Limit sets of cyclic quaternionic Kleinian groups

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Abstract
In this paper, we consider the natural action of $\text{SL}(3, \mathbb{H})$ on the quaternionic projective space $\mathbb{P}^2_{\mathbb{H}}$. Under this action, we investigate limit sets for cyclic subgroups of $\text{SL}(3, \mathbb{H})$. We compute two types of limit sets, which were introduced by Kulkarni and Conze-Guivarc’h, respectively.

Keywords Quaternions · Projective transformations · Kleinian groups · Kulkarni limit sets · Conze-Guivarc’h limit sets

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1 Introduction
Classically, the Kleinian groups arise as discrete subgroups of $\text{SL}(2, \mathbb{C})$ acting on the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$, which is the boundary of the three-dimensional hyperbolic space. The action is in terms of the Möbius transformations. The Kleinian groups played a significant role in several areas of mathematics due to their algebraic, geometric, and dynamical significance, see [5, 13]. The limit sets are the starting points toward understanding the Kleinian group’s dynamics. A Kleinian group acts properly discontinuously on the complement of its limit set. In the classical setup, the structure of the limit sets analogously related to the holomorphic dynamics of iterated functions of the Riemann sphere via the Sullivan dictionary, cf. [14]. In an attempt to generalise the Sullivan dictionary in the higher dimension, Seade and Verjovsky [20–22] initiated the investigation of limit sets of discrete subgroups of $\text{SL}(3, \mathbb{C})$ that act on the two-dimensional complex projective space $\mathbb{P}^2_{\mathbb{C}}$. The theory of complex Kleinian group is also closely related to the understanding of discrete subgroups acting on the two-dimensional complex hyperbolic space because of the embedding of $\text{SU}(2, 1)$ inside $\text{SL}(3, \mathbb{C})$. A com-
plex Kleinian group means a discrete subgroup of PSL(3, C) with a non-empty domain of discontinuity. It is a problem of broad interest to obtain a maximal domain of discontinuity for a complex Kleinian group. On the other hand, in [12], Kulkarni developed the notion of a generalised limit set for any group acting on a locally compact Hausdorff topological space. Cano et al. have systematically investigated the Kulkarni limit sets and the Kulkarni domain of discontinuity for a wide class of complex Kleinian groups, see [8–10, 16].

The aim of this paper is to initiate an investigation of “quaternionic Kleinian groups”, that is, discrete subgroups of PSL(3, H) with a non-empty domain of discontinuity. A starting point to do that is to understand the limit sets of the cyclic subgroups of PSL(3, H). In this paper, we compute the Kulkarni limit sets of cyclic subgroups of PSL(3, H). We have worked following the spirit of Navarette [17] who computed the Kulkarni limit sets of cyclic subgroups of PSL(3, C).

We shall describe the limit sets in three tables. Before that, in the following, we recall the notion of the Kulkarni limit set.

**Definition 1.1** Let $P_X = \{ A_\beta \mid \beta \in B \}$ be a family of subsets of $X$, where $B$ is an infinite indexing set. A point $p$ in $X$ is called a cluster point of $P_X$ if every neighbourhood of $p$ intersects $A_\beta$ for infinitely many $\beta \in B$.

Consider the natural action of a subgroup $G$ of SL(3, H) on the two-dimensional quaternionic projective space $X = \mathbb{P}^2_H$. The isotropy subgroup of any $x \in X$ is defined as $G_x = \{ g \in G \mid gx = x \}$. Consider the following three sets

(a) $L_0(G) :=$ the closure of the set of points of $X$ which have an infinite isotropy group,
(b) $L_1(G) :=$ the closure of the cluster points of orbits of points in $X \setminus L_0(G)$, and
(c) $L_2(G) :=$ the closure of the cluster points of $\{ g(K) \}_{g \in G}$, where $K$ runs over all the compact subsets of $X \setminus \{ L_0(G) \cup L_1(G) \}$.

**Definition 1.2** With $G$ and $X$ as defined above, the Kulkarni limit set of $G$ is

$$\Lambda(G) := L_0(G) \cup L_1(G) \cup L_2(G). \quad (1.1)$$

The Kulkarni domain of discontinuity of $G$ is defined as $\Omega(G) = X \setminus \Lambda(G)$.

We remark that the Definition 1.2 is valid for any group $G$ acting on a locally compact Hausdorff space $X$ and Kulkarni proved the following proposition in [12].

**Proposition 1.3** (cf. [12, Proposition 1.3]) Let $X$ be a locally compact Hausdorff space and $G$ be a group acting on $X$, then $G$ acts properly discontinuously on $\Omega(G)$. Also, $\Omega(G)$ is a $G$-invariant open subset of $X$. Further, if $\Omega(G) \neq \phi$, then $G$ is discrete.

**Definition 1.4** A discrete subgroup $G \subset$ PSL(3, H) is called quaternionic Kleinian if there exists a non-empty open invariant set where the action is properly discontinuous, i.e., $\Omega(G) \neq \phi$.

Since SL(3, H) is a double cover of PSL(3, H) by $\{ \pm I_3 \}$, we often lift an element from the projective group PSL(3, H) to SL(3, H) and will consider its matrix representation. To classify the limit sets, we pick up a Jordan form in SL(3, H) and compute the limit set for the cyclic subgroup generated by that Jordan form. Note that each eigenvalue class of a
quaternionic matrix contains a unique complex representative with non-negative imaginary part. This gives us the following classification of elements of SL(3, \mathbb{H}) into three mutually exclusive classes.

**Definition 1.5** Let \( g \) be an element in SL(3, \mathbb{H}).

(i) \( g \) is called elliptic if it is semisimple and the eigenvalue classes are represented by unit modulus complex numbers. In other words, the complex representatives of the eigenvalues are \( e^{i\alpha}, e^{i\beta}, e^{i\gamma} \), where \( \alpha, \beta, \gamma \in [0, \pi] \).

(ii) \( g \) is called loxodromic if not all the eigenvalue classes are represented by unit modulus complex numbers.

(iii) \( g \) is called parabolic if it is not semisimple, and the eigenvalue classes are represented by unit modulus complex numbers.

The above classes can be divided further into several subclasses: an elliptic element is rational elliptic if all of \( \alpha, \beta, \gamma \) are rational. A loxodromic element is regular loxodromic if all its eigenvalues have distinct moduli. If all the eigenvalues of a parabolic element are 1, we call it unipotent. Otherwise, we call it ellipto-parabolic or ellipto-translation according if all its eigenvalues have distinct moduli. If all the eigenvalues of a parabolic element are 1, we call it unipotent. Otherwise, we call it ellipto-parabolic or ellipto-translation according

We have summarized the Kulkarni limit sets for the cyclic subgroups of PSL(3, \mathbb{H}) in Tables 1–3.

There is another notion of a limit set for the action of a discrete subgroup on a linear space that was introduced by Conze and Guivarc’h, cf. [6]. The notion of Conze-Guivarc’h limit set is inspired by the ideas of proximal transformations introduced by Abels, Margulis, and Soifer in [1].

**Definition 1.6** An element \( g \) in SL(3, \mathbb{H}) is called proximal if it has a maximal norm eigenvalue. In the case of PSL(3, \mathbb{H}) we say that an element is proximal if its lift is proximal.

| Kulkarni Set            | \( L_0(G) \) | \( L_1(G) \) | \( L_2(G) \) | \( \Lambda(G) \) |
|-------------------------|--------------|--------------|--------------|-----------------|
| **Rational elliptic**   | \( \phi \)   | \( \phi \)   | \( \phi \)   | \( \phi \)      |
| **Simple irrational elliptic** | \( \mathbb{P}_C^2 \) | \( \mathbb{P}_H^2 \) | \( \phi \) | \( \mathbb{P}_H^2 \) |
| **Compound irrational elliptic** | set of fixed points | \( \mathbb{P}_H^2 \) | \( \phi \) | \( \mathbb{P}_H^2 \) |

| Kulkarni Set               | \( L_0(G) \) | \( L_1(G) \) | \( L_2(G) \) | \( \Lambda(G) \) |
|---------------------------|--------------|--------------|--------------|-----------------|
| **Regular loxodromic**    | \{e₁, e₂, e₃\} | \{e₁, e₂, e₃\} | \( \mathbb{L}[e₁, e₂] \cup \mathbb{L}[e₂, e₃] \) | \( \mathbb{L}[e₁, e₂] \cup \mathbb{L}[e₂, e₃] \) |
| **Screw**                 | \{e₁, e₂, e₃\} | \( \mathbb{L}[e₁, e₂] \cup \{e₃\} \) | \( \mathbb{L}[e₁, e₂] \cup \{e₃\} \) | \( \mathbb{L}[e₁, e₂] \cup \{e₃\} \) |
| **Homothety**             | set of fixed points | \( \mathbb{L}[e₁, e₂] \cup \{e₃\} \) | \( \mathbb{L}[e₁, e₂] \cup \{e₃\} \) | \( \mathbb{L}[e₁, e₂] \cup \{e₃\} \) |
| **Loxo-parabolic**        | \{e₁, e₃\} | \{e₁, e₃\} | \( \mathbb{L}[e₁, e₂] \cup \mathbb{L}[e₁, e₃] \) | \( \mathbb{L}[e₁, e₂] \cup \mathbb{L}[e₁, e₃] \) |
on the dual space of $P$ elements. In [4], Barrera et al. considered the action of discrete subgroups of $PSL$ for complex Kleinian groups. These have been further studied in [19]; also see [8] and the references therein. We now extend the definition from [4] to the quaternionic setup.

**Definition 1.7** Let $G \subset PSL(3, \mathbb{H})$ that contains proximal elements and whose action on $\mathbb{P}^2_{\mathbb{H}}$ is strongly irreducible, i.e., there does not exist any proper non-zero subspace of $\mathbb{P}^2_{\mathbb{H}}$ invariant under the action of a subgroup of finite index in $G$. The Conze-Guivarc’h limit set is the closure of the subset of $\mathbb{P}^2_{\mathbb{H}}$ consisting of all the attracting fixed points of proximal elements of $G$.

Note that the Conze-Guivarc’h limit set is only applicable to those discrete subgroups of $PSL(3, \mathbb{H})$ with proximal elements, but not every discrete subgroup contains proximal elements. In [4], Barrera et al. considered the action of discrete subgroups of $PSL(3, \mathbb{C})$ on the dual space of $\mathbb{P}^2_{\mathbb{C}}$ and introduced the notion of extended Conze-Guivarc’h limit set for complex Kleinian groups. These have been further studied in [19]; also see [8] and the references therein. We now extend the definition from [4] to the quaternionic setup.

**Definition 1.8** Let us consider $G \subset PSL(3, \mathbb{H})$, acting on the dual space $(\mathbb{P}^2_{\mathbb{H}})^*$ which is the space of lines in $\mathbb{P}^2_{\mathbb{H}}$. We say that $q \in (\mathbb{P}^2_{\mathbb{H}})^*$ is a limit point of $G$ if there exists an open subset $U \subset (\mathbb{P}^2_{\mathbb{H}})^*$ and there exists a sequence $\{g_n\} \subset G$ of distinct elements such that for every $p \in U$, $\lim_{n \to \infty} g_n \cdot p = q$. The set of limit points will be called the extended Conze-Guivarc’h limit set, denoted by $\hat{L}(G)$.

In Sect. 6, we have discussed the extended Conze-Guivarc’h limit set for cyclic subgroups of $PSL(3, \mathbb{H})$. Table 4 gives the extended Conze-Guivarc’h limit sets for cyclic subgroups of $PSL(3, \mathbb{H})$.

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Table 4  Extended Conze-Guivarch limit sets for cyclic subgroups of $\text{PSL}(3, \mathbb{H})$

| Elliptic                        | $\hat{L}(G)$ | Parabolic                  | $\hat{L}(G)$ | Loxodromic                | $\hat{L}(G)$ |
|--------------------------------|--------------|----------------------------|--------------|----------------------------|--------------|
| Rational elliptic              | $\phi$       | Vertical translation       | $\mathbb{L}\{e_1, e_3\}$ | Regular loxodromic         | $\mathbb{L}\{e_1, e_2\} \cup \mathbb{L}\{e_2, e_3\}$ |
| Simple irrational elliptic     | $(\mathbb{P}^2\mathbb{H})^*$ | Non-vertical translation  | $\mathbb{L}\{e_1, e_2\}$ | Screw                      | $\mathbb{L}\{e_1, e_2\}$ |
| Compound irrational elliptic   | $(\mathbb{P}^2\mathbb{H})^*$ | Ellipto-parabolic         | $\mathbb{L}\{e_1, e_3\}$ | Homothety                  | $\mathbb{L}\{e_1, e_2\}$ |
|                                |              | Ellipto-translation       | $\mathbb{L}\{e_1, e_2\}$ | Loxo-parabolic              | $\mathbb{L}\{e_1, e_2\} \cup \mathbb{L}\{e_1, e_3\}$ |
1.1 Structure of the paper

We discuss the preliminaries in the Sect. 2. The third, fourth, and fifth sections are devoted to proving our results about the Kulkarni limit sets for cyclic subgroups of $\text{PSL}(3, \mathbb{H})$ generated by elliptic, loxodromic and parabolic elements, respectively. In the Sect. 6, we have studied the extended Conze-Guivarc’h limit sets for cyclic subgroups of $\text{PSL}(3, \mathbb{H})$ and proved Table 4.

2 Preliminaries

Let $\mathbb{H}$ denote the division ring of Hamilton’s quaternions. We recall that every element in $\mathbb{H}$ can be expressed as $a = a_0 + a_1i + a_2j + a_3k$, where $i^2 = j^2 = k^2 = ijk = -1$, and $a_0, a_1, a_2, a_3 \in \mathbb{R}$. The conjugate of $a$ is given by $\bar{a} = a_0 - a_1i - a_2j - a_3k$. We identify the real subspace $\mathbb{R} \oplus \mathbb{R}i$ with the usual complex plane $\mathbb{C}$. For an elaborate discussion on the theory of matrices over the quaternions, see [18], [23].

Definition 2.1 Let $A$ be an arbitrary element of the algebra $M(n, \mathbb{H})$ of all $n \times n$ matrices over $\mathbb{H}$. A non-zero vector $v \in \mathbb{H}^n$ is said to be a (right) eigenvector of $A$ corresponding to a (right) eigenvalue $\lambda \in \mathbb{H}$ if the equality $Av = v\lambda$ holds.

Eigenvalues of $A$ occur in similarity classes, i.e., if $v$ is an eigenvector corresponding to the eigenvalue $\lambda$, then $v\mu \in v\mathbb{H}$ is an eigenvector corresponding to the eigenvalue $\mu^{-1}\lambda\mu$. Each similarity class of eigenvalues contains a unique complex number with non-negative imaginary part. Here we shall represent each similarity class of eigenvalues by the unique complex representative with non-negative imaginary part and refer to them as eigenvalues.

Let $A \in M(n, \mathbb{H})$ and write $A = A_1 + A_2 j$, where $A_1, A_2 \in M(n, \mathbb{C})$. Now consider the embedding $\Phi : M(n, \mathbb{H}) \longrightarrow M(2n, \mathbb{C})$ defined as

$$\Phi(A) = \left(\begin{array}{cc} A_1 & A_2 \\ -\bar{A}_2 & \bar{A}_1 \end{array}\right),$$

where $\bar{A}_i$ denotes the complex conjugate of $A_i$.

Definition 2.2 For $A \in M(n, \mathbb{H})$, the determinant of $A$ is denoted by $\det_{\mathbb{H}}(A)$ and is defined as the determinant of the corresponding matrix $\Phi(A)$, i.e., $\det_{\mathbb{H}}(A) \equiv \det(\Phi(A))$, see [18, §5.9].

Consider the Lie groups $\text{GL}(n, \mathbb{H}) = \{g \in M(n, \mathbb{H}) \mid \det_{\mathbb{H}}(g) \neq 0\}$ and $\text{SL}(n, \mathbb{H}) = \{g \in \text{GL}(n, \mathbb{H}) \mid \det_{\mathbb{H}}(g) = 1\}$.

Definition 2.3 (cf. [18, p. 94]) A Jordan block $J(\lambda, m)$ is a $m \times m$ matrix with $\lambda$ on the diagonal entries, 1 on all of the super-diagonal entries, and zero elsewhere. We will refer to a block diagonal matrix where each block is a Jordan block as Jordan form.

Next we recall the Jordan decomposition in $M(n, \mathbb{H})$, see [18, Theorem 5.5.3].

Lemma 2.4 (Jordan forms in $M(n, \mathbb{H})$, cf. [18]) For every $A \in M(n, \mathbb{H})$ there is an invertible matrix $S \in \text{GL}(n, \mathbb{H})$ such that $SAS^{-1}$ has the form

$$SAS^{-1} = J(\lambda_1, m_1) \oplus \cdots \oplus J(\lambda_k, m_k)$$

(2.1)

where $\lambda_1, \ldots, \lambda_k$ are complex numbers (not necessarily distinct) and have non-negative imaginary parts. The form (2.1) is uniquely determined by $A$ up to a permutation of Jordan blocks.
2.1 The Quaternionic Projective Space $P^2_H$

Consider the (right) quaternionic space $\mathbb{H}^3$. The two-dimensional quaternionic projective space $P^2_H$ is formed by taking the quotient of $\mathbb{H}^3 \setminus \{0\}$ under the equivalence relation $\sim$ in $\mathbb{H}^3 \setminus \{0\}$, where $z \sim w \iff z = w \alpha$ for some non-zero quaternion $\alpha$. Let $P : \mathbb{H}^3 \setminus \{0\} \to P^2_H$ be the corresponding quotient map. A non-empty set $W \subseteq P^2_H$ is said to be a projective subspace of dimension $k$ if there exists a $(k + 1)$-dimensional $\mathbb{H}$-linear subspace $\widetilde{W}$ of $\mathbb{H}^3$ such that $P(\widetilde{W} \setminus \{0\}) = W$. The quaternionic projective subspaces of dimension 1 are called lines. Let $p = (x, y, z) \in \mathbb{H}^3$ then we will use notation $[x : y : z]$ to denote $P(p)$. Note that $[x : y : z] = [x \alpha : y \alpha : z \alpha]$ for all $(x, y, z) \in \mathbb{H}^3$ and non-zero quaternion $\alpha \in \mathbb{H}$. Given a set of points $S$ in $P^2_H$, we define

$$\langle S \rangle = \bigcap \{W \subseteq P^2_H : W \text{ is a projective subspace containing } S\}.$$ Clearly, $\langle S \rangle$ is a projective subspace of $P^2_H$. If $p, q$ are distinct points of $P^2_H$ then $\langle \{p, q\} \rangle$ is the unique proper quaternionic projective subspace passing through $p$ and $q$. Such a subspace will be called a quaternionic (projective) line, and we will denote $\langle \{p, q\} \rangle$ by $L(p, q)$.

2.2 Projective Transformations

Action of an arbitrary element $\gamma \in GL(3, \mathbb{H})$ on $P^2_H$ is given by

$$\gamma(P(z)) = P(\gamma(z)) \text{ for all } z \in \mathbb{H}^3 \setminus \{0\}.$$ Note that for any non-zero $r \in \mathbb{R}$, we have $(r \gamma)(P(z)) = \gamma(P(z))$ for all $z \in \mathbb{H}^3 \setminus \{0\}$. We denote the corresponding quotient map by $\pi : GL(3, \mathbb{H}) \to PGL(3, \mathbb{H})$. The group of projective transformations of $P^2_H$ may be identified with

$$\text{PSL}(3, \mathbb{H}) := \text{SL}(3, \mathbb{H}) / Z(\text{SL}(3, \mathbb{H})), $$
where the center $Z(\text{SL}(3, \mathbb{H})) = \{\pm I_3\}$ and it acts by the usual scalar multiplication on $\mathbb{H}^3$. Given $\tilde{\gamma} \in \text{PSL}(3, \mathbb{H})$, we say that $\gamma \in \text{SL}(3, \mathbb{H})$ is a lift of $\tilde{\gamma}$ if $\pi(\gamma) = \tilde{\gamma}$. There are exactly two lifts $\tilde{\gamma}, -\tilde{\gamma} \in \text{SL}(3, \mathbb{H})$ for each projective transformation $\tilde{\gamma} \in \text{PSL}(3, \mathbb{H})$.

Let $\tilde{g} \in \text{PSL}(3, \mathbb{H})$ be a projective transformation, then it is called elliptic, loxodromic and parabolic according as its lift $g \in \text{SL}(3, \mathbb{H})$, see Definition 1.5. Note that this classification is well defined in view of the Jordan decomposition of matrices over the quaternions, see Lemma 2.4.

2.3 Pseudo-projective Transformations

The concept of pseudo-projective transformations was introduced by P. J. Myrberg [15], see [3] for a brief explanation. This has been very useful for understanding the complex Kleinian groups. We extend this notion over the quaternions here.

Let $\tilde{M} : \mathbb{H}^3 \to \mathbb{H}^3$ be a non-zero (right) linear transformation which is not necessarily invertible. Let $\text{Ker}(\tilde{M}) \subseteq \mathbb{H}^3$ be its kernel. Then $\tilde{M}$ induces a well-defined transformation $M : P^2_H \setminus \text{Ker}(M) \to P^2_H$ given by $M(P(v)) = P(\tilde{M}(v))$, where $\text{Ker}(M) \subseteq P^2_H$ is the image of $\text{Ker}(\tilde{M}) \setminus \{0\}$ under the projective map $P$. Note that $M$ is well defined because $\tilde{M}(v) \neq 0$, and $M$ is a projective transformation on its domain: for every non-zero quaternion $\alpha \in \mathbb{H}$, $M(P(\alpha \alpha)) = P(\tilde{M}(\alpha \alpha))$ and coincide with $P(\tilde{M}(v))$ in $P^2_H$.
We call the map \( M \) a pseudo-projective transformation, and we denote by \( \text{QP}(3, \mathbb{H}) \) the space of all pseudo-projective transformations of \( \mathbb{H}^2 \). Therefore,

\[
\text{QP}(3, \mathbb{H}) = \{ M \mid \tilde{M} \text{ is a non-zero linear transformation of } \mathbb{H}^3 \}.
\]

Note that \( \text{PSL}(3, \mathbb{H}) \subseteq \text{QP}(3, \mathbb{H}) \). We will use pseudo-projective transformations and classification of the elements of the lifts of \( \text{PSL}(3, \mathbb{H}) \) (see Definition 1.5) for computing the limit sets for different cyclic subgroups of \( \text{PSL}(3, \mathbb{H}) \).

### 2.4 Dense subsets of \( S^1 \) and \( T^2 \)

We state without proof some of the important well-known results of unit circle and torus.

**Lemma 2.5** (cf. [11, Proposition 1.3.3]) Let \( S^1 = \{ x \in \mathbb{C} \mid |x| = 1 \} \) and \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \). Then the set \( A = \{ e^{2\pi in\alpha} \in S^1 \mid n \in \mathbb{Z} \} \) is dense in \( S^1 \).

Note that for each \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \), there exists a sequence \( (n_k)_{k \in \mathbb{N}} \) such that \( e^{2\pi in_k\alpha} \) converges to 1 as \( k \to \infty \). Further, if \( r \in \mathbb{Q} \) then we can also find a suitable sequence \( (n_k)_{k \in \mathbb{N}} \) such that \( (e^{2\pi in_k\alpha}, e^{2\pi inkr\alpha}) \) converges to \( (1, 1) \) as \( k \to \infty \).

**Lemma 2.6** (cf. [11, Proposition 1.4.1]) Let \( T^2 = S^1 \times S^1 \) and \( \alpha, \beta \in \mathbb{R} \) such that \( \frac{\alpha}{\beta} \in \mathbb{R} \setminus \mathbb{Q} \). Then the set \( A = \{ e^{2\pi in\alpha}, e^{2\pi in\beta} \in T^2 \mid n \in \mathbb{Z} \} \) is dense in \( T^2 \).

**Remark 2.7** It follows that if \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) and \( \beta \in \mathbb{R} \) then there exists a sequence \( (n_k)_{k \in \mathbb{N}} \) such that \( (e^{2\pi in_k\alpha}, e^{2\pi in_k\beta}) \xrightarrow{k \to \infty} (1, 1) \). We can also deduce that there exists a sequence \( (n_k)_{k \in \mathbb{N}} \) such that \( (e^{2\pi i\alpha n_k}, e^{2\pi i\beta n_k}, e^{2\pi i\gamma n_k}) \xrightarrow{k \to \infty} (1, 1, 1) \) whenever \( \alpha, \beta, \gamma \in \mathbb{R} \setminus \mathbb{Q} \), see [11, 17]. \( \square \)

### 2.5 Useful Lemmas

In this subsection, we shall discuss two useful lemmas we need in our later sections. The proofs of these lemmas follow in a similar line of argument as in the corresponding complex analogue, see [2, Lemma 3.2], [9, Proposition 3.1] and [17, Lemma 5.3].

**Lemma 2.8** Let \( (g_n)_{n \in \mathbb{N}} \subseteq \text{PSL}(3, \mathbb{H}) \) be a sequence. Then there exists a subsequence \( (g_{n_k})_{k \in \mathbb{N}} \) and an element \( g \in \text{QP}(3, \mathbb{H}) \) such that \( g_n \xrightarrow{n \to \infty} g \) uniformly on the compact subsets of \( \mathbb{P}^2_\mathbb{H} \setminus \text{Ker}(g) \).

**Proof** Let us take a lift \( \tilde{g}_n \in \text{SL}(3, \mathbb{H}) \) of \( g_n \), where \( \tilde{g}_n = (g_{ij}^{(n)})_{1 \leq i, j \leq 3} \). Also, consider the supremum norm of \( g_n \) given by \( |g_n| = \max\{|g_{ij}^{(n)}| \mid i, j = 1, 2, 3\} \). Then the sequence \( \left( \frac{1}{|g_n|} \tilde{g}_n \right)_{n \in \mathbb{N}} \) is a bounded sequence of matrices over quaternions. It gives two bounded sequences of matrices over complex numbers since one can write \( q \in \mathbb{H} \) as \( q = q_1 + q_2j \), \( q_1, q_2 \in \mathbb{C} \). Recall that every bounded sequence of complex matrices has a convergent subsequence. This implies that the sequence of matrices \( \left( \frac{1}{|g_n|} \tilde{g}_n \right)_{n \in \mathbb{N}} \) has a convergent subsequence, that is, there is a subsequence \( \left( \frac{1}{|g_{n_k}|} \tilde{g}_{n_k} \right)_{k \in \mathbb{N}} \) which converges to some non-zero \( \tilde{g} \in \text{M}(3, \mathbb{H}) \). Assume that \( g \in \text{QP}(3, \mathbb{H}) \) be the pseudo-projective transformation corresponding to \( \tilde{g} \).
Now, let \( K \subset \mathbb{P}_H^2 \setminus Ker(g) \) be a compact set and \( \tilde{K} = \{ k \in \mathbb{H}^3 \mid P(k) \in K \} \cap \{ u \in \mathbb{H}^3 \mid |u| = 1 \} \). Then \( P(\tilde{K}) = K \) and \( \frac{1}{|g_{nk}|} \tilde{g}_{nk} \) converges to \( \tilde{g} \) on \( \tilde{K} \) in the compact-open topology. As for each \( k \in \mathbb{N} \), \( \frac{1}{|g_{nk}|} \tilde{g}_{nk} \) is a lift of \( g_{nk} \), hence \( g_{nk} \to g \) on \( K \) uniformly.

The complex version of the following lemma was introduced by Navarrete in [17].

**Lemma 2.9** Let \( G \) be a subgroup of \( \text{PSL}(3, \mathbb{H}) \). If \( C \subset \mathbb{P}_H^2 \) is a closed subset such that for every compact subset \( K \subset \mathbb{P}_H^2 \setminus C \), the cluster points of the family \( \{ g(K) \}_{g \in G} \) of compact sets are contained in \( L_0(G) \cup L_1(G) \), then \( L_2(G) \subset C \).

**Proof** Suppose there is a point \( x \in L_2(G) \setminus C \). Then there exists a compact subset \( K \subset \mathbb{P}_H^2 \setminus L_0(G) \cup L_1(G) \) such that \( x \) is a cluster point of the orbit \( \{ g(K) \}_{g \in G} \). This implies that there exists a sequence \( (g_n)_{n \in \mathbb{N}} \subset G \) such that \( g_n(k_n) \xrightarrow{n \to \infty} x \), where \( (k_n)_{n \in \mathbb{N}} \subset K \) and \( k_n \xrightarrow{n \to \infty} k \in K \subset \mathbb{P}_H^2 \setminus \{ L_0(G) \cup L_1(G) \} \). Since \( x \notin C \) and \( C \) is a closed set, so by discarding a finite number of terms, if needed, we can assume \( g_n(k_n) \in \mathbb{P}_H^2 \setminus C \) for all \( n \in \mathbb{N} \).

Note that \( g_n^{-1}(g_n(k_n)) = k_n \) for each \( n \in \mathbb{N} \). Then the hypothesis applied to the compact subset \( \{ g_n(k_n) \}_{n \in \mathbb{N}} \cup \{ x \} \subset \mathbb{P}_H^2 \setminus C \) yields that \( k \in L_0(G) \cup L_1(G) \), which is a contradiction.

\[ \square \]

3 Kulkarni Limit Sets for Elliptic Transformations

In this section, we shall find the Kulkarni limit set for the cyclic subgroups generated by the elliptic elements of \( \text{PSL}(3, \mathbb{H}) \). Proof of Table 1 given in section 1 follows from the next theorem.

**Theorem 3.1** Let \( \tilde{g} \in \text{PSL}(3, \mathbb{H}) \) be an elliptic transformation given by

\[
g = \begin{pmatrix} e^{2\pi i\alpha} & 0 & 0 \\ 0 & e^{2\pi i\beta} & 0 \\ 0 & 0 & e^{2\pi i\gamma} \end{pmatrix}, \text{ where } \alpha, \beta, \gamma \in \mathbb{R}. \text{ The Kulkarni sets } L_0(G), \ L_1(G), \ L_2(G) \text{ for cyclic subgroup } G := \langle \tilde{g} \rangle \text{ are as follows:}
\]

(i) **Rational elliptic:** If \( \alpha, \beta, \gamma \in \mathbb{Q} \), then \( L_0(G) = L_1(G) = L_2(G) = \phi \).

(ii) **Simple irrational elliptic:** If \( \alpha = \beta = \gamma \in \mathbb{R} \setminus \mathbb{Q} \), then \( L_0(G) = \mathbb{P}_C^2 \), \( L_1(G) = \mathbb{P}_H^2 \) and \( L_2(G) = \phi \).

(iii) **Compound irrational elliptic:** If \( \alpha, \beta, \gamma \) not all are rational and all are not equal, then \( L_0(G) = \{ x \in \mathbb{P}_H^2 \mid x \text{ is a fixed point} \} \), \( L_1(G) = \mathbb{P}_C^2 \) and \( L_2(G) = \phi \).

**Proof** We can divide our calculations into three parts depending on the rationality of \( \alpha, \beta \) and \( \gamma \).

(i) **Rational elliptic:** Let \( \alpha, \beta, \gamma \in \mathbb{Q} \). Then there exists a \( n_0 \in \mathbb{N} \) such that \( e^{2\pi in_0\alpha} = e^{2\pi in_0\beta} = e^{2\pi in_0\gamma} = 1 \), that is, \( G = \langle \tilde{g} \rangle \) is a finite group. Therefore, there is no \( x \in \mathbb{P}_H^2 \), which has an infinite isotropy group. Thus \( L_0(G) = \{ \phi \} \). Similarly, we have \( L_1(G) = \phi \) and \( L_2(G) = \phi \).

(ii) **Simple irrational elliptic:** Let \( \alpha = \beta = \gamma \in \mathbb{R} \setminus \mathbb{Q} \). Recall that \( w = j\tilde{w} \) for each \( w \in \mathbb{C} \). Consider the copy of the complex projective space inside \( \mathbb{P}_H^2 \) that we have identified as \( \mathbb{P}_C^2 := \{ [z]_C \mid z \in \mathbb{C}^3 \setminus \{0\} \} \), where \( [z]_C = \{ w \in \mathbb{C}^3 \mid w = za \text{ for some } a \in \mathbb{C} \setminus \{0\} \} \). Then \( L_0(G) = \mathbb{P}_C^2 \), since by the definition of the projective space, for all \( [x : y : z] \in \mathbb{P}_C^2 \)
we have
\[
g^n \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} e^{2\pi n i \alpha} & & \\ e^{2\pi n i \beta} & e^{2\pi n i \gamma} & \\ e^{2\pi n i \gamma} & e^{2\pi n i \gamma} & e^{2\pi n i \gamma} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = e^{2\pi n i \alpha} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = e^{2\pi n i \alpha} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.
\]

Since \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \), there exists a sequence \((n_k)_{k \in \mathbb{N}} \subset \mathbb{N}\) such that \(e^{2\pi i n_k \alpha} \xrightarrow{k \to \infty} 1\), see Lemma 2.5. This implies \(g^{n_k} [x : y : z] \xrightarrow{k \to \infty} [x : y : z]\) for every point \([x : y : z] \in \mathbb{P}_2^2 \setminus L_0(G)\). Therefore, \(\mathbb{P}_2^2 \setminus L_0(G) \subset L_1(G)\). Again, for every point \(p \in L_0(G)\) there exists a point \(q \in \mathbb{P}_2^2 \setminus L_0(G)\) such that \(g^{n_k}(q) \xrightarrow{k \to \infty} p\) (one can assume \(q = p\)). Hence, \(L_1(G) = \mathbb{P}_2^2\). Consequently, \(L_2(G) = \{\phi\}\).

(iii) **Compound irrational elliptic**: When all the eigenvalues of an elliptic transformation are not equal and not all are rational numbers, we call them compound irrational elliptic. We have the following cases.

(a) **Compound irrational elliptic type I**: Let \(\alpha, \beta \in \mathbb{Q}\) and \(\gamma \in \mathbb{R} \setminus \mathbb{Q}\). Note that \(L_0(G)\) is the set of fixed points of \(g\) as these points have an infinite isotropy group. Since \(\alpha, \beta \in \mathbb{Q}\), there exists a \(n_0 \in \mathbb{N}\) such that \(e^{2\pi i n_0 \alpha} = e^{2\pi i n_0 \beta} = 1\). Further, using Lemma 2.5 we get a sequence \((n_k)_{k \in \mathbb{N}}\) such that \((e^{2\pi i n_k \alpha}, e^{2\pi i n_k \beta}, e^{2\pi i n_k \gamma}) \xrightarrow{k \to \infty} (1, 1, 1)\). With the above argument in knowledge, we can prove that for every point \(p \in \mathbb{P}_2^2\) there exists a point \(q \in \mathbb{P}_2^2 \setminus L_0(G)\) such that \(g^{n_k}(q) \xrightarrow{k \to \infty} p\). To see this, for each point \(p = [x : y : z] \in \mathbb{P}_2^2 \setminus L_0(G)\), choose \(q = p\), then \(g^{n_k}(q) = g^{n_k} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = e^{2\pi i n_k \alpha} e^{2\pi i n_k \beta} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{k \to \infty} \begin{bmatrix} x \\ y \\ z \end{bmatrix}\). Observe that when \(p \in L_0(G)\), then depending upon the point \(p\) we can also find a suitable point \(q \in \mathbb{P}_2^2 \setminus L_0(G)\) such that \(g^{n_k}(q)\) converges to \(p\) as \(k \to \infty\). Therefore, we have \(L_1(G) = \mathbb{P}_2^2\) and consequently \(L_2(G) = \{\phi\}\). A similar line of argument will hold if, instead of \(\gamma\), either \(\alpha\) or \(\beta\) is irrational.

(b) **Compound irrational elliptic Type II**: Let \(\alpha \in \mathbb{Q}\) and \(\beta, \gamma \in \mathbb{R} \setminus \mathbb{Q}\). Now \(L_0(G)\) is the set of fixed points of \(g\) as these points have an infinite isotropy group. In this case we can find a sequence \((n_k)_{k \in \mathbb{N}}\) such that \(e^{2\pi i n_k \beta} \xrightarrow{k \to \infty} 1\) and \(e^{2\pi i n_k \gamma} \xrightarrow{k \to \infty} 1\), see Remark 2.7. Also, since \(\alpha \in \mathbb{Q}\), there exists a \(n_0 \in \mathbb{N}\) such that \(e^{2\pi i n_0 \alpha} = 1\). Therefore for every \([x : y : z] \in \mathbb{P}_2^2 \setminus L_0(G)\) we have a sequence \((n_k)_{k \in \mathbb{N}}\) such that \(g^{n_k} [x : y : z] \xrightarrow{k \to \infty} [x : y : z]\) and so \(\mathbb{P}_2^2 \setminus L_0(G) \subset L_1(G)\). Again for any \(p \in L_0(G)\) we can choose a suitable point \(q \in \mathbb{P}_2^2 \setminus L_0(G)\) such that \(g^{n_k}(q) \to p\) as \(k \to \infty\). Therefore, \(L_1(G) = \mathbb{P}_2^2\). Consequently, \(L_2(G) = \{\phi\}\). A similar line of argument will hold if, instead of \(\alpha\), either \(\beta\) or \(\gamma\) is rational.

(c) **Compound irrational elliptic Type III**: Let \(\alpha, \beta, \gamma \in \mathbb{R} \setminus \mathbb{Q}\) and not all are equal. Then \(G\) is an infinite group, so \(L_0(G) = \{x \in \mathbb{P}_2^2 \mid x \text{ is a fixed point}\}\). To find \(L_1(G)\), we can use Remark 2.7 and show \(L_1(G) = \mathbb{P}_2^2\). Consequently, \(L_2(G) = \{\phi\}\).

This completes the proof. \(\square\)
4 Kulkarni Limit Sets for Loxodromic Transformations

In this section, we consider loxodromic transformations of $SL(3, \mathbb{H})$, see Definition 1.5. The following theorem proves Table 2.

**Theorem 4.1** Let $\tilde{g} \in PSL(3, \mathbb{H})$ be a loxodromic transformation and $g \in SL(3, \mathbb{H})$ be a lift of $\tilde{g}$. Then the Kulkarni sets for $G = \langle \tilde{g} \rangle$ are the following:

(i) **Regular loxodromic:** If $g = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \xi \end{pmatrix}$ with $|\lambda| < |\mu| < |\xi|$ then $L_0(G) = L_1(G) = \{e_1, e_2, e_3\}$ and $L_2(G) = \mathbb{L}[e_1, e_2] \cup \mathbb{L}[e_2, e_3]$.

(ii) **Screw loxodromic:** If $g = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \xi \end{pmatrix}$ with $|\lambda| = |\mu| \neq 1$ such that $\lambda \neq \mu$ and $|\xi| = \frac{1}{|\lambda|^2}$ then $L_0(G) = \{e_1, e_2, e_3\}$ and $L_1(G) = \mathbb{L}[e_1, e_2] \cup \{e_3\} = L_2(G)$.

(iii) **Homothety:** If $g = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \xi \end{pmatrix}$ with $|\lambda| \neq 1$ and $|\xi| = \frac{1}{|\lambda|^2}$ then $L_1(G) = L_2(G) = \mathbb{L}[e_1, e_2] \cup \{e_3\} \cup \{ \mathbb{L}[e_1, e_2], \text{ if } \lambda \in \mathbb{R} \} = \mathbb{L}[e_1, e_2] \cup \{e_3\}, \text{ if } \lambda \in \mathbb{C}\backslash\mathbb{R}$.

(iv) **Loxo-parabolic:** If $g = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \xi \end{pmatrix}$ with $|\lambda| \neq 1$ and $|\xi| = \frac{1}{|\lambda|^2}$ then $L_0(G) = \{e_1, e_3\} = L_1(G)$ and $L_2(G) = \mathbb{L}[e_1, e_2] \cup \mathbb{L}[e_1, e_3]$.

**Proof** We will consider various cases depending on the eigenvalues to determine the Kulkarni sets.

(i) **Regular loxodromic:** First, we will find $L_0(G)$. A point $p = [x : y : z] \in \mathbb{P}^2_\mathbb{H}$ has an infinite isotropy group if $g^n(p) = p$ for infinitely many $n \in \mathbb{Z}$ and $g^n(p) = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \mu^n & 0 \\ 0 & 0 & \xi^n \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda^n x \\ \mu^n y \\ \xi^n z \end{pmatrix}$ is true only when $p = e_1, e_2$ or $e_3$ (as $\mathbb{P}^2_\mathbb{H}$ is a projective space). Hence, $L_0(G) = \{e_1, e_2, e_3\}$. For calculating $L_1(G)$, we have to find the set of the cluster points of orbits of points in $\mathbb{P}^2_\mathbb{H} \setminus L_0(G)$. Recall that for each non-zero $r \in \mathbb{R}$, $rg$ is a lift of $g$, and both induce the same projective transformation on $\mathbb{P}^2_\mathbb{H}$. Now consider $|\xi|^{-n} g^n = \begin{pmatrix} |\xi|^{-n} |\lambda|^n e^{i n \arg(\lambda)} & 0 & 0 \\ 0 & |\xi|^{-n} |\mu|^n e^{i n \arg(\mu)} & 0 \\ 0 & 0 & e^{i n \arg(\xi)} \end{pmatrix}$, where $n \in \mathbb{N}$. Note that there is a subsequence $(n_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ such that $e^{i n_k \arg(\xi)} \xrightarrow{k \to \infty} 1$. Since $|\lambda| \frac{1}{|\xi|^2} < 1$ and $|\mu| \frac{1}{|\xi|^2} < 1$, there exists a subsequence of $(|\xi|^{-n} g^n)_{n \in \mathbb{N}^*}$ still denoted by $(|\xi|^{-n} g^n)_{n \in \mathbb{N}^*}$, which converges to $D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ as $n \to \infty$. Note that $Ker(D_1) = \mathbb{L}[e_1, e_2] = \{[x : y : z] \in \mathbb{P}^2_\mathbb{H} \mid z = 0\}$. Thus for every point $p \in \mathbb{P}^2_\mathbb{H} \setminus \mathbb{L}[e_1, e_2]$, $|\xi|^{-n} g^n(p)_{n \in \mathbb{N}^*}$ converges to $e_3 \in \mathbb{P}^2_\mathbb{H}$ as $n \to \infty$. Hence, $e_3 \in L_1(G)$. Similarly, by considering a lift $|\lambda|^n g^{-n}$ of $g^{-n}$, we can show that there is a subsequence of $(|\lambda|^n g^{-n})_{n \in \mathbb{N}}$, which converges to $D_2 = \text{diag}(1, 0, 0)$ as $n \to \infty$. 

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Note that $Ker(D_2) = \mathbb{L}\{e_2, e_3\} = \{x : y : z \in \mathbb{P}_2^2 | x = 0\}$. Thus for every point $p \in \mathbb{P}_2^2 \setminus \mathbb{L}\{e_2, e_3\}$, $(|\lambda|^n g^n(p))_{n \in \mathbb{N}} \xrightarrow{n \to \infty} e_1 \in \mathbb{P}_2^2$ and hence $e_1 \in L_1(G)$. Now consider the sequences $(\mu^{-n}g^n(p))_{n \in \mathbb{N}}$ and $(\mu^n g^{-n}(q))_{n \in \mathbb{N}}$. Then there are subsequences of $\mu^{-n}g^n(p)$ and $\mu^n g^{-n}(q)$ that converge to $e_2 \in \mathbb{P}_2^2$ whenever $p \in \mathbb{L}\{e_1, e_2\}$ and $q \in \mathbb{L}\{e_2, e_3\}$. Therefore, $e_2 \in L_1(G)$. Also, from the above discussion, it follows that for every point $p \in \mathbb{P}_2^2 \setminus L_0(G)$, the set of cluster points of the orbits $\{g^n(p)\}_{n \in \mathbb{N}}$ lies in the set $\{e_1, e_2, e_3\}$. Consequently, $L_1(G) = \{e_1, e_2, e_3\}$. To find $L_2(G)$, consider a closed set $C = \mathbb{L}\{e_1, e_2\} \cup \mathbb{L}\{e_2, e_3\}$ in $\mathbb{P}_2^2$. Now note that $D_1$ and $D_2$ induce pseudo-projective transformations in $\mathbb{Q}_P(3, \mathbb{H})$ such that $\mathbb{L}\{e_1, e_2\} = Ker(D_1)$ and $\mathbb{L}\{e_2, e_3\} = Ker(D_2)$. Therefore, using Lemma 2.8 we can show that the set of cluster points of orbits of compact subsets of $\mathbb{P}_2^2 \setminus C$ lies in $L_0(G) \cup L_1(G) = \{e_1, e_2, e_3\}$. Then, using Lemma 2.9 we have $L_2(G) \subset C$, that is, $L_2(G) \subset \mathbb{L}\{e_1, e_2\} \cup \mathbb{L}\{e_2, e_3\}$ further, to show that $\mathbb{L}\{e_2, e_3\} \subset L_2(G)$, choose a compact set $K = \{x : y : z \in \mathbb{P}_2^2 | |x|^2 = |y|^2 + |z|^2\}$ which is a subset of $\mathbb{P}_2^2 \setminus L_0(G) \cup L_1(G)$. For a point $[0 : y : z] \in \mathbb{L}\{e_2, e_3\}$, take a sequence $(k'_n)_{n \in \mathbb{N}} \subset K$ such that $k'_n = [\mu^{-n}y : |\lambda|^n(\xi)z]$ converges to a point as $n \to \infty$ as $\lambda \neq 1$ and $\lambda \neq 1$. Hence, $\mathbb{L}\{e_1, e_2\} \cup \mathbb{L}\{e_2, e_3\} \subset L_2(G)$. Consequently, $L_2(G) = \mathbb{L}\{e_1, e_2\} \cup \mathbb{L}\{e_2, e_3\}$.

(ii) **Screw loxodromic:** In this case, we have $|\lambda| = |\mu| \neq 1, \lambda \neq 1$ and $|\xi| = 1/|\lambda|^2$. Further, we will assume that $1 < |\lambda|$ since proof for the case $|\lambda| < 1$ follows using a similar line of argument. If $g^n = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \lambda^n x \\ \mu^n y \\ \xi^n z \end{bmatrix}$, then the only points with infinite isotropy groups are $\{e_1, e_2, e_3\}$. Hence $L_0(G) = \{e_1, e_2, e_3\}$. Consider a sequence $(|\lambda|^{-n}g^n)_{n \in \mathbb{N}}$ such that

$$
|\lambda|^{-n}g^n = \begin{pmatrix} e^{in \arg(\lambda)} & 0 & 0 \\ 0 & e^{in \arg(\mu)} & 0 \\ 0 & 0 & 1/|\lambda|^{2n}e^{in \arg(\xi)} \end{pmatrix}.
$$

Then we get a subsequence of $(|\lambda|^{-n}g^n)_{n \in \mathbb{N}}$ which converges to $D_3 = \text{diag}(1, 1, 0)$. Note that $Ker(D_3) = \{e_3\} = \{x : y : z \in \mathbb{P}_2^2 | x = y = 0\}$. Thus for every point $p \in \mathbb{P}_2^2 \setminus \{e_3\}$, $(|\lambda|^{-n}g^n(p))_{n \in \mathbb{N}}$ converges to a point of $\mathbb{L}\{e_1, e_2\} \subset \mathbb{P}_2^2$ as $n \to \infty$. Therefore, for each point $q = [x : y : 0] \in \mathbb{L}\{e_1, e_2\}$, we can choose a point $p = [x : y : 1] \in \mathbb{P}_2^2 \setminus L_0(G)$ such that $g^n(p) \xrightarrow{n \to \infty} q$ and hence $q \in L_4(G)$. Consequently,
$L_1(G)$. Now consider the sequence $(|\xi|^n g^{-n})_{n \in \mathbb{N}}$ such that

$$|\xi|^n g^{-n} = \text{diag} \left( \frac{1}{|\lambda|^3} e^{-i n \arg(\lambda)}, \frac{1}{|\lambda|^3} e^{-i n \arg(\mu)}, e^{-i n \arg(\varepsilon)} \right).$$

Since $1 < |\lambda|$, we get a subsequence of $(|\xi|^n g^{-n})_{n \in \mathbb{N}}$ which converges to $D_1 = \text{diag}(0, 0, 1)$, where $\text{Ker}(D_1) = \mathbb{L}[e_1, e_2]$. Thus for every point $p \in \mathbb{P}_3^2 \setminus \mathbb{L}[e_1, e_2]$, $(|\xi|^n g^{-n}(p))_{n \in \mathbb{N}}$ converges to $e_3 \in \mathbb{P}_3^2$ as $n \to \infty$. Therefore, $e_3 \in L_1(G)$. Also, from the above discussion, it follows that for every point $p \in \mathbb{P}_3^2 \setminus L_0(G)$, the set of cluster points of the orbits $\{g^n(p)\}_{n \in \mathbb{Z}}$ lies in the set $L[e_1, e_2] \cup \{e_3\}$. Consequently, $L_1(G) = \mathbb{L}[e_1, e_2] \cup \{e_3\}$. To find $L_2(G)$, consider a closed set $C = \mathbb{L}[e_1, e_2] \cup \{e_3\} \subset \mathbb{P}_3^2$. Now note that $D_1$ and $D_3$ induce pseudo-projective transformations in $\mathbb{P}(3, \mathbb{H})$ such that $\mathbb{L}[e_1, e_2] = \text{Ker}(D_1)$ and $\mathbb{L}[e_3] = \text{Ker}(D_3)$. Therefore, using Lemma 2.8 we can show that the set of cluster points of orbits of compact subsets of $\mathbb{P}_3^2 \setminus C$ lies in $L_0(G) \cup L_1(G) = \mathbb{L}[e_1, e_2] \cup \{e_3\}$. Then, using Lemma 2.9 we have that $L_2(G) \subset \mathbb{L}[e_1, e_2] \cup \{e_3\}$. Further, we have already observed in the calculation of $L_1(G)$ that points of $\mathbb{P}_3^2 \setminus \mathbb{L}[e_1, e_2] \cup \{e_3\}$ converges to a point of $\mathbb{L}[e_1, e_2]$ for positive iterates of $g$, and converges to $\{e_3\}$ for negative iterates of $g$. Therefore, we can easily show that $\mathbb{L}[e_1, e_2] \cup \{e_3\} \subset L_2(G)$. Hence, $L_2(G) = \mathbb{L}[e_1, e_2] \cup \{e_3\}$.

(iii) **Homothety:** In this case, we have $|\lambda| \neq 1$ and $|\xi| = 1/|\lambda|^2$. Let $\lambda = r e^{i \arg(\lambda)}$, where $r \neq 1$. Since each point $[x : y : z] \in L_0(G)$ has an infinite isotropy group and satisfy the equation $g^n([x : y : z]) = [\lambda^n x : \lambda^n y : \xi^n z] = [x : y : z]$ for infinitely many values of $n \in \mathbb{Z}$. It is easy to note that $e_3 \in L_0(G)$. Recall that $wj = j\bar{w}$ for each $w \in \mathbb{C}$. Thus if $\lambda = r$, then $g^n[x : y : 0] = [xr^n : yr^n : 0] = [x : y : 0]$ for each $x, y \in \mathbb{H}$. If $\lambda \neq r$ then, $g^n[x : y : 0] = [x\lambda^n : y\lambda^n : 0] = [x : y : 0]$ for each $x, y \in \mathbb{C}$. Therefore,

$$L_0(G) = \{e_3\} \cup \begin{cases} \mathbb{L}[e_1, e_2], & \text{if } \lambda \in \mathbb{R} \\ \mathbb{L}[e_1, e_2], & \text{if } \lambda \in \mathbb{C} \setminus \mathbb{R} \end{cases}$$

where $\mathbb{L}[e_1, e_2]$ denotes the complex projective line joining $e_1$ and $e_2$. The proof of $L_1(G) = \mathbb{L}[e_1, e_2] \cup \{e_3\} = L_2(G)$ follows from a similar line of argument as in the case of **screw loxodromic**, and we omit it.

(iv) **Loxo-parabolic:** In the loxo-parabolic case $g = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \xi \end{pmatrix}$. $|\xi| = |\lambda|^{-2} \neq 1$. Further, we will assume that $1 < |\lambda|$ since proof for the case $|\lambda| < 1$ follows using a similar line of argument. Note the following equation

$$g^n \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & 0 \\ 0 & \lambda^n & 0 \\ 0 & 0 & \xi^n \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda^n x + n\lambda^{n-1}y \\ \lambda^n y \\ \xi^n z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$ 

Clearly, the above equation is satisfied by infinitely many values of $n \in \mathbb{Z}$ only when $[x : y : z] = e_1$ or $e_3$. Hence, $L_0(G) = \{e_1, e_3\}$. Now consider lifts $\frac{1}{|\lambda|^n} g^n$ of $g^n$ and $|\xi|^n g^{-n}$ of $g^{-n}$ for each $n \in \mathbb{N}$ such that $\lambda^n \neq 0$.
Then there is a subsequence of \( \left( \frac{1}{n|\lambda|^n} g^n \right)_{n \in \mathbb{N}} \) which converges to \( D_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), where \( \text{Ker}(D_4) = \mathbb{L}\{e_1, e_3\} \). Similarly, there exists a subsequence of \( \left( \langle \xi \rangle^n g^{-n} \right)_{n \in \mathbb{N}} \) which converges to \( D_1 = \text{diag}(0, 0, 1) \) where \( \text{Ker}(D_1) = \mathbb{L}\{e_1, e_2\} \). Thus for every point \( p \in \mathbb{P}^2_{\mathbb{H}} \setminus \mathbb{L}\{e_1, e_3\} \) and \( q \in \mathbb{P}^2_{\mathbb{H}} \setminus \mathbb{L}\{e_1, e_2\} \), we have \( \left( \frac{1}{n|\lambda|^n} g^n(p) \right)_{n \in \mathbb{N}} \overset{n \to \infty}{\to} e_1 \in \mathbb{P}^2_{\mathbb{H}} \) and \( \langle \xi \rangle^n g^{-n}(q) \rangle_{n \in \mathbb{N}} \overset{n \to \infty}{\to} e_3 \in \mathbb{P}^2_{\mathbb{H}} \). Therefore, \( \{e_1, e_3\} \subset L_1(G) \). Also, from the above discussion, it follows that for every point \( p \in \mathbb{P}^2_{\mathbb{H}} \setminus L_0(G) \), the set of cluster points of the orbits \( \{g^n(p)\}_{n \in \mathbb{Z}} \) lies in the set \( \{e_1, e_3\} \) and hence \( L_1(G) = \{e_1, e_3\} \). To find \( L_2(G) \), consider a closed set \( C = \mathbb{L}\{e_1, e_2\} \cup \mathbb{L}\{e_2, e_3\} \in \mathbb{P}^2_{\mathbb{H}} \). Now note that \( D_1 \) and \( D_4 \) induce pseudo-projective transformations in \( \mathbb{Q}(3, \mathbb{H}) \) such that \( \mathbb{L}\{e_1, e_2\} = \text{Ker}(D_1) \) and \( \mathbb{L}\{e_1, e_3\} = \text{Ker}(D_2) \). Therefore, using Lemma 2.8 we can show that the set of cluster points of orbits of compact subsets of \( \mathbb{P}^2_{\mathbb{H}} \setminus C \) lies in \( L_0(G) \cup L_1(G) = \{e_1, e_3\} \). Then, using Lemma 2.9 we have \( L_2(G) \subset C \), that is, \( L_2(G) \subset \mathbb{L}\{e_1, e_2\} \cup \mathbb{L}\{e_1, e_3\} \). Further, to show that \( \mathbb{L}\{e_1, e_3\} \subset L_2(G) \), consider a sequence \( \{k_n\}_{n \in \mathbb{N}} \) such that \( k_n = \left[ 1 : 1 : \frac{-n}{|\lambda|^{3n+1}w} \right] \), where \( w \) is a non-zero quaternion. Note that as \( 1 < |\lambda|, k_n \) is a sequence in the compact subset \( K = \{k_n \mid n \in \mathbb{N}\} \cup \{(1 : 1 : 0)\} \) of \( \mathbb{P}^2_{\mathbb{H}} \setminus \{e_1, e_3\} \), where \( \{e_1, e_3\} = L_0(G) \cup L_1(G) \). Then we have
\[
g^{-n}(k_n) = g^{-n}\left(1 : 1 : \frac{-n}{|\lambda|^{3n+1}w}\right) = \lambda^{-n} - n\lambda^{-1} : \lambda^{-n} : \frac{-n\xi^{-2n}}{|\lambda|^{3n+1}w}.
\]
This implies, \( \frac{|\lambda|^{n+1}}{n} g^{-n}(k_n) = \left[ \frac{|\lambda|^i e^{-in arg(\lambda)}}{n} - e^{-i(n+1) arg(\lambda)} : \frac{|\lambda|^i e^{-in arg(\lambda)}}{n} - e^{-i2n arg(\xi)} w \right] \).

Then, there exists a subsequence of \( \left( g^{-n}(k_n) \right)_{n \in \mathbb{N}} \) which converges to a point \( w : 0 : 1 \in \mathbb{L}\{e_1, e_3\} \). Thus \( \mathbb{L}\{e_1, e_3\} \subset L_2(G) \). Now consider a sequence \( \{k_n\}_{n \in \mathbb{N}} \) such that \( k_n = \left[ \frac{e^{-i arg(\lambda)}}{|\lambda|} + \frac{\nu}{n} : 1 : \frac{1}{|\lambda|} \right] \), where \( \nu \) is a non-zero quaternion. Note that \( k_n \) is a sequence in the compact subset \( K = \{k_n \mid n \in \mathbb{N}\} \cup \{(\nu : 1 : \xi) : 0 : 1\} \) of \( \mathbb{P}^2_{\mathbb{H}} \setminus L_0(G) \cup L_1(G) \). Then we have
\[
g^n(k_n) = \frac{1}{|\lambda|^n} g^n\left(\frac{e^{-i arg(\lambda)}}{|\lambda|} + \frac{\nu}{n} : 1 : \frac{1}{|\lambda|} \right) = \left[ \frac{e^{-in arg(\lambda)}}{n} : \frac{e^{-in arg(\lambda)}}{n} : \frac{1}{|\lambda|^{3n+1} e^{-in arg(\xi)}} \right].
\]
This implies, \( g^n(k_n) = \frac{n}{|\lambda|^n} g^n(k_n) = \left[ \frac{e^{in arg(\lambda)}}{|\lambda|^n} : \frac{e^{in arg(\lambda)}}{|\lambda|^n} : \frac{n}{|\lambda|^{3n+1} e^{in arg(\xi)}} \right] \).

Therefore, there exists a subsequence of \( \left( g^n(k_n) \right)_{n \in \mathbb{N}} \) which converges to a point \( \{\nu : 1 : 0\} \in \mathbb{L}\{e_1, e_2\} \). Thus \( \mathbb{L}\{e_1, e_2\} \subset L_2(G) \). Hence, \( L_2(G) = \mathbb{L}\{e_1, e_2\} \cup \mathbb{L}\{e_1, e_3\} \).

This completes the proof. \( \square \)
5 Kulkarni Limit Sets for Parabolic Transformations

In this section, we work out the Kulkarni limit sets for the cyclic groups generated by the parabolic elements of \( \text{PSL}(3, \mathbb{H}) \). We have several subcases of parabolic elements, such as unipotent, ellipto-parabolic and ellipto-translation. The following theorem proves Table 3.

**Theorem 5.1** Let \( \tilde{g} \in \text{PSL}(3, \mathbb{H}) \) be a parabolic transformation and \( g \in \text{SL}(3, \mathbb{H}) \) be a lift of \( \tilde{g} \). Then the Kulkarni sets for \( G = (\tilde{g}) \) are given by the following:

(i) **Vertical translation:** If \( g = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \), then \( L_0(G) = \mathbb{L}[e_1, e_3] \), \( L_1(G) = \{ e_1 \} \) and \( L_2(G) = \{ e_1 \} \).

(ii) **Non-vertical translation:** If \( g = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \), then \( L_0(G) = \{ e_1 \} \), \( L_1(G) = \{ e_1 \} \) and \( L_2(G) = \mathbb{L}[e_1, e_2] \).

(iii) **Ellipto-parabolic:** If \( g = \begin{pmatrix} e^{2\pi i \alpha} & 1 & 0 \\ 0 & e^{2\pi i \beta} & 0 \\ 0 & 0 & e^{2\pi i \beta} \end{pmatrix} \), \( e^{2\pi i \alpha} \neq 1 \), then \( L_2(G) = \{ e_1 \} \) and \( L_1(G) = \{ e_1 \} \) and \( L_2(G) = \mathbb{L}[e_1, e_2] \).

(a) **Rational ellipto-parabolic:** if \( \alpha, \beta \in \mathbb{Q} \) then \( L_0(G) = \mathbb{L}[e_1, e_3] \), \( L_1(G) = \{ e_1 \} \).

(b) **Irrational ellipto-parabolic:** if either or both of the \( \alpha \) and \( \beta \) are in \( \mathbb{R} \setminus \mathbb{Q} \), then \( L_0(G) = \{ e_1, e_3 \} \) and \( L_1(G) = \mathbb{L}[e_1, e_3] \).

(iv) **Ellipto-translation:** If \( g = \begin{pmatrix} e^{2\pi i \alpha} & 1 & 0 \\ 0 & e^{2\pi i \alpha} & 1 \\ 0 & 0 & e^{2\pi i \alpha} \end{pmatrix} \), then \( L_0(G) = L_1(G) = \{ e_1 \} \) and \( L_2(G) = \mathbb{L}[e_1, e_2] \).

**Proof** If all the eigenvalues of the transformation are 1, then it is called a unipotent. Unipotent elements of \( \text{SL}(3, \mathbb{H}) \) are divided into two cases depending upon their minimal polynomials. A unipotent element is called vertical, resp. non-vertical translations if the corresponding minimal polynomial is \( (x-1)^2 \), resp. \( (x-1)^3 \).

(i) **Vertical translation:** In this case, consider the following equation

\[
g^n \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^n \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + ny \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix},
\]

which gives us \( y = 0 \). This implies \( L_0(G) = \mathbb{L}[e_1, e_3] = \{ [x : y : z] \in \mathbb{P}^2_\mathbb{H} : y = 0 \} \).

For the calculation of \( L_1(G) \), consider the sequences \( \left( \frac{1}{n} g^n \right)_{n \in \mathbb{N}} \) and \( \left( -\frac{1}{n} g^{-n} \right)_{n \in \mathbb{N}} \). Then for every point \( p = [x : y : z] \in \mathbb{P}^2_\mathbb{H} \setminus L_0(G) \) we have

\[
g^n \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{n} g^n \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{n} x + y \\ \frac{1}{n} y \\ \frac{1}{n} z \end{bmatrix} \rightarrow \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
\]

Similarly, we can show \( \left( -\frac{1}{n} g^{-n}(p) \right)_{n \in \mathbb{N}} \rightarrow e_1 \) for every point \( p \in \mathbb{P}^2_\mathbb{H} \setminus L_0(G) \). Hence, \( L_1(G) = e_1 \). Note that for each \( n \in \mathbb{N} \), \( \frac{1}{n} g^n \) and \( -\frac{1}{n} g^{-n} \) are lifts of \( g^n \) and \( g^{-n} \), respectively. Then in view of Lemma 2.8, we have that both the sequences \( g^n \)
and $g^{-n}$ converges to a pseudo-projective transformation $D_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \text{QP}(3, \mathbb{H})$

uniformly on the compact subsets of $\mathbb{P}^2_{\mathbb{H}} \setminus \mathbb{L}(e_1, e_3)$, where $\mathbb{L}(e_1, e_3) = \text{Ker}(D_4) = \{ [x : y : z] \in \mathbb{P}^2_{\mathbb{H}} | y = 0 \}$. Since $\mathbb{L}(e_1, e_3) = L_0(G) \cup L_1(G)$, it follows that $e_1 = [1 : 0 : 0] \in \mathbb{P}^2_{\mathbb{H}}$ is the only cluster point of the orbits $\{g^n(K)\}_{n \in \mathbb{Z}}$ for every compact subsets $K$ of $\mathbb{P}^2_{\mathbb{H}} \setminus L_0(G) \cup L_1(G)$. As a consequence $L_2(G) = \{e_1\}$.

(ii) **Non-vertical translation:** In this case, we have $g = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Then for each $n \in \mathbb{N}$, we have $g^n \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $g^{-n} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & -n & 1 \\ 0 & 1 & -n \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Therefore, if the equation $g^n(p) = p$ satisfied by a point $p = [x : y : z] \in \mathbb{P}^2_{\mathbb{H}}$ for infinitely many values of $n \in \mathbb{Z}$ then $p = [1 : 0 : 0] = e_1$ and hence $L_0(G) = \{e_1\}$. Note that for each $n \in \mathbb{N}$, $\frac{2}{n(n-1)}g^n$ and $\frac{2}{n(n+1)}g^{-n}$ are lifts of $g^n$ and $g^{-n}$, respectively. Thus it follows that both the sequences $g^n$ and $g^{-n}$ converge to $D_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where $\text{Ker}(D_5) = \mathbb{L}(e_1, e_2) = \{ [x : y : z] \in \mathbb{P}^2_{\mathbb{H}} | z = 0 \}$.

Thus for each point $p \in \mathbb{P}^2_{\mathbb{H}} \setminus \mathbb{L}(e_1, e_2)$, sequences $g^n(p)$ and $g^{-n}(p)$ converge to $e_1$ as $n \to \infty$. Since $L_0(G) = \{e_1\} \subset \mathbb{L}(e_1, e_2)$, it follows that $e_1$ is the only cluster point of orbits $\{g^n(p)\}_{n \in \mathbb{Z}}$ for every point $p \in \mathbb{P}^2_{\mathbb{H}} \setminus L_0(G)$. Hence, $L_1(G) = \{e_1\}$. To find $L_2(G)$, consider a closed set $C = \mathbb{L}(e_1, e_2) \subset \mathbb{P}^2_{\mathbb{H}}$. Now note that $D_5$ induces a pseudo-projective transformation in $\text{QP}(3, \mathbb{H})$ such that $\mathbb{L}(e_1, e_2) = \text{Ker}(D_5)$. Therefore, using Lemma 2.8 we can show that the set of cluster points of orbits of compact subsets of $\mathbb{P}^2_{\mathbb{H}} \setminus C$ lies in $L_0(G) \cup L_1(G) = \{e_1\}$. Then, using Lemma 2.9 we have that $L_2(G) \subset \mathbb{L}(e_1, e_2)$.

Now to show that $\mathbb{L}(e_1, e_2) \subset L_2(G)$, consider a compact subset $K = \{ [x : -\frac{n-1}{n} : \frac{2}{n}] \in \mathbb{P}^2_{\mathbb{H}} | n \in \mathbb{N} \} \cup \{ [x : -1 : 0] \}$ of $\mathbb{P}^2_{\mathbb{H}} \setminus \{e_1\}$, where $\{e_1\} = L_0(G) \cup L_1(G)$. Then the sequence $g^n([x : -\frac{n-1}{n} : \frac{2}{n}]_n) = [x : 1 + \frac{2}{n}]$ converges to a point $[x : 1 : 0] \in \mathbb{L}(e_1, e_2)$ as $n \to \infty$. This implies $\mathbb{L}(e_1, e_2) \subset L_2(G)$. Hence, $L_2(G) = \mathbb{L}(e_1, e_2)$.

(iii) **Ellipto-parabolic:** In this case, for each $n \in \mathbb{N}$, we have

$$g^n \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} e^{i\alpha n} & 0 & 0 \\ 0 & e^{i\beta n} & 0 \\ 0 & 0 & e^{i\gamma n} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} e^{i\alpha n} x + e^{i\beta n} y \\ e^{i\beta n} y \\ e^{i\gamma n} z \end{pmatrix},$$

and

$$g^{-n} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} e^{-i\alpha n} & 0 & 0 \\ 0 & e^{-i\beta n} & 0 \\ 0 & 0 & e^{-i\gamma n} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} e^{-i\alpha n} x - e^{-i\beta n} y \\ e^{-i\beta n} y \\ e^{-i\gamma n} z \end{pmatrix}.$$
as $n \to \infty$. Now to find $L_0(G)$ and $L_1(G)$, we can consider the following subcases depending on the rationality of $\alpha$ and $\beta$.

(a) **Rational ellipto-parabolic:** Here $\alpha$ and $\beta$ both are rational numbers. So there exists a $n_0 \in \mathbb{N}$ such that $e^{2\pi i n_0 \alpha} = e^{2\pi i n_0 \beta} = 1$. Thus a point $p = [x : y : z] \in \mathbb{P}_{\mathbb{H}}^2$ satisfy $g^n(p) = p$ for infinitely many values of $n \in \mathbb{Z}$ only if $y = 0$. Hence, $L_0(G) = \mathbb{L}\{e_1, e_3\}$. Further, if $p = [x : y : z] \in \mathbb{P}_{\mathbb{H}}^2 \setminus \mathbb{L}\{e_1, e_3\}$, then $y \neq 0$. Therefore, $e_1$ is the only cluster point of orbits $\{g^n(p)\}_{n \in \mathbb{Z}}$ for every point $p \in \mathbb{P}_{\mathbb{H}}^2 \setminus L_0(G)$. Hence, $L_1(G) = \{e_1\}$.

(b) **Irrational Ellipto-Parabolic:** In this case, either $\alpha$ or $\beta$ or both of them are irrational numbers. The points which give infinite isotropy groups are only $e_1$ and $e_3$. Therefore, $L_0(G) = \{e_1, e_3\}$. Now note that for each point $p = [x : y : z] \in \mathbb{P}_{\mathbb{H}}^2 \setminus L_0(G)$ with $y \neq 0$, $e_1$ is the only cluster point of orbits $\{g^n(p)\}_{n \in \mathbb{Z}}$. Also, $g$ acts as an elliptic transformation on $\mathbb{L}\{e_1, e_3\} = \{[x : y : z] | y = 0\}$ and hence $\mathbb{L}\{e_1, e_3\} \subset L_1(G)$ (cf. elliptic case). Therefore, $L_1(G) = \mathbb{L}\{e_1, e_3\}$.

Therefore, we have $L_0(G) \cup L_1(G) = \mathbb{L}\{e_1, e_3\}$. Now consider a compact set $K \subset \mathbb{P}_{\mathbb{H}}^2 \setminus L_0(G) \cup L_1(G)$. Using a similar argument as in the case of vertical translation, we can show that $e_1$ is the unique cluster point of the family of compact sets $\{g^n(K)\}_{n \in \mathbb{Z}}$. Hence, $L_2(G) = \{e_1\}$.

(iv) **Ellipto-translation:** In this case, $g^n = \begin{pmatrix} e^{2\pi i n \alpha} & n e^{2\pi i (n-1) \alpha} & n(n-1) e^{2\pi i (n-2) \alpha} \\ 0 & e^{2\pi i n \alpha} & 2 n e^{2\pi i (n-1) \alpha} \\ 0 & 0 & e^{2\pi i n \alpha} \end{pmatrix}$. Thus for each $n \in \mathbb{N}$, we have

\[
\begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix}
= \begin{bmatrix}
    e^{2\pi i n \alpha} x + n e^{2\pi i (n-1) \alpha} y + \frac{n(n-1)}{2} e^{2\pi i (n-2) \alpha} z \\
    e^{2\pi i n \alpha} y + n e^{2\pi i (n-1) \alpha} z \\
    e^{2\pi i n \alpha} z
\end{bmatrix},
\]

and

\[
\begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix}
= \begin{bmatrix}
    e^{-2\pi i n \alpha} x - n e^{-2\pi i (n+1) \alpha} y + \frac{n(n+1)}{2} e^{-2\pi i (n+2) \alpha} z \\
    e^{-2\pi i n \alpha} y - n e^{-2\pi i (n+1) \alpha} z \\
    e^{-2\pi i n \alpha} z
\end{bmatrix}.
\]

Therefore, if the equation $g^n(p) = p$ satisfied by a point $p = [x : y : z] \in \mathbb{P}_{\mathbb{H}}^2$ for infinitely many values of $n \in \mathbb{Z}$ then $p = [1 : 0 : 0] = e_1$ and hence $L_0(G) = \{e_1\}$. Note that there exist subsequences of $\{\frac{2}{n(n-1)} g^n(p)\}_{n \in \mathbb{N}}$ and $\{\frac{2}{n(n+1)} g^{-n}(p)\}_{n \in \mathbb{N}}$ which converge to $D_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where $Ker(D_5) = \mathbb{L}\{e_1, e_2\}$. Thus for each point $p = [x : y : z] \in \mathbb{P}_{\mathbb{H}}^2$ with $z \neq 0$, both the sequences $\{\frac{2}{n(n-1)} g^n(p)\}_{n \in \mathbb{N}}$ and $\{\frac{2}{n(n+1)} g^{-n}(p)\}_{n \in \mathbb{N}}$ converge to $e_1 \in \mathbb{P}_{\mathbb{H}}^2$ as $n \to \infty$. It follows that $e_1$ is the only cluster point of orbits $\{g^n(p)\}_{n \in \mathbb{Z}}$ for every point $p \in \mathbb{P}_{\mathbb{H}}^2 \setminus L_0(G)$ and so $L_1(G) = \{e_1\}$.

To find $L_2(G)$, consider a closed set $C = \mathbb{L}\{e_1, e_2\} \subset \mathbb{P}_{\mathbb{H}}^2$. Now note that $D_5$ induces a pseudo-projective transformation in $QP(3, \mathbb{H})$ such that $\mathbb{L}\{e_1, e_2\} = Ker(D_5)$. Therefore, using Lemma 2.8 we can show that the set of cluster points of orbits of compact subsets of $\mathbb{P}_{\mathbb{H}}^2 \setminus C$ lies in $L_0(G) \cup L_1(G) = \{e_1\}$. Then, using Lemma 2.9 we have $L_2(G) \subset \mathbb{L}\{e_1, e_2\}$. Now to show that $\mathbb{L}\{e_1, e_2\} \subset L_2(G)$, we can suitably modify the example given in the case of non-vertical translation. Consider a sequence $(k_n)_{n \in \mathbb{N}}$ such that $k_n = [x : (-1 + \frac{1}{n}) : \frac{2}{n} e^{2\pi i \alpha}]$. Note that sequence $(k_n)_{n \in \mathbb{N}} \subset K$, where
For the loxodromic case, it is:

\[
g^n(k_n) = g^n \left( x : \left(1 + \frac{1}{n} \right) e^{2\pi i n} : \frac{2}{n} e^{2\pi i n} \right).
\]

Then there exists a subsequence of \( g^n(k_n) \) which converges to a point \([x : 1 : 0] \in L_2(G)\) as \( n \to \infty \). This implies \( L_1 e_1, e_2 \subset L_2(G) \). Hence, \( L_2(G) = \mathbb{L}\{e_1, e_2\} \).

This completes the proof. \( \square \)

### 6 Extended Conze-Guivarc'h Limit Set

In this section, we take the natural action of \( \text{PSL}(3, \mathbb{H}) \) on the dual space \( (\mathbb{P}_1^2)^* \). This action is given by \( g : l \mapsto (g^{-1})^T \begin{bmatrix} a \\ b \\ c \end{bmatrix} \), where \( g \in \text{PSL}(3, \mathbb{H}) \) and \( l \in (\mathbb{P}_1^2)^* \) is the quaternionic projective line with polar vector \([a : b : c] \in \mathbb{P}_1^2 \mathbb{H} \). We define lines \( l_1, l_2, l_3 \in (\mathbb{P}_1^2)^* \) as \( \lambda_i = \{x_i = 0 \} \subset \mathbb{P}_1^2 \mathbb{H} \), where \( i \in \{1, 2, 3\} \). Note that for each \( i \), \( l_i \) is polar to the point \( e_i \) of \( \mathbb{P}_1^2 \mathbb{H} \). Define the open sets \( U_i \) as \( U_i = \{ z_i : z_i \neq 0 \} \subset (\mathbb{P}_1^2)^* \), where \( i = 1, 2, 3 \).

Now recall the notion of the extended Conze-Guivarc'h limit set, see Definition 1.8. In the next theorem, we will prove Table 4 for the extended Conze-Guivarc'h limit sets.

**Theorem 6.1** The extended Conze-Guivarc'h limit sets for the cyclic subgroups of \( \text{PSL}(3, \mathbb{H}) \) are the following:

(i) For the elliptic case, it is \( \hat{L}(G) = \mathbb{L}\{e_1, e_3\} \) for vertical translation.

(ii) For the parabolic case, it is:

(a) \( \hat{L}(G) = \mathbb{L}\{e_1, e_3\} \) for vertical translation,

(b) \( \hat{L}(G) = \mathbb{L}\{e_1, e_2\} \) for non-vertical translation,

(c) \( \hat{L}(G) = \mathbb{L}\{e_1, e_3\} \) for ellipto-parabolic and

(d) \( \hat{L}(G) = \mathbb{L}\{e_1, e_2\} \) for ellipto-parabolic.

(iii) For the loxodromic case, it is:

(a) \( \hat{L}(G) = \mathbb{L}\{e_1, e_2\} \cup \mathbb{L}\{e_2, e_3\} \) for regular loxodromic,

(b) \( \hat{L}(G) = \mathbb{L}\{e_1, e_2\} \) for screw,

(c) \( \hat{L}(G) = \mathbb{L}\{e_1, e_2\} \) for homothety and

(d) \( \hat{L}(G) = \mathbb{L}\{e_1, e_2\} \cup \mathbb{L}\{e_1, e_3\} \) for loxo-parabolic.

**Proof** We will consider various cases depending on the conjugacy classes to determine the extended Conze-Guivarc'h limit sets.

(i) **Elliptic element:** In this case, we have \( g = \begin{bmatrix} e^{2\pi i \alpha} & 0 & 0 \\ 0 & e^{2\pi i \beta} & 0 \\ 0 & 0 & e^{2\pi i \gamma} \end{bmatrix} \), \( \alpha, \beta, \gamma \in \mathbb{R} \). Thus an elliptic element \( g \) is of finite order only if \( \alpha, \beta, \gamma \in \mathbb{Q} \). In this case

\[
g^n \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = (g^{-n})^T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} e^{-2\pi i n \alpha} & 0 & 0 \\ 0 & e^{-2\pi i n \beta} & 0 \\ 0 & 0 & e^{-2\pi i n \gamma} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} e^{-2\pi i n \alpha}a \\ e^{-2\pi i n \beta}b \\ e^{-2\pi i n \gamma}c \end{bmatrix}.
\]
For finite order case \( \hat{L}(G) = \phi \) and for infinite order case it is \((P^2_{H})^*\). To see this, for each \( q = [a : b : c] \in (P^2_{H})^*\), consider open set \( U = (P^2_{H})^*\).

(ii) **Parabolic element:**

(a) **Vertical translation:** In this case, \( g = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \). This case is similar to the corresponding complex case (cf. [4]), therefore \( \hat{L}(G) = \mathbb{L}\{e_1, e_3\} \subset P^2_{H} \). To see this, consider \( q = [0 : 1 : 0] \in (P^2_{H})^* \) and open set \( U = U_1 \).

(b) **Non-vertical translation:** In this case, \( g = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \). This case is similar to the corresponding complex case (cf. [4]), so \( \hat{L}(G) = \mathbb{L}\{e_1, e_2\} \subset P^2_{H} \). To see this, consider \( q = [0 : 0 : 1] \in (P^2_{H})^* \) and open set \( U = U_1 \).

(c) **Ellipto-parabolic:** In this case, \( g = \begin{pmatrix} e^{2\pi i \alpha} & 1 & 0 \\ 0 & e^{2\pi i \alpha} & 0 \\ 0 & 0 & e^{2\pi i \beta} \end{pmatrix} \). Then for each line \( p = [a : b : c] \in (P^2_{H})^* \), we have \( g^n \cdot p = (g^{-n})^T \begin{pmatrix} a \\ b \\ c \end{pmatrix} \). Then for each line \( p = [a : b : c] \in (P^2_{H})^* \), we have

\[
\begin{pmatrix}
e^{-2\pi in\alpha} & 0 & 0 \\
ne^{-2\pi i(n-1)\alpha} & e^{2\pi in\alpha} & 0 \\
0 & 0 & e^{-2\pi in\beta}
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix}
= 
\begin{pmatrix}
e^{-2\pi in\alpha} a \\
ne^{-2\pi i(n-1)\alpha} + e^{2\pi in\alpha} b \\
e^{-2\pi in\beta} c
\end{pmatrix}.
\]

Now consider the open set \( U = U_1 \subset (P^2_{H})^* \). Note that sequence \( \frac{1}{n}(g^{-n})^T \begin{pmatrix} a \\ b \\ c \end{pmatrix} \) converges to the line \( q = [0 : 1 : 0] \in (P^2_{H})^* \) as \( n \to \infty \) whenever \( a \neq 0 \). Therefore, \( \hat{L}(G) = \mathbb{L}\{e_1, e_3\} \subset P^2_{H} \).

(d) **Ellipto-translation:** In this case, \( g = \begin{pmatrix} e^{2\pi i \alpha} & 1 & 0 \\ 0 & e^{2\pi i \alpha} & 1 \\ 0 & 0 & e^{2\pi i \alpha} \end{pmatrix} \). This implies

\[
(g^{-n})^T =
\begin{pmatrix}
e^{-2\pi i\alpha} & 0 & 0 \\
-ne^{-2\pi i(n+1)\alpha} & e^{-2\pi i\alpha} & 0 \\
n(n+1) & e^{-2\pi i(n+2)\alpha} & -ne^{-2\pi i(n+1)\alpha} & e^{-2\pi i\alpha}
\end{pmatrix}
\]

Thus for each line \( p = [a : b : c] \in (P^2_{H})^* \), we have

\[
g^n \cdot p = (g^{-n})^T \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix}
e^{-2\pi in\alpha} a \\
n^{-2\pi i(n+1)\alpha} + e^{-2\pi in\alpha} b \\
\frac{n(n+1)}{2} e^{-2\pi i(n+2)\alpha} a ne^{-2\pi i(n+1)\alpha} b - e^{-2\pi in\alpha} c\end{pmatrix}.
\]

Now consider the open set \( U = U_1 \subset (P^2_{H})^* \). Note that sequence \( \frac{2}{n(n+1)}(g^{-n})^T \begin{pmatrix} a \\ b \\ c \end{pmatrix} \) converges to the line \( q = [0 : 0 : 1] \in (P^2_{H})^* \) as \( n \to \infty \) whenever \( a \neq 0 \). Therefore, \( \hat{L}(G) = \mathbb{L}\{e_1, e_2\} \subset P^2_{H} \).
(iii) Loxodromic element:

(a) Regular loxodromic: In this case, \( g = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \xi \end{pmatrix} \), \(|\lambda| < |\mu| < |\xi|\). Then \( g^n \cdot p = (g^{-n})^T \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \lambda^{-n}a & 0 & 0 \\ 0 & \mu^{-n}b & 0 \\ 0 & 0 & \xi^{-n}c \end{pmatrix} \). Now consider the open set \( U_1 = \{ [a : b : c] \in (\mathbb{P}^2_\mathbb{H})^* \mid a \neq 0 \} \) and lift \(|\lambda|^n(g^{-n})^T\) of \((g^{-n})^T\). Then the sequence \( g^n \cdot p \) converges to the line \( q = [1 : 0 : 0] \in (\mathbb{P}^2_\mathbb{H})^* \) as \( n \to \infty \) for every line \( p = [a : b : c] \in U_1 \). Therefore, \( \mathbb{L}(e_2, e_3) \subset \hat{L}(G) \). Further, consider the open set \( U_3 = \{ [a : b : c] \in (\mathbb{P}^2_\mathbb{H})^* \mid c \neq 0 \} \) and lift \( \frac{1}{|\xi|^n} (g^n)^T \) of \((g^n)^T\). Then the sequence \( g^{-n} \cdot p \) converges to the line \( q = [0 : 0 : 1] \in (\mathbb{P}^2_\mathbb{H})^* \) as \( n \to \infty \) for every line \( p = [a : b : c] \in U_3 \). Therefore, \( \mathbb{L}(e_1, e_2) \subset \hat{L}(G) \). Hence, we have \( \hat{L}(G) = \mathbb{L}(e_1, e_2) \cup \mathbb{L}(e_2, e_3) \).

(b) Screw loxodromic: In this case, \( g = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \xi \end{pmatrix} \), where \(|\lambda| = |\mu| \neq 1\), \( \lambda \neq \mu \) and \(|\xi| = 1/|\lambda|^2\). Then \( g^n \cdot p = (g^{-n})^T \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \lambda^{-n}a & 0 & 0 \\ 0 & \mu^{-n}b & 0 \\ 0 & 0 & \xi^{-n}c \end{pmatrix} \). Consider the open set \( U_3 = \{ [a : b : c] \in (\mathbb{P}^2_\mathbb{H})^* \mid c \neq 0 \} \) and lift \( \frac{1}{|\xi|^n} (g^{-n})^T \) of \((g^{-n})^T\). Now it is not hard to check that \( g^n \cdot p \) converges to the line \( q = [0 : 0 : 1] \in (\mathbb{P}^2_\mathbb{H})^* \) as \( n \to \infty \) for every line \( p = [a : b : c] \in U_3 \). Therefore, \( \hat{L}(G) = \mathbb{L}(e_1, e_2) \).

(c) Homothety: In this case, \( g = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \xi \end{pmatrix} \), \(|\lambda| \neq 1\), \(|\xi| = 1/|\lambda|^2\). Then \((g^{-n})^T = \begin{pmatrix} \lambda^{-n} & 0 & 0 \\ 0 & \lambda^{-n} & 0 \\ 0 & 0 & \xi^{-n} \end{pmatrix} \). Now we can show that \( \hat{L}(G) = \mathbb{L}(e_1, e_2) \) by using a similar argument as in the case of screw loxodromic.

(d) Loxo-parabolic: In this case, \( g = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda \alpha \xi & 0 \\ 0 & 0 & \xi \end{pmatrix} \), \(|\xi| = |\lambda|^{-2} \neq 1\). Then

\[
g^n \cdot p = (g^{-n})^T \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \lambda^{-n}a \\ -n\lambda^{-n-1}a + \lambda^{-n}b \\ |\lambda|^{2n} e^{-in\arg(\xi)}c \end{pmatrix}
\]

Now consider the open set \( U_3 = \{ [a : b : c] \in (\mathbb{P}^2_\mathbb{H})^* \mid c \neq 0 \} \) and lift \( \frac{1}{|\lambda|^{-n}} (g^{-n})^T \) of \((g^{-n})^T\). Then the sequence \( g^n \cdot p \) converges to the line \( q = [0 : 0 : 1] \in (\mathbb{P}^2_\mathbb{H})^* \) as \( n \to \infty \) for every line \( p = [a : b : c] \in U_3 \). Therefore, \( \mathbb{L}(e_2, e_3) \subset \hat{L}(G) \). Further, consider the open set \( U_1 = \{ [a : b : c] \in (\mathbb{P}^2_\mathbb{H})^* \mid a \neq 0 \} \) and lift \( \frac{1}{n|\lambda|^{-n}} (g^n)^T \) of \((g^n)^T\). Then the sequence \( g^{-n} \cdot p \) converges to the line \( q = [0 : 1 : 0] \in (\mathbb{P}^2_\mathbb{H})^* \) as \( n \to \infty \) for every line \( p = [a : b : c] \in U_1 \). Therefore, \( \mathbb{L}(e_1, e_3) \subset \hat{L}(G) \). Hence, we have \( \hat{L}(G) = \mathbb{L}(e_1, e_2) \cup \mathbb{L}(e_1, e_3) \).

This completes the proof. \( \square \)
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Declarations

Competing Interests The authors declare no competing interests.

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