On Siegel invariants of certain CM-fields

JA KYUNG KOO, GILLES ROBERT, DONG HWA SHIN AND DONG SUNG YOON∗

Abstract

We first construct Siegel invariants of some CM-fields in terms of special values of theta constants, which would be a generalization of Siegel-Ramachandra invariants of imaginary quadratic fields. And, we further describe Galois actions on these invariants and provide some numerical examples to show that this invariant really generates the ray class field of a CM-field.

1 Introduction

Let $K$ be a number field and $\mathcal{O}_K$ be its ring of integers. For a proper nontrivial ideal $\mathfrak{f}$ of $\mathcal{O}_K$ we denote by $\text{Cl}(\mathfrak{f})$ and $K_{\mathfrak{f}}$ the ray class group of $K$ modulo $\mathfrak{f}$ and its corresponding ray class field, respectively (see [4]). Suppose that there is a family $\{\Psi_{\mathfrak{f}}(C)\}_{C \in \text{Cl}(\mathfrak{f})}$ of algebraic numbers, which we shall call a Siegel family, such that

(R1) each $\Psi_{\mathfrak{f}}(C)$ belongs to $K_{\mathfrak{f}}$,

(R2) $\Psi_{\mathfrak{f}}(C)^{\sigma_{\mathfrak{f}}(D)} = \Psi_{\mathfrak{f}}(CD)$ for all $D \in \text{Cl}(\mathfrak{f})$, where $\sigma_{\mathfrak{f}} : \text{Cl}(\mathfrak{f}) \to \text{Gal}(K_{\mathfrak{f}}/K)$ is the Artin reciprocity map for $\mathfrak{f}$.

Then, every algebraic number $\Psi_{\mathfrak{f}}(C)$ becomes a primitive generator of $K_{\mathfrak{f}}$ over $K$ under some conditions (Proposition 2.1). In particular, if $K$ is an imaginary quadratic field, then the Siegel-Ramachandra invariants form such a Siegel family having the properties (R1) and (R2) (see [2]).

Let $K$ be a CM-field and $\mathfrak{f}$ be a proper nontrivial ideal of $\mathcal{O}_K$ satisfying certain conditions (Assumption 5.1). In this paper, we shall first construct a meromorphic Siegel modular function of level $N \geq 2$ by making use of theta constants (Definition 4.2 and Proposition 4.5). Furthermore, we shall assign the special value $\Theta_{\mathfrak{f}}(C)$ of this function to each ray class $C$ in $\text{Cl}(\mathfrak{f})$, and call it the Siegel invariant modulo $\mathfrak{f}$ at $C$ (Definition 5.4). This value depends only on $\mathfrak{f}$ and the class

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∗Corresponding author.

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C (Propositions 6.1 and 6.2), essentially by the fact that the Siegel modular variety is a moduli space for principally polarized abelian varieties. Finally, we are able to show by applying Shimura’s reciprocity law that the Siegel invariant $\Theta_f(C)$, as a possible ray class invariant (Conjecture 7.4), satisfies the transformation formula (R2), that is,

$$\Theta_f(C)^{\sigma(D)} = \Theta_f(CD) \quad \text{for all } D \in \text{Cl}(f)$$

(Theorem 7.3). By making use of Maple software we also present a nontrivial numerical example for which $K_f$ is generated by $\Theta_f(C)$ over $K$ (Example 7.5).

2 Siegel-Ramachandra invariants

As a criterion for a primitive generator of a ray class field, we shall present a precise proof of some known result ([10, Theorem 3 in Chapter 22]) which motivates this paper.

**Proposition 2.1.** Let $K$ be a number field and $f$ be a proper nontrivial ideal of $\mathcal{O}_K$. Let $\{\Psi_f(C)\}_{C \in \text{Cl}(f)}$ be a Siegel family satisfying the properties (R1) and (R2). Assume that for each ray class $D$ in $\text{Cl}(f)$ there exists a character $\chi_D$ of $\text{Cl}(f)$ such that

- (S1) $\chi_D(D) \neq 1$,
- (S2) the Stickelberger element

$$S_f(\chi_D) = \sum_{C \in \text{Cl}(f)} \chi_D(C) \ln |\Psi_f(C)|$$

of $\chi_D$ is nonzero.

Then, $\Psi_f(C)$ generates $K_f$ over the base field $K$ for any class $C$ in $\text{Cl}(f)$.

**Proof.** By (R2) it suffices to prove the proposition when $C$ is the identity class $C_0$ of $\text{Cl}(f)$.

Suppose on the contrary that $\Psi_f(C_0)$ does not generate $K_f$ over $K$. Then, there is a nonidentity class $D$ whose corresponding Galois element $\sigma(D)$ leaves $\Psi_f(C_0)$ fixed. Take a character $\chi_D$ of $\text{Cl}(f)$ satisfying (S1) and (S2). If $S$ is a subset of $\text{Cl}(f)$ consisting of right coset representatives of the subgroup $\langle D \rangle$, then

$$\text{Cl}(f) = \bigsqcup_{A \in S} \langle D \rangle A.$$

Hence we derive that

$$S_f(\chi_D) = \sum_{A \in S} \sum_{B \in \langle D \rangle} \chi_D(BA) \ln |\Psi_f(BA)|$$

$$= \sum_{A \in S} \sum_{B \in \langle D \rangle} \chi_D(B) \chi_D(A) \ln |\Psi_f(C_0)^{\sigma(D)\sigma(A)}| \quad \text{by (R2)}$$

$$= \sum_{A \in S} \chi_D(A) \sum_{B \in \langle D \rangle} \chi_D(B) \ln |\Psi_f(C_0)^{\sigma(A)}| \quad \text{since } \sigma(D) \text{ leaves } \Psi_f(C_0) \text{ fixed}.$$
\[= \sum_{A \in S} \chi_D(A) \sum_{B \in \langle D \rangle} \chi_D(B) \ln |\Psi_f(A)| \text{ by (R2)}\]
\[= \sum_{A \in S} \chi_D(A) \ln |\Psi_f(A)| \sum_{B \in \langle D \rangle} \chi_D(B) \]
\[= 0 \text{ because (S1) implies that } \chi_D \text{ is a nonprincipal character of the subgroup } \langle D \rangle.\]

But this contradicts (S2). Therefore, \(\Psi_f(C)\) generates \(K_f\) over \(K\) for any class \(C\) in \(\text{Cl}(f)\).

Let \(\mathbf{v} = \begin{bmatrix} r \\ s \end{bmatrix} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2\). The Siegel function \(g_v(\tau)\) on the complex upper half-plane \(\mathbb{H} = \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}\) is given by the infinite product
\[
g_v(\tau) = -q^{\mathbf{B}_2(r)/2} e^{\pi i (r-1)} (1 - q^r e^{2\pi i s}) \prod_{n=1}^{\infty} (1 - q^n + r e^{2\pi i s}) (1 - q^n - r e^{-2\pi i s}), \tag{1}\]
where \(\mathbf{B}_2(x) = x^2 - x + 1/6\) is the second Bernoulli polynomial and \(q = e^{2\pi i \tau}\). If \(N \geq 2\) is a positive integer so that \(N \mathbf{v} \in \mathbb{Z}^2\), then \(g_v(\tau)^{12N}\) is a meromorphic modular function of level \(N\) which has neither a zero nor a pole on \(\mathbb{H}\) ([6, Theorem 1.2 in Chapter 2]).

Let \(K\) be an imaginary quadratic field. Let \(f\) be a proper nontrivial ideal of \(\mathcal{O}_K\) in which \(N\) is the smallest positive integer, and let \(C \in \text{Cl}(f)\). Take any integral ideal \(c\) in \(C\), and let \(\omega_1, \omega_2 \in \mathbb{C}\) and \(a, b \in \mathbb{Z}\) such that
\[
f c^{-1} = \mathbb{Z} \omega_1 + \mathbb{Z} \omega_2 \quad \text{with } \omega = \omega_1 / \omega_2 \in \mathbb{H},
N = a \omega_1 + b \omega_2.\]

The Siegel-Ramachandra invariant \(g_f(C)\) modulo \(f\) at \(C\) is defined as
\[
g_f(C) = g_{\left[\frac{a}{N} / \frac{b}{N}\right]}(\omega)^{12N}.
\]
This value depends only on \(f\) and the class \(C\), not on the choices of \(c\) and \(\omega_1, \omega_2\) ([6, Chapter 11, §1]). Furthermore, it lies in \(K_f\) and satisfies
\[
g_f(C)^{\sigma(D)} = g_f(CD) \quad (D \in \text{Cl}(f))\]
([6, Theorem 1.1 in Chapter 11]).

**Proposition 2.2.** Let \(f = \prod p | f \ p^{e_p}\) be the prime ideal factorization of \(f\), and let
\[
G_p = (\mathcal{O}_K / p^{e_p})^\times / \{\mu + p^{e_p} \mid \mu \in \mathcal{O}_K^\times\}.
\]
If \(|G_p| > 2\) for every \(p | f\), then any nonzero power of \(g_f(C)\) generates \(K_f\) over \(K\).

**Proof.** See [8, Theorem 4.6].

**Remark 2.3.** (i) This result is obtained by utilizing the second Kronecker limit formula ([18, Theorem 9 in Chapter II] or [19, Theorem 2 in Chapter 22]).
(ii) Without any condition, Ramachandra ([22]) constructed a primitive generator of \(K_f\) over \(K\) as a high power product of Siegel-Ramachandra invariants and singular values of the modular discriminant \(\Delta\)-function.
3 Actions on Siegel modular functions

In this section we shall briefly recall the action of an idele group on the field of meromorphic Siegel modular functions due to Shimura.

For a positive integer $g$ and a commutative ring $R$ with unity, we let

$$\text{GSp}_2^g(R) = \{ \alpha \in \text{GL}_2^g(R) \mid \alpha^T J \alpha = \nu J \text{ for some } \nu \in R^\times \}$$

where $J = \begin{bmatrix} O & -I_g \\ I_g & O \end{bmatrix}$, $\text{Sp}_2^g(R) = \{ \alpha \in \text{GL}_2^g(R) \mid \alpha^T J \alpha = J \}$.

Here, $\alpha^T$ stands for the transpose of the matrix $\alpha$. Observe that the relation $\alpha^T J \alpha = \nu J$ implies $\det(\alpha) = \nu^g$ ([17, (1.11)]). If $\alpha$ belongs to either $\text{GSp}_2^g(R)$ or $\text{Sp}_2^g(R)$, then $\alpha^T$ also belongs to the same group ([17, p. 17]).

The symplectic group $\text{Sp}_2^g(\mathbb{Z})$ acts on the Siegel upper half-space

$$\mathbb{H}_g = \{ Z \in M_g(\mathbb{C}) \mid Z^T = Z, \text{ Im}(Z) \text{ is positive definite} \}$$

by

$$\gamma(Z) = (AZ + B)(CZ + D)^{-1} \quad (\gamma \in \text{Sp}_2^g(\mathbb{Z}), \ Z \in \mathbb{H}_g),$$

where $A, B, C, D$ are $g \times g$ block matrices of $\gamma$ ([5, Proposition 1 in §1]). For a positive integer $N$ let

$$\Gamma(N) = \{ \gamma \in \text{Sp}_2^g(\mathbb{Z}) \mid \gamma \equiv I_{2g} \text{ (mod } N \cdot M_{2g}(\mathbb{Z})) \}.$$ 

We call a holomorphic function $f : \mathbb{H}_g \to \mathbb{C}$ a Siegel modular form of weight $k$ and level $N$ if

(M1) $f(\gamma(Z)) = \det(CZ + D)^k f(Z)$ for every $\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma(N),$

(M2) $f$ is holomorphic at every cusp only when $g = 1$.

Every Siegel modular form $f$ can be expressed as

$$f(Z) = \sum_\beta c(\beta)e(\text{tr}(\beta Z)/N) \quad (c(\beta) \in \mathbb{C}),$$

where $\beta$ runs over all $g \times g$ positive semi-definite symmetric matrices over half integers with integer diagonal entries and $e(x) = e^{2\pi i x}$ ($x \in \mathbb{R}$) ([5, Theorem 1 in §4]). Here, we call $c(\beta)$ the Fourier coefficients of $f$. For a subfield $F$ of $\mathbb{C}$ we set

$$\mathcal{M}_k(\Gamma(N), F) = \text{ the } F\text{-vector space of all Siegel modular forms of weight } k \text{ and level } N \text{ with Fourier coefficients in } F,$$

$$\mathcal{M}_k(F) = \bigcup_{N=1}^\infty \mathcal{M}_k(\Gamma(N), F),$$

$$\mathcal{A}_0(\Gamma(N), F) = \text{ the field of all meromorphic Siegel modular functions of the form } g/h, \text{ with } g \in \mathcal{M}_k(F) \text{ and } h \in \mathcal{M}_k(F) \setminus \{0\} \text{ for some } k,$$
which is invariant under the group $\Gamma(N)$,

$$A_0(F) = \bigcup_{N=1}^{\infty} A_0(\Gamma(N), F).$$

In particular, let

$$\mathcal{F}_N = A_0(\Gamma(N), \mathbb{Q}(\zeta_N)),$$
$$\mathcal{F} = A_0(\mathbb{Q}_{ab}),$$

where $\zeta_N = e(1/N)$ and $K_{ab}$ denotes the maximal abelian extension of a number field $K$.

On the other hand, we let

$$G = \text{GSp}_{2g}(\mathbb{Q}),$$
$$G_+ = \{ \alpha \in G \mid \alpha^T J \alpha = \nu J \text{ for some } \nu > 0 \},$$
$$G_\mathbb{A} = \text{the adelization of } G \text{ with } G_0 \text{ and } G_\infty \text{ the non-archimedean part and the archimedean part, respectively},$$
$$G_{\mathbb{A}+} = G_0 G_{\infty+}, \text{ where } G_{\infty+} \text{ is the identity component of } G_\infty.$$

Shimura presented in [17, Theorem 8.10] a group homomorphism

$$\tau : G_{\mathbb{A}+} \to \text{Aut}(\mathcal{F})$$

satisfying the following properties: Let $f \in \mathcal{F}$.

(A1) $f^{\tau(\alpha)} = f \circ \alpha$ for all $\alpha \in G_+$, where $\alpha$ acts on $\mathbb{H}_g$ by the same way as in (2).

(A2) $f^{\tau(\iota(s))} = f[s, \mathbb{Q}]$ for all $s \in \prod_p \mathbb{Z}_p^\times$, where $\iota(s) = \begin{bmatrix} I_g & O \\ O & s^{-1} I_g \end{bmatrix}$. Here, the action of $[s, \mathbb{Q}]$ on $f$ is understood as the action of it on the Fourier coefficients of $f$ (see also [14, Theorem 5]).

Note that the mapping $s \mapsto [s, \mathbb{Q}]$ yields an isomorphism of $\prod_p \mathbb{Z}_p^\times$ onto $\text{Gal}(\mathbb{Q}_{ab}/\mathbb{Q})$ ([17, §8.1]). Then, $\mathcal{F}_N$ coincides with the fixed field of $\mathcal{F}$ by the subgroup

$$\mathbb{Q}^\times \cdot \{ \alpha \in G_{\mathbb{A}+} \mid \alpha_p \in \text{GL}_{2g}(\mathbb{Z}_p) \text{ and } \alpha_p \equiv I_{2g} \pmod{N \cdot M_{2g}(\mathbb{Z}_p)} \text{ for all rational primes } p \}$$
of $G_{\mathbb{A}+}$ ([14, Theorem 3] or [17, Theorem 8.10 (6)]).

4 Siegel modular functions in terms of theta constants

Let $g$ and $N$ be positive integers, and let $r, s \in (1/N)\mathbb{Z}^g$. The (classical) theta constant $\theta([s], Z)$ is defined by

$$\theta([s], Z) = \sum_{n \in \mathbb{Z}^g} e \left( \frac{1}{2} (n + r)^T Z (n + r) + (n + r)^T s \right) \quad (Z \in \mathbb{H}_g). \quad (3)$$

For a matrix $E \in M_g(\mathbb{Z})$ we mean by $\{E\}$ the $g$-vector whose components are the diagonal entries of $E$.  

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LEMMA 4.1. We have the following properties of theta constants.

(i) $\theta([T], Z)$ is identically zero if and only if $r, s \in (1/2)Z^g$ and $e(2r^Ts) = -1$.

(ii) $\theta([-T], Z) = \theta([T], Z)$.

(iii) If $a, b \in Z^g$, then $\theta([T] + [a], Z)^N = \theta([T], Z)^N$.

(iv) $\theta([T], Z)$ is identically zero if and only if $\theta([T], Z)$ is identically zero.

Proof. (i) See [3, Theorem 2].

(ii) This is immediate from the definition [3].

(iii) See [15, p. 676 (13)].

(iv) See [15, Proposition 1.3] or [1, Theta Transformation Formula 8.6.1].

(v) The function $\theta([T], Z)/\theta([0], Z)$ belongs to $F_{2N^2}$. Furthermore, if $\alpha \in G_{2K+} \cap \prod_p \mathrm{GL}_2(Z_p)$ such that $\alpha_p = \begin{bmatrix} I_g & O \\ O & tI_g \end{bmatrix} (\mod 2N^2 \cdot M_2(Z_p))$ for all rational primes $p$ with a positive integer $t$, then

$$
\left( \frac{\theta([T], Z)}{\theta([0], Z)} \right)^{\tau(\alpha)} = \frac{\theta([T], Z)}{\theta([0], Z)}.
$$

Proof. (i) See [3, Theorem 2].

(ii) This is immediate from the definition [3].

(iii) See [15, p. 676 (13)].

(iv) See [15, Proposition 1.3] or [1, Theta Transformation Formula 8.6.1].

(v) See [15, Proposition 1.7].

Proof. Let

$$
S_- = \{ [a] | a, b \in \{0, 1/2\}^g \text{ such that } e(2a^Tb) = -1 \},
$$

$$
S_+ = \{ [c] | c, d \in \{0, 1/2\}^g \text{ such that } e(2c^Td) = 1 \}.
$$

By Lemma 4.1 (i) and (iv) one can regard each element $\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_{2g}(Z)$ as a permutation of the set $S_-$ (and $S_+$) so that

$$
\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \gamma^T \begin{bmatrix} a \\ b \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \{A^TC\} \\ \{B^TD\} \end{bmatrix} (\mod Z_{2g}).
$$

DEFINITION 4.2. We define a function

$$
\Theta([T], Z) = 2^{4N} e\left(-2^g N(2^g - 1)(2^g + 1) r^T s \right) \frac{\prod_{[a] \in S_-} \theta([a] - [T], Z)^{4N(2^g + 1)}}{\prod_{[c] \in S_+} \theta([c], Z)^{4N(2^g - 1)}} (Z \in H_g).
$$

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REMARK 4.3.  
(i) One can easily check that 
\[ |S_-| = 2^{g-1}(2^g - 1) \quad \text{and} \quad |S_+| = 2^{g-1}(2^g + 1). \]

Hence we have 
\[ \text{lcm}(|S_-|, |S_+|) = 2^{g-1}(2^g - 1)(2^g + 1) = |S_-|(2^g + 1) = |S_+|(2^g - 1). \]

(ii) When \( g = 1 \), let \( N \geq 2 \) and \( \left[ \frac{r}{s} \right] \in (1/N)\mathbb{Z}^2 \setminus \mathbb{Z}^2 \). By using Jacobi’s triple product identity 
([2] (17.3)) which reads 
\[ \sum_{n \in \mathbb{Z}} a^n q^{n^2/2} = \prod_{n=1}^{\infty} (1 - q^n)(1 + a q^{n-1/2})(1 + a^{-1} q^{n-1/2}) \quad (a \in \mathbb{C} \setminus \mathbb{Q}), \]

one can justify 
\[ \Theta([\tau], \tau) = q^{\tau^2/2} \tau^{12N} \quad (\tau \in \mathbb{H}). \]

This shows that the function in Definition 4.2 would be a multi-variable generalization of the Siegel function described in Definition 4.2.

LEMMA 4.4. We obtain the following transformation formulas for \( \Theta([\frac{a}{b}], Z) \).

(i) \( \Theta([\frac{a}{b}] + [\frac{c}{d}], Z) = \Theta([\frac{a}{b}], Z) \) for all \( a, b \in \mathbb{Z}^g \).

(ii) \( \Theta([\frac{a}{b}], \gamma(Z)) = \Theta(\gamma T [\frac{a}{b}], Z) \) for all \( \gamma \in \text{Sp}_{2g}(\mathbb{Z}). \)

(iii) If \( \alpha \in G_{a+} \cap \prod_p \text{GL}_{2g}(\mathbb{Z}_p) \) for which \( \alpha_p \equiv \begin{bmatrix} I_g & O \\ O & t I_g \end{bmatrix} \pmod{2N^2 \cdot M_{2g}(\mathbb{Z}_p)} \) for all rational primes \( p \) with a positive integer \( t \), then \( \Theta([\frac{a}{b}], Z)^{T(\alpha)} = \Theta([\frac{a}{b}], Z). \)

PROOF. (i) This is immediate from Lemma 4.1 (iii) and the Definition 4.2

(ii) Let \( \gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{Sp}_{2g}(\mathbb{Z}). \) For \( x, y \in \mathbb{Q}^g \), we set \( \begin{bmatrix} x' \\ y' \end{bmatrix} = \gamma T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A^T x + C^T y \\ B^T x + D^T y \end{bmatrix}. \) Here we observe that
\[
\prod_{[b] \in S_-} e^{2N(2^g + 1)((a - r)^T(b - s) - (a' - r')^T(b' - s'))} \\
\prod_{[c] \in S_+} e^{2N(2^g - 1)(c^T d - (c')^T d'))}
\]
\[ = \begin{bmatrix} a & b \\ a' & b' \end{bmatrix} \]
\[ = \begin{bmatrix} a^T b - (a')^T b' \\ a^T AB^T a + b^T CD^T b \end{bmatrix} \]
\[ = \begin{bmatrix} a^T AB^T a + b^T CD^T b \end{bmatrix} \]
because \( AD^T + BC^T = I_g + 2BC^T \)
\[ = \begin{bmatrix} a^T AB^T a + b^T CD^T b \end{bmatrix} \]
\[ = \begin{bmatrix} a^T AB^T a + b^T CD^T b \end{bmatrix} \]
\[ = \begin{bmatrix} a^T AB^T a + b^T CD^T b \end{bmatrix} \]

\[ \prod_{[c] \in S_+} e^{2N(2^g - 1)(c^T d - (c')^T d'))} \]

because \( AD^T + BC^T = I_g + 2BC^T \)
\[ = \begin{bmatrix} a^T AB^T a + b^T CD^T b \end{bmatrix} \]
\[ = \begin{bmatrix} a^T AB^T a + b^T CD^T b \end{bmatrix} \]
\[ = \begin{bmatrix} a^T AB^T a + b^T CD^T b \end{bmatrix} \]

\[ 1_{[b] \in \{0,1/2\}^{2g}} \]
because $S_+ = \{0,1/2\}^{2g} \setminus S_-
\begin{align*}
&= e\left(2^g N(2^g - 1)(2^g + 1)(r^T s - (r')^T s')\right) \prod_{a \in \{0,1/2\}^g} e\left(-2^{g+1} N (a^T (AB^T + CD^T)a)\right) \\
&= e\left(2^g N(2^g - 1)(2^g + 1)(r^T s - (r')^T s')\right).
\end{align*}
Hence we derive

$$
\Theta([s], \gamma(Z)) = 2^{4N} e\left(-2^g N(2^g - 1)(2^g + 1)r^T s\right) \prod_{[a,b] \in S_+} \frac{\theta([a] - [b] - [r], \gamma(Z))^{4N(2^g+1)}}{\prod_{[c,d] \in S_+} \theta([c] - [d], \gamma(Z))^{4N(2^g-1)}}
$$

by Lemma 4.1 (iv)

$$
= 2^{4N} e\left(-2^g N(2^g - 1)(2^g + 1)((r')^T s')\right) \prod_{[a,b] \in S_+} \frac{\theta([a] - [b] - [r'], \gamma(Z))^{4N(2^g+1)}}{\prod_{[c,d] \in S_+} \theta([c] - [d], \gamma(Z))^{4N(2^g-1)}}
$$

by Lemma 4.1 (iii) and the fact that $\gamma$ is a permutation of $S_-$ (and $S_+$)

$$
= \Theta([r'], Z).
$$

(iii) Since $t$ is odd, $[a, b] \mapsto [a, t b]$ (mod $\mathbb{Z}^{2g}$) gives rise to a permutation of $S_-$ (and $S_+$). Furthermore, it follows from [17, §8.1] that

$$
e(1/N)^{\tau(a)} = e(t/N).
$$

Hence we see by Lemma 4.1 (v) that

$$
\Theta([s], Z)^{\tau(a)} = 2^{4N} e\left(-2^g t N(2^g - 1)(2^g + 1)t^T s\right) \prod_{[a,b] \in S_+} \frac{\theta([a] - [b] - [r], \gamma(Z))^{4N(2^g+1)}}{\prod_{[c,d] \in S_+} \theta([c] - [d], \gamma(Z))^{4N(2^g-1)}}
$$

by Lemma 4.1 (iii)

$$
= \Theta([s], Z).
$$

\[\Box\]

**Proposition 4.5.** The function $\Theta([s], Z)$ belongs to $F_N$.

**Proof.** By Lemma 4.1 (v), $\Theta([s], Z)$ belongs to $F_{2N^2}$.

For any $\gamma \in \Gamma(N)$ we achieve that

$$
\Theta([s], \gamma(Z)) = \Theta(\gamma^T [s], Z) \text{ by Lemma 4.4 (ii)}
$$

$$
= \Theta([s], Z) \text{ by the fact } \gamma^T [s] \equiv [s] \pmod{\mathbb{Z}^{2g}} \text{ and Lemma 4.4 (i)}.
$$
This claims that $\Theta([s], Z)$ lies in $\mathcal{A}_0(\Gamma(N), \mathbb{Q}(\zeta_{2N^2}))$.

Let $s$ be an element of $\prod_p \mathbb{Z}_p$ such that $[s, \mathbb{Q}]$ is the identity on $\mathbb{Q}(\zeta_N)$. Take a positive integer $t$ for which
\[
Ig = \begin{bmatrix} I_g & O \\ O & t^{-1}I_g \end{bmatrix} \equiv \begin{bmatrix} I_g & O \\ O & tI_g \end{bmatrix} \pmod{2N^2 \cdot M_{2g}(\mathbb{Z}_p)} \quad \text{for all rational primes } p.
\]
Since $s_p \equiv 1 \pmod{N \cdot \mathbb{Z}_p}$ for all rational primes $p$, we have $t \equiv 1 \pmod{N}$. We then obtain
\[
\Theta([s], Z)[s, \mathbb{Q}] = \Theta([s], Z) \tau(\iota(s)) \quad \text{by (A2)}
\]
\[
= \Theta([t], Z) \quad \text{by Lemma 4.4 (iii)}
\]
\[
= \Theta([s], Z) \quad \text{by the fact } t \equiv 1 \pmod{N} \text{ and Lemma 4.4 (i)}.
\]
This implies that every Fourier coefficient of $\Theta([s], Z)$ lies in $\mathbb{Q}(\zeta_N)$.

Therefore, we conclude that $\Theta([s], Z)$ belongs to $\mathcal{F}_N$. \qed

5 Abelian varieties associated with CM-fields

Let $n$ be a positive integer and $K$ be a CM-field with $[K : \mathbb{Q}] = 2n$, that is, $K$ is a totally imaginary quadratic extension of a totally real number field. Fix a set $\{\varphi_1, \ldots, \varphi_n\}$ of embeddings of $K$ into $\mathbb{C}$ such that $\varphi_1, \ldots, \varphi_n, \overline{\varphi}_1, \ldots, \overline{\varphi}_n$ are all the embeddings of $K$ into $\mathbb{C}$, which is called a CM-type of $K$. Take a finite Galois extension $L$ of $\mathbb{Q}$ containing $K$ and set
\[
S = \{\sigma \in \text{Gal}(L/\mathbb{Q}) \mid \sigma|_K = \varphi_i \text{ for some } 1 \leq i \leq n\},
\]
\[
S^* = \{\sigma^{-1} \mid \sigma \in S\},
\]
\[
H^* = \{\gamma \in \text{Gal}(L/\mathbb{Q}) \mid \gamma S^* = S^*\}.
\]
Let $K^*$ be the subfield of $L$ corresponding to the subgroup $H^*$ of $\text{Gal}(L/\mathbb{Q})$ and $\{\psi_1, \ldots, \psi_g\}$ be the set of all the embeddings of $K^*$ into $\mathbb{C}$ obtained by the elements of $S^*$. Then
\[
K^* = \mathbb{Q}\left(\sum_{i=1}^n a^{\varphi_i} \mid a \in K\right)
\]
and it is also a CM-field with a (primitive) CM-type $\{\psi_1, \ldots, \psi_g\}$ (\cite{16} Proposition 28 in §8.3).

We define an embedding
\[
\Psi : K^* \to \mathbb{C}^g
\]
\[
a \mapsto \begin{bmatrix} a^{\psi_1} \\ \vdots \\ a^{\psi_g} \end{bmatrix}.
\]

For an element $c$ of $K^*$ which is purely imaginary, define an $\mathbb{R}$-bilinear form $E_c : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{R}$ as
\[
E_c(u, v) = \sum_{j=1}^g c^{\psi_j}(u_j \overline{v}_j - \overline{u}_j v_j) \quad (u = \begin{bmatrix} u_1 \\ \vdots \\ u_g \end{bmatrix}, v = \begin{bmatrix} v_1 \\ \vdots \\ v_g \end{bmatrix} \in \mathbb{C}^g).
\]
Then we know that
\[ E_c(\Psi(a), \Psi(b)) = \text{Tr}_{K^*/Q}(cab) \quad \text{for all } a, b \in K^*. \] (4)

**Assumption 5.1.** We assume the following two conditions:

(C1) The complex torus \( \mathbb{C}^g/\Psi(O_{K^*}) \) can be given a structure of a principally polarized abelian variety.

(C2) \( (K^*)^* = K \).

**Remark 5.2.**
(i) It is well known that a complex torus can be equipped with a structure of an abelian variety if and only if there is a non-degenerate Riemann form on the torus in the sense of [16, §3.1]. See also [1, §4.2].

(ii) The assumption (C1) is equivalent to saying that there is an element \( \xi \) of \( K^* \) satisfying the following properties:

(P1) \( \xi^{\psi_i} \) lies on the positive imaginary axis for every \( 1 \leq i \leq g \).

(P2) The map \( E_\xi \) yields a Riemann form on \( \mathbb{C}^g/\Psi(O_{K^*}) \).

(P3) \( \delta_{K^*}^{-1} = \xi O_{K^*} \), where \( \delta_{K^*} \) is the different ideal of \( K^* \).

See [16, Theorem 4 in §6.2]. In this case, we call the pair \( (\mathbb{C}^g/\Psi(O_{K^*}), E_\xi) \) a principally polarized abelian variety.

(iii) The assumption (C2) holds if and only if \( (K; \{\varphi_i\}_{i=1}^n) \) is a primitive CM-type, that is, the abelian varieties of this CM-type are simple (16, §8.2, Proposition 26).

(iv) Throughout this paper, we fix an element \( \xi \) of \( K^* \) satisfying (P1)∼(P3) so that \( (\mathbb{C}^g/\Psi(O_{K^*}), E_\xi) \) becomes a principally polarized abelian variety.

By Assumption 5.1 (C2), one can define a group homomorphism
\[ \varphi : K^* \to (K^*)^* \]
\[ a \mapsto \prod_{i=1}^n a^{\varphi_i}, \]
and extend it naturally to a homomorphism of idele groups \( \varphi : K_\mathbb{A}^\times \to (K^*)_\mathbb{A}^\times \). It is also known that for a fractional ideal \( a \) of \( K \) there exists a fractional ideal \( \varphi(a) \) of \( K^* \) such that
\[ \varphi(a)O_L = \prod_{i=1}^n (aO_L)^{\varphi_i} \] (5)

([16, Proposition 29 in §8.3]).

For a number field \( F \) and a nonzero integral ideal \( a \) of \( F \) let \( N_F(a) \) be the absolute norm of \( a \), namely, \( N_F(a) = \mid O_F/a \mid \) (so, \( N_{F/Q}(a) = N_F(a)Z \)). In general, for a fractional ideal \( b \) of \( F \) with
prime ideal factorization \( b = \prod_p p^{e_p} \) we define \( N_F(b) = \prod_p N_F(p)^{e_p} \). Furthermore, let \( D_{F/Q}(b) \) be the discriminant ideal of \( b \) and \( d_{F/Q}(b) \) be its positive generator in \( Q \). We then have the relation

\[
d_{F/Q}(b) = N_F(b)^2 d_{F/Q}(O_F)
\]

(9, Proposition 13 in Chapter III).

Let \( K_0 \) be the fixed field of \( L \) by the subgroup

\[
\langle \sigma \in \text{Gal}(L/Q) \mid \sigma|_K = \varphi_i \text{ for some } i \rangle
\]

of \( \text{Gal}(L/Q) \). One can readily check that \( K_0 \) becomes either an imaginary quadratic subfield of \( K \) and \( K^* \), or \( Q \). In particular, we see from the assumption (C2) that \( K_0 = Q \) when \( g \geq 2 \) (\[11\], Remark (1) in p. 213) or \[16\] Theorem 3 in \[6.2\]).

From now on, we let \( f = f_0O_K \) for a proper nontrivial ideal \( f_0 \) of \( O_{K_0} \). Let \( C \) be a given ray class in \( \text{Cl}(f) \). For an integral ideal \( \mathfrak{c} \) in \( C \) we set

\[
m_\mathfrak{c} = \sqrt[2]{N_{K^*}(f^{*})^{-1} \varphi(\mathfrak{c})},
\]

where \( f^* = f_0O_{K^*} \). Let \( d_0 = 2/[K_0 : Q] \). Then we get

\[
m_\mathfrak{c} = \sqrt{N_{K^*}(f^{*})^{-1/2}N_{K^*}(\varphi(\mathfrak{c})\overline{\varphi(\mathfrak{c})})^{1/2}}
\]

\[
= \sqrt{N_{K^*}(N_{K_0}(f_0)^{d_0}O_{K^*})^{-1/2}N_{K^*}(N_{K}(\mathfrak{c}))^{1/2}}
\]

\[
= N_{K_0}(f_0)^{-d_0}N_{K}(\mathfrak{c}).
\]

**Proposition 5.3.** \( (\mathbb{C}^g/\Psi(f^*\varphi(\mathfrak{c})^{-1}), E_{\xi m_\mathfrak{c}}) \) is also a principally polarized abelian variety.

**Proof.** Since \( m_\mathfrak{c} \) is a positive rational number, we see that \( \xi m_\mathfrak{c} \) lies on the positive imaginary axis and \( E_{\xi m_\mathfrak{c}} = m_\mathfrak{c} E_\xi \). So, we achieve that

\[
E_{\xi m_\mathfrak{c}}(\Psi(f^*\varphi(\mathfrak{c})^{-1}), \Psi(f^*\varphi(\mathfrak{c})^{-1})) = \text{Tr}_{K^*/Q}(\xi m_\mathfrak{c} f^* \varphi(\mathfrak{c})^{-1} f^* \overline{\varphi(\mathfrak{c})}) \quad \text{by (3)}
\]

\[
= \text{Tr}_{K^*/Q}(\xi m_\mathfrak{c} f^* (\varphi(\mathfrak{c}) \overline{\varphi(\mathfrak{c})})^{-1})
\]

\[
= \text{Tr}_{K^*/Q}(\xi m_\mathfrak{c} N_{K_0}(f_0)^{d_0} N_{K}(\mathfrak{c})^{-1} O_{K^*})
\]

\[
= \text{Tr}_{K^*/Q}(\xi O_{K^*}) \quad \text{by (3)}
\]

\[
= E_\xi(\Psi(O_{K^*}), \Psi(O_{K^*}))
\]

\[
\subseteq \mathbb{Z} \quad \text{because } E_\xi \text{ is a Riemann form on } \mathbb{C}^g/\Psi(O_{K^*}).
\]

Thus \( E_{\xi m_\mathfrak{c}} \) is a Riemann form on \( \mathbb{C}^g/\Psi(f^*\varphi(\mathfrak{c})^{-1}) \).

By Assumption 5.1(C1) we can take a symplectic basis \( \{\mathbf{a}_1, \ldots, \mathbf{a}_{2g}\} \) of the principally polarized abelian variety \( (\mathbb{C}^g/\Psi(O_{K^*}), E_\xi) \) in such a way that

\[
\Psi(O_{K^*}) = \sum_{j=1}^{2g} \mathbb{Z} \mathbf{a}_j \quad \text{and} \quad [E_\xi(\mathbf{a}_i, \mathbf{a}_j)]_{1 \leq i, j \leq 2g} = \begin{bmatrix} O & -I_g \\ I_g & O \end{bmatrix}.
\]
Let \( x_1, \ldots, x_{2g} \) be elements of \( \mathcal{O}_{K^*} \) such that \( a_j = \Psi(x_j) \) \((1 \leq j \leq 2g)\). On the other hand, let \( \{b_1, \ldots, b_{2g}\} \) be a symplectic basis of the abelian variety \((\mathbb{C}^g/\Psi(\varphi(\epsilon)^{-1}), E_{\xi m_e})\) such that

\[
\Psi(\varphi(\epsilon)^{-1}) = \sum_{j=1}^{2g} \mathbb{Z}b_j \quad \text{and} \quad \left[ E_{\xi m_e}(b_i, b_j) \right]_{1 \leq i, j \leq 2g} = \begin{bmatrix} O & -\mathcal{E} \\ \mathcal{E} & O \end{bmatrix},
\]

where \( \mathcal{E} = \begin{bmatrix} \varepsilon_1 & 0 & \cdots & 0 \\ 0 & \varepsilon_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon_g \end{bmatrix} \) for some positive integers \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_g \) satisfying \( \varepsilon_1 \mid \varepsilon_2 \mid \cdots \mid \varepsilon_g \).

Furthermore, let \( y_1, \ldots, y_{2g} \) be elements of \( \varphi(\epsilon)^{-1} \) such that \( b_j = \Psi(y_j) \) \((1 \leq j \leq 2g)\), and hence \( \{y_1, \ldots, y_{2g}\} \) is a \( \mathbb{Z} \)-basis for \( \varphi(\epsilon)^{-1} \). Since \( \mathcal{O}_{K^*} \) and \( \varphi(\epsilon)^{-1} \) are free \( \mathbb{Z} \)-modules of rank \( 2g \) in \( K^* \), there is a matrix \( \alpha \in \text{GL}_{2g}(\mathbb{Q}) \) so as to have

\[
\begin{bmatrix} x_1 \\ \vdots \\ x_{2g} \end{bmatrix} = \alpha \begin{bmatrix} y_1 \\ \vdots \\ y_{2g} \end{bmatrix}. \tag{9}
\]

We then obtain

\[
\begin{bmatrix} x_1^\psi_1 & \cdots & x_1^\psi_g & x_2^\psi_1 & \cdots & x_2^\psi_g \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{2g}^\psi_1 & \cdots & x_{2g}^\psi_g \end{bmatrix} = \alpha \begin{bmatrix} y_1^\psi_1 & \cdots & y_1^\psi_g & y_2^\psi_1 & \cdots & y_2^\psi_g \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ y_{2g}^\psi_1 & \cdots & y_{2g}^\psi_g \end{bmatrix}.
\]

Taking determinant and squaring yields

\[
\Delta_{K^*/\mathbb{Q}}(x_1, \ldots, x_{2g}) = \det(\alpha)^2 \Delta_{K^*/\mathbb{Q}}(y_1, \ldots, y_{2g})
\]

\((\text{cf. } \S 1.13)\), and so it follows from \((6)\) and the definition \((7)\) that

\[
det(\alpha)^2 = \frac{\Delta_{K^*/\mathbb{Q}}(x_1, \ldots, x_{2g})}{\Delta_{K^*/\mathbb{Q}}(y_1, \ldots, y_{2g})} = \frac{d_{K^*/\mathbb{Q}}(\mathcal{O}_{K^*})}{d_{K^*/\mathbb{Q}}(\varphi(\epsilon)^{-1})} = \frac{\mathcal{N}_{K^*/\mathbb{Q}}((\varphi(\epsilon)^{-1})^{-2})}{\mathcal{N}_{K^*/\mathbb{Q}}((\varphi(\epsilon)^{-1})^2)} = m_e^{2g}. \tag{10}
\]

Since

\[
\begin{bmatrix} a_1 & \cdots & a_{2g} \end{bmatrix} = \begin{bmatrix} b_1 & \cdots & b_{2g} \end{bmatrix} \alpha^T
\]

by \((9)\), we deduce that

\[
m_e \begin{bmatrix} O & -I_g \\ I_g & O \end{bmatrix} = \left[ m_e E_{\xi}(a_i, a_j) \right]_{1 \leq i, j \leq 2g} = \left[ E_{\xi m_e}(a_i, a_j) \right]_{1 \leq i, j \leq 2g}.
\]
\[ E \xi m_c \left( \sum_{k=1}^{2g} a_{ik} b_k \sum_{\ell=1}^{2g} a_{j\ell} b_\ell \right)_{1 \leq i,j \leq 2g} \] where \( \alpha = \left[ a_{ij} \right]_{1 \leq i,j \leq 2g} \)

\[ = \alpha E \xi m_c \left( b_i, b_j \right)_{1 \leq i,j \leq 2g} \alpha^T \]

By taking determinant we get
\[ m_\xi^{2g} = \det(\alpha)^2 (\varepsilon_1 \cdots \varepsilon_g)^2, \]
which yields by \((\ref{10})\) that \(\varepsilon_1 = \cdots = \varepsilon_g = 1\), and so \(E = I_g\). Therefore, \(\left( \mathbb{C}^g / \Psi(f^* \varphi(c)^{-1}), E \xi m_c \right)\) is a principally polarized abelian variety as desired.

As in the proof of Proposition \ref{5.3} let \(\{ b_1, \ldots, b_{2g} \}\) be a symplectic basis of the principally polarized abelian variety \(\left( \mathbb{C}^g / \Psi(f^* \varphi(c)^{-1}), E \xi m_c \right)\), and let \(y_1, \ldots, y_{2g}\) be elements of \(f^* \varphi(c)^{-1}\) satisfying \(b_j = \Psi(y_j) \ (1 \leq j \leq 2g)\). As is well known (\cite{I} Proposition 8.1.1), the \(g \times g\) matrix
\[ Z_*^* = \left[ \begin{array}{cccc} b_{g+1} & \cdots & b_{2g} \\ \vdots & \ddots & \vdots \\ b_1 & \cdots & b_g \end{array} \right]^{-1} \left[ \begin{array}{cccc} b_1 & \cdots & b_g \end{array} \right] \]
belongs to \(\mathbb{H}_g\), which we call a CM-point. Since the smallest positive integer \(N\) in \(f = f_0 \mathcal{O}_K\) also belongs to \(f^* \varphi(c)^{-1} = f_0 \varphi(c)^{-1}\), we can express \(N\) as
\[ N = \sum_{j=1}^{2g} r_j y_j \quad \text{for some unique integers } r_1, \ldots, r_{2g}. \]  \((\ref{11})\)

**Definition 5.4.** We define the Siegel invariant \(\Theta_f(C)\) modulo \(f\) at \(C\) by
\[ \Theta_f(C) = \Theta \left( \begin{array}{c} r_1/N \\ \vdots \\ r_{2g}/N \end{array} \right), Z_*^*. \]

**6 Well-definedness of Siegel invariants**

Following the same notations and assumptions as in \cite{I} we shall show in this section that the Siegel invariant \(\Theta_f(C)\) is well defined, independent of the choices of a symplectic basis of the principally polarized abelian variety \(\left( \mathbb{C}^g / \Psi(f^* \varphi(c)^{-1}), E \xi m_c \right)\) and an integral ideal \(c\) in \(C\).

**Proposition 6.1.** \(\Theta_f(C)\) does not depend on the choice of a symplectic basis \(\{ b_1, \ldots, b_{2g} \}\) of \(\left( \mathbb{C}^g / \Psi(f^* \varphi(c)^{-1}), E \xi m_c \right)\).

**Proof.** Let \(\{ \bar{b}_1, \ldots, \bar{b}_{2g} \}\) be another symplectic basis of \(\left( \mathbb{C}^g / \Psi(f^* \varphi(c)^{-1}), E \xi m_c \right)\). Then we have
\[ \left[ \begin{array}{cccc} \bar{b}_1 & \cdots & \bar{b}_{2g} \end{array} \right] = \left[ \begin{array}{cccc} b_1 & \cdots & b_{2g} \end{array} \right] \beta \quad \text{for some } \beta = \left[ \begin{array}{cc} A & B \\ C & D \end{array} \right] \in \text{GL}_{2g}(\mathbb{Z}). \]  \((\ref{12})\)
Let \( \tilde{Z}_0^* \) be the CM-point in \( \mathbb{H}_g \) associated with \( \{\tilde{b}_1, \ldots, \tilde{b}_{2g}\} \). Then we obtain

\[
\tilde{Z}_0^* = \left[ \tilde{b}_{g+1} \cdots \tilde{b}_{2g} \right]^{-1} \left[ \tilde{b}_1 \cdots \tilde{b}_g \right] = (\left[ \tilde{b}_1 \cdots \tilde{b}_g \right] B + \left[ \tilde{b}_{g+1} \cdots \tilde{b}_{2g} \right] D)^{-1} (\left[ \tilde{b}_1 \cdots \tilde{b}_g \right] A + \left[ \tilde{b}_{g+1} \cdots \tilde{b}_{2g} \right] C) \text{ by (12)}
\]

\[
= \left( A^T \left[ \tilde{b}_1 \cdots \tilde{b}_g \right]^T + C^T \left[ \tilde{b}_{g+1} \cdots \tilde{b}_{2g} \right]^T \right) \left( B^T \left[ \tilde{b}_1 \cdots \tilde{b}_g \right]^T + D^T \left[ \tilde{b}_{g+1} \cdots \tilde{b}_{2g} \right]^T \right)^{-1}
\]

because \( (\tilde{Z}_0^*)^T = \tilde{Z}_0^* \)

\[
= \left( A^T (Z_0^*)^T + C^T (B^T (Z_0^*)^T + D^T)^{-1} \right)
\]

\[
= \left( A^T Z_0^* + C^T (B^T Z_0^* + D^T)^{-1} \right) \text{ since } (Z_0^*)^T = Z_0^*
\]

\[
= \beta^T (Z_0^*).
\]

On the other hand, we deduce that

\[
\begin{bmatrix}
O & -I_g \\
I_g & O
\end{bmatrix}
= \left[ E_{\xi m_i}(\tilde{b}_{l_i}, \tilde{b}_{j}) \right]_{1 \leq i, j \leq 2g}
= \beta^T \left[ E_{\xi m_i}(\tilde{b}_{l_i}, \tilde{b}_{j}) \right]_{1 \leq i, j \leq 2g} \beta \text{ by (12)}
\]

\[
= \beta^T \begin{bmatrix} O & -I_g \\ I_g & O \end{bmatrix} \beta,
\]

which claims that \( \beta \) belongs to \( \text{Sp}_{2g}(\mathbb{Z}) \), and hence \( \beta^T \in \text{Sp}_{2g}(\mathbb{Z}) \).

Let \( \tilde{y}_1, \ldots, \tilde{y}_{2g} \) be elements of \( \mathfrak{f}^* \varphi(\mathcal{C})^{-1} \) such that \( \tilde{b}_j = \Psi(\tilde{y}_j) \) \( (1 \leq j \leq 2g) \). Together with (11) we can express \( N \) as

\[
N = \sum_{j=1}^{2g} r_j \tilde{y}_j = \sum_{j=1}^{2g} \tilde{r}_j \tilde{y}_j \text{ for some unique integers } \tilde{r}_1, \ldots, \tilde{r}_{2g}.
\]

Taking \( \Psi \) we achieve by (12) that

\[
\Psi(N) = \left[ \begin{array}{ccc} b_1 & \cdots & b_{2g} \\ r_1 \\ \vdots \\ r_{2g} \end{array} \right] = \left[ \begin{array}{ccc} \tilde{b}_1 & \cdots & \tilde{b}_{2g} \\ \tilde{r}_1 \\ \vdots \\ \tilde{r}_{2g} \end{array} \right] = \left[ \begin{array}{ccc} b_1 & \cdots & b_{2g} \end{array} \right] \beta \left[ \begin{array}{ccc} \tilde{r}_1 \\ \vdots \\ \tilde{r}_{2g} \end{array} \right],
\]

from which we have

\[
\begin{bmatrix} \tilde{r}_1 \\ \vdots \\ \tilde{r}_{2g} \end{bmatrix} = \beta^{-1} \begin{bmatrix} r_1 \\ \vdots \\ r_{2g} \end{bmatrix}.
\]
We then derive that
\[
\Theta\left(\begin{bmatrix}
\tilde{r}_1/N \\
\vdots \\
\tilde{r}_{2g}/N 
\end{bmatrix}, \tilde{Z}_0^*\right) = \Theta\left(\beta^{-1}\begin{bmatrix}
r_1/N \\
\vdots \\
r_{2g}/N 
\end{bmatrix}, \beta^T(Z_0^*)\right)
\]
by (13) and (14)
\[
= \Theta\left(\begin{bmatrix}
r_1/N \\
\vdots \\
r_{2g}/N 
\end{bmatrix}, Z_0^*\right)
\]
by the fact \(\beta^T \in \text{Sp}_{2g}(Z)\) and Lemma 4.4 (ii)
\[
= \Theta\left(\begin{bmatrix}
r_1/N \\
\vdots \\
r_{2g}/N 
\end{bmatrix}, \tilde{Z}_0^*\right).
\]
This completes the proof.

**Proposition 6.2.** \(\Theta_f(C)\) does not depend on the choice of an integral ideal \(c\) in \(C\).

**Proof.** Let \(c'\) be another integral ideal in the class \(C\), and so
\[
c' = \nu c \quad \text{for some } \nu \in K^\times \text{ such that } \nu \equiv 1 \pmod{f}.
\]
Then we may write \(\nu\) as
\[
\nu = 1 + x \quad \text{for some } x \in fa^{-1}, \text{ where } a \text{ is an integral ideal of } K \text{ prime to } f.
\]
Since \(1 \in c^{-1}\) and \(\nu \in c'c^{-1} \subseteq c^{-1}\) by (15), we get
\[
x = -1 + \nu \in c^{-1}.
\]
Thus we derive that
\[
xac \subseteq fc \cap a \quad \text{by the fact } x \in fa^{-1} \text{ and (16)}
\]
\[
\subseteq f \cap a
\]
\[
= fa \quad \text{since } f \text{ and } a \text{ are relatively prime,}
\]
and hence
\[
x \in fc^{-1}.
\]
Let
\[
y_j' = \varphi(\nu^{-1})y_j \quad \text{and} \quad b_j' = \Psi(y_j') \quad (1 \leq j \leq 2g).
\]
We know that \(\{y_1', \ldots, y_{2g}\}\) is a \(\mathbb{Z}\)-basis for \(f^*\varphi(c')^{-1}\) and
\[
b_j' = Db_j \quad \text{where } D = \begin{bmatrix}
\varphi(\nu^{-1})\psi_1 & 0 & \cdots & 0 \\
0 & \varphi(\nu^{-1})\psi_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \varphi(\nu^{-1})\psi_g
\end{bmatrix}.
\]
Moreover, we obtain that
\[
\left[ E_{\xi m, c}(b'_i, b'_j) \right]_{1 \leq i, j \leq 2g} = \left[ \text{Tr}_{K^*/Q}(\xi m, c)[y_i, y_j]^\nu \right]_{1 \leq i, j \leq 2g} \quad \text{by (11)}
\]
\[
= \left[ \text{Tr}_{K^*/Q}(\xi m, c, \varphi(\nu^{-1})y_i, \varphi(\nu^{-1})y_j) \right]_{1 \leq i, j \leq 2g}
\]
\[
= \left[ \text{Tr}_{K^*/Q}(\xi m, c, N_K/Q(\nu^{-1})y_i, y_j) \right]_{1 \leq i, j \leq 2g}
\]
\[
= \left[ \text{Tr}_{K^*/Q}(\xi m, c, \varphi(\nu^{-1})y_i, y_j) \right]_{1 \leq i, j \leq 2g}
\]
by (8) and the fact $N_K/Q(\nu) > 0$
\[
= \left[ E_{\xi m, c}(b_i, b_j) \right]_{1 \leq i, j \leq 2g} \quad \text{by (4)}
\]
\[
= \left[ O - I_g \right]_{1 \leq i, j \leq 2g}
\]
\[
= \left[ D_{b_{g+1}} \cdots D_{b_{2g}} \right]^{-1} \left[ b_1 \cdots b_g \right] = Z^*_g.
\]

Hence, \{b'_1, \ldots, b'_{2g}\} is a symplectic basis of $(C^g/\Psi, E_{\xi m, c})$, and the associated CM-point is given by (18)
\[
\left[ b'_{g+1} \cdots b'_{2g} \right]^{-1} \left[ b'_1 \cdots b'_g \right] = \left[ D_{b_{g+1}} \cdots D_{b_{2g}} \right]^{-1} \left[ D_{b_1} \cdots D_{b_g} \right] = Z^*_g.
\]

Since $K_0$ is the subfield of $K$ fixed by the subgroup $\langle \sigma \in \text{Gal}(L/Q) \mid \sigma|_K = \varphi_i \text{ for some } i \rangle$ of $\text{Gal}(L/Q)$, we have $(f_O L)^{\varphi_i} = (f_0 O_L)^{\varphi_i} = f_0 O_L = f^* O_L$. We then achieve by the fact $x \in f^{-1}$ that
\[
\varphi(\nu) = \varphi(1 + x) = \prod_{i=1}^n (1 + x)^{\varphi_i} \in K^* \cap (1 + f^* \varphi(c)^{-1} O_L) = 1 + f^* \varphi(c)^{-1}.
\]

Since $N \in f^* \varphi(c)^{-1}$, one can express $N$ as
\[
N = \sum_{j=1}^{2g} r'_j / \varphi(\nu^{-1})y_j \quad \text{for some integers } r'_1, \ldots, r'_{2g}.
\]

Hence we have
\[
\varphi(\nu) = \sum_{j=1}^{2g} (r'_j / N)y_j,
\]
which implies by (11) and (19)
\[
\left[ \begin{array}{c} r'_1 / N \\ \vdots \\ r'_{2g} / N \end{array} \right] \in \left[ \begin{array}{c} r_1 / N \\ \vdots \\ r_{2g} / N \end{array} \right] + \mathbb{Z}^{2g}.
\]

Therefore, the proposition follows from Lemma 4.1 (iii). \qed
7 Galois conjugates of Siegel invariants

Finally, we shall show that under the Assumption 5.1 the Siegel invariant $\Theta_{\mathfrak{f}}(C)$ lies in the ray class field $K_{\mathfrak{f}}$ and satisfies the natural transformation formula via the Artin reciprocity map for $\mathfrak{f}$.

Let $h : K^* \to M_{2g}(\mathbb{Q})$ be the regular representation with respect to the $\mathbb{Q}$-basis $\{y_1, \ldots, y_{2g}\}$ of $K^*$, that is, $h$ is the map given by the relation

$$h(a) \begin{bmatrix} y_1 \\ \vdots \\ y_{2g} \end{bmatrix} = a \begin{bmatrix} y_1 \\ \vdots \\ y_{2g} \end{bmatrix} \quad (a \in K^*).$$

(20)

We naturally extend $h$ to the map $(K^*)_{\mathbb{A}} \to M_{2g}(\mathbb{Q}_{\mathbb{A}})$, and also denote it by $h$.

**Proposition 7.1** (Shimura’s Reciprocity Law). Let $f \in F$. If $f$ is finite at $Z_0^* \in \mathbb{H}_g$, then $f(Z_0^*)$ belongs to $K_{ab}$. Moreover, if $s \in K_{ab}^*$, then we get $h(\varphi(s)) \in G_{\mathbb{A}}$ and

$$f(Z_0^*[s,K]) = f^{r(h(\varphi(s)^{-1}))}(Z_0^*).$$

**Proof.** See [17, Lemma 9.5 and Theorem 9.6].

**Remark 7.2.** Observe that we are assuming $(K^*)^* = K$.

**Theorem 7.3.** If $\Theta_{\mathfrak{f}}(C)$ is finite, then it lies in $K_{\mathfrak{f}}$. Furthermore, it satisfies

$$\Theta_{\mathfrak{f}}(C)^{\sigma(D)} = \Theta_{\mathfrak{f}}(CD) \quad \text{for all} \ D \in \text{Cl}(\mathfrak{f}),$$

where $\sigma$ is the Artin reciprocity map for $\mathfrak{f}$.

**Proof.** Since $\Theta_{\mathfrak{f}}(C)$ belongs to $K_{ab}$ by Propositions 4.5 and 7.1, there is a positive integer $M$ such that $2N^2 \mid M$ and $\Theta_{\mathfrak{f}}(C) \in K_{\mathfrak{g}}$, where $\mathfrak{g} = M\mathcal{O}_K$.

We can take integral ideals $\mathfrak{c} \subset C$ (as before) and $\mathfrak{d} \subset D$ which are relatively prime to $\mathfrak{g}$ by using the surjectivity of the natural map $\text{Cl}(\mathfrak{g}) \to \text{Cl}(\mathfrak{f})$. Let $\mathbf{b}_1, \ldots, \mathbf{b}_{2g}$ and $\mathbf{d}_1, \ldots, \mathbf{d}_{2g}$ be symplectic bases of the principally polarized abelian varieties $(C^g/\Psi(f^*\varphi(\mathfrak{c})^{-1}), E_{\xi_{m_1}})$ and $(C^g/\Psi(f^*\varphi(\mathfrak{d})^{-1}), E_{\xi_{m_{m_2}}})$, respectively. Furthermore, let $y_1, \ldots, y_{2g}$ and $z_1, \ldots, z_{2g}$ be elements of $f^*\varphi(\mathfrak{c})^{-1}$ and $f^*\varphi(\mathfrak{d})^{-1}$, respectively, such that $\mathbf{b}_j = \Psi(y_j)$ and $\mathbf{d}_j = \Psi(z_j)$ ($1 \leq j \leq 2g$).

Since $f^*\varphi(\mathfrak{c})^{-1} \leq f^*\varphi(\mathfrak{d})^{-1} = f^*\varphi(\mathfrak{c})^{-1}\varphi(\mathfrak{d})^{-1}$, we have

$$\begin{bmatrix} y_1 \\ \vdots \\ y_{2g} \end{bmatrix} = \delta \begin{bmatrix} z_1 \\ \vdots \\ z_{2g} \end{bmatrix} \quad \text{for some} \ \delta \in M_{2g}(\mathbb{Z}) \cap \text{GL}_{2g}(\mathbb{Q}),$$

(21)

and hence

$$\begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_{2g} \end{bmatrix} = \begin{bmatrix} \mathbf{d}_1 & \cdots & \mathbf{d}_{2g} \end{bmatrix} \delta^T.$$

If we let $Z_0^*$ and $Z_1^*$ be the CM-points associated with $\{\mathbf{b}_1, \ldots, \mathbf{b}_{2g}\}$ and $\{\mathbf{d}_1, \ldots, \mathbf{d}_{2g}\}$, respectively, then we obtain

$$Z_1^* = \delta^{-1}(Z_0^*)$$

(22)
in a similar way as in the proof of Proposition 6.1. We also obtain
\[
\begin{bmatrix}
O & -I_g \\
I_g & O
\end{bmatrix} = \left[ E_{\xi_m}(b_i, b_j) \right]_{1 \leq i, j \leq 2g} = \delta \left[ E_{\xi_m}(d_i, d_j) \right]_{1 \leq i, j \leq 2g} \delta^T = \delta \left[ m_\epsilon m_{\epsilon_0}^{-1} E_{\xi_m}(d_i, d_j) \right]_{1 \leq i, j \leq 2g} \delta^T = m_\epsilon m_{\epsilon_0}^{-1} \delta \begin{bmatrix}
O & -I_g \\
I_g & O
\end{bmatrix} \delta^T.
\]

This shows that
\[
\delta \in M_{2g}(\mathbb{Z}) \cap G_+ \text{ with } \det(\delta) = (m_\epsilon^{-1} m_{\epsilon_0})^g = N_K(\mathfrak{d})^g \text{ by (8).}
\]

Let \( s = (s_p)_p \) be an idele of \( K \) such that
\[
\begin{align*}
  s_p &= 1 \quad \text{if } p \mid M, \\
  s_p(O_K)_p &= \mathfrak{d}_p \quad \text{if } p \nmid M.
\end{align*}
\] (24)

If we denote by \( \widetilde{D} \) the ray class in \( \text{Cl}(\mathfrak{g}) \) containing \( \mathfrak{d} \), then we get by (24) that
\[
[s, K]|_{K_\mathfrak{d}} = \sigma_\mathfrak{d}(\widetilde{D}),
\] (25)
\[
\varphi(s)^{-1}_p(\mathcal{O}_K)_p = \varphi(\mathfrak{d})^{-1}_p \text{ for all rational primes } p.
\] (26)

By (25)~(26), we deduce that for each rational prime \( p \), the components of each of the vectors
\[
h(\varphi(s)^{-1})_p \begin{bmatrix} y_1 \\ \vdots \\ y_{2g} \end{bmatrix} \text{ and } \delta^{-1} \begin{bmatrix} y_1 \\ \vdots \\ y_{2g} \end{bmatrix}
\]
form a basis of \( \mathfrak{f}^* \varphi(\mathfrak{d})^{-1}_p = \mathfrak{f}^* \varphi(\mathfrak{c})^{-1}_p \varphi(\mathfrak{d})^{-1}_p \). Thus there is a matrix \( u = (u_p)_p \in \prod_p \text{GL}_{2g}(\mathbb{Z}_p) \) satisfying
\[
h(\varphi(s)^{-1})_p = u \delta^{-1}.
\] (27)

On the other hand, since \( \delta \) and
\[
\begin{bmatrix}
I_g & O \\
O & N(\mathfrak{d}) I_g
\end{bmatrix}
\]
can be viewed as elements of \( \text{GSp}_{2g}(\mathbb{Z}/M\mathbb{Z}) \) by (23), there exists a matrix \( \gamma \in \text{Sp}_{2g}(\mathbb{Z}) \) such that
\[
\delta \equiv \begin{bmatrix}
I_g & O \\
O & N(\mathfrak{d}) I_g
\end{bmatrix} \gamma \quad \text{(mod } M \cdot M_{2g}(\mathbb{Z})).
\] (28)

due to the surjectivity of the reduction \( \text{Sp}_{2g}(\mathbb{Z}) \to \text{Sp}_{2g}(\mathbb{Z}/M\mathbb{Z}) \). Since \( h(\varphi(s)^{-1})_p = I_{2g} \) for all \( p \mid M \) by (24), we achieve \( u_p = \delta \) for all \( p \mid M \) by (27). Hence we obtain by (28) that
\[
u_p \gamma^{-1} \equiv \begin{bmatrix}
I_g & O \\
O & N(\mathfrak{d}) I_g
\end{bmatrix} \quad \text{(mod } M \cdot M_{2g}(\mathbb{Z}_p)).
\] (29)
If we write
\[ N = \sum_{j=1}^{2g} r_j y_j \]
for some integers \( r_1, \ldots, r_{2g} \), then we see by (21) that
\[
N = \begin{bmatrix} r_1 & \cdots & r_{2g} \end{bmatrix} \begin{bmatrix} y_1 \\
\vdots \\
y_{2g} \end{bmatrix} = \begin{bmatrix} r_1 & \cdots & r_{2g} \end{bmatrix} \delta(\delta^{-1}) \begin{bmatrix} y_1 \\
\vdots \\
y_{2g} \end{bmatrix} = \begin{bmatrix} r_1 & \cdots & r_{2g} \end{bmatrix} \delta \begin{bmatrix} z_1 \\
\vdots \\
z_{2g} \end{bmatrix}. \tag{30}
\]

Letting \( v = \begin{bmatrix} r_1/N \\
\vdots \\
r_{2g}/N \end{bmatrix} \) we derive that
\[
\Theta_f(C)^{\tau_g(\overline{D})} = \Theta_f(C)^{[s,K]} \ \text{by (25)}
\]
\[ = \Theta(v, Z_0^*[s,K]) \ \text{by Definition 5.4}
\]
\[ = \Theta(v, Z)^{\tau(h(\varphi(s)^{-1}))}|_{Z=Z_0^*} \ \text{by Proposition 7.1}
\]
\[ = \Theta(v, Z)^{\tau(u\delta^{-1})}|_{Z=Z_0^*} \ \text{by (27)}
\]
\[ = \Theta(v, Z)^{\tau(u\gamma^{-1})\tau(\gamma)(\delta^{-1})}|_{Z=Z_0^*} \ \text{by (28) and Lemma 4.4 (iii)}
\]
\[ = \Theta(\begin{bmatrix} I_g & O \\
O & N(\delta)I_g \end{bmatrix} v, Z)^{\tau(\gamma)(\delta^{-1})}|_{Z=Z_0^*} \ \text{by (29) and Lemma 4.4 (ii)}
\]
\[ = \Theta(\gamma^T v, Z)^{\tau(\delta^{-1})}|_{Z=Z_0^*} \ \text{by Lemma 4.4 (i)}
\]
\[ = \Theta(\delta^T v, \delta^{-1}(Z_0^*)) \ \text{owing to the fact } \delta \in G_+ \text{ and (A1)}
\]
\[ = \Theta_f(C D) \ \text{by (22), (30) and Definition 5.4}
\]

In particular, suppose that \( d = dO_K \) for some \( d \in O_K \) such that \( d \equiv 1 \mod f \), and so \( D \) is the identity class of \( \text{Cl}(f) \). The above observation yields that \( \sigma_\delta(\overline{D}) \) leaves \( \Theta_f(C) \) fixed, from which we conclude that \( \Theta_f(C) \) lies in \( K_f \) as desired. This proves the theorem.

Lastly, we expect that under the Assumption 5.1 the following conjecture will turn out to be affirmative.

**Conjecture 7.4.** The Siegel invariant \( \Theta_f(C) \) discussed here is a primitive generator of the fixed field of \( \ker(\overline{\varphi}) \) in the ray class field \( K_f \) of a CM-field \( K \), where \( \overline{\varphi} : \text{Cl}(f) \rightarrow \text{Cl}(f^*) \) is the natural homomorphism induced from the map \( \varphi \) defined in (5). Here, \( \text{Cl}(f^*) \) is the ray class group of \( K^* \) modulo \( f^* \).

**Example 7.5.** Let \( \ell \) be an odd prime and \( g = (\ell - 1)/2 \). Let \( K = \mathbb{Q}(\zeta_\ell) \) with \( \zeta_\ell = e^{2\pi i/\ell} \). Then \( [K : \mathbb{Q}] = 2g \). For each \( 1 \leq i \leq g \), let \( \varphi_i \) be the element of \( \text{Gal}(K/\mathbb{Q}) \) determined by \( \zeta_\ell^{\varphi_i} = \zeta_i \). Then \( (K; \{\varphi_1^{-1}, \varphi_2^{-1}, \ldots, \varphi_g^{-1}\}) \) is a primitive CM-type and its reflex is \( (K; \{\varphi_1, \varphi_2, \ldots, \varphi_g\}) \) (16) p.
that is, such that

Thus CM-point is

Assume that the class number of $K$ is 1. Let $f = NO_K$ for a positive integer $N$, and let $C \in \text{Cl}(f)$. Take an integral ideal $c$ of $K$ in $C$. Then $a = \lambda O_K$ for some $\lambda \in O_K$. Let

$$x_j = \begin{cases} \zeta_j^{2j} & \text{for } 1 \leq j \leq g \\ \sum_{k=1}^{j-g} \zeta_k^{2k-1} & \text{for } g + 1 \leq j \leq 2g, \end{cases}$$

and $\varphi(a) = \prod_{i=1}^{g} a_i^{2i-1}$ for $a \in K$. Then $\{N\varphi^{-1}(x_j)\}_{j=1}^{2g}$ forms a free $\mathbb{Z}$-basis of $f\varphi(c)^{-1}$ and

$$[E_{\xi_{m_i}} (\Psi(N\varphi^{-1} x_i), \Psi(N\varphi^{-1} x_j))]_{1 \leq i,j \leq 2g} = \begin{bmatrix} O & -I_g \\ I_g & O \end{bmatrix}.$$ 

Thus $\{\Psi(N\varphi^{-1} x_j)\}_{j=1}^{2g}$ is a symplectic basis of $(\mathbb{C}^g / \Psi(f\varphi(c)^{-1}), E_{\xi_{m_k}})$ and the corresponding CM-point is

$$Z^*_f = \left[ \begin{array}{c} \Psi(N\varphi^{-1} x_{g+1}) & \cdots & \Psi(N\varphi^{-1} x_{2g}) \\ \Psi(x_{g+1}) & \cdots & \Psi(x_{2g}) \end{array} \right]^{-1} \left[ \begin{array}{c} \Psi(N\varphi^{-1} x_1) \\ \Psi(x_1) \end{array} \right].$$

Note that $Z^*_f$ does not depend on a ray class $C$. On the other hand, there exist integers $r_1, r_2, \ldots, r_{2g}$ such that

$$N = \sum_{j=1}^{2g} r_j (N\varphi^{-1} x_j),$$

that is, $\varphi(x) = \sum_{j=1}^{2g} r_j x_j$. Then we obtain

$$\Theta_f(C) = \Theta \left[ \begin{array}{c} r_1/N \\ r_2/N \\ \vdots \\ r_{2g}/N \end{array} \right] \left( Z^*_f \right).$$

Now, consider the case where $K = \mathbb{Q}(\zeta_5)$.

(i) Let $f = 5O_K$. One can readily show that $[K_f : K] = 5$ and

$$\text{Cl}(f) = \langle C_1 \rangle \cong \mathbb{Z}/5\mathbb{Z},$$

where $C_1$ denotes the ray class in Cl(f) containing the ideal $(2 + \zeta_5)O_K$. Let $C_k = C_1^k$ for an integer $k$. Then we have

$$\varphi(2 + \zeta_5) = (2 + \zeta_5)(2 + \zeta_5^3)$$

$$= -2\zeta_5 - 4\zeta_5^2 - 2\zeta_5^3 - 3\zeta_5^4 \quad \text{since } \sum_{k=0}^{4} \zeta_5^k = 0$$

20
\[ \equiv x_1 + 2x_2 + 0 \cdot x_3 + 3x_4 \pmod{f}, \]

where
\[ x_1 = \zeta_5^2, \quad x_2 = \zeta_5^4, \quad x_3 = \zeta_5, \quad x_4 = \zeta_5 + \zeta_5^3. \]

Hence we see from (31), (32) and Lemma 4.4 (i) that
\[ \Theta_f(C_1) = \Theta \begin{bmatrix} 1/5 \\ 2/5 \\ 0 \\ 3/5 \end{bmatrix} (Z_5^*) \approx -2.13359 \times 10^{-69} + 4.17297 \times 10^{-70}i, \]

where
\[ Z_5^* = \begin{bmatrix} \zeta_5 & \zeta_5 + \zeta_5^3 \\ \zeta_5^2 & \zeta_5^2 + \zeta_5 \end{bmatrix}^{-1} \begin{bmatrix} \zeta_5^2 & \zeta_5^4 \\ \zeta_5 & \zeta_5^3 \end{bmatrix}. \]

In like manner, we obtain
\[ \Theta_f(C_2) = \Theta \begin{bmatrix} 3/5 \\ 1/5 \\ 0 \\ 2/5 \end{bmatrix} (Z_5^*) \approx (4.16089 - 1.58401i) \times 10^{-50}, \]
\[ \Theta_f(C_3) = \Theta \begin{bmatrix} 3/5 \\ 2/5 \\ 0 \\ 2/5 \end{bmatrix} (Z_5^*) \approx (4.16089 + 1.58401i) \times 10^{-50}, \]
\[ \Theta_f(C_4) = \Theta \begin{bmatrix} 2/5 \\ 2/5 \\ 0 \\ 4/5 \end{bmatrix} (Z_5^*) \approx -2.13359 \times 10^{-69} - 4.17297 \times 10^{-70}i, \]
\[ \Theta_f(C_5) = \Theta \begin{bmatrix} 1/5 \\ 1/5 \\ 0 \\ 1/5 \end{bmatrix} (Z_5^*) \approx 4.85930 \times 10^{-254}. \]

Here we estimate these values with the aid of Maple software. Observe that
\[ \Theta_f(C_k) = \overline{\Theta_f(C_{5-k})} \quad \text{for } k \in \mathbb{Z}. \]

Since all conjugates of \( \Theta_f(C_1) \) are distinct, we get
\[ K_f = K(\Theta_f(C)) \quad \text{for } C \in \text{Cl}(f). \]

(ii) Let \( f = 6 \mathcal{O}_K \). Then we have \( [K_f : K] = 10 \) and
\[ \text{Cl}(f) = \langle C_1 \rangle \cong \mathbb{Z}/10\mathbb{Z}. \]
In a similar way as in (i), one can estimate

\[
\Theta_f(C_1) = \Theta \begin{bmatrix} 2/6 \\ 3/6 \\ 0 \\ 4/6 \end{bmatrix} (Z_5^*) \approx (-1.68219 - 1.88870i) \times 10^{-66}
\]

\[
\Theta_f(C_2) = \Theta \begin{bmatrix} 2/6 \\ 5/6 \\ 0 \\ 2/6 \end{bmatrix} (Z_5^*) \approx (9.08964 + 7.01165i) \times 10^{-135}
\]

\[
\Theta_f(C_3) = \Theta \begin{bmatrix} 3/6 \\ 4/6 \\ 4/6 \end{bmatrix} (Z_5^*) \approx (-3.16257 + 1.88358i) \times 10^{-65}
\]

\[
\Theta_f(C_4) = \Theta \begin{bmatrix} 3/6 \\ 4/6 \\ 0 \\ 3/6 \end{bmatrix} (Z_5^*) \approx (2.29176 + 1.51419i) \times 10^{-93}
\]

\[
\Theta_f(C_5) = \Theta \begin{bmatrix} 5/6 \\ 0 \\ 0 \\ 4/6 \end{bmatrix} (Z_5^*) \approx 8.33316 \times 10^{-136}
\]

\[
\Theta_f(C_6) = \Theta \begin{bmatrix} 5/6 \\ 2/6 \\ 0 \\ 2/6 \end{bmatrix} (Z_5^*) \approx (2.29176 - 1.51419i) \times 10^{-93}
\]

\[
\Theta_f(C_7) = \Theta \begin{bmatrix} 4/6 \\ 2/6 \\ 1/6 \\ 3/6 \end{bmatrix} (Z_5^*) \approx (-3.16257 - 1.88358i) \times 10^{-65}
\]

\[
\Theta_f(C_8) = \Theta \begin{bmatrix} 3/6 \\ 1/6 \\ 0 \\ 3/6 \end{bmatrix} (Z_5^*) \approx (9.08964 - 7.01165i) \times 10^{-135}
\]

\[
\Theta_f(C_9) = \Theta \begin{bmatrix} 5/6 \\ 3/6 \\ 0 \\ 1/6 \end{bmatrix} (Z_5^*) \approx (-1.68219 + 1.88870i) \times 10^{-66}
\]

\[
\Theta_f(C_{10}) = \Theta \begin{bmatrix} 5/6 \\ 5/6 \\ 0 \\ 5/6 \end{bmatrix} (Z_5^*) \approx 3.26284 \times 10^{-348},
\]

which are distinct. Note that

\[
\Theta_f(C_k) = \Theta_f(C_{10-k}) \quad \text{for } k \in \mathbb{Z}.
\]

Therefore, we conclude

\[
K_f = K(\Theta_f(C)) \quad \text{for every } C \in \text{Cl}(f).
\]

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Department of Mathematical Sciences  
KAIST  
Daejeon 34141  
Republic of Korea  
E-mail address: jkkoo@math.kaist.ac.kr

Laboratoire de Mathématiques  
Institut Fourier  
B.P. 74  
F-38402 Saint-Martin-d’Hères  
France  
E-mail address: gilles.rbzt@gmail.com

Department of Mathematics  
Hankuk University of Foreign Studies  
Yongin-si, Gyeonggi-do 17035  
Republic of Korea  
E-mail address: dhshin@hufs.ac.kr

Department of Mathematical Sciences  
KAIST  
Daejeon 34141  
Republic of Korea  
E-mail address: math_dsyoon@kaist.ac.kr