Blow-up Rate Estimates and Blow-up Set for a System of Two Heat Equations with Coupled Nonlinear Neumann Boundary Conditions

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Abstract

This paper deals with the blow-up properties of positive solutions to a parabolic system of two heat equations, defined on a ball in \( \mathbb{R}^n \), associated with coupled Neumann boundary conditions of exponential type. The upper bounds of blow-up rate estimates are derived. Moreover, it is proved that the blow-up in this problem can only occur on the boundary.

Keywords: Heat equation; Neumann boundary conditions; Blow-up set; Blow-up rate estimate; Green function.

1. Introduction

In this paper, we consider the following parabolic system of two heat equations associated with Neumann boundary conditions:

\[
\begin{align*}
    u_t &= \Delta u, & v_t &= \Delta v, & (x, t) &\in B_R \times (0, T), \\
    \frac{\partial u}{\partial \eta} &= \lambda_1 e^{v^p}, & \frac{\partial v}{\partial \eta} &= \lambda_2 e^{u^q}, & (x, t) &\in \partial B_R \times (0, T), \\
    u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), & x &\in B_R,
\end{align*}
\]

(1)

where \( p, q > 1; \lambda_1, \lambda_2 > 0; B_R \) is a ball in \( \mathbb{R}^n \); \( \eta \) is the outward normal; \( u_0, v_0 \) are both smooth functions, radially symmetric, nonzero, nonnegative and satisfy the condition:

\[
(\Delta u_0, \eta) \geq 0, \quad u_0(\xi), v_0(\xi) \geq 0, \quad \text{for } \xi \in \bar{B}_R,
\]

(2)

Since the last decades, many authors have studied the blow-up properties to solutions of parabolic problems, defined on bounded domains [see for instance 1, 2]. One of the studied problems is the system of two heat equations defined in a ball, associated with coupled Neumann boundary conditions:

\[
\begin{align*}
    u_t &= \Delta u, & v_t &= \Delta v, & (x, t) &\in B_R \times (0, T), \\
    \frac{\partial u}{\partial \eta} &= f(v), & \frac{\partial v}{\partial \eta} &= g(u), & (x, t) &\in \partial B_R \times (0, T), \\
    u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), & x &\in B_R,
\end{align*}
\]

(3)

This problem was previously studied [3-6] : in case of the nonlinear functions \( f \) and \( g \) take one of the two forms:

\[
\begin{align*}
    f(v) &= v^p, & g(u) &= u^q, & p, q > 1, \\
    f(v) &= e^{pv}, & g(u) &= e^{qu}, & p, q > 0.
\end{align*}
\]

(4)

(5)

For both cases, it was shown that if the initial data \((u_0, v_0)\) are nonzero and nonnegative, then the blow-up can only occur on the boundary.

In addition to that, with case 4, it was proved that the blow-up rate estimates take the form:

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\[
c \leq \max_{x \in \Omega} u(x,t)(T-t)^{\frac{p+1}{2(pq-1)}} \leq C, \quad t \in (0,T),
\]
\[
c \leq \max_{x \in \Omega} v(x,t)(T-t)^{\frac{q+1}{2(pq-1)}} \leq C, \quad t \in (0,T)
\]
where \(c\) and \(C\) are positive constants.

While, with case 5, it was proved that the blow-up rate estimates take the form:
\[
C_1 \leq e^{qu(R,t)}(T-t)^{\frac{1}{2}} \leq C_2,
\]
\[
C_3 \leq e^{pv(R,t)}(T-t)^{1/2} \leq C_4
\]
where \(C_1, C_2, C_3\) and \(C_4\) are positive constants.

In this paper, firstly, we show that the upper blow-up rate estimates for problem 1 are as follows
\[
\max_{\overline{B}_R} u(x,t) \leq \log C_1 - \frac{\alpha}{2} \log(T-t), \quad 0 < t < T,
\]
\[
\max_{\overline{B}_R} v(x,t) \leq \log C_2 - \frac{\beta}{2} \log(T-t), \quad 0 < t < T,
\]
where \(\alpha = \frac{p+1}{pq-1}, \beta = \frac{q+1}{pq-1}\).

Secondly, we prove that the blow-up in problem 1 can only occur on the boundary.

2. Preliminaries
It is well known that with any smooth initial functions \((u_0, v_0)\), satisfying the compatibility condition 2, there exists a unique local classical solution to problem 1 [7]. On the other hand, it is easy to show that every nontrivial solution blows up simultaneously in finite time and that due to the known blow-up results of problem 3 with 4 and the comparison principle [2,3].

The next lemma, which was previously proved [2], states some properties of the classical solutions of problem 1.

For simplicity, we denote \(u(r, t) = u(x, t)\).

**Lemma 2.1** Let \((u, v)\) be a classical solution to problem 1. Then
1. \(u, v\) are positive, radial. Moreover, \(u_r, v_r \geq 0\) in \([0, R] \times (0, T)\).
2. \(u_t, v_t > 0\) in \(\overline{B}_R \times (0, T)\).

3. Blow-up Upper Rate Estimates
The next Lemmas and theorem, proved in other articles [5,8], will be used in this section to derive the upper blow-up rate estimates for problem 1.

**Lemma 3.1** [5]: Let \(A\) and \(B\) be positive and differentiable functions in \([0, T)\), such that they satisfy the two inequalities:
\[
A'(t) \geq c \frac{B'(t)}{\sqrt{T-t}}, \quad B'(t) \geq c \frac{A'(t)}{\sqrt{T-t}}
\]
for \(t \in [0, T)\),
\[
A(t) \rightarrow +\infty \quad or B(t) \rightarrow +\infty \quad as t \rightarrow T^-,
\]
where \(p, q > 0, c > 0\) and \(pq > 1\).

Then there exists \(C > 0\) such that
\[
A(t) \leq C(T-t)^{-\frac{\alpha}{2}}, \quad B(t) \leq C(T-t)^{-\frac{\beta}{2}}, \quad t \in [0, T),
\]
where \(\alpha = \frac{p+1}{pq-1}, \beta = \frac{q+1}{pq-1}\).

**Lemma 3.2** [6]: Let \(x \in \overline{B}_R\). If \(0 \leq a < n - 1\). Then there exists \(C > 0\) such that
\[
\int_{S_R} |x-y|^a \leq C.
\]

**Theorem 3.3** (Jump relation, [8]) Let \(\Gamma(x, t)\) be the fundamental solution of heat equation, namely
\[
\Gamma(x, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp[-\frac{|x|^2}{4t}].
\]

Let \(\varphi\) be a continuous function on \(S_R \times [0, T]\). Then for any \(x \in B_R, x^0 \in S_R, 0 < t_1 < t_2 \leq T\), for some \(T > 0\), the function
\[
U(x, t) = \int_{t_1}^{t_2} \int_{S_R} \Gamma(x - y, t - z) \varphi(y, z) ds dy d\tau
\]
satisfies the jump relation
\[ \frac{\partial}{\partial \eta} U(x, t) \rightarrow -\frac{1}{2} \varphi(x^0, t) + \frac{\partial}{\partial \eta} U(x^0, t) \quad \text{as} \ x \rightarrow x^0. \]

**Theorem 3.4** Let \((u, v)\) be a blow-up solution to problem 1, and \(T > 0\) is the blow-up time. Then there exist two positive constants \(C_1, C_2\) such that
\[ \max_{\overline{B}_R} u(x, t) \leq \log C_1 - \frac{\alpha}{2} \log(T - t), \quad 0 < t < T, \]
\[ \max_{\overline{B}_R} v(x, t) \leq \log C_2 - \frac{\beta}{2} \log(T - t), \quad 0 < t < T. \]

**Proof:** In order to prove this theorem, we follow the technique used in a previous work [5].

Define the functions \(M\) and \(M_b\) as follows:
\[ M(t) = \max_{\overline{B}_R} u(x, t), \quad \text{and} \quad M_b(t) = \max_{\overline{S}_R} u(x, t). \]

Similarly,
\[ N(t) = \max_{\overline{B}_R} v(x, t), \quad \text{and} \quad N_b(t) = \max_{\overline{S}_R} v(x, t). \]

Depending on Lemma 2.1, both of \(M, M_b\) are monotone increasing functions.

Since \(u\) is a solution of heat equation, it cannot attain interior maximum without being constant. Therefore,
\[ M(t) = M_b(t). \quad \text{Similarly} \quad N(t) = N_b(t). \]

Moreover, since \(u, v\) blow up simultaneously, we have
\[ M(t) \rightarrow +\infty, \quad N(t) \rightarrow +\infty \quad \text{as} \ t \rightarrow T^- \]

According to the second Green’s identity [5, 7, 9], with considering the Green function:
\[ G(x, y; z_1, t) = \Gamma(x - y, t - z_1), \]
where \(\Gamma\) is defined in 6, the integral equation to problem 1, with respect to \(u\), takes the form:
\[ u(x, t) = \int_{\overline{B}_R} \Gamma(x - y, t - z_1) u(y, z_1) dy + \int_{z_1}^{t} \int_{\overline{S}_R} \lambda_z e^{P(y, \tau)} \Gamma(x - y, t - \tau) ds_y d\tau \]
\[ - \int_{z_1}^{t} \int_{\overline{S}_R} u(y, \tau) \frac{\partial}{\partial \eta_y} (x - y, t - \tau) ds_y d\tau, \]

By applying Theorem 3.3 on the third term in the right-hand side of the last equation and with letting \(x \rightarrow S_R\), we obtain
\[ \frac{1}{2} u(x, t) = \int_{\overline{B}_R} \Gamma(x - y, t - z_1) u(y, z_1) dy + \int_{z_1}^{t} \int_{\overline{S}_R} \lambda_z e^{P(y, \tau)} \Gamma(x - y, t - \tau) ds_y d\tau \]
\[ - \int_{z_1}^{t} \int_{\overline{S}_R} u(y, \tau) \frac{\partial}{\partial \eta_y} (x - y, t - \tau) ds_y d\tau, \]

for \(x \in S_R, 0 < z_1 < t < T\).

Depending on Lemma 2.1, \(u, v\) are both radial and positive functions.

Therefore,
\[ \int_{\overline{B}_R} \Gamma(x - y, t - z_1) u(y, z_1) dy > 0, \]
\[ \int_{z_1}^{t} \int_{\overline{S}_R} \lambda_z e^{P(y, \tau)} \Gamma(x - y, t - \tau) ds_y d\tau = \int_{z_1}^{t} \lambda_z e^{P(R, \tau)} \int_{\overline{S}_R} \Gamma(x - y, t - \tau) ds_y d\tau. \]

This leads to
\[ \frac{1}{2} M(t) \geq \int_{z_1}^{t} \lambda_z e^{N(t)} \int_{\overline{S}_R} \Gamma(x - y, t - \tau) ds_y d\tau - \int_{z_1}^{t} M(\tau) \int_{\overline{S}_R} \left| \frac{\partial}{\partial \eta_y} (x - y, t - \tau) \right| ds_y d\tau, \]

\[ x \in S_R, 0 < z_1 < t < T. \]

It is known that (see [8]) there exists \(C_0 > 0\), such that \(\Gamma\) satisfies
\[ \left| \frac{\partial}{\partial \eta_y} (x - y, t - \tau) \right| \leq \frac{C_0}{(t - \tau)^{\mu}} \frac{1}{|x - y|^{|n+1-2\mu-\sigma|^r}} \]
\[ x, y \in S_R, \sigma \in (0, 1). \]

Choose \(1 - \frac{\sigma}{2} < \mu < 1\), from Lemma 3.2, there exists \(C^* > 0\) such that
\[ \int_{\overline{S}_R} \frac{ds_y}{|x - y|^{|n+1-2\mu-\sigma|}} < C^*. \]

Moreover, for \(0 < t_1 < t_2\) and \(t_1\) is closed to \(t_2\), there exists \(c > 0\), such that
\[ \int_{S_R} \Gamma(x, y, t_2 - t_1)ds_y \geq \frac{c}{\sqrt{t_2 - t_1}} \]

Thus
\[ \frac{1}{2} M(t) \geq c \int_{t_1}^{t} \frac{\lambda_1 e^{NP(r)}}{\sqrt{T - r}} \, dr - c \int_{t_1}^{t} \frac{M(r)}{|r|^{1-\mu}} \, dr. \]

Since for \( 0 < z_1 < t_0 < t < T \), it follows that \( M(t_0) \leq M(t) \), thus the last equation becomes
\[ \frac{1}{2} M(t) \geq c \int_{z_1}^{t} \frac{\lambda_1 e^{NP(r)}}{\sqrt{T - r}} \, dr - C_1^1 M(t) |T - z_1|^{1-\mu}. \]

Similarly, for \( 0 < z_2 < t < T \), we have
\[ \frac{1}{2} N(t) \geq c \int_{z_2}^{t} \frac{\lambda_2 e^{MQ(r)}}{\sqrt{T - r}} \, dr - C_2^1 N(t) |T - z_2|^{1-\mu}. \]

Taking \( z_1, z_2 \) so that
\[ C_1^1 |T - z_1|^{1-\mu} \leq 1/2, \quad C_2^1 |T - z_2|^{1-\mu} \leq 1/2, \]

it follows
\[ M(t) \geq c \int_{z_1}^{t} \frac{\lambda_1 e^{NP(r)}}{\sqrt{T - r}} \, dr, \quad N(t) \geq c \int_{z_2}^{t} \frac{\lambda_2 e^{MQ(r)}}{\sqrt{T - r}} \, dr. \] (8)

Since \( M, N \) are both increasing functions and by 7, we can find \( T_1 \in (0, T) \), such that
\[ M(t) \geq q^{(p-1)}, \quad N(t) \geq p^{(p-1)}, \quad \text{for} \ T_1 \leq t < T. \]

Thus
\[ e^{M(t)} \geq e^{qM(t)}, \quad e^{NP(t)} \geq e^{pN(t)}, \quad T^* \leq t < T. \]

Therefore, if we choose \( z_1, z_2 \) in \((T^*, T)\), then 8 becomes
\[ e^{M(t)} \geq c \int_{z_1}^{t} \frac{\lambda_1 e^{NP(r)}}{\sqrt{T - r}} \, dr \equiv I_1(t), \]
\[ e^{N(t)} \geq c \int_{z_2}^{t} \frac{\lambda_2 e^{MQ(r)}}{\sqrt{T - r}} \, dr \equiv I_2(t). \]

Clearly,
\[ I_1'(t) = c \frac{\lambda_1 e^{NP(t)}}{\sqrt{T - t}} \geq c \frac{\lambda_1 I_2^p}{\sqrt{T - t}}, \quad I_2'(t) = c \frac{\lambda_2 e^{MQ(t)}}{\sqrt{T - t}} \geq c \frac{\lambda_2 I_1^q}{\sqrt{T - t}}. \]

By Lemma 3.1, it follows that
\[ I_1(t) \leq \frac{c \lambda_1}{(T - t)^{\frac{p}{2}}} \quad I_2(t) \leq \frac{c \lambda_2}{(T - t)^{\frac{q}{2}}} \] (9)

\( t \in \max(z_1, z_2), T \).

On the other hand, with assuming that \( t \) is close to \( T \), we have
\[ I_1(t) \geq c \int_{t^*}^{t} \frac{\lambda_1 e^{NP(r)}}{\sqrt{T - r}} \, dr \geq c \lambda_1 e^{pN(t^*)} \int_{t^*}^{t} \frac{1}{2t - r} \, dr = 2c \lambda_1 (\sqrt{2} - 1) \sqrt{T - t} e^{pN(t^*)} \]

where \( t^* = 2t - T \).

Combining the last inequality with 9 yields
\[ e^{N(t^*)} \leq \frac{C}{2c^{p+1}(\sqrt{2} - 1)^{\frac{1}{2p}}(T - t)^{\frac{1}{2p}}} \leq \frac{q^{(p-1)}}{2c^{(p-1)}(T - t)^{\frac{1}{2(p-1)}}}. \]

It follows, there exists a constant \( c_1 > 0 \) such that
\[ e^{N(t^*)} \leq c_1. \]

Similarly, we can find \( c_2 > 0 \) such that
\[ e^{M(t^*)} \leq c_2. \]

This leads to, there exist \( C_1, C_2 > 0 \) such that
\[ \max_{B_R}(x, t) \leq \log C_1 - \frac{a}{2} \log(T - t), \quad \max_{B_R}(x, t) \leq \log C_2 - \frac{b}{2} \log(T - t). \] (10)

\[ \max_{B_R}(x, t) \leq \log C_3 - \frac{a}{2} \log(T - t) \] (11)
for $0 < t < T$

4. Blow-up Set

In this section, we study the blow-up set for problem 1, showing that the blow-up can only occur on the boundary. To prove this result, we recall the following lemma proved in a previous article [6].

**Lemma 4.1.** Let $w$ be a continuous function on the domain $B_R \times [0, T)$ and satisfies

\[
\begin{align*}
    w_t &= \Delta w, & (x, t) \in B_R \times (0, T), \\
    w(x, t) &\leq \frac{C}{(T-t)^m}, & (x, t) \in S_R \times (0, T), \ m > 0
\end{align*}
\]

Then for any $0 < a < R$,

\[
\sup\{w(x, t) : 0 \leq |x| \leq a, 0 \leq t < T\} < \infty.
\]

**Proof:** Set

\[
\begin{align*}
    h(x) &= (R^2 - r^2)^2, \quad r = |x|, \\
    z(x, t) &= \frac{c_1}{h(x)^{\frac{c_1}{m}}},
\end{align*}
\]

We can show that:

\[
\begin{align*}
    \Delta h - \frac{(m+1)|\nabla h|^2}{h} &= 8r^2 - 4n(R^2 - r^2) - (m+1)16r^2 \geq -4nR^2 - 16R^2(m+1), \\
    z_t - \Delta z &= \frac{c_1m}{[h(x)+c_2(T-t)]^m} (C_2 + \Delta h - \frac{(m+1)|\nabla h|^2}{h+c_2(T-t)}) \geq \frac{c_1m}{[h(x)+c_2(T-t)]^m} (C_2 - 4nR^2 - 16R^2(m+1)).
\end{align*}
\]

Let $C_2 = 4nR^2 + 16R^2(m+1) + 1$, and take $C_1$ to be large such that $z(x, 0) \geq w(x, 0), \ x \in B_R.
$ Let $C_1 \geq C(C_2)^m$, which implies that

\[
\begin{align*}
    z(x, t) &\geq w(x, t) \quad \text{on} \ S_R \times [0, T).
\end{align*}
\]

Then from the maximum principle [10], it follows that

\[
\begin{align*}
    z(x, t) &\geq w(x, t), \quad (x, t) \in \overline{B}_R \times (0, T)
\end{align*}
\]

and hence

\[
\sup\{w(x, t) : 0 \leq |x| \leq a, 0 \leq t < T\} \leq C_1 (R^2 - a^2)^{-2m} < \infty,
\]

for $0 \leq a < R$.

**Theorem 4.2** Let $(u, v)$ be a blow-up solution to problem 1, and $T > 0$ is the blow-up time. Then $(u, v)$ can only blow-up on the boundary.

**Proof:** By using equations 10 and 11, we obtain

\[
\begin{align*}
    u(R, t) &\leq \frac{c_1}{(T-t)^{\frac{c_1}{m}}}, \quad v(R, t) \leq \frac{c_2}{(T-t)^{\frac{c_2}{m}}}
\end{align*}
\]

for $t \in (0, T)$.

From Lemma 4.1, it follows that

\[
\begin{align*}
    \sup\{u(x, t) : (x, t) \in B_a \times [0, T]\} &\leq C_1 (R^2 - a^2)^{-a} < \infty, \\
    \sup\{v(x, t) : (x, t) \in B_a \times [0, T]\} &\leq C_1 (R^2 - a^2)^{-\beta} < \infty,
\end{align*}
\]

for $a < R$. Therefore, if $x \in B_R$, it cannot be a blow-up point.

5. Conclusions

This paper is concerned with the blow-up properties of positive solutions to a system of two heat equations defined on a ball in $\mathbb{R}^n$ associated with coupled Neumann boundary conditions of exponential type. Firstly, the upper bounds of blow-up rate estimates are derived. Secondly, the blow-up set is considered. The main conclusion is that the blow-up in this problem only occurs on the boundary.

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