Title:

Complexes of $C$-projective modules

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COMPLEXES OF $C$-PROJECTIVE MODULES

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ABSTRACT. Inspired by a recent work of Buchweitz and Flenner, we show that, for a semidualizing bimodule $C$, $C$–perfect complexes have the ability to detect when a ring is strongly regular. It is shown that there exists a class of modules which admit minimal resolutions of $C$–projective modules.

Keywords: Semidualizing, $C$–projective, $P_C$–resolution, $C$–perfect complex, strongly regular.

MSC(2010): Primary: 13D05; Secondary: 16E05, 16E10.

1. Introduction

Let $R$ be a left and right noetherian ring (not necessarily commutative), all modules left $R$–modules and $C$ a semidualizing $(R, R)$–bimodule (Definition 2.1). A complex $X_\bullet$ of $R$–modules is said to be $C$–perfect if it is quasiisomorphic to a finite complex

$$T_\bullet = 0 \rightarrow C \otimes_R P_n \rightarrow C \otimes_R P_{n-1} \rightarrow \cdots \rightarrow C \otimes_R P_1 \rightarrow C \otimes_R P_0 \rightarrow 0,$$

where each $P_i$ is a finite (i.e. finitely generated) projective $R$–module. The width of such a $C$–perfect complex $X_\bullet$, denoted by $\text{wd}(X_\bullet)$, is defined to be the minimal length $n$ of a complex $T_\bullet$ satisfying the above conditions. Recall from [3], a ring $R$ is called strongly regular whenever there exists a non-negative integer $r$ such that every $R$–perfect complex is quasiisomorphic to a direct sum of $R$–perfect complexes of width $\leq r$. Buchweitz and Flenner, in [3], characterize the commutative noetherian rings which are strongly regular.

Our first objective is to detect when a ring is strongly regular by means of $C$–perfect complexes (Theorem 3.8). We also prove that $C$–projective modules (i.e., modules of the form $C \otimes_R P$ with $P$ projective) have the ability to detect when a ring is hereditary (Proposition 3.1).
Our second goal is to find a class of $R$–modules which admit minimal resolutions of $C$–projective modules (see Theorem 3.10).

2. Preliminaries

Throughout, $R$ is a left and right noetherian ring (not necessarily commutative) and let all $R$–modules be left $R$–modules. Right $R$–modules are identified with left modules over the opposite ring $R^{op}$. An $(R, R)$–bimodule $M$ is both left and right $R$–module with compatible structures.

**Definition 2.1.** [9, Definition 2.1] An $(R, R)$–bimodule $C$ is semidualizing if it is a finite $R$–module, finite $R^{op}$–module, and the following conditions hold.

1. The homothety map $R \rightarrow \text{Hom}_{R^{op}}(C, C)$ is an isomorphism.
2. The homothety map $R \rightarrow \text{Hom}_R(C, C)$ is an isomorphism.
3. $\text{Ext}^{\geq 1}_R(C, C) = 0$.
4. $\text{Ext}^{\geq 1}_{R^{op}}(C, C) = 0$.

Assume that $R$ is a commutative noetherian ring, then the above definition agrees with the definition of semidualizing $R$–module (see e.g. [9, 2.1]). Also, every finite projective $R$–module of rank 1 is semidualizing (see [11, Corollary 2.2.5]).

**Definition 2.2.** [9, Definition 3.1] A semidualizing $(R, R)$–bimodule $C$ is said to be faithfully semidualizing if it satisfies the following conditions

(a) If $\text{Hom}_R(C, M) = 0$, then $M = 0$ for any $R$–module $M$;
(b) If $\text{Hom}_{R^{op}}(C, N) = 0$, then $N = 0$ for any $R^{op}$–module $N$.

Note that over a commutative noetherian ring, all semidualizing modules are faithfully semidualizing, by [9, Proposition 3.1].

For the remainder of this section $C$ denotes a semidualizing $(R, R)$–bimodule. The following class of modules, is already appeared in, for example, [8], [9], and [13].

**Definition 2.3.** An $R$–module is called $C$–projective if it has the form $C \otimes_R P$ for some projective $R$–module $P$. The class of (resp. finite) $C$–projective modules is denoted by $\mathcal{P}_C$ (resp. $\mathcal{P}^f_C$).

A complex $A$ of $R$–modules is called $\text{Hom}_R(\mathcal{P}_C, -)$–exact if $\text{Hom}_R(C \otimes_R P, A)$ is exact for each projective $R$–module $P$. The term $\text{Hom}_R(\mathcal{P}_C, -)$–exact is defined dually.

For the notations in the next fact one may see [12, Definitions 1.4 and 1.5].

**Fact 2.1.** A $\mathcal{P}_C$–resolution of an $R$–module $M$ is a complex $X$ in $\mathcal{P}_C$ with $X_{-n} = 0 = H_n(X)$ for all $n > 0$ and $M \cong H_0(X)$. The following exact sequence is the augmented $\mathcal{P}_C$–resolution of $M$ associated to $X$:

$$X^+ = \cdots \rightarrow C \otimes_R P_1 \rightarrow C \otimes_R P_0 \rightarrow M \rightarrow 0.$$
A $\mathcal{P}_C$-resolution $X$ of $M$ is called proper if in addition $X^+ = \text{Hom}_R(\mathcal{P}_C, -)$-exact.

The $\mathcal{P}_C$-projective dimension of $M$ is the quantity

$$\mathcal{P}_C - \text{pd}(M) = \inf \{ \sup \{ n \geq 0 \mid X_n \neq 0 \} \mid X \text{ is an } \mathcal{P}_C - \text{resolution of } M \}.$$ 

The objects of $\mathcal{P}_C$-projective dimension 0 are exactly $\mathcal{P}_C$-projective $R$-modules.

The notion (proper) $\mathcal{P}_C$-coresolution is defined dually. The augmented $\mathcal{P}_C$-coresolution $Y$ associated to a $\mathcal{P}_C$-coresolution $X$ is denoted by $\mathcal{P}_C + Y$.

In [13], the authors proved the following proposition for a commutative ring $R$. However, by an easy inspection, one can see that it is true even if $R$ is non-commutative.

**Proposition 2.4.** Assume that $C$ is a faithfully semidualizing $(R, R)$-bimodule and that $M$ is an $R$-module. The following statements hold true.

(a) [13, Corollary 2.10(a)] The inequality $\mathcal{P}_C - \text{pd}(M) \leq n$ holds if and only if there is a complex

$$0 \rightarrow C \otimes_R P_n \rightarrow \cdots \rightarrow C \otimes_R P_0 \rightarrow M \rightarrow 0$$

which is $\text{Hom}_R(\mathcal{P}_C, -)$-exact.

(b) [13, Theorem 2.11(a)] $\text{pd}_R(M) = \mathcal{P}_C - \text{pd}_R(C \otimes_R M)$.

(c) [13, Theorem 2.11(c)] $\mathcal{P}_C - \text{pd}_R(M) = \text{pd}_R(\text{Hom}_R(C, M))$.

**Remark 2.5.** By [9, Proposition 5.3] the class $\mathcal{P}_C$ is precovering, that is, for an $R$-module $M$, there exists a projective $R$-module $P$ and a homomorphism $\phi : C \otimes_R P \rightarrow M$ such that, for every projective $Q$, the induced map

$$\text{Hom}_R(C \otimes_R Q, C \otimes_R P) \xrightarrow{\text{Hom}_R(C \otimes_R Q, \phi)} \text{Hom}_R(C \otimes_R Q, M)$$

is surjective. Then one can iteratively take precovers to construct a complex

$$W = \cdots \rightarrow C \otimes_R P_n \xrightarrow{\partial_2^X} C \otimes_R P_0 \rightarrow M \rightarrow 0 \quad (2.5.1)$$

such that $W^+$ is $\text{Hom}_R(\mathcal{P}_C, -)$-exact, where

$$W^+ = \cdots \rightarrow C \otimes_R P_1 \xrightarrow{\partial_2^X} C \otimes_R P_0 \xrightarrow{\phi} M \rightarrow 0.$$ 

For the notions precovering, covering, preenveloping and enveloping one can see [6].

Note that if $C$ is faithfully semidualizing $(R, R)$-bimodule and $M$ is an $R$-module, then, by Proposition 2.4(a), $\mathcal{P}_C - \text{pd}(M)$ is equal to the length of the shortest complex as (2.5.1). Thus for any $R$-module $M$, the quantity $\mathcal{P}_C$-projective dimension of $M$, defined in [9] and [13], is equal to $\mathcal{P}_C - \text{pd}(M)$ in Fact 2.1.
3. Results

A ring $R$ is (left) hereditary if every left ideal is projective. The Cartan-Eilenberg theorem [10, Theorem 4.19] shows that $R$ is hereditary if and only if every submodule of a projective module is projective. We show that the quality of being hereditary can be detected by $C$--projective modules, which is interesting on its own.

**Proposition 3.1.** Assume that $C$ runs through the class of faithfully semidualizing $(R, R)$--bimodules. The following statements are equivalent.

(i) $R$ is left hereditary.

(ii) For any $C$, every submodule of a $C$--projective $R$--module is also $C$--projective.

(iii) There exists a $C$ such that every submodule of a $C$--projective $R$--module is also $C$--projective.

**Proof.** (i)$\Rightarrow$(ii). Let $C$ be a faithfully semidualizing bimodule and $N$ a submodule of $C \otimes_R P$, where $P$ is a projective $R$--module. Then one gets the exact sequence $0 \to \text{Hom}_R(C, N) \to P$. As $R$ is left hereditary, $\text{Hom}_R(C, N)$ is a projective $R$--module. By Proposition 2.4(c), $\mathcal{P}_C$--pd$(N) = \text{pd}(\text{Hom}_R(C, N)) = 0$.

(ii)$\Rightarrow$(iii) is immediate.

(iii)$\Rightarrow$(i). As every submodule of a $C$--projective $R$--module is $C$--projective, for any $R$--module $M$ one has $\mathcal{P}_C$--pd$(M) \leq 1$. Then for any $R$--module $N$ one gets $\text{pd}(N) = \mathcal{P}_C$--pd$(C \otimes_R N) \leq 1$, by Proposition 2.4(b). It follows that every submodule of a projective is projective and so, by [10, Theorem 4.19], $R$ is left hereditary. \qed

**Definition 3.2.** A complex $X_\bullet$ of $R$--modules is called $C$--perfect if it is quasiisomorphic to a finite complex

$$T_\bullet = 0 \to C \otimes_R P_n \to C \otimes_R P_{n-1} \to \cdots \to C \otimes_R P_1 \to C \otimes_R P_0 \to 0,$$

where $P_i$ are finite projective $R$--modules. The width of such a $C$--perfect complex $X_\bullet$, denoted by $\text{wd}(X_\bullet)$, is defined to be the minimal length $n$ of a complex $T_\bullet$ satisfying the above conditions. A $C$--perfect complex $X_\bullet$ is called indecomposable if it is not quasiisomorphic to a direct sum of two non-trivial $C$--perfect complexes.

**Definition 3.3.** [3, Definition 1.1] A ring $R$ is called strongly $r$--regular if every perfect complex over $R$ is quasiisomorphic to a direct sum of perfect complexes of width $\leq r$. If $R$ is strongly $r$--regular for some $r$ then it will be called strongly regular.

**Remark 3.4.** As Professor Ragnar-Olaf Buchweitz kindly pointed out in his personal communication with the authors, in [3] it should be added the blanket statement that rings are noetherian and modules are finite. Thus Definition 3.3
agrees with [3, Definition 1.1]. Indeed, over a noetherian ring every perfect complex has bounded and finite homology.

Note that a hereditary ring \( R \) is strongly 1-regular, see [3, Remark 1.2].

In order to bring the results Theorem 3.8 and Proposition 3.9, we quote some preliminaries.

**Definition 3.5.** [7, III.3.2(b)] and [4, Definition 2.2.8] Let \( \alpha : A \to B \) be a morphism of \( R \)-complexes. The mapping cone \( \text{Cone}(\alpha) \), is a complex which is given by

\[
(\text{Cone}(\alpha))_n = B_n \oplus A_{n-1} \quad \text{and} \quad \partial_n^{\text{Cone}(\alpha)} = \begin{pmatrix} \partial_n^B & \alpha_{n-1} \\
0 & -\partial_n^A \end{pmatrix}.
\]

It easy to see that the following lemma is also true if \( R \) is non-commutative.

**Lemma 3.6.** Let \( \alpha : A \to B \) be a morphism of \( R \)-complexes and \( M \) be an \( R \)-module. The following statements hold true.

(a) [4, Lemma 2.2.10] The morphism \( \alpha \) is a quasiisomorphism if and only if \( \text{Cone}(\alpha) \) is acyclic.

(b) [4, Lemma 2.3.11] \( \text{Cone}(\text{Hom}_R(M, \alpha)) \cong \text{Hom}_R(M, \text{Cone}(\alpha)) \).

(c) [4, Lemma 2.4.11] \( \text{Cone}(M \otimes_R \alpha) \cong M \otimes_R \text{Cone}(\alpha) \).

**Remark 3.7.** Let \( C \) be a semidualizing \((R, R)\)-bimodule. Assume that 
\[
X = 0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to 0
\]
is an exact complex of \( R \)-modules.

(a) If each \( X_i \) is a projective \( R \)-module, then it is easy to see that the induced complex \( C \otimes_R X \) is exact.

(b) If each \( X_i \) is a \( C \)-projective \( R \)-module, then the induced complex \( \text{Hom}_R(C, X) \) is exact, since \( \text{Ext}_R^{\geq 1}(C, X_i) = 0 \).

**Theorem 3.8.** The following statements are equivalent.

(i) \( R \) is strongly \( r \)-regular.

(ii) For any faithfully semidualizing bimodule \( C \), every \( C \)-perfect complex is quasiisomorphic to a direct sum of \( C \)-perfect complexes of width \( \leq r \).

(iii) There exists a faithfully semidualizing bimodule \( C \) such that every \( C \)-perfect complex is quasiisomorphic to a direct sum of \( C \)-perfect complexes of width \( \leq r \).

**Proof.** (i)\(\Rightarrow\)(ii). Let \( R \) be strongly \( r \)-regular, \( C \) a faithfully semidualizing bimodule. Assume that \( X_\bullet \) is a \( C \)-perfect complex. Then, by Definition 3.2, there exists a finite complex 
\[
T_\bullet = 0 \to C \otimes_R P_n \to C \otimes_R P_{n-1} \to \cdots \to C \otimes_R P_0 \to 0,
\]
such that each \( P_i \) is a finite projective \( R \)-module and \( X_\bullet \) is quasiisomorphic to \( T_\bullet \). Therefore \( \text{Hom}_R(C, T_\bullet) \cong 0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to 0 \) is a perfect complex. By Definition 3.3, there is a quasiisomorphism \( \alpha :
\[
\text{Hom}_R(C, T_{\bullet}) \cong \bigoplus_{i=1}^{r} F^{(i)}_{\bullet},
\]
where each \( F^{(i)}_{\bullet} \) is a perfect complex of width \( \leq r \). We may assume that each \( F^{(i)}_{\bullet} \) is a perfect complex of finite projective \( R \)-modules. By Lemma 3.6(a), \( C_{\bullet} \) is a perfect complex of projective \( R \)-modules, Remark 3.7 implies that the complex \( C \otimes_R C_{\bullet} \) is acyclic. By Lemma 3.6, the complex \( \text{Cone}(C \otimes_R C_{\bullet}) \) is acyclic too and so \( C \otimes_R C_{\bullet} \) is quasi-isomorphic. Therefore \( T_{\bullet} \) is quasi-isomorphic to \( \bigoplus_{i=1}^{r} C \otimes_R F^{(i)}_{\bullet} \). Note that each \( C \otimes_R F^{(i)}_{\bullet} \) is a \( C \)-perfect complex of width \( \leq r \).

(ii)\Rightarrow(iii) is immediate.

(iii)\Rightarrow(i). Let \( Y_{\bullet} \) be a perfect complex. Then, by Definition 3.2, there is a finite complex \( F_{\bullet} = 0 \to P_m \to P_{m-1} \to \cdots \to P_0 \to 0 \) of finite projective modules which is quasi-isomorphic to \( Y_{\bullet} \). As \( C \otimes_R F_{\bullet} \) is a \( C \)-perfect complex, our assumption implies that there is a quasi-isomorphism \( \beta : C \otimes_R F_{\bullet} \to \bigoplus_{i=1}^{r} T^{(i)}_{\bullet} \), where each \( T^{(i)}_{\bullet} \) is a \( C \)-perfect complex of width \( \leq r \). We may assume that, for each \( i \),

\[
T^{(i)}_{\bullet} = 0 \to C \otimes_R P^{(i)}_{n_i} \to \cdots \to C \otimes_R P^{(i)}_0 \to 0
\]

where each \( P^{(i)}_j \) is a finite projective \( R \)-module. Similar to the proof of (i)\Rightarrow(ii), one observes that \( \text{Hom}_R(C, \beta) \) is a quasi-isomorphism. Therefore \( F_{\bullet} \) is quasi-isomorphic to \( \bigoplus_{i=1}^{r} \text{Hom}_R(C, T^{(i)}_{\bullet}) \). Note that each \( \text{Hom}_R(C, T^{(i)}_{\bullet}) \) is a perfect complex of width \( \leq r \). Thus \( R \) is strongly \( r \)-regular.

In [2, Section 1], Avramov and Martsinkovsky define a general notion of minimality for complexes: A complex \( X \) is \textit{minimal} if every homotopy equivalence \( \sigma : X \to X \) is an isomorphism. In [14, Lemma 4.8], it is proved that, over a commutative local ring \( R \) with maximal ideal \( m \), a complex \( X \) consisting of modules in \( P^d_C \) is minimal if and only if \( \partial^X(X) \subseteq mX \).

In consistent to [3, Lemma 1.6] we prove the following proposition.

**Proposition 3.9.** Let \( R \) be a commutative noetherian local ring and \( C \) a semidualizing \( R \)-module. The following statements hold true.

(a) Every \( C \)-perfect complex \( X_{\bullet} \) is quasi-isomorphic to a minimal finite complex \( T_{\bullet} = 0 \to C \otimes_R F_n \to C \otimes_R F_{n-1} \to \cdots \to C \otimes_R F_1 \to C \otimes_R F_0 \to 0 \), where each \( F_i \) is finite free \( R \)-module.

(b) If two minimal finite complexes of modules of the form \( C^m = \bigoplus^m C \) are quasi-isomorphic, then they are isomorphic.

**Proof.** (a). By Definition 3.2, a \( C \)-perfect complex \( X_{\bullet} \) is quasi-isomorphic to a finite complex \( T_{\bullet} = 0 \to C \otimes_R P_n \to C \otimes_R P_{n-1} \to \cdots \to C \otimes_R P_1 \to C \otimes_R P_0 \to 0 \), where each \( P_i \) is a finite free \( R \)-module. The complex \( \text{Hom}_R(C, T_{\bullet}) \) is a perfect complex and so, by [3, Lemma 1.6(1)], there exists a minimal finite complex.
of finite free $R$–modules and a quasiisomorphism $\alpha : \text{Hom}_R(C, T_\bullet) \xrightarrow{\sim} F_\bullet$. As in the proof of Theorem 3.8, it follows that $C \otimes_R \alpha : C \otimes_R \text{Hom}_R(C, T_\bullet) \rightarrow C \otimes_R F_\bullet$ is a quasiisomorphism. As $C \otimes_R F_\bullet$ is a minimal finite complex, we are done.

(b). Let $T_\bullet$ and $L_\bullet$ be two minimal finite complexes of modules of the form $C^m$. Assume that $\alpha : T_\bullet \rightarrow L_\bullet$ is a quasiisomorphism. Then, by Remark 3.7 and Lemma 3.6, $\text{Hom}_R(C, \alpha) : \text{Hom}_R(C, T_\bullet) \rightarrow \text{Hom}_R(C, L_\bullet)$ is a quasiisomorphism of minimal finite complexes of finite free $R$–modules. Thus, by the proof of [3, Lemma 1.6(2)], $\text{Hom}_R(C, \alpha)$ is an isomorphism. Now, there is a commutative diagram of complexes and morphisms

\[
\begin{array}{ccc}
T_\bullet & \xrightarrow{\sim} & L_\bullet \\
\uparrow{\cong} & & \uparrow{\cong} \\
C \otimes_R \text{Hom}_R(C, T_\bullet) & \xrightarrow{\cong} & C \otimes_R \text{Hom}_R(C, L_\bullet),
\end{array}
\]

where the vertical morphisms are natural isomorphisms. This implies that $\alpha$ itself must be an isomorphism. \qed

It is proved in [14, Lemma 4.9] that every finite module $M$ over a commutative noetherian local ring $R$ with $P_C$–pd$(M) < \infty$ admits a minimal $P_C$–resolution. Now we show that every finite $R$–module which has a proper $P_C$–resolution, admits a minimal proper one. Note that if $P_C$–pd$(M) < \infty$ then $M$ admits a proper $P_C$–resolution (see proof of [13, Corollary 2.10]).

**Theorem 3.10.** Assume that $R$ is a commutative noetherian local ring and that $C$ is a semidualizing $R$–module. Then $P_C$ is covering in the category of finite $R$–modules. For any finite $R$–module $M$, there is a complex $X = \cdots \rightarrow C^{n_1} \rightarrow C^{n_0} \rightarrow 0$ with the following properties.

1. $X^+ = \cdots \rightarrow C^{n_1} \rightarrow C^{n_0} \rightarrow M \rightarrow 0$ is $\text{Hom}_R(P_C, -)$–exact.
2. $X$ is a minimal complex.

If $M$ admits a proper $P_C$–resolution, then $X^+$ is exact and so $X$ is a minimal proper $P_C$–resolution of $M$.

**Proof.** Let $M$ be a finite $R$–module. Assume that $n_0 = \nu(\text{Hom}_R(C, M))$ denotes the number of a minimal set of generators of $\text{Hom}_R(C, M)$ and that $\alpha : R^{n_0} \rightarrow \text{Hom}_R(C, M)$ is the natural epimorphism. As $\alpha$ is a $P_C$–cover of $\text{Hom}_R(C, M)$, the natural map $\beta = C \otimes_R R^{n_0} \xrightarrow{C \otimes_R \alpha} C \otimes_R \text{Hom}_R(C, M) \xrightarrow{\nu} M$ is a $P_C$–cover of $M$. Set $M_1 = \text{Ker} \beta$ and $n_1 = \nu(\text{Hom}_R(C, M_1))$. Thus there is a $P_C$–cover $\beta_1 : C \otimes_R R^{n_1} \rightarrow M_1$. Proceeding in this way one obtains a complex

\[
X = \cdots \xrightarrow{\partial_2 = \epsilon_2 \beta_2} C \otimes_R R^{n_1} \xrightarrow{\partial_1 = \epsilon_1 \beta_1} C \otimes_R R^{n_0} \rightarrow 0,
\]
where $\epsilon_i : M_i \to C \otimes_R R^{n_i-1}$ is the inclusion map for all $i \geq 1$. As the maps in $X$ are obtained by $P_C^1$-covers, the complex $X^+$ is $\text{Hom}_R(P_C, -)$-exact. It is easy to see that $\text{Hom}_R(C, X)$ is minimal free resolution of $\text{Hom}_R(C, M)$. Now we show that $X$ is a minimal complex. Let $f : X \to X$ be a morphism which is homotopic to $\text{id}_X$. It is easy to see that the morphism $\text{Hom}_R(C, f)$ is homotopic to $\text{id}_{\text{Hom}_R(C, X)}$. As the complex $\text{Hom}_R(C, X)$ is minimal, by [2, Proposition 1.7], the morphism $\text{Hom}_R(C, f)$ is an isomorphism. The commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X \\
\downarrow \cong & & \downarrow \cong \\
C \otimes_R \text{Hom}_R(C, X) & \xrightarrow{\cong} & C \otimes_R \text{Hom}_R(C, X),
\end{array}
$$

with vertical natural isomorphisms, implies that $f$ is an isomorphism. Therefore, by [2, Proposition 1.7], $X$ is minimal. If $M$ admits a proper $P_C$-resolution, then by [13, Corollary 2.3], $X^+$ is exact. □

The proof of the next lemma is similar to [13, Corollary 2.3].

**Lemma 3.11.** Let $R$ be a commutative noetherian ring and let $M$ be a finite $R$-module. Assume that $C$ is a semidualizing $R$-module. The following are equivalent.

(i) $M$ admits a proper $P_C^1$-coresolution.

(ii) Every $\text{Hom}_R(-, P_C^1)$-exact complex of the form

$$
0 \to M \to C \otimes_R Q_0 \to C \otimes_R Q_{-1} \to \cdots
$$

is exact, where $Q_i$ is an object of $P_C^1$ for all $i \leq 0$.

(iii) The natural homomorphism $M \to \text{Hom}_R(\text{Hom}_R(M, C), C)$ is an isomorphism and $\text{Ext}_R^{\geq 1}(\text{Hom}_R(M, C), C) = 0$.

**Proposition 3.12.** Assume that $R$ is a commutative noetherian local ring and that $C$ is a semidualizing $R$-module. Then $P_C^1$ is enveloping in the category of finite $R$-modules. For any finite $R$-module $M$, there is a complex $Y = 0 \to C^{m_0} \to \cdots$ with the following properties.

(1) $Y^+ = 0 \to M \to C^{m_0} \to C^{m_1} \to \cdots$ is $\text{Hom}_R(-, P_C)$-exact.

(2) $Y$ is a minimal complex.

If $M$ admits a proper $P_C^1$-coresolution, then $Y^+$ is exact and so $Y$ is a minimal proper $P_C^1$-coresolution of $M$.

**Proof.** Let $M$ be a finite $R$-module. Assume that $m_0 = \nu(\text{Hom}_R(M, C))$ denotes the number of a minimal set of generators of $\text{Hom}_R(M, C)$ and that $\alpha : R^{m_0} \to \text{Hom}_R(M, C)$ is the natural $P_C^1$-cover of $\text{Hom}_R(M, C)$. It follows that $\gamma = M \xrightarrow{\delta m} \text{Hom}_R(\text{Hom}_R(M, C), C) \xrightarrow{\text{Hom}_R(\alpha, C)} \text{Hom}_R(R^{m_0}, C)$ is a $P_C^1$-envelope of $M$. Set $M_{-1} = \text{Coker}\gamma$ and $m_1 = \nu(\text{Hom}_R(M_{-1}, C))$. As
mentioned, there is a $\mathcal{P}_C^I$-envelope $\gamma_1 : M_{-1} \longrightarrow \text{Hom}_R(R^{m_1}, C)$. Proceed-
ing in this way one obtains a complex $Y = 0 \longrightarrow \text{Hom}_R(R^{m_0}, C) \xrightarrow{\partial_0 = \gamma_1 \pi_1} \text{Hom}_R(R^{m_1}, C) \xrightarrow{\partial_1 = \gamma_2 \pi_2} \cdots$, where $\pi_i$ is the natural epimorphism for all $i \geq 1$. Since the maps in $Y$ are obtained by $\mathcal{P}_C^I$-envelopes, the complex $^+Y$ is $\text{Hom}_R(\mathcal{P}_C^I, \bullet)$-exact. It is easy to see that $\text{Hom}_R(Y, C)$ is minimal free resolution of $\text{Hom}_R(M, C)$. Similar to the proof of Theorem 3.10, we find that $Y$ is a minimal complex. If $M$ admits a proper $\mathcal{P}_C^I$-coresolution, then, by Lemma 3.11, $^+Y$ is exact. 

In the following example we find an $R$–module $M$ with $\mathcal{P}_C^I$–pd($M$) = $\infty$ which admits a minimal proper $\mathcal{P}_C^I$–resolution. This example shows that a commutative noetherian local ring which admits an exact zero-divisor is not a strongly regular ring.

Example 3.13. Let $R$ be a commutative noetherian local ring and $C$ a semidualizing $R$–module. Assume that $x, y$ form a pair of exact zero-divisors on both $R$ and $C$ (e.g. see [1, Example 3.2]). Then $\mathcal{P}_C^I$–pd($C/xC$) = $\text{pd}(R/xR) = \infty$. The complex

$$T_\bullet = \cdots \xrightarrow{x} C \xrightarrow{y} C \xrightarrow{x} C \longrightarrow 0 \ (\text{resp. } L_\bullet = 0 \longrightarrow C \xrightarrow{x} C \xrightarrow{y} C \xrightarrow{x} \cdots)$$

is a minimal $\mathcal{P}_C^I$–resolution (resp. $\mathcal{P}_C^I$–coresolution) of $C/xC$. By [1, Proposition 3.4], $C/xC$ is a semidualizing $R/xR$–module. By [5, Proposition 2.13], there are isomorphisms

$$\text{Hom}_R(C, C/xC) \cong \text{Hom}_{R/xR}(C/xC, C/xC) \cong R/xR,$$

$$\text{Hom}_R(C/xC, C) \cong \text{Hom}_{R/xR}(C/xC, C/xC) \cong R/xR.$$ Applying $\text{Hom}_R(C, -)$ and $\text{Hom}_R(-, C)$ on the above complexes, respectively, would result the isomorphisms $\text{Hom}_R(C, T^*_\bullet) \cong F^*_\bullet$ and $\text{Hom}_R(^+L_\bullet, C) \cong F^*_\bullet$, where $F^*_\bullet$ is the exact complex $\cdots \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \longrightarrow R/xR \longrightarrow 0$. Therefore $T_\bullet$ (resp. $L_\bullet$) is a minimal proper $\mathcal{P}_C^I$–resolution (resp. $\mathcal{P}_C^I$–coresolution) of $C/xC$.

For each $n$, one obtains a $C$–perfect complex of length $n$ as

$$T_\bullet^{(n)} = 0 \longrightarrow C \longrightarrow C \longrightarrow \cdots \xrightarrow{x} C \xrightarrow{y} C \xrightarrow{x} C \longrightarrow 0,$$

where $T_\bullet^{(n)} = T_\bullet$ for all $0 \leq i \leq n$ and $T_i^{(n)} = 0$ otherwise. Note that the induced map $d_i : T_i^{(n)}/\text{Ker} d_i \rightarrow T_{i-1}^{(n)}$ is injective, where $\text{Ker} d_i$ is equal to $yC$ or $xC$. As $C$ is indecomposable $R$–module, $T_\bullet^{(n)}$ is indecomposable which has a similar proof to [3, Proposition 1.5].
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