AN EXTENSION OF THE RAINBOW ERDŐS-ROTHSCHILD PROBLEM

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Abstract. Given integers \( r \geq 2, k \geq 3 \) and \( 2 \leq s \leq \binom{k}{2} \), and a graph \( G \), we consider \( r \)-edge-colorings of \( G \) with no copy of a complete graph \( K_k \) on \( k \) vertices where \( s \) or more colors appear, which are called \( P_{k,s} \)-free \( r \)-colorings. We show that, for large \( n \) and \( r \geq r_0(k,s) \), the \((k-1)\)-partite Turán graph \( T_{k-1}(n) \) on \( n \) vertices yields the largest number of \( P_{k,s} \)-free \( r \)-colorings among all \( n \)-vertex graphs, and that it is the unique graph with this property.

1. Introduction

Given a fixed graph \( F \), the well-known Turán problem for \( F \) is concerned with the maximum number \( \text{ex}(n,F) \) of edges over all \( F \)-free \( n \)-vertex graphs, namely over all \( n \)-vertex graphs that do not contain \( F \) as a subgraph. The graphs that achieve this maximum are called \( F \)-extremal. When \( F = K_k \) is the complete graph on \( k \geq 3 \) vertices, the unique \( F \)-extremal graph on \( n \) vertices is the balanced, complete, \((k-1)\)-partite graph \( T_{k-1}(n) \), known as the Turán graph for \( K_k \) [24]. For general graphs (and hypergraphs) \( F \), determining \( \text{ex}(n,F) \) and the corresponding extremal graphs is a very important problem and there is a vast literature related with it (more information may be found in Füredi and Simonovits [11], and in the references therein).

An \( r \)-coloring of a graph \( G \) is a function \( f: E(G) \rightarrow [r] \) that associates a color in \([r] = \{1,\ldots,r\}\) with each edge of \( G \). Erdős and Rothschild [9] were interested in \( n \)-vertex graphs that admit the largest number of \( r \)-colorings such that every color class is \( F \)-free. In particular, they conjectured that, for all \( n \geq n_0(k) \), the number of \( K_k \)-free 2-colorings is maximized by the Turán graph \( T_{k-1}(n) \). Note that \( F \)-extremal graphs are natural candidates for maximality, as their edges may be colored arbitrarily, which leads to \( r^{\text{ex}(n,F)} \) colorings. It is clear that the number of colorings might increase if we have more than \( \text{ex}(n,F) \) edges to color, but additional edges also produce copies of \( F \), placing constraints on colorings of their edges.

Regarding the Erdős-Rothschild Conjecture, Yuster [25] gave an affirmative answer for \( k = 3 \) and any \( n \geq 6 \), while Alon, Balogh, Keevash and Sudakov [1] showed that, for \( r \in \{2,3\} \) and \( n \geq n_0 \), where \( n_0 \) is a constant depending on \( r \) and \( k \), the Turán graph \( T_{k-1}(n) \) is the unique optimal \( n \)-vertex graph for the number of \( K_k \)-free \( r \)-colorings. Recently, Hán and Jiménez [12] obtained better bounds on \( n_0 \) using the Container Method. For \( r \geq 4 \), the answer is more complicated. Pikhurko and Yilma [21] found the graphs that admit the largest number of such colorings for \( r = 4 \) and \( k \in \{3,4\} \), which turn out to be balanced, complete, multipartite graphs that are not \( K_k \)-free.

This work was partially supported by CAPES and DAAD via Probral (CAPES Proc. 88881.143993/2017-01 and DAAD 57391132). The first author acknowledges the support of CNPq 308054/2018-0), Conselho Nacional de Desenvolvimento Científico e Tecnológico.
Botler et al. [7] characterized the extremal graphs for $k = 3$ and $r = 6$, and they gave an approximate result for $k = 3$ and $r = 5$. Pikhurko, Staden and Yilma [20] showed that at least one of the graphs with the largest number of colorings is complete multipartite.

Balogh [3] was the first to consider $r$-colorings that avoid a copy of a graph $F$ colored in a non-monochromatic way. A similar problem was investigated by Hoppen and Lefmann [14] and by Benevides, Hoppen and Sampaio [6], who considered edge-colorings of a graph avoiding a copy of $F$ with a prescribed pattern. Given a graph $F$, a pattern $P$ of $F$ is a partition of its edge set. An edge-coloring of a graph $G$ is said to be $(F, P)$-free if $G$ does not contain a copy of $F$ in which the partition of the edge set induced by the coloring is isomorphic to $P$. For instance, if the partition $P$ consists of a single class, $(F, P)$-free colorings avoid monochromatic copies of $F$. On the other hand, if $P$ is the pattern where each edge of $F$ lies in a different class, $(F, P)$-free colorings avoid rainbow copies of $F$. These colorings are known as Gallai colorings when $F = K_3$.

Given the number of colors $r ≥ 1$, a graph $F$ and a pattern $P$ of $F$, let $C_{r, (F, P)}(G)$ be the set of all $(F, P)$-free $r$-colorings of a graph $G$. We write

$$c_{r, (F, P)}(n) = \max \left\{ |C_{r, (F, P)}(G)| : |V(G)| = n \right\},$$

and we say that an $n$-vertex graph $G$ is $(r, F, P)$-extremal if $|C_{r, (F, P)}(G)| = c_{r, (F, P)}(n)$. Most results about $c_{r, (F, P)}(n)$ involve monochromatic or rainbow patterns, more information may be found in [1, 8, 13, 16, 17] and in the references therein. In particular, the work of [6] implies that, for any such pattern, there is an extremal $(r, F, P)$-extremal graph that is complete multipartite.

Here, we generalize this problem to colorings that avoid a family of patterns. Let $k ≥ 3$, $r ≥ 2$ and $s ≤ \binom{k}{2}$ be positive integers. Given a graph $G$, we are interested in $r$-edge-colorings of $G$ with no copy of $K_k$ colored with $s$ or more colors, which are called $P_{k, s}$-free $r$-colorings. Let $C_{r, P_{k, s}}(G)$ denote this set of $r$-colorings and let

$$c_{r, P_{k, s}}(n) = \max \left\{ |C_{r, P_{k, s}}(G)| : |V(G)| = n \right\}. \quad (1)$$

If $s = 1$, finding the $n$-vertex graphs that achieve $c_{r, P_{k, 1}}(n)$ colorings is just a re-statement of the Turán problem, as it is equivalent to finding a $K_k$-free $n$-vertex graph with the largest number of edges. If $s = \binom{k}{2}$, this is precisely the problem of finding $(r, K_k, P)$-extremal graphs, where $P$ is the rainbow pattern of $K_k$.

We note that previous results already give the full solution of this problem for $k = 3$, at least for large $n$. Note that $1 ≤ s ≤ 3$ in this case. The case $s = 1$ corresponds to the Turán problem, so that $T_2(n)$ is the unique extremal configuration for all $n, r ≥ 2$. In the case $s = 2$, every triangle in a graph $G$ has to be monochromatic in a $P_{3, 2}$-free $r$-coloring of $G$. So, if any edge contained in a triangle is removed from $G$, the number of colorings does not decrease, which immediately implies that $c_{r, P_{3, 2}}(n) = |C_{r, P_{3, 2}}(T_2(n))|$ for any $r ≥ 2$ and $n ≥ 2$. It is easy to show that $T_2(n)$ is the only $n$-vertex graph with this property. For $s = 3$, Balogh and Li [4] proved that, for $n$ sufficiently large, the complete graph is the unique extremal configuration for $r ≤ 3$ and $T_2(n)$ is the unique extremal configuration for $r ≥ 4$. Bastos, Benevides and Han [3] obtained related results and Hoppen, Lefmann and Odermann [16] had previously established the extremality of $T_2(n)$ for $r ≥ 5$.

*Analogously $c_{r, P_{k, z}}(n) = |C_{r, P_{k, z}}(T_{k-1}(n))|$ for any $r ≥ 2$ and $n ≥ 2$. 
The following states two easy facts about determining \( \binom{k}{2} \) and the \( n \)-vertex graphs that achieve extremality.

**Lemma 1.1.** Let \( n \geq k \geq 3, s \leq \binom{k}{2} \) and \( r \geq 2 \) be integers.

(a) If \( r < s \), then \( c_{r,P_{k,s}}(n) = \left| C_{r,P_{k,s}}(K_n) \right| = r^{\binom{k}{2}} \).

(b) If \( c_{r,P_{k,s}}(n) = \left| C_{r,P_{k,s}}(T_{k-1}(n)) \right| \) and \( 1 \leq s' < s \), then

\[
\left| C_{r,P_{k,s'}}(n) \right| = \left| C_{r,P_{k,s'}}(T_{k-1}(n)) \right|.
\]

**Proof.** Part (a) is trivial, as no \( r \)-coloring can produce a copy of \( K_k \) colored with \( s \) or more colors if \( r < s \). In part (b), the hypothesis tells us that for any \( n \)-vertex graph \( G \)

\[
c_{r,P_{k,s}}(n) = \left| C_{r,P_{k,s}}(T_{k-1}(n)) \right| \geq \left| C_{r,P_{k,s}}(G) \right|
\]

The conclusion now follows from the fact that \( C_{r,P_{k,s}}(G) \subseteq C_{r,P_{k,s}}(G) \) for any graph \( G \) and that \( C_{r,P_{k,s}}(T_{k-1}(n)) = C_{r,P_{k,s}}(T_{k-1}(n)) \). \( \square \)

The work [17, Theorem 1.2] shows that, given \( k \geq 4 \) and the rainbow pattern \( P \) of \( K_k \), there is \( r_0 \) such that \( c_{r,(K_n,P)}(n) = \left| C_{r,(K_n,P)}(T_{k-1}(n)) \right| \) for all \( r \geq r_0 \) and \( n \geq n_0(r,k) \).

With Lemma 1.1 (b), we deduce that, for any \( k \geq 4 \) and \( s \leq \binom{k}{2} \), there is \( r_0 \) such that \( c_{r,P_{k,s}}(n) = \left| C_{r,P_{k,s}}(T_{k-1}(n)) \right| \) for all \( r \geq r_0 \) and \( n \geq n_0(r,k,s) \). However, this value of \( r_0 \) is superexponential in \( k \), and the authors of [17] believed that this result should hold for much smaller values of \( r_0 \).

By addressing a more general problem, we are able to obtain much better bounds for smaller values of \( s \); moreover, our results lead to better bounds on \( r_0 \) in the case \( s = \binom{k}{2} \), i.e., when only rainbow copies of \( K_k \) are avoided. The main result in this paper is the following. For simplicity, we write it in terms of functions that will be defined in the next subsection.

**Theorem 1.2.** Let \( k \geq 4 \) and \( 2 \leq s \leq \binom{k}{2} \) be integers. Fix \( r \geq r_0(k,s) \), defined in \([3],[6]\) and \([7]\) below for \( s \leq s_0(k) \), \( s_0(k) < s \leq s_1(k) \) and \( s > s_1(k) \), respectively. There is \( n_0 = n_0(r,k,s) \) for which the following holds. Every graph \( G = (V,E) \) on \( n > n_0 \) vertices satisfies

\[
\left| C_{r,P_{k,s}}(G) \right| \leq r^{\text{ex}(n,K_k)}.
\]

Moreover, equality holds if and only if \( G \) is isomorphic to \( T_{k-1}(n) \).

To prove Theorem 1.2, we use a stability method that relies on the following result.

**Theorem 1.3.** Let \( k \geq 4 \) and \( 2 \leq s \leq \binom{k}{2} \) be integers. Fix \( r \geq r_0(k,s) \), defined in \([3],[6]\) and \([7]\) below for \( s \leq s_0(k) \), \( s_0(k) < s \leq s_1(k) \) and \( s > s_1(k) \), respectively. For any \( \delta > 0 \), there is \( n_0 = n_0(\delta,r,k,s) \) as follows. If \( G = (V,E) \) is a graph on \( n > n_0 \) vertices such that \( \left| C_{r,P_{k,s}}(G) \right| \geq r^{\text{ex}(n,K_k)} \), then there is a partition \( V = W_1 \cup \cdots \cup W_{k-1} \) such that at most \( \delta n^2 \) edges have both endpoints in a same class \( W_i \).

It turns out that the value of \( r_0(k,s) \) in the above statements is needed in our proof of Theorem 1.3, and we do not believe that it is best possible. However, our results give much better dependency on \( s \) and \( k \) than the bound in [17]. Indeed, if \( s \leq s_0(k) \), the value given for \( r_0(k,s) \) is less than \( (s-1)^2 \). Moreover, if \( s \leq s_1(k) \), the quantity \( r_0(k,s) \) is less than \( (s-1)^7 \).
1.1. The functions $r_0(k, s)$, $s_0(k)$ and $s_1(k)$. In order to specify the quantities given in the statement of Theorems 1.2 and 1.3, we shall define some additional functions. For $j \in \{2, \ldots, k-1\}$, let

$$A(k, j) = \frac{k}{2} - \text{ex}(k, K_{j+1}) = \left(\left\lfloor \frac{k}{j} \right\rfloor \cdot \frac{j}{2} \right) + \left(\left\lfloor \frac{k}{j} \right\rfloor \cdot \frac{k}{2} \right),$$

(2)

which, by Turán’s Theorem, is the minimum number of edges that must be deleted from a complete graph $K_k$ to make it $j$-partite. Let

$$s_0(k) = A(k, 2) + 2 = \frac{k}{2} - \left\lfloor \frac{k}{2} \right\rfloor \cdot \left\lfloor \frac{k}{2} \right\rfloor + 2$$

$$s_1(k) = \frac{k}{2} - \frac{k}{2} + 2.$$

For $s \leq s_0(k)$, let $i^*$ be the least value of $i$ such that $A(k, k - i) \geq s - 2$. As it turns out, we have $i^* \leq \min\{s - 2, k - 2\}$. Let $r_0(k, s)$ be the least integer greater than

$$(s - 1)^{k-1} \frac{p^*}{2} \prod_{i=2}^{i^*} (s - A(k, k - i + 1) - 1) \frac{1}{(k - i + 1)(k - i)} \cdot (s - 1)^{k-2}. \quad (3)$$

For any fixed $s > s_0(k)$, we consider additional parameters. Let $j \in [k - 1]$ and $2 \leq p \leq k - 1$ be integers satisfying the following condition:

$$b(k, p, j) = \min \left\{ j \frac{p}{2}, \left\lfloor \frac{k}{p} \right\rfloor \left(\frac{p}{2}\right) + \left(\frac{k}{2} - \left\lfloor \frac{k}{p} \right\rfloor \right) \right\} \leq \left(\frac{k}{2}\right) - s + 2,$$

(4)

and define

$$L(k, s, p, j) = 1 + \frac{2p(k - 1)}{j(p - 1)}. \quad (5)$$

If $s_0(k) < s \leq s_1(k)$, let $p^*$ be the largest $p \geq 2$ such that (4) holds for $j = k - 1$. Our choice of $s$ ensures that there is such a $p$. We define $r_0(k, s)$ as the least integer greater than

$$(s - A(k, 2) - 1)^{L(k,s,p^*,k-1)} \cdot \prod_{i=2}^{i^*} (s - A(k, k - i + 1) - 1) \frac{1}{(k - i + 1)(k - i)} \cdot (s - 1)^{k-2}. \quad (6)$$

If $s > s_1$, let $j^*$ be the largest $j \geq 1$ such that (4) holds for $p = 2$. We define $r_0(k, s)$ as the least integer greater than

$$(s - A(k, 2) - 1)^{L(k,s,2j^*,k-1)} \cdot \prod_{i=2}^{i^*} (s - A(k, k - i + 1) - 1) \frac{1}{(k - i + 1)(k - i)} \cdot (s - 1)^{k-2}. \quad (7)$$

Table 1.1 provides the values of $r_0(k, s)$ for a few values of $k$ and $s$. The symbols $\ast$ and $\ast$ are used to indicate the first value of $s$ such that $s > s_0$ and such that $s > s_1$, respectively.

For comparison, it is easy to see that, if $r \leq r_1(k, s) = [(s - 1)^{k-1}/(k - 2) - 1]$, then $|C_{r,p_{k,s}}(K_n)| > |C_{r,p_{k,s}}(T_{k-1}(n))|$ for large values of $n$, so that Theorem 1.2 cannot

\footnote{For completeness, we added values of $r_0$ known to hold for $k = 3$.}
possibly be extended to such values of \( r \). In particular, these two tables imply that the values for \( r_0(k, s) \) are best possible for \( s = 3 \) and for \((k, s) = (5, 4)\).

The paper is structured as follows. In Section 2, we introduce the tools needed to prove our main results. We then prove Theorem 1.3 and 1.2 in Sections 3 and 4, respectively.

2. Preliminaries

In this section, we fix the notation and introduce concepts and results used to prove our main results. We first state a well-known auxiliary lemma.

**Lemma 2.1.** If \( \ell \geq 2 \) and \( G \) is a graph with \( m \) edges, then \( G \) contains an \( \ell \)-partite subgraph with more than \((\ell - 1)m/\ell \) edges.

The next lemma generalizes a result of Alon and Yuster [2].

**Lemma 2.2.** Fix \( 1 \leq j \leq k - 1 \) and \( 2 \leq p \leq k - 1 \). Let \( 0 < \gamma \leq \frac{j(p-1)}{2p(k-1)^2} \) and let \( H'' \) be a \((k-1)\)-partite graph on \( m \) vertices with partition \( V(H'') = U_1 \cup \cdots \cup U_{k-1} \) and at least \( \text{ex}(m, K_k) - \gamma m^2 \) edges. If we add at least \( \left( \frac{p(k-1)}{j(p-1)} + 1 \right) \gamma m^2 \) new edges to \( H'' \), then in the resulting graph there is a copy of \( K_k \) with at most \( b(k, p, j) = \min\{j(p)^2, k/p, (k/p)^2 + \frac{(k-1)^2}{2p}\} \) new edges. Every such copy contains at most \( p \) vertices in each class \( U_i \) and each new edge in the copy connects two vertices of \( K_k \) that lie in a same vertex class \( U_i \) of \( H'' \).

**Proof.** Let \( H'' \) be as in the statement of the lemma. Consider adding at least

\[
\left( \frac{p(k-1)}{j(p-1)} + 1 \right) \gamma m^2
\]

new edges to \( H'' \) to produce a graph \( H' \). At least \( \frac{p(k-1)}{j(p-1)} \gamma m^2 \) new edges have both endpoints in a same partition class. By an averaging argument, there exist \( j \) classes
$U_i, \ldots, U_j$ containing at least $\frac{p}{p-1}\gamma m^2$ new edges. Indeed, if this were false, we would have
\[
\binom{k-2}{j-1} (e_H(U_1) + \cdots + e_H(U_{k-1})) = \sum_{1 \leq i_1 < \cdots < i_j \leq k-1} (e_H(U_{i_1}) + \cdots + e_H(U_{i_j})) < \binom{k-1}{j} \frac{p}{p-1}\gamma m^2,
\]
which implies that $e_H(U_1) + \cdots + e_H(U_{k-1}) < \frac{p(k-1)}{j(p-1)}\gamma m^2$, a contradiction.

Let $\Gamma$ be the spanning subgraph of $H'$ with edges in $E_{H'}(U_i) \cup \cdots \cup E_{H'}(U_j)$, so that $\Gamma$ contains at least $\frac{p}{p-1}\gamma m^2$ edges. By Lemma 2.1, $\Gamma$ has a $p$-partite subgraph $\Gamma'$ with more than $\gamma m^2$ edges. We will refer to the edges of $\Gamma'$ as the new edges. The sum of the number of edges in $\Gamma'$ with the number of edges in $H''$ is greater than $\text{ex}(m, K_k)$, hence there exists a copy of $K_k$ in the union of $H''$ and $\Gamma'$. Note that there cannot be $(p+1)$ or more vertices of this copy in a same class $U_i$, as this would produce a copy of $K_{p+1}$ where all edges are new, a contradiction. Therefore the copy of $K_k$ contains at most $j\binom{p}{2}$ new edges. On the other hand, at most $\lfloor k/p \rfloor$ classes may contain $p$ vertices, so that the number of new edges is at most $\lfloor k/p \rfloor \binom{p}{2} + \binom{k-\lfloor k/p \rfloor}{2}$. As a consequence, the number of new edges is at most $\min\{j\binom{p}{2}, \lfloor k/p \rfloor \binom{p}{2} + \binom{k-\lfloor k/p \rfloor}{2}\}$, as required. \qed

For later reference, we state Lemma 2.2 in the special case $p = 2$ and $j = 1$, which is precisely the result of Alon and Yuster \cite{AY}.

**Corollary 2.3.** Let $0 < \gamma \leq 1/(4(k-1)^2)$ be fixed and let $H''$ be a $(k-1)$-partite graph on $m$ vertices with partition $V(H'') = U_1 \cup \cdots \cup U_{k-1}$ with at least $\text{ex}(m, K_k) - \gamma m^2$ edges. If we add at least $(2k-1)\gamma m^2$ new edges to $H''$, then in the resulting graph there is a copy of $K_k$ with exactly one new edge, which connects two vertices of $K_k$ in the same vertex class $U_i$ of $H''$.

### 2.1. Regularity Lemma

To prove our results we use an approach similar to the one from \cite{A1}, which is based on the Szemerédi Regularity Lemma \cite{S}. Let $G = (V, E)$ be a graph, and let $A$ and $B$ be two subsets of $V(G)$. If $A$ and $B$ are non-empty, define the density of edges between $A$ and $B$ by
\[
d(A, B) = \frac{e(A, B)}{|A||B|},
\]
where $e(A, B)$ is the number of edges with one vertex in $A$ and the other in $B$. (When $A = B$, we write $e(A, A) = e(A)$.) For $\varepsilon > 0$ the pair $(A, B)$ is called $\varepsilon$-regular if, for every subsets $X \subseteq A$ and $Y \subseteq B$ satisfying $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$, we have
\[
|d(X, Y) - d(A, B)| < \varepsilon.
\]

An equitable partition of a set $V$ is a partition of $V$ into pairwise disjoint classes $V_1, \ldots, V_m$ of almost equal size, i.e., $||V_i| - |V_j|| \leq 1$ for all pairs $i, j$. An equitable partition of the set $V$ of vertices of $G$ into the classes $V_1, \ldots, V_m$ is called $\varepsilon$-regular if at most $\varepsilon\binom{m}{2}$ of the pairs $(V_i, V_j)$ are not $\varepsilon$-regular.

We shall use the following colored version of the Regularity Lemma \cite{S} that will be particularly useful for our purposes.
Lemma 2.4. For every \( \varepsilon > 0 \) and every positive integer \( r \), there exists a constant \( M = M(\varepsilon, r) \) such that the following property holds. If the edges of a graph \( G \) of order \( n > M \) are \( r \)-colored \( E(G) = E_1 \cup \cdots \cup E_r \), then there is a partition of the vertex set \( V(G) = V_1 \cup \cdots \cup V_m \), with \( 1/\varepsilon \leq m \leq M \), which is \( \varepsilon \)-regular simultaneously with respect to all graphs \( G_i = (V_i, E_i) \) for all \( i \in [r] \).

A partition as in Lemma 2.4 will be called a \textit{multicolored \( \varepsilon \)-regular partition}. Given such a partition \( V_1 \cup \cdots \cup V_m \) and a constant \( \eta > 0 \), we may also define the \textit{multicolored cluster graph} \( \hat{H} = H(\eta) \) associated with this partition and with a constant \( \eta > 0 \): the vertex set is \([m]\) and \( e = \{i, j\} \) is an edge of \( \hat{H} \) if the pair \((V_i, V_j)\) is \( \varepsilon \)-regular with respect to all colors and the density between \( V_i \) and \( V_j \) is at least \( \eta \) for at least one of the colors. Each edge \( e = \{i, j\} \) in \( H \) is assigned the list \( L_e \) of colors \( c \) such that \( c \) appears with density at least \( \eta \) between \( V_i \) and \( V_j \) in \( G \).

Given a colored graph \( \hat{F} \), we say that a multicolored cluster graph \( H \) contains \( \hat{F} \) if \( H \) contains a copy of the (uncolored) graph induced by \( \hat{F} \) for which the color of each edge of \( \hat{F} \) is contained in the list of the corresponding edge in \( H \). More generally, if \( F \) is a graph with color pattern \( P \), we say that \( H \) contains \((F, P)\) if it contains some colored copy of \( F \) with pattern \( P \).

Given colored graphs \( \hat{F} \) and \( \hat{H} \), a function \( \psi : V(\hat{F}) \to V(\hat{H}) \) is called a \textit{colored homomorphism} of \( \hat{F} \) in \( \hat{H} \) if, for every edge \( e = \{i, j\} \in E(\hat{F}) \), the pair \( (\psi(i), \psi(j)) \) is an edge of \( \hat{H} \) with the color of \( e \). If \( H \) is a multicolored cluster graph, it suffices that the color of \( e \) lies in the list associated with the edge \( (\psi(i), \psi(j)) \). In connection with these definitions, the following embedding result holds (for a proof, see [17]).

Lemma 2.5. For every \( \eta > 0 \) and all positive integers \( k \) and \( r \), there exist \( \varepsilon = \varepsilon(r, \eta, k) > 0 \) and a positive integer \( n_0(r, \eta, k) \) with the following property. Suppose that \( \hat{G} = (V, E) \) is an \( r \)-colored graph on \( n > n_0 \) vertices with a multicolored \( \varepsilon \)-regular partition \( V = V_1 \cup \cdots \cup V_m \) which defines the multicolored cluster graph \( H = H(\eta) \). Let \( \hat{F} \) be a \( k \)-vertex graph colored with \( t \leq r \) colors. If there exists a colored homomorphism \( \psi \) of \( \hat{F} \) into \( H \), then the graph \( \hat{G} \) contains \( \hat{F} \).

The following standard embedding result will also be useful.

Lemma 2.6. Let \( k \geq 2 \) be an integer and fix a constant \( 0 < \alpha \leq 3/4 \). Let \( G \) be a graph whose vertex set contains mutually disjoint sets \( W_1, \ldots, W_k \) with the following property. For every pair \( \{i, j\} \subseteq [k] \), where \( i \neq j \), and all subsets \( X_i \subseteq W_i \), where \( |X_i| \geq \alpha^k |W_i| \), and \( X_j \subseteq W_j \), where \( |X_j| \geq \alpha^k |W_j| \), there are at least \( \alpha |X_i||X_j| \) edges between \( X_i \) and \( X_j \) in \( G \). Then \( G \) contains a copy of \( K_k \) with one vertex in each set \( W_i \).

Proof. Our proof is by induction on \( k \). For \( k = 2 \) the result is trivial. Assume by induction that the statement holds for \( k - 1 \), where \( k \geq 3 \). Now, fix

\[
\alpha \leq \frac{3}{4} < \min \left\{ (\ell - 1)^{-\frac{1}{k}} : 2 \leq \ell \leq k \right\}
\]  

and let \( G \) be a graph whose vertex set contains mutually disjoint sets \( W_1, \ldots, W_k \) as in the statement of the lemma.

For all \( i \in [k-1] \), let \( W_k^i \subseteq W_k \) contain all vertices in \( W_k \) with fewer than \( \alpha |W_i| \) neighbors in \( W_i \). Then we have \( e(W_k^i, W_i) < \alpha |W_k^i||W_i| \), so that \( |W_k^i| < \alpha^k |W_k| \) by
hypothesis. Our choice of $\alpha$ implies that
$$\sum_{i=1}^{k-1} W_k^i < (k - 1) \cdot \alpha^k \cdot |W_k| < |W_k|.$$ 

Let $v$ be a vertex in $W_k \setminus \bigcup_{i=1}^{k-1} W_k^i$ and, for $i \in [k-1]$, let $W_k^i$ be the set of neighbors of $v$ in $W_i$, so that $|W_k^i| \geq \alpha|W_k|$. Observe that, for all subsets $X_i \subseteq W_k^i$ and $X_j \subseteq W_k^j$ such that $|X_i| \geq \alpha^{k-1}|W_k^i| \geq \alpha^k|W_k|$ and $|X_j| \geq \alpha^{k-1}|W_k^j| \geq \alpha^k|W_j|$, there are at least $\alpha^k|X_i||X_j|$ edges between $X_i$ and $X_j$ in $G$. By induction, $G$ contains a copy of $K_{k-1}$ with one vertex in each set $W_i^i$, where $1 \leq i \leq k-1$. Adding $v$ creates a copy of $K_k$ in $G$ with one vertex in $W_i$ for each $i \in [k]$.

2.2. Stability. Another concept that will be particularly useful in our paper are stability results in the sense of Erdős and Simonovits [22]. It will be convenient to use the following theorem by Füredi [10].

**Theorem 2.7.** Let $G = (V, E)$ be a $K_k$-free graph on $m$ vertices. If $|E| = \text{ex}(m, K_k) - t$ for some $t \geq 0$, then there exists a partition $V = V_1 \cup \cdots \cup V_{k-1}$ with $\sum_{i=1}^{k-1} e(V_i) \leq t$.

We recall the following bounds on the number of edges in the Turán graph $T_{k-1}(m)$:
$$\frac{(k-2)m^2}{2(k-1)} - k + 1 < \text{ex}(m, K_k) \leq \frac{(k-2)m^2}{2(k-1)}.$$ 

(9)

For later use, we state the following fact about the size of the classes in a $(k-1)$-partite graph with a large number of edges (the easy proof is in [17]).

**Proposition 2.8.** Let $G = (V, E)$ be a $(k-1)$-partite graph on $m$ vertices with $(k-1)$-partition $V = V_1 \cup \cdots \cup V_{k-1}$. If, for some $t \geq (k-1)^2$, the graph $G$ contains at least $\text{ex}(m, K_k) - t$ edges, then for each $i \in [k-1]$ we have
$$\left| |V_i| - \frac{m}{k-1} \right| < \sqrt{2t}.$$ 

We also consider the entropy function $H : [0, 1] \to [0, 1]$ given by $H(x) = -x \log_2 x - (1-x) \log_2 (1-x)$ with $H(0) = H(1) = 0$. It is used in the well-known inequality
$$\binom{n}{\alpha n} \leq 2^{H(\alpha)n}$$ 

(10)

for all $0 \leq \alpha \leq 1$. It turns out that, for $x \leq 1/8$, we have:
$$H(x) \leq -2x \log_2 x.$$ 

(11)

3. Proof of Theorem 1.3

In this section, we prove Theorem 1.3 For convenience, we restate it here.

**Theorem 1.3.** Let $k \geq 4$ and $2 \leq s \leq \left(\frac{k}{2}\right)$ be integers. Fix $r \geq r_0(k, s)$, defined in (3), (6) and (7) below for $s \leq s_0(k)$, $s_0(k) < s \leq s_1(k)$ and $s > s_1(k)$, respectively. For any $0 > \delta > 0$, there is $n_0 = n_0(\delta, r, k, s)$ as follows. If $G = (V, E)$ is a graph on $n > n_0$ vertices such that $|\mathcal{G}_{r, r, s}(G)| \geq \text{ex}(n, K_{k})$, then there is a partition $V = W_1 \cup \cdots \cup W_{k-1}$ such that at most $\delta n^2$ edges have both endpoints in a same class $W_i$. 

Proof. Fix positive integers \( k \geq 4, 2 \leq s \leq \binom{k}{2} \) and \( r \geq r_0(k, s) \) according to (3), (6) or (7), respectively, depending on the value of \( s \). Let \( \delta > 0 \). We shall fix a positive constant \( \beta_0 < \frac{\delta}{8k+2} \) and consider \( \eta > 0 \) sufficiently small to satisfy

\[
\eta \eta < \frac{\delta}{2} \tag{12}
\]

and inequality (27).

Let \( n_0 = n_0(r, \eta, k) \) and \( \varepsilon = \varepsilon(r, \eta, k) > 0 \), where \( \varepsilon < \eta/2 \), be given by Lemma 2.4 and let \( M = M(\varepsilon, r) \) be defined by Lemma 2.5. Let \( G \) be a graph on \( n > \max\{n_0, M\} \) vertices with at least \( n \cdot \varepsilon(\eta, r, k) \) distinct \((K_k, \geq s)\)-free \( r \)-colorings. Fix one such coloring.

By Lemma 2.4 there exists a partition of \( V(G) \) into \( 1/\varepsilon \leq m \leq M \) parts that is \( \varepsilon \)-regular with respect to all \( r \) colors. Let \( H = H(\eta) \) be the \( m \)-vertex multicolored cluster graph associated with this partition, where each edge \( e \) has a non-empty list \( L_e \) of colors. We write \( E_i = E_i(H) \) for the set of edges of \( H \) for which \( |L_e| = i \), and we let \( e_i(H) = |E_i(H)| \), for \( i \in [r] \).

We shall bound the number of \( r \)-colorings of \( G \) that lead to the partition \( V(G) = V_1 \cup \cdots \cup V_m \) and to the multicolored cluster graph \( H \). Given a color \( c \in [r] \), the number of \( \varepsilon \)-irregular pairs \((V_i, V_j)\) with respect to the spanning subgraph \( G_c \) of \( G \) with edge set given by all edges of color \( c \) is at most \( \varepsilon(\frac{m}{2}) \). This leads to at most

\[
r \cdot \varepsilon \cdot \left( \frac{m}{2} \right) \cdot \left( \frac{n}{m} \right)^2 \leq \frac{r \cdot \varepsilon}{2} \cdot n^2 \tag{13}
\]

edges between \( \varepsilon \)-irregular pairs with respect to some color. By the definition of an \( \varepsilon \)-regular partition and the fact that \( m \geq 1/\varepsilon \), there are at most

\[
m \cdot \left( \frac{n}{m} \right)^2 = \frac{n^2}{m} \leq \varepsilon n^2 \tag{14}
\]

edges with both endpoints in the same class \( V_i \) for some \( i \in [m] \). Finally, the number of edges with some color \( c \) connecting a pair \((V_i, V_j)\) such that the density of the pair in \( G_c \) is less than \( \eta \) is bounded above by

\[
r \cdot \eta \cdot \left( \frac{m}{2} \right) \cdot \left( \frac{n}{m} \right)^2 \leq \frac{r \cdot \eta}{2} \cdot n^2. \tag{15}
\]

Combining equations (13) – (15) and using that \( \varepsilon < \eta/2 \), there are less than \( r \eta n^2 \) edges of any of these three types. They may be chosen in the \( r \)-coloring of \( G \) in at most \( \left( \frac{n^2}{2r\eta} \right) \) ways and may be colored in at most \( r^{r \eta n^2} \) ways.

As a consequence, the number of \( r \)-colorings of \( G \) that give rise to the partition \( V(G) = V_1 \cup \cdots \cup V_m \) and the multicolored cluster graph \( H(\eta) \) is bounded above by

\[
\left( \frac{n^2}{2r\eta} \right)^2 \cdot r^{r \eta n^2} \cdot \left( \prod_{e \in E(H)} |L_e| \right)^{(\frac{n}{m})^2} \leq 2^{H(2r\eta) \frac{n^2}{2r\eta}^2} \cdot r^{r \eta n^2} \cdot \left( \prod_{i=1}^{r} e_i(H) \right)^{(\frac{n}{m})^2}, \tag{16}
\]

where \( e_i(H) \) is the number of edges of \( H \) whose lists have size equal to \( i \).

Since \( m \leq M \), there are at most \( M^m \) distinct \( \varepsilon \)-regular partitions \( V(G) = V_1 \cup \cdots \cup V_m \). Summing (16) over all possible partitions and all possible multicolored cluster graphs
$H$, the number of $(K_k, \geq s)$-free $r$-colorings of $G$ is at most

$$M^n \cdot 2^{H(2m)^2} \cdot r^{rp^2} \cdot \sum_H \left( \prod_{i=1}^{r} i^{e_i(H)} \right)^{(n/m)^2}.$$  

(17)

Note that, in this expression, we have $m = m(H) = |V(H)|$. We wish to bound the value of $\prod_{i=1}^{r} i^{e_i(H)}$.

First observe that

$$e_s(H) + \cdots + e_r(H) \leq \text{ex}(m, K_k).$$

Otherwise, by Turán’s Theorem, the multicolored cluster graph $H$ would contain a copy of $K_k$ such that every edge has a list of size at least $s$. This clearly induces a colored homomorphism of some pattern of $K_k$ with at least $s$ classes into $H$, so that by Lemma 2.5 $G$ would contain a copy of $K_k$ whose set of edges is colored with $s$ colors, a contradiction. More generally, we prove the following:

**Claim 3.1.** Let $H$ be a $(K_k, \geq s)$-free multicolored graph where $E_j = E_j(H)$ is the number of edges in $H$ with list of size $j$ and $e_j(H) = |E_j(H)|$ for any $j \in [r]$. For any fixed $i$ such that $A(k, k-i) \leq s - 1$, there is no copy of $K_k$ for which all edges lie in $E_{s-A(k,k-i)} \cup \cdots \cup E_r$ and at least $A(k, k-i)$ edges lie in $E_s \cup \cdots \cup E_r$. Therefore, if $i \leq k - 2$ and $A(k, k-i) \leq s - 1$, the following inequality holds:

$$\frac{k - i - 1}{k - i} \cdot \left( e_{s-A(k,k-i)}(H) + \cdots + e_{s-1}(H) \right) + e_s(H) + \cdots + e_r(H) \leq \text{ex}(m, K_k).$$

Note that $A(k, k-i) = i$ for $i \leq [k/2]$, so in this case the inequality in the claim becomes

$$\frac{k - i - 1}{k - i} \cdot \left( e_{s-i}(H) + \cdots + e_{s-1}(H) \right) + e_s(H) + \cdots + e_r(H) \leq \text{ex}(m, K_k).$$

**Proof.** Assume that there is a copy of $K_k$ for which all edges lie in $E_{s-A(k,k-i)} \cup \cdots \cup E_r$ and at least $A(k, k-i)$ edges lie in $E_s \cup \cdots \cup E_r$. Let $p$ be the number of edges in $E_{s-A(k,k-i)} \cup \cdots \cup E_{s-1}$ in this copy of $K_k$. Proceeding greedily (and starting with the edges in $E_{s-A(k,k-i)} \cup \cdots \cup E_{s-1}$), we may find a copy of $K_k$ such that $\alpha \geq \min\{s - A(k, k-i), p\}$ additional distinct colors appear in the edges $E_{s-A(k,k-i)} \cup \cdots \cup E_{s-1}$. If $\alpha \geq s - A(k, k-i)$, then at least $s - \alpha$ distinct colors may be chosen in the edges in $E_{s} \cup \cdots \cup E_r$, as this union contains at least $A(k, k-i)$ edges, each with a list of size at least $s$. If $\alpha = p < s - A(k, k-i)$, then the number of edges of the copy in $E_{s} \cup \cdots \cup E_r$ is $(\binom{k}{2}) - p \geq s - p$, so that at least $s - \alpha$ additional distinct colors may be chosen for edges in this set. In both cases, we get a copy of $K_k$ colored with $s$ or more colors, the desired contradiction.

Next, assuming that $k-i \geq 2$ and that $A(k, k-i) \leq s - 1$, let $E' \subseteq E_{s-A(k,k-i)} \cup \cdots \cup E_{s-1}$ be maximum with the property that the edges in $E'$ induce a $(k-i)$-partite subgraph of $H$. The number of edges of $E'$ in a copy of $K_k$ is at most $\binom{k}{2} - A(k, k-i)$, so that $|E'| + e_s(H) + \cdots + e_r(H) \leq \text{ex}(m, K_k)$ by the previous discussion. By Lemma 2.1, we know that $|E'| \geq (k-i) \cdot |E_{s-A(k,k-i)} \cup \cdots \cup E_{s-1}|/(k-i)$, which gives the desired result. \qed
For a multicolored cluster graph $H$, let
\[
\beta = \beta(H) = \frac{1}{m(H)^2}
\left( \text{ex}(m(H), K_k) - \sum_{j=s}^{r} e_j(H) \right) \geq 0. \tag{19}
\]

To find an upper bound in (17) on the number of $r$-colorings of $H$, we use (18) and (19) in the product
\[
\left( \prod_{e \in E(H)} |L_e| \right)^{(\frac{\beta}{m})^2} = \prod_{i=2}^{r} e_i(H) \leq 2^{e_2(H)} \cdot 3^{e_3(H)} \cdots (s - 1)^{e_{s-1}(H)} \cdot r^{\text{ex}(n, K_k) - \beta(H)n^2}. \tag{20}
\]

Maximizing this product is the same as maximizing
\[
\log \left( 2^{e_2(H)} \cdot 3^{e_3(H)} \cdots (s - 1)^{e_{s-1}(H)} \right) = \log 2 \cdot e_2(H) + \cdots + \log(s - 1) \cdot e_{s-1}(H). \tag{21}
\]

Moreover, with (19), the inequalities in Claim 3.1 lead to the following constraints. For all $i \in [k - 1]$ such that $A(k, k - i) \leq s - 1$, we get inequalities of the form
\[
\frac{k - i - 1}{k - i} \cdot (e_{s-A(k,k-i)}(H) + \cdots + e_{s-1}(H)) \leq \beta m^2. \tag{22}
\]

For $s \leq s_0(k)$, let $i^*$ be the least value of $i$ such that
\[
s - A(k, k - i) \leq 2,
\]
as defined in the introduction. The fact that $s \leq s_0(k)$ implies that $i^* \leq k - 2$. The constraints (22) for $i \in [i^*]$ may be written as
\[
\begin{align*}
\frac{k - 2}{k - 1} \cdot (e_{s-A(k,k-1)}(H) + \cdots + e_{s-1}(H)) & \leq \beta m^2 \\
\frac{k - 3}{k - 2} \cdot (e_{s-A(k,k-2)}(H) + \cdots + e_{s-1}(H)) & \leq \beta m^2 \\
& \vdots \\
\frac{k - i^* - 1}{k - i^*} \cdot (e_{s-A(k,k-i^*)}(H) + \cdots + e_{s-1}(H)) & \leq \beta m^2.
\end{align*}
\tag{23}
\]

This leads to a linear program with objective function (21), and constraints (23) and (24) for $i \in \{2, \ldots, i^*\}$. It is easy to see that the optimum is obtained for $e_{s-1}(H) = \frac{k-1}{k-2} \cdot \beta m^2$ and $e_{s-A(k,k-i+1)-1}(H) = (\frac{k-i}{k-i-1} - \frac{k-i+1}{k-i}) \cdot \beta m^2$ for $i \in \{2, \ldots, i^*\}$. Note that in this case we have
\[
\sum_{i=1}^{i^*} e_{s-A(k,k-i+1)-1}(H) = 2\beta m^2. \tag{24}
\]

To simplify the expressions below, for $k \geq 3$ and $2 \leq i \leq k - 2$, define the quantities
\[
s_i = s - A(k, k - i + 1) - 1 \quad \text{and} \quad \xi_i = \frac{1}{(k-i-1)(k-i)}. \tag{25}
\]
Plugging the optimal solution of the linear program into (17), we obtain (for \( s \leq s_0(k) \)),

\[
M^n \cdot 2^{H(2\eta)n^2} \cdot r^{\eta m^2} \cdot \sum_H \left( \prod_{i=1}^r \frac{\beta_i(H)}{r} \right)^{\frac{(\eta m)^2}{r}} \leq 
\]

\[
r^{r(H(2\eta)+2r\eta)} \cdot \sum_H \left( \frac{\prod_{i=2}^r s_i^{\xi_i}}{r^2} \right)^{\frac{(\eta m)^2}{r}} \beta(H) \eta^2 \leq r^{\text{ex}(n,K_k)}. \tag{26}
\]

By our choice of \( r_0 = r_0(k,s) \) (see [3]), given any \( \beta_0 > 0 \), there is \( \eta > 0 \) such that

\[
r^{\gamma'(H(2\eta)+2r\eta)} \cdot \left( \frac{\prod_{i=2}^r s_i^{\xi_i}}{r^2} \right)^{\frac{(\eta m)^2}{r}} \beta_0 \leq r^{\text{ex}(n,K_k)} \leq 1. \tag{27}
\]

Recall that we are using \( \beta_0 < \delta/(8k+2) \) and that \( \eta > 0 \) satisfies (27) for this value of \( \beta_0 \).

We claim that there exists a multicolored cluster graph \( H \) such that \( \beta(H) < \beta_0 \). Indeed, if this does not happen, the inequality (26) would be bounded above by

\[
r^{r(H(2\eta)+2r\eta)} \cdot 2^{rM^2/2} \left( \frac{\prod_{i=2}^r s_i^{\xi_i}}{r^2} \right)^{\frac{(\eta m)^2}{r}} \beta_0 \leq r^{\text{ex}(n,K_k)} \leq 1. \tag{28}
\]

a contradiction (we are using that the number of distinct multicolored cluster graphs is bounded above by \( 2^{rM^2/2} \), which is less than \( r^{H(2\eta)n^2/2} \) for \( n \) sufficiently large).

So, let \( H \) be a multicolored cluster graph for which \( \beta = \beta(H) < \beta_0 \). Consider the spanning subgraph \( H' \) of \( H \) with edge set \( E_s \cup \cdots \cup E_r \). This graph contains \( \text{ex}(m,K_k) - \beta m^2 \) edges. We may apply Theorem 2.7 with \( t = \beta m^2 \). By removing at most \( \beta m^2 \) edges of \( H' \), we produce a \((k-1)\)-partite subgraph \( H'' \). Let \( U_1 \cup \cdots \cup U_{k-1} \) be the resulting partition of \( V(H'') = V(H) \) such that

\[
\sum_{i=1}^{k-1} e_{H'}(U_i) \leq \beta m^2. \tag{29}
\]

By Claim 3.1 for \( i = k - 2 \), we know that \( e_2(H) + \cdots + e_{s-1}(H) \leq 2\beta m^2 \) for \( s \leq s_0 \). To bound \( e_1(H) \), we apply Corollary 2.3 to the \((k-1)\)-partite graph \( H'' \). This lemma ensures that we may not have \( e_1(H) \geq (2k-1) \cdot 2\beta m^2 = (4k-2) \beta m^2 \), otherwise adding \( E_1 \) to \( H'' \) would produce a copy of \( K_k \) for which exactly one of the edges would have a list of size one and all other edges would have lists with \( s \) or more colors, leading to the forbidden pattern. Therefore we must have \( e_1(H) < (4k-2) \beta m^2 \). This gives us an upper bound on the number of edges of \( H \) with color lists of size up to \( s-1 \):

\[
e_1(H) + \cdots + e_{s-1}(H) \leq 4k \beta m^2. \tag{30}
\]

Let \( W_i = \bigcup_{j \in U_i} V_j \), where \( i \in [k-1] \). We shall prove that \( V(G) = W_1 \cup \cdots \cup W_{k-1} \) satisfies the conclusion of the theorem. Edges of \( G \) with both endpoints in a same class \( W_i \) may come from three sources: edges of \( G \) that are not represented in \( H \); edges of
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$G$ in pairs $(V_s, V_t)$ such that $\{s,t\} \in E(H) \setminus E(H')$; edges of $G$ in pairs $(V_s, V_t)$ that correspond to edges in $E(H')$ with both endpoints in a same class $U_j$. By (13) – (15) and (30), we obtain

$$\sum_{i=1}^{k-1} e_G(W_i) \leq \eta n^2 + \left(\sum_{i=1}^{k-1} e_{H'}(U_i) + e_1(H) + \cdots + e_{s-1}(H)\right) \cdot \left(\frac{n}{m}\right)^2$$

$$\leq \eta n^2 + \left(\beta_0 m^2 + 4k\beta_0 m^2\right) \cdot \left(\frac{n}{m}\right)^2 \leq \eta n^2,$$

by our choice of $\beta_0$ and $\eta > 0$.

We now consider the case when $s > s_0(k)$. All the inequalities in (23) hold up to $i^* = k-2$, but in this case $s-A(k,2) > 2$, so that the variables $e_2(H), \ldots, e_{s-A(k,2)-1}(H)$ are not bounded by the linear constraints. The constraints become

$$\begin{align*}
\frac{k-2}{k-1} \cdot e_{s-1}(H) &\leq \beta m^2 \\
\frac{k-3}{k-2} \cdot (e_{s-A(k-2)}(H) + e_{s-1}(H)) &\leq \beta m^2 \\
: &\vdots \\
\frac{1}{2} \cdot (e_{s-A(k,2)}(H) + \cdots + e_{s-1}(H)) &\leq \beta m^2.
\end{align*}$$

(31)

Consider $1 \leq j \leq k-1$ and $2 \leq p \leq k-1$. Assume that $s \leq \binom{k}{2} - b(k,p,j) + 2$, where $b(k,p,j)$ comes from Lemma 2.2. We divide the set of multicolored cluster graphs into two classes, according to whether $\beta(H) \geq j(p-1)/(4p(k-1)^2)$ or $\beta(H) < j(p-1)/(4p(k-1)^2)$.

If $\beta = \beta(H) \geq j(p-1)/(4p(k-1)^2)$, we have, for $s_i$ and $\xi_i$ as in (25),

$$\prod_{i=1}^{r} i^{\xi_i(H)} \leq (s-A(k,2)-1)^{\binom{m}{2} - \text{ex}(m,K_k) + \beta m^2} \cdot \left(\prod_{i=2}^{k-2} s_i^{\xi_i} \cdot \left(\frac{s-1}{r}\right)^{k-1} \right) \cdot \text{ex}(m,K_k)^{\beta m^2}$$

$$\leq (s-A(k,2)-1)^{\sum_{i=2}^{k-2} \xi_i} \cdot \left(\frac{s-1}{r}\right)^{k-1} \cdot \text{ex}(m,K_k)^{\beta m^2}$$

$$\leq \left(\frac{(s-A(k,2)-1)^{2m(k-1)} \cdot \prod_{i=2}^{k-2} \xi_i \cdot (s-1)^{k-1}}{(s-A(k,2)-1) \cdot r} \right)^{\beta m^2} \cdot \text{ex}(m,K_k)^{\beta m^2}.$$
where \( \beta \leq \gamma \leq 2\beta \). In particular, the set \( \tilde{E} = E(H') - E(H'') \subset E_s \cup \cdots \cup E_r \) has cardinality \( (\gamma - \beta)m^2 \). Let \( E' = E_2 \cup \cdots \cup E_{s-A(k,2)-1} \). We may apply Lemma 2.2 to the edge set \( \tilde{E} \cup E' \) for our values of \( j \) and \( p \) to this value of \( \gamma \). If \( |E'| + |\tilde{E}| \geq (p(k-1)/(j(p-1)) + 1) \gamma m^2 \), then the multicolored cluster graph obtained by adding the edges in \( E' \cup \tilde{E} \) to \( H'' \) would contain a copy of \( K_k \) with at most \( b(k,p,j) \) edges in \( E' \). If \( |E' \cup \tilde{E}| = 1 \), we would greedily build an \( s \)-colored copy of \( K_k \), starting with the edge in \( E' \cup \tilde{E} \), a contradiction. If \( |E' \cup \tilde{E}| \geq 2 \), we obtain a copy of \( K_k \) with at least \((2 + \min\{s - 2, (k^2) - b(k,p,j)\}) \) colors. This is a contradiction whenever \( s \leq (k^2) - b(k,p,j) + 2 \), which is precisely the hypothesis in this case.

As a consequence, we have

\[
e_2(H) + \cdots + e_{s-A(k,2)-1}(H) + |\tilde{E}| \leq \left( \frac{p(k-1)}{j(p-1)} + 1 \right) \gamma m^2,
\]

so that

\[
e_2(H) + \cdots + e_{s-A(k,2)-1}(H) \leq \left( \frac{p(k-1)}{j(p-1)} + 1 \right) \gamma m^2 - |\tilde{E}|
= \frac{p(k-1)}{j(p-1)} \gamma m^2 + \beta m^2 \\
\leq \left( \frac{2p(k-1)}{j(p-1)} + 1 \right) \beta m^2.
\]

(33)

Using an argument as in (26), but applying the additional inequality (33) to bound \( e_2(H), \ldots, e_{s-A(2)-1}(H) \), we get, again with the notation in (25) and using the fact that \( e_{s-A(k,2)}(H) + \cdots + e_{s-1}(H) = 2\beta m^2 \), when the optimum solution of the linear program is achieved, it follows that

\[
\prod_{i=1}^{r} e_i(H) \leq \left( \frac{(s - A(k,2) - 1) 2p(k-1)}{j(p-1)} + 1 \right) \gamma m^2 \cdot \left( \prod_{i=2}^{k-2} s_i \cdot (s - 1)^{k-1} \right)^{1/2} r^{ex(m, K_k)}.
\]

(34)

It is clear that (32) is less than (33) for any fixed pair \((j,p)\). Therefore, by choosing \( r_0(k,s) \) greater than

\[
(s - A(k,2) - 1)^{L_{opt}(k,s)} \cdot \left( \prod_{i=2}^{k-2} (s - A(k, k-i + 1) - 1)^{1/((k-i-1)(k-i))} \right) \cdot (s - 1)^{k-1/2},
\]

where \( L_{opt}(k,s) \) is the least possible value of \( L(k,s,p,j) = 1 + \frac{2p(k-1)}{j(p-1)} \) subject to conditions (30) and (37), we may proceed as in the case \( s \leq s_0(k) \). That is, we may again fix \( \beta_0 = \delta/(8k + 2) \) and choose \( \eta > 0 \) appropriately, and consider whether \( \beta(H) \geq \beta_0 \) for all multicolored cluster graphs \( H \) (in which case we reach a contradiction) or whether there is a multicolored cluster graph \( H \) for which \( \beta < \beta_0 \) (in which case we get the desired partition), see the inequalities (26), (27) and (28).

To conclude the proof of Theorem 1.3, we determine the value of \( L_{opt}(k,s) \). This is the least value of

\[
L(k,s,p,j) = 1 + \frac{2p(k-1)}{j(p-1)}.
\]

(35)
where \( j \in [k - 1] \), \( p \in \{2, \ldots, k - 1\} \) and

\[
s \in \left[ \left( \frac{k}{2} \right) - \left\lfloor \frac{k}{2} \right\rfloor + 3, \left( \frac{k}{2} \right) \right]
\]

satisfy

\[
b(k, p, j) = \min \left\{ j \left( \frac{p^2}{2} \right), \left\lfloor \frac{k}{p} \right\rfloor \left( \frac{p^2}{2} \right) + \left( k - \left\lfloor \frac{k}{p} \right\rfloor p \right) \right\} \leq \left( \frac{k}{2} \right) - s + 2.
\]

Clearly, choosing \( j \) and \( p \) large would be good to minimize (35). The following claim summarizes our result.

**Claim 3.2.** For all \( s > s_0 \), the quantity \( L_{opt}(k, s) \) satisfies the following.

(a) If \( s \leq s_1 \), then there is \( p^* \in \{2, \ldots, k - 1\} \) such that \( L_{opt}(k, s) = L(k, s, p^*, k - 1) \).

Moreover,

\[
3 < L_{opt}(k, s) \leq 5
\]

(b) If \( s > s_1 \), then there is \( j^* \in \{2, \ldots, k - 2\} \) such that

\[
L(k, s, 2, j^*) = L_{opt}(k, s) = 1 + \frac{4(k - 1)}{\left( \frac{k}{2} \right) - s + 2} > 9.
\]

To prove the claim, we start with part (b), where \( s \geq \left( \frac{k^2}{2} \right) - \left\lfloor \frac{k^2}{2} \right\rfloor + 3 \). First note that

\[
\left\lfloor \frac{k}{p} \right\rfloor \left( \frac{p^2}{2} \right) + \left( k - \left\lfloor \frac{k}{p} \right\rfloor p \right) \leq \left\lfloor \frac{k}{2} \right\rfloor \left( \frac{p^2}{2} \right) + \left( \frac{k}{2} - \left\lfloor \frac{k}{2} \right\rfloor \right) \left( \left( \frac{k}{2} \right) - s + 2 \right).
\]

In particular, to satisfy (37), we need \( b(k, p, j) = j \left( \frac{p^2}{2} \right) \leq \left( \frac{k}{2} \right) - s + 2 \). As a consequence, for any pair \((p, j)\) such that (37) holds, we have

\[
L(k, s, p, j) \geq 1 + \frac{2p(k - 1)}{(p - 1) \left( \frac{k}{2} - s + 2 \right) \left( \frac{k}{2} \right)} = 1 + \frac{p^2(k - 1)}{\left( \frac{k}{2} \right) - s + 2} \geq L(k, s, 2, \left( \frac{k}{2} \right) - s + 2).
\]

Since the pair \((p, j) = (2, \left( \frac{k}{2} \right) - s + 2)\) satisfies (37), we deduce that \( L_{opt}(k, s) = L(k, s, 2, \left( \frac{k}{2} \right) - s + 2) \), so that

\[
L_{opt}(k, s) = 1 + \frac{4(k - 1)}{\left( \frac{k}{2} \right) - s + 2}
\]

\[
\geq 1 + \frac{4(k - 1)}{\left( \frac{k^2}{2} \right) - \left\lfloor \frac{k^2}{2} \right\rfloor + 3 + 2}
\]

\[
\geq 1 + \frac{8(k - 1)}{k - 2} > 9.
\]

This proves part (b).

We now consider part (a). Fix \( p \geq 2 \) and fix a pair \((p, j)\) that satisfies (37). Since

\[
\left\lfloor \frac{k}{p} \right\rfloor \left( \frac{p^2}{2} \right) + \left( k - \left\lfloor \frac{k}{p} \right\rfloor p \right) \leq \left\lfloor \frac{k(p - 1)}{2} \right\rfloor,
\]

if the minimum in (37) is attained by \( j \left( \frac{p^2}{2} \right) \), then

\[
j \left( \frac{p^2}{2} \right) \leq \frac{k(p - 1)}{2},
\]
which implies $j \leq k/p$. This implies that, if there exists $j > k/p$ such that $(p, j)$ satisfies (37), then we must have

$$\left\lfloor \frac{k}{p} \right\rfloor \left( \frac{p}{2} \right) + \left( k - \left\lfloor \frac{k}{p} \right\rfloor p \right) \leq \left( \frac{k}{2} \right) - s + 2,$$

so that the pair $(p, k - 1)$ satisfies (37) and leads to $L(k, s, p, k - 1) \leq L(k, s, p, j)$.

For $j = k - 1$, let $p^*$ be the largest value of $p$ such that $(p, k - 1)$ satisfies (37). This is well defined because $(2, k - 1)$ satisfies (37) for $s \leq \left( \frac{k}{2} \right) - \left\lfloor \frac{k}{2} \right\rfloor + 2 = s_1(k)$. Towards finding a suitable pair $(p, j)$ that minimizes $L(k, s, p, j)$, the only other candidates are pairs $(j', p')$, satisfying (37), such that $j' \leq k/2$ and $p^* \leq p' \leq k - 1$.

The inequality

$$\frac{p \cdot k/2}{(p - 1)(k - 1)} \leq \frac{p'}{p' - 1}. \quad (38)$$

holds because the left-hand side is at most $k/(k - 1)$, while the right-hand side is greater than this, as $p' \leq k - 1$. We conclude that

$$L(k, s, p^*, k - 1) = \frac{2p^*(k - 1)}{(k - 1)(p^* - 1)} \leq \frac{2p'(k - 1)}{j' \cdot (p' - 1)} = L(k, s, p', j').$$

Thus, to compute $L_{opt}(k, s)$, it remains to find the right value of $p^*$.

For $j = k - 1$ and $p \geq 2$, we have

$$(k - 1)\left( \frac{p}{2} \right) \geq \frac{k(p - 1)}{2},$$

hence in this case

$$b(k, p, k - 1) = \left\lfloor \frac{k}{p} \right\rfloor \left( \frac{p}{2} \right) + \left( k - \left\lfloor \frac{k}{p} \right\rfloor p \right) \leq \left\lfloor \frac{k(p - 1)}{2} \right\rfloor.$$

This means that (37) becomes

$$\left\lfloor \frac{k}{p} \right\rfloor \left( \frac{p}{2} \right) + \left( k - \left\lfloor \frac{k}{p} \right\rfloor p \right) \leq \left( \frac{k}{2} \right) - s + 2 \quad (39)$$

and that

$$L_{opt}(k, s) = L(k, s, p, k - 1) = 1 + \frac{2p^*}{p^* - 1}$$

where $p^*$ is the largest $p$ satisfying (39).

On the one hand,

$$L(k, s, p^*, k - 1) = 1 + \frac{2p^*}{p^* - 1} > 3$$

for any $p^* \geq 2$. On the other hand, we have

$$L_{opt}(k, s) \leq L(k, s, 2, k - 1) = 5.$$

This establishes part (a) of our claim, and finishes the proof of Theorem 1.3. \qed
4. Proof of Theorem 1.2

To prove Theorem 1.2 we shall use the following special case of an auxiliary result [10], whose proof uses tools as in [6, Theorem 1.1].

**Theorem 4.1.** Let $n, r, k \geq 2$ with $2 \leq s \leq \binom{k}{2}$ be integers. If there exists an $(r, P_{k,s})$-extremal graph on $n$ vertices that is not complete multipartite, then there exist at least two non-isomorphic $(r, P_{k,s})$-extremal complete multipartite graphs on $n$ vertices.

We now show that, for $k \geq 4$, $2 \leq s \leq \binom{k}{2}$, $r > r_0(k, s)$ and sufficiently large $n$, there is actually a single $(r, P_{k,s})$-extremal graph on $n$ vertices, the Turán graph $T_{k-1}(n)$.

**Proof of Theorem 1.2.** Let $k \geq 4$ and $2 \leq s \leq \binom{k}{2}$ be integers. Fix $r \geq r_0(k, s)$, with $r_0$ defined in (3), (6) and (7) for $s \leq s_0(k)$, $s_0(k) < s \leq s_1(k)$ and $s > s_1(k)$, respectively. Consider a constant $0 < \alpha < 1$ such that

$$2^{2H(\alpha)} \cdot (r - 1) < r,$$

and fix $\delta > 0$ such that

$$\delta < \frac{1}{2(k-1)^8} \quad \text{and} \quad r^\delta < \left[ \frac{r}{(r-1) \cdot 2^{2H(\alpha)}} \right]^{\frac{\alpha^2(k-1)}{r^{x(k-1)^p}}}.$$

Let $n_0 = n_0(r, k, s)$ from Theorem 1.3. We shall further assume that $n_0$ is large enough so that all the inequalities marked with $n \gg 0$ are satisfied. Fix $n_1 \geq n_0^2$.

To reach a contradiction, suppose that there is an $n$-vertex graph $G = (V, E)$ that is $(r, K_{k, s})$-extremal, but $G \neq T_{k-1}(n)$. We may assume that $G$ is a complete multipartite graph (if it is not, replace it by an $(r, K_{k, s})$-extremal graph that is complete multipartite and different from $T_{k-1}(n)$, which exists by Theorem 4.1). Let $V = V_1' \cup \cdots \cup V_p'$ be the multipartition of $G$, where $p \geq k$.

Let $V = V_1 \cup \cdots \cup V_{k-1}$ be a partition of the vertex set of $G$ such that $\sum_{i=1}^{k-1} e(V_i)$ is minimized, so that

$$\sum_{i=1}^{k-1} e(V_i) \leq \delta n^2$$

by Theorem 1.3. The minimality of this partition ensures that, if $v \in V_i$, then $|V_j \cap N(v)| \geq |V_i \cap N(v)|$, for every $j \in [k-1]$, where $N(v)$ denotes the set of neighbors of $v$. Moreover, by Proposition 2.8 we must have

$$\left| |V_i| - \frac{n}{k-1} \right| < \sqrt{2\delta} \cdot n.$$

Finally, given that $p > k - 1$, there must be an edge $\{x, y\} \in E$ whose endpoints are contained in the same class of the partition; assume without loss of generality that they lie in $V_{k-1}$ and that $|N(x) \cap V_{k-1}| \geq |N(y) \cap V_{k-1}|$. Since $x$ and $y$ are in different classes of $V_1' \cup \cdots \cup V_p'$, any $z \in V_{k-1} - \{x, y\}$ must be adjacent to $x$ or $y$. We conclude that, for any $i \in [k-1],$

$$|N(x) \cap V_i| \geq |N(x) \cap V_{k-1}| \geq \frac{|V_{k-1}| - 2}{2} + 1 \geq \frac{n}{2(k-1)} - \frac{\sqrt{2\delta} \cdot n}{2} \geq \frac{n}{(k-1)^2}.$$

For simplicity, we write $W_i = N(x) \cap V_i$ for $i \in [k-1]$.

We shall consider the cases $2 \leq s \leq \binom{k-1}{2} + 1$ and $s > \binom{k-1}{2} + 1$ separately.
Case 1. Assume that $2 \leq s \leq \binom{k-1}{2} + 1$. Let $\mathcal{C}$ be the family of $\mathcal{P}_{k,s}$-free $r$-colorings of $G$.

Fix a coloring $\widehat{G} \in \mathcal{C}$. For each $i \in [k-1]$ and each color $c \in [r]$, let $W_{c,i}^\widehat{G}$ be the set of vertices in $N(x) \cap V_i$ that are connected to $x$ by an edge of color $c$ in $\widehat{G}$. By the pigeonhole principle, for each $i \in [k-1]$, there must be a color $c_i \in [r]$ such that

$$|W_{c,i}^\widehat{G}| \geq \frac{|W_i|}{r} \geq \frac{n}{r(k-1)^3}.$$

We say that color $c$ is rare with respect to a pair $\{i, j\} \in \binom{[k-1]}{2}$ if there exist subsets $X_i \subseteq W_{c,i}^\widehat{G}$ and $X_j \subseteq W_{c,j}^\widehat{G}$, where $|X_i| \geq \alpha^{k-1}|W_{c,i}^\widehat{G}|$ and $|X_j| \geq \alpha^{k-1}|W_{c,j}^\widehat{G}|$, for which the number of edges of color $c$ between $X_i$ and $X_j$ is less than $\alpha |X_i||X_j|$. Otherwise, $c$ is said to be abundant for the pair $\{i, j\}$.

We claim that, for any fixed $\widehat{G}$, there must be a pair $\{i, j\} \in \binom{[k-1]}{2}$ and a color $c \in [r]$ such that $c$ is rare with respect to $\{i, j\}$. To see why this is true, assume on the contrary that every color is abundant with respect to every pair. Since $r \geq s$ and $s \leq \binom{k-1}{2} + 1$, we may choose $s - 1$ colors in $[r] \setminus \{c_{k-1}\}$ and assign them arbitrarily to pairs $\{i, j\} \in \binom{[k-1]}{2}$ in a way that each pair is assigned a color and all colors appear. This leads to an edge-coloring $\widehat{K}_{k-1}$ of a copy of $K_{k-1}$ with vertex set $\{v_1, \ldots, v_{k-1}\}$ where exactly $s - 1$ colors appear and they are all different from $c_{k-1}$. By Lemma 2.6 $\widehat{G}$ contains a copy of $\widehat{K}_{k-1}$ with vertex set $\{x_1, \ldots, x_{k-1}\}$ with the property that $x_i \in W_{c,i}^\widehat{G}$ for all $i \in [k-1]$. By construction, we see that $\widehat{G}[x_1, x_2, \ldots, x_{k-1}]$ induces a copy of $K_k$ where at least $s$ colors appear, the desired contradiction.

We are now ready to find an upper bound on $\mathcal{C}$. To do this, we shall bound the number of $\mathcal{P}_{k,s}$-free $r$-colorings that may be associated with a pair $(X_i, X_j)$ and color $c$ as above. There are $r$ choices for the color $c$ and at most $2^{2n}$ choices for the pair of sets $X_i, X_j$. Once $c$ and the sets $X_i, X_j$ are fixed, we have at most

$$\binom{|X_i||X_j|}{r-1}|X_i||X_j| \cdot 2^{H(\alpha)|X_i||X_j|} \geq 2^{H(\alpha)|X_i||X_j|}(r - 1)^{|X_i||X_j|}$$

ways to color the edges between $X_i$ and $X_j$. Note that $|X_i||X_j| \geq \frac{k^2 n}{r(k-1)^3}$. Assuming towards an upper bound that the at most $\exp(n, K_k) + \delta n^2 - |X_i||X_j|$ remaining edges may be colored arbitrarily, we obtain

$$|\mathcal{C}| \leq r \cdot 2^{2n} \cdot \exp(n, K_k) + \delta n^2 - |X_i||X_j| \cdot 2^{H(\alpha)|X_i||X_j|}(r - 1)^{|X_i||X_j|} \leq \left(2^{2H(\alpha)} \cdot \frac{r-1}{r}\right)^{|X_i||X_j|} \cdot r^{\delta n^2} \cdot \exp(n, K_k) \leq \left(2^{2H(\alpha)} \cdot \frac{r-1}{r}\right)^{|X_i||X_j|} \cdot r^{\delta n^2} \cdot \exp(n, K_k) = \gamma^{n^2} \exp(n, K_k) \leq \exp(n, K_k)$$

where $\gamma < 1$ is a constant\footnote{with respect to $n$.} by our choice of $\delta > 0$ in (11). This contradicts our choice of $G$. 

Case 2. Now assume that \( s > \binom{k-1}{2} + 1 \). Let \( s' = s - \binom{k-1}{2} \), so that \( 2 \leq s' \leq k - 1 \). To get a contradiction, we assume that the complete multipartite \((r, (K_1, \geq s))\)-extremal graph \( G \) has \( r^{ex(n, K_{k-1}) + m} \) distinct \((K_1, \geq s)\)-free colorings, where \( m \geq 0 \). We shall prove that the graph \( G - x \) must have at least \( r^{ex(n-1, K_k) + m+1} \) such colorings. This conclusion will lead to the desired contradiction, as we could apply this argument iteratively until we obtain a graph \( G' \) on \( n_0 \) vertices and at least \( r^{ex(n_0, K_k) + m + n - n_0} > r^{n_0^2} \geq r^{\mid E(G) \mid} \).

Let \( C \) be the family of \( P_{k,s} \)-free \( r \)-colorings of \( G \), and let \( C_1 \) be the subfamily containing all colorings \( \hat{G} \) for which there is a choice of distinct indices \( i_1, \ldots, i_{s'} \in [k - 1] \) and distinct colors \( c_{i_1}, \ldots, c_{i_{s'}} \in [r] \) such that, for each \( p \in [s'] \), the set \( W_{i_p, c_{i_p}}^\hat{G} \subseteq V_p \cap N(x) \) of neighbors of \( x \) in \( V_p \) through edges of color \( c_{i_p} \) (with respect to \( \hat{G} \)) satisfies \( |W_{i_p, c_{i_p}}^\hat{G}| \geq n/[r(k-1)^3] \). For any \( i \in [k - 1] \setminus \{i_1, \ldots, i_{s'}\} \), we fix an arbitrary color \( c_i \) such that the set \( W_{i, c_i}^\hat{G} \) of neighbors of \( x \) in \( V_i \) through edges of color \( c_i \) satisfies \( |W_{i, c_i}^\hat{G}| \geq n/[r(k-1)^3] \) (this color exists by the pigeonhole principle).

As in Case 1, we say that color \( c \) is rare with respect to a pair \( \{i, j\} \in \binom{k-1}{2} \) if there exist subsets \( X_i \subseteq W_{i, c_i}^\hat{G} \) and \( X_j \subseteq W_{j, c_j}^\hat{G} \), where \( |X_i| \geq \alpha^{k-1}|W_{i, c_i}^\hat{G}| \) and \( |X_j| \geq \alpha^{k-1}|W_{j, c_j}^\hat{G}| \), for which the number of edges of color \( c \) between \( X_i \) and \( X_j \) is less than \( \alpha|X_i||X_j| \). Otherwise, \( c \) is said to be abundant for the pair \( \{i, j\} \). We claim that, for any fixed \( \hat{G} \), there must be a pair \( \{i, j\} \in \binom{k-1}{2} \) and a color \( c \in [r] \) such that \( c \) is rare with respect to \( \{i, j\} \). If this was not the case, we would be able to fix \( s - s' = \binom{k-1}{2} \) distinct colors in \( [r] \setminus \{c_{i_1}, \ldots, c_{i_{s'}}\} \) to be assigned to the edges of a copy of \( K_{k-1} \), which, with Lemma 2.6 would lead to a contradiction. Using the arguments in (12), we conclude that \( |C_1| \leq \gamma n^{2r^{ex(n, K_k)}} \), where \( \gamma < 1 \) is a constant.

As a consequence the family \( C_2 = C \setminus C_1 \) contains at least \( r^{ex(n, K_k) + m} - \gamma n^{2r^{ex(n, K_k)}} \geq r^{ex(n, K_k) + m - 1} \) colorings. Fix a coloring \( \hat{G} \in C_2 \). We define a bipartite graph \( B^\hat{G} \) with bipartition \([k - 1] \cup [r]\) such that \( \{i, c\} \) is an edge if the set \( W_{i, c}^\hat{G} \subseteq V_i \cap N(x) \) of neighbors of \( x \) in \( V_i \) through edges of color \( c_i \) satisfies \( |W_{i, c}^\hat{G}| \geq n/[r(k-1)^3] \). Note that \( \hat{G} \) lies in \( C_1 \) if and only if \( B^\hat{G} \) contains a matching of size \( s' \). Since \( \hat{G} \notin C_1 \), the following holds by Hall’s Theorem. There is an integer \( h \), where \( 1 \leq h \leq s' - 1 \), a set of distinct indices \( I = \{i_1, \ldots, i_{h+1}\} \subseteq [k - 1] \) and a set \( C \subseteq [r] \) of colors, where \( |C| \leq h \), such that \( |W_{i_p, c}^\hat{G}| \geq n/[r(k-1)^3] \) only if \( c \in C \). We shall associate each \( \hat{G} \in C_2 \) with such a triple \((h, I, C)\). Note that we may suppose that \( |C| = h \) by adding arbitrary new elements to \( C \) if necessary.

Let \( \phi(C_2) \) be the family that contains the projection of each element of \( C_2 \) onto the edges incident with \( x \). In other words, \( \phi(C_2) \) contains all \( r \)-edge colorings of the edges incident with \( x \) that may be extended to a coloring in \( C_2 \). We are now ready to find an upper bound on the cardinality of \( \phi(C_2) \). To do this, we shall bound the number of colorings that may be associated with a triple \((h, I, C)\) as above. Given \( h \), there are \( \binom{k-1}{h+1} \leq 2^{k-1} \) ways to fix \( I = \{i_1, \ldots, i_{h+1}\} \) and \( \binom{r}{h} \leq 2^r \) ways to choose \( C \). Once \((h, I, C)\) is fixed, an upper bound on the number of ways to color the edges between \( x \) and \( V_i \), where \( i \in I \), is

\[
\left( \frac{|V_i|}{n/[r(k-1)^3]} \right)^r \leq 2^H \binom{\frac{\gamma}{r(k-1)^3}}{n/|V_i|} \leq \left( r^{k-1} \right) \frac{2^h}{(k-1)^{r}} \frac{1}{h^{\frac{1}{k-1} + \sqrt{2d}}}. \]
The first term in the product is an upper bound on the number of ways to color edges using each of the at most \( r \) colors that are used at most \( n/\lfloor r(k-1)^3 \rfloor \) times, the second term is an upper bound on the number of ways to color edges with the remaining colors, which lie in \( C \). For \( i \notin I \), the number of ways to color the edges between \( x \) and \( V_i \) will be bounded with the trivial bound \( r^{|V_i|} \leq r^{(\frac{1}{k+\sqrt{2}d})} \).

Combining this information, we obtain

\[
|\phi(C_2)| \leq \sum_{h=1}^{s' \cdot 2^{k-1} \cdot 2^r \cdot \lfloor r(k-1)^3 \rfloor^{\frac{2h+1}{(k-1)^3} \cdot \frac{r^{h+1} \cdot r^{k-h-2}}{(\frac{1}{k+\sqrt{2}d})}}}
\]  

(43)

Consider the function \( f(h) = h^{h+1}r^{-h} \). With \( 1 \leq h \leq k-2 \), since \( r > (k-1)^4 > c(k+1) \) for \( k \geq 4 \), we have

\[
\frac{f(h+1)}{f(h)} = \left( 1 + \frac{1}{h} \right)^h \cdot \frac{(h+1)^2}{rh} \leq \frac{e}{r} \left( h + 2 + \frac{1}{h} \right) < 1.
\]

This means that \( f(h) \) is maximized for \( h = 1 \). On the other hand, it is clear that

\[
\lfloor r(k-1)^3 \rfloor^{\frac{2h+1}{(k-1)^3} \cdot \frac{r^{h+1} \cdot r^{k-h-2}}{(\frac{1}{k+\sqrt{2}d})}} \leq \lfloor r(k-1)^3 \rfloor^{\frac{2s'}{(k-1)^3} \cdot \frac{r^{k-3} \cdot (\frac{1}{k+\sqrt{2}d})}{(k-1)^3}}.
\]

Finally, the minimum degree \( \delta_{k-1}(n) \) of a vertex in the Turán graph \( T_{k-1}(n) \) satisfies \( \delta_{k-1}(n) \geq \frac{k-2}{k-1}n - 1 \). Using this in (43) leads to

\[
|\phi(C_2)| \leq s' \cdot 2^{k-1} \cdot 2^r \cdot \lfloor r(k-1)^3 \rfloor^{\frac{2s'}{(k-1)^3} \cdot \frac{r^{k-3} \cdot (\frac{1}{k+\sqrt{2}d})}{(k-1)^3}} \cdot r^{\delta_{k-1}(n)}.
\]

(44)

We claim that

\[
\frac{(k-1)^6s'}{r(k-1)^2 - 2s' - \frac{k-2}{k-1}} < 1.
\]

(45)

Before proving this, we argue that it leads to the desired result. Using this, inequality (44) implies that

\[
|\phi(C_2)| \leq r^{\delta_{k-1}(n) - 2}
\]

for sufficiently large \( n \), and therefore

\[
|C_{r,p_{k,s}}(G-x)| \geq \frac{|C_2|}{|\phi(C_2)|} \geq \frac{r^{ex(n,K_k)+m-1}}{r^{\delta_{k-1}(n) - 2}} = r^{ex(n-1,K_k)+m+1},
\]

as required.

To conclude the proof, we show that (45) holds. We start with the case \( k \geq 5 \), where we first show that \( r_0(k,s) > (k-1)^4 \) when \( s \geq \left( \frac{k-1}{2} \right)^2 \). Indeed, by the definition of \( r_0(s,t) \) (for \( s > s_0 \)) and by the lower bounds on \( L_{opt}(k,s) \) given in Claim 3.2, we deduce
that

\[ r_0(k, s) > (s - A(k, 2) - 1)^3 \cdot (s - A(k, 2))^2 - \frac{k-1}{k-2} \cdot (s - 1)^{\frac{k-1}{k-2}} \]

\[ \geq (s - A(k, 2) - 1)^5 \cdot \left( \binom{k-1}{2} + 1 \right) \]

\[ > \left( \binom{k-1}{2} - \left( \binom{k}{2} - \left\lfloor \frac{k}{2} \right\rfloor \right) + 1 \right) \cdot \left( \binom{k-1}{2} \right)^5 \cdot \left( \frac{k-1}{2} \right) \]

It turns out that

\[ \left\lfloor \frac{k}{2} \right\rfloor \cdot \left\lfloor \frac{k}{2} \right\rfloor - k + 2 \geq \frac{k^2 - 1}{4} - k + 2 \geq k - 2. \]

Therefore, the inequality (46) leads to

\[ r_0(k, s) > \frac{(k-2)^6(k-1)}{2} > (k-1)^4, \]  

(47)

where the last part may be easily proved by induction (for \( k \geq 4 \)).

Coming back to (45), we first consider the case \( k \geq 5 \). First note that

\[ \frac{(k-1)^{6s'}}{r^{(k-1)^2-2s'-\frac{k-1}{k-2}}} \leq \frac{(k-1)^{4s'/2}}{r^{(k-1)^2-2s' - 1}}. \]

Since \( r > (k-1)^4 \), inequality (45) holds if we show that \( 3s'/2 \leq (k-1)^2 - 2s' - 1 \), which is equivalent to

\[ \frac{7s'}{2} \leq k^2 - 2k. \]

The left-hand side of this inequality is at most \( 7(k-1)/2 \), and it is indeed the case that \( 7(k-1) \leq 2k^2 - 4k \) for all \( k \geq 5 \).

In the case \( k = 4 \), inequality (45) holds if and only if

\[ 3^{6s'} \leq r^{9-2s'-\frac{5}{4}}. \]  

(48)

Since we are in Case 2, we need to consider the cases \( s = 5 \) (so \( s' = 2 \)) and \( s = 6 \) (so \( s' = 3 \)). In the case \( s = 5 \), the inequality holds if \( 3^{12} \leq r^{14/3} \), which holds for \( r \geq 17 \). According to Table 1.1, \( r_0(4, 5) = 222 \). In the case \( s = 6 \), inequality (45) holds if \( 3^{18} \leq r^{8/3} \), holds for \( r \geq 1662 \). According to Table 1.1, \( r_0(4, 6) = 5434 \). This concludes the proof of Theorem 1.2. \( \square \)

5. Final remarks and open problems

In this paper, given integers \( r \geq 2 \), \( n \geq k \geq 3 \) and \( 2 \leq s \leq \binom{k}{2} \), we were interested in characterizing \( n \)-vertex graphs \( G \) for which the number of \( r \)-edge-colorings with no copy of \( K_k \) colored with \( s \) or more colors satisfies

\[ |\mathcal{C}_{r,P_k,s}(G)| = c_{r,P_k,s}(n) = \max \left\{ |\mathcal{C}_{r,P_k,s}(G')| : |V(G')| = n \right\}. \]

(49)

This problem is a common generalization of the Turán problem and of the rainbow Erdős-Rothschild problem, i.e., the problem of finding \( n \)-vertex graphs \( G \) with the largest number of \( r \)-edge-colorings with no rainbow copy of \( K_k \).

More precisely, we have found functions \( r_1(k, s) \) and \( r_0(k, s) \) such that
(a) If \( r \geq r_0(k, s) \) and \( n \) is sufficiently large, then \( |C_{r, \mathcal{P}_{k,s}}(G)| = c_{r, \mathcal{P}_{k,s}}(n) \) if and only if \( G \) is isomorphic to \( T_{k-1}(n) \).

(b) If \( r \leq r_1(k, s) \), then \( |C_{r, \mathcal{P}_{k,s}}(K_n)| > |C_{r, \mathcal{P}_{k,s}}(T_{k-1}(n))| \).

We should mention that reference [17] gives a function \( r'_0(k) \) such that the statement of part (a) holds for all \( r \geq r'_0(k) \geq \left(\frac{k}{2}\right)^8 k^{-4} \) when \( s = \left(\frac{k}{2}\right) \). With Lemma 1.1, this implies the existence of a function \( r'_0(k, s) \) such that (a) is satisfied. The results of the current paper are such that \( r_0(k, \left(\frac{k}{2}\right)) \leq (k^2/4)^4 k \), that \( r_0(k, s) \leq (s - 1)^2 \) for \( s \leq s_0(k) = \left(\frac{k}{2}\right) - \left\lfloor \frac{k}{2} \right\rfloor + 2 + 2 \) and that \( r_0(k, s) \leq (s - 1)^7 \) for \( s \leq s_1(k) = \left(\frac{k}{2}\right) - \left\lfloor \frac{k}{2} \right\rfloor + 2 \).

For general values of \( k \) and \( s \), there is a significant gap between the functions \( r_0 \) and \( r_1 \) in (a) and (b), but it turns out that \( r_0(k, s) = r_1(k, s) + 1 \) for all pairs \((k, 3)\) with \( k \geq 4 \) and for infinitely many other pairs \((k, s)\). Computing the values of \( r_0 \) and \( r_1 \), we see that this happens for the following pairs if \((k, s)\) where \( 4 \leq k < 10 \) and \( s \geq 4 \):

\[(k, s) \in \{(5, 4), (7, 4), (7, 5), (8, 4), (8, 5), (9, 3), (9, 4), (9, 6)\}.

In fact, the following holds.

**Proposition 5.1.** Let \( s \geq 3 \) be an integer. There exists \( k_0 \) such that, for all \( k \geq k_0 \), the pair \((k, s)\) satisfies \( r_0(k, s) = s \) and \( r_1(k, s) = s - 1 \).

**Proof.** Let \( s \geq 3 \). First notice that \((s - 1)^{1/(k-2)} \) is not an integer for \( k > 2 + \log_2(s - 1) \).

Fix \( k_1 \) such that \((s - 1)^{1/(k-2)} \) is not an integer and \((s - 1)^{\frac{k-1}{s-2}} < s - \frac{1}{2} \) for all integers \( k \geq k_1 \). In particular, we have

\[ s - 1 = \left\lfloor (s - 1)^{\frac{k-1}{s-2}} - 1 \right\rfloor = r_1(k, s). \]

Next, let \( k_2 \) be such that, for all integers \( k \geq k_2 \), we have \( s \leq s_0(k) \) and \( i^* = s - 2 \), see [3]. In particular, \( r_0(k, s) \) is the least integer greater than

\[ (s - 1)^{\frac{k-1}{s-2}} \cdot (s - 2)^{\frac{s-3}{(k-2)(s-1)}} \cdot (s - 3)^{\frac{s-3}{(k-4)(s-1)}} \cdots 2^{\frac{s-3}{(k-s+1)(s-1)}} \leq (s - 1)^{\frac{s-3}{(k-s+1)(s-2)}}. \]

We may choose \( k_3 \) such that, for all \( k \geq k_3 \), it holds that

\[ \left(1 + \frac{1}{2(s - 1)^{3/2}}\right)^{(k-s+1)(k-2)} > (s - 2)^{s-3}. \]

This implies that, for all \( k \geq \max\{k_1, k_2, k_3\} \), it is

\[ (s - 1)^{\frac{k-1}{s-2}} (s - 2)^{\frac{s-3}{(k-s+1)(k-2)}} < (s - 1)^{\frac{k-1}{s-2}} \left(1 + \frac{1}{2(s - 1)^{3/2}}\right) \leq (s - 1)^{\frac{k-1}{s-2}} + \frac{1}{2} < s. \]

As a consequence, we have

\[ s \leq r_0(k, s) \leq \left\lfloor (s - 1)^{\frac{k-1}{s-2}} (s - 2)^{\frac{s-3}{(k-s+1)(k-2)}} \right\rfloor + 1 < s + 1, \]

so that \( r_0(k, s) = s \). \( \square \)

By definition, it is obvious that, if \( r \leq s - 1 \), then \( |C_{r, \mathcal{P}_{k,s}}(G)| = c_{r, \mathcal{P}_{k,s}}(n) \) if and only if \( G = K_n \). This implies that, for pairs \((k, s)\) such that \( s = 3 \) or Proposition 5.1 is satisfied, the functions \( r_1 \) and \( r_0 \) are best possible in (a) and (b) and \( K_n \) is \((r, \mathcal{P}_{k,s})\)-extremal for \( r \leq 2 \) and \( T_{k-1}(n) \) is \((r, \mathcal{P}_{k,s})\)-extremal for \( r \geq 3 \).

We propose the following questions.

\footnote{For sufficiently large \( n \).}
Question 1. We say that a pair \((k, s)\) such that \(k \geq 3\) and \(3 \leq s \leq \binom{k}{2}\) satisfies Property A if there exists \(r^* = r^*(k, s)\) such that, for any fixed \(r \geq 2\), there exists \(n_0\) for which the following holds: \(G\) is an \((r, P_{k,s})\)-extremal \(n\)-vertex graph with \(n \geq n_0\) if and only if \(r < r^*\) and \(G = K_n\), or \(r \geq r^*\) and \(G = T_{k-1}(n)\). Is it true that all such pairs \((k, s)\) satisfy Property A?

The above paragraph ensures that all pairs that fulfil Proposition 5.1 also satisfy Property A. Moreover, previous work for triangles [4, 15] implies that the pair \((3, s)\) satisfies Property A for any \(s\), while the current paper shows that the pair \((k, 3)\) satisfies Property A for any \(k\).

Question 2. Is it true that, for any pair \((k, s)\) such that \(k \geq 3\) and \(3 \leq s \leq \binom{k}{2}\) and any \(r \geq s\), it holds that

\[
|C_{r,P_{k,s}}(K_n)| = \left(\frac{r}{s-1}\right) + o_n(1) \left(s-1\right)^{\binom{n}{2}} \tag{50}
\]

Equation (50) holds for all \(r \geq 3\) in the case \(k = 3\), see [4, 5] (the second reference actually shows that this relation holds for some functions \(r = r(n)\) such that \(r \to \infty\) as \(n \to \infty\)). We observe that, if a pair \((k, s)\) satisfies Property A and equation (50) holds for \((k, s)\) and any \(r > r_1(k, s)\), then we must have \(r^*(k, s) = r_1(k, s) + 1\).

In a different direction, given integers \(r \geq 2\), \(n \geq k \geq 3\) and \(1 \leq s \leq \binom{k}{2}\), it would be interesting to consider \(r\)-edge-colorings with no copy of \(K_k\) colored with \(s\) or less colors, which would lead to a common generalization of the Turán problem and of the (monochromatic) Erdős-Rothschild problem. The class of extremal configurations for large \(n\) and \(r\) will be much richer, as \(K_n\) does not admit any such coloring by Ramsey’s Theorem and \(T_{k-1}(n)\) admits fewer colorings than other constructions (see [1, 20]).

References

1. N. Alon, J. Balogh, P. Keevash and B. Sudakov, The number of edge colorings with no monochromatic cliques, J. London Math. Soc. 70(2) (2004), 273–288.
2. N. Alon and R. Yuster, The number of orientations having no fixed tournament, Combinatorica 26 (2006), 1–16.
3. J. Balogh, A remark on the number of edge colorings of graphs, European Journal of Combinatorics 27 (2006), 565–573.
4. J. Balogh and L. Li, The typical structure of Gallai colorings and their extremal graphs, SIAM Journal on Discrete Mathematics 33(4) (2019), 2416–2443.
5. J.O. Bastos, F.S. Benevides, and J. Han, The number of Gallai \(k\)-colorings of complete graphs, Journal of Combinatorial Theory, Series B 144 (2020), 1–13.
6. F.S. Benevides, C. Hoppen and R.M. Sampaio, Edge-colorings of graphs avoiding a prescribed coloring pattern, Discrete Mathematics 240(9) (2017), 2143–2160.
7. F. Botler, J. Corsten, A. Dankovics, N. Frankl, H. Hán, A. Jiménez and J. Skokan, Maximum number of triangle-free edge colourings with five and six colours, Acta Mathematica Universitatis Comenianae 88(3) (2019), 495–499.
8. L. Colucci, E. Győri and A. Methuku, Edge colorings of graphs without monochromatic stars arXiv:1903.04541
9. P. Erdős, Some new applications of probability methods to combinatorial analysis and graph theory, Proc. of the Fifth Southeastern Conference on Combinatorics, Graph Theory and Computing (1974), 39–51.
10. Z. Füredi, A proof of the stability of extremal graphs, Simonovits’s stability from Szemerédi’s regularity, Journal of Combinatorial Theory, Series B 115 (2015), 66–71.
11. Z. Füredi and M. Simonovits, The history of degenerate (bipartite) extremal graph problems, *Erdös Centennial*, Bolyai Society of Mathematics 25, Springer Verlag (2013), 169–213.

12. H. Hán and A. Jiménez, Improved bound on the maximum number of clique-free colorings with two and three colors, *SIAM Journal on Discrete Mathematics* 32(2) (2018), 1364–1368.

13. C. Hoppen, Y. Kohayakawa and H. Lefmann, Edge-colorings of graphs avoiding fixed monochromatic subgraphs with linear Turán number, *European Journal of Combinatorics* 35(1) (2014), 354–373.

14. C. Hoppen and H. Lefmann, Edge-colorings avoiding a fixed matching with a prescribed color pattern, *European Journal of Combinatorics* 47 (2015), 75–94.

15. C. Hoppen and H. Lefmann, Remarks on an edge-coloring problem, Proceedings of the X Latin-American Algorithms, Graphs, and Optimization Symposium (LAGOS 2019), Electronic Notes in Theoretical Computer Science 346 (2019), 511–521.

16. C. Hoppen, H. Lefmann and K. Odermann, On graphs with a large number of edge-colorings avoiding a rainbow triangle, *European Journal of Combinatorics* 66 (2017), 168–190.

17. C. Hoppen, H. Lefmann and K. Odermann, A rainbow Erdős-Rothschild problem, *SIAM Journal on Discrete Mathematics* 31 (2017), 2647–2674.

18. J. Komlós and M. Simonovits, Szemerédi’s regularity lemma and its applications in graph theory, Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993), 295–352, Bolyai Soc. Math. Stud., 2, János Bolyai Math. Soc., Budapest (1996), 295–352.

19. D. Nolibos, On extremal configurations of generalized Erdős-Rothschild problems, *PhD thesis*, preprint.

20. O. Pikhurko, K. Staden and Z.B. Yilma, The Erdős-Rothschild problem on edge-colourings with forbidden monochromatic cliques, *Mathematical Proceedings of the Cambridge Philosophical Society* 163(2) (2017), 341–356.

21. O. Pikhurko and Z.B. Yilma, The maximum number of $K_3$-free and $K_4$-free edge 4-colorings, *J. London Math. Soc.* 85(2) (2012), 593–615.

22. M. Simonovits, *A Method for Solving Extremal Problems in Graph Theory, Stability Problems*, Theory of Graphs (Proc. Colloq., Tihany, 1966), Academic Press, New York (1968), 279–319.

23. E. Szemerédi, *Regular partitions of graphs*, Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), Colloq. Internat. CNRS, vol. 260, CNRS, Paris (1978), 399–401.

24. P. Turán, On an extremal problem in graph theory (in Hungarian), *Matematikai és Fizikai Lapok* 48 (1941), 436–452.

25. R. Yuster, The number of edge colorings with no monochromatic triangle, *J. Graph Theory* 21 (1996), 441–452.