Large deviations for Lévy diffusions in the small noise regime

Pedro Catuogno * André de Oliveira Gomes †

Abstract

This article concerns the large deviations regime and the consequent solution of the Kramers problem for a two-time scale stochastic system driven by a common jump noise signal perturbed in small intensity $\varepsilon > 0$ and with accelerated jumps by intensity $\frac{1}{\varepsilon}$. We establish Freidlin-Wentzell estimates for the slow process of the multiscale system in the small noise limit $\varepsilon \to 0$ using the weak convergence approach to large deviations theory. The core of our proof is the reduction of the large deviations principle to the establishment of a stochastic averaging principle for auxiliary controlled processes. As consequence we solve the first exit time/exit locus problem from a bounded domain containing the stable state of the averaged dynamics for the family of the slow processes in the small noise limit.

Keywords: Large deviations principle; multi-scale stochastic differential equations with jumps; stochastic averaging principle; weak convergence approach; 2010 Mathematical Subject Classification: 60H10; 60F10; 60J75.

1 Introduction

A large deviations principle (LDP for short) is a refinement of the Law of Large Numbers in the sense it encodes a much finer asymptotic analysis concerning the exponential decay of probabilities of unlikely events with respect to a certain parameter and exhibiting the rate of decay in terms of a deterministic functional that is denoted commonly in the literature as good rate function. Historically large deviations theory (LDT) made its first appearance in 1877 [9, 33] in the context of Boltzmann’s studies of the second law of thermodynamics. Other landmark on the field was given by the seminal result by Crâmer in the context of risk probabilities and rapidly this mathematical branch evolved with a diversity of applications and interactions with other fields especially after the groundbreaking contributions from Varadhan [70, 74, 28, 75, 76]. As excellents and exemplary monographs on the field we refer [27, 26, 49]. With physical examples in mind large deviations principles can refer to extreme events such as systems that exchange from one equilibrium state to another and that occur with small probability. We refer to [41, 32, 66] where applications of large deviations regimes to statistical mechanics and to the study of metastable systems are respectively developed. In this work we import several techniques from LDT for Poisson random measures in order to understand asymptotically, as the source of noise vanishes, qualitative features concerning the following
multiscale stochastic system driven by a pure jump noise signal. Fixed $T > 0$ and $\varepsilon > 0$ let

$$
\begin{align*}
    dX_t^\varepsilon &= a(X_t^\varepsilon, Y_t^\varepsilon)dt + \varepsilon \int_{R^d \backslash \{0\}} c(X_t^\varepsilon, z) d\tilde{N}^{\varepsilon}(ds); \\
    dY_t^\varepsilon &= \frac{1}{\varepsilon} f(X_t^\varepsilon, Y_t^\varepsilon)ds + \int_{R^d \backslash \{0\}} h(X_t^\varepsilon, Y_t^\varepsilon, z) d\tilde{N}^{\varepsilon}(dz); \\
    X_0^\varepsilon &= x \in R^d; \quad Y_0^\varepsilon = y \in R^k.
\end{align*}
$$

For every $\varepsilon > 0$ the stochastic process $(X_t^\varepsilon, Y_t^\varepsilon)_{t \in [0, T]}$ takes values in $R^n := R^d \times R^k$ and the driving signal $\tilde{N}^{\varepsilon}$ is a compensated Poisson random measure with intensity $\frac{1}{\varepsilon} \nu \otimes ds$, where $ds$ stands for the Lebesgue measure on the real line and $\nu$ is a Lévy measure on $(R^d \backslash \{0\}, B(R^d \backslash \{0\}))$. We consider $\nu$ possibly with infinite total mass but satisfying an exponential integrability assumption that reads as the big jumps of the underlying Lévy process that drives $Y$ having exponential moments of order 2. The assumptions on the coefficients of (1) and on the measure $\nu$ will be precised with full detail in the following section.

Multi-scale systems as (1) are nowadays very trendy in mathematical and physical disciplines since they succesfully capture phenomena that exhibit different levels of heterogeneity/homogeneity categorized by a scaling parameter. Typical examples are multi-factor stochastic volatility models in Finance \[36, 37\] and dynamics of proxy-data in Climatology \[55\]. Usually such systems are highly complex and difficult to simulate \[61\]. Due to that complexity and the goal of approximating the dynamics of such systems by much simpler ones the idea of stochastic averaging was introduced by Khasminkii \[54\] in different contexts. As examples of works on stochastic averaging principles we mention \[19, 20, 21\] concerning multi-scale stochastic partial differential equations (SPDEs for short) driven by Gaussian signals and \[45, 60, 80, 81\] for multi-scale systems driven by jump diffusions.

Although the averaging principle (3) gives an approximation result for small $\varepsilon > 0$ of the slow variable process $(X_t^\varepsilon)_{t \in [0, T]}$ by the averaged dynamics of $\bar{X}$ nothing is said on the rate of convergence. A large deviations principle provide sharper estimates within the identification of the rate of convergence for (4) in an exponentially small scale $\varepsilon \to 0$ in terms of the good rate function. We refer to \[16, 29, 58, 79\] as examples of stochastic averaging principles under the large deviations regime. The first goal of this work is to derive the large deviations principle (LDP for short) for $(X^\varepsilon)_{\varepsilon > 0}$ given by (1). We show that $(X^\varepsilon)_{\varepsilon > 0}$ satisfies a large deviations principle in $D([0, T]; R^d)$ the space of càdlàg functions endowed with the Skorokhod topology (Section 12 in \[8\]) and the good rate function $J : D([0, T]; R^d) \to [0, \infty]$ given by

$$I(\psi) := \inf_{g \in B} \int_0^T \int_{R^d \backslash \{0\}} (g(s, z) \ln g(s, z) - g(s, z) + 1) \nu(dz)ds.$$
where
\[ S := \bigcup_{M>0} S^M := \bigcup_{M>0} \left\{ g : [0, T] \times \mathbb{R}^d \setminus \{0\} \to [0, \infty) \mid \int_0^T \int_{\mathbb{R}^d \setminus \{0\}} (g(s, z) \ln g(s, z) - g(s, z) + 1)\nu(dz)ds \leq M \right\}, \]

and for every \( x \in \mathbb{R}^d \) and \( g \in S \) the function \( \psi \in \mathcal{D}([0, T]; \mathbb{R}^d) \) solves uniquely the skeleton equation
\[
\psi(t) = x + \int_0^t \tilde{a}(\psi(s))ds + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} c(X^0_s, z)(g(s, z) - 1)\nu(dz)ds, \quad t \in [0, T].
\]

This means that \( J \) has compact sublevel sets in the Skorokhod topology \( \mathcal{D}([0, T]; \mathbb{R}^d) \) and that for every open set \( G \in \mathcal{B}(\mathcal{D}([0, T]; \mathbb{R}^d)) \) and closed set \( F \in \mathcal{B}(\mathcal{D}([0, T]; \mathbb{R}^d)) \) the following holds
\[
\lim_{\varepsilon \to 0} \inf \varepsilon \ln \mathbb{P}(X^\varepsilon \in F) \geq - \inf_{\psi \in F} J(\psi) \quad \text{and} \quad \lim_{\varepsilon \to 0} \sup \varepsilon \ln \mathbb{P}(X^\varepsilon \in G) \leq - \inf_{\psi \in G} J(\psi).
\]

The second goal of this work is, under stricter assumptions on the measure \( \nu \), to use the LDP \( \text{(6)} \) in order to solve the asymptotic behavior \( \varepsilon \to 0 \) of the law and the expectation of the first exit time and location (known as Kramers problem in the literature)
\[
\sigma^\varepsilon(x) := \inf \{ t \geq 0 \mid X^\varepsilon_t \notin D \} \quad \text{and} \quad X^\varepsilon_{\sigma^\varepsilon(x)} \quad \text{respectively,} \quad \text{(7)}
\]

where, for every \( \varepsilon > 0 \), we stress the dependence on the initial condition \( x \in D \) that lives on a neighborhood \( D \subset \mathbb{R}^d \) of 0 which we assume to be a stable state of the averaged dynamical system \( \bar{X} \).

The Kramers problem arose firstly in the context of chemical reaction kinetics \( [2, 34, 57] \). Nowadays this is a classical problem in Probability theory and provides crucial insights in many areas ranging from statistical mechanics, statistics, risk analysis, population dynamics, fluid dynamics to neurology. It was also a driving force in the development of large deviations theory in the small noise limit for Gaussian dynamical systems in many different settings and effects derived such as metastability and stochastic resonance. Classical texts with detailed exposition include \( [5, 6, 10, 11, 18, 42, 44, 73] \) and examples of the recent active research on the field include \( [22, 43, 62, 63, 77] \). The establishment of large deviations principles in the small noise limit and its use to solve the Kramers problem and the metastable behavior of the perturbed dynamical systems by Gaussian signals is nowadays commonly designated by Freidlin-Wentzell theory.

Nevertheless, the literature on large deviations theory and the application to the study of the Kramers problem for Markovian systems with jumps is more fragmented and recent. One reason for the use of a Freidlin-Wentzell theory for dynamical systems perturbed in low intensity by jump signals is the variety of Lévy processes, including processes with heavy tails and the resulting lack of moments. LDPs for certain classes of Lévy noises and Poisson random measures are given in \( [11, 12, 55, 60, 63, 64, 68] \). The first exit time problem for small jump processes starts with the seminal work \( [47] \) for certain classes of Lévy noises and Poisson random measures where \( \mathbb{E} \int_0^T \int_{\mathbb{R}^d \setminus \{0\}} e^{-|z|^\alpha} \nu(dz)ds \) for the explicit Lévy measure \( \nu(dz) = e^{\beta|z|^\beta}dz \) with \( \alpha > 1, \beta \in (0, 1) \).

Our method to prove the large deviations principle for the family of slow variables \( (X^\varepsilon)_{\varepsilon>0} \) in \( \text{(1)} \) relies on the weak convergence approach of Dupuis, Ellis, Budhiraja and collaborators, specifically on the works \( [14, 15] \). The weak convergence approach to LDT builds up on the equivalence in Polish
More precisely the difficult part is to prove directly the following. Fix $M > t \in \mathbb{R}$ where $\bar{\rho}$ in law we show that the family $(X^\varepsilon)_{\varepsilon > 0}$ of SDEs:

$$
X^\varepsilon_t = x + \int_0^t \left( a(X^\varepsilon_s, Y^\varepsilon_s) + \int_{\mathbb{R}^d \setminus \{0\}} c(X^\varepsilon_s, z)(\varphi^\varepsilon(s, z) - 1) \nu(dz) \right) ds + \varepsilon \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} c(X^\varepsilon_s, z) \tilde{N}^\varepsilon \varphi^\varepsilon (ds, dz)
$$

and

$$
Y^\varepsilon_t = y + \frac{1}{\varepsilon} \int_0^t \left( f(X^\varepsilon_s, Y^\varepsilon_s) + \int_{\mathbb{R}^d \setminus \{0\}} h(X^\varepsilon_s, Y^\varepsilon_s, z)(\varphi^\varepsilon(s, z) - 1) \nu(dz) \right) ds + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} h(X^\varepsilon_s, Y^\varepsilon_s, z) \tilde{N}^\varepsilon \varphi^\varepsilon (ds, dz)
$$

where, for any $\varepsilon > 0$, the random measure $\tilde{N}^\varepsilon \varphi^\varepsilon$ is a controlled random measure that under a change of probability measure has the same law of $\tilde{N}^\varepsilon$ under the original probability measure. This will be rigorously stated in Section 4.

Under the following setting, the main task in the derivation of the LDP is to prove that $X^\varepsilon \Rightarrow \bar{X}$ where $\bar{X}$ solves (5) uniquely in $C([0, T]; \mathbb{R}^d)$ for the control $\varphi \in S^M$. In order to prove that convergence in law we show that the family $(X^\varepsilon)_{\varepsilon > 0}$ satisfies a tightened averaging principle, i.e. for every $\delta > 0$ the following holds

$$
\limsup_{\varepsilon \to 0} \mathbb{P} \left( \sup_{t \in [0, T]} |X^\varepsilon(t) - \bar{X}(t)| > \delta \right) = 0,
$$

where $(\bar{X}^\varepsilon)_{\varepsilon > 0}$ is defined for every $\varepsilon > 0$ and $t \in [0, T]$ by

$$
\bar{X}^\varepsilon_t = x + \int_0^t \left( \bar{a}(\bar{X}^\varepsilon_s) + \int_{\mathbb{R}^d \setminus \{0\}} c(\bar{X}^\varepsilon_s, z)(\varphi^\varepsilon(s, z) - 1) \nu(dz) \right) ds + \varepsilon \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} c(\bar{X}^\varepsilon_s, z) \tilde{N}^\varepsilon \varphi^\varepsilon (ds, dz).
$$

This will imply by Shutzky’s theorem (Theorem 4.1 in [5]) that $(X^\varepsilon)_{\varepsilon > 0}$ has the same weak limit of $(\bar{X}^\varepsilon)_{\varepsilon > 0}$. And therefore we are conducted to the (easier) task to show that $X^\varepsilon \Rightarrow \bar{X}$ (since the dynamics of (11) is decoupled from the dynamics of the fast variable of the original stochastic system (4)).

The proof of the tightened controlled averaging principle (10) is inspired on the classical Khasminkii’s technique introduced in [54]. In a nutshell the procedure relies on a discretization of the time interval $[0, T]$ in a finite number of intervals with same length $\Delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ satisfying some growth
conditions that will interplay with the ergodic properties of the averaged dynamics via the construction of auxiliary processes \((\tilde{X}^\varepsilon)^{\varepsilon > 0}\) and \((\hat{X}^\varepsilon)^{\varepsilon > 0}\).

Our main result shows in particular that \((X^\varepsilon)^{\varepsilon > 0}\) obeys the same LDP of \((\tilde{X}^\varepsilon)^{\varepsilon > 0}\) where we define the averaged process \(\tilde{X}^\varepsilon\) for every \(\varepsilon > 0\) and \(t \in [0, T]\) by

\[
\tilde{X}^\varepsilon_t = x + \int_0^t \tilde{a}(\tilde{X}^\varepsilon_s)ds + \varepsilon \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} c(\tilde{X}^\varepsilon_s - z)\tilde{N}^\varepsilon_{s}(ds, dz).
\] (12)

One could firstly derive the LDP for \((\tilde{X}^\varepsilon)^{\varepsilon > 0}\) and secondly show that the families \((X^\varepsilon)^{\varepsilon > 0}\) and \((\tilde{X}^\varepsilon)^{\varepsilon > 0}\) are exponentially equivalent, i.e. for every \(\delta > 0\) we have

\[
\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P} \left( \sup_{0 \leq t \leq T} |X^\varepsilon_t - \tilde{X}^\varepsilon_t| > \delta \right) = -\infty.
\] (13)

This would imply that \((X^\varepsilon)^{\varepsilon > 0}\) obeys the same LDP of \((\tilde{X}^\varepsilon)^{\varepsilon > 0}\) as \(\varepsilon \to 0\). However verifying the exponential equivalence of those families is in general hard. The reasoning employed in this work illustrates the robustness of the weak convergence approach providing a way of reducing the proof of the LDP to the verification of properties concerning continuity and tightness of certain auxiliary processes associated to \((X^\varepsilon)^{\varepsilon > 0}\). Such reduction of complexity in such endeavour can be appreciated immediately by the contrast between the zero scale of the limit (3) with the exponential negligibility demanded in the establishment of the limit (13).

Due to the exponential equivalence of \((X^\varepsilon)^{\varepsilon > 0}\) and \((\tilde{X}^\varepsilon)^{\varepsilon > 0}\) the LDP of \((X^\varepsilon)^{\varepsilon > 0}\) is written with the good rate function of the LDP of \((\tilde{X}^\varepsilon)^{\varepsilon > 0}\) which has important implications. It follows that the LDP of \((X^\varepsilon)^{\varepsilon > 0}\) is given as an optimization problem under the averaged dynamics solved by continuous controlled paths with a nonlocal component due to the pure jump noise. Therefore in the solution of the Kramers problem for \((X^\varepsilon)^{\varepsilon > 0}\) we are therefore allowed to write the potential height for the description of the law and the expected first exit time of \((X^\varepsilon)^{\varepsilon > 0}\) from the domain \(D \subset \mathbb{R}^d\) containing the stable state of the averaged dynamics (1).

The solution of the Kramers problem for \((X^\varepsilon)^{\varepsilon > 0}\) follows as this point analogously to what was done by the authors in the work [24]. Analogously to the classical Freidlin-Wentzell theory we solve the Kramers problem with a pseudo-potential given in terms of the good rate function of the LDP. In the Brownian case, under very mild assumptions on the coefficients of the SDE the respective controlled dynamics exhibits continuity properties that are crucial in the characterization of the first exit times. This differs strongly from the pure jump case. In this context, obtaining a closed form for the rate function is a hard task since the class of minimizers are scalar functions that represent shifts of the compensator of \(\varepsilon \tilde{N}^\varepsilon\) on the nonlocal (possibly singular) component of the underlying controlled dynamics. This is an additional difficulty in the characterization of the first exit time in terms of the pseudo-potential. However, in the case of finite and symmetric jump measures we can solve the first exit time problem with the help of explicit formulas that we obtain for the controls. In other words, on an abstract level the physical intuition remains intact; however, since the control is given as a density w.r.t. the Lévy measure \(\nu\), it is often hard to calculate the energy minimizing paths.

Analogously to the Brownian case [27] [41] we construct for the lower bound of the first exit time a (modified) Markov chain approximation that takes into account the topological particularities of the Skorokhod space on which we have the LDP. Here the symmetry of the measure \(\nu\) plays an essential role since it enables us to derive the lower bound of the probabilities of exit in terms of probabilities of excursions from neighborhoods of the stable state of the deterministic dynamical system. Otherwise we would need to control the trajectories of perturbations from the deterministic dynamical system including the non-vanishing compensator of \(\varepsilon \tilde{N}^\varepsilon\).

The proof of [2] in Theorem [1] follows the lines of the proof for the Brownian case but rests on several auxiliary results which derivation take into account the specific large deviations principle for \((X^\varepsilon)^{\varepsilon > 0}\), particularities of the Skorokhod topology, the reduction of the dynamics of \((X^\varepsilon)^{\varepsilon > 0}\) from
the bounded domain $D$ to estimates on excursions from certain balls contained in the $D$ and the exponential equivalence between $(X^\varepsilon)_\varepsilon>0$ and $(\bar{X}^\varepsilon)_\varepsilon>0$ that permits to describe the potential for the solution of the Kramers problem as the potential associated to the averaged (and simplified) dynamics of $(\bar{X}^\varepsilon)_\varepsilon>0$.

**Notation.** The arrow $\Rightarrow$ means convergence in distribution. Throughout the article we use when convenient the shorthand notation $A(\varepsilon) \lesssim_{\varepsilon} B(\varepsilon)$ to mean that there exist a constant $c > 0$ independent of $\varepsilon > 0$ and $\varepsilon_0 > 0$ such that $A(\varepsilon) \leq cB(\varepsilon)$ for every $\varepsilon < \varepsilon_0$. We write $A(\varepsilon) \asymp_{\varepsilon} B(\varepsilon)$ as $\varepsilon \to 0$ to mean that $A(\varepsilon) \lesssim_{\varepsilon} B(\varepsilon)$ and $B(\varepsilon) \lesssim_{\varepsilon} A(\varepsilon)$ as $\varepsilon \to 0$.

## 2 The multi-scale system and statement of the main results

### 2.1 Notation and the probabilistic setting

Let $T > 0$. Let $\mathbb{M}$ be the space of locally finite measures defined on $([0, T] \times \mathbb{R}^d \setminus \{0\}, \mathcal{B}([0, T] \times \mathbb{R}^d \setminus \{0\}))$ and let us fix $\nu \in \mathbb{M}$. Let $\mathbb{P}$ be the unique probability measure defined on $(\mathbb{M}, \mathcal{B}(\mathbb{M}))$ such that the canonical map $N$ turns out to be a Poisson random measure with intensity $\nu \otimes ds$ where $ds$ stands for the Lebesgue measure on $[0, T]$. For every $\varepsilon > 0$ let $N^\varepsilon$ be the Poisson random measure defined on $(\mathbb{M}, \mathcal{B}(\mathbb{M}), \mathbb{P})$ with intensity $\frac{1}{\varepsilon} \nu(dz) \otimes ds$ and $\bar{N}^\varepsilon$ its compensated version.

In what follows we augment the probability space in order to register not only the instant and the size of the jumps given by the Poisson random measure $N$ but also their intensity. Consider $[0, T] \times \mathbb{R}^d \setminus \{0\} \times [0, \infty)$ and let $\mathbb{M}$ be the space of all locally finite measures defined on the Borel sets of that Cartesian product. There exists a unique probability measure $\bar{\mathbb{P}}$ under which the canonical map

\[ \bar{N} : \mathbb{M} \to \bar{\mathbb{M}} \]

\[ \bar{N}(\bar{m}) = \bar{m} \]

turns out to be a Poisson random measure with intensity given by $ds \otimes \nu \otimes dr$ where $dr$ stands for the Lebesgue measure on $[0, \infty)$. We remark that for every $\varepsilon > 0$ the Poisson random measure $N^\varepsilon$ can be seen as a controlled random measure with respect to $\bar{N}$ in the following way:

\[ N^\varepsilon([0, s] \times A)(\omega) = \int_0^s \int_A \int_0^\infty 1_{[0, \varepsilon]}(r)\bar{N}(ds, dz, dr), \quad s \geq 0; A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}) \]

Given $t \in [0, T]$, let

\[ \mathcal{F}_t := \sigma\left(\bar{N}(0, s] \times A) \mid 0 \leq s \leq t, \quad A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\} \times [0, \infty))\right) \]

Let $\bar{\mathcal{F}} := \{\bar{\mathcal{F}}_t\}_{t \in [0, T]}$ be the completion of $(\mathcal{F}_t)_{t \in [0, T]}$ under $\bar{\mathbb{P}}$ and consider $\bar{\mathcal{P}}$ the predictable $\sigma$-field on $[0, T] \times \bar{\mathbb{M}}$ with respect to the filtration $\{\bar{\mathcal{F}}_t\}_{t \in [0, T]}$ on $(\mathbb{M}, \mathcal{B}(\mathbb{M}), \bar{\mathbb{P}})$.

### 2.2 Hypothesis on the coefficients

Fix $T > 0$, $n = d + k$ with $d, k \in \mathbb{N}$ and let $\nu \in \mathbb{M}$.

For every $T > 0$ and $\varepsilon > 0$ we consider the following system of stochastic differential equations

\[
\begin{align*}
X^\varepsilon_t &= x + \int_0^t a(X^\varepsilon_s, Y^\varepsilon_s)ds + \varepsilon \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} c(X^\varepsilon_s, z)\bar{N}^\varepsilon(ds, dz) \\
Y^\varepsilon_t &= y + \frac{1}{\varepsilon} \int_0^t f(X^\varepsilon_s, Y^\varepsilon_s)ds + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} h(X^\varepsilon_s, Y^\varepsilon_s)\bar{N}^\varepsilon(ds, dz), \quad t \in [0, T].
\end{align*}
\]
In order to guarantee existence and uniqueness of solution for (14) we assume that the coefficients are deterministic measurable functions \(a : \mathbb{R}^d \times \mathbb{R}^n \rightarrow \mathbb{R}^d, \ c : \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{R}^d, \ f : \mathbb{R}^d \times \mathbb{R}^n \rightarrow \mathbb{R}^n\) and \(h : \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{R}^n\) satisfying the following.

**Hypothesis A.**
1. There exists \(L > 0\) such that for every \(x, \bar{x} \in \mathbb{R}^d\) and \(y, \bar{y} \in \mathbb{R}^n\) the following holds
\[
|a(x, y) - a(\bar{x}, \bar{y})| \leq L \left( |x - \bar{x}| + |y - \bar{y}| \right);
\]
\[
\int_{\mathbb{R}^d \setminus \{0\}} |c(x, z) - c(\bar{x}, z)| \nu(dz) \leq L |x - \bar{x}|;
\]
\[
|f(x, y) - f(\bar{x}, \bar{y})| \leq L \left( |x - \bar{x}| + |y - \bar{y}| \right);
\]
\[
\int_{\mathbb{R}^d \setminus \{0\}} |c(x, y, z) - c(\bar{x}, \bar{y}, z)| \nu(dz) \leq L \left( |x - \bar{x}| + |y - \bar{y}| \right);
\]
\[
(15)
\]
2. The functions \(c(0, z)\) and \(h(0, 0, z)\) are in \(L^1(\nu)\).

**Remark 2.1.** Hypothesis (A) implies obviously that the coefficients of (14) have sublinear growth.

**Definition 2.1.** Given \(T > 0, \ \varepsilon > 0, \ x \in \mathbb{R}^d\) and \(y \in \mathbb{R}^k\) we consider the stochastic basis \((\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}), \mathbb{F}, \bar{\mathbb{P}})\). A strong solution of (14) is a stochastic process \((X^\varepsilon_t, Y^\varepsilon_t)_{t \in [0,T]}\) that is \((\bar{\mathcal{F}}_t)_{t \in [0,T]}\)-adapted and that solves (14) \(\bar{\mathbb{P}}\)-a.s.

Given \(T > 0, \ m \in \mathbb{N}\) and \(\mathbb{F} = \{\bar{\mathcal{F}}_t\}_{t \in [0,T]}\) we define the space
\[
S^2(\mathbb{F}; [0,T]; \mathbb{R}^m) := \left\{ \varphi : \bar{\mathbb{M}} \times [0,T] \rightarrow \mathbb{R}^k \ | \ \varphi \text{ is } \bar{\mathbb{F}} \text{ - adapted with càdlàg paths such that} \right\}
\[
E \left[ \sup_{t \in [0,T]} |\varphi(t)|^2 \right] < \infty.
\]

The existence and uniqueness of the solution process \((X^\varepsilon_t, Y^\varepsilon_t)_{t \in [0,T]} \in S^2(\mathbb{F}; [0,T]; \mathbb{R}^d) \times S^2(\mathbb{F}; [0,T]; \mathbb{R}^n)\) of (14) in the sense of Definition 2.1 follows from Lemma V.2 and Theorem V.7 of [69]. This is the content of the following result.

**Theorem 2.1.** Fix \(\nu \in \bar{\mathbb{M}}, \ T, \varepsilon > 0, \ x \in \mathbb{R}^d\) and \(y \in \mathbb{R}^k\). Let us assume that Hypothesis (A) hold. Then there exists a stochastic process
\[
(X^\varepsilon_t, Y^\varepsilon_t)_{t \in [0,T]} \in S^2(\mathbb{F}; [0,T]; \mathbb{R}^d) \times S^2(\mathbb{F}; [0,T]; \mathbb{R}^n)
\]
that solves uniquely (14) in the sense of Definition 2.1.

### 2.3 The averaged dynamics

We make the further boundedness dissipativity assumptions on the coefficients of (14) that yield the existence and uniqueness of solution for the averaged dynamics given by (4).

**Hypothesis B.**
1. The function \(a\) satisfies \(a(0, y) = 0\) for any \(y \in \mathbb{R}^k\) and there exists \(\Lambda > 0\) such that
\[
|h(x, y, z)| \leq \Lambda |z|, \text{ for every } x \in \mathbb{R}^d, y \in \mathbb{R}^k, z \in \mathbb{R}^d \setminus \{0\}.
\]
\[
(16)
\]
2. There exist constants \(\beta_1, \beta_2 > 0\), such that, for any \(x, \bar{x} \in \mathbb{R}^d, y, \bar{y} \times \mathbb{R}^k\) one has
\[
\langle a(x, y) - a(\bar{x}, y), x - \bar{x} \rangle \leq -\beta_1 |x - \bar{x}|^2;
\]
\[
(17)
\]
The proof of the following result concerning the Lipschitz continuity of \( \overline{\mu} \) adaptation of Theorem 11.4.2 in \([53]\) asserts that there exists a constant \( C > 0 \) such that for all \( 0 < \varepsilon < 1 \) we have

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X^\varepsilon(t)|^2 \right] + \sup_{0 \leq t \leq T} \mathbb{E} \left[ |Y^\varepsilon(t)|^2 \right] \leq C_1.
\]

(21)

We consider the equation for the fast variable of (14) whenever the slow component is frozen and given by \( x \in \mathbb{R}^d \) in the regime \( \varepsilon = 1 \), i.e. fix \( y \in \mathbb{R}^k \); for every \( t \geq 0 \) let

\[
Y^{x,y}_t = y + \int_0^t f(x, Y^{x,y}_s) ds + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} h(x, Y^{x,y}_s, z) \tilde{N}_2(ds, dz).
\]

(22)

We assume that Hypotheses \([A13]\) hold. We follow closely \([19, 20]\) in the argumentation below.

Fixed \( x \in \mathbb{R}^d \) we define the transition semigroup on the space \( \mathcal{B}_{0}(\mathbb{R}^k) \) of the bounded measurable functions associated with the jump diffusion defined by the strong solution of (22) by

\[
P^\varepsilon_t f(y) := \mathbb{E}[f(Y^{x,y}_t)], \quad t \geq 0, \quad y \in \mathbb{R}^k.
\]

(23)

In what follows we discuss the existence and uniqueness of an invariant measure for the family of linear operators \( (P^\varepsilon_t)_{t \geq 0} \), i.e. a probability measure \( \mu^\varepsilon \in \mathcal{P}(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k)) \) such that

\[
\int_{\mathbb{R}^k} P^\varepsilon_t f(y) \mu^\varepsilon(dy) = \int_{\mathbb{R}^k} f(y) \mu^\varepsilon(dy), \quad t \geq 0, f \in \mathcal{B}_{0}(\mathbb{R}^k).
\]

(24)

The dissipativity assumption given in \([19]\) yields some \( C > 0 \) such that, for any \( T_0 \geq 0 \), the following bound holds:

\[
\sup_{T \geq T_0} \mathbb{E}[|Y^{x,y}(T)|^2] \leq C e^{-2\beta_1 T} (1 + ||\xi||^2 + |y|^2).
\]

(25)

The estimate \([26]\) implies that the family of the laws of the process \( \{\mathcal{L}(Y^{x,y}(T))\}_{T \geq T_0} \) is tight in \( \mathcal{P}(\mathbb{R}^k; \mathcal{B}(\mathbb{R}^k)) \) when \( T_0 \to \infty \). Prokhorov’s theorem (Section 5 in \([8]\)) implies the existence of a weak limit \( \mu^\varepsilon \) as \( T_0 \to \infty \) and an indirect use of Krylov-Bogliobov’s theorem (such as a straightforward adaptation of Theorem 11.4.2 in \([53]\)) asserts that \( \mu^\varepsilon \) is the unique invariant measure of \((P^\varepsilon_t)_{t \geq 0}\), in the sense of (24). Due to the estimate \([25]\) and the definition of \( \mu^\varepsilon \) in (24), the simple application of monotone convergence shows, as in Lemma 3.4. in \([20]\), that there exists \( C > 0 \) such that

\[
\int_{\mathbb{R}^k} |y|^2 \mu^\varepsilon(dy) \leq C(1 + |x|^2 + |y|^2).
\]

(26)

For any \( x \in \mathbb{R}^d \) we can define the averaged mixing coefficient

\[
\tilde{a}(x) := \int_{\mathbb{R}^k} a(x, y) \mu^\varepsilon(dy).
\]

(27)

The proof of the following result concerning the Lipschitz continuity of \( \tilde{a} \) follows as in the arguments used to prove the stochastic averaging principle in \([81]\).
Proposition 2.2. Fix $T > 0$ and $y \in \mathbb{R}^k$. Let Hypothesis $A$, $B$ hold for some $\nu \in \mathcal{M}$. Then the function $\bar{a}$ defined by (27) is Lipschitz continuous.

Proposition 2.3 ensures that the averaged differential equation with initial condition $x \in \mathbb{R}^d$,
\[
\begin{cases}
\frac{d}{dt}X_t^{0,x} = \bar{a}(X_t^{0,x}), \\
X_{t_0}^{0,x} = x
\end{cases}
\]  
has a unique solution $X_t^{0,x} \in C([0,T];\mathbb{R}^d)$.

The following proposition reads as a strong mixing property of the averaged coefficient $\bar{a}$ given by (27) and it plays a crucial role in the establishment of the large deviations principle for the family $(X^x)_{\epsilon > 0}$. The proof follows is straightforward and we refer the reader to [31].

Proposition 2.3. Fix $T > 0$ and $y \in \mathbb{R}^k$. Let Hypothesis $A$, $B$ hold for some $\nu \in \mathcal{M}$. Then there exists some function $\alpha : [0, \infty) \rightarrow [0, \infty)$ such that $\alpha(T) \rightarrow 0$ as $T \rightarrow \infty$ and satisfying for any $t \in [0,T]$  
\[ 
\mathbb{E}_{\bar{a}} \frac{1}{T} \int_t^{t+T} a(\zeta, Y_s^x,y) ds - \bar{a}(x) \bigg| \leq \alpha(T)(1 + |x|^2 + |y|^2) 
\]  
where the averaged coefficient $\bar{a}$ is defined by (27).

3 The main theorems

3.0.1 The large deviations principle

We make the following assumption on $\nu \in \mathcal{M}$ that is used in the derivation of the large deviations principle for $(X^x)_{\epsilon > 0}$.

Hypothesis C. The measure $\nu \in \mathcal{M}$ is a Lévy measure on $(\mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^d \setminus \{0\}))$, i.e. such that  
\[ 
\int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |z|^2) \nu(dz) < \infty 
\]  
that satisfies  
\[ 
\int_{|z| \geq 1} e^{\alpha |z|^2} \nu(dz) < \infty, \quad \text{for some } \alpha > 0. 
\]  

In order to state the large deviations principle for the family of slow components $(X^x)_{\epsilon > 0}$ given by [14], we fix some notation following mainly [14] and [15]. Fix $T > 0$ and a measurable function $g : [0,T] \times \mathbb{R}^d \setminus \{0\} \rightarrow [0,\infty)$. We define the entropy functional by  
\[ 
\mathcal{E}_T(g) := \int_0^T \int_{\mathbb{R}^d \setminus \{0\}} (g(s,z) \ln g(s,z) - g(s,z) + 1) \nu(dz) ds. 
\]  

For every $M \geq 0$ we define the sublevel sets of the functional $\mathcal{E}_T$ by  
\[ 
\mathcal{S}^M := \left\{ g : [0,T] \times \mathbb{R}^d \setminus \{0\} \rightarrow [0,\infty) \text{ measurable } | \mathcal{E}_T(g) \leq M \right\} 
\]  
and set  
\[ 
\mathcal{S} := \bigcup_{M \geq 0} \mathcal{S}^M. 
\]  

Given $T > 0$, $x \in \mathbb{R}^d$ and $g \in \mathcal{S}$ we consider the controlled integral equation  
\[ 
U^g(t;x) = x + \int_0^t \bar{a}(U^g(s;x)) ds + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} c(U^g(s;x),z)(g(s,z) - 1) \nu(dz) ds, \ t \in [0,T]. 
\]  

It is a standard fact that the equation (33) has a unique solution $U^g \in C([0,T], \mathbb{R}^d)$ and it satisfies the uniform bound  
\[ 
\sup_{t \in [0,T]} \sup_{g \in \mathcal{S}^M} |U^g(t;x)| < \infty \quad \text{for all } M > 0. 
\]
In particular, the map $G^{0,x} : S \to C([0,T], \mathbb{R}^d)$, $g \mapsto G^{0,x}(g) := U^y(\cdot; x)$ is well-defined for any fixed $x \in \mathbb{R}^d$.

For $\varphi \in C([0,T], \mathbb{R}^d)$ we define $S_{\varphi,x} := \{g \in S \mid \varphi = G^{0,x}(g)\}$ and set $J_{x,T} : D([0,T], \mathbb{R}^d) \to [0, \infty]$ to
\[
J_{x,T}(\varphi) := \inf_{g \in S_{\varphi,x}} E_T(g),
\]
with the convention that $\inf \emptyset = \infty$.

**Theorem 3.1.** Let Hypotheses \textbf{A}-\textbf{C} be satisfied for some $\nu \in \mathbb{M}$, $T > 0$, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^k$ fixed. Let $X^{\varepsilon} = (X^{\varepsilon}_t)_{t \in [0,T]}, \varepsilon > 0$, be the slow component of the strong solution of (14) given in Theorem 2.1. Then the family $(X^{\varepsilon})_{\varepsilon > 0}$ satisfies a LDP with the good rate function $J_{x,T}$ given by (28) in the Skorokhod space $D([0,T], \mathbb{R}^d)$. This means that, for any $a \geq 0$ the sublevel set $\{J_{x,T} \leq a\}$ is compact in $D([0,T], \mathbb{R}^d)$ and for any $G \subset D([0,T], \mathbb{R}^d)$ open and $F \subset D([0,T], \mathbb{R}^d)$ closed,
\[
\begin{align*}
\lim \inf_{\varepsilon \to 0} \varepsilon \ln \bar{P}(X^{\varepsilon} \in G) & \geq - \inf_{\varphi \in G} J_{x,T}(\varphi) \text{ and } \\
\lim \sup_{\varepsilon \to 0} \varepsilon \ln \bar{P}(X^{\varepsilon} \in F) & \leq - \inf_{\varphi \in F} J_{x,T}(\varphi).
\end{align*}
\]
We prove Theorem 3.1 in Section 6.

**Comment:** The reader can appreciate in the development of Section 4 the bypass of the usual exponential tightness between the family $(X^{\varepsilon})_{\varepsilon > 0}$ given by (14) and $(X^{\varepsilon})_{\varepsilon > 0}$ given by (12), that would be of much harder verification, with the key establishment of a controlled version of a stochastic averaging principle for the family $(X^{\varepsilon})_{\varepsilon > 0}$ defined in (35). The establishment of such stochastic averaging principle is the key result to prove Theorem 3.1 using the weak convergence approach.

3.0.2 The first exit time problem in the small noise limit

In this subsection we state the second main result of this work concerning the solution of the Kramers problem for $(X^{\varepsilon})_{\varepsilon > 0}$.

**Further assumptions.** We make the additional assumptions on the deterministic dynamical system (28) and on the measure $\nu \in \mathbb{M}$ as follows.

**Hypothesis D.** 1. Let us consider a bounded domain $D \subset \mathbb{R}^d$ with $0 \in D$, $\partial D \in C^1$ and that $a$ is inward-pointing on $\partial D$, that is,
\[
\langle a(z), n(z) \rangle < 0, \quad \text{for all } z \in \partial D,
\]

2. The vector field $\bar{a}$ satisfies the following for some $c_1 > 0$ dissipativity condition
\[
\langle \bar{a}(x) - \bar{a}(\bar{x}), x - \bar{x} \rangle \leq -c_1 |x - \bar{x}|^2, \quad x, \bar{x} \in D. \quad (36)
\]

**Remark 3.1.** 1. Hypothesis \textbf{A} implies that $0 \in \mathbb{R}^d$ is a critical point of (28).

2. The assumption (39) on Hypothesis \textbf{D} implies that $D\bar{a}(x)$ is strictly negative definite for any $x \in D$. In the case (28) is a gradient system given by $\bar{a} = \nabla U$ for some potential $U : \mathbb{R}^d \to [0, \infty)$ this is equivalent to uniform convexity. As a consequence of (36) it follows that there exists $c_2 > 0$ such that $e^{-c_2 t} X^{0,x}_t \to 0$ as $t \to \infty$ for any $x \in D$.

3. Hypothesis \textbf{D} implies that the solution of (28) is positive invariant on $\bar{D}$, that is, for all $x \in \bar{D}$, we have $X^{0,x}_t \in D$ for all $t \geq 0$ and $X^{0,x}_t \to 0$ as $t \to \infty$.  

10
For every \( \varepsilon > 0 \) the process \( \varepsilon \tilde{N}^N \) is a compensated Poisson random measure defined on \( (\bar{M}, \mathcal{B}(\bar{M}), \bar{P}) \) with compensator given by \( ds \otimes \frac{1}{\varepsilon} \nu(dz) \) where \( \nu \in M \) satisfies the following assumption which replaces Hypothesis C.

**Hypothesis E.** The measure \( \nu \in M \) is non-atomic and satisfies the following conditions.

**E.1:** The measure \( \nu \) is a Lévy measure, i.e. \( \int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |z|^2) \nu(dz) < \infty. \)

**E.2:** \( \nu \in M \) satisfies \( \int_{B^1(0)} e^{\Gamma|z|^2} \nu(dz) < \infty \) for some \( \Gamma > 0. \)

**E.3:** \( \nu \in M \) is symmetric.

**Comment:** We refer the reader to [24] for a discussion on the assumptions made on the measure \( \nu \) in Hypothesis [24]

The following hypothesis is a continuity assumption on the controlled differential equation [33] that is essential for solving the first exit time problem.

**Hypothesis F.** For every \( \rho_0 > 0 \) there exist a constant \( M > 0 \) and a non-decreasing function \( \xi : [0, \rho_0] \to \mathbb{R}^+ \) with \( \lim_{\rho \to 0} \xi(\rho) = 0 \) satisfying the following. For all \( x_0, y_0 \in \mathbb{R}^d \) such that \( |x_0 - y_0| \leq \rho_0 \) there exist \( \Phi \in C([0, \xi(\rho_0)], \mathbb{R}^d) \) and \( g \in S^M \) such that \( \Phi(\xi(\rho_0)) = y_0 \) and solving

\[
\Phi(t) = x_0 + \int_0^t \bar{a}(\Phi(s))ds + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} c(\Phi(s), z)(g(s, z) - 1)\nu(dz)ds, \quad t \in [0, \xi(\rho_0)].
\]

**Remark 3.2.** Hypothesis [24] is satisfied for a general class of Lévy measures \( \nu \in M \) that satisfy Hypothesis [24] under the stricter assumptions that

**B.1:** \( \nu \) is a finite measure, \( \nu(\mathbb{R}^d \setminus \{0\}) < \infty. \)

**B.2:** The measure \( \nu \) is absolutely continuous with respect to the Lebesgue measure \( dz \) on the measurable space \( (\mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^d \setminus \{0\})) \) and \( \frac{d\nu}{dz} \neq 0 \) for every \( z \in \mathbb{R}^d \setminus \{0\}. \)

**B.3:** We have that \( \nu \) is symmetric.

We refer the reader to Proposition 15 in [24] where it is explained how the set of assumptions in Hypothesis [24] imply the solution of Kramers problem.

Given \( \varepsilon > 0 \), \( x \in D \) and \( \nu \in M \) satisfying Hypotheses A [13] [15] and E, we define the first exit time of the slow component \( X^{\varepsilon, x} \) of \( \varepsilon\tilde{N}^N \) from \( D \)

\[
\sigma^\varepsilon(x) := \inf\{t \geq 0 \mid X^{\varepsilon, x}_t \notin D\}.
\]

and the first exit location \( X^{\varepsilon, \sigma^\varepsilon(x)} \).

The function \( V \) quantifying the cost of shifting the intensity jump measure by a scalar control \( g \) and steering \( U^g(t; x) \) from its initial position \( x \) to some \( z \in \mathbb{R}^d \) in cheapest time is defined as

\[
V(x, z) := \inf_{T > 0} \left\{ \mathbb{E}(x, T, \varphi) \mid \varphi \in \mathbb{D}([0, T], \mathbb{R}^d) : \varphi(T) = z \right\} \quad \text{for } x, z \in \mathbb{R}^d.
\]

The function \( V(0, z) \) is called the quasi-potential of the stable state 0 with potential height

\[
\tilde{V} := \inf_{z \notin D} V(0, z).
\]

The following result solves the Kramers problem for \( (X^{\varepsilon})_{\varepsilon > 0} \).

**Theorem 3.2.** Let Hypotheses A [13] [15] [16] and E be satisfied. Then \( \tilde{V} < \infty \) and we obtain the following result.
1. For any $x \in D$ and $\delta > 0$, we have
\[
\lim_{\varepsilon \to 0} \mathbb{P}\left( e^{\frac{V}{\varepsilon}} < \sigma^\varepsilon(x) < e^{\frac{V}{\varepsilon} + \delta} \right) = 1.
\] (41)

Furthermore, for all $x \in D$ it follows $\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{E}[\sigma^\varepsilon(x)] = \bar{V}$.

2. For any closed set $F \subset D^c$ satisfying $\inf_{z \in F} V(0, z) > \bar{V}$ and any $x \in D$, we have
\[
\lim_{\varepsilon \to 0} \mathbb{P}\left( X^{\varepsilon,x}_{\sigma^\varepsilon(x)} \in F \right) = 0.
\] (42)

In particular, if $\bar{V}$ is taken by a unique point $z^* \in D^c$, it follows, for any $x \in D$ and $\delta > 0$, that
\[
\lim_{\varepsilon \to 0} \mathbb{P}\left( |X^{\varepsilon,x}_{\sigma^\varepsilon(x)} - z^*| < \delta \right) = 1.
\] (43)

The proof of (1) in Theorem 1 follows as the proof of Theorem 3 in [24]. We present the proof of (2) from Theorem 1 in Subsection 6.2.

**Comment:** We emphasize that the cost functional $V$ and the height potential $\bar{V}$ given by (39) and (40) respectively are written in terms of the good rate function $J$ defined in (38), that is given by the minimization of the entropy functional $\mathcal{E}$ over the controlled paths that solve (36). This is clearly an example of complexity reduction in the solution of the Kramers problem for $(X^\varepsilon)_{\varepsilon > 0}$ using the averaged stochastic dynamics of $(X^\varepsilon)_{\varepsilon > 0}$.

**3.1 Examples**

**Strongly tempered exponentially light Lévy measures.** Hypothesis $E$ covers a wide class of Lévy measures and we point out the following special benchmark cases.

1. Our setting covers the simplest case of finite intensity super-exponentially light jump measures given by $\nu(dz) = e^{-|z|^2}$ for some $\alpha > 1$. For every $\varepsilon > 0$ the corresponding stochastic process $L_{\varepsilon} := \int_0^t \int_{\mathbb{R}} z \tilde{N}^\varepsilon(ds, dz)$, $t \geq 0$ is a compensated compound Poisson process.

2. More generally Hypothesis $E$ covers a class of Lévy measures that mimics the class of strongly tempered exponentially light measures introduced by Rosiński in [71], however, with a Gaussian damping in order to satisfy (39). For the polar coordinate $r = |z|$ and any $A \in \mathcal{B}(\mathbb{R})$ we define
\[
\nu(A) = \int_{\mathbb{R}^{d_{\{0\}}} \setminus \{0\}} \int_0^\infty 1_A(rz) \frac{e^{-r^2}}{r^{\alpha' + 1}} dr R(dz), \quad \alpha' \in (0, 2),
\]
for some measure $R \in \mathcal{M}$ such that $\int_{\mathbb{R}^{d_{\{0\}}} \setminus \{0\}} |z|^\alpha R(dz) < \infty$. We point out that, for every $\varepsilon > 0$, the corresponding Lévy process $(L_{\varepsilon,t})_{t \geq 0}$ differs from the compound Poisson process of the paragraph before not only from the fact that the corresponding jump measure has infinite total mass but also from the fact that although a compound Poisson process with positive jumps has almost surely nondecreasing paths, it does not have paths that are almost surely strictly increasing.

**Invariant measures for the Markov semigroup associated to the fast variable.** For every $\varepsilon > 0$, $T > 0$, $c > 0$, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^k$ let us consider the multi-scale system where the fast variable $(Y^{\varepsilon}_t)_{t \in [0,T]}$ is decoupled from the slow component $(X^\varepsilon_t)_{t \in [0,T]}$ and is given by
\[
\begin{cases}
X^{\varepsilon}_t = x + \int_0^t a(X^{\varepsilon}_s, Y^{\varepsilon}_s) ds + \varepsilon \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} c(X^{\varepsilon}_{\varepsilon - }, z) \tilde{N}^\varepsilon(ds, dz); \\
Y^{\varepsilon}_t = y - \frac{c}{\varepsilon} \int_0^t Y^{\varepsilon}_s ds + L_{\varepsilon}.
\end{cases}
\]
under Hypotheses A/B. Here the stochastic process \( (L^ε_t)_{t \in [0,T]} \) is a pure jump process given by

\[
L^ε_t := \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} z \tilde{N}^\perp(ds, dz)
\]

where \( \tilde{N}^\perp \) is the compensated version of the Poisson random measure defined on \( (\mathcal{M}, \mathcal{B}(\mathcal{M})) \) with intensity given by \( \frac{1}{2} \nu \otimes ds \) for some \( \nu \in \mathcal{M} \) satisfying Hypothesis C. Therefore due to Theorem 17.5 in [72], freezing \( \varepsilon = 1 \), there exists a unique invariant measure \( \mu \) of \( (Y^1_t)_{t \in [0,T]} \) given by its Fourier transform by

\[
\hat{\mu}(dz) = \exp \left( \int_0^\infty \psi(e^{-\varepsilon z}) ds \right)
\]

where \( \psi \) (cf. Corollary 2.5. in [56]) is the Lévy symbol of \( (L^1_t)_{t \in [0,T]} \) given by

\[
\mathbb{E} \left[ \exp(i\xi L^1_t) \right] = e^{-\psi(\xi)} \quad \text{for every } \xi \in \mathbb{R}^d.
\]

## 4 Proof of the large deviations principle

### 4.1 The weak convergence in a nutshell

The weak convergence approach to large deviations theory builds up in the equivalence between the definition of large deviations principle for a family \((X^ε)_{ε > 0}\) defined on some common probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with values in a Polish (complete separable) space \(E\) and the following definition.

**Definition 4.1.** Let \((X^ε)_{ε > 0}\) be a family of \(E\)-valued random variables. The family \((X^ε)_{ε > 0}\) is said to satisfy the Laplace-Varadhan principle in \(E\) with the good rate function \(I : E \rightarrow [0, \infty]\) if for every \(h \in C_b(E)\) the following holds:

\[
\begin{align*}
\limsup_{ε \to 0} & \epsilon \int E \left[ e^{-\frac{1}{\epsilon} L^ε(X^ε)} \right] \leq - \inf_{x \in E} \{ h(x) + I(x) \} \quad \text{and} \\
\liminf_{ε \to 0} & \epsilon \int E \left[ e^{-\frac{1}{\epsilon} L^ε(X^ε)} \right] \geq - \inf_{x \in E} \{ h(x) + I(x) \}.
\end{align*}
\]

For a proof of this equivalence given by Varadhan we refer the reader to [30].

Now, writing for every \(ε > 0\) the shifted measure \(\mathbb{P}^ε := \mathbb{P} \circ (X^ε)^{-1}\) we have by Donsker-Varadhan’s theorem [30] that for every \(ε > 0\) and every \(h \in C_b(E)\) the following variational formula holds

\[
- \epsilon \ln \mathbb{E} \left[ e^{-\frac{1}{\epsilon} h(X^ε)} \right] = \inf_{Q \in \mathcal{P}(E)} \left\{ h(x) Q(dx) + R(Q||\mathbb{P}^ε) \right\}
\]

where \(\mathcal{P}(E)\) is the set of probability measures defined on \(E\) and \(R(Q||\mathbb{P}^ε)\) is the relative entropy of the measure \(Q\) with respect to \(\mathbb{P}^ε\), i.e.

\[
R(Q||\mathbb{P}^ε) := \begin{cases} 
\int_E \ln \left( \frac{dQ}{d\mathbb{P}^ε} \right) d\mathbb{P}^ε & \text{if } Q \ll \mathbb{P}^ε; \\
\infty & \text{otherwise}.
\end{cases}
\]

Due to the mentioned foundational results in order to show a large deviations principle to \((X^ε)_{ε > 0}\) in the Polish space \(E\) one has to show

\[
\inf_{Q \in \mathcal{P}(E)} \left\{ h(x) Q(dx) + R(Q||\mathbb{P}^ε) \right\} \to \inf_{x \in E} \{ h(x) + I(x) \}.
\]

We consider the setup of Subsection 2.1. Let \(\mathcal{A}\) be the class of \(\bar{\mathcal{P}} \otimes \mathcal{B}(\mathbb{R}_d \setminus \{0\}) \otimes B([0,\infty)))\) measurable scalar functions (random controls) \(\varphi : [0,T] \times \mathbb{R}_d \mathcal{M} \rightarrow [0,\infty)\) and let

\[
L_T(\varphi) := \int_0^T \int_{\mathbb{R}_d \setminus \{0\}} (\varphi \ln \varphi - \varphi + 1) \nu(dz) ds.
\]
We define the controlled random measure $N^\varphi$ with respect to $\bar{N}$ under $\bar{\mathbb{P}}$:

$$N^\varphi([0, T] \times A)(\omega) := \int_0^T \int_A \int_0^\infty 1_{[0, \varphi(s, z)(\omega)](r)} \bar{N}(ds, dz, dr) \, dr, \quad A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}). \quad (45)$$

We refer the interested reader to [14]. In virtue of (45) we note that, for every $\varepsilon > 0$, the Poisson random measure $N^\varepsilon$ can be interpreted as a controlled random measure by the scalar control function $\varphi(s, z)(\omega) = \frac{1}{\varepsilon}$. In this particular case the constant scalar control $\frac{1}{\varepsilon}$ is selecting in a non-anticipative way the jumps registered by the Poisson random measure that have intensity at most $\frac{1}{\varepsilon}$.

We let $g : [0, T] \times \mathbb{R}^d \setminus \{0\} \rightarrow [0, \infty)$ be any measurable function (deterministic control) and $M > 0$. We define the controlled random measure $N^\varphi$ with respect to $\bar{\mathbb{P}}$.

We denote $S^M$ the sublevel $M > 0$ for the energy (44), i.e. we say $g \in S^M$ iff $LT(g) \leq M$. Let $\mathcal{S} := \bigcup_{M \geq 0} S^M$. We identify $g \in S^M$ with the measure $\nu^{g} \in \mathcal{M}$ given by

$$\nu^{g}(A) := \int_A g(s, z) \nu(dz) ds, \quad A \in \mathcal{B}(\mathcal{M}). \quad (46)$$

Under the identification $S^M \simeq \{ \nu^{g} \mid g \in S^M \}$ the space $S^M$ turns to be compact for the topology described by the vague convergence. We refer the reader to [15] for more details.

We write for any $M > 0$

$$\mathcal{U}^M := \{ \varphi \in \bar{A} \mid \varphi \in S^M \quad \bar{\mathbb{P}} \text{-a.s.} \}. \quad (47)$$

In [14] the authors prove the following variational formula, for every $F \in M_0(\mathcal{M})$ measurable bounded function defined on $\mathcal{M}$,

$$-\ln \bar{\mathbb{E}}[e^{F(N^\varphi)}] = \inf_{\varphi \in \bar{A}} \bar{\mathbb{E}}[LT(\varphi) + F(N^\varphi)].$$

With the help of the variational formula given above in [14], the authors introduced a sufficient abstract criteria for large deviations principles.

**Hypothesis G.** For any $\varepsilon > 0$ we consider measurable maps $\mathcal{G}^0 : \mathcal{S} \rightarrow \mathbb{D}([0, T]; \mathbb{R}^d)$ and $\mathcal{G}^\varepsilon : \mathcal{M} \rightarrow \mathbb{D}([0, T]; \mathbb{R}^d)$ such that the following two conditions hold.

1. **Continuity statement of the limiting map.** Given $M \geq 0$ let $(g_n)_{n \in \mathbb{N}} \subset S^M$ and $g \in S^M$ such that $\nu^{g_n} \rightarrow \nu^g$ as $n \rightarrow \infty$ in the vague convergence. Then up to a subsequence it holds

$$\mathcal{G}^0(g_n) \rightarrow \mathcal{G}^0(g) \quad \text{as } n \rightarrow \infty.$$

2. **Weak law of large numbers for the random maps of shifted noises.** Given $M \geq 0$ let $(\varphi^{\varepsilon})_{\varepsilon > 0} \subset \mathcal{U}^M$ and $\varphi \in \mathcal{U}^M$ such that $\varphi^{\varepsilon} \Rightarrow \varphi$ as $\varepsilon \rightarrow 0$. Then up to a subsequence the following holds:

$$\mathcal{G}^{\varepsilon} \left( \varepsilon N^\varphi \right) \Rightarrow \mathcal{G}^0(\varphi) \quad \text{as } \varepsilon \rightarrow 0.$$

For every $\varepsilon > 0$ let $Z^{\varepsilon} := \mathcal{G}^{\varepsilon} \left( \varepsilon N^\varphi \right)$. The next theorem states that Hypothesis G is sufficient to retrieve the large deviations principle for the family of random variables $(Z^{\varepsilon})_{\varepsilon > 0}$. We refer the reader to [14].

**Theorem 4.1.** Under the Hypothesis G the family $(Z^{\varepsilon})_{\varepsilon > 0}$ satisfies a large deviations principle in the Skorokhod space $\mathbb{D}([0, T]; \mathbb{R}^d)$ with the good rate function

$$\mathbb{J} : \mathbb{D}([0, T]; \mathbb{R}^d) \rightarrow [0, \infty]$$

$$\mathbb{J}(\eta) := \inf_{\mathcal{G}^0(\eta) = \eta} LT(g)$$

$$= \inf_{\mathcal{G}^0(\eta) = \eta} \int_0^T \int_{\mathbb{R}^d \setminus \{0\}} (g(s, z) \ln g(s, z) - g(s, z) + 1) \nu(dz) ds.$$
4.2 Verifying the abstract sufficient criteria.

Let us fix $T > 0$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^k$. We assume Hypotheses \( A \) \( B \) and \( C \) to hold. For every \( \varepsilon > 0 \) let \((X^\varepsilon_{t,x,y}, Y^\varepsilon_{t,x,y})_{t \in [0,T]}\) be the unique strong solution of (44) in the sense of Definition 2.1 given by Theorem 2.1. We drop the dependence on \( x \) and \( y \). For any \( \varepsilon > 0 \), Yamada-Watanabe’s theorem ensures the existence of a measurable map \( G^\varepsilon : \mathbb{M} \to \mathcal{D}([0,T];\mathbb{R}^d) \) such that

\[
X^\varepsilon := G^\varepsilon(\varepsilon N^\varepsilon).
\]

The proof of Theorem 3.1 consists in checking the conditions (1) and (2) of Hypothesis \( C \) for \( (G^\varepsilon)_{\varepsilon > 0} \) and \( G^0 : S \to C([0,T];\mathbb{R}^d) \), \( G^0(g) = U^g \), with \( U^g \in C([0,T];\mathbb{R}^d) \) defined by the skeleton equation (33). Hence Theorem 3.1 allows us to conclude.

4.2.1 The skeleton equations and the compactness condition

For any \( x \in \mathbb{R}^d \) and \( g \in S \) let us denote by \( U^g = U^{n,x} \in C([0,T];\mathbb{R}^d) \) the unique solution of (33). By definition we have

\[
G^0(g) = U^g.
\]

Proposition 4.1. For every \( M < \infty \) one has that the set

\[
\mathbb{K}_M := \left\{ G^0(g) \mid g \in S^M \right\}
\]

is compact in \( C([0,T];\mathbb{R}^d) \).

The proof is straightforward and we omit it. For details we refer the reader to Proposition 11 in [24].

Remark 4.1. Proposition 4.1 can be rewritten as follows. Fix \( 0 \leq M < \infty \). Let \((g_n)_{n \in \mathbb{N}} \subset S^M \) such that \( g_n \to g \) in the topology of the vague convergence as \( n \to \infty \) under the identification \( S^M \approx \{ \nu^g \mid g \in S^M \} \) given above with \( \nu^g \) defined in (45) for any \( g \in S \). Therefore

\[
G^0(g_n) \to G^0(g) \quad \text{as } n \to \infty \quad \text{in the uniform topology}.
\]

This form of restating Proposition 4.1 corresponds to the verification of the first condition in Hypothesis \( C \) for \( G^0 \).

4.2.2 Strategy of the proof of the second condition in Hypothesis \( C \)

In order to apply Theorem 1.1 and conclude the large deviations principle given in Theorem 3.1 we proceed verifying the second condition in Hypothesis \( C \) for \( G^0 \) and the family \( \{G^\varepsilon : \mathbb{M} \to \mathcal{D}([0,T];\mathbb{R}^d)\}_{\varepsilon > 0} \). For every \( M > 0 \) recall the random sublevel set \( U^M \) given by (47) and for every \( \varepsilon > 0 \) let \( \varphi^\varepsilon \in U^M \). Set \( \tilde{\varphi}^\varepsilon = \frac{\varphi^\varepsilon}{\varphi} \). The definition of \( \tilde{\varphi}^\varepsilon \) makes sense since one has that \( \tilde{\varphi} \)-a.s. \( \varphi^\varepsilon \in \mathbb{A}_b \) holds. For any \( t \in [0,T] \) we define the \( \mathbb{F} \)-martingale

\[
\mathcal{E}(\tilde{\varphi}^\varepsilon)(t) := \exp \left( \int_0^t \int_{\mathbb{R}^2 \setminus \{0\}} \int_0^{\tilde{\varphi}^\varepsilon(s,z)N(ds,dz,dr)} \ln \tilde{\varphi}^\varepsilon(s,z)N(ds,dz,dr) \right)
\]

\[
+ \int_0^t \int_{\mathbb{R}^2 \setminus \{0\}} \int_0^{\tilde{\varphi}^\varepsilon(s,z)+1} ds \nu(dz,dr).
\]

Girsanov’s theorem stated in the form of Theorem III.3.24 of [52] ensures that \( \mathcal{E}(\varphi^\varepsilon)(t) \) \( t \in [0,T] \) is an \( \mathbb{F} \)-martingale. Hence the probability measures defined on \( (\mathbb{M},\mathcal{B}(\mathbb{M})) \) by

\[
\mathbb{Q}^\varepsilon_T(G) := \int_G \mathcal{E}(\varphi^\varepsilon)(T)d\tilde{\mathbb{P}}, \quad \text{for all } G \in \mathcal{B}(\mathbb{M})
\]

15
are absolutely continuous with respect to \( \bar{P} \). Under \( Q^\varepsilon \), the stochastic process \( \varepsilon N^\varepsilon \) is a random measure with the same law of \( N^\varepsilon \) under \( \bar{P} \). We recall that
\[
N^\varepsilon((0,t] \times U) := \int_0^t \int_U 1_{[0, \varepsilon^2]}(r) \bar{N}(ds,dz), \quad t \geq 0, U \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).
\]
For any \((x,y) \in \mathbb{R}^d \times \mathbb{R}^k\), \( M > 0\), \( \varepsilon > 0\) and \( \varphi \in \mathcal{U}^M\) we define the slow controlled process \((X^\varepsilon(t))_{t \in [0,T]}\) and the fast controlled process \((\tilde{Q}^\varepsilon(t))_{t \in [0,T]}\) given as the strong solutions with respect to \( \bar{P} \) (since \( Q^\varepsilon \ll \bar{P} \)) of the following stochastic differential system of equations
\[
\begin{align*}
\dot{X}^\varepsilon_t &= x + \frac{1}{\varepsilon} \int_0^t \left( a(X^\varepsilon_s, Y^\varepsilon_s) + \int_{\mathbb{R}^d \setminus \{0\}} c(X^\varepsilon_s, z)(\varphi^\varepsilon(s, z) - 1) \nu(dz) \right) ds + \varepsilon \int_0^t c(X^\varepsilon_{s-}, z) \bar{N}^\varepsilon(ds,dz) \\
\dot{Y}^\varepsilon_t &= y + \frac{1}{\varepsilon} \int_0^t \left( f(X^\varepsilon_s, Y^\varepsilon_s) + \int_{\mathbb{R}^d \setminus \{0\}} h(X^\varepsilon_s, Y^\varepsilon_s, z)(\varphi^\varepsilon(s, z) - 1) \nu(dz) \right) ds + \int_0^t h(X^\varepsilon_{s-}, Y^\varepsilon_{s-}, z) \bar{N}^\varepsilon(ds,dz).
\end{align*}
\]
(49)
For every \( T > 0\), \( x \in \mathbb{R}^d\), \( M > 0\), \( \varepsilon > 0\) and \( \varphi \in \mathcal{U}^M\), we define \((\bar{X}^\varepsilon(t))_{t \in [0,T]}\) the fast averaged controlled process as the strong solution under \( \bar{P} \) of the controlled stochastic differential equation
\[
\bar{X}^\varepsilon_t = x + \int_0^t \left( \bar{a}(X^\varepsilon_s) + \int_{\mathbb{R}^d \setminus \{0\}} c(X^\varepsilon_s, z)(\varphi^\varepsilon(s, z) - 1) \right) ds + \varepsilon \int_0^t c(X^\varepsilon_{s-}, z) \bar{N}^\varepsilon(ds,dz) \tag{50}
\]
For every \( \varepsilon > 0\) and \( M > 0\) let \( (\varphi^\varepsilon)_{\varepsilon > 0} \subset \mathcal{U}^M\) such that \( \varphi^\varepsilon \Rightarrow \varphi \) as \( \varepsilon \to 0\) in the vague convergence topology in \( S^M\). The conclusion in the second statement in Hypothesis \( \mathbb{G}\) for \( (G^\varepsilon)_{\varepsilon > 0}\) and \( G^0\) reads as \( \bar{X}^\varepsilon \Rightarrow \bar{X}\), as \( \varepsilon \to 0\) in law where \( \bar{X} \in C([0,T]; \mathbb{R}^d)\) solves uniquely
\[
\bar{X}^\varphi(t) = \int_0^t \bar{a}(\bar{X}(s)) ds + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} c(\bar{X}(s), z)(\varphi(s, z) - 1) \nu(dz) ds \tag{51}
\]
In order to prove that \( X^\varepsilon \Rightarrow \bar{X}\), as \( \varepsilon \to 0\) we proceed as follows.

1. This step passes through two intermediary tasks. Firstly one shows that the laws of \((\bar{X}^\varepsilon)_{\varepsilon > 0}\) are tight in \( \mathcal{P}(C([0,T]; \mathbb{R}^d))\) (since compact sets in the topology generated by the uniform convergence are also compact sets in the Skorokhod topology). Then it follows that there exists \( \bar{X} \in C([0,T]; \mathbb{R}^d)\) such that \( \bar{X}^\varepsilon \Rightarrow \bar{X}\) as \( \varepsilon \to 0\). Passing to the pointwise limit in the equation \((51)\) satisfied by \( \bar{X}^\varepsilon\) and due to the uniqueness of solution of \((51)\) we conclude that \( \bar{X} = \bar{X}^\varphi\).

2. We prove the following strong (controlled) averaging principle:
\[
\lim_{\varepsilon \to 0} \bar{P}\left( \sup_{t \in [0,T]} |X^\varepsilon(t) - \bar{X}^\varphi(t)| > \delta \right) = 0, \quad \text{for any } \delta > 0. \tag{52}
\]
From the limit above and Theorem 4.1. in [8], commonly known as Slutsky’s theorem, we can identify \( \bar{X} \) as the weak limit of \((X^\varepsilon)_{\varepsilon}\) as \( \varepsilon \to 0\).

The first point in the strategy announced above is proved in the following proposition. The proof follows analogous as the proof of Proposition 12 in [24].

**Proposition 4.2.** Given \( M > 0\) and \( \varepsilon > 0\) let \( \varphi \in \mathcal{U}^M\) such that \( \varphi^\varepsilon \Rightarrow \varphi \) as \( \varepsilon \to 0\) in the vague convergence in \( S^M\). Therefore \( G^0(\varphi) = \bar{X}^\varphi \) is a limit point in law of \( G^\varepsilon(\varepsilon N^\varepsilon) \) in \( D([0,T]; \mathbb{R}^d)\).

### 4.2.3 The controlled averaging principle

In order to conclude one has resumed to prove \((52)\). For that purpose we use a localization technique based in the following useful result. The proof uses a Bernstein-type inequality for càdlàg martingales given in [31] and the proof follows analogously as the proof of Proposition 9 in [24].
Proposition 4.4. Let the hypotheses of Theorem 3.3 be satisfied. For every $M > 0$, $(\varphi^\varepsilon)_{\varepsilon > 0} \subset U^M$, any function $R : (0, 1) \to (0, \infty)$ such that $R(\varepsilon) \to \infty$ as $\varepsilon R^2(\varepsilon) \to 0$ as $\varepsilon \to 0$, $x \in \mathbb{R}^d$ and $T > 0$ we have that there exists some $\varepsilon_0 \in (0, 1)$ and $C > 0$ such that for every $\varepsilon < \varepsilon_0$ the following holds

$$\mathbb{P} \left( \sup_{t \in [0, T]} |X_t^\varepsilon| \vee |\hat{X}_t^\varepsilon| > R(\varepsilon) \right) \leq 2e^{-\frac{1}{2} R(\varepsilon)} + C \varepsilon R(\varepsilon).$$

For every $\varepsilon > 0$ and any $R : (0, 1) \to (0, \infty)$ satisfying the limits $R(\varepsilon) \to \infty$ and $\varepsilon R^2(\varepsilon) \to 0$ as $\varepsilon \to 0$ such as in the statement of Proposition 4.3 let us define the $\mathbb{F}$-stopping time

$$\bar{\tau}_{R(\varepsilon)} := \inf \{ t \geq 0 \mid X_t^\varepsilon \notin B_{R(\varepsilon)}(0) \} \wedge \inf \{ t \geq 0 \mid \hat{X}_t^\varepsilon \notin B_{R(\varepsilon)}(0) \} \wedge T. \quad (54)$$

The following a-priori bounds are used in the sequel for the proof of the controlled averaging principle (52). The proof is straightforward and follows from several applications of the Ito’s formula and the Burkholder-Dqvis-Gundy’s inequalities.

Proposition 4.4. Let $M > 0$. Fix a function $R : (0, 1) \to [0, \infty)$ satisfying the assumptions of Proposition 4.3 and for every $\varepsilon > 0$ let $\bar{\tau}_{R(\varepsilon)}$ be defined in (54). Under the assumptions of Theorem 3.3 we have that there exists $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$ implying

$$\Gamma_1(M) := \sup_{0 < \varepsilon < \varepsilon_0} \sup_{\varphi \in U^M} \mathbb{E} \left[ \sup_{0 \leq t \leq \bar{\tau}_{R(\varepsilon)}} |X_t^\varepsilon|^2 \right] + \sup_{0 \leq t \leq T} \mathbb{E} \left[ |Y_t^\varepsilon|^2 I_{\bar{\tau}_{R(\varepsilon)}} \right] < \infty \quad (55)$$

and

$$\Gamma_2(M) := \sup_{0 < \varepsilon < \varepsilon_0} \sup_{\varphi \in U^M} \mathbb{E} \left[ \sup_{0 \leq t \leq \bar{\tau}_{R(\varepsilon)}} |\hat{X}_t^\varepsilon|^2 \right] < \infty. \quad (56)$$

We follow the technique introduced in [54] with the the required modifications to our setting in order to deal with the nonlocal components of the auxiliary processes $(X^\varepsilon)_{\varepsilon > 0}$ and $(Y^\varepsilon)_{\varepsilon > 0}$ given by (49).

Let $[0, T]$ be divided into intervals of the same length parametrized for every $\varepsilon > 0$

$$\Delta = \Delta(\varepsilon) := \varepsilon \gamma |\ln \varepsilon|^p, \quad \text{for some } \gamma \in \left( 0, \frac{1}{2} \right) \text{ and } p > 0. \quad (57)$$

We note the following convergences

$$\Delta(\varepsilon) \to 0; \quad \text{and} \quad \frac{\Delta(\varepsilon)}{\varepsilon} \to \infty \quad \text{as} \quad \varepsilon \to 0. \quad (58)$$

For any $t \in [0, T]$ we denote $t_{\Delta} := \left[ \frac{t}{\Delta} \right] \Delta$.

We construct the auxiliary processes $(\hat{Y}^\varepsilon(t))_{t \in [0, T]}$ and $(\hat{X}^\varepsilon(t))_{t \in [0, T]}$ by means of the following equations: for any $t \in [0, T]$ let

$$\begin{align*}
\hat{Y}^\varepsilon_t &= Y^\varepsilon_{t_{\Delta}} + \frac{1}{\varepsilon} \int_{t_{\Delta}}^t \left( f(X^\varepsilon_{s_{\Delta}}, \hat{Y}^\varepsilon_s) + \int_{\mathbb{R}^d \setminus \{0\}} h(X^\varepsilon_{s_{\Delta}}, \hat{Y}^\varepsilon_s, z)(\varphi^\varepsilon(s, z) - 1) \nu(dz) \right) ds \\
&+ \int_{t_{\Delta}}^t \int_{\mathbb{R}^d \setminus \{0\}} h(X^\varepsilon_{s_{\Delta}}, \hat{Y}^\varepsilon_s, z) \tilde{N}^{R^2(\varepsilon)}(ds, dz)
\end{align*}$$

and

$$\begin{align*}
\hat{X}^\varepsilon_t &= x + \int_0^t \left( a(X^\varepsilon_{s_{\Delta}}, Y^\varepsilon_s) + \int_{\mathbb{R}^d \setminus \{0\}} c(X^\varepsilon_{s_{\Delta}}, z)(\varphi^\varepsilon(s, z) - 1) \nu(dz) \right) ds \\
&+ \varepsilon \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} c(X^\varepsilon_{s_{\Delta}}, z) \tilde{N}^{R^2(\varepsilon)}(ds, dz).
\end{align*} \quad (59)$$

and

$$\begin{align*}
\hat{X}^\varepsilon_t &= x + \int_0^t \left( a(X^\varepsilon_{s_{\Delta}}, Y^\varepsilon_s) + \int_{\mathbb{R}^d \setminus \{0\}} c(X^\varepsilon_{s_{\Delta}}, z)(\varphi^\varepsilon(s, z) - 1) \nu(dz) \right) ds \\
&+ \varepsilon \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} c(X^\varepsilon_{s_{\Delta}}, z) \tilde{N}^{R^2(\varepsilon)}(ds, dz).
\end{align*} \quad (60)$$

17
Comment: The modifications from the continous Gaussian regime to the pure jump noise regime in (59) and (60) are natural with the respective frozen variables done in the same way. We enunciate the following list of results that are used in the sequel to prove (52). The proofs are modifications from the arguments used in (54) to the Poissonian case.

Lemma 4.1. For every \( \varepsilon > 0 \) let \( R(\varepsilon) > 0 \) and \( \Delta(\varepsilon) > 0 \) fixed as above. Then for every \( \delta > 0 \) the following

\[
\bar{\mathbb{P}} \left( \sup_{0 \leq t \leq T} |X^\varepsilon_t - \hat{X}^\varepsilon_t| > \delta \right) \leq \varepsilon \Xi(\varepsilon) \to 0, \quad \text{as } \varepsilon \to 0,
\]

(61)

Lemma 4.2. For every \( \varepsilon > 0 \) let \( R(\varepsilon) \) fixed as in Proposition 4.3 and \( \Delta(\varepsilon) \) given by (57). Then the following convergence holds:

\[
\sup_{0 \leq t \leq T} \bar{\mathbb{E}} \left[ |Y^\varepsilon(t) - \hat{Y}^\varepsilon(t)| I_{\{T < T^\varepsilon_{R(\varepsilon)}\}} \right] \leq \varepsilon C(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0.
\]

(62)

for some \( C(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \) uniformly in the initial condition \((x, y) \in \mathbb{R}^d \times \mathbb{R}^k\).

The previous a-priori bounds (61) and (62) imply the following.

Proposition 4.5. For any \( \delta > 0 \) we have

\[
\limsup_{\varepsilon \to 0} \bar{\mathbb{P}} \left( \sup_{0 \leq t \leq T} |X^\varepsilon_t - \hat{X}^\varepsilon_t| > \frac{\delta}{2} \right) = 0.
\]

(63)

Proof. The definitions of \( (Y^\varepsilon(t))_{t \in [0, T]} \) and \( (\hat{Y}^\varepsilon(t))_{t \in [0, T]} \) given in (49) and (57) respectively combined with Hypothesis 3 yield for every \( \varepsilon > 0 \) and \( t \in [0, T] \) that

\[
\hat{X}^\varepsilon_t - X^\varepsilon_t = \int_0^t \left( a(X^\varepsilon_{s\Delta}, \hat{Y}^\varepsilon_s) - a(X^\varepsilon_s, Y^\varepsilon_s) \right) ds
\]

\[
\leq L \int_0^t |X^\varepsilon_{s\Delta} - X^\varepsilon_s| ds + L \int_0^t |\hat{Y}^\varepsilon_s - Y^\varepsilon_s| |ds.
\]

The asymptotic behaviour (58) of \( \Delta(\varepsilon) > 0 \) fixed in (57) combined with Lemma 4.1 of Lemma 4.2 yield some \( C = C(L, T) > 0 \) such that

\[
\bar{\mathbb{P}} \left( \sup_{0 \leq t \leq T} |\hat{X}^\varepsilon_t - X^\varepsilon_t| > \frac{\delta}{2L} \right) \leq \bar{\mathbb{P}} \left( \int_0^{T\wedge T^\varepsilon_{R(\varepsilon)}} |a(X^\varepsilon_{s\Delta}, \hat{Y}^\varepsilon_s) - a(X^\varepsilon_s, Y^\varepsilon_s)| ds > \frac{\delta}{2L} \right)
\]

\[
+ \bar{\mathbb{P}} \left( \int_0^T |\hat{Y}^\varepsilon_s - Y^\varepsilon_s|^2 1_{\{T < T^\varepsilon_{R(\varepsilon)}\}} ds > \frac{\delta}{2L} \right)
\]

\[
\leq \varepsilon \Xi(\varepsilon) + \frac{2}{\delta} \int_0^T \bar{\mathbb{E}} \left[ |\hat{Y}^\varepsilon_s - Y^\varepsilon_s|^2 1_{\{T < T^\varepsilon_{R(\varepsilon)}\}} ds \right]
\]

\[
\leq \varepsilon \Xi(\varepsilon) + C(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0.
\]

This finishes the proof of (63).

\[\Box\]

Proposition 4.6. For any \( \delta > 0 \) we have

\[
\limsup_{\varepsilon \to 0} \bar{\mathbb{P}} \left( \sup_{0 \leq t \leq T} |\hat{X}^\varepsilon_t - X^\varepsilon_t| > \frac{\delta}{2} \right) = 0.
\]

(64)

Proof. For every \( \varepsilon > 0, t \in [0, T], \xi \in \mathbb{R}^d \) and \( \varphi^\varepsilon \in \mathcal{U}^M \), we define the function

\[
b^\varepsilon(\xi)(t) := \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} c(\xi, z)(\varphi^\varepsilon(s, z) - 1) \nu(dz) ds.
\]
The definitions of \((\mathcal{A}_t(t))_{t \in [0,T]}\) and \((\hat{\mathcal{A}}_t(t))_{t \in [0,T]}\) given in [49] and respectively in [60] combined with the definition of \(b^c\) given above imply for every \(t \in [0,T]\) and \(\varepsilon > 0\) the following identity \(\bar{P}\)-a.s. on the event \(\{T < \hat{\tau}_{\mathcal{R}(\varepsilon)}\}\):

\[
\hat{X}^\varepsilon_t - X^\varepsilon_t = \int_0^t \left( b^c(\hat{X}^\varepsilon_s) - b^c(X^\varepsilon_s) \right) ds \\
+ \int_0^t \left( a(X^\varepsilon_{s+}, \hat{Y}^\varepsilon_s) - \bar{a}(X^\varepsilon_s) \right) ds \\
+ \int_0^t \left( \bar{a}(X^\varepsilon_s) - \bar{a}(\hat{X}^\varepsilon_s) \right) ds + \int_0^t \left( \bar{a}(\hat{X}^\varepsilon_s) - \bar{a}(\bar{X}^\varepsilon_s) \right) ds \\
+ \varepsilon \int_0^t \int_{\mathbb{R}} \left( c(\hat{X}^\varepsilon_s, z) - c(\bar{X}^\varepsilon_s, z) \right) \hat{N}(\cdot, \varepsilon) (ds, dz). \tag{65}
\]

Hypothesis [22] Proposition [22] and [65] yield some constant \(C = C(L, T) > 0\) such that on the event \(\{T < \hat{\tau}_{\mathcal{R}(\varepsilon)}\}\) we have \(\bar{P}\)-a.s.

\[
\sup_{0 \leq s \leq t} |\hat{X}^\varepsilon_s - X^\varepsilon_s|^2 \leq C \left( \sup_{0 \leq u \leq s} |\hat{X}^\varepsilon_u - X^\varepsilon_u|^2 ds + \sup_{0 \leq s \leq t} \left| \int_0^s \left( a(X^\varepsilon_{u+}, \hat{Y}^\varepsilon_u) - \bar{a}(X^\varepsilon_u) \right) du \right|^2 \right. \\
+ \sup_{t \in [0,T]} |J^\varepsilon(t)|^2 \left| 1_{\{T < \hat{\tau}_{\mathcal{R}(\varepsilon)}\}} \right) \right),
\]

where for any \(\varepsilon > 0\) we write

\[
J^\varepsilon(t) := \varepsilon \int_0^t \int_{\mathbb{R}} \left( c(\hat{X}^\varepsilon_{s+}, z) - c(\bar{X}^\varepsilon_{s+}, z) \right) \hat{N}(\cdot, \varepsilon) (ds, dz).
\]

Gronwall’s lemma implies for any \(\varepsilon > 0\) that

\[
\sup_{-T \leq t \leq T} |X^\varepsilon_t - X^\varepsilon_t|^2 \| 1_{\{T < \hat{\tau}_{\mathcal{R}(\varepsilon)}\}} \leq e^{CT} \left( \sup_{0 \leq s \leq t} \left| \int_0^s \left( a(X^\varepsilon_{u+}, Y^\varepsilon_u) - \bar{a}(X^\varepsilon_u) \right) du \right|^2 \right) 1_{\{T < \hat{\tau}_{\mathcal{R}(\varepsilon)}\}} + \sup_{0 \leq t \leq T} |J^\varepsilon(t)|^2. \tag{66}
\]

The estimate [60] yields for any \(\delta > 0\)

\[
\bar{P} \left( \sup_{0 \leq t \leq \hat{\tau}_{\mathcal{R}(\varepsilon)}} |\hat{X}^\varepsilon_t - X^\varepsilon_t| > \frac{\delta}{2} \right) \leq \bar{P} \left( \sup_{0 \leq s \leq t} \left| \int_0^s \left( a(X^\varepsilon_{u+}, Y^\varepsilon_u) - \bar{a}(X^\varepsilon_u) \right) du \right| \right] 1_{\{T < \hat{\tau}_{\mathcal{R}(\varepsilon)}\}} > \frac{\delta^2 e^{-CT}}{8} \right)
\]

\[
\bar{P} \left( \sup_{0 \leq t \leq \hat{\tau}_{\mathcal{R}(\varepsilon)}} |J^\varepsilon(t)|^2 > \frac{\delta^2 e^{-CT}}{8} \right).
\]

Burkholder-Davis-Gundy’s inequalities and Lemma 8 in [21] imply that there exists some constant \(C_1 = C_1(\delta, \mathcal{C}_1, \Gamma_1, \Gamma_2, M) > 0\), where \(\Gamma_1, \Gamma_2 > 0\) are given by [55] and [50] of Proposition 4.4 that may change from line to line, such that

\[
\bar{P} \left( \sup_{0 \leq t \leq T \wedge \hat{\tau}_{\mathcal{R}(\varepsilon)}} |J^\varepsilon(t)|^2 > \frac{\delta^2 e^{-CT}}{8} \right) \leq \frac{8e^{-CT}}{\delta^2} \bar{P} \left( \sup_{0 \leq t \leq T \wedge \hat{\tau}_{\mathcal{R}(\varepsilon)}} |J^\varepsilon(t)|^2 \right) \leq \frac{\varepsilon}{C_1 \sup_{g \in \mathcal{S}_M^{\mathcal{A}}} \int_0^T \int_{\mathbb{R} \setminus \{0\}} |z|^2 g(s, z) \nu(dz) ds \leq C\varepsilon \to 0. \tag{68}
\]

We estimate now the first term in the right-hand-side of [67]. For every \(\varepsilon > 0\) and \(t \in [0, T]\) we write \(\bar{P}\)-a.s. on the event \(\{T < \hat{\tau}_{\mathcal{R}(\varepsilon)}\}\)

\[
\int_0^t \left( a(X^\varepsilon_{s+}, Y^\varepsilon_s) - \bar{a}(X^\varepsilon_s) \right) ds = \sum_{k=0}^{\left\lfloor \frac{t}{\Delta} \right\rfloor - 1} \int_{k\Delta}^{(k+1)\Delta} \left( a(X^\varepsilon_{s+}, Y^\varepsilon_s) - \bar{a}(X^\varepsilon_{s+}) \right) ds
\]

19
It follows from (69) that
\[
\mathbb{P}\left( \sup_{0 \leq s \leq t} \left| \int_0^s \left( a(X_{\epsilon s}^\epsilon, Y_{\epsilon s}^\epsilon) - \tilde{a}(X_{\epsilon s}^\epsilon) \right) du \right|^2 \mathbf{1}_{\{T < \tau_{K(\epsilon)}\}} \right) + \frac{\delta^2 e^{-CT}}{8} \leq \mathbb{P}\left( \sup_{0 \leq t \leq T} |I_1^\epsilon(t)| \mathbf{1}_{\{T < \tau_{K(\epsilon)}\}} > \frac{\delta e^{-CT}}{2\sqrt{2}} \right) + \mathbb{P}\left( \sup_{0 \leq t \leq T} |I_2^\epsilon(t)| \mathbf{1}_{\{T < \tau_{K(\epsilon)}\}} > \frac{\delta e^{-CT}}{2\sqrt{2}} \right) + \mathbb{P}\left( \sup_{0 \leq t \leq T} |I_3^\epsilon(t)| \mathbf{1}_{\{T < \tau_{K(\epsilon)}\}} > \frac{\delta a(\varepsilon) e^{-CT}}{2\sqrt{2}} \right).
\]

**Estimating** \( I_2^\epsilon \). We observe that for any \( \varepsilon > 0 \)
\[
I_2^\epsilon = \int_0^{t_{\Delta}} \left( \tilde{a}(X_{\epsilon s}^\epsilon) - \tilde{a}(X_{\epsilon s}^\epsilon) \right) ds.
\]

Proposition 2.2 and Lemma 4.4 imply for some \( C_2 = C(T) > 0 \), any \( \delta > 0 \) and \( \varepsilon > 0 \) small enough that
\[
\mathbb{P}\left( \sup_{t \in [0, T]} |I_2^\epsilon(t)| \mathbf{1}_{\{T < \tau_{K(\epsilon)}\}} > \frac{\delta a(\varepsilon) e^{-CT}}{2\sqrt{2}} \right) \leq C_2 \mathbb{E} \left[ \left( \sup_{0 \leq t \leq \tau_{K(\epsilon)}} \left| \int_0^t \left( a(X_{\epsilon s}^\epsilon, Y_{\epsilon s}^\epsilon) - \tilde{a}(X_{\epsilon s}^\epsilon) \right) ds \right|^2 \right) \mathbf{1}_{\{T < \tau_{K(\epsilon)}\}} \right] \leq C_2 \mathbb{E} \left[ \int_0^T \left( 1 + |X_{\epsilon s}^\epsilon|^2 + |X_{\epsilon s}^\epsilon|_\infty^2 + |Y_{\epsilon s}^\epsilon|^2 \right) \mathbf{1}_{\{T < \tau_{K(\epsilon)}\}} ds \right] \lesssim \varepsilon \Delta(\varepsilon) \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.
\]

**Estimating** \( I_3^\epsilon \). Hypothesis \( \Lambda \), Proposition 2.2 and Proposition 4.4 yield some constant \( C_3 = C_3(L, \Gamma_1(M)) > 0 \) that may change from line to line such that, for every \( \varepsilon > 0 \) small enough and any \( \delta > 0 \), one has
\[
\mathbb{P}\left( \sup_{t \in [0, T]} |I_3^\epsilon(t)| \mathbf{1}_{\{T < \tau_{K(\epsilon)}\}} > \frac{\delta a(\varepsilon) e^{-CT}}{2\sqrt{2}} \right) \leq C_3 \mathbb{E} \left[ \left( \sup_{0 \leq t \leq \tau_{K(\epsilon)}} \left| \int_0^t \left( a(X_{\epsilon s}^\epsilon, Y_{\epsilon s}^\epsilon) - \tilde{a}(X_{\epsilon s}^\epsilon) \right) ds \right|^2 \right) \mathbf{1}_{\{T < \tau_{K(\epsilon)}\}} \right] \leq C_3 \mathbb{E} \left[ \int_0^T \left( 1 + |X_{\epsilon s}^\epsilon|^2 + |X_{\epsilon s}^\epsilon|_\infty^2 + |Y_{\epsilon s}^\epsilon|^2 \right) \mathbf{1}_{\{T < \tau_{K(\epsilon)}\}} ds \right] \lesssim \varepsilon \Delta(\varepsilon) \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0,
\]
due to (58).

**Estimating** \( I_1^\epsilon \). We construct a new process \( Z := Y^\epsilon(X_{\epsilon K(\epsilon)}^\epsilon, Y^\epsilon(k\Delta)) \) where the notation that is displayed here stresses out that the process is the fast variable process \( Y^\epsilon \) with frozen slow component \( X_{\epsilon K(\epsilon)}^\epsilon \) and initial condition \( Y^\epsilon(k\Delta) \). It is a classical fact in the course of the Khasminskii’s technique employed in [54] for the proof of the strong averaging principle that for every \( s \in [0, \Delta] \) we have
\[
(X_{\epsilon s}^\epsilon, Y_{\epsilon s}^\epsilon, Y^\epsilon(k\Delta)) =^d \left( X_{\epsilon K(\epsilon)}^\epsilon, Y^\epsilon(X_{\epsilon K(\epsilon)}^\epsilon, Y^\epsilon(k\Delta)) \left( \frac{s}{\varepsilon} \right) \right).
\]

We may assume in addition that the fabricated noises above are independent of \( X_{\epsilon K(\epsilon)}^\epsilon \) and \( Y^\epsilon(k\Delta) \). For the proof of the statements above we refer the reader to Section 5 in [81].
Hence Proposition 4.3 together with the Markov property of \((X^\varepsilon_t, Y^\varepsilon(t))_{t \in [0,T]}\) implies for every \(k = 0, \ldots, \lfloor \varepsilon \rfloor\) the following:

\[
\mathbb{E}\left[\int_{k\Delta}^{(k+1)\Delta} \left(a(X^\varepsilon_{k\Delta}, \hat{Y}^\varepsilon(s)) - \bar{a}(X^\varepsilon_{k\Delta})\right) ds\right] \leq \Delta \mathbb{E}\left[\int_{\Delta}^{\varepsilon} \left(a(X^\varepsilon_\zeta, Z(\zeta)) - \bar{a}(X^\varepsilon_\zeta)\right) d\zeta\right]
\]

\[
= \Delta \mathbb{E}\left[\int_0^{\frac{\varepsilon}{\Delta}} a(\zeta, Z^\varepsilon(\zeta)) - \bar{a}(\zeta) \left| (\zeta, y) = (X^\varepsilon_{k\Delta}, Y^\varepsilon(k\Delta))\right|\right]
\]

\[
\leq \Delta \alpha \left(\frac{\Delta}{\varepsilon}\right) \left(1 + \mathbb{E}\|X^\varepsilon_{k\Delta}\| + \mathbb{E}\|Y^\varepsilon(k\Delta)\|\right). \tag{73}
\]

Proposition 2.3, Proposition 4.4, (58) and (73) yield, for any \(\delta > 0\) and \(\varepsilon > 0\) sufficiently small, that

\[
\mathbb{P}\left(\sup_{0 \leq t \leq T} |I^\varepsilon_t| 1_{\{T < \tau^\varepsilon_{\mathcal{E}(\varepsilon)}\}} > \frac{\delta a(\varepsilon)e^{-CT}}{2\sqrt{2}}\right) \leq \varepsilon \mathbb{E}\left[\sup_{0 \leq t \leq T} |I^\varepsilon_t(t)|^2 1_{\{T < \tau^\varepsilon_{\mathcal{E}(\varepsilon)}\}}\right]
\]

\[
\leq \varepsilon \sum_{k=0}^{\lfloor \frac{\varepsilon}{\Delta} \rfloor} \mathbb{E}\left[\int_{k\Delta}^{(k+1)\Delta} \left(a(X^\varepsilon_{k\Delta}, \hat{Y}^\varepsilon_s) - \bar{a}(X^\varepsilon_{k\Delta})\right) 1_{\{T < \tau^\varepsilon_{\mathcal{E}(\varepsilon)}\}} ds\right]^2
\]

\[
\leq \varepsilon \Delta(\varepsilon) \alpha \left(\frac{\Delta(\varepsilon)}{\varepsilon}\right) \to 0 \text{ as } \varepsilon \to 0. \tag{74}
\]

The convergence above follows from the choice of the parametrization \(\Delta = \Delta(\varepsilon)\) fixed in \((57)\) and \(\alpha\) constructed in Proposition 2.3.

**Theorem 4.2.** Let the hypotheses of Theorem 3.1 to hold. Then we have

\[
\limsup_{\varepsilon \to 0} \mathbb{P}\left(\sup_{0 \leq t \leq \tau^\varepsilon_{\mathcal{E}(\varepsilon)}} |X^\varepsilon_t - \hat{X}^\varepsilon_t| > \delta\right) = 0.
\]

**Proof.** For any \(\varepsilon > 0\) fix \(\mathcal{E}(\varepsilon) > 0\) such as in Proposition 4.3 and recall the definition of \(\tau^\varepsilon_{\mathcal{E}(\varepsilon)}\) in \((53)\).

For any \(\delta > 0\) we have

\[
\limsup_{\varepsilon \to 0} \mathbb{P}\left(\sup_{0 \leq t \leq T} |X^\varepsilon(t) - \hat{X}^\varepsilon(t)| > \delta\right) \leq \limsup_{\varepsilon \to 0} \mathbb{P}\left(\sup_{0 \leq t \leq \tau^\varepsilon_{\mathcal{E}(\varepsilon)}} |X^\varepsilon_t - \hat{X}^\varepsilon_t| > \delta\right)
\]

\[
+ \limsup_{\varepsilon \to 0} \mathbb{P}\left(\sup_{0 \leq t \leq \tau^\varepsilon_{\mathcal{E}(\varepsilon)}} |\hat{X}^\varepsilon_t - \hat{X}^\varepsilon_t| > \delta\right)
\]

\[
+ \limsup_{\varepsilon \to 0} \mathbb{P}\left(\tau^\varepsilon_{\mathcal{E}(\varepsilon)} \leq T\right) = 0,
\]

due to Proposition 4.3, Proposition 4.4 and Proposition 4.6.

**5 Proof of Theorem 3.1**

We recall the collection of measurable maps \((G^\varepsilon)_{\varepsilon > 0}\) introduced in \((58)\) and \(G^0\) defined by means of the skeleton equation \((53)\). We note that Proposition 4.1 reads as the Condition 1 of Hypothesis \(\mathcal{C}\) for \((G^\varepsilon)_{\varepsilon > 0}\) and \(G^0\). Proposition 4.2 combined with Theorem 4.2 yield, due to Slutsky’s theorem, that Condition 2 of Hypothesis \(\mathcal{C}\) is verified for \((G^\varepsilon)_{\varepsilon > 0}\) and \(G^0\). Hence, the result follows from Theorem 4.1.
6 The Kramers problem - the exit locus

6.1 Auxiliary results

In the sequel we use the continuity of the LDP of \((X^{\varepsilon,x})_{\varepsilon>0}\) with respect to the initial condition \(x \in \mathbb{R}^d\) that follows directly from Theorem 4.4 in [65].

**Proposition 6.1.** Given \(T > 0\) and \(x \in D\) let \(F \subset \mathbb{D}([0,T], \mathbb{R}^d)\) be closed and \(G \subset \mathbb{D}([0,T], \mathbb{R}^d)\) open with respect to the Skorokhod topology. Then we have

\[
\limsup_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}(X^{\varepsilon,x} \in F) \leq -\inf_{f \in F} \mathbb{J}_{x,T}(f),
\]

\[
\liminf_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}(X^{\varepsilon,x} \in G) \geq -\inf_{g \in G} \mathbb{J}_{x,T}(g).
\]

**Lemma 6.2.** For any \(X \in \mathbb{D}([0,T], \mathbb{R}^d)\) the uniform LDP for \((X^{\varepsilon,x})_{\varepsilon>0}\) implies the uniform LDP for \((X^{\varepsilon,x})_{\varepsilon>0}\) when the initial state \(x \in K\) for \(K \subset D\) a closed (and bounded) set. The proof is virtually the same as the one given in the Brownian case and we omit it. We refer the reader to Corollary 5.6.15 in [27].

**Corollary 6.1.** Let \(T > 0, K \subset D\) be compact, \(F \subset \mathbb{D}([0,T], \mathbb{R}^d)\) closed and \(G \subset \mathbb{D}([0,T], \mathbb{R}^d)\) open with respect to the \(J_1\) topology and \(x \in D\). Then it follows

\[
\limsup_{\varepsilon \to 0} \sup_{y \in K} \varepsilon \ln \mathbb{P}(X^{\varepsilon,x} \in F) \leq -\inf_{y \in K, f \in F} \mathbb{J}_{y,T}(f),
\]

\[
\liminf_{\varepsilon \to 0} \sup_{y \in K} \varepsilon \ln \mathbb{P}(X^{\varepsilon,x} \in G) \geq -\inf_{y \in K, g \in G} \mathbb{J}_{y,T}(g).
\]

In the sequel this result is applied to the first exit time problem of \(X^{\varepsilon,x}\) from \(D\).

**Lemma 6.1.** For any \(x \in D\) and \(\rho > 0\) such that \(\bar{B}_\rho(0) \subset D\) we have

\[
\lim_{\varepsilon \to 0} \mathbb{P}(X^{\varepsilon,x}_0 \in \bar{B}_\rho(0)) = 1.
\]

For a proof of this lemma we refer the reader to Lemma 20 in [24].

For a given \(\rho > 0\) such that \(\bar{B}_\rho(0) \subset D\) we define

\[
\vartheta^\rho(x) := \inf\{t \geq 0 \mid |X_t^\varepsilon - x| \leq \rho \text{ or } X_t^\varepsilon \in D^c\}.
\]

**Lemma 6.2.** For any \(\rho > 0\) and \(c > 0\) there exists \(\xi(\rho) > 0\) such that \(t \in [0,\xi(\rho)]\) implies

\[
\limsup_{\varepsilon \to 0} \varepsilon \ln \sup_{x \in D} \mathbb{P}(\sup_{t \in [0,\xi(\rho)]} |X_t^\varepsilon - x| \geq \rho) < -c.
\]

For a proof of the next result we refer the reader Lemma 22 in [24].

**Lemma 6.3.** Let \(F \subset D^c\) closed. Then

\[
\lim_{\rho \to 0} \limsup_{\varepsilon \to 0} \varepsilon \ln \sup_{x \in \bar{B}_\rho(0)} \mathbb{P}(X^\varepsilon_{\vartheta^\rho} \in F) \leq -\inf_{z \in F} V(0, z).
\]

For a proof of the previous lemma we refer the reader Lemma 23 in [24].

6.2 Proof of (2) in Theorem 1

**Proof.** 1. Let \(F \subset D^c\) be a given closed set such that

\[
V_F := \inf_{z \in F} V(0, z) > \bar{V}.
\]
The case \( \inf_{z \in F} V(0, z) = \infty \) is covered by some \( V_F \in (\tilde{V}, \infty) \) instead.

Let \( x \in D \). For some \( \rho > 0 \) let \( \vartheta^\rho_L(x) \) be the \( \tilde{P} \)-stopping time given by (75).

By definition of \( \sigma^\varepsilon \) and \( \vartheta^\rho_L(x) \) one has the elementary estimate

\[
\tilde{P}\left(X_{\sigma^\varepsilon(x)} \in F\right) \leq \tilde{P}\left(X_{\vartheta^\rho_L(x)} \notin B_\rho(0)\right) + \sup_{y \in B_\rho(0)} \tilde{P}\left(X_{\vartheta^\rho_L(y)} \in F\right).
\]

The first term of the r.h.s. of the last estimate converges to 0 as \( \varepsilon \to 0 \) by Lemma 6.3. It remains to prove that

\[
\lim_{\varepsilon \to 0} \sup_{y \in B_\rho(0)} \tilde{P}\left(X_{\vartheta^\rho_L(y)} \in F\right) = 0.
\]

Fix now \( \eta \in (0, \frac{V_0 - \tilde{V}}{2}) \) and according to Lemma 6.3 we choose \( \rho, \varepsilon_0 > 0 \) such that

\[
\sup_{|z| \leq 2\rho} \tilde{P}\left(X_{\vartheta^\rho_L(z)} \in F\right) \leq e^{-\frac{V_0 - \tilde{V}}{2}}, \quad \text{for every } \varepsilon < \varepsilon_0. \tag{79}
\]

We now consider the following auxiliary stopping times constructed as follows. Due to Hypothesis 10 there is \( \rho' > 0 \) such that \( B_{\rho'}(0) \subset D \) and \( \sup_{x \in B_{\rho'}(0)} \langle b(x), n(x) \rangle < 0 \) and let \( \rho > 0 \) such that \( B_{\rho}(0) \subset B_{\rho'}(0) \). For every \( x \in D \) we define recursively

\[
\zeta^\varepsilon_k := 0 \quad \text{and for any } k \in \mathbb{N} \quad \vartheta^\rho_L k, \rho := \inf\{ t \geq \zeta^\varepsilon_k \mid X_t^\varepsilon \in \tilde{B}_\rho(0) \cup D^c \},
\]

\[
\zeta^\varepsilon_{k+1} := \begin{cases} \infty, & \text{if } X_{\vartheta^\rho_L k, \rho}^{\varepsilon} \in D^c, \\ \inf\{ t \geq \vartheta^\rho_L k, \rho \mid X_t^\varepsilon \in \tilde{B}_\rho(0) \}, & \text{if } X_{\vartheta^\rho_L k, \rho}^{\varepsilon} \in \tilde{B}_\rho(0). \end{cases} \tag{80}
\]

**Comment:** This modified Markov chain approximation from the Gaussian case takes into account the topological particularities of the Skorokhod space on which we have the LDP. In addition, the effect of the \( \frac{1}{\varepsilon} \) acceleration of the jump intensity enters as follows. The asymptotically exponentially negligible error estimates concerning the stickyness of the diffusion to its initial value, which in the classical Brownian case are valid for time intervals of order 1, in our case only hold for time intervals of order \( \varepsilon \). We account for a tilting between entering the balls of radius \( \rho \) and \( \rho' \) centered on the stable state of the deterministic dynamical system and the excursions outside the domain \( D \).

By construction \((\zeta^\varepsilon_k)_{k \in \mathbb{N}}\) and \((\vartheta^\rho_L k, \rho)_{k \in \mathbb{N}}\) we have \( \tilde{P} \)-a.s. for all \( k \in \mathbb{N} \)

\[
\zeta^\varepsilon_k \leq \vartheta^\rho_L k, \rho \leq \zeta^\varepsilon_{k+1} \leq \vartheta^\rho_L k+1, \rho.
\]

Since \( \rho' > \rho \) we have that \( \zeta^\varepsilon_{k+1} > \vartheta^\rho_L k, \rho \) if \( X_{\vartheta^\rho_L k, \rho}^{\varepsilon} \in \tilde{B}_\rho(0) \). Hence \((\vartheta^\rho_L k, \rho)_{k \in \mathbb{N}}\) is an increasing sequence of \((F_t)_{t \geq 0}\) stopping times. Since the process \((X_t^{\varepsilon \text{, } x})_{t \geq 0}\) has the strong Markov property with respect to \((F_t)_{t \geq 0}\) it follows that \((X_{\vartheta^\rho_L k, \rho}^{\varepsilon \text{, } x})_{k \in \mathbb{N}}\) is a Markov chain and \( \sigma^\varepsilon \) = \( \vartheta^\rho_L \) for some (random) \( \ell \in \mathbb{N} \) with the convention \( X_{\vartheta^\rho_L \ell, \rho}^{\varepsilon \text{, } x} := X_{\sigma^\varepsilon(x)}^\varepsilon \) if \( \vartheta^\rho_L \ell, \rho = \infty \).

From the strong Markov property of \((X_t)_{t \in [0, T]}\) it follows for all \( T > 0 \) and \( k \in \mathbb{N} \) that

\[
\sup_{z \in D} \tilde{P} \left( \vartheta^\rho_L k, \rho \leq kT \right) \leq k \sup_{z \in D} \tilde{P} \left( \sup_{0 \leq t \leq T} |X_t^\varepsilon - z| \geq \rho \right).
\]
In what follows we apply Lemma \[6.2\] with the choice of \( \rho > 0 \) done as before and \( c := V_F - \eta \). One can choose \( T := T(\rho, V_F, \eta) < \infty \) such that

\[
\sup_{z \in D} \mathbb{P} \left( \sup_{0 \leq t \leq T} |X^\varepsilon_t - z| \geq \rho \right) \leq e^{-\frac{V_F - V}{\varepsilon}}, \quad \varepsilon \leq \varepsilon_0.
\]

Consequently there exists \( T' < \infty \) such that for every \( \varepsilon > \varepsilon_0 \) with \( \varepsilon_0 > 0 \) small enough one has

\[
\sup_{z \in D} \mathbb{P} \left( \partial_{\rho,k}^z \leq KT' \right) \leq ke^{-\frac{V_F - \eta}{\varepsilon}}.
\]  

Due to the strong Markov property one observes that

\[
\{ \sigma^\varepsilon(y) > \vartheta_{m-1,\rho} \} \in \mathcal{F}_{\vartheta_{m-1,\rho}} \subset \mathcal{F}_{\varepsilon_0}.
\]

and \( \partial_{m,\rho}^y = \zeta_m + \vartheta_{\rho}(X_{m}^\varepsilon y) \). Therefore for any \( y \in D \) it follows that

\[
\bar{\mathbb{P}} \left( X_{\sigma^\varepsilon}^\varepsilon(y) \in F \right) = \int \int \bar{\mathbb{P}} \left( X_{\sigma^\varepsilon}^\varepsilon(y) \in F \right) d\mathbb{P}
\]

\[
\leq \sup_{|z| \leq 2\rho} \bar{\mathbb{P}} \left( \sigma^\varepsilon(y) > \vartheta_{m-1,\rho} \right) \psi_{\vartheta_{m-1,\rho}}(z) \psi_{\vartheta_{m-1,\rho}} \psi_{\vartheta_{k,\rho}}(z) \psi_{\vartheta_{k,\rho}} + 2ke^{-\frac{V_F - \eta}{\varepsilon}} + \sup_{|z| \leq 2\rho} \bar{\mathbb{P}} \left( \sigma^\varepsilon(y) > \vartheta_{m-1,\rho} \right) \psi_{\vartheta_{m-1,\rho}}(z) \psi_{\vartheta_{k,\rho}}(z) \psi_{\vartheta_{k,\rho}}.
\]

For all \( y \in B_{\rho}(0), k \in \mathbb{N} \) and \( \varepsilon > \varepsilon_0 \), where \( \varepsilon_0 > 0 \) is fixed as earlier, due to the fact \( \sigma^\varepsilon(y) > \vartheta_{0}^y = 0 \), the definitions of \( \partial_{k,\rho}(x) \), \( \sigma^\varepsilon \) and the strong Markov property of \( X^\varepsilon \) yield

\[
\bar{\mathbb{P}} \left( X_{\sigma^\varepsilon}^\varepsilon(y) \in F \right) \leq \sup_{|z| \leq 2\rho} \bar{\mathbb{P}} \left( \sigma^\varepsilon(y) > \vartheta_{k,\rho} \right) \psi_{\vartheta_{k,\rho}}(z) \psi_{\vartheta_{k,\rho}} + 2ke^{-\frac{V_F - \eta}{\varepsilon}} + \sup_{|z| \leq 2\rho} \bar{\mathbb{P}} \left( \sigma^\varepsilon(y) > \vartheta_{m-1,\rho} \right) \psi_{\vartheta_{m-1,\rho}}(z) \psi_{\vartheta_{m-1,\rho}}(z) \psi_{\vartheta_{k,\rho}}(z) \psi_{\vartheta_{k,\rho}}.
\]

In the last estimate above we combined \([4]\) with \([8]\).

The first statement in Theorem \([1]\) implies that for every \( \delta > 0 \) and \( \varepsilon < \varepsilon_0 \) with \( \varepsilon_0 > 0 \) small enough we have

\[
\sup_{x \in D} \mathbb{E} \left[ \sigma^\varepsilon(x) \right] \leq e^{-\frac{\varepsilon}{\delta}}.
\]

We parametrize \( k = k_\varepsilon = \left( e^{\frac{\varepsilon + 2\eta}{\varepsilon}} \right) \). By the previous choice of \( \eta > 0 \) one has \( \tilde{V} - V_F + 3\eta < 0 \). Hence \([82]\) and \([83]\) yield

\[
\limsup_{\varepsilon \to 0} \sup_{y \in B_{\rho}(0)} \bar{\mathbb{P}} \left( X_{\sigma^\varepsilon}^\varepsilon(y) \in F \right) \leq \limsup_{\varepsilon \to 0} \left( 2k(\varepsilon)e^{-\frac{\varepsilon}{\delta}} + \frac{1}{k(\varepsilon)T} e^{\frac{\varepsilon + 3\eta}{\delta}} \right) \leq \limsup_{\varepsilon \to 0} \left( 2e^{-\frac{V_F - V}{\varepsilon}} + \frac{1}{T} e^{-\frac{V_F - V}{\varepsilon}} \right) = 0,
\]

which concludes the proof of the first statement.

The second statement follows taking the closed set

\[
F := \left\{ z \in D^\varepsilon \mid |X_{\sigma^\varepsilon(z)}^\varepsilon - z| < \delta \right\}.
\]

24
The previous proof implies immediately that
\[
\lim_{\epsilon \to 0} \tilde{\mathbb{P}} \left( |X_{\sigma(\epsilon)^*} - z^*| < \delta \right) = 1.
\]

\[\square\]

7 Appendix: the large deviations regime for a pure-jump process

In this subsection we prove that rescaling in a inverse way the size and the intensity of the jumps of a pure jump process produces a large deviations principle in the small noise limit. For any \( \epsilon > 0 \) and \( T > 0 \) let
\[
L_\epsilon^T := \int_0^T \int_{\mathbb{R}^d \setminus \{0\}} \varepsilon z N^0_\epsilon(ds, dz), \hspace{1em} t \in [0, T]
\]
be the pure jump process driven by the Poisson random measure \( N^0_\epsilon \) defined on \( (\Omega, \mathcal{F}, \mathbb{P}) \) with intensity measure \( \frac{1}{\epsilon^d} \nu \otimes ds \) where \( ds \) stands for the Lebesgue measure on the positive real line and \( \nu \) is the Lévy measure on \( (\mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^d \setminus \{0\})) \) given by
\[
\nu(dz) = \frac{1}{|z|^{d+\beta}} e^{-|z|^\alpha} dz, \hspace{1em} \alpha > 1, \beta \in [0, 1).
\]
The following proposition implies (by Theorem 1.3.7 in [30]) that there exists a subsequence \( \epsilon(n) \to 0 \) as \( n \to \infty \) such that the family \( (L_\epsilon^{\varepsilon(n)})_{\varepsilon(n) > 0} \) satisfies a large deviations principle along for some good rate function.

**Proposition 7.1.** For any \( M > 0 \) we have
\[
\lim_{\epsilon \to 0} \epsilon \ln \mathbb{P} \left( L_\epsilon^T > M \right) = -\infty. \tag{84}
\]

**Proof.** For every \( \epsilon > 0 \) and \( T > 0 \) we write
\[
L_\epsilon^T = I_\epsilon^T + J_\epsilon^T
\]
\[
:= \int_0^T \int_{0<|z|<1} \varepsilon z N^0_\epsilon(ds, dz) + \int_0^T \int_{|z|\geq 1} \varepsilon z N^0_\epsilon(ds, dz).
\]
The stochastic process \( (J_\epsilon^T)_{t \in [0,T]} \) that reads as the big jumps of \( (L_\epsilon^T)_{t \in [0,T]} \) is a compound Poisson process. We represent
\[
J_\epsilon^T = \varepsilon \sum_{k=1}^{\infty} |W_k|_{\{T_k^\varepsilon \leq T\}}
\]
where the sequence of jumps \( (W_k)_{k \in \mathbb{N}} \) is i.i.d. with law \( \frac{\nu}{\beta} \) where we define \( \nu(A) := \nu(A \cap |z| \geq 1) \) and \( \beta = \nu(B_1^\varepsilon) < \infty \). The jump times \( (T_k^\varepsilon)_{k \in \mathbb{N}} \) are defined recursively as
\[
T_0^\varepsilon := 0
\]
\[
T_k^\varepsilon := \inf \{ s > T_{k-1}^\varepsilon \mid |\Delta_s L_\varepsilon| > 1 \} \hspace{1em} k \geq 1.
\]
The waiting times \( (\tau_k^\varepsilon)_{k \in \mathbb{N}} \) are defined for any \( \epsilon > 0 \) and \( k \in \mathbb{N} \) by
\[
\tau_k^\varepsilon := T_k^\varepsilon - T_{k-1}^\varepsilon \sim EXP \left( \frac{\beta}{\varepsilon} \right).
\]

25
For every $\varepsilon > 0$ let $(N^\varepsilon_t)_{t \in [0,T]}$ be the Poisson clock with intensity $\frac{\beta}{\varepsilon}$. For every $\varepsilon > 0$, $T > 0$ and $M > 0$ we have
\[
P(J^\varepsilon_T > M) = \mathbb{P} \left( \sum_{k=1}^{N^\varepsilon_T} |W_k| 1\{T_k \leq T\} > M \right)
= \mathbb{P} \left( \sum_{k=1}^{N^\varepsilon_T} |W_k| > \frac{M}{\varepsilon} \right)
\leq \sum_{n=1}^{\infty} \sum_{k=1}^{n} \mathbb{P} \left( |W_k| > \frac{M}{\varepsilon} \right) e^{-\frac{\beta}{\varepsilon}T} \frac{1}{n!} \left( \frac{\beta T}{n} \right)^n.
\] (85)

We estimate the deviation of one single jump, for any $u \geq 0$, by
\[
P(|W_1| > u) = \frac{1}{\beta} \int B_u (dz) \leq \frac{1}{\beta} \int B_{u \wedge |z| \geq 1} e^{-|z|^\alpha} dz
\leq c_d \int_u^\infty e^{-x^\alpha} x^{d-1} dx
\leq \frac{c_d}{\alpha} \left( \frac{d}{\alpha} u^\alpha \right)
\]
for some normalizing constant $c_d > 0$ and where the Euler’s Gamma function $\Gamma$ is defined by
\[
\Gamma(s, y) = \int_y^\infty z^{s-1} e^{-z} ds, \quad s, y \in \mathbb{R}.
\]

We use the well-known asymptotic behaviour of the function $\Gamma$
\[
\lim_{y \to \infty} \frac{\Gamma(s, y)}{y^{s-1} e^{-y}} = 1
\]
and we conclude for some $C > 0$ and every $u > 0$ that
\[
P(|W_1| > u) \leq \frac{C}{\beta} u^{d-\alpha} e^{-\frac{u^\alpha}{\alpha}}.
\] (86)

Combining (85) and (86) yields for any $M > 0$ that
\[
P(J^\varepsilon_T > M) \leq \frac{C}{\beta} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \left( \frac{M}{\varepsilon} \right)^{d-\alpha} e^{-\frac{\beta}{\varepsilon}T} \left( \frac{\beta T}{\varepsilon} \right)^n.
\]

Therefore for every $\varepsilon > 0$ we have at a logarithmic scale that
\[
\varepsilon \ln P(J^\varepsilon_T > M) \lesssim \varepsilon - \frac{M^\alpha}{\varepsilon^{\alpha-1}} - \beta T.
\] (87)

We proceed now to examine the logarithmic deviations of the small jumps family $(I^\varepsilon_T)_{\varepsilon > 0}$. For any $\lambda = \lambda(\varepsilon) > 0$ such that $\lambda(\varepsilon) \varepsilon^2 \to 0$ as $\varepsilon \to 0$ we have, due to Campbell’s inequality, for any $\varepsilon > 0$ and $M > 0$ that
\[
P(I^\varepsilon_T > M) = \mathbb{P} \left( e^{\lambda(I^\varepsilon_T)^2} > e^{\lambda M^2} \right)
\leq e^{-\lambda M^2} \mathbb{E} \left[ e^{\lambda(I^\varepsilon_T)^2} \right]
\leq e^{-\lambda M^2} \exp \left( \frac{T}{\varepsilon} \int_{\mathbb{R}^d \setminus \{0\}} \left( e^{\lambda \varepsilon^2 |z|^2} - 1 \right) \nu(dz) \right)
\]

26
\[ \leq e^{-\lambda M^2} \exp \left( \frac{T}{\varepsilon} \lambda \varepsilon^2 \int_{0<|z|<1} |z|^2 \nu(dz) \right). \] (88)

In the last estimate we used the fact that
\[ \frac{e^{\lambda^2 |z|^2} - 1}{\lambda \varepsilon^2} \to 1 \]
since \( \lambda \varepsilon^2 \to 0 \) as \( \varepsilon \to 0 \). We fix the parametrization \( \lambda(\varepsilon) = \frac{1}{\varepsilon^{1+p}} \) for some \( p \in (0,1) \) for this purpose. The estimate (88) yields for every \( M > 0 \) that
\[ \varepsilon \ln P \left( I_T^\varepsilon > M \right) \lesssim -\lambda(\varepsilon) \varepsilon M^2 + \lambda(\varepsilon) \varepsilon^2 \to 0 \] as \( \varepsilon \to 0 \) (89)
due to the choice of \( \lambda = \lambda(\varepsilon) > 0 \) fixed above.
Hence combining (87) and (88) we conclude the desired limit (84).

Acknowledgments. The author acknowledge and thanks the financial support from the FAPESP grant number 2018/06531-1 at the University of Campinas (UNICAMP) SP-Brazil, the FAPESP grant number 2019/21324-5 at ENSTA-ParisTech, Palaiseau-France and the FAPESP grant number 2020/04426-6.

References

[1] A. de Acosta. Large deviations for vector-valued Lévy processes. Stoch. Proc. and Appl. vol. 51, 143-156 (1994)

[2] Arrhenius, S. A. Über die Dissociationswärme und den Einfluß der Temperatur auf den Dissociationsgrad der Elektrolyte. Z. Phys. Chem. 4, 96–116 (1889) Doi: 10.1515/zpch-1889-0408.

[3] F. Barret and A. Bovier and S. Méléard. Uniform estimates for metastable transitions in a coupled bistable system, Electronic Journal of Probability, 15, 323–345, 2010

[4] N. Berglund. Kramers’ law: Validity, derivations and generalisations, Markov Processes and Related Fields, 19 (3), 459–490 (2013)

[5] N. Berglund and B. Gentz. On the noise-induced passage through an unstable periodic orbit I: Two-level model, Journal of Statistical Physics, 114 (5-6) 1577–1618 (2004)

[6] N. Berglund and B. Gentz. The Eyring–Kramers law for potentials with nonquadratic saddles, Markov Processes and Related Fields, 3 (16) 549–598 (2010)

[7] C.-H. Rhee, J. Blanchet, B. Zwart. Sample path large deviations for Lévy processes and random walks with regularly varying increments. eprint arXiv:1606.02795

[8] P. Billingsley. Convergence of Probability Measures. Wiley-Interscience 2nd edition (1999)

[9] L. Boltzmann, Über die Beziehung zwischen dem zweiten Hauptsätze der mechanischen Wärmetheorie und der Wahrscheinlichkeitsrechnung respektive den Sätzen über das Wärmegleichgewicht (On the relationship between the second law of the mechanical theory of heat and the probability calculus), Wiener Berichte (1877)

[10] Bovier, A. and Eckhoff, M. and Gayrard, V. and Klein, M. Metastability and low lying spectra in reversible Markov chains, Communications in Mathematical Physics, 228, 219–255, 2002.

[11] A. Bovier, M. Eckhoff, V. Gayrard, M. Klein. Metastability in reversible diffusion processes I. Sharp asymptotics for capacities and exit times. J. Eur. Math. Soc., 6(4):399–424, (2004).
[12] A. A. Borovkov. Boundary-value problems for random walks and large deviations in function spaces. Theory Probab. Appl. vol. 12, 575-595 (1967) particles. Comm. in Math. Physics, vol. 56(2), 101–113 (1977)

[13] A. Budhiraja, P. Dupuis. A variational representation for positive functionals of infinite Brownian motion. Probab. Math. Stat., 20(1, Acta Univ. Wratislav. No. 2246), 39–61 (2000)

[14] A. Budhiraja, P. Dupuis, V. Maroulas. Variational representations for continuous time processes. Ann. de l’Inst. Henri Poinc. (B) Probabilités et Statistiques. vol. 47(3), 725–747 (2011)

[15] A. Budhiraja, J. Chen, P. Dupuis. Large deviations for stochastic partial differential equations driven by a Poisson random measure. Stochastic Process. Appl. vol. 123(2), 523–560 (2013)

[16] A. Budhiraja, P. Dupuis, A. Ganguly. Large deviations for small noise diffusions in a fast Markovian environment. Electron. J. Probab. Vol.23, paper no. 112, 33 pp. (2018)

[17] A. Budhiraja, P. Dupuis. Analysis and Approximation of Rare Events. Representations and Weak Convergence Methods. Series Prob. Theory and Stoch. Modelling vol.94 Springer (2019)

[18] Cerrai, S. and Roeckner, M., Large deviations for stochastic reaction-diffusion systems with multiplicative noise and non-Lipshitz reaction term Ann. Probab., 1B (32), 1100–1139, 2004.

[19] S. Cerrai. A Khasminskii type of Averaging Principle for Stochastic Reaction Diffusion Equations. The Annals of Appl. Prob. vol.19(3), pp. 899-948 (2009)

[20] S. Cerrai. M. Freidlin. Averaging principle for a class of stochastic reaction-diffusion equations. Probab. Theory and Related Fields vol.144, pp. 137-177 (2009)

[21] S. Cerrai. Normal deviations from the averaged motion for some reaction-diffusion equations with fast oscillating perturbation. Journal de Mathématiques Pures et Appliquées vol. 91, pp. 614-647

[22] Cerrai, S., Debussche, A., Large deviations for the two-dimensional stochastic Navier-Stokes equation with vanishing noise correlation. Ann. Inst. Henri Poincaré Probab. Stat. 55 (2019), no. 1, 211–236.

[23] H. Cramér. Sur un nouveau théorème-limite de la théorie des probabilités, Actualités scientifiques et industrielles, Hermann et Cie, Paris. 736 (277) 2-23 (1938).

[24] A.de Oliveira Gomes, M. Högele. The Kramers problem for SDEs driven by small, accelerated Lévy noise with exponentially light jumps.

[25] A. Debussche, M. Högele, P. Imkeller. The dynamics of nonlinear reaction-diffusion equations with small Lévy noise. Springer Lect. Notes in Maths. vol. 2085 (2013)

[26] Deuschel, J.-D. and Stroock, D.,Large deviations, Pure and Applied Mathematics, Academic Press, 1989.

[27] A. Dembo, O. Zeitoni. Large Deviations Techniques and Applications. Applications of Mathematics vol. 38. Springer New York, 2nd ed. (1998)

[28] Donsker, M. D.; Varadhan, S. R. S. Large deviations for Markov processes and the asymptotic evaluation of certain Markov process expectations for large times. Probabilistic methods in differential equations (Proc. Conf., Univ. Victoria, Victoria, B. C., 1974), pp. 82–88. Lecture Notes in Math., Vol. 451, Springer, Berlin, 1975.

[29] J. Duan, W. Wang, A.J. Roberts. Large deviations and approximations for slow–fast stochastic reaction–diffusion equations. J. Diff. Eqs. vol. 253, pp. 3501-3522 (2012)

[30] P. Dupuis, R. S. Ellis. A Weak Convergence Approach to the Theory of Large Deviations. Wiley Series in Probability and Statistics. Wiley and Sons, New York (1997)

[31] K. Dzhaparidze, J.H. van Zantem. On Bernstein-type inequalities for martingales. Stoch. Proc. Appl. 93, pp 109–117 (2001)
[32] R. S. Ellis. Entropy, Large Deviations and Statistical Mechanics. Grundlheren der Mathematischen Wis-
senschaften. Springer, New York (1985)

[33] R . Ellis. The theory of large deviations: from Boltzmann’s 1877 calculation to equilibrium macrostates
in 2D turbulence. Physica D 133, 106-136 (1999)

[34] Eyring, H., The activated complex in chemical reactions, "The Journal of Chemical Physics, 3, 107–115
(1935)

[35] D. Florens, H. Pham. Large deviations probabilities in estimation of Poisson random measures. Stoch.
Proc. and Appl, vol. 76, 117-139 (1998)

[36] J.-P. Fouque, G. Papanicolaou, K. R. Sircar. Derivatives in Financial Markets with Stochastic Volatility.
Cambridge Univ. Press (2000)

[37] J.-P. Fouque, G. Papanicolaou, K. R. Sircar. K. Solna. Multiscale stochastic volatility asymptotics. Mul-
tiscale Model. Simul. vol. 2, pp. 22-42 (2003)

[38] W. H. Fleming. A stochastic control approach to some large deviations problems, in C. Dolcetta, W.H.
Fleming, T. Zoletti (Eds.), Recent Mathematical Methods in Dynamic Programming. Springer Lecture
notes in Math. vol.1119, pp. 52-66 (1985)

[39] W.H. Fleming. Stochastic control and large deviations. In: Bensoussan A., Verjus J.P. (eds) Future
Tendencies in Computer Science, Control and Applied Mathematics. INRIA 1992. Lecture Notes in
Computer Science, vol 653. Springer, Berlin, Heidelberg (1992)

[40] M. Freidlin. The Averaging Principle and Theorems on Large Deviations. Russian Math. Surveys
vol.33(5), pp. 117-176 (1978)

[41] M. I. Freidlin, A. D. Wentzell. Random Perturbations of Dynamical Systems. Grundlehren der Mathe-
matischen Wissenschaften 260. Springer New York, 2nd ed. (1998)

[42] M. I. Freidlin. Quasideterministic approximation, metastability and stochastic ressonance. Physics D 137,
pp. 333–352 (2000)

[43] F. Gamboa, J. Nagel, A. Rouault. Sum rules and large deviations for spectral matrix measures. Bernoulli
25 (2019), no.1, 712-741.

[44] A. Galves. E. Olivieri. M. E. Vares. Metastability for a Class of Dynamical Systems Subject to Small
Random Perturbations. The Annals of Probability 15(4) (1987)

[45] D. Givon. Strong Convergence Rate for Two-Time-Scale Jump Diffusion Stochastic Differential Systems.
Multiscale Model Simul. vol.6(2), pp. 577-594

[46] Godovanchuk, V. V., Asymptotic probabilities of large deviations due to large jumps of a Markov process,
Theory of Probability and its Applications, 26, 314–327 (1982)

[47] Godovanchuk, V. V., Probabilities of large deviations for sums of independent random variables attracted
to a stable law, Theory of Probability and its Applications, 23, 603–608 (1979)

[48] M. Högele and I. Pavlyukevich. The exit problem from a neighborhood of the global attractor for dynam-
ical systems perturbed by heavy-tailed Lévy processes Journal of Stochastic Analysis and Applications,
32(1), 163–190 (2013)

[49] F. den Hollander. Large Deviations. Fields Institute Monographs vol. 14. American Mathematical Society,
Providence, RI (2000)

[50] P. Imkeller, I. Pavlyukevich. First exit times of SDEs driven by stable Lévy processes. Stoch. Proc. Appl.
vol. 116(4), 611-642 (2006)
[51] P. Imkeller, I. Pavlyukevich. Meta-stable behavior of small noise Lévy-driven diffusions. ESAIM Probab. Stat. 12, 412-437 (2008)

[52] J. Jacod, A.N. Shiryaev. Limit Theorems for Stochastic Processes. Springer-Verlag (1987)

[53] Gopinath Kallianpur, P Sundar, Stochastic Analysis and Diffusion Processes, series Oxford Graduate Texts in Mathematics (2013)

[54] R. Z. Khasminskii. On the principle of averaging the Ito’s stochastic differential equations. Kybernetika (Prague) vol.4, pp. 260–279. MR0260052 (1968)

[55] Y. Kifer. Averaging and climate models in Stochastic Climate Models. Eds. P. Imkeller, J.-S.g Storch. Progress in Probability Vol. 49, Birkhäuser Verlag (2001)

[56] D. Khoshnevisan, R. Schilling. From Lévy-Type Processes to Parabolic SPDEs. Advanced Courses in Mathematics. CRM Barcelona Birkhäuser (2015)

[57] Kramers, H. A., Brownian motion in a field of force and the diffusion model of chemical reactions, Physica, 7, 284–304 (1940)

[58] R. Kumar, L. Popovic. Large deviations for multi-scale jump-diffusion processes. Stoch. Proc. and their Appl. vol. 127, pp. 1297-1320 (2017)

[59] C. Leonard. Large deviations for Poisson random measures. Stoch. Proc. and Appl. vol. 85, 93-121 (2000)

[60] D. Liu. Strong convergence rate of principle of averaging for jump-diffusion processes. Front. Math. China vol.7(2), pp. 305-320 (2012)

[61] E. Weinan. S. Liu. E. Vanden-Eijnden. Analysis of multiscale methods of stochastic differential equations. Comm. Pure and Appl. Math. vol. IVIII, pp. 1544–1585 (2005)

[62] Logachov, A., Logachova, O., Yambartsev, A., Large deviations in a population dynamics with catastrophes. Statist. Probab. Lett. 149 (2019), 29–37

[63] E. Löcherbach. Large deviations for cascades of diffusions arising in oscillating systems of interacting Hawkes processes. J. Theoret. Probab. 32 (2019), no.1, 131–162

[64] J. Lynch, J. Sethuraman. Large deviations for processes with independent increments. Ann. Probab. 15, no.2, 610-627 (1987)

[65] V. Maroulas. Large Deviations for Infinite Dimensional System with Jumps. Mathematika 57, pp. 175–192 (2011)

[66] E. Olivieri. M.E. Vares. Large Deviations and Metastability (Encyclopedia of Mathematics and its Applications, Band 100 (2005)

[67] I. Pavlyukevich. First exit times of solutions of stochastic differential equations driven by multiplicative Lévy noise with heavy tails. Stoch. and Dyn. vol.11(2-3), 495-519 (2011)

[68] A. Puhalskii. Large deviations of semimartingales via convergence of the predictable characteristics. Stochastics and Stochastic Reports, 49(1-2), pp. 27-85 (1994)

[69] E. Protter. Stochastic Integration and Differential Equations. Stochastic Modelling and Applied Probability vol.21 (2004)

[70] S. Ramasubramanian. Large deviations: An introduction to 2007 Abel prize. Proc. Math. Sci. vol. 118(2), 161-182 (2008)

[71] J. Rosiński. Tempering stable processes. Stoch. Proc. and Appl. vol. 177(6), pp. 677–707 (2007)

[72] K. Sato. Lévy Processes and Infinitely Divisible Distributions. Cambridge Studies in Advanced Mathematics 2nd ed. (2013)
[73] W. Siegert. Local Lyapunov Exponents: Sublimiting Growth Rates of Linear Random Differential Equations, Lecture notes in Mathematics, vol. 1961, Springer Verlag (2009)

[74] S.R.S. Varadhan. Asymptotic probabilities and differential equations. Comm. Pure Appl. Math 19, 261-286 (1966)

[75] S.R.S. Varadhan. Large Deviations and Applications. CBMS-NSF Regional Conference Series in Applied Mathematics vol.46, Soc. for Ind. and Appl. Math. (1984)

[76] S.R.S Varadhan. Large deviations. Ann. Prob. vol. 36(2), 397-419 (2008)

[77] Shi, W. A note on large deviation probabilities for empirical distribution of branching random walks. Statist. Probab. Lett. 147 (2019), 18–28.

[78] A. Y. Veretennikov. On the Averaging Principle for Systems of Stochastic Differential Equations. Math. USRR-Sbornik vol. 69, pp. 271-284 (1991)

[79] A. Y. Veretennikov. A. Yu., On large deviations for SDEs with small diffusion and averaging, Stochastic Processes and their Applications, vol. 89(1), pp. 69-79 (2000).

[80] J. Xu. Y. Miao. J. Liu. Strong averaging principle for slow-fast SPDEs with Poisson random measures. Discrete and Continuous Dyn. Systems Series B. vol.20(7), pp. 2233-2256 (2015)

[81] J.Xu. Lp-strong convergence of the averaging principle for slow-fast SPDEs with jumps. Journal of Math. Analysis and Appl. vol. 445, pp. 342-373 (2017)