Numerical assessment for Poisson image blending problem using MSOR iteration via five-point Laplacian operator

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Abstract. The demand for image editing in the field of image processing has been increased throughout the world. One of the most famous equations for solving image editing problem is Poisson equation. Due to the advantages of the Successive Over Relaxation (SOR) iterative method with one weighted parameter, this paper examined the efficiency of the Modified Successive Over Relaxation (MSOR) iterative method for solving Poisson image blending problem. As we know, this iterative method requires two weighted parameters by considering the Red-Black ordering strategy, thus comparison of Jacobi, Gauss-Seidel and MSOR iterative methods in solving Poisson image blending problem is carried out in this study. The performance of these iterative methods to solve the problem is examined through assessing the number of iterations and computational time taken. Based on the numerical assessment over several experiments, the findings had shown that MSOR iterative method is able to solve Poisson image blending problem effectively than the other two methods which it requires fewer number of iterations and lesser computational time.

1. Introduction
Gradient domain techniques has becoming an important tool in image processing. For instance in seamless cloning, seamless stitching and shadow removal. According to [1], by employing the gradient domain approach, eventually a large sparse linear system will be solved by linear solvers which was generated from Poisson equation. The idea of relating Poisson equation into image processing was motivated from [2]. In their research, they named this method as Poisson Image Editing (PIE), where this method is applying gradient domain approach. Besides, [3] provided the reason that PDEs are able to analyze the image in continuous space, thus it had motivated the mathematicians and engineers to formulate new methods in image processing problem.

In this paper, we are focusing in Poisson image blending problem. The intention of this study is to solve the problem by applying five-point Laplacian operator based on finite difference approach and then the linear system which is generated from Poisson equation is solved by selected linear solvers, namely the Jacobi, Gauss-Seidel and MSOR iterative methods. The performance of these iterative methods via five-point Laplacian operator is examined through the number of iterations used and the computational time taken when solving the Poisson image blending problem. In addition, MSOR iterative methods and Poisson equation had been used to solve for other problems as well, [4,5,6].
2. Related Work

Image processing problems based Poisson equations had well spread since the invention from [2]. Researchers from mainly computer science and mathematics have discovered numerous methods to improve the original method (PIE) while solving the Poisson image editing problems. In addition to Poisson based, other image processing problems, for instance, [7,8,9,10,11] are also being developed. There are also many applications of image processing in other fields, for instance [12,13]. Here, we briefly review some selected related works in this research.

A similar yet different idea to implement PIE has been proposed by [14] to overcome the issue of colour inconsistency in the original PIE method. This suggested method added an additional inner Dirichlet boundary condition and enlarge the Laplacian values when solving the Poisson equation with Dirichlet boundary condition. The output images generated by this modified Poisson equation looks more natural without seams.

Apart from that, for the ease of editing the image based Poisson equation, [15] proposed to optimize the boundary condition of the selected region in order to achieve seamless image composition. A blended guidance field is defined to preserve the fractional object boundary as well. Meanwhile, [16] implemented an interpolation based method with optimized gradient. The PIE method is solving a large sparse linear system generated by Poisson equation, while in this method, this step is avoided since the cloning area is reconstructed with optimizes gradient. The experimental results obtained by this approach is more satisfying compared to other methods.

Instead of manipulating the pixel intensities, the gradient of an image is more preferable for image editing. However, [17] found that gradient domain approach suffers from the colour bleeding and bleeding artefacts problems. Thus, a modified Poisson blending (MPB) method is invented which is concerning the pixels at the boundary of both target and source images to overcome the bleeding artefacts and added an alpha compositing step to tackle the colour bleeding problem. Experimental results had shown that this method worked well in minimizing the problems.

Besides, [18] proposed another efficient Poisson image editing method to minimize the processing time while obtaining the most satisfying results as compared to other methods. This method applied image pyramid and divide-and-conquer method to obtain the unique solution of the Poisson equation. In this method, the unknown region is partitioned into thin slice and small square blocks. The results had shown that it reduced the processing time.

Another approach of solving Poisson image editing problem had been discovered by [19], which is known as Fourier solver. The advantage of this method is that the desired region from source image is selected by an algorithm instead of manual selection and with this surplus, the composition time is shorter compared to other methods. Recently, a review paper from [20] discussed two numerical implementations on Poisson image editing problem, the finite difference approach and Fourier approach. Based on the experimental results, in general, finite difference approach performs very well in most seamless cloning examples while Fourier approach is superior in contrast enhancement.

3. Description of the Model

The process of Poisson image blending is seamlessly clone a desired region from source image $g$ into a target image $f^*$ to create a new output image $f$. Let $O$ represent the desired region with $\partial O$ as the boundary. According to [2], the problem of this Poisson image editing is to compute the new intensity values $f$ for all the pixels in the desired region that minimize the difference between the gradient of the new image and the guidance vector field $\nu$, which is defined in $O$. This minimization problem is stated as follow:

$$\min_f \int_O |\nabla f - \nu|^2 \text{ with } f|_{\partial O} = f^*|_{\partial O}$$  \hspace{1cm} (1)

By obtaining the unique solution of Poisson equation with Dirichlet boundary condition,

$$\Delta f = \text{div } \nu \text{ at } 0 \text{ with } f|_{\partial O} = f^*|_{\partial O}$$  \hspace{1cm} (2)
the equation (1) can be solved. By selecting the vector field directly from the source image [Perez],
equation (2) now reads

$$\Delta f = \Delta g \text{ at } O \text{ with } f|_O = f^*|_O$$

(3)

where $\Delta$ is the Laplacian operator. By solving the linear system generated formed from equation (3)
for three different color (RGB) channel separately, the output images is built by merging the solved
color channel values. The finite grid network of the pixels in an image is shown in figure 1.

3.1. Discretization of Poisson Equation

As mentioned, the solution of the minimization problem generated from the Poisson image blending is
equivalent to the unique solution of the Poisson equation. Thus, the standard 2-Dimensions (2D)
Poisson PDE is defined as follow:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = f(x, y)$$

(4)

Thus, the Poisson image blending problem can be solved by obtaining the approximate solution of
equation (4) with Dirichlet boundary condition. Then, equation (4) is discretized by applying finite
difference approach with central difference scheme in second order, which is presented as follow:

$$\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{\Delta x^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{\Delta y^2} \cong f_{i,j}$$

(7)

To simplify equation (7), let $h^2 = \Delta x^2 = \Delta y^2$ and then the simplified approximate equation for
Poisson PDE is as follow:

$$U_{i,j} \cong \frac{1}{4} \left[ U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - h^2 f_{i,j} \right]$$

(8)

Discretizing Poisson PDE by second order central difference scheme will formed a five-point
Laplacian operator to obtain the approximate solution with $U_{i,j}$ as the reference point. The five-point
Laplacian operator is illustrated in figure 2. In order to solve the Poisson PDE, equation (8) forms a
system of linear equations and then is solved by Jacobi, Gauss-Seidel and MSOR iterative methods.

**Figure 1.** Finite grid network of pixels in an image.

**Figure 2.** Five-point Laplacian operator.
3.2. Implementation of Jacobi, Gauss-Seidel and MSOR iterative methods

In order to solve the Poisson image blending problem, linear system is formed from the approximation equation (8), which can be defined in general as follow:

\[ Au = b \] (9)

To generate these three proposed iterative methods, matrix \( A \) which is large and sparse is first decomposed into,

\[ A = D - L - U \] (10)

where \( D \), \( L \) and \( U \) are diagonal matrix, strictly lower triangular matrix and strictly upper triangular matrix respectively. Then, by substituting equation (10) into equation (9) and manipulating it, the general scheme for Jacobi iterative method is defined as,

\[ u^{(k+1)} = D^{-1}[(L + U)u^{(k)} + b] \quad k = 1, 2, 3, \ldots, n \] (11)

and the general scheme for Gauss-Seidel iterative method is stated as,

\[ u^{(k+1)} = (D - L)^{-1}[Uu^{(k)} + b] \quad k = 1, 2, 3, \ldots, n \] (12)

In addition, MSOR iterative method is invented by Kincaid and Young [21]. The idea of the implementation of MSOR is identical to the SOR iterative method [22,23] with the concept of red-black ordering, which can be defined as,

\[ u^{(k+1)} = (D - \omega_r L)^{-1}[(1 - \omega_r D) + \omega_r U]u^{(k)} + \omega_r b] \quad k = 1, 2, 3, \ldots, n \] (13)

\[ u^{(k+1)} = (D - \omega_b L)^{-1}[(1 - \omega_b D) + \omega_b U]u^{(k)} + \omega_b b] \quad k = 1, 2, 3, \ldots, n \] (14)

where the relaxation parameter, \( \omega_r \) is used for red equations and \( \omega_b \) is used for black equations.

4. Results and Discussion

The experimental results based on the three proposed methods are presented in this section. Three sets of source and target images are being selected for this evaluation [24], refer to figure 3. The evaluation criteria used in this paper are the number of iterations used and composing time taken by the three proposed iterative methods. Besides, we employed Root of Sum of Square (RSS) as stopping criterion with 1.0 threshold. RSS is used when we are combining the sources of errors.

(i)  
(ii)
According to figure 4, MSOR used the least number of iterations as compared to Jacobi and Gauss-Seidel iterative methods. The number of iterations reduced significantly in every example by approximately 97% between MSOR and Jacobi iterative methods and a huge reduction by approximately 93% between MSOR and Gauss-Seidel iterative methods. Figure 5 presents the results on the composing time for the three proposed iterative methods. As compared to Jacobi iterative method, MSOR improved the composing time by approximately 93%, while when it is compared to Gauss-Seidel iterative method, an improvement of approximately 94% is obtained. In addition, visually there are no significant differences spotted between the output images by the three iterative methods, as shown in figure 6.

![Figure 3.](image)

**Figure 3.** (a) Target images. (b) Source images.

![Figure 4.](image)

**Figure 4.** Number of iterations taken.

![Figure 5.](image)

**Figure 5.** Composing time taken.
5. Conclusion

In this study, three iterative methods which are derived based on finite difference approach via five-point Laplacian operator are selected and examined for their efficiency in solving Poisson image blending problem. MSOR iterative method is shown to be more superior in solving the problem than the Jacobi and Gauss-Seidel iterative methods in terms of number of iterations and computational time taken. Besides, the quality of the generated images from Poisson blending process is identical among the three proposed methods.

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[24] Test examples are available at https://pixabay.com.