A DISCONTINUOUS GALERKIN PRESSURE CORRECTION SCHEME FOR THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS: STABILITY AND CONVERGENCE

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Abstract. A discontinuous Galerkin pressure correction numerical method for solving the incompressible Navier–Stokes equations is formulated and analyzed. We prove unconditional stability of the proposed scheme. Convergence of the discrete velocity is established by deriving a priori error estimates. Numerical results verify the convergence rates.

1. Introduction

The numerical simulation of the incompressible Navier-Stokes equations presents a challenging computational task primarily because of two reasons: (a) the coupling of the velocity and pressure by the incompressibility constraint and (b) the nonlinearity of the convection term [14, 18]. The development of splitting schemes aims to overcome these difficulties by decoupling the nonlinearity in the convection term from the pressure term. For an overview of such methods, we refer to the works of Glowinski [15] and of Guermond, Minev, and Shen [18]. In this paper, we will focus on pressure correction schemes. The basic idea of a non-incremental pressure correction scheme in time was first proposed by Chorin and Temam [5, 28]. This scheme was subsequently modified by several mathematicians leading to two major variations: (1) the incremental scheme where a previous value of the pressure gradient is added [16,30] and (2) the rotational scheme where the non-physical boundary condition for the pressure is corrected by using the rotational form of the Laplacian [29].

The main contribution of our work is the theoretical analysis of a discontinuous Galerkin (dG) discretization of the pressure correction approach. We derive stability and a priori error bounds on a family of regular meshes. The discrete velocities are approximated by discontinuous piecewise polynomials of degree $k_1$ and the discrete potential and pressure by polynomials of degree $k_2$. Stability of the solutions is obtained under the constraint $k_1 - 1 \leq k_2 \leq k_1 + 1$ whereas the convergence of the scheme is obtained for the case $k_2 = k_1 - 1$ because of approximation properties. The proofs are technical and rely on several tools including special lift operators.

The semi-discrete error analysis of pressure correction schemes has been extensively studied, see for example the work by Shen and Guermond [21,27]. The use
of finite element approximations in conjunction with pressure correction schemes is also well studied and a priori error estimates are established. Without being exhaustive, we refer to the work by Guermond and Quartapelle [19, 20], and to the work by Nochetto and Pyo [24] for the analysis of finite element methods.

More recently, the combination of dG spatial approximations with pressure correction formulation in time to solve the incompressible Navier-Stokes equations has been the subject of several computational papers. These dG methods have multiple important features including local mass conservation and high order convergence. In what follows, we mention a non-exhaustive list of such publications. Botti and Di Pietro employ a dG approximation to the velocity and a continuous Galerkin approximation to the pressure [2]. Liu et al. formulate an interior penalty dG method with the pressure correction approach for the computational estimation of rock permeability [23]. Piatkowski et al. use a modified upwind scheme based on Vijayasundaram numerical flux, postprocess the projected velocity so that it is discretely divergence free, and perform numerical experiments of large scale 3D problems [25]. We also mention the work by Fehn et al. where the stability of pressure correction and velocity correction dG methods is numerically investigated for small time step sizes, and the robustness of the methods is demonstrated for laminar high Reynolds number flow problems [9].

Error analysis of dG methods for the steady state Navier-Stokes equations has been established, see the work by Girault, Riviere and Wheeler [13]. The authors also prove a priori error estimates for the time dependent Navier-Stokes equations where an operator splitting scheme is combined with dG methods in [14]. To the best of our knowledge, the a priori error analysis for a pressure correction dG approximation of the incompressible Navier-Stokes equations is missing from the literature.

The outline of this paper is as follows. In Section 2, the model problem and the splitting scheme in time are presented. Section 3 introduces the dG forms and summarizes their properties. We present the fully discrete scheme and show existence and uniqueness of solutions in time. Unconditional stability is established in Section 5, and convergence of the discrete velocity is shown in Section 6. Conclusions follow.

2. Model problem and time discretization

Consider the incompressible time dependent Navier-Stokes equations with homogenous Dirichlet boundary condition for the velocity and a zero average constraint for the pressure.

\[
\begin{align*}
\partial_t \mathbf{u} - \mu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p &= \mathbf{f}, & \text{in } & \Omega \times (0, T], \\
\nabla \cdot \mathbf{u} &= 0, & \text{in } & \Omega \times (0, T], \\
\mathbf{u} &= \mathbf{u}^0, & \text{in } & \Omega \times \{0\}, \\
\mathbf{u} &= \mathbf{0}, & \text{on } & \partial \Omega \times (0, T], \\
\int_{\Omega} p &= 0, & \forall t \in (0, T].
\end{align*}
\]

Here, \( \mathbf{u} \) is the fluid velocity, \( p \) is the pressure, \( \mathbf{f} \) is the external force, \( \Omega \) is an open bounded polyhedral domain in \( \mathbb{R}^d \) where \( d \in \{2, 3\} \), and \( \mu \) is the positive constant viscosity. Let \( \tau \) denote the time step size and consider a uniform partition of the
time interval \([0, T]\) into \(N_T\) subintervals. Throughout the paper, we use the notation \(g^n = g(t^n)\) and \(q^n = q(t^n)\) for given functions \(g\) and \(q\) evaluated at \(t^n = n\tau\). We now formulate a variation of the consistent splitting scheme that splits the operators and introduces additional velocity \(v^n\) and potential \(\phi^n\) \cite{18, 23}. Initially, set \(p^0 = 0\). For \(n = 1, \ldots, N_T\), given \(u^{n-1}\) and \(p^{n-1}\), compute an intermediate velocity \(v^n\) such that
\[
\begin{align*}
\nabla v^n - \tau \mu \Delta v^n + \tau u^{n-1} \cdot \nabla v^n + \tau \nabla p^{n-1} &= u^{n-1} + \tau f^n, & \text{in } \Omega, \\
\n
abla v^n &= 0, & \text{on } \partial \Omega.
\end{align*}
\]

Given \(v^n\), compute \(\phi^n\) such that,
\[
\begin{align*}
-\Delta \phi^n &= -\frac{1}{\tau} \nabla \cdot v^n, & \text{in } \Omega, \\
\nabla \phi^n \cdot n &= 0, & \text{on } \partial \Omega, \\
\int_{\Omega} \phi^n &= 0.
\end{align*}
\]

Given \((v^n, p^{n-1}, \phi^n)\), update the pressure \(p^n\) and the velocity \(u^n\) as follows.
\[
\begin{align*}
p^n &= p^{n-1} + \phi^n - \delta \mu \nabla \cdot v^n, \\
u^n &= v^n - \tau \nabla \phi^n.
\end{align*}
\]

In (2.11), \(\delta\) is a positive parameter that can be chosen in the interval \([0, 1/(4d)]\), as stated in the stability and convergence results proved in this work. The vector \(n\) is the unit outward normal vector to \(\partial \Omega\).

### 3. Notation and discontinuous Galerkin forms

For a given non-negative integer \(m\) and real number \(r \geq 1\), define the Sobolev space on a domain \(\Omega \subset \mathbb{R}^d\):
\[
W^{m,r}(\Omega) = \{ w \in L^r(\Omega) : \forall |\alpha| \leq m, \ D^\alpha w \in L^r(\Omega) \},
\]
where \(\alpha\) is a multi-index and \(D^\alpha w\) is the corresponding weak partial derivative of \(w\).

The usual Sobolev norms and semi-norms are denoted by \(|\cdot|_{W^{m,r}(\Omega)}\) and \(|\cdot|_{W^{m,r}(\Omega)}\) respectively. If \(r = 2\), we denote \(H^m(\Omega) = W^{m,2}(\Omega)\), \(|\cdot|_{H^m(\Omega)} = |\cdot|_{W^{m,2}(\Omega)}\) and \(|\cdot|_{H^m(\Omega)} = |\cdot|_{W^{m,2}(\Omega)}\). The \(L^2\) inner-product over \(\Omega\) is denoted by \((\cdot, \cdot)\) and the resulting \(L^2\) norm by \(|\cdot|\). Let \(E_h = \{ E_k \}\) denote a family of regular partitions of the domain \(\Omega\) \cite{6}. That is, there exists a constant \(\rho\) independent of \(h\) such that
\[
\frac{h_E}{\rho_E} \leq \rho, \quad \forall E \in E_h,
\]
where \(h_E = \text{diam}(E)\) and \(\rho_E\) is the maximum diameter of a ball inscribed in \(E\). Let \(h\) denote the maximum diameter of the elements in \(E_h\). We define the following broken Sobolev spaces.
\[
\begin{align*}
X &= \{ v \in L^2(\Omega)^d : \forall E \in E_h, \ v|_E \in W^{2,2d/(d+1)}(E)^d \}, \\
M &= \{ q \in L^2(\Omega) : \forall E \in E_h, \ q|_E \in W^{1,2d/(d+1)}(E)^d \}.
\end{align*}
\]

With this choice of spaces, we have that the trace of \(v \in X\) and of each component of its gradient belongs to \(L^2(\partial E)\) for all \(E \in E_h\). Similarly, the trace of \(q \in M\) belongs to \(L^2(\partial E)\) for all \(E \in E_h\). To see that this holds, we refer to Theorem 5.36 in \cite{1}. In
In the analysis, it will be useful to separate the upwind terms from the form
\[ C(3.10) \]
\[ a(3.8) \]
It follows that:

We also define the upwind terms for the spatial discretization of (2.6)-(2.12). For the convection term in (2.6), we use the same discretization form, the dG formulations for the spatial discretization of (2.6)-(2.12). For the convection term in (2.6), we use the same discretization form, the dG formulations for the spatial discretization of (2.6)-(2.12). For the convection term in (2.6), we use the same discretization form, the dG formulations for the spatial discretization of (2.6)-(2.12).

Denote by \( \Gamma_h \) the set of all interior faces of the subdivision \( E_h \). For an interior face \( e \in \Gamma_h \), we associate a normal \( n_e \), and we denote by \( E^1_e \) and \( E^2_e \) the two elements that share \( e \), such that \( n_e \) points from \( E^1_e \) to \( E^2_e \). Define the average and jump for a function \( \theta \in X \) as such,

\[
\{ \theta \} = \frac{1}{2}(\theta|_{E^1_e} + \theta|_{E^2_e}), \quad [\theta] = \theta|_{E^1_e} - \theta|_{E^2_e}, \quad \forall e = \partial E^1_e \cap \partial E^2_e.
\]

For a boundary face, \( e \in \partial \Omega \), the vector \( n_e \) is chosen as the unit outward vector to \( \partial \Omega \). The definition of the average and jump in this case are extended as such,

\[
\{ \theta \} = [\theta] = \theta|_{E^e}, \quad \forall e = \partial E^e \cap \partial \Omega.
\]

Similar definitions are used for scalar valued functions, \( q \in M \). We now introduce the dG formulations for the spatial discretization of (2.6)-(2.12). For the convection term in (2.6), we use the same discretization form, \( a_C \), as in [13]. To define this form, we use the following notation: the vector \( n_E \) denotes the outward normal to \( E \), the trace of a function \( v \) on the boundary of \( E \) coming from the interior (resp. exterior) of \( E \) is denoted by \( v^{\text{int}} \) (resp. \( v^{\text{ext}} \)). By convention, \( v^{\text{ext}}|_e = 0 \) if \( e \) is a boundary face \( (e \subset \partial \Omega) \). We also introduce the notation for the inflow boundary of \( E \) with respect to a function \( z \in X \):

\[
\partial E^z = \{ x \in \partial E : \langle z(x) \rangle \cdot n_E < 0 \}.
\]

With this notation, we define for \( z, w, v, \theta \in X \),

\[
a_C(z; w, v, \theta) = \sum_{E \in \mathcal{E}_h} \left( \int_E (w \cdot \nabla v) \cdot \theta + \frac{1}{2} \int_E (\nabla \cdot w) v \cdot \theta \right) - \frac{1}{2} \sum_{e \in \Gamma_h \cup \partial \Omega} \int_{\partial E^z} [w] \cdot n_e \{ v \cdot \theta \} + \sum_{E \in \mathcal{E}_h} \int_{\partial E^z} [\{ w \} \cdot n_E](v^{\text{int}} - v^{\text{ext}}) \cdot [\theta^{\text{int}}].
\]

We recall the positivity property satisfied by \( a_C \) (see (1.18) in [13]):

\[
a_C(w; w, v, v) \geq 0, \quad \forall w, v \in X.
\]

In the analysis, it will be useful to separate the upwind terms from the form \( a_C(w; w, v, \theta) \). To this end, for \( z, w, v, \theta \in X \), we define:

\[
C(w, v, \theta) = \sum_{E \in \mathcal{E}_h} \left( \int_E (w \cdot \nabla v) \cdot \theta + \frac{1}{2} \int_E (\nabla \cdot w) v \cdot \theta \right) - \frac{1}{2} \sum_{e \in \Gamma_h \cup \partial \Omega} \int_{\partial E^z} [w] \cdot n_e \{ v \cdot \theta \}.
\]

We also define the upwind terms

\[
U(z; w, v, \theta) = \sum_{E \in \mathcal{E}_h} \int_{\partial E^z} [w] \cdot n_E(v^{\text{int}} - v^{\text{ext}}) \cdot [\theta^{\text{int}}].
\]

It follows that:

\[
a_C(w; w, v, \theta) = C(w, v, \theta) - U(w, w, v, \theta).
\]
The discretization for the elliptic operator, \(-\Delta \mathbf{v}\), is given as follows [26]. For \(\mathbf{v}, \theta \in X\),

\[
\begin{align*}
(3.13) \quad a_e(\mathbf{v}, \theta) &= \sum_{E \in \mathcal{E}_h} \int_E \nabla \mathbf{v} : \nabla \theta - \sum_{e \in \Gamma_h} \int_e \{\nabla \mathbf{v}\} n_e \cdot [\theta] \\
& \quad + \epsilon \sum_{e \in \Gamma_h \cup \partial \Omega} \int_e \{\nabla \theta\} n_e \cdot [v] + \sum_{e \in \Gamma_h \cup \partial \Omega} \frac{\sigma}{h_e} [v] \cdot [\theta].
\end{align*}
\]

In the above form, \(h_e = |e|^{1/(d-1)}\), \(\epsilon \in \{-1, 0, 1\}\), \(\sigma > 0\) is a user specified penalty parameter. The discretization for the term \(-\nabla p\) is given as follows. For \(\theta \in X\) and \(q \in M\), define

\[
(3.14) \quad b(\theta, q) = \sum_{E \in \mathcal{E}_h} \int_E (\nabla \cdot \theta) q - \sum_{e \in \Gamma_h \cup \partial \Omega} \int_e \{q\} [\theta] \cdot n_e.
\]

To approximate \(u\) and \(p\), we introduce discrete function spaces \(X_h \subset X\) and \(M_{h0} \subset M_h \subset M\). For any integers \(k_1 \geq 1, k_2 \geq 0\),

\[
(3.15) \quad X_h = \{v_h \in (L^2(\Omega))^d : \forall E \in \mathcal{E}_h, \ v_h|_E \in (P_{k_1}(E))^d\},
\]

\[
(3.16) \quad M_h = \{q_h \in L^2(\Omega) : \forall E \in \mathcal{E}_h, \ q_h|_E \in P_{k_2}(E)\},
\]

\[
(3.17) \quad M_{h0} = \{q_h \in M_h : \int_\Omega q_h = 0\}.
\]

In the above, for \(k \in \mathbb{N}\), \(P_k(E)\) denotes the space of polynomials of degree at most \(k\). We will assume that (for reasons that will be evident in the analysis)

\[
k_1 - 1 \leq k_2 \leq k_1 + 1.
\]

To discretize the elliptic operator \(-\Delta \phi\), we define for \(\phi_h, q_h \in M_h\),

\[
(3.18) \quad a_{\text{ellip}}(\phi_h, q_h) = \sum_{E \in \mathcal{E}_h} \int_E \nabla \phi_h \cdot \nabla q_h - \sum_{e \in \Gamma_h} \int_e \{\nabla \phi_h\} \cdot n_e [q_h] \\
- \sum_{e \in \Gamma_h \cup \partial \Omega} \int_e \{\nabla q_h\} \cdot n_e [\phi_h] + \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} \int_e [\phi_h]|q_h|.
\]

Here, \(\tilde{\sigma} > 0\) is a penalty parameter. For \(\theta \in X\), define the energy norm as follows:

\[
(3.19) \quad \|\theta\|_{DG}^2 = \sum_{E \in \mathcal{E}_h} \|\nabla \theta\|_{L^2(E)}^2 + \sum_{e \in \Gamma_h \cup \partial \Omega} \frac{\sigma}{h_e} \|\theta\|_{L^2(e)}^2.
\]

For \(q \in M\), the energy semi-norm is defined as such:

\[
(3.20) \quad |q_h|_{DG}^2 = \sum_{E \in \mathcal{E}_h} \|\nabla q\|_{L^2(E)}^2 + \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} \|q\|_{L^2(e)}^2.
\]

Clearly, \(|\cdot|_{DG}\) is a norm for the space \(M_{h0}\). We recall the following coercivity properties.

\[
(3.21) \quad a_e(\theta_h, \theta_h) \geq \kappa \|\theta_h\|_{DG}^2, \quad \forall \theta_h \in X_h,
\]

\[
(3.22) \quad a_{\text{ellip}}(q_h, q_h) \geq \frac{1}{2} |q_h|_{DG}^2, \quad \forall q_h \in M_h.
\]

It is shown that (3.21) holds with \(\kappa = 1\) if \(\epsilon = 1\) and with \(\kappa = 1/2\) if \(\epsilon \in \{-1, 0\}\) and \(\sigma\) is large enough [26]. Property (3.22) holds for \(\tilde{\sigma}\) large enough [26]. In what follows, we will assume that (3.22) and (3.21) are satisfied. Further, we recall that
\[ a_c(\theta_h, z_h) \leq C\|\theta_h\|_{DG} \|z_h\|_{DG}, \quad \forall \theta_h, z_h \in X_h, \]

\[ a_{ellip}(q_h, \xi_h) \leq C|q_h|_{DG} |\xi_h|_{DG}, \quad \forall q_h, \xi_h \in M_h. \]

We now state an equivalent expression for the form \( b \).

\textbf{Lemma 3.1.} We have the following equivalent form for \( b(\theta, q) \),

\[ b(\theta, q) = - \sum_{E \in \mathcal{E}_h} \int_E \theta \cdot \nabla q + \sum_{e \in \Gamma_h} \int_e \{\theta\} \cdot n_e[q], \quad \forall (\theta, q) \in X \times M. \]

\textit{Proof.} The result is obtained by applying Green’s theorem to the first term in (3.14) and by using the identity

\[ \sum_{e \in \Gamma_h} \int_e [\theta \cdot n_e] = \sum_{e \in \Gamma_h} \int_e \{\theta\} \cdot n_e + \sum_{e \in \Gamma_h} \int_e \{q\} \cdot \nabla n_e. \]

We will also make use of lift operators, which are useful tools in the theoretical analysis of dG methods. Given \( e \in \Gamma_h \cup \partial \Omega \), we introduce a new lift operator \( r_e : (L^2(e))^d \to M_h \) as follows

\[ \int_{\Gamma_h} r_e(\zeta) q_h = \int_e \{q_h\} \zeta \cdot n_e, \quad \forall q_h \in M_h. \]

We next recall the lift operator \( g_e \) introduced in [7]: given an interior face \( e \in \Gamma_h \), the operator \( g_e : L^2(e) \to X_h \) satisfies

\[ \int_{\Omega} g_e(\zeta) \cdot \theta_h = \int_e \{\theta_h\} \cdot n_e \zeta, \quad \forall \theta_h \in X_h. \]

With these definitions, we construct two operators, \( R_h : X_h \to M_h \) and \( G_h : M_h \to X_h \)

\[ R_h(\theta_h) = \sum_{e \in \Gamma_h \cup \partial \Omega} r_e(\theta_h), \quad \theta_h \in X_h, \]

\[ G_h(\beta_h) = \sum_{e \in \Gamma_h} g_e(\beta_h), \quad \beta_h \in M_h. \]

It is easy to check that \( R_h \) and \( G_h \) are linear operators. We next show boundedness of the lift operators.

\textbf{Lemma 3.2.} There exist constants \( M_{k_2}, \tilde{M}_{k_1} > 0 \) independent of \( h \) but depending on the polynomial degrees \( k_2 \) and \( k_1 \) respectively, such that the following bounds hold:

\[ \|R_h(\theta_h)\| \leq M_{k_2} \left( \sum_{e \in \Gamma_h \cup \partial \Omega} h_e^{-1} \|\theta_h\|_{L^2(e)}^2 \right)^{1/2}, \quad \forall \theta_h \in X_h, \]

\[ \|G_h(q_h)\| \leq \tilde{M}_{k_1} \left( \sum_{e \in \Gamma_h} h_e^{-1} \|q_h\|_{L^2(e)}^2 \right)^{1/2}, \quad \forall q_h \in M_h. \]
Proof: We will show (3.31). We use the definitions of $R_h$ (3.29) and of $r_e$ (3.27). We have
\[
\|R_h([\theta_h])\|^2 = \int_{\Omega} R_h([\theta_h]) R_h([\theta_h]) = \sum_{e \in \Gamma_h \cup \partial \Omega} \int_{\partial \Omega} r_{e}(\theta_h) R_h([\theta_h])
\]
\[
= \sum_{e \in \Gamma_h} \int_{e} \frac{1}{2} \left( R_h([\theta_h]) |_{E_1} + R_h([\theta_h]) |_{E_2} \right) \theta_h \cdot n_e + \sum_{e \in \partial \Omega} \int_{\partial \Omega} R_h([\theta_h]) |_{E_e} [\theta_h] \cdot n_e.
\]
Cauchy-Schwarz’s inequality yields
\[
\|R_h([\theta_h])\|^2 \leq \frac{1}{2} \left( \sum_{e \in \Gamma_h} h_{E_1} \| R_h([\theta_h]) |_{E_1} \|^2 \right)^{1/2} \left( \sum_{e \in \Gamma_h} h_{E_1}^{-1} \| \theta_h \cdot n_e \|^2 \right)^{1/2}
\]
\[
+ \frac{1}{2} \left( \sum_{e \in \Gamma_h} h_{E_2} \| R_h([\theta_h]) |_{E_2} \|^2 \right)^{1/2} \left( \sum_{e \in \Gamma_h} h_{E_2}^{-1} \| \theta_h \cdot n_e \|^2 \right)^{1/2}
\]
\[
+ \left( \sum_{e \in \partial \Omega} h_{E_e} \| R_h([\theta_h]) |_{E_e} \|^2 \right)^{1/2} \left( \sum_{e \in \partial \Omega} h_{E_e}^{-1} \| \theta_h \cdot n_e \|^2 \right)^{1/2}.
\]
With a local trace inequality and the fact that $h_E \geq h_e$ for any element $E$ having a face $e$, we obtain the result with $M_k$, depending on a trace constant. The proof of (3.32) follows a similar argument and is omitted for brevity.

Further, with the definitions of the lift operators (3.29) and (3.30), we have the following equivalent forms to (3.14) and (3.25) respectively.
\begin{align}
(3.33) \quad b(\theta_h, q_h) &= (\nabla_h \cdot \theta_h, q_h) - (R_h([\theta_h]), q_h), \quad \forall \theta_h \in X_h, \forall q_h \in M_h, \\
(3.34) \quad b(\theta_h, q_h) &= -(\nabla h q_h, \theta_h) + (G_h([q_h]), \theta_h), \quad \forall \theta_h \in X_h, \forall q_h \in M_h,
\end{align}
where $\nabla_h$ and $\nabla$ denote the broken gradient and divergence operators respectively.

4. Numerical scheme

We start with setting $p_h^0 = \phi_h^0 = 0$. We let $u_h^0$ be the local $L^2$ projection of $u^0$ onto $X_h$.
\[
(4.1) \quad \int_{E} (u_h^0 - u^0) \cdot \theta_h = 0, \quad \forall \theta_h \in \mathbb{P}_{k}(E)^d, \quad \forall E \in \mathcal{E}_h.
\]
For $n = 1, \ldots, N_T$, given $(u_h^{n-1}, p_h^{n-1}) \in X_h \times M_h$ compute $v_h^n \in X_h$ such that for all $\theta_h \in X_h$,
\[
(4.2) \quad (v_h^n, \theta_h) + \tau a_C(u_h^{n-1}, u_h^{n-1}, v_h^n, \theta_h) + \tau \mu a_e(v_h^n, \theta_h) = (u_h^{n-1}, \theta_h) + \tau b(\theta_h, p_h^{n-1}) + \tau (f^n, \theta_h).
\]
Next, compute $\phi_h^n \in M_{ho}$ such that for all $q_h \in M_h$,
\[
(4.3) \quad a_{clip}(\phi_h^n, q_h) = -\frac{1}{\tau} b(v_h^n, q_h).
\]
Finally, compute $p_h^n \in M_h$ and $u_h^n \in X_h$ such that for all $q_h \in M_h$ and for all $\theta_h \in X_h$,
\[
(4.4) \quad (p_h^n, q_h) = (p_h^{n-1}, q_h) + (\phi_h^n, q_h) - \delta \mu b(v_h^n, q_h),
\]
\[
(4.5) \quad (u_h^n, \theta_h) = (v_h^n, \theta_h) + \tau b(\theta_h, \phi_h^n).
\]
Using (3.33) and (3.34), steps (4.4) and (4.5) can be written as follows. For all \( q_h \in M_h \) and \( \theta_h \in X_h \),
\begin{align}
(4.6) \quad (p_h^n, q_h) &= (p_h^{n-1}, q_h) + (\phi_h^n, q_h) - \delta \mu (\nabla_h \cdot v_h^n - R_h([v_h^n]), q_h), \\
(4.7) \quad (u_h^n, \theta_h) &= (u_h^n, \theta_h) - \tau (\nabla_h \phi_h^n - G_h([\phi_h^n]), \theta_h).
\end{align}

**Lemma 4.1.** Let \( n \geq 0 \) and \( p_h^n \in M_h \) be defined by (4.6), then \( p_h^n \in M_{h0} \).

**Proof.** We present a proof by induction on \( n \). For \( n = 0 \), the statement trivially holds. Assume \( p_h^{n-1} \in M_{h0} \). Let \( q_h = 1 \) in (4.4) and use (3.34). We obtain:
\begin{equation}
(4.8) \quad \int_{\Omega} p_h^n = \int_{\Omega} p_h^{n-1} + \int_{\Omega} \phi_h^n.
\end{equation}
We conclude the result by using the induction hypothesis and the fact that \( \phi_h^n \in M_{h0} \).

**Lemma 4.2.** Given \((v_h^{n-1}, u_h^{n-1}, p_h^{n-1}) \in X_h \times X_h \times M_{h0}\), there exists a unique solution \((v_h^n, u_h^n, p_h^n) \in X_h \times X_h \times M_{h0}\) to the dG scheme given by (4.2) - (4.7).

**Proof.** Existence of the intermediate velocity \( v_h^n \): Since this is a linear problem in finite dimension, it suffices to show uniqueness of the solution. Suppose there exist two solutions \( v_h^n \) and \( \tilde{v}_h^n \) to (4.2) and let \( \chi_h^n \) denote the difference between the two solutions,
\[ \chi_h^n = v_h^n - \tilde{v}_h^n. \]
Then, recalling that \( a_C \) is linear in the third argument, and choosing \( \theta_h = \chi_h^n \), we obtain
\[ (\chi_h^n, \chi_h^n) + \tau a_C(u_h^{n-1}; u_h^{n-1}, \chi_h^n, \chi_h^n) + \tau \mu a_\varepsilon(\chi_h^n, \chi_h^n) = 0. \]
With the positivity property of \( a_C \) (3.9) and the coercivity of \( a_\varepsilon \) (3.21), we conclude that
\begin{equation}
(4.9) \quad \|\chi_h^n\|^2 + \kappa \tau \mu \|\chi_h^n\|_{DG}^2 \leq 0.
\end{equation}
This implies that \( \chi_h^n = 0 \). Thus, the solution to (4.2) is unique.

Existence of \( u_h^n \) and \( p_h^n \): The existence of \( \phi_h^n \in M_{h0} \) follows by similar arguments, the coercivity property of \( a_{\text{clip}}(\cdot, \cdot) \), and the fact that \( \cdot \mid_{DG} \) is a norm for the space \( M_{h0} \). The existence of \( p_h^n \in M_{h0} \) follows directly from (4.6) and Lemma 4.1. The existence of \( u_h^n \in X_h \) follows from (4.7).

## 5. Stability

In this section, we present a stability analysis of the proposed scheme. We proceed by noting the following identity.

**Lemma 5.1.** Let \((u_h^n, v_h^n, \phi_h^n) \in X_h \times X_h \times M_{h0}\) be defined by algorithm (4.2)-(4.7). For all \( q_h \in M_h \) and \( n \geq 1 \), the following holds.
\begin{align}
(5.1) \quad b(u_h^n, q_h) &= b(u_h^n, q_h) + \tau a_{\text{clip}}(\phi_h^n, q_h) \\
&= -\tau \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} \int_e [\phi_h^n][q_h] + \tau (G_h([\phi_h^n]), G_h([q_h])).
\end{align}
In addition, we have
\begin{align}
(5.2) \quad b(u_h^n, q_h) &= -\tau \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} \int_e [\phi_h^n][q_h] + \tau (G_h([\phi_h^n]), G_h([q_h])).
\end{align}
Proof. Let \( q_h \in M_h \) be arbitrary but fixed. Take \( \theta_h = \nabla_h q_h \) in (4.7). We have,
\[
(u_h^n, \nabla_h q_h) = (v_h^n, \nabla_h q_h) - \tau (\nabla_h \phi_h^n - G_h([\phi_h^n]), \nabla_h q_h).
\]
Next, take \( \theta_h = G_h([q_h]) \) in (4.7). We have,
\[
(u_h^n, G_h([q_h])) = (v_h^n, G_h([q_h])) - \tau (\nabla_h \phi_h^n - G_h([\phi_h^n]), G_h([q_h])).
\]
Subtracting (5.3) from (5.4) and using (3.34), we obtain
\[
b(u_h^n, q_h) = b(v_h^n, q_h) + \tau (\nabla_h \phi_h^n - G_h([\phi_h^n]), \nabla_h q_h - G_h([q_h])).
\]
Observe that
\[
(\nabla_h \phi_h^n - G_h([\phi_h^n]), \nabla_h q_h - G_h([q_h])) = a_{ellip}(\phi_h^n, q_h)
\]
\[- \sum_{e \in \Gamma_h} \frac{\bar{\sigma}}{h_e} \int_e [\phi_h^n][q_h] + (G_h([\phi_h^n]), G_h([q_h])).
\]
Substituting (5.6) in (5.5) implies (5.1). To see that (5.2) holds, define \( \beta_h \in M_{h0} \) as \( \beta_h = q_h - (q_h) \) with \( (q_h) = (1/|\Omega|) \int_\Omega q_h \). Since \( b(v_h^n, (q_h)) = a_{ellip}(\phi_h^n, (q_h)) = 0 \), we obtain
\[
b(u_h^n, q_h) = b(v_h^n, \beta_h) + \tau a_{ellip}(\phi_h^n, \beta_h) - \tau \sum_{e \in \Gamma_h} \frac{\bar{\sigma}}{h_e} \int_e [\phi_h^n][q_h] + \tau (G_h([\phi_h^n]), G_h([q_h])).
\]
By applying (4.3), we obtain (5.2). \( \square \)

We recall the following \( L^q \) bound for the broken Sobolev spaces. Since \( X \subset H^1(\mathcal{E}_h)^d \), the proof of this lemma can be found in Lemma 6.2 in [13] in 2D and in Corollary 2.2 in 3D in [22].

**Lemma 5.2.** There exists a constant \( C_P \) independent of \( h \) and \( \tau \) but depending on \( q \) such that
\[
\|\theta\|_{L^q(\Omega)} \leq C_P\|\theta\|_{DG}, \quad \forall \theta \in X,
\]
where \( 2 \leq q < \infty \) in 2D \( (d = 2) \) and \( 2 \leq q \leq 6 \) in 3D \( (d = 3) \).

To state and prove our stability result, for \( n \geq 0 \), we introduce auxiliary functions \( S_h^n \in M_h \) and \( \xi_h^n \in M_h \).
\[
\begin{align*}
S_h^0 &= 0, & S_h^n &= \delta \mu \sum_{i=1}^n (\nabla_h \cdot v_h^i - R_h([v_h^i])), & n &\geq 1, \\
\xi_h^0 &= 0, & \xi_h^n &= p_h^n + S_h^n, & n &\geq 1.
\end{align*}
\]
Note that \( S_h^n \in M_{h0} \) since with the equivalent definition (3.33), we have
\[
\int_{\Omega} S_h^n = \delta \mu \sum_{i=1}^n b(v_h^i, 1) = 0, \quad n \geq 1.
\]
This implies that \( \xi_h^n \in M_{h0} \) since \( p_h^n \in M_{h0} \), see Lemma 4.1. We are now ready to prove the main result of this section.
Theorem 5.3. Assume that $\sigma \geq M_{k_2}^2/d$, $\tilde{\sigma} \geq \tilde{M}_{k_1}^2$, $\delta \leq \kappa/(2d)$, and $u^0 \in L^2(\Omega)^d$. Then, the pressure correction scheme (4.2)-(4.7) is unconditionally stable. For all $\tau > 0$ and $1 \leq m \leq N_T$,

\[
\|u_h^m\|^2 + \frac{\kappa \mu}{2} \tau \sum_{n=1}^{m} \|v_h^n\|^2_{DG} + \frac{1}{2} \tau^2 \|\xi_h^n\|^2_{DG} + \frac{1}{\delta \mu} \tau \|S_h^n\|^2 \\
\leq \|u^0\|^2 + \frac{2C^2}{\kappa \mu} \tau \sum_{n=1}^{m} \|f^n\|^2.
\]

Proof. Let $\theta_h = v_h^n$ in (4.2). We use the positivity property of $a_c$ (3.9) and the coercivity of $a_c$ (3.21). We obtain the following.

\[
\frac{1}{2} (\|v_h^n\|^2 - \|v_h^{n-1}\|^2 + \|v_h^n - u_h^{n-1}\|^2) + \kappa \mu \tau \|v_h^n\|^2_{DG} \\
\leq \tau b(v_h^n, \phi_h^{n-1}) + \tau (f^n, v_h^n).
\]

Next, let $\theta_h = u_h^n$ in (4.5),

\[
\frac{1}{2} (\|u_h^n\|^2 - \|u_h^{n-1}\|^2 + \|u_h^n - v_h^n\|^2) = \tau b(u_h^n, \phi_h^n).
\]

Let $\theta_h = u_h^n - v_h^n$ in (4.5). We obtain,

\[
\|u_h^n - v_h^n\|^2 = \tau b(u_h^n - v_h^n, \phi_h^n).
\]

Using (5.1) in Lemma 5.1, we find

\[
\|u_h^n - v_h^n\|^2 = \tau^2 a_{\text{ellip}}(\phi_h^n, \phi_h^n) - \tau^2 \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} \|\phi_h^n\|_{L^2(e)} + \tau^2 \|G_h([\phi_h^n])\|^2.
\]

Similarly, using (5.2) in Lemma 5.1, we obtain

\[
b(u_h^n, \phi_h^n) = -\tau \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} \|\phi_h^n\|_{L^2(e)} + \tau \|G_h([\phi_h^n])\|^2.
\]

Substituting the above expressions in (5.13), we obtain

\[
\frac{1}{2} (\|u_h^n\|^2 - \|v_h^n\|^2) + \frac{\tau^2}{2} a_{\text{ellip}}(\phi_h^n, \phi_h^n) \\
+ \frac{\tau^2}{2} \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} \|\phi_h^n\|_{L^2(e)} = \frac{\tau^2}{2} \|G_h([\phi_h^n])\|^2.
\]

The last term is estimated using (3.32):

\[
\|G_h([\phi_h^n])\| \leq M_{k_1} \left( \sum_{e \in \Gamma_h} h_e^{-1} \|\phi_h^n\|_{L^2(e)}^2 \right)^{1/2}.
\]

Using the above estimate in (5.14) yields

\[
\frac{1}{2} (\|u_h^n\|^2 - \|v_h^n\|^2) + \frac{\tau^2}{2} a_{\text{ellip}}(\phi_h^n, \phi_h^n) + \frac{\tau^2}{2} \sum_{e \in \Gamma_h} (\tilde{\sigma} - \tilde{M}_{k_1}^2) h_e^{-1} \|\phi_h^n\|_{L^2(e)}^2 \leq 0.
\]

With the assumption on the penalty parameter, $\tilde{\sigma} \geq \tilde{M}_{k_1}^2$, we obtain

\[
\frac{1}{2} \|u_h^n\|^2 - \frac{1}{2} \|v_h^n\|^2 + \frac{\tau^2}{2} a_{\text{ellip}}(\phi_h^n, \phi_h^n) \leq 0.
\]
We add (5.17) to (5.12),

\[ (5.18) \quad \frac{1}{2} (\| u_h^n \|^2 - \| u_h^{n-1} \|^2 + \| v_h^n - u_h^{n-1} \|^2) + \frac{\kappa \mu \tau}{2} \| v_h^n \|^2_{DG} + \frac{\tau^2}{2} a_{ellip}(\phi_h^n, \phi_h^n) \leq \tau b(v_h^n, p_h^{n-1}) + \tau(f^n, v_h^n). \]

To handle the first term on the right-hand side of the above inequality, we use the auxiliary function $S_h^n$ defined in (5.8) and write

\[ (5.19) \quad b(v_h^n, p_h^{n-1}) = b(v_h^n, p_h^{n-1} + S_h^{n-1}) - b(v_h^n, S_h^{n-1}). \]

With the equivalent expression (3.33), the second term in (5.19) is rewritten:

\[ (5.20) \quad b(v_h^n, S_h^{n-1}) = (\nabla_h \cdot v_h^n - R_h([v_h^n]), S_h^{n-1}) \]

\[ = \frac{1}{\delta h} (S_h^n - S_h^{n-1}, S_h^{n-1}) = \frac{1}{2 \delta h} (\| S_h^n \|^2 - \| S_h^{n-1} \|^2 - \| S_h^n - S_h^{n-1} \|^2). \]

Recall the definition of $\xi_h^n$ in (5.9) and note that (4.6) implies

\[ (5.21) \quad \xi_h^n - \xi_h^{n-1} = \phi_h^n, \quad \forall n \geq 1. \]

Since $S_h^n, p_h^n \in M_h$, we can apply (4.3). With (5.21), we have

\[ (5.22) \quad b(v_h^n, \xi_h^{n-1}) = -\tau a_{ellip}(\phi_h^n, \xi_h^{n-1}) = -\tau a_{ellip}(\xi_h^n - \xi_h^{n-1}, \xi_h^{n-1}). \]

Since $a_{ellip}(\cdot, \cdot)$ is symmetric, we have

\[ b(v_h^n, \xi_h^{n-1}) = -\frac{\tau}{2} \left( a_{ellip}(\xi_h^n, \xi_h^n) - a_{ellip}(\xi_h^{n-1}, \xi_h^{n-1}) - a_{ellip}(\xi_h^n - \xi_h^{n-1}, \xi_h^{n-1}) \right) \]

\[ = -\frac{\tau}{2} \left( a_{ellip}(\xi_h^n, \xi_h^n) - a_{ellip}(\xi_h^{n-1}, \xi_h^{n-1}) - a_{ellip}(\phi_h^n, \phi_h^n) \right). \]

Substituting (5.23) and (5.20) in (5.19) yields

\[ (5.24) \quad b(v_h^n, p_h^{n-1}) = -\frac{\tau}{2} \left( a_{ellip}(\xi_h^n, \xi_h^n) - a_{ellip}(\xi_h^{n-1}, \xi_h^{n-1}) - a_{ellip}(\phi_h^n, \phi_h^n) \right) \]

\[ - \frac{1}{2 \delta h} (\| S_h^n \|^2 - \| S_h^{n-1} \|^2 - \| S_h^n - S_h^{n-1} \|^2). \]

With the above expression, (5.18) becomes

\[ (5.25) \quad \frac{1}{2} \left( \| u_h^n \|^2 - \| u_h^{n-1} \|^2 + \| v_h^n - u_h^{n-1} \|^2 \right) + \frac{\kappa \mu \tau}{2} \| v_h^n \|^2_{DG} + \frac{\tau^2}{2} \left( a_{ellip}(\xi_h^n, \xi_h^n) - a_{ellip}(\xi_h^{n-1}, \xi_h^{n-1}) \right) + \frac{\tau}{2 \delta h} (\| S_h^n \|^2 - \| S_h^{n-1} \|^2) \]

\[ \leq \tau (f^n, v_h^n) + \frac{\tau}{2 \delta h} \| S_h^n - S_h^{n-1} \|^2. \]

To handle the second term on the right-hand side of (5.25), recall that from the definition of $S_h^n$, see (5.8),

\[ S_h^n - S_h^{n-1} = \delta h (\nabla_h \cdot v_h^n - R_h([v_h^n])). \]
Using (3.31), the assumption that \( \delta \leq \kappa/(2d) \) and \( \sigma \geq M_{k_2}^2/d \), we have
\[
\frac{1}{2\delta \mu} \| S_h^n - S_h^{n-1} \|^2 \leq \delta \mu \| \nabla h \cdot v_h^n \|^2 + \delta \mu \| R_h([u_h^n]) \|^2,
\]
\[
\leq d \delta \mu \| \nabla h v_h^n \|^2 + \delta \mu M_{k_2}^2 \sum_{e \in \Gamma_h \cup \partial \Omega} h_{e}^{-1} \| [u_h^n] \|^2_{L^2(e)},
\]
(5.26)
\[
\leq \frac{\kappa \mu}{2} \| \nabla h v_h^n \|^2 + \frac{\kappa \mu}{2} \sum_{e \in \Gamma_h \cup \partial \Omega} \sigma h_{e}^{-1} \| [u_h^n] \|^2_{L^2(e)}.
\]
Substituting the above bound in (5.25) yields,
\[
\frac{1}{2} (\| u_h^n \|^2 - \| u_h^{n-1} \|^2 + \| v_h^n - u_h^{n-1} \|^2) + \frac{\kappa \mu}{2} \tau \| v_h^n \|^2_{\text{DG}}
\]
\[
+ \frac{\tau^2}{2} (a_{\text{ellip}}(\xi_h^n, \xi_h^n) - a_{\text{ellip}}(\xi_h^{n-1}, \xi_h^{n-1})) + \frac{\tau}{2 \delta \mu} (\| S_h^n \|^2 - \| S_h^{n-1} \|^2) \leq \tau (f^n, v_h^n).
\]
Using Cauchy-Schwarz’s inequality, Poincare’s inequality (5.7), and Young’s inequality, we have
\[
\| (f^n, v_h^n) \| \leq \frac{C_{\mu}^2}{\kappa \mu} \| f^n \|^2 + \frac{\kappa \mu}{4} \| v_h^n \|^2_{\text{DG}}.
\]
We substitute the above bound in (5.27), multiply by 2 and sum the resulting inequality from \( n = 1 \) to \( n = m \). We obtain
\[
\| u_h^n \|^2 + \frac{\kappa \mu}{2} \tau \sum_{n=1}^{m} \| v_h^n \|^2_{\text{DG}} + \tau^2 a_{\text{ellip}}(\xi_h^m, \xi_h^n) + \frac{1}{\delta \mu} \tau \| S_h^m \|^2
\]
\[
\leq \| u_h^0 \|^2 + \frac{2C_{\mu}^2}{\kappa \mu} \tau \sum_{n=1}^{m} \| f^n \|^2 + \tau^2 a_{\text{ellip}}(\xi_h^0, \xi_h^n) + \frac{1}{\delta \mu} \tau \| S_h^0 \|^2.
\]
(5.29)
Recall that \( \xi_h^0 = S_h^0 = 0 \). With the coercivity property of \( a_{\text{ellip}} \) (3.22) and the stability of the \( L^2 \) projection, we attain the result. \( \square \)

**Remark 5.4.** The particular choice for the upper bound on \( \delta \) in the statement of Theorem 5.3 is chosen to simplify the write-up. However, it can be altered and a similar result would hold. For example, the condition on \( \delta \) can be chosen as \( \delta < \kappa/d \). This will lead to a larger constant \( C \) resulting from Young’s inequality in (5.28).

6. **Error estimates for the velocity**

In this section, we denote \( k = k_1 \) and consider the case \( k_2 = k - 1 \). To simplify notation, for any function \( v \in (W^{1,3}(\Omega) \cap L^\infty(\Omega))^d \), we define the norm:
\[
\| v \| = \| v \|_{L^\infty(\Omega)} + \| v \|_{W^{1,3}(\Omega)}.
\]
(6.1)
We assume that \( u^0 \in H^1_0(\Omega)^d \) and that \( \nabla \cdot u^0 = 0 \). Throughout the rest of this paper, we denote by \( C \) a constant independent of \( \mu, h, \) and \( \tau \) and by \( C_\mu \) a constant independent of \( h \) and \( \tau \) but depending on \( e^{1/\mu} \). These constants may take different values when used in different places and may depend on certain norms of the true solution \( (u, p) \). These norms are associated with the spaces specified by the regularity assumptions of Theorem 6.7.
6.1. Approximation. We will make use of the operator \( \Pi_h : H^1_0(\Omega)^d \to X_h \) that preserves the discrete divergence [4, 26]:

\[
(6.2) \quad b(\Pi_h u(t), q_h) = b(u(t), q_h) = 0, \quad \forall q_h \in M_h, \forall 0 \leq t \leq T.
\]

This operator satisfies the following approximation properties [4]. For \( E \in \mathcal{E}_h, 1 \leq r \leq \infty, 1 \leq s \leq k + 1, 0 \leq t \leq T \), and \( u(t) \in X \cap (W^{s,r}(E) \cap H^1_0(\Omega))^d \), we have

\[
(6.3) \quad \|\Pi_h u(t) - u(t)\|_{L^r(E)} \leq C h_E^{s/r} |u(t)|_{W^{s,r}(E)},
\]

\[
(6.4) \quad \|\nabla (\Pi_h u(t) - u(t))\|_{L^r(E)} \leq C h_E^{s/r} |u(t)|_{W^{s,r}(\Delta_E)},
\]

where \( \Delta_E \) is a macro element that contains \( E \). For \( 0 \leq t \leq T \), if \( u(t) \in (W^{s,r}(\Omega) \cap H^1_0(\Omega))^d \), then bounds (6.3) and (6.4) yield the global estimates:

\[
(6.5) \quad \|\Pi_h u(t) - u(t)\|_{L^\infty(\Omega)} \leq C h_E^{s} |u(t)|_{W^{s,r}(\Omega)},
\]

\[
(6.6) \quad \|\Pi_h u(t) - u(t)\|_{D^G} \leq C h_E^{s-1} |u(t)|_{H^s(\Omega)}.
\]

In the subsequent sections, the following stability bound for the operator \( \Pi_h \) will be used.

**Lemma 6.1.** Fix \( t \in [0, T] \). Assume that \( u(t) \in (L^\infty(\Omega) \cap W^{1,3}(\Omega) \cap H^1_0(\Omega))^d \). Then, we have the following bound.

\[
(6.7) \quad \|\Pi_h u(t)\|_{L^\infty(\Omega)} \leq C \|u(t)\|.
\]

**Proof.** Let \( I_h u(t) \in X_h \) be a Lagrange interpolant of \( u(t) \). Fix \( E \in \mathcal{E}_h \). With Minkowski's inequality, an inverse estimate, stability and approximation properties of the Lagrange interpolant and (6.3), we have

\[
\|\Pi_h u(t)\|_{L^\infty(E)} \leq \|\Pi_h u(t) - I_h u(t)\|_{L^\infty(E)} + \|I_h u(t)\|_{L^\infty(E)},
\]

\[
\leq C h_E^{-d/3} \|\Pi_h u(t) - I_h u(t)\|_{L^3(E)} + \|u(t)\|_{L^\infty(E)},
\]

\[
\leq C h_E^{1-d/3} |u(t)|_{W^{1,3}(\Delta_E)} + \|u(t)\|_{L^\infty(E)},
\]

\[
\leq C \|u(t)\|.
\]

Given that \( \|\Pi_h u(t)\|_{L^\infty(E)} = \max_{E \in \mathcal{E}_h} \|\Pi_h u(t)\|_{L^\infty(E)} \) and the above bound is uniform in \( E \), we obtain the result. \( \square \)

Define the local \( L^2 \) projection \( \pi_h : L^2(\Omega) \to M_h \) as follows. For \( 0 \leq t \leq T \), a given function \( p(t) \in L^2(\Omega) \), and any \( E \in \mathcal{E}_h \),

\[
(6.8) \quad \int_E (\pi_h p(t) - p(t)) q_h = 0, \quad \forall q_h \in P_{k-1}(E).
\]

For \( 0 \leq t \leq T \) and \( p(t) \in H^s(\Omega) \), the following estimate holds. For all \( E \in \mathcal{E}_h \),

\[
(6.9) \quad \|\pi_h p(t) - p(t)\|_{L^2(E)} + h_E \|\nabla (\pi_h p(t) - p(t))\|_{L^2(E)} \leq C h_E^{\min(k,s)} |p(t)|_{H^s(\Omega)}.
\]

We also recall that the local \( L^2 \) projection is stable in the dG norm [10].

**Lemma 6.2.** Fix \( t \in [0, T] \). Assume that \( p(t) \in H^1(\Omega) \). Then,

\[
(6.10) \quad |\pi_h p(t)|_{D^G} \leq C |p(t)|_{H^1(\Omega)}.
\]
6.2. Error equations. We recall that the dG discretization is consistent, in the sense that the true solution \((u, p)\) of (2.1)-(2.5) satisfies for \(n \geq 1\)

\[
(\partial_t u^n, \theta_h) + a_C(u^n; u^n, u^n, \theta_h) + \mu a_e(u^n, \theta_h) = b(\theta_h, p^n) + (f^n, \theta_h), \quad \forall \theta_h \in X_h.
\]

For readability, we define the following discretization errors, \(e^n_h \in X_h, e^n_0 \in X_h:\)

\[
e^n_h = v^n_h - \Pi_h u^n, \quad e^n_0 = u^n_0 - \Pi_h u^n, \quad \forall n \geq 0.
\]

In the above, we set \(v^n_0 = u^n_0\). Multiplying (6.11) by \(\tau\) and subtracting it from (4.2), we have for all \(n \geq 1\)

\[
(e^n_0, \theta_h) + \tau a_C(u^n; u^{n-1}, e^n_h, \theta_h) + \tau R_C(\theta_h) + \tau \mu a_e(e^n_h, \theta_h)
= (e^n_0, \theta_h) - \tau \mu a_e(\Pi_h u^n - u^n, \theta_h) + \tau b(\theta_h, p^{n-1} - p^n) + R_l(\theta_h), \quad \forall \theta_h \in X_h.
\]

In the above, we set

\[
R_C(\theta_h) = a_C(u^{n-1}; u^{n-1}, \Pi_h u^n, \theta_h) - a_C(u^n; u^n, \theta_h),
\]

\[
R_l(\theta_h) = \tau((\partial_t u^n, \theta_h) + (\Pi_h u^n - u^n, \theta_h)).
\]

With (6.2) and (3.33), we have for all \(n \geq 1\)

\[
b(\Pi_h u^n, q_h) = (\nabla_h \cdot \Pi_h u^n - R_h([\Pi_h u^n]), q_h) = 0, \quad \forall q_h \in M_h.
\]

Using (6.16) and (4.3), we obtain for all \(n \geq 1\)

\[
a_{\text{clip}}(\phi^n_h, q_h) = -\frac{1}{\tau} b(e^n_h, q_h), \quad \forall q_h \in M_h.0.
\]

Inserting \(\Pi_h u^n\) in (4.7) yields for all \(n \geq 1\)

\[
(e^n_h, \theta_h) = (e^n_0, \theta_h) - \tau(\nabla_h \phi^n_h - G_h([\phi^n_h]), \theta_h), \quad \forall \theta_h \in X_h.
\]

6.3. Intermediate results. We proceed by showing some intermediate properties of the error functions.

**Lemma 6.3.** Fix \(q_h \in M_h\). The following holds for \(n \geq 1\),

\[
b(e^n_h, q_h) = b(e^n_0, q_h) + \tau a_{\text{clip}}(\phi^n_h, q_h)
\]

\[
- \tau \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} \int_e \phi^n_h \cdot q_h + \tau(G_h([\phi^n_h]), G_h([q_h])).
\]

Further, for \(n \geq 1\),

\[
b(e^n_0, q_h) = -\tau \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} \int_e \phi^n_h \cdot q_h + \tau(G_h([\phi^n_h]), G_h([q_h])).
\]

In addition, for \(n \geq 1\),

\[
||e^n_h - e^{n-1}_h||^2 = ||e^n_h - e^{n-1}_h||^2 + \tau^2(||\nabla_h \phi^n_h||^2 + ||G_h([\phi^n_h])||^2) + \tau^2(A^n - A^{n-1})
\]

\[
+ \tau^2 \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} ||\phi^n_h - \phi^{n-1}_h||^2 - ||G_h([\phi^n_h - \phi^{n-1}_h])||^2
\]

\[
- 2\tau^2(\nabla_h \phi^n_h, G_h([\phi^n_h])) + 2\delta_{n,1} \tau b(e^n_0, \phi^n_h).
\]
where $\delta_{n,1}$ is the Kronecker delta and $A_1^n$ and $A_2^n$ for $n \geq 1$ are given by

\begin{align}
(6.22) & \quad A_1^n = \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} ||[\phi^n_h]||^2_{L^2(e)} - \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} ||[\phi^n_{h-1}]| |^2_{L^2(e)},
(6.23) & \quad A_2^n = ||G_h([\phi^n_h])||^2 - ||G_h([\phi^n_{h-1}])||^2.
\end{align}

**Proof.** Using (6.16) in (5.1), we obtain (6.19) for $n \geq 1$. Substituting (6.17) in (6.19), we obtain (6.20) for $n \geq 1$. To show (6.21), we use (6.18):

\begin{equation}
(6.24) \quad \|\tilde{e}_h^n - e_h^{n-1}\|^2 = (\tilde{e}_h^n - e_h^{n-1}, \tilde{e}_h^n - e_h^{n-1})
= (e_h^n - e_h^{n-1} + \tau(\nabla_h \phi_h^n - G_h([\phi_h^n])), e_h^n - e_h^{n-1} + \tau(\nabla_h \phi_h^n - G_h([\phi_h^n]))
= \|e_h^n - e_h^{n-1}\|^2 + 2\tau(e_h^n - e_h^{n-1}, \nabla_h \phi_h^n - G_h([\phi_h^n])) + \tau^2\|\nabla_h \phi_h^n - G_h([\phi_h^n])\|^2.
\end{equation}

For the second term, we use the equivalent definition (3.34).

\begin{equation}
(6.25) \quad (e_h^n - e_h^{n-1}, \nabla_h \phi_h^n - G_h([\phi_h^n])) = -b(e_h^n - e_h^{n-1}, \phi_h^n).
\end{equation}

Since $\phi_h^0 = 0$, expanding the last term in (6.24) and using (6.20) yield (6.21) for $n = 1$. For $n \geq 2$, we subtract (6.20) at time step $n$ from (6.20) at time step $n - 1$. We obtain for $n \geq 2$

\begin{equation}
- b(e_h^n - e_h^{n-1}, \phi_h^n) = \tau \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} \int_e ([\phi^n_h] - [\phi^{n-1}_h]) [\phi^n_h]
- \tau(G_h([\phi^n_h]) - G_h([\phi^{n-1}_h]), G_h([\phi^n_h])).
\end{equation}

Thus, for $n \geq 2$, we have

\begin{equation}
(6.26) \quad (e_h^n - e_h^{n-1}, \nabla_h \phi_h^n - G_h([\phi_h^n])) = \frac{\tau}{2} \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} (||[\phi^n_h]||^2_{L^2(e)} - ||[\phi^{n-1}_h]| |^2_{L^2(e)}
+ ||[\phi^n_h - \phi^{n-1}_h]| |^2_{L^2(e)}) - \frac{\tau}{2} (||G_h([\phi^n_h])||^2 - ||G_h([\phi^{n-1}_h])||^2 + ||G_h([\phi^n_h - \phi^{n-1}_h])||^2).
\end{equation}

Substituting (6.26) in (6.24) and expanding the last term in (6.24) yield (6.21) for $n \geq 2$.

We proceed by presenting some bounds for the forms $a_c, \mathcal{U},$ and $C$. The proof of these bounds is inspired from the proof of proposition 4.1 in [14]. However, the proof differs since in our scheme, we do not have that $b(e_h^n, q_h) = 0$ for all $q_h \in M_h$.

**Lemma 6.4.** Fix $\phi_h \in M_h$. Assume that there is $w_h \in X_h$ such that

\begin{equation}
(6.27) \quad b(w_h, q_h) = -\tau \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} \int_e [\phi_h] [q_h] + \tau(G_h([\phi_h]), G_h([q_h])), \quad \forall q_h \in M_h.
\end{equation}

Then, there exists a constant $C$, independent of $h$, $\tau$, $w_h, z, v,$ and $\theta_h$, such that the following estimates hold.

(i) If $v \in (W^{1,3}(\Omega) \cap H_0^1(\Omega) \cap L^\infty(\Omega))^d$, then for any $z \in X$ and $\theta_h \in X_h$:

\begin{equation}
(6.28) \quad |a_c(z; w_h, v, \theta_h)| \leq C \left( ||w_h|| + \tau \left( \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} ||[\phi_h]| |^2_{L^2(e)} \right)^{1/2} \right) ||v|| \|\theta_h\|_{DG}.
\end{equation}
(ii) If \( \mathbf{v} \in (H^{k+1}(\Omega) \cap H^1_0(\Omega))^d \), then for any \( \mathbf{\theta}_h \in \mathbf{X}_h \):

\[
|C(w_h, \mathbf{v} - \Pi_h \mathbf{v}, \mathbf{\theta}_h)| \leq C \left( \|w_h\| + \tau \left( \sum_{e \in \Gamma_h} \frac{\delta}{h_e} \|\phi_{e}\|_{L^2(e)}^2 \right)^{1/2} \right) \|\mathbf{v}\| \|\mathbf{\theta}_h\|_{DG}.
\]

**Proof.** (i) Since \( \mathbf{v} \) belongs to \( H^1(\Omega)^d \) and vanishes on the boundary, we have:

\[
a_c(z; w_h, \mathbf{v}, \mathbf{\theta}_h) = \sum_{E \in \mathcal{E}_h} \int_E (w_h \cdot \nabla \mathbf{v}) \cdot \mathbf{\theta}_h + \frac{1}{2} b(w_h, \mathbf{v} \cdot \mathbf{\theta}_h).
\]

With Hölder’s inequality and (5.7), we have

\[
\left| \sum_{E \in \mathcal{E}_h} \int_E (w_h \cdot \nabla \mathbf{v}) \cdot \mathbf{\theta}_h \right| \leq \|w_h\| \|\nabla \mathbf{v}\|_{L^3(\Omega)} \|\mathbf{\theta}_h\|_{L^6(\Omega)} \leq C_F \|w_h\| \|\mathbf{v}\|_{W^{1,3}(\Omega)} \|\mathbf{\theta}_h\|_{DG}.
\]

To bound the second term in (6.30), we use a similar argument as in [14]. Define \( c_1, c_2 \) as piecewise constant vectors where on each element \( E \in \mathcal{E}_h \),

\[
c_1|_E = \frac{1}{|E|} \int_E \mathbf{v}, \quad c_2|_E = \frac{1}{|E|} \int_E \mathbf{\theta}_h.
\]

We then write,

\[
\frac{1}{2} b(w_h, \mathbf{v} \cdot \mathbf{\theta}_h) = \frac{1}{2} b(w_h, \mathbf{v} \cdot \mathbf{\theta}_h - c_1 \cdot c_2) + \frac{1}{2} b(w_h, c_1 \cdot c_2).
\]

The first term is bounded in Proposition 4.1 in [14] in 2D domains. We follow a similar technique to obtain a bound in 2D/3D domains. We have,

\[
\frac{1}{2} b(w_h, \mathbf{v} \cdot \mathbf{\theta}_h - c_1 \cdot c_2) = \frac{1}{2} \sum_{E \in \mathcal{E}_h} \int_E (\nabla \cdot w_h)(\mathbf{v} \cdot \mathbf{\theta}_h - c_1 \cdot c_2)
\]

\[-\frac{1}{2} \sum_{e \in \Gamma_h \cup \partial \Omega} \int_e (\mathbf{v} \cdot \mathbf{\theta}_h - c_1 \cdot c_2) |w_h| \cdot n_e.
\]

For the first term, note that by Hölder’s, inverse and triangle inequalities, we have:

\[
\left| \int_E (\nabla \cdot w_h)(\mathbf{v} \cdot \mathbf{\theta}_h - c_1 \cdot c_2) \right|
\]

\[
\leq C h^{-1}_E \|w_h\|_{L^2(E)} \left( \|\mathbf{v} - c_1\| \cdot \|\mathbf{\theta}_h\|_{L^2(E)} + \|c_1 \cdot \mathbf{\theta}_h - c_2\|_{L^2(E)} \right)
\]

\[
\leq C h^{-1}_E \|w_h\|_{L^2(E)} \left( \|\mathbf{v} - c_1\| \cdot \|\mathbf{\theta}_h\|_{L^6(E)} + \|c_1 \cdot \mathbf{\theta}_h - c_2\|_{L^6(E)} \right).
\]

To proceed, we recall Poincaré’s inequality, see Theorems 1 and 2 in Section 5.8 in [8]. For \( r \in [1, \infty], E \in \mathcal{E}_h \) and \( \mathbf{\theta} \) in \( W^{1,r}(E) \),

\[
\|\mathbf{\theta} - \frac{1}{|E|} \int_E \mathbf{\theta}\|_{L^r(E)} \leq C h_E \|\nabla \mathbf{\theta}\|_{L^r(E)}.
\]

Applying (6.35), we find

\[
\left| \int_E (\nabla \cdot w_h)(\mathbf{v} \cdot \mathbf{\theta}_h - c_1 \cdot c_2) \right| \leq C \|w_h\|_{L^2(E)} \|\mathbf{v}\|_{W^{1,3}(E)} \|\mathbf{\theta}_h\|_{L^6(E)}
\]

\[
+ \|\mathbf{v}\|_{L^\infty(E)} \|\nabla \mathbf{\theta}_h\|_{L^2(E)}.
\]
To handle the second term in the right-hand side of (6.34), consider a face $e \in \Gamma_h$ and let $E_e^1$ and $E_e^2$ denote the elements sharing $e$. Terms involving faces $e \in \partial\Omega$ are handled similarly. With Cauchy-Schwarz’s inequality, we have

$$
\left| \int_e \{v \cdot \partial \theta - c_1 \cdot c_2 \} |w_h| \cdot n_e \right| \leq \frac{1}{2} \sum_{i,j=1}^2 \| (v \cdot \partial \theta - c_1 \cdot c_2) |E_e^1 \|_{L^2(e)} \| w_h |_{E_e^2} \|_{L^2(e)}.
$$

Let us consider the case $i = 1$ and $j = 2$; the other terms are handled similarly:

$$
\| (v \cdot \partial \theta - c_1 \cdot c_2) |E_e^1 \|_{L^2(e)} \| w_h |_{E_e^2} \|_{L^2(e)} \leq \| (v - c_1) |E_e^1 \cdot \partial \theta \|_{E_e^2} \| w_h |_{E_e^2} \|_{L^2(e)}
$$

$$
+ \| c_1 \cdot (\theta - c_2) |E_e^1 \|_{L^2(e)} \| w_h |_{E_e^2} \|_{L^2(e)}.
$$

With a trace inequality and Hölder’s inequality, we have

$$
\| (v \cdot \partial \theta - c_1 \cdot c_2) |E_e^1 \|_{L^2(e)} \| w_h |_{E_e^2} \|_{L^2(e)} \leq C \| v - c_1 |E_e^1 \|_{L^2(e)} \| w_h |_{E_e^2} \|_{L^2(e)}
$$

$$
+ h_{E_e^1} \| \nabla v |L^2(e)\| \| \partial \theta |L^2(e)\| + \| v |L^\infty(e)\| \| \theta - c_2 |L^2(e)\|.
$$

With (6.35), this becomes

$$
\| (v \cdot \partial \theta - c_1 \cdot c_2) |E_e^1 \|_{L^2(e)} \| w_h |_{E_e^2} \|_{L^2(e)} \leq C \| v - c_1 |E_e^1 \|_{L^2(e)} \| w_h |_{E_e^2} \|_{L^2(e)}
$$

$$
+ h_{E_e^1} \| \nabla v |L^2(e)\| \| \partial \theta |L^2(e)\| + \| v |L^\infty(e)\| \| \theta - c_2 |L^2(e)\|.
$$

(6.37)

Therefore, with discrete Hölder’s inequality and (5.7), we obtain

$$
\frac{1}{2} \sum_{e \in \Gamma_h \cup \partial \Omega} \| (v \cdot \partial \theta - c_1 \cdot c_2) |E_e^1 \|_{L^2(e)} \| w_h |_{E_e^2} \|_{L^2(e)} \leq C \| w_h \|_{DG} \| v \|_{DG} \| \theta |DG\|.
$$

We bound the other terms similarly, and combining the bounds, we obtain

$$
\frac{1}{2} \| b(w_h, v \cdot \partial \theta - c_1 \cdot c_2) \| \leq C \| w_h \|_{DG} \| v \|_{DG} \| \theta |DG\|.
$$

(6.38)

It remains to handle the second term in the right-hand side of (6.33). To this end, we use (6.27).

$$
\frac{1}{2} b(w_h, c_1 \cdot c_2) = - \frac{\tau}{2} \sum_{e \in \Gamma_h} \frac{\partial}{\partial e} \int_e [\phi_h |c_1 \cdot c_2] + \frac{\tau}{2} (G_h([\phi_h]), G_h([c_1 \cdot c_2])).
$$

(6.39)

Using Cauchy-Schwarz’s inequality and (3.32), we have

$$
\frac{1}{2} \| b(w_h, c_1 \cdot c_2) \| \leq C \tau \left( \sum_{e \in \Gamma_h} \frac{\partial}{\partial e} \| \phi_h \|_{L^2(e)}^2 \right)^{1/2} \left( \sum_{e \in \Gamma_h} \frac{\partial}{\partial e} \| [c_1 \cdot c_2] \|_{L^2(e)}^2 \right)^{1/2}.
$$

(6.40)

We write

$$
\sum_{e \in \Gamma_h} \frac{\partial}{\partial e} \| [c_1 \cdot c_2] \|_{L^2(e)}^2 = \sum_{e \in \Gamma_h} \frac{\partial}{\partial e} \| [c_1 \cdot (c_2 - \theta) \| + \| [c_1 \cdot \theta] \|_{L^2(e)}^2
$$

$$
\leq 2 \sum_{e \in \Gamma_h} \frac{\partial}{\partial e} \| [c_1 \cdot (c_2 - \theta)] \|_{L^2(e)}^2 + 2 \sum_{e \in \Gamma_h} \frac{\partial}{\partial e} \| [c_1 \cdot \theta] \|_{L^2(e)}^2
$$

(6.41)

$$
= T_1 + T_2.
$$
To bound $T_1$, we apply a trace inequality and (6.35), as it was done in (6.37). Consider the elements $E^1_e$ and $E^2_e$ that share $e$. We have, for $i = 1, 2$:

$$
\frac{1}{\sqrt{h_e}} \|c_1 \cdot (c_2 - \theta_h)\|_{E^i_e} \leq C|e|^{\frac{1}{2} - \frac{1}{\gamma - 1}} \|E^1_e\|^{-\frac{1}{2}} \|E^2_e\| \|v\|_{L^\infty(E^i_e)} \|
\n\n\n\leq C\|v\|_{L^\infty(E^i_e)} \|\nabla \theta_h\|_{L^2(E^i_e)}.
$$

Hence, we obtain the following bound on $T_1$.

$$
(6.42) \quad T_1 \leq C\|v\|^2_{L^\infty(\Omega)} \|\theta_h\|^2_{DG}.
$$

To handle $T_2$, we split it as such.

$$
|c_1 \cdot \theta_h| = |(c_1 - v) \cdot \theta_h| + |v \cdot \theta_h|.
$$

Thus, we have

$$
(6.43) \quad T_2 \leq 4 \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} \|((c_1 - v) \cdot \theta_h)\|^2_{L^2(e)} + 4 \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} \|v \cdot \theta_h\|^2_{L^2(e)} = T^1_2 + T^2_2.
$$

To bound $T^1_2$, we will use a trace inequality, Hölder’s inequalities, (6.35), and (5.7). We skip the details as the argument is similar to the one used in (6.37).

$$
(6.44) \quad T^1_2 \leq C(\|v\|^2_{W^{1,\infty}(\Omega)} + \|v\|^2_{L^\infty(\Omega)}) \|\theta_h\|^2_{DG}.
$$

The term $T^2_2$ is simply bounded as such,

$$
(6.45) \quad T^2_2 \leq 4\|v\|^2_{L^\infty(\Omega)} \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} \|\theta_h\|^2_{L^2(e)} \leq 4\|v\|^2_{L^\infty(\Omega)} \|\theta_h\|^2_{DG}.
$$

Substituting bounds (6.42), (6.44) and (6.45) in (6.40), we obtain

$$
(6.46) \quad \frac{1}{2} |b(w_h, c_1 \cdot c_2)| \leq C\tau \|v\| \|\theta_h\|_{DG} \left( \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} \|\phi_h\|^2_{L^2(e)} \right)^{1/2}.
$$

With (6.38) and (6.46), we have the following bound:

$$
(6.47) \quad \frac{1}{2} |b(w_h, v \cdot \theta_h)| \leq C \left( \|w_h\| + \left( \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} \|\phi_h\|^2_{L^2(e)} \right)^{1/2} \right) \|v\| \|\theta_h\|_{DG}.
$$

We conclude the proof of (6.28) by combining bounds (6.31) and (6.47).

(ii) First, we remark that by a Sobolev embedding, $v \in (W^{1,\infty}(\Omega) \cap L^\infty(\Omega))^d$ since $k \geq 1$. We have

$$
(6.48) \quad C(w_h, v - \Pi_h v, \theta_h) = \sum_{E \in \mathcal{E}_h} \int_E (w_h \cdot \nabla (v - \Pi_h v)) \cdot \theta_h
$$

$$
+ \frac{1}{2} b(w_h, (v - \Pi_h v) \cdot \theta_h).
$$

The first term is bounded exactly like in the proof of (i) (see the derivation of bound (6.31)) with $v$ replaced by $v - \Pi_h v$. By the stability of the interpolant, we have

$$
(6.49) \quad \sum_{E \in \mathcal{E}_h} \left| \int_E (w_h \cdot \nabla (v - \Pi_h v)) \cdot \theta_h \right| \leq C\|w_h\| \|v\|_{W^{1,\infty}(\Omega)} \|\theta_h\|_{DG}.
$$
The second term is split in the same way as in the proof of (i).

\[(6.50) \quad \frac{1}{2} b(w_h, (v - \Pi_h v) \cdot \theta_h) = \frac{1}{2} b(w_h, (v - \Pi_h v) \cdot \theta_h - \hat{c}_1 \cdot c_2) + \frac{1}{2} b(w_h, \hat{c}_1 \cdot c_2),\]

where \(c_2\) is defined in (6.32) and \(\hat{c}_1\) is a piecewise constant vector defined on each element \(E \in \mathcal{E}_h\) as

\[\hat{c}_1|_E = \frac{1}{|E|} \int_E (v - \Pi_h v).\]

Similar to (6.38) and using the stability of the interpolant (6.7), we have the bound

\[(6.51) \quad \frac{1}{2} |b(w_h, (v - \Pi_h v) \cdot \theta_h - \hat{c}_1 \cdot c_2)| \leq C\|v - \Pi_h v\|_{L^2(\Omega)} + \|v - \Pi_h v\|_{L^\infty(\Omega)} \|\theta_h\|_{DG} \leq C\|v\|_1 \|\theta_h\|_{DG}.

With similar arguments as equations (6.40), (6.41) and (6.43), we have

\[(6.52) \quad \frac{1}{2} b(w_h, \hat{c}_1 \cdot c_2) \leq C\tau \left( \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} |||\phi_h|||^2_{L^2(e)} \right)^{1/2} (\tilde{T}_1 + \tilde{T}_2 + \tilde{T}_3)^{1/2}.\]

Here \(\tilde{T}_1, \tilde{T}_2, \text{ and } \tilde{T}_3\) are defined in a similar way to \(T_1, T_2, \text{ and } T_3\) with \(c_1\) replaced by \(\hat{c}_1\) and \(v\) replaced by \(v - \Pi_h v\). The terms \(\tilde{T}_1\) and \(\tilde{T}_2\) are bounded exactly like \(T_1\) (6.42) and \(T_2\) (6.44) with \(v\) replaced by \(v - \Pi_h v\). With the stability of the interpolant, we have

\[|\tilde{T}_1| \leq C\|v - \Pi_h v\|_{L^\infty(\Omega)} \|\theta_h\|_{DG} \leq C\|v\|_1 \|\theta_h\|_{DG},\]

\[|\tilde{T}_2| \leq C \left( \|\nabla_h (v - \Pi_h v)\|_{L^2(\Omega)} + \|v - \Pi_h v\|_{L^\infty(\Omega)} \|\theta_h\|_{DG} \right) \|\theta_h\|_{DG} \leq C\|v\|_1 \|\theta_h\|_{DG}^2.

The only term that differs is \(\tilde{T}_3\). We have

\[(6.53) \quad \tilde{T}_3 = 4 \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} |||(v - \Pi_h v) \cdot \theta_h|||_{L^2(e)}^2.\]

Consider an edge \(e\) and an adjacent element \(E^1_{e}\). Using trace and Hölder’s inequalities, we have

\[\|(v - \Pi_h v) \cdot \theta_h|_{E^1_{e}}\|_{L^2(e)} \leq Ch_{E^1_{e}}^{-1/2} \|v - \Pi_h v\|_{L^3(E^1_{e})} \|\theta_h\|_{L^6(E^1_{e})} \leq Ch_{E^1_{e}}^{-1/2} \|v - \Pi_h v\|_{L^3(E^1_{e})} \|\theta_h\|_{L^6(E^1_{e})} + Ch_{E^1_{e}}^{1/2} \|v - \Pi_h v\|_{L^\infty(E^1_{e})} \|\nabla \theta_h\|_{L^2(E^1_{e})},\]

With the approximation property (6.3), we have:

\[(6.54) \quad \|v - \Pi_h v\|_{L^3(E^1_{e})} \leq Ch_{E^1_{e}} \|v\|_{W^{1,3}(\Delta_{E^1_{e}})}.

Hence, with discrete Hölder’s inequality, (5.7), (6.4), and (6.7), \(\tilde{T}_3\) is bounded as follows.

\[(6.55) \quad |\tilde{T}_3| \leq C\|v\|_{W^{1,3}(\Omega)}^2 \|\theta_h\|_{DG}^2.

Then, we have

\[(6.56) \quad \frac{1}{2} |b(w_h, \hat{c}_1 \cdot c_2)| \leq C\tau \left( \sum_{e \in \Gamma_h} \frac{\tilde{\sigma}}{h_e} |||\phi_h|||^2_{L^2(e)} \right)^{1/2} \|v\|_1 \|\theta_h\|_{DG}.\]
Combining bounds (6.49), (6.51), (6.56), we obtain the bound (6.29). □

In addition, we have the following bounds on $a_C$ and $C$ presented in the next lemma.

**Lemma 6.5.** There exists a constant $C$, independent of $h, r, w, z, v, w_h$, and $\theta_h$ such that the following bounds hold.

(i) If $w \in (H^{k+1}(\Omega) \cap H_0^1(\Omega))^d$ satisfies $b(w, q_h) = 0$, $\forall q_h \in M_h$ and if $v \in (W^{1,3}(\Omega) \cap H_0^1(\Omega) \cap L^\infty(\Omega))^d$, then for any $z \in X$ and $\theta_h \in X_h$:

$$|a_C(z; \Pi_h w - w, v, \theta_h)| \leq Ch^{k+1} |w|_{H^{k+1}(\Omega)} \|v\|_\theta \|\theta_h\|_{DG},$$

(ii) If $w \in (H^{k+1}(\Omega) \cap H_0^1(\Omega))^d$ and $v \in (W^{1,3}(\Omega) \cap H_0^1(\Omega) \cap L^\infty(\Omega))^d$, then for any $z \in X$ and $\theta_h \in X_h$:

$$|C(\Pi_h v, w - \Pi_h w, \theta_h)| \leq Ch^k |w|_{H^{k+1}(\Omega)} \|v\|_\theta \|\theta_h\|_{DG},$$

(iii) If $v \in H^{k+1}(\Omega)^d$, then for any $z \in X$ and $w_h, \theta_h \in X_h$:

$$|U(z; \Pi_h v, w - \Pi_h w, \theta_h)| \leq Ch^{k-1/2} \|w_h\|_\theta \|v\|_{H^{k+1}(\Omega)} \|\theta_h\|_{DG}.$$

**Proof.** (i) Note that since $v$ has zero jumps and vanishes on the boundary, the upwind term vanishes. We have

$$a_C(z; \Pi_h w - w, v, \theta_h) = \sum_{E \in \mathcal{E}_h} \int_E ((\Pi_h w - w) \cdot \nabla v) \cdot \theta_h + \frac{1}{2} b(\Pi_h w - w, v \cdot \theta_h).$$

The first term is bounded similarly as before (with (5.7) and (6.5)):

$$\left| \sum_{E \in \mathcal{E}_h} \int_E ((\Pi_h w - w) \cdot \nabla v) \cdot \theta_h \right| \leq \|\Pi_h w - w\|_{W^{1,3}(\Omega)} \|\theta_h\|_{DG} \leq Ch^{k+1} |w|_{H^{k+1}(\Omega)} \|v\|_{W^{1,3}(\Omega)} \|\theta_h\|_{DG}. \tag{6.61}$$

Recall the definition of $c_1, c_2$ in (6.32). Since $b(\Pi_h w - w, q_h) = 0$ for any $q_h \in M_h$, we have

$$b(\Pi_h w - w, v \cdot \theta_h) = b(\Pi_h w - w, v \cdot \theta_h - c_1 \cdot c_2). \tag{6.62}$$

Using Hölder’s inequality and (6.35), we have the following bounds. For $E \in \mathcal{E}_h$,

$$\|v \cdot \theta_h - c_1 \cdot c_2\|_{L^2(E)} \leq \|(v - c_1) \cdot \theta_h\|_{L^2(E)} + \|c_1 \cdot (c_2 - \theta_h)\|_{L^2(E)} \leq \|v - c_1\|_{L^3(E)} \|\theta_h\|_{L^6(E)} + \|c_1\|_{L^\infty(E)} \|c_2 - \theta_h\|_{L^2(E)} \leq Ch_E |v|_{W^{1,3}(E)} \|\theta_h\|_{L^6(E)} + Ch_E \|v\|_{L^\infty(E)} \|\nabla \theta_h\|_{L^2(E)}.$$

Hence, with the help of the discrete Hölder’s inequality and (5.7), the volume term in the right-hand side of (6.62) is bounded as follows.

$$\left| \sum_{E \in \mathcal{E}_h} \int_E (\nabla \cdot (\Pi_h w - w))(v \cdot \theta_h - c_1 \cdot c_2) \right| \leq C \sum_{E \in \mathcal{E}_h} \|\nabla (\Pi_h w - w)\|_{L^2(E)} \|v \cdot \theta_h - c_1 \cdot c_2\|_{L^2(E)} \leq Ch^{k+1} |w|_{H^{k+1}(\Omega)} \|v\|_\theta \|\theta_h\|_{DG}. \tag{6.63}$$
The face terms in the right-hand side of (6.62) are handled in a slightly different way as in (6.37), with \( w_h \) replaced by \( \Pi_h w - w \). For instance, one of the terms is bounded as follows:

\[
\|(v - c_1) \cdot \theta_h\|_{E^1_\ell} \leq (\Pi_h w - w)_{E^2_\ell} \leq C|\epsilon|^\frac{1}{2}|E^2_\ell|^\frac{1}{2} h E^2_\ell
\]

\[
\times (\|\nabla v\|_{L^2(E^3)} \|	heta_h\|_{L^2(E^3)} + \|v\|_{L^\infty(E^3)} \|
abla \theta_h\|_{L^2(E^3)}).
\]

With (6.3), (6.4), we have

\[
(6.66)
\frac{1}{2} \sum_{e \in \Gamma_h \cup \partial \Omega} \int_e (\Pi_h w - w) \cdot n_e (v \cdot \theta_h - c_1 \cdot c_2)
\leq C h^{k+1} |w|_{H^{k+1}(\Omega)} \|v\| \|	heta_h\|_{DG}.
\]

With bounds (6.63) and (6.64), we obtain:

\[
(6.65)
\frac{1}{2} b(\Pi_h w - w, v \cdot \theta_h) \leq C h^{k+1} |w|_{H^{k+1}(\Omega)} \|v\| \|	heta_h\|_{DG}.
\]

Combining bounds (6.61) and (6.65) yields the result.

(ii) To show (6.58), we write

\[
(6.66)
C(\Pi_h v, w - \Pi_h w, \theta_h) = \sum_{E \in \mathcal{E}_h} \int_E (\Pi_h v \cdot \nabla (w - \Pi_h w)) \cdot \theta_h
+ \frac{1}{2} b(\Pi_h v, (w - \Pi_h w) \cdot \theta_h).
\]

With Hölder's inequality, stability of the interpolant and (6.5), the first term is bounded by:

\[
(6.67)
\|\Pi_h v\|_{L^\infty(\Omega)} \|\nabla (w - \Pi_h w)\| \|	heta_h\| \leq C|\epsilon|^k |w|_{H^{k+1}(\Omega)} \|	heta_h\|_{DG}.
\]

To handle the second term, we will use (3.25). We have:

\[
(6.68)
b(\Pi_h v, (w - \Pi_h w) \cdot \theta_h) = - \sum_{E \in \mathcal{E}_h} \int_E \Pi_h v \cdot \nabla ((w - \Pi_h w) \cdot \theta_h)
+ \sum_{e \in \Gamma_h \cup \partial \Omega} \int_e \Pi_h v \cdot n_e ((w - \Pi_h w) \cdot \theta_h) = B_1 + B_2.
\]

We bound \( B_1 \) with Hölder’s inequality, stability of the interpolant (6.7), and the approximation properties (6.5)-(6.6). We have

\[
|B_1| \leq \|\Pi_h v\|_{L^\infty(\Omega)} \|\nabla (w - \Pi_h w)\| \|	heta_h\| + \|w - \Pi_h w\| \|\nabla \theta_h\|
\leq C |\epsilon|^k \|v\|_{H^{k+1}(\Omega)} \|	heta_h\|_{DG}.
\]

For \( B_2 \), let \( e \) be an interior face in \( \Gamma_h \) shared by \( E^1_\ell \) and \( E^2_\ell \). We have,

\[
\|(w - \Pi_h w) \cdot \theta_h\|_{L^1(\ell)} \leq \|(w - \Pi_h w)\|_{E^1_\ell} \cdot \theta_h|_{E^1_\ell} + \|(w - \Pi_h w)\|_{E^2_\ell} \cdot \theta_h|_{E^2_\ell} \|L^1(\ell)\|
\]

\[
(6.69)
B_2 \leq C h^{k+1} |w|_{H^{k+1}(\Omega)} \|	heta_h\|_{DG}.
\]

For \( B_2 \), let \( e \) be an interior face in \( \Gamma_h \) shared by \( E^1_\ell \) and \( E^2_\ell \). We have,
With Hölder’s inequality and a trace estimate, we have for $i = 1, 2$:
\[
\| (w - \Pi_h w)_{E_1} \cdot \theta_h_{E_1} \|_{L^1(E_1)} \leq C h^{-1/2}_{E_1} \| (w - \Pi_h w)_{E_1} \|_{L^2(E_1)} \| \theta_h \|_{L^2(E_1)}
\]
\[
\leq C h^{-1}_{E_1} \| w - \Pi_h w \|_{L^2(E_1)} + \| \nabla (w - \Pi_h w) \|_{L^2(E_1)} \| \theta_h \|_{L^2(E_1)}.
\]

We apply the approximation properties, sum over the faces, use (5.7), and obtain
\[
|B_2| \leq C h^k \| \Pi_h v \|_{L^\infty(\Omega)} \| w \|_{H^{k+1}(\Omega)} \| \theta_h \|_{DG} \leq C h^k \| v \|_{H^{k+1}(\Omega)} \| \theta_h \|_{DG}.
\]

Bounds (6.67), (6.69), and (6.70) yield (6.58). To handle the upwind term, we consider the faces’ contributions to the upwind terms. For details on such contributions, see Proposition 4.10 in [12]. We bound it by:
\[
C \| \Pi_h v \|_{L^\infty(\Omega)} \sum_{i,j=1}^{2} \sum_{e \in \Gamma_h} \| w - \Pi_h w \|_{E_1} \| \theta_h \|_{E_1} \| \theta_h \|_{E_1}.
\]

Consider the case $e \in \Gamma_h$. With a trace inequality and approximation properties (6.3)-(6.4), we have
\[
\| w - \Pi_h w \|_{E_1} \| \theta_h \|_{E_1} \leq C |e|^{-1/4} \| E_1 \|^{-1/4} \| \theta_h \|_{L^1(E_1)},
\]
\[
\| \theta_h \|_{L^1(E_1)} \leq C |E_1|^{-1/4} \| \theta_h \|_{L^2(E_1)},
\]
\[
\forall \theta_h \in X_h.
\]

We also consider the faces’ contributions to handle the upwind term. With Hölder’s inequality, we bound the upwind term, $U(z; w_h, v - \Pi_h v, \theta_h)$ by:
\[
C \sum_{i,j,k=1}^{2} \sum_{e \in \Gamma_h} \| w_h \|_{E_1} \| \theta_h \|_{E_1} \| (v - \Pi_h v) \|_{E_1} \| \theta_h \|_{E_1} \| \theta_h \|_{E_1}.
\]

Consider the case when $e \in \Gamma_h$ and $j = k$. The other cases are handled similarly. We use (6.71), (6.72), and (6.3)-(6.4):
\[
\| w_h \|_{E_1} \| \theta_h \|_{E_1} \| (v - \Pi_h v) \|_{E_1} \| \theta_h \|_{E_1} \| \theta_h \|_{E_1} \| \theta_h \|_{E_1}
\]
\[
\leq C |e|^{-1/2} E_1^{-3/4} h^{k+1} \| w_h \|_{L^2(E_1)} \| v \|_{H^{k+1}(E_1)} \| \theta_h \|_{L^2(E_1)},
\]

Since $|e|^{-1/2} E_1^{-3/4} h^{k+1} \leq C$ and $|E_1| \geq C (1/\rho)^d h^d E_1$, we obtain
\[
\| w_h \|_{E_1} \| \theta_h \|_{E_1} \| (v - \Pi_h v) \|_{E_1} \| \theta_h \|_{E_1} \| \theta_h \|_{E_1}
\]
\[
\leq C h^{-d/4} \| w_h \|_{L^2(E_1)} \| v \|_{H^{k+1}(E_1)} \| \theta_h \|_{L^2(E_1)}.
\]
Note that we have
\[
\left( \sum_{e \in \Gamma_h} |v|^2_{H^{k+1}(\Delta e_h)} \right)^{1/4} \leq \left( \sum_{e \in \Gamma_h} |v|^2_{H^{k+1}(\Delta e_h)} \right)^{1/2} \leq C |v|_{H^{k+1}(\Omega)}.
\]
This bound along with similar bounds to (6.73), discrete Hölder’s inequality, and (5.7) show estimate (6.60).

With the bounds established in Lemma 6.4 and Lemma 6.5, we show a bound on \( R_C(\tilde{e}_h^n) \).

**Lemma 6.6.** Assume that \( u \in L^\infty(0,T;H^{k+1}(\Omega)^d) \) and \( \partial_t u \in L^2(0,T;L^2(\Omega)^d) \). There exist positive constants \( \gamma \) and \( C \) independent of \( h \), \( \tau \) and \( \mu \) such that if \( \tau \leq \gamma \mu \), the following bound holds. For \( n \geq 1 \),

\[
(6.74) \quad |R_C(\tilde{e}_h^n)| \leq \frac{C}{\mu} \tau \int_{t_{n-1}}^{t_n} \|\partial_t u\|^2 + \frac{C}{\mu} h^{2k} + \frac{C}{\mu} \|e_h^{n-1}\|^2 + \frac{\kappa^\mu}{16} \|\tilde{e}_h^n\|^2_{\text{DG}}
\]

\[
+ \frac{1}{16} \sum_{e \in \Gamma_h} \frac{1}{h_e} \left\| \phi_h^n \right\|^2_{L^2(e)} + \frac{1}{4} \tau \sum_{e \in \Gamma_h} \frac{1}{h_e} \left\| \phi_h^n - \phi_h^{n-1} \right\|^2_{L^2(e)} + \frac{\delta_n h_{e}}{2} |b(\tilde{e}_h^n; \Pi_h u^n, \tilde{e}_h^n)|.
\]

**Proof.** We follow a similar technique to [14]. Since \( u^n \) belongs to \( H^1_0(\Omega)^d \), we can write

\[
ac(u^n; u^n, u^n, \tilde{e}_h^n) = ac(u_h^{n-1}; u^n, u^n, \tilde{e}_h^n).
\]

Therefore, we have

\[
R_C(\tilde{e}_h^n) = ac(u_h^{n-1}; u_h^{n-1}, \Pi_h u^n - u^n, \tilde{e}_h^n) + ac(u_h^{n-1}; e_h^{n-1}, \tilde{e}_h^n)
\]

\[
+ ac(u_h^{n-1}; \Pi_h u^{n-1} - u^n, u^n, \tilde{e}_h^n) = W_1^n + W_2^n + W_3^n.
\]

With the help of (3.12), we write \( W_1^n \) as follows.

\[
W_1^n = C(e_h^{n-1}, \Pi_h u^n - u^n, \tilde{e}_h^n) + C(\Pi_h u^{n-1}, \Pi_h u^n - u^n, \tilde{e}_h^n)
\]

\[
- \mathcal{U}(u_h^{n-1}, e_h^{n-1}, \Pi_h u^n - u^n, \tilde{e}_h^n) - \mathcal{U}(u_h^{n-1}, \Pi_h u^{n-1}, \Pi_h u^n - u^n, \tilde{e}_h^n).
\]

For \( n = 1 \), with (6.48) and (6.49), we have:

\[
C(e_h^0, \Pi_h u^1 - u^1, \tilde{e}_h^1) \leq C \|e_h^0\|_1 \|u^1\|^1_{W^{1,\infty}(\Omega)} \|\tilde{e}_h^1\|_{\text{DG}} + \frac{1}{2} b(e_h^0, \Pi_h u^1 - u^1) \cdot \tilde{e}_h^1.
\]

Since (6.20) holds, we can apply (6.29) in Lemma 6.4 to bound the first term in \( W_1^n \) for \( n \geq 2 \). We also use estimates (6.58) for the second term, (6.60) for the third term, and (6.59) along with (5.7) for the fourth term. For \( n \geq 1 \), we obtain

\[
W_1^n \leq C \|e_h^{n-1}\|_1 \|u^n\|_1 \|\tilde{e}_h^n\|_{\text{DG}} + C \tau \left( \sum_{e \in \Gamma_h} \frac{1}{h_e} \left\| \phi_h^{n-1} \right\|^2_{L^2(e)} \right)^{1/2} \|u^n\|_1 \|\tilde{e}_h^n\|_{\text{DG}}
\]

\[
+ Ch^{k-1/2} \|u^n\|_{H^{k+1}(\Omega)} \|e_h^{n-1}\|_1 \|\tilde{e}_h^n\|_{\text{DG}} + Ch^k \|u^n\|_{H^{k+1}(\Omega)} \|u^{n-1}\|_1 \|\tilde{e}_h^n\|_{\text{DG}}
\]

\[
+ \frac{1}{2} \delta_{n} |b(e_h^0, \Pi_h u^1 - u^1) \cdot \tilde{e}_h^1|.
\]

Further, for \( k \geq 1 \), with Morrey’s inequality and a Sobolev embedding (see Theorem 1.4.6 in [3] and Theorem 2 in section 5.6.1 [8]), we have for \( n \geq 0 \):

\[
(6.75) \quad \|u^n\|_1 \leq C \|u^n\|_{H^{k+1}(\Omega)} \leq C \|u\|_{L^\infty(0,T;H^{k+1}(\Omega))}.
\]
With the help of (6.75), the assumption that \( \mathbf{u} \in L^\infty(0,T; H^{k+1}(\Omega)^d) \), and Young’s inequality, we obtain for constants \( C \) and \( C_1 \):

\[
W^n_1 \leq \frac{C}{\kappa \mu} \| \mathbf{e}_h^{n-1} \|^2 + \frac{C}{\kappa \mu} \| \mathbf{e}_h \|^2 \| h^k + C_1 \tau \| \| \mathbf{e}_h^n \|^2_{DG} + \frac{\tau}{32} \sum_{e \in \Gamma_h} \frac{\sigma}{h_e} \| [\phi_h^n - \phi_h^{n-1}] \|^2_{L^2(e)} + \frac{\tau}{8} \sum_{e \in \Gamma_h} \frac{\sigma}{h_e} \| [\phi_h^n - \phi_h^{n-1}] \|^2_{L^2(e)} + \frac{1}{2} \delta_{n,1} \| \mathbf{b}(\mathbf{e}_h^n, (\Pi \mathbf{u}^1 - \mathbf{u}^1) \cdot \mathbf{e}_h^n) \|.
\]

With (6.30) and (6.31), we have

\[
a_C(\mathbf{u}_h^0, \mathbf{e}_h^n, \mathbf{u}^1, \mathbf{e}_h^1) \leq C \| \mathbf{e}_h^n \| \| \mathbf{u}^1 \|_{W^{1,3}(\Omega)} \| \mathbf{e}_h^n \|_{DG} + \frac{1}{2} \delta_{n,1} \| \mathbf{b}(\mathbf{e}_h^n, \mathbf{u}^1, \mathbf{e}_h^1) \|.
\]

Since (6.20) holds, we can apply (6.28) to bound \( W_2^n \) for \( n \geq 2 \). Thus, we have

\[
W_2^n \leq C \| \mathbf{e}_h^{n-1} \| \| \mathbf{u}^n \| \| \mathbf{e}_h^n \|_{DG} + C \tau \left( \sum_{e \in \Gamma_h} \frac{\sigma}{h_e} \| [\phi_h^{n-1}] \|^2_{L^2(e)} \right)^{1/2} \| \mathbf{u}^n \| \| \mathbf{e}_h^n \|_{DG} + \frac{1}{2} \delta_{n,1} \| \mathbf{b}(\mathbf{e}_h^n, \mathbf{u}^1, \mathbf{e}_h^1) \|.
\]

With (6.75) and Young’s inequality, we obtain for constants \( C \) and \( C_2 \)

\[
W_2^n \leq \frac{C}{\kappa \mu} \| \mathbf{e}_h^{n-1} \|^2 + \left( \frac{\kappa \mu}{96} + C_2 \tau \right) \| \mathbf{e}_h^n \|^2_{DG} + \frac{\tau}{32} \sum_{e \in \Gamma_h} \frac{\sigma}{h_e} \| [\phi_h^{n-1}] \|^2_{L^2(e)} + \frac{\tau}{8} \sum_{e \in \Gamma_h} \frac{\sigma}{h_e} \| [\phi_h^{n-1}] \|^2_{L^2(e)} + \frac{1}{2} \delta_{n,1} \| \mathbf{b}(\mathbf{e}_h^n, \mathbf{u}^1, \mathbf{e}_h^1) \|.
\]

To bound \( W_3^n \), we split it into

\[
W_3^n = a_C(\mathbf{u}_h^{n-1}; \Pi \mathbf{u}^n - \mathbf{u}^n, \mathbf{u}^n, \mathbf{e}_h^n) + a_C(\mathbf{u}_h^{n-1}; \mathbf{u}^n - \mathbf{u}_h^n, \mathbf{u}^n, \mathbf{e}_h^n).
\]

Since \( b(\mathbf{u}^{n-1}, q_h) = 0, \forall q_h \in M_h, \forall n \geq 1 \), we apply (6.57) and the assumption that \( \mathbf{u} \in L^\infty(0,T; H^{k+1}(\Omega)^d) \) to bound the first term.

\[
|a_C(\mathbf{u}_h^{n-1}; \Pi \mathbf{u}^n - \mathbf{u}^n, \mathbf{u}^n, \mathbf{e}_h^n)| \leq C h^{k+1} \| \mathbf{u}^n \|_{H^{k+1}(\Omega)^d} \| \mathbf{u}^n \| \| \mathbf{e}_h^n \|_{DG} \leq \frac{C}{\kappa \mu} h^{k+2} + \frac{\kappa \mu}{96} \| \mathbf{e}_h^n \|^2_{DG}.
\]

Since \( \mathbf{u}^{n-1} \) and \( \mathbf{u}^n \) belong to \( H^1_0(\Omega)^d \) and their divergence is zero, we have

\[
a_C(\mathbf{u}_h^{n-1}; \mathbf{u}^{n-1} - \mathbf{u}_h^n, \mathbf{u}^n, \mathbf{e}_h^n) = \sum_{E \in \mathcal{I}_h} \int_E ((\mathbf{u}^{n-1} - \mathbf{u}^n) \cdot \nabla \mathbf{u}^n) \cdot \mathbf{e}_h^n.
\]

By Hölder’s inequality, (5.7), and (6.75), we obtain

\[
|a_C(\mathbf{u}_h^{n-1}; \mathbf{u}^{n-1} - \mathbf{u}_h^n, \mathbf{u}^n, \mathbf{e}_h^n)| \leq \| \mathbf{u}^{n-1} - \mathbf{u}_h^n \| \| \mathbf{u}^n \|_{W^{1,3}(\Omega)} \| \mathbf{e}_h^n \|_{DG} \leq \frac{C}{\kappa \mu} \| \mathbf{u}^{n-1} - \mathbf{u}_h^n \|^2 \| \mathbf{u}^n \|^2_{W^{1,3}(\Omega)} + \frac{\kappa \mu}{96} \| \mathbf{e}_h^n \|^2_{DG} \leq \frac{C}{\kappa \mu} \tau \int_{t^{n-1}}^{t^n} \| \partial_t \mathbf{u} \|^2 + \frac{\kappa \mu}{96} \| \mathbf{e}_h^n \|^2_{DG}.
\]

The result follows by combining all the above bounds, and by assuming \((C_1 + C_2) \tau \leq \kappa \mu / 96\). \( \square \)

We are now ready to present and prove the main convergence result.
Theorem 6.7. Assume that $\sigma \geq M^2_{-1}/d$, $\bar{\sigma} \geq 4\tilde{M}^2$, and $\delta \leq \kappa/(2d)$. There exists a positive constant $\gamma_1$, independent of $h, \tau$ and $\mu$, such that if $\tau \leq \gamma_1 \mu$, the following estimate holds. For $0 \leq m \leq N_T$,

\begin{equation}
\|u_h^m - u^m\|^2 + \frac{\mu \tau}{4} \sum_{n=1}^m \|u_h^n - u^n\|_{DG}^2 \leq C_{\mu} \left(1 + \frac{1}{\mu}\right) (\tau + h^2).
\end{equation}

The above estimate holds under the following regularity assumptions:

$u \in L^\infty(0, T; H^{k+1}(\Omega)^d)$, $\partial_t u \in L^2(0, T; H^k(\Omega)^d)$, $\partial_t u \in L^2(0, T; L^2(\Omega)^d)$, and

$\theta \in L^\infty(0, T; H^{k}(\Omega))$.

Proof. Let $\theta_h = \bar{e}^n_h$ in (6.13). Using the positivity property of $a_\bar{e}$ (3.9) and the coercivity property of $a_\bar{e}$ (3.21), we have

\begin{equation}
\frac{1}{2}(\|\bar{e}^n_h\|^2 - \|e_h^{n-1}\|^2 + \|e_h^{n} - e_h^{n-1}\|^2) + \kappa \mu \tau \|\bar{e}^n_h\|_{DG}^2 + \tau R_C(\bar{e}^n_h) \leq \tau b(e^n_h, p^n_h - p^n) - \tau \mu a_\bar{e}(\Pi_h u^n - u^n, \bar{e}^n_h) + R_t(\bar{e}^n_h).
\end{equation}

Substituting (6.21) in the above expression, we obtain

\begin{equation}
\frac{1}{2}(\|\bar{e}^n_h\|^2 - \|e_h^{n-1}\|^2 + \|e_h^{n} - e_h^{n-1}\|^2) + \kappa \mu \tau \|\bar{e}^n_h\|_{DG}^2 + \frac{\tau^2}{2} \sum_{e \in \Gamma_h} \|\phi_h^n - \phi_h^{n-1}\|_{L^2(e)}^2 \leq \tau b(e^n_h, p^n_h - p^n) - \tau \mu a_\bar{e}(\Pi_h u^n - u^n, \bar{e}^n_h) + R_t(\bar{e}^n_h) + \frac{\tau^2}{2} \|G_h([\phi_h^n - \phi_h^{n-1}])\|^2 + \tau R_C(\bar{e}^n_h).
\end{equation}

We begin by handling the fifth and sixth terms. With (3.32) and the assumption that $\bar{\sigma} \geq 2\tilde{M}^2$, we have:

\begin{equation}
\|G_h([\phi_h^n - \phi_h^{n-1}])\|^2 \leq \sum_{e \in \Gamma_h} \frac{\tilde{M}^2}{h_e} \|[\phi_h^n - \phi_h^{n-1}]\|_{L^2(e)}^2 \leq \frac{1}{2} \sum_{e \in \Gamma_h} \frac{\bar{\sigma}}{h_e} \|[\phi_h^n - \phi_h^{n-1}]\|_{L^2(e)}^2.
\end{equation}

Using Cauchy-Schwarz’s inequality, Young’s inequality, (3.32), and the assumption that $\bar{\sigma} \geq 4\tilde{M}^2$, we obtain

\begin{equation}
|\langle \nabla_h \phi_h^n, G_h([\phi_h^n]) \rangle| \leq \frac{1}{4} \|\nabla_h \phi_h^n\|^2 + \|G_h([\phi_h^n])\|^2 \leq \frac{1}{4} \|\nabla_h \phi_h^n\|^2 + \frac{1}{4} \sum_{e \in \Gamma_h} \frac{\bar{\sigma}}{h_e} \|[\phi_h^n]\|_{L^2(e)}^2.
\end{equation}

Next, let $\theta_h = e_h^n$ in (6.18) and use (3.34).

\begin{equation}
\frac{1}{2} (\|e_h^n\|^2 - \|\bar{e}^n_h\|^2) + \frac{1}{2} \|e_h^n - \bar{e}^n_h\|^2 = \tau b(e_h^n, \phi_h^n).
\end{equation}

Using (6.20), we obtain

\begin{equation}
\frac{1}{2} (\|e_h^n\|^2 - \|\bar{e}^n_h\|^2) + \frac{1}{2} \|e_h^n - \bar{e}^n_h\|^2 + \tau^2 \sum_{e \in \Gamma_h} \frac{\bar{\sigma}}{h_e} \|[\phi_h^n]\|_{L^2(e)}^2 = \tau^2 \|G_h([\phi_h^n])\|^2.
\end{equation}
Since the operator \( \pi \), we let \( \mathbf{\theta}_h = \mathbf{e}^n_h - \bar{\mathbf{e}}^n_h \) in (6.18) and use (6.19):

\[
(6.82) \quad ||\mathbf{e}^n_h - \bar{\mathbf{e}}^n_h||^2 = \tau b(\mathbf{e}^n_h - \bar{\mathbf{e}}^n_h, \phi^n_h) = \tau^2 a_{\text{ellip}}(\phi^n_h, \phi^n_h)
\]

By using trace inequalities and (6.9), we obtain

\[
(6.87)
\]

Hence, (6.81) reads

\[
(6.83) \quad \frac{1}{2} (||\mathbf{e}^n_h||^2 - ||\bar{\mathbf{e}}^n_h||^2) + \frac{\tau^2}{2} a_{\text{ellip}}(\phi^n_h, \phi^n_h)
\]

We add (6.83) to (6.77). With bounds (6.78) and (6.79), we obtain

\[
(6.84)
\]

Since the operator \( \pi_h \) preserves cell averages and \( p^n \) satisfies (2.5), we know that \( \pi_h p^n \in M_{h0} \). By (6.17), continuity of \( a_{\text{ellip}}(\cdot, \cdot) \) on \( M_h \times M_h \) see (3.24), Young’s inequality, and (6.10), we obtain

\[
(6.85) \quad | - b(\bar{\mathbf{e}}^n_h, \pi_h p^n)| = |\tau a_{\text{ellip}}(\phi^n_h, \pi_h p^n)| \leq C \tau |\phi^n_h|_{\text{DG}} |\pi_h p^n|_{\text{DG}}
\]

By using trace inequalities and (6.9), we obtain

\[
(6.86) \quad |b(\bar{\mathbf{e}}^n_h, \pi_h p^n - p^n)| \leq C \mu h^{2k} |p^n|_{H^k(\Omega)} + \frac{\kappa \mu}{16} ||\bar{\mathbf{e}}^n_h||^2_{\text{DG}}.
\]
Note that by (6.2), 
\[ b(\tilde{e}_h^n, p_h^{n-1}) = b(v_h^n, p_h^{n-1}). \]
Hence, from (5.24), we have

\[ (6.88) \quad b(\tilde{e}_h^n, p_h^{n-1}) = -\frac{\tau}{2} \left( a_{\text{ellip}}(\xi_h^n, \xi_h^n) - a_{\text{ellip}}(\xi_h^{n-1}, \xi_h^{n-1}) - a_{\text{ellip}}(\phi_h^n, \phi_h^n) \right) \]

\[-\frac{1}{2\delta \mu} \left( \|S_h^n\|^2 - \|S_h^{n-1}\|^2 - \|S_h^n - S_h^{n-1}\|^2 \right).\]

With this expression, (6.87) becomes

\[ (6.89) \]

\[ \frac{1}{2} \left( \|e_h^n\|^2 - \|e_h^{n-1}\|^2 + \|e_h^n - e_h^{n-1}\|^2 \right) + \frac{\tau^2}{2} \left( a_{\text{ellip}}(\xi_h^n, \xi_h^n) - a_{\text{ellip}}(\xi_h^{n-1}, \xi_h^{n-1}) \right) \]

\[ + \frac{\tau^2}{8} \|\phi_h^n\|^2_{DG} + \kappa \mu \tau \|\tilde{e}_h^n\|^2_{DG} + \frac{\tau^2}{4} \sum_{e \in \Gamma_h} \|\phi_h^n - \phi_h^{n-1}\|^2_{L^2(e)} + \frac{\tau^2}{2} (A_h^n - A_h^2) \]

\[ + \frac{\tau}{2\delta \mu} \left( \|S_h^n\|^2 - \|S_h^{n-1}\|^2 \right) \leq -\tau \mu a_{\text{Pi}}(\Pi_h u^n - u^n, \tilde{e}_h^n) - \tau R_C(\tilde{e}_h^n) + C \tau^2 |p_h^n|_{H^1(\Omega)}^2 \]

\[ + \frac{\tau}{2\delta \mu} \|S_h^n - S_h^{n-1}\|^2 + \frac{C}{\mu} \tau h^{2k} |p_h^n|^2_{H^k(\Omega)} + \frac{K \mu}{16} \|\tilde{e}_h^n\|^2_{DG} + R_t(\tilde{e}_h^n) \]

\[-\delta_n \tau b(e_h^n, \phi_h^n).\]

We now consider \( R_t(\tilde{e}_h^n) \) (see (6.15)). Using Cauchy-Schwarz’s inequality, Young’s inequality and (5.7), we obtain

\[ |R_t(\tilde{e}_h^n)| = |\tau \left( (\partial_t u)^n, \tilde{e}_h^n \right) + (\Pi_h u^n - \Pi_h u^n, \tilde{e}_h^n) | \]

\[ \leq \frac{C}{\mu} \tau^{-1} \|\tau (\partial_t u)^n - (u^n - u^{n-1})\|^2 \]

\[ + \frac{C}{\mu} \tau^{-1} \|\Pi_h u^n - u^n\|^2 + \frac{K \mu}{16} \|\tilde{e}_h^n\|^2_{DG}.\]

With a Taylor expansion and approximation property (6.5), we have

\[ (6.90) \quad |R_t(\tilde{e}_h^n)| \leq \frac{C}{\mu} \tau^2 \int_{t_{n-1}}^{t_n} \|\partial_t u\|^2 + \frac{C}{\mu} \int_{t_{n-1}}^{t_n} \|\partial_t (\Pi_h u - u)\|^2 + \frac{K \mu}{16} \tau \|\tilde{e}_h^n\|^2_{DG} \]

\[ \leq \frac{C}{\mu} \tau^2 \int_{t_{n-1}}^{t_n} \|\partial_t u\|^2 + \frac{C}{\mu} h^{2k} \int_{t_{n-1}}^{t_n} \|\partial_t u\|^2_{H^k(\Omega)} + \frac{K \mu}{16} \|\tilde{e}_h^n\|^2_{DG}.\]

With the definition of \( S_h^n \) (5.8) and (6.16), we have

\[ (6.91) \quad S_h^n - S_h^{n-1} = \delta \mu (\nabla \cdot \tilde{e}_h^n - R_h(\tilde{e}_h^n)). \]

With the assumptions that \( \delta \leq \kappa / (2d) \) and \( \sigma \geq M^2_{k-1} / d \) and with (3.31), we have

\[ \frac{1}{2\delta \mu} \|S_h^n - S_h^{n-1}\|^2 \leq \delta \mu \|\nabla \cdot \tilde{e}_h^n\|^2 + \delta \mu \|R_h(\tilde{e}_h^n)\|^2 \]

\[ \leq \delta \mu d \|\nabla \tilde{e}_h^n\|^2 + \delta \mu M^2_{k-1} \sum_{e \in \Gamma_h \cup \partial \Omega} h_{e}^{-1} \|\tilde{e}_h^n\|^2_{L^2(e)} \]

\[ \leq \frac{K \mu}{2} \|\nabla \tilde{e}_h^n\|^2 + \frac{K \mu}{2} \sum_{e \in \Gamma_h \cup \partial \Omega} \sigma h_{e}^{-1} \|\tilde{e}_h^n\|^2_{L^2(e)}. \]
To handle \( a_e(\Pi_h u^n - u^n, \tilde{e}_h^n) \), we write:

\[
    a_e(\Pi_h u^n - u^n, \tilde{e}_h^n) = \sum_{E \in \mathcal{E}_h} \int_E \nabla (\Pi_h u^n - u^n) : \nabla \tilde{e}_h^n
    - \sum_{e \in \Gamma_h \cup \partial \Omega} \int_e \{ \nabla (\Pi_h u^n - u^n) \} n_e \cdot [\tilde{e}_h^n] + \epsilon \sum_{e \in \Gamma_h \cup \partial \Omega} \int_e \{ \nabla \tilde{e}_h^n \} n_e \cdot [\Pi_h u^n - u^n]
    + \frac{\sigma}{h_e} \sum_{e \in \Gamma_h \cup \partial \Omega} ||\Pi_h u^n - u^n|| |\tilde{e}_h^n| = Q_1 + Q_2 + Q_3 + Q_4.
\]

The terms \( Q_1, Q_3 \) and \( Q_4 \) are handled via standard arguments. By using trace inequalities, we have

\[
(6.93) \quad |Q_1| + |Q_3| + |Q_4| \leq \frac{K}{32} \| \tilde{e}_h^n \|^2_{\text{DG}} + Ch^{2k} |u^n|_{H^{k+1}(\Omega)}^2.
\]

The term \( Q_2 \) is more delicate. We follow an approach similar to the one in Chapter 6 of [26] and [13]. By Cauchy-Schwarz’s inequality, we have

\[
(6.94) \quad |Q_2| \leq Ch^{1/2} \left( \sum_{e \in \Gamma_h \cup \partial \Omega} \| \{ \nabla (\Pi_h u^n - u^n) \} \|_{L^2(e)}^2 \right)^{1/2} \\| \tilde{e}_h^n \|_{\text{DG}}.
\]

Consider one element \( E_e^1 \) adjacent to \( e \). By the trace theorem, we have

\[
(6.95) \quad \| \nabla (\Pi_h u^n - u^n) \|_{L^2(e)} \leq Ch_e^{-1/2} \left( \| \nabla (\Pi_h u^n - u^n) \|_{L^2(E_e^1)} + h_{E_e^1} \right) \| \nabla^2 (\Pi_h u^n - u^n) \|_{L^2(E_e^1)}
\]

Let \( \tilde{u}^n \in \mathbf{X}_h \) be an interpolant of \( u^n \), see Theorem 2.6 in [26], satisfying:

\[
\| \nabla (u^n - \tilde{u}^n) \|_{L^2(E_e^1)} + h_{E_e^1} \| \nabla^2 (u^n - \tilde{u}^n) \|_{L^2(E_e^1)} \leq Ch_{E_e^1}^{-k} |u^n|_{H^{k+1}(E_e^1)}, \quad \forall E \in \mathcal{E}_h.
\]

By the triangle inequality, an inverse estimate, the above bound and (6.6), we obtain:

\[
\| \nabla^2 (\Pi_h u^n - u^n) \|_{L^2(E_e^1)} \leq \| \nabla^2 (\Pi_h u^n - \tilde{u}^n) \|_{L^2(E_e^1)} + \| \nabla^2 (\tilde{u}^n - u^n) \|_{L^2(E_e^1)} \leq Ch_{E_e^1}^{-1} \| \nabla (\Pi_h u^n - \tilde{u}^n) \|_{L^2(E_e^1)} + Ch_{E_e^1}^{-k-1} |u^n|_{H^{k+1}(E_e^1)} \leq Ch_{E_e^1}^{-k-1} |u^n|_{H^{k+1}(E_e^1)}.
\]

Hence, (6.95) becomes

\[
(6.96) \quad \| \nabla (\Pi_h u^n - u^n) \|_{E_e^1} \|_{L^2(e)} \leq Ch_{E_e^1}^{-k-1/2} |u^n|_{H^{k+1}(E_e^1)}.
\]

The term involving the neighboring element \( E_e^2 \) is handled similarly. Thus, we can conclude that

\[
(6.97) \quad |Q_2| \leq \frac{K}{32} \| \tilde{e}_h^n \|^2_{\text{DG}} + Ch^{2k} |u^n|_{H^{k+1}(\Omega)}^2.
\]

We employ the above bounds on \( Q_1-Q_4 \), the bound on the nonlinear terms \( |R_c(\tilde{e}_h^n)| \) (6.74), (6.92), and the bound on \( |R_d(\tilde{e}_h^n)| \) (6.90) in (6.89). With these bounds and the assumptions that \( u \in L^\infty(0,T;H^{k+1}(\Omega)^d) \) and \( p \in L^\infty(0,T;H^k(\Omega)) \), (6.89)
becomes:
(6.98)
\[
\frac{1}{2} (\|e_h^n\|^2 - \|e_h^{n-1}\|^2 + \|e_h^n - e_h^{n-1}\|^2) + \frac{\tau^2}{2} (a_{\text{elli}}(\xi_h^n, \xi_h^n) - a_{\text{elli}}(\xi_h^{n-1}, \xi_h^{n-1})) \\
+ \frac{\tau^2}{16} \phi_h^1\|e_h^n\|_{DG}^2 + \frac{K\mu}{4} \tau \|e_h^n\|_{DG}^2 + \frac{\tau^2}{2} (A_1^n - A_2^n) + \frac{\tau}{2\delta\mu} (\|S_h^n\|^2 - \|S_h^{n-1}\|^2) \\
\leq \frac{C}{\mu} \tau^2 \int_{t_{n-1}}^{t_n} (\|\partial_t u\|^2 + \|\partial_t u\|^2) + \frac{C}{\mu} \tau^2 \sum_{t_{n-1}}^{t_n} \|\partial_t u\|_{H^k(\Omega)}^2 + C \left( \mu + \frac{1}{\mu} \right) \tau h^{2k} \\
+ \frac{C}{\mu} \tau^2 \|e_h^{n-1}\|^2 + C \tau^2 - \delta_n, \tau b(e_h^0, \phi_h^1) + \frac{1}{2} \delta_n, \tau \|b(e_h^0, \Pi_h u^1 \cdot \hat{e}_h^1)\|.
\]

We sum (6.98) from \(n = 1\) to \(n = m\), multiply by 2, and use the regularity assumptions. We obtain

(6.99)
\[
\|e_h^n\|^2 + \sum_{n=1}^{m} \|e_h^n - e_h^{n-1}\|^2 + \tau^2 a_{\text{elli}}(\xi_h^n, \xi_h^n) + \frac{\tau^2}{8} \sum_{n=1}^{m} \|e_h^n\|_{DG}^2 + \frac{\tau}{2} \sum_{n=1}^{m} \|e_h^n\|_{DG}^2 \\
+ \tau^2 \sum_{n=1}^{m} (A_1^n - A_2^n) + \frac{\tau}{\delta\mu} \|S_h^n\|^2 \leq C \left( 1 + \mu + \frac{1}{\mu} \right) (\tau + h^{2k}) + C \tau \sum_{n=0}^{m-1} \|e_h^n\|^2 \\
+ \|e_h^0\|^2 + 2\tau \|b(e_h^0, \phi_h^1)\| + \tau \|b(e_h^0, \Pi_h u^1 \cdot \hat{e}_h^1)\|.
\]

We recall that \(\phi_h^0 = 0\) and the definitions of \(A_1^n\) and \(A_2^n\) (see (6.22) - (6.23)). We use (3.32) with the assumption that \(\tilde{\sigma} \geq M_k^2\) to obtain

\[
\sum_{n=1}^{m} (A_1^n - A_2^n) = \sum_{\gamma \in \Gamma_n} \frac{\tilde{\sigma}}{h_e} \|[(\phi_h^m)]_{L^2(\gamma)} - \|G_h((\phi_h^m))\|_{L^2(\gamma)}^2 \\
\geq \sum_{\gamma \in \Gamma_n} \frac{\tilde{\sigma} - M_k}{h_e} \|[(\phi_h^m)]_{L^2(\gamma)}^2 \geq 0.
\]

With (3.34), (3.32), and approximation properties we have

\[
|b(e_h^0, \Pi_h u^1 \cdot \hat{e}_h^1)| \leq C \|e_h^0\| \|\phi_h^1\|_{DG} \leq C \tau^{-1} h^{2k+2} \|u_0^0\|_{H^{k+1}(\Omega)} + \frac{\tau}{16} \|\phi_h^1\|_{DG}^2.
\]

With Hölder’s inequality, trace inequality, stability of the interpolant, and (5.7), we obtain

\[
|b(e_h^0, \Pi_h u^1 \cdot \hat{e}_h^1)| \leq C \|\nabla_h e_h^0\| \|\Pi_h u^1\|_{L^3(\Omega)} \|\hat{e}_h^1\|_{DG} + C \sum_{\gamma \in \Gamma_h \cup \partial \Omega} \left( \sum_{i=1}^{2} \|\Pi_h u^1\|_{L^3(E_i)} \|\hat{e}_h^1\|_{L^6(E_i)} h^{-1/2}_{E_i} \|e_h^0\|_{L^6(\gamma)} \right) \\
\leq C \|e_h^0\|_{DG} \|u^1\|_{W^{1,3}(\Omega)} \|\hat{e}_h^1\|_{DG} \leq C \frac{h^{2k}}{\kappa \mu} \|u^1\|_{H^{k+1}(\Omega)} + \frac{\kappa \mu}{4} \|\hat{e}_h^1\|_{DG}.
\]

The above bounds, approximation properties, the coercivity property of \(a_{\text{elli}}\) (3.22), discrete Gronwall’s inequality, and the triangle inequality yield bound (6.76). \(\square\)
Two immediate results of the above arguments are as follows. Under the same regularity assumptions of Theorem 6.7 and for $\tau \leq \gamma_1 \mu$ and for $1 \leq m \leq N_T$,

$$\sum_{n=1}^{m} \| e^n_h - e^{n-1}_h \|^2 + \frac{1}{16} \tau^2 \sum_{n=1}^{m} \| \phi^n_h \|^2_{DG} + \frac{\kappa \mu}{4} \tau \sum_{n=1}^{m} \| \tilde{\omega}^n_h \|^2_{DG}$$

$$\leq C_\mu \left( 1 + \mu + \frac{1}{\mu} \right) (\tau + h^{2k}).$$

In addition, with (4.7) and (3.32), we have

$$\| u^n_h - u^n \|^2 \leq 2 \| u^n_h - u^n_h \|^2 + 2 \| u^n_h - u^n \|^2$$

$$\leq C \tau^2 |\phi^n_h|^2_{DG} + 2 \| u^n_h - u^n \|^2 \leq C_\mu \left( 1 + \mu + \frac{1}{\mu} \right) (\tau + h^{2k}).$$

**Remark 6.8.** Assume that the mesh is quasi-uniform. If we use (4.7) and the following inverse inequality.

$$\| \theta_h \|_{DG} \leq C h^{-1} |\theta_h|, \quad \forall \theta_h \in X_h,$$

we obtain the following

$$\mu \tau \sum_{n=1}^{m} \| u^n_h - u^n \|^2_{DG} \leq \mu \tau \sum_{n=1}^{m} \| u^n_h - u^n \|^2_{DG} + \mu \tau \sum_{n=1}^{m} \tau^2 \| \nabla \phi^n_h + G_h(\phi^n_h) \|^2_{DG}$$

$$\leq C_\mu \left( 1 + \mu + \frac{1}{\mu} \right) (\tau + h^{2k}) + C \mu \tau h^{-2} \sum_{n=1}^{m} \tau^2 |\phi^n_h|^2_{DG}.$$ 

Therefore, under a CFL condition, $\tau \leq \gamma_2 h^2$, we have the following error bound:

$$\mu \tau \sum_{n=1}^{m} \| u^n_h - u^n \|^2_{DG} \leq C_\mu \left( 1 + \mu + \frac{1}{\mu} \right) (\tau + h^{2k}).$$

7. Numerical experiments

In this section, we use the manufactured solution method to compute the convergence rates. Let $\Omega = (0,1)^3$ denote the computational domain and let $T = 1$ be the end time for all numerical experiments. The domain $\Omega$ is partitioned into cubic elements. We select the following Beltrami flow with parameter $\mu = 1$ as the prescribed solution.

$$u_x(x, y, z, t) = - \exp (-t + x) \sin (y + z) - \exp (-t + z) \cos (x + y),$$

$$u_y(x, y, z, t) = - \exp (-t + y) \sin (x + z) - \exp (-t + x) \cos (y + z),$$

$$u_z(x, y, z, t) = - \exp (-t + z) \sin (x + y) - \exp (-t + y) \cos (x + z),$$

$$p(x, y, z, t) = - \exp (-2t) \left( \exp (x + z) \sin (y + z) \cos (x + y) + \exp (y + z) \sin (x + z) \cos (y + z) + \frac{1}{2} \exp (2x) + \frac{1}{2} \exp (2y) + \frac{1}{2} \exp (2z) - \tilde{p} \right).$$

Here $\tilde{p} = 7.63958172715414$ is used to force the average pressure over $\Omega$ equal to zero (up to machine precision). The Beltrami flow has the property that the velocity and vorticity vectors are parallel to each other and that the nonlinear convection term
is balanced by the pressure gradient, namely, $\partial_t \mathbf{u} \Delta \mathbf{u} = 0$ and $\mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0$, which implies $f = 0$. The initial condition and Dirichlet boundary condition on $\partial \Omega$ for velocity are imposed by the exact solution.

First, we derive temporal rates of convergence by computing the solutions with time step size $\tau \in \{1/2^3, 1/2^4, \ldots, 1/2^7\}$. The $P_2 - P_1$ dG scheme with mesh resolution $h_e = 1/160$ is employed to guarantee the error in space does not dominate the error in time. We set $\epsilon = -1$, $\sigma = 64$ on $\Gamma_h$ and $\sigma = 128$ on $\partial \Omega$. The other penalty parameter $\tilde{\sigma}$ is equal to 2. If $err$ denotes the error when time step size is equal to $\tau$, the convergence rate is defined by $\ln(err_{\tau}/err_{\tau/2})/\ln(2)$. Table 1 displays the errors and convergence rates for the velocities and pressure. We observe second order convergence rate in time, which is higher than the expected first order rate.

Next, we obtain spatial rates of convergence by computing the solutions on a sequence of uniformly refined meshes (see the second column of Table 2 for $h_e$). We fix $\tau = 1/2^{10}$ for the $P_1 - P_0$ dG scheme, we fix $\tau = 1/2^{13}$ for the $P_2 - P_1$ dG scheme, and we fix $\tau = 1/2^{15}$ for the $P_3 - P_2$ dG scheme to guarantee the spatial error dominates. For $P_1 - P_0$, we set $\tilde{\sigma} = 1$, $\epsilon = -1$, $\sigma = 8$ on $\Gamma_h$ and $\sigma = 16$ on $\partial \Omega$. For $P_2 - P_1$, we set $\tilde{\sigma} = 2$, $\epsilon = -1$, $\sigma = 64$ on $\Gamma_h$ and $\sigma = 128$ on $\partial \Omega$. For $P_3 - P_2$, we set $\tilde{\sigma} = 8$, $\epsilon = -1$, $\sigma = 128$ on $\Gamma_h$ and $\sigma = 256$ on $\partial \Omega$. If $err_{h_e}$ is the error on a mesh with resolution $h_e$, then the rate is defined by $\ln(err_{h_e}/err_{h_e/2})/\ln(2)$. We show the errors and rates in Table 2. The convergence rates are optimal.

In all numerical experiments, the values $\|v_h^{N_T} - \mathbf{u}(T)\|$ and $\|u_h^{N_T} - \mathbf{u}(T)\|$ are identical for at least nine digits.

| $k$ | $\tau$ | $\|v_h^{N_T} - \mathbf{u}(T)\|$ rate | $\|u_h^{N_T} - \mathbf{u}(T)\|$ rate | $\|p_h^{N_T} - p(T)\|$ rate |
|-----|--------|-----------------------------------|-----------------------------------|-----------------------------|
| 2   | $1/2^3$ | 1.911E+2                           | --                                | 2.196E-1                    |
|     | $1/2^4$ | 6.437E-3                           | 1.570                             | 1.025E-1                    |
|     | $1/2^5$ | 1.699E-3                           | 1.922                             | 3.093E-2                    |
|     | $1/2^6$ | 4.506E-4                           | 1.915                             | 7.790E-3                    |
|     | $1/2^7$ | 1.207E-4                           | 1.901                             | 1.919E-3                    |

Table 1. Errors and temporal convergence rates of velocity and pressure.

8. Conclusion

The main contribution of this work is a theoretical error analysis of a dG pressure correction scheme for solving the incompressible Navier-Stokes equations. We establish stability of the method and derive a priori error estimates for the discrete velocity that are optimal in the broken gradient norm. The convergence analysis is technical and relies on lift operators and appropriate local approximations. Future work will consist of deriving error bounds for the pressure and optimal error estimates for the discrete velocity in the $L^2$ norm.

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| $k$ | $h_k$ | $\|v_h^{N_T} - u(T)\|$ rate | $\|u_h^{N_T} - u(T)\|$ rate | $\|p_h^{N_T} - p(T)\|$ rate |
|-----|------|-----------------|-----------------|-----------------|
| 1   | $1/2^1$ | 2.849E−3 — | 2.849E−3 — | 5.255E−2 — |
|     | $1/2^2$ | 7.204E−4 1.984 | 7.204E−4 1.984 | 2.152E−2 1.288 |
|     | $1/2^3$ | 1.815E−4 1.989 | 1.815E−4 1.989 | 9.752E−3 1.142 |
|     | $1/2^4$ | 4.572E−5 1.989 | 4.572E−5 1.989 | 4.706E−3 1.051 |
|     | $1/2^5$ | 1.215E−5 1.912 | 1.215E−5 1.912 | 2.330E−3 1.014 |
| 2   | $1/2^1$ | 6.322E−3 — | 6.322E−3 — | 1.925E−1 — |
|     | $1/2^2$ | 7.874E−4 3.005 | 7.874E−4 3.005 | 2.274E−2 3.082 |
|     | $1/2^3$ | 9.717E−5 3.019 | 9.717E−5 3.019 | 3.850E−3 2.562 |
|     | $1/2^4$ | 1.198E−5 3.020 | 1.198E−5 3.020 | 8.130E−4 2.244 |
|     | $1/2^5$ | 1.502E−6 2.996 | 1.502E−6 2.996 | 1.918E−4 2.084 |
| 3   | $1/2^1$ | 5.129E−3 — | 5.129E−3 — | 5.021E−1 — |
|     | $1/2^2$ | 4.272E−4 3.586 | 4.272E−4 3.586 | 4.376E−2 3.521 |
|     | $1/2^3$ | 2.692E−5 3.988 | 2.692E−5 3.988 | 3.690E−3 3.568 |
|     | $1/2^4$ | 1.611E−6 4.063 | 1.611E−6 4.063 | 3.170E−4 3.541 |
|     | $1/2^5$ | 1.066E−7 3.918 | 1.066E−7 3.918 | 2.920E−5 3.441 |

Table 2. Errors and spatial convergence rates of velocity and pressure.

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