Abstract. Steiner symmetrization along \( n \) linearly independent directions transforms every compact subset of \( \mathbb{R}^n \) into a set of finite perimeter. The perimeter of a non-convex set of finite perimeter decreases strictly under Steiner symmetrization in most directions, but not necessarily in all of them [3].

We seek to bound the perimeter of an arbitrary compact set \( A \subset \mathbb{R}^n \) after a finite sequence of Steiner symmetrizations. Our main result is that \( n \) consecutive Steiner symmetrization in linearly independent directions suffice to transform \( A \) into a set of finite perimeter.

Theorem 1 (Perimeter estimate). If \( A \subset \mathbb{R}^n \) is a compact set and \( u_1, \ldots, u_n \) are linearly independent unit vectors in \( \mathbb{R}^n \), then

\[
\text{Per} \left( S_{u_n} \ldots S_{u_1} A \right) \leq \frac{a_n R^{n-1}}{|\det (u_1, \ldots, u_n)|},
\]

where \( a_n = 2n \omega_{n-1} \), and \( R \) is the outradius of \( A \).

The theorem is motivated by the special case of the coordinate directions \( e_1, \ldots, e_n \). The set \( S_{e_n} \ldots S_{e_1} A \) is symmetric under reflection at each coordinate hyperplane, and its intersection with the positive cone lies under the graph of a monotone function \( x_n = f(x_1, \ldots, x_{n-1}) \). The perimeter of such a set is bounded by twice the sum of the area of its projections onto the coordinate hyperplanes, which cannot exceed \( a_n R^{n-1} \), see Figure 1.

We start with some definitions and notation. The dimension \( n \geq 2 \) will be fixed throughout the paper. The \( n \)-dimensional volume of a Lebesgue measurable set \( A \subset \mathbb{R}^n \) is denoted by \( \text{Vol}(A) \). By \( \text{Per}(A) \),
we mean the Caccioppoli perimeter of $A$, defined by

$$\operatorname{Per} (A) = \sup_{\|F\| \leq 1} \int_A \text{div } F(x) \, dx,$$

where the supremum runs over all smooth compactly supported vector fields $F$ on $\mathbb{R}^n$. If $\operatorname{Per} (A) < \infty$, then its value agrees with the $(n-1)$-dimensional Hausdorff measure of the essential boundary of $A$.

We denote by $B$ the closed unit ball in $\mathbb{R}^n$, centered at the origin, and by $\omega_n$ its volume. The closed centered ball of radius $\rho > 0$ will be denoted by $B_\rho$. The Minkowski sum of two subsets $A, C \subset \mathbb{R}^n$ is given by

$$A + C = \{a + c : a \in A, c \in C\}.$$

Their Minkowski difference is the largest set whose Minkowski sum with $C$ lies in $A$,

$$A - C = \{x \in \mathbb{R}^n : x + c \in A \forall c \in C\}.$$

The sets $A + B_\rho$ and $A - B_\rho$ will be called the outer and inner parallel sets of $A$. The Hausdorff distance between $A$ and $C$ is given by

$$d_H(A, C) = \inf\{\delta > 0 : A \subset C + B_\delta \text{ and } C \subset A + B_\delta\}.$$  

Let $A \subset \mathbb{R}^n$ be a compact set, and let $u \in \mathbb{R}^n$ be a unit vector. The Steiner symmetrization of $A$ in the direction of $u$ is defined by the following property. For each point $x \perp u$, we compute the length of the intersection of $A$ with the inverse image of $x$ under the orthogonal projection onto the hyperplane $u \perp$, and then replace it with the closed interval of the same one-dimensional measure centered on $u \perp$. If the intersection is empty, then the interval is empty; if it is a nonempty set of measure zero, then the interval consists of a single point. The resulting set will be denoted by $S_u A$. Note that $S_u A$ is compact and symmetric under reflection at $u \perp$. By Cavalieri’s principle, $S_u A$ has the
same volume as $A$, i.e., Steiner symmetrization is a volume-preserving rearrangement. The **symmetric rearrangement** of $A$ is the closed centered ball $A^*$ of the same volume as $A$. If $A$ is empty, we take $A^*$ to be empty; if $A$ is a non-empty set of measure zero, then $A^* = \{0\}$. We will refer to the radius of $A^*$ as the **volume radius** of $A$.

The corresponding symmetrizations of functions are defined by symmetrizing their level sets. Let $f$ be a nonnegative continuous function with compact support. Its Steiner symmetrization $S_u f$ is determined by the property that

$$\{ x : S_u f(x) \geq t \} = S_u \{ x : f(x) \geq t \}$$

for every $t > 0$, and its symmetric decreasing rearrangement $f^*$ is the unique radially decreasing continuous function that is equimeasurable to $f$. These symmetrizations improve the modulus of continuity and contract distances in the space of continuous functions.

It is useful to think of Steiner symmetrization as an operation on the one-dimensional cross sections

$$A(x) = \{ t \in \mathbb{R} : x + tu \in A \}$$

for $x \in u^\perp$. By definition,

$$(S_u A)(x) = (A(x))^*,$$

where $(A(x))^*$ is the one-dimensional symmetric rearrangement of $A(x)$. Since one-dimensional symmetrization preserves the subset relation, Steiner symmetrization preserves it as well, and therefore

$$S_u A \cap S_u C \supset S_u (A \cap C),$$

$$S_u A \cup S_u C \subset S_u (A \cup C).$$

In particular, the outradius of $S_u A$ is no larger than the outradius of $A$.

Consider a pair of non-empty cross sections $A(x)$ and $C(y)$. Let $a(x)$ be the leftmost point in $A(x)$, and let $c(y)$ be the rightmost point in $C(y)$. Clearly,

$$A(x) + C(y) \supset (a(x) + C(y)) \cup (A(x) + c(y)),$$

with equality when $A(x)$ and $C(y)$ are intervals. Since the two sets on the right hand side have only the point $a(x) + c(y)$ in common, the one-dimensional measure of $A(x) + C(y)$ is at least as large as the sum of the measures of $A(x)$ and $C(y)$. (This is the Brunn-Minkowski inequality in one dimension). It follows that $(A(x))^* + (C(y))^*$ is contained in $(A(x) + C(y))^*$, and therefore

$$(2) \quad S_u A + S_u C \subset S_u (A + C)$$
for every pair of compact sets \( A, C \subset \mathbb{R}^n \). By definition of the Minkowski difference, this in turn implies that

\[
S_u A - S_u C \supset S_u (A - C).
\]

In particular, Steiner symmetrization reduces the volume of outer parallel sets and increases the volume of inner parallel sets.

In the proof of Theorem 1, we will bound the perimeter of \( S_u A \) in terms of the volume of its parallel sets. Specifically, we will establish Eq. (1) for the outer Minkowski perimeter, given by

\[
\text{Per}_M^+(A) = \limsup_{\delta \to 0} \frac{1}{\delta} (\text{Vol} (A + B_\delta) - \text{Vol} (A))
\]

and then argue that \( \text{Per}_M^+(A) \geq \text{Per} (A) \) for every compact set \( A \). The first lemma concerns the Minkowski sum and difference of a compact set \( A \) with a line segment.

**Lemma 1.** Let \( A \subset \mathbb{R}^n \) be a compact set, let \( u \) be a unit vector in \( \mathbb{R}^n \), and fix \( \beta > 0 \). Assume that \( S_u A = A \), and consider the line segment \( L_\beta u = \{ tu, |t| \leq \beta \} \). Then, for every \( R > 0 \),

\[
\begin{align*}
\text{Vol} ((A + L_\beta u) \cap B_R) &\leq \text{Vol} (A \cap B_R) + 2\omega_{n-1} R^{n-1} \beta, \\
\text{Vol} ((A - L_\beta u) \cap B_R) &\geq \text{Vol} (A \cap B_R) - 2\omega_{n-1} R^{n-1} \beta.
\end{align*}
\]

**Proof.** By assumption, each non-empty cross section \( A(x) \) is either a centered interval of positive length \( \ell(x) \), or a single point, in which case we set \( \ell(x) = 0 \). The corresponding cross section of \( A + L_\beta u \) is a line segment of length \( \ell(x) + 2\beta \). The corresponding cross section of \( A - L_\beta u \) is either a centered interval of length \( \ell(x) - 2\beta \), a single point, or empty. The claims follow upon integration over \( x \in u^\perp \cap B_R \).

**Lemma 2.** Let \( A \subset \mathbb{R}^n \) be a compact set, and let \( R > 0 \). For a finite collection of unit vectors \( u_1, \ldots, u_k \in \mathbb{R}^n \) and \( \beta_1, \ldots, \beta_k \geq 0 \), set

\[
A_k = S_{u_k} \ldots S_{u_1} A, \quad C_k = \sum_{i \leq k} S_{u_k} \ldots S_{u_{i+1}} L_{\beta_i u_i},
\]

where \( L_{\beta_i u_i} \) is a line segment as in Lemma 1. Then

\[
\begin{align*}
\text{Vol} ((A_k + C_k) \cap B_R) &\leq \text{Vol} (A \cap B_R) + 2\omega_{n-1} R^{n-1} \sum_{i \leq k} \beta_i, \\
\text{Vol} ((A_k - C_k) \cap B_R) &\geq \text{Vol} (A \cap B_R) - 2\omega_{n-1} R^{n-1} \sum_{i \leq k} \beta_i.
\end{align*}
\]
Proof. We proceed by induction on \(k\). Lemma 1 settles the base case \(k = 1\). Let \(1 < k \leq n\), and suppose the first claim holds for \(k - 1\). By Eq. (2),
\[
A_k + C_k \subset S_{u_k}(A_{k-1} + C_{k-1}) + L_{\beta_k u_k}.
\]
We combine this with the first inequality of Lemma 1 and then apply the inductive hypothesis to obtain
\[
\text{Vol} \left( (A_k + C_k) \cap B_R \right) \leq \text{Vol} \left( (A_{k-1} + C_{k-1}) \cap B_R \right) + 2\omega_n R^{n-1} \beta_k
\leq \text{Vol} \left( A \cap B_R \right) + 2\omega_n R^{n-1} \sum_{i \leq k} \beta_i.
\]
This completes the induction. For the second claim, we argue similarly, using Eq. (3) and the second inequality of Lemma 1. \(\square\)

The next lemma gives a lower bound for the inradius of the parallelepiped \(C_n = \sum_{i \leq n} S_{u_i} \ldots S_{u_{i+1}} L_{\beta u_i}\).

**Lemma 3.** Let \(u_1, \ldots, u_n\) be linearly independent unit vectors in \(\mathbb{R}^n\), and let \(\beta, \rho > 0\). If \(\beta \det (u_1, \ldots, u_k) \geq \rho\), then
\[
B_\rho \subset \sum_{i \leq n} S_{u_i} \ldots S_{u_{i+1}} L_{\beta u_i}.
\]

**Proof.** Denote by \(V_k\) the subspace spanned by \(u_1, \ldots, u_k\), and set
\[
C_k = \sum_{i \leq k} S_{u_i} \ldots S_{u_{i+1}} L_{\beta u_i}, \quad k = 1, \ldots, n.
\]
Let \(\rho_k\) be the inradius of \(C_k\) (considered as a subset of \(V_k\)), and let \(\lambda_k\) be the \(k\)-dimensional measure of the parallelepiped spanned by \(u_1, \ldots, u_k\). We will show by induction on \(k\) that \(\rho_k \geq \beta \lambda_k\) for \(k = 1, \ldots, n\).

In the base case \(k = 1\), we have \(C_1 = L_{\beta u_1}, \rho_1 = \beta\), and \(\lambda_1 = 1\). Let now \(1 < k \leq n\), and suppose we have already shown that \(\rho_{k-1} \geq \beta \lambda_{k-1}\). By definition,
\[
C_k = S_{u_k} C_{k-1} + L_{\beta u_k}.
\]
The Steiner symmetrization \(S_{u_k}\) acts on subsets of \(V_{k-1}\) as the orthogonal projection onto \(u_k^\perp\). Let \(\theta_k\) be the angle between \(V_{k-1}\) and \(u_k\). The projection onto \(u_k^\perp\) shrinks the length of vectors in \(V_{k-1}\) by a factor that is no smaller than \(\sin \theta_k\), and shrinks the \((k-1)\)-dimensional volume of subsets exactly by a factor \(\sin \theta_k\). By the inductive assumption,
\[
\rho_k \geq \rho_{k-1} \sin \theta_{k-1} \geq \beta \lambda_{k-1} \sin \theta_{k-1} = \beta \lambda_k,
\]
completing the induction. \(\square\)
Theorem 2 (Volume estimate). If $A \subset \mathbb{R}^n$ is a compact set with outradius $R$ and $u_1, \ldots, u_n$ are linearly independent unit vectors in $\mathbb{R}^n$, then

$$
\text{Vol} \left( S_{u_n} \ldots S_{u_1}A + B_\delta \right) \leq \text{Vol} \left( A \right) + \frac{a_n(R + \delta)^{n-1}\delta}{|\det (u_1, \ldots, u_n)|},$$

$$
\text{Vol} \left( S_{u_n} \ldots S_{u_1}A - B_\delta \right) \geq \text{Vol} \left( A \right) - \frac{a_nR^{n-1}\delta}{|\det (u_1, \ldots, u_n)|},
$$

for every $\delta > 0$. Here, $a_n = 2n\omega_{n-1}$.

Proof. We apply Lemma 3 with $\beta = \delta/|\det (u_1, \ldots, u_n)|$ to see that

$$
B_\delta \subset \sum_{i \leq n} S_{u_n} \ldots S_{u_{i+1}}L_{\beta u_i} =: C.
$$

It follows from the first inequality in Lemma 2 that

$$
\text{Vol} \left( S_{u_n} \ldots S_{u_1}A + B_\delta \right) \leq \text{Vol} \left( (S_{u_n} \ldots S_{u_1}A + C) \cap B_{R+\delta} \right)
\leq \text{Vol} \left( A \right) + 2n\omega_{n-1}(R + \delta)^{n-1}\beta,
$$

proving the first claim. Similarly, we obtain from the second inequality in Lemma 2

$$
\text{Vol} \left( S_{u_n} \ldots S_{u_1}A - B_\delta \right) \geq \text{Vol} \left( (S_{u_n} \ldots S_{u_1}A - C) \cap B_R \right)
\geq \text{Vol} \left( A \right) - 2n\omega_{n-1}R^{n-1}\beta.
$$

The next lemma is not needed for the proof of the main result. It will be used at the end of the paper to turn the volume estimate from Theorem 2 into an inequality for the volume radius of parallel sets.

Lemma 4. Let $A$ be a non-empty compact set in $\mathbb{R}^n$ with $n \geq 2$. For $\delta > 0$, let $\rho(\delta)$ be the volume radius of $A + B_\delta$, let $\rho(-\delta)$ be the volume radius of $A - B_\delta$, and let $r$ be the volume radius of $A^*$. Assume that

$$
\text{Vol} \left( A + B_\delta \right) \leq \text{Vol} \left( A \right) + b(R + \delta)^{n-1}\delta,$$

$$
\text{Vol} \left( A - B_\delta \right) \geq \text{Vol} \left( A \right) - bR^{n-1}\delta
$$

for all $\delta > 0$, where $b \geq 2\omega_n r^n/R^n$ and $R \geq r$ are constants. Then

$$
|\rho(\pm\delta) - r| \leq c\delta,
$$

where $c = bR^{n-1}/(\omega_n r^{n-1})$. 
Proof. Note that \( c \geq 2r/R \). By Jensen’s inequality,

\[
\operatorname{Vol}(A^* + B_{c\delta}) - \operatorname{Vol}(A) = n\omega_n \int_0^{c\delta} (r + s)^{n-1} ds \\
\geq cn\omega_n r^{n-1} \delta \left(1 + \frac{c\delta}{2r}\right)^{n-1} \\
\geq bR^{n-1} \delta \left(1 + \frac{\delta}{R}\right)^{n-1} \\
\geq \operatorname{Vol}(A + B_\delta) - \operatorname{Vol}(A) \\
= \operatorname{Vol}(B_{\rho(\delta)}) - \operatorname{Vol}(A),
\]

where the last two steps used the assumption on \( A + B_\delta \) and the definition of \( \rho(\delta) \). It follows that \( A^* + B_{c\delta} \supset B_{\rho(\delta)} \), which gives the claim for \( t > 0 \). On the other hand, the assumption on \( A - B_\delta \) implies that

\[
\omega_n r^{n-1} (r - \rho(-\delta)) \leq \operatorname{Vol}(B_{\rho(-\delta)}) \leq bR^{n-1} \delta,
\]

which gives the \( \rho(-\delta) \). \( \square \)

**Proof of Theorem 7**. The first inequality of Theorem 2 yields for the outer Minkowski perimeter

\[
\operatorname{Per}^+(S_{u_n} \ldots S_{u_1} A) = \limsup_{\delta \to 0} \frac{1}{\delta} (\operatorname{Vol}(S_{u_n} \ldots S_{u_1} A + B_\delta) - \operatorname{Vol}(A)) \\
\leq \frac{a_n R^{n-1}}{|\det(u_1, \ldots, u_n)|}.
\]

The proof is completed with the lemma below. \( \square \)

**Lemma 5.** If \( A \subset \mathbb{R}^n \) is compact, then \( \operatorname{Per}(A) \leq \operatorname{Per}^+(A) \).

**Proof (L. Ambrosio).** Apply the coarea formula (see [4, Theorem 13.1]) to the function \( f(x) = \operatorname{dist}(x, A) \), which is clearly Lipschitz continuous, and hence differentiable almost everywhere. Since \( |\nabla f| = 1 \) a.e. outside \( A \) and vanishes a.e. on \( A \),

\[
\operatorname{Vol}(A + B_\delta) - \operatorname{Vol}(A) = \int_{A + B_\delta} |\nabla f(x)| dx \\
= \int_0^{\delta} \operatorname{Per}(A + B_t) dt \\
\geq \delta \cdot \inf_{0 < t < \delta} \operatorname{Per}(A + B_t).
\]

In the second step, we have used the coarea formula and observed that \( f^{-1}(t) = \partial(A + B_t) \) for \( t > 0 \). We now divide by \( \delta \) and take \( \delta \to 0 \). Since \( A \) is compact, the parallel set \( A + B_\delta \) converges to \( A \) in
symmetric difference. It follows from the lower semicontinuity of the perimeter that
\[
\Per^+_M(A) = \limsup_{\delta \to 0} \frac{1}{\delta} (\Vol (A + B_\delta) - \Vol (A)) \\
\geq \liminf_{\delta \to 0} \Per (A + B_\delta) \\
\geq \Per (A).
\]
This concludes the proof of the main result. \hfill \qed

There are various notions of perimeter, which agree for open sets with smooth boundary but may differ for less regular sets (see [1] for some recent results). In particular, \(\Per (A)\) can be much smaller than the \((n-1)\)-dimensional Hausdorff measure of the topological boundary of \(A\). Having established Eq. (1) for the Caccioppoli perimeter, we wish to extend the inequality to another commonly used measure of the size of the boundary.

The **two-sided Minkowski perimeter** of a compact set \(A\) is defined by
\[
\Per_M(A) = \limsup_{\delta \to 0} \frac{1}{2\delta} \Vol (\partial A + B_\delta),
\]
where \(\partial A\) is the boundary of \(A\). It is not hard to show, using a Vitali covering argument, that the \((n-1)\)-dimensional Hausdorff measure of \(\partial A\) is bounded by \(2 \cdot 3^{n-1} (\omega_{n-1}/\omega_n) \Per_M(A)\), but it is not clear (to us) whether the bound holds without the constant factor. The last lemma will be used to relate \(\partial A\) to the outer and inner parallel sets of \(A\).

**Lemma 6.** For any pair of compact sets \(A, C \subset \mathbb{R}^n\) and every \(\delta > 0\),
\[
\partial A + C \subset (A + C) \setminus \text{interior} \ (A - C^-),
\]
where \(C^- = \{-c : c \in C\}\) is the reflection of \(C\) through the origin. If \(C\) is connected, then the converse inclusion also holds.

**Proof.** Clearly, \(\partial A + C \subset A + C\). We need to show that \(\partial A + C\) does not intersect the interior of \(A - C^-\). Suppose that \(x\) lies in the interior of \(A - C^-\). Then there exists \(\delta > 0\) such that \(B_\delta(x) \subset A - C^-\). This means that \(B_\delta(x - c) \subset A\), i.e., \(\text{dist} \ (x - c, \partial A) \geq \delta\) for every \(c \in C\). We conclude that \(x\) cannot lie in \(\partial A + C\).

For the reverse inclusion, assume furthermore that \(C\) is connected. Let \(x \in (A + C) \setminus \text{interior} \ (A - C^-)\), and consider
\[
C_1 = \{c \in C : x - c \in A\}, \\
C_2 = \{c \in C : x - c \not\in \text{interior} \ (A)\}.
\]
By definition, \( C_1 \) and \( C_2 \) are closed and cover \( C \). Furthermore, \( C_1 \) is non-empty because \( x \in A + C \), and \( C_2 \) is non-empty because \( x \notin \text{interior}(A - C^-) \). Since \( C \) is connected, \( C_1 \) and \( C_2 \) cannot be disjoint. Pick \( c \in C_1 \cap C_2 \). Then \( x - c \in \partial A \), i.e., \( x \in \partial A + C^- \), as claimed. \( \square \)

In the special case where \( C = B_\delta \), the lemma implies that
\[
\text{Vol}(\partial A + B_\delta) = \text{Vol}(A + B_\delta) - \text{Vol}(A - B_\delta),
\]
because the boundary of \( A - B_\delta \), which consists of all points having distance exactly \( \delta \) from the complement of \( A \), is a set of volume zero. Combining Eq. (4) with Theorem 2, we obtain
\[
\text{Vol}(\partial S_{u_n} \ldots S_{u_1} A + B_\delta)
= \text{Vol}(S_{u_n} \ldots S_{u_1} A + B_\delta) - \text{Vol}(S_{u_n} \ldots S_{u_1} A - B_\delta)
\leq \frac{4n\omega_{n-1}(R + \delta)^{n-1}\delta}{|\det(u_1, \ldots, u_n)|}.
\]
Dividing by \( 2\delta \) and taking \( \delta \to 0 \) extends Eq. (1) to the two-sided Minkowski perimeter.

**Corollary 1.** Under the assumptions of Theorem 4
\[
\text{Per}_M(S_{u_n} \ldots S_{u_1} A) \leq \frac{a_n R^{n-1}}{|\det(u_1, \ldots, u_n)|}.
\]
Since \( \text{Per}(A) \leq \text{Per}_M(A) \) by the same reasoning as in Lemma 5, this improves upon Theorem 1.

Finally, we discuss an application to random sequences of Steiner symmetrizations. Consider a non-empty compact set \( A \subset \mathbb{R}^n \), let \( r \) be its volume radius, and assume that \( A \subset B_R \). Let \( \{U_k\}_{k \geq 0} \) be a sequence of unit vectors picked independently, uniformly at random from the unit sphere in \( \mathbb{R}^n \), and define recursively
\[
A_0 = A, \quad A_{k+1} = S_{U_k} A_k \quad (k \geq 0).
\]
It was recently shown by Burchard and Fortier that the expectation of the symmetric difference from \( A_k \) to \( A^* \) satisfies
\[
E(A_k \triangle A^*) \leq n\omega_n 2^{n+1} R^n k^{-1}
\]
for all \( k > 0 \) [2, Proposition 5.2]. Under suitable regularity assumptions on \( \partial A \), this can be used to bound the Hausdorff distance \( d_H(\partial A_k, \partial A^*) \), which controls how much the outradius and inradius of \( A_k \) differ from its volume radius.

We briefly describe the tools developed in [2, Section 7]. The authors consider the auxiliary function
\[
f(x) = \text{dist}(x, \mathbb{R}^n \setminus A) + (R - \text{dist}(x, A))_+.
\]
and its symmetrizations

\[ F_0 = f, \quad F_{k+1} = S_{U_k} F_k \quad (k \geq 0). \]

By construction, \( A_k = \{ x : F_k(x) > R \} \). Using Eqs. (2) and (3), they show that

\[ d_H(\partial A_k, \partial A^*) \leq \max_{\pm} |\rho(\pm||F_k - f^*||_\infty) - r|, \tag{6} \]

where \( \rho(\pm\delta) \) is the volume radius of the parallel set \( A \pm B_\delta \). It follows from [2, Proposition 5.2] that

\[ E(||F_k - f^*||_\infty) \leq 12R^k \tag{7} \]

for \( k > 0 \). Under the assumption that \( A \) has finite Minkowski perimeter, they differentiate \( \rho \) at \( \delta = 0 \) and obtain from Eqs. (6) and (7) a sequence of Steiner symmetrizations along which \( d_H(\partial A_k, \partial A^*) = O(k^{-n/2}) \) as \( k \to \infty \). The rate of convergence estimates in Eqs. (5) and (7) are proved by comparing Steiner symmetrization with polar-ization, a simpler rearrangement that preserves perimeter as well as volume [2, Section 5].

We will use Theorems 1 and 2 to obtain a stronger bound on the perimeter of \( A_{n+1} \) that results in stronger bounds on \( \rho \) and, through Eq. (6), on \( d_H(\partial A_k, \partial A^*) \). By Theorem 1, the perimeter of \( A_n \) is almost surely finite, because the probability that the vectors \( U_0, \ldots, U_{n-1} \) lie in a common hyperplane is zero. We next argue that \( \text{Per}(A_{n+1}) \) has finite expectation. Since \( \text{Per}(A_{n+1}) \leq \text{Per}(A_n) \), we can apply Theorem 1 to \( A_n \) and \( A_{n+1} \) to obtain

\[ \text{Per}(A_{n+1}) \leq a_n R^{n-1} Y_n, \]

where \( a_n = 2n\omega_{n-1} \), and the random variable \( Y_n \) is given by

\[ Y_n = \min \left\{ |\det(U_1, \ldots, U_n)|^{-1}, |\det(U_0, \ldots, U_{n-1})|^{-1} \right\}. \tag{8} \]

As in the proof of Lemma 3, we expand \( |\det(U_1, \ldots, U_n)| = \prod_{k=2}^{n} X_k \) and \( |\det(U_0, \ldots, U_{n-1})| = X'_n \cdot \prod_{k=2}^{n-1} X_k \), where \( X_k \) is the Euclidean distance of \( U_k \) to the subspace of \( \mathbb{R}^n \) spanned by \( U_1, \ldots, U_{k-1} \), and \( X'_n \) is the distance of \( U_0 \) to the subspace spanned by \( U_1, \ldots, U_{n-1} \). Then

\[ Y_n = (\max\{X_n, X'_n\})^{-1} \cdot \prod_{k=2}^{n-1} X_k^{-1}. \]

By rotational invariance, \( X_k \) has the same distribution as the distance of a random point on the sphere from a \((k-1)\)-dimensional coordinate plane, \( X'_n \) has the same distribution as \( X_n \), and \( X_2, \ldots, X_n, X'_n \) are independent. Since the sphere is compact and intersects the coordinate planes transversally, there exist constants \( b_{n,k} \) such that \( P(X_k \leq t) \leq \)
By the independence of $X_n$ and $X'_n$, it follows that $P(\max\{X_n, X'_n\} \leq t) \leq (b_{n,t})^2$. Therefore,

$$E(Y_n) = E((\max\{X_n, X'_n\})^{-1}) \cdot \prod_{k=2}^{n-1} E(X_k^{-1}) < \infty.$$  

We have proved the following inequality.

**Corollary 2.**

$$E(\text{Per}(A_{n+1})) \leq b_n R^{n-1},$$

where $b_n = 2\omega_n E(Y_n)$ depends only on the dimension.

We want to apply Eqs. (6) and (7) to the conditional expectation $E(\cdot | A_{n+1})$. Let $f, F_k, \rho$ be the functions corresponding to $f, F_k$, and $\rho$ with $A_{n+1}$ in place of $A$. Replacing Theorem 1 with Theorem 2 in the proof of Corollary 2, we obtain for every $\delta > 0$,

$$\text{Vol}(A_{n+1} + B_\delta) \leq \text{Vol}(A) + a_n (R + \delta)^{n-1} \delta Y_n,$$

$$\text{Vol}(A_{n+1} + B_\delta) \geq \text{Vol}(A) - a_n R^{n-1} \delta Y_n,$$

where $a_n = 2n\omega_n$, and $Y_n$ is the random variable from Eq. (8). Since $a_n \geq 2\omega_n$, $Y_n \geq 1$, and $R \geq r$, the assumptions of Lemma 4 are satisfied with $b = a_n Y_n$, and so

$$|\tilde{\rho}(\pm \delta) - r| \leq \frac{a_n R^{n-1}}{\omega_n r^{n-1}} \delta.$$

It follows that

$$d_H(\partial A_{n+1+k}, \partial A^*) = \max_{\pm} |\tilde{\rho}(\pm \|\tilde{F}_k - f^*\|_\infty) - r|$$

$$\leq \frac{a_n R^{n-1}}{\omega_n r^{n-1}} Y_n \|\tilde{F}_k - f^*\|_\infty$$

for $k > 0$, see Eq. (6). Since $Y_n$ is independent of $U_k$ for $k > n$ and $\tilde{F}_k$ depends on $U_0, \ldots, U_n$ only through $A_{n+1}$, we can invoke the Markov property to obtain

$$E(d_H(\partial A_{n+1+k}, \partial A^*)) = E(E(d_H(\partial A_{n+1+k}, \partial A^*) | U_0, \ldots, U_n))$$

$$\leq \frac{a_n R^{n-1}}{\omega_n r^{n-1}} E(Y_n) E(\|\tilde{F}_k - f^*\|_\infty | A_{n+1}))$$

$$\leq \frac{12 a_n R^{n-1}}{\omega_n r^{n-1}} E(Y_n) R k^{-\frac{1}{m+1}}$$

for $k > 0$. In the last line, we have applied Eq. (7) to $\tilde{F}_k$. By Corollary 2, the expected value of $Y_n$ is finite. We shift the index and adjust the constant to obtain the desired bound on the rate of convergence.
Corollary 3.

\[ E(d_H(\partial A_k, \partial A^*)) \leq c_n (R/r)^{n-1} R^{-\frac{1}{n+1}}, \]

where \( c_n = 25n\omega_{n-1}E(Y_n)/\omega_n \).

We close with an explicit bound on the constants \( b_n \) and \( c_n \) that appear in Corollaries 2 and 3. We consider separately each of the expected values in Eq. (9). A routine spherical integral (conveniently evaluated as a Gaussian integral over \( \mathbb{R}^n \)) gives

\[ E(X^{-1}_k) = \frac{(n-k+1)\omega_{n-k+1}}{(n-k)\omega_{n-k}} \frac{(n-1)\omega_{n-1}}{n\omega_n} \]

for \( 2 \leq k < n \). Collecting terms, we obtain

\[ \prod_{k=2}^{n-1} E(X^{-1}_k) = \frac{(n-1)\omega_{n-1})^{n-1}}{2(n\omega_n)^{n-2}}. \]

A similar integral yields

\[ P(X_n \leq \sin \alpha) = \frac{(n-1)\omega_{n-1}}{n\omega_n} \int_0^\alpha (\cos t)^{n-2} dt. \]

Using that \( X_n \) and \( X'_n \) are independent, we estimate for \( n \geq 3 \)

\[ E((\max\{X_n, X'_n\})^{-1}) \leq 1 + \left( \frac{2(n-1)\omega_{n-1}}{n\omega_n} \right)^2. \]

When \( n = 3 \), this equation holds with equality, resulting in \( E(Y_3) = \pi \). In two dimensions, we find that \( E(Y_2) \leq 2 \), and for \( n \to \infty \), we have \( \lim n^{-1} \log E(Y_n) = \sqrt{2} e \). Since \( \lim n^{-1} \log (n\omega_{n-1}) = -\infty \) and \( \lim n^{-1} \log (n\omega_{n-1}/\omega_n) = 1 \), we conclude that \( b_n \) converges to zero and \( c_n \) grows exponentially.

Acknowledgments. This work was partially supported by an NSERC Discovery Grant (Burchard) and an NSERC Canadian Graham Bell Graduate Scholarship (Chambers). We would also like to thank Luigi Ambrosio for the proof of Lemma 5.

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