On $p - q$ Duality and Explicit Solutions in $c \leq 1$ 2d Gravity Models

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Abstract

We study the integral representation for the exact solution to nonperturbative $c \leq 1$ string theory. A generic solution is determined by two functions $W(x)$ and $Q(x)$ which behave at the infinity like $x^p$ and $x^q$ respectively. The integral formula for arbitrary $(p, q)$ models is derived which explicitly demonstrates $p - q$ duality of the minimal models coupled to gravity. We discuss also the exact solutions to string equation and reduction condition and present several explicit examples.


1 Introduction

Recent years brought us to a great progress in understanding of non-perturbative string theory. The key idea, established at least for the most simple set of $c \leq 1$ conformal theories interacting with two-dimensional gravity, is the appearance of the structure of integrable hierarchy in the description of generating function for physical correlators in these models [1, 2].

Fortunately, the particular solutions to non-perturbative string theory can be singled from the whole set of solutions to the Kadomtsev-Petviashvili (KP) or rather Toda lattice hierarchy by an additional requirement usually known in the form of the string equation, which allows one to present these particular solutions in a form, based on the integral representations. In the papers [3, 4, 5, 6, 7, 8] it was shown, that there exists even a certain matrix-integral representation, describing the particular subset of solutions to (reduced) KP-hierarchy satisfying at the same time the string equation. The proposed matrix theory can be considered ideologically as unifying theory for $c \leq 1$ coupled to 2d gravity string models [1], allowing one to interpolate among them [3, 4, 8], thus being a sort of effective string field theory [7].

Below, we are going to investigate solutions to various $(p, q)$-models (with central charges $c_{p,q} = 1 - 6 \frac{(p-q)^2}{pq}$) coupled to 2d gravity in more details. Moreover, we would stress the advantages of their integral (or better multiple integral) representation, proposed in [8] for the particular “stringy” solutions to KP hierarchy.

In particular, we are going to argue, that for higher critical points the integral representation still makes sense, though it does not give at the moment the final ”string-field-theory” answer. Shortly, higher critical points can be described using the same “action” principle for the Douglas equations [1], based on study of the quasiclassical limit [11, 12], but the exact answer has much more complicated form and depends in general on two functions $W(x) = x^p + ...$ and $Q(x) = x^q + ...$. In contrast to the simple $(p, 1)$ situation (with $Q(x) = x$), the integral representation for these solutions besides the “action” functional has very complicated structure of the integration measure. Nevertheless, this integral representation obeys the basic property of $p - q$ duality in the spirit of [13] and

\footnote{and honestly for $(p, 1)$ theories}
might turn to be useful for studying the exact solutions in various examples.

The sense of the $p - q$-duality is no longer a simple symmetry of the theory. Indeed, there exists a kind of transformation, connecting the solution to the $p - q$-problem with the solution to the $q - p$-problem.

In sect.2 we are going to repeat the main results of [3] on $(p, 1)$ solutions and speculate on naive “analytic continuation” to higher critical points. In sect.3 we will formulate the general prescription and derive an integral formula, valid in the case of arbitrary $(p, q)$-solutions. In sect.4 we consider $p - q$ symmetry in the formulation using the reduction condition and the action of the Kac-Schwarz operator [20, 21]. Sect.5 contains several examples of $c < 1$ exact $(p, q)$-solutions and sect.6—some comments on what is supposed to be a particular example of $c = 1$ situation. In sect.7 we give several concluding remarks.

## 2 Review of $(p, 1)$ models

First, we remind that the partition function is defined [3, 4] as a matrix integral

$$Z^{(N)}[V|M] \equiv C^{(N)}[V|M] e^{TrV(M) - TrMV'(M)} \int DX \ e^{-TrV(X)+TrV'(M)X}$$

over $N \times N$ “Hermitean” matrices, with the normalizing factor given by Gaussian integral

$$C^{(N)}[V|M]^{-1} \equiv \int DY \ e^{-TrV_2[M,Y]},$$

$$V_2 \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} Tr[V(M + \epsilon Y) - V(M) - \epsilon YV'(M)]$$

and $Z$ actually depends on $M$ only through the invariant variables

$$T_k = \frac{1}{k} Tr M^{-k}, \ k \geq 1 ;$$

moreover, if rewritten in terms of $T_k$, $Z[V|T] = Z^{(N)}[V|M]$ is actually independent of the size $N$ of the matrices.

As a function of $T_k$, $Z[V|T_k]$ is a $\tau$-function of KP-hierarchy, $Z[V|T_k] = \tau_V[T_k]$, while the potential $V$ specifies (up to certain invariance) the relevant point of the infinite-dimensional Grassmannian.

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\footnote{Similar in a sense to the $N - k$ duality for the $SU(N)_k$ Yang-Mills theory (Nahm transformation for instantons etc.)}
For various choices of the potential $V(X)$ the model (1) formally reproduces various $(p, q)$-series: the potential $V(X) = \frac{X^{p+1}}{p+1}$ can be associated with the entire set of $(p, q)$-minimal string models with all possible $q$'s. In order to specify $q$ one needs to make a special choice of $T$-variables: all $T_k = 0$, except for $T_1$ and $T_{p+q}$ (the symmetry between $p$ and $q$ is implicit in this formulation).

However, this is only a formal consideration. For the potential $V(X) = \frac{X^{p+1}}{p+1}$ the partition function $Z[V|T_k] = \tau_V[T_k] \equiv \tau_p[T_k]$ satisfies the string equation which looks like

$$\sum_{k=1}^{p-1} k(p-k)T_k T_{p-k} + \sum_{k=1}^{\infty} (p+k)(T_{p+k} - \frac{p}{p+1} \delta_{k,1}) \frac{\partial}{\partial T_k} \log \tau_p[T] = 0$$

(4)

i.e. $\tau$-function is defined with all Miwa times (8) around zero values (in $1/M$ decomposition like in original Kontsevich model) with the only exception - $T_{p+1}$ is shifted what corresponds obviously to $(p, 1)$ model. Thus, we see that the matrix integral gives an explicit solution only to $(p, 1)$ string models which must be nothing but particular topological matter coupled to topological gravity.

Of course, we still have an opportunity for analytic continuation in string equation, using the definition of Miwa’s times (3). We have to satisfy the following conditions:

$$T_1 = x$$
$$T_2 = 0$$
$$...$$
$$T_{p+1} - \frac{p}{p+1} = 0$$
$$T_{p+q} = t_{p+q} = fixed$$
$$T_{p+q+1} = 0$$
$$...$$

(5)

which is a system of equations on the Miwa parameters $\{\mu_i\}$, $i = 1, ..., N$. So, to do this analytical continuation one has to decompose the whole set

$$\{\mu_i\} = \{\xi_a\} \oplus \{\mu'_s\}$$

$$T_x = \frac{1}{T_x} M^{-k} = \frac{1}{2} \sum_{k=1}^{N} \mu_k^{-k} = \frac{1}{2} \sum \xi^{-k} + \frac{1}{2} \sum \mu'_{-k} = T^{(cl)} + T'$$

(6)
into “classical” and “quantum” parts respectively. In principle it is clear that we have now to solve the equations

$$T_{k}^{(cl)} = \frac{1}{k} \sum \xi_{a}^{-k} - t_{p+q}\delta_{k,p+q} - \frac{p}{p+1}\delta_{k,p+1}$$ (7)

and this can be done adjusting a certain block form of the matrix $M$ \[4, 7\]. However, in such a way we can only vanish several first times, and the rest ones can be vanished only adjusting correct behaviour in the limit $N \to \infty$. The most elegant way \(^3\) is to use the formula

$$\exp(-\sum_{k=1}^{\infty} \lambda^{k}T_{k}^{(cl)}) = \lim_{K \to \infty} (1 - \frac{1}{K} \sum_{k=1}^{\infty} \lambda^{k}T_{k}^{(cl)})^{K} = \prod_{a}(1 - \frac{\lambda}{\xi_{a}})$$ (8)

and then the solution to (7) will be given by $K$ sets of roots of the equation

$$\sum_{k=1}^{\infty} \lambda^{k}T_{k}^{(cl)} - K = t_{p+q}\lambda^{p+q} - \frac{p}{p+1}\lambda^{p+1} - K = 0$$ (9)

Obviously, the eigenvalues $\xi_{a}$ will now depend on the size of the matrix $N = (p+q)K + N'$ through explicit $K$-dependence ($\xi_{a} \sim K^{1/(p+q)}$) and we lose one of the main features of $(p,1)$ theories – trivial dependence of the size of the matrix. Now we can consider only matrices of infinite size and deal only with the infinite determinant formulas.

That is why we call such way to get higher critical points as a formal one. Below we will try to understand an alternative way of thinking, connected with so-called $p$-times. Indeed, it was noticed in \[3\] that there exists a priori another integrable structure in the model \[1\], connected with the time variables, related to the non-trivial coefficients of the potential $V$. As a results, the cases of monomial potential $V_{p}(X) = \frac{X^{p+1}}{p+1}$ and arbitrary polynomial of the same degree $(p + 1)$ are closely connected with each other. The direct calculation \[3\] shows

$$Z[V|T_{k}] = \tau_{V}[T_{k}] = \exp \left( -\frac{1}{2} \sum A_{ij}(t)(\tilde{T}_{i} - t_{i})(\tilde{T}_{j} - t_{j}) \right) \tau_{p}[\tilde{T}_{k} - t_{k}] ,$$ (10)

where

$$V(x) = \sum_{i=0}^{p} \frac{v_{i}}{t^{i}},$$

$$\tilde{T}_{k} = \frac{1}{k}Tr\tilde{M}^{-k} ,$$
\[ \tilde{M}^p = V'(M) \equiv W(M) , \]
\[ A_{ij} = \text{Res}_\mu W^{i/p} dW^{j/p} , \quad (11) \]

where \( f(\mu)_+ \) denotes the non-negative part of the Laurent series \( f(\mu) = \sum f_i \mu^i \) and

\[ \tau_p[T] \equiv \tau_{V_p}[T] \quad (12) \]

– is the \( \tau \)-function of \( p \)-reduction. The parameters \( \{ t_k \} \) are certain linear combinations of the coefficients \( \{ \nu_k \} \) of the potential \[18, 11\]

\[ t_k = -\frac{p}{k(p-k)} \text{Res} W^{1-k/p}(\mu) d\mu \quad (13) \]

Formula (10) means that “shifted” by flows along \( p \)-times (13) \( \tau \)-function is easily expressed through the \( \tau \)-function of \( p \)-reduction, depending only on the difference of the time-variables \( \tilde{T}_k \) and \( t_k \). The change of the spectral parameter in (5) \( M \to \tilde{M} \) (and corresponding transformation of times \( T_k \to \tilde{T}_k \)) is a natural step from the point of view of equivalent hierarchies.

The \( \tau \)-functions in (10) are defined by formulas

\[ \tau_V[T] = \frac{\det \phi_i(\mu_j)}{\Delta(\mu)} \quad (14) \]

and

\[ \frac{\tau_p[\tilde{T} - t]}{\tau_p[t]} = \frac{\det \hat{\phi}_i(\tilde{\mu}_j)}{\Delta(\tilde{\mu})} \quad (15) \]

with the corresponding points of the Grassmannian determined by the basic vectors

\[ \phi_i(\mu) = [W'(\mu)]^{1/2} \exp(V(\mu) - \mu W(\mu)) \int x^{i-1} e^{-V(x) + x W(\mu)} dx \quad (16) \]

and

\[ \hat{\phi}_i(\tilde{\mu}) = [p^p \tilde{\mu}^{p-1}]^{1/2} \exp \left(-\sum_{j=1}^{p+1} t_j \tilde{\mu}_j \right) \int x^{i-1} e^{-V(x) + x \tilde{\mu}^p} dx \quad (17) \]

respectively. Then it is easy to show that \( \hat{\tau}_p(T) \) satisfies the \( L_{-1} \)-constraint with shifted KP-times in the following way

\[ \sum_{k=1}^{p-1} k(p-k)(\tilde{T}_k - t_k)(\tilde{T}_{p-k} - t_{p-k}) + \sum_{k=1}^\infty (p+k)(\tilde{T}_{p+k} - t_{p+k}) \frac{\partial}{\partial \tilde{T}_k} \log \hat{\tau}_p[\tilde{T} - t] = 0 \quad (18) \]

where \( \hat{\tau}_p \) defined by (17) is identically equal to zero for \( i \geq p+2 \).
The formulas (10,18) demonstrate at least two things. First, the partition function in the case of deformed monomial potential (≡ polynomial of the same degree) is expressed through the equivalent solution (in the sense [13, 16]) of the same p-reduced KP hierarchy, second – not only $t_{p+1}$ but all $t_k$ with $k \leq p + 1$ are not equal to zero in the deformed situation. We will call such theories as topologically deformed $(p, 1)$ models (in contrast to pure $(p, 1)$ models given by monomial potentials $V_p(X)$), the deformation is “topological” in the sense that it preserves all the features of topological models we discussed above. Moreover, this “topological” deformation preserves almost all features of 2d Landau-Ginzburg theories and from the point of view of continuum theory they should be identified with the twisted Landau-Ginzburg topological matter interacting with gravity.

These topologically deformed $(p, 1)$ models as we already said preserve all properties of $(p, 1)$ models. Indeed, according to [2] shifting of first times $t_1, ..., t_{p+1}$ is certainly not enough to get higher critical points. To do this one has to obtain $t_{p+q} \neq 0$, but this cannot be done using above formulas naively, because it is easily seen from definition (13) of $p$-times, that $t_k \equiv 0$ for $k \geq p + 2$. To do this we have to modify the above procedure and we are going to this in next section.

3 General description

The above scheme has a natural quasiclassical interpretation. Indeed, the solution to $(p, 1)$ theories given by the partition function (1) can be considered as a “path integral” representation of the solution to Douglas equations

\[ [\hat{P}, \hat{Q}] = 1 \]  

(19)

where $\hat{P}$ and $\hat{Q}$ are certain differential operators (of order $p$ and $q$) respectively and obviously $p - th$ order of $\hat{P}$ dictates $p$-reduction, while $q$ stands for $q - th$ critical point. Quasiclassically, (19) turns into Poisson brackets relation

\[ \{P, Q\} = 1 \]  

(20)

where $P(x)$ and $Q(x)$ are now certain (polynomial) functions. It is easily seen that the above equations can be solved in the same way with $\hat{P}(x)$ replaced through the
polynomial $P(x)$ should be identified with $W(x) \equiv V'(x)$ \[1\]. Thus, the exponentials in (1), (16) and (17) acquire an obvious sense of action functionals

$$S_{P,1}(x, \mu) = -V(x) + xW(\mu) = -\int_0^x dy W(y)Q'(y) + Q(x)W(\mu)$$

$$W(x) = V'(x) = x^p + \sum_{k=1}^p v_k x^{k-1}$$

$$Q(x) = x$$

and we claim that the generalization to arbitrary $(p, q)$ case must be

$$S_{W,Q} = -\int_0^x dy W(y)Q'(y) + Q(x)W(\mu)$$

$$W(x) = V'(x) = x^p + \sum_{k=1}^p v_k x^{k-1}$$

$$Q(x) = x^q + \sum_{k=1}^q \tilde{v}_k x^{k-1}$$

Now the “true” co-ordinate is $Q$, therefore the extreme condition of action (22) is still

$$W(x) = W(\mu)$$

having $x = \mu$ as a solution, and for extreme value of the action one gets

$$S_{W,Q}|_{x=\mu} = \int_0^\mu dy W'(y)Q(y) =$$

$$= \sum_{k=-\infty}^{p+q} t_k \tilde{\mu}^k$$

where $\tilde{\mu}^p = W(\mu)$ and

$$t_k \equiv t_k^{(W,Q)} = -\frac{p}{k(p-k)} \text{Res} W^{1-k/p}dQ .$$

We should stress that the extreme value of the action (22), represented in the form (24), determines the quasiclassical (or dispersionless) limit of the $p$-reduced KP hierarchy [11, 12] with $p + q - 1$ independent flows. We have seen that in the case of topologically deformed $(p, 1)$ models the quasiclassical hierarchy is exact in the strict sense: topological solutions satisfy the full KP equations and the first basic vector is just the Baker-Akhiezer function of our model (1) restricted to the small phase space. Unfortunately, this is not
to find the basic vectors in the explicit form one should solve the original problem and find the exact solutions of the full KP hierarchy along first \( p + q - 1 \) flows. Nevertheless, we argue that the presence of the “quasiclassical component” in the whole integrable structure of the given models is of importance and it can give, in principle, some useful information, for example, we can make a conjecture that the coefficients of the basic vectors are determined by the derivatives of the corresponding quasiclassical \( \tau \)-function.

Returning to eq.(25) we immediately see, that now only for \( k \geq p + q + 1 \) \( p \)-times are identically zero, while

\[
t_{p+q} \equiv t_{p+q}^{(W,Q)} = \frac{p}{p + q}
\]

and we should get a correct critical point adjusting all \( \{t_k\} \) with \( k < p + q \) to be zero. The exact formula for the Grassmannian basis vectors in general case acquires the form

\[
\phi_i(\mu) = [W'(\mu)]^{1/2} \exp(- S_{W,Q}|_{x=\mu}) \int dM_Q(x) f_i(x) \exp S_{W,Q}(x, \mu)
\]

where \( dM_Q(x) \) is the integration measure. We are going to explain, that the integration measure for generic theory determined by two arbitrary polynomials \( W \) and \( Q \) has the form

\[
dM_Q(z) = [Q'(z)]^{1/2} dz
\]

by checking the string equation. For the choice (28) to insure the correct asymptotics of basis vectors \( \phi_i(\mu) \) we have to take \( f_i(x) \) being functions (not necessarily polynomials) with the asymptotics

\[
f_i(x) \sim x^{i-1}(1 + O(1/x))
\]

4 \( p \)-reduction and the Kac-Schwarz operator

To satisfy the string equation, one has to fulfill two requirements: the reduction condition

\[
W(\mu)\phi_i(\mu) = \sum_j C_{ij} \phi_j(\mu)
\]

and the Kac-Schwarz \([20, 21]\) operator action

\[
A^{(W,Q)} \phi_i(\mu) = \sum A_{ij} \phi_j(\mu)
\]
with
\[
A^{(W,Q)} \equiv N^{(W,Q)}(\mu) \frac{1}{W'(\mu)} \frac{\partial}{\partial \mu} [N^{(W,Q)}(\mu)]^{-1} = \\
= \frac{1}{W'(\mu)} \frac{\partial}{\partial \mu} \frac{1}{2 W'(\mu)^2} + Q(\mu)
\]
and
\[
N^{(W,Q)}(\mu) \equiv [W'(\mu)]^{1/2} \exp(-S_{W,Q}|_{x=\mu})
\]
These two requirements are enough to prove string equation (see [4] for details). The structure of action immediately gives us that
\[
A^{(W,Q)} \phi_i(\mu) = N^{(W,Q)}(\mu) \int d\mathcal{M}_Q(z) Q(z) f_i(z) \exp S_{W,Q}(z,\mu)
\]
and the condition [31] can be reformulated as a \(Q\)-reduction property of basis \(\{f_i(z)\}\)
\[
Q(z) f_i(z) = \sum A_{ij} f_j(z)
\]

Let us check now the reduction condition. Multiplying \(\phi_i(\mu)\) by \(W(\mu)\) and integrating by parts we obtain
\[
W(\mu) \phi_i(\mu) = \\
= N^{(W,Q)}(\mu) \int d\mathcal{M}_Q(z) f_i(z) \frac{1}{Q'(z)} \frac{\partial}{\partial z} [\exp Q(z) W(\mu)] \exp[- \int_0^z dy W(y) Q'(y)] = \\
= - N^{(W,Q)}(\mu) \int d\mathcal{M}_Q(z) \exp[S_{W,Q}(z,\mu)] \left( \frac{1}{Q'(z)} \frac{\partial}{\partial z} \frac{1}{2 Q'(z)^2} - W(z) \right) f_i(z) = \\
\equiv - N^{(W,Q)}(\mu) \int d\mathcal{M}_Q(z) \exp[S_{W,Q}(z,\mu)] A^{(Q,W)} f_i(z)
\]
Therefore, in the “dual” basis \(\{f_i(z)\}\) the condition (31) turns to be
\[
A^{(Q,W)} f_i(z) = - \sum C_{ij} f_j(z)
\]
with \(A^{(Q,W)}(\neq A^{(W,Q)})\) being the “dual” Kac-Schwarz operator
\[
A^{(Q,W)} = \frac{1}{Q'(z)} \frac{\partial}{\partial z} - \frac{1}{2 Q'(z)^2} W(z)
\]
The representation [27], [28] is an exact integral formula for basis vectors solving the \((p,q)\) string model. It has manifest property of \(p-q\) duality (in general \(W - Q\)), turning the \((p,q)\)-string equation into the equivalent \((q,p)\)-string equation.
Now let us transform (27), (28) into a little bit more explicit $p-q$ form. As before for $(p,1)$ models we have to make substitutions, leading to equivalent KP solutions:

$$\tilde{\mu}^p = W(\mu), \tilde{z}^q = Q(z)$$

(38)

Then we can rewrite (27) as

$$\hat{\phi}_i(\tilde{\mu}) = \left[p+q\right]^{1/2} \exp \left(-\sum_{k=1}^{p+q} t_k \tilde{\mu}^k\right) \int d\tilde{z}[q\tilde{z}^{q-1}]^{1/2} \hat{f}_i(\tilde{z}) \exp S_{W,Q}(\tilde{z}, \tilde{\mu})$$

(39)

where action is given now by

$$S_{W,Q}(\tilde{z}, \tilde{\mu}) = \left[-\int_0^{\tilde{z}} d\tilde{y} q\tilde{y}^{q-1} W(y(\tilde{y}))\right] + \tilde{z}^q \tilde{\mu}^p$$

(40)

In new coordinates the reduction conditions are

$$\tilde{\mu}^p \hat{\phi}_i(\tilde{\mu}) = \sum_j C_{ij} \hat{\phi}_j(\tilde{\mu})$$

$$\tilde{z}^q \hat{f}_i(\tilde{z}) = \sum_j A_{ij} \hat{f}_j(\tilde{z})$$

(41)

and for the Kac-Schwarz operators one gets conventional formulas [20, 21, 4]

$$\tilde{A}^{(p,q)} = \frac{1}{p\tilde{\mu}^{p-1}} \frac{\partial}{\partial \tilde{\mu}} - p - \frac{1}{2p} \frac{1}{\tilde{\mu}^p} + \frac{1}{p} \sum_{k=1}^{p+q} k t_k \tilde{\mu}^{k-p}$$

$$\tilde{A}^{(q,p)} = \frac{1}{q\tilde{z}^{q-1}} \frac{\partial}{\partial \tilde{z}} - q - \frac{1}{2q} \frac{1}{\tilde{z}^q} + \frac{1}{q} \sum_{k=1}^{p+q} k \bar{t}_k \tilde{z}^{k-q}$$

(42)

where for $(q,p)$ models we have introduced the “dual” times:

$$\bar{t}_k \equiv t_k^{(q,W)} = \frac{q}{k(q-k)} \text{Res} Q^{1-k/q} dW$$

(43)

in particularly, $\bar{t}_{p+q} = -\frac{q}{p+1} \frac{t_{p+q}}{p+q}$. Now string equations give correspondingly

$$\tilde{A}^{(p,q)} \hat{\phi}_i(\tilde{\mu}) = \sum_j \tilde{A}_{ij} \hat{\phi}_j(\tilde{\mu})$$

$$\tilde{A}^{(q,p)} \hat{f}_i(\tilde{z}) = -\sum_j \tilde{C}_{ij} \hat{f}_j(\tilde{z})$$

(44)

By these formulas we get a manifestation of $p-q$ duality if solutions to $2d$ gravity.
5 Examples

Now, let us consider several explicit examples. First, for monomials $W(x) = x^p$ and $Q(x) = x^q$, $\bar{\mu} \equiv \mu$, $\bar{z} \equiv z$, $\hat{\phi}_i \equiv \phi_i$ and $\hat{f}_i \equiv f_i$, thus, the formulas of the previous section will be

$$\phi_i(\mu) = [p\mu^{p-1}]^{1/2} \exp \left( -\frac{p}{p + q} \mu^{p+q} \right) \times$$

$$\times \int dz[qz^{q-1}]^{1/2} f_i(z) \exp \left( -\frac{q}{p + q} z^{p+q} + z^q \mu^p \right)$$

(45)

and the Kac-Schwarz operators acquire the most simple form

$$A^{(p,q)} = \frac{1}{p\mu^{p-1}} \frac{\partial}{\partial \mu} - \frac{p - 1}{2p} \frac{1}{\mu^p} + \mu^q$$

$$A^{(q,p)} = \frac{1}{qz^{q-1}} \frac{\partial}{\partial z} - \frac{q - 1}{2q} \frac{1}{z^q} - z^p$$

(46)

For any $(p, q)$ theory with $q > p$ the formula (45) maps it onto the corresponding “dual” theory with $q < p$ and vice versa.

In such way one can easily consider the $(p, 1)$ topological theories as dual to the “higher critical points” of the $(1, p)$ theory with the potential $V_2(x) = \frac{1}{2} x^2$, $W_2 = x$. For this theory the “topological” solution is trivial (for example, the partition function is given by a Gaussian integral and equals to unity) so the basis vectors are

$$f^{(1,p)}_i(z) = z^{i-1}$$

(47)

and the Kac-Schwarz operator

$$A^{(1,p)} = \frac{\partial}{\partial z} - z^p$$

(48)

preserves reduction of the corresponding $(p, 1)$ model in a trivial way

$$A^{(1,p)} f^{(1,p)}_i(z) = \left[ \frac{\partial}{\partial z} - z^p \right] z^{i-1} =$$

$$= -z^{i+p-1} + (i - 1)z^{i-2} = -f^{(1,p)}_{i+p-1}(z) + (i - 1)f^{(1,p)}_{i-1}(z)$$

(49)

In this particular case we see how the duality formula turns the problem of finding non-trivial basis of $[20, 3, 4, 10]$ to the trivial basis in the Grassmannian (47), corresponding to sphere.

In general case, we have no more the situation when a non-trivial problem reduces to...
reduces to an equation of generic hypergeometric series giving rise to (linear combinations of) generalized hypergeometric functions [22, 23, 24].

Indeed, we can obtain some particular solutions of the conditions (31) as follows. Let us consider the \((p,q)\) model with \(q = pn + \alpha, \alpha = 1, ..., p - 1; n = 0,1,2, ...\) Using condition of \(p\)-reduction we can choose the whole basis in the form

\[
\phi_{i+pk} = \mu^{pk} \varphi_i, \ i = 1, ..., p
\]  

(50)

and therefore eq.(31) give the system of equations for first \(p\) vectors \(\{\varphi_i(\mu)\}\) [21]:

\[
A \varphi_i = \sum_j A_{ij}(\mu^p) \varphi_j, \ i = 1, ..., p
\]

(51)

where in the case under consideration

\[
A = \frac{1}{p \mu^{p-1}} \frac{\partial}{\partial \mu} - \frac{p-1}{2p} \frac{1}{\mu^p} + \sum \ k t_k \mu^k \equiv \equiv N(\mu) \frac{1}{p \mu^{p-1}} \frac{\partial}{\partial \mu} [N(\mu)]^{-1}
\]

(52)

and

\[
N(\mu) = [p \mu^{p-1}]^{1/2} \exp\left( - \sum t_k \mu^k \right)
\]

(53)

\[q = pn + \alpha\]

After the substitution

\[
\varphi_i = N(\mu) u_i(\mu)
\]

(54)

the system (51) acquires the form of

\[
\frac{\partial}{\partial \mu^p} u_i = \sum_j A_{ij}(\mu^p) u_j, \ i = 1, ..., p
\]

(55)

Now, let us present several explicit formulas for the simplest case of \(p = 2, q = 2m - 1\) solutions

\[
A_{ij}(\lambda) = \begin{pmatrix}
A_{11}(\lambda) & A_{12}(\lambda) \\
A_{21}(\lambda) & A_{22}(\lambda)
\end{pmatrix}
\]

(56)

with \(A_{11}(\lambda), A_{22}(\lambda)\) and \(A_{12}(\lambda)\) being the polynomials of degree \(m-1\) while the polynomial \(A_{21}(\lambda)\) has the degree \(m\).
For the case of topological gravity $m = 1$ (56) looks like

$$\begin{pmatrix}
\beta & \alpha \\
\lambda + \gamma & -\beta
\end{pmatrix}$$

which can be by means of triangular transformations of the basis brought to the form

$$\begin{pmatrix}
0 & \alpha \\
\lambda + \tilde{\gamma} & 0
\end{pmatrix}$$

(58)

and for the case of pure gravity ($m = 2$) instead of (57) and (58) one gets

$$\begin{pmatrix}
m\lambda + b & l\lambda + a \\
a_2\lambda^2 + a_1\lambda + a_0 & -m\lambda - b
\end{pmatrix}$$

(59)

which again, using the triangular transformations can be brought to the form

$$\begin{pmatrix}
\tilde{b} & l\lambda + a \\
a_2\lambda^2 + \tilde{a}_1\lambda + \tilde{a}_0 & -\tilde{b}
\end{pmatrix}$$

(60)

Now, the eigenvalues of matrix (56) given by $^6$

$$\mathcal{A}_\pm = -\frac{1}{2\lambda} \pm 2\lambda \left( A_{11}^2 + A_{12}A_{21} + \frac{A_{11}}{2\lambda^2} + \frac{1}{16\lambda^4} \right)^{1/2}$$

(61)

are actually related to the "Krichever" times

$$\mathcal{A}(\lambda)\pm = \sum k_t k \lambda^{k-1}$$

(62)

which follows from the asymptotics of the solutions to the Kac-Schwarz equations. It allows one to fix in (58)

$$\alpha \sim t_3$$

$$\tilde{\gamma} \sim t_1$$

(63)

while in (59) and (60) the equations are more complicated and even fixing $t_1 = 0 = t_3, \ t_5 = 2/5$ one ends up with a nontrivial matrix

$^4$Implied by $TrA = A_{11} + A_{22} = 0$, because as one can check $TrA$ gives contributions only to the even times in (62), and the eigenvalues (oddly) only to odd times, with the help of recursion conditions.
\[
\left( \begin{array}{cc} \tilde{b} & \lambda + a \\ \lambda^2 - a\lambda + a^2 & -\tilde{b} \end{array} \right) \quad (64)
\]

Now the system of equations (51) with the matrix \( A_{ij} \) given by (56) can be diagonalized giving rise to

\[
A_{12}u''_1 - A'_{12}u'_1 - (A^2_{12}A_{21} + A'_{11}A_{12} - A_{11}A'_{12})u_1 = 0 \quad (65)
\]

which for the case \( m = 1 \) has as a solution

\[
u_1(\lambda) = \lambda^0 F_1 \left[ \frac{4}{3}; -\lambda^3 \right] = \lambda^{1/2} J_{1/3}(2\lambda^{3/2}) = Ai(\lambda) \quad (66)
\]

while for the case \( m = 2 \)

\[
u_1(\lambda) = \lambda^2 F_1 \left[ \frac{7}{5}; \frac{1}{5}\lambda^5 \right] = \lambda J_{2/5}(\frac{2}{5}\lambda^{5/2}) \quad (67)
\]

The sense of these parameters, their relation to monodromy properties of the solutions and relation to [19] deserves further investigation.

6 Remarks on \( c \to 1 \) limit

Let us now make some comments on \( c = 1 \) situation. From basic point of view we need in generic situation to get the most general (unreduced) KP or Toda-lattice tau-function satisfying some (unreduced) string equation. In a sense this is not a limiting case for \( c < 1 \) situation but rather a sort of "direct sum" for all \((p,q)\) models. This reflects that in conformal theory coupled to 2d gravity there is, in a sense, less difference between \( c < 1 \) and \( c = 1 \) situations than this coupling.

However, there are several particular cases when one can construct a sort of direct \( c \to 1 \) limit and which should correspond to certain highly "degenerate" \( c = 1 \) theories. From the general point of view presented above these are nothing but very specific cases of \((p,q)\) string equations, and they could correspond only to a certain very reduced subsector of \( c = 1 \) theory.

\(^5\)provided \( a = 0, \tilde{b} = 0 \)
Indeed, it is easy to see, that for two special cases \( p = \pm q \) the equations (51) can be simplified drastically, actually giving rise to a single equation instead of a system of them. Of course, these two cases don’t correspond to minimal series where one needs \((p, q)\) being coprime numbers. However, we still can fulfill both reduction and Kac-Schwarz condition and these solutions to our equations using naively the formula for the central charge, one might identify with \( c = 1 \) for \( p = q \) and \( c = 25 \) for \( p = -q \).

Now, the simplest theories should be again with \( q = 1 \). For such case “\( c = 1 \)” turns to be equivalent to a discrete matrix model \[5\] while “\( c = 25 \)” is exactly what one would expect from generalization of the Penner approach \[7, 25\]. Indeed, taking non-polynomial functions, like

\[
W(x) = x^{-\beta} \\
Q(x) = x^\beta
\]

the action would acquire a logarithmic term

\[
S_{-\beta, \beta} = -\beta \log x + \frac{x^\beta}{\mu^\beta}
\]

while equations (51) give rise just to rational solutions. It is very easy to see that \( \beta = 1 \) immediately gives the Penner model in the external field, which rather corresponds to “dual” to \( c = 1 \) situation with matter central charge being \( c_{\text{matter}} = 25 \) with a highly non-unitary realization of conformal matter \[6\].

On the other hand, \( p = q = 1 \) solution is nothing but a trivial theory, which however becomes a nontrivial discrete matrix model for unfrozen zero-time. Moreover, these particular \( p = \pm q \) solutions become nontrivial only if one considers the Toda-lattice picture with negative times being involved into dynamics of the effective theory. On the contrary, we know that \( c < 1 \) \((p, q)\)-solutions in a sense trivially depends on negative times with the last ones playing the role of symmetry of string equation \[5\]. It means, that we don’t yet understand enough the role of zero and negative times in the Toda-lattice formulation.

From the point of view of the duality formula one can, however, try to identify \( c = 1 \) situation with the fixed point of the duality transformation (27), i.e. to put \( W = Q \) and \( \phi_i = f_i \) in (27):

\[6\] This \( c = 1 - c = 25 \) duality might be also connected with the known fact that there exists a Legendre
\[
\phi_i(\mu) = [W'(\mu)]^{1/2} \exp\left(-\frac{1}{2}W^2(\mu)\right) \int dz [W'(z)]^{1/2} \phi_i(z) \exp\left(-\frac{1}{2}W^2(z) + W(z)W(\mu)\right)
\]

(70)

In is obvious that (70) has trivial solutions (for \( W = x^2 \)) related to the discrete matrix model, however, it is more interesting if there are more sensible solutions for \( c = 1 \) situation.

7 Conclusion

Let us make some conclusive remarks. We tried to present in the paper the exact mechanism of transitions among different \((p, q)\) solutions of non-perturbative 2d gravity in the framework of general scheme proposed in papers \([3, 4, 5, 6, 7]\). We demonstrated that a naive analytic continuation in the space of Miwa parameters though correct formally leads to certain practical difficulties in explicit description of higher critical points even in trivial situation. Instead, we proposed a concrete scheme, which allow one to shift “classical” counterparts of the KP times, determined by the coefficients of the potential and by the choice of right variable.

The corresponding integral representation is a direct consequence of the action principle and in principle can be interpreted as a certain field theory integral with a highly nontrivial measure. It obeys manifest \( p-q \) symmetry which is evident and restores equivalence in motion along naively two different \( p \)- and \( q \)- directions. Moreover, the appearance of higher degrees of polynomials can be obtained by transformation from the higher critical points of lower \( p \) models. Various examples demonstrate that in principle one might look for a self-consistent multiple integral description of the generic \((p, q)\) models though in contrast to the \( q = 1 \) situation this is still an open question.

One might also find some other questions to be answered. Even in a dual to topological \((p, 1)\) series model there exists nontriviality after \( \alpha \log X \) term (and negative times terms) are added to the potential. For the \( p = 1 \) model this gives rise to a separate interesting problem – the discrete Hermitean matrix model \([4]\) and the question is about interpretation of such generalizations of nontrivial theories.
not reduced to particular “degenerate” cases considered in sect.6) and the role of negative
times: symmetry between positive and negative times, the “dissappearing” of negative
times in $c < 1$ case etc. It is also quite interesting to study the quasiclassical limit of
general $(p,q)$ solutions and to compare them with topological theories. This might shed
light to the underlying topological structure of generic $(p,q)$ models.

All these problems deserves further investigation and we are going to return to them
elsewhere.

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