Laboratory realization of KP-solitons

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Abstract. Kodama and his colleagues presented a classification theorem for exact soliton solutions of the quasi-two-dimensional Kadomtsev-Petviashvili (KP) equation. The classification theorem is related to non-negative Grassmann manifold, Gr($N, M$) that is parameterized by a unique chord diagram based on the derangement of the permutation group. The cord diagram can infer the asymptotic behavior of the solution with arbitrary number of line solitons. Here we present the realization of a variety of the KP soliton formations in the laboratory environment. The experiments are performed in a water tank designed and constructed for precision experiments for long waves. The tank is equipped with a directional-wave maker, capable of generating arbitrary-shaped multi-dimensional waves. Temporal and spatial variations of water-surface profiles are captured using the Laser Induces Fluorescent method – a nonintrusive optical measurement technique with sub-millimeter precision. The experiments yield accurate anatomy of the KP soliton formations and their evolution behaviors. Physical interpretations are discussed for a variety of KP soliton formations predicted by the classification theorem.

1. Introduction

Based on the assumption of weakly nonlinear, weakly dispersive water waves that have small deviation in the transverse direction $y$ from the main wave propagation direction $x$, we take the following orders in the effects of wave nonlinearity, dispersion and directivity:

\[
\frac{a_0}{h_0} = \mathcal{O}(\epsilon), \quad \left( \frac{h_0}{\lambda_0} \right)^2 = \mathcal{O}(\epsilon), \quad \tan^2 \psi_0 = \mathcal{O}(\epsilon),
\]

respectively, where $a_0$ is the wave amplitude, $h_0$ is the constant water depth at the quiescent state, $\lambda_0$ is the wavelength scale, and $\psi_0$ is the small oblique angle from the $x$ direction. Then, the Euler formulation is approximated by the Kadomtsev-Petviashvili (KP) equation of the form

\[
\left( \eta_t + c_0 \eta_x + \frac{3c_0}{2h_0} \eta \eta_x + \frac{c_0^2 h^2}{6} \eta_{xxx} \right)_x + \frac{c_0}{2} \eta_{yy} = 0,
\]

in which $\eta$ is the water-surface elevation from $h_0$, and $c_0 = \sqrt{gh_0}$. The KP equation (1) can be expressed in the following canonical non-dimensionalized form,

\[
(4u_T + 6uu_x + u_{\xi\xi})_\xi + 3u_{YY} = 0,
\]
where the non-dimensional variables \((\xi, Y, T)\) and \(u\) are defined by
\[
\eta = \frac{2h_0}{3} u, \quad x - c_0 t = h_0 \xi, \quad y = h_0 Y, \quad t = \frac{3h_0}{2c_0} T.
\]
With the form of the KP equation (2), we consider the solution in the form
\[
u(\xi, Y, T) = 2\partial^2_\xi (\ln \tau(\xi, Y, T)),
\]
where \(\tau\) is referred to as the \(\tau\)-function defined in the Wronskian determinant. For a 2-soliton case, \(\tau\)-function can be written as
\[
\tau = \text{Wronskian}(f_1, f_2) = \begin{vmatrix}
f_1 & \partial_\xi f_1 \\
f_2 & \partial_\xi f_2
\end{vmatrix}.
\]
In this paper, the term '2-soliton' is used to describe a wave condition with the presence of two solitons each as \(y \to \pm\infty\). The functions \(f_1\) and \(f_2\) satisfy the linear equations: \(\partial_Y f_i = \partial^2_\xi f_i\) and \(\partial_T f_i = -\partial^3_\xi f_i\) (see e.g. Hirota [1]). Soliton solutions can be written by exponential functions of the form:
\[
E_j = \exp(\theta_j) = \exp(k_j \xi + k_j^2 Y - k_j^3 T).
\]
Therefore,
\[
\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{pmatrix} = A \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{pmatrix}.
\]
Here the coefficient matrix \(A = (a_{ij})\) is a constant \(2 \times 4\) matrix. Thus each solution \(\nu(\xi, Y, T)\) is completely determined by the \(A\)-matrix and the \(k\)-parameters. The foregoing is generalized for arbitrary number \(N\) \(f\)-functions and \(M\) \(E\)-exponential functions. The \(\tau\)-function leads to the notion of Grassmannian variety: \(\text{Gr}(N, M)\). When constraints are applied for the regular soliton solutions, the \(\tau\)-function is identified as a point of the totally nonnegative Grassmannian cell. The asymptotic soliton solutions for \(y \to \pm\infty\) can be parameterized by the permutations, which lead to the introduction of chord diagram to express the classification of soliton solutions as a chord joining a pair of \(k_j\)'s following its permutation representation. Detailed mathematical descriptions are given by Chakravarty and Kodama [2] and Kodama [3]. Figure 1 shows seven possible types of \(N = 2\) solutions.

![Figure 1](image-url)

**Figure 1.** The chord diagrams for seven different types of 2-soliton solutions. The number in the parenthesis is the permutation.

As for the simplest case, consider the solution of single line-soliton that can be expressed as
\[
\tau = E_1 + aE_2
\]
\[
= 2\sqrt{a} \exp \frac{1}{2}(\theta_1 + \theta_2) \cosh \frac{1}{2}(\theta_1 - \theta_2 - \ln a).
\]
Therefore,

\[ u = 2\partial_\xi^2(\ln \tau) = \frac{1}{2}(k_1 - k_2)^2\text{sech}^2\frac{1}{2}(\theta_1 - \theta_2 - \ln a), \]

\[ = A_{[i,j]}^2\text{sech}^2 \sqrt{\frac{A_{[i,j]}}{2}}(\xi + Y \tan \Psi_{[i,j]} - C_{[i,j]}T - x_{[i,j]}^0), \]

where \( A_{[i,j]} = \frac{1}{2}(k_j - k_i)^2; \tan \Psi_{[i,j]} = k_i + k_j; C_{[i,j]} = k_i^2 + k_i k_j + k_j^2 = \frac{1}{2}A_{[i,j]} + \frac{3}{4}\tan^2 \Psi_{[i,j]}, \) and \( \Psi_{[i,j]} \) is the direction of wave propagation. The soliton formation and the corresponding code diagram is shown in Fig. 2.

\[ \text{Figure 2.} \quad \text{A line-soliton contour plot and the corresponding chord diagram.} \]

2. Laboratory Experiments

Laboratory experiments are conducted in the wave tank, 7.3 m long, 3.6 m wide, and 0.30 m deep: see Fig. 3. The tank is elevated 1.2 m above the laboratory floor to enable us to make measurements with optical instruments. The bottom and sidewalls are made of 12.7 mm thick glass plates. The wave basin is equipped with a 16-axis directional-wavemaker system along the 3.6-m long headwall, capable of generating arbitrary-shaped, multi-directional waves. Each wave paddle is pushed through hinge connections by two adjacent linear-servo-motor motion devices. The paddles are made of PVDF (Polyvinylidene fluoride) plates that are driven horizontally in piston-like motions; they are optimally flexible to form a smooth curve between the driving axes. Each paddle has a maximum horizontal stroke of 55 cm – more than adequate to generate very long and multiple waves in a water depth. This wavemaker system, together with the precise wave tank, is ideal for investigation of nonlinear dynamics of long-wave motion. It is emphasized that the generated waves are precisely replicable: for example, the maximum error is less than 0.06 mm (or 0.1% of the depth) for a solitary wave with amplitude \( a = 1.73 \text{ cm} \) in the water depth of \( h_0 = 6.0 \text{ cm} \). Note that, the horizontal dimensions of the tank scaled by the depth are \( 120h_0 \times 60h_0 \), which is considered extremely large. Incidentally, the wave Reynolds number is on the order of 12,000 that we consider large enough to circumvent serious viscous scale effects.

The Laser Induced Fluorescent (LIF) method is used to examine temporal and spatial variations of water-surface profiles. Figure 4 (left) shows a setup for the LIF method. A laser beam (from a 5W diode-pumped solid-state laser) is converted to a thin laser sheet using a cylindrical lens. Two front-surface mirrors direct the laser sheet to illuminate the vertical plane. With the aid of fluorescein dye dissolved in water, the vertical laser-sheet illumination induces the dyed water to fluoresce and identifies the water-surface profile directly and non-intrusively.
Figure 3. Schematic drawings of the laboratory apparatus: (upper) a plan view showing the waveguide that creates an oblique wave reflection and the tank frame that can be seen through the bed that is made of glass plates; (lower) an elevation view.

The illuminated profiles, as the wave passes through the light sheet, are recorded by a high-speed high-resolution video camera. The captured images are rectified and processed with the calibrated image so that the resulting images can be analyzed quantitatively. The transparent glass bed of the tank minimizes the reflection of the laser illumination that could have caused contamination in the image analysis for the wave profiles. The temporal and spatial variations of the water-surface profiles captured with the LIF technique are shown in Fig. 4 (right).

Figure 4. (Left) Setup for the Laser-Induced Fluorescent (LIF) method. (Right) Temporal variation of the water surface profile along the vertical wall at $y = 0$. The profiles were constructed by making a montage of three LIF images. The time interval of each profile is $1/100$ sec.

One of the difficulties associated with the LIF technique for measuring long waves is the
limitation in resolution. Unlike experiments for capillary waves or breaking waves, long waves have an inherently small vertical-to-horizontal scale ratio. This causes insufficient resolution in the vertical direction. Our laboratory experiments require measurements of small wave amplitudes (a few centimeters) in a large horizontal span (more than 100 cm). To circumvent this difficulty, LIF water-surface profiles are repeated on approximately 30 cm segments, creating a montage of the segment profiles.

3. Results

We first demonstrate the relation between a wave pattern of two solitons and the corresponding chord diagram. Let us consider, for example, two line solitons at \( y \to \pm \infty \): namely the one has the wave amplitude \( a_{[1,2]} = 0.35 \) cm with the inclination angle \( \psi_{[1,2]} = -20^\circ \), and the other has the amplitude \( a_{[3,4]} = 0.70 \) cm with \( \psi_{[3,4]} = 30^\circ \) in the water depth \( h_0 = 6.0 \) cm: see Fig. 5 (left). Using the four relations \( A_{i,j} = \frac{1}{2} (k_j - k_i)^2 \) and \( \tan \Psi_{i,j} = k_i + k_j \), we determine four \( k \)-parameters: \( k_1 = -0.3785, k_2 = 0.01457, k_3 = 0.03250, \) and \( k_4 = 0.5448 \). Because the asymptotic line-solitons are presented with \([1,2]\) and \([3,4]\) for \( y \to \pm \infty \), the corresponding chord diagram is \( \pi = (2143) \), which is also called the “O-type” 2-soliton because this solution was “originally” found to express the two-soliton solution [2]. Because the chords are closed independently, representing two individual line-solitons, the two solitons are not resonantly interacting and the waveform simply translates in time, maintaining the small phase shift at the intersection of the two solitons. It is noted that the chord diagram does not contain the initial phase information, which is represented in \( A = (a_{ij}) \) matrix.

![Figure 5](image)

**Figure 5.** Theoretical water-surface profile of O-type 2-soliton solution of the KP equation, and the corresponding chord diagram with \( \pi = (2143) \).

3.1. \( \pi = (2143) \) and \( \pi = (3142) \)

In the laboratory, we create 2-soliton waveforms that are symmetric about the \( x \)-axis by utilizing the wave reflection at the vertical wall: in other words, only a half of the domain, \( 0 < y < \infty \), is realized in the laboratory taking the advantage of the symmetric wave pattern. Figure 6 shows the measured 2-soliton patterns of \( \pi = (2143) \) (O-type) and \( \pi = (3142) \), respectively. The tank sidewall is located along the top edge of the plot. The horizontal direction in the plot represents time that increases to the right. Negligibly small capillary-wave noise emanating from the wall can be detected in the figure. As shown in the figure, we observe non-resonant interaction in \( \pi = (2143) \) and the stem-like formation along the wall remains unchanged, because this stem wave is a consequence of the phase shift resulted from the wave intersection.
Figure 6. (Upper) O-type wave-profile evolution \(a_i/h_0 = 0.076 (\kappa = 1.395)\) and (lower) \((3142)\)-type \(a_i/h_0 = 0.277 (\kappa = 0.731)\). \(\psi_i = 30^\circ, x/h_0 = 71.1, 96.1,\) and 121.1 from the left panel to the right. Positive \(t\)-axis points to the right and positive \(y\)-axis points downward.

On the other hand, in the case of the \((3142)\)-type, the stem-wave along the wall continually grows as it propagates. According to Miles [4], \(\pi = (2143)\) type and \(\pi = (3142)\) type are distinguished by the parameter \(\psi_i/\sqrt{2A_0}\); recently, Li et al. [5] found that the parameter should be presented by \(\kappa = \frac{\tan \psi_i}{\cos \psi_i \sqrt{3a_i/h_0}}\) for real-world solitons, in which \(a_i\) is the physical wave amplitude. When \(\kappa > 1\), the wave pattern is represented by \(\pi = (2143)\) type, and when \(\kappa < 1, \pi = (3142)\). (Note that Kodama [6] formally derived the higher-order correction for the parameter \(\kappa\), which includes the correction in terms of \(a_i\).)

Figure 7 depicts the chord diagram of \(\pi = (2143)\) and \(\pi = (3142)\), respectively. The amplitude of \([1,4]\)-soliton (the stem wave) can be inferred from the chord diagram of \(\pi = (3142)\). According to Miles [4], the stem-wave amplification is predicted by \((1 + \sqrt{\alpha_r})^2\) where \(\alpha_r\) is the reflected wave amplification of the Mach reflection or equivalently, the ratio of the smaller wave amplitude to the larger one: i.e. \(\alpha_r = A_{[3,4]}/A_{[1,3]}\) at \(y \to +\infty\) or \(A_{[1,2]}/A_{[2,4]}\) at \(y \to -\infty\). The theoretical amplitude prediction of \([1,4]\)-soliton can be found in the chord diagram as

\[
A_{[1,4]} = \frac{1}{2} (k_4 - k_1)^2 = \frac{1}{2} \left[ (k_3 - k_1) + (k_4 - k_3) \right]^2 = \frac{1}{2} \left[ k_3 - k_1 \right] \left( 1 + \sqrt{\frac{1}{2} \left( k_4 - k_3 \right)^2} \right)^2,
\]

which represents the amplification of in fact \((1 + \sqrt{\alpha_r})^2\). The transition between \((2143)\)-type and \((3142)\)-type can be depicted as the condition \((k_3 - k_2) \to 0\). At this critical state, the

Figure 7. Chord diagrams of \(\pi = (2143)\) and \(\pi = (3142)\) types and their critical state.
amplification of the stem wave can be correctly interpreted from the chord diagram: namely the 4-fold amplification, $A_{[1,4]} = \frac{1}{2}(k_4 - k_1)^2 = 4 \times \frac{1}{2}(k_2 - k_1)^2$.

3.2. $\pi = (3412)$
According to Chakravarty and Kodama [2], $\pi = (3412)$-type solution is called the T-type solution, and the T-type solution has four asymptotic solitons that form an exact “X” shape near the origin. Nonetheless, the wave pattern deviates from the exact “X” shape near the soliton intersection. Figure 8 compares the KP solution with our laboratory measurements for the T-type solitons. At $t = 0$, the exact X-shaped wave pattern is created. The continual growth of [1,4] and [2,3] waves (see Fig. 1) is observed in both theoretical prediction and the laboratory observation, and the observed “box-shaped” wave pattern is consistent with the prediction.

![Figure 8](image)

**Figure 8.** Laboratory (upper) and theoretical (lower) wave patterns for $\pi = (3412)$-type (T-type). $A_i = 0.280; \psi_i = 20^\circ$.

The exact X-shaped wave pattern is made in the laboratory wave tank as shown in Fig. 9. Unlike (3412)-type, this is the O-type solution. Immediately after the formation of the exact X-shaped waveform at $t = 0$, the waves adjacent to the intersection are bent so as to accommodate the formation of the phase shift. Once it happens, the waveform remains steady.

3.3. V-Shaped Initial Waves
Initially V-shaped waves are generated along the sidewall in the laboratory with the aid of directional wavemaker system (see Fig. 10). This initial waveform creates the incomplete chord diagrams. Kao and Kodama [7] conducted the numerical experiments for the similar cases based on the KP theory. Here we verify their numerical findings in the real-fluid environment.

Figure 11 shows the case with $\kappa = \frac{\tan \psi}{\cos \psi \sqrt{3a_{[2,3]}/h_0}} = 1$, and $a_{[1,3]} / a_{[2,3]} = 4.0$. This is the
critical condition of the Mach reflection [4]. The observation shows that the V-shaped wave is quickly transformed to the Y-shaped wave by completing the chord diagram by linking \( k_2 \) to \( k_1 \) (Fig. 11). Once the Y-shaped wave is formed by completing the chord diagram, the wave becomes stable maintaining the constant crest length of the stem wave in the laboratory setup. The actual wave amplitudes measured at \( x/h_0 = 50 \) are \( a_{[2,3]}/h_0 = 0.133 \), \( a_{[1,2]}/h_0 = 0.130 \), and \( a_{[1,3]}/h_0 = 0.459 \), consequently \( \kappa = 1.05 \). Note that \( a_{[2,3]}/a_{[1,2]} \approx 1 \) as expected. The discrepancy must be due to viscous dissipation that is unavoidable in the real-fluid experiments. Because of the amplitude attenuation, the value of \( \kappa \) continually but slowly increases in time.

Figure 12 shows the cases for \( \kappa > 1 \) and \( \kappa < 1 \), respectively. In the case of \( \kappa > 1 \), the crest length of the stem wave continually decreases as it propagates, which means that the origin of \( k \) in the chord diagram practically shifts to the right and \( \psi_{[1,3]} \) is now slightly negative to make the Y-shaped wave formation steady; in other words, the observation coordinates should be rotated slightly in the clockwise direction. Also observed is \( a_{[2,3]} < a_{[1,2]} \); the actual wave amplitudes measured at \( x/h_0 = 50 \) are \( a_{[2,3]}/h_0 = 0.090 \) and \( a_{[1,2]}/h_0 = 0.124 \), consequently \( a_{[2,3]}/a_{[1,2]} < 1 \).
Figure 11. Incomplete Y-shaped soliton generated in the laboratory for the case of $\kappa = 1$; the generated wave amplitudes are $a_{[2,3]}/h_0 = 0.148$ and $a_{[1,3]}/h_0 = 0.592$ with the oblique angle $\tan \psi_{[2,3]} = 0.577$.

Figure 12. Incomplete Y-shaped soliton generated in the laboratory for the case of (top) $\kappa = 1.22 > 1$: the generated wave amplitudes are $a_{[2,3]}/h_0 = 0.10$ and $a_{[1,3]}/h_0 = 0.491$ with the oblique angle $\tan \psi_{[2,3]} = 0.577$; (bottom) $\kappa = 0.92 < 1$: the generated wave amplitudes are $a_{[2,3]}/h_0 = 0.175$ and $a_{[1,3]}/h_0 = 0.645$ with the oblique angle $\tan \psi_{[2,3]} = 0.577$.

In the case of $\kappa < 1$, the crest length of the stem wave continually increases as it propagates. To make the Y-shaped wave pattern steady, the origin $k = 0$ in the chord diagram should be
shifted to the left and $\psi_{[1,3]}$ is now slightly positive, or the observation coordinate system is rotated slightly in the counterclockwise direction. Opposite to the case of $\kappa > 1$, this case yields $a_{[2,3]} > a_{[1,2]}$: the actual wave amplitudes measured at $x/h_0 = 50$ are $a_{[2,3]}/h_0 = 0.155$ and $a_{[1,2]}/h_0 = 0.129$, consequently $a_{[2,3]}/a_{[1,2]} > 1$. Those observations are in accord with the KP theory and well described in the chord diagram.

4. Summary
The classification theorem for 2-soliton solutions of the KP theory is validated by performing the experiments in the precisely controlled laboratory apparatus. The chord diagrams resulted from the classification theorem do not only represent the asymptotic solitons at $|y| \to \infty$, but also heuristically imply the interaction behaviors, even quantitatively in some cases.

We first demonstrated the difference between the (2143)-type (i.e. O-type) and the (3142)-type. The stem-wave-like formation is a consequence of the phase-shift associated with the (2143)-type and the intersecting wave formation remains constant and steady. On the other hand, the (3142)-type shows the resonant interaction behavior; consequently, the interacting stem wave continually grows its crest length. The chord diagram can provide the quantitative wave amplitude of the stem wave. Furthermore, the critical four-fold amplification at $\kappa = 1.0$ can be illustrated by the chord diagram.

The (3412)-type (i.e. T-type) soliton is realized in the laboratory, which creates the box-shaped wave formation at the intersection. The box-shaped wave formation grows in time in accordance with the KP theory. On the other hand, the exact X-shaped initial wave formation does not necessarily lead to the (3412)-type. When the soliton amplitude is sufficiently small, the X-shaped wave formation becomes the (2143)-type (i.e. O-type). In this case, the wave crests near the intersection are bent so as to accommodate the necessary phase shift associated with the (2143)-type.

Lastly, we studied the V-shaped KP-soliton formation at $t = 0$. This V-shaped soliton formation can be interpreted as an incomplete Y-shaped wave with missing one upper arm, or the corresponding chord diagram is not completely closed. It is observed that immediately after the generation, the V-shaped wave becomes the Y-shaped wave formation by completing the chord diagram. Three distinct cases of the incomplete Y-shaped wave formation are demonstrated in the laboratory, presenting the wave transformation behaviors in accordance with the three sets of the chord diagrams.

In short, we demonstrated the KP soliton formations in the real-fluid environment, and presented the physical implications of the chord diagram without solving the KP equation.

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