Revisiting the electroelastic solution for an FGPM thick-walled cylinder subjected to mechanical and electric loadings

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Abstract

Theoretical analysis for an empty thick-walled FGPM cylinder exposed to electric and mechanical loadings are investigated. The cylinder is a composite material composed of PZT4 and PVDF and the volume fraction of PZT4 is given by the power law with three controllable parameters which can cover more complex circumstances. The hypergeometric equation of the radial displacement is acquired by utilizing the Voigt method, and the solutions of the stresses and the electric potential are obtained after solving the radial displacement. The method in this paper is appropriate for real functionally graded piezoelectric materials and can avoid assumptions about unknown overall material parameters appeared in previous references. Finally, the impacts of the parameter \( n \) in volume fraction of FGPM cylinder on mechanical and electric behaviors are examined. Furthermore, the distinction between the hoop stress and radial stress is discussed to decrease the pressure concentration in FGPM cylinder.

Nomenclature

- \( a, b \) - inner and outer radii
- \( r \) - radial coordinate
- \( c(r) \) - volume fraction of material A
- \( c_0, k, n \) - material parameters in the volume fraction \( c(r) \)
- \( P_a, P_b \) - internal and external pressures
- \( \varphi_a, \varphi_b \) - internal and external electric potentials
- \( C_{ijkl}^i (i = 1, 2) \) - elastic modulus of the component
- \( e_{ij}^i (i = 1, 2) \) - piezoelectric tensor modulus of the component
- \( k_{in}^i (i = 1, 2) \) - dielectric modulus of the component
- \( u \) - radial displacement
- \( \varphi \) - electric potential
- \( \varepsilon_r^{(i)}, \varepsilon_g^{(i)} \) - radial and hoop strains of the component
- \( \bar{\varepsilon}_r, \bar{\varepsilon}_g \) - average radial and hoop strains of the cylinder
- \( \sigma_r^{(i)}, \sigma_g^{(i)} \) - radial, hoop, and axial stresses of the component
- \( \sigma_r, \sigma_g, \sigma_z \) - average radial, hoop, and axial stresses of the cylinder
- \( E_r^{(i)}, D_r^{(i)} \) - radial electric field and electric displacement of the component
- \( E_r, D_r \) - average radial electric field and electric displacement of the cylinder
- \( a(r), b(r), d(r), g(r), f(r), h(r), k(r) \) - functions related to the volume fraction \( c(r) \)
\[ x \quad \text{new variables about } r \]
\[ F \quad \text{Hypergeometric function} \]
\[ \alpha, \beta, \delta \quad \text{constants in hypergeometric function} \]
\[ \phi_i (i = 1, 2, \ldots, 6) \quad \text{coefficients in equation (19)} \]
\[ P(r), Q(r) \quad \text{two linearly independent solutions of equation (22)} \]
\[ G(r) \quad \text{a particular solution of equation (22)} \]
\[ \bar{a}, \bar{b}, \delta \quad \text{constants in } G(r) \]
\[ A(\alpha, \beta, \delta) \quad \text{functions in electric potential} \]
\[ C_i (i = 1, 2, \ldots, 4) \quad \text{integral constants} \]
\[ C_{11}, C_{12}, C_{21}, C_{22}, C_0, \bar{C}_0 \quad \text{constants in integral constants} \]
\[ \bar{r} \quad \text{non-dimensional radial coordinate} \]
\[ \bar{a}, \bar{b} \quad \text{non-dimensional inner and outer radii} \]
\[ \bar{u} \quad \text{non-dimensional radial displacement} \]
\[ \bar{\sigma}_{ij} \quad \text{non-dimensional stresses} \]
\[ \varphi \quad \text{non-dimensional electric potential} \]

1. Introduction

Piezoelectric material, a so-called smart material, can generate electrical activity in response to minute mechanical deformation, which is broadly used due to their direct and inverse effect in many applications such as transducers, sensors, actuators, and biological equipment [1–3]. However, these applications are greatly limited in some respects due to the shortcomings of piezoelectric materials such as its brittleness and low power consumption. Last few years, functionally graded materials (FGMs) have been explored because of their capacity to modify their mechanical behaviors by designing the material parameters with certain functional forms. And this material has been applied in many designing fields [4–6]. In this manner, with such appealing and viable points of interest, combing FGMs with piezoelectric materials creates functionally graded piezoelectric materials (FGPMs) with more thorough applications.

With respect to the FGPM cylinders under different loading conditions, there have been a few work focuses its electroelastic responses by utilizing various techniques. Authors [7] assumed power laws for both mechanical and thermal properties and acquired a correct solution for FGPM cylinder under thermal and mechanical loads. For example, the elastic stiffness was assumed as \( C_{ij}(r) = C_{ij}^0 r^a \), piezoelectric coefficient was assumed as \( e_{ij}(r) = c_{ij}^0 r^a \), and the dielectric constant was assumed as \( \eta_{ij}(r) = \eta_{ij}^0 r^a \). The elastic behaviors of a FGPM cylinder which is subjected to thermal, electric and mechanical loads under a uniform magnetic field also are presented by assuming the material parameters as power functions [8]. Similar idea of using power function material properties can be found in numerous articles [9–18]. A few groups proposed different assumptions that [19–21] the material parameters were assumed as exponential function. For instance, the elastic stiffness was \( C_{ij}(r) = C_{ij}^0 e^{\alpha r} \), piezoelectric coefficient was \( e_{ij}(r) = c_{ij}^0 e^{\alpha r} \), and the dielectric constant was \( \eta_{ij}(r) = \eta_{ij}^0 e^{\alpha r} \).

In the above papers, material parameters (the elastic stiffness, piezoelectric coefficient and the dielectric constant) of FGPM cylinder are assumed to be power functions or exponential functions, which imply FGPM cylinder comprises of multi-layers. This point may violate the definition of FGM shaped of at least two constituent phases. On the other hand, all material parameters have a similar index value \((\alpha or \alpha)\), which appears to be unreasonable in practice. Besides, there are two variable constants in power or exponential functions, which are not adequate for depicting more complex cases.

With the goal to take care of the previously mentioned issues, starting from the definition of FGMs we think about that the FGPM cylinder comprises of two phases (PZT4 and PVDF). The volume fraction of PZT4 in FGPM cylinder can be given instead of assuming material properties of the FGPM cylinder in previous references. Simultaneously, the volume fraction of PZT4 change with three-variable parameters is not quite the same as two-variable parameters in previous reports, which can cover more complex circumstances. The general material properties can’t be given straightforwardly, yet they can be gotten as far as their parameters of components and the volume fractions in a specific control manner [22, 23]. There are some methods can be used to analyze the mechanical behaviors of composites, such as DQ (Differential quadrature) in [24], DSC (Discrete singular convolution) in [25], FEM (Finite element method) in [26, 27] and higher-order shear deformation theory in [28]. Different from the methods above, the micromechanical method is used in this work to obtain the equivalent material parameters of composites. The method presented in this paper has clear physical meaning.
and can avoid the assumption of the distribution regularities of unknown overall material parameters appeared in existing papers. The constitutive model is acquired dependent on the Voigt method, and then the electroelastic behaviors of the FGPM cylinder are obtained. In this work, two boundary conditions are studied, one is stress boundary condition corresponding to the positive piezoelectric effect, and the other is electric boundary condition corresponding to the inverse piezoelectric effect. Finally, the impact of volume fraction on displacement, stresses along different directions and electric potential of the FGPM cylinder are considered.

2. Theoretical analysis

An empty radially polarized FGPM cylinder is considered here. Denote the thickness of the cylinder as \( b-a \) with inner radius \( a \) and outer radius \( b \). It is assumed that the cylinder is adequately long and exposed to both axisymmetric mechanical and electric loadings on its surfaces (Figure 1). The stress boundary conditions are

\[
\sigma_{r|b=a} = -p_a \quad \text{and} \quad \sigma_{r|b=b} = -p_b
\]

and the electric boundary conditions are \( \varphi_{b=a} = \varphi_a \) and \( \varphi_{b=b} = \varphi_b \).

The FGPM cylinder is composed of two materials A (such as PZT4) and B (such as PVDF). PZT4 is piezoelectric ceramic and PVDF is polyvinylidene difluoride. The volume fraction \( c(r) \in [0, 1] \) of material A is given by

\[
c(r) = c_0 [1 - k (r/b)^n]
\]

where \( r \) represent the radius and \( c_0, k \) and \( n \) represent the material parameters. In particular, \( c_0 = 0 \) and \( c_0 = 1 \), \( k = 0 \) denote the cylinder totally consists of material B and material A, separately.

As is known to all, the material properties of this cylinder will show transversely isotropic properties if the polarization direction of the piezoelectric material is along a certain direction. For cylindrically orthotropic homogeneous piezoelectric materials polarized in the radial direction, the tensor forms of constitutive equations can be given as following [18]

\[
\sigma_{ij}^{(i)} = C_{ijkl}^{(i)} e_{ij}^{(i)} - e_{ij}^{(i)} E_{m}^{(i)}
\]

\[
D_{m}^{(i)} = e_{ij}^{(i)} E_{m}^{(i)} + k_{ijk}^{(i)} E_{m}^{(i)}
\]

the superscript \( i = 1, 2 \) represents material A and B, separately; \( \sigma_{ij}^{(i)} \) and \( e_{ij}^{(i)} \) are the stress and strain tensors, separately; \( E_{m}^{(i)} \) and \( D_{m}^{(i)} \) are electric field and electric displacement vectors, separately; \( C_{ijkl}^{(i)} \), \( e_{ij}^{(i)} \) and \( k_{ijk}^{(i)} \) are elastic, piezoelectric tensor and dielectric moduli, separately.

For convenience, the component forms of equation (2) for the cylinder can be written as

\[
\sigma_r^{(i)} = C_{11}^{(i)} e_r^{(i)} + C_{13}^{(i)} e_z^{(i)} - e_{33}^{(i)} E_r^{(i)}
\]

\[
\sigma_{\theta}^{(i)} = C_{11}^{(i)} e_\theta^{(i)} + C_{13}^{(i)} e_z^{(i)} - e_{33}^{(i)} E_{\theta}^{(i)}
\]

\[
\sigma_z^{(i)} = C_{12}^{(i)} e_\theta^{(i)} + C_{13}^{(i)} e_z^{(i)} - e_{33}^{(i)} E_z^{(i)}
\]

\[
D_r^{(i)} = e_{33}^{(i)} e_r^{(i)} + e_{33}^{(i)} e_z^{(i)} + k_{33}^{(i)} E_r^{(i)}
\]

where \( \sigma_r^{(i)}, \sigma_\theta^{(i)} \) and \( \sigma_z^{(i)} \) represent the radial, hoop and axial stresses, separately; \( e_r^{(i)} \) and \( e_\theta^{(i)} \) are the radial and hoop strains, separately; \( E_r^{(i)} \) and \( D_r^{(i)} \) are electric field and electric displacement in radial direction, separately.

Equation (3) can be easily revised as

\[
E_r^{(i)} = \frac{D_r^{(i)} - e_{33}^{(i)} e_\theta^{(i)} - e_{33}^{(i)} e_z^{(i)}}{k_{33}^{(i)}}
\]
Substituting equation (4) into the stress components of equation (3), we can get

\[
\begin{align*}
\sigma_r^{(i)} &= \left( C_{11}^i + \frac{e_{11}^i e_{11}^i}{k_{33}^i} \right) e_r^{(i)} + \left( C_{33}^i + \frac{e_{33}^i e_{33}^i}{k_{33}^i} \right) e_r^{(i)} - \frac{e_{33}^i}{k_{33}^i} D_r^{(i)} \\
\sigma_{\theta}^{(i)} &= \left( C_{44}^i + \frac{e_{44}^i e_{44}^i}{k_{33}^i} \right) e_{\theta}^{(i)} + \left( C_{55}^i + \frac{e_{55}^i e_{55}^i}{k_{33}^i} \right) e_{\theta}^{(i)} - \frac{e_{55}^i}{k_{33}^i} D_{\theta}^{(i)} \\
\sigma_z^{(i)} &= \left( C_{12}^i + \frac{e_{12}^i e_{12}^i}{k_{33}^i} \right) e_z^{(i)} + \left( C_{33}^i + \frac{e_{33}^i e_{33}^i}{k_{33}^i} \right) e_z^{(i)} - \frac{e_{33}^i}{k_{33}^i} D_z^{(i)}
\end{align*}
\]

The average stress tensor, strain tensor, electric field tensor and electric displacement tensor over a representative volume element (RVE) \( V \) are characterized as [29]

\[
\begin{align*}
\sigma &= \frac{1}{V} \int_V \hat{\sigma}(x) \, dx, \quad \sigma^{(i)} &= \frac{1}{V} \int_{V_i} \hat{\sigma}^{(i)}(x) \, dx \\
\varepsilon &= \frac{1}{V} \int_V \hat{\varepsilon}(x) \, dx, \quad \varepsilon^{(i)} &= \frac{1}{V} \int_{V_i} \hat{\varepsilon}^{(i)}(x) \, dx \\
E &= \frac{1}{V} \int_V \hat{E}(x) \, dx, \quad E^{(i)} &= \frac{1}{V} \int_{V_i} \hat{E}^{(i)}(x) \, dx \\
D &= \frac{1}{V} \int_V \hat{D}(x) \, dx, \quad D^{(i)} &= \frac{1}{V} \int_{V_i} \hat{D}^{(i)}(x) \, dx
\end{align*}
\]

where \( \hat{\sigma}, \hat{\varepsilon}, \hat{E} \) and \( \hat{D} \) are stress tensor, strain tensor, electric field tensor and electric displacement tensor at arbitrary over a RVE, separately; \( \sigma, \varepsilon, E \) and \( D \) are general volume average stress tensor, strain tensor, electric field tensor and electric displacement tensor of composite, separately; \( \hat{\sigma}^{(i)}, \hat{\varepsilon}^{(i)}, \hat{E}^{(i)} \) and \( \hat{D}^{(i)} \) are stress tensor, strain tensor, electric field tensor and electric displacement tensor at arbitrary over the constituents of composite, separately; and \( \sigma^{(i)}, \varepsilon^{(i)}, E^{(i)} \) and \( D^{(i)} \) are general volume average stress tensor in their subvolumes \( V_i \), separately.

For FGPM cylinder comprises of materials A \( (i = 1) \) and B \( (i = 2) \), equation (6) leads to

\[
\begin{align*}
\sigma &= c(r) \sigma^{(1)} + [1 - c(r)] \sigma^{(2)} \\
\varepsilon &= c(r) \varepsilon^{(1)} + [1 - c(r)] \varepsilon^{(2)} \\
D_r &= c(r) D_r^{(1)} + (1 - c(r)) D_r^{(2)} \\
E_r &= c(r) E_r^{(1)} + (1 - c(r)) E_r^{(2)}
\end{align*}
\]

Utilizing the Voigt method, the components of strain tensor and electric displacement tensor components are

\[
\varepsilon_{\theta}^{(1)} = \varepsilon_{\theta}^{(2)} = \varepsilon_{\theta}, \quad \varepsilon_r^{(1)} = \varepsilon_r^{(2)} = \varepsilon_r, \quad D_r^{(1)} = D_r^{(2)} = D_r
\]

The strain-displacement relations are [30]

\[
\varepsilon_r = \frac{du}{dr}, \quad \varepsilon_{\theta} = \frac{u}{r}
\]

with \( u \) is the displacement in the radial direction.

Substituting equations (5) into (7) and using equations (8) and (9), the equivalent stresses of FGPM cylinder are

\[
\begin{align*}
\sigma_r &= a(r) \frac{u}{r} + b(r) \frac{du}{dr} - f(r) D_r \\
\sigma_{\theta} &= d(r) \frac{u}{r} + a(r) \frac{du}{dr} - h(r) D_r \\
\sigma_z &= g(r) \frac{u}{r} + a(r) \frac{du}{dr} - h(r) D_r
\end{align*}
\]
By utilizing equations (4) and (7), the equivalent radial electric field of the cylinder can be acquired as

$$E_r = k(r)D_r \quad - h(r)\varepsilon_\theta - f(r)\varepsilon_r$$

where $k(r) = c(r)\frac{\varepsilon_1 \varepsilon_3}{k_{33}} + \frac{1 - c(r)}{k_{33}}$.

Considering the electric potential $\varphi$, the radial electric field is [7]

$$E_r = -\frac{d\varphi}{dr}$$

Equilibrium equation and electrostatic charge equation in the cylindrical coordinate system for axisymmetric deformation, separately, are [8]

$$\frac{d\sigma_r}{dr} + \sigma_r - \sigma_0 = 0$$

and

$$\frac{dD_r}{dr} + \frac{D_r}{r} = 0$$

Equation (15) can be easily comprehended to acquire the radial electric displacement as

$$D_r = C_1$$

where $C_1$ is integral constant.

The radial electric field can be gotten when substituting equations (16) into (12) and utilizing equation (9) as

$$E_r = C_1 \frac{k(r)}{r} - h(r)\frac{u}{r} - f(r)\frac{du}{dr}$$

From the radial electric field, the electric potential $\varphi$ is solved by comparing equations (13) and (17) as

$$\varphi(r) = \int_a^r \left[ \frac{h(r)}{r} - \frac{df(r)}{dr} \right]udr + f(r)u - C_3 \int_a^r \frac{k(r)}{r}dr + C_2$$

where $C_2$ is integral constant.

From equation (18), the particular type of the electric potential $\varphi$ can be gotten once the displacement along the radial direction $u$ is given. The displacement along the radial displacement $u$ is given and considered below.

Substituting equations (10) into (14) and utilizing equation (16), we have

$$r\frac{d}{dr}\left[ r(\phi_1 - \phi_2r^n)\frac{du}{dr} \right] + \left( \phi_4r^n - \phi_3 \right)u = C_3(\phi_6r^n - \phi_3)$$
where
\[
\phi_1 = c_0(C_1 + e_1^2/k_1^2) + (1 - c_0)(C_2 + e_2^2/k_2^2)
\]
\[
\phi_2 = c_0(k(C_1 + e_1^2/k_1^2) - C_2 - e_2^2/k_2^2)/b^n
\]
\[
\phi_3 = c_0(C_1 + e_1^2/k_1^2) + (1 - c_0)(C_2 + e_2^2/k_2^2)
\]
\[
\phi_4 = c_0k[C_1 + e_1^2/k_1^2 - C_1 - e_2^2/k_2^2 - n(C_1 + e_1^2/k_1^2 - C_2 - e_2^2/k_2^2)]/b^n
\]
\[
\phi_5 = c_0e_1/k_1^2 + (1 - c_0)e_2/k_2^2
\]
\[
\phi_6 = c_0k[e_1^2/k_1^2 - e_2^2/k_2^2 - n(e_1^2/k_1^2 - e_2^2/k_2^2)]/b^n
\]

(20)

The estimation of \( \phi_1 \) usually does not equal to zero because the volume fraction ranges from zero to one, then equation (19) can be reworked as
\[
r \frac{d}{dr} \left[ r \left( 1 - \frac{\phi_r}{\phi_1} \right) \frac{du}{dr} \right] + \left( \frac{\phi_r}{\phi_1} - \frac{\phi_5}{\phi_1} \right) u = G \left( \frac{\phi_5}{\phi_1} r^n - \frac{\phi_5}{\phi_1} \right)
\]

(21)

For notational simplification, another variable \( x = \chi(r) = \frac{\phi_r}{\phi_1} \) is presented here, equation (21) becomes
\[
x^2(x - 1) \frac{d^2u}{dx^2} + x(2x - 1) \frac{du}{dx} + \frac{1}{n^2} \left( \frac{\phi_1}{\phi_1} - \frac{\phi_1}{\phi_2} \right) u = G \left( \frac{\phi_5}{\phi_1} - \frac{\phi_5}{\phi_2} \right)
\]

(22)

Equation (22) is a nonhomogeneous ordinary differential equation and the result is the whole of the general result \( u_g(r) \) of the relating homogeneous ordinary differential equation and a particular result \( u_p(r) \). The general result \( u_g(r) \) can be examined in the range \( 0 < x < 1 \) as \[30\]
\[
u_g(r) = C_r x^{1/2 - 1/2} F(\alpha, \beta; x) + C_4 x^{1/2 - \delta/2} F(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; x)
\]

(23)

with \( C_3 \) and \( C_4 \) are integral constants. The function \( F \) denote for the hypergeometric function based on equation (11) in \[30\] and the constants in \( F \) are
\[
\delta = 1 + \frac{1}{4} \sqrt{\frac{\phi_4}{n^2\phi_1}}, \quad \alpha = \frac{\delta}{2} + \frac{1}{4} \sqrt{\frac{\phi_4}{n^2\phi_2}}, \quad \beta = \delta - \alpha
\]

(24)

Note that \( x = \chi(r) \) predominantly lies in the interval \( 0 < x < 1 \) for different material parameters. Here we do not examine the solutions of equation (22) for other intervals.

For convenience, rewriting \( u_p(r) \) as
\[
u_p(r) = C_3 P(r) + C_4 Q(r)
\]

(25)

where \( P(r) \), \( Q(r) \) and their corresponding derivatives \( P'(r) \), \( Q'(r) \) are
\[
P(r) = x^{1/2 - 1/2} F(\alpha, \beta; x)
\]
\[
Q(r) = x^{1/2 - \delta/2} F(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; x)
\]
\[
P'(r) = n^{-1} \left[ \frac{\delta - 1}{2} P(r) + \frac{\alpha \beta}{\delta} x^{\delta + 1/2} F(\alpha + 1, \beta + 1, 1 + \delta; x) \right]
\]
\[
Q'(r) = n^{-1} \left[ \frac{1 - \delta}{2} Q(r) + \frac{(\alpha - \delta + 1)(\beta - \delta + 1)}{2 - \delta} x^{\delta + 1/2} F(\alpha - \delta + 2, \beta - \delta + 2, 3 - \delta; x) \right]
\]

(26)

A particular result can be easily acquired as
\[
u_p(r) = C_3 G(r)
\]

(27)

where \( G(r) \) and its derivative \( G'(r) \) are
\[
G(r) = \frac{\phi_5}{\phi_3} + \frac{\phi_4}{\phi_3} \left( \frac{\phi_5}{\phi_3} - \frac{\phi_5}{\phi_1} \right) + \frac{n}{\phi_3} \left( \frac{\phi_5}{\phi_3} - \frac{\phi_5}{\phi_1} \right)
\]
\[
G'(r) = n^{-1} \left[ \frac{\phi_5}{\phi_3} \left( \frac{\phi_5}{\phi_3} - \frac{\phi_5}{\phi_1} \right) \right] + \frac{n}{\phi_3} \left( \frac{\phi_5}{\phi_3} - \frac{\phi_5}{\phi_1} \right)
\]

(28)

with \( C \) is combination symbol. The constants in \( G(r) \) and \( G'(r) \) are
\[
\delta = \delta/2 - 1/2, \quad \alpha = \delta - \alpha, \quad \beta = -\alpha - 1
\]

(29)

The result of equation (22) is the whole of equations (25) and (27), specifically
\[
u(r) = C_3 P(r) + C_4 Q(r) + C_3 G(r)
\]

(30)
Then the stress components can be acquired substituting equations (30) into (10) as

\[
\sigma_r = a(r)[C_1 P(r) + C_4 Q(r) + C_6 G(r) + b(r)[C_2 P'(r) + C_5 Q'(r) + C_7 G'(r) - C_9 f'(r)/r
\]

\[
\sigma_\theta = d(r)[C_1 P(r) + C_4 Q(r) + C_6 G(r) + [a(r)[C_2 P'(r) + C_5 Q'(r) + C_7 G'(r) - C_9 h(r)/r
\]

\[
\sigma_z = g(r)[C_1 P(r) + C_4 Q(r) + C_6 G(r) + a(r)[C_2 P'(r) + C_5 Q'(r) + C_7 G'(r) - C_9 h(r)/r
\]

(31)

And after that the electric potential \(\phi\) can be obtained substituting equations (30) into (18) as

\[
\phi(r) = C_1[A(r) + P(r)f(r)] + C_4[B(r) + Q(r)f(r)]
\]

\[
+ C_1 [C(r) + G(r)f(r) - \left(\frac{a_0}{k_{33}} + \frac{1 - a_0}{k_{33}^2}\right) \ln r + \frac{a_0 k(k_{33}^2 - k_{33})_r}{nbk_{11}^1 k_{33}^2} r^n] + C_2
\]

(32)

where

\[
A(r) = \int_0^r \left[ \frac{h(r)}{r} - \frac{df(r)}{dr} \right] P(r) dr
\]

\[
B(r) = \int_0^r \left[ \frac{h(r)}{r} - \frac{df(r)}{dr} \right] Q(r) dr
\]

\[
C(r) = \int_0^r \left[ \frac{h(r)}{r} - \frac{df(r)}{dr} \right] G(r) dr
\]

(33)

and the constants in \(A(r), B(r)\) and \(C(r)\) are

\[
A_5 = \frac{c_0 e_{11}}{k_{33}^1} + \left(1 - c_0\right)e_{33}^2,
\]

\[
A_6 = \frac{c_0 k}{b} e_{11}^2 - \frac{e_{33}^1}{k_{33}^1} n e_{33}^1 - n\left(\frac{e_{33}^2}{k_{33}^2} - \frac{e_{33}^1}{k_{33}^1}\right),
\]

\[
A_1 = \frac{A_5}{n} \left(\frac{\phi_0}{\phi_1}\right)^\frac{b}{a},
\]

\[
A_2 = \frac{A_6}{n} \left(\frac{\phi_0}{\phi_1}\right)^\frac{b}{a},
\]

\[
A_3 = \frac{A_5}{n} \left(\frac{\phi_0}{\phi_1}\right)^\frac{b}{a},
\]

\[
A_4 = \frac{A_6}{n} \left(\frac{\phi_0}{\phi_1}\right)^\frac{b}{a}
\]

(34)

Until now, the displacement along the radial direction, the stress components and the electric potential are all acquired. In the following, the integral constants \(C_1, C_2, C_3\) and \(C_4\) are solved utilizing the boundary conditions.

With stress boundary conditions \(\sigma_r|_{r=a} = -p_a\) and \(\sigma_r|_{r=b} = -p_b\), the connections between \(C_{33}, C_{4}\) and \(C_1\) are

\[
C_3 = C_1 C_4 + C_{12}
\]

\[
C_4 = C_{21} C_4 + C_{22}
\]

(35)
with various parameters. In the case of 

- \[ f(b) = b(a)G(b) + b(b)G'(b) \]
- \[ f(a) = a(a)G(a) + b(b)Q'(b) \]

C_{12} = \[ \{ p_a[a(a)Q(a)/a + b(a)Q'(a)] - p_b[a(b)Q(b)/b + b(b)Q'(b)]\} / C_0 \]

C_{21} = \[ \{ f(b) - a(a)G(b) + b(b)G'(b)\} / C_0 \]

C_{22} = \[ \{ f(b) - a(a)G(b) - b(b)G'(b)\} / C_0 \]

\[ C_0 = [a(a)P(a)/a + b(a)P'(a)] [a(b)Q(b)/b + b(b)Q'(b)] \]

\[ - [a(a)Q(a)/a + b(a)Q'(a)] [a(b)P(b)/b + b(b)P'(b)] \]}

Then the potential electric boundary conditions \( \varphi_{|r=a} = \varphi_a \) and \( \varphi_{|r=b} = \varphi_b \) are used for solving \( C_1 \) and \( C_2 \), and we have

\[ C_1 = (\varphi_b - \varphi_a - C_{12} [A(b) + P(b)f(b) - P(a)f(a)]) \]

\[ C_{22} = \left( [B(b) + Q(b)f(b) - Q(a)f(a)] \right) / C_0 \]

\[ C_2 = \varphi_b - C_{21} / C_{12} P(a)f(a) - (C_{22} + C_{C11}) Q(a)f(a) \]

\[ C_0 = C_{11} [A(b) + P(b)f(b) - P(a)f(a)] + C_{21} [Q(b)f(b) + C(b) + G(b)f(b) - G(a)f(a)] \]

\[ - \left( \frac{c_0}{k_{33}^2} + \frac{1}{k_{23}^2} \right) \ln \frac{b}{a} \right] + \frac{c_0 k(a^n - a^n) (k_{33}^2 - k_{13}^2)}{n a^2 k_{33}^2 k_{33}^2} \]

In particular, the cylinder will be homogeneous material if the volume fraction is zero or one. For instance, \( c_0 = 0 \) means the cylinder is purely material B; \( c_0 = 1 \) and \( k = 0 \) means the cylinder totally consists of material A. Then equation (20) reduce to

\[ r \frac{d^2 u}{dr^2} + \frac{dr}{d\bar{\varphi}_i} u = -C_i \frac{\phi_i}{\phi_1} \]

where the constants in equation (41) are listed below and \( i = 1 \) and 2 represent materials A and B, separately.

\[ \phi_1 = C_{33} + e_{33}^i / k_{33} \]

\[ \phi_3 = C_{11} + e_{11}^i / k_{33} \]

\[ \phi_5 = e_{33}^i / k_{33} \]

Additionally, the results of the displacement along the radial direction, the stresses, and the electric potential can simplify from equations (30)–(32). Since the results are tedious and nothing but a routine work, they are not introduced in detail here.

### 3. Results and discussion

In the numerical part, the dimensionless expressions for the radial coordinate, inner radius, elastic potential, stress and displacement along the radial displacement are characterized as \( \bar{r} = r/b, \bar{\varphi} = \varphi(r)/\varphi_0 \), \( \bar{\varphi}_0 = \varphi_0(r)/\varphi_0 \), and \( \bar{u} = u(r) / (r C_{11}/(b_p)^2 \), separately. \( a \) and \( b \) represent the inner and outer radii, separately. \( p_a \) is the internal pressure and \( C_{11} \) is one of elastic modulus of material A. The inner radius \( a \) is 0.7, which is acceptable for a FGPM cylinder. Here, the internal pressure \( p_a \) is taken as 100 KPa and the internal electric potential \( \varphi_0 \) is taken as 100 V. Elastic, piezoelectric and dielectric properties of homogeneous PZT4 [31] and PVDF [32] are listed in table 1.

As appeared in figure 2, we can see the development of \( c(\bar{r}) \) with various parameters. In the case of \( c_0 = 1 \), \( k = 0 \), \( c(\bar{r}) \) equal to one and the cylinder is homogeneous and is fully constituted by purely PZT4. In the FGPM cylinder simulation, the parameters in \( c(\bar{r}) \) are picked as \( c_0 = 1, k = 1 \), throughout.

To illustrate the distinction between FGPM and homogeneous piezoelectric material, the elastic results of homogeneous PZT4 are also compared in figures as reference. In the following simulation, two representative boundary conditions are applied.
Example 1. Consider the mechanical and electric potential boundary conditions of FGPM cylinder are, separately, taken as

\[ p_0 = 100 \text{ KPa}, \quad p_b = 0, \quad \varphi_a = 0, \quad \varphi_b = 0 \]  \hspace{1cm} (42)

As appeared in figure 3, the development of displacement along the radial direction is plotted with various parameter \( n \). The greatest and least estimations of the displacement along the radial direction appear at the internal and external radii, separately. Comparing with the homogeneous PZT4 we know the estimations of displacement along the radial displacement for various parameter \( n \) of FGPM cylinder are greater than homogeneous PZT4. Additionally, the bigger the parameter \( n \) is, the closer to the curve of homogeneous PZT4. This phenomenon is mainly attributed to the change in volume fraction. We can see the curve of \( n = 5 \) is closer to the curve of homogeneous PZT4 (\( c_0 = 1, k = 0 \)). Also, the figure shows that the bigger the estimation of parameter \( n \) is, the less the relative change happens in displacement along the radial direction.

The stresses along the radial, hoop and axial directions are shown in figures 4–6, individually. Contrasted with the consequences for displacement along the radial direction, the impacts of parameter \( n \) on the stresses

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Table 1. Elastic, piezoelectric and dielectric properties of homogeneous material A (PZT4) and material B (PVDF).

| Moduli   | PZT4 \((I = 1)\) | PVDF \((I = 2)\) |
|----------|-----------------|-----------------|
| \( C_{11} \) (GPa) | 139             | 3               |
| \( C_{22} \) | 139             | 3               |
| \( C_{33} \) | 115             | 3               |
| \( C_{12} \) | 77.8            | 1.5             |
| \( C_{13} \) | 74.3            | 1.5             |
| \( C_{23} \) | 74.3            | 1.5             |
| \( C_{44} \) | 25.6            | 0.75            |
| \( C_{55} \) | 25.6            | 0.75            |
| \( C_{66} \) | 30.6            | 0.75            |
| \( e_{31} \) (C m \(^{-1}\)) | -5.2           | -0.0015         |
| \( e_{32} \) | -5.2            | 0.0285          |
| \( e_{33} \) | 15.1            | -0.051          |
| \( e_{14} \) | 12.7            | —               |
| \( e_{24} \) | 12.7            | —               |
| \( k_{11} \) (F m \(^{-1}\)) | 3.27e-9         | 0.1062e-9       |
| \( k_{22} \) | 3.27e-9         | 0.1062e-9       |
| \( k_{33} \) | 5.62e-9         | 0.1062e-9       |

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Figure 2. Development of volume fraction with various parameters.
along different directions are not conspicuous. In spite of the fact that parameter \( n \) has little impact on stresses of FGPM cylinder, it is unique in relation to homogeneous piezoelectric material. The stress boundary conditions \( \bar{\sigma}_r |_{r=a} = -1 \) and \( \bar{\sigma}_r |_{r=b} = 0 \) are satisfied and the radial stress along the radial direction of homogeneous piezoelectric material looks closer to linearity (see figure 4). Figure 5 demonstrates that the hoop stress along the radial direction monotonically diminishes from inward surface to outward surface of the cylinder, and this additionally applies to the axial stress in figure 6.

In this work, we further focus on the contrast \( \bar{\sigma}_h - \bar{\sigma}_r \) between the hoop stress and radial stress. Figure 7 demonstrates the advancement of \( \bar{\sigma}_h - \bar{\sigma}_r \) with various parameter \( n \). Besides, the curve \( \bar{\sigma}_h - \bar{\sigma}_r \) of homogeneous PZT4 is plotted to illustrate the distinction between FGPM and homogeneous material. We can see that \( \bar{\sigma}_h - \bar{\sigma}_r \) of homogeneous PZT4 is gentler than that of FGPM cylinder, which implies FGPM cylinder is easier to enter plasticity than that of homogeneous cylinder in this boundary conditions.

Figure 8 depicts the change in electric potential with parameter \( n \). It is not hard to see that the electric potential complies with the electric boundary conditions \( \varphi |_{r=a} = 0 \) and \( \varphi |_{r=b} = 0 \). The greatest esteem of the electric potential is located near the middle of the cylinder. This greatest esteem expand with an expanding parameter \( n \), which means if we want to get a large or small voltage in the middle position, we should choose the

*Figure 3.* Development of radial displacement along the radial direction with various parameter \( n \).

*Figure 4.* Development of radial stress along the radial direction with various parameter \( n \).
right value of parameter $n$ accordingly. Additionally, in spite of the fact that the most extreme estimation of electric potential happens at an inward position, the position somewhat moves with parameter $n$ differing. The bigger value of parameter $n$ is, the closer the peak is to the outside of the cylinder.

**Example 2.** Consider the mechanical and electric potential boundary conditions of FGPM cylinder are, separately, taken as

$$p_a = 0, \quad p_b = 0, \quad \varphi_a = 100 \text{ V}, \quad \varphi_b = 0$$

(43)

Figure 9 demonstrates displacement along the radial direction as a function of radius. It tends to be seen from the figure 9 that the displacement of the uniform piezoelectric material step by step expands to outer surface from the inner surface of the cylinder, while the FGPM cylinder expands first and then decreases along the radial direction. In addition, the estimation of displacement for FGPM cylinder is significantly smaller than that of uniform material. As parameter $n$ expands, the magnitude of the displacement also expands.

Figures 10–12 demonstrate the stresses along the radial, hoop and axial directions in the radial direction. The estimations of these three stress components are less at the electric potential boundary conditions than
the stress boundary conditions. For the radial stress, the stress boundary conditions \( \bar{\sigma}_r |_{r=a} = 0 \) and \( \bar{\sigma}_r |_{r=b} = 0 \) satisfy equation (43) no matter it is FGPM or homogeneous piezoelectric cylinder. The expansive estimation of stress along the radial direction is found in FGPM compared with the homogeneous piezoelectric material. The maximum value of stress along the radial direction is roughly exhibited amidst the cylinder and shifts with various parameter \( n \)-esteem areas. Also, as \( n \) expands, the greatest estimation of stress along the radial direction shifts to the inner surface of the cylinder. As appeared in figure 11, stress along the hoop direction continuously expands from the center along each ray. An intriguing phenomenon can be seen here is the estimation of stress along the hoop direction is nearly equivalent at external surface. The stress along the hoop direction of the uniform material alters significantly along the radial direction, and the contrast between inner surface and outer surface reaches 34.15. The change of stress along the axial direction (figure 12) and stress along hoop direction are similar as the radius expands. Additionally, a fascinating phenomenon can be discovered, which the estimation stress along the axial direction is fundamentally equivalent at the outer surface of the cylinder.

In this example, we discuss the distinction \( \bar{\sigma}_0 - \bar{\sigma}_r \) between stress along the hoop direction \( \bar{\sigma}_0 \) and stress along the radial direction \( \bar{\sigma}_r \) as well. Figure 13 demonstrates the development of \( \bar{\sigma}_0 - \bar{\sigma}_r \) with various parameter.
n for FGPM and homogeneous material. The curve of $\sigma_0 - \sigma_1$ is steep for homogeneous PZT4 yet delicate for FGPM cylinder. So the designer can control the value of parameter $n$ to make the different positions of the cylinder reach plastic in the meantime, which is exceptionally useful for engineering practice.

Figure 14 plots the variation of electric potential with parameter $n$. Clearly, the electric boundary conditions $\varphi_{r=a} = 1$ and $\varphi_{r=b} = 0$ are same with equation (43). When the estimation of parameter $n$ is small, the electric potential change is near a straight line; and as the estimation of parameter $n$ expands, the nonlinear pattern of the electric potential turns out to be increasingly self-evident. The electric potential curve of the homogeneous piezoelectric material is close to the instance of $n=1.5$ of FGPM material.

4. Conclusions

The exact solutions of the related mechanical quantities of a FGPM empty thick-walled cylinder under axisymmetric mechanical and electric loadings are conducted in current work. The present method considers that the cylinder comprises of two materials (such as PZT4 and PVDF) and the content of the components is controlled by the volume fraction, which is different from other works that consider the cylinder comprises of
multi-layers. The method in this work can avoid the assumption of the distribution regularities of unknown overall material parameters appeared in existing papers such as elastic properties, piezoelectric tensor and dielectric moduli. This method is appropriate for real building gradient piezoelectric materials, and the volume fraction function can cover more complex circumstances. Numerical simulation compared the solutions of the FGPM cylinder and homogeneous PZT4 cylinder and a few conclusions can be acquired as follows: (a) For stress-dominated boundary conditions: the impacts of parameter $n$ on the stresses are not obvious but are gigantic in displacement along the radial direction and electric potential. Besides, the greatest estimation of the electric potential happens at an inner position and the position moves as parameter $n$ fluctuates. The bigger the estimation of parameter $n$ is, the closer the pinnacle is to the outside of the cylinder. (b) For electric potential-dominated boundary conditions: the estimation of displacement along the radial direction for FGPM is significantly smaller than that of the uniform material. The greatest estimation of stress along the radial direction is around exhibited amidst the cylinder and changes with various parameter $n$ areas. And also, the most extreme estimation of the stress along the radial stress moves to inner surface of the cylinder as $n$ expands. In addition, the contrast between the hoop stress and radial stress is acquired and the designer can pick reasonable parameter $n$ to make the different positions of the cylinder reach plastic in the meantime.
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