On the asymptotic shape of solutions to Neumann problems for non-cooperative parabolic systems

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Abstract

We consider a class of nonautonomous parabolic competition-diffusion systems on bounded radial domains under Neumann boundary conditions. We show that, if the initial profiles satisfy a reflection inequality with respect to a hyperplane, then global positive solutions are asymptotically (in time) foliated Schwarz symmetric with respect to antipodal points. Additionally, a related result for (positive and sign changing solutions) of a scalar equation with Neumann boundary conditions is given. The asymptotic shape of solutions to cooperative systems is also discussed.

Keywords: Lotka-Volterra systems, competition, rotating plane method

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1 Introduction

The focus of the present paper is the asymptotic shape of positive global solutions of parabolic systems with competition on bounded radial domains with Neumann boundary conditions. The problem which mainly motivates our study is the following Lotka-Volterra System of two equations:

\begin{align*}
(u_1)_t - \mu_1 \Delta u_1 &= a_1(t)u_1 - b_1(t)u_1^2 - \alpha_1(t)u_1u_2, \quad x \in B, \ t > 0, \\
(u_2)_t - \mu_2 \Delta u_2 &= a_2(t)u_2 - b_2(t)u_2^2 - \alpha_2(t)u_1u_2, \quad x \in B, \ t > 0, \\
\frac{\partial u_i}{\partial \nu} &= 0 \quad \text{on } \partial B \times (0, \infty), \\
u_i(x, 0) &= u_{0,i}(x) \geq 0 \quad \text{for } x \in B, \ i = 1, 2.
\end{align*}

Here and in the remainder of the paper, $B$ denotes a ball or an annulus in $\mathbb{R}^N$ with $N \geq 2$, and $\nu$ denotes the unit outer normal on $\partial B$. Moreover, $\mu_1$ and $\mu_2$...
are positive constants and
\[ a_i, b_i, \alpha_i \in L^\infty((0, \infty)) \text{ satisfy } a_i(t), b_i(t) \geq 0 \text{ for } t > 0 \text{ and } \inf_{t>0} \alpha_i(t) > 0 \text{ for } i = 1, 2. \tag{1.2} \]

The Lotka-Volterra System is commonly used to model the competition between two different species, and the coefficients \( \mu_i, a_i, b_i, \alpha_i \) represent diffusion, birth, saturation, and competition rates respectively (see \[11\]). In the literature, the system is mostly considered with constant coefficients for matters of simplicity, whereas it is more natural to assume time-dependence as e.g. in \[1, 13, 15\] in order to model the effect of different time periods on the birth rates, the movement, or the aggressiveness of the species. Even in the case of constant coefficients, the possible dynamics of the system has a very rich structure and depend strongly on relationships between these constants, see e.g. \[1, 3, 5, 6, 15\]. In the case of time-dependent coefficients, a full understanding of the asymptotic dynamics is out of reach, but one can still guess that the shape of the underlying domain has some effect on the shape of solutions for large positive times. In the present paper we study this question on a radial bounded domain \( B \). More precisely, for a solution \( u = (u_1, u_2) \) of \((1.1)\), we study symmetry and monotonicity properties of elements in the associated omega limit set, which is defined as
\[
\omega(u) = \omega(u_1, u_2) := \{(z_1, z_2) \in C(B) \times C(B) \mid \max_{i=1,2} \lim_{n \to \infty} \|u_i(\cdot, t_n) - z_i\|_{L^\infty(B)} = 0 \text{ for some sequence } t_n \to \infty \}.
\]

For global solutions which are uniformly bounded and have equicontinuous semiorbits \( \{u_i(\cdot, t) : t \geq 1 \} \), the set \( \omega(u) \) is nonempty, compact, and connected. The equicontinuity can be obtained under mild boundedness and regularity assumptions on the equation and using boundary and interior Hölder estimates (see Lemma \[24\] below). To present our results we need to introduce some notation. Let \( S^{N-1} = \{ x \in \mathbb{R}^N : |x| = 1 \} \) be the unit sphere in \( \mathbb{R}^N \), \( N \geq 2 \). For a vector \( e \in S^{N-1} \), we consider the hyperplane \( H(e) := \{ x \in \mathbb{R}^N : x \cdot e = 0 \} \) and the half domain \( B(e) := \{ x \in B : x \cdot e > 0 \} \). We write also \( \sigma_e : \overline{B} \to \overline{B} \) to denote reflection with respect to \( H(e) \), i.e. \( \sigma_e(x) := x - 2(x \cdot e)e \) for each \( x \in B \).

Following \[22\], we say that a function \( u \in C(B) \) is foliated Schwarz symmetric with respect to some unit vector \( p \in S^{N-1} \) if \( u \) is axially symmetric with respect to the axis \( \mathbb{R}p \) and nonincreasing in the polar angle \( \theta := \arccos(\frac{r}{|x|} \cdot p) \in [0, \pi] \).

We refer the reader to the survey article \[24\] for a broad discussion of symmetry properties of this type. Our main result concerning \((1.1)\) is the following.

**Theorem 1.1.** Suppose that \((1.2)\) holds, and let \( u = (u_1, u_2) \) be a classical solution of \((1.1)\) such that \( \|u_i\|_{L^\infty(B \times (0, \infty))} < \infty \) for \( i = 1, 2 \). Moreover, assume that
\[
(h0) \quad u_{0,1} \geq u_{0,1} \circ \sigma_e, \quad u_{0,2} \leq u_{0,2} \circ \sigma_e \quad \text{in } B(e) \quad \left\{ \begin{array}{l} \text{for some } e \in S^{N-1} \text{ with } \ \ u_{0,i} \neq u_{0,i} \circ \sigma_e \text{ for } i = 1, 2. \end{array} \right.
\]
Then there is some $p \in \mathbb{S}^{N-1}$ such that every $(z_1, z_2) \in \omega(u)$ has the property that $z_1$ is foliated Schwarz symmetric with respect to $p$ and $z_2$ is foliated Schwarz symmetric with respect to $-p$.

This theorem is a direct consequence of a more general result which we state in Theorem 1.3 below. Note that the inequality condition (h0) does not seem very strong, but it is a key assumption in order to obtain the result. Indeed, for general positive initial data, foliated Schwarz symmetry cannot be expected, as one may see already by looking at equilibria (i.e., stationary solutions) in special cases. Consider e.g. the elliptic system

$$
-\Delta u_1 = \lambda u_1 - u_1 u_2 \quad \text{in } B, \\
-\Delta u_2 = \lambda u_2 - u_1 u_2 \quad \text{in } B, \\
\partial_\nu u_1 = \partial_\nu u_2 = 0 \quad \text{on } \partial B.
$$

Using bifurcation theory, one can detect values $\lambda > 0$ and $\varepsilon > 0$ such that (1.3) admits positive solutions in the annulus $B = \{ x \in \mathbb{R}^2 : 1 - \varepsilon < |x| < 1 \}$ such that the angular derivatives of $u_1, u_2$ change sign multiple times and therefore neither of the components is foliated Schwarz symmetric, see Theorem 7.1 below in the appendix. Theorem 1.1 is somewhat related to our previous work [21] on scalar nonlinear parabolic equations under Dirichlet boundary conditions. The main idea of both [21] and the present paper is to obtain the symmetry of elements in the omega limit set by a rotating plane argument. However, different tools are required to set up the method under Neumann boundary conditions. In particular, the main result of [21] does not extend in a straightforward way to the scalar nonlinear Neumann problem

$$
u t - \mu (|x|, t) \Delta u = f(t, |x|, u) \quad \text{in } B \times (0, \infty), \\
\partial_\nu u = 0 \quad \text{on } \partial B \times (0, \infty), \\
u(x, 0) = u_0(x) \quad \text{for } x \in B.
$$

It therefore seems appropriate to include a symmetry result for (positive and sign changing) solutions of (1.4) in the present paper. This result will be easier to prove than Theorem 1.1. We need the following hypotheses on the nonlinearity $f$ and the diffusion coefficient $\mu$. In the following, we put $I_B := \{ x : x \in \overline{B} \}$.

(H1) The nonlinearity $f : [0, \infty) \times I_B \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous in $u$ uniformly in $r$ and $t$, i.e.,

$$
\sup_{r \in I_B, t > 0, u, \bar{u} \in K, u \neq \bar{u}} \frac{|f(t, r, u) - f(t, r, \bar{u})|}{|u - \bar{u}|} < \infty \quad \text{for any compact subset } K \subset [0, \infty).
$$

(H2) $\sup_{r \in I_B, t > 0} |f(t, r, 0)| < \infty$.

(H3) $\mu \in C^1(I_B \times (0, \infty))$ and there are constants $\mu^* \geq \mu_* > 0$ such that $\| \mu \|_{C^1(I_B \times (0, \infty))} \leq \mu^*$ and $\mu(t, r) \geq \mu_*$ for all $r \in I_B, t > 0$. 

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The following is our main result on (1.4).

**Theorem 1.2.** Assume that (H1)-(H3) are satisfied, and let $u \in C^{2,1}(\overline{B} \times (0, \infty)) \cap C(\overline{B} \times [0, \infty))$ be a classical solution of (1.4) such that

$$\|u\|_{L^\infty(B \times (0, \infty))} < \infty.$$  

(1.5)

Suppose furthermore that assumption (H4) holds.

Moreover, for every $v \in \mathbb{S}^{N-1}$ such that $v_0 = u_0 \circ \sigma_v$ in $B(v)$ and $u_0 \neq u_0 \circ \sigma_v$. Then, there is some $p \in \mathbb{S}^{N-1}$ such that every element of the omega limit set

$$\omega(u) := \{ z \in C(\overline{B}) | \lim_{n \to \infty} \| u(\cdot, t_n) - z \|_{L^\infty(B)} = 0 \text{ for some sequence } t_n \to \infty \}$$

is foliated Schwarz symmetric with respect to $p$.

We now turn to a general class of two-component nonlinear competitive systems which includes (1.1). More precisely, we consider, for $i = 1, 2$,

$$(u_i)_t - \mu_i(|x|, t) \Delta u_i = f_i(t, |x|, u_i) - \alpha_i(|x|, t) u_i u_2, \quad x \in B, \ t > 0,$$

$$\partial_r u_i = 0, \quad x \in \partial B, \ t > 0, \quad (1.6)$$

$$u_i(x, 0) = u_{0,i}(x) \geq 0 \quad \text{for } x \in B.$$

On the data, we assume the following.

(h1) For $i = 1, 2$, the function $f_i : [0, \infty) \times I_B \times [0, \infty) \to \mathbb{R}$, $(t, r, v) \mapsto f(t, r, v)$ is locally Lipschitz continuous in $v$ uniformly with respect to $r$ and $t$, i.e.

$$\sup_{r \in I_B, t > 0, \ v, \tilde{v} \in K, v \neq \tilde{v}} \frac{|f_i(t, r, v) - f_i(t, r, \tilde{v})|}{|v - \tilde{v}|} < \infty \text{ for any compact subset } K \subset [0, \infty).$$

Moreover $f_1(t, r, 0) = f_2(t, r, 0) = 0$ for all $r \in I_B, t > 0$.

(h2) $\mu_i \in C^{2,1}(I_B \times (0, \infty))$ and there are constants $\mu^* \geq \mu_+ > 0$ such that $\mu_t(\cdot, t) \leq \mu^*$ and $\mu_i(r, t) \geq \mu_+$ for all $r \in I_B, t > 0$, and $i = 1, 2$.

(h3) $\alpha_i \in L^\infty(I_B \times (0, \infty))$ and there are constants $\alpha^* \geq \alpha_+ > 0$ such that $\alpha_t \leq \alpha_+ \leq \alpha^*$ for all $r \in I_B, t > 0$, and $i = 1, 2$.

Then we have the following result.

**Theorem 1.3.** Let (h1)-(h3) be satisfied, and let $u_1, u_2 \in C^{2,1}(\overline{B} \times (0, \infty)) \cap C(\overline{B} \times [0, \infty))$ be functions such that $u = (u_1, u_2)$ solves (1.6) and

$$\|u_i\|_{L^\infty(B \times (0, \infty))} < \infty \quad \text{for } i = 1, 2.$$  

(1.7)

Suppose furthermore that assumption (h0) of Theorem (1.4) holds. Then there is some $p \in \mathbb{S}^{N-1}$ such that every $(z_1, z_2) \in \omega(u)$ has the property that $z_1$ is foliated Schwarz symmetric with respect to $p$ and $z_2$ is foliated Schwarz symmetric with respect to $-p$. 

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As mentioned before, Theorem 1.1 is an immediate consequence of Theorem 1.3. As far as we know, there is no previous result on the asymptotic symmetry of competition-diffusion parabolic systems. For a related class of Dirichlet problems for elliptic competing systems with a variational structure, Tavares and the second author proved recently in [23] that the ground state solutions are foliated Schwarz symmetric with respect to antipodal points. Note that, in contrast, the elliptic counterpart of (1.1) has no variational structure which could lead to symmetry information. More is known in the case of Dirichlet problems for cooperative systems. In particular, for a class of parabolic cooperative systems, Földes and Poláčik [9] proved that, in the case where the underlying domain is a ball, positive solutions are asymptotically radially symmetric and radially decreasing. Moreover, for elliptic cooperative systems with variational structure and some convexity properties of the data, Damascelli and Pacella [4] proved foliated Schwarz symmetry of solutions having Morse index less or equal to the dimension of the domain.

To prove Theorems 1.2 and 1.3, we follow the strategy of our previous work [21] on a scalar Dirichlet problem, using a rotating plane argument. However, the proofs in [21] rely strongly on parabolic maximum principles for small domains due to Poláčik [17], and these are only available under Dirichlet boundary conditions. In the present paper, we replace this tool by a Harnack-Hopf type estimate, Lemma 3.1 below, which yields information up to the nonsmooth part of the boundary of cylinders over half balls and half annuli. With the help of this tool we show a stability property of reflection inequalities with respect to small perturbations of a hyperplane, see Lemma 3.3 below.

The adjustment of the rotating plane method to systems gives rise to a further difficulty. When dealing with the so-called semi-trivial limit profiles, that is, elements of $\omega(u_1, u_2)$ of the form $(z, 0)$ and $(0, z)$, the perturbation argument within the rotating plane method cannot be carried out directly. To overcome this obstacle, we apply a new normalization procedure and distinguish different cases for the asymptotics of the normalized profile. We remark that a similar normalization argument can be made for the Dirichlet problem version of system (1.6). In this case, the estimates given in [12] play a decisive role, and the argument is somewhat more technical. To keep this paper short we do not include the Dirichlet case here. We note that the occurrence and nature of semi-trivial limit profiles have been studied extensively in recent years, see e.g. [1, 5, 6, 13, 15].

It is natural to ask whether similar symmetry properties are available for the cooperative version of problem (1.6), i.e.,

\[
\begin{align*}
(u_i)_t - \mu_i(|x|, t)\Delta u_i &= f_i(t, |x|, u_1) + \alpha_i(|x|, t)u_1u_2 & \text{in } B \times (0, \infty), \\
\partial_{\nu}u_i &= 0 & \text{on } \partial B \times (0, \infty), \\
u_i(x, 0) &= u_{0,i}(x) \geq 0 & \text{for } x \in B, i = 1, 2.
\end{align*}
\]

Indeed, the proof of Theorem 1.3 can easily be adjusted to deal with (1.8). More precisely, we have the following result.

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Theorem 1.4. Let \((h1)\)–\((h3)\) be satisfied, and let \(u_1, u_2 \in C^{2,1}(\overline{B} \times (0, \infty)) \cap C(\overline{B} \times [0, \infty))\) be functions such that \(u = (u_1, u_2)\) solves \((1.8)\) and satisfies \((1.7)\). Suppose furthermore that
\[
(h0)' \quad u_{0,i} \geq u_{0,i} \circ \sigma_e \quad \text{in } B(e) \quad \text{for some } e \in S^{N-1} \text{ with } u_{0,i} \not\equiv u_{0,i} \circ \sigma_e \quad \text{for } i = 1, 2.
\]
Then there is some \(p \in S^{N-1}\) such that every \((z_1, z_2) \in \omega(u)\) has the property that \(z_1, z_2\) are foliated Schwarz symmetric with respect to \(p\).

The paper is organized as follows. In Section 2 we collect some preliminary tools which are rather easy consequences of already established results. In Section 3 we derive a Harnack-Hopf type estimate for scalar equations in a half cylinder under mixed boundary conditions, a related version of the Hopf Lemma for cooperative systems and a perturbation lemma for hyperplane reflection inequalities. In Section 4 we complete the proof of Theorem 1.2, and in Section 5 we complete the (more difficult) proof of Theorem 1.3. In Section 6 we first provide the proof of Theorem 1.4 and then briefly discuss further classes of competitive and cooperative systems (see (6.1) and (6.2) below).

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2 Preliminaries

First we fix some notation. Throughout the paper, we assume that \(B\) is a ball or an annulus in \(\mathbb{R}^N\) centered at zero, and we fix \(0 \leq A_1 < A_2 < \infty\) such that
\[
B := \begin{cases} 
\{x \in \mathbb{R}^N : A_1 < |x| < A_2\}, & \text{if } A_1 > 0, \\
\{x \in \mathbb{R}^N : |x| < A_2\}, & \text{if } A_1 = 0.
\end{cases}
\] (2.1)

Note that \(I_B = [A_1, A_2]\). For \(\Omega \subset \mathbb{R}^N\), we let \(\Omega^o\) denote the interior of \(\Omega\). For two sets \(\Omega_1, \Omega_2 \subset \mathbb{R}^N\), we put \(\text{dist}(\Omega_1, \Omega_2) := \inf\{|x - y| : x \in \Omega_1, y \in \Omega_2\}\). If \(\Omega_1 = \{x\}\) for some \(x \in \mathbb{R}^N\), we simply write \(\text{dist}(x, \Omega_2)\) in place of \(\text{dist}(\{x\}, \Omega_2)\).

We will need equicontinuity properties of uniformly bounded global solutions of \((1.4)\) and \((1.5)\) and their gradients. These properties are derived from standard uniform regularity estimates as collected in the following lemma.

Lemma 2.1. Let \(\Omega \subset \mathbb{R}^N\) be a smooth bounded domain, \(I \subset \mathbb{R}\) open, \(\mu \in C^1(\Omega \times I), g \in L^\infty(\Omega \times I)\), and let \(v \in C^{2,1}(\overline{\Omega} \times I) \cap C(\overline{\Omega} \times I)\) be a classical solution of
\[
\begin{align*}
u_t - \mu(x,t) \Delta v &= g(x,t) & \text{in } \Omega \times I, \\
\partial_\nu v &= 0 & \text{on } \partial \Omega \times I.
\end{align*}
\]

Suppose moreover that
\[
\mu_* := \inf_{\Omega \times I} \mu(x, t) > 0,
\]
\[
K := \|v\|_{L^\infty(\Omega \times I)} + \|\nu\|_{C^1(\Omega \times I)} + \|g\|_{L^\infty(\Omega \times I)} < \infty.
\]
Let \( h \in \{v, \nu_q : j = 1, \ldots, N\} \) and \( I \subset I \) with \( \text{dist}(I, \partial I) \geq 1 \). Then there exist positive constants \( C \) and \( \gamma \), depending only on \( \Omega, \mu_*, \) and \( K \), such that
\[
\text{dist}(v, K) \leq C.
\]  

Proof. Fix \( t_0 \in I \) and set \( Q := \Omega \times [t_0, t_0 + 1] \). Then, by \cite{article} Theorem 7.35, p.185 there is a constant \( C > 0 \), which depends only on \( \Omega, \mu_* \), and \( K \) such that
\[
\|D^2 v\|_{L^{N+3}(Q)} + \|u_i\|_{L^{N+3}(Q)} \leq C\|g\|_{L^{N+3}(Q)} + \|v\|_{L^{N+3}(Q)} \leq 2C|\Omega|K.
\]

In particular, there is some constant \( \bar{K} > 0 \) independent of \( t_0 \) such that
\[
\|v\|_{W^{2,1}_{\bar{K}+3}(Q)} \leq \bar{K}.
\]

Next, fix \( 0 < \gamma < 1 - \frac{\alpha + \beta}{N+3} \in (0, 1) \) By Sobolev embeddings (see, for example, \cite{article} embedding (1.2)) there exists a constant \( \hat{C} > 0 \) which only depends on \( \Omega \) such that
\[
\|v\|_{C^{1,\gamma}([t_0, t_0 + 1], Q)} \leq \hat{C}\|v\|_{W^{2,1}_{\bar{K}+3}(Q)} \leq \hat{C}\bar{K},
\]
where
\[
\|v\|_{C^{1,\gamma}([t_0, t_0 + 1], Q)} := \|v\|_{L^{\infty}(Q)} + \|u_i\|_{\gamma; Q} + \|\nu u\|_{L^{\infty}(Q)} + \|\nabla u\|_{\gamma; Q}
\]
and
\[
|v|_{\gamma; Q} := \sup \left\{ \frac{|v(x, t) - v(y, s)|}{|x - y|^\gamma + |t - s|^\beta} : (x, t), (y, s) \in Q, (x, t) \neq (y, s) \right\}
\]
for functions \( v : Q \to \mathbb{R} \) resp. \( v : Q \to \mathbb{R}^N \). Since the constant \( \hat{C}\bar{K} \) in (2.3) does not depend on the choice of \( t_0 \), we obtain \cite{article}.

Remark 2.2. If \( u = (u_1, u_2) \) is a nonnegative solution of \cite{article} with \( u_1, u_2 \in C^{2,1}(\overline{B} \times (0, \infty)) \cap C(\overline{B} \times [0, \infty)) \), then \( u_i \) satisfies
\[
(u_i)_t - \mu_i(|x|, t)\Delta u_i = g_i(x, t), \quad x \in B, \ t > 0,
\]
with \( g_i : B \times (0, \infty) \to \mathbb{R} \) given by
\[
g_i(x, t) = f_i(t, |x|, u_i(x, t)) - \alpha_i(|x|, t)u_1(x, t)u_2(x, t) \quad \text{for } i = 1, 2.
\]

If, moreover, (h1)-(h3) and \cite{article} are satisfied, then we have \( \|u_i\|_{L^\infty(B \times (0, \infty))} < \infty \) and \( \|g_i\|_{L^\infty(B \times (0, \infty))} < \infty \) for \( i = 1, 2 \). Since also the diffusion coefficients
μ, satisfy the assumptions of Lemma 2.1, we conclude that \( u_i \) and \( \partial_j u_i \) for all \( j = 1, \ldots, N, \ i = 1, 2 \), satisfy (2.2) with \( I = (1, \infty) \). As a consequence of (2.2), the semiorbits \( \{u_i(\cdot, t) : t \geq 1\} \) are precompact sets for \( i = 1, 2 \). Hence, by a standard compactness argument, the omega limit sets of \( u = (u_1, u_2) \) and its components are related as follows:

\[
\omega(u_i) = \{z_i : z = (z_1, z_2) \in \omega(u)\} \quad \text{for } i = 1, 2. \tag{2.4}
\]

Next we define extensions of solutions to second order Neumann problems on \( B \) to a larger domain via inversion at the boundary. Recalling (2.1), we define

\[
\tilde{B} := \left\{ \begin{array}{l}
\{x \in \mathbb{R}^N : \frac{A_1}{2} < |x| < \frac{A_2}{2}, & \text{if } A_1 > 0, \\
\{x \in \mathbb{R}^N : |x| < 2A_2, & \text{if } A_1 = 0.
\end{array} \right.
\]

and for \( x \in \tilde{B} \setminus B \) we put

\[
\hat{x} := \left\{ \begin{array}{l}
\frac{A_1}{|x|}x, & \text{if } |x| \geq A_2, \\
\frac{A_2}{|x|}x, & \text{if } |x| \leq A_1.
\end{array} \right.
\]

Lemma 2.3. Let \( I \subset \mathbb{R} \) be an open interval, \( \mu, g : B \times I \to \mathbb{R} \) be given functions and let \( u \in C^{2,1}(\overline{B} \times I) \cap C(\overline{B} \times I) \) be a solution of

\[
\begin{cases}
\begin{aligned}
u_t - \mu(x,t) \Delta u &= g(x,t) & & \text{in } B \times I, \\
\partial_r u &= 0 & & \text{on } \partial B \times I.
\end{aligned}
\end{cases}
\]

Then the function

\[
\tilde{u} : \tilde{B} \to \mathbb{R}, \quad \tilde{u}(x,t) := \left\{ \begin{array}{l}
u(x,t), & x \in \overline{B}, \ t \in I, \\
u(\hat{x},t), & x \in \overline{\tilde{B} \setminus B}, \ t \in I.
\end{array} \right.
\]

satisfies that \( \tilde{u} \in W^{2,1}_{p,1}(\overline{\tilde{B}} \times I) \cap C(\overline{\tilde{B}} \times I) \) for any \( p \geq 1 \), and it is a strong solution of the equation

\[
\tilde{u}_t - \tilde{\mu}(x,t) \Delta \tilde{u} - \tilde{b}(x,t) \partial_r \tilde{u} = \tilde{g}(x,t) \quad \text{in } \tilde{B} \times I. \tag{2.7}
\]

Here \( \partial_r = \frac{1}{|x|} \sum_{j=1}^N x_j \partial_j \) is the radial derivative and

\[
\tilde{\mu}(x,t) := \begin{cases}
\mu(x,t), & x \in B, \ t \in I, \\
\frac{1}{|x|^2} \mu(\hat{x},t), & x \in \tilde{B} \setminus B, \ t \in I,
\end{cases}
\]

\[
\tilde{b}(x,t) := \begin{cases}
0, & x \in B, \ t \in I \\
\frac{(4-2N)|x|}{|x|^2} \mu(\hat{x},t), & x \in \tilde{B} \setminus B, \ t \in I,
\end{cases}
\]

\[
\tilde{g}(x,t) := \begin{cases}
g(x,t), & x \in B, \ t \in I \\
g(\hat{x},t), & x \in \tilde{B} \setminus B, \ t \in I.
\end{cases}
\]
Proof. As a consequence of the Neumann boundary conditions we have \( \tilde{u} \in C^1(\tilde{B} \times I) \). Fix \( p \geq 1 \). By assumption, \( \|u\|_{W^{2,1}_p(\tilde{B} \times J)} < \infty \) for any subinterval \( J \subset I \). Since the map \( x \mapsto \tilde{x} \) has uniformly bounded first and second derivatives in \( \tilde{B} \setminus B \), it follows that \( \|\tilde{u}\|_{W^{2,1}_p(\tilde{B} \times J)} < \infty \). Finally, it is easy to check by direct calculation that (2.7) holds for \( x \in \tilde{B} \setminus \partial B \) and \( t \in I \). Combining these facts, we find that \( \tilde{u} \) is a strong solution of (2.7). \( \square \)

Remark 2.4. A similar extension property is valid in half balls and half annuli under mixed boundary conditions. More precisely, let \( B_+ := \{ x \in \overline{B} : x_N > 0 \} \), \( I \subset \mathbb{R} \) be an open interval, let \( \mu, g : B_+ \times I \to \mathbb{R} \) given functions and let \( u \in C^{2,1}(\overline{B_+} \times I) \cap C(\overline{B_+} \times I) \) be a solution of

\[
\begin{align*}
    u_t - \mu(x,t)\Delta u &= g(x,t) \quad \text{in } B^2_+ \times I,
\end{align*}
\]
satisfying \( u = 0 \) on \( \Sigma_1 \times I \) and \( \partial_\nu u = 0 \) on \( \Sigma_2 \times I \), where

\[
\begin{align*}
    \Sigma_1 := \{ x \in \partial B_+ : x_N = 0 \}, \quad \Sigma_2 := \{ x \in \partial B_+ : x_N > 0 \}. \quad (2.8)
\end{align*}
\]

Let \( \tilde{B}_+ := \{ x \in \tilde{B} : x_N > 0 \} \) and define \( \tilde{u} : \tilde{B}_+ \to \mathbb{R} \) by (2.6) for \( x \in \tilde{B}_+ \). Then \( \tilde{u} \in W^{2,1}_{p,\text{loc}}(\tilde{B}_+ \times I) \cap C(\tilde{B}_+ \times I) \) for any \( p > N + 2 \) and it is a strong solution of (2.7) in \( \tilde{B}_+ \) with coefficients defined analogously as in Lemma 2.3.

The final preliminary tool we need is a geometric characterization of a set of foliated Schwarz symmetric functions. We first recall the following result from [21] Proposition 3.2.

Proposition 2.5. Let \( \mathcal{U} \) be a set of continuous functions defined on a radial domain \( B \subset \mathbb{R}^N \), \( N \geq 2 \), and suppose that there exists

\[
\tilde{e} \in \mathcal{M}_\mathcal{U} := \{ e \in S^{N-1} \mid z(x) \geq z(\sigma_e(x)) \text{ for all } x \in B(e) \text{ and } z \in \mathcal{U} \}. \quad (2.9)
\]

If for all two dimensional subspaces \( P \subset \mathbb{R}^N \) containing \( \tilde{e} \) there are two different points \( p_1, p_2 \) in the same connected component of \( \mathcal{M}_\mathcal{U} \cap P \) such that \( z \equiv z \circ \sigma_{p_1} \) and \( z \equiv z \circ \sigma_{p_2} \) for every \( z \in \mathcal{U} \), then there is \( p \in S^{N-1} \) such that every \( z \in \mathcal{U} \) is foliated Schwarz symmetric with respect to \( p \).

Instead of applying this Proposition directly, we will rather use the following corollary.

Corollary 2.6. Let \( \mathcal{U} \) be a set of continuous functions defined on a radial domain \( B \subset \mathbb{R}^N \), \( N \geq 2 \), and suppose that the set \( \mathcal{M}_\mathcal{U} \) defined in (2.6) contains a nonempty subset \( \mathcal{N} \) with the following properties:

(i) \( \mathcal{N} \) is relatively open in \( S^{N-1} \);

(ii) For every \( e \in \partial \mathcal{N} \) and \( z \in \mathcal{U} \) we have \( z \leq z \circ \sigma_e \) in \( B(e) \). Here \( \partial \mathcal{N} \) denotes the relative boundary of \( \mathcal{N} \) in \( S^{N-1} \).

Then there is \( p \in S^{N-1} \) such that every \( z \in \mathcal{U} \) is foliated Schwarz symmetric with respect to \( p \).
Proof. By assumption, there exists \( \tilde{e} \in \mathcal{N} \subset \mathcal{M}_U \). Let \( P \subseteq \mathbb{R}^N \) be a two-dimensional subspace containing \( \tilde{e} \), and let \( L \) denote the connected component of \( \mathcal{N} \cap P \) containing \( \tilde{e} \). Since \( \mathcal{M}_U \) is closed, \( L \) is a subset of the connected component of \( \mathcal{M}_U \cap P \) containing \( \tilde{e} \). By Proposition 2.5, it suffices to show that there are different points \( p_1, p_2 \in L \) such that \( z \equiv z \circ \sigma_{p_1} \) and \( z \equiv z \circ \sigma_{p_2} \) for every \( z \in \mathcal{U} \).

We distinguish two cases. If \( L = S^{N-1} \cap P \), then we have \( z \equiv z \circ \sigma_p \) in \( B \) for every \( p \in L, z \in \mathcal{U} \) by the definition of \( \mathcal{M}_U \) and since \( L \subset \mathcal{M}_U \).

If \( L \neq S^{N-1} \cap P \), then there exists two different points \( p_1, p_2 \) in the relative boundary of \( L \) in \( S^{N-1} \cap P \). Since \( \mathcal{N} \) is relatively open in \( S^{N-1} \), these points are contained in \( \partial \mathcal{N} \subset \mathcal{M}_U \), and by assumption and the definition of \( \mathcal{M}_U \) we have \( z \equiv z \circ \sigma_{p_1} \) and \( z \equiv z \circ \sigma_{p_2} \) in \( B \) for every \( z \in \mathcal{U} \), as required. \( \square \)

3 A Harnack-Hopf type lemma and related estimates

The first result of this section is an estimate related to a linear parabolic boundary value problem on a (parabolic) half cylinder. The estimate can be seen as an extension of both the Harnack inequality and the Hopf lemma since it also gives information on a “tangential” derivative at corner points. A somewhat related (but significantly weaker) result for supersolutions of the Laplace equation was given in [10, Lemma A.1].

Lemma 3.1. Let \( a, b \in \mathbb{R}, a < b \), \( I := (a, b), B_+ := \{ x \in \overline{B} : x_N > 0 \} \). Suppose that \( v \in C^{2,1}(B_+ \times I) \cap C(B_+ \times I) \) satisfies

\[
vt - \mu \Delta v - cv \geq 0 \quad \text{in } B_+^0 \times I, \\
\frac{\partial v}{\partial \nu} = 0 \quad \text{on } \Sigma_2 \times I, \\
v = 0 \quad \text{on } \Sigma_1 \times I, \\
v(x, a) \geq 0 \quad \text{for } x \in B_+, 
\]

where the sets \( \Sigma_i \) are given in (2.8) and the coefficients satisfy

\[
\frac{1}{M} \leq \mu(x, t) \leq M \quad \text{and} \quad |c(x, t)| \leq M \quad \text{for } (x, t) \in B_+ \times I
\]

with some positive constant \( M > 0 \). Then \( v \geq 0 \) in \( B_+ \times (a, b) \). Moreover, if \( v(\cdot, a) \neq 0 \) in \( B_+ \), then

\[
v > 0 \text{ in } B_+ \times I \quad \text{and} \quad \frac{\partial v}{\partial e_N} > 0 \text{ on } \Sigma_1 \times I. \tag{3.1}
\]

Furthermore, for every \( \delta_1 > 0, \delta_2 \in (0, \frac{b-a}{4}] \), there exist \( \kappa > 0 \) and \( p > 0 \) depending only on \( \delta_1, \delta_2, B \) and \( M \) such that

\[
v(x, t) \geq \kappa \left( \int_{Q_{(\delta_1, \delta_2)}} v^p \, dx \, dt \right)^{\frac{1}{p}} \quad \text{for every } x \in B_+, t \in [a + 3\delta_2, a + 4\delta_2], \tag{3.2}
\]
with \( Q(\delta_1, \delta_2) := \{(x, t) : x \in B_+, x_N \geq \delta_1, a + \delta_2 \leq t \leq a + 2\delta_2\} \).

Proof. We begin by showing that by Remark 2.4, there exists an extension \( \tilde{v} \) with \( \text{Remark 2.4. Without loss, we may assume that \( \tilde{v} \) satisfies} \)

\[
\varphi_t - \mu \Delta \varphi - (c - 2M) \varphi \geq \varepsilon(2M - c) \geq \varepsilon M, \quad x \in B_+^*, t \in I,
\]

\[
\varphi_t - \mu \Delta \varphi - (c - 2M) \varphi \geq \varepsilon(2M - c) \geq \varepsilon M, \quad x \in B_+^*, t \in I,
\]

\[
\varphi(x, t) = \varepsilon, \quad x \in \Sigma_1, t \in I,
\]

\[
\varphi(x, a) \geq \varepsilon, \quad x \in B_+.
\]

Suppose by contradiction that \( \bar{t} := \sup\{t \in [a, b] : \varphi > 0 \text{ in } B_+ \times [a, t]\} < b \). By continuity, we have \( \bar{t} > a, \varphi(\cdot, \bar{t}) \geq 0 \text{ in } B_+ \) and \( \varphi(\bar{x}, \bar{t}) = 0 \) for some \( \bar{x} \in B_+ \). As a consequence of the Neumann boundary conditions on \( \Sigma \times I \) and the boundary point lemma (see for example [14, Lemma 2.8]), we find that \( \bar{x} \in B_+^\circ \). But then

\[
0 \geq \phi_t(\bar{x}, \bar{t}) - \mu \Delta \varphi(\bar{x}, \bar{t}) - (c - 2M) \varphi(\bar{x}, \bar{t}) \geq \varepsilon M > 0
\]

a contradiction. Therefore \( \varphi > 0 \text{ in } B_+ \times I \). Since \( \varepsilon > 0 \) was chosen arbitrarily, we conclude that \( v \geq 0 \text{ in } B_+ \times I \). Then the first claim in (3.1) follows by the strong maximum principle and the boundary point lemma (see e.g. [14, Theorem 2.7 and Lemma 2.8]).

Next we note that the second claim in (3.1) is a consequence of the first claim and the inequality (3.2) (for suitably chosen \( \delta_1, \delta_2 \)). It thus remains to prove (3.2). Let \( \delta_1 > 0, \delta_2 \in (0, \frac{\delta_1}{2}) \) and consider \( B, B_+ \) as defined in (2.5) and Remark 2.4. Without loss, we may assume that

\[
\delta_1 < \min\left\{ \frac{\delta_2}{2}, \frac{\text{dist}(B, \partial B)}{3} \right\}.
\]

By Remark 2.4, there exists an extension \( \tilde{v} \in W^{2,1}_{N+1,loc}(\bar{B}_+ \times I) \) of \( v \) which satisfies \( \mathcal{L}(t, x)\tilde{v} \geq 0 \) in \( B_+ \times I \) in the strong sense. Here the linear differential operator \( \mathcal{L} \) is given by

\[
\mathcal{L}(t, x)w := w_t - \tilde{\mu}(x, t)\Delta w - \tilde{b}(x, t)\partial_r w - \tilde{c}(x, t)w
\]

with \( \tilde{\mu}, \tilde{b}, \tilde{c} \) given as in Lemma 2.3 and

\[
\tilde{c}(x, t) := \begin{cases} c(x, t), & x \in B_+, t \in I, \\ c(\hat{x}, t), & x \in B_+ \setminus B_+, t \in I. \end{cases}
\]

Moreover, there is a positive constant \( \beta_0 \) which only depends on \( B \) and \( M \) such that \( \tilde{\mu}, \tilde{b}, \text{ and } \tilde{c} \) are uniformly bounded by \( \beta_0 \), and \( \tilde{\mu} \) is bounded below by \( \beta_0^{-1} \). Next, we define the compact sets

\[
K_{\delta_1} := \{x \in B_+ : x_N \geq \frac{\delta_1}{2}\} \quad \text{and} \quad \hat{K}_{\delta_1} := \{x \in B_+ : x_N \geq \frac{\delta_1}{2}, \text{dist}(x, \partial B) \geq \delta_1\}.
\]
By the parabolic Harnack inequality given in \cite[Lemma 3.5]{17}, there exist \(\kappa_1 > 0\) and \(p > 0\), depending only on \(\delta_1, \delta_2, B\) and \(M\), such that
\[
\inf_{x \in \tilde{K}_{\delta_1}} \tilde{v}(x,t) \geq \kappa_1 \left( \int_{\tilde{K}_{\delta_1} \times [a + \delta_2, a + 2\delta_2]} \tilde{v}^p \, dx \, dt \right)^{\frac{1}{p}} \geq \kappa_1 \left( \int_{Q(\delta_1, \delta_2)} v^p \, dx \, dt \right)^{\frac{1}{p}}. \tag{3.4}
\]

Here we used in the last step that \(Q(\delta_1, \delta_2) \subset K_{\delta_1} \times [a + \delta_2, a + 2\delta_2] \subset \tilde{K}_{\delta_1} \times [a + \delta_2, a + 2\delta_2] \).

Next, we define
\[
D := \{(x,t) : t < 0, x < \frac{\delta_1}{2}, |x - \delta_1 e_N|^2 + t^2 < \delta_1^2 \};
\]
\[
\Gamma_1 := \{(x,t) : t \leq 0, x < \frac{\delta_1}{2}, |x - \delta_1 e_N|^2 + t^2 = \delta_1^2 \};
\]
\[
\Gamma_2 := \{(x,t) : t \leq 0, |x - \delta_1 e_N|^2 + t^2 \leq \delta_2^2, x_N = \frac{\delta_1}{2} \}.
\]

Note that \(\Gamma_1 \cup \Gamma_2\) equals \(\partial_p D\), the parabolic boundary of \(D\). Let \(x_0 \in \Sigma_1\) and \(t_0 \in [a + 3\delta_2, a + 4\delta_2]\). By construction and \eqref{3.3}, we then have
\[
\{(x_0 + x, t_0 + t) : (x,t) \in D\} \subset \tilde{B}_+^{\circ} \times [a + \frac{5}{2} \delta_2, a + 4\delta_2]
\]
and
\[
\{(x_0 + x, t_0 + t) : (x,t) \in \Gamma_2\} \subset \tilde{K}_{\delta_1} \times [a + \frac{5}{2} \delta_2, a + 4\delta_2]. \tag{3.5}
\]

Next we fix \(k > 0\) such that
\[
k \geq \frac{2\beta_0 [\delta_1 + \beta_0 N (1 + \delta_1)]}{\delta_1^2}.
\]

Moreover, we define the function
\[
z : \bar{D} \to \mathbb{R}, \quad z(x,t) := \left( e^{-k(|x - \delta_1 e_N|^2 + t^2)} - e^{-k\delta_1^2} \right) e^{-\beta_0 t}
\]
Let also
\[
\varepsilon := \frac{\min_{(x,t) \in \Gamma_2} \tilde{v}(x_0 + x, t_0 + t)}{\max_{(x,t) \in \Gamma_2} z(x,t)} > 0
\]
and consider
\[
w : \bar{D} \to \mathbb{R}, \quad w(x,t) := \tilde{v}(x_0 + x, t_0 + t) - \varepsilon z(x,t)
\]
Then \( w \geq 0 \) on \( \Gamma_2 \) and also \( w \geq 0 \) on \( \Gamma_1 \), since \( z \equiv 0 \) on \( \Gamma_1 \). Moreover, for \((x,t) \in D\) we have

\[
\mathcal{L}(t_0 + t, x_0 + x)z(x,t) = \left[-\beta_0 - \tilde{c}(t_0 + t, x_0 + x)\right]z(x,t) + 2k e^{-k(\beta + t_0 + t, x_0 + x)} \left[\tilde{\mu}(t_0 + t, x_0 + x)(N - 2k|x - \delta_1 e_N|)\right] \\
- t - \tilde{b}(t_0 + t, x_0 + x) \frac{x_0 + x}{|x_0 + x|} \cdot (x - \delta_1 e_N)
\]

\[
\leq 2k e^{-k(\beta + t_0 + t, x_0 + x)} \left[\delta_1 - \frac{2k}{\beta_0} \left(\frac{\delta_1}{2}\right)^2 + \beta_0 N(1 + \delta_1)\right] \leq 0,
\]

by the definition of \( k \). Therefore we have

\[
\mathcal{L}(t_0 + t, x_0 + x)w(x,t) \geq 0 \quad \text{for} \quad (x,t) \in D \quad \text{and} \quad w \geq 0 \quad \text{on} \quad \partial_x D = \Gamma_1 \cup \Gamma_2.
\]

By the maximum principle for strong solutions, we conclude that \( w \geq 0 \) in \( \overline{D} \) and thus in particular

\[
\tilde{v}(x_0 + s \epsilon e_N, t_0) \geq \epsilon z(se_N,0) \quad \text{for} \quad s \in (0, \frac{\delta_1}{2}).
\]

Since moreover

\[
z(se_N,0) = e^{-k(s - \delta_1)^2} - e^{-k\delta_1^2} \geq s \epsilon_1 \max_{(x,t) \in \Gamma_2} z(x,t) \quad \text{for} \quad s \in (0, \frac{\delta_1}{2})
\]

with a constant \( \epsilon_1 \in (0, \frac{1}{\text{diam}(B)}) \) depending only on the function \( z \) and on \( B \), it follows that

\[
\tilde{v}(x_0 + s \epsilon e_N, t_0) \geq s \epsilon_1 \epsilon \max_{(x,t) \in \Gamma_2} z(x,t) = \epsilon_1 s \min_{(x,t) \in \Gamma_2} \tilde{v}(x_0 + x, t_0 + t) \quad \text{for} \quad s \in (0, \frac{\delta_1}{2}).
\]

By (3.3) and since \( x_0 \in \Sigma_1, t_0 \in [a + 3\delta_2, a + 4\delta_2] \) were chosen arbitrarily, we conclude that

\[
v(x,t) \geq \epsilon_1 x_N \inf_{y \in K_\delta} \inf_{\tau \in [a + \frac{1}{2} \delta_2, a + 4\delta_2]} \tilde{v}(y,\tau) \quad \text{for} \quad x \in B_+ \quad \text{with} \quad x_N < \frac{\delta_1}{2}
\]

and \( t \in [a + 3\delta_2, a + 4\delta_2] \). By definition of \( K_\delta \) and since \( 0 \leq \epsilon_1 x_N \leq 1 \) for \( x \in B_+ \), the latter estimate holds also without the restriction \( x_N < \frac{\delta_1}{2} \). Combining this fact with (3.2), we obtain that

\[
v(x,t) \geq \kappa \epsilon_1 x_N \left( \int_{Q(\delta_1, \delta_2)} v^p \, dxdt \right)^{\frac{1}{p}} \quad \text{for} \quad x \in B_+ \quad \text{and} \quad t \in [a + 3\delta_2, a + 4\delta_2],
\]

so that (3.2) holds with \( \kappa := \kappa_1 \epsilon_1. \)
Next, we prove a related but weaker Hopf Lemma for a class of cooperative systems under mixed boundary conditions. The argument is essentially the same as in the scalar case, but we include it for completeness since we could not find the result in this form in the literature. We use the notation of Lemma 3.1.

**Lemma 3.2.** Let \( a, b \in \mathbb{R}, a < b, I := (a, b), J := \{1, 2, \ldots, n\} \) for some \( n \in \mathbb{N} \), and \( w = (w_1, w_2, \ldots, w_n) \) with \( w_i \in C^{2,1}(\bar{B}_+ \times I) \cap C(\bar{B} \times I) \) be a classical solution of

\[
(w_i)_t - \mu_i \Delta w_i = \sum_{j \in J} c_{ij} w_j \quad \text{in } B_+^o \times I,
\]

\[
\frac{\partial w_i}{\partial \nu} = 0 \quad \text{on } \Sigma_2 \times I,
\]

\[
w_i = 0 \quad \text{on } \Sigma_1 \times I,
\]

\[
w_i(x, \tau) \geq 0 \quad \text{for } x \in B_+,
\]

for \( i \in J \) with coefficient functions \( \mu_i, c_{ij} \in L^\infty(B_+ \times I) \). Suppose moreover that \( \inf_{B_+ \times I} \mu_i > 0 \) and \( \inf_{B_+ \times I} c_{ij} \geq 0 \) for \( i, j \in J, i \neq j \). Then

\[
w_i \geq 0 \quad \text{in } B_+ \times I \text{ for } i \in J. \tag{3.6}
\]

Moreover, if \( w_i(x, \tau) \neq 0 \) for some \( i \in J \), then

\[
w_i > 0 \text{ in } B_+ \times I \quad \text{and} \quad \frac{\partial w_i}{\partial e_N} > 0 \text{ on } \Sigma_1 \times I. \tag{3.7}
\]

**Proof.** To prove (3.6), we fix \( \lambda > \max_{i \in J} \sum_{j \in J} \|c_{ij}\|_{L^\infty(B_+ \times I)} \) and let \( \varepsilon > 0 \). We define \( v_i(x, t) := e^{-\lambda t}w_i(x, t) + \varepsilon \) for \( x \in \bar{B}_+, t \in T \) and \( i \in J \). Then

\[
(v_i)_t - \mu_i \Delta v_i - (c_{ii} - \lambda)v_i \geq \sum_{j \in J \setminus \{i\}} c_{ij} v_j \quad \text{in } B_+^o \times I,
\]

\[
\frac{\partial v_i}{\partial \nu} = 0 \quad \text{on } \Sigma_2 \times I, \quad \text{and}
\]

\[
v_i \geq \varepsilon > 0 \quad \text{on } \Sigma_1 \times I \cup B_+ \times \{a\}.
\]

As in the proof of Lemma 3.1 we show that \( v_i > 0 \) in \( B_+ \times I \) for all \( i \in J \). Suppose by contradiction that

\( \bar{t} := \sup\{t \in [a, b] : v_i > 0 \text{ in } B_+ \times [a, t] \text{ for all } i \in J\} < b \).

By continuity, we have \( \bar{t} > a \), \( v_i(\cdot, \bar{t}) \geq 0 \) in \( B_+ \) for all \( i \in J \) and \( v_j(\bar{x}, \bar{t}) = 0 \) for some \( \bar{x} \in B_+ \) and some \( j \in J \). The Neumann boundary conditions on \( \Sigma_2 \times I \) and the boundary point lemma (see for example \cite{13} Lemma 2.8) then imply that \( \bar{x} \in B_+^o \). But then

\[
0 \geq (v_j)_t(\bar{x}, \bar{t}) - \mu_i \Delta v_i(\bar{x}, \bar{t}) - (c_{ii} - \lambda)v_i(\bar{x}, \bar{t}) \geq \sum_{j \in J \setminus \{i\}} c_{ij} v_j(\bar{x}, \bar{t}) \geq 0,
\]

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a contradiction. Therefore \( v_i > 0 \) in \( B_+ \times I \) for all \( i \in J \). Since \( \varepsilon > 0 \) was chosen arbitrarily, we conclude that (3.6) holds. Consequently, the non-negativity of \( c_{ij} \) for \( i \neq j \) implies that

\[
(w_i)_t - \mu_i(x,t)\Delta w_i - c_{ii}(x,t)w_i = \sum_{j \in J \setminus \{i\}} c_{ij}w_j \geq 0 \quad \text{in } B_+ \times I \text{ for } i \in J.
\]

Hence (3.7) follows from Lemma 3.1.

For the last lemma of this section, we need to fix additional notation. For \( e \in S^{N-1} \), let \( \sigma_e : \overline{B} \to \overline{B} \) and \( B(e) \subset B \) be defined as in the introduction. We also put

\[
\Sigma_1(e) := \{ x \in \partial B(e) : x \cdot e = 0 \} \quad \text{and} \quad \Sigma_2(e) := \{ x \in \partial B(e) : x \cdot e > 0 \}. \tag{3.8}
\]

For a subset \( I \subset \mathbb{R} \) and a function \( v : \overline{B} \times I \to \mathbb{R} \), we define

\[
v^e : \overline{B} \times I \to \mathbb{R}, \quad v^e(x,t) := v(x,t) - v(\sigma_e(x),t).
\]

To implement the rotating plane technique for the boundary value problems considered in our main results, we need to analyze under which conditions positivity of \( v^e(t,\cdot) \) in \( B(e) \) at some time \( t \in I \) induces positivity of \( v^{e'}(t',\cdot) \) in \( B(e') \) for a slightly perturbed direction \( e' \) at a later time \( t' > t \). The following perturbation lemma is sufficient for our purposes.

**Lemma 3.3.** Let \( I = (0,1) \), let \( v \in C^{2,1}(\overline{B} \times I) \), and consider a function \( \chi : [0, \sqrt{1 + \text{diam}(B)^2}] \to [0, \infty) \) such that

\[
(E\chi) \begin{cases}
\lim_{\theta \to 0} \chi(\theta) = 0 \\
|v(x,t) - v(y,s)| + |\nabla v(x,t) - \nabla v(y,s)| \leq \chi(|(x,t) - (y,s)|) \\
\text{for all } (x,t), (y,s) \in \overline{B} \times I.
\end{cases}
\]

Moreover, let \( d, k, M > 0 \) be given constants. Then there exists \( \rho > 0 \), depending only on \( B, d, k, M \), and the function \( \chi \), with the following property: If \( e \in S^{N-1} \) is such that

(i) the function \( v^e \) satisfies

\[
v^e_i - \mu(x,t)\Delta v^e - c(x,t)v^e \geq 0 \quad \text{in } B(e) \times I
\]

with some coefficient functions \( \mu, c \) satisfying

\[
\frac{1}{M} \leq \mu(x,t) \leq M \quad \text{and} \quad |c(x,t)| \leq M \quad \text{for } (x,t) \in B(e) \times I,
\]

and

\[
\frac{\partial v^e}{\partial v} = 0 \text{ on } \Sigma_2(e) \times I, \quad v^e = 0 \text{ on } \Sigma_1(e) \times I, \quad v^e \geq 0 \text{ on } B(e) \times \{0\},
\]

then

\[
\begin{split}
\|v^e(t,\cdot)|_i &= \frac{1}{M} \|v(x_0,t)\|_{L^1(B(e))} + \int_0^t M |v^e_i - \mu(x,t)\Delta v^e - c(x,t)v^e| \, ds \\
&\leq \frac{1}{M} \|v(x_0,t)\|_{L^1(B(e))} + \int_0^t M |v^e_i - \mu(x,t)\Delta v^e - c(x,t)v^e| \, ds
\end{split}
\]

for all \( i \in J \) and \( t \in I \).
(ii) \( \sup \{ v^\varepsilon(x, \frac{1}{2}) : x \in B(e), x \cdot e \geq d \} \geq k, \)

then

\[ v^{e'}(\cdot, 1) > 0 \quad \text{in } B(e') \quad \text{for all } e' \in S^{N - 1} \text{ with } |e - e'| < \rho. \]

**Remark 3.4.** The result obviously remains true if \( v^\varepsilon \) is replaced by \(-v^\varepsilon\), and we will use this fact later on.

**Proof.** Let \( e \in S^{N - 1} \) be such that (i) and (ii) are satisfied, and let \( \kappa > 0 \) and \( p > 0 \) be the constants given by Lemma 3.1 applied to \( a = 0, b = 1, \delta_1 = d \) and \( \delta_2 = \frac{1}{4} \). We first note that condition \((E\chi)\) and hypothesis (ii) imply that there exists \( C_1 > 0 \), depending only on \( B, d, k, M, \) and \( \chi \), such that

\[ \kappa \left( \int_{Q^\varepsilon} (v^\varepsilon)^p \, dx \, dt \right)^{\frac{1}{p}} \geq C_1, \]

where \( Q^\varepsilon := \{(x, t) : x \in B(e), x \cdot e \geq d, \frac{1}{4} < t < \frac{1}{2} \} \). Then, by Lemma 3.1 it follows that

\[ |\nabla v^\varepsilon(x, 1)| = \nabla v^\varepsilon(x, 1) \cdot e \geq C_1 \quad \text{for all } x \in \Sigma_1(e). \]

By condition \((E\chi)\), there is some \( \rho_0 > 0 \), depending only on \( B, d, k, M, \) and \( \chi \), such that

\[ |\nabla v^\varepsilon(x, 1)| = \nabla v^\varepsilon(x, 1) \cdot e' \geq \frac{3}{4} C_1 \]

\[ \begin{cases} \text{for } e' \in S^{N - 1} \text{ with } |e - e'| < \rho_0 \text{ and } x \in \Sigma_1(e'). & (3.9) \end{cases} \]

Again by \((E\chi)\), we then find \( \rho_1 \in (0, \rho_0) \), depending only on \( B, d, k, M, \) and \( \chi \), such that

\[ \nabla v^{e'}(x, 1) \cdot e' \geq \frac{C_1}{2} \]

\[ \begin{cases} \text{for } e' \in S^{N - 1} \text{ and } x \in B \text{ with } |e - e'| < \rho_0 \text{ and } |x \cdot e'| \leq \rho_1. & (3.10) \end{cases} \]

By Lemma 3.1 there is some \( \eta_1 > 0 \) which only depends on \( B, d, k, M, \) and \( \chi \), such that

\[ v^\varepsilon(x, 1) \geq \eta_1 \quad \text{for } x \in \overline{B(e)} \text{ with } x \cdot e \geq \frac{\rho_1}{2}. \]

Again by \((E\chi)\), we may fix \( \rho \in (0, \rho_1) \), depending only on \( B, d, k, M \) and \( \chi \), such that for all \( e' \in S^{N - 1} \) with \( |e - e'| < \rho \),

\[ v^{e'}(x, 1) \geq \frac{\eta_1}{2} \quad \text{for } x \in \overline{B(e')} \text{ with } x \cdot e' \geq \frac{\rho_1}{2}. \] \( (3.11) \)

For fixed \( e' \in S^{N - 1} \) with \( |e - e'| < \rho \), \((3.11)\) ensures that

\[ v^{e'}(x, 1) = v(x, 1) - v(\sigma_{e'}(x), 1) > 0 \quad \text{for } x \in B(e') \text{ with } x \cdot e' \leq \frac{\rho_1}{2}. \]

Combining this with \((3.11)\), we find that

\[ v^{e'}(x, 1) > 0 \quad \text{for } x \in B(e'), \]

as claimed. \( \square \)
4 The scalar Neumann problem

This section is devoted to the proof of Theorem 1.2. Let \( u \in C^{2,1}(\mathbb{B} \times (0, \infty)) \cap C(\mathbb{B} \times [0, \infty)) \) be a (possibly sign changing) solution of (1.4) such that the hypothesis (H1)-(H4) and (1.5) of Theorem 1.2 are fulfilled. We first note that

\[
\begin{align*}
\frac{\partial}{\partial t} u - \mu(|x|, t) \Delta u - c(x, t) u &= f(t, |x|, 0) \quad \text{in } B \times (0, \infty), \\
\partial_{\nu} u &= 0 \quad \text{on } \partial B \times (0, \infty)
\end{align*}
\]

with

\[
c(x, t) := \begin{cases} 
\frac{f(t, |x|, u(x, t)) - f(t, |x|, 0)}{u(x, t)}, & \text{if } u(x, t) \neq 0, \\
0, & \text{if } u(x, t) = 0
\end{cases}
\]

for \( x \in \mathbb{B}, t > 0 \). By (H1) and (1.5) we have \( c \in L^\infty(\mathbb{B} \times (0, \infty)) \), and thus (H2) and Lemma 2.1 imply that the functions

\[
B \times [0, 1] \rightarrow \mathbb{R}, \quad (x, t) \mapsto u(x, \tau + t), \quad \tau \geq 1
\]

and \( B \times [0, 1] \rightarrow \mathbb{R}^N, (x, t) \mapsto \nabla u(x, \tau + t), \tau \geq 1 \) are uniformly equicontinuous. Hence there exists a function \( \chi : [0, 1] \rightarrow [0, \infty) \) with \( \lim_{\vartheta \rightarrow 0} \chi(\vartheta) = 0 \) and such that \((E\chi)\) of Lemma 3.3 holds for all of the functions in (4.1). Next, we set

\[
u^e(x, t) := u(x, t) - u(\sigma_e(x), t) \quad \text{for } x \in \mathbb{B}, t > 0, \text{ and } e \in \mathbb{S}^{N-1}.
\]

We wish to apply Corollary 2.6 to the sets \( \mathcal{U} := \omega(u) \) and

\[
\mathcal{N} := \{ e \in \mathbb{S}^{N-1} | \exists \ T > 0 \text{ such that } \nu^e(x, t) > 0 \text{ for all } x \in B(e), t > T \}
\]

With \( \mathcal{M}_U \) defined as in (2.9), it is obvious that \( \mathcal{N} \subset \mathcal{M}_U \). We note that the function \( \nu^e \) satisfies

\[
\begin{align*}
\frac{\partial}{\partial t} \nu^e - \mu(|x|, t) \Delta \nu^e &= c^e(x, t) \nu^e \quad \text{in } B(e) \times (0, \infty), \\
\frac{\partial \nu^e}{\partial \nu} &= 0 \quad \text{on } \Sigma_2(e) \times (0, \infty), \\
\nu^e &= 0 \quad \text{on } \Sigma_1(e) \times (0, \infty),
\end{align*}
\]

with \( \Sigma_i(e) \) as defined in (3.8) and

\[
c^e(x, t) := \begin{cases} 
f(t, |x|, u(x, t)) - f(t, |x|, u(\sigma_e(x), t)), & \text{if } \nu^e(x, t) \neq 0, \\
0, & \text{if } \nu^e(x, t) = 0
\end{cases}
\]

By (H1), there exists \( M > 0 \) with

\[
\|c^e\|_{L^\infty(B \times (0, \infty))} \leq M \quad \text{for all } e \in \mathbb{S}^{N-1}.
\]
Moreover, by making $M$ larger if necessary and using (H3), we may also assume that
\[
\frac{1}{M} \leq \mu(|x|, t) \leq M \quad \text{for all } x \in B, \ t > 0.
\]
By (H4), there exists $\hat{e} \in S^{N-1}$ such that $u^e(\cdot, 0) \geq 0$, $u^e(\cdot, 0) \neq 0$ on $B(\hat{e})$ and thus $u^e(x, t) > 0$ in $B(\hat{e}) \times (0, \infty)$ by Lemma 3.1 so that $\hat{e} \in \mathcal{N}$. Moreover, it easily follows from Lemmas 3.1 and 3.3 that $\mathcal{N}$ is a relatively open subset of $S^{N-1}$. By Corollary 2.6, it therefore only remains to prove that $z \leq z \circ \sigma_e$ in $B(\hat{e})$ for every $z \in \omega(u)$ and $e \in \partial \mathcal{N}$. We argue by contradiction. Assume there is $\hat{e} \in \partial \mathcal{N}$ and $z \in \omega(u)$ such that $z \not\leq z \circ \sigma_{\hat{e}}$ in $B(\hat{e})$. Define
\[
z^e : \overline{B} \to \mathbb{R} \quad \text{by} \quad z^e(x) := z(x) - z(\sigma_e(x))
\]
for $e \in S^{N-1}$. Then there exist constants $d, k > 0$ such that
\[
\sup\{z^e(x) : x \in B, x \cdot \hat{e} \geq d\} > k
\]
We now let $\rho > 0$ be given by Lemma 3.3 corresponding to the choices of $d, k, M$ and $\chi$ made above. By continuity and since $\hat{e} \in \partial \mathcal{N}$, there exists $\tilde{e} \in \mathcal{N}$ such that $|e - \tilde{e}| < \rho$ (4.2) and
\[
\sup\{z^e(x) : x \in B, x \cdot e \geq d\} > k \quad (4.3)
\]
Let $(t_n) \subset (0, \infty)$ be a sequence with $t_n \to \infty$ and $u(t_n, \cdot) \to z$ in $L^\infty(\overline{B})$. By (4.3), there exists $n_0 \in \mathbb{N}$ such that
\[
\sup\{u^e(t_n, x) : x \in B, x \cdot e \geq d\} > k \quad \text{for all } n \geq n_0.
\]
Moreover, by the definition of $\mathcal{N}$ there exists $T > 0$ such that $u^e(\cdot, t) > 0$ in $B(e)$ for $t \geq T$. Next, fixing $n \in \mathbb{N}$ such that $t_n \geq \max\{T + \frac{1}{4}, t_{n_0}\}$ and applying Lemma 3.3 to the function
\[
\overline{B} \times [0, 1] \to \mathbb{R}, \quad (x, t) \mapsto u(x, t_n - \frac{1}{4} + t),
\]
we find, using (4.2), that $u^e(x, t_n + \frac{3}{4}) > 0$ for all $x \in B(\hat{e})$. Hence $\hat{e} \in \mathcal{N}$. Since $\mathcal{N}$ is relatively open in $S^{N-1}$, this contradicts the fact that $\hat{e} \in \partial \mathcal{N}$. The proof of Theorem 1.2 is thus finished.

5 Proof of the main result for competitive systems

In this section we will complete the proof of Theorem 1.3. For the remainder of this section, let $u_1, u_2 \in C^{2,1}(\overline{B} \times (0, \infty)) \cap C(\overline{B} \times [0, \infty))$ be functions such that $u = (u_1, u_2)$ solves (1.6) and such that assumptions (h0)-(h3), (1.7) from
the introduction are fulfilled. A key ingredient of the proof is the following quotient estimate which compares the values of the components of $u$ at different times. Similar estimates were obtained by J. Hůška, P. Poláčik, and M. V. Safonov in [12, Corollary 3.10] for positive solutions of scalar parabolic Dirichlet problems. We point out that the Neumann boundary conditions on $\partial B$ allow to obtain a stronger result in the present setting with a much simpler proof. In the following, for matters of simplicity, we sometimes omit the arguments $(x, t)$ and $(|x|, t)$.

**Lemma 5.1.** There exists a constant $\eta > 1$ such that

$$\frac{1}{\eta} \leq \frac{u_i}{\|u_i(\cdot, \tau)\|_{L^\infty(B)}} \leq \eta \text{ in } B \times [\tau - 3, \tau + 3]$$

for all $\tau \geq 5$ and $i = 1, 2$.

**Proof.** We only prove the estimate for $i = 1$, the proof for $i = 2$ is the same. For simplicity, we write $u$ in place of $u_1$, and we note that

$$u_t - \mu_1 \Delta u = cu \text{ in } B \times (0, \infty)$$

with

$$c(x, t) := \alpha_1(t, |x|) u_2(x, t) + \begin{cases} f_1(t, |x|, u(x, t)) \div u(x, t), & \text{if } u(x, t) \neq 0, \\ 0, & \text{if } u(x, t) = 0. \end{cases}$$

By (h1), (h3), and (1.7), we have that $c \in L^\infty(B \times (0, \infty))$. Let $\tilde{u}$ denote the extension of $u$ to $\tilde{B}$ as defined in (2.6). Then Lemma 2.3 implies that $\tilde{u}$ is a strong solution of

$$(\tilde{u})_t - \tilde{\mu} \Delta \tilde{u} - \tilde{b} \partial_r \tilde{u} = \tilde{c} \tilde{u} \text{ in } \tilde{B} \times (0, \infty).$$

Here $\tilde{\mu}, \tilde{b} \in L^\infty(\tilde{B} \times (0, \infty))$ are defined as in Lemma 2.3 with $\mu$ replaced by $\mu_1$, and $\tilde{c} \in L^\infty(\tilde{B} \times (0, \infty))$ is defined by

$$\tilde{c}(x, t) := \begin{cases} c(x, t), & x \in B, t \in (0, \infty), \\ \tilde{c}(\tilde{x}, t), & x \in \tilde{B} \setminus B, t \in (0, \infty). \end{cases}$$

We also note that $\inf_{B \times (0, \infty)} \tilde{\mu} > 0$ as a consequence of (h2). Next, we fix $\tau \geq 5$, and we apply the Harnack inequality for strong solutions given in [17, Lemma 3.5] (with $p = \infty$, $U = \tilde{B}$, $D = B$, and $v = \tilde{u}$). The application yields $\kappa_1 > 0$ independent of $\tau$ such that

$$\inf_{B \times (\tau - 3, \tau + 3)} u \geq \kappa_1 \|u(\cdot, \tau - 4)\|_{L^\infty(B)}, \quad (5.1)$$

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Let \( x \) (see for example [14, Lemma 7.1]) and the uniform bounds on the coefficients, there exists \( \kappa_2 > \kappa_1 \) independent of \( \tau \) such that
\[
\| u(\cdot, s) \|_{L^\infty(B)} \leq \kappa_2 \| u(\cdot, \tau - 4) \|_{L^\infty(B)} \quad \text{for } s \in [\tau - 3, \tau + 3].
\]
Let \( x \in B \) and \( t \in [\tau - 3, \tau + 3] \). Then, by (5.1) and (5.2),
\[
\frac{u(x, t)}{\| u(\cdot, \tau) \|_{L^\infty(B)}} \geq \frac{\kappa_1}{\kappa_2} \| u(\cdot, \tau - 4) \|_{L^\infty(B)} \geq \frac{\kappa_1}{\kappa_2},
\]
and
\[
\frac{u(x, t)}{\| u(\cdot, \tau) \|_{L^\infty(B)}} \leq \frac{\kappa_2}{\kappa_1} \| u(\cdot, \tau - 4) \|_{L^\infty(B)} \leq \frac{\kappa_2}{\kappa_1}.
\]
Thus the claim follows with \( \eta = \frac{\kappa_2}{\kappa_1} \).

Next, we slightly change some notation used in previous sections in order to deal with competitive systems of two equations. For \( e \in S^{N-1} \), a radial domain \( B \subset \mathbb{R}^N, I \subset \mathbb{R} \) and a pair \( v = (v_1, v_2) \) of functions \( v_i : \overline{B} \times I \rightarrow \mathbb{R}, i = 1, 2 \), we set
\[
\begin{align*}
\hat{u}_1^e(x, t) &:= v_1(x, t) - v_1(\sigma_e(x), t), \quad x \in \overline{B}, \ t > 0, \\
\hat{u}_2^e(x, t) &:= v_2(\sigma_e(x), t) - v_2(x, t), \quad x \in \overline{B}, \ t > 0,
\end{align*}
\]
(5.3)
The same notation is used if the functions do not depend on time. More precisely, for a pair \( z = (z_1, z_2) \) of functions \( z_i : \overline{B} \rightarrow \mathbb{R}, i = 1, 2 \), we set
\[
\begin{align*}
\hat{z}_1^e(x) &:= z_1(x) - z_1(\sigma_e(x)), \quad x \in \overline{B}, \\
\hat{z}_2^e(x) &:= z_2(\sigma_e(x)) - z_2(x), \quad x \in \overline{B}.
\end{align*}
\]
(5.4)
Since \( u = (u_1, u_2) \) solves (1.6), for fixed \( e \in S^{N-1} \) we have
\[
\begin{align*}
(u_1^e)_t - \mu_1 \Delta u_1^e - \hat{c}_1^e(x, t) u_1^e &= \alpha_1 [\hat{u}_1^e - u_1^e], \\
(u_2^e)_t - \mu_2 \Delta u_2^e - \hat{c}_2^e(x, t) u_2^e &= \alpha_2 [\hat{u}_2^e - u_2^e],
\end{align*}
\]
in \( B \times (0, \infty) \) with \( \hat{u}_i(x, t) := u_i(\sigma_e(x), t) \) and
\[
\hat{c}_i^e(x, t) := \begin{cases} 
\frac{f_i(t, |x|, u_i(x, t)) - f_i(t, |x|, u_i(\sigma_e(x), t))}{u_i(x, t) - u_i(\sigma_e(x), t)}, & \text{if } u_i^e(x, t) \neq 0, \\
0, & \text{if } u_i^e(x, t) = 0
\end{cases}
\]
for \( i = 1, 2 \). Setting
\[
\begin{align*}
\hat{c}_1^e(x, t) &= \hat{c}_1^e(x, t) - \alpha_1 |x| u_2(\sigma_e(x), t), \\
\hat{c}_2^e(x, t) &= \hat{c}_2^e(x, t) - \alpha_2 |x| u_1(\sigma_e(x), t)
\end{align*}
\]
for \( i = 1, 2 \). Setting
for $x \in B$, $t > 0$, we thus obtain the system
\[
(u_1^e)_t - \mu_1 \Delta u_1^e - c_1^e u_1^e = \alpha_1 u_1 u_2^e \quad \text{in } B(e) \times (0, \infty) \tag{5.5}
\]
\[
(u_2^e)_t - \mu_2 \Delta u_2^e - c_2^e u_2^e = \alpha_2 u_2 u_1^e \quad \text{in } B(e) \times (0, \infty)
\]
together with the boundary conditions
\[
\frac{\partial u_i^e}{\partial \nu} = 0 \quad \text{on } \Sigma_2(e) \times (0, \infty), \quad u_i^e = 0 \quad \text{on } \Sigma_1(e) \times (0, \infty), \tag{5.6}
\]
where the sets $\Sigma_i(e)$ are given as in (3.8) for $i = 1, 2$. As a consequence of (h1),(h3), and (1.7), we have
\[
\|c_i^e\|_{L^\infty(B \times (0, \infty))} \leq M \quad \text{and} \quad \|c_2^e\|_{L^\infty(B \times (0, \infty))} \leq M \quad \text{for all } e \in S^{N-1}
\]
with some constant $M > 0$. Moreover, by making $M$ larger if necessary and using (h2), we may also assume that
\[
\frac{1}{M} \leq \mu_i(|x|, t) \leq M \quad \text{for } x \in B, t > 0, \text{ and } i = 1, 2. \tag{5.8}
\]
We note that, by (h3) and since $u_1, u_2 \geq 0$ in $B \times (0, \infty)$, system (5.5) is a (weakly coupled) cooperative parabolic system. For these systems a variety of estimates are available (see for example [18] and [9]). In particular, Lemma 3.2 can be applied to study the boundary value problem (5.5), (5.6).

To prove Theorem 1.3 we wish to apply Corollary 2.6 to the sets
\[
\mathcal{U} := \omega(u_1) \cup -\omega(u_2) = \{z_1, -z_2 : z \in \omega(u)\} \tag{5.9}
\]
and
\[
\mathcal{N} := \{e \in S^{N-1} : \exists T > 0 \text{ s.t. } u_i^e > 0 \text{ in } B(e) \times [T, \infty) \text{ for } i = 1, 2\}. \tag{5.10}
\]
Note that the equality in (5.9) is a consequence of (2.4). In this case the associated set $\mathcal{M}_U$, defined in (2.4), can also be written as
\[
\mathcal{M}_U = \{e \in S^{N-1} : z_i^e \geq 0 \text{ in } B(e) \text{ for all } z \in \omega(u), i = 1, 2\}.
\]
Thus we obviously have $\mathcal{N} \subset \mathcal{M}_U$. Moreover, for $e \in S^{N-1}$ as in (h0), we have
\[
u_i^e(\cdot, 0) \geq 0, \quad u_i^e(\cdot, 0) \equiv 0 \quad \text{in } B(e) \text{ for } i = 1, 2.
\]
Lemma 3.2 then implies that $u_i^e > 0$ in $B(e) \times (0, \infty)$ for $i = 1, 2$, so that $e \in \mathcal{N}$ and thus $\mathcal{N}$ is nonempty. We also note the following.

**Lemma 5.2.** $\mathcal{N}$ is relatively open in $S^{N-1}$.

**Proof.** Let $e \in \mathcal{N}$. Then $(u_1^e, u_2^e)$ is a solution of (5.5), and there is $T > 0$ such that $u_1^e$ and $u_2^e$ are positive in $B(e) \times (T, \infty)$. Thus
\[
(u_1^e)_t - \mu_1 \Delta u_1^e - c_1^e u_1^e = \alpha_1 u_1 u_2^e \geq 0, \quad x \in B(e), \ t > T,
\]
\[
(u_2^e)_t - \mu_2 \Delta u_2^e - c_2^e u_2^e = \alpha_2 u_2 u_1^e \geq 0, \quad x \in B(e), \ t > T,
\]

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since \(\alpha_1\) and \(\alpha_2\) are non-negative by hypothesis (h3). Applying Lemma 5.3 and Remark 6.1 to the functions
\[
\mathcal{B} \times [0, 1] \to \mathbb{R}, \quad (x, t) \mapsto u_i(x, T + t), \quad i = 1, 2,
\]
we find that there exists \(\rho > 0\) such that \(u_i^\prime(\cdot, T + 1) \in B(e') \) for \(e' \in S^{N-1}\) with \(|e' - e| < \rho\). Hence, by Lemma 5.2 \(e' \in N\) for \(e' \in S^{N-1}\) with \(|e' - e| < \rho\), and thus \(N\) is open. \(\square\)

In order to apply Corollary 2.6, it now suffices to prove the following.

**Lemma 5.3.** For every \(e \in \partial N\) and every \(z \in \omega(u)\) we have \(z_1^e \equiv z_2^e \equiv 0\) in \(B(e)\).

**Proof.** Let \(z = (z_1, z_2) \in \omega(u)\), and consider an increasing sequence \(t_n \to \infty\) with \(t_1 > 5\) and such that \(u_i(\cdot, t_n) \to z_i\) uniformly in \(\mathcal{B}\) for \(i = 1, 2\). We will only show that \(z_2^e \equiv 0\) in \(B(e)\) for all \(e \in \partial N\), since the same argument shows that \(z_1^e \equiv 0\) in \(B(e)\) for all \(e \in \partial N\). Since, as noted in Remark 2.2 \(u_2\) and its first derivatives satisfy the Hölder condition \((E\chi)\) of Lemma 5.3 holds for all of the functions
\[
\mathcal{B} \times [0, 1] \to \mathbb{R}, \quad (x, t) \mapsto u_2(x, \tau + t), \quad \tau \geq 1. \quad (5.11)
\]

Arguing by contradiction, we now assume that \(z_2^e \not\equiv 0\) in \(B(\hat{e})\) for some \(\hat{e} \in \partial N\). By the equicontinuity of the functions in \((5.11)\), there are \(\zeta \in (0, \frac{1}{2})\), a nonempty open subset \(\Omega \subset B(\hat{e})\), and \(k_1 > 0\) such that, after passing to a subsequence,

\[
u_2^e \geq k_1 \quad \text{on} \quad \Omega \times [t_n - \zeta, t_n + \zeta] \quad \text{for all} \quad n \in \mathbb{N}. \quad (5.12)
\]

We now apply a normalization procedure for \(u_1\), since we cannot exclude the possibility that \(u_1(\cdot, t_n) \to 0\) as \(n \to \infty\). Define, for \(n \in \mathbb{N}\),
\[
I_n := [t_n - 2, t_n + 2] \subset \mathbb{R}, \quad \beta_n := \|u_1(\cdot, t_n)\|_{L^\infty(B)}/\beta_n
\]
and the functions
\[
v_n : \mathcal{B} \times I_n \to \mathbb{R}, \quad v_n(x, t) = \frac{u_1(x, t)}{\beta_n}.
\]

By Lemma 5.3 there exists \(\eta > 1\) such that
\[
\frac{1}{\eta} \leq v_n \leq \eta \quad \text{in} \quad B \times I_n \quad \text{for all} \quad n \in \mathbb{N}. \quad (5.13)
\]

Moreover, we have
\[
\sup_{x, \bar{x} \in \mathcal{B}, t, \bar{t} \in [s, s+1], x \not\equiv \bar{x}, t \not\equiv \bar{t}, s \in [-1, 1]} \frac{|v_n(x, t_n + t) - v_n(\bar{x}, t_n + \bar{t})|}{|x - \bar{x}|^\gamma + |t - \bar{t}|^\gamma} < K, \quad (5.14)
\]
and

\[
\sup_{x, \bar{x} \in \mathcal{B}, t, \tilde{t} \in [s, s+1], x \neq \bar{x}, t \neq \tilde{t}, s \in [-1,1]} \frac{|\nabla v_n(x, t_n + t) - \nabla v_n(\bar{x}, t_n + \tilde{t})|}{|x - \bar{x}|^\gamma + |t - \tilde{t}|^\gamma} < K
\]

for all \( n \in \mathbb{N} \) with positive constants \( \gamma \) and \( K \). This follows from Lemma 2.1 and the fact that \( v_n \) satisfies

\[
(v_n)_t - \mu_1 \Delta v_n = cv_n - \alpha_1 v_n u_2 \quad \text{in } B \times I_n, \\
\partial_\nu v_n = 0 \quad \text{on } \partial B \times I_n
\]

with

\[
c \in L^\infty(B \times (0, \infty)), \quad c(x, t) := \begin{cases} \frac{f_1(t, |x|, u_1(t, x))}{u_1(t, x)}, & \text{if } u_1(t, x) \neq 0, \\ 0, & \text{if } u_1(t, x) = 0. \end{cases}
\]

As a consequence, by adjusting the function \( \chi \) above, we may also assume that all of the functions

\[
B \times [0, 1] \to \mathbb{R}, \ (x, t) \mapsto v_n(x, \tau + t), \ |t_n - \tau| \leq 1 \text{ for some } n \in \mathbb{N}
\]
satisfy the equicontinuity condition \((E \chi)\) of Lemma 3.3. For \( \epsilon \in \mathbb{S}^{N-1} \), \( n \in \mathbb{N} \) we also consider

\[
v_n^\epsilon : B(\epsilon) \times I_n \to \mathbb{R}, \quad v_n^\epsilon(x, t) := v_n(x, t) - v_n(\sigma_\epsilon(x), t),
\]

and we note that

\[
\begin{align*}
(v_n^\epsilon)_t - \mu_1 \Delta v_n^\epsilon - c_1^\epsilon v_n^\epsilon &= \alpha_1 v_n^\epsilon u_2^\epsilon \quad \text{in } B(\epsilon) \times I_n, \\
(u_2^\epsilon)_t - \mu_2 \Delta u_2^\epsilon - c_2^\epsilon u_2^\epsilon &= \alpha_2 \beta_n u_2 v_n^\epsilon \quad \text{in } B(\epsilon) \times I_n, \\
\partial_\nu v_n^\epsilon &= \partial_\nu u_2^\epsilon = 0 \quad \text{on } \Sigma_2(\epsilon) \times I_n, \\
v_n^\epsilon(x, t) &= u_2^\epsilon(x, t) = 0 \quad \text{on } \Sigma_1(\epsilon) \times I_n
\end{align*}
\]

with \( \Sigma_\iota(\epsilon) \) as defined in 3.3. We now distinguish two cases.

**Case 1:** \( \limsup_{n \to \infty} \|v_n^\epsilon\|_{L^\infty(B(\epsilon) \times |t_n - \zeta, t_n + \zeta|)} > 0. \)

In this case, by (5.14), there are \( d \in (0, 1), k_2 > 0 \), and \( t^* \in [-\zeta, \zeta] \) such that, after passing to a subsequence,

\[
\sup\{v_n^\epsilon(x, t_n + t^*) : x \in B(\epsilon), x \cdot \hat{\epsilon} \geq d\} \geq k_2 \quad \text{for } n \in \mathbb{N}.
\]

Without loss, we may assume that \( d < \min\{x \cdot \hat{\epsilon} : x \in \Omega\} \), so that also

\[
\sup\{u_2^\epsilon(x, t_n + t^*) : x \in B(\epsilon), x \cdot \hat{\epsilon} \geq d\} \geq k_1 \quad \text{for } n \in \mathbb{N}
\]
In this case we fix a nonnegative function $\bar{e}$ by Lemma 3.3 for $M$ satisfying (5.14), (5.8) and $d$, $k$, $\chi$ as chosen above. Since $\hat{e} \in \partial \mathcal{N}$, there exists $e \in \mathcal{N}$ such that $|e - \hat{e}| < \frac{3}{4}$ and, by equicontinuity,

$$
\begin{align*}
\sup \{ u_n^e(x, t_n + t^*) : x \in B(e), x \cdot e \geq d \} &\geq k, \\
\sup \{ u_n^e(x, t_n + t^*) : x \in B(e), x \cdot e \geq d \} &\geq k
\end{align*}
$$

for all $n \in \mathbb{N}$. Since $e \in \mathcal{N}$ we can fix $n \in \mathbb{N}$ such that

$$
v_n^e(x, t_n + t^* - \frac{1}{4}) \geq 0, \quad u_n^e(x, t_n + t^* - \frac{1}{4}) \geq 0 \quad \text{for all } x \in B(e).
$$

Then applying Lemma 3.3 to the functions

$$
\mathbb{B} \times [0, 1] \to \mathbb{R}, \quad (x, t) \mapsto u_2(x, t_n + t^* - \frac{1}{4} + t), \quad (x, t) \mapsto v_n(x, t_n + t^* - \frac{1}{4} + t),
$$

we conclude that

$$
u_n^e(\cdot, t_n + t^* + \frac{3}{4}) > 0 \quad \text{and} \quad v_n^e(\cdot, t_n + t^* + \frac{3}{4}) > 0 \quad \text{in } B(\bar{e})
$$

for all $\bar{e} \in \mathbb{S}^{N-1}$ with $|\bar{e} - e| < \rho$, and thus in particular for $\bar{e} = \hat{e}$. This yields $u_i^e(\cdot, t_n + t^* + \frac{3}{4}) > 0$ in $B(\bar{e})$ for $i = 1, 2$, and thus $\hat{e} \in \mathcal{N}$ by Lemma 5.2. Since $\mathcal{N} \subset \mathbb{S}^{N-1}$ is relatively open by Lemma 5.2, this contradicts the hypothesis that $\hat{\hat{e}} \in \partial \mathcal{N}$.

\textbf{Case 2:} \quad $\lim_{n \to \infty} \|v_n^e\|_{L^\infty(B(\bar{e}) \times [t_n - \zeta, t_n + \zeta])} = 0$. \quad (5.16)

In this case we fix a nonnegative function $\varphi \in C_c^\infty(B(\hat{e}) \times (-\zeta, \zeta))$ with $\varphi \equiv 1$ on $\Omega \times (-\frac{1}{2}, \frac{1}{2})$. Moreover, let $\Omega_n := B(\hat{e}) \times (t_n - \zeta, t_n + \zeta)$ and $\varphi_n \in C_c^\infty(\Omega_n)$, $\varphi_n(x, t) := \varphi(x, t_n + t)$. Setting $(u_n^\varphi)^+ := \max\{u_n^\varphi, 0\}$ and $(u_n^\varphi)^- := -\min\{-u_n^\varphi, 0\}$, we find by (h3), (5.12) and (5.13) that

$$
A_n := \int_{\Omega_n} \alpha_1 v_n u_n^\varphi \varphi_n d(x, t) = \int_{\Omega_n} \alpha_1 v_n [(u_n^\varphi)^+ - (u_n^\varphi)^-] \varphi_n d(x, t)
$$

\begin{align*}
&\geq \frac{\alpha_s}{\eta} \int_{\Omega_n} (u_n^\varphi)^+ \varphi_n d(x, t) - \alpha^s \eta \| (u_n^\varphi)^- \|_{L^\infty(\Omega_n)} \| \varphi_n \|_{L^1(\Omega_n)}, \\
&\geq \frac{\alpha_s}{\eta} \int_{\Omega_n} (u_n^\varphi)^+ \varphi_n d(x, t) - \alpha^s \eta \| (u_n^\varphi)^- \|_{L^\infty(\Omega_n)} \| \varphi_n \|_{L^1(\Omega_n)},
\end{align*}

for $n \in \mathbb{N}$, whereas $\lim_{n \to \infty} \| (u_n^\varphi)^- \|_{L^\infty(\Omega_n)} = 0$ since $\hat{e} \in \partial \mathcal{N}$. Hence $\lim_{n \to \infty} A_n > 0$. On the other hand, integrating by parts, we have by (5.15) that

$$
A_n = \int_{\Omega_n} [(v_n^\varphi)_\varphi - \mu_1 \Delta v_n^\varphi - c_1 v_n^\varphi] \varphi_n d(x, t)
$$

\begin{align*}
&= -\int_{\Omega_n} [v_n^\varphi(\varphi_n)_\varphi + v_n^\varphi \Delta (\mu_1 \varphi_n) + c_1 v_n^\varphi \varphi_n] d(x, t) \\
&\leq \|v_n^\varphi\|_{L^\infty(\Omega_n)} \int_{\Omega_n} (|\varphi_n|_\varphi + |\Delta(\mu_1 \varphi_n)| + M \varphi_n) d(x, t)
\end{align*}

\text{24}
for \( n \in \mathbb{N} \). Invoking (h2) and (5.16), we conclude that \( \limsup_{n \to \infty} A_n \leq 0 \). So we have obtained a contradiction again, and thus the claim follows.

**Proof of Theorem 1.3 (completed).** By Lemmas 5.2 and 5.3 and the remarks before Lemma 5.2, the assumptions of Corollary 2.6 are satisfied with \( U \) and \( N \) as defined in (5.9) and (5.10). Consequently, there exists \( p \in \mathbb{S}^{N-1} \) such that every \( z \in U \) is foliated Schwarz symmetric with respect to \( p \). By definition of \( U \), this implies that every \( z = (z_1, z_2) \in \omega(u) \) has the property that \( z_1 \) is foliated Schwarz symmetric with respect to \( p \) and \( z_2 \) is foliated Schwarz symmetric with respect to \( -p \).

### 6 The cooperative case and other problems

In this section we first complete the

**Proof of Theorem 1.4.** Let \( u_1, u_2 \in C^{2,1}(\mathbb{B} \times (0, \infty)) \cap C(\overline{\mathbb{B}} \times [0, \infty)) \) be functions such that \( u = (u_1, u_2) \) solves (1.8), and suppose that (h0)', (h1)–(h3) and (1.7) are satisfied. The proof is almost exactly the same as the one of Theorem 1.3 with only two changes. The first change concerns the definitions of \( v_{e}^i \) and \( z_{e}^i \) in (5.3) and (5.4). More precisely, we now set

\[
v_{e}^i(x, t) = v^i(x, t) - v^i(\sigma_e(x), t) \quad \text{and} \quad z_{e}^i(x) = z^i(x) - z^i(\sigma_e(x)) \quad \text{for} \quad i = 1, 2.
\]

With this change, we again arrive at the linearized system (5.5). Considering now the sets

\[
U := \omega(u_1) \cup \omega(u_2) = \{z_1, z_2 : z \in \omega(u)\}
\]

in place of (5.9) and

\[
N := \{e \in \mathbb{S}^{N-1} : \exists T > 0 \text{ s.t. } u_{e}^i > 0 \text{ in } B(e) \times [T, \infty) \text{ for } i = 1, 2\},
\]

we may now validate the assumptions of Corollary 2.6 in exactly the same way as in Section 5. Hence the proof is complete.

**Remark 6.1.** (i) Note that both in Theorem 1.3 and in Theorem 1.4 we assume that the components \( u_i \) are non-negative, and this assumption is essential for the cooperativity of the linearized system (5.5). Without the sign restriction, systems (1.6) and (1.8) arise from each other by replacing \( u_i \) by \( -u_i \) for \( i = 1, 2 \) and adjusting \( f \) accordingly.

(ii) As a further example, we wish to mention the cubic system

\[
\begin{align*}
(u_1)_t - \Delta u_1 &= \lambda_1 u_1 + \gamma_1 u_1^3 - \alpha_1 u_2^2 u_1 \quad \text{in } B \times (0, \infty), \\
(u_2)_t - \Delta u_2 &= \lambda_2 u_2 + \gamma_2 u_2^3 - \alpha_2 u_1^2 u_2 \quad \text{in } B \times (0, \infty), \\
\partial_{\nu} u_1 &= \partial_{\nu} u_2 = 0 \quad \text{on } \partial B \times (0, \infty), \\
u_i(x, 0) &= u_{0,i}(x) \geq 0 \quad \text{for } x \in B, \ i = 1, 2,
\end{align*}
\]

where \( \lambda_i, \gamma_i, \) and \( \alpha_i \) are positive constants. The elliptic counterpart of this system is being studied extensively due to its relevance in the study of binary
mixtures of Bose-Einstein condensates, see [7]. The asymptotic symmetry of uniformly bounded classical solutions of this problem satisfying the initial reflection inequality condition \((h0)\) can be characterized in the same way as in Theorem 1.3. To see this, minor adjustments are needed in the proof of Theorem 1.3 to deal with a slightly different linearized system. Details will be given in [20]. Symmetry aspects of the elliptic counterpart of (6.1) have been studied in [23].

(iii) Our method breaks down if the coupling term has different signs in the components, as e.g. in a predator-prey type system

\[
\begin{align*}
(1) & : \quad (u_1)_t - \mu_1 \Delta u_1 = f_1(t, |x|, u_1) + \alpha_1 u_1 u_2 & \text{in } B \times (0, \infty), \\
(2) & : \quad (u_2)_t - \mu_2 \Delta u_2 = f_2(t, |x|, u_2) - \alpha_2 u_1 u_2 & \text{in } B \times (0, \infty).
\end{align*}
\]

In this case, there seems to be no way to derive a cooperative linearized system of the type (6.2) for difference functions related to hyperplane reflections. The asymptotic shape of solutions for this system (satisfying Dirichlet or Neumann boundary conditions) remains an interesting open problem.

(iv) Consider general systems of the form

\[
\begin{align*}
(3) & : \quad (u_i)_t - \Delta u_i = f_i(t, |x|, u) & \text{in } B \times (0, \infty), \\
(4) & : \quad \partial_{\nu} u_i = 0 & \text{on } \partial B \times (0, \infty),
\end{align*}
\]

for \(i = 1, 2\), where the nonlinearities \(f_i : [0, \infty) \times I_B \times \mathbb{R}^2 \to \mathbb{R}\) are locally Lipschitz in \(u = (u_1, u_2)\) uniformly with respect to \(r \in I_B\) and \(t > 0\). We call system (6.2) an irreducible cooperative system if for every \(m > 0\) there is a constant \(\sigma > 0\) such that

\[
\frac{\partial f_i(t, r, u)}{\partial u_j} \geq \sigma \quad \text{for every } i, j \in \{1, 2\}, i \neq j, r \in I_B, t > 0, \ |u| \leq m
\]

such that the derivative exists.

For this class of systems a symmetry result similar to Theorem 1.4 can be derived even for sign changing solutions, and in fact the proof is simpler. The precise statement and detailed arguments are given in [20], while we only discuss the key aspects here. We first note that, for a given uniformly bounded classical solution \(u = (u_1, u_2)\) of (6.2) and \(\epsilon \in \mathbb{S}^{N-1}\), we can use the Hadamard formulas as in [9] to derive a cooperative system for the functions \((x, t) \mapsto u_i^\epsilon(x, t) := u_i(x, t) - u_i(\sigma_\epsilon(x), t)\). This system has the form

\[
\begin{align*}
(5) & : \quad (u_i^\epsilon)_t - \Delta u_i^\epsilon = \sum_{j=1}^{2} c_{ij}^\epsilon u_j^\epsilon & \text{in } B(\epsilon) \times (0, \infty)
\end{align*}
\]

with functions \(c_{ij}^\epsilon \in L^\infty(B \times (0, \infty))\), \(i, j = 1, 2\) such that

\[
\inf_{B(\epsilon) \times (0, \infty)} c_{ij}^\epsilon > 0 \quad \text{for } i \neq j.
\]

With the help of the latter property, one can prove that for every sequence of positive times \(t_n\) with \(t_n \to \infty\) and every \(\epsilon \in \mathbb{S}^{N-1}\) we have the equivalence

\[
\lim_{n \to \infty} \|u_1^\epsilon(\cdot, t_n)\|_{L^\infty(B(\epsilon))} = 0 \quad \iff \quad \lim_{n \to \infty} \|u_2^\epsilon(\cdot, t_n)\|_{L^\infty(B(\epsilon))} = 0.
\]

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As a consequence, semitrivial limit profiles \((z_1, 0), (0, z_2) \in \omega(u)\) have the property that the nontrivial component must be a radial function, and hence no normalization procedure as in Section 3 is needed to deal with these profiles. This is the reason why the positivity of components is not needed in this case.

Details are given in [20]. Note that the cooperative system (1.8) is not irreducible.

The arguments and results for irreducible cooperative systems sketched in (iv) also apply to a corresponding system with \(n \geq 3\) equations. On the other hand, one may also consider cooperative systems of the form

\[
(\mu_i - \mu_i)(|x|, t)u_i = f_i(t, |x|, u_i) + \sum_{j \neq i}^n \alpha_{ij}(|x|, t)u_i u_j, \quad i = 1, \ldots, n. \tag{6.3}
\]

with \(n \geq 3\) equations which are not irreducible. Assume that (h1) and (h2) hold for \(f_i\) and \(\mu_i\), \(i = 1, \ldots, n\), and that \(\alpha_{ij} \in L^\infty(I_B \times (0, \infty))\) are nonnegative functions for \(i, j = 1, \ldots, n, i \neq j\). It is then an open question which additional positivity assumptions on the coefficients \(\alpha_{ij}\) are required for the corresponding generalization of Theorem 1.4. Similar arguments as in Section 5 apply in the case where \(\alpha_{ij} \geq \alpha_i > 0\) for \(i, j = 1, \ldots, n, i \neq j\), but we do not think that this assumption is optimal. We thank the referee for pointing out this question.

\section{Appendix}

Here we show the existence of positive solutions of the elliptic system (1.3) without foliated Schwarz symmetric components. More precisely, we have the following result.

\textbf{Theorem 7.1.} Let \(k \in \mathbb{N}\). Then there exists \(\varepsilon, \lambda > 0\) such that (1.3) admits a positive classical solution \((u_1, u_2)\) in \(B := B_\varepsilon = \{x \in \mathbb{R}^2 : 1 - \varepsilon < |x| < 1\} \subset \mathbb{R}^2\) such that the angular derivatives \(\partial_\theta u_i\) of the components change sign at least \(k\) times on every circle contained in \(B_\varepsilon\).

\textbf{Proof.} We apply a classical bifurcation result of Crandall and Rabinowitz, see [2] Lemma 1.1. Let \(\tilde{X}\) denote the space of functions \(u \in C(B_\varepsilon)\) which are symmetric with respect to reflection at the \(x_1\)-axis and \(\tilde{Y}\) the space of all \(u \in \tilde{X} \cap C^2(B_\varepsilon)\) with \(\partial_\nu u = 0\) on \(\partial B_\varepsilon\). Then \(\tilde{X}\) and \(\tilde{Y}\) are Banach spaces with respect to the norms of \(C^2(B_\varepsilon)\), \(C(B_\varepsilon)\), respectively. Let \(X := \tilde{X} \times \tilde{X}\), \(Y := \tilde{Y} \times \tilde{Y}\), and let \(F : (0, \infty) \times X \to Y\) be given by

\[
F(\lambda, u) = \left(\frac{\Delta u_1 + \lambda(u_1 + \lambda) - (u_1 + \lambda)u_2}{\Delta u_2 + \lambda(u_2 + \lambda) - (u_1 + \lambda)u_2}, \frac{\Delta u_1 - (u_1 + \lambda)u_2}{\Delta u_2 - (u_2 + \lambda)u_1}\right).
\]

Then we have \(F(\lambda, 0) = 0\) for all \(\lambda > 0\). Moreover, \(u = (u_1, u_2) \in X\) solves (1.3) if and only if \(F(\lambda, u_1 - \lambda, u_2 - \lambda) = 0\). We consider the partial derivative

\[
\partial_u F : (0, \infty) \times X \to \mathcal{L}(X, Y), \quad \partial_u F(\lambda, u)v = \left(\frac{\Delta v_1 - u_2v_1 - (u_1 + \lambda)v_2}{\Delta v_2 - u_1v_2 - (u_2 + \lambda)v_1}\right).
\]
For \( \lambda > 0 \) we put
\[
A_\lambda := \partial_\nu F(\lambda, 0) \in \mathcal{L}(X, Y), \quad A_\lambda v = \left( \Delta v_1 - \lambda v_2 \right).
\]
and we let \( N(A_\lambda) \) resp. \( R(A_\lambda) \) denote the kernel and the image of \( A_\lambda \), respectively. If \( v \in N(A_\lambda) \), then \( c := v_1 + v_2 \) satisfies \(-\Delta c + \lambda c = 0\) in \( B_\varepsilon \) and \( \partial_\nu c = 0 \) on \( \partial B_\varepsilon \), which easily implies that \( c \equiv 0 \) since \( \lambda > 0 \). Consequently, \( v \in N(A_\lambda) \) if and only if \( v_2 = -v_1 \) and
\[-\Delta v_1 = \lambda v_1 \quad \text{in} \quad B_\varepsilon, \quad \partial_\nu v_1 = 0 \quad \text{on} \quad \partial B_\varepsilon.\]
By separation of variables, there exists \( k \in \mathbb{N} \cup \{0\} \) such that in polar coordinates we have \( v_1(r, \theta) = \varphi(r) \cos(k \theta) \), where \( \varphi \in C^2([1 - \varepsilon, 1]) \) satisfies
\[-\Delta_r \varphi + \frac{k^2}{r^2} \varphi = \lambda \varphi \quad \text{in} \quad (1 - \varepsilon, 1), \quad \partial_r \varphi(1 - \varepsilon) = \partial_r \varphi(1) = 0 \quad (7.1)\]
with \( \Delta_r = \partial_{rr} + \frac{1}{r} \partial_r \). Let \( \lambda_j(k, \varepsilon) \geq 0 \) denote the \( j \)-th eigenvalue of \((7.1)\), counted with multiplicity in increasing order. By Sturm-Liouville theory, these eigenvalues are simple. It is easy to see that, for fixed \( k \in \mathbb{N} \cup \{0\} \), we have \( \lambda_1(k, \varepsilon) \to k^2 \) and \( \lambda_j(k, \varepsilon) \to \infty \) for \( j \geq 2 \) as \( \varepsilon \to 0 \). Moreover, \( \lambda_j(k, \varepsilon) \) is strictly increasing in \( k \) for fixed \( \varepsilon > 0 \). We now fix \( k \in \mathbb{N} \), and we choose \( \varepsilon = \varepsilon(k) > 0 \) small enough such that \( \lambda_2(0, \varepsilon) > \lambda_1(k, \varepsilon) \). We then set \( \lambda_* = \lambda_1(k, \varepsilon) > 0 \), and we let \( \varphi \) denote the unique positive eigenfunction of \((7.1)\) for \( \lambda = \lambda_* \) with \( \|\varphi\|_\infty = 1 \). It then follows that \( N(A_{\lambda_*}) \) is spanned by \( (\psi, -\psi) \in X \) with \( \psi(r, \theta) = \varphi(r) \cos(k \theta) \). Moreover, it easily follows from integration by parts that \( \int_{B_\varepsilon} \psi(v_1 - v_2) \, dx = 0 \) for every \( v = (v_1, v_2) \in R(A_{\lambda_*}) \). Since \( A_{\lambda_*} \) is a Fredholm operator of index zero, we thus conclude that
\[
R(A_{\lambda_*}) = \left\{ v \in Y : \int_{B_\varepsilon} \psi(v_1 - v_2) \, dx = 0 \right\}.
\]
In particular, since \( \frac{d}{d\lambda} A_\lambda v = (-v_2) \) for \( v \in X \) and \( \lambda > 0 \), we find that \( \frac{d}{d\lambda} A_{\lambda_*} (\psi, -\psi) = (0, \psi) \notin R(A_{\lambda_*}) \). Hence the assumptions of [2] Lemma 1.1 are satisfied, and thus there exists \( \delta > 0 \) and \( C^1 \)-functions \( \lambda : (-\delta, \delta) \to (0, \infty) \) and \( u : (-\delta, \delta) \to X \) such that \( \lambda(0) = \lambda_* \), \( F(\lambda(t), u(t)) = 0 \) for all \( t \in (-\delta, \delta) \) and \( u(t) = t \psi, -\psi + o(t) \) in \( X \). Hence, fixing \( t \in (-\delta, \delta) \setminus \{0\} \) sufficiently close to zero and considering \( \lambda := \lambda(t) \), we find that \( u = (u_1, u_2) \) with \( u_1 = u_1(t) + \lambda(t), u_2 = u_2(t) + \lambda(t) \) is a positive solution of \((1.3)\) such that the angular derivatives \( \frac{\partial u_i}{\partial \theta} \) of the components change sign at least \( k \) times on every circle contained in \( B_\varepsilon \).

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