Hierarchical of Information Scrambling,
Thermalization, and Hydrodynamic Flow in Graphene

Markus J. Klug,1 Mathias S. Scheurer,1,2 and Jörg Schmalian1,3
1Institut für Theorie der Kondensierten Materie, Karlsruher Institut für Technologie, 76131 Karlsruhe, Germany
2Department of Physics, Harvard University, Cambridge MA 02138, USA
3Institut für Festkörperphysik, Karlsruher Institut für Technologie, 76344 Karlsruhe, Germany
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We determine the information scrambling rate $\lambda_L$ due to electron-electron Coulomb interaction in graphene. $\lambda_L$ characterizes the growth of chaos and has been argued to give information about the thermalization and hydrodynamic transport coefficients of a many-body system. We demonstrate that $\lambda_L$ behaves for strong coupling similar to transport and energy relaxation rates. A weak coupling analysis, however, reveals that scrambling is related to dephasing or single particle relaxation. Furthermore, $\lambda_L$ is found to be parametrically larger than the collision rate relevant for hydrodynamic processes, such as electrical conduction or viscous flow, and the rate of energy relaxation, relevant for thermalization. Thus, while scrambling is obviously necessary for thermalization and quantum transport, it does generically not set the time scale for these processes. In addition we derive a quantum kinetic theory for information scrambling that resembles the celebrated Boltzmann equation and offers a physically transparent insight into quantum chaos in many-body systems.

The emergence of chaos is the most plausible explanation for the thermalization of closed quantum many-body systems. An efficient concept to quantify quantum chaos is through the scrambling rate $\lambda_L$ [1, 2]. After the time $\lambda_L^{-1}$ of the evolution of an out-of-equilibrium initial state, quantum entanglement has spread across the system. Then, the initial state cannot be recovered, i.e. unscrambled, via local measurements. The system has lost its memory, a key prerequisite for thermalization. Still, it is unclear whether the scrambling rate is indeed the characteristic scale that determines the return to thermal equilibrium. Similar to the scrambling rate with respect to temporal evolution, the butterfly velocity $v_B$ characterizes the corresponding spread of entanglement in space after a local perturbation. Formally, these quantities can be determined from the growth in time or space of commutators or anticommutators of local operators $A$ and $B$,

$$\langle \|A(x, t), B(0, 0)\|^2 \rangle \sim e^{2\lambda_L \left( t - \frac{\langle x \rangle}{v_B} \right)}.$$  \hspace{1cm} (1)

A transparent interpretation of this squared commutator exists in the quasi-classical limit for the motion of a particle with initial coordinate $q(0)$ and conjugate momentum $p(t)$ for which $\langle p(t), p(0) \rangle^2 = \langle \frac{\partial^2 p(t)}{\partial t^2} \rangle^2$.[1] Exponential growth behavior of this correlator is e.g. realized by electrons in a weakly disordered metal [3]. Thus, the expectation value measures the dependence of the momentum at time $t$ as one changes the initial coordinates, a key measure for chaotic motion. The square in the commutator ensures that positive and negative momenta at time $t$ do not average to zero. The corresponding spread of an initial state of two nearby electrons in graphene is sketched in Fig. 1.

While the formal interpretation of scrambling is established in the information theoretic sense, with key applications in quantum circuits [4], it is not obvious how to measure $\lambda_L$ or $v_B$ for a generic solid-state system. In fact it is unclear which specific physical observable might essentially probe these quantities and how this relates to thermalization. For example a close link to transport quantities was proposed and $v_B^2/\lambda_L$ related to the charge [5, 6] or heat [7] diffusivities. The relation between scrambling and transport seems consistent with the bound on chaos $\lambda_L \leq 2\pi k_B T/\hbar$, derived under rather general conditions [8], a bound that is saturated by the Sachdev-Ye-Kitaev model [9–11] and more generally in models with holographic duals. Together with the Planckian transport rate $\tau_T^{-1} \approx k_B T/\hbar$ that emerges in some strongly correlated systems [12, 13], this suggests a connection between scrambling and transport processes. On the other hand, the analysis of weakly interacting diffusive electrons revealed that $\lambda_L$ is rather determined by the single particle scattering rate [14]. In addition, the scrambling rate of a bad metal, coupled to long-lived phonons was shown to be determined by phonons and not by the short transport time [15].

Figure 1. Exponentially diverging electron trajectories (orange) in graphene resulting from Coulomb interaction (red).
Graphene is a unique condensed matter system owed to its Dirac spectrum. On the one hand, recent experiments demonstrated hydrodynamic flow of the electron fluid with giant magneto-drag [16], a breakdown of the Wiedemann-Franz law [17], super-ballistic transport [18, 19], and negative local resistance [20, 21]. Even nonlinear phenomena, like hydrodynamic hot spot relaxation from out-of-equilibrium configurations have been proposed [22, 23]. Rapid local thermalization is crucial for the applicability of hydrodynamic descriptions. On the other hand, graphene, like weakly interacting diffusive electrons discussed in [14], displays a number of distinct scattering rates – for single particle excitations $\tau_q^{-1}(\epsilon)$, for energy relaxations $\tau_{E}^{-1}(\epsilon)$, or for transport processes $\tau_{tr}^{-1}(\epsilon)$; see Ref. [24]. In Fig. 2 we plot these characteristic time scales as function of the particle energy $\epsilon$. While the single-particle rate is always the largest, it depends on the characteristic energy whether the rate of energy relaxation or the transport rate is larger. For a summary of these time scales see also Appendix A. The origin for these distinct scales is the infrared-singular collision kernel. It allows to identify to what extend scrambling and hydrodynamic collisions are related. In addition, it allows to distinguish thermalization, that should be governed by the energy relaxation rate at $\epsilon \sim k_B T$, from scrambling.

In the first part of this work, we determine the scrambling rate $\lambda_L$ for electrons in graphene with electron-electron Coulomb interaction using a diagrammatic approach, presented e.g. in Ref. [7, 14, 15]. Details of the considered microscopic model are presented in Sec. I, whereas the computation of $\lambda_L$ is found in Sec. II. The analysis is done for large $N$, where $N$ is the number of fermion flavors ($N = 4$ in graphene). This allows the determination of $\lambda_L$ for arbitrary values of the effective fine structure constant $\alpha = Ne^2/\hbar v$ of graphene, which is formally considered to be $N$-independent [25]. There $v$ and $e$ denote the bare Fermi velocity and the electron charge, respectively. At strong coupling we find

$$\lambda_L (\alpha \gg 1) \approx 0.8 \frac{4 \pi k_B T}{N},$$

(2)

a behavior that is consistent with other large-$N$ calculations [26]. Due to the large-$N$ expansion this result is parametrically far from the above bound. However, it does behave like the transport relaxation rate $\tau_{tr}^{-1}(k_B T) \sim k_B T/\hbar N$ that occurs in the same limit [27]. For large $\alpha$ the only characteristic scale is $k_B T$ making a clear association of $\lambda_L$ with a specific time scale difficult. Therefore, our analysis is more revealing in the weak coupling limit, where we find

$$\lambda_L (\alpha \ll 1) \approx 0.37 \frac{4 \pi k_B T}{N} \frac{1}{\hbar}.$$  

(3)

This result is parametrically larger than the transport rate $\tau_{tr}^{-1}(k_B T) \sim 2k_B T/\hbar N$ that occurs in the d.c. conductivity [28, 29] or the electron viscosity [30]. Scrambling processes at weak coupling are therefore a lot faster than the collisions that give rise to the hydrodynamic behavior of graphene, and also faster than the energy relaxation rate that one would expect to govern thermalization. Instead, the scrambling time in graphene is closely related to the dephasing scattering rate.

In order to obtain a clear physical understanding of information scrambling in many-body systems, we also present an alternative approach to quantum chaos using non-equilibrium techniques in Sec. III, similar in spirit to the methods presented in Ref. [31, 32]. We derive a kinetic equation similar to the Boltzmann equation in form of an integro-differential equation describing the growth and spread of a small, localized perturbation. It is shown explicitly that this approach reproduces the results obtained within the diagrammatic formalism.

### I. THE MODEL

We consider the following Hamiltonian for electrons in graphene near the Dirac point with electron-electron...
Coulomb interaction (setting $\hbar = 1$)

$$H = v \sum_{i=1}^{N} \int d^2x \psi^\dagger_i(x) \left( \frac{1}{i} \nabla \cdot \sigma \psi_i(x) \right) \tag{4}$$

$$+ \frac{e^2}{2} \sum_{\alpha=1}^{N} \int d^2x d^2x' \frac{\psi^\dagger_i(x) \psi^\dagger_j(x') \psi_j(x') \psi_i(x)}{|x - x'|}.$$ 

Here, $\psi_i = (\psi_{i1}, \psi_{i2})^T$ is a two component spinor and $\sigma = (\sigma_x, \sigma_y)^T$ are Pauli matrices acting in pseudo-spin space. $i = 1 \ldots N$ is an additional flavor index that includes spin and valley degrees of freedom. While $N = 4$ for graphene, we keep $N$ arbitrary to be able to perform a controlled expansion in $1/N$ [25, 33]. A key justification to use this approach for the description of graphene comes from experiment. Several measurements clearly reveal interaction effects [34–36], but of a kind that is fully consistent with renormalization group assisted perturbation theory [37].

The retarded fermionic propagator, on the bare level given by

$$(G^R)^{-1} = (\omega + i0^+) \delta^0 - \mathbf{v} \cdot \mathbf{\sigma},$$

is dressed in leading order in $1/N$ by the usual rainbow diagram for the retarded self energy

$$\Sigma^R_{\alpha\beta}(k, \omega) = \frac{i}{2N} (D^R \circ G^K_{\alpha\beta} + D^K \circ G^K_{\alpha\beta})(k, \omega),$$

where the superscripts $R$, $K$, and later $A$ stand for retarded, Keldysh, and advanced components of the Green’s functions, and $\circ$ stands for a convolution with regards to frequencies and momenta. $D^{-1} = D_0^{-1} + \Pi$ is the collective plasmon propagator with bare Coulomb interaction $D_0^R = \frac{2\pi e^2}{q_0}$ and the bosonic self energy

$$\Pi^R(k, \omega) = \frac{i}{2} (G^R_{\alpha\beta} \circ G^K_{\beta\alpha} + G^K_{\alpha\beta} \circ G^K_{\beta\alpha})(k, \omega),$$

which is of order $N^0$.

II. DIAGRAMMATIC APPROACH

We start from the ‘regularized’ version of the squared anticommutator

$$J_{\gamma\delta}^{\alpha\beta}(x_1, x_2, x_3, x_4) = \frac{1}{N^2} \sum_{i,j=1}^{N} \text{tr} \left( \left\{ \psi_{i\alpha}(x_1), \psi^\dagger_{j\beta}(x_2) \right\} \right)$$

$$\times \sqrt{\rho} \left\{ \psi_{j\delta}(x_3), \psi^\dagger_{i\gamma}(x_4) \right\} \sqrt{\rho}$$

where the regularization amounts to splitting the density matrix $\rho = Z^{-1} e^{-H/\hbar k_BT}$ between the two anticommutators [8, 38]. Otherwise the exponent would depend explicitly on the UV cut-off of our effective field theory and would therefore be ill defined. $x = (x, t)$ stands for space and time coordinates. In order to determine the scrambling time, we analyze

$$C(t) = \theta(t) \int d^2x \sum_{\alpha\beta} f_{\alpha\beta}(x, 0, x),$$

which contains a correlator with non-trivial time order denoted out-of-time-order correlator (OTOC), i.e. sequences of operators which cannot be represented on a conventional Keldysh contour evolving back and forth in time.

A. Out-of-time-order formalism

To determine the function $f$, we use the out-of-time-order formalism and the corresponding four-branch (two-loop) Keldysh contour [31, 39]. In this case, the thermal expectation value is expressed as expectation value of four-component Grassmann fields $\psi_{i\alpha}(t) = (\psi^{u+}, \psi^{u-}, \psi^{d+}, \psi^{d-})_{i\alpha}(t)$, and analogously $\bar{\psi}_{i\alpha}$, placed on the Keldysh contour according to their relative causal position. The position of the fields is specified, besides the time parameter $t$, by the loop index $\sigma = \{u, l\}$ for the upper and lower loop, and the branch index $\{+, -, -\}$ denoting the branches propagating forward and backward in time, respectively. The contour including the position of the fractions of density matrices placed between the anticommutators is depicted in Fig. 3.

We perform the standard Keldysh rotation [40] for each of the two loops $\sigma = \{u, l\}$ separately,

$$\psi^{\sigma u} = \frac{1}{\sqrt{2}} \left( \psi^{\sigma +} + \psi^{\sigma -} \right), \psi^{\sigma q} = \frac{1}{\sqrt{2}} (\psi^{\sigma +} - \psi^{\sigma -}),$$

$$\bar{\psi}^{\sigma u} = \frac{1}{\sqrt{2}} \left( \psi^{\sigma +} + \psi^{\sigma -} \right), \bar{\psi}^{\sigma q} = \frac{1}{\sqrt{2}} (\psi^{\sigma +} + \psi^{\sigma -}),$$

where Keldysh indices $s = \{u, q\}$ denote ‘classical’- (cl) and ‘quantum’- (q) field components, respectively.

An effective Keldysh action is obtained by introducing real plasmon fields $\phi$ which couple linearly to a pair of fermion fields. For this, we carry out the standard Hubbard-Stratonovich transformation to decouple the interaction term of the Hamilton operator in Eq. 4 in the

$$\sqrt{\rho} \rightarrow \text{upper loop} \quad +$$

$$\sqrt{\rho} \rightarrow \text{lower loop} \quad +$$

Figure 3. Two-loop Keldysh contour. Index $u$ and $l$ refer to the fields residing on the upper and lower Keldysh loop, index + and − to fields residing on the forward and backward propagating branch, respectively. The square roots of density matrices are placed at the left turning points of the contour denoted by solid black dots.
charge channel. Consequently, the quadratic part of the Keldysh action is given by (pseudo-spin and flavor indices are dropped):

\[
S_0 [\psi, \bar{\psi}, \phi] = \sum_{\sigma_i, \sigma'_i} \left( \int_k \bar{\psi}_k^{\sigma_i} G_{\sigma'_i, \sigma_i, ss'}^{-1}(k) \psi_k^{\sigma'_i} + \int_q \bar{\phi}_q^{-\sigma_i} D_{\sigma_i, ss'}^{-1}(q) \phi_q^{\sigma_i} \right),
\]

where \( k = (\epsilon, \mathbf{k}) \) and \( \int_k = \int \frac{d\epsilon}{2\pi} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \). The intra-loop components of the fermionic and bosonic plasmon propagators (\( \sigma = \sigma' \)) have the usual causal structure

\[
G_{\sigma, ss'} = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix}_{ss'}, \hspace{1cm} D_{\sigma, ss'} = \begin{pmatrix} D^K & D^R \\ D^A & 0 \end{pmatrix}_{ss'},
\]

where superscripts \( R/A \) denote retarded and advanced components, and the fermionic and bosonic Keldysh components are given by \( G^R(k) = 2i \tanh(\frac{\epsilon}{2k_B T}) \text{Im} G^R(k) \) and \( D^R(k) = 2i \coth(\frac{\epsilon}{2k_B T}) \text{Im} D^R(k) \). In the case of inter-loop correlations (\( \sigma' = \bar{\sigma} \) where \( \bar{u} = l \) and vice versa),

\[
G_{\sigma, ss'} = \begin{pmatrix} 0 & G^K_{\sigma, ss'} \\ G^R_{\sigma, ss'} & 0 \end{pmatrix}, \hspace{1cm} D_{\sigma, ss'} = \begin{pmatrix} D^K & 0 \\ 0 & D^R \end{pmatrix}_{ss'},
\]

there is only a Keldysh components which relates to the retarded components as

\[
D^R_{ul}(\omega, q) = \frac{2 \text{Im} D^R(\omega, q)}{\sinh \left( \frac{\omega}{2k_B T} \right)}, \hspace{1cm} G^K_{ul, a}(\epsilon, k) = -G^K_{ul, a}(\epsilon, k).
\]

Here, we use the band basis where the first term in Eq. 4 is diagonal and \( G^R = (\omega + i0^+ - \epsilon \sigma \hat{v} | k) \) \(^{-1} \) with \( a = \pm 1 \).

The fermionic and bosonic fields couple via the term

\[
S_{\text{int}} [\psi, \bar{\psi}, \phi] = \frac{1}{\sqrt{N}} \sum \int q \bar{\psi}_q^{\sigma_i} \psi_k^{\sigma'_i} \phi_q^{\sigma_i} \gamma_{ss'}^{\sigma_i \sigma_i} \gamma_{ss'}^{\sigma'_i \sigma_i},
\]

which is weighted by a factor of \( 1/\sqrt{N} \). The coupling is diagonal in Keldysh-loops as well as all dropped indices.

The coupling vertices acting in Keldysh component space are \( \gamma_{ss'}^{cl} = \delta_{ss'} \) and \( \gamma_{ss'}^{q} = (\sigma_1)_{ss'} \) with \( \sigma_1 \) being the first Pauli matrix.

Within this framework, the squared anticommutator Eq. 8 is recast as the expectation value of ‘classical’- and ‘quantum’-Keldysh field components

\[
f_{\gamma\beta}^{\alpha\alpha}(x_1, x_2, x_3, x_4) = -\frac{1}{N^2} \sum_{n,j=1}^N \langle \psi_{\alpha a}^{cl}(x_1) \bar{\psi}_{\beta b}^{cl}(x_2) \psi_{\gamma c}^{unq}(x_3) \bar{\psi}_{\delta d}^{unq}(x_4) \rangle_{\gamma \beta}
\]

which are evaluated with respect to the two-loop Keldysh action \( \langle \ldots \rangle_{K} = \int \mathcal{D} (\bar{\psi}, \psi, \phi) \ldots e^{iS_K} \) with \( S_K = S_0 + S_{\text{int}} \). Eventually, contributions of \( S_{\text{int}} \) to \( f \) are incorporated perturbatively in orders of \( 1/N \).

\[\begin{align*}
\int \mathcal{D} (\bar{\psi}, \psi, \phi) \ldots e^{iS_K} & = f_{\gamma\beta}^{\alpha\alpha}(x_1, x_2, x_3, x_4) \\
& = \int \mathcal{D} (\bar{\psi}, \psi, \phi) \ldots e^{iS_K} \beta(\epsilon, k) \end{align*}\]

with \( k = (\epsilon, k) \), \( \int_k = \int \frac{d\epsilon}{2\pi} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \) and \( \int_x = \int dt \int d^2 \mathbf{x} \).

The resulting Bethe-Salpeter equation, which determines

\[\begin{align*}
\int \mathcal{D} (\bar{\psi}, \psi, \phi) \ldots e^{iS_K} & = f_{\gamma\beta}^{\alpha\alpha}(x_1, x_2, x_3, x_4) \\
& = \int \mathcal{D} (\bar{\psi}, \psi, \phi) \ldots e^{iS_K} \beta(\epsilon, k) \end{align*}\]
\[ f^\alpha_\beta (\omega, k) = \frac{1}{N} \sum_{\gamma} G^R_{\alpha \gamma}(\omega + \epsilon, k) G^A_{\gamma \beta}(k) + \]
\[ \frac{1}{N} \int_{k^\prime} \sum_{\gamma^\prime, \delta} f^\gamma_\delta (\omega, k^\prime) \Gamma^{\gamma^\prime \delta}_{\beta \gamma}(\omega, k, k^\prime) G^R_{\beta \gamma}(\omega + \epsilon, k) G^A_{\gamma \beta}(k) \]

and depicted in Fig. 4d as diagrammatic representation. The inter-loop scattering vertex \( \Gamma \) contains one-rung (first term) and two-rung (second term) contributions,

\[ \Gamma^{\alpha \beta}_{\gamma \delta}(\omega, k, k^\prime) = i \delta_{\alpha \delta} \delta_{\gamma \beta} D^K_{ul}(k - k^\prime) \]
\[ + \int_k G^a_{\alpha \gamma}(k - \tilde{k}) G^b_{\beta \gamma}(k^\prime - \tilde{k}) D_R(\omega + \epsilon, \tilde{k}) D_A(\tilde{k}), \]

shown in Fig. 4(a) and (b), respectively.

Focusing on the leading contribution to \( \lambda_\varepsilon \) in our large-\( N \) expansion allows to perform a series of simplifications of Eq. 18. First, we set the fermionic propagator on mass-shell which requires a representation in diagonal basis and which implicitly assumes that a quasi-particle description is applicable. We therefore introduce the projection operator \( \mathcal{P}^a_{\alpha \beta}(k) \) for the two bands \( a = \pm 1 \) which connects the pseudo-spin and band-basis by \( G^R_{\alpha \beta}(k) = \sum_a \mathcal{P}^a_{\alpha \beta}(k) G^R_{\alpha \beta}(k). \) The projection operator has the properties \( \sum_a \mathcal{P}^a_{\alpha \beta}(k) = 1 \) and \( \sum_b \mathcal{P}_{\alpha \beta}^a(k) \mathcal{P}_{\beta \gamma}^b(k^\prime) = \frac{1}{2} \mathcal{P}_{\alpha \gamma}(k) (1 + ab |k^\prime| |k|), \) for \( a, b = \pm 1. \) This allows us to replace the product of Green’s functions by

\[ G^R_{\alpha \gamma}(\omega + \epsilon, k) G^A_{\beta \gamma}(\epsilon, k) = 2\pi i \mathcal{P}^a_{\alpha \gamma}(k) \mathcal{P}^b_{\beta \gamma}(k) \delta(\epsilon - \omega |k|) \]n\omega - (a - b) |k|| + i0^\mp. \]

To focus on the most rapidly growing term, we restrict Eq. 20 to \( a = b \) and set the frequency of the scattering vertex in Eq. 19 to zero, \( \Gamma^{\alpha \beta}_{\gamma \delta}(\omega, k, k^\prime) \omega = 0. \)

Furthermore, to leading order in \( 1/N, \) the squared anticommutator is expressed by one band index only

\[ f^\alpha_\beta (\omega, k) = \sum_{a = \pm 1} f_a(\omega, k) \mathcal{P}^a_{\alpha \beta}(k) 2\pi \delta(\epsilon - \omega |k|). \]

Exploiting particle-hole symmetry, determining the Lyapunov exponent is eventually reduced to solving the integral equation

\[ f(\omega, k) = \frac{i}{\omega N} \left( 1 + \int_{k^\prime} M(k, k^\prime) f(\omega, k^\prime) \right) \]

where \( f(\omega, k) = \int \sum_\alpha f^\alpha_\alpha(\omega, k) \) with the kernel \( M = M_+ + M_- \) comprised out of band-preserving (\( + \)) and band-changing (\( - \)) processes

\[ M_{\pm}(k, k^\prime) = i K_{\pm}(k, k^\prime) D^K_{ul}(|k^\prime| + |k|, k^\prime \mp k) \]
\[ + \sum_{\omega^\prime} \int_{k^\prime} K_{\pm}(k^\prime, k^\prime - \tilde{k}) K_{\pm}(k, k - \tilde{k}) \]
\[ \times G^u_{ul}(\pm a |k| - \epsilon, k - \tilde{k}) G^u_{ul}(b |k^\prime| - \epsilon, k^\prime - \tilde{k}) \]
\[ \times D_R(\tilde{k}, \mu) D_A(\tilde{k}, \mu) \]

where \( K_{ab}(k, k^\prime) \equiv \frac{1}{2} (1 + ab |k||k^\prime|) \). The first term represents one-rung (denoted by \( M_{\pm}(k, k^\prime) \)) and the second term two-rung scattering, see Fig. 4d.

The Lyapunov exponent is finally determined by finding the set of eigenvalues \( \{ \lambda \} \) and corresponding eigenfunctions \( \{ f \} \) of \( M(k, k^\prime) \). Being real and symmetric, it is possible to bring the kernel in diagonal form,

\[ M^d(k, k^\prime) = \int_{\mathbb{Q}^2} V(k, q) M(q, q^\prime) V(q^\prime, k^\prime) = N \lambda k \delta(k - k^\prime) \]

where the linear orthonormal transformation is described by the orthogonal matrix \( V. \) Using Eq. 22, we get

\[ f^d(k, \omega) = \frac{i}{\omega N} \left[ \epsilon_k + N \lambda k f^d(k, \omega) \right] \]

where \( f^d(k, \omega) = \int_q V(k, q) f(q, \omega) \) and \( \epsilon_k = \int_q V(k, q). \)

Applying the inverse Fourier transform \( \int \frac{d\omega}{2\pi} e^{-i\omega t} \int \frac{d^2k}{2\pi^2} \) we find

\[ f^d(k, t) = \frac{\epsilon_k}{N} \sum_{n=0}^{\infty} \frac{(\lambda k t)^n}{n!} = \frac{\epsilon_k}{N} e^{\lambda k t} \]

which represents the desired exponential growth behavior described by the spectrum of growth exponents \( \lambda_k \) (see also Appendix D).

The Lyapunov exponent is now defined as \( \lambda_L = \max[\lambda_k], \) and the corresponding eigenfunction is denoted \( f_L(k). \) Consequently, the Lyapunov exponent can be efficiently determined as the largest eigenvalue of the eigenvalue equation

\[ \lambda f(k, \omega) = \int_{k^\prime} M(k, k^\prime) f(k^\prime, \omega) \]

instead of solving the inhomogeneous Eq. 22. An explicit representation of the homogeneous Bethe-Salpeter equation, which is solved numerically, can be found in App. B.

C. Results

The Lyapunov exponent as a function of coupling \( \alpha \) is depicted in Fig. 5. It saturates for strong coupling to
the asymptotic value given in Eq. 2. Even if we extrapolate the number of fermion flavors to its physical value \( N = 4 \) the mentioned bound is not saturated. For weak coupling, we obtain Eq. 3.

For our subsequent discussion it is important to determine the corresponding eigenfunctions of the kernel \( M(k, k') \). We find that in the case of strong coupling the eigenfunction \( f_t(\ell) \) is peaked at energies of order of the temperature, \( v|k| \approx k_B T \), which is the only energy scale present, see Fig. 6. In the weak coupling regime, however, the peak shifts due to the finiteness of the coupling to \( v|k| \propto \alpha k_B T \) which is the scale associated with the thermal screening of the Coulomb interaction [24].

As shown in Fig. 5, the dominant contribution to scrambling in graphene is the one-ring band-preserving scrambling process \( \lambda^{(1)} \). The Bethe-Salpeter equation only taking into account \( \lambda^{(1)} \) is given by

\[
\lambda f(\omega, K) = \frac{2\pi k_B T}{N} \frac{2}{2\pi} \int_0^\infty dK' \int [K + K']^1 QdQ \frac{QdQ}{2\pi} \sqrt{(K + K')^2 - Q^2} \text{Im}D^R(\ell_K, k') f(\omega, K') \]

where we introduced the dimensionless momenta \( K = \frac{v|k|}{2\pi k_B T} \) and the dimensionless imaginary part of the propagator \( \text{Im}D^R(\ell_K, k') \) (see Appendix B for its definition). In the co-scattering limit \( K \sim K' \), and transferred momenta smaller than the thermal screening scale \( Q < L_s^{-1} \ll 1 \) where \( L_s^{-1} = \frac{1}{\alpha} \frac{\alpha k_B T}{\ln 2} \), the kernel to the characteristic energy scales discussed in [24]. This discussion is most transparent if we focus on Eq. 28. The kernel behaves qualitatively the same as the one for the relaxation rate \( \tau^{-1}(\ell) \) of [24], i.e. there are no \( 1 - \cos^2 \theta_{k,k'} \) back-scattering corrections that enter the transport rate \( \tau^{(1)}(\ell) \) or energy weights \( (K - K')^2 \) that determine the energy relaxation rate \( \tau^{(1)}(\ell) \), respectively. For details see also Fig. 2 and the Appendix A. Furthermore, our eigenfunctions vary on a scale \( \alpha k_B T \), see Fig. 6. Projection to energy scales \( \alpha k_B T \) amounts essentially to setting the typical energy scale \( \epsilon \approx \alpha k_B T \). In this limit follows indeed from Ref. [24] that \( \tau^{-1}(\ell) \approx 0.58\alpha k_B T/\hbar N \) similar to our scrambling rate. While there are differences in the detailed numerical prefactors - the coefficient of \( \lambda_L \) is about 16 times larger, see Fig. 5 - the scrambling rate in graphene behaves as a dephasing rate. For weak coupling this scale is much faster than the characteristic transport collision rate of the hydrodynamic
regime \( \tau_{\text{tr}}^{-1} \sim \alpha^2 k_B T / \hbar N \), guaranteeing local thermalization which is a key prerequisite of a hydrodynamic description. Since \( \lambda_L \gg \tau_E^{-1} \) for energies between \( \alpha k_B T \) and \( k_B T \) we also find that actual thermalization is a much slower process than information scrambling.

### III. KINETIC EQUATION

In this section we present an alternative approach to describe the spread of information in time and space in many-body systems. In contrast to the diagrammatic expansion conducted in the previous section, we show that scrambling is described by an integro-differential equation similar to the well-known Boltzmann equation. It describes the growth of an initially small, localized perturbation.

The process of information scrambling is governed by two scales: The Lyapunov exponent \( \lambda_L \) and the Butterfly velocity \( v_B = \sqrt{D \Delta L} \) which gives rise to an additional length scale \( L_B = \frac{\Delta L}{\lambda_L} \) associated with the spatial spreading of information. For the system discussed in this work, the scrambling parameters are small \( \lambda_L, L_B^{-1} \sim O(1/N) \). Based on the smallness of these parameters we propose that the spreading of information and the exponential growth signaling chaotic behavior is described by a quantity \( f(t, x) \) which is governed by the partial differential equation

\[
\partial_t f - D \Delta_x f = f_0 \delta(t) \delta(x) + \lambda_L f, \quad (30)
\]

where higher order gradient terms are suppressed in higher orders of \( \lambda_L \) and \( L_B^{-1} \). The LHS represents a diffusion equation characterized by diffusion constant \( D \), whereas the RHS contains a source term and a term \( \sim \lambda_L \) that indicates that \( f \) is not a conserved quantity: A perturbing term \( f_0 \ll 1 \) triggering the onset of growth and a second term causing the characteristic exponential growth behavior. Eq. 30 is valid for early times, i.e.

\[
0 < t - \frac{|x|}{v_B} < t^* \quad \text{with} \quad t^* \sim \lambda^{-1} \log N.
\]

For times \( t - \frac{|x|}{v_B} > t^* \), non-linear terms are relevant causing \( f \) to saturate against its asymptotic long-time value.

The solution of Eq. 30 is obtained by Fourier transform and is given by

\[
f(x, t) = \frac{f_0}{4 \pi D t} e^\lambda_L t - \frac{x^2}{2 D t} \approx \frac{f_0}{4 \pi D t} e^{2 \lambda_L (t - \frac{|x|}{v_B})} \quad (31)
\]

where the approximative result is obtained for \( |x| \approx v_B t \). It suggests that information spreads diffusively. However, due to the additional source term in Eq. 30 spreading is enhanced and the perturbation propagates 'quasiballistically' as indicated by the approximative solution Eq. 31.

In the following section we derive Eq. 30 microscopically for the specific case of graphene. The approach is however more general and also applicable to other systems which can be treated perturbatively.

![Figure 7. Leading order self-energy diagram in the large-N, weak-coupling (\( \alpha \ll 1 \)) regime. Dashed blue lines represent functional derivatives applied to obtain the linearized kinetic equations in Eq. 42.](image)
ence of the generating field by

\[-[G_{a}^{-1}; H_{a}^{\alpha}]] = \frac{W_{\alpha}}{N} + \Sigma_{\alpha}^{R} - \Sigma_{\alpha}^{A}, \quad (35a)
\]

\[-[G_{a}^{-1}; H_{a}^{\alpha}]] = \Sigma_{\alpha}^{R} - \Sigma_{\alpha}^{A} \cdot \frac{1}{N} \cdot \left( \Sigma_{\alpha}^{R} - \Sigma_{\alpha}^{A} \right), \quad (35b)
\]

The expression for the inter-loop Keldysh self-energy is given by

\[\Sigma_{\alpha}^{R} = \frac{1}{2N} G_{\alpha}^{R} (x_1, x_2) D_{\beta}^{\alpha} (x_1, x_2),\]

whereas the retarded and advanced components are indicated in Sec. I. For convenience, we replace the bosonic Keldysh components by

\[D_{\beta}^{\alpha} = \frac{1}{2} \frac{1}{N} \left( \Sigma_{\beta}^{R} - \Sigma_{\beta}^{A} \right),\]

which is obtained by exchanging the 'collision-integral' \[I[h^{\alpha}, h^{\beta}],\] and conduct the frequency integration which is identical to the mass-shell approximation conducted in the previous section with the diagrammatic approach, see Eq. 20 and 21.

We eventually arrive at the following set of coupled partial differential equations

\[\left( \partial_{T} + \mathbf{v}_{\mathbf{p}}^{*} \cdot \nabla_{\mathbf{X}} \right) h_{\mathbf{a}}^{\mathbf{p}} (T, \mathbf{X}) - \frac{W_{\mathbf{a}}}{N} I \left[ h^{\alpha}, h^{\beta} \right] = \frac{1}{\pi} \int_{\mathbf{X}} e^{i\omega (t - T)} Q (T, \mathbf{X}), \quad (38a)
\]

\[\mathbf{F} \left[ h_{\mathbf{p}}^{\mathbf{a}} (T, \mathbf{X}) \right] = I \left[ h^{\mathbf{a}}, h^{\mathbf{p}} \right] + \frac{1}{\pi} \int_{\mathbf{X}} e^{i\omega (t - T)} Q (T, \mathbf{X}), \quad (38b)
\]

where the mass-shell restricted generating field reads

\[W_{\mathbf{a}} (T, \mathbf{X}) = i \int_{\mathbf{X}} 2 A_{\mathbf{a}} (\mathbf{p}, \mathbf{p}) (T, \mathbf{X}, \mathbf{p}) \mathbf{F} (T, \mathbf{X}) \mathbf{F} (T, \mathbf{X}), \mathbf{p}) \mathbf{p} ; \mathbf{q}
\]

\[\hbar \mathbf{F} \left[ h_{\mathbf{p}}^{\mathbf{a}} (T, \mathbf{X}) \right] = I \left[ h^{\mathbf{a}}, h^{\mathbf{p}} \right] + \frac{1}{\pi} \int_{\mathbf{X}} e^{i\omega (t - T)} Q (T, \mathbf{X}), \quad (38b)
\]

The single-particle spectral function \[A_{\mathbf{a}} (\mathbf{p}, \mathbf{p}) = -\text{Im} \Sigma_{\mathbf{a}}^{R} (\mathbf{p})\] is peaked for \[\epsilon \approx \epsilon_{\mathbf{p}}.\] If the momentum dependence of \[\text{Im} \Sigma_{\mathbf{a}}^{R} (\mathbf{p})\] is negligible, which is the case for graphene [24], the spectrum contains no incoherent background and the quasi-particle description applies [41]. This allows us to integrate out the frequency dependence to define the quasi-particle distribution function

\[h_{\mathbf{a}}^{\mathbf{p}} (T, \mathbf{X}) = \int_{\mathbf{X}} 2 A_{\mathbf{a}} (\mathbf{p}, \mathbf{p}) \mathbf{F} (T, \mathbf{X}, \mathbf{p}), \quad (37)
\]

In the following we, approximate \[A_{\mathbf{a}} (\mathbf{p}, \mathbf{p}) \approx \text{const} \delta (\mathbf{p})\] and conduct the frequency integration which is identical to the mass-shell approximation conducted in the previous section with the diagrammatic approach, see Eq. 20 and 21.

We eventually arrive at the following set of coupled partial differential equations

\[\left( \partial_{T} + \mathbf{v}_{\mathbf{p}}^{*} \cdot \nabla_{\mathbf{X}} \right) h_{\mathbf{a}}^{\mathbf{p}} (T, \mathbf{X}) - \frac{W_{\mathbf{a}}}{N} I \left[ h^{\alpha}, h^{\beta} \right] = \frac{1}{\pi} \int_{\mathbf{X}} e^{i\omega (t - T)} Q (T, \mathbf{X}), \quad (38a)
\]

\[\mathbf{F} \left[ h_{\mathbf{p}}^{\mathbf{a}} (T, \mathbf{X}) \right] = I \left[ h^{\mathbf{a}}, h^{\mathbf{p}} \right] + \frac{1}{\pi} \int_{\mathbf{X}} e^{i\omega (t - T)} Q (T, \mathbf{X}), \quad (38b)
\]

where the mass-shell restricted generating field reads

\[W_{\mathbf{a}} (T, \mathbf{X}) = i \int_{\mathbf{X}} 2 A_{\mathbf{a}} (\mathbf{p}, \mathbf{p}) (T, \mathbf{X}, \mathbf{p}) \mathbf{F} (T, \mathbf{X}) \mathbf{F} (T, \mathbf{X}), \mathbf{p}) \mathbf{p} ; \mathbf{q}
\]

\[\hbar \mathbf{F} \left[ h_{\mathbf{p}}^{\mathbf{a}} (T, \mathbf{X}) \right] = I \left[ h^{\mathbf{a}}, h^{\mathbf{p}} \right] + \frac{1}{\pi} \int_{\mathbf{X}} e^{i\omega (t - T)} Q (T, \mathbf{X}), \quad (38b)
\]

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\[h_{\mathbf{a}}^{\mathbf{p}} (T, \mathbf{X}) = \int_{\mathbf{X}} 2 A_{\mathbf{a}} (\mathbf{p}, \mathbf{p}) \mathbf{F} (T, \mathbf{X}, \mathbf{p}), \quad (37)
\]

In the following we, approximate \[A_{\mathbf{a}} (\mathbf{p}, \mathbf{p}) \approx \text{const} \delta (\mathbf{p})\] and conduct the frequency integration which is identical to the mass-shell approximation conducted in the previous section with the diagrammatic approach, see Eq. 20 and 21.

We eventually arrive at the following set of coupled partial differential equations

\[\left( \partial_{T} + \mathbf{v}_{\mathbf{p}}^{*} \cdot \nabla_{\mathbf{X}} \right) h_{\mathbf{a}}^{\mathbf{p}} (T, \mathbf{X}) - \frac{W_{\mathbf{a}}}{N} I \left[ h^{\alpha}, h^{\beta} \right] = \frac{1}{\pi} \int_{\mathbf{X}} e^{i\omega (t - T)} Q (T, \mathbf{X}), \quad (38a)
\]

\[\mathbf{F} \left[ h_{\mathbf{p}}^{\mathbf{a}} (T, \mathbf{X}) \right] = I \left[ h^{\mathbf{a}}, h^{\mathbf{p}} \right] + \frac{1}{\pi} \int_{\mathbf{X}} e^{i\omega (t - T)} Q (T, \mathbf{X}), \quad (38b)
\]

where the mass-shell restricted generating field reads

\[W_{\mathbf{a}} (T, \mathbf{X}) = i \int_{\mathbf{X}} 2 A_{\mathbf{a}} (\mathbf{p}, \mathbf{p}) (T, \mathbf{X}, \mathbf{p}) \mathbf{F} (T, \mathbf{X}) \mathbf{F} (T, \mathbf{X}), \mathbf{p}) \mathbf{p} ; \mathbf{q}
\]

\[\hbar \mathbf{F} \left[ h_{\mathbf{p}}^{\mathbf{a}} (T, \mathbf{X}) \right] = I \left[ h^{\mathbf{a}}, h^{\mathbf{p}} \right] + \frac{1}{\pi} \int_{\mathbf{X}} e^{i\omega (t - T)} Q (T, \mathbf{X}), \quad (38b)
\]

The single-particle spectral function \[A_{\mathbf{a}} (\mathbf{p}, \mathbf{p}) = -\text{Im} \Sigma_{\mathbf{a}}^{R} (\mathbf{p})\] is peaked for \[\epsilon \approx \epsilon_{\mathbf{p}}.\] If the momentum dependence of \[\text{Im} \Sigma_{\mathbf{a}}^{R} (\mathbf{p})\] is negligible, which is the case for graphene [24], the spectrum contains no incoherent background and the quasi-particle description applies [41]. This allows us to integrate out the frequency dependence to define the quasi-particle distribution function

\[h_{\mathbf{a}}^{\mathbf{p}} (T, \mathbf{X}) = \int_{\mathbf{X}} 2 A_{\mathbf{a}} (\mathbf{p}, \mathbf{p}) \mathbf{F} (T, \mathbf{X}, \mathbf{p}), \quad (37)
\]
to intra-loop self-energy contributions (see Eq. 36), is of particular significance. It does not directly contribute to the process of scrambling, but it is necessary to establish equilibrium. In contrast to the collision-integrals of the conventional Boltzmann equation, the products of distribution functions differ which results, e.g., in the absence of an equivalent of the H-theorem, or conservation laws [31].

To obtain the final result, we perform the functional derivative and introduce

$$f_a(X, p) = \sum_k \int \frac{\delta h_{ap}(X)}{\delta W_{bk}(0)} \bigg|_{W=0} \, ,$$

(41)

which relates to the quantity $f_a(\omega, k)$ found in Eq. 21 of the previous section. It relates to the initial correlation Eq. 33 where one half of the indices is traced over, and external legs are restricted to the same band, which contributes predominantly to the exponential growth behavior as shown in the previous section. Applied to the kinetic equations we obtain

$$\delta(T + v^* \cdot \nabla_X) f_{ap}^{ul}(T, X) - \frac{\delta(T) \delta(X)}{N} = I[f_{ap}^{ul}, f_{ap}^{ul}], \quad (42a)$$

$$\left( \partial_T + v^* \cdot \nabla_X \right) f_{ap}^{ul}(T, X) = -I[f_{ap}^{ul}, f_{ap}^{ul}], \quad (42b)$$

where the linearized collision-integral is given by

$$I[f_{ap}^{ul}, f_{ap}^{ul}] = \frac{\pi}{2N} \sum_{a'bb'} \int \delta(\epsilon_{ap} - \epsilon_{bk} + \epsilon_{b'q+k} - \epsilon_{a'p+q}) |T_{a'b}^{ab'}(p, k, q)|^2 \times \left[ h_{b'q+k}^{(0)} h_{b}^{(0)} f_{a'p+q}^{ul} + h_{a'b}^{(0)} h_{b}^{(0)} f_{a'p+q}^{ul} + h_{b'q+k}^{(0)} h_{a'b}^{(0)} f_{a'p+q}^{ul} - f_{a'p+q}^{ul} f_{a'b}^{ul} - f_{a'p+q}^{ul} h_{b}^{(0)} h_{b'q+k}^{(0)} \right] \, ,$$

(43)

and $h_{ap}^{(0)} = h_{ap}^{ul,(eq)} = -h_{ap}^{ul,(eq)}$. The first three terms entering the collision integral are diagrammatically represented in Fig. 7 where each contribution is obtained by performing the functional derivative, i.e. by ’cutting’ one solid line, respectively. In comparison to our diagrammatic approach, the first term represents the one-run contribution, whereas the second and third term the two-run contribution. The last term of the collision term is due to intra-loop contributions and does not contribute to scrambling.

**B. Connection to the diagrammatic approach**

To make the connection to the diagrammatic approach more explicit, we show the identity of the first term in the collision integral with the one-run diagrammatic contribution as discussed in Eq. 22. By performing the Fourier transform and dropping the spatial gradient term as well as all terms except the first term of the collision integral, we find by using $K_{ab}(p, k) = (U_p U_k^\dagger)_{ab}(U_k U_p^\dagger)_{ba}$:

$$-i\omega f_{ap}^{ul}(\omega) = \frac{1}{N} \left( 1 + \sum_q K_{aa'}(p, p + q) \right. \times \left. \tilde{D}_{ul}^\alpha(q, \epsilon_{a'p+q} - \epsilon_{ap}) f_{a'p+q}^{ul}(\omega) \right) \, ,$$

(44)

where we introduced

$$\tilde{D}_{ul}^\alpha(q, \omega) = \frac{\pi}{2} |D(q)|^2 \sum_{bb'} \int \delta(\epsilon_{b'q+k} - \epsilon_{bk} - \omega) K_{bb'}(k, k + q) h_{b'q+k}^{(0)} h_{bk}^{(0)} \, .$$

(45)

This is identical to Eq. 22 obtained in the diagrammatic approach.

To connect Eq. 42 to 30, one has to solve for $f^{ul}$ and $f^{lu}$ which is, as outlined in the previous section, a demanding task. The solution consequently gives expressions for the parameters $\lambda_1$ and $D$ and the corresponding eigenfunction $f$ describing the process of information scrambling. Note that the order one gradient terms on the LHSs of Eq. 42 vanish when taking the average of the external momentum angle: By assuming $f_p = f_p[\pi, \psi]$, which is legitimate for a rotational symmetric initial perturbation, and $\psi_p \propto \frac{p}{|p|}$, averaging yields

$$\int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{a'b'} v_{ap} \cdot \nabla f_{p}^{ul}(T, X) = 0,$n and Eq. 30 is recovered.

In this section, we reproduced the results obtained earlier in a diagrammatic approach using non-equilibrium techniques. This puts the previously obtained results on firmer ground, but also gives a deeper insight into the theoretical description of information scrambling in many-body systems: The free term entering the Bethe-Salpether equation can be interpreted as perturbing source term of Eq. 30. This quantum kinetic approach can be applied to other weak coupling problems as well.

We also comment on the experimental accessibility of scrambling. As shown in the kinetic equation-based formulation, the inter-loop distribution functions $h^{ul}$ and $h^{lu}$ are sensitive to the processes of scrambling. Their evolution is determined by Eq. 38 where $h^{ul}$ and $h^{lu}$ as well as the intra-loop distribution function $h$ are coupled in the collision integral. The evolution of $h$ however is determined by a conventional kinetic equation (see e.g. Ref. [40]) and is therefore not affected by the inter-loop distribution functions. Measuring $h^{ul}$ and $h^{lu}$ by measuring $h$ via physical observables is therefore not possible.

**IV. CONCLUSION**

In this work, we determined the information scrambling rate $\lambda_1$ of graphene as a function of the coupling constant $\alpha$ of the Coulomb interaction within a large-$N$ expansion. We showed that for strong coupling ($\alpha \gg 1$), scrambling saturates and is solely determined by temperature which is the only energy scale present.
In contrast at weak coupling ($\alpha \ll 1$), new scales, such as the thermal screening length $l_T^{-1}$ as discussed previously, emerge. These additional scales cause physical quantities to scale differently with temperature and coupling constant rendering them distinguishable. In this regime, the scrambling rate scales as $\lambda_L \sim \alpha k_B T/N$ and is consequently much larger than the transport rate $\tau_{\text{tr}}^{-1} \sim \alpha^2 k_B T/N$ that occurs in the d.c. conductivity or the electron viscosity. Instead, the scrambling rate in graphene is closely related to the quantum or dephasing scattering rate $\tau_{\text{q}}^{-1}(\epsilon, T)$ for characteristic energies $\epsilon \sim \alpha k_B T$. Scrambling processes at weak coupling are therefore a lot faster than the collisions that give rise to the hydrodynamic behavior of graphene implying that graphene is a comparatively fast information scrambler which is characterized by single particle decay. However, $\lambda_L$, as defined by Eq. 8, is not relevant for local thermalization as scrambling probes the system only on a rather small range of excitation energies around $\epsilon \sim \alpha k_B T$ and is parametrically larger than the energy relaxation rate $\tau_{\text{r}}^{-1} \sim \alpha^2 \log(\alpha^{-1}) k_B T/\hbar N$ that one would expect to govern thermalization.

In the second part of this work we presented an alternative approach towards scrambling in many-body systems. We showed that the results obtained in a diagrammatic approach are reproduced by using non-equilibrium techniques yielding a partial differential equation describing the growth and spread of an initially small, localized perturbation. This approach allows a physically deeper insight into the process of information scrambling in many-body systems.

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Appendix A: Scattering rates of electrons in graphene

In the strong coupling limit ($\alpha \gg 1$) all characteristic single particle scattering rates are of order $k_B T/\hbar N$. In contrast in the weak coupling limit ($\alpha \ll 1$), electrons in graphene display a number of distinct scattering rates, such as the dephasing or quantum rate $\tau_q^{-1}$, the rate for energy relaxations $\tau_E^{-1}$, or the rate relevant for transport processes $\tau_{tr}^{-1}$, which scale differently with coupling constant and temperature. The origin of these scaling behaviors is the infrared-singular collision kernel, a behavior to some extent similar to that of weakly interacting diffusive electrons discussed in Ref. [14]. A detailed analysis of these frequency and energy-dependent scattering rates was performed in Ref. [24]. The obtained results, which are relevant for this work, are reviewed shortly in the following.

The scattering rates are given by

$$\tau_i^{-1}(\epsilon, T) = \pi \int_\omega \left( \coth \left( \frac{\omega}{2 k_B T} \right) + f(\epsilon - \omega) \right) \int_q \text{Im} D^R(\omega, q) \times K_i(p, q, \omega) \sum_{s=\pm} \delta(\epsilon - \omega - v_l |p - q|).$$

(A1)

where the specific rates distinguish themselves through different kernels $K_i = K_{\{q, x, tr\}}(p, q, \omega)$. Here, $p$ is the external momentum that enters the analysis through $\epsilon = v |p|$ on the mass shell. The kernel for the dephasing rate is $K_q = 1$ and yields the single particle scattering rate $\tau_q^{-1}(\epsilon, T) = -2 \text{Im} \Sigma^R(\epsilon, \tilde{p})$ with single particle self-energy $\Sigma^R(\epsilon, \tilde{p})$. The energy relaxation rate is determined by $K_E = (\omega/k_B T)^2$. It is the relevant rate to determine the energy diffusion coefficient. Finally, the transport scattering rate that enters transport coefficients such as the electrical conductivity or the shear viscosity is determined by $K_{tr} = \sin^2 \theta_{p, q}$ where $\theta_{p, q}$ is the angle between the momenta $p$ and $q$.

The main results of Ref. [24] are summarized as follows: First, it is necessary to carefully distinguish between the relevant energy regimes. For the dephasing rate holds that

$$\tau_q^{-1}(\epsilon, T) \sim \begin{cases} \frac{1}{N} T \sqrt{\frac{T}{\epsilon}} & \text{if} \quad \epsilon \ll \alpha^2 T \\ \frac{\alpha T}{\sqrt{\epsilon}} & \text{if} \quad \epsilon \gg \alpha^2 T \end{cases}$$

(A2)

where we suppressed numerical coefficients of order unity, and $\hbar = k_B = 1$ for the sake of representation. As shown in detail in Ref. [24], the numerical coefficient in front of $\frac{1}{N} T$ for $\epsilon \gg \alpha^2 T$ depends on whether $\epsilon$ is smaller or larger than the scale $\alpha T$, which for small $\alpha$ is large compared to $\alpha^2 T$. This behavior is owed to the screening length $l_s^{-1} \sim \alpha T/v$ due to thermally excited carriers.

The situation is significantly richer for the transport rate

$$\tau_{tr}^{-1}(\epsilon, T) \sim \begin{cases} \frac{1}{N} T \sqrt{\frac{T}{\epsilon}} & \text{if} \quad \epsilon \ll \alpha^2 T \\ \frac{\alpha T}{\sqrt{\epsilon}} & \text{if} \quad \alpha^2 T \ll \epsilon \ll \alpha T \\ \frac{\alpha^2 T}{(T/\epsilon)} & \text{if} \quad T \ll \epsilon \end{cases}$$

(A3)

If one uses this scattering rate as relevant input in the collision integral of a kinetic equation, it holds that transport coefficients are governed by the rate for the typical energies $\epsilon \sim T$, where $\tau_{tr}^{-1} \sim \alpha^2 T$. This behavior is owed to the screening length $l_s^{-1} \sim \alpha T/v$ due to thermally excited carriers.

Finally, for the energy relaxation rate holds that

$$\tau_E^{-1}(\epsilon, T) \sim \begin{cases} \frac{1}{N} T \sqrt{\frac{T}{\epsilon}} & \text{if} \quad \epsilon \ll \alpha^2 T \\ \frac{\alpha T}{\sqrt{T/\epsilon}} \log \left( \frac{\epsilon}{\alpha^2 T} \right) & \text{if} \quad \alpha^2 T \ll \epsilon \ll T \end{cases}$$

(A4)

The origin of the additional logarithm is the singular phase space in collinear scattering processes, an effect that does not enter the transport rate because of the forward scattering kernel $K_{tr}$. 
At lowest energies $\epsilon \ll \alpha^2 T$, all scales behave the same. For $\epsilon \sim \alpha T$, $\tau_q^{-1} \sim \frac{1}{\alpha} \sim \frac{1}{\alpha} T \gg \tau_E^{-1} \sim \frac{1}{N} \alpha^{3/2} \log \alpha^{-1} T$ implying that energy relaxation is the slowest process. For $\epsilon \sim T$, the dephasing rate is the largest scale $\tau_q^{-1} \sim \frac{1}{N} \alpha^{-1} T \gg \tau_E^{-1} \sim \frac{1}{\alpha} T \log \frac{1}{\alpha} \sim \frac{1}{N} \alpha^2 T$. A representation of these scattering rates as function of energy is depicted in Fig. 2.

Appendix B: The homogeneous Bethe-Salpeter equation

For further analysis, we express the homogeneous Bethe-Salpeter equation (see, Eq. 27) in terms of dimensionless variables $K = \frac{\sqrt{|k|}}{2k_B T}$ and replace the angle integrations by an additional momentum integration. We obtain the homogeneous integral equation

$$\lambda f(\omega, K) = \frac{4 \pi k_B T}{\hbar} \int_0^\infty \frac{K'dK'}{2\pi} M(K, K') f(\omega, K')$$

(B1)

with $M = M_+ + M_-$. The one-rung contributions (superscript (1)) are given by

$$M_+^{(1)}(K, K') = \frac{2}{KK'} \int_{|K-K'|}^{K+K'} QdQ \sqrt{(K+K')^2 - Q^2} \frac{\text{Im} D^{(1)}_{\omega}(\{K-K'\}, Q)}{\sinh(\{K-K'\})}$$

(B2)

$$M_-^{(1)}(K, K') = \frac{2}{KK'} \int_{|K-K'|}^{K+K'} QdQ \sqrt{(K+K')^2 - Q^2} \frac{\text{Im} D^{(1)}_{\omega}(K+K', Q)}{\sinh(K+K')}$$

(B3)

where we introduce the dimensionless imaginary part of the bosonic propagator as

$$\text{Im} D^{(1)}_{\omega}(x, y) = \frac{(\frac{\omega}{2})^2 \mathcal{I}_F(x, y)}{Q + \frac{\alpha T(x, y)}{2}} + \left(\frac{\alpha T(x, y)}{2}\right)^2.$$  

(B4)

The dimensionless functions $\mathcal{I}_F$ and $\mathcal{I}_G$ are defined via the real and imaginary part of the polarization operator $\text{Im} \Pi^R(\omega, q) = \frac{k_B T}{2\pi} \mathcal{I}_F(\frac{\omega}{2k_B T}, \frac{\sqrt{|q|}}{2k_B T})$, and $\text{Re} \Pi^R(\omega, q) = \frac{k_B T}{2\pi} \mathcal{I}_G(\frac{\omega}{2k_B T}, \frac{\sqrt{|q|}}{2k_B T})$, respectively. Their explicit expressions are

$$\mathcal{I}_F(x, y) = \frac{\sinh x \sqrt{\frac{\sqrt{x^2 - y^2}}{\cosh x + \cosh y} \sinh y}}{\sqrt{x^2 - y^2} \cosh x + \cosh y}$$

(B5)

$$\mathcal{I}_G(x, y) = -\frac{2}{\pi} \int_y^{\infty} d\xi \int_0^\infty d\eta \left(\frac{\sqrt{\xi^2 - y^2}}{\sqrt{\xi^2 - x^2}} \frac{\eta^2}{\eta^2 - \xi^2} \frac{\sinh \eta}{\cosh \eta + \cosh \xi} + \frac{\sqrt{y^2 - \eta^2}}{\sqrt{\xi^2 - \eta^2}} \frac{\xi^2}{\eta^2 - \xi^2} \frac{\sinh \eta}{\cosh \eta + \cosh \xi}\right).$$

(B6)

The two-rung contributions (superscript (2)) are given by

$$M_+^{(2)}(K, K') = \frac{4 \pi}{KK'} \int \frac{KdK}{2\pi}$$

$$\times \left(\int_{\min\{2|K-K'|+K'\}}^{\max\{2K+K'-K\}+K'} dQ \sqrt{\frac{K^2 - (2K - Q)^2}{K^2 - K^2}} \sqrt{\frac{K^2 - (2K' - Q)^2}{K^2 - K^2}} \frac{|D(Q, \hat{K})|^2}{\cosh(K - Q) \cosh(K' - Q)}\right)$$

(B7)

and

$$M_-^{(2)}(K, K') = \frac{4 \pi}{KK'} \int \frac{KdK}{2\pi}$$

$$\times \left(\int_{\min\{2|K-K'|+K'\}}^{\max\{2K+K'-K\}+K'} dQ \sqrt{\frac{(2K + Q)^2 - K^2}{K^2 - Q^2}} \sqrt{\frac{(2K' + Q)^2 - K^2}{K^2 - Q^2}} \frac{|D(Q, \hat{K})|^2}{\cosh(K + Q) \cosh(K' + Q)}\right)$$

(B8)
where

\[ |D(Q, \tilde{K})|^2 = \frac{\left( \frac{\alpha}{2} \right)^2}{Q + \frac{\alpha I_C(x,y)}{2} + \left( \frac{\alpha I_F(x,y)}{2} \right)^2}. \]  

(B9)

**Appendix C: Numerical procedure**

The integral equation B1 is solved numerically by discretizing the area of integration. We use a homogeneous grid with up to $2^{9} \times 2^{9}$ grid points in $K \times K'$-space and diagonalize the obtained matrix. We compare our results to results obtained by solving the integral equation recursively for the one-rung $M_{+}^{(1)}$-contribution only where the kernel is evaluated analytically. In this case, the one-dimensional $K$-domain is discretized using $10^4$ grid points. It turns out that this process contributes predominantly to $\lambda_L$ and serves as a lower bound on the exponent for weak coupling.

**Appendix D: Lyapunov spectrum**

The spectrum of exponents for a specific coupling is depicted in Fig. 8. It is qualitatively the same for all couplings. We observe that the largest eigenvalue ($\lambda_L$) is well separated from the next-to-largest eigenvalue. This justifies the discussion about one specific Lyapunov exponent $\lambda_L$.

![Figure 8. Spectrum of Lyapunov exponents $\{\lambda_i\}$ in the strong coupling regime $\alpha \gg 1$. Lyapunov exponents are binned in intervals of length 0.02.](image)