Vortex-like Structures in the Skyrme - Einstein Chiral Model

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September 21, 2017

Abstract

A regular method is suggested for constructing vortex-like solutions with cylindrical symmetry in the Skyrme - Einstein chiral model. The method is based on the expansion of metric and field functions in power series with respect to the two small parameters entering the model. The length mass density of the vortex is estimated.

1 Introduction

The Skyrme chiral model [1] proved its efficiency in modelling the structure of baryons and light nuclei [2] via soliton-like configurations endowed with the topological charge (winding number), the latter being interpreted as the baryonic one. Recent years the interest to general relativistic description of Skyrmions arose in view of possible astrophysical applications of topological solitons coupled to gravity [3-5]. The economy of the Skyrme - Einstein model in treating nuclear and gravitational interactions of matter being its very attractive feature, one should however indicate the nonintegrability of the equations of motion that forced the authors of the aforementioned papers to use numerical methods. In lieu of keeping up with this tradition we elaborate in the present work an analytic method of solving the equations for vortex configurations with the aim to approximate three-dimensional structures by closed vortices and to expand, in future, this method to the configurations with spherical or axial symmetry.
2 Equations of the Skyrme - Einstein model: neutral vortex case

The Lagrangian density of matter in the model in question can be constructed from the chiral current $l_\mu = U^+ \partial_\mu U$, $\mu = 0, 1, 2, 3$, $U \in SU(2)$, and reads

$$\mathcal{L}_m = -\frac{1}{4\lambda^2} \text{Sp}(l_\mu l_\mu) + \frac{\epsilon^2}{16} \text{Sp}([l_\mu, l_\nu][l_\mu, l_\nu]),$$

(1)

where $\epsilon, \lambda$ are constant parameters. For the configurations with cylindrical symmetry the metric is chosen in the form

$$ds^2 = e^{2\mu} dt^2 - e^{2\alpha} d\xi^2 - e^{2\beta} d\phi^2 - e^{2\gamma} dz^2,$$

(2)

where the functions $\mu, \alpha, \beta, \gamma$ depend on the radial coordinate $\xi \in (-\infty, +\infty)$. As a first step we consider the simplest case of the neutral vortex, when the chiral matrix $U$ is taken in the form

$$U = \exp(i\tau \Theta(\xi)), \quad \tau = \begin{bmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{bmatrix}.$$  

(3)

Putting (3) into (1) and taking into account of the Einsteinian gravity Lagrangian density $\mathcal{L}_g = R/2\alpha$, where $R$ is the scalar curvature and $\alpha$ is the Einstein gravitational constant, one finds the following expression for the action functional:

$$A = 2\pi \int dt \int dz \int d\xi e^{\mu+\gamma} \left[ \frac{1}{\alpha} e^{\beta-\alpha}(\beta' \gamma' + \mu' \beta' + \mu' \gamma') - \frac{1}{2\lambda^2} \left( e^{\beta-\alpha} \Theta'^2 + e^{\alpha-\beta} \sin^2 \Theta \right) - e^2 e^{-(\alpha+\beta)} \Theta'^2 \sin^2 \Theta \right].$$

(4)

One infers from (4) that the theory admits the discrete symmetry $\mu \leftrightarrow \gamma$. Therefore making the substitution $\mu = \gamma$ and imposing the harmonic coordinate condition [6]

$$\alpha = \beta + 2\gamma,$$

(5)

one transforms the functional (4) as follows:

$$A = 2\pi \int dt \int dz \int d\xi \left[ \frac{1}{16\alpha} (u'^2 - v'^2) - \frac{1}{2\lambda^2} \Theta'^2 \times \right.$$

$$\times (1 + 2e^2 \lambda^2 e^{-\frac{\epsilon}{2} \sin^2 \Theta}) - \frac{1}{2\lambda^2} e^{u-v} \sin^2 \Theta \left].$$

(6)
where \( u = 4(\beta + \gamma), \ v = 4\beta. \)

The functional (6) admits the evident scale symmetry:

\[
\xi \to \sigma \xi, \ z \to \sigma z, \ u \to u - 2 \ln \sigma, \ v \to v,
\]

that will be used later for simplifying the asymptotic behavior of solutions at space infinity.

Now we introduce the dimensionless parameters in the problem by scale dilation:

\[
\xi \to \sigma x, \ z \to \sigma z, \ v \to v + 2 \ln \sigma, \ u \to u + \ln \sigma^2,
\]

where \( \sigma^2 = 2\lambda^2. \) Then we come to the quasimechanical problem with \( x \) playing the role of "time" and with the Hamiltonian "action" (measured in the units \( 2\pi/\lambda^2 \)) of the form

\[
S = \frac{1}{2} \int dx \left[ \frac{1}{\nu} (u'^2 - v'^2) - \Theta'^2 (1 + 2e^{-v/2} \sin^2 \Theta) - e^{u-v} \sin^2 \Theta \right],
\]

where \( \nu = 8\alpha/\lambda^2. \) If one uses the natural units \( \hbar = c = 1 \) and the experimental value of the parameter \( \lambda = 2/F_\pi, \ F_\pi = 186 \text{ MeV}, \) then one gets \( \nu \approx 1.26 \times 10^{-38}. \) The other dimensionless constant \( \epsilon \) appearing in the functional (8) takes the experimental value \([7]\) \( \epsilon \approx 0.13. \) Thus there exist two small parameters in the problem: \( \nu \sim 10^{-38} \) and \( \epsilon^2 \sim 10^{-2}, \) that allows one to utilize the perturbation scheme.

First of all we write down the variational equations corresponding to (8):

\[
\frac{1}{\nu} u'' = -\frac{1}{2} e^{u-v} \sin^2 \Theta,
\]

\[
\frac{1}{\nu} v'' = -\frac{1}{2} e^{u-v} \sin^2 \Theta - \frac{\epsilon^2}{4} e^{-v/2} \Theta'^2 \sin^2 \Theta,
\]

\[
\Theta'' (1 + \epsilon^2 e^{-v/2} \sin^2 \Theta) - \frac{\epsilon^2}{2} e^{-v/2} \Theta' \sin^2 \Theta = \frac{1}{2} \sin 2\Theta (e^{u-v} - \epsilon^2 e^{-v/2} \Theta'^2),
\]

and the "energy" integral

\[
\frac{1}{\nu} (u'^2 - v'^2) + e^{u-v} \sin^2 \Theta = \Theta'^2 (1 + \epsilon^2 e^{-v/2} \sin^2 \Theta).
\]

Using the scale symmetry (7) we impose the regularity conditions at space infinity:

\[
x \to +\infty, \ v \to +\infty, \ u - v \to 0, \ \Theta \to O(e^{-x}),
\]

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and on the axis [6]:

\[
x \to -\infty, \quad v \to -\infty, \quad \Theta \to \pi,
| u - v | < \infty, \quad v' \exp \left(\frac{v - u}{2}\right) \to 4.
\]  

(14)

As follows from the equations (9) - (12), the regularity conditions (13) and (14) imply the asymptotic behavior of solutions at space infinity:

\[
v = u = 4x, \quad \Theta = Ae^{-x}, \quad A = \text{const},
\]  

(15)

and on the symmetry axis:

\[
v = v_a + 4Dx, \quad u = u_a + 4Dx, \quad \Theta = \pi - Be^{Dx},
\]  

(16)

with the integration constants \(v_a, u_a, D, B\) satisfying the constraint

\[
D \exp \left(\frac{v_a - u_a}{2}\right) = 1.
\]  

(17)

3 Structure of vortex-like solutions

Now we take into account of the mechanical analogy and rewrite the Eq. (12) in the form of Hamilton - Jacobi equation

\[
S^2_{\Theta} = \left(1 + \epsilon^2 e^{-\frac{2}{2} \sin^2 \Theta}\right) \times
\times \left[\nu (S^2_u - S^2_v) + e^{u-v} \sin^2 \Theta\right].
\]  

(18)

Due to small value of the parameter \(\nu\) we can represent the solution to the Eq. (18) as the power series expansion

\[
S = \frac{1}{\nu} F + G + \nu H + \cdots.
\]  

(19)

Inserting (19) into (18) and equating the terms with the same powers in \(\nu\) order by order, we get the equations for the unknown functions \(F, G, H, \cdots\). The equations for the function \(F\) read

\[
F_{\Theta} = 0, \quad (F_u - F_v)(F_u + F_v) = 0.
\]  

(20)

The asymptotic behavior (15) being equivalent to the relations (as \(x \to \infty\)):

\[
F_u = u' = v' = -F_v,
\]

one concludes that the general solution to (20) turns out to be

\[
F = F(u - v).
\]  

(21)
The form (21) of the function $F$ implies the following equation satisfied by the function $G$:

$$G^2 \Theta = (1 + e^2 e^{-\frac{x}{2}} \sin^2 \Theta) \times \left[ 2F'(u-v)(G_u + G_v) + e^{u-v} \sin^2 \Theta \right]. \quad (22)$$

The variables in the Eq. (22) can be separated by the substitution

$$F(u-v) = 8e^{(u-v)/2}, \quad (23)$$

$$G = e^{(u-v)/2} \sum_{n=0}^{\infty} e^{2n} e^{-nv/2} V_n(\Theta), \quad (24)$$

where the choice (23) is motivated by the nonuniform term in (22) and by the asymptotic behavior (15). Inserting (23) and (24) in (22) one obtains the recurrence relations satisfied by the functions $V_n(\Theta)$:

$$\sum_{k=0}^{n} V'_k V'_{n-k} + 4(n-1) \sin^2 \Theta V_{n-1} = \delta_{n0} \sin^2 \Theta + \delta_{n1} \sin^4 \Theta. \quad (25)$$

For $n = 0$ one derives from (25) and (15) that

$$V_0 = 1 - \cos \Theta. \quad (26)$$

For $n = 1$ one gets, in view of (26), the equation

$$2 \sin \Theta V'_1 + 4V_1 = \sin^4 \Theta. \quad (27)$$

Taking into account of the relations (15) and (16), one concludes that the solution $V_1(\Theta)$ to the Eq. (27) has the two branches $V_1^\pm(\Theta)$ with the domains

$$D^+ = \{0 \leq \Theta \leq \frac{\pi}{2}\}, \quad D^- = \{\frac{\pi}{2} \leq \Theta \leq \pi\}.$$ 

The corresponding functions attain the forms:

$$V_1^- = -\frac{1}{6}(1 + \cos \Theta)^2 \left( \frac{4}{1 - \cos \theta} + 3 - \cos \Theta \right),$$

$$V_1^+ = \frac{1}{6}(1 - \cos \Theta) \sin^2 \Theta. \quad (28)$$
For $n \geq 2$ one obtains the following integral representation for the solution to the Eq. (25):

$$V_n^\pm = -\left(\frac{1 + \cos \Theta}{1 - \cos \Theta}\right)^n \int_{a^\pm}^\Theta d\Theta \left(\frac{1 - \cos \Theta}{1 + \cos \Theta}\right)^n \times \left(1 + \cos \Theta\right)_{a^\pm} - \cos \Theta_{a^\pm} + 2(n - 1) \sin^2 \Theta V_{n-1} \right), \quad (29)$$

where $a_+ = 0$, $a_- = \pi$. In particular, for $n = 2$ from (28) and (29) one deduces that

$$V_2^\pm = \left(\frac{1 + \cos \Theta}{1 - \cos \Theta}\right)^2 \left(-\frac{8}{9} \ln \left((1 \pm \cos \Theta)/2\right) + A_0 \delta^\pm + \frac{4}{9} \tan^2 \Theta + \sum_{n=1}^5 A_n^\pm (1 - \cos \Theta)^n\right), \quad (30)$$

where the following numerical coefficients are introduced:

$$A_0 = -\frac{22}{135}, A_1^\pm = \pm\frac{2}{3}, A_2^\pm = \frac{2}{9}, A_3^\pm = \frac{5}{54}, A_4^\pm = \frac{1}{24}, A_5^\pm = -\frac{1}{40}, \delta^\pm = (1 \mp 1)/2.$$

Within the scope of the first two terms in the expansion (19) the following normal system of equations emerges:

$$\frac{1}{\nu} u' = S_u = \frac{1}{2} e^{(u-v)/2} \left(\frac{8}{\nu} + \sum_{n=0}^\infty e^{2n} e^{-\nu} V_n(\Theta)\right),$$

$$\frac{1}{\nu} v' = -S_v = \frac{1}{2} e^{(u-v)/2} \left(\frac{8}{\nu} + \sum_{n=0}^\infty e^{2n} (n+1) e^{-\nu} V_n(\Theta)\right), \quad (31)$$

$$-\Theta'(1 + e^2 e^{-\nu/2} \sin^2 \Theta) = S_\Theta = e^{(u-v)/2} \sum_{n=0}^\infty e^{2n} e^{-\nu} V_n(\Theta).$$

The system (31) should be complemented by the smooth matching condition at the point $x = x_0$, where $\Theta(x_0) = \pi/2$.
$$\sum_{n=1}^{\infty} e^{2n} e^{-n v(x_0)/2} \left( V_+^{n'}(\pi/2) - V_-^{n'}(\pi/2) \right) = 0. \quad (32)$$

In particular, within the confines of \( n = 2 \) approximation the Eq. (32) allows us to find, through the use of (28) and (30), the approximate value of the effective expansion parameter

$$\zeta = \frac{1}{3} e^{2} e^{-v(x_0)/2} = \frac{2}{16 \ln 2 - 37/15} \approx 0.232. \quad (33)$$

Now we can use (33) to estimate the mass length density of the vortex in search defined as

$$M = -\frac{2\pi}{\lambda^2} \Delta S, \quad (34)$$

where \( \Delta S \) denotes the total variation of \( S(x) \) in the domains \( D^\pm \):

\[
\Delta S = S^+(\infty) - S^+(x_0) + S^-(x_0) - S^-(\infty) = \]
\[
= -2 - \frac{1}{\nu} e^{(n-2)} \sum_{n=1}^{\infty} e^{2n} e^{-\nu} (S^+_n - S^-_n)|_{x=x_0} \quad (35)
\]

Using (31) one gets in the first approximation in \( \nu \) that

$$u = v = 4x + \mathcal{O}(\nu). \quad (36)$$

Inserting (36) into (35) and taking into account of (28) and (30), one finds up to \( \zeta^2 \) terms that

$$\Delta S \approx -2 - 4\zeta - 2\zeta^2 \left( \frac{41}{15} - 8 \ln 2 \right). \quad (37)$$

Substituting (37) into (34) one gets, in view of (33), the approximate value of the vortex mass

$$M \approx \frac{4\pi}{\lambda^2} 1.31. \quad (38)$$

As for the structure of the function \( \Theta(x) \) defining the radial distribution of matter inside the vortex, as well as those of the functions \( u(x) \) and \( v(x) \) describing the gravitational field, they can be derived from the system (31) through the use of the standard iteration technique.

### 4 Discussion

In conclusion it is worthwhile to underline that we considered here the case of the neutral vortex not endowed with the topological charge.
density. Therefore after closing the vortex piece we come to the configuration with zero baryon number $Q$, this fact ensuing from the formula for the winding number

$$Q = -\frac{1}{24\pi^2} \varepsilon^{ijk} \int d^3x Sp(l_i, l_j, l_k). \quad (39)$$

However, this shortcoming can be cured by generalizing the substitution (3) with the aim to generate the topological charge density in the vortex. To this end one can try the following ansatz:

$$U = \cos \psi \exp (i\sigma_3 \chi) + i\sigma_1 \sin \psi \exp (i\sigma_3 \delta), \quad (40)$$

where $\sigma_1, \sigma_3$ are the Pauli matrices and $\psi, \chi, \delta$ stand for the chiral angles represented in the cylindrical coordinates $\rho, \varphi, z$ as follows:

$$\psi = \psi(\rho), \quad \chi = -kz\eta(a - \rho), \quad \delta = m\varphi, \quad (41)$$

with $k = \text{const}$, $m \in \mathbb{Z}$. The radius $a$, entering the argument of the Heaviside step function $\eta$, is chosen to satisfy the piece-wise smooth matching condition $\cos \psi(a) = 0$. In the simplest case we can impose the following boundary conditions:

$$\psi(0) = \pi, \quad \psi(a) = \frac{\pi}{2}, \quad \psi(\infty) = 0, \quad (42)$$

and also the closure one:

$$kl = 2\pi n, \quad n \in \mathbb{Z}, \quad z \in [-\frac{l}{2}, \frac{l}{2}], \quad (43)$$

where $l$ stands for the length of the vortex piece. Inserting (40) into (39) yields, in virtue of (41), (42) and (43), the nontrivial baryon number

$$Q = \frac{1}{4\pi^2} \int d^3x \sin 2\psi (\nabla \psi \nabla \chi \nabla \delta) = mn. \quad (44)$$

Thus, the suggested substitution meets all necessary requirements. The analysis of the Skyrme - Einstein vortex configurations endowed with the topological charge density will be exhibited in the subsequent paper.

**References**

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