Dual Forms on Supermanifolds and Cartan Calculus

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Abstract

The complex of “stable forms” on supermanifolds is studied. Stable forms on $M$ are represented by certain Lagrangians of “copaths” (formal systems of equations, which may or may not specify actual surfaces) on $M \times \mathbb{R}^D$. Changes of $D$ give rise to stability isomorphisms. The Cartan-de Rham complex made of stable forms extends both in positive and negative degree and its positive half is isomorphic to the complex of forms defined as Lagrangians of paths. Considering the negative half is necessary, in particular, for homotopy invariance.

We introduce analogs of exterior multiplication by covectors and of contraction with vectors. We find (anti)commutation relations for them. An analog of Cartan’s homotopy identity is proved. Before stabilization it contains a stability operator $\sigma$.

Introduction

The crucial difference of “exterior algebra” in the super case from the usual case is that the analog of the “top exterior power” for a $\mathbb{Z}_2$-graded vector space cannot be obtained by tensor operations. This is because the determinant in the super case (the Berezinian) is not a polynomial expression, but a fraction whose numerator and denominator separately are not multiplicative. Thus the space $\text{Ber} V$ (which corresponds to the usual $\det V$) enters
independently of the “naive” generalization of exterior multiplication by the sign rule. A complete theory of “exterior forms” has to be built upon the Berezinian from the beginning. This fact has far reaching consequences.

“Naive” differential forms on a supermanifold $M^{n|m}$ are, of course, (locally) polynomials in $dx^A$, where $x^A$ are coordinates. Experts know that there are two possible conventions for the parity and commutation relations for the differentials (see [9]). According to one of them, $dx^A$ is assigned the same parity as $x^A$ and the differentials anticommute: the flip of $dx^A$ and $dx^B$ results in the factor $-(-1)^{\tilde{A}\tilde{B}}$. The other convention assigns to $dx^A$ the parity opposite to that of $x^A$ and the differentials are regarded as commuting variables. We shall refer to them in the sequel as to skew-commutative and commutative conventions, respectively.

From the viewpoint of integration, the fatal drawback of such naive forms is that they can’t be integrated over $M = M^{n|m}$ (unless $m = 0$). Because of that, some remedies were suggested.

Bernstein and Leites [3] defined “integral forms” as tensor products of multivector fields with Berezin volume forms. This permitted integration over $M^{n|m}$ and an analog of Gauss-Ostrogradsky formula. If we are integration-minded, we expect that the correct forms on supermanifolds should be graded by super dimensions $r\mid s$ (dimensions of surfaces or chains over which a form can be integrated). Thus, integral forms should correspond to “$r\mid m$-forms” ($s = m$) and volume forms to “$n\mid m$-forms”. Naive differential forms from this point of view correspond to “$r\mid 0$-forms” ($s = 0$). What about other values of $r$, $s$?

For non-polynomial functions of $dx^A$ (with the commutative convention) Bernstein and Leites [4] showed that they also can be integrated over $M^{n|m}$ provided they sufficiently rapidly decrease in $d\xi^\mu$, where $\xi^\mu$ are odd coordinates. Such “pseudodifferential forms” are very beautiful. However, since they do not have any grading (and, in fact, are good for integration only for a particular type of orientation and not good for others, see [11]) they do not solve the problem.

A crucial step towards the theory of “$r\mid s$-forms” was made by A.S. Schwarz, M.A. Baranov, A.V. Gajduk, O.M. Khudaverdian and A.A. Rosly in the beginning of 1980-s and was motivated by quantum field theory. They based their investigation of the “objects of integration” on supermanifolds directly on the notion of Berezinian and studied Lagrangians of parameterized surfaces $\Gamma : I^{r\mid s} \rightarrow M^{n|m}$ which induce volume forms on $r\mid s$-dimensional space $U^{r\mid s}$. They are called densities. The key result was the concept of “closed-
ness” of a density \([1, 5, 11]\): a density is said to be *closed* if the corresponding action is identically stationary. (On ordinary manifolds, for densities corresponding to closed forms this follows from the Stokes’ formula.)

As the author discovered, this notion of “closedness” precisely follows from a certain construction of a differential in terms of variational derivatives. Densities, initially defined only for embedded surfaces (hence \(0 \leq r \leq n\), \(0 \leq s \leq m\)), should be replaced by more general “covariant Lagrangians”, for which \(r \geq 0\) can exceed \(n\), and a certain system of differential equations with respect to the components of tangent vectors is imposed upon Lagrangians. Roughly speaking, this system (see Eq. (2) below) is a non-trivial analog of multilinearity/skew symmetry property of the usual exterior forms. (The odd-odd part of the system amazingly coincides with the equations introduced by F. John [7] and Gelfand-Shapiro-Gindikin-Graev (see [3]) for the description of the image of the Radon-like transforms in integral geometry.) The theory of \(r|s\)-forms in this sense was developed by the author with A.V. Zorić [14, 15, 16, 17] and the author [11]. The differential has degree +1, so \(r|s\)-forms are mapped to \(r+1|s\)-forms. The complex obtained in this way possesses all natural properties of the usual Cartan-de Rham complex like functoriality in a suitable category, Stokes’ formula and homotopy invariance, and also has some similarity with extraordinary cohomology (an analog of the Atiyah-Hirzebruch spectral sequence), see [11]. For \(s = 0\), it naturally incorporates the “naive” generalization of differential forms. For \(s = m\) and \(r \geq 0\) it also incorporated integral forms of Bernstein and Leites.

However, an *ad hoc* augmentation of the complex had to be introduced [11] to achieve homotopy invariance. The existence of Bernstein-Leites integral forms of negative degree has also hinted to “hidden” \(r|s\)-forms with \(r < 0\).

Such objects were indeed discovered in [12]. Together with forms considered in [11] they give a desired de Rham complex stretching both in positive and negative directions.

The solution is based on the idea of a *dual form* [12] (important results were independently obtained in [8]). Geometrically, dual forms are Lagrangians of surfaces specified by maps \(M^{n|m} \supset U^{n|m} \rightarrow \mathbb{R}^{p|q}\) (copaths) rather than maps \(I^{r|s} \rightarrow M^{n|m}\) (paths). To define a complex, dual forms are not enough. One has to introduce new independent parameters and to allow to increase their number. An intermediate product is labeled “mixed form”. A whole bunch of isomorphisms enters the stage, and the final picture is the result of a stabilization (see [12] and subsection 1.1 below). (Geometrically, one gets a sort of “virtual surfaces”, which can have both negative
In the current paper we develop the algebraic and differential theory of stable forms (the unified complex). We do not touch integration. The main result of the paper is an analog of Cartan calculus that includes module structures for forms and the relation between the differential, Lie derivative and a “contraction operator” with a vector field (which is defined in this paper). All results are new. They will be used to study the homotopy properties of stable forms and the de Rham cohomology of supermanifolds.

The paper is organized as follows.

In Section 1 we define dual and mixed forms on a superspace $V$, the stability isomorphisms and isomorphisms with forms considered in [11]. Operators $e(\alpha)$ and $e(u)$ are introduced, where $u \in V$, $\alpha \in \Lambda^*(V)$. We prove that they are stable (commute with the stability isomorphisms) and relate them with operators on forms of [11] (Theorem 1.3). Then we find the relations that they obey. We get a “skew-commutative” version of a Clifford algebra involving a stability operator $\sigma$ as an additional central element (Theorem 1.4). As a corollary, we obtain module structures over the exterior algebras $\Lambda(\Lambda^*(V))$ and $\Lambda(V)$ (the skew-commutative versions).

In Section 2 we consider the complex of stable forms on a supermanifold $M$. We prove the Leibniz identity (=differential module structure) for the multiplication by naive differential forms $\omega \in \Omega^*(M)$ (Theorem 2.1). Then we consider the Lie derivative for mixed forms. We prove that the anticommutator of the differential and the operator $e(X)$, where $X$ is a vector field, equals the Lie derivative multiplied by the operator $\sigma$ (Theorem 2.2). It immediately implies a “Cartan’s homotopy identity” for stable forms.

The results are discussed in Section 3.

We mainly follow the notation and terminology of the book [11].

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1 Algebraic theory

1.1 Construction of forms. Stability isomorphisms

Consider a superspace $V$ over $\mathbb{R}$ of dimension $\dim V = n|m$. We identify vector superspaces with the corresponding supermanifolds. By $\text{Vol} V :=$
Ber $V^*$ we denote the space of volume forms on $V$. In the following we consider functions whose arguments are vectors or covectors. Components of vectors are written as rows, components of covectors as columns.

Recall the following definition.

**Definition 1.1** (see [14, 17, 11]). A form on $V$ of degree $r|s$ is a smooth map $L :_V \times \cdots \times V \times \Pi V \times \cdots \times \Pi V \rightarrow \mathbb{R}$ satisfying the following conditions (1) and (2):

$$L(gv) = L(v) \Ber g,$$

(1)

for all $g \in \GL(r|s)$ and

$$\frac{\partial^2 L}{\partial v_A^G \partial v_B^G} + (-1)^{\hat{F}G+(\hat{F}+\hat{G})B} \frac{\partial^2 L}{\partial v_A^G \partial v_F^B} = 0.$$  

(2)

In our notation the argument of the function $L$ is written as a matrix $v = (v_F^A)$ whose rows $v_F$ are vectors (written in components). The condition (1) implies that $L(v)$ is defined only if odd vectors $v_K, \hat{K} = 1$, are linearly independent. Hence $0 \leq s \leq m$, while $r \geq 0$ can be arbitrary.

Though this definition provides no efficient description of forms, such a description can be given in special cases (corresponding to naive differential forms and to Bernstein-Leites integral forms) and in other cases various examples can be provided. See [11]. In particular, if $m > 0$, for $s \neq m$ there are nonzero forms with $r > n$. We shall give here an illustrative example of an $r|s$-form.

**Example 1.1.** Let $\alpha^F \in V^*$ be an array of covectors of suitable parity. Then from the properties of the Berezinian it follows that the function $L(v) := \Ber(\langle v_F, \alpha^G \rangle)$ satisfies (1), (2). So it is a form. If $s > 0$, $L$ has a pole at those odd vectors whose linear span is not transverse to the annihilator of the linear span of the odd part of $(\alpha^G)$. If $s = 0$, then $L(v) = \det(\langle v_i, \alpha^j \rangle)$, where $i, j = 1, \ldots, r$, so $L$ is nothing else than the exterior product $\alpha^1 \wedge \cdots \wedge \alpha^r$. In general, this form with a singularity should be regarded as a “nonlinear analog” of the exterior product of an array of even and odd covectors $\alpha^F$. It naturally appears in physical context (e.g., [8],[2]).
As shown in [12], the above construction of forms is not sufficient and must be supplemented in order to obtain \( r|s \)-forms with \( r \in \mathbb{Z} \) arbitrary, including negative values. This is achieved by the following “dualization” and the subsequent “stability argument”. When we shall need to distinguish forms in the sense of Definition 1.1, we shall call them “straight forms”. We shall denote the space of (straight) \( r|s \)-forms on \( V \) by \( \Lambda^{r|s}(V) \).

**Definition 1.2.** A dual form on \( V \) of codegree \( p|q \) is a smooth map \( \mathcal{L} : V^* \times \cdots \times V^* \Pi \times \cdots \times V^* \Pi \rightarrow \text{Vol } V \) satisfying the conditions

\[
\mathcal{L}(ph) = \mathcal{L}(p) \text{Ber } h, \tag{3}
\]

for all \( h \in \text{GL}(p|q) \) and

\[
\frac{\partial^2 \mathcal{L}}{\partial p_A^K \partial p_B^L} + (-1)^{\overline{A}B + (\overline{A} + \overline{B})L} \frac{\partial^2 \mathcal{L}}{\partial p_B^K \partial p_A^L} = 0. \tag{4}
\]

The arguments of \( \mathcal{L} \) (covectors) are written as vector-columns, and they are organized in a matrix \( p = (p_A^K) \). Notice that due to the condition (4), odd covectors \( p^K, \overline{K} = 1 \), should be linearly independent, hence \( 0 \leq q \leq m \).

Fix a dimension \( r|s \) and consider \( V \oplus \mathbb{R}^{r|s} \).

**Definition 1.3.** A mixed form on \( V \) of codegree \( p|q \) and additional degree \( r|s \) is a smooth map

\[
\mathcal{L} : (V \oplus \mathbb{R}^{r|s})^* \times \cdots \times (V \oplus \mathbb{R}^{r|s})^* \times (V \oplus \mathbb{R}^{r|s}) \Pi \times \cdots \times (V \oplus \mathbb{R}^{r|s}) \Pi \rightarrow \text{Vol } V
\]

satisfying the following conditions (5)–(9):

\[
\mathcal{L}(ph, wh) = \mathcal{L}(p, w) \text{Ber } h, \tag{5}
\]

for all \( h \in \text{GL}(p|q) \),

\[
\mathcal{L}(p + aw, gw) = \mathcal{L}(p, w) \text{Ber } g, \tag{6}
\]
for all \( g \in \text{GL}(r|s) \) and all \( a \in \text{Mat}(r|s \times n|m) \), and

\[
\frac{\partial^2 L}{\partial p_A K \partial p_B L} + (-1)^{\hat{A} + \hat{B}} L \frac{\partial^2 L}{\partial p_B K \partial p_A L} = 0, \tag{7}
\]

\[
\frac{\partial^2 L}{\partial p_A K \partial w_F L} + (-1)^{\hat{A} + \hat{F}} L \frac{\partial^2 L}{\partial w_F K \partial p_A L} = 0, \tag{8}
\]

\[
\frac{\partial^2 L}{\partial w_F K \partial w_G L} + (-1)^{\hat{F} + \hat{G}} L \frac{\partial^2 L}{\partial w_G K \partial w_F L} = 0, \tag{9}
\]

where \( p = (p_A^K) \), \( w = (w_F^L) \) and for a given \( K \) the entries \( p_A^K \), \( w_F^K \) are the components of a covector on \( V \oplus \mathbb{R}^r|s \) (where \( K \) is the number of the covector). Matrix notation suggests placing \( p \) over \( w \) in the argument of \( L \), but for typographic reasons we shall do it only when convenient. Notice that \( s \leq q \leq m + s \) because of (5),(6).

Examples of dual and mixed forms can be mimicked from the examples of straight forms (since they are defined via similar conditions), and we skip them.

Notation: \( \Lambda_{p/q}(V) \) and \( \Lambda_{p/q}^{r|s}(V) \) for the spaces of dual and mixed forms on \( V \), respectively. We shall omit the indication to \( V \) when no confusion is possible. Notice that \( \Lambda_{p/q}(V) = \Lambda_{p/q}^{0|0}(V) \).

Consider the following homomorphisms: \( \sigma = \sigma_{k|l} : \Lambda_{p/q}^{r|s} \to \Lambda_{p+k|q+l}^{r+k|s+l} \) and \( \sigma^{-1} = \sigma_{k|l}^{-1} : \Lambda_{p+k|q+l}^{r+k|s+l} \to \Lambda_{p/q}^{r|s} \).

\[
(\sigma L) \begin{pmatrix} p_1 & p_2 \\ w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} := L \begin{pmatrix} p_1 - p_2 w_{22}^{-1} w_{21} \\ w_{11} - w_{12} w_{22}^{-1} w_{21} \end{pmatrix} \cdot \text{Ber} w_{22}, \tag{10}
\]

\[
(\sigma^{-1} L^*) \begin{pmatrix} p \\ w \end{pmatrix} := L^* \begin{pmatrix} p & 0 \\ w & 0 \\ 0 & 1 \end{pmatrix}, \tag{11}
\]

where \( L \in \Lambda_{p/q}^{r|s}, L^* \in \Lambda_{p+k|q+l}^{r+k|s+l} \). (We write arguments of forms as matrices and subdivide them into blocks corresponding to the “first” and “last” rows and columns.)

**Theorem 1.1** ([12]). Maps \( \sigma \) and \( \sigma^{-1} \) are well-defined (in particular, \( \sigma \) uniquely extends to all admissible arguments of \( L \)) and are indeed mutually
inverse isomorphisms of the spaces $\Lambda^r|s$ and $\Lambda^{r+k}|s+l$. The equality $\sigma_k|\sigma_{k'} = \sigma_{k+k'|l+l'}$ holds.

Define $\Lambda^{k|l}(V) := \lim_{N,M} \Lambda^{r+N|s+M}(V)$, where $k|l = r + n - p|s + m - q$ and call it the space of stable $k|l$-forms on $V$. Note that $k \in \mathbb{Z}$ (may be negative), while $l = 0, \ldots, m$. It’s not hard to produce an example of a stable $k|l$-form with negative $k$ (if $l > 0$). Take as a representative a dual form with the number of even arguments greater that $n$ (exactly as in examples of straight $r|s$-forms with $r > n$, cf. [11]). Similarly, if $l < m$, there are nonzero $k|l$-forms with $k > n$.

Obviously, $\Lambda^{k|l}(V) \cong \Lambda^{r|s}(V)$ if $k = r + n - p$, $l = s + m - q$, for all $r, s, p \geq 0$ and $s \leq q \leq s + m$.

Corollary 1.1. $\Lambda^{k|l}(V) \cong \Lambda^{n-k|m-l}(V)$ for $k \leq n$.

Consider the following homomorphisms: $\tau = \tau_{r|s} : \Lambda^{r|s} \to \Lambda^{r|s}_{n|m}$ and $\tau^{-1} = \tau_{r|s}^{-1} : \Lambda^{r|s}_{n|m} \to \Lambda^{r|s}$,

\[
(\tau L) \left( \begin{array}{c} p \\ w \end{array} \right) := L(wp^{-1}) \cdot \text{Ber } p, \quad (12)
\]

\[
(\tau^{-1} L)(v) := \mathcal{L} \left( \begin{array}{c} 1 \\ v \end{array} \right), \quad (13)
\]

where $\mathcal{L} \in \Lambda^{r|s}_{n|m}$, $L \in \Lambda^{r|s}$.

Theorem 1.2 ([12]). Maps $\tau$ and $\tau^{-1}$ are well-defined (in particular, $\tau$ uniquely extends to all admissible arguments of $L$) and are indeed mutually inverse isomorphisms of the spaces $\Lambda^{r|s}_{n|m}$ and $\Lambda^{r|s}$.

Corollary 1.2. $\Lambda^{k|l}(V) \cong \Lambda^{k|l}(V)$ for $k \geq 0$.

Remark 1.1. In view of Theorems 1.1 and 1.2 one may regard it excessive to consider all spaces of mixed forms. Indeed, it is sufficient to consider only $\Lambda^{r|s}$ and $\Lambda^{p|q}$ together with the isomorphism $\Lambda^{r|s} \cong \Lambda^{n-r|m-s}$ defined in the range $0 \leq r \leq n$. However, it would be practically restrictive. It is easier to work with various operations in terms of mixed forms.
1.2 Operators $e(\alpha)$, $e(u)$. Commutation relations and the module structure

Consider a covector $\alpha \in V^*$. We introduce an operator $e(\alpha) : \Lambda^{r|s}_{p|q} \to \Lambda^{r+1|s}_{p|q}$ by the following formula:

$$e(\alpha) \mathcal{L} := (-1)^r \alpha_A w_{r+1}^K (1-1)_{\bar{A}}^A \frac{\partial \mathcal{L}}{\partial p_A^K},$$

(14)

where $\alpha = e^A \alpha_A$.

Likewise, consider a vector $u \in V$. Define $e(u) : \Lambda^{r|s}_{p|q} \to \Lambda^{r+1|s}_{p+1|q}$ by the formula

$$e(u) \mathcal{L} := (-1)^r u_A \left( p_A^{p+1} - (-1)^{\bar{B}K} p_A^K p_B^{p+1} \frac{\partial}{\partial p_B^K} - (-1)^{\bar{F}K} p_A^K w_{r+1}^F \frac{\partial}{\partial w_{r+1}^F} \right) \mathcal{L},$$

(15)

where $u = u^A e_A$. Here $(e_A)$ and $(e^A)$ are dual bases of $V$ and $V^*$.

Remark 1.2. On dual forms, $e(u) : \Lambda^{p|q}_{p|q} \to \Lambda^{p+1|q}_{p+1|q}$,  

$$e(u) \mathcal{L} = (-1)^r u_A \left( p_A^{p+1} - (-1)^{\bar{B}K} p_A^K p_B^{p+1} \frac{\partial}{\partial p_B^K} \right) \mathcal{L}. \quad (16)$$

The proof that $e(\alpha)$ and $e(u)$ indeed map forms to forms and do not depend on the choice of bases is postponed until Section 2. The parities of $e(\alpha)$ and $e(u)$ are the same as the respective parities of $\alpha$ and $u$; operators $e(\alpha)$ and $e(u)$ depend on $\alpha$ and $u$ linearly.

Theorem 1.3. The operators $e(\alpha)$ and $e(u)$ are stable, i.e., they commute with all isomorphisms $\sigma_{k|l}$. Under the isomorphism \([12]\), the operator $e(\alpha)$ corresponds to the operator $e_\alpha : \Lambda^{r|s}_{p|q} \to \Lambda^{r+1|s}_{p|q}$,

$$e_\alpha = (-1)^r \left( v_{r+1}^A \alpha_A - (-1)^{\bar{F}+\bar{B}} v_F^A \alpha_A v_{r+1}^B \frac{\partial}{\partial v_{r+1}^B} \right)$$

(17)

and if $r > 0$ the operator $e(u)$ corresponds to the operator $i_u : \Lambda^{r|s}_{p|q} \to \Lambda^{r-1|s}_{p|q}$,

$$i_u = (-1)^{r-1} u_A \frac{\partial}{\partial v_{r-1}^A},$$

(18)

the substitution of $u \in V$ into the last even slot of $L \in \Lambda^{r|s}$. Here $L = L(v)$, $v = (v_F^A)$. (The operators $e_\alpha$, $i_u$ were introduced in \([11]\).)
Proof. Consider $e(u)$. We have to check that $e(u)$ commutes with $\sigma_{1|0}$ and $\sigma_{0|1}$. We shall consider $\sigma_{1|0}$ (the case of $\sigma_{0|1}$ is similar, but simpler). Denote $\sigma := \sigma_{1|0}$. It is sufficient to give proof for $\mathcal{L} \in \Lambda_{p|q}$, then the general case will follow. Consider the diagram

$$
\begin{array}{ccc}
\Lambda_{p|q} & \xrightarrow{\sigma} & \Lambda^{1|0}_{p+1|q} \\
e(u) \downarrow & & \downarrow e(u) \\
\Lambda_{p+1|q} & \xrightarrow{\sigma} & \Lambda^{1|0}_{p+2|q}
\end{array}
$$

(19)

Take $\mathcal{L} \in \Lambda_{p|q}$. Apply $\sigma$. We get $\mathcal{L}^{*} \in \Lambda^{1|0}_{p+1|q}$, where $\mathcal{L}^{*} \left( \begin{array}{c} p \\ w \\ p' \\ w' \end{array} \right) = \mathcal{L} \left( \begin{array}{c} p \\ p' \\ w' \\ w' \end{array} \right) w'$. Here $p = (p_{A} K)$, $w = (w_{F} K)$, $p' = (p_{A}^{p+1})$, $w' = w^{p+1}$. Apply $e(u)$. We obtain

$$
\begin{align*}
(e(u)\mathcal{L}^{*}) \left( \begin{array}{c} p \\ w \\ p' \\ w' \end{array} \right) &= -u^{A} \left( p_{A}^{p+2} - (-1)(b_{K}^{*} p_{A} K p_{B}^{p+2} \frac{\partial}{\partial p_{B} K^{*}}) \right) \\
p_{A} K^{*} w^{p+2} \frac{\partial}{\partial w K^{*}} \mathcal{L}^{*} &= -u^{A} \left( p_{A}^{p+2} - (-1)(b_{K}^{*} p_{A} K p_{B}^{p+2} \frac{\partial}{\partial p_{B} K}) \right) \\
&= -u^{A} \left( p_{A}^{p+2} \mathcal{L} w^{p+1} - (-1)(b_{K}^{*} p_{A} K p_{B}^{p+2} \frac{\partial}{\partial p_{B} K} w^{p+1}) \right) + \\
p_{A}^{p+2} p_{B}^{p+2} K \frac{\partial}{\partial p_{B} K} + p_{A} K^{p+2} p_{A}^{p+1} \frac{\partial}{\partial w K} & \mathcal{L} \left( \begin{array}{c} p \\ p' \\ w' \\ w' \end{array} \right) w^{p+1} \\
&= \left( -\frac{1}{(w^{p+1})^{2}} \right) w^{p+1} - p_{A}^{p+1} w^{p+2} \mathcal{L}
\end{align*}
$$

(20)

where in the last expression the argument of $\mathcal{L}$ and $\partial \mathcal{L}/\partial p$ is $p - p' w^{p-1} w$ and we denote $p'' := (p_{A}^{p+2})$, $w'' := (w^{p+2})$. Now let us apply first $e(u)$, then $\sigma$. Calculate:

$$
(e(u)\mathcal{L})(p \quad p'') = u^{A} \left( p_{A}^{p+2} - (-1)(b_{K}^{*} p_{A} K p_{B}^{p+2} \frac{\partial}{\partial p_{B} K}) \right) \mathcal{L}(p); 
$$

(21)
applying $\sigma$ we obtain

$$(\sigma e(u)L) \left( \begin{array}{ccc} p & p' & p'' \\ w & w' & w'' \end{array} \right) = (e(u)L) \left( \begin{array}{ccc} p - p'w^{r-1} & p'' - p'(w^{p+1} - 1)w^{p+2} \\ w & w' & w'' \end{array} \right) = u^A \left( p_B^{p+2} - p_A^{p+1}(w^{p+1} - 1)w^{p+2} \right) \frac{\partial L}{\partial p_B^K} \right) w^{p+1},$$

(22)

where the argument of $L$ and $\partial L/\partial p$ in the last expression is $p - p'w^{r-1}w$. Multiplying through, we obtain exactly the same terms as in (20) with the opposite sign. Notice that $\sigma e(u)L$ as a form is skew-symmetric in even columns. Thus we can swap $\left( p' w' \right)$ and $\left( p'' w'' \right)$, cancelling the minus sign, and obtain

$$(\sigma e(u)L) \left( \begin{array}{ccc} p & p' & p'' \\ w & w' & w'' \end{array} \right) = (e(u)\sigma L) \left( \begin{array}{ccc} p & p' & p'' \\ w & w' & w'' \end{array} \right),$$

(23)

as desired. Stability of $e(\alpha)$ is proved in the same way, and we omit the calculation.

Let us turn to the relation with the isomorphisms (12). Consider the following diagram.

The claim is that it is commutative. To check this, take $L \in \Lambda^{|r|}_{n|m}$. We have:

$$(i_u\tau^{-1}L)(v) = (-1)^{r-1}u^A \frac{\partial}{\partial v^A}(\tau^{-1}L)(v) = (-1)^{r-1}u^A \frac{\partial L}{\partial v^A} \left( \begin{array}{c} 1 \\ v \end{array} \right) = (-1)^{r-1}u^A \frac{\partial L}{\partial w^A} \left( \begin{array}{c} 1 \\ v \end{array} \right);$$
now,

\[
(e(u)\mathcal{L}) \begin{pmatrix} p \\ w \\ w^{n+1} \end{pmatrix} = (-1)^{r} u^{A} \left( p_{A}^{n+1} - (-1) \hat{B} \hat{K} p_{A}^{K} p_{B}^{p+1} \frac{\partial}{\partial p_{B}^{K}} - (-1) \hat{F} \hat{K} p_{A}^{K} w_{F}^{n+1} \frac{\partial}{\partial w_{F}^{K}} \right) \mathcal{L} \begin{pmatrix} p \\ w \end{pmatrix};
\]

\[
(\sigma^{-1} e(u)\mathcal{L}) \begin{pmatrix} p \\ w^{*} \end{pmatrix} = e(u)\mathcal{L} \begin{pmatrix} p \\ w \\ p^{n+1} \\ w^{n+1} \end{pmatrix} \bigg|_{w_{r}^{n+1}=1} =
\]

\[
\begin{pmatrix} w_{r}^{K}=0 \quad (K \neq n+1) \\
 w_{F}^{n+1}=0 \quad (F \neq r) \\
 p_{A}^{n+1}=0 
\end{pmatrix}
\]

\[
(-1)^{r} u^{A} \left( 0 - (-1)^{0} p_{A}^{K} \frac{\partial}{\partial w_{r}^{K}} \right) \mathcal{L} \begin{pmatrix} p \\ w^{*} \\ 0 \end{pmatrix} = (-1)^{r} u^{A} \left( -p_{A}^{K} \frac{\partial \mathcal{L}}{\partial w_{r}^{K}} \left( \begin{pmatrix} p \\ w \end{pmatrix} \right) \right);
\]

hence

\[
(\tau^{-1} \sigma^{-1} e(u)\mathcal{L})(v) = (-1)^{r} \left( -u^{A} \frac{\partial \mathcal{L}}{\partial w_{A}^{K}} \left( \begin{pmatrix} 1 \\ v \end{pmatrix} \right) \right) = i_{u} \tau^{-1} \mathcal{L}(v),
\]

as desired. (Here \( w^{*} \) stands for \( w \) without the row \( w_{r} \).) In a similar way the equality \( e(\alpha)\tau = \tau e_{\alpha} : \Lambda^{r+1|s} \rightarrow \Lambda^{r+1|s} \) is checked.

**Corollary 1.3.** For exterior forms on a purely even space \( V \) the operator \( e(\alpha) \) corresponds to the usual exterior multiplication \( \alpha \wedge \). The operator \( e(u) \) corresponds to the usual interior multiplication or contraction \( i_{u} = u \wedge \).

Note that in our mixed description both operators increase respective degrees and thus have appearance of “exterior” products.

**Theorem 1.4.** The operators \( e(\alpha) \) and \( e(u) \) obey the following relations:

\[
e(u)e(v) + (-1)^{\bar{\alpha} \bar{\beta}} e(v)e(u) = 0,
\]

\[
e(\alpha)e(\beta) + (-1)^{\alpha \beta} e(\beta)e(\alpha) = 0,
\]

\[
e(u)e(\alpha) + (-1)^{\bar{\alpha} \bar{\alpha}} e(\alpha)e(u) = \langle u, \alpha \rangle \sigma.
\]

Here \( u, v \in V, \alpha, \beta \in V^{*}, \) and \( \sigma = \sigma_{1|0} : \Lambda_{p|q}^{r|s} \rightarrow \Lambda_{p+1|q}^{r+1|s} \) is the stability isomorphism \( \mathbb{I}^{10} \).
Proof. To find relations between $e(u)$ and $e(v)$, for $u, v \in V$, it is sufficient to consider the case $r = s = 0$. (The general case is formally reduced to it by considering dual forms on extended space $V \oplus \mathbb{R}^{|s|}$ and by setting $u^F = v^F = 0$.) Then for $\mathcal{L} \in \Lambda_{p|q}$ we have

$$e(u) e(v) \mathcal{L}$$

$$= u^A \left( p_A^{p+2} - (-1)^{\tilde{b} \tilde{K}} K_B^{p+2} \frac{\partial}{\partial p_B^R} \right) v^C \left( p_C^{p+1} - (-1)^{\tilde{D} \tilde{L}} p_C^{p+1} \frac{\partial}{\partial p_D^R} \right) \mathcal{L}$$

$$= u^A v^C (-1)^{\tilde{v} + \tilde{C}} \left( p_A^{p+2} p_C^{p+1} - p_A^{p+1} p_C^{p+1} - (-1)^{\tilde{D} \tilde{L}} p_A^{p+2} p_D^{p+1} p_C^{p+1} \frac{\partial}{\partial p_D^R} - (-1)^{\tilde{D} \tilde{L} + \tilde{A} \tilde{C} + \tilde{B} \tilde{K}} p_D^{p+1} p_C^{p+1} \frac{\partial}{\partial p_D^R} + (-1)^{\tilde{D} \tilde{L} + \tilde{A} \tilde{C} + \tilde{B} \tilde{K}} p_D^{p+1} p_C^{p+1} \frac{\partial}{\partial p_D^R} \right) \mathcal{L},$$

(27)

where $a = \tilde{B} \tilde{C} + \tilde{B} \tilde{L} + \tilde{B} \tilde{D} + \tilde{C} \tilde{K} + \tilde{K} \tilde{L} + \tilde{A} \tilde{B} + \tilde{A} \tilde{D} + \tilde{C} \tilde{D}$. Notice that the range of $K$ in the first line of (27) contains $p + 1$. Simultaneously interchanging $u$ and $v$ and the indices $A$ and $C$, we obtain

$$e(v) e(u) \mathcal{L}$$

$$= (-1)^{\tilde{u} \tilde{e}} u^A v^C (-1)^{\tilde{v} + \tilde{C}} \left( p_A^{p+1} p_C^{p+2} - p_A^{p+2} p_C^{p+1} - (-1)^{\tilde{D} \tilde{L}} p_A^{p+1} p_D^{p+1} p_C^{p+1} \frac{\partial}{\partial p_D^R} - (-1)^{\tilde{D} \tilde{L} + \tilde{A} \tilde{C} + \tilde{B} \tilde{K}} p_A^{p+2} p_D^{p+1} p_C^{p+1} \frac{\partial}{\partial p_D^R} + (-1)^{\tilde{D} \tilde{L} + \tilde{A} \tilde{C} + \tilde{B} \tilde{K}} p_A^{p+2} p_D^{p+1} p_C^{p+1} \frac{\partial}{\partial p_D^R} \right) \mathcal{L},$$

(28)

where $b = \tilde{C} \tilde{K} + \tilde{A} \tilde{B} + \tilde{K} \tilde{L} + \tilde{B} \tilde{C} + \tilde{C} \tilde{D} + \tilde{A} \tilde{D} + \tilde{L} \tilde{D}$. Now we see that all terms except for the last one in $(-1)^{\tilde{u} \tilde{e}} e(v) e(u) \mathcal{L}$ would cancel the similar terms in $e(u) e(v) \mathcal{L}$. Notice that $a + b = \tilde{B} \tilde{D} + (\tilde{B} + \tilde{D}) \tilde{L}$. It follows that

$$e(u) e(v) + (-1)^{\tilde{u} \tilde{e}} e(v) e(u) \mathcal{L} = (-1)^{a} p_B^{p+2} p_D^{p+1} p_A^K p_C^L \left( \frac{\partial^2 \mathcal{L}}{\partial p_B^K \partial p_D^L} + (-1)^{\tilde{B} \tilde{D} + (\tilde{B} + \tilde{D}) \tilde{L}} \frac{\partial^2}{\partial p_B^K \partial p_D^L} \right),$$

(29)
which equals zero by the equation (4).

Consider now $e(\alpha)$ and $e(\beta)$. For $L \in \Lambda_{p|q}^{r|s}$ we readily have

$$e(\alpha)e(\beta) L = (-1)^{r+1} \alpha_A w^K_{r+2} \frac{\partial}{\partial p^A K} \left( (-1)^r \beta_B w^L_{r+2} \frac{\partial L}{\partial p^B L} (-1)^{\beta \tilde{B}} \right) =$$

$$- (-1)^{\tilde{A} + \tilde{B}} \alpha_A \beta_B w^K_{r+2} w^L_{r+1} \frac{\partial^2 L}{\partial p^A K \partial p^B L} (-1)^{\tilde{B} + \tilde{A} + (\tilde{A} + \tilde{B}) L}. \quad (30)$$

Similarly, for $e(\beta)e(\alpha)$ we obtain

$$e(\beta)e(\alpha) L = - (-1)^{\tilde{B} + \tilde{A}} \alpha_A \beta_B w^K_{r+2} w^L_{r+1} \frac{\partial^2 L}{\partial p^A K \partial p^B L} =$$

$$- (-1)^{\tilde{B}} e(\alpha)e(\beta) L, \quad (31)$$

again by the equation (4).

Finally, let us find the relation between operators $e(u)$ and $e(\alpha)$. Notice that $e(u)e(\alpha), e(\alpha)e(u) : \Lambda_{p|q}^{r|s} \to \Lambda_{p+1|q}^{r+1|s}$. For $L \in \Lambda_{p|q}^{r|s}$ by a direct calculation similar to (27). (30) using the equations (7). (8), we obtain the equality

$$(e(u)e(\alpha) + (-1)^{\tilde{a} \tilde{u}} e(\alpha)e(u)) L =$$

$$u^A \alpha_A \left( w^{p+1}_{r+1} - (-1)^{\tilde{a} \tilde{K}} w^K_{r+1} p^A_{p+1} \frac{\partial}{\partial p^A K} - (-1)^{\tilde{K}} w^A_{r+1} w^F_{p+1} \frac{\partial}{\partial w^F K} \right) L. \quad (32)$$

Apply now the transformation $\sigma^{-1} : \Lambda_{p+1|q}^{r+1|s} \to \Lambda_{p|q}^{r|s}$. That means setting $w^{p+1}_{r+1} := 1, w^K_{r+1} := 0, p^A_{p+1} := 0, w^F_{p+1} := 0$. We arrive at

$$\sigma^{-1} \left( e(u)e(\alpha) + (-1)^{\tilde{a} \tilde{u}} e(\alpha)e(u) \right) L = (u, \alpha) L, \quad (33)$$

from where (26) follows. Notice that by this calculation we showed that the operator in the r.h.s. of (32) gives another expression for the isomorphism $\sigma_{1|0} : \Lambda_{p|q}^{r|s} \to \Lambda_{p+1|q}^{r+1|s}$.

\[ \square \]

**Corollary 1.4.** (1) The space $\Lambda^1_{p|q}(V)$ is a module over exterior algebras $\Lambda(V)$ and $\Lambda(V^*)$ defined by relations $uv = -(1)^{\tilde{a} \tilde{u}} vu$ and $\alpha \beta = -(1)^{\tilde{a} \tilde{b}} \beta \alpha$.

(2) The space of stable forms $\Lambda^1_{p|q}(V)$ is a module over a Clifford algebra $\text{Cliff}(V \oplus V^*)$ defined by relations $uv = -(1)^{\tilde{a} \tilde{u}} vu$, $\alpha \beta = -(1)^{\tilde{a} \tilde{b}} \beta \alpha$ and $u \alpha + (1)^{\tilde{a} \tilde{u}} \alpha u = (u, \alpha)$. 

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Remark 1.3. Notice that we arrive at the relations of exterior and Clifford algebras (in “skew” versions) not as conventions but as actual identities between linear operators. It is also worth noting that the anticommutation relations obtained here for $e(u)$ and $e(\alpha)$ are not at all obvious. While under the isomorphism with straight or dual forms one of the operators $e(u)$ or $e(\alpha)$ can be interpreted as a substitution into a suitable even slot (hence the anticommutativity between such operators will become transparent), the other one will remain an “exterior product” defined by a formula like (17), which involves both even and odd slots. By duality $e(u)$ transforms into $e(\alpha)$ and vice versa. However, this can be exploited only in the common range $0 \leq r \leq n$ where dual and straight forms are both good. Hence a certain portion of tedious calculations is unavoidable to get all the relations (24)–(26).

2 Cartan calculus

2.1 Differential

Consider a supermanifold $M = M^{n|m}$. For forms on $M$, i.e., sections of the corresponding vector bundles associated with $TM$, we shall use the notation $\Omega^r|s$, $\Omega^r_{p|q}$, $\Omega^r_{p|q}$ and $\Omega^{r+s}$. By $\Omega^r = \oplus \Omega^r$ we shall denote the algebra of “naive” differential forms with the skew-commutative convention (and the even differential, cf. [9]). A differential $d : \Omega^r_{p|q} \rightarrow \Omega^{r+1}_{p|q}$ is defined by the formula

$$d\mathcal{L} := (-1)^r u^K_{r+1} (1) \partial\frac{\partial\mathcal{L}}{\partial A^K}$$

(34)

(see [12]). In [12] it is proved that the operator $d$ is stable, hence we have a complex $d : \Omega^r \rightarrow \Omega^{r+1}$. For $\cdot \geq 0$, this complex is isomorphic to the “straight” complex $d : \Omega^r \rightarrow \Omega^{r+1}$ studied in [11] and for $\cdot \leq n$ to the complex of dual forms $\delta : \Omega_n \rightarrow \Omega_{n+1}$ introduced in [12]:

$$\begin{array}{cccccccc}
0 & \longrightarrow & \Omega^0 & \longrightarrow & \Omega^1 & \longrightarrow & \cdots & \longrightarrow & \Omega^n & \longrightarrow & \Omega^{n+1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \cdots \\
\cdots & \longrightarrow & \Omega^{-1} & \longrightarrow & \Omega^0 & \longrightarrow & \Omega^1 & \longrightarrow & \cdots & \longrightarrow & \Omega^n & \longrightarrow & \Omega^{n+1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \cdots \\
\cdots & \longrightarrow & \Omega_{n+1} & \longrightarrow & \Omega_n & \longrightarrow & \Omega_{n-1} & \longrightarrow & \cdots & \longrightarrow & \Omega_0 & \longrightarrow & 0
\end{array}$$
Consider a mixed form $L$ and a function $f$. Calculate $\bar{d}(fL)$:

$$\bar{d}(fL) = (-1)^r w^K_{r+1} (-1)^{\tilde{A}K} \frac{\partial}{\partial x^A} \frac{\partial}{\partial p_A^K} (fL) =$$

$$(-1)^r w^K_{r+1} (-1)^{\tilde{A}K} \frac{\partial}{\partial x^A} f \frac{\partial L}{\partial p_A^K} (-1)^{\tilde{F}(\tilde{A}+\tilde{K})} =$$

$$(-1)^r w^K_{r+1} (-1)^{\tilde{A}K} \left((-1)^{\tilde{F} (\tilde{A}+\tilde{K})} \frac{\partial A}{\partial A} f \frac{\partial L}{\partial p_A^K} + (-1)^{\tilde{F} \tilde{K}} f \frac{\partial L}{\partial p_A^K} \frac{\partial L}{\partial p_A^K} \right) =$$

$$f \frac{dL}{dx} + (-1)^r \frac{\partial}{\partial p_A^K} f w^K_{r+1} (-1)^{\tilde{F} \tilde{A}} = f \frac{dL}{dx} + e(df) L, \quad (35)$$

where $df = dx^A \partial_A f$ is considered as an element of $\Omega^1(M)$. We stress that the algebra with the even differential is considered. Since $d(fL)$ is a form and $fL$ is a form, it follows that $e(df) L$ is a well-defined form. We can conclude that for arbitrary 1-form $\alpha$ the operation $e(\alpha)$ is also well-defined, i.e., does not depend on the choice of coordinates and maps mixed forms into mixed forms. The formula (14) is extracted from this calculation. Similar calculation gives the formula (17) for $e(\alpha)$ on straight forms; by duality it can be rewritten to produce a formula (16) for $e(u)$ on dual forms, from which we get our formula (15) on mixed forms. Thus it follows that both operators $e(u)$, $e(\alpha)$ on mixed forms are well-defined, which justifies our consideration in the previous section. It is not easy to give a purely algebraic proof of this fact.

**Remark 2.1.** The stability of $e(u)$, $e(\alpha)$ as well can be deduced from the stability of $d$.

In the previous Section we got the module structure of mixed forms over $\Omega^1(M)$.

**Theorem 2.1.** Leibniz formula holds:

$$d(\omega L) = d\omega L + (-1)^k \omega dL, \quad (36)$$

for $\omega \in \Omega^k$ and $L \in \Omega^r_{p|q}$.

**Proof.** Since $\Omega^1(M)$ is a differential graded algebra, generated by elements $df$ over $C^\infty(M)$ (locally), it is sufficient to check the formula (36) for two cases:
\[ \omega = f \text{ and } \omega = df, \text{ where } f \text{ is a function. The first case was considered above. Consider } \omega = df. \text{ Then, by definition,} \]

\[ df \mathcal{L} = d(f \mathcal{L}) - f \, d\mathcal{L}. \quad (37) \]

Apply \( \bar{d} \). We obtain

\[ \bar{d}(df \mathcal{L}) = \bar{d}(f \mathcal{L}) - \bar{d}(f \, d\mathcal{L}) = 0 - df \, d\mathcal{L} = ddf \mathcal{L} + (-1)^1 df \, d\mathcal{L}, \quad (38) \]

as desired.

Therefore, \( \Omega^{|s} \) is a graded differential module over \( \Omega^r \) for all \( s \).

**Remark 2.2.** Notice that \( \Lambda^r \cong \Lambda^{|0}, \Omega^r \cong \Omega^{|0} \) as modules.

### 2.2 Homotopy identity

Consider a vector field \( X \in \text{Vect} \, M \) and the corresponding infinitesimal transformation:

\[ x^A \mapsto x^A + \varepsilon X^A(x), \varepsilon^2 = 0. \]

By a straightforward calculation we obtain the following formula for the Lie derivative on mixed forms:

\[ \delta_X \mathcal{L} = X^A \frac{\partial \mathcal{L}}{\partial x^A} - (-1)^{\hat{a}X} \frac{\partial X^B}{\partial x^A} p_{B}^{K} \frac{\partial \mathcal{L}}{\partial p^{K}_{A}} + (-1)^{\hat{a}(\hat{X} + 1)} \frac{\partial X^A}{\partial x^A} \mathcal{L}, \quad (39) \]

where we picked the notation \( \delta_X \) to avoid overloading the letter ‘\( \mathcal{L} \)’. The Lie derivative \( \delta_X \) has the same parity as \( X \). It preserves all degrees and is obviously a derivation for all kinds of natural multiplications. Operation \( \delta_X \) commutes with the stability isomorphisms \((10)\) and with the isomorphisms \((12)\).

**Theorem 2.2.** For mixed forms on a supermanifold \( M \), the following identity holds:

\[ d \, e(X) + e(X) \, d = \delta_X \sigma, \quad (40) \]

where \( \sigma = \sigma_{1|0} : \Omega^{r|s}_{p|q} \to \Omega^{r+1|s}_{p+1|q} \) is the stability isomorphism.

**Proof.** Let \( \mathcal{L} \) be in \( \Omega^{r|s}_{p|q} \). Consider \( \sigma^{-1} : \Omega^{r+1|s}_{p+1|q} \to \Omega^{r|s}_{p|q} \). Recall that the action of this operator consists in setting \( p_{A}^{p+1} = 0, w_{F}^{p+1} = 0, w_{r+1}^{K} = 0, w_{r+1}^{p+1} = 1 \) in
the argument. We shall find \( \sigma^{-1} e(X) d \mathcal{L} \) and \( \sigma^{-1} d e(X) \mathcal{L} \). Directly from (15):
\[
\sigma^{-1} e(X) d \mathcal{L} = (-1)^{r+1} X^A \left( -p_A \frac{\partial}{\partial w_{r+1}^A} d \mathcal{L} \right) =
\]
\[
(-1)^r X^A p_A K \frac{\partial}{\partial w_{r+1}^A} \left( (-1)^r w_{r+1}^r (-1) \oint \frac{\partial \mathcal{L}}{\partial x^B} \frac{\partial}{\partial p_B} \right) =
\]
\[
X^A p_A K (-1) \oint \frac{\partial \mathcal{L}}{\partial x^B} \frac{\partial}{\partial p_B};
\]
now,
\[
\sigma^{-1} d e(X) \mathcal{L} = (-1)^r w_{r+1}^{K^*} (-1) \oint \frac{\partial}{\partial x^A} \frac{\partial}{\partial p_A} (e(X) \mathcal{L}) \bigg|_{w_0^{L+1}=0, p_0^{L+1}=0} =
\]
\[
(-1)^r \left( \frac{\partial}{\partial x^A} \frac{\partial}{\partial p_A} (e(X) \mathcal{L}) \right) \bigg|_{p_0^{L+1}=0, w_0^{L+1}=0} = \left( \frac{\partial}{\partial x^B} \frac{\partial}{\partial p_A} \right) X^A \mathcal{L} \bigg|_{p_0^{L+1}=0, w_0^{L+1}=0}
\]
\[
- (-1) \oint \frac{\partial}{\partial x^A} \frac{\partial}{\partial p_A} (e(X) \mathcal{L}) \bigg|_{p_0^{L+1}=0, w_0^{L+1}=0} =
\]
\[
\frac{\partial}{\partial x^B} \left( X^A (-1) \oint \frac{\partial}{\partial x^A} \frac{\partial}{\partial p_A} \right) \bigg|_{p_0^{L+1}=0, w_0^{L+1}=0}
\]
Comparing with (11), we immediately conclude that
\[
\sigma^{-1} (e(X) d + d e(X)) \mathcal{L} =
\]
\[
(1) \oint \frac{\partial}{\partial x^A} \frac{\partial}{\partial p_A} \mathcal{L} + X^B \frac{\partial}{\partial x^B} \mathcal{L} - \frac{\partial}{\partial x^B} \mathcal{L} (-1) \oint \frac{\partial}{\partial x^A} \frac{\partial}{\partial p_A} \mathcal{L} = \delta \mathcal{L}.
\]

Applying \( \sigma \) to both sides of (13), we obtain the desired identity (10). (Notice that \( \sigma \) and \( \delta \) commute.)

**Corollary 2.1.** In the complex of stable forms \( \Omega^1 \) we have the usual form of “Cartan’s homotopy identity”:
\[
d e(X) + e(X) d = \delta X.
\]
3 Discussion

We introduced the operators $e(u)$ and $e(\alpha)$ on the space of mixed forms, where $u$ is a vector and $\alpha$ is a covector. They are analogs of the contraction $u \cdot$ and of the exterior product $\alpha \wedge$ for usual forms on purely even vector space. Though these operations change only even part of degrees, their construction involves all (even and odd) arguments. We proved that these operations are stable, hence they induce the corresponding operations on the space of stable forms. We established the anticommutation relations for the operators $e(u)$ and $e(\alpha)$. They yield the relations of a super Clifford algebra (or, before stabilization, with an additional central element $\sigma$). It is remarkable that a “skew-commutative” version of Clifford relations (anticommutators without parity reversion) rather than more popular choice of commutators and reversed parity naturally appears here.

The main incentive of considering these operators was the necessity to straighten out the Cartan calculus for forms on supermanifolds. The homotopy identity found in [11] was valid only for $r|s$-forms with $r > 0$; the case $r = 0$ had to be mended with the help of an ad hoc augmentation. The existence of Bernstein-Leites integral forms of negative degree has given another hint to a “hidden” part of the super Cartan-de Rham complex. This hidden part was discovered in [12]. The entire complex (incorporating positive and negative halves) is made up by stable forms, for which mixed forms are representatives. In the current paper we established the relation between the differential and the operator $e(X)$, where $X$ is a vector field. Again, for mixed forms it contains the element $\sigma$ and after stabilization an analog of the usual form of the homotopy identity is reproduced. Thus, the introduction of the stable complex indeed solves the problem.

What is next? We need to check the functorial behaviour of stable forms and get a “generalized” version of the homotopy identity, which will imply the homotopy invariance of the complex (note that $\delta_X$ in (40,44) corresponds to an infinitesimal diffeomorphism; we need perturbations of arbitrary maps), hence an analog of the Atiyah-Hirzebruch sequence (cf. [11]). The investigation of “point cohomology” of stable forms will require more detailed analysis of their algebraic properties. Another topic, which we did not touch here at all, is, of course, integration. We hope to consider these subjects elsewhere. In the paper [13], the author showed that the variational differential can be used to make a complex of arbitrary Lagrangians of paths, not just forms. It would be interesting to combine this fact with the results of [12] and of the
current paper.

References

[1] M.A. Baranov and A.S. Schwarz. Cohomology of supermanifolds. *Funk. Anal. Pril.*, 18(3):69–70, 1984.

[2] A. Belopolsky. Picture changing operators in supergeometry and superstring theory. [hep-th/9706033](https://arxiv.org/abs/hep-th/9706033).

[3] J.N. Bernstein and D.A. Leites. Integral forms and Stokes formula on supermanifolds. *Funk. Anal. Pril.*, 11(1):55–56, 1977.

[4] J.N. Bernstein and D.A. Leites. How to integrate differential forms on supermanifolds. *Funk. Anal. Pril.*, 11(3):70–71, 1977.

[5] A.V. Gajduk, O.M. Khudaverdian, and A.S. Schwarz. Integration on surfaces in superspace. *Teor. Mat. Fiz.*, 52(3):375–383, 1982.

[6] I.M. Gelfand, S.G. Gindikin, and M.I. Graev. Integral geometry in affine and projective spaces, volume 16 of *Itogi Nauki i Tekhn. Sovrem. Problemy Matem.* VINITI, Moscow, 1980. 53-226.

[7] Fritz John. The ultrahyperbolic differential equation with four independent variables. *Duke Math. J.*, 4:300–322, 1938.

[8] O.M. Khudaverdian. Batalin-Vilkovisky formalism and odd symplectic geometry. In P.N. Pyatov and S.N. Solodukhin, editors, *Proceedings of the Workshop “Geometry and Integrable Models”, Dubna, Russia, 4-8 October 1994*. World Scientific Publ., 1995. [hep-th 9508174](https://arxiv.org/abs/hep-th/9508174).

[9] Yu.I. Manin. *Gauge fields and complex geometry*. Moscow, Nauka, 1984.

[10] A.A. Rosly, O.M. Khudaverdian, and A.S. Schwarz. *Supersymmetry and complex geometry*, volume 9 of *Itogi Nauki i Tekhn. Sovrem. Problemy Matem. Fundam. Napravl.* VINITI, Moscow, 1986. 247-284.

[11] T[heodore] Voronov. *Geometric Integration Theory on Supermanifolds*, volume 9 of *Sov. Sci. Rev. C. Math. Phys*. Harwood Academic Publ., 1992.
[12] Theodore Voronov. Supermanifold forms and integration. A dual theory. In V.M.Buchstaber and S.P.Novikov, editors, *Solitons, Geometry, and Topology: On the Crossroad*, pages 153–172. AMS Translations, ser. 2, vol. 179, 1997. Advances in the Math. Sciences. dg-ga/9603009.

[13] Th.Th. Voronov. Complex generated by variational derivatives. Lagrangian formalism of infinite order and a generalized Stokes’ formula. *Uspekhi Matem. Nauk*, 51(6):195–196, 1996. math.DG/9711113.

[14] Th.Th. Voronov and A.V. Zorich. Complexes of forms on supermanifold. *Funk. Anal. Pril.*, 20(2):58–59, 1986.

[15] Th.Th. Voronov and A.V. Zorich. Integral transformations of pseudodifferential forms. *Uspekhi Matem. Nauk*, 41(6):167–168, 1986.

[16] Th.Th. Voronov and A.V. Zorich. Bordism theory and homotopy properties of supermanifolds. *Funk. Anal. Pril.*, 21(3):77–78, 1987.

[17] Th.Th. Voronov and A.V. Zorich. Cohomology of supermanifolds and integral geometry. *Doklady Akademii Nauk*, 298(3):528–533, 1988.