TRANSVERSELY PRODUCT SINGULARITIES OF FOLIATIONS IN PROJECTIVE SPACES

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Abstract. We prove that a transversely product component of the singular set of a holomorphic foliation on $\mathbb{P}^n$ is necessarily a Kupka component.

1. Introduction

Let $U$ be an open set of a complex manifold $M$ and let $k \in \mathbb{N}$. Let $\eta$ be a holomorphic $k$-form on $U$ and let $\text{Sing} \eta = \{p \in U : \eta(p) = 0\}$ denote the singular set of $\eta$. We say that $\eta$ is integrable if each point $p \in U \setminus \text{Sing} \eta$ has a neighborhood $V$ supporting holomorphic 1-forms $\eta_1, \ldots, \eta_k$ with $\eta|_V = \eta_1 \wedge \cdots \wedge \eta_k$, such that $d\eta_j \wedge \eta = 0$ for each $j = 1, \ldots, k$. In this case the distribution

$$D_\eta: D_\eta(p) = \{v \in T_pM : i_v \eta(p) = 0\}, \quad p \in U \setminus \text{Sing} \eta$$

defines a holomorphic foliation of codimension $k$ on $U \setminus \text{Sing} \eta$. A singular holomorphic foliation $\mathcal{F}$ of codimension $k$ on $M$ can be defined by an open covering $(U_j)_{j \in J}$ of $M$ and a collection of integrable $k$-forms $\eta_j \in \Omega^k(U_j)$ such that $\eta_j = g_{ij} \eta_i$ for some $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ whenever $U_i \cap U_j \neq \emptyset$. The singular set $\text{Sing} \mathcal{F}$ is the proper analytic subset of $M$ given by the union of the sets $\text{Sing} \eta_j$. From now on we only consider foliations $\mathcal{F}$ such that $\text{Sing} \mathcal{F}$ has no component of codimension one.

Given a singular holomorphic foliation $\mathcal{F}$ of codimension $k$ on $M$ as above, the Kupka singular set of $\mathcal{F}$, denoted by $K(\mathcal{F})$, is the union of the sets

$$K(\eta_j) = \{p \in U_j : \eta_j(p) = 0, d\eta_j(p) \neq 0\}.$$

This set does not depend on the collection $(\eta_j)$ of $k$-forms used to define $\mathcal{F}$. It is well known that, given $p \in K(\mathcal{F})$, the germ of $\mathcal{F}$ at $p$ is holomorphically equivalent to the product of a one-dimensional foliation with an isolated singularity by a regular foliation of dimension $(\dim \mathcal{F} - 1)$. More precisely, if $\dim M = k + m + 1$, there exist a holomorphic vector field $X = X_1 \partial_{x_1} + \cdots + X_{k+1} \partial_{x_{k+1}}$ on $\mathbb{D}^{k+1}$ with a unique singularity at the origin, a neighborhood $V$ of $p$ in $M$ and a biholomorphism $\psi: V \to \mathbb{D}^{k+1} \times \mathbb{D}^m$, $\psi(p) = 0$, which conjugates $\mathcal{F}$ with the foliation $\mathcal{F}_X$ of $\mathbb{D}^{k+1} \times \mathbb{D}^m$ generated by the commuting vector fields $X, \partial_{y_1}, \ldots, \partial_{y_m}$, where $y = (y_1, \ldots, y_m)$ are the coordinates in $\mathbb{D}^m$. If $\mu = dx_1 \wedge \cdots \wedge dx_{k+1}$, the foliation $\mathcal{F}_X$ is also defined by the $k$-form $\omega = i_X \mu$ and the Kupka condition $d\omega(0) \neq 0$ is equivalent to the inequality $\text{div} X(0) \neq 0$.

Following [7], we say that $\mathcal{F}$ is a transversely product at $p \in \text{Sing} \mathcal{F}$ if as above there exist a holomorphic vector field $X$ and a biholomorphism $\psi: V \to \mathbb{D}^{k+1} \times \mathbb{D}^m$ conjugating $\mathcal{F}$ with $\mathcal{F}_X$, except that it is not assumed that $\text{div} X(0) \neq 0$. We say that $\Gamma$ is a local transversely product component of $\text{Sing} \mathcal{F}$ if $\Gamma$ is a compact irreducible component of $\text{Sing} \mathcal{F}$ and $\mathcal{F}$ is a transversely product at each $p \in \Gamma$. In particular, if $\Gamma \subset K(\mathcal{F})$ we say that $\Gamma$ is a Kupka component — for more
information about Kupka singularities and Kupka components we refer the reader to \[8, 6, 1, 2, 3, 4, 5\]. If $\Gamma$ is a transversely product component of Sing $\mathcal{F}$, we can cover $\Gamma$ by finitely many normal coordinates like $\psi$, with the same vector field $X$; that is, there exist a holomorphic vector field $X$ on $\mathbb{D}^{k+1}$ with a unique singularity at the origin and a covering of $\Gamma$ by open sets $(V_\alpha)_{\alpha \in A}$ such that each $V_\alpha$ supports a biholomorphism $\psi_\alpha: V_\alpha \to \mathbb{D}^{k+1} \times \mathbb{D}^m$ that maps $\Gamma \cap V_\alpha$ onto $\{0\} \times \mathbb{D}^m$ and conjugates $\mathcal{F}$ with the foliation $\mathcal{F}_X$. The sets $(V_\alpha)$ can be chosen arbitrarily close to $\Gamma$.

In \[7\], the author proves that a local transversely product component of a codimension one foliation on $\mathbb{P}^n$ is necessarily a Kupka component. The goal of the present paper is to generalize this theorem to foliations of any codimension.

**Theorem 1.** Let $\mathcal{F}$ a holomorphic foliation of dimension $\geq 2$ and codimension $\geq 1$ on $\mathbb{P}^n$. Let $\Gamma$ be a transversely product component of Sing $\mathcal{F}$. Then $\Gamma$ is a Kupka component.

This theorem is a corollary of the following result.

**Theorem 2.** Let $\mathcal{F}$ a holomorphic foliation of dimension $\geq 2$ and codimension $k \geq 1$ on a complex manifold $M$. Suppose that $\mathcal{F}$ is defined by an open covering $(U_j)_{j \in J}$ of $M$ and a collection of $k$-forms $\eta_j \in \Omega^k(U_j)$. Let $L$ be the line bundle defined by the cocycle $(g_{ij})$ such that $\eta_1 = g_{ij} \eta_j$, $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$. Let $\Gamma$ be a transversely product component of Sing $\mathcal{F}$ that is not a Kupka component. Then, if $V$ is a tubular neighborhood of $\Gamma$, we have that $c_1(L|_V) = 0$.

**2. Proof of the results**

**Proof of Theorem 2.** Let $V$ be a tubular neighborhood of $\Gamma$. Then the map

$$\Theta \in H^2_{\text{dR}}(V) \mapsto \Theta|_\Gamma \in H^2_{\text{dR}}(\Gamma)$$

is an isomorphism and so it suffices to prove that $c_1(L|_\Gamma) = 0$. Let $\dim M = k + m + 1$. As explained in the introduction, there exist a holomorphic vector field $X$ on $\mathbb{D}^{k+1}$ with a unique singularity at the origin and a covering of $\Gamma$ by open sets $(V_\alpha)_{\alpha \in A}$ such that each $V_\alpha$ is contained in $V$ and supports a biholomorphism $\psi_\alpha: V_\alpha \to \mathbb{D}^{k+1} \times \mathbb{D}^m$ that maps $\Gamma \cap V_\alpha$ onto $\{0\} \times \mathbb{D}^m$ and conjugates $\mathcal{F}$ with the foliation $\mathcal{F}_X$ generated by the commuting vector fields $X, \partial_{y_1}, \ldots, \partial_{y_m}$. Notice that $\text{div}(X)(0) = 0$, because $\Gamma$ is not a Kupka component. Since $\mathcal{F}_X$ is defined by the $k$-form $\omega = i_X \mu$, where $\mu = dx_1 \wedge \cdots \wedge dx_{k+1}$, we have that $\mathcal{F}|_{V_\alpha}$ is defined by the $k$-form $\psi_\alpha^*(\omega)$. If $V_\alpha \cap V_\beta \neq \emptyset$, there exists $\theta_{\alpha \beta} \in \mathcal{O}^*(V_\alpha \cap V_\beta)$ such that

$$\psi_\alpha^*(\omega) = \theta_{\alpha \beta} \psi_\beta^*(\omega).$$

We can assume that the $k$-forms $\psi_\alpha^*(\omega)$ belong to the family of $k$-forms $(\eta_j)_{j \in J}$ defining $\mathcal{F}$. Therefore the cocycle $(\theta_{\alpha \beta})$ define the line bundle $L$ restricted to some neighborhood of $\Gamma$. Thus, in order to prove that $c_1(L|_\Gamma) = 0$ it is enough to show that each $\theta_{\alpha \beta}|_{\Gamma}$ is locally constant. Fix some $\alpha, \beta \in A$ such that $V_\alpha \cap V_\beta \neq \emptyset$. If we set $\psi = \psi_\alpha \circ \psi_\beta^{-1}$ and $\theta = \theta_{\alpha \beta} \circ \psi_\beta^{-1}$, from (2.1) we have that $\psi^*(\omega) = \theta \omega$, which means that $\psi$ preserves the foliation $\mathcal{F}_X$. It suffices to prove that the derivatives $\partial_{y_1}(p), \ldots, \partial_{y_m}(p)$ vanish if $p \in \{0\} \times \mathbb{D}^m$. Since $\partial_{y_1}$ is tangent to $\mathcal{F}_X$, then the vector field $\psi_\alpha(\partial_{y_1})$ is tangent to $\mathcal{F}_X$ and so we can express

$$\psi_\alpha(\partial_{y_1}) = \lambda X + \lambda_1 \partial_{y_1} + \cdots + \lambda_m \partial_{y_m},$$
where \( \lambda, \lambda_1, \ldots, \lambda_m \) are holomorphic. Then
\[
\mathcal{L}_{\psi_* (\partial_{\psi})} \omega = \mathcal{L}_{\mathcal{X}} \omega = \lambda \mathcal{L}_{\mathcal{X}} \omega + d \lambda \wedge i_{\mathcal{X}} \omega = \lambda \mathcal{L}_{\mathcal{X}} \omega = \lambda \text{div}(\mathcal{X}) \omega,
\]
where the last equality follows from the identity \( \omega = i_{\mathcal{X}} \mu \). Thus, since
\[
\psi^* \left( \mathcal{L}_{\psi_* (\partial_{\psi})} \omega \right) = \mathcal{L}_{\partial_{\psi_1}} \psi^* \omega = \mathcal{L}_{\partial_{\psi_1}} (\theta \omega) = \theta \psi_1 \omega,
\]
we obtain that
\[
\theta \psi_1 \omega = \psi^* (\lambda \text{div}(\mathcal{X}) \omega) = \lambda (\psi) \text{div}(\mathcal{X})(\psi) \theta \omega
\]
and therefore \( \theta \psi_1 (p) = 0 \) if \( p \in \{0\} \times \mathbb{D}^m \), because \( \text{div}(\mathcal{X}) \) vanishes along \( \{0\} \times \mathbb{D}^m \). In the same way we prove that \( \theta \psi_2 (p) = \cdots = \theta \psi_m (p) = 0 \) if \( p \in \{0\} \times \mathbb{D}^m \), which finishes the proof.

\[ \square \]

**Proof of Theorem 1.** Suppose that \( \Gamma \) is not a Kupka component. Let \( L \) be the line bundle associated to \( \mathcal{F} \) as in the statement of Theorem 2. We notice that \( c_1(L) \neq 0 \), otherwise \( \mathcal{F} \) will be defined by a global \( k \)-form on \( \mathbb{P}^n \), which is impossible. Then, if we take an algebraic curve \( C \subset \Gamma \), we have \( c_1(L) \cdot C \neq 0 \). Therefore, if \( \Omega \) is a 2-form on \( \mathbb{P}^n \) in the class \( c_1(L) \) and \( V \) is a tubular neighborhood of \( \Gamma \),
\[
c_1(L|_V) \cdot C = \int_C \Omega|_V = \int_C \Omega = c_1(L) \cdot C \neq 0,
\]
which contradicts Theorem 2.

\[ \square \]

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