A Proof On Arnold’s Chord Conjecture On Cotangent Bundles *

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Abstract

In this article, we prove that there exists at least one chord which is characteristic of Reeb vector field connecting a given Legendre submanifold in a contact manifolds of induced type in the cotangent bundles of any smooth open manifolds which confirms the Arnold conjecture in cotangent bundles.

Keywords J-holomorphic curves, Legendre submanifolds, Reeb chord.
2000 MR Subject Classification 32Q65, 53D35, 53D12

1 Introduction and results

Let Σ be a smooth closed oriented manifold of dimension $2n - 1$. A contact form on Σ is a 1–form such that $\lambda \wedge (d\lambda)^{n-1}$ is a volume form on Σ. Associated to $\lambda$ there is the so-called Reeb vectorfield $X_\lambda$ defined by

\[
i_X \lambda \equiv 1, \quad i_X d\lambda \equiv 0.
\]

*Project 19871044 Supported by NSF
Concerning the dynamics of Reeb flow, there is a well-known conjecture raised by Arnold in [2] which concerned the Reeb orbit and Legendre submanifold in a contact manifold. If \((\Sigma, \lambda)\) is a contact manifold with contact form \(\lambda\) of dimension \(2n - 1\), then a Legendre submanifold is a submanifold \(L\) of \(\Sigma\), which is \((n - 1)\)-dimensional and everywhere tangent to the contact structure \(\ker \lambda\). Then a characteristic chord for \((\lambda, L)\) is a smooth path \(x : [0, T] \to \Sigma, T > 0\) with \(\dot{x}(t) = X_\lambda(x(t))\) for \(t \in (0, T), x(0), x(T) \in L\). Arnold raised the following conjecture:

**Conjecture 1 (see [2])**. Let \(\lambda_0\) be the standard tight contact form \(\lambda_0 = \frac{1}{2}(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)\) on the three sphere \(S^3 = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 | x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1\}\). If \(f : S^3 \to (0, \infty)\) is a smooth function and \(L\) is a Legendre knot in \(S^3\), then there is a characteristic chord for \((f \lambda_0, L)\).

This conjecture was completely solved in [16, 17].

In this paper we improve the Gromov’s proof on that there exists at least one intersection point for the weakly exact Lagrangian submanifold under the weakly Lagrangian isotopy [10, 23, B3−4] to prove:

**Theorem 1.1** Let \((\Sigma, \lambda)\) be a contact manifold with contact form \(\lambda\) of induced type or Weinstein type in the cotangent bundles of any open smooth manifold with symplectic form \(\sum_{i=1}^n dp_i \wedge dq_i\) induced by Liouville form \(\alpha = \sum_{i=1}^n p_i dq_i\), i.e., there exists a transversal vector field \(Z\) to \(\Sigma\) such that \(L_Z \omega = \omega, \lambda = i_Z \omega\). Let \(X_\lambda\) its Reeb vector field and \(L\) a closed Legendre submanifold. Then \(p_i dq_i - \lambda\) defines an element \([p_i dq_i - \lambda] \in H^1(\Sigma)\). If \([p_i dq_i - \lambda] = 0\), then there exists at least one characteristic chord for \((X_\lambda, L)\).

**Sketch of proofs**: We work in the framework as in [10, 16]. In Section 2, we study the linear Cauchy-Riemann operator and sketch some basic properties. In section 3, first we construct a Lagrangian submanifold \(W\) under the assumption that there does not exist Reeb chord connecting the Legendre submanifold \(L\); second, we study the space \(D(V, W)\) consisting of contractible disks in manifold \(V\) with boundary in Lagrangian submanifold \(W\) and construct a Fredholm section of tangent bundle of \(D(V, W)\). In section 4, following [10], we construct a non-proper Fredholm section by using a special anti-holomorphic section as in [10, 16]. In section 5, we transform the non-homogenous Cauchy-Riemann equations to \(J\)-holomorphic curves. In section 6, we finish the proof of Theorem 1.1. as in [10].

## 2 Linear Fredholm Theory

For \(100 < k < \infty\) consider the Hilbert space \(V_k\) consisting of all maps \(u \in H^{k,2}(D, C^n)\), such that \(u(z) \in R^n \subset C^n\) for almost all \(z \in \partial D\). \(L_{k-1}\)
denotes the usual Hilbert $L_{k-1}$-space $H_{k-1}(D,C^n)$. We define an operator
\[ \partial : V_p \mapsto L_p \]
by
\[ \partial u = u_s + iu_t \quad (2.1) \]
where the coordinates on $D$ are $(s,t) = s+it$, $D = \{ z \parallel z \parallel \leq 1 \}$. The following result is well known (see [4, 21]).

**Proposition 2.1** $\partial : V_p \mapsto L_p$ is a surjective real linear Fredholm operator of index $n$. The kernel consists of the constant real valued maps.

Let $(C^n, \sigma = -\text{Im}(\cdot, \cdot))$ be the standard symplectic space. We consider a real $n$-dimensional plane $R^n \subset C^n$. It is called Lagrangian if the skew-scalar product of any two vectors of $R^n$ equals zero. For example, the plane $\{(p,q) | p = 0\}$ and $\{(p,q) | q = 0\}$ are Lagrangian subspaces. The manifold of all (nonoriented) Lagrangian subspaces of $R^{2n}$ is called the Lagrangian-Grassmanian $\Lambda(n)$. One can prove that the fundamental group of $\Lambda(n)$ is free cyclic, i.e. $\pi_1(\Lambda(n)) = \mathbb{Z}$. Next assume $\Gamma(z)$ is a smooth map associating to a point $z \in \partial D$ a Lagrangian subspace $\Gamma(z)$ of $C^n$, i.e. $(\Gamma(z))_{z \in \partial D}$ defines a smooth curve $\alpha$ in the Lagrangian-Grassmanian manifold $\Lambda(n)$. Since $\pi_1(\Lambda(n)) = \mathbb{Z}$, one have $[\alpha] = ke$, we call integer $k$ the Maslov index of curve $\alpha$ and denote it by $m(\alpha)$, see ([2]).

Now let $z : S^1 \mapsto R^n \subset C^n$ be a smooth curve. Then it defines a constant loop $\alpha$ in Lagrangian-Grassmanian manifold $\Lambda(n)$. This loop defines the Maslov index $m(\alpha)$ of the map $z$ which is easily seen to be zero.

Now let $(V, \omega)$ be a symplectic manifold and $W \subset V$ a Lagrangian submanifold. Let $u : (D, \partial D) \to (V,W)$ be a smooth map homotopic to constant map $u_0 : (D, \partial D) \to p \in W$. Then $u^*TV$ is a symplectic vector bundle and $(u|_{\partial D})^*TW$ be a Lagrangian subbundle in $u^*TV$. Since $u$ is homotopic to $u_0$ by $h(t,z)$ with $h(0,\cdot) = u_0$ and $h(1,\cdot) = u$, we can take a trivialization of $h^*TV$ as

\[ \Phi(h^*TV) = [0,1] \times D \times C^n \]

and

\[ \Phi(h|_{\partial D})^*TW \subset [0,1] \times S^1 \times C^n. \]

Let
\[ \pi_2 : [0,1] \times D \times C^n \to C^n \]
then
\[ h : (t,z) \in [0,1] \times S^1 \to \pi_2 \Phi(h|_{\partial D})^*TW \mid (t,z) \in \Lambda(n). \]

**Lemma 2.1** Let $u : (D^2, \partial D^2) \to (V,W)$ be a $C^k-$map $(k \geq 1)$ as above. Then,
\[ m(\bar{u}) = 0 \]
Proof. Since the homotopy $h(t, z)$ induces a homotopy $\bar{h}$ in Lagrangian-Grassmanian manifold. Note that $m(\bar{h}(0, \cdot)) = 0$. By the homotopy invariance of Maslov index, we know that $m(\bar{u}) = 0$.

Consider the partial differential equation
\[
\bar{\partial}u + A(z)u = 0 \text{ on } D
\]  
\[
u(z) \in \Gamma(z)R^n \text{ for } z \in \partial D
\]  
\[
\Gamma(z) \in GL(2n, R) \cap Sp(2n)
\]  
\[
m(\Gamma) = 0
\]  

For $100 < k < \infty$ consider the Banach space $\bar{V}_k$ consisting of all maps $u \in H^{k,2}(D, C^n)$ such that $u(z) \in \Gamma(z)$ for almost all $z \in \partial D$. Let $L_{k-1}$ the usual Hilbert space $H_{k-1}(D, C^n)$.

We define an operator $P: \bar{V}_k \to L_{k-1}$ by
\[
P(u) = \bar{\partial}u + Au
\]  

where $D$ as in (2.1).

**Proposition 2.2** $\bar{\partial}: \bar{V}_k \to L_{k-1}$ is a real linear Fredholm operator of index $n$.

Proof: see [4, 10, 21].

### 3 Nonlinear Fredholm Theory

#### 3.1 Construction of Lagrangian Submanifold

Let $M$ be an open manifold and $(T^*M, p_i dq_i)$ be the cotangent bundle of open manifold with the Liouville form $p_i dq_i$. Since $M$ is open, there exists a function $g : M \to R$ without critical point. The translation by $tTd\bar{g}$ along the fibre gives a hamiltonnian isotopy of $T^*M$:

\[
h^T_i(q, p) = (q, p + tTd\bar{g}(q))
\]  
\[
h^{T^*}_i(p_i dq_i) = p_i dq_i + tTd\bar{g}.
\]

**Lemma 3.1** For any given compact set $K \subset T^*M$, there exists $T = T_K$ such that $h^T_1(K) \cap K = \emptyset$. 

\[ 4 \]
Proof. Similar to [10, 15]

Let \( \Sigma \subset T^*M \) be a closed hypersurface, if there exists a vector field \( V \) defined in the neighbourhood \( U \) of \( \Sigma \) transversal to \( \Sigma \) such that \( L_V \omega = \omega \), here \( \omega = dp_i \wedge dq_i \) is a standard symplectic form on \( T^*M \) induced by the Liouville form \( p_i dq_i \), we call \( \Sigma \) the contact manifold of induced type in \( T^*M \) with the induced contact form \( \lambda = i_V \omega \).

Let \( (\Sigma, \lambda) \) be a contact manifold of induced type or Weinstein’s type in \( T^*M \) with contact form \( \lambda \) and \( X \) its Reeb vector field, then \( X \) integrates to a Reeb flow \( \eta_s \) for \( s \in R \).

By using the transversal vector field \( V \), one can identify the neighbourhood \( U \) of \( \Sigma \) foliated by flow \( f_t \) of \( V \) and \( \Sigma \), i.e., \( U = \cup_t f_t(\Sigma) \) with the neighbourhood of \( \{0\} \times \Sigma \) in the symplectization \( R \times \Sigma \) by the exact symplectic transformation(see[16]).

Consider the form \( d(e^a \lambda) \) at the point \( (a, x) \) on the manifold \( (R \times \Sigma) \), then one can check that \( d(e^a \lambda) \) is a symplectic form on \( R \times \Sigma \). Moreover One can check that

\[
\begin{align*}
    i_X(e^a \lambda) &= e^a \\
    i_X(d(e^a \lambda)) &= -de^a
\end{align*}
\]

So, the symplectization of Reeb vector field \( X \) is the Hamilton vector field of \( e^a \) with respect to the symplectic form \( d(e^a \lambda) \). Therefore the Reeb flow lifts to the Hamilton flow \( h_s \) on \( R \times \Sigma \)(see[3, 6]).

Let \( L \) be a closed Legendre submanifold in \( (\Sigma, \lambda) \), i.e., there exists a smooth embedding \( Q : L \to \Sigma \) such that \( Q^*\lambda|_L = 0 \). Let

\[
(V', \omega') = (T^*M, dp_i \wedge dq_i)
\]

and

\[
W' = L \times R, \quad W'_s = L \times \{s\}
\]

define

\[
G' : W' \to V' \quad \quad G'(w') = G'(l, s) = (0, \eta_s(Q(l)))
\]

Lemma 3.2 There does not exist any Reeb chord connecting Legendre submanifold \( L \) in \( (\Sigma, \lambda) \) if and only if \( G'(W'(s)) \cap G'(W'(s')) \) is empty for \( s \neq s' \).

Proof. Obvious.

Lemma 3.3 If there does not exist any Reeb chord for \( (X_\lambda, L) \) in \( (\Sigma, \lambda) \) then there exists a smooth embedding \( G' : W' \to V' \) with \( G'(l, s) = (0, \eta_s(Q(l))) \) such that

\[
G'_K : L \times (-K, K) \to V'
\]
is a regular open Lagrangian embedding for any finite positive $K$.

Proof. One check

$$G'^* (d(e^\alpha \lambda)) = \eta(\cdot, \cdot)^* d\lambda = (\eta^*_s d\lambda + i_X d\lambda \wedge ds) = 0$$

(3.8)

This implies that $G'$ is a Lagrangian embedding, this proves Lemma3.3.

Note that $d\lambda = dp_i \wedge dq_i$ on $\Sigma$ by the definition of induced contact type and by assumption $[p_i dq_i - \lambda] = 0 \in H^1(\Sigma)$, we know that

$$p_i dq_i = \lambda + d\beta(\sigma) \text{ on } \Sigma$$

(3.9)

Then by the proof of Lemma3.3, one computes

$$G'^* (p_i dq_i) = G'^* (\lambda) + G'^* (\eta^*_s ds + d\eta^* \beta) = ds + d\beta$$

here we also use $\beta$ denote the $\eta^* \beta$.

All above construction is contained in [16]. Now we introduce the upshot construction in [17]:

$$F'_0 : \mathcal{L} \times R \times R \to (R \times \Sigma)
F'_0(((l, s, a) = (a, \eta_s(l))))$$

(3.10)

Now we embed an elliptic curve $E$ long along $s-axis$ and thin along $a-axis$ such that $E \subset [-K, K] \times [0, \varepsilon]$. We parametrize the $E$ by $t' \in S^1$.

**Lemma 3.4** If there does not exist any Reeb chord in $(\Sigma, \lambda)$, then

$$F_0 : \mathcal{L} \times S^1 \to (R \times \Sigma)
F_0(l, t') = (a(t'), \eta_s(l))$$

(3.11)

is a compact Lagrangian submanifold. Moreover

$$l(R \times \Sigma, F_0(\mathcal{L} \times S^1), d(e^\alpha \lambda))
= \inf \{ \int_D f^* d\alpha \lambda > 0 | f : (D, \partial D) \to (R \times \Sigma, F_0(\mathcal{L} \times S^1)) \}
= \text{area}(E)$$

(3.12)

Proof. We check that

$$F_0^* (e^\alpha \lambda) = e^{a(t')} ds(t')$$

(3.13)

So, $F_0$ is a Lagrangian embedding.
If the circle $C_1$ homotopic to $C_1 \subset L \times s_0$ then we compute
\[
\int_C F_0^* (e^a \lambda) = \int_{C_1} F_0^* (e^a \lambda) = 0. \tag{3.14}
\]
since $\lambda|_{C_1} = 0$ due to $C_1 \subset L$ and $L$ is Legendre submanifold.

If the circle $C$ homotopic to $C_1 \subset l_0 \times S^1$ then we compute
\[
\int_C F_0^* (e^a \lambda) = \int_{C_1} F_0^* (e^{a(t')}) ds(t') = n(\text{area}(E)). \tag{3.15}
\]
This proves the Lemma.

Now we modify the above construction as follows:

\[
F' : L \times R \times R \to ([0, \varepsilon] \times \Sigma) \subset T^* M

F'(l, s, a) = (a, \eta_s(l)) \tag{3.16}
\]

Now we embed a elliptic curve $E$ long along $s$–axis and thin along $b$–axis such that $E \subset [-s_1, s_2] \times [0, \varepsilon]$. We parametrize the $E$ by $t'$.

**Lemma 3.5** If there does not exist any Reeb chord in $(\Sigma, \lambda)$, then

\[
F : L \times S^1 \to ([0, \varepsilon] \times \Sigma) \subset T^* M

F(l, t') = (a(t'), \eta_s(t')(l)) \tag{3.17}
\]

is a compact Lagrangian submanifold. Moreover

\[
l(V', F(L \times S^1), d(p_i dq_i)) = \text{area}(E) \tag{3.18}
\]

Proof. We check that

\[
F^*(p_i dq_i) = F^*(e^a \lambda + d\beta). \tag{3.19}
\]

This proves the Lemma.

Now we construct an isotopy of Lagrangian embeddings as follows:

\[
F' : L \times S^1 \times [0, 1] \to V'

F'(l, t', t) = h_t^T (a(t'), \eta_s(t')(l))

F'_t(l, t') = F'(l, t', t). \tag{3.20}
\]

**Lemma 3.6** If there does not exist any Reeb chord for $X_{\lambda}$ in $(\Sigma, \lambda)$ then $F'$ is an weakly exact isotopy of Lagrangian embeddings. Moreover for the choice of $T = T_\Sigma$ satisfying $[0, \varepsilon] \times \Sigma \cap h_T^T ([0, \varepsilon] \times \Sigma) = \emptyset$, then $F_0^*(L \times S^1) \cap F'_1(L \times S^1) = \emptyset$. 

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Proof. By Lemma 3.1-3.5 and below.

Let $(V', \omega') = (T^*M, dp_i \wedge dq_i)$, $W' = F(\mathcal{L} \times S^1)$, and $(V, \omega) = (V' \times C, \omega' \oplus \omega_0)$. As in [10], we use figure eight trick invented by Gromov to construct a Lagrangian submanifold in $V$ through the Lagrange isotopy $F'$ in $V'$. Fix a positive $\delta < 1$ and take a $C^\infty$-map $\rho : S^1 \to [0, 1]$, where the circle $S^1$ is parametrized by $\Theta \in [-1, 1]$, such that the $\delta$–neighborhood $I_0$ of $0 \in S^1$ goes to $0 \in [0, 1]$ and $\delta$–neighbourhood $I_1$ of $\pm 1 \in S^1$ goes 1 in $[0, 1]$. Let $h^T_{\rho}(t, w') = h^T_{\rho(0)}(w')$ and

$$\tilde{l} = h^T_{\rho}(p, dq_i) = p, dq_i - \rho(\Theta) T dg$$

$$= e^{a(t')} ds(t') + d\beta - \rho T dg = e^{a(t')} ds(t') + d\beta + d\rho T g + T g dp$$

$$= e^{a(t')} ds(t') + d\beta + d\rho T g - T g \rho'(\Theta) d\Theta$$

$$= e^{a(t')} ds(t') + d\beta + d\rho T g - \Phi d\Theta$$

be the pull-back of the form $\tilde{l} = e^{a(t')} ds(t') + d\beta + d\rho T g - \psi(s, t) dt$ to $W' \times S^1$ under the map $(w', \Theta) \to (w', \rho(\Theta))$ and assume without loss of generality $\Phi$ vanishes on $W' \times (I_0 \cup I_1)$. Since $[\tilde{l}]|W' \times \{t\} = [e^{a(t')} ds(t')]$ is independent of $t$, so $F'$ is weakly exact. It is crucial here $| - \psi(s, t)| \leq M_0$ and $M_0$ is independent of $area(E)$.

Next, consider a map $\alpha$ of the annulus $S^1 \times [\Phi_-, \Phi_+]$ into $R^2$, where $\Phi_-$ and $\Phi_+$ are the lower and the upper bound of the function $\Phi$ correspondingly, such that

(i) The pull-back under $\alpha$ of the form $dx \wedge dy$ on $R^2$ equals $-d\Phi \wedge d\Theta$.

(ii) The map $\alpha$ is bijective on $I \times [\Phi_-, \Phi_+]$ where $I \subset S^1$ is some closed subset, such that $I \cup I_0 \cup I_1 = S^1$; furthermore, the origin $0 \in R^2$ is a unique double point of the map $\alpha$ on $S^1 \times 0$, that is

$$0 = \alpha(0, 0) = \alpha(\pm 1, 0),$$

and $\alpha$ is injective on $S^1 \times 0$ minus $\{0, \pm 1\}$.

(iii) The curve $S^0 = \alpha(S^1 \times 0) \subset R^2$ “bounds” zero area in $R^2$, that is $\int_{S^0} xdy = 0$, for the 1–form $xdy$ on $R^2$.

**Proposition 3.1** Let $V'$, $W'$ and $F'$ as above. Then there exists an exact Lagrangian embedding $F : W' \times S^1 \to V' \times R^2$ given by $F(w', \Theta) = (F'(w', \rho(\Theta)), \alpha(\Theta, \Phi))$. Denote $F(W' \times S^1)$ by $W$. $W \subset T^*M \times B_{r_0}(0)$ with $4\pi r_0^2 = 8M_0$.

**Proof.** Similar to [10, 2.3B_3].

### 3.2 Formulation of Hilbert manifolds

Let $(\Sigma, \lambda)$ be a closed $(2n - 1)$–dimensional manifold with a contact form $\lambda$ of induced type in $T^*M$, it is well-known that $T^*M$ is a Stein manifold, so it is

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exhausted by a proper pluri-subharmonic function. In fact since $M$ is an open manifold one can take a proper Morse function $g$ on $M$ and let $f = \frac{|p|^2}{2} + \pi^* g$. Then $f$ is pluri-subharmonic function on $T^*M$ for some complex structure $J'$ on $T^*M$ tamed by $dp_i \wedge dq_i$ (see [5]). Since $\Sigma$ is compact and $W' = G'((L \times R)$ is contained in $\Sigma$, by our construction we have $W'$ is contained in a compact set $f_c$ for $c$ large enough.

Let $V' = T^*M$ and we choose an almost complex structure $J'$ on $T^*M$ tamed by $\omega' = dp_i \wedge dq_i$ and the metric $g' = \omega'(\cdot, J'^{\cdot})$ (see [10]). By above discussion we know that all mechanism such as $W'$ or $\Sigma$ contained in $f_c$ for $c$ large enough, i.e., contained in a compact set $V'_c$ in $T^*M$. Then we expanding near $\partial f^{-1}(c)$ to get a complete exact symplectic manifold with a complete Riemann metric with injective radius $r_0 > 0$(see [16]).

In the following we denote by $(V, \omega) = (V' \times R^2, dp_i dq_i \oplus dx \wedge dy)$ with the metric $g = g' \oplus g_0$ induced by $\omega(\cdot, J^{\cdot})(J = J' \oplus i$ and $W \subset V$ a Lagrangian submanifold which was constructed in section 3.1, moreover we can slightly perturb the $J' \oplus i$ near $p$ such that $J \oplus i$ is integrable near $p$.

Let

$$D^k(V, W, p) = \{ u \in H^k(D, V) | u(x) \in W \, a.e \, for \, x \in \partial D \, and \, u(1) = p \}$$

for $k \geq 100$.

**Lemma 3.7** Let $W$ be a Lagrangian submanifold in $V$. Then,

$$D^k(V, W, p) = \{ u \in H^k(D, V) | u(x) \in W \, a.e \, for \, x \in \partial D \, and \, u(1) = p \}$$

is a pseudo-Hilbert manifold with the tangent bundle

$$TD^k(V, W, p) = \bigcup_{u \in D^k(V, W, p)} \Lambda^{k-1}(u^*TV, u^*|_{\partial D}TW, p) \quad (3.22)$$

here

$$\Lambda^{k-1}(u^*TV, u^*|_{\partial D}TW, p) = \{ H^{k-1} - sections of (u^*(TV), (u^*)|_{\partial D}^*TL) which \, vanishes \, at \, 1 \}$$

Proof: See [4, 14].

Now we consider a section from $D^k(V, W, p)$ to $TD^k(V, W, p)$ follows as in [4, 10], i.e., let $\bar{\partial} : D^k(V, W, p) \to TD^k(V, W, p)$ be the Cauchy-Riemann section

$$\bar{\partial} u = \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} \quad (3.23)$$

for $u \in D^k(V, W, p)$. 

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Theorem 3.1  The Cauchy-Riemann section $\bar{\partial}$ defined in (3.23) is a Fredholm section of Index zero.

Proof. According to the definition of the Fredholm section, we need to prove that $u \in \mathcal{D}^k(V, W, p)$, the linearization $D\bar{\partial}(u)$ of $\bar{\partial}$ at $u$ is a linear Fredholm operator. Note that

$$D\bar{\partial}(u) = D\bar{\partial}_{[u]}$$

(3.24)

where

$$(D\bar{\partial}_{[u]})v = \frac{\partial v}{\partial s} + J\frac{\partial v}{\partial t} + A(u)v$$

(3.25)

with

$$v|_{\partial D} \in (u|_{\partial D})^*TW$$

here $A(u)$ is $2n \times 2n$ matrix induced by the torsion of almost complex structure, see [4, 10] for the computation.

Observe that the linearization $D\bar{\partial}(u)$ of $\bar{\partial}$ at $u$ is equivalent to the following Lagrangian boundary value problem

$$\frac{\partial v}{\partial s} + J\frac{\partial v}{\partial t} + A(u)v = f, \quad v \in \Lambda^k(u^*TV)$$

$$v(t) \in T_{u(t)}W, \quad t \in \partial D$$

(3.26)

One can check that (3.26) defines a linear Fredholm operator. In fact, by Proposition 2.2 and Lemma 2.1, since the operator $A(u)$ is a compact, we know that the operator $\bar{\partial}$ is a nonlinear Fredholm operator of the index zero.

Definition 3.1 Let $X$ be a Banach manifold and $P : Y \to X$ the Banach vector bundle. A Fredholm section $F : X \to Y$ is proper if $F^{-1}(0)$ is a compact set and is called generic if $F$ intersects the zero section transversally, see [4, 7, 10].

Definition 3.2 $\deg(F, y) = \sharp\{F^{-1}(0)\}\mod 2$ is called the Fredholm degree of a Fredholm section (see [4, 7, 10]).

Theorem 3.2 The Fredholm section $F = \bar{\partial} : \mathcal{D}^k(V, W, p) \to T(\mathcal{D}^k(V, W, p))$ constructed in (3.23) is proper near $F^{-1}(0)$ and

$$\deg(F, 0) = 1$$

Proof: We assume that $u : D \to V$ be a $J$–holomorphic disk with boundary $u(\partial D) \subset W$ and by the assumption that $u$ is homotopic to the constant map $u_0(D) = p$. Since almost complex structure $J$ tamed by the symplectic form $\omega$, by stokes formula, we conclude $u : D \to V$ is a constant map. Because
\( u(1) = p \). We know that \( F^{-1}(0) = p \) which implies the properness. Next we show that the linearization \( DF(p) \) of \( F \) at \( p \) is an isomorphism from \( T_pD(V,W,p) \) to \( E \). This is equivalent to solve the equations

\[
\frac{\partial v}{\partial s} + J \frac{\partial v}{\partial t} + Av = f
\]

\[v|_{\partial D} \subset T_pW \]  

(3.27)

where \( J = J(p) = i \) and \( A(=0) \) a constant zero matrix. By Lemma 2.1, we know that \( DF(p) \) is an isomorphism. Therefore \( \deg(F,0) = 1 \).

4 Non-properness of a Fredholm section

In this section we shall construct a non-proper Fredholm section \( F_1 : D \rightarrow E \) by perturbing the Cauchy-Riemann section as in [4, 10].

4.1 Anti-holomorphic section

Let \((V',\omega') = (T^*M,\omega_M)\) and \((V,\omega) = (V' \times C, \omega' \oplus \omega_0)\), and \( W \) as in section 3 and \( J = J' \oplus i \), \( g = g' \oplus g_0 \), \( g_0 \) the standard metric on \( C \).

Now let \( c \in C \) be a non-zero vector. We consider the equations

\[ v = (v', f) : D \rightarrow V' \times C \]

\[ \bar{\partial}_v v' = 0, \bar{\partial} f = c \]

\[ v|_{\partial D} : \partial D \rightarrow W \]  

(4.1)

here \( v \) homotopic to constant map \( \{p\} \) relative to \( W \). Note that \( W \subset V' \times B_{r_0}(0) \).

**Lemma 4.1** Let \( v \) be the solutions of (4.1), then one has the following estimates

\[
E(v) = \int_D (g'(\frac{\partial v'}{\partial x}, J' \frac{\partial v'}{\partial y}) + g'(\frac{\partial v'}{\partial y}, J' \frac{\partial v'}{\partial y})
\]

\[+g_0(\frac{\partial f}{\partial x}, i \frac{\partial f}{\partial y}) + g_0(\frac{\partial f}{\partial y}, i \frac{\partial f}{\partial y}))d\sigma \leq 4\pi r_0^2. \]  

(4.2)

Proof: Since \( v(z) = (v'(z), f(z)) \) satisfy (4.1) and \( v(z) = (v'(z), f(z)) \in V' \times C \) is homotopic to constant map \( v_0 : D \rightarrow \{p\} \subset W \) in \((V, W)\), by the Stokes formula

\[ \int_D v^*(\omega' \oplus \omega_0) = 0 \]  

(4.3)
Note that the metric $g$ is adapted to the symplectic form $\omega$ and $J$, i.e.,

$$g = \omega(\cdot, J \cdot)$$  \hspace{1cm} (4.4)

By the simple algebraic computation, we have

$$\int_D v^*\omega = \frac{1}{4} \int_{D^2} (|\partial v|^2 - |\bar{\partial} v|^2) = 0$$  \hspace{1cm} (4.5)

and

$$|\nabla v|^2 = \frac{1}{2}(|\partial v|^2 + |\bar{\partial} v|^2)$$  \hspace{1cm} (4.6)

Then

$$E(v) = \int_D |\nabla v|^2$$

$$= \int_D \left\{ \frac{1}{2}(|\partial v|^2 + |\bar{\partial} v|^2) \right\} d\sigma$$

$$= \frac{\pi |e|^2}{g_0}$$  \hspace{1cm} (4.7)

By the equations (4.1), one get

$$\bar{\partial} f = c \text{ on } D$$  \hspace{1cm} (4.8)

We have

$$f(z) = \frac{1}{2}c \bar{z} + h(z)$$  \hspace{1cm} (4.9)

here $h(z)$ is a holomorphic function on $D$. Note that $f(z)$ is smooth up to the boundary $\partial D$, then, by Cauchy integral formula

$$\int_{\partial D} f(z)dz = \frac{1}{2}c \int_{\partial D} \bar{z}dz + \int_{\partial D} h(z)dz$$

$$= \frac{1}{\pi i c} \int_{\partial D} \bar{z}dz$$  \hspace{1cm} (4.10)

So, we have

$$|c| = \frac{1}{\pi} \left| \int_{\partial D^2} f(z)dz \right|$$  \hspace{1cm} (4.11)

Therefore,

$$E(v) \leq \pi |c|^2 \leq \frac{1}{\pi} \left| \int_{\partial D} f(z)dz \right|^2$$

$$\leq \frac{1}{\pi} \left| \int_{D^2} |f(z)||dz|^2 \right|$$

$$\leq 4\pi |\text{diam}(pr_2(W))|^2$$

$$\leq 4\pi r_0^2.$$  \hspace{1cm} (4.12)

This finishes the proof of Lemma.
Proposition 4.1 For $|c| \geq 3r_0$, then the equations (4.1) has no solutions.

Proof. By (4.11), we have
\[
|c| \leq \frac{1}{\pi} \int_{\partial D} |f(z)||dz| \\
\leq \frac{1}{2} \int_{\partial D} \text{diam}(pr_2(W))|dz| \\
\leq 2r_0.
\] (4.13)

It follows that $c = 3r_0$ can not be obtained by any solutions.

4.2 Modification of section $c$

Note that the section $c$ is not a section of the Hilbert bundle in section 3 since $c$ is not tangent to the Lagrangian submanifold $W$, we must modify it as follows:

Let $c$ as in section 4.1, we define
\[
c_{\chi,\delta}(z, v) = \begin{cases} 
  c & \text{if } |z| \leq 1 - 2\delta, \\
  0 & \text{otherwise}
\end{cases}
\] (4.14)

Then by using the cut off function $\varphi_h(z)$ and its convolution with section $c_{\chi,\delta}$, we obtain a smooth section $c_\delta$ satisfying
\[
c_\delta(z, v) = \begin{cases} 
  c & \text{if } |z| \leq 1 - 3\delta, \\
  0 & \text{if } |z| \geq 1 - \delta.
\end{cases}
\] (4.15)

for $h$ small enough, for the convolution theory see [12].

Now let $c \in C$ be a non-zero vector and $c_\delta$ the induced anti-holomorphic section. We consider the equations
\[
v = (v', f) : D \rightarrow V' \times C \\
\bar{\partial}_J v' = 0, \bar{\partial}f = c_\delta \\
v|_{\partial D} : \partial D \rightarrow W
\] (4.16)

Note that $W \subset V \times B_{r_0}(0)$. Then by repeating the same argument as section 4.1., we obtain

Lemma 4.2 Let $v$ be the solutions of (4.16) and $\delta$ small enough, then one has the following estimates
\[
E(v) \leq 4\pi r_0^2.
\] (4.17)

and

Proposition 4.2 For $|c| \geq 3r_0$, then the equations (4.16) has no solutions.
4.3 Modification of $J \oplus i$

Let $(\Sigma, \lambda)$ be a closed contact manifold with a contact form $\lambda$ of induced type in $T^*M$. Let $J_M$ be an almost complex structure on $T^*M$ and $J_1 = J_M \oplus i$ the almost complex structure on $T^*M \times \mathbb{R}^2$ tamed by $\omega' \oplus \omega_0$. Let $J_2$ be any almost complex structure on $T^*M \times \mathbb{R}^2$.

Now we consider the almost complex structure on the symplectic fibration $D \times V \to D$ which will be discussed in detail in section 5.1., see also [10].

\[
J_{x,\delta}(z, v) = \begin{cases} 
  i \oplus J_M \oplus i & \text{if } |z| \leq 1 - 2\delta, \\
  i \oplus J_2 & \text{otherwise}
\end{cases} \quad (4.18)
\]

Then by using the cut off function $\varphi_h(z)$ and its convolution with section $J_{x,\delta}$, we obtain a smooth section $J_\delta$ satisfying

\[
J_\delta(z, v) = \begin{cases} 
  i \oplus J_M \oplus i & \text{if } |z| \leq 1 - 3\delta, \\
  i \oplus J_2 & \text{if } |z| \geq 1 - \delta.
\end{cases} \quad (4.19)
\]

as in section 4.2.

Then as in section 4.2, one can also reformulation of the equations (4.16) and get similar estimates of Cauchy-Riemann equations, we leave it as exercises to reader.

**Theorem 4.1** The Fredholm sections $F_1 = \bar{\partial} + c_\delta : \mathcal{D}_k(V, W, p) \to T(\mathcal{D}_k(V, W, p))$ is not proper for $|c|$ large enough.

Proof. See [4, 10].

5 $J$–holomorphic section

Recall that $W \subset T^*M \times B_{r_0}(0)$ as in section 3. The Riemann metric $g$ on $M \times \mathbb{R}^2$ induces a metric $g|W$.

Now let $c \in C$ be a non-zero vector and $c_\delta$ the induced anti-holomorphic section. We consider the nonlinear inhomogeneous equations (4.16) and transform it into $J$–holomorphic map by considering its graph as in [4, 10].

Denote by $Y^{(1)} \to D \times V$ the bundle of homomorphisms $T_s(D) \to T_v(V)$. If $D$ and $V$ are given the disk and the almost Kähler manifold, then we distinguish the subbundle $X^{(1)} \subset Y^{(1)}$ which consists of complex linear homomorphisms and we denote $\hat{X}^{(1)} \to D \times V$ the quotient bundle $Y^{(1)}/X^{(1)}$.

Now, we assign to each $C^1$-map $v : D \to V$ the section $\bar{\partial}v$ of the bundle
over the graph $\Gamma_v \subset D \times V$ by composing the differential of $v$ with the quotient homomorphism $Y^{(1)} \to \bar{X}^{(1)}$. If $c_\delta : D \times V \to \bar{X}$ is a $H^k-$ section we write $\bar{\partial}v = c_\delta$ for the equation $\bar{\partial}v = c_\delta|_{\Gamma_v}$.

**Lemma 5.1 (Gromov[10])** There exists a unique almost complex structure $J_g$ on $D \times V$ (which also depends on the given structures in $D$ and in $V$), such that the (germs of) $J_\delta-$holomorphic sections $v : D \to D \times V$ are exactly and only the solutions of the equations $\bar{\partial}v = c_\delta$. Furthermore, the fibres $z \times V \subset D \times V$ are $J_\delta-$holomorphic (i.e. the subbundles $T(z \times V) \subset T(D \times V)$ are $J_\delta-$complex) and the structure $J_\delta|z \times V$ equals the original structure on $V = z \times V$. Moreover $J_\delta$ is tamed by $k\omega_0 \oplus \omega$ for $k$ large enough which is independent of $\delta$.

### 6 Proof of Theorem 1.1

**Theorem 6.1** There exists a non-constant $J-$holomorphic map $u : (D, \partial D) \to (T^*M \times C, W)$ with $E(u) \leq 4\pi r_0^2$.

Proof. By Gromov’s $C^0-$convergence theorem and the results in section 4 shows the solutions of equations (4.16) must denegerate to a cusp curves, i.e., we obtain a Sacks-Uhlenbeck’s bubble, i.e., $J-$holomorphic sphere or disk with boundary in $W$, the exactness of the symplectic form on $T^*M \times R^2$ rules out the possibility of $J-$holomorphic sphere. For the more detail, see the proof of Theorem 2.3.B in [10].

**Proof of Theorem 1.1.** By Theorem 6.1, we know that

\[
l(V', F(\mathcal{L} \times S^1), d(p_i dq_i)) = area(E) \leq 4\pi r_0^2
\]

But if $K$ large enough, $area(E) > 8\pi r_0^2$. This implies the assumption that $\mathcal{L}$ has no self-intersection point under Reeb flow does not hold.

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