Stability in the Erdős–Gallai Theorem on cycles and paths

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Dedicated to the memory of G. N. Kopylov

Abstract

The Erdős–Gallai Theorem states that for $k \geq 2$, every graph of average degree more than $k - 2$ contains a $k$-vertex path. This result is a consequence of a stronger result of Kopylov: if $k$ is odd, $k = 2t + 1 \geq 5$, $n \geq (5t - 3)/2$, and $G$ is an $n$-vertex 2-connected graph with at least $h(n, k, t) := \binom{k-1}{2} + t(n-k+t)$ edges, then $G$ contains a cycle of length at least $k$ unless $G = H_{n,k,t} := K_n - E(K_{n-1})$.

In this paper we prove a stability version of the Erdős–Gallai Theorem: we show that for all $n \geq 3t > 3$, and $k \in \{2t+1, 2t+2\}$, every $n$-vertex 2-connected graph $G$ with $e(G) > h(n, k, t-1)$ either contains a cycle of length at least $k$ or contains a set of $t$ vertices whose removal gives a star forest. In particular, if $k = 2t+1 \neq 7$, we show $G \subseteq H_{n, k, t}$. The lower bound $e(G) > h(n, k, t-1)$ in these results is tight and is smaller than Kopylov’s bound $h(n, k, t)$ by a term of $n - t - O(1)$.

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1 Introduction

A cornerstone of extremal combinatorics is the study of Turán-type problems for graphs. One of the fundamental questions in extremal graph theory is to determine the maximum number of edges in an $n$-vertex graph with no $k$-vertex path. According to [10], this problem was posed by Turán. A solution to the problem was obtained by Erdős and Gallai [7]:

Theorem 1.1 (Erdős and Gallai [7]). Let $G$ be an $n$-vertex graph with more than $\frac{1}{2}(k - 2)n$ edges, $k \geq 2$. Then $G$ contains a $k$-vertex path $P_k$.

This result is best possible for $n$ divisible by $k - 1$, due to the $n$-vertex graph whose components are cliques of order $k - 1$. To obtain Theorem 1.1 Erdős and Gallai observed that if $H$ is an $n$-vertex graph without a $k$-vertex path $P_k$, then adding a new vertex and joining it to all other vertices we have a graph $H'$ on $n + 1$ vertices $e(H) + n$ edges and containing no cycle $C_{k+1}$ or longer. Then Theorem 1.1 is a consequence of the following:

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Theorem 1.2 (Erdős and Gallai [7]). Let $G$ be an $n$-vertex graph with more than $\frac{1}{2}(k-1)(n-1)$ edges, $k \geq 3$. Then $G$ contains a cycle of length at least $k$.

This result is best possible for $n-1$ divisible by $k-2$, due to any $n$-vertex graph where each block is a clique of order $k-1$. Let $\text{ex}(n, P_k)$ be the maximum number of edges in an $n$-vertex graph with no $k$-vertex path; Theorem 1.1 shows $\text{ex}(n, P_k) \leq \frac{1}{2}(k-2)n$ with equality for $n$ divisible by $k-1$. Several proofs and sharpenings of the Erdős-Gallai theorem were obtained by Woodall [16], Lewin [12], Faudree and Schelp [8, 9] and Kopylov [11] – see [10] for further details. The strongest version was proved by Kopylov [11]. To describe his result, we require the following graphs. Suppose $n \geq k$, $(k/2) > a \geq 1$. Define the $n$-vertex graph $H_{n,k,a}$ as follows. The vertex set of $H_{n,k,a}$ is partitioned into three sets $A, B, C$ such that $|A| = a$, $|B| = n - k + a$ and $|C| = k - 2a$ and the edge set of $H_{n,k,a}$ consists of all edges between $A$ and $B$ together with all edges in $A \cup C$. Let

$$h(n, k, a) := e(H_{n,k,a}) = \left(\frac{k-a}{2}\right) + a(n-k+a).$$

Theorem 1.3 (Kopylov [11]). Let $n \geq k \geq 5$ and $t = \lceil \frac{k-1}{2} \rceil$. If $G$ is an $n$-vertex 2-connected graph with no cycle of length at least $k$, then

$$e(G) \leq \max \{h(n, k, 2), h(n, k, t)\}$$

with equality only if $G = H_{n,k,2}$ or $G = H_{n,k,t}$.

In this paper, we prove a stability version of Theorems 1.1 and 1.3. A star forest is a vertex-disjoint union of stars.

Theorem 1.4. Let $t \geq 2$ and $n \geq 3t$ and $k \in \{2t+1, 2t+2\}$. Let $G$ be a 2-connected $n$-vertex graph containing no cycle of length at least $k$. Then $e(G) \leq h(n, k, t-1)$ unless

(a) $k = 2t+1$, $k \neq 7$, and $G \subseteq H_{n,k,t}$ or
(b) $k = 2t+2$ or $k = 7$, and $G - A$ is a star forest for some $A \subseteq V(G)$ of size at most $t$.

This result is best possible in the following sense. Note that $H_{n,k,t-1}$ contains no cycle of length at least $k$, is not a subgraph of $H_{n,k,t}$, and $H_{n,2t+2,t-1} - A$ has a cycle for every $A \subseteq V(H_{n,2t+2,t-1})$ with $|A| = t$. Thus the claim of Theorem 1.4 does not hold for $G = H_{n,k,t-1}$. Therefore the condition $e(G) \leq h(n, k, t-1)$ in Theorem 1.4 is best possible. Since

$$h(n, 2t+2, t) = \left(\frac{t}{2}\right) + t(n-t) + 1 = h(n, 2t+1, t) + 1$$

and

$$h(n, 2t+2, t-1) = \left(\frac{t}{2}\right) + (t-1)(n-t) + 6 = h(n, 2t+1, t-1) + 3,$$

the difference between Kopylov’s bound and the bound in Theorem 1.4 is

$$h(n, k, t) - h(n, k, t-1) = \begin{cases} 
  n - t - 3 & \text{if } k = 2t+1 \\
  n - t - 5 & \text{if } k = 2t+2.
\end{cases}$$

(2)
It is interesting that for a fixed $k$, the difference in $\frac{2}{3}$ divided by $h(n, k, t)$ does not tend to 0 when $n \to \infty$.

We will need to prove a more detailed version of Theorem 1.4. This version, Theorem 4.1, will yield the following cleaner claim for 3-connected graphs.

**Corollary 1.5.** Let $k \geq 11$, $t = \lfloor \frac{k-1}{2} \rfloor$, and $n \geq \frac{3k}{2}$. If $G$ is an $n$-vertex 3-connected graph with no cycle of length at least $k$, then $e(G) \leq h(n, k, t - 1)$ unless $G \subseteq H_{n,k,t}$.

In the same way that Theorem 1.2 implies Theorem 1.1, Theorem 1.4 applies to give a stability theorem for paths:

**Theorem 1.6.** Let $t \geq 2$ and $n \geq 3t - 1$ and $k \in \{2t, 2t + 1\}$, and let $G$ be a connected $n$-vertex graph containing no $k$-vertex path. Then $e(G) \leq h(n + 1, k + 1, t - 1) - n$ unless

\begin{itemize}
  \item[(a)] $k = 2t$, $k \neq 6$, and $G \subseteq H_{n,k,t-1}$ or
  \item[(b)] $k = 2t + 1$ or $k = 6$, and $G - A$ is a star forest for some $A \subseteq V(G)$ of size at most $t - 1$.
\end{itemize}

Indeed, let $G'$ be obtained from an $n$-vertex connected graph $G$ with more than $h(n+1,k+1,t-1)-n$ edges by adding a vertex adjacent to all vertices in $G$. Then $G'$ is 2-connected and $G'$ has more than $h(n+1,k+1,t-1)$ edges. If $G$ has no $k$-vertex path, then $G'$ has no cycle of length at least $k + 1$. By Theorem 1.4 $G'$ satisfies (a) or (b) in Theorem 1.4, which means $G$ satisfies (a) or (b) in Theorem 1.6. Repeating this argument, Corollary 1.5 implies the following.

**Corollary 1.7.** Let $k \geq 11$, $t = \lfloor \frac{k-1}{2} \rfloor$, and $n \geq \frac{3k}{2}$. If $G$ is an $n$-vertex 2-connected graph with no $k$-vertex paths, then $e(G) \leq h(n + 1, k + 1, t - 1) - n$ unless $G \subseteq H_{n,k,t-1}$.

**Organization.** The proof of Theorem 1.4 will use a number of classical results listed in Section 2 and some lemmas on contractions proved in Section 3. Then in Section 4 we describe several families of extremal graphs and state and prove a more technical Theorem 4.1, implying Theorem 1.4 for $k \geq 9$. Finally, in Section 5 we prove the analog of our technical Theorem 1.1 for $4 \leq k \leq 8$. In particular, we describe all 2-connected graphs with no cycles of length at least 6.

**Notation.** We use standard notation of graph theory. Given a simple graph $G = (V,E)$, the neighborhood of $v \in V$, i.e. the set of vertices adjacent to $v$, is denoted by $N_G(v)$ or $N(v)$ for short, and the closed neighborhood is $N[v] := N(v) \cup \{v\}$. The degree of vertex $v$ is $d_G(v) := |N_G(v)|$. Given $A \subseteq V$ we also use $N_G(v,A)$ for $N(v) \cap A$, $d(v,A)$ for $|N(v) \cap A|$, and $N(A) := \bigcup_{v \in A} N(v) \setminus A$. For an edge $xy$ in $G$, let $T_G(xy)$ denote the number of triangles containing $xy$ and $T(G) := \min\{T_G(xy) : xy \in E\}$. The minimum degree of $G$ is denoted by $\delta(G)$. For an edge $xy$ in $G$, $G/xy$ denotes the graph obtained from $G$ by contracting $xy$. We frequently use $x * y$ for the new vertex. The length of the longest cycle in $G$ is denoted by $c(G)$, and $e(G) := |E|$. Denote by $K_n$ the complete $n$-vertex graph, and $K(A,B)$ the complete bipartite graph with parts $A$ and $B$ ($A \cap B = \emptyset$). Given vertex-disjoint graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$, the graph $G_1 + G_2$ has vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup E(K(V_1, V_2))$. If $G$ is a graph, then $\overline{G}$ denotes the complement of $G$ and for a positive integer $\ell$, $\ell G$ denotes the graph consisting of $\ell$ components, each isomorphic to $G$. For disjoint sets $A,B \subseteq V(G)$, let $G(A,B)$ denote the bipartite graph with parts $A$ and $B$ consisting of all edges of $G$ between $A$ and $B$, and for $A \subseteq V(G)$, let $G[A]$ denote the subgraph induced by $A$. 
2 Classical theorems

We require a number of theorems on long paths and cycles in dense graphs. The following is an extension to 2-connected graphs of the well-known fact that an \( n \)-vertex non-hamiltonian graph has at most \( \binom{n-1}{2} + 1 \) edges:

**Theorem 2.1** (Erdős [6]). Let \( d \geq 1 \) and \( n > 2d \) be integers, and

\[
\ell_{n,d} := \max \left\{ \left( \frac{n-d}{2} \right)^{2} + d^2, \left( \frac{n+1}{2} \right)^{2} + \left\lfloor \frac{n-1}{2} \right\rfloor^2 \right\}.
\]

Then every \( n \)-vertex graph \( G \) with \( \delta(G) \geq d \) and \( e(G) > \ell_{n,d} \) is hamiltonian.

The bound on \( \ell_{n,d} \) is sharp, due to the graphs \( H_{n,n,2} \) and \( H_{n,n,[(n-1)/2]} \). Since \( \delta(G) \geq 2 \) for every 2-connected \( G \), this has the following corollary.

**Theorem 2.2** (Erdős [6]). If \( n \geq 5 \) and \( G \) is an \( n \)-vertex 2-connected non-hamiltonian graph, then \( e(G) \leq \binom{n-2}{2} + 4 \), with equality only for \( G = H_{n,n,2} \).

It is well-known that every graph of minimum degree at least \( d \geq 2 \) contains a cycle of length at least \( d + 1 \). A stronger statement was proved by Dirac for 2-connected graphs:

**Theorem 2.3** (Dirac [4]). If \( G \) is 2-connected then \( c(G) \geq \min\{n, 2\delta\} \).

This theorem was strengthened as follows by Kopylov [11], based on ideas of Pósa [14]:

**Theorem 2.4** (Kopylov [11]). If \( G \) is 2-connected, \( P \) is an \( x, y \)-path of \( \ell \) vertices, then \( c(G) \geq \min\{\ell, d(x,P) + d(y,P)\} \).

**Theorem 2.5** (Chvátal [3]). Let \( n \geq 3 \) and \( G \) be an \( n \)-vertex graph with vertex degrees \( d_1 \leq d_2 \leq \ldots \leq d_n \). If \( G \) is not hamiltonian, then there is some \( i < n/2 \) such that \( d_i \leq i \) and \( d_{n-i} < n - i \).

The \( k \)-closure of a graph \( G \) is the unique smallest graph \( H \) of order \( n := |V(G)| \) such that \( G \subseteq H \) and \( d_H(u) + d_H(v) < k \) for all \( uv \notin E(H) \). The \( k \)-closure of \( G \) is denoted by \( Cl_k(G) \), and can be obtained from \( G \) by a recursive procedure which consists of joining nonadjacent vertices with degree-sum at least \( k \).

**Theorem 2.6** (Bondy and Chvátal [1]). If \( Cl_n(G) \) is hamiltonian, then so is \( G \). Therefore if \( Cl_n(G) = K_n, n \geq 3 \), then \( G \) is hamiltonian.

Concerning long paths between prescribed vertices in a graph, Lovász [13] showed that if \( G \) is a 2-connected graph in which every vertex other than \( u \) and \( v \) has degree at least \( k \), then there is a \( u, v \)-path of length at least \( k + 1 \). This result was strengthened by Enomoto. The following theorem immediately follows from Corollary 1 in [5]:

**Theorem 2.7** (Enomoto [5]). Let \( 5 \leq s \leq n \) and \( \ell := 2(n-3)/(s-4) \). Suppose \( H \) is a 3-connected \( n \)-vertex graph with \( d(x) + d(y) \geq s \) for all non-adjacent distinct \( x, y \in V(H) \). Then for every distinct vertices \( x \) and \( y \) of \( H \), there is an \( x, y \)-path of length at least \( s - 2 \). Moreover, if for some distinct \( x, y \in V(H) \), there is no \( x, y \)-path of length at least \( s - 1 \), then either

\[
\frac{K_{s/2}}{\sqrt{2}} + \frac{K_{n-s/2}}{\sqrt{2}} \leq H \leq K_{s/2} + K_{n-s/2}
\]
or \( \ell \) is an integer and
\[
K_3 + \ell K_{s/2-2} \subseteq H \subseteq K_3 + \ell K_{s/2-2}.
\]

A further strengthening of this result was given by Bondy and Jackson [2]. Finally, we require some results on cycles containing prescribed sets of edges. The following was proved by Pósa [15]:

**Theorem 2.8** (Pósa [15]). Let \( n \geq 3 \), \( k < n \) and let \( G \) be an \( n \)-vertex graph such that
\[
d(u) + d(v) \geq n + k \quad \text{for every non-edge } uv \text{ in } G. \tag{3}
\]
Then for every linear forest \( F \) with \( k \) edges contained in \( G \), the graph \( G \) has a hamiltonian cycle containing all edges of \( F \).

The analog of Pósa’s Theorem for bipartite graphs below is a simple corollary of Theorem 7.3 in [17].

**Theorem 2.9** (Zamani and West [17]). Let \( s \geq 3 \) and \( K \) be a subgraph of the complete bipartite graph \( K_{s,s} \) with partite sets \( A \) and \( B \) such that for every \( x \in A \) and \( y \in B \) with \( xy \notin E(K) \),
\[
d(x) + d(y) \geq s + 1 + i.
\]
Then for every linear forest \( F \subseteq K \) with at most \( 2i \) edges, there is a hamiltonian cycle in \( K \) containing all edges of \( F \).

We will use only the following partial case of Theorem 2.9:

**Corollary 2.10.** Let \( s \geq 4, 1 \leq i \leq 2 \) and \( K \) be a subgraph of \( K_{s,s} \) with at least \( s^2 - s + 2 + i \) edges. If \( F \subseteq K \) is a linear forest with at most \( 2i \) edges and at most two components, then \( K \) has a hamiltonian cycle containing all edges of \( F \).

### 3 Lemmas on contractions

An essential part of the proof of Theorem 1.4 is to analyze contractions of edges in graphs. Specifically, we shall start with a graph \( G \) and contract edges according to some basic rules. Let us mention that the extensive use of contractions to prove the Erdős–Gallai Theorem was introduced by Lewin [12]. In this section, we present some basic structural lemmas on contractions.

**Lemma 3.1.** Let \( n \geq 4 \) and let \( G \) be an \( n \)-vertex 2-connected graph. Let \( v \in V(G) \) and \( W(v) := \{w \in N(v) : N[v] \not\subseteq N[w]\} \). If \( W(v) \neq \emptyset \), then there is \( w \in W(v) \) such that \( G/vw \) is 2-connected.

**Proof.** Let \( w \in W(v), G_w = G/vw \). Recall that \( v \ast w \) is the vertex in \( G_w \) obtained by contracting \( v \) with \( w \). Since \( G \) is 2-connected, \( G_w \) is connected. If \( x \neq v \ast w \) is a cut vertex in \( G_w \), then it is a cut vertex in \( G \), a contradiction. So, the only cut vertex in \( G_w \) can be \( v \ast w \). Thus, if the lemma does not hold, then for every \( w \in W(v) \), \( v \ast w \) is the unique cut vertex in \( G_w \). This means that for every \( w \in W(v) \), \( \{v,w\} \) is a separating set in \( G \).

Choose \( w \in W(v) \) so that to minimize the order of a minimum component in \( G - v - w \). Let \( C \) be the vertex set of such a component in \( G - v - w \) and \( C' = V(G) \setminus (C \cup \{v,w\}) \). Since \( G \) is 2-connected, \( v \) has a neighbor \( u \in C \) and a neighbor \( u' \in C' \). Since \( uu' \notin E(G), u \in W(v) \). But the vertex set of every component of \( G - v - u \) not containing \( w \) is contained in \( C \). This contradicts the choice of \( w \). \( \square \)
This lemma yields the following fact.

**Lemma 3.2.** Let $n \geq 4$ and let $G$ be an $n$-vertex 2-connected graph. For every $v \in V(G)$, there exists $w \in N(v)$ such that $G/vw$ is 2-connected.

**Proof.** If $W(v) \neq \emptyset$, this follows from Lemma 3.1. Suppose $W(v) = \emptyset$. This means $G[N(v)]$ is a clique. Then contracting any edge incident with $v$ is equivalent to deleting $v$. Let $G' = G - v$. Since $d(v) \geq 2$ and $G[N(v)]$ is a clique, any cut vertex in $G'$ is also a cut vertex in $G$. \hfill \Box

For an edge $xy$ in a graph $H$, let $T_H(xy)$ denote the number of triangles containing $xy$. Let $T(H) := \min\{T_H(xy) : xy \in E(H)\}$. When we contract an edge $uv$ in a graph $H$, the degree of every $x \in V(H) \setminus \{u, v\}$ either does not change or decreases by 1. Also the degree of $u + v$ in $H/uv$ is at least $\max\{d_H(u), d_H(v)\} - 1$. Thus

$$\delta(H/uv) \geq \delta(H) - 1 \text{ for every graph } H \text{ and } uv \in E(H).$$

Similarly,

$$T(H/uv) \geq T(H) - 1 \text{ for every graph } H \text{ and } uv \in E(H).$$

Suppose we contract edges of a 2-connected graph one at a step, choosing always an edge $xy$ so that

(i) the new graph is 2-connected and,

(ii) $xy$ is in the fewest triangles;

(iii) the contracted edge $xy$ is incident to a vertex of degree as small as possible up to (ii).

**Lemma 3.3.** Let $h$ be a positive integer. Suppose a 2-connected graph $G$ is obtained from a 2-connected graph $G'$ by contracting edge $xy$ into $x*y$ using the above rules (i)–(iii). If $G$ has at least $h$ vertices of degree at most $h$, then either $G' = K_{h+2}$ or $G'$ also has a vertex of degree at most $h$.

**Proof.** Since $G$ is 2-connected, $h \geq 2$. If $G$ has a vertex of degree less than $h$, the lemma holds by (1). So, let $A_h$ denote the set of vertices of degree exactly $h$ in $G$, and assume $|A_h| \geq h$. Let $A'_h = A_h \setminus \{x*y\}$. Suppose the lemma does not hold. Then we have

each $v \in A'_h$ has degree $h + 1$ in $G'$ and is adjacent to both, $x$ and $y$ in $G'$. (6)

**Case 1:** $|A'_h| \geq h$. Then by (6), $xy$ belongs to at least $h$ triangles in which the third vertex is in $A_h$. So by (ii) and the symmetry between $x$ and $y$, we may assume $d_{G'}(x) = h + 1$. This in turn yields $N_{G'}(x) = A_h \cup \{y\}$. Since $G'$ is 2-connected each $v \in A'_h$ is not a cut vertex. Even more, $xv$ is not a cut edge. Indeed, $y$ is a common neighbor of all neighbors of $x$ so all neighbors of $x$ must be in the same component as $y$ in $G' - x - v$. It follows that

for every $v \in A'_h$, $G'/vx$ is 2-connected. (7)

If $uv \notin E(G)$ for some $u, v \in A_h$, then by (6) and (ii), we would contract the edge $xu$ and not $xy$. Thus $G'[A'_h \cup \{x, y\}] = K_{h+2}$ and so either $G' = K_{h+2}$ or $y$ is a cut vertex in $G'$, as claimed.

**Case 2:** $|A'_h| = h - 1$. Then $x*y \in A_h$. We obtain that $d_{G'}(x) = d_{G'}(y) = h + 1$ and $N_{G'}[x] = N_{G'}[y]$. So by (2), there is $z \in V(G)$ such that $N_{G'}[x] = N_{G'}[y] = A'_h \cup \{x, y, z\}$. Again (7) holds (for the
same reason that \( N_{G'}[x] \subseteq N_{G'}[y] \). Thus similarly \( vu \in E(G') \) for every \( v \in A'_h \) and every \( u \in A'_h \cup \{z\} \). Hence \( G'[A'_h \cup \{x, y, z\}] = K_{h+2} \) and either \( G' = K_{h+2} \) or \( z \) is a cut vertex in \( G' \), as claimed. \( \square \)

**Lemma 3.4.** Suppose that \( G \) is a 2-connected graph and \( C \) is a longest cycle in it. Then no two consecutive vertices of \( C \) form a separating set.

**Proof.** Indeed, if for some \( i \) the set \( \{v_i, v_{i+1}\} \) is separating, then let \( H_1 \) and \( H_2 \) be two components of \( G - \{v_i, v_{i+1}\} \) such that \( V(C) \cap V(H_1) \neq \emptyset \). Then \( V(C) \setminus \{v_i, v_{i+1}\} \subseteq V(H_1) \). Let \( x \in V(H_2) \). Since \( G \) is 2-connected, it contains two paths from \( x \) to \( \{v_i, v_{i+1}\} \) that share only \( x \). Since \( \{v_i, v_{i+1}\} \) separates \( V(H_2) \) from the rest, these paths are fully contained in \( V(H_2) \cup \{v_i, v_{i+1}\} \). So adding these paths to \( C - v_iv_{i+1} \) creates a cycle longer than \( C \), a contradiction. \( \square \)

### 4 Proof of the main result, Theorem 1.4, for \( k \geq 9 \)

In this section, we give a precise description of the extremal graphs for Theorem 1.4 for \( k \geq 9 \). The description for \( k \leq 8 \) is postponed to Section 5. For Theorem 1.4(a), when \( k = 2t + 1 \) and \( t \neq 3 \), these are simply subgraphs of the graphs \( H_{n,k,t} \); recall that \( H_{n,k,a} \) has a partition into three sets \( A, B, C \) such that \( |A| = a, |B| = n - k + a \) and \( |C| = k - 2a \) and the edge set of \( H_{n,k,a} \) consists of all edges between \( A \) and \( B \) together with all edges in \( A \cup C \). For Theorem 1.4(b), when \( k = 2t + 2 \) or \( k = 7 \), the extremal graphs \( G \) contain a set \( A \) of size at most \( t \) such that \( G - A \) is a star forest. In this case a more detailed description is required.

**Classes \( \mathcal{G}_i(n,k) \) for \( i \leq 3 \).** Let \( \mathcal{G}_1(n,k) := \{H_{n,k,t}\} \). Each \( G \in \mathcal{G}_2(n,k) \) is defined by a partition \( V(G) = A \cup B \cup J \), \( |A| = t \) and a pair \( a_1 \in A, b_1 \in B \) such that \( G[A] = K_t \), \( G[B] \) is the empty graph, \( G(A,B) \) is a complete bipartite graph and for every \( c \in J \) one has \( N(c) = \{a_1, b_1\} \). Every member of \( G \in \mathcal{G}_3(n,k) \) is defined by a partition \( V(G) = A \cup B \cup J \), \( |A| = t \) such that \( G[A] = K_t \), \( G(A,B) \) is a complete bipartite graph, and

- \( G[J] \) has more than one component
- all components of \( G[J] \) are stars with at least two vertices each
- there is a 2-element subset \( A' \) of \( A \) such that \( N(J) \cap (A \cup B) = A' \)
- for every component \( S \) of \( G[J] \) with at least 3 vertices, all leaves of \( S \) are adjacent to the same vertex \( a(S) \) in \( A' \).

The class \( \mathcal{G}_4(n,k) \) is empty unless \( k = 10 \). Each member of \( \mathcal{G}_4(n,10) \) has a 3-vertex set \( A \) such that \( G[A] = K_3 \) and \( G - A \) is a star forest such that if a component \( S \) of \( G - A \) has more than two vertices then all its leaves are adjacent to the same vertex \( a(S) \) in \( A \). These classes are illustrated below:
Statement of main theorem. Having defined the classes $G_i(n,k)$ for $i \leq 4$, we now state a theorem which implies Theorem 1.4 for $k \geq 9$ and shows that the extremal graphs are the graphs in the classes $G_i(n,k)$:

**Theorem 4.1.** (Main Theorem) Let $k \geq 9$, $n \geq \frac{3k}{2}$ and $t = \lfloor \frac{k-1}{2} \rfloor$. Let $G$ be an $n$-vertex 2-connected graph with no cycle of length at least $k$. Then $e(G) \leq h(n,k,t-1)$ or $G$ is a subgraph of a graph in $G(n,k)$, where

1. if $k$ is odd, then $G(n,k) := G_1(n,k) = \{H_{n,k,t}\}$;
2. if $k$ is even and $k \neq 10$, then $G(n,k) := G_1(n,k) \cup G_2(n,k) \cup G_3(n,k)$;
3. if $k = 10$, then $G(n,k) := G_1(n,10) \cup G_2(n,10) \cup G_3(n,10) \cup G_4(n,10)$.

We prove this theorem in this section. We also observe that if $k \geq 11$, then the only graph in the classes $G_i(n,k)$ that is 3-connected is $H_{n,k,t}$. Therefore Theorem 4.1 implies Corollary 1.5.

The idea of the proof is to take a graph $G$ satisfying the conditions of the theorem with $c(G) < k$, and to contract edges while preserving the average degree and 2-connectivity of $G$. A key fact is that if a graph contains a cycle of length at least $k$ and is obtained from another graph by contracting edges, then that other graph also contains a cycle of length at least $k$. The process terminates with an $m$-vertex graph $G_m$ such that $G_m$ is 2-connected, $m \geq k$, and if $m > k$ then
$G_m$ has minimum degree at least $t - 1$. If $m > k$, then we apply Theorem 2.7 to show that $G_m$ is a dense subgraph of $H_{m,k,t}$. If $m = k$, then we apply Theorem 2.1 and 2.6 to show that $G_m$ is a dense subgraph of $H_{k,k,t}$. Using this, we show that $G_m$ contains a dense nice subgraph. Analyzing contractions, we then show that $G$ itself contains a dense nice subgraph. Finally, we show that every dense $n$-vertex graph containing a dense nice subgraph but not containing a cycle of length at least $k$ must be a subgraph of a graph in one of the classes described in Theorem 1.1.

### 4.1 Basic Procedure

Let $k, n$ be positive integers with $n \geq k$. Let $G$ be an $n$-vertex 2-connected graph with $e(G) < k$ and $e(G) \geq h(n, k, t - 1) + 1$. We denote $G$ as $G_n$ and run the following procedure.

**Basic Procedure.** At the beginning of each round, for some $j : k \leq j \leq n$, we have a $j$-vertex 2-connected graph $G_j$ with $e(G_j) \geq h(j, k, t - 1) + 1$.

1. If $j = k$, then we stop.
2. If there is an edge $xy$ with $T_{G_j}(xy) \leq t - 2$ such that $G_j/xy$ is 2-connected, choose one such edge so that
   - $T_{G_j}(xy)$ is minimum, and subject to this
   - $xy$ is incident to a vertex of minimum possible degree.
   Then obtain $G_{j-1}$ by contracting $xy$.
3. If (R2) does not hold, $j \geq k + t - 1$ and there is $uv \in E(G_j)$ such that
   - $G_j - u - v$ has at least 3 components and one of the components, say $H_1$ is a $K_{t-1}$, then let $G_{j-t+1} = G_j - V(H_1)$.
4. If neither (R2) nor (R3) occurs, then we stop.

**Remark 1.** By construction, every obtained $G_j$ is 2-connected and has $e(G_j) < k$. Let us check that

$$e(G_j) \geq h(j, k, t - 1) + 1$$

for all $m \leq j \leq n$. For $j = n$, (8) holds by assumption. Suppose $j > m$ and (8) holds. If we apply (R2) to $G_j$, then the number of edges decreases by at most $t - 1$, and $(h(j, k, t - 1) + 1) - (h(j - 1, k, t - 1) + 1) = t - 1$. If we apply (R3) to $G_j$, then the number of edges decreases by at most $\binom{t+1}{2} - 1$, and $(h(j, k, t - 1) + 1) - (h(j - (t - 1)), k, t - 1) + 1) = (t - 1)^2$. But for $k \geq 9$, $(t - 1)^2 \geq \binom{t+1}{2} - 1$. Thus every step of the basic procedure preserves (8).

Let $G_m$ denote the graph with which the procedure terminates.

**Remark 2.** Note that if the rule (R3) applies for some $G_j$, then the set $\{u, v\}$ is still separating in $G_{j-t+1}$, and $T_{G_{j-t+1}}(xy) \geq t - 1$ for every edge $xy$ such that $G_{j-t+1}/xy$ is 2-connected. In particular, $\delta(G_{j-t+1}) \geq t$. So after any application of (R3), rule (R2) does not apply, and $\delta(G_m) \geq t$.

### 4.2 The structure of $G_m$

In the next two subsections, we prove Proposition 4.2 below, considering the cases $m = k$ and $m > k$ separately. Let $F_4$ be the graph obtained from $K_{3,6}$ by adding three independent edges in
the part of size six. In this section we usually suppose that \( n \geq 3t, t \geq 4 \), although many steps work for smaller values as well.

**Proposition 4.2.** The graph \( G_m \) satisfies the following properties:

\[
\begin{align*}
(1) & \quad G_m \subseteq H_{m,k,t} \text{ or} \\
(2) & \quad m > k = 10 \text{ and } G_m \supseteq F_4.
\end{align*}
\]

### 4.2.1 The case \( m = k \)

If \( G_k \) is hamiltonian, then \( c(G) \geq k \), a contradiction. So \( G_k \) is not hamiltonian.

By Theorem 2.5 for every non-hamiltonian \( n \)-vertex graph \( G \) with vertex degrees \( d_1 \leq d_2 \leq \ldots \leq d_n \), we define

\[
r(G) := \min \{ i : d_i \leq i \text{ and } d_{n-i} < n - i \}.
\]

**Lemma 4.3.** Let \( t \geq 4, n \geq 3t \). If the vertex degrees of \( G_k \) are \( d_1 \leq d_2 \leq \ldots \leq d_k \), then \( r(G_k) = t \).

**Proof for** \( k = 2t + 2 \). Note that \( r(G_k) \leq t \) since \( r(G) < n/2 \) (see Theorem 2.5). Suppose \( r := r(G_k) \leq t - 1 \). Then by Remark 2, Rule (R3) never applied, and \( G_k \) was obtained from \( G \) by a sequence of \( n - m \) edge contractions according (R2). We may assume that for all \( m \leq j < n \), graph \( G_j \) was obtained from \( G_{j+1} \) by contracting edge \( x_jy_j \). Then conditions for (R2) imply

\[
T_{G_j}(x_{j-1}y_{j-1}) \leq t - 2 \quad \text{for every} \quad m + 1 \leq j \leq n.
\]

By Lemma 3.3 \( \delta(G_{m+1}) \leq r \). This together with (9) and (3) yield that for every \( m < j \leq n \),

\[
\delta(G_j) \leq r + j - m - 1 \quad \text{and so} \quad T_{G_j}(x_{j-1}y_{j-1}) \leq \min \{ r + j - m - 2, t - 2 \}. \tag{10}
\]

Contracting edge \( x_{j-1}y_{j-1} \) in \( G_j \), we lose \( T_{G_j}(x_{j-1}y_{j-1}) + 1 \) edges. Since \( e(G) \geq h(n,k,t-1) + 1 \), by (5) we obtain,

\[
e(G_k) \geq h(n,k,t-1) + 1 - \sum_{j=m+1}^{n} \min\{t-1, r+j-m-1\} \tag{11}
\]

\[
= \left(\frac{t+3}{2}\right) + (t-1)(n-t-3) + 1 - \sum_{j=m+1}^{n} \min\{t-1, r+j-m-1\}
\]

\[
= \left(\frac{t+3}{2}\right) + (t-1)(n-t-3) + 1 - (t-1)(n-m) + \sum_{j=m+1}^{n} \max\{0, m+t-r-j\}
\]

\[
= \frac{3t^2 + t + 10}{2} + \sum_{j=m+1}^{n} \max\{0, 3t + 2 - r - j\}.
\]

Since \( n \geq 3t \), \( \{ \max\{0, 3t + 2 - r - j\} : m + 1 \leq j \leq n \} = \{ 0, 1, 2, \ldots, t - 1 - r \} \). Therefore

\[
e(G_k) \geq \frac{3t^2 + t + 10}{2} + \sum_{i=1}^{t-1-r} i = \frac{3t^2 + t + 10}{2} + \left( \frac{t-r}{2} \right). \tag{12}
\]
On the other hand, by the definition of $r$, $G_m$ has at most $r^2$ edges incident with the $r$ vertices of the smallest degrees and at most $\binom{m-r}{2}$ other edges. Thus $e(G_m) \leq r^2 + \binom{2t+2-r}{2}$. Hence

$$\frac{3t^2 + t + 10}{2} + \binom{t - r}{2} \leq r^2 + \binom{2t + 2 - r}{2}. \tag{13}$$

Expanding the binomial terms in (13) and regrouping we get

$$t(r - 3) \leq r^2 - 2r - 4. \tag{14}$$

If $r = 3$, then the left hand side of (14) is 0 and the right hand side is $-1$, a contradiction. If $r \geq 4$, then dividing both sides of (14) by $r - 3$ we get $t \leq r + 1 - 1/(r - 3)$, which yields $r \geq t$, as claimed.

So suppose $r = 2$ and let $v_1, v_2$ be two vertices of degree 2 in $G_k$. Then by (12), the graph $H = G_k - v_1 - v_2$ has at least

$$\frac{3t^2 + t + 10}{2} + \binom{t - 2}{2} - 2(2) = 2t^2 - 2t + 4$$

edges. So the complement of $H$ has at most $t - 4$ edges and thus, for $u, w \in V(H)$:

$$d_H(u) + d_H(w) \geq 2(2t - 1) - (t - 4) - 1 = 3t + 1 = |V(H)| + t + 1.$$ 

Hence by Theorem 2.8

for each linear forest $F \subseteq H$ with $e(F) \leq t + 1$, $H$ has a spanning cycle containing $E(F).$ \tag{15}

If $N(v_i) = \{u_i, w_i\}$ for $i = 1, 2$ and $v_1v_2 \in E(G_k)$, say $u_1 = v_2$ and $u_2 = v_1$, then $w_1 \neq w_2$ since $H$ is 2-connected. Thus by (15), graph $H' = H + w_1w_2$ has a spanning cycle containing $w_1w_2$, and this cycle yields a hamiltonian cycle in $G_k$, a contradiction. So $v_1v_2 \notin E(G_k)$. Similarly, if $N(v_1) \neq N(v_2)$, then by (15), graph $H'' = H + u_1w_1 + u_2w_2$ has a spanning cycle containing $u_1w_1$ and $u_2w_2$. Again this yields a hamiltonian cycle in $G_k$. Thus we may assume $N(v_1) = N(v_2) = \{u, w\}$. Let

$$H_0 = H + uw \text{ if } uw \notin E(G) \text{ and } H_0 = H \text{ otherwise.} \tag{16}$$

If $x_m \ast y_m \notin N[v_1] \cup N[v_2]$, then $T_{G_{m+1}}(x_my_m) \leq 1$ (since $T_{G_{m+1}}(v_1u_1) \leq 1$) and $G_{m+1}$ contains vertices $v_1$ and $v_2$ of degree 2. So by Lemma 3.3 for $h = 2$, $G_{m+2}$ also has a vertex of degree 2. Thus by (4) for $r = 2$ instead of (10) we have for every $m + 2 \leq j \leq n$,

$$\delta(G_j) \leq \min\{j - m, t - 1\} \text{ and so } T_{G_j}(x_{j-1}y_{j-1}) \leq \min\{j - m - 1, t - 2\}. \tag{17}$$

Plugging (17) instead of (10) into (11) for $r = 2$, we will instead of (13) get the stronger inequality

$$\frac{3t^2 + t + 10}{2} + (t - 3) + \binom{t - 2}{2} \leq 2^2 + \binom{2t + 2 - 2}{2}. \tag{18}$$

Thus instead of (14) we have for $r = 2$ the stronger inequality $t(2 - 3) + (t - 3) \leq 2^2 - 4 - 4$, which does not hold. This contradiction implies $x_m \ast y_m \in N[v_1] \cup N[v_2]$. By symmetry we have two cases.
Case 1: \( x_m \ast y_m = v_1 \). As above, graph \( H_0 \) has a spanning cycle \( C \) containing \( uv \). If

\[
x_m u, y_m w \in E(G_{m+1}),
\]

then \( C \) extends to a \( k \)-cycle in \( G_{m+1} \) by replacing \( uw \) with path \( u, x_m, y_m, w \). A similar situation holds if

\[
x_m w, y_m u \in E(G_{m+1}).
\]

But by degree conditions each of \( x_m, y_m \) has a neighbor in \( \{ u, w \} \). By definition, each of \( u, w \) has a neighbor in \( \{ x_m, y_m \} \). So at least one of (19) and (20) holds.

Case 2: \( x_m \ast y_m = u \). If \( d_{G_{m+1}}(v_1) = d_{G_{m+1}}(v_2) = 2 \), then as before we get (18) instead of (14) and get a contradiction. So by symmetry we may assume that \( x_m \) is adjacent to both \( x_m \) and \( y_m \) in \( G_{m+1} \). Since \( G_m \) is 2-connected, vertex \( w \) does not separate \( \{ v_1, v_2, u \} \) from the rest of the graph. Thus by symmetry we may assume that \( y_m \) has a neighbor \( z \in V(G_{m+1}) \setminus \{ x_m, v_1, v_2, w \} \). Again by (15), graph \( H_0 \) defined by (16) has a spanning cycle containing edges \( uw \) and \( uz \), and again this cycle yields a \( k \)-cycle in \( G_{m+1} \) (using path \( w, v_1, x_m, y_m, z \)), a contradiction.

Proof for \( k = 2t + 1 \). We repeat the argument for \( k = 2t + 2 \), but instead of (12) and (13), we get

\[
\frac{3t^2 - t + 6}{2} + \binom{t - r}{2} \leq e(G_k) \leq r^2 + \binom{2t + 1 - r}{2}.
\]

Expanding the binomial terms and regrouping, similarly to (14), we get

\[
t(r - 2) \leq r^2 - r - 3.
\]

The analysis of this inequality is simpler than that of (14): If \( r = 2 \), then the left hand side is 0 and the right hand side is \(-1\), while if \( r \geq 3 \), then dividing both sides by \( r - 2 \) we get \( t \leq r + 1 - 1/(r - 2) \), which yields \( r \geq t \), as claimed.

\[
\Box
\]

Lemma 4.4. Under the conditions of Lemma 4.3, \( G_k \) is a subgraph of the graph \( H_{k,k,t} \).

Proof for \( k = 2t + 2 \). By Lemma 4.3, \( r(G_k) = t \). Let \( G' \) be the \( k \)-closure of \( G_k \) and \( d_1' \leq d_2' \leq \ldots \leq d_k' \) be the vertex degrees in \( G' \). By the definition of the \( k \)-closure,

\[
d(u) + d(v) \leq k - 1 \quad \text{for every non-edge } uv \text{ in } G'.
\]

Since \( d_i' \geq d_i \) for every \( i \) and \( G' \) is also non-hamiltonian, \( r(G') \geq r(G_k) = t \). Since \( r(G') \leq t \) from \( r(G) < n/2 \), \( r(G') = t \). Let \( V(G') = \{ v_1, \ldots, v_k \} \) where \( d_{G'}(v_i) = d_i' \) for all \( i \). By the definition of \( r(G') \), on the one hand \( d_i' \leq t \) and \( d_{k-t} = k - t - 1 = t + 1 \), on the other hand either \( d_{t-1}' > t - 1 \) or \( d_{k-(t-1)}' \geq k - (t - 1) = t + 3 \). In any case, \( d_{t+3}' \geq t \). Summarizing,

\[
d_{t+3}' \geq t, \quad d_i' \leq t \quad \text{and} \quad d_{t+1}' \leq d_{t+2}' \leq t + 1.
\]

Let \( B = \{ v_1, \ldots, v_{t+2} \} \) and \( A = V(G') \setminus B \). If \( d_{t+4}' \leq t + 2 \), then

\[
\sum_{i=1}^{k} d_i' \leq (t|B| + 2) + (t + 2)2 + (2t + 1)(t - 2) = 3t^2 + t + 4.
\]
a contradiction to \( e(G_k) \geq h(k, k, t - 1) + 1 \). Thus \( d'_{t+4} \geq t + 3 \), and by (21) and (22), \( G'[A] = K_t \).

In summary,

\[
d'_{t+4} \geq t + 3 \quad \text{and} \quad G'[A] = K_t.
\]

(23)

Suppose that there are distinct \( v_{i_1}, v_{i_2} \in B \) and distinct \( v_{j_1}, v_{j_2} \in A \) such that \( v_{i_1}v_{j_1} \) and \( v_{i_2}v_{j_2} \) are non-edges in \( G' \). Then by (21) and (22),

\[
\sum_{i=1}^{2t+2} d'_i \leq (2t + 1)2 + t(|B| - 2) + 2 + (2t + 1)(|A| - 2)
= 4t + 2 + t^2 + 2t^2 - 3t - 2 = 3t^2 + t + 2.
\]

This contradicts \( e(G_k) > h(k, k, t - 1) \). So, some \( v_j \) is incident with all non-edges of \( G' \) connecting \( A \) with \( B \).

**Case 1:** \( j \leq t + 2 \), i.e. \( v_j \in B \). Then each \( v \in B - v_j \) has \( t \) neighbors in \( A \). Thus each \( v \in B \setminus \{v_j, v_{t+1}, v_{t+2}\} \) has no neighbors in \( B \), and each of \( v_{t+1}, v_{t+2} \) has at most one neighbor in \( B \). If each of \( v_{t+1}, v_{t+2} \) is adjacent to \( v_j \), then \( G' \) has a hamiltonian cycle using edges \( v_{t+1}v_j \) and \( v_{t+2}v_j \). Otherwise \( G'[B] \) has at most one edge, as claimed.

**Case 2:** \( j \geq t + 3 \), i.e. \( v_j \in A \). Together with (23), this yields that \( G' \) contains \( K_{t-1, t+3} \) with partite sets \( A \setminus \{v_j\} \) and \( B \cup \{v_j\} \). In particular, all pairs of vertices in \( A \setminus \{v_j\} \) are adjacent. So, \( G' \) is obtained from \( K_{2t+2} - E(K_{t+3}) \) by adding at least \( e(G') - \left(\binom{2t+2}{2} + \binom{t+3}{2}\right) \geq 7 \) edges. If \( G'[B \cup \{v_j\}] \) contains a linear forest with four edges, then \( G' \) has a hamiltonian cycle. So suppose

\[
G'[B \cup \{v_j\}] \text{ contains no linear forests with four edges,}
\]

(24)

**Case 2.1:** \( G'[B \cup \{v_j\}] \) contains a cycle \( C \). By (24), \( |C| \leq 4 \) and if \( |C| = 4 \), then each other edge in \( G'[B \cup \{v_j\}] \) has both ends in \( V(C) \). Thus \( G'[B \cup \{v_j\}] \) has at most 6 edges, a contradiction. So suppose \( C = (x, y, z) \). If no other edge is incident with \( V(C) \), then the set of the remaining at least four edges in \( G'[B \cup \{v_j\}] \) contains a linear forest with two edges, a contradiction to (24). Thus we may assume that \( G'[B \cup \{v_j\}] \) has an edge \( xu \) where \( u \notin \{y, z\} \). Then by (24) and the fact that \( G'[B \cup \{v_j\}] \) contains no 4-cycles, none of \( u, y, z \) is incident with other edges. On the other hand, if \( G'[B \cup \{v_j\}] \) has an edge not incident with \( V(C) \), this would contradict (24). Hence \( G'[B \cup \{v_j\}] \setminus \{x\} \) has only the edge \( yz \), as claimed.

**Case 2.2:** \( G'[B \cup \{v_j\}] \) is a forest. By (24), there is \( x \in B \cup \{v_j\} \) of degree at least 3 in \( G'[B \cup \{v_j\}] \). If there is another vertex \( y \) of degree at least 3 in \( G'[B \cup \{v_j\}] \), then we can choose two edges incident with \( x \) and two edges incident with \( y \) that together form a linear forest with four edges. So \( G'[B \cup \{v_j\}] \setminus \{x\} \) is a linear forest, call it \( F \), and thus has at most 3 edges. Each edge of \( F \) has at most one end adjacent to \( x \), and the degree of \( x \) in \( G'[B \cup \{v_j\}] \) is at least four. So if \( F \) has exactly \( m \in \{2, 3\} \) edges, then we can choose \( 4 - m \) edges incident with \( x \) so that together with \( F \) they form a linear forest. And if \( F \) has at most one edge, then the lemma holds.

**Proof for** \( k = 2t + 1 \). The proof is almost identical to the case \( k = 2t + 2 \). By Lemma 1.3, \( r(G_k) = t \). Let \( G' \) be the \( k \)-closure of \( G_k \) and \( d'_1 \leq d'_2 \leq \ldots \leq d'_k \) be the vertex degrees in \( G' \). As
in [21], we have
\[ d(u) + d(v) \leq k - 1 = 2t \quad \text{for every non-edge } uv \text{ in } G'. \] (25)

As in the proof in the case \( k = 2t + 2, r(G') = t \). Let \( V(G') = \{v_1, \ldots, v_k\} \) where \( d_{G'}(v_i) = d_i' \) for all \( i \). Instead of [22], we get the stronger claim
\[ d_{i+2}' \geq t \text{ and } d_i' \leq d_{i+1}' = t. \] (26)

Let \( B = \{v_1, \ldots, v_{t+1}\} \) and \( A = V(G') \setminus B \). If \( d_{t+3}' \leq t + 1 \), then
\[ \sum_{i=1}^{2t+1} d_i' \leq t|B| + (t + 1)2 + (2t)(t - 2) = 3t^2 - t + 2 \leq h(k, k, t - 1), \]
a contradiction. Thus,
\[ d_{t+3}' \geq t + 2 \quad \text{so by (25) and (26), } G'[A] = K_t. \] (27)

If there are distinct \( v_{i_1}, v_{i_2} \in B \) and distinct \( v_{j_1}, v_{j_2} \in A \) such that \( v_{i_1}v_{j_1} \) and \( v_{i_2}v_{j_2} \) are non-edges in \( G' \), then by [25] and [26],
\[ \sum_{i=1}^{k} d_i' \leq (2t)2 + t(|B| - 2) + (2t)(|A| - 2) = 4t + t^2 - t + 2t^2 - 4t = 3t^2 - t \leq h(k, k, t - 1), \]
a contradiction. So, some \( v_j \) is incident with all non-edges of \( G' \) connecting \( A \) with \( B \).

**Case 1:** \( j \leq t + 1 \), i.e. \( v_j \in B \). Then each \( v \in B - v_j \) has \( t \) neighbors in \( A \). Thus by [26], each \( v \in B - v_j \) has no neighbors in \( B \), hence \( B \) is independent, as claimed.

**Case 2:** \( j \geq t + 2 \), i.e. \( v_j \in A \). Together with [27], this yields that \( G' - v_j \) contains \( K_{t-1,t+2} \) with partite sets \( A \setminus \{v_j\} \) and \( B \cup \{v_j\} \). In particular, each vertex in \( A \setminus \{v_j\} \) is all-adjacent. So, \( G' \) is obtained from \( K_k - E(K_{t+2}) \) by adding at least four edges. If \( G'[B \cup \{v_j\}] \) contains a linear forest with three edges, then \( G' \) has a hamiltonian cycle. Every graph with at least four edges not containing a linear forest with three edges is a star plus isolated vertices. And if \( G'[B \cup \{v_j\}] \) is a star plus isolated vertices, then \( G' \subseteq H_{k,k,t} \). \hfill \Box

### 4.2.2 The case \( m > k \).

**Lemma 4.5.** Let \( m > k \geq 9 \).

1. If \( k \neq 10 \), then \( G_m \subseteq H_{m,k,t} \).
2. If \( k = 10 \) then \( G_m \subseteq H_{m,k,t} \) or \( G_m \supseteq F_4 \).

**Proof** for \( k = 2t + 2 \). \( G_m \) is an \( m \)-vertex 2-connected graph with \( e(G_m) \leq 2t + 1 \) satisfying \( e(G) \geq h(n, k, t - 1) + 1 \). Since (R2) is not applicable,
\[ T_{G_m}(xy) \geq t - 1 \quad \text{for every non-separating edge } xy. \] (28)
By Lemmas 3.2 and 3.1 (28) implies
\[ \delta(G_m) \geq t \quad \text{and for each } v \in V(G_m) \text{ with } d(v) = t, G_m[N(v)] = K_{t+1}. \] (29)

Let \( C = (v_1, \ldots, v_q) \) be a longest cycle in \( G_m \). Since \( \delta(G_m) \geq t \), Dirac’s Theorem (Theorem 2.3) yields \( q \geq 2t \). Obviously, \( q \leq 2t + 1 \).

By (28) and Lemma 3.4 each edge of \( C \) is in at least \( t-1 \) triangles. By the maximality of \( C \), the third vertex of each such triangle is in \( V(C) \). So
\[ \text{the minimum degree of } G_m[V(C)] \text{ is at least } t. \] (30)

We now prove that
\[ G_m[V(C)] \text{ is } 3\text{-connected}. \] (31)

Indeed, assume (31) fails and \( G_m[V(C)] \) has a separating set \( S \) of size 2. By symmetry, we may assume that \( S = \{v_1, v_j\} \) and that \( j \leq \lfloor q/2 \rfloor + 1 \leq t + 1 \). Then by (30), \( j = t + 1 \) and \( G_m[\{v_1, \ldots, v_{t+1}\}] = K_{t+1} \). In particular,
\[ v_1v_{t+1} \in E(G_m). \] (32)

Let \( H_1 = G_m[\{v_1, \ldots, v_{t+1}\}] \) and \( H_2 = G_m[\{v_{t+1}, \ldots, v_q, v_1\}] \). Similarly to \( H_1 \), graph \( H_2 \) is either \( K_{t+1} \) (when \( q = 2t \)) or is obtained from \( K_{t+2} \) by deleting some matching (when \( q = 2t + 1 \)).

Concerning almost complete graphs we need the following statement which is an easy consequence of Theorem 2.8 (or one can prove it directly).
\[ \text{For } p \geq 6 \text{ and for any matching } M \subseteq K_p, \text{ every two edges of } K_p - M \text{ are in a common hamiltonian cycle of } K_p - M. \] (33)

Since \( G_m \) is 2-connected, each component \( F \) of \( G_m - V(C) \) has at least two neighbors, say \( y(F) \) and \( y'(F) \), in \( C \). If at least one of them, say \( y'(F) \), is not in \( S = \{v_1, v_{t+1}\} \), then we can construct a cycle longer than \( C \) as follows.

If \( y(F) \in V(H_1) \setminus \{v_1, v_{t+1}\} \) and \( y'(F) \in V(H_2) \setminus \{v_1, v_{t+1}\} \), then \( H_1 - v_{t+1} \) has a hamiltonian \( v_1, y(F) \)-path \( P_1 \) (recall that \( H_1 - v_{t+1} \) is a complete graph), and \( H_2 \) has a hamiltonian \( v_1, y'(F) \)-path \( P_2 \), by (33) and since \( k \geq 4 \). So \( P_1 \cup P_2 \) and a \( y(F), y'(F) \)-path through \( F \) form a longer than \( C \) cycle in \( G_m \).

If both, \( y(F) \) and \( y'(F) \) are in the same \( H_j \), then we let \( H_j' \) be the graph obtained from \( H_j \) by adding the edge \( y(F)y'(F) \). Recall that by (32), \( v_1v_{t+1} \in E(H_j) \). If we have a hamiltonian cycle \( C' \) in \( H_j' \) containing \( y(F)y'(F) \) and \( v_1v_{t+1} \), then let \( P \) be the \( v_1, v_{t+1} \)-path obtained from \( C' \) by deleting edge \( v_1v_{t+1} \) and replacing edge \( y(F)y'(F) \) with a \( y(F), y'(F) \)-path \( P' \) through \( F \), and then replace in \( C \) the \( v_1, v_{t+1} \)-path through \( V(H_j) \) with the longer path \( P \). There is such a \( C' \) if \( |V(H_j)| \geq 6 \) by (33), and also if \( |V(H_j)| = 5 \) because in the latter case \( |V(H_j)| = t + 1 \) with \( t = 4 \) and it is a complete graph.

Thus every component \( F \) of \( G_m - V(C) \) is adjacent only to \( S \), and \( S \) is a separating set in \( G_m \). In particular, \( H_1 - S = K_{t-1} \) and \( H_2 - S \) are components of \( G_m - S \). So, if \( m \geq 3t + 1 \), then Rule
Similarly, if there are two fully disjoint paths, say $P$ and $Q$, in $G$.

Case 1: Suppose that $n > m$, then $q = 2t$, $H_2 = K_{t+1}$ and $H_3 := G_m - (V(C) - S) = K_{t+1}$. Hence

$$e(G_m) - h(m, k, t - 1) - 1 = 3\left(\frac{t+1}{2}\right) - 2 - h(3t - 1, k, t - 1) - 1$$

$$= \frac{3t^2 + 3t - 4}{2} - \frac{5t^2 - 7t + 16}{2} = -t^2 + 5t - 10 < 0.$$

Similarly, if $m = 3t$, then the component sizes of $G_m - S$ are $t, t - 1, t - 1$. Thus in this case

$$e(G_m) - h(m, k, t - 1) - 1 \leq t^2 + t + \left(\frac{t+2}{2}\right) - 2 - h(3t, k, t - 1) - 1$$

$$= \frac{3t^2 + 5t}{2} - 1 - \frac{5t^2 - 5t + 14}{2} = -t^2 + 5t - 8 < 0.$$

These contradictions prove (31).

By (31) and Theorem 2.7 for $n = q, s = 2t$ and $H = G_m[V(C)]$, one of three cases below holds:

**Case 1:** $K_t + K_{q-t} \subseteq G_m[V(C)] \subseteq K_t + K_{q-t}$. Let $B$ be the independent set of size $q-t$ in $G_m[V(C)]$ and $A = V(C) \setminus B$. In this case, since $G_m[V(C)]$ has hamiltonian cycle $C$ and an independent set $B$ of size $q-t$, we need $q = 2t$.

Suppose that $G_m - V(C)$ has a component $D$ with at least two vertices. By Menger’s Theorem, there are two fully disjoint paths, say $P_1$ and $P_2$, connecting some two distinct vertices, say $u$ and $v$, of $D$ with two distinct vertices, say $x$ and $y$, of $C$. Since $G_m[V(C)]$ contains $K_{t,t}$, it has an $x, y$-path with at least $2t - 1$ vertices. This path together with $P_1, P_2$ and a $u,v$-path in $D$ form a cycle of length at least $2t + 1$, a contradiction to the maximality of $C$. Thus each component of $G_m - V(C)$ is a single vertex and is adjacent either only to vertices in $A$ or only to vertices in $B$. Moreover, by (29), each such vertex has degree exactly $t$, and thus its neighborhood is a complete graph. Since $B$ is independent, each $v \in V(G_m) - C$ is adjacent only to vertices in $A$. Thus $G_m = K_m - E(K_{m-t}) = H_{m,k,1,t} \subseteq H_{m,k,t}$.

**Case 2:** $K_3 + tK_{t-2} \subseteq G_m[V(C)] \subseteq K_3 + tK_{t-2}$, where $\ell = 2(q - 3)/(2t - 4)$. Again, since $G_m[V(C)]$ has hamiltonian cycle $C$ and a separating set of size 3 (call this set $A$), $\ell \leq 3$. If $\ell \leq 2$, then $q \leq 3 + 2(t - 2) < 2t$, a contradiction. Thus, $\ell = 3$ and $q = 3 + 3(t - 2) = 3t - 3$. Since $2t \leq q \leq 2t + 1$, we get $t \in \{3, 4\}$. Since $t \geq 4$ by assumption, we obtain that $t = 4$ and $F_4 \subseteq G_m$.

**Case 3:** For every two distinct $x, y \in V(C)$, the graph $G_m[V(C)]$ contains an $x, y$-path with at least $2t$ vertices. Let $W = V(G_m) - V(C)$. Repeating the argument of the second paragraph of Case 1, we obtain that in our case

$$each component of G_m[w] is a singleton and so N(w) \subseteq V(C) for each w \in W. \quad (34)$$

Since no $w \in W$ is adjacent to two consecutive vertices of $C$ (by the maximality of $C$) and $q \leq 2t+1$, by (29),

$$d_{G_m}(w) = t \text{ for every } w \in W. \quad (35)$$
Fix some $w_1 \in W$. Then we may relabel the vertices of $C$ so that $N_{G_m}(w_1) = \{v_1, v_3, v_5, \ldots, v_{2t-1}\}$. By (29), this also yields $G_m[\{v_1, v_3, \ldots, v_{2t-1}\}] = K_t$ and thus $d_{G_m}(v_i) \geq t+1$ for all $i \in \{1, 3, \ldots, 2t-1\}$. In particular,

$$d_{G_m}(v) \geq t + 1 \text{ for every } v \in N_{G_m}(w_1).$$

Then for every $j \in \{2, 4, \ldots, 2t-2\}$ (and for $j = 2t$ in the case $q = 2t$) we can replace $v_j$ with $w_1$ in $C$ and obtain another longest cycle. By (35) and (34), this yields $d_{G_m}(v_j) = t$ and

$$N_{G_m}(v_j) \subseteq V(C) \text{ for all } j \in \{2, 4, \ldots, 2t-2\} \text{ (and for } j = 2t \text{ in the case } q = 2t).$$

**Case 3.1:** $q = 2t$. Switching the roles of $w_1$ with $v_j$ together with (36) yields

$$N_{G_m}(v_j) = \{v_1, v_3, v_5, \ldots, v_{2t-1}\} \text{ for all } j = 2, 4, \ldots, 2t.$$ By (35) and (38), $N_{G_m}(w) = \{v_1, v_3, v_5, \ldots, v_{2t-1}\}$ for all $w \in V(G_m) - \{v_1, v_3, v_5, \ldots, v_{2t-1}\}$. This means $G_m \subseteq H_{m,2t+2,t}$, as claimed.

**Case 3.2:** $q = 2t+1$. Since $m \geq 2t + 3$, there is $w_2 \in W - w_1$. By (37), vertex $w_2$ is not adjacent to $v_j$ for $j \in \{2, 4, \ldots, 2t-2\}$. Suppose that $w_2$ is adjacent to $v_2t$ or $v_{2t+1}$, say $w_2v_{2t} \in E(G_m)$. Then by the maximality of $C$, $w_2v_{2t+1}, w_2v_{2t-1} \not\in E(G_m)$. So the only possible $t$-element set of neighbors of $w_2$ is $\{v_1, v_3, \ldots, v_{2t-1}, v_{2t}\}$. But then $G_m$ has the $(2t+2)$-cycle $(w_2, v_3, v_4, v_5, \ldots, v_{2t-1}, v_1, v_1, v_{2t+1}, v_{2t}, w_2)$, a contradiction. Thus

$$N_{G_m}(w) = \{v_1, v_3, v_5, \ldots, v_{2t-1}\} \text{ for all } w \in W.$$ Since we can replace in $C$ any $v_j$ for $j \in \{2, 4, \ldots, 2t-2\}$ with $w_1$, (30) yields $N_{G_m}(v_j) = \{v_1, v_3, v_5, \ldots, v_{2t-1}\}$ for all $j = 2, 4, \ldots, 2t - 2$. It follows that $\{v_1, v_3, v_5, \ldots, v_{2t-1}\}$ covers all edges in $G_m$ apart from edge $v_2v_{2t+1}$. This means $G_m \subseteq H_{m,2t+2,t}$, as claimed.

**Proof for** $k = 2t + 1$. Similarly to the proof for $k = 2t + 2$, we have (28) and (29). Let $C = (v_1, \ldots, v_q)$ be a longest cycle in $G_m$. Since $\delta(G_m) \geq t$, by Theorem 2.3, $q \geq 2t$, so $e(G_m) < k$ yields $q = 2t$. Then repeating the argument for $k = 2t + 2$, we obtain (30) and finally (31). So by Theorem 2.7 for $n = s = 2t$ and $H = G_m[V(C)]$, one of three cases below holds:

**Case 1:** $\overline{K_t} + \overline{K_t} \subseteq G_m[V(C)] \subseteq K_t + \overline{K_t}$. As in the proof for $k = 2t + 2$, we derive $G_m = K_m - E(K_{m-t}) = H_{m,k,t}$.

**Case 2:** $\overline{K_3} + \ell K_{t-2} \subseteq G_m[V(C)] \subseteq K_3 + \ell K_{t-2}$, where $\ell = 2(2t - 3)/(2t - 4)$. Again, since $G_m[V(C)]$ has hamiltonian cycle $C$ and a separating set of size three (call this set $A$), $\ell \leq 3$. Since $t \geq 4, \ell \neq 3$. If $\ell \leq 2$, then $q \leq 3 + 2(t - 2) < 2t$, a contradiction.

**Case 3:** For every two distinct $x, y \in V(C)$, graph $G_m[V(C)]$ contains a hamiltonian $x,y$-path. Then for any component $H$ of $G_m - V(C)$, let $x$ and $y$ be neighbors of $H$ in $V(C)$. By the case, $G_m[V(C)]$ contains a $2t$-vertex path, say $P$. Then $P$ together with an $x,y$-path through $H$ forms a cycle with at least $k$ vertices, a contradiction. But since $m > k$, such a component $H$ does exist. □
4.3 Subgraphs of $G_m$

In this section, we define classes of graphs which we shall show are subgraphs of $G_m$, and these subgraphs will have the important property that they have many long paths and are preserved by the reverse of the contraction process in the Basic Procedure.

For a graph $F$ and a nonnegative integer $s$, we denote by $K^{-s}(F)$ the family of graphs obtained from $F$ by deleting at most $s$ edges.

Let $F_0 = F_0(t)$ denote the complete bipartite graph $K_{t,t+1}$ with partite sets $A$ and $B$ where $|A| = t$ and $|B| = t + 1$. Let $F_0 := K^{-t+3}(F_0)$, i.e., the family of subgraphs of $K_{t,t+1}$ with at least $t(t+1) - t + 3$ edges.

Let $F_1 = F_1(t)$ denote the complete bipartite graph $K_{t,t+2}$ with partite sets $A$ and $B$ where $|A| = t$ and $|B| = t + 2$. Let $F_1 := K^{-t+4}(F_1)$, i.e., the family of subgraphs of $K_{t,t+2}$ with at least $t(t+2) - t + 4$ edges.

Let $F_2$ denote the family of graphs obtained from a graph in $K^{-t+4}(F_1)$ by subdividing an edge $a_1b_1$ with a new vertex $c_1$, where $a_1 \in A$ and $b_1 \in B$. Note that any member $H \in F_2$ has at least $|A||B| - (t - 3)$ edges between $A$ and $B$ and the pair $a_1b_1$ is not an edge.

Let $F_3 = F_3(t,t')$ denote the complete bipartite graph $K_{t,t'}$ with partite sets $A$ and $B$ where $|A| = t$ and $|B| = t'$. Take a graph from $K^{-t+4}(F_3)$, select two non-empty subsets $A_1, A_2 \subseteq A$ with $|A_1 \cup A_2| \geq 3$ such that $A_1 \cap A_2 = \emptyset$ if $\min\{|A_1|, |A_2|\} = 1$, add two vertices $c_1$ and $c_2$, join them to each other and add the edges from $c_1$ to the elements of $A_i$, $(i = 1, 2)$. The class of obtained graphs is denoted by $F(A, B, A_1, A_2)$. The family $F_3$ consists of these graphs when $|A| = |B| = t$, $|A_1| = |A_2| = 2$ and $A_1 \cap A_2 = \emptyset$. In particular, for $t = 4$ the family $F_3$ consists of exactly one graph, call it $F(3, 4)$.

Recall that $F_4$ is a 9-vertex graph with vertex set $A \cup B$, $A = \{a_1, a_2, a_3\}$, $B := \{b_1, b_2, \ldots, b_6\}$ and edges of the complete bipartite graph $K(A, B)$ and three extra edges $b_1b_2, b_3b_4$, and $b_5b_6$. Define $F'_4$ as the (only) member of $F(A, B, A_1, A_2)$ where $|A| = |B| = t = 4$, $A_1 = A_2$, and $|A_i| = 3$. Let $F_4 := \{F_4, F'_4\}$, which is defined only for $t = 4$.

In this subsection we will prove two useful properties of graphs in $F_0 \cup \cdots \cup F_4$: First we show that $G_m$ contains one of them (Proposition 4.6) and then show that such graphs have long paths with given end-vertices (Lemma 4.8).

**Proposition 4.6.** Let $k \geq 9$. If $k$ is odd, then $G_m$ contains a member of $F_0 \cup \cdots \cup F_4$, and if $k$ is even then $G_m$ contains a member of $F_1 \cup \cdots \cup F_4$.

**Proof.** By Proposition 4.2, $G_m \subseteq H_{m,k,t}$ or $m > k = 10$ and $F_4 \subseteq G_m$. In the latter case, the proof is complete. So assume $G_m \subseteq H_{m,k,t}$ and $A, B, C$ are as in the definition of $H_{m,k,t}$. First suppose $k$ is even and $C = \{c_1, c_2\}$. If $m = k$ then by \[2\],

$$e(H_{m,k,t}) - e(G_m) \leq h(m, k, t) - h(m, k, t - 1) - 1 = t - 4,$$

i.e. $G_m \in K^{-t+4}(H_{m,k,t})$. Since $F_1(t) \subseteq H_{m,k,t}$, $G_m$ contains a subgraph in $F_1$. If $m > k$ then by (R2) and Lemma 3.2 we have $\delta(G_m) \geq t$. So, each $v \in B$ is adjacent to every $u \in A$ and each of $c_1, c_2$ has at least $t - 1$ neighbors in $A$. Since $|B \cup \{c_1\}| \geq m - t - 1 \geq t + 2$, $G_m$ contains a
member of $K^{-1}(F_1(t))$. Thus $G_m$ contains a member of $F_1$ unless $t = 4$, $m = 2t + 3$ and $c_1$ has a nonneighbor $x \in A$. But then $c_1c_2 \in E(G_m)$, and so $G_m$ contains either $F_3(4)$ or $F'_4$.

Similarly, if $k$ is odd and $m = k$, then by $[2]$, $G_m \in K^{-t+3}(H_{m,k,t})$. Thus, since $H_{m,k,t} \supseteq F_0(t)$, $G_m$ contains a subgraph in $F_0$. If $k$ is odd and $m > k$ then by (R2) we have $\delta(G_m) \geq t$. So, each $v \in V(G_m) - A$ is adjacent to every $u \in A$. Hence $G_m$ contains $K_{t,m-t}$.

In order to prove Lemma 4.8, we will use Corollary 2.10 and the following implication of it.

Lemma 4.7. Let $t \geq 4$ and $H \in F(A,B,A_1,A_2)$ with $|B| \geq t - 1$, $|A| = t$. Let $P$ be a path $a_1c_1c_2a_2$ and $L$ be a subgraph of $H$ with $|E(L)| \leq 2$ such that $P \cup L$ form a linear forest. Then

$H$ has a cycle $C$ of length $2t + 1$ containing $P \cup L$. (40)

**Proof.** Choose some $B' \subseteq B$ with $|B'| = t - 1$ such that $B \cap V(L) \subseteq B'$. Let $Q$ be the bipartite graph whose $t$-element partite sets are $A$ and $B' \cup \{c\}$ where $c$ is a new vertex, and the edge set consists of $H[A \cup B']$ and all edges joining $c$ to $A$. By the conditions of the lemma, the set $E' := E(L) \cup \{a_1c,ca_2\}$ forms a linear forest in $Q$. Since $Q$ misses at most $t - 4$ edges connecting $A$ with $B' \cup \{c\}$, by Corollary 2.10 with $s = t$ and $i = 2$, $Q$ has a hamiltonian cycle $C'$ containing $E'$. Then the $(2t+1)$-cycle $C$ in $H$ obtained from $C'$ by replacing path $a_1ca_2$ with $P$ satisfies (40).

**Lemma 4.8.** Let $H \in F_0 \cup F_1 \cup \cdots \cup F_4$ and $x, y \in V(H)$.

- (a) $H$ contains an $x, y$-path of length at least $2t - 2$;
- (b) if $H$ does not contain an $x, y$-path of length at least $2t - 1$, then
  - (b0) $H \in F_0$ and $\{x, y\} \subseteq A$, or
  - (b1) $H \in F_1$ and $\{x, y\} \subseteq A$, or
  - (b2) $H = F_4 \in F_4$ and $\{x, y\} \subseteq A$;
- (c) if $H$ does not contain an $x, y$-path of length at least $2t$, then
  - (c0) $H \in F_0$, or
  - (c1) $H \in F_1$ and at least one of $x, y$ is in $A$, or
  - (c2) $H \in F_2$ and either $\{x, y\} \subseteq A$ or $\{x, y\} = \{a_1, b_1\}$, or
  - (c3) $H \in F_3$ and $\{x, y\} \subseteq A$, or
  - (c4) $H \in F_4$ and $\{x, y\} \subseteq A$.

**Proof.** The statements concerning $H \in F_0 \cup F_1$ are the easiest. Using Corollary 2.10 (or just using induction on $t$) it is easy to prove a bit more. Suppose that $H \in K^{-t+2}_{t+1}(A,B)$, $t \geq 2$. Then every pair $x, y \in A \cup B$ is joined by a path of maximum possible length. This means that every pair of vertices $b_1, b_2 \in B$ is joined by a path of length $2t$, every pair $a \in A$, $b \in B$ is joined by a path of length $2t - 1$, and every pair $a_1, a_2 \in A$ is joined by a path of length $2t - 2$. For example, the proof for $H \in F_0$, $a \in A$ and $b \in B$ is as follows. Consider $H'$ obtained from $H$ by adding edge $ab$ if $ab \notin E(H)$ and deleting any $b' \in B - b$. Then by Corollary 2.10 $H'$ has a hamiltonian cycle containing $ab$, which yields an $a, b$-path in $H$ of length $2t - 1$.

The cycle $(b_1b_2a_1b_3a_2b_5a_3b_1)$ and path $b_1b_2a_1b_3a_2b_4a_3b_5b_6$ in $F_4$ prove (b2) and the part of (c4) related to $F_4$. 

19
Suppose now that \( H \in \mathcal{F}_2 \cup \mathcal{F}_3 \cup \{F'_1\} \); even in a more general setting suppose that \( H \in \mathcal{F}(A, B, A_1, A_2) \) with \( |B| = |A| = t, |A_1 \cup A_2| \geq 3, |A_2| \geq |A_1| \geq 1 \) (and in case of \( |A_1| = 1 \) one has \( A_1 \cap A_2 = \emptyset \)). We prove the statements in reverse order, first (c2) and (c3), then (b), finally (a). When we comment below "Case BC" or "Case AA", this means that we consider paths from \( B \) to \( C \) or from \( A \) to \( B \), respectively.

By Lemma 4.7, we already knew that \( c_1c_2 \) is contained in a cycle of length \( 2t + 1 \) so these two vertices are joined by a path of length \( 2t \) (Case CC). If \( b \in B \), and \( a_i \in A_i \), then the almost complete bipartite subgraph \( H[A \cup B] \) contains a \( b, a_i \)-path of length \( 2t - 1 \), so \( b \) and \( c_{3-i} \) is joined in \( H \) by a path of length \( 2t + 1 \) (Case BC). Concerning \( b_1b_2 \in B \) we can define \( H^+ \) by adding an extra vertex \( a_{t+1} \) to \( B \) and joining it to each vertex of \( B \). Applying Lemma 4.7 to \( H^+ \) (with \( t + 1 \) in place of \( t \)) we get that it has a cycle \( C_{2t+3} \) through \( b_1a_{t+1}b_2 \). This cycle gives a \( b_1b_2 \)-path of length \( 2t + 1 \) in \( H \) (Case BB). In case of \( x \in A \), \( y \in A \) the high edge density of \( H \) implies that \( x \) and \( y \) have a common neighbor \( b \in B \). One can find a path \( P = a_1c_1c_2a_2 \) such that \( P \) and \( xby \) form a linear forest. Then Lemma 4.7 yields a cycle \( C_{2t+1} \) through all these edges. Leaving out \( b \) one gets an \( x,y \)-path of length \( 2t - 1 \) in \( H \) (Case AA). In case of \( x \in A \), \( y \in B \) maybe we have to add the edge \( xy \) to obtain a cycle \( C_{2t+1} \) through it by Lemma 4.7. This yields an \( x,y \)-path of length \( 2t \) (Case AB). Finally, if \( x \in A \), \( y = c_i \) one uses a path \( c_i, c_{3-i}, x' \) and an \( x,x' \)-path of length \( 2t - 2 \) in \( A \cup B \) to get an \( x,y \)-path of length \( 2t \), if this can be done. If such an \( x' \neq x \) does not exists, then \( x = a_1 \in A_1 \), \( |A_1| = 1 \), and \( y = c_2 \). This is the case described in (c2) (Case AC). \( \square \)

### 4.4 Reversing contraction

The aim of this section is to prove Lemma 4.9 below on preserving certain subgraphs during the reverse of the Basic Procedure.

**Lemma 4.9** (Main lemma on contraction). Let \( k \geq 9 \) and suppose \( F \) and \( F' \) are 2-connected graphs such that \( F = F' / xy \) and \( c(F') < k \).

If \( k \) is even and \( F \) contains a subgraph \( H \in \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_4 \), then \( F' \) has a subgraph \( H' \in \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_4 \).

If \( k \) is odd and \( F \) contains a subgraph \( H \in \mathcal{F}_0 \), then \( F' \) has a subgraph \( H' \in \mathcal{F}_0 \).

**Proof for \( k \) even.** Case 1. \( H \in \mathcal{F}_1 \). Let \( u = x \ast y \). If \( u \notin V(H) \) then \( H \subseteq F' \) and we are done. In case of \( u \in A \) consider the sets \( X := N_F(x) \cap B \) and \( Y := N_F(y) \cap B \). If \( X = X \cup Y \) then \( F' \) restricted to \( (A \setminus \{u\}) \cup \{x\} \cup B \) contains a copy of \( H \). If \( X = X \cup Y \setminus \{y\} \) for \( y' \in V(H') \), then \( F' \) restricted to \( (A \setminus \{u\}) \cup \{x\} \cup B \cup \{y\} \) contains a copy of a graph from \( \mathcal{F}_2 \) (with \( a_1 := x, b_1 := y', \) and \( c_1 := y \)). We proceed in the same way if \( Y = X \cup Y \) or if \( |Y| = |X \cup Y| - 1 \). In the remaining case \( |Y| \leq 2 \) and \( |X \setminus Y| \geq 2 \), so one can choose five distinct elements \( b_0, x_1, x_2, y_1, y_2 \) from \( B \) such that \( \{x_1, x_2\} \subseteq X \setminus Y \) and \( \{y_1, y_2\} \subseteq Y \setminus X \). Then the bipartite subgraph \( Q_0 \) of \( F' \) generated by the sets \( A \setminus \{u\} \cup \{x, y\} \) and \( B \setminus \{b_0\} \) contains the linear forest \( L \) consisting of the paths \( x_1x_2 \) and \( y_1y_2 \). If we define the graph \( Q \) by adding to \( Q_0 \) all edges joining \( x \) and \( y \) to \( B \setminus \{b_0\} \), then \( Q \) has at least \( (t + 1)^2 - (t - 4) \) edges. So by Corollary 2.10 for \( s = t + 1 \) and \( i = 2 \), \( Q \) has a hamiltonian cycle \( C_{2t+2} \) containing all edges of \( L \), and this cycle also appears in \( F' \), contradicting \( c(F') < k \).

In case of \( u \in B \) consider the sets \( X := N_F(x) \cap A \) and \( Y := N_F(y) \cap A \). If \( |X \setminus Y| \leq 1 \) or \( |Y \setminus X| \leq 1 \), then we proceed as above and find a subgraph \( H' \) of \( F \) either isomorphic to \( H \) or
belonging to \( F_2 \). If \(|X \setminus Y| \geq 2\) and \(|Y \setminus X| \geq 2\), then we have four distinct elements \( x_1, x_2, y_1, y_2 \) in \( A \) such that \( \{x_1, x_2\} \subseteq X \setminus Y \) and \( \{y_1, y_2\} \subseteq Y \setminus X \). Then \( F' \) contains a member of \( F_3 \) with \((c_1, c_2) = (x, y)\), \( A_1 := \{x_1, x_2\} \), and \( A_2 := \{y_1, y_2\} \).

**Case 2.** \( H \in F_2 \cup F_3 \cup \{F'_4\} \). The proof in this case follows from two claims. We say that the graph \( H \) has the Property \((W_\ell)\) if the following holds.

\[
(W_\ell) \quad \text{For all } z \in V(H) \text{ there exists } w \in N(z) \text{ such that for all } w' \in N(z) \setminus \{w\}, \text{ the graph } H \text{ has a cycle } C_\ell \text{ containing the path } wz w'.
\]

**Claim 1.** Suppose that the graph \( F \) contains a subgraph \( H \) satisfying Property \((W_\ell)\), and \( c(F') \leq \ell \). Then \( F' \) has a subgraph \( H' \) isomorphic to \( H \).

Let \( z = x \ast y \) and \( V = V(F) - z = V(F') - x - y \). If \( V(H) \subseteq V \), then there is nothing to prove.

Suppose that \( z \in V(H) \subseteq V(F) \) and define \( X := N_{F'}(x) \cap N_H(z) \) and \( Y := N_{F'}(y) \cap N_H(z) \). Then \( X \cup Y = N_H(z) \). Let \( w \in N(z) \) be the vertex from the definition of the Property \((W_\ell)\). Since \( N_H(z) = X \cup Y \), we may assume by symmetry that \( w \in X \).

We claim that \( Y - w = \emptyset \). Indeed, suppose there is \( w' \in Y - w \). By Property \((W_\ell)\), \( H \) has a cycle \( C_\ell \) containing the path \( wz w' \). Then the path \( C_\ell - z \) in \( F' \) together with the edges \( w'y, yx \) and \( xw \) forms a cycle of length \( \ell + 1 \), contradicting \( c(F') \leq \ell \).

This implies that \( N_{F'}(x) \) contains \( N_H(z) \). So \( F' \) contains a copy of \( H \) with the vertex set \( (V(H) \setminus \{z\}) \cup \{x\} \).

**Claim 2.** If \( H \in F_2 \cup F_3 \) or \( H = F'_4 \), then \( H \) satisfies Property \((W_{2t+1})\).

We prove a bit more: every \( H \in \mathcal{F}(A, B, A_1, A_2) \) with \(|B| \geq t - 1\), \(|A| = t \) satisfies \((W_{2t+1})\). Indeed, for \( z = c_i \) we can choose a \( w := c_{3-i} \). For \( z \in B \) we can choose a \( w \in A \) arbitrarily. For \( z \in A \) we can choose \( w \in N(z) \subseteq B \) arbitrarily, except if \( z \in A_1 \) and \(|A_1| = 1 \). In this latter case we can use \( w := c_i \). In each of these cases, given \( L := wz w' \) one can find a path \( P := a_1 c_1 c_2 a_2 \) such that \( P \cup L \) is a linear forest. Then Lemma \[4.7\] yields that \( H \) has a cycle \( C_{2t+1} \) through \( wz w' \).

Since each \( H \in F_2 \cup F_3 \cup \{F'_4\} \) belongs to such \( \mathcal{F}(A, B, A_1, A_2) \), this completes the proof of Claim 2.

**Case 3.** \( H = F_4 \). Let \( u = x \ast y \). By symmetry, we can consider only two cases: \( u = a_1 \) and \( u = b_1 \). First, suppose \( u = a_1 \) and \( x b_1 \in E(F') \). Then since \( c(F') \leq 9 \), \( y \) is not adjacent to any of \( b_3, b_4, b_5, b_6 \). Thus \( x \) is adjacent to all of them, and if \( y b_2 \in E(F') \), then the cycle \((y b_2 b_1 a_2 b_3 a_3 b_5 b_6 x y)\) contradicts \( c(F') \leq 9 \). So \( x b_2 \in E(F') \) and the subgraph of \( F' \) with vertex set \( V(H) \setminus \{u\} \cup \{x\} \) contains \( F_4 \).

Similarly, suppose \( u = b_1 \) and \( x b_2 \in E(F') \). Then to avoid a 10-cycle in \( F' \), \( y \) has no neighbors in \( A \) and thus \( z \) is adjacent to all of \( A \). So, again the subgraph of \( F' \) with vertex set \( V(H) \setminus \{u\} \cup \{x\} \) contains \( F_4 \).

**Proof for \( k \) odd.** First we prove the following statement \[(41)\] which is true for every \( t \geq 2 \). Let \( H \in \mathcal{K}^{-t+2}(K(A, B)) \) with \(|A| = t\), \(|B| = t + 1\). Let \( P \) be a path of length two in \( H \). Then

\[
H \text{ has a cycle } C \text{ of length } 2t \text{ containing } P.
\]
If every vertex of $B \setminus P$ is joined to all vertices of $A$, then one can find a $C_{2t}$ through $P$ directly. Otherwise, there is a vertex $v \in B \setminus P$ of degree at most $t - 1$, so $H \setminus \{v\}$ is a subgraph of $K_{t,t}$ with at least $t^2 - t + 3$ edges. Then the statement follows from Corollary 2.10 for $s = t$ and $i = 1$.

Now suppose that $H \in \mathcal{F}_0$, $H \subseteq F$, $F = F'/xy$, and $H,F,F'$ satisfy the constraints of Lemma 4.9. Then (11) implies that $H$ satisfies property $(W_{2t})$. Thus by Claim 1, $F'$ has a subgraph $H'$ isomorphic to $H$.

4.5 Completing the proof of Theorem 4.1

Proof for $k$ even. Proposition 4.6 and Lemma 4.9 imply that there is a subgraph $H$ of $G = G_n$ such that $H \in \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_4$. Let $G' = G - V(H)$ and $S_1, \ldots, S_k$ be the components of $G'$. Each of $S_i$ has at least two neighbors, say $x_i$ and $y_i$ in $V(H)$. Let $\ell_i$ denote the length of a longest $x_i,y_i$-path in $G[V(S_i) \cup \{x_i,y_i\}]$. Since $c(G) < k$, by Lemma 4.8(a) and (b),

$$\text{for all } i, \quad \ell_i \leq 3 \quad \text{and if } H \in \mathcal{F}_2 \cup \mathcal{F}_3 \cup \{\mathcal{F}_4\}, \text{ then } \ell_i \leq 2. \quad (42)$$

Case 1: $H \in \mathcal{F}_3 \cup \{\mathcal{F}_4\}$. By (42), $\ell_i \leq 2$ for all $i$ and all choices of $x_i$ and $y_i$. Since $G$ is 2-connected, this yields that each $S_i$ is a singleton, say $v_i$. Moreover, Lemma 4.8(c3) and (c4) imply $N(v_i) \subseteq A$ for all $i$. So $G$ is contained in a graph in $\mathcal{G}_1(n,k)$, and the only edge outside $A$ is $c_1c_2$.

Case 2: $H \in \mathcal{F}_2$. Again, by (42), $\ell_i \leq 2$ for all $i$ and all choices of $x_i$ and $y_i$. So again this yields that each $S_i$ is a singleton, say $v_i$. But now Lemma 4.8(c2) implies that for all $i$, either $N(v_i) \subseteq A$ or $N(v_i) = \{a_1,b_1\}$. Thus $G$ is contained in a graph in $\mathcal{G}_2(n,k)$, where the only possible star component of $G - A$ with at least three vertices is a star with center $b_1$ and $c_1$ a leaf.

Case 3: $H \in \mathcal{F}_1$. Suppose first that some $x_i$ is in $B$. Then by Lemma 4.8(c3), $y_i \in A$ and by Lemma 4.8(b), $\ell_i = 2$. So, denoting the common neighbor of $x_i$ and $y_i$ in $S_i$ by $c_1$, we get Case 2. Thus it is enough to consider below only the situation when

$$N(S_i) \cap V(H) \subseteq A \text{ for every } i. \quad (43)$$

We consider three cases.

Case 3.1: For some $i \neq j$, $\ell_i \geq 3$ and $\ell_j \geq 3$, say $\ell_1 \geq 3$ and $\ell_2 \geq 3$. Then by (42), $\ell_1 = \ell_2 = 3$. For $i = 1,2$, let $(x_i,v_1,v'_i,y_i)$ denote an $x_i,y_i$-path of length three in $G[V(S_i) \cup \{x_i,y_i\}]$. Also, by (43), $x_1,y_1,x_2,y_2 \in A$. Suppose first that $\{x_1,y_1\} \neq \{x_2,y_2\}$. We proceed as in the beginning of the proof of Lemma 4.9. Choose a $(t - 2)$-element subset $B' \subseteq B$ and add two new vertices $b'_1$ and $b'_2$ and join them to all vertices of $A$. Then the obtained bipartite graph $H'$ has at least $t^2 - t + 4$ edges so there is a hamiltonian cycle $C'$ containing the linear forest $x_1b'_1y_1 \cup x_2b'_2y_2$ by Corollary 2.10. This $C'$ corresponds to a cycle of length $k$ in $G$, a contradiction.

It follows that every component $S_i$ with $\ell_i \geq 3$ has exactly two neighbors in $V(H)$ and these two neighbors, say $x_1,y_1$, are the same for all such components; furthermore $x_1,y_1 \in A$. Furthermore, in order to have $\ell_i \leq 3$, all leaves of $S_i$ have the same neighbor in $A$. Thus $G$ is contained in a graph in $\mathcal{G}_3(n,k)$.

Case 3.2: There exists exactly one $i$ with $\ell_i \geq 3$, say $\ell_1 \geq 3$. Then by (42), $\ell_1 = 3$. Let
(x_1, v_1, v_1', y_1) be an x_1, y_1-path of length 3 in \( G[V(S_i) \cup \{x_1, y_1\}] \). By (43), every other component \( S_i \) is a singleton, say \( v_i \) with \( N(v_i) \subseteq A \). As in Case 3.2, in order to have \( \ell_i \leq 3 \), \( S_1 \) should be a star, and if \( S_1 \neq K_2, K_1 \), then all leaves of \( S_1 \) are adjacent to the same vertex in \( A \). Thus \( G \) is contained in a graph in \( \mathcal{G}_1(n, k) \cup \mathcal{G}_2(n, k) \).

**Case 3.3:** \( \ell_i \leq 2 \) for all \( i \). Here \( G \) is contained in a graph in \( \mathcal{G}_1(n, k) \). Then each \( S_i \) is a singleton with all neighbors in \( A \). It follows that \( G - A \) is an independent set.

**Case 4:** \( H = F_4 \). By Lemma 4.8(c4), (43) holds. Together with (42), this yields that every component \( S \) of \( G - A \) is a star and if \( |S| \geq 3 \), then all leaves of \( S \) have the same neighbor in \( A \). It follows that \( G \in \mathcal{G}_4(n, k) \).

**Proof for \( k \) odd.** By Proposition 1.6 and Lemma 4.9, \( G_n \) contains some \( H \in \mathcal{F}_0 \). Let \( G' = G_n - H \) and \( S_1, \ldots, S_8 \) be the components of \( G' \). Each of \( S_i \) has at least two neighbors, say \( x_i \) and \( y_i \) in \( V(H) \). Let \( \ell_i \) denote the length of a longest \( x_i, y_i \)-path in \( G_n[V(S_i) \cup \{x_i, y_i\}] \). Since \( c(G_n) \leq 2t \), by Lemma 4.8

\[
\text{for all } i, \quad \ell_i \leq 2 \quad \text{and} \quad \{x_i, y_i\} \subseteq A. \tag{44}
\]

Then each \( S_i \) is a singleton with all neighbors in \( A \). It follows that \( G - A \) is an independent set. This completes the proof of Theorem 4.1 for odd \( k \).

**5 Proof of Theorem 1.4 for \( k \leq 8 \)**

Recall that Theorem 4.1 describes for \( k \geq 9 \) and \( n \geq 3k/2 \) the \( n \)-vertex 2-connected graphs with no cycle of length at least \( k \) and more than \( h(n, k, t - 1) \) edges. In this section, we will do the same for \( 4 \leq k \leq 8 \) and \( n \geq k \). We will use for this the classes \( \mathcal{G}_i(n, k) \) defined in Section 4 and the notion of a \( J_3 \)-bridge. For \( A \subseteq V(G) \) and \( S \subseteq V(G) \setminus A \), \( S \) forms a \( J_3 \)-bridge of \( A \) with endpoints \( a_1, a_2 \) if \( a_1, a_2 \in A \), \( A' := \{a_1, a_2\} \) is a cutset of \( G \), \( G[S \cup A'] \cup \{a_1a_2\} \) is a 2-connected graph, \( G[S] \) is connected, and the length of the \( a_1, a_2 \)-path in \( G[S \cup A'] \) is three.

Furthermore, since the description (but not the proof) for \( k = 8 \) is more sophisticated, we will need four more special graph classes for \( k = 8 \): Each of the graph classes \( \mathcal{G}_i(n, 8) \) (\( 5 \leq i \leq 8 \)) contains 2-connected \( n \)-vertex graphs \( G \) with \( c(G) < 8 \) and having a special vertex set \( A = \{a_1, a_2, \ldots, a_s\} \) with \( G[A] \) being a complete graph and such that \( G \setminus A \) consists of \( J_3 \)-bridges and isolated vertices having exactly two neighbors in \( A \).

If \( G \in \mathcal{G}_5(n, 8) \), then \( s = 3 \) and \( a_1 \) is adjacent to each component in \( G \setminus A \). So the edge \( a_2a_3 \) is contained in a unique triangle, namely \( a_1a_2a_3 \).

If \( G \in \mathcal{G}_6(n, 8) \cup \mathcal{G}_7(n, 8) \), then \( s = 4 \) and the endpoints of all \( J_3 \)-bridges are \( \{a_1, a_2\} \) while one of the neighbors of some isolated vertex \( c \) of \( G \setminus A \) is \( a_1 \) in case of \( \mathcal{G}_6(n, 8) \) and \( N(c) = \{a_3, a_4\} \) for all \( c \) in case of \( \mathcal{G}_7(n, 8) \).

If \( G \in \mathcal{G}_8(n, 8) \), then \( s = 5 \) and \( N(S) = \{a_1, a_2\} \) for each component \( S \) of \( G \setminus A \).

**Theorem 5.1.** Let \( 4 \leq k \leq 8 \) and \( n \geq k \). Let \( G \) be an \( n \)-vertex 2-connected graph with no cycle of length at least \( k \). Then either \( 7 \leq k \leq 8 \) and \( e(G) \leq h(n, k, t - 1) \) edges or \( G \) is a subgraph of a graph in \( \mathcal{G}(n, k) \), where
\( \mathcal{G}(n, 4) = \emptyset, \)
\( \mathcal{G}(n, 5) := \mathcal{G}_1(n, 5), \)
\( \mathcal{G}(n, 6) := \mathcal{G}_1(n, 6) \cup \mathcal{G}_2(n, 6), \)
\( \mathcal{G}(n, 7) := \{ H_{n,7,3} \} \cup \mathcal{G}_1(n, 6) \cup \mathcal{G}_2(n, 6) \cup \mathcal{G}_3(n, 6), \)
\( \mathcal{G}(n, 8) := \bigcup_{1 \leq i \leq 8, i \neq 4} \mathcal{G}_i(n, 8). \)

The proof scheme is that we consider a graph \( G \) satisfying the conditions of the theorem and take a longest cycle \( C \) with vertex set, say \( X := \{x_0, x_1, x_2, \ldots, x_r\} \). Moreover, we will assume that \( C \) has the maximum sum of the degrees of its vertices among the longest cycles in \( G \). Analyzing possibilities, we will derive that \( G \in \mathcal{G}(n, k) \).

A bridge of \( C \) is the vertex set of a component of \( G - X \).

We start from a sequence of simple claims on the structure of bridges and the edges between \( X \) and the bridges. For brevity we denote by \( d_C(i, j) \) the distance on \( C \) between \( x_j \) and \( x_i \), i.e. \( \min \{ |j - i|, r + 1 - |j - i| \} \). For a bridge \( S \) and neighbors \( x, x' \) of \( S \) on \( C \), an \( (x, x', S) \)-path is an \( x, x' \)-path whose all internal vertices are in \( S \).

The maximality of \( |C| \) implies our first claim:

**Claim 5.2.** For every bridge \( S \) and any \( x_i, x_j \in N(S) \cap X \), the length of any \( (x_i, x_j, S) \)-path is at most \( d_C(i, j) \). In particular, if \( S \) contains distinct \( c_1, c_2 \) such that \( x_i c_1, x_j c_2 \in E(G) \), then \( d_C(i, j) \geq 3 \).

If \( |S| \geq 2 \), then by the 2-connectedness of \( G \), there are two vertex-disjoint \( S, X \)-paths. Thus if \( G[S] \) contains a cycle, then for some \( x_i, x_j \in N(S) \cap X \), the length of the longest \( (x_i, x_j, S) \)-path is at least 4. Hence, since \( |C| \leq k - 1 \leq 7 \), by Claim 5.2 we get the next claim:

**Claim 5.3.** For every bridge \( S \) of \( X \) and any distinct \( x_i, x_j \in N(S) \cap X \), the length of any \( (x_i, x_j, S) \)-path is at most 3. In particular, \( G[S] \) is acyclic (a tree).

Suppose that for some bridge \( S \), and two leaves \( c_1, c_2 \) of the tree \( G[S] \), there is a \( c_1, c_2 \)-path \( P \) in \( G[S] \) of length at least 3. Then by Claim 5.3 each of \( c_1 \) and \( c_2 \) has exactly one neighbor in \( X \), and this is the same vertex, say \( x_i \). Again by the 2-connectedness of \( G \), there is \( x_j \in X \cap N(S) \setminus \{x_i\} \). Then there is an \( (x_j, x_i, S) \)-path of length at least 4 through either \( c_1 \) or \( c_2 \), which contradicts Claim 5.3. Thus we get:

**Claim 5.4.** For every bridge \( S \) of \( X \), \( G[S] \) is a star. Moreover, if \( |S| \geq 3 \), then all leaves of \( G[S] \) have degree 2 in \( G \) and the same neighbor, \( x(S) \), in \( X \).

Suppose \( |S| \geq 2 \) and \( |N(S) \cap X| \geq 3 \), say \( \{x, x', x''\} \subseteq N(S) \cap X \). Let \( c_1 \) be a leaf of \( G[S] \). If \( |S| \geq 3 \), then by Claim 5.3 it has a unique neighbor in \( X \), say \( x \). It follows that there are an \( (x, x', S) \)-path and an \( (x, x'', S) \)-path of length at least 3. Also there is an \( (x', x'', S) \)-path of length at least 2. Then by Claim 5.2 the distance on \( C \) from \( x \) to \( x' \) and to \( x'' \) is at least 3 and between \( x' \) and \( x'' \) is at least 2. Thus \( |X| \geq 3 + 3 + 2 = 8 \), a contradiction. Similarly, if \( S = \{c_1, c_2\} \), then by symmetry we may assume that \( x \in N(c_1) \cap X \) and \( \{x', x''\} \subseteq N(c_2) \cap X \). In this case again by Claim 5.2 \( |X| \geq 3 + 3 + 2 = 8 \), a contradiction. Thus summarizing this with the previous claims, we have proved the following.

24
Claim 5.5. For every bridge $S$ of $X$ with $|S| \geq 2$, $|N(S) \cap X| = 2$. Moreover, if $|S| \geq 3$, then $G[S]$ is a star and all leaves of $G[S]$ have degree 2 in $G$ and the same neighbor, $x(S)$, in $X$. In other words, each bridge $S$ with $|S| \geq 2$ is a $J_3$-bridge of $X$.

From Claims 5.2 and 5.5 we deduce:

Claim 5.6. For every $J_3$-bridge $S$ of $X$ with endpoints $x_i$ and $x_j$, $d_C(i, j) \geq 3$.

If there are $i_1 < i_2 < i_3 < i_4 \leq r$ and bridges $S_1$ and $S_2$ such that $G$ contains an $(x_{i_1}, x_{i_3}, S_1)$-path $P_1$ and an $(x_{i_2}, x_{i_4}, S_2)$-path $P_2$, then we can construct two new cycles $C_1$ and $C_2$ such that each of them contains the edges of $P_1$ and $P_2$ and each edge of $C$ belongs to exactly one of $C_1$ and $C_2$. Then the total length of $C_1$ and $C_2$ is at least $|E(C)| + 2(|E(P_1)| + |E(P_2)|) \geq (k - 1) + 8 \geq 2k - 1$. Thus at least one of them is longer than $C$, a contradiction. Thus we have:

Claim 5.7. There are no $i_1 < i_2 < i_3 < i_4 \leq r$ and bridges $S_1$ and $S_2$ of $X$ such that $G$ contains an $(x_{i_1}, x_{i_3}, S_1)$-path and an $(x_{i_2}, x_{i_4}, S_2)$-path. In particular, since $k - 1 \leq 7$, any two $J_3$-bridges share an endpoint.

We now can prove Theorem 5.1. Indeed, by Claim 5.2, $|X| \geq 4$. This proves $G(n, 4) = \emptyset$, i.e., Part 1 of the theorem.

We will consider 3 cases according to the value of $|X|$. As mentioned above, $|X| \geq 4$.

Case 1: $4 \leq |X| \leq 5$. Then by Claims 5.5 and 5.6, each bridge is a singleton. Furthermore, by Claim 5.2 each such singleton has exactly two (necessarily nonconsecutive) neighbors in $X$. If $|X| = 4$, Claim 5.7 yields that this pair of neighbors is the same for all bridges, say it is $\{x_0, x_2\}$. Then $G$ is contained in $H_{n,5,2}$ with $A = \{x_0, x_2\}$, as claimed. This proves Part 2.

Let $|X| = 5$. If also each bridge has the same pair of neighbors in $X$, say $\{x_0, x_2\}$, then since $n \geq |X| + 1 = 6$, $x_1$ is not adjacent to $\{x_3, x_4\}$ to avoid a 6-cycle. Thus in this case, $G$ is contained in $H_{n,6,2}$ with $A = \{x_0, x_2\}$, and so $\epsilon(G) \leq h(n, 6, 2)$. Otherwise by Claim 5.7 there are exactly two distinct pairs of neighbors of the bridges, and they share a vertex. Suppose these pairs are $\{x_0, x_2\}$ and $\{x_0, x_3\}$ and for $j \in \{2, 3\}$, $Y_j$ is the set of vertices adjacent to $x_0$ and $x_j$. Then to avoid a 6-cycle, edges $x_1x_4, x_1x_3$ and $x_2x_4$ are not present in $G$. Then $G \in G_2(n, 6)$ with $A = \{x_0, x_2\}$, $B = Y_2 \cup \{x_3\}$ and $J = Y_3 \cup \{x_4\}$. Since $H_{n,5,2}$ contains $H_{n,5,2}$, this together with the previous paragraph proves Part 3 of the theorem.

Case 2: $|X| = 6$. By Claims 5.5, 5.7 it is enough to consider the following three subcases.

Case 2.1: $X$ has a bridge $S$ with $|N(S) \cap X| \geq 3$. By Claim 5.5, $S$ is a single vertex, say $z$, and by Claim 5.2, $z$ has exactly 3 (nonconsecutive) neighbors on $C$, say $x_0, x_2$ and $x_4$. In view of the cycle $x_0x_2x_3x_4x_5$ and the maximality of the degree sum of $C$, $d(x_1) \geq d(z) \geq 3$. By Claim 5.7, $x_1$ has no neighbors outside of $C$. In order to avoid a 7-cycle in $G$, $x_1x_3, x_1x_5 \notin E(G)$. So $x_1x_4 \notin E(G)$. Similarly, $x_2x_5, x_0x_3 \in E(G)$, so $G$ contains $K_{3,4}$ with parts $A = \{x_0, x_2, x_4\}$ and $B = \{x_1, x_3, x_5, z\}$. Moreover, $B$ is independent. Let $C$ be the vertex set of any component of $G - A - B$. If $C$ has a neighbor in $B$ or is not a singleton, then $G[A \cup B \cup C]$ has a cycle of length at least 7. Thus each component of $G - A - B$ is a singleton and has no neighbors in $B$. This means $A$ meets all edges and so $G$ is a subgraph of $H_{n,7,3}$. 

25
Case 2.2: $X$ has a $J_3$-bridge $S$. Then by Claim 5.2 and symmetry, we may assume $N(S) = \{x_0, x_3\}$. In this case, $G$ has 3 internally disjoint $x_0, x_3$-paths of length 3. Thus to have $c(G) \leq 6$, $\{x_0, x_3\}$ separates internal vertices of distinct paths. It follows that $G - \{x_0, x_3\}$ is a collection of $J_3$-bridges of $\{x_0, x_3\}$ and isolated vertices each having only $x_0$ and $x_3$ as endpoints. Thus $G$ is a subgraph of a graph in $\mathcal{G}_3(n, 6)$.

Case 2.3: $V \setminus X$ is independent and each $z \in V \setminus X$ has degree 2. By Theorem 1.3 for each $z \in V \setminus X$, graph $G[X \cup \{z\}]$ has at most $h(7, 7, 2) = 14$ edges, which yields $e(G) \leq 2n = h(n, 7, 2)$. This proves Part 4 of Theorem 5.1.

Case 3: $|X| = 7$. By Claims 5.5 and 5.7 it is enough to consider the following four subcases.

Case 3.1: $X$ has a bridge $S$ with $|N(S) \cap X| \geq 3$. As in Case 2.1, $S$ is a single vertex, say $z$, and we may assume $N(S) \cap X = \{x_0, x_2, x_4\}$. Again, similarly to Case 2.1, in view of the 7-cycle $x_0x_2x_3x_4x_5x_6$, we obtain that $d(x_1) \geq d(z) \geq 3$, and that (to avoid a long cycle in $G$) the third neighbor of $x_1$ is $x_4$. Similarly, $x_0x_3 \in E(G)$. Thus, $G$ has a subgraph consisting of $K_{3,3}$ with parts $A := \{x_0, x_2, x_4\}$ and $B := \{x_1, x_3, x_5\}$ and an attached 3-path $x_0x_2x_4x_6$. Moreover, $d(x_1) = d(x_2) = d(z) = 3$ and these are isolated vertices in $G \setminus A$. Let $Y$ be the vertex set of the component of $G - A$ containing $\{x_5, x_6\}$. If there is another component $Y'$ of $G - A$ with $|Y'| \geq 2$, then to avoid a $\geq 8$-cycle, $G$ must be a subgraph of a graph in $\mathcal{G}_3(n, 8)$. If all the bridges of $A$ apart from $A$ are singletons, then $G$ is a subgraph of a graph in either $\mathcal{G}_1(n, 8)$ (if $|Y| = 2$) or $\mathcal{G}_2(n, 8)$ (if $|Y| \geq 3$).

Case 3.2: $G$ has $J_3$-bridges $S_1$ and $S_2$ of $X$ with $N(S_1) \neq N(S_2)$. By Claims 5.7 and 5.6 we may assume $N(S_1) = \{x_0, x_3\}$ and $N(S_2) = \{x_0, x_4\}$. By the 2-connectivity of $G$, we may assume that there is an $(x_0, x_3, S_1)$-path $x_0y_2y_3x_3$ and an $(x_4, x_0, S_2)$-path $x_4y_5y_6x_0$. Let $A = \{x_0, x_3, x_4\}$. Then the edges $y_1y_2, y_3y_5, x_1x_2, x_5x_6$ belong to distinct components of $G \setminus A$. Thus to avoid long cycles in $G$, no bridge of $A$ is adjacent to both, $x_3$ and $x_4$ and none of the bridges $S$ of $A$ contains an $(x_0, x_3, S)$-path or an $(x_0, x_4, S)$-path of length at least 4. It follows that $G$ is a subgraph of a graph in $\mathcal{G}_5(n, 8)$.

Case 3.3: $G$ has a $J_3$-bridge $S$ of $X$, and every other $J_3$-bridge of $X$ (if exists) has the same neighbors as $S$ in $X$. We may assume that $N(S) \cap X = \{x_0, x_4\}$ and $G$ contains an $(x_0, x_4, S)$-path $x_0y_6y_5x_4$. Then the edges $y_5y_6, x_1x_2, x_5x_6$ belong to three distinct components of $G \setminus \{x_0, x_4\}$. Let $Y$ be the component of $G \setminus \{x_0, x_4\}$ containing $\{x_1, x_2, x_3\}$. By the case, all other components are either isolated vertices or $J_3$-bridges of $\{x_0, x_4\}$. Also, every vertex $y \in (Y \setminus \{x_1, x_2, x_3\})$ has only neighbors in $X$ (i.e., $N(y) \subseteq \{x_0, x_1, \ldots, x_4\}$).

If $|Y| = 3$ we obtain that $G$ is a subgraph of a member of $\mathcal{G}_8(n, 8)$ with $A = \{x_0, x_1, x_2, x_3, x_4\}$. Suppose $|Y| \geq 4$. If there is $y \in Y \setminus \{x_1, x_2\}$ with $N_G(y) = \{x_0, x_3\}$, then to avoid an 8- or 9-cycle, $x_1x_4 \notin E(G)$ and no $y' \in Y \setminus \{x_2, x_3\}$ has $N_G(y') = \{x_1, x_4\}$. So, either $\{x_0, x_3\}$ is a cut set in $G$ or $x_3x_4 \in E(G)$. In the former case, $G$ is a subgraph of a graph in $\mathcal{G}_6(n, 8)$ with $A = \{x_0, x_3, x_4\}$ and $a_1 = x_0$. In the latter case, in order to avoid an $(x_0, x_4, Y)$-path of length $\geq 5$, graph $G[\{x_1, x_2, x_3, x_4, y\}]$ has only the 5 edges we already know and no vertex $y' \in Y \setminus \{x_1\}$ has $N(y') \subseteq \{x_1, x_2, x_3, x_4, y\}$. This means $G$ is a subgraph of a graph in $\mathcal{G}_6(n, 8)$ with $A = \{x_0, x_4, x_2, x_3\}$, where $a_1 = x_0$ and $a_2 = x_4$. The case of $y \in Y \setminus \{x_1, x_2\}$ with $N_G(y) = \{x_1, x_4\}$
is symmetrical. If there is \( y \in Y \setminus \{x_1\} \) with \( N(y) = \{x_0, x_2\} \), then in order to avoid an \((x_0, x_4, Y)\)-path of length \( \geq 5 \), \( x_1 x_3 \notin E(G) \) and every \( y' \in Y - X \) is adjacent to \( x_2 \). This means \( G \) is a subgraph of a graph in \( G_2(n,8) \cup G_3(n,8) \) with \( A = \{x_2, x_4, x_0\} \). The last possibility is that \( N(y) = \{x_1, x_3\} \) for every \( y \in Y - X \). Since \( |Y| \geq 4 \), this yields \( x_2 x_0, x_2 x_4 \notin E(G) \). Thus \( G \) is a subgraph of a member of \( G_7(n,8) \) with \( \{a_1, a_2\} := \{x_0, x_4\} \) and \( \{a_3, a_4\} := \{x_1, x_3\} \).

Case 3.4: \( G \setminus X \) consists of isolated vertices only, each having degree 2 in \( G \). By Theorem 1.3, for each \( z \in V \setminus X \), graph \( G[X \cup \{z\}] \) has at most \( h(8,8,2) = 19 \) edges, which yields \( e(G) \leq 2n + 3 = h(n, 8, 2) \).

Theorem 5.1 yields the following analog of Theorem 4.1(1) for a smaller range of \( e(G) \).

**Corollary 5.8.** Suppose that \( G \) is a 2-connected, \( n \)-vertex graph with \( c(G) < 7 \), \( n \geq 8 \). If \( e(G) \geq \lfloor (5n - 6)/2 \rfloor \) then \( G \) is a subgraph of \( H_{n,7,3} \), and this bound is best possible. \( \square \)

### 6 Concluding remarks

It could be that for \( k \geq 11 \), Theorem 1.4 holds already for \( n \geq 5k/4 \). Note that by Theorem 1.3, it does not hold for \( n < 5k/4 \). It may also be possible, albeit complicated, to describe the structure of 2-connected \( n \)-vertex graphs with no cycles of length at least \( k = 2t + 1 \) and at least \( h(n, k, t - 2) \) edges. We leave these as avenues for further research.

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