CANCELLATION FOR THE MULTILINEAR HILBERT TRANSFORM

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Abstract. For any natural number $k$, consider the $k$-linear Hilbert transform

$$H_k(f_1, \ldots, f_k)(x) := \text{p.v.} \int \frac{f_1(x+t) \ldots f_k(x+kt)}{t} \, dt$$

for test functions $f_1, \ldots, f_k : \mathbb{R} \to \mathbb{C}$. It is conjectured that $H_k$ maps $L^{p_1}(\mathbb{R}) \times \cdots \times L^{p_k}(\mathbb{R}) \to L^p(\mathbb{R})$ whenever $1 < p_1, \ldots, p_k, p < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_k}$. This is proven for $k = 1, 2$, but remains open for larger $k$.

In this paper, we consider the truncated operators

$$H_{k,r,R}(f_1, \ldots, f_k)(x) := \int_{r \leq |t| \leq R} f_1(x+t) \ldots f_k(x+kt) \, \frac{dt}{t}$$

for $R > r > 0$. The above conjecture is equivalent to the uniform boundedness of $|H_{k,r,R}|_{L^{p_1}(\mathbb{R}) \times \cdots \times L^{p_k}(\mathbb{R}) \to L^p(\mathbb{R})}$ in $r, R$, whereas the Minkowski and Hölder inequalities give the trivial upper bound of $2 \log \frac{R}{r}$ for this quantity. By using the arithmetic regularity and counting lemmas of Green and the author, we improve the trivial upper bound on $|H_{k,r,R}|_{L^{p_1}(\mathbb{R}) \times \cdots \times L^{p_k}(\mathbb{R}) \to L^p(\mathbb{R})}$ slightly to $o(\log \frac{R}{r})$ in the limit $\frac{R}{r} \to \infty$ for any admissible choice of $k$ and $p_1, \ldots, p_k, p$. This establishes some cancellation in the $k$-linear Hilbert transform $H_k$, but not enough to establish its boundedness in $L^p$ spaces.

1. Introduction

For any natural number $k$ and test functions $f_1, \ldots, f_k : \mathbb{R} \to \mathbb{C}$, define the $k$-linear Hilbert transform $H_k(f_1, \ldots, f_k) : \mathbb{R} \to \mathbb{C}$ by the formula

$$H_k(f_1, \ldots, f_k)(x) := \text{p.v.} \int f_1(x+t) \ldots f_k(x+kt) \, \frac{dt}{t},$$

or more explicitly

$$H_k(f_1, \ldots, f_k)(x) = \lim_{r \to 0, R \to \infty} H_{k,r,R}(f_1, \ldots, f_k)(x)$$

where $H_{k,r,R}$ is the truncated $k$-linear Hilbert transform

$$H_{k,r,R}(f_1, \ldots, f_k)(x) := \int_{r \leq |t| \leq R} f_1(x+t) \ldots f_k(x+kt) \, \frac{dt}{t}. \quad (1.2)$$

The operator $H_1$ is the classical Hilbert transform, which as is well known (see e.g. [10]) is bounded on $L^p(\mathbb{R})$ for every $1 < p < \infty$. The operator $H_2$ is the bilinear Hilbert transform; it was shown by Lacey and Thiele [8, 9] using time-frequency analysis techniques that $H_2$ maps $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$ whenever $1 < p, p_1, p_2 < \infty$ and $\frac{1}{p} + \frac{1}{p_1} + \frac{1}{p_2} = \frac{2}{3}$, in fact, they were able to relax the constraint $p > 1$ to $p > 2/3$, however

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in this paper it will be convenient to restrict\(^1\) to the “Banach space case” when all
exponents are greater than 1. The same argument shows the corresponding bounds for
\(H_{2,r,R}\) that are uniform in \(r,R\); that is to say, one has
\[
\|H_{2,r,R}(f_1, f_2)\|_{L^p(\mathbb{R})} \leq C_{p,p_1,p_2} \|f_1\|_{L^{p_1}(\mathbb{R})} \|f_2\|_{L^{p_2}(\mathbb{R})}
\]
whenever \(1 < p, p_1, p_2 < \infty, \frac{1}{p} + \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}\), \(f_1 \in L^{p_1}(\mathbb{R})\), \(f_2 \in L^{p_2}(\mathbb{R})\), and \(0 < r < R\),
where \(C_{p,p_1,p_2}\) is a quantity independent of \(r,R\). Note that the condition \(\frac{1}{p} + \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}\) is
necessary from dimensional analysis (or scaling) considerations.

From these facts, one may make the following conjecture.

**Conjecture 1.1.** Let \(k \geq 1\) and \(1 < p_1, \ldots, p_k, p < \infty\) be such that \(\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_k}\).
Then one has
\[
\|H_k(r_1, \ldots, k)\|_{L^p(\mathbb{R})} \leq C_{k,p,p_1,\ldots,p_k} \|f_1\|_{L^{p_1}(\mathbb{R})} \cdots \|f_k\|_{L^{p_k}(\mathbb{R})}
\]
whenever \(1 < p, p_1, \ldots, p_k < \infty, \frac{1}{p} + \cdots + \frac{1}{p_k} = \frac{1}{p}\), \(f_i \in L^{p_i}(\mathbb{R})\) for \(i = 1, \ldots, k\), and
\(0 < r < R\), where \(C_{k,p,p_1,\ldots,p_k}\) is a quantity independent of \(r,R\). In particular, from (1.1) and Fatou’s lemma we have
\[
\|H_k(f_1, \ldots, f_k)\|_{L^p(\mathbb{R})} \leq C_{k,p,p_1,\ldots,p_k} \|f_1\|_{L^{p_1}(\mathbb{R})} \cdots \|f_k\|_{L^{p_k}(\mathbb{R})}
\]
for all test functions \(f_1, \ldots, f_k : \mathbb{R} \to \mathbb{C}\).

As mentioned above, this conjecture is established for \(k = 1, 2\), but is completely open
for larger values of \(k\). For instance, in the case \(k = 3\) of the trilinear Hilbert transform
\(H_3\), no \(L^p\) bounds whatsoever are known. Although it is not needed to motivate our main
results, we also remark that the implication of (1.4) from (1.3) can be reversed (with
some loss in the multiplicative constant); if (1.4) holds, then by restricting \(f_1, \ldots, f_k\) to
intervals of length \(R\), applying (1.4) to these restrictions, and averaging over all such
intervals (using Minkowski’s inequality and Hölder’s inequality to estimate some error
terms) it is not difficult to show that
\[
\lim_{r \to 0} \|H_k(r_1, \ldots, k)\|_{L^p(\mathbb{R})} \leq C'_{k,p,p_1,\ldots,p_k} \|f_1\|_{L^{p_1}(\mathbb{R})} \cdots \|f_k\|_{L^{p_k}(\mathbb{R})}
\]
for some constant \(C'_{k,p,p_1,\ldots,p_k}\) and test functions \(f_1, \ldots, f_k\), and then on subtracting this
bound for two different choices of \(R,r\) and using a limiting argument we obtain (1.3)
(with a slightly worse constant). We leave the details to the interested reader.

One can approach Conjecture 1.1 by introducing the operator norm \(C_{k,p,p_1,\ldots,p_k}(R/r)\) of
\(H_{k,r,R}\), defined as the best constant for which one has
\[
\|H_{k,r,R}(f_1, \ldots, f_k)\|_{L^p(\mathbb{R})} \leq C_{k,p,p_1,\ldots,p_k}(R/r) \|f_1\|_{L^{p_1}(\mathbb{R})} \cdots \|f_k\|_{L^{p_k}(\mathbb{R})}
\]
for all \(f_1 \in L^{p_1}(\mathbb{R}), \ldots, f_k \in L^{p_k}(\mathbb{R})\). Note from scaling that the operator norm of \(H_{k,\lambda r,\lambda R}\)
is the same as that of \(H_{k,r,R}\) for any \(\lambda > 0\), which is why we write the operator norm
\(C_{k,p,p_1,\ldots,p_k}(R/r)\) as a function of the ratio \(R/r\) rather than of \(R, r\) separately. Conjecture
1.1 is then equivalent to the assertion that \(C_{k,p,p_1,\ldots,p_k}(R/r)\) remains bounded in the limit

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\(^1\)See also the negative results of [2] showing that \(H_3\) can be unbounded for certain values of \(p\) below 1.
R/r \to \infty$. On the other hand, from (1.2), Minkowski’s integral inequality and Hölder’s inequality we have the trivial bound

\[ C_{k,p,p_1,\ldots,p_k}(R/r) \leq \int_{|t| \leq R} \frac{dt}{|t|} = 2 \log \frac{R}{r}. \]

Our main result is the following slight improvement of the trivial bound.

**Theorem 1.2** (Improvement over trivial bound). Let \( k \geq 1 \) and \( 1 < p_1, \ldots, p_k, p < \infty \) be such that \( \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_k} \), and let \( \varepsilon > 0 \). Then, if \( R/r \) is sufficiently large depending on \( \varepsilon, k, p_1, \ldots, p_k, p \), one has

\[ C_{k,p,p_1,\ldots,p_k}(R/r) \leq \varepsilon \log \frac{R}{r}. \]

This falls well short of Conjecture 1.1 but it does show that some cancellation is occurring in the \( k \)-linear Hilbert transform.

A novel feature\(^2\) in our arguments is the introduction of tools from arithmetic combinatorics, particularly the theory of higher degree Gowers uniformity that was initially developed in [5, 6] to provide a new proof of Szemerédi’s theorem [11] on arithmetic progressions. Such tools are known to be useful for controlling expressions such as

\[ \sum_{x,t \in \mathbb{Z}} f_0(x) f_1(x + t) \cdots f_k(x + kt) \]

for various bounded, compactly supported functions \( f_0, \ldots, f_k : \mathbb{Z} \to \mathbb{C} \), so it is not so surprising in retrospect that they should also be able to say something non-trivial about integral expressions such as (1.2). Unfortunately, at the current state of development of the theory of higher degree uniformity, the quantitative bounds arising from these tools are quite poor for \( k \geq 3 \) (with no explicit bounds whatsoever in the current literature for \( k \geq 5 \)), and so one would need a significant quantitative strengthening of the arithmetic combinatorics results, or the introduction of additional techniques, if one were to hope to make substantial progress towards Conjecture 1.1 beyond Theorem 1.2.

Roughly speaking, the strategy of proof of Theorem 1.2 is as follows. After some reductions reminiscent of those in [8, 9], as well as a discretisation step in which one replaces the real line \( \mathbb{R} \) with the integers \( \mathbb{Z} \), one reduces matters to establishing a “tree estimate” in which one demonstrates non-trivial cancellation in expressions roughly of the form

\[ \sum_{x,t \in \mathbb{Z}} f_0(x) f_1(x + t) \cdots f_k(x + kt) \psi(t/2^n) \varphi(2^{-n}x - j) \tag{1.5} \]

for some smooth compactly supported functions \( \psi, \varphi \), some parameters \( n, j \), and some bounded functions \( f_0, \ldots, f_k : \mathbb{Z} \to \mathbb{R} \). Crucially, the function \( \psi \) will be odd, reflecting the odd nature of the Hilbert kernel \( \frac{dt}{t} \) appearing in (1.4). A standard “generalised von Neumann theorem” from arithmetic combinatorics tells us that expressions of the

\(^2\)See also [1] and [7] for previous appearances of methods from arithmetic combinatorics in bounding multilinear operators related to \( H_k \).
form (1.5) are negligible if at least one of the functions \( f_i \) is very small in a certain Gowers uniformity norm. We then apply an arithmetic regularity lemma from [4] that asserts, roughly speaking, that any bounded function \( f_i \) can be approximated (up to errors small in Gowers uniformity norm, plus an additional error small in an \( L^2 \) sense) with a special type of function, namely an irrational virtual nilsequence. This effectively allows one to replace all the functions \( f_0, \ldots, f_k \) in (1.5) with such nilsequences. The point of this reduction is that irrational nilsequences enjoy a counting lemma (also from [4]) that allows one to obtain good asymptotics for expressions such as (1.5). At this point, the fact that \( \psi \) is odd ensures that the main term in those asymptotics vanish, and the surviving error terms turn out to be small enough to eventually obtain the required conclusion in Theorem 1.2.

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1.2. Notation. We use the asymptotic notation \( X \lesssim Y, \; Y \gtrsim X, \; \text{or} \; X = O(Y) \) to denote the assertion that \( |X| \leq CY \) for some absolute constant \( C \), which we call the implied constant. We will sometimes need to allow the implied constant to depend on additional parameters, in which case we indicate this by subscripts, e.g. \( X \lesssim_\delta Y, \; Y \gtrsim_\delta Y, \; \text{or} \; X = O_\delta(Y) \) denotes the assertion that \( |X| \leq C_\delta Y \) for some \( C_\delta \) depending on \( \delta \). For brevity we will sometimes fix some basic parameters (e.g. \( k \)) and allow all implied constants to depend on such parameters (so that, for instance, \( X = O_\delta(Y) \) is now short for \( X = O_{\delta,k}(Y) \)). We also write \( X \sim Y \) for \( X \lesssim Y \lesssim X \).

We also use the asymptotic notation \( X = o_{N \to \infty}(Y) \) to denote the assertion \( |X| \leq c(N)Y \) where \( c(N) \) is a quantity depending on a parameter \( N \) that goes to zero as \( N \) goes to infinity. Again, if we need \( c(N) \) to depend on external parameters, we will indicate this by subscripts; for instance, \( X = o_{N \to \infty;k}(Y) \) denotes the assertion that \( |X| \leq c_k(N)Y \) where \( c_k(N) \) goes to zero as \( N \to \infty \) for each fixed choice of \( k \).

2. Initial reductions

We begin the proof of Theorem 1.2. For technical reasons (having to do with the fact that the arithmetic regularity and counting lemmas in the literature are phrased in a discrete setting rather than a continuous one) we will need to transfer Theorem 1.2 from the reals \( \mathbb{R} \) to the integers \( \mathbb{Z} \), giving up the scale invariance of the problem in the process. Namely, we will derive Theorem 1.2 from the following discrete version of that theorem.

**Theorem 2.1** (Discrete version of main theorem). Let \( k \geq 1 \) and \( 1 < p_1, \ldots, p_k, p < \infty \) be such that \( \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_k} \), and let \( \varepsilon > 0 \). Then, if \( R \geq r \geq 1 \) and \( R/r \) is sufficiently
large depending on \(\varepsilon, k, p_1, \ldots, p_k, p\), one has
\[
\left\| \sum_{t \in \mathbb{Z}; \lambda r \leq |t| \leq \lambda R} \frac{f_1(x + t) \cdots f_k(x + kt)}{t} \right\|_{\ell^p(\mathbb{Z})} \leq \varepsilon \log \frac{R}{r} \prod_{i=1}^{k} \|f_i\|_{\ell^p(\mathbb{Z})}
\]
for all \(f_i \in \ell^p(\mathbb{Z})\), \(i = 1, \ldots, k\), where the \(\ell^p\) norm on the left-hand side is in the \(x\) variable.

Let us assume Theorem 2.1 for the moment and see how it implies Theorem 1.2. We will use a standard transference argument. Let \(k, p_1, \ldots, p_k, p, \varepsilon, R, r\) be as in Theorem 1.2 with \(R/r\) assumed large enough depending on \(p_1, \ldots, p_k, p, \varepsilon, R, r\). Let \(\lambda > 0\) be a large quantity (depending on \(k, p_1, \ldots, p_k, p, \varepsilon, R, r\)) to be chosen later. For \(\lambda\) large enough, we have \(\lambda R \geq \lambda r \geq 1\), and so by Theorem 2.1 we have
\[
\left\| \sum_{t \in \mathbb{Z}; \lambda r \leq |t| \leq \lambda R} \frac{f_1(x + t) \cdots f_k(x + kt)}{t} \right\|_{\ell^p(\mathbb{Z})} \leq \varepsilon \log \frac{R}{r} \prod_{i=1}^{k} \|f_i\|_{\ell^p(\mathbb{Z})}
\]
for all \(f_i \in \ell^p(\mathbb{Z})\). In particular, given \(f_i \in L^p(\mathbb{R})\) for \(i = 1, \ldots, k\), we have
\[
\left\| \sum_{t \in \mathbb{Z}; \lambda r \leq |t| \leq \lambda R} \frac{f_1(x + t + \theta) \cdots f_k(x + kt + \theta)}{t} \right\|_{\ell^p(\mathbb{Z})} \leq \varepsilon \log \frac{R}{r} \prod_{i=1}^{k} \left( \sum_{x \in \mathbb{Z}} |f_i(x + \theta)|^{p_i} \right)^{1/p_i}
\]
for any \(0 \leq \theta \leq 1\). Averaging (in \(L^p\)) over all such \(\theta\) and using Hölder’s inequality and Fubini’s theorem, we conclude that
\[
\left\| \sum_{t \in \mathbb{Z}; \lambda r \leq |t| \leq \lambda R} \frac{f_1(x + t) \cdots f_k(x + kt)}{t} \right\|_{L^p(\mathbb{R})} \leq \varepsilon \log \frac{R}{r} \prod_{i=1}^{k} \|f_i\|_{L^p(\mathbb{R})}.
\]
Rescaling by \(\lambda\), we conclude that
\[
\left\| \frac{1}{\lambda} \sum_{t \in \mathbb{Z}; \lambda r \leq |t| \leq \lambda R} \frac{f_1(x + t) \cdots f_k(x + kt)}{t} \right\|_{L^p(\mathbb{R})} \leq \varepsilon \log \frac{R}{r} \prod_{i=1}^{k} \|f_i\|_{L^p(\mathbb{R})}.
\]
Sending \(\lambda \to \infty\) and using Riemann integrability and Fatou’s lemma, we conclude that
\[
\left\| \int_{r \leq |t| \leq R} f_1(x + t) \cdots f_k(x + kt) \frac{dt}{t} \right\|_{L^p(\mathbb{R})} \leq \varepsilon \log \frac{R}{r} \prod_{i=1}^{k} \|f_i\|_{L^p(\mathbb{R})}
\]
if the \(f_i\) are continuous and compactly supported. Applying a limiting argument, we obtain Theorem 1.2.

**Remark 2.2.** In the converse direction, one can derive Theorem 2.1 from Theorem 1.2 by applying the latter to functions of the form \(x \mapsto f_i([x])\); we leave the details to the interested reader.

It remains to establish Theorem 2.1. We will perform a number of preliminary reductions analogous to those in [8, 9], namely a reduction to a dyadic restricted weak-type
estimate, and the construction of various “trees” of dyadic intervals, with the key nontrivial input being a tree estimate (see Proposition 2.4 below) that improves over the trivial bound coming from the triangle inequality.

We turn to the details. By duality, it will suffice to show that
\[ \left| \sum_{t \in \mathbb{Z}, r \leq |t| \leq R} \sum_{x \in \mathbb{Z}} \frac{f_0(x) f_1(x + t) \ldots f_k(x + (k-1)t)}{t} \right| \leq \varepsilon \log \frac{R}{r} \prod_{i=0}^{k} \|f_i\|_{L^p(\mathbb{R})} \]
whenever \( k \geq 1, \varepsilon > 0, 1 < p_0, \ldots, p_k < \infty \) are such that \( \frac{1}{p_0} + \cdots + \frac{1}{p_k} = 1 \), \( f_i \in L^{p_i}(\mathbb{R}) \) for \( i = 0, \ldots, k \), \( R \geq r \geq 1 \), and \( R/r \) is sufficiently large depending on \( \varepsilon \). By multilinear interpolation (and modifying \( \varepsilon, p_0, \ldots, p_k \) as necessary), we may replace the strong Lebesgue norms \( L^{p_i}(\mathbb{Z}) \) here by the Lorentz norms \( L^{p_i,1}(\mathbb{Z}) \), and then by convexity we may reduce to the case where each of the \( f_i \) are indicator functions, thus it will suffice to show that
\[ \left| \sum_{t \in \mathbb{Z}, r \leq |t| \leq R} \sum_{x \in \mathbb{Z}} \frac{1_{E_0}(x) 1_{E_1}(x + t) \ldots 1_{E_k}(x + (k-1)t)}{t} \right| \leq \varepsilon \log \frac{R}{r} \prod_{i=0}^{k} |E_i|^{1/p_i} \tag{2.1} \]
whenever \( \varepsilon > 0, 1 < p_0, \ldots, p_k < \infty \) are such that \( \frac{1}{p_0} + \cdots + \frac{1}{p_k} = 1 \), \( E_i \) are subsets of \( \mathbb{Z} \) with finite cardinality \( |E_i| \), and \( R \geq r \geq 1 \) with \( R/r \) is sufficiently large depending on \( \varepsilon \). Here of course \( 1_E \) denotes the indicator function of \( E \).

Henceforth \( k, p_0, \ldots, p_k \) will be fixed, and all implied constants will be allowed to depend on these parameters. From Hölder’s inequality we have
\[ \sum_{x \in \mathbb{Z}} 1_{E_0}(x) 1_{E_1}(x + t) \ldots 1_{E_k}(x + (k-1)t) \, dx \leq \min_{0 \leq i \leq k} |E_i| \]
for any \( t \), and thus
\[ \left| \sum_{t \in \mathbb{Z}, r \leq |t| \leq R} \sum_{x \in \mathbb{Z}} \frac{1_{E_0}(x) 1_{E_1}(x + t) \ldots 1_{E_k}(x + (k-1)t)}{t} \right| \leq \log \frac{R}{r} \min_{0 \leq i \leq k} |E_i| \]
Comparing this with (2.1), we see that we are done unless
\[ |E_i| \sim_\varepsilon |E_0| \tag{2.2} \]
for all \( 0 \leq i \leq k \). Henceforth we will assume that (2.2) holds. It will now suffice (after adjusting \( \varepsilon \) if necessary) to show that
\[ \left| \sum_{t \in \mathbb{Z}, r \leq |t| \leq R} \sum_{x \in \mathbb{Z}} \frac{1_{E_0}(x) 1_{E_1}(x + t) \ldots 1_{E_k}(x + (k-1)t)}{t} \right| \leq \varepsilon \log \frac{R}{r} |E_0| \tag{2.3} \]
if \( R/r \) is sufficiently large depending on \( \varepsilon \).

The next step is a decomposition into dyadic intervals. Let \( \psi : \mathbb{R} \to \mathbb{R} \) be a fixed smooth odd function supported on \([-2,-1/2] \cup [1/2,2]\) with the property that
\[ \sum_{n \in \mathbb{Z}} 2^{-n} \psi(2^{-n}t) = \frac{1}{t} \]
for all $t \neq 0$; such a function can be constructed by taking $\psi(t) := \phi(t) - \phi(t/2)$ for some smooth even $\phi : \mathbb{R} \to \mathbb{R}$ supported on $[-1, 1]$ and equaling 1 on $[-1/2, 1/2]$. Henceforth we allow implied constants to depend on $\psi$. Then the function $\mathbb{1}_{t \leq |t| \leq R}$ differs from $\sum_{n,r \leq 2^n \leq R} 2^{-n} \psi(t/2^n)$ only when $t \sim R$ or $t \sim r$, where both functions are $O(1/R)$ and $O(1/r)$ respectively. From this and the triangle inequality one sees that (2.3) is equivalent to

$$\sum_{n,r \leq 2^n \leq R} 2^{-n} \sum_{x, t \in \mathbb{Z}} E_0(x) 1_{E_1}(x+t) \ldots 1_{E_k}(x + kt) \psi(t/2^n) \leq \varepsilon \log \frac{R}{r} |E_0|$$

since the left-hand side here differs from that of (2.3) by $O(1)$, which is acceptable if $R/r$ is large enough depending on $\varepsilon$. By the triangle inequality, it thus suffices to show that

$$\sum_{n,r \leq 2^n \leq R} 2^{-n} \left| \sum_{x, t \in \mathbb{Z}} E_0(x) 1_{E_1}(x+t) \ldots 1_{E_k}(x + kt) \psi(t/2^n) \right| dx dt \leq \varepsilon \log \frac{R}{r} |E_0|.$$  

We now introduce a further smooth function $\phi : \mathbb{R} \to \mathbb{R}$ supported on $[-1, 1]$ such that

$$\phi(x) = 1$$

for all $x \in \mathbb{R}$; indeed one can take $\phi(x) := \eta(x) - \eta(x + 1)$ for some smooth $\eta : \mathbb{R} \to \mathbb{R}$ equal to 1 for negative $x$ and 0 for $x > 1$. We allow implied constants to depend on $\phi$. For each $I$ (discrete) dyadic interval $I = \{x \in \mathbb{Z} : j2^n < x \leq (j + 1)2^n \}$ with $n \geq 0$, we define the quantity

$$a_I := 2^{-2n} \left| \sum_{x, t \in \mathbb{Z}} E_0(x) 1_{E_1}(x + t) \ldots 1_{E_k}(x + (k - 1)t) \psi(t/2^n) \phi(2^{-n}x - j) \right|$$  

so by the triangle inequality it suffices to show that

$$\sum_{I, r \leq |I| \leq R} a_I |I| \leq \varepsilon \log \frac{R}{r} |E_0|$$  

where the sum is over dyadic intervals $I$ of length between $r$ and $R$.

From the triangle inequality and (2.4) we have the bound

$$a_I \leq 1$$  

for all $I$. We also have the following estimate:

**Lemma 2.3.** We have

$$\sum_{I, r \leq |I| \leq R} a_I^{p/2} |I| \leq \varepsilon_p \log \frac{R}{r} |E_0|$$

for any $1 < p < \infty$.  

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As we are working on the integers, we do not consider dyadic intervals of length less than 1.
Proof. We bound

\[ a_I \lesssim 2^{-2n} \int_{|I|^{-2^n}} \int_{x=(j+O(1))2^n} 1_{E_0}(x) 1_{E_1}(x + t) \, dx \, dt \]

\[ \lesssim 2^{-2n} \left( \int_{x=(j+O(1))2^n} 1_{E_0}(x) \, dx \right) \left( \int_{y=(j+O(1))2^n} 1_{E_1}(y) \, dy \right) \]

\[ \lesssim \inf_{x \in I} M1_{E_0}(x) M1_{E_1}(x) \]

where \( Mf(x) := \sup_{r \geq 1} \frac{1}{2r+1} \sum_{y \in \mathbb{Z} : |y-x| \leq r} |f(y)| \) is the Hardy-Littlewood maximal operator on the integers \( \mathbb{Z} \). Thus

\[ a_I^{p/2} |I| \lesssim \sum_{x \in I} M1_{E_0}(x)^{p/2} M1_{E_1}(x)^{p/2} \]

and so (since each \( x \in \mathbb{Z} \) is contained in \( O(\log R) \) dyadic intervals \( I \) with \( r \leq |I| \leq R \))

\[ \sum_{I : r \leq |I| \leq R} a_I^{p/2} |I| \lesssim \left( \log \frac{R}{r} \right) \sum_{x \in \mathbb{Z}} M1_{E_0}(x)^{p/2} M1_{E_1}(x)^{p/2} \]

and the claim follows from the Cauchy-Schwarz inequality, the Hardy-Littlewood maximal inequality, and \((2.2)\). \( \square \)

From the above lemma with \( p = 3/2 \) (say) we see in particular that

\[ \sum_{I : r \leq |I| \leq R, a_I \leq \delta} a_I |I| \lesssim \delta^{1/4} \log \frac{R}{r} |E_0| \]

so to prove \((2.5)\) it suffices by \((2.6)\) to show that

\[ \sum_{I : r \leq |I| \leq R, a_I > \delta} |I| \lesssim \varepsilon \log \frac{R}{r} |E_0| \]

for any \( \delta > 0 \), whenever \( R/r \) is sufficiently large depending on \( \varepsilon, \delta \).

In the next section, we will establish the following result.

**Proposition 2.4** (Cancellation in a tree). Let \( I_0 \) be a dyadic interval, and let \( \delta > 0 \). Then there exists a quantity \( 1 \leq A \ll 1 \) (which can depend on \( I_0 \)) such that

\[ \sum_{I : I \subset I_0, |I| \ll |I_0|, a_I \leq \delta} a_I |I| \lesssim \delta |I_0| \log A. \]

The point here, of course, is the gain of \( \delta \) on the right-hand side, since otherwise the claim is immediate from the trivial bound \((2.6)\).

Let us assume this proposition for the moment and see how to conclude \((2.7)\). Call a dyadic interval \( I \) bad if \( r \leq |I| \leq R \) and \( a_I > \delta \). From the proof of Lemma 2.3 (bounding \( M1_{E_1} \) crudely by 1) we see that

\[ M1_{E_0}(x) \gg \delta \]

whenever \( x \) lies in a bad interval. In particular, from the Hardy-Littlewood maximal inequality we see that there are only finitely many bad intervals.
Let $I$ denote the collection of bad dyadic intervals. Using a greedy algorithm (starting with the largest bad intervals and only moving on to the smaller bad intervals once all the largest ones have been covered), as well as Proposition 2.4 (with $\delta$ replaced by $\varepsilon \delta$), we may cover $I$ by a family $T$ of disjoint “trees” $T$, each of the form $T = \{I : I \subset I_T; |I_T|/A_T \leq |I| \leq |I_T|\}$ for some “tree top” $I_T \in I$ and some quantity $1 \leq A_T \ll_{\varepsilon, \delta} 1$, with the property that
\[ \sum_{I \in T} a_I |I| \lesssim \varepsilon \delta |I_T| \log A_T. \]

Note that we only require that the top $I_T$ of the tree $T$ lie in $I$; the other elements of $T$ may lie outside $I$.

Summing over all trees $T \in \mathcal{T}$, we conclude that
\[ \sum_{I : r \leq |I| \leq R} |I| \lesssim \varepsilon \delta \sum_{T \in \mathcal{T}} |I_T| \log A_T. \]

On the other hand, we have
\[ |I_T| \log A_T \ll \int_{\mathbb{R}} \sum_{I \in T} 1_I(x) \, dx \]
for each tree $T \in \mathcal{T}$, and thus (by the disjointness of the trees $T$)
\[ \sum_{I : r \leq |I| \leq R} |I| \lesssim \varepsilon \delta \sum_{x \in \mathbb{Z}} \sum_{I \in \bigcup_{T \in \mathcal{T}}} 1_I(x) \, dx. \]

For each $x$ in the support of $\sum_{I \in \bigcup_{T \in \mathcal{T}}} 1_I(x)$, we have $x \in I_T$ for some tree top $I_T$, and thus $M 1_{E_0}(x) \gg \delta$. By the Hardy-Littlewood maximal inequality, we thus see that $x$ is contained in a set of measure $O(|E_0|/\delta)$. Finally, by construction, every interval $I$ in a tree $T \in \mathcal{T}$ has size at most $R$ and at least $r/A_T \gg_{\varepsilon, \delta} r$, and so each $x$ is contained in at most $O(\log \frac{R}{r})$ intervals if $R/r$ is sufficiently large depending on $\varepsilon, \delta$. Putting all this together we obtain (2.7) as required.

It remains to establish Proposition 2.4. This will be accomplished in the next section.

3. Applying the arithmetic regularity and counting lemmas

By translation we may assume that
\[ I_0 = \{1, \ldots, N\} =: [N] \]
for some natural number $N$ which is a power of 2. Our task is to find $1 \ll A \ll_{\delta} 1$ such that
\[ \sum_{n : N/A \leq 2^n \leq N} \sum_{j=0}^{N/2^n-1} 2^{-n} \left| \sum_{x \in \mathbb{Z}} 1_{E_0}(x) 1_{E_1}(x + t) \cdots 1_{E_k}(x + (k-1)t) \psi(t/2^n) \varphi(2^{-n}x - j) \right| \ll \delta N \log A. \]

(3.1)

We will in fact produce an $A$ with $A \geq C_{\delta}$, where $C_{\delta}$ is a sufficiently large quantity depending on $\delta$. We may assume that $N$ is sufficiently large depending on $\delta$, since
otherwise we can use the trivial bound of $O(N \log N)$ on the left-hand side (coming from the fact that there are only $O(\log N)$ choices for $n$) to conclude, after choosing $A$ large enough.

Let $E'_i := E_i \cap [N]$. For each $n$ with $N/A \leq 2^n \leq N$, we may replace the $E_i$ by $E'_i$ in (3.1) for all but $O(1)$ choices of $j$ (coming from those $j$ near 0 or $N/2^n$). The total error in replacing $E_i$ by $E'_i$ is thus

$$\sum_{n : N/A \leq 2^n \leq N} O(2^{-n}2^{2n}) = O(N)$$

which is acceptable since $A \geq C_\delta$ for some large $C_\delta$. It thus suffices to show that

$$\sum_{n : N/A \leq 2^n \leq N} \sum_{j = 0}^{N/2^n - 1} 2^{-n} \left| \sum_{x, t \in \mathbb{Z}} 1_{E_i'}(x)1_{E_i}(x + t)\ldots1_{E_i}(x + kt)\psi(t/2^n)\varphi(2^{-n}x - j) \right| \ (3.2)$$

for some $C_\delta \leq A \ll 1$.

To control this expression we recall the Gowers uniformity norms from [5, 6]. If $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ is a function on a cyclic group $\mathbb{Z}/N\mathbb{Z}$ and $d \geq 1$, we define the Gowers uniformity norm $\|f\|_{U^d(\mathbb{Z}/N\mathbb{Z})}$ by the formula

$$\|f\|_{U^d(\mathbb{Z}/N\mathbb{Z})} = \left( \frac{1}{N^{d+1}} \sum_{h_1, \ldots, h_d, x \in \mathbb{Z}/N\mathbb{Z}} \Delta_{h_1} \ldots \Delta_{h_d} f(x) \right)^{1/2d}$$

where $\Delta_h f(x) := f(x + h)f(x)$. Given instead a function $f : [N] \to \mathbb{C}$ on the interval $[N]$, we define the Gowers norm $\|f\|_{U^d([N])}$ by the formula

$$\|f\|_{U^d([N])} = \|f\|_{U^d(\mathbb{Z}/N\mathbb{Z})}/\|1_{[N]}\|_{U^d(\mathbb{Z}/N\mathbb{Z})}$$

for any $N' \geq 2^d N$, where we embed $[N]$ into $\mathbb{Z}/N'\mathbb{Z}$ and extend $f$ by zero outside of $[N]$; it is easy to see that this definition does not depend on the choice of $N$. We have the generalised von Neumann theorem

$$\frac{1}{N^2} \left| \sum_{x, t \in \mathbb{Z}} f_0(x)f_1(x + t)\ldots f_k(x + kt) \right| \leq \inf_{0 \leq i \leq k} \|f_i\|_{U^k([N])} \ (3.3)$$

for any $f_0, \ldots, f_k : [N] \to \mathbb{C}$ bounded in magnitude by 1 (extending by zero outside of $[N]$); see e.g. [12] Lemma 11.4 (after applying the embedding of $[N]$ into some suitable $\mathbb{Z}/N'\mathbb{Z}$). We have the following variant:

**Lemma 3.1.** Let $d := \max(k, 2)$. Then for any $n$ with $2^n \leq N$, any $j \in \mathbb{Z}$ and any functions $f_0, \ldots, f_k : [N] \to \mathbb{C}$ bounded in magnitude by 1 (and extended by zero outside of $[N]$), we have

$$\frac{1}{N^2} \left| \sum_{x, t \in \mathbb{Z}} f_0(x)f_1(x + t)\ldots f_k(x + kt)\psi(t/2^n)\varphi(2^{-n}x - j) \right| \leq \inf_{0 \leq i \leq k} \|f_i\|_{U^k([N])}.$$
In particular, by the triangle inequality we have
\[
\sum_{n: N/A \leq 2^n \leq N} \sum_{j=0}^{N/2^n-1} 2^{-n} \left| \sum_{x, t \in \mathbb{Z}} f_0(x) f_1(x + t) \ldots f_k(x + kt) \psi(t/2^n) \varphi(2^{-n} x - j) \right| \leq A \inf_{0 \leq i \leq k} \| f_i \|_{U^d([N])}.
\] (3.4)

**Proof.** By adding a dummy function \( f_{k+1} \) if necessary we may assume that \( k \geq 2 \), so that \( d = k \). By Fourier inversion we may write
\[
\psi(t) = \int_{\mathbb{R}} e^{2\pi i t u} \hat{\psi}(u) \, du
\]
and
\[
\varphi(x) = \int_{\mathbb{R}} e^{2\pi i x v} \hat{\varphi}(v) \, dv
\]
for some rapidly decreasing (and in particular, absolutely integrable) functions \( \hat{\psi}, \hat{\varphi} : \mathbb{R} \to \mathbb{C} \). By the triangle inequality, it thus suffices to show that
\[
\frac{1}{N^2} \left| \sum_{x, t \in \mathbb{Z}} f_0(x) f_1(x + t) \ldots f_k(x + kt) e^{2\pi i t u} e^{2\pi i x v} \right| \leq \inf_{0 \leq i \leq k} \| f_i \|_{U^k([N])}
\]
uniformly for all \( u, v \in \mathbb{R} \). Writing
\[
f_0(x) f_1(x + t) e^{2\pi i t u} e^{2\pi i x v} = f_0(x) e^{2\pi i x (v-u)} \times f_1(x + t) e^{2\pi i (x + t)u}
\]
and applying (3.3), we see that
\[
\frac{1}{N^2} \left| \sum_{x, t \in \mathbb{Z}} f_0(x) f_1(x + t) \ldots f_k(x + kt) e^{2\pi i t u} e^{2\pi i x v} \right| \leq \inf_{0 \leq i \leq k} \| \tilde{f}_i \|_{U^k([N])}
\]
where \( \tilde{f}_i \) is \( f_i \) modulated by a Fourier character \( x \mapsto e^{2\pi i x v} \) for some \( v \in \mathbb{R} \). But if \( k \geq 2 \), one easily verifies that \( \tilde{f}_i \) has the same \( U^k \) norm as \( f_i \) (because \( \Delta_{h_1} \Delta_{h_2} \tilde{f}_i = \Delta_{h_1} \Delta_{h_2} f_i \) for any \( h_1, h_2 \)), and the claim follows. \( \square \)

We also need a similar statement in which the \( U^d \) norm is replaced by the \( L^2 \) norm:

**Lemma 3.2.** For any \( f_0, \ldots, f_k : [N] \to \mathbb{C} \) bounded in magnitude by 1, we have
\[
\sum_{n: N/A \leq 2^n \leq N} 2^{-n} \left| \sum_{x, t \in \mathbb{Z}} f_0(x) f_1(x + t) \ldots f_k(x + kt) \psi(t/2^n) \varphi(2^{-n} x - j) \right| \leq N^{1/2} (\log A) \inf_{0 \leq i \leq k} \| f_i \|_{L^2([N])}.
\]

**Proof.** Let \( 0 \leq i \leq k \). Observe that for each integer \( y \) and each \( n \) with \( N/A \leq 2^n \leq N \), there are at most \( O(1) \) choices of \( j \) for which the sum \( \sum_{x, t \in \mathbb{Z}} f_0(x) f_1(x + t) \ldots f_k(x + \)
for all \( n \) in the Mal’cev basis a sequence \( g \) (Polynomial sequence) \( \psi \). From this and the triangle inequality we see that

\[
\sum_{n: N/A \leq 2^n \leq N} \sum_{j=0}^{N/2^n-1} 2^{-n} \left| \sum_{x, t \in \mathbb{Z}} f_0(x) f_1(x + t) \ldots f_k(x + kt) \psi(t/2^n) \varphi(2^{-n}x - j) \right| 
\]

and the claim now follows from the Cauchy-Schwarz inequality. \qed

The above lemmas show, roughly speaking, that we may freely modify each of the \( 1_{E^i} \) by errors that are either small in (normalised) \( \ell^2([N]) \) norm, or extremely small in \( U^d([N]) \) norm. To exploit this phenomenon, we will need the arithmetic regularity lemma from \cite{3} that asserts that all bounded functions on \([N]\), up to errors of the above form, can be expressed as a very well distributed nilsequence. To make this statement precise, we need to recall a large number of definitions from \cite{4} (although for the purposes of this paper, many of the concepts defined here can be taken as “black boxes”).

**Definition 3.3** (Filtered nilmanifold). Let \( s \geq 1 \) be an integer. A **filtered nilmanifold** \( G/\Gamma = (G/\Gamma, G_\cdot) \) of degree \( \leq s \) consists of the following data:

1. A connected, simply-connected nilpotent Lie group \( G \);
2. A discrete, cocompact subgroup \( \Gamma \) of \( G \) (thus the quotient space \( G/\Gamma \) is a compact manifold, known as a nilmanifold);
3. A filtration \( G_\cdot = (G(i))_{i=0}^\infty \) of closed connected subgroups

\[
G = G(0) \supseteq G(1) \supseteq G(2) \supseteq \ldots
\]

of \( G \), which are rational in the sense that the subgroups \( \Gamma(i) := \Gamma \cap G(i) \) are cocompact in \( G(i) \), such that \([G(i), G(j)] \subseteq G(i+j)\) for all \( i, j \geq 0 \), and such that \( G(i) = \{ \text{id} \} \) whenever \( i > s \);
4. A Mal’cev basis \( \mathcal{X} = \{ X_1, \ldots, X_{\dim(G)} \} \) adapted to \( G_\cdot \), that is to say a basis \( X_1, \ldots, X_{\dim(G)} \) of the Lie algebra of \( G \) that exponentiates to elements of \( \Gamma \), such that \( X_j \) span a Lie algebra ideal for all \( j \leq i \leq \dim(G) \), and \( X_{\dim(G)-\dim(G(i)+1)}, \ldots, X_{\dim(G)} \) spans the Lie algebra of \( G(i) \) for all \( 1 \leq i \leq s \).

One may use a Mal’cev basis to define a metric \( d_{G/\Gamma} \) on the nilmanifold \( G/\Gamma \), as per \cite{3} Definition 2.2].

**Definition 3.4** (Complexity). Let \( M \geq 1 \). We say that a filtered nilmanifold \( G/\Gamma = (G/\Gamma, G_\cdot) \) has complexity \( \leq M \) if the dimension of \( G \), the degree of \( G_\cdot \), and the rationality of the Mal’cev basis \( \mathcal{X} \) (cf. \cite{3} Definition 2.4]) are bounded by \( M \).

**Definition 3.5** (Polynomial sequence). Let \( (G/\Gamma, G_\cdot) \) be a filtered nilmanifold, with filtration \( G_\cdot = (G(i))_{i=0}^\infty \). A **polynomial sequence** adapted to this filtered nilmanifold is a sequence \( g : \mathbb{Z} \to G \) with the property that

\[
\partial_{h_1} \ldots \partial_{h_i} g(n) \in G(i)
\]

for all \( i \geq 0 \) and \( h_1, \ldots, h_i, n \in \mathbb{Z} \), where \( \partial_h g(n) := g(n + h)g(n)^{-1} \) is the derivative of \( g \) with respect to the shift \( h \).
Definition 3.6 (Orbits). Let \( s \geq 1 \) be an integer, and let \( M, A > 0 \) be parameters. A polynomial orbit of degree \( \leq s \) and complexity \( \leq M \) is any function \( n \mapsto g(n)\Gamma \) from \( \mathbb{Z} \to G/\Gamma \), where \( (G/\Gamma, G_\star) \) is a filtered nilmanifold of complexity \( \leq M \), and \( g : \mathbb{Z} \to G \) is a polynomial sequence.

Definition 3.7 (Nilsequences). A (polynomial) nilsequence of degree \( \leq s \) and complexity \( \leq M \) is any function \( f : \mathbb{Z} \to \mathbb{C} \) of the form \( f(n) = F(g(n)\Gamma, n \bmod q, n/N) \), where \( 1 \leq q \leq M \) is an integer, \( n \mapsto g(n)\Gamma \) is a polynomial orbit of degree \( \leq s \) and complexity \( \leq M \), and \( F : G/\Gamma \times \mathbb{Z}/q\mathbb{Z} \to \mathbb{C} \) is a function of Lipschitz norm\(^4\) at most \( M \).

Definition 3.8 (Virtual nilsequences). Let \( N \geq 1 \). A virtual nilsequence of degree \( \leq s \) and complexity \( \leq M \) at scale \( N \) is any function \( f : [N] \to \mathbb{C} \) of the form \( f(n) = F(g(n)\Gamma, n \bmod q, n/N) \), where \( 1 \leq q \leq M \) is an integer, \( n \mapsto g(n)\Gamma \) is a polynomial orbit of degree \( \leq s \) and complexity \( \leq M \), and \( F : G/\Gamma \times \mathbb{Z}/q\mathbb{Z} \times \mathbb{R} \to \mathbb{C} \) is a function of Lipschitz norm at most \( M \). (Here we place a metric on \( G/\Gamma \times \mathbb{Z}/q\mathbb{Z} \times \mathbb{R} \) in some arbitrary fashion, e.g. by embedding \( \mathbb{Z}/q\mathbb{Z} \) in \( \mathbb{R}/\mathbb{Z} \) and taking the direct sum of the metrics on the three factors.) We define a vector valued virtual nilsequence \( f : [N] \to \mathbb{C}^d \) similarly, except that \( F \) now takes values in \( \mathbb{C}^d \) instead of \( \mathbb{C} \).

We now have almost all the definitions needed to state the arithmetic regularity lemma:

Theorem 3.9 (Arithmetic regularity lemma). Let \( f : [N] \to [0,1]^d \) be a function for some \( d \geq 0 \), let \( s \geq 1 \) be an integer, let \( \varepsilon > 0 \), and let \( \mathcal{F} : \mathbb{R}^+ \to \mathbb{R}^+ \) be a monotone increasing function with \( \mathcal{F}(M) \geq M \) for all \( M \). Then there exists a quantity \( M = O_{s,\varepsilon,\mathcal{F},d}(1) \) and a decomposition

\[
 f = f_{\text{nil}} + f_{\text{sml}} + f_{\text{unf}}
\]

of \( f \) into functions \( f_{\text{nil}}, f_{\text{sml}}, f_{\text{unf}} : [N] \to [-1,1]^d \) of the following form:

1. \((f_{\text{nil}}, \text{structured})\) \( f_{\text{nil}} \) is a \((\mathcal{F}(M), N)\)-irrational vector-valued virtual nilsequence of degree \( \leq s \), complexity \( \leq M \), and scale \( N \), where the notion of \((A,N)\)-irrationality is defined in\(^1\) Definition A.6;
2. \((f_{\text{sml}}, \text{small})\) Each of the \( d \) components of \( f_{\text{sml}} \) has an \( \ell^2([N]) \) norm of at most \( \varepsilon N^{1/2} \);
3. \((f_{\text{unf}}, \text{very uniform})\) Each of the \( d \) components of \( f_{\text{unf}} \) has a \( U^{s+1}([N]) \) norm of at most \( 1/\mathcal{F}(M) \);
4. \((\text{Nonnegativity})\) \( f_{\text{nil}} \) and \( f_{\text{nil}} + f_{\text{sml}} \) take values in \([0,1]^d\).

Proof. See\(^2\) Theorem 1.2. Strictly speaking, this theorem is only stated in the scalar case \( d = 1 \), but the same argument extends without difficulty to the vector-valued case \( d \geq 1 \) (note we allow our bound on \( M \) to depend on \( d \)). We remark that the bounds on \( M \) in the above theorem are extremely poor (at least tower-exponential or worse, in

\(^4\)The (inhomogeneous) Lipschitz norm \( |F|_{\text{Lip}} \) of a function \( F : X \to \mathbb{C} \) on a metric space \( X = (X,d) \) is defined as

\[
 |F|_{\text{Lip}} := \sup_{x \in X} |F(x)| + \sup_{x,y \in X, x \neq y} \frac{|F(x) - F(y)|}{|x - y|}.
\]
practice) for a variety of reasons, including the lack of good (or indeed any) quantitative bounds for the inverse theorem for higher order Gowers uniformity norms.

We apply this lemma with $\varepsilon$ replaced by $\delta^2$, $s + 1$ replaced by $d$, $d$ replaced by $k + 1$, and $\mathcal{F}$ a rapidly increasing function to be chosen later, to obtain decompositions

$$1_{E_1} = f_{\text{nil},i} + f_{\text{sml},i} + f_{\text{unf},i}$$

with $f_{\text{nil},i}, f_{\text{sml},i}, f_{\text{unf},i} : [N] \rightarrow [-1, 1]$ being the components of functions $f_{\text{nil}}, f_{\text{sml}}, f_{\text{unf}} : [N] \rightarrow [-1, 1]^{k+1}$ obeying the conclusions of Theorem 3.9. By the triangle inequality, the left-hand side of (3.2) can be written as the sum of $3^{k+1} = O(1)$ terms, in which each of the $1_{E_1}$ has been replaced by one of $f_{\text{nil},i}, f_{\text{sml},i}, f_{\text{unf},i}$. By Lemma 3.1, the contribution of any term involving one of the $f_{\text{unf},i}$ is at most $O_A(N/\mathcal{F}(M))$, while from Lemma 3.2 the contribution of any term involving one of the $f_{\text{sml},i}$ is $O(\delta N \log A)$. Thus, one may bound the left-hand side of (3.2) by

$$\left( \sum_{n: N/A \leq 2^n \leq N} X_n \right) + O_A(N/\mathcal{F}(M)) + O(\delta N \log A)$$

where $X_n$ is the quantity

$$X_n := \sum_{j=0}^{N/2^n-1} 2^{-n} \left| \sum_{x \in \mathbb{Z}} f_{\text{nil},0}(x) f_{\text{nil},1}(x + t) \ldots f_{\text{nil},k}(x + kt) \psi(t/2^n) \varphi(2^{-n}x - j) \right|.$$ 

We now turn to the estimation of $X_n$. Bounding each $f_{\text{nil},i}$ by $O(1)$, we have the trivial bound

$$X_n \lesssim N$$

which we will use for values of $2^n$ that are close to $N$. For the remaining values of $n$, we argue as follows. By Definition 3.8, we have

$$f_{\text{nil},i}(x + it) = F_i \left( g(x + it) \Gamma, x + it \mod q, \frac{x + it}{N} \right)$$

whenever $0 \leq i \leq k$ and $x + it \in [N]$, for some positive integer $q = O_M(1)$, some $(\mathcal{F}(M), N)$-irrational polynomial orbit $n \mapsto g(n) \Gamma$ of degree $\leq d - 1$ and complexity $O_M(1)$ into a filtered nilmanifold $G/\Gamma$ of degree $\leq d - 1$ and complexity $O_M(1)$, and a function $F_i : G/\Gamma \times \mathbb{Z}/q\mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{C}$ of Lipschitz norm at most $O_M(1)$.

For all but $O(1)$ values of $0 \leq j \leq N/2^n - 1$, one has $x, x + t, \ldots, x + kt \in [N]$ in the support of $\psi(t/2^n) \varphi(2^{-n}x - j)$. The exceptional values of $j$ contribute $O(2^n)$ to $X_n$; thus

$$X_n \ll \sum_{j=0}^{N/2^n-1} 2^{-n} \left| \sum_{x \in \mathbb{Z}} \prod_{i=0}^{k} F_i \left( g(x + it) \Gamma, x + it \mod q, \frac{x + it}{N} \right) \psi(t/2^n) \varphi(2^{-n}x - j) \right| + 2^n.$$ 

By the triangle inequality, we thus have

$$X_n \ll 2^{-2nN} \left| \sum_{x \in \mathbb{Z}} \prod_{i=0}^{k} F_i(g(x + it) \Gamma, x + it \mod q, \frac{x + it}{N}) \psi(t/2^n) \varphi(2^{-n}x - j) \right| + 2^n$$
for some $0 \leq j < N^{1/2^n}$. Splitting $x, t$ into residue classes modulo $q = O_M(1)$ and using
the triangle inequality, we thus have

$$X_n \lesssim_M 2^{-2n}N \sum_{x,t \in \mathbb{Z} : x=a \mod q,t=b \mod q} \prod_{i=0}^k F_i(g(x+it)\Gamma, a+ib \mod q, \frac{x+it}{N})\psi(t/2^n)\varphi(2^{-n}x-j) + 2^n$$

for some $0 \leq a,b < q$. Next, on the support of $\psi(t/2^n)\varphi(2^{-n}x-j)$ we have $\frac{x+it}{N} = \frac{j2^n}{N} + O(\frac{2^n}{N})$, hence by the Lipschitz property

$$F_i(g(x+it)\Gamma, a+ib \mod q, \frac{x+it}{N}) = F'_i(g(x+it)\Gamma) + O_M\left(\frac{2^n}{N}\right)$$

where $F'_i = F_{i,j,a,h,n,N} : G/\Gamma \to \mathbb{R}$ is the function

$$F'_i(x) := F_i\left(x, a+ib \mod q, \frac{j2^n}{N}\right).$$

Note that $F'_i$ has a Lipschitz norm of $O_M(1)$. We conclude that

$$X_n \lesssim_M 2^{-2n}N \sum_{x,t \in \mathbb{Z} : x=a \mod q,t=b \mod q} \prod_{i=0}^k F_{i,j}(g(x+it)\Gamma)\psi(t/2^n)\varphi(2^{-n}x-j) + 2^n.$$

Making the substitution $x = qx' + a$, $t = qt' + b$, this becomes

$$X_n \lesssim_M 2^{-2n}N \sum_{x',t' \in \mathbb{Z}} \prod_{i=0}^k F_{i,j}(g(qx' + iqt' + a+ib)\Gamma)\psi(\frac{qt' + a}{2^n})\varphi(2^{-n}qx' + 2^{-n}a-j) + 2^n.$$ 

At this point, we use the counting lemma from [4, Theorem 1.11]. This gives the asymptotic

$$\sum_{x',t' \in \mathbb{Z}} \prod_{i=0}^k F_{i,j}(g(qx' + iqt' + a+ib)\Gamma) = |I||J|\alpha + o_{F(M)\to x;M}(N^2) + o_{N\to x;M}(N^2)$$

for any intervals $I, J \subset [-N,N]$, where $\alpha$ is a quantity independent of $I, J$ (it is given by
an explicit integral of a certain Lipschitz function on a certain filtered nilmanifold, but
its precise value is immaterial for the current argument). Since the left-hand side of this
asymptotic is $O_M(|I||J|)$, we may assume without loss of generality that $\alpha = O_M(1)$. A
routine Riemann sum argument using the bound $2^n \geq N/A$ and the smooth nature of
$\psi, \varphi$ (decomposing the $x', t'$ variables into intervals of length $[N/A^{100}]$) then shows that

$$\sum_{x',t' \in \mathbb{Z}} \prod_{i=0}^k F_{i,j}(g(qx' + iqt' + a+ib)\Gamma)\psi(\frac{qt' + a}{2^n})\varphi(2^{-n}qx' + 2^{-n}a-j)$$

$$= \alpha \int_{\mathbb{R}^2} \psi(\frac{qt' + a}{2^n})\varphi(2^{-n}qx' + 2^{-n}a-j) \, dx'dt' + O(A^{-90}N^2)$$

$$+ o_{F(M)\to x;M,A}(N^2) + o_{N\to x;M,A}(N^2).$$

Since $\psi$ is odd, the integral here vanishes. We conclude (since $2^{-2n}N \ll A^2N^{-1}$) that

$$X_n \ll_M o_{F(M)\to x;M,A}(N) + o_{N\to x;M,A}(N) + A^{-80}N + 2^n.$$
Using this bound for $2^n \leq A^{-\delta} N$, and the trivial bound \((3.5)\) for $A^{-\delta} N < n \leq N$, we have
\[
\sum_{n: N/2^n \leq N} X_n \ll o_{F(M)}(A) + o_{N \to A}(A(N)) + O_M(A^{-\delta} N) + \delta N \log A
\]
and so we may bound the left-hand side of \((3.2)\) by
\[
o_{F(M)}(A) + o_{N \to A}(A(N)) + O_M(A^{-\delta} N) + O(\delta N \log A).
\]
If we choose $A$ sufficiently large depending on $\delta, M$, and then $F$ sufficiently rapidly growing, and then $N$ sufficiently large depending on $\delta, F$ (recalling that $M = O_{k,F}(1)$), we can make this expression $O(\delta N \log A)$, giving \((3.2)\) as required.

**Remark 3.10.** The above arguments also provide some non-trivial cancellation for (truncations of) other variants of the multilinear Hilbert transform. For instance, one replace $H_k$ by the maximal truncated $k$-linear Hilbert transform
\[
sup_{r<R} |H_{k,r,R} f(x)|;
\]
one could also consider the multilinear Hilbert transform
\[
p.v. \int_{\mathbb{R}} f_1(x + c_1 t) \ldots f_k(x + c_k t) \frac{dt}{t} \tag{3.6}
\]
with rational coefficients $c_1, \ldots, c_k$; we leave the modification of the above arguments to these operators to the interested reader. Curiously, there appears to be some difficulty extending the arguments to the final variant \((3.6)\) if the $c_1, \ldots, c_k$ are not rationally commensurate, as one cannot easily discretise in this case to deploy additive combinatorics tools. A somewhat similar phenomenon appeared previously in [1]. It may be possible to get around this difficulty by developing a continuous version of the arithmetic regularity and counting lemmas. It is also possible, in principle at least, to obtain a corresponding result for the maximal multilinear Hilbert transform or by a polynomial Carleson type operator
\[
sup_{p} p.v. \int_{\mathbb{R}} f_1(x + t) \ldots f_k(x + kt) e^{2\pi i p(t)} \frac{dt}{t}
\]
where $P$ ranges over all polynomials $P : \mathbb{R} \to \mathbb{R}$ of degree bounded by some fixed $d$, although this may require a more sophisticated counting lemma than the one given in [4].

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