1. Introduction. Let $G_n$ be a compact Lie group from the list: $\text{SO}(2n+1)$, $\text{Sp}(2n)$, $\text{O}(2n)$. Consider the adjoint action of $G_n$ on its Lie algebra $\mathfrak{g}_n$. Since $G_n$ is a compact group, for each orbit of the action there exists a unique $G_n$-invariant probability measure supported on this orbit. We shall call this measure the orbital measure. Each orbit can be parametrized by an $n$-tuple $X = (x_1 \leq \ldots \leq x_n)$, $x_1 \geq 0$ of weakly decreasing numbers, which corresponds to the canonical form of the matrix from $\mathfrak{g}_n$. Let us denote by $X_n$ the set of such $n$-tuples.

Now we consider a natural projection map $p^n_\ell : \mathfrak{g}_n \to \mathfrak{g}_\ell$. Let $\mu_X$ be a $G_n$-orbital measure; then the measure $p^n_\ell(\mu_X)$ is invariant under the action of the group $G_\ell$. Each invariant measure can be represented as a continuous combination of orbital measures: in fact, for each Borel subset $S \in \mathfrak{g}_\ell$ we have

$$p^n_\ell(\mu_X)(S) = \int_{Y \in X_\ell} \mu_Y^{(k)}(S) \cdot \nu_{X,k}(dY),$$

where $\mu_Y^{(k)}$, $Y \in X_\ell$ are $G_\ell$-orbital measures, while $\nu_{X,k}$ is a certain probability measure on the set $\nu_{X,k}$ of $\ell$-tuples.

The measure $\nu_{X,k}$ is called the radial part of the measure $p^n_\ell(\mu_X)$. In the case of the unitary group $U(n)$ the measure $\nu_{X,k}$ was computed by Olshanski [5] and Faraut [3]. Using the method of Faraut, we compute these measures for the groups $\text{SO}(2n+1)$, $\text{Sp}(2n)$, $\text{O}(2n)$, expressing the radial parts of the orbital measures in terms of determinants of $B$-splines with the knots that are symmetric with respect to 0.

2. Main result Before we formulate the main result, let us give the necessary definitions.

According to Curry and Schoenberg, [1], the $B$-spline with $n$ knots $t_1 < \ldots < t_n$ is a $C^{n-3}$-smooth function $M_n(t_1, \ldots, t_n; t)$ on $\mathbb{R}$ with the following properties:

1) $\text{supp}(M_n(t_1, \ldots, t_n; t)) = [t_1, t_n]$;
2) The function $M_n(t_1, \ldots, t_n; t)$ is the polynomial in $t$ of degree $n-2$ on each subinterval $(t_i, t_{i+1})$;
3) $\int_{\mathbb{R}} M_n(t_1, \ldots, t_n; t) dt = 1$.

Note that the conditions above define the B-spline $M_n(t_1, \ldots, t_n; t)$ in a unique way.

Recall also that the divided differences of a function $f$ are defined by induction as follows:

$$f[t_1, t_2] = \frac{f(t_1) - f(t_2)}{t_1 - t_2}; \ldots; f[t_1, \ldots, t_n] = \frac{f[t_1, t_2, \ldots, t_{n-1}] - f[t_2, \ldots, t_{n-1}, t_n]}{t_1 - t_n}.$$

The Hermite-Genocchi formula connects B-splines with divided differences. This formula tells that for a function $f$ which has a piecewise continuous derivative of degree $n-1$
there is an equality

\[ f[t_1, ..., t_n] = \frac{1}{(n-1)!} \int_\mathbb{R} f^{(n-1)}(t)M_n(t_1, ..., t_n; t)dt. \]

Using the method of Faraut \[3\], for the groups SO(2\(n + 1\)), Sp(2n), SO(2n) we obtain the formulas for \(\nu_{X,k}\) in terms of determinants of B-splines (Theorem 1).

Let us denote by

\[ V_n(T) := V_n(t_1, ..., t_n) = \prod_{1 \leq i < j \leq m} (t_i - t_j) \]

the Vandermonde determinant in the variables \(T = (t_1, ..., t_n)\), and let us also set

\[ v_n(dT) = V_n(T^2) \cdot \prod_{i=1}^{n} dt_i = V_n(t_1^2, ..., t_n^2) \cdot \prod_{i=1}^{n} dt_i. \]

Finally, for arbitrary \(j, m \in \mathbb{N}\) we shall use a shortened notation

\[ M_{2m+2}(\pm t|_{j+1}^{m+j}; t) := M_{2m+2}(-t_{m+j}, ..., -t_j, t_j, ..., t_{m+j}; t). \]

**Theorem 1.** The radial part \(\nu_{X,k}\) of projection of the \(G_n\)-orbital measure \(\mu_X\) is given by the following formula:

\[ \nu_{X,k}(dY) = \frac{c(n, k)}{\prod_{j-i \geq n-k+1} (x_j^2 - x_i^2)} \det \left[ \Delta M_{2n-2k+2}(\pm x|_{j}^{j+n-k}; y_i) \right]_{i,j=1}^{k} v_k(dY), \]

where \(\Delta\) is the following differential operator

\[ \Delta = -y \frac{d}{dy} + \kappa(n, k), \]

and \(c(n, k), \kappa(n, k)\) are the constants that depend on \(n\) and \(k\).

If \(G_n = \text{SO}(2n + 1)\) or \(G_n = \text{Sp}(2n)\), then \(\kappa(n, k) = 0\) and

\[ c(n, k) = \frac{(2n - 2k)!!}{(2n)!!} \prod_{i=0}^{k-1} \left( 2n - 2k + 2i + 2 \right). \]

In the case when \(G_n = \text{O}(2n)\) we have \(\kappa(n, k) = 2(n - k)\) and

\[ c(n, k) = \frac{(2n - 2k - 1)!!}{(2n - 1)!!} \prod_{i=0}^{k-1} \left( 2n - 2k + 2i + 1 \right). \]

**Remark.** Note that the derivative of the B-spline \(M_m, m \in \mathbb{N}\) can be expressed as the difference of two splines of order \(m - 1\); namely, for any \(m\) points \(t_1, ..., t_m \in \mathbb{R}\) the following equality holds (see \[3\]):

\[ \frac{t_m - t_1}{m - 1} \cdot \frac{d}{dt} M_m(t_1, ..., t_m; t) = M_{m-1}(t_1, ..., t_{m-1}; t) - M_{m-1}(t_2, ..., t_m; t). \]
3. Proof of Theorem 1.

3.1. The Laplace transform of orbital measures. We define the Laplace transform of the orbital measure \( \mu \) on the Lie algebra \( \mathfrak{g}_n \) of the group \( G_n \) as the orbital integral

\[
\widehat{\mu}_X(T) = \int_{\text{Orbit}(X)} e^{ \text{tr}(TX) } \mu(dX) = \int_{G_n} e^{ \text{tr}(T \Lambda_0(X)) } dg,
\]

where \( dg \) is the Haar measure on \( G_n \).

In general, the Laplace transform for the measures with a compact support is defined on the complexification of the Lie algebra. But let us notice that the function \( \widehat{\mu}_X(T) \) is invariant under the adjoint action of \( G_n \). Thus we can think of \( T \) as of the matrix of canonical form and consider the Laplace transform of the orbital measure as a function on the coordinate space \( \mathbb{C}^n \).

Due to the Harish-Chandra theorem (see [4], Theorem 2), one can obtain the following formulas for the Laplace transform of the orbital measure \( \mu_X \) for \( G_n \):

\[
\widehat{\mu}_X^{(B_n,C_n)}(t_1, ..., t_n) = \frac{(2n-1)!(2n-3)!...1!}{V_n(X^2)V_n(T^2)} \det \left[ \frac{\sinh(t_i t_j)}{t_i t_j} \right]_{i,j=1}^n, \tag{1}
\]

\[
\widehat{\mu}_X^{(D_n)}(t_1, ..., t_n) = \frac{(2n-2)!...2!}{V_n(X^2)V_n(T^2)} \det \left[ \cosh(t_i t_j) \right]_{i,j=1}^n. \tag{2}
\]

In the formulas (1), (2) \( B_n, C_n \) and \( D_n \) are the standard notations for the classical series of Lie algebras of groups \( \text{SO}(2n+1), \text{Sp}(2n) \) and \( \text{O}(2n) \) respectively.

These formulas imply that, up to multiplicative constants, the Laplace transform of the orbital measure \( \mu_X \) has the form

\[
D_n(f; T; X) = \frac{\det \left[ f(t_i t_j) \right]_{i,j=1}^n}{V_n(T^2)V_n(X^2)},
\]

where \( f \) is an analytic function with the Taylor series

\[
f(z) = \sum_{m=0}^{\infty} \frac{z^{2m}}{2^m m!(\alpha + 1)_m}.
\]

Here \((\alpha + 1)_m = (\alpha + 1)(\alpha + 2)...(\alpha + m)\) is the Pochhammer symbol, and the parameter \( \alpha \) is equal to 1/2 in the case of the groups \( \text{SO}(2n+1) \) and \( \text{Sp}(2n) \) and \( \alpha = -1/2 \) when \( G_n = \text{O}(2n) \).

3.2. Projections and the Laplace transform. For any Borel measure \( \mu \) on \( \mathfrak{g}_n \) the restriction of the Laplace transform to \( \mathfrak{g}_k \) is equal to the Laplace transform of the measure \( \mu^{(k)} \) which is the image of \( \mu \) under projection onto \( \mathfrak{g}_k \). Thus the problem is reduced to the computation of the quantities \( D_n(f; t_1, ..., t_k, 0, ..., 0; X) \).

Lemma 1. If \( f \) is an even analytic function in some neighbourhood of 0, then the quantity \( D_n(f; t_1, ..., t_k, 0, ..., 0; X) \) can be expressed in terms of divided differences of the functions \( \varphi_i(y) = f(t_i \sqrt{y}) \) so that

\[
D_n(f; t_1, ..., t_k, 0, ..., 0; X) = \frac{a_{n-k}(f)}{V_n(X^2)V_k(T^2)(t_1...t_k)^{2(n-k)}} \times
\]
\begin{equation}
\times \prod_{1 \leq j-i \leq n-k} (x_j^2 - x_i^2) \cdot \det \left[ \varphi_i [x_j^2, \ldots, x_{j+n-k}^2] \right]_{i,j=1}^{k}.
\end{equation}

Here \( a_n(f) = \prod_{j=0}^{n-1} c_{2j} \), where \( c_{2j} \) are the even coefficients of the Taylor series of \( f \).

The proof of this lemma is similar to the ones of Theorem 4.1 and 5.3 in the paper [3].

\[ \square \]

3.3. Doubling the knots. Since the functions \( \varphi_i \) from Lemma 1 depend on \( \sqrt{z} \), the formula (3) becomes inconvenient for applying the Hermite-Genocchi formula. But we can overcome this difficulty, if we double the the number of knots.

**Lemma 2.** Let \( f(z) \) be an analytic function in a certain neighbourhood of \( 0 \), and let \( \varphi(z) = f(\sqrt{z}) \). Then, given \( m \) points \( 0 < z_1 < \ldots < z_m \), the following equality for the divided differences holds:

\[ \varphi[z_1^2, \ldots, z_m^2] = g[ -z_m, \ldots, -z_1, z_1, \ldots, z_m ], \]

where \( g(z) = zf(z) \).

**Proof.** By the definition of divided differences, we have

\[ \varphi[z_1^2, \ldots, z_m^2] = \sum_{l=1}^{m} f(z_l) \prod_{r \neq l} \frac{1}{z_l^2 - z_r^2} = \]

\[ = 2 \sum_{l=1}^{m} \frac{z_l f(z_l)}{2z_l^2} \prod_{r \neq l} \frac{1}{z_l^2 - z_r^2} = g[ -z_m, \ldots, -z_1, z_1, \ldots, z_m ], \]

where \( g(z) = zf(z) \)

3.4. Applying the Hermite-Genocchi formula.

Due to Lemmas 1 and 2 the computation of \( D_n(f; t_1, \ldots, t_k, 0, \ldots, 0; X) \) has been reduced to the computation of the determinant of divided differences of the functions \( g_i(z) = zf(t_iz) \).

Applying the Hermite-Genocchi formula, we get the following equality:

\[ g_i[-x_{j+n-k}, \ldots, -x_j, x_j, \ldots, x_{j+n-k}] = \frac{1}{(2n - 2k + 1)!} \int_{\mathbb{R}} (g_i(z))^{(2n-2k+1)} M_{2n-2k+2} (-x_{j+n-k}, \ldots, -x_j, x_j, \ldots, x_{j+n-k}; z) dz = \]

\[ = \frac{-1}{(2n - 2k + 1)!} \int_{\mathbb{R}} (g_i(z))^{(2n-2k)} M'_{2n-2k+2} (\pm x_j^{j+n-k}; z) dz. \]

Now we want to express \( (g_i(z))^{(2n-2k)} = (zf_i(z))^{(2n-2k)} \) in terms of \( f_i(z) = f(t_iz) \).

By the definition of \( f_i \), for any \( m \in \mathbb{N} \cup \{0\} \) we have

\[ (g_i(z))^{(2n-2k)} = \sum_{m=0}^{\infty} \frac{t_i^{2m} z^{2m+1}}{2^{2m} m! (\alpha + 1)_m} \cdot \gamma_{\alpha,m}, \]
where
\[
\gamma_{\alpha,m} = \gamma_{\alpha}(m) := \frac{(m + \frac{3}{2})_{n-k}}{(m + \alpha + 1)_{n-k}}.
\]

The problem is now divided into two cases:

(B-C) \( G_n = \text{Sp}(2n) \) or \( \text{SO}(2n+1) \). In this case \( \alpha = 1/2, \gamma_{\alpha,m} = 1 \) and
\[
(g_i(z))^{(2n-2k)} = t_i^{2n-2k} z f(t_i z).
\]

(D) \( G_n = \text{SO}(2n) \). In this case \( \alpha = -1/2, \gamma_{\alpha,m} = 1 + \frac{2(n-k)}{2m+1} \) and therefore,
\[
(g_i(z))^{(2n-2k)} = t_i^{2n-2k} \cdot (zf(t_i z) + (2n - 2k)t_i^{-1} f^{(-1)}(t_i z)),
\]
where \( f^{(-1)}(\cdot) \) is the primitive of \( f \) such that \( f^{(-1)}(0) = 0 \).

In both cases the divided differences of \( g_i \)'s can be expressed as follows:
\[
g_i[-x_{j+n-k}, \ldots, -x_j, \ldots, x_{j+n-k}] =
\]
\[
\frac{t_i^{2n-2k}}{(2n - 2k + 1)!} \cdot \int_{\mathbb{R}} f(t_i y) \Delta M_{2n-2k+2}(\pm x_j^{j+n-k}; y) \, dy,
\]
where the operator \( \Delta \) has the form
\[
\Delta = \Delta_{1/2} = -y \frac{d}{dy} \quad \text{(the (B-C) case)}
\]
or
\[
\Delta = \Delta_{-1/2} = -y \frac{d}{dy} + 2(n-k) \quad \text{(the (D) case)}.
\]

Now let us apply the Binet-Cauchy identity to the formula (4):
\[
\det [g_i[-x_j, \ldots, -x_{j+n-k}, x_{j+n-k}, \ldots, x_j]] = \prod_{i=1}^{k} t_i^{2n-2k-1} \cdot \frac{1}{((2n - 2k + 1)!)^k} \times
\]
\[
	imes \int_{\mathbb{R}_+^k} \det [f(t_i y_j)]_{i,j=1}^{k} \det [\Delta M_{2n-2k+2}(\pm x_j^{j+n-k}; y_i)]_{i,j=1}^{k} \, dy_1 \ldots dy_k.
\]

To complete the proof of Theorem 1 it suffices to compare the LHS and RHS in (3) and to apply the inverse Laplace transform to both sides of that equation. Theorem 1 is proved. □.

Remark. After this work was finished, the author became aware of an interesting paper of Defosseux [2], where she considered, for the matrix \( X \in \mathfrak{g}_n \), the array \( M(X) \) of the union of spectra of \( X \) and of all its images under projections \( p_k^n, k = 1, \ldots, n-1 \). It turns out that, the radial part of projection of the orbital measure \( \mu_X \) can be derived as
a correlation function of the determinantal point process on arrays $M(X)$ with the kernel given explicitly (see Theorem 6.3 of the work of Defosseux).

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