WHY AND WHENCE THE HILBERT SPACE IN QUANTUM THEORY?

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ABSTRACT. We explain why and how the Hilbert space comes about in quantum theory. The axiomatic structures of vector space, of scalar product, of orthogonality, and of the linear functional are derivable from the statistical description of quantum micro-events and from Hilbertian sum of squares $|a_1|^2 + |a_2|^2 + \ldots$. The latter leads (non-axiomatically) to the standard writing of the Born formula $f = |\langle \psi | \phi \rangle|^2$. As a corollary, the status of Pythagorean theorem, the concept of a length, and the 6-th Hilbert problem undergo a quantum ‘revision’. An issue of deriving the norm topology may no have a short-length solution (too many abstract math-axioms) but is likely solvable in the affirmative; the problem is reformulated as a mathematical one.

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Key words: quantum foundations, non-axiomaticity, bases of observables, orthogonality, scalar product, topology, Hilbert space, Pythagoras theorem, hierarchy of math/physical languages

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1. INTRODUCTION

The machinery of quantum theory (QT) starts with the Hilbert space and self-adjoint operators acting on it. The question of whence these very mathematical constructs come as axioms [18, 37] has been actively discussing in the literature hitherto [28, 14, 9, 31, 13]; because of its profound importance for the theory itself [33, 24].

At the same time, leaving aside the quantization of models, the most part of the theory is self-sufficiently described only by a linear vector space (LVS), i.e., turns into an empirical theorem [5] known as the principle of linear superposition of quantum states over the field of complex numbers $\mathbb{C}$. In doing so, the ‘quantum field’ $\mathbb{C}$ is naturally equipped with the complex-conjugation operation $(n + im) \mapsto (n - im)$, and for understanding the ‘primary/derivative’ in the quantum-math abstractions—vector space, linear operators, and the like—the following is necessary. The abstractions come from the number entities [27], and even the notion of the number itself is a nontrivial point in foundations of QT. Accordingly, ideology of deducing the QT-structures should be considered ‘through the lenses’ of coordinate representations of LVS, inasmuch as initially we have no motives to introduce any abstracta. For instance, the complex scalar product, along with the $\mathbb{C}$-numbers themselves, do still leave the issue of its own nature [2, 14], while the real case is seen as something quite natural. See, however, discussion about ‘identity’ between the classical Pythagoras’ theorem and the quantum Born rule in [7].

Yet a further salient feature lies in the fact that the genesis of quantum structures cannot in principle be physical—a point, long championed by G. Ludwig [30, Foreword], [29] and voiced clearly by other writers (E. Jaynes, A. Peres, D. Mermin; see references and quotations in ref. [5]). By this, one should understand that the justification in the physics’ tongue—temporal $t$-evolution, interaction with environment, collapses of states, notion of the ‘state $|\text{before}\rangle/|\text{after}\rangle’$, modelling the measuring process [8], etc.—will always suffer from the circular logic when attempting to derive mathematical apparatus of QT.

The situation is well known in the literature as the ‘never-ending’ debate over interpretations [28, 24, 1, 36]. This has an impact on physicality of argument and even on the ‘physical level of rigor’ in reasoning; this term should also be excluded.

* The search for underpinning of quantum theory—unusual as it may seem—is largely not a question of physics [30], much less of phenomenology: particles, phenomena, fields and their interactions.

The similar argument is already present in the literature [1, p. 220], [24]. For example, in abstract of the article [12], one openly claims that “mathematization of the physical theory … must contain no physical primitives. In provocative words: ‘physics from no physics’”.

Attempts to justify the formalism of QT [2, 12] had already begun under von Neumann as a programme called “continuous geometry” [32]. However, it is now recognized [25] that the programme had not been successful, though it gave birth to the three mathematically nice theories: algebraic approach to QM, quantum logic [14], and rings of operators (von Neumann algebras). In this connection we emphasize that the underpinning cannot
constitute the ready-made mathematics [5], because the mathematical structures, even commonly-used, are not delivered from above. The questions ‘whence/why?’ can hardly be answered in the sequence [35]

$$\text{[terminology, definienda]} \to \text{[statement]} \to \text{[proof]} \quad (1)$$

$$\big( + \text{[philosophical/physical ‘support’]} \big) .$$

As we will see below, the math-structures have an origin, and axiomatizations, so prevalent in QT [14, 24, 28, 30, 33, 36], are not necessary and should ideally be minimized; an orthodox opinion [2, 9, 13, 18, 28, 31]. Not only should the mathematical and physical phrasing not be leaned upon the postulates, but the language itself should be severely limited [30] and far from being free as in the classical theories. The problems with interpretations of quantum mathematics in terms of observables [28, 24] do not then arise since their language has not yet in place.

The answer to Neumann’s doubts quoted above (and oft mentioned in the literature [14]) lies in the fact that Hilbert’s space is not a starting point in quantum elements. For example, Ludwig has devoted the whole monograph [29] to the ‘quantum’ derivation (different from ours) of this structure. The different views of the Hilbert-space problem and exhaustive references along the lines can be found in the works [13, 23, 31]. In the first place, the space is just a linear space (superposition principle), of which the empirical nature manifests initially as a commutative group with operator automorphisms [26, sect. I.1.2]. These are the numbers, and focusing on them above is no accident.

Such a (re)formulation, at first glance, might have been seen as a full abstraction (see [5, sects. 7–8] for details), but one may go further. It is the quantum paradigm and ‘building’ the quantum mathematics from scratch, rather than a reliance upon the ready-to-use formal constructs, that provide the most convincing answer to the question of why and whence the complex numbers [9] and the very vector space. The number in quantum foundations, not only the complex, is not a matter of course in the context of the accustomed arithmetic [27]. The subsequent putting the question about observables as of entities and of their numerical values will result in answer to the question posed in the title of the work. With that, the observables arise alongside the state space but not yet as (Hermitian) operators.

The objective of this work is to focus on an end-point for the Hilbert-space problem; stated on p. 20 of the previous study [6]. The case in point is a derivation of what defines the Hilbertian add-ons over quantum LVS: scalar product $\langle \psi | \varphi \rangle$, orthogonality $|\psi \rangle \perp |\varphi \rangle$, norm $\|\psi\|$, topology, and bases of observables. In doing so, we do not assume—even in a hidden form—any additional mathematical structures or physical preconceptions/arguments: axioms/theorems of probability, quantum logic, “assertions about the physical world”, notably the word ‘measurement’, physical observations, etc. More precisely, one deduces an a priori unknown mathematics that stems from quantumness of the state-space and is dictated by quantum paradigm itself. The follow-up relationship between quantum Hilbert’ space and physics is discussed in the final section of the work.

### 2. Structures on Quantum-State Space

The basis for a vector space does always exist [17, 22], but the LVS-axioms, in and of itself, contains neither such a concept nor a motivation as to why/what-for the basis may/needs-to be changed. Similarly, the numeric quantities that are associated with quantum vectors—lengths and projections—do not follow from anywhere and hence may arise in the LVS’ theory only from the outside.
2.1. Bases in quantum theory (why?). Every LVS possesses the infinitely many bases but quantum space \(\mathbb{H}\) comes into being at the outset together with these objects [5]. More than that, it arises through the special kind bases, termed the \(\mathcal{A}\)-bases, such that each vector \(\mathcal{A}\)-representation

\[
|\Psi\rangle = a_1 \cdot |\alpha_1\rangle + a_2 \cdot |\alpha_2\rangle + \cdots \in \mathbb{H}
\]

has an associated number characteristic—\(\{f_j\}\)—statistics of quantum \(\mathcal{A}\)-micro-events \(a_j\) (pre-images of \(|\alpha_j\rangle\)-kets [5]). They should be read as the detector responses at a collider, the interferometer flashes, and the like. It is these \(|\alpha_j\rangle \neq |\alpha_k\rangle\) that implement the empirical distinguishability of \(a\)-clicks \(a_j \neq a_k\) from each other, and accumulation thereof into numerical arrays is formalized by the objects \(a_j \in \mathbb{C}\).

The statistical weights \(\{f_j\}\) is an observable entity and, hence, the \(\mathcal{A}\)-bases have a conventional terminology—the basis of an observable \(\mathcal{A}\) with the eigen states \(|\alpha_j\rangle\). The presence of a numerical concept \(\{f_j\}\) is an integral part of the superposition theorem [5]. Otherwise, we would have ‘an abstract bare’ LVS, and all the other and the familiar theoretical structures would have had to be postulated in their own rights. This would run counter to the task of ascertaining the nature of quantum axioms. Interrelation between \(\{f_j\}\) and bases of quantum LVS may be set forth more mathematically.

Being a collection of experimental numbers, the statistics \(\{f_j\}\) may have a source in LVS (‘to be calculated from’) only from its primary numeric quantities. But ‘bare’ LVS contains nothing but vectors, field \(\mathbb{C}\), and dimension \(\text{dim} \mathbb{H}\). Therefore, for quantum observations these quantities may only be the \(\mathcal{C}\)-coefficients of superpositions

\[
c_1 \cdot |e_1\rangle + c_2 \cdot |e_2\rangle + \cdots = |\Psi\rangle.
\]

The meaningfulness/uniqueness of the \(c\)’s, for a given vector \(|\Psi\rangle\), is possible only if the family \(\{|e_1\rangle, |e_2\rangle, \ldots\}\) forms a basis of linearly independent vectors. Coefficients \(c\)’s in the arbitrary abstract superpositions

\[
c_1 \cdot |\Phi_1\rangle + c_2 \cdot |\Phi_2\rangle
\]

would not do for this.

Since there are no restrictions on the bases of LVS, the \(c\)-coefficients in (3)–(4) may be any collections; i.e., any coordinates may be assigned to every vector. Hence, formula for computation of frequencies \(\{f_j\}\) from coefficients \(c_j\) may exist only if \(\{|f_j\rangle\}\) is a basis but it is not free. What is required is a basis, for which both its ket-vectors and the expansion coefficients in (3) keep the nature of the origin of the basis as a concept—accumulating the (in)distinguishable micro-events \(a_j\). Therefore it is permissible to rely only on coefficients \(a_j\) in \(|\alpha\rangle\)-expansions (2), i.e., on bases, to which the \(f\)-statistics has been attached. This is implemented by a core object—the statistical length [6]

\[
\text{StatLength}(a_1 \cdot |\alpha_1\rangle + a_2 \cdot |\alpha_2\rangle + \cdots).
\]

The function \(\text{StatLength}(\cdots) =: \mathcal{N}[\cdots]\) is utterly minimalistic in its derivation—it requires neither Hilbert’s space nor physics, is unique, and has a sum-of-squares form [6]

\[
\mathcal{N}[a_1 \cdot |\alpha_1\rangle + a_2 \cdot |\alpha_2\rangle + \cdots] = |a_1|^2 + |a_2|^2 + \cdots.
\]

From this, there immediately follows the formula

\[
f_k = \frac{|a_k|^2}{|a_1|^2 + |a_2|^2 + \cdots}.
\]
which will be further turned into the standard writing \( t = |\langle \psi | \phi \rangle|^2 \) of the famous Born rule [3]. Expressed another way, the formula to be directly deduced is not a Neumann–Hilbertian version of this rule like \( t = |P \psi|^2 \), but rather the purely numerical full-ratio (7). The derivation exploits [6] only the minimal structure—the vector space, but numerical form (7) should be represented invariantly.

In consequence, whereas the concept of a quantum \( \mathcal{A} \)-basis has been present in the superposition theorem [5] and has been exploited when deriving (7), formalization of this term has yet to be created.

2.2. \( \| \psi \| \) and \( \langle \psi | \varphi \rangle \). As for the scalar product and norm on LVS, the self-evident usage of these structures seems also illogical in keeping with the axiom-free building the QT. At the moment, they have no empirical grounds. For example, empiricism of experiment is inherently unable to tell us anything about property \( \langle \psi | \varphi \rangle = |\langle \varphi | \psi \rangle| \) for a certain map \((\mathbb{H} \times \mathbb{H}) \to \mathbb{C}\); this point is often discussed in quantum logic [14].

The habitual classical and physically illustrative argumentation is not an exception. More to the point, as noted above, such argumentation has actually been banned [5], because it is a source of confusions and of logical inconsistencies [36]. Say, the typical use of the aforementioned map \(|\langle \text{in} | \text{out} \rangle|^2\) for calculation of the ‘transition probabilities between states |before\rangle and |after\rangle’ should be regarded in a very conditional way, since each of the words (and combinations thereof) in this sentence is still a subject of discussion [28, 24, 1, 18], especially the philosophical.

If, in contrast, we draw on these structures as on the ready-made ones, then one will be required the guessing and substantiating their quite specific properties such as parallelogram rule and the triangle inequality

\[
\| \psi \oplus \varphi \|^2 + \| \psi \ominus \varphi \|^2 = 2 \| \psi \|^2 + 2 \| \varphi \|^2 , \quad \| \psi \| + \| \varphi \| \geq \| \psi \oplus \varphi \| ,
\]

the polarization procedure

\[
4 \langle \psi | \varphi \rangle = \| \psi \oplus \varphi \|^2 - \| \psi \ominus \varphi \|^2 + i \| \psi \oplus i \cdot \varphi \|^2 - i \| \psi \ominus i \cdot \varphi \|^2 ,
\]

the companion von Neumann–Jordan–Fréchet theorems, equivalence of norms on LVS, and the like [17, 22, 21]. The questions “whence/why?” do not go away here (just being shifted to another domain), and the familiar quantum ‘difficulties in relationship’ (physics \( \rightarrow \) mathematics) are aggravated inasmuch as the motivating is substituted for definienda and the inferencing—for proofs. Beyond the matter of quantum bases, the typical example is a slightly disconcerting Riesz’ theorem on the representation of a (bounded) linear functional by a scalar product [17, 21]. Functionals on the LVS are more primary, as they do not require for their own definition anything but LVS itself, whilst the scalar product is a quite nontrivial external add-on over it. Not to wonder why and whether the concept of a linear functional should come into play (see sect. 5.2).

2.3. Summary. Now, the logic in foundations of QT should, ideally, avoid the sequencing (1), and we will adhere to the scheme back toward (1):

\[
([\text{common ‘philosophical background’}] \oplus ) \quad [\text{motivation}] \to [\text{inference}] \to [\text{construct}] \to [\text{formalization, terminology}] .
\]

This is because the search precisely for such a scheme [18, Introduction (!)] constitutes the dominant bulk of the subject matter of so extensive literature on math of quantum foundations [14, 36, 13]. That is why nearly all the quantum terminology we have been accustomed
[17, 21], e. g.,

\[
\begin{align*}
&\text{[scalar product]}, \quad \text{[norm]}, \quad \text{[dual space]}, \\
&\text{[linear operators/functionals]}, \quad \text{[self-adjointness]}, \quad \text{[spectra]}, \\
\end{align*}
\]

expressions and devices like

\[
\left[ \langle \psi | \varphi \rangle, \quad \hat{P}, \quad a_1 b_1^* + a_2 b_2^* + \cdots, \quad \ldots \right]
\]

should be viewed methodologically as non-existing at the beginning. The usage of such terms is forbidden until they have been created explicitly without their implication a priori and without ‘guessing or fitting underneath’ the familiar constructs.

A few words about the LVS itself. As was the case with deriving the rule (7), the whole discourse does not depend on “mechanism for the emergence of LVS” [6, p. 4]. If desired, the statement about quantum linearity [5] may be forgotten/ignored as a theorem, and one may keep within the orthodox/postulational view on the superposition. Then the questions are:

- Is such a non-Hilbertian LVS-version of the superposition principle sufficient to produce the remaining (counterintuitive for comprehension) quantum math? Will that minimalism call for extra premises or axioms?

The answers are ‘yes and no’, respectively, as soon as this LVS is supplemented with the ‘observable’ function (6).

Given what we have to work with, the problem as to how a calculus on the $\mathbb{H}$-space will look like, i.e. which mathematical structures are entailed by quantum paradigm, is determined in our exposition only by the data pair

\[
\text{[vector space $\mathbb{H}$]} \oplus \text{[the number function (6)]}
\]

(and nothing more). In other words, mathematics of QT should be created not from axioms, calculus acquires the character of a linear-algebra calculus [22, 17], but emergence of the structures—operators, their spectra, hermicity, quadratic/bilinear forms, the informational conceptions, quantum dynamics, and the like—is not postulated a priori. None of these terminologies will be further required.

The numerical nature of Born’s rule—function $\text{StatLength}$ on abstract LVS—has already been described in [6], and we may therefore characterize subsequent actions also in the ‘vein of the numerical ideology’. This is because the abstract linear space of the $|\Psi\rangle$-vectors is isomorphic [22, 17] to its model of the number collections $(a_1, a_2, \ldots) \in \mathbb{C}^n$:

1° The numerical form of relations in $\mathcal{A}$-bases. Orthogonality (sect. 3).
2° The numerical form of a new object—product of vectors (sect. 4).
3° Axiomatization of the scalar product $\langle \Psi | \Phi \rangle$ (sect. 5).
4° Revision of orthogonality and of length in classical geometry (sect. 6).
5° Back to the abstraction $\mathbb{H}$. The unitary LVS.
6° Topology and norm. The quantum Hilbert space (sect. 7).

Thus, the appearance sequence of what follows, i.e.

\[
\begin{align*}
&\text{[statistics]} \rightarrow \text{[LVS-bases]} \rightarrow \text{[unitarity]} \rightarrow \text{[‘orthogonality’]} \rightarrow \text{[observables]} \rightarrow \\
&\text{[scalar product]} \rightarrow \text{[axiomatization]} \rightarrow \text{[unitary space]} \rightarrow \text{[topology]} \rightarrow \text{[Hilbert space]}
\end{align*}
\]
is pretty much the opposite of the typical QT-axiomatics followed by interpreting: [axioms/math of Hilbert space] → [(statistical) treatment].

3. WHAT IS (QUANTUM) ORTHOGONALITY?

Our next immediate task is to find out a formulation of distinguishability of ℂ-bases from the others, and in addition to the LVS itself, the only thing we may exploit is the additive nature of the statistical-length function (6):

$$\mathcal{N}[\alpha_1 \cdot |\alpha_1\rangle \pm \alpha_2 \cdot |\alpha_2\rangle \pm \cdots] = \mathcal{N}[\alpha_1 \cdot |\alpha_1\rangle] + \mathcal{N}[\alpha_2 \cdot |\alpha_2\rangle] + \cdots \quad \forall \alpha_i.$$  \hspace{1cm}(9)

It is admitted only to such bases or, to be precise, it is through this nature that determines them [6]. One notices that we do not even assume a priori that the belonging to a basis of an observable should be numerical in character and binary, i.e., that it should be given by some numerical relation between its two members.

At the same time, additivity (9) and arbitrariness of coefficients αi say that the N-function is defined, as a minimum, by a sum of only two terms. The general case (9) is processed recursively. Therefore the question of belonging to ℂ-basis reduces to the pairwise relations between its elements. This gives rise to the term binary superstructure in theory.

In order not to burden notation with indices, let us write

$$\mathcal{N}[a \cdot |\alpha\rangle + b \cdot |\beta\rangle] = \mathcal{N}[a \cdot |\alpha\rangle] + \mathcal{N}[b \cdot |\beta\rangle] \quad \forall a, b \in \mathbb{C},$$

assuming that vectors |α⟩ and |β⟩ belong to a certain ℂ-basis.

3.1. The nature of orthogonality. In line with ideology 1°–6°, let us switch over a Cⁿ-number equivalent of the space H. That is, in accord with (2), we identify all its vectors with their number ℂ-representatives:

$$\{\text{vectors } |\Psi\rangle \in H\} \xrightarrow{\text{def}} \{\text{collections } (a_1, a_2, \ldots) \in \mathbb{C}^n\} =: H_{\mathbb{C}}.$$  \hspace{1cm}(11)

Because of isomorphism between H and Cⁿ [17], the algebraic operations on the space H_{\mathbb{C}} will be denoted by the same symbols {+, ·}:

$$(a_1, a_2, \ldots) + (b_1, b_2, \ldots) := (a_1 + b_1, a_2 + b_2, \ldots), \quad (a_1, a_2, \ldots) \cdot (c_1, c_2, \ldots) := (c_1 a_1, c_2 a_2, \ldots).$$ \hspace{1cm}(12)

The statistical-length function N, as a function of an ℂ-representation of vector (5), turns into a function Nₙ that is fully defined on all the H_{\mathbb{C}}-vectors by formula

$$\mathcal{N}_n[(a_1, a_2, \ldots)] = |a_1|^2 + |a_2|^2 + \cdots \quad \forall (a_1, a_2, \ldots) \in H_{\mathbb{C}};$$

now, without reference to the concept of a basis. But since the N was being derived as an ℂ-invariant construct—this is one of its axioms [6]—the change of bases* should be carried over to the space H_{\mathbb{C}}. This amounts to a transition to the same but one more space Hₙ. To put it differently, invariance speaks about a consistent universality of the squares formula

$$\mathcal{N}_n[(b_1, b_2, \ldots)] = |b_1|^2 + |b_2|^2 + \cdots \quad \forall (b_1, b_2, \ldots) \in H_{\mathbb{C}},$$ \hspace{1cm}(13)

i.e., about the equality $\mathcal{N}_n[(a_1, a_2, \ldots)] = \mathcal{N}_n[(b_1, b_2, \ldots)]$, when $(a_1, a_2, \ldots) \in H_{\mathbb{C}}$ and $(b_1, b_2, \ldots) \in H_{\mathbb{C}}$ do represent the one vector |Ψ⟩ ∈ H. Then one may forget the statistical treatment to the ℂ-expansions (2) and regard the N as an abstract and well-defined function on all the spaces Hₙ.

*The 'change of bases and representations', as a mathematical action, is involved not just for a formalization, but is a fundamental point for the emergence of QT itself [5, sect. 5.4].
The two $\mathcal{A}$-base vectors $|\alpha\rangle$ and $|\beta\rangle$ in (10) play a dedicated role at the moment. Their coordinates have special form

$$
\mathbb{H} \ni |\alpha\rangle \xrightarrow{} (1, 0, 0, \ldots) \in \mathbb{H}_\mathcal{A},
$$

$$
\mathbb{H} \ni |\beta\rangle \xrightarrow{} (0, 1, 0, \ldots) \in \mathbb{H}_\mathcal{A}.
$$

(14)

What can be said about their representations in other $\mathbb{H}_\mathcal{A}$’s? Let their $\mathcal{B}$-representatives, after transition to a different $\mathcal{A}$-basis $\mathcal{B}$, be designated as

$$
\mathbb{H} \ni |\alpha\rangle \xrightarrow{\mathcal{B}} (\mathfrak{A}_1, \mathfrak{A}_2, \ldots) \in \mathbb{H}_\mathcal{A},
$$

$$
\mathbb{H} \ni |\beta\rangle \xrightarrow{\mathcal{B}} (\mathfrak{B}_1, \mathfrak{B}_2, \ldots) \in \mathbb{H}_\mathcal{A}.
$$

These have already a considerable numerical freedom $\mathfrak{A}_1, \ldots, \mathfrak{B}_1, \ldots \in \mathbb{C}$.

Invoke now the additivity, i.e., let us write down (10) in this new basis:

$$
\mathcal{N}_\mathcal{A}[a \cdot (\mathfrak{A}_1, \mathfrak{A}_2, \ldots) + b \cdot (\mathfrak{B}_1, \mathfrak{B}_2, \ldots)] = \mathcal{N}_\mathcal{A}[a \cdot (\mathfrak{A}_1, \mathfrak{A}_2, \ldots)] + \mathcal{N}_\mathcal{A}[b \cdot (\mathfrak{B}_1, \mathfrak{B}_2, \ldots)].
$$

According to algebra (12), we have

$$
\mathcal{N}_\mathcal{A}[(a\mathfrak{A}_1 + b\mathfrak{B}_1)(a\mathfrak{A}_1 + b\mathfrak{B}_1)^* + (a\mathfrak{A}_2 + b\mathfrak{B}_2)(a\mathfrak{A}_2 + b\mathfrak{B}_2)^* + \cdots] = \{ (a\mathfrak{A}_1)(a\mathfrak{A}_1)^* + (a\mathfrak{A}_2)(a\mathfrak{A}_2)^* + \cdots \} + \{ (b\mathfrak{B}_1)(b\mathfrak{B}_1)^* + (b\mathfrak{B}_2)(b\mathfrak{B}_2)^* + \cdots \}.
$$

By expanding and canceling, one arrives at equation (recalling the arbitrariness of $a, b$)

$$
(ab^*)(\mathfrak{A}_1\mathfrak{A}_1^* + \mathfrak{A}_2\mathfrak{A}_2^* + \cdots) + (ab)(\mathfrak{A}_1\mathfrak{B}_1 + \mathfrak{A}_2\mathfrak{B}_2 + \cdots) = 0 \quad \forall a, b \in \mathbb{C},
$$

that is

$$
\text{Re}\{c(\mathfrak{A}_1\mathfrak{A}_1^* + \mathfrak{A}_2\mathfrak{A}_2^* + \cdots)\} = 0 \quad \forall c \in \mathbb{C}.
$$

It follows that

- for every two $\mathcal{A}$-base vectors $|\alpha\rangle$ and $|\beta\rangle$, their coordinates in a basis of any other observable must satisfy the numeric relation

$$
\mathfrak{A}_1\mathfrak{A}_1^* + \mathfrak{A}_2\mathfrak{A}_2^* + \cdots = 0.
$$

(15)

Whilst this relation is coordinate, its meaning is absolute for all the $\mathcal{A}$-bases, and we call it orthogonality relation $|\alpha\rangle \perp |\beta\rangle$. The point 1° has been completed. The nature of orthogonality as of a concept is identical to that of Born’s rule—the numerical.

3.2. Observable = Orthogonal Basis. In view of the fact that the associating a StatLength with expansions (2) is the only thing that distinguishes the quantum space from merely linear $V$, the mechanism for making an abstract LVS the quantum $\mathbb{H}$ is not axiomatic and is as follows.

One considers an abstract $V$. Any vector $|\mathbf{e}_i\rangle \in V$ may be viewed as an eigen one for certain observable $\mathcal{A}$; this is fully aligned with the statistical genesis of ket-objects [5, sect. 6.2]. Let us declare this vector to be an $|\alpha\rangle$-vector for the $\mathcal{A}$. Similarly for any other linearly independent vectors $\{ |\mathbf{e}_2\rangle, \ldots \} = \{ |\mathbf{e}_2\rangle, \ldots \}$ down to exhausting the dimension $\text{dim} V$. One obtains an LVS with a dedicated (‘good’) basis—the (orthogonal) basis of a certain (any) observable $\mathcal{A}$. Such an act does always have a set-theoretic character and always is a matter of the declaration/appointment. Despite the title ‘observable’ in this appointment, there is no sense in looking for its origin in the temporal processes like [quantum $\implies$ classical] or for
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a physical underlying reason through the natural-language meaning to the word ‘observable’. Similarly, the geometric ‘meaning’ of the word orthogonal. An analogy: in a set, no its element is ‘innately good/bad’ (say, by physical reasons) until over this set there has been created a math superstructure-criterion, according to which some set-members differ from the others. The role of a superstructure over \( LVS \) is played here by the very basis \( \mathcal{A} \), with the ascribed function \( \text{StatLength} \)–\( (5)–(6) \), and in which the orthogonality property \( (15) \) is seen to be ‘hard-wired’. Orthogonality for \( \mathcal{A} \)-elements does automatically hold due to \( (14) \).

Let us carry out all possible transformations \( U \) of \( \mathcal{A} \) that preserve the property ‘to be an \( \mathcal{A} \)-basis’. Emergence of the \( U \)-transformations \( ^* \) has been described in \([6]\). One gets all the other \( \mathcal{A} \)-bases and, thereby, they become special (not arbitrary) since the \( U \)-matrices are not arbitrary:

\[
U^*U = 1 .
\]

These may be normally referred to as complex-orthogonal. Due to group property, the transformations \( U \) ‘depersonalize’ the initial \( \mathcal{A} \), and it ceases to be a dedicated basis or, according to the physical terminology, the preferable one \([8, \text{“pointer state”}] \). Such \( \mathcal{A} \)-bases—an oft-discussed subject in quantum measurement theory \([8]\)–should not be present in QT \([6, \text{sect. 6a}] \), because this has been demanded by the basic principle of invariance and a prohibition on the physical meaning to \( |\text{ket}\rangle \)-vectors \([19]\). The remaining bases \( \{ |e_j\rangle \} \) drop out of the \( U \)-series—they are not orthogonal, non-\( \mathcal{A} \)-bases—and expansions \( (3) \) over them are ‘bad’ in the context of the statistical function \( \mathcal{N} \); its values are ill-defined. The \( \z_j \)-numbers in \( (3) \) simply do not have any observable meaning, just like it is not in arbitrary \( (\hat{\cdot}) \)-superpositions \( (4) \).

As an outcome, the concept of quantum observable—we are not talking about its numerical values \( \alpha_j \) —is identical at the moment to the orthogonal basis, and vice versa; though the term “orthogonal” has not yet been completed as a mathematical structure. By nature, such a meaning to the \( \mathcal{A} \)-vectors (spectra) and linear operators are absent in discourse at the moment (they are not required and have not yet come into being), we have deal merely with \( \mathcal{A} \)-numbers in expansions \( (2) \). Being the carriers of \( \mathcal{I} \)-statistics \( (7) \), these numbers will represent the subject matter of the \( H \)-calculus, and it must be independent—put this in a definition of the word \( H \)-calculus—of the coordinate spaces-representations \( H_\mathcal{A} \).

4. Whence the scalar product?

If quantum \( |\alpha\rangle \)-representations produce statistics, then what do we multiply the abstract \( |\Psi\rangle \)-states for \([14]\)? Why scalarly?

4.1. Statement of problem. Why binarity? Quantum theory is a numerical one, which is why upon establishing the properties of \( \mathcal{A} \)-bases, it is necessary to address the arbitrary vectors in order to build the formal calculus on \( H \). We still do not have it per se and what it should consist of is undefined at the moment.

Because the spectral labels of \( |\alpha\rangle \)-vectors (spectra) and linear operators are absent in discourse at the moment (they are not required and have not yet come into being), we have deal merely with \( \alpha \)-numbers in expansions \( (2) \). Being the carriers of \( \mathcal{I} \)-statistics \( (7) \), these numbers will represent the subject matter of the \( H \)-calculus, and it must be independent—put this in a definition of the word \( H \)-calculus—of the coordinate spaces-representations \( H_\mathcal{A} \).

\* Unitarity \( U \) at the moment is (and arises originally as) but a property of a numerical collection \( U \), i.e., it is not an algebraic object and is not of the invariant nature.
The aforesaid means that the space of quantum states $\mathcal{H}$ is supplemented with the task of deducing an invariant formula

$$a_j = \mathcal{N}(\langle \psi \rangle, |\alpha_j\rangle)$$

(17)

as an operation of ‘extracting’ the $a$-coefficient from expansion (2) when $|\alpha_j\rangle$ is a member of an $\mathcal{A}$-basis. Invariance of the formula is also called for because the $\mathcal{H}$-space itself owes its existence to the availability of at least two instruments $\mathcal{A}$ and $\mathcal{B}$, and none of them are preferable. The formula is necessary also for obtaining the invariant writing of Born’s rule (7). Notice that any different way of searching-for or the ‘right proof’ of the rule will knowingly have not met with success because, in experiment, there are neither operators nor the $\langle \text{bra}\mid \text{ket}\rangle$-abstracta; there is no even the arithmetic there [5, sect. 2.3].

Inasmuch as not only is the $|\psi\rangle$ arbitrary, but every vector of the space may serve as an eigen one $|\alpha_j\rangle$, then the problem (17) should be solved through a certain (yet another) binary superstructure over the entire $\mathcal{H}$: $(\mathcal{H} \times \mathcal{H}) \overset{\mathcal{N}}{\rightarrow} \mathbb{C}$. That is just what the theory of function $\mathcal{N}$ is. It is appended to the $\mathcal{H}$-space, which remains as is without the need for its (over/re)defining.

How to search for $\mathcal{N}$? Has it had, being a function of unnecessarily nonorthogonal vectors, a link with orthogonality of the $\mathcal{A}$-basis ones? To be totally precise, we do not even have the concept of nonorthogonality, since orthogonality (15) is currently a structure not over the entire $\mathcal{H}$ but is an exclusive property of some special collections—bases of observables.

4.2. Product of arbitrary vectors. While remaining within ‘numerical ideology’ $1^\circ–6^\circ$, let us write an equality of the two $\mathcal{A}$-representations for $|\psi\rangle$:

$$a_1 \cdot |\alpha_1\rangle + a_2 \cdot |\alpha_2\rangle + \cdots = |\psi\rangle = b_1 \cdot |\beta_1\rangle + b_2 \cdot |\beta_2\rangle + \cdots .$$

(18)

To solve the task (17), one suffices to solve it at first in coordinates, i.e., to express coordinates $a_j$ through coordinates $\{b_j\}$ of arbitrary vector $|\psi\rangle$ in any new $\mathcal{A}$-basis $\{|\beta_j\rangle\}_\mathcal{A}$. Having obtained an answer, this is the pt. $2^\circ$, we come back from the coordinate language of spaces $\{\mathcal{H}_\alpha, \mathcal{H}_\beta, \ldots \}$ to the initial $\mathcal{H}$; this will be done in sect. 5 as required by pts. $3^\circ$ and $5^\circ$.

By virtue of unitarity (16), the relationship between bases $\{|\alpha_j\rangle\}$ and $\{|\beta_j\rangle\}$ is as follows:

$$|\beta_1\rangle = U_{11} \cdot |\alpha_1\rangle + U_{12} \cdot |\alpha_2\rangle + \cdots \quad |\alpha_1\rangle = U_{11}^\dagger \cdot |\beta_1\rangle + U_{12}^\dagger \cdot |\beta_2\rangle + \cdots$$

$$|\beta_2\rangle = U_{21} \cdot |\alpha_1\rangle + U_{22} \cdot |\alpha_2\rangle + \cdots \quad |\alpha_2\rangle = U_{21}^\dagger \cdot |\beta_1\rangle + U_{22}^\dagger \cdot |\beta_2\rangle + \cdots .$$

(19)

Properties of unitary matrices include the algebraic relations between $U_{1k}$ and $U_{2k}$ [20, sect. IX.7], but we will view of them also as non-existent. However, it is clear that they are corollaries of formulas (14), (16), and of the unit statistical length

$$\mathcal{N}[|\alpha_j\rangle] \rightarrow \mathcal{N}_{|\alpha_j\rangle}[(0, \ldots, 0, 1, 0, \ldots)] = \mathcal{N}_{|\beta_j\rangle} = \cdots = 1 .$$

(20)

Substitution of $|\beta_j\rangle$ from (19) into (18) gives the commonly known rule of conversion between the $a_j$- and $b_j$-coordinates of one and the same vector $|\psi\rangle$:

$$a_1 = b_1 U_{11} + b_2 U_{21} + \cdots$$

$$a_2 = b_1 U_{12} + b_2 U_{22} + \cdots .$$

(21)
Right hand part of these formulas needs to be realized in the $\mathbb{H}_\sigma$-space. Here, according to the task (17), there must appear only the vectors $|\Psi\rangle$ and $|a_i\rangle$; more precisely, coordinates of these objects. Is this possible?

For simplicity, consider the 1-st formula in (21). In it, the row $(b_1, \ldots)$ is already an $\mathbb{H}_\sigma$-image of the first $\mathcal{N}$-argument in (17), i.e. $|\Psi\rangle$:

$$ (18) \quad \Rightarrow \quad |\Psi\rangle \overset{\mathcal{B}}{\mapsto} (b_1, b_2, \ldots) =: \Psi \in \mathbb{H}_\sigma , $$

whereas the row $(\mathcal{U}_{11}, \mathcal{U}_{21}, \ldots)$, on the face of it, is a free aggregate of side numbers. However, according to the second module in representations (19), one has

$$ |a_1\rangle \overset{\mathcal{B}}{\mapsto} (\mathcal{U}_{11}, \mathcal{U}_{21}, \ldots) =: A \in \mathbb{H}_\sigma , $$

i.e., the aggregate $(\mathcal{U}_{11}, \mathcal{U}_{21}, \ldots)$ is exactly the complex conjugation of coordinates of the second $\mathcal{N}$-argument in (17)—coordinates of vector $|a_1\rangle$ in basis $\mathcal{B}$. Thus the right hand side of formulas (21)

$$ b_1 (\mathcal{U}_{11})^* + b_2 (\mathcal{U}_{21})^* + \cdots = \cdots $$

turns, as required, into the coordinate expression of vectors (i.e. $\mathcal{U}$ disappears):

$$ \cdots = \Psi_1 A_1^* + \Psi_2 A_2^* + \cdots =: a_1 \text{ in } \mathcal{B}\text{-representation}. $$

In its turn, this very expression coincides with the orthogonality structure (15). With that, by contrast to (15), there is nothing special about vectors $\Psi, A \in \mathbb{H}_\sigma$. Both of them may be arbitrary, with only one restriction on $\text{StatLength}$ of vector $A$:

$$ (20) \quad \Rightarrow \quad \mathcal{U}_{11} \mathcal{U}_{11} + \mathcal{U}_{21} \mathcal{U}_{21} + \cdots = 1 \quad \Rightarrow \quad A_1 A_1^* + A_2 A_2^* + \cdots = 1 \quad (= N([a_1])) . $$

As a result, the vector $|a_1\rangle \overset{\mathcal{B}}{\mapsto} (1, 0, \ldots) \in \mathbb{H}_\sigma$, being a fixed one, has turned into an arbitrary $\mathbb{H}_\sigma$-vector (22) thanks to the freedom in $\mathcal{U}$. In other words, both the orthogonality form (15) and the coordinate’s calculation of any vector are implemented in all the $\mathbb{H}_\sigma$-spaces through the unique numeric expression $a_1 b_1^* + a_2 b_2^* + \cdots$. It becomes universal, and $\Psi$ and $A$ may be replaced by the two arbitrary vectors $A, B$:

$$ (a_1, a_2, \ldots) =: A \in \mathbb{H}_\sigma , \quad (b_1, b_2, \ldots) =: B \in \mathbb{H}_\sigma . $$

This freedom allows us to forget that one of the vectors was a preimage of the $|a_i\rangle$-eigen one in the map (11) and consisted of zeroes/unities (14). Even the restriction (23) becomes fictitious since $\text{StatLength}$ of any vector $A$, once it has been declared to be an $\sigma$-basic one, can always be scaled to the unity by (20). No orthogonality (15) with other $|a\rangle$-vectors gets lost by this.

4.3. Résumé. All the $\mathbb{H}_\sigma$-spaces are certain $\sigma$-realizations of the space $\mathbb{H}$. Therefore, by virtue of arbitrariness of $\{a_i, b_j\}$, we introduce a designation for the $\mathcal{C}$-numerical and new form-‘multiplication’:

$$ a_1 b_1^* + a_2 b_2^* + \cdots =: (A, B) , \quad \forall A, B \in \mathbb{H}_\sigma . $$

It becomes an $\mathbb{H}_\sigma$-representative of the invariant binary construct $\mathcal{N}$, now on the entire $\mathbb{H}$, and does prepare solution to the problem (17). Moreover, orthogonality (15) completely fits in the structure of this multiplication when the value of this form vanishes. Any vectors $\Psi$ and $A$ may now be multiplied according to the rule (24), yielding their indecomposability $(\Psi, A) = 0$ through each other or, in accord with (17), the expansion coefficients with respect to orthogonal bases:

$$ \Psi = a \cdot A \overset{\text{[A'-component]}}{\mapsto} \forall \Psi, A \in \mathbb{H}_\sigma . $$

$$ a = \frac{(\Psi, A)}{(A, A)} , \quad \forall (\Psi, A) = 0 $$
Orthogonality of basis \( \{ A^{(i)} \} \) may then be consistently thought of as \(( A^{(i)}, A^{(j)} ) \sim \delta_{jk} \), although it is, as before, independent of the construction \((24)\).

The main conclusion now is a conclusion about the appearance sequence. It is a weaker and a particular structure of orthogonality \((15)\), not multiplication of arbitrary vectors \((24)\), that is primary when creating the quantum mathematics.

In turn, the very orthogonality is preceded by the \( \text{StatLength} \) \((6)\). Had we not stated the task \((17)\), the property \((15)\) and unitarity would still exist as a property ‘to be an \(\mathcal{A}\)-basis’. Orthogonality does not depend on the task \((17)\) and ‘knows nothing’ about calculating the \(\alpha\)-coefficients in expansions \((2)\) by formulas like \((25)\) or much less by the familiar versions thereof through Neumann’s projectors \(\hat{P}_x\). When following a different sequence, the formal postulation of quantum-vector multiplication will always call for motivation of binarity, of scalarness, and of linearity (see below); not to mention the complex-conjugation operation in axioms of the scalar product.

Certainly, all these commentaries can be carried over to the pure mathematics—introduction of the additional math structures on vector spaces at all \([21]\). For example, S. Gudder introduces the concept of orthogonal additivity \([21\text{, sect.} \, 5.2]\), whilst additivity \((9)\) is primary in itself \([6]\) and orthogonality \((15)\) is its consequence.

Generality of the multiplication-form \((24)\) allows us to cast away not only the restriction on \( \text{StatLength} \) of basis vectors \(N_\nu[A^{(i)}] = 1\) but even to forget the concept itself. For all the \(\mathcal{A}\)-instruments, the \(\text{StatLength}\) is replaced now by the structure \((A, B)\) with redefinition \(N_\nu[A] = \langle A, A \rangle\). Then formula \((7)\), having regard to \((25)\), is obviously modified:

\[
f_k = \frac{N_\nu[\alpha_k \cdot A^{(i)}]}{N_\nu[\alpha_1 \cdot A^{(i)} + \cdots]} = \frac{|\alpha_k|^2 \langle A^{(i)}, A^{(i)} \rangle}{N_\nu[\psi]} \langle \psi, \psi \rangle = \frac{\langle \psi, A^{(i)} \rangle \langle A^{(i)}, A^{(i)} \rangle}{\langle \psi, \psi \rangle} = \frac{\langle \psi, A^{(i)} \rangle \langle A^{(i)}, A^{(i)} \rangle}{\langle \psi, \psi \rangle}.
\]

However we will remain within the framework \(N[|\alpha\rangle] = 1\) in order not to replace the well-established definition of unitarity. See ref. \([7]\) for the full discussion on this unity and the values \(N[|\alpha\rangle]\)’s.

### 5. Space \(\mathbb{H}^\delta\)

Let us take up the ‘returning’ from \(\{ \mathbb{H}_\mathcal{A}, \mathbb{H}_\mathcal{A}, \ldots \} \) to the \(|\Psi\rangle\)-abstracta of the space \(\mathbb{H}\); i.e. pt. \(3^5\).

#### 5.1. Axiomatization

The players of the construct \((24)\) are a non-ordered pair \(\{ A, B \} \) and the point that both of these vectors are the LVS-algebra’s elements. This is all we need when formalizing the multiplication \((24)\).

The orthogonality relation is symmetric—from \(A \perp B\) there follows \(B \perp A\). At the same time, the ‘equal quantum rights’ of \(A\) and \(B\) say that in expansion of one vector along a basis wherein the second is its element, i.e. in formulas

\[
A = c \cdot B + [B^-\text{-component}], \quad B = c' \cdot A + [A^-\text{-component}],
\]

each of the vectors may be deemed first or second. We should therefore consider a permutation of words [first \(\rightarrow\) second] in \((24)\) and its degeneration—a case [the first] = \(\perp\) the second. Obviously, the permutation a \(\leftrightarrow\) b entails the properties

\[
(A, B) = (B, A)^*, \quad \{ (A, A) \geq 0, \ (A, A) = 0 \implies A = 0 \}.
\]
Notice that (27) is not merely a side property, but the one that will define the axiomatization. It is the aforesaid ‘equal-rights’ that demand for this.

One is left with analyzing the algebraic operations \( \{ \hat{+}, \cdot \} \) of the vector space itself:

\[
(A \hat{+} B, C) = ?, \quad (\lambda \cdot A, B) = ?.
\]

Application of addition/multiplication (12) to (24) gives the linearity rules:

\[
(A \hat{+} B, C) = (A, C) + (B, C) , \quad (\lambda \cdot A, B) = \lambda \times (A, B) \quad \forall \lambda \in C . \tag{28}
\]

Algebra of LVS does not contain any other structures such as the order relation or continuity, which is why no quantum paradigm requires any further steps for formalizing the construct (24).

It is known that nonisomorphic models of the abstract \( V \), under the fixed dimension \( \operatorname{dim} V \), do not exist [17, 22]. Therefore we may forget about \( C^\omega, \mathbb{H}_2 \) and the word ‘model’ at all and turn the properties (27)–(28) for \( \{ \mathbb{H}_1, \mathbb{H}_2, \ldots \} \) into the abstract axioms of a new abstract binary superstructure \( (\mathbb{H} \times \mathbb{H}) \mapsto C) \).

**Remark 1.** Conversely, if a physical problem is realized by a certain model for \( \mathbb{H} \)—functional, the matrix-based, by \( \psi \)-functions, etc.—i.e. not necessarily by the \( C^\omega \)-model, then the model will in no way contradict the deduced axioms. At the same time, the question about interpreting the vectors of the model must not appear, since identifying the wave functions \( \phi(x, t) \) with some spatio-temporal processes/phenomena will be problematic [36], [5, sect. 10.2]. Indeed, the concepts of a field \( \psi \) and of the very \( \{x, t\}-\)space are as yet absent in theory. The real/experimental meaning is attached only to the mod-squares \( |\psi|^2 \) in expansions (2).

Non-physicality of quantum states has been often discussed and repeatedly pointed out in the literature [30], [19, “quantum state does not represent an element of physical reality”]. It is significant that the interpretation, in mathematical logic [16], is yet a further theory in its own right. This is not a reconstruction or an ‘intrinsic reshuffle’ of the initial theory but the new mappings. In our setting, the physical interpretation may arise only after the Born statistics (7). These points are fully discussed in sect. 8.

As a result, transforming the notation \( (\Psi, \Phi) \mapsto (|\Psi\rangle, |\Phi\rangle) \mapsto \langle \Psi | \Phi \rangle \), we claim

\[
\langle \Psi \hat{+} \Phi | \Theta \rangle = \langle \Psi | \Theta \rangle + \langle \Phi | \Theta \rangle , \quad \langle \lambda \cdot \Psi | \Phi \rangle = \lambda \times (\Psi, \Phi) \quad \forall \lambda \in C ,
\]

\[
\langle \Psi | \Phi \rangle = (\Phi, \Psi)^*, \quad \{ \langle \Psi | \Psi \rangle \geq 0, \quad \langle \Psi | \Psi \rangle = 0 \Rightarrow \langle \Psi | \Psi \rangle = 0 \}
\]

and call construction \( (\Psi, \Phi) \in C \) the scalar/inner product of vectors \( |\Psi\rangle, |\Phi\rangle \in \mathbb{H} \). The matrix unitarity (16) may now be turned into an algebraic object on LVS, i.e., be redefined invariantly as a reversible abstract transformation (operator) that preserves the scalar product:

\[
\mathbb{H} \mapsto \hat{U} : \quad \langle \hat{U} | \Psi \rangle \Phi = \langle \Psi | \Phi \rangle . \tag{30}
\]

Upon supplementation of the space \( \mathbb{H} \) with that product, i.e. following the creation of the structure

\[
[\mathbb{H}\text{-space}] + [(29)–(30)] =: \mathbb{H}^0 , \tag{31}
\]

there arises a question about isomorphism of the \( \mathbb{H}^0 \)-space models. Whilst realization of the abstract axioms (29) on one space \( \mathbb{H} \) is already multiple, the categoricity of structures (31) under the fixed dimension \( \operatorname{dim} \mathbb{H} < \infty \) is well known [22, 17]. In detail, for the two models \( \mathbb{H}_1, \mathbb{H}_2 \) of space \( \mathbb{H} \), one can move any basis of \( \mathbb{H}_1 \) into any other basis of \( \mathbb{H}_2 \) by a one-to-one transformation. By virtue of this equivalence, every orthogonal basis in \( \mathbb{H}_2^0 \) has a preimage in \( \mathbb{H}_1^0 \), which is also an \( \mathcal{A} \)-basis there. Meanwhile, transition between the \( \mathcal{A} \)-bases in \( \mathbb{H}_1 \) is a main property of unitarity \( U \)—invariance of \( \text{StatLength} \). Therefore, subsequent to bases, there transform uniquely both the vectors and the values of all of the \( \langle \cdot | \cdot \rangle \)-products in \( \mathbb{H}_1^0 \) and in \( \mathbb{H}_2^0 \). All the models for (31) are isomorphic.
5.2. Hilbert space. Now, solution to the problem (17) acquires the form

$$a_j = \frac{\langle \Psi | \alpha_j \rangle}{\langle \alpha_j | \alpha_j \rangle}, \quad (32)$$

and the scalar product, as well as its isometry (30), may be viewed as an invariant, yet auxiliary tool for calculating the $a$’s. It has no other motivation for emergence and it, along with the known normalization requirement $\langle \Psi | \Psi \rangle = 1$, is not a question about inner axiomatics of the quantum $|\Psi\rangle$-state set. Incidentally, it does not appeal to the concept of a dual space and its (bra)-vectors. These objects would require introducing not only a certain linear function on $\mathbb{H}$ but also creating one more (why, what for?) $L$-$V_S$; a typical treatment of observables as functionals on the first $L$-$V_S$.

Terminology with new product can be continued by introducing the projecting operation

$$\langle \Psi | \alpha \rangle \langle \alpha | \alpha \rangle =: \langle \langle \Psi, \alpha \rangle \rangle,$$

and by rewriting the rule (32):

$$a_j = \langle \langle \Psi, \alpha_j \rangle \rangle. \quad (33)$$

It is easy to see that the projection satisfies (29) except for the transposition axiom:

$$\langle \langle \Psi + \Phi, \alpha \rangle \rangle = \langle \langle \Psi, \alpha \rangle \rangle + \langle \langle \Phi, \alpha \rangle \rangle, \quad \langle \langle \lambda \cdot \Psi, \alpha \rangle \rangle = \lambda \times \langle \langle \Psi, \alpha \rangle \rangle \quad \forall \lambda \in \mathbb{C}. \quad (34)$$

This is nothing but the axioms of the linear functional $P_n[|\Psi\rangle] = \langle \Psi, \alpha \rangle$ with a parameter $|\alpha\rangle$; this completes a comment on the Riesz theorem mentioned in sect. 2.2.

Let us reverse the discourse. For the invariant producing the $a$-coefficient from (2), one can easily get towards the linearity properties (34). By this, however, the whole theory of such a ‘functional $P$-calculus’ would be accomplished vacuously, for the formal linear functional $P_n[|\Psi\rangle]$ ‘extracts’ $c$-coefficients from any expansions (3)–(4). It contains no idea of a quantum basis, i.e., of a quantitative observability $\{f_j\}$, and additivity of this $P$ does not care even the linear dependence of vectors. The idea, in contrast, is implemented through the auxiliary and unary function $\text{StatLength}$, and nonlinear at that.

Thus non-axiomatic (without treatments and interpretations) description of the quantum LVS necessitates separating the notions pertaining to the structural properties of the QM-state set $\mathbb{H}$ per se from the calculus add-ons over it. Attention should be paid to the fact that the emergence of $\mathbb{H}^\|$, usually perceived as a quantum analog of the classical phase-space, calls for neither ‘observable’ Born’s $f$-numbers (7) nor even a physics that accompanies them. The space $\mathbb{H}^\|$ is not a space of physical quantum states with familiar ‘illustrations’ $|\text{alive}\rangle + |\text{dead}\rangle$ but is merely the space of quantum states. It cannot have a physical analog because the physics is required neither for $\mathbb{H}^\|$-space nor for its ‘bare’ LVS-version $\mathbb{H}$.

We call mathematical construction $\mathbb{H}^\|$ the unitary space [22] or the Hilbert space, stipulating the point that questions of topology and of infinities will be considered in their own rights. Assuming that the question of the (normed) topology on $\mathbb{H}$ is solved in the affirmative (sect. 7), we arrive at the final result.

- **The 3-rd theorem of quantum empiricism (on Hilbert’s space)**
  1. The space of quantum states $\mathbb{H}^\|$ is an abstract vector space over $\mathbb{C}^*$, which has been equipped with the statistical-length function (6) and the concept of the quantum observable $\alpha$ as an orthogonal denumerable basis. All the models to this structure are isomorphic.
2. The inner-product superstructure (29) and normalization \( \langle \Psi | \Psi \rangle = 1 \), not being a necessity for \( \mathbb{H}^0 \), do formalize the general \( \mathbb{H}^0 \)-calculus. Vector coordinates (2) in every \( \mathcal{A} \)-base are calculated according the projection rule (33); no use is required of the concept of operator.

3. Statistical weights (7) and (26) admit an invariant writing, which has the standard form of the Born rule

\[
 f_a = \frac{|a_1|^2}{|a_1|^2 + |a_2|^2 + \cdots} \quad \Rightarrow \quad f_a = \frac{\langle \Psi | a \rangle \langle a | \Psi \rangle}{\langle \Psi | \Psi \rangle} ;
\]

or \( f_a = |\langle \Psi | a \rangle|^2 \) under the normalization convention.

4. Topology on \( \mathbb{H}^0 \) is defined by function (6) entailing the concept of a norm \( \| \Psi \|^2 = \mathcal{N}(\| \Psi \|) \) and of metric \( q(\langle \Psi, \Phi \rangle) = \| \Psi \|^2 = |\Phi \|^2 \).

5.3. Comment on statistics and physical interpreting the \( \langle \Psi | \varphi \rangle \). The symmetry of the quantity \( f_a \) with respect to \( \Psi \) and \( \Phi \) allows us to forget that it was being created for the \( \mathcal{A} \)-bases as for collections of \( N \) vectors \( \{|a_1\}, \ldots, |a_N\} \). In this regard, the very problem (17) implied, strictly speaking, not the pair of data \( \{|\Psi\}, |\alpha\} \) but the whole \( \mathcal{A} \)-basis. But now, one may speak about the statistical weight of one abstract state in the other:

\[
f = \langle \Psi, \Phi \rangle \langle \Phi, \Psi \rangle \quad \text{(symmetry} \| \Psi \| = |\Phi \| \).
\]

Thereby the observable number \( f \) has acquired not merely an invariant definition (no reference to the word basis) but has turned into a binary symmetrical structure on the entire \( \mathbb{H}^0 \). The structure ignores the orthogonal remainders in linear expansions

\[
 |\Psi\rangle = c_1 \cdot |\Phi_1\rangle + \cdots .
\]

There is no contradiction here, since each vector \( |\Phi_1\rangle \in \mathbb{H} \) may get to be an eigen one for a certain \( \mathcal{A} \)-instrument and each vector \( |\Psi\rangle \in \mathbb{H} \) is a carrier of all the statistics \( \{f_1, f_2, \ldots\} \) under the arbitrary \( |\Phi_1\rangle \) and \( \{ |\Phi_2\rangle, \ldots \} = |\Phi_1\rangle^\perp \).

Drawing an analogy to the classical statistics or to the probability theory, a contrast is in place.

- The classical physics has no underlying linear theory—theory of space \( \mathbb{H}^0 \),

which is why such binarity is impossible. This point is known as a problem with understanding/comprehending/defining and making sense (‘physicalization’) of the quantum probability [28, 24, 33]. For example, it is obvious that for the purely statistical/classical analog to the left/right hand part of (37), i.e., for the data-set \( \{f \text{-statistics}\} \oplus \{\alpha \text{-spectra}\} \) such as

\[
 \{(\tilde{f}_1, \tilde{a}_1), (\tilde{f}_2, \tilde{a}_2), \ldots\}, \quad \{(f_1, a_1), (f_2, a_2), \ldots\} \quad (= \text{StatData}) ,
\]

formula for calculating the \( f_1 \) by ‘data-vector’ \( \{(\tilde{f}_1, \tilde{a}_1), \ldots\} \) cannot exist. The stat-distributions \( \{(\tilde{f}_1, \tilde{f}_2, \ldots) \) and \( \{(f_1, f_2, \ldots) \) are not related in any way. There is no an \( \mathbb{H}^0 \)-theory analog in between.

In the physical theories, the aforesaid binarity makes it possible to give meaning to such wording as \( |\text{before}\rangle \), \( |\text{after}\rangle \), transitions \( |\text{in}\rangle \rightarrow |\text{out}\rangle \), etc. However these mathematical equal-rights do not, as before, bear on the reversibility \( t_1 \rightleftharpoons t_2 \) in time that is associated usually with the ‘physical processes’ \( |\Psi_{t_1}\rangle \rightleftharpoons |\Psi_{t_2}\rangle \). The nature of the quantum Hilbert space is free of (also the physical/empirical) notion of time, because the abstract states are not tied to the chrono-geometrical notions of causality/(non)locality/propagation/speed (of something, say of light c) and to “the objective determination of space-time phenomena” (W. Heisenberg, Nobel Lecture). Indeed, the \( \mathbb{H}^0 \)-theory does not yet have a collocation
“causality of the classical events”; none of these concepts have been defined. Say, it is meaningless to bring the two states \(|\Psi_1\rangle, |\Psi_2\rangle\) into correlation with each other in the context of causality/locality/determinism without conception ‘the observable’. But again, creation of the latter has not yet been completed.

As remarked previously, unitarity \(\hat{U}\) may be weakened merely to a preserving the \(\mathfrak{f}\)-structure (36) and orthogonality: ['unitarity \(\ast\) scalability']. Isomorphism of the \(\mathbb{H}^\mathfrak{f}\)-space models remains under that modification.

6. ON (A QUANTUM) PYTHAGORAS THEOREM*

Having written the additive property (10) in the \(\mathbb{H}_\mathfrak{f}\)-notation

\[
\mathcal{N}_\mathfrak{f} [a \cdot X \hat{+} b \cdot Y] = \mathcal{N}_\mathfrak{f} [a \cdot X] + \mathcal{N}_\mathfrak{f} [b \cdot Y] \quad \forall a, b \in \mathbb{C},
\]

one can go further and adopt for the statistical length, by etymology of this term and the quadratic form (6), the new notation \(\mathcal{N}_\mathfrak{f} [X] =: \|X\|^2\). Then

\[
\|a \cdot X \hat{+} b \cdot Y\|^2 = \|a \cdot X\|^2 + \|b \cdot Y\|^2 \quad \forall a, b \in \mathbb{C},
\]

where the two in \(||\cdot\|^2\) does not yet mean the squaring a number. Recalling now that the statistical content of vectors implies their free scalability \(\{X \mapsto a \cdot X =: x, Y \mapsto b \cdot Y =: y\}\) with preserving the orthogonality and the observable meaning, we get

\[
\|x \hat{+} y\|^2 = \|x\|^2 + \|y\|^2; \quad \text{(40)}
\]

i.e., identity with the standard writing the Pythagoras theorem \(|\vec{x} + \vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2\), in which the first sign of ‘addition’ should however be denoted by a different plus. The passage (39) \(\mapsto\) (40) may seem to be a formal ‘concealment’ of quantum field \(\mathbb{C}\) into the \(\mathbb{R}\)-reality of Pythagorean theorem only at first glance. Therefore, let us get back to the concept of orthogonality (sect. 3).

6.1. Quantum ‘inspection’ of Pythagorean theorem. The vocabulary that is involved in the theorem comprises the following terms: triangle, side, direction, length, perpendicular, addition, geometric square, angle, sum, the orthogonal, the right, distance, area, diagonal, squaring, cathetus/hypotenuse, numerical operations, etc. Bearing in mind the fact that all of them appeal to the familiar model on a plane, the list should be supplemented with words about rotations, translation, and about reflections, since handling the squares implies their geometric transporting; squares are compared only ‘with the help’ of group of motions.

In the natural/scholastic language, all these notions are considered as real entities that accompany triangles, squares, and the like. Though labeled by numerals, noon of these entities are the number itself. For instance, addition of the directed line segments (forming a triangle), addition of areas (‘square meters’), and addition of usual numbers is far from being the same addition; not to mention subtraction. See, e.g., commentaries and emphasis in italics by Mordukhai-Boltovskoi concerning the ancient-Greek perception of the “idea of a number” on p. 375 in [15, Russian translation] and notably the explanation to the effect that the “Euclidean \(AB \times AC\) is not in any way . . . not in the sense as understood in arithmetic . . . not the multipicare”; and also a comment on arithmetizing the geometric images and “quantities at all” in section “2. Product of segments” on pages 297, 248, and 317. The “area is a quantity, the essence of which as a primary notion is not defined by [Euclid]. . . . he does not give a measuring the area by number” [15, pp. 286–7].

* Content of this section is an exhaustive math-explanation to the popular and illustrated exposition [7].
However, it is clear that Pythagorean theorem is a quantitative statement about the aforementioned segments, which ’visually add up’ to triangles with scalable sides. And this is what we call the model of the vector space. Its operations \{\hat{+}, \cdot\}, due to difference between their nature and the arithmetical operations \{+, \times\}, are not numerical but abstract in principle; especially the unary (not binary) ’multiplication’ symbol \cdot. Thus,

- the accurate (re)formulation of the theorem, one way or another, demands proceeding to the abstract vectors of the abstract LVS (the associating an affine space to this LVS is not essential in our context). As this takes place, no such quantities as squares/…/angles should be implied. These should be built over the LVS, and relations between them should be ascertained. Put another way, the ‘pure algebra of bare’ LVS is supplemented with ‘calculus of the real entities’. But this is almost the same situation that took place when deriving the Born rule [6, thesis (•)].

If, as usual for LVS, we introduce the scalar product then the theorem (40) would boil down to the necessary and sufficient condition of orthogonality of vectors

$$x \perp y \iff (x \hat{+} y, x \hat{+} y) = (x, x) + (y, y) \quad (\text{field } \mathbb{R})$$  \hspace{1cm} (41)

The theorem is thereby simplified; there remains a single identity between the four entities \{⊥, (·, ·), \hat{+}, +\} without invoking the vocabulary mentioned above. Thus formalization discloses excessiveness of the usual terminology. The necessary and the redundant elements are intertwined in meaning with each other, however, the consideration does not end with this.

Orthogonality and scalar product are the derivative constructs of the quadratic function (6). By tracking its emergence in [6], it is a simple matter to see that the derivation procedure remains the same (and even simpler [7, p. 3]) for the field \(\mathbb{R}\) as well, provided that the complex \((+)-\)involution is replaced with \(a \mapsto (-a)\). In this respect, the nature of the scalar product and of the companion concepts—angle and orthogonality—does not depend on the number field (cf. [22, sect. 60]). Besides, the numerical \(\mathbb{N}\)-structure (6) is so minimal construct [6] that there is no need to introduce terminology of lengths/…/angles. Clearly, this is not about the quantum scheme of things’ since vocabulary of realities in the normal wording the theorem is automatically recovered.

Meanwhile, in quantum course of action over linear manifold, the issue of the intertwined terminology does not even arise. Logic of QT calls for severe separation of LVS-abstracta from observable entities at the outset [5]. Thus,

- we arrive at the general conclusion about a single nature of the classical Pythagorean theorem and the Born rule [7].

The analogies between Pythagorean theorem and Hilbertian sum of squares were self-apparent at all times; they are encountered throughout the literature. But the issue lies in straightening out the concepts, whereupon the rule ceases, as we have seen, to be a postulate, and the physical ’rationale behind’ the StatLength can be cast away. Consider now some details, without attaching much distinction to Pythagorean and Bornian cases. All the more so the origin of the square \(^2\) has already been ascertained and it was ascertained in the quantum context.

\*\*Although the main quantum ideology is easily visible: one requires a “controlling the language over itself” [5, sect. 11.1]. In quantum situation, the natural language creates the LVS and then the language of physics. In classical Pythagoras’ geometry, the language of squares/…/angles in theorem itself follows from the LVS-language; see below.
The very first real quantity in theory is the \(\text{StatLength}\); this is a theoretical representative of the \(\hat{a}\)-click number \([6]\). Because the scalar-product structure is fully derivable and orthogonality is independent of it \(\text{(sects. 3–4)}\) and derivable as well, we are dealing with not anything else but ‘remaking’ the theorem \((41)\) to definition—deducing and formalizing the structural properties of the object \((5)\) \([6, \text{Definition}]\), from which formula \((6)\) emerges. Therefore quantum statistics of Born \((7)\) is not provable within an extended Hilbertian version of the classical Pythagoras theorem \((41)\) under generalizing \((41)\) to the field \(\mathbb{C}\):

\[
x \perp y \iff [(x + y, x + y) = (x, x) + (y, y) = (i \cdot x + y, i \cdot x + y)] \quad (\text{field } \mathbb{C})
\]

(cf. exercise 4(c) in \([22, \text{p. 123}]\)). The Born formula was being derived rather than being proved \([6]\).

Simplification of the aforesaid leads to fact that the quadratic notation \((6)\) begins to be associated with the words ‘geometry, Pythagoras, theorem’, and symbol \(|x|^2 := (x, x)\) loses the square. With this dropping, this symbol is subconsciously equipped with a meaning being called the length of vector \(|\ldots|\) and having some properties in triangles. But remaking of theorem to definition should not disappear. Notice that proofs of the (direct/converse) Pythagoras theorem, wherein lengths do not appear, are well known. Such is the proof \(\text{(complicated)}\) and wording of Euclid himself \([15, \text{Proposition 47 and 48, pp. 46–48}]\) in the language of adding the plain areas.

In point of fact, the aforementioned ‘different additions’ say that the classical formulation—‘the square of the (length of) hypotenuse . . .’—by the quantum viewpoint, abounds with empirical inaccuracies. It is utterly fundamental to claim: how and what’s being added, what is defined through what, whence the length, what we have, what we have no, and what’s being deduced. Following thus the quantum spirit, we need to line up, as accurately as possible, the strict hierarchy of ‘what from what’, including the precise indication ‘which addition’, ‘which multiplication’, and what is understood by them at all.

6.2. Additivity and scalability. What is length? The language intension of the additivity of \(\text{StatLength} \text{(9–10)}\) is a core point both in Born’s rule and in Pythagoras’ theorem because the natural language is always necessary \([30, \text{sect. 3.1.1}, [35]\); it is primary even for foundations of mathematics \([16, 27, 10], [34, \text{Chs. 1, 4, 19, 21}]\). Being a concept that is inseparable from \(|a\)-vectors of observable \(\mathcal{A}\), the additivity of \(\text{StatLength}\) gives birth to their orthogonality and, then, to the scalar product. There appears the space \(\mathbb{H}^0\) and, as a consequence of this definition, the habitual language of the triangle sides should be reorganized \(\text{(see also sect. 6.4.1)}\) to the language of linearly independent vectors of LVS. Expressed differently,

- what the ‘side of triangle’ is supplemented by under the term length/. . ./square is defined as an additive function \(\mathcal{N}\) \((\text{property } 38)\), which is, upon action by number, expressed only through itself \((\text{non-multiplied})\):

\[
\mathcal{N}[\hat{c}|\Psi\rangle] = \mathcal{C}(\mathcal{N}[|\Psi\rangle]), \quad \mathcal{N}[\hat{c}|\Psi\rangle]^2 = \text{const} \times \mathcal{N}[|\Psi\rangle].
\]

That is to say, quantification \(\mathcal{N}\) of real things \([5, \text{sects. 7.1–2 and Remark 16}]\) is inseparable from the operator nature \(|\Psi\rangle \mapsto \hat{c}|\Psi\rangle = c \cdot |\Psi\rangle\) of the abstract number \(c\) \([26], [5, \S7.2]\).

The unit character of the quantity \(\mathcal{N}\) under creation calls for ascertainment of its ‘multiplicativity’ property \((42)\), while the fact that function \(\mathcal{C}\) ought to become a multiplying does not follow from anywhere. Planning to call \(\mathcal{N}\), say, the length of vector/side or to create the

\(^{\ast}\)Metres, square metres, and the like. This is also a part of the natural-language definition of \(\mathcal{N}\), but it is subject to ‘revision’ through the rule \((42)\) too. In this regard, the term \(\text{StatSize}\) or \(\text{StatNumber}\) \((\text{statistical number})\) would be better suited for \(\text{StatLength}\).
concept of a square/volume, we may not postulate in advance the character of its scalability since, in LVS, there has been present an axiom that combines the action of multiplication \( \cdot \) and of addition \( \hat{+} \). This is the distributive law
\[
c \cdot (|\Psi\rangle \hat{+} |\Phi\rangle) = c \cdot |\Psi\rangle \hat{+} c \cdot |\Phi\rangle \implies \mathcal{N}[c \cdot (|\Psi\rangle \hat{+} |\Phi\rangle)] = \mathcal{N}[c \cdot |\Psi\rangle \hat{+} c \cdot |\Phi\rangle],
\]
and it is it that dictates what the rule (42) has to be [6, p. 9]. This fact alone says that lengths, squares, and volumes do exist not on their own account but are tied to the abstraction LVS, which is not so obvious when proceeding from their everyday understanding or from the classical physics.

After having ascertained the quadraticity of the scaling, i.e. once eqs. (42)–(43) together with involution \( \mathcal{N}[c \cdotp |\Psi\rangle] = \mathcal{N}[|\Psi\rangle] \) have led to the expression
\[
\mathcal{N}[c \cdotp |\Psi\rangle] = |c|^2 \times \mathcal{N}[|\Psi\rangle]
\]
(see [6, sect. 5] for details), one reveals a distinction in operations \( \{\cdot, \times\} \) and disadvantage of the term \( (\cdot) \)-multiplication.

- The intuitive perception of words ‘to scale vector by a number’ does not furnish the naturally anticipated ‘to change its length’. The construct \( \mathcal{N} \) gives rise, in case of field \( \mathbb{R} \), the square on a vector and, in the quantum \( \mathbb{C} \)-case, the StatLength.

The question of length still stands. Thus, leaving aside vectors and the LVS itself, we conclude that the square (generalized) rather than the length is a primary quantitative entity both for the theorem and for the rule. But in both the cases, there has been ‘hardwired for free’ the concept of orthogonality.

The reason of this phenomenon is of course the multidimensionality of LVS because, in case of \( \dim H = 1 \), the concept of linear independence goes away, and additivity and length are trivialized just into a number. Once \( \dim H > 1 \), there arise nonequivalent bases and arbitrariness in coordinates, and intuitive ‘1-dimensional (= quantitative) concept’ the length must be created for vector. This is a nontrivial action since vector—even the scholastic—is an abstraction far from merely the number. The nature of the latter is roughly speaking the ‘number of something’ [5, sect. 7.2], while the ‘number of a multidimensional’ is an ill-defined semantics. Revision of the theorem is exactly what gives meaning to that.

As a result, the length ceases to be a 1-dimensional structure existing irrespective of the ‘2-dimensional concepts’ of the right angle, orthogonality, or square. Attention is drawn to the fact that the concept of a square, nevertheless, does not arise as an object on two vectors; it is not a binary construction. Speaking more loosely, length is defined through a square root of something more primary; cf. definition \( d\ell = \sqrt{g_{\mu\nu}dx^\mu dx^\nu} \) in geometry or in gravitation theory. Therefore, when the term orthogonality is dismissed in the classical case, there should disappear not only the ‘subject of proving’ in theorem (41) but also the length as a notion. In doing so, vectors, their coordinates, and the visual images-segments still stand.

The ideology of non-axiomaticity prohibits introducing the length through a norm since it relies entirely on the above-listed intertwined terms. Indeed, what is the motivation for arising a (new) concept of triangle \( x \pm y = z \) and whence the associated familiar inequality [17, p. 333], [22, p. 127], had we not possessed the concept of a length? The geometric (interpretative) intuition of the primary lengths/norms is correct neither in QT nor in the classical geometry; more precisely, under the empirical arithmetizing these theories.

The last structural property—invariance with respect to involutions—is obvious from the natural language. For example, in the \( \mathbb{R} \)-case, function \( \mathcal{N} \) should not depend on changing
the direction in space \( \vec{x} \mapsto -\vec{x} \). Indeed, a line segment and its quantitative measure \( N \), being a numerical add-on over vector, do not care the notion of an origin/terminus inherent to geometric vector. There arises the requirement \( N[(-1) \cdot x] = N[x] \).

The quantum constituent of the theory has already been elaborated at length, therefore let us sum up with a focus on its ‘Pythagorean part’.

6.3. Theorem --- definition. Let there be a problem, the model of which does in some way employ a vector space. For us, this is the scholastic geometry and quantum states. In view of autonomy of the LVS-axioms, any further theory may only be built upon this LVS\(^*\) [6, thesis (●)]. If the case in point is a quantitative theory—and this is our situation—then these new number objects must be supplemented with due regard to a numerical part of axioms of the very LVS \([17, 22]\). These numerical quantities have (must have by the nature of the task) the language/semantic descriptions, which are subject to mathematization. In the cases considered—theory of Born and of Pythagoras, these descriptions are formalized into the one minimalistic thesis [6, thesis (●●)]:

- ●● Equip the linear independence \((3)\) with a numerical additive function \( N \) (of coordinate representatives of vectors), which is invariant under the number involutions.

Does the function exist and, if yes, are all bases (linear independence) that allow this?

Among other things, this wording and the preceding rationale do in fact ‘exorcize’ the notion of a physical/illustrative/geometric interpretation from the theory fundamentals, because the structural properties of function \( N \), in and of themselves, is what we really mean by the word interpretation. See Remark 4.1 in [6] for more detailed comments.

In the language of formulas, the proposed minimalism turns into the rules, which we write down as applied to the plain Pythagorean case (dim \( V = 2 \)).

An operator character of the notion of the number, i.e. \((42)\), has been implied at all times.

i) For the two scale-related vectors \( \alpha \mapsto \hat{c} \alpha \), the quantities \( N[\alpha] \) and \( N[\hat{c} \alpha] = N[c \cdot \alpha] \) must be related to each other:

\[
N[c \cdot \alpha] = C(N[\alpha]) .
\]

Then the two claimed properties follow.

ii) Additivity on the linearly independent vectors \( \{\alpha, \beta\} \):

\[
N[a \cdot \alpha + b \cdot \beta] = N[a \cdot \alpha] + N[b \cdot \beta] \quad \forall a, b \in \mathbb{R} .
\]  \((44)\)

iii) Involutory symmetry:

\[
N[-a \cdot \alpha] = N[a \cdot \alpha] \quad \forall a, \alpha .
\]

As earlier, the words triangle/length/…/angle are considered now non-existent. Certainly, the theory is meaningless without invariance with respect to the changes of bases.

iv) Well-definiteness (= meaningfulness of the quantity \( N \)):

\[
a \cdot \alpha + b \cdot \beta = a' \cdot \alpha' + b' \cdot \beta' \quad \Rightarrow \quad N[a \cdot \alpha + b \cdot \beta] = N[a' \cdot \alpha' + b' \cdot \beta'] .
\]

\(^*\)And perhaps the copies of LVS, when we create, say, the tensor products of vector spaces for describing the multi-particle quantum problems.
Though technically important \([6, \text{pp. } 12-13], [7, \text{p. } 4]\), the latter point might well have been omitted. Clearly, the introducing a function \(N\) on vectors \(\{\alpha, \ldots\}\) is absurd if the \(N\) is incompatible with the concept of equality \((\alpha = \beta) \Rightarrow (N[\alpha] = N[\beta])\). One immediately reveals that not each linear independence \((3)\), i.e., not all of the \(\{\alpha, \beta\}\)- and \(\{\{e\}\}\)-bases admit the function \(N\) but only some special ones. The construct does automatically produce what they have to be. All possible bases of LVS are separated into the orthogonal—the \(\alpha\)-bases—and all the remaining abstract ones (sect. 3.2).

As a result, the points i)–iii) entail not only the Pythagorean square\(^2\) per se and, formalization \((41)\) but also the angles and, literally, the entire elementary geometry around the theorem. The attaching to this math the vocabulary from sect. 6.1 is a matter of harmonizing the terminology with ordinary verbal vehicles. Similarly the geometry of Born’s space \(\mathbb{H}^0\).

Let us elucidate the aforesaid geometrically. We simulate the relation \(\alpha \pm \beta = \gamma, \text{ in the plane and assign}\) to this the words triangle of a general position (generic). The triangle should be thought of as abstract, without terms the lengths of sides and (right) angle between them. In virtue of involution, those triangles that admit the function \(N[\alpha \pm \beta] = N[\alpha] + N[\beta] = N[\gamma]\) should be pictured as having the equal \(N[\alpha]\) and \(N[\beta]\). Consequently, these triangles ought to be parts of a quadrilateral having equal diagonals. It is implied that the words diagonal and quadrilateral have been defined through symbols \(\pm\); we are continuing to avoid the word ‘length’. Let us call that triangles the rectangular and label them by the standard symbol of the right angle \(\blacktriangle\). A most intelligible illustrations of this material is given in \([7]\). This accomplishes the commentary both to the abstract and to the ‘actual’ ‘theorem’ of Pythagoras, which appears here as nothing more than definition i)–iii). Incidentally, the \(N\)-calculus mathematics, as opposed to the familiar Pythagorean extension of numbers \(\sqrt{1^2 + 1^2} = \sqrt{2}\), is not beyond the scope of the number-rationality domain: \(N[\alpha] + N[\beta] = 1 + 1 = N[\gamma] = 2\).

The situation has a parallel in topology when creating the concept of ‘a line’. An arbitrary (continual) point set, being initially considered as ‘merely a set’ (‘dispersed anywhere and ad libitum’, the generic), ceases to be ‘arbitrarily dispersed’, and we portray it as \(=\) it becomes) a line only after the set has been equipped with an algebraic axiom of ordering \(a < b\). By analogy, the nature of the geometric term right angle is algebraic and lies solely in existence and in equipping the LVS with the (quadratic) function \(N\). It is this function that creates the right angle—rather than the reverse, and it arises even prior to the notion of an angle or of its numerical characteristic like \(\cos(\hat{\alpha}\hat{\gamma})\).

6.3.1. Additivity, again. Now, the ideology of an additive function on linear independence is a key to ascertain an analogy in the ‘brace’ Pythagoras–Born.

- Having had in a single theory the two different pluses—the abstract/multidimensional \(\pm\) and the number/one-dimensional \(+\), we should declare a rule of their compatibility. This is what the formulas \((9)\) and \((44)\) do via function \(N\); a number add-on over the LVS-abstraction \([6, 7]\). The ‘Pythagorean’ and the ‘quantum’ are not distinctive in this regard.

This fact, along with uncovering the meaning to the Pythagorean theorem through the quantum theory, seems to be lacking in the literature\(^*\). The geometry reduces to the pure algebra

\*Likely, had the origin of Pythagorean squares \((40)\) through the additivity \((44)\) been known, its quantum C-counterpart \((6)\) would not be a postulate and would long have appeared in the literature as a normal formula. In consequence, the familiar Gleason theorem (and his “frame functions”) would become a self-suggested corollary of this point when involving the concepts of the mean and of the linear operator.
without intertwining the terminology [22, sects. 59–62]. The theorem itself turns roughly speaking into a definition of the additivity, and the Born quantum postulate—into a theorem-corollary of this definition and into the structure of the Hilbert space.

This is the main conclusion one should draw for understanding the theorem and the postulate, regardless of whether we are looking at them in a quantum or in a classical manner, i.e., regardless of the number field C or R. An oddity is that the physical/quantum theory not only updated the standard formulation of LVS (see Remark 15 in [5] and sect. 1) but also ‘compelled us’ to look more carefully at such an ancient theorem, turning it into a definition. Here, we do not touch upon the topic about relationship of the theorem with the 5-th Euclidean parallel postulate—the theorem depends on it—and with the familiar Riemannian discourse on empirical bases of geometry. Notice that the concept of the length of vector is still up in the air, there is no need for it. The matter will remain the same when considering the topology in sect. 7.

Remark 2. The $\mathcal{N}$-additivity (44) resembles an analogous property of the additive measure $\mu$ on a set [29]:

$$\mu(A \cup B) = \mu(A) + \mu(B) \quad \text{for} \quad A \cap B = \emptyset.$$ 

However our $\mathcal{N}$-object is being sought not as function on (sub)sets/spaces of LVS [14] but on coordinate developments (3). The numbers are produced form the other ones: $\lceil$abstract numbers of the field $\mathbb{R}\rceil \rightarrow \lceil$quantities/magnitudes\rceil$. The more so as the function exists not for all of the bases (3).

6.4. Quantum and classical language, revisited. The emergence of further (mathematically unnecessary) terms—perpendicularity, length, angles, distances, etc—reflects a property of the ordinary language ‘to simplify itself’ [10], introducing the larger and derived concepts to avoid the repeating and heaping the primary primitives and long collocations.

Remark 3. One might say that the turning a certain (lengthy) verbal vehicle into the integrated whole, the ‘naming’ it a single term (say, square), or identifying ‘the self-similars, the likeness’s, …’ are, in and of themselves, an act of abstracting, which permanently has been present in thinking, giving birth to the primitive elements of the math language: sets, families, addition/union, quantities, abstract numbers, etc [5, sect. 7.2].

Therefore when working backwards [mathematics $\rightarrow$ explanation], there arise the word ‘interpretation’ and the problem of treatments in terms of observable quantities. The nature of observables in QT is the known and long-standing polemical matter [28, 12]. This is due to the fact that the natural language, having its free reducing and reproducing the phrases, oftentimes does not conform to requirement of coordination/consistency that is a must in theory. As can be seen, even the classical Pythagoras theorem does, in a sense, ‘acquire a problem’ of interpretation, since its accurate re-enunciation changes the way of looking at the notion of the length. The commentaries by Mordukhay-Boltovskoil to the language of Euclid’s Elements [15, Russian translation] elucidate this point very well.

The natural language ‘perceives the length’ as the first and evident observable entity, while there is no place for it in the correct statement of problem—[$\lceil$the LVS$\rceil \overset{(*)}{\rightarrow} \lceil$ii–iii$\rceil$]. The object ‘the square’ should be in its own right rather than a square of a length. It can in no way be declared as the ‘two segments with equal lengths and a right-angle in between’, although when visualizing the geometry of LVS this cannot be avoided because this is a part of the interpretation language. But in quantum Pythagoras’ theorem—the statistical-length rule (6)—the situation is opposite and simpler. The sum of squares is the number one observable, and the length of quantum vector is absent as a notion. Here, the observables and the abstractions have been severely distanced, and in doing so, it is futile to introduce the former prior to the latter [5, sect. 10.2].

A more formal view of the situation delivers, strange as it may seem, the most precise explanation. The point is that the quantum or Pythagorean vectors \{|$\Upsilon$\rangle, $c \cdot$|$\Upsilon$\rangle\}, from the
vector-space ‘standpoint’, are merely different vectors, the different elements of a set. The mathematical structure the LVS is not ‘aware of’ our geometric ways for visualizing the number as an operator: \( |\Psi\rangle \leftrightarrow \epsilon |\Psi\rangle = \epsilon \cdot |\Psi\rangle \). Hence, the picturing this operation as a co- or non-codirectional dilatation (over \( \mathbb{R} \) and notably over \( \mathbb{C} \)) does amount to a bringing the new ‘illegal’ words—shortening/stretching/.../rotation—into the theory of LVS as an abstraction and to an implicit interpreting and introducing the concept of the length.

6.4.1. Numbers and observables, again. As for the numbers, the situation is analogous. These, as operators, are applicable both to abstractions and to anything just as we apply the numbers to various units [5, Remark 16]. But in theories, they both are being created.

- The essential difference in between the abstract \( \{\mathbb{R}, \mathbb{C}\} \)-numbers and the quantitative \( \mathbb{R}^+ \)-entities should not be neglected.

Say, when handling the expansions (3) and even (2), we must not think/imagine they constitute something the treatable, the real, or simultaneously existing in writings like \( 0.6 \cdot |\text{here}\rangle + 0.8 \cdot |\text{there}\rangle \). The quantum foundations do forbid the numerical explanations a priori [30, 36, 5], and this has an analog, as we have seen, in classical geometry. The language for that explanations may be created only after the numerical object \( \text{StatLength}/\text{square} \), and only after it the quantitative objects under creation may be consistently ‘accompanied’ by the physical adjectives: observable, spectral readings, real, measurable, etc (see sect. 8).

Even if we were to pursue the aforementioned goals, then the word ‘constitute’ should be implemented through a mechanism in its own right—introducing the concept of ‘the observable quantity’ and explanation as to why the linear operator (if any) comes about here. As a consequence, to take an illustration, for the notorious ‘problem of Schrödinger’s cat’—ignore for a moment its meaningless∗—the statement of ‘the problem’ must be corrected. The mechanism of the introducing ought to bring the basis invariance (= quantum noncommutativity) into play. That is, apart from observable \( \{|\text{alive}\rangle, |\text{dead}\rangle\} \) (‘cat’s momentum’), one requires at least one more “pointer-state” set [8]; e.g., the ‘cat coordinate’ \( \{|\text{left}\rangle, |\text{right}\rangle\} \)-corner in the box. Formalization of these two ‘observables’ is the pt. iv).

7. Topology on Quantum States

Non-physicality and non-mathematicity of prerequisites for the theory (sect. 1) have an impact on the question of a quantum-state topology, which seems to be entirely a formal problem. Why and how should we norm the quantum vectors [29, 30]? To measure (?) the length (?) of quantum state?

7.1. Numbers and open sets. It is not correct to say that the two states physically differ little from each other [29] (or, e.g., the one approaches the other), because in the natural language, the phrase ‘differ little’ implies the handling of observable entities, and the word ‘little’ means the ‘reified \( \mathbb{R} \)-numbers’. However the states are in no way producible and comparable physically [19]. This is a task of quantum (meta)mathematics, and it consists in transporting the natural-language notions of the ‘smallness, approximation, smoothness’, etc from empirical language to the arisen \( \mathbb{H}^\ast \)-abstracta. It is common knowledge that this is implemented by topology [4, Introduction]: neighborhoods, closed/open sets, limit/boundary points, etc.

∗“When I hear of Schrödinger’s cat, I reach for my gun”, says S. Hawking in interview by T. Ferris (Pasadena, California, 4 April 1983).
The ideology of non-axiomaticity does not allow us to employ the familiar methods of turning the LVS into a topological space by metric or by an isomorphism between $\mathbb{H}$ and $\mathbb{C}^N$ with an automatical importing the natural (product) topology $\mathbb{C}^N$ onto $\mathbb{H}$. All this are the mathematical rather than empirical ways to topologize the $\mathbb{H}$. For the same reason we are not concerned with topological equivalence of norms on LVS, including their compatibility with scalar product. The necessity of the very concept of a norm for quantum states—in effect, the question of a length in sect. 6.2—is the subject matter of the present section.

Topology is needed not only for introducing the continuity or continuing maps of vectors $|\Psi\rangle$ into something, but also is a necessity for the internal needs of the $\mathbb{H}$-space itself: continuity of algebraic operations $\{\mp, \cdot\}$ and the making sense to infinite sums of abstractions (2) when $\dim \mathbb{H} = \infty$. The latter point has been commented in [6, sect. 6b]—the topology must be determined by function (6). By virtue of its uniqueness, it is ‘topologically’ necessary for $\dim \mathbb{H} < \infty$ too.

The reducibility to numbers—the values of function (6)—is to be a subject of topology inasmuch as we have no criteria for a number-free way of declaring the open-set systems or neighborhoods in the quantum $\mathbb{H}$-abstraction. Therefore, even if we were to introduce these objects, then the $|\Psi\rangle$-states might enter the condition that specifies such sets only through the only numeric characteristic the quantum vector has possessed—its own statistical length $\mathcal{N}(\|\Psi\|) = \|\Psi\|^2$ (erasing the $|\text{ket}\rangle$). Moreover, the $\|\Psi\|$ is a real number, which is why we get towards the $\mathbb{R}$-topology. Some comments are now in order.

Given what we have said about $\text{StatLength}$, the task of introducing a topology should be associated with a task of the numerical convergence of a chain $(\ldots, S_n, S_{n+1}, \ldots)$ of the partial sums $S_n = |a_1|^2 + \cdots + |a_n|^2$; complexness of numbers $a_n$ is of no significance here. That is to say, the question of an abstract topology per se is converted for our $\mathbb{H}$ into the question about convergence of the real-number sequence $\{S_n\}_{n=\infty}$. And this is always the subject of mathematizing the phrase ‘differ little from’ [27]. With the natural understanding of the number, this phrase is formalized into the familiar inequality $|S_{n+1} - S_n| < \epsilon$. In turn, such a difference $|b - a|$ can be reformulated without usage of arithmetic—the subtraction operation—but only with using the natural ordering $<: |b - a| < \epsilon \Rightarrow a < x < b$

\[ [\epsilon\text{-neighborhood}, \ x \text{ is an element of the open set } (a, b)] \quad (45) \]

It may be added that the natural ordering $<$ had entered a definition of $\mathbb{R}$-numbers when they were as yet arising in quantum theory; through the ensemble-accumulation procedure [5, sects. 2.5, 7.3]. If needed, that ordinal and quantitative definition may be formalized into the set-theoretic inclusion $\subset$. Then such notions as ‘little/nearly/…/almost’ are represented mathematically by the ‘small $\epsilon$-quantities (cardinalities of sets) that are contained in between’ numbers $a$ and $b$; the writing $a \subset \cdots \subset b$. Without such an intensity of the 1-dimensional intervals (45) and of their lengths $|\cdot|$, even the natural language has been blurring.

**Remark 4.** More formally, one may draw on the following point, of which the proof is omitted. Having had a (well-defined, single-valued) function/map $\mathcal{N}$ from (as yet) non-topological $\mathbb{H}$ onto the numbers $\mathbb{R}$ (with natural topology), the preimage $\mathcal{N}^{-1}$ of open intervals on $\mathbb{R}$ delivers automatically a family of open sets in $\mathbb{H}$. Clearly, the $\mathcal{N}$ is a function to be naturally identified with the quantum one (6), and precisely a function map into the numeric domain, being the only function that has been motivated empirically. In its turn, the uniqueness of the natural topology on $\mathbb{R}$ is also known, as each open set on the real line is a countable union of intervals (45). If a given $\mathbb{H}$ has been equipped, as in our case, with algebra like $\{\mp, \cdot, +, \times\}$, then the checking it for continuity is a math problem in its own right. This should also include the supplementary questions of the topology axiomatics—the countability/separation $T$-axioms [4, sect. I.8], which are essential for the numerical
convergence and for the meaningful concept of a limit [4, sect. I.7.3]. Notice that for \( \mathbb{R} \)-numbers with the natural topology, all these axioms are met.

Now, let us adopt that the notions ‘little difference, continuity’, and the like, empirical as they are, have to be introduced and applied to the \( |\Psi| \)-abstracta. That is called for by ‘rigorising’ [25] the \( \mathbb{H} \)-calculus’ constructed above: topological completeness of the \( \mathbb{H} \)-space, linear operator as a ‘smooth’ map \( \mathbb{H} \mapsto \mathbb{H} \), its matrix elements, etc. When reasoning about continuity, the technically precise and universal term the abstract open-set system is then substituted for the ‘small quantities/numbers’, i.e., for the \( \varepsilon \)-neighborhoods (45).

### 7.2. Are the norm and metric necessary?

Thus the two possibilities are available. The first one—a commonplace in physics—is the axiomatical introducing the concept of a length/\( \| \Psi \| \)/ and postulating its relation to Born’s square \( |\cdot|^2 \), since each sum of squares is a certain square:

\[
|\Psi⟩ = a_1 \cdot |a_1⟩ \pm a_2 \cdot |a_2⟩ \pm \cdots = b \cdot |β⟩ \quad \Rightarrow \quad |a_1|^2 + |a_2|^2 + \cdots = |b|^2 = |\Psi|^2 .
\]

The \( \mathbb{H} \)-space mathematics is accomplished then by the scheme:

\[
\text{norm } \| \Psi \| \quad \mapsto \quad \text{metric } ρ(⟨\Psi, |Φ⟩) = \| \Psi \| \cdot |Φ⟩ \quad \mapsto \quad \varepsilon \text{-topology (45)} .
\]

The second option is to justify step-by-step a way of regular deducing the axioms of norm, i.e., to ‘take the square root’ of the \( \text{StatLength} (6) \) and respect the algebra \{ \( \pm, \cdot \) \}.

It is not improbable that despite the fundamental multi-dimensionality of the \( \mathbb{H} \)-mathematics, one can even make do without superstructure \( \| \Psi \| \) and, as the \( \mathbb{R} \)-numbers guide us (Remark 4), reduce topological examination for the \{ \( \Psi \), \( a, \pm, \cdot \) \}-algebra solely to the 1-dimensional language \( \varepsilon, δ \). Speaking a little simplistically, the very concepts of an abstract open set and of norm can turn out to be an artifact for the quantum-state topology. This, of course, is not to say that it is not worth adopting the norm, terminologically, as a usable concept.

We are inclined to believe that one can get around the first option in quantum mathematics and reduce the second. To all appearances, all the ingredients of Hilbertian abstracta we have considered in the present work may be viewed not as the postulated structures but rather as the deducible ones. It is generally tempting to infer that the norm \( \ell^2 \)-topology is, in a sense, natural and inevitable. At least, it does really stand out from the others, in particular, from \( \ell^p \).

Let us now return to Remark 4, i.e., to reservations about the countability axioms, about the topological indistinguishability of norms on LVS, and to the fact that the total number of axioms that pertain to the topology on \( \mathbb{H} \) is likely more than a dozen” [4, Ch. I]. All of them are essentially abstract and far from empiricism, much less from physics. These will need to be (re)considered, but this is a solely mathematical problem.

In the physical literature, the issues of this sort are often just ignored [31, 1, 36], and quite clear why; see, however [37]. It is obvious that the concern, in its full generality, will move deeply into the domain of mathematical logic or even the foundations of mathematics [16, 27, 34] rather than of (math)physics; not to mention the quantum foundations. One must have stopped somewhere because the questions will simply lose their significance for these very foundations. Indeed, should discussions on the QT-elements take up the separation \( T \)-axioms (Hausdorff, Kolmogorov, ...), the Urysohn theorems on metrizability/immersion,

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*Including, e.g., a numerical axiom of Archimedes [4]. We mention this example, inasmuch as the dismissal of quite low-level axioms is known not only in mathematics—non-Archimedean fields—but also in the \( p \)-adic redeveloping the quantum theory itself [23].*
or the countable base of topology, etc? All these are the things that ensure separability of $\mathcal{H}$, uniqueness of the limit, and, in general, meaningfulness of the math we have been accustomed.

Let us summarize briefly the stuff above. The origin of an abstract structure of the Hilbert QT-space, as well as of a Born-rule fromat (35), may be thought of as the resolved question. The sought-for procedure is described, as was stated in the previous study [6], by the axiom-free continuation of the axiom-free scheme (8):

$$[\text{micro-events’ accumulation}] \Rightarrow [\text{LVS}+ (6)] \Rightarrow [\text{Hilbert space } \mathcal{H}^1].$$

8. CONCLUDING REMARKS (PHYSICS)

Non-axiomaticity and non-physicality of the $\mathcal{H}^1$-construct and the ‘observable’ Stat-Length lead us to the general conclusion in the context of Hilbert’s sixth problem [2, 11].

On the one hand, the problem calls for ‘quantization’; on the other, recalling von Neumann’s programme [32], the straightforward applying and adhering to the ideology [axioms $\rightarrow \cdots \rightarrow$ interpretations] is not possible. The statement of the problem cannot disregard the language semantics because semantical circularities are very well known [28, 36], and informal semantic analyses are needed prior to any formal reconstruction [35, 10]. Semantics, in turn, begins with empiricism of quantum micro-events [5, sect. 2]. More to the point, we have seen from sects. 3.2, 4 and 5.1 that one may dismiss not only the bulk of mathematical axiomatics but also the physical aspects: quantum measurements [33, p. xiii; “there can be no quantum measurement theory”], quantum probability, quantum transitions/leaps, dynamics of observables, interaction, preferable bases, etc. Instead of the habitual ‘backbone’ [math] $\mathcal{H}^1$ [physical principles], axiomatics of physics is replaced by a single underlying construct free from the words “axiomatische Methode/Behandlung” (Hilbert) [11].

The situation is close to that of interpreting the formal languages in mathematic logic [16, 30]. Therefore the Hilbert problem is not solvable without streamlining the nomenclature and without hierarchy and splitting the language in use into the

1* Meta-language: micro-events, ensembles thereof, theoretical primitive $\Psi \rightarrow \mathcal{U}$, setting the macro-environment by the conception ‘the same’ [5, sect. 5.4], operational/quantitative/ordinal meaning to the number, …

2* Object language: $\{\mathbb{R}, \mathbb{C}\}$-numbers, $\mathcal{H}$- and $\mathcal{H}^1$-structures (Hilbert space), spectra, mixture of orthogonal states $\{|\alpha_1\rangle(\mathbb{I}), |\alpha_2\rangle(\mathbb{I}), \ldots\} \rightarrow$ statistical operator $\hat{q}$, quantum statistics, the concept of a mean, linear operators, self-adjointness, the notion of the quantum system with an inner composition (multiparticleness and tensor products of $\mathcal{H}^1$-spaces), …

3* Math-physical theories [30]: instrumental readings, numerical measurement, space-time continuum, quantum nature of the metric tensor $g_{\alpha\beta}dx^\alpha dx^\beta$, (non)observables, causality, locality, dynamical equations/variables/fields/potentials, symmetries and $\hat{U}$-operators, the concepts of an interaction and a closed system, the ideology of quantizing a model, of the Hamiltonian, of the Lagrangian, of action, …

4* Language of (physical) interpretations: tangible bodies, masses, forces, waves, observable phenomena, analogies in between, their descriptions through each other, the explanation language, …

In this, each language is created from the previous ones through the natural language. The interpretation problems, paradoxes of QT, and questions of reality [19], [24, Introduction],
Why Hilbert space?

[33, p. 10] disappear in the sense that they become a task of the axiom-free creating the languages of the (math)physical reasoning 3* and 4*. Their terminology—this we stress with emphasis—is forbidden in languages 1* and 2*. Understandably, the very meaning of the words ‘explaining, the explanation language’ does always imply a hierarchy of vocabularies in use [30, 10]. The main difficulty here is that the separation [pre-math, math, pre-phys, physics] in transition [1* → · · · → 4*] breaks the habit of ratiocinating in the seemingly inevitable language of realistic notions and of human intuition.

All this much having been said, “the formal frame for quantum theory” (von Neumann) does not seem to require axiomatics in the ordinary sense of the word, if we do not postulate initially the concepts like metric and the space-time as a topological continuum with a dimension \( D + 1 \) \( (D = 3?) \). In particularly, the turning the Hilbertian unitarity \( \hat{U} \) (sect. 5.1) into the unitary dynamics \( \hat{U}(t) \) (and introducing a Hamiltonian) is a quite nontrivial act that should be motivated in the same manner as the initial emergence of unitarity itself [6, sect. 5]. We are speaking here of quantum mechanics, although this notion should also be (re)created. It is not difficult to foresee that the completion of the language 2* is a more or less technical task; not addressed in the present work (4-th theorem).

To sum up, the coherent strategy for constructing the physics of QT-foundamentals consists in setting up the languages 3* and 4* and should not recourse to the classical conceptions. In other words, this needs not be a relativistic QFT-generalization of quantum mechanics followed by a quantizing the gravity; to be renormalizable? by quantizing the fields? Instead, the strategy must constitute an ab initio direct creation of a framework for an entirely (Poincaré/generally) covariant theory, within which the familiar ingredients—the concepts of the gauge/observable fields and of a particle, equations of motion, unitary/Hermitian operators, Wightman’s axioms [37], representations of the (local) invariance groups—are being created on a regular basis of the abstract Hilbert \( \mathbb{H}^4 \)-space and a brace thereof with the \((x, t)\)-continuum. By this we mean the following premise.

- **The quantum origin of coordinates and fields**
  The classical and the quantum continual objects—the spacetime coordinates \((x, t)\) and fields \(\{u(x, t), A(x, t), \ldots, \hat{\psi}(x, t)\}\)—may come to theory only from continuality of the quantum-base changes \(\{\{\alpha\} \overset{C\mathcal{L}(C)}{\longrightarrow} \{\alpha'\}, \alpha \overset{C\mathcal{L}(C)}{\longrightarrow} \alpha'\}\). From this, there also springs the (Einstein’s idea of) ‘liberation’ from coordinates: the follow-up concept of covariance of the dynamical field equations (the relativity principle), the concept of the vector/tensor/spinor representations of the specific (Poincaré, Lorentz, …) groups, and the like.

See also the endings of sects. 8.3 and 9.3 in [5]. These questions will be addressed elsewhere.

**REFERENCES**

[1] Aaronson S. Quantum Computing since Democritus. Cambridge University Press (2013).
[2] Accardi L., Degond P. & Gorban A. (eds) Hilbert’s sixth problem. Phil. Trans. Royal Soc. A (2018) 376(2118), whole (theme) issue.
[3] Born M. Zur Quantenmechanik der Stoßvorgänge. Zeit. Phys. (1926) XXXVII, 863–867.
[4] Bourbaki N. Topologie générale. Springer (2007).
[5] Brezihney Yu. V. Linear superposition as a core theorem of quantum empiricism. Universe (2022) 8(4), 217 (78).
[6] Brezihney Yu. V. The Born rule as a statistics of quantum micro-events. Proc. Royal Soc. A (2020) 476(2244), 20200028 (22).
[7] Brezihney Yu. V. Pythagoras–Born: on a relationship between the quantum and the classical. Preprint https://www.researchsquare.com/article/rs-2288224/v1 (2022), 1–6.
[8] Busch P., Lahti P. J. & Mittelstaedt P. The Quantum Theory of Measurement. Springer (1996).
[9] Cassinelli G. & Lahti P. Quantum mechanics: why complex Hilbert space? Phil. Trans. Royal Soc. A (2017) 375(2106), 20160393 (9).
[10] Chomsky N. Language and Mind. Cambridge University Press (2008).
[11] Corry L. David Hilbert and the Axiomatization of Physics (1898–1918). Archimedes. New Studies in the History and Philosophy of Science and Technology 10. Springer Science+Business Media Dordrecht (2004).
[12] D’Ariano G. M. The solution of the sixth Hilbert problem: the ultimate Galilean revolution. In: [2], 20170224 (8).
[13] D’Ariano G. M., Chiribella G. & Perinotti P. Quantum Theory from First Principles. An Informational Approach. Cambridge University Press (2017).
[14] Engesser K., Gabbay D. M. & Lehmann D. Handbook of Quantum Logic and Quantum Structures. Elsevier (2007).
[15] Euclid. Elements. Books 1–13 (The Greek text of J. L. Heiberg and modern English translation). Richard Fitzpatrick (2008). Russian translation by Mordukhai-Boltovskoi: Евклид. Начала. Книги I–VI. Гос. Изд. Техн. Тетр. Литературы (1950).
[16] Fraenkel A. A. & Bar-Hillel Y. Foundations of Set Theory. North-Holland Publishing Company (1958).
[17] Friedberg S., Insel A. & Spence L. Linear Algebra. Pearson (2002).
[18] Fuchs C. A. Quantum mechanics as quantum information, mostly. Journ. Mod. Optics (2003) 50(6–7), 987–1023.
[19] Fuchs C. A. & Schack R. QBism and the Greeks: why a quantum state does not represent an element of physical reality. Phys. Scripta (2014) 90(1), 015104 (6).
[20] Gantmacher F. R. The theory of matrices I. Chelsea Publishing Co. (1959).
[21] Gudder S. P. Stochastic Methods in Quantum Mechanics. North Holland (1979).
[22] Halmos P. R. Finite-Dimensional Vector Spaces. Springer (1987).
[23] Khrennikov A. Interpretations of Probability. Walter de Gruyter (2009).
[24] Khrennikov A. Beyond Quantum. Taylor & Francis Group, LLC (2014).
[25] Kronz F. & Lupher T. Quantum Theory and Mathematical Rigor. Stanford Encyclopedia. https://plato.stanford.edu/entries/qt-nvd/ (2019).
[26] Kurosh A. G. Lectures on general algebra. Chelsea Publishing Co. (1963).
[27] Lakoff G. & Núñez R. E. Where Mathematics Comes from. How the Embodied Mind Brings Mathematics into Being. Basic Books (2000).
[28] Laloe F. Do We Really Understand Quantum Mechanics? Cambridge University Press (2012).
[29] Ludwig G. An Axiomatic Basis for Quantum Mechanics 1. Derivation of Hilbert Space Structure. Springer (1985).
[30] Ludwig G. & Thurler G. A New Foundation of Physical Theories. Springer (2005).
[31] Müller M. P. Probabilistic theories and reconstructions of quantum theory. SciPost Phys. Lect. Notes (2021) 28, 1–41.
[32] von Neumann J. Continuous geometry with a transition probability (edited by I. Halperin). Memoirs of AMS (1981).
[33] Peres A. Quantum Theory: Concepts and Methods. Kluwer Academic Publishers (2002).
[34] Russell B. Principles of Mathematics. Routledge Classics (2010).
[35] Salmon W. Informal Analytic Approaches to the Philosophy of Science. In: Asquith P. D. & Kyburg H. E. (eds) Current Research in the Philosophy of Science, pp. 3–15. East Lansing, MI: Philosophy of Science Association (1979).
[36] Silverman M. P. Quantum Superposition. Counterintuitive Consequences of Coherence, Entanglement, and Interference. Springer (2008).
[37] Wightman axioms. https://ncatlab.org/nlab/show/Wightman+axioms (2019).

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