ON THE SUPPORTS IN THE HUMILIÈRE COMPLETION
AND
γ-COISOTROPIC SETS

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ABSTRACT. The symplectic spectral metric on the set of Lagrangian submanifolds or Hamiltonian maps can be used to define a completion of these spaces. For an element of such a completion, we define its γ-support. We also define the notion of γ-coisotropic set, and prove that a γ-support must be γ-coisotropic together with many properties of the γ-support and γ-coisotropic sets. We give examples of Lagrangians in the completion having large γ-support and we study those (called "regular Lagrangians") having small γ-support. Finally we try to understand which singular Hamiltonians (i.e. a Hamiltonian continuous on the complement of a set) have a well-defined flow in the Humilière completion.

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1. Introduction

In [Vit92], a metric, denoted $\gamma$, was introduced on the set $\mathcal{L}_0(T^* N)$ of Lagrangians Hamiltonianly isotopic to the zero section in $T^* N$, where $N$ is a compact manifold and on $\mathcal{D}\mathcal{H}\mathcal{a}\mathcal{m}_c(T^* N)$ the group of Hamiltonian maps with compact support for $N = \mathbb{R}^n$ or $T^n$ (in [Vit06] a similar metric was defined on $\mathcal{D}\mathcal{H}\mathcal{a}\mathcal{m}_c(T^* N)$ for general compact $N$). This metric was extended by Schwartz and Oh to general symplectic manifolds $(M, \omega)$ using Floer cohomology for the Hamiltonian case (see [Sch00; Oh05]), and by [Lec08; LZ18] for the Lagrangian case. In particular this distance is defined for elements in $\mathcal{L}_L(M, \omega)$ the set of exact Lagrangians in $(M, \omega)$ and can sometimes be extended to $\mathcal{L}(M, \omega)$ the space of all exact Lagrangians, notably for $M = T^* N$.

The completion of the spaces $\mathcal{L}_0(M, \omega)$ and $\mathcal{D}\mathcal{H}\mathcal{a}\mathcal{m}_c(M, \omega)$ for the metric $\gamma$ have been studied for the first time in [Hum08b]. We shall denote these completions by $\hat{\mathcal{L}}(M, \omega)$ and $\hat{\mathcal{D}}\mathcal{H}\mathcal{a}\mathcal{m}(M, \omega)$ and call them the Humilière completions.

Our goal in this paper is, among others, to pursue the study of these completions. Our $\gamma$-distance is the one defined using Floer cohomology, but we shall see that in $T^* N$ it can also be defined using sheaves (see Section 4) and this will be useful when applying results from [Vit22].

We wish to point out that elements in the Humilière completion occur naturally in symplectic topology. One example is symplectic homogenization (see [Vit23]) where flows of Hamiltonians which are only continuous appear as homogenized Hamiltonians. Another example is the graph of $df$ where $f$ is only a continuous function. This yields a notion of subdifferential, that will be proved in [AGHIV] to be equal to the one defined by Vichery ([Vic13]) using the microlocal theory of sheaves. More generally a continuous Hamiltonian with singularities, if the singular set is small, has a flow in $\hat{\mathcal{D}}\mathcal{H}\mathcal{a}\mathcal{m}_c(T^* N)$ (see [Hum08b] and Section 10).

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1. Assuming the aspherical assumptions $|\omega|\pi_2(M, L) = 0, \mu_1, \pi_2(M, L) = 0$.

2. Mostly in the Hamiltonian case and for $\mathbb{R}^{2n}$, but most the results in [Hum08b] still hold in this more general case.
In Section 6 we shall define the notion of \( \gamma \)-support for a Lagrangian in \( \mathcal{L}(M,\omega) \) (see Definition 6.1) and prove some of their properties. In Section 7 we define \( \gamma \)-coisotropic subsets (see Definition 7.1) - originally defined in a slightly different version by M. Usher in [Ush19] under the name of “locally rigid”. We will then prove a number of their properties (see Proposition 7.5). In particular we prove that the \( \gamma \)-support of an element in \( \mathcal{L}(M,\omega) \) is \( \gamma \)-coisotropic, and many \( \gamma \)-coisotropic subsets can be obtained as \( \gamma \)-supports. Characterizing the \( \gamma \)-coisotropic that are \( \gamma \)-supports remains however an open problem.

In Section 8 we study elements in \( \mathcal{L}(M,\omega) \) having minimal \( \gamma \)-support, i.e. such that their support is Lagrangian. It is legitimate to ask, at least when the \( \gamma \)-support is a smooth, whether \( L \) coincides with \( \gamma \)-supp(\( L \)).

We were only able to prove this for a certain class of manifolds. This point of view also yields a new definition of \( C^0 \)-Lagrangian: these are the \( n \)-dimensional topological submanifolds which are \( \gamma \)-supports of elements of \( \mathcal{L}(M,\omega) \). We show that our definition of \( \gamma \)-coisotropic is more restrictive than previously defined ones (see Proposition 7.10). In Section 10 we try to explain which singular Hamiltonians have a flow defined in \( \mathcal{D}\mathcal{S}\mathcal{A}\mathcal{M}(M,\omega) \), extending previous results by [Hum08b].

The ideas introduced here have been applied in a number of other works.

In a joint paper with Stéphane Guillermou ([GVit22a]) we prove that singular supports of sheaves in \( D^b(N) \) are \( \gamma \)-coisotropic, replacing the “involutive” notion from [KS90] by a condition invariant by symplectic homeomorphism. Together with Tomohiro Asano, Stéphane Guillermou, Vincent Humilière and Yuichi Ike ([AGHIV]) we prove that for the quantization \( \mathcal{F}_L \) of an element \( \bar{L} \) in \( \hat{\mathcal{D}}(T^*\mathcal{N}) \) (see [GVit22a] for the definition), the \( \gamma \)-support of \( \bar{L} \) coincides with the singular support of \( \mathcal{F}_L \).

Finally in a joint work with M.-C. Arnaud and Vincent Humilière (see [AHV24]), we study some connections between the \( \gamma \)-support and conformal symplectic dynamics and their invariant sets, showing that the \( \gamma \)-supports provide a natural generalization of the Birkhoff attractor in dimension 2.

2. Comments and Acknowledgements

This paper precedes, both logically and historically the papers [Vit22] and [GVit22a] which are of course related.

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3. Notations

\( \mathcal{FD}(M, \omega) \) the Fukaya-Donaldson category of Lagrangian branes
\( \text{FD}(M, \omega) \) the set of isomorphism classes in the Fukaya-Donaldson category
\((T^*N, d\lambda)\) or just \( T^*N \) the cotangent bundle of a closed manifold, \( \lambda \) its Liouville form
\( (T^*N, -d\lambda) \) or just \( \overline{T^*N} \) the cotangent bundle of a closed manifold, with the opposite symplectic form
\( (T^*N \setminus 0_N, d\lambda) \) or \( \check{T}^*N \) the cotangent bundle of a closed manifold, with the zero section removed
\( (M, \omega) \) an aspherical symplectic manifold, i.e. \([\omega] \pi_2(M) = 0\) and \( c_1(TM) \pi_2(M) = 0 \) either closed or convex at infinity
\( (M, d\lambda) \) an exact symplectic manifold with Liouville form \( \lambda \), convex at infinity
\( c_+(\overline{L}_1, \overline{L}_2), c_-(\overline{L}_1, \overline{L}_2) \) spectral numbers associated to pairs of elements in \( \mathcal{L}(M, \omega) \)
\( \mathcal{L}(M, \omega) \) the set of compact Lagrangians in \( (M, \omega) \)
\( \gamma \) the spectral metric, defined either for all pairs of compact exact Lagrangians when \( M = T^*N \) or those such that \([\omega] \pi_2(M, L) = 0\) and \( FH_* \mathcal{L}(M, \omega) \cong H_*(L_0) \) (in particular those Hamiltonianly isotopic to \( L_0 \))
\( \mathcal{\hat{L}}(M, \omega) \) the Humilière completion of the above (i.e. completion for \( \gamma \))
\( \mathcal{L}_0(M, \omega) \) a component of the set of Lagrangians in \( (M, \omega) \) on which \( \gamma \) is well-defined and endowed with the metric \( \gamma \). The Lagrangian \( L_0 \) is implicit (see Definition 5.1)
\( \mathcal{\hat{L}}_0(M, \omega) \) the Humilière completion of the above (i.e. completion for \( \gamma \))
\( \mathcal{L}(M, d\lambda) \) the set of compact Lagrangian branes \( \check{L} \) in \( (M, d\lambda) \) (see Definition 5.1)
\( \mathcal{\hat{L}}(M, d\lambda) \) the Humilière completion of the above (i.e. completion for \( c \))
\( \mathcal{\hat{L}}_0(M, d\lambda) \) the set of \( \check{L} \) in \( \mathcal{\hat{L}}(M, d\lambda) \) such that \( L \in \mathcal{L}_0(M, d\lambda) \)
\( \mathcal{DHam}_c(M, \omega) \) the group of time-one maps of compact supported Hamiltonian flows endowed with the \( \gamma \)-metric
\( \mathcal{\overline{DHam}}(M, \omega) \) its Humilière completion (i.e. completion for \( \gamma \))
\( \mathcal{DHam}_c(M, \omega) \) the group of compact supported Hamiltonian maps (see page 5).
\( \mathcal{\overline{DHam}}(M, \omega) \) its Humilière completion (i.e. completion for \( c \))

4. Spectral invariants for sheaves and Lagrangians

For \( M \) a symplectic manifold, if \( \Lambda(M) \) is the bundle of Lagrangians subspaces of the tangent bundle to \( M \), with fiber the Lagrangian Grassmanian \( \Lambda(T_xM) = \Lambda(n) \), we denote by \( \overline{\Lambda}(M) \) the bundle induced by the universal cover \( \overline{\Lambda}(n) \rightarrow \Lambda(n) \).
When using coefficients in a field of characteristic different from 2, given a Lagrangian \( L \), we assume we have a lifting of the Gauss map \( G_L : L \to \Lambda(M) \) given by \( x \mapsto T_x L \) to a map \( \bar{G}_L : L \to \bar{\Lambda}(M) \). This is called a grading of \( L \) (see [Sei00]). Given a graded \( L \), the canonical automorphism of the covering induces a new grading and we denote it as \( T(L) \) or \( L[1] \), and its \( q \)-th iteration as \( T^q(L) \) or \( L[q] \). The grading yields an absolute grading for the Floer homology of a pair \((L_1, L_2)\) and hence for the complex of sheaves in the Theorem stated below. We shall almost never mention explicitly the grading, but notice that for exact Lagrangians in \( T^* N \), a grading always exists since the obstruction to its existence is given by the Maslov class, and for exact Lagrangians in \( T^* N \) the Maslov class vanishes, as was proved by Kragh and Abouzaid (see [Kra13], and also the sheaf-theoretic proof by [Gui12]). For \((M, d\lambda)\) we consider the set \( \mathcal{L}(M, d\lambda) \) of Lagrangian branes, that is triples \( \bar{L} = (L, f_L, \bar{G}_L) \) where \( L \) is a compact exact graded Lagrangian, and \( f_L \) a primitive of \( \lambda_L \). We sometimes talk about an exact Lagrangian, and this is just the pair \((L, f_L)\).

When \( f_L \) is implicit we only write \( L \), for example graph\((d f)\) means \((\text{graph}(d f), f)\) and in particular \( 0 \) means \((0, 0, 0)\). In both cases there is an obvious canonical grading. For \( \bar{L} = (L, f_L) \) and \( c \) a real constant, we write \( \bar{L} + c \) for \((L, f_L + c)\). Considering compact supported Hamiltonian diffeomorphisms as special correspondences, that is Lagrangians in \( \mathcal{M} \times M \) we can consider the corresponding branes, and denote this space by \( \mathcal{D} \mathcal{H}am_c(M, \omega) \).

More generally if \( \mathcal{F}D(M, d\lambda) \) is the Fukaya-Donaldson category, with objects elements of \( \mathcal{L}(M, d\lambda) \) and morphisms \( \text{Mor}_{\mathcal{F}D}(L, L') = FH^*(L, L') \) and \( \mathcal{F}D(M, d\lambda) \) any subcategory containing a single object for each isomorphisms class in \( \mathcal{F}D(M, d\lambda) \) : this is also called the bf skeleton of \( \mathcal{F}D(M, d\lambda) \). By abuse of language if \( L \) is in \( \mathcal{L}(M, d\lambda) \) and \( A \) is an object in \( \mathcal{F}D(M, d\lambda) \) we write \( L \in A \) if \( L \) is isomorphic to \( A \).

**4.1. Sheaf theoretic approach in \( T^* N \).** Let \((L, f_L)\) be an exact Lagrangian in \( \mathcal{L}(T^* N) \) and

\[
\hat{L} = \{ (q, \tau p, f_L(q, p), \tau) \mid (q, p) \in L, \tau > 0 \}
\]

the homogenized Lagrangian in \( T^*(N \times \mathbb{R}) \). We denote by \( D^b(N) \) the derived category of bounded complexes of sheaves on \( N \). On \( D^b(N \times \mathbb{R}) \) we define \( * \) as follows. First we set \( s : N \times \mathbb{R} \times N \times \mathbb{R} \to N \times N \times \mathbb{R} \) given by \( s(x_1, t_1, x_2, t_2) = (x_1, x_2, t_1 + t_2) \) and \( d : N \times \mathbb{R} \to N \times N \times \mathbb{R} \) given by \( d(x, t) = (x, x, t) \) and now

\[
\mathcal{F}^* \star \mathcal{G}^* = (Rs)_! d^{-1}(\mathcal{F}^* \boxtimes \mathcal{G}^*)
\]
Then $R\mathcal{H}om^*$ is the adjoint of $\star$ in the sense that
\[
\text{Mor}_{D^b}(\mathcal{F}^*, R\mathcal{H}om^*(\mathcal{G}^*, \mathcal{H}^*)) = \text{Mor}_{D^b}(\mathcal{F}^* \star \mathcal{G}^*, \mathcal{H}^*)
\]

According to [Gui12] (for (1), (2), (4)) and [Vit19] (for the other properties), we have

**Theorem 4.1.** To each $\tilde{L} \in \mathcal{L}(T^*N)$ we can associate $\mathcal{F}_L^* \in D^b(N)$ such that

1. $SS(\mathcal{F}_L^*) \cap \tilde{T}^*(N \times \mathbb{R}) = \tilde{L}$
2. $\mathcal{F}_L^*$ is pure (cf. [KS90] page 309), $\mathcal{F}_L = 0$ near $N \times \{-\infty\}$ and $\mathcal{F}_L = k_N$ near $N \times \{+\infty\}$
3. We have an isomorphism
\[
F\mathcal{H}^*(L_0, L_1; a, b) = H^* \left( N \times [a, b], R\mathcal{H}om^*(\mathcal{F}_L^*, \mathcal{F}_L^*) \right)
\]
4. $\mathcal{F}_L^*$ is unique satisfying properties (1) and (2).
5. There is a natural product map
\[
R\mathcal{H}om^*(\mathcal{F}_{L_1}^*, \mathcal{F}_{L_2}^*) \otimes R\mathcal{H}om^*(\mathcal{F}_{L_2}^*, \mathcal{F}_{L_3}^*) \rightarrow R\mathcal{H}om^*(\mathcal{F}_{L_1}^*, \mathcal{F}_{L_3}^*)
\]
inducing in cohomology a map
\[
H^* (N \times [\lambda, +\infty], R\mathcal{H}om^*(\mathcal{F}_{L_1}^*, \mathcal{F}_{L_2}^*)) \otimes H^* (N \times [\mu, +\infty], R\mathcal{H}om^*(\mathcal{F}_{L_2}^*, \mathcal{F}_{L_3}^*)) \]
\[
\downarrow \cup_{\star}
\]
\[
H^* (N \times [\lambda + \mu, +\infty], R\mathcal{H}om^*(\mathcal{F}_{L_1}^*, \mathcal{F}_{L_3}^*))
\]
that coincides through the above identifications to the triangle product in Floer cohomology.

**Remark 4.2.** The grading $\tilde{G}_L$ defines the grading of $\mathcal{F}_L^*$, hence of the Floer cohomology.

Note that for $X$ open, we denoted by $H^* (X \times [\lambda, \mu[, \mathcal{F}^*)$ the relative cohomology of sections on $X \times ]-\infty, \mu[$ vanishing on $X \times ]-\infty, \lambda[ and fitting in the exact sequence
\[
H^* (X \times [\lambda, \mu[, \mathcal{F}^*) \rightarrow H^* (X \times ]-\infty, \mu[, \mathcal{F}^*) \rightarrow H^* (X \times ]-\infty, \lambda[, \mathcal{F}^*)
\]
It is also equal to the cohomology associated to the derived functor $R\Gamma_Z$ where $Z$ is the locally closed set $X \times [\lambda, \mu[$. We should write $H^*_X (X \times \mathbb{R}, \mathcal{F}^*)$ but this is too cumbersome. We should write $\mathcal{F}_L^*$ instead of $\mathcal{F}_L^*$ but this abuse of notation should be harmless.

**Definition 4.3.** We denote by $\mathcal{F}_0(N)$ the set of $\mathcal{F}^*$ in $D^b(X \times \mathbb{R})$ such that $SS(\mathcal{F}^*) \subset \{\tau \geq 0\}$, $\mathcal{F}^* = 0$ near $N \times \{-\infty\}$ and $\mathcal{F}^* = k_N$ near $N \times \{+\infty\}$.

Note that the set of $\mathcal{F}^*$ such that $SS(\mathcal{F}^*) \subset \{\tau \geq 0\}$ contains the Tamarkin category.
**Definition 4.4** (see [Vic12], Section 8.3). Let $\mathcal{F}^*$ be an element in $\mathcal{T}_0(N)$. Let $\alpha \in H^*(N \times \mathbb{R}, \mathcal{F}^*) \simeq H^*(N)$ be a nonzero class. We define

$$c(\alpha, \mathcal{F}^*) = \sup \{ t \in \mathbb{R} \mid \alpha \in \text{Im}(H^*(N \times [t, +\infty[, \mathcal{F}^*)) \}$$

Note that $\mathcal{F}^*_L$ satisfies (2), so $H^*(N \times \mathbb{R}, \mathcal{F}^*) \simeq H^*(N)$ and thus we have, using the canonical map

$$H^*(N \times [t, +\infty[, \mathcal{F}^*) \to H^*(N \times \mathbb{R}, \mathcal{F}^*)$$

and Theorem [4.1](#).

**Proposition 4.5.** Then $c(\alpha, \mathcal{F}^*_L)$ coincides with the spectral invariant $c(\alpha, L)$ associated to $\alpha$ by using Floer cohomology (see [Sch00, Oh05]).

As a consequence the $c(\alpha, \mathcal{F}^*)$ satisfy the properties of the Floer homology Lagrangian spectral invariants, and in particular the triangle inequality, since this holds in Floer homology (see [HLS16b], theorem 17). However we shall sometimes need to extend the triangle inequality to situations where $\mathcal{F}^*$ is in $\mathcal{T}_0(X)$ but does not necessarily correspond to an exact embedded Lagrangian.

**Proposition 4.6** (Triangle inequality for sheaves (see [Vic12], proposition 8.13)). Let $\mathcal{F}_1^*, \mathcal{F}_2^*, \mathcal{F}_3^*$ be sheaves on $X \times \mathbb{R}$ such that $\mathcal{F}_i^* \in \mathcal{T}_0(X)$. Then we have

$$c(\alpha \cup_\star \beta; \mathcal{F}_1^*, \mathcal{F}_3^*) \geq c(\alpha; \mathcal{F}_1^*, \mathcal{F}_2^*) + c(\beta; \mathcal{F}_2^*, \mathcal{F}_3^*)$$

**Proof.** We set $\mathcal{F}_{i,j}^* = R\mathcal{H}om^*(\mathcal{F}_i^*, \mathcal{F}_j^*)$ and we have a product

$$\cup_\star : \mathcal{F}_{1,2}^* \otimes \mathcal{F}_{2,3}^* \to \mathcal{F}_{1,3}^*$$

inducing the cup-product

$$H^*(X \times [s, +\infty[, \mathcal{F}_{1,2}^*) \otimes H^*(X \times [t, +\infty[, \mathcal{F}_{2,3}^*) \to H^*(X \times [s + t, +\infty[, \mathcal{F}_{1,3}^*)$$

(see [Vit19], section 9). Then we have the diagram

$$H^*(X \times [s, +\infty[, \mathcal{F}_{1,2}^*) \otimes H^*(X \times [t, +\infty[, \mathcal{F}_{2,3}^*) \to H^*(X \times [s + t, +\infty[, \mathcal{F}_{1,3}^*)$$

where horizontal arrows are cup-products and vertical arrows restriction maps. So if $\alpha \otimes \beta$ is in the image of the left-hand side, which is equivalent to $s \leq c(\alpha, \mathcal{F}_{1,2}^*), t \leq c(\beta, \mathcal{F}_{2,3}^*)$, we have $\alpha \cup_\star \beta$ is in the image of the right hand side, so that $s + t \leq c(\alpha \cup_\star \beta, \mathcal{F}_{1,3}^*)$. This proves our claim. 

Similarly we have
Proposition 4.7 (Lusternik-Shnirelman for sheaves). Let $\mathcal{F}^\ast$ as above. Let $\alpha \in H^\ast(N \times \mathbb{R}, \mathcal{F}^\ast)$ and $\beta \in H^\ast(N \times \mathbb{R}, \mathcal{G}^\ast)$. Then we have a product $\alpha \cup \beta \in H^\ast(N \times \mathbb{R}, \mathcal{F}^\ast \star \mathcal{G}^\ast)$. Then
\[
c(\alpha \cup \beta, \mathcal{F}^\ast \star \mathcal{G}^\ast) \geq c(\alpha, \mathcal{F}^\ast)
\]
and equality implies that $\beta \neq 0$ in $H^\ast(\pi(\text{SS}(\mathcal{F}^\ast)) \cap \{t = c\}), \mathcal{G}^\ast)$ where $c$ is the common critical value. In particular if $c(1, \mathcal{F}^\ast) = c(\mu_N, \mathcal{F}^\ast) = c$ and $\mathcal{F}^\ast$ is constructible, then $\text{SS}(\mathcal{F}^\ast) \supset 0_N \times T^\ast_c \mathbb{R}$.

Proof. By assumption $\alpha$ vanishes in $H^\ast(N \times] - \infty, c - \varepsilon[, \mathcal{F}^\ast)$ but not in $H^\ast(N \times] - \infty, c + \varepsilon[, \mathcal{F}^\ast)$. Assume $\beta$ vanishes in $(N \setminus U) \times] - \infty, c - \varepsilon, c + \varepsilon[$. Then $\alpha \cup \beta$ vanishes in $H^\ast(N \times] - \infty, c - \varepsilon[ \cup (N \setminus U) \times] - \infty, c + \varepsilon[, \mathcal{F}^\ast \star \mathcal{G}^\ast)$ and by assumption does not vanish in
\[
H^\ast(N \times] - \infty, c + \varepsilon[, \mathcal{F}^\ast \star \mathcal{G}^\ast)
\]
But deforming $N \times] - \infty, c - \varepsilon[ \cup (N \setminus U) \times] - \infty, c + \varepsilon[$ can be done through a family of hypersurfaces bounding $W_t$ such that
\[
W_0 = N \times] - \infty, c - \varepsilon[ \cup (N \setminus U) \times] - \infty, c + \varepsilon[ \text{ while } W_1 = N \times] - \infty, c + \varepsilon[,
\]
and we assume
\[
\text{SS}(\mathcal{F}^\ast \star \mathcal{G}^\ast) \cap \{(x, p) \mid x \in \bigcap_{t > 1} W_{\frac{t}{2}} \setminus W_t\} \subset 0_N \times \mathbb{R}
\]
According to the microlocal deformation lemma ([KS90], lemma 2.7.2 page 117 and corollary 5.4.19 page 239), this implies that the natural map $H^\ast(W_1, \mathcal{F}^\ast \star \mathcal{G}^\ast) \to H^\ast(W_0, \mathcal{F}^\ast \star \mathcal{G}^\ast)$ is an isomorphism. In our case this implies that $\alpha \cup \beta$ is zero in
\[
H^\ast(N \times] - \infty, c + \varepsilon[, \mathcal{F}^\ast \star \mathcal{G}^\ast)
\]
hence $c(\alpha \cup \beta; \mathcal{F}^\ast \star \mathcal{G}^\ast) \geq c + \varepsilon$ a contradiction. □

Obviously when $\mathcal{F}^\ast = \mathcal{F}^\ast_L$ the equality $c(1, \mathcal{F}^\ast_L) = c(\mu_N, \mathcal{F}^\ast_L)$ implies $\hat{L} \supset 0_N \times T^\ast_c \mathbb{R}$ hence $L = 0_N$.

Corollary 4.8. As a result $\gamma$ defines a pseudo-metric on $\mathcal{F}_0(N)$. It restricts to a metric on the image of $\mathcal{I}(T^\ast N)$ by the embedding $L \to \mathcal{F}^\ast_L$ which yields a bi-Lipschitz embedding.

Proof. We have
\[
c(1; \mathcal{F}^\ast_1, \mathcal{F}^\ast_3) \geq c(1; \mathcal{F}^\ast_1, \mathcal{F}^\ast_2) + c(1; \mathcal{F}^\ast_2, \mathcal{F}^\ast_3)
\]
and
\[
0 = c(\mu; \mathcal{F}^\ast_1, \mathcal{F}^\ast_1) \geq c(1; \mathcal{F}^\ast_1, \mathcal{F}^\ast_2) + c(\mu; \mathcal{F}^\ast_2, \mathcal{F}^\ast_1)
\]
\[
c(\mu, \mathcal{F}^\ast_1, \mathcal{F}^\ast_2) \geq c(1; \mathcal{F}^\ast_1, \mathcal{F}^\ast_2) + c(\mu, \mathcal{F}^\ast_2, \mathcal{F}^\ast_2) = c(1; \mathcal{F}^\ast_1, \mathcal{F}^\ast_2)
\]
so that $\gamma(\mathcal{F}^\ast_1, \mathcal{F}^\ast_2) \geq 0$. □
Remark 4.9. We refer to [GVit22a, AI20] for different definitions of the metric \( \gamma \) on \( D^b(N \times \mathbb{R}) \). As pointed out in [GVit22a], the map \( Q : L \to \mathcal{F}_L \) extends to a bi-Lipschitz map \( \hat{Q} : \hat{\mathcal{F}}(T^*N) \to D^b(N \times \mathbb{R}) \). In [AGH14] it is proved that \( \gamma \)-supp\((L) = SS(\hat{Q}(L)) \), where for a homogeneous set \( X \) in \( T^*(N \times \mathbb{R}), \hat{X} = X \cap \{ \tau = 1 \} / (t, \tau) \) are the canonical coordinates in \( T^*\mathbb{R} \).

Let \( \mu_N \in H^n(N) \) be the fundamental class of \( N \) and \( 1_N \in H^0(N) \) the degree 0 class.

**Definition 4.10.** We set for \( \mathcal{F}^* \) in \( \mathcal{F}_0(N) \)

\[
c_+(\mathcal{F}^*) = c(\mu_N, \mathcal{F}^*)
\]

\[
c_-(\mathcal{F}^*) = c(1_N, \mathcal{F}^*)
\]

\[
\gamma(\mathcal{F}^*) = c_+(\mathcal{F}^*) - c_-(\mathcal{F}^*)
\]

We set \( \mathbb{D}\mathcal{F}^* \) to be the Verdier dual of \( \mathcal{F}^* \) and \( s(x, t) = (x, -t) \) and \( \hat{\mathcal{F}}^* \) is quasi-isomorphic to \( 0 \to k_{N \times \mathbb{R}} \to s^{-1}(\mathbb{D}\mathcal{F}^*) \to 0 \).

We notice that \( SS(\mathbb{D}\mathcal{F}^*) = -SS(\mathcal{F}^*) \) where for \( A \subset T^*(N \times \mathbb{R}) \), we set \( -A = \{(x, -p, t, -\tau) \mid (x, p, t, \tau) \in A \} \) (see [KS90] Exercise V.13, p. 247). As a result, \( \hat{\mathcal{F}}^*_L = \hat{\mathcal{F}}^*_{-L} \) where \( -L = \{(q, -p) \mid (q, p) \in L \} \). The triangle inequality then implies

**Proposition 4.11.** We have for \( \mathcal{F}^* \) constructible in \( \mathcal{F}_0(N) \)

1. \( c_+(\mathcal{F}^*) \geq c_-(\mathcal{F}^*) \)
2. \( c_+(\mathcal{F}^*) = -c_-(\mathcal{F}^*) \) \( c_-(\mathcal{F}^*) = -c_+(\hat{\mathcal{F}}^*) \) so that \( \mathcal{F}^* \to \hat{\mathcal{F}}^* \) is an \( \gamma \)-isometry.

And of course if \( L \in \mathcal{L}(T^*N) \) we have \( c_{\pm}(\mathcal{F}^*_L) = c_{\pm}(L) \) and \( \gamma(\mathcal{F}^*_L) = \gamma(L) \).

**Proof.** Note that for \( \mathcal{F}^* \) cohomologically constructible, we have \( \hat{\mathcal{F}}^* = \mathcal{F}^* \) \( \square \)

Of course if \( L \in \mathcal{L}(T^*N) \) the \( c_{\pm}(L) \) are not well defined (they are only defined up to constant), however \( \gamma(L) \) is well-defined.

4.2. **Persistence modules and Barcodes as sheaves on the real line.** By a **Persistence module**, we mean a constructible sheaf on \( (\mathbb{R}, \leq) \). We refer to [Bar94, Cha+09, ELZ02, KS18, ZC05] for the theory and applications. Such a Persistence module is uniquely defined by the spaces \( V_t = \mathcal{F}([t, \infty), t) \) and the linear maps \( r_{s,t} : V_t \to V_s \), defined for \( s \leq t \), such that

1. for \( s < t < u \) we have \( r_{u,t} \circ r_{s,t} = r_{s,u} \)
2. \( \lim_{t \to s} V_t = V_s \) where the limit is that of the directed system given by the \( r_{s,t} \)
3. \( r_{t,t} = \text{Id} \)
Obviously $k_{[a,b]}$ is such a persistence module. As a consequence of Gabriel’s theorem on quivers (see [Gab72]), we have

**Proposition 4.12** ([Cra15]). *Any persistence module is isomorphic to a unique sum*

$$\bigoplus_j k_{[a_j,b_j]}$$

*where $a_j \in \mathbb{R} \cup \{-\infty\}, b_j \in \mathbb{R} \cup \{+\infty\}$ and $a_j < b_j$.*

It will be useful to remind the reader that

**Lemma 4.13** (see [KS18], (1.10)).

$$\text{Mor}(k_{[a,b]}, k_{[c,d]}) = \begin{cases} k & \text{for } a \leq c < b \leq d \\ 0 & \text{otherwise} \end{cases}$$

There is a graded version, when we consider $D^b((\mathbb{R}, \leq))$. We denote by $k_{[a,b]}[n]$ the element in $D^b((\mathbb{R}, \leq))$ given by the complex $0 \to k_{[a,b]}[n] \to 0$ concentrated in degree $n$. We set $D^b_c(N)$ to be the category of constructible sheaves on $N$

**Proposition 4.14** (see [Gui19; KS18]). *Any element $\mathcal{F}^\bullet$ in $D^b_c((\mathbb{R}, \leq))$ is isomorphic to a unique sum*

$$\bigoplus_j k_{[a_j,b_j]}[n_j]$$

*where $a_j \in \mathbb{R} \cup \{-\infty\}, b_j \in \mathbb{R} \cup \{+\infty\}, a_j < b_j$ and $n_j \in \mathbb{Z}$. If $\mathcal{F}^\bullet$ vanishes at $-\infty$ and is equal to $k_{[c,d]}^d$ at $+\infty$, then all the $a_j$ are different from $-\infty$ and the number of $j$ such that $b_j = +\infty$ is exactly $d$.*

We refer to [GVit22a] for extensions to the non-constructible case. It is quite clear that elements of $D^b_c((\mathbb{R}, \leq))$ are obtained by considering graded persistence modules $V^d_t$ for $d$ in a finite range (in $\mathbb{Z}$) and with the maps $r_{s,t}$ we have maps $\delta_t : V^d_t \to V^{d+1}_t$ so that

1. $\delta_t^2 = 0$
2. $\delta_s \circ r_{s,t} = r_{s,t} \circ \delta_t$

**Remark 4.15.** One has to be careful, it is **not true** that

$$\text{Mor}_{D^b_c((\mathbb{R}, \leq))}(k_{[a,b]}[m], k_{[c,d]}[n]) = 0$$

for $m \neq n$!

In particular remember that given two elements $L_0, L_1 \in \mathcal{L}(M, \omega)$ the Floer complex of $L_0, L_1$ is generated by the intersection points of $L_0 \cap L_1$. This complex is filtered by $f_{L_0,L_1}(x) = f_{L_1}(x) - f_{L_0}(x)$ for $x \in L_0 \cap L_1$. Since the boundary map is given by counting holomorphic strips, it decreases the filtration, and we can define $V_t = FC^\bullet(L_0, L_1; t)$ the subspace of the Floer complex generated by the elements with filtration less than $t$. There is a natural map for $s < t$ from $V_t = FC^\bullet(L_0, L_1; t)$ to $V_s = FC^\bullet(L_0, L_1; t)$
which defines a persistence module. By the previous Corollary, this is also
the persistence module associated to \( W_t = FC^*(L_0, L_1; t) \). By the previous
theorem, this yields an isomorphism between this persistence module and some \( \bigoplus_j k_{[a_j, b_j]}(n_j) \).

Since \( V_t \) vanishes for \( t \) small enough, the \( a_j \) must all be finite. Since up
to a shift in grading we have by Floer’s theorem \( FH^k(L_0, L_1; t) = H^k(L_0) = H^j(L_1) \) for \( t \) large enough, the number of \( j \) such that \( b_j = +\infty \) and \( n_j = k \)
is given by \( \dim H^k(L_0) = \dim H^k(L_1) \). Finally, if \( F_j^* \) is associated to \( L_j \) by
Theorem 4.1 we have that the above persistence module is given by the
sheaf \( (R\mathcal{H}om^*(\mathcal{F}_1^*, \mathcal{F}_2^*)) \). We thus get

**Proposition 4.16.** Let us set (according to Proposition 4.14) for \( F_j^* = F_{L_j}^* \)
and

\[
(Rt)_*(R\mathcal{H}om^*(\mathcal{F}_1^*, \mathcal{F}_2^*)) = \bigoplus_j k_{[a_j, b_j]}(n_j)
\]

Then there is a unique \( j_- \) such that \( b_{j_-} = +\infty \) and \( n_{j_-} \) is minimal and then
\( a_{j_-} = c_-(L_0, L_1) \) and a unique \( j_+ \) such that \( b_{j_+} = +\infty \) and \( n_{j_+} \) is maximal
and then \( a_{j_+} = c_+(L_0, L_1) \). Moreover \( n_{j_+} - n_{j_-} = n = \dim(L_0) = \dim(L_1) \). In
particular if \( \Omega \) is a connected open set with smooth boundary, we have

\[
H^k(\Omega \times \mathbb{R}, R\mathcal{H}om^*(\mathcal{F}_1^*, \mathcal{F}_2^*)) = H^{k-n_{j_-}}(\Omega)
\]

and

\[
H^c_k(\Omega \times \mathbb{R}; R\mathcal{H}om^*(\mathcal{F}_1^*, \mathcal{F}_2^*)) = H^c_k(\Omega \times \mathbb{R}, \partial\Omega \times \mathbb{R}; R\mathcal{H}om^*(\mathcal{F}_1^*, \mathcal{F}_2^*)) = \nabla
H^{k-n_{j_-}}(\Omega)
\]

**Remark 4.17.** The actual values of \( n^+_j, n^-_j \) depend on the grading of \( L_1, L_2 \):
shifting the grading shifts the values by the same quantity. Note that \( \mathcal{F}^* = R\mathcal{H}om^*(\mathcal{F}_1^*, \mathcal{F}_2^*) \) satisfies \( SS(\mathcal{F}^*) \subset \{ \tau \geq 0 \} \) and \( \mathcal{F}^* \) equals \( k_X[n_{j-}] \) over
\( X \times [+\infty] \) (we will shorten this by writing \( \mathcal{F}^* = k_X[n_{j-}] \) at \( +\infty \). Note
that we sometimes assume \( \mathcal{F}^* = k_X \), which is equivalent to normalizing
\( n_{j-} = 0 \).

**Proof.** Only the last statement needs a proof. But by assumption

\[
SS(R\mathcal{H}om^*(\mathcal{F}_1^*, \mathcal{F}_2^*)) \subset \{ \tau \geq 0 \}
\]

so

\[
H^k(\Omega \times \mathbb{R}, R\mathcal{H}om^*(\mathcal{F}_1^*, \mathcal{F}_2^*)) = H^k(\Omega \times [+\infty], R\mathcal{H}om^*(\mathcal{F}_1^*, \mathcal{F}_2^*)) = H^{k-n_{j-}}(\Omega)
\]

\[\square\]
5. VARIOUS SPACES, METRICS AND COMPLETIONS

We remind the reader that according to \cite{FSS08, Kra13}, $FD(T^*N, d\lambda)$ has a single object. We shall use the following.

**Definition 5.1.** We set for $A \in FD(M, d\lambda)$

1. $\mathcal{L}_A(M, d\lambda)$ to be the set of triples $\tilde{L} = (L, f_L, G_L)$ of graded exact closed Lagrangians branes $L \in A$ where $f_L$ is a primitive of $\lambda|_L$, i.e. $df_L = \lambda|_L$.

2. The action of $\mathbb{R}$ on $\mathcal{L}_A(M, d\lambda)$ given by $(L, f_L, G_L) \mapsto (L, f_L + c, G_L)$ is denoted $\lambda|_L$. The action of the generator of the deck transformation $\tilde{L}(M, d\lambda) \mapsto \tilde{L}(M, d\lambda)$ is denoted by $\tilde{L} \mapsto \tilde{L}[1]$.

3. For $\tilde{L}_1, \tilde{L}_2 \in \mathcal{L}(T^*N)$, and $\alpha \in H^*(N)$, we denote the spectral invariants obtained using Floer cohomology (see \cite{HLS16}) by $c(\alpha, \tilde{L}_1, \tilde{L}_2)$. They are usually shortened in $c(\alpha; L_1, L_2)$ if $f_{L_1}, f_{L_2}$ are implicit.

4. On $\mathcal{L}(M, d\lambda)$ we define $c_\pm(\bullet, \bullet) = c(\mu; \bullet, \bullet)$, $c_\cdot(\bullet, \bullet) = c(\cdot; \bullet, \bullet)$ and $\gamma(\bullet, \bullet) = c_+(\bullet, \bullet) - c_-(\bullet, \bullet)$. The metric $c$ given by

\[ c(\tilde{L}_1, \tilde{L}_2) = \max\{c_+(\tilde{L}_1, \tilde{L}_2), 0\} - \min\{c_-(\tilde{L}_1, \tilde{L}_2), 0\} \]

5. We denote by $\mathcal{L}(M, d\lambda)$ the set of exact closed Lagrangians. There is a forgetful functor $\text{unf} : \mathcal{L}(M, d\lambda) \mapsto \mathcal{L}(M, d\lambda)$ obtained by forgetting $f_L$ and the grading. The quantity $\gamma$ descends to a metric on $\mathcal{L}(M, d\lambda)$, still denoted $\gamma$.

**Remark 5.2.** We let the reader check that $c(\tilde{L}_1, \tilde{L}_2)$ indeed defines a metric.

As we mentioned, Proposition\ref{prop:spectral-invariants} implies that for complexes of sheaves of the form $\mathcal{F}_L$, the spectral invariants defined by Definition\ref{def:invariants} coincide with those defined using Floer cohomology. It will be useful to define some modified metrics and completions.

In the sequel we fix a class $A \in FD(M, d\lambda)$ and write $\mathcal{L}(M, d\lambda)$ and $\mathcal{L}_A(M, d\lambda)$ instead of $\mathcal{L}_A(M, d\lambda)$ and $\mathcal{L}_A(M, d\lambda)$.

We may now state

**Proposition 5.3.** The following holds

1. Set $\mathcal{Ham}_c(M, \omega)$ to be the set of compact supported Hamiltonians on $[0,1] \times M$ and $\mathcal{D}\mathcal{H}am_c(M, \omega)$ the space of compact supported Hamiltonian diffeomorphisms. Then the compact supported isotopy $(\varphi^t_H)_{t \in [0,1]}$ uniquely defines the Hamiltonian, so there is a fibration $\mathcal{Ham}_c(M, \omega) \mapsto \mathcal{D}\mathcal{H}am_c(M, \omega)$ (which is essentially the path-space fibration). It factors through $\mathcal{D}\mathcal{H}am_c(M, \omega)$, so we have in fact

\[ \mathcal{Ham}_c(M, \omega) \mapsto \mathcal{D}\mathcal{H}am_c(M, \omega) \mapsto \mathcal{D}\mathcal{H}am_c(M, \omega) \]
Definition 5.4.

(2) There is an action of \( \mathcal{H} \text{am}_c(M, d\lambda) \) on \( \mathcal{L}(M, d\lambda) \) given by

\[
\varphi_H(L, f_L) = (\varphi_H(L), H \# f_L)
\]

where

\[
(H \# f_L)(z) = f_L(z) + \int_0^1 [p(t)q(t) - H(t, q(t), p(t))] dt
\]

where \( q(t), p(t) = \varphi_H^t(z) \) factors through \( \mathcal{D}H \text{am}_c(M, \omega) \). The action does is the obvious action on the grading (and of course commutes with the shift \( L \mapsto L[k] \)). This action descends to the canonical action of \( \mathcal{D}H \text{am}_c(M, d\lambda) \) on \( \mathcal{L}(M, d\lambda) \).

(3) The above action commutes with \( T_c \). We have

\[
c_\pm(T_c \tilde{L}_1, \tilde{L}_2) = c_\pm(\tilde{L}_1, T_{-\epsilon} \tilde{L}_2) = c_\pm(\tilde{L}_1, \tilde{L}_2) + c
\]

(4) The metric \( c \) on \( \mathcal{L}(M, d\lambda) \) and the metric \( \gamma \) on \( \mathcal{L}(M, d\lambda) \) are related by the following formula

\[
\gamma(L_1, L_2) = \inf \{ c(\tilde{L}_1, \tilde{L}_2) \mid \text{unf}(\tilde{L}_1) = L_1, \text{unf}(\tilde{L}_2) = L_2 \}
\]

(5) For \( H \leq K \) we have \( c_+(\varphi_H(L_1), L_2) \leq c_+(\varphi_K(L_1), L_2) \) and the same holds for \( c_- \).

Proof: Proofs of the first three statements are left to the reader. For the fourth, we just notice that we may change \( f_{L_1}, f_{L_2} \) to \( f_{L_1} + c_1, f_{L_2} + c_2 \) for any two constants \( c_1, c_2 \). Then \( c_\pm(\tilde{L}_1, \tilde{L}_2) \) is changed to \( c_\pm(\tilde{L}_1, \tilde{L}_2) + c_1 - c_2 \), and \( c(\tilde{L}_1, \tilde{L}_2) \) is changed to

\[
\max[c_+(\tilde{L}_1, \tilde{L}_2) + c_1 - c_2, 0] - \min[c_-(\tilde{L}_1, \tilde{L}_2) + c_1 - c_2, 0]
\]

and this is minimal when \( c_-(\tilde{L}_1, \tilde{L}_2) < c_2 - c_1 < c_+(\tilde{L}_1, \tilde{L}_2) \) and takes the value

\[
c_+(\tilde{L}_1, \tilde{L}_2) - c_-(\tilde{L}_1, \tilde{L}_2) = \gamma(L_1, L_2)
\]

The last statement is equivalent to \( c(\alpha, \varphi_H^t(\tilde{L}_1), \tilde{L}_2) \) is increasing for \( H \geq 0 \). But this follows from the formula (see [Vit92], prop 4.6 and Lemma 4.7) for almost all \( t \in [0, 1] \) we have

\[
\frac{d}{dt} c(\alpha, \varphi_H^t(\tilde{L}_1), \tilde{L}_2) = \gamma(t, \tilde{L}_2)
\]

for some point \( z \) of \( \varphi_H^t(L_1) \cap L_2 \). \( \square \)

We may now set

Definition 5.4.

(1) \( \mathcal{D}(M, d\lambda) \) is the \( c \)-completion of \( \mathcal{L}(M, d\lambda) \) and \( \hat{\mathcal{L}}(M, \omega) \) the Humilière completion of \( \mathcal{L}(M, \omega) \).

(2) \( \mathcal{D}\mathcal{H}\text{am}(M, \omega) \) is the Humilière completion of \( \mathcal{D}\mathcal{H}\text{am}_c(M, \omega) \) and \( \hat{\mathcal{D}\mathcal{H}\text{am}}(M, \omega) \) is the Humilière completion of \( \mathcal{D}\mathcal{H}\text{am}_c(M, \omega) \).
Remarks 5.5.

(1) Elements in \( \widehat{\mathcal{F}}(M, d\lambda) \) or \( \widehat{\mathcal{D}_0}(M, \omega) \) are not necessarily compact supported: they could be limits of sequences with larger and larger support.

(2) It is not clear whether the connected component of the zero section in \( \widehat{\mathcal{L}}(M, d\lambda) \) containing \( L_0 \) coincides with the completion of \( \mathcal{L}_0(M, d\lambda) \).

(3) According to the continuity of the spectral distance in terms of the \( C^0 \)-distance, proved in [BHS21], an element in the group of Hamiltonian homeomorphisms, that is a \( C^0 \)-limit of elements of \( \mathcal{D}_0(M, \omega) \), belongs to \( \widehat{\mathcal{D}_0}(M, \omega) \). Moreover an element in \( \widehat{\mathcal{L}}(T^*N) \) (or \( \widehat{\mathcal{D}_0}(M, \omega) \)) has a barcode, as follows from the Kislev-Shelukhin theorem (see [Ki-Sh22] and the Appendix in [Vit22]) and was pointed out in [BHS21].

The relationship between the two completions \( \widehat{\mathcal{F}}(M, d\lambda) \) and \( \widehat{\mathcal{L}}(M, d\lambda) \) is clarified by the following:

**Proposition 5.6.**

1. There is a continuous map 
   \[ \widehat{\text{unf}} : \widehat{\mathcal{F}}(M, d\lambda) \to \widehat{\mathcal{L}}(M, d\lambda) \]
   extending \( \text{unf} \) and a continuous map 
   \[ \widehat{T}_c : \widehat{\mathcal{F}}(M, d\lambda) \to \widehat{\mathcal{F}}(M, d\lambda) \]
   extending \( T_c \).

2. If \( L \in \widehat{\mathcal{L}}(M, d\lambda) \), there exists \( \tilde{L} \in \widehat{\mathcal{F}}(M, d\lambda) \) such that \( \widehat{\text{unf}}(\tilde{L}) = L \) (i.e. \( \text{unf} \) is onto).

3. \( \widehat{\text{unf}}(\tilde{L}_1) = \widehat{\text{unf}}(\tilde{L}_2) \) if and only if \( \tilde{L}_2 = \tilde{T}_c\tilde{L}_1 \) for some \( c \in \mathbb{R} \).

**Proof.**

1. This is obvious since Lipschitz maps extend to the completion. Now \( \text{unf} \) is 1-Lipschitz, while \( T_c \) is an isometry.

2. Let \( (L_j) \) be a Cauchy sequence in \( \mathcal{L}(M, d\lambda) \). We may assume that 
   \( \gamma(L_j, L_{j+1}) < 2^{-j} \). There is a lift \( \tilde{L}_j \) of \( L_j \), well determined up to a constant, and we can recursively adjust \( \tilde{L}_{j+1} \) so that \( c(L_j, L_{j+1}) < 2^{-j} \). Indeed, assume the \( \tilde{L}_j \) are defined for \( 1 \leq j \leq k \). Since \( \tilde{L}_{k+1} \) is in the preimage of \( L_{k+1} \), we have 
   \( \gamma(L_k, L_{k+1}) = \inf_{c \in \mathbb{R}} c(\tilde{L}_k, \tilde{L}_{k+1} + c) \) and we just choose \( \tilde{L}_{k+1} = \tilde{L}_{k+1} + c_{k+1} \), so that \( c(\tilde{L}_k, \tilde{L}_{k+1}) < 2^{-k} \). As a result \( (\tilde{L}_j)_{j \in \mathbb{N}} \) is a Cauchy sequence so defines an element in \( \widehat{\mathcal{F}}(T^*N) \). Moreover its limit \( \tilde{L} \) projects on \( L \).

3. Set \( L_j = \widehat{\text{unf}}(\tilde{L}_j) \) with \( L_2 = L_1 \). Let \( (\tilde{L}_{k+1}^1, \tilde{L}_{k+1}^2) \) be Cauchy sequences in \( \mathcal{L}(T^*N) \), converging to \( \tilde{L}_1, \tilde{L}_2 \), so that the \( L_j^k \) converge...
to $L_i$. We showed that $\gamma(L^k_1, L^k_2) = \inf_{c \in \mathbb{R}} c(\tilde{L}^k_1, \tilde{L}^k_2 + c)$ and since by assumption, $\gamma(L^k_1, L^k_2)$ goes to zero as $k$ goes to infinity, we have a sequence $c_k$ such that $\lim_k c(\tilde{L}^k_1, \tilde{L}^k_2 + c_k) = 0$. But $c(\tilde{L}^k_1, \tilde{L}^k_2 + c_k) = c(\tilde{L}^k_1, \tilde{L}^k_2) + c_k$, and since $\lim_k c(\tilde{L}^k_1, \tilde{L}^k_2) = c(\tilde{L}_1, \tilde{L}_2)$, we must have $\lim_k c_k = -c(\tilde{L}_1, \tilde{L}_2)$. As a result
\[
\tilde{L}_1 = \tilde{L}_2 - c(\tilde{L}_1, \tilde{L}_2)
\]

\[\square\]

**Remark 5.7.** Notice that to a pair $\tilde{L}_1, \tilde{L}_2$ in $\widehat{\mathcal{D}}(M, \omega)$ we may associate a Floer homology as a filtered vector space. Indeed, by the Kislev-Shelukhin inequality (see [Ki-Sh22], or the Appendix of [Vit22]), the bottleneck distance between the persistence module, denoted $V(L^k_1, L^k_2)$ associated to $FH^*(L^k_1, L^k_2; a, b)$ satisfies
\[
\beta(V(L^k_1, L^k_2), V(L^l_1, L^l_2)) \leq 2\gamma(L^k_1, L^l_1) + \gamma(L^k_2, L^l_2)
\]
so we get a Cauchy sequence of persistence modules, and this has a limit as well (however if we started with persistence modules with finite barcodes, the limit will only have finitely many bars of size $> \varepsilon > 0$, but possibly an infinite number in total).

6. **Defining the $\gamma$-Support**

An element $L$ in $\widehat{\mathcal{L}}(M, \omega)$ is not a subset of $M$ (and an element in $\widehat{\mathcal{D}}\mathcal{Ham}(M, \omega)$ does not define a map), but we may define its $\gamma$-**support** as follows:

**Definition 6.1 (The $\gamma$-support).**

1. Let $L \in \widehat{\mathcal{L}}(M, \omega)$. Then $x \in \gamma\text{-}\text{supp}(L)$ if for any neighbourhood $U$ of $x$ there exists a Hamiltonian map $\varphi$ supported in $U$ such that $\gamma(\varphi(L), L) > 0$.
2. Let $\tilde{L}$ is in $\widehat{\mathcal{D}}(M, d\lambda)$. Then $x \in c - \text{supp}(\tilde{L})$ if for any neighbourhood $U$ of $x$ there exists a Hamiltonian $H$ supported in $U$ such that $c_+(\varphi_H(\tilde{L}), \tilde{L}) > 0$.

**Remarks 6.2.**

1. The definition automatically implies that $\gamma\text{-}\text{supp}(L)$ (and $c - \text{supp}(\tilde{L})$ is closed.
2. Notice that for $\text{sup}(\tilde{L}) = L$ we may have $c(\varphi_H(\tilde{L}), \tilde{L}) > 0$ but $\gamma(\varphi_H(L), L) = 0$. For example if $H \equiv c$ on $L$(but $H$ is compact supported) then $c_+(\varphi_H(\tilde{L}), \tilde{L}) = c, c_-(\varphi_H(\tilde{L}), \tilde{L}) = c$ so that $c(\varphi_H(\tilde{L}), \tilde{L}) = 2c$, while of course $\gamma(\varphi_H(L), L) = 0$. Note that in this case $\varphi_H(\tilde{L}) = \tilde{L} + c$ hence $c_+(\varphi_H^k(\tilde{L}), \tilde{L}) = k \cdot c$.

We can however get rid of the second definition.
Proposition 6.3. We have for $\widehat{\text{unf}}(\bar{L}) = L$,
\[ c - \text{supp}(\bar{L}) = \gamma\text{-supp}(L) \]

Proof. Indeed, if this was not the case, we would have a Hamiltonian $H$ supported near $z$ such that $c(\varphi_H(\bar{L}), \bar{L}) > 0$ while $L = \varphi_H(L)$. According to Proposition 5.6 (3), this implies that $\varphi_H(\bar{L}) = \bar{L} + c$. This is impossible for $H$ supported in a displaceable set, since we would have $\varphi_H^k(\bar{L}) = L + k \cdot c$. and then, possibly replacing $H$ by $-H$, we may assume $c > 0$. Using the triangle inequality and that for a displaceable set, $\text{supp}(c_+(\varphi_H) \mid \text{supp}(H) \subset U) = c(U) < +\infty$ (see [Vit92], prop. 4.12 which is valid in any manifold) we have
\[ k \cdot c = c_+(\varphi_H^k(\bar{L}), \bar{L}) \leq c_+(\varphi_H^k) \leq c(\text{supp}(H)) < +\infty \]
a contradiction. \hfill \Box

We may now define compact supported Lagrangians and Hamiltonians in the completion

Definition 6.4. (1) We define $\widehat{\mathcal{L}}(M, \omega)$ (resp. $\widehat{\mathcal{C}}(M, \omega)$) to be the set of $L \in \mathcal{L}(M, \omega)$ (resp. $L \in \mathcal{C}(M, \omega)$) such that $\gamma\text{-supp}(L)$ is bounded (hence compact).

(2) We define $\mathcal{DHam}(M, \omega)$ to be the set of $\varphi \in \mathcal{DHam}(M, \omega)$ such that the $\gamma$-support of $\Gamma(\varphi)$ is contained in the union of $\Delta_M$ and a bounded subset of $M \times M$.

Remark 6.5. For an element in $\mathcal{DHam}(M, \omega)$, the $\gamma$-support is of course just the closure of the $\gamma$-support of its graph in $(M \times M, \omega \oplus \omega)$ with the diagonal removed. However there is a natural smaller support that one can define, using the action of $\mathcal{DHam}(M, \omega)$ by conjugation on $\mathcal{DHam}(M, \omega)$. We say that $z$ is in the restricted $\gamma$-support of $\varphi \in \mathcal{DHam}(M, \omega)$ if for all $\epsilon > 0$ there exist $\rho \in \mathcal{DHam}(B(z, \epsilon))$ such that $\rho \varphi \rho^{-1} \neq \varphi$. It is easy to show that the restricted $\gamma$-support is contained in the $\gamma$-support.

Clearly, by Proposition 6.17 given a sequence $\bar{L}_k$ such that the $\gamma\text{-supp}(L_k)$ are contained in a fixed bounded set and $\gamma$-converge to $\bar{L}_\infty$. $\bar{L}_\infty$ is in $\widehat{\mathcal{C}}(M, d\lambda)$. However the converse is not clear.

Question 6.6. Is an element in $\widehat{\mathcal{C}}(M, d\lambda)$ the limit of a sequence $(L_k)_{k \geq 1}$ in $\mathcal{C}(M, d\lambda)$ such that their support is uniformly bounded? Same question for $\mathcal{DHam}(M, \omega)$.

We shall make repeated use of the fragmentation lemma

Lemma 6.7 ([Ban78] Lemma III.3.2). Let $(M, \omega)$ be a closed symplectic manifold and $(U_j)_{j \in [1, N]}$ an open cover of $M$. Then any Hamiltonian isotopy $(\varphi^t)_{t \in [0, 1]}$ can be written as a product of Hamiltonian isotopies $(\varphi^t_j)_{t \in [0, 1]}$
with Hamiltonian supported in some $U_{k(j)}$. The same holds for compact supported Hamiltonian isotopies.

**Remarks 6.8.**

1. The number of isotopies is not bounded by the number of open sets: we may have more than one isotopy for each open set.

2. The lemma is stated in [Ban78] for compact manifolds, and for $U_j$ symplectic images of ball, but a covering can always be replaced by a finer one by balls. Moreover the proof works for compact supported isotopies with fixed support inside an open manifold and this is how it is stated in [Ban97], p. 110.

3. In the sequel, by support of an isotopy we mean the closure of the set $\{z \in M \mid \exists t, \varphi^t(z) \neq z\}$. If the complement of the support is connected and the isotopy is generated by a Hamiltonian $H(t, z)$, this is also the projection on $M$ of the support of $H$ in $[0,1] \times M$. When the isotopy is implicit, we still write $\text{supp}(\varphi)$ for the support of the (implicit) isotopy and $\tilde{\text{supp}}(\varphi)$ (or $\text{supp}(\varphi_H)$ or $\text{supp}(H)$) for the support of the implicit Hamiltonian. Note that in Banyaga’s theorem we may assume the support of the Hamiltonians are in the $U_{k(j)}$ (since the complement of a small ball is always connected). This implies

**Lemma 6.9.** We have the following properties:

1. Let $L$ be an element in $\mathcal{L}(M, \omega)$ and $\varphi \in \mathcal{D}\mathcal{Y}\mathcal{A}\mathcal{M}_c(M, \omega)$ be such that $\gamma(\varphi(L), L) > 0$. Then $\text{supp}(\varphi) \cap \gamma\text{-supp}(L) \neq \emptyset$.

2. Let $\tilde{L}$ be an element in $\mathcal{D}(M, \omega)$ and $H$ be a compact supported Hamiltonian such that $c(\varphi_H(\tilde{L}), \tilde{L}) > 0$. Then $\text{supp}(H) \cap \gamma\text{-supp}(\tilde{L}) = \text{supp}(H) \cap \gamma\text{-supp}(L) \neq \emptyset$.

**Proof.**

1. Indeed, if this was not the case, for each $x \in \text{supp}(\varphi)$ there would be an open set $U_x$ such that for all isotopies $\psi^t$ supported in $U_x$ we have $\gamma(\psi^t(L), L) = 0$. But by a compactness argument, we may find finitely many $x_j$ and $U_j = U_{x_j}$ ($1 \leq j \leq k$) such that $\text{supp}(\varphi)$ is covered by the $U_j$. Then $\varphi^1$ is a product of $\psi_j$ supported in $U_{k(j)}$, but since $\gamma(\psi_j(L), L) = 0$, we get by induction $\gamma(\psi_1 \circ \psi_2 \circ ... \circ \psi_{j-1} \circ \psi_j(L), L) = 0$ and finally $\gamma(\varphi_H(L), L) = 0$, a contradiction.

2. The second statement is analogous, since $\varphi_H$ is the product of $\varphi_{H_j}$ with $H_j$ supported in $U_{k(j)}$. If we had $c(\varphi_{H_j}(\tilde{L}))) = \tilde{L}$ for all $j$, then we would have $\varphi_H(\tilde{L}) = \tilde{L}$, a contradiction. Note that here we do not assume that the support of $H$ is small, so we may well have $\tilde{\text{supp}}(H) \neq \text{supp}(\varphi_H)$.
Proposition 6.10. Let \( L_1 \in \mathcal{L}(M, d\lambda), L_2 \in \mathcal{L}_c(M, d\lambda) \). Then \( \gamma\text{-supp}(L_1) \cap \gamma\text{-supp}(L_2) \neq \emptyset \). In particular \( \gamma\text{-supp}(L) \) is not displaceable, and intersects any exact Lagrangian. If \( L \in \mathcal{L}_c(M, d\lambda) \) it also intersects any fiber \( T^*_x N \).

Remark 6.11. Of course, unless we know some singular support which does not contain -or is not a \( C^0\)-limit- of exact smooth Lagrangians, this does not add anything to the known situation, that any two closed exact Lagrangians intersect, a consequence of the Fukaya-Seidel-Smith (see [FSS08]).

The proof of Proposition 6.10 will make use of the following Lemma

Lemma 6.12. Let \( L \in \mathcal{L}(T^* N) \) and \( H \) be a Hamiltonian equal to a constant \( a \) on \( \gamma\text{-supp}(L) \). Then \( \varphi_H(\tilde{L}) = \tilde{L} + a \) (hence \( \varphi_H(L) = L \)).

Proof. Let \((\tilde{L}_k)_{k \geq 1}\) be a sequence in \((T^* N)\) converging to \( \tilde{L} \) and assume first \( H = a \) on a domain containing all the \( L_k = \text{unf}(\tilde{L}_k) \). Then obviously since \( \varphi_H(\tilde{L}_k) = \tilde{L}_k + a = T_a(\tilde{L}_k) \), by continuity we get \( \varphi_H(\tilde{L}) = \tilde{L} + a \). Now let \( H_1 \) be equal to \( a \) on a domain containing all the \( L_k \) (hence necessarily containing \( \gamma\text{-supp}(L) \)). Then \( \varphi_{H_1}^{-1} \circ \varphi_H \) is generated by

\[
K(t, z) = H(t, \varphi_{H_1}^1(z)) - H_1(z, \varphi_{H_1}^1(z))
\]

and \( K \) vanishes on \( \gamma\text{-supp}(L) \), since \( \varphi_{H_1}^1 \) preserves \( \gamma\text{-supp}(L) \) and \( H = H_1 = a \) on \( \gamma\text{-supp}(L) \). Now we claim that if \( K = 0 \) on \( \gamma\text{-supp}(L) \) then we have \( \varphi_K(\tilde{L}) = \tilde{L} \). Indeed, if this was not the case, we would have by Lemma 6.9 that \( \text{supp}(K) \cap c - \text{supp}(\tilde{L}) \neq \emptyset \), a contradiction. Hence \( \varphi_{H_1}^{-1} \circ \varphi_H(L) = \tilde{L} \) and we conclude that \( \varphi_H(\tilde{L}) = \varphi_{H_1}(\tilde{L}) = \tilde{L} + a \) and this proves our claim.

Proof of Proposition 6.10. Let \( \tilde{L}_1, \tilde{L}_2 \) be elements in \( \mathcal{L}(M, d\lambda) \) having image by unf equal to \( L_1, L_2 \). Assume their \( \gamma \)-supports are disjoint and the \( \gamma \)-support of \( L_2 \) is compact. Let \( H \) be a compact supported Hamiltonian, and assume \( H \) has support in the complement of \( \gamma\text{-supp}(L_1) \) and equals \( a < 0 \) in a neighbourhood of \( \gamma\text{-supp}(L_2) \). Then

\[
c_+((\varphi_H^1(\tilde{L}_1)), \tilde{L}_1) \geq c_+((\varphi_H^1(\tilde{L}_1)), \tilde{L}_2) - c_+(\tilde{L}_1, \tilde{L}_2) = c_+(\tilde{L}_1, \varphi_{H_1}^{-1}(\tilde{L}_2)) - c_+(\tilde{L}_1, \tilde{L}_2) = -a
\]

since \( \varphi_{H_1}^{-1}(\tilde{L}_2) = \tilde{L}_2 - a \) by the previous Lemma. As a result \( \text{supp}(H) \) intersects \( \gamma\text{-supp}(L_1) \) and this implies that \( \gamma\text{-supp}(L_1) \cap \gamma\text{-supp}(L_2) \neq \emptyset \).

It is easy to prove that for \( L \in \mathcal{L}_c(T^* N) \), we have that \( \gamma\text{-supp}(L) \) intersects any vertical fiber \( T^*_x N \). Indeed, if this was not the case, we could find a small ball \( B(x, \varepsilon) \) such that \( T^*(B(x, \varepsilon)) \cap \gamma\text{-supp}(L) = \emptyset \). Now if \( f \) is
a smooth function such that all critical points of $f$ are in $B(x, \varepsilon)$, then for any bounded set $W$ contained in the complement of $T^*(B(x, \varepsilon))$ we have $\text{graph}(tdf) \cap W = \emptyset$ for $t$ large enough, so $\text{graph}(tdf) \cap L = \emptyset$. But this contradicts our first statement.

\[\square\]

**Question 6.13.** What can $\gamma$-supp($L$) be?

One of the goals of this paper is to partially answer this question.

**Example 6.14.** Let $f \in C^0(N, \mathbb{R})$. Then $\gamma$-supp($\text{graph}(df)$) = $\partial f$ where $\partial f$ is the subdifferential defined by Vichery in [Vic13] (see [AGHIV] for a proof). Therefore $\partial f$ intersects any exact Lagrangian $L$. If $L$ is isotopic to the zero section, then $L$ has a G.F.Q.I. and $\partial f \cap L$ is given by the critical points of $S(x, \xi) - f(x)$.

In [Hum08b] section 2.3.1. a different notion of support was presented:

**Definition 6.15 (H-support).** A point $x \in M$ is in the complement of the H-support of $L \in \mathcal{L}(M, \omega)$ if there is a sequence of smooth Lagrangians $(L_k)_{k \geq 1}$ converging to $L$ and a neighbourhood $U$ of $x$ such that $L_k \cap U = \emptyset$.

We shall first prove that our definition of the support yields an a priori smaller set than Humilière’s. We shall actually prove something slightly more general.

**Definition 6.16.** Let $X_k$ be a sequence of subsets in the topological space $Z$. We define its topological upper limit as

$$\limsup_j X_j = \bigcap \bigcup_{n} X_j$$

$$= \left\{ x \in Z \mid \exists (x_j)_{j \geq 1}, \ x_j \in X_j \text{ for infinitely many } j, \lim x_j = x \right\}$$

and its topological lower limit as

$$\liminf_j X_j = \left\{ x \in Z \mid \exists (x_j)_{j \geq 1}, \ x_j \in X_j, \lim x_j = x \right\}$$

Note that if $\liminf_j X_j = \limsup_j X_j$ then this is the topological limit, which for a compact metric space coincides with the Hausdorff limit (see [Kec95], p.25-26), but we shall not need this here. Also $x \in \liminf_j X_j$ if and only if $\lim_j d(x, X_j) = 0$. Note that this limit could in principle be empty, even if the $X_j$ are non-empty. An easy result is now
Proposition 6.17. Let \((L_k)_{k \geq 1}\) be sequence in \(\hat{\mathcal{L}}(M, \omega)\) of Lagrangians such that \(\gamma - \lim_k L_k = L\). Assume \(\gamma\)-\text{supp}(L_k) \cap U = \emptyset\) for infinitely many \(k\). Then
\[
\gamma\text{-supp}(L) \cap U = \emptyset
\]
In other words
\[
\gamma\text{-supp}(L) \subset \liminf_k (\gamma\text{-supp}(L_k))
\]

Proof. For the first statement let \(\varphi\) be supported in \(U\). Then for \(k\) in the subsequence \(U \cap \gamma\text{-supp}(L_k) = \emptyset\). This implies that \(\gamma(\varphi(L_k), L_k) = 0\), and passing to the limit, that \(\gamma(\varphi(L), L) = 0\).

Now if \(x \notin \liminf_k (\gamma\text{-supp}(L_k))\), there must be a subsequence such that \(d(x, \liminf_k (\gamma\text{-supp}(L_k))) > \epsilon_0 > 0\). But then on this subsequence \(B(x, \epsilon_0/2) \cap \gamma\text{-supp}(L_k) = \emptyset\) and applying the previous result, we have \(B(x, \epsilon_0/2) \cap \gamma\text{-supp}(L) = \emptyset\), so \(x \notin \gamma\text{-supp}(L)\). \(\square\)

For the case where \(L\) and the \(L_k\) are smooth, this remark was previously made by Seyfaddini and the author (see [Vic13]) and is also a consequence of lemma 7 in [HLS15]. From Proposition 6.17 we immediately conclude

Corollary 6.18. We have for \(L \in \hat{\mathcal{L}}(M, \omega)\) the inclusion
\[
\gamma\text{-supp}(L) \subset H - \text{supp}(L)
\]

Our definition of support has one advantage compared to Humilière’s definition: if \(W \cap H - \text{supp}(L) = \emptyset\), we do not know whether there is a sequence \((L_k)_{k \geq 1}\) \(\gamma\)-converging to \(L\) and such that \(L_k \cap W = \emptyset\), we only know that \(W\) can be covered by sets \(W_j\) such that for each \(j\) there is a sequence \((L^j_k)_{k \geq 1}\) such that \(\gamma - \lim_k L^j_k = L\) and \(L^j_k \cap W_j = \emptyset\). On the other hand if \(W \cap \gamma\text{-supp}(L) = \emptyset\), we know that for all \(\varphi\) supported in \(W\) we have \(\gamma(\varphi(L), L) = 0\). This still leaves open the following

Question 6.19. Do we have equality in Corollary 6.18 in other words is any \(L \in \hat{\mathcal{L}}(M, \omega)\) the limit of a sequence \((L_k)_{k \geq 1}\) contained in a neighbourhood of \(\gamma\text{-supp}(L)\)?

Note that the Hamiltonians we need to consider to determine whether a point is in the support can be restricted to a rather small family. Let \(\chi(r)\) be a non-negative smooth function function equal to 1 on \([-1/2, 1/2]\) and supported in \([-1, 1]\). We set \(H^\chi_{\rho, \varepsilon}(z) = \rho(1/\varepsilon) d(z, z_0)\) and \(\varphi^\chi_{\rho, \varepsilon}\) to be the flow associated to \(H^\chi_{\rho, \varepsilon}\). Since for any \(\varphi\) supported in \(B(z_0, \varepsilon/2)\) we can find a positive constant \(c\) so that \(\varphi \leq \varphi^\chi_{\rho, \varepsilon}\), that is \(c_-(\varphi^\chi_{\rho, \varepsilon} \circ \varphi^{-1}) = 0\), we have for a lift \(\tilde{L}\) of \(L\), \(c_+ (\varphi(\tilde{L}), \tilde{L}) \leq c_+ (\varphi^\chi_{\rho, \varepsilon}(\tilde{L}), \tilde{L})\), so \(0 < c_+(\varphi(\tilde{L}), \tilde{L})\) implies \(0 < c_+(\varphi^\chi_{\rho, \varepsilon}(\tilde{L}), \tilde{L})\). Note that if \(c_-(\varphi(\tilde{L}), \tilde{L}) < 0\) we have \(0 < c_+(\tilde{L}, \varphi(\tilde{L})) = c_+(\varphi^{-\chi}(\tilde{L}), \tilde{L})\), so applying the same argument to \(\varphi^{-1}\) we almost get
Proposition 6.20 (Criterion for the γ-support). A point z is in the γ-support of $\mathcal{L} \in \mathcal{L}(M, \omega)$ if and only if we have

$$\forall \varepsilon > 0, \ c_+ (\varphi_{z_0, \varepsilon}(\mathcal{L}), \mathcal{L}) > 0$$

where $\varphi_{z_0, \varepsilon}$ is the flow associated to $H^1_{z_0, \varepsilon}$.

**Proof.** The only point missing is to show that we can take $c = 1$. But let $n$ be an integer such that $c < n$. Assume we had $c_+ (\varphi_{z_0, \varepsilon}(\mathcal{L}), \mathcal{L}) = 0$. Then, since $\varphi_{z_0, \varepsilon} = \text{Id}$, we have

$$c_+ (\varphi_{z_0, \varepsilon}^c (\mathcal{L}), \mathcal{L}) \leq c_+ (\varphi_{z_0, \varepsilon}^n (\mathcal{L}), \mathcal{L}) \leq \sum_{j=1}^{n} c_+ (\varphi_{z_0, \varepsilon}^j (\mathcal{L}), \varphi_{z_0, \varepsilon}^{j-1}(\mathcal{L})) = nc_+ (\varphi_{z_0, \varepsilon}(\mathcal{L}), \mathcal{L}) = 0$$

This contradicts our assumption that $c_+ (\varphi_{z_0, \varepsilon}^c (\mathcal{L}), \mathcal{L}) > 0$. \[ \square \]

Let $A \subset M_1 \times M_2, B \subset M_2 \times M_3$. We set $A \circ B$ to be the projection of $(A \times B) \cap (M_1 \times M_2 \times M_3)$. We then have

Proposition 6.21. We have

1. For $L$ a smooth Lagrangian in $(M, \omega)$ we have $\gamma\text{-supp}(L) = L$.
2. $\gamma\text{-supp}(L)$ is non-empty
3. For $\psi$ a symplectic map, $\gamma\text{-supp}(\psi(L)) = \psi(\gamma\text{-supp}(L))$
4. $\gamma\text{-supp}(L_1 \times L_2) \subset \gamma\text{-supp}(L_1) \times \gamma\text{-supp}(L_2)$
5. Let $K$ be coisotropic in $(M, \omega)$ and $\mathcal{K}$ its coisotropic foliation. If $K/\mathcal{K} = P$ is a symplectic manifold, and $L \in (M, \omega)$ having reduction $L_K \in \mathcal{L}(P, \omega_K)$, then

$$\gamma\text{-supp}(L_K) \subset \gamma\text{-supp}(L)_K$$

6. Let $\Lambda_1, \Lambda_2$ be correspondences, that is Lagrangians in $\mathcal{L}(M_1 \times M_2)$ and $\mathcal{L}(M_2 \times M_3)$ respectively. Then $\Lambda_1 \circ \Lambda_2 \subset M_1 \times M_3$ satisfies

$$\gamma\text{-supp}(\Lambda_1 \circ \Lambda_2) \subset \gamma\text{-supp}(\Lambda_1) \cap \gamma\text{-supp}(\Lambda_2)$$

**Proof.** For (1), the inclusion $\gamma\text{-supp}(L) \subset L$ follows from Corollary 6.18. The converse can be reduced to the case of the zero section by using a Darboux chart $U$ in which $(U, L \cap U)$ is identified with $(B^{2n}(0, r), B^{2n}(0, r) \cap \mathbb{R}^n)$. But then if $f = 0$ outside a neighbourhood of $z$ and the oscillation of $f$ is $\varepsilon^2, |\nabla f| \leq \varepsilon$ we can locally deform the zero section to $\Gamma_{df}$ in a neighbourhood of $z$, and obtain $L'$ such that $\gamma(L', L) \geq \varepsilon^2$.

For (2), this follows from Proposition 6.10.

For (3), we have $\gamma(\varphi_\psi(L), \psi(L)) > 0 \iff \gamma(\psi^{-1}\varphi_\psi(L), L) > 0$ and since $\text{supp}(\psi^{-1}\varphi_\psi) = \psi(\text{supp}(\varphi))$ this proves our statement.

For (4), we note that if $\gamma(\varphi(L), L) > 0$ we either have $c_+ (\varphi(L), L) > 0$ or $c_- (\varphi(L), L) > 0$. Changing $\varphi$ to $\varphi^{-1}$, we can always assume $c_+ (\varphi(L), L) > 0$. 
Then suppose there is a $\psi$ supported in $U_1 \times T^*N$ such that $c_+(\psi(L_1 \times L_2), L_1 \times L_2) > 0$. Then we can find $\varphi_1$ supported in $U_1$ such that

$$\varphi_1 \times \text{Id} \geq \psi$$

just replace the Hamiltonian generating $\psi$, $K(z_1, z_2)$ by $H_1(z_1)$ such that $K(z_1, z_2) \leq H(z_1)$ (note that $K$ is compact supported). Then

$$c_+(\psi(L_1 \times L_2), L_1 \times L_2) \leq c_+((\varphi_1 \times \text{Id})(L_1 \times L_2), L_1 \times L_2) =$$

$$c_+(\varphi_1(L_1), L_1) + c_+(L_2, L_2) = c_+(\varphi_1(L_1), L_1) = 0$$

a contradiction. So we proved that $U_1 \cap \text{supp}(L_1) = \emptyset$ implies $U_1 \times T^*N \cap \text{supp}(L_1 \times L_2) = \emptyset$. The same holds by exchanging the two variables, so we get

$$\text{supp}(L_1 \times L_2) \subset \text{supp}(L_1) \times \text{supp}(L_2)$$

For [5] let $z \in \gamma$-$\text{supp}(L_K)$ so there exists $\varphi \in \mathcal{D}\mathcal{J}am_c(B(z, \varepsilon))$ such that $\gamma(\varphi(L), L) > 0$. Let $U_\varepsilon$ be the preimage of $B(z, \varepsilon)$ in $K$ and $\tilde{\varphi} \in \mathcal{D}\mathcal{J}am(M, \omega)$ be a lift of $\varphi$, i.e. $\tilde{\varphi}$ preserves $K$ and projects to $\varphi$. If $\varphi$ is the flow of $H$ we set $\tilde{H}$ to be an extension of $H \circ \pi$ (where $\pi : K \rightarrow K/K$ is the projection) and set $\tilde{\varphi}$ to be the flow of $\tilde{H}$. Now the neighbourhood of the leaf $I = \mathcal{K}_z$ can be identified to a neighbourhood of $I \times \{z\}$ in $T^*I \times B(z, \varepsilon)$. Then by the reduction inequality [Vit21], Prop 7.43 and the equality $(\tilde{\varphi}(L))_K = \varphi(L_K)$, we have

$$\gamma(\tilde{\varphi}(L), L) \geq \gamma(\varphi(L_K), L_K)$$

Indeed, if $L'$ $\gamma$-converges to $L$, the sequence $L_K'$ $\gamma$-converges to $L_K$ and $\gamma(\varphi(L_K'), L_K') > \delta_0 > 0$ implies $\gamma(\tilde{\varphi}(L'), L') > \delta_0 > 0$ that is $\gamma((\tilde{\varphi}(L), L) > \delta_0 > 0$. 

**Figure 1.** $L$ and $L' = \varphi(L)$ with $\varphi$ supported in the interior of the blue circle. The hatched region has area $\varepsilon^2$.
For (6) we notice that $\Lambda_1 \circ \Lambda_2$ is the composition of the product $\Lambda_1 \times \Lambda_2$ and a reduction of $M_1 \times \bar{M}_2 \times M_2 \times \bar{M}_3$ by $M_1 \times \Delta_{M_2} \times \bar{M}_3$, where $\Delta_{M_2}$ is the diagonal of $M_2$.

Thus it is enough to prove that the $\gamma$-support of the reduction is contained in the reduction of the $\gamma$-support, which we just proved.

Let us consider the group $\mathcal{H}_\gamma(M,\omega)$ of homeomorphisms preserving $\gamma$. In particular since $\gamma$ is $C^0$-continuous (see [BHS21]) this group contains the group of homeomorphisms which are $C^0$-limits of symplectic diffeomorphisms, denoted $\text{Homeo}(M,\omega)$, usually called the group of symplectic homeomorphisms. We have $\text{Homeo}(M,\omega) \subset \mathcal{DHam}(M,\omega)$

**Corollary 6.22.** Let $\psi \in \mathcal{H}_\gamma(M,\omega)$. Then we have

$$\gamma\text{-supp}(\psi(L)) = \psi(\gamma\text{-supp}(L))$$

**Proof.** Indeed, let $\psi_j$ be a sequence $C^0$-converging to $\psi$. By [BHS21] the sequence $\gamma$-converges to $\psi$. Then

$$\gamma\text{-supp}(\psi(L)) \subset \liminf_j \gamma\text{-supp}(\psi_j(L)) = \liminf_j \psi_j(\gamma\text{-supp}(L)) = \psi(\gamma\text{-supp}(L))$$

This yields one inclusion. Applying the same to $\psi^{-1}$, gives the other inclusion. $\square$

**Remark 6.23.** When $L$ is smooth $\mathcal{DHam}_c(M,\omega)$ acts transitively on $\gamma\text{-supp}(L)$, and even better, the restriction map $\mathcal{DHam}_c(M,\omega) \to \text{Diff}_0(L)$ is onto. This is not at all the case in general as the following example shows.

**Example 6.24.** Let us consider the sequence of smooth Lagrangians represented below

![Figure 2](image)

**Figure 2.** The sequence $(L_k)_{k \geq 1}$ and the $\gamma$-support of the limit $\Lambda$ of the sequence $(L_k)_{k \geq 1}$

(1) Quite clearly there is a singular point, so $\mathcal{DHam}_c(M,\omega)$ cannot act transitively on $\gamma\text{-supp}(\Lambda)$. 
(2) We can arrange that $\Lambda$ is symmetric with respect to the involution $(x, p) \mapsto (x, -p)$, i.e. $\gamma$-supp$(\Lambda) = \gamma$-supp$(\Lambda)$. However $\Lambda \neq \Lambda$ because as we see from Figure 3, $\gamma(L_k, \Lambda_k)$ remains bounded from below (by twice the area of the loop). As a result the elements $\Lambda$ and $\Lambda$ in $\mathcal{L}(T^* N)$ have the same $\gamma$-support while being distinct.

\[ \text{Figure 3. The sequence } (L_k)_{k \geq 1} \]

(3) We conjecture that the set of elements in $\mathcal{L}(T^* S^1)$ such that $\gamma$-supp$(L) = \gamma$-supp$(L)$ is $\{\Lambda, \bar{\Lambda}\}$

Finally note that we can define a kind of density of $L \in \mathcal{L}(M, \omega)$ as follows

**Definition 6.25.** Let $L \in \mathcal{L}(M, \omega)$. We define

$$\rho_\varepsilon(z, L) = \sup \{ \gamma(\varphi(L), L) \mid \varphi \in \mathcal{D}\text{Ham}_c(B(z, \varepsilon)) \}$$

and $\rho(z, L)$ to be the limit

$$\limsup_{\varepsilon \to 0} \frac{1}{-2\log(\varepsilon)} \log \rho_\varepsilon(z, L)$$

It is easy to show that for $\psi \in \mathcal{D}\text{Ham}_c(M, \omega)$ we have $\rho(\psi(z), \psi(L)) = \rho(z, L)$ so that for a smooth Lagrangian, $\rho(z, L) = 1$, since this is the case for a Lagrangian vector space in $\mathbb{R}^{2n}$. We leave the proof of the following to the reader

**Proposition 6.26.** Let $(L_k)_{k \geq 1}$ be a converging sequence in $\hat{\mathcal{L}}(M, \omega)$ having limit $L$. Assume there exists $\varepsilon, \delta > 0$ such that $\rho_\varepsilon(z, L_k) \geq \delta$ for all $k$. Then, $z \in \gamma$-supp$(L)$ and $\rho_\varepsilon(z, L) \geq \delta$.

7. The $\gamma$-coisotropic subsets in a symplectic manifold: definitions and first properties

We now define the notion of $\gamma$-coisotropic subset in a symplectic manifold. Of course this notions will coincide with usual coisotropy in the case of smooth submanifolds. We start with a new definition that will
play a central role in this paper. After this paper was written, we real-
ized that the analogue of \( \gamma \)-coisotropic with \( \gamma \) replaced by the Hofer dis-
tance (and under the name of “locally rigid”) had already been defined in [Ush19]. Using such a notion, the property that for a submanifold, the equivalence between local rigidity and being coisotropic was stated by Usher and he proved that the Humilière-Leclercq-Seyfaddini theorem ([HLS15]) follows from the definition and its invariance by symplectic homeomorphism.

7.1. Basic definitions.

**Definition 7.1.** Let \( V \) be a subset of \((M, \omega)\) and \( x \in V \). We shall say that \( V \) is non-\( \gamma \)-coisotropic at \( x \) if for any ball \( B(x, \varepsilon) \) there exists a ball \( B(x, \eta) \subset B(x, \varepsilon) \) and a sequence \((\varphi_k)_{k \geq 1}\) of Hamiltonian maps supported in \( B(x, \varepsilon) \) such that \( \gamma - \lim \varphi_k = \text{Id} \) and we have \( \varphi_k(V) \cap B(x, \eta) = \emptyset \).

In other words a set \( V \) is \( \gamma \)-coisotropic at \( x \in V \) if there exists \( \varepsilon > 0 \) such that for any ball \( B(x, \eta) \) with \( 0 < \eta < \varepsilon \) there is a \( \delta(\eta) > 0 \) such that for all \( \varphi \in D\mathfrak{Ham}_c(B(z, \varepsilon)) \) such that \( \varphi(V) \cap B(x, \eta) = \emptyset \) we have \( \gamma(\varphi) > \delta(\eta) \).

We shall say that \( V \) is \( \gamma \)-coisotropic if it is non-empty and \( \gamma \)-coisotropic at each \( x \in V \). It is nowhere \( \gamma \)-coisotropic if each point \( x \in V \) is non-\( \gamma \)-coisotropic.

**Remark 7.2.** (1) A variation of this definition assumes in the definition of non-\( \gamma \)-coisotropic, that \( \varphi(V) = V' \) is fixed. This would be better in some instances, but the locally hereditary property, explained below becomes non-obvious (if at all true). Note that our definition should make the empty set to be \( \gamma \)-coisotropic, however we explicitly excluded this case.

(2) Note that we do not need to define the spectral distance on \((M, \omega)\) to define \( \gamma \)-coisotropic subsets. Indeed, we only need to consider elements in \( D\mathfrak{Ham}_c(B(x, \varepsilon)) \) for \( \varepsilon \) small enough. But for this, using a Darboux chart, it is enough to have \( \gamma \) defined in \( D\mathfrak{Ham}_c(\mathbb{R}^{2n}) \).

**Examples 7.3.** (1) A point in \( \mathbb{R}^2 \) is non-\( \gamma \)-coisotropic. Let \( H(q, p) \) be such that \( \frac{\partial H}{\partial q_1}(0, p) \) is sufficiently large, but \( H \) is \( C^0 \)-small. Then the flow of \( H \) takes the origin outside of \( B(0, \eta) \). By truncation, we can do this while being compact supported in \( B(0, \varepsilon) \).

(2) If \( I = [0, 1] \) in \( \mathbb{R}^2 \) it is easy to see that interior points of the interval are \( \gamma \)-coisotropic, while the boundary points are not. Thus \( I \) is not \( \gamma \)-coisotropic.

(3) It is not hard to see that \( \{(0, 0)\} \times \mathbb{R}^{2n-2} \) is not \( \gamma \)-coisotropic, and neither is \( I \times \{0\} \times \mathbb{R}^{2n-2} \). However this last set is \( \gamma \)-coisotropic at the point \( (x, 0, \ldots, 0) \) if and only if \( 0 < x < 1 \). More generally \( D^{n+r}(1) \times \{0\} \) is \( \gamma \)-coisotropic for \( 1 \leq r \leq n \) but not for \( r = 0 \).
(4) A smooth Lagrangian submanifold without boundary is \( \gamma \)-coisotropic.

**Question 7.4.** Let \( V \) be a \( \gamma \)-coisotropic subset. Does \( V \) have positive displacement energy? Even better, is it true that \( \gamma(V) = \inf \{ \gamma(\psi) | \psi(V) \cap V = \emptyset \} > 0 \) (see [Boi98; Gin07; Ion10] for results in this direction in the smooth case).

One of the goals of this paper is to show a number of natural occurrences of \( \gamma \)-coisotropic sets, starting from the \( \gamma \)-support of elements of \( \mathcal{L}(M, \omega) \) (see Theorem 7.12 and [GVit22a]).

We denote by \( \mathcal{L}_\gamma(M, \omega) \) the set of images of smooth Lagrangians by elements of \( \mathcal{H}_\gamma(M, \omega) \) and by \( \mathcal{S}_2(M, \omega) \) the set of images of symplectic submanifolds (not necessarily closed) of codimension greater or equal to 2 by elements of \( \mathcal{H}_\gamma(M, \omega) \). Note that these are subsets (and even topological submanifolds) of \( M \), which is not the case for elements in \( \mathcal{L}(M, \omega) \).

The following is quite easy:

**Proposition 7.5.** We have the following properties

1. Being \( \gamma \)-coisotropic is invariant by \( \mathcal{H}_\gamma(M, \omega) \), hence by \( \text{Homeo}(M, \omega) \) (and also by conformally symplectic homeomorphisms).
2. Being \( \gamma \)-coisotropic is a local property in \( M \). It only depends on a neighbourhood of \( V \) in \( (M, \omega) \).
3. If \( X, Y \) are \( \gamma \)-coisotropic then \( X \cup Y \) is \( \gamma \)-coisotropic.
4. \( \gamma \)-coisotropic implies locally rigid (in the sense of [Ush19]).
5. Being \( \gamma \)-coisotropic is locally hereditary in the following sense: if through every point \( x \in V \) there is a \( \gamma \)-coisotropic submanifold \( V_x \subset V \), then \( V \) is \( \gamma \)-coisotropic. In particular if through any point of \( V \) there is a Lagrangian germ, then \( V \) is \( \gamma \)-coisotropic. If any point in \( V \) has a neighbourhood contained in an element of \( \mathcal{S}_2(M, \omega) \) then \( V \) is non-\( \gamma \)-coisotropic.

**Proof.** The first two statements as well as the third are obvious from the definition. The fifth follows immediately from the inequality \( \gamma \leq d_H \) between Hofer distance and spectral norm. For the last one, the only non-obvious statement is that a smooth Lagrangian germ is coisotropic at any interior point, while a codimension 2 symplectic submanifold is everywhere non-\( \gamma \)-coisotropic.

For a Lagrangian, if \( B(x_0, 1) \) is a ball and \( L \cap B(x_0, 1) = \mathbb{R}^n \times \{0\} \) and \( L' \cap B(x_0, 1) = \emptyset \), then \( \gamma(L, L') \geq \pi/2 \). Indeed it is easy to construct a Hamiltonian isotopy supported in \( B(x_0, 1) \) such that \( \gamma(L_t, L) = \pi/2 \) and \( L_t = L \) outside \( B(1) \) (see Figure 1).

As a result

\[
\pi/2 = c_+(L_1, L') \leq c_+(L_1, L') + c_+(L', L) \leq c_+(L, L') - c_-(L, L') = \gamma(L, L')
\]
But this implies \( \gamma(\varphi) \geq \gamma(L', L) \geq \pi/2 \) and this concludes the proof in the Lagrangian case, since by a conformal map we can always reduce to this case.

Now let \( S \) be symplectic of codimension 2, we want to prove that \( S \) is nowhere coisotropic. Up to taking a subset of \( S \) and after a symplectic change of coordinates (and possibly a dilation), we may assume that \( S, U \) are identified locally to

\[
S_0 = \left\{ (0, 0, \tilde{q}, \tilde{p}) \in D^{2k}(1) \times D^{2n-2k}(1) \right\} \subset D^{2k}(1) \times D^{2n-2k}(1) = U
\]

where \( D^k(r) \) (resp. \( D^{2n-2k}(r) \)) are the symplectic balls of radius \( r \) in \( \mathbb{R}^2, \sigma_{2k} \) (resp. in \( \mathbb{R}^{2n-2k}, \sigma_{2n-2k} \)). Consider the isotopy

\[
t \mapsto (ta(\tilde{q}, \tilde{p}), 0, \tilde{q}, \tilde{p})
\]

where \( a(\tilde{q}, \tilde{p}) = A(|\tilde{q}|^2 + |\tilde{p}|^2) \) and \( A \) is a compact supported function bounded by 1, and equal to 1 in \( D^{2n-2k}(1) \). This is a Hamiltonian isotopy generated by

\[
H(\tilde{q}, \tilde{q}, \tilde{p}, \tilde{p}) = \chi(\tilde{q}, \tilde{p}) a(\tilde{q}, \tilde{p}), (\tilde{q}, \tilde{p}) \in \mathbb{R}^{2k} \text{ and } (\tilde{q}, \tilde{p}) \in \mathbb{R}^{2n-2k},
\]

sending \( S_0 \) to \( S_1 \), where

\[
S_1 = \left\{ (1, 0, \tilde{q}, \tilde{p}) \in D^{2k}(1) \times D^{2n-2k}(1) \right\}
\]

and in particular \( S_0 \cap S_1 = \emptyset \). Let

\[
H(q_1, \tilde{q}, p_1, \tilde{p}) = \chi(q_1, p_1) \cdot a(\tilde{q}, \tilde{p})
\]

be such that \( \frac{\partial(\chi(0), 0)}{\partial q_1} = 0 \) and \( \frac{\partial(\chi(0), 0)}{\partial p_1} = 1 \) for \( |q_1| \leq 1 \) and such that \( \|\chi\|_{C^0} \leq \varepsilon \).

The flow of \( H \) is given by

\[
\begin{aligned}
\dot{q}_1 &= \frac{\partial \chi}{\partial p_1}(q_1(t), p_1(t))A(|\tilde{q}(t)|^2 + |\tilde{p}(t)|^2) \\
\dot{p}_1 &= -\frac{\partial \chi}{\partial q_1}(q_1(t), p_1(t))A(|\tilde{q}(t)|^2 + |\tilde{p}(t)|^2) \\
\dot{\tilde{q}}(t) &= 2\chi(q_1(t), p_1(t)) A'(|\tilde{q}(t)|^2 + |\tilde{p}(t)|^2) \tilde{p}(t) \\
\dot{\tilde{p}}(t) &= -2\chi(q_1(t), p_1(t)) A'(|\tilde{q}(t)|^2 + |\tilde{p}(t)|^2) \tilde{q}(t)
\end{aligned}
\]

The last two equations imply that \( |\tilde{q}|^2 + |\tilde{p}|^2 \) is constant, hence \( a(\tilde{q}(t), \tilde{p}(t)) \) is constant. If we start from \( p_1 = q_1 = 0 \), we have \( p_1(t) = 0 \) and \( q_1(t) = tA(|\tilde{q}(0)|^2 + |\tilde{p}(0)|^2) \). As a result we have \( \varphi^1_H(S_0) = S_1 \). Since \( \|H\|_{C^0} \leq \varepsilon \) this proves our claim.

\[\square\]

A first consequence is

**Proposition 7.6.** Let \( V \) be a smooth submanifold. Then it is \( \gamma \)-coisotropic if and only if it is coisotropic in the usual sense, i.e. for all points \( x \in V \) we have \( (T_x V)^{\omega} \subset T_x V \).
Proof. Assume \( C \) is coisotropic. Locally \( C \) can be identified to
\[
\{(x_1, \ldots, x_n, p_1, \ldots, p_k, 0, \ldots, 0) \mid x_j, p_j \in \mathbb{R}\}
\]
This contains the Lagrangian \( p_1 = \ldots = p_n = 0 \) which we proved to be \( \gamma \)-coisotropic, hence \( C \) is \( \gamma \)-coisotropic.

Conversely assume \( V \) is smooth but not coisotropic. Then locally we can embed \( V \) in a codimension 2 symplectic submanifold as follows: according to the Decomposition theorem ([Vit21] thm 2.14), we can write \( T_x V = I \oplus S \) where \( I \) is isotropic, \( S \) symplectic, and there exists \( K \), uniquely defined by the choice of \( S \), such that \((K \oplus I)\) is symplectic and contained in \( S^0 \). Moreover \( K \oplus I \oplus S = D(x) \) is symplectic, so choosing a continuously varying \( S \), the same will hold for \( D \). If \( V \) is not coisotropic then
\[
D(x) \neq T_x M.
\]
We thus have a symplectic distribution \( D \) near \( x_0 \) such that \( T_x V \subset D(x) \). Since being symplectic is an open condition, \( D(x) \) will be symplectic in a neighbourhood of \( x_0 \). Then, using the exponential map, we may find a symplectic manifold \( W \) defined in a neighbourhood of \( V \) such that \( T_x W = D(x) \). Since we proved that a codimension 2 symplectic submanifold is non-coisotropic, and being \( \gamma \)-coisotropic is locally hereditary, we conclude that \( V \) is not \( \gamma \)-coisotropic. \( \square \)

7.2. Links with other notions of coisotropy. There are other definitions of coisotropic subsets.

**Definition 7.7** (Poisson coisotropic). We shall say that the set \( V \) is Poisson coisotropic if \( \mathcal{P}_V = \{ f \in C^\infty(M) \mid f = 0 \text{ on } V \} \) is closed for the Poisson bracket. In other words if \( f, g \) vanish on \( V \) so does \( \{f, g\} \).

Finally we define, following Bouligand (see [Bou32]), for a subset \( V \) in a smooth manifold, two cones:

**Definition 7.8.** The paratingent cone of a set \( V \) at \( x \) is
\[
C^+(x, V) = \left\{ \lim_n c_n (x_n - y_n) \mid x_n, y_n \in V, c_n \in \mathbb{R}, \lim_n x_n = \lim_n y_n = x, \lim_n c_n = +\infty \right\}
\]

The contingent cone of a set \( V \) at \( x \) is
\[
C^-(x, V) = \left\{ \lim_n c_n (x_n - x) \mid x_n \in V, c_n \in \mathbb{R}, \lim_n x_n = x, \lim_n c_n = +\infty \right\}
\]

Clearly \( C^-(x, V) \subset C^+(x, V) \). Note that \( C^+(x, V) \) is invariant by \( \nu \mapsto -\nu \), while it is not necessarily the case for \( C^-(x, V) \). We then have the following definition, for which we refer to Kashiwara and Schapira

**Definition 7.9** (Cone-coisotropic, see [KS90], theorem 6.5.1 p. 271). We shall say that \( V \) is cone-coisotropic if whenever a hyperplane \( H \) is such...
that $C^+(x, V) \subset H$ then the symplectic orthogonal of $H$, $H^\omega$ is contained in $C^-(x, V)$.

Note that what we call here cone-coisotropic is called involutivity in [KS90].

![Figure 4](image_url) Two cones $C \subset H$. The one on the left is coisotropic, the one on the right is not.

It is an elementary fact that in both cases a smooth submanifold is Poisson coisotropic or cone coisotropic if and only if it is coisotropic in the usual sense. We proved in [GVit22a] the first part of

**Proposition 7.10.** For a subset $V$ in $(M, \omega)$ we have the following implications

$$
\gamma\text{-coisotropic} \implies \text{cone-coisotropic} \implies \text{Poisson coisotropic}
$$

**Proof of the second implication of the Proposition.** We argue by contradiction. Assume there are two functions $f, g$ vanishing on $V$ such that $\{f, g\} \neq 0$. Let $x_0$ be a point such that $\{f, g\}(x_0) \neq 0$. Consider the set $S = f^{-1}(0) \cap g^{-1}(0)$ near $x_0$. Then since $df(x_0), dg(x_0)$ are non-zero and linearly independent $S$ is a codimension 2 submanifold near $x_0$ and since $X_f, X_g$ are normal to $S$ and have non-zero symplectic product, we see that $(T_{x_0}S)^\omega = \langle X_f(x_0), X_g(x_0) \rangle$ is symplectic, so $S$ is symplectic and we may conclude that $V$ is not cone-coisotropic, since in local coordinates, $S$ is given by $\{q_n = p_n = 0\}$ and $T_{x_0}S \subset \{q_n = 0\}$ but $\frac{\partial}{\partial p_n} \notin T_{x_0}S$ and since in this case $T_{x_0}S = C^+(x_0, S) = C^-(x_0, S)$ this proves that $S$ is not cone-coisotropic. □

There are Poisson-coisotropic sets which are not cone-coisotropic. In [GVit22a] we give an example of a cone-coisotropic set which is not $\gamma$-coisotropic.

**Example 7.11.** Let $V = \{(q, p) \mid p = 0, q \geq 0\}$. It is Poisson coisotropic, as this is obvious on $V \cap \{q > 0\}$ and the set of points where $V$ is Poisson coisotropic is closed. But $V$ is not cone-coisotropic at $(0, 0)$ because $C^+(0, V) = \mathbb{R} \frac{\partial}{\partial q}$ while $C^-(0, V) = \mathbb{R} \frac{\partial}{\partial p}$ so the cone condition is violated.
Our main result in this section is

**Theorem 7.12 (Main Theorem).** We have

1. For any \( L \in \hat{\mathcal{L}}(M, \omega) \), \( \gamma\)-\( \text{supp}(L) \) is \( \gamma \)-coisotropic.
2. (Peano Lagrangian) For any \( r \in \{1, \ldots, n\} \) we may find \( L \in \hat{\mathcal{L}}_c(T^* N) \) such that \( \text{supp}(L) \) contains the \( \gamma \)-coisotropic set
   \[
   K_r = \{(q, p) \mid |q| \leq 1, |p| \leq 1, p_r = p_{r+1} = \ldots = p_n = 0\}
   \]
3. There exists \( L \neq L' \) in \( \hat{\mathcal{L}}(T^* N) \) such that \( \gamma\)-\( \text{supp}(L) = \gamma\)-\( \text{supp}(L') \).

**Proof of Theorem 7.12 (1).** Assume \( V = \text{supp}(L) \) is non-\( \gamma \)-coisotropic at \( z \in V \). Then there exists a sequence of Hamiltonian maps \( \varphi_k \) going to \( \text{Id} \) (again for \( \gamma \)), supported in \( B(z, \varepsilon) \) and such that \( \varphi_k(V) \cap B(z, \eta) = \emptyset \). Now on one hand \( \gamma - \lim_k \varphi_k(L) = L \) since \( \gamma - \lim_k \varphi_k = \text{Id} \). Thus \( \gamma - \lim_k \varphi_k(L_k) = L \) has support \( \text{supp}(L) \). On the other hand \( \varphi_k(L) \) has support \( \varphi_k(\text{supp}(L)) \) and since \( \varphi_k(\text{supp}(L)) \cap B(z, \eta) = \emptyset \), Proposition 6.17 implies that \( \text{supp}(L) \cap B(z, \eta) = \emptyset \). We thus get a contradiction.

2. Let us consider a “cube” that is in local coordinates
   \[
   K_r = \{(q, p) \in T^* \mathbb{R}^n \mid |q| \leq 1, |p| \leq 1, p_r = p_{r+1} = \ldots = p_n = 0\}
   \]
   Any embedding of \( D^n \) in \( N \) yields such a cube. Our goal is first to construct \( L \in \hat{\mathcal{L}}(T^* S^1) \) such that \( \text{supp}(L) \supset K_r \).

**Theorem 7.13.** There exists an element \( L \in \mathcal{L}(T^* N) \) with support satisfying \( \text{supp}(L) = K_n \cup 0_N \).

**Proof.** Consider \( L = \psi(0_N) \) where \( \psi \) is supported in \( K \). Let \( A \) be a finite set in \( K \setminus L \), \( z \) a point in \( K \setminus L \) not contained in \( A \), \( U \subset K \setminus (L \cup A) \) be the symplectic image of the product of two Lagrangian balls \( \sigma : B^n(\varepsilon) \times B^n(1) \to K \setminus L \cup A \). Notice that \( B^n(\varepsilon) \times B^n(1) \) is symplectically isotopic to \( B^n\left(\sqrt{\frac{\varepsilon}{3}}\right) \times B^n\left(\sqrt{\frac{\varepsilon}{3}}\right) \). We assume \( z = \sigma(0) \) and set \( \Delta = \sigma(B^n(\varepsilon) \times \{0\}) \) be the image of the first Lagrangian ball.

**Lemma 7.14.** Let \( L, K, A, U, \Delta \) as above. Then there is a constant \( c \) such that for all \( 0 < \varepsilon < c \), there is a symplectic isotopy \( \varphi \) such that

1. \( \rho \) is supported in \( \hat{K} \setminus A \)
2. \( \rho(L) \cap U = \Delta \)
3. \( \gamma(\rho) < \varepsilon \) hence \( \gamma(\rho(L), L) < \varepsilon \)
4. there exists \( \rho \) supported in \( U \) such that \( \gamma(\varphi \rho(L), \rho(L)) > \frac{\varepsilon}{5} \)

**Proof.** First of all we may apply an isotopy sending \( L \) to the zero section. Then, since \( A \) is discrete, we may push all its points by a symplectic isotopy in a neighbourhood of the boundary of \( K \) and at the same time move \( z \) to \( q_1 = \ldots = q_n = 0, p_1 = \frac{1}{2}, \ldots, p_n = \frac{1}{2} \), since \( K \setminus 0_N \) is connected (except in
dimension 2, in which case we have two connected components, and $z$ is either $(0, \frac{1}{2})$ or $(0, -\frac{1}{2})$. As a result we may assume that $z$ is the center of a translate of $B^n(\frac{\rho}{2}) \times B^n(\frac{1}{3})$ contained in $K$ and avoiding both $A$ and $0_N$. We claim that we can move the zero section by a Hamiltonian isotopy $\tau_{\epsilon}$ generated by a Hamiltonian supported in $K$, with norm $\|H\|_{C^0} \leq \epsilon$ and such that

$$\tau_{\epsilon}(0_N) \cap \left( B^n(\frac{\rho}{2}) \times B^n(\frac{1}{3}) \right) = z + \left( B^n(\frac{\rho}{2}) \times \{0\} \right)$$

Taking $\rho = \tau_{\epsilon}$, this proves (1), (2), (3). Finally to prove (4), it is enough to construct a Hamiltonian isotopy supported in $B^n(\frac{\rho}{2}) \times B^n(\frac{1}{3})$ as is done in the proof of Proposition 6.21, see Figure 1, such that if $L \cap \left( B^n(\frac{\rho}{2}) \times B^n(\frac{1}{3}) \right) = B^n(\frac{\rho}{2}) \times \{0\}$ we have $\gamma(\varphi(L), L) \geq \frac{\epsilon}{5}$. For this it is enough to replace $B^n(\frac{\rho}{2}) \times \{0\}$ by the graph of $df$ where $f$ is supported in $B^n(\frac{\rho}{2})$ and $|df| < \frac{1}{3}$ and osc$f > \frac{\epsilon}{5}$. This induces a Hamiltonian isotopy such that $\gamma(\varphi(L), L) > \frac{\epsilon}{5}$.

\[\begin{array}{c}
\text{Figure 5. Illustration of Lemma 7.14}
\end{array}\]
Figure 6. The Lagrangian $\rho(0_N) \cap U$ and the path $\rho_k(\Delta)$.
The pink region has area less than $\varepsilon_k$.

Let now $(z_j)_{j \geq 1}$ be a dense subset in $\tilde{K}$. We apply the above Proposition inductively with $L = L_k, A_k = \{z_1, ..., z_k\}, \varepsilon = \varepsilon_k$. We shall determine the sequence $(\varepsilon_k)_{k \geq 1}$ later. We thus get sequences $(\rho_k)_{k \geq 1}, (\phi_k)_{k \geq 1}$ such that properties [1]-[4] of the former Proposition hold. In other words we have

1. $\rho_k$ is supported in $\tilde{K} \setminus A_k$
2. $\gamma(\rho_k(L_k), L_k) < \varepsilon_k$
3. there exists $\phi_k$ supported in $U_k$ such that $\gamma(\phi_k(\rho_k(L_k)), \rho_k(L_k)) > \varepsilon_k / 5$

We then set $L_{k+1} = \rho_k(L_k)$. According to Property [4] this sequence will be $\gamma$-Cauchy if the series $\sum_j \varepsilon_j$ converges. More precisely $\gamma(L_k, L_\infty) < \sum_{j=k}^{+\infty} \varepsilon_j$. Now assume the sequence $(\varepsilon_j)_{j \geq 1}$ satisfies $\varepsilon_{j+1} < \varepsilon_j / 20$ then denoting by $L_\infty$ the limit of $L_k$, we have

$$\gamma(L_{k+1}, L_\infty) < \varepsilon_{k+1} \sum_{j=0}^{+\infty} \frac{1}{20^j} = \frac{20}{19} \varepsilon_{k+1} < \frac{1}{19} \varepsilon_k$$

Then we claim

**Lemma 7.15.** Consider the sequence $L_k$ just defined and let $L_\infty$ be its $\gamma$-limit. Then we have for all $k$ large enough $\gamma(\phi_k(L_\infty), L_\infty) > \frac{1}{11} \varepsilon_k$. 

Proof. Let us indeed use the triangle inequality to compute
\[ \gamma(\varphi_k(L_{k+1}), L_{k+1}) \leq \gamma(\varphi_k(L_{k+1}), \varphi_k(L_\infty)) + \gamma(\varphi_k(L_\infty), L_\infty) + \gamma(L_\infty, L_{k+1}) \]
so that
\[ \gamma(\varphi_k(L_\infty), L_\infty) \geq \gamma(\varphi_k(L_{k+1}), L_{k+1}) - \gamma(\varphi_k(L_{k+1}), \varphi_k(L_\infty)) - \gamma(L_\infty, L_{k+1}) \]
and since \( \varphi_k \) is an isometry for \( \gamma \) we have
\[ \gamma(\varphi_k(L_\infty), L_\infty) \geq \gamma(\varphi_k(L_{k+1}), L_{k+1}) - 2\gamma(L_\infty, L_{k+1}) \geq \frac{\varepsilon_k}{5} - \frac{2}{19}\varepsilon_k > \frac{9}{95}\varepsilon_k > \frac{1}{11}\varepsilon_k \]
\[ \square \]

Proof. (Proof of Theorem 7.13) We start with (2). We now consider the case where the sequence \( (z_k)_{k \geq 1} \) is just the sequence of points with rational coordinates, and \( U_k \) are neighbourhoods of \( z_k \) making a basis of open sets for the topology of \( K \). Then \( \text{supp}(L_\infty) \) is a closed set, meeting all the \( \text{supp}(\varphi_k) \) and since \( \gamma(\varphi_k h(L_\infty), L_\infty) > 0 \), it meets all the \( U_k \). This implies that \( \text{supp}(L_\infty) \) is dense in \( K \), hence contains \( K \).

As for (3), we can construct two sequences \((L_k)_{k \geq 1}\) and \((L'_k)_{k \geq 1}\) having \( \gamma \)-limits \( L, L' \) as above so that \( \gamma-\text{supp}(L) = \gamma-\text{supp}(L') \) and \( L \neq L' \). Indeed, we may create the first “tongue” in different directions, so that \( \gamma(L_1, L'_1) \geq \varepsilon_0 \) and if we choose the \( \varepsilon_k \) such that \( \varepsilon_0 > \sum_{j=1}^{+\infty} \varepsilon_j \) we get \( \gamma(L, L') > \varepsilon_0 - \sum_{j=1}^{+\infty} \varepsilon_j \). Thus the support does not, in general, determine a unique element in \( \widehat{\mathcal{L}}(M, \omega) \).

Remark 7.16. It is easy to see that \( \text{supp}(L_\infty) \subset 0_N \cup K \) so that \( \text{supp}(L_\infty) = 0_N \cup K \).

To conclude the proof of the case \( 1 \leq r < n \) of Theorem 7.12 we first use

Proposition 7.17. Let \( L_1 \in \widehat{\mathcal{L}}(M_1, \omega_1) \) and \( L_2 \in \mathcal{L}(M_2, \omega_2) \). Then we have
\[ \gamma-\text{supp}(L_1 \times L_2) = \gamma-\text{supp}(L_1) \times L_2 \]

Proof. We already proved the inclusion \( \gamma-\text{supp}(L_1 \times L_2) \subset \gamma-\text{supp}(L_1) \times \gamma-\text{supp}(L_2) = \gamma-\text{supp}(L_1) \times L_2 \). The opposite inclusion follows from the following two claims

(1) \( \gamma-\text{supp}(L_1 \times L_2) \cap \{z_1\} \times L_2 \neq \emptyset \) for all \( z_1 \in \gamma-\text{supp}(L_1) \)

(2) \( \gamma-\text{supp}(L_1 \times L_2) \) is invariant by \( \text{id} \times \varphi \) for any \( \varphi \in \text{Diff}(M_2, \omega_2) \) such that \( \varphi(L_2) = L_2 \)

Let us first show that these two claims imply the Proposition. Indeed, for \( (z_1, z_2) \in \gamma-\text{supp}(L_1) \times L_2 \), we have by (1) that there exists \( z'_2 \) such that \( (z_1, z'_2) \in \gamma-\text{supp}(L_1 \times L_2) \). Now by (2), choosing \( \varphi \) such that \( \varphi(z'_2) = z_2 \),
which is possible since the Hamiltonian maps preserving $L_2$ act transitively on $L_2$, we get that $(z_1, z_2) \in \gamma \text{-supp}(L_1 \times L_2)$. For [1], let $\varphi_1$ be supported in $B(z_1, \varepsilon)$ such that $\varphi(L_1, L_1) > 0$. Then we have $\gamma(\varphi_1 \times \text{Id})(L_1 \times L_2), L_1 \times L_2) = \gamma(\varphi_1(L_1) \times L_2, L_1 \times L_2) = \gamma(\varphi(L_1), L_1) > 0$, where the last equality follows from corollary 7.53 from [Vit21]. This implies that $\text{supp}(\varphi \times \text{Id})$ intersects $\gamma \text{-supp}(L_1 \times L_2)$, and letting $\varepsilon$ go to 0 we get 

$$\gamma \text{-supp}(L_1 \times L_2) \cap \{z_1\} \times M_2 \neq \emptyset$$

Since $\gamma \text{-supp}(L_1 \times L_2) \subset \gamma \text{-supp}(L_1) \times L_2$, we have

$$\gamma \text{-supp}(L_1 \times L_2) \cap \{z_1\} \times M_2 \subset \gamma \text{-supp}(L_1) \times L_2 \cap \{z_1\} \times M_2 = \{z_1\} \times L_2$$

Therefore $\gamma \text{-supp}(L_1 \times L_2) \cap \{z_1\} \times L_2 \neq \emptyset$.

Claim [1] then follows from the fact that $\text{id} \times \varphi$ preserves $L_1 \times L_2$ hence preserves $\gamma \text{-supp}(L_1 \times L_2)$. \hfill \Box

We may now conclude the proof of the case $1 \leq r < n$. Consider $L \subset T^*N$ where $N$ has dimension $r$ such that $\gamma \text{-supp}(L_1) = 0_N \cup [-1, 1]^{2r}$. Let $M$ be $n - r$ dimensional and containing an embedding of $[-1, 1]^{n-r}$ (this is of course always the case). Then $\gamma - \text{supp}(L_1 \times 0_M) = \gamma \text{-supp}(L) \times 0_M = (0_L \cup [-1, 1]^{2r}) \times 0_M = 0_{L \times M} \cup K_r$. \hfill \Box

**Remarks 7.18.** (1) Similarly we can find $L \in \widehat{\mathcal{L}}(T^* N)$ such that $\gamma \text{-supp}(L) = 0_N \cup T^* D^u(1)$.

**Question 7.19.** Can a coisotropic submanifold containing “no Lagrangian”, for example such that the coisotropic foliation has a dense leaf, be the $\gamma$-support of an element $L \in \widehat{\mathcal{L}}(T^* N)$? One should be careful about the meaning of “Lagrangian” as this should be understood in a weak sense, since for a $C^0$ function $f$, the graph $\text{graph}(df)$ will usually not contain a smooth Lagrangian.

We consider a sequence of $\gamma$-coisotropic sets, $V_k$. If the $V_k$ are compact and contained in a bounded set, then, up to taking a subsequence, they have a Hausdorff limit and setting $V = \lim_k V_k$ we may ask whether $V$ is $\gamma$-coisotropic. The answer is obviously negative: take $V_k$ to be sphere of center 0 and radius $\frac{1}{k}$. Then $V_k$ has for limit $\{0\}$ which is not coisotropic. However the following question is more sensible

**Question 7.20.** Let $L_k$ be a sequence in $\widehat{\mathcal{L}}(M, \omega)$ and $V_k = \gamma \text{-supp}(L_k)$. Assume $V = \lim_k V_k$ where the limit is a Hausdorff limit. Is $V$ a $\gamma$-coisotropic set?

Note that the sequence cannot collapse to a set of Hausdorff dimension less than $n$ because of the intersection properties of the $\gamma$-supports (Proposition 6.10).
Remarks 7.21. (1) For the connection between Hausdorff and $\gamma$-convergence (for Lagrangians) in the presence of Riemannian constrains we refer to [Cha21].

(2) Note that if all $\gamma$-supports contain an exact Lagrangian, then Proposition 6.10 becomes obvious. Of course this can not be true in the usual sense, i.e. all $\gamma$-supports do not contain a smooth Lagrangian: if $f$ is a $C^0$ function, graph$(df)$ cannot contain a smooth Lagrangian. On the other hand if this is not the case, then we get a really new class of subsets, invariant by symplectic isotopy, having intersection properties.

Note that one could hope that if $u_L$ is the graph selector associated to $L$, we have $\gamma$-supp(graph$(du_L)) \subset \gamma$-supp$(L)$. This is however not the case as we see from the following example

Example 7.22. Let $u(x) = |x|$. It is then easy to see by using smooth approximations of $u$ that $\gamma$-supp(graph$(du_L))$ is the union of $\{(x,-1) | x \leq 0\}, \{(x,1) | x \geq 0\}$ and $\{0\} \times [-1,1]$. However $u$ is the selector for the Lagrangian represented on Figure 7.

![Figure 7. The Lagrangian from Exemple 7.22](image)

However it is not difficult to prove that

$$\gamma$$-supp$(\text{graph}(du_L)) \subset \text{Conv}_p(\gamma$$-supp$(L))$$

where $\text{Conv}_p$ is the $p$-convex hull, that is

$$\text{Conv}_p(X) = \left\{ (q,p) \in T^*N | \exists (p_1,...,p_r) \in T^*_qN, t_j \geq 0, \sum t_j = 1, p = \sum_{j=1}^n t_j p_j \right\}$$

Question 7.23. Does $\gamma$-supp$(du_L) \cap \gamma$-supp$(L)$ contain the extremal points of $\text{Conv}_p(\gamma$$-supp(L))$?

8. Regular Lagrangians

In this section $N$ will be a closed $n$-dimensional manifold.
Definition 8.1. An element \( L \) in \( \hat{\mathcal{L}}(M, \omega) \) is said to be regular if \( \text{supp}(L) \) is a smooth \( n \)-dimensional manifold. Such a manifold is then Lagrangian by Theorem 7.12 and Proposition 7.6. It is topologically regular if \( \text{supp}(L) \) is a topological \( n \)-dimensional manifold.

Conjecture 8.2. A regular Lagrangian coincides with an element of \( \mathcal{L}(M, \omega) \). More precisely if \( L \in \hat{\mathcal{L}}(M, \omega) \) is such that \( \gamma\text{-supp}(L) = V \) is an smooth \( n \)-manifold, then \( V \) is an exact Lagrangian and \( L = V \).

It follows from [AGHIV-2] (written after this paper was posted) that

Theorem 8.3 (Regular Lagrangians in \( T^*N \)). If \( L \) is regular in \( T^*N \) and \( \gamma\text{-supp}(L) \) is exact, then \( L = \gamma\text{-supp}(L) \).

Remark 8.4. This means that the Lagrangian \( L \in \hat{\mathcal{L}}(T^*N) \) coincides with \( \gamma\text{-supp}(L) \in \mathcal{L}(T^*N) \subset \hat{\mathcal{L}}(T^*N) \).

Following this theorem, it makes sense to set

Definition 8.5. A topological Lagrangian in \( (M^{2n}, \omega) \) is a \( C^0 \)-submanifold of dimension \( n \), \( V \), such that there exists \( L \in \hat{\mathcal{L}}(M, \omega) \) with \( V = \gamma\text{-supp}(L) \).

Remark 8.6. There are other possible definitions for topological Lagrangians. For example we could define a topological Lagrangian as an \( n \)-dimensional topological manifold that is \( \gamma \)-coisotropic.

It follows from [BHS21], where it is proved that \( C^0 \)-convergence implies \( \gamma \)-convergence, that if \( \varphi_k \) is a sequence of smooth symplectic maps converging \( C^0 \) to \( \varphi \), then \( \varphi = \gamma - \lim_k \varphi_k \), so \( \varphi(L) = \gamma - \lim \varphi_k(L) \) and from Corollary 6.18 \( \gamma\text{-supp}(\varphi(L)) = \varphi(L) \) so that \( \varphi(L) \) is a topological Lagrangian.

Question 8.7. How far can we extend this result? In particular is the property \( \dim(\text{supp}(L)) = n \) enough if we only assume \( L \) is a topological manifold? Or if we assume \( \text{supp}(L) \) contains no proper coisotropic? Or if \( \text{supp}(L) \) is minimal for inclusion among the supports of elements in \( \mathcal{L}(T^*N) \)?

It is probably useful at this point to remind the reader that smooth Lagrangians are indeed minimal among \( \gamma \)-coisotropic sets.

Proposition 8.8 (see [GVit22a], proposition 9.13). Let \( L = \varphi(L_0) \) where \( L_0 \) is smooth Lagrangian and \( \varphi \in \mathcal{H}_\gamma(M, \omega) = \hat{\mathcal{H}} \text{am}(M, \omega) \cap \text{Homeo}(M) \). Then any closed proper subset of \( L \) is not \( \gamma \)-coisotropic.

Here is another related question

Question 8.9. For \( L_1, L_2 \) such that \( L_1 \) is smooth, \( L_2 \in \hat{\mathcal{L}}(T^*N) \) and \( L_1 \subset \gamma\text{-supp}(L_2) \), what is the relation between \( L_1 \) and \( L_2 \)?
We may generalise this to the following

**Question 8.10.** For which sets $V$ is the set of $L \in \mathcal{L}(T^* N)$ such that $\gamma$-supp($L$) = $V$ is compact?

**Remark 8.11.** Let $(L_k)$ be a sequence in $\mathcal{L}(T^* N)$ and assume it converges for the Hausdorff topology to a smooth submanifold $L$. Necessarily $L$ must have dimension at least $n$ since otherwise for $k$ large enough, $L_k$ would not intersect some vertical fibre. Assume it has dimension exactly $n$, we then we claim that $L$ is Lagrangian. Indeed, $L_k \times S^1 \subset T^* N \times T^* S^1$ converges in the Hausdorff topology to $L \times 0$. By [LS94], if $L$ is not Lagrangian, $L \times 0$ is displaceable by a Hamiltonian isotopy and so will be $L_k \times 0$ for $k$ large enough. But this is impossible. So $L$ is Lagrangian, and moreover it is exact by [MO21]. Now remember the Conjecture from [Vit23], proved in some special cases in [She18; She19; GVic22; Vit22].

**Conjecture 8.12** (Geometrically bounded implies spectrally bounded). Let $N$ be a closed Riemannian manifold. There exists a constant $C_N$ such that for all exact Lagrangian $L$ contained in

$$DT^* L = \{(q, p) \in T^* N \mid |p|_g \leq 1\}$$

we have $\gamma(L) \leq C_N$.

This naturally extends to

**Conjecture 8.13.** There exists a constant $C_N$ such that for any $L \in \mathcal{L}(T^* N)$ with $\gamma$-supp($L$) $\subset$ $DT^* N$, we have

$$\gamma(L) \leq C_N$$

Note that this does not immediately follow from the conjecture in the smooth case: even though $L$ is the limit of smooth $L_k$, we cannot claim that the $L_k$ are contained in a neighbourhood of $\gamma$-supp($L$). The above conjecture is proved for a certain class of manifolds in [Vit22].

Now we claim

**Proposition 8.14.** Let $\varphi \in \mathcal{D}\mathcal{H}am_c(T^* T^n)$ such that $\gamma$-supp($\Gamma(\varphi)$) = $\Delta_{T^* T^n}$. Then $\varphi = Id$.

**Proof.** Consider the $\mathbb{Z}^n$ symplectic covering from $T^*(\Delta_{T^* T^n})$ to $T^* T^n \times \overline{T^* T^n}$. Let $\Gamma(\varphi)$ be the graph of $\varphi$. For $\varphi \in \mathcal{D}\mathcal{H}am_c(T^* T^n)$, $\Gamma(\varphi)$ has a
Remark 8.16 that \( \phi \in \text{DHam}_\psi(T^*T^n) \) are diffeomorphic to \( T^*T^n \). Now the projection of the covering yields a diffeomorphism between \( \hat{\Gamma}(\phi) \) and \( \Gamma(\phi) \)

\[
\gamma(\hat{\Gamma}(\phi_1), \hat{\Gamma}(\phi_2)) = \gamma(\Gamma(\phi_1), \Gamma(\phi_2))
\]

and this implies that an element \( \varphi \in \text{DHam}_\psi(T^*T^n) \) defines a unique \( \hat{\Gamma}(\varphi) \) in \( \hat{\mathcal{L}}(T^*(\Delta_{T^*T^n})) \). Now \( \gamma\text{-supp}(\hat{\Gamma}(\varphi)) = \Delta_{T^*T^n} \) so, according to the previous result, we have \( \hat{\Gamma}(\varphi) = \Delta \). But this means that \( \varphi = \text{Id} \).

\( \square \)

The same argument yields

**Corollary 8.15.**

1. We have the same result for \( \text{DHam}(T^{2n}) \)
2. The center of \( \text{DHam}_\psi(T^*T^n) \) is \( \{\text{Id}\} \). The same holds for the center of \( \text{DHam}(T^{2n}) \).

**Proof.** The first result is obtained using the covering map

\[
T^*\Delta_{T^{2n}} \to T^{2n} \times \overline{T}^{2n}
\]

\[
(x, y, X, Y) \mapsto (x + \frac{Y}{2}, y - \frac{X}{2}, x - \frac{Y}{2}, y + \frac{X}{2})
\]

and repeating the above argument. For the second, we see that if \( \varphi \in \text{DHam}_\psi(T^{2n}) \) commutes with all elements of \( \text{DHam}_\psi(T^{2n}) \), then for all \( \psi \in \text{DHam}_\psi(T^{2n}) \) we have \( (\psi \times \psi)(\Gamma(\varphi)) = \Gamma(\varphi) \) hence the same holds for \( \gamma\text{-supp}(\Gamma(\varphi)) \). But the only subsets of \( M \times M \) invariants by all maps \( \psi \times \psi \) are \( M \times M \) and the subsets of the diagonal. However since \( \gamma\text{-supp}(\Gamma(\varphi)) \) is \( \gamma \)-coisotropic and contained in the diagonal, it must be equal to the diagonal. As a result \( \gamma\text{-supp}(\Gamma(\varphi)) = \Delta_{T^{2n}} \) and it follows from Proposition 8.14 that \( \varphi = \text{Id} \).

\( \square \)

**Remark 8.16.** We do not know if there exists \( \varphi \in \text{DHam}(M, \omega) \) such that \( \gamma\text{-supp}(\Gamma(\varphi)) = M \times \overline{M} \). This justifies the next definition.

**Definition 8.17.** The set \( \{ \varphi \in \text{DHam}(M, \omega) \mid \gamma\text{-supp}(\Gamma(\varphi)) = \Delta_M \} \) is denoted \( \mathcal{N}_\Delta(M, \omega) \) and is actually a normal closed subgroup in \( \text{DHam}(M, \omega) \).

Note that the closeness follows from the fact that if \( \varphi_k \to \varphi \) and \( \gamma\text{-supp}(\varphi) = \Delta_M \) then \( \gamma\text{-supp}(\varphi_k) \subseteq \Delta_M \) but since \( \Delta_M \) is Lagrangian, and no proper

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4 The following equality is easy to see in the context of Floer cohomology, since there is a one-to-one correspondence between holomorphic strips on the base and on the covering space, as strips are simply connected.

5 The above inequality implies that the map \( \text{DHam}(T^*T^n) \to \hat{\mathcal{L}}(T^*(\Delta_{T^*T^n})) \) is obtained by taking the completion of the isometric embedding \( \text{DHam}_\psi(T^*T^n) \to \mathcal{L}(T^*(\Delta_{T^*T^n})) \). It is therefore injective: two elements having the same graph are equal!
subset of a Lagrangian is $\gamma$-coisotropic (by [GVit22a], proposition 9.13, see Proposition 8.8), we must have $\gamma$-supp($\varphi$) = $\Delta_M$.

We have the following

**Conjecture 8.18.** For any symplectic manifold $M$ the only element $\varphi \in \widehat{DHam}_c(M,\omega)$ such that $\gamma$-supp($\Gamma(\varphi)$) = $\Delta_M$ is $\varphi = \text{Id}$, i.e. $\mathcal{N}_\Delta(M,\omega) = \{\text{Id}\}$. As a result the center of $\widehat{DHam}_c(M,\omega)$ is $\{\text{Id}\}$.

Of course a stronger conjecture would be that $\widehat{DHam}(M,\omega)$ is a simple group.

**9. ON THE EFFECT OF LIMITS AND REDUCTION**

As we know in the smooth case the property of being coisotropic is closed for the $C^1$ topology, and according to [HLS15] even in the $C^0$ case (actually slightly less: to apply [HLS15] we need that the sequence $C_k$ of coisotropic to be given by $\varphi_k(C)$ where the sequence $(\varphi_k)_{k \geq 1}$ $C^0$ converges to $\varphi \in Homeo(M,\omega)$). Also being coisotropic is a property preserved by symplectic reduction.

**Proposition 9.1.** There exists a sequence of Lagrangians such that its Hausdorff limit is non-$\gamma$-coisotropic.

*Proof.* Indeed let us consider the sequence represented on Figure 6 below. The Hausdorff limit of the curve is a half-line. But a half-line is not $\gamma$-coisotropic at its endpoint, since it is easy to move $p = 0, q \geq 0$ by $H(q,p)$ such that $\frac{\partial H}{\partial p}(q,0) > 1$ outside the unit ball and $H$ is arbitrarily small (or check that it is not cone-coisotropic). □

Note that for the same reason the red curve does not belong to the support of the $\gamma$-limit of the $L_n$ (it is easy to see that the sequence $(L_n)_{n \geq 1}$ is $\gamma$-Cauchy) since it is not $\gamma$-coisotropic.
Figure 8. A Hausdorff limit (in red) of Lagrangians (in black) that is not $\gamma$-coisotropic.

Now let us examine the effect of reduction on $\mathcal{L}(T^*N)$. For example consider $V$ a closed submanifold in $X$. The symplectic reduction $L_V = L \cap T^*_V X)/\simeq$ where

$$(q, p_1) = (q, p_2) \Leftrightarrow p_1 - p_2 = 0$$

of $L \in \mathcal{L}(T^*X)$ is well-defined for $L$ transverse to $T^*_V X$ and such that the projection is an embedding. Then $L_V \in \mathcal{L}(T^*V)$, and transversality, the set of $L$ for which $L_V$ is well-defined and embedded is open for the $C^1$ topology. We denote by $\mathcal{L}_V(T^*X)$ the set of such Lagrangians.

**Proposition 9.2.** Let $V$ be a closed submanifold in $X$. The map $\cdot_V : L \mapsto L_V$ defined on $\mathcal{L}_V(T^*X)$ with image in $\mathcal{L}(T^*V)$ extends to a (well-defined) map $\hat{\cdot}_V : \hat{\mathcal{L}}_V(T^*X) \longrightarrow \hat{\mathcal{L}}(T^*V)$ where $\hat{\mathcal{L}}_V(T^*X)$ is the closure of $\mathcal{L}_V(T^*X)$ in $\hat{\mathcal{L}}(T^*X)$.

**Proof.** Given a map $f$ defined on a subset $C$ of a metric space $(A, d_A)$, to $(B, d_B)$ and $f$ is Lipschitz, it extends to a continuous map from the closure of $C$ to the completion of $B$, $(\hat{B}, \hat{d}_B)$ since $f$ sends Cauchy sequences to Cauchy sequences. Now we just need to apply this to $C = \mathcal{L}_V(T^*X)$, the subset of elements in $\mathcal{L}(T^*X)$ having a $V$-reduction and $B$ the space $\hat{\mathcal{L}}(T^*V)$, both endowed with the metric $\gamma$. The proof is concluded by invoking proposition 5.2 in [Vit22] which claims

$$\gamma((L_1)_V, (L_2)_V) \leq \gamma(L_1, L_2)$$
We shall by abuse of language denote by $L_V$ the reduction of $L \in \mathcal{L}(T^*X)$ to $\hat{\mathcal{L}}(T^*V)$. Now let $C$ be a subset in $T^*(X \times Y)$ and $C_x \subset T^*Y$ the reduction of $C$ by $\{x\} \times Y$ that is $C \cap T^*_X X \times T^*_Y Y / T^*_X X \simeq T^*_Y Y$.

Remark 9.3. While the elements of $\mathcal{L}_V(T^*X)$ all yield by reduction, an embedded Lagrangian in $\mathcal{L}(T^*V)$, it is possible that an element of $\hat{\mathcal{L}}_V(T^*V)$ reduces to a non-embedded Lagrangian in $T^*V$. This if for example the case for a $C^0$ limit of Lagrangians, but one could hope for more interesting examples.

**Proposition 9.4.** For $L \in \hat{\mathcal{L}}_V(T^*(X \times Y))$, we have

$$\gamma\text{-supp}(L_V) \subset (\gamma\text{-supp}(L))_V$$

**Proof.** As this is a local result, it is enough to prove it for $T^*\mathbb{R}^n$ with $V = \{x = 0\}$ where $x \in \mathbb{R}$, $z \in \mathbb{R}^{n-1}$ are coordinates in $\mathbb{R}^n$, and we denote by $H_u$ the hyperplane $x = u$. Assume $\varphi_0 \in \mathcal{D}\text{Ham}(H_0/H_0^0)$ is supported in a neighbourhood of $y_0 = 0$ and $\gamma(\varphi_0(L_0), L_0) > 0$. Then we may find $\varphi$ defined in a neighbourhood of $(0, z_0)$ in $M$ such that $\gamma(\varphi(L), L) > \epsilon_0$. Indeed, we may choose $\varphi_0$ as in Proposition 6.20, generated by $\chi(z^2 + p_z^2)$. Extending $\varphi_0$ by considering the flow of $\chi(\{x\}) \chi(\gamma^2 + p_y^2)$, we get $\varphi$ such that the reduction of $\varphi(L)$ at $x = 0$ is $\varphi_0(L_0)$. By the reduction inequality we get

$$\gamma(\varphi(L), L) \geq \gamma(\varphi_0(L_0), L_0) > 0$$

Since the support of $\varphi$ is a neighbourhood of $T^*\mathbb{R} \times B(z_0, \epsilon)$, applying Lemma 6.9 we have a point $(0, p, y, p_y)$ contained in $\text{supp}(L)$. This implies $\text{supp}(L_0) \subset \text{supp}(L_0)$. 

We let prove the opposite inclusion at least for elements in $\hat{\mathcal{L}}(T^*X)$.

**Proposition 9.5.** Let $L \in \hat{\mathcal{L}}_V(T^*X) \cap \hat{\mathcal{L}}_V(T^*X)$. Then there is a sequence $V_j$ converging (for the $C^\infty$ topology) to $V$ such that

$$(\gamma\text{-supp}(L))_V \subset \liminf \gamma\text{-supp}(L_{V_j})$$

**Proof.**

Attention: il faut remplacer $V$ par $V'$ proche de $V$ et vérifier qu'on reste dans $\hat{\mathcal{L}}_V(T^*X)$

Set $\chi_\delta(t) = \chi(t^2)$. Again this is a local result and we only deal with the case $X = \mathbb{R}^n$ and $V = \{x = 0\}$. Let $(y, p_y)$ in $\gamma\text{-supp}(L)_x$, so there exists $p_0$ such that $(0, p_0, y_0, p_{y_0})$ in $\gamma\text{-supp}(L)$. We can consider the Hamiltonian $H_{\delta, \alpha}(x, p_x, y, p_y) = \chi_\delta(|x|^2) \chi_\alpha((y-y_0)^2 + (p_y - p_{y_0})^2)$, with flow $\varphi_\delta^t$ and using Proposition 6.20 we must have $c_\alpha(\varphi_\delta^t(L), L) > 0$ for $t > 0$. This flow preserves the submanifolds $\{(x, p_x, y, p_y) \mid x = x_0\}$, so that $(\varphi_\delta^t(L))_{x_0}^t = \cdots$
\( \varphi^{t}_{\delta,a,x_{0}}(L_{x_{0}}) \) where \( \varphi_{\delta,a,x_{0}} \) is the flow of \( \chi_{\delta}(|x_{0}|^{2})\chi_{a}((y - y_{0})^{2} + (p_{y} - p_{y_{0}})^{2}) \) on \( T^{*}\mathbb{R}^{n-1} \).

Then by the inverse reduction inequality ([Vit22], thm. 4.16)

\[ 0 < \varepsilon_{0} < c_{+}(\varphi_{\delta,a}(L), L) \leq \sup_{x} c_{+}(\varphi_{\delta,a}(L)_{x}, L_{x}) + C_{d} \sup_{x} \gamma((\varphi_{\delta,a}(L))_{x}, L_{x}) \]

The right-hand side is equal to

\[ \sup_{x} c_{+}(\varphi_{\delta,a,x}(L_{x}), L_{x}) + C_{d} \sup_{x} \gamma((\varphi_{\delta,a}(L))_{x}, L_{x}) \]

So there is \( x \in B_{\delta}(0) \) such that \( B_{\delta}(\gamma, p_{y}) \cap \gamma\text{-}\text{supp}(L_{x}) \neq \emptyset \). In other words \((\gamma, p_{y}) \in \limsup_{x \to 0} \gamma\text{-}\text{supp}(L_{x}) \) so

\[ \gamma\text{-}\text{supp}(L_{0}) \subset \limsup_{x \to 0} \gamma\text{-}\text{supp}(L_{x}) \]

\[ \square \]

For \( \gamma\text{-}\text{coisotropic sets} \), we have

**Proposition 9.6.** Let \( C \) be a \( \gamma\text{-}\text{coisotropic set in } T^{*}(X \times Y) \). Then its symplectic reduction \( C_{x} \) is \( \gamma\text{-}\text{coisotropic for } x \text{ outside a nowhere dense subset of } X \).

We will need two lemmata.

**Lemma 9.7.** Let \( \varphi \in \mathcal{D}\mathcal{S}\mathcal{A}_c\mathcal{M}(B^{2n-2}(R)) \) and \( 0 < r' < r \). Then there is \( \Phi \in \mathcal{D}\mathcal{S}\mathcal{A}_c\mathcal{M}(B^{2n-2}(R \times ]-r, r[ \times \mathbb{R})) \) and a constant \( C_{n} \) depending only on \( n \) such that

1. \( \gamma(\Phi) \leq C_{n}\gamma(\varphi) \)
2. \( \Phi(u, q_{n}, p_{n}) = (\varphi(u), q_{n}, \chi(u, q_{n}, p_{n})) \) for \( q_{n} \in [-r', r'[, \quad p_{n} \in \mathbb{R} \)

**Proof.** See [Vit22], prop 8.3.

**Lemma 9.8** (see [Sey15] [HLS16a] [GT]). Let \( (\varphi_{j})_{1 \leq j \leq N} \) be in \( \mathcal{D}\mathcal{S}\mathcal{A}_c(U_{j}) \) where the \( U_{j} \) are symplectically separated domains. Then

\[ \gamma(\varphi_{1} \circ \ldots \circ \varphi_{N}) \leq 2 \max_{1 \leq j \leq N} \{\gamma(\varphi_{j})\} \]

**Proof.** Remember that “symplectically separated” means the distance between \( \psi(U_{i}) \) and \( \psi(U_{j}) \) can be made arbitrarily large by some symplectic isotopy \( \psi \). Applying theorem 44 in [HLS16a] we get

\[ c_{+}(\varphi_{1} \circ \ldots \circ \varphi_{N}) = \max_{1 \leq j \leq N} \{c_{+}(\varphi_{j})\} \]
\[ c_{-}(\varphi_{1} \circ \ldots \circ \varphi_{N}) = \min_{1 \leq j \leq N} \{c_{-}(\varphi_{j})\} \]

so taking the difference, and using that \( c_{-}(\varphi) \leq 0 \leq c_{+}(\varphi) \) we get

\[ \gamma(\varphi_{1} \circ \ldots \circ \varphi_{N}) = \max_{1 \leq j \leq N} \{c_{+}(\varphi_{j})\} - \min_{1 \leq j \leq N} \{c_{-}(\varphi_{j})\} \leq 2 \max_{1 \leq j \leq N} \{\gamma(\varphi_{j})\} \]
Proof of Proposition 9.6. We shall prove that if the reduction $C_x$ is non-$\gamma$-coisotropic for $x$ in a dense set, then $C$ is non-$\gamma$-coisotropic.

Indeed, since the problem is local, and we can work by induction on the codimension of the reduction, it is enough to deal with the case where the reduction is by a hyperplane $H_x$ with $x \in \mathbb{R}$. We can also assume $C \cap H_x$ is connected, since if $A \cup B$ is non-$\gamma$-coisotropic at $x$, then both $A$ and $B$ are non-$\gamma$-coisotropic at $x$ (according to Proposition 7.5 (3)).

The following argument is inspired from the proof of Proposition 2.1 from [Vit00]. Let $(z_0, x_0, p_0) \in C$ with $z_0 \in \mathbb{R}^{2n-2}, x_0, p_0 \in \mathbb{R}$.

1. Assume for $x$ in a dense set near $x_0$ there is a Hamiltonian map $\varphi_x$ in $H_x/H_x^\circ = \mathbb{R}^{2n-2}$ supported in $B^{2n-2}(z_0, \epsilon)$ and such that

$$\varphi_x(C_x) \cap B^{2n-2}(z_0, \eta) = \emptyset$$

Pick an extension of $\varphi_x$, $\Phi_x$ given by corollary 8.10 in [Vit22], such that $\gamma(\Phi_x) \leq C_n \gamma(\varphi_x)$. We claim that for $\alpha_x > 0$ small enough,

$$\Phi_x(C) \cap (B^{2n-2}(z_0, \eta/2) \times [x - \alpha_x, x + \alpha x]) = \emptyset$$

Indeed, if this was not the case, we would have

$$\varphi_x(C_{x'}) \cap B^{2n-2}(z_0, \eta/2) \neq \emptyset$$

for some $x'$ such that $|x - x'| \leq \alpha_x$. But since $x' \mapsto C_{x'}$ is upper semi-continuous for the Hausdorff distance, i.e. $C_{x_0} \supset \lim_{x \to x_0} C_x$, this is impossible. So for each $x \in [-R, R]$ there is a $\Phi_x$ such that $\Phi_x(B^{2n-2}(z_0, \eta/2) \times B^2(\alpha_x)) \cap C \cap H_y = \emptyset$ for $y \in I_x$ where $I_x$ is an open neighbourhood of $x$.

2. By compactness we may cover $[-R, R]$ by a finite number of such intervals, $I_1, ..., I_p$ and we may assume $I_j \cap I_k = \emptyset$ for $|j - k| > 1$ and $I_j \cap I_j + 1$ is as small as we wish. Let us set $\Psi = \Phi_1 \circ ... \circ \Phi_p$. Then obviously if $y \in I_j \setminus I_k$ for all $k \neq j$ then $\Phi(B^{2n}(R) \cap C \cap H_y) = \emptyset$. For $y \in I_j \cap I_j + 1$ set $\psi_j$ to be the flow of the Hamiltonian $\chi_j(y)$ where $\chi_j(y) \geq 2$ for $y \in I_j \cap I_j + 1$ and $\chi_j(y) = 0$ for $y \in I_k$ $k \neq j, j + 1$. It is then easy to check that $\Phi = \Psi \circ \psi_1 \circ ... \circ \psi_k$ satisfies $\Phi(B^{2n}(R)) \cap C = \emptyset$.

3. Since balls are Liouville domains, we may assume by proposition 9.3 in [Vit22] that $\gamma(\Phi) \leq C_n \gamma(\varphi)$, if we had a sequence $\varphi_k$ such that $\gamma(\varphi_k)$ converges to $0$ and such that

$$\varphi_k(C_x) \cap B^{2n-2}(z, \eta) = \emptyset$$

a standard extension (in the sense of Definition 9.5 of [Vit22]). Thus if the reduction of $C$ was not $\gamma$-coisotropic, we would have
a sequence $(\varphi_k)_{k \geq 1}$ with $\gamma(\varphi_k)$ going to zero. But then $\Phi_k$ is supported in $B(z, \varepsilon) \times B^2(\varepsilon)$ and $\Phi_k$ moves $C$ away from $B(z, \eta) \times B^2(\eta)$ and $\gamma(\Phi_k) \leq C_d \gamma(\varphi_k)$ which converges to 0 and $C$ is non-$\gamma$-coisotropic at $z$.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{chi_j}
\caption{Illustration of $\chi_j$}
\end{figure}

Note that $x \mapsto L_x$ is continuous for the $\gamma$-topology. However $L \mapsto \gamma$-supp$(L)$ is not continuous for the Hausdorff topology on closed sets. It may be continuous for the $\gamma$-topology on sets that we shall define later but we have no idea on how to prove such a statement. Let us mention the following application

**Proposition 9.9.** Let $\mathcal{H}^k$ be the $k$-dimensional Hausdorff measure. Let $V$ be a set in $(M^{2n}, d\lambda)$ such that $\mathcal{H}^n(V) = 0$. Then $V$ is nowhere $\gamma$-coisotropic.

*Proof.* We argue by induction: assume this holds in a symplectic space of dimension less than $2n - 2$ and let us prove it in dimension $2n$.

As this is a local result, we may assume we are in a symplectic vector space with symplectic basis $(e_1, f_1, ..., e_n, f_n)$. We set for $x \in \mathbb{R}$, $H_x = xe_1 + \mathbb{R}^{2n-1}$ where $\mathbb{R}^{2n-1} = \langle f_1, e_2, f_2, ..., e_n, f_n \rangle$. Then the reduction $V_x$ of $V$ at $H_x$ is given by

$$(V \cap H_x) / \langle f_1 \rangle = (V / \langle f_1 \rangle) \cap (H_x / \langle f_1 \rangle)$$

We denote in the sequel $K_x$ and $W$ for the hyperplane $H_x / \langle f_1 \rangle$ and the subset $V / \langle f_1 \rangle$ in $\mathbb{R}^{2n-1}$. Moreover we set $W(x) = K_x \cap W$ that is the reduction of $V$ at $x$. We assume $\dim_H(V) < n$ which implies that $\dim_H(W) < n$.

Now according to [Mar54] (see [Mat19], prop. 6.2.3)

$$\dim \{x \in \mathbb{R} \mid \dim(W(x)) \geq n - 1 \} < 1$$

This implies that the set of $x$ such that $\dim_H(W(x)) < n - 1$ is dense. By induction $W(x)$ is nowhere $\gamma$-coisotropic (at least in a neighbourhood of $z_0$)
and therefore there is a dense set of \( x \) such that the \( W(x) \) are nowhere \( \gamma \)-coisotropic. According to Proposition 9.6, this implies that \( V \) is nowhere \( \gamma \)-coisotropic.

\[ \square \]

Remark 9.10. We prove in [AHV24] that for \( V = \gamma \)-supp(\( L \)) then \( V \) carries an \( n \)-dimensional cohomology class, so its dimension is at least \( n \).

10. SINGULARITIES OF HAMILTONIANS IN \( D\text{Ham}(M, \omega) \)

Our goal here is to deal with Hamiltonians with singularities and understand these in the framework of the Humilière completion. This was already explained in [Hum08b], but here we show how \( \gamma \)-coisotropic sets enter the picture. We first define

**Definition 10.1.** Let \( U \) be open and \( W \) be closed in \((M, \omega)\).

1. An element \( \varphi \in \overset{\sim}{D\text{Ham}}(M, \omega) \) equals \( \text{Id} \) on \( U \) if
   \[
   \gamma \text{-supp}(\Gamma(\varphi)) \cap (U \times M) = \gamma \text{-supp}(\Gamma(\varphi)) \cap (M \times U) \subset \Delta_M
   \]
   We denote the set of such \( \varphi \) by \( \overset{\sim}{D\text{Ham}}(M, W, \omega) \). We then set
   \[
   \overset{\sim}{D\text{Ham}}(M, W, \omega) = \bigcup_{W \subset U} \overset{\sim}{D\text{Ham}}(M, U, \omega)
   \]

2. We say that a sequence \((\varphi_k)_{k \geq 1}\) in \( \overset{\sim}{D\text{Ham}}(M, \omega) \) \( \gamma \)-converges to \( \text{Id} \) on \( U \) if there exists a sequence \((\psi_k)_{k \geq 1}\) of elements of \( \overset{\sim}{D\text{Ham}}(M, U, \omega) \) such that \( \gamma(\varphi_k, \psi_k) \rightarrow 0 \). In other words
   \[
   \lim_k \gamma(\varphi_k, \overset{\sim}{D\text{Ham}}(M, U, \omega)) = 0
   \]

3. We define \( \overset{\sim}{D\text{Ham}}_M(U, \omega) \) as the set of equivalence classes of sequences \((\varphi_k)_{k \geq 1}\) such that for any sequence \((l_k)_{k \geq 1}\) with \( l_k \geq k \), the sequence \((\varphi_k \varphi_{l_k}^{-1})_{k \geq 1}\) \( \gamma \)-converges to \( \text{Id} \) on \( W \).

   Two sequences \((\varphi_k)_{k \geq 1}\) and \((\psi_k)_{k \geq 1}\) are equivalent if \((\varphi_k \psi_k^{-1})_{k \geq 1}\) \( \gamma \)-converges to \( \text{Id} \) on \( U \). We define
   \[
   \overset{\sim}{D\text{Ham}}_M(W, \omega) = \bigcap_{W \subset U} \overset{\sim}{D\text{Ham}}_M(U, \omega)
   \]

**Remarks 10.2.**

1. Since in general \( \omega \) is fixed we shall omit it from the notation.
2. In \( U \), we do not assume that the sequence \( \gamma \)-converges !
3. If \( \varphi \) is smooth, saying that \( \varphi \) equals \( \text{Id} \) on \( U \) or \( W \) in the sense of Definition 10.1 is equivalent to its usual meaning since \( \gamma \)-supp(\( \Gamma(\varphi) \)) = \( \Gamma(\varphi) \).
(4) If $\gamma$-supp($\Gamma(\varphi)) \cap W \times \overline{M} \subset \Delta_M$ and $W$ is the closure of its interior, we must have

$$\text{supp}(\Gamma(\varphi)) \cap W \times \overline{M} = \Delta_W$$

Indeed, if $L$ is smooth Lagrangian, no proper subset of $L$ is $\gamma$-coisotropic (see Proposition 8.8).

(5) There is of course an embedding from $\overline{\mathcal{DHam}_c(U)}$ to $\overline{\mathcal{DHam}_M(U)}$ since an element in $\mathcal{DHam}_c(U)$ extends automatically to $\mathcal{DHam}_c(M)$ and this extension equals $\text{Id}$ on $U$ if and only if we started from $\text{Id}_U$.

(6) There is a bi-invariant metric still denoted by $\gamma$ on $\overline{\mathcal{DHam}(M,U)}$

$$\gamma(\varphi, \text{Id}) = \lim_k \gamma(\varphi_k, \mathcal{DHam}(M,W))$$

where $\varphi$ is represented by the sequence $(\varphi_k)_{k \geq 1}$. The existence of the limit follows from the inequality

$$\left| \gamma(\varphi_k, \mathcal{DHam}(M,W,\omega)) - \gamma(\varphi_l, \mathcal{DHam}(M,W)) \right| \leq \gamma(\varphi_k \varphi_l^{-1}, \mathcal{DHam}(M,W))$$

so $\gamma(\varphi_k, \mathcal{DHam}(M,U,\omega))$ is a Cauchy sequence in $\mathbb{R}$. There is then a topology on $\mathcal{DHam}_M(W)$ given by $\varphi_k \gamma \to \varphi$ if and only if this holds in $\mathcal{DHam}_M(K,\omega)$ for some neighbourhood $U$ of $W$. It is easy to check that this limit does not depend on the choice of the sequence and that the embedding of $\mathcal{DHam}_c(U)$ in $\mathcal{DHam}_M(U)$ yields an isometric embedding, hence an isometric embedding $\mathcal{DHam}(U) \to \mathcal{DHam}_M(U)$.

**Question 10.3.** Is the map $\overline{\mathcal{DHam}(U)} \to \overline{\mathcal{DHam}_M(U)}$ bijective?

**Proposition 10.4.** We have the following properties

(1) The set $\overline{\mathcal{DHam}(M,U)}$ is a closed subgroup in $\overline{\mathcal{DHam}(M,\omega)}$

(2) We have an exact sequence

$$1 \to \overline{\mathcal{DHam}(M,U)} \to \overline{\mathcal{DHam}(M)} \to \overline{\mathcal{DHam}_M(U)} \to 1$$

(3) If an element $\varphi \in \overline{\mathcal{DHam}(M,\omega)}$ equals $\text{Id}$ on $U$ and $V$ then it is equal to $\text{Id}$ on $V \cup W$. If a sequence $(\varphi_k)_{k \geq 1}$ $\gamma$-converges to $\text{Id}$ on $U$ and $V$ then it $\gamma$-converges to $\text{Id}$ on $U \cup V$. In other words

$$\overline{\mathcal{DHam}(M,U)} \cap \overline{\mathcal{DHam}(M,V)} = \overline{\mathcal{DHam}(M,U \cup V)}$$

(4) If $\psi \in \overline{\mathcal{DHam}(M)}$ sends $U$ to $U'$, then $\varphi \mapsto \psi \varphi \psi^{-1}$ sends $\overline{\mathcal{DHam}(M,U)}$ to $\overline{\mathcal{DHam}(M,U')}$.
(5) If Conjecture 8.18 holds for $M$ (in particular for $M = T^*T^n$ or $T^{2n}$), an element $\varphi \in \mathcal{DH}_\text{Ham}(M,\omega)$ equals $\text{Id}$ on $M$, then $\varphi = \text{Id}$ in $\mathcal{DH}_\text{Ham}(M,\omega)$.

(6) If Conjecture 8.18 holds for $M$ and if the sequence $(\varphi_k)_{k \geq 1}$ $\gamma$-converges to $\text{Id}$ on $M$, then it $\gamma$-converges to $\text{Id}$ in the usual sense.

Proof. (1) For the first statement, indeed, if $(\varphi_k)_{k \geq 1}$ is a $\gamma$-converging sequence such that $\gamma$-supp($\varphi_k$) $\cap$ $U \times \overline{M} \subset \Delta_M$, then, since according to Proposition 6.17 we have $\gamma$-supp($\varphi$) $\subset$ $\lim_k \gamma$-supp($\varphi_k$) we deduce

$$\gamma$$-supp($\varphi$) $\cap$ $U \times \overline{M} \subset \Delta_M$

(2) Let $(\varphi_k)_{k \geq 0}$ be a $\gamma$-Cauchy sequence in $\mathcal{DH}_\text{Ham}(M)$. Then $\varphi_k \varphi^{-1}_k$ converges to $\text{Id}$ on $M$, hence on $U$ which defines the map

$$\mathcal{DH}_\text{Ham}(M,\omega) \rightarrow \mathcal{DH}_\text{Ham}_M(U)$$

If such a sequence is in the kernel, this means that $(\varphi_k)_{k \geq 0}$ converges to $\text{Id}$ on $W$, i.e. there is a sequence $(\psi_k)_{k \geq 1}$ such that $\psi_k = \text{Id}$ on $U$ and $\gamma(\varphi_k, \psi_k)$ goes to 0 as $k$ goes to $+\infty$. But then

$$\gamma - \lim_k \psi_k = \gamma - \lim_k \varphi_k = \varphi$$

and since

$$\gamma$$-supp($\Gamma(\psi_k)$) $\cap$ $M \times U = \gamma$$-supp($\Gamma(\psi_k)$) $\cap$ $W \times U \subset \Delta_M$$

and $\gamma$-supp($\Gamma(\psi)$) $\subset$ $\lim_k \gamma$-supp($\Gamma(\psi_k)$) we have $\varphi = \text{Id}$ on $U$, i.e. $\varphi \in \mathcal{DH}_\text{Ham}(M,\omega)$.

(3) Indeed if

$$\gamma$$-supp($\Gamma(\varphi)$) $\cap$ $M \times U = \gamma$$-supp($\Gamma(\varphi)$) $\cap$ $U \times M \subset \Delta_M$$

and

$$\gamma$$-supp($\Gamma(\varphi)$) $\cap$ $M \times V = \gamma$$-supp($\Gamma(\varphi)$) $\cap$ $V \times M \subset \Delta_M$$

then

$$\gamma$$-supp($\Gamma(\varphi)$) $\cap$ $M \times (U \cup V) \subset \Delta_M$$

(4) Is obvious from the definition

(5) If $\gamma$-supp($\Gamma(\varphi)$) = $\Delta_M$, we have $\varphi = \text{Id}$.

(6) Same as above

Remark 10.5. We can also work in the general case, i.e. without assuming $M$ satisfies Conjecture 8.18 by replacing $\mathcal{DH}_\text{Ham}_c(M,\omega)$ by the quotient $\mathcal{DH}_\text{Ham}_c(M,\omega)/\mathcal{N}_\Delta(M,\omega)$.

We now set
Definition 10.6. Let $V$ be a locally closed subset in $(M, \omega)$. We shall say that $V$ is $f$-coisotropic at $x \in V$ if for any closed neighbourhood $W$ of $x$ in $M$ there is an element $\varphi$ in $\hat{\mathcal{D}}\mathcal{H}am_M(M \setminus (U \cap V))$ such that $\varphi$ is not the restriction of an element in $\hat{\mathcal{D}}\mathcal{H}am(M, \omega)$.

We say that $V$ is nowhere $f$-coisotropic, if for all $x$ in $V$, $V$ is not $f$-coisotropic at $x$.

Even though in a different formulation, Humilière proved in [Hum08b] that if $\dim(V) < n$ then $V$ is nowhere $f$-coisotropic.

Note that the definition of $f$-coisotropic is reminiscent of that of a domain of holomorphy: this is a domain such that there exist holomorphic functions on $\Omega$ that cannot be extended to a bigger set. Such sets also have a local geometric characterization as being Levi convex. Here the objects that cannot be extended are element of $\hat{\mathcal{D}}\mathcal{H}am(V, \omega)$ and the extension is to $\hat{\mathcal{D}}\mathcal{H}am(M, \omega)$.

Proposition 10.7. If $V$ is nowhere $\gamma$-coisotropic, any element in $\hat{\mathcal{D}}\mathcal{H}am_M(M \setminus V)$ defines a unique element in $\hat{\mathcal{D}}\mathcal{H}am(M, \omega) / \mathcal{N}_\Delta$. In particular any $H \in C^0(M \setminus V)$ defines a unique element in $\hat{\mathcal{D}}\mathcal{H}am(M, \omega) / \mathcal{N}_\Delta$.

Uniqueness follows from the following Lemma.

Lemma 10.8. Let $V$ be nowhere $\gamma$-coisotropic. If $\varphi \in \hat{\mathcal{D}}\mathcal{H}am(M, \omega)$ and $\varphi = \text{Id}$ on $M \setminus V$ then $\varphi = \text{Id}$ on $M$ (i.e. $\varphi \in \mathcal{N}_\Delta$). Similarly if $(\varphi_k)_{k \geq 1}$ $\gamma$-converges to Id on $M \setminus V$, then it converges to Id on $M$ (i.e. $\lim_k \gamma(\varphi_k, \mathcal{N}_\Delta) = 0$).

Proof. Let $x \in V$ and $\psi_k$ be a sequence such that $\gamma - \lim_k \psi_k = \text{Id}$ and $\psi_k(V) \subset M \setminus B(x, \eta)$. We denote by $V_\varepsilon$ an $\varepsilon$-neighbourhood of $V$. Then there exists $\varepsilon_k$ such that $\psi_k(M \setminus V_{\varepsilon_k}) \supset B(x, \eta/2)$ so that $\varphi \leftrightarrow \psi_k \varphi \psi_k^{-1}$ sends $\hat{\mathcal{D}}\mathcal{H}am(M, M \setminus V_{\varepsilon_k})$ to $\hat{\mathcal{D}}\mathcal{H}am(M, B(x, \eta/2))$. But since $\gamma - \lim_k \psi_k = \text{Id}$ and $\varphi \in \hat{\mathcal{D}}\mathcal{H}am(M, M \setminus V_{\varepsilon_k})$ for all $k$, then $\varphi \in \hat{\mathcal{D}}\mathcal{H}am(M, B(x, \eta/2))$, hence $\varphi \in \hat{\mathcal{D}}\mathcal{H}am(M, B(x, \eta/2) \cup (M \setminus V))$. As a result $\varphi$ equals Id on $M \setminus V \cup B(x, \eta/2)$. By taking a covering of $V$ by open balls and iterating this argument, we get that $\varphi = \text{Id}$ on $M$ that is $\varphi \in \hat{\mathcal{D}}\mathcal{H}am(M, M) = \mathcal{N}_\Delta$.

Proof of Proposition 10.7. Indeed, $(\varphi_k)_{k \geq 1}$ is Cauchy in $\hat{\mathcal{D}}\mathcal{H}am_M(M \setminus V)$ if and only if for any subsequence $(l_k)_{k \geq 1}$ going to infinity, the sequence $\varphi_k \varphi_l^{-1}$ converges to Id on $M \setminus V$. But then we just proved that the sequence converges to Id in $\hat{\mathcal{D}}\mathcal{H}am(M, \omega)$. As a result $(\varphi_k)_{k \geq 1}$ is Cauchy in $\hat{\mathcal{D}}\mathcal{H}am(M, \omega) / \mathcal{N}_\Delta$ hence converges. Finally we want to prove that $H \in C^0(M \setminus V)$ defines an element in $\hat{\mathcal{D}}\mathcal{H}am_M(M \setminus V)$. We can write $H$ as the $C^0$-limit on compact sets of a sequence $H_k$ of Hamiltonians in $C^0_c(M \setminus V)$.
with $H_k = H_l$ on an exhausting sequence of open subsets. Clearly for any given compact set $W \subset M \setminus V$ and $k, l$ large enough, their flow $\varphi_k \varphi_l^{-1}$ equals Id on $W$, so that the sequence is Cauchy in $\overline{D\mathfrak{Ham}}_M(W)$. By definition it yields an element in $\overline{D\mathfrak{Ham}}_M(M \setminus V)$. □

**Corollary 10.9.** If $V$ is $f$-coisotropic at $x$ then it is $\gamma$-coisotropic at $x$.

**Proof.** The statement is an obvious consequence of the Proposition. □

From Proposition 9.9 we infer the following reinforcement of Humilière's result

**Corollary 10.10.** If $\dim_H(V) < n$ then $V$ is nowhere $f$-coisotropic.

**Proposition 10.11.** We have the following properties

1. Being $f$-coisotropic is invariant by $\mathcal{H}_\gamma(M, \omega)$, hence by Homeo$(M, \omega)$.
2. Being $f$-coisotropic is a **local property** in $M$. It only depends on a neighbourhood of $V$ in $(M, \omega)$.
3. Being $f$-coisotropic is **locally hereditary** in the following sense: if through every point $x \in V$ there is an $f$-coisotropic submanifold $V_x \subset V$, then $V$ is $f$-coisotropic. In particular if through any point of $V$ there is an element of $\mathfrak{L}(M, \omega)$ then $V$ is coisotropic. If any point in $V$ has a neighbourhood contained in an element of $S_2(M, \omega)$ then $V$ is not $f$-coisotropic.

**Proof.** The first two statements are obvious from the definition. For the third one, if $x \in C_x \subset V$ is $f$-coisotropic, then there is an element in $\varphi$ in $\overline{D\mathfrak{Ham}}(M \setminus (C_x \cap U))$ which does not extend to $\overline{D\mathfrak{Ham}}(M)$. But since $\varphi$ belongs to $\overline{D\mathfrak{Ham}}(M \setminus (V \cap U))$, this implies that $V$ is $f$-coisotropic at $x$. □

**APPENDIX A. THE SPACE $\mathfrak{L}(T^* N)$ IS NOT A POLISH SPACE**

The question studied in this section is due to Michele Stecconi. I thank him for the suggestion and for his help with the proof. We shall prove

**Proposition A.1.** The space $(\mathfrak{L}(T^* N), \gamma)$ is not a Polish space.

Remember that a Baire space is a space where Baire's theorem holds: a countable intersection of open dense sets is dense. A topological space is a Polish space if its topology can be defined by a complete metric. Equivalently the space is a countable intersection of open dense sets in its completion. So if a space is not a Polish space, its completion really adds a lot of points.
Proof. Let $\text{gr} \circ d : C^0(N, \mathbb{R}) \to \mathcal{L}(T^*N)$ be the extension of the isometric embedding $\text{gr} \circ d : (C^\infty(N, \mathbb{R}), d_{C^\infty}) \to (\mathcal{L}(T^*N), \gamma)$ given by $f \mapsto \text{graph}(df)$. Let

$$\mathcal{G}(T^*N) = \{\text{graph}(df) \mid f \in C^0(N, \mathbb{R}), \text{graph}(df) \in \mathcal{L}(T^*N)\}$$

Thus $\mathcal{G}(T^*N)$ is the set of continuous functions such that $\text{graph}(df)$ is a smooth Lagrangian.\footnote{This is not the set of smooth functions! For example the submanifold $x = p^3$ in $T^*\mathbb{R}$ is smooth and is the graph of the differential of $f(x) = \frac{1}{3}x^3$ that is not smooth.} Note that the image of $\text{gr} \circ d$ is closed, since it is an isometry (for the natural $C^0$ and $\gamma$ norms) and both spaces are complete. Then $\mathcal{G}(T^*N)$ is closed in $\mathcal{L}(T^*N)$ since it is the intersection of the image of $\text{gr} \circ d$ - which is closed in $\mathcal{L}(T^*N)$, as the isometric image of a complete space- and $\mathcal{L}(T^*N)$. As a result, if $(\mathcal{L}(T^*N), \gamma)$ is Polish, so is $\mathcal{G}(T^*N)$, since a closed subset of a Polish space is Polish (see [Kec95], thm 3.11, p.17). Now let us consider the open sets

$$U_n(x_0) = \left\{L \in \mathcal{G}(T^*N) \mid L = \text{graph}(df), \text{and } \exists t \in ]0, \frac{1}{n}] \inf_{x \in S(x_0, t)} f(x) > f(x_0) \right\}$$

where $S(x_0, t)$ is the sphere of radius $t$ for some Riemannian metric on $N$. We claim that $U_n(x_0)$ is dense in $\mathcal{G}(T^*N)$. Indeed, we may modify $f$ by adding a $C^0$-small smooth function $g$ so that $f + g$ is in $U_n(x_0)$, since $\gamma(\text{graph}(df), \text{graph}(df + dg)) = \|g\|_{C^0}$. Note that $U_n(x_0)$ is open, since the set of functions such that $\inf_{x \in S(x_0, t)} f(x) > f(x_0)$ is open for the $C^0$-topology, hence $U_n(x_0)$ is open for the $\gamma$-topology (since it coincides with the $C^0$-topology on graphs).

Then if $\text{gr}(df) \in U_n(x_0)$ and $f$ is smooth in $B(x_0, \frac{1}{n})$ there must be a local minimum $y$ of $f$ in $B(x_0, \frac{1}{n})$ so that $df(y) = 0$. Now let $(z_k)_{k \geq 1}$ be a dense sequence of points in $N$. We claim that

$$\bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} U_n(z_k)$$

is the zero section, since if $\text{gr}(df)$ belongs to this intersection and is smooth on the open set $W \subset N$ of full measure, then $df$ must vanish on some point in $B(z_k, \frac{1}{n})$ whenever $B(z_k, \frac{1}{n}) \subset W$. But this implies that $df$ is identically zero on $W$, so $f$ is a constant. Our last argument uses that if $\text{gr}(df)$ is in $\mathcal{L}(T^*N)$ we have that $f$ is smooth on an open set of full measure. Indeed, we proved in [OV94] (see also the Appendix 2 in [Vit18]) that the selector $c(1_x, L)$ is smooth on an open set of full measure, but since obviously $c(1_x, \text{gr}(df)) = f(x)$, this implies that $f$ is smooth on an open set of full measure. As a result, the intersection of the open and dense set

\footnote{Remember that $\mathcal{L}(T^*N)$ is a set of smooth manifolds but that $\mathcal{G}(T^*N) \neq \{\text{graph}(df) \mid f \in C^\infty(N, \mathbb{R})\}$}
\[ U_n(z_k) \] is the singleton \{0_N\}. Thus \( \mathcal{S}(T^*N) \) is not even a Baire space (i.e. a space where a countable intersection of open dense sets is dense) and Polish spaces are obviously Baire.

Note that there is no obvious explicit description of \( \mathcal{S}(T^*N) \) in terms of the singularities of \( f \). Requiring \( f \) to be smooth everywhere is too strong while only requiring \( C^1 \) is too weak. One possibility would be that \( f \) must be \( C^1 \) everywhere and smooth on an open set of full measure but even though the condition is necessary, as we saw above, we have no idea as to whether it would be sufficient.

Remark A.2. Since on \( \mathcal{S}(T^*N) \) the metric \( \gamma \) coincides with the Hofer metric, we may conclude that \( L(T^*N) \) endowed with the Hofer metric is not a Polish space either.

Appendix B. The set of pseudo-graphs: an example of a closed set in \( \hat{\mathcal{L}}(T^*N) \).

Let us now describe a closed set in \( \hat{\mathcal{L}}(T^*N) \). Remember that \( FH^*(L_1, L_2; t) \), the Floer homology of \( L_1, L_2 \) with action filtration below \( t \) (i.e. generated by the intersection points in \( L_1 \cap L_2 \) such that \( f_{L_1}(z) - f_{L_2}(z) < t \)) yields a persistence module, and as such we can associate to it a barcode.

We now set

**Definition B.1.** A Lagrangian \( L \) in \( \hat{\mathcal{L}}(T^*N) \) is a pseudo-graph if and only if for all \( x \), the barcode of \( FH^*(L, V_x) \) is reduced to a single bar \([c(1_x, L), +\infty[\)

**Proposition B.2.** The set of pseudo-graphs is a closed subset of \( \hat{\mathcal{L}}(T^*N) \) and contains \( \text{gr} \circ \text{d}(C^0(N, \mathbb{R})) \) the set of graphs of differentials of continuous functions.

**Proof.** First of all if \( L_n \xrightarrow{\gamma} L \), we claim that for all \( x \in N \), denoting by \( V_x \) the vertical fibre \( T^*_x N \), the persistence modules \( t \mapsto FH^*(L_n, V_x; t) \) converge to the persistence module \( t \mapsto FH^*(L, V_x; t) \), i.e. the barcode of the persistence module \( t \mapsto FH^*(L_n, V_x, t) \) converges to the barcode of the persistence module \( t \mapsto FH^*(L, V_x, t) \) for the bottleneck distance. This is an immediate consequence of the Kislev-Shelukhin inequality (see [Ki-Sh22]). Here we use the formulation in [Vit22], Proposition A.3, which applies provided \( t \mapsto FH^*(L_n, V_x, t) \) satisfies properties\( (1)-(3) \), with conditions \( (1)-(4) \) stated after Proposition A.1 in [Vit22]. Now only the existence of the PSS units (i.e. Property \( 3 \)) is non-trivial, but since \( FH^*(L, V_x) = FH^*(0_N, V_x) = \mathbb{K} \cdot u \), this and the conditions \( (1)-(4) \) are easily checked.

Now if \( L_n \) is a pseudo-graph, the barcode of \( t \mapsto FH^*(L_n, V_x; t) \) is made of a single bar, \([c(1_x, L_n), +\infty[\), so its limit is necessarily a barcode made of a single bar, and since \( \lim_n c(1_x, L_n) = c(1_x, L) \) we have that the barcode
Proposition B.3. We have equality: the set of pseudographs coincides with the set of graph(s) of graphs such that $f \in C^0(N, \mathbb{R})$. Now for $f \in C^0(N, \mathbb{R})$, we can find a sequence of smooth functions, $f_n$ such that $C^0 - \lim_n f_n = f$. But this implies that $\gamma - \lim_n \text{graph}(df_n) = \text{gr}(df)$, and since the set of pseudo-graphs is $\gamma$-closed, this implies that graph$(df)$ is also a pseudo-graph. \hfill \Box

**Proposition B.3.** We have equality: the set of pseudographs coincides with the set of graph$(df)$ for $f \in C^0(N, \mathbb{R})$.

**Proof.** Indeed, let $f$ be the selector for the pseudograph $L$. Then $f$ is continuous, and $L - \text{graph}(df)$ is a pseudograph such that $c(1_x, L) = 0$. The proposition is then an obvious consequence of the following Lemma.

**Lemma B.4.** Let $L \in \mathcal{F}(T^* N)$ be such that for each $x$ we have $FH^*(L, V_x; a, b) = 0$ for given $a < b$. Then we have $FH^*(L, 0_N; a, b) = 0$. In particular if this holds for all $a < b$ such that $0 \not\in [a, b]$, then $L = 0_N$.

**Proof.** Let $\mathcal{F}^* \in D^b(N \times \mathbb{R})$ associated to $L$. According to \cite{Gui12}, proposition 9.9, there is an element $\mathcal{F}_L$ in $D_{1c}(N \times \mathbb{R})$ such that it extends the Lagrangian quantization map $Q : \mathcal{L}(T^* N) \longrightarrow D^b(N \times \mathbb{R})$ from \cite{Gui12, Vit19}. In particular

$$FH^*(L_1, L_2, a, b) = H^*(N \times [a, b], R\text{Hom}^*(\mathcal{F}_1^*, \mathcal{F}_2^*))$$

Then $H^*(N \times [a, b], \mathcal{F}^*) = FH^*(L, 0_N; a, b)$ since this holds for $\mathcal{F}^* = \mathcal{F}_L^*$ in the smooth case. There is a spectral sequence with $E_2^{p,q} = H^p(N, \mathcal{H}^q([a, b], \mathcal{F}_x^*))$, but $\mathcal{H}^q([a, b], \mathcal{F}_x^*) = H^q(L, V_x; a, b)$ which is zero by assumption, so $E_2^{p,q} = 0$ and

$$FH^*(L, 0_N; a, b) = H^*(N \times [a, b], \mathcal{F}^*) = 0$$

Finally if $FH^*(L, 0_N; a, b) = 0$ whenever $0 \not\in [a, b]$, then we must have $c_1(L) = c_\infty(L) = 0$ and $L = 0_N$. \hfill \Box

**Remark B.5.** One can prove directly that $\text{gr}(\circ \bar{d}(C^0(N, \mathbb{R})))$ is closed in $\mathcal{L}(T^* N)$. Indeed, $\text{gr}(\circ \bar{d})$ is an isometric embedding between $(C^0(N, \mathbb{R}), C^0)$ to $(\mathcal{L}(T^* N), \gamma)$. But an isometric embedding between complete spaces must have closed image. Notice that the non-obvious statement in the Proposition is that all pseudographs are actually graphs.

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