A critical analysis of thermodynamic properties of braneworld black holes in anti-de Sitter spacetime

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Thermodynamic properties of Schwarzschild-Anti de-Sitter (Sch-AdS) and Reissner-Nordström-Anti de-Sitter (RN-AdS) blackholes in 3+1 dimensional spacetime are studied critically with special reference to the warped braneworld black holes with non-vanishing cosmological constant on the brane. Explicit dependence of the thermodynamic variables on the parameters of the braneworld model such as the induced three brane cosmological constant as well as the bulk cosmological constant have been determined. Hawking-Page phase transition has been discussed for both Sch-AdS and RN-AdS black holes. At the phase transition point it is shown that the parameters mass, charge and cosmological constant get related by an inequality relation which originates from the background warped geometry model.

I. INTRODUCTION

Standard model of elementary particles, despite it’s success in explaining physics up to a scale near TeV, has the well known fine tuning/gauge hierarchy problem in connection with the only scalar in the theory namely Higgs. Supersymmetry resolves this problem at the expense of introducing large number of superpartners in the theory, none of which has been detected so far. Alternatively theories of extra dimensions were developed recently. The warped geometry model proposed by Randall and Sundrum was particularly successful in resolving the gauge hierarchy problem without bringing in any intermediate mass scale in the theory. In this model the only extra dimension in a 5 dimensional anti-de Sitter bulk space-time is compactified on a \( S_1 / \mathbb{Z}_2 \) orbifold and two 3-branes are placed at the two orbifold fixed points at \( y = 0 \) and \( y = \pi \) separated by a distance \( r \). Because of the bulk gravity it turns out that the two branes have hierarchal mass scales even when the bulk cosmological constant and brane separation modulus are at the Planck scale. In the effective 4D theory the lower energy ( TeV ) brane is flat with zero cosmological constant. Generalizing this model further, it has been shown recently that the gauge hierarchy problem can be resolved in the case of a non-flat 3-brane [1] also, contrary to the Randall-Sundrum model with flat 3-brane [2]. The metric for the above spacetime is given by :

\[
ds^2 = e^{-2A(y)} g_{\mu\nu} dx^\mu dx^\nu + r^2 dy^2
\]

(1.1)

where

\[
e^{-2A} = \omega^2 \cosh^2 \left( \ln \frac{\omega}{c_1} + ky \right)
\]

Here \( \Omega \) is the brane cosmological constant of the \((3+1)\)-spacetime and \( \omega^2 = -\frac{\Omega}{\rho} \) and Depending on the signature of \( \omega^2 \) the brane can be either asymptotically AdS or dS. \( c_1 = 1 + \sqrt{1 - \omega^2} \) and \( k \) is related to the Bulk cosmological constant(\( \Lambda \)) as \( k \sim \sqrt{-\Lambda / 12M^4} \). The matter content on the four dimensional hypersurface can give rise to Schwarzschild-anti de Sitter (Sch-AdS henceforth) or Reissner-Nordström-anti de Sitter (RN-AdS) blackholes on the brane for negative values of \( \Omega \). Note that in all the cases the hierarchy problem is solved in the gravitational weak field limit. There are several possibilities that can occur in the above braneworld scenario. The brane tension can be either positive or negative for \( \omega^2 \) < 0. For negative brane cosmological constant there is an upperbound on the value
of $\omega^2$ for which the hierarchy issues can be resolved. Moreover it has been shown that the $\omega^2$ plays a significant role in determining the stability of the spacetime [3] and in localizing a bulk fermion near the brane [4].

In the present article our primary intention is to investigate the implications of the presence of a brane cosmological constant $\omega^2$ to the thermodynamics of four dimensional asymptotically AdS black holes. We show that the constraints that arise from the requirement of the resolution of the gauge hierarchy problem have interesting implications in the black hole physics on the brane. The quest for a consistent description of black hole physics and gravitational collapse in braneworlds [3] has been a challenge to theoretical physicists. The first attempt in this direction was to provide a black string solution [6], with extended singularity in higher dimensions, which turned out to be neither stable nor localized on the brane. Later on, it was found that the simplest localized black hole solution on the brane is Reissner-Nördstrom type with the usual charge replaced by a geometric quantity called ‘tidal charge’ [7]. Subsequently, black hole solutions with non-vacuum brane and the black hole intersecting the bulk [8, 9] and for charged rotating black holes were obtained [10, 11]. Some good results in this direction can also be found in [12]. Oppenheimer-Snyder type [13] gravitational collapse of spherically symmetric objects was also studied in [14] and the new results were formulated as a no-go theorem leading to a non-static exterior for the collapsing sphere on the brane. Subsequent generalizations that include, among others, the Gauss-Bonnet gravity for the bulk spacetime have been carried out in [15, 16]. However, later on, it was demonstrated in [17] that a static exterior of the Schwarzschild can be obtained by relaxing certain assumptions, thereby re-establishing Birkhoff’s theorem for braneworld black holes as well. All these results lead to the conclusion that we are yet to make a strong conclusive comment about the physics of black holes in the braneworld scenario.

On the other hand, in the general relativistic framework, different issues related to thermodynamics of black holes have been addressed over the years. The reader may go through [18] for a useful review of the subject. Though the thermodynamics of asymptotically anti de Sitter black holes have always drawn much attention [19], there has been a renewed interest in this area mainly due to their string theoretical connections [20]. Some recent works in this direction include investigation for quasinormal modes for such black holes. For example, [21] deal with Sch-AdS black holes whereas [22] discuss the case for RN-AdS black holes. However, black hole thermodynamics is somewhat less explored in the braneworld scenario. Although there have been some works in the literature that deal with certain thermodynamic features of braneworld black holes [23], they are mainly developed in the context of single brane scenario. Our aim in this article is to extend the analysis in the framework of curved two-brane scenario proposed in [1]. As it will turn up, our analysis itself has certain interesting features that worth studying. The salient features of our analysis are the following: The study of black hole thermodynamics for curved two-brane models is the first of its kind in two-brane context. To this end, we have obtained exact analytical solutions for different thermodynamic quantities, both for Sch-AdS as well as for RN-AdS black holes. We have further established certain link of the major black hole parameters (mass, charge and cosmological constant to be precise) for asymptotically AdS black holes on the brane to the bulk cosmological constant via thermodynamic analysis. This interpretation in turn leads to the interesting finding that an upper bound for the black hole mass results from an upper bound for the cosmological constant in this particular warped braneworld context. In a nutshell, here we have a curved brane scenario which not only addresses issues like gauge hierarchy, stability and fermion hierarchy, but also provides important insights to the thermodynamics properties of asymptotically AdS black holes on the brane.

II. SCHWARZSCHILD-ADS BLACK HOLE

The four dimensional brane metric representing a Sch-AdS spacetime is given by

$$\text{ds}^2(4) = -\left(1 - \frac{2m/M_{Pl}^2}{r} + k^2\omega^2 r^2\right)dt^2 + \frac{dr^2}{1 - \frac{2m/M_{Pl}^2}{r} + k^2\omega^2 r^2} + r^2 d\Omega^2$$  \hspace{1cm} (2.1)

with $\omega^2 = -\frac{\Omega}{4kr}$ where $\Omega$ is the induced brane-cosmological constant and $m$ corresponds to the mass parameter of the Sch-AdS blackhole. For $\Omega > 0$ we obtain Sch-AdS solution on the brane. Parameter $k$ is related to the bulk cosmological constant($\Lambda$) and hence encodes the bulk effects on the brane black hole parameters. Considering the above metric in Eq. (2.1) one can easily find the expression for the horizon. There exist three roots for the equation $g_{tt}(r) = 0$, out of which two are complex and only one is real. The real root determines the horizon $r_H$ which is explicitly given by

$$r_H = -\frac{1}{3\sqrt{3} \left(\frac{mk^4\omega^4}{M_{Pl}^4} + \sqrt{3} \sqrt{k^6\omega^6} + \sqrt{27m^2k^8\omega^8}ight)^\frac{1}{3}} + \frac{\left(\frac{mk^4\omega^4}{M_{Pl}^4} + \sqrt{3} \sqrt{k^6\omega^6} + \sqrt{27m^2k^8\omega^8}ight)^\frac{1}{3}}{3\sqrt{k^2}\omega^2}$$  \hspace{1cm} (2.2)
The requirement, \( r_H > 0 \) gives rise to the condition \((m/M_{Pl}^2)k^7\omega^7 > 0\) which is automatically satisfied and puts no extra constraints on the parameters of the theory. This implies that there will always exist a horizon for the Schwarzschild-AdS metric. In Fig. 1 we plot the variation of \( r_H \) with respect to \( \omega \).

![Graph](image)

**FIG. 1:** \( r_H \) is plotted against \( \omega \) for different values of \( m \)

It is interesting to note that if we consider \( \omega^2 \) to be very very small then the expression for horizon in the leading order is given as,

\[
r_H = 2\left(\frac{m}{M_{Pl}^2}\right) - 8\left(\frac{m^3}{M_{Pl}^6}\right)k^2\omega^2
\]

For \( \Omega = 0 \), \( r_H = 2(m/M_{Pl}^2) \) which is same as an asymptotically flat Schwarzschild black hole in four dimensions. The non-zero cosmological constant therefore reduces the horizon radius from that of the Schwarzschild value. We now estimate other thermodynamic quantities such as the Hawking temperature \( (T_H) \), Entropy \( (S) \) and the specific heat \( C_v \) for this class of black holes. For brevity let us introduce a function \( P(\omega) \) as,

\[
P(\omega) = \left(9\left(\frac{m}{M_{Pl}^2}\right)k^4\omega^4 + \sqrt{3}\left(k^6\omega^6 + 27\left(\frac{m^2}{M_{Pl}^2}\right)k^8\omega^8\right)\right)^{1/3}
\]

In what follows we shall express the quantities in terms of this newly-introduced function, which contains the brane cosmological constant \( \omega \) as well as the bulk cosmological constant through the parameter \( k \).

The Hawking temperature for the above metric (2.1) turns out to be

\[
T = \frac{-3\frac{4}{3}k^2\omega^2 + (P(\omega))^{\frac{3}{2}} + 9\left(\frac{m}{M_{Pl}^2}\right)k^4\omega^4(P(\omega))^{\frac{3}{2}}}{3\pi\frac{4}{3}(P(\omega))^{\frac{3}{2}}}
\]

The temperature profile in Eq.(2) has been plotted in figure 2. The Hawking temperature turns out to be a increasing function with respect to \( \omega \). As it has been shown earlier that the horizon radius decreases with increasing cosmological constant, the Hawking temperature, which is inversely related to the Hawking radius for purely Schwarzschild case, therefore predictably grows with cosmological constant.

Employing the thermodynamic relations we obtain the expression for the Bekenstein-Hawking entropy as

\[
S = \pi \left(-\frac{1}{3^{1/3}P(\omega)^{1/3}} + \frac{P(\omega)^{1/3}}{3^{2/3}k^2\omega^2}\right)^2
\]

The variation of Entropy with respect to \( \omega \) (in units of \( M_{Pl} = 1 \) and \( k = M_{Pl} \)) has been shown in the figure 3.

The entropy thus falls with increase in the value of the brane cosmological constant as shown in the figure. For a given value of \( \omega \), the entropy is larger for a larger value of the mass parameter \( m \) as is expected. For large value of \( \omega \), the role of the mass parameter becomes less significant and curves for different \( m \) slowly converges.
Finally the expression for the specific heat $C_v$ is found to be,

$$C_v = -\frac{1}{9k^6\omega^6} \left[ 2\pi \left( -243 \frac{3^{1/3}}{2^3} \frac{m^3}{M_{Pl}^6} k^6 \omega^6 (P(\omega))^{1/3} ight) ight. 
- \left. 3^{1/6} \left( k^6 \omega^6 + 27(m^2/M_{Pl}^4)k^8 \omega^8 (P(\omega))^{1/3} \left( 2 \frac{3^{2/3}}{2^{2/3}} + 9(m/m_{Pl}^2) (P(\omega))^{1/3} \right) ight) \right] 
+ k^2 \omega^2 \left( 27 \frac{3^{5/6}}{2^3} (m^2/M_{Pl}^4) \sqrt{k^6 \omega^6 + 27(m^2/M_{Pl}^4)k^8 \omega^8 (P(\omega))^{1/3} - 2 \frac{3^{2/3}}{2^{2/3}} (P(\omega))^{2/3} \right) 
+ 3k^4 \omega^4 \left( 4 + 9 \frac{3^{1/3}}{2^{1/3}} (m/M_{Pl}^2) (P(\omega))^{1/3} \left( 1 + 2 \frac{3^{1/3}}{2^{1/3}} (m/M_{Pl}^2) (P(\omega))^{1/3} \right) \right) \right] \right]$$

(2.7)

The plot has been shown in Figure 3.

In contrary to the ordinary Schwarzschild metric where $C_v$ is always negative and no phase transition is possible at any temperature, here we obtain the possibility of phase transition. The phase transition (for a given value of $m$) occurs at a particular value of $\omega$. The corresponding value of the temperature can be determined from the $\omega$ versus $T$ plot given in the previous figure. For larger value of $m$ the phase transition takes place at a lower value of the cosmological constant and hence at a lower value of temperature.

Therefore while the Sch-black hole in a flat space-time intrinsically unstable, an Ads-Sch black hole can become stable at a temperature when it’s specific heat becomes positive. From the expression of specific heat in Eq. (2.7) one can see that at a critical value $m/M_{Pl}^2 = \frac{2}{3\sqrt{3}k\omega}$, the specific heat blows up. The entropy at that point is found to be positive. It is apparent from the Fig. (3) that above the critical value of the mass the specific heat is found to be positive. Therefore in AdS-Schwarzschild case depending on the values of the mass and the cosmological constant we can have a stable black hole.
III. REISSNER-NORDSTROM -ADS BLACK HOLE

We now focus our attention to the thermodynamics properties of the Reissner-Nordstrom-AdS blackhole. Let us start with the following metric which when embedded in five dimensions gives a constant curvature hypersurface.

\[ ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2 \]

\[ f(r) = 1 - \frac{2m}{M_p r} + \frac{q^2}{M_p r^2} + k^2 \omega^2 r^2 \]  

(3.1)

The mass and charge of the blackhole is given by \( m \) and \( q \) respectively whereas the AdS space curvature is determined by \( \omega \). The horizons are determined by solving equation \( f(r_H) = 0 \). In this case two of the four roots are necessarily complex and we are left with only two physical (real) roots \( r_+ \) and \( r_- \) which correspond to the outer and inner horizon. It is customary to consider the outer radius \( r_+ \) as the horizon \( r_H \). Using Eq. (3.1) we obtain the following expression for the outer horizon.

\[ r_H = \frac{1}{2} \sqrt{X + \frac{1}{2} \sqrt{-(X - \frac{6}{3 k^2 \omega^2} + \frac{4m}{M_p^2 k^2 \omega^2 \sqrt{X}})}} \]  

(3.2)

where we have used the following definitions:

\[ X = -\frac{2}{3 k^2 \omega^2} + A + B \]

\[ A = \frac{21}{3} \left(1 + 12\left(\frac{q}{M_p}\right)^2 k^2 \omega^2\right) \]

\[ B = \frac{\left[2 + 108 \frac{m^2}{M_p^2} k^2 \omega^2 - 72q^2 k^2 \omega^2 + \sqrt{\left(2 + 108 \frac{m^2}{M_p^2} k^2 \omega^2 - 72q^2 k^2 \omega^2\right)^2 - 4 \left(1 + 12q^2 k^2 \omega^2\right)^3}\right]^{1/3}}{32^{1/3} k^2 \omega^2} \]  

(3.3)

From the above expressions in Eq. (3.2) and (3.3) it is apparent that for the existence of a horizon we must have

\[ f = -X - \frac{2}{k^2 \omega^2} + \frac{4m}{M_p^2 k^2 \omega^2 \sqrt{X}} > 0 \]  

(3.4)

This in turn tells us that the charge and \( \omega^2 \) gives a bound to the blackhole mass for the existence of the horizon. This is the generalization to the condition that one obtains between mass and charge for an asymptotically flat Reissner-Nordstrom solution which in turn determines the extremality condition when the two horizons coincide. In fig. (4) we have plotted the variation of the horizon with respect to \( \omega \) for a fixed charge.
Let us determine various thermodynamic quantities for these class of blackholes and their dependence on the cosmological constant $\omega$. We first write down the expression of the Hawking temperature explicitly in terms of mass, charge and $\omega$ as follows.

$$T_H = \frac{4}{\pi} \frac{1}{16} \frac{\sqrt{X} + \sqrt{\frac{-2 - 4m^2}{M_{Pl}^2} + Xk^2\omega^2}}{3} \left(\sqrt{X} + \sqrt{\frac{-2 - 4m^2}{M_{Pl}^2} + Xk^2\omega^2}\right)^4$$

In the figures below we have shown the variation of blackhole temperature with respect $\omega$ for a definite charge. (Once again we have taken $k \simeq M_{Pl}$ and $M_{Pl} = 1$).

![Figure 5](image1)

**FIG. 5:** $r_H$ is plotted against $\omega$ and $q = 1$

We have shown earlier that for anti-de Sitter Schwarzschild black hole, the Hawking temperature is an increasing function of the cosmological constant. It may however be recalled that in flat space-time the Hawking temperature corresponding to Reissner-Nordstrom solution is a decreasing function of the charge. This explains the presence of a turning point in the $T_H - \omega$ (for a fixed value of the charge and mass) graph in the present scenario where the Hawking temperature starts decreasing with cosmological constant after certain value which depends on the charge. Expectedly the turning point shifts to the right side as the charge decreases.

The Bekenstein-Hawking entropy for this black hole is given by the expression:

$$S = \frac{1}{4} \left(\sqrt{X} + \sqrt{\frac{-2 - 4m^2}{M_{Pl}^2} + Xk^2\omega^2}\right)^2$$

![Figure 6](image2)

**FIG. 6:** $T_H$ is plotted against $\omega$ for $q = 3$
for a fixed value of $q$ the variation of entropy w.r.t $\omega$ is plotted in the figure (7).

![Graph of entropy against $\omega$ for different masses](image)

**FIG. 7:** $S$ is plotted against $\omega$ for $q = 3$

Finally we obtain the specific heat of RN-AdS blackhole as

$$C_v = \frac{2\pi r_H^2 (1 - \frac{q^2}{M_{Pl} y^2} + 3k^2 \omega^2 y^2)}{-1 + 3\frac{q^2}{M_{Pl} y^2} + 3k^2 \omega^2 y^2}$$

(3.7)

For a fixed value of $\omega$ and $q$ the variation of specific heat with respect to mass is plotted below in Fig (8).

![Graph of specific heat against mass for different $\omega$](image)

**FIG. 8:** $C_v$ is plotted against $\omega$ for $q = 3$

From the expression of the specific heat in Eq. (3.7) we see that for $C_v \to \infty$ the following condition emerges,

$$\frac{q^2}{M_{Pl} y^2} + k^2 \omega^2 y^2 = \frac{1}{3}$$

(3.8)

where $y = r_H$. This further leads to the following condition

$$0 < k\omega(q/M_{Pl}) < \frac{1}{6}$$

(3.9)

So we see that the reality condition for the horizon also determines the discontinuity of the specific heat. From the graph it is clear that we have Hawking-Page phase transition in RN-AdS black hole. Since at the blowing up point the entropy is positive, the phase transition is necessarily of second order.

**IV. SUMMARY AND OUTLOOK**

This work brings out in details the exact correlations among the various parameters like mass, charge, cosmological constant in the context of thermodynamic properties of black holes in anti-de Sitter space time. Without resorting to
any approximation our analysis reveals the interesting behaviours of thermodynamics variables like Hawking temperature, entropy, specific heat in an anti-de Sitter space-time. The parametric relations for the occurrence of Hawking-Page phase transitions and different bounds on the parameters are determined. The exact extremality condition for the Ads Reissner-Nordstrom black hole has been determined in terms of the mass, charge and cosmological constant. Our study specially emphasizes the role of cosmological constant in determining the thermodynamic properties. Such cosmological constant may have its origin in a two brane generalized Randall-Sundrum model with bulk cosmological constant appears through the parameter $k$ in the Physics on the brane. In the context of such a two brane generalized Randall-Sundrum model, the existence of an upper bound in the value of the brane cosmological constant puts further constraints and bounds on the black hole parameters like mass and charge. We have derived the exact correlation between the black hole mass and charge with an inequality relation so that a horizon can be formed on the brane. Our work thus brings out various interesting details of the thermodynamics of black holes in an anti-de Sitter space-time, embedded in a background 5-dimensional warped braneworld model.

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