Design and Simulation of Trajectory Tracking Guidance Law Based on LQR for Target Missile

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Abstract. Aiming at the trajectory tracking problem of the target projectile, a ballistic tracking guidance law of the target projectile is designed based on the linear quadratic regulator (LQR) theory. Using time as an independent variable, the motion equation of the center of mass of the target bomb was linearized using the small disturbance method to establish the center of mass motion of the Space expressions; The LQR theory was used to design the trajectory tracking guidance law of the target projectile. Finally, the semi-physical simulation test was conducted to verify the trajectory tracking guidance law of the designed target projectile.

1. Introduction
The target missile is a commonly used target for evaluating the performance of air defense weapons in the process of development, type setting, batch sampling and identification. It is used to simulate the external dimensions, ballistic characteristics, and confrontation characteristics of the incoming missile target [1,2].

The accuracy of the target's ballistic tracking program is one of the important indexes that affect the target's elastic energy. During the actual flight of the target bomb, the deviation of the actual flight trajectory of the target missile from the project trajectory will be greatly affected by factors such as aerodynamic deviation, position deviation of the start and control, and wind. Therefore, it is necessary to design an appropriate trajectory tracking guidance law to ensure its accuracy. Trajectory flight along the program can still be performed with high accuracy under various disturbance conditions [3]. The early program trajectory is mainly achieved through program control, and its control accuracy is greatly affected by disturbance factors.

In the design of today's target guidance control method, the control method of position control and feedback of position deviation is often used [4,5]. With the development and application of modern control theory such as synovial control, optimal control, and intelligent control, many new methods have emerged for the trajectory tracking of guided weapons. Literature [3] applied the method of dynamic inversion and synovial control in the ballistic tracking of missiles. Tang Shantong designed a new air defense missile trajectory tracking guidance law based on Lyapunov stability principle in Literature [6]. Greg Dukeman introduced the LQR theory in the design of ballistic tracking control methods for reentry vehicles, effectively improving the ballistic tracking accuracy of reentry vehicle programs [7]. In this paper, a certain type of rocket missile reconstructed target missile is taken as the...
research object. The LQR theory is used to design the trajectory tracking guidance law of this project. The designed guidance law is verified by semi-physical simulation.

2. Target Centroid Movement Model
In the design of the trajectory tracking guidance law of the target project, the main consideration is to change the spatial position of its center of mass, and the rotation process of the target around the center of mass can be ignored. The target bomb is regarded as a manipulable particle during the entire flight. With the assumption of instantaneous equilibrium, it is assumed that the target control system is an ideal control system with no error and no time delay. Ignoring the lateral plane motion of the target bomb, the equation of motion for the center of mass of the target bomb in the longitudinal plane is [8]:

\[
\begin{align*}
\frac{d}{dt}mV &= -X - mg \sin \theta \\
\frac{d}{dt}mV \theta &= Y - mg \cos \theta \\
\frac{d}{dt}x &= V \cos \theta \\
\frac{d}{dt}y &= V \sin \theta \\
\delta_z &= -\frac{m^2 \alpha}{m^2 \gamma} \\
m &= m_0 \\
\epsilon &= 0
\end{align*}
\]  

(1)

Among them, \( m \) is the target projectile mass; \( V \) is the target projectile velocity; \( \alpha \) is the angle of attack; \( \theta \) is the ballistic inclination; \( X \) and \( Y \) is the air resistance and lift force.

The calculation formula for the lift and resistance of the target is:

\[
\begin{align*}
X &= C_x q S = \frac{1}{2} \rho V^2 S C_x \\
Y &= C_y q S = \frac{1}{2} \rho V^2 S C_y
\end{align*}
\]  

(2)

In the formula, \( \rho \) is the air density; \( S \) is the reference area of the airfoil; \( C_x \) and \( C_y \) is the drag coefficient and the lift coefficient. Both the drag coefficient and the lift coefficient are functions of the angle of attack and the Mach number. The corresponding aerodynamic parameters can be obtained from the wind tunnel test.

The approximate formula for air density and gravitational acceleration:

\[
\begin{align*}
\rho &= \rho_0 \exp\left(-\frac{h}{h_p}\right) \\
g &= g_0 \frac{R_e^2}{(R_e + h)^2}
\end{align*}
\]  

(3)

Among them, \( \rho_0 \) is the air density at the surface; \( g_0 \) is the acceleration of gravity at sea level; \( R_e \) is the average radius of the earth.
3. Linear Quadratic Regulator (LQR) Theory

For a linear time-varying system with the following formal state space equations:

$$\dot{x} = A(t)x + B(t)u$$ (4)

Where, $x$ is the state variables, $u$ is the control vectors, $A(t)$ and $B(t)$ are $n \times n$ matrices that satisfy unique conditions for solutions.

Given quadratic performance index functions for state variables and control variables:

$$J = \frac{1}{2} \int_0^T \left[ x^T(t)Q(t)x + u^T(t)R(t)u \right] dt$$ (5)

According to the LQR theory, there exists a comprehensive optimal control variable, and the system state is adjusted from the initial state to the equilibrium state while ensuring that the above performance index takes a minimum value.

The optimal control volume calculation method is:

$$\begin{cases}
u^* = -K(t)x \\ K(t) = -R^{-1}(t)B^T(t)P(t) 
\end{cases}$$ (6)

Among them, $P(t)$ is a $n \times n$ positive semidefinite matrix satisfies the following matrix Riccati differential equation:

$$P(t) + P(t)A(t) + A^T(t)P(t) - P(t)B(t)R^{-1}(t)B^T(t)P(t) + Q(t) = 0$$ (7)

4. LQR program ballistic tracking guidance law design

4.1 Linearization of the motion equation of the center of mass of the target bomb

Using the small perturbation assumption, linearize the equation of motion of the center of mass in the longitudinal plane of the target bomb. After linearization, the mass disturbance perturbation equations of motion are:

$$\begin{aligned}
\frac{d\Delta V}{dt} &= -\frac{X^V}{m}\Delta V - \frac{X^a}{m}\Delta \alpha - g \cos \theta \Delta \theta - \\
& \quad \frac{X^\gamma}{m} \Delta y
\end{aligned}$$

$$\begin{aligned}
\frac{d\Delta \theta}{dt} &= \frac{Y^V}{mV} \Delta V + \frac{Y^a}{mV} \Delta \alpha + g \sin \theta \Delta \theta + \\
& \quad \frac{Y^\gamma}{mV} \Delta y + \frac{Y^\delta}{mV} \Delta \delta
\end{aligned}$$

$$\begin{aligned}
\frac{d\Delta x}{dt} &= \cos \theta \Delta V - V \sin \theta \Delta \theta \\
\frac{d\Delta y}{dt} &= \sin \theta \Delta V + V \cos \theta \Delta \theta
\end{aligned}$$ (8)

Where, $X^V, X^a$ and $X^\gamma$ are the partial derivative values of the undisturbed movement resistance at the corresponding moments for speed, angle of attack, and the target missile's flight altitude. $Y^V, Y^a$
and \( Y^\alpha \) are the partial derivative values of the undisturbed movement lift for speed, angle of attack, and altitude at the corresponding time.

\[
\begin{align*}
Y^\alpha &= \frac{\rho V^2 S C^\alpha_y}{2} \\
Y^{\delta_z} &= \frac{\rho V^2 S C^{\delta_z}_y}{2} \\
Y^\gamma &= \frac{1}{2c} \rho V^2 SC_y \frac{\partial C^\alpha_y}{\partial Ma} + \rho VSC_y
\end{align*}
\]  

(9)

Approximate calculation formula of gravitational acceleration can obtain the partial derivative of gravity to height in the course of target missile flight:

\[ G^\gamma = -2mg_0 \frac{R_e^2}{(R_e + y)^3} \]  

(10)

The partial lift of the target missile to the height of flight:

\[ Y^\gamma = -\frac{1}{2} V^2 SC_y \frac{p_0}{h_p} \exp(-\frac{y}{h_p}) \]  

(11)

In the ballistic coordinate system, the longitudinal overload of the target during flight:

\[ \Delta n_y = \frac{Y^\alpha \Delta \alpha + Y^{\delta_z} \Delta \delta_z}{G} \]  

(12)

Since the target's velocity characteristics have been designed during the ballistic design phase of the target projectile, the speed of the target projectile is almost uncontrollable after it is launched at a certain angle. When designing the trajectory tracking guidance law of the target bomb, the control of the flying speed is not considered, so the equation concerning the speed can be saved.

Substituting equation (15) into equation (9), taking the disturbance state variables:

\[ x = [\Delta \theta \quad \Delta y] \]  

(13)

Control variables: \( u = \Delta n_y \). The linearized state equation of the center of mass movement in the longitudinal plane of the target can be obtained:

\[ \dot{x} = A(t)x + B(t)u \]  

(14)

Where:

\[ A(t) = \begin{bmatrix}
\frac{g \sin \theta}{V} & \frac{Y^\gamma - G^\gamma \cos \theta}{mV} \\
V \cos \theta & 0
\end{bmatrix} \]  

(15)

\[ B(t) = \begin{bmatrix}
g \\
V \\
0
\end{bmatrix} \]  

(16)
4.2. Determining Weight Matrix
In practical applications, the weight matrix \( Q \), \( R \) is generally selected as a diagonal matrix. Therefore, the matrix can be written as:

\[
Q = \begin{bmatrix}
Q_1 & 0 \\
0 & Q_2
\end{bmatrix}
\]

(17)

\[
R = [R_1]
\]

(18)

The corresponding quadratic performance index can be written as:

\[
J = \frac{1}{2} \int_{t_0}^{t_f} [Q_1 (\Delta \theta)^2 + Q_2 (\Delta y)^2 + R_1 (\Delta u_n)^2] dt
\]

(19)

In order to simplify the calculation, it is also possible to ensure that the actual flight trajectory height \( y \) of the target missile tracks the program ballistic altitude in real time. According to the Bryson principle, we have:

\[
Q_2 (\Delta y_{\text{max}})^2 = R_1 (\Delta u_{\text{max}})^2
\]

(20)

Where, the subscript " max " represents the maximum allowable deviation of the corresponding variable. Since the selection of any fixed-weighted weight matrix does not affect the final result, so can let \( Q_2 = 1 \), then:

\[
R_1 = \frac{(\Delta y_{\text{max}})^2}{(\Delta u_{\text{max}})^2}
\]

(21)

4.3. Design of ballistic tracking guidance law for target project
During the process of the ballistic flight of the target projectile, the dynamic characteristics of the target projectile with the flight altitude, speed, and other conditions vary greatly, and it is difficult to accurately calculate the dynamic characteristic parameters at any time during the flight process. In order to simplify the calculation considering the linearization accuracy of the target's center-of-mass equation of motion, several trajectory feature points that can fully represent the full trajectory dynamics are selected in the trajectory of the target projectile, and the linearized state equation of the target missile at the trajectory feature point is calculated.

The disturbance motion of the target centroid near the ballistic feature point can be approximated as a linear time-invariant system. Its state equation is:

\[
x = A_0 x + B_0 u
\]

(22)

Among them, \( A_0 \), \( B_0 \) is determined by the dynamic coefficient of the target bomb at the feature point of the undisturbed motion trajectory.

According to the infinite-time LQR theory, for the linear time-invariant system shown in Equation (22), in Equation (5) \( t_j \rightarrow \infty \), the terminal constraint \( x^T(t_j)Fx(t_j)/2 \) loses its meaning. Therefore, the quadratic performance index function can be simplified as:

\[
J = \frac{1}{2} \int_{t_0}^{t_f} [x^T Q(t)x + u^T R(t)u] dt
\]

(23)

From the weight matrix determined by 3.2, the optimal state feedback gain \( K \) at each trajectory feature point is calculated according to Equations (7) and (8).

The feature points on the program trajectory and the optimal state feedback gain \( K \) are bound to the on board computer. During the flight of the target bomb, the integrated state navigation system is used to acquire the state information such as the position and velocity of the target bomb, and the
real-time state deviation of the actual flight trajectory and the project trajectory of the target bomb is obtained:

\[
\begin{align*}
\Delta \theta &= \theta - \theta_{bc} \\
\Delta y &= y - y_{bc}
\end{align*}
\]  

Using flight time as an independent variable, the optimal state feedback gain at this moment is obtained from the computer on board by interpolation method. The corresponding optimal feedback guidance law is:

\[
u^* = K[\Delta \theta \quad \Delta y]^T
\]

5. Simulation

In the simulation process, the numerical integration method of the fourth-order Runge-Kutta was used to solve differential equations. The ballistic solution step length was chosen to be 0.01 s, the target missile was launched with the initial emission angle \( \theta_0 = 60^\circ \), and the engine thrust was \( 6 \times 10^4 \) N, work Time \( t = 6.4 \) s, 10 s after the target launches, the corresponding control law is applied. Before the control law is inserted, only the target missile attitude is controlled stably. The range of target height for target bombs is 2.5~20 km in the descending phase, and the target height error should not exceed 100 m.

Under the non-interference conditions, the results of semi-physical simulation are shown in the figure 1 to figure 3. It can be seen from figure 1 that the LQR scheme’s trajectory tracking guidance law and classical control method can effectively track the trajectory of the program under no-jamming conditions. From figure 2 and figure 3, it can be seen that the trajectory accuracy of the classic control method tracking scheme is greatly influenced by the aerodynamic parameter change during the flight of the target bomb. The maximum height deviation is up to 120 m, the maximum ballistic angle deviation is \(-0.803^\circ\), and the target segment height deviation range is \(-52.8\) to \(80.5\) m, and the deviation range of ballistic inclination is \(-0.803\) to \(-0.115^\circ\). The LQR guidance law keeps the trajectory height and trajectory inclination error of the tracking program within a relatively small range during the entire flight of the target bomb. The maximum height deviation is 17.5 m, for the target segment height deviation range -1.7 to 12.5 m, ballistic inclination deviation range \(-0.069\) to \(-0.057^\circ\). Therefore, LQR guidance law, which is available under no interference conditions, provides higher accuracy than the classical control method.

![Figure 1. Interference free conditional time-height curve](image-url)
The results of the lower physical simulation with the starting and controlling position deviation and the 10% aerodynamic deviation are shown in figure 4 to figure 6. It can be seen from figure 4 that under the disturbance conditions, the trajectory of the classical control method tracking scheme produces large errors, and the accuracy of the LQR guidance law tracking scheme is higher than the classical control method. Figure 5 and figure 6 show that the classical control method provides a target range height deviation range of -642.9 to 294.1 m, a ballistic range deviation range of -0.752 to 5.205°, and the height deviation exceeding the allowable target height tolerance range; LQR guidance law The height deviation range of the target segment is -98.5 to 41.78 m, and the deviation range of the ballistic inclination is 0 to 0.434°. Compared with the classical control method, the target accuracy is higher, and the target height error is controlled within the allowable range.
Fig. 5. Start to control position deviation time-height curve

Fig. 6. Start to control position deviation time-ballistic inclination deviation curve

6. Conclusion

Targeting high-performance target missiles requires the actual flight trajectory to track the flight trajectory of the program with a high degree of accuracy. After applying the small disturbance method to linearize the linear center of mass motion equation of the target bomb, the LQR theory is used to design the target bomb in the longitudinal plane. The program's trajectory tracking guidance law, and the use of semi-physical simulation method for simulation verification. The simulation results show that the LQR-based trajectory tracking guidance law of the target projectile can effectively track the trajectory of the project, and the performance of suppressing the position deviation of the start-control and aerodynamic parameter deviation is better than that of the classical control method. It provides a target trajectory tracking for the target missile. The control problem provides a new solution to improve the target ballistic accuracy of the target bomb.

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