The 3-way intersection problem for $S(2, 4, v)$
designs

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Abstract

In this paper the 3-way intersection problem for $S(2, 4, v)$ designs is investigated. Let $b_v = \frac{(v-1)(v-3)}{12}$ and $I_3[v] = \{0, 1, \ldots, b_v\} \setminus \{b_v - 7, b_v - 6, b_v - 5, b_v - 4, b_v - 3, b_v - 2, b_v - 1\}$. Let $J_3[v] = \{k\}$ there exist three $S(2, 4, v)$ designs with $k$ same common blocks. We show that $J_3[v] \subseteq I_3[v]$ for any positive integer $v \equiv 1, 4 \pmod{12}$ and $J_3[v] = I_3[v]$, for $v \geq 49$ and $v = 13$. We find $J_3[16]$ completely.
Also we determine some values of $J_3[v]$ for $v = 25, 28, 37$ and 40.

KEYWORDS: 3-way intersection; $S(2, 4, v)$ design; GDD; trade

1 Introduction

A Steiner system $S(2, 4, v)$ is a pair $(\mathcal{V}, \mathcal{B})$ where $\mathcal{V}$ is a $v$-element set and $\mathcal{B}$ is a family of 4-element subsets of $\mathcal{V}$ called blocks, such that each 2-element subsets of $\mathcal{V}$ is contained in exactly one block.

Two Steiner systems $S(2, 4, v)$, $(\mathcal{V}, \mathcal{B}_1)$ and $(\mathcal{V}, \mathcal{B}_2)$ are said to intersect in $s$ blocks if $|\mathcal{B} \cap \mathcal{B}_1| = s$. The intersection problem for $S(2, 4, v)$ designs can be extended in this way: determine the sets $J_\mu[v](J_\mu[v])$ of all integers $s$ such that there exists a collection of $\mu \geq 2$ $S(2, 4, v)$ designs mutually intersecting in $s$ blocks (in the same set of $s$ blocks). This generalization is called $\mu$-way intersection problem. Clearly $J_2[v] = J_2[v] = J[v]$ and $J_\mu[v] \subseteq J_\mu[v] \subseteq J[v]$.

The intersection problem for $\mu = 2$ was considered by Colbourn, Hoffman, and Lindner in \cite{8}. They determined the set $J[v](J_2[v])$ completely.
for all values \( v \equiv 1, 4 \pmod{12} \), with some possible exceptions for \( v = 25, 28 \) and 37. Let \( [a, b] = \{a, a + 1, \ldots, b, b + 1\}, \) \( b_v = \frac{v(v-1)}{12} \), and \( I[v] = \{0, b_v\} \setminus ((b_v - 5, b_v - 1) \cup \{b_v - 7\}) \). It is shown in [9]; that:

1. \( J[u] \subseteq I[v] \) for all \( v \equiv 1, 4 \pmod{12} \).
2. \( J[u] = I[v] \) for all admissible \( v \geq 40 \).
3. \( J[13] = I[13] \) and \( J[16] = I[16] \setminus \{7, 9, 10, 11, 14\} \).
4. \( I[25] \setminus \{31, 33, 34, 37, 39, 40, 41, 42, 44\} \subseteq J[25] \) and \( \{42, 44\} \nsubseteq J[25] \).
5. \( I[28] \setminus \{44, 46, 49, 50, 52, 53, 54, 57\} \subseteq J[28] \).
6. \( I[37] \setminus \{64, 66, 76, 82, 84, 85, 88\} \cup \{90, 94\} \cup \{96, 101\} \) \( \subseteq J[37] \).

Also Chang, Feng, and Lo Faro investigate another type of intersection which is called triangle intersection (See [4]). Milici and Quattrocchi [15] determined \( J_3[v] \) for \( STSs \). Other results about the intersection problem can be found in [11, 12, 13, 14, 15, 16, 17, 18]. In this paper we investigate the three way intersection problem for \( S(2, 4, v) \) designs. We set \( I_3[v] = \{0, b_v\} \setminus \{b_v - 7, b_v - 1\} \). As our main result, we prove the following theorem.

**Theorem 1.1**

1. \( J_3[v] \subseteq I_3[v] \) for all \( v \equiv 1, 4 \pmod{12} \).
2. \( J_3[40] \setminus \{b_{40} - 15, b_{40} - 14\} \subseteq J_3[40] \).
3. \( J_3[13] = I_3[13] \) and \( J_3[16] = I_3[16] \setminus \{7, 9, 10, 11, 12\} \).
4. \( J_3[25] \setminus \{0, 11\} \cup \{13, 15, 17, 20, 29, 50\} \cup \{22, 24\} \subseteq J_3[25] \) and \( \{42\} \nsubseteq J_3[25] \).
5. \( I[24] \cup \{27, 28, 33, 37, 39, 63\} \subseteq J_3[28] \).
6. \( I[37] \setminus \{18, 19, 78, 79, 81, 87, 102, 103, 111\} \cup \{21, 32\} \cup \{34, 36\} \cup \{38, 43\} \cup \{45, 48\} \cup \{52, 54\} \cup \{58, 63\} \cup \{67, 71\} \) \( \subseteq J_3[37] \).

## 2 Necessary conditions

In this section we establish necessary conditions for \( J_3[v] \). For this purpose, we use another concept that is relative to intersection problem: A \((v, k, t)\) trade of volume \( s \) consists of two disjoint collections \( T_1 \) and \( T_2 \), each of \( s \) blocks, such that for every \( t \)-subset of blocks, the number of blocks containing these elements \((t\)-subset\) are the same in both \( T_1 \) and \( T_2 \). A \((v, k, t)\) trade of volume \( s \) is Steiner when for every \( t \)-subset of blocks, the number of blocks containing these elements are at most one. A \( \mu \)-way \((v, k, t)\) trade \( T = \{T_1, T_2, \ldots, T_{\mu}\} \), \( \mu \geq 2 \) is a set of pairwise disjoint \((v, k, t)\) trade.

In every collection the union of blocks must cover the same set of elements. This set of elements is called the foundation of the trade. Its notation is found \((T)\) and \( r_x = \) no. of blocks in a collection which contain the element \( x \).

By definition of the trade, if \( b_v - s \) is in \( J_3[v] \), then it is clear that there exists a 3-way Steiner \((v, 4, 2)\) trade of volume \( s \). Consider three \( S(2, 4, v) \) designs (systems) intersecting in \( b_v - s \) same blocks (of size four). The remaining set of blocks (of size four) form disjoint partial quadruple systems, containing
precisely the same pairs, and each has \( s \) blocks. Rashidi and Soltankhah in [10] established that there do not exist a 3-way Steiner \((v, 4, 2)\) trade of volume \( s \), for \( s \in \{1, 2, 3, 4, 5, 6, 7\} \). So we have the following lemma:

**Lemma 2.1** \( J_4[v] \subseteq I_3[v] \).

### 3 Recursive constructions

In this section we give some recursive constructions for the 3-way intersection problem. The concept of GDDs plays an important role in these constructions. Our aim of common blocks is the same common blocks in the sequel.

Let \( K \) be a set of positive integers. A group divisible design \( K \)-GDD (as GDD for short) is a triple \((\mathcal{X}, \mathcal{G}, \mathcal{A})\) satisfying the following properties: (1) \( \mathcal{G} \) is a partition of a finite set \( \mathcal{X} \) into subsets (called groups); (2) \( \mathcal{A} \) is a set of subsets of \( \mathcal{X} \) (called blocks), each of cardinality from \( K \), such that a group and a block contain at most one common element; (3) every pair of elements from distinct groups occurs in exactly one block.

If \( \mathcal{G} \) contains \( u_i \) groups of size \( g_i \), for \( 1 \leq i \leq s \), then we denote by \( g_1^{a_1} g_2^{a_2} \ldots g_s^{a_s} \) the group type (or type) of the GDD. If \( K = \{k\} \), we write \( \{k\} \)-GDD as \( k \)-GDD. A \( k \)-GDD of type \( 1^s \) is commonly called a pairwise balanced design, denoted by \((v, k, 1)\)-PBD. When \( K = \{k\} \) a PBD is just a Steiner system \( S(2, k, v) \).

The following construction is a variation of Willson’s Fundamental Construction.

**Theorem 3.1** (Weighting construction). Let \((\mathcal{X}, \mathcal{G}, \mathcal{A})\) be a GDD with groups \( G_1, G_2, \ldots, G_s \). Suppose that there exists a function \( w : X \to \mathbb{Z}^+ \cup \{0\} \) (a weight function) so that for each block \( A = \{x_1, \ldots, x_k\} \in \mathcal{A} \) there exist three \( K \)-GDDs of type \([w(x_1), \ldots, w(x_k)]\) with \( b_A \) common blocks. Then there exist three \( K \)-GDDs of type \( [\sum_{x \in G_1} w(x), \ldots, \sum_{x \in G_s} w(x)] \) which intersect in \( \sum_{A \in \mathcal{A}} b_A \) blocks.

**proof.** For every \( x \in \mathcal{X} \), let \( S(x) \) be a set of \( w(x) \) “copies” of \( x \). For any \( \mathcal{Y} \subset \mathcal{X} \), let \( S(\mathcal{Y}) = \bigcup_{y \in \mathcal{Y}} S(y) \). For every block \( A \in \mathcal{A} \), there exist three \( K \)-GDDs: \((S(A), \{S(x) : x \in A\}, B_A), (S(A), \{S(x) : x \in A\}, \hat{B}_A), (S(A), \{S(x) : x \in A\}, \check{B}_A)\), which intersect in \( b_A \) blocks. Then it is readily checked that there exist three, \( K \)-GDDs: \((S(\mathcal{X}), \{S(G) : G \in \mathcal{G}\}, \cup_{A \in \mathcal{A}} B_A), (S(\mathcal{X}), \{S(G) : G \in \mathcal{G}\}, \cup_{A \in \mathcal{A}} \hat{B}_A), (S(\mathcal{X}), \{S(G) : G \in \mathcal{G}\}, \cup_{A \in \mathcal{A}} \check{B}_A)\), which intersect in \( \sum_{A \in \mathcal{A}} b_A \) blocks. \( \square \)

**Theorem 3.2** (Filling construction (ii)). Suppose that there exist three \( 4 \)-GDDs of type \( g_1 g_2 \ldots g_s \) which intersect in \( b \) blocks. If there exist three
\(S(2,4,g_i+1)\) designs with \(b_i\) common blocks for \(1 \leq i \leq s\), then there exist three \(S(2,4,\sum_{i=1}^{s} g_i + 1)\) designs with \(b + \sum_{i=1}^{s} b_i\) common blocks.

**proof.** It is obvious. \[\blacksquare\]

**Theorem 3.3** (Filling construction (ii)). Suppose that there exist three 4-GDDs of type \(g_1 g_2 \ldots g_s\) which intersect in \(b\) blocks. If there exist three \(S(2,4,g_i+4)\) designs containing \(b_i\) common blocks for \(1 \leq i \leq s\). Also all designs containing a block \(y\). Then there exist three \(S(2,4,\sum_{i=1}^{s} g_i + 4)\) designs with \(b + \sum_{i=1}^{s} b_i - (s - 1)\) common blocks.

**proof.** Let \((\mathcal{X}, \mathcal{G}, A_1)\), \((\mathcal{X}, \mathcal{G}, A_2)\) and \((\mathcal{X}, \mathcal{G}, A_3)\) be three 4-GDDs of type \(g_1 g_2 \ldots g_s\) which intersect in \(b\) blocks. Let \(\mathcal{Y} = \{y_1, y_2, y_3, y_4\}\) be a set of cardinality 4 such that \(\mathcal{X} \cap \mathcal{Y} = \phi\).

For \(1 \leq i \leq s\), there exist three \(S(2,4,g_i+4)\) designs \((g_i \cup \mathcal{Y}, \varepsilon_{1i}), (g_i \cup \mathcal{Y}, \varepsilon_{2i})\) and \((g_i \cup \mathcal{Y}, \varepsilon_{3i})\) containing the same block \(y = y_1, y_2, y_3, y_4\) with \(b_i\) common blocks. It is easy to see that \((\mathcal{X} \cup \mathcal{Y}, A_1 \cup (\bigcup_{1 \leq i \leq s-1} (\varepsilon_{1i} - y))) \cup \varepsilon_{14}\), \((\mathcal{X} \cup \mathcal{Y}, A_2 \cup (\bigcup_{1 \leq i \leq s-1} (\varepsilon_{2i} - y))) \cup \varepsilon_{24}\) and \((\mathcal{X} \cup \mathcal{Y}, A_3 \cup (\bigcup_{1 \leq i \leq s-1} (\varepsilon_{3i} - y))) \cup \varepsilon_{34}\) are three \(S(2,4,\sum_{i=1}^{s} g_i + 4)\) designs with \(b + \sum_{i=1}^{s} b_i - (s - 1)\) common blocks.

We apply another type of recursive constructions that explained in the following.

Let there be three \(S(2,4,v)\) designs with a common parallel class, then \(J_{p,v}\) for \(v \equiv 4\) (mod 12) denotes the number of blocks shared by these \(S(2,4,v)\) designs, in addition to those shared in the parallel class.

**Lemma 3.4** Let \(G\) be a GDD on \(v = 3s + 6t\) elements with \(b\) blocks of size 4 and group type \(3^s 6^t\). Then there exist three \(S(2,4,4v + 4)\) designs with precisely \(\sum_{i=1}^{b} a_i + \sum_{i=1}^{s} c_i + \sum_{i=1}^{t} d_i\) blocks in common.

**proof.** The proof is similar to Lemma 3.3 in [8]. \[\blacksquare\]

The *flower* of an element is the set of blocks containing that element. Let \(J_{p,v}\) denote the number of blocks shared by three \(S(2,4,v)\) designs, in addition to those in a required common flower.

**Lemma 3.5** Let \(G, B\) be a GDD of order \(v\) with \(b_i\) blocks of size 4, \(b_5\) blocks of size 5 and group type \(4^s 5^t\). Then there exist three \(S(2,4,3v + 1)\) designs intersecting in precisely \(\sum_{i=1}^{b_4} a_i + \sum_{i=1}^{b_5} c_i + \sum_{i=1}^{s} d_i + \sum_{i=1}^{t} e_i\) blocks.

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proof. The proof is similar to Lemma 3.5 in [8].

Lemma 3.6. The necessary and sufficient conditions for the existence of a 4-GDD of type \( g^n \) are: (1) \( n \geq 4 \), (2) \( (n - 1)g \equiv 0 \pmod{3} \), (3) \( n(n - 1)g^2 \equiv 0 \pmod{12} \), with the exception of \((g, n) \in \{(2, 4), (6, 4)\}\), in which case no such GDD exists.

Lemma 3.7. There exists a \((v, \{4, 7\}, 1)\)-PBD with exactly one block of size 7 for any positive integer \( v \equiv 7, 10 \pmod{12} \) and \( v \neq 10, 19 \).

Lemma 3.8. A 4-GDD of type \( 12^u m^1 \) exists if and only if either \( u = 3 \) and \( m = 12 \), or \( u \geq 4 \) and \( m \equiv 0 \pmod{3} \) with \( 0 \leq m \leq 6(u - 1) \).

4 Ingredients

In this section we discuss some small cases needed for general constructions.

Lemma 4.1. \( J_3[13] = J_3[13] \).

proof. Construct an \( S(2, 4, 13) \) design, \((\mathcal{V}, \mathcal{B})\) with \( \mathcal{V} = \mathbb{Z}_{10} \cup \{a, b, c\} \). All blocks of \( \mathcal{B} \) are listed in the following, which can be found in Example 1.26 in [9].

\[
\begin{align*}
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 5 \\
1 & 2 & 4 & 6 & 2 & 5 & 7 & 3 & 6 & 4 & 7 & 8 & 9 \\
3 & 8 & 5 & a & 4 & 6 & b & 5 & 7 & 6 & 8 & 9 & a \\
9 & c & 7 & b & a & 8 & c & b & 9 & c & a & b & c
\end{align*}
\]

Consider the following permutations on \( \mathcal{V} \).

| \( \pi_1 \) | \( \pi_2 \) | \( \pi_3 \) | int. no. |
|-----------|-----------|-----------|---------|
| id        | (0, 1, 2, 3, 4, 5) | (5, 4, 3, 2, 1, 0) | 0       |
| id        | (8, 5)(a, b)(3, 7)(1, 6) | (8, 6)(1, 5)(a, 3, b, 7) | 1       |
| id        | (7, b, c, 6) | (8, 5, 6, 7) | 2       |
| id        | (4, 7)(9, 2)(1, 8) | (3, 9, c)(1, 8) | 3       |
| id        | (3, 7)(c, 0, 2)(1, 6)(9, b) | (9, 4)(3, 7)(0, 2)(1, 6) | 4       |
| id        | (a, b)(4, 5) | (a, b)(c, 8) | 5       |
| id        | id | id | 13      |

Lemma 4.2. \( J_3[16] = J_3[16] \setminus \{7, 9, 10, 11, 12\} \).
proof. The proof has three steps:

(1) $J_3[16] \subseteq J[16] = \{0, 1, 2, 3, 4, 5, 6, 8, 12, 20\}$.

(2) Construct an $S(2, 4, 16)$ design, $(\mathcal{V}, \mathcal{B})$ with $\mathcal{V} = \mathcal{Z}_{10} \cup \{a, b, c, d, e, f\}$. All 20 blocks of $\mathcal{B}$ are listed in the following, which can be found in Example 1.31 in [9].

\[
\{0, 1, 2, 3\}, \{0, 4, 5, 6\}, \{0, 7, 8, 9\}, \{0, a, b, c\}, \{0, d, e, f\}, \{1, 4, 7, a\}, \{1, 5, b, d\}, \{1, 6, 8, e\}, \{1, 9, c, f\}, \{2, 4, c, e\}, \{2, 5, 7, f\}, \{2, 6, 9, b\}, \{2, 8, a, d\}, \{3, 4, 9, d\}, \{3, 5, 8, c\}, \{3, 6, a, f\}, \{3, 7, b, e\}, \{4, 8, b, f\}, \{5, 9, a, e\}, \{6, 7, c, d\}.
\]

Consider the following permutations on $\mathcal{V}$.

| $\pi_1$ | $\pi_2$ | $\pi_3$ | int. no. |
|---------|---------|---------|---------|
| id      | (0, 1, 4, 9, d)(6, 5)(b, f) | (2, 3, 7, 6, e)(5, 8)(a, b)(f, 4) | 0       |
| id      | (0, 1, 2, 3)(8, 9, 5, b, c) | (d, f, c, a)(4, 6, 7)(c, 9) | 1       |
| id      | (1, 2)(a, b)(7, f)(c, e)(6, 8) | (a, b, 7, f, c, e, 6, 8)(4, d) | 2       |
| id      | (c, e)(5, 6)(b, f, 7, d, 9) | (8, b, 7, a, d, 9, f)(1, 3) | 3       |
| id      | (0, 1, 2, 3) | (4, 8, b, f) | 4       |
| id      | (4, f)(d, 7)(a, 9, 5) | (7, c, d)(f, b, 4) | 5       |
| id      | (0, 1, 2)(a, e) | (f, b)(1, 0, 2) | 6       |
| id      | (6, 7, c) | (6, e, 7) | 8       |
| id      | id | id | 20      |

Hence we have $\{0, 1, 2, 3, 4, 5, 6, 8, 20\} \subseteq J_3[16]$.

(3) $b_{16} - 8 \notin J_3[16]$:
If $b_{16} - 8 = 20 - 8 = 12 \in J_3[16]$, then a 3-way $(v, 4, 2)$ trade of volume 8 is contained in the $S(2, 4, 16)$ design. Let $T$ be this trade.
If all elements in found (T) appear 3 times in $T_1$, then for one block as $a_1a_2a_3a_4$, there exist 8 more blocks, so $|T_1| \geq 9$. Hence there exists $x \in$ found (T), with $r_x = 2$. Without loss of generality, let $xa_1a_2a_3$ and $xb_1b_2b_3$ be in $T_1$. But $T$ is Steiner trade so there exist (for example): $xb_1a_2a_3$ and $xa_1b_2b_3$ in $T_2$ and there exist $xa_1b_3a_2$ and $xb_1b_2a_3$ in $T_3$. Now $T_1$ must contains at least 6 pairs: $a_1b_2$, $a_1b_3$, $a_2b_1$, $a_3b_1$, $a_3b_3$, $a_2b_2$ which those come in disjoint blocks, since $T$ is Steiner. So we have:
Consider the following permutations on $V$.

| $T_1$   | $T_2$   | $T_3$   |
|---------|---------|---------|
| $xa_1a_2a_3$ | $xb_1a_2a_3$ | $xa_1b_3a_3$ |
| $xb_1b_2b_3$ | $xa_1b_2b_3$ | $xb_1b_2a_2$ |
| $a_1b_2$     | $a_1b_2$     | $a_1b_2$     |
| $a_1b_3$     | $a_1b_3$     | $a_1b_3$     |
| $a_2b_1$     | $a_2b_1$     | $a_2b_1$     |
| $a_2b_2$     | $a_2b_2$     | $a_2b_2$     |
| $a_3b_1$     | $a_3b_1$     | $a_3b_1$     |
| $a_3b_3$     | $a_3b_3$     | $a_3b_3$     |

We know that the $S(2,4,16)$ design is unique (See [11]). Without loss of generality, we can assume $x, a_1, a_2, a_3 = 0,1,2,3$ and $x, b_1, b_2, b_3 = 0,4,5,6$ (two blocks of the $S(2,4,16)$ design). Hence $T$ has the following form:

| $T_1$   | $T_2$   | $T_3$   |
|---------|---------|---------|
| 0123    | 0423    | 0163    |
| 0456    | 0156    | 0452    |
| 15bd    |        |         |
| 168e    |        |         |
| 24ce    |        |         |
| 257f    |        |         |
| 349d    |        |         |
| 36af    |        |         |

Therefore $r_7 = 1$, and by Lemma 3 in [11] this is impossible. ■

**Lemma 4.3** \{0,23,29,50\} $\subseteq J_3[25]$.

**proof.** Construct an $S(2,4,25)$ design, $(V,B)$ with $V = Z_{10} \cup \{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o\}$. All 50 blocks of $B$ are listed in the following, which can be found in Example 1.34 in [9].

\{0,1,2,i\}, \{0,l,3,6\}, \{0,4,8,a\}, \{0,a,5,9\}, \{0,7,g,h\}, \{0,b,d,n\}, \{0,c,f,g\}, \{0,k,m,e\}, \{1,3,a,b\}, \{1,4,7,m\}, \{1,5,6,o\}, \{1,8,f,h\}, \{1,9,e,l\}, \{1,c,k,n\}, \{1,d,g,j\}, \{2,3,7,o\}, \{2,4,b,9\}, \{2,8,5,n\}, \{2,6,f,g\}, \{2,a,c,m\}, \{2,d,k,l\}, \{2,e,h,9\}, \{3,4,5,j\}, \{3,8,c,d\}, \{3,9,k,f\}, \{3,c,g,n\}, \{3,h,i,m\}, \{4,6,d,e\}, \{4,a,g,k\}, \{h,4,c,l\}, \{4,i,n,f\}, \{5,7,c,e\}, \{5,b,h,k\}, \{5,d,f,m\}, \{5,g,i,l\}, \{6,7,8,k\}, \{6,9,c,i\}, \{6,a,h,n\}, \{6,b,j,m\}, \{9,7,j,n\}, \{7,a,d,i\}, \{7,b,f,l\}, \{m,g,8,9\}, \{8,a,j,l\}, \{8,b,e,i\}, \{9,d,h,o\}, \{a,e,f,o\}, \{b,c,g,o\}, \{i,j,k,o\}, \{l,m,n,o\}. Consider the following permutations on $V$. 

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Hence we have \( \{23, 29, 50\} \subset J_3[25] \).
By taking the 5th, 6th and 8th designs, of Table 1.34 in [9], we have \( 0 \in J_3[25] \).

**Lemma 4.4** \( \{1, 63\} \subset J_3[28] \).

**proof.** \( 63 \in J_3[28] \) by taking an \( S(2, 4, 28) \) design thrice. We obtain \( 1 \in J_3[28] \), by applying the following permutations on the design of Lemma 6.3 (Step 1) in the last section.
\( \pi_1 \) is identity, \( \pi_2 = (0, 1, 3, 5, 6, 7, 12, 17, 15, 18, 19, 20, 11)(25, 26, 27) \), and \( \pi_3 = \pi_2^{-1} \).

**Lemma 4.5** There exist three 4-GDDs of type \( 4^4 \) with \( i \) common blocks, \( i \in \{0, 1, 2, 4, 16\} \).

**proof.** Take the \( S(2, 4, 16) \) design, \( (V, B) \) constructed in Lemma 4.2. Consider the parallel class \( P = \{\{0, 1, 2, 3\}, \{4, 8, b, f\}, \{5, 9, a, e\}, \{6, 7, c, d\}\} \) as the groups of GDD to obtain a 4-GDD of type \( 4^4 \) \( (X, G, B') \), where \( X = V \), \( G = P \) and \( B' = B \setminus P \).
Consider the following permutations on \( X \), which keep \( G \) invariant.

| \( \pi_1 \) | \( \pi_2 \) | \( \pi_3 \) | int. no. |
|---|---|---|---|
| id | \( (6, 7, c) \) | id | 16 |
| id | \( (0, 1, 2)(a, e) \) | \( (f, b)(1, 0, 2) \) | 4 |
| id | \( (4, f)(d, 7)(a, 9, 5) \) | \( (7, c, d)(f, b, 4) \) | 2 |
| id | \( (0, 1, 2, 3) \) | \( (4, 8, b, f) \) | 1 |

In fact \( J_{p3}[16] \) is precisely the intersection sizes of three 4-GDDs of group type \( 4^4 \) having all groups in common.

**Corollary 4.6** \( \{0, 1, 2, 4, 16\} \subset J_{p3}[16] \).

**Lemma 4.7** There exist three 4-GDDs of type \( 3^5 \) with \( i \) common blocks, \( i \in \{0, 1, 3, 15\} \).

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proof. Take the $S(2,4,16)$ design, $(\mathcal{V}, \mathcal{B})$ constructed in Lemma 4.2. Delete the element 0 from this design to obtain a 4-GDD of type $3^5 (\mathcal{X}, \mathcal{G}, \mathcal{B'})$, where $\mathcal{X} = \mathcal{V} \setminus \{0\}$,

$\mathcal{G} = \{\{1,2,3\}, \{4,5,6\}, \{7,8,9\}, \{a,b,c\}, \{d,e,f\}\}$ and $\mathcal{B'} = \mathcal{B} \setminus \{B \in \mathcal{B} : 0 \in B\}$. Consider the following permutations on $\mathcal{X}$, which keep $\mathcal{G}$ invariant.

| $\pi_1$ | $\pi_2$ | $\pi_3$ | int. no. |
|--------|--------|--------|--------|
| id     | id     | id     | 15     |
| id     | (a,c)(1,3)(6,5) | (7,9)(d,e)(2,3)(a,c) | 15     |
| id     | (2,3)(5,6)(7,8)(a,c)(d,f) | (1,2)(4,6)(8,9)(a,c)(d,e) | 15     |

If we delete $d$ then we have a 4-GDD of type $3^5 (\mathcal{X}, \mathcal{G}, \mathcal{B'})$, where $\mathcal{X} = \mathcal{V} \setminus \{d\}$,

$\mathcal{G} = \{\{6,7,c\}, \{2,8,a\}, \{1,5,b\}, \{3,4,9\}, \{0,e,f\}\}$ and $\mathcal{B'} = \mathcal{B} \setminus \{B \in \mathcal{B} : d \in B\}$. When the following permutations act on $\mathcal{X}$ then we obtain 3 as intersection number.

$\pi_1 = \text{id}$, $\pi_2 = (6,7,c)$, $\pi_3 = (6,c,7)$.

We have $\{0,1,3,15\} \subseteq J_{f3}[16]$ because $J_{f3}[16]$ is precisely the intersection sizes of three 4-GDDs of group type $3^5$ having all groups in common.

Lemma 4.8 There exist three 4-GDDs of type $3^4$ with $i$ common blocks, $i \in \{0,1,9\}$.

proof. Take the $S(2,4,13)$ design, $(\mathcal{V}, \mathcal{B})$ constructed in Lemma 4.1. Delete the element 0 from the design to obtain a 4-GDD of type $3^4 (\mathcal{X}, \mathcal{G}, \mathcal{B'})$, where $\mathcal{X} = \mathcal{V} \setminus \{0\}$,

$\mathcal{G} = \{\{1,3,9\}, \{2,8,c\}, \{4,5,7\}, \{6,a,b\}\}$ and $\mathcal{B'} = \mathcal{B} \setminus \{B \in \mathcal{B} : 0 \in B\}$. Consider the following permutations on $\mathcal{X}$, which keep $\mathcal{G}$ invariant.

| $\pi_1$ | $\pi_2$ | $\pi_3$ | int. no. |
|--------|--------|--------|--------|
| id     | id     | id     | 9      |
| id     | (a,b)(4,5) | (a,b)(8,9) | 1      |

If we delete 8 then we have a 4-GDD of type $3^4 (\mathcal{X}, \mathcal{G}, \mathcal{B'})$, where $\mathcal{X} = \mathcal{V} \setminus \{8\}$,

$\mathcal{G} = \{\{0,2,c\}, \{1,5,6\}, \{3,7,a\}, \{b,4,9\}\}$ and $\mathcal{B'} = \mathcal{B} \setminus \{B \in \mathcal{B} : 8 \in B\}$. When the following permutations act on $\mathcal{X}$ then we obtain 0 as intersection number.

$\pi_1 = \text{id}$, $\pi_2 = (3,7)(c,0,2)(1,6)(9,b)$, $\pi_3 = (9,4)(3,7)(0,2)(1,6)$.

We obtain $\{0,1,9\} \subseteq J_{f3}[13]$ because $J_{f3}[13]$ is precisely the intersection sizes of three 4-GDDs of group type $3^4$ having all groups in common.
sizes of three 4-GDDs of group type 3⁴ having all groups in common.

Corollary 4.9 \( \{0, 1, 9\} \subseteq J_{3}[13] \).

5 Applying the recursions

In this section, we prove the main theorem for all \( v \geq 40 \). First we treat the (easier) case \( v \equiv 1 \) (mod 12).

**Theorem 5.1** For any positive integer \( v = 12u + 1, \ u \equiv 0, 1 \) (mod 4) and \( u \geq 4 \), \( J_{3}[v] = I_{3}[v] \).

**proof.** Start from a 4-GDD of type 3⁴ from Lemma 3.6. Give each element of the GDD weight 4. By Lemma 4.5 there exist three 4-GDDs of type 4⁴ with \( \alpha \) common blocks, \( \alpha \in J_{p3}[16] \). Then apply construction 3.1 to obtain three 4-GDDs of type 12 with \( \sum_{i=1}^{b} \alpha_i \) common blocks, where \( b = \frac{3u(u-1)}{4} \) and \( \alpha_i \in J_{p3}[16] \) for \( 1 \leq i \leq b \). By construction 3.2 filling in the groups by three \( S(2, 4, 13) \) designs with \( \beta_j(1 \leq j \leq u) \) common blocks, we have three \( S(2, 4, 12u + 1) \) designs with \( \sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{u} \beta_j \) common blocks, where \( \beta_j \in J_{3}[13] \) for \( 1 \leq j \leq u \). It is checked that for any integer \( n \in I_{3}[v] \), \( n \) can be written as the form of \( \sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{u} \beta_j \), where \( \alpha_i \in J_{p3}[16] \) and \( \beta \in J_{3}[25] \).

**Theorem 5.2** For any positive integer \( v = 12u + 1, \ u \equiv 2, 3 \) (mod 4) and \( u \geq 7 \), \( J_{3}[v] = I_{3}[v] \).

**proof.** There exists a \( (3u + 1, \{4, 7^*\}, 1) \)-PBD from Lemma 3.7 which contains exactly one block of size 7. Take an element from this block of size 7. Delete this element to obtain a 4-GDD of type 3⁴⁻²6¹. Give each element of the GDD weight 4. By Lemma 4.5 there exist three 4-GDDs of type 4⁴ with \( \alpha \) common blocks, \( \alpha \in J_{p3}[16] \). Then apply construction 3.1 to obtain three 4-GDDs of type 12 with \( \sum_{i=1}^{b} \alpha_i \) common blocks, where \( b = \frac{3u^2-u-2}{4} \) and \( \alpha_i \in J_{p3}[16] \) for \( 1 \leq i \leq b \). By construction 3.2 filling in the groups by three \( S(2, 4, 13) \) designs with \( \beta_j(1 \leq j \leq u - 2) \) common blocks, and three \( S(2, 4, 25) \) designs with \( \beta \) common blocks, we have three \( S(2, 4, 12u + 1) \) designs with \( \sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{u-2} \beta_j + \beta \) common blocks, where \( \beta_j \in J_{3}[13] \) for \( 1 \leq j \leq u - 2 \) and \( \beta \in J_{3}[25] \). It is checked that for any integer \( n \in I_{3}[v] \), \( n \) can be written as the form of \( \sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{u-2} \beta_j + \beta \), where \( \alpha_i \in J_{p3}[16] \) and \( \beta_j \in J_{3}[13] \) and \( \beta \in J_{3}[25] \).
Theorem 5.3 \( J_3[73] = I_3[73] \).

**Proof.** Start from an \( S(2, 5, 25) \) design. Delete an element from this design to obtain a 5-GDD of type \( 4^6 \). Give each element of the GDD weight 3. By Lemma 4.7 there exist three 4-GDDs of type \( 3^5 \) with \( \alpha \) common blocks, \( \alpha \in \{0, 1, 3, 15\} \). Then apply construction 5.1 to obtain three 4-GDDs of type \( 12^6 \) with \( \sum_{i=1}^{24} \alpha_i \) common blocks, where \( \alpha_i \in \{0, 1, 3, 15\} \) for \( 1 \leq i \leq 24 \). By construction 3.2 filling in the groups by three \( S(2, 4, 13) \) designs with \( B_j (1 \leq j \leq 6) \) common blocks, \( \beta_j \in J_3[13] \) we have three \( S(2, 4, 73) \) designs with \( \sum_{i=1}^{24} \alpha_i + \sum_{j=1}^{6} \beta_j \) common blocks. It is checked that for any integer \( n \in I_3[73] \), \( n \) can be written as the form of \( \sum_{i=1}^{24} \alpha_i + \sum_{j=1}^{6} \beta_j \). \( \blacksquare \)

For the case \( v = 12u + 4 \) we have the following Theorems:

Theorem 5.4 \( I_3[40] \setminus \{b_{40} - 15, b_{40} - 14\} \subseteq J_3[40] \).

**Proof.** We use \( "v \to 3v + 1" \) rule. (See 13). Let \( (\mathcal{V}, \mathcal{B}) \) be an \( S(2, 4, v) \) design, and let \( \mathcal{V}' \) be a set such that \( |\mathcal{V}'| = 2v + 1 \), \( \mathcal{V}' \cap \mathcal{V} = \phi \). Let \( (\mathcal{V}', \mathcal{C}) \) be a resolvable \( STS(2v + 1) \) and let \( \mathcal{R} = \{R_1, \ldots, R_v\} \) be a resolution of \( (\mathcal{V}', \mathcal{C}) \), that is, let \( (\mathcal{V}', \mathcal{C}, \mathcal{R}) \) be a Kirkman triple system of order \( 2v + 1 \); since \( v \equiv 1, 4 \pmod{12} \), such a system exists. Form the set of quadruples \( D_i = \{\{v_i, x, y, z\} : v_i \in \mathcal{V}, \{x, y, z\} \in R_i\} \), and put \( \mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \). Then \( (\mathcal{V} \cup \mathcal{V}', \mathcal{B} \cup \mathcal{D}) \) is an \( S(2, 4, 3v + 1) \) design.

Now let \( v = 13 \) and \( (\mathcal{V}', \mathcal{C}) \) be a \( KTS(27) \) containing three disjoint Kirkman triple systems of order 9. Let \( R_1, \ldots, R_4, R_5, \ldots, R_{13} \) are the 13 parallel classes of the \( KTS(27) \) so that \( R_1, \ldots, R_4 \) each induce parallel classes in the three \( KTS(9) \)'s. We add 13 elements \( a_1, \ldots, a_4, b_1, \ldots, b_9 \) to this \( KTS(27) \) and form blocks by adding \( a_i \) to each triple in \( R_i \) (\( i = 1, \ldots, 4 \)) and \( b_i \) to each triple in \( R_i \) (\( i \geq 5 \)). Finally, place an \( S(2, 4, 13) \) design on the 13 new elements. Consider each ingredient in turn. On the \( S(2, 4, 13) \) design we can get any intersection size from \( J_3[13] \). On the \( (b_i, R_i) \) blocks, we can permute the \( R_i \) to obtain intersection numbers \( \{0, 9, 18, 27, 36, 45, 54, 81\} \). We do not have 63 in this set because there exist three designs for intersection and we must permute at least three parallel classes. On the \( (a_i, R_i) \) blocks, we can permute the parallel classes of each of the \( KTS(9) \)'s to obtain intersection numbers \( \{0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 36\} \). It is checked that for any integer \( n \in I_3[40] \), \( n \) can be written as the sum of these numbers except \( b_{40} - 16, b_{40} - 15, \) and \( b_{40} - 14 \).

For \( b_{40} - 16 \):

Start from a 4-GDD of type \( 3^4 \) from Lemma 5.8. Give each element of the GDD weight 3. By Lemma 4.8 there exist three 4-GDDs of type \( 3^4 \) with \( \alpha \) common blocks, \( \alpha \in \{0, 1, 9\} \). Then apply construction 5.1 to obtain three 4-GDDs of type \( 9^9 \) with \( \sum_{i=1}^{9} \alpha_i \) common blocks, where \( \alpha_i \in \{0, 1, 9\} \) for \( 1 \leq i \leq 9 \). By construction 3.3 filling in the groups by three \( S(2, 4, 13) \) designs with \( \beta_j \) (\( 1 \leq j \leq 4 \)) common blocks, \( \beta_j \in J_3[13] \). We have three
$S(2, 4, v)$ designs with $\sum_{i=1}^{9} \alpha_i + \sum_{j=1}^{4} \beta_j - 3$ common blocks. $b_{40} - 16$ can be written as this form. 

\begin{proof}
Using a 5-GDD of type $5^5$ (the 2-(25,5,1) design itself): Give each element of the GDD weight 3. By Lemma 4.7 there exist three 4-GDDs of type $3^5$ with $\alpha$ common blocks, $\alpha \in \{0, 1, 3, 15\}$. Then apply construction 4.1 to obtain three 4-GDDs of type $15^5$ with $\sum_{i=1}^{25} \alpha_i$ common blocks, where $\alpha_i \in \{0, 1, 3, 15\}$ for $1 \leq i \leq 25$. By construction 3.2 filling in the groups by three $S(2, 4, 16)$ designs with $B_j (1 \leq j \leq 5)$ common blocks, we have three $S(2, 4, 76)$ designs with $\sum_{i=1}^{25} \alpha_i + \sum_{j=1}^{5} \beta_j$ common blocks, where $\alpha_i \in \{0, 1, 3, 15\}$ for $1 \leq i \leq 5$ and $\beta_j \in J_4[16]$. It is checked that for any integer $n \in I_5[76]$, $n$ can be written as the form of $\sum_{i=1}^{25} \alpha_i + \sum_{j=1}^{5} \beta_j$, except $b_{76} - 8, b_{76} - 9, b_{76} - 10, b_{76} - 11, b_{76} - 13, b_{76} - 21, b_{76} - 22, b_{76} - 23$. Now we must handle the remaining values. Rees and Stinson (See [17]) proved that if $v \equiv 1, 4 \pmod{12}$, $w \equiv 1, 4 \pmod{12}$ and $v \geq 3w + 1$, then there exists an $S(2, 4, v)$ design contains an $S(2, 4, w)$ subdesign. By taking all blocks not in the subdesign identically, and three copies of the subdesign intersecting in all but $s$ blocks we have that $b_v - s \in J_4[v]$ if $b_w - s \in J_4[v]$. Using this result with $w = 13$, we obtain intersection numbers $b_v - 8, b_v - 9, b_v - 10, b_v - 11, b_v - 13$ for $v \geq 40$. Similarly using $w = 25$, we obtain $b_v - 21 \in J_4[v]$ for $v \geq 76$. There exists a 4-GDD of type $12^4 15^1$ from Lemma 3.8 Filling in the groups with five $S(2, 4, 13)$ designs and one $S(2, 4, 16)$ design. Hence we have an $S(2, 4, 76)$ design. This design has five $S(2, 4, 13)$ subdesigns intersecting in a single element. By choosing suitable intersection sizes from $J_4[13]$ we can obtain $\{b_{76} - 22, b_{76} - 23\} \subset J_4[76]$.
\end{proof}

Lemma 5.6 (i) $\{b_v - 21, b_v - 22, b_v - 23, b_v - 25\} \subset J_3[v]$, for $v = 52$ and 64.
(ii) $\{b_v - 22, b_v - 23, b_v - 25\} \subset J_3[v]$, for $v = 88, 100$, and 112.

\begin{proof}
For $v = 52$, observe that there exists a GDD on 52 elements with block size 4 and group type $13^4$ (See [8]). Construct three $S(2, 4, 52)$ designs, take the blocks of GDD identically. Replace each of the four groups by three $S(2, 4, 13)$ designs. By choosing suitable intersection sizes from $J_4[13]$, we get $\{b_{52} - 21, b_{52} - 22, b_{52} - 23, b_{52} - 25\} \subset J_3[52]$. Consider $v = 64$. Let $G, B$ be a GDD on 21 elements with block size 4 and 5, and group type $5^1 4^1$ (See [8]). Apply Lemma 4.8 to produce $S(2, 4, 64)$ design. This design has four $S(2, 4, 13)$ subdesigns intersecting in a single element, and by choosing suitable intersection sizes from $J_3[13]$ we have

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\{b_{b_4} - 21, b_{b_4} - 22, b_{b_4} - 23, b_{b_4} - 25\} \subset J_3[64].

ii. There exists a 4-GDD of type 12u−15 \(1\) from Lemma 3.8 for \(u = 7, 8,\) and 9. Filling in the groups with \(S(2, 4, 13)\) designs and one \(S(2, 4, 16)\) design. Hence we have an \(S(2, 4, 12u + 4)\) design. This design has \(u - 1, S(2, 4, 13)\) subdesigns intersecting in a single element. By choosing suitable intersection sizes from \(J_3[13]\) we can obtain \(\{b_{12u+4} - 22, b_{12u+4} - 23, b_{12u+4} - 25\} \subset J_3[12u + 4],\) for \(u = 7, 8,\) and 9.

\[\text{Theorem 5.7} \quad \text{For any positive integer } v = 12u+4, u \equiv 0, 1 \pmod{4}, u \ge 4, J_3[v] = I_3[v]. \quad \text{proof.} \quad \text{There exists a 4-GDD of type } 3^u \text{ with } b = 3u(u-1) \text{ blocks. By Lemma 3.4 we have three } S(2, 4, 12u + 4) \text{ designs with } \sum_{i=1}^{b_4} \alpha_i + \sum_{j=1}^{u} \beta_j \text{ common blocks, where } \beta_j + 1 \in J_3[16] \text{ for } 1 \le j \le u-1, \beta_u \in J_3[16] \text{ and } \alpha_i \in J_3[16] \text{ for } 1 \le i \le b. \text{ This produce all values except } b_v - 8, b_v - 9, b_v - 10, b_v - 11, b_v - 13, b_v - 21, b_v - 22, b_v - 23, \text{ and } b_v - 25. \text{ By a similar argument as in Lemma 5.5 we have } \{b_v - 8, b_v - 9, b_v - 10, b_v - 11, b_v - 13\} \subset J_3[v](v \ge 40), \text{ and } b_v - 21 \in J_3[v](v \ge 76). \text{ But } b_v - 22, b_v - 23, \text{ and } b_v - 25: \text{ We know, } I_3[40] - \{b_{40} - 14, b_{40} - 15\} \subset J_3[40]. \text{ By Ress and Stinson theorem } \{b_v - 22, b_v - 23, b_v - 25\} \subset J_3[12u + 4], \text{ for all } u \text{ that } 12u + 4 \ge 3 \times 40 + 1 \Rightarrow u \ge 10. \text{ Hence it remains to prove that } \{b_v - 22, b_v - 23, b_v - 25\} \subset J_3[12u + 4] \text{ for } 4 \le u \le 9 (u \equiv 0, 1 \pmod{4}), \text{ that it is proved in Lemma 5.6.} \]

\[\text{Theorem 5.8} \quad \text{For any positive integer } v = 12u+4, u \equiv 2, 3 \pmod{4}, u \ge 7, J_3[v] = I_3[v]. \quad \text{proof.} \quad \text{By proof of Theorem 5.2, there exists a 4-GDD of type } 3^u - 2^6. \text{ From Lemma 3.4 we have three } S(2, 4, 12u + 4) \text{ designs with } \sum_{i=1}^{b_4} \alpha_i + \sum_{j=1}^{u} \beta_j + d_1 \text{ common blocks, where } \beta_j + 1 \in J_3[16] \text{ for } 1 \le j \le u - 3, \beta_{u-2} \in J_3[16], \alpha_i \in J_3[16] \text{ for } 1 \le i \le b \text{ and } d_1 + 1 \in J_3[28]. \text{ Like the previous case we have all intersection numbers except } b_v - 22, b_v - 23, \text{ and } b_v - 25: \text{ By a similar argument as in Theorem 5.7 we have } \{b_v - 22, b_v - 23, b_v - 25\} \subset J_3[12u + 4], \text{ for all } u \text{ that } 12u + 4 \ge 3 \times 40 + 1 \Rightarrow u \ge 10. \text{ Hence it remains to prove that } \{b_v - 22, b_v - 23, b_v - 25\} \subset J_3[12u + 4] \text{ for } 7 \le u \le 9 (u \equiv 2, 3 \pmod{4}), \text{ that it is proved in Lemma 5.6.} \]

6 Small Orders

Three small orders, \{25, 28, 37\}, remain. We use some techniques to determine situation of half of numbers that those can be as intersection numbers.
In the next example we discuss a method which may help in understanding a general method in the following theorems.

**Example 6.1** Construct an \(S(2,4,25)\) design, \((\mathcal{V}, \mathcal{B})\) with \(\mathcal{V} = \mathbb{Z}_{25}\). In this \(S(2,4,25)\) design, the elements \(\{1, 2, 3, 5, 6, 8, 9\}\) induce an \(STS(7)\). All blocks of \(\mathcal{B}\) are listed in the following, which can be found in [19] (design 17).

We divide these blocks to three parts \(A, B\) and \(C\). \(A\) contains the blocks that induce the \(STS(7)\). \(B\) contains the blocks that do not contain any element of the \(STS(7)\) and \(C\) contains the remanded blocks.

\[
A: \quad 1, 2, 3, 4 \quad 1, 5, 6, 7 \quad 1, 8, 9, 10 \quad 2, 5, 8, 11 \\
2, 6, 9, 14 \quad 3, 5, 9, 24 \quad 3, 6, 8, 22.
\]

\[
B: \quad 4, 10, 16, 25 \quad 4, 11, 17, 21 \quad 4, 15, 22, 24 \quad 7, 10, 19, 24 \\
7, 12, 14, 18 \quad 10, 13, 14, 22 \quad 11, 14, 20, 24 \quad 7, 11, 22, 23.
\]

\[
C: \quad 1, 11, 12, 13 \quad 1, 14, 15, 16 \quad 1, 17, 18, 19 \quad 1, 20, 21, 22 \quad 1, 23, 24, 25 \\
2, 7, 15, 17 \quad 2, 10, 12, 20 \quad 2, 13, 16, 23 \quad 2, 18, 21, 24 \quad 2, 19, 22, 25 \\
3, 7, 13, 25 \quad 3, 10, 11, 18 \quad 3, 12, 15, 19 \quad 3, 14, 21, 23 \quad 3, 16, 17, 20 \\
5, 14, 4, 19 \quad 5, 10, 17, 23 \quad 5, 12, 21, 25 \quad 5, 13, 15, 20 \quad 5, 16, 18, 22 \\
9, 4, 13, 18 \quad 9, 7, 16, 21 \quad 9, 11, 15, 25 \quad 9, 12, 17, 22 \quad 9, 19, 20, 23 \\
8, 4, 7, 20 \quad 8, 12, 16, 24 \quad 8, 13, 19, 21 \quad 8, 14, 17, 25 \quad 8, 15, 18, 23 \\
6, 4, 12, 23 \quad 6, 10, 15, 21 \quad 6, 11, 16, 19 \quad 6, 13, 17, 24 \quad 6, 18, 20, 25.
\]

Consider the permutation \(\pi = (1, 2, 3)(18, 17, 16, 13, 12, 11)\). This permutation consists of two parts the first part \(\pi_1 = (1, 2, 3)\) contains some elements of the \(STS(7)\) and the second part \(\pi_2 = (18, 17, 16, 13, 12, 11)\) does not contain any element of the \(STS(7)\). When \(\pi\) and \(\pi^{-1}\) act on \(A\) we have 1 as intersection number on \(A\), \(\pi(A)\), and \(\pi^{-1}(A)\).

| \(A\)   | \(\pi(A)\)  | \(\pi^{-1}(A)\) |
|--------|-------------|-----------------|
| 1, 2, 3, 4 | 1, 2, 3, 4 | 1, 2, 3, 4     |
| 1, 5, 6, 7 | 2, 5, 6, 7 | 3, 5, 6, 7     |
| 1, 8, 9, 10 | 2, 8, 9, 10 | 3, 8, 9, 10   |
| 2, 5, 8, 11 | 3, 5, 8, 18 | 1, 5, 8, 12   |
| 2, 6, 9, 14 | 3, 6, 9, 14 | 1, 6, 9, 14   |
| 3, 5, 9, 24 | 1, 5, 9, 24 | 2, 5, 9, 24   |
| 3, 6, 8, 22 | 1, 6, 8, 22 | 2, 6, 8, 22   |

But when \(\pi\) and \(\pi^{-1}\) act on \(B \setminus A\), we have 6 as intersection number on \(B \setminus A\), \(\pi(B \setminus A)\), and \(\pi^{-1}(B \setminus A)\). Then we get intersection number \(7 = 1 + 6\) on \(B\), \(\pi(B)\) and \(\pi^{-1}(B)\).
In fact we obtain two intersection numbers, the first number is obtained when \( \pi_1 \) and \( \pi_1^{-1} \) act on \( A \). The second number is obtained when \( \pi_2 \) and \( \pi_2^{-1} \) act on \( B \setminus A \). Then we add these numbers and obtain intersection number of \( B, \pi(B), \) and \( \pi^{-1}(B) \). Since the common blocks of \( A, \pi_1(A), \) and \( \pi_1^{-1}(A) \) do not contain any element of \( \pi_2 \) and common blocks of \( B \setminus A, \pi_2(B \setminus A), \) and \( \pi_2^{-1}(B \setminus A) \) do not contain any element of \( \pi \).

Note, we choose some permutations which change at most two elements of each block. Also the design is Steiner, so when the block \( b \) changes, it is commuted to different block from the other blocks. Hence by applying permutations, no new common block form in \( \pi(B) \) and \( \pi^{-1}(B) \). (We separate

| \( B \setminus A \) | \( \pi(B \setminus A) \) |
|---|---|
| 1, 11, 12, 13 | 1, 14, 15, 16 2, 17, 18, 19 1, 20, 21, 22 1, 23, 24, 25 |
| 2, 7, 15, 17 | 2, 10, 12, 20 2, 13, 16, 23 2, 18, 21, 24 2, 19, 22, 25 |
| 3, 7, 13, 25 | 3, 10, 11, 18 3, 12, 15, 19 3, 14, 21, 23 3, 16, 17, 20 |
| 5, 14, 4, 19 | 5, 10, 17, 23 5, 12, 21, 25 5, 13, 15, 20 5, 16, 18, 22 |
| 9, 4, 13, 18 | 9, 7, 16, 21 9, 11, 15, 25 9, 12, 17, 22 9, 19, 20, 23 |
| 8, 4, 7, 20 | 8, 12, 16, 24 8, 13, 19, 21 8, 14, 17, 25 8, 15, 18, 23 |
| 6, 4, 12, 23 | 6, 10, 15, 21 6, 11, 16, 19 6, 13, 17, 24 6, 18, 20, 25 |
| 4, 10, 16, 25 | 4, 11, 17, 21 | 4, 15, 22, 24 | 7, 10, 19, 24 | 7, 11, 22, 23 |
| 7, 12, 14, 18 | 10, 13, 14, 22 | 11, 14, 20, 24 | 11, 14, 20, 24 |

| \( \pi(B \setminus A) \) | \( \pi^{-1}(B \setminus A) \) |
|---|---|
| 2, 18, 11, 12 | 2, 14, 15, 13 2, 16, 17, 19 2, 20, 21, 22 2, 23, 24, 25 |
| 3, 7, 15, 16 | 3, 10, 11, 20 3, 12, 13, 23 3, 17, 21, 24 3, 19, 22, 25 |
| 1, 7, 12, 25 | 1, 10, 18, 17 1, 11, 15, 19 1, 14, 21, 23 1, 13, 16, 20 |
| 6, 4, 11, 23 | 6, 10, 15, 21 6, 18, 13, 19 6, 12, 16, 24 6, 17, 20, 25 |
| 5, 14, 4, 19 | 5, 10, 16, 23 5, 11, 21, 25 5, 12, 15, 20 5, 13, 17, 22 |
| 8, 11, 13, 24 | 8, 12, 19, 21 8, 14, 16, 25 8, 15, 17, 23 | 8, 4, 7, 20 |
| 9, 4, 12, 17 | 9, 7, 13, 21 9, 18, 15, 25 9, 11, 16, 22 | 9, 19, 20, 23 |
| 4, 10, 13, 25 | 4, 18, 16, 21 | 4, 15, 22, 24 | 7, 10, 19, 24 | 7, 18, 22, 23 |
| 7, 11, 14, 17 | 10, 12, 14, 22 | 18, 14, 20, 24 | 18, 14, 20, 24 |
We have \( 42 \in J_3[25] \) of Example 6.1 on \( \{5,8\} \cup \{11,13,15,17,20,24,29\} \) by the method of Example 6.1 on \( (V,B) \). Also we get these intersection numbers: \( 0,10 \cup \{22,23,29\} \), with applying straight permutations on \( B \).

We have \( 42 \notin J_3[25] \) since \( 42 \notin J_2[25] \). This completes proof.

**Lemma 6.3** \( [1,24] \cup \{27,28,33,37,39,63\} \subseteq J_3[28] \).

**proof.** We obtain these intersection numbers in two steps.

**Step 1:**
Construct an \( S(2,4,28) \) design, \( (V,B) \) with \( V = Z_{28} \). All blocks of \( B \) are listed in the following, which can be found in Theorem 20 in [14]. In this \( S(2,4,28) \) design the elements \( \{2,4,16,22,25,26,27\} \) induce an \( STS(7) \).

By a similar argument in Example 6.1 we obtain these intersection numbers: \( [2,7] \cup [10,12] \cup [16,19] \cup [21,24] \cup \{14,27,28,33,37,39\} \).

\[
\begin{array}{cccccccc}
2,0,1,3 & 4,5,6,7 & 16,17,18,19 & 22,20,21,23 & 25,7,8,1 \\
2,7,18,23 & 4,11,14,23 & 16,3,11,13 & 22,3,6,19 & 25,5,14,18 \\
2,10,12,17 & 4,0,8,12 & 16,0,5,20 & 22,5,10,15 & 25,9,19,20 \\
2,13,20,24 & 4,1,17,21 & 16,1,6,23 & 22,0,7,17 & 25,0,15,23 \\
2,5,19,21 & 4,15,19,24 & 16,9,15,21 & 22,9,12,18 & 25,6,13,17 \\
2,8,6,15 & 4,3,18,20 & 16,7,12,24 & 22,1,14,24 & 25,3,12,21 \\
26,1,15,18 & 27,3,15,17 & 8,9,10,11 & 0,6,9,24 & 24,25,26,27 \\
26,7,10,19 & 27,5,12,23 & 12,13,14,15 & 0,10,13,18 & 9,2,4,27 \\
26,9,17,23 & 27,1,10,20 & 3,7,9,14 & 1,11,12,19 & 11,2,22,25 \\
26,3,8,5 & 27,0,14,19 & 3,10,23,24 & 6,10,14,21 & 14,2,16,26 \\
26,12,6,20 & 27,11,6,18 & 5,11,17,24 & 7,11,15,20 & 10,4,16,25 \\
26,0,11,21 & 27,7,13,21 & 1,5,9,13 & 8,14,17,20 & 13,4,22,26 \\
8,16,22,27 & 8,13,19,23 & 8,18,21,24.
\end{array}
\]

**Step 2:**
Take an \( S(2,4,28) \) design, \( (V,B) \) with \( V = Z_{28} \). In this \( S(2,4,28) \) design, the elements \( \{4,5,6,13,14,15,19,20,21\} \) induce an \( STS(9) \). All blocks of \( B \) are listed in the following, which can be found in Theorem 21 in [14].
By a similar argument in Example 6.1, we obtain these intersection numbers \([5, 17] \cup [19, 21] \cup \{23, 24, 28, 33, 39\}\), in this step.

Also, we obtain 1 as intersection number in Lemma 4.3.

\[\text{Lemma 6.4} \quad \{18, 19, 78, 79, 81, 87, 102, 103, 111\} \cup [21, 32] \cup [34, 36] \cup [38, 43] \cup [45, 48] \cup [52, 54] \cup [58, 63] \cup [67, 71] \subseteq J_3[37]\]

\[\text{proof.} \quad \text{In this Lemma we have three steps.} \]

**Step 1:**
Take an \(S(2, 4, 37)\) design, \((\mathcal{V}, \mathcal{B})\) with \(\mathcal{V} = \{a_0, \ldots, a_8, b_0, \ldots, b_4, c_0, \ldots, c_8, d_0, \ldots, d_8, \infty\}\). Develop the following base blocks over \(\mathcal{Z}_9\) to obtain all blocks of \(\mathcal{B}\) (See [12]). In this \(S(2, 4, 37)\) design the elements \(\{a_0, a_3, a_6, b_0, b_3, b_6, c_0, c_3, c_6\}\) induce an \(STS(9)\).

\[\{\infty, a_0, a_3, a_6\}, \{\infty, b_0, b_3, b_6\}, \{\infty, c_0, c_3, c_6\}, \{\infty, d_0, d_3, d_6\} \]
\[\{a_0, a_3, b_0, c_0\}, \{a_0, a_6, b_3, c_6\}, \{a_0, a_7, d_0, d_1\}, \{a_0, b_0, b_4, c_3\}\]
\[\{a_1, c_3, c_8, d_0\}, \{a_2, c_6, c_7, d_0\}, \{a_3, b_8, d_0, d_7\}, \{a_4, b_2, b_3, d_0\}\]
\[\{b_0, c_1, d_0, d_2\}, \{b_5, b_7, c_0, d_0\}, \{b_6, c_2, c_4, d_0\}\]

By a similar argument in Example 6.1 we obtain these intersection numbers:
\[\{18, 19, 21, 22, 69, 70, 78, 81\} \cup [24, 32] \cup [34, 36] \cup [38, 43] \cup [45, 48] \cup [52, 54] \cup [60, 62]\]

**Step 2:**
Construct an \(S(2, 4, 37)\) design, \((\mathcal{V}, \mathcal{B})\) with \(\mathcal{V} = \mathcal{Z}_{11} \times \{1, 2, 3\} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}\). In this \(S(2, 4, 37)\) design the elements \(\{0_1, 1_1, 2_2, 10_2, 3_3, 4_3, 5_3\}\) induce an \(STS(7)\). Develop the following base blocks over \(\mathcal{Z}_{11}\) to obtain all blocks of \(\mathcal{B}\) (\(\infty_1, \infty_2, \infty_3\) and \(\infty_4\) are constants) (See [14]). By a similar argument in Example 6.1 we obtain these intersection numbers:
\{23, 26, 29, 32, 35, 36, 38, 39, 42, 43, 45, 47, 48, 53, 54, 60, 61, 68, 69, 78\}.

\{0_1, 0_2, 0_3, \infty\}, \{0_1, 1_2, 2_3, \infty\}, \{0_1, 2_2, 5_3, \infty\},
\{0_1, 8_2, 6_3, \infty\}, \{0_1, 1_1, 5_1, 10\}, \{0_2, 2_2, 5_2, 7_3\},
\{8_1, 0_3, 1_3, 5_3\}, \{0_1, 3_1, 6_2, 7_2\}, \{0_2, 4_2, 8_3, 10_3\},
\{2_1, 4_1, 0_3, 3_3\}

Also the design contains the block \{\infty, \infty, \infty, \infty\}.

Step 3: Take an \(S(2, 4, 37)\) design, \((\mathcal{V}, \mathcal{B})\) with \(\mathcal{V} = \{\infty\} \cup \{x, y, z\} \times \mathcal{Z}_{12}\). By developing the following base blocks over \(\mathcal{Z}_{12}\) we get the main part of the blocks (See [8]):

\{z_0, x_0, y_0, \infty\}, \{x_0, x_4, y_11, z_5\}, \{x_2, z_0, z_1, z_3\},
\{x_7, y_0, y_1, z_9\}, \{x_{10}, y_0, y_2, z_4\}, \{x_3, y_0, y_4, z_7\},
\{x_2, y_0, y_5, z_{10}\}, \{x_5, y_1, z_0, z_2\},
and the short orbits:
\{y_0, y_3, y_6, y_9\}, \{z_0, z_3, z_6, z_9\}.

Call the resulting set of 102 blocks \(\mathcal{B}\) and call the other set of blocks \(\mathcal{C}\). \(\mathcal{C}\) contains nine blocks which covers the remaining pairs. In fact \(\mathcal{C}\) comes from \(S(2, 4, 13)\) design with omitting one flower. This enable us to replace \(\mathcal{C}\) by a different set \(\mathcal{C}'\) or \(\mathcal{C}''\) of blocks covering the same pairs, So in this part we can have intersection number \(\mathcal{C} \cap \mathcal{C}' \cap \mathcal{C}''\). Recall that \(\mathcal{C} \cap \mathcal{C}' \cap \mathcal{C}''\) can be any of \(\{0, 1, 9\} \subseteq J_{13}[13]\). Also we consider some permutations on \(\mathcal{B}\) which be used in [8] and those are suitable for three designs. Let \(\pi\) be one of them. We construct \(\mathcal{B}' = \pi(\mathcal{B})\) and \(\mathcal{B}'' = \pi^{-1}(\mathcal{B})\). Hence we obtain intersection sizes \(|\mathcal{B} \cap \mathcal{B}' \cap \mathcal{B}''| + i, i \in \{0, 1, 9\}\). Now we get in this step these intersection numbers \(\{58, 59, 62, 63, 67, 78, 79, 87, 102, 103, 111\} \cup \{69, 71\}\). ■

7 conclusion

In this paper, we have obtained the complete solution of the intersection problem for three \(S(2, 4, v)\) designs with \(v = 13, 16\) and \(v \geq 49\).

Proof of Theorem 1.1
(1): By Lemma 2.1 we have \(J_3[v] \subseteq I_3[v]\).
(2): By combining the results of Theorems 5.1, 5.2, 5.3, 5.5, 5.7, and 5.8 we have \(J_3[v] = I_3[v]\) for all admissible \(v \geq 49\).
(3): By Theorem 5.4 we obtain \(I_3[40] \setminus \{b_{40} - 15, b_{40} - 14\} \subseteq J_3[40]\).
(4): It holds by Lemmas 4.1 and 4.2
(5), (6), and (7): We prove these sentences in the last section.
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