Adaptive synchronization of fractional-order complex-valued neural networks with time-varying delays

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ABSTRACT In this paper, the adaptive synchronization of fractional-order complex-valued neural networks with time-varying delays (FOCVNNNTDs) is investigated. First, two novel fractional-order differential inequalities with time delays are established, which can be seen as an extension of Halanay inequality. Besides, complete synchronization and quasi-projective synchronization of FOCVNNNTDs are investigated based on the two novel inequalities using a novel adaptive controller. In addition, instead of separating the complex-valued neural networks into two real-valued networks, a non-decomposition method is adopted to study the adaptive synchronization of FOCVNNNTDs, which avoids the difficulty and complexity of theoretical analysis. Finally, some numerical simulation examples are provided to demonstrate the validity of our theory.

INDEX TERMS adaptive synchronization, complex-valued neural networks, fractional-order, time-varying delays

I. INTRODUCTION

Artificial neural networks have always been a research hot spot in artificial intelligence due to their properties of simulating biological neural networks. They have been widely applied in pattern recognition [1], signal processing [2] and other hot fields. However, time delays are inevitable in both biological neural networks and their hardware implementations. Additionally, time-varying delays may cause chaotic, oscillating, or unstable behaviors of the system. Thus, it is necessary and meaningful to take time delays into consideration when studying neural networks. And numerous papers involving neural networks with time delays have been published [3]–[7].

Fractional-order calculus was initially proposed as a purely mathematical theory, but scholars are constantly exploring the research value of fractional systems in practical applications over the years. So far, fractional calculus has been extensively utilized in fluid mechanics [8], finance [9], infectious diseases [10], [11] and other fields. In contrast to neural network models with integer-order, fractional-order models can more accurately describe the change process of the memory and historical characteristics of the natural system. Hence, it is of great significance to investigate fractional-order neural networks (FONNs).

Recently, the dynamic behaviors of FONNs such as dissipativity [12], synchronization [13]–[15] and stability [16], [17] have attracted increasing interest. Among all these dynamic behaviors, synchronization refers to a certain relative relationship between two or more time-varying systems under external control. Based on this property, synchronization is widely applied in image processing [18], security communications [19] and other fields. Currently, various synchronizations have been investigated such as exponential synchronization [20], [21], Mittag–Leffler synchronization [22]–[24], complete synchronization [25] and projective synchronization [26]. In [27]–[30], the finite-time synchronization of FONNs was studied. Moreover, in [31], [32], the authors attempt to investigate the fixed-time synchronization of FONNs via sliding mode control and integer-order methods. Among these synchronizations, the projective synchroniza-
tion is of essential research and application value due to its characteristic, which is that the drive and response systems can achieve proportional synchronization more quickly.

To synchronize systems, numerous control methods have been proposed, including feedback control [29], [33], impulse control [34], adaptive control [24] and sliding mode control [32]. Above all these strategies, the adaptive control is an excellent choice to study the synchronization of FONNs. Because it can update the control gains by itself and is easier to be applied in the existing systems. Therefore, it is valuable to design a suitable adaptive controller to study synchronization problems of neural networks.

It is worth noting that complex-valued signals are commonly found in pattern recognition [35], nonlinear filtering [36] and image reconstruction [37]. Compared with real-valued systems, complex-valued systems have more complex properties and wider practical applications. When studying complex-valued neural networks, a number of researchers choose to separate the networks into two real-valued systems [13], [38]–[40]. Although it is feasible, the number of dimensions of the system is doubled, which greatly increases the difficulty and complexity of theoretical analysis. Therefore, in [29], [33], [41], the authors use a non-decomposition method, which is based on the proposed complex-valued inequalities, to avoid this problem.

In [33], the author studies the quasi-projective synchronization of fractional-order complex-valued neural networks without accounting for time delays in the model. For neural networks with time delays, researchers usually use the following methods to deal with the delays. One approach is to utilize a controller with a delay term to eliminate the time delays in the networks [24], [42]. However, the controller designed to solve the time delay problem is not simple enough. The second method is to use the inequality scaling technique to eliminate terms with time delays [43]. The third method is to use the existing inequality lemmas to solve the time delay problem. Therefore, in the research of synchronization of neural networks, it is worth studying which method to use to deal with time delays. And it is necessary to apply a simple controller and rigorous mathematical proofs to achieve synchronization of neural networks.

Following the analysis above, this article study the adaptive synchronization of FOCVNNTDs. A summary of the main contributions of this paper is:

(1) Firstly, a novel fractional differential inequality with time delays (Lemma 7) is established. Different from the results in [33], [43], this novel inequality contains time delays, which are inevitable in natural systems. Based on this inequality, we can design a controller without time delay, which saves control costs. Moreover, in the proof of this paper, as long as the condition of this inequality is satisfied, the synchronization result can be obtained.

(2) Secondly, another novel fractional differential inequality (Lemma 8) is established. Compared to Lemma 7, this inequality has a more common form. Under certain conditions, Lemma 8 can degenerate into Lemma 7, which extends our theoretical results. In addition, as a generalization of fractional Halanay inequality in [44], these two novel inequalities are suitable for investigating the adaptive synchronization problem of fractional-order systems.

(3) Thirdly, complete synchronization and quasi-projective synchronization of FOCVNNTDs are investigated based on novel adaptive controllers and these two novel fractional differential inequalities. Compared to the results in [24], [43], the controllers we designed are more straightforward and the theoretical proof process in the paper is rigorous.

This paper will be organized in the following structure. Section 2 of presents some definitions, known lemmas and two new fractional differential inequalities. In Section 3, two adaptive controllers are designed and some adaptive synchronization criteria of FOCVNNTDs are established. Section 4 will provide some numerical simulation examples to demonstrate the validity of our theory. In Section 5, we reach a conclusion.

**Notation:** In this article, the sets $\mathbb{C}$ and $\mathbb{C}^n$, respectively, denote all complex numbers and a space containing all n-dimensional complex vectors, while $\mathbb{R}$ and $\mathbb{R}^+$ denote the set of all real numbers and the set of all real numbers that are non-negative. For $z \in \mathbb{C}$, the real part of $z$ is $\text{Re}(z)$ whereas the imaginary part is $\text{Im}(z)$, $ar{z}$ is the conjugate of $z$, $|z| = \sqrt{\text{Re}(z)^2 + \text{Im}(z)^2}$ represents the modulus of $z$. For $Z = (z_1, z_2, \ldots, z_n)^T \in \mathbb{C}^n$, $\|Z\|_2 = \left(\sum_{i=1}^{n} |z_i|^2\right)^{\frac{1}{2}}$.

## II. PRELIMINARIES AND MODEL DESCRIPTION

### A. PRELIMINARIES

A few basic definitions and necessary lemmas will be presented in this part.

**Definition 1.** [45] The Riemann-Liouville fractional integral with fractional-order $\alpha$ for an integrable function $f(t)$ is

$$I^\alpha_0 f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where $\alpha > 0$ and $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt$.

**Definition 2.** [45], [46] When $\alpha \in (0, 1)$, the Caputo fractional derivative with fractional-order $\alpha$ for a differentiable function $f(t) \in C^\alpha([0, +\infty), \mathbb{R})$ is

$$D^\alpha_0 f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds,$$

and the Laplace transform of $D^\alpha_0 f(t)$ is given as

$$\mathcal{L} \left\{ D^\alpha_0 f(t) \right\} = s^{\alpha} f(s) - s^{\alpha-1} f(0)$$

where $f(s)$ is the Laplace transform of $f(t)$.

**Definition 3.** [45], [46] The Mittag-Leffler function $E_{p,q}(\cdot)$ is defined as

$$E_{p,q}(\varphi) = \sum_{m=0}^{\infty} \frac{\varphi^m}{\Gamma(pm + q)}.$$
where $p > 0$, $q > 0$ and $\varphi \in \mathbb{C}$. And if $q = 1$,

$$E_p(\varphi) = E_{p,1}(\varphi) = \sum_{m=0}^{\infty} \frac{\varphi^m}{p^m}.$$  

**Lemma 1.** [45] Let $t > 0$, then the Laplace transform of function $t^{\alpha-1}E_{p,q}(\lambda t^p)$ is

$$\mathcal{L}\{t^{\alpha-1}E_{p,q}(\lambda t^p)\} = \frac{s^{p-q} - \lambda}{s^p - \lambda}$$

where $\lambda \in \mathbb{C}$ and $|\lambda s^{-p}| < 1$. In addition, the function $E_{p,q}(\lambda t^p)$ is monotonic and non-increasing and $0 \leq E_{p,q}(\lambda t^p) \leq 1$ for $\lambda \leq 0$.

**Lemma 2.** [47] For any two complex numbers $\alpha$ and $\beta$, and any real constant $\varepsilon > 0$, then this inequality holds:

$$\alpha\beta + \varepsilon\beta \leq \varepsilon\alpha + \frac{1}{\varepsilon}\beta\beta.$$  

**Lemma 3.** [48] Let function $f(t) \in \mathbb{R}$ be continuous and differentiable on $t \in [0, +\infty)$, in this case, for any constant $\lambda \in \mathbb{R}$,

$$\int_0^\infty \frac{1}{2\alpha} D_{t}^{\alpha}(f(t) - \lambda)^2 \leq (f(t) - \lambda)^2 \tau_0 D_t^\alpha f(t)$$

in which $t \geq 0$ and $0 \prec \lambda \prec 1$.

**Lemma 4.** [33] The function $f(t) \in \mathbb{C}$ is analytic and continuous, then the following inequality is established:

$$C_0^\alpha D_t^\alpha f(t)f(t) \leq f(t)C_0^\alpha D_t^\alpha f(t) + \frac{1}{\alpha} D_t^\alpha f(t)$$

where $t \geq 0$ and $0 \prec \alpha \prec 1$.

**Lemma 5.** [49] If $\psi : [\tau, +\infty) \to \mathbb{R}^+$ is continuous on $[0, +\infty)$ and bounded on $[\tau, 0]$, $a, K : \mathbb{R}^+ \to \mathbb{R}$ are two continuous functions and satisfy $t \to \infty$ $a(t) = 0$, $K(t) = 0$, $\lim_{t \to +\infty} K(t) = 0$ and $K(t) \in L^1(\mathbb{R}^+)$. $\eta > 0$ is a constant, then this inequality holds:

$$\psi(t) \leq a(t) + \eta \int_0^t K(t - s) \sup_{s \leq \xi \leq \eta} \psi(s) ds, \quad t \geq 0.$$

If $\|K\|_{L^1(\mathbb{R}^+)} < 1$, then $\lim_{t \to +\infty} \psi(t) = 0$.

**Lemma 6.** [44] For $t \in \mathbb{R}$, if the function $\omega(t) \geq 0$ is continuous and satisfies that

$$\left\{\begin{array}{l}
\omega(t) \leq c_1 + c_2 \sup_{-\tau(t) \leq \tau(t) \leq t} \omega(s), \quad t \in [0, +\infty) \\
\omega(t) = |\psi(t)|, \quad t \in [-\sigma, 0]
\end{array}\right.$$

where the function $\psi(t)$ is continuous and bounded, $\sigma$ is a positive constant. The coefficients satisfy $c_1 \geq 0$, $0 < c_2 \leq 1$ and $-\sigma \leq -\tau(t) \leq t$. Let $M_0 = \sup_{-\sigma \leq \xi \leq 0} |\psi(\xi)|$. Then we have

$$\omega(t) \leq \frac{c_1}{1 - c_2} + M_0, \quad t \geq 0.$$  

Moreover, if $\lim_{t \to +\infty} (t - \tau(t)) = +\infty$, then for any given $\varepsilon > 0$, there exists $t^*_s = t^*_s(M_0, \varepsilon) > 0$ such that

$$\omega(t) \leq \frac{c_1}{1 - c_2} + \varepsilon, \quad t \geq t^*_s.$$  

In addition, the inequality still holds if the conditions is modified into that

$$\left\{\begin{array}{l}
\omega(t) \leq c_1 + c_2 \sup_{-\tau \leq \xi \leq t} \omega(\xi), \quad t \in [0, +\infty) \\
\omega(t) = |\psi(t)|, \quad t \in [-\sigma, 0]
\end{array}\right.$$  

**Lemma 7.** Let $V(t)$ be bounded on $[-\tau, 0]$ and continuous on $[0, +\infty)$. $(t)$ and $U(t)$ are non-negative and satisfy

$$\int_0^t \lambda^q (V(t) + U(t)) \leq \rho V(t) + \mu \sup_{-\tau \leq h \leq 0} V(t + h), t \geq 0,$$

where $0 < \alpha < 1$, $\rho > \mu > 0$ and $h$ is the time delay. Then, $\lim_{t \to +\infty} V(t) = 0$.

Proof. According to (1), there exists a non-negative function $\varphi(t)$ such that

$$\int_0^t \lambda^q (V(t) + U(t)) + \varphi(t) = \rho V(t) + \mu \sup_{-\tau \leq h \leq 0} V(t + h),$$

Let $H(t) = \sup_{-\tau \leq h \leq 0} V(t + h)$, according to the Laplace transform in Definition 2,

$$s^\alpha V(s) + s^\alpha U(s) - s^{\alpha-1} (V(0) + U(0)) + \varphi(s) = \rho V(s) + \mu H(s).$$

where $V(s) = \mathcal{L}\{V(t)\}, U(s) = \mathcal{L}\{U(t)\}, \varphi(s) = \mathcal{L}\{\varphi(t)\}$ and $H(s) = \mathcal{L}\{H(t)\}$. Then

$$V(s) = \frac{s^{\alpha-1} (V(0) + U(0)) - (s^\alpha + \mu) U(s) - \varphi(s)}{s^\alpha + \mu}$$

Based on the inverse Laplace transform in Lemma 1, we can get

$$V(t) = (V(0) + U(0)) E_\alpha(-\rho t^\alpha) - U(t) * (1 - t^{\alpha-1} E_\alpha, t^{\alpha-1} E_\alpha),$$

where $*$ is convolution operator. Based on the properties of Mittag-Leffler function, we know that

$$\rho t^{\alpha-1} E_\alpha, t^{\alpha-1} E_\alpha \leq \frac{\rho t^{\alpha-1}}{\Gamma(\alpha)}, \quad t \geq 0,$$

Assume that $t_1 > 0$, such that $\rho t_1^{\alpha-1} E_\alpha, t^{\alpha-1} E_\alpha \geq 0$

for all $t \geq t_1$. Based on (3), and as $U(t), \varphi(t), t^{\alpha-1} E_\alpha$ are non-negative functions, we know

$$V(t) \leq (V(0) + U(0)) E_\alpha(-\rho t^\alpha) + \mu \int_0^t (t - s)^{\alpha-1} E_\alpha, (t - s)^\alpha \sup_{-\tau \leq h \leq 0} V(s + h) ds$$

for $t \geq t_1$ and $t_1 = \left(\frac{\Gamma(\alpha)}{\rho}\right)^{1/\alpha} \rho^{-1}$. Denote

$$a(t) = (V(0) + U(0)) E_\alpha(-\rho t^\alpha), b(t) = t^{\alpha-1} E_\alpha, (-\rho t^\alpha).$$
Obviously, \( \lim_{t \to +\infty} a(t) = 0 \), \( K(t) \geq 0 \) and \( \lim_{t \to +\infty} K(t) = 0 \). According to Lemma 5, we need to prove the condition \( \mu \| k \|_{L^1(R^+)} < 1 \) so that \( \lim_{t \to +\infty} V(t) = 0 \). On the one hand, we know that

\[
\frac{d}{dt} [t^\alpha E_{\alpha, \alpha+1}(-\rho t^\alpha)] = t^{\alpha-1} E_{\alpha, \alpha}(-\rho t^\alpha),
\]

which implies that

\[
\int_0^t s^{\alpha-1} E_{\alpha, \alpha}(-\rho s^\alpha) \, ds = t^\alpha E_{\alpha, \alpha+1}(-\rho t^\alpha). \tag{5}
\]

On the other hand, from Definition 3,

\[
t^\alpha E_{\alpha, \alpha+1}(-\rho t^\alpha) = t^\alpha \sum_{m=0}^{\infty} \frac{(-\rho t^\alpha)^m}{\Gamma(\alpha m + \alpha + 1)}
\]

\[
= -\frac{1}{\rho} \sum_{m=0}^{\infty} \frac{(-\rho)^{m+1} \rho^m}{\Gamma(\alpha (m + 1) + 1)}
\]

\[
= -\frac{1}{\rho} \sum_{m=0}^{\infty} \frac{(-\rho)^m}{\Gamma(\alpha (m + 1))} - 1
\]

\[
= -\frac{1}{\rho} E_{\alpha}(-\rho t^\alpha) + \frac{1}{\rho}.
\]

We get

\[
\lim_{t \to +\infty} t^\alpha E_{\alpha, \alpha+1}(-\rho t^\alpha) = \frac{1}{\rho}. \tag{7}
\]

According to (5), (7) and \( \rho > \mu \),

\[
\mu \| K \|_{L^1(R^+)} = \mu \int_0^{+\infty} s^{\alpha-1} E_{\alpha, \alpha}(-\rho s^\alpha) \, ds
\]

\[
= \lim_{t \to +\infty} \mu t^\alpha E_{\alpha, \alpha+1}(-\rho t^\alpha) = \frac{\mu}{\rho} < 1.
\]

From Lemma 5, \( \lim_{t \to +\infty} V(t) = 0 \), the proof is completed. \( \square \)

**Lemma 8.** If the function \( V(t) \) is continuous on \( [0, +\infty) \) and bounded on \( [-\tau, 0] \). \( V(t) \) and \( U(t) \) are non-negative and satisfy

\[
\frac{\partial}{\partial t} [t^\alpha E_{\alpha, \alpha+1}(t^\alpha)] (V(t) + U(t)) \leq -\rho V(t) + \mu \sup_{t-\tau(t) \leq h \leq t} V(h) + C,
\]

where \( 0 < \alpha < 1, C \geq 0, \rho > \mu > 0 \) and \( 0 \leq \tau(t) \leq \tau \). Let \( M_0 = \sup_{-\tau \leq h \leq 0} V(h) \), then

\[
V(t) \leq \left( V(0) + U(0) - \frac{C}{\rho} \right) E_{\alpha}(-\rho t^\alpha) + \frac{C}{\rho} + M_0. \tag{9}
\]

Moreover, if \( \lim_{t \to +\infty} (t - \tau(t)) = +\infty \), then for any given \( \varepsilon > 0 \), there exists \( t_* = t_*(M_0, \varepsilon) > 0 \) such that

\[
V(t) \leq \left( V(0) + U(0) - \frac{C}{\rho} \right) E_{\alpha}(-\rho t^\alpha) + \frac{C}{\rho} + \varepsilon. \tag{10}
\]

**Proof.** According to (8), there exists a non-negative function \( \varphi(t) \) such that

\[
\varphi(t) = -\rho V(t) + \mu \sup_{t-\tau(t) \leq h \leq t} V(h) + C.
\]

Let \( H(t) = \sup_{t-\tau(t) \leq h \leq t} V(h) \), according to the Laplace transform in Definition 2,

\[
s^\alpha V(s) + s^\alpha U(s) - s^{\alpha-1} (V(0) + U(0)) + \varphi(s) = -\rho V(s) + \mu H(s) + \frac{C}{s}
\]

where \( V(s) = \mathcal{L} \{ V(t) \}, U(s) = \mathcal{L} \{ U(t) \} \), \( \varphi(s) = \mathcal{L} \{ \varphi(t) \} \) and \( H(s) = \mathcal{L} \{ H(t) \} \). We can obtain that

\[
V(s) = \frac{s^{\alpha-1} (V(0) + U(0)) - \varphi(s)}{s^\alpha + \rho - s^{\alpha-1} C} + \frac{\varphi(s) + \mu H(s)}{s^\alpha + \rho} + \frac{s^{-1} C}{s^\alpha + \rho}
\]

Based on the inverse Laplace transform in Lemma 1,

\[
V(t) = (V(0) + U(0)) E_{\alpha}(-\rho t^\alpha) - \rho U(t) + \frac{\mu}{\rho} \sup_{s-\tau(s) \leq h \leq s} V(h) + \frac{C}{\rho}
\]

\[
+ \mu \int_0^t (t - s)^{\alpha-1} E_{\alpha, \alpha}(-\rho (t - s)^\alpha) \sup_{s-\tau(s) \leq h \leq s} V(h) \, ds
\]

\[
\leq \left( V(0) + U(0) - \frac{C}{\rho} \right) E_{\alpha}(-\rho t^\alpha) + \frac{C}{\rho} + \mu \int_0^t (t - s)^{\alpha-1} E_{\alpha, \alpha}(-\rho (t - s)^\alpha) \sup_{-\tau \leq h \leq t} V(h) \, ds
\]

\[
\leq \left( V(0) + U(0) - \frac{C}{\rho} \right) E_{\alpha}(-\rho t^\alpha) + \frac{C}{\rho} + \mu \sup_{-\tau \leq h \leq t} V(h).
\]

Therefore, \( t^\alpha E_{\alpha, \alpha+1}(-\rho t^\alpha) \leq \frac{1}{\rho} \).

Substituting (5) and (13) into inequality (12),

\[
V(t) \leq \left( V(0) + U(0) - \frac{C}{\rho} \right) E_{\alpha}(-\rho t^\alpha) + \frac{C}{\rho} + \mu \sup_{-\tau \leq h \leq t} V(h) t^\alpha E_{\alpha, \alpha+1}(-\rho t^\alpha)
\]

\[
\leq \left( V(0) + U(0) - \frac{C}{\rho} \right) E_{\alpha}(-\rho t^\alpha) + \frac{C}{\rho} + \mu \sup_{-\tau \leq h \leq t} V(h).
\]
By Lemma 6, we obtain that
\[ V(t) \leq \frac{V(0) + U(0) - C}{\rho} E_\alpha(-\rho t^\alpha) + C \]
\[ + M_0, \quad t \geq t_1. \tag{14} \]
Moreover, if \( t \to +\infty \), then for any given \( \varepsilon > 0 \), there exists \( t_\varepsilon > t_0 \) such that
\[ V(t) \leq \frac{V(0) + U(0) - C}{\rho} E_\alpha(-\rho t^\alpha) + C \]
\[ + \varepsilon, \quad t \geq \max\{t_1, t_\varepsilon\}. \tag{15} \]
The proof is completed. \( \square \)

Remark 1. When \( C = 0 \), from inequality (15) we can get \( \lim_{t \to +\infty} V(t) \leq \varepsilon \). For \( V(t) \) is a non-negative function and \( \varepsilon \) can be selected as a positive number that is infinitely close to zero, we get \( \lim_{t \to +\infty} V(t) = 0 \). That means Lemma 8 can be reduced to Lemma 7.

Remark 2. In the proof of Lemma 7 and Lemma 8, some existing lemmas and some properties of Laplace transform and Mittag–Leffler function are applied. The proof is rigorous and the proof techniques are versatile.

Remark 3. Unlike the Lemma 5 in [43], Lemma 7 and Lemma 8 in this paper contain the terms with time-varying delays, which are inevitable in real systems. Therefore, our results are more general. In addition, as a generalization of Fractional Halanay inequality in [44], Lemma 7 and Lemma 8 in this paper are suitable for investigating the adaptive synchronization problem of FONNs.

B. MODEL DESCRIPTION
In this article, the following FOCVNNTDs is considered:
\[ C^\alpha_0 D^\alpha_t x_i(t) = -d_ix_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) \]
\[ + \sum_{j=1}^n b_{ij} g_j(x_j(t - \tau_j(t))) + I_i(t) \tag{16} \]
for \( t \geq 0 \), where \( 0 < \alpha < 1, j \in \{1, 2, ..., n\} \), \( n \) represents the number of neurons, \( f_j(\cdot), g_j(\cdot) : \mathbb{C} \to \mathbb{C} \) respectively denote the activation functions without and with delays, \( x_i(t) \in \mathbb{C} \) denotes the state of the \( i \)th neuron at time \( t \), \( d_i \in \mathbb{R} \) denotes the self-inhibition rate of the \( i \)th neuron, \( a_{ij}, b_{ij} \in \mathbb{C} \) represent the connection weight of the \( i \)th neuron and \( j \)th neuron, \( \tau_j(t) \) is time-varying delay satisfying \( 0 \leq \tau_j(t) \leq \tau, I_i(t) \in \mathbb{C} \) denotes the external input.

Let system (16) be the drive system, and the response system is described as:
\[ C^\alpha_0 D^\alpha_t y_i(t) = -d_i y_i(t) + \sum_{j=1}^n a_{ij} f_j(y_j(t)) + I_i(t) \]
\[ + \sum_{j=1}^n b_{ij} g_j(y_j(t - \tau_j(t))) + u_i(t) \tag{17} \]
where \( y_i(t) \in \mathbb{C} \) denotes the state variable of the \( i \)th neuron of the system (17) at time \( t \), and \( u_i(t) \in \mathbb{C} \) denotes the controller.

To study the synchronization between system (16) and system (17), some assumptions are needed:

**Assumption 1.** For any \( i, j \in \{1, 2, ..., n\} \), \( u \in \mathbb{C} \), there exist real numbers \( m_{ij}, m_{2j} \) such that
\[ |f_j(u)| \leq m_{ij}, \quad |g_j(u)| \leq m_{2j}. \]
For the external input \( I_i(t) \), we have
\[ |I_i(t)| \leq \gamma_i \]
where \( \gamma_i \) is a real number.

**Assumption 2.** The activation functions \( f_j(\cdot) \) and \( g_j(\cdot) \) are Lipschitz continuous, and for any \( u, v \in \mathbb{C} \), there exist real numbers \( l_{ij}, l_{2j} \) such that
\[ |f_j(u) - f_j(v)| \leq l_{ij} |u - v|, \]
\[ |g_j(u) - g_j(v)| \leq l_{2j} |u - v|. \]

### III. MAIN RESULTS
In this part, based on the new lemmas, the synchronization problem of FOCVNNTDs is investigated by adaptive controller.

**A. THE COMPLETE SYNCHRONIZATION OF FOCVNNTDs BY ADAPTIVE CONTROL**

The synchronization error is \( e_i(t) = y_i(t) - x_i(t) \). Design the adaptive controller as:
\[ u_i(t) = -m_i(t) e_i(t), \]
\[ \mu C^\alpha_0 D^\alpha_t |m_i(t)| = k_i e_i(t) e_i(t), \tag{18} \]
where \( m_i(t) \) is the adaptive coefficient and \( k_i \) is a positive constant. Based on the above conditions, we get that
\[ C^\alpha_0 D^\alpha_t e_i(t) = -[d_i + m_i(t)] e_i(t) \]
\[ + \sum_{j=1}^n a_{ij} [f_j(y_j(t)) - f_j(x_j(t))] \]
\[ + \sum_{j=1}^n b_{ij} [g_j(y_j(t - \tau_j(t))) - g_j(x_j(t - \tau_j(t)))]. \tag{19} \]

**Theorem 1.** Under Assumption 2 and the adaptive controller (18), system (16) and (17) are completely synchronized if \( \rho_1 > \mu \geq 0 \), where
\[ \rho_1 = \min_{1 \leq i \leq n} \left\{ 2d_i + 2m_i - \sum_{j=1}^n (a_{ij}a_{ij} + b_{ij}b_{ij}) - n \mu \right\} \]
and
\[ \mu = n \max_{1 \leq i \leq n} \{l_{2i}\}. \]

**Proof.** Construct the Lyapunov function as:
\[ V_1(t) = \sum_{i=1}^n e_i(t) e_i(t) + \sum_{i=1}^n \frac{1}{k_i} (m_i(t) - m_i)^2, \tag{20} \]
\[
\begin{aligned}
\frac{C}{0} D^\alpha_t V_1(t) &\leq -2 \sum_{i=1}^{n} (d_i + m_1) e_i(t) e_i(t) \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ a_{ij} \overline{a_{ij}} e_i(t) e_i(t) + (f_j(y_j(t)) - f_j(x_j(t))) (f_j(y_j(t)) - f_j(x_j(t))) \right] \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} \overline{b_{ij}} e_i(t) e_i(t) \\
&+ (g_j(y_j(t - \tau_j(t))) - g_j(x_j(t - \tau_j(t)))) (g_j(y_j(t - \tau_j(t))) - g_j(x_j(t - \tau_j(t)))) \\
&\leq -2 \sum_{i=1}^{n} (d_i + m_1) e_i(t) e_i(t) + \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ a_{ij} \overline{a_{ij}} e_i(t) e_i(t) + b_{ij} \overline{b_{ij}} e_i(t) e_i(t) \right] \\
&+ n \sum_{i=1}^{n} l_{1i}^2 e_i(t) e_i(t) + n \sum_{i=1}^{n} l_{2i}^2 e_i(t - \tau_i(t)) e_i(t - \tau_i(t)) \\
&\leq -\rho_1 V_{11}(t) + \mu \sup_{t - \tau(t) \leq h \leq t} V_{11}(h).
\end{aligned}
\]  

where \( m_1 \) satisfies the inequality (23). Let \( V_{11}(t) = \sum_{i=1}^{n} e_i(t) e_i(t) \) and \( V_{12}(t) = \sum_{i=1}^{n} \frac{1}{k_i} (m_i(t) - m_1)^2 \), according to Lemma 2 and Lemma 3, the \( \alpha \)-order Caputo derivative of \( V_1(t) \) is calculated as

\[
\begin{aligned}
\frac{C}{0} D^\alpha_t V_1(t) \\
&\leq \sum_{i=1}^{n} \left[ e_i(t) \frac{C}{0} D^\alpha_t e_i(t) + e_i(t) \frac{C}{0} D^\alpha_t e_i(t) \right] \\
&+ \sum_{i=1}^{n} \frac{2}{k_i} [m_i(t) - m_1] \frac{C}{0} D^\alpha_t m_i(t) \\
&= -2 \sum_{i=1}^{n} (d_i + m_1) e_i(t) e_i(t) \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ a_{ij} e_i(t) (f_j(y_j(t)) - f_j(x_j(t))) \\
&+ a_{ij} e_i(t) (f_j(y_j(t)) - f_j(x_j(t))) \right] \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ b_{ij} e_i(t) (g_j(y_j(t - \tau_j(t))) - g_j(x_j(t - \tau_j(t)))) \\
&+ b_{ij} e_i(t) (g_j(y_j(t - \tau_j(t))) - g_j(x_j(t - \tau_j(t)))) \right] \\
&\leq -\rho_1 V_{11}(t) + \mu \sup_{t - \tau(t) \leq h \leq t} V_{11}(h).
\end{aligned}
\]  

According to Lemma 2 and Assumption 2, then we can get the inequality (21). That is

\[
\begin{aligned}
\frac{C}{0} D^\alpha_t (V_{11}(t) + V_{12}(t)) &\leq -\rho_1 V_{11}(t) + \mu \sup_{t - \tau(t) \leq h \leq t} V_{11}(h),
\end{aligned}
\]

\[
\rho_1 = \min_{1 \leq i \leq n} \left\{ 2d_i + 2m_1 - \sum_{j=1}^{n} (a_{ij} \overline{a_{ij}} + b_{ij} \overline{b_{ij}} - nl_{1i}^2) \right\}
\]

and \( \mu = n \max_{1 \leq i \leq n} \{ l_{2i}^2 \} \). We choose appropriate parameters to meet

\[
\rho_1 > \mu \geq 0,
\]

then according to Lemma 7, we have

\[
\lim_{t \to +\infty} V_{11}(t) = 0.
\]

Therefore, we can obtain that

\[
\lim_{t \to +\infty} \|e(t)\|_2 = 0.
\]

Based on the above analysis, it is clear that system (16) and (17) are completely synchronized. The proof is completed. \( \square \)

\section*{B. QUASI-PROJECTIVE SYNCHRONIZATION OF FOCVNNTDS BY ADAPTIVE CONTROL}

This part investigates the quasi-projective synchronization of FOCVNNTDs by adaptive control. Define the synchronization error as \( \hat{e}_i(t) = y_i(t) - \beta x_i(t) \), where \( \beta \in \mathbb{C} \) is the projection coefficient and \( \beta \neq 0 \). The adaptive controller is:

\[
\begin{aligned}
&\begin{cases}
u_i(t) &= -m_i(t) \hat{e}_i(t), \\
\frac{C}{0} D^\alpha_t m_i(t) &= k_i \hat{e}_i(t) \hat{e}_i(t),
\end{cases}
\end{aligned}
\]

where \( m_i(t) \) is adaptive coefficient and \( k_i \) is a positive constant. According to these conditions, we get

\[
\begin{aligned}
\frac{C}{0} D^\alpha_t \hat{e}_i(t) &\leq -d_i \hat{e}_i(t) + (1 - \beta) I_i(t) - m_i(t) \hat{e}_i(t) \\
&+ \sum_{j=1}^{n} a_{ij} [f_j(y_j(t)) - f_j(\beta x_j(t))] \\
&+ \sum_{j=1}^{n} b_{ij} [g_j(y_j(t - \tau_j(t))) - g_j(\beta x_j(t - \tau_j(t)))]
\end{aligned}
\]
\[ +\sum_{j=1}^{n} a_{ij} \left[ f_j \left( \beta x_j(t) \right) - \beta f_j \left( x_j(t) \right) \right] \]
\[ +\sum_{j=1}^{n} b_{ij} \left[ g_j \left( \beta x_j(t - \tau_j(t)) \right) - \beta g_j \left( x_j(t - \tau_j(t)) \right) \right]. \tag{26} \]

For convenience, we define:
\[ \rho_2 = \min_{1 \leq i \leq 2} \left\{ 2d_i + 2m_2 - \gamma_i - 2l_i^2 \right\} \]
\[ - \sum_{j=1}^{2} \left( a_{ij} \bar{a}_{ij} + b_{ij} \bar{b}_{ij} \right) \]}
\[ C = n(1 - \beta) \left( 1 - \beta \right) + 2n \sum_{i=1}^{n} \left( 1 + \beta \beta \right) m_{1i}^2 \]
\[ + 2n \sum_{i=1}^{n} \left( 1 + \beta \beta \right) m_{2i}^2. \]

**Theorem 2.** According to Assumption 1 and Assumption 2, system (16) and (17) can achieve quasi-projectively synchronized under the adaptive controller (25) if \( \rho_2 > \mu \geq 0 \) and \( C \geq 0 \). In addition, we can obtain that
\[ \lim_{t \to +\infty} \| e(t) \|_2 \leq \sqrt{\frac{C}{\rho_2 - \mu}} + \varepsilon \]
where the number \( \varepsilon \) is positive.

**Proof.** Construct the Lyapunov function as:
\[ V_2(t) = \sum_{i=1}^{n} \hat{e}_i(t) \hat{e}_i(t) + \sum_{i=1}^{n} \frac{1}{k_i} (m_i(t) - m_2)^2, \tag{27} \]
where \( m_2 \) satisfies the inequality (35). Let \( V_{21}(t) = \sum_{i=1}^{n} \hat{e}_i(t) \hat{e}_i(t) \) and \( V_{22}(t) = \sum_{i=1}^{n} \frac{1}{k_i} (m_i(t) - m_2)^2 \), according to Lemma 2 and Lemma 3, the \( \alpha \)-order Caputo derivative of \( V_2(t) \) is
\[ C^\alpha D^\alpha V_2(t) \leq \sum_{i=1}^{n} \left[ \hat{e}_i(t) C^\alpha D^\alpha \hat{e}_i(t) + \hat{e}_i(t) C^\alpha D^\alpha \hat{e}_i(t) \right] \]
\[ + \sum_{i=1}^{n} \frac{2}{k_i} [m_i(t) - m_2] C^\alpha m_i(t) \]
\[ = -2 \sum_{i=1}^{n} \left( d_i + m_2 \right) \hat{e}_i(t) \hat{e}_i(t) \]
\[ + \sum_{i=1}^{n} \left[ (1 - \beta) \hat{e}_i(t) I_i(t) + (1 - \beta) \hat{e}_i(t) I_i(t) \right] \]
\[ + \sum_{i=1}^{n} \left( a_{ij} \hat{e}_i(t) \left( f_j \left( y_j(t) \right) - f_j \left( \beta x_j(t) \right) \right) \right] \]
\[ + \sum_{i=1}^{n} \left( a_{ij} \hat{e}_i(t) \left( f_j \left( y_j(t) \right) - f_j \left( \beta x_j(t) \right) \right) \right] \]
\[ + \sum_{i=1}^{n} \left( a_{ij} \hat{e}_i(t) \left( f_j \left( \beta x_j(t) \right) - \beta f_j \left( x_j(t) \right) \right) \right] \]
\[ + \sum_{i=1}^{n} \left( a_{ij} \hat{e}_i(t) \left( f_j \left( \beta x_j(t) \right) - \beta f_j \left( x_j(t) \right) \right) \right] \]
\[ + \sum_{i=1}^{n} \left( a_{ij} \hat{e}_i(t) \left( f_j \left( \beta x_j(t) \right) - \beta f_j \left( x_j(t) \right) \right) \right] \]

From Assumption 2 and applying Lemma 2,
\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ a_{ij} \hat{e}_i(t) \left( f_j \left( y_j(t) \right) - f_j \left( \beta x_j(t) \right) \right) \right] \]
\[ + \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ b_{ij} \hat{e}_i(t) \left( g_j \left( y_j(t) \right) - g_j \left( \beta x_j(t) \right) \right) \right] \]
\[ + \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ b_{ij} \hat{e}_i(t) \left( g_j \left( y_j(t) \right) - g_j \left( \beta x_j(t) \right) \right) \right] \]
\[ + \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ b_{ij} \hat{e}_i(t) \left( g_j \left( y_j(t) \right) - g_j \left( \beta x_j(t) \right) \right) \right]. \tag{28} \]

Similarly, we have
\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ b_{ij} \hat{e}_i(t) \left( g_j \left( y_j(t) \right) - g_j \left( \beta x_j(t) \right) \right) \right] \]
\[ + \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ b_{ij} \hat{e}_i(t) \left( g_j \left( y_j(t) \right) - g_j \left( \beta x_j(t) \right) \right) \right] \]
\[ + \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ b_{ij} \hat{e}_i(t) \left( g_j \left( y_j(t) \right) - g_j \left( \beta x_j(t) \right) \right) \right] \]
\[ + \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ b_{ij} \hat{e}_i(t) \left( g_j \left( y_j(t) \right) - g_j \left( \beta x_j(t) \right) \right) \right]. \tag{29} \]

According to Assumption 1 and applying Lemma 2, we derive
\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ a_{ij} \hat{e}_i(t) \left( f_j \left( \beta x_j(t) \right) - \beta f_j \left( x_j(t) \right) \right) \right] \]
\[ + \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ a_{ij} \hat{e}_i(t) \left( f_j \left( \beta x_j(t) \right) - \beta f_j \left( x_j(t) \right) \right) \right] \]
\[ \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ a_{ij} \bar{a}_{ij} \hat{e}_i(t) \hat{e}_i(t) \right] \]
\[ + \sum_{i=1}^{n} \sum_{j=1}^{n} \left( 2 \left( f_j \left( \beta x_j(t) \right) \right) \bar{f}_j \left( \beta x_j(t) \right) + \beta \bar{f}_j \left( x_j(t) \right) \bar{f}_j \left( x_j(t) \right) \right] \]
\[ \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ a_{ij} \bar{a}_{ij} \hat{e}_i(t) \hat{e}_i(t) + 2n \sum_{i=1}^{n} \left( 1 + \beta \beta \right) m_{1i}^2 \right]. \tag{31} \]
According to Lemma 2, we can get
\[
\sum_{i=1}^{n} \left[ (1-\beta)\frac{\dot{e}_i(t)}{L_i(t)} I_i(t) + (1-\beta)\frac{\dot{e}_i(t)\dot{I}_i(t)}{L_i(t)} \right] \\
\leq n(1-\beta)(1-\beta) + \sum_{i=1}^{n} \gamma_i^2 \frac{\dot{e}_i(t)}{L_i(t)}.
\]  
(33)

Submitting (29)-(33) into (28), it has
\[
\sum_{i=1}^{n} \left[ (1-\beta)\frac{\dot{e}_i(t)}{L_i(t)} I_i(t) + (1-\beta)\frac{\dot{e}_i(t)\dot{I}_i(t)}{L_i(t)} \right] \\
\leq n(1-\beta)(1-\beta) + \sum_{i=1}^{n} \gamma_i^2 \frac{\dot{e}_i(t)}{L_i(t)}.
\]  
(34)

Then, choose \( m_2 \) such that
\[
\rho_2 > \mu \geq 0,
\]  
(35)

and \( C \geq 0 \). According to Lemma 8, there exists \( t_2 = \left( V'(0) \right)^{-1} \) such that
\[
V_{21}(t) \leq \left( V_{21}(0) + U_{21}(0) - \frac{C}{p_2} \right) E_a (\rho_2 t^{\alpha}) + \frac{C}{p_2} \\
+ M_0, \quad t \geq t_2
\]  
(36)

where \( M_0 = \sup_{\tau \leq h \leq 0} V_{21}(h) \).

Moreover, if \( \lim_{t \to +\infty} (t - \tau(t)) = +\infty \), then for any given \( \varepsilon > 0 \), there exists \( t_3 = t_3 (M_0, \varepsilon) > t_0 \) such that
\[
V_{21}(t) \leq \left( V_{21}(0) + U_{21}(0) - \frac{C}{p_2} \right) E_a (\rho_2 t^{\alpha}) + \frac{C}{p_2} \\
+ \varepsilon, \quad t \geq \max \{ t_2, t_3 \}
\]  
(37)

Thus, based on (27),
\[
\|\dot{e}(t)\|_2 \leq \sqrt{\left( V_{21}(0) + U_{21}(0) - \frac{C}{p_2} \right) E_a (\rho_2 t^{\alpha}) + \frac{C}{p_2} + \varepsilon},
\]  
(38)

where \( t \geq \max \{ t_2, t_3 \} \).

Based on the property of Mittag-Leffler function, we can finally get
\[
\lim_{t \to +\infty} \|\dot{e}(t)\|_2 \leq \sqrt{\frac{C}{\rho_2 - \mu} + \varepsilon},
\]  
(39)

where \( \varepsilon \) is a positive number. Therefore, system (16) and system (17) can achieve quasi-projective synchronization with the error bound \( \sqrt{\frac{C}{\rho_2 - \mu} + \varepsilon} \). The proof process is finished.

\( \square \)

**Remark 4.** Instead of separating the complex-valued networks into two real-valued systems [38]–[40], the proofs of Theorem 1 and Theorem 2 apply a non-decomposing method based on complex functions to deal with complex-valued systems, which simplifies the proof process and reduces the difficulty of theoretical analysis.

**Remark 5.** In [24], the author designs an adaptive controller with time delays to study the Mittag–Leffler synchronization of delayed FONNs. In contrast, the controllers used in Theorem 1 and 2 do not contain time delay terms. This means that our controller is simpler and can save control costs to some extent.

**Remark 6.** On the processing of time delays in [43], the author gives \( \xi > 1 \) such that \( V_1(t - \tau) \leq \xi V_1(t) \). In this way, the items with time delays in the proof are eliminated. This seems feasible, but is not rigorous enough because we do not know how to determine the value of \( \xi \). Based on the conclusions of Lemma 7 and 8, we do not need to eliminate the time delay terms during the proofs of Theorem 1 and 2. Therefore, Theorem 1 and 2 are more rigorous and reasonable in handling time delays due to the two novel inequalities.

**IV. NUMERICAL SIMULATION**

In this part, some numerical simulation examples will be provided to demonstrate the validity of our theory. We consider the FOCVNNTDs as follows:
\[
C_0 D_t^{0.98} x_i(t) = -d_i x_i(t) + \sum_{j=1}^{2} a_{ij} f_j (x_j (t)) + \sum_{j=1}^{2} b_{ij} g_j (x_j (t - \tau_j (t))) + I_i (t)
\]  
(40)

The response system is given by
\[
C_0 D_t^{0.98} y_i (t) = -d_i y_i (t) + \sum_{j=1}^{2} a_{ij} f_j (y_j (t)) + I_i (t) + \sum_{j=1}^{2} b_{ij} g_j (y_j (t - \tau_j (t))) + u_i (t),
\]  
(41)

where \( i = 1, 2 \), \( d_1 = d_2 = 1 \), \( I_1 (t) = I_2 (t) = 0 \), \( \tau_1 (t) = \frac{\tau_1}{\tau_{1+1}} \) and \( \tau_2 (t) = 0.8 \times \frac{1}{\tau_{1+1}} \). Obviously, we know
that \( \tau = \max_{j=1,2} \{ \tau_j(t) \} = 1 \). The initial values \( x(s) = (1 + 0.4i, 1.5 + 3i)^T \) and \( y(s) = (-1 - 0.2i, -3 - i)^T \) for \( s \in [-1,0] \). The connection weights are given as
\[
A = (a_{ij})_{2 \times 2} = \begin{pmatrix} 2 + 2i & -0.2 - 0.1i \\ -1.8 - 2.5i & 1 + i \end{pmatrix},
\]
\[
B = (b_{ij})_{2 \times 2} = \begin{pmatrix} -1.2 - 1.4i & -0.1 - 0.1i \\ -0.1 - 0.1i & 2 + 3i \end{pmatrix}.
\]

The activation functions are set as follows
\[
f_j(x_j(t)) = g_j(x_j(t)) = \frac{1 - \exp(-\text{Re}(x_j(t)))}{1 + \exp(-\text{Re}(x_j(t)))} + \frac{1}{1 + \exp(-\text{Im}(x_j(t)))}, \quad j = 1, 2.
\]

It is evident that Assumption 1, 2 are satisfied and \( \gamma_1 = \gamma_2 = 0, m_{11} = m_{12} = m_{21} = m_{22} = \sqrt{2}, l_{11} = l_{12} = l_{21} = l_{22} = 2 \) by simple calculation.

The phase trajectories of real part \( \text{Re}(x_i(t)) \) and imaginary part \( \text{Im}(x_i(t)) \) of system (40) are shown in Figs. 1 and 2.

A. COMPLETE SYNCHRONIZATION

To make system (40) and system (41) achieve complete synchronization, we choose \( m_1 = 15 \). It can be calculated that
\[
\rho_1 = \min_{1 \leq i \leq 2} \left\{ 2d_i + 2m_1 - \sum_{j=1}^{2} (a_{ij}a_{ij} + b_{ij}b_{ij}) - 2l_{2i}^2 \right\} = 14.53 \quad \text{and} \quad \mu = 2 \max_{1 \leq i \leq 2} \{ l_{2i}^2 \} = 8. \quad \text{Hence,} \quad \rho_1 > \mu > 0.
\]

systems (40) and (41) can achieve complete synchronization under the controller (18) based on Theorem 1. The state trajectories and error curves of the system (40), (41) are shown in Fig. 3-5.

Remark 7. Keeping the system parameters unchanged, controller (18) was changed to the traditional negative feedback controller used in [33], [43]. Fig. 6 shows the error curves of the system (40), (41) under feedback control. Comparing Fig. 5 and Fig. 6, it can be seen that systems (40) and (41) can achieve complete synchronization faster with the adaptive controller. This proves the validity of Theorem 1 and the superior of our adaptive controller.

Fig. 3: The state trajectories of x1 and y1 under the controller (18).

Fig. 4: The state trajectories of x2 and y2 under the controller (18).
value when $m_2$ takes infinite values as long as the condition $\rho_2 > \mu > 0$ is satisfied. The error curve of the system (40),(41) is depicted in Fig.9 and it proves that the theoretical analysis of Theorem 2 is correct and the estimated error bound is valid.

Fig. 5: Synchronization error under the adaptive controller (18).

Fig. 6: Synchronization error under feedback controller.

**B. QUASI-PROJECTIVE SYNCHRONIZATION**

We choose $m_2 = 150$, the projective coefficient $\beta = 0.9 + 0.4i$. Then we get

$$\rho_2 = \min_{1 \leq i \leq 2} \left\{ 2d_i + 2m_2 - \gamma_i^2 - 2l_i^2 \right\} = 269.49,$$

$$\mu = 2 \max_{1 \leq i \leq 2} \left\{ l_{2i}^2 \right\} = 8,$$

$$C = 2 \left( 1 - \beta \right) \left( 1 - \beta \right) + 4 \sum_{j=1}^{2} \left( 1 + \beta \beta \right) m_{1j}^2 + 4 \sum_{i=1}^{2} \left( 1 - \beta \bar{\beta} \right) m_{2i}^2 = 66.44.$$

It is obvious that $\rho_2 > \mu > 0$ and $C > 0$. According to Theorem 2, system (40) and system (41) can achieve quasi-projective synchronization under the adaptive controller (25), which is depicted in Fig.7-8. Furthermore, the error bound is estimated as $\sqrt{\frac{C}{\rho_2 - \mu} + \varepsilon} = 0.514$, where $\varepsilon = 0.01$. Theoretically, the estimation error bound is equal to a minimal

Fig. 7: The real and imaginary part of x1 and y1 under the controller (25).

Fig. 8: The real and imaginary part of x2 and y2 under the controller (25).

Fig. 9: The norm of error $\|\hat{e}(t)\|_2$ under the controller (25).
V. CONCLUSION
In this paper, two novel fractional differential inequalities are established, which can be regarded as the generalization of the results in [15], [33], [43]. Based on these two novel inequalities and a novel adaptive controller, the complete synchronization and quasi-projective synchronization of FOCVNNTs are investigated. Compared with the results in [24], [43], Theorem 1 and Theorem 2 are more rigorous and reasonable in handling time delays due to the two novel inequalities. Additionally, different from [38]–[40], the proofs of Theorem 1 and Theorem 2 apply a non-decomposing method, which is based on some complex-valued inequalities. In this way, the proof process is simplified and the difficulty of theoretical analysis is reduced. It should be noted that the complete synchronization and quasi-projective synchronization of FOCVNNTs discussed in this paper are all infinite-time synchronization. In future research, we will consider the finite-time synchronization of FOCVNNTs and it would be a meaningful job. As a result of the diffusion phenomenon inherent in artificial neural networks, the reaction-diffusion term will also be considered in the synchronization study of FOCVNNTs.

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