UMBILIC SINGULARITIES AND LINES OF CURVATURE
ON ELLIPSOIDS OF $\mathbb{R}^4$

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Abstract. The topological structure of the lines of principal curvature, the umbilic and partially umbilic singularities of all tridimensional ellipsoids of $\mathbb{R}^4$ is described.

1. Introduction

In 1796 G. Monge [20] determined the first example of a principal curvature configuration on a surface in $\mathbb{R}^3$, consisting of the umbilic points (at which the principal curvatures coincide) and, outside them, the foliations by the minimal and maximal principal curvature lines. See also [10] [14]. This configuration was achieved for the case of the ellipsoid with 3 different axes defined by $q(x,y,z) = x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, $a > b > c > 0$, where the positive orientation is defined by the normal toward the interior, given by $-\nabla q$. See the illustration in figure 1. The cases of the ellipsoids with two different axes (with rotational symmetry, with two umbilic points at the poles) defined by $q(x,y,z) = x^2/a^2 + y^2/a^2 + z^2/b^2 = 1$ and only one axis (sphere, which is totally umbilic) are illustrated in figure 2.

In this paper will be determined the principal configurations for the ellipsoids in $\mathbb{R}^4$, extending in one dimension the results for $\mathbb{R}^3$ outlined above and illustrated in figures 1 and 2.

By an ellipsoid in $\mathbb{R}^4$ is meant the unit level hypersurface $E_{a,b,c,d}$ defined (implicitly) by a positive definite quadratic form $Q$, which, after orthonormal diagonalization, can be written as $Q(x,y,z,w) = x^2/a^2 + y^2/b^2 + z^2/c^2 + w^2/d^2 = 1$, with positive semi-axes: $a$, $b$, $c$, $d$. The positive orientation will be defined by the unit inner normal $N = \frac{\nabla Q}{|\nabla Q|}$.

The principal configuration on an oriented hypersurface in $\mathbb{R}^4$, with euclidean scalar product $\langle \cdot, \cdot \rangle$, consists on the umbilic points, at which the 3 principal curvatures coincide, the partially umbilic points, at which only 2
**Figure 1.** Two views of the global behavior of 4 umbilic points and principal foliations $\mathcal{F}_i$ on the ellipsoid with 3 different axes. Positive orientation given by the inner normal: minimal (maximal) curvature lines in red (blue). When the positive orientation is given by the outer normal, names and colors must be interchanged. Umbilic points with 1 umbilic separatrix, said of type $D_1$, in green.

**Figure 2.** Global behavior of the principal foliations on an ellipsoid of revolution in $\mathbb{R}^3$ (left and middle), and the totally umbilic sphere (right, green). Left: $b > a > 0$ (*prolate ellipsoid*), the leaves of the maximal principal foliation are the parallels (blue) and the minimal leaves are the meridians (red). Center: $a > b > 0$ (*oblate ellipsoid*), the leaves of the maximal foliation are the meridians (blue) and the minimal leaves are the parallels (red). When the positive orientation is given by the outer normal names and colors must be interchanged. Umbilic points, of center type, in green. Right: $a = b = c$, *sphere*, totally umbilic, green.

Principal curvatures are equal, and the integral foliations of the three principal line fields on the complement of these sets of points, which will be referred to as the *regular part*. 
Recall that the principal curvatures $0 < k_1 \leq k_2 \leq k_3$ are the eigenvalues of the automorphism $D(-N)$ of the tangent bundle of the hypersurface, taken here as $\mathbb{E}_{a,b,c,d}$ with positive unit normal $N$.

Thus the set partially umbilic points is the union of $\mathcal{P}_{12}$, where $k_1 = k_2 < k_3$, and of $\mathcal{P}_{23}$, where $k_3 = k_2 > k_1$. The set $\mathcal{U}$ of umbilic points is defined by $k_1 = k_2 = k_3$.

The eigenspaces corresponding to the eigenvalues $k_i$ will be denoted by $\mathcal{L}_i, i = 1, 2, 3$. They are line fields, well defined and, for ellipsoids, also analytic on the regular part. In fact $\mathcal{L}_1$ is defined and analytic on the complement of $\mathcal{U} \cup \mathcal{P}_{12}$, $\mathcal{L}_3$ is defined and analytic on the complement of $\mathcal{U} \cup \mathcal{P}_{23}$ and $\mathcal{L}_2$ is defined and analytic on the complement of $\mathcal{U} \cup \mathcal{P}_{12} \cup \mathcal{P}_{23}$.

Given a principal direction $e_i \in \mathcal{L}_i$, consider the plane tangent to the hypersurface passing through $q$, having $e_i(q)$ as the normal vector:

\begin{equation}
\Pi_i(q) = \{(du_1, du_2, du_3); \langle (du_1, du_2, du_3), G \cdot (e_i(q))^T \rangle = 0\},
\end{equation}

where $G = [g_{ij}]_{3 \times 3}$ is the first fundamental form and $(u_1, u_2, u_3)$ is a local chart.

Therefore there are three 2-plane fields which are singular at the umbilic and partially umbilic sets. In general these plane fields are not Frobenius integrable. Special cases where these plane distributions are integrable are the hypersurfaces that belong to a quadruply orthogonal system, as is the case of all ellipsoids studied here.

The work of Garcia established the generic properties of principal configurations on smooth hypersurfaces. There was determined the principal configuration on the ellipsoid $\mathbb{E}_{a,b,c,d}$ with four different axes, $a > b > c > d > 0$, in $\mathbb{R}^4$. It was proved that it has four closed regular curves of generic partially umbilic points whose transversal structures are of type $D_1$, as at the umbilic points in the ellipsoid with 3 different axes in $\mathbb{R}^3$. See Fig. 1.

A different proof of this result will be given in Theorem 2 and its conclusions will be illustrated in more detail in figures 11 and 12.

A complete description of the principal configurations on all tridimensional ellipsoids is established in this paper.

Propositions 1 and 4 and Theorem 1, with their pertinent illustrations establish the principal configurations on the other ellipsoids in $\mathbb{R}^4$.

The totally umbilic ellipsoid: $\mathbb{E}_{a,a,a,a}$ trivially consists in the whole sphere of radius $a$. An illustration would be the same as that in Figure 2, right.

The principal configurations increase in complexity as follows:
The types $E_{b,a,a,a}$ and $E_{a,a,a,b}$, corresponding to 3 equal axes, are studied in Proposition 1.

The case of two pairs of equal axes $E_{a,a,b,b}$, found nowhere in the literature, is treated in Proposition 2. See illustration in Figure 4.

The ellipsoids $E_{a,b,c,c}$ and $E_{c,c,a,b}$ of one pair of equal axes, disjoint from the interval of distinct axes, $b < a$ are established in Propositions 3 and 4. See the illustration in figs. 5 and 6. The case $E_{a,c,c,b}$, treating the case of the double axis inside the interval $(b,a)$ of distinct axes, exhibiting isolated umbilic points, novel in the literature, is proved in Theorem 1. See the illustration in figs. 7 and 8.

Comments and references concerning other crucial steps focusing on additional aspects of principal configurations of hypersurfaces in $\mathbb{R}^4$ have been given in section 3.

2. Principal Configurations on Ellipsoids in $\mathbb{R}^4$

2.1. Color and Print Conventions for Illustrations in this Paper. The color convention for ellipsoids in $\mathbb{R}^3$ in figures 1 and 2 has been upgraded for $\mathbb{R}^4$ as follows.

- Black (− · − · −·): integral curves of line field $L_1$,
- Blue (− − − − −): integral curves of line field $L_3$,
- Red (__________): integral curves of line field $L_2$,
- Green (− − − − −): Partially umbilic arcs $P_{12}$,
- Light Blue (− − − − −): Partially umbilic arcs $P_{23}$,
- Purple (●): Umbilic Points $U$.

The following dictionary has been adopted for illustrations of integral curves appearing in Figs. 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12 in this paper when printed in black and white: dashed, for blue; dotted-dashed-dotted, for black; full trace, for red.

2.2. Open Book Structures and Hopf Bands in Ellipsoidal Principal Configurations. In Propositions 1, 2, 3 and 4 and Theorem 1 the ellipsoids exhibit a special structure of foliations with singularities by a family of two-dimensional ellipsoids, all crossing along a circle or an ellipse, this structure is called a book structure with binding, in this case along the circle or ellipse, with ellipsoidal pages. See [11], [12] and [27].

Let $M$ be a 3-manifold and $S$ a regular curve on $M$, i.e., a submanifold of codimension two.
An open book structure on $M$ is a smooth fibration $p : M \setminus S \to \mathbb{S}^1$ satisfying the following conditions: i) For all $\theta \in \mathbb{S}^1$, the closure of $p^{-1}(\theta)$ contains $S$ and is a regular surface, a page. ii) The submanifold $S$ coincides with $\bigcap_{\theta \in \mathbb{S}^1} cl[p^{-1}(\theta)]$ and is called the binding.

The following geometric structure will appear in the description of partially umbilic separatrix surfaces in the principal configurations in subsections 2.5, 2.6 and 2.7.

A Hopf band in $E_{a,b,c,d}$ is an embedding $\beta : \mathbb{S}^1 \times [0,1] \to E_{a,b,c,d}$ such that $S_1 = \beta(\mathbb{S}^1 \times \{0\})$, $S_2 = \beta(\mathbb{S}^1 \times \{1\})$ are linked curves with linking number equal to $\pm 1$, see [12].

2.3. Three equal axes: $E_{a,a,a,b}$ and $E_{b,a,a,a}$.

**Proposition 1.** Consider the ellipsoid $E_{a,a,a,b}$ defined by

$$\frac{x^2 + y^2 + z^2}{a^2} + \frac{w^2}{b^2} = 1, \ a^2 \neq b^2 \neq 0.$$ 

Then the umbilic set consists of two points $(0,0,0,\pm b)$, the partially umbilic set is the open set $E_{a,a,a,b} \setminus \{(0,0,0,\pm b)\}$. One principal foliation is regular on $E_{a,a,a,b} \setminus \{(0,0,0,\pm b)\}$ and consists of the integral curves of the gradient of the $w-$ projection. The distribution defined by the partially umbilic 2–planes has as integral foliation the level spheres of this projection. Figure 3, right, illustrates this principal configuration. The same figure, left, illustrates the ellipsoid $E_{b,a,a,a}$.

**Figure 3.** Global behavior of the regular principal configurations for Ellipsoids $E_{b,a,a,a}$, left, and $E_{a,a,a,b}$, right. Foliations $\mathcal{F}_1$ (left, black $-\cdot-\cdot-\cdot$) and $\mathcal{F}_3$ (right, blue, $-\cdot-\cdot-\cdot-\cdot$).
Proof. Consider the parametrization \( \alpha : [0, 2\pi] \times (0, \pi) \times (-b, b) \to E_{a,a,a,b} \) defined by:
\[
\alpha(u, v, w) = \left( \frac{a\sqrt{b^2 - w^2}}{b} \cos u \sin v, \frac{a\sqrt{b^2 - w^2}}{b} \sin u \sin v, \frac{a\sqrt{b^2 - w^2}}{b} \cos v, w \right).
\]
The principal curvatures are given by:
\[
k(u, v, w) = l(u, v, w) = \frac{b^2}{a\Delta}, \quad m(u, v, w) = \frac{ab^4}{\Delta^3}, \quad \Delta = \sqrt{(a^2 - b^2)w^2 + b^4}.
\]
It follows that \( k_1 = k = l = k_2 < k_3 = m \) when \( a > b > 0 \) and that \( k_3 = k_2 = l > m = k_1 \) when \( 0 < a < b \).

The parametrization \( \alpha \) does not cover the ellipse \( x = 0, y = 0 \) in \( E_{a,a,a,b} \).

To analyze the principal configuration around it consider the parametrization
\[
\beta(u, v, t) = (u, v, 0, 0) + \sqrt{a^2 - u^2 - v^2}(0, 0, \cos t, \frac{b}{a} \sin t).
\]
Calculation gives the following expressions for the principal curvatures:
\[
k_1(0, 0, t) = k_2(0, 0, t) = \frac{b}{a\Delta}, \quad k_3(0, 0, t) = \frac{ab}{\Delta^3}, \quad \Delta = \sqrt{b^2 \cos^2 t + a^2 \sin^2 t}.
\]
For \( t = \pm \pi/2 \) it follows that \( k_1 = k_2 = k_3 = \frac{b}{a^2} \) that correspond to the two umbilic points \((0, 0, 0, \pm b)\).

In the parametrization \( \alpha \) the two fundamental forms \( g_{ij} \) and \( b_{ij} \) are diagonal and only one principal foliation is regular outside the two umbilic points, thus it follows that the partially umbilic plane field, orthogonal to the principal regular direction, is integrable and the spheres given by \( w = \text{cte} \) are its integral leaves. Therefore the integral curves of the regular principal foliation are the trajectories of the gradient vector field of the projection \( \pi(x, y, z, w) = w \) with respect to the metric \( g = (g_{ij}), \langle g_{ij} = \langle \partial\alpha/\partial u_i, \partial\alpha/\partial u_j \rangle \rangle \) induced by \( \alpha \).

\( \square \)

2.4. Two pairs of equal axes: \( E_{a,a,b,b} \).

Proposition 2. Consider the ellipsoid \( E_{a,a,b,b} \) defined by
\[
\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1, \quad a > b > 0.
\]

Then it follows that:

i) The umbilic set is empty and the partially umbilic set is the union of two regular circular curves \( \mathcal{P}_{23} = (a \cos u, a \sin u, 0, 0) \) and \( \mathcal{P}_{12} = (0, 0, b \cos v, b \sin v) \). The curves \( \mathcal{P}_{12} \) and \( \mathcal{P}_{23} \) are linked in \( E_{a,a,b,b} \).
ii) The behavior of the principal foliations near the partially umbilic curves is illustrated in Fig. [4]. The intermediate foliation $\mathcal{F}_2$ is singular on $\mathcal{P}_{12} \cup \mathcal{P}_{23}$, while $\mathcal{F}_1$ is singular only on $\mathcal{P}_{12}$ and $\mathcal{F}_3$ is singular only on $\mathcal{P}_{23}$.

iii) All regular leaves of $\mathcal{F}_1$ and $\mathcal{F}_3$ are circles and the leaves of $\mathcal{F}_2$ are arcs of ellipses with boundary points located at $\mathcal{P}_{12} \cup \mathcal{P}_{23}$.

\textbf{Figure 4.} Top: Global behavior of the Principal Foliations $\mathcal{F}_i$. Bottom, center left: Ellipsoid of revolution whose poles slide along one of partially umbilic closed line (green) and whose equator contains the other partially umbilic closed line (light blue). They are leaves of integral foliation of the plane distributions spanned by $\mathcal{L}_2$ and $\mathcal{L}_3$, attaching the poles (green), illustrated locally in the extreme left. Observe the book structure with binding along the light blue circle, whose pages are these ellipsoids. Bottom, center right: Ellipsoid of revolution whose poles slide along one of partially umbilic closed line (light blue) and whose equator contains the other partially umbilic closed line (green). They are leaves of the integral foliation of the plane distribution spanned by $\mathcal{L}_2$ and $\mathcal{L}_1$, attaching the poles (light blue), illustrated locally in the extreme right. Observe the book structure with binding along the green circle, whose pages are these ellipsoids.
Proof. Consider the parametrization
\[ \alpha(u, v, t) = (a \cos u \cos t, a \sin u \cos t, b \cos v \sin t, b \sin v \sin t) \]
with \(0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi\) and \(0 < t < \pi\).

A non unitary normal field is given by:
\[ N(u, v, w) = (b \cos u \cos t, b \sin u \cos t, a \cos v \sin t, a \sin v \sin t). \]

Direct analysis shows that \(\alpha\) is a regular parametrization, the coordinate curves are principal lines and are given by
\[ k_1(u, v, t) = \frac{b}{a\Delta}, \quad k_2(u, v, t) = \frac{a}{b\Delta}, \]
\[ k_3(u, v, t) = \frac{ab}{\Delta}, \quad \Delta = [a^2 \sin^2 t + b^2 \cos^2 t]^{\frac{1}{2}}. \]

Therefore it follows that \(k_1 < k_2 < k_3\) when \(a > b > 0\) and \(0 < t < \pi\).

For \(t = t_0 \in (0, \pi)\) it follows that \(\alpha_{t_0}(u, v) = \alpha(u, v, t_0)\) is a parametrization of a torus invariant by two principal foliations.

For \(u = u_0 \in [0, 2\pi]\) it follows that \(\alpha_{u_0}(v, t) = \alpha(u_0, v, t)\) is a parametrization of an ellipsoid of revolution, contained in the hyperplane \(u_0x - \cos u_0y = 0\), invariant by two principal foliations. Analogous conclusion holds when \(v = v_0 \in [0, 2\pi]\).

The ellipsoid is decomposed as follows, the open set \(E_{a, a, b, b} \setminus (P_{12} \cup P_{23})\) is foliated by tori and the two linked curves \(P_{12}\) and \(P_{23}\) are singular leaves of this decomposition. Each torus is foliated by two one dimensional principal ones. The intermediate foliation \(F_2\) is singular on \(P_{12} \cup P_{23}\), while \(F_1\) is singular only on \(P_{12}\) and \(F_3\) is singular only on \(P_{23}\).

For \(w = 0\) and \(w = \pi\) the parametrization above is singular. To carry out the analysis near the curve \(P_{12}\), given by \(\alpha(u, v, 0) = (a \cos u, a \sin u, 0, 0)\), consider the following parametrization.
\[ \tilde{\alpha}(u, v, w) = (a \cos u \frac{\sqrt{b^2 - v^2 - w^2}}{b}, a \sin u \frac{\sqrt{b^2 - v^2 - w^2}}{b}, v, w). \]

It follows that \(k_1(u, 0, 0) = \frac{1}{a}, \quad k_2(u, 0, 0) = k_3(u, 0, 0) = \frac{a}{b^2}. \)

The principal directions \(L_i(\tilde{\alpha})\) are defined by the following differential equation:
\[ du = 0, \quad -vw(dv^2 - dw^2) + (v^2 - w^2)dv = 0, \quad L_2(\tilde{\alpha}) \text{ and } L_3(\tilde{\alpha}) \]
\[ dw = 0, \quad dv = 0, \quad L_4(\tilde{\alpha}). \]

For each \(u = u_0\) fixed, \(\tilde{\alpha}_{u_0}(v, w) = \tilde{\alpha}(u_0, v, w)\) is a surface contained in the hyperplane \(u_0x - \cos u_0y = 0\), invariant by two principal foliations and
the principal configuration on $\alpha_{u_0}$ is equivalent to that of an ellipsoid of revolution of $\mathbb{R}^3$.

Similar analysis is valid to establish the principal configuration near $\mathcal{P}_{23}$.

The two partially umbilic curves $\mathcal{P}_{12}$ and $\mathcal{P}_{23}$ are linked since they are contained in the planes $(x, y, 0, 0)$ and $(0, 0, z, w)$, respectively. □

2.5. One pair of equal axes: $E_{a,b,c,c}$.

**Proposition 3.** Consider the ellipsoid $E_{a,b,c,c}$ defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \frac{w^2}{c^2} = 1, \quad a > b > c > 0.$$ 

Let $E_0 = \{(x, y, 0, 0) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$.

Then it follows that:

i) The umbilic set is empty and the partially umbilic set is the union of three regular curves $\mathcal{P}_{12}$, $\mathcal{P}_{23}^1$, $\mathcal{P}_{23}^2$. The curves $\mathcal{P}_{23}^1$ and $\mathcal{P}_{23}^2$ are circles contained in the planes $(\pm a \sqrt{\frac{a^2-b^2}{a^2-c^2}}, 0, z, w)$. The pairs of curves $\{\mathcal{P}_{12}, \mathcal{P}_{23}^1\}$ and $\{\mathcal{P}_{12}, \mathcal{P}_{23}^2\}$ are linked while the pair $\{\mathcal{P}_{23}^1, \mathcal{P}_{23}^2\}$ is not linked.

ii) The ellipsoid $E_{a,b,c,c}$ is foliated by a pencil of two dimensional surfaces, bidimensional ellipsoids, all passing through the ellipse $\mathcal{P}_{12}$.

Each surface is an integral leaf of the distribution by planes generated by $\mathcal{L}_1$ and $\mathcal{L}_3$, and there the restricted principal configuration is principally equivalent to that of the bi-dimensional ellipsoid, with 3 different axes.

iii) The principal foliation $\mathcal{F}_2$ is singular on $\mathcal{P}_{12} \cup \mathcal{P}_{23}^1 \cup \mathcal{P}_{23}^2$ and all its regular leaves are closed curves orthogonal to the pencil of surfaces with book structure around $\mathcal{P}_{12}$. The behavior of the principal foliations near the partially umbilic curves is illustrated in Fig. 5.

**Proof.** Consider the parametrization $\alpha : [0, 2\pi] \times \{(v, w) : v^2 + w^2 < 1\} \to E_{a,b,c,c}$ defined by

$$\alpha(u, v, w) = (a \cos u \sqrt{1 - v^2 - w^2}, b \sin u \sqrt{1 - v^2 - w^2}, cv, cw).$$

Consider a normal positive vector

$$N(u, v, w) = -\left(\frac{c \cos u \sqrt{1 - v^2 - w^2}}{a}, \frac{c \sin u \sqrt{1 - v^2 - w^2}}{b}, v, w\right).$$
Figure 5. Behavior of the Principal Foliations $\mathcal{F}_i$ near the partially umbilic curves $\mathcal{P}_{23}^1$, $\mathcal{P}_{23}^2$ (light blue lines) and $\mathcal{P}_{12}$ (dotted green line), top. Bottom, left: Integral foliations by ellipsoids of plane distributions spanned by $\mathcal{L}_2$ and $\mathcal{L}_1$ or $\mathcal{L}_3$, which are ellipsoids. Bottom, right: ellipsoid with 3 different axes whose 4 umbilics slide along the partially umbilic closed lines (horizontal, light blue) and whose equator contains the other partially umbilic closed lines (vertical, green dotted print). Observe the book structure with binding along the green circle, whose pages are these ellipsoids.

The curve $\alpha(u,0,0)$ is a partially umbilic ($k_1 = k_2 < k_3$). In fact, the principal curvatures are given by:

$$k_1(u,0,0) = k_2(u,0,0) = \frac{ab}{\Delta^3}, \quad k_3(u,0,0) = \frac{ab}{c^2 \Delta},$$

$$\Delta = \sqrt{a^2 \sin^2 u + b^2 \cos^2 u}.$$

Let

$$\gamma_{\pm}(t) = (\pm a \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, 0, c \sqrt{\frac{b^2 - c^2}{a^2 - c^2}} \cos t, c \sqrt{\frac{b^2 - c^2}{a^2 - c^2}} \sin t).$$
Along $\gamma_{\pm}$ (green lines in Fig. 5) the principal curvatures $k_1 < k_2 = k_3$ are given by:

$$k_1(t) = \frac{ab}{c^2 \Delta}, \quad k_2(t) = k_3(t) = \frac{a}{b \Delta}, \quad \Delta = \frac{b}{c} \sqrt{b^2 \sin^2 t + c^2 \cos^2 t}.$$ 

Next consider the parametrization $\beta : [0, 2\pi] \times \{(u,v) : \frac{u^2}{a^2} + \frac{v^2}{b^2} < 1\} \to \mathbb{E}_{a,b,c,c}$ defined by

$$\beta(u,v,t) = (au,bv,c \cos t \sqrt{1 - u^2 - v^2}, c \sin t \sqrt{1 - u^2 - v^2}).$$

The positive normal vector (oriented inward) $N_{\beta} = -\beta_u \wedge \beta_v \wedge \beta_t$ is given by:

$$N_{\beta} = -(bc^2 u, ac^2 v, abc \sqrt{1 - u^2 - v^2} \cos t, abc \sqrt{1 - u^2 - v^2} \sin t).$$

In this parametrization the partially umbilic set is the union of two regular curves $\beta(u-,0,t), \beta(u+,0,t)$ where $u_{\pm} = \pm \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}$ and $a > b > c > 0$.

The principal curvatures restricted to these curves satisfy $k_1 < k_2 = k_3$ and are given by:

$$k_1(u_{\pm},0,t) = \frac{ac}{b^3}, \quad k_2(u_{\pm},0,t) = k_3(u_{\pm},0,t) = \frac{a}{bc}.$$

The structure of the principal lines follows from the rotational symmetry of $\mathbb{E}_{a,b,c,c}$. In fact, consider the pencil of hyperplanes $\pi_\theta$ given by $\cos \theta z + \sin \theta w = 0$. For all $\theta$, $\pi_\theta \cap \mathbb{E}_{a,b,c,c}$ is a two dimensional ellipsoid that contains the ellipse $E_0$. Moreover $\pi_\theta \cap \mathbb{E}_{a,b,c,c}$ is an orthogonal intersection and therefore the principal lines of this two dimensional ellipsoid are principal lines of the three dimensional ellipsoid $\mathbb{E}_{a,b,c,c}$.

The third family of principal lines are the orthogonal trajectories to the family of ellipsoids $\pi_\theta \cap \mathbb{E}_{a,b,c,c}$ and all are circles. It turns out that these circles are the leaves of $\mathcal{F}_2$. \hfill \Box

2.6. One pair of equal axes: $\mathbb{E}_{c,c,a,b}$.

Proposition 4. Consider the ellipsoid $\mathbb{E}_{c,c,a,b}$ defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2 + w^2}{c^2} = 1, \quad c > a > b > 0.$$

Let $E_0 = \{(x,y,0,0) : x^2/a^2 + y^2/b^2 = 1\}$.

Then the umbilic set is empty and the partially umbilic set is the union of three regular curves $\mathcal{P}_{23} \cup \mathcal{P}_{12} \cup \mathcal{P}_{12}^2$.

The partially umbilic curves $\mathcal{P}_{12}^1$ and $\mathcal{P}_{12}^2$ are circles contained in the hyperplanes $(0, \pm b \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, z,w)$. The pairs of curves $\{\mathcal{P}_{12}^1, \mathcal{P}_{23}\}$ and $\{\mathcal{P}_{12}^2, \mathcal{P}_{23}\}$ are linked while the pair $\{\mathcal{P}_{12}^1, \mathcal{P}_{12}^2\}$ is not linked.
Then $E_{c,c,a,b}$ is foliated by a pencil of two dimensional surfaces, bidimensional ellipsoids, all passing through the ellipse $E_0 = P_{23}$.

Each surface is an integral leaf of the distribution by planes generated by $L_1$ and $L_3$, and there the restricted principal configuration is principally equivalent to that of the bi-dimensional ellipsoid, with 3 different axes. The principal foliation $F_2$ is singular on $P_{12}^1 \cup P_{12}^2 \cup P_{23}$ and all its regular leaves are closed curves orthogonal to the pencil of surfaces with book structure around $P_{23}$.

The behavior of the principal foliations near the partially umbilic curves are illustrated in Fig. 6.

Figure 6. Behavior of the Principal Foliations $F_i$ near the partially umbilic curves $P_{12}^1$, $P_{12}^2$ (green lines) and $P_{23}$ (light blue line), top. Bottom, left: Integral foliations by ellipsoids of plane distributions spanned by $L_2$ and $L_1$ or $L_3$, which are ellipsoids. Bottom, right: ellipsoid with 3 different axes whose 4 umbilics slide along the partially umbilic closed lines (horizontal, green dotted print) and whose equator contains the other partially umbilic closed lines (vertical, light blue dotted). Observe the book structure with binding along the blue circle, whose pages are these ellipsoids.
Proof. Consider the parametrization $\alpha : [0, 2\pi] \times \{ (v, w) : v^2 + w^2 < 1 \} \to \mathbb{E}_{c, c, a, b}$ defined by

$$\alpha(u, v, w) = (a \cos u \sqrt{1 - v^2 - w^2}, b \sin u \sqrt{1 - v^2 - w^2}, cv, cw)$$

A positive non unitary normal vector is given by:

$$N(u, v, w) = \left( -\frac{c \cos u \sqrt{1 - v^2 - w^2}}{a} , \frac{c \sin u \sqrt{1 - v^2 - w^2}}{b} , v, w \right).$$

In this parametrization the curve $\alpha(u, 0, 0) = (a \cos u, b \sin u, 0, 0)$ is a partially umbilic line ($k_1 < k_2 = k_3$) and the principal curvatures are given by:

$$k_1(u, 0, 0) = \frac{ab}{c^2 \Delta}, \quad k_2(u, 0, 0) = k_3(u, 0, 0) = \frac{ab}{\Delta^3}$$

$$\Delta = \sqrt{a^2 \sin^2 u + b^2 \cos^2 u}.$$  

The set defined by $\cos(u) = 0$ and $v^2 + w^2 = \frac{c^2(a^2 - u^2)}{c^2 - b^2}$ is a partially umbilic line.

Let

$$\gamma_{\pm}(\theta) = (0, \pm b \sqrt{\frac{a^2 - b^2}{c^2 - b^2} \frac{c^2 - a^2}{c^2 - b^2}} \cos \theta, \frac{c}{c^2 - b^2} \frac{c^2 - a^2}{c^2 - b^2} \sin \theta).$$

Along $\gamma_{\pm}$ the principal curvatures $k_1 = k_2 < k_3$ are given by:

$$k_1(\theta) = \frac{ab}{c^2 \Delta}, \quad k_2(\theta) = k_3(\theta) = \frac{a}{b \Delta}, \quad \Delta = \frac{b}{c} \sqrt{b^2 \sin^2 \theta + c^2 \cos^2 \theta}.$$  

Consider the parametrization $\beta : [0, 2\pi] \times \{ (u, v) : u^2 + v^2 < 1 \} \to \mathbb{E}_{c, c, a, b}$ defined by

$$\beta(u, v, t) = (au, bv, c \cos t \sqrt{1 - u^2 - v^2}, c \sin t \sqrt{1 - u^2 - v^2}).$$

In this parametrization the partially umbilic set is the union of two regular curves $\beta(0, v_{-}, t), \beta(0, v_{+}, t)$ where $v_{\pm} = \pm \sqrt{\frac{a^2 - b^2}{c^2 - b^2}}$.

Since the umbilic set is empty, the structure of the principal lines can be explained as in the proof of Proposition $\Box$.
2.7. One pair of equal axes: \( E_{a,c,c,b} \).

**Theorem 1.** Consider the ellipsoid \( E_{a,c,c,b} \) defined by
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \frac{w^2}{c^2} = 1, \quad a > c > b > 0.
\]
Let \( E_0 = \{(x,y,0,0) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\} \). Then it follows that:

i) The umbilic set is contained in the ellipse \( E_0 \) and consists of four points \((a \cos u_{\pm}, b \sin u_{\pm}, 0, 0)\), \( \cos u_{\pm} = \sqrt{a^2 - c^2} / a^2 \); the partially umbilic set consists of the four complementary arcs contained in \( E_0 \).

ii) The ellipsoid \( E_{a,c,c,b} \) is foliated by a pencil of two dimensional surfaces, bidimensional ellipsoids, all passing through the ellipse \( E_0 \). Each surface is an integral leaf of the distribution by planes generated by \( L_1 \) and \( L_3 \), and there the restricted principal configuration is principally equivalent to that of the bi-dimensional ellipsoid, with 3 different axes.

iii) The principal foliation \( F_2 \) is singular on \( E_0 \) and all its regular leaves are circles orthogonal to the pencil of surfaces with book structure around \( E_0 \). The behavior of the principal foliations near the partially umbilic curves and umbilic points are illustrated in Fig. 7 and Fig. 8.

Figure 7. Behavior of the three principal foliations \( F_i \) near the partially umbilic and umbilic points located on the ellipse \( E_0 \).
LINES OF PRINCIPAL CURVATURE ON ELLIPSOIDS OF $\mathbb{R}^4$

Figure 8. Global behavior of the principal foliations $F_i$.

The foliation $F_2$ is singular along the ellipse $E_0$ and all its leaves are circles. Observe the book structure with binding along the ellipse $E_0$, the pages are two dimensional ellipsoids with three distinct axes. The ellipse contains four umbilic points and four arcs of partially umbilic points. The plane distributions spanned by $L_2$ and $L_1$ or $L_3$ are integrable.

Proof. Consider the parametrization $\alpha : [0, 2\pi] \times \{(v, w) : v^2 + w^2 < 1\} \to \mathbb{E}_{a,c,c,b}$ defined by

$$\alpha(u, v, w) = (a \cos u \sqrt{1 - v^2 - w^2}, b \sin u \sqrt{1 - v^2 - w^2}, cv, cw).$$

In this chart the first fundamental form is given by:

$$g_{11} = (1 - v^2 - w^2)(b^2 \cos^2 u + a^2 \sin^2 u),$$
$$g_{12} = (a^2 - b^2)v \sin u \cos u, \quad g_{13} = (a^2 - b^2)w \sin u \cos u$$
$$g_{22} = c^2 + \frac{v^2(a^2 \cos^2 u + b^2 \sin^2 u)}{(1 - v^2 - w^2)}, \quad g_{23} = \frac{vw(a^2 \cos^2 u + b^2 \sin^2 u)}{(1 - v^2 - w^2)}$$
$$g_{33} = c^2 + \frac{w^2(a^2 \cos^2 u + b^2 \sin^2 u)}{(1 - v^2 - w^2)}$$

With respect to the positive normal vector (oriented inward)

$$N(u, v, w) = -\left(\frac{c \cos u \sqrt{1 - v^2 - w^2}}{a}, \frac{c \sin u \sqrt{1 - v^2 - w^2}}{b}, v, w\right).$$
the second fundamental form \( b_{ij} = \langle \alpha_{ij}, N \rangle \) \((\alpha_{11} = \alpha_{uu}, \ \alpha_{12} = \alpha_{uv}, \ etc.)\) is given by
\[
\begin{align*}
b_{11} &= c(1 - v^2 - w^2), \quad b_{12} = 0, \quad b_{13} = 0 \\
b_{22} &= \frac{c(1 - w^2)}{c^2 - v^2 - w^2}, \quad b_{23} = \frac{vw}{1 - v^2 - w^2} \\
b_{33} &= \frac{c(1 - v^2)}{c^2 - v^2 - w^2}
\end{align*}
\]

Let \( p(k) = \det(b_{ij} - kg_{ij}) \). Since \( a > c > b \), write \( a^2 = c^2 + s, \ c^2 = b^2 + t \) with \( s > 0 \) and \( t > 0 \).

Also let \( v = R \cos \gamma \) and \( w = R \sin \gamma \).

The polynomial \( p(k) \) has double roots on \( v = 0, \ w = 0 \) and also when resultant \( p(k), p'(k), k = 0 \).

Recall that the resultant of a cubic polynomial \( P(k) = Ak^3 + Bk^2 + Ck + 1 \) and its derivative \( P'(k) = 3Ak^2 + 2Bk + C \), called the discriminant of \( P(k) \), is given by (see [1]):
\[
27A^2 - 18ABC - B^2C^2 + 4B^3 + 4AC^3.
\]

Algebraic manipulation shows that the zeros of the discriminant of \( p(k) \) is equivalent to the equation:
\[
PU(R, u) = (s \cos^2 u - t \sin^2 u)^2 R^4 + 2[(s + t)^2 \cos^2 u \sin^2 u + st]R^2 \\
(t \cos^2 u - s \sin^2 u)^2 = 0.
\]

Under the hypothesis \((a > c > b \iff s > 0, t > 0)\) the equation \( PU(R, u) = 0 \) has real roots only when \( R = 0 \) and so \( p(k) \) has double or triple roots only on \( v = 0, w = 0 \).

Thus the partially umbilic set is contained in \( \{v = 0, w = 0\} \) and there the principal curvatures are given by:
\[
k_1(u, 0, 0) = k_2(u, 0, 0) = \frac{ab}{c^2} < k_3(u, 0, 0) = \frac{ab}{\Delta^3}, \text{ if } \cos^2 u < \frac{a^2 - c^2}{a^2 - b^2} \\
k_1(u, 0, 0) = \frac{ab}{\Delta^3} < k_2(u, 0, 0) = k_3(u, 0, 0) = \frac{ab}{c^2}, \text{ if } \cos^2 u > \frac{a^2 - c^2}{a^2 - b^2} \\
\Delta = \sqrt{a^2 \sin^2 u + b^2 \cos^2 u}.
\]

It follows that \( k_1 = k_2 = k_3(u, 0, 0) \) when \( \cos^2 u = \frac{a^2 - c^2}{a^2 - b^2} = \frac{s}{s + t} \).

Next consider the parametrization \( \beta : [0, 2\pi] \times \{(u, v) : u^2 + v^2 < 1\} \to \mathbb{E}_{a, c, c, b} \) defined by
\[
\beta(u, v, \theta) = (au, bv, c \cos \theta \sqrt{1 - u^2 - v^2}, c \sin \theta \sqrt{1 - u^2 - v^2}).
\]
Similar analysis gives that the umbilic and partially umbilic points are defined by the equation

\[ PU(u, v, \theta) = (su^2 - tv^2)^2 - 2(s + t)(su^2 + tv^2) + (s + t)^2 = 0, \quad u^2 + v^2 < 1. \]

\[ = (1 - v^2)s^2 + (1 - u^2)t^2 + 2(1 - u^2v^2 - v^2 - u^2)st = 0 \]

It follows that this equation above does not have real solutions. In fact, in the region \( u^2 + v^2 < 1 \) the above equation, seen as a quadratic form in the variables \( (s, t) \), is positive definite. \( \square \)

2.8. Four distinct axes: \( E_{a,b,c,d} \). In this section will be established the global behavior of principal lines in the ellipsoid of four different axes.

**Lemma 1.** The ellipsoid \( Q_0 = E_{a,b,c,d} \) given by

\[ Q(x, y, z, w) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \frac{w^2}{d^2} - 1 = 0, \quad a > b > c > d > 0, \]

has sixteen principal charts \( (u, v, t) = \varphi_i(x, y, z, w) \), where

\[ \varphi_i^{-1} : (d^2, c^2) \times (c^2, b^2) \times (b^2, a^2) \{ (x, y, z, w) : xwyz \neq 0 \} \cap Q^{-1}(0) \]

is defined by equation (4).

\[ x^2 = \frac{a^2(a^2 - u)(a^2 - v)(a^2 - t)}{(a^2 - b^2)(a^2 - c^2)(a^2 - d^2)}, \quad y^2 = \frac{b^2(b^2 - u)(b^2 - v)(b^2 - t)}{(b^2 - a^2)(b^2 - c^2)(b^2 - d^2)}, \]

\[ z^2 = \frac{c^2(c^2 - u)(c^2 - v)(c^2 - t)}{(c^2 - a^2)(c^2 - b^2)(c^2 - d^2)}, \quad w^2 = \frac{d^2(d^2 - u)(d^2 - v)(d^2 - t)}{(d^2 - a^2)(d^2 - b^2)(d^2 - c^2)}. \]

For all \( p \in \{ (x, y, z, w) : xwyz \neq 0 \} \cap Q^{-1}(0) \) and \( Q_0 = Q^{-1}(0) \) positively oriented, the principal curvatures satisfy \( 0 < k_1(p) < k_2(p) < k_3(p) \).

**Proof.** It will be shown that the ellipsoid \( Q_0 = E_{a,b,c,d} \) belongs to a quadruply orthogonal family of quadrics. For \( p = (x, y, z, w) \in \mathbb{R}^4 \) and \( \lambda \in \mathbb{R} \), let

\[ Q(p, \lambda) = \frac{x^2}{(a^2 - \lambda)} + \frac{y^2}{(b^2 - \lambda)} + \frac{z^2}{(c^2 - \lambda)} + \frac{w^2}{(d^2 - \lambda)}. \]

For \( p = (x, y, z, w) \in Q_0 \cap \{ (x, y, z, w) : xwyz \neq 0 \} \) let \( u, v \) and \( t \) be the solutions of the quartic equation in \( \lambda \), \( Q(p, \lambda) - 1 = 0 \) with \( u \in (d^2, c^2), \)
\( v \in (c^2, b^2) \) and \( t \in (b^2, a^2) \). So the map \( \varphi : Q_0 \cap \{ (x, y, z, w) : xwyz \neq 0 \} \rightarrow (d^2, c^2) \times (c^2, b^2) \times (b^2, a^2) \) is well defined.

By definition of \( (u, v, t) \) it follows that:
\[
Q(p, \lambda) - 1 = \frac{-\lambda(\lambda - u)(\lambda - v)(\lambda - t)}{\xi(\lambda)}
\]

(5)

\[
\xi(\lambda) = (a^2 - \lambda)(b^2 - \lambda)(c^2 - \lambda)(d^2 - \lambda)
\]

Differentiating equation (5) with respect to \(\lambda\) and evaluating at \(\lambda = 0\), \(\lambda = u\), \(\lambda = v\) and \(\lambda = t\), which are the four simple roots of the equation, it follows that:

\[
\frac{uvt}{a^2b^2c^2d^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} + \frac{w^2}{d^4} \quad \text{and}
\]

\[
\frac{-u(v-u)(u-t)}{\xi(u)} = \frac{x^2}{(a^2 - u)^2} + \frac{y^2}{(b^2 - u)^2} + \frac{z^2}{(c^2 - u)^2} + \frac{w^2}{(d^2 - u)^2}
\]

\[
\frac{-v(v-u)(v-t)}{\xi(v)} = \frac{x^2}{(a^2 - v)^2} + \frac{y^2}{(b^2 - v)^2} + \frac{z^2}{(c^2 - v)^2} + \frac{w^2}{(d^2 - v)^2}
\]

\[
\frac{-t(t-u)(t-v)}{\xi(t)} = \frac{x^2}{(a^2 - t)^2} + \frac{y^2}{(b^2 - t)^2} + \frac{z^2}{(c^2 - t)^2} + \frac{w^2}{(d^2 - t)^2}
\]

The solution of the linear system above in the variables \(x^2\), \(y^2\), \(z^2\) and \(w^2\) is given by:

\[
x^2 = \frac{a^2(a^2 - u)(a^2 - v)(a^2 - t)}{(a^2 - b^2)(a^2 - c^2)(a^2 - d^2)}, \quad y^2 = \frac{b^2(b^2 - u)(b^2 - v)(b^2 - t)}{(b^2 - a^2)(b^2 - c^2)(b^2 - d^2)}
\]

\[
z^2 = \frac{c^2(c^2 - u)(c^2 - v)(c^2 - t)}{(c^2 - a^2)(c^2 - b^2)(c^2 - d^2)}, \quad w^2 = \frac{d^2(d^2 - u)(d^2 - v)(d^2 - t)}{(d^2 - a^2)(d^2 - b^2)(d^2 - c^2)}
\]

(6)

The map

\(\varphi : Q_0 \cap \{(x, y, z, w) : xyzw \neq 0\} \to (d^2, c^2) \times (c^2, b^2) \times (b^2, a^2)\),

\(\varphi(x, y, z, w) = (u, v, t)\) is a regular covering which defines a chart in each orthant of the ellipsoid \(Q_0\).

So, equations in (6) define parametrizations \(\psi(u, v, t) = (x, y, z, w)\) of the connected components of the region \(Q_0 \cap \{(x, y, z, w) : xyzw \neq 0\}\). By symmetry, it is sufficient to consider only the positive octant \(\{(x, y, z, w) : x > 0, y > 0, z > 0, w > 0\}\).

Consider the parametrization \(\psi(u, v, t) = \varphi^{-1}(u, v, t) = (x, y, z, w)\), with \((x, y, z, w)\) in the positive orthant. The fundamental forms of \(Q_0\) will be evaluated and expressed in terms of the function

\[
\xi(\lambda) = (a^2 - \lambda)(b^2 - \lambda)(c^2 - \lambda)(d^2 - \lambda).
\]

(7)
Evaluating \( g_{11} = (x_u)^2 + (y_u)^2 + (z_u)^2 + (w_u)^2 \), \( g_{22} = (x_v)^2 + (y_v)^2 + (z_v)^2 + (w_v)^2 \) and \( g_{33} = (x_t)^2 + (y_t)^2 + (z_t)^2 + (w_t)^2 \) and observing that \( g_{ij} = 0, \ i \neq j \), it follows that the first fundamental form is given by:

\[
I = -\frac{1}{4} \left[ \frac{u(u-v)(u-t)}{\xi(u)} du^2 + \frac{v(u-v)(v-t)}{\xi(v)} dv^2 + \frac{t(t-u)(t-v)}{\xi(t)} dt^2 \right].
\]

The normal vector \( N = (\psi_u \wedge \psi_v \wedge \psi_t)/|\psi_u \wedge \psi_v \wedge \psi_t| \) is oriented inward. In fact, it follows that \( \langle N(u,v,t), \psi(u,v,t) \rangle < 0 \).

Similar and straightforward calculation shows that the second fundamental form with respect to \( N \) is given by:

\[
II = -\frac{1}{4} \frac{abcd}{(uvt)^2} \left[ \frac{(u-v)(u-t)}{\xi(u)} du^2 + \frac{(v-u)(v-t)}{\xi(v)} dv^2 + \frac{(t-u)(t-v)}{\xi(t)} dt^2 \right].
\]

Therefore the coordinate lines are principal curvature lines and the principal curvatures \( b_{ii}/g_{ii}, \ i = 1, 2, 3 \) are given by:

\[
l = \frac{1}{u} \left( \frac{abcd}{\sqrt{uvt}} \right), \quad m = \frac{1}{v} \left( \frac{abcd}{\sqrt{uvt}} \right), \quad n = \frac{1}{t} \left( \frac{abcd}{\sqrt{uvt}} \right).
\]

Since \( u \leq v \leq t \) it follows that \( l = m \) if, and only if, \( u = v = c^2 \). Also \( m = n \) if, and only if, \( v = t = b^2 \). Therefore, for \( p \in Q_0 \cap \{(x,y,z,w) : xyzw \neq 0\} \) it follows that the principal curvatures satisfy \( k_1(p) < k_2(p) < k_3(p) \).

**Lemma 2.** Consider the ellipsoid \( E_{a,b,c} \) given in \( \mathbb{R}^3 \) by:

\[
x^2/a^2 + y^2/b^2 + z^2/c^2 = 1, \quad a > b > c > 0.
\]

Let

\[
s_1 = \frac{1}{2} \int_{a^2}^{b^2} \sqrt{\frac{-u}{(a^2-u)(c^2-u)}} du < \infty \quad \text{and} \quad s_2 = \frac{1}{2} \int_{c^2}^{b^2} \sqrt{\frac{-v}{(a^2-v)(c^2-v)}} dv < \infty.
\]

There exists a parametrization \( \varphi : [-s_1,s_1] \times [-s_2,s_2] \to E_{a,b,c} \cap \{(x,y,z), y \geq 0\} \) such that the principal lines are the coordinate curves and \( \varphi \) is conformal in the interior of the rectangle.

Moreover \( \varphi(s_1,s_2) = U_1, \ \varphi(-s_1,s_2) = U_2, \ \varphi(-s_1,-s_2) = U_3, \) and \( \varphi(s_1,-s_2) = U_4 \).

By symmetry considerations the same result holds for the region \( E_{a,b,c} \cap \{(x,y,z), y \leq 0\} \). See Fig. 2.

**Proof.** The ellipsoid \( E_{a,b,c} \) has a principal chart \((u,v)\) defined by the parametrization \( \psi : [b^2,a^2] \times [c^2,b^2] \to \{(x,y,z) : x > 0, \ y > 0, \ z > 0\} \)
given by:

$$
\psi(u, v) = \left( a \sqrt{(a^2 - v)(a^2 - u)} b \sqrt{(b^2 - v)(b^2 - u)} c \sqrt{(c^2 - v)(c^2 - u)} \right) \left( a \sqrt{a^2 - b^2} a \sqrt{a^2 - c^2} c \sqrt{b^2 - c^2} \right).
$$

The fundamental forms in this chart are given by:

$$
I = Edu^2 + Gdv^2 = \frac{1}{4} \frac{u(v - u)}{h(u)} du^2 - \frac{1}{4} \frac{v(u - v)}{h(v)} dv^2,
$$

$$
II = Edu^2 + Gdv^2 = \frac{abc(u - v)}{4 \sqrt{uvh(u)}} du^2 - \frac{abc(u - v)}{4 \sqrt{uvh(v)}} dv^2
$$

$$
h(t) = (a^2 - t)(b^2 - t)(c^2 - t).
$$

The principal curvatures are given by $k_2(u, v) = \frac{abc}{v \sqrt{uv}}$, $k_1(u, v) = \frac{abc}{u \sqrt{uv}}$. Therefore, $k_1(u, v) = k_2(u, v)$ if and only if $u = v = b^2$.

Considering the change of coordinates defined by $ds_1 = \sqrt{E} du$, $ds_2 = \sqrt{G} dv$ obtain a conformal parametrization $\varphi : [0, s_1] \times [0, s_2] \rightarrow \{(x, y, z) : x > 0, y > 0, z > 0\}$ in which the coordinate curves are principal lines and the fundamental forms are given by $I = ds_1^2 + ds_2^2$ and $II = k_1 ds_1^2 + k_2 ds_2^2$.

From the symmetry of the ellipsoid $E_{a,b,c}$ with respect to coordinate plane reflections, consider an analytic continuation of $\varphi$ from the rectangle $R = [−s_1, s_1] \times [−s_2, s_2]$ and to obtain a conformal chart $(U, V)$ of $R$ covering the region $E_{a,b,c} \cap \{y \geq 0\}$.

By construction $\varphi(\partial R)$ is the ellipse in the plane $xz$ and the four vertices of the rectangle $[−s_1, s_1] \times [−s_2, s_2]$ are mapped by $\varphi$ to the four umbilic points $U_i$ given by $\left( \pm a \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, 0, \pm c \sqrt{\frac{b^2 - c^2}{a^2 - c^2}} \right)$.

This lemma has been included here with a proof for convenience of the reader. No explicit proof of it in the literature is known to the authors. It will be essential in the formulation and proof of Lemma 3.
Lemma 3. Let $\lambda \in (d^2, c^2)$ and consider the intersection of the quadric
\[ Q_{\lambda}(x, y, z, w) = \frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} + \frac{z^2}{c^2 - \lambda} + \frac{w^2}{d^2 - \lambda} = 1, \quad a > b > c > d > 0, \]
with the ellipsoid $Q_0 = Q_0^{-1}(0)$. Let $Q_{\lambda} = Q_{\lambda}^{-1}(0) \cap Q_0$. Then $Q_{\lambda}$ is a compact quartic surface with two connected components, both being diffeomorphic to $\mathbb{S}^2$ and there exists a principal parametrization $\varphi_{\lambda} : [-s_1(\lambda), s_1(\lambda)] \times [-s_2(\lambda), s_2(\lambda)] \rightarrow Q_{\lambda} \cap \{(x, y, z, w), y \geq 0, w > 0\}$ such that the principal lines are the coordinate curves and $\varphi$ is conformal in the interior of the rectangle. Here,
\[ s_1(\lambda) = \frac{1}{2} \int_{b_2}^{a_2} \sqrt{\frac{(u - \lambda)u}{\xi(u)}} \, du < \infty \quad \text{and} \quad s_2(\lambda) = \frac{1}{2} \int_{c_2}^{b_2} \sqrt{\frac{(v - \lambda)v}{\xi(v)}} \, dv < \infty, \]
where $\xi(t) = (a^2 - t)(c^2 - t)(d^2 - t)$.

Moreover $\varphi_{\lambda}([s_1(\lambda), s_2(\lambda))] = P_{12}^1(\lambda)$, $\varphi_{\lambda}([-s_1(\lambda), s_2(\lambda)]) = P_{12}^2(\lambda)$, $\varphi_{\lambda}([-s_1(\lambda), -s_2(\lambda)]) = P_{12}^3(\lambda)$, $\varphi_{\lambda}([s_1(\lambda), -s_2(\lambda)]) = P_{12}^4(\lambda)$.

By symmetry considerations the same result holds for the region $Q_{\lambda} \cap \{(x, y, z, w), y \leq 0, w > 0\}$. See Fig. 10. Similar conclusions hold for $\lambda \in (b^2, a^2)$.

**Figure 10.** Principal lines of the quartic $Q_{\lambda}$ represented in a conformal chart.

**Proof.** The principal chart defined by equation (11) has the first and second fundamental forms given by equations (8) and (9). Therefore, for $\lambda \in (d^2, c^2)$, $\psi_{\lambda}(u, v) = \psi(u, v, \lambda)$, $\psi : (b^2, c^2) \times (b^2, a^2) \rightarrow Q_{\lambda}$ is a parametrization of $Q_{\lambda}$ in the region $Q_{\lambda} \cap \{(x, y, z, w), w > 0\}$ by principal curvature lines. Since the quadratic form $Q_{\lambda}(x, y, z, w)$ has signature $1 (+ + + -)$ in the sense of Morse, it follows that $Q_{\lambda}$ has two connected components, one contained in the region $w > 0$ and the other in the region $w < 0$. The conclusion of the proof is similar to that of Lemma 2.  \[\square\]
Lemma 4. Let \( \lambda \in (c^2, b^2) \) and consider the quartic surface \( Q_\lambda = Q_\lambda^{-1}(0) \cap Q_0 \).

Then \( Q_\lambda \) is diffeomorphic to a bidimensional torus of revolution and there exists a conformal principal parametrization of \( Q_\lambda \) such that the principal lines are the coordinates curves.

Proof. Similar to the proof of Lemma 3. For \( \lambda \in (c^2, b^2) \) the quadratic form \( Q_\lambda(x, y, z, w) \) has signature 2 \((++--)\), so it follows that \( Q_\lambda \) has only one connected component diffeomorphic to a torus.

Remark 1. Lemmas 2, 3 and 4 establish that all regular leaves of all principal foliations \( F_i \) \((i = 1, 2, 3)\) of the ellipsoid \( E_{a,b,c,d} \) are closed (no recurrences occur) and also that the singularities (partially umbilic and umbilic points) are contained in the coordinate planes.

This geometric structure is crucial to obtain the global description of the principal foliations of the ellipsoid \( E_{a,b,c,d} \) with four different axes. Examples of recurrent principal lines on surfaces diffeomorphic to a torus or to the ellipsoid can be made explicit, see [10] and [15].

Proposition 5. For \( \lambda \in (b^2, a^2) \cup (d^2, c^2) \), the principal configuration \( \{F_1, F_2, P_{12}\} \) or \( \{F_2, F_3, P_{23}\} \) on each connected component of \( Q_\lambda = Q_1^1 \cup Q_2^2 \) is principally equivalent to an ellipsoid with three distinct axes in \( \mathbb{R}^3 \), i.e., there exists a homeomorphism \( h_i : Q_\lambda^i \to E_{a,b,c} \) preserving both principal foliations and singularities.

For \( \lambda \in (c^2, b^2) \), \( Q_\lambda \) is principally equivalent to a torus of revolution in \( \mathbb{R}^3 \).

Proof. It follows from Lemmas 2 and 3 that the principal configuration of the ellipsoid with three different axes in \( \mathbb{R}^3 \) is topologically equivalent to the principal configuration of the quartic surface \( Q_\lambda \subset E_{a,b,c,d} \) when \( \lambda \in (d^2, c^2) \cup (b^2, a^2) \).

By Lemma 4 the principal configuration of \( Q_\lambda \subset E_{a,b,c,d} \), with \( \lambda \in (c^2, b^2) \), is topologically equivalent to the principal configuration of a torus of revolution in \( \mathbb{R}^3 \).

The construction of the homeomorphism can be done by the method of canonical regions. Traditionally, as explained by M. C. Peixoto and M. M. Peixoto [21], this procedure partitions the domain of a foliation with singularities into cells with a parallel structure and other disk-cells and annuli containing, respectively, the singularities and periodic leaves of the foliation. These are the canonical regions. Then the homeomorphism preserving
lines of principal curvature on ellipsoids of \( \mathbb{R}^4 \)

leaves between two foliations with similar partitions of canonical regions is carried out first on the parallel regions and then extending it to the disk-cells and annuli. The method of canonical regions was adapted from the case of single foliations with singularities to that of nets of transversal foliations with singularities on surfaces, such as the principal configurations on surfaces, in [14].

In the present case the principal configuration of the ellipsoid, see Fig. 9 in lemma 2, is equivalent to that of the quartic surface diffeomorphic to the ellipsoid, see Fig. 10 in Lemma 3. The canonical regions in each hemisphere are product of parallel foliations. In the case of the torus, Lemma 4, both principal configurations consist of foliations by closed curves, circles in the torus of revolution and quartic algebraic curves on the surface \( Q_{\lambda} \). In this case the canonical region is the entire surface.

Take the parametrizations

\[
\alpha_\pm(u, v, w) = (au, bv, cw, \pm d\sqrt{1 - u^2 - v^2 - w^2})
\]

of the ellipsoid \( E_{a,b,c,d} \) defined by

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \frac{\omega^2}{d^2} = 1, \quad a > b > c > d > 0.
\]

**Theorem 2.** The umbilic set of \( E_{a,b,c,d} \) is empty and its partially umbilic set consists of four closed curves \( P_{112}, P_{122}, P_{123} \) and \( P_{223} \), that, assuming the notation \( c^2 = d^2 + t, \ b^2 = d^2 + t + s, \ a^2 = d^2 + t + s + r, \)
are defined in the chart \((u, v, w)\) by:

\[
\begin{align*}
  w &= 0, \quad s(r + s + t)u^2 + (t + s)(r + s)v^2 - s(r + s) = 0, \quad \text{ellipse} \\
  v &= 0, \quad (st + sr + s^2)u^2 - trw^2 - sr = 0, \quad \text{hyperbole}.
\end{align*}
\]

i) The principal foliation \( F_1 \) is singular on \( P_{112} \cup P_{122}, \) \( F_3 \) is singular on \( P_{123} \cup P_{223} \) and \( F_2 \) is singular on \( P_{121} \cup P_{212} \cup P_{123} \cup P_{232} \).

The partially umbilic curves \( P_{112} \cup P_{122} \) whose transversal structures are of type \( D_1 \), the partially umbilic separatrix surfaces of \( P_{112} \) and \( P_{122} \) span a cylinder \( C_{12} \) such that \( \partial C_{12} = P_{112} \cup P_{122} \).

Also, the partially umbilic curves \( P_{123} \) and \( P_{223} \) are of type \( D_1 \), its partially umbilic separatrix surfaces span a cylinder \( C_{23} \) such that \( \partial C_{23} = P_{223} \cup P_{232} \).

All the leaves of \( F_1 \) outside the cylinder \( C_{12} \) are compact and diffeomorphic to \( S^1 \). Analogously for the principal foliation \( F_3 \) outside the cylinder \( C_{23} \).

See illustration in Fig. [11].
ii) The principal foliation $F_2$ is singular at $P_{12}$, $P_{13}$, $P_{23}$, and there are Hopf bands $H_{123}$ and $H_{123}^2$ such that $\partial H_{123} = P_{12} \cup P_{23}$ and $\partial H_{123}^2 = P_{12} \cup P_{23}^2$, which are partially umbilic separatrix surfaces.

All the leaves of $F_2$, outside the partially umbilic surfaces separatrices, are closed. See illustration in Fig. 12.

![Figure 11. Global behavior of the principal foliations $F_i$, ($i = 1, 2, 3$). The cylinder $C_{12}$ is foliated by principal lines of $F_1$ with boundary two partially umbilic lines (dotted green lines). The cylinder $C_{23}$ is foliated by principal lines of $F_3$ with boundary two partially umbilic lines (dotted blue lines).](image)

**Proof.** It will be considered only the parametrization $\alpha_+$. The other case follows by symmetry.

In the parametrization $\alpha_+$, the first fundamental form ($g_{ij}$) is given by:

\[
\begin{align*}
g_{11} &= \frac{a^2(u^2 + v^2 + w^2 - 1) - d^2w^2}{u^2 + v^2 + w^2 - 1}, \\
g_{12} &= -\frac{d^2uv}{u^2 + v^2 + w^2 - 1}, \\
g_{22} &= \frac{b^2(u^2 + v^2 + w^2 - 1) - d^2v^2}{u^2 + v^2 + w^2 - 1}, \\
g_{13} &= -\frac{d^2uw}{u^2 + v^2 + w^2 - 1}, \\
g_{33} &= \frac{c^2(u^2 + v^2 + w^2 - 1) - d^2w^2}{u^2 + v^2 + w^2 - 1}, \\
g_{23} &= -\frac{d^2vw}{u^2 + v^2 + w^2 - 1}.
\end{align*}
\]

The positive normal field (oriented inward) $N_+ = \frac{1}{abcd} \alpha_u \wedge \alpha_v \wedge \alpha_w$ is given by:

\[
N_+(u, v, w) = -\left(\frac{u}{a\Delta}, \frac{v}{b\Delta}, \frac{w}{c\Delta}, \frac{1}{d}\right), \quad \Delta = \sqrt{1 - u^2 - v^2 - w^2}.
\]
The second fundamental form \( (b_{ij}) \), \( b_{ij} = \langle \alpha_{ij}, N_+ \rangle \), relative to \( N_+ \), is given by:

\[
\begin{align*}
  b_{11} &= \frac{1 - v^2 - w^2}{(1 - u^2 - v^2 - w^2)^{\frac{3}{2}}}, \\
  b_{12} &= \frac{uv}{(1 - u^2 - v^2 - w^2)^{\frac{3}{2}}}, \\
  b_{22} &= \frac{1 - u^2 - w^2}{(1 - u^2 - v^2 - w^2)^{\frac{3}{2}}}, \\
  b_{13} &= \frac{uw}{(1 - u^2 - v^2 - w^2)^{\frac{3}{2}}}, \\
  b_{23} &= \frac{vw}{(1 - u^2 - v^2 - w^2)^{\frac{3}{2}}}, \\
  b_{33} &= \frac{1 - u^2 - v^2}{(1 - u^2 - v^2 - w^2)^{\frac{3}{2}}}.
\end{align*}
\]

Calculation shows that the principal curvatures, which are the roots of
\[
\text{det}(b_{ij} / |N_+| - kg_{ij}) = 0,
\]
are given by the zeros of

\[
p(k) = 1 - [(a^2 - d^2)u^2 + (b^2 - d^2)v^2 + (c^2 - d^2)w^2 - a^2 - b^2 - c^2]k
+ [(d^2 - a^2)(b^2 + c^2)u^2 + (a^2 + c^2)(d^2 - b^2)v^2
+ (d^2 - c^2)(a^2 + b^2)w^2 + b^2c^2 + a^2c^2 + a^2b^2]k^2
- [b^2c^2(a^2 - d^2)u^2 + a^2c^2(b^2 - d^2)v^2 + a^2b^2(c^2 - d^2)w^2 - a^2b^2c^2]k^3.
\]
Denote by $R(u,v,w) = \text{resultant}(p(k), p'(k), k)$, the discriminant of $p(k)$. Consider the polynomials $p_i$ defined below.

$p_1 = -s[(2t + 2s)v^2 - 2tw^2 - s] + [(s + t)v^2 + tw^2]^2,$

$p_2 = (s + t)(s + r)v^2 + rtw^2 + r(r + s)$

$p_3 = [(s + t + r)u^2 + tw^2]^2 - (s + r)[2(t + s + r)u^2 - 2tw^2 - s - r],$

$p_4 = s(t + s + r)u^2 - rtw^2 - rs$

$p_5 = -r[2(t + s + r)u^2 - 2(s + t)v^2 - r] + [(s + t + r)u^2 + (s + t)v^2]^2,$

$p_6 = s(s + t + r)u^2 + (s + t)(s + r)v^2 - s(s + r)$

It follows that $R(u,v,0) = -\frac{1}{6}p_5p_6^2$, $R(u,0,w) = -\frac{1}{6}p_3p_4^2$ and $R(0,v,w) = -\frac{1}{6}p_1p_2^2.$

By Lemma [1] there are no umbilics nor partially umbilic points outside the coordinate hyperplanes.

Therefore, the partially umbilic and umbilic sets are defined by the real branches of the equations $R(u,v,0) = 0$, $R(0,v,w) = 0$, $R(0,v,w) = 0$.

Now, by elementary analysis, it follows that the real zeros of the equations above are given by:

$\{(u,0,w) : p_4 = 0\} \cup \{(u,v,0) : p_6 = 0\}.$

The explicit solutions are given by:

$w = 0, s(r + s + t)u^2 + (t + s)(r + s)v^2 - s(r + s) = 0, \text{ (ellipse)}$

$v = 0, (st + sr + s^2)u^2 - trw^2 - sr = 0, \text{ (hyperbole)}.$

Let $P_{23}^1 = \alpha_+(\text{ellipse})$ and $P_{23}^2 = \alpha_- (\text{ellipse}).$ It follows that $P_{23}^1$ and $P_{23}^2$ are partially umbilic curves and they are singularities of $\mathcal{F}_2$ and $\mathcal{F}_3$ and regular leaves of $\mathcal{F}_1$.

The hyperbole $v = 0, (st + sr + s^2)u^2 - trw^2 - sr = 0$ is only a part of the other two connected components of the partially umbilic set defined in the domain $u^2 + v^2 + w^2 < 1$.

To compute the other two connected components of the partially umbilic set in a single local chart consider the parametrizations

$\beta_\pm(u,v,w) = (\pm a \sqrt{1 - u^2 - v^2 - w^2}, bv, cu, dw).$

Similar analysis as that for $\alpha_\pm(u,v,w)$ gives that the partially umbilic set is defined by:


\[ v = 0, (t + s)(r + s)u^2 + s(r + s + t)w^2 - s(t + s) = 0, \text{ (ellipse)} \]
\[ u = 0, rtv^2 - s(t + r + s)w^2 + st = 0, \text{ (hyperbole)}. \]

Let \( P_{1_{12}} = \beta_+ (\text{ellipse}) \) and \( P_{1_{12}} = \beta_- (\text{ellipse}) \). It follows that \( P_{1_{12}} \) and \( P_{2_{12}} \) are partially umbilic curves and they are singularities of \( F_1 \) and \( F_2 \) and regular leaves of \( F_3 \).

The two partially umbilic lines \( P_{1_{12}} \) and \( P_{2_{12}} \) (respectively \( P_{1_{23}} \) and \( P_{2_{23}} \)) contained in the plane \( v = 0 \) (respectively in the plane \( w = 0 \)) bound a cylinder \( C_{12} \) (respectively bound a cylinder \( C_{23} \)). The cylinder \( C_{12} \) is diffeomorphic to the cylinder \( S^1 \times [0, 1] \) and the boundary curves are not linked in \( E_{a,b,c,d} \). It is foliated by principal lines of \( F_1 \). Observe that the book structure in Proposition 3 near a double partially umbilic green curve \( P_{12} \), in Fig. 5 changes its transversal structure as the principal configuration of an ellipsoid of revolution changes into one with three different axes.

In a similar way \( C_{23} \) is diffeomorphic to \( C_{12} \) and it is foliated by principal lines of \( F_3 \), see Fig 11.

The pair of curves \( \{P_{1_{12}}, P_{1_{23}}\} \) is linked with linking number equal to \( \pm 1 \). This follows from Proposition 2 and by the invariance of the linking number by homotopy, see 4.

Also \( \{P_{1_{12}}, P_{1_{23}}\} \) bounds a Hopf band \( H_{1_{123}} \), i.e. there is an embedding \( \beta : S^1 \times [0, 1] \to E_{a,b,c,d} \) such that \( P_{1_{12}} = \beta(S^1 \times \{0\}) \) and \( P_{2_{23}} = \beta(S^1 \times \{1\}) \) and the surface \( H_{1_{123}} \) is foliated by leaves of the principal foliation \( F_2 \).

This structure of Hopf bands follows since the ellipsoid \( E_{a,b,c,d} \) with four different axes belongs to a quadruply orthogonal family of quadrics and its principal lines are closed curves, obtained intersecting three hypersurfaces of this family, see the proof of Lemma 4.

Near each partially umbilic curve \( P_{1_{12}}, P_{1_{12}}, P_{2_{12}}, P_{2_{23}} \) and \( P_{2_{23}} \) the principal foliation \( F_2 \) has its transversal structure equivalent to the principal configuration of a Darbouxian umbilic point \( D_1 \), see Figs. 5, 6 and 12 and Propositions 3 and 4. So the umbilic separatrix surface of \( F_2 \) of the partially umbilic curves \( P_{1_{12}} \) and \( P_{1_{23}} \) is a cylinder having these curves in its boundary. As the pair \( \{P_{1_{12}}, P_{2_{23}}\} \) is linked, it follows that this invariant cylinder by the principal foliation \( F_2 \) is a Hopf band.

In similar way the pair \( \{P_{1_{12}}, P_{2_{23}}\} \) bounds a Hopf band \( H_{1_{123}}^2 \) which is foliated by leaves of \( F_2 \), see Fig. 12. \( \square \)
3. Concluding Comments

In this work has been established the geometric structure (principal configuration) defined on an ellipsoid in $\mathbb{R}^4$ by its umbilic and partially umbilic points and by the integral foliations of its principal plane and line fields. The results have been proved analytically and illustrated graphically.

The configurations established in Proposition 2 and Theorem 1 respectively, for the bi and triaxial ellipsoids $E_{a,a,b,b}$ and $E_{a,c,c,b}$, have no parallel in the literature. The second case establishing the existence of four umbilic points which separate four arcs of partially umbilic points, altogether forming a closed regular curve, as illustrated in figure 7, maybe regarded as the most novel contribution of this paper.

The explanation for the quadriaxial ellipsoid $E_{a,b,c,d}$ in subsection 2.8 makes explicit and gives more analytic and geometric details not found in previous studies on this matter found in literature: [6], [8]. More details not formulated explicitly neither illustrated in previous works are provided in figures 11 and 12.

A crucial step in the study of lines of principal curvature on surfaces in $\mathbb{R}^3$ was given by Darboux who in 1887 established the local principal configurations at generic umbilic points on analytic surfaces. See [2] and its extension from analytic to smooth generic surfaces by Gutierrez and Sotomayor in 1982 [14].

The determination of the local principal configurations around the generic singularities of the principal configuration of a smooth hypersurface in $\mathbb{R}^4$ was achieved by Garcia in 1989 [6]. See also [7] and [8]. This can be considered as the analogue for $\mathbb{R}^4$ of Darboux result mentioned above for $\mathbb{R}^3$. A new proof of Garcia’s result, which also allows an extension to study the generic bifurcations of the singularities of principal configurations in families of hypersurfaces depending generically on one parameter will appear in [18]. See also [17].

The local and global study of principal configurations of surfaces in $\mathbb{R}^3$ has been developed in several directions. A very partial list of references is given as a sample: [9], [10], [13], [14], [16], [19], [22], [23], [24].

Meanwhile, for the case of hypersurfaces in $\mathbb{R}^4$ it seems that, besides the few papers cited in this work, a wide horizon of possibilities wait to be explored.
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LINES OF PRINCIPAL CURVATURE ON ELLIPSOIDS OF $\mathbb{R}^4$

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