Teaching General Relativity with sector models: the field equations

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Abstract. The field equations of the general theory of relativity state the connection between spacetime curvature and matter, often summarized as "matter curves spacetime". In this contribution we analyze the structure of the field equations from a pedagogical point of view and describe a formulation that is accessible in secondary school and undergraduate physics. This formulation of the field equations requires the concept of curvature in three dimensions and we show how this notion can be introduced using sector models. The field equations are illustrated using the black hole as an example.

1. Introduction

General relativity is one of the fundamental advancements of physics in the 20th century. Well established and tested to high accuracy, it is the basis of modern astrophysics and cosmology. Besides its major importance for physics, general relativity and its applications are also especially fascinating for many students, both in school and at university. Contemporary physics classes should therefore give all students the opportunity to get acquainted with general relativity. This means introducing the topic no later than at secondary school level as well as generally including it in undergraduate university studies in physics. However, the teaching of general relativity at the secondary school and undergraduate levels is a challenge for physics education. Novel and abstract concepts must be explained. The mathematical framework of the theory is involved and is not accessible to learners at these levels. New approaches must therefore be developed that make it possible to teach general relativity using elementary mathematics only. In view of this goal we have developed a novel approach, based on a class of models representing curved spaces and spacetimes true to scale ([1]). These models that we call sector models, are realized as physical models built from paper or cardboard. The teaching strategy relies on the fact that general relativity is a geometric theory. The approach is focused on developing geometric insight. It uses measurements and graphic constructions performed on sector models as an alternative to setting up and solving equations.

In previous work we have described how sector models can be computed using elementary mathematics and how they can be used as tools to construct geodesics with pencil and ruler ([2]). Here we report on another aspect of teaching general relativity. The field equations are an important and mathematically involved part of the general theory of relativity. They state the connection between curvature and matter, often summarized as "matter curves spacetime". This contribution discusses how the field equations can be described in more detail, showing how curvature and matter, respectively,
enter in these equations. This is made possible on an elementary mathematical level by using sector models to visualize the curvature properties of three-dimensional curved space.

2. The field equations
In this section we will give an account of the quantities entering the field equations and describe the basic structure of the equations. This is done with a view to identifying the requirements for providing a conceptual understanding of the meaning and the essential structure of the Einstein equations. The goal is to convey an idea of how the curvature of a spacetime is linked to its matter content according to the general theory of relativity. This very basic account of the field equations is subdivided into three parts: the description of matter, the description of curvature, and the connection between the two. The description of matter includes the energy density (i.e. the mass density, by the equivalence of mass and energy), the momentum density, and the stress. These add up to a total of ten quantities: energy density (1), momentum density (3, momentum being a vector with three components), and stress (6, 3x3 symmetric matrix of the stress tensor components, stress being the force exerted across a surface). These ten quantities describing matter make up the 4x4 symmetric matrix of the components of the energy-momentum-stress tensor.

The description of curvature is given in terms of the Riemann curvature tensor. In the two-dimensional case of a curved surface, the intrinsic curvature is fully determined by a scalar, the Gaussian curvature. The corresponding Riemann curvature tensor (16 components) has one independent component determined by the Gaussian curvature. The symmetry properties of the tensor then fix the values of all other components. In three dimensions, curvature is described by a Riemann curvature tensor with six independent components (out of a total of 81), and for a four-dimensional spacetime the number of independent curvature components amounts to twenty (out of a total of 256).

The field equations state the connection between the energy-momentum-stress tensor and the Riemann curvature tensor. They are a set of ten coupled equations corresponding to the ten quantities in the description of matter. To demonstrate the essence of this connection, we use a specific choice of coordinates that simplifies the form of the equations ([3]): We consider a point of the spacetime and a local inertial system in the vicinity of this point. Using the space and time coordinates of this local inertial system, the ten equations take the following general form: On the one side there is one of the components of the energy-momentum-stress tensor (multiplied by the constant $K = \frac{8\pi G}{c^4}$, $G$ the gravitational constant, $c$ the speed of light), and on the other side the sum of three components of the Riemann curvature tensor.

We focus on a single equation out of the set of ten equations in order to illustrate the way in which the field equations state the connection between curvature and matter. We propose to visualize the equation for the energy density. This equation is the easiest to explain, on both sides: the energy density is the most familiar of the quantities describing matter. And the curvature components involved are purely spatial as opposed to spatiotemporal components that enter the other equations. The major requirement clearly is an understanding of curvature in three dimensions. This understanding must include the notion that curvature in three dimensions is represented not by a single number, but by several components. We need a mental image of three-dimensional curved space that allows us to visualize its curvature. This image should also enable us to compare different three-dimensional spaces with respect to their curvature components.

3. The visualization of curved spaces with sector models
Though curved surfaces are familiar, it is not at all evident how to extend the notion of curvature to three-dimensional curved spaces. The reason is that we picture curved surfaces as being embedded in three-dimensional space. It is, however, not possible for us to picture a region of three-dimensional space as being embedded in a four- or higher-dimensional environment. What we need is a mental image of three-dimensional curved space that is accessible to three-dimensional thinking. Sector models provide such an image as we show below.
Figure 1. Sector models of the Euclidean plane (left column), the sphere (middle column), and the saddle (right column). a, b, c: subdivision into quadrangles, d, e, f: sector models, g, h, i: test for curvature.

3.1. Sector Models in two dimensions

The purpose of this section is the introduction of sector models using flat and curved surfaces as examples. It will be summarized how the Gaussian curvature of a surface can be determined by measurements taken on its sector model ([1]).

Fig. 1 illustrates the construction of sector models for three examples: the Euclidean plane, the sphere, and the saddle. First, the surface is subdivided into small parts, here chosen to be quadrangles (Fig. 1a to c). Next, the lengths of all edges of the quadrangles are determined. Finally, quadrangles with the same edge lengths are constructed in the Euclidean plane (Fig. 1d to f): these are the sectors. The sectors of one surface as a whole form its sector model. The model is an approximation to the surface; by using more and smaller sectors the approximation can be improved.

The curvature of the surface can be read off from the sector model as shown in Fig. 1g to i. The sectors of the Euclidean plane fit together without gaps: zero curvature. The sectors of the other two surfaces do not fit without gaps, and this indicates non-zero curvature. The type of gap allows to discriminate between positive and negative curvature. The sphere is the prototype of a surface with positive curvature: when a piece of the sphere is flattened onto the plane, it tears. The tearing shows up as a gap in the sector model when sectors are assembled around their common vertex (Fig. 1h). The saddle is the prototype of a surface with negative curvature: flattened onto the plane it buckles.
Translated into the sector model, buckling means that sectors overlap, when assembled around their common vertex (Fig. 1 i). By measuring the deficit angle that characterizes the gap or the overlap, respectively, the Gaussian curvature can be determined quantitatively.

3.2. **Sector Models in three dimensions**

The construction of sector models and the determination of curvature from the model can be translated from two-dimensional surfaces to three-dimensional spaces ([1]). The crucial point about sector models that allows the transition to three-dimensional space is the fact that no extra dimension of an embedding space comes into play. A two-dimensional curved surface has a two-dimensional sector model and so a three-dimensional curved space has a three-dimensional sector model and is thus accessible to our imagination (and can be built as a physical model).

Fig. 2 illustrates the transition to three dimensions: A three-dimensional space is subdivided into small blocks, here adapted to a space with spherical symmetry. The lengths of all the edges are measured and blocks with the same edge lengths are constructed in Euclidean space. These are the three-dimensional sectors. The sectors of Euclidean space (Fig. 2, left) can be assembled without gaps, the sectors of the curved space around a black hole (Fig. 2, right) cannot. To measure curvature, the sectors are tested for tearing or buckling, and the deficit angles are measured. This criterion is transferred from the two-dimensional case: In two dimensions, flat sectors are assembled around a common vertex; in three dimensions, blocks are assembled around a common edge (Figures 3 and 4). Fig. 4 shows tearing (left) and buckling (middle, right) in the three-dimensional case.

It is apparent from the model that there is a major difference between curvature in two dimensions and in three dimensions. When testing for curvature at a point in three-dimensional space, there are several edges with different orientations. In general, curvature is different for different edges at the same point. Thus, the description of curvature at a point in curved space consists of several curvature components. Sector models provide a visualization of three-dimensional curvature. The sign of a curvature component can be determined by inspection, and the value can be computed from the measured deficit angle ([1]).

![Figure 2. Sector models of Euclidean space (left) and of the space around a black hole (right), ([1]).](image)
4. Visualization of the field equations
The visualization of curved space around a black hole shows that its curvature must be described in terms of the three components along the radial, the poloidal and the azimuthal axes. Given this visualization we come back to the exemplary field equation described above. It equates the energy density of matter (times the constant K) with the sum of the three spatial curvature components as visualized in the sector models shown above.

The field equation can be illustrated by the sector models shown in Figures 2, 3, and 4: Euclidean space has a sector model with zero deficit angle in all directions. According to the field equations, this means zero energy density: Space is Euclidean in the absence of matter. The black hole sector model describes space in the exterior region where the density of matter is zero. Visual inspection of the model shows that two of the curvature components are negative, and one is positive. The positive component is larger than either of the negative components. By virtue of the spherical symmetry of the black hole space, the two negative components are equal. It is clear that the positive and negative components will at least partially cancel in the sum. The precise values show that the sum is indeed zero. By virtue of the field equation, this means zero density, as expected.

Figure 3. Test for curvature in three dimensions: Euclidean space (Fig. 2a) has zero curvature.

Figure 4. Test for curvature in three dimensions: Space near a black hole (Fig. 2b) has positive curvature along a radial edge (left) and negative curvature along poloidal and azimuthal edges (middle and right), ([1]).

5. Conclusions
We have shown that the basic structure of the field equations can be described as equating a sum of curvature components with a quantity describing matter, such as the energy density. Sector models provide a way of visualizing curvature as a physical quantity described by several components, thus paving the way for an understanding of the structure of the field equations. The equation involving the energy density, in particular, can be visualized using a three-dimensional spatial sector model, because it involves three purely spatial curvature components (as opposed to spatiotemporal components
involved in the other equations) and these can be read off from the sector model of a three-dimensional curved space.

As this contribution shows, sector models and the conclusions drawn from them have a close correspondence to the mathematical formulation of the theory. For this reason, this visualization of a field equation can be used not only for an elementary introduction to general relativity but also within a standard course to complement the mathematical formulation and strengthen geometric insight.

References
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