Effective interaction in an unbalanced Fermion mixture

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Abstract

A one–dimensional Fermi mixture with delta–interaction is investigated in the limit of extreme imbalance. In particular we consider the cases of only one or two minority Fermions which interact with the Fermi-sea of the majority Fermions. We calculate dispersion relation and polaron mass for the minority Fermions as well as equal time density-density correlators. Within a cluster expansion we derive an expression for the effective interaction potential between minority Fermions. For our calculations we use a reformulation of the exact wave functions, originally obtained by Yang and Gaudin by a nested Bethe ansatz, in terms of determinants.

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I. INTRODUCTION

Imbalanced mixtures of two different species of Fermions or of Fermions and Bosons have raised considerable interest during the recent years [1, 2]. Experimentally they have become feasible in traps of laser cooled Fermi gases, where a smaller number of $^{40}$K atoms is moving in a sea of $^{6}$Li atoms or in partially polarized ensembles of $^{6}$Li atoms. Long-standing questions of solid state physics about the coexistence of a normal and a superfluid in a partially polarized fermionic systems or about the emergence of a spatially varying order parameter giving rise to a Fulde–Ferrell–Larkin–Ovchinnikov phase [3, 4] could thereby be addressed experimentally [5]. Since a magnetic field is expelled from charged Fermionic superfluids due to the Meissner effect similar experiments to date could not be performed in solid state systems.

A related question arises in the highly imbalanced limit. It was predicted [6, 7] that in this regime a screening of the minority Fermions by a cloud of the majority Fermions leads to Fermi–polaron physics. A polaron quasiparticle peak emerging from a broad background in the radio–frequency adsorption spectrum was indeed measured in experiments [8, 9]. Similar results were obtained as well in experiments with imbalanced ultracold Boson mixtures [10]. The polaron problem in highly imbalanced Fermi mixtures was addressed theoretically by a variety of methods like Bethe–Goldstone equations [11], diagrammatic Monte Carlo [7], variation of trial wave functions [6], functional renormalization group [12] and others [13], both for weak and strong coupling.

Recently experiments were performed on imbalanced Fermi–mixtures in one dimensional [14] renewing the interest in 1d systems. Since in one–dimension essentially any interaction is strong, the regime of strong interaction is particularly easy accessible in one–dimensional traps where the confinement into two directions [15, 16] is much stronger than in the third one. In the limit of vanishing Fermi wave vector $k_F$ of the majority Fermions and diverging scattering length $c^{-1}$ with the dimensionless interaction strength $a = c/k_F$ finite, the details of the interaction–potential become irrelevant and the interaction can be assumed delta–like. The system of spin $1/2$ Fermions with delta–interaction, often called Yang–Gaudin model, in one dimension is exactly solvable by Bethe’s ansatz [17, 18]. However the resulting wave–functions are considerably more complicated than that of its Bosonic counterpart, the widely used Lieb–Liniger model [19]. Even in the hardcore–limit $a \to \infty$, which corresponds to the
Tonks–Girardeau gas in the Bosonic case, exact results for the dynamical density–density correlation and for the single–particle Green’s function were achieved only rather recently [20].

The study of an isolated minority Fermion (assumed a spin–up Fermion in the following) in a sea of spin–down Fermions was initiated by McGuire [21, 22] and has in particular in the recent years attracted much interest. The polaron problem was addressed in [23]. In [24] numerically exact results were obtained for few particles.

In the present work we investigate the Yang–Gaudin model in the highly imbalanced limit. Our approach relies on a reformulation of the exact many–body wave function in terms of determinants. This form seems at least in the imbalanced limit more suitable than the original one, obtained by Yang with a nested Bethe Ansatz. Using this wave function we treat the case of two minority Fermions analytically. We achieve exact results for the two–point density function and even for higher order density correlators.

The system’s energy can be expressed exactly as a function of the free momenta of the minority Fermions. This yields the polaron’s dispersion relation, from which its effective mass is derived. These results together with the results obtained in previous works [21, 22, 25] yield a quite complete picture of the single repulsive polaron dynamics in the Yang–Gaudin model. Polaron–polaron interaction is investigated within a cluster expansion of the energy. For weak coupling an effective two–body potential between the minority Fermions is derived, which is solely induced by the sea of majority Fermions.

The paper is organized as follows. In Section II we define the model and fix notation. The reformulation of the wave–function is described in Sec. III. The spectrum of the minority Fermions is analyzed in section IV. From this spectrum the effective potential between minority Fermions is derived in Sec. V. Density–correlation functions are investigated in Sec. VI.

II. MODEL

We consider \(N + M\) Fermions on a line which interact via a repulsive \(\delta\)-potential. While \(N\) of the particles are supposed to be spin–down Fermions we assume the remaining \(M\)
Fermions to carry spin-up polarization. The Hamiltonian reads

$$\hat{H} = -\sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2} - \sum_{m=1}^{M} \frac{\partial^2}{\partial y_m^2} + 4c \sum_{n=1}^{N} \sum_{m=1}^{M} \delta(x_n - y_m),$$

where the units are chosen such that $\hbar = 1$ and all masses are equal to 1/2. Furthermore $c \geq 0$ denotes the interaction strength and the coordinates $x = \{x_n\}_{n=1,\ldots,N}$ and $y = \{y_m\}_{m=1,\ldots,M}$ refer to the positions of the spin-down and spin-up Fermions, respectively.

The exact eigenfunctions of the Hamiltonian (1) can be constructed by means of Bethe’s Ansatz. For the cases $M = 1$ and $M = 2$ they have been found by McGuire [21, 22] and by Flicker & Lieb [26], respectively. The generalization of these results to an arbitrary number $M$ of spin-up particles was overcome by Gaudin [18] and Yang [17] via a nested Bethe-Ansatz. The eigenfunctions of $H$ constructed by this method are of the form

$$\Psi(x, \Lambda, y, k) \propto \sum_{Q,P \in S_{N+M}} \sum_{R \in S_M} [P|Q|R] \exp \left( i \sum_{n=1}^{N+M} k_{Pn} X_{Qn} \right) \prod_{n=0}^{N+M} \Theta(X_{Q(n+1)} - X_{Qn}),$$

where the set $X = \{x, y\}$ comprises all coordinates and $X_{Q0} = -\infty$ and $X_{Q(N+M+1)} = +\infty$. A sector, that is an ordering of particles, is labeled by the permutations $Q$. The ordering of particles corresponding to a permutation $Q$ is given by

$$-\infty < X_{Q1} < X_{Q2} < \cdots < X_{Q(N+M)} < +\infty .$$

In each sector the wave function (2) is a superposition of plane waves where the coefficients $[P|Q|R]$ are coordinate independent within a sector. They are usually written as follows (see e.g. [27]): Let all the $N + M$ particles be ordered and the spin–up particles be located at the integer positions

$$1 \leq f_1 < f_2 < \cdots < f_M \leq (N + M).$$

Then solution for the amplitudes $[P|Q|R]$ can be cast into the from

$$[P|Q|R] = \text{sgn}(R) \prod_{j<l}^{M} (\Lambda_{Rj} - \Lambda_{Rl} - i2c) \prod_{j=1}^{M} F_P(f_j, \Lambda_{Rj}),$$

where

$$F_P(f, \Lambda) = \prod_{i=1}^{f-1} ((k_{P_i} - \Lambda + ic) \prod_{l=f+1}^{N+M} ((k_{P_l} - \Lambda - ic).$$
Thus the full wave function is a sum over the product of three permutation groups $S_{N+M} \times S_M$. Although the Ansatz (2) is relatively simple the full wave functions turns out to be a rather cumbersome object due to the involved structure of the amplitudes (4) and the summations over the permutation groups in Eq. (2).

III. EIGENFUNCTIONS AS DETERMINANTS

We cast Eq. (2) in a determinantal form with is particularly suited for the case when the thermodynamic limit is taken only for one species. For the simplest case $M = 1$ this has been achieved in [25]. Here we state the generalization of this result:

**Theorem 1** The eigenfunctions of the Hamiltonian in Eq. (2) can be cast into the form

$$
\Psi(x, k, y, \Lambda) \propto \sum_{R \in S_M} \text{sgn}(R) \prod_{j<l} [i(\Lambda_{Rj} - \Lambda_{Rl}) + 2\text{sgn}(y_l - y_j)] \Phi(x, k, y, \Lambda),
$$

where $\Phi(x, k, y, \Lambda)$ is given by the $(N + M) \times (N + M)$ determinant

$$
\Phi(x, k, y, \Lambda) = \det \left[ \prod_{s=1}^{M} A_j(\Lambda_{Rs}, x_l - y_s) e^{ik_l x_l} \prod_{s \neq m}^{M} A_j(\Lambda_{Rs}, y_m - y_s) e^{ik_m y_m} \right]_{j=1, \ldots, N+M, \atop l=1, \ldots, N, \atop m=1, \ldots, M},
$$

and

$$A_j(\Lambda, x) = i(k_j - \Lambda) + \text{sgn}(x).$$

The wave functions (6) are eigenfunctions of the Hamiltonian (2) and of the center of mass momentum operator

$$\hat{K} = \frac{i}{\hbar} \left( \sum_{n=1}^{N} \frac{\partial}{\partial x_n} + \sum_{m=1}^{M} \frac{\partial}{\partial y_m} \right),$$

to the eigenvalues $E = \sum_{j=1}^{N+M} k_j^2$ and $K = \sum_{j=1}^{N+M} k_j$.

We prove Theorem 1 in App. A. Comparing Eq. (6) with the original form (2) shows that essentially the summations over two of the three permutation groups were replaced by a $(N + M) \times (N + M)$ determinant. This is at least in principle more convenient than Eq. (2) since it allows to employ the powerful methods of matrix algebra to manipulate determinants.

We briefly discuss the symmetries of the functions $\Psi(x, k, y, \Lambda)$ in (6). Due to the determinantal form of $\Phi(x, k, y, \Lambda)$ the antisymmetry of $\Psi(x, k, y, \Lambda)$ in $x$ and in $k$ is
obvious. To show that $\Psi$ is antisymmetric in $y$ we act with an arbitrary permutation $P$ on $y$ as $P(y) : y_\mu \leftrightarrow y_{P_\mu}$ and write Eq. (6) as

$$\Psi(x, k, P(y), \Lambda) = \sum_{R \in S_M} \text{sgn}(P) \text{sgn}(R) \Phi(x, k, P(y), P(\Lambda))$$

$$\prod_{j<l}^{M} \left[ t(\Lambda_{PR_j} - \Lambda_{PR_l}) + 2c \text{sgn}(y_{Pl} - y_{Pj}) \right]$$ (10)

We observe that $\Phi$ as well as the second line in Eq. (10) are antisymmetric under the simultaneous action of the permutation $P(y)$ and $P(\Lambda) : \Lambda_{R\mu} \leftrightarrow \Lambda_{PR\mu}$. Thus a minus sign is picked up by the sign of the permutation $\text{sgn}(P) = -1$. The antisymmetry $\Psi(x, k, y, P(\Lambda)) = -\Psi(x, k, y, \Lambda)$ can be proven similarly. Finally we note that the wave function (6) has no well defined symmetry when a spin-up and a spin-down particle are exchanged.

Imposing periodic boundary conditions on (10) yields a set of coupled algebraic equations, which are known as Bethe Ansatz equations [17]

$$k_j L = 2\pi n_j - 2 \sum_{m=1}^{M} \arctan \left( \frac{k_j - \Lambda_m}{c} \right), \quad j = 1, \ldots, N + M ,$$

$$2\pi J_\mu = 2 \sum_{j=1}^{N+M} \arctan \left( \frac{k_j - \Lambda_\mu}{c} \right) + 2 \sum_{\nu=1}^{M} \arctan \left( \frac{\Lambda_\mu - \Lambda_\nu}{2c} \right), \quad \mu = 1, \ldots, M .$$ (12)

The quantum numbers $J_\mu$ are integers for $N$ odd and half-odd integers for $N$ even. The quantum numbers $n_j$ are integers for $M$ even and half-odd integers for $M$ odd. For convenience we will always assume in the following $N$ to be odd and $M < N$. The values for $J_\mu$ are bounded by

$$- \frac{N + 2M - 1}{2} \leq J_\mu \leq \frac{N + 2M - 1}{2} .$$ (13)

In the ground state the $n_j$ are adjacent integers or half odd integers ranging from $(N + M - 1)/2$ to $-(N + M - 1)/2$ and for an odd number of spin-up particles the $J_\mu$ are chosen as

$$\{J_\mu\}_{\mu=1,\ldots,M} = \left\{ 0, \pm 1, \ldots, \pm \left( \frac{M}{2} - 1 \right), \frac{M}{2} \right\} .$$ (14)

In the following we will assume that for $c = 0$ the spin up particles have free momenta $-k_{\uparrow\mu}$, which lie in the interval $-k_F < k_{\uparrow\mu} < k_F$, where $k_F = \pi N/L$ is the Fermi momentum of the non–interacting spin down particles. In this range the quantum numbers $J_\mu$ can be identified with the momentum of the non–interacting spin up particle. They indicate the
single particle states, which for $c \to 0^+$ become doubly occupied. This means in the limit $c \to 0$: $\Lambda_{\mu}(c) \to -k_{\tau_{\mu}}$ and $k_{\tau_{\mu}} = 2\pi J_{\mu}/L$. For the systems overall momentum $K$ follows from Eqs. (11) and (12)

$$K = \sum_{j=1}^{N+M} k_j = \frac{2\pi}{L} \left( \sum_{\mu=1}^{N+M} n_j - \sum_{\mu=1}^{M} J_{\mu} \right). \quad (15)$$

If the set of quantum numbers $n_j$ is determined by the ground state the first sum vanishes and $K = -\sum_{\mu=1}^{M} k_{\tau_{\mu}}$.

IV. THERMODYNAMIC LIMIT

In a highly imbalanced system the density of the minority Fermions is very low. Thus one can assume that their thermodynamics is well approximated by a virial expansion.

In this approach the thermodynamical limit is taken only for the spin–down particles. The number of spin–up Fermions is finite. Since all spin–up particles have free momenta smaller than $k_F$, the quantum numbers $n_j$ are those of the ground state. The density of the quasi-momenta is obtained by taking the derivative of Eq. (11)

$$\varrho(k) = \frac{L}{2\pi} + \sum_{\mu=1}^{M} \frac{1}{\pi} \frac{c}{(k - \Lambda_{\mu})^2 + c^2}. \quad (16)$$

This is in leading order in $L$ to the momentum distribution of a sea of free Fermions. The quasi-momenta distribute themselves with the constant density $L/(2\pi)$ between two values $k_{\pm}$. The $M$ additional terms on the right hand side of Eq. (16) then might be interpreted as momentum distribution for the spin-up particles. The momenta $k_{\pm}$ are defined through the conditions

$$N + M = \frac{1}{k_+} \int_{-k_-}^{k_+} dk \varrho(k)$$

$$K = \frac{1}{k_-} \int_{-k_-}^{k_+} dk k \varrho(k) \quad (17)$$

which are transcendental equations. Assuming that the non–interacting Fermi sea is at rest, the solutions can be expanded in inverse powers of the system size $L$

$$k_{\pm} = \pm k_F \pm \frac{2}{L} \sum_{\mu=1}^{M} \left( \frac{\pi}{2} - \arctan \left( \frac{k_F \mp \Lambda_{\mu}}{c} \right) \right) + \mathcal{O}(L^{-2}) \quad (18)$$
where we introduce the notation

\[ v(\Lambda, a) = \frac{1}{\pi} \int_{-a}^{a} dk \arctan \left( \frac{k - \Lambda}{c} \right), \tag{19} \]

Using Eq. (16) the second Bethe-Ansatz equation (12) can be written as

\[ k_{\uparrow \mu} = v(\Lambda_{\mu}, k_F) + w_1(\Lambda_{\mu}) + \sum_{\nu=1}^{M, \nu \neq \mu} w_2(\Lambda_{\mu}, \Lambda_{\nu}), \tag{20} \]

where the functions \( w_1(\Lambda_{\mu}) \) and \( w_2(\Lambda_{\mu}, \Lambda_{\nu}) \) scale \( \mathcal{O}(L^{-1}) \) with the system size

\[ w_1(\Lambda_{\mu}) = \frac{\pi}{L} \frac{\partial v(\Lambda_{\mu}, k_F)}{\partial k_F}, \]

\[ w_2(\Lambda_{\mu}, \Lambda_{\nu}) = \frac{2c}{L\pi} \int_{-k_F}^{+k_F} dk \frac{\arctan \left( \frac{k - \Lambda_{\mu}}{c} \right)}{(k - \Lambda_{\nu})^2 + c^2} + \frac{2}{L} \arctan \left( \frac{\Lambda_{\mu} - \Lambda_{\nu}}{2c} \right). \tag{21} \]

Thus in leading order in \( L \) the quantity \( \Lambda_{\mu} \) is determined by \( \hat{J}_{\mu} \) only. Terms that couple different \( \Lambda_{\mu} \)'s are of order \( 1/L \). For \( c > 0 \) the function \( v \) in (19) is monotonously decreasing in \( \Lambda \). Thus it can be inverted. To this end the identity \( \arctan(x) = \text{sgn}(x)\pi/2 - \arctan(x^{-1}) \) is plugged into equation (19). The first term containing the sign–function can be integrated. One obtains an implicit equation for \( \Lambda_{\mu} \) which can be iterated. The inverted function reads to first order in the coupling strength

\[ \Lambda_{\mu} \approx -k_{\uparrow \mu} - \frac{c}{\pi} \ln \left| \frac{k_F + k_{\uparrow \mu}}{k_F - k_{\uparrow \mu}} \right| + \mathcal{O}(c^2). \tag{22} \]

For \( c \to \infty \) the quantity \( \Lambda_{\mu} \) scales as \( \Lambda_{\mu} = c\lambda_{\mu} \). The integral (19) becomes trivial and \( \Lambda_{\mu} = -c \tan(\pi k_{\uparrow \mu}/2k_F) \). The total energy is given by

\[ E = \int_{k_-}^{k_+} dk k^2 \varrho(k) = \frac{L(k_+^3 - k_-^3)}{6\pi} + \frac{1}{\pi} \sum_{\mu=1}^{M} \int_{k_-}^{k_+} dk \frac{ck^2}{(k - \Lambda_{\mu})^2 + c^2}, \tag{23} \]

This equation can be expanded in inverse powers of the system size using equation (18). All terms \( \mathcal{O}(L^{-2}) \) are neglected. The standard integral on the right hand side of Eq. (23) can be evaluated and the total energy can be written as

\[ E = E_F + \sum_{\mu=1}^{M} E^{(1)}(\Lambda_{\mu}) + \sum_{\mu<\nu}^{M} E^{(2)}(\Lambda_{\mu}, \Lambda_{\nu}), \tag{24} \]
where $E_F = L k_F^3 / (3\pi)$ denotes the energy of the non-interacting sea particles. The single particle energy $E^{(1)}(\Lambda_\mu)$ reads in leading order
\[ E^{(1)}(\Lambda_\mu) = k_F^2 + \frac{c}{\pi} \left( 2k_F + \Lambda_\mu \ln \left( \frac{c^2 + (\Lambda_\mu - k_F)^2}{c^2 + (\Lambda_\mu + k_F)^2} \right) \right) \]
\[ + \left( k_F^2 - \Lambda_\mu^2 + c^2 \right) \frac{\partial \nu(\Lambda_\mu, k_F)}{\partial \Lambda_\mu} + \mathcal{O}(L^{-1}). \]

The two particle energy can be expanded in the same way. Here the leading order term is $\mathcal{O}(L^{-1})$
\[ E^{(2)}(\Lambda_\mu, \Lambda_\nu) = \sum_{\sigma = \pm 1} \frac{2k_F}{\pi L} \left( 2 - k_F \frac{d}{dk_F} \right) \]
\[ \left[ \frac{\pi}{2} - \arctan \left( \frac{k_F + \sigma \Lambda_\mu}{c} \right) \right] \left[ \frac{\pi}{2} - \arctan \left( \frac{k_F + \sigma \Lambda_\nu}{c} \right) \right] + \mathcal{O}(L^{-2}). \]

Since the energy of the non-interacting system is $E_0 = E_F + \sum_{\mu=1}^{M} k_{\tau\mu}^2$, the total energy shift $\Delta E_M = E - E_0$ for $M$ spin-up particles can in leading order in the system size be expressed as
\[ \Delta E_M(k_\uparrow) = \sum_{\mu=1}^{M} \left( E^{(1)}(\Lambda_\mu) - k_{\tau\mu}^2 \right). \]

where $\Lambda_\mu$ is determined by $k_{\tau\mu}$ only. The energy shift is additive, that is each contribution corresponds to the energy shift caused by the interaction of a single spin-up particle with the Fermi-sea. In subleading order $\Lambda_\mu = \Lambda_\mu(k_\uparrow)$ depends on all initial momenta $k_{\tau\mu}$, $\mu = 1, \ldots, M$.

V. EFFECTIVE INTERACTION OF TWO SPIN-UP FERMIONS

The single particle energy $E^{(1)}(\Lambda_\mu)$ is additive as a function of $\Lambda_\mu$, however it is not additive as a function of the free momenta $k_{\tau\mu}$. Thus the functional form of the total energy shift changes when a spin-up particle is added to the system. The total energy shift of a system of $M$ spin-up particles $\Delta E_M$ can be expanded in a cluster expansion as
\[ \Delta E_1(k_{\tau1}) = E^{(1)}(\Lambda_1)|_{\Lambda_1=\nu^{-1}(k_{\tau1})} - k_{\tau1}^2 \]
\[ \Delta E_2(k_{\tau1}, k_{\tau2}) = \Delta E_1(k_{\tau1}) + \Delta E_1(k_{\tau2}) + W_2(k_{\tau1}, k_{\tau2}) \]
\[ \Delta E_3(k_{\tau1}, k_{\tau2}, k_{\tau3}) = \sum_{\mu=1}^{3} \Delta E_1(k_{\tau\mu}) + \sum_{\mu>\nu}^{3} W_2(k_{\tau\mu}, k_{\tau\nu}) + W_3(k_{\tau1}, k_{\tau2}, k_{\tau3}) \]
\[ \vdots \]
In the following we focus on the first two terms in this expansion, which contain the information about the single particle dispersion and on the interaction between two spin–up particles in the presence of the Fermi–sea. The single particle energy $E^{(1)}(k^\uparrow)$ given in equation (25) is plotted in figure 1. It is seen that the dispersion relation is at least for small values of $k^\uparrow$ still approximately quadratic $E(k) \approx \Sigma + k^2/m^*$. Self–energy and renormalized mass are given in terms of $a = c/k_F$ by

$$\Sigma = k_F^2 \left( \frac{2a}{\pi} - a^2 + \frac{2}{\pi} (1 + a^2) \arctan a \right)$$

$$m^* = \frac{\pi/2 - 2 \arctan a + (2/\pi) \arctan^2 a}{\pi/2 - \arctan a - a/(1 + a^2)}$$  \hspace{1cm} (29)$$

For finite interaction strength $c > 0$ and for larger momenta $k^\uparrow$ the dispersion relation deviates from the quadratic behavior, for $k^\uparrow = k_F$ the first derivative vanishes, leading to a van Hove like singularity in the density of states. Formally the Fermi momentum of the majority Fermions $k_F$ plays the role of an inverse lattice spacing as momentum cutoff. For $c = \infty$ the mass becomes formally infinite and the energy momentum independent.
The effective interaction energy $W_2(k_{\uparrow 1}, k_{\uparrow 2})$ is a function of $k_{\uparrow 1}$ and $k_{\uparrow 2}$ rather than of $\Lambda_1$ and $\Lambda_2$. Therefore it is not just given by the sum of the two–particle energies $E^{(2)}(\Lambda_1, \Lambda_2)$ and $E^{(2)}(\Lambda_2, \Lambda_1)$ but also the single particle energies $E^{(1)}(\Lambda_\mu)$ contribute. Using Eqs. (20) and (21) one finds

$$W_2(k_{\uparrow 1}, k_{\uparrow 2}) = -\frac{\partial E^{(1)}(\Lambda_1)}{\partial \Lambda_1} \left( \frac{\partial v(\Lambda_1, k_F)}{\partial \Lambda_1} \right)^{-1} w_2(\Lambda_1, \Lambda_2) \bigg|_{\Lambda_1=v^{-1}(k_{\uparrow 1})} - \frac{\partial E^{(1)}(\Lambda_2)}{\partial \Lambda_2} \left( \frac{\partial v(\Lambda_2, k_F)}{\partial \Lambda_2} \right)^{-1} w_2(\Lambda_2, \Lambda_1) + E^{(2)}(\Lambda_1, \Lambda_2) \bigg|_{\Lambda_2=v^{-1}(k_{\uparrow 2})}.$$

which has for general $c > 0$ to be treated numerically, since the function $v(\Lambda, a)$ in Eq. (19) can only be inverted numerically. An expansion of Eq. (30) for small interaction strength is possible and yields for $k_{\uparrow 1} < k_F$ and $k_{\uparrow 2} < k_F$

$$\lim_{c \to 0^+} W_2(k_{\uparrow 1}, k_{\uparrow 2}) = -\frac{4ck_{\uparrow 1}k_{\uparrow 2}}{L} \left( \frac{1}{k_F^2 - k_{\uparrow 1}^2} + \frac{1}{k_F^2 - k_{\uparrow 2}^2} \right).$$

This expression becomes singular if either of the two momenta approaches the Fermi momentum. An asymptotic expansion of Eq. (30) for strong interaction yields

$$W_2(k_{\uparrow 1}, k_{\uparrow 2}) = \frac{2\pi k_F}{L} \left\{ 1 + \frac{k_{\uparrow 1}k_{\uparrow 2}}{k_F^2} - \frac{2k_F}{c\pi} \left[ \cos^2 \left( \frac{\pi k_{\uparrow 1}}{2k_F} \right) + \cos^2 \left( \frac{\pi k_{\uparrow 2}}{2k_F} \right) \right] \right\} + O(c^{-2}).$$

Figures 1 show the plots of the interaction potential as a function of $a = c/k_F$ for zero center–of–mass momentum and for different relative momentum $k = k_{\uparrow 1} = -k_{\uparrow 2}$. The curves depend crucially on the relative momentum $k$. Whereas for small momentum the interaction energy increases monotonously with $c$, for higher momenta the interaction energy has a maximum for small interaction strength $c$ and decays for strong coupling strength to a value given by equation (32). Only for $k = k_F$ it decays to zero.

The interaction energy depends non–trivially on both arguments. This means that translation invariance of the reduced system of minority Fermions is broken by the Fermi sea. The dependence on the total momentum becomes most striking in the small coupling limit (31). For $k_{\uparrow 1}k_{\uparrow 2} < 0$ the interaction energy $W_2(k_{\uparrow 1}, k_{\uparrow 2})$ is always positive but for $k_{\uparrow 1}k_{\uparrow 2} > 0$ and for small values of $c$ it becomes negative. This can be seen in figure 3, where $W_2(k_{\uparrow 1}, k_{\uparrow 2})$ is plotted for zero relative momentum and different values of the center of mass momentum. Although the dependence of the interaction on the center of mass momentum is certainly an interesting feature, we focus in the sequel mainly on $K = 0$. 

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Neglecting the higher order interactions in the cluster expansion \cite{28} the thermodynamical limit can now be taken for the spin–up particles as well. In the ground state the $M < N$ quantum numbers $J_n$ are adjacent integers distributed around zero. We introduce an imbalance parameter $\eta = M/N$, which varies between zero and one for the balanced case. The ground state energy per unit length becomes

\[
\frac{E_0}{L} = \frac{k_F^3}{3\pi} + \frac{1}{2\pi} \int_{-\eta k_F}^{\eta k_F} E^{(1)}(\Lambda(x)) dx + \frac{L}{8\pi^2} \int_{-\eta k_F}^{\eta k_F} dx \int_{-\eta k_F}^{\eta k_F} dx' E^{(2)}(\Lambda(x), \Lambda(x')) .
\]

(33)

It can be checked that the expansion of this expression in powers of the imbalance parameter $\eta$ coincides up to second order with the result of Suzuki \cite{28}. The energies of the second order cluster expansion \cite{28} constitute the spectrum of an effective few body– Hamiltonian for the spin–up particles

\[
\hat{H}_{\uparrow\uparrow} = \sum_{|k|<k_F} \left( k^2 + \Delta E_1(k) \right) \hat{c}_{\downarrow k}^\dagger \hat{c}_{\uparrow k} + \sum_{|k|<k_F} \sum_{|k'|<k_F} \hat{c}_{\uparrow k}^\dagger \hat{c}_{\uparrow k} W_2(k, k') \hat{c}_{\uparrow k'}^\dagger \hat{c}_{\uparrow k'} .
\]

(34)
where the anticommuting operators $\hat{c}^\dagger_{\uparrow k} (\hat{c}_{\uparrow k})$ create (annihilate) a spin–up polaron with free momentum $k$. The interaction energy $W_2(k_1, k_2)$ can be related to an effective interaction potential of the minority Fermions via

$$W_2(k_1^\uparrow, k_2^\uparrow) = \int_0^L dy_1 \int_0^L dy_2 V_{\uparrow\uparrow}(y_1, y_2) R_0(y_1, y_2; k_1^\uparrow, k_2^\uparrow) ,$$

(35)

where $R_0(y_1, y_2; k_1^\uparrow, k_2^\uparrow)$ is the density–density correlation function of two minority Fermions. Its precise definition is given in equation (38) in the next section, where it is calculated exactly for arbitrary momenta $k_1^\uparrow, k_2^\uparrow$ and interaction strength $c$. The simplest approximation for the interaction potential, which is essentially a Born approximation, is obtained by replacing $R_0$ by its non–interacting value

$$R_0(y_1, y_2; k_1^\uparrow, k_2^\uparrow) = \frac{2}{L^2} (1 - \cos ((k_1^\uparrow - k_2^\uparrow)(y_1 - y_2))) .$$

(36)

Focussing on $K = 0$ implies $V_{\uparrow\uparrow}(y_1, y_2) = V_{\uparrow\uparrow}(y_1 - y_2)$. Thus for small coupling $c$ the effective interaction is essentially the Fourier transform of the interaction energy. Using the small $c$
expansion (31) of the interaction energy the potential is given by

\[ V(x) = c\delta(x) + ck_F \sin(2k_Fx) \]  

(37)

The Hamilton–operator (34) is bounded. The Fermi momentum acts as a momentum cutoff. Thus also the interaction potential can be determined only up to a length scale of order \( k_F \). This uncertainty is taken into account by convolution with a proper distribution of width \( k_F^{-1} \). The oscillatory term in the potential (37) cancels. Choosing for definiteness the characteristic function \( \chi_{[-k_F,k_F]} \) for the convolution, the interaction between two minority Fermions is zero everywhere but for distances smaller than \( k_F \), where it is constant \( V(x) = 2ck_F \).

VI. DENSITY-DENSITY CORRELATION FUNCTIONS

Using the determinantal representation (6) of the many–body wave function the density-density correlation function of two minority Fermions and even the three point correlation function of two minority Fermions and one majority Fermion can be calculated exactly. The latter yields insight to what extent two impurities affect the otherwise flat density profile of the Fermi sea.

The general density-density correlation function for two spin-up and \( n \)-spin-down Fermions is defined in coordinate representation through the multiple integral

\[ R_n(y_1, y_2, x_1, \ldots, x_n; k_1\uparrow, k_2\uparrow) := 4N^n \int_0^L dx_{n+1} \cdots \int_0^L dx_N |\Psi(x, k, y_1, y_2, \Lambda_1, \Lambda_2)|^2 . \]  

(38)

With the explicit form (6) of the eigenfunctions for \( M = 2 \) this can be written as

\[ R_n \propto \sum_{R,R' \in S_2} \text{sgn} (R + R') \left[ i(\Lambda_{R1} - \Lambda_{R2}) - 2c\text{sgn} (y_1 - y_2) \right] \]

\[ \times \left[ i(\Lambda_{R'2} - \Lambda_{R'1}) - 2c\text{sgn} (y_1 - y_2) \right] \]

\[ \times \int_0^L dx_{n+1} \cdots \int_0^L dx_N \Phi(x, y_1, y_2, \Lambda_{R1}, \Lambda_{R2}) \Phi^*(x, y_1, y_2, \Lambda_{R'1}, \Lambda_{R'2}) . \]  

(39)

Note that the quantities \( \Lambda_{1,2} \) and therefore the correlation function itself depends on the momenta \( k_{\uparrow1}, k_{\uparrow2} \) of the free spin–up particles. We are interested in \( n = 0, 1 \). For both cases, the correlation function can be cast into the unified form. We define \( \tilde{R}_n = L^{2+n}R_n/4N^n \) and assume \( y_1 \leq y_2 \) without loss of generality. Then \( \tilde{R}_n \) can be written as

\[ \tilde{R}_n = 2^{-n} \mathcal{R} \left\{ \det[I^{(n)}(\Lambda_1, \Lambda_2)] - \text{Re} \left( e^{2i\arctan(\frac{\Lambda_{12}}{\Lambda_1})} \det[J^{(n)}(\Lambda_1, \Lambda_2)] \right) \right\} , \]  

(40)
where the normalization constant $\mathcal{R}$ reads

$$\mathcal{R}^{-1} = \prod_{j=1}^{2} \int_{-k_F}^{k_F} dk \frac{k_F}{(k - \Lambda_j)^2 + c^2} - \text{Re} \left( \int_{-k_F}^{k_F} dk \frac{k_F e^{i \arctan \left( \frac{\Lambda - \Lambda_j}{2c} \right)}}{-i(k - \Lambda_1) + c}[i(k - \Lambda_2) + c] \right)^2$$

The quantities $I^{(n)}(\Lambda_1, \Lambda_2) = [J^{(n)}_{jl}(\Lambda_1, \Lambda_2)]_{j,l=1,...,n+2}$ and $J^{(n)}(\Lambda_1, \Lambda_2) = [J^{(n)}_{jl}(\Lambda_1, \Lambda_2)]_{j,l=1,...,n+2}$ in Eq. (40) denote $(n + 2) \times (n + 2)$ matrices. Their explicit form will be stated below.

Beside its coordinate dependence $\tilde{R}_n$ depends on the interaction strength $c$ and $\Lambda_1$ and $\Lambda_2$. The latter two quantities are determined by Eq. (20). Details of the derivation of Eq. (40) are presented in App. B. In the following we treat the two cases $n = 0, 1$ separately.

A. Two particle density-density correlation function

The density–density correlation function of the two spin up particles corresponds to $n = 0$ in the general expression (40). We give the explicit form of the entries of the matrix $I^{(0)}(\Lambda_1, \Lambda_2)$ and $J^{(0)}(\Lambda_1, \Lambda_2)$. As shown in App. B these are

$$I^{(0)}_{jj}(\Lambda_1, \Lambda_2) = \int_{-k_F}^{+k_F} dk \frac{k_F}{(k - \Lambda_j)^2 + c^2}, \quad j = 1, 2,$$

$$I^{(0)}_{12}(\Lambda_1, \Lambda_2) = \int_{-k_F}^{+k_F} dk \frac{k_F e^{ik(y_1 - y_2)}}{(i(k - \Lambda_1) + c)(i(k - \Lambda_2) + c)},$$

$$J^{(0)}_{jj}(\Lambda_1, \Lambda_2) = \int_{-k_F}^{+k_F} dk \frac{k_F}{(-i(k - \Lambda_1) + c)(i(k - \Lambda_2) + c)}, \quad j = 1, 2,$$

$$J^{(0)}_{12}(\Lambda_1, \Lambda_2) = \int_{-k_F}^{+k_F} dk \frac{k_F e^{ik(y_1 - y_2)} - 2i \arctan((k - \Lambda_2)/c)}{(k - \Lambda_1)^2 + c^2}. \quad (41)$$

Moreover $I^{(0)}_{21}(\Lambda_1, \Lambda_2) = [I^{(0)}_{12}(\Lambda_1, \Lambda_2)]^*$ and $J^{(0)}_{21}(\Lambda_1, \Lambda_2) = [J^{(0)}_{12}(\Lambda_2, \Lambda_1)]^*$. The integrals in Eq. (41) reveal that $\tilde{R}_0(y_1, y_2)$ is a function of the difference $y_1 - y_2$ only, as expected from translation invariance. For $c \to 0^+$ the integrals in Eq. (41) can be evaluated using

$$\lim_{c \to 0^+} \int \frac{k_F}{(k - \Lambda)^2 + c^2} = \delta(k - \Lambda). \quad (42)$$

This yields in the limit of vanishing interaction strength the density-density correlation function of two free Fermions with momenta $k_{\uparrow 1}$ and $k_{\uparrow 2}$ is given by Eq. (36). For hardcore
interaction the integrals in Eqs. (41) and (41) become trivial and we obtain

$$\lim_{c \to +\infty} \tilde{R}_0(y_1, y_2) = 1 - \left( \frac{\sin(k_F(y_1 - y_2))}{k_F(y_1 - y_2)} \right)^2.$$  \hfill (43)

This is the density-density correlation function of a non–interacting Fermi-sea with the typical decay $\tilde{R}_0 \sim (y_1 - y_2)^{-2}$ for large distances. This large distance behavior remains unchanged for finite $c$.

In Fig. 4 $\tilde{R}_0(y_1, y_2; k_{\uparrow 1}, k_{\uparrow 2})$ is plotted for the choice $k_{\uparrow 1} = -k_{\uparrow 2} = \pi k_F/8$ and for different values of $c$. As $c$ varies from zero to infinity the two particle density-density correlation function undergoes a transition from that one of two free Fermions to the one of a Fermi-sea. For other choices of the quantum numbers $k_{\uparrow 1}$ and $k_{\uparrow 2}$ the picture remains similar.
B. Three particle density-density correlation function

In contrast to the two particle case now $I^{(1)}(\Lambda_1, \Lambda_2)$ and $J^{(1)}(\Lambda_1, \Lambda_2)$ represents a $3 \times 3$ matrix. The entries are $I^{(1)}_{ij} = I^{(0)}_{ij}$ respectively $J^{(1)}_{ij} = J^{(0)}_{ij}$ for $i, j = 1, 2$. Moreover $I^{(1)}_{33} = 2$ and

$$J^{(1)}_{33}(\Lambda_{R1}, \Lambda_{R2}) = \begin{cases} \frac{1}{k_F} \int_{-k_F}^{k_F} \frac{dk}{k_F} \prod_{n=1}^{2} e^{-2i \arctan((k-\Lambda_n)/c)} & \text{if } x \in [y_2, y_1], \\ 2 & \text{otherwise.} \end{cases}$$

(44)

The other entries are given by

$$I^{(1)}_{13}(\Lambda_1, \Lambda_2) = \frac{1}{k_F} \int_{-k_F}^{k_F} \frac{dk}{k_F} \frac{-i(k - \Lambda_1) + \text{sgn}(x - y_1)[-i(k - \Lambda_2) + \text{sgn}(x - y_2)]}{[(k - \Lambda_1)^2 + c^2]i(k - \Lambda_2) + c} e^{ik(y_1 - x)} ,$$

$$I^{(1)}_{23}(\Lambda_1, \Lambda_2) = \frac{1}{k_F} \int_{-k_F}^{k_F} \frac{dk}{k_F} \frac{-i(k - \Lambda_1) + \text{sgn}(x - y_1)[-i(k - \Lambda_2) + \text{sgn}(x - y_2)]}{[(k - \Lambda_2)^2 + c^2]i(k - \Lambda_1) + c} e^{ik(y_2 - x)} ,$$

$$J^{(1)}_{13}(\Lambda_1, \Lambda_2) = \frac{1}{k_F} \int_{-k_F}^{k_F} \frac{dk}{k_F} \frac{-i(k - \Lambda_1) + \text{sgn}(x - y_1)[-i(k - \Lambda_2) + \text{sgn}(x - y_2)]}{[(k - \Lambda_1)^2 + c^2]i(k - \Lambda_2) + c} e^{ik(y_1 - x)} ,$$

$$J^{(1)}_{23}(\Lambda_1, \Lambda_2) = \frac{1}{k_F} \int_{-k_F}^{k_F} \frac{dk}{k_F} \frac{-i(k - \Lambda_1) + \text{sgn}(x - y_1)[-i(k - \Lambda_2) + \text{sgn}(x - y_2)]}{[(k - \Lambda_2)^2 + c^2]i(k - \Lambda_1) + c} e^{ik(y_2 - x)} ,$$

(45)

moreover $J^{(1)}_{3n}(\Lambda_1, \Lambda_2) = [I^{(1)}_{n3}(\Lambda_1, \Lambda_2)]^*$, $J^{(1)}_{13}(\Lambda_1, \Lambda_2) = [J^{(1)}_{n3}(\Lambda_2, \Lambda_1)]^*$ for $n = 1, 2$. For $c \to 0^+$ we make use of relation (42) to evaluate the normalization constant (41) and the integrals in Eq. (45). The outcome is

$$\lim_{c \to 0^+} \tilde{R}_1(y_1, y_2, x) = 1 - \cos ((k_{r1} - k_{r2})(y_1 - y_2)) .$$

(46)

In the hardcore limit the integrals in Eqs. (45) become elementary and can be evaluated. The resulting expression for $\tilde{R}_1(\tilde{y}_1, \tilde{y}_2, x)$ can be cast into the form

$$\lim_{c \to +\infty} \tilde{R}_1(y_1, y_2, x) = w(\lambda_1, \lambda_2) \tilde{R}_{FF}(\tilde{y}_1, \tilde{y}_2, x) ,$$

(47)

where we have introduced the quantity

$$\tilde{R}_{FF}(y_1, y_2, x) = 1 - \left( \frac{\sin(x - y_1)}{y_1 - y_2} \right) - \left( \frac{\sin(x - y_2)}{y_2 - y_1} \right) - \left( \frac{\sin(y_1 - y_2)}{y_1 - y_2} \right) + 2 \frac{\sin(x - y_1) \sin(x - y_2) \sin(y_1 - y_2)}{(y_1 - y_2)^2}$$

(48)
which corresponds to the three particle density-density correlation function of free Fermions. We recall the definition of $\lambda_n = \lim_{c \to \infty} \Lambda_n/c = -\tan(\pi k_{\gamma}\mu/2k_F)$. It varies from $-\infty$ to $+\infty$ as the $k_{\gamma}\mu$ vary from $k_F$ to $-k_F$. The factor $w(\lambda_1, \lambda_2)$ in Eq. (47) is given by

$$w(\lambda_1, \lambda_2) = \frac{1 - \Re uv}{1 - \Re u}$$

with

$$u = \frac{(1 + \lambda_1^2)(1 + \lambda_2^2)}{(\lambda_1 - \lambda_2)^2 + 4} \left( \frac{i(\lambda_1 - \lambda_2) + 2}{(1 + i\lambda_1)(1 - i\lambda_2)} \right)^2,$$

$$v = \frac{(i\lambda_1 + \sgn (x - y_2))(i\lambda_2 + \sgn (x - y_1))}{(i\lambda_1 + \sgn (x - y_1))(i\lambda_2 + \sgn (x - y_2))}.$$  

According to the equations above $v(\lambda_1, \lambda_2)$ and hence also $w(\lambda_1, \lambda_2)$ is a piecewise constant function of $x$. If $x$ lies outside the interval $[y_1, y_2]$, $v = 1$ such that $\tilde{R}_1(y_1, y_2, x) = \tilde{R}_{FF}(y_1, y_2, x)$ for $x \notin (y_1, y_2)$. However, if $x$ lies between the two spin-up particles the function $w(\lambda_1, \lambda_2)$ yields a weight for the density of the Fermi-sea inside the interval $(y_1, y_2)$ which crucially depends on the quantities $\lambda_1$ and $\lambda_2$. We first consider the case $k_{\gamma_1} = -k_{\gamma_2}$. This implies $\lambda_1 = -\lambda_2$ and Eq. (49) simplifies further

$$w(\lambda_1, -\lambda_1) = \frac{(\lambda_1^2 - 3)^2}{(1 + \lambda_1^2)^2}.$$  

Thus if $k_{\gamma_1} = \pm k_F$ borders the Fermi-sea such that $\lambda_1 = \pm \infty$ we have $w(\lambda_1, -\lambda_1) = 1$ and consequently the three particle density-density correlation function coincides with that of free Fermions for all values of $x$. Most interestingly from Eq. (52) follows that $w(\lambda_1, -\lambda_1) = 0$ for $\lambda_1 = \sqrt{3}$ which corresponds to the choice $k_{\gamma_1} = -k_{\gamma_2} = 2k_F/3$. This implies that the density of the Fermi-sea between $y_1$ and $y_2$ vanishes identically.

Next we consider the choice $k_{\gamma_2} = 0$ to be in the core of the Fermi-sea. The quantity $w(\lambda_1, 0)$ diverges as $k_{\gamma_1} \to \pm k_F$ approaches the border of the Fermi-sea. Consequently the density of the spin-down particles in the region $(y_1, y_2)$ increases as $k_{\gamma_1} \to \pm k_F$ and finally diverges for $k_{\gamma_1} = \pm k_F$.

For finite $c$ the integrals (45) are evaluated numerically. For fixed $y_1$ and $y_2$ the three particle density-density correlation function corresponds to the density profile of the Fermi-sea. Figure 5 shows the plot of $\tilde{R}_1(-k_F, k_F, x)$ for different values of $c$ where the particles momenta $k_{\gamma_1} = -k_{\gamma_2} = \pi k_F/8$ are chosen symmetrically around zero. While for $c = 0$ the density profile of the Fermi-sea is constant it changes when interaction is switched on. The density at the positions of the two spin-up particles decreases as the interaction
FIG. 5. Three particle density-density correlation function for fixed $y_1 = -k_F^{-1}$ and $y_2 = k_F^{-1}$ as function of $x$ for the quantum numbers $k_{↑1} = -k_{↑2} = \pi k_F/8$. The values for the interaction strength are: $c = 0$ short dashed line (orange), $c = 0.5k_F$ solid line (black), $c = k_F$ dashed line (red), $c = 2k_F$ dotted line (blue) and $c = +\infty$ dot-dash line (green).

increases and finally vanishes for $c \to +\infty$. Figure 6 shows the same as Fig. 5 but for higher quantum numbers $k_{↑1} = -k_{↑2} = \pi/4$ and slightly different values of the interaction strength. Comparing Fig. 6 with Fig. 5 reveals that with increasing $c$ the suppression of the density in between the two spin-up particles is the stronger the higher quantum numbers $k_{↑1}$ and $k_{↑2}$ are. A qualitatively similar picture emerges when the $k_{↑1}$ and $k_{↑2}$ are inside the Fermi-sea but chosen in a non symmetric way. The situation changes rather drastically if one of the quantum numbers, say $k_{↑1}$ is set to unity such that it borders the Fermi-sea and the other one is chosen in the core of the Fermi-sea, that is $k_{↑2} = 0$. Figure 7 shows the corresponding plots. Now the density between the two spin-up particles is enhanced with respect to the density in the outer regions. With increasing interaction strength also the density inside the interval $(y_1, y_2)$ increases and finally for $c \to \infty$ diverges in accordance with the discussion following Eq. (52).
FIG. 6. Three particle density-density correlation function for fixed $y_1 = -k_F$ and $y_2 = k_F$ as function of $x$ for the quantum numbers $k_{\uparrow 1} = -k_{\uparrow 2} = \pi k_F/4$. The values for the interaction strength are: $c = 0$ short dashed line (orange), $c = 0.1k_F$ solid line (black), $c = 0.3k_F$ dashed line (red), $c = 0.5k_F$ dotted line (blue) and $c = +\infty$ dot-dash line (green).

VII. SUMMARY & CONCLUSION

The Bethe-Ansatz solvable wave function of the Yang Gaudin model was expressed as a sum of determinants. Especially in highly imbalanced systems with only a small number of minority particles the eigenfunctions acquire a simple form.

Equilibrium properties of one and two minority Fermions in a sea of majority Fermions were investigated. Even for small coupling constants the minority Fermions become strongly entangled with the sea particles and form composite particles (polarons). The dispersion relation being approximately quadratic in a large range of momenta, estimates for the polaron mass and self energy were made. Essentially the Fermi sea establishes via $k_F^{-1}$ an additional length scale. On this scale the minority Fermions interact among each other giving rise to a non–vanishing interaction energy.

The density-density correlation function of two minority Fermions shows even for small $c$ the quadratic decay with distance which is typical for Fermionic systems.
FIG. 7. Three particle density-density correlation function for fixed $y_1k_F = -1$ and $y_2k_F = +1$ as function of $x$ for the quantum numbers $k_{\uparrow 1} = k_F$ and $k_{\uparrow 2} = 0$. The values for the interaction strength are: $c = 0$ short dashed line (orange), $c = 5k_F$ solid line (black), $c = 3k_F$ dashed line (red), $c = k_F$ dotted line (blue) and $c = 0.1k_F$ dot-dash line (green).

Particularly interesting are results obtained for the three–point function. They show how the Fermi-sea is locally deformed by the interaction with the minority Fermions which act like dynamical impurities. The details of the deformation depend crucially on the free momenta $k_{\uparrow 1}$ and $k_{\uparrow 2}$ of the minority Fermions. Depending on $k_{\uparrow 1}$ and $k_{\uparrow 2}$ the density of sea particles between them can be either suppressed or enhanced.

For infinite strong interaction this becomes most evident. In this case our analysis revealed that the density in between the two spin-up particles diverges if the quantum numbers are chosen as $k_{\uparrow 1} = 1$ and $k_{\uparrow 2} = 0$. On the other hand for $k_{\uparrow 1} = -k_{\uparrow 2} = 2k_F/3$ the density between the two minority Fermions vanishes identically.
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Appendix A: Proof of Theorem

In principal one could act directly with the Hamiltonian on $\Psi(x, k, y, \Lambda)$ in Eq. (6) to prove part one of theorem. However, a more handy way to prove theorem is to show that the wave function fulfills the following conditions: 1.) $\Psi(x, k, y, \Lambda)$ fulfills in every sector the free Schrödinger equation, 2.) $\Psi(x, k, y, \Lambda)$ is continues everywhere and 3.) let $y_m$ and $x_n$ be adjacent. Then the first derivative of $\Psi(x, k, y, \Lambda)$ evaluated at $x_n = y_m$ is discontinues such that

$$\left( \frac{\partial}{\partial x_n} - \frac{\partial}{\partial y_m} \right) \Psi(x, k, y, \Lambda) \bigg|_{x_n - x_m = 0^+} = 4c \Psi(x, k, y, \Lambda) \bigg|_{x_n = y_m},$$

where $0^\pm$ has the meaning that zero is approached from above/below.

As can be seen right from its definition the wave function is continuous and fulfills the free Schrödinger within each sector. The continuity of $\Psi(x, k, y, \Lambda)$ at $x_n = x_j$ is obvious, too. To show the continuity of the wave function at $x_n = y_m$, it is written

$$\Psi(x, k, y, \Lambda) \bigg|_{x_n - y_m = 0^\pm} \propto \sum_{R \in S_M} \text{sgn}(R) \prod_{j<l} \left[ i(\Lambda_{Rj} - \Lambda_{Rl}) + 2c \text{sgn}(y_l - y_j) \right] (A2)$$

$$\det \left[ \ldots \left[ i(k_j - \Lambda_{Rm}) \pm c \right] \prod_{s \neq m} A_j(\Lambda_{Rs}, y_m - y_s) e^{ik_j y_m} \right. \ldots \right],$$

where the dots indicate that all other columns of the determinant remain unchanged. Using multilinearity of the determinant it is seen that in the difference the $m$-th and the $(M + n)$-th column are linearly depended and consequently the determinant vanishes. This proves the continuity of $\Psi(x, k, y, \Lambda)$ at $x_n = y_m$. The continuity of the wave-function at $y_m = y_v$ can be shown similarly.

To prove the third condition we take the derivative of Eq. (1) with respect to $x_n$ on both
sides of the border. This yields for the difference

$$\frac{\partial \Psi(x, k, y, A)}{\partial x_n} \bigg|_{x_n - y_m = 0^+} \propto \sum_{R \in S_M} \text{sgn}(R) \prod_{j < l}^M [i(\Lambda_{R_j} - \Lambda_{R_l}) + 2\text{sgn}(y_l - y_j)]$$

(A3)

$$\text{det} \left[ \ldots |2ck \prod_{i \neq m} A_j(\Lambda_{R_i}, y_m - y_i)e^{ik_jy_m}| \ldots \right].$$

Taking the derivative of $\Psi(x, k, y, A)$ with respect to $y_m$ yields

$$\frac{\partial \Psi(x, k, y, A)}{\partial y_m} \bigg|_{x_n - y_m = 0^+} \propto$$

$$\sum_{R \in S_M} \text{sgn}(R) \left( \frac{\partial}{\partial y_m} \prod_{j < l}^M [i(\Lambda_{R_j} - \Lambda_{R_l}) + 2\text{sgn}(y_l - y_j)] \bigg) \Phi(x, k, y, A)$$

(A4)

$$\prod_{j < l}^M [i(\Lambda_{R_j} - \Lambda_{R_l}) + 2\text{sgn}(y_l - y_j)] \frac{\partial \Phi(x, k, y, A)}{\partial y_m} \bigg|_{x_n - y_m = 0^+}.$$

(A5)

$$\prod_{j < l}^M [i(\Lambda_{R_j} - \Lambda_{R_l}) + 2\text{sgn}(y_l - y_j)] \frac{\partial \Phi(x, k, y, A)}{\partial y_m} \bigg|_{x_n - y_m = 0^+}.$$

(A6)

Performing the derivative of the prefactor in line (A4) yield an factor $\delta(y_n - y_m), n = 1, \ldots, M \neq m$ multiplied with $\Phi(x, k, y, A)$ evaluated at $x_n - y_m = 0^\pm$. However, as follows from its definition in Eq. (7) $\Phi(x, k, y, A)$ is continues at $x_n = y_m$ and hence this term vanishes when the difference is taken. Thus

$$\frac{\partial \Psi(x, k, y, A)}{\partial y_m} \bigg|_{x_n - y_m = 0^+} = \frac{\partial \Phi(x, k, y, A)}{\partial y_m} \bigg|_{x_n - y_m = 0^+}.$$  

(A7)

and it is sufficient to consider the derivative of $\Phi(x, k, y, A)$ with respect to $y_m$. Using

$$\frac{\partial}{\partial y_m} \prod_{s=1}^M [i(k_j - \Lambda_{R(s)}) + 2\text{sgn}(x_l - y_s)] e^{ik_jx_l} \bigg|_{x_n - y_m = 0^\pm}$$

(A8)

$$= \begin{cases} 
0 & \text{for } l \neq n \\
-2c\delta(x_n - y_m) \prod_{s \neq m} A_j(\Lambda_{R_s}, y_m - y_s)e^{ik_jy_m} & \text{for } l = n
\end{cases}$$

as well as properties of the determinant reveals that the derivative on the first $N$ columns of $\Phi(x, k, y, A)$ with respect to $y_m$ vanishes and only the derivatives of the last $M$ columns have to taken into account. We obtain

$$\frac{\partial \Phi(x, k, y, A)}{\partial y_m} \bigg|_{x_n - y_m = 0^\pm} \propto 4c \sum_{i \neq m}^M \delta(y_m - y_i) \det \left[ \ldots |\prod_{s=1}^M A_j(\Lambda_{R_s}, y_m - y_s)e^{ik_jy_m}| \ldots \right] +$$

$$\text{det} \left[ \ldots |ki \prod_{s=1}^M A_j(\Lambda_{R_s}, y_m - y_s)e^{ik_jy_m}| \ldots \right] \pm c \prod_{s \neq m} A_j(\Lambda_{R_s}, y_m - y_s)e^{ik_jy_m}.$$
Using properties of the determinant it can be shown that the terms in line (A9) proportional to \( \delta(y_m - y_l) \) vanishes when the difference is taken. Hence we have

\[
\frac{\partial \Phi(x, k, y, \Lambda)}{\partial y_m} \bigg|_{x_n - y_m = 0^+} = \operatorname{det} \left[ \ldots \left| \frac{1}{2} \pi k_j \prod_{i \neq m, l} A_j(L_{Ri}, y_m - y_i) e^{ik_j y_m} \right| \right] \tag{A9}
\]

Observing that

\[
\Psi(x, k, y, \Lambda) \big|_{x_n = y_m} \propto \sum_{R \in S(M)} \operatorname{sgn}(R) \prod_{j < l}^{M} \left[ i(\Lambda_{R_j} - \Lambda_{R_l}) + 2c \operatorname{sgn}(y_l - y_j) \right] \tag{A10}
\]

leads in combination with Eqs. (A3), (A7) and (A9) to Eq. (A1). This completes the proof that the wave function in Eq. (6) is an eigenfunction to the Hamiltonian (1). The corresponding eigenvalue is given by \( E = \sum_{n=1}^{N+1} k_n^2 \), the eigenvalues of the momentum operator are \( K = \sum_{n=1}^{N+1} k_n \). This follows from the fact that \( \Psi(x, k, y, \Lambda) \) fulfills the free Schrödinger equation in each sector.

**Appendix B: Derivation of two and three particle correlators**

According to theorem (1) the eigenfunctions acquire for \( M = 2 \) the form

\[
\Psi(x, k, y_1, y_2, \Lambda_1, \Lambda_2) \propto \sum_{R \in S_2} \operatorname{sgn}(R) \left[ i(\Lambda_{R_1} - \Lambda_{R_2}) + 2c \operatorname{sgn}(y_2 - y_1) \right] \Phi(x, k, y_1, y_2, \Lambda_{R_1}, \Lambda_{R_2}) \tag{B1}
\]

We use in this appendix the convention \( A_j(\Lambda) \equiv A_j(\Lambda, 1) = i(k_j - \Lambda) + c \). To evaluate Eq. (39) for \( n = 0 \) we shift the integration variables in Eq. (39) by \( x_l \rightarrow x_l + y_1 \) and expand the determinant in Eq. (7) with respect to the last two columns. This yields

\[
\Phi(x - y_1, y_1, y_2, \Lambda_{R_1}, \Lambda_{R_2}) = \prod_{j=1}^{N+2} e^{ik_j y_1} \sum_{n=m}^{N+2} (-1)^{n+m} A_n(\Lambda_{R_1}) A_m^*(\Lambda_{R_1}) e^{ik_n y_1} \tag{B2}
\]

\[
\times \det \left[ A_j(\Lambda_{R_1}, x_l) A_j(\Lambda_{R_2}, x_l - (y_2 - y_1)) e^{ik_j x_l} \right]_{j=1,\ldots,N+2 \neq n,m} \].

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Then by employing properties of the determinant the integral in the last line of Eq. (39) can be cast into the form
\[
\int_0^L dx_1 \cdots \int_0^L dx_N \Phi(x, y_1, y_2, \Lambda_{R1}, \Lambda_{R2}) \Phi^*(x, y_1, y_2, \Lambda_{R1'}, \Lambda_{R2'}) = N! \sum_{n \neq m} \sum_{s \neq t} (-1)^{n+m} \left( -1 \right)^{s+t} \prod_{j=1}^{N+2} A_j(\Lambda_{R1}) A_{j}^*(\Lambda_{R1'}) \frac{A_m^*(\Lambda_{R2}) A_t(\Lambda_{R2'})}{A_m(\Lambda_{R1}) A_t^*(\Lambda_{R1'})} e^{i(k_n-k_s)y^{-}} \det [Q_{jl}]_{j,l=1,\ldots,N+2} \quad (B3)
\]
where the quantities \( Q_{jl} \) in Eq. (B3) are given by
\[
Q_{jl} = \int_0^L dx A_j(\Lambda_{R2}, x - y^{-}) A_{j}^*(\Lambda_{R2'}, x - y^{-}) e^{i(k_j-k_l)x} . \quad (B4)
\]
The integral is elementary. Evaluating it we obtain
\[
Q_{jl} = \left[ LA_j(\Lambda_{R2}) A_{j}^*(\Lambda_{R2'}) + 2ct (\Lambda_{R2} - \Lambda_{R2'}) y^{-} \right] \delta_{jl} - 2c \left( 1 - \frac{\Lambda_{R2} - \Lambda_{R2'}}{k_j - k_l} \right) e^{i(k_j-k_l)y^{-}} (1 - \delta_{jl}) + \text{B.T.} \quad ,
\]
where B.T. stands for the expression
\[
\text{B.T.} = \frac{1}{i(k_j - k_l)} \left[ ((k_j - \Lambda_{R2}) (k_l - \Lambda_{R2'}) + c^2) (e^{i(k_j-k_l)L} - 1) + ic (k_j - k_l - \Lambda_{R2} + \Lambda_{R2'}) (e^{i(k_j-k_l)L} + 1) \right] (1 - \delta_{jl})
\]
which corresponds to terms which arise from the boundaries when the off-diagonal terms with \( j \neq l \) in Eq. (B4) are integrated. However, using the Bethe-Ansatz equations it shown these terms vanish identically as a consequence of translational invariance. The expression (B5) reveals that the diagonal terms where \( j = l \) scale like \( L \) while the off-diagonal terms scale like \( c \). Therefore, in the thermodynamic limit the off-diagonal terms are negligible and the entries can be approximated by the diagonal terms only. In leading order of \( L \) therefore
\[
\det [Q_{jl}]_{j,l=1,\ldots,N+2} = \left( \prod_{j \neq n,m} L A_j(\Lambda_{R2}) A_{j}^*(\Lambda_{R2'}) \right) \det \begin{bmatrix} \delta_{ns} & \delta_{nt} \\ \delta_{ms} & \delta_{mt} \end{bmatrix} . \quad (B7)
\]
Combining the expression (B7) with Eq. (B3) the two particle density-density correlation function acquires the form
\[
R_0(y_1, y_2) \propto \sum_{R,R' \in S_2} \text{sgn} (R + R') \left[ i(\Lambda_{R1} - \Lambda_{R2}) + 2c \right] [-i(\Lambda_{R1} - \Lambda_{R2'}) + 2c] 
\]
\[
\sum_{n \neq m} \sum_{s \neq t} A_m^*(\Lambda_{R1}) A_t(\Lambda_{R1'}) A_n(\Lambda_{R2}) A_m(\Lambda_{R2'}) A_s^*(\Lambda_{R2'}) A_{t}^*(\Lambda_{R2'}) e^{i(k_n-k_s)y^{-}} \det \begin{bmatrix} \delta_{ns} & \delta_{nt} \\ \delta_{ms} & \delta_{mt} \end{bmatrix} .
\]
Since the terms in the second line of Eq. (B8) factorize in all summation indices we can write each term into the corresponding row or column of the determinant. Furthermore, due to the Kronecker-$\delta$'s two of the four summations drop out. Thus the second line of the equation above can be expressed by a determinant whose entries are one-fold sums. Now it is straightforward to take the thermodynamic limit. Assuming the Fermi-sea to be in the ground state i.e. at zero temperature, the quasi-momenta distribute themselves uniformly between $\pm k_F$ with a density $\varrho(k) = L/(2\pi)$. In the usual way replace the sums in Eq. (B8) over the quasi-momenta by integrals. Then Eq. (B8) can be cast into the form

$$R_0(y_1, y_2) \propto \left[ (\Lambda_1 - \Lambda_2)^2 + 4c^2 \right] \det [I^{(0)}(\Lambda_1, \Lambda_2)] - 2\text{Re} (i(\Lambda_1 - \Lambda_2) + 2c)^2 \det [J^{(0)}(\Lambda_1, \Lambda_2)].$$

(B9)

The two matrices $I^{(0)}(\Lambda_1, \Lambda_2) = [I^{(0)}_{jl}(\Lambda_1, \Lambda_2)]_{j,l=1,2}$ and $J^{(0)}(\Lambda_1, \Lambda_2) = [J^{(0)}_{jl}(\Lambda_1, \Lambda_2)]_{j,l=1,2}$ are given in (H). Together with the normalization this yields to the form (40) of the two particle density-density correlation function. It remains to determine the normalization constant $R$. The normalization condition reads

$$4 = \int_0^L dy_1 \int_0^L dy_2 \ R_0(y_1, y_2).$$

(B10)

To evaluate it we use the form (B8) of $R_0(y_1, y_2)$. The integration over the exponential there yields in leading order of $L$

$$\int_0^L dy_1 \int_0^L dy_2 e^{i(k_n - k_s)(y_2 - y_1)} = L^2 \delta_{ns}$$

(B11)

and thus only the diagonal terms of the entries contribute to the normalization. This immediately leads to Eq. (H).

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