Should Bank Stress Tests Be Fair?

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Abstract

Regulatory stress tests have become one of the main tools for setting capital requirements at the largest U.S. banks. The Federal Reserve uses confidential models to evaluate bank-specific outcomes for bank-specific portfolios in shared stress scenarios. As a matter of policy, the same models are used for all banks, despite considerable heterogeneity across institutions; individual banks have contended that some models are not suited to their businesses. Motivated by this debate, we ask, what is a fair aggregation of individually tailored models into a common model? We argue that simply pooling data across banks treats banks equally but is subject to two deficiencies: it may distort the impact of legitimate portfolio features, and it is vulnerable to implicit misdirection of legitimate information to infer bank identity. We compare various notions of regression fairness to address these deficiencies, considering both forecast accuracy and equal treatment. In the setting of linear models, we argue for estimating and then discarding centered bank fixed effects as preferable to simply ignoring differences across banks. We present evidence that the overall impact can be material. We also discuss extensions to nonlinear models.

1 Introduction

In the aftermath of the 2008 financial crisis, U.S. banking regulators adopted stress testing as a primary tool for monitoring the capital adequacy of the largest banks. For each round of annual stress tests, the Federal Reserve announces a “severely adverse stress scenario,” defined by a hypothetical path of economic variables over the next several quarters. A typical path includes an increase in unemployment, a decline in GDP, and projections for the level and volatility of the stock market, among other variables. The largest banks provide the Fed with detailed information about their loan portfolios and other assets. The Fed then applies internally developed models to project revenues and losses for each bank through the stress scenario. Banks are required to have sufficient capital to weather the projected losses.

The Fed does not disclose details of the models it uses to project revenues and losses. The Fed makes clear that to ensure consistent treatment for different banks it uses “industry models,” as opposed to models tailored to individual banks. As a matter of policy, the same models are applied to all banks. Quoting Board of Governors [9] (p.3), “two firms with the
same portfolio receive the same results for that portfolio.” We will refer to this statement as the Fed’s principle of equal treatment.

Banks have countered that the Fed’s models fail to capture bank-specific features that could lower projected losses. They have made these arguments in requests for reconsideration of stress test results. Of course, banks are not objective critics of the Fed’s supervision; but significant heterogeneity among the largest banks is indisputable. The banks subject to annual stress testing include universal banks, investment banks, large regional banks, the U.S. subsidiaries of certain foreign banks, and a variety of more specialized financial firms. It is certainly possible that bank-specific models would produce more accurate forecasts than a single industry model, in which case using a single model entails a trade-off between forecast accuracy and consistency across banks.

The heterogeneity among large banks motivates the questions we study: What is the best way to aggregate bank-specific models into an industry model? How should the Fed’s principle of equal treatment be interpreted and implemented? Is simply ignoring bank identity in estimating and applying models the best way to achieve fairness? To what extent is fairness at odds with accuracy? Although the heterogeneity of large banks is widely recognized, we know of no prior work that seeks to address this property within the constraints of the Fed’s policy of equal treatment. We will argue that addressing heterogeneity is preferable to ignoring it.

The question of fairness in algorithms and models has received a great deal of renewed interest in recent years, in some cases reviving earlier debates over fairness in testing and related policies that were not explicitly “algorithmic;” see, for example, the overviews in Barocas, Hardt, and Narayanan [6] and Hutchinson and Mitchell [31]. We draw on this literature, but our setting differs in important ways from most discussions of fairness.

Algorithmic fairness is usually concerned with ensuring that certain protected attributes — race or gender, for example — do not influence outcomes such as hiring decisions or loan approvals. Different methods can be compared based on alternative measures of influence and the degree to which sensitive attributes are indeed protected.

The counterpart of a protected attribute in our setting is a bank’s identity; but this attribute is not so much protected (in the sense that race and gender are) as inadmissible for the Fed’s purpose. In stating that “two firms with the same portfolio receive the same results for that portfolio,” the Fed is stating that bank identity is not a legitimate predictor of losses. Perhaps, then, fairness is achieved as long as the Fed uses the same model for all banks. In other words, perhaps “fairness through unawareness,” paraphrasing Dwork et al. [13], is sufficient in this setting. Moreover, in questioning whether the Fed’s models apply to them, banks are not claiming discrimination; on the contrary, they are asking for discrimination — asking that the
Fed change its models to recognize ways in which an individual bank differs from other banks. To investigate these issues, we focus primarily on a simple setting in which the “true” loss rate for each bank is described by a bank-specific regression on portfolio features and scenario features. The regulator’s goal is to aggregate these bank-specific models into a single model. A natural interpretation of an “industry” model in this setting is a pooled regression based on combining results across banks. The pooled model treats banks equally, but we show that it has at least two significant deficiencies: when applied to heterogeneous banks, it can produce poor measures of the marginal impact of individual features, even resulting in the wrong sign; and it implicitly misdirects legitimate information in portfolio features to infer (or proxy for) bank identity in forecasting losses. The second of these deficiencies works against the spirit of equal treatment of banks, even if bank identity is not explicitly used in the model.

We then investigate the application of ideas from algorithmic fairness in our setting. The fairness literature has mainly focused on classification problems (hiring decisions and credit approvals, for example), with regression problems getting somewhat less attention. Chzhen et al. [12] and Le Gouic et al. [37] developed a method of particular importance for regression that Le Gouic et al. [37] call “projection to fairness.” This method produces optimal forecasts (in the least-squares sense) subject to a fairness constraint known as demographic parity. We examine the application of this approach in our setting and conclude that it goes too far in leveling results across banks.

The pooled method ignores fairness and the projection method goes too far in imposing fairness, so we seek an intermediate solution. Johnson, Foster, and Stine [33] introduce a variety of methods for introducing fairness considerations in regression. These include methods they call “full equality of opportunity” (FEO) and “substantive equality of opportunity” (SEO). We examine these methods in our setting and conclude that the FEO method provides an attractive solution. In particular, we show that it addresses the two deficiencies of the pooled method highlighted above: it removes the distortion in the pooled coefficients that results from bank heterogeneity, and it prevents the misdirection of legitimate information to infer bank identity. Indeed, we show that the only way to achieve lower forecast errors than the FEO method is through such misdirection, a result that sheds light on the trade-off between accuracy and fairness.

Moreover, the method is easy to interpret and implement: fit a pooled model with centered bank fixed effects, and then discard the centered fixed effects to forecast losses. Including the fixed effects prevents misdirection of legitimate information; discarding them is necessary to treat banks equally; centering ensures that the overall mean forecast remains unchanged. Although we mainly work with linear models, we show that these ideas can be extended to
We also derive an extension of FEO to remove certain interaction effects, as opposed to just fixed effects.

We then investigate the empirical relevance of these considerations. We regress the loss rates of loan portfolios (credit cards, first lien mortgages, commercial real estate, and commercial and industrial loans) on measures of portfolio quality (past-due rates and allowances for losses) and macroeconomic variables. We document significant heterogeneity across banks in their estimated coefficients, and we show that the differences between pooled and FEO estimates can be material. This investigation is limited by the information banks make public — the Federal Reserve has access to far more granular data in forecasting losses. We cannot claim to approximate the Fed’s forecasts; our goal is to provide evidence of the potential importance of heterogeneity.

To help position our work, we briefly discuss some other research on bank stress tests. Covas, Rump, and Zakrajsek [14], Kapinos and Mitnik [34], and Kupiec [36] find strong evidence of heterogeneity in banks’ responses to macroeconomic shocks, and Kapinos and Mitnik [34] argue that ignoring heterogeneity can substantially underestimate projected capital requirements. The related models of Hirtle et al. [32] and Guerrieri and Welch [28] forecast aggregate results and are therefore not concerned with differences among banks. Heterogeneity in the accuracy of the Fed’s models for different banks is suggested by the comparisons in Agarwal et al. [1], Bassett and Berrospide [7], and Flannery, Hirtle, and Kovner [18] between the Fed’s results and results based on the banks’ own models.

A separate line of research considers the design of stress scenarios. Several studies (including Breuer et al. [11], Flood and Korenko [20], Glasserman et al. [22], Pritsker [44], and Schuermann [46]) have advocated the use of multiple scenarios to capture different combinations of risk factors. Cope et al. [13] and Flood et al. [19] recommend designing scenarios to reflect bank heterogeneity. Parlatore and Philippon [41] propose a theoretical framework for scenario design as a problem of optimal information acquisition.

Several studies have investigated the information content of stress test results, either through market responses (as in Fernandes, Igan, and Pinheiro [16], Flannery, Hirtle, and Kovner [18], Georgescu et al. [21], Glasserman and Tangirala [23], Guerrieri and Modugno [27], Morgan, Peristiani, and Savino [40], and Sahin, de Haan, and Neretina [45]) or through subsequent bank performance (as in Kupiec [36] and Philippon, Pessarossi, and Camara [42]). Flannery [17] discusses just how much information the Fed should disclose about stress testing procedures and outcomes. For perspectives on the effectiveness of the Fed’s stress tests, see Kohn and Liang [35] and Schuermann [46].

We provide additional background on the Federal Reserve’s stress tests in Section 2.
tion 3 lays out our modeling framework and analyzes the pooled industry model within this framework. Section 4 analyzes various ways to introduce fairness considerations, including the projection-to-fairness and FEO methods. Section 5 considers nonlinear models. Section 6 presents our empirical results. Proofs of our main results appear in the appendix. Additional supporting theoretical (Sections EC.1–EC.3) and empirical (Sections EC.4–EC.6) material is included in the Electronic Companion. Most of our discussion considers loss models, but we consider revenue models in Section EC.6.

2 Background

This section provides background on the Federal Reserve’s stress testing process and on the heterogeneity of the participating banks.

2.1 Regulatory Bank Stress Tests

In early 2009, in the depths of the Global Financial Crisis, the Federal Reserve launched a stress test of the 19 largest U.S. bank holding companies to gauge how much more capital they would need if economic conditions continued to worsen. The results of the stress test were made public, and the transparency and credibility of the process have been credited with restoring public confidence and helping to end the crisis.

The Dodd-Frank Act, the package of reforms that followed the crisis, codified the use of stress testing for bank supervision. The number of banks subject to DFAST (Dodd-Frank Act Stress Tests) has varied over time. The current requirement applies annually to banks with over $250 billion in assets and every other year to banks with assets between $100 billion and $250 billion. The 2022 DFAST covered 34 banks. We refer to the participating firms as “banks,” but they are more precisely holding companies, including the U.S. subsidiaries of some foreign banks.

The inputs to the stress test analysis are the stress scenario, which is common to all banks, and bank-specific balance sheet information. A scenario is specified through a hypothetical path of economic variables over the next 13 quarters. The 2022 DFAST specified paths for 28 variables, including GDP, inflation, unemployment, stock market and real estate indexes, interest rates, exchange rates, and measures of overseas economic activity. Each bank submits detailed information on its loans and other assets.

The Fed uses 21 models to integrate the stress scenarios with bank-level information to make bank-level projections. For example, one model applies to credit cards, one to first lien residential mortgages, one to commercial real estate loans, and another to commercial and
industrial loans. These models project losses in each of these portfolios. Some other models project revenues.

The Fed does not disclose details of its models, either to banks or the general public. But it does describe its general modeling approach in public documents. At a high level, a model assigns a loss rate to a set of bank-specific loan portfolio features $x$ and a common set of scenario variables $z$ through a function $f(x, z)$. The function $f$ is estimated from past observations of the macro variables and portfolio features for multiple banks. Thus, $f$ is estimated as an industry-wide model and then applied individually to each bank.

This approach is described, for example, on p.3 of Board of Governors [9], where we read, “The Federal Reserve generally develops its models under an industry-level approach calibrated using data from many financial institutions... The Federal Reserve models the response of specific portfolios and instruments to variations in macroeconomic and financial scenario variables such that differences across firms are driven by differences in firm-specific input data, as opposed to differences in model parameters and specifications. As a result, two firms with the same portfolio receive the same results for that portfolio in the supervisory stress test, facilitating the comparability of results.”

As noted in the introduction, we refer to the principle that banks with the same portfolio receive the same results as equal treatment.

2.2 Bank Heterogeneity

The appropriateness of equal treatment seems incontrovertible. But the right notion of consistency across firms becomes less clear when portfolios vary widely, and the largest U.S. banks are a highly heterogeneous group. We may not expect a regional bank to have an investment bank’s skill in the capital markets, nor do we expect the investment bank to have the regional bank’s skill in making single-family residential loans.

Heterogeneity among large banks is illustrated in Figure 2.1. The left panel applies to the banks that participated in the Federal Reserve’s 2022 stress test. It shows the distribution of the banks by their Global Industry Classification Standard sub-industry classifications. This group includes diversified banks (such as JPMorgan Chase and Bank of America); regional banks (like PNC Financial and Citizens Financial); consumer finance companies (including American Express and Discover); custody banks (such as Bank of New York Mellon and State Street); investment banks (including Goldman Sachs and Morgan Stanley); and intermediate holding companies comprising the U.S. subsidiaries of foreign banks (such as TD Group and Credit Suisse USA). The distribution of banks across categories reflects important differences in their areas of specialization.
The right panel of Figure 2.1 applies to the U.S. Global Systemically Important Banks (G-SIBs). It shows heterogeneity in the fractions of loans the banks hold in each of four categories. For example, for Wells Fargo (WFC) first lien mortgages are a relatively large fraction of its loans, whereas for Citigroup (C), credit cards make up a relatively large fraction. The figure suggests different areas of specialization in lending, even among the largest U.S. banks.

Beginning in 2020, the Federal Reserve allowed banks to submit requests for reconsideration of the stress capital buffer set by the Fed through the stress testing process. (The capital buffer is set through the Comprehensive Capital Analysis and Review, or CCAR, process, which accompanies the stress test.) The banks’ requests are confidential, but the Fed’s responses to these requests are public. The responses show that the banks were arguing for reconsideration at least in part based on claims that the Fed’s models do not capture distinctive features of the banks’ businesses. For example, Regions Financial claimed that the Fed’s models overlook the bank’s hedging of interest rate risk. Goldman Sachs claimed that the Fed’s models omit information relevant to the credit quality of the bank’s mortgage loans. Citizens Financial claimed that the Fed’s models overlook the bank’s loss-sharing agreements in its retail portfolio.

Five firms requested reconsideration in 2020, and all five requests were rejected. In its response to Goldman Sachs, the Fed wrote, “the Board has determined that it will follow its published principles for stress testing, including the principle of creating industry-level models, and not modify the existing results of these models. In particular, models used in the supervisory stress test are generally developed according to an industry-level approach, calibrated using data from many institutions.” Similar statements appear in all five rejections. These exchanges point to a debate in which the banks highlight their heterogeneity and the Fed asserts the importance of consistency.

1https://www.federalreserve.gov/supervisionreg/files/goldman-sachs-group-inc-20200904.pdf
2.3 Heterogeneity and Fairness

Read narrowly, the principle of equal treatment — the Fed’s statement that “two firms with the same portfolio receive the same results for that portfolio” — is easy to satisfy. It holds in any model that forecasts losses based only on portfolio features and the stress scenario, without using any other bank-specific information. Even this narrow reading has important implications. For example, the quality of a bank’s IT systems or the strength of its “culture”\(^2\) may be important factors in determining a bank’s losses under stress, but as they are not features of a loan portfolio, the Fed’s modeling principle would preclude incorporating them into the Fed’s models. Matters like the quality of a bank’s internal governance and controls must be addressed in other parts of the overall bank supervision process, outside of stress testing\(^3\).

Within the Basel framework, these considerations are part of the Pillar 2 supervisory process, as described in BCBS \([8]\).

Under a broader interpretation of the principle of equal treatment, a regulatory model should also exclude indirect proxies for bank identity. Suppose, for example, that a bank with outdated IT systems had a particularly large number of loans to the energy sector. Suppose further that because of its weak IT the bank was a poor monitor of its borrowers and suffered abnormally large losses in downturns. With information about IT excluded, a predictive model of losses that uses this history would likely overstate the risk of loans to the energy sector. This outcome is arguably unfair to all banks making energy loans, in that they would be indirectly penalized for one bank’s weak IT. Addressing these types of indirect effects drives our investigation. In its narrow sense, equal treatment requires an indifference to which banks hold which portfolios once a model is selected; the broader interpretation seeks to remove the influence of bank identity in the design of the model.

The Fed’s stated principle implicitly responds to concerns for *disparate treatment* of banks. The broader interpretation — precluding proxies for bank identity — aligns with a concern for a particular notion of *disparate impact* used in the literature on algorithmic fairness (see, for example, Chapter 6 of Barocas et al. \([6]\), Section 3 of Lipton et al. \([38]\), and Prince and Schwartz \([43]\)), sometimes called “proxy discrimination.” The banks’ objections, as reflected in their reconsideration requests, can be seen as concerns for a different type of disparate impact: even if the same model is applied to all banks, and even if the model is free of bank-identity proxies, some banks may claim to be more adversely affected than others by the model’s limitations.

\(^2\)For a perspective on the importance of culture, see, for example, “Enhancing Financial Stability by Improving Culture in the Financial Services Industry,” a speech given by then president of the Federal Reserve Bank of New York, William C. Dudley, on October 20, 2014, https://www.newyorkfed.org/newsevents/speeches/2014/dud141020a.html.

\(^3\)The Fed’s stress tests previously included a qualitative component, but this component was dropped in 2019.
Most of the objections raised by banks can be understood as pointing to omitted variables — features omitted from the Fed’s models that a bank believes would result in a more favorable outcome if included in the models. The Fed’s responses suggest a reluctance to incorporate overly narrow features into models, particularly features that might affect only a single bank. Model limitations, of the type claimed by the banks are likely inevitable, given the limited data available on bank performance in scenarios of severe stress. The Fed should strive to continue to improve its models, but our concern is not primarily for the banks’ objections. Our focus is rather on how best to interpret and implement the Fed’s stated principle of equal treatment, particularly under the broader interpretation that addresses elements of both disparate treatment and disparate impact, within the overarching goal of accurately forecasting stressed losses for each bank.

The fairness literature distinguishes notions of individual fairness and group fairness, where the members of a group often share a sensitive or protected attribute. More abstractly, an individual is defined by a fixed set of features, and a group is characterized by a probability distribution over features; see, for example, the characterizations of individuals and groups in Sections 2 and 3 of Dwork et al. [15]. From this perspective, individual fairness is concerned with fairness conditional on a set of features, whereas measures of group fairness incorporate distributions over features. Individual fairness typically requires that individuals with similar features be treated similarly. For a portfolio loss model, this condition is satisfied if the predicted loss is a suitably smooth function of the features of individual portfolios. But group fairness is more relevant to our setting than individual fairness because we think of each bank, with its particular mix of businesses and areas of focus, not as one portfolio but as a distribution over portfolios the bank might hold at different times. We are interested in accuracy and fairness with respect to these distributions of bank portfolio features. We therefore view each bank as a group of individual portfolios that share the attribute of bank identity. In contrast, individual fairness would be relevant to evaluating accuracy and fairness conditional on a specific portfolio for each bank. We will make this formulation of groups and individuals more explicit in the next section after introducing our basic model.

3 Pooling: Fairness Through Unawareness?

3.1 Basic Model

To capture bank heterogeneity, we consider a market with multiple banks, indexed by $s = 1, \ldots, S$. The loss rate (or net charge-off rate) $Y_s$ for bank $s$ is given by

$$Y_s = \alpha_s + \beta_s^\top X_s + \epsilon_s,$$

(1)
with \( \alpha_s \in \mathbb{R} \) and \( \beta_s \in \mathbb{R}^d \). Here, \( X_s \) is a \( d \)-dimensional vector of predictive variables; at this point, we do not distinguish between portfolio characteristics and macro variables. The portfolio characteristics include information about a bank’s borrowers and loan terms. We use a linear specification in (1) because it offers the simplest setting to explore the interaction of heterogeneity and fairness; we discuss nonlinear extensions in Section 5. We take (1) to be the true relationship between the loss rate \( Y_s \) for bank \( s \) over the forecast horizon and characteristics \( X_s \) known at the date the forecast is made. Loss rates are normalized by loan balances to make values of \( Y_s \) comparable across banks of different sizes.

We think of \( X_s \) as a draw from some distribution with

\[
\mu_s = \mathbb{E}[X_s] \in \mathbb{R}^d, \quad \Sigma_s = \text{var}[X_s] \in \mathbb{R}^{d \times d}.
\]  

(2)

The randomness in \( X_s \) can be interpreted as reflecting the variation in the characteristics for bank \( s \) (and the macro variables) over time — in particular, times of stress. We assume throughout that each \( \Sigma_s \) is nonsingular. The error \( \epsilon_s \) in (1) is assumed to satisfy, for each \( s \),

\[
\mathbb{E}[\epsilon_s] = 0 \quad \text{and} \quad \text{cov}[X_s, \epsilon_s] = 0.
\]  

(3)

The regulator’s problem is to choose a model \( g \) that forecasts the loss rate \( g(x, s) \) for bank \( s \) if the bank’s portfolio characteristic vector is \( x \). The forecasts should, at a minimum, satisfy the following narrow property, which prohibits the regulator from applying different models to different banks:

**Definition 3.1 (Equal treatment).** Model \( g : \mathbb{R}^d \times \{1, \ldots, S\} \to \mathbb{R} \) satisfies equal treatment if

\[
g(x, s) = g(x, s'), \quad \text{for all} \ x \in \mathbb{R}^d, \ \text{for all} \ s, s' \in \{1, \ldots, S\}.
\]

As the true relationship for each bank is linear in (1), we mainly focus on the case of a linear industry-wide model. The regulator’s problem is then to choose a single \( \alpha \in \mathbb{R} \) and \( \beta \in \mathbb{R}^d \) that it will use to form a forecast

\[
\hat{Y}(x) = \alpha + \beta^T x,
\]  

(4)

given portfolio characteristics \( x \). The forecast (4) satisfies equal treatment because it has no functional dependence on bank identity \( s \). The parameters of the industry model (4) may depend on the bank-specific parameters \( (\alpha_s, \beta_s) \) and on the mean and variance in (2), but they should not depend on the realized features \( X_s \).

The regulator would like the forecast loss \( \hat{Y}(X_s) \) to be close to the actual loss \( Y_s \) in (1) for every bank \( s \). To aggregate errors across banks, we introduce a random variable \( S \) that picks a bank according to a distribution

\[
P(S = s) = p_s, \quad s = 1, \ldots, \bar{S},
\]  

(5)
with the probabilities \( p_s \) summing to 1. In the simplest case, all banks get equal weight, and the \( p_s \) are all equal; but the \( p_s \) could also reflect relative asset sizes or other weighting schemes. When we replace a bank label \( s \) with the random variable \( S \), we get a mixture over banks. In particular, we can combine the bank-specific models (1) into a mixture or hierarchical model by writing

\[
Y_S = \alpha_S + \beta_S^\top X_S + \epsilon_S. \tag{6}
\]

In choosing parameters \( \alpha \) and \( \beta \) in (4), the regulator would like to make the forecast errors small for all banks. A natural way to aggregate forecast errors across banks is to consider the average squared error, in which case the regulator’s problem becomes choosing \( \alpha \) and \( \beta \) in (4) to solve

\[
\min_{\alpha, \beta} E[(\hat{Y}(X_S) - Y_S)^2]. \tag{7}
\]

The objective in (7) averages squared forecast errors over banks. It can also be written as \( \sum_s p_s E[(\hat{Y}(X_s) - Y_s)^2] \).

**Remark 3.1.** Before solving (7), we make several comments on our problem formulation.

(i) **Targets versus estimators.** The problem posed by (7), like the more general problem of choosing industry parameters in (4), is one of characterizing ideal coefficients \( \alpha \) and \( \beta \). This is a question of choosing the correct targets of estimation, rather than a question of choosing estimators. In particular, \( \alpha \) and \( \beta \) are population quantities rather than sample quantities. In practice, the regulator may have a panel of time series of observations across banks. Estimation methods for panel data ordinarily focus on coefficients that are common to all units and exploit the panel structure to estimate these shared values. Our concern is precisely with the case of heterogeneous coefficients, where we need to identify suitable targets before we can consider their estimation.

(ii) **Groups versus individuals.** As discussed in Section 2.3, individual fairness is concerned with ensuring that if two feature vectors \( x \) and \( x' \) are close, then the predicted losses \( \hat{Y}(x) \) and \( \hat{Y}(x') \) are also close. Our concern is for accuracy and fairness with respect to the distributions of \( (X_s, Y_s), s = 1, \ldots, \bar{S} \), and not just for individual outcomes; we do not want the choice of industry model to depend on the realization of \( X_s, s = 1, \ldots, \bar{S} \). Each \( (X_s, Y_s) \) reflects a distribution over individual portfolio features and losses — individuals that share the bank identity attribute \( s \). Each bank thus represents a group of potential individual portfolios, and we are interested in accuracy and fairness with respect to the distributions that define these groups.

(iii) **Stressed versus unstressed.** For the application to stress testing, it is helpful to think of the portfolio features and scenario variables in \( X_s \) and the losses \( Y_s \) in (1) as having their con-
ditional distributions given stress conditions. The regulator is then interested in forecasting the conditional mean loss for each bank, given stress conditions. By focusing on the conditional mean, this formulation makes the squared error (7) a reasonable benchmark for studying accuracy and fairness. In our empirical work in Section 6 we approximate the restriction to stress conditions by giving more weight to data from periods of stress in the squared error objective.

(iv) Regulator versus banks. As discussed in Section 2.3, some of the objections raised by banks can be understood as pointing to features missing from (1) and (4), features that are rejected by the Fed as overly narrow. Our investigation assumes the bank-specific models (1)–(3) are correct; in particular, under (3) any relevant omitted features are uncorrelated with included features. We focus on the regulator’s problem of how best to aggregate the bank-specific models (assuming their correctness) into an industry model, considering both accuracy and fairness; we do not address the banks’ claims regarding which features should be included in the models.

For the solution to (7), write

$$\bar{\mu} = E[X_S] = E[\mu_S] = \sum_s p_s \mu_s \in \mathbb{R}^d,$$

and

$$\text{var}[X_S] = E[(X_S - \bar{\mu})(X_S - \bar{\mu})^\top] = E[W_S] = \sum_s p_s W_s,$$

with

$$W_s = \Sigma_s + \mu_s \mu_s^\top - \bar{\mu} \bar{\mu}^\top \in \mathbb{R}^{d \times d}. \quad (10)$$

Similarly,

$$\text{cov}[\alpha_S, \mu_S] = \sum_s p_s \alpha_s (\mu_s - \bar{\mu}) \in \mathbb{R}^d.$$

**Proposition 3.1.** Problem (7) is solved by

$$\beta_{\text{Pool}} = E[W_S]^{-1} (\text{cov}[\alpha_S, \mu_S] + E[W_S \beta_S]) \quad (11)$$

and

$$\alpha_{\text{Pool}} = E[Y_S] - \beta_{\text{Pool}}^\top \bar{\mu}. \quad (12)$$

Loss forecasts using $\alpha_{\text{Pool}}$ and $\beta_{\text{Pool}}$ in (4) provide *fairness through unawareness*, in that they ignore bank identity. They satisfy equal treatment in the narrow sense of Definition 3.1. Given our starting point (1), problem (7) would seem to be the most direct interpretation of the Fed’s policy of developing an “industry-level approach calibrated using data from many financial institutions.”
However, the solution in (11) is not a satisfactory target. Indeed, (11) shows where heterogeneity is most problematic. If the intercepts $\alpha_s$ covary with the means $\mu_s$, this effect can distort $\beta_{Pool}$ through what is commonly known as Simpson’s paradox. As an extreme example, consider the case that $\beta_s = 0$ for all $s$; in other words, none of the features in $X_s$ is predictive of losses for any of the banks. The regulator’s model (4) using $\beta_{Pool}$ would nevertheless forecast losses based on these features if $\text{cov}[\alpha_s, \mu_S]$ is nonzero. This covariation would create the illusion of predictability. In applying (11), we would be forecasting losses based on irrelevant features, purely as a consequence of the way we aggregated the bank-specific models.

Even in a less extreme setting in which the $\beta_s$ are nonzero, the presence of the $\text{cov}[\alpha_s, \mu_S]$ term in (11) reflects an indirect influence of bank identity on loss forecasts. If the bank-level mean characteristics $\mu_s$ positively covary with the bank-level intercepts $\alpha_s$, then in the pooled model this covariance will lead to a higher loss forecast for a bank with a higher value of $X_s$. This is arguably unfair, in the sense that the loss forecast is not based on the legitimate influence of the feature $X_s$. We will formalize the idea that the pooled method misdirects legitimate information in Sections 4.2 and 4.5.

This effect is reminiscent of the bias incurred in panel regressions when fixed effects are present in the data but omitted from a model. As we emphasized in Remark 3.1(i), in our setting the primary objective is to define the appropriate target of estimation, given the heterogeneity in the coefficients. We cannot say the term $\text{cov}[\alpha_S, \mu_S]$ introduces bias until we have decided what we are trying to estimate.

3.2 Average Treatment Effects

We can gain additional insight by considering the case of scalar $X_s$. In this case, the pooled coefficient $\beta_{Pool}$ in (11) becomes

$$\beta_{Pool} = \frac{\text{cov}[\alpha_S, \mu_S] + \sum_s p_s(\sigma^2_s + \mu^2_s - \bar{\mu}\mu_s)\beta_s}{\sum_s p_s(\sigma^2_s + \mu^2_s - \bar{\mu}\mu_s)}.$$  \hspace{1cm} (13)

In the special case that $\text{cov}[\alpha_S, \mu_S] = 0$ and $\sigma^2_s + \mu^2_s - \bar{\mu}\mu_s \geq 0$, for all $s$, (13) becomes a convex combination of the individual $\beta_s$. In Section EC.3 we state some simple properties that an aggregation of the individual $\beta_s$ into a single industry value should satisfy, and we show that only a convex combination satisfies these properties. Equation (13) thus shows a further potential problem with the pooled method. Even if $\text{cov}[\alpha_S, \mu_S] = 0$, the coefficient on some $\beta_s$ could be negative, which would mean that a reduction in $\beta_s$ would increase $\beta_{Pool}$. This could mean that an improvement in risk management by one bank increases predicted losses at all banks. We investigate these types of cross-bank effects further in Section EC.1.
We will refer to any convex combination of the $\beta_s$ as a *weighted average treatment effect* or WATE parameter. This terminology is suggested by thinking of a unit increase in a portfolio characteristic $X_s$ as a treatment, and $\beta_s$ as the response to that treatment. The (ordinary) average treatment effect is the expected coefficient,

$$\beta_{ATE} = \mathbb{E}[\beta_S] = \sum_{s} p_s \beta_s,$$

but weighting the individual coefficients allows other combinations. In particular, if the $\mu_s$ are all equal, the pooled coefficient $\beta_{Pool}$ becomes

$$\beta_{Pool} = \frac{\sum_s p_s \sigma^2_s \beta_s}{\sum_s p_s \sigma^2_s}.$$  

(15)

We will say more about these cases in subsequent sections.

To translate a WATE coefficient into a loss projection $\hat{Y}$, as in (4), we also need to specify an intercept. Setting

$$\alpha_{WATE} = \mathbb{E}[Y_S] - \beta_{WATE} \bar{\mu},$$

ensures that the forecasts

$$\hat{Y}_{WATE}(X_s) = \alpha_{WATE} + \beta_{WATE}^T X_s, \quad s = 1, \ldots, \bar{S},$$

have zero expected error, in the sense that

$$\mathbb{E}[\hat{Y}_{WATE}(X_S) - Y_S] = \sum_s p_s (\alpha_{WATE} + \beta_{WATE}^T \mu_s) - \mathbb{E}[Y_S] = 0.$$

4 Fair Regressions

We have seen that if the regulator’s sole objective is to minimize average squared forecast errors subject to equal treatment, then the solution is given by the pooled coefficients in (11) and (12). However, we have also seen that (11) has consequences that are undesirable and even unfair, in the sense that it is indirectly influenced by bank identity. In this section, we turn to methods that expand the squared loss minimization objective (7) to include fairness considerations. Because the pooled method minimizes (7), any method that addresses fairness will entail a loss of accuracy as measured by (7).

4.1 Projection to Fairness

In the literature on fairness in classification methods, *demographic parity* is among the most widely discussed fairness principles; see, for example, Chapter 3 of Barocas et al. [6]. In the simplest classification setting, the counterpart of our forecast is a binary outcome $\hat{Y} \in \{0,1\}$.  

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For example, $\hat{Y} = 1$ may indicate a hiring decision, a loan approval, or a school admission decision. The decision is to be based on certain features of a candidate that are deemed legitimate. Demographic parity requires that the event $\{\hat{Y} = 1\}$ be statistically independent of a protected attribute, such as race or gender. This objective is difficult to achieve when legitimate features covary with the protected attribute.

Chzhen et al. [12] and Le Gouic et al. [37] extend the notion of demographic parity to the regression setting by requiring that model predictions be independent of a protected attribute. These two articles solve the problem of finding the model that minimizes mean squared prediction errors while achieving demographic parity. We will use the term projection to fairness (PTF), coined in Le Gouic et al. [37], for the method in these papers.

Both papers reduce the problem of regression fairness to one of finding the Wasserstein barycenter of a set of distributions, in the sense of Agueh and Carlier [2]. The barycenter is the distribution closest to the set of distributions in an average sense. For a squared error and one-dimensional distributions, the barycenter can be described as the distribution whose quantile function is a weighted average of the individual quantile functions. (The quantile function is the inverse of the cumulative distribution function.)

In the setting of Section 3.1, the resulting solution can be interpreted as follows. For bank $s$, the regulator first forms the forecast $\hat{Y}_s(x) = \alpha_s + \beta_s^\top x$, using the bank-specific coefficients and the realized features $X_s = x$. Suppose $\hat{Y}_s$ falls at the 80th percentile of the forecast distribution for bank $s$. The regulator then takes a weighted average of the 80th percentile forecast for all of the bank-specific models. That weighted average becomes the forecast for bank $s$.

To make this procedure more explicit and to specialize the general framework of Chzhen et al. [12] and Le Gouic et al. [37] to our setting, we consider the case (for this section only) that each feature vector $X_s$ has a multivariate normal distribution $N(\mu_s, \Sigma_s)$. Write $\Sigma_s^{1/2}$ for the symmetric square root of $\Sigma_s$, and define the standardized feature vectors

$$Z_s = \Sigma_s^{1/2}(X_s - \mu_s);$$

(16)

each $Z_s$ has a multivariate standard normal distribution. Write the basic identity (1) using standardized variables as

$$Y_s = \alpha_o^s + \beta_o^s Z_s + \epsilon_s,$$

with standardized coefficients

$$\beta_o^s = \Sigma_s^{1/2} \beta_s, \quad \alpha_o^s = \alpha_s + \beta_s^\top \mu_s.$$ 

(17)

Suppose $\|\beta_s^o\| \neq 0$, for all $s$, with $\|\cdot\|$ denoting the usual Euclidean norm. Consider the model
that assigns, to each bank $s = 1, \ldots, \bar{S}$, with features $X_s = x$ the forecast

$$
\hat{Y}^o(x, s) = \sum_i p_i \alpha^o_i + \sum_i p_i \|\beta^o_i\| \frac{\beta^o_i \top z_s}{\|\beta^o_i\|}, \quad z_s = \Sigma_s^{-1/2}(x - \mu_s).
$$

(18)

If there exists a $\beta \in \mathbb{R}^d$ and scalars $a_s > 0$ for which

$$
\beta^o_s = a_s \beta, \quad s = 1, \ldots, \bar{S},
$$

(19)

then we will see that (18) simplifies to the weighted average

$$
\hat{Y}^o(x, s) = \bar{\alpha}^o + \bar{\beta}^o \top z_s, \quad \bar{\alpha}^o = \sum_i p_i \alpha^o_i, \quad \bar{\beta}^o = \sum_i p_i \beta^o_i.
$$

(20)

In the case of scalar $X_s$, (19) holds whenever all $\beta_s$ have the same sign.

**Proposition 4.1.** Suppose that the $X_s$ are multivariate normal and $\|\beta_s\| \neq 0$, for all $s = 1, \ldots, \bar{S}$. Then (18) is the projection-to-fairness of the bank-specific models (4), meaning that (18) minimizes $E[(\hat{Y}^o(X_S, S) - Y_S)^2]$ among all models (whether linear or not) that satisfy demographic parity. If (19) holds, the projection-to-fairness is given by (20).

We can see from (18) that the PTF model does not satisfy equal treatment: to calculate the loss forecast for a bank, we need to know its identity $s$. We have included the special case of (20) because it more nearly parallels the type of model we seek in (4). The coefficients in (20) are weighted averages of bank-specific coefficients. The model in (20) satisfies equal treatment with respect to the standardized features $Z_s$, rather than the raw features $X_s$: two banks with the same standardized features will receive the same forecasts. But the means for the two banks could be very different — the standardization is done separately for each bank — indicating that one bank’s portfolio may be much riskier than the other bank’s. In treating standardized characteristics for different banks as comparable, the PTF model implicitly evaluates the riskiness of each bank relative to the distribution for that bank. The suitability of PTF in our setting is therefore questionable.

The root of the problem is that demographic parity is too strong a property for our setting. Ensuring that a hiring decision is independent of race or gender is important; but forcing the distribution of loss projections to be independent of bank identity ignores relevant differences in banks’ portfolios. Whereas the pooled model (11)–(12) does too little to address heterogeneity across banks, the PTF model goes too far in leveling differences. The next section provides a better balance.
4.2 Formal Equality of Opportunity

Johnson, Foster, and Stine [33] introduce the concept of formal equality of opportunity (FEO) in regression, based on the use of the term in political philosophy, for which they cite the review in Arneson [5]. According to Arneson [5], FEO means that “positions and posts that confer superior advantages should be open to all applicants. Applications are assessed on their merits.”

In adapting this idea to our setting, it is helpful to make a contrast with the previous section: whereas demographic parity requires that loss forecasts be independent of bank identity, FEO allows bank-dependence, but only through legitimate portfolio characteristics — through the bank’s “merits.” This notion aligns well with the Fed policy, quoted earlier, that “two firms with the same portfolio receive the same results.” The objective of FEO in regression, as developed by Johnson et al. [33], is to ensure that a protected attribute (for us, bank identity) has no direct or “causal” impact on a model’s predictions. The predictions may be correlated with bank identity if different banks tend to have different levels of exposure to legitimate portfolio features.

To develop this idea in our setting, we introduce the centered dummy variables

\[ U_i(s) = 1\{s = i\} - p_i, \quad i = 1, \ldots, S - 1, \quad s = 1, \ldots, S. \]  

(21)

We discuss the implications of centering below. For any coefficients \( \alpha, \delta_1, \ldots, \delta_{S-1} \in \mathbb{R} \) and \( \beta \in \mathbb{R}^d \), and any \( x \in \mathbb{R}^d \), let

\[ \hat{Y}(x, s) = \alpha + \sum_i \delta_i U_i(s) + \beta^T x. \]  

(22)

We have included the bank label \( s \) as an argument of \( \hat{Y} \) because \( U_i \) depends on \( s \). Let \( \alpha_F, \{\delta_i, i = 1, \ldots, S - 1\} \), and \( \beta_F \) solve the error minimization problem

\[ \min_{\alpha, \{\delta_i\}, \beta} \mathbb{E}[(\hat{Y}(X_S, S) - Y_S)^2]. \]  

(23)

With the coefficients that minimize (7), (22) becomes the linear projection of \( Y_S \) onto the span of \( \{1, U_1(S), \ldots, U_{S-1}(S), X_S\} \), evaluated at \( S = s \) and \( X_S = x \). Now drop the centered dummy variables \( U_i \) and define

\[ \hat{Y}_F(x) = \alpha_F + \beta_F^T x. \]  

(24)

The FEO loss forecast for bank \( s \) is \( \hat{Y}_F(X_s) \).

Steps (22)–(24) result from applying the definition of an impartial estimate (their Definition 2) in Johnson et al. [33]. (More precisely, steps (22)–(24) define a population counterpart of the sample formulation in [33,]). The procedure in (22)–(24) can be interpreted as follows: pool
losses and portfolio features across banks; regress losses on portfolio features with bank fixed-effects included; throw away the fixed effects in forecasting future losses. The resulting model (24) is an equal-treatment model, with no explicit dependence on bank identity. Centering the discarded variables $U_i$ ensures that $E[\hat{Y}_F(X_S)] = E[Y_S]$, so dropping the fixed effects does not introduce an overall bias.

We will say more about the implications of this approach, but we first show that our setting allows an explicit expression for the FEO coefficients:

**Proposition 4.2.** (i) The FEO coefficients are given by

$$\beta_F = E[\Sigma S]^{-1} E[\Sigma S \beta_S],$$

and

$$\alpha_F = E[Y_S] - \beta_F^\top \bar{\mu}.$$  

In particular, in the scalar case,

$$\beta_F = \sum_s p_s \sigma_s^2 \beta_s \sum_s p_s \sigma_s^2.$$  

(ii) We also have

$$\beta_F = \text{var}[X_S - \mu_S]^{-1} \text{cov}[X_S - \mu_S, Y_S],$$

so $\beta_F^\top (X_S - \mu_S)$ is the linear projection of $Y_S - E[Y_S]$ onto $X_S - \mu_S$.

We encountered (27) in (15) as a special case of the pooled coefficient when the bank means $\mu_s$ are constant. The general case in (25) similarly coincides with the pooled coefficient in (11) when the means are constant. In other words, introducing the bank-level fixed effects in (22) purges $\beta_F$ of the effect of different feature means across banks; dropping these fixed effects in (24) ensures that the regulator’s model has no explicit dependence on bank identity and satisfies equal treatment.

In what sense is this procedure fair? We adapt the interpretation in Johnson et al. [33] to our setting. Write $U = (U_1, \ldots, U_{S-1})^\top$ for the vector of centered dummy variables. Write $\text{cov}[X_S, U(S)]$ for the $d \times (S - 1)$ matrix of covariances between the components of $X_S$ and $U(S)$. Let

$$\Lambda = (\text{var}[X_S])^{-1} \text{cov}[X_S, U(S)].$$

This matrix minimizes $E[||U(S) - \Lambda^\top (X_S - \bar{\mu})||^2]$, so $\Lambda^\top (X_S - \bar{\mu})$ is the linear projection of the bank-identity variables $U(S)$ onto the centered portfolio features $X_S - \bar{\mu}$. The relationship between $\beta_{\text{pool}}$ and $\beta_F$ can be expressed as follows.
Proposition 4.3. The coefficients $\beta_{\text{Pool}}$ and $\beta_F$ satisfy

$$\beta_{\text{Pool}} = \beta_F + \Lambda \delta,$$

(30)

where $\delta = (\delta_1, \ldots, \delta_{S-1})^\top$ is the vector of coefficients from (22)–(23). In particular,

$$\delta_s = (\alpha_s + \beta_s \mu_s) - (\alpha_{\bar{S}} + \beta_{\bar{S}} \mu_{\bar{S}}) - \beta_F^\top (\mu_s - \mu_{\bar{S}}), \quad s = 1, \ldots, \bar{S} - 1.$$

(31)

We can write the forecast in (22), using the optimal coefficients from (23) as

$$\hat{Y}(x, s) = \mathbb{E}[Y_S] + \delta^\top U(s) + \beta_F^\top (x - \bar{\mu});$$

(32)

This is the linear projection of $Y_S$ onto $(1, U(S), X_S)$, evaluated at $S = s$, $X_s = x$. Let $\hat{Y}_P(x) = \alpha_{\text{Pool}} + \beta_{\text{Pool}}^\top x$ denote the forecast based on the pooled coefficients (11) and (12). Decomposing $U(S)$ into its projection onto $X_S - \bar{\mu}$ and an orthogonal component leads to the following contrast of these forecasts:

$$\hat{Y}(x, s) = \mathbb{E}[Y_S] + \delta^\top \Lambda^\top (x - \bar{\mu}) + \delta^\top [U(s) - \Lambda^\top (x - \bar{\mu})] + \beta_F^\top (x - \bar{\mu});$$

(33)

$$\hat{Y}_P(x) = \mathbb{E}[Y_S] + \delta^\top \Lambda^\top (x - \bar{\mu}) + \beta_F^\top (x - \bar{\mu});$$

(34)

$$\hat{Y}_F(x) = \mathbb{E}[Y_S] + \beta_F^\top (x - \bar{\mu}).$$

(35)

The term $\delta^\top [U(s) - \Lambda^\top (x - \bar{\mu})]$ in (33) affects the forecast through information in bank identity that is orthogonal to the legitimate features $x$. This would be disparate treatment, as in Johnson et al. [33]. Through “unawareness” (meaning that it has no functional dependence on bank identity) the pooled forecast (34) drops this term, but it retains $\delta^\top \Lambda^\top (x - \bar{\mu})$, as can be seen from (30).

The term $\delta^\top \Lambda^\top (x - \bar{\mu})$ is the problematic component of the pooled method. Although it does not explicitly use bank identity, this term relies on the fact that bank identity is to some extent predictable from portfolio features. Imagine the regulator forming loss forecasts from blinded data — the regulator does not know the identity of the bank. The term $\Lambda^\top (x - \bar{\mu})$ is the least-squares prediction of $U(s)$ from $x - \bar{\mu}$. In the pooled forecast (34), the regulator is implicitly “misdirecting” the data in the features $x - \bar{\mu}$ to try to identify the bank and then to adjust the forecast based on the inferred identity. The FEO forecast (35) removes this effect and retains only the direct effect of portfolio features on the loss rate.

In the terminology of Section 2.3, dropping $\delta^\top [U(s) - \Lambda^\top (x - \bar{\mu})]$ ensures the narrow sense of equal treatment — that loss forecasts not depend explicitly on bank identity. Dropping $\Lambda^\top (x - \bar{\mu})$ ensures a broader sense of equal treatment — that loss forecasts not depend on proxies for bank identity. Johnson et al. [33] refer to their counterpart of $\Lambda^\top (x - \bar{\mu})$ as disparate treatment.
impact, which is consistent with the notion of “proxy discrimination” as a particular type of disparate impact (as in Prince and Schwarz [43]). In our setting, as noted in Section 2.3, the disparate impact of most immediate concern to banks is the omission of features from the Fed’s models that might otherwise benefit individual banks. Omitted features may contribute to $\Lambda^\top (x - \bar{\mu})$, but dropping this term does not necessarily dispel banks’ complaints. The banks’ disagreements with the Fed concern the scope of portfolio features that should be modeled. We therefore prefer to associate $\delta^\top [U(s) - \Lambda^\top (x - \bar{\mu})]$ and $\Lambda^\top (x - \bar{\mu})$ with narrow and broad interpretations of the Fed’s own principle of equal treatment (as discussed in Section 2.3), rather than with separate concerns for disparate treatment and disparate impact by the Fed and the banks. We emphasize the interpretation of $\Lambda^\top (x - \bar{\mu})$ as a misdirection of legitimate information, rather than as a contributor to disparate impact.

The FEO method offers a further advantage over the pooled method. Recall again from Section 2.3 (and Remark 3.1(iv)) that we interpret the banks’ objections as calls for the inclusion of features that are omitted from the Fed’s models. Under the condition $\text{cov}[X_s, \epsilon_s] = 0$ in (3), the FEO coefficients of included features are unaffected by the omission of other features. The pooled coefficients do not in general have this property.

We will conclude in Section 4.5 that the FEO forecast is, in a precise sense, the best way to aggregate the bank-specific models into a single regulatory model. The FEO forecast has no direct dependence on bank identity; but it also removes the indirect dependence that results when bank identity is partly predictable from portfolio features. We discuss other methods for comparison.

4.3 Conditional Expectation Model

A similar misdirection of information occurs if we project the bank-specific models to an industry model in the sense of conditional expectation, rather than least squares. Suppose $X_s$ has density $g_s$, and suppose $\mathbb{E}[\epsilon_s|X_s] = 0$, $s = 1, \ldots, \bar{S}$. Then, by Bayes’ rule,

$$\hat{Y}_C(x) \equiv \mathbb{E}[Y_S|X_S = x] = \frac{\sum_s p_s g_s(x) (\alpha_s + \beta_s^\top x)}{\sum_s p_s g_s(x)}.$$ (36)

This model satisfies equal treatment — $\hat{Y}_C(x)$ depends on the portfolio features $x$ but not on a bank’s identity. However, the point of the weights $p_s g_s(x)$ is to infer the identity of the bank from the features. Indeed, as discussed in Section 5, the conditional expectation $\mathbb{E}[Y_S|X_S = x]$ can be viewed as a nonlinear generalization of the pooled method, with some of the same shortcomings.
### 4.4 Substantive Equality of Opportunity

As discussed in Arneson [5], a system in which admission decisions are made through a competitive exam open to everyone achieves formal equality of opportunity; but if only the wealthy have access to the preparation required for the exam, the system fails to achieve substantive equality of opportunity (SEO). In the regression setting, Johnson et al. [33] interpret SEO to mean that any influence of protected attributes should be removed from other variables included in a regression model. In the analogy with Arneson’s [5] example, SEO would seek to remove the effect of economic status from performance on the exam, whereas FEO would accept exam scores as a legitimate basis for decision-making. (Our use of “SEO” follows Johnson et al. [33].)

For a broader interpretation of substantive equality in algorithmic fairness, see Green [24].

To apply these ideas to our setting, define the $(\bar{S} - 1) \times d$ matrix

$$M = \text{var}[U(S)]^{-1} \text{cov}[U(S), X_S];$$

(37)

then $M$ minimizes $E[||X_S - \bar{\mu} - M^T U(S)||^2]$. In accordance with Definition 2 of Johnson et al. [33], define

$$\hat{Y}_{\text{SEO}}(x, s) = \alpha_F + \beta_F^T (x - M^T U(s)),$$

(38)

with $\alpha_F$ and $\beta_F$ defined by (23). The SEO forecast adjusts the portfolio features $x$ to remove the linear projection onto the centered bank dummy variables $U$. We can write (38) somewhat more explicitly as follows:

**Proposition 4.4.** With $M$ as in (37)

$$M^T U(s) = \sum_i (\mu_i - \bar{\mu}) U_i(s) = \mu_s - \bar{\mu},$$

(39)

so the SEO forecast (38) is given by

$$\hat{Y}_{\text{SEO}}(x, s) = \alpha_F + \beta_F^T (x - \mu_s + \bar{\mu}).$$

(40)

The SEO forecast is the linear projection of $Y_S$ onto a constant and $X_S - \mu_S$.

Recall from Section 4.1 that a model satisfies demographic parity if its forecasts are independent of bank identity. Let us say that a model satisfies weak demographic parity if its forecasts are uncorrelated with the bank-identity variables $U_i(S)$. The centered features $X_S - \mu_S$ are uncorrelated with the $U_i(S)$. It therefore follows from Proposition 4.4 that SEO forecasts are uncorrelated with the $U_i(S)$. In other words, we have the following result:

**Corollary 4.1.** The SEO forecast satisfies weak demographic parity.
Under additional conditions, we get a stronger conclusion:

**Corollary 4.2.** If the covariance matrix $\Sigma_s$ and the distribution of $Z_s$ in (16) are the same for all $s$, then the SEO model coincides with the standardized model (20), and both satisfy demographic parity.

Under the conditions in the corollary, the mean adjustment in (40) is sufficient to give $\hat{Y}_{SEO}(X_s, s)$ the same distribution for all $s$. Put differently, PTF considers only the quantile of $\alpha_s + \beta_s^\top X_s$, relative to the distribution for bank $s$, to be legitimate information; SEO considers $X_s - \mu_s$ to be legitimate information. Under the conditions of the corollary, the two concepts coincide.

The mean adjustment in (40) requires knowledge of the bank identity $s$, so (38) does not satisfy Definition 3.1. The intent of the mean adjustment is to achieve a greater degree of equality. Consider the example with which began this section. If $x$ represents an exam score and $\mu_1 > \mu_0$ are the mean scores among wealthy and non-wealthy exam takers, (40) adjusts scores downward for wealthy exam takers and upward for non-wealthy exam takers.

Such an adjustment may be appropriate when the individuals or firms under evaluation are, in some sense, not responsible for their mean characteristic (or the mean in their peer group) and are therefore evaluated based on deviations from the mean. This type of consideration does not seem applicable to the stress-test setting, but it could arise more generally in settings where capital regulation intersects with other policy objectives.

One such example is suggested by the Paycheck Protection Program Lending Facility (PPPL) launched by the Federal Reserve early in the COVID crisis. The PPPL provided for loans to small businesses to be made by banks and guaranteed by the Small Business Administration. Under normal circumstances, the loans would increase participating banks’ balance sheets and thus potentially increase their capital requirements. To promote use of the facility, banking regulators issued a rule excluding PPPL loans from capital requirements, thus “neutralizing the effects of participating in the PPPL Facility on regulatory capital requirements.”

This “neutralizing” action is somewhat analogous to the SEO adjustment in that it removes responsibility for the larger balance sheet from the bank. The adjustments differ in that SEO adjusts for the mean whereas the PPPL adjustment removes the amount lent through the program.

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4 Federal Register, Vol. 85, No. 71, p.20389, April 13, 2020.
Form Constraint Forecast
(i) $\hat{Y}_s = \alpha + \beta^\top X_s$ Pooled (11)–(12)
(ii) $\hat{Y}_s = g(X_s, s)$, some $g$ $\hat{Y}_S$ independent of $S$ PTF (12, 37)
(iii) $\hat{Y}_s = g(X_s)$, some $g$, $E[\epsilon_s | X_s] = 0$ Cond. exp. (36)
(iv) $\hat{Y}_s = \alpha + \beta^\top X_s$ $\text{cov}[Y_S - \hat{Y}_S, X_S - \mu_S] = 0$ FEO (24)
(v) $\hat{Y}_s = \alpha + \lambda^\top U(s) + \beta^\top X_s$ $\text{cov}[\hat{Y}_S, U(S)] = 0$ SEO (38)

Table 4.1: Summary of forecast model forms and constraints.

4.5 A Unified Perspective: Legitimate Information

All of the methods we have discussed can be seen as ways of choosing forecasts $\hat{Y}_s, s = 1, \ldots, S$, (of the form $\hat{Y}(X_s)$ or $\hat{Y}(X_s, s)$) to minimize

$$E[(\hat{Y}_S - Y_S)^2],$$

subject to additional considerations. Table 4.1 summarizes the cases we have considered. In rows (i), (iv), and (v), we minimize (41) over the indicated coefficients. In (ii) and (iii), we allow $g$ to be an arbitrary (suitably measurable) function of the indicated arguments. In (iii) we strengthen the condition on the errors $\epsilon_s$.

Proposition 4.5. In each row of Table 4.1, the squared loss (41) is minimized over forecasts of the form in the first column, subject to the constraint in the second column, by the model in the last column.

The constraint in Table 4.1 (v) is weak demographic parity. SEO implicitly takes the view that the only legitimate information in forecasting losses for bank $s$ is the deviation $X_s - \mu_s$. In contrast, FEO takes the full set of features $X_s$ as legitimate information. Through the constraint in Table 4.1 (iv), it enforces a requirement we call no misdirection of legitimate information. FEO uses all of $X_s$ in forecasting losses; but it chooses the coefficient $\beta_F$ to be the coefficient in a regression of $Y_S$ on $X_S - \mu_S$, which is the part of $X_S$ orthogonal to bank identity. This condition ensures that the information in $X_S$ is not misdirected to infer bank identity.

To make this idea precise, consider any model of the form (4). If we assume the intercept is chosen to match the unconditional mean, we may write the model as

$$\hat{Y}_\gamma(x) = E[Y_S] + (\beta_F + \gamma^\top (x - \bar{\mu}),$$

for some $\gamma \in \mathbb{R}^d$. With $\gamma = 0$, we get the FEO forecast (24).

Proposition 4.6. If $\gamma$ reduces errors in the sense that $E[(\hat{Y}_\gamma(X_S) - Y_S)^2] < E[(\hat{Y}_F(X_S) - Y_S)^2]$, then the forecast $\hat{Y}_\gamma$ misdirects legitimate information in the sense that
(i) $\text{cov}[\gamma^\top X_S, \delta^\top \Lambda^\top X_S] > 0$, and

(ii) $\text{cov}[\gamma^\top M^\top U(S), \delta^\top U(S)] > 0$.

Recall that $\Lambda^\top (X_S - \bar{\mu})$ is the linear projection of the centered bank identity variables $U(S)$ onto the centered portfolio features $X_S - \bar{\mu}$. The condition in (i) therefore indicates that $\gamma$ misdirects some of the legitimate information in $X_S$ toward inferring bank identity. Thus, deviating from $\beta_F$ in (42) either increases errors or misdirects information.

Property (ii) has a similar interpretation. The term $\delta^\top U(S)$ is the direct influence of bank identity on losses $Y_S$. The proposition states that any deviation $\gamma$ that reduces forecast errors (relative to $\gamma = 0$) implicitly picks up some of the information in bank identity.

To further illustrate the contrast between FEO and SEO consider a simple example in which some component of $X_s$ measures exposure to community development projects. Suppose for simplicity that this feature is uncorrelated with other features. In the SEO forecast, the only legitimate information from this exposure is a bank’s deviation from its own mean. Years in which a bank had above average exposure would lead to higher loss forecasts, but the bank’s average exposure to community development would not directly inform the forecasts — it is neutralized. In contrast, FEO treats the bank’s total exposure (mean plus deviation) as legitimate information. Like SEO, in evaluating the impact of this exposure — that is, in estimating the coefficient on the exposure — it relies only on the within-bank variation. This ensures that the information in the exposure is not misdirected toward inferring the bank’s identity, as could happen in the pooled regression.

4.6 Extension of FEO for Interaction Effects

Recall that the FEO forecast controls for bank fixed effects. One might similarly consider controlling for interactions between bank indicators and components of the feature vectors. This leads to a family of extensions of FEO that differ in which interactions they include. We will show that with a full set of interactions, the extended FEO model becomes the ATE model.

To examine this case, suppose the feature vector for each bank $s$ is partitioned into two components, $X_s$ and $V_s$. We extend FEO by including interactions with components of $V_s$ but not with components of $X_s$. (Thus, in our discussion of FEO, $V_s$ was empty.) We assume that for every bank $s$, the components of $X_s$ are uncorrelated with the components of $V_s$. This allows a clear delineation between variables with and without interactions. Let $\nu_s = \mathbb{E}[V_s]$. The bank-specific models now take the form

$$Y_s = \alpha_s + \beta_s^\top X_s + \gamma_s^\top V_s + \epsilon_s,$$  \hspace{1cm} (43)
with $\epsilon_s$ uncorrelated with $X_s$ and $V_s$.

We extend FEO to the following procedure:

1) Project $Y_S$ linearly onto $1, U_1(S), \ldots, U_{S-1}(S), X_S - \mu_S, V_S - \nu_S, U_1(S)V_S, \ldots, U_{S}(S)V_S$.

Let $\beta_F$ denote the coefficient of $X_S - \mu_S$ and let $\gamma_F$ denote the coefficient of $V_S - \nu_S$.

2) Set $\hat{Y}_F(x, v) = \alpha_F + \beta_T^Tx + \gamma_T^Tv$, with $\alpha_F$ chosen so that $E[\hat{Y}(X_S, V_S)] = E[Y_S]$.

If $V_S$ is empty, then we know from (28) that these steps do indeed reduce to the original FEO forecast. We have included the interaction $U_S(S)V_S$ in the first step (even though we omitted $U_{S-1}(S)$) to simplify the derivation of $\gamma_F$. Including this term means that the coefficients on the interactions $U_i(S)V_S$ are determined only up to constant, because $U_1(S)V_S + \cdots + U_{S-1}(S)V_S = 0$.

These coefficients are dropped in the second step, so their value is immaterial.

**Proposition 4.7.** Suppose $\text{var}[X_s]$ and $\text{var}[V_s]$ have full rank and $X_s$ and $V_s$ are uncorrelated, for each $s = 1, \ldots, \bar{S}$. Then $\beta_F$ is given by (25) and (28), and $\gamma_F = \bar{\gamma} = \sum_s p_s \gamma_s$. In particular, if interactions with $U(S)$ are included for all features, the FEO vector of coefficients reduces to the average treatment effect (14).

This result allows us to interpret the ATE forecast as a version of the FEO forecast that removes the effects of certain interactions. As a convex combination of the bank-specific coefficients, the ATE coefficient retains some of the advantages of the FEO coefficient, particularly for the cross-bank effects studied in Section [EC.1].

However, we do not see a compelling case for controlling for interactions between bank identity and portfolio features. When we control for the bank-identity variables in FEO, we are ensuring that the industry $\beta$ for legitimate features is not affected by heterogeneity in the banks’ constants (the fixed effects). This reasoning does not necessarily extend to removing the influence of heterogeneity in exposures to portfolio features.

## 5 Nonlinear Models

Most of the ideas developed in previous sections for linear regressions extend to generalized linear models through a transformation of the response variable. For example, instead of working with the loss rate $Y_s$, we could specify a linear model for its logit transformation $\log\left(\frac{Y_s}{1-Y_s}\right)$.

But we can also extend ideas from previous sections to more fully nonlinear models. Replace the mixture model in (6) with a general representation of the form

$$Y_S = g(S, X_S) + \epsilon_S, \quad E[\epsilon_S|S, X_S] = 0.$$  

(44)
In other words, the loss for bank \( s \) is given by \( g(s, X_s) + \epsilon_s \). We assume that \( g(S, X_S) \) and \( \epsilon_S \) are square-integrable.

The counterpart of the pooled estimate becomes

\[
f_{Pool}(x) \equiv E[Y_S | X_S = x] = E[g(S, X_S) | X_S = x].
\]

This rule satisfies equal treatment — it has no functional dependence on \( S \) — but we argued earlier (in Section 4.3) that this forecast implicitly uses the information in the portfolio features \( x \) to infer bank identity.

To introduce a nonlinear version of the FEO forecast, we will make the relatively modest assumption that (44) admits a decomposition of the form

\[
Y_S = f_0 + f_1(S) + f_2(X_S) + \epsilon,
\]

with \( f_0 = E[Y_S], f_1 : \{1, \ldots, \bar{S}\} \to \mathbb{R}, f_2 : \mathbb{R}^d \to \mathbb{R}, \) and

\[
\begin{align*}
E[Y_S - f_0 - f_1(S)|X_S] &= f_2(X_S) \\
E[Y_S - f_0 - f_2(X_S)|S] &= f_1(S),
\end{align*}
\]

Equations (46)–(47) are population versions of the backfitting algorithm in Hastie and Tibshirani [29], which is a special case of the alternating conditional expectations algorithm of Breiman and Friedman [10]. Given an initial choice of \( f_1 \) (and known \( f_0 \)), (46) defines an initial choice of \( f_2 \) through the regression of the residual \( Y_S - f_0 - f_1(S) \) on \( X_S \). Equation (47) then defines an updated choice of \( f_1 \). The algorithm iterates over (46) and (47). In writing (45), we are positing that this algorithm has a fixed point. Convergence of the backfitting algorithm is established under widely applicable conditions in Ansley and Kohn [4].

We now introduce

\[
\hat{Y}_F(x) = f_0 + f_2(x)
\]

as a nonlinear counterpart of the FEO forecast. We justify this interpretation by showing that \( \hat{Y}_F \) exhibits properties that are nonlinear counterparts of the key properties of the FEO forecast in Section 4.2 and 4.5. To state the result, consider forecasts of the form

\[
\hat{Y}_\gamma(x) = f_0 + f_2(x) + \gamma(x),
\]

for some \( \gamma : \mathbb{R}^d \to \mathbb{R} \) with \( E[\gamma(X_S)^2] < \infty \).
Proposition 5.1. The nonlinear FEO forecast \((48)\) satisfies
\[
\text{cov} \{ \hat{Y}_F(X_S) - Y_S, X_S - \mathbb{E}[X_S|S] \} = 0.
\] (50)

For \(\hat{Y}_\gamma\) as in \((49)\), if \(\gamma\) reduces errors, in the sense that \(\mathbb{E}[(\hat{Y}_\gamma(X_S) - Y_S)^2] < \mathbb{E}[(\hat{Y}_F(X_S) - Y_S)^2]\), then it misdirects legitimate information, in the sense that
\[
\text{cov} \left\{ \gamma(X_S), \mathbb{E}[f_1(S)|X_S] \right\} > 0
\] (51)
and
\[
\text{cov} \left\{ \mathbb{E}[\gamma(X_S)|S], f_1(S) \right\} > 0.
\] (52)

Property (50) parallels the condition in row (iv) of Table 4.1 that characterizes the FEO forecast in the linear setting. It says that the forecast error \(\hat{Y}_F(X_S) - Y_S\) is uncorrelated with the legitimate information \(X_S - \mathbb{E}[X_S|S]\), which is the component of \(X_S\) orthogonal to bank identity \(S\). Properties (51)–(52) parallel conditions (i) and (ii) in Proposition 4.6. In particular, in (51), \(\mathbb{E}[f_1(S)|X_S]\) is the expected impact of bank identity inferred from portfolio features; the positive covariance with \(\gamma(X_S)\) thus indicates that \(\gamma\) misdirects some of the information in \(X_S\) to inferring \(S\).

We briefly contrast our FEO forecast in (48) with an alternative approach to extending fairness concerns to complex, nonlinear models. The alternative seeks to strip \(X_S\) of any protected attributes before a model is estimated. Examples of this general approach include Grünewälder and Khaleghi [25] and Madras et al. [39]. This approach is primarily concerned with ensuring demographic parity: if a model has no access — not even indirect access — to a protected attribute, its forecasts will be independent of the attribute. But we argued previously that demographic parity is too strong a condition for our setting. Our FEO forecast in (48) treats all the information in \(X_S\) as legitimate information — even elements that could help infer \(S\) — but it ensures that the information is not in fact misdirected to infer \(S\).

6 Empirical Evidence

In this section, we document empirical evidence of heterogeneity in bank-specific models of loss rates, and we examine the implications of this heterogeneity for the choice of an industry-wide model. We find strong evidence of statistically significant differences in model parameters across banks. These differences can lead to material differences between pooled and FEO coefficients in an industry model.

We must emphasize, however, that our investigation is constrained by the very limited information made publicly available by banks about the risk characteristics and losses in their
loan portfolios. The Federal Reserve has far more granular information about banks’ loans and losses. Our results can therefore provide only a rough indication of the impact of bank heterogeneity in the Fed’s stress tests.

6.1 Data

We use two types of data and data sources: historical macroeconomic data and loan information for individual banks.

6.1.1 Macroeconomic Data

We use data on seven of the macro variables used in the Federal Reserve’s stress tests: real disposable income growth, real GDP growth, house price index level, inflation rate, unemployment rate, Dow Jones total stock index level, and the Treasury spread. The Federal Reserve provides historical data on its website for all variables used in forming stress scenarios, including these. We use the values reported by the Fed for these variables in the June 2020 stress test; these values run from 1990 through 2019.

We aggregate these variables into a single macro variable by taking the first principal component of their correlation matrix. Table 6.1 shows the corresponding loadings. We see that an increase in the principal component corresponds to decreases in income growth and GDP growth and an increase in unemployment, suggesting that this composite variable serves as a reasonable measure of overall economic conditions. Figure 6.1 plots the level of this variable over time and shows a sharp climb around 2008 and 2020.

| Macro Factor                  | PC1 Loading |
|-------------------------------|-------------|
| Real disposable income growth | -0.229      |
| Real GDP growth               | -0.525      |
| Change House Price Index      | -0.467      |
| CPI inflation rate            | -0.079      |
| Change unemployment           | 0.529       |
| Change Dow                    | -0.293      |
| Change Treasury Spread        | 0.287       |

Table 6.1: Loadings of first principal component on macro variables.

The loadings in Table 6.1 are calculated using data through 2019, and we use these loadings to extend PC1 through the end of 2021. When we include the COVID period in the calculation of the principal components, PC1 becomes harder to interpret. For example, the coefficients for income growth and unemployment have the same sign.
6.1.2 Loan Information for Individual Banks.

Bank holding companies publicly report financial information quarterly through the Federal Reserve’s form Y-9C. We use these filings to collect information on four loan types that are treated separately in the Fed’s stress tests: credit cards, first lien mortgages, commercial real estate loans, and commercial and industrial loans. For each category, each bank, and each quarter, we collect loan balances, charge-offs, recoveries, and total amounts past due serving as our proxy measures of loan portfolio risk.

We collect this data from 2001 to 2021 for the thirty-five largest banks by total assets (as of December 2021). The banks are listed in Table EC.1. The stress test focuses on adverse economic conditions; we weight each observation by the level of stress in each quarter and each bank’s load balance: for each bank $s$, each quarter $t$, and loan category $p$, we weight the observations by

$$w_{s,t}^p = e^{\lambda MacroPC_t} \times Loan_{s,t}^p,$$

where $Loan_{s,t}^p$ is the size of the loan portfolio of type $p$ for bank $s$ in quarter $t$. We choose $\lambda$ so that for the same loan level, the worst economic quarter (as measured by $MacroPC_t$) is given twice the weight as the best quarter.

We would prefer to conduct our analysis using data from stress periods only, but that would leave us with too few observations. Weighting by the level of stress in a quarter allows us to approximate the effect of conditioning on stress while making greater use of the available data. This approach relies on the assumption that data from non-stressful periods is relevant to forecasting losses in periods of stress.

We merger-adjust all bank data. For example, Truist Financial, one of the banks in Table EC.1, was formed from the 2019 merger of BB&T and SunTrust, so our data for Truist in
earlier years combines data from those two banks. We repeat this process as we work backwards in time. We obtain information on mergers and acquisitions from the Federal Financial Institutions Examination Council website. (We have also run our analysis without merger-adjusting the data; doing so does not change our conclusions and generally increases heterogeneity across banks.)

In each loan category, we calculate a loss rate (net charge-off rate) for each bank \( s \) and each quarter \( t \) as the ratio

\[
\text{LossRate}_{s,t} = \frac{\text{Charge-offs}_{s,t} - \text{Recoveries}_{s,t}}{\text{Total Loans in Category}_{s,t-1}}.
\]

This measure is commonly used in stress testing; see, for example, Guerrieri and Welch \[28\], Hirtle et al. \[32\], and Kapinos and Mitnik \[34\]. We similarly normalize the amounts past due to get a \( \text{PastDueRate}_{s,t} \) for each bank-quarter. We remove values less than \(-50\%\) or greater than \(50\%\) of \( \text{LossRate} \) and values greater than \(20\%\) of \( \text{PastDueRate} \). We winsorize \( \text{PastDueRate} \) at the upper and lower \(5\%\) levels. To attain a mostly balanced panel for more reliable estimates, in each loan category we include only banks with at least \(18\) years (72 quarters) of history from 2001 Q1 to 2021 Q4.

Table 6.2 shows descriptive statistics for these variables. Loss rates and past due rates are shown by loan category — credit cards (CC), first liens (FL), commercial real estate (CRE), and commercial and industrial (CI). Columns 2–4 of the table summarize time-averaged values across banks. Columns 5–8 summarize observations across all banks and quarters.

|                  | bank averages |                      | all observations |                      |
|------------------|---------------|----------------------|------------------|----------------------|
|                  | min | mean | max | lower 5% | mean | upper 5% | std  |
| Loss Rate: CC    | -0.20 | 2.50 | 3.36 | 0.31 | 3.00 | 5.88 | 2.01 |
| Loss Rate: FL    | 0.01 | 0.22 | 0.66 | -0.01 | 0.27 | 1.03 | 0.51 |
| Loss Rate: CRE   | 0.06 | 0.19 | 0.46 | -0.04 | 0.16 | 1.00 | 0.41 |
| Loss Rate: CI    | 0.05 | 0.41 | 1.00 | 0.00 | 0.42 | 1.56 | 0.52 |
| Past Due Rate: CC| 1.10 | 2.90 | 4.23 | 1.06 | 3.30 | 5.74 | 1.53 |
| Past Due Rate: FL| 0.71 | 4.01 | 8.02 | 0.45 | 6.20 | 11.80 | 4.60 |
| Past Due Rate: CRE| 1.07 | 2.15 | 3.80 | 0.36 | 2.12 | 6.52 | 1.88 |
| Past Due Rate: CI| 0.05 | 1.65 | 2.85 | 0.05 | 1.74 | 3.97 | 1.23 |

Table 6.2: Descriptive statistics in percent. Columns 2–4 are calculated from banks’ time averages, and columns 5–8 are calculated from all observations, with mean and standard deviation are stressed time and loan balance weighted.

Figure 6.2 plots the mean past due rate (\(\pm 1.96\) standard errors) for each bank in each loan category. The banks are identified by their stock tickers. The figure illustrates substantial heterogeneity across banks in their loan portfolios. For example, Bank of America (BAC)
has among the highest past due rates for credit card loans, but in the commercial real estate
category it has among the lowest. This type of pattern is consistent with the idea that banks
have different areas of specialization and may target different markets.

The widths of the bars in Figure 6.2 show differences across loan categories and banks in the
volatility of their past due rates. We again observe significant heterogeneity among different
banks.

Figure 6.2: Past due rates (winsorized) by bank and loan category. The dots show mean values
and each horizontal bar corresponds to ±1.96 standard errors.
6.2 Heterogeneity in Slopes and Intercepts

We use the bank data to approximate our theoretical framework through the specification

\[
\text{LossRate}_{s,t} = \alpha_s + \beta_s \text{PastDueRate}_{s,t-1} + \gamma_s \text{MacroPC}_{t-1} + \epsilon_{s,t},
\]

for bank \( s \) in quarter \( t \), where \( \text{MacroPC} \) is the principal component of the macro variables introduced in Section 6.1.1. (In Section EC.5.1, we also include allowances in (55) as a robustness check.) The lag \( l \) is four quarters to mimic the stress testing’s forward-looking framework. We estimate separate coefficients for each of the four loan categories, for each bank, and the observations are loan balance and stress weighted using (53). Because these are bank-specific regressions, we do not add bank-specific controls.

For each loan category, we want to test for heterogeneity in parameters across banks. When we test for heterogeneity, the null hypothesis states that slopes for all banks are equal,

\[
H_0: \beta_1 = \cdots = \beta_S,
\]

or that the intercepts are equal,

\[
H_0: \alpha_1 = \cdots = \alpha_S.
\]

The alternative hypothesis in each case states that the indicated parameters are not identical across banks. We will run these tests with different subsets of the variables in (55) included and interpret the coefficients in (56) accordingly.

To test these hypotheses for a particular loan category, let \( X_s \) be the \( n_s \) by \( k \) data matrix for bank \( s \), where \( n_s \) is the number of observations for bank \( s \) in the loan category, and \( k = 1 \) or \( 2 \) is the number of variables included on the right side of (55). Let \( \tilde{X}_s = (1, X_s) \) be \( X_s \) concatenated with a column of 1s, and let \( X^* \) be the diagonal block matrix \( X^* = \text{diag}(\tilde{X}_1, \ldots, \tilde{X}_S) \). Let \( \theta^* = (\alpha_1, \beta_1^T, \ldots, \alpha_S, \beta_S^T)^T, \epsilon^* = (\epsilon_1, \ldots, \epsilon_S) \), where \( \epsilon_s \) is a column vector of length \( n_s \). Our unrestricted model can be written as

\[
Y = X^* \theta^* + \epsilon^*,
\]

and the restrictions in (56) and (57) impose linear constraints on the parameter \( \theta^* \).

We apply the Wald test to test linear constraints on \( \theta^* \) in (58) under various assumptions on the error covariance matrix. We consider (i) bank-clustered errors, which allows correlation in errors across quarters for each bank, but no correlation across banks; (ii) time-clustered errors, which allows correlation in errors across banks in each quarter, but no correlation across time.

Table 6.3 reports \( p \)-values for the tests when different subsets of variables are included on the right side of (55), for a forecast horizon of one year. All tests indicate strong evidence of heterogeneity in the intercepts, the coefficients for past due rates, and macro variables.
Next we examine the impact of heterogeneity. Table 6.4 compares pooled and FEO coefficients for *PastDueRate* using a one-year lag when *MacroPC* is and is not included in (55). We estimate $\beta_{Pool}$ in a pooled panel regression, and $\beta_F$ in a panel regression with bank fixed effects included. Both regressions are weighted using (53).

In Table 6.4 the columns labeled “diff” show the difference in estimates $\beta_F - \beta_{Pool}$, serving as a measure of the impact of addressing heterogeneity in choosing an industry model. The table also shows $p$-values for tests of $H_0: \beta_{Pool} = \beta_F$ (or $\gamma_{Pool} = \gamma_F$ in the case of the macro variable). To calculate these $p$-values, we estimate the pooled and FEO models simultaneously, as follows. Let $\tilde{X}_{Pool} = (1, X)$ be $X$ concatenated with a column of 1s, and let $\tilde{X}_F = (U, X)$ be $X$ concatenated with columns corresponding to centered bank identity variables $U$. Let $X_* = \text{diag}(\tilde{X}_{Pool}, \tilde{X}_F)$ and $\theta_* = (\alpha_{Pool}, \beta_{Pool}^T, \delta_1, \delta_2, \ldots, \delta_S, \beta_F^T)^T$. Then we have

$$Y = X_* \theta_* + \epsilon_*,$$

and testing $H_0$ is equivalent to testing linear constraints on the parameters $\theta_*$, for which we apply the Wald test. The macro variable captures common variability over time, so we cluster errors by bank.

The results in Table 6.4 show that the differences between the pooled and FEO estimates are significant in three of the four loan categories. Moreover, the differences can be material. For example, for first lien loans in the top panel of Table 6.4 an absolute difference of 0.015 translates to a relative difference of 24% ($= |\beta_{Pool} - \beta_F|/\beta_F$), which can have a large relative impact on predicted losses. From Table 6.2 we see that the average bank has a past due rate of 4.01% on FL loans. The difference $0.015 \times 4.01\% = 0.060\%$ is 27% of the average FL loss rate of 0.22% in Table 6.2. The additional capital required to offset the higher predicted loss rate would be 27% of the capital required to offset the average loss rate.

To further analyze the differences in forecasts, we consider the relative prediction differences given by $|\tilde{Y}_{Pool}(x) - \tilde{Y}_F(x)}/Y(x)|$, where the denominator is the observed loss rate. Table 6.5 reports the mean and median of these relative differences in each of the four loan categories, for all banks in all quarters. In each loan category, the mean relative difference is large; moreover, in each case the mean is appreciably larger than the median, reflecting the presence of some very large relative prediction differences, which could be particularly important. To illustrate the differences, Figure 6.3 plots histograms of the FEO and pooled fitted values for Citigroup’s first lien loans. The comparison shows, in particular, that the frequency of the largest and smallest predicted loss rates differ between the two methods.

Our main results use data through 2021. As a robustness check, we run our analysis using data through 2019. This truncation serves two purposes. It ensures that our conclusions are
Table 6.3: $P$-values for heterogeneity tests. In each loan category, the first two rows are for a model with PastDueRate only, and the last two rows are for a model with PastDueRate and MacroPC. The two rows for each model show results under alternative assumptions on the error covariance matrix.

not driven by a few extreme values during the COVID period 2020–2021, and it accounts for a change in how banks measure allowances (the Current Expected Credit Losses methodology) beginning at the end of 2019. We also consider including banks’ allowances for losses as another proxy for the portfolio risk. Because allowances are not consistently reported separately by loan category, we use banks’ total allowances across all loan types. That is, for each bank-quarter we calculate AllowanceRate$_{s,t}$ using (54), but normalizing by the total loans in all categories.

We repeat our tests with pre-COVID data and the addition of allowance rates. The results, reported in Section EC.5.1, are similar to those reported in this section.

Our results document evidence of bank heterogeneity and its potential impact on loss forecasts. We have not sought to identify the drivers of heterogeneity; that would require a very different investigation, particularly since some of the most interesting potential drivers (a bank’s management, the quality of its IT systems) are difficult to measure. Guerrieri and Harkrader [26], for example, find that bank-specific factors account for a sizable fraction of the variation in bank performance. But they measure the bank-specific component as the residual in a regression that removes the effect of macroeconomic and banking-wide factors; they do not identify specific bank features that influence performance. Some examples of bank features used as controls in stress testing models can be found in Hirtle et al. [32], Kapinos and Mittnic [34], and Kupiec [36]. These are balance sheet features, and they are usually found to be more relevant to forecasting revenues than losses. We discuss revenue models in Section EC.6.

6.3 Nonlinear Models

Building on Section 5 we consider generalized additive models (GAMs) in which the effect of the past due rates and the macro variable are not restricted to be linear. More specifically, we consider the specifications

$$Y_{s,t}^{Pool} = f_0^P + f_1^P(PDR_{s,t-l}) + f_2^P(MacroPC_{t-l}) + f_3^P(PDR_{s,t-l} \times MacroPC_{t-l}) + \epsilon_{s,t}^P \quad (59)$$
| Loan Type | Past Due Rate $\beta_{Pool}$ | Past Due Rate $\beta_F$ | $\beta_{Pool} - \beta_F$ | p-value | Macro PC $\gamma_{Pool}$ | Macro PC $\gamma_F$ | $\gamma_{Pool} - \gamma_F$ | p-value |
|-----------|------------------|------------------|-----------------|--------|------------------|------------------|-----------------|--------|
| CC        | 0.782            | 0.833            | -0.050          | 0.009***| 0.040            | 0.037            | 0.003           | 0.000***|
| FL        | 0.047            | 0.062            | -0.015          | 0.001***| 0.018            | 0.019            | -0.001          | 0.176   |
| CRE       | 0.131            | 0.129            | 0.002           | 0.464   | 0.011            | 0.011            | 0.000           | 0.813   |
| CI        | 0.208            | 0.219            | -0.011          | 0.040** | 0.022            | 0.022            | 0.000           | 0.127   |

| Loan Type | Past Due Rate $\beta_{Pool}$ | Past Due Rate $\beta_F$ | $\beta_{Pool} - \beta_F$ | p-value | Macro PC $\gamma_{Pool}$ | Macro PC $\gamma_F$ | $\gamma_{Pool} - \gamma_F$ | p-value |
|-----------|------------------|------------------|-----------------|--------|------------------|------------------|-----------------|--------|
| CC        | 0.774            | 0.823            | -0.049          | 0.009***| 0.040            | 0.037            | 0.003           | 0.000***|
| FL        | 0.046            | 0.061            | -0.015          | 0.001***| 0.018            | 0.019            | -0.001          | 0.176   |
| CRE       | 0.131            | 0.129            | 0.002           | 0.485   | 0.011            | 0.011            | 0.000           | 0.813   |
| CI        | 0.205            | 0.216            | -0.010          | 0.044** | 0.022            | 0.022            | 0.000           | 0.127   |

Table 6.4: Comparison of coefficients for one-year forecasts. We regress LossRate on (i) PastDueRate and (ii) PastDueRate and MacroPC. Difference is calculated as $\beta_{Pool} - \beta_F$. p-values test $H_0: \beta_{Pool} = \beta_F$ for PastDueRate or $H_0: \gamma_{Pool} = \gamma_F$ for MacroPC.

| CC | FL | CRE | CI |
|----|----|-----|----|
| mean | 6.1 | 510.1 | 52.0 | 24.0 |
| median | 2.4 | 89.5 | 4.2 | 3.7 |

Table 6.5: Mean and median of relative prediction differences between the pooled and the FEO estimates (in %).

Figure 6.3: Pooled and FEO predicted loss rates for Citigroup’s first lien loans.
and

\[ Y_{s,t}^F = f_0^F + f_1^F(PDR_{s,t-1}) + f_2^F(MacroPC_{t-1}) + f_3^F(PDR_{s,t-1} \times MacroPC_{t-1}) + f_4^F(s) + \epsilon_{s,t}^F, \]  

(60)
in which \( f^P_i, \ i = 1, 2, 3, \) are (possibly nonlinear) centered functions of PastDueRate, MacroPC, and their interaction PastDueRate \( \times \) MacroPC, respectively, and \( f_4^F \) measures centered bank fixed effects. The pooled model (59) omits \( f_4^P \); the FEO model (60) estimates \( f_4^F \) but discards it in forecasting loss rates to satisfy equal treatment. We consider the modeling of loss rates four quarters ahead, so \( l = 4 \) in both cases.

We use the R package \textit{gam} (Hastie [30]) to fit (59) and (60), taking the \( f^P_i, \ i = 1, 2, 3, \) to be smoothing splines with 4 degrees of freedom. We choose \textit{gam} because it is a direct implementation of the backfitting algorithm in Hastie and Tibshirani [29], which underpins the framework in Section 5. In particular, Proposition 5.1 applies to FEO forecasts based on (60).

Model (59) is nested within model (60), so we can use an \( F \)-test to compare the two. In all four loan categories, the test rejects (with \( p \)-values smaller than 0.01) the restriction to equal bank fixed effects \( (f_4^P \equiv 0) \) imposed in the pooled model (59). Figure 6.4 plots the centered bank fixed effects \( f_4^F \) for all four loan categories, expressed in percent. We observe significant variability within each loan type. For example, for credit card loans, JPM’s fixed effect is three percentage points larger than WFC’s. We also observe variability across loan types for individual banks. For example, CFG has the highest fixed effect for first lien loans, but one of the lowest for credit card loans. These observations again reflect the notable heterogeneity among the bank holding companies.

Table 6.6 reports the mean and median of the relative prediction differences between the FEO and the pooled predictions, given, as in Section 6, by \(|(\hat{Y}_{Pool}(x) - \hat{Y}_F(x))/Y(x)|\). These summary statistics show that the relative prediction differences can indeed be very large. To further illustrate this point, Figure 6.5 contrasts the prediction distributions for Citigroup’s first lien loans using the pooled and FEO methods. As in Proposition 5.1, the pooled method may yield smaller prediction errors overall, but it does so by implicitly misdirecting legitimate information.

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Table 6.6: Mean and median of relative prediction differences between the pooled and the FEO estimates (in %) for GAMs.

|    | CC | FL  | CRE | CI  |
|----|----|-----|-----|-----|
| mean| 25.5 | 559.0 | 106.6 | 88.5 |
| median| 10.4 | 120.8 | 14.2 | 9.8 |

Figure 6.4: Banks’ generalized fixed effects. Y-axis is in %.
7 Concluding Remarks

The current practice of regulatory stress testing ignores bank heterogeneity in loss models as a matter of policy and principle. We have argued that simply pooling banks can distort coefficients on legitimate features and is vulnerable to implicit misdirection of legitimate information to infer bank identity. We have examined various ways of incorporating fairness considerations and shown that estimating and discarding centered bank fixed effects addresses the deficiencies of pooling — and it does so in an optimal sense.

Beyond this specific recommendation, the broader conclusion to be drawn from our analysis is that accuracy and equal treatment can more effectively be addressed by accounting for bank heterogeneity rather than ignoring it. Although we have focused on the stress testing application, our analysis applies more generally to settings requiring the fair aggregation of individually tailored models into a single common model.

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A  Proofs

**Proposition 3.1.** Problem (7) is solved by the linear projection of $Y_S$ onto the span of 1 and $X_S$. If $\text{var}[X_S]$ is invertible, then the coefficients of the linear projection are given by (12) and

$$\beta_{Pool} = \text{var}[X_S]^{-1}\text{cov}[Y_S, X_S];$$

see, for example, Wooldridge [47], p.25. In (9)–(10) we can write

$$\text{var}[X_S] = \sum_s p_s W_s = \sum_s p_s \Sigma_s + \text{var}[\mu_S].$$

This matrix is positive definite because we assumed that each $\Sigma_s$ is positive definite, so $\text{var}[X_S] = E[W_S]$ is indeed invertible. To evaluate $\text{cov}[Y_S, X_S]$ for $Y_S$ in (6), we first note that

$$\text{cov}[X_S, \epsilon_S] = E[\text{cov}[X_S, \epsilon_S|S]] + \text{cov}[E[X_S|S], E[\epsilon_S|S]] = E[0] + \text{cov}[\mu_S, 0] = 0.$$

It follows that

$$\text{cov}[Y_S, X_S]
= E[\text{cov}[\alpha_S, X_S|S]] + \text{cov}[E[\alpha_S|S], E[X_S|S]] + E[\text{cov}[\beta_S^\top X_S, X_S|S]] + \text{cov}[E[\beta_S^\top X_S|S], E[X_S|S]]
= 0 + \text{cov}[\alpha_S, \mu_S] + E[\Sigma_S \beta_S] + E[\text{var}[\mu_S] \beta_S]
= \text{cov}[\alpha_S, \mu_S] + E[W_S \beta_S].$$
**Proposition 4.1.** By Proposition 3.4 of Chzhen et al. [12] or Theorem 6 of Le Gouic et al. [37], the expected squared error is minimized subject to demographic parity by the rule that assigns to bank \( s \) with features \( x \) the loss forecast
\[
\hat{Y}_{PTF}(x, s) = \sum_i p_i F_i^{-1}(F_s(\alpha_s + \beta_s^T x)),
\]
where \( F_s \) is the cumulative distribution function of \( \alpha_s + \beta_s^T X_s \). By construction, \( F_s \) is then also the cumulative distribution function of \( \alpha^0_s + \beta^0_s Z_s \), which is normal with mean \( \alpha^0_s \) and variance \( ||\beta^0_s||^2 \). Writing \( \Phi \) for the standard normal distribution function, we get
\[
F_s(y) = \Phi\left( \frac{y - \alpha^0_s}{||\beta^0_s||} \right), \quad F_i^{-1}(q) = \alpha^0_i + ||\beta^0_i|| \Phi^{-1}(q).
\]
Making these substitutions in (61) and writing \( \alpha^0_s + \beta^0_s^T z_s \) for \( \alpha_s + \beta_s^T x \), with \( z_s = \Sigma^{-1}_s(x - \mu_s) \), we get
\[
\hat{Y}_{PTF}(x, s) = \sum_i p_i F_i^{-1}(F_s(\alpha^0_s + \beta^0_s^T z_s))
\]
\[
= \sum_i p_i F_i^{-1}(\Phi(\beta^0_s^T z_s/||\beta^0_s||))
\]
\[
= \sum_i p_i \{ \alpha^0_i + ||\beta^0_i|| \Phi^{-1}(\Phi(\beta^0_s^T z_s/||\beta^0_s||)) \}
\]
\[
= \sum_i p_i \{ \alpha^0_i + ||\beta^0_i|| \beta^0_s^T z_s/||\beta^0_s|| \},
\]
which is (18). (Demographic parity holds because the distribution of \( \beta^0_s^T Z_s/||\beta^0_s|| \) does not depend on \( s \).) Under (19), \( ||\beta^0_s||/||\beta^0_s|| = ||a_s\beta||/||a_s\beta|| = a_s\beta = \beta^0_s \), and we get (20).

**Proposition 4.2.** We can rewrite \( \hat{Y}(x, s) \) in (22) as
\[
\hat{Y}(x, s) = \sum_{i=1}^S a_i 1\{s = i\} + \beta^T x,
\]
for suitable \( a_i \). Minimizing (23) over the \( a_i \) and \( \beta \) yields the same value for \( \beta \) as minimizing (23) using (22) because the indicators \( 1\{s = i\} \) have the same span as the \( U_i(s) \) and a constant. Thus, the \( \beta_F \) defined by (23) is the coefficient of \( X_S \) in the regression of \( Y_S \) on \( X_S \) and the indicators \( 1\{S = i\} \). By the Frisch-Waugh-Lovell Theorem (as in Angrist and Pischke [3], pp.35–36), we can therefore evaluate \( \beta_F \) as the coefficient in the regression of \( Y_S \) on the component of \( X_S \) orthogonal to the other variables, which in our case are the indicators. The projection of \( X_S \) onto the indicators is given by \( \sum_i \mu_i 1\{S = i\} = \mu_S \), so the orthogonal component is \( X_S - \mu_S \). We may therefore evaluate \( \beta_F \) as the coefficient in the regression of \( Y_S - E[Y_S] \) on \( X_S - \mu_S \), which is (28). For the first factor in (28), we have
\[
\text{var}[X_S - \mu_S] = E[\text{var}[X_S - \mu_S|S]] + \text{var}[E[X_S - \mu_S|S]] = E[\Sigma_S] + 0.
\]
For the second factor, we similarly have

$$\text{cov}[X_S - \mu_S, Y_S] = \mathbb{E}[\text{cov}[X_S - \mu_S, Y_S|S]] = \mathbb{E}[\text{cov}[X_S - \mu_S, \beta_F^\top X_S|S]] = \mathbb{E}[\Sigma_S \beta_S],$$

so (25) follows. The optimal $\alpha_F$ in (23) ensures that $\mathbb{E}[\bar{Y}(X_S, S)] = \mathbb{E}[Y_S]$, which yields (26).

**Proposition 4.3.** The minimization in (23) yields coefficients $\alpha_F$, $\delta$, and $\beta_F$, with which we can write

$$Y_S = \alpha_F + \sum_{i=1}^{\bar{s}-1} \delta_i U_i(S) + \beta_F^\top X_S + u,$$

where the error $u$ has mean zero and is uncorrelated with $U(S)$ and $X_S$. We thus have

$$\beta_{\text{Pool}} = \text{var}[X_S]^{-1} \text{cov}[X_S, Y_S]$$

$$= \text{var}[X_S]^{-1}\{\text{cov}[X_S, \beta_F^\top X_S] + \text{cov}[X_S, \delta^\top U(S)]\}$$

$$= \text{var}[X_S]^{-1}\{\text{cov}[X_S, \beta_F] + \text{cov}[X_S, U(S)]\} + \Lambda \delta,$$

using the expression for $\Lambda$ in (29) for the last step.

Next, we evaluate $\delta$. Using (62), we can derive $\delta$ as the vector of coefficients in a regression of $Y_S - \beta_F^\top X_S$ on $U(S)$. Thus,

$$\delta = \text{var}[U(S)]^{-1} \text{cov}[U(S), Y_S - \beta_F^\top X_S]$$

$$= \text{var}[U(S)]^{-1} \text{cov}[U(S), Y_S] - \text{var}[U(S)]^{-1} \text{cov}[U(S), X_S] \beta_F. \tag{63}$$

To evaluate $\text{var}[U(S)]^{-1}$, we first note that

$$\text{var}[U(S)] = \begin{pmatrix}
p_1 - p_1^2 & -p_1 p_2 & \cdots & -p_1 p_{\bar{s}-1} \\
-p_2 p_1 & p_2 - p_2^2 & \cdots & -p_2 p_{\bar{s}-1} \\
& \ddots & \ddots & \ddots \\
-p_{\bar{s}-1} p_1 & -p_{\bar{s}-1} p_2 & \cdots & p_{\bar{s}-1} - p_{\bar{s}-1}^2
\end{pmatrix};$$

direct multiplication then verifies that

$$\text{var}[U(S)]^{-1} = \begin{pmatrix}
1/p_1 + 1/p_{\bar{s}} & 1/p_{\bar{s}} & \cdots & 1/p_{\bar{s}} \\
1/p_{\bar{s}} & 1/p_2 + 1/p_{\bar{s}} & \cdots & 1/p_{\bar{s}} \\
& \ddots & \ddots & \ddots \\
1/p_{\bar{s}} & 1/p_{\bar{s}} & \cdots & 1/p_{\bar{s}-1} + 1/p_{\bar{s}}
\end{pmatrix}.$$
and row $s$ of the matrix $\text{cov}[U(S), X_S]$ is given by $p_s(\mu_s - \bar{\mu})^\top$. Thus, for $s = 1, \ldots, \tilde{S} - 1$, we have the vector elements

$$
(\text{var}[U(S)]^{-1}\text{cov}[U(S), Y_S])_s = (E[Y_s] - E[Y_S]) + \sum_{i=1}^{\tilde{S}-1} p_i(E[Y_i] - E[Y_S])/p_S = E[Y_s] - E[Y_S],
$$

and similarly row $s$ of the matrix $\text{var}[U(S)]^{-1}\text{cov}[U(S), X_S]$ is given by

$$
(\beta_s - \bar{\mu})^\top + \sum_{i=1}^{\tilde{S}-1} p_i(\mu_i - \bar{\mu})^\top/p_S = (\beta_s - \bar{\mu})^\top.
$$

Combining these terms in (63) yields (31).

**Proposition 4.4.** We derived an expression for the rows of $M$ in (64), and (39) follows from that expression. By applying expression (28) for $\beta_F$ in (40), we see that $\bar{Y}_{SEO}$ is the claimed projection.

**Corollary 4.2.** From (25) we know that if $\Sigma_s = \Sigma$ then $\beta_F = E[\beta_S]$. From (20), we get $\bar{\beta}^o = \sum_i p_i \beta_i^o = \sum_i p_i \Sigma^{1/2} \beta_i = \Sigma^{1/2} E[\beta_S] = \Sigma^{1/2} \beta_F$. Thus, $\beta_F^\top (x - \mu_s) = \bar{\beta}^o \Sigma^{-1/2} (x - \mu_s) = \bar{\beta}^o z_s$. It follows that (20) and (40) coincide because they have the same overall mean. If the distribution of $Z_s$ does not depend on $s$, then (20) satisfies demographic parity.

**Proposition 4.3.** The claim for (i) simply restates Proposition 3.1. The constraint in (ii) is demographic parity, so the optimizer follows from the definition of projection to fairness. The constraint in (v) requires $\text{cov}[\lambda^\top U(S) + \beta^\top X_S, U(S)] = 0$. Rearranging this equation, we get $\lambda = -(\text{var}[U(S)])^{-1}\text{cov}[U(S), X_S]\beta$; i.e., $\lambda = -M \beta$. Making this substitution in the form of $\bar{Y}_S$ in row (v), (41) becomes

$$
E[(Y_S - \alpha - \lambda^\top U(S) - \beta^\top X_S)^2] = E[(Y_S - \alpha - \beta^\top [X_S - M^\top U(S)])^2].
$$

Minimizing this expression over $\alpha$ and $\beta$ yields the coefficients in a linear regression of $Y_S$ on a constant $X_S - M^\top U(S)$. In light of Proposition 4.4, the optimal $\beta$ in (65) is then the coefficient on $X_S - \mu_S$ in a regression of $Y_S$ on a constant $X_S - \mu_S$. It follows from (28) that the optimal $\beta$ in (65) is therefore $\beta_F$. Because $E[U(S)] = 0$, the minimizing $\alpha$ in (65) is the $\alpha_F$ defined by (23). We have thus shown that the optimal forecast in row (v) is

$$
\bar{Y}_s = \alpha_F + \beta_F^\top (X_s - M^\top U) = \alpha_F + \lambda^\top U(s) + \beta_F^\top X_s.
$$

In case (iv), by applying (39) we see that the constraint requires $\text{cov}[Y_S - \beta^\top X_S, X_S - M^\top U(S)] = 0$, so $\beta = (\text{cov}[X_S, X_S - M^\top U(S)])^{-1} \text{cov}[Y_S, X_S - M^\top U(S)]$. Using the fact that $X_S - M^\top U(S)$
is orthogonal to $U(S)$, we get

$$
\text{cov}[X_S, X_S - M^\top U(S)]
= \text{cov}[X_S - M^\top U(S), X_S - M^\top U(S)] + \text{cov}[M^\top U(S), X_S - M^\top U(S)]
= \text{var}[X_S - M^\top U(S)],
$$

and therefore $\beta = (\text{var}[X_S - M^\top U(S)])^{-1}\text{cov}[Y_S, X_S - M^\top U(S)]$. In other words, the optimal $\beta$ in (iv) is the coefficient in a linear regression of $Y_S$ on $X_S - M^\top U(S)$. As noted in the discussion of (v), this is $\beta_F$, and it follows from $E[U(S)] = 0$ that the optimal $\alpha$ in (iv) is $\alpha_F$. 

**Proposition 4.6.** By construction, the least-squares projection of $Y_S$ onto a constant and $X_S$ is given by the pooled forecast, so

$$
Y_S = E[Y_S] + \beta^\top_{\text{Pool}}(X_S - \bar{\mu}) + \epsilon_P,
$$

for some orthogonal error $\epsilon_P$ with a variance $\sigma^2_P$ that does not depend on $\gamma$. We therefore have

$$
E[(\hat{Y}_\gamma(X_S) - Y_S)^2] = E[(\hat{Y}_\gamma(X_S) - E[Y_S] - \beta^\top_{\text{Pool}}(X_S - \bar{\mu}))^2] + \sigma^2_P
= E[((\gamma - \Lambda\delta)^\top(X_S - \bar{\mu}))^2] + \sigma^2_P,
$$

from which (i) follows.

Using the linear projection of $Y_S$ onto $(1, U(S), X_S)$ in (32), we can write

$$
Y_S = E[Y_S] + \delta^\top U(S) + \beta^\top_F(X_S - \bar{\mu}) + \epsilon,
$$

for some orthogonal error $\epsilon$ with a variance $\sigma^2_\epsilon$ that does not depend on $\gamma$. We therefore have

$$
E[(\hat{Y}_\gamma(X_S) - Y_S)^2] = E[(\gamma^\top(X_S - \bar{\mu}) - \delta^\top U(S))^2] + \sigma^2_\epsilon
= E[(\gamma^\top(X_S - \bar{\mu}) + \gamma^\top(\bar{\mu} - \bar{\mu}) - \delta^\top U(S))^2] + \sigma^2_\epsilon
= E[(\gamma^\top(X_S - \bar{\mu} + \gamma^\top\bar{\mu} - \delta^\top U(S))^2] + \sigma^2_\epsilon
= E[(\gamma^\top(X_S - \mu_S))^2] + E[((\gamma^\top M^\top - \delta^\top)U(S))^2] + \sigma^2_\epsilon,
$$

where the third equality uses (39), and the last equality uses the orthogonality of $X_S - \mu_S$ and $U(S)$. If this expression is smaller than the corresponding value with $\gamma = 0$, then (ii) must hold.

**Proposition 4.7.** We saw in the proof of Proposition 4.2 that $X_S - \mu_S$ is uncorrelated with the centered indicators $U_i(S)$. It is also uncorrelated with $V_S - \nu_S$ because

$$
E[(X_S - \mu_S)(V_S - \nu_S)] = \sum_s p_s E[(X_s - \mu_s)(V_s - \nu_s)] = 0,
$$

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Thus, other words, including \( X \) span of which expresses the elements of \( X \) so under our assumption that \( X \) and \( S \) is in the span of \( Y \) is uncorrelated with all the elements of \( O = \{1, U(S), V_S - \nu_S, U_1(S)V_S, \ldots, U_S(S)V_S\} \).

Starting from the representation of (43) as

\[
Y_S = \sum_{i=1}^{S} 1\{S = i\}\{\alpha_i + \beta_i^T X_S + \gamma_i^T V_S + \epsilon_i\},
\]

we may write

\[
Y_S = \beta_S^T (X_S - \mu_S) + \sum_{i=1}^{S} 1\{S = i\}(\alpha_i + \beta_i^T \mu_i) + \sum_{i=1}^{S} 1\{S = i\}\gamma_i^T V_S + \epsilon_S
\]

\[
= \beta_S^T (X_S - \mu_S) + \tilde{Y} + \epsilon_S,
\]

which expresses \( Y_S \) as the sum of three mutually orthogonal terms. As \( X_S - \mu_S \) is uncorrelated with \( O \), and \( \tilde{Y} \) is uncorrelated with \( X_S - \mu_S \), we may calculate the projection of \( Y_S \) onto the span of \( X_S - \mu_S \) and \( O \) by projecting \( \beta_S^T (X_S - \mu_S) \) onto \( X_S - \mu_S \) and projecting \( \tilde{Y} \) onto \( O \).

We know from (28) that the projection of \( \beta_S^T (X_S - \mu_S) \) onto \( X_S - \mu_S \) is \( \beta_F^T (X_S - \mu_S) \); in other words, including \( V_S \) and the interaction terms does not change \( \beta_F \).

For the projection of \( \tilde{Y} \) onto \( O \), let \( a_i = \alpha_i + \beta_i^T \mu_i + \gamma_i^T \nu_i \) and \( \bar{a} = \sum_i p_i a_i \). Then,

\[
\tilde{Y} = \sum_{i=1}^{S} 1\{S = i\}(\alpha_i + \beta_i^T \mu_i) + \sum_{i=1}^{S} 1\{S = i\}\gamma_i^T V_S
\]

\[
= \sum_{i=1}^{S} 1\{S = i\}(\alpha_i + \beta_i^T \mu_i) + \sum_{i=1}^{S} U_i(S)\gamma_i^T V_S + \sum_{i=1}^{S} p_i \gamma_i^T V_S
\]

\[
= \sum_{i=1}^{S} 1\{S = i\}(\alpha_i + \beta_i^T \mu_i + \gamma_i^T \nu_i) + \sum_{i=1}^{S} U_i(S)\gamma_i^T V_S + \sum_{i=1}^{S} p_i \gamma_i^T (V_S - \nu_S)
\]

\[
= \bar{a} + \sum_{i=1}^{S-1} U_i(S)(a_i - \bar{a}) + \sum_{i=1}^{S} U_i(S)\gamma_i^T V_S + \gamma_i^T (V_S - \nu_S).
\]

Thus, \( \tilde{Y} \) is in the span of \( O \), and its coefficient on \( V_S - \nu_S \) is \( \tilde{\gamma} \). With all \( \text{var}[V_S] \) having full rank, \( V_S - \nu_S \) is not spanned by the other elements of \( O \), so its coefficient \( \tilde{\gamma} \) is uniquely determined.

**Proposition 5.1.** For the first claim, we have

\[
\text{cov}[\hat{Y}_F(X_S) - Y_S, X_S - E[X_S|S]] = -E[(f_i(S) + \epsilon)(X_S - E[X_S|S])]
\]

\[
= -E[f_i(S)(X_S - E[X_S|S])] - E[\epsilon X_S] + E[\epsilon E[X_S|S]]
\]

\[
= 0 + E[\epsilon E[S][E[X_S|S]]] - E[\epsilon E[X_S|S]X_S] = 0.
\]

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For the second claim, we have

\[
E[(\hat{\gamma}(X_S) - Y_S)^2] = E[(\gamma(X_S) - f_1(S) - \epsilon)^2] \\
= E[(\gamma(X_S) - f_1(S))^2] + E[\epsilon^2].
\]

The last step uses

\[
E[(\gamma(X_S) - f_1(S))\epsilon] = E[(\gamma(X_S) - f_1(S))E[\epsilon|S]] = 0.
\]

It now follows that if \( \gamma \) reduces the expected squared forecast error then \( E[\gamma(X_S)f_1(S)] > 0 \), which implies (51) and (52). \( \square \)
This Electronic Companion covers the following topics. Section EC.1 examines how changes in bank-specific parameters affect parameters in an aggregated model. We show that in many cases these parameter externalities favor FEO over a pooled model; Section EC.2 provides supporting analysis. Section EC.3 argues that the only parameter aggregation rules satisfying some simple conditions are convex combinations, again supporting the FEO model over the pooled model. Section EC.4 provides some additional information on the data used in Section 6 and Section EC.5 provides some robustness checks supporting Section 6. Section EC.6 discusses bank heterogeneity in revenue models.

EC.1 Cross-Bank Parameter Externalities

As a consequence of aggregating bank-specific results into a single industry model, changes at one bank can affect loss forecasts at other banks, and the results are sometimes counterintuitive. In this section, we argue that these cross-bank externalities are generally more reasonable under FEO forecasts than under the pooled method.

For simplicity, we consider a setting with a single scalar feature $x$. More generally, we can think of this as a feature that is uncorrelated with all other features. We adopt the convention that this feature is nonnegative, and that higher values of $x$ are associated with higher losses. Thus, for each bank $s$ we assume $\mu_s \geq 0$ and $\beta_s \geq 0$. In reducing $\mu_s$, a bank improves its portfolio quality; in reducing $\beta_s$, a bank improves its ability to manage portfolio risk; and in reducing $\alpha_s$, a bank improves unobserved features to reduces its losses. We examine how these improvements — reductions in $\mu_s$, $\alpha_s$, and $\beta_s$ — affect stress test results for bank $s$ and other banks $l$.

We can write the FEO loss forecast (24) for bank $l$ evaluated at $X_l = x$ as

$$\hat{Y}_{F,l}(x) = \hat{Y}_{F}(x) = \sum_s p_s (\alpha_s + \beta_s \mu_s) + \beta_F (x - \bar{\mu}),$$

with $\beta_F = \sum_i p_i \sigma_i^2 \beta_i / \sum_i p_i \sigma_i^2$, as in (27). The forecast is the same for all banks $l$ because FEO satisfies equal treatment. It is now easy to see that

$$\frac{\partial \hat{Y}_F(x)}{\partial \mu_s} = p_s \beta_s - p_s \beta_F \geq 0, \quad \text{if and only if } \beta_s \geq \beta_F;$$

with $\beta_F = \sum_i p_i \sigma_i^2 \beta_i / \sum_i p_i \sigma_i^2$, as in (27). The forecast is the same for all banks $l$ because FEO satisfies equal treatment. It is now easy to see that
\[
\frac{\partial \hat{Y}_F(x)}{\partial \alpha_s} = p_s \geq 0; \quad (68)
\]
and
\[
\frac{\partial \hat{Y}_F(x)}{\partial \beta_s} = p_s \mu_s + (x - \bar{\mu}) p_s \sigma_s^2 / \sum_i p_i \sigma_i^2 \geq 0, \quad \text{if } x > \bar{\mu}. \quad (69)
\]

In (67) we see that if bank \( s \) has above-average (relative to \( \beta_F \)) sensitivity to feature \( x \), then reducing its average exposure to that feature \( \mu_s \) reduces loss forecasts for all banks. Equation (68) shows a similar overall benefit if bank \( s \) improves on the other dimensions captured by \( \alpha_s \). In (67) we see that an improvement in risk management at bank \( s \), corresponding to a reduction in \( \beta_s \), reduces loss forecasts at above-average levels of \( x \). If \( x \) is part of the stress scenario, then large values of \( x \) are particularly relevant.

The directional effects in (67)–(69) are fairly simple and reasonable, considering that cross-bank effects are inevitable in an industry model. If the industry improves its performance (perhaps because of improvements at one bank) we generally expect loss forecasts to decrease. (A decrease in a forecast corresponds to a positive derivative because we are considering a decrease \( \mu_s, \alpha_s, \) or \( \beta_s \).) Counterparts to (67)–(69) continue to hold if we replace \( \beta_F \) in (66) with any convex combination of the \( \beta_s \), as in the WATE model. However, the pooled method behaves quite differently.

The pooled forecast \( \hat{Y}_P(x) \) can be written in the same form as (66) but with \( \beta_F \) replaced by \( \beta_{Pool} \) in (11). We now get
\[
\frac{\partial \hat{Y}_P(x)}{\partial \mu_s} = p_s (\beta_s - \beta_{Pool}) + (x - \bar{\mu}) \frac{\partial \beta_{Pool}}{\partial \mu_s}. \]
The sign of the last term is not determined by a simple condition, so the overall directional effect is difficult to predict. The sign of
\[
\frac{\partial \hat{Y}_P(x)}{\partial \alpha_s} = p_s + p_s \left( \frac{(\mu_s - \bar{\mu})(x - \bar{\mu})}{\sum_i p_i \sigma_i^2 + \text{var}(\mu)} \right) \beta_s,
\]
depends on the magnitudes of \( \mu_s \) and \( x \), relative to \( \bar{\mu} \). For the sensitivity to \( \beta_s \), we can write
\[
\frac{\partial \hat{Y}_P(x)}{\partial \beta_s} = p_s \mu_s + (x - \bar{\mu}) \frac{\partial \beta_{Pool}}{\partial \beta_s}, \quad \frac{\partial \beta_{Pool}}{\partial \beta_s} = \frac{p_s (\sigma_s^2 + \mu_s (\mu_s - \bar{\mu}))}{\sum_i p_i \sigma_i^2 + \text{var}(\mu)}.
\]
Among the most troubling aspects of the pooled model is that the last term could be negative: a reduction in \( \beta_s \) could produce an increase in \( \beta_{Pool} \). In particular, \( \sigma_s^2 + \mu_s (\mu_s - \bar{\mu}) \) is negative for a bank with below-average exposure to feature \( x \) (so \( \mu_s < \bar{\mu} \)) and low variability \( \sigma_s^2 \) in this exposure. Under the pooled model, it is therefore possible for an improvement in risk management at one bank (a reduction in \( \beta_s \)) to produce an \textit{increase} in loss forecasts at all banks.
Table EC.1: Sensitivity of results for bank \( l \) in response to a decrease in parameter \( \mu_s \), \( \alpha_s \), or \( \beta_s \) for bank \( s \). Sensitivities shown are for predicted loss \( \hat{Y}(x) \) (top), mean predicted loss \( E[\hat{Y}(X_l)] \) (middle), and the bias \( E[\hat{Y}(X_l)−Y_l] \).

The top panel of Table EC.1 shows sufficient conditions for positive sensitivities of \( \hat{Y}_F(x) \) and \( \hat{Y}_P(x) \). The middle and bottom panels show corresponding results for the expected forecasts \( E[\hat{Y}] = E[\hat{Y}(X_l)] \) and for the bias \( E[\hat{Y}(X_l)−Y_l] \). Supporting details for the second and third cases are provided in Section EC.2. We have tried to provide simple sufficient conditions, and in most cases the conditions are not necessary. All of the conditions for FEO extend to WATE with \( \beta_F \) replaced by the weighted average coefficient.

Some counterintuitive and undesirable cases can arise at empirically plausible parameter values. For example, in equation (72) we derive an expression for \( \partial E[\hat{Y}_P(X_l)]/\partial \alpha_s \). Using estimated parameters for the credit card data in Section 6, we find that this derivative is negative when \( l \) is Citigroup and \( s \) is JPMorgan Chase. In other words, an improvement at JPMorgan Chase would result in a higher expected loss forecast at Citigroup under the pooled model.

The bias sensitivities in Table EC.1 are more complicated than the other cases because the bias involves the difference between the predicted and actual loss rates. A reduction in the predicted loss rate can increase or decrease bias, depending on whether the initial forecast is too low or too high.
EC.2 Sensitivity Analysis

This section provides supporting details for Section EC.1, particularly the conclusions summarized in the middle and bottom panels of Table EC.1. We begin with an analysis of forecast bias that is of independent interest.

EC.2.1 Forecast Bias

If losses at different banks are described by different models, then forecast bias becomes inevitable when we apply a single model to all banks. But the distribution of bias across banks may differ under different choices of the single model.

Let $\hat{Y}_s$ be any of the forecasts for bank $s$ in Table 4.1, and, as in (1), let $Y_s$ denote the actual loss rate for bank $s$. Both $\hat{Y}_s$ and $Y_s$ are evaluated at $X_s$. Define the forecast bias for bank $s$ to be

$$\text{bias}(s) = E[\hat{Y}_s - Y_s].$$

(70)

The expectation integrates over the distribution of the error $\epsilon_s$ in (1) and the features $X_s$.

**Proposition EC.2.1.** For each forecast in Table 4.1 the bias is as follows.

(i) Pooled: $\text{bias}(s) = E[Y_S] - E[Y_s] + \beta^T_{Pool}(\mu_s - \bar{\mu})$;

(ii) PTF in (18): $\text{bias}(s) = E[Y_S] - E[Y_s]$;

(iii) Conditional expectation: $\text{bias}(s) = E[\hat{Y}_C(X_s)] - E[Y_s]$;

(iv) FEO: $\text{bias}(s) = E[Y_S] - E[Y_s] + \beta^T_F(\mu_s - \bar{\mu})$;

(v) SEO: $\text{bias}(s) = E[Y_S] - E[Y_s]$.

**Proof.** For (i), we have, using the definition of $\alpha_{Pool}$ in (12),

$$E[\hat{Y}_s - Y_s] = E[\alpha_{Pool} + \beta^T_{Pool}X_s - Y_s] = (E[Y_S] - \beta^T_{Pool}\bar{\mu}) + \beta^T_{Pool}\mu_s - E[Y_s] = E[Y_S] - E[Y_s] + \beta^T_{Pool}(\mu_s - \bar{\mu}).$$

For the PTF forecast, (17) and (18) yield

$$E[\hat{Y}_s] = \alpha^o = \sum_s p_s \alpha^o_s = \sum_s p_s E[Y_s] = E[Y_S],$$

and the bias in (ii) follows. The expression in (iii) holds by definition. The argument for (iv) is the same as the argument for (i). The bias in (v) follows from (iv) because we see from (38) that the SEO forecast for bank $s$ subtracts $\beta^T_F(\mu_s - \bar{\mu})$ from the FEO forecast. \qed
In every case of Proposition EC.2.1, the average bias $\sum_s p_s \text{bias}(s)$ is zero, but the methods differ in how they distribute bias across banks. We saw previously that the PTF and SEO methods go the farthest in equalizing differences; we now see that the bias for each of these methods is the difference $E[Y_S] - E[Y_s]$ between the average loss rate for all banks and the average for an individual bank.

Using the relationship $\beta_{Pool} = \beta_F + \Lambda \delta$ from (30), we see that the difference between the expressions in (i) and (iv) is

$$\text{bias}_{Pool}(s) - \text{bias}_{FEO}(s) = \delta^T \Lambda^T (\mu_s - \bar{\mu}).$$

In light of the discussion in Section 4.2, this difference is the expected disparate impact on bank $s$ of using the pooled model.

**EC.2.2 Improvement in Intercept $\alpha_s$**

By taking the expectation of (66), we get

$$E[\hat{Y}_F(X_l)] = \sum_s p_s (\alpha_s + \beta_s \mu_s) + \beta_F (\mu_l - \bar{\mu}),$$

and the same holds for the expected pooled forecast with $\beta_F$ replaced by $\beta_{Pool}$. It follows that, for any banks $s$ and $l$,

$$\frac{\partial E[\hat{Y}_F(X_l)]}{\partial \alpha_s} = p_s > 0.$$

In other words, all expected forecasts decrease following a reduction in $\alpha_s$.

In contrast, for the pooled model we get

$$\frac{\partial E[\hat{Y}_P(X_l)]}{\partial \alpha_s} = p_s - \frac{\partial \text{cov}(\alpha_S, \mu_S)}{\partial \alpha_s} \frac{\beta_s (\bar{\mu} - \mu_l)}{s} = p_s + p_s \frac{(\mu_s - \bar{\mu})(\mu_l - \bar{\mu})}{\sum_s p_s \sigma_s^2 + \text{var}(\mu_S) \beta_s}. $$

Bank $s$ benefits from its reduction of $\alpha_s$, in the sense that the derivative with $l = s$ is positive. For $l \neq s$, the sign of (72) does not admit a simple description. In particular, it may be negative when $\mu_s$ and $\mu_l$ are on opposite sides of $\bar{\mu}$, meaning that one bank’s loans are riskier than average and the other bank’s loans are less risky than average.

For the bias under FEO we have

$$\frac{\partial \text{bias}_F(l)}{\partial \alpha_s} = p_s - 1 \{l = s\}$$

It is then immediate that

$$\frac{\partial \text{bias}_F(l)}{\partial \alpha_s} > 0 \text{ if } l \neq s \quad \text{and} \quad \frac{\partial \text{bias}_F(s)}{\partial \alpha_s} < 0.$$
The direction of change makes sense. If the bias for a bank is positive, meaning that the industry model overestimates its losses, then improvements at other banks will reduce loss forecasts and thus reduce the bias. The bank’s own improvements will increase the bias by reducing the bank’s own losses by more than they reduce the model’s forecasts. The situation is reversed for a bank with a negative bias.

However, for the pooled regression method,

$$\frac{\partial \text{bias}_P(l)}{\partial \alpha_s} = ps + ps \sum_s ps \sigma_s^2 + \text{var}(\mu_S) \beta_s - 1\{l = s\},$$

and the direction of change is unclear.

**EC.2.3 Improvement in Loan Quality**

Now suppose bank $s$ improves the quality of its loan portfolio, resulting in a smaller $\mu_s$. This has no effect on $\beta_F$, which makes sense — changing one bank’s loan quality should not change the sensitivity of losses to loan quality. However, it is evident from (11) that $\beta_{Pool}$ does change with $\mu_s$.

Under FEO, the mean the mean predicted loss rate satisfies

$$\frac{\partial \hat{E}Y_F(X_l)}{\partial \mu_s} = ps \beta_s + \beta_F(1\{l = s\} - ps),$$

which is always positive if $l = s$. This means that an improvement in bank $l$’s loan quality (a reduction in $\mu_l$) reduces bank $l$’s mean predicted losses. In the pooled model,

$$\frac{\partial \hat{E}Y_P(X_l)}{\partial \mu_s} = ps \beta_s + \beta_{Pool}(1\{l = s\} - ps) + (\mu_s - \bar{\mu}) \frac{\partial \beta_{Pool}}{\partial \mu_s};$$

this expression could be negative, even with $l = s$, meaning that a bank could be penalized (through a higher mean predicted loss rate) as a result of improving its loan quality.

The sensitivity of the bias under FEO is given by

$$\frac{\partial \text{bias}_F(l)}{\partial \mu_s} = (1\{l = s\} - ps)(\beta_F - \beta_s);$$

in particular, the bias for bank $l$ moves in opposite directions with respect to changes in $\mu_l$ and $\mu_s$, $s \neq l$. Suppose industry model overestimates bank $l$’s losses, in the sense that the bias is positive, and suppose the industry model overestimates bank $l$’s sensitivity to loan quality, in the sense that $\beta_F > \beta_l$. Then bank $l$ will benefit (in the sense of reducing the bias) from improving its loan quality by reducing $\mu_l$.

For the pooled regression,

$$\frac{\partial \text{bias}_P(l)}{\partial \mu_s} = (\beta_{Pool} - \beta_s)(1\{l = s\} - ps) + (\mu_s - \bar{\mu}) \frac{\partial \beta_{Pool}}{\partial \mu_s}.$$

The sign of this expression does not admit a simple condition.
EC.2.4 Improvement in Loan Management

Now suppose bank $s$ improves its abilities in loan management, resulting in a reduction in $\beta_s$. The mean predicted loss rate under FEO satisfies
\[
\frac{\partial E[\hat{Y}_F(X_l)]}{\partial \beta_s} = p_s (\mu_s + \mu_l - \bar{\mu}),
\]
and is positive if $\mu_s + \mu_l > \bar{\mu}$. In the pooled model
\[
\frac{\partial E[\hat{Y}_P(X_l)]}{\partial \beta_s} = p_s \mu_s + \frac{p_s (\sigma^2_s + \mu_s (\mu_s - \bar{\mu}))}{\sum_i p_i (\sigma^2_i + \mu_i (\mu_i - \bar{\mu}))} (\mu_l - \bar{\mu}),
\]
so $[\sigma^2_s + \mu_s (\mu_s - \bar{\mu})](\mu_l - \bar{\mu}) > 0$ is a sufficient condition for the sensitivity to be positive. Regardless of the value of $\mu_l$, the sensitivity is positive for all sufficiently large $\mu_s$.

For $l \neq s$, the sensitivity of the bias for bank $l$ with respect to $\beta_s$ equals the sensitivity of the mean predicted loss because the actual expected loss $E[Y_l]$ is unaffected by $\beta_s$. We therefore focus on the case $l = s$. Under FEO,
\[
\frac{\partial \text{bias}_F(s)}{\partial \beta_s} = (p_s - 1) \mu_s + p_s (\mu_s - \bar{\mu}),
\]
which is guaranteed to be negative if $\mu_s < \bar{\mu}$. Under the pooled model, the sign of
\[
\frac{\partial \text{bias}_P(s)}{\partial \beta_s} = (p_s - 1) \mu_s + \frac{p_s (\sigma^2_s + \mu_s (\mu_s - \bar{\mu}))}{\sum_i p_i (\sigma^2_i + \mu_i (\mu_i - \bar{\mu}))} (\mu_s - \bar{\mu})
\]
does not admit a simple characterization.

EC.3 Convex Combinations of Coefficients

Equation (13) aggregates the individual scalar slopes $\beta_s$ into a single value. We can generalize this perspective and ask what properties we would like in an aggregation function, meaning a function $f : \mathbb{R}^S \rightarrow \mathbb{R}$,
\[
\beta_s = f(\beta_1, \ldots, \beta_S),
\]
that combines bank-specific coefficients $\beta_s$ into an “industry” parameter $\beta_s$.

We consider the following properties:

(i) $f(kb_1, \ldots, kb_S) = kf(b_1, \ldots, b_S)$, for all $k, b_1, \ldots, b_S \in \mathbb{R}$;

(ii) $f(b, \ldots, b) = b$, for at least one nonzero $b \in \mathbb{R}$;

(iii) $b_s > 0$, for all $s$, implies $f(b_1, \ldots, b_S) \geq 0$;

(iv) $f$ is differentiable at zero.
Property (i) is needed for the aggregation to perform sensibly under a change of units in the measurement of $X_s$: if we divide each $X_s$ by $k$, each $\beta_s$ increases by a factor of $k$, and it is natural to require that $\beta_s$ scale accordingly. Properties (ii) and (iii) are also very modest requirements. Property (iv) is harder to motivate but not unreasonable. These properties constrain the aggregation function as follows:

**Proposition EC.3.1.** If (i)–(iv) hold, then $f(\beta_1, \ldots, \beta_S)$ is a convex combination of its arguments.

**Proof.** Fix $\beta \in \mathbb{R}^S$. Let $g(t) = f(t\beta)$. By condition (iv), $g'(0) = \beta^T f'(0)$. Condition (i) and (ii) imply $g(t) = tf(\beta)$, so $g'(t) = f(\beta)$ for any $t$. Thus, $f(\beta) = g'(0) = \beta^T f'(0) = \sum_{i=1}^S f'_i(0)\beta_i$. Condition (ii) now implies $\sum_{i=1}^S f'_i(0) = 1$, and condition (iii) implies $f'_i(0) \geq 0$, for all $i$. Thus, $f(\beta) = \sum_{i=1}^S f'_i(0)\beta_i$ is a convex combination of the components of $\beta$. 

The scalar FEO coefficient in (27) is a convex combination of the bank-specific coefficients $\beta_s$, but the pooled coefficient (13) is generally not. This property of the FEO model extends to the multivariate case under additional conditions. If all the bank-specific covariance matrices $\Sigma_s$, $s = 1, \ldots, S$, coincide, then in (25) we get $\beta_F = E[\beta_S] = \sum_s p_s \beta_s$. If all $\Sigma_s$ are diagonal (but not necessarily identical), then the representation of the scalar FEO coefficient in (27) applies to each coordinate of $\beta_F$. If all $\Sigma_s$ have the same eigenvectors, then we can transform the original features $X_s$ into uncorrelated features using principal components. Using these transformed features, each coordinate of $\beta_F$ is a convex combination of bank-specific coefficients.

**EC.4 Additional Information on Empirical Analysis**

Table EC.1 lists the bank holding companies included in our empirical analysis and the symbols we use to refer to them. The companies are listed in order of size by total assets.
| Ticker | Bank Name                                                   |
|--------|-------------------------------------------------------------|
| JPM    | JPMORGAN CHASE & CO.                                       |
| BAC    | BANK OF AMERICA CORPORATION                                |
| C      | CITIGROUP INC.                                             |
| WFC    | WELLS FARGO & COMPANY                                      |
| GS     | GOLDMAN SACHS GROUP, INC., THE                             |
| MS     | MORGAN STANLEY                                             |
| SCHW   | CHARLES SCHWAB CORPORATION, THE                            |
| USB    | U.S. BANCORP                                               |
| PNC    | PNC FINANCIAL SERVICES GROUP, INC., THE                     |
| TFC    | TRUIST FINANCIAL CORPORATION                               |
| TD     | TD GROUP US HOLDINGS LLC                                   |
| BK     | BANK OF NEW YORK MELLON CORPORATION, THE                   |
| COF    | CAPITAL ONE FINANCIAL CORPORATION                          |
| STT    | STATE STREET CORPORATION                                   |
| HSBC   | HSBC NORTH AMERICA HOLDINGS INC.                           |
| SIVB   | SVB FINANCIAL GROUP                                        |
| FITB   | FIFTH THIRD BANCORP                                        |
| USAA   | UNITED SERVICES AUTOMOBILE ASSOCIATION                     |
| BMO    | BMO FINANCIAL CORP.                                        |
| CFG    | CITIZENS FINANCIAL GROUP, INC.                             |
| AXP    | AMERICAN EXPRESS COMPANY                                   |
| KEY    | KEYCORP                                                    |
| NTRS   | NORTHERN TRUST CORPORATION                                 |
| ALLY   | ALLY FINANCIAL INC.                                        |
| AMP    | AMERIPRISE FINANCIAL, INC.                                 |
| RY     | RBC US GROUP HOLDINGS LLC                                  |
| HBAN   | HUNTINGTON BANCSHARES INCORPORATED                         |
| RF     | REGIONS FINANCIAL CORPORATION                              |
| MUFG   | MUFG AMERICAS HOLDINGS CORPORATION                         |
| BCS    | BARCLAYS US LLC                                            |
| SAN    | SANTANDER HOLDINGS USA, INC.                               |
| MTB    | M&T BANK CORPORATION                                       |
| BNPQY  | BNP PARIBAS USA, INC.                                      |
| DB     | DB USA CORPORATION                                         |
| DFS    | DISCOVER FINANCIAL SERVICES                                 |

Table EC.1: Symbols and names of included bank holding companies.

We construct the loss rates, past due rates, and allowance rates using the entries in FR Y-9C forms outlined in Table EC.2.
| Variables   | Loan Types | 2007Q1 – Present | 2003Q1 – 2006Q4 |
|------------|------------|------------------|-----------------|
| Loan Amount| CC         | BHCKB538         | BHCKB538        |
|            | FL         | BHDM5367         | BHDM5367        |
|            | CRE        | Owned: BHCKF160  | BHD1480         |
|            | CRE        | Other: BHCKF161  | BHD1480         |
|            | CI         | BHCK1763         | BHCK1763        |
|            | Total      | BHCK2122         | BHCK2122        |
| Charge-Offs| CC         | BHCKB514         | BHCKB514        |
|            | FL         | BHCKC234         | BHCKC234        |
|            | CRE        | Owned: BHCKC895  | BHCK3590        |
|            | CRE        | Other: BHCKC897  | BHCK3590        |
|            | CI         | BHCK4645         | BHCK4645        |
| Recoveries | CC         | BHCKB515         | BHCKB515        |
|            | FL         | BHCKC217         | BHCKC217        |
|            | CRE        | Owned: BHCKC896  | BHCK3591        |
|            | CRE        | Other: BHCKC898  | BHCK3591        |
|            | CI         | BHCK4617         | BHCK4617        |
| Past Due: 30-89 days and accruing | CC    | BHCKB575         | BHCKB575        |
|            | FL         | BHCKC236         | BHCKC236        |
|            | CRE        | Owned: BHCKF178  | BHCK3502        |
|            | CRE        | Other: BHCKF179  | BHCK3502        |
|            | CI         | BHCK1606         | BHCK1606        |
| Past Due: 90 days and accruing | CC    | BHCKB576         | BHCKB576        |
|            | FL         | BHCKC237         | BHCKC237        |
|            | CRE        | Owned: BHCKF180  | BHCK3503        |
|            | CRE        | Other: BHCKF181  | BHCK3503        |
|            | CI         | BHCK1607         | BHCK1607        |
| Past Due: non-accrual | CC     | BHCKB577         | BHCKB577        |
|            | FL         | BHCKC229         | BHCKC229        |
|            | CRE        | Owned: BHCKF182  | BHCK3504        |
|            | CRE        | Other: BHCKF183  | BHCK3504        |
|            | CI         | BHCK1608         | BHCK1608        |

Table EC.2: Loan variables and FR Y-9C form correspondence.

**EC.5 Robustness Checks**

**EC.5.1 Pre-COVID Data and Allowance Rate**

We repeat the analysis of Section 6, limiting the data to 2001–2019. This serves two purposes. It addresses the possibility that our results are driven by a few extreme values during the COVID period 2020–2021. It also accounts for a change in how banks measure allowances (the Current Expected Credit Losses methodology) that began to take effect at the end of 2019. We also consider adding allowance rate as another proxy for portfolio risks. Tables EC.1 and EC.2
report the results for heterogeneity tests and differences of the parameter estimates under this setting. The evidence for heterogeneity and its impact is generally at least as strong using the pre-COVID data as using data through 2021.
| Covariance Estimation | $\alpha$ | $\beta_{PDR}$ | $\beta_{AR}$ | $\gamma$ |
|-----------------------|-----------|--------------|--------------|---------|
| bank clustered        | 0.00 0.00 | 0.00 0.00    | 0.00 0.00    | 0.00    |
| time clustered        | 0.00 0.00 | 0.00 0.00    | 0.00 0.00    | 0.00    |
| bank clustered        | 0.00 0.00 | 0.00 0.00    | 0.00 0.00    | 0.00    |
| time clustered        | 0.00 0.00 | 0.00 0.00    | 0.00 0.00    | 0.00    |
| bank clustered        | 0.00 0.00 | 0.00 0.00    | 0.00 0.00    | 0.00    |
| time clustered        | 0.00 0.00 | 0.00 0.00    | 0.00 0.00    | 0.00    |

Table EC.1: Heterogeneity tests using pre-COVID data with allowance rate as an additional proxy for banks’ portfolio risks.
| Loan Type | β_Pool | β_F | p-value | Pool | β_Pool | β_F | p-value | Pool | β_Pool | β_F | p-value | Pool | β_Pool | β_F | p-value |
|-----------|--------|-----|---------|------|--------|-----|---------|------|--------|-----|---------|------|--------|-----|---------|
| CC        | 0.781  | 0.838 | 0.006*** | FL   | 0.045  | 0.061 | 0.002*** | CRE  | 0.135  | 0.133 | 0.002 | 0.449 |
| CI        | 0.207  | 0.217 | 0.069*   | FL   | 0.010  | 0.021 | 0.002*** | CRE  | 0.095  | 0.068 | 0.026 | 0.052* |
|           | 0.178  | 0.196 | -0.011   | FL   | 0.037  | 0.052 | 0.002*** | CRE  | 0.129  | 0.126 | 0.002 | 0.477 |
|           | 0.155  | 0.155 | 0.001    | FL   | 0.006  | 0.018 | -0.012   | CRE  | 0.097  | 0.078 | 0.019 | 0.035* |
|           | 0.140  | 0.153 | -0.012   | FL   | 0.009  | 0.029 | -0.032   | CRE  | 0.129  | 0.126 | 0.002 | 0.477 |
|           | 0.155  | 0.155 | 0.001    | FL   | 0.006  | 0.018 | -0.012   | CRE  | 0.097  | 0.078 | 0.019 | 0.035* |
|           | 0.140  | 0.153 | -0.012   | FL   | 0.009  | 0.029 | -0.032   | CRE  | 0.129  | 0.126 | 0.002 | 0.477 |
|           | 0.155  | 0.155 | 0.001    | FL   | 0.006  | 0.018 | -0.012   | CRE  | 0.097  | 0.078 | 0.019 | 0.035* |
|           | 0.140  | 0.153 | -0.012   | FL   | 0.009  | 0.029 | -0.032   | CRE  | 0.129  | 0.126 | 0.002 | 0.477 |
|           | 0.155  | 0.155 | 0.001    | FL   | 0.006  | 0.018 | -0.012   | CRE  | 0.097  | 0.078 | 0.019 | 0.035* |
|           | 0.140  | 0.153 | -0.012   | FL   | 0.009  | 0.029 | -0.032   | CRE  | 0.129  | 0.126 | 0.002 | 0.477 |
|           | 0.155  | 0.155 | 0.001    | FL   | 0.006  | 0.018 | -0.012   | CRE  | 0.097  | 0.078 | 0.019 | 0.035* |
|           | 0.140  | 0.153 | -0.012   | FL   | 0.009  | 0.029 | -0.032   | CRE  | 0.129  | 0.126 | 0.002 | 0.477 |
|           | 0.155  | 0.155 | 0.001    | FL   | 0.006  | 0.018 | -0.012   | CRE  | 0.097  | 0.078 | 0.019 | 0.035* |
|           | 0.140  | 0.153 | -0.012   | FL   | 0.009  | 0.029 | -0.032   | CRE  | 0.129  | 0.126 | 0.002 | 0.477 |
|           | 0.155  | 0.155 | 0.001    | FL   | 0.006  | 0.018 | -0.012   | CRE  | 0.097  | 0.078 | 0.019 | 0.035* |
|           | 0.140  | 0.153 | -0.012   | FL   | 0.009  | 0.029 | -0.032   | CRE  | 0.129  | 0.126 | 0.002 | 0.477 |
|           | 0.155  | 0.155 | 0.001    | FL   | 0.006  | 0.018 | -0.012   | CRE  | 0.097  | 0.078 | 0.019 | 0.035* |
|           | 0.140  | 0.153 | -0.012   | FL   | 0.009  | 0.029 | -0.032   | CRE  | 0.129  | 0.126 | 0.002 | 0.477 |
|           | 0.155  | 0.155 | 0.001    | FL   | 0.006  | 0.018 | -0.012   | CRE  | 0.097  | 0.078 | 0.019 | 0.035* |
|           | 0.140  | 0.153 | -0.012   | FL   | 0.009  | 0.029 | -0.032   | CRE  | 0.129  | 0.126 | 0.002 | 0.477 |
|           | 0.155  | 0.155 | 0.001    | FL   | 0.006  | 0.018 | -0.012   | CRE  | 0.097  | 0.078 | 0.019 | 0.035* |
|           | 0.140  | 0.153 | -0.012   | FL   | 0.009  | 0.029 | -0.032   | CRE  | 0.129  | 0.126 | 0.002 | 0.477 |
|           | 0.155  | 0.155 | 0.001    | FL   | 0.006  | 0.018 | -0.012   | CRE  | 0.097  | 0.078 | 0.019 | 0.035* |
EC.6 Revenue Models

The Federal Reserve’s stress testing framework includes models of revenues as well as models of losses. We have focused on loan portfolio loss models because they fit most clearly within the Fed’s policy of equal treatment and its preference for industry models. In this section, we show that the heterogeneity documented for loss models in Section 6 extends to revenue models, referred to in the Fed’s framework as models of pre-provision net revenue or PPNR.

The Fed uses a suite of PPNR models to forecast difference sources of revenue. These models differ from the loss models in at least two important respects: they are typically autoregressive (AR) models, and, unlike the portfolio loss models, they do not rule out bank fixed effects; see [9]. This feature points to the presence of unmodeled bank heterogeneity in the revenue forecasts. Our goal in this section is to check for heterogeneity in a simple PPNR model and to compare coefficient estimates in the pooled and fixed-effect models.

We consider the modeling of trading revenue, which is one of the PPNR components in the Fed’s framework. We compare AR models with fixed-effects,

\[ Y_{s,t}^{FE} = \alpha_s + \rho_{FE} Y_{s,t-1} + \beta_{FE} X_{s,t} + \gamma_{FE} VIX_t + \epsilon_{s,t}^{FE} \]  

or pooled without fixed effects,

\[ Y_{s,t}^{P} = \alpha_P + \rho_{P} Y_{s,t-1} + \beta_{P} X_{s,t} + \gamma_{P} VIX_t + \epsilon_{s,t}^{P}. \]  

In both models, \( Y \) is trading revenue normalized by total trading assets; this choice of normalization is consistent with [9]. For the AR term, we use either a one-quarter lag \( Y_{s,t-1} \) or a four-quarter average lag, in which case we use \( 1/4 \sum_{j=1}^{4} Y_{s,t-j} \) in place of \( Y_{s,t-1} \) in (73) and (74) to capture average performance over the past year.

For \( X_{s,t} \) we use the size of bank \( s \) in quarter \( t \), as measured by the log of total assets. For the macro variable, we use the \( VIX_t \), the market volatility index taken from the Federal Reserve’s stress testing historical dataset. Market volatility is expected to have a direct impact on trading revenue, and indeed we observe a more significant effect of \( VIX_t \) than MacroPC\(_t\) (from Section 6) in this setting.

Models (73) and (74) differ in their intercept terms: (73) captures banks’ fixed effects, but (74) requires the same intercept across all banks. The fixed-effect coefficient estimates \( \rho_{FE}, \beta_{FE} \) and \( \gamma_{FE} \) are identical to those of FEO; the methods differ in their forecasts: FEO uses the average fixed effect, rather than bank-specific fixed effects in its forecasts.

As in Section 6, we use Y-9C financial reporting data for the top 35 banks by total asset size (as of year-end 2021), and we include only banks with at least 18 years of data to ensure our panel is mostly balanced. We fit the models using weighted least squares, weighting each
observation by quarter stress and bank asset balance. As in [53], we choose the weights so that, for the same asset level, the quarters with the highest market volatility get twice the weight as the quarters with the lowest market volatility. As in the AR models in [32], we cluster standard errors by time.

Table EC.1 reports the results. The first two columns correspond to the one-quarter AR setting, and the last two columns are the one-year-average AR setting. The numbers in parentheses are standard errors.

As expected, both the lagged response and the VIX term are statistically significant. The volatility term is more significant in the one-year-average AR setting, presumably because the market environment changes less over one quarter, and its effect is partly captured by the lagged response.

The pooled and fixed-effect methods result in different estimates of \( \rho \), and the differences in estimates in all settings are more than 20%. We tested the hypotheses that all banks share the same (i) intercept, (ii) AR coefficient term, (iii) VIX coefficient, or (iv) coefficients of total asset size, following the approach used in Section 6.2; all tests strongly reject that banks have identical model coefficients, with \( p \)-values less than 0.01, extending what we found for loss models. We have also examined the forecasts of trading interest income and trading interest expense (two other components of PPNR), and observed similar patterns across all three components.

|                | Normalized Trading Revenue |
|----------------|---------------------------|
| AR             | 0.466***           | 0.588***           | 0.406***           | 0.596***           |
|                | (0.134)           | (0.111)           | (0.078)           | (0.094)           |
| VIX            | -0.024**          | -0.022*           | -0.027***         | -0.022***         |
|                | (0.011)           | (0.012)           | (0.008)           | (0.008)           |
| Log Assets     | -0.003            | -0.007***         | 0.000             | -0.006***         |
|                | (0.004)           | (0.001)           | (0.003)           | (0.002)           |
| Bank FE        | included          | –                 | included          | –                 |

\* \( p < 0.1 \); \** \( p < 0.05 \); \*** \( p < 0.01 \)

Table EC.1: Coefficient estimates for the AR models. The first two columns use a one-quarter lag, and the last two use a one-year average lag. Columns 1 and 3 correspond to AR models with bank fixed effects, and columns 2 and 4 are pooled AR models.