ON ASYMPTOTIC BEHAVIOR OF THE PREDICTION ERROR FOR A CLASS OF DETERMINISTIC STATIONARY SEQUENCES

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Abstract. We study the prediction problem for deterministic stationary processes \(X(t)\) possessing spectral density \(f\). We describe the asymptotic behavior of the best linear mean squared prediction error \(\sigma_n^2(f)\) in predicting \(X(0)\) given \(X(t), -n \leq t \leq -1\), as \(n\) goes to infinity. We consider a class of spectral densities of the form \(f = f_dg\), where \(f_d\) is the spectral density of a deterministic process that has a very high order contact with zero due to which the Szegő condition is violated, while \(g\) is a nonnegative function that can have arbitrary power type singularities. We show that for spectral densities \(f\) from this class the prediction error \(\sigma_n^2(f)\) behaves like a power as \(n \to \infty\). Examples illustrate the obtained results.

1. Introduction

Let \(X(t), t \in \mathbb{Z} := \{0, \pm 1, \ldots\}\), be a centered discrete-time second-order stationary process. The process is assumed to have an absolutely continuous spectrum with spectral density function \(f(\lambda), \lambda \in [-\pi, \pi]\). The ‘finite’ linear prediction problem is as follows. Suppose we observe a finite realization of the process \(X(t): \{X(t), -n \leq t \leq -1\}, n \in \mathbb{N} := \{1, 2, \ldots\}\). We want to make an one-step ahead prediction, that is, to predict the unobserved random variable \(X(0)\), using the linear predictor \(Y = \sum_{k=1}^{n} c_k X(-k)\).

The coefficients \(c_k, k = 1, 2, \ldots, n,\) are chosen so as to minimize the mean-squared error: \(\mathbb{E}|X(0) - Y|^2\), where \(\mathbb{E}[\cdot]\) stands for the expectation operator. If such minimizing constants \(\hat{c}_k := \hat{c}_{k,n}\) can be found, then the random

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variable $\hat{X}_n(0) := \sum_{k=1}^{n} \hat{c}_k X(-k)$ is called the best linear one-step ahead predictor of $X(0)$ based on the observed finite past: $X(-n), \ldots, X(-1)$. The minimum mean-squared error: $\sigma^2_n(f) := \mathbb{E}|X(0) - \hat{X}_n(0)|^2 \geq 0$ is called the best linear one-step ahead prediction error of $X(t)$ based on the past of length $n$.

One of the main problems in prediction theory of second-order stationary processes is to describe the asymptotic behavior of the prediction error $\sigma^2_n(f)$ as $n \to \infty$. This behavior depends on the regularity nature (deterministic or nondeterministic) of the observed process $X(t)$.

Observe that $\sigma^2_{n+1}(f) \leq \sigma^2_n(f)$ ($n \in \mathbb{N}$), and hence, the limit of $\sigma^2_n(f)$ as $n \to \infty$ exists. Denote by $\sigma^2(f) := \sigma^2_\infty(f)$ the prediction error of $X(0)$ by the entire infinite past: $\{X(t), t \leq -1\}$.

From the prediction point of view it is natural to distinguish the class of processes for which we have error-free prediction by the entire infinite past, that is, $\sigma^2(f) = 0$. Such processes are called deterministic or singular. Processes for which $\sigma^2(f) > 0$ are called nondeterministic (for more about these terms see, e.g., Babayan et al. [4], and Grenander and Szegő [9, p. 176]).

Define the relative prediction error $\delta_n(f) := \sigma^2_n(f) - \sigma^2(f)$, and observe that $\delta_n(f)$ is non-negative and tends to zero as $n \to \infty$. But what about the speed of convergence of $\delta_n(f)$ to zero as $n \to \infty$? The paper deals with this question. Specifically, the prediction problem we are interested in is to describe the rate of decrease of $\delta_n(f)$ to zero as $n \to \infty$, depending on the regularity nature (deterministic or nondeterministic) of the observed process $X(t)$.

The prediction problem stated above goes back to classical works of Kolmogorov, Szegő and Wiener. It was then considered by many authors for different classes of nondeterministic processes (see, e.g., the survey papers Bingham [5] and Ginovyan [8], and references therein).

We focus in this paper on deterministic processes, that is, when $\sigma^2(f) = 0$. This case is not only of theoretical interest, but is also important from the point of view of applications. For example, as pointed out by Rosenblatt [14] (see also Pierson [10]), situations of this type arise in Neumann’s theoretical model of storm-generated ocean waves.

Only few works are devoted to the study of the speed of convergence of $\delta_n(f) = \sigma^2_n(f)$ to zero as $n \to \infty$, that is, the asymptotic behavior of the prediction error for deterministic processes. One needs to go back to the classical work of Rosenblatt [14], where the asymptotic behavior of the prediction error $\sigma^2_n(f)$ was investigated in the following two cases:

(a) the spectral density $f(\lambda)$ is continuous and positive on an interval of the segment $[-\pi, \pi]$ and zero elsewhere,

(b) the spectral density $f(\lambda)$ has a very high order of contact with zero at points $\lambda = 0, \pm \pi$, and is strictly positive otherwise.

Acta Mathematica Hungarica 167, 2022
For the case (a) above, Rosenblatt [14] proved that the prediction error \( \sigma_n^2(f) \) decreases to zero exponentially as \( n \to \infty \). Later, the problem (a) was studied by Babayan [1,2] and Babayan et al. [4] (see also Davisson [7]), where some extensions of Rosenblatt’s result have been obtained.

Concerning the case (b) above, for a specific deterministic process \( X(t) \), Rosenblatt proved in [14] that the prediction error \( \sigma_n^2(f) \) decreases to zero like a power as \( n \to \infty \). More precisely, the deterministic process \( X(t) \) considered in Rosenblatt [14] has the spectral density

\[
(1.1) \quad f_a(\lambda) := \frac{e^{(2\lambda-\pi)\varphi(\lambda)}}{\cosh(\pi\varphi(\lambda))}, \quad f_a(-\lambda) = f_a(\lambda), \quad 0 \leq \lambda \leq \pi,
\]

where \( \varphi(\lambda) = (a/2)\cot\lambda \) and \( a \) is a positive parameter.

Using the technique of orthogonal polynomials on the unit circle and Szegő’s results, Rosenblatt [14] proved the following theorem.

**Theorem A (Rosenblatt [14]).** Suppose that the process \( X(t) \) has spectral density \( f_a \) given by (1.1). Then the following asymptotic relation for the prediction error \( \sigma_n^2(f_a) \) holds:

\[
(1.2) \quad \frac{\sigma_n^2(f_a)}{\Gamma^2((a+1)/2)} \sim \frac{\Gamma^2((a+1)/2)}{\pi 2^{2-a}} n^{-a} \quad \text{as} \quad n \to \infty.
\]

Note that the function in (1.1) was first considered by Pollaczek [11], and then by Szegő [16], as a weight-function of a class of orthogonal polynomials that serve as illustrations for certain ‘irregular’ phenomena in the theory of orthogonal polynomials. For the function \( f_a \) in (1.1), we have the following asymptotic relation (for details see Szegő [16] and Example 6.3 below):

\[
(1.3) \quad f_a(\lambda) \sim \begin{cases} 2e^a \exp\{-a\pi/|\lambda|\} & \text{as} \quad \lambda \to 0, \\ 2\exp\{-a\pi/(|\lambda|-\pi)\} & \text{as} \quad \lambda \to \pm\pi. \end{cases}
\]

Thus, the function \( f_a \) in (1.1) has a very high order of contact with zero at points \( \lambda = 0, \pm\pi \), due to which the process with spectral density \( f_a \) is deterministic and the prediction error \( \sigma_n^2(f_a) \) in (1.2) decreases to zero like a power as \( n \to \infty \).

In Babayan et al. [4] (see also Babayan and Ginovyan [3]) it was proved that if the spectral density \( f \) is such that the sequence \( \{\sigma_n(f)\} \) is weakly varying (a term defined in Section 2.3) and if, in addition, \( g \) is a nonnegative function that can have polynomial type singularities, then the sequences \( \{\sigma_n(fg)\} \) and \( \{\sigma_n(f)\} \) have the same asymptotic behavior as \( n \to \infty \) (see Theorem B in Section 3). Using this result, Rosenblatt’s Theorem A was extended in Babayan et al. [4] to a class of spectral densities of the form \( f = f_0g \), where \( f_0 \) is as in (1.1) and \( g \) is a nonnegative function that can have polynomial type singularities (see Theorem C in Section 3).
In this paper, we extend the above quoted results to a broader class of spectral densities, for which the function \( g \) can have arbitrary power type singularities.

Throughout the paper we will use the following notation. The standard symbols \( \mathbb{N}, \mathbb{Z}, \mathbb{R} \) and \( \mathbb{C} \) denote the sets of natural, integer, real and complex numbers, respectively. Also, we denote \( \Lambda := [-\pi, \pi], \mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \} \). For a point \( \lambda_0 \in \Lambda \) and a number \( \delta > 0 \) by \( O_\delta(\lambda_0) \) we denote a \( \delta \)-neighborhood of \( \lambda_0 \), that is, \( O_\delta(\lambda_0) := \{ \lambda \in \Lambda : |\lambda - \lambda_0| < \delta \} \). By \( L^p(\mu) := L^p(\mathbb{T}, \mu) \ (p \geq 1) \) we denote the weighted Lebesgue space with respect to the measure \( \mu \), and by \( \| \cdot \|_{p, \mu} \) we denote the norm in \( L^p(\mu) \). In the special case where \( \mu \) is the Lebesgue measure, we will use the notation \( L^p \) and \( \| \cdot \|_p \), respectively. For a function \( f \geq 0 \) by \( G(f) \) we denote the geometric mean of \( f \). For two functions \( f(\lambda) \geq 0 \) and \( g(\lambda) \geq 0, \lambda \in \Lambda \), we will write \( f(\lambda) \sim g(\lambda) \) as \( \lambda \to \lambda_0 \) if \( \lim_{\lambda \to \lambda_0} f(\lambda)/g(\lambda) = 1 \), and \( f(\lambda) \simeq g(\lambda) \) as \( \lambda \to \lambda_0 \) if \( \lim_{\lambda \to \lambda_0} f(\lambda)/g(\lambda) = c > 0 \). We will use similar notation for sequences: for two sequences \( \{a_n \geq 0, n \in \mathbb{N}\} \) and \( \{b_n > 0, n \in \mathbb{N}\} \), we will write \( a_n \sim b_n \) if \( \lim_{n \to \infty} a_n/b_n = 1 \), \( a_n \simeq b_n \) if \( \lim_{n \to \infty} a_n/b_n = c > 0 \); \( a_n = O(b_n) \) if \( a_n/b_n \) is bounded, and \( a_n = o(b_n) \) if \( a_n/b_n \to 0 \) as \( n \to \infty \). The letters \( C, c, M \) and \( m \) with or without indices are used to denote positive constants.

The paper is organized as follows. In Section 2 we present some necessary notions and preliminary results that are used throughout the paper. In Section 3 we state some auxiliary results. In Section 4 we state the main results of the paper. Section 5 contains the proofs of the main results. In Section 6 we discuss some examples illustrating the obtained results.

2. Preliminaries

In this section we present some notions and auxiliary results, which will be used in the sequel: the Kolmogorov–Szegő Theorem, formulas and some properties of the finite prediction error, properties of the geometric mean of a function, and definition and properties of weakly varying sequences.

2.1. Kolmogorov–Szegő’s Theorem. Let \( X(t) \) be a centered discrete-time stationary process defined on a probability space \((\Omega, \mathcal{F}, P)\) with covariance function \( r(t) \), \( t \in \mathbb{Z} \). By the Herglotz theorem (see, e.g., Brockwell and Davis [6], pp. 117-118), there is a finite measure \( \mu \) on \( \Lambda \) such that the covariance function \( r(t) \) admits the following spectral representation:

\[
(2.1) \quad r(t) = \int_{-\pi}^{\pi} e^{-it\lambda} d\mu(\lambda), \quad t \in \mathbb{Z}.
\]

The measure \( \mu \) in (2.1) is called the spectral measure of the process \( X(t) \). If \( \mu \) is absolutely continuous (with respect to the Lebesgue measure), then
the function \( f(\lambda) := \frac{d\mu(\lambda)}{d\lambda} \) is called the spectral density of \( X(t) \). We assume that \( X(t) \) is a non-degenerate process, that is, \( \text{Var}[X(0)] = E|X(0)|^2 = r(0) > 0 \) and, without loss of generality, we may take \( r(0) = 1 \). Also, to avoid the trivial cases, we assume that the spectral measure \( \mu \) is non-trivial, that is, the support of \( \mu \) has positive Lebesgue measure.

**Remark 2.1.** The parametrization of the unit circle \( \mathbb{T} \) by the formula \( z = e^{i\lambda} \) establishes a bijection between \( \mathbb{T} \) and the interval \( [-\pi, \pi) \). By means of this bijection the measure \( \mu \) on \( \Lambda \) generates the corresponding measure on the unit circle \( \mathbb{T} \), which we also denote by \( \mu \). Thus, depending on the context, the measure \( \mu \) will be supported either on \( \Lambda \) or on \( \mathbb{T} \). We use the standard Lebesgue decomposition of the measure \( \mu \):

\[
d\mu(\lambda) = d\mu_a(\lambda) + d\mu_s(\lambda) = f(\lambda) \, d\lambda + d\mu_s(\lambda),
\]

where \( \mu_a \) is the absolutely continuous part of \( \mu \) (with respect to the Lebesgue measure) and \( \mu_s \) is the singular part of \( \mu \), which is the sum of the discrete and continuous singular components of \( \mu \).

The next result describes the asymptotic behavior of the prediction error \( \sigma_n^2(\mu) \) for a stationary process \( X(t) \) with spectral measure \( \mu \) of the form (2.2) and gives a spectral characterization of deterministic and nondeterministic processes (see, e.g., Grenander and Szegö [9, p. 44]).

**Theorem (Kolmogorov–Szegö’s Theorem).** Let \( X(t) \) be a non-degenerate stationary process with spectral measure \( \mu \) of the form (2.2). The following relations hold.

\[
\lim_{n \to \infty} \sigma_n^2(\mu) = \lim_{n \to \infty} \sigma_n^2(f) = \sigma^2(f) = 2\pi G(f),
\]

where \( G(f) \) is the geometric mean of \( f \), namely

\[
G(f) := \begin{cases} 
\exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln f(\lambda) \, d\lambda \right\} & \text{if } \ln f \in L^1(\Lambda) \\
0 & \text{otherwise}.
\end{cases}
\]

It is remarkable that the limit in (2.3) is independent of the singular part \( \mu_s \).

The condition \( \ln f \in L^1(\Lambda) \) in (2.4) is equivalent to the Szegö condition:

\[
\int_{-\pi}^{\pi} \ln f(\lambda) \, d\lambda > -\infty
\]

(this equivalence follows because \( \ln f(\lambda) \leq f(\lambda) \)). The Szegö condition (2.5) is also called the non-determinism condition.

In this paper we consider the class of deterministic processes with absolutely continuous spectra. We will assume that the corresponding spectral
Using Kolmogorov’s isometric isomorphism \( V \), in view of (2.6), for the prediction error very high order of contact with zero at a point \( a \) tended to ahead prediction error:

\[
(2.7)-(2.8) \text{ is called the one-step ahead linear prediction problem.}
\]

The result by Szegő solves the minimum problem (2.7)-(2.8) (see, e.g., Grenander et al. [4]), in view of (2.6), for the prediction error \( \sigma^2_n(\mu) \) we can write

\[
(2.7) \quad \sigma^2_n(\mu) = \min_{\{c_k\}} \left\| 1 - \sum_{k=1}^{n} c_k e^{-ik\lambda} \right\|_{2,\mu}^2 = \min_{\{q_n \in \mathcal{Q}_n\}} \|q_n\|_{2,\mu}^2,
\]

where \( \| \cdot \|_{2,\mu} \) is the norm in \( L^2(\mathbb{T}, \mu) \), and

\[
(2.8) \quad \mathcal{Q}_n := \{ q_n : q_n(z) = z^n + c_1 z^{n-1} + \cdots + c_n \}
\]

is the class of monic polynomials (i.e. with \( c_0 = 1 \)) of degree \( n \). Thus, the problem of finding \( \sigma^2_n(\mu) \) becomes to the problem of finding the solution of the minimum problem (2.7)-(2.8).

The polynomial \( p_n(z) := p_n(z, \mu) \) which solves the minimum problem (2.7)-(2.8) is called the optimal polynomial for \( \mu \) in the class \( \mathcal{Q}_n \). The next result by Szegő solves the minimum problem (2.7)-(2.8) (see, e.g., Grenander and Szegő [9, p. 38]).

**Proposition 2.1.** The unique solution of the minimum problem (2.7)-(2.8) is given by \( p_n(z) = \kappa_n^{-1} \varphi_n(z) \), and the minimum in (2.7) is equal to \( \|p_n\|_{2,\mu}^2 = \kappa_n^{-2} \), where \( \varphi_n(z) = \kappa_n z^n + \cdots + l_n \) is the \( n^{th} \) orthogonal polynomial on the unit circle associated with the measure \( \mu \), and \( \kappa_n \) is the leading coefficient of \( \varphi_n(z) \).

Thus, for the prediction error \( \sigma^2_n(\mu) \) we have the following formula:

\[
(2.9) \quad \sigma^2_n(\mu) = \min_{\{q_n \in \mathcal{Q}_n\}} \|q_n\|_{2,\mu}^2 = \|p_n(\mu)\|_{2,\mu}^2 = \|\kappa_n^{-1} \varphi_n(\mu)\|_{2,\mu}^2 = \kappa_n^{-2}.
\]
Denote by $D_n = D_n(\mu) := \det[r(t - s), t, s = 0, 1, \ldots n]$ the $n^{th}$ Toeplitz determinant generated by the measure $\mu$, where $r(t)$ is the covariance function given by (2.1). Taking into account that $\kappa_n^2 = D_{n-1}/D_n$ (see, e.g., Grenander and Szegö [9, p. 38]), in view of (2.9) we obtain the following formula for the prediction error $\sigma_n^2(\mu)$:

\begin{equation}
\sigma_n^2(\mu) = \frac{D_n(\mu)}{D_{n-1}(\mu)}.
\end{equation}

In what follows we assume that the spectral measure $\mu$ is absolutely continuous with spectral density $f$, and instead of $\sigma_n^2(\mu), p_n(\mu)$ and $D_n(\mu)$ we use the notation $\sigma_n^2(f), p_n(f)$ and $D_n(f)$, respectively. The next proposition contains some properties of the prediction error $\sigma_n^2(f)$.

**Proposition 2.2.** The prediction error $\sigma_n^2(f)$ possesses the following properties.

(a) $\sigma_n^2(f)$ is a non-decreasing functional of $f$: $\sigma_n^2(f_1) \leq \sigma_n^2(f_2)$ when $f_1(\lambda) \leq f_2(\lambda), \lambda \in [-\pi, \pi]$.

(b) If $f(\lambda) = g(\lambda)$ almost everywhere on $[-\pi, \pi]$, then $\sigma_n^2(f) = \sigma_n^2(g)$.

(c) For any positive constant $c$ we have $\sigma_n^2(cf) = c\sigma_n^2(f)$.

(d) If $f(\lambda) = f(\lambda - \lambda_0), \lambda_0 \in [-\pi, \pi]$, then $\sigma_n^2(f) = \sigma_n^2(\bar{f})$.

**Proof.** To prove assertion (a), observe that by the definition of optimal polynomials $p_n(z, f_1)$ and $p_n(z, f_2)$, corresponding to spectral densities $f_1$ and $f_2$, respectively, and formula (2.9) we have

\[ \sigma_n^2(f_1) = \|p_n(f_1)\|_{2,f_1}^2 \leq \|p_n(f_2)\|_{2,f_1}^2 \leq \|p_n(f_2)\|_{2,f_2}^2 = \sigma_n^2(f_2). \]

In the last relation the first inequality follows from the optimality of the polynomial $p_n(z, f_1)$, while the second inequality follows from assumption that $f_1(\lambda) \leq f_2(\lambda), \lambda \in \Lambda$.

We show that assertions (b)–(d) follow from formula (2.10). Indeed, observing that the elements of the Toeplitz determinant $D_n(f)$ being integrals are invariant with respect to null sets, in view of (2.10) we obtain assertion (b). To prove assertion (c), observe that for the Toeplitz determinants generated by the functions $cf$ and $f$ we have $D_n(cf) = c^nD_n(f)$ (see Grenander and Szegö [9, pp. 64-65]). Hence in view of (2.10) we have

\[ \sigma_n^2(cf) = \frac{D_n(cf)}{D_{n-1}(cf)} = \frac{c^nD_n(f)}{c^{n-1}D_{n-1}(f)} = c\sigma_n^2(f). \]

Finally, the assertion (d) follows from (2.10) and the fact that $D_n(\bar{f}) = D_n(f)$ (see Grenander and Szegö [9, pp. 64-65]). □

Recall that for a function $h \geq 0$ by $G(h)$ we denote the geometric mean of $h$ (see formula (2.4)). In the next proposition we list some properties of the geometric mean $G(h)$ (see Babayan et al. [4]).
Proposition 2.3. The following assertions hold.

(a) Let $c > 0$, $\alpha \in \mathbb{R}$, $f(\lambda) \geq 0$ and $g(\lambda) \geq 0$. Then

\begin{equation}
G(c) = c, \quad G(fg) = G(f)G(g), \quad G(f^\alpha) = G^\alpha(f).
\end{equation}

(b) $G(f)$ is a non-decreasing functional of $f$: if $0 \leq f(\lambda) \leq g(\lambda)$, then $0 \leq G(f) \leq G(g)$. In particular, if $0 \leq f(\lambda) \leq 1$, then $0 \leq G(f) \leq 1$.

(c) If $t(\lambda)$ is a nonnegative trigonometric polynomial, then $G(t^\alpha) > 0$ for $\alpha \in \mathbb{R}$.

2.3. Weakly varying sequences. We recall the notion of weakly varying sequences and state some of their properties (see Babayan et al. [4]). This notion will be used in the specification of the class of deterministic processes to be considered in this paper.

Definition 2.1. A sequence of non-zero numbers \( \{a_n, n \in \mathbb{N}\} \) is said to be weakly varying if $\lim_{n \to \infty} a_{n+1}/a_n = 1$.

In the next proposition we list some simple properties of the weakly varying sequences (see Babayan et al. [4]).

Proposition 2.4. The following assertions hold.

(a) If $\{a_n, n \in \mathbb{N}\}$ is a weakly varying sequence, then $\lim_{n \to \infty} a_{n+\nu}/a_n = 1$ for any $\nu \in \mathbb{N}$.

(b) If $\{a_n, n \in \mathbb{N}\}$ is a sequence such that $a_n \to a \neq 0$ as $n \to \infty$, then $\{a_n\}$ is a weakly varying sequence.

(c) If $\{a_n, n \in \mathbb{N}\}$ and $\{b_n, n \in \mathbb{N}\}$ are weakly varying sequences, then $ca_n$ ($c \neq 0$), $a_n^\alpha$ ($\alpha \in \mathbb{R}, a_n > 0$), $a_nb_n$ and $a_n/b_n$ also are weakly varying sequences.

(d) If $\{a_n, n \in \mathbb{N}\}$ is a weakly varying sequence, and $\{b_n, n \in \mathbb{N}\}$ is a sequence of non-zero numbers such that $\lim_{n \to \infty} b_n/a_n = c \neq 0$, then $\{b_n, n \in \mathbb{N}\}$ is also a weakly varying sequence.

3. Asymptotic behavior of the prediction error. Auxiliary results

In this section we state some auxiliary results concerning asymptotic behavior of the prediction error (cf. Babayan et al. [4]).

In what follows we consider the class of deterministic processes possessing spectral densities $f$ for which the sequence of prediction errors $\{\sigma_n(f)\}$ is weakly varying, and denote by $\mathcal{F}$ the class of the corresponding spectral densities:

\begin{equation}
\mathcal{F} := \left\{ f \in L^1(\Lambda) : f \geq 0, \ G(f) = 0, \ \lim_{n \to \infty} \frac{\sigma_{n+1}(f)}{\sigma_n(f)} = 1 \right\}.
\end{equation}
Remark 3.1. According to Rakhmanov’s theorem (see Rakhmanov [12, 13], Babayan et al. [4], Simon [15, p. 5]) a sufficient condition for \( f \in \mathcal{F} \) is that \( f > 0 \) almost everywhere on \( \Lambda \) and \( G(f) = 0 \). Thus, the considered class \( \mathcal{F} \) includes all deterministic processes \( (G(f) = 0) \) with almost everywhere positive spectral densities \( (f > 0 \text{ a.e.}) \). On the other hand, according to Theorem 3.2 and Remark 3.7 of Babayan et al. [4], the class \( \mathcal{F} \) does not contain spectral densities, which vanish on an entire segment of \( \Lambda \) (or on an arc of the unit circle \( T \)).

Definition 3.1. Let \( \mathcal{F} \) be the class of spectral densities defined by (3.1). For \( f \in \mathcal{F} \) denote by \( \mathcal{M}_f \) the class of nonnegative functions \( g(\lambda) \ (\lambda \in \Lambda) \) satisfying the following three conditions: \( G(g) > 0, \ f g \in L^1(\Lambda), \) and

\[
\lim_{n \to \infty} \frac{\sigma_n^2(fg)}{\sigma_n^2(f)} = G(g),
\]

that is,

\[
\mathcal{M}_f := \left\{ g \geq 0, \ G(g) > 0, \ f g \in L^1(\Lambda), \ \lim_{n \to \infty} \frac{\sigma_n^2(fg)}{\sigma_n^2(f)} = G(g) \right\}.
\]

The next proposition shows that the class \( \mathcal{F} \) is close under multiplication by functions from the class \( \mathcal{M}_f \).

Proposition 3.1. If \( f \in \mathcal{F} \) and \( g \in \mathcal{M}_f \), then \( fg \in \mathcal{F} \).

Proof. According to the definition of the class \( \mathcal{F} \), we have to show that \( f g \in L^1(\Lambda), \ G(fg) = 0, \) and

\[
\lim_{n \to \infty} \frac{\sigma_{n+1}(fg)}{\sigma_n(fg)} = 1.
\]

The assertion \( fg \in L^1(\Lambda) \) follows from the condition \( g \in \mathcal{M}_f \), while \( G(fg) = 0 \) follows from the condition \( f \in \mathcal{F} \) and Proposition 2.3(a): \( G(fg) = G(f)G(g) = 0 \). As for the relation (3.4), we have

\[
\lim_{n \to \infty} \frac{\sigma_{n+1}(fg)}{\sigma_n(fg)} = \lim_{n \to \infty} \frac{\sigma_{n+1}(f)}{\sigma_n(f)} \cdot \frac{\sigma_{n+1}(f)}{\sigma_n(f)} \cdot \frac{\sigma_n(f)}{\sigma_n(fg)} = \lim_{n \to \infty} \frac{\sigma_{n+1}(f)}{\sigma_n(f)} \cdot \lim_{n \to \infty} \frac{\sigma_{n+1}(f)}{\sigma_n(f)} \cdot \lim_{n \to \infty} \frac{\sigma_n(f)}{\sigma_n(fg)} = \sqrt{G(g)} \cdot 1/\sqrt{G(g)} = 1,
\]

and the result follows. □

The next result shows that the class \( \mathcal{M}_f \) in a certain sense is close under multiplication.
Proposition 3.2. Let \( f \in \mathcal{F} \). If \( g_1 \in \mathcal{M}_f \) and \( g_2 \in \mathcal{M}_{fg_1} \), then \( g := g_1g_2 \in \mathcal{M}_f \) and \( fg \in \mathcal{F} \). In particular, if \( g \in \mathcal{M}_f \cap \mathcal{M}_{fg} \), then \( g^2 \in \mathcal{M}_f \).

Proof. By the definition of the classes \( \mathcal{M}_f \) and \( \mathcal{M}_{fg_1} \), we have \( G(g_1) > 0, G(g_2) > 0 \), and hence, by Proposition 2.3(a), \( G(g) > 0 \). By Proposition 3.1 we have \( fg_1 \in \mathcal{F} \). Hence, taking into account that \( g_2 \in \mathcal{M}_{fg_1} \), we have \( fg_1g_2 \in L^1(\Lambda) \), and

\[
\lim_{n \to \infty} \frac{\sigma_n^2(fg_1g_2)}{\sigma_n^2(fg_1)} = G(g_2).
\]

Then, we can write

\[
\lim_{n \to \infty} \frac{\sigma_n^2(fg)}{\sigma_n^2(f)} = \lim_{n \to \infty} \frac{\sigma_n^2(fg_1g_2)}{\sigma_n^2(fg_1)} = \lim_{n \to \infty} \frac{\sigma_n^2(fg_1g_2)}{\sigma_n^2(fg_1)} \cdot \frac{\sigma_n^2(fg_1)}{\sigma_n^2(f)}
\]

\[
= \lim_{n \to \infty} \frac{\sigma_n^2(fg_1g_2)}{\sigma_n^2(fg_1)} \cdot \lim_{n \to \infty} \frac{\sigma_n^2(fg_1)}{\sigma_n^2(f)} = G(g_1) \cdot G(g_2) = G(g).
\]

In the last relation the fourth equality follows from (3.5) and the condition \( g_1 \in \mathcal{M}_f \), while the fifth equality follows from Proposition 2.3(a). Thus, we have proved that \( g \in \mathcal{M}_f \), from this and Proposition 3.1 it follows that \( fg \in \mathcal{F} \). \( \square \)

In the next definition we introduce certain classes of bounded functions.

Definition 3.2. We define the class \( B \) to be the set of all nonnegative, Riemann integrable on \( \Lambda = [-\pi, \pi] \) functions \( h(\lambda) \). Also, we define the following subclasses:

\( B_+ := \{ h \in B : h(\lambda) \geq m \} \), \( B^- := \{ h \in B : h(\lambda) \leq M \} \), \( B^-_+ := B_+ \cap B^- \),

where \( m \) and \( M \) are some positive constants.

In the next proposition we list some obvious properties of the classes \( B_+ \), \( B^- \) and \( B^-_+ \).

Proposition 3.3. The following assertions hold.

a) If \( h \in B_+(B^-) \), then \( 1/h \in B^-(B^+)_+ \).

b) If \( h_1, h_2 \in B_+(B^-) \), then \( h_1 + h_2 \in B_+(B^-) \) and \( h_1h_2 \in B_+(B^-) \).

c) If \( h_1, h_2 \in B^- \) and \( h_1/h_2 \) is bounded, then \( h_1/h_2 \in B^- \).

d) If \( h_1, h_2 \in B_+, \) then \( h_1 + h_2 \in B_+, \) \( h_1h_2 \in B_+ \) and \( h_1/h_2 \in B^- \).

The following theorem, proved in Babayan et al. [4], describes the asymptotic behavior of the ratio \( \sigma_n^2(fg)/\sigma_n^2(f) \) as \( n \to \infty \), and essentially states that if the spectral density \( f \) is from the class \( \mathcal{F} \) (see (3.1)), and \( g \) is a nonnegative function, which can have polynomial type singularities, then the sequences \( \{ \sigma_n(fg) \} \) and \( \{ \sigma_n(f) \} \) have the same asymptotic behavior as \( n \to \infty \) up to a positive numerical factor.
Theorem B (Babayan et al. [4]). Let \( f \) be an arbitrary function from the class \( \mathcal{F} \), and let \( g \) be a function of the form:

\[
\tag{3.6}
g(\lambda) = h(\lambda) \cdot \frac{t_1(\lambda)}{t_2(\lambda)}, \quad \lambda \in \Lambda,
\]

where \( h \in B_+^\infty \), \( t_1 \) and \( t_2 \) are nonnegative trigonometric polynomials, such that \( fg \in L^1(\Lambda) \). Then \( g \in M_f \) and \( fg \in \mathcal{F} \), that is, \( fg \) is the spectral density of a deterministic process with weakly varying prediction error, and the relation (3.2) holds.

Taking into account that the sequence \( \{n^{-\alpha}, n \in \mathbb{N}, \alpha > 0\} \) is weakly varying, as a consequence of Theorem B, we have the following result.

Corollary 3.1 (Babayan et al. [4]). Let the functions \( f \) and \( g \) be as in Theorem B, and let \( \sigma_n(f) \sim cn^{-\alpha} \) \( (c > 0, \alpha > 0) \) as \( n \to \infty \). Then

\[
\sigma_n(fg) \sim cG(g)n^{-\alpha} \quad \text{as} \quad n \to \infty,
\]

where \( G(g) \) is the geometric mean of \( g \).

The next result, which immediately follows from Theorem B and Corollary 3.1, extends Rosenblatt’s Theorem A.

Theorem C (Babayan et al. [4]). Let \( f = f_ag \), where \( f_a \) is defined by (1.1) and \( g \) satisfies the assumptions of Theorem B. Then

\[
\sigma^2_n(f) \sim \frac{\Gamma^2\left(\frac{a+1}{2}\right)G(g)}{\pi^{2-a}} n^{-a} \quad \text{as} \quad n \to \infty,
\]

where \( G(g) \) is the geometric mean of \( g \).

We thus have the same limiting behavior for \( \sigma^2_n(f) \) as in the Rosenblatt’s relation (1.2) up to an additional positive factor \( G(g) \).

4. Asymptotic behavior of the prediction error. Main results

In this section we state the main results of this paper, extending the above stated Theorems B and C to a broader class of spectral densities, for which the function \( g \) can have arbitrary power type singularities.

Theorem 4.1. Let \( f \) be an arbitrary function from the class \( \mathcal{F} \), and let \( g \) be a function of the form:

\[
\tag{4.1}
g(\lambda) = h(\lambda) \cdot |t(\lambda)|^\alpha, \quad \alpha > 0, \ \lambda \in \Lambda,
\]

where \( h \in B_+^\infty \) and \( t \) is an arbitrary trigonometric polynomial. Then \( g \in M_f \) and \( fg \in \mathcal{F} \), that is, \( fg \) is the spectral density of a deterministic process with weakly varying prediction error, and the relation (3.2) holds.
Corollary 4.1. The conclusion of Theorem 4.1 remains valid if the function \( g \) has the following form:

\[
g(\lambda) = h(\lambda) \cdot |t_1(\lambda)|^{\alpha_1} \cdot |t_2(\lambda)|^{\alpha_2} \cdots |t_m(\lambda)|^{\alpha_m}, \quad \lambda \in \Lambda,
\]

where \( h \in B_+^\infty \), \( t_1, t_2, \ldots, t_m \) are arbitrary trigonometric polynomials, \( \alpha_1, \alpha_2, \ldots, \alpha_m \) are arbitrary positive numbers, and \( m \in \mathbb{N} \).

Theorem 4.2. Let \( f \) be an arbitrary function from the class \( \mathcal{F} \), and let \( g \) be a function of the form:

\[
g(\lambda) = h(\lambda) \cdot t^{-\alpha}(\lambda), \quad \alpha > 0, \quad \lambda \in \Lambda,
\]

where \( h \in B_+^\infty \) and \( t \) is a nonnegative trigonometric polynomial. Then the following assertions hold.

(a) \( g \in \mathcal{M}_f \) and \( fg \in \mathcal{F} \) provided that \( \alpha \in \mathbb{Z} \) and \( ft^{-\alpha} \in L^1(\Lambda) \).

(b) \( g \in \mathcal{M}_f \) and \( fg \in \mathcal{F} \) provided that \( \alpha \notin \mathbb{Z} \) and \( ft^{-(k+1)} \in L^1(\Lambda) \), where \( k := \lfloor \alpha \rfloor \) is the integer part of \( \alpha \).

To state the next result we need the following definition.

Definition 4.1. Let \( E_1 \) and \( E_2 \) be two numerical sets such that for any \( x \in E_1 \) and \( y \in E_2 \) we have \( x < y \). We say that the sets \( E_1 \) and \( E_2 \) are separated from each other if \( \sup E_1 < \inf E_2 \). Also, we say that a numerical set \( E \) is separated from infinity if it is bounded from above.

Theorem 4.3. Let \( f(\lambda) \) and \( \hat{f}(\lambda) \) \( (\lambda \in \Lambda) \) be spectral densities of stationary processes satisfying the following conditions:

1) \( f, \hat{f} \in B^- \);

2) the functions \( f(\lambda) \) and \( \hat{f}(\lambda) \) have \( k \) common essential zeros \( \lambda_1, \lambda_2, \ldots, \lambda_k \in \Lambda \) \((-\pi < \lambda_1 < \lambda_2 < \cdots < \lambda_k \leq \pi, k \in \mathbb{N})\),

that is,

\[
(4.3) \quad \lim_{\lambda \to \lambda_j} f(\lambda) = \lim_{\lambda \to \lambda_j} \hat{f}(\lambda) = 0, \quad j = 1, 2, \ldots, k;
\]

3) the functions \( f(\lambda) \) and \( \hat{f}(\lambda) \) are infinitesimal of the same order in a neighborhood of each point \( \lambda_j \) \((j = 1, 2, \ldots, k)\), that is,

\[
(4.4) \quad \lim_{\lambda \to \lambda_j} \frac{\hat{f}(\lambda)}{f(\lambda)} = c_j > 0, \quad j = 1, 2, \ldots, k;
\]

4) the functions \( f(\lambda) \) and \( \hat{f}(\lambda) \) are bounded away from zero outside any neighborhood \( O_\delta(\lambda_j) \) \((j = 1, 2, \ldots, k)\), which is separated from the neighboring zeros \( \lambda_{j-1} \) and \( \lambda_{j+1} \) of \( \lambda_j \), that is, there is a number \( m := m_\delta > 0 \) such that
\( f(\lambda) \geq m \) and \( \hat{f}(\lambda) \geq m \) for almost all \( \lambda \not\in \bigcup_{j=1}^{k} O_{\delta}(\lambda_j) \). Then the following assertions hold:

a) \( h(\lambda) := \frac{\hat{f}(\lambda)}{f(\lambda)} \in B^{-}_{+}; \)

b) the processes with spectral densities \( f \) and \( \hat{f} \) either both are deterministic or both are nondeterministic;

c) if one of the functions \( f \) and \( \hat{f} \) is from the class \( F \), then so is the other, and the following relation holds:

\[
\lim_{n \to \infty} \frac{\sigma^2_n(\hat{f})}{\sigma^2_n(f)} = G(h) > 0.
\]

**Remark 4.1.** The conditions of Theorem 4.3 mean that the points \( \lambda_j \) \((j = 1, 2, \ldots, k)\) are the only common zeros of functions \( f(\lambda) \) and \( \hat{f}(\lambda) \). Besides, in the case of deterministic processes, at least one of these zeros should be of sufficiently high order. Also, notice that the conditions 1) and 4) of Theorem 4.3 will be satisfied if the functions \( f(\lambda) \) and \( \hat{f}(\lambda) \) are continuous on \( \Lambda \).

**Theorem 4.4.** Let \( f \) be an arbitrary function from the class \( F \), and let \( g \) be a function of the form:

\[
g(\lambda) = h(\lambda) \cdot |q(\lambda)|^{\alpha}, \quad \alpha \in \mathbb{R}, \quad \lambda \in \Lambda,
\]

where \( h \in B^{-}_{+}, q \) is an arbitrary algebraic polynomial with real coefficients, and \( fg \in L^{1}(\Lambda) \). Then \( fg \in F \) and \( g \in \mathcal{M}_{f} \).

The next result extends Theorem C to a broader class of functions \( g \).

**Corollary 4.2.** Let \( f = f_{a}g \), where \( f_{a} \) is defined by (1.1), and let \( g \) be a function satisfying the conditions of one of Theorems 4.1, 4.2, 4.4 or Corollary 4.1. Then

\[
\delta_n(f) = \sigma^2_n(f) \sim \frac{\Gamma^2\left(\frac{a+1}{2}\right)G(g)}{\pi^{2a-2}} n^{-a} \quad \text{as} \quad n \to \infty,
\]

where \( G(g) \) is the geometric mean of \( g \).

**Remark 4.2.** In view of Remark 3.1 it follows that all the above stated results remain true if the condition \( f \in \mathcal{F} \) is replaced by the slightly strong but more constructive condition: ‘the spectral density \( f \) is positive \((f > 0)\) almost everywhere on \( \Lambda \) and \( G(f) = 0 \).’

5. **Proofs**

In this section we prove the main results of this paper stated in Section 4. We first establish a number of lemmas.
5.1. Lemmas.

**Lemma 5.1.** Let $f$ be an arbitrary function from the class $\mathcal{F}$, and let $t$ be a nonnegative trigonometric polynomial. Then $t^{1/2} \in \mathcal{M}_f$ and $ft^{1/2} \in \mathcal{F}$.

**Proof.** Observe first that, by Proposition 3.1, the second assertion of the lemma ($ft^{1/2} \in \mathcal{F}$) follows from the first assertion ($t^{1/2} \in \mathcal{M}_f$). So, to complete the proof of the lemma we have to prove the relation $t^{1/2} \in \mathcal{M}_f$.

To this end, we first verify the first two conditions for $t^{1/2}$ to belong to the class $\mathcal{M}_f$, that is, the conditions: $G(t^{1/2}) > 0$ and $ft^{1/2} \in L^1(\Lambda)$ (see (3.3)).

First, the condition $G(t^{1/2}) > 0$ follows from Proposition 2.3(c). Next, observing that the function $t^{1/2}$ is continuous, and hence is bounded, in view of the condition $f \in \mathcal{F}$, we conclude that $ft^{1/2} \in L^1(\Lambda)$. Also, observe that by Proposition 2.3(a), we have $G(ft^{1/2}) = G(f)G(t^{1/2}) = 0$, showing that $ft^{1/2}$ is the spectral density of a deterministic process.

Now we proceed to verify the third condition for $t^{1/2}$ to belong to the class $\mathcal{M}_f$, that is, the relation:

\[
\lim_{n \to \infty} \frac{\sigma_n^2(ft^{1/2})}{\sigma_n^2(f)} = G(t^{1/2}) > 0. \tag{5.1}
\]

To do this, observe first that since $t^{1/2} \in B^-$, we can apply Lemma 4.5 of Babayan et al. [4] to get

\[
\limsup_{n \to \infty} \frac{\sigma_n^2(ft^{1/2})}{\sigma_n^2(f)} \leq G(t^{1/2}). \tag{5.2}
\]

So, we have to prove the inverse inequality:

\[
\liminf_{n \to \infty} \frac{\sigma_n^2(ft^{1/2})}{\sigma_n^2(f)} \geq G(t^{1/2}) > 0. \tag{5.3}
\]

To do this, we consider the function:

\[
\hat{f}(\lambda) := f(\lambda)t(\lambda), \quad \lambda \in \Lambda. \tag{5.4}
\]

Applying Theorem B with $h(\lambda) \equiv t_2(\lambda) \equiv 1$, $t_1(\lambda) = t(\lambda)$ we conclude that $t \in \mathcal{M}_f$ and $\hat{f} \in \mathcal{F}$, and, in particular,

\[
\lim_{n \to \infty} \frac{\sigma_n^2(\hat{f})}{\sigma_n^2(f)} = G(t) > 0. \tag{5.5}
\]

*Acta Mathematica Hungarica* 167, 2022
Next, taking into account that \( t^{-1/2} \in B_+ \), we can apply Lemma 4.6 of Babayan et al. [4] to obtain

\[
\liminf_{n \to \infty} \frac{\sigma_n^2(f \cdot t^{-1/2})}{\sigma_n^2(\hat{f})} \geq G(t^{-1/2}) > 0.
\]

Now we can write

\[
\liminf_{n \to \infty} \frac{\sigma_n^2(f t^{1/2})}{\sigma_n^2(f)} = \liminf_{n \to \infty} \frac{\sigma_n^2(f \cdot t^{-1/2})}{\sigma_n^2(f)} \cdot \frac{\sigma_n^2(\hat{f})}{\sigma_n^2(f)} = G(t^{-1/2}) \cdot G(t) = G(t^{1/2}).
\]

In the last relation, the third equality follows from (5.4), the inequality after that follows from (5.5) and (5.6), and the last equality follows from Proposition 2.3(a). Thus, the inequality (5.3) is proved.

Combining the inequalities (5.2) and (5.3) we obtain the desired equality (5.1), which together with the above obtained facts: \( G(t^{1/2}) > 0 \) and \( ft^{1/2} \in L^1(\Lambda) \) means that \( t^{1/2} \in \mathcal{M}_f \).

**Lemma 5.2.** Let \( f \) be an arbitrary function from the class \( \mathcal{F} \), and let \( t \) be a nonnegative trigonometric polynomial. Then \( t^{1/2m} \in \mathcal{M}_f \) and \( ft^{1/2m} \in \mathcal{F} \) for any \( m \in \mathbb{N} \).

**Proof.** As in the proof of Lemma 5.1, we have only to prove the relation \( t^{1/2m} \in \mathcal{M}_f \). To this end, similar to Lemma 5.1, we first verify the first two conditions for \( t^{1/2m} \) to belong to the class \( \mathcal{M}_f \), that is, \( G(t^{1/2m}) > 0 \) and \( ft^{1/2m} \in L^1(\Lambda) \). Also, we observe that \( G(ft^{1/2m}) = 0 \), showing that \( ft^{1/2m} \) is the spectral density of a deterministic process.

Next, we use induction on \( m \) to prove the third condition for \( t^{1/2m} \) to belong to the class \( \mathcal{M}_f \), that is, the relation:

\[
\lim_{n \to \infty} \frac{\sigma_n^2(ft^{1/2m})}{\sigma_n^2(f)} = G(t^{1/2m}) > 0.
\]

Observe first that for \( m = 1 \) the relation (5.8) coincides with (5.1). Now assuming that it is satisfied for \( m = \nu \), that is,

\[
\lim_{n \to \infty} \frac{\sigma_n^2(ft^{1/2\nu})}{\sigma_n^2(f)} = G(t^{1/2\nu}) > 0,
\]
we prove that it remains valid for \( m = \nu + 1 \), that is,

\[
\lim_{n \to \infty} \frac{\sigma_n^2(fft^{1/2(\nu+1)})}{\sigma_n^2(f)} = G(t^{1/2(\nu+1)}) > 0.
\]

To this end, observe first that arguments similar to those used in the proof of Lemma 5.1 can be applied to obtain (cf. (5.2)):

\[
\limsup_{n \to \infty} \frac{\sigma_n^2(fft^{1/2(\nu+1)})}{\sigma_n^2(f)} \leq G(t^{1/2(\nu+1)}).
\]

The proof of the inverse inequality

\[
\liminf_{n \to \infty} \frac{\sigma_n^2(fft^{1/2(\nu+1)})}{\sigma_n^2(f)} \geq G(t^{1/2(\nu+1)})
\]

is similar to that of the inequality (5.3). Indeed, observe that \( t^{-1/2(\nu+1)} \in B_+ \), and by the inductive assumption the function

\[
\hat{f}(\lambda) := f(\lambda)t^{1/2\nu}(\lambda), \quad \lambda \in \Lambda
\]

belongs to the class \( \mathcal{F} \). Hence, we can apply Lemma 4.6 of Babayan et al. [4] to obtain

\[
\liminf_{n \to \infty} \frac{\sigma_n^2(\hat{f}t^{-1/2(\nu+1)})}{\sigma_n^2(\hat{f})} \geq G(t^{-1/2(\nu+1)}) > 0.
\]

Next, we can write

\[
\liminf_{n \to \infty} \frac{\sigma_n^2(fft^{1/2(\nu+1)})}{\sigma_n^2(f)} = \liminf_{n \to \infty} \frac{\sigma_n^2(fft^{1/2\nu}, t^{-1/2(\nu+1)})}{\sigma_n^2(f)}
\]

\[
= \liminf_{n \to \infty} \frac{\sigma_n^2(fft^{1/2\nu}, t^{-1/2(\nu+1)})}{\sigma_n^2(f)} \cdot \frac{\sigma_n^2(f)}{\sigma_n^2(\hat{f})}
\]

\[
= \liminf_{n \to \infty} \frac{\sigma_n^2(\hat{f} \cdot t^{-1/2(\nu+1)})}{\sigma_n^2(\hat{f})} \cdot \liminf_{n \to \infty} \frac{\sigma_n^2(fft^{1/2\nu})}{\sigma_n^2(f)}
\]

\[
\geq G(t^{-1/2(\nu+1)}) \cdot G(t^{1/2\nu}) = G(t^{1/2(\nu+1)}).
\]

In the last relation, the third equality follows from (5.13), the inequality after that follows from (5.14) and inductive assumption (5.9), and the last equality follows from Proposition 2.3(a). Thus, the inequality (5.12) is proved.

Combining the inequalities (5.11) and (5.12) we obtain the desired equality (5.10), which together with the above obtained facts: \( G(t^{1/2(\nu+1)}) > 0 \) and \( ft^{1/2(\nu+1)} \in L^1(\Lambda) \), means that \( t^{1/2(\nu+1)} \in \mathcal{M}_f \). \( \square \)
Lemma 5.3. Let \( f \) be an arbitrary function from the class \( \mathcal{F} \), and let \( t \) be a nonnegative trigonometric polynomial. Then \( t^{k/2^m} \in \mathcal{M}_f \) and \( ft^{k/2^m} \in \mathcal{F} \) for any \( k, m \in \mathbb{N} \).

**Proof.** We use induction on \( k \), and observe first that for \( k = 1 \), the assertion of the lemma coincides with that of Lemma 5.2. Now assuming that it is satisfied for \( k = \nu \), that is, \( t^{\nu/2^m} \in \mathcal{M}_f \) and \( ft^{\nu/2^m} \in \mathcal{F} \), we prove it for \( k = \nu + 1 \), that is, \( t^{(\nu+1)/2^m} \in \mathcal{M}_f \) and \( ft^{(\nu+1)/2^m} \in \mathcal{F} \).

To this end, we set \( g_1 := t^{\nu/2^m} \) and \( g_2 := t^{1/2^m} \), and observe that by inductive assumption, we have \( g_1 \in \mathcal{M}_f \) and \( fg_1 \in \mathcal{F} \). Taking into account the last relation, we can apply Lemma 5.2 with \( fg_1 \) as \( f \), and conclude that \( g_2 \in \mathcal{M}_{fg_1} \).

Therefore, by Proposition 3.2, we have \( g_1g_2 = t^{(\nu+1)/2^m} \in \mathcal{M}_f \) and \( fg_1g_2 = ft^{(\nu+1)/2^m} \in \mathcal{F} \). \( \Box \)

**Lemma 5.4.** Let \( f \) be an arbitrary function from the class \( \mathcal{F} \), and let \( t \) be a nonnegative trigonometric polynomial. Then \( t^{-k} \in \mathcal{M}_f \) and \( ft^{-k} \in \mathcal{F} \) for any \( k \in \mathbb{N} \), provided that \( ft^{-k} \in L^1(\Lambda) \). In particular, we have

\[
(5.16) \quad \lim_{n \to \infty} \frac{\sigma_n^2(ft^{-k})}{\sigma_n^2(f)} = G(t^{-k}) > 0.
\]

**Proof.** We use induction on \( k \), and show that the result follows from Theorem B by using Proposition 3.2. Observe first that for \( k = 1 \) the assertion of the lemma coincides with Theorem B applied to \( h(\lambda) \equiv t_1(\lambda) \equiv 1 \) and \( t_2(\lambda) = t(\lambda) \). Now assuming that it is satisfied for \( k = \nu \), that is, \( t^{-\nu} \in \mathcal{M}_f \) and \( ft^{-\nu} \in \mathcal{F} \) provided that \( ft^{-\nu} \in L^1(\Lambda) \), we prove it for \( k = \nu + 1 \), that is, \( t^{-(\nu+1)} \in \mathcal{M}_f \) and \( ft^{-(\nu+1)} \in \mathcal{F} \) provided that \( ft^{-(\nu+1)} \in L^1(\Lambda) \).

To this end, we set \( g_1(\lambda) := t^{-\nu}(\lambda) \), \( g_2(\lambda) = t^{-1}(\lambda) \) and observe that by inductive assumption, we have \( g_1 \in \mathcal{M}_f \) and \( fg_1 \in \mathcal{F} \). We show that \( g_2 \in \mathcal{M}_{fg_1} \). Indeed, by Proposition 2.3(c) we have \( G(g_2) = G(t^{-1}) > 0 \). Besides, \( fg_1g_2 = ft^{-(\nu+1)} \in L^1(\Lambda) \) by assumption. The obtained relations allow to apply Theorem B with \( fg_1 \) as \( f \) and \( g_2 \) as \( g \), and conclude that \( g_2 \in \mathcal{M}_{fg_1} \).

Thus, the functions \( g_1 \) and \( g_2 \) satisfy Proposition 3.2, and hence \( g_1g_2 = t^{-(\nu+1)} \in \mathcal{M}_f \) and \( fg_1g_2 = ft^{-(\nu+1)} \in \mathcal{F} \). \( \Box \)

**5.2. Proof of the theorems.**

**Proof of Theorem 4.1.** We first verify the first two conditions for a function \( g(\lambda) \) to belong to the class \( \mathcal{M}_f \), that is, the conditions: \( G(g) > 0 \) and \( fg \in L^1(\Lambda) \). From the condition \( h \in B_+ \) it follows that \( G(h) > 0 \), while by Proposition 2.3(c) we have \( G(|t|^{\alpha}) > 0 \). Therefore, \( G(g) = G(h)G(|t|^{\alpha}) > 0 \).

Since both functions \( h(\lambda) \) and \( |t(\lambda)|^{\alpha} \) are bounded, the function \( g(\lambda) = h(\lambda)|t(\lambda)|^{\alpha} \) is also bounded, and \( fg \in L^1(\Lambda) \). Besides, we have \( G(fg) = \)
$G(f)G(g) = 0$, showing that $fg$ is the spectral density of a deterministic process.

Taking into account Proposition 3.1, to complete the proof of the theorem it remains to verify the relation (3.2). The proof of relation (3.2) we split into four steps.

**Step 1.** We prove the relation (3.2) in the special case where $h(\lambda) \equiv 1$ and $t(\lambda)$ is a nonnegative trigonometric polynomial satisfying the condition:

$$t(\lambda) \leq 1, \quad \lambda \in \Lambda. \quad (5.17)$$

For an arbitrary natural number $m$ by $k_m$ we denote the integer part of the number $2^m \cdot \alpha$, that is, $k_m := \lfloor 2^m \cdot \alpha \rfloor$. Then we have the following inequality:

$$k_m \leq 2^m \cdot \alpha < k_m + 1,$$

or, equivalently

$$\frac{k_m}{2^m} \leq \alpha < \frac{k_m + 1}{2^m}. \quad (5.18)$$

In view of (5.17) and (5.18) we can write

$$t^{(k_m+1)/2^m}(\lambda) < t^\alpha(\lambda) \leq t^{k_m/2^m}(\lambda), \quad \lambda \in \Lambda, \quad (5.19)$$

implying that

$$f(\lambda)t^{(k_m+1)/2^m}(\lambda) < f(\lambda)t^\alpha(\lambda) \leq f(\lambda)t^{k_m/2^m}(\lambda), \quad \lambda \in \Lambda. \quad (5.20)$$

Taking into account that by Proposition 2.2(a) the prediction error $\sigma_n^2(f)$ is a non-decreasing functional of $f$ from (5.20), we obtain

$$\sigma_n^2(f t^{(k_m+1)/2^m}) \leq \sigma_n^2(f t^\alpha) \leq \sigma_n^2(f t^{k_m/2^m}). \quad (5.21)$$

Dividing the last inequality by $\sigma_n^2(f)$ and passing to the limit as $n \to \infty$, we obtain

$$\lim_{n \to \infty} \frac{\sigma_n^2(f t^{(k_m+1)/2^m})}{\sigma_n^2(f)} \leq \liminf_{n \to \infty} \frac{\sigma_n^2(f t^\alpha)}{\sigma_n^2(f)} \leq \limsup_{n \to \infty} \frac{\sigma_n^2(f t^{k_m/2^m})}{\sigma_n^2(f)} \leq \sigma_n^2(f). \quad (5.22)$$

By Lemma 5.3, the first and the last limits in (5.21) are equal to $G(t^{(k_m+1)/2^m})$ and $G(t^{k_m/2^m})$, respectively. Hence (5.21) can be written as follows:

$$G(t^{(k_m+1)/2^m}) \leq \liminf_{n \to \infty} \frac{\sigma_n^2(f t^\alpha)}{\sigma_n^2(f)} \leq \limsup_{n \to \infty} \frac{\sigma_n^2(f t^\alpha)}{\sigma_n^2(f)} \leq G(t^{k_m/2^m}).$$
On the other hand, taking into account that the geometric mean $G(f)$ is a non-decreasing functional of $f$ (see Proposition 2.3(b)), from (5.19) we obtain

$$G\left( t^{(k_m+1)/2m} \right) \leq G(t^\alpha) \leq G\left( t^{k_m/2m} \right).$$  

(5.23)

Thus, to complete the proof in the considered case, it remains to show that for large enough $m$ the quantities $G\left( t^{(k_m+1)/2m} \right)$ and $G\left( t^{k_m/2m} \right)$ are arbitrarily close. To do this, observe that in view of (5.17) and the properties of the geometric mean (see Proposition 2.3(a),(b)), we can write

$$1 \leq \frac{G\left( t^{k_m/2m} \right)}{G\left( t^{(k_m+1)/2m} \right)} = G\left( t^{-1/2m} \right) = G^{-1/2m}(t).$$  

(5.24)

Therefore, in view of the limiting relation $\lim_{n \to \infty} c^{1/n} = 1$ ($c > 0$), it follows that

$$\lim_{m \to \infty} G^{-1/2m}(t) = 1.$$  

(5.25)

Finally, passing to the limit in (5.22) as $m \to \infty$ and taking into account the relations (5.23)-(5.25), we obtain the desired relation (3.2).

Step 2. Now we prove the relation (3.2) without assuming the condition (5.17), that is, in the case where $h(\lambda) \equiv 1$ and $t(\lambda)$ is an arbitrary nonnegative trigonometric polynomial. To this end, we denote $c := \max_{\lambda \in \Lambda} t(\lambda) > 0$, and consider the trigonometric polynomial $\hat{t}(\lambda) := (1/c)t(\lambda)$. Observe that the polynomial $\hat{t}(\lambda)$ satisfies the condition (5.17), that is, $\hat{t}(\lambda) \leq 1$ for all $\lambda \in \Lambda$. Therefore for $\hat{t}(\lambda)$ the relation (3.2) is satisfied, that is, we have

$$\lim_{n \to \infty} \frac{\sigma_n^2(\hat{t}^\alpha)}{\sigma_n^2(\hat{t})} = G(\hat{t}^\alpha).$$  

(5.26)

On the other hand, by Propositions 2.2(c) and 2.3(a), we have

$$\sigma_n^2(\hat{t}^\alpha) = c^{-\alpha} \sigma_n^2(\hat{t})$$  

and  

$$G(\hat{t}^\alpha) = c^{-\alpha} G(\hat{t}).$$  

(5.27)

From (5.26) and (5.27) we get

$$\frac{1}{c^\alpha} \lim_{n \to \infty} \frac{\sigma_n^2(\hat{t})}{\sigma_n^2(f)} = c^{-\alpha} G(t^\alpha),$$

which is equivalent to (3.2).

Step 3. Now we prove the relation (3.2) in the case where $h(\lambda) \equiv 1$ and $t(\lambda)$ is an arbitrary trigonometric polynomial. Denoting $\tilde{t}(\lambda) := t^2(\lambda)$, we can write

$$|t(\lambda)|^\alpha = (t^2(\lambda))^{\alpha/2} = (\tilde{t}(\lambda))^{\alpha/2}, \quad \lambda \in \Lambda.$$  

(5.28)
Since \( \hat{t}(\lambda) \) is a nonnegative trigonometric polynomial, according to Step 2, for \( t(\lambda) \) the relation (3.2) is satisfied, that is, we have

\[
\lim_{n \to \infty} \frac{\sigma^2_n(f|t|^{\alpha_2})}{\sigma^2_n(f)} = G(|t|^\alpha),
\]

and, in view of (5.28), we get

\[
\lim_{n \to \infty} \frac{\sigma^2_n(f|t|^\alpha)}{\sigma^2_n(f)} = G(|t|^\alpha),
\]

and the result follows.

**Step 4.** Finally, we prove the relation (3.2), and thus the theorem, in the general case, that is, when \( h(\lambda) \) is an arbitrary function from the class \( B_+ \) and \( t \) is an arbitrary trigonometric polynomial. By Step 3, we have \( g_1 := |t|^\alpha \in M_f \) and \( fg_1 = f|t|^\alpha \in F \). Also, according to Theorem B with \( fg_1 \) as \( f \), the function \( g_2 := h \) belongs to the class \( M_{fg_1} \). Thus, the functions \( g_1 \) and \( g_2 \) satisfy the conditions of Proposition 3.2, and hence \( g := g_1g_2 = h|t|^\alpha \in M_f \) and \( fg = fh|t|^\alpha \in F \). This completes the proof of Theorem 4.1. \( \square \)

**Proof of Corollary 4.1.** For \( m = 1 \) the corollary coincides with Theorem 4.1. The result then follows from inductive arguments and Proposition 3.2. \( \square \)

**Proof of Theorem 4.2.** Observe first that by Proposition 2.3(c) we have \( G(t^{-\alpha}) > 0 \). Next, if \( \alpha \in \mathbb{Z} \) and \( ft^{-\alpha} \in L^1(\Lambda) \), then according to Lemma 5.4 we have \( t^{-\alpha} \in M_f \) and \( ft^{-\alpha} \in F \).

Now let \( \alpha \notin \mathbb{Z} \), \( [\alpha] = k \) and \( ft^{-(k+1)} \in L^1(\Lambda) \). Since \( k + 1 - \alpha > 0 \), from the equality \( ft^{-\alpha} = ft^{-(k+1)}t^{(k+1)-\alpha} \), we have \( ft^{-\alpha} \in L^1(\Lambda) \). Denote \( g_1 := t^{-(k+1)} \) and \( g_2 := t^{(k+1)-\alpha} \). Then by Lemma 5.4 the function \( g_1 \) satisfies the conditions: \( g_1 \in M_f \) and \( fg_1 \in F \). Also, according to Theorem 4.1, we have \( g_2 \in M_{fg_1} \). Therefore, by Proposition 3.2, we have \( g_1g_2 = t^{-\alpha} \in M_f \) and \( fg_1g_2 = ft^{-\alpha} \in F \).

Thus, in both cases (a) and (b) we have \( t^{-\alpha} \in M_f \) and \( ft^{-\alpha} \in F \). Applying Proposition 3.2 now to the functions \( t^{-\alpha} \) and \( h \), we conclude that \( g = ht^{-\alpha} \in M_f \) and \( fg \in F \). Theorem 4.2 is proved. \( \square \)

**Proof of Theorem 4.3.** We first prove assertion a), that is, that the function \( h := \hat{f}/f \) belongs to the class \( B_+ \). To do this we denote \( c := \min_{1 \leq j \leq k} c_j \), where \( c_j \) is as in (4.4). Then, in view of (4.4), for \( \varepsilon = c/2 \) and each \( j = 1, 2, \ldots, k \), there is a number \( \delta_j = \delta_j(\varepsilon) > 0 \) such that \( |h(\lambda) - c_j| < \varepsilon \) for all \( \lambda \in O_{\delta_j}(\lambda_j) \), or equivalently

\[
(5.30) \quad c_j - c/2 < h(\lambda) < c_j + c/2 \quad \text{for all } \lambda \in O_{\delta_j}(\lambda_j), \ j = 1, 2, \ldots, k.
\]
Denote $C := \max_{1 \leq j \leq k} c_j$, $\lambda_0 = -\pi$, $\Delta \lambda_j = \lambda_j - \lambda_{j-1}$, and
$$\delta := \min_{1 \leq j \leq k} \{\delta_j, \Delta \lambda_j/3\}.$$ Then, in view of the inequalities $c \leq c_j \leq C$ and $\delta \leq \delta_j$, from (5.30) we obtain

$$c/2 < h(\lambda) < 3C/2 \quad \text{for all} \quad \lambda \in \bigcup_{j=1}^{k} O_\delta(\lambda_j).$$

Thus, the function $h(\lambda)$ is bounded away from zero and infinity in the set $\bigcup_{j=1}^{k} O_\delta(\lambda_j)$. On the other hand, according to the conditions 1) and 4) of the theorem, positive constants $m$ and $M$ can be found to satisfy

$$f(\lambda) \leq M, \quad \hat{f}(\lambda) \leq M \quad \text{for all} \quad \lambda \in \Lambda$$

and

$$f(\lambda) \geq m, \quad \hat{f}(\lambda) \geq m \quad \text{for all} \quad \lambda \in \Lambda \setminus \bigcup_{j=1}^{k} O_\delta(\lambda_j).$$

Therefore

$$m/M \leq h(\lambda) \leq M/m \quad \text{for all} \quad \lambda \in \Lambda \setminus \bigcup_{j=1}^{k} O_\delta(\lambda_j).$$

Hence, denoting $\hat{m} := \min\{c/2, m/M\}$ and $\hat{M} := \max\{3C/2, M/m\}$, in view of (5.31) and (5.34) we obtain

$$\hat{m} < h(\lambda) < \hat{M} \quad \text{for all} \quad \lambda \in \Lambda.$$ Thus, the function $h(\lambda)$ is bounded away from zero and infinity. Therefore, $h(\lambda)$, being a ratio of two functions from the class $B^-$, by Proposition 3.3 c), also belongs to the class $B^-$. Therefore, $h(\lambda) \in B^+_\lambda$, and hence $G(h) > 0$. To prove assertion b), assume that the process with spectral density $f(\lambda)$ is nondeterministic, that is, $G(f) > 0$. Then taking into account that $\hat{f}(\lambda) = f(\lambda)h(\lambda)$, we have $G(\hat{f}) = G(f)G(h) > 0$, implying that the process with spectral density $\hat{f}(\lambda)$ is also nondeterministic. So, in view of (2.3), we have

$$\lim_{n \to \infty} \sigma_n^2(\hat{f}) = 2\pi G(\hat{f}) \quad \text{and} \quad \lim_{n \to \infty} \sigma_n^2(f) = 2\pi G(f).$$

From (5.35) and Proposition 2.3(a) we easily obtain the relation (4.5). Besides, from the first relation in (5.35) and Proposition 2.4(b), we infer that
the sequence \( \{\sigma_n(\hat{f})\} \) is weakly varying. This completes the proof of assertion b) of the theorem.

The assertion c) of the theorem immediately follows from Theorem 4.1. Indeed, if \( f \in \mathcal{F} \), then applying Theorem 4.1 with \( t(\lambda) \equiv 1 \) (that is, \( g(\lambda) = h(\lambda) \)), we conclude that \( \hat{f} = fh \) is the spectral density of a deterministic process with a weakly varying sequence \( \{\sigma_n(\hat{f})\} \) of prediction errors and the relation (4.5) holds. This completes the proof of Theorem 4.3. □

Proof of Theorem 4.4. Let the polynomial \( q(\lambda) \) in (4.6) be of degree \( m \geq 1 \), and let \( \lambda_i \in (-\pi, \pi) \) be the zeros of \( q(\lambda) \) of multiplicities \( m_i \in \mathbb{N} \) \((i = 1, 2, \ldots, k, k \in \mathbb{N})\), respectively. Then by the Fundamental Theorem of Algebra we can write

\[
q(\lambda) = c_0(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k} \hat{q}(\lambda), \quad \lambda \in \Lambda,
\]

where \( \hat{q}(\lambda) \) consists of a product of linear binomial factors with zeros outside \((-\pi, \pi)\) and quadratic trinomial factors, which are positive on \( \mathbb{R} \), implying that \( \hat{q}(\lambda) \in B_+^\infty \).

Consider the trigonometric polynomial:

\[
t(\lambda) = \sin^{m_1}(\lambda - \lambda_1)\sin^{m_2}(\lambda - \lambda_2) \cdots \sin^{m_k}(\lambda - \lambda_k), \quad \lambda \in \Lambda,
\]

and the function

\[
(5.36) \quad \hat{h}(\lambda) := \frac{|q(\lambda)|^\alpha}{|t(\lambda)|^\alpha}, \quad \lambda \in \Lambda.
\]

From the relation \( \sin(\lambda - \lambda_i) \sim (\lambda - \lambda_i) \) as \( \lambda \to \lambda_i \) it follows that with some positive constants \( c_i, \ |q(\lambda)|^\alpha \sim c_i|t(\lambda)|^\alpha \) as \( \lambda \to \lambda_i \) \((i = 1, 2, \ldots, k)\). Hence the functions \( |q(\lambda)|^\alpha \) and \( |t(\lambda)|^\alpha \), with regard to their continuity, satisfy the conditions of Theorem 4.3. Therefore, \( \hat{h}(\lambda) \in B_+^\infty \) and, according to Proposition 3.3 d), we have \( h(\lambda)\hat{h}(\lambda) \in B_+^\infty \).

Now we can apply Theorem 4.2 to conclude that the function \( g(\lambda) \) in (4.6) is the spectral density of a nondeterministic process and to obtain

\[
\lim_{n \to \infty} \frac{\sigma_n^2(\hat{f}g)}{\sigma_n^2(\hat{f})} = \lim_{n \to \infty} \frac{\sigma_n^2(fh|q|^\alpha)}{\sigma_n^2(f)} = \lim_{n \to \infty} \frac{\sigma_n^2(fh\hat{h}|t|^\alpha)}{\sigma_n^2(f)}
\]

\[
= G(h\hat{h}|t|^\alpha) = G(h|q|^\alpha) = G(g) > 0.
\]

Here the first and the fifth relations follow from (4.6), the second and the fourth relations follow from (5.36), and the third relation follows from Theorem 4.2.

This completes the proof of Theorem 4.4. □
6. Examples

In this section we discuss examples demonstrating the obtained results.

Example 6.1. Let the function $g(\lambda)$ ($\lambda \in \Lambda$) be as in (4.2) with $h(\lambda) = 1$ and $t(\lambda) = \sin(\lambda - \lambda_0)$, where $\lambda_0$ is an arbitrary point from $[-\pi, \pi]$, that is, $g(\lambda) = |\sin(\lambda - \lambda_0)|^\alpha$, $\alpha \in \mathbb{R}$. Then, according to Example 4.4 of Babayan et al. [4], for the geometric mean of $\sin^2(\lambda - \lambda_0)$ we have

\begin{equation}
G(\sin^2(\lambda - \lambda_0)) = \frac{1}{4}.
\end{equation}

According to Proposition 2.3(a) and (6.1), for the geometric mean of $g(\lambda)$, we obtain

\begin{equation}
G(g) = G(|\sin(\lambda - \lambda_0)|^\alpha) = G^{\alpha/2}(\sin^2(\lambda - \lambda_0)) = \frac{1}{2^\alpha},
\end{equation}

and in view of (3.2), we get

$$
\lim_{n \to \infty} \frac{\sigma_n^2(fg)}{\sigma_n^2(f)} = G(g) = \frac{1}{2^\alpha}.
$$

Thus, multiplying the spectral density $f(\lambda)$ by the function $g(\lambda) = |\sin(\lambda - \lambda_0)|^\alpha$ yields a $2^\alpha$-fold asymptotic reduction of the prediction error.

Example 6.2. Let the function $g(\lambda)$ be as in (4.6) with $h(\lambda) = 1$ and $q(\lambda) = \lambda$, that is, $g(\lambda) = |\lambda|^\alpha$, $\alpha \in \mathbb{R}$. By direct calculation we obtain

\begin{equation}
\ln G(g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln |\lambda|^\alpha d\lambda = \frac{\alpha}{\pi} \int_0^{\pi} \ln \lambda d\lambda = \alpha \ln(\pi/e).
\end{equation}

Therefore

$$
G(g) = (\pi/e)^\alpha \approx (1.156)^\alpha,
$$

and in view of (3.2), we get

$$
\lim_{n \to \infty} \frac{\sigma_n^2(fg)}{\sigma_n^2(f)} = G(g) = \left(\frac{\pi}{e}\right)^\alpha \approx (1.156)^\alpha.
$$

Thus, multiplying the spectral density $f(\lambda)$ by the function $g(\lambda) = |\lambda|^\alpha$ multiplies the prediction error asymptotically by $(\pi/e)^\alpha \approx (1.156)^\alpha$.

It follows from Proposition 2.2(d) that the same asymptotic is true for the prediction error with spectral density $\bar{g}(\lambda) = |\lambda - \lambda_0|^\alpha$, $\lambda_0 \in [-\pi, \pi]$. 

Acta Mathematica Hungarica 167, 2022
Example 6.3. We first analyze the Pollaczek–Szegő function \( f_a(\lambda) \) given by (1.1) (cf. Pollaczek [11] and Szegő [16]). We have

\[
f_a(\lambda) = \frac{2e^{2\lambda \varphi(\lambda)}e^{-\pi \varphi(\lambda)}}{e^{\pi \varphi(\lambda)} + e^{-\pi \varphi(\lambda)}} = \frac{2e^{2\lambda \varphi(\lambda)}}{e^{2\pi \varphi(\lambda)} + 1}, \quad 0 \leq \lambda \leq \pi, \quad \varphi(\lambda) = (a/2) \cot \lambda.
\]

Observe that \( \varphi(\lambda) \to +\infty \) as \( \lambda \to 0^+ \), and we have

\[
\varphi(\lambda) \sim a/(2\lambda), \quad e^{2\lambda \varphi(\lambda)} \sim e^a, \quad e^{2\pi \varphi(\lambda)} + 1 \sim e^{a\pi/\lambda} \quad \text{as} \quad \lambda \to 0^+.
\]

Taking into account that \( f_a(\lambda) \) is an even function, from (6.3) and (6.4) we obtain the following asymptotic relation for \( f_a(\lambda) \) in a vicinity of the point \( \lambda = 0 \).

\[
f_a(\lambda) \sim 2e^a \exp\{-a\pi/|\lambda|\} \quad \text{as} \quad \lambda \to 0.
\]

Next, observe that \( \varphi(\lambda) \to -\infty \) as \( \lambda \to \pi \), and we have

\[
\varphi(\lambda) = -\varphi(\pi - \lambda) \sim (-a/2)(\pi - \lambda), \quad 2\lambda \varphi(\lambda) \sim -a\pi/(\pi - \lambda), \quad \text{as} \quad \lambda \to \pi.
\]

In view of (6.3) and (6.6) we obtain the following asymptotic relation for the function \( f_a(\lambda) \) in a vicinity of the point \( \lambda = \pi \).

\[
f_a(\lambda) \sim 2e^{2\lambda \varphi(\lambda)} \sim 2\exp\{-a\pi/(\pi - \lambda)\} \quad \text{as} \quad \lambda \to \pi.
\]

Putting together (6.5) and (6.7), and taking into account evenness of \( f_a(\lambda) \), we conclude that

\[
f_a(\lambda) \sim \begin{cases} 2e^a \exp\{-a\pi/|\lambda|\} & \text{as} \quad \lambda \to 0, \\ 2\exp\{-a\pi/(\pi - |\lambda|)\} & \text{as} \quad \lambda \to \pm\pi, \end{cases}
\]

Thus, the function \( f_a(\lambda) \) is positive everywhere except for points \( \lambda = 0, \pm\pi \), and has a very high order of contact with zero at these points, so that Szegő’s condition (2.5) is violated implying that \( G(f_a) = 0 \). Also, observe that \( f_a(\lambda) \) is infinitely differentiable at all points of the segment \([-\pi, \pi]\) including the points \( \lambda = 0, \pm\pi \), and attains its maximum value of 1 at the points \( \pm\pi/2 \). For some specific values of the parameter \( a \) the graph of the function \( f_a(\lambda) \) is represented in Fig. 1a).

For \( a > 0 \) and \( \lambda \in [-\pi, \pi] \), consider the pair of functions \( \hat{f}_1(\lambda) \) and \( \hat{f}_2(\lambda) \) defined by formulas:

\[
\hat{f}_1(\lambda) := \exp\{-a\pi/|\lambda|\}, \quad \hat{f}_2(\lambda) := \exp\{-a\pi/(\pi - |\lambda|)\}.
\]

Observe that the function \( \hat{f}_1(\lambda) \) is positive everywhere except for point \( \lambda = 0 \) at which it has the same order of contact with zero as \( f_a(\lambda) \), and hence
ON ASYMPTOTIC BEHAVIOR OF THE PREDICTION ERROR

**Fig. 1:** a) Graph of the function $f_a(\lambda)$. b) Graph of the function $\hat{f}_a(\lambda)$.

**Fig. 2:** a) Graph of the function $\hat{f}_1(\lambda)$. b) Graph of the function $\hat{f}_2(\lambda)$.

$G(\hat{f}_1) = 0$. Also, $\hat{f}_1(\lambda)$ is infinitely differentiable at all points of the segment $[-\pi, \pi]$ except for the points $\lambda = \pm \pi$, where it attains its maximum value equal to $e^{-a}$. As for the function $\hat{f}_2(\lambda)$, it is positive everywhere except for points $\lambda = \pm \pi$, at which it has the same order of contact with zero as $f_a(\lambda)$, and hence $G(\hat{f}_2) = 0$. Also, $\hat{f}_2(\lambda)$ is infinitely differentiable at all points of the segment $[-\pi, \pi]$ except for the point $\lambda = 0$, where it attains its maximum value equal to $e^{-a}$. For some specific values of the parameter $a$ the graphs of functions $\hat{f}_1(\lambda)$ and $\hat{f}_2(\lambda)$ are represented in Fig. 2.

Denote by $\hat{f}_a(\lambda)$ the product of functions $\hat{f}_1(\lambda)$ and $\hat{f}_2(\lambda)$ defined in (6.9) and normalized by the factor $e^{4a}$:

$$
\hat{f}_a(\lambda) := e^{4a} \hat{f}_1(\lambda) \hat{f}_2(\lambda) = e^{4a} \exp \left\{ -a\pi^2/(|\lambda| (\pi - |\lambda|)) \right\},
$$

(6.10)
and observe that \( \hat{f}_a(\lambda) \) behaves similar to \( f_a(\lambda) \). Indeed, the function \( \hat{f}_a(\lambda) \) also is positive everywhere except for points \( \lambda = 0, \pm \pi \), it is infinitely differentiable at all points of the segment \([-\pi, \pi]\) including the points \( \lambda = 0, \pm \pi \), and attains it maximum value of 1 at the points \( \pm \pi/2 \). Also, in view of (6.8) and (6.10), at points \( \lambda = 0, \pm \pi \) the function \( \hat{f}_a(\lambda) \) has the same order of zeros as \( f_a(\lambda) \), and hence \( G(\hat{f}_a) = 0 \). Thus, the process \( X(t) \) with spectral density \( \hat{f}_a(\lambda) \) is deterministic. For some specific values of the parameter \( a \) the graph of the function \( \hat{f}_a(\lambda) \) is represented in Fig. 1b).

The functions \( f_a(\lambda) \) and \( \hat{f}_a(\lambda) \) defined by (1.1) and (6.10), respectively, satisfy the conditions of Theorem 4.3. Therefore, we have (see (4.5))

\[
(6.11) \quad \lim_{n \to \infty} \frac{\sigma_n^2(\hat{f}_a)}{\sigma_n^2(f_a)} = G(\hat{f}_a/f_a) := \hat{C}(a) > 0.
\]

In view of (1.2) and (6.11) we have

\[
(6.12) \quad \sigma_n^2(\hat{f}_a) \sim C(a) \cdot n^{-a} \quad \text{as} \quad n \to \infty.
\]

where

\[
(6.13) \quad C(a) := \frac{\Gamma^2((a + 1)/2)}{\pi^{2a}} \cdot \hat{C}(a).
\]

The values of the constants \( \hat{C}(a) \) and \( C(a) \) for some specific values of the parameter \( a \) are given in Table 1.

| \( a \) | \( \frac{\Gamma^2((a + 1)/2)}{\pi^{2a}} \) | \( \hat{C}(a) \) | \( C(a) \) |
|---|---|---|---|
| 0.1 | 0.223 | 0.797 | 0.178 |
| 0.5 | 0.169 | 1.113 | 0.188 |
| 1.0 | 0.159 | 2.545 | 0.406 |
| 1.5 | 0.185 | 6.446 | 1.193 |
| 2.0 | 0.250 | 16.830 | 4.214 |
| 3.0 | 0.637 | 119.220 | 76.379 |
| 3.3 | 0.902 | 215.715 | 194.656 |
| 3.4 | 1.020 | 263.173 | 268.375 |
| 5.0 | 10.186 | 6128.990 | 62429.000 |
| 10.0 | 223256 | 1.104 \cdot 10^8 | 2.428 \cdot 10^{13} |

Table 1: The values of constants \( \hat{C}(a) \) and \( C(a) \)
Now we compare the prediction errors $\sigma_n^2(\hat{f}_1)$ and $\sigma_n^2(\hat{f}_2)$ with $\sigma_n^2(f_a)$. To this end, observe first that the function $g_1(\lambda) := f_a(\lambda)/\hat{f}_1(\lambda)$ has a very high order of contact with zero at points $\lambda = \pm \pi$, so that Szegö’s condition (2.5) is violated implying that $G(g_1) = 0$. Besides, the function $g_1(\lambda)$ is continuous on $[-\pi, \pi]$, and hence $g_1 \in B^-$. Therefore, according to Corollary 4.5 of Babayan et al. [4], we have

\begin{equation}
\sigma_n^2(f_a) = o(\sigma_n^2(\hat{f}_1)) \quad \text{as } n \to \infty. \tag{6.14}
\end{equation}

Similar arguments applied to the function $f_2(\lambda)$ yield

\begin{equation}
\sigma_n^2(f_a) = o(\sigma_n^2(\hat{f}_2)) \quad \text{as } n \to \infty. \tag{6.15}
\end{equation}

The relations (6.14) and (6.15) show that the rate of convergence to zero of the prediction errors $\sigma_n^2(\hat{f}_1)$ and $\sigma_n^2(\hat{f}_2)$ is less than the one for $\sigma_n^2(f_a)$, that is, the power rate of convergence $n^{-a}$ (see (6.12)). Thus, the rate of convergence $n^{-a}$ is due to the joint contribution of all zeros $\lambda = 0, \pm \pi$ of the function $f_a(\lambda)$, whereas each of these zeros separately does not guarantee the rate of convergence $n^{-a}$.

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