On the coefficients of the Báez-Duarte criterion for the Riemann hypothesis and their extensions

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(Received 2006)
September 6, 2006

Abstract

We present analytic properties and extensions of the constants $c_k$ appearing in the Báez-Duarte criterion for the Riemann hypothesis. These constants are the coefficients of Pochhammer polynomials in a series representation of the reciprocal of the Riemann zeta function. We present generalizations of this representation to the Hurwitz zeta and many other special functions. We relate the corresponding coefficients to other known constants including the Stieltjes constants and present summatory relations. In addition, we generalize the Maślanka hypergeometric-like representation for the zeta function in several ways.

Key words and phrases
Báez-Duarte criterion, Riemann hypothesis, Stieltjes constants, Pochhammer polynomial, Riemann and Hurwitz zeta functions, polygamma function, complete Bell polynomials, Maślanka representation, Dirichlet $L$ function
Introduction

As reformulated by Báez-Duarte, the Riemann hypothesis (RH) is equivalent to a certain growth condition on constants $c_k$ that appear in a series representation of the reciprocal of the Riemann zeta function $\zeta(s)$. In this paper we analytically investigate these constants and introduce some parametrized extensions. We point out the importance of such extensions for future work. We relate parametrized coefficients $c_k(b,a)$ to other important constants of analytic number theory including the Stieltjes constants and present summatory relations for the former.

With

$$c_k \equiv \sum_{j=0}^{\infty} (-1)^j \binom{k}{j} \frac{1}{\zeta(2j+2)}, \quad k \geq 0, \quad (1)$$

unconditionally $c_k = O(k^{-1/2})$ and on the RH we have $c_k = O(k^{-3/4+\epsilon})$ for any $\epsilon > 0$. The constants $c_k$ are known to consist of a relatively rapidly decreasing term $\propto -1/k^2$ and an oscillatory contribution, as can be shown by Rice’s integrals or other asymptotic methods. Not surprisingly, it is the detailed behaviour of the oscillatory contribution upon which the validity of the RH depends. Numerical results for $c_k$ are presented in Refs. [5, 6, 14, 20] and the first billion values have been reported. These values are consistent with the Báez-Duarte criterion under the RH.

If further $c_k = O(k^{-3/4})$, then the complex zeros of $\zeta(s)$ are on the critical line $\text{Re } s = 1/2$ and are simple.

Summatory relations for $c_k$

In this section we relate $c_k$ to the Stieltjes constants $\gamma_k$. The latter are the coeffi-
cients of the Laurent series of the Riemann zeta function about \( s = 1 \). In preparation we have Lemma 1 concerning the derivatives of Pochhammer polynomials.

Let \( P_k(s) \equiv (1 - s)_k/k! = (-1)^k \Gamma(s)/k! \Gamma(s - k) \), where \((a)_n = \Gamma(a + n)/\Gamma(a)\) is the Pochhammer symbol. Let \( \psi = \Gamma'/\Gamma \) be the digamma function, where \( \Gamma \) is the Gamma function, and \( \psi^{(j)} \) the polygamma function \([2]\).

**Lemma 1.** Set

\[
g(s) \equiv \frac{1}{2} [\psi(1 - s/2) - \psi(k + 1 - s/2)],
\]

with

\[
g^{(\ell)}(s) = \frac{(-1)^\ell}{2^{\ell+1}} \left[ \psi^{(\ell)}(1 - s/2) - \psi^{(\ell)}(k + 1 - s/2) \right],
\]

Then we have

\[
\frac{d}{ds} P_k \left( \frac{s}{2} \right) = P_k \left( \frac{s}{2} \right) g(s),
\]

and

\[
\left( \frac{d}{ds} \right)^j P_k \left( \frac{s}{2} \right) = P_k \left( \frac{s}{2} \right) Y_j \left[ g(s), g'(s), \ldots, g^{(j-1)}(s) \right],
\]

where \( Y_j \) are (exponential) complete Bell polynomials \([11]\).

**Proof.** Equation 3 follows from the definition of the Pochhammer symbol and Eq. (4) by Lemma 1 of Ref. \([7]\).

A very small subset of the relations between the constants \( c_k \) and the Stieltjes constants is contained in the following. Equation (5c) presents how the Euler constant \( \gamma = -\psi(1) \) may be written in terms of \( c_j \).

**Proposition 1.** We have

\[
\sum_{k=0}^{\infty} \frac{\Gamma(k + 1/2)}{k!} c_k = 0,
\]

\[
(5a)
\]
\[- \frac{1}{2} \sqrt{\pi} \sum_{k=0}^{\infty} \frac{\Gamma(k + 1/2)}{k!} \psi(k + 1/2) c_k = 1, \quad (5b)\]

and
\[\sum_{k=0}^{\infty} \frac{\Gamma(k + 1/2)}{k!} [\psi^2(k + 1/2) + \psi'(k + 1/2)] c_k = -4\sqrt{\pi}(\gamma - 2 \ln 2). \quad (5c)\]

As prelude to the proof of Proposition 1, we know that the representation
\[\frac{1}{\zeta(s)} = \sum_{k=0}^{\infty} c_k P_k(s/2), \quad (6)\]

holds unconditionally in the half plane \(\Re(s) > 1\), converging uniformly on compact sets \(3\).

**Proof of Proposition 1.** We combine the definition of the Stieltjes constants (e.g., [12]) with Eq. (6) to write
\[\frac{1}{\zeta(s)} = \left[ \frac{1}{s - 1} + \sum_{k=0}^{\infty} \frac{\gamma_k k!}{s - 1} \right]^{-1} = \sum_{k=0}^{\infty} c_k P_k(s/2) \quad (7a)\]

\[= s - 1 - \gamma (s - 1)^2 + (\gamma^2 + \gamma_1)(s - 1)^3 + (-\gamma^3 - 2\gamma \gamma_1 - \gamma_2/2)(s - 1)^4 + O((s - 1)^5). \quad (7b)\]

Here we used \(\gamma_0 = \gamma\). We then Taylor expand the right side of Eq. (7a) about \(s = 1\) using Lemma 1. In particular, we have
\[\left( \frac{d}{ds} \right)^j \left. P_k\left( \frac{s}{2} \right) \right|_{s=1} = \frac{\Gamma(k + 1/2)}{\sqrt{\pi} k!} Y_j \left[ g(s), g'(s), \ldots, g^{(j-1)}(s) \right]_{s=1}, \quad (8)\]

\[g(1) = \frac{1}{2} [\psi(1/2) - \psi(k + 1/2)] = -\frac{\Gamma(k + 1/2)}{2\sqrt{\pi} k!} \sum_{\ell=0}^{k-1} \frac{1}{\ell + 1/2}, \quad (9a)\]

and
\[g^{(\ell)}(1) = \frac{(-1)^\ell}{2^{\ell+1}} \left[ \psi^{(\ell)}(1/2) - \psi^{(\ell)}(k + 1/2) \right] = -\frac{\ell!}{2^{\ell+1}} \sum_{j=0}^{k-1} \frac{1}{(j + 1/2)^{\ell+1}}. \quad (9b)\]
Equations (9a) and (9b) follow from the functional equations of the digamma and polygamma functions respectively. We next equate successive like powers of $s - 1$ on both sides of Eq. (7). Effectively, we evaluate the successive derivatives of the representation (6) as $s \to 1^+$. We first obtain Eq. (5a). We then use it to obtain Eq. (5b). We then use both of these to obtain Eq. (5c) and Proposition 1 follows.

Remarks. The continuation of the process just described yields the explicit relation between $\gamma_k$ and sums over the constants $c_j$. The appearance of the low order Bell polynomials $Y_0 = 1$, $Y_1(x_1) = x_1$, and $Y_2(x_1, x_2) = x_1^2 + x_2$ is implicit in writing Eqs. (5a)-(5c). They do not appear in the final results there since we performed successive manipulations.

Equation (5a) is a reflection of the simple pole of $\zeta(s)$ at $s = 1$ and of the relation $\sum_{n=1}^{\infty} \mu(n)/n = 0$, where $\mu$ is the Möbius function. Once results such as Eqs. (5) have been derived, they may be directly verified using the alternative expression

$$c_k = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \left(1 - \frac{1}{n^2}\right)^k. \quad (10)$$

For instance, we recover

$$\sum_{k=0}^{\infty} \frac{\Gamma(k + 1/2)}{k!} c_k = \sqrt{\pi} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} n = 0. \quad (11)$$

**Corollary 1.** In Eq. (5a), the ratio $\Gamma(k + 1/2)/k! \sim k^{-1/2}$ for $k \to \infty$ while in Eq. (5b) the factor $\psi(k + 1/2) \sim \ln k$ as $k \to \infty$. Therefore, the former equation shows that if the $c_k$’s did not change sign, they would have to decrease at least as fast as $k^{-1/2}$ as $k \to \infty$. 

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The following expression may be combined with the Stieltjes constant expansion of $1/\zeta(s)$ given in Eq. (7) to write $\gamma_j$ in terms of sums of the constants $c_k$ and Stirling numbers of the first kind $s(k, \ell)$ [11, 16, 17].

**Lemma 2.** The coefficient of $(s - 1)^j$ on the right side of Eq. (7a) is given by

$$
\sum_{k=0}^{j} (-1)^k \frac{c_k}{k!} \sum_{\ell=j}^{k} \frac{s(k, \ell)}{2^\ell} (-1)^{\ell-j} \binom{\ell}{j}.
$$

(12)

**Proof.** We first re-express the Pochhammer polynomials using the Stirling numbers $s(k, \ell)$, then binomially expand and reorder sums:

$$
\sum_{k=0}^{\infty} \frac{c_k}{k!} (1 - s/2)_k = \sum_{k=0}^{\infty} \frac{c_k}{k!} \sum_{\ell=0}^{k} (-1)^{k+\ell} s(k, \ell) (1 - s/2)^\ell
$$

$$
= \sum_{k=0}^{\infty} (-1)^k \frac{c_k}{k!} \sum_{\ell=0}^{k} \frac{s(k, \ell)}{2^\ell} \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} (s - 1)^j
$$

$$
= \sum_{k=0}^{\infty} (-1)^k \frac{c_k}{k!} \sum_{j=0}^{\infty} \sum_{\ell=j}^{k} \frac{s(k, \ell)}{2^\ell} (-1)^{\ell-j} \binom{\ell}{j} (s - 1)^j.
$$

(13)

**First extension of the constants $c_k$**

The Hurwitz zeta function $\zeta(s, a) = \sum_{k=0}^{\infty} (k + a)^{-s}$, $a \notin \mathbb{N}_0^-$ for $\text{Re} \ s > 1$ extends to an analytic function with only a simple pole at $s = 1$. Here we demonstrate the representation

**Corollary 2.** For $\text{Re} \ s > 1$ we have

$$
\frac{1}{\zeta(s, a)} = \sum_{k=0}^{\infty} c_k(a) P_k(s/2),
$$

(14)

with

$$
c_k(a) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\zeta(2j + 2, a)}, \quad k \geq 0.
$$

(15)
In particular, there results at $a = 1/2$

**Corollary 3.**

$$\frac{1}{\zeta(s)} = (2^s - 1) \sum_{k=0}^{\infty} c_k(1/2)P_k(s/2).$$  \hspace{1cm} (16)$$

Corollary 2 follows from

**Proposition 2.** Define for $b > 1 + \delta$ with $\delta > 0$ and Re $a > 0$ the functions

$$F(x, b, a) \equiv \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\zeta(bk, a)} \frac{x^k}{(k-1)!}$$  \hspace{1cm} (17)$$

and

$$\varphi(s, b, a) \equiv \int_{0}^{\infty} x^{-(s/b+1)} F(x, b, a) dx, \quad 1 < \text{Re } s < b.$$  \hspace{1cm} (18)$$

Put

$$c_k(b, a) \equiv \sum_{j=0}^{k} \frac{(-1)^j}{\zeta(bj + b, a)}.$$  \hspace{1cm} (19)$$

Then we have the series and integral representations for $\text{Re } s > 1$

$$\frac{1}{\zeta(s, a)} = \sum_{k=0}^{\infty} c_k(b, a)P_k(s/b) = \frac{\varphi(s, b, a)}{\Gamma(1-s/b)}.$$  \hspace{1cm} (20)$$

In the proof we use the infinite series

**Lemma 3**

$$\sum_{k=j}^{\infty} \frac{1}{k!} \binom{k}{j} x^k = \sum_{k=0}^{\infty} \frac{1}{(k+j)!} \binom{k+j}{j} x^k = \frac{x^j}{j!} e^x,$$  \hspace{1cm} (21)$$

and

**Lemma 4**

$$\sum_{n=0}^{\infty} F\left[\frac{x}{(n+a)^b}, b, a\right] = xe^{-x}.$$  \hspace{1cm} (22)$$

The series (17) is uniformly convergent on compact sets of the complex $x$ plane so that the interchange of sums used to show Eq. (22) is valid.
We have
\[ \int_0^\infty x^{-(s/b+1)} F \left[ \frac{x}{(n+a)^b}, b, a \right] dx = (n + a)^{-s} \varphi(s, b, a). \] (23)

Without assuming the RH, \( \varphi \) defined in Eq. (18) converges absolutely and uniformly in the strip specified. The behaviour \( F(x, b, a) \sim x/\zeta(bk, a) \) as \( x \to 0 \) dictates the condition \( \Re s < b \). The behaviour \( F(x, b, a) = o(x^{1/b}) \) as \( x \to \infty \) without the RH gives the requirement \( \Re s > 1 \). Then summing both sides of Eq. (23) on \( n \) from 0 to \( \infty \) and using Lemma 4 gives
\[ \zeta(s, a) \varphi(s, b, a) = \int_0^\infty x^{-s/b} e^{-x} dx = \Gamma(1 - s/b), \] (24)
so that we have obtained the 'outer' equality of Eq. (20).

We next re-express the function \( \varphi \) in terms of Pochhammer polynomials. We have from Eqs. (17) and (18)
\[ \varphi(s, b, a) = \int_0^\infty \sum_{j=0}^\infty \frac{(-1)^j}{\zeta(bj + b, a)} \frac{1}{j!} x^{j-s/b} dx \] (25a)
\[ = \int_0^\infty \sum_{j=0}^\infty \frac{(-1)^j}{\zeta(bj + b, a)} \frac{1}{j!} e^x x^{j-s/b} dx. \] (25b)

We next apply Lemma 3 so that
\[ \varphi(s, b, a) = \int_0^\infty \sum_{k=0}^\infty \frac{1}{k!} \sum_{j=0}^k \frac{(-1)^j}{\zeta(bj + b, a)} \binom{k}{j} e^{-x} x^{k-s/b} dx \] (26a)
\[ = \int_0^\infty \sum_{k=0}^\infty \frac{1}{k!} \sum_{j=0}^k \frac{(-1)^j}{\zeta(bj + b, a)} \binom{k}{j} e^{-x} x^{k-s/b} dx \] (26b)
\[ = \sum_{k=0}^\infty \frac{1}{k!} c_k(b, a) \Gamma(k + 1 - s/b). \] (26c)
In obtaining Eq. (26b) from (26a) we reordered the double series and (26c) used the definition (19). We have therefore found that

$$\varphi(s, b, a) = \Gamma(1 - s/b) \sum_{k=0}^{\infty} c_k(b, a) P_k(s/b),$$

and Proposition 2 is completed.

We have shown a way to directly relate the RH criteria of Báez-Duarte [3] and of Riesz [15]. The summatory function appearing in the Riesz criterion corresponds to $R(x) \equiv F(x, 2, 1)$ in Eq. (17) and under the RH it is $O(x^{1/4+\epsilon})$ for $\epsilon > 0$.

A recent construction for the function $1/\zeta$ similar to Proposition 2 has been given in Ref. [5]. The authors of that reference used the Möbius function in that development, whereas we have proceeded differently and obtained a result also applying to the reciprocal of the Hurwitz zeta function.

Arguing as we have in Proposition 2 gives many extensions. An example is

**Corollary 4.** Putting

$$G(x) \equiv \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{x^k}{\zeta(2k + 1)},$$

$$\varphi_G(x) = \int_{0}^{\infty} x^{-(s+1)/2} G(x) dx,$$

and

$$c_k^G \equiv \sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{1}{\zeta(2j + 1)},$$

we have for Re $s > 1$ the representations

$$\frac{1}{\zeta(s)} = \sum_{k=1}^{\infty} c_k^G P_k((s + 1)/2) = \frac{\varphi_G(s)}{\Gamma((1 - s)/2)}.$$
Given Corollary 2 and Proposition 2 Eqs. (7) extend to

\[
\frac{1}{\zeta(s, a)} = \left[ \frac{1}{s - 1} + \sum_{k=0}^{\infty} \frac{\gamma_k(a)}{k!} (s - 1)^k \right]^{-1} = \sum_{k=0}^{\infty} \frac{c_k(b, a) P_k(s/b)}{s - 1 + \psi(a)(s-1)^2 + [\psi^2(a) + \gamma_1(a)](s-1)^3 + [-\psi^3(a) + 2\psi(a)\gamma_1(a) - \gamma_2(a)/2](s-1)^4 + O[(s-1)^5]},
\]

wherein \(\gamma_0(a) = -\psi(a)\) has been used. The Stieltjes constants \(\gamma_k(a)\) may be written in terms of sums containing the Bernoulli numbers \(B_j\) and elementary constants such as powers of \(\ln 2\). Further properties of \(\gamma_k(a)\) are given in the very recent Refs. [8] and [9].

**Remark 1.** Analogous to Eqs. (14) and (15) it is not difficult to show that the Maślanka representation for \(\zeta(s)\) [13] may be extended to

\[
\zeta(s, a) = \frac{1}{s - 1} \sum_{k=0}^{\infty} (1 - s/2)_k A_k(a) \frac{k!}{k!},
\]

where

\[
A_k(a) \equiv \sum_{j=0}^{k} (-1)^j \binom{k}{j} (2j + 1) \zeta(2j + 2, a).
\]

As a Corollary, we obtain as special cases a representation for Bernoulli polynomials \(B_n(x) = -n\zeta(1 - n, x)\) and polygamma functions \(\psi^{(m)}(z) = (-1)^{m+1} m! \zeta(m + 1, z)\). Additional cases include representations for the alternating Hurwitz zeta function, the digamma function, and the function \(\beta(x) = (1/2)[\psi[(x + 1)/2] - \psi(x/2)] = \sum_{k=0}^{\infty} (-1)^k / (x + k)\).

**Remark 2.** We believe that it is very useful to have the constants \(c_k\) extended to include one or more parameters. In that case manipulations on \(c_k(a)\) and \(1/\zeta(s, a)\)
for instance may be performed with respect to $a$ and/or $s$ and then the Riemann zeta function case recovered by putting $a = 1$ or $1/2$. For instance, we have from Eq. (15)

$$\frac{d}{da} c_k(a) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(2j + 2)\zeta(2j + 3, a)}{\zeta^2(2j + 2, a)}. \quad (35)$$

Remark 3. The representation (33) may be further extended to the Hurwitz-Lerch zeta function $\Phi(z, s, a) = \sum_{n=0}^{\infty} z^n/(n + a)^s$, where $s \in \mathbb{C}$ for $|z| < 1$, $\text{Re } s > 1$ when $|z| = 1$. In this case we have

**Proposition 3**

$$\Phi(z, s, a) = \frac{1}{s - 1} \sum_{k=0}^{\infty} (1 - s/2)_k A_k(z, a) \frac{A_k(z, a)}{k!}, \quad (36)$$

where

$$A_k(z, a) \equiv \sum_{j=0}^{k} (-1)^j \binom{k}{j} (2j + 1)\Phi(z, 2j + 2, a). \quad (37)$$

The proof of Proposition 3 and an extension of it is contained as a special case of Proposition 4 proved below. As a Corollary, we obtain Maślanka-type representations for polylogarithm functions (Jonquières function)

$$\text{Li}_s(z) = z\Phi(z, s, 1), \quad (38)$$

where $s \in \mathbb{C}$ for $|z| < 1$, $\text{Re } s > 1$ when $|z| = 1$.

Remark 4. We have

**Conjecture 1.** For a class $\mathcal{M}$ of analytic functions expressible as a Dirichlet series and possessing at most polar singularities in the complex plane such that for $f \in \mathcal{M}$ and $q > 1$ we have the representation

$$f(s) = \frac{1}{s - 1} \sum_{k=0}^{\infty} A_k P_k(s/q), \quad (39)$$
holding in a half plane of $C$ with

$$A_k \equiv \sum_{j=0}^{k} (-1)^j \binom{k}{j} (qj + q - 1) f(qj + q). \quad (40)$$

We believe that a demonstration of some form of this Conjecture is possible by an appropriate application of sampling theory (cf. the Appendix). However, a more expedient approach may be to apply known results for approximating analytic functions in terms of the zeta function. Then given the representation (33) Conjecture 1 would follow.

In support of Conjecture 1 we have the following

**Proposition 4.** Let

$$f(s, a) = \sum_{n=0}^{\infty} \frac{f_n}{(n+a)^s}, \quad (41)$$

where it is assumed that $\{f_n\}_{0}^{\infty}$ is such that the series converges in a half-plane $\Re s > \sigma > 1$ and $a \in C/\{0, -1, \ldots\}$. Then we have for $p > 1$

$$f(s, a) = \frac{1}{s-1} \sum_{k=0}^{\infty} A_k(a) P_k(s/p), \quad (42)$$

with

$$A_k(a) \equiv \sum_{j=0}^{k} (-1)^j \binom{k}{j} (pj + p - 1) f(pj + p, a). \quad (43)$$

**Proof.** We proceed as in Ref. [4] in the case $f_n = 1$, $a = 1$, and $p = 2$, forming

$$\alpha^{-s} (s-1)f(s, a) = -\frac{\partial}{\partial \alpha} \alpha^{1-s} \sum_{n=0}^{\infty} \frac{f_n}{(n+a)^s}, \quad (44)$$

After term-by-term differentiation of the series we evaluate at $\alpha = 1$. The interchange of various operations is justified by the assumption on the sequence $f_n$ and the estimates of Ref. [4].
As a Corollary, we obtain generalized Maślanka-type representations for other important special functions. These include the multiple zeta function
\[
\zeta_n(s, z) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \frac{1}{(k_1 + k_2 + \ldots + k_n + z)^s} = \sum_{k=0}^{\infty} \frac{1}{(k + z)^s} \binom{k + n - 1}{n - 1},
\]
and the multiple Gamma function \(\Gamma_n\). The latter function has a product representation and may be defined by the recurrence-functional equation \(\Gamma_n(z + 1) = \Gamma_n(z)/\Gamma_n(z)\), \(\Gamma_1(z) = \Gamma(z)\), \(\Gamma_n(1) = 1\), for \(z \in \mathbb{C}\) and \(n \in \mathbb{N}^+\). The multiple Gamma function may be expressed in terms of derivatives of the multiple zeta function [19]:
\[
\ln \Gamma_n(z) = \lim_{s \to 0} \frac{\partial \zeta_n(s, z)}{\partial s} + \sum_{k=1}^{n} (-1)^k \binom{z}{k-1} R_{n+1-k},
\]
where
\[
R_n = \sum_{k=1}^{n} \lim_{s \to 0} \frac{\partial \zeta_n(s, 1)}{\partial s}.
\]

**Second extension of the constants \(c_k\)**

Another possible extension of the constants \(c_k\) would be to write
\[
c_k(a, b) \equiv \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\zeta(a j + b)}, \quad k \geq 0,
\]
wherein we considered \(a = b\) in Eq. (19). This extension has also been observed in Ref. [5] and numerical experiments presented. However, we remain with the case \(a = b = 2\) and instead note that
\[
\zeta(2j + 2) = \frac{(2\pi)^{2j+2}(-1)^j B_{2j+2}}{2(2j + 2)!} = \frac{(2\pi)^{2j+2}(-1)^j}{2(2j + 2)!} B_{2j+2}(x)|_{x=0}.
\]
Then we may consider
\[
c_k(x) = \sum_{j=0}^{k} \binom{k}{j} \frac{2(2j + 2)!}{(2\pi)^{2j+2}} \frac{1}{B_{2j+2}(x)},
\]
such that \( B_n'(x) = nB_{n-1}(x) \). If \( f(x) = 1/B_{2j+2}(x) \), then we have
\[
\frac{df}{dx} = -f(x)(2j + 2)\frac{B_{2j+1}(x)}{B_{2j+2}(x)},
\]
(50)
and Lemma 1 of Ref. [7] applies for the higher order derivatives of \( f(x) \) in terms of (exponential) complete Bell polynomials. When evaluated at \( x = 0, B_{2j+1}(0) = 0 \) unless \( j = 0 \) when \( B_1 = -1/2 \), and Ref. [18] describes the Bell polynomials when the odd-indexed variables are set to zero.

**Connection with the \( \eta_j \) constants**

We first mention in passing the following that recovers a result of [3], but in a different way. We have

**Lemma 3**

\[
\lim_{k \to \infty} P_k(s)(k + 1)^s = \frac{1}{\Gamma(1 - s)},
\]
(51)

**Proof.** We write \( P_k(s) = (1 - s)_k/k! = \Gamma(k + 1 - s)/k!\Gamma(1 - s) \), apply the known asymptotic form of \( \Gamma(z + a)/\Gamma(z + b) \) (e.g., [1]), and take the limit.

We now introduce the constants \( \eta_j \) of the Laurent expansion
\[
\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s - 1} - \sum_{p=0}^{\infty} \eta_p(s - 1)^p, \quad |s - 1| < 3,
\]
(52)
with \( \eta_0 = -\gamma \). These coefficients are important in the theory of the function \( \ln\zeta(s) \); hence they are connected with the behaviour of the prime counting function \( \pi(x) \). The alternating binomial sum \( S_2(n) \equiv \sum_{j=1}^{k} (-1)^j \binom{k}{j} |\eta_{j-1}| \) is key in the Li criterion for the RH; its sublinearity in \( n \) would imply the latter conjecture (e.g., [10]).
We apply the identity
\[ \frac{d}{ds} \frac{1}{\zeta(s)} = -\frac{1}{\zeta(s)} \frac{\zeta'(s)}{\zeta(s)}, \] (53)
and evaluate the derivatives as \( s \to 1 \) from the right. We obtain

**Proposition 5.** Put \( a_0 = \Gamma(k + 1/2)/\sqrt{\pi k!} \) and

\[ a_q = \frac{1}{q!} \frac{\Gamma(k + 1/2)}{\sqrt{\pi k!}} Y_q \left[ g(s), g'(s), \ldots, g^{(q-1)}(s) \right] \big|_{s=1}, \quad q \geq 1, \] (54)
where the function \( g \) and its derivatives are given in Eq. (2). We then have for \( q \geq 1 \)

\[ qa_{q+1} \sum_{k=0}^{q+1} c_k = \sum_{j=1}^{q} a_j \sum_{k=0}^{j} c_k \eta_{q-j}. \] (55)

**Proof.** We first recall from the representation (6) that

\[ \frac{1}{\zeta(s)} = \sum_{k=0}^{\infty} c_k \sum_{q=0}^{k} \frac{1}{q!} \left( \frac{d}{ds} \right)^q P_k \left( \frac{s}{2} \right) \big|_{s=1} (s-1)^q \] (56a)
\[ = \sum_{k=0}^{\infty} c_k \sum_{q=1}^{k} a_q (s-1)^q. \] (56b)

For Eq. (56a) we have kept in mind that \( P_k(s) \) is a polynomial of degree \( k \) in \( s \). For Eq. (56b) we have used Lemma 1 together with Eq. (5a). We then reorder the double summation, obtaining

\[ \frac{1}{\zeta(s)} = \sum_{q=1}^{\infty} a_q \sum_{k=0}^{q} c_k (s-1)^q, \] (57a)

and

\[ \frac{d}{ds} \frac{1}{\zeta(s)} = \sum_{q=0}^{\infty} (q+1) a_{q+1} \sum_{k=0}^{q+1} c_k (s-1)^q. \] (57b)

We then apply identity (53), a form of which is

\[ \sum_{k=0}^{\infty} c_k \frac{d}{ds} P_k \left( \frac{s}{2} \right) = \sum_{k=0}^{\infty} c_k P_k \left( \frac{s}{2} \right) \left[ \frac{1}{s-1} + \sum_{p=0}^{\infty} \eta_p (s-1)^p \right]. \] (58)
We carry out the necessary multiplication of series on the right side of Eq. (58) and reorder the second term. We then equate coefficients of like powers of $s - 1$ on both sides and Eq. (55) follows.

**Other summatory relations**

The authors of the extremely recent Ref. [6] derived the identity

$$\sum_{k=0}^{\infty} c_k s^k = \frac{1}{1 - s} \sum_{k=0}^{\infty} \left( -\frac{s}{1 - s} \right)^k \frac{1}{\zeta(2k + 2)}, \quad -1 \leq \text{Re } s < 1/2 \tag{59}$$

and noted the value $\sum_{k=0}^{\infty} (-1)^k c_k = \sum_{k=1}^{\infty} 2^{-k} / \zeta(2k) \approx 0.7825279853$.

We first illustrate that Eq. (59) can provide the basis of a family of summatory relations and have

**Proposition 6.** For $-1 \leq \text{Re } t \leq 1/2$ we have

$$c_0 \ln(1 - t) + \sum_{k=1}^{\infty} \frac{c_k}{k} t^k \sum_{k=1}^{\infty} \frac{(-1)^k}{k \zeta(2k + 2)} (\frac{t}{1 - t})^k, \tag{60}$$

giving

**Corollary 5.**

$$c_0 \ln(2/3) + \sum_{k=1}^{\infty} \frac{c_k}{k} \frac{1}{3^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k \zeta(2k + 2)} \frac{1}{2^k} \approx -0.369410468, \tag{61a}$$

$$c_0 \ln 2 + \sum_{k=1}^{\infty} \frac{(-1)^k c_k}{k} \frac{1}{2^k} = \sum_{k=1}^{\infty} \frac{1}{k \zeta(2k + 2)} \frac{1}{2^k} \approx 0.65279901499, \tag{61b}$$

$$-c_0 \ln 2 + \sum_{k=1}^{\infty} \frac{c_k}{k} \frac{1}{2^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k \zeta(2k + 2)} \approx -0.624463294, \tag{61c}$$

and

$$\sum_{k=1}^{\infty} \frac{c_k}{k} \left[ (-1)^k + \frac{1}{2^k} \right] = \sum_{k=1}^{\infty} \frac{1}{k \zeta(2k + 2)} \left[ (-1)^k + \frac{1}{2^k} \right] \approx 0.0283357. \tag{61d}$$
Proof of Proposition 6. We write Eq. (59) in the form

\[
\frac{c_0 s}{s-1} + \sum_{k=1}^{\infty} c_k s^k = \frac{1}{1-s} \sum_{k=1}^{\infty} \left( \frac{-s}{1-s} \right)^k \frac{1}{\zeta(2k+2)}. \tag{62}
\]

We then divide both sides by \(s\), integrate on \(s\) from 0 to \(t\), and Eq. (60) obtains.

Special cases of \(t\) in Eq. (60) yield Corollary 5.

Equations (59) and (60) may represent the only so far known series associated with the zeta function where reciprocal zeta values at integer argument occur in the summand. Equation (59) is extended by

**Proposition 7.** For \(b > 1\) and \(\text{Re } a > 0\) we have

\[
\sum_{k=0}^{\infty} c_k(b, a) s^k = \frac{1}{1-s} \sum_{k=1}^{\infty} \left( \frac{-s}{1-s} \right)^k \frac{1}{\zeta(bk+b, a)}, \quad -1 \leq \text{Re } s < 1/2, \tag{63}
\]

where \(c_k(b, a)\) is defined in Eq. (19). In particular, we have

**Corollary 6.**

\[
\sum_{k=0}^{\infty} (-1)^k c_k(b, a) = \sum_{k=1}^{\infty} \frac{1}{2^k \zeta(bk+b, a)}. \tag{64}
\]

**Proof.** We use the definition (19), reorder a double sum, and apply the binomial expansion:

\[
\sum_{k=0}^{\infty} c_k(b, a) s^k = \sum_{k=0}^{\infty} s^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\zeta(bj+b, a)}
\]

\[
= \sum_{j=0}^{\infty} \left( \frac{-1)^j}{\zeta(bj+b, a)} \right) \sum_{k=j}^{\infty} \binom{k}{j} s^j
\]

\[
= \sum_{j=0}^{\infty} \left( \frac{-1)^j s^j}{\zeta(bj+b, a)} \right) \sum_{k=0}^{\infty} \binom{k+j}{j} s^k
\]

\[
= \sum_{j=0}^{\infty} \left( \frac{-1)^j s^j}{\zeta(bj+b, a) (1-s)^j+1} \right). \tag{65}
\]
The alternating sum (64) obtains at \( s = -1 \).

Similarly, Proposition 6 and Corollary 5 may be extended to include the values \( c_k(b, a) \). We have

**Proposition 8.** For \(-1 \leq \Re t \leq 1/2, b > 1, \) and \( \Re a > 0 \) there holds

\[
c_0(b, a) \ln(1 - t) + \sum_{k=1}^{\infty} \frac{c_k(b, a)}{k} t^k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k \zeta(bk + b, a)} \left( \frac{t}{1 - t} \right)^k.
\]

(66)

We omit the proof.

**Third extension of the constants \( c_k \)**

Let \( \chi \) be a Dirichlet character mod \( k \) and \( L(s, \chi) \) the corresponding Dirichlet \( L \)-function (e.g., [13])

\[
L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \Re s > 1.
\]

(67)

We recall that

\[
\frac{1}{L(s, \chi)} = \sum_{n=1}^{\infty} \frac{\chi(n) \mu(n)}{n^s}, \quad \Re s > 1.
\]

(68)

We have

**Proposition 9.** For \( b > 1 \) there holds

\[
\frac{1}{L(s, \chi)} = \sum_{k=0}^{\infty} c_k(b, \chi) P_k(s/b), \quad \Re s > 1,
\]

(69)

where

\[
c_k(b, \chi) \equiv \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{L(bj + b, \chi)}.
\]

(70)

**Proof.** We proceed as in Ref. [3]. By simply noting that \( |\chi(n)| \leq 1 \) the estimates given there justify the interchange of infinite summations.
Final remarks

The representation of Proposition 9 is expected to extend to automorphic $L$ functions. Accordingly we expect a criterion on the rate of growth of $|c_k(b,\chi)|$ and its generalization to be equivalent to the extended and generalized Riemann hypothesis, respectively.

Based upon a special case of Proposition 3 (or 4) there is an extended Maślanka type representation of Dirichlet $L$ functions. This follows since Dirichlet $L$ functions may be written as a combination of Hurwitz zeta functions.

The analog of Stieltjes constants and the constants $\eta_j$ exist for Dirichlet $L$ functions (e.g., Appendix E of Ref. 10) and our method of Proposition 1 or 5 would equally well apply for relating them to $c_k(b,\chi)$.

If we insert the Euler product for $\zeta(s)$ into the expression (1) for $c_k$ we have

$$c_k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \prod_p \left[ 1 - p^{-(2j+2)} \right]$$

$$= \delta_{0k} - \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left[ \sum_p p^{-(2j+2)} + \ldots \right].$$  \hfill (71)

In Eq. (71) the product or sum over $p$ is over all primes and $\delta_{jk}$ is the Kronecker symbol. The first sum in brackets on the right side of this equation may be estimated as $\sum_{p \leq x} p^{-(2j+2)} \sim \text{Ei}[-(2j + 1) \ln x]$, where $\text{Ei}$ is the exponential integral.

An approximate expression for $c_k$ for large values of $k$ is given by 6

$$c_k \approx \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} e^{-k/n^2}.$$  \hfill (72)
We note that alternatively this approximation may be written as a Fourier transform:

\[ c_k \approx \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \int_{0}^{\infty} \frac{\cos kt \, dt}{n^2 t^2 + n^{-2}}. \]  

(73)

Summary

Our results generalize both the Báez-Duarte representation of the reciprocal of the Riemann zeta function [3] and the Maślanka representation of the zeta function itself [14]. The Maślanka representation has been generalized to the Hurwitz zeta function, the Hurwitz-Lerch zeta function, the multiple zeta function, and other important special functions. We have further extended the Báez-Duarte representation of \( 1/\zeta \) to the representation of the reciprocal of Dirichlet L functions and it is anticipated that this may be generalized to automorphic L functions. By way of our generalization of the Báez-Duarte representation in terms of Pochhammer polynomials, we have effectively demonstrated the equivalence of the Riesz [15] and Báez-Duarte criteria for the Riemann hypothesis. We have obtained summatory relations for the coefficients \( c_k \) of the Báez-Duarte criterion for the Riemann hypothesis and related them to important constants of analytic number theory. In describing the relation of \( c_k \) to the Stieltjes and other constants we have made use of the (exponential) complete Bell polynomials \( Y_j \).

Acknowledgement

I thank Prof. L. Báez-Duarte for his comments upon reading the manuscript, in particular in clarifying the statement of Conjecture 1.
Appendix: Interpolating binomial sums from the Faà di Bruno formula

A significant source of alternating binomial sums is the Faà di Bruno formula, a generalization of the chain rule. Put \( D_z \equiv d/dz \) and \( x = x(z) \). Then we have

\[
D_z^n f(x) = \sum_{k=0}^{n} D_x^k f(x) \frac{(-1)^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} x^{k-j} D_z^n x^j. \tag{A.1}
\]

In particular, we have for real \( a \)

\[
D_z^n x^{-a} = a \binom{a+n}{n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{1}{a+j} x^{-a-j} D_z^n x^j, \tag{A.2}
\]

or equivalently

\[
x^a D_z^n x^{-a} = \sum_{j=0}^{n} \frac{(-a)^j}{j!} \binom{n+a}{n-j} x^{-j} D_z^n x^j. \tag{A.3}
\]

In connection with developing alternative representations of analytic functions, we point out that Eq. (A.3) can be viewed as an immediate consequence of Lagrange interpolation. This follows from

\[
\binom{-a}{j} \binom{n+a}{n-j} = \prod_{\substack{k=0 \ \text{to} \ n \ \text{except} \ j \ \text{and} \ k \neq j}} \frac{k+a}{k-j}, \quad 0 \leq j \leq n. \tag{A.4}
\]

With the Faà di Bruno formula the exponential Bell polynomials again make an appearance [11].
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