Analytical determination of the distributed L and C parameters of metamaterial transmission lines using transmission ABCD matrices representation

E Coskuner and J G Garcia

Grupo de Aplicaciones Electromagnéticas Industriales (GAEMI), Departamento de Ingeniería Electrónica, Universitat Autònoma de Barcelona, Spain

Corresponding author’s e-mail address: joan.garcia@uab.es

Abstract. This work point out the possibility to identify the metamaterial behaviour of a transmission line by analysing the matrix elements of the ABCD representation. The method allows to identify the distributed $L$ and $C$ of the transmission lines, and therefore the relative permittivity of the metamaterial transmission line, and the characteristic impedance. To illustrate the Method, a metamaterial periodic structure is analysed using the proposed technique.

1. Introduction

The utilization of metamaterial transmission lines offers the possibility of having additional degrees of freedom for the designers. Since the metamaterial introduction in 1999 by Pendry [1], many efforts have been devoted to developing equivalent circuit models for the design of metamaterial structures [2,3]. Metamaterial equivalent circuit models based on lumped elements are limited to narrowband descriptions, generally around the resonance frequencies, which does not make them very useful from the point of view of the designer. On this regard, the distributed equivalent circuit model results more convenient to describe these structures. This paper establishes an easy way to obtain the distributed $L$ and $C$ parameters of a left-handed transmission lines from measured transmission and reflection parameters (S-parameter, ABCD-parameters or equivalent). The analysis also offers an easy way to identify the metamaterial transmission behavior. Work is in progress to develop an equivalent circuit model based on the distributed $L$ and $C$ identified parameters.

2. Theoretic background

One of the main drawback in many metamaterial transmission lines is the difficult to identify the metamaterial behaviour and the value of their negative characteristic permeability and permittivity. In general, the description of selective high frequency transmission lines is done by using the S-parameters. The design of standard filter response use to be done in terms of these parameters, however, the transmission ABCD matrices offer advantages from the point of view of the identification and characterization of metamaterial devices. Fortunately, the S-parameters and the ABCD representation are univocally linked by analytical expressions [4,5,6]. The ABCD transmission line formalism allows to obtain the transfer matrix of a composite network as the product of the transfer matrices of the cascade subnetworks. By combination the corresponding transmission matrices of the constituent shunt and
series elements, the ABCD matrix corresponding to an ideal left-handed transmission matrix of an infinitesimal length \( dx \) is given by expression (1).

\[
\begin{pmatrix}
V(x) \\
I(x)
\end{pmatrix} = ABCD_{LH} \begin{pmatrix}
V(x + dx) \\
I(x + dx)
\end{pmatrix}
\]

where
\[
ABCD_{LH} = \begin{bmatrix}
1 & \frac{1}{j\omega L} \\
\frac{1}{j\omega C} & 1
\end{bmatrix}
\]

\[ (1) \]

**Figure 1.** (a) Scheme of the Left-handed transmission line and (b) ABCD transmission matrix of a left-handed transmission line infinitesimal section.

where \( L \) and \( C \) are the distributed inductance and capacitance of the left-handed transmission line. On the other hand, the ABCD matrix of a transmission line can be obtained from the direct measurement or by the transformation of the measured S-parameters. As can be observed in equation (1), the non-diagonal elements of the transmission matrices are negative and pure imaginary, in the case of ideal left-handed transmission lines. From the equation (1) it can also be observed that the ABCD parameters offers a clear method for the identification of the characteristic distributed elements of both right and left-handed the transmission lines. In the case of the left-handed transmission line the non-diagonal elements of the ABCD transmission line allows to evaluate the \( L \) and \( C \) distributed elements as follow.

\[
L = \text{imag} \left[ \frac{1}{2\pi f C_{LH}} \right] \quad C = \text{imag} \left[ \frac{1}{2\pi f B_{LH}} \right] \quad (2)
\]

where \( f \) is the frequency variable in Hz, \( L \) and \( C \) are the distributed inductance and capacitance of the line, and \( C_{LH} \) and \( L_{LH} \) are the corresponding non-diagonal elements of the left-handed ABCD transmission line matrix.

In the case of a discrete line the \( dx \) in figure 1 should be substituted by the periodicity lattice parameter \( a \), equal to the metamaterial unit cell length. The equivalent transfer matrix in this case can be evaluated as the multiplication of the ABCD matrices of the unit cells

\[
ABCD_{LH} = ABCD_{UC_1} \cdot ABCD_{UC_2} \cdots = \prod_{i=1}^{N} ABCD_{UC_i} \quad (3)
\]

where \( N \) is the total number unit cells in the discrete metamaterial structure and \( ABCD_{UC} \) is the transmission matrix of the metamaterial unit cell. The resulting transmission matrix, although complicate, keep the main characteristics of the distributed matrix (1). For instance, it can be shown that the elements outside of the resulting matrix \( ABCD_{LH} \) in (3) are always purely imaginary, and the diagonal elements are real. This fact, by itself constitute an easy way to check the metamaterial behavior in the left-handed transmission lines formed by periodic structures. Moreover, the analysis of specific cases allows to analytically determine the distributed \( L, C \) parameters of the left-handed transmission line as will be shown in the next section. The equation (3) points out the potential complexity of the four elements of the ABCD matrix of a discrete transmission line. In general, each matrix element will be a \( N \)-order polynomial fraction. The zeros and the poles of these fraction will introduce divergent points in the transmission line that should tend to disappear in the effective medium limit.
3. Analysis of a left-handed transmission line

Figure 2. (a). shows a periodic structure with three cells between the ports 1 and 2. Each cell is formed by two rectangular patches with a 4mm diameter grounded via centred in the external side of each patch. The unit cell is a variation of the mushroom structure modified to enhance the symmetry in the transmission direction and minimize the loses. The black area corresponds to the top metallic layer of the microstrip structure in a 50mil thick RO3010 substrate. Figure 2. (b) shows the ADS-Momentum simulated response.

![Figure 2](image)

**Figure 2.** (a) Layout of the microstrip top metallic layer of a left-handed transmission line formed by three-unit cells being a the lattice parameter including the more significant dimensions. (b) ADS-Momentum S-parameter simulated response of the LH structure.

The dimensions of the structure showed in figure 2 (a) have been optimized to obtain the bandpass between 1.2 and 1.25 GHz showed in figure 2 (b) Due to the structure of the unit cells of the structure the transmission band is purely left handed, and therefore, in the limit of effective media, it should be described by the equations (1).

![Figure 3](image)

**Figure 3.** Phase of the non-diagonal elements of the transmission ABCD matrix
The ABCD transmission matrix can be analytically calculated from the S-parameters [3,4,5,6]. Figure 3 shows the phase of the non-diagonal elements of the ABCD transmission matrix of the structure. It can be observed that the phase of these elements is oscillate + and − 90° indicating the complex nature of these non-diagonal elements.

**Figure 4.** Graphs of the S-parameter simulated response of the structure described in figure 2, calculated total capacitance (C) and inductance (L) of the transmission line and the characteristic impedance of the left-handed transmission line (Z_{LHTL}).

Figure 4 shows the result of the proposed analysis applied to the structure described in figure 1. The S-parameters situate a 0.4 GHz bandpass centered at 1.217 GHz. The L and C graphs are obtained applying the expressions (2) to the non-diagonal elements of the ABCD transfer matrix. The results are C=3.60 pF and L=8.05 nH for the full length of the transmission line. The average distributed C and L can be
evaluated by dividing the value by $3 \cdot a$ since there are three unit cells in the structure showed in figure 1. Finally, the $Z_{LHTL}$ parameter is the characteristic impedance or the left-handed transmission line, obtained as the square root of the $L$ and $C$ ratio.

4. Conclusions
A simple way to identify the left-handed behaviour in a transmission line by the examination of the ABCD matrix elements has been exposed and tested in this work. The transmission ABCD matrix can be analytically obtained from measured S-parameter matrix, and it has been shown how the evaluation of their elements allows to identify the left-handed transmission line distributed $L$ and $C$ characteristics parameters. To illustrate the method, a mushroom like unidirectional transmission line is analyzed. Work is in progress to develop distributed equivalent circuit models of the left-handed transmission line based on the distributed parameters evaluated following the exposed method.

'References'
[1] J.B. Pendry, A.J. Holden, D.J. Robbins and W.J. Stewart, “Magnetism from conductors and enhanced nonlinear phenomena”, IEEE Transactions Microwave Theory Tech., vol. 47, pp. 2075-2084, November 1999.
[2] N. Engheta, R. W. Ziolkowski, “Metamaterials Physics and Engineering Esplorations” Wiley & Sons, 2006
[3] F. Capolino, “Metamaterials Handbook. Applications of Metamaterials”, CRC Press, 2009.
[4] D. M. Pozar, “Microwave Engineering”, John Willey & Sons, Inc. New Cork 1998.
[5] K. C. Gupta, R. Garg, I. Bahl, and P. Bhartis, Microstrip Lines and Slotlines, Second Edition, Artech House, Boston, 1996.
[6] Jia-Sheng Hong & M.L. Lancaster, “Microstrip Filters for RF/Microwave Applications”, John Willey & Sons, Inc., New York 2001.