Level Statistics and Localization for Two Interacting Particles in a Random Potential

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We consider two particles with a local interaction $U$ in a random potential at a scale $L_1$ (the one particle localization length). A simplified description is provided by a Gaussian matrix ensemble with a preferential basis. We define the symmetry breaking parameter $\mu \propto U^{-2}$ associated to the statistical invariance under change of basis. We show that the Wigner-Dyson rigidity of the energy levels is maintained up to an energy $E_\mu$. We find that $E_\mu \approx 1/\sqrt{\pi}$ when $\Gamma$ (the inverse lifetime of the states of the preferential basis) is smaller than $\Delta_4$ (the level spacing), and $E_\mu \propto 1/\mu$ when $\Gamma > \Delta_2$. This implies that the two-particle localization length $L_2$ first increases as $|U|$ before eventually behaving as $U^2$.

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For a single particle diffusing in a disordered system of size $L$ smaller than the one particle localization length $L_1$, there are two characteristic energies: the Thouless energy $E_c = \hbar D/L^2$ and the level spacing $\Delta_1 \approx B_1/L^d$ ($B_1, D$ and $d$ are the band width, the diffusion constant and the system dimension, respectively). If one writes the distribution of energy levels as a Gibbs factor of a fictitious Coulomb gas, the corresponding pairwise interaction for levels with separation $\epsilon < E_c$ coincides \[2\] with the logarithmic repulsion characteristic of the matrix ensembles which are statistically invariant under change of basis, e. g. the Gaussian Orthogonal Ensemble (GOE). For $\epsilon > E_c$, the level repulsion vanishes more or less quickly, depending on the system dimension. The dimensionless conductance $g_1$ is given by $E_c/\Delta_1$. This ratio is the single relevant parameter in the scaling theory of localization. In quasi-one dimension, the size where $g_1 \approx 1$ defines $L_1$. In three dimensions, the mobility edge is characterized by $g_1 \approx g_c$ where $g_c$ is of order 1.

We shall generalize those concepts to two particles with a local (repulsive or attractive) interaction. This two interacting particle (TIP) problem has received a particular attention since Shepelyansky \[4\] pointed out that certain TIP states may extend over a scale $L_2$ much larger than $L_1$. Shepelyansky’s original reasoning consists in mapping the problem for $L \gg L_1$ onto a random band matrix model with a superimposed diagonal matrix (SBRM-model). Imry \[3\] used later the Thouless scaling block picture to arrive at precisely the same results as Shepelyansky. The smearing due to the interaction of the energy levels within $L_1$ was estimated using Fermi’s golden rule, yielding $L_2 \propto U^2$. This delocalization effect has been confirmed by transfer matrix studies \[3\], \[6\], and unambiguously illustrated from numerical studies \[8\] of rings threaded by an $AB$-flux. However, in one dimension, for system sizes which can be numerically investigated, one obtains \[8\] $L_2 \propto |U|$ contrary to Fermi’s golden rule, and a disorder dependence \[8\] $L_2 \propto L_1^{\alpha}$ with $\alpha \approx 1.5 - 1.7$ and not 2, as predicted by Shepelyansky and Imry.

To understand those contradictory results, we study the TIP energy level statistics at a scale $L_1$ in order to identify the energy which plays the role of $E_c$ in this case, and to determine its dependence on $U$. For the original TIP-problem, we assume a tight-binding model \[2\] on a $d$-dimensional lattice ($L^d$ sites $p$ where the random potential is taken with a box distribution of width $2W$). The nearest neighbor hopping term takes a constant value $V = 1$ and $U$ is the on-site interaction. Assuming two electrons with opposite spins, we consider only the symmetric states. The TIP-Hamiltonian \[8\] can be written in the basis of the $N = L^d_1(L^d_1 + 1)/2$ (symmetrized) products of one particle states $|AB\rangle$. We denote by $R_{pA}$ the value on site $p$ of the one particle eigenstate with energy $\epsilon_A$. In this basis, the diagonal terms are dominated by one particle contributions $\epsilon_A + \epsilon_B$ and the interaction Hamiltonian yields a full matrix (for $L \leq L_1$) with entries $U \cdot Q_{AB AB'} = U \sum_p R_{pA} R_{pB} R_{p'A} R_{p'B'}$. The magnitude of those terms is of order $U / L_1^{3d/2}$ with a random sign which does not preserve the sign of the interaction.

Before considering the TIP-Hamiltonian, it is instructive to discuss a simplified matrix model where the correlations between matrix elements are neglected: an ensemble of real symmetric matrices $G$ with independent entries, characterized by Gaussian distributions with variances $< G^2_{ii} > = B_1^2/3$ ($B_1 = 4Vd + 2W$) and $< G^2_{ij} > \approx U^2/L_1^d$ for the diagonal and off-diagonal terms, respectively. The averages are set to zero, which neglects a shift of the diagonal terms by an amount $U \cdot Q_{AB AB} \approx U / L_1^d$ assumed to be much smaller than $B_1$. These shifts preserve the sign of $U$ and for large $U$, eventually split the energy band into two parts. For the sake of simplicity, we ignore them, restricting us to small $U$ and to a Gaussian matrix with preferential basis (GMPB)-model which has been used previously \[8\] to study the GOE to Poisson crossover for the level statistics, and to define a maximum entropy model \[8\] where the range of the level interaction depends on a parameter. When $< G^2_{ii} >$ is very large. 

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We introduce a symmetry breaking parameter $\mu$ in the probability density
\[ \rho(G) \propto \exp \left( -\frac{1}{2} \sigma^2 \right), \]  
(2)
with $\sigma^2 \approx B^2/3$ and $\sigma^2/(2(1+\mu)) \approx U^2/L^3$. When $\mu = 0$, one recovers the GOE ensemble with $\rho_{GOE}(G) \propto \exp(-tr(G^2)/2\sigma^2)$. When $\mu \neq 0$, there is a factor $\rho_{\mu}(G)$ which removes the statistical invariance under change of basis. Expressed \( \mathbf{8} \) in eigenvalue-eigenvector coordinates, it reads
\[ \rho_{\mu}(G) \propto \prod_{\alpha<\beta}^{N} \exp \left( -\frac{\mu}{2\sigma^2} (E_{\alpha} - E_{\beta})^2 \sum_{p} O_{p\alpha}^2 O_{p\beta}^2 \right). \]  
(3)

The question is to understand how this additional factor, after integration over the matrices $O$ (distributed with Haar measure $\mu(dO)$ over the orthogonal group) can destroy the logarithmic level repulsion coming from the measure $\mu(dG) = \prod_{\alpha<\beta}^{N} |E_{\alpha} - E_{\beta}| \prod_{\alpha}^{N} dE_{\alpha} \mu(dO)$, This will allow us to identify the characteristic scale $E_{\mu}$ below which one recovers the GOE rigidity, and above which the levels become uncorrelated. Two cases have to be considered:

(i) $\Gamma < \Delta_2$. The $G_{ij}$ are so small that one can just consider the coupling between two nearest neighbor diagonal entries: i.e. a $2 \times 2$ matrix which can be diagonalized by a rotation of an angle $\theta$. One finds \( \mathbf{8} \) \int d\theta \rho_{\mu}(G) = f(x) = \exp(-x) \cdot I_0(x)$ where $x = \mu \cdot \epsilon^2/(8\sigma^2)$, $\epsilon$ denoting the separation of the two coupled levels. For $x < 1$, $f(x) \approx 1$ and decreases as $1/\sqrt{x}$ for $x > 1$. This gives
\[ \frac{E_{\mu}}{\Delta_2} = \frac{\sqrt{8\sigma^2/\mu}}{\Delta_2} \propto \frac{N}{\sqrt{\mu}}. \]  
(4)

For $\epsilon < E_{\mu}$, one has the GOE statistics, while for $\epsilon > E_{\mu}$, the levels are uncorrelated.

(ii) $\Gamma > \Delta_2$. Many neighboring $G_{ij}$ are coupled by the off-diagonal terms. First, we consider the case where $\epsilon = |E_{\alpha} - E_{\beta}| < \Gamma$: i.e. the case where the two corresponding eigenvectors have a strong overlap. Assuming that the eigenvectors $|O_{\alpha}\rangle$ have non-zero coordinates of order $O_{\alpha\alpha}^2 \approx \Delta_2/\Gamma$ over $\Gamma/\Delta_2$ neighboring basis states only, one gets $\sum_{p=1}^{N} O_{p\alpha}^2 O_{p\beta}^2 \approx \Delta_2/\Gamma$, and $\exp \left( -\mu^2 (2\Delta_2/(2\sigma^2\Gamma)) \right) \approx 1$, independent on $\epsilon(< \Gamma)$. Writing $O = \exp A$, with $A$ a real antisymmetric matrix ($\mu(dO) = \prod_{\alpha<\beta} dA_{\alpha\beta}$), one can see that the small fluctuations of the $A_{\alpha\alpha}$ around their typical values will not yield a correction to the GOE level repulsion. This means that there is no coupling between eigenvalues and eigenvectors as far as $\epsilon < \Gamma \equiv E_{\mu}$ with
\[ \frac{E_{\mu}}{\Delta_2} \propto \frac{N^2}{\mu}. \]  
(5)
now, instead of $N/\sqrt{\mu}$ previously. When $|\epsilon| > \Gamma$, the
eigenvectors do not overlap and the levels should become
uncorrelated. In Ref. [8], it was noted that if $O_{p\alpha} \approx$
$\delta_{p\alpha} + A_{p\alpha}$ where $A_{p\alpha} \ll 1$, $\sum_{p=1}^{N} O_{p\alpha}^{2} O_{p\beta}^{2} \approx 2A_{\alpha\beta}^{2}$, which
gives a $1/|\epsilon|$ factor, after integration over $A_{\alpha\beta}$. This level
attraction exactly compensates the level repulsion due to
$\mu(dG)$. Qualitatively, one can adapt this reasoning to
produce the requested level attraction, after integration
over the eigenvectors. Quantitatively, the calculation of
the exact form of the level interaction as a function of $\epsilon$
taking into account the Breit-Wigner form of the eigen-
 vectors, is postponed to a further study.

We have carried out a numerical study of the GMPB-
ensemble ($N = 500$) as a function of $\mu$, to illustrate
the two regimes. The number variance $\Sigma_{2}(E)$ (vari-
ance of the number of levels in an energy interval $E$) is
shown in Fig. 2. For small energy intervals, $\Sigma_{2}(E)$
coincides with the GOE-logarithmic increase observed when
$\mu = 0$. For larger energy intervals, $\Sigma_{2}(E)$ can be fitted
by $(E/E_{\mu})^{\alpha(\mu)}$, which gives a first method for calculating
$E_{\mu}$. A second method consists in calculating the energy
interval where $\Sigma_{2}$ is above the GOE-curve by a certain
threshold (e. g. 20 percent). Note that those methods
give a non zero $E_{\mu}$ (depending on the chosen threshold)
even for uncorrelated levels, which has been subtracted
from the data. The inset of Fig. 2 confirms that the
two methods are in agreement and exhibits the predicted
crossover for $E_{\mu}$ when $\Gamma \approx \Delta_{2}$ ($\mu \approx N^{2}$), from an $N^{2}/\mu$-
dependence (small $\mu$) towards a $N/\sqrt{\mu}$-dependence. The
$\mu$-dependence of the exponent $\alpha$ (see inset in Fig. 4)
depends on the exact form of the paired level repulsion.

We now turn our attention to the TIP-Hamiltonian in
two dimensions. The corresponding $\Sigma_{2}$ is shown in Fig. 3,
for different $U$. We have obtained the same curves

for $U = 1$ and $U = -1$, and consider in more detail
repulsive interactions. We took rings containing $10 \times 10$
sites threaded by a magnetic flux $\Phi = \Phi_{0}/4$. Diamonds, squares
and triangles are for $U = 0.25, 0.75$ and $2.0$, respectively. The
inset shows how $E_{\mu}$ depends on $U$. The data are obtained as
described in the caption of Fig. 2. The dotted line (solid line)
corresponds to $E_{\mu}/\Delta_{2} = 1.35U/V$ ($E_{\mu}/\Delta_{2} = 2.1(U/V)^{2}$).

A similar study in one dimension is very instructive.
When $W = V = 1$, we have $L_{1} \approx 25$, which gives again
$U_{c} \approx 1$. As expected, one can see in Fig. 4 that $E_{U} \propto |U|
when |U| < U_{c}$, but when $U > U_{c}$, the splitting of the energy
band occurs, and $E_{U}$ decreases. Note that one recovers a Poisson
statistics when $U$ is very large (for $d = 2$, there is only a saturation of $E_{U}$). For $d = 1$, this
means that one can couple only two basis states within
$L_{1} \approx 25$, with a small enough value of $U$ to justify the
simplified GMPB ensemble. The observation of the $U^{2}$
behavior of $E_{U}$ requires larger values of $L_{1}$ in $d = 1$ than
considered in the numerical studies [3, 4].
We now follow the argument developed by Imry \cite{3} to estimate the localization length $L_2$. First, we consider a series of building blocks of size $L_1^d$. $\Gamma_U$ is the smearing of the TIP levels of one of the blocks, due to the interaction-induced coupling with the neighboring block. For such a quasi-1d wire, the dimensionless conductance at scale $L_1$ is given by

$$g_2(L_1) = \frac{L_2}{L_1} \approx \frac{1}{2} + A \frac{\Gamma_U}{\Delta_2}. \quad (6)$$

The factor $1/2$ gives the right limit when $U \to 0$ and $A$ is a constant. Obviously, one should have $\Gamma_U \equiv E_U$. When $\Gamma_U > \Delta_2$, $\Gamma_U$ is given by Fermi’s golden rule, the case considered in Ref. \cite{3} and we only discuss the case $\Gamma_U < \Delta_2$, where $\Gamma_U \approx \sqrt{\nu^2/L_1^d}$. Physically, this means that $U$ is so small that it couples only a single TIP state in one of the blocks to another TIP state in the next block, giving rise to “Rabi oscillations” between those two coupled states. The inverse life time is no longer given by the square of the coupling term, as in Fermi’s golden rule, but by its absolute value. In addition, we have shown that this inverse life time gives the scale below which one has a GOE spectral rigidity, extending the known results for non-interacting particles to the TIP problem.

We continue by discussing a few implications for TIP localization. For $d = 1$ and $U < U_c$ ($U_c \approx 1$ when $W \approx V \approx 1$, see Fig. 4), one gets ($L_2/L_1$) $\approx 1/2 + A(|U|/B_1)\sqrt{L_1}$ which is in agreement with the dependence on $U$ observed in the numerical studies \cite{4}. The conjecture proposed in Ref. \cite{3} gives $L_1$ instead of $\sqrt{L_1}$. As noted in Ref. \cite{3}, the distribution of the $Q_{ABA'B'}$ is far from being Gaussian, which can matter as far as the description of the $L_1$-dependence by the GMPB-model is concerned. However $L_2 \approx L_1^{3/2}$ is close to the behavior observed in Ref. \cite{4}. The $U$-dependence is not affected by this consideration and is correctly described by the GMPB-model.

If one considers two quasi-particles above a Fermi sea, one should replace $\Delta$ in Eq. $\Delta_2$ by $\Delta_2(E) \approx \Delta_2/E$ where $E$ is the total excitation energy. One immediately obtains that the quasi-particle conductance $g_2(E, L_1)$ is of order of $g_2(L_1)$ when $E \approx B_1$, which gives $L_2(E) \approx B_1 = L_2$, in agreement with Ref. \cite{3}. Similarly, in three dimensions, Imry’s relation $(E_{m_2} \approx (B_1^2/U)E_{m_1}^{3/2})$ between the one quasi-particle mobility edge $E_{m1}$ and the two quasi-particle mobility edge $E_{m2}$ does not change when $U < U_c$ ($\nu$ denotes the critical exponent associated with $L_1$).

In summary, we have shown that the basic concepts developed for non interacting particles can be naturally extended to $M = 2$ interacting particles, after the changes $E \to E_U$ and $\Delta_1 \to \Delta_2$. A similar conclusion has been obtained from a non linear $\sigma$ model description of the TIP-Hamiltonian \cite{4}, when $L > L_1$. Moreover, our approach can be easily extended to an arbitrary number $M$ of particles.

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