NON-ARCHIMEDEAN SENDOV'S CONJECTURE

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Abstract. We prove non-archimedean analogue of Sendov’s conjecture. We also provide complete list of polynomials over an algebraically closed non-archimedean field $K$ that satisfy the optimal bound in the Sendov’s conjecture.

1. INTRODUCTION

Sendov’s conjecture [2] can be stated as follows:

**Conjecture 1** (Sendov). Let $f(z) = \prod_{k=1}^{n}(z - z_k)$ be a monic polynomial over $\mathbb{C}$ where all zeros are in the closed unit disk centered at zero, i.e. $|z_k| \leq 1$ for all $1 \leq k \leq n$. Then, for each $k$, we can find a zero $w$ of $f'(z)$ such that $|w - z_k| \leq 1$.

The conjecture is known to be true for polynomials of degree $\leq 8$ [11] and polynomials of sufficiently large degree [4], but still widely open in general. In this paper, we study the analogue of the Sendov’s conjecture over non-archimedean fields [4]. By enlarging the radius of the discs appear in the conjecture (centered at each zero $z_k$), we show that the Sendov’s conjecture holds for an algebraically closed non-archimedean field $K$. The theorem reduces to the direct analogue of the Sendov’s conjecture for polynomials of degree $n$ with norm $|n| = 1$. At last, we give the necessary and sufficient conditions for the polynomials to meet the tight bound of the theorem.

2. THE NON-ARCHIMEDEAN SENDOV’S CONJECTURE.

We first prove the following analogue of Sendov’s conjecture, which has a weaker bound on the radius than the original conjecture over complex numbers. From now, let $(K, \cdot |)$ be an algebraically closed non-archimedean field, $\mathcal{O}_K = \{x \in K : |x| \leq 1\}$ be its valuation ring, and $\mathfrak{m}_K = \{x \in K : |x| < 1\}$ be its maximal ideal.

**Theorem 1.** Let let $r = r_n = |n|^{-1/(n-1)}$. Then the following variation of Sendov’s conjecture holds for all $f(z) \in K[z]$: assume that all zeros $z_i$ of $f(z)$ are in the closed unit disk centered at zero. Then for each $z_i$, the closed disk $\overline{D}(z_i, r_n) := \{w \in K : |w - z_i| \leq r_n\}$ contains at least one zero of $f'(z)$.

**Proof.** Let $f(z) = \prod_{i=1}^{n-1}(z - z_i) \in K[z]$ with $|z_i| \leq 1$. Let $\{w_1, \ldots, w_{n-1}\}$ be the zeros of $f'(z)$, so that $f'(z) = n \prod_{i=1}^{n-1}(z - w_i)$. Assume that Sendov’s conjecture is false for $f(z)$. Then, without loss of generality, we have $|z_1 - w_i| > r_n$ for all $1 \leq i \leq n - 1$. This gives

$$f'(z) = n \prod_{i=1}^{n-1}(z - w_i) \Rightarrow |f'(z_1)| = |n| \prod_{i=1}^{n-1}|z_1 - w_i| = |n|r_n^{n-1} > 1.$$

However, since $z_1$ is a zero of $f(z)$, we also have

$$f'(z) = \sum_{i=1}^{n} \prod_{j \neq i}(z - z_j) \Rightarrow |f'(z_1)| = \prod_{i=2}^{n}|z_1 - z_i| \leq \prod_{i=2}^{n}\max\{|z_1|, |z_i|\} \leq 1,$$

which gives a contradiction. $\square$

**Remark.** Note that since $r_n \geq 1$ and $|z_1| = 1$, $\overline{D}(z_1, r_n) = \overline{D}(0, r_n)$ for every $i$. Hence, every closed disks occur at the end of Theorem 1 are in fact all same. Therefore, the Theorem 1 is equivalent to $|w| \leq r_n$ for some zero $w$ of $f'$.

For example, when $K = \overline{\mathbb{Q}_p}$ with $p \nmid n$, we have $|n| = |n|_p = 1$ and so $r_n = 1$. In this case, the direct analogue of Sendov’s conjecture is actually true. However, when $n$ is a multiple of $p$, there is a counterexample for the original version.
Proposition 1. Let \( p \) be an integer prime and \( n \) be a multiple of \( p \). Then \( f_n(z) = z^n - z \) is a counterexample for the original analogue of Sendov’s conjecture over \( K = \overline{\mathbb{Q}}_p \). More precisely, for all zeros \( z_i \) of \( f_n \), the closed unit disc \( \mathbb{D}(z_i, 1) = \{ w \in K : |w - z_i| \leq 1 \} \) does not contain any zero of \( f'_n(z) \).

Proof. Let \( \zeta_{n-1} \) be \( (n-1) \)-th root of unity in \( \overline{\mathbb{Q}}_p \). Then the roots of \( f_n(z) \) are

\[
z_j = \zeta_{n-1}^j \quad (1 \leq j \leq n-1), \quad z_n = 0,
\]

where the roots of \( f'_n(z) = nz^{n-1} - 1 \) are

\[
w_j = n^{-1/(n-1)} \zeta_{n-1}^j \quad (1 \leq j \leq n-1).
\]

Then we have \( |z_i - w_j|_p > 1 \) for all \( 1 \leq i \leq n \) and \( 1 \leq j \leq n-1 \). In fact, we have

\[
|z_i - w_j|_p = |\zeta_{n-1}^i - n^{-1/(n-1)} \zeta_{n-1}^j|_p = |n|_p^{-1/(n-1)} \cdot |n|_p^{1/(n-1)} |\zeta_{n-1}^{i-j} - 1|_p,
\]

for \( 1 \leq i \leq n-1 \). Since \( n \) is a multiple of \( p \), we have \( |n|_p < 1 \) and so \( |n|_p^{1/(n-1)} |\zeta_{n-1}^{i-j} - 1|_p = |n|_p^{-1/(n-1)} < 1 \), which gives

\[
|z_i - w_j|_p = |n|_p^{-1/(n-1)} > 1.
\]

Note that \( |\alpha + \beta|_p = \max\{|\alpha|_p, |\beta|_p\} \) whenever \( |\alpha|_p \neq |\beta|_p \). Similarly, for \( z_n = 0 \) we have

\[
|z_n - w_j|_p = |w_j|_p = |n|_p^{-1/(n-1)} > 1.
\]

\( \square \)

3. Optimality criteria

The counterexample given in the Proposition 1 actually satisfies the optimal bound of the Theorem 1. In other words, the zero \( w_j \) of \( f'(z) \) contained in the closed disk \( \mathbb{D}(z_i, r_n) \) actually lies in the boundary of the disk, i.e. \( |z_i - w_j| = r_n \). In this section, we give simple criteria to determine whether a given polynomial satisfies the optimal bound of the Theorem 1. Let \( f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 \in K[z] \) be a polynomial where all the zeros of \( f \) lies in a closed unit disk centered at origin. Note that this is equivalent to \( a_i \in \mathcal{O}_K \) for every \( 0 \leq i \leq n-1 \) since \( \mathcal{O}_K \) is integrally closed. Define \( I(f) \) as

\[
I(f) = \max_{z : f(z) = 0} \min_{w : f'(w) = 0} |z - w|.
\]

By the Theorem 1, \( I(f) \leq r_n \). We will call \( f(z) \) satisfies the optimal bound of the Theorem 1 if \( I(f) = r_n \). The following theorem gives necessary and sufficient conditions that a given polynomial over \( K \) satisfy the optimal bound of the Theorem 1 when degree \( n \) has norm strictly smaller than 1, i.e. \( v(n) > 0 \).

Theorem 2. Assume that \( n = \deg f \) satisfies \( |n| < 1 \). Then \( f(z) \) satisfies the optimal bound of the Theorem 1 if and only if

\[
(1) \quad v(a_j) \geq \max \left\{ 0, \frac{j - 1}{n - 1} v(n) - v(j) \right\} \quad (2 \leq j \leq n), \quad v(a_1) = 0, \quad v(a_0) \geq 0,
\]

where \( v = v_K : K \to \mathbb{R} \cup \{\infty\} \) is a valuation corresponds to \( |\cdot| \).

We give two different proofs of the valuation. First proof uses the following property about Newton polygon.

Proposition 2. Let \( K \) be a Henselian valued field with valuation \( v = v_K \), and let \( p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \in K[z] \). Define \( N(f) \), the Newton polygon of \( f \), as a lower convex hull of the set of points \( \{(i, v(a_i)) : 0 \leq i \leq n\} \). Let \( \mu_1, \ldots, \mu_r \) be the slopes of the line segment of \( N(f) \) arranged in increasing order, and let \( \lambda_1, \ldots, \lambda_r \) be the corresponding lengths of the line segments projected onto the \( x \)-axis. Then for all \( 1 \leq k \leq r, f(z) \) has exactly \( \lambda_k \) roots of valuation \( -\mu_k \).

See \( \square \) for the proof of Proposition 2.

First proof of Theorem 2 First, we show that the condition is necessary. Let’s write \( f(z) \) as \( f(z) = \prod_{i=1}^n (z - z_i) \) and \( f'(z) = n \prod_{j=1}^{n-1} (z - w_j) \). Without loss of generality, assume that \( I(f) = |n|^{-1/(n-1)} = \)
$|z_1 - w_1|, |z_1 - w_j| \geq |n|^{-1/(n-1)}$ and $\min_{1 \leq j \leq n-1} |z_i - w_j| \leq |n|^{-1/(n-1)}$ for all $i \neq 1$. We have

$$1 \leq |n| \prod_{1 \leq j \leq n-1} |z_1 - w_j| = |f'(z_1)| = \left| \prod_{2 \leq i \leq n} (z_1 - z_i) \right| \leq \prod_{2 \leq i \leq n} \max \{ |z_1|, |z_i| \} \leq 1,$$

and all the inequalities should be equality, so $|z_1 - w_j| = |n|^{-1/(n-1)}$ for $1 \leq j \leq n - 1$. Since $|z_1| \leq 1$ and $|n|^{-1/(n-1)} > 1$, we get $|w_j| = |n|^{-1/(n-1)}$, i.e. $v(w_j) = -\frac{1}{n-1} v(n)$. From $w_1 \cdots w_{n-1} = (-1)^{n-1} a_n$, 

$$-v(n) = \sum_{1 \leq i \leq n-1} v(w_j) = v(a_1) - v(n)$$

and so $v(a_1) = 0$. Now, consider the Newton polygon $N(f')$ of $f'(z) = n z^{n-1} + (n-1) a_n - z^{n-2} + \cdots + 2a_2 z + a_1$, which is a convex hull of $(j - 1, v(j) + v(a_j))$ for $1 \leq j \leq n$. The slope of the segment that connects $(0, v(a_1)) = (0, 0)$ and $(n - 1, v(n))$ is $\frac{1}{n-1}$, which equals to $-v(a_j)$ for all $j$. So the Newton polygon itself become the segment $\{ t \frac{v(n)}{n-1} : 0 \leq t \leq n - 1 \}$, and all other points should be located above this line, which is equivalent to

$$\frac{v(j) + v(a_j)}{j - 1} \geq \frac{v(n)}{n - 1} \Rightarrow v(a_j) \geq \frac{j - 1}{n - 1} v(n) - v(j).$$

For the Newton polygon of $f(z)$, since all zeros lie in the unit disk, $v(z_i) \geq 0$ for all $1 \leq i \leq n$. This means that the slopes of the line segments of the Newton polygon $N(f)$ is not positive. Since $v(a_n) = 0$, all the points $(i, v(a_i))$ should lie above x-axis, i.e. $v(a_i) \geq 0$ for all $i$. Hence we get (1).

Conversely, suppose that the coefficients of $f(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \in [z]$ satisfy (1). Then the Newton polygon $N(f')$ is the line segment $\{ t \frac{v(n)}{n-1} : 0 \leq t \leq n - 1 \}$ and all the zeros $w_1, \ldots, w_{n-1}$ of $f'(z)$ have valuation $-\frac{v(n)}{n-1}$, i.e. $|w_j| = |n|^{-1/(n-1)}$. From $|z_1| \leq 1$, we have $|z_i - w_j| = |n|^{-1/(n-1)}$ for $1 \leq i \leq n, 1 \leq j \leq n - 1$ and so $I(f) = |n|^{-1/(n-1)}$.

**Second proof of Theorem**. It is enough to show that $f(z)$ satisfies the optimal bound of the Theorem 1 if and only if

$$|a_j| \leq \min \left\{ 1, \frac{n^{1-\frac{1}{n}}}{j} \right\} \quad (2 \leq j \leq n), \quad |a_1| = 1, \quad |a_0| \leq 1.$$ 

As we noted above, $|z_i| \leq 1$ for all $1 \leq i \leq n$ if and only if $|a_i| \leq 1$. We will prove the equivalence under this assumption.

For any $1 \leq i \leq n$ and $1 \leq j \leq n - 1$, $|z_i - w_j| \leq \max\{|z_i|, |w_j|\} \leq \max\{1, |w_j|\}$, so if $I(f) = r_n > 1$, then $r_n \leq |w_j|$ for any $1 \leq j \leq n - 1$. Since

$$|n|^{-1} = r_n^{n-1} \leq \prod_{j=1}^{n-1} |w_j| = \left| (-1)^{n-1} a_1 \right| / n \leq |n|^{-1},$$

all the inequalities should be equality. Hence $|w_j| = r_n$ for every $1 \leq j \leq n - 1$. Also, if $|w_j| = r_n$ for all $1 \leq j \leq n - 1$, then trivially $I(f) = r_n$. Hence $f(z)$ satisfies the optimal bound of Theorem 1 if and only if $|w_j| = r_n$ for all $1 \leq j \leq n - 1$. Now define $h(z) \in [z]$ as

$$h(z) = f'(n^{-\frac{1}{n-1}} z) = z^{n-1} + \sum_{j=1}^{n-1} \frac{j a_j}{n} z^{j-1}.$$ 

Then zeros of $h$ are $n^{\frac{1}{n-1}} w_j$, so $|w_j| = r_n$ for all $1 \leq j \leq n - 1$ if and only if every zero of $h$ has valuation 1. Since $h$ is a monic polynomial, it is easy to see that this occurs exactly when all the coefficients of $h(z)$ are in $O_K$ and the constant term is in $O_K^*$, which is equivalent to

$$|a_j| \leq \frac{n^{\frac{1}{n-1}}}{j} \quad (2 \leq j \leq n), \quad |a_1| = 1.$$
Example 1. We already saw that the polynomial $f_n(z) = z^n - z$ satisfies the optimal bound, and the coefficients of $f_n$ satisfies (1) (note that $v(0) = \infty$).

Example 2. Let $p$ be an odd prime and $f(z) = z^{2p} - p^{-1/2}z^p$. Then $f(z)$ fails to satisfy (1) since $v(a_p) = -\frac{1}{2} < -\frac{p}{2p-1}$. Indeed, the zeros of $f(z)$ and $f'(z) = 2pz^{2p-1} - p^{1/2}z^{p-1}$ are

\[ z_1 = \cdots = z_p = 0, z_{p+1} = p^{-1/2}p, \ldots, z_{2p} = p^{-1/2}p^{p-1} \]

\[ w_1 = \cdots = w_{p-1} = 0, w_p = (4p)^{-1/2}p, \ldots, w_{2p-1} = (4p)^{-1/2}p^{p-1} \]

and one can check that $I(f) = p^{1/2p} < r_{2p} = p^{1/(2p-1)}$.

When $|n| = 1$, we express optimality condition as a non-divisibility condition of a reduced polynomial over a residue field of $K$. For any elements $g \in \mathcal{O}_K[z]$ (resp. $x \in \mathcal{O}_K$), let $\overline{g}$ (resp. $\overline{x}$) be the corresponding element (mod $\mathfrak{m}_K$ reduction) in $\mathcal{O}_K/\mathfrak{m}_K[z]$ (resp. $\mathcal{O}_K/\mathfrak{m}_K$).

Theorem 3. Assume $n = \deg f$ satisfies $|n| = 1$. Then $f(z)$ satisfies the optimal bound of the Theorem 1, i.e. $I(f) = |n|^{-1/(n-1)} = 1$ if and only if $\overline{f}$ does not divides $\overline{f}^n$.

Proof. Since $f' \in \mathcal{O}_K[z]$ and the leading coefficient of $f'$ is an unit of $\mathcal{O}_K$, every zeros of $f'$ are in $\mathcal{O}_K$. For any polynomial $g$, let $V(g)$ be the set of zeros of $g$. By definition, $f$ does not satisfy the optimal bound of Theorem 1 if and only if for every $z \in V(f)$ there exists $w \in Z(f')$ such that $|z - w| < 1$. Since $|z - w| < 1$ if and only if $\overline{z} = \overline{w}$, this is equivalent to for any $z \in V(f)$ there exist $w \in V(f')$ such that $\overline{z} = \overline{w}$. Since $\{z : z \in V(g)\} = V(\overline{g})$ for any polynomial $g \in \mathcal{O}_K[z]$ with unit leading coefficient, this is equivalent to $V(\overline{f}) \subseteq V(\overline{f}^n)$. By considering the linear factorization of $\overline{f}$ and $\overline{f}^n$, this is equivalent to $\overline{f} \nmid \overline{f}^n$.

By the long division of polynomials, the condition $\overline{f} \nmid \overline{f}^n$ can be expressed as a set of inequalities of valuations of certain polynomials whose variables are $a_1, \ldots, a_n$. Hence, in this sense, Theorem 3 has a similar spirit as Theorem 2 (although the criterion is a little more complicated).

Example 3. Consider $f(z) = z^2 + z + \frac{1-p^2}{4} = (z + \frac{1+p}{2}) (z - \frac{1-p}{2}) \in K[z]$ with $K = \mathbb{Q}_p$. Then $f'(z) = 2z + 1$ and $f'(z)^2 = 4z^2 + 4z + 1 \equiv 4f(z) \pmod{\mathfrak{m}_K}$ so $f'(z)^2$ is divisible by $f(z)$ in $\mathcal{O}_K/\mathfrak{m}_K[z]$. Indeed, the distances between zeros $z = (-1 \pm p)/2$ of $f(z)$ and the zero $w = -1/2$ of $f'(z)$ is $|p| = p^{-1}$, and $I(f) = p^{-1} < 1$.

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