PARAMETER ESTIMATION IN DIAGONALIZABLE STOCHASTIC HYPERBOLIC EQUATIONS

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Abstract. A parameter estimation problem is considered for a linear stochastic hyperbolic equation driven by additive space-time Gaussian white noise. The damping/amplification operator is allowed to be unbounded. The estimator is of spectral type and utilizes a finite number of the spatial Fourier coefficients of the solution. The asymptotic properties of the estimator are studied as the number of the Fourier coefficients increases, while the observation time and the noise intensity are fixed.

1. Introduction

A typical example of a parabolic equation is the heat equation
\[ u_t = u_{xx}; \]
a typical example of a hyperbolic equation is the wave equation
\[ u_{tt} = u_{xx}. \]
In a more abstract setting, if \( A \) is a linear operator such that \( \dot{u} + Au = 0 \) is a parabolic equation, then
\[ \dot{u} + Au = 0 \]
is natural to call a hyperbolic equation; \( \dot{u} \) and \( \ddot{u} \) are the first and second time derivatives of \( u \).

Damping in a hyperbolic equations is introduced via a term depending on the first time derivative of the solution. For example, a damped wave equation is
\[ u_{tt} = u_{xx} - au_t, \quad a > 0. \]
Indeed, if we define the total energy \( E(t) = \int (u_t^2(t,x) + u_{xx}^2(t,x))dx \), then integration by parts shows that
\[ \frac{d}{dt} E(t) = -a \int u_t^2(t,x)dx; \]
it also shows that \( a < 0 \) (negative damping) corresponds to amplification. More generally, we write a damped linear hyperbolic equation in an abstract form
\[ \ddot{u} + Au = Bu, \tag{1.1} \]
where \( A \) and \( B \) are linear operators on a separable Hilbert space \( H \); depending on the properties of the operator \( B \), the result can be either damping or amplification.

In this paper, we consider a stochastic version of \((1.1)\), perturbed by additive space-time white noise and with operators \( A \) and \( B \) specified up to an unknown parameter:
\[ \ddot{u} + (A_0 + \theta_1 A_1)u = (B_0 + \theta_2 B_1)u + \dot{W}, \quad 0 < t \leq T. \tag{1.2} \]

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The objectives are

- to determine the conditions on the operators so that the equation has a generalized solution that is a square-integrable random element with values in a suitable Hilbert space;
- to construct a maximum likelihood estimator of the unknown parameters $\theta_1, \theta_2$ using a finite-dimensional projection of the solution, and to study the asymptotic properties of the estimator as the dimension of the projection increases.

For stochastic parabolic equations with one unknown parameter, a similar problem was first suggested by Huebner, Khasminskii and Rozovskii [4] and was further investigated by Huebner and Rozovskii [5]. Estimation of several parameters in parabolic equations has also been studied [3, 10]. For stochastic hyperbolic equations, most of these problems remain open. Since the equation is second-order in time, it is natural to start with two unknown parameters. In the case of the wave equation, these parameters correspond to the propagation speed of the wave and the damping coefficient [9].

With precise definitions to come later, at this point we interpret $\dot{W}(t)$ as a formal sum

$$\dot{W}(t) = \sum_{k \geq 1} h_k \dot{w}_k(t),$$

where $\{h_k, \ k \geq 1\}$ is an orthonormal basis in the Hilbert space $H$, and $w_k(t)$ are independent standard Brownian motions. We look for the solution of (1.2) as a Fourier series

$$u(t) = \sum_{k \geq 1} u_k(t) h_k,$$

and call it a generalized solution. If the trajectories of $u_k(t)$ are observed for $1 \leq k \leq N$ and all $0 < t < T$, then there exists a closed-form expression for maximum likelihood estimator of $(\theta_1, \theta_2)$ in terms of $u_k$ and $\dot{u}_k$; see Section 3 below.

The main technical assumptions about the equation are

- zero initial conditions (to simplify the presentation);
- the ability to write equation (1.2) as an infinite system of uncoupled stochastic ordinary differential equation (this is essential in the construction and the analysis of the estimator). In other words, we assume that the equation is diagonalizable: the operators $A_0, A_1, B_0$ and $B_1$ have a common system of eigenfunctions $\{h_k, \ k \geq 1\}$:

$$A_0 h_k = \kappa_k h_k, \quad B_0 = \rho_k h_k, \quad A_1 h_k = \tau_k h_k, \quad B_1 = \nu_k h_k$$

and this system is an orthonormal basis in the Hilbert space $H$.
- hyperbolicity, that is,

  (1) there exist positive numbers $C^*, c_1, c_2$ such that $\{\kappa_k + \theta \tau_k + C^*, \ k \geq 1\}$ is a positive, non-decreasing, and unbounded sequence for all $\theta \in \Theta_1$ and

$$c_1 \leq \frac{\kappa_k + \theta \tau_k + C^*}{\kappa_k + \theta' \tau_k + C^*} \leq c_2$$

for all $\theta, \theta' \in \Theta_1$;
(2) there exist positive numbers $C, J$ such that, for all $k \geq J$ and all $\theta_1 \in \Theta_1$, $\theta_2 \in \Theta_2$,
\begin{equation}
T(\rho_k + \theta_2 \nu_k) \leq \ln(\kappa_k + \theta_1 \tau_k) + C.
\end{equation}

If equation (1.2) is diagonalizable and the solution has the form (1.3), then the Fourier coefficient $u_k$ satisfies
\begin{equation}
\ddot{u}_k - (\rho_k + \theta \nu_k) \dot{u}_k + (\kappa_k + \theta_1 \tau_k) u_k = \dot{w}_k, \quad u_k(0) = \dot{u}_k(0) = 0.
\end{equation}

We show in Section 2 that if the equation is also hyperbolic and $X$ is a Hilbert space such that $H \subset X$ and the embedding operator $j : H \rightarrow X$ is Hilbert-Schmidt, then $u$ is an $X$-valued process.

The maximum likelihood estimators of $\theta_1$ and $\theta_2$ are constructed in Section 3 using the processes $u_k, \dot{u}_k, k = 1, \ldots, N$ (the corresponding formulas are too complicated to present in the Introduction). Analysis of these estimators in the limit $N \rightarrow \infty$ is the main objective of the paper and is carried out in Sections 4 and 5. Here is the main result of the paper for the case when $A_i, B_i$ are (pseudo)differential elliptic operators.

**Theorem 1.1.** Assume that equation (1.2) is diagonalizable and hyperbolic and that $A_i, B_i$ are positive-definite elliptic self-adjoint differential or pseudo-differential operators on a smooth bounded domain in $\mathbb{R}^d$ with suitable boundary conditions or on a smooth compact $d$-dimensional manifold. Then

(1) the maximum likelihood estimator of $\theta_1$ is consistent and asymptotically normal in the limit $N \rightarrow \infty$ if and only if
\begin{equation}
\text{order}(A_1) \geq \frac{\text{order}(A_0 + \theta_1 A_1) + \text{order}(B_0 + \theta_2 B_1) - d}{2};
\end{equation}

(2) the maximum likelihood estimator of $\theta_2$ is consistent and asymptotically normal in the limit $N \rightarrow \infty$ if and only if
\begin{equation}
\text{order}(B_1) \geq \frac{\text{order}(B_0 + \theta_2 B_1) - d}{2}.
\end{equation}

Similar to the parabolic case (Huebner [3]), the results of the paper extend to a more general estimation problem
\begin{equation}
\ddot{u} + \sum_{i=0}^n \theta_{1i} A_i u = \sum_{j=0}^m \theta_{2j} B_j \dot{u} + \dot{W},
\end{equation}
as long as all the operators $A_i, B_j$ have a common system of eigenfunctions. For example, in the setting similar to Theorem 1.1, the coefficient $\theta_{1p}$ can be consistently estimated if and only if
\begin{equation}
\text{order}(A_p) \geq \frac{\text{order}(\sum_{i=0}^n \theta_{1i} A_i) + \text{order}(\sum_{j=0}^m \theta_{2j} B_j) - d}{2}.
\end{equation}

Throughout the presentation below, we fix a stochastic basis
\begin{equation}
\mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})
\end{equation}
with the usual assumptions (completeness of $\mathcal{F}_0$ and right-continuity of $\mathcal{F}_t$). We also assume that $\mathbb{F}$ is large enough to support countably many independent standard Brownian motions. For a random variable $\xi$, $\mathbb{E}\xi$ and $\text{Var} \xi$ denote the expectation and variance respectively. The time derivative of a function is denote either by a dot on top (as in $\dot{u}$) or by a subscript $t$ (as in $u_t$).

The following notations are used for two non-negative sequences $a_n, b_n$, $n \geq 1$:

\begin{enumerate}
\item $a_n \ll b_n$ if there exist positive numbers $c_1, c_2$ such that $c_1 \leq a_n/b_n \leq c_2$ for all sufficiently large $n$;
\item $a_n \asymp b_n$ if \begin{equation}
\lim_{k \to \infty} \frac{a_k}{b_k} = c \quad \text{for some} \quad c > 0;\end{equation}
\item $a_n \sim b_n$ if (1.11) holds with $c = 1$. Note that if $a_n \sim b_n$ and $\sum_n a_n$ diverges, then $\sum_{k=1}^n a_k \sim \sum_{k=1}^n b_k$.
\end{enumerate}

Finally, we recall that a cylindrical Brownian motion $W = W(t)$, $t \geq 1$, over (or on) a Hilbert space $H$ is a linear mapping $W : f \mapsto W_f(\cdot)$ from $H$ to the space of zero-mean Gaussian processes such that, for every $f, g \in H$, $t, s > 0$,

\begin{equation}
\mathbb{E}(W_f(t)W_g(s)) = \min(t, s)(f, g)_H.
\end{equation}

A cylindrical Brownian motion $W$ is often written as a generalized Fourier series

\begin{equation}
W(t) = \sum_{k \geq 1} w_k(t)h_k,
\end{equation}

where $w_k = W_{h_k}$. The corresponding space-time white noise is written as

\begin{equation}
\dot{W}(t) = \sum_{k \geq 1} \dot{w}_k(t)h_k.
\end{equation}

\section{Diagonalizable Stochastic Hyperbolic Equations}

We start by introducing the following objects:

(1) $H$, a separable Hilbert space with an orthonormal basis $\{h_k, k \geq 1\}$;

(2) $X$, a separable Hilbert space such that $H$ is densely and continuously embedded into $X$ and

\begin{equation}
\sum_{k \geq 1} \|h_k\|_X^2 < \infty
\end{equation}

(in other words, the embedding operator from $H$ to $X$ is Hilbert-Schmidt);

(3) $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{B}_0$, $\mathcal{B}_1$, linear operators on $H$;

(4) $\Theta_1$, $\Theta_2$, two compact sets in $\mathbb{R}$;

(5) $\theta_1$, $\theta_2$, two real numbers, $\theta_1 \in \Theta_1$, $\theta_2 \in \Theta_2$;
(6) \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 1}, \mathbb{P})\), a stochastic basis with the usual assumptions and a countable collection of independent standard Brownian motions \(\{w_k = w_k(t), k \geq 1\}\).

In this setting, a cylindrical Brownian motion \(W = W(t)\) on \(H\) is a continuous \(X\)-valued Gaussian process with representation
\[
W(t) = \sum_{k \geq 1} h_k w_k(t).
\]
The process \(W\) indeed has values in \(X\) rather than \(H\) because
\[
\mathbb{E}\|W(t)\|^2_X = t \sum_{k \geq 1} \|h_k\|^2_X < \infty.
\]

For fixed non-random \(T > 0\), consider the second-order stochastic evolution equation
\[
u_{tt}(t) + (A_0 + \theta_1 A_1)u(t) = (B_0 + \theta_2 B_1)u_t(t) + \dot{W}(t), \quad 0 < t \leq T,
\]
with zero initial conditions \(u(0) = u_t(0) = 0\).

**Definition 2.1.** Equation (2.3) is called **diagonalizable** if the operators \(A_0, A_1, B_0, \) and \(B_1\) have a common system of eigenfunctions \(\{h_k, k \geq 1\}\).

We will refer to \(A = A_0 + \theta_1 A_1\) and \(B = B_0 + \theta_2 B_1\) as the evolution and dissipation operators, respectively, and use notations (1.4) for the eigenvalues of the operators \(A_i, B_i\). Hyperbolicity of the equation means that the evolution operator is bounded from below and dominates, in some sense, the dissipation operator. More precisely, we have

**Definition 2.2.** A diagonalizable equation (2.3) is called **hyperbolic** on the time interval \([0, T]\) if

1. there exist positive numbers \(C^*, c_1, c_2\) such that \(\{\kappa_k + \theta \tau_k + C^*, k \geq 1\}\) is a positive, non-decreasing, and unbounded sequence for all \(\theta \in \Theta_1\) and
\[
c_1 \leq \frac{\kappa_k + \theta \tau_k + C^*}{\kappa_k + \theta' \tau_k + C^*} \leq c_2
\]
for all \(\theta, \theta' \in \Theta_1\);

2. there exist positive numbers \(C, J\) such that, for all \(k \geq J\) and all \(\theta_1 \in \Theta_1, \theta_2 \in \Theta_2\),
\[
T(\rho_k + \theta_2 \nu_k) \leq \ln(\kappa_k + \theta_1 \tau_k) + C.
\]

Condition (2.5) means that there is no restriction on the strength of dissipation, but amplification must be weak. For example, let \(\Delta\) be the Laplace operator in a smooth bounded domain \(G \subset \mathbb{R}^d\) with zero boundary conditions, and \(H = L_2(G)\). Then each of the following equations is diagonalizable and hyperbolic on \([0, T]\) for all \(T > 0\):

\[
u_{tt} = \Delta u + u_t + \dot{W}, \quad \nu_{tt} = \Delta u - u_t + \dot{W},
\]
\[
u_{tt} = \Delta (u + u_t) + \dot{W}, \quad \nu_{tt} = \Delta u - \Delta^2 u_t + \dot{W},
\]
while equations
\[
u_{tt} = \Delta (u - u_t) + \dot{W}, \quad \nu_{tt} = \Delta u + \Delta^2 u_t + \dot{W}.
are diagonalizable but not hyperbolic on any \([0, T]\). To construct an example of an equation that is hyperbolic on every time interval \([0, T]\) and has unbounded amplification, take \(\theta_1 = \theta_2 = 1\) and consider the operators with eigenvalues \(\kappa_k = \rho_k = 0\), \(\tau_k = e^k\), \(\nu_k = \ln k\).

The following result shows that, in a hyperbolic equation, the evolution operator is uniformly bounded from below.

**Proposition 2.3.** If equation (2.3) is diagonalizable and hyperbolic, then

\[
\lim_{k \to \infty} (\kappa_k + \theta \tau_k) = +\infty
\]

uniformly in \(\theta \in \Theta_1\), and there exists an index \(J \geq 1\) and a number \(c_0\) such that, for all \(k \geq J\) and \(\theta \in \Theta_1\),

\[
\kappa_k + \theta \tau_k > 1,
\]

\[
\frac{|\tau_k|}{\kappa_k + \theta \tau_k} \leq c_0.
\]

**Proof.** To simplify the notations, define

\[
\lambda_k(\theta) = \kappa_k + \theta \tau_k.
\]

Since \(\{\lambda_k(\theta) + C^*, \ k \geq 1\}\) is a positive, non-decreasing, and unbounded sequence for all \(\theta \in \Theta_1\) and (2.4) holds, we have (2.7), and then (2.8) follows.

To prove (2.9), we argue by contradiction. Assume that the sequence \(\{\frac{|\tau_{k_j}|}{\lambda_{k_j}(\theta)}\}, j \geq 1\) is not uniformly bounded. Then there is a sequence \(\{\frac{|\tau_{k_j}|}{\lambda_{k_j}(\theta)}\}, j \geq 1\) such that

\[
\lim_{j \to \infty} \frac{|\tau_{k_j}|}{\lambda_{k_j}(\theta)} = +\infty.
\]

With no loss of generality, assume that \(\tau_{k_j} > 0\), and, since \(\Theta_1\) is compact, we also assume that \(\lim_{j \to \infty} \theta_j = \theta^o \in \Theta_1\) (if not, extract a further sub-sequence).

Then (2.10) implies

\[
\lim_{j \to \infty} \frac{\kappa_{k_j}}{\tau_{k_j}} = -\theta^o.
\]

Note that \(\lim_{j \to \infty} |\tau_{k_j}| = +\infty\), because \(\lim_{j \to \infty} (\theta^o \tau_{k_j} + \kappa_{k_j}) = +\infty\). Consequently,

\[
\lim_{j \to \infty} \frac{\lambda_{k_j}(\theta) + C^*}{\lambda_{k_j}(\theta^o) + C^*} = \frac{\theta - \theta^o}{\theta^o + \lim_{j \to \infty} (\kappa_{k_j}/\tau_{k_j})} = \infty, \quad \theta \neq \theta^o.
\]

As a result, if (2.9) fails, then so does (2.4) for \(\theta \neq \theta^o, \theta^o = \theta^o\). \(\square\)

To state the result about existence and uniqueness of solution for (2.3), note that we do not have enough information about the operators \(A_i\) and \(B_i\) to define the traditional variational solution because we are not assuming that the operators act in a normal triple of Hilbert spaces — the usual setting to define a variational solution (see, for example, Chow [2, Section 6.8]). On the other hand, if the operators \(A_i, B_i\) were bounded and if the process \(W\) were \(H\)-valued, then there would be a unique process \(v = v(t)\) with continuous trajectories in \(H\) such that \(u(t) = \int_0^t v(s)ds\) and

\[
v(t) + \int_0^t (A_0 + \theta_1 A_1)u(s)ds = \int_0^t (B_0 + \theta_2 B_1)v(s)ds + W(s);
\]
see Chow [2, Theorem 6.8.2]. If, in addition, equation (2.3) is diagonalizable, then \( u \) would have the following expansion in the basis \( \{ h_k, k \geq 1 \} \):

\[
(2.12) \quad u(t) = \sum_{k \geq 1} u_k(t) h_k,
\]

\[
(2.13) \quad \ddot{u}_k(t) - (\rho_k + \theta_2 \nu_k) \dot{u}_k(t) + (\kappa_k + \theta_1 \tau_k) u_k(t) = \dot{w}_k(t), \quad u_k(0) = \dot{u}_k(0) = 0.
\]

The basis \( \{ h_k, k \geq 1 \} \) thus becomes a natural collection of test functions.

Since the operators \( A_i, B_i \) are in general not bounded on \( H \) and the process \( W \) is not \( H \)-valued, we use the auxiliary space \( X \) and establish the following result.

**Theorem 2.4.** Assume that equation (2.3) is diagonalizable and hyperbolic. Then there is a unique adapted \( X \)-valued process \( u = u(t) \) with representation (2.12), (2.13); we call the process \( u \) a **generalized solution** of (2.3). If, in addition, there exists a real number \( C_0 \) such that \( \theta_2 \nu_k + \rho_k \leq C_0 \) for all \( k \), then \( v(t) = \dot{u}(t) \) is also an \( X \)-valued process.

Let us make a few comments about the result.

1. By Lemma 2.3, we know that \( \kappa_k + \theta_1 \tau_k > 1 \) for all sufficiently large \( k \). Condition (2.3) means certain subordination of the dissipation operator \( B_0 + \theta_2 B_1 \) to the evolution operator \( A_0 + \theta_1 A_1 \). In particular, any dissipation (negative \( \mu_k = \rho_k + \theta_2 \nu_k \)) is admissible, as well as certain unbounded amplification (positive and unbounded \( \mu_k \)), as long as the sequence \( \{ \mu_k, k \geq 1 \} \) does not grow too fast; the critical growth rate depends on the length of the time interval. This possibility to have an **unbounded dissipation operator** makes the result different from those considered in the literature, such as [2, Theorem 6.8.4].

2. The resulting generalized solution is **weak in the PDE sense, but is strong in the probabilistic sense**, being constructed on a given stochastic basis;

3. The solution is defined by its Fourier coefficients and therefore does not depend on the choice of the space \( X \). The role of \( X \) is to ensure that the equation is **well-posed** in the sense that the output process (the solution \( u \)) takes values in the same space as the “input” process \( W \). Given the special form of \( W \), we are not discussing any **continuous dependence** of \( u \) on \( W \).

**Proof of Theorem 2.4** To simplify the presentation, introduce the notations

\[
(2.14) \quad \lambda_k = \kappa_k + \theta_1 \tau_k, \quad \mu_k = \rho_k + \theta_2 \nu_k.
\]

For a fixed \( k \geq 1 \), let us consider the process \( u_k \) defined by (2.13). Equation (2.13) has a unique solution, and direct computations show that

\[
(2.15) \quad u_k(t) = \int_0^t \dot{f}_k(t - s) dw_k(s),
\]

where the **fundamental solution** \( f_k \) satisfies

\[
(2.16) \quad \ddot{f}_k(t) - \mu_k \dot{f}_k(t) + \lambda_k f_k(t) = 0, \quad f_k(0) = 0, \quad \dot{f}_k(0) = 1;
\]

see Appendix for details. Thus, \( \mathbb{E} u_k(t) = 0 \) and, since the processes \( u_k \) are independent for different \( k \), the series (2.12) defines an \( X \)-valued process if

\[
(2.17) \quad \sup_{k \geq 1} \sup_{t \in [0, T]} \mathbb{E} |u_k(t)|^2 < \infty.
\]
By direct computation using (2.15) and the Itô isometry,
\begin{equation}
E|u_k(t)|^2 = \int_0^t f_k^2(t-s)ds = \int_0^t f_k^2(s)ds.
\end{equation}
The proof of the theorem is thus reduced to the study of the fundamental solution $f_k$ for sufficiently large $k$. More precisely, we will show that
\begin{equation}
\sup_{t \in [0,T]} \sup_k f_k^2(t) < \infty,
\end{equation}
which, by (2.18), implies (2.17).

The solution of equation (2.16) is determined by the roots $r_\pm$ of the characteristic equation
\begin{equation}
r^2 - \mu kr + \lambda_k = 0: \quad r_\pm = \frac{\mu_k \pm \sqrt{\mu_k^2 - 4\lambda_k}}{2}.
\end{equation}
By Lemma 2.3, $\lim_{k \to \infty} \lambda_k = +\infty$, and, in particular, $\lambda_k > 0$ for all sufficiently large $k$. Also, condition (2.5) means that if $\mu_k > 0$, then $\mu_k \leq (\ln \lambda_k + C)/T$, and therefore $\mu_k < 2\sqrt{\lambda_k}$ for all sufficiently large $k$. Accordingly, we assume that $\lambda_k > 0$ and consider two cases: $|\mu_k| < 2\sqrt{\lambda_k}$ and $\mu_k \leq -2\sqrt{\lambda_k}$.

If $|\mu_k| < 2\sqrt{\lambda_k}$, then equation (2.20) has complex conjugate roots, and, with $\ell_k = \sqrt{\lambda_k - |\mu_k|^2/4},$
\begin{equation}
f_k^2(t) = t^2 e^{\mu_k t} \left( \frac{\sin(\ell_k t)}{\ell_k t} \right)^2.
\end{equation}
If $\mu_k \leq 0$, then $f_k^2 \leq T^2$ for all $t \in [0,T]$ and (2.19) follows. If $\mu_k > 0$, then, for sufficiently large $k$, condition (2.5) ensures that $e^{\mu_k t} \leq \lambda_k e^{C}$ and $\lambda_k/\ell_k^2 < 2$. Then $f_k^2 \leq 2T^2 e^{C}$ and (2.19) follows.

If $\mu_k \leq -2\sqrt{\lambda_k}$, then (2.20) has real roots (a double root if $\mu_k = -2\sqrt{\lambda_k}$), and, using the notations $\ell_k = \sqrt{\mu_k^2 - 4\lambda_k}, a = \mu_k + \ell_k,$
\begin{equation}
f_k^2(t) = t^2 e^{at} \left( \frac{1 - e^{-\ell_k t}}{\ell_k t} \right)^2;
\end{equation}
the case of the double root corresponds to the limit $\ell_k \to 0$. By assumption, $a \leq 0$, so that $f_k^2(t) \leq T^2$ and (2.19) follows.

Similarly, $v(t) = \sum_k v_k(t)h_k, v_k(t) = \int_0^t f_k(t-s)dw_k(s)$, and $E|v_k^2(t)| = \int_0^t |f_k(s)|^2ds$. By direct computation, if $\mu_k \leq C_0$, then $\sup_{t \in [0,T]} \sup_k |f_k(t)|^2 < \infty$, and therefore $v(t) \in X$.

This completes the proof of Theorem 2.4. \hfill \Box

3. Estimation of Parameters

Assume that the solution of equation (2.3) is observed so that the measurements of $u_k(t)$ and $v_k(t) = \dot{u}_k$ are available for all $t \in [0,T]$ and $k = 1, \ldots, N$. The objective is to estimate the parameters $\theta_1, \theta_2$. We keep notations (1.4), and also define
\begin{equation}
\lambda_k(\theta) = \kappa_k + \theta \tau_k, \quad \mu_k(\theta) = \rho_k + \theta \nu_k.
\end{equation}
Since
\begin{equation}
(3.2) \quad dv_k(t) = (-\lambda_k(\theta_1)u_k + \mu_k(\theta_2)v_k)dt + dw_k(t)
\end{equation}
(see (2.13)), and \( u_k(t) = \int_0^t v_k(s)ds \), the vector process \( \mathbf{v} = (v_1, \ldots, v_N) \) is a diffusion-type process in the sense of Liptser and Shiryaev; see [8, Definition 4.2.7]. Therefore, by Theorem 7.6 in [8] (see also Section 7.2.7 of the same reference), the measure \( \mathbf{P}^\mathbf{v} \) generated by the process \( \mathbf{v} \) in the space of \( \mathbb{R}^N \)-valued continuous functions on \([0,T]\) is absolutely continuous with respect to the measure \( \mathbf{P}^\mathbf{w} \), generated in the same space by the \( N \)-dimensional standard Brownian motion \( \mathbf{w} = (w_1, \ldots, w_N) \). Moreover, the density \( Z = d\mathbf{P}^\mathbf{v}/d\mathbf{P}^\mathbf{w} \) has a representation
\[
Z(\mathbf{v}) = \exp \left( \sum_{k=1}^N \left( \int_0^T (-\lambda_k(\theta_1)u_k(t) + \mu_k(\theta_2)v_k(t))dv_k(t) \right) - \frac{1}{2} \int_0^T \left( -\lambda_k(\theta_1)u_k(t) + \mu_k(\theta_2)v_k(t) \right)^2 dt \right).
\]
Define
\[
\mathfrak{h} = \ln Z(\mathbf{v}).
\]
Note that \( \mathfrak{h} \) is a function of \( \theta_1, \theta_2 \), and the maximum likelihood estimator of the parameters \( \theta_1, \theta_2 \) is computed by solving the system of equations
\begin{equation}
(3.3) \quad \frac{\partial \mathfrak{h}}{\partial \theta_1} = 0, \quad \frac{\partial \mathfrak{h}}{\partial \theta_2} = 0,
\end{equation}
with unknowns \( \theta_1, \theta_2 \). This system can be written as
\begin{equation}
(3.4) \quad F_{1,N} + L_{1,N} + K_{1,N} \theta_1 + K_{12,N} \theta_2 = A_{1,N}, \\
F_{2,N} + L_{2,N} + K_{12,N} \theta_1 + K_{2,N} \theta_2 = A_{2,N},
\end{equation}
where
\begin{equation}
(3.5) \quad A_{1,N} = -\sum_{k=1}^N \int_0^T \tau_k u_k(t)dv_k(t), \quad A_{2,N} = \sum_{k=1}^N \int_0^T \nu_k v_k(t)dv_k(t), \\
F_{1,N} = -\sum_{k=1}^N \int_0^T \kappa_k \tau_k u_k^2(t)dt, \quad F_{2,N} = \sum_{k=1}^N \int_0^T \rho_k \nu_k v_k^2(t)dt, \\
K_{1,N} = \sum_{k=1}^N \int_0^T \tau_k^2 u_k^2(t)dt, \quad K_{2,N} = \sum_{k=1}^N \int_0^T \nu_k^2 v_k^2(t)dt, \quad K_{12,N} = -\sum_{k=1}^N \int_0^T \nu_k \tau_k u_k(t)v_k(t)dt, \\
L_{1,N} = -\sum_{k=1}^N \int_0^T \rho_k \tau_k u_k(t)v_k(t)dt, \quad L_{2,N} = -\sum_{k=1}^N \int_0^T \kappa_k \nu_k u_k(t)v_k(t)dt.
\end{equation}
All the numbers \( A, F, L \) and \( K \) are computable from the observations of \( u_k(t) \) and \( v_k(t) \), \( k = 1, \ldots, N, t \in [0,T] \).
Note that
\[ K_{12,N} = -\frac{1}{2} \sum_{k=1}^{N} \tau_k \nu_k u_k^2(T), \quad L_{1,N} = -\frac{1}{2} \sum_{k=1}^{N} \rho_k \tau_k u_k^2(T), \quad L_{2,N} = -\frac{1}{2} \sum_{k=1}^{N} \kappa_k \nu_k u_k^2(T), \]
because, by assumption, \( u_k(0) = 0 \) and thus
\[ \int_{0}^{T} u_k v_k(t) dt = \int_{0}^{T} u_k(t) d\mu_k(t) = \frac{1}{2} u_k^2(T). \]

By the Cauchy-Schwartz inequality, \( K_{1,N} K_{2,N} - K_{12,N}^2 > 0 \) with probability one, because the process \( u_k \) is not a scalar multiple of \( v_k \). Therefore (3.4) has a unique solution
\[
\hat{\theta}_{1,N} = \frac{K_{2,N}(A_{1,N} - F_{1,N} - L_{1,N}) - K_{12,N}(A_{2,N} - L_{2,N} - F_{2,N})}{K_{1,N} K_{2,N} - K_{12,N}^2},
\]
\[
\hat{\theta}_{2,N} = \frac{K_{1,N}(A_{2,N} - F_{2,N} - L_{2,N}) - K_{12,N}(A_{1,N} - F_{1,N} - L_{1,N})}{K_{1,N} K_{2,N} - K_{12,N}^2}.
\]

With notations (3.5) in mind, formulas (3.6) provide explicit expressions for the maximum likelihood estimators of \( \theta_1 \) and \( \theta_2 \). To study asymptotic properties of these estimators, we need expressions for \( \hat{\theta}_{i,N} - \theta_i \), \( i = 1, 2 \):
\[
\hat{\theta}_{1,N} - \theta_1 = \frac{1}{1 - D_N} \left( \frac{\iota_{1,N}}{K_{1,N}} - \frac{\iota_{2,N} K_{12,N}}{K_{1,N} K_{2,N}} \right),
\]
\[
\hat{\theta}_{2,N} - \theta_2 = \frac{1}{1 - D_N} \left( \frac{\iota_{2,N}}{K_{2,N}} - \frac{\iota_{1,N} K_{12,N}}{K_{1,N} K_{2,N}} \right),
\]
where
\[
\iota_{1,N} = -\sum_{k=1}^{N} \int_{0}^{T} \tau_k u_k(t) d\mu_k(t), \quad \iota_{2,N} = \sum_{k=1}^{N} \int_{0}^{T} u_k v_k(t) d\mu_k(t), \quad D_N = \frac{K_{12,N}^2}{K_{1,N} K_{2,N}}.
\]

It follows that, as \( N \to \infty \), asymptotic behavior of the estimators is determined by \( \iota_{i,N}/K_{i,N}, \ i = 1, 2 \), and \( K_{12,N}/(K_{1,N} K_{2,N}) \). Note that each of \( \iota_{i,N}, K_{i,N}, K_{12,N} \) is a sum of independent random variables. Moreover,
\[
\mathbb{E} \iota_{i,N}^2 = \mathbb{E} K_{i,N}, \ i = 1, 2.
\]

If \( f_k \) is the function satisfying
\[
\tilde{f}_k(t) - \mu_k(\theta_2) \tilde{f}_k(t) + \lambda_k(\theta_1) f_k(t) = 0, \ f_k(0) = 0, \ \tilde{f}_k(0) = 1,
\]
then, by direct computation, \( u_k(t) = \int_{0}^{t} f_k(t-s) d\mu_k(s) \) (see Appendix for more details), so that
\[
\mathbb{E} u_k^2(t) = \int_{0}^{t} |\tilde{f}_k(s)|^2 ds, \ \mathbb{E} v_k^2(t) = \int_{0}^{t} |\tilde{f}_k(s)|^2 ds,
\]
and
\[
\Psi_{1,N} := \mathbb{E} K_{1,N} = \sum_{k=1}^{N} \tau_{k}^{2} \int_{0}^{T} \int_{0}^{t} |f_{k}(s)|^{2} ds dt,
\]
\[
(3.11)
\Psi_{2,N} := \mathbb{E} K_{2,N} = \sum_{k=1}^{N} \nu_{k}^{2} \int_{0}^{T} \int_{0}^{t} |\dot{f}_{k}(s)|^{2} ds dt,
\]
\[
\Psi_{12,N} := \mathbb{E} K_{12,N} = -\frac{1}{2} \sum_{k=1}^{N} \tau_{k} \nu_{k} \int_{0}^{T} |f_{k}(s)|^{2} ds.
\]

The following is a necessary conditions for the consistency of the estimators.

**Proposition 3.1.** If \(\lim_{N \to \infty} \hat{\theta}_{i,N} = \theta_{i}\) in probability, then \(\lim_{N \to \infty} \Psi_{1,N} = +\infty\). Similarly, if \(\lim_{N \to \infty} \hat{\theta}_{2,N} = \theta_{i}\) in probability, then \(\lim_{N \to \infty} \Psi_{2,N} = +\infty\).

**Proof.** Each of the sequences \(\{\Psi_{i,N}, N \geq 1\}\) is monotonically increasing and thus has a limit, finite or infinite. If \(\lim_{N \to \infty} \Psi_{i,N} < \infty\), then \(\lim_{N \to \infty} \nu_{i,N}/K_{i,N}\) exists with probability one and is a non-degenerate random variable. Equalities \((3.1)\) then implies that \(\hat{\theta}_{i,N}\) cannot converge to \(\theta_{i}\). \(\square\)

Under the assumptions of Theorem 2.4, we derived a bound \(|f_{k}(t)|^{2} \leq \text{const.} \cdot T^{2}\), which was enough to establish existence and uniqueness of solution of \((2.3)\). To study estimators \(\hat{\theta}_{i,N}\), and, in particular, convergence/divergence of the sequences \(\{\Psi_{i,N}, N \geq 1\}\), we need more delicate bounds on both \(|\dot{f}_{k}(t)|^{2}\) and \(|\ddot{f}_{k}(t)|^{2}\). The computations, while relatively straightforward, are rather long and lead to the following relations (see \((1.10)\) for the definition of \(\sim\)):

\[
(3.12) \quad \mathbb{E} u_{k}^{2}(T) \sim \frac{e^{\mu_{k}(\theta_{2})T} - 1}{2 \mu_{k}(\theta_{2}) \lambda_{k}(\theta_{1})}, \quad \text{Var} u_{k}^{2}(T) \sim 3 \left( \frac{e^{\mu_{k}(\theta_{2})T} - 1}{2 \mu_{k}(\theta_{2}) \lambda_{k}(\theta_{1})} \right)^{2};
\]
\[
(3.13) \quad \mathbb{E} \int_{0}^{T} u_{k}^{2}(t) dt \sim \frac{T^{2} M(T \mu_{k}(\theta_{2}))}{\lambda_{k}(\theta_{1})}, \quad \text{Var} \int_{0}^{T} u_{k}^{2}(t) dt \sim \frac{T^{4} V(T \mu_{k}(\theta_{2}))}{\lambda_{k}^{2}(\theta_{1})},
\]
\[
(3.14) \quad \mathbb{E} \int_{0}^{T} \nu_{k}^{2}(t) dt \sim T^{2} M(T \mu_{k}(\theta_{2})), \quad \text{Var} \int_{0}^{T} \nu_{k}^{2}(t) dt \sim T^{4} V(T \mu_{k}(\theta_{2})),
\]

where

\[
(3.15) \quad M(x) = \begin{cases} 
\frac{e^{x} - x - 1}{2x^{2}}, & \text{if } x \neq 0, \\
\frac{1}{4}, & \text{if } x = 0;
\end{cases}
\]
\[
(3.16) \quad V(x) = \begin{cases} 
\frac{e^{2x} + 4e^{x} - 4xe^{x} - 2x - 5}{4x^{4}}, & \text{if } x \neq 0, \\
\frac{1}{24}, & \text{if } x = 0.
\end{cases}
\]

Note that the functions \(M\) and \(V\) are continuous and positive on \(\mathbb{R}\), and

\[
(3.17) \quad M(x) \sim \begin{cases} 
(2|x|)^{-1}, & x \to -\infty, \\
2(2x)^{-2} e^{x}, & x \to +\infty;
\end{cases} \quad V(x) \sim \begin{cases} 
4(2|x|)^{-3}, & x \to -\infty, \\
4(2x)^{-4} e^{2x}, & x \to +\infty.
\end{cases}
\]
The computations leading to (3.12)–(3.14) rely on the fact that $u_k$ and $v_k$ are Gaussian processes, so that, for example,

$$\text{Var} \int_0^T u_k^2(t)dt = 4 \int_0^T \int_0^t \left( \mathbb{E}(u(t)u(s)) \right)^2 ds dt.$$ 

It follows from (3.13) and (3.14) that if $\lim_{N \to \infty} \Psi_{i,N} = +\infty$, then

$$\Psi_{1,N} \sim T^2 \sum_{k=1}^{N} \frac{\tau_k^2 M(T \mu_k(\theta_2))}{\lambda_k(\theta_1)}, \Psi_{2,N} \sim T^2 \sum_{k=1}^{N} \nu_k^2 M(T \mu_k(\theta_2)).$$

Relations (3.18) show that conditions for consistency and asymptotic normality of the estimators require additional assumptions on the asymptotical behavior of the eigenvalues of the operators $A_i, B_i$.

The asymptotic behavior of the eigenvalues of an operator is well-known when the operator is elliptic and self-adjoint. For example, let $\mathcal{D}$ be an operator defined on smooth functions by

$$\mathcal{D}f(x) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial f(x)}{\partial x_j} \right),$$

in a smooth bounded domain $G \subset \mathbb{R}^d$, with zero Dirichlet boundary conditions. Assume that the functions $a_{ij}$ are all infinitely differentiable in $G$ and are bounded with all the derivatives, and the matrix $(a_{ij}(x), i,j = 1, \ldots, d)$ is symmetric and uniformly positive-definite for all $x \in G$. Then the eigenvalues $d_k$ of $\mathcal{D}$ can be enumerated so that

$$d_k \asymp k^{2/d}$$

in the sense of notation (1.10). More generally, for a positive-definite elliptic self-adjoint differential or pseudo-differential operator $\mathcal{D}$ of order $m$ on a smooth bounded domain in $\mathbb{R}^d$ with suitable boundary conditions or on a smooth compact $d$-dimensional manifold, the asymptotic of the eigenvalues $d_k$, $k \geq 1$, is

$$d_k \asymp k^{m/d},$$

note that $m$ can be an arbitrary positive number. This result is well-known; see, for example, Safarov and Vassiliev [1, Section 1.2]. An example of $\mathcal{D}$ is $(1-\Delta)^{m/2}$, $m > 0$, where $\Delta$ is the Laplace operator; note also that, for this operator, relation (3.20) holds even when $m \leq 0$.

In our setting, when the operators are defined by their eigenvalues and eigenfunctions, more exotic eigenvalues are possible, for example, $\tau_k = e^{k}$ or $\nu_k = (-1)^{k}/k$. On the other hand, it is clear that the analysis of the estimators should be easier when all the eigenvalues in the equation are of the type (3.20). Accordingly, we make the following

**Definition 3.2.** Equation (2.3) is called algebraically hyperbolic if it is diagonalizable, hyperbolic, and the eigenvalues $\lambda_k(\theta) = \kappa_k + \theta \tau_k$, $\mu_k(\theta) = \rho_k + \theta \nu_k$ have the following properties:

1. There exist real numbers $\alpha, \alpha_1$ such that, for all $\theta \in \Theta_1$,

$$\lambda_k(\theta) \asymp k^{\alpha}, \quad |\tau_k| \asymp k^{\alpha_1},$$

where $\Theta_1$ is a suitable subset of $\mathbb{R}$.
(2) Either \( |\mu_k(\theta)| \leq C \) for all \( \theta \in \Theta_2 \) or there exist numbers \( \beta > 0, \beta_1 \in \mathbb{R} \) such that, for all \( \theta \in \Theta_2 \),
\[
-\mu_k(\theta) \approx k^\beta; |\nu_k| \approx k^{\beta_1}.
\]

To emphasize the importance of the numbers \( \alpha \) and \( \beta \), we will sometimes say that the equation is \((\alpha, \beta)\)-algebraically hyperbolic; \( \beta = 0 \) includes the case of uniformly bounded \( \mu_k(\theta) \).

The reader can easily verify that
- under hyperbolicity assumption, \( \alpha > 0 \) and no unbounded amplification is possible;
- each of the equations in (2.6) is algebraically hyperbolic.

4. Analysis of Estimators: Algebraic Case

**Theorem 4.1.** Assume that equation (2.3) is \((\alpha, \beta)\)-algebraically hyperbolic in the sense of Definition 3.2.

(1) If
\[
\alpha_1 \geq \frac{\alpha + \beta - 1}{2},
\]
then the estimator \( \hat{\theta}_{1,N} \) is strongly consistent and asymptotically normal with rate \( \sqrt{\Psi_{1,N}} \) as \( N \to \infty \):

\[
\lim_{N \to \infty} \sqrt{\Psi_{1,N}}(\hat{\theta}_{1,N} - \theta_1) = \xi_1 \text{ in distribution},
\]
where \( \xi_1 \) is a standard Gaussian random variable.

(2) If
\[
\beta_1 \geq \frac{\beta - 1}{2},
\]
then the estimator \( \hat{\theta}_{2,N} \) is strongly consistent and asymptotically normal with rate \( \sqrt{\Psi_{2,N}} \) as \( N \to \infty \):

\[
\lim_{N \to \infty} \sqrt{\Psi_{2,N}}(\hat{\theta}_{2,N} - \theta_2) = \xi_2 \text{ in distribution},
\]
where \( \xi_2 \) is a standard Gaussian random variable.

(3) If both (4.1) and (4.4) hold, then the random variables \( \xi_1, \xi_2 \) are independent.

**Remark 4.2.** (a) In terms of the orders of the operators (see (3.20)), condition (4.1) becomes
\[
\text{order}(A_1) \geq \frac{\text{order}(A_0 + \theta_1 A_1) + \text{order}(B_0 + \theta_2 B_1) - d}{2},
\]
and condition (4.4) becomes
\[
\text{order}(B_1) \geq \frac{\text{order}(B_0 + \theta_2 B_1) - d}{2}.
\]
(b) The condition for consistency of \( \hat{\theta}_{2,N} \) does not depend on the evolution operator and is similar to the consistency condition in the parabolic case [5, Theorem 2.1].

The intuition behind conditions (4.7) and (4.8) is as follows. The information about the numbers \( \theta_1, \theta_2 \) is carried by the terms \( A_1 u \) and \( B_1 u_t \), respectively, and these terms must be irregular enough to be distinguishable in the noise \( \dot{W} \) during a finite observation window \([0, T]\). The higher the orders of the operators, the more irregular these terms, the easier the estimation.

**Proof of Theorem 4.1.** Note that if \( \beta > 0 \), then \( \lim_{k \to \infty} \mu_k(\theta) = -\infty \), and therefore, by (3.17),

\[
M_k(T \mu_k(\theta)) \sim \frac{1}{2T|\mu_k(\theta)|} \preceq k^{-\beta}, \quad V_k(T \mu_k(\theta)) \sim \frac{1}{2T|\mu_k(\theta)|^3} \preceq k^{-3\beta}.
\]

Let \( \gamma_1 = 2\alpha_1 - \alpha - \beta, \quad \gamma_2 = 2\beta_1 - \beta, \quad \gamma_{12} = \alpha_1 - \alpha + \beta_1 - \beta \).

We have (see (1.9) for the definition of \( \triangleright \))

\[
\tau^2_k \mathbb{E} \int_0^T u^2_k(t) dt \triangleright \kappa^{\gamma_1}, \quad \tau^4_k \text{Var} \int_0^T u^2_k(t) dt \triangleright k^{2\gamma_1 - \beta},
\]

\[
\nu^2_k \mathbb{E} \int_0^T v^2_k(t) dt \triangleright \kappa^{\gamma_2}, \quad \nu^4_k \text{Var} \int_0^T v^2_k(t) dt \triangleright k^{2\gamma_2 - \beta},
\]

\[
|\nu_k \tau_k| \mathbb{E} u^2_k(T) \triangleright \kappa^{\gamma_{12}}, \quad \nu^2_k \tau^2_k \text{Var} u^2_k(T) \triangleright k^{2\gamma_{12}},
\]

and therefore

\[
\Psi_{1,N} \sim \begin{cases} 
\text{const.}, & \text{if } \gamma_1 < -1, \\
\ln N, & \text{if } \gamma_1 = -1, \\
N^{\gamma_1+1}, & \text{if } \gamma_1 > -1,
\end{cases}
\]

\[
\Psi_{2,N} \sim \begin{cases} 
\text{const.}, & \text{if } \gamma_2 < -1, \\
\ln N, & \text{if } \gamma_2 = -1, \\
N^{\gamma_2+1}, & \text{if } \gamma_2 > -1.
\end{cases}
\]

\[
|\Psi_{12,N}| \sim \begin{cases} 
\text{const.}, & \text{if } \gamma_{12} < -1, \\
\ln N, & \text{if } \gamma_{12} = -1, \\
N^{\gamma_{12}+1}, & \text{if } \gamma_{12} > -1.
\end{cases}
\]

Next, we show that condition (4.1) implies

\[
\lim_{N \to \infty} \frac{K_{1,N}}{\Psi_{1,N}} = 1 \quad \text{with probability one},
\]

condition (4.4) implies

\[
\lim_{N \to \infty} \frac{K_{2,N}}{\Psi_{2,N}} = 1 \quad \text{with probability one},
\]

and either (4.1) or (4.4),

\[
\lim_{N \to \infty} D_N = 0 \quad \text{with probability one}.
\]

Indeed, convergence (4.15) follows from (3.13) and (A.10), because (4.1) implies

\[
\sum_n \frac{n^{2\gamma_1 - \beta}}{\Psi_{n,1}^2} < \infty.
\]
Similarly, (4.16) follows from (3.14) and (A.10), because (4.4) implies
\[ \sum n^{2\gamma_2} \Psi_{n,2}^2 < \infty. \]

For (4.17), we first observe that \( \lim_{N \to \infty} K_{12,N}/\Psi_{12,N} \) exists with probability one. If \( \gamma_{12} < -1 \), the the limit is a \( \mathbb{P} \)-a.s finite random variable. If \( \gamma_{12} \geq -1 \), the (3.12) and (A.10) imply that limit is 1. Then direct analysis shows that
\[ \lim_{N \to \infty} \frac{\Psi_{12,N}^2}{\Psi_{1,N} \Psi_{2,N}} = 0 \]
if at least one of \( \Psi_{1,N}, \Psi_{2,N} \) is unbounded.

Next, we show that (4.1) implies
\[ \lim_{N \to \infty} \frac{\iota_{1,N}}{\Psi_{1,N}} = 0 \]
with probability one,
and
\[ \lim_{N \to \infty} \frac{\iota_{1,N}}{\sqrt{\Psi_{1,N}}} = \xi_1 \]
in distribution,
whereas (4.4) implies
\[ \lim_{N \to \infty} \frac{\iota_{2,N}}{\Psi_{2,N}} = 0 \]
with probability one,
and
\[ \lim_{N \to \infty} \frac{\iota_{2,N}}{\sqrt{\Psi_{2,N}}} = \xi_2 \]
in distribution.

Indeed, (4.18) follows from (3.13) and (A.5), because (4.1) implies that \( \sum_k k^{\gamma_1} = +\infty \).
Similarly, (4.15) follows from (3.14) and (A.5).

Both (4.19) and (4.21) follow from Corollary A.4. Together with (4.17), the same Corollary also implies independence of \( \xi_1 \) and \( \xi_2 \) if both (4.1) and (4.4) hold.

To complete the proof of the theorem it remains to show that
\[ \lim_{N \to \infty} \frac{\iota_{i,N} K_{12,N}}{K_{1,N} K_{2,N}} = 0, \ i = 1, 2, \]
with probability one,
and
\[ \lim_{N \to \infty} \frac{\sqrt{\Psi_{i,N}} \iota_{i,N} K_{12,N}}{K_{1,N} K_{2,N}} = 0, \ i = 1, 2, \]
in probability.

We leave to the interested reader to verify that (A.5) implies (4.22), and (4.17), (4.19), (4.21) imply (4.23).

This completes the proof of Theorem 4.1. \( \square \)

**Remark 4.3.** From Proposition 3.1, we see that condition (4.1) is both necessary and sufficient for consistency and asymptotic normality of estimator \( \hat{\theta}_{1,N} \). Similarly, condition (4.4) is necessary and sufficient for consistency and asymptotic normality of estimator \( \hat{\theta}_{2,N} \).
Since in the algebraic case the sum $\sum_{k=1}^{N} k^{\gamma}$ appears frequently, we introduce a special notation to describe the asymptotic of this sum as $N \to \infty$ for $\gamma \geq -1$:

\begin{equation}
\Upsilon_{N}(\gamma) = \begin{cases} 
N^{\gamma+1}, & \text{if } \gamma > -1, \\
\ln N, & \text{if } \gamma = -1.
\end{cases}
\end{equation}

With this notation, $\sum_{k=1}^{N} k^{\gamma} \asymp \Upsilon_{N}(\gamma)$, $\gamma \geq -1$.

Let us consider several examples, in which $\Delta$ is the Laplace operator in a smooth bounded domain $G$ in $\mathbb{R}^d$ with zero boundary conditions; $H = L_2(G)$. We start with these three equations:

\begin{align*}
(4.25) & \quad u_{tt} = \theta_1 \Delta u + \theta_2 u_t + \dot{W}, & \theta_1 > 0, & \theta_2 \in \mathbb{R}; \\
(4.26) & \quad u_{tt} = \Delta (\theta_1 u + \theta_2 u_t) + \dot{W}, & \theta_1 > 0, & \theta_2 > 0; \\
(4.27) & \quad u_{tt} = \theta_1 \Delta u - \theta_2 \Delta^2 u_t + \dot{W}, & \theta_1 > 0, & \theta_2 > 0.
\end{align*}

The following table summarizes the results:

| Asymptotic | Eq. (4.25) | Eq. (4.26) | Eq. (4.27) |
|------------|------------|------------|------------|
| $\Psi_{1,N}$ | $N^{\frac{2}{d}+1}$ | $N$ | $\Upsilon_{N}(-2/d)$, $d \geq 2$ |
| $\Psi_{2,N}$ | $N$ | $N^{\frac{2}{d}+1}$ | $N^{\frac{4}{d}+1}$ |

In equations (4.25)–(4.27), $\mathcal{A}_1$ and $\mathcal{B}_1$ are leading operators, that is, $\alpha = \alpha_1$ and $\beta = \beta_1$. This, in particular, ensures that the estimator $\hat{\theta}_{2,N}$ is always consistent.

Let us now consider examples when $\mathcal{A}_1$ and $\mathcal{B}_1$ are not the leading operators:

\begin{align*}
(4.28) & \quad u_{tt} = (\Delta u + \theta_1 u) + (\Delta u_t + \theta_2 u_t) + \dot{W}, & \theta_1 \in \mathbb{R}, & \theta_2 \in \mathbb{R}; \\
(4.29) & \quad u_{tt} + (\Delta^2 u + \theta_1 u) = (\theta_2 \Delta u_t - \Delta^2 u_t) + \dot{W}, & \theta_1 \in \mathbb{R}, & \theta_2 \in \mathbb{R}; \\
(4.30) & \quad u_{tt} + (\Delta^2 u + \theta_1 \Delta u) = (\theta_2 u_{tt} - \Delta^2 u_{tt}) + \dot{W}, & \theta_1 \in \mathbb{R}, & \theta_2 \in \mathbb{R}.
\end{align*}

The following table summarizes the results:

| Asymptotic | Eq. (4.28) | Eq. (4.29) | Eq. (4.30) |
|------------|------------|------------|------------|
| $\Psi_{1,N}$ | $\Upsilon_{N}(-4/d)$, $d \geq 4$ | $\Upsilon_{N}(-8/d)$, $d \geq 8$ | $\Upsilon_{N}(-4/d)$, $d \geq 4$ |
| $\Psi_{2,N}$ | $\Upsilon_{N}(-2/d)$, $d \geq 2$ | $N$ | $\Upsilon_{N}(-4/d)$, $d \geq 4$ |

As was mentioned in the Introduction, an interested reader can investigate a four-parameter estimation problem, such as

$u_{tt} + (\theta_{11} \Delta^2 u + \theta_{12} \Delta u) = (\theta_{21} u_t - \theta_{22} \Delta^2 u_t) + \dot{W}$. 
5. Analysis of Estimators: General Case

While the algebraic case, corresponding to elliptic partial differential operators, seems the most natural, we believe that a more general case, allowing eigenvalues such as \( \lambda_k \sim e^k \) or \( \mu_k \sim \ln k \), is also worth considering, not only as a mathematical curiosity, but also as an example of a model with observations coming from independent but not identical channels (see Korostelev and Yin [6]).

As the proof of Theorem 4.1 shows, the key arguments involve a suitable law of large numbers. Verification of the corresponding conditions is straightforward in the algebraic case, but is impossible in the general case unless we make additional assumptions about the eigenvalues of the operators. Indeed, as we work with weighted sums of independent random variables, we need some conditions on the weights for a law of large numbers to hold. In particular, the weights should not grow too fast: if \( \xi_k, k \geq 1 \), are iid standard Gaussian random variables, then the sequence \( \{n^{-2} \sum_{k=1}^{n} n \xi_k^2, n \geq 1\} \) converges with probability one to \( 1/2 \), but \( \{e^{-n} \sum_{k=1}^{n} e^{k} \xi_k^2, n \geq 1\} \) does not have a limit, even in probability.

Theorem A.2 in Appendix summarizes some of the laws of large numbers, and leads to the following

**Definition 5.1.** The sequence \( \{a_n, n \geq 1\} \) of positive numbers is called slowly increasing if

\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{n} a_n^2}{\left(\sum_{k=1}^{n} a_k\right)^2} = 0.
\]

The purpose of this definition is to simplify the statement of the main theorem (Theorem 5.2 below). It was not necessary in the algebraic case because the sequence \( \{n^\gamma, n \geq 1\} \) is slowly increasing if and only if \( \gamma \geq -1 \). The reason for the terminology is that the sequence \( \{e^{n^r}, n \geq 1\} \) has property (5.1) if and only if \( r < 1 \). Further discussion of (5.1), including the connections with the weak law of large numbers, is after the proof of Theorem A.2.

In general, we have to replace (4.1) with

**Condition 1.** The sequence \( \{\tau_k^2 M(T \mu_k(\theta_2))/\lambda_k(\theta_1), k \geq 1\} \) is slowly increasing, and (4.4), with

**Condition 2.** The sequence \( \{\nu_k^2 M(T \mu_k(\theta_2)), k \geq 1\} \) is slowly increasing.

**Theorem 5.2.** Assume that equation (2.3) is diagonalizable and hyperbolic.

(1) If Condition 1 holds, then

\[
\lim_{N \to \infty} \hat{\theta}_{1,N} = \theta_1 \quad \text{in probability};
\]

\[
\lim_{N \to \infty} \sqrt{\Psi_{1,N}}(\hat{\theta}_{1,N} - \theta_1) = \xi_1 \quad \text{in distribution},
\]

where \( \xi_1 \) is a standard Gaussian random variable.
If Condition 2 holds then

\[
\lim_{N \to \infty} \hat{\theta}_{2,N} = \theta_2 \text{ in probability;}
\]

\[
\lim_{N \to \infty} \sqrt{\Psi_{2,N}} \left( \hat{\theta}_{2,N} - \theta_2 \right) = \xi_2 \text{ in distribution,}
\]

where \( \xi_2 \) is a standard Gaussian random variable.

If both Conditions 1 and 2 hold, then the random variables \( \xi_1, \xi_2 \) are independent.

Proof. The main steps are the same as in the algebraic case (Theorem 4.1). In particular, (4.18) and (4.20) continue to hold as long as \( \Psi_{1,N} \to \infty \) and \( \Psi_{2,N} \to \infty \), respectively. The only difference is that Conditions 1 and 2 do not provide enough information about the almost sure behavior of \( K_{12,N}/\mathbb{E}K_{12,N} \), and, in this general setting, there is no natural condition that would do that. As a result, in (4.17), the convergence is in probability rather than with probability one, and then, in both (4.15) and (4.16), convergence in probability will suffice. Conditions 1 and 2 ensure (4.15) and (4.16), respectively, but with convergence in probability rather than almost sure. This is a direct consequence of the weak law of large numbers.

In the case of (4.17), we have

\[
\mathbb{E} |K_{12,N}| \leq \sum_{k=1}^{N} |\tau_k\nu_k| \mathbb{E} u_k^2(T)
\]

and, for all sufficiently large \( k \),

\[
\mathbb{E} u_k^2(T) \leq \frac{4T}{\lambda_k(\theta_1)} M(T\mu_k(\theta_2)) \left( 1 + \max \left( 0, T\mu_k(\theta_2) \right) \right),
\]

because \( xe^x - x \leq 4(e^x - x - 1)(1 + \max(0, x)) \) for all \( x \in \mathbb{R} \). Then

\[
\lim_{N \to \infty} \frac{\mathbb{E} |K_{12,N}|}{\sqrt{\Psi_{1,N} \Psi_{2,N}}} = 0.
\]

Indeed, under Condition 1, (5.6) follows from

\[
\frac{(\mathbb{E} |K_{12,N}|)^2}{\Psi_{1,N} \Psi_{2,N}} \leq \frac{16 \sum_{k=1}^{N} \tau_k^2 \mathbb{E} M(T\mu_k(\theta_2)) \left( 1 + \max \left( 0, T\mu_k(\theta_2) \right) \right)^2}{\lambda_k(\theta_1)}
\]

(Cauchy-Schwartz inequality) and

\[
\lim_{k \to \infty} \frac{\left( 1 + \max \left( 0, T\mu_k(\theta_2) \right) \right)^2}{\lambda_k(\theta_1)} = 0
\]

(hyperbolicity condition), while, under Condition 2, (5.6) follows from

\[
\frac{(\mathbb{E} |K_{12,N}|)^2}{\Psi_{1,N} \Psi_{2,N}} \leq \frac{16 \sum_{k=1}^{N} \nu_k^2 M(T\mu_k(\theta_2)) \left( 1 + \max \left( 0, T\mu_k(\theta_2) \right) \right)^2}{\lambda_k(\theta_1)}
\]

(Cauchy-Schwartz inequality with a different arrangement of terms) and (5.7).

The interested reader can fill in the details in the rest of the proof. □
As an example, consider the operators with eigenvalues \( \kappa_k = e^{2k}, \tau_k = e^k, \rho_k = 0, \nu_k = \ln \ln(k + 3) \) and assume that \( \theta_1 > 0, \theta_2 > 0 \). Then
\[
\lambda_k = e^{2k} + \theta_1 e^k, \quad \mu_k = \theta_2 \ln(k + 3),
\]
so that \( \tau_k^2 / \lambda_k \sim 1 \). Next, for all sufficiently large \( k \),
\[
(\ln(k + 3))^{T \theta_2 / 2} < M(T \mu_k) < (\ln(k + 3))^{T \theta_2},
\]
and also \( \nu_k^2 M(T \mu_k) \sim (\ln(k + 3))^{T \theta_2} \). Using integral comparison, we conclude that, for all \( r > 0 \),
\[
\sum_{k=1}^{N} (\ln k)^r \sim N(\ln N)^r.
\]
Thus, both Condition 1 and Condition 2 hold. By Theorem 5.2, both \( \hat{\theta}_{1,N} \) and \( \hat{\theta}_{2,N} \) are consistent and asymptotically normal. An interested reader can verify that \( \Psi_{1,N} \sim N(\ln N)^{T \theta_2}, \Psi_{2,N} \sim N(\ln N)^{T \theta_2} \).

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Appendix

First, let us recall some basic facts about second-order stochastic ordinary differential equations with constant coefficients. Consider the initial value problem
\[
\ddot{y}(t) - 2b \dot{y}(t) + a^2 y(t) = 0, \quad y(0) = 0, \quad \dot{y}(0) = 1.
\]
With \( 2b = \mu_k(\theta_2) \) and \( a^2 = \lambda_k(\theta_1) \), we recover (2.16); recall that \( \lambda_k(\theta_1) > 0 \) for all sufficiently large \( k \). If \( \ell = \sqrt{|b^2 - a^2|} \), then
\[
y(t) = \begin{cases}
\frac{\sin(\ell t)}{\ell} e^{bt}, & a^2 > b^2; \\
\frac{te^{bt}}{\ell}, & a^2 = b^2; \\
\frac{\sinh(\ell t)}{\ell} e^{bt}, & a^2 < b^2;
\end{cases}
\]
as usual, \( \sinh x = (e^x - e^{-x})/2 \). Note that if \( b < 0 \) and \( b^2 > a^2 \), then \( b + \ell < 0 \). The solution of the inhomogeneous equation
\[
\ddot{x}(t) - 2b \dot{x}(t) + a^2 x(t) = f(t), \quad x(0) = \dot{x}(0) = 0
\]
is then \( x(t) = \int_0^t y(t-s) f(s) ds \).

Next, we formulate the laws of large numbers and the central limit theorem used in the proof of consistency and asymptotic normality of the estimators.

To begin, let us recall Kolmogorov’s strong law of large numbers.
Theorem A.1. Let \( \{\xi_k, k \geq 1\} \) be a sequence of independent random variables with \( E\xi_n^2 < \infty \). If \( \{b_n \geq 1\} \) is an unbounded increasing sequence of real numbers \( (b_n \nearrow +\infty) \) and
\[ \sum_{n \geq 1} b_n^{-2} \text{Var}(\xi_n) < \infty, \]
then
\[ \lim_{n \to \infty} b_n^{-1} \sum_{k=1}^{n} (\xi_k - E\xi_k) = 0 \]
with probability one.

Proof. See, for example, Shiryaev [12, Theorem IV.3.2]. \( \square \)

The following laws of large numbers, both strong and weak, are often used in the current paper.

Theorem A.2 (Several Laws of Large Numbers). Let \( \chi_k, k \geq 1, \) be independent random variables, each with zero mean and positive finite variance. If
\[ \sum_{k \geq 1} E\chi_k^2 = +\infty, \]
then
\[ \lim_{N \to \infty} \frac{\sum_{k=1}^{N} \chi_k}{\sum_{k=1}^{N} E\chi_k^2} = 0 \]
with probability one.

Next, assume in addition that
\[ E\chi_k^4 \leq c_1 \left( E\chi_k^2 \right)^2 \]
for all \( k \geq 1, \) with \( c_1 > 0 \) independent of \( k. \) Then
\[ \lim_{n \to \infty} \frac{\sum_{k=1}^{n} \left( E\chi_k^2 \right)^2}{\left( \sum_{k=1}^{n} E\chi_k^2 \right)^2} = 0 \]
implies
\[ \lim_{N \to \infty} \frac{\sum_{k=1}^{N} \chi_k^2}{\sum_{k=1}^{N} E\chi_k^2} = 1 \]
in probability
and
\[ \sum_{n \geq 1} \left( \frac{E\chi_n^2}{\left( \sum_{k=1}^{n} E\chi_k^2 \right)^2} \right) < \infty, \]
implies
\[ \lim_{N \to \infty} \frac{\sum_{k=1}^{N} \chi_k^2}{\sum_{k=1}^{N} E\chi_k^2} = 1 \]
with probability one.
Proof. To prove (A.5), we take \( \xi_n = \chi_n \) and \( b_n = \sum_{k=1}^{n} \mathbb{E} \chi_k^2 \) and apply Theorem A.1; note that convergence of \( \sum_{n} b_n^{-2} \mathbb{E} \chi_n^2 \) follows from divergence of \( \sum_{k \geq 1} \mathbb{E} \chi_k^2 \):\[
\sum_{n} \frac{\mathbb{E} \chi_n^2}{b_n^2} \leq \sum_{n} \frac{1}{b_{n-1}} - \frac{1}{b_n}.
\]
To prove (A.10), we take \( \xi_n = \chi_n^2 \) and \( b_n = \sum_{k=1}^{n} \mathbb{E} \chi_k^2 \), and again apply Theorem A.1; this time, we have to assume convergence of the series \( \sum_{n} \text{Var} \xi_n b_n^{-2} \). Finally, (A.8) follows from (A.7) and Chebyshev’s inequality. \( \square \)

In other words, normalizing a sum of zero-mean random variables by the total variance will give in the limit zero with probability one as long as the total variance is unbounded, while normalizing a sum of positive random variables by the total mean will give in the limit one only under some additional assumptions. Given a collection of iid standard normal random variables \( \{\xi_k, k \geq 1\} \), an interested reader can verify that the sequence \( \left( \sum_{k=1}^{n} e^k \xi_k^2 \right) / \left( \sum_{k=1}^{n} e^k \right) \) does not converge in probability as \( n \to \infty \).

To understand the meaning of conditions (A.7) and (A.9), note that if \( \xi_k, k \geq 1 \), are iid non-negative random variables with \( \mathbb{E} \xi_1 = A > 0 \), then, taking in Theorem A.1 \( b_n = \sum_{k=1}^{n} \mathbb{E} \xi_k \), we recover the classical strong law of large numbers:
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \xi_k = A
\]
with probability one. In the second part of Theorem A.2, we want to establish a similar result when the random variables \( \xi_k \) are positive and independent, but not identically distributed. Condition (A.6) (which holds, for example, for Gaussian random variables) allows us to apply Theorem A.1 with \( b_n = \sum_{k=1}^{N} \mathbb{E} \xi_k \). If \( a_k := \mathbb{E} \xi_k > 0 \) for all \( k \), then conditions (A.6) and (A.9) become, respectively,
\[
\sum_{n \geq 1} a_n = +\infty,
\]
\[
\sum_{n \geq 1} \frac{a_n^2}{(\sum_{k=1}^{n} a_k)^2} < \infty.
\]
On the other hand, if
\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{n} a_k^2}{(\sum_{k=1}^{n} a_k)^2} = 0,
\]
then Chebyshev’s inequality leads to a weak law of large numbers.

In general, (A.11) does not imply (A.12) or (A.13) (take \( a_n = e^n \)), nor does (A.12) imply (A.11) (take \( a_n = 1 / n^2 \)), but obviously (A.13) implies (A.11). An interested reader can also verify that the sequence \( \{e^{n^2}, n \geq 1\} \) satisfies (A.13) but not (A.12). On the other hand, since we use (A.11) and (A.12) to prove a strong law of large numbers, and use (A.13) to prove a weak law of large numbers, it will be natural to expect that conditions (A.11) and (A.12) together are stronger than (A.13). Kronecker’s Lemma (see [12, Lemma IV.3.2]) with \( b_n = (\sum_{k=1}^{n} a_n)^2, x_n = a_n^2 / b_n \) shows that this is indeed the case: (A.11) and (A.12) imply (A.13).
We say that a sequence of positive numbers \( \{a_n, n \geq 1\} \) is \textbf{slowly increasing} if condition \((A.13)\) holds. The notion of a slowly increasing sequence simplifies the conditions for consistency and asymptotic normality of the estimators in the general (non-algebraic) setting. Related conditions in the context of the law of large numbers can be found, for example, in the paper [1]. If \( a_n = n^\gamma, \gamma \in \mathbb{R} \) (algebraic case), then \((A.11)\) (that is, \( \gamma \geq -1 \)) implies \((A.12)\), which is the reason for the strong consistency in Theorem 4.1.

The following theorem is used to prove asymptotic normality of the estimators.

\[ \textbf{Theorem A.3} \] \textbf{(A Martingale Central Limit Theorem).} Let \( M_{i,n} = M_{i,n}(t), t \geq 0, \ n \geq 1, \ i = 1, 2, \) be two sequences of continuous square–integrable martingales. If, for some \( T > 0 \),

\[ \lim_{n \to \infty} \frac{\langle M_{i,n}(T) \rangle}{\langle M_n(T) \rangle} = 1, \ i = 1, 2, \quad \text{in probability}, \]

and

\[ \lim_{n \to \infty} \frac{\langle M_{1,n}, M_{2,n}(T) \rangle}{\left( \langle M_{1,n}(T) \rangle \right)^{1/2} \left( \langle M_{2,n}(T) \rangle \right)^{1/2}} = 0 \quad \text{in probability}, \]

then

\[ \lim_{n \to \infty} \left( \begin{array}{c} M_{1,n}(T) \left( \langle M_{1,n}(T) \rangle \right)^{-1/2} \\ M_{2,n}(T) \left( \langle M_{2,n}(T) \rangle \right)^{-1/2} \end{array} \right) = \mathcal{N}(0, I) \quad \text{in distribution}, \]

where \( \mathcal{N}(0, I) \) is a two-dimensional vector whose components are independent standard Gaussian random variables.

\[ \textbf{Proof.} \] If \( X_n \) and \( X \) are continuous square-integrable martingales with values in \( \mathbb{R}^d \) such that \( X \) is a Gaussian process and \( \lim_{n \to \infty} \langle X_n(T) \rangle = \langle X(T) \rangle \) in probability, then \( \lim_{n \to \infty} X_n(T) = X(T) \) in distribution; recall that, for a vector-valued martingale \( X = (X^{(1)}, \ldots, X^{(d)}), \) \( \langle X(t) \rangle \) is the symmetric matrix with entries \( \langle X^{(i)}(t), X^{(j)}(t) \rangle \). This is one of the central limit theorems for martingales; see, for example, Lipster and Shiryaev [7, Theorem 5.5.11]. The result now follows if we take

\[ X_n(t) = \left( \begin{array}{c} X_{1,n} \\ X_{2,n} \end{array} \right), \quad X(t) = \left( \begin{array}{c} w_1(t)/\sqrt{T} \\ w_2(t)/\sqrt{T} \end{array} \right), \]

where

\[ X_{i,n} = \frac{M_{i,n}(t)}{\left( \langle M_{i,n}(T) \rangle \right)^{1/2}}, \ i = 1, 2, \]

and \( w_1, w_2 \) are independent standard Brownian motions. \( \square \)

\[ \textbf{Corollary A.4.} \] Let \( f_{i,k} = f_{i,k}(t), t \geq 0, \ i = 1, 2, \ k \geq 1 \) be continuous, square-integrable processes and \( w_k = w_k(t) \) be independent standard Brownian motions. Define

\[ \eta_{i,N} = \frac{\sum_{k=1}^{N} \int_{0}^{T} f_{i,k}(t) dw_k(t)}{\left( \sum_{k=1}^{N} \mathbb{E} \int_{0}^{T} f_{i,k}^2(t) dt \right)^{1/2}}, \ i = 1, 2. \]
If
\[
\lim_{N \to \infty} \frac{\sum_{k=1}^{N} f_1^T f_2^2 (t) dt}{\sum_{k=1}^{N} E \int_0^T f_2^2 (t) dt} = 1 \quad \text{in probability, and}
\]
\[
\lim_{N \to \infty} \frac{\sum_{k=1}^{N} E \left| \int_0^T f_1 f_2 (t) f_2 (t) dt \right|}{\left( \sum_{k=1}^{N} E \int_0^T f_2^2 (t) dt \right)^{1/2}} = 0,
\]
then
\[
\lim_{N \to \infty} \begin{pmatrix} \eta_{1,N} \\ \eta_{2,N} \end{pmatrix} = N(0, I) \quad \text{in distribution,}
\]
where \(N(0, I)\) is a two-dimensional vector whose components are independent standard Gaussian random variables.

Proof. This follows from Theorem \[A.3\] by taking
\[
M_{i,n}(t) = \frac{\sum_{k=1}^{n} \int_0^t f_{i,k} (s) dw_k (s)}{\left( \sum_{k=1}^{N} E \int_0^T f_2^2 (t) dt \right)^{1/2}},
\]
because \(E(M_{i,n}) (T) = 1\) and
\[
\langle M_{1,n}, M_{2,n} \rangle (T) = \frac{\sum_{k=1}^{N} f_1^T f_1 f_2 (t) dt}{\left( \sum_{k=1}^{N} E \int_0^T f_2^2 (t) dt \right)^{1/2}}.
\]

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