Intrinsic Formulation of Geometric Integrability and Associated Riccati System Generating Conservation Laws

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Abstract
An intrinsic version of the integrability theorem for the classical Bäcklund theorem is presented. It is characterized by a one-form which can be put in the form of a Riccati system. It is shown how this system can be linearized. Based on this, a procedure for generating an infinite number of conservation laws is given.

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1. INTRODUCTION.

The development of the study of nonlinear evolution equations, their integrability and associated soliton solutions has produced many fascinating results. Such equations have Bäcklund transformations [1] and moreover their solutions can be associated with the generation of such geometrical objects as surfaces. This is the case for both constant mean curvature surfaces as well as surfaces which have constant Gaussian curvature [2]. It has been shown that constant mean curvature surfaces play an important role in soliton theory by means of the generalized Weierstrass representation of Konopelchenko [2]. Moreover, Sasaki [3, 4] established a geometrical interpretation for the inverse-scattering problem which was originally formulated by Ablowitz [5] and associates in terms of pseudospherical surfaces [6]. This encompasses a large class of nonlinear evolution equation. The Bäcklund transformation was originally introduced as a transformation which maps one pseudospherical surface into another. These ideas were developed and extended by Chern and Tenenblat [7] who obtained a systematic procedure to determine a linear problem for which a given equation is the integrability condition. They also realized how the geometrical properties of a surface can provide analytic input for such equations. Subsequently, Cavalcante and Tenenblat [8] gave a method to derive conservation laws for evolution equations that describe pseudospherical surfaces.

It is to be understood here that in the Chern Tenenblat approach to integrability, a type of geometric integrability is implied. To formulate this more precisely, we call an evolution equation \( \Xi = 0 \) geometrically integrable if it describes a nontrivial, one-parameter family of pseudospherical surfaces. There are, in addition to this, other formulations of integrability such as formal integrability and the existence of a Lax pair [9] for the system. It is hoped that more insight into the relationship between geometric integrability and other types of integrability can be achieved by investigating the types of conservation laws which can be obtained.

Thus the purpose here is to study the formulation of Bäcklund transformations intrinsically based on a Pfaffian system for the case of nonlinear evolution equations which describe pseudospherical surfaces. Arising out of this, we return to the study and determination of conservation
laws for such equations. A different procedure from that of Cavalcante and Tenenblat is applied to yield a more standard form for these conservation laws. How these different formulations of integrability correspond can be seen by studying the conservation laws which can be produced. Reyes [9,10] has studied these different types of integrability and generalized the approach to conservation laws, but in a different form from that here. The general approach is to take the Pfaffian system which generates Bäcklund transformations and show that it can be put in Riccati form in terms of one-forms which satisfy the structure equations of the surface. Once this is done, some further properties of the system will be developed which lead to two conserved one-forms. It will be shown how these Riccati equations can be transformed, or put in correspondence with, a linear system of one-forms as well.

Finally, it will be shown how these results can be used to generate infinite classes of conservation laws. By introducing an appropriate expansion for the quantity in the relevant Riccati equation in terms of a parameter that appears in the one-forms for the surface, an infinite number of conservation laws can be produced. The form of the conservation equations which are generated by this procedure are seen to correspond to those given by other approaches to integrability. As an example, the conservation laws for a specific nonlinear equation, the MKdV equation, will be determined at the end.

2. GEOMETRIC FORMULATION OF INTEGRABLE PF AFFIAN.

Chern and Tenenblat [6] introduced the idea of a differential equation for a function $u(x,t)$ that describes a pseudospherical surface. There is one such surface for each solution, or if the equation is geometrically integrable, there is a one-parameter family of pseudo-spherical surfaces for each solution. This idea of geometric integrability can be regarded as a bridge between kinematic and formal integrability. It will be seen that geometrical properties of a pseudospherical surface provide a systematic method for obtaining an infinite number of conservation laws and often Bäcklund transformations [11]. The classical Bäcklund theorem originated in the study of pseudospherical surfaces relating solutions of the sine-Gordon equation, and given three solutions of the same equation, the superposition formula provides a new solution algebraically.
Given a nonlinear differential equation for a function \( u(x,t) \), suppose operators \( F \) and \( G \) defined on an appropriate space of functions exist such that
\[
\frac{\partial}{\partial t} F(u(x,t)) + \frac{\partial}{\partial x} G(u(x,t)) = 0,
\]
for all solutions \( u \) of the initial equation. Then (2.1) is called a conservation law of the differential equation. The functional defined by
\[
I(u) = \int_{-\infty}^{\infty} F(u(x,t)) \, dx,
\]
is called a conserved quantity, since \( I(u) \) is independent of \( t \) for each solution \( u \) which satisfies appropriate conditions as \( x \to \pm \infty \).

Let \( M \) be a two-dimensional manifold, or surface, and suppose \( \{ e_1, e_2 \} \) constitute a local orthonormal frame field and \( \{ \omega_1, \omega_2 \} \) is the dual coframe with \( \omega_{12} \) the connection form. The structure equations [12] for \( M \) are given by
\[
d\omega_1 = \omega_{12} \wedge \omega_2,
\]
\[
d\omega_2 = \omega_1 \wedge \omega_{12},
\]
\[
d\omega_{12} = -K \omega_1 \wedge \omega_2,
\]
where \( K \) is the Gaussian curvature of \( M \). Then \( M \) is a pseudospherical surface when \( K = -1 \).

**Definition 2.1** A differential equation for \( u(x,t) \) describes a pseudospherical surface if it is a necessary and sufficient condition for the existence of differentiable functions \( f_{\alpha \beta} \) with \( 1 \leq \alpha \leq 3 \) and \( 1 \leq \beta \leq 2 \) depending on \( u \) and its derivatives such that the one forms
\[
\omega_{\alpha} = f_{\alpha 1} \, dx + f_{\alpha 2} \, dt,
\]
where \( \omega_3 \equiv \omega_{12} \) satisfy structure equations (2.3).

**Theorem 2.1** Let the forms \( \omega_{\alpha} \) be defined by (2.4), then the functions \( f_{\alpha \beta} \) satisfy the following system of equations
\[
-f_{11,t} + f_{12,x} = f_{31} f_{22} - f_{21} f_{32},
\]
\[
-f_{21,t} + f_{22,x} = f_{11} f_{32} - f_{12} f_{31},
\]
\[ -f_{31,t} + f_{32,x} = f_{11}f_{22} - f_{12}f_{21}. \]

Partial derivatives may be indicated by variable subscripts. To prove this, simply substitute one-forms (2.4) into structure equations (2.3) and then equate the coefficients of the corresponding two-forms on both sides.

As a consequence of Definition 2.1, each solution of the differential equation provides a metric on \( M \) whose Gaussian curvature \( K = -1 \). The definition assures that the equation for \( u \) is the required integrability condition. The following Proposition will be crucial in what follows. Due originally to Chern and Tenenblat, a modified proof is given.

**Proposition 2.1** Let \( M \) be a \( C^\infty \) Riemannian surface. \( M \) is pseudospherical if and only if given any unit vector \( v_0 \) tangent to \( M \) at \( p_0 \in M \), there exists an orthonormal frame field \( v_1, v_2 \) locally defined, such that \( v_1(p_0) = v_0 \) and the associated one-forms \( \theta_1, \theta_2 \) and \( \theta_{12} \) satisfy

\[
\theta_{12} + \theta_2 = 0. \tag{2.6}
\]

In this case, \( \theta_1 \) is a closed form.

Proof: It has to be shown that equation (2.6) is completely integrable if and only if \( M \) is a pseudospherical surface. Let \( \mathcal{I} \) be the ideal generated by the form \( \gamma = \theta_{12} + \theta_2 \). Then it follows from the structure equations (2.3) that

\[
d\gamma = d\theta_{12} + d\theta_2 = d\theta_{12} + \theta_1 \wedge \theta_2 = d\theta_{12} + \theta_1 \wedge \gamma - \theta_1 \wedge \theta_2 = d\theta_{12} - \theta_1 \wedge \theta_2,
\]

modulo \( \mathcal{I} \). Therefore, \( \mathcal{I} \) is closed under exterior differentiation if and only if \( M \) is a pseudospherical surface. Hence, the first part follows from the Frobenius Theorem. The fact that \( \theta_1 \) is closed follows from (2.6) upon exterior differentiation and using the structure equations

\[
d\theta_1 = \theta_{12} \wedge \theta_2 = -\theta_2 \wedge \theta_2 = 0.
\]

This is the intrinsic version of the integrability theorem for the classical Bäcklund Theorem. It follows that the integral curves of \( v_1 \) and \( v_2 \) are geodesics and horocycles of \( M \). ♣

Now it is necessary to present an analytic form of Proposition 2.1. To do this, relations between the different one-forms for different orthonormal frames on \( M \) are established. Again \( M \)
is a Riemannian surface, $e_1$, $e_2$ and $v_1$, $v_2$ are two orthonormal frames with $\omega_1$, $\omega_2$, $\omega_{12}$ and $\theta_1$, $\theta_2$ and $\theta_{12}$ the associated one-forms.

Consider both frames with the same orientation, then

$$e_1 = \cos \varphi v_1 + \sin \varphi v_2, \quad e_2 = - \sin \varphi v_1 + \cos \varphi v_2,$$

(2.7)

and therefore,

$$\omega_1 = \cos \varphi \theta_1 + \sin \varphi \theta_2,$$

$$\omega_2 = - \sin \varphi \theta_1 + \cos \varphi \theta_2,$$

$$\omega_{12} = \theta_{12} + d\varphi,$$

(2.8)

where $\varphi$ is the angle of rotation of the frames. The representation in which $f_{21} = \eta$, where $\eta$ is a constant parameter which appears in $\omega_2$ of (2.4), is considered here.

**Proposition 2.2** Let $\Xi = 0$ be a differential equation which describes a pseudospherical surface with associated one-forms given in (2.4) with $f_{21} = \eta$. Then for each solution $u$ of $\Xi = 0$, the system of equations for $\varphi(x,t)$, namely

$$\varphi_x - f_{31} - f_{11} \sin \varphi - \eta \cos \varphi = 0,$$

$$\varphi_t - f_{32} - f_{12} \sin \varphi - f_{22} \cos \varphi = 0,$$

(2.9)

is completely integrable. Moreover, for each solution $u$ of $\Xi = 0$ and corresponding solution $\varphi$, the one-form

$$(f_{11} \cos \varphi - \eta \sin \varphi) \, dx + (f_{12} \cos \varphi - f_{22} \sin \varphi) \, dt = 0,$$

(2.10)

is a closed one-form.

Proof: From (2.8), we have that $\theta_{12} = \omega_{12} - d\theta$, and we can write $\theta_1$, and $\theta_2$ in terms of $\omega_1$, $\omega_2$ as follows

$$\theta_1 = \cos \varphi \omega_1 - \sin \varphi \omega_2,$$

$$\theta_2 = \sin \varphi \omega_1 + \cos \varphi \omega_2,$$

(2.11)

From Proposition 2.1, $u$ is a solution of $\Xi = 0$ if and only if $\theta_{12} = - \theta_2$, hence from (2.8), $\omega_{12}$ takes the form

$$\omega_{12} - d\varphi + \sin \varphi \omega_1 + \cos \varphi \omega_2 = 0,$$

(2.12)

and is completely integrable for $\varphi$. Moreover, $\theta_1$ in (2.11) is a closed form.
By writing $d\varphi = \varphi_x \, dx + \varphi_t \, dt$ in coordinates and using (2.4), equation (2.12) becomes

\[ f_{31} \, dx + f_{32} \, dt - \varphi_x \, dx - \varphi_t \, dt + \sin \varphi f_{11} \, dx + \sin \varphi f_{12} \, dt + \eta \cos \varphi \, dx + \cos \varphi f_{22} \, dt = 0. \]

Equating the coefficients of $dx$ and $dt$ to zero, we obtain the pair (2.9), with integrability condition $\Xi = 0$. The closed form $\theta_1$ can be written as

\[ \theta_1 = \cos \varphi (f_{11} \, dx + f_{12} \, dt) - \sin \varphi (\eta \, dx + f_{22} \, dt) = (f_{11} \cos \varphi - \eta \sin \varphi) \, dx + (f_{12} \cos \varphi - f_{22} \sin \varphi) \, dt. \]

Lemma 2.1 The differential forms

\[ \theta_1 = \cos \varphi \omega_1 - \sin \varphi \omega_2, \quad \Phi = \omega_{12} - d\varphi + \sin \varphi \omega_1 + \cos \varphi \omega_2 \]

are closed one-forms modulo (2.3),

The proof of complete integrability of the $\varphi$ system when $\omega_\alpha$ are given by (2.4) can be formulated as follows.

Theorem 2.2 Let $f_{\alpha\beta}$, with $1 \leq \alpha \leq 3$ and $1 \leq \beta \leq 2$ be differentiable functions of $(x,t)$ which satisfy system (2.5). Then the system

\[ \varphi_x = f_{31} + f_{11} \sin \varphi + f_{21} \cos \varphi, \quad \varphi_t = f_{32} + f_{12} \sin \varphi + f_{22} \cos \varphi, \quad (2.13) \]

is completely integrable for $\varphi$.

Proof: From (2.13), we calculate both derivatives $\varphi_{xt}$ and $\varphi_{tx}$, then subtract to obtain,

\[ \varphi_{xt} - \varphi_{tx} = f_{32,x} - f_{31,t} + (f_{12,x} - f_{11,t}) \sin \varphi + (f_{12} f_{31} - f_{11} f_{32}) \cos \varphi + f_{12} f_{21} \cos^2 \varphi - f_{11} f_{22} \cos^2 \varphi \]

\[ + (f_{22,x} - f_{21,x}) \cos \varphi - (f_{22} f_{31} - f_{21} f_{32}) \sin \varphi - (f_{22} f_{11} - f_{21} f_{12}) \sin^2 \varphi. \]

Substituting system (2.5) into this, it is found that the right-hand side vanishes, so $\varphi_{xt} = \varphi_{tx}$. ♣
In certain cases, (2.9) provides Bäcklund transformations [13,14] for the equation $\Xi = 0$. Suppose it is possible to eliminate $u$ from (2.9), then an expression of the form

$$u = H(\varphi),$$

(2.14)

is obtained, as well as a differential equation for $\varphi$,

$$L(\varphi) = 0.$$  

(2.15)

These last two equations (2.14) and (2.15) are equivalent to (2.9). Thus, Proposition 2.3 follows from Proposition 2.2.

**Proposition 2.3** Let $\Xi = 0$ be a differential equation which describes pseudospherical surfaces with associated one-forms given by (2.4). Suppose (2.9) is equivalent to a system of equations (2.14) and (2.15). Given a solution $u$ of $\Xi = 0$, the set of equations (2.9) is completely integrable and $\varphi$ is a solution of (2.15). Conversely, if $\varphi$ is a solution of (2.15), then $u$, which is defined by (2.14), is a solution of the equation $\Xi = 0$.

3. ASSOCIATED RICCATI SYSTEM AND ITS LINEARIZATION.

It is shown that the one-form (2.12) can be put in Riccati form. In this section, it is not required that the forms $\omega_\alpha$ be given as in (2.4). However they must satisfy the structure equations in (2.3).

**Proposition 3.1.** (i) Let $\Xi = 0$ be a differential equation describing pseudospherical surfaces with associated one-forms $\{\omega_1, \omega_2, \omega_{12}\}$. Under a change of variable $\Gamma = \tan(\varphi/2)$, the completely integrable Pfaffian system (2.12) and the closed one-form $\theta_1$ can be expressed as

$$2d\Gamma = \omega_{12} + \omega_2 + 2\Gamma \omega_1 + \Gamma^2(\omega_{12} - \omega_2),$$

(3.1)

$$\Theta_1 = \omega_1 + \Gamma(\omega_{12} - \omega_2).$$

(3.2)

(ii) Let $\Xi = 0$ be a differential equation describing pseudospherical surfaces with associated one-forms $\{\omega_1, \omega_2, \omega_{12}\}$. Under the change of variable $\hat{\Gamma} = \cot(\varphi/2)$, the completely integrable Pfaffian system (2.12) and closed one-form $\theta_1$ can be expressed as

$$-2d\hat{\Gamma} = \omega_{12} - \omega_2 + 2\hat{\Gamma} \omega_1 + \hat{\Gamma}^2(\omega_{12} + \omega_2),$$

(3.3)
\[ \Theta_2 = \omega_1 + \hat{\Gamma}(\omega_{12} + \omega_2). \]  

(3.4)

Proof: Since \( 2d\Gamma = \sec^2(\varphi/2) \, d\varphi \), the Pfaffian system becomes

\[ 2d\Gamma = \sec^2(\frac{\varphi}{2}) (\omega_{12} + \omega_2) + 2\tan(\frac{\varphi}{2}) \omega_1 - 2\tan^2(\frac{\varphi}{2}) \omega_2 \]

\[ = \omega_{12} + \omega_2 + 2\tan(\frac{\varphi}{2}) \omega_1 + \tan^2(\frac{\varphi}{2}) (\omega_{12} - \omega_2) = \omega_{12} + \omega_2 + 2\Gamma \omega_1 + \Gamma^2 (\omega_{12} - \omega_2). \]

Now,

\[ \cos^2 \frac{\varphi}{2} = \frac{1}{1 + \Gamma^2}, \quad \sin^2 \frac{\varphi}{2} = \frac{\Gamma^2}{1 + \Gamma^2}. \]

Substituting into \( \theta_1 \), we find that

\[ \theta_1 \equiv \theta_\Gamma = \frac{1 - \Gamma^2}{1 + \Gamma^2} \omega_1 - \frac{2\Gamma}{1 + \Gamma^2} \omega_2. \]

Define the one-form \( \Theta_1 \) in the following way,

\[ \Theta_1 = \theta_\Gamma + d\ln(1 + \Gamma^2) \]

\[ = \frac{1 - \Gamma^2}{1 + \Gamma^2} \omega_1 - \frac{2\Gamma}{1 + \Gamma^2} \omega_2 + \frac{\Gamma}{1 + \Gamma^2} (\omega_{12} + \omega_2 + 2\Gamma \omega_1 + \Gamma^2 (\omega_{12} - \omega_2)) \]

\[ = \omega_1 - \Gamma \omega_2 + \Gamma \omega_{12} = \omega_1 + \Gamma(\omega_{12} - \omega_2). \]

This finishes \((i)\). The proof of \((ii)\) proceeds in the same way starting with the definition of \( \hat{\Gamma} \).

**Proposition 3.2.** Define the following two completely integrable Pfaffian equations in terms of the set of one-forms \( \omega_1, \omega_2 \) and \( \omega_{12} \) which satisfy structure equations (2.3),

\[ 0 = \gamma_1 = -2d\Gamma + \omega_{12} + \omega_2 + 2\Gamma \omega_1 + \Gamma^2 (\omega_{12} - \omega_2), \]  

(3.5)

\[ 0 = \gamma_2 = 2d\hat{\Gamma} + \omega_{12} - \omega_2 + 2\hat{\Gamma} \omega_1 + \hat{\Gamma}^2 (\omega_{12} + \omega_2). \]  

(3.6)

Then the one-forms \( \gamma_1 \) and \( \gamma_2 \) satisfy the following pair of equations

\[ d\gamma_1 = -\gamma_1 \wedge (\omega_1 + \Gamma(\omega_{12} - \omega_2)), \]  

(3.7)

\[ d\gamma_2 = \gamma_2 \wedge (\omega_1 + \hat{\Gamma}(\omega_{12} + \omega_2)), \]  

(3.8)

in which the one-forms \( \Theta_1 = \omega_1 + \Gamma(\omega_{12} - \omega_2) \) and \( \Theta_2 = \omega_1 + \hat{\Gamma}(\omega_{12} + \omega_2) \) are closed one-forms.
Proof: Differentiating $\gamma_1$, we obtain upon using (2.3)

$$d\gamma_1 = d\omega_{12} + d\omega_2 + 2d\Gamma \wedge \omega_1 + 2\Gamma d\omega_1 + 2\Gamma d\Gamma \wedge (\omega_{12} - \omega_2) + \Gamma^2 (d\omega_{12} - d\omega_2)$$

$$= \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_{12} + 2d\Gamma \wedge \omega_1 + 2\Gamma d\omega_{12} \wedge \omega_2 + 2\Gamma d\Gamma \wedge (\omega_{12} - \omega_2) + \Gamma^2 (\omega_1 \wedge \omega_2 - \omega_1 \wedge \omega_{12}).$$

Now modulo $2d\Gamma = -\gamma_1 + \omega_{12} + \omega_2 + 2\Gamma \omega_1 + \Gamma^2 (\omega_{12} - \omega_2)$, we obtain after simplifying

$$d\gamma_1 = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_{12} + (-\gamma_1 + \omega_{12} + \omega_2 + 2\Gamma \omega_1 + \Gamma^2 (\omega_{12} - \omega_2)) \wedge \omega_1 + 2\Gamma \omega_{12} \wedge \omega_2$$

$$+ \Gamma(\omega_{12} - \omega_2) + \Gamma \omega_{12} \wedge \omega_2 + \Gamma \omega_2 \wedge \omega_{12} + 2\Gamma^2 \omega_1 \wedge (\omega_{12} - \omega_2) + \Gamma^2 \omega_1 \wedge \omega_2 - \Gamma^2 \omega_1 \wedge \omega_{12}$$

$$= -\gamma_1 \wedge (\omega_1 + \Gamma(\omega_{12} - \omega_2)).$$

This is exactly statement (3.7). Equation (3.8) is obtained upon differentiating $\gamma_2$ and using (2.3) modulo $2d\hat{\Gamma}$. ♣

Now it is shown that a linear problem can be formulated which is equivalent to (3.1) and (3.3).

**Proposition 3.3** Define a linear problem

$$d\Psi = \Omega \Psi,$$  (3.9)

by means of the matrix of one-forms $\Omega$ and $\Psi$ a matrix of zero-forms which are defined to be

$$\Omega = \frac{1}{2} \begin{pmatrix} -\omega_1 & \omega_2 - \omega_{12} \\ \omega_2 + \omega_{12} & \omega_1 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. $$  (3.10)

Then the linear problem defined by (3.9) and (3.10) is equivalent to the nonlinear Riccati system (3.1) and (3.3) under the rational transformations

$$\Gamma = \frac{\psi_2}{\psi_1}, \quad \hat{\Gamma} = \frac{\psi_1}{\psi_2}. $$  (3.11)

Proof: The linear system (3.9) has the pair of components

$$d\psi_1 = -\frac{1}{2}\psi_1 \omega_1 + \frac{1}{2}\psi_2 (\omega_2 - \omega_{12}), \quad d\psi_2 = \frac{1}{2}\psi_1 (\omega_2 + \omega_{12}) + \frac{1}{2}\psi_2 \omega_1. $$  (3.12)
Differentiating \( \Gamma \) given in (3.11), then using (3.12) we obtain

\[
2\psi_1^2 d\Gamma = 2\psi_1 d\psi_2 - 2\psi_2 d\psi_1
\]

\[
= 2\psi_1 \left( \frac{1}{2} \psi_1 (\omega_2 + \omega_{12}) + \frac{1}{2} \psi_2 \omega_1 \right) - 2\psi_2 \left( -\frac{1}{2} \psi_1 \omega_1 + \frac{1}{2} \psi_2 (\omega_2 - \omega_{12}) \right)
\]

\[
= \psi_1^2 (\omega_2 + \omega_{12}) + 2\psi_1 \psi_2 \omega_1 + \psi_2^2 (\omega_{12} - \omega_1).
\]

Dividing both sides of this by \( \psi_1^2 \) and using the definition of \( \Gamma \) given in (3.11), we obtain equation (3.1).

Similarly, differentiating both sides of \( \hat{\Gamma} \) in (3.11), there results the equations

\[
2\psi_2^2 d\hat{\Gamma} = 2\psi_2 d\psi_1 - 2\psi_1 d\psi_2
\]

\[
= 2\psi_2 \left( -\frac{1}{2} \psi_1 \omega_1 + \frac{1}{2} \psi_2 (\omega_2 - \omega_{12}) \right) - 2\psi_1 \left( \frac{1}{2} \psi_1 (\omega_{12} + \omega_2) + \frac{1}{2} \psi_2 \omega_1 \right)
\]

\[
= \psi_2^2 (\omega_2 - \omega_{12}) - 2\psi_1 \psi_2 \omega_1 - \psi_1^2 (\omega_{12} + \omega_2).
\]

Dividing both sides of this by \( \psi_2^2 \), the second Riccati equation (3.3) appears. Thus the Riccati system (3.1) and (3.3) is equivalent to the linear system (3.9) and (3.10).

4. CONSERVATION LAWS.

The one-forms \( \Theta_1 \) and \( \Theta_2 \) which appeared in Proposition 3.1 can be thought of as conservation laws in some sense. In fact, the Riccati system derived there can be put in explicit conservation law form, and to do so, we will write the one-forms \( \omega_\alpha \) in terms of coordinates in a different way from that of (2.4). In doing so, the Riccati system of equations will take a form which is very convenient from the point of view of obtaining and expressing the conservation laws.

Let the system of one-forms be written as

\[
\omega_1 = -\eta dx - 2A dt, \quad \omega_2 = (r + q) dx + (C + B) dt, \quad \omega_{12} = (r - q) dx + (C - B) dt. \quad (4.1)
\]

In (4.1), \( \eta \) is a parameter and the other coefficients are functions of the coordinates. Substituting (4.1) into structure equations (2.3), it is found that the functions which appear as coefficients must satisfy the following system of equations

\[
A_x = qC - rB.
\]
\[ q_t = B_x + 2Aq - \eta B, \quad r_t = C_x - 2Ar + \eta C. \] (4.2)

This system is the analogue of (2.5) using the representation (4.1). In terms of the functions which appear in (4.1), the following result assumes a concise form.

**Proposition 4.1** Let the one-forms \( \omega_\alpha \) be given by (4.1), then modulo (4.2), Riccati system (3.1) and (3.3) can be rewritten in the following conservation law form

\[
(q\Gamma)_t = (A + B\Gamma)_x, \quad (r\hat{\Gamma})_t = (-A + C\hat{\Gamma})_x.
\] (4.3)

Proof: Writing \( d\Gamma \) in terms of coordinates \((x, t)\) and substituting (4.1), equation (3.1) becomes

\[
\frac{\partial \Gamma}{\partial x} = r - \eta \Gamma - q\Gamma^2, \quad \frac{\partial \Gamma}{\partial t} = C - 2A\Gamma - B\Gamma^2.
\] (4.4)

Equations (4.4) imply that

\[
B \frac{\partial \Gamma}{\partial x} - q \frac{\partial \Gamma}{\partial t} = Br - qC + (2qA - \eta B)\Gamma.
\] (4.5)

Adding \(-q_t\Gamma\) to both sides and using the expression \(B_x = q_t - 2Aq + \eta B\) from (4.2), equation (4.5) takes the form

\[
-(qB)_t = Br - qC - B_x\Gamma - B\Gamma_x = -(A + B\Gamma)_x.
\] (4.6)

This is the first equation in (4.3)

Similarly, from (3.3), we can write

\[
\frac{\partial \hat{\Gamma}}{\partial x} = q + \eta \hat{\Gamma} - r\hat{\Gamma}^2, \quad \frac{\partial \hat{\Gamma}}{\partial t} = B + 2A\hat{\Gamma} - C\hat{\Gamma}^2.
\] (4.7)

Therefore, equation (4.7) implies that

\[
C \frac{\partial \hat{\Gamma}}{\partial x} - r \frac{\partial \hat{\Gamma}}{\partial t} = Cq + \eta C\hat{\Gamma} - rB - 2rA\hat{\Gamma} = qC - rB + (\eta C - 2rA)\hat{\Gamma}.
\] (4.8)

Adding \(-r_t\hat{\Gamma}\) to both sides of (4.8) and using (4.2) for \(r_t\), this reduces to

\[
(r\hat{\Gamma})_t = (-A + C\hat{\Gamma})_x,
\]
as required.

Let us show how an infinite number of conservation laws result from these results. The Riccati equations for \( \Gamma \) and \( \hat{\Gamma} \) in the \( x \)-variable can be rearranged to take the form
\[
\eta q \Gamma = rq - (q \Gamma)^2 - q[\frac{\partial}{\partial x}(\frac{q \Gamma}{q})], \quad \eta r \hat{\Gamma} = -rq + (r \hat{\Gamma})^2 + r[\frac{\partial}{\partial x}(\frac{q \hat{\Gamma}}{q})].
\] (4.9)

A similar pair of equations can be obtained for the \( t \) derivatives.

Expand \( q \Gamma \) into a power series in inverse powers of \( \eta \) so that
\[
q \Gamma = \sum_{n=1}^{\infty} g_n \eta^{-n}.
\] (4.10)

The \( g_n \) are unknown at this point, however a recursion relation can be obtained for the \( g_n \) by using (4.9). Substituting (4.10) into the \( \Gamma \) equation in (4.9), we find that
\[
\sum_{n=1}^{\infty} g_n \eta^{-n+1} = qr - \left( \sum_{n=1}^{\infty} g_n \eta^{-n} \right)^2 - q \sum_{n=1}^{\infty} (\frac{g_n}{q})_{x} \eta^{-n}.
\] (4.11)

Applying the Cauchy product formula to the square in (4.11), it then takes the form
\[
g_1 + g_2 \eta^{-1} + \sum_{n=2}^{\infty} g_{n+1} \eta^{-n} = qr - \sum_{n=2}^{\infty} \left( \sum_{j=1}^{n-1} g_j g_{n-j} \right) \eta^{-n} - q(\frac{g_1}{q})_{x} \eta^{-1} - q \sum_{n=2}^{\infty} (\frac{g_n}{q})_{x} \eta^{-n}.
\]

Now equate powers of \( \eta \) on both sides of this expression to produce the set of recursions,
\[
g_1 = qr,
\]
\[
g_2 = -q(\frac{g_1}{q})_{x} = -qr_x,
\] (4.12)
\[
g_{n+1} = -\sum_{k=1}^{n-1} g_k g_{n-k} - q(\frac{g_n}{q})_{x}, \quad n \geq 2.
\]

Substituting (4.10) into (4.3), the following system of conservation laws appears
\[
\sum_{n=1}^{\infty} \frac{\partial g_n}{\partial t} \eta^{-n} = \frac{\partial}{\partial x}(A + B \sum_{n=1}^{\infty} \frac{g_n}{q} \eta^{-n}).
\] (4.13)

In general, \( A \) and \( B \) will depend on parameter \( \eta \), the function \( q \) and higher derivatives of \( q \). Substituting \( A \) and \( B \) into (4.13) for a particular case, (4.13) will simplify under relations (4.12), and then like powers of \( \eta \) can be equated on both sides of (4.13). This procedure generates an infinite number of conservation laws for the equation under examination.
To obtain conservation laws using (4.13) in a few particular examples using this procedure, let us consider the sine-Gordon and MKdV systems.

1. For the sine-Gordon equation,

\[ q = -r = u_x / 2, \quad A = \frac{1}{2\eta} \cos u, \quad B = C = -\frac{1}{2\eta} \sin u. \] (4.14)

Substituting (4.14) into (4.2), the first equation in (4.2) reduces to an identity, and the remaining two hold modulo the sine-Gordon equation

\[ u_{xt} = \sin u. \] (4.15)

Putting (4.14) into (4.13), it is found that

\[ \sum_{n=1}^{\infty} \frac{\partial g_n}{\partial t} \eta^{-n} = \frac{\partial}{\partial x} \left( \frac{1}{2\eta} \cos u - \frac{1}{2\eta} \sin u \sum_{n=1}^{\infty} \frac{g_n}{q} \eta^{-n} \right). \]

Since \( g_1 = -q^2 \) and \( g_2 = q q_x \), the \( n = 1 \) term cancels on both sides given that \( u(x,t) \) satisfies (4.15), and we are left with

\[ \sum_{n=2}^{\infty} \frac{\partial g_n}{\partial t} \eta^{-n} = -\frac{1}{2} \frac{\partial}{\partial x} \left( \sin u \sum_{n=2}^{\infty} \frac{g_{n-1}}{q} \eta^{-n} \right). \]

Using \( q = u_x / 2 \) and equating powers of \( \eta \) on both sides of this, the following set of conservation laws results for \( n \geq 2 \),

\[ \frac{\partial g_n}{\partial t} = -\frac{\partial}{\partial x} \left( \frac{\sin u}{u_x} g_{n-1} \right). \] (4.16)

In fact, taking \( n = 2 \) in (4.16) reproduces (4.15) as well.

2. Consider the case of the MKdV equation for which \( r = -q \) and

\[ A = -\frac{1}{2} \eta^3 - \eta q^2, \quad B = -q_{xx} - \eta q_x - \eta^2 q - 2q^3, \quad C = q_{xx} - \eta q_x + \eta^2 q + 2q^3. \] (4.17)

Putting (4.17) into (4.2), the first equation in (4.2) reduces to an identity, and the remaining two generate the MKdV equation

\[ q_t + 6q^2q_x + q_{xxx} = 0. \] (4.18)

Finally substituting (4.17) into (4.13), we find that

\[ \sum_{n=1}^{\infty} \frac{\partial g_n}{\partial t} \eta^{-n} = \frac{\partial}{\partial x} \left( -\eta q^2 - \left( \frac{q_{xx}}{q} + 2q^2 \right) \sum_{n=1}^{\infty} g_n \eta^{-n} - \frac{q_x}{q} g_1 - \frac{q_x}{q} \sum_{n=1}^{\infty} g_{n+1} \eta^{-n} - \eta g_1 - g_2 - \sum_{n=1}^{\infty} g_{n+2} \eta^{-n} \right). \]
However, from the recursions in (4.12), it follows that $g_1 + q^2 = 0$ and $-q_x g_1 - q g_2 = 0$. Using these to simplify this, the remaining coefficients of $\eta^{-n}$ can be equated on both sides, and the following set of conservation laws are obtained for $n \geq 1$,

$$
\frac{\partial g_n}{\partial t} = -\frac{\partial}{\partial x} \left[ (\frac{q_{xx}}{q} + 2q^2)g_n + \frac{q_x}{q} g_{n+1} + g_{n+2} \right].
$$

(4.19)

It should be stated that a similar set of equations can be developed from $\hat{\Gamma}$ based on this Riccati system (3.3) and (4.3).

5. CONCLUSIONS.

It has been shown how the integrability theorem for the classical Bäcklund Theorem can be formulated intrinsically, and how this can lead to the determination of Bäcklund transformations. More importantly, a corresponding Riccati system is given in intrinsic form as well for which it is shown there exists a linearization. Finally, by taking the one-forms for the structure equations in a particular way, conservation laws which correspond to those obtained other ways are obtained.
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