The Vanishing of Two-Point Functions for Three-Loop Superstring Scattering Amplitudes

Samuel Grushevsky1, Riccardo Salvati Manni2

1 Mathematics Department, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544, USA. E-mail: sam@math.princeton.edu
2 Dipartimento di Matematica, Università “La Sapienza”, Piazzale A. Moro 2, Roma, I 00185, Italy. E-mail: salvati@mat.uniroma1.it

Abstract: In this paper we show that the two-point function for the three-loop chiral superstring measure ansatz proposed by Cacciatori, Dalla Piazza, and van Geemen [2] vanishes. Our proof uses the reformulation of the ansatz given in [8], theta functions, and specifically the theory of the $\Gamma_{00}$ linear system, introduced by van Geemen and van der Geer [6], on Jacobians.

At the two-loop level, where the amplitudes were computed by D’Hoker and Phong [11–14,17,18], we give a new proof of the vanishing of the two-point function (which was proven by them). We also discuss the possible approaches to proving the vanishing of the two-point function for the proposed ansatz in higher genera [3,8,25].

1. Introduction

An investigation of the problem of computing the superstring measure explicitly for arbitrary genus of the worldsheet was begun by the work of Green and Schwarz [7], who gave an explicit formula in genus 1 using operator methods. D’Hoker and Phong in a series of papers [11–14] introduced a gauge-fixing procedure and computed from first principles the genus 2 superstring measure, verifying that it satisfied the physical constraints, e.g. the vanishing of the 1,2,3-point functions. They also proposed in [15,16] to search for an ansatz for the superstring measure in arbitrary genus as the product of the bosonic measure and a modular form.

The ansatz for three-loop measure in this form was then proposed by Cacciatori, Dalla Piazza, and van Geemen in [2]. The genus $g \leq 3$ ansatze were reformulated in terms of syzygetic subspaces by the first author in [8], where an ansatz for general genus was proposed, under the assumption on holomorphicity of certain $2^r$-roots. Cacciatori, Dalla Piazza, and van Geemen in [3] give the genus 4 ansatz in terms of quadrics in the theta constants. The second author in [25] showed that the proposed ansatz is holomorphic in genus 5. Dalla Piazza and van Geemen in [4] proved the uniqueness of the

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modular form in genus 3 satisfying the factorization constraints. Morozov in [23] surveyed this work and gave an alternative proof that factorization constraints are satisfied for the ansatz; in [24] he has also investigated the 1,2,3-point functions of the proposed ansatz, proving under certain non-trivial mathematical assumption that they vanish on the hyperelliptic locus.

In this paper we use the techniques of theta functions, and especially the \(\Gamma_{00}\) sublinear system of the linear system \(|\Theta|\) introduced by van Geemen and van der Geer [6] to prove the vanishing of the 2-point function in genus 3. We also obtain a new proof of the vanishing of the 2-point function in genus 2.

2. Notations and Definitions

We denote by \(A_g\) the moduli space of complex principally polarized abelian varieties (ppav for short) of dimension \(g\), and by \(H_g\) the Siegel upper half-space of symmetric complex matrices with positive-definite imaginary part, called period matrices. The space \(H_g\) is the universal cover of \(A_g\), with the deck group \(\text{Sp}(2g, \mathbb{Z})\), so that we have \(A_g = H_g/\text{Sp}(2g, \mathbb{Z})\) for a certain action of the symplectic group. A function \(f : H_g \to \mathbb{C}\) is called a (scalar) modular form of weight \(k\) with respect to a subgroup \(\Gamma \subset \text{Sp}(2g, \mathbb{Z})\) if

\[
f(\gamma \circ \tau) = \det(C \tau + D)^k f(\tau) \quad \forall \gamma \in \Gamma, \forall \tau \in H_g,
\]

where \(C\) and \(D\) are the lower blocks if we write \(\gamma\) as four \(g \times g\) blocks.

For a period matrix \(\tau \in H_g\) the principal polarization \(\Theta_\tau\) on the abelian variety \(A_\tau := \mathbb{C}^g/(\mathbb{Z}^g + \tau \mathbb{Z}^g)\) is the divisor of the theta function

\[
\theta(\tau, z) := \sum_{n \in \mathbb{Z}^g} \exp(\pi i (n^t \tau n + 2n^t z)).
\]

Notice that for fixed \(\tau\) theta is a function of \(z \in \mathbb{C}^g\), and its automorphy properties under the lattice \(\mathbb{Z}^g + \tau \mathbb{Z}^g\) define the bundle \(\Theta_\tau\).

Given a point of order two on \(A_\tau\), which can be uniquely represented as \(\frac{\varepsilon \tau + \delta}{2}\) for \(\varepsilon, \delta \in \mathbb{Z}_2^g\) (where \(\mathbb{Z}_2 = \{0, 1\}\) is the additive group), the associated theta function with characteristic is

\[
\theta \left[ \begin{matrix} \varepsilon & \delta \\ \end{matrix} \right](\tau, z) := \sum_{n \in \mathbb{Z}^g} \exp(\pi i ((n + \varepsilon)^t \tau (n + \varepsilon) + 2(n + \varepsilon)^t (z + \delta))).
\]

As a function of \(z\), \(\theta \left[ \begin{matrix} \varepsilon & \delta \\ \end{matrix} \right]\) is odd or even depending on whether the scalar product \(\varepsilon \cdot \delta \in \mathbb{Z}_2\) is equal to 1 or 0, respectively. The theta function with characteristic is the generator of the space of sections of the bundle \(\Theta_\tau + \frac{\varepsilon \tau + \delta}{2}\) (where we have implicitly identified the principally polarized abelian variety with its dual, and think of points as bundles of degree 0). Thus the square of any theta function with characteristic is a section of \(2\Theta_\tau\), and the basis for the space of sections of this bundle is given by theta functions of the second order

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