On multi-solitons for the energy-critical wave equation in dimension 5

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Abstract
We construct \( K \)-solitons of the focusing energy-critical nonlinear wave equation in five-dimensional space, i.e. solutions \( u \) of the equation such that
\[
\|\nabla_{t,x} u(t) - \nabla_{t,x} \left( \sum_{k=1}^{K} W_k(t) \right) \|_{L^2} \to 0 \quad \text{as} \ t \to \infty,
\]
where \( K \geq 2 \) and for any \( k \in \{1, \ldots, K\} \), \( W_k \) is the Lorentz transform of the explicit standing soliton \( W(x) = (1 + |x|^2/15)^{-3/2} \), with any speed \( \ell_k \in \mathbb{R}^5 \), \( |\ell_k| < 1 \) satisfying \( \ell_{k'} \neq \ell_k \) for \( k' \neq k \), and an explicit smallness condition. The proof extends the refined method of construction of asymptotic multi-solitons from Martel and Merle (2016 Arch. Ration. Mech. Anal. 222 1113–60; Martel and Merle 2018 Inventiones Math. 214 1267–363).

Keywords: multi-solitons, wave equation, nonlinear model
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1. Introduction
1.1. Statement of the main result
In this paper, we are interested in the construction of multi-soliton solutions for the focusing energy-critical wave equation in five-dimensional space:
\[
\begin{cases}
\partial^2_t u - \Delta u - |u|^4 u = 0, & (t,x) \in [0, \infty) \times \mathbb{R}^5, \\
u|_{t=0} = u_0 \in H^1, & \partial_t u|_{t=0} = u_1 \in L^2.
\end{cases}
\]

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Recall that the Cauchy problem for equation (1.1) is locally well-posed in the energy space $H^1 \times L^2$, using suitable Strichartz estimates. See e.g. [8] and references therein. In particular, for any initial data $(u_0, u_1) \in H^1 \times L^2$, there exists a maximal interval of existence $(T_-, T_+)$, $-\infty \leq T_- < T_+ \leq +\infty$, and a unique solution $(u, \partial_t u) \in C((T_-, T_+); H^1 \times L^2) \cap L^1_{loc}((T_-, T_+); L^2(R^3))$. For $H^1 \times L^2$ solution, the energy $E(u(t), \partial_t u(t))$ and momentum $M(u(t), \partial_t u(t))$ are conserved, where

$$E(u, v) = \frac{1}{2} \int \nabla u^2 + \frac{1}{2} \int |\nabla v|^2 - \frac{3}{10} \int |u|^4, \quad M(u, v) = \int v \Delta u.$$

For a function $u : \mathbb{R}^3 \to \mathbb{R}$ and $\lambda > 0$, we denote

$$u_{\lambda}(x) = \frac{1}{\lambda^2} u\left(\frac{x}{\lambda}\right).$$

A change of variables shows that

$$E((u_0)_\lambda, \lambda^{-1}(u_1)_\lambda) = E(u_0, u_1).$$

Equation (1.1) is called energy-critical since it is invariant under the same scaling: if $(u, \partial_t u)$ is a solution of (1.1) and $\lambda > 0$, then

$$(t, x) \to \left(\frac{1}{\lambda^2} u \left(\frac{t}{\lambda}, \frac{x}{\lambda}\right), \frac{1}{\lambda^2} \partial_t u \left(\frac{t}{\lambda}, \frac{x}{\lambda}\right)\right)$$

is also a solution with initial data $((u_0)_\lambda, \lambda^{-1}(u_1)_\lambda)$ at time $t = 0$.

Recall that the function $W$ defined by

$$W(x) = \left(1 + \frac{|x|^2}{15}\right)^{-\frac{1}{2}}, \quad \Delta W + W^2 = 0, \quad x \in \mathbb{R}^3,$$

is a stationary solution of (1.1), called here ground state, or soliton. By scaling, translation invariances and change of sign, we obtain a family of stationary solutions of (1.1) defined by $W_{\lambda, x_0, \pm}(x) = \pm \lambda^{-\frac{2}{3}} W(\lambda^{-1}(x - x_0))$, where $\lambda > 0$ and $x_0 \in \mathbb{R}^3$.

Using the Lorentz transformation, we obtain traveling waves. For $\ell \in \mathbb{R}^3$, with $|\ell| < 1$, let

$$W_\ell(x) = W \left(\frac{1}{\sqrt{1 - |\ell|^2}} - 1 \right) \frac{\ell \cdot x}{|\ell|^2} + x;$$

then $u(t, x) = \pm W_\ell(x - \ell t)$ is solution of (1.1).

In this paper, we prove the existence of solutions of (1.1) which display non trivial asymptotic behavior in the nonradial case. Indeed, multi-solitons are canonical objects behaving exactly as the sum of decoupled traveling solitons as $t \to +\infty$.

**Theorem 1.1 (Existence of multi-solitons).** Let $K \geq 2$. For $k \in \{1, \ldots, K\}$, let $\lambda^\infty_k > 0, y_k^\infty \in \mathbb{R}^3, c_k \in \{-1\}$, $\ell_k = \sum_{i=1}^5 \ell_{k,i} e_i$, with $|\ell_k| < 1$. Assume that $\ell_k \neq \ell_{k'}$ for $k \neq k'$, and

$$\left(\sum_{i=1}^5 \max_{k \in \{1, \ldots, K\}} |\ell_{k,i}|^2\right)^{\frac{1}{2}} < \frac{3}{5}. \quad (1.4)$$
Then, there exist $T_0 > 0$ and a solution $u$ of (1.1) on $[T_0, +\infty)$ in the energy space such that

$$
\lim_{t \to +\infty} \left\| u(t) - \sum_{k=1}^{K} \frac{e_k}{(\lambda_k^{\infty})^{3/2}} W_{\ell_k} \left( \frac{-\ell_k t - y_k^{\infty}}{\lambda_k^{\infty}} \right) \right\|_{H^1} = 0, \quad (1.5)
$$

$$
\lim_{t \to +\infty} \left\| \partial_t u(t) + \sum_{k=1}^{K} \frac{e_k}{(\lambda_k^{\infty})^{5/2}} (\ell_k \cdot \nabla W_{\ell_k}) \left( \frac{-\ell_k t - y_k^{\infty}}{\lambda_k^{\infty}} \right) \right\|_{L^2} = 0. \quad (1.6)
$$

The construction of multi-solitons for non integrable dispersive and wave equations has been the subject of several previous works. First, the existence of multi-solitons for the nonlinear Schrödinger (NLS) and the generalized Korteweg–de Vries (gKdV) equations was studied in the mass critical and subcritical cases by Merle [14], Martel [9], and Martel and Merle [10]. The proofs are partly based on standard stability properties of solitary waves (see references in these articles). A key point in [9] and [10] is the introduction of localized versions of the energy and mass functionals to deal with solutions containing several solitons. Such functionals are reminiscent of Martel et al. [13], where the stability of the sum of several solitons was studied in the energy space. For gKdV, the technique also allows to prove the uniqueness of multi-solitons [9], property which is not available for NLS. Next, the strategy of these works was extended to the case of exponentially unstable solitons: see Côte et al. [3] for the construction of multi-solitons for supercritical gKdV and NLS, and Combet [2] for a classification result for supercritical gKdV. In these papers, the exponential instability of each soliton is controlled through a simple topological argument. For the nonlinear Klein–Gordon equation, the strategy was adapted by Côte and Muñoz [4] (for real-valued, unstable solitons) and Bellazzini et al. [1] (for complex-valued, stable solitons). For the water-wave system, see the work of Ming et al. [15].

Closely related to the present work, for the energy-critical wave equation in dimension 5, Martel and Merle [11] proved the existence of $K$-solitons for any $K \geq 2$ but only in the case where the speeds $\{\ell_k\}_{k \in \{1, \ldots, K\}}$ are collinear. The Lorentz transform allows to remove this assumption in the special case $K = 2$. Technically, such constructions for the energy critical wave equation are more challenging than for gKdV and NLS (for which no limitation on the speeds appears) because of the weak decay of the solitons (1.2). Specific localized energy-momentum functionals are introduced to control solitons interactions in this context. In [11], the nature of the localization does not allow non-collinear multi-solitons, even under a smallness condition. The main improvements in the present work are the following: (1) the use of a refined approximate solution from [12]; (2) the introduction of a new localized energy functional to deal with non-collinear solitons. The smallness condition (1.4) is still probably technical and we conjecture that the result holds for any choice of speeds. However, by our method, (1.4) seems to be the best condition using the refined approximate solution from [12]. Moreover, constructing approximate solutions improving the one in [12] seems a difficult task.

The construction of $K$-solitons for the energy critical wave equation is especially motivated by the soliton resolution conjecture, fully solved in the 3D radial case by Duyckaerts et al. [6], and for a subsequence of time in 3, 4 and 5D by Duyckaerts et al. [5]. Indeed, the existence of such multi-solitons complements the statements in [5, 6] by exhibiting examples of solutions containing $K$ solitons as $t \to +\infty$ for any $K \geq 2$. In the same direction, recall that for the energy-critical wave equation in dimension 6, Jendrej [7] proved the existence of radial two-bubble solutions.
1.2. Outline of the proof

The strategy of the proof of theorem 1.1 combines the use of a refined approximate solution for the $K$-soliton problem from [12], with uniform estimates and compactness, as in [11]. See also references in [11, 12] for earlier works.

First, we introduce the refined approximate solution of the form

$$\vec{W} = \sum_{k=1}^{K} (\vec{W}_k + c_k \vec{v}_k)$$

for large $t > 0$, where for any $k \in \{1, \ldots, K\}$, $\vec{W}_k$ is a soliton with two time-dependent parameters regarding scaling and translation respectively, and $\vec{v}_k$ is a correction term [12] improving the simpler approximate solution used in [11]. These correction terms of size $t^{-2}$ in the energy space are solutions of non-homogeneous wave equations whose source terms are the main order of the nonlinear interactions of size $t^{-3}$ between the $K$-solitons. In this way, $\vec{W}$ is an approximate solution of the $K$-solitons problem at order $t^{-4}$. This is decisive for the construction under the smallness condition (1.4).

Let $S_n \to +\infty$ as $n \to +\infty$ and, for each $n$, let $u_n$ be the backwards solution of (1.1) with initial data at time $S_n$

$$\vec{u}_n(S_n, x) \sim \sum_{k=1}^{K} \left( \vec{W}_k^{\infty}(S_n, x) + c_k \vec{V}_k^{\infty}(S_n, x) \right).$$

(See (5.3) for a precise definition of $\vec{u}_n(S_n)$). The goal is to prove that there exists a time $T_0$ independent of $n$, such that the following uniform estimates (1.8) hold on $[T_0, S_n]$,

$$\left\| \vec{u}_n(t) - \sum_{k=1}^{K} \vec{W}_k^{\infty}(t) - \sum_{k=1}^{K} c_k \vec{V}_k^{\infty}(t) \right\|_{\dot{H}^1 \times L^2} \lesssim \frac{1}{t}.$$  

(1.8)

The existence of a multi-soliton then follows easily from standard compactness arguments (note that $\vec{u}_n(T_0)$ converges strongly in the energy space $\dot{H}^1 \times L^2$; see section 5). Thus, we now focus on the proof of (1.8).

We introduce

$$\vec{e} = \left( \epsilon \eta \right), \quad \vec{u}_n = \begin{pmatrix} u_n \\ \partial_t u_n \end{pmatrix} = \vec{W} + \vec{e},$$

where

$$\vec{W} = \begin{pmatrix} W \\ X \end{pmatrix} = \sum_{k=1}^{K} \left( \vec{W}_k + c_k \vec{v}_k \right).$$

By a standard procedure, we choose modulation parameters $\lambda_k(t)$ and $y_k(t)$ close to $\lambda_k^{\infty}$ and $y_k^{\infty}$, and obtain suitable orthogonality conditions on $\vec{e}$. The equation of $\vec{e}$ is thus coupled with equations of $\lambda_k(t)$ and $y_k(t)$. See lemma 4.1.

To prove (1.8), we introduce the following energy functional

$$\mathcal{H} = \int \left\{ |\nabla \epsilon|^2 + |\eta|^2 - 2(F(W + \epsilon) - F(W) - f(W)\epsilon) \right\} + 2 \int \left( \chi \cdot \nabla \epsilon \right) \eta,$$

where the bounded function $\chi$ is equal to $\ell_k$ in a neighborhood of the soliton $W_k$ and close to $\frac{1}{\xi}$ in 'transition regions' between $K$ solitons (see section 5.3 for a precise definition). The specific choice of function $\chi$ is the main novelty of this paper compared to [11, 12].

The functional $\mathcal{H}(t)$ has the following two important properties (see lemma 5.4 for more precise statements):
(1) $\mathcal{H}$ is coercive, in the sense that (up to unstable directions, to be controlled separately) it controls the size of $\tilde{\varepsilon}$ in the energy space

$$\mathcal{H} \sim \|\varepsilon\|^2_{\tilde{H}^1} + \|\eta\|^2_{L^2}.$$  

(2) The variation of $\mathcal{H}(t)$ is controlled on $[T_0, S_n]$ in the following sense, for any $\delta > 0$ small enough,

$$-\frac{d}{dt} (t^{\delta - 3\delta} \mathcal{H}) \lesssim \frac{1}{t^{1+\delta}}. \quad (1.9)$$

Therefore, integrating (1.9) on $[t, S_n]$, from (1.7), we find the uniform bound, for any $t \in [T_0, S_n]$,

$$\|\varepsilon\|^2_{\tilde{H}^1} + \|\eta\|^2_{L^2} \lesssim t^{-3+\delta}. \quad (1.10)$$

Using time integration of the equation of the parameters, we can obtain (1.11) from the above estimate

$$|\lambda_k(t) - \lambda_k^\infty| \lesssim \frac{1}{t}, \quad |y_k(t) - y_k^\infty| \lesssim \frac{1}{t}. \quad (1.11)$$

Then (1.8) follows from (1.10) and (1.11).

2. Notation and preliminaries

2.1. Notation

We denote

$$(g, \tilde{g})_{L^2} = \int g \tilde{g}, \quad \|g\|^2_{L^2} = \int |g|^2, \quad (g, \tilde{g})_{\tilde{H}^1} = \int \nabla g \cdot \nabla \tilde{g}, \quad \|g\|^2_{\tilde{H}^1} = \int |\nabla g|^2.$$  

For

$$\tilde{g} = \left(\begin{array}{c} \tilde{g}_h \\ \tilde{g}_\tilde{h} \end{array}\right), \quad \tilde{g} = \left(\begin{array}{c} g \\ \tilde{h} \end{array}\right),$$

set

$$(\tilde{g}, \tilde{g})_{L^2} = (g, \tilde{g})_{L^2} + (h, \tilde{h})_{L^2}, \quad (\tilde{g}, \tilde{g})_{\tilde{H}^1 \times L^2} = (g, \tilde{g})_{\tilde{H}^1} + (h, \tilde{h})_{L^2}.$$  

Set $(x) = (1 + |x|^2)^{\frac{1}{2}}$. Let

$$\Lambda = \frac{3}{2} + x \cdot \nabla, \quad \tilde{\Lambda} = 2 + x \cdot \nabla, \quad \tilde{\Lambda} \nabla = \nabla \Lambda, \quad \tilde{\Lambda} = \left(\begin{array}{c} \tilde{\Lambda} \\ \Lambda \end{array}\right).$$

Let

$$J = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right).$$

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For \( \ell \in \mathbb{R}^5 \) with \(|\ell| < 1\), denote
\[
(g, \tilde{g})_{H^1_{\ell}} = \int (\nabla g \cdot \nabla \tilde{g} - (\ell \cdot \nabla g)(\ell \cdot \nabla \tilde{g})), \quad \|g\|_{H^1_{\ell}}^2 = \|g\|^2 - \|\ell \cdot \nabla g\|_{L^2}.
\]
Note that if we define
\[
g_\ell(x) = g \left( \frac{1}{\sqrt{1 - |\ell|^2}} - 1 \right) \frac{\ell \cdot x}{|\ell|^2} + x
\]
and similarly for \( \tilde{g}, \tilde{g}_\ell \), then
\[
(g_\ell, \tilde{g}_\ell)_{H^1_{\ell}} = \left( 1 - |\ell|^2 \right) (g, \tilde{g})_{H^1}.
\]
Set
\[
x_\ell = \left( \frac{1}{\sqrt{1 - |\ell|^2}} - 1 \right) \frac{\ell \cdot x}{|\ell|^2} + x - \frac{\ell t}{\sqrt{1 - |\ell|^2}}, \quad A_\ell = \partial_t - \ell \cdot \nabla,
\]
\[
B_\ell = \partial_t^2 - (\ell \cdot \nabla) \cdot (\ell \cdot \nabla) = A_\ell^2 - 2(\ell \cdot \nabla)A_\ell,
\]
\[
\Lambda_\ell = \frac{3}{2} + (x - \ell t) \cdot \nabla, \quad \Delta_\ell = \Delta - (\ell \cdot \nabla)(\ell \cdot \nabla).
\]
For \( \alpha > 0 \) small to be fixed later, set
\[
\varphi_\alpha(x) = (1 + |x|^2)^{-\alpha}.
\]
We recall standard Sobolev and Hölder inequalities
\[
\|u\|_{L^{30/7}} \lesssim \|u\|_{H^\alpha}, \quad \|u\|_{L^{30}} \lesssim \|u\|_{H^{1/2}}, \quad \|u\|_{L^{30/7}} \lesssim \|u\|_{L^{30/3}} \|w\|_{L^{4/3}} \lesssim \|u\|_{H^{1/3}} \|v\|_{H^{1/3}} \|w\|_{H^{1/3}}, \quad \|u\|_{L^{10/7}} \lesssim \|u\|_{L^{10/3}} \|v\|_{L^{7/3}} \|w\|_{L^{10/7}} \lesssim \|u\|_{L^{10/3}} \|v\|_{L^{10/3}} \|w\|_{L^{10}}.
\]
Denote
\[
f(u) = |u|^2 u, \quad F(u) = \frac{3}{10} |u|^{10/3}.
\]

2.2. Energy linearization around \( W \)

Let
\[
L = -\Delta - \frac{7}{3} W^4, \quad (Lg, g) = \int \left( |\nabla g|^2 - \frac{7}{3} W^4 g^2 \right),
\]
\[
H = \begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix}, \quad (H\tilde{g}, \tilde{g}) = (Lg, g) + \|h\|^2_{L^2} \quad \text{for } \tilde{g} = \begin{pmatrix} g \\ h \end{pmatrix}.
\]
For $\vec{g}$ small in the energy space, we recall the expansion of the energy

$$E(W + g, h) = E(W, 0) + \frac{1}{2} (H_{\vec{g}}, \vec{g})_{L^2} + O(\|g\|^3_{H^1}).$$

(2.5)

We recall some technical properties of the operator $L$.

**Lemma 2.1 ([11]).**

(i) Spectrum. The operator $L$ on $L^2$ with domain $H^2$ is a self-adjoint operator with essential spectrum $[0, +\infty)$, no positive eigenvalue and only one negative eigenvalue $-\lambda_0$ with a smooth radial positive eigenfunction $Y \in S(\mathbb{R}^3)$. Moreover,

$$L(\lambda W) = L(\partial_{\lambda} W) = 0, \quad \text{for any } j = 1, \ldots, 5.$$  

(ii) Localized coercivity. For $\alpha > 0$ small enough, there exists $\mu > 0$ such that, for all $g \in H^1$,

the following holds.

$$\int |\nabla g|^2 \varphi^2 - f'(W) g^2 \geq \mu \int |\nabla g|^2 \varphi^2 - \frac{1}{\mu} \left( (g, \Lambda W)_{H^1} + \sum_{j=1}^{5} (g, \partial_j W)_{H^1} + (g, Y)^2_{L^2} \right).$$

2.3. Energy linearization around $W_\ell$

For $\ell \in \mathbb{R}^3$ with $|\ell| < 1$, $W_\ell$ defined in (1.3) solves

$$\Delta W_\ell - \ell \cdot \nabla (\ell \cdot \nabla W_\ell) + W_\ell^2 = 0$$

(2.6)

so that $u(t, x) = W_\ell(x - \ell t)$ is a solution of (1.1). Note that

$$E(W_\ell, -\ell \cdot \nabla W_\ell) - \int |\ell \cdot \nabla W_\ell|^2 = (1 - |\ell|^2)^{\frac{1}{2}} E(W, 0).$$

(2.7)

The following operators are related to the linearization of the energy around $W_\ell$. Let

$$L_\ell = -\Delta + \ell \cdot \nabla (\ell \cdot \nabla) - f'(W_\ell),$$

$$(L_\ell g, g)_{L^2} = \int \nabla g \cdot \nabla g - (\ell \cdot \nabla g) \cdot (\ell \cdot \nabla g) - f'(W_\ell) g^2.$$

$$H_\ell = \begin{pmatrix} -\Delta - f'(W_\ell) & -\ell \cdot \nabla \\ \ell \cdot \nabla & \text{Id} \end{pmatrix}, \quad (H_{\ell, g}, g)_{L^2} = (L_\ell g, g)_{L^2} + \|\ell \cdot \nabla + h\|_{L^2}^2.$$

Indeed, proceeding as in (2.5), using (2.6) and (2.7), we obtain

$$E(W_\ell + g, -\ell \cdot \nabla W_\ell + h) + \int \ell \cdot \nabla (W_\ell + g)(-\ell \cdot \nabla W_\ell + h)$$

$$= (1 - |\ell|^2)^{\frac{1}{2}} E(W, 0) + \frac{1}{2} (H_{\ell, g}, g)_{L^2} + O(\|g\|^3_{H^1}).$$

Set

$$Z_\ell^0 = \begin{pmatrix} \Lambda W_\ell \\ -\ell \cdot \nabla (\Lambda W_\ell) \end{pmatrix}, \quad Z_\ell^\cdot = \begin{pmatrix} \partial_\mu W_\ell \\ -\ell \cdot \nabla (\partial_\mu W_\ell) \end{pmatrix}, \quad Z_\ell^w = \begin{pmatrix} W_\ell \\ -\ell \cdot \nabla W_\ell \end{pmatrix}.$$
\( Y_\epsilon = Y \left( \left( \frac{1}{\sqrt{1 - |\ell|^2}} - 1 \right) \ell \cdot x + x \right) \), \( \tilde{Z}_\epsilon^\pm = \left( \ell \cdot \nabla Y_\epsilon \pm \sqrt{\frac{\kappa}{1 - |\ell|^2}} Y_\epsilon \right) e^{\pm \sqrt{\frac{\kappa}{1 - |\ell|^2}} \ell \cdot x} \).

We recall several technical facts.

**Lemma 2.2 ([12])**. For \( \ell \in \mathbb{R}^5 \) with \( |\ell| < 1 \),

(i) Properties of \( L_\ell \).

\[ L_\ell (\lambda W_\ell) = L_\ell (\partial_0 W_\ell) = 0, \quad L_\ell Y_\epsilon = -\lambda \phi_\epsilon, \quad L_\ell W_\ell = -\frac{4}{3} W_\ell^2. \]

(ii) Properties of \( H_\ell \) and \( H_\ell J \).

\[ H_\ell \tilde{Z}_\ell^\pm = H_\ell \tilde{Z}_\ell^\pm = 0, \quad H_\ell \tilde{Z}_\ell^\pm = -\frac{4}{3} \begin{pmatrix} W_\ell^2 \\ 0 \end{pmatrix}, \quad \left( H_\ell \tilde{Z}_\ell^\pm \right)_{H^1_\ell \times L^2} = \left( \tilde{Z}_\ell^\pm \right)_{H^1_\ell \times L^2} = 0, \quad \left( \tilde{Z}_\ell^\pm \right)_{H^1_\ell \times L^2} = 0. \]

(iii) Coercivity. For \( \alpha > 0 \) small enough, there exists \( \mu > 0 \) such that, for all \( \tilde{g} \in H^1 \times L^2 \),

\[ \int \left( |\nabla \tilde{g}|^2 \varphi^2 - \frac{7}{3} W_\ell^2 \tilde{g}^2 + h_\ell^2 \varphi^2 + 2(\ell \cdot \nabla \tilde{g}) h_\ell \varphi^2 \right) \geq \mu \int \left( |\nabla \tilde{g}|^2 + h_\ell^2 \varphi^2 - \frac{1}{\mu} \left\{ (g, \lambda W_\ell)_{H^1_\ell}^2 + |(g, \nabla W_\ell)_{H^1_\ell}|^2 + (g, \tilde{Z}_\ell^\pm)^2_{L^2} + (g, \tilde{Z}_\ell^\pm)^2_{L^2} \right\} \right). \]

2.4. Approximate solution to a non-homogeneous linearized equation

Let \( |\ell| < 1 \) and \( F, G \) be defined by

\[ F = W_\ell^4 + \kappa_\ell \Lambda W, \quad G = \kappa_\ell (1 - |\ell|^2)^{-\frac{1}{2}} \ell \cdot \nabla \Lambda W, \quad \kappa_\ell = -(1 - |\ell|^2)^{-\frac{1}{2}} \frac{(W_\ell^4, \Lambda W)}{||\lambda W||_{L^2}^2} > 0. \]

Set

\[ w_\ell(t,x) = W(x_\ell), \quad F_\ell(t,x) = F(x_\ell), \quad G_\ell(t,x) = G(x_\ell). \]

We recall following technical lemma in [12].

**Lemma 2.3**. There exists a smooth function \( \varphi_\ell \) such that, for all \( 0 < \delta < 1 \) and all \( t \geq 1 \),

\[ \| (\varphi_\ell, \partial_0 \varphi_\ell)(t) \|_{H^1 \times L^2} \lesssim t^{-2}, \quad \| \varphi_\ell(t) \|_{L^2} \lesssim t^{-2+\delta}, \quad \| \varphi_\ell(t) \|_{H^1} \lesssim t^{-4+\delta}, \]

where

\[ \varphi_\ell = \partial_0^2 \varphi_\ell - \Delta \varphi_\ell - \frac{7}{3} W_\ell^4 \varphi_\ell - f_\ell - g_\ell, \quad f_\ell = t^{-3} F_\ell, \quad g_\ell = t^{-2} G_\ell. \]
Moreover, for all $m \geq 0$, $|\alpha| = 1$, $|\alpha'| \geq 2$, $t \geq 1$, $x \in \mathbb{R}^5$,

$$
\begin{align*}
|A^m_\ell u_s(t, x)| &\lesssim (t + \langle x_\ell \rangle)^{-1} t^{-(1+m)} \langle x_\ell \rangle^{-2+\delta}, \\
|A^m_\ell \partial^\alpha u_s(t, x)| &\lesssim (t + \langle x_\ell \rangle)^{-1} t^{-(1+m)} \langle x_\ell \rangle^{-3+\delta}, \\
|A^m_\ell \partial^\alpha v(t, x)| &\lesssim (t + \langle x_\ell \rangle)^{-1} t^{-(1+m)} \langle x_\ell \rangle^{-4+\delta},
\end{align*}
$$

(2.8)

and

$$
\begin{align*}
|A^m_\ell \partial^\alpha v(t, x)| &\lesssim t^{-(4+m)+\delta} \langle x_\ell \rangle^{-3}, \\
|A^m_\ell \partial^\alpha E(t, x)| &\lesssim t^{-(4+m)+\delta} \langle x_\ell \rangle^{-4}, \\
|A^m_\ell \partial^\alpha E(t, x)| &\lesssim t^{-(4+m)+\delta} \langle x_\ell \rangle^{-5}.
\end{align*}
$$

(2.9)

**Proof.** See proof of lemma 3.1 in [12]. Although this lemma is proved only for $\ell = \ell_1$, by rotation, we see that it holds for any $\ell \in \mathbb{R}^5$ with $|\ell| < 1$. \square

3. Refined approximate solution of K-solitons problem

In this section, we construct a refined approximate solution to the $K$-soliton problem. Let $K \geq 2$ and for any $k \in \{1, \ldots, K\}$, let

$$
\lambda_\ell^\infty > 0, \quad y_\ell^\infty \in \mathbb{R}^5, \quad \epsilon_\ell = \pm 1, \quad \ell_\ell \in \mathbb{R}^5, \quad \text{where } |\ell_\ell| < 1.
$$

Let $C_0 \gg 1$ and $T_0 \gg 1$ to be fixed and $I \subset [T_0, +\infty)$ be an interval of $\mathbb{R}$. We assume that these functions satisfy, for all $t \in I$,

$$
|\lambda_\ell(t) - \lambda_\ell^\infty| + |y_\ell(t) - y_\ell^\infty| \leq C_0 t^{-1}.
$$

(3.1)

For $k \in \{1, \ldots, K\}$, we consider $C^1$ functions $\lambda_k > 0, y_k \in \mathbb{R}^5$ defined on $I$. For $\overline{G} = (G, H)$, define

$$
(\theta_\ell G)(t, x) = \frac{\ell_\ell}{\lambda_k^\ell(t)} G \left( x - \ell_\ell t + y_k(t) \right), \quad \overline{\theta_\ell \ell G} = \left( \frac{\partial G}{\partial \lambda_k H} \right), \quad \overline{\theta_\ell G} = \left( \frac{\partial G}{\partial \lambda_k H} \right).
$$

In particular, set

$$
W_k = \theta_\ell W_k, \quad X_k = -\ell_\ell \cdot \nabla W_k, \quad \overline{W}_k = \left( \begin{array}{c} W_k \\ X_k \end{array} \right),
$$

(3.2)

3.1. Main interaction terms

Expanding the nonlinearity $|u|^4 u$ at $u = \sum_{k=1}^K W_k$, we prove that the main order of the nonlinear interactions is equivalent to the form $t^{-3} \sum_{k=1}^K c_k |W_k|^4$. The remaining error term is of size $t^{-4}$.

**Lemma 3.1.** Assume (3.1). For $k, k' \in \{1, 2, \ldots, K\}$, $k \neq k'$, let

$$
\sigma_{k,k'} = \left( \frac{1}{\sqrt{1 - |\ell_\ell'|^2}} - 1 \right) \frac{\ell_\ell' \cdot (\ell_k - \ell_{k'})}{|\ell_k|^2} + \ell_k - \ell_{k'},
$$

(3.3)
and \( c_k = \frac{1}{5}(15)^{\frac{3}{2}} \sum_{k' \neq k} e^k (\lambda_k^2)^2 |\sigma_{k,k'}|^{-\frac{3}{2}} \). Then,

\[
\left| \sum_{k=1}^{K} W_k \right|^4 \left( \sum_{k=1}^{K} W_k \right) - \sum_{k=1}^{K} |W_k|^4 W_k = t^{-3} \sum_{k=1}^{K} c_k |W_k|^4 + R_\Sigma,
\]

where, for all \( t \in I \),

\[
\|R_\Sigma\|_{H^n} \lesssim t^{-4}. \tag{3.3}
\]

**Proof.** The proof is similar to that of lemma 3.1 in [12]. Let \( \sigma = \frac{1}{10} \min_{k \neq k'} |\ell_k - \ell_{k'}| \) and \( B_k(t) = \{ x, |x - \ell_k t| \leq \sigma t \} \), \( B(t) = \cup_k B_k(t) \).

First, we claim, \( \forall k \in \{1, \ldots, K\} \) and \( k \neq k' \)

\[
\|W_k^2\|_{L^2(B_k)} \lesssim t^{-4}, \quad \|W_k^2\|_{L^2(B_k)} \lesssim t^{-\frac{3}{2}}. \tag{3.4}
\]

and

\[
\|W_k^2\|_{L^2(B_k)} \lesssim t^{-\frac{3}{2}}, \quad \|W_k^2\|_{L^2(B_k)} \lesssim t^{-\frac{3}{2}}. \tag{3.5}
\]

Note that, \( \forall x \in B_k \), we have

\[
|W_k|^2 \lesssim \langle x - \ell_k t \rangle^{-7} \lesssim t^{-4} \langle x - \ell_k t \rangle^{-3}, \quad |W_k|^2 \lesssim \langle x - \ell_k t \rangle^{-4} \lesssim t^{-1} \langle x - \ell_k t \rangle^{-3},
\]

and from \( x \mapsto \langle x \rangle^{-3} \in L^2(\mathbb{R}^3) \), we obtain (3.4). Note that, \( \forall x \in B_k \) and \( k' \neq k \)

\[
|W_k|^2 \lesssim t^{-6} \langle x - \ell_k t \rangle, \quad |W_k|^2 \lesssim t^{-\frac{1}{2}}. \tag{3.6}
\]

Integrating (3.6) on \( B_k(t) \), we obtain (3.5).

Second, we claim, \( \forall k \in \{1, \ldots, K\} \),

\[
\left\| \sum_{k' = 1}^{K} W_{k'}^2 \right\|^4 \left( \sum_{k' = 1}^{K} W_{k'} \right) - \left| \sum_{k' = 1}^{K} W_{k'}^2 \right| W_k = \frac{7}{3} |W_k|^4 \sum_{k' \neq k} W_{k'}^2 \|L^2(B_k)\| \lesssim t^{-\frac{3}{2}}. \tag{3.7}
\]

From Taylor expansion, we have,

\[
\left| \sum_{k' = 1}^{K} W_{k'} \right|^4 \left( \sum_{k' = 1}^{K} W_{k'} \right) = |W_k|^2 W_k + \frac{7}{3} |W_k|^2 \sum_{k' \neq k} W_{k'} + O \left( |W_k|^2 \sum_{k' \neq k} |W_{k'}|^2 \right) + O \left( \sum_{k' \neq k} |W_{k'}|^2 \right). \tag{3.8}
\]
Using (3.5) and (3.8), we obtain (3.7).

Third, we claim, \( k' \neq k \),

\[
\left\| W_k \right\|_{L^2(B_\ell)} \lesssim t^{-\frac{4}{3}}.
\]

Indeed, for \( x \in B_\ell \),

\[
| W_{k'}(t,x) - W_{k'}(t,\ell_k t) | \lesssim \sup_{B_\ell} | \nabla W_{k'}(t) | \cdot | x - \ell_k t | \lesssim t^{-\frac{4}{3}} | x - \ell_k t |,
\]

and so,

\[
| W_k |^{-\frac{4}{3}} | W_{k'}(t,x) - W_{k'}(t,\ell_k t) | \lesssim t^{-\frac{4}{3}} | x - \ell_k t |^{-3},
\]

which implies (3.9).

Then, note that, from the explicit expression (1.2) of \( W \), we have,

\[
| W(x) - 15\frac{2}{3} | x |^{-3} | \lesssim | x |^{-5}, \quad \text{for} \ | x | \gg 1.
\]

Thus, using the assumption on the parameters (3.1) and the definition (3.2) of \( W_{k'} \) and (3.10), we obtain

\[
W_{k'}(t,\ell_k t) = \frac{\epsilon_{k'}}{\lambda^{k'}_{\ell}(t)} W_{k'}, \quad \left( (\ell_k - \ell_{k'}) t - y_{k'}(t) \right) = \frac{\epsilon_{k'}}{\lambda^{k'}_{\ell}(t)} \left( \frac{((\ell_k - \ell_{k'}) t)}{\lambda^{k'}_{\ell}} \right) + O(t^{-4})
\]

\[
= \frac{\epsilon_{k'}}{(\lambda^{k'}_{\ell})^{\frac{3}{2}}} W \left( \frac{\sigma_{k,k',t}}{\lambda^{k'}_{\ell}} \right) + O(t^{-4}) = 15\frac{2}{3} \left( \frac{\epsilon_{k'}}{\lambda^{k'}_{\ell}} \right)^{\frac{3}{2}} | \sigma_{k,k'} |^{-3} t^{-3} + O(t^{-4}).
\]

Gathering above estimates, we obtain the \( L^2 \) estimate of \( \mathbf{R}_\Sigma \) in (3.3).

Now, we observe for \( j \in \{1, \cdots, 5\} \),

\[
\partial_j \mathbf{R}_\Sigma = \frac{7}{3} \sum_{k=1}^{K} W_k \left( \sum_{k'=1}^{K} \partial_{j'} W_{k'} \right) - \frac{7}{3} \sum_{k=1}^{K} \left( | W_k |^{-\frac{4}{3}} \partial_{j} W_k \right)
\]

\[
- \frac{4}{3} t^{-3} \sum_{k=1}^{K} c_k \left( | W_k |^{-\frac{4}{3}} W_k \right) \partial_j W_k.
\]

From Taylor expansion and the decay of \( W \) and \( \partial_j W \), we have, for \( k \in \{1, \cdots, K\} \),

\[
\left| \sum_{k'=1}^{K} W_{k'} \right| \left( \sum_{k'=1}^{K} \partial_{j'} W_{k'} \right) \left( \sum_{k'=1}^{K} \partial_{j'} W_{k'} \right) = | W_k |^{-\frac{4}{3}} \partial_j W_k + \frac{4}{3} \left( \left( | W_k |^{-\frac{4}{3}} W_k \right) \partial_j W_k \right) \left( \sum_{k' \neq k} W_{k'} \right)
\]

\[
+ O\left( \left| \sum_{k' \neq k} | W_{k'} | \right| \right) + O\left( \left| \sum_{k' \neq k} \left| \partial_j W_{k'} \right| \right| \right) + O\left( \left| \sum_{k' \neq k} | W_{k'} |^{\frac{4}{3}} \right| \right).
\]

And then, using similar arguments, we prove \( \dot{H}^1 \) estimate of \( \mathbf{R}_\Sigma \) in (3.3).
3.2. The approximate solution $\mathbf{W}$

We recall a result on approximate solution $\mathbf{\tilde{W}}$ in 5D from [12]. To remove the main interaction terms $\sum_{k=1}^{K} c_k t^{-3} |W_k|^5$ computed in lemma 3.1, we define suitably rescaled versions of the function $v_k$ given by lemma 2.3. Let

$$v_k(t, x) = \frac{1}{\lambda_k^2} v_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right),$$

$$z_k(t, x) = \frac{1}{\lambda_k^2} (\partial_t v_k) \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right) + \frac{\kappa_{k} c_k}{2 \lambda_k^2} \Lambda_k W_k(t, x) \tag{3.12}$$

where $\Lambda_k = \frac{3}{2} + (x - \ell_k t - y_k) \cdot \nabla$, and

$$\mathbf{a}_k = \left( v_k, z_k \right), \quad \mathbf{\kappa}_k = \left( 1 - |\ell_k|^2 \right) \frac{\langle W_k^4, AW \rangle}{\|AW\|^2_{L^2}}, \quad \mathbf{a}_{k} = \frac{c_k \kappa_k}{2}.$$

Set

$$\mathbf{\tilde{W}} = \begin{pmatrix} \mathbf{\tilde{W}}_1 & \cdots & \mathbf{\tilde{W}}_K \end{pmatrix} = \sum_{k=1}^{K} \left( \mathbf{\tilde{W}}_k + c_k \mathbf{a}_k \right). \tag{3.13}$$

**Lemma 3.2.** The function $\mathbf{\tilde{W}}$ satisfies on $I \times \mathbb{R}^5$

$$\left\{ \begin{array}{l}
\partial_t \mathbf{W} = \mathbf{X} - \text{Mod}_W - \text{Mod}_V \\
\partial_t \mathbf{X} = \Delta \mathbf{W} + |\mathbf{W}|^4 \mathbf{W} - \text{Mod}_X - \text{Mod}_Z - \mathbf{R} \mathbf{X} 
\end{array} \right. \tag{3.14}$$

where

$$\text{Mod}_W = \sum_{k=1}^{K} \left( \frac{\lambda_k}{\lambda_k} - \frac{a_k}{\lambda_k^2} \right) \Lambda_k W_k + \sum_{k=1}^{K} \mathbf{\dot{y}}_k \cdot \nabla W_k$$

$$\text{Mod}_X = - \sum_{k=1}^{K} \left( \frac{\lambda_k}{\lambda_k} - \frac{a_k}{\lambda_k^2} \right) (\ell_k \cdot \nabla) \Lambda_k W_k - \sum_{k=1}^{K} \left( \mathbf{\dot{y}}_k \cdot \nabla \right) (\ell_k \cdot \nabla) W_k,$$

and

$$\text{Mod}_V = \sum_{k=1}^{K} \left( 3 \frac{\lambda_k}{\lambda_k} v_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right) + \frac{\lambda_k}{\lambda_k^2} \Lambda_k v_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right) \right)$$

$$+ \sum_{k=1}^{K} \left( \frac{\lambda_k}{\lambda_k^2} \left( \lambda_k - \frac{\ell_k t - y_k}{\lambda_k} \right) \cdot \nabla v_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right) + \frac{\lambda_k}{\lambda_k^2} \nabla v_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right) \right),$$

$$\text{Mod}_Z = \sum_{k=1}^{K} \left( \mathbf{\dot{y}}_k \cdot \nabla \partial_t v_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right) + 4 \frac{\lambda_k}{\lambda_k^2} \partial_t v_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right) \right)$$

$$+ \sum_{k=1}^{K} \left( \frac{\lambda_k}{\lambda_k^2} \Lambda_k \partial_t v_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right) + \frac{\lambda_k}{\lambda_k^2} \left( \lambda_k - \frac{\ell_k t - y_k}{\lambda_k} \right) \cdot \nabla \partial_t v_k \left( \frac{t}{\lambda_k}, \frac{x - y_k}{\lambda_k} \right) \right)$$

$$+ \sum_{k=1}^{K} \left( \frac{\kappa_k c_k}{2 \lambda_k^2} \frac{\lambda_k}{\lambda_k} \left( \frac{1}{2} \Lambda_k W_k + \Lambda_k^2 W_k \right) + \frac{\kappa_k c_k}{2 \lambda_k^2} \mathbf{\dot{y}}_k \cdot \nabla \Lambda_k W_k \right).$$

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\[ \tilde{R} = \begin{pmatrix} 0 \\ R_x \end{pmatrix}, \quad \| \tilde{R} \|_{H^s \times L^2} + \| \nabla \tilde{R} \|_{H^s \times L^2} \lesssim t^{-4+\delta}. \] (3.15)

Moreover, for all \( 0 < \delta < 1 \),
\[ |W| + |x - \ell t| |\nabla W| \lesssim \sum_{k=1}^{K} (|x - \ell t|^{-3} + t^{-1}|x - \ell t|^{-3+\delta}), \] (4.1)
\[ |X| \lesssim \sum_{k=1}^{K} (|x - \ell t|^{-4} + t^{-2}|x - \ell t|^{-3+\delta}). \] (3.16)

**Proof.** See proof of lemma 4.3 in [12]. Although the original argument in this lemma only holds for \( \ell_k = \ell_k e_1 \) and \( K = 2 \), after using lemmas 2.3 and 3.2, we easily check that the argument still holds for any \( K \geq 2 \) and any \( \ell_k \in \mathbb{R}^5 \) with \( |\ell_k| < 1 \). \( \square \)

### 4. Decomposition around refined approximate solution

We prove in this section a general decomposition around refined approximate solution. Let \( K \geq 2 \) and for any \( k \in \{1, \ldots, K\} \), let \( \lambda_k^\infty > 0 \), \( y_k^\infty \in \mathbb{R}^5 \), \( \ell_k \in \mathbb{R}^5 \), \( |\ell_k| < 1 \) with \( \ell_k \neq \ell_{k'} \) for \( k' \neq k \). For \( \tilde{G} = (G, H) \), set
\[ \theta_k^\infty = \frac{\epsilon_k}{(\lambda_k^\infty)^3} G \left( \frac{x - \ell_k t - y_k^\infty}{\lambda_k^\infty} \right), \quad \tilde{\theta}_k^\infty = \frac{\theta_k^\infty}{X_k^\infty}, \quad W_k^\infty = \theta_k^\infty W_k, \quad X_k^\infty = -\ell_k \cdot \nabla W_k^\infty, \quad \tilde{W}_k^\infty = \left( \frac{W_k^\infty}{X_k^\infty} \right), \quad \Lambda_k^\infty = \frac{3}{2} + (x - \ell_k t - y_k^\infty). \]

We also set
\[ \psi_k^\infty = \frac{1}{(\lambda_k^\infty)^3} \psi_k \left( \frac{t}{\lambda_k^\infty}, \frac{x - y_k^\infty}{\lambda_k^\infty} \right), \]
\[ \zeta_k^\infty = \frac{1}{(\lambda_k^\infty)^3} (\partial_{\ell_k} \psi_k) \left( \frac{t}{\lambda_k^\infty}, \frac{x - y_k^\infty}{\lambda_k^\infty} \right) + \frac{k_k \ell_k}{2(\lambda_k^\infty)^3} \Lambda_k^\infty W_k^\infty(t,x), \quad \tilde{\psi}_k^\infty = \left( \frac{\psi_k^\infty}{\zeta_k^\infty} \right). \]

**Lemma 4.1 (Properties of the decomposition).** There exist \( T_0 \gg 1 \) and \( 0 < \delta_0 \ll 1 \) such that if \( u(t) \) is a solution of (1.1) which satisfies on \( I \),
\[ \| \bar{u} - \sum_{k=1}^{K} \left( \tilde{W}_k^\infty + c_k \tilde{V}_k^\infty \right) \|_{H^s \times L^2} < \delta_0, \] (4.1)
then there exist \( \mathcal{C}^1 \) functions \( \lambda_k > 0 \), \( y_k \) on \( I \) such that, \( \tilde{\varepsilon}(t) \) being defined by
\[ \tilde{\varepsilon} = \left( \begin{array}{c} \varepsilon \\ \eta \end{array} \right), \quad \bar{u} = \left( \begin{array}{c} u \\ \partial_{\ell_k} u \end{array} \right) = \tilde{W} + \tilde{\varepsilon}, \] (4.2)
the following hold on \( I \), for \( k \in \{1, \ldots, K\} \).
\(\lambda_k \approx 0\) and \(\partial H = \frac{\lambda_k}{\lambda_k^2} + |\hat{y}_k| \leq \|\hat{e}\|_{\dot{H}^1} + \frac{1}{\bar{r}}.\) (4.6)

(iv) Unstable directions. Let \(z_k^+ = (\hat{e}, \hat{v}_k, \hat{\lambda}_k^+).\) Then, for any \(0 < \delta < 1,\)
\[
\left| \frac{d}{dr} z_k^\pm + \frac{\sqrt{\lambda_0}}{\lambda_k} (1 - |\lambda_k|^2)^{1/2} z_k^\pm \right| \leq \|\hat{e}\|_{\dot{H}^1}^2 + r^{-1} \|\hat{e}\|_{\dot{H}^1}^2 + r^{-4+\delta}.\] (4.7)

**Proof.**

**Step 1.** Decomposition. The existence of parameters \(\lambda_k\) and \(y_k\) such that (4.3) and (4.4) hold is proved similarly as (i) of lemma 3.1 in [11].

**Step 2.** Equation of \(\tilde{e}\). We formally derive the equation of \(\tilde{e}(t), \lambda_k(t)\) and \(y_k(t)\) from (1.1) and (3.14).

\[\begin{align*}
\varepsilon_t &= \partial_t u - \partial_t W = \eta + \dot{X} = \partial_t W = \eta + \text{Mod}_W + \text{Mod}_V.
\end{align*}\]

Second, since \(\eta = \partial_t u - \dot{X},\) we have
\[\begin{align*}
\partial_t \eta &= \partial_t^2 u - \partial_t X = \Delta u + |u|^4 u - \Delta W - |W|^4 W + \text{Mod}_X + \text{Mod}_Z + R_X \\
&= \Delta \varepsilon + |W + \varepsilon|^4 (W + \varepsilon) - |W|^4 W + \text{Mod}_X + \text{Mod}_Z + R_X.
\end{align*}\]

We also denote
\[
\begin{align*}
R_{NL} &= |W + \varepsilon|^4 (W + \varepsilon) - |W|^4 W - \frac{7}{3} \sum_k |W_k|^4 \varepsilon = R_1 + R_2, \\
R_1 &= \frac{7}{3} \left( |W|^4 - \sum_k |W_k|^4 \right) \varepsilon, \\
R_2 &= |W + \varepsilon|^4 (W + \varepsilon) - |W|^4 W - \frac{7}{3} |W|^4 \varepsilon,
\end{align*}\]
and
\[
\begin{align*}
\hat{L} &= \begin{pmatrix} 0 & 1 \\ \Delta + \frac{y}{\tilde{X}} \sum_k |W_k|^4 & 0 \end{pmatrix}, \\
\hat{\text{Mod}}_1 &= \begin{pmatrix} \text{Mod}_W \\ \text{Mod}_X \end{pmatrix}, \\
\hat{\text{Mod}}_2 &= \begin{pmatrix} \text{Mod}_V \\ \text{Mod}_Z \end{pmatrix}.
\end{align*}
\]
\[ \hat{R} = \begin{pmatrix} 0 \\ R_x \end{pmatrix}, \quad \hat{R}_{NL} = \begin{pmatrix} 0 \\ R_{NLx} \end{pmatrix}, \quad \hat{R}_1 = \begin{pmatrix} 0 \\ R_1 \end{pmatrix}, \quad \hat{R}_2 = \begin{pmatrix} 0 \\ R_2 \end{pmatrix}. \]

With this notation, the system (4.5) rewrites
\[ \partial_t \hat{\varepsilon} = \hat{L} \varepsilon + \text{Mod}_1 + \text{Mod}_2 + \hat{R} + \hat{R}_{NL} + \hat{L} \varepsilon + \text{Mod}_1 + \hat{R} + \hat{R}_1 + \hat{R}_2, \quad (4.8) \]

We claim the following estimates on \( R_1 \) and \( R_2 \)
\[ \|R_1\|_{L^\infty} \lesssim t^{-1} \|\varepsilon\|_{L^\infty}, \quad \|R_2\| \lesssim \|W\|_1^2 \varepsilon^2 + \|\varepsilon\|^2, \quad \|R_2\|_{L^\infty} \lesssim \|\varepsilon\|_{L^\infty}^2 \lesssim \|\varepsilon\|_{L^1}^2. \quad (4.9) \]

The estimate on \( R_2 \) follows from (2.4). To prove the estimate on \( R_1 \), we first recall the inequality, for \( p > 1 \), for any reals \( r_k \),
\[ \left| \sum_k r_k - \sum_{k'} r_{k'} \right|^p \lesssim \sum_k |r_k|r_{k'}^{p-1}. \]

Therefore,
\[
\left| W \right|^\frac{4}{3} - \sum_{k=1}^K |W_k|^\frac{4}{3} \lesssim \left| W \right|^\frac{4}{3} - \left| \sum_{k=1}^K W_k \right|^\frac{4}{3} + \left| \sum_{k=1}^K W_k \right|^\frac{4}{3} - \left| \sum_{k' \neq k} |W_k| |W_{k'}| \right|^\frac{1}{3} \\
\lesssim \left( \sum_{k=1}^K |v_k| \right) \left( \sum_{k=1}^K (|W_k| + |v_k|) \right)^\frac{1}{3} + \sum_{k' \neq k} |W_k||W_{k'}|^\frac{1}{3},
\]

and thus
\[ |R_1| \lesssim |\varepsilon| \left( \sum_{k=1}^K |v_k| \right) \left( \sum_{k=1}^K (|W_k| + |v_k|) \right)^\frac{1}{3} + |\varepsilon| \sum_{k' \neq k} |W_k||W_{k'}|^\frac{1}{3}. \]

By (2.4), we obtain
\[ \|R_1\|_{L^\infty} \lesssim \|\varepsilon\|_{L^\infty} \left( \sum_{k=1}^K |v_k| \|L\|_{L^\infty} \right) \left( \sum_{k=1}^K \left( |W_k| |\varepsilon| \right)^\frac{1}{3} + |\varepsilon| \sum_{k' \neq k} |W_k| |W_{k'}|^\frac{1}{3} \right) + \|\varepsilon\|_{L^\infty} \sum_{k' \neq k} \left( |W_k| |W_{k'}|^\frac{1}{3} \right)^{\frac{3}{2}}. \]

By (2.8), we have \( \|v_k\|_{L^\infty} \lesssim t^{-2} \). Moreover \( \|W_k||W_{k'}|^\frac{1}{3} \lesssim t^{-1} \) is a consequence of the following technical result.

**Claim 4.2 (Claim 2 in [11])**. Let \( 0 < r_2 \leq r_1 \) be such that \( r_1 + r_2 > \frac{3}{2} \). For \( t \) large, if \( r_1 > \frac{3}{2} \) then \( \int |W_1|^{r_1} |W_2|^{r_2} \lesssim t^{-3r_2} \), whereas if \( r_1 \leq \frac{3}{2} \) then \( \int |W_1|^{r_1} |W_2|^{r_2} \lesssim t^{3-3(r_1+r_2)} \).

**Step 3.** Parameter estimates. Now, we derive the equations of \( \lambda_k \) and \( y_k \) from the orthogonality conditions (4.3). First,
\[
\frac{d}{dt} (\varepsilon, \Lambda_1 W_1)_{\hat{\mu}_k} = (\partial_\theta \varepsilon, \Lambda_1 W_1)_{\hat{\mu}_k} + (\varepsilon, \partial_k (\Lambda_1 W_1))_{\hat{\mu}_k} = 0.
\]

Thus, using the first line of (4.5), and the expression of \(\text{Mod}_W\) in lemma 3.2,

\[
0 = (\eta, \Lambda_1 W_1)_{\hat{\mu}_k} - (\varepsilon, \ell_1 \cdot \nabla (\Lambda_1 W_1))_{\hat{\mu}_k} - \frac{\dot{\lambda}_1}{\lambda_1} (\varepsilon, \Lambda_1^2 W_1)_{\hat{\mu}_k} - (\varepsilon, y_1 \cdot \nabla \Lambda_1 W_1)_{\hat{\mu}_k} \\
+ \sum_{k=2}^K \left( \frac{\dot{\lambda}_k}{\lambda_k} - \frac{a_k}{\lambda_k^3 t^2} \right) (\Lambda_k W_k, \Lambda_1 W_1)_{\hat{\mu}_k} + \sum_{k=2}^K (\dot{y}_k \cdot \nabla W_k, \Lambda_1 W_1)_{\hat{\mu}_k} + (\text{Mod}_V, \Lambda_1 W_1)_{\hat{\mu}_k}.
\]

By the decay properties of \(W\), we obtain

\[
\left| (\eta, \Lambda_1 W_1)_{\hat{\mu}_k} \right| + \left| (\varepsilon, \nabla (\Lambda_1 W_1))_{\hat{\mu}_k} \right| + \left| (\varepsilon, \Lambda_1^2 W_1)_{\hat{\mu}_k} \right| + \left| (\varepsilon, \nabla \Lambda_1 W_1)_{\hat{\mu}_k} \right| \lesssim \| \varepsilon \|_{H^1 \times L^2}.
\]

Next, \((\Lambda_1 W_1, \Lambda_1 W_1)_{\hat{\mu}_k} = (1 - |\ell_1|^2)\|\Lambda W\|_{H^1}^2\) and by parity, \((\nabla W_1, \Lambda_1 W_1)_{\hat{\mu}_k} = 0\). Using claim 4.2, we have, for \(k \in \{2, \ldots, K\}\),

\[
\left| (\Lambda_k W_k, \Lambda_1 W_1)_{\hat{\mu}_k} \right| + \left| (\nabla W_k, \Lambda_1 W_1)_{\hat{\mu}_k} \right| \lesssim r^{-3}.
\]

And then, using the expression of \(\text{Mod}_V\) and (2.8), we obtain

\[
\left| (\text{Mod}_V, \Lambda_1 W_1)_{\hat{\mu}_k} \right| \lesssim \frac{1}{r^2} \sum_{k=1}^K \left( \left| \frac{\dot{\lambda}_k}{\lambda_k} \right| + |\dot{y}_k| \right) \lesssim \frac{1}{r^2} \sum_{k=1}^K \left( \left| \frac{\dot{\lambda}_k}{\lambda_k} - \frac{a_k}{\lambda_k^3 t^2} \right| + |\dot{y}_k| \right) + r^{-4}.
\]

In conclusion of the previous estimates, the orthogonality condition \((\varepsilon, \Lambda_1 W_1)_{\hat{\mu}_k} = 0\), gives the following

\[
\left| \frac{\dot{\lambda}_1}{\lambda_1} - \frac{a_1}{\lambda_1^3 t^2} \right| \lesssim \| \varepsilon \|_{H^1 \times L^2} + (\| \varepsilon \|_{H^1 \times L^2} + r^{-2}) \sum_{k=1}^K \left( \left| \frac{\dot{\lambda}_k}{\lambda_k} - \frac{a_k}{\lambda_k^3 t^2} \right| + |\dot{y}_k| \right) + r^{-4}.
\]

Using the other orthogonality conditions, we obtain similarly,

\[
\sum_{k=1}^K \left( \left| \frac{\dot{\lambda}_k}{\lambda_k} - \frac{a_k}{\lambda_k^3 t^2} \right| + |\dot{y}_k| \right) \lesssim \| \varepsilon \|_{H^1 \times L^2} + (\| \varepsilon \|_{H^1 \times L^2} + r^{-2}) \sum_{k=1}^K \left( \left| \frac{\dot{\lambda}_k}{\lambda_k} - \frac{a_k}{\lambda_k^3 t^2} \right| + |\dot{y}_k| \right) + r^{-4}.
\]

Therefore, for \(\delta_0\) small enough and \(T_0\) large enough, we find (4.6).

**Step 4.** Equations of the unstable directions. We prove the case of \(z_1^\pm\). The other case is similar. Recall that \(\hat{Z}_k^\pm \in S\) by their definition in section 2.3. By (4.8), we have
\[
\frac{d}{dt} \mathcal{E} (e, \theta_i \bar{Z}_k^+) = \left( \partial_t e, \tilde{\theta}_i \bar{Z}_k^+ \right) + \left( e, \partial_t \tilde{\theta}_i \bar{Z}_k^+ \right)
\]
\[
= \left( \mathcal{E} e, \tilde{\theta}_i \bar{Z}_k^+ \right) - \frac{\xi_i}{\lambda_1} \left( e, \tilde{\theta}_i \bar{Z}_k^+ \right) - \frac{\lambda_i}{\lambda_1} \left( e, \tilde{\theta}_i \bar{Z}_k^+ \right) - \frac{y_i}{\lambda_1} \left( e, \tilde{\theta}_i \bar{Z}_k^+ \right) + \left( \tilde{\mathcal{M}}_1, \tilde{\theta}_i \bar{Z}_k^+ \right) + \left( \tilde{\mathcal{M}}_2, \tilde{\theta}_i \bar{Z}_k^+ \right) + \left( \tilde{\mathcal{R}}, \tilde{\theta}_i \bar{Z}_k^+ \right) + \left( \tilde{\mathcal{R}}_{\text{NL}}, \tilde{\theta}_i \bar{Z}_k^+ \right).
\]

First, by direct computations, using (ii) of lemma 2.2,
\[
\left( \mathcal{E} e, \tilde{\theta}_i \bar{Z}_k^+ \right) - \frac{\xi_i}{\lambda_1} \left( e, \tilde{\theta}_i \bar{Z}_k^+ \right) = \frac{1}{\lambda_1} \left( e, \tilde{\theta}_i \left( -H_{\bar{X}} + \bar{Z}_k^+ \right) \right) + \sum_{k \geq 2} \left( e, f' W_k \left( \theta_i \bar{Z}_k^+ \right) \right)
\]
\[
= \frac{\sqrt{\lambda_0}}{\lambda_1} \left( 1 - |\xi_i|^2 \right) \xi_i^+ + \sum_{k \geq 2} \left( e, f' W_k \left( \theta_i \bar{Z}_k^+ \right) \right).
\]

By the decay properties of \( \bar{Z}_k^+ \) and claim 4.2,
\[
\sum_{k \geq 2} \left| \left( e, f' W_k \left( \theta_i \bar{Z}_k^+ \right) \right) \right| \lesssim t^{-4} \| e \|_{H^1}.
\]

Next, by (4.6),
\[
\left| \frac{\lambda_i}{\lambda_1} \left( e, \tilde{\theta}_i \bar{Z}_k^+ \right) \right| + \left| \frac{y_i}{\lambda_1} \left( e, \tilde{\theta}_i \bar{Z}_k^+ \right) \right| \lesssim \| e \|_{H^1 L^2}^2 + t^{-2} \| e \|_{H^1 L^2}.
\]

Concerning the term with \( \tilde{\mathcal{M}}_1 \). From (ii) of lemma 2.2, we obtain
\[
(\bar{\theta}_i \bar{Z}_k^+, \tilde{\theta}_i \bar{Z}_k^+) = (\bar{Z}_k^+, \bar{Z}_k^+) = 0, \quad (\bar{\theta}_i \bar{Z}_k^+, \tilde{\theta}_i \bar{Z}_k^+) = (\bar{Z}_k^+, \bar{Z}_k^+) = 0.
\]

Moreover, by claim 4.2, we have
\[
\sum_{k \geq 2} \left( \left| \left( \bar{\theta}_i \bar{Z}_k^+, \tilde{\theta}_i \bar{Z}_k^+ \right) \right| + \left| \left( \bar{\theta}_i \bar{Z}_k^+, \tilde{\theta}_i \bar{Z}_k^+ \right) \right| \right) \lesssim t^{-3}.
\]

and thus, by (4.6),
\[
\sum_{k \geq 2} \left( \frac{\lambda_i}{\lambda_k} - \frac{\lambda_i}{\lambda_k^2} \right) \left| \left( \bar{\theta}_i \bar{Z}_k^+, \tilde{\theta}_i \bar{Z}_k^+ \right) \right| + \left| \left( \bar{\theta}_i \bar{Z}_k^+, \tilde{\theta}_i \bar{Z}_k^+ \right) \right| \lesssim t^{-3} \| e \|_{H^1 L^2} + t^{-7}.
\]

Concerning the term with \( \tilde{\mathcal{M}}_2 \). From (2.8), (4.6) and the definition of \( \tilde{\mathcal{M}}_2 \), we obtain
\[
\left| \left( \tilde{\mathcal{M}}_2, \tilde{\theta}_i \bar{Z}_k^+ \right) \right| \lesssim \frac{1}{t^2} \sum_{k=1}^{K} \left( \frac{\lambda_i}{\lambda_k} \right) + \left| y_i \right| \lesssim t^{-2} \| e \|_{H^1 L^2} + t^{-4}.
\]

Finally, we claim
\[
\left| \left( \bar{\mathcal{R}}, \tilde{\theta}_i \bar{Z}_k^+ \right) \right| + \left| \left( \bar{\mathcal{R}}_{\text{NL}}, \tilde{\theta}_i \bar{Z}_k^+ \right) \right| \lesssim t^{-4+\delta} + t^{-1} \| e \|_{H^1} + \| e \|_{H^1}^2.
\]

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Indeed, from (3.15), we have \(|(\mathbf{R}, \mathbf{t}, \mathbf{Z}_k^\pm)| \lesssim t^{-3+\delta}\). Second, by (4.9) and the decay of \(Y\), we have
\[
\left| \left( \mathbf{R}_1, \mathbf{t}, \mathbf{Z}_k^\pm \right) \right| \lesssim \|\mathbf{R}_1\|_\infty \lesssim t^{-1} \|\varepsilon\|_{H^1} \quad \text{and} \quad \left| \left( \mathbf{R}_2, \mathbf{t}, \mathbf{Z}_k^\pm \right) \right| \lesssim \|\mathbf{R}_2\|_\infty \lesssim \|\varepsilon\|_{H^1}^2.
\]

Gathering these estimates, we obtain (4.7). The proof of lemma 4.1 is complete. \(\square\)

5. Proof of theorem 1.1

To construct the \(K\)-soliton solution at \(+\infty\), we follow the strategy of [11] using the refined approximate solution \(\mathbf{W}\) defined in the previous section. We argue by compactness and obtain the solution \(u(t)\) as the limit of a sequence of approximate multi-solitons \(u_n(t)\).

**Proposition 5.1.** There exist \(T_0 > 0\) and a solution \(u(t)\) of (1.1) on \([T_0, +\infty)\) satisfying, for all \(0 < \delta < 1\), for all \(t \in [T_0, +\infty)\),
\[
\|\nabla u(t) - \nabla \mathbf{W}(t)\|_{L^2} + \|\partial_t u(t) - \mathbf{X}(t)\|_{L^2} \lesssim t^{-3+\delta}\]
(5.1)
where \(\lambda_k(t), y_k(t)\) are such that, for all \(t \in [T_0, +\infty)\),
\[
|\lambda_k(t) - \lambda_k^\infty| + |y_k(t) - y_k^\infty| \lesssim t^{-1}.
(5.2)
\]

This section is devoted to the proof of proposition 5.1. Note that proposition 5.1 implies theorem 1.1.

Let \(S_n \to +\infty\). Let \(\zeta_{k,n}^\pm \in \mathbb{R}\) small to be determined later. These free parameters correspond to two exponentially stable/unstable directions for each soliton—see statements of proposition 5.2, claim 5.3 and lemma 5.6. For any large \(n\), we consider the solution \(u_n\) of
\[
\begin{aligned}
\partial_t^2 u_n - \Delta u_n - |u_n|^4 u_n &= 0 \\
(u_n(S_n), \partial_t u_n(S_n))^T &= \sum_{k=1}^K \left( \mathbf{W}_k^\infty(S_n) + c_k \tilde{\mathbf{W}}_k^\infty(S_n) \right) + \sum_{k=1}^K \zeta_k^{\pm}(\tilde{\partial}_t \mathbf{Z}_k^\pm(S_n)).
\end{aligned}
(5.3)
\]
Note that since \((u_n(S_n), \partial_t u_n(S_n)) \in H^1 \times L^2\), the solution \(\tilde{u}_n\) is well-defined in \(H^1 \times L^2\) at least on a small interval of time around \(S_n\).

Now, we state uniform estimates on \(u_n\) backwards in time up to some uniform \(T_0 \gg 1\).

**Proposition 5.2.** There exist \(n_0 > 0\) and \(T_0 > 0\) such that, for any \(n \geq n_0\), there exist \((\zeta_{k,n}^\pm)_{k \in \{1, \ldots, K\}} \in \mathbb{R}^K \times \mathbb{R}^K\), with
\[
\sum_{k=1}^K |\zeta_{k,n}^+|^2 + \sum_{k=1}^K |\zeta_{k,n}^-|^2 \lesssim S_n^{-7},
\]
and such that the solution \(\tilde{u}_n = (u_n, \partial_t u_n)^T\) of (5.3) is well-defined in \(H^1 \times L^2\) on the time interval \([T_0, S_n]\) and satisfies, for all \(t \in [T_0, S_n]\),
\[
\left\| \tilde{u}_n(t) - \mathbf{W}_n(t) \right\|_{H^1 \times L^2} \lesssim t^{-3+\delta},
(5.4)
\]
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and for all $\nu > 0$, there exists $M > 0$ such that, for all $n \geq n_0$,
\[
\|\bar{u}_n(t_0)\|_{(H^1 \times L^2)(|x| > M)} < \nu,
\]
where $\bar{W}_n(t,x) = \bar{W}(t,x; \{\lambda_k,\nu\}, \{y_k\}_n(t))$ is defined in section 4, and
\[
|\lambda_k(t) - \lambda_k^\infty| + |y_k(t) - y_k^\infty| \lesssim t^{-1}, \quad |\bar{\lambda}_k(t)| + |\bar{y}_k(t)| \lesssim t^{-2}.
\]
Moreover, $\bar{u}_n \in C([T_0,S_n], H^2 \times H^1)$ and satisfies, for all $t \in [T_0,S_n]$, $\|\bar{u}_n(t)\|_{H^2 \times H^1} \lesssim 1$.

5.1. Proof of proposition 5.1, assuming proposition 5.2

In view of the estimates obtained in proposition 5.2 on $(\bar{u}_n(t_0))$ and (5.5), up to the extraction of a subsequence, $(\bar{u}_n(t_0))$ converges strongly in $H^1 \times L^2$ to some $(\bar{u}_0, \bar{u}_1)^T$ as $n \to +\infty$. Consider the solution $u(t)$ of (1.1) associated to the initial data $(\bar{u}_0, \bar{u}_1)^T$ at $t = T_0$. Then, by the uniform bounds (5.4) and the continuous dependence of the solution of (1.1) with respect to its initial data in the energy space $H^1 \times L^2$ (see e.g. [8] and references therein), the solution $u$ is well-defined in the energy space on $[T_0, \infty)$.

Recall that we denote by $\lambda_{k,n}$ and $y_{k,n}$ the parameters of the decomposition of $u_0$ on $[T_0,S_n]$. By the uniform estimates in (5.6), using Ascoli’s theorem and a diagonal argument, it follows that there exist continuous functions $\lambda_k$ and $y_k$ such that up to the extraction of a subsequence, $\lambda_{k,n} \to \lambda_k$, $y_{k,n} \to y_k$ uniformly on compact sets of $[T_0, +\infty)$. Moreover on $[T_0, +\infty)$,
\[
|\lambda_k(t) - \lambda_k^\infty| \lesssim t^{-1}, \quad |y_k(t) - y_k^\infty| \lesssim t^{-1}.
\]

Passing to the limit in (5.4) for any $t \in [T_0, +\infty)$, we finish the proof of proposition 5.1.

The rest of this section is devoted to the proof of proposition 5.2.

5.2. Bootstrap setting

We denote by $B_{\mathbb{R}^k}(r)$ (respectively, $S_{\mathbb{R}^k}(r)$) the open ball (respectively, the sphere) of $\mathbb{R}^k$ of center 0 and of radius $r > 0$, for the norm $\|\xi\| = \left(\sum_{j=1}^k |\xi_j|^2\right)^{1/2}$.

For $t = S_0$ and for $t < S_0$ as long as $u_0(t)$ is well-defined in $H^1 \times L^2$ and satisfies (4.1), we decompose $u_0(t)$ as in lemma 4.1. In particular, we denote by $(\varepsilon, \eta)$, $(\lambda_k)_k$, $(y_k)_k$, $(\bar{\varepsilon}_k)_k$ as the parameters of the decomposition of $u_0$.

We start with a technical result similar to lemma 3 in [3]. This claim will allow us to adjust the initial values of $(z_{\pm}^k(S_n))_k$ from the choice of $\zeta_{\pm}^k$ in (5.3).

Claim 5.3 (Choosing the initial unstable modes). There exist $n_0 > 0$ and $C > 0$ such that, for all $n \geq n_0$, for any $(\xi_k)_{k \in \{1, \ldots, K\}} \in B_{\mathbb{R}^{2K}}(S_n^{-1/2})$, there exists a unique $(\zeta_{\pm}^k)_{k \in \{1, \ldots, K\}} \in B_{\mathbb{R}^{2K}}(CS_n^{-1/2})$ such that the decomposition of $u_0(S_n)$ satisfies
\[
z_{\pm}^k(S_n) = \xi_k, \quad z^+_k(S_n) = 0,
\]
\[
|\lambda_k(S_n) - \lambda_k^\infty| + |y_k(S_n) - y_k^\infty| + \|\varepsilon(S_n)\|_{H^1 \times L^2} + \|\bar{\varepsilon}(S_n)\|_{H^1 \times H^1} \lesssim S_n^{-7/2}.
\]

Sketch of the proof of claim 5.3. The proof of existence of $(\zeta_{\pm}^k)_k$ in claim 5.3 is similar to lemma 3 in [3] and we omit it. Estimates in (5.8) are consequences of (4.4).
The proof of proposition 5.2 is based on the following bootstrap estimates, for some $0 < \delta < 1$ to be fixed later, $k \in \{1, \ldots, K\}$ and $C_0$ to be chosen,

\[
\begin{aligned}
|\lambda_k(t) - \lambda_k^\infty| & \leq C_0 t^{-1}, \\
|y_k(t) - y_k^\infty| & \leq C_0 t^{-1}, \\
|\xi_k^\Delta(t)|^2 & \leq t^{-2}, \\
\|\varepsilon(t)\|_{H^\ell \cap L^2} & \leq C_0 t^{-3+\delta}. \\
\end{aligned}
\tag{5.9}
\]

Set
\[
T^* = T_0^*((\xi_k)_k) = \inf \{ t \in [T_0, S]\} \colon u_0 \text{ satisfies (4.1) and (5.9) holds on } [t, S]\).
\tag{5.10}
\]

In what follows, we will prove that there exists $T_0$ large enough and at least one choice of $(\xi_k)_k \in \mathcal{B}_{\mathbb{R}^k}(S_n^{7/2})$ so that $T^* = T_0$, which is enough to finish the proof of proposition 5.2. For this, we derive general estimates for any $(\xi_k)_k \in \mathcal{B}_{\mathbb{R}^k}(S_n^{7/2})$ (see lemma 5.5) and use a topological argument (see lemma 5.6) to control the unstable directions, in order to strictly improve (5.9) on $[T^*, S_0]$. As a consequence of the bootstrap estimates (4.6) and (5.9), we have, for $k \in \{1, \ldots, K\},$

\[
\left| \frac{\lambda_k}{\lambda_k^\infty} - \frac{a_k}{\lambda_k^\infty^2} \right| + |y_k| \lesssim \|\varepsilon(t)\|_{H^\ell \cap L^2} + t^{-4+\delta} \lesssim C_0 t^{-3+\delta}.
\]

In particular, from the expression of $\text{Mod}_V$ and $\text{Mod}_Z$ in lemma 3.2, (2.8), (2.9), (4.6) and (5.9), we have

\[
\|\text{Mod}_V\|_{H^\ell \cap L^2} \leq t^{-4} \quad \text{and} \quad \|\text{Mod}_Z\|_{H^\ell} \leq t^{-4+\delta}.
\tag{5.11}
\]

From the expression of $\text{Mod}_W$ and $\text{Mod}_X$ in lemma 3.2, for all $\alpha \in \mathbb{N}^5$,

\[
|\partial_\alpha^p \text{Mod}_W(t)| \lesssim t^{-3+\delta} \sum_{k=1}^K |W_k|^{1+|\alpha|}, \quad |\partial_\alpha^p \text{Mod}_X(t)| \lesssim t^{-3+\delta} \sum_{k=1}^K |W_k|^{1+|\alpha|}. 
\tag{5.12}
\]

5.3. Energy functional

Recall that $f(u) = |u|^\gamma u$ and $F(u) = \frac{1}{\gamma+1} |u|^{\gamma+1}$. Let

\[
\forall k \in \{1, \ldots, K\}, \quad \varepsilon_k = \sum_{i=1}^5 \varepsilon_{i,k} e_i \quad \text{where} \quad |\varepsilon_k| < 1,
\tag{5.13}
\]

and denote (see (1.4))

\[
\tau = \left( \sum_{i=1}^5 \left( \max_{k \in \{1, \ldots, K\}} \varepsilon_{i,k}^2 \right) \right)^{\frac{1}{2}} < \frac{3}{5}.
\tag{5.14}
\]

We fix $\delta \leq \frac{1}{100}$ small enough, such that

\[
\begin{aligned}
\tau \leq \left( \frac{(2 - 4\delta)(8 - 4\delta)}{(6 - 4\delta)^2} \right)^{\frac{1}{2}}, \quad \tau \leq \left( \frac{(1 - 4\delta)(9 - 4\delta)}{(5 - 4\delta)^2} \right)^{\frac{1}{2}}, \\
\tau \leq \left( \frac{(5 - 4\delta)}{(6 - 4\delta)} \right)^{\frac{1}{2}}, \quad \tau \leq \left( \frac{(4 - 4\delta)}{(6 - 4\delta)} \right)^{\frac{1}{2}}, \quad \tau \leq \left( \frac{(3 - 4\delta)(7 - 4\delta)}{(6 - 4\delta)^2} \right)^{\frac{1}{2}}. \\
\end{aligned}
\tag{5.15}
\]

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Moreover, we set, 

\[\forall i \in \{1, \ldots, 5\} - 1 < \ell_{i,j} < \cdots < \ell_{K_i,j} < 1\] such that \(\{\ell_{1,j}, \ldots, \ell_{K_i,j}\} = \{\ell_{1,i}, \ldots, \ell_{K_i,i}\}\). We denote \(I = \{i | i \in \{1, \ldots, 5\} \text{ and } K_i \geq 2\}\). For

\[0 < \sigma < \frac{1}{10} \min_{i \in I} (\ell_{k+1,i} - \ell_{k,i})\]

small enough to be fixed, we set

for \(k = 1, \ldots, K_i - 1\), \(\ell_{k,i}^+ = \ell_{k,i} + \sigma(\ell_{k+1,i} - \ell_{k,i})\),

for \(k = 2, \ldots, K_i\), \(\ell_{k,i}^- = \ell_{k,i} - \sigma(\ell_{k,i} - \ell_{k-1,i})\),

and for \(t > 0\), we denote,

\[\Omega_i(t) = ((\ell_{1,i}^- t, \ell_{1,i}^+ t) \cup \cdots \cup (\ell_{K_i,i}^- t, \ell_{K_i,i}^+ t)), \quad \Omega_{0,i}(t) = \mathbb{R} \setminus \Omega_i(t) \quad \text{for } i \in I\]

and

\[\Omega_i(t) = \emptyset, \quad \Omega_{0,i}(t) = \mathbb{R} \quad \text{for } i \notin I.\]

When \(i \in I\), we consider the continuous function \(\chi_i(t, x) = \chi_i(t, x_i)\) defined as follows, for all \(t > 0\),

\[
\begin{cases}
\chi_i(t, x) = \ell_{i,j} \text{ for } x_j \in (-\infty, \ell_{i,j}], \\
\chi_i(t, x) = \ell_{k,i}^- \text{ for } x_k \in [\ell_{k,i}^- t, \ell_{k,i}^+ t], \text{ for } k \in \{2, \ldots, K_i - 1\}, \\
\chi_i(t, x) = \ell_{K_i,i}^- \text{ for } x_{K_i} \in [\ell_{K_i,i}^- t, +\infty], \\
\chi_i(t, x) = \frac{\ell_{K_i,i}^- t}{1 - 2\sigma} \left(\ell_{K_i,i}^- t + \ell_{K_i,i}^-\right) \text{ for } x_{K_i} \in [\ell_{K_i,i}^+ t, \ell_{K_i,i}^- t], k \in \{1, \ldots, K_i - 1\}.
\end{cases}
\] (5.16)

In particular,

\[
\begin{cases}
\partial_t \chi_i(t, x) = 0, \quad \nabla \chi_i(t, x) = 0, \quad \text{on } \mathbb{R}^{i-1} \times \Omega_{0,i}(t) \times \mathbb{R}^{5-i}, \\
\partial_t \chi_i(t, x) = \frac{1}{\gamma} \frac{\partial}{\partial \gamma} \chi_i(t, x), \quad \text{for } x \in \mathbb{R}^{i-1} \times \Omega_i(t) \times \mathbb{R}^{5-i}, \\
\partial_t \chi_i(t, x) = -\frac{1}{\gamma} \frac{\partial}{\partial \gamma} \chi_i(t, x), \quad \text{for } x \in \mathbb{R}^{i-1} \times \Omega_{0,i}(t) \times \mathbb{R}^{5-i}.
\end{cases}
\] (5.17)

When \(i \notin I\), we consider the continuous function \(\chi_i(t, x) = \chi_i(t, x_i) = \ell_{i,j}\) for all \(t > 0\).

We denote

\[\Omega = \bigcup_{i,j \neq 0} \Omega_{i,j} \times \cdots \times \Omega_{i,j} \quad \text{and} \quad \bar{\Omega} = (\chi_1, \ldots, \chi_5).\]

The choice of \(\bar{\Omega}\) in this paper is different from that in [11, 12] to take into account non-colinear speeds.

We define (see [4, 11–13] for similar energy functional)

\[\mathcal{H} = \int \left\{ |\nabla \varepsilon|^2 + |\eta|^2 - 2\left(F(\mathbf{W} + \varepsilon) - F(\mathbf{W}) - F'(\mathbf{W})\varepsilon\right) \right\} + 2 \int (\bar{\chi} \cdot \nabla \varepsilon) \eta.\]

**Lemma 5.4.** There exists \(\mu > 0\) such that, for \(t \in [T^*, S_\alpha]\), the following hold.

(i) Bound.

\[|\mathcal{H}(t)| \leq \frac{\|\mathbf{P}(t)\|^2_{H^1 \times L^2}}{\mu}.\] (5.18)
(ii) Coercivity.

\[ \mathcal{H}(t) \geq \mu \| \vec{\varepsilon}(t) \|_{H^1 \times L^2}^2 - \frac{t^{-7}}{\mu}. \]  

(5.19)

(iii) Time variation. For all \( 0 < \delta < \frac{1}{100} \) small enough to satisfy (5.15),

\[ -\frac{d}{dt} (t^{\delta-3\delta} H) \leq \frac{1}{\mu} C_0 t^{-1-\delta}. \]  

(5.20)

Proof of lemma 5.4.

Proof of (5.18). Since

\[ |F(W + \varepsilon) - F(W) - f(W)\varepsilon| \lesssim |\varepsilon|^2 + |\varepsilon|^3 |W|^2, \]

estimate (5.18) on \( H \) follows from (2.2), (2.3) and \( \| \vec{\varepsilon} \|_{H^1 \times L^2} + \| W \|_{H^1} \lesssim 1. \)

Proof of (5.19). Set

\[ N_{\Omega}(t) = \int_{\Omega} (|\nabla \varepsilon(t)|^2 + \eta^2(t) + 2(\bar{\chi}(t) \cdot \nabla \varepsilon(t)) \eta(t)) , \quad N_{\Omega'}(t) = \int_{\Omega'} (|\nabla \varepsilon|^2 + \eta^2(t)). \]

Note that, since \( |\bar{\chi}| < \ell \),

\[ N_{\Omega} \geq \int_{\Omega} (|\nabla \varepsilon|^2 + \eta^2) - 2\ell \int_{\Omega} |\nabla \varepsilon| |\eta| \geq (1 - \ell) \int_{\Omega} (|\nabla \varepsilon|^2 + \eta^2). \]  

(5.21)

To obtain (5.19), we claim the following estimates, for some small \( \alpha > 0 \)

\[ \mathcal{H}(t) \geq N_{\Omega}(t) + \mu N_{\Omega'}(t) - \frac{t^{-7}}{\mu} - \frac{t^{-4\alpha}}{\mu} \| \vec{\varepsilon} \|_{H^1 \times L^2}^2 - \frac{t^{-1}}{\mu} \| \vec{\varepsilon} \|_{H^1 \times L^2}^2 - \frac{1}{\mu} \| \vec{\varepsilon} \|_{H^1 \times L^2}^3. \]  

(5.22)

To prove (5.22), we decompose \( \mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 \), where

\[ \mathcal{H}_1 = \int (|\nabla \varepsilon|^2 - \sum_{k=1}^K \int f'(W_k) \varepsilon^2 + \int \eta^2 + 2 \int (\bar{\chi} \cdot \nabla \varepsilon) \eta), \]

\[ \mathcal{H}_2 = -2 \int \left( F(W + \varepsilon) - F(W) - f(W) \varepsilon - \frac{1}{2} f'(W) \varepsilon^2 \right), \]

\[ \mathcal{H}_3 = \int \left( \sum_{k=1}^K f'(W_k) - f'(W) \right) \varepsilon^2. \]

We claim the following estimates

\[ \mathcal{H}_1 \geq N_{\Omega} + \mu N_{\Omega'} - \frac{t^{-7}}{\mu} - \frac{t^{-4\alpha}}{\mu} \| \vec{\varepsilon} \|_{H^1 \times L^2}^2, \]  

(5.23)

\[ |\mathcal{H}_2| + |\mathcal{H}_3| \lesssim \| \vec{\varepsilon} \|^3_{H^1 \times L^2} + t^{-1} \| \vec{\varepsilon} \|^2_{H^1 \times L^2}. \]  

(5.24)

which imply (5.22) for \( T_0 \) large enough.
Proof of (5.23). For $\varphi_{\alpha}$ defined in (2.1), set

$$\varphi_{\alpha}(t, x) = \varphi_{\alpha}\left(\frac{x - \ell x t - y(x)}{\lambda x(t)}\right).$$

We decompose $\mathcal{H}_1$ as follows

$$\mathcal{H}_1 = N_\Omega + \sum_{k=1}^{K} \left( \int |\nabla \varepsilon|^2 \varphi_k^2 - \int f'(W_k)\varepsilon^2 + \int \eta^2 \varphi_k^2 + 2 \int (\chi \cdot \nabla \varepsilon) \eta \varphi_k \right) + \int_{\Omega^c} (|\nabla \varepsilon|^2 + \eta^2 + 2(\chi \cdot \nabla \varepsilon) \eta) \left(1 - \sum_{k=1}^{K} \varphi_k^2\right)$$

$$- \int_{\Omega} (|\nabla \varepsilon|^2 + \eta^2 + 2(\chi \cdot \nabla \varepsilon) \eta) \left(\sum_{k=1}^{K} \varphi_k^2\right) + 2 \sum_{k=1}^{K} \int ((\chi - \ell_k) \cdot \nabla \varepsilon) \eta \varphi_k^2 = N_\Omega + \mathcal{H}_{1,1} + \mathcal{H}_{1,2} + \mathcal{H}_{1,3} + \mathcal{H}_{1,4}.$$ 

By lemma 2.2(iii), the orthogonality conditions on $\varepsilon$ and a change of variable, we have

$$\mathcal{H}_{1,1} \geq \mu_0 \int (|\nabla \varepsilon|^2 + \eta^2) \left(\sum_{k=1}^{K} \varphi_k^2\right) - \frac{1}{\mu_0} \sum_{k=1}^{K} ((z_k^-)^2 + (z_k^+)^2).$$

Thus, using (5.9),

$$\mathcal{H}_{1,1} \geq \mu_0 \int (|\nabla \varepsilon|^2 + \eta^2) \left(\sum_{k=1}^{K} \varphi_k^2\right) - \frac{1}{\mu_0} \sum_{k=1}^{K} ((z_k^-)^2 + (z_k^+)^2) - \frac{1}{\mu_0} t^{-\gamma}.$$

Next, note that if $x$ is such that $\varphi_k(t, x) > \frac{1}{\mathbf{r}}$, then $\varphi_k^2(x) \lesssim r^{-4\alpha}$ for $k' \neq k$. Thus, there exists $\mu_1$ the estimate $1 - \sum_{k=1}^{K} \varphi_k^2 \geq \frac{r^{-4\alpha}}{\mu_1}$ holds on $\mathbb{R}$. By direct computations (with the notation $v_+ = \max(0, v)$),

$$\mathcal{H}_{1,2} = \int_{\Omega^c} \left(\frac{\chi \cdot \nabla \varepsilon}{r} + \eta\right)^2 \left(1 - \sum_{k=1}^{K} \varphi_k^2\right) + (1 - 7) \int_{\Omega^c} \eta^2 \left(1 - \sum_{k=1}^{K} \varphi_k^2\right)$$

$$+ \int_{\Omega^c} \left(|\nabla \varepsilon|^2 - \frac{|\chi \cdot \nabla \varepsilon|^2}{r}\right) \left(1 - \sum_{k=1}^{K} \varphi_k^2\right) \geq (1 - 7) \int_{\Omega^c} (|\nabla \varepsilon|^2 + \eta^2) \left(1 - \sum_{k=1}^{K} \varphi_k^2\right) + \frac{\|\varepsilon\|_{H^1_x L^2}^2}{\mu_1} t^{-4\alpha}.$$

Last, by the definition of $\chi$, the decay property of $\varphi_{\alpha}$ and (5.9), we have

$$\|\varphi_k^2\|_{L^\infty(\Omega)} \lesssim t^{-4\alpha} \quad \text{and} \quad \|\chi - \ell_k\varphi_k^2\|_{L^\infty} \lesssim t^{-4\alpha}.$$ 

Thus, $|\mathcal{H}_{1,3}| \lesssim t^{-4\alpha} \|\varepsilon\|_{H^1_x L^2}^2$ and $|\mathcal{H}_{1,4}| \lesssim t^{-4\alpha} \|\varepsilon\|_{H^1_x L^2}^2$. Therefore, for some $\mu > 0$, and $T_0$ large enough, we have
\[ \mathcal{H}_{t,1} + \mathcal{H}_{t,2} + \mathcal{H}_{t,3} + \mathcal{H}_{t,4} \geq \mu \mathcal{N}_{\partial \Omega} - \frac{1}{\mu} t^{-\gamma} - \frac{1}{\mu} t^{-4\alpha} \| \varepsilon \|_{L^2}^2. \]

**Proof of (5.24).** Using (2.2), (2.3), (3.16) and (5.9), we have
\[ |\mathcal{H}_2| \lesssim \int |\varepsilon|^4 + |\varepsilon|^3 |\mathbf{W}| \lesssim \| \varepsilon \|_{H^1}^3. \]

Last, we observe that by (4.9), \( |\mathcal{H}_3| \lesssim \| \mathbf{R}_1 \|_{L^{\infty}} \| \varepsilon \|_{L^2}^2 \lesssim t^{-1} \| \varepsilon \|_{H^1}^2. \)

**Proof of (5.20).** We decompose
\[ \frac{d}{dt} \mathcal{H} = \int \partial_t \{ |\nabla \varepsilon|^2 + |\eta|^2 - 2(F(W + \varepsilon) - F(W)) \varepsilon \} \]
\[ + 2 \int \chi \cdot \partial_t ((\nabla \varepsilon) \eta) + 2 \int (\partial_t \chi) \cdot (\nabla \varepsilon) \eta = g_1 + g_2. \]

We claim the following estimates
\[ g_1 = 2 \int (\partial_t \varepsilon) (-\Delta \text{Mod}_w - f'(W) \text{Mod}_w) + 2 \int \eta \text{Mod}_x \]
\[ + 2 \int \sum_{k=1}^5 \eta_k \cdot \nabla W_k (f(W + \varepsilon) - f(W) - f'(W) \varepsilon) + O(C_d \delta^{7+2\delta}). \]

\[ g_2 = \int (\eta^2 + |\nabla \varepsilon|^2) \text{div} \chi - 2 \sum_{j=1}^5 \sum_{k=1}^5 (\partial_{\xi_j} \varepsilon) (\partial_{\xi_k} \varepsilon) \partial_{\xi_j} \chi_k + 2 \sum_{k=1}^5 (\partial_{\xi_k} \varepsilon) \eta \partial_{\xi_k} \]
\[ - 2 \int \chi \cdot \sum_{k=1}^5 \nabla W_k (f(W + \varepsilon) - f(W)) \]
\[ + 2 \int (\chi \cdot \nabla \text{Mod}_w) \eta - 2 \int \varepsilon (\chi \cdot \nabla \text{Mod}_x) + O(C_d \delta^{7+2\delta}). \]

**Estimate on \( g_1. \)** From direct differentiation and integration by parts, we have
\[ g_1 = 2 \int (\partial_t \varepsilon) (-\Delta \varepsilon - (f(W + \varepsilon) - f(W))) + 2 \int (\partial_t \eta) \eta \]
\[ - 2 \int (\partial_t W) (f(W + \varepsilon) - f(W) - f'(W) \varepsilon). \]

Using (3.14) and (4.5)
\[ g_1 = 2 \int (-\Delta \varepsilon - f'(W) \varepsilon) \text{Mod}_w + 2 \int \eta \text{Mod}_x \]
\[ + 2 \int (-\Delta \varepsilon - f'(W) \varepsilon) \text{Mod}_v + 2 \int \eta \text{R}_x + 2 \int \eta \text{Mod}_z \]
\[ - 2 \int \text{X} (f(W + \varepsilon) - f(W) - f'(W) \varepsilon) = g_{1.1} + g_{1.2} + g_{1.3}. \]

We integrate by parts terms in \( g_{1.1}. \) Next, by (2.2), (2.3), (3.15), (3.16), (5.9) and (5.11), we obtain
Thus, by (5.9) and integrating by parts and using (5.17), it follows that
\[ |g_{1,3} - 2 \int \left( \sum_{k=1}^{K} \ell_k \cdot \nabla W_k \right) \left( f(W + \varepsilon) - f(W) - f'(W)\varepsilon \right) | \]
\[ \lesssim \int \left( \sum_{k=1}^{K} |z_k| \right) \left( |\varepsilon|^2 + |\varepsilon|^2 |W|^2 \right) \lesssim \sum_{k=1}^{K} \|z_k\|_{L^\infty} \|\varepsilon\|^2 + \|\varepsilon\|^2 \lesssim C_0 r^{-7+2\delta}. \]

**Estimate on \( g_2 \):**

\[ g_2 = 2 \int (\chi \cdot \nabla \partial_t \varepsilon) \eta + 2 \int (\chi \cdot \nabla \varepsilon) \partial_t \eta + 2 \int (\partial_t \chi) \cdot (\nabla \varepsilon) \eta \]
\[ = 2 \int (\chi \cdot \nabla \eta) \eta + 2 \int (\chi \cdot \nabla \varepsilon) \Delta \varepsilon + 2 \int (\partial_t \chi) \cdot (\nabla \varepsilon) \eta \]
\[ + 2 \int (\chi \cdot \nabla W) \eta + 2 \int (\chi \cdot \nabla \varepsilon) W + 2 \int (\chi \cdot \nabla \varepsilon) W + 2 \int (\chi \cdot \nabla \varepsilon) W. \]

Note that by integration by parts
\[ 2 \int (\chi \cdot \nabla \eta) \eta + 2 \int (\chi \cdot \nabla \varepsilon) \Delta \varepsilon + 2 \int (\partial_t \chi) \cdot (\nabla \varepsilon) \eta \]
\[ = (-\Delta + |\nabla \varepsilon|^2) \text{div} \chi - 2 \sum_{j,k=1}^{S} (\partial_{x_j} \varepsilon) (\partial_{x_k} \varepsilon) \partial_{x_j} \chi_k + 2 \sum_{k=1}^{S} (\partial_{x_k} \varepsilon) \eta \partial_t \chi_k. \]

Next, we observe
\[ \int (\chi \cdot \nabla \varepsilon) (f(W + \varepsilon) - f(W)\varepsilon) \]
\[ = \int \chi \cdot (F(W + \varepsilon) - F(W) - f(W)\varepsilon) - \int \chi \cdot (\nabla W) (f(W + \varepsilon) - f(W) - f'(W)\varepsilon). \]

Integrating by parts and using (5.17),
\[ | - \int \chi \cdot \nabla (F(W + \varepsilon) - F(W) - f(W)\varepsilon) | = \left| \int_{\Omega} (\text{div} \chi) (F(W + \varepsilon) - F(W) - f(W)\varepsilon) \right| \]
\[ \lesssim \frac{1}{(1 - 2\sigma)^2} \int_{\Omega} |F(W + \varepsilon) - F(W) - f(W)\varepsilon|. \]

Thus, by (5.9) and
\[ \|W\|_{L^2(\Omega)} \lesssim \sum_{k=1}^{K} \left( \|W_k\|_{L^2(\Omega)} + \|\varphi_k\|_{H^1} \right) \lesssim r^{-2}. \]
we obtain
\[
\left| \int \chi \cdot \nabla (F(W + \varepsilon) - F(W) - f(W)\varepsilon) \right| \lesssim t^{-1} \int_\Omega \left( |\varepsilon|^2 + |W + \varepsilon|^2 \right) \\
\lesssim t^{-1} \left( \|\varepsilon\|_{L^\infty}^2 + \|\varepsilon\|_{L^\infty}^2 \|W\|_{L^\infty(\Omega)}^2 \right) \\
\lesssim t^{-1} \left( C_0 \|\varepsilon\|_{L^\infty}^2 t^{-10 + \frac{\delta}{2}} + (C_0 t^{-8 + 2\delta}) \right) \lesssim C_0 t^{-7 + 2\delta}.
\]

Moreover, again by (2.8) and (5.9)
\[
\left| \int \chi \cdot \left( \nabla W - \nabla \sum_{k=1}^K W_k \right) (f(W + \varepsilon) - f(W) - f'(W)\varepsilon) \right| \\
= \left| \int \chi \cdot \nabla \left( \sum_{k=1}^K Q_k \right) (f(W + \varepsilon) - f(W) - f'(W)\varepsilon) \right| \\
\lesssim \sum_{k=1}^K \left| \int \nabla Q_k \left( |\varepsilon|^2 W^2_k + |\varepsilon|^2 \right) \right| \lesssim \left( \sum_{k=1}^K \|\nabla Q_k\|_{L^\infty}^2 \right) \|\varepsilon\|_{L^\infty}^2 \lesssim (C_0)^2 t^{-8 + 2\delta} \lesssim C_0 t^{-7 + 2\delta}.
\]

Next, integrating by parts,
\[
2 \int (\chi \cdot \nabla \varepsilon) \text{Mod}_X = -2 \int (\chi \cdot \nabla \text{Mod}_X)\varepsilon + O \left( C_0 t^{-7 + 2\delta} \right),
\]

since by (5.9), (5.12) and (5.17)
\[
\left| \int (\text{div}\chi)\varepsilon \text{Mod}_X \right| \lesssim t^{-4 + \delta} \int_\Omega |\varepsilon| \left( \sum_{k=1}^K |W_k| \right)^\frac{2}{3} \lesssim t^{-4 + \delta} \|\varepsilon\|_{L^\infty}^2 \sum_{k=1}^K \|W_k\|_{L^\infty(\Omega)}^\frac{2}{3} \lesssim C_0 t^{-7 + 2\delta}.
\]

We finish the estimate of \( g_2 \) by observing that (3.15), (5.9) and (5.11) yield
\[
\left| \int (\chi \cdot \nabla \text{Mod}_Y)\eta \right| + \left| \int (\chi \cdot \nabla \varepsilon) \text{Mod}_Z \right| + \left| \int (\chi \cdot \nabla \varepsilon) \text{R}_X \right| \\
\lesssim \|\eta\|_{L^2} \|\nabla \text{Mod}_Y\|_{L^2} + \|\nabla \varepsilon\|_{L^2} \|\text{Mod}_Z\|_{L^2} + \|\nabla \varepsilon\|_{L^2} \|\text{R}_X\|_{L^2} \lesssim C_0 t^{-7 + 2\delta}.
\]

Gathering the estimates on \( g_1 \) and \( g_2 \), we rewrite
\[
\frac{d}{dt} h = h_1 + h_2 + h_3 + h_4 + O \left( C_0 t^{-7 + 2\delta} \right),
\]

where
\[
h_1 = \int (-\eta^2 + |\nabla\varepsilon|^2) \text{div}\chi - 2 \sum_{j=1}^5 (\partial_k \varepsilon)(\partial_k \eta)\partial_k \chi_k + 2 \sum_{k=1}^5 (\partial_k \varepsilon)\eta \partial_k\chi_k,
\]
\[
h_2 = 2 \int \left( \sum_{k=1}^K (\chi_k - \chi) \cdot \nabla W_k \right) (f(W + \varepsilon) - f(W) - f'(W)\varepsilon).
\]
\[
h_3 = 2 \int \eta \left( \text{Mod}_X + \chi \cdot \nabla \text{Mod}_W \right).
\]
\[
h_4 = 2 \int \varepsilon \left( -\Delta \text{Mod}_W - \chi \cdot \nabla \text{Mod}_X - f'(W)\text{Mod}_W \right).
\]
Estimate on $h_1$. We claim the following estimate

$$-(1 - 2\sigma)h_1 \leq (6 - 4\delta + C\sigma)N_\Omega(t).$$

(5.25)

Let

$$I = -(1 - 2\sigma)_t \left( -\eta^2 + |\nabla\varepsilon|^2 \text{div} \bar{\chi} - 2 \sum_{j=1}^3 (\partial_{ij}\varepsilon)(\partial_i\varepsilon)\partial_{i\chi} + 2 \sum_{k=1}^2 (\partial_{ik}\varepsilon)\eta\partial_{i\chi} \right).$$

To obtain (5.25), we will actually prove the following stronger property

$$I \leq (6 - 4\delta) \left( |\nabla\varepsilon|^2 + \eta^2 + 2(\bar{\chi} \cdot \nabla\varepsilon)\eta \right) + C\sigma(|\nabla\varepsilon|^2 + \eta^2) \quad \forall x \in \Omega(t).$$

(5.26)

Without loss of generality, we consider the following five cases,

**Case 1:** Let $x \in \Omega_{1.1} \times \Omega_{0.2} \times \Omega_{0.3} \times \Omega_{0.4} \times \Omega_{0.5}$. From (5.17), we obtain

$$\begin{cases}
\partial_{i\chi_1}(t,x) = \frac{4}{(1-2\sigma)^\beta}, & \partial_{i\chi_1}(t,x) = -\frac{2}{7} \frac{1}{(1-2\sigma)^\beta}, \\
\partial_{i\chi_1}(t,x) = 0, & \nabla\chi_i(t,x) = 0 \quad \forall i \neq 1.
\end{cases}$$

(5.27)

From direct computations and (5.27), we obtain

$$I = \eta^2 + |\nabla\varepsilon|^2 + 2(\delta_{ik}\varepsilon)(\partial_i\varepsilon)\eta - 2(5 - 4\delta)(\partial_{i\varepsilon} \cdot \chi_1) \eta - 2(6 - 4\delta) \sum_{i=1}^3 (\partial_{i\varepsilon} \cdot \chi_i) \eta + 2\eta \partial_{i\varepsilon} \left( \frac{4}{7} - \chi_1 \right).$$

(5.28)

Finally, using (5.14)–(5.16), (5.28) and Cauchy–Schwarz inequality, we obtain

$$I \leq \eta^2 + |\nabla\varepsilon|^2 + 2(6 - 4\delta)(\bar{\chi} \cdot \nabla\varepsilon)\eta + C\sigma(|\nabla\varepsilon|^2 + \eta^2)$$

$$+ (6 - 4\delta)\|\bar{\ell}\| \left( \frac{5 - 4\delta}{\|\bar{\ell}\|} |\nabla\varepsilon|^2 + \|\bar{\ell}\| (6 - 4\delta) \right) \eta^2
\leq (6 - 4\delta) \left( |\nabla\varepsilon|^2 + \eta^2 + 2(\bar{\chi} \cdot \nabla\varepsilon)\eta \right) + C\sigma(|\nabla\varepsilon|^2 + \eta^2).$$

**Case 2:** Let $x \in \Omega_{1.1} \times \Omega_{0.2} \times \Omega_{0.3} \times \Omega_{0.4} \times \Omega_{0.5}$. From (5.17), we obtain

$$\begin{cases}
\partial_{i\chi_i}(t,x) = \frac{4}{(1-2\sigma)^\beta}, & \partial_{i\chi_i}(t,x) = -\frac{2}{7} \frac{1}{(1-2\sigma)^\beta} \quad \text{for } i \in \{1, 2\}, \\
\partial_{i\chi_1}(t,x) = 0, & \nabla\chi_i(t,x) = 0 \quad \text{for } i \in \{3, 4, 5\}.
\end{cases}$$

(5.29)

From direct computations and (5.29), we obtain

$$I = 2\eta^2 - 2|\nabla\varepsilon|^2 + 2\sum_{i=1}^2 |\partial_{i\varepsilon}|^2 + 2(6 - 4\delta)(\bar{\chi} \cdot \nabla\varepsilon)\eta - 2(5 - 4\delta) \sum_{i=1}^2 (\partial_{i\varepsilon} \cdot \chi_i) \eta$$

$$- 2(6 - 4\delta) \sum_{i=3}^5 (\partial_{i\varepsilon} \cdot \chi_i) \eta + 2 \sum_{i=1}^2 \eta \partial_{i\varepsilon} \left( \frac{4}{7} - \chi_i \right).$$

(5.30)

Finally, using (5.14)–(5.16), (5.30) and Cauchy–Schwarz inequality, we obtain

$$I \leq 2\eta^2 + 2(6 - 4\delta)(\bar{\chi} \cdot \nabla\varepsilon)\eta + (6 - 4\delta)\|\bar{\ell}\| \left( \frac{1}{\|\bar{\ell}\|} |\nabla\varepsilon|^2 + \|\bar{\ell}\| \eta^2 \right) + C\sigma(|\nabla\varepsilon|^2 + \eta^2)$$

$$\leq (6 - 4\delta) \left( |\nabla\varepsilon|^2 + \eta^2 + 2(\bar{\chi} \cdot \nabla\varepsilon)\eta \right) + C\sigma(|\nabla\varepsilon|^2 + \eta^2).$$
Case 3: Let \( x \in \Omega_{1,1} \times \Omega_{1,2} \times \Omega_{1,3} \times \Omega_{0,4} \times \Omega_{0,5} \). From (5.17), we obtain
\[
\begin{align*}
\partial_t \chi_i(t, x) & = \frac{1}{(1 - 2 \sigma)^7}, \\
\partial_x \chi_i(t, x) & = -\frac{\eta}{(1 - 2 \sigma)^7} \\
\partial_l \chi_i(t, x) & = 0, \\
\nabla \chi_i(t, x) & = 0 
\end{align*}
\]
for \( i \in \{1, 2, 3\} \), (5.31)

From direct computations and (5.31), we obtain
\[
I = 3|\eta|^2 - 3|\nabla \chi|^2 + 2 \sum_{i=1}^3 |\partial_\eta \epsilon|^2 + 2(6 - 4 \delta)(\chi \cdot \nabla \epsilon) \eta \\
- 2(5 - 4 \delta) \sum_{i=1}^3 (\partial_\eta \epsilon \cdot \chi_i) \eta - 2(6 - 4 \delta) \sum_{i=4}^5 (\partial_\eta \epsilon \cdot \chi_i) \eta + 2 \sum_{i=1}^3 \eta \partial_\eta \epsilon (\frac{\eta}{\chi} - \chi_i) .
\]
(5.32)

Finally, using (5.14)–(5.16), (5.32) and Cauchy–Schwarz inequality, we obtain
\[
I \leq 3|\eta|^2 - |\nabla \epsilon|^2 + 2(6 - 4 \delta)(\chi \cdot \nabla \epsilon) \eta + C \sigma (|\nabla \epsilon|^2 + \eta^2) \\
+ (6 - 4 \delta) |\nabla \chi|^2 + \frac{(7 - 4 \delta)}{(6 - 4 \delta)} |\nabla \chi|^2 + \frac{(6 - 4 \delta)}{(7 - 4 \delta)} \eta^2 \\
\leq (6 - 4 \delta) (|\nabla \epsilon|^2 + \eta^2 + 2(\chi \cdot \nabla \epsilon) \eta) + C \sigma (|\nabla \epsilon|^2 + \eta^2) .
\]

Case 4: Let \( x \in \Omega_{1,1} \times \Omega_{1,2} \times \Omega_{1,3} \times \Omega_{1,4} \times \Omega_{0,5} \). From (5.17), we obtain
\[
\begin{align*}
\partial_t \chi_i(t, x) & = \frac{1}{(1 - 2 \sigma)^7}, \\
\partial_x \chi_i(t, x) & = -\frac{\eta}{(1 - 2 \sigma)^7} \\
\partial_l \chi_i(t, x) & = 0, \\
\nabla \chi_i(t, x) & = 0 
\end{align*}
\]
for \( i \in \{1, 2, 3, 4\} \), (5.33)

From direct computations and (5.33), we obtain
\[
I = 4|\eta|^2 - 4|\nabla \epsilon|^2 + 2 \sum_{i=1}^3 |\partial_\eta \epsilon|^2 + 2(6 - 4 \delta)(\chi \cdot \nabla \epsilon) \eta - 2(5 - 4 \delta) \sum_{i=1}^4 (\partial_\eta \epsilon \cdot \chi_i) \eta \\
- 2(6 - 4 \delta)(\partial_\eta \epsilon \cdot \chi_5) \eta + 2 \sum_{i=1}^4 \eta \partial_\eta \epsilon (\frac{\eta}{\chi} - \chi_i) .
\]
(5.34)

Finally, using (5.14)–(5.16), (5.34) and Cauchy–Schwarz inequality, we obtain
\[
I \leq 4|\eta|^2 - 2|\nabla \epsilon|^2 + 2(6 - 4 \delta)(\chi \cdot \nabla \epsilon) \eta + C \sigma (|\nabla \epsilon|^2 + \eta^2) \\
+ (6 - 4 \delta) |\nabla \chi|^2 + \frac{(8 - 4 \delta)}{(6 - 4 \delta)} |\nabla \chi|^2 + \frac{(6 - 4 \delta)}{(8 - 4 \delta)} \eta^2 \\
\leq (6 - 4 \delta) (|\nabla \epsilon|^2 + \eta^2 + 2(\chi \cdot \nabla \epsilon) \eta) + C \sigma (|\nabla \epsilon|^2 + \eta^2) .
\]

Case 5: Let \( x \in \Omega_{1,1} \times \Omega_{1,2} \times \Omega_{1,3} \times \Omega_{1,4} \times \Omega_{1,5} \). From (5.17), we obtain
\[
\partial_t \chi_i(t, x) = \frac{1}{(1 - 2 \sigma)^7}, \\
\partial_x \chi_i(t, x) = -\frac{\eta}{t (1 - 2 \sigma)^7} \\
\partial_l \chi_i(t, x) = 0, \\
\nabla \chi_i(t, x) = 0 
\]
for \( i \in \{1, 2, 3, 4, 5\} \), (5.35)

From direct computations and (5.35), we obtain
\[
I = 5|\eta|^2 - 5|\nabla \epsilon|^2 + 2 \sum_{i=1}^5 |\partial_\eta \epsilon|^2 + 2(6 - 4 \delta)(\chi \cdot \nabla \epsilon) \eta \\
- 2(5 - 4 \delta) (\nabla \epsilon \cdot \chi_i) \eta + 2 \sum_{i=1}^5 \eta \partial_\eta \epsilon (\frac{\eta}{\chi} - \chi_i) .
\]
(5.36)

Finally, using (5.14)–(5.16), (5.32) and Cauchy–Schwarz inequality, we obtain
\[
I \leq 5|\eta|^2 - 3|\nabla \epsilon|^2 + 2(6 - 4 \delta)(\chi \cdot \nabla \epsilon) \eta + C \sigma (|\nabla \epsilon|^2 + \eta^2) \\
+ (5 - 4 \delta) |\nabla \chi|^2 + \frac{(9 - 4 \delta)}{|\nabla \chi|^2} + \frac{|\nabla \chi|^2}{(9 - 4 \delta)} \eta^2 \\
\leq (6 - 4 \delta) (|\nabla \epsilon|^2 + \eta^2 + 2(\chi \cdot \nabla \epsilon) \eta) + C \sigma (|\nabla \epsilon|^2 + \eta^2) .
\]
Estimate on $h_2$. We observe that by the definition of $\tilde{\chi}$ in (5.17) and the decay of $\nabla W$ and $W$,

$$
\| (\ell_k - \tilde{\chi}) \cdot \nabla W_k \|_{L^\infty} \lesssim t^{-\frac{1}{2}}.
$$

Thus, by (2.3), $|h_2| \lesssim t^{-\frac{1}{2}} \| \varepsilon \|_{L^2}^2 \lesssim C_0 t^{-\frac{1}{2} + 2\delta} \lesssim C_0 t^{-\frac{7}{2} + 2\delta}$.

Estimate on $h_3$. Denote

$$
M_k = \left( \frac{\lambda_k}{\chi_k} - \frac{a_k}{\lambda_k^2} \right) \Lambda W_k + \tilde{y}_k \cdot \nabla W_k
$$

so that $\text{Mod}_W = \sum_k M_k$ and $\text{Mod}_X = - \sum_k \ell_k \cdot \nabla M_k$. Using (3.16) and the definition of $\tilde{\chi}$ (see (5.17)), we have $\| (\ell_k - \tilde{\chi}) \cdot \nabla M_k \|_{L^2} \lesssim t^{-\frac{1}{2} + \delta}$. It follows from (4.6) that

$$
\| \text{Mod}_X + \tilde{\chi} \cdot \nabla \text{Mod}_W \|_{L^2} \lesssim t^{-\frac{1}{2} + \delta},
$$

and thus

$$
|h_3| = \left| \int \eta (\text{Mod}_X + \tilde{\chi} \cdot \nabla \text{Mod}_W) \right| \lesssim t^{-\frac{1}{2} + \delta} \| \eta \|_{L^2} \lesssim C_0 t^{-\frac{7}{2} + 2\delta}.
$$

Estimate on $h_4$. By (i) of lemma 2.2, $-\Delta M_k + (\ell_k \cdot \nabla)(\ell_k \cdot \nabla)M_k - f'(W_k)M_k = 0$. Thus,

$$
\| -\Delta M_k + (\tilde{\chi} \cdot \nabla) \cdot (\ell_k \cdot \nabla)M_k - f'(W_k)M_k \|
\lesssim \| ((\tilde{\chi} - \ell_k) \cdot \nabla) \cdot (\ell_k \cdot \nabla)M_k \| + |f'(W_k) - f'(W)| \| M_k \|.
$$

As before, by (3.16), $\| ((\tilde{\chi} - \ell_k) \cdot \nabla) \cdot (\ell_k \cdot \nabla)M_k \|_{L^\infty} \lesssim t^{-\frac{1}{2} + \delta}$. Moreover, by (2.8)

$$
\left\| |W|^\frac{3}{4} - |W_k|^\frac{3}{4} \right\| |M_k|
\lesssim C_0 t^{-\frac{3}{2} + \delta} \left( \sum |W_k| + \sum |D_k| \right)^\frac{1}{2} \left( \sum |W_{k'}| + \sum |D_{k'}| \right) |W_k|
\lesssim C_0 t^{-\frac{3}{2} + \delta} \left[ \sum_{k' \neq k} |W_{k'}|^2 |W_{k'}| + t^{-\frac{3}{2} + \delta} \sum |W_{k'}|^2 \right.
\left. + t^{-\frac{3}{2} + \delta} \sum_{k' \neq k} |W_{k'}|^2 |W_{k'}| + t^{-\frac{3}{2} + \delta} \sum |W_{k'}|^2 \right],
$$

and from claim 4.2,

$$
\| (f'(W) - f'(W_k)) M_k \|_{L^\infty} \lesssim t^{-5 + 2\delta}.
$$

Therefore, using (4.6),

$$
\| -\Delta \text{Mod}_W - \tilde{\chi} \cdot \nabla \text{Mod}_X - f'(W) \text{Mod}_W \|_{L^\infty} \lesssim t^{-\frac{3}{2} + \delta}.
$$

It follows that by (5.9),

$$
|h_4| \lesssim \| \varepsilon \|_{L^\infty} \| -\Delta \text{Mod}_W - \tilde{\chi} \cdot \nabla \text{Mod}_X - f'(W) \text{Mod}_W \|_{L^\infty} \lesssim C_0 t^{-\frac{7}{2} + 2\delta}.
$$
In conclusion, using (5.19), for \( \sigma \) small, and \( T_0 \) large,
\[
- \frac{d}{dt} \mathcal{H} \leq \frac{(6 - 4 \delta + C \sigma)}{t} N_\Omega + O(C_0 t^{-7+2\delta}) \leq \frac{6 - 3 \delta}{t} \mathcal{H} + O(C_0 t^{-7+2\delta})
\]
and the proof of lemma 5.4 is complete.

5.4. Parameters and energy estimates

The following result, mainly based on lemma 5.4, improves all the estimates in (5.9), except the ones on \((z_k^\infty)\).

**Lemma 5.5 (Closing estimates except \((z_k^\infty)\)).** For \( C_0 > 0 \) large enough, for all \( t \in [T^*, S_n] \),
\[
\begin{align*}
|\lambda_k(t) - \lambda_k^\infty| &\leq \frac{C_0}{2t} t^{-1}, \\
|y_k(t) - y_k^\infty| &\leq \frac{C_0}{2t} t^{-1}, \\
|z_k^\infty(t)| &\leq \frac{1}{2} t^{-7}, \\
\|\bar{\varepsilon}(t)\|_{L^2} &\leq \frac{C_0}{2t} t^{-3+\delta}.
\end{align*}
\] (5.37)

**Proof.**

**Step 1.** Estimate on \((\lambda_k(t), y_k(t))_{k \in \{1, \ldots, K\}}\). From (4.6) and (5.9), we have
\[
\left| \frac{\lambda_k}{\lambda_k^\infty} \right| + |y_k| \leq C t^{-2}
\] (5.38)

where the constant \( C \) depends on the parameters of the \( K \) solitons, but not on \( C_0 \). Using (5.8) and (5.38) and taking \( C_0 \) large enough, we obtain
\[
|\lambda_k(t) - \lambda_k^\infty| \leq |\lambda_k(t) - \lambda(S_n)| + |\lambda(S_n) - \lambda_k^\infty| \\
\leq \int_t^{S_n} \frac{C}{s^2} ds + \frac{1}{S_n^2} \leq C_0 \frac{1}{2t}.
\]

Using again (5.8) and (5.38) and taking \( C_0 \) large enough, we obtain
\[
|y_k(t) - y_k^\infty| \leq |y_k(t) - y(S_n)| + |y(S_n) - y_k^\infty| \\
\leq \int_t^{S_n} \frac{C}{s^2} ds + \frac{1}{S_n^2} \leq C_0 \frac{1}{2t}.
\]

**Step 2.** Bound on the energy norm. From (5.8) and (5.18), we have
\[
\mathcal{H}(S_n) \lesssim S_n^{-7}.
\] (5.39)

Integrating (5.20) on \([t, S_n]\), and using (5.39), we obtain,
\[
\int_t^{S_n} \frac{d}{ds} (s^{6-3\delta} \mathcal{H}(s)) ds + s_n^{6-3\delta} \mathcal{H}(S_n) \lesssim \int_t^{S_n} \frac{C_0}{s^{1+\delta}} ds \lesssim C_0 \frac{1}{t^{1+\delta}}.
\]

It follows that, \( \mathcal{H}(t) \lesssim C_0 t^{-6+2\delta} \). Using (5.19), we conclude that \( \|\bar{\varepsilon}\|_{L^2} \lesssim \sqrt{C_0} t^{-3+\delta} \).

The estimate follows for \( C_0 \) large enough.

**Step 3.** Estimate on \((z_k^\infty(t))_{k \in \{1, \ldots, K\}}\). Let \( \beta_k = \frac{\sqrt{2}}{\lambda_k^\infty} (1 - |\mathbf{E}_k|^2)^{1/2} > 0 \). Then, from (4.7) and (5.9),
\[
\frac{d}{dt} \left( e^{-\beta t z_k^+} \right) \lesssim C_0 e^{-\beta t} t^{-4+\delta}.
\] (5.40)

Integrating (5.40) on \([t, S_n]\), and using (5.7), we obtain \(-z_k^+(t) \lesssim t^{-4+\delta}\). Using the same estimate for \(-e^{-\beta t z_k^+}\), we obtain the conclusion for \(T_0\) large enough. \(\square\)

As in [3, 4, 11, 12], the parameters \(z_k^-\) require a specific argument.

**Lemma 5.6 (Control of unstable directions in [12]).** There exist \((\xi_k, n) \in B_{R^n}(S_n^{-\gamma/2})\) such that, for \(C^* > 0\) large enough, \(T^*(\xi_k, n) = T_0\). In particular, let \((\zeta_k^+)\) be given by claim 5.3 from such \((\xi_k, n)\), then the solution \(u_0\) of (5.3) satisfies (5.4).

**Proof.** See proof of lemma 5.8 in [12]. \(\square\)

Estimates (5.4) follow directly from the estimates (5.9) on \(\varepsilon(t), \lambda_k(t), y_k(t)\).

5.5. End of the proof of proposition 5.2

We refer to [12] for the proofs of the \(\dot{H}^2 \times \dot{H}^1\) bound and (5.5) (sections 5.4 and 5.5 of [12]).

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