On the hurricane collective molecular dynamics

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Abstract. We study the molecular dynamics underlying the formation and the evolution of hurricanes. The giant global structure of hurricanes cannot be solely the results of random molecular kinematics. We show that the non-stationary, topologically non-trivial hurricane behavior is the manifestation of the collective dynamics at the molecular level. This is described in terms of spontaneous breakdown of the rotational symmetry of the water molecule electrical dipoles. The long range molecular correlations generated as a consequence of the spontaneous breakdown of symmetry are responsible of the large scale collective motion resulting in the hurricane structure. The critical non-stationary dynamics is analyzed in terms of the non-stationary time dependent Ginzburg-Landau equation leading to the vortex equation. Phase transition, criticality, temperature dependence, energy exchanges and entropy are studied. Energy storage and dissipation processes in the coherent state dynamics are analyzed. Finally the self-similar fractal structure of the hurricane is shown to be the manifestation of the coherence of the collective molecular dynamics.

1. Introduction

In this paper we report some results obtained in the study of the molecular dynamics in the hurricane formation and interaction with its environment [1]. In tropical regions, large-scale storms, called hurricanes in the Atlantic ocean, form over warm ocean waters. A general consensus on the physical processes involved in cyclone formation and intensification or weakening is still lacking, although it is clear that the temperature of the ocean surface, the troposphere humidity and vorticities play crucial roles in their genesis [2, 3, 4, 5, 6]. Of great interest is for obvious reasons the understanding of the mechanisms by which they may be getting stronger or weaker on their way in crossing the ocean. Heat and angular momentum are any way main sources of energy for tropical cyclones [3, 7]. Indeed the hurricane may get more heat in the regions where more water will evaporate from the ocean, due to the warmer ocean’s surface in those regions. Flowing upward, the vapor cools down and latent heat is released from its condensation. It is also thought that an increase of the hurricane angular momentum may occur when it captures wind currents blowing in the proper direction (in this connection see the so called balanced vortex models and other numerical models [2, 8]).

Evidently, the hurricane formation and evolution result from highly non-trivial non-equilibrium thermodynamic and hydrodynamic processes and the complexity of the problem requires heavy use of numerical simulations. It is, however, also evident that the hurricane cannot be solely the result of random molecular kinematics; a giant collective molecular dynamics must
be at the basis of the hurricane formation and evolution. Aim of our work is in fact the study of the molecular dynamics responsible of large scale collective molecular behaviors turning into the hurricane topologically non-trivial structure, its energy storage and dissipation, its propagation speed as a macroscopic system, the criticality of its non-equilibrium dynamics, temperature dependence, etc. In Section 2 we introduce few elements of the fluid dynamics underlying the hurricane Burgers model description. In Section 3 we show how the central features of the Burgers model can be obtained in the canonical formalism for molecular dynamics. In Section 4 we discuss the collective molecular dynamics in the quantum field theory (QFT) framework. Criticality, vorticity and non-stationary properties are analyzed in Section 5 by using the time dependent Ginzburg-Landau equation. In Section 6 the fractal self-similarity of the logarithmic spiral structure of hurricane is shown to be the macroscopic manifestation of the coherence of the collective molecular dynamics. Conclusions and final comments on energy storage in coherent systems are presented in Section 7.

2. The hurricane fluid dynamics

In fluid dynamics studies of the hurricane phenomenon, one considers the Navier-Stokes equation for a fluid in a given space region with fixed density $\rho$ and viscosity $\nu$:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{\rho} \nabla p + \nu \Delta \mathbf{u}. \quad (1)$$

$\mathbf{u}$ is the velocity field and $p$ the pressure. The rotor of (1) gives:

$$\frac{\partial \mathbf{\omega}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{\omega}) = \nu \Delta \mathbf{\omega}. \quad (2)$$

where $\mathbf{\omega} = \nabla \times \mathbf{u}$ is the vorticity. Since in the considered region the fluid is assumed to be incompressible, the continuity equation reduces to $\nabla \cdot \mathbf{u} = 0$ and the vorticity equation is:

$$\frac{\partial \mathbf{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{\omega} = (\mathbf{\omega} \cdot \nabla) \mathbf{u} + \nu \Delta \mathbf{\omega}. \quad (3)$$

Within such a frame, we consider the Burgers model [9] describing how a vortex may stretch so as to concentrate its vorticity in a smaller region (see the Appendix A of [1] for details). We use cylindrical coordinates with the $z$-axis along the axis of the vortex and we look for a flow such that the vertical component of the velocity is $u_z = 2Cz$ (with $C$ a constant), and the radial component $u_r$ and $u_\theta$ are functions of $r$ only. Then, one obtains that the radial velocity must be equal to $u_r = -Cr + A/r$. We will choose $A = 0$ for convenience and thus we have

$$u_z = 2Cz; \quad u_r = -Cr. \quad (4)$$

In general $C$ depends on several space-time dependent variables, such as temperature $T$, pressure, density and viscosity of the medium, its molecular composition, the density and charge of present particles, atoms, molecules, ions, etc.. Thus, $C$ actually denotes its effective value $C(r,t)$ resulting from its dependence on the said variables. However, for definiteness in the model it is considered the approximation of constant $C$ over a relatively wide space region and relatively long time intervals. Putting $z = 0$ at the ground and limiting ourselves to the region $z > 0$, we see that $C > 0$ corresponds to a radially inward and vertically upward flow, whereas $C < 0$ corresponds to a radially outward and vertically downward flow (flow reversal). As mentioned, in the core of a hurricane, there is a vertical current of warm and wet air going upward. Unless differently stated, in the following the case $C > 0$ is considered.
Note that \( C = 0 \) in the absence of incoming or outgoing flux. On the other hand, the fact that for positive non-vanishing \( C \) the velocity increases without bond with the distance from the origin indicates that this model is only locally valid.

The conditions imposed on the flow imply that the vorticity is parallel to the \( z \)-axis: \( \omega(r, t) = \omega(r, t)u_z \). The vorticity equation becomes:

\[
\frac{\partial \omega}{\partial t} = 2C\omega + Cr \frac{\partial \omega}{\partial r} + \frac{\nu}{r} \frac{\partial}{\partial r} (r \frac{\partial \omega}{\partial r}).
\]

Observe that for \( C = 0 \) Eq. (5) reduces to the diffusion equation:

\[
\frac{\partial \omega}{\partial t} = \nu \Delta \omega,
\]

which has the solution:

\[
\omega(r, t) = \frac{\Gamma}{4\pi \nu t} \exp\left(-\frac{r^2}{4\nu t}\right).
\]

Here \( \Gamma \) denotes the circulation along a horizontal circle with infinite radius or, equivalently, to the flux of vorticity across the entire \( xy \)-plane:

\[
\Gamma = \lim_{r \to \infty} \oint_{C_r} u_\theta r \, d\theta = \int_{xy \text{-plane}} \omega \, dS
\]

In order to solve the case \( C \neq 0 \), we look for a Gaussian shaped solution of the form

\[
\omega(r, t) = \frac{1}{r} \frac{\partial}{\partial r} \left[ ru_\theta(r, t)\right] = \frac{\Gamma}{2\pi \delta^2(t)} \exp\left(-\frac{r^2}{2\delta^2(t)}\right),
\]

Eq. (9) used in (5) yields:

\[
(\delta(t) \dot{\delta}(t) + C\delta^2(t) - \nu)(2 - \frac{r^2}{\delta^2(t)}) = 0.
\]

We look for a solution \( \delta(t) \) independent of \( r \). For \( 2 - r^2/\delta^2(t) \neq 0 \) we obtain the first order linear differential equation for \( \delta^2(t) \) and find the central result:

\[
\delta^2(t) = \frac{\nu}{C} + (\delta_0^2 - \frac{\nu}{C}) e^{-2Ct},
\]

where \( \delta_0 \) is the initial value of \( \delta \), and \( \delta^2(t = \infty) = \nu/C \). The sign of \( \delta^2(t) - \nu/C \) is determined by the sign of \( \delta_0^2 - \nu/C \) (in principle \( \delta_0^2 \) can be larger or smaller than \( \nu/C \)). Using (9) we finally obtain [1]

\[
u_\theta = \frac{\Gamma}{2\pi r} \left[1 - \exp\left(-\frac{r^2}{2\delta^2(t)}\right)\right].
\]

Summarizing, we have obtained

\[
\begin{cases}
  u_z = 2Cz \\
  u_r = -Cr \\
  u_\theta = \frac{\Gamma}{2\pi r} \left[1 - \exp\left(-\frac{r^2}{2\delta^2(t)}\right)\right]
\end{cases}
\]

The flow is therefore a combination of two motions. An irrotational part controlled by the parameter \( C \), with radial and vertical velocity components \( u_r \) and \( u_z \) and a rotational part...
characterized by the angular velocity $u_{\theta}$. It describes an axis-symmetric, vertically uniform vortex flow with strength and size controlled by the parameters $\Gamma$ and $\delta$ respectively.

The energy associated to the vortex flow (per vertical unit) is:

$$E_v = \int u_{\theta}^2 \pi r \, dr = \frac{\Gamma^2}{4\pi} \int_0^{\delta} \frac{1}{r} [1 - \exp\left(-\frac{r^2}{2\delta^2}\right)]^2 \, dr \quad (14)$$

This integral is well defined in the limit $r \to 0$ but it diverges as $\ln(r)$ for $r \to \infty$. Therefore, the natural cutoff $\delta$ is introduced in the integration. The necessity of the cutoff confirms that this model is only locally valid. We also recognize the relevance of the role played by $\delta$.

**Figure 1.** $u_{\theta}$ and $E_{\text{density}}$ is plotted for two different values of $\delta$. The energy maximum occurs for a slightly larger $r$-value than the velocity maximum.

We verified numerically how the energy (14) depends on $\delta$. In figure 2, for two values of $\delta$, the velocity and energy distribution are shown as a function of the distance from the axis of the vortex. The maximum of energy and the radius at which this maximum occurs are shown as a function of the vortex size in figure 2. The integrated energy is clearly independent of $\delta$. The energy density instead depends on $\delta$ as follows:

$$E_{\text{density}} = \frac{\Gamma^2}{4\pi r^2} [1 - \exp\left(-\frac{r^2}{2\delta^2}\right)]^2 \quad (15)$$

The storage or dissipation of energy by hurricanes during their time evolution is of course a question of great practical interest.

3. The hurricane molecular dynamics

We want now to study the underlying molecular dynamics in the hurricane-environment interactions. As already observed, $\delta$ plays a crucial role in the dynamics described by the Navier-Stokes equation. It evolves in time according to Eq. (11), which shows that there is a competition between the viscosity, which tends to make the vortex diffuse, and the stretching flow, which tends to make the vortex more concentrated around its axis.
Figure 2. Plot of the energy maximum (blue) and radius where the maximum occurs (green) for different values of $\delta$. The maximum of energy increases without bound for small $\delta$ and the radius at the maximum increases linearly with $\delta$ (like $\approx 2\delta$). The energy integrated over five $\delta$ (red) is shown to be clearly independent of $\delta$.

Define $\delta^2(t) = |\delta^2(t) - \nu/C|$, $\delta^2_0 = |\delta^2_0 - \nu/C|$. Eq. (11) is then written as

$$\dot{\rho}(\theta) = \rho_0 e^{-\theta}$$

where we put $\theta \equiv \theta(t) = C t$. The associated parametric equations in polar coordinates are

$$\begin{align*}
\xi &= \rho(\theta) \cos \theta = \rho_0 e^{-\theta} \cos \theta , \\
\eta &= \rho(\theta) \sin \theta = -\rho_0 e^{-\theta} \sin \theta .
\end{align*}$$

The completeness of the (hyperbolic) basis $\{e^{-\theta}, e^{+\theta}\}$ requires that both elements $q = e^{\pm \theta}$ must be taken into account in the complex $z$-plane $z = \xi + i \eta$.

We thus consider both the points $z_1 = \rho_0 e^{-\theta} e^{-i \theta}$ and $z_2 = \rho_0 e^{+\theta} e^{+i \theta}$, and for convenience both signs also for the imaginary exponent $\pm i \theta$.

We note that $z_1(t) = \rho_0 e^{-i \Omega t} e^{-C t}$ and $z_2(t) = \rho_0 e^{+i \Omega t} e^{+C t}$ solve the equations

$$\begin{align*}
m \ddot{z}_1 + \gamma \dot{z}_1 + \kappa z_1 &= 0 , \\
m \ddot{z}_2 - \gamma \dot{z}_2 + \kappa z_2 &= 0 ,
\end{align*}$$

respectively, with positive real $m$, $\gamma$ and $\kappa$, $C \equiv \gamma/2m$; $\Omega^2 \equiv C^2 = (1/m)(\kappa - \gamma^2/4m)$, with $\kappa > \gamma^2/4m$; “dot” denotes derivative with respect to $t$. Note that no new parameters have been introduced. In fact the ratios $\gamma/m$ and $\kappa/m$ are proportional to $C$ and $C^2$, respectively.

By putting $x(t) \equiv [z_1(t) + z_2(-t)]/2$ and $y(t) \equiv [z_1(-t) + z_2(t)]/2$, Eqs. (19) reduce to the couple of damped and amplified harmonic oscillators

$$\begin{align*}
m \ddot{x} + \gamma \dot{x} + kx &= 0 , \\
m \ddot{y} - \gamma \dot{y} + ky &= 0 ,
\end{align*}$$
respectively. Note that \( C = \gamma / 2m \) controls the dissipation (amplification) terms. Eqs. (21) and (22) are one the time-reversed image of the other one (\( \gamma \rightarrow - \gamma \)) (the \( \pm \) sign of \( C \) is taken into account and shifted to the \( \pm \) of time \( t \)). The physical meaning of this is that total energy is conserved. The global system \((x, y)\) is indeed a closed system. The canonical formalism can only describe closed systems. The \( y \) system plays indeed the role of the environment for the \( x \) system (or vice-versa). This shows thus how important it is to consider the role of the interaction with the environment, as indeed stressed in hurricane studies [3].

We remark that the domain of surface \( \delta^2(t) = |\delta^2(t) - \nu / C| \) has blurred borders, i.e. not sharply defined linear dimension \( \delta \), due to fluctuating molecular movements arising from the hurricane interaction with the environment molecules. The hurricane exchanges indeed energy by interacting at a molecular level with the atmosphere molecules (mostly nitrogen (\( N_2 \)) and oxygen (\( O_2 \)), but also carbon dioxide (\( CO_2 \)) and Argon (\( Ar \)) and water molecules and droplets largely present in the environment due to moist convection, cumulus and clouds. These brownian-like molecular motions may be represented by two coordinates \( x_+(t) \) (going forward in time) and \( x_-(t) \) (going backward in time)

\[
x_\pm \equiv x \pm \frac{y}{2}.
\]

Eqs. (21) and (22) are then rewritten as

\[
m\dddot{x}_+ + \gamma \ddot{x}_- + kx_+ = 0,
\]

\[
m\dddot{x}_- + \gamma \ddot{x}_+ + kx_- = 0.
\]

The Lagrangian from which they can be derived and the Hamiltonian are [10, 11]

\[
L = \frac{m}{2} \dddot{x}_+^2 - \frac{k}{2} x_+^2 - \frac{m}{2} \dddot{x}_-^2 + \frac{k}{2} x_-^2 + \frac{\gamma}{2} (\dot{x}_+ x_- - \dot{x}_- x_+),
\]

\[
H = H_+ - H_- = \frac{1}{2m} (p_+ - \gamma x_-)^2 + \frac{k}{2} x_+^2 - \frac{1}{2m} (p_- + \gamma x_+)^2 - \frac{k}{2} x_-^2.
\]

respectively, with conjugate momenta \( p_+ = m \dot{x}_+ + (\gamma / 2)x_- \), \( p_- = -m \dot{x}_- - (\gamma / 2)x_+ \). It is interesting that the damping manifests itself as a correction in the kinetic energy and that the dissipative constant acts as a coupling between \( x_+ \) and \( x_- \) (cf. Eq. (26)).

From Eq. (27), the forward and backward in time velocities \( v_\pm \) are

\[
v_\pm = \frac{\partial H}{\partial p_\pm} = \pm \frac{1}{m} (p_\pm \mp \frac{\gamma}{2} x_\pm).
\]

By using as usual \( p_\pm = -i\hbar d/dx_\pm \), we have

\[
[v_+, v_-] = i\hbar \frac{\gamma}{m^2}.
\]

Provided that we set \( \gamma \equiv q B \delta / c \), Eq. (29) is similar to the usual commutators for the velocities \( \mathbf{v} = (p - qA/c)/m \) of a charged particle moving in a magnetic field \( \mathbf{B} \); i.e.,

\[
[v_1, v_2] = (i\hbar q B_3 / m^2 c) \quad \text{(also analogous to the Bohm-Aharonov phase interference for charged particles)}.
\]

The commutator (29) implies a topologically non-trivial (noncommutative) geometry in the \((v_+, v_-)\) plane, reflecting of course the non-trivial topology of the hurricane vortex.

These remarks and the non-vanishing density of ionized particles suggest to us to write \( H \) as

\[
H = H_+ - H_- = \frac{1}{2m} (p_+ - \frac{q}{c} A_+)^2 + q \Phi_+ - \frac{1}{2m} (p_- + \frac{q}{c} A_-)^2 + q \Phi_-.
\]
with
\[ A_+ = \frac{B}{2} x_-, \quad A_- = \frac{B}{2} x_+, \quad B = \frac{c}{q}, \] (31)
namely, \((\gamma/2)x_\pm\) in Eq. (27) are represented as the components \(A_+ \equiv A_1\) and \(A_- \equiv A_2\) of the em vector potential \(A\) [11, 12, 13] with the magnetic field \(B = \nabla \times A = -\mathbf{B}\).

The Hamiltonian \(H\) then describes two particles with opposite charge \(q_+ = -q_- = q\) in the (oscillator) potential \(\Phi \equiv (k/2q)(x_+^2 - x_-^2) \equiv \Phi_+ - \Phi_-\), \(\Phi_\pm \equiv (k/2q)x_\pm^2\), and in the constant magnetic field \(B\). One particle thus moves in the em field whose source is the other particle. Eqs. (24) and (25) are recognized to describe nothing but the Lorentz forces on particles with charge \(q_+ = -q_- = q\), in the magnetic field \(B = -\mathbf{B}\) and in the electric field \(E = -\nabla \Phi\): \(\mathcal{F}_i = m \ddot{x}_i = q_i[E_i + \frac{1}{c} (\mathbf{v} \times \mathbf{B})_i]\), (32)

\(i = 1, 2 \equiv +, -\), no sum on \(i\), respectively, and \(\mathbf{v} = (\dot{x}_+, \dot{x}_-, 0)\).

In conclusion, we have expressed the dynamical content of Eq. (11) of the Burgers model by the above Lagrangian and Hamiltonian which account for the dynamics of molecular motion and, with the identification (31), also of charged particles motion in the hurricane-environment interaction.

4. Molecular collective modes
The role played by cloud water droplets, cumulus and moist convection in hurricane genesis and evolution has been much discussed in the literature by the help of numerical and analytical studies [2, 3]. Our standpoint is that in the region where the hurricane is generated, a weak perturbation, e.g. the em field generated by ionized particles, or perturbations of other kind, mechanical, chemical, etc., may break the rotational symmetry of the electric dipole of the water molecules present in that region. The molecular dynamics will be then conditioned by the spontaneous breakdown of the dipole rotational symmetry (SBS). As a consequence, the system components undergo an in phase, or coherent collective motion resulting in ordered states. The ordering is generated by long range correlations among them (the Goldstone theorem in QFT) [14, 15]. Ion-dipole interactions may be indeed observed to induce ordering of water molecular electric dipole even at very low ion concentrations over coherent domains of nanoscopic length scales [16, 17, 18] with observable effects in surface tension experiments at the macroscopic scale. The degree of ordering is described by the “order parameter”, characterizing the macroscopic behavior of the system. The change of scale, from the dynamics of the elementary components to the system’s macroscopic behavior, is thus dynamically obtained and is expressed by classical field equations and observables. In the following we will always refer to such a classical level of description, which emerges out of the molecular QFT dynamics.

We thus consider the non-homogeneous polarization density \(P(r, t)\) that arises as a dynamical effect of the SBS. The consequent collective correlations among the in phase oscillating molecular dipoles are described by dipole fields (the Nambu-Goldstone (NG) fields) whose coherent condensation in the ground state manifests itself in the classical, macroscopic properties and behavior of the system.

Let \(\rho(r, t)\) denote the charge density and \(\ell\) the (average) dipole length. The symmetry under rotations by a constant phase \(\lambda\) around the dipole axis (the global \(U(1)\) symmetry) is broken since it is impossible to change by a constant amount the phase of the dipole field simultaneously at every space-point. Then, one can show [19] that under quite general conditions the order parameter \(\sigma(r, t)\) is proportional to \(P(r, t)\): \(|\sigma(r, t)|^2 \propto 2P(r, t) = \rho(r, t) \ell\).

By adopting a standard recipe [14, 15, 20], the order parameter is described in terms of the charge density wave function \(\sigma(r, t)\): \(\sigma(r, t) = \sqrt{\rho(r, t)} e^{i\varphi(r, t)}\), (33)
where the real phase $\varphi(r,t)$ field plays the role of the NG quantum field. The phase transformation for $\sigma(r,t)$, i.e. $\varphi(r,t) \rightarrow \varphi(r,t) + (q/\hbar c) \lambda(r,t)$, describes the coherent NG boson condensation process. In full generality,

$$\varphi(r,t) \rightarrow \varphi(r,t) + f(r,t), \quad (34)$$

with $f(r,t)$ playing the role of a form factor in the $\varphi$ non-homogeneous (i.e. space-time dependent) condensation. A topologically non-trivial structure is obtained when $f(r,t)$ carries topological singularities [21, 22]. For example, vortices appear by using

$$f(x) = \arctan \left( \frac{x_2}{x_1} \right), \quad (35)$$

which indeed carries a singularity on the line $r = 0$, with $r^2 = x_1^2 + x_2^2$.

In order to ensure the invariance under local phase transformation, as usual in the canonical formalism one considers the em field gauge transformation

$$\mathbf{A}(r,t) \rightarrow \mathbf{A}' = \mathbf{A}(r,t) + \nabla \lambda(r,t). \quad (36)$$

When using $\varphi$ as gauge function in $\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + (\hbar c/q) \nabla \varphi$, the transformation $\mathbf{A}' \rightarrow \mathbf{A}' + \nabla \lambda$ is induced by $\varphi(r,t) \rightarrow \varphi(r,t) + (q/\hbar c) \lambda(r,t)$.

We adopt the gauge condition $\nabla \cdot \mathbf{A} = 0$ ($\nabla \cdot \mathbf{A}' = 0$), which requires $\nabla^2 \varphi(r,t) = 0$ and $\nabla^2 \lambda(r,t) = 0$ (and $\nabla^2 f(r,t) = 0$). When the polarization current $\mathbf{J}_p = \rho(r,t) \mathbf{v}$, with $\mathbf{v} = \partial \mathbf{E}/\partial t$, is assumed to be the only relevant current, the Maxwell equation $\square \mathbf{A}(r,t) = \mathbf{J}_p(r,t)$ is

$$(\square + M^2) \mathbf{A}(x) = \frac{2\mathcal{P}}{m \ell \hbar} \nabla \varphi(r,t). \quad (37)$$

where $M^2 \equiv 2q \mathcal{P}/(m \ell c)$. In obtaining (37) we used

$$\mathbf{J}_p(r,t) = \frac{1}{m} \rho(r,t)(\hbar \nabla \varphi(r,t) - \frac{q}{c} \mathbf{A}(r,t)), \quad (38)$$

derived in a standard fashion by using (33) in the definition of $\mathbf{J}_p$ and the usual minimal coupling derivative ($-i\hbar \nabla - q\mathbf{A}$). From (38) we see that $m \mathbf{v} = \hbar \nabla \varphi - (q/c) \mathbf{A}$, with $\nabla \cdot \mathbf{v} = 0$ and $\nabla \cdot \mathbf{J}_p = 0$. In condensed matter physics, $(q/mc) \rho(r,t) \mathbf{A}(r,t)$ is the Meissner current term and $(1/m)\rho(r,t)h \nabla \varphi(r,t)$ is the boson current.

The SBS thus dynamically produces the non-vanishing mass $M$ for the field $\mathbf{A}$ [21], implying its damping by a factor $\propto \exp(-c M s/\hbar)$ and its self-focusing propagation with a transversal size of the order of $s \sim 1/M$ [20].

Self-focusing propagation of the em field is a well-studied phenomenon in classical nonlinear optics [23, 24, 25] and it can be shown [20] that the present analysis accounts for phenomenological aspects of em wave propagation in nonlinear non-homogeneous media, as it is indeed the medium of the hurricane.

The dipole of the water molecule is due to the displacement of a pair of electrons; thus $q = 2e$ and $m = 2m_e$ in the above expression for $M^2$. Then $\mathcal{P} = 2e\ell/c \times n$ with $n$ the number of oriented dipoles. In the extreme case where all dipoles are oriented, $n = N/18$, where $N$ is the Avogadro number. We then obtain [20] $M_H c^2 = \hbar c (16 \pi r_0 n)^{1/2} \approx 13.60$ eV, with $r_0 = (1/4 \pi \epsilon_0)(e^2/m_e c^2) = 2.8 \times 10^{-15}$ m the “classical radius” of the electron, $\hbar = 0, 6582 \times 10^{-15}$ eV sec; $c$ the light velocity and $1 \ m^3$ the unit volume. We thus see that the mass acquired by the self-focusing em field in such a case corresponds approximately to the hydrogen ionization energy.
A single photon with energy $h\nu < M_H c^2$ is thus not able to propagate through the correlated medium. A highly energetic photon ($h\nu \gg M_H c^2$) would instead produce the breakdown of the correlation modes, thus restoring the masslessness of em field (symmetry restoration). In general the polarization density values are smaller than the maximum value above considered. This allows, however, an interesting mechanism of energy storage in coherent states.

Energies from different processes (mechanical, chemical, electromagnetic), less than the hydrogen ionization energy, could be stored indeed as polarization modes in the coherent state, provided that they can produce excitations in the correlated medium. When the accumulated energy is enough to reach the ionization value, the whole polarization mode would manifest as an em field propagating as in (37) with the corresponding energy value $M c^2$.

Indeed, in the quasi-stationary approximation, the minimization of the free energy, $dF = dE - \frac{1}{\beta}dS = 0$, with $\beta = 1/k_BT$, implies that the rate of change of the number $N_\kappa$ of condensate polarization modes is related to the changes of the ‘internal energy’ $dE$ and entropy $dS$ [10, 14, 15]

$$dE = \sum_\kappa E_\kappa \dot{N}_\kappa(t)dt = \frac{1}{\beta}dS,$$

with $\dot{N}_\kappa \equiv dN_\kappa/dt$. The exchanged heat is $dQ = (1/\beta)dS$, which enters in the balance of the hurricane heating/cooling.

Most interesting is, however, the non-equilibrium dynamics characterized by criticality and phase transitions (assumed anyway to occur between thermodynamic equilibrium states to which the fluctuation theorem applies [26, 27, 28]).

5. Criticality, the Ginzburg-Landau non-stationary regime and the vortex equation

In order to study the variations of the charge density wave function $\sigma(r, t)$ occurring in the non-equilibrium phase transition processes, we start by considering the free energy density Ginzburg-Landau (GL) functional $F(\sigma, \sigma^*, A)$ [29]. Extremizing $F(\sigma, \sigma^*, A)$ with respect to $\sigma^*(r, t)$ gives the stationary Ginzburg-Landau (GL) equation: $\partial F/\partial \sigma^* = 0$. In full generality, we may write:

$$\frac{\partial F}{\partial \sigma^*} = \left[\frac{1}{2m}(-ih\nabla - \frac{q}{c}A)^2 + \mu^2 + \lambda|\sigma(r, t)|^2\right]\sigma(r, t),$$

(40)

where the explicit expressions of $\mu^2(T)$ and $\lambda(T)$ depend on the system under study. They act as mass term and positive coupling constant term, respectively. The quantity $(\mu^2 + \lambda|\sigma|^2)\sigma$ may be considered as derived from the potential $V(\sigma, \sigma^*) = -(\lambda/2)(|\sigma|^2 + \mu^2/\lambda)^2$.

The value of $\sigma$ minimizing $V(\sigma, \sigma^*)$ is zero (disordered or symmetric ground state) for $\mu^2 \geq 0$, and non-zero for $\mu^2 < 0$, with $|\sigma|^2 = -\mu^2/\lambda \neq 0$, playing the role of the order parameter (the ordered or asymmetric ground state). The dependence on a critical temperature $T_G$ may be taken into account through the dependence of $\mu^2$ on $T_G$ in a form, e.g., proportional to $(T - T_G)/T$, assuming $T$ varies linearly with time near the transition region.

By tuning $\mu^2$, the pump parameter in the Haken interpretation [30], the system may be carried in the laserizing process between the disordered and the ordered (laser) state (the phase transition), $|\sigma|^2$ changing in time from zero to a non-vanishing value proportional to $\mu^2$.

In the non-stationary regime, $\partial F/\partial \sigma^*$ is non-vanishing and expresses the rate at which $\sigma(r, t)$ approaches its stationary value at the minimum of the free energy. The generalized time dependent GL (TDGL) equation is:

$$i\hbar \frac{\partial \sigma}{\partial t} = \hat{H}\sigma - i\frac{\partial F}{\gamma \partial \sigma^*}.$$

(41)

The second term in the r.h.s. of Eq. (41) describes the dissipative contribution coming from incoherent relaxation processes with $\gamma$ the relaxation parameter [29, 31].
In the limit of the stationary regime \((1/\gamma)dF/d\sigma^i \rightarrow 0\), the equilibrium wave function is denoted by \(\sigma^i(t) = |\sigma^0|\exp(-i\epsilon t/\hbar)\), with \(\hat{H}\sigma^0 = \epsilon\sigma^0\).

By using \(\sigma = \sqrt{n}\exp(i\varphi)\), one can show that the TDGL equation becomes

\[
\frac{1}{D_{GL}} \frac{d\chi}{dt} = \nabla^2 \chi + \frac{1}{\xi_{GL}^2} \left[ (1 - \chi^2) - \xi_{GL}^2 \left( \frac{mv}{\hbar} \right)^2 \right] \chi,
\]

where \(\chi = |\sigma|/|\sigma^0| \equiv (n/n^0)^{1/2}\) is the normalized wave function, \(D_{GL} \equiv \xi_{GL}^2/\tau_{GL}\), is the diffusion coefficient, with \(\xi_{GL}\) and \(\tau_{GL}\) denoting the GL correlation length and the GL relaxation life-time, respectively (see [1] for details).

Eq. (42), for \((1/D_{GL})d\chi/dt = \hbar/\gamma d\chi/dt \approx 0\) is indeed the vortex equation [32]. It is associated with the singularity at \(r = 0\).

For numerical analysis and simulations of Eq. (42) see, e.g. [33] and refs. therein quoted.

One may derive from the TDGL equation that the vortex core size is in the first approximation of the order \(\xi_{GL}^2 \sim [\mu^2]^{-1}\), and thus it increases as \(T\) increases, approaching to \(T_C\) from below; the vortex envelope disappears at \(T_C\). Symmetry restoration is obtained for temperature rising above \(T_C\). Vice-versa, going back to \(T_C\) from above the unstructured (fully symmetric) background activity exhibits an undefined phase at \(T_C\), i.e. the singularity at the center line \(r = 0\). The critical regime starts as \(T\) goes below \(T_C\), vortices appear, and their core shrinks as temperature decreases. The vortex formation starts form the singularity, at the vortex core, in the process of phase transition due to SBS (“where the vortex comes from”) [32].

The cooling itself is a manifestation of the process of symmetry breakdown. The non-homogeneous condensation and ordered localized patterns are lower in energy and separated by an energy gap from the symmetric state. The phase transition is non-instantaneous and begins with cooling [34].

In the TDGL formalism it is assumed that a time-dependent temperature might still be defined in non-instantaneous phase transition processes. In the general picture the transition is assumed to start at the critical temperature \(T_C\). Time evolution leads then to the stable configuration at the so-called “Ginzburg temperature” \(T_G\), with \(T_G < T_C\).

The critical regime is defined to be the one evolving in time between \(T_C\) and \(T_G\), and is supposed to be always a transition between different thermodynamic equilibrium states, regardless of the non-equilibrium processes occurring in the course of the transition process.

Thus the general thermodynamic description derived from the fluctuation theorem [26] applies to the ‘critical dynamics’ of our system. As already mentioned, in the process of reaching new equilibrium configurations below the critical temperature \(T_C\), local exchanges of heat with environments turn into entropy changes of the vortex, with subsequent rearrangements of its configurations and (internal) energy density contents.

We remark that the departure from the stationary regime (at \(T_C\)), namely the start of the critical regime, is driven by fluctuations which can trigger a phase transition, if the necessary transition energy is provided by some external input [35]. These ground state fluctuations turn into temperature fluctuations since in our dissipative model the ground state is in fact a thermal state [32, 36]. At the end of the critical regime (at \(T_G\)) the system arrives at a “new” ground state configuration and the phase transition is thus completed. The system thus undergoes a continuous sequence of phase transitions going through a path, or trajectory, through the (infinitely) many coherent ground states (in each of them free energy is minimized).

Remarkably, these trajectories in the manifold of the coherent states turn out to be classical chaotic trajectories [37], which shows the sensibility of the molecular dynamics and of the hurricane evolution in general to slight changes in the environmental conditions.
6. The hurricane fractal structure

Our discussion in the previous Sections has shown that coherent state structures characterize the molecular dynamics out of which hurricanes originate. Since an isomorphism exists between (deformed) coherent states and fractals [38, 39], this suggests to us to search for fractal self-similarity properties of the hurricane. We observe indeed that Eq. (16) may be regarded as the logarithmic spiral equation in polar coordinates [40] (see Fig. 3).

In order to see this, suppose that \( \Omega \) may be scaled by tuning a real parameter, say \( d \), so that \( \Omega = C/d \). By putting \( \alpha = \theta/d \), we write then \( \xi = \varrho_0 e^{-\theta} \cos \alpha, \eta = -\varrho_0 e^{-\theta} \sin \alpha \) (cf. Eqs. (17)) and Eq. (16) is written as \( \varrho(\alpha) = \varrho_0 e^{-d\alpha} \), which is represented in a log-log plot with abscissa \( \alpha = \ln e^\alpha \) by the straight line of slope \( d \):

\[
\ln \frac{\varrho}{\varrho_0} = -d \alpha.
\]

The constancy of the angular coefficient \( \tan^{-1}d \) represents the self-similarity property of \( \varrho \).

The parameter \( d \) is called the fractal or self-similarity dimension. Rescaling \( t \to nt \) changes \( \varrho/\varrho_0 = e^{-d\alpha} \) by the power \((\varrho/\varrho_0)^n\). At time \( \tau = 2\pi d/C \) it is \( (C/d) \tau = 2\pi \) and at \( t = n \tau \), \( z_1 = \varrho_0 (e^{-2\pi d})^n, z_2 = \varrho_0 (e^{2\pi d})^n \), with integer \( n = 1, 2, 3... \)

Such a macroscopic self-similarity feature of the logarithmic-spiral-like hurricane finds its correspondence in the \( d \)-deformed coherent states structure at the molecular level previously discussed. For brevity we do not show here how such an isomorphism is proved. See [38] for details.

7. Conclusions

We have shown that central results of the Burgers model for hurricanes may be derived in the QFT of the collective molecular dynamics in the hurricane-environment interaction. Macroscopic hurricane behaviors and properties, such as vorticity, transitions between different dynamical regimes (phase transitions), criticality and chaotic behavior, energy storage and dissipation, free energy and entropy changes, etc. are thus the result not of random kinetic molecular motions,
but of collective coherent molecular dynamics of water molecules in the hurricane interaction with moist convection flows, cumulus and cloud water droplets.

The QFT spontaneous breakdown of dipole rotational symmetry of water molecules is induced by weak perturbations due, e.g., to ionized particles, or to other sources, mechanical, chemical, etc. The consequent generation of long range molecular correlations is responsible of collective coherent molecular motions characterized by non-trivial topology and fractal self-similar structures which manifest at the hurricane macroscopic level as vortices and logarithmic spiral configurations. The time dependent Ginzburg-Landau equation has been used to describe such phenomena.

Energy storage and dissipation by hurricanes has been related with the properties of the coherent state molecular structure. Energy can be accumulated on the coherent net of long range correlations and when it reaches a threshold (energy gap) depending on the system under study, it may be then released, as discussed in Section 4. For example, the total hamiltonian

\[ H = H_0 + H_F, \]

with \( H_0 = \sum_k \hbar \omega_k (c_k^\dagger c_k + 1/2) \) and \( H_F \) the contribution (the energy gap \( \Delta E \)) due to the external force \([15, 41]\). Consider the coherent state \( |\alpha\rangle = \Pi_k |\alpha_k\rangle \), where \( |\alpha_k\rangle \) is the \( k \)-mode coherent state, \( |\alpha_k\rangle = \exp(-|\alpha_k|^2/2) \sum_n (1/n_k!) (\alpha_k)^n_k (c_k^\dagger)^n_k |0\rangle \), and \( \alpha_k \) is the \( k \)-mode coherence strength of \( |\alpha_k\rangle \). \( |0\rangle \) denotes the vacuum, or ground state, for the \( c_k \)-oscillators, \( c_k |0\rangle = 0 \) for any \( k \). The number of coherently condensed \( c_k \)-modes in \( |\alpha\rangle \) is \( \langle \alpha | c_k^\dagger c_k | \alpha \rangle = |\alpha_k|^2 \). The expectation value of \( H_T \) in \( |\alpha\rangle \) is \( \langle \alpha | H_T | \alpha \rangle = \sum_k [\hbar \omega_k |\alpha_k|^2 - F_k \alpha - F_k^\dagger \alpha^*] \), without considering the zero point energy. It is minimized for \( (1/\hbar \omega_k) F_k = \alpha_k^2 \), namely when the external action matches the coherence phase of the net. The energy “reservoir” in the coherent state is thus \(-2 \sum_k \hbar \omega_k |\alpha_k|^2 \), which is clearly not due to random kinematical motion, but to the coherent dynamics. Energy supplied to the coherent net (below of course a critical threshold) does not thermalize the net components increasing their kinetic energy, it is instead stored “in the net”.

Note that divergences arise in the case \( \omega_k = 0 \). This corresponds to the absence of restoring forces opposing to \( F_k \) with increasingly unbounded acceleration for the system as a whole \([41]\). As already observed in the previous Sections, coherence provides also the bridge to go from microscopic (quantum) to macroscopic (classical) dynamics, namely from the quantum operator fields \( c_k \) to the c-number (classical) \( \alpha_k \) functions. For example, the probability of finding \( n_k \) quantum modes in the coherent state \( |\alpha\rangle \) is given by the classical distribution

\[ \langle n_k |\alpha\rangle^2 = \exp(-|\alpha_k|^2/(1/n_k!)) |\alpha_k|^{2n_k}, \]

which is recognized to be the Poisson distribution.

The above remarks shed some light on the fact that the hurricane can deliver its giant amounts of energy as a whole, not just as the uncorrelated molecular energy bits delivered by the non-coherent molecular masses of a “normal” or “simple”, although energetic and speedy, wind. In this last case, damages are produced by the intensity of the kinematical (uncorrelated) molecular wind flows (currents).

In conclusion, we have shown that QFT may reproduce features of the Burgers model; however, classical fluid dynamics alone cannot account for the coherent molecular quantum dynamics. Hurricanes appear to be macroscopic quantum systems, in the sense, indeed, that some of their properties cannot be accounted without recourse to quantum dynamics.

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