On the existence of $d$-homogeneous 3-way Steiner trades

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Abstract

A $\mu$-way $(v, k, t)$ trade $T = \{T_1, T_2, ..., T_\mu\}$ of volume $m$ consists of $\mu$ disjoint collections $T_1, T_2, ..., T_\mu$, each of $m$ blocks of size $k$, such that for every $t$-subset of $v$-set $V$ the number of blocks containing this $t$-subset is the same in each $T_i$ (for $1 \leq i \leq \mu$). A $\mu$-way $(v, k, t)$ trade is called $\mu$-way $(v, k, t)$ Steiner trade if any $t$-subset of $\text{found}(T)$ occurs at most once in $T_i$ ($T_j$, $j \geq 2$). A $\mu$-way $(v, k, t)$ trade is called $d$-homogeneous if each element of $V$ occurs in precisely $d$ blocks of $T_1$ ($T_j$, $j \geq 2$). In this paper we characterize the 3-way 3-homogeneous $(v, 3, 2)$ Steiner trades of volume $v$. Also we show how to construct a 3-way $d$-homogeneous $(v, 3, 2)$ Steiner trade for $d \in \{4, 5, 6\}$, except for seven small values of $v$.

Key words: Steiner trade, $\mu$-way trade, Homogeneous trade.

Subject classification: 05B05.

1 Introduction and preliminary results

Let $V$ be a set of $v$ elements and $k$ and $t$ be two positive integers such that $t < k < v$. A $(v, k, t)$ trade $T = \{T_1, T_2\}$ of volume $m$ consists of two disjoint

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collections \(T_1\) and \(T_2\), each of containing \(m\), \(k\)-subsets of \(V\), called blocks, such that every \(t\)-subset of \(V\) is contained in the same number of blocks in \(T_1\) and \(T_2\). A \((v, k, t)\) trade is called a Steiner trade if any \(t\)-subset of \(V\) occurs in at most once in \(T_1(T_2)\). In a \((v, k, t)\) trade, both collections of blocks must cover the same set of elements. This set of elements is called the foundation of the trade and is denoted by \(\text{found}(T)\).

The concept of \(\mu\)-way \((v, k, t)\) trade, was defined recently in [6].

**Definition:** A \(\mu\)-way \((v, k, t)\) trade \(T = \{T_1, T_2, \ldots, T_\mu\}\) of volume \(m\) consists of \(\mu\) disjoint collections \(T_1, T_2, \ldots, T_\mu\), each of \(m\) blocks of size \(k\), such that for every \(t\)-subset of \(v\)-set \(V\) the number of blocks containing this \(t\)-subset is the same in each \(T_i\) (for \(1 \leq i \leq \mu\)). In other words any pair of collections \(\{T_i, T_j\}\), \(1 \leq i < j \leq \mu\) is a \((v, k, t)\) trade of volume \(m\). It is clear by the definition that a trade is a 2-way trade. A \(\mu\)-way \((v, k, t)\) trade is called \(\mu\)-way \((v, k, t)\) Steiner trade if any \(t\)-subset of \(\text{found}(T)\) occurs at most once in \(T_i\) (\(T_j, j \geq 2\)). A \(\mu\)-way \((v, k, t)\) trade is called \(d\)-homogeneous if each element of \(V\) occurs in precisely \(d\) blocks of \(T_1\) (\(T_j, j \geq 2\)). Let \(x \in \text{found}(T)\), the number of blocks in \(T_i\) (for \(1 \leq i \leq \mu\)) which contains \(x\) is denoted by \(r_x\). The set of blocks in \(T_i\) (for \(1 \leq i \leq \mu\)) which contains \(x\) is denoted by \(T_{ix}\) (for \(1 \leq i \leq \mu\)). It is easy to see that \(T_x = \{T_{1x}, \ldots, T_{\mu x}\}\) is a \(\mu\)-way \((v, k, t - 1)\) trade of volume \(r_x\). If we remove \(x\) from the blocks of \(T_x\), then the result will be a \(\mu\)-way \((v - 1, k - 1, t - 1)\) trade which is called the derived trade of \(T\). Also it is easy to show that if \(T\) is a Steiner trade then its derived trade is also a Steiner trade. A trade \(T' = \{T'_1, T'_2, \ldots, T'_\mu\}\) is called a subtrade of \(T = \{T_1, T_2, \ldots, T_\mu\}\), if \(T'_i \subseteq T_i\) for \(1 \leq i \leq \mu\).

For \(\mu = 2\), Cavenagh et al. [4], constructed minimal \(d\)-homogeneous \((v, 3, 2)\) Steiner trades of foundation \(v\) and volume \(dv/3\) for sufficiently large values of \(v\), (specifically, \(v > 3(1.75d^2 + 3)\) if \(v\) is divisible by 3 and \(v > d(4d^2/3 + 1) + 1\) otherwise).

Generally we can ask the following question.

**Question 1.** For given \(\mu\), \(d\) and \(v\), does there exist a \(\mu\)-way \(d\)-homogeneous \((v, 3, 2)\) Steiner trade?

In this paper, we aim to construct 3-way \(d\)-homogeneous \((v, 3, 2)\) Steiner trades. The Latin trades are a useful tool for building these trades when \(v \equiv 0\) (mod 3), so we use some obtained results on 3-way \(d\)-homogeneous Latin trades.

A Latin square of order \(n\) is an \(n \times n\) array \(L = (\ell_{ij})\) usually on the set
$N = \{1, 2, ..., n\}$ where each element of $N$ appears exactly once in each row and exactly once in each column. We can represent each Latin square as a subset of $N \times N \times N$, $L = \{(i, j; k)\}$ element $k$ is located in position $(i, j)$. A partial Latin square of order $n$ is an $n \times n$ array $P = (p_{ij})$ of elements from the set $N$ where each element of $N$ appears at most once in each row and at most once in each column. The set $S_P = \{(i, j) \mid (i, j; k) \in P\}$ of the partial Latin square $P$ is called the shape of $P$ and $|S_P|$ is called the volume of $P$.

A $\mu$-way Latin trade, $(L_1, L_2, ..., L_\mu)$, of volume $s$ is a collection of $\mu$ partial Latin squares $L_1, L_2, ..., L_\mu$ containing exactly the same $s$ filled cells, such that if cell $(i, j)$ is filled, it contains a different entry in each of the $\mu$ partial Latin squares, and such that row $i$ in each of the $\mu$ partial Latin squares contains, set-wise, the same symbols and column $j$, likewise. Adams et al.\cite{1} studied $\mu$-way Latin trades. A $\mu$-way Latin trade which is obtained from another one by deleting its empty rows and empty columns, is called a $\mu$-way $d$-homogeneous Latin trade (for $\mu \leq d$) or briefly a $(\mu, d, m)$ Latin trade, if it has $m$ rows and in each row and each column $L_r$ for $1 \leq r \leq \mu$, contains exactly $d$ elements, and each element appears in $L_r$ exactly $d$ times. Bagheri et al.\cite{2}, studied the $\mu$-way $d$-homogeneous Latin trades and their main result is as follows:

**Theorem A.**\cite{2} All $(3, d, m)$ Latin trades (for $4 \leq d \leq m$) exist, for

(a) $d = 4$, except for $m = 6$ and $7$ and possibly for $m = 11$,
(b) $d = 5$, except possibly for $m = 6$,
(c) $6 \leq d \leq 13$,
(d) $d = 15$,
(e) $d \geq 4$ and $m \geq d^2$,
(f) $m$ a multiple of $5$, except possibly for $m = 30$,
(g) $m$ a multiple of $7$, except possibly for $m = 42$ and $(3, 4, 7)$ Latin trade.

All 3-way $(v, 3, 2)$ Steiner trades are characterized in \cite{6}. The authors proved that there is no 3-way $(v, 3, 2)$ Steiner trade of volumes $1, 2, 3, 4, 5$ and $7$. Also they showed that the 3-way $(v, 3, 2)$ Steiner trade of volume $6$ is unique (where the number of occurrences of each element is not the same). So the following proposition is clear.
Proposition 1. The 3-way $d$-homogeneous $(v, 3, 2)$ Steiner trade of volume $m$ does not exist for $m \in \{1, 2, ..., 7\}$.

Remark 1. Since the volume of a $\mu$-way $d$-homogeneous $(v, 3, 2)$ Steiner trade is $dv/3$, at least one of $d$ or $v$ should be multiple of 3.

Remark 2. In a $\mu$-way $d$-homogeneous $(v, 3, 2)$ Steiner trade, since every element should belong to $d$ blocks and the other elements of these blocks should be different, so $v \geq 2d + 1$.

Lemma 1. If there exist two 3-way $d$-homogeneous $(v_1, 3, 2)$ and $(v_2, 3, 2)$ Steiner trades of volume $m_1$ and $m_2$, respectively, then we have a 3-way $d$-homogeneous $(v_1 + v_2, 3, 2)$ Steiner trade of volume $m_1 + m_2$.

Proof. Let $T = \{T_1, T_2, T_3\}$ be a 3-way $d$-homogeneous $(v_1, 3, 2)$ Steiner trade of volume $m_1$ and $T' = \{T'_1, T'_2, T'_3\}$ be a 3-way $d$-homogeneous $(v_2, 3, 2)$ Steiner trade of volume $m_2$. It is enough to relabel the elements of $\text{found}(T')$, such that $\text{found}(T) \cap \text{found}(T') = \emptyset$. It is clear that $S = \{T_1 \cup T'_1, T_2 \cup T'_2, T_3 \cup T'_3\}$ is a 3-way $d$-homogeneous $(v_1 + v_2, 3, 2)$ Steiner trade of volume $m_1 + m_2$. □

The following lemma which is similar to Lemma 2 of [4], shows how to construct a 3-way $d$-homogeneous $(3m, 3, 2)$ Steiner trade, by using 3-way $d$-homogeneous Latin trade of order $m$.

Lemma 2. Let $(L_1, L_2, L_3)$ be a 3-way $d$-homogeneous Latin trade of order $m$. For each $a \in \{1, 2, 3\}$, define $T_a = \{\{i_1, j_2, k_3\} | (i, j, k) \in L_a\}$. Then $T = \{T_1, T_2, T_3\}$ is a 3-way $d$-homogeneous $(3m, 3, 2)$ Steiner trade.

The following theorem is a consequence of Theorem A and Lemma 2.

Theorem 1. All 3-way $d$-homogeneous $(3m, 3, 2)$ Steiner trades (for $4 \leq d \leq m$) exist for

(a) $d = 4$, except possibly for $m = 6, 7$ and 11,

(b) $d = 5$, except possibly for $m = 6,$

(c) $6 \leq d \leq 13,$
(d) \(d = 15\),
(e) \(d \geq 4\) and \(m \geq d^2\),
(f) \(m \equiv 0 \pmod{5}\), except possibly for \(m = 30\),
(g) \(m \equiv 0 \pmod{7}\), except possibly for \(m = 7\) (where \(d = 4\)) and \(m = 42\).

The existence of 3-way \(d\)-homogeneous \((v, 3, 2)\) Steiner trades when \(v\) is not multiple of 3, will be investigated later.

A Steiner triple system of order \(v\) (briefly \(STS(v)\)) is a pair \((X, B)\) where \(X\) is a \(v\)-set and \(B\) is a collection of 3-subsets of \(X\) (called triple) such that every pair of distinct elements of \(X\) belongs to exactly one triple of \(B\). It is well known that a \(STS(v)\) exists if and only if \(v \equiv 1, 3 \pmod{6}\). A Kirkman triple system of order \(v\) (briefly \(KTS(v)\)) is a Steiner triple system of order \(v\), \((X, B)\) together with a partition \(R\) of the set of triples \(B\) into subsets \(R_1, R_2, ..., R_n\) called parallel classes such that each \(R_i\) (for \(i = 1, 2, ..., n\)) is a partition of \(X\).

**Theorem 2.** There exists a 3-way \(((v-1)/2)\)-homogeneous \((v, 3, 2)\) Steiner trade of volume \(v(v-1)/6\) for every \(v \equiv 1, 3 \pmod{6}\), \(v \neq 7\).

**Proof.** In [5], it is shown that there exist 3 disjoint \(STS(v)\) for every \(v \equiv 1, 3 \pmod{6}, v \neq 7\). It is obvious that the 3-way trade \(T = \{T_1, T_2, T_3\}\), where \(T_1, T_2\) and \(T_3\) are the disjoint \(STS(v)\)s, is the desired trade. \(\square\)

## 2 3-way 3-homogeneous \((v, 3, 2)\) Steiner trades

In this section we answer Question 1 when \(\mu = d = 3\). Note that by Remark 1 there is no restriction for \(v\) in this case.

**Lemma 3.** For every \(v = 9\ell\), where \(\ell \in \{1, 2, 3, \ldots\}\), there exists a 3-way 3-homogeneous \((v, 3, 2)\) Steiner trade of volume \(v\).

**Proof.** According to Corollary 1 of [2], there exists a \((3, 3, m)\) Latin trade if and only if \(3|m\). By Lemma 2 we can obtain a 3-way 3-homogeneous \((v, 3, 2)\) Steiner trade of volume \(v\) from a \((3, 3, m)\) Latin trade, where \(v = 3m\). \(\square\)
Lemma 4. For every $v = 8\ell$, where $\ell \in \{1, 2, 3, \cdots \}$, there exists a 3-way $3$-homogeneous $(v, 3, 2)$ Steiner trade of volume $v$.

Proof. The following trade is a 3-way $3$-homogeneous $(8, 3, 2)$ Steiner trade.

$$
\begin{array}{cccccccc}
T_1 & 123 & 147 & 158 & 248 & 267 & 357 & 368 \\
T_2 & 124 & 138 & 157 & 237 & 268 & 467 & 458 \\
T_3 & 127 & 135 & 148 & 246 & 238 & 367 & 457 \\
\end{array}
$$

So the proof is obvious by Lemma 1. $\square$

The last two lemmas and Lemma 1 yields the following theorem.

Theorem 3. For every nonzero $v = 9\ell + 8\ell'$, where $\ell, \ell' \in \{0, 1, 2, 3, \cdots \}$, there exists a 3-way $3$-homogeneous $(v, 3, 2)$ Steiner trade of volume $v$.

The following lemma can be used for characterizing 3-way $3$-homogeneous $(v, 3, 2)$ Steiner trades of volume $v$.

Lemma 5. There exist only two non-isomorphic 3-way $(v, 2, 1)$ Steiner trade of volume 3.

Proof. Let $T = (T_1, T_2, T_3)$ be a 3-way $(v, 2, 1)$ Steiner trade of volume 3. Since $T$ is a Steiner trade, $|\text{found}(T)| = 6$ and let $\text{found}(T) = \{1, 2, \ldots, 6\}$. Without loss of generality we can assume that $T_1 = \{12, 34, 56\}$ and $T_2 = \{13, 26, 45\}$. Now if we consider the blocks of the trade as edges in a 6 vertex graph and color the edges of each trade with a different color, then the first two trades form an alternating colored 6-cycle. Up to isomorphism, there are only two different one factors in $K_6$ to complete this graph to a 3-regular graph. So all trademates put together either form a bipartite or a non-bipartite 3-regular graph. So $T = (T_1, T_2, T_3)$ up to isomorphism, should be as one of the following trades:

$$
\begin{array}{ccc|ccc}
T_1 & T_2 & T_3 & T_1 & T_2 & T_3 \\
12 & 13 & 14 & 12 & 13 & 15 \\
34 & 26 & 25 & 34 & 26 & 24 \\
56 & 45 & 36 & 56 & 45 & 36 \\
\end{array}
$$

$\square$
**Theorem 4.** Every 3-way 3-homogeneous \((v, 3, 2)\) Steiner trade of volume \(v\) contains a 3-way 3-homogeneous \((u, 3, 2)\) Steiner trade of volume 8 or 9, as a subtrade.

**Proof.** Let \(T = \{T_1, T_2, T_3\}\) be a 3-way 3-homogeneous \((v, 3, 2)\) Steiner trade of volume \(v\) with \(\text{found}(T) = \{1, 2, ..., v\}\). Let \(x \in \text{found}(T)\), then \(T_x = \{T_{1x}, T_{2x}, T_{3x}\}\) is a \((v, 3, 1)\) trade of volume \(r_x = 3\). Without loss of generality we can assume that \(\text{found}(T_x) = \{x, 1, 2, ..., 6\}\). According to Lemma 5 there exist only two cases for \(T_x\).

| \(T_{1x}\) | \(T_{2x}\) | \(T_{3x}\) | \(T_{1x}\) | \(T_{2x}\) | \(T_{3x}\) |
|---|---|---|---|---|---|
| \(x12\) | \(x13\) | \(x14\) | \(x12\) | \(x13\) | \(x15\) |
| \(x34\) | \(x26\) | \(x25\) | \(x34\) | \(x26\) | \(x24\) |
| \(x56\) | \(x45\) | \(x36\) | \(x56\) | \(x45\) | \(x36\) |

Let \(T\) contains \(T_x\) which is as the first form.

Since \(T = \{T_1, T_2, T_3\}\) is a 3-homogeneous trade, each of \(T_1, T_2\) and \(T_3\) should contain two other blocks containing element 1. According to definition of Steiner trades, \(T_1\) cannot contain block 134 (since it has block \(x34\)). So there exist three possible cases:

1. Only one of \(T_2\) and \(T_3\) contains block 123 or 124:
   - Let \(T_2\) contains block 123 (If \(T_1\) contains block 124, then the same result will be achieved).

   | \(T_1\) | \(x12\) | \(x34\) | \(x56\) | \(13a\) | \(14b\) |
   | \(T_2\) | \(x14\) | \(x25\) | \(x36\) | \(123\) | \(1ab\) |
   | \(T_3\) | \(x13\) | \(x26\) | \(x45\) | \(12b\) | \(14a\) |

   There are four possible cases for elements \(a\) and \(b\):

   1.1. \(a, b \notin \{5, 6\}\)

      The other blocks which contain 4 should be as follows. The pair 3b exists in \(T_3\) and does not exist in \(T_2\), it is a contradiction with definition of trade.

   | \(T_1\) | \(x12\) | \(x34\) | \(x56\) | \(13a\) | \(14b\) | \(45a\) |
   | \(T_2\) | \(x14\) | \(x25\) | \(x36\) | \(123\) | \(1ab\) | \(45b\) | \(4a3\) |
   | \(T_3\) | \(x13\) | \(x26\) | \(x45\) | \(12b\) | \(14a\) | \(43b\) |
1.2. $a = 6$ and $b = 5$

The other blocks of $T_1$ and $T_2$ which contain 6 should be as follows. It is a contradiction with definition of trade.

|   |   |   |   |   |
|---|---|---|---|---|
| $T_1$ | x12 | x34 | x56 | 136 | 145 | 624 |
| $T_2$ | x14 | x25 | x36 | 123 | 156 | 624 |
| $T_3$ | x13 | x26 | x45 | 125 | 146 |   |

1.3. $a \notin \{5, 6\}$ and $b = 5$

A 3-way 3-homogeneous $(8, 3, 2)$ is achieved.

|   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|
| $T_1$ | x12 | x34 | x56 | 13a | 145 | 52a | 236 | 46a |
| $T_2$ | x14 | x25 | x36 | 123 | 15a | 546 | 26a | 34a |
| $T_3$ | x13 | x26 | x45 | 125 | 14a | 56a | 23a | 364 |

1.4. $a = 6$ and $b \notin \{5, 6\}$

Only one of the remaining blocks of $T_2$ can contain 6, but we should have $62, 64, 65$ is $T_2$ which is impossible.

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| $T_1$ | x12 | x34 | x56 | 13a | 14b |
| $T_2$ | x14 | x25 | x36 | 123 | 16b |
| $T_3$ | x13 | x26 | x45 | 12b | 14a |

2. $T_2$ and $T_3$ contain block 123 and 124, respectively:

The other blocks of $T_2$ and $T_3$ which contain 1 should be as follows. It is a contradiction with definition of trade.

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| $T_1$ | x12 | x34 | x56 | 13a | 14b |
| $T_2$ | x14 | x25 | x36 | 123 | 1ab |
| $T_3$ | x13 | x26 | x45 | 12b | 1ab |

3. $T_2$ and $T_3$ do not contain the blocks 123 and 124:

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| $T_1$ | x12 | x34 | x56 | 13a | 14b |
| $T_2$ | x14 | x25 | x36 | 13b | 12a |
| $T_3$ | x13 | x26 | x45 | 14a | 12b |

There are four possible cases for elements $a$ and $b$:

3.1. $a, b \notin \{5, 6\}$

A 3-way 3-homogeneous $(9, 3, 2)$ is achieved.

|   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|
| $T_1$ | x12 | x34 | x56 | 13a | 14b | 36b | 45a | 62a | 52b |
| $T_2$ | x14 | x25 | x36 | 13a | 12b | 34a | 45a | 65a | 62b |
| $T_3$ | x13 | x26 | x45 | 14a | 12b | 34b | 3a6 | 65b | 52a |
3.2. \( a = 6 \) and \( b = 5 \)

The other blocks of \( T_1 \), \( T_2 \) and \( T_3 \) which contain 6 should be as follows. So the other blocks of \( T_2 \) and \( T_3 \) which contain 4 should be 423. It is a contradiction with definition of trade.

\[
\begin{array}{cccccccc}
T_1 & x_{12} & x_{34} & x_{56} & 136 & 145 & 624 \\
T_2 & x_{14} & x_{25} & x_{36} & 135 & 126 & 645 & 423 \\
T_3 & x_{13} & x_{26} & x_{45} & 146 & 125 & 635 & 423 \\
\end{array}
\]

3.3. \( a \notin \{5,6\} \) and \( b = 5 \)

The other blocks which contain 5 should be as follows. So the other blocks of \( T \) which contain 2 should be as follows. 36 appears two times in \( T_2 \). It is a contradiction with definition of Steiner trade.

\[
\begin{array}{cccccccc}
T_1 & x_{12} & x_{34} & x_{56} & 13a & 145 & 523 & 26a \\
T_2 & x_{14} & x_{25} & x_{36} & 135 & 12a & 546 & 236 \\
T_3 & x_{13} & x_{26} & x_{45} & 14a & 125 & 536 & 23a \\
\end{array}
\]

3.4. \( a = 6 \) and \( b \notin \{5,6\} \)

The other blocks which contain 6 should be as follows. There is a block in \( T_2 \) which contain 45. 4 appears three times in \( T_1 \) but the pair 45 does not appear in blocks of \( T_1 \). It is a contradiction with definition of trade.

\[
\begin{array}{cccccccc}
T_1 & x_{12} & x_{34} & x_{56} & 13b & 14b & 624 \\
T_2 & x_{14} & x_{25} & x_{36} & 13b & 12b & 645 \\
T_3 & x_{13} & x_{26} & x_{45} & 14b & 12b & 635 \\
\end{array}
\]

For the other case, the same result can be obtained by a similar argument. So it can be deduced that if there exists a 3-way 3-homogeneous \((v,3,2)\) Steiner trade of volume \( v \) (for \( v \geq 8 \)), then it contains a 3-way 3-homogeneous \((8,3,2)\) or \((9,3,2)\) Steiner trade of volume 8 or 9, respectively. \( \square \)

The following corollary is the direct result of Theorem 4.

**Corollary 1.** If there exists a 3-way 3-homogeneous \((v,3,2)\) Steiner trade of volume \( v \), then it can be represented as a union of disjoint 3-way 3-homogeneous \((8,3,2)\) or \((9,3,2)\) Steiner trades of volume 8 or 9, respectively.

Define \([a, b] = \{c \in Z| a \leq c \leq b\}\).
**Theorem 5.** The 3-way 3-homogeneous \((v, 3, 2)\) Steiner trade of volume \(v\) does not exist for \(v \in [1, 7] \cup [10, 15] \cup [19, 23] \cup [28, 31] \cup [37, 39] \cup \{46, 47, 55\} \).

**Proof.** By Proposition 1, the 3-way 3-homogeneous \((v, 3, 2)\) Steiner trade of volume \(v\) does not exist for \(v \in [1, 7]\). Let there exist a 3-way 3-homogeneous \((v, 3, 2)\) Steiner trade \(T\) of volume \(v\) for \(v \in [10, 15]\). By Theorem 4, it should contain a 3-way 3-homogeneous \((8, 3, 2)\) or \((9, 3, 2)\) Steiner trade \(T'\) of volume 8 or 9, respectively. Therefore, \(T \setminus T'\) is a 3-way 3-homogeneous \((u, 3, 2)\) Steiner trade of volume \(u\), where \(u \in [1, 7]\), which is impossible. By same argument, let there exist a 3-way 3-homogeneous \((v, 3, 2)\) Steiner trade \(T\) of volume \(v\) for \(v \in [19, 23]\). By Theorem 4, it should contain a 3-way 3-homogeneous \((8, 3, 2)\) or \((9, 3, 2)\) Steiner trade \(T'\) of volume 8 or 9, respectively. So \(T \setminus T'\) is a 3-way 3-homogeneous \((u, 3, 2)\) Steiner trade of volume \(u\), where \(u \in [10, 15]\), which is a contradiction. The same way, we can prove non-existence of the other mentioned trades. 

**Theorem 6.** For every \(v \geq 8\), there exists a 3-way 3-homogeneous \((v, 3, 2)\) Steiner trade of volume \(v\), except for \(v \in [10, 15] \cup [19, 23] \cup [28, 31] \cup [37, 39] \cup \{46, 47, 55\}\).

**Proof.** According to Theorem 3, it is enough to represent every \(v \geq 8\) in the form \(9\ell + 8\ell'\), where \(\ell, \ell' \geq 0\) as follows:
\[
\begin{align*}
v & = 9k, \text{ where } k \geq 1 \\
v & = 9k + 1 = 9(k - 7) + 64, \text{ where } k - 7 \geq 0 \\
v & = 9k + 2 = 9(k - 6) + 56, \text{ where } k - 6 \geq 0 \\
v & = 9k + 3 = 9(k - 5) + 48, \text{ where } k - 5 \geq 0 \\
v & = 9k + 4 = 9(k - 4) + 40, \text{ where } k - 4 \geq 0 \\
v & = 9k + 5 = 9(k - 3) + 32, \text{ where } k - 3 \geq 0 \\
v & = 9k + 6 = 9(k - 2) + 24, \text{ where } k - 2 \geq 0 \\
v & = 9k + 7 = 9(k - 1) + 16, \text{ where } k - 1 \geq 0 \\
v & = 9k + 8, \text{ where } k \geq 0
\end{align*}
\]
Using Theorem 5 completes the proof. 

# 3 3-way \(d\)-homogeneous \((v, 3, 2)\) Steiner trade for \(d \in \{4, 5, 6\}\)

In this section we answer Question 1 when \(\mu = 3\) and \(d = 4, 5\) and 6.
### 3.1 $d = 4$

In this subsection we completely answer Question 1 for $d = 4$. Note that since $d = 4$ by Remark 1, $v$ should be a multiple of 3.

**Proposition 2.** For $v = 9, 18, 21$ and $33$, there exist 3-way 4-homogeneous $(v, 3, 2)$ Steiner trades of volume 12, 24, 28 and 44, respectively.

**Proof.** According to Theorem 2, there exists a 3-way 4-homogeneous $(9, 3, 2)$ Steiner trade of volume 12. By Theorem 1, there exist 3-way 4-homogeneous $(12, 3, 2)$ and $(15, 3, 2)$ Steiner trades of volume 16 and 20, respectively. Since $18 = 9 + 9$, $21 = 12 + 9$ and $33 = 15 + 18$, regarding to Lemma 1, the result follows. □

**Theorem 7.** There exists a 3-way 4-homogeneous $(v, 3, 2)$ Steiner trade if and only if $v \geq 9$ and $v \equiv 0 \pmod{3}$.

**Proof.** The result is followed by Theorem 1, Proposition 2 and Remarks 1 and 2. □

### 3.2 $d = 5$

In this subsection we solve Question 1 for $d = 5$, except for $v = 18$. By Remark 1, $v$ should be a multiple of 3.

**Proposition 3.** There exists a 3-way 5-homogeneous $(12, 3, 2)$ Steiner trade of volume 20.

**Proof.** It is enough to use three disjoint decomposition of $K_{12} - I$ (the graph obtained from $K_{12}$ by removing the edges of a perfect matching) into triples. In other words, in the literature of block designs, it is enough to consider three disjoint compatible $(2, 3)$-packings on 12 points (see [3]). □

**Theorem 8.** Except possibly for $v = 18$, there exists a 3-way 5-homogeneous $(v, 3, 2)$ Steiner trade if and only if $v \geq 12$ and $v \equiv 0 \pmod{3}$.

**Proof.** The result follows by Theorem 1, Proposition 3 and Remarks 1 and 2. □
### 3.3 \( d = 6 \)

In this subsection we solve Question 1 for \( d = 6 \), only six values of \( v \) are left undecided. Note that by Remark 1 there is no restriction for \( v \) in this case.

**Proposition 4.** There exist a 3-way 6-homogeneous \((v, 3, 2)\) Steiner trade of volume \( 2v \), where \( v = 13, 14, 15 \) and 16.

**Proof.** By Theorem 2 there exist a 3-way 6-homogeneous \((13, 3, 2)\) Steiner trade of volume 26.

Let \( X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, f\} \) and let \((X, B)\) be a \( KTS(15)\), where \( B = \{123, 48e, 5ae, 6bd, 79f, 145, 28a, 3df, 69e, 7bc, 167, 29b, 3ce, 4af, 58d, 189, 2cf, 356, 4be, 7ad, 2de, 347, 59e, 68f, 1cd, 246, 39a, 5bf, 78e, 1ef, 257, 38h, 49d, 6ac\}. We define the following permutations on \( X \):

\[
\begin{align*}
\pi_1 &= (1\,e)(2\,4)(3\,7)(5\,6)(8\,b)(a\,c)(9\,d), \\
\pi_2 &= (9\,7)(3\,d)(4\,a)(2\,c)(6\,8)(5\,b)(1\,e).
\end{align*}
\]

The intersection of three sets \( B, \pi_1(B) \) and \( \pi_2(B) \) is \( C = \{79f, 3df, 4af, 2cf, 68f, 5bf, 1ef\} \). So \( T = \{T_1, T_2, T_3\} \) is a 3-way 6-homogeneous \((14, 3, 2)\) Steiner trade of volume 28, where \( T_1 = B \setminus C \), \( T_2 = \pi_1(B) \setminus C \) and \( T_3 = \pi_2(B) \setminus C \).

Let \( X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, f\} \) and let \((X, B_1 \cup B_2)\) be a \( KTS(15)\), where \( B_1 = \{12f, 345, 678, 9ab, cde\} \) and \( B_2 = \{36f, 15e, 24a, 7bc, 89d, 9cf, 147, 2be, 38a, 56d, 7af, 16c, 258, 3bd, 49e, 4df, 18b, 269, 37e, 5ac, 8ef, 1ad, 23c, 46b, 579, 5bf, 139, 27d, 48c, 6ae\} \).

We define permutations \( \pi = (6\,7\,8)(9\,b\,a)(c\,d\,e) \) on \( X \):

\( B_2, \pi(B_2) \) and \( \pi(\pi(B_2)) \) are disjoint. So \( T = \{B_2, \pi(B_2), \pi(\pi(B_2))\} \) is a 3-way 6-homogeneous \((15, 3, 2)\) Steiner trade of volume 30.

Let \( Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, f, g\} \) and \( D = \{59e, 9cg, 7af, 8bd, 5bf, 6ad, 7ce, 89g, 19f, 2bg, 3cd, 4ae, 1ag, 2cf, 3be, 49d, 15d, 28e, 36f, 47g, 16e, 27d, 35g, 48f, 17b, 269, 38a, 45c, 18c, 25a, 379, 46b\} \). In fact \( D \) is the block set of a Kirkman frame of type \( \left(4^4, 2^{12}\right) \). We define the following permutations on \( Y \):

\[
\begin{align*}
\pi_3 &= (1\,2)(3\,4)(5\,6)(7\,8), \\
\pi_4 &= (1\,3)(2\,4)(5\,7)(6\,8)(9\,a)(b\,c).
\end{align*}
\]

\( D, \pi_3(D) \) and \( \pi_4(D) \) are disjoint. So \( T = \{D, \pi_3(D), \pi_4(D)\} \) is a 3-way 6-homogeneous \((16, 3, 2)\) Steiner trade of volume 32. \( \square \)
Theorem 9. There exists a 3-way 6-homogeneous \((v, 3, 2)\) Steiner trade of volume \(2v\) for every \(v \geq 13\), except possibly for \(v \in \{17, 19, 22, 23, 25\}\).

Proof. By Remark 2, \(v \geq 13\). We investigate three cases for \(v\). For \(v = 3\ell\), where \(\ell \geq 6\), the result clearly follows by Theorem 1. For cases \(v = 3\ell + 1 = 3(\ell - 4) + 13\) and \(v = 3\ell + 2 = 3(\ell - 4) + 14\), where \(\ell \geq 10\), we use Theorem 1 and Proposition 4.

For \(v \in \{13, 14, 15, 16, 26, 28, 29\}\), by Proposition 4 and Lemma 1, there exists a 3-way 6-homogeneous \((v, 3, 2)\) Steiner trade of volume \(2v\).

Our results are summarized below:

Main Theorem. All 3-way \(d\)-homogeneous \((v, 3, 2)\) Steiner trades exist for

(I) \(v = 3m\):
   \(\begin{align*}
   (a) & \quad d = 4, \ m \geq 3, \text{ (by Theorem 7)} \\
   (b) & \quad d = 5, \ m \geq 4 \text{ except possibly for } m = 6, \text{ (by Theorem 8)} \\
   (c) & \quad 7 \leq d \leq 13, \ m \geq d, \text{ (by Theorem 1)} \\
   (d) & \quad d = 15, \ m \geq d, \text{ (by Theorem 1)} \\
   (e) & \quad d \geq 4 \text{ and } m \geq d^2, \text{ (by Theorem 1)} \\
   (f) & \quad m \equiv 0 \text{ (mod 5)} \text{ and } m \geq d, \text{ except possibly for } m = 30, \text{ (by Theorem 1)} \\
   (g) & \quad m \equiv 0 \text{ (mod 7)} \text{ and } m \geq d, \text{ except possibly for } m = 7 \text{ (where } d = 4) \text{ and } m = 42, \text{ (by Theorem 1)}
   \end{align*}\)

(II) \(v = 6m + 1\):
   \(d = 3m, \ m \geq 2, \text{ (by Theorem 2)}\)

(III) \(v = 6m + 3\):
   \(d = 3m + 1, \ m \geq 1, \text{ (by Theorem 2)}\)

(IV) All \(v\):
   \(\begin{align*}
   (a) & \quad d = 3, \ v \geq 8 \text{ except for } v \in \{10, 15\} \cup \{19, 23\} \cup \{28, 31\} \cup \{37, 39\} \cup \{46, 47, 55\}, \text{ (by Theorem 6)} \\
   (b) & \quad d = 6, \ v \geq 13 \text{ except possibly for } v \in \{17, 19, 20, 22, 23, 25\}. \text{ (by Theorem 9)}
   \end{align*}\)
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