ISING-LINK QUANTUM GRAVITY

Tom Fleming ¹ and Mark Gross ²
Dept. of Physics and Astronomy, California State Univ., Long Beach, CA 90840

Ray Renken ³
Department of Physics, University of Central Florida, Orlando, Florida 32816

Abstract

We define a simplified version of Regge quantum gravity where the link lengths can take on only 2 possible values, both always compatible with the triangle inequalities. This is therefore equivalent to a model of Ising spins living on the links of a regular lattice with somewhat complicated, yet local interactions. The measure corresponds to the natural sum over all $2^{\# \text{links}}$ configurations, and numerical simulations can be efficiently implemented by means of look-up tables. In three dimensions we find a peak in the “curvature susceptibility” which grows with increasing system size. However, the value of the corresponding critical exponent as well as the behavior of the curvature at the transition differ from that found by Hamber and Williams for the Regge theory with continuously varying link lengths.

PACS number(s): 04.60.+n,11.15.Ha

¹Electronic address: fleming@physics1.natsci.csulb.edu
²Electronic Address: mgross@csulb.edu
³Electronic Address: rlr@phys.physics.ucf.edu
I. INTRODUCTION

To date, two main formulations of lattice quantum gravity have been considered, the so-called “Regge gravity” \cite{1, 2} and “simplicial gravity” approaches \cite{3, 4, 5}. While it could be argued that both formulations involve Regge calculus and simplexes, the distinguishing feature is that the former has a fixed incidence matrix and varying link lengths while the latter has a varying incidence matrix and fixed link lengths.

Both formulations are technically and computationally demanding. For example, the Regge approach involves calculating areas and deficit angles involving general d-simplexes. In the simplicial approach these take on only a limited number of possible values, but the updating moves involve complicated interchanges of several simplexes at once. We introduce here a third lattice gravity approach which is structurally and computationally much simpler than either Regge or simplicial gravity, and may as a result be amenable to analytic attack in more than two dimensions.

We call our formulation “Ising-link quantum gravity”. It is easy to define. The incidence matrix is fixed exactly as in the Regge approach. But the link lengths can only take on two values,

\[ l_i = 1 + bs_i \]  

with \( s_i = \pm 1 \) and \( b \) a positive constant. \( i \) is a link label. In order that the triangle inequality (or its higher-dimensional generalization - that the simplex volume is real and positive) is always satisfied, it is straightforward to show that we must take \( b < \frac{1}{3} \) in two dimensions, \( b < 3 - \sqrt{8} \approx .17 \) in three dimensions, etc. (See Section II.) We restrict \( b \) to satisfy this inequality so that all \( 2^{N_1} \) configurations are allowed. \( N_1 \) is the number of links.) This is quite different from either Regge or simplicial gravity where most potential updates either violate the triangle inequalities or violate the manifold property. Furthermore it provides us with a natural measure which gives all \( 2^{N_1} \) configurations equal weight. It is clear that our model is completely equivalent to a (regular lattice) Ising model with spins \( (s_i) \) living on the links. We will see that the spin interactions are local, albeit somewhat complicated.

The Ising-link model is analytically and computationally much simpler than either the Regge or simplicial gravity approaches. But is it too simple? In Section III we present mean field theory results on the model in three-dimensions and in Section IV we give corresponding Monte Carlo results. We compare to results obtained by Hamber and Williams for the Regge theory in 3-d.

II. ISING MODEL FORMULATION

In this section we discuss how to compute the Ising action corresponding to the discrete form of

\[ l_i = c(1 + bs_i) \]  

\(^4 \) is no more general, as \( c \) can be absorbed into the definitions of \( \lambda \) and \( k \) in (2).
\[ S = \lambda V - \frac{k}{2} \int d^d x \sqrt{g} R , \] (2)

where \( V \) is the d-dimensional volume, \( \int d^d x \sqrt{g} \), and \( R \) is the scalar curvature. The lattice is formed out of hypercubes plus face, cubic \((d \geq 3)\) and hypercubic diagonals \((d \geq 4)\), with periodic boundary conditions \([2]\). First we will treat two dimensions, then three. Four dimensions is just like three, only harder.

**Two Dimensions:** Consider a triangle with link lengths \( l_1, l_2 \) and \( l_3 \). Define \( l_i = 1 + b s_i \) as in (1). Since \( s_i^2 = 1 \) and the formula for the area of the triangle must be symmetric in the 3 spins, the most general form for the area is

\[ A_{123} = C_0 + C_1 (s_1 + s_2 + s_3) + C_2 (s_1 s_2 + s_2 s_3 + s_3 s_1) + C_3 s_1 s_2 s_3 . \] (3)

There are only 4 possible values for the area of the triangle, corresponding to 0, 1, 2 or all 3 of its spins being equal to +1. Computing these 4 areas and comparing to (3) gives four linear equations for the \( C_\alpha \) in terms of the parameter \( b \). Their solution is

\[ 32C_0 = 2\sqrt{3}(1 + b^2) + 3f(b) + 3g(b) \]
\[ 32C_1 = 4b\sqrt{3} + f(b) - g(b) \]
\[ 32C_2 = 2\sqrt{3}(1 + b^2) - f(b) - g(b) \]
\[ 32C_3 = 4b\sqrt{3} - 3f(b) + 3g(b) , \] (4)

where \( f(b) \equiv |1 - b|\sqrt{(1 + 3b)(3 + b)} \) and \( g(b) \equiv (1 + b)\sqrt{(1 - 3b)(3 - b)} \). For example, \( b = .1 \) gives \( C_1 \approx .0291, C_2 \approx .0039 \) and \( C_3 \approx -.0008 \). As stated in the Introduction, it is seen that we must have \( b < 1/3 \) for the triangle areas to be real and positive.

Since the Einstein term in (2) is a topological invariant, it is not relevant to the case of fixed topology being considered here. Thus, summing over all triangles and dropping the irrelevant constant term, the two-dimensional action is

\[ S = \lambda (2C_1 \sum_i s_i + C_2 \sum_{<ij>} s_i s_j + C_3 \sum_{<ijk>} s_i s_j s_k) , \] (5)

where \( i, j \) and \( k \) are link labels. \(<ij>\) indicates \( i \) and \( j \) are two of three links forming a triangle. In this case \( s_i \) and \( s_j \) may be termed nearest-neighbor links. \(<ijk>\) means that \( i, j \) and \( k \) are three links which form a triangle. The \( C_2 \) term is a “nearest-neighbor” ferromagnetic interaction. The \( C_1 \) term is a magnetic field term and the \( C_3 \) term is an additional symmetry-breaking term.

One might hope that the continuum limit of Ising-link quantum gravity would correspond to a second-order magnetization phase transition. But with the explicit symmetry-breaking terms in (5), it is clear that this transition can not be from order \(<s>\neq 0\) to disorder \(<s>=0\). It would have to be an \(<s>\neq 0\) to \(<s>\neq 0\) transition. In two
dimensions, as expected, we found no evidence of such a transition, at least in the mean field theory approximation.

Three Dimensions: We will now go on to discuss the form of the theory in three dimensions. In the next section we will compare numerical results in 3-d to results for the unconstrained Regge theory.

Consider the labeled tetrahedron of Fig. 1. The volume $V_{tet}$ is given by the formula [2]

$$144 V_{tet}^2 = 4l_1^2 l_2^2 l_4^2 - l_1^4 (l_3^2 - l_6^2 + l_4^2)^2 - l_3^4 (l_1^2 + l_4^2 - l_5^2)^2 - l_4^4 (l_1^2 - l_2^2 + l_3^2)^2 + (l_3^2 - l_6^2 + l_4^2)(l_1^2 + l_4^2 - l_5^2)(l_1^2 - l_2^2 + l_3^2),$$

(6)

where the $l_i$ may be written in terms of spins $s_i$ using Eq. (1). There are only 11 distinct possible values for the volume of the tetrahedron, corresponding to the 11 a priori unknown constants in the most general equation for $V_{tet}$ compatible with the symmetries of the labeled tetrahedron:

$$V_{tet} = C_0 + C_1 \sum_i s_i + C_2 \sum_{<ij>} s_is_j + C_3 \sum_{[ij]} s_is_j + C_4 \sum_{<ijk>} s_is_js_k + C_5 \sum_{(ijk)} s_is_js_k + C_6 \sum_{[ijk]} s_is_js_k + \left( \prod_{l=1}^6 s_l (C'_0 + C'_1 \sum_i s_i + C'_2 \sum_{<ij>} s_is_j + C'_3 \sum_{[ij]} s_is_j) \right).$$

(7)

Here $<i,j>$ are again two of three links that form a triangle in Fig. 1. $[i,j]$ are the remaining pairs of links. $<i,j,k>$ form a triangle, $(i,j,k)$ share a common site and $[i,j,k]$ are the remaining triplets of links. Because $s_i^2 = 1$, the last four terms involve 4, 5 and 6-link interactions. Evaluating (7) for each of the 11 distinct volumes results in 11 linear equations for the $C_i$ and $C'_i$. As in (4) for the 2-d case, these can easily be solved to determine the $C_i$ and $C'_i$ as functions of $b$. The result is not particularly illuminating and we will not reproduce it here. It is worth noting, however, that the the volumes are always real and positive if we choose $b < 3 - \sqrt{8} \approx .17$. In the mean field and numerical results described in the next two sections, $b$ is held equal to 0.1.

We see that after summing up the volumes of all the tetrahedrons, the volume term in (2) will consist of only local interactions of the spins, involving up to 6-spin interactions.

---

5One reason for not dwelling on the two-dimensional theory is that there is one (though only one) known disagreement between 2-d Regge quantum gravity (which appears most closely related to the 2-link model) and continuum results [8], namely the critical exponents for Ising spins coupled to 2-d Regge gravity [6]. One of the authors (M.G., unpublished) has examined coupling Ising spins within a fixed invariant distance rather than those connected by a link, since the latter coupling (used in [7]) has no invariant meaning. But even with the invariant coupling there was still no agreement between Regge gravity and continuum results; despite the fact that the total area was held fixed, enough of the links got sufficiently small for a finite fraction of the spins to all become coupled together. As a result the free energy of that model is proportional to the number of degrees of freedom squared - a fatal illness.
The second (Einstein) term in (2), \(-\frac{k}{2} \int d^3 x \sqrt{g} R\), takes the form [2]

\[ S_E = k \sum_i l_i \left( \sum_{t/i} \theta_{t/i} - 2\pi \right) \]  

where \( t/i \) denotes a tet containing the link \( i \) and \( \theta_{t/i} \) is the corresponding dihedral angle at link \( i \). For the tetrahedron shown in Fig. 1, \( \theta_{t/5} \) is given by

\[ \cos(\theta_{t/5}) = \frac{1}{16A_{145}A_{256}} \left[ 2(l_4^2 + l_6^2 - l_3^2)l_5^2 - (l_4^2 + l_3^2 - l_5^2)(l_5^2 + l_6^2 - l_2^2) \right] , \]  

where \( A_{ijk} \) is the triangle formed by links \( i, j \) and \( k \). The term \( S_E \) can also be written in terms of local spin interactions, but we shall omit the details here.

III. MEAN FIELD THEORY IN THREE DIMENSIONS

The Ising-link model is quite accessible to mean field theory (MFT) techniques. We write

\[ Z = \sum_{\{s\}} \exp(-\beta H[s]) , \]  

where here \( \beta \equiv 1 \) and

\[ H \equiv S = \lambda V - \frac{k}{2} \int d^3 x \sqrt{g} R . \]  

This is a functional of the spins since \( l_i = 1 + bs_i \), by Eq. (1). We wish to minimize the free energy,

\[ F = < H > - S/\beta , \]  

for the spin probability distribution function, \( P[s] \), where \( S \) is the entropy.

The mean field approximation [8] consists of replacing the true probability distribution for the spins by a factorized form:

\[ P[s] \rightarrow p(s_1)p(s_2)p(s_3)\ldots p(s_{N_1}) , \]  

where \( N_1 \) is the number of links. If all links were equivalent, we could write \( p(s_i) = \frac{1 + ms_i}{2} \Rightarrow \sum_s p(s_i) = 1 \) and \( < s_i > = m \). But there are three different kinds of links in the lattice formed out of cubes with body and face diagonals: the body diagonals, the cube edges, and the face diagonals. Links of the same type have the same geometrical
environment; links of different types don’t. As a result we must use the more general
distribution,

\[ p_j(s_i) \equiv \frac{1 + m_j s_i}{2}, \]

where \( j = 1, 2 \) and 3 for body diagonals, cube edges and face diagonals respectively.

A straightforward calculation allows us to determine \( < H > \) and \( S \) as functions of \( m_1, m_2 \) and \( m_3 \). Let

\[ P_V \equiv p_2(s_1)p_3(s_2)p_2(s_3)p_3(s_4)p_1(s_5)p_2(s_6) \]
\[ P_{R1} \equiv p_1(s_1)p_3(s_2)p_2(s_3)p_4(s_4)p_2(s_5)p_2(s_6) \]
\[ P_{R2} \equiv p_3(s_1)p_2(s_2)p_1(s_3)p_2(s_4)p_2(s_5)p_3(s_6) \]
\[ P_{R3} \equiv p_2(s_1)p_1(s_2)p_3(s_3)p_2(s_4)p_3(s_5)p_2(s_6) \] (15)

We find that

\[ < V > = 6N_0 \sum_{s_1} \ldots \sum_{s_6} P_V(s)V_{tet}(s) , \]

\[ \frac{1}{2} \int d^3x \sqrt{g} R = 2\pi N_0[7 + b(m_1 + 3m_2 + 3m_3)] \]
\[ - 6N_0 \sum_{s_1} \ldots \sum_{s_6} l_5(s) \theta_{l/5}(s)[P_V(s) + 2P_{R1}(s) + P_{R2}(s) + 2P_{R3}(s)] \] (17)

and

\[ S \equiv - < lnP[s] > = -N_0[h(m_1) + 3h(m_2) + 3h(m_3)] , \]

where \( V_{tet} \) and \( \theta_{l/5} \) are given by (6) and (9) respectively, \( N_0 \) is the number of lattice sites and \( h(x) \equiv \frac{1+x}{2} \ln(\frac{1+x}{2}) + \frac{1-x}{2} \ln(\frac{1-x}{2}) \). \( s \) is shorthand for \( s_1, s_2, ..., s_6 \). Now it is a simple matter to numerically minimize the free energy (12) as a function of \( m_1, m_2 \) and \( m_3 \). Then

\[ < s > = \frac{m_1 + 3m_2 + 3m_3}{7} , \]

and

\[ R \equiv < l^2 > \frac{\int d^3x \sqrt{g} R}{< V >} = (1 + b^2 + 2b < s >) \frac{\int d^3x \sqrt{g} R}{< V >} . \] (20)

Here we follow the notation of Hamber and Williams [9]. Also the “curvature susceptibility” is defined as

\[ \chi_R = \frac{2}{< V >} \frac{\partial}{\partial k} < \int d^3x \sqrt{g} R > . \] (21)
Results. Fig. 2 shows typical MFT results for the case $\lambda = 1$. There is sharp cross-over behavior seen in $<s>$, $v$ and $R$ at $k$ slightly negative. $R$ and $<\int d^3x \sqrt{g} R>$ (not shown in Fig. 2) are related by (20) and exhibit similar behavior in this region. $\chi_R$ was evaluated using (21), by taking a numerical derivative of $<\int d^3x \sqrt{g} R>$ with increment $\Delta k = 1$. We see a peak in $\chi_R$ here, as expected from the rapid cross over behavior in $R$. Hamber and Williams [9] found a second order phase transition in the 3-d Regge theory exhibiting $\chi_R \sim |k_c - k|^\delta$ (22) for $k < k_c$, with $\delta = 0.80 \pm 0.06$, a very weak second order phase transition. To investigate whether this kind of non-analyticity is seen in the MFT approximation to the Ising-link model, we varied $\Delta k$ from 1 downward. ($\Delta k$ is the increment used to take the numerical derivative of $<\int d^3x \sqrt{g} R>$.) The behavior (22) would result in the peak of $\chi_R$ growing with $\Delta k$ like $(\Delta k)^{\delta-1}$. $\delta = 0.80$ would imply that the peak would grow by 58% in height for each factor of 10 decrease in $\Delta k$. However, for all values of $\lambda$ considered (up through $\lambda = 75$), there was no increase in the peak height as $\Delta k$ was decreased from 1 down to .01. As a result we have no evidence of (22) with $\delta - 1 < 0$ in the mean field approximation to the 3-d Ising-link model. Nevertheless, in the next section we will present evidence Monte Carlo evidence for (22) with $\delta - 1 < 0$ at large values of $\lambda$.

For $\lambda \neq 1$ the situation looks similar to that shown in Fig. 2. There is a finite peak in $\chi_R$ at $k$ slightly negative and a first order phase transition at large positive $k$. The height of the peak in $\chi_R$ and the size of the discontinuity at the first order phase transition vary appreciably with $\lambda$. Fig. 3 shows a dashed curve in $\lambda - k$ space where there is a peak in $\chi_R$ and a solid curve of first order phase transitions. For $k$ to the right of the first order phase transition, $R$ rapidly approaches zero from below as the system approaches a state with $m_1 = m_3 = -m_2 = 1$. As $k \to -\infty$, the system approaches a state with $m_1 = m_2 = -m_3 = 1$.

IV. NUMERICAL RESULTS IN THREE DIMENSIONS

The 3-d Ising-link model was also analyzed by the Monte Carlo method. The discrete form of (2) was used, with toroidal topology and without higher derivative terms, as discussed in the previous section. As in that section, $b$ was chosen to be 0.1 thoughtout. (See Eq. (1).)

Since there are only two possible lengths per link, there are at most $2^6 = 64$ possible configurations for a particular tetrahedron. (Actually there are significantly fewer due to symmetry.) Because all terms in the action are determined solely by the link lengths in the lattice, the limited number of distinct tetrahedrons allows many of the calculations to be performed only once at program entry and stored for later use in the form of “look-up tables”. These tables are accessed during the Monte Carlo updating. As a result, the Ising-link model proved to be quite computationally efficient; run times were reduced by as much as a factor of ten over continuous-link (Regge gravity) simulations.
Except for the reliance on look-up tables, the simulations were carried out in the usual way. An initial random configuration of link lengths is chosen, generating a particular initial geometry. A link update consists of choosing a particular link in the lattice, calculating the change in the action if the link takes on its other possible value, and accepting the new link value with probability proportional to the exponential of the negative change in the action (heat bath). Link updates are performed for each link in the lattice; this constitutes one sweep. The quantities of interest are calculated after each sweep of the lattice, and the values for each new geometry are binned for statistical analysis. Runs of up to 100k sweeps on the $4^3$ and $8^3$ lattices, and 80k sweeps on the $16^3$ lattice were performed for various values of $\lambda$ and $k$.

The two physical quantities of greatest interest were $R$ and $\chi_R$, defined in Eqs. (20) and (21). Hamber and Williams found that in the Regge theory, the curvature susceptibility diverges at points where $R$ vanishes [9]. Thus, a portion of the curve $R = 0$ was first identified (Fig. 4), and the behavior of the model was studied along that curve. But peaks in $\chi_R$ did not appear along the $R = 0$ curve, but rather were located at values of $k$ that correspond to local inflection points in $R$. This behavior is consistent with Eq. (21), which relates $\chi_R$ to the first derivative of $R$ with respect to $k$. Also plotted in Fig. 4 is a dashed curve of peaks in $\chi_R$. The case $\lambda = 1$ was studied for values of $k$ close to that curve, but no growth of the peak with increasing system size was observed, indicating the absence of a second-order transition in this region of parameter space. At $\lambda = 75$, however, we did find growth in the $\chi_R$ peak with increasing system size indicative of a second order phase transition (see below). We expect that the dashed curve becomes a line of second order phase transitions somewhere between $\lambda = 1$ and $\lambda = 75$.

Fig. 4 may be compared with Fig. 3 determined by MFT. The location of the peak in $\chi_R$ is the same in both plots to within Monte Carlo statistical errors. However no second order phase transition occured in the MFT approximation for any value of $\lambda$. The agreement for large $k$ between MFT and Monte Carlo is poorer. As discussed in the previous section, MFT exhibited a first order phase transition along the solid curve of Fig. 3, and for $k$ to the right of that curve, $R$ asymptotically approached 0 from below. But no first order phase transition was found in Monte Carlo, and $R$ went from negative to positive values at the location of the solid curve in Fig. 4.

The comparison with MFT is also seen by comparing Monte Carlo $\lambda = 1$ data displayed in Fig. 5 with corresponding MFT data shown in Fig. 2. Note the difference in the scales for $<s>$ and $\chi_R$. For $k$ negative the agreement is quite good; in fact the Monte Carlo results agree completely with MFT at $k \to -\infty$; both indicate that the system freezes into a state with the body diagonals and cube edges long ($s = 1$) and the face diagonals short ($s = -1$). The main negative $k$ disagreement occurs at the peak in $\chi_R$ which is much lower in the MFT approximation than even on a $4^3$ lattice. For $k$ positive the agreement is much poorer. For large $k$ in MFT, $R$ never goes positive, a spurious first order phase transition is predicted ($k \approx 48$) and $<s>$ is off by about a factor of 3.

We now return to the evidence for a second order phase transition at large $\lambda$. The desired signature for critical behavior would be
\[ R \approx R_0 + A|k_c - k|^\delta \]  

(23)

and

\[ \chi_R \approx B|k_c - k|^{\delta^{-1}} \]  

(24)

for \( k \approx k_c \), where \( k_c \) is the critical point for fixed \( \lambda \) and \( \delta \) is the critical exponent characteristic of the transition [9].

Previous work on the full Regge theory in three dimensions at \( \lambda = 1 \) determined that \( R_0 \approx 0 \) and \( \delta = .80 \pm .06 \) [9]. In that theory \( k \) had to approach \( k_c \) from below; the theory was sick for \( k > k_c \). Here we found that the Ising-link model has \( R_0 \neq 0 \) at the peaks in \( \chi_R \). The curvature susceptibility, though, does show behavior expected of a second-order phase transition. Near criticality there is an observed narrowing in the curvature susceptibility and an increase in peak height with increasing system size. This is readily seen in Fig. 6. Following Hamber and Williams [9], the finite-size scaling relation for the peak of the curvature susceptibility is

\[ \ln(\chi_R) \sim c + \frac{\alpha}{\nu} \ln L, \]  

(25)

where \( L \) is the system length and \( \alpha/\nu = d(1 - \delta)/(1 + \delta) \), with \( d = 3 \). Using this relation, the critical exponent was determined from the curvature susceptibility data (Fig. 6) to be \( \delta = .55 \pm .02 \) for the Ising-link model with \( \lambda = 75 \).

We conclude that at least at this one value of \( \lambda \), the Ising-link model does appear to undergo a second order phase transition of the form considered by Hamber and Williams. However the two values of \( \delta \) are statistically incompatible, indicating two distinct universality classes for the two models.

V. DISCUSSION AND SUGGESTIONS FOR FURTHER WORK

In three dimensions we have uncovered critical behavior in the Ising-link model for \( \lambda = 75 \) but not for \( \lambda = 1 \). The region between should be carefully investigated. At \( \lambda = 75 \) the critical behavior takes the form (23) and (24) as found by Hamber and Williams for the full Regge theory in 3-d. But \( R_0 \) was 0 in their model and not in ours. Their model was sick for \( k > k_c \); ours was not. At \( \lambda = 75 \) our critical exponent \( \delta \) is statistically different from that seen by Hamber and Williams at \( \lambda = 1 \) (where we found no phase transition). Does this difference persist at other values of \( \lambda \) or do both models exhibit universality in \( \lambda \)? Is there universality in \( b \) for the Ising-link model? Clearly more work in three dimensions is needed.

The Ising-link model can and should be investigated in four dimensions as well. The mean field approximation is somewhat more difficult but still quite feasible, and it should be
more accurate in four dimensions than in three. Other analytic methods should also be considered. 4-d Monte Carlo computations can still be performed using look-up tables and as a result should be many times faster than for the full Regge theory. If different universal behavior between the two theories persists in four dimensions, more work would be needed to determine which, if either theory is relevant to the universe we live in.

ACKNOWLEDGEMENTS

M.G. is very grateful to H. Hamber for many helpful discussions. This work was supported in part by the National Science Foundation through Grant No. NSF-PHY-9007497, and California State University.

Note added: While writing up this work we received two related preprints by W. Bierl, H. Markum, and J. Riedler [10]. They independently define the Ising-link model and investigate its behavior in two dimensions.

References

[1] T. Regge, Nuovo Cimento 19, 558 (1961).
[2] H.W. Hamber, in Proc. 1984 Les Houches Summer School, Session XLIII, ed. K. Osterwalder and R. Stora (North-Holland, Amsterdam, 1986).
[3] M. Gross, Nucl. Phys. B (Proc. Suppl.) 20, 724 (1991); N. Godfrey and M. Gross, Phys. Rev. D 43, R1749 (1991).
[4] J. Ambjørn, B. Durhuus and T. Jonsson, Mod. Phys. Lett. A 6, 1133 (1991).
[5] M.E. Agishtein and A.A. Migdal, Mod. Phys. Lett. A 6, 1863 (1991).
[6] V.G. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov, Mod. Phys. Lett. A 3, 819 (1988).
[7] M. Gross and H. Hamber, Nucl. Phys. B364, 703 (1991).
[8] A nice treatment is found in G. Parisi, ‘Statistical Field Theory’, Addison Wesley, 1988.
[9] H.W. Hamber, Nucl. Phys. B (Proc. Suppl.) 99A, 1 (1991); H.W. Hamber and R.M. Williams, Phys. Rev. D 47, 510 (1993).
[10] W. Beirl, H. Markum and J. Riedler, hep-lat/9312054 and ibid. 9312055, Dec. 1993.
**Figure Captions**

Fig. 1. A labeled tetrahedron.

Fig. 2. A plot of the indicated quantities versus $k$ for $\lambda = 1$ as calculated in MFT. $v$ is the average volume per site, $<V>/N_0$.

Fig. 3. The ‘phase diagram’ of the 3-d Ising-link model in the MFT approximation. The dashed curve shows the location of the peak in $\chi_R$ and the solid curve is a first order phase transition.

Fig. 4. The ‘phase diagram’ of the 3-d Ising-link model as determined by Monte Carlo simulations. The dashed curve shows the location of the peak in $\chi_R$ which is found to scale with system size at large $\lambda$. The solid curve is $\mathcal{R} = 0$. Lattice sizes up to $16^3$ were used, and error bars are of order the size of the data points.

Fig. 5. Monte Carlo data for $\lambda = 1$ on a $4^3$ lattice. Data was taken every $\Delta k = 1$, and error bars are 0.013 or smaller.

Fig. 6. The peak in $\chi_R$ for lattices of length 4, 8 and 16. The largest statistical errors of the data points are respectively, 0.3, 1.3 and 3.3.