REDUCIBILITY OF RELATIVISTIC SCHRÖDINGER EQUATION WITH UNBOUNDED PERTURBATIONS

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Abstract. In this paper, we prove a reducibility result for a relativistic Schrödinger equation on torus with time quasi-periodic unbounded perturbations of order $\frac{1}{2}$, and finally conjugate the original equation to a time independent, $2 \times 2$ block diagonal one.

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1. INTRODUCTION

In this paper, we study the reducibility of a relativistic Schrödinger equation with unbounded quasi-periodic perturbations on the torus $\mathbb{T}$,

\begin{equation}
    i\partial_t u = (-\partial_{xx} + m^2)^{\frac{1}{2}} u + \epsilon W(\omega t) u, \quad x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \quad t \in \mathbb{R}.
\end{equation}

The operator $(-\partial_{xx} + m^2)^{\frac{1}{2}}$, defined via its symbol $(\xi^2 + m^2)^{\frac{1}{2}}$ under Fourier transform, is the kinetic energy operator of a relativistic particle of mass $m$, $0 \leq m \leq \frac{1}{4}$. Regarding more information about the operator, we refer readers [13]. $W(\omega t)$ is a pseudo-differential operator of order $\frac{1}{2}$, and quasi-periodic in time with frequencies $\omega \in \Omega = [1, 2]^d$. Informally speaking, we seek a time quasi-periodic bounded mapping (from $H^s$ to $H^s$), which conjugates the original equation (1.1) to a block diagonal, time independent one.

Key words and phrases. KAM theory, pseudo-differential operator, Sobolev norms.
In the PDEs context, the reducibility results deal with the infinite dimensional systems which are a diagonal operator under small perturbation of the form,

\[ i\omega \cdot \partial_\theta + D + \varepsilon W(\omega t), \quad \omega \in \mathbb{R}^d, \]

where \( D \) is a diagonal operator, \( \varepsilon \) is small and \( \omega \) is in some cantor sets. In the past two decades, the reducibility problems of such systems have been studied by many scholars. The literature can be divided into two cases. The first is the diagonal operator with bounded perturbations, we mention [15, 20, 22, 30, 31]. The second is with unbounded perturbations, which is the focus of the present paper.

It needs to be emphasized that the reducibility procedure becomes very complicated in the case of unbounded perturbations, and the adapted classical KAM method ([3, 23]) can only handle the system where the unperturbed part \( D \) has order \( n \) and the perturbation \( W(\omega t) \) is of order \( \delta \leq n - 1 \). The new method for further development is obtained by [2], which is based on some pseudo-differential operator calculus. The new strategy is applying a series of bounded transformation before KAM iteration, and converting the original problem to a new one

\[ i\omega \cdot \partial_\theta + D^* + \varepsilon W^*(\omega t), \quad \omega \in \mathbb{R}^d, \]

where the new perturbation \( W^* \) is of low order. Such strategies have been used to study one dimensional PDEs by several authors [4–7, 16, 18, 28].

Moreover, the dimension of space domains and the frequencies of unperturbed parts are also related to the reducibility results. They can induce new problems in counting the number of non-resonance condition. Interestingly, the pseudo-differential operator technique is also a powerful tool to deal with this problem. The idea is that the smoothing character of the perturbation can be used to recover a smoothness loss due to the small denominators, we refer readers [7, 19, 32].

In this paper, we conjugate the original problem (1.1) to a new one based on the abstract pseudo-differential operator technique, such that the new perturbation is sufficiently smooth. However, the measure estimate in KAM iteration still faces some problems when the gap of eigenvalues cannot be greater than any constant. Hence, we put some reasonable restrictions on the original perturbation in Theorem 2.6 so that some dilemmas in measure estimate can be avoided. Furthermore, if the space torus can be introduced as a new parameter, these restrictions can be avoided. The details have been discussed in Appendix A.

**Remark 1.1.** From the mathematical viewpoint, the reducibility results of equation (1.1) imply that the Sobolev norms of solutions stay bounded for all time. In the context of non-small perturbations (without the small parameter \( \varepsilon \)), the dynamic behavior of the solution of equation (1.1) is rich. In [3, 26], the authors show that if \( \omega \) satisfies some non resonant conditions, then only a weak upper bound can be obtained, i.e., \( \forall \varepsilon \geq 0 \), there exists a constant \( C_\varepsilon \) such that

\[ \|u(t, x)\|_{H^s} \leq C_\varepsilon t^{\varepsilon} \|u(t, x)\|_{H^s}. \]  

Furthermore, if \( \omega \) is resonant, Maspero [24] constructs some perturbations which provoke Polynomial growth of Sobolev norms. The conclusion in this paper is supplement to the previous results. It further shows that stability of Sobolev norms is a non-resonant phenomenon.

**Remark 1.2.** In this paper, we use the abstract PDO technique in [19] to regularize the perturbation, instead of the quantization technique in [1]. The main advantage is that we can deal with much more general unbounded perturbations and generalize to high dimensional manifolds. Without much changes, we can also deal with the following two models.
1: Relative Schrödinger equation on $S^2$,

\[ i\partial_t u = \sqrt{-\Delta_g + m^2} u + \varepsilon W(\omega t,x)(-i\partial^3_x + V(\omega t,x))u, \quad u = u(t,x), \quad x \in S^2. \]

Here $i\partial_\phi = i(x_1 \partial_{x_2} - x_2 \partial_{x_1})$ is the $x_3$ component of the orbital angular momentum (and the generator of rotations about the $x_3$ axis.) Regarding more information about the perturbation, we recommend readers [17].

2: Relative Schrödinger equation on Zoll manifold of dimension $n \in \mathbb{N}$.

\[ i\partial_t u = \sqrt{-\Delta_g + m^2} u + \varepsilon W(\omega t,u), \quad u = u(t,x), \quad x \in M^n. \]

Here $-\Delta_g$ is the positive Laplace-Beltrami operator on $M^n$ and the linear operator $W(\omega t)$ is a time quasi-periodic pseudo-differential operator of order $0$ with frequency $\omega \in [1,2]^d$.

The paper is organized as follows: In section 2, we introduce some important notions and definitions to precisely state our main results. In section 3, we introduce the abstract pseudo-differential operator (PDO) technique used in [7, 9], such that the original unbounded perturbation can be reduced to a smoothing operator. In section 4, we give a KAM reducibility result. In section 5, we emphasize the difference of relativistic Schrödinger equations on $\mathbb{T}$ and on $\mathbb{T}_z$. In section 6, we give some important technical lemmas used in this paper.

Notations: In the present paper, we denote the notation $A \lesssim B$ as $A \leq CB$, where $C$ is a constant number depending on the fixed number $d, m$.

2. MAIN RESULTS

In order to state the main results of the paper precisely, we introduce some important notations and definitions in this section.

2.1. Function space and pseudo-differential operators.

Given any function $u \in L^2(\mathbb{T})$, it can be expressed as

\[ u(x) = \sum_{j \in \mathbb{Z}} \hat{u}(j)e^{ij \cdot x}, \quad \hat{u}(j) = \frac{1}{2\pi} \int_{\mathbb{T}} u(x)e^{-ij \cdot x} \, dx. \]

The Sobolev space on $\mathbb{T}$ defined by

\[ H^r(\mathbb{T}) := \left\{ u \in L^2(\mathbb{T}) : \|u\|^2_{H^r(\mathbb{T})} := \sum_{j \in \mathbb{Z}} \|j\|^{2r} \hat{u}(j)^2 < \infty \right\}, \]

where $\langle j \rangle = \max\{1,|j|\}$.

For a function $a : \mathbb{T} \times \mathbb{Z} \to \mathbb{R}$, define the difference operator $\Delta a(x,j) := a(x,j+1) - a(x,j)$ and let $\Delta^\beta = \Delta \circ \ldots \circ \Delta$ be the composition $\beta$ times of $\Delta$. Then, we have the following definitions:

**Definition 2.1.** Let $m \in \mathbb{R}$, we say that a function $a : \mathbb{T} \times \mathbb{Z} \to \mathbb{R}$ is a symbol of class $S^m$ if for any $j \in \mathbb{Z}$ the map $x \mapsto a(x,j)$ is smooth and, furthermore, for any $\alpha, \beta \in \mathbb{N}$, there exists $C_{\alpha,\beta} > 0$ such that

\[ \left| \partial^\alpha_{x} \Delta^\beta a(x,j) \right| \leq C_{\alpha,\beta} \langle j \rangle^{m-\beta}, \quad \forall x \in \mathbb{T}. \]

**Definition 2.2.** Given a symbol $a \in S^m$, we called $Op(a) \in OPS^m$ is its associated pseudo-differential operator if for any $u \in L^2(\mathbb{T})$

\[ Op(a)[u](x) = \sum_{j \in \mathbb{Z}} a(x,j) \hat{u}(j)e^{ij \cdot x}. \]
We can endow the operator \( Op(a) \in OPS^m \) a family of seminorms
\[
\chi^m_{\rho,s}(Op(a)) := \sum_{\alpha + \beta \leq \rho} \sup_{x,j \in \mathbb{Z}} \langle \xi \rangle^{-m + \beta} |\partial_x^\alpha \Delta^\beta a(x,j)|.
\]

**Definition 2.3.** Considering the pseudo-differential operator \( A(\theta) \) depending the angel variable \( \theta \in \mathbb{T}^d \) in smooth way. Then the operator \( A(\theta) \) can be expressed as
\[
A(\theta) = \sum_{\ell \in \mathbb{Z}^d} \hat{A}(\ell)e^{i\ell \cdot \theta}
\]
where \( \hat{A}(\ell) \in OPS^m \). We denote them by \( C^\infty(\mathbb{T}^d, OPS^m) \). If the operator \( A(\theta) \) is also Lipschitz-way depending on the parameter \( \omega \in \Omega \subseteq \mathbb{R}^d \), we denote them by \( Lip(\Omega, C^\infty(\mathbb{T}^d, OPS^m)) \).

**Remark 2.4.** The symbol of the pseudo-differential operator \( A(\theta) \) can be represent as
\[
a(\theta, x, j) = a(x, j)(\ell)e^{i\ell \cdot \theta},
\]
where \( a(x, j)(\ell) \) is the symbol of the pseudo-differential operator \( \hat{A}(\ell) \).

**Definition 2.5.** Let \( s > \frac{d}{2} \), the operator \( A(\theta) \in C^\infty(\mathbb{T}^d, OPS^m) \) can be endowed a family of seminorms:
\[
\chi^m_{\rho,s}(A(\theta)) := \left( \sum_{\ell \in \mathbb{Z}^d} \left( \langle \xi \rangle^{2s} \chi^m_{\rho,\ell}(\hat{A}(\ell)) \right)^2 \right)^{\frac{1}{2}}.
\]
Moreover, we can endow the operator \( A(\theta, \omega) \in Lip(\Omega, C^\infty(\mathbb{T}^d, OPS^m)) \) a family of Lipschitz seminorms:
\[
\chi^m_{\rho,s,Lip}(A(\theta, \omega)) := \chi^m_{\rho,s,Lip}(A(\theta, \omega)) + \chi^m_{\rho,s,Lip}(A(\theta, \omega))
\]
\[
= \sup_{\omega \in \Omega} \chi^m_{\rho,s}(A(\theta, \omega)) + \sup_{\omega_1, \omega_2 \in \Omega} \frac{\chi^m_{\rho,s}(A(\omega_1) - A(\omega_2))}{|\omega_1 - \omega_2|}.
\]

**2.2. Main results.**

The perturbation \( W(\omega t) \) is a quasi-periodic driving pseudo-differential operator, which satisfies following two conditions:

- **(C1):** \( W(\omega t) \) is an Hermitian operator, and belongs to \( C^\infty(\mathbb{T}^d, OPS^+^\infty) \).
- **(C2):** Set the symbol of pseudo-differential operator \( W(\omega t) \) as \( w(\theta, x, j) \), then
\[
\int_{\mathbb{T}^d} \int_{\mathbb{T}} w(\theta, x, j) dxd\theta = a(j)^{\frac{1}{2}} + b(j), \quad j \in \mathbb{Z},
\]
where \( a \) is independent of \( j \) and \( b \) is dependent on \( j \). Moreover, there exists a constant \( C \) such that
\[
b(j) \leq C, \quad \forall j \in \mathbb{Z}.
\]

**Theorem 2.6.** Consider the equation (1.1) and assume conditions **(C1)** and **(C2)**. For any \( s \geq 0 \), there exists \( \varepsilon^* > 0 \). \( \forall \varepsilon \in (0, \varepsilon^*) \), there exists a closed asymptotically full Lebesgue set \( \Omega_\varepsilon \subseteq \Omega \). Then \( \forall \omega \in \Omega_\varepsilon \), there exist a family linear and invertible bounded operator \( U(\theta, \omega) \in L(H^\infty) \), conjugate the equation (1.1) to
\[
i \partial_t u = H^\infty u, \quad H^\infty = \text{diag}\left\{ \Lambda^\infty_j(\omega) | j \in \mathbb{N} \right\}.
\]
Here \( \Lambda^\infty_j, j \geq 1 \) is a \( 2 \times 2 \) Hermitian matrix, and \( \Lambda_0^\infty \) is a real number close to \( m \).

As a consequence, we can get a Sobolev norms control of the flow generated by the equation (1.1).
Corollary 2.7. For any \( s \geq 0 \) and \( \omega \in \Omega_{\varepsilon} \), the solution \( u(t, x) \) of equation (1.1) with initial condition \( u(0, x) \in H^s \) satisfies

\[
(3.1) \quad \| u(0, x) \|_{H^s} \leq \| u(t, x) \|_{H^s} \leq C_s \| u(0, x) \|_{H^s}.
\]

It needs to be emphasized that the condition (C1) is indispensable. Inspired by the author obtained a family of analytical solutions of elliptic equation by taking the space torus as frequency parameter. We can also introduce the space torus as frequency parameter to avoid the condition (C1). Hence, we consider the following equation:

\[
(2.11) \quad i\partial_t u = (-\partial_{xx} + m^2)^{\frac{1}{2}} u + \varepsilon \mathcal{W}(\omega_t)u, \quad x \in T_\beta = \mathbb{R}/2\pi\beta\mathbb{Z}, \quad t \in \mathbb{R},
\]

where \( \mathcal{W}(\omega_t) \) is a pseudo-differential operator of order \( \frac{1}{2} \), and quasi-periodic in time with frequencies \( \omega \in [1, 2]^d \). The space domain changes with the parameter \( \beta \in \left[ \frac{1}{2}, 1 \right] \).

Then, we can prove the following reducibility result.

Theorem 2.8. Set \( \mathcal{W}(\omega_t) \) is an Hermitian operator and belongs to \( C^\infty(T_\beta, \mathcal{OPS}_{\mathbb{R}}^+) \). For any \( s \geq 0 \), there exist \( \varepsilon^* > 0. \forall \varepsilon \in (0, \varepsilon^*) \), there exists a closed asymptotically full Lebesgue set \( \Omega_{\varepsilon} \subset \Omega = [1, 2]^{d+1} \). \( \forall \omega = (\omega_0, \frac{1}{2}) \subset \Omega_{\varepsilon} \), there exist a family time quasi-periodic and invertible bounded operator \( \mathcal{U}(\theta, \omega) \in \mathcal{L}(H^s) \), conjugate the equation (2.11) to

\[
(2.12) \quad i\partial_t u = H^\infty u, \quad H^\infty = \text{diag}\left\{ A^\infty_j(\omega)\right\} j \in \mathbb{N}.
\]

Here \( A^\infty_j, j \geq 1 \) is a \( 2 \times 2 \) Hermitian matrix, and \( A^\infty_0 \) is a real number close to \( m \).

Remark 2.9. The proof of Theorem 2.8 has the same framework as Theorem 2.6. The main differences are explained in detail in Appendix A.

3. Block matrix representation

Let \( H^\infty := \cap_{r \in \mathbb{R}} H^r \) and \( H^{-\infty} := \cup_{r \in \mathbb{R}} H^r \). For any linear operator \( A: H^\infty \rightarrow H^{-\infty} \), we take its matrix representation of block coefficients \( (A^{[n]}_{[m]})_{m,n \in \mathbb{N}} \) as

\[
(3.1) \quad A^{[n]}_{[m]} = \left( \begin{array}{cc} A^m_n & A^{-n}_m \\ A^{-m}_n & A^m_{-n} \end{array} \right)
\]

on the basis \( \{ \hat{e}_j := e^{ij\varphi} \}_{j \in \mathbb{Z}} \), defined for \( m, n \in \mathbb{Z} \) as

\[
A^m_n \equiv \langle A \hat{e}_m, \hat{e}_n \rangle_{H^0}.
\]

The matrix \( A^{[n]}_{[m]} \) can be seen as a linear operator in \( \mathcal{L}(E_m, E_n) \) for any \( m, n \in \mathbb{N} \), where \( E_m \) is defined as

\[
(3.2) \quad E_m := \text{span}\{ e^{im\varphi}, e^{-im\varphi} \}.
\]

In this paper we also consider the \( \theta \)-depending linear operator

\[
T^d \ni \theta \rightarrow A = A(\theta) = \sum_{\ell \in \mathbb{Z}^d} \hat{A}(\ell)e^{i\ell \cdot \theta},
\]

where \( \hat{A}(\ell) \in \mathcal{L}(H^\infty, H^{-\infty}) \). \( A(\theta) \) can be regarded as an operator acting on function \( u(\theta, x) \) of space-time as

\[
(Au)(\theta, x) = (A(\theta)u(\theta, \cdot))(x).
\]

Having the infinite dimensional matrix \( A \) and \( A(\theta) \), we can define the following \( s \)-decay norms.
Remark 3.2

Definition 3.1. I: The $s$-decay norms of infinite dimensional matrix $A$ defined as

\[ \|A\|_{s,s} = \left( \sum_{h \in \mathbb{N}} (h)^{2s} \sup_{|i-j|=h} \|A[i][j]\|^2 \right)^{\frac{1}{2}}, \]

where $\|A[i][j]\|$ is the $L^2$ operator norm of $L(E_j, E_i)$.

II: Considering a $\theta$-depending infinite dimensional matrix $A(\theta)$, we define its norms as

\[ \|A(\theta)\|_{s,s} = \left( \sum_{\ell \in \mathbb{Z}^d, h \in \mathbb{N}} (\ell, h)^{2s} \sup_{|i-j|=h} \|A[i][j](\ell)\|^2 \right)^{\frac{1}{2}}. \]

We denote by $M^s$ the space matrices finite $s$-decay norm.

III: If the linear operator $A(\theta)$ is a family Lipschitz map from $\mathbb{R}^d \ni \Omega \ni \omega$ to $M^s$, we define the Lipschitz $s$-decay norm as

\[ \|A(\theta)\|_{s,Lip, \Omega} = \sup_{\omega \in \Omega} \|A(\omega)\|_{s,s} + \sup_{\omega_1, \omega_2 \in \Omega} \frac{\|A(\omega_1) - A(\omega_2)\|_{s,s}}{\|\omega_1 - \omega_2\|}. \]

We denote by $M^{s,Lip, \Omega}$ the family Lipschitz map from $\mathbb{R}^d \ni \Omega \ni \omega$ to $M^s$ with finite Lipschitz $s$-decay norm. For notationally convenience, drop the range of $\omega$, $M^{s,Lip, \Omega}$ denoted as $M^{s,Lip}$.

Remark 3.2. In the present paper, we claim that the $\theta$-depending linear operator $A(\theta)$ is an Hermitian operator, if and only if

\[ A = A^* \iff A(\ell)^* = A(-\ell), \forall \ell \in \mathbb{Z}^d \iff (A[m][n](\ell))^* = A[n][m](\ell), \forall \ell \in \mathbb{Z}^d, m, n \in \mathbb{N}. \]

It is crucial to investigate the tame or algebra property of $s$-decay norm. Thus, we need the following lemmas.

Lemma 3.3. For any $s \geq s_0 > \frac{d+1}{2}$, the following results hold:

A: there is a constant $C(s)$ such that

\[ \|AB(\theta)\|_{s,s} \leq C(s)(\|A\|_{s_0,s_0} \|B\|_{s,s} + \|A\|_{s,s} \|B\|_{s_0,s_0}). \]

B: given an infinite dimension matrix $A(\theta)$, for any $N \in \mathbb{N}$, we define the cutoff matrix $\Pi_N A$ as

\[ (\Pi_N A)[i][j](\ell) = \begin{cases} A[i][j](\ell), & \text{if } |i-j| < N \text{ and } |\ell| < N, \\ 0, & \text{otherwise}. \end{cases} \]

Denote $\Pi_N^\perp A$ as $A - \Pi_N A$, we have

\[ \|\Pi_N A\|_{s,s} \leq CN_N^{-\beta} \|A\|_{s+s, s+s+\beta}. \]

The bounds of $\Pi_N^\perp A$ as $A - \Pi_N A$, we have

\[ \|\Pi_N A\|_{s,s} \leq \|\Pi_N A\|_{s,s} \leq \|A\|_{s,s}. \]

The items $[3.7],[3.8],[3.9]$ are valid by replacing $\|\cdot\|_{s,s}$ as $\|\cdot\|_{s,Lip}$.

Proof. The proof of $[3.7]$ can be found in [10]. The items $[3.8],[3.9]$ can be obtained from their definitions.

Lemma 3.4. Let $s_0 > \frac{d+1}{2}$, one has

\[ \|A(\theta)\|_{L(H^s)} \leq C(s)\|A(\theta)\|_{s,s} \leq C(s)\|A(\theta)\|_{s+s_0,s+s_0}. \]

Here $\|A(\theta)\|_{s,s} = \left( \sum_{h \in \mathbb{N}} (h)^{2s} \sup_{\theta \in \mathbb{Z}^d} \|A[i][j](\theta)\|^2 \right)^{\frac{1}{2}}$.

Proof. The proof can be found in Lemma 2.4 [2].
In the KAM procedure of section 4, the smoothing operator plays an important role. Hence, we introduce the following norms.

**Definition 3.5.** Considering a linear operator $A(\theta)$ in $\mathcal{L}(H^{s+m}, H^{s+n})$, we introduce a new $s$-Decay norm as

$$\|A(\theta)\|^s_{s+m,s+n} = \left( \sum_{\ell \in \mathbb{Z}^d, h \in \mathbb{N}} |\langle \ell, h \rangle|^{2s} \sup_{|i-j|=h} \langle i \rangle^{-2n} \|\hat{A}_{[i,j]}(\ell)\|^{2} \langle j \rangle^{2m} \right)^{\frac{1}{2}}.$$  

(3.11)

We denote $\mathcal{M}_{s+m,s+n}^s$ as the space matrices with finite $s$-Decay norm. Moreover, if the linear operator $A(\theta)$ is a family Lipschitz map from $\mathbb{R}^d \supset \Omega \ni \omega$ to $\mathcal{M}_{s+m,s-n}^s$, we can define the Lipschitz $s$-Decay norm in the same way as Definition 3.1.

**Remark 3.6.** Define a $\theta$-independent diagonal operator $D$, acting on $u \in H^0$ as

$$Du(x) = \sum_{k \in \mathbb{Z}} \langle k \rangle \hat{u}_k e^{ikx}.$$  

For any $m, n \in \mathbb{R}$, $A(\theta) \in \mathcal{M}_{s+m,s+n}^s$, there exists a linear operator $Q(\theta) \in \mathcal{M}_{s,s}^s$ such that

$$\hat{Q}_{[i,j]} = \frac{A_{[i,j]}(\theta)}{\langle i \rangle^{n}}.$$  

Moreover,

$$\|A(\theta)\|^s_{s+m,s+n} = \|\langle D \rangle^n Q(\theta) (D)^{-m} \|^s_{s+m,s+n} = \|Q(\theta)\|^s_{s,s}.$$  

**Lemma 3.7.** Fix $s \geq s_0 > \frac{d+1}{2}$, for any linear operator $A \in \mathcal{M}_{s+m,s+l}^s$ and $B \in \mathcal{M}_{s+l,s+n}^s$, there exists constant $C(s)$ such that

$$\|AB\|^s_{s+m,s+n} \leq C(s) \left( \|A\|^s_{s+m,s+l} \|B\|^{s_0} + \|A\|^{s_0} \|B\|^s_{s+m,s+n} + \|A\|^{s_0} \|B\|^s_{s+l,s+n} \right).$$  

(3.12)

The assertion holds true by replacing $\|\cdot\|^s_{s+m,s+n}$ by $\|\cdot\|^{\mathcal{L}ip}_{s+m,s+n}$.

**Proof.** These bounds can be obtained from Remark 3.6 and Lemma 3.3.

**Lemma 3.8.** Assume that $s_0 > \frac{d+1}{2}$ and $C(s)\|A\|^{s_0,\mathcal{L}ip}_{s_0+m,s_0+m} \leq \frac{1}{2}$ for some $m \in \mathbb{R}$ and large $C(s) > 0$ depending on $s \geq s_0$, then the map $\Phi := \text{Id} + \Psi$ defined as $\Phi = e^{iA} = \sum_{p \geq 0} \frac{1}{p!} (iA)^p$ satisfies

$$\|\Psi\|^{s,\mathcal{L}ip}_{s+m,s+m} \leq C\|A\|^{s,\mathcal{L}ip}_{s+m,s+m},$$  

(3.13)

where $C$ is a constant depending on $s, d, m$.

**Proof.** From Lemma 3.7, for some $C(S) \geq 0$,

$$\|A^n\|^{s,\mathcal{L}ip}_{s+m,s+m} \leq n[C(S)\|A\|^{s_0,\mathcal{L}ip}_{s_0+m,s_0+m}]^{n-1} C(s)\|A\|^{s,\mathcal{L}ip}_{s+m,s+m}.$$  

(3.14)

Hence,

$$\|\Psi\|^{s,\mathcal{L}ip}_{s+m,s+m} \leq \|A\|^{s,\mathcal{L}ip}_{s+m,s+m} \sum_{p \geq 1} \frac{C(s)^p}{(p-1)!} \|A\|^{s_0,\mathcal{L}ip}_{s_0+m,s_0+m}.$$  

(3.15)

for some large $C(s) > 0$. The bounds 3.13 can be obtained from the small condition of $C(s)\|A\|^{s_0,\mathcal{L}ip}_{s_0+m,s_0+m}$.

□
4. Reduce the order of perturbations

The main goal of this section is to convert the original problem \((1.1)\) to a new one, which the new perturbation is a sufficiently smoothing operator. By direct calculation, the equation \((1.1)\) can be rewritten as
\[
\partial_t u = Ku + Qu + \varepsilon W(\omega t)[u],
\]
where \(K = (-\partial_{xx})^{1/2}, \ K \xi = |j| \xi, \forall j \in \mathbb{Z}\). \(Q\) is a pseudo-differential operator of \(-1\) order, and
\[
Q \xi = c(m, j) \langle j \rangle \xi.
\]
Here \(c(m, j)\) is depending on \(m, j\).

**Lemma 4.1.** Given a linear operator \(Z : H^\infty \mapsto H^{-\infty}\). If \([Z, K] = 0\), the block matrix representation of \(Z\) satisfies
\[
Z[i][j] = 0, \quad \forall i \neq j.
\]
**Proof.** From \([Z, K] = 0\), for any \(j, i \in \mathbb{N}\), one gets that
\[
Z[i][j](i - j) = 0.
\]
Hence, for any \(i \neq j\), \((4.3)\) implies that
\[
Z[i][j] = 0.
\]
\(\square\)

**Lemma 4.2.** Given a pseudo-differential operator \(B \in OPS^\eta\), the corresponding linear operator \(e^{i\kappa \cdot K} Be^{-i\kappa \cdot K}\) is \(2\pi\) periodic to \(\kappa\).

**Proof.** The spectrum of \(K\) is integer, thus \(e^{i\kappa \cdot K} Be^{-i\kappa \cdot K} = e^{i(\kappa + 2\pi) \cdot K}\).

The following lemma plays an important role in the regularization process.

**Lemma 4.3.** Set the cantor set \(\Omega_{0, \alpha} \subseteq \Omega\) as
\[
\Omega_{0, \alpha} := \{ \omega \in \Omega : |\omega \cdot \ell + m| \geq \frac{\alpha}{1 + |\ell|^{d+1}}, \forall (\ell, m) \in \mathbb{Z}^{d+1} \setminus \{0\} \}\.
\]
Let \(W\) be an Hermitian operator and belongs to \(\text{Lip}(\Omega, C^\infty(\mathbb{T}^d, OPS^\eta)), \eta \leq 1\). Then, the homological equation
\[
\omega \cdot \partial_\theta B + i[K, B] = W - \langle W \rangle
\]
with
\[
\langle W \rangle := \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{T}^d} \int_{\mathbb{T}} e^{i\kappa \cdot K} W e^{-i\kappa \cdot K} d\kappa d\theta
\]
has a solution \(B \in \text{Lip}(\Omega_{0, \alpha}, C^\infty(\mathbb{T}^d, OPS^\eta))\). Moreover, the operator \(B\) is an Hermitian operator too.

**Proof.** For \(W(\theta) \in \text{Lip}(\Omega_{0, \alpha}, C^\infty(\mathbb{T}^d, OPS^\eta))\), we define \(W(\theta, \kappa) = e^{i\kappa \cdot K} W(\theta) e^{-i\kappa \cdot K}\). From Remark \(8.4\) we know that
\[
W(\theta, \kappa) \in \text{Lip}(\Omega_{0, \alpha}, C^\infty(\mathbb{T}^{d+1}, OPS^\eta))\).
\]
Since \(W(\theta, \kappa)\) is defined on \(\mathbb{T}^{d+1}\), it can be expanded by its Fourier series as
\[
W(\theta, \kappa) = \sum_{(\ell, m) \in \mathbb{Z}^{d+1}} \hat{W}_{\ell, m} e^{i(\ell \cdot \theta + m \cdot \kappa)}
\]
where
\[ W(\theta) = W(\theta, 0) = \sum_{(\ell, 0) \in \mathbb{Z}^{d+1}} \hat{W}_{\ell, 0} e^{i\ell \cdot \theta}. \]

The homological equation (4.6) can be extended as
\[ (4.8) \quad \omega \cdot \partial_t B(\theta, \kappa) + i[K, B(\theta, \kappa)] = W(\theta, \kappa) - \langle W(\theta, \kappa) \rangle. \]

Obviously, if \( B(\theta, \kappa) \) is the solution of equation (4.8), then \( B(\theta, 0) \) is the solution of equation (4.6). Notice that
\[ (4.9) \quad i[K, B(\theta, \kappa)] = \frac{d}{ds}{\bigg |}_{s=0} e^{is \cdot K} B(\theta, \kappa) e^{-is \cdot K} \]
\[ (4.10) \quad = \frac{d}{ds}{\bigg |}_{s=0} B(\theta, \kappa + s) = \sum_{(\ell, m) \in \mathbb{Z}^{d+1}} \hat{B}_{\ell, m} \frac{d}{ds}{\bigg |}_{s=0} e^{i\ell \cdot \theta + \imath \cdot m(\kappa + s)} \]
\[ (4.11) \quad = \sum_{(\ell, m) \in \mathbb{Z}^{d+1}} \imath \hat{B}_{\ell, m} e^{i\ell \cdot \theta + \imath \cdot m \cdot \kappa}. \]

The homological equation (4.8) is equivalent to
\[ (4.12) \quad i(\omega \cdot \ell + m) \hat{B}_{\ell, m} = \hat{W}_{\ell, m}, \quad (\ell, m) \neq (0, 0) \quad \text{and} \quad \hat{B}_{0, 0} = 0. \]

Since the operator \( W(\theta, \kappa) \) belongs to \( \mathcal{L}(\Omega_{0, \alpha}, C^\infty(\mathbb{T}^{d+1}, OPS^n)) \), the seminorms of \( \hat{W}_{\ell, m} \) decay faster than any power of \( |\ell| + |m| \). From the definition of \( \Omega_{0, \alpha} \), we see that \( \hat{B}_{\ell, m} \) also decay faster than any power of \( |\ell| + |m| \). See that \( B(\theta) = B(\theta, 0) \), thus \( B(\theta) \in C^\infty(\mathbb{T}^d, OPS^n) \).

Furthermore, for any \( \omega_1, \omega_2 \in \Omega_{0, \alpha} \), one has
\[ (4.13) \quad \hat{B}_{\ell, m}(\omega_1) - \hat{B}_{\ell, m}(\omega_2) = \frac{\hat{W}_{\ell, m}(\omega_1)(\omega_2 - \omega_1)\ell}{\omega_1 \ell + m} \left( \frac{\hat{W}_{\ell, m}(\omega_1) - \hat{W}_{\ell, m}(\omega_2)}{\omega_2 \ell + m} \right). \]

Hence, from the non-resonance condition (4.20), we can obtain the Lipschitz regular of \( B \) to the parameter \( \omega \).

Moreover, from
\[ W - W^* = e^{-iK} (W(\theta, \kappa) - W^*(\theta, \kappa)) e^{iK}, \]
\[ B - B^* = e^{-iK} (B(\theta, \kappa) - B^*(\theta, \kappa)) e^{iK}, \]
we know that \( W(\text{resp } B) \) is an Hermitian operator, if and only if \( W(\theta, \kappa)(\text{resp } B(\theta, \kappa)) \) is an Hermitian operator. From \( \hat{B}_{\ell, m} = \frac{\hat{W}_{\ell, m}}{n(\ell \cdot m)} \) and Remark 3.2, we can obtain that \( B \) is an Hermitian operator.

**Theorem 4.4.** \( \forall M > 0 \), there exists a sequence of symmetric maps \( \{B_i(\theta, \omega)\}_{i=0}^M \) with \( B_i(\theta, \omega) \in \mathcal{L}(\Omega_{0, \alpha}, C^\infty(\mathbb{T}^d, OPS^{1/2}) \}) \) such that the change of variables
\[ u = e^{-iB_0(\theta, \omega)} \cdots e^{-iB_I(\theta, \omega)} \]
conjugates the Hamiltonian \( H_0 = K + Q + W(\omega t) \) to
\[ (4.14) \quad H_{i+1} = K + Q + \varepsilon Z^{i+1} + \varepsilon W^{i+1}, \]
where \( Z^{i+1} \) is time-independent and fulfills
\[ (4.15) \quad [Z^{i+1}, K] = 0. \]

Also,
\[ (4.16) \quad Z^{i+1}(\omega) \in \mathcal{L}(\Omega_{0, \alpha}, OPS^{1/2}), \]
\[ (4.17) \quad W^{i+1}(\theta, \omega) \in \mathcal{L}(\Omega_{0, \alpha}, C^\infty(\mathbb{T}^d, OPS^{-1/2})). \]

Furthermore, \( Z^{i+1}, W^{i+1} \) are Hermitian operators.
Remark 4.5. From Lemma 8.1 for all $j = 1, 2, \cdots, M$, the operator $e^{\pm iB_j} \in \mathcal{L}(H^s), \forall s \geq 0$, and
\begin{equation}
\|e^{\pm iB_j} - \text{Id}\|_{\mathcal{L}(H^s, H^{s-(\frac{d}{2}+j)})} \lesssim \varepsilon \|B_j\|_{\mathcal{L}(H^{s-(\frac{d}{2}+j)})}.
\end{equation}

Moreover, we also show that the closed set $\Omega_{0,\alpha}$ is asymptotically full Lebesgue.

**Proposition 4.6.**
\[\text{meas}(\Omega \setminus \Omega_{0,\alpha}) \leq C\alpha.\]

**Proof.** Set $Q_{\ell, m}$ as
\begin{equation}
\{ \omega \in \Omega : |\omega \cdot \ell + m| < \frac{\alpha}{1 + |\ell|^d + 2} \}.
\end{equation}

If $|\ell| < \frac{|m|}{2}$ the set $Q_{\ell, m}$ is empty.
If $|\ell| \geq \frac{|m|}{2}$, one gets that
\begin{equation}
\text{meas}(Q_{\ell, m}) \leq \frac{2\alpha}{1 + |\ell|^d + 2}.
\end{equation}
Finally, we have
\[
\text{meas}(\Omega \setminus \Omega_{0,\alpha}) \leq \sum_{|m| \leq 2|\ell|, \ell \in \mathbb{Z}^d} \text{meas}(Q_{\ell,m}) \leq C\alpha.
\]
\[\square\]

5. KAM REDUCIBILITY

5.1. The reducibility theorem.

Fix the number of regularization step in Theorem 4.3 as
\[
M := 4m + 1.
\]
After \(M\) steps of regularization in previous section, we get the new equation
\[
(5.2) \quad \omega \cdot \partial_\theta u = \mathcal{H}^M u = \Lambda_0 u + P_0 u,
\]
where \(\Lambda_0 = K + Q + \varepsilon Z^M\) and \(P_0 = \varepsilon \mathcal{W}^M\).

The equation (5.2) satisfies the following assumptions:

(A1) The linear operator \(\Lambda_0\) is an Hermitian operator, and block diagonal, independent of \(\theta\), Lipschitz on \(\omega \in \Omega_{0,\alpha}\). Denoting \((\lambda_{j,k})_{k=1,2}\) as the eigenvalue of the 2 \(\times\) 2 block \((\Lambda_0)^{[j]}\), for any \(\omega \in \Omega_{0,\alpha}\), there exists a constant \(c_0\) such that
\[
|\lambda_{i,k} - \lambda_{j,k'}| \geq c_0|i - j|, \quad \forall k, k', 1, 2, \text{ and } i \neq j,
\]
\[
|\lambda_{j,k}(\omega)|^{lip,\Omega_{0,\alpha}} = \sup_{\omega_1,\omega_2 \in \Omega_{0,\alpha}} \frac{|\lambda_{j,k}(\omega_1) - \lambda_{j,k}(\omega_2)|}{|\omega_1 - \omega_2|} \leq \frac{1}{8}, \quad \forall j \in \mathbb{N}, k = 1, 2.
\]

(A2) The linear operator \(P_0\) is an Hermitian operator and belongs to \(\mathcal{M}_{S-m,S+m}^{\mathbb{R},lip}\), \(S \geq s_0 > \frac{d+1}{4}\).

Remark 5.1. The assumption (A2) can be obtained from the Theorem 4.4 and Prop 8.5. Regarding assumption (A1), we need the following Lemma.

Lemma 5.2. Suppose that \(\mathcal{W}(\omega \theta) \in C^\infty(\mathbb{T}^n, OPS^{1/2}_\ell)\), the eigenvalues \((\lambda_{j,k})_{k=1,2}\) of the 2 \(\times\) 2 block \((\Lambda_0)^{[j]}\) have the asymptotic expression
\[
\lambda_{j,k} = (j^2 + m^2)^{1/2} + \varepsilon a(j)^{1/2} + r_{j,k},
\]
where \(|r_{j,k}|^{lip,\Omega_{0,\alpha}} \leq C\varepsilon\).

Proof. From Theorem 4.4, one gets
\[
\Lambda_0 = K + Q + \varepsilon Z^M,
\]
where \(Z^M = \langle \mathcal{W}^0 \rangle + \langle \mathcal{W}^1 \rangle + \cdots + \langle \mathcal{W}^{M-1} \rangle\). Here, \(\langle \mathcal{W}^0 \rangle \in OPS^{1/2}_\ell\), and \(\langle \mathcal{W}^1 \rangle + \langle \mathcal{W}^2 \rangle + \cdots + \langle \mathcal{W}^{M-1} \rangle \in OPS^0\). The symbol \(w(\theta, x, j)\) of pseudo-differential operator \(\mathcal{W}\) can be written as
\[
w(\theta, x, j) = \sum_{\ell \in \mathbb{Z}^d} w(x, j)(\ell) e^{i\ell \cdot \theta} = \sum_{(\ell, k) \in \mathbb{Z}^{d+1}} w_{\ell,k}(j) e^{i\ell \cdot \theta} e^{ik \cdot x}.
\]
From (5.1) and (5.7), one has
\[
(5.8) \quad \langle \mathcal{W}(\theta) \rangle^{[j]} = \begin{pmatrix} w_{0,0}(j) & w_{-2j,0}(j) \\ w_{2j,0}(-j) & w_{0,0}(-j) \end{pmatrix}.
\]
These four elements in the matrix are independent of $\omega$. From definition (2.1) and condition (C1), one gets
\[
\omega_{0,0}(\pm j) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{T}^{d+1}} w(\theta, x, \pm j) dx d\theta = a(j)^{\frac{1}{2}} + b(j),
\]
\[
|\omega_{2j,0}(\pm j)| \leq C \sum_{\beta \leq 1} \sup_{x \in T} |\partial_x^\beta w(\pm z, \pm j)(0)| \leq \tilde{C} \langle j \rangle^{\frac{1}{2}} + 1 + |2j|.
\]

Rewrite $\langle W \rangle$ as $\langle W \rangle_r + \langle W \rangle_l$, where
\[
(5.9) \quad \langle W(\theta) \rangle^{[j]}_r = \begin{pmatrix} a(j)^{\frac{1}{2}} & 0 \\ 0 & a(j)^{\frac{1}{2}} \end{pmatrix}, \quad \langle W(\theta) \rangle^{[j]}_l = \begin{pmatrix} b(j) & w_{2j,0}(j) \\ w_{2j,0}(-j) & b(-j) \end{pmatrix}.
\]

Denoting $(\mu_{j,k})_{k=1,2}$ as the eigenvalues of the block $[K + Q + \langle W(\theta) \rangle_s^{[j]}]$, one has
\[
\mu_{j,k} = (j^2 + m^2)^{\frac{1}{2}} + a(j)^{\frac{1}{2}}.
\]

Let $R = \langle W(\theta) \rangle_s + \langle W \rangle_l + \cdots + \langle W \rangle_M^{[j]}$, from Theorem 4.1 and prop 8.5, for any $S \geq s_0 > \frac{d+1}{4}$, one has $R \in \mathcal{M}_S^{L, \text{lip}}$. From prop 8.6 and Corollary A.7 in [17], the Lipschitz variation of the eigenvalues of an Hermitian matrix is controlled by the Lipschitz variation of the matrix. Then, we can get
\[
(5.10) \quad |r_{j,k}|^{L, \text{lip}} = |\lambda_{j,k} - \mu_{j,k}|^{L, \text{lip}} \leq \|R_{[j]}^{[j]}\|^{L, \text{lip}} \leq C \varepsilon.
\]

Hence, this lemma is proved. \hfill \square

The main goal of this section is the following theorem.

**Theorem 5.3. (The Reducibility Theorem)** Let $s \in [s_0, S - R]$ and $\alpha \in (0, 1)$, there exists positive $\varepsilon_0 = \varepsilon_0(s, \alpha)$ such that, if $\varepsilon \leq \varepsilon_0$, there exists a cantor subset $\Omega_\varepsilon \subseteq \Omega_{0,\alpha}$ with
\[
\text{meas}(\Omega_{0,\alpha} \setminus \Omega_\varepsilon) \leq C \alpha.
\]

For any $\omega \in \Omega_\varepsilon$, there exist a family bounded and invertible operators $\Phi_\infty = \Phi_\infty(\omega, \theta) \in \mathcal{L}(\mathcal{H}^s)$, conjugating the linear equation (5.2) to
\[
(5.11) \quad i\omega \cdot \partial_\theta u = H^\infty u,
\]
where $H^\infty$ is a time independent and block-diagonal Hamiltonian operator. Moreover, we have
\[
(5.12) \quad \sup_{\theta \in \mathbb{T}^d} \|\Phi_{[1]}^[-1](\theta) - \text{Id}\|_{\mathcal{L}(\mathcal{H}^s)} \leq C \varepsilon_0, \quad \forall \omega \in \Omega_\varepsilon.
\]

The procedure of KAM iteration is well known. For the convenience of reader, we show an outline of one step of the KAM reducibility.

We conjugate the linear equation
\[
i \partial_t u = H(t) u = Au + P(t) u
\]
through a transformation $u = e^{-iG}v$, so that the new equation is
\[
i \partial_t v = H^+(t)v,
\]
where
\[
H^+(t) = e^{iG(\omega t)}H(t)e^{-iG(\omega t)} - \int_0^t e^{isG(\omega t)} \dot{G} e^{-i\omega s G(\omega t)} ds,
\]
\[
H^+ = A + i[G, A] + \Pi N P - \dot{G} + P^+,
\]

Set $\varepsilon_0 = \|P_0\|_{S, \text{lip}}^{S, \text{lip}, S, \text{lip}}$. 

Theorem 5.3. (The Reducibility Theorem) Let $s \in [s_0, S - R]$ and $\alpha \in (0, 1)$, there exists positive $\varepsilon_0 = \varepsilon_0(s, \alpha)$ such that, if $\varepsilon \leq \varepsilon_0$, there exists a cantor subset $\Omega_\varepsilon \subseteq \Omega_{0,\alpha}$ with 
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\text{meas}(\Omega_{0,\alpha} \setminus \Omega_\varepsilon) \leq C \alpha.
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For any $\omega \in \Omega_\varepsilon$, there exist a family bounded and invertible operators $\Phi_\infty = \Phi_\infty(\omega, \theta) \in \mathcal{L}(\mathcal{H}^s)$, conjugating the linear equation (5.2) to
\[
(5.11) \quad i\omega \cdot \partial_\theta u = H^\infty u,
\]
where $H^\infty$ is a time independent and block-diagonal Hamiltonian operator. Moreover, we have
\[
(5.12) \quad \sup_{\theta \in \mathbb{T}^d} \|\Phi_{[1]}^[-1](\theta) - \text{Id}\|_{\mathcal{L}(\mathcal{H}^s)} \leq C \varepsilon_0, \quad \forall \omega \in \Omega_\varepsilon.
\]

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\[
i \partial_t v = H^+(t)v,
\]
where
\[
H^+(t) = e^{iG(\omega t)}H(t)e^{-iG(\omega t)} - \int_0^t e^{isG(\omega t)} \dot{G} e^{-i\omega s G(\omega t)} ds,
\]
\[
H^+ = A + i[G, A] + \Pi N P - \dot{G} + P^+,
\]
and

\begin{equation}
(5.16) \quad \mathbf{P}^+ = e^{iG(\omega t, \omega)} \mathbf{A} e^{-iG(\omega t, \omega)} - (\mathbf{A} + i[\mathbf{G}, \mathbf{A}] + \mathbf{P}) + e^{iG(\omega t, \omega)} \mathbf{P} e^{-iG(\omega t, \omega)}
\end{equation}

\begin{equation}
(5.17) \quad - \left( \int_0^1 e^{isG(\omega t, \omega)} \mathbf{G} e^{-isG(\omega t, \omega)} ds - \mathbf{G} \right) + \Pi_N^k \mathbf{P}.
\end{equation}

Our goal is determine the operator \( \mathbf{G} \) by solving the homological equation

\begin{equation}
(5.18) \quad \omega \cdot \partial_\theta \mathbf{G} = i[\mathbf{G}, \mathbf{A}] + \Pi_N^k \mathbf{P} - \text{diag}([\mathbf{P}^j_l](\omega)|j \in \mathbb{N}).
\end{equation}

Here \([\mathbf{P}^j_l](\omega)\) denotes

\begin{equation}
(5.19) \quad [\mathbf{P}^j_l](\omega) = \int_{\mathbb{R}^d} \mathbf{P}^j_l(\theta, \omega) d\theta.
\end{equation}

The new Hamiltonian is

\begin{equation}
(5.20) \quad \mathbf{H}^{k+1}(t) = \Lambda^+ + \mathbf{P}^+, \quad \Lambda := \text{diag}\{\Lambda_j|j \in \mathbb{N}\}, \quad \Lambda^+ = \Lambda + \text{diag}\{[\mathbf{P}^j_l](\omega)|j \in \mathbb{N}\}.
\end{equation}

As we all known, the crucial of KAM iteration is to estimate the solution \( \mathbf{G} \) of homological equation \((5.18)\). In order to deal with the notorious small divisor problem, it is necessary to impose some non-resonance condition on the eigenvalue of diagonal operator \( \Lambda \).

Denoting \((\lambda_{j,v})_{v=1,2}\) as the eigenvalues of the \(2 \times 2\) block \( \Lambda_j \), we define the non-resonance set \( \Omega^{k+1}_\alpha(\omega) \) at the \( k + 1 \) step reducibility,

\begin{equation}
(5.21) \quad \Omega^{k+1}_\alpha := \left\{ \omega \in \Omega^{k}_\alpha : |\omega \cdot \ell + \lambda^k_{j,\ell,\nu} - \lambda^k_{j,\ell',\nu'}| \geq \frac{\alpha}{N_\omega^{\ell}(i)\nu}(j)\sigma \right\}
\end{equation}

\( \forall i, j \in \mathbb{N}, \quad |\ell| \leq N_k, \quad v, v' = 1, 2, \quad (\ell, i, j) \neq (0, i, i). \}

In the following lemmas, we will estimate the solution \( \mathbf{G}^{k+1} \) of equation \((5.18)\) and the new perturbation \( \mathbf{P}^{k+1} \) in the KAM procedure.

5.2. The homological equation.

**Lemma 5.4.** For any \( \omega \in \Omega^{k+1}_\alpha \) and \( s \in [s_0, S - \beta] \), the homological equation

\begin{equation}
(5.22) \quad \omega \cdot \partial_\theta \mathbf{G}^{k+1} + i[\mathbf{A}^k, \mathbf{G}^{k+1}] = \Pi_N^k \mathbf{P}^k - \text{diag}[\mathbf{P}^k]
\end{equation}

has a solution \( \mathbf{G}^{k+1} \) defined on \( \Omega^{k+1}_\alpha \) with

\begin{equation}
(5.23) \quad \|\mathbf{G}^{k+1}\|_{s, \text{Lip}^+_{s+m, s+m}} \lesssim N_k^{2\tau + 2\sigma + 2}\|\mathbf{P}\|_{s, \text{Lip}^+_{s-m, s-m}},
\end{equation}

\begin{equation}
(5.24) \quad \|\mathbf{G}^{k+1}\|_{s+\beta, \text{Lip}^+_{s+\beta+m, s+\beta+m}} \lesssim N_k^{2\tau + 2\sigma + 2}\|\mathbf{P}\|_{s+\beta, \text{Lip}^+_{s+\beta-m, s+\beta-m}}.
\end{equation}

**Proof.** For notation simplicity, we rename \( \mathbf{A}^k, \mathbf{G}^{k+1}, \mathbf{P}^k, \lambda^k_j, N_k \) as \( \mathbf{A}, \mathbf{G}, \mathbf{P}, \lambda_j, N \). Considering the matrix representation and Fourier coefficients of these linear operator, the homological equation \((5.22)\) is equivalent to

\begin{equation}
(5.25) \quad i\omega \cdot \mathbf{G}^j_l(\ell) + i\lambda_j \mathbf{G}^j_l(\ell) - i\hat{G}^j_l(\ell) \Lambda_j = \hat{P}^j_l(\ell), \quad \forall 0 < |i - j| < N, \quad |\ell| < N, \quad (\ell, i, j) \neq (0, i, i).
\end{equation}

and \( \hat{G}^j_l(0) = 0 \).
From [5.21] and prop [5.7] for any \(|i - j| < N\), one has

\[
\| \hat{\mathbf{G}}_{[i]}^{[j]}(\ell) \| \lesssim \frac{\| \hat{\mathbf{P}}_{[i]}^{[j]}(\ell) \| \| N^{\tau}(i)^{\sigma}(j)^{\sigma} \|}{\alpha} \\
\lesssim \alpha^{-1} \| \hat{\mathbf{P}}_{[i]}^{[j]}(\ell) \| \| N^{\tau}(j)^{\sigma}(j)^{\sigma} + |i - j|^\sigma \| \\
\lesssim \alpha^{-1} \| \hat{\mathbf{P}}_{[i]}^{[j]}(\ell) \| \| N^{\tau}(j)^{\sigma}(j)^{\sigma} + N^{\sigma} \| \\
\lesssim \alpha^{-1} \| \hat{\mathbf{P}}_{[i]}^{[j]}(\ell) \| N^{\tau+\sigma}(j)^{2\sigma}.
\]

From the definition of the norm \(\| \cdot \|_{s+m,s+m}^s\), we can get

\[
\| G \|_{s+m,s+m}^s = \sum_{\ell \in \mathbb{Z}^d, h \in \mathbb{N}} \langle \ell, h \rangle^{2s} \sup_{h \in \mathbb{N}} \| \mathbf{G}_{[i]}^{[j]}(\ell) \|^2 (j)^{2m} \langle i \rangle^{2m}
\]

\[
\lesssim \alpha^{-2} N^{2\sigma + 2\tau} \sum_{\ell \in \mathbb{Z}^d, h \in \mathbb{N}} \langle \ell, h \rangle^{2s} \sup_{h \in \mathbb{N}} \| \mathbf{P}_{[i]}^{[j]}(\ell) \|^2 (j)^{2m} (j)^{2\sigma} \langle j \rangle^{2m}
\]

\[
\lesssim \alpha^{-2} N^{2\sigma + 2\tau} \langle \| \mathbf{P} \|_{s-m,s+m} \rangle^2
\]

The inequality (5.29) is valid, because \(\sigma \leq m\) and \(4\sigma - 2m \leq 2m\). By the same way, we also have

\[
\| G \|_{s-m,s-m}^s \lesssim \alpha^{-2} N^{2\sigma + 2\tau} \langle \| \mathbf{P} \|_{s-m,s+m} \rangle^2.
\]

There is no difference in estimating the norm of \(\| \mathbf{G} \|_{s+\beta \pm m,s+\beta \pm m}^s \) with \(\| \mathbf{G} \|_{s+\beta \pm m,s+\beta \pm m}^s\).

Regarding the Lipschitz semi-norm of \(\mathbf{G}\), we introduce the difference operator \(\Delta\). Given the operator \(\mathbf{G}\) of \(\omega\), set \(\Delta \mathbf{G} = \mathbf{G}(\omega_1) - \mathbf{G}(\omega_2)\). Applying the difference operator \(\Delta\) to equation (5.24), we have

\[
i \omega \cdot \ell (\Delta \mathbf{G}_{[i]}^{[j]}(\ell)) + j \mathbf{A}_j (\Delta \mathbf{G}_{[i]}^{[j]}(\ell)) - i (\Delta \mathbf{G}_{[i]}^{[j]}(\ell)) \mathbf{A}_i = \Delta (\mathbf{P}_{[i]}^{[j]}(\ell)) - i \Delta \omega \cdot \ell \mathbf{G}_{[i]}^{[j]}(\ell)
\]

\[
+ i (\Delta \mathbf{A}_j) \mathbf{G}_{[i]}^{[j]}(\ell) - i \mathbf{G}_{[i]}^{[j]}(\ell) (\Delta \mathbf{A}_i).
\]

Applying prop [5.7] again, we have

\[
\| \Delta \mathbf{G}_{[i]}^{[j]}(\ell) \| \lesssim \frac{N^{\tau}(i)^{\sigma}(j)^{\sigma}}{\alpha} \left( \| \Delta \mathbf{P}_{[i]}^{[j]}(\ell) \| \| \mathbf{G}_{[i]}^{[j]}(\ell) \| \| \mathbf{G}_{[i]}^{[j]}(\ell) \| (i + j) \right)
\]

\[
\lesssim \frac{N^{\tau}(i)^{\sigma} \| \mathbf{P}_{[i]}^{[j]}(\ell) \|}{\alpha} + \frac{N^{2\tau+1}(j)^{2\sigma+1}(j)^{2\sigma}}{\alpha^2} \| \mathbf{P}_{[i]}^{[j]}(\ell) \|.
\]

Now, we can get

\[
\| \Delta \mathbf{G} \|_{s+\beta \pm m,s+\beta \pm m} \lesssim \frac{N^{\tau+\sigma}}{\alpha} \| \Delta \mathbf{P} \|_{s-m,s+m} + \frac{N^{2\tau+2\sigma+2}}{\alpha^2} \| \mathbf{P} \|_{s-m,s+m}
\]

It is similar to consider \(\| \Delta \mathbf{G} \|_{s+\beta \pm m,s+\beta \pm m} \). Respectively, we can get (5.23) and (5.24).

Next, we consider the new perturbation \(\mathbf{P}^{k+1}\).

\[
\square
\]
Lemma 5.5. The new perturbation $P^{k+1}$ is defined on $\Omega_{\alpha}^{k+1}$, and satisfies the following quantities bounds:

\[
\|P^{k+1}\|_{s-m,s+m} \leq C \left( N_k^{-\beta} \|P^k\|_{s-m,s+m} + N_k^{2\tau+2\sigma+2} (\|P^k\|_{s-m,s+m})^2 \right),
\]

(5.35)

\[
\|P^{k+1}\|_{s+\beta,m,s+m+\beta} \leq C \left( \|P^k\|_{s+\beta-m,s+m+\beta} + N_k^{2\tau+2\sigma+2} \|P^k\|_{s-m,s+m} \|P^k\|_{s+\beta-m,s+\beta} \right).
\]

(5.36)

$C$ is a constant depending on $s, m, \sigma, \tau$.

Proof. Recall the definition of $P^{k+1}$, we have

\[
P^{k+1} = \Pi_N^k P^k + \int_0^1 e^{isG^{k+1}} [G^{k+1}, P^k] e^{-isG^{k+1}} ds
\]

(5.37)

\[
\quad + \int_0^1 ds \int_0^s e^{isG^{k+1}} [G^{k+1}, \Pi_N^k P^k - [P^k]] e^{-isG^{k+1}} ds.
\]

(5.38)

The Lemma 3.3 implies that

\[
\|\Pi_N^k P^k\|_{s-m,s+m} \leq N_k^{-\beta} \|P^k\|_{s+\beta-m,s+\beta+m}.
\]

(5.39)

From Lemma 3.7 and (5.23), we can get

\[
\|G^{k+1}, P^k\|_{s-m,s+m} \leq N_k^{2\tau+2\sigma+2} \left( \|P^k\|_{s-m,s+m} \right)^2,
\]

(5.40)

and

\[
\|G^{k+1}, P^k\|_{s+\beta-m,s+\beta+m} \leq N_k^{2\tau+2\sigma+2} \|P^k\|_{s-m,s+m} \|P^k\|_{s+\beta-m,s+\beta+m}.
\]

(5.41)

The estimation of $[G^{k+1}, \Pi_N^k P^k - [P^k]]$ is no difference from $[G^{k+1}, P^k]$. Summing up the contribution of these operators and following the spirit of Lemma 3.8, we can obtain (5.35) and (5.36) respectively.

5.3. Proof of reducibility theorem.

5.3.1. Iterative lemma.

The proof of the reducibility theorem 5.3 is depending on the following iterative lemma. Some constants should be fixed before the following lemma. Given $\tau > d, \sigma > 1$, we fix

\[
m = 2\sigma + 2, \quad \alpha = 6\tau + 6\sigma + 7, \quad \beta = \alpha + 1.
\]

(5.42)

Moreover, we fix the scale on which we perform the reducibility scheme as

\[
N_k = (N_0) \left( \frac{1}{2} \right)^k, \quad \forall k \in \mathbb{N}, \quad N_{-1} = 1.
\]

(5.43)

Proposition 5.6. Let $s \in [s_0, S - \beta]$, if there exists a constant $c_0(s, d)$ big enough such that

\[
\epsilon N_0^{c_0} \leq \frac{1}{2},
\]

(5.44)

then we can recursively define a family of non-resonance set $\{\Omega_k\}_{k \geq 0}$. For any $\omega \in \Omega_k$, we can iteratively define a Lipschitz family linear operator

\[
\mathcal{L}_k = \omega \cdot \partial_x - A^k - P^k, \quad k \geq 0,
\]

(5.45)

such that the following items hold true for any $k \geq 0$:

A: There exists a Lipschitz family transformation operator $e^{-iG^{k+1}}$ defined on $\Omega_{k+1}$, which conjugate the linear operator $\mathcal{L}_k$ to

\[
\mathcal{L}_{k+1} = e^{iG^{k+1}} \mathcal{L}_k e^{-iG^{k+1}}.
\]

(5.46)
Moreover, for any \( s \in [s_0, S - \beta] \)
\[
\| G^{k+1} \|_{s,M,s,M} \leq C_* N_{k_1}^{2\tau+2\sigma+2} N_{k-1}^{-\alpha} \epsilon.
\]

**B:** \( A^{k+1} \) is \( 2 \times 2 \) block diagonal and time independent. Denoting \((\lambda_{j,v}^{k+1})_{v=1,2}\) as the eigenvalues of block \(A_j^{k+1}\), for any \( \omega \in \Omega_{k+1}\), there exists a positive constant \( c_0 \) such that
\[
|\lambda_{i,v}^{k+1} - \lambda_{j,v'}^{k+1}| \geq \frac{c_0}{2}|i-j|, \quad \forall i \neq j, \; v, v' = 1, 2,
\]
and
\[
|\lambda_{j,v}^{k+1}| \leq \|A_j^{k+1}\| \leq \frac{1}{4}.
\]

**C:** For any \( s \in [s_0, S - \beta] \), the perturbation \( P^{k+1} \) defined on \( \Omega_{k+1} \), and satisfies
\[
\| P^{k+1} \|_{s,M,s,M} \leq C_* N_{k}^{-\alpha} \epsilon
\]
\[
\| P^{k+1} \|_{s+\beta,M,s+\beta,M} \leq C_* N_{k}
\]
The constant \( C_* \) is depending on \( m, \sigma, \tau, s, d \).

**Proof.** We prove this proposition by induction.

From the assumption (A1) and (A2), the conditions (A),(B),(C) are valid for \( n = 0 \). Hence, we assume that conditions (A),(B),(C) hold for \( 1 \leq n \leq k \). We prove that they also hold for \( n = k + 1 \).

From Lemma 5.4, for any \( s \in [s_0, S - \beta] \) and \( \omega \in \Omega_{k+1} \), we have
\[
\| G^{k+1} \|_{s,M,s,M} \leq N_{k}^{2\tau+2\sigma+2} \| P^{k} \|_{s,M,s,M} \leq C_* N_{k}^{2\tau+2\sigma+2} N_{k-1}^{-\alpha} \epsilon.
\]

Hence, the condition (A) holds true for \( n = k + 1 \).

From Lemma 5.5, for any \( s \in [s_0, S - \beta] \) and \( \omega \in \Omega_{k+1} \), one gets
\[
\| P^{k+1} \|_{s,M,s,M} \leq C_* N_{k}^{-\alpha} \epsilon,
\]
\[
\| P^{k+1} \|_{s+\beta,M,s+\beta,M} \leq C_* N_{k}^{-\alpha} \epsilon,
\]
provided
\[
2CN_{k+1}^{\alpha-\beta} N_{k-1} \leq 1, \quad 2CN_{k-1}^{-\alpha} N_{k}^{2\tau+2\sigma+2} \epsilon \leq 1.
\]
These conditions can be verified by (5.42) and (5.43). Furthermore, we have
\[
\| P^{k+1} \|_{s+\beta,M,s+\beta,M} \leq C_* N_{k}^{-\alpha} \epsilon,
\]
provided \( N_0 \) is big enough.

Hence, the condition (C) is valid for \( n = k + 1 \).
Regarding the new diagonal operator \( \Lambda^{k+1} = \Lambda^k + \text{diag}[P^k] \), from prop 8.7, \( \forall i \neq j \), one has

\[
|\lambda_{i,v}^{k+1} - \lambda_{j,v'}^{k+1}| \geq |\lambda_{i,v} - \lambda_{j,v'}| - \left( \sum_{n=0}^{k} (P^n)[i][i] + \sum_{n=0}^{k} (P^n)[j][j] \right)
\]

\[
\geq c_0 |i - j| - 2 \sum_{n=0}^{k} \|P^n\|_{\text{Lip}} \| \mu_i^n \|_{\text{Lip}} + 2 \sum_{n=0}^{k} \|P^n\|_{\text{Lip}} \| \mu_j^n \|_{\text{Lip}}
\]

\[
\geq c_0 |i - j| - 2C_\star \sum_{n=0}^{k} N_n^{-\alpha} \epsilon
\]

\[
\geq c_0 \frac{1}{2} |i - j|.
\]

Since the Lipschitz variation of the eigenvalues of an Hermitian matrix is controlled by the Lipschitz variation of the matrix, one has

\[
|\lambda_{j,v}^{k+1}|^{\text{Lip}} \leq \left\| (\Lambda^{k+1})[j][j] \right\|^{\text{Lip}} \leq \frac{1}{8} + \sum_{n=0}^{k} \left\| (P^n)[i][i] \right\|^{\text{Lip}}
\]

\[
\leq \frac{1}{8} + C_\star \sum_{n=0}^{k} N_n^{-\alpha} \epsilon \leq \frac{1}{4}.
\]

Hence, the condition (B) is verified for \( n = k + 1 \). \( \square \)

Moreover, we need estimate the set of parameter excluded in the KAM process. Thus, we need the following assertions.

5.3.2. Measure Estimates.

In this section, we show that the set excluded in the KAM iteration is asymptotic full measure. In the iteration procedure, we have recursively defined the set \( \{ \Omega_{k,\alpha} \}, k \geq 0 \), where \( \Omega_{k+1,\alpha} \subseteq \Omega_{k,\alpha}, k \geq 1 \).

Set \( \Omega_{\infty,\alpha} = \bigcap_{k=0}^{\infty} \Omega_{k,\alpha} \), we prove the following assertions.

**Theorem 5.7.**

\[
\text{meas}(\Omega_{0,\alpha} \setminus \Omega_{\infty,\alpha}) \leq C_\alpha.
\]

Since \( \Omega_{k+1} \subseteq \Omega_k, k \geq 0 \), we can decompose \( \Omega_{0,\alpha} \setminus \Omega_{\infty,\alpha} \) as

\[
\Omega_{0,\alpha} \setminus \Omega_{\infty,\alpha} = \bigcup_{k=0}^{\infty} (\Omega_{k,\alpha} \setminus \Omega_{k+1,\alpha}).
\]

The estimate of \( \text{meas}(\Omega_{k,\alpha} \setminus \Omega_{k+1,\alpha}) \) is crucial. From the definition of \( \Omega_k \), one has

\[
\Omega_k \setminus \Omega_{k+1} \subseteq \bigcup_{\ell \in \mathbb{Z}^d, |\ell| \leq N_k} \bigcup_{(i,j) \neq (0,j,j)} R_{\ell_{ij} v v'}
\]

and

\[
R_{\ell_{ij} v v'} = \left\{ \omega \in \Omega_{k,\alpha} : |\omega \cdot \ell + \lambda_{i,v}^k - \mu_{j,v'}^k| \leq \frac{\alpha}{N_k^2 (\ell^2/j^2)^{\alpha}} \right\}.
\]

**Lemma 5.8.**

\[
\text{meas}(\Omega_{k,\alpha} \setminus \Omega_{k+1,\alpha}) \leq C_\alpha N_k^{-1}.
\]
Proof. If $\ell = 0$ and $i \neq j$, we have
\begin{equation}
|\lambda_{i,v}^k - \mu_{j,v}^k| \geq \frac{c_0}{2}|i - j| \geq \alpha.
\end{equation}
Hence, $R_{0ijvv'}$ is an empty set.

Regarding other cases, we consider the Lipschitz function $g(\omega)$
\begin{equation}
g(\omega) = \omega \cdot \ell + \lambda_{i,v}^k(\omega) - \lambda_{j,v}^k(\omega).
\end{equation}
For any $\ell \neq 0$, we write
\begin{equation}
\omega = \frac{\ell}{|\ell|} s + \omega_1, \quad \omega_1 \in \mathbb{R}^d, \quad \omega_1 \cdot \ell = 0,
\end{equation}
and
\begin{equation}
g(s) = |\ell| \cdot s + \lambda_{i,v}^k(\omega(s)) - \lambda_{j,v}^k(\omega(s)).
\end{equation}
From [5.49], we can obtain
\begin{equation}
|g(s_1) - g(s_2)| \geq (|\ell| - \frac{1}{4})|s_1 - s_2| \geq \frac{1}{2}|s_1 - s_2|,
\end{equation}
which implies
\begin{equation}
\text{meas}\{ s : g_{ijvv'}(s) < \frac{\alpha}{N_k^\sigma(i)^\sigma(j)^\sigma} \} < \frac{2\alpha}{N_k^\sigma(i)^\sigma(j)^\sigma}
\end{equation}
By the Fubini theorem, we can get
\begin{equation}
\text{meas}(R_{ijvv'}) \leq \frac{2\alpha}{N_k^\sigma(i)^\sigma(j)^\sigma}.
\end{equation}
Finally, we have
\begin{align}
\text{meas}(\Omega_{k,\alpha} \setminus \Omega_{k+1,\alpha}) & \leq \sum_{\ell \in \mathbb{Z}^d, |\ell| \leq N_k} \sum_{i,j \in \mathbb{Z}} \text{meas}(R_{ijvv'}) \\
& \leq \sum_{\ell \in \mathbb{Z}^d, |\ell| \leq N_k} \frac{8\alpha}{N_k^\sigma(i)^\sigma(j)^\sigma} \\
& \leq CN_k^{-1}\alpha
\end{align}
\begin{proof}
(Proof of Theorem 5.7) From Lemma 5.8, provided $N_0$ big enough, we have
\begin{equation}
\text{meas}(\Omega_{0,\alpha} \setminus \Omega_{\alpha,\infty}) \leq \sum_{k=0}^\infty CN_k^{-1}\alpha \leq C\alpha.
\end{equation}
\end{proof}

From proposition 5.6 and Theorem 5.7, we can give a precise proof the reducibility theorem 5.3.

Proof. (Proof of Theorem 5.3) For any $k \geq 0$, we can define a sequence linear operator
\begin{equation}
\Phi_k = e^{-iG_1} \circ e^{-iG_2} \circ \cdots \circ e^{iG_k}
\end{equation}
on the set $\Omega_{\alpha,\infty}$. The sequence of linear operator $\{\Phi_k\}_{k \geq 1}$ is converges to an invertible operator $\Phi_\infty$, and satisfies
\begin{equation}
\|\Phi_\infty^\pm - \text{Id}\|_{\text{op}} \leq C(s)N_0^{2\tau + 2\sigma + \epsilon}.
\end{equation}
For any \( s' \in \[0, S - \beta]\) and \( s \in \[s_0, S - \beta]\), we have
\[
\sup_{\theta \in \mathbb{T}^d} \| \Phi^\pm_{\infty} - \text{Id} \|_{\mathcal{L}(\mathbf{H}^s)} \leq \| \Phi^1_{\infty} - \text{Id} \|_{\mathbf{H}^{s_{+m}, s_{-m}}} \leq C(s) N_0^{2\tau + 2\sigma + 2\epsilon}.
\]
Passing the iterative lemma 5.6 to the limit, the operator \( \mathcal{L}_0 \) is conjugated to
\[
\mathcal{L}_\infty = i\omega \cdot \partial_\theta - \Lambda^\infty
\]
where \( \Lambda^\infty \) is a \( \theta \) independent, block diagonal, Hermitian operator. \( \square \)

6. PROOF OF MAIN RESULTS

Proof. (Proof of Theorem 2.6) Considering the composition operator
\[
\mathcal{N}(\theta) = e^{-iB_0(\theta)} \circ \cdots \circ e^{-iB_{M-1}(\theta)} \circ \Phi_\infty(\theta)
\]
defined on \( \Omega_{\infty, \alpha} \). From Theorem 5.7 and prop 4.6 one gets
\[
\text{meas}(\Omega \setminus \Omega_{\infty, \alpha}) \leq \text{meas}(\Omega \setminus \Omega_{0, \alpha}) + \text{meas}(\Omega_{0, \alpha} \setminus \Omega_{\infty, \alpha}) \leq C\alpha.
\]
The coordinate changes \( u = \mathcal{N}(\theta)v \) transforms the equation (1.1) to
\[
i\partial_t v = \Lambda^\infty v
\]
where \( \Lambda^\infty = \mathcal{K} + \mathcal{Q} + \mathcal{Z} \).

From Lemma 8.1 and Theorem 5.7 for any \( s \geq 0 \), there exists a finite constant \( C_0 \) such that
\[
\sup_{\theta \in \mathbb{T}^d} \| \mathcal{N}(\theta) \|_{\mathcal{L}(\mathbf{H}^s)} \leq \| e^{-iB_0(\theta)} \circ \cdots \circ e^{-iB_{M-1}(\theta)} \|_{\mathcal{L}(\mathbf{H}^s)} \| \Phi_\infty(\theta) \|_{\mathcal{L}(\mathbf{H}^s)} \leq C_0
\]
and
\[
\sup_{\theta \in \mathbb{T}^d} \| \mathcal{N}^{-1}(\theta) \|_{\mathcal{L}(\mathbf{H}^s)} \leq \| \Phi^{-1}_\infty(\theta) \|_{\mathcal{L}(\mathbf{H}^s)} \| e^{iB_0(\theta)} \circ \cdots \circ e^{iB_{M-1}(\theta)} \|_{\mathcal{L}(\mathbf{H}^s)} \leq C_0.
\]
Hence, the theorem 2.6 is proved. \( \square \)

7. APPENDIX A

For the convenience of reader, we emphasize the distinction of reducibility theorem 5.3 that we proved on an irrational torus \( \mathbb{T}_\beta \) and a tours \( \mathbb{T} \).

**The difference in functional space:**
The Sobolev space \( \mathcal{H}^r(\mathbb{T}_\beta) \) defined by
\[
\mathcal{H}^r(\mathbb{T}_\beta) := \left\{ u = \sum_{\xi \in \mathbb{Z}} \hat{u}(\xi)e^{i\frac{2\pi}{\ell^\beta} \xi} : \| u \|^2_{\mathcal{H}^r(\mathbb{T}^d)} := \sum_{\xi \in \mathbb{Z}} \langle \xi \rangle^{2r} \hat{u}(\xi)^2 < \infty \right\}.
\]
Similarly, we can define the pseudo-differential operator on the irrational torus.

**Definition 7.1.** Given \( m \in \mathbb{R} \), a function \( a(x, \xi) \in C^\infty(\mathbb{T}_\beta \times \mathbb{Z}) \) is called a symbol of class \( S^m \) if for any \( \alpha, \beta \in \mathbb{N} \), there exists \( C_{\alpha, \beta} > 0 \) such that
\[
|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m-\beta}, \quad \forall (x, \xi) \in \mathbb{T}_\beta \times \mathbb{Z}.
\]

**Definition 7.2.** Given a symbol \( a \in S^m \), we called \( Op(a) \) is its associated pseudo-differential operator if for any \( u \in L^2(\mathbb{T}_\beta) \)
\[
Op(a)[u](x) = \sum_{\xi \in \mathbb{Z}} a(x, \xi) \hat{u}(\xi)e^{i\frac{2\pi}{\ell^\beta} \xi}.
\]
Since the space tours has been introduced as a new parameter, the pseudo-differential operator $\mathcal{W}(\omega t)$ changes with the parameter. We should establish an equivalence relation between pseudo-differential operators on different irrational torus, and prove that this relation does not change with algebraic operation.

**Definition 7.3.** Given two symbol $a \in C^\infty(\mathbb{T}_{\beta_1} \times \mathbb{Z})$, $b \in C^\infty(\mathbb{T}_{\beta_2} \times \mathbb{Z})$, we call the associated pseudo-differential operators $Op(a)$ and $Op(b)$ in the same class, if

\begin{equation}
(7.3) \quad a(x, \xi) = b\left(\frac{\beta_2}{\beta_1}x, \xi\right),
\end{equation}

namely, $a \approx b$.

**Remark 7.4.** Given two pseudo-differential operators $Op(a)$ and $Op(b)$ in the same class, if $c(x) \approx d(x)$, we have

\begin{equation}
(7.4) \quad Op(a)c(x) \approx Op(b)d(x).
\end{equation}

**Lemma 7.5.** Given the following four symbol, $a, b \in C^\infty(\mathbb{T}_{\beta_1} \times \mathbb{Z})$ and $c, d \in C^\infty(\mathbb{T}_{\beta_2} \times \mathbb{Z})$. If $a \approx c$ and $b \approx d$, the composition of pseudo-differential operators $Op(a) \circ Op(b)$ and $Op(c) \circ Op(d)$ are in the same class.

**Proof.** Notice that $Op(a) \circ Op(b) = Op(ab\overline{b})$, one gets

\begin{equation}
(7.5) \quad ab\overline{b}(x, \xi) = \sum_{j \in \mathbb{Z}} a(x, \xi + j)\overline{b}(j)e^{i\frac{\pi}{2}j}.
\end{equation}

Also, let $Op(c) \circ Op(d) = Op(cd\overline{d})$, we have

\begin{equation}
(7.6) \quad cd\overline{d}(x, \xi) = \sum_{j \in \mathbb{Z}} c(x, \xi + j)\overline{d}(j)e^{i\frac{\pi}{2}j}.
\end{equation}

From definition (7.3) $b \approx d$ implies that $\hat{b}_j(\xi) = \hat{d}_j(\xi)$. Finally, we can get

\begin{equation}
(7.7) \quad ab\overline{b}(x, \xi) \approx cd\overline{d}(x, \xi).
\end{equation}

\[\square\]

**The difference in reduce the order of perturbation:**

The equation (2.11) can be rewritten as

\begin{equation}
(7.8) \quad i\partial_t u = v \cdot \mathcal{K}u + \mathcal{Q}(\omega t)u + \varepsilon \mathcal{W}(\omega t)[u],
\end{equation}

where $\mathcal{K}e^{ij\xi} = |j|e^{ij\xi}$, $v = \frac{1}{2} \in [1, 2]$. $\mathcal{Q}$ is a pseudo-differential operator of $-1$ order, and

\[\mathcal{Q}e^{ij\xi} = \frac{c(m, v, j)}{j}e^{ij\xi},\]

where

\[|c(m, v, j)|e^{jv} \leq \frac{1}{4}, \quad \forall j \in \mathbb{N}, \quad 0 < m < \frac{1}{4}, \quad v \in [1, 2].\]

Moreover, we define a new parameter set $\hat{\Omega}_{0,\alpha} \in [1, 2]^{d+1}$, where

\begin{equation}
(7.9) \quad \hat{\Omega}_{0,\alpha} = \left\{\hat{\omega} = (\omega, v) \in [1, 2]^{d+1} : |\omega \cdot \ell + v \cdot k| \geq \frac{\alpha}{(|\ell| + |k|)^{d+1}}, \quad \forall (\ell, k) \in \mathbb{Z}^{d+1} \setminus \{0\}\right\}.
\end{equation}

The Lemma (4.3) can be replaced by the following lemma.
Lemma 7.6. Let $W$ be an Hermitian operator and belongs to $\text{Lip}(\tilde{\Omega}_{0,\alpha}, C^\infty(\mathbb{T}^d, \text{OPS}^n)), \eta \leq 1$. Then, the homological equation
\begin{equation}
\omega \cdot \partial_\theta B + i [v \cdot K, B] = W - \langle W \rangle
\end{equation}
with
\begin{equation}
\langle W \rangle := \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{T}^d} \int_{\mathbb{T}} e^{i \kappa \cdot K} W e^{-i \kappa \cdot K} d\kappa d\theta
\end{equation}
has a solution $B \in \text{Lip}(\tilde{\Omega}_{0,\alpha}, C^\infty(\mathbb{T}^d, \text{OPS}^n))$. Moreover, the operator $B$ is an Hermitian operator too.

Proof. The proof is almost the same with Lemma 4.3. The only change is the homological equation (7.10) is equivalent to
\begin{equation}
i(\omega \cdot \ell + v \cdot k) \hat{B}_{\ell,k} = \hat{W}_{\ell,k}, \quad (\ell, k) \neq (0,0).
\end{equation}
Hence, we can obtain the conclusion by the same way with Lemma 4.3. \qed

From Lemma 7.5 and Lemma 7.6, we can repeat the process of Theorem 4.4 without significant changes. Fix the number $M$ as
\[M = 1 + 4m.\]
After $M$ steps of regularization, the original equation (7.8) is transformed to
\begin{equation}
i\omega \cdot \partial_y u = \Lambda_0 u + P_0 u,
\end{equation}
where $\Lambda_0 = K + Q + \varepsilon Z^M$ and $P_0 = \varepsilon W^M$. Denoting $(\mu_{j,n})_{n=1,2}$ as the eigenvalue of the block $(\Lambda_0)^{[j]}_{[j]}$, it has the asymptotic expression
\begin{equation}
\mu_{j,n} = v |j| + z(j, m, v) + p_{j,n}(\omega),
\end{equation}
where $|z(j, m, v)|_{\text{Lip}} \leq \frac{1}{4}$ and $|p_{j,n}|_{\text{Lip}} \leq C \varepsilon$.

The difference in KAM reducibility:
After finite times KAM iteration, the equation (7.13) is converted to
\begin{equation}
i\omega \cdot \partial_y u = \Lambda^k u + P^k u.
\end{equation}
Furthermore, denoting $(\mu_{j,n}^k)_{n=1,2}$ as the eigenvalues of the block $(\Lambda^k)^{[j]}_{[j]}$, it has the asymptotic expression
\begin{equation}
\mu_{j,n}^k = v |j| + z(j, m, v) + p_{j,n}(\omega),
\end{equation}
where $|p_{j,n}^k|_{\text{Lip}} \leq C(\varepsilon + \varepsilon)$.
Hence, we can define the non-resonance set $\hat{\Omega}_{k+1}(\omega)$ at the $k + 1^{th}$ step reducibility.
\begin{equation}
\Omega_{k+1} := \{ \omega \in \hat{\Omega}_k : \omega \cdot \ell + \mu_{i,n}^k - \mu_{j,n'}^k \geq \frac{\alpha}{N_k(\ell)^a(j)^b} \}
\end{equation}
\begin{equation}
\forall i, j \in \mathbb{N}, \quad |\ell| \leq N_k, \quad n, n' = 1, 2, \quad (k, i, j) \neq (0, i, i).
\end{equation}

Remark 7.7. In the proof of Lemma 5.7, we should ensure that the gap of eigenvalues greater than some constant. Thus, the conditions (C2) is indispensable. However, by introducing space torus $\beta$ as new parameter, there are some new phenomenons.

Lemma 7.8.
\[\text{meas}(\Omega_{k,\alpha} \setminus \Omega_{k+1,\alpha}) \leq C \alpha N_k^{-1}.\]
Proof. Considering the Lipschitz function $g(\omega)$,

\begin{equation}
(7.18) \quad g(\omega) = \omega \cdot \ell + \mu_{i,n}^k - \mu_{i,n'}^k = \omega \cdot (\ell, i-j) + z(i, v) - z(j, v) + p_{i,n}^k(\omega) - p_{j,n'}^k(\omega).
\end{equation}

For any $(\ell, i-j) \neq 0$, we can write

\begin{equation}
(7.19) \quad \omega = \frac{\ell, i-j}{|\ell, i-j|} s + \omega_1, \quad \omega_1 \in \mathbb{R}^{d+1}, \quad (\ell, i-j) = 0,
\end{equation}

and

\begin{equation}
(7.20) \quad g(s) = |(\ell, i-j)s + z(i, v(s)) - z(j, v(s)) + p_{i,n}^k(\omega(s)) - p_{j,n'}^k(\omega(s)).
\end{equation}

Subsequently, we have

\begin{equation}
(7.21) \quad |g(s_1) - g(s_2)| \geq |(\ell, i-j)||s_1 - s_2| - \left(|z(i, v(s_1)) - z(i, v(s_2))| + |z(j, v(s_1)) - z(j, v(s_2))|\right)
\end{equation}

\begin{equation}
(7.22) \quad \geq (1 - \frac{1}{2} - C\varepsilon)|s_1 - s_2| \geq \frac{1}{4}|s_1 - s_2|.
\end{equation}

which implies

\begin{equation}
(7.23) \quad \text{meas}\left\{ s : g_{ij,j'}(s) < \frac{\alpha}{N_k(i)(j)} \right\} < \frac{4\alpha}{N_k(i)(j)}.\end{equation}

The rest of proof is the same as Lemma 5.7. \hfill \Box

8. Appendix B

8.1. Properties of pseudo-differential operators.

Lemma 8.1. Let $\eta < 1$ and $G \in C^\infty(\mathbb{T}^d, OPS^\eta)$ be such that $G(\theta) + G^*(\theta) = 0$ and let $e^{tG}$ be the flow of the autonomous PDE

\begin{equation}
\partial_t u = G(t)u, \quad t \in [-1, 1]
\end{equation}

1: $\forall \sigma > 0, e^{tG} \in L(H^\sigma)$.

2: $\forall \sigma > 0, \forall \alpha \in \mathbb{N}^n, \partial_\theta^\alpha e^{tG}(\theta) \in L(H^\sigma, H^{\sigma - |\alpha|}).$

3: If $G \in Lip(\Omega, C^\infty(\mathbb{T}^d, OPS^\eta))$, $\partial_\theta^\alpha e^{tG}(\theta, \omega) \in Lip(\Omega, L(H^\sigma, H^{\sigma - |\alpha|+1})), \forall \sigma > 0, \alpha \in \mathbb{N}^d$.

Proof. The item 1 is well known, and can be found in [29]. Items 2 and 3 follow as in Lemma A.3 in [11]. \hfill \Box

Remark 8.2. Let $A(\theta) \in Lip(\Omega, C^\infty(\mathbb{T}^d, OPS^\eta))$ and $G \in Lip(\Omega, C^\infty(\mathbb{T}^d, OPS^\eta))$ with $\eta < 1$. If $\forall j \in \mathbb{N}$, we define

\begin{equation}
Ad_G^0 A = A, \quad Ad_G^1 A = [G, Ad_G],
\end{equation}

then $Ad_G^j A \in Lip(\Omega, C^\infty(\mathbb{T}^d, OPS^{\eta-j(1-\eta)}))$.

Lemma 8.3. Let $A(\theta) \in Lip(\Omega, C^\infty(\mathbb{T}^d, OPS^\eta))$ and $G \in Lip(\Omega, C^\infty(\mathbb{T}^d, OPS^\eta))$ with $\eta < 1$ such that $G(\theta) + G^*(\theta) = 0$. Then

\begin{equation}
(8.1) \quad e^{tG}Ae^{-tG} \in Lip(\Omega, C^\infty(\mathbb{T}^d, OPS^\eta)).
\end{equation}

Proof. This is a simple version of Egorov theorem. The details of proof can be found in [29]. \hfill \Box

Remark 8.4. From Theorem A.0.9 in [29], one has that if $A(\theta) \in Lip(\Omega, C^\infty(\mathbb{T}^d, OPS^\eta))$, then $\forall \alpha \in \mathbb{N}$,

\begin{equation}
(8.2) \quad e^{i\alpha K}Ae^{-i\alpha K}, \quad \partial_\alpha^\varepsilon (e^{i\alpha K}Ae^{-i\alpha K}) \in Lip(\Omega, C^\infty(\mathbb{T}^d, OPS^\eta)).
\end{equation}
In the next Proposition, we essentially prove that pseudo-differential operators as in Definition 8.2 have matrix presentation, which belong to the classes $\mathcal{M}^{s, L^p}_{\gamma, \delta}$ extended from Definition 8.5.

**Proposition 8.5.** Let $F \in \mathcal{L}ip(\Omega, C^\infty(\mathbb{T}^d, OPS^0))$. For any $s > \frac{d+1}{2}$, the matrix of the operator $(D)^\gamma F(D)^\zeta$, $\gamma + \zeta + \mu \geq 0$ belongs to $\mathcal{M}^{s, L^p}_{\gamma, \delta}$. Moreover, there exists $\sigma > 0$, such that

$$\|\langle D \rangle^\gamma F(D)^\zeta\|_{s, s}^s \leq C \chi^0_{\gamma+\sigma, s+\sigma}(F).$$

**Proof.** We start by proving the case $\gamma = \zeta = 0$. Fix $s > \frac{d+1}{2}$, for any $m, n \in \mathbb{Z}$, we have

$$\hat{F}_m^n(\ell) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^{d+1}} \hat{F}(\ell) e^{inx} e^{-inx} dx$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^{d+1}} f(\ell, x, m) e^{i(m-n)x} dx.$$

For the case $m \neq n$. Integrating by parts $\tilde{s}$ times in $x$, with $\tilde{s} = \lfloor s \rfloor + 2$, one gets for any $n, m \in \mathbb{N}$, $n \neq m$, $\ell \in \mathbb{Z}^d$,

$$\|\hat{F}_m^n(\ell)\|_{L^2} \leq \sup_{|k|=m, |k'|=n} |\hat{F}_k^{k'}(\ell)| \leq \frac{1}{|m-n|^\sigma} \chi^0_{0}(\hat{F}(\ell))$$

For the case $m = n$, we can prove $\|\hat{F}_m^n(\ell)\|_{L^2} \leq \chi^0_{0}(\hat{F}(\ell))$ in a similar way. Thus, we can get

$$\|F\|_{s, s}^s \leq C \chi^0_{0, \tilde{s}}(F) \leq C^* \chi^0_{0, \tilde{s}}(F).$$

For the other cases, the operator $(D)^\gamma F(D)^\zeta$ belongs to $\mathcal{L}ip(\Omega, C^\infty(\mathbb{T}^d, OPS^0))$, so we have

$$\|\langle D \rangle^\gamma F(D)^\zeta\|_{s, s}^s \leq C^* \chi^0_{0, \tilde{s}}((D)^\gamma F(D)^\zeta) \leq C^m \chi^\mu_{\tilde{s}, \tilde{s}}(F).$$

\[ \square \]

### 8.2. Properties of Hermitian matrix.

In this section, we recall some well known facts about Hermitian operator in the finite dimension Hilbert space $\mathcal{H}$. Let $\mathcal{H}$ be a finite dimensional Hilbert space of dimension $n$ equipped by the inner product $(\cdot, \cdot)_\mathcal{H}$. For any Hermitian operator $A$, we order its eigenvalues as $\text{spec}(A) := \lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)$.

**Proposition 8.6.** (Weyl’s Perturbation Theorem) Let $A$ and $B$ be Hermitian matrices. Then

$$|\lambda_k(A) - \lambda_k(B)| \leq \|A - B\|_{\mathcal{L}^2(\mathcal{H})}, \forall k \in 1, \cdots, n.$$

**Proof.** The proof can be found in Theorem III.2.1 \[12\]. \[ \square \]

**Proposition 8.7.** Let $A$ and $B$ be Hermitian matrices, and let $\delta = \text{dist}(\sigma(A), \sigma(B))$. Then the solution $X$ of the equation $AX - XB = Y$ satisfies the inequality

$$\|X\|_{\mathcal{L}^2(\mathcal{H})} \leq \frac{C}{\delta} \|Y\|_{\mathcal{L}^2(\mathcal{H})}.$$

**Proof.** The proof can be found in Theorem VII.2.8 \[12\]. \[ \square \]

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