean ramified $F'/F$ (Rapoport–Smithling–Z. ’15, ’16): an arithmetic transfer (AT) conjecture, and the case $n \leq 3$ is proved.

Question: what about archimedean $F'/F$?
Ramified quadratic extension $F'/F$

- Non-archimedean ramified $F'/F$ (Rapoport–Smithling–Z. ’15, ’16): an arithmetic transfer (AT) conjecture, and the case $n \leq 3$ is proved.
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