Robust error bounds for the Navier-Stokes equations using implicit-explicit second order BDF method with variable steps

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Abstract
This paper studies fully discrete finite element approximations to the Navier-Stokes equations using inf-sup stable elements and grad-div stabilization. For the time integration two implicit-explicit second order backward differentiation formulae (BDF2) schemes are applied. In both the laplacian is implicit while the nonlinear term is explicit, in the first one, and semi-implicit, in the second one. The grad-div stabilization allow us to prove error bounds in which the constants are independent of inverse powers of the viscosity. Error bounds of order $r$ in space are obtained for the $L^2$ error of the velocity using piecewise polynomials of degree $r$ to approximate the velocity together with second order bounds in time, both for fixed time step methods and for methods with variable time steps. A CFL-type condition is needed for the method in which the nonlinear term is explicit relating time step and spatial mesh sizes parameters.

Keywords: Incompressible Navier-Stokes equations; Variable step BDF2; Implicit-explicit methods; Grad-div stabilization; Robust error Bounds

1 Introduction
The numerical simulation of the Navier-Stokes equations is still a challenge in which there are several questions that deserve some research. In the present paper we try to contribute to the following two aspects. On the one hand, we consider the possibility of optimize the cost of the temporal integration. On the other, we consider spatial approximations that allow to prove error bounds with constants independent of inverse powers of the viscosity. Concerning the first question, following the recent reference [9], we consider implicit-explicit methods with explicit or semi-implicit treatment of the nonlinear term, which reduces the cost of every single time step. Also, we include the error analysis of the variable step case. As stated in [13]: “Adaptive time stepping is an important tool in Computational Fluid Dynamics for controlling the accuracy of simulations and for enhancing their efficiency”. Concerning the second question, we add grad-div stabilization in the spatial discretization of the method. As in [7], adding this stabilization we are able to get bounds with constants independent of inverse powers of the viscosity. This is one of the methods considered in [12] in which methods with robust bounds (error bounds with constants independent of the

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Reynolds number are studied). Methods for whom robust estimates can be derived enable stable flow simulations for small viscosity coefficients on comparatively coarse grids.

Let \( \Omega \subset \mathbb{R}^d \), \( d \in \{2, 3\} \), be a bounded domain with polyhedral and Lipschitz boundary \( \partial \Omega \). The incompressible Navier–Stokes equations model the conservation of linear momentum and the conservation of mass (continuity equation) by

\[
\begin{aligned}
\partial_t u - \nu \Delta u + (u \cdot \nabla)u + \nabla p &= f \quad \text{in} \quad (0, T) \times \Omega, \\
\nabla \cdot u &= 0 \quad \text{in} \quad (0, T) \times \Omega, \\
 u(0, \cdot) &= u_0(\cdot) \quad \text{in} \quad \Omega,
\end{aligned}
\]  

(1)

where \( u \) is the velocity field, \( p \) the kinematic pressure, \( \nu > 0 \) the kinematic viscosity coefficient, \( u_0 \) a given initial velocity, and \( f \) represents the external body accelerations acting on the fluid. The Navier–Stokes equations (1) are equipped with homogeneous Dirichlet boundary conditions \( u = 0 \) on \( \partial \Omega \).

In [7] a stabilized method using only grad-div stabilization to approach the evolutionary Navier-Stokes equations is considered and analyzed. Error bounds with constants independent of inverse powers of the viscosity are proved. Using piecewise polynomials of degree \( r \) for the velocity bounds of order \( r \) are obtained for the \( L^2 \) error of the velocity. Both the continuous-in-time case and the fully discrete scheme with the fully implicit backward Euler method as time integrator are analyzed.

Recently, in [9], implicit-explicit (IMEX) variable stepsize methods for the evolutionary Navier-Stokes equations are analyzed. In these algorithms the nonlinear term is treated fully explicitly while the remaining terms are treated implicitly. Stability and convergence is proved for a variable stepsize first order method. However, the error bounds in [9] depend strongly on inverse powers of \( \nu \). In particular, a CFL-type condition is needed for the time step depending on \( \nu^{-1} \). The authors state that the analysis including grad-div stabilization remains an open problem.

In the present paper, as in [7], we add grad-did stabilization to the mixed finite element approximation to be able to get bounds independent on \( \nu^{-1} \). We analyze a second order BDF2 method in time, with explicit treatment of the nonlinear term. We consider both the case of fixed time step and the case of variable time step. A CFL-type condition is needed in our error analysis. This condition is stronger than the usual one in which \((\Delta t)h^{-1}\) has to be bounded (\( h \) being the mesh size) because we required \((\Delta t)h^{-2}\) to be bounded (a similar CFL condition is obtained for the variable step case). However, we improve the condition in [9] in which the quantity that has to be bounded is \((\Delta t)h^{-1}\nu^{-1}\). Then, in principle, the method we consider could be applied in case of high Reynolds numbers without the need of reducing the size of the time step. Due to the stronger CFL condition of the method with explicit treatment of the nonlinear term, depending on the problem, it could be worth to use instead a semi-implicit form for the nonlinear term. In this paper, we also include the error analysis of such a method in which the nonlinear term is semi-implicit. For this method the error analysis is obtained with the same tools as the previous analysis but it is much simpler. Also, no CFL condition is needed. The same kind of bounds as for the previous method are proved in which the constants are also independent on inverse powers of the viscosity.

Concerning the BDF2 method with variable time step, we did not found in the literature any reference with the error analysis for the Navier-Stokes equations. In [2] a second order backward difference method with variable steps is analyzed for a linear parabolic problem. The error analysis for semilinear parabolic problems can be found in [10]. The convergence of the variable two-step BDF time discretisation of nonlinear evolution problems governed by a monotone potential operator is studied in [11] while the analysis for the Cahn-Hilliard equation appears in [4].

The outline of the paper is as follows. In Section 2 we introduce some notation and state some preliminaries. In Section 3 we carry out the error analysis of the fully discrete methods, both for the fixed step and the variable time step cases. Some technical lemmas
are stated and proved along the section. Although the error analysis of the fixed step can be obtained as a particular case of the variable step we decided to start with the analysis of the simpler case since most of the ideas are common and in this way, in our opinion, the skeleton of the analysis can be better understood. We prove optimal convergence of second order in time for the fully discrete methods (both for the fixed and variable time step size cases and both for explicit and semi-implicit treatment of the nonlinear term) and the same rate of convergence in space as in [7]. In Section 4 we show some numerical experiments while some conclusions are stated in the last section.

2 Preliminaries and notation

Throughout the paper we will denote by $W^{s,p}(D)$ the Sobolev space of real valued functions defined on the domain $D \subset \mathbb{R}^d$ with distributional derivatives of order up to $s$ in $L^p(D)$, endowed with the usual norm which is denoted by $\| \cdot \|_{W^{s,p}(D)}$. If $s$ is not a positive integer, $W^{s,p}(D)$ is defined by interpolation [1]. If $s=0$ we understand that $W^{0,p}(D) = L^p(D)$. As it is standard, $W^{s,p}(D)^d$ will be endowed with the product norm that, if no confusion can arise, will be denoted again by $\| \cdot \|_{W^{s,p}(D)}$. We will distinguish the case $p=2$ using $H^s(D)$ to denote the space $W^{s,2}(D)$. We will make use of the space $H_0^1(D)$, the closure in $H^1(D)$ of the set of infinitely differentiable functions with compact support in $D$. For simplicity, we use $\| \cdot \|_s$ (resp. $| \cdot |_s$) to denote the norm (resp. seminorm) both in $H^s(D)$ or $H^s(D)^d$. The exact meaning will be clear by the context. The inner product of $L^2(\Omega)$ or $L^2(\Omega)^d$ will be denoted by $\langle \cdot , \cdot \rangle$ and the corresponding norm by $\| \cdot \|_2$. The norm of the space of essentially bounded functions $L^\infty(\Omega)$ will be denoted by $\| \cdot \|_\infty$. For vector valued function we will use the same conventions as before. We represent by $\| \cdot \|_{-1}$ the norm of the dual space of $H_0^1(\Omega)$ which is denoted by $H^{-1}(\Omega)$. As usual, we always identify $L^2(\Omega)$ with its dual so we have $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$ with compact injection.

Using the function spaces $V = H_0^1(\Omega)^d$, and

$$ Q = L_0^2(\Omega) = \{ q \in L^2(\Omega) : (q,1) = 0 \}, $$

the weak formulation of problem [1] is: Find $(u,p) : (0,T] \to V \times Q$ such that for all $(v,q) \in V \times Q$,

$$ (\partial_t u, v) + \nu(\nabla u, \nabla v) + ((u \cdot \nabla)u) v - (\nabla \cdot v, p) + (\nabla \cdot u, q) = (f, v). \quad (2) $$

The Hilbert space

$$ H^{\text{div}} = \{ u \in L^2(\Omega)^d \mid \nabla \cdot u = 0, u \cdot n|_{\partial \Omega} = 0 \} $$

will be endowed with the inner product of $L^2(\Omega)^d$ and the space

$$ V^{\text{div}} = \{ u \in V \mid \nabla \cdot u = 0 \} $$

with the inner product of $V$.

The following Sobolev’s embedding [1] will be used in the analysis: For $1 \leq p < d/s$ let $q$ be such that $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$. There exists a positive constant $C$, independent of $s$, such that

$$ \| v \|_{L^q(\Omega)} \leq C \| v \|_{W^{s,p}(\Omega)}, \quad \frac{1}{q'} \geq \frac{1}{q}, \quad v \in W^{s,p}(\Omega). \quad (3) $$

If $p > d/s$ the above relation is valid for $q' = \infty$. A similar embedding inequality holds trivially for vector valued functions.

Let $V_h \subset V$ and $Q_h \subset Q$ be two families of finite element spaces composed of piecewise polynomials of degrees at most $k$ and $l$, respectively, that correspond to a family of partitions
of $\Omega$ into mesh cells with maximal diameter $h$. In the following, we consider pairs of finite element spaces that satisfy the inf-sup condition

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{(\nabla \cdot v_h, q_h)}{\|\nabla v_h\|_0 \|q_h\|_0} \geq \beta_0,$$  \hspace{1cm} (4)$$

with $\beta_0$ a constant independent of the mesh size $h$.

It will be assumed that the meshes are quasi-uniform and that the following inverse inequality holds for each $v_h \in V_h$, see e.g., [3] Theorem 3.2.6,

$$\|v_h\|_{W^{m, p}(K)} \leq C_{inv} h_K^{-m-\alpha \left(\frac{n}{2} - \frac{d}{2}\right)} \|v_h\|_{W^{n, q}(K)},$$  \hspace{1cm} (5)$$

where $0 \leq n \leq m \leq 1$, $1 \leq q \leq p \leq \infty$ and $h_K$ is the size (diameter) of the mesh cell $K \in T_h$.

The space of discrete divergence-free functions is denoted by

$$V_h^{div} = \{v_h \in V_h \mid (\nabla \cdot v_h, q_h) = 0 \quad \forall q_h \in Q_h\}.$$  

Denoting by $\pi_h$ the $L^2(\Omega)$ projection of the pressure $p$ onto $Q_h$, we have that for $0 \leq m \leq 1$

$$\|p - \pi_h\|_m \leq C h^{l+1-m}\|p\|_{l+1} \quad \forall p \in H^{l+1}(\Omega).$$  \hspace{1cm} (6)$$

In the error analysis, the Poincaré–Friedrichs inequality

$$\|v\|_0 \leq C_{\Omega}^{1/d} \|\nabla v\|_0, \quad \forall v \in H^1_0(\Omega)^d,$$  \hspace{1cm} (7)$$

will be used.

In the analysis, the Stokes problem

$$-\nu \Delta u + \nabla p = g \quad \text{in} \Omega,$$

$$u = 0 \quad \text{on} \partial \Omega,$$

$$\nabla \cdot u = 0 \quad \text{in} \Omega,$$  \hspace{1cm} (8)$$

will be considered. Let us denote by $(u_h, p_h) \in V_h \times Q_h$ the mixed finite element approximation to (8), given by

$$\nu(\nabla u_h, \nabla v_h) - (\nabla \cdot v_h, p_h) = (g, v_h) \quad \forall v_h \in V_h$$

$$(\nabla \cdot u_h, q_h) = 0 \quad \forall q_h \in Q_h.$$  \hspace{1cm} (9)$$

Following [13], one gets the estimates

$$h^{-1}\|u - u_h\|_1 + \|u - u_h\|_1 \leq C \left(\inf_{v_h \in V_h} \|u - v_h\|_1 + h^{-1} \inf_{q_h \in Q_h} \|p - q_h\|_0\right),$$  \hspace{1cm} (10)$$

$$\|p - p_h\|_0 \leq C \left(\nu \inf_{v_h \in V_h} \|u - v_h\|_1 + \inf_{q_h \in Q_h} \|p - q_h\|_0\right).$$  \hspace{1cm} (11)$$

It can be observed that the error bounds for the velocity depend on negative powers of $\nu$.

For the analysis, it will be advantageous to use a projection of $(u, p)$ into $V_h \times Q_h$ with uniform in $\nu$, optimal, bounds for the velocity. In [6] a projection with this property was introduced. Let $(u, p)$ be the solution of the Navier–Stokes equations (1) with $u \in V \cap H^{k+1}(\Omega)^d$, $p \in Q \cap H^k(\Omega)$, $k \geq 1$, and observe that $(u, 0)$ is the solution of the Stokes problem [8] with right-hand side

$$g = f - \partial_t u - (u \cdot \nabla) u - \nabla p.$$  \hspace{1cm} (12)$$
Denoting the corresponding Galerkin approximation in $V_h \times Q_h$ by $(s_h, l_h)$, one obtains from (10)-(11)
\[
\|u - s_h\|_0 + h\|u - s_h\|_1 \leq C h^{k+1}\|u\|_{k+1},
\]
\[
\|l_h\|_0 \leq C \nu h^k\|u\|_{k+1},
\]
where the constant $C$ does not depend on $\nu$.

**Remark 1** Assuming the necessary smoothness in time and considering (8) with
\[
g = \partial_t (f - \partial_t u - (u \cdot \nabla) u - \nabla p),
\]
one can derive an error bound of the form (13) also for $\partial_t(u - s_h)$. One can proceed similarly
for higher order derivatives in time.

Following [3], one can also obtain the following bounds for $s_h$
\[
\|u - s_h\|_\infty \leq C_\infty h \left( \log(|\Omega|^{1/d} / h) \right) \|\nabla u\|_\infty,
\]
\[
\|\nabla (u - s_h)\|_\infty \leq C_\infty \|\nabla u\|_\infty,
\]
where $C_\infty$ does not depend on $\nu$ and $\bar{r} = 0$ for linear elements and $\bar{r} = 1$ otherwise.

## 3 The IMEX and semi-implicit BDF2 methods in time with grad-div stabilization in space

In the sequel we well call IMEX to the method with explicit treatment of the nonlinear term
and semi-implicit to the method with semi-implicit nonlinear term.

In this section we state and analyze a fixed step and a variable step versions of both
an IMEX and a semi-implicit BDF2 methods with grad-div stabilization in space. We first
introduce the grad-div stabilization and a couple of Gronwall lemmas that will be applied
for the error analysis of the methods. Then, we introduce the fully discrete schemes. Some
lemmas are then proved that will be used both for the fixed and variable step cases analyzed
in next two subsections. In Section 3.1 we carry out the error analysis for the case in which
the step size is fixed. The variable step size case is analyzed in Section 3.2. The ideas in
both sections are similar, the second one having more technical difficulties. For this reason,
we think it is convenient to show first the analysis for the fixed step size, although this case
can be obviously obtained as a particular case of the error analysis of Section 3.2.

The spatial discretization that will be studied for the approximation of the solution of the
Navier–Stokes equations (1) is obtained by adding to the Galerkin equations a control of the
divergence constraint (grad-div stabilization). More precisely, the following grad-div method
will be considered: Find $(u_h, p_h) : (0, T) \to V_h \times Q_h$ such that for all $(v_h, q_h) \in V_h \times Q_h$ one has
\[
(\partial_t u_h, v_h) + \nu(\nabla u_h, \nabla v_h) + b(u_h, u_h, v_h) - (p_h, \nabla \cdot v_h)
+ (\nabla \cdot u_h, q_h) + \mu(\nabla \cdot u_h, \nabla \cdot v_h) = (f, v_h),
\]
with $u_h(0)$ given. Here, and in the rest of the paper,
\[
b(u, v, w) = (B(u, v), w) \quad \forall u, v, w \in H_0^1(\Omega)^d,
\]
where,
\[
B(u, v) = (u \cdot \nabla) v + \frac{1}{2}(\nabla \cdot u) v \quad \forall u, v \in H_0^1(\Omega)^d.
\]
Notice the well-known property
\[
b(u, v, w) = -b(u, w, v) \quad \forall u, v, w \in V,
\]
such that, in particular, $b(u, w, w) = 0 \forall u, w \in V$. We will use the following discrete
Gronwall inequality whose proof can be found in [15].
Lemma 1 Let $k, B, a_j, b_j, c_j, \gamma_j$ be nonnegative numbers such that
\[
a_n + k \sum_{j=0}^{n} b_j \leq k \sum_{j=0}^{n} \gamma_j a_j + k \sum_{j=0}^{n} c_j + B, \quad \text{for} \quad n \geq 0.
\]
Suppose that $k\gamma_j < 1$, for all $j$, and set $\sigma_j = (1 - k\gamma_j)^{-1}$. Then
\[
a_n + k \sum_{j=0}^{n} b_j \leq \exp \left( k \sum_{j=0}^{n} \sigma_j \gamma_j \right) \left\{ k \sum_{j=0}^{n} c_j + B \right\}, \quad \text{for} \quad n \geq 0.
\]
We will also apply the following Gronwall lemma.

Lemma 2 Let $B, a_j, b_j, c_j, \gamma_j, \delta_j$ be nonnegative numbers such that
\[
a_n + \sum_{j=0}^{n} b_j \leq \gamma_n a_n + \sum_{j=0}^{n-1} \gamma_j a_j + \sum_{j=0}^{n} c_j + B, \quad \text{for} \quad n \geq 0.
\]
Suppose that $\gamma_j < 1$, for all $j$, and set $\sigma_j = (1 - \gamma_j)^{-1}$. Then
\[
a_n + \sum_{j=0}^{n} b_j \leq \exp \left( \sigma_n \gamma_n + \sum_{j=0}^{n-1} \sigma_j \gamma_j + \delta_j \right) \left\{ \sum_{j=0}^{n} c_j + B \right\}, \quad \text{for} \quad n \geq 0.
\]

Proof We follow the proof of [15] Lemma 5.1. Let $d_n$ be defined by the relation
\[
a_n + \sum_{j=0}^{n} b_j + d_n = \gamma_n a_n + \sum_{j=0}^{n-1} \gamma_j a_j + \sum_{j=0}^{n} c_j + B, \quad \text{for} \quad n \geq 0,
\]
and let us denote by $S_n$ the left hand side above. Notice then
\[
S_n - S_{n-1} = \gamma_n a_n + \delta_{n-1} a_{n-1} + c_n \leq \gamma_n S_n + \delta_{n-1} S_{n-1} + c_n, \quad n \geq 1.
\]
For $n = 0$, we have
\[
S_0 \leq (1 + \gamma_0)^{-1} (c_0 + B) = (1 + \sigma_0 \gamma_0) (c_0 + B) \leq \exp(\sigma_0 \gamma_0) (c_0 + B).
\]
Assume that
\[
S_k \leq \exp \left( \sigma_k \gamma_k + \sum_{j=0}^{k-1} \sigma_j \gamma_j + \delta_j \right) \left( \sum_{j=0}^{k} c_j + B \right), \quad k = 0, \ldots, n.
\]
and let us see that the inequality also holds for $k = n + 1$, which will finish the proof. We have
\[
S_{n+1} \leq (1 - \gamma_{n+1})^{-1} (1 + \delta_n) S_n + c_{n+1} \leq (1 + \sigma_{n+1} \gamma_{n+1}) (\exp(\delta_n) S_n + c_{n+1})
\]
\[
\leq \exp(\sigma_{n+1} \gamma_{n+1}) \left( \exp \left( \sum_{k=0}^{n} (\sigma_k \gamma_k + \delta_k) \right) \left( \sum_{k=0}^{n} c_k + B \right) + c_{n+1} \right).
\]
\[
\square
\]

Now, we consider a fully discrete approximation to the solution of [17]. Let us consider time levels $0 = t_0 < t_1 < \ldots < t_N = T$, and let us denote
\[
\Delta t_n = t_{n+1} - t_n, \quad n = 0, 1, \ldots, N - 1,
\]
and
\[
\omega_n = \Delta t_n / \Delta t_{n-1}, \quad n = 1, \ldots, N - 1.
\]
In the sequel we will assume that for $\gamma \in (0,1)$ and $\Gamma > 1$, we have
$$\gamma \leq \omega_n \leq \Gamma, \quad n = 0, 1, \ldots, N - 1. \quad (19)$$

We will also use the following notation,
$$Du^n = u^n - u^{n-1}, \quad \hat{u}^n = u^n + \omega_n Du^n,$$
and
$$D\hat{u}^n = Du^n + \frac{\omega_n-1}{1+\omega_n-1} (u^n - \hat{u}^{n-1}).$$

Observe that $\hat{u}^n$ is the extrapolated value at $t_{n+1}$ of the linear interpolant taking values $u^n$ and $u^{n-1}$ at $t_n$ and $t_{n-1}$, respectively. Notice also that for fixed step size, that is when $\Delta t_n = T/N$, $n = 0, 1, \ldots, N - 1$, we have
$$\hat{u}^n = 2u^n - u^{n-1} \quad \text{and} \quad D\hat{u}^n = (D + \frac{1}{2}D^2)u^n = \frac{3}{2}u^n - 2u^{n-1} + \frac{1}{2}u^{n-2}.$$

Now, we consider the following implicit-explicit (IMEX) approximation to (17) based on the BDF2 method. Given $u_h^n, u_h^{n+1}$ solve, for $n \geq 1$ and for all $(v_h, q_h) \in V_h \times Q_h$

$$\left( \frac{1}{\Delta t_n} Du_h^{n+1}, v_h \right) + \nu (\nabla u_h^{n+1}, \nabla v_h) + b(u_h^n, \hat{u}_h^n, v_h) - (p_h^{n+1}, \nabla \cdot v_h)$$
$$+ (\nabla \cdot u_h^{n+1}, q_h) + \mu (\nabla \cdot u_h^{n+1}, \nabla \cdot v_h) = (f^{n+1}, v_h), \quad (20)$$

where, here and in the sequel, the notation $f^{n+1}$ means $f(t_{n+1})$. We also consider the semi-implicit method obtained when the third term on the left-hand side of (20) is replaced by
$$b(u_h^n, u_h^{n+1}, v_h)$$
that is,
$$\left( \frac{1}{\Delta t_n} Du_h^{n+1}, v_h \right) + \nu (\nabla u_h^{n+1}, \nabla v_h) + b(u_h^n, u_h^{n+1}, v_h) - (p_h^{n+1}, \nabla \cdot v_h)$$
$$+ (\nabla \cdot u_h^{n+1}, q_h) + \mu (\nabla \cdot u_h^{n+1}, \nabla \cdot v_h) = (f^{n+1}, v_h), \quad (22)$$

Taking $v_h \in V_h^{\text{div}}$ in (20) we get
$$\left( \frac{1}{\Delta t_n} Du_h^{n+1}, v_h \right) + \nu (\nabla u_h^{n+1}, \nabla v_h) + b(u_h^n, \hat{u}_h^n, v_h) + \mu (\nabla \cdot u_h^{n+1}, \nabla \cdot v_h) = (f^{n+1}, v_h). \quad (23)$$

To get $u_h^n$ we apply the IMEX Euler method with the same structure as BDF2 (explicit non-linear term) for both the IMEX and the semi-implicit methods. In fact, formulae (20) and (23) are valid for $n = 0$ if we set
$$\hat{u}^0 = u^0, \quad D\hat{u}^1 = Du^1. \quad (24)$$

We will compare $u_h^n$ with $s_h^n$. For simplicity we will assume $u_h^n = s_h^n$. It is easy to prove that for $n \geq 1$ and $v_h \in V_h^{\text{div}}$

$$\left( \frac{1}{\Delta t_n} Ds_h^{n+1}, v_h \right) + \nu (\nabla s_h^{n+1}, \nabla v_h) + b(s_h^n, s_h^n, v_h) + \mu (\nabla \cdot s_h^{n+1}, \nabla \cdot v_h)$$
$$= (f^{n+1}, v_h) + (p^{n+1} - \gamma v_h^{n+1}, \nabla \cdot v_h) + \mu (\nabla \cdot (s_h^{n+1} - u^{n+1}), \nabla \cdot v_h)$$
$$+ \left( \frac{1}{\Delta t_n} Ds_h^{n+1} - u_h^{n+1}, v_h \right) + b(s_h^n, \hat{s}_h^n, v_h) - b(u^{n+1}, u^{n+1}, v_h). \quad (25)$$

Then, denoting by
$$e_h^n = s_h^n - u_h^n.$$
and subtracting (23) from (25) we get
\[
\left( \frac{1}{\Delta t_n} D e^{n+1}_h, v_h \right) + \mu (\nabla e^{n+1}_h, \nabla v_h) + b(\hat{s}^n_h, \hat{s}^n_h, v_h) - b(\hat{\mu}^n_h, \hat{\mu}^n_h, v_h) + \mu (\nabla \cdot e^{n+1}_h, \nabla \cdot v_h) = (\tau^{n+1}_1 + \tau^{n+1}_3, v_h) + (\tau^{n+1}_2, \nabla \cdot v_h)
\]
(26)
where
\[
\begin{align*}
\tau^{n+1}_1 &= \frac{1}{\Delta t_n} D s^{n+1}_h - u^{n+1}_i, \\
\tau^{n+1}_2 &= (\mu^{n+1} - \mu^{n+1}_0) + \mu (\nabla (s^{n+1}_h - u^{n+1})), \\
(\tau^{n+1}_3, v_h) &= b(\hat{s}^n_h, \hat{s}^n_h, v_h) - b(\mu^{n+1}_0, \mu^{n+1}_0, v_h).
\end{align*}
\]
Similarly, for the semi-implicit method we get (23) with the third and fourth terms replaced by \(b(\hat{s}^n_h, \hat{s}^n_h, v_h) - b(\hat{u}^n_h, \hat{u}^n_h, v_h)\), and \(\tau_3\) replaced by \(\tau_4\) defined by
\[
(\tau^{n+1}_4, v_h) = b(\hat{s}^n_h, \hat{s}^n_h, v_h) - b(u^{n+1}_i, u^{n+1}_i, v_h).
\]
In the following lemma we bound the difference of the nonlinear terms for the IMEX method.

**Lemma 3** Considering the fixed stepsize case \(\Delta t_n = \Delta t = T/N\), the following bound holds
\[
|b(\hat{s}^n_h, \hat{s}^n_h, e^{n+1}_h) - b(\hat{u}^n_h, \hat{u}^n_h, e^{n+1}_h)| \leq \left( \frac{||\nabla \hat{s}^n_h||_{\infty}}{2} + \frac{9}{2\mu} ||\hat{s}^n_h||_2 \right) \|e^{n+1}_h\|_0^2 \\
+ 3||\nabla \hat{s}^n_h||_{\infty} \|e^n_h\|_2^2 + \frac{3}{2} ||\nabla \hat{s}^n_h||_{\infty} \|e^{n-1}_h\|_2^2 + \frac{\mu}{3} \|\nabla \cdot e^n_h\|_0^2 + \frac{\mu}{6} \|\nabla \cdot e^{n-1}_h\|_0^2 \\
+ 2\|\hat{u}^n_h\|_{\infty} ||\nabla \hat{s}^n_h||_{\infty} \|e^{n+1}_h - \hat{e}^n_h\|_0 \|e^{n+1}_h\|_0.
\]
(29)
In view of the upper bound in (19), for the variable stepsize case the bound for the difference of the nonlinear terms is
\[
|b(\hat{s}^n_h, \hat{s}^n_h, e^{n+1}_h) - b(\hat{u}^n_h, \hat{u}^n_h, e^{n+1}_h)| \leq \left( \frac{||\nabla \hat{s}^n_h||_{\infty}}{2} + \frac{9\alpha}{2\mu} ||\hat{s}^n_h||_2 \right) \|e^{n+1}_h\|_0^2 \\
+ \frac{1 + 2\Gamma}{2} \|\nabla \hat{s}^n_h||_{\infty} \|e^n_h\|_2^2 + \frac{\Gamma(1 + 2\Gamma)}{2} \|\nabla \hat{s}^n_h||_{\infty} \|e^{n-1}_h\|_2^2 \\
+ \frac{\mu(1 + 2\Gamma)(1 + \Gamma)}{18\alpha} \|\nabla \cdot e^n_h\|_0^2 + \frac{\Gamma(1 + 2\Gamma)}{18\alpha} \|\nabla \cdot e^{n-1}_h\|_0^2 \\
+ 2\|\hat{u}^n_h\|_{\infty} ||\nabla \hat{s}^n_h||_{\infty} \|e^{n+1}_h - \hat{e}^n_h\|_0 \|e^{n+1}_h\|_0,
\]
(30)
where \(\alpha\) is a positive constant.

**Proof** We start by proving (29). Adding \(\pm b(\hat{u}^n_h, \hat{u}^n_h, e^{n+1}_h)\), and noticing that \(b(\hat{u}^n_h, e^{n+1}_h, e^{n+1}_h) = 0\), we have
\[
|b(\hat{s}^n_h, \hat{s}^n_h, e^{n+1}_h) - b(\hat{u}^n_h, \hat{u}^n_h, e^{n+1}_h)| = |b(\hat{e}^n_h, \hat{e}^n_h, e^{n+1}_h) - b(\hat{u}^n_h, e^{n+1}_h - \hat{e}^n_h, e^{n+1}_h)|.
\]
(31)
For the first term on the right-hand side above we write
\[
|b(\hat{e}^n_h, \hat{e}^n_h, e^{n+1}_h)| \leq \|\hat{e}^n_h\|_0 \|\nabla \hat{s}^n_h||_{\infty} \|e^{n+1}_h\|_0 + \|\hat{s}^n_h\|_{\infty} \|\nabla \cdot \hat{e}^{n+1}_h\|_0 \|e^{n+1}_h\|_0 \\
\leq \left( \frac{||\nabla \hat{s}^n_h||_{\infty}}{2} + \frac{9}{2\mu} ||\hat{s}^n_h||_2 \right) \|e^{n+1}_h\|_0^2 + \|\nabla \hat{s}^n_h||_{\infty} \|e^{n+1}_h\|_0 \|\hat{e}^n_h\|_0 + \frac{\mu}{18} \|\nabla \cdot \hat{e}^n_h\|_0^2.
\]
(32)
Let us now observe that,
\[
\|\hat{e}^n_h\|_0^2 \leq 6\|e^n_h\|_0^2 + 3\|e^{n-1}_h\|_0^2.
\]

and a similar expression for \( \| \nabla \cdot \hat{e}_h^n \|_0^2 \), so that,

\[
|b(\hat{e}_h^n, \hat{s}_h^n, e_h^{n+1})| \leq \left( \frac{\| \nabla s_h^n \|_2}{2} + \frac{9}{2\mu} \| s_h^n \|_\infty^2 \right) \| e_h^{n+1} \|_0 + \frac{\mu}{3} \| \nabla \cdot e_h^n \|_0^2 + \frac{\mu}{6} \| \nabla \cdot e_h^{n-1} \|_0^2 + 3 \| \nabla s_h^n \|_\infty \| e_h^n \|_3^2 + \frac{3}{2} \| \nabla s_h^{n+1} \|_\infty \| e_h^{n-1} \|_0 + \frac{\mu}{3} \| \nabla \cdot e_h^n \|_0^2 + \frac{\mu}{6} \| \nabla \cdot e_h^{n-1} \|_0^2.
\]

Finally, from the definition of the nonlinear term and using inverse inequality (19) it is easy to check that

\[
|b(\hat{u}_h^n, e_h^{n+1} - \hat{e}_h^n, e_h^{n+1})| \leq 2\| \hat{u}_h^n \|_c \| e_h^{n+1} - \hat{e}_h^n \|_0 \| e_h^{n+1} \|_0.
\]

The proof of (32) can be obtained arguing in the same way with only two differences. The first one is that we include a parameter \( \alpha \) to bound the second term on the first line of the right-hand side of (32). The second one is that we apply (19) to bound both \( \| e_h^n \|_3^2 \) and \( \| \nabla \cdot e_h^n \|_0^2 \).

For the semi-implicit method we have the following result.

**Lemma 4** The following bound holds in the fixed stepsize case \( \Delta t_n = \Delta t = T/N \),

\[
|b(\hat{s}_h^n, s_h^{n+1}, e_h^{n+1}) - b(\hat{u}_h^n, u_h^{n+1}, e_h^{n+1})| \leq \left( \frac{\| \nabla s_h^n \|_\infty}{2} + \frac{9}{2\mu} \| s_h^n \|_\infty^2 \right) \| e_h^{n+1} \|_0^2 + 3 \| \nabla s_h^n \|_\infty \| e_h^n \|_3^2 + \frac{3}{2} \| \nabla s_h^{n+1} \|_\infty \| e_h^{n-1} \|_0 + \frac{\mu}{3} \| \nabla \cdot e_h^n \|_0^2 + \frac{\mu}{6} \| \nabla \cdot e_h^{n-1} \|_0^2.
\]

and, in the variable stepsize case, for \( \Gamma \) the constant in (19) and \( \alpha \) any positive constant, the following bound holds:

\[
|b(\hat{s}_h^n, s_h^{n+1}, e_h^{n+1}) - b(\hat{u}_h^n, u_h^{n+1}, e_h^{n+1})| \leq \left( \frac{\| \nabla s_h^n \|_\infty}{2} + \frac{9}{2\mu} \| s_h^n \|_\infty^2 \right) \| e_h^{n+1} \|_0^2 + (1 + 2\Gamma)(1 + \Gamma) \| \nabla s_h^n \|_\infty \| e_h^n \|_3^2 + \frac{\Gamma(1 + 2\Gamma)}{2} \| \nabla s_h^{n+1} \|_\infty \| e_h^{n-1} \|_0 + \frac{\Gamma(1 + 2\Gamma)}{18\alpha} \| \nabla \cdot e_h^n \|_0^2 + \frac{\mu}{18\alpha} \| \nabla \cdot e_h^{n+1} \|_0^2.
\]

**Proof** By adding \( \pm b(\hat{u}_h^n, s_h^{n+1}, e_h^{n+1}) \) and taking into account that \( b(\hat{u}_h^n, e_h^{n+1}, e_h^{n+1}) = 0 \), we have

\[
|b(\hat{s}_h^n, s_h^{n+1}, e_h^{n+1}) - b(\hat{u}_h^n, u_h^{n+1}, e_h^{n+1})| = |b(\hat{e}_h^n, s_h^{n+1}, e_h^{n+1})|
\]

which can be bounded as the term \( |b(\hat{e}_h^n, \hat{s}_h^n, e_h^{n+1})| \) in the previous lemma.

**Remark 2** We notice the similarity between the bounds in Lemmas 4 and 4. Indeed, bounds (33) and (34) differ from (29) and (30) in that \( \hat{s}_h^n \) in (29) and (30) is replaced by \( s_h^{n+1} \) in (33) and (34) and, more importantly, the last term, \( 2\| \hat{u}_h^n \|_c \| e_h^{n+1} - \hat{e}_h^n \|_0 \| e_h^{n+1} \|_0 \) on the right-hand side of (29) and (30) is not present in (33) and (34). We will see that this term gives rise to a CFL type condition that affects the IMEX (20) method but not the semi-implicit method (22).

We now estimate the truncation errors. For \( \tau_2^{n+1} \), in view of (9) and (13), it easily follows that

\[
\| \tau_2^{n+1} \|_0 \leq C(\mu \| u_{n+1} \|_k h^k + \| p_{n+1} \|_{l+1} h^{l+1}).
\]
Thus, applying (13) and Hölder's inequality an easy calculation shows that (36) holds.

We also notice that

\[ \| \tau_n^{n+1} \|_0 \leq 4 \frac{h^{2k}}{\Delta t_n} \int_{t_n}^{t_{n+1}} \| u(t) \|_q^2 \, dt + \frac{2h^{2k}}{\Delta t_{n-1}} \int_{t_{n-1}}^{t_n} \| u(t) \|_q^2 \, dt \]

\[ + \frac{(1 + \omega_n)^2}{10} (\Delta t_n)^3 \int_{t_n}^{t_{n+1}} \| u_{t(t)}(t) \|_q^2 \, ds + \frac{\omega_n^2}{10} (\Delta t_n + \Delta t_{n-1})^3 \int_{t_{n-1}}^{t_n} \| u_{t(t)}(t) \|_q^2 \, dt. \]

Lemma 5 The following bounds hold for \( n \geq 1 \).

\[ \| \tau_n^{n+1} \|_0 \leq C \left( (1 + \omega_n) \| u \|_{L^\infty(W^{1,\infty})} \right) \]

\[ + \| u \|_{L^\infty(W^{1,\infty})} \| \dot{u}^n - u^{n+1} \|_0 + \| u \|_{L^\infty(L^\infty)} \| \dot{u}^n - u^{n+1} \|_1 \| \nu_n \|_0, \]

\[ \| \tau_u \| \leq C (1 + \omega_n) \| u \|_{L^\infty(W^{1,\infty})} \| u \|_{L^\infty(H^{k+1})} \]

\[ + \| u \|_{L^\infty(W^{1,\infty})} \| \dot{u}^n - u^{n+1} \|_0 \| \nu_n \|_0, \]

where

\[ \| u^{n+1} - \dot{u}^n \|_2 \leq \frac{1 + \omega_n}{\sqrt{3}} \left( \int_{t_n}^{t_{n+1}} \| u_{t(t)}(t) \|_q^2 \, dt \right)^{1/2} \]

\[ + \frac{\omega_n}{\sqrt{3}} (\Delta t_n + \Delta t_{n-1})^{3/2} \left( \int_{t_{n-1}}^{t_n} \| u_{t(t)}(t) \|_q^2 \, dt \right)^{1/2}, \quad l = 0, 1. \]

Proof We first write \( \tau_1 = (\Delta t_n)^{-1} D(s_h^{n+1} - u^{n+1}) + (\Delta t_n)^{-1} (D u^{n+1} - u^{n+1}) \) and, further, we notice that we can write for the first term

\[ \frac{1}{\Delta t_n} D(s_h^{n+1} - u^{n+1}) = (1 + \frac{\omega_n}{1 + \omega_n}) \frac{1}{\Delta t_n} D(s_h^{n+1} - u^{n+1}) - \frac{\omega_n}{1 + \omega_n} \frac{1}{\Delta t_n} D(s_h^n - u^n) \]

\[ = (1 + \frac{\omega_n}{1 + \omega_n}) \frac{1}{\Delta t_n} \int_{t_n}^{t_{n+1}} d(s_h(t) - u(t)) \, dt \]

\[ - \frac{\omega_n}{1 + \omega_n} \frac{1}{\Delta t_{n-1}} \int_{t_{n-1}}^{t_n} d(s_h(t) - u(t)) \, dt. \]

Furthermore, Taylor expansion with integral remainder shows that the second term can be bounded by

\[ \frac{1}{\Delta t_n} (D u^{n+1} - u^{n+1}) = (1 + \omega_n) \frac{1}{2 \Delta t_n} \int_{t_n}^{t_{n+1}} (t - t_n)^2 u_{t(t)}(t) \, dt \]

\[ - \frac{\omega_n^2}{2(1 + \omega_n)} \frac{1}{\Delta t_{n-1}} \int_{t_{n-1}}^{t_n} (t - t_{n-1})^2 u_{t(t)}(t) \, dt. \]

We also notice that

\[ \frac{\omega_n^2}{2(1 + \omega_n)} \frac{1}{\Delta t_{n-1}} = \frac{\omega_n}{2(1 + \omega_n)} \frac{1}{\Delta t_{n-1}} = \frac{\omega_n}{2(\Delta t_n + \Delta t_{n-1})}. \]

Thus, applying (13) and Hölder’s inequality an easy calculation shows that (36) holds.

For \( \tau_3^{n+1} \), adding \( \pm b(\dot{u}^n, s_h^n, \nu_h) \pm b(\ddot{u}^n, s_h^n, \nu_h) \pm b(\dddot{u}^n, s_h^n, \nu_h) \), we can write

\[ \tau_3^{n+1} + \nu_h = b(s_h^n - u^n, s_h^n, \nu_h) + b(u^n, s_h^n - u^n, \nu_h) + b(u^n, u^n - u^{n+1}, \nu_h) \]

\[ + b(u^n - u^{n+1}, u^{n+1}, \nu_h). \]
Thus, by noticing that \( \| \hat{s}_h^n - \hat{u}^n \|_l \leq 2(1 + \omega_n) \| s_h - u \|_{L^\infty(H)} \), \( l = 0, 1 \), applying (13), Hölder’s inequality and bounding \( h \leq |\Omega|^{1/4} \), when necessary, we get (37). For \( \tau_4 \), adding \( \pm b(\hat{u}^n, s_h^{n+1}, u_h) \pm b(u^n, s_h^{n+1}, v_h) \) one gets

\[
(\tau_4^{n+1}, u_h) = b(\hat{u}^n - \hat{u}_h, s_h^{n+1}, v_h) + b(u^n - u^{n+1}, s_h^{n+1}, v_h) + b(u^{n+1}, s_h^{n+1} - u^{n+1}, v_h),
\]

(41)

from where, arguing as with \( \tau_3^{n+1} \), the bound (35) follows.

Also, by means of Taylor expansion with integral remainder and applying Hölder’s inequality, it is easy to show (39). \( \square \)

We now deal with the truncation errors in the first step. Recall that

\[
\tau_1^{1} = \frac{1}{\Delta t_0} D s_h^0 - u_t^1
\]

and \( \tau_1^{1} \) are defined as in (27) but with \( \hat{s}_h^0 = s_h^0 \).

**Lemma 6** We have the following bounds,

\[
\| \tau_1^{1} \|^2_0 \leq C \left( \frac{1}{\Delta t_0} h^{2k} \| u_t \|^2_{L^2(H)} + (\Delta t_0)^2 \| u_{tt} \|^2_{L^\infty(L^2)} \right),
\]

(42)

\[
\| \tau_1^{1} \|^2_0 \leq C \left( |\Omega|^{2/4} \| u_t \|_{L^\infty(W^{1,\infty})} h^{2k} \right.
\]

\[
+ (\Delta t_0)^2 \| u_t \|_{L^\infty(W^{1,\infty})} \| u_{tt} \|^2_{L^\infty(L^2)} + \| u_t \|^2_{L^\infty(L^\infty)} \| u_{tt} \|^2_{L^\infty(H)} \right)
\]

(43)

**Proof** For \( \tau_1^{1} \), by arguing as in (36) we have

\[
\| \tau_1^{1} \|^2_0 \leq C \frac{1}{\Delta t_0} h^{2k} \int_{t_0}^{t_1} \| u_t \|^2_{L^2} dt + C \Delta t_0 \int_{t_0}^{t_1} \| u_{tt} \|^2_{L^0}, dt,
\]

from where (42) follows. To prove (43) we go back to (40), and repeat the argument there, but now taking into account that for \( n = 0 \), \( \hat{s}_h^0 = s_h^0 \) and \( \hat{u}_t^0 = u_t^0 \), so that \( \| \hat{s}_h^n - u_t^n \| \leq C h^{k+1-\ell} \| u_t \|_{L^\infty(L^\infty)} \) and

\[
\| \hat{u}_t^{n+1} - u_t^{n+1} \|_l = \left| \int_{t_0}^{t_1} u_t dt \right|_l \leq \Delta t_0 \| u_t \|_{L^\infty(H)}.
\]

Then (45) follows easily. \( \square \)

### 3.1 Fixed steplsize

We consider now the case where \( \Delta t_0 = \Delta t = T/N, n = 0, 1, \ldots, N - 1 \).

**Theorem 1** Fix \( \kappa > 0 \) and let \( \Delta t \) and \( h \) satisfy the following CFL-type condition

\[
64 e_{inv}^2 \| u_t \|^2_{L^\infty(L^\infty)} \frac{\Delta t}{\kappa^2} \leq \kappa \frac{T}{\Delta t},
\]

(44)

and

\[
\Delta t \left( 4 \frac{\kappa + 1}{T} + M^k \right) \leq \frac{1}{2}, \quad k = 2, \ldots, n + 1,
\]

(45)

with \( M^k \) defined in (54). Then, there exist a positive constant \( h_0 \), such that for \( h < h_0 \) the error \( e_{n+1} = s_h^{n+1} - u_h^{n+1} \) of the IMEX method (20), satisfies the following bound for \( 1 \leq n \leq N - 1 \),

\[
\| e_{n+1} \|^2_0 + \sum_{k=2}^{n+1} \nu \Delta t \| \nabla e_h \|^2_0 + \mu \sum_{k=2}^{n+1} \Delta t \| \nabla \cdot e_h \|^2_0 \leq
\]

\[
e^{8(n+1) + 2T \max_j M^j} \left( C_0^2 + TC_0^2 \right) (\Delta t)^3 + T \left( C_0^2 h^{2k} + C_0^2 h^{2l+2} \right)
\]

(46)

where the constants \( C_0^2 \) are defined in (59) and (63).
Proof. Taking $v_h = e_h^{n+1}$ in (26), we obtain
\begin{align}
\frac{1}{4} \frac{1}{\Delta t} \| e_h^{n+1} \|^2_0 + \frac{1}{4} \| e_h^{n+1} \|^2_0 + \frac{1}{4} \| e_h^{n+1} - e_h^n \|^2_0 \\
- \frac{1}{4} \frac{1}{\Delta t} \| e_h^n \|^2_0 - \frac{1}{4} \| e_h^{n+1} + \nu \| \nabla e_h^{n+1} \|_0^2 \\
+ \mu \| \nabla \cdot e_h^{n+1} \|_0^2 \\
\leq b(\hat{\mathbf{s}}_h^n, \mathbf{s}_h^n, e_h^{n+1}) - b(\mathbf{u}_h^n, \mathbf{u}_h^n, e_h^{n+1}) + \frac{T}{2} \| \nu_1^{n+1} \|_0^2 + \frac{T}{2} \| \tau_3^{n+1} \|_0^2 \\
+ \frac{1}{T} \| e_h^{n+1} \|^2_0 + \frac{9}{2 \mu} \| \tau_2^{n+1} \|_0^2 + \frac{\mu}{18} \| \nabla \cdot e_h^{n+1} \|_0^2.
\end{align}
(47)

We now apply (29) from Lemma 3 and, further, we bound
\begin{align}
2 \| \hat{\mathbf{u}}_h^n \|_{C_{inv}} h^{-1} \| e_h^{n+1} - e_h^n \|_0 \leq 4 c_{inv}^2 \| \hat{\mathbf{u}}_h^n \|_{C_{inv}}^2 \Delta t h^2 | e_h^{n+1} \|_0^2 + \frac{\| e_h^{n+1} - e_h^n \|_0^2}{4 \Delta t}.
\end{align}
(49)

Then
\begin{align}
b(\hat{\mathbf{s}}_h^n, \mathbf{s}_h^n, e_h^n) - b(\mathbf{u}_h^n, \mathbf{u}_h^n, e_h^n) & \leq \left( \| \nabla \hat{s}_h^n \|_\infty + \frac{9}{2 \mu} \| \hat{s}_h^n \|_\infty^2 + 4 c_{inv}^2 \| \hat{\mathbf{s}}_h^n \|_{C_{inv}}^2 \frac{\Delta t}{h^2} \right) \| e_h^{n+1} \|_0^2 \\
+ 3 \| \nabla \hat{s}_h^n \|_\infty \| e_h^n \|_0 \| e_h^{n+1} \|_0^2 \\
+ \frac{3}{2} \| \nabla \hat{s}_h^n \|_\infty \| e_h^{n-1} \|_0^2 \\
+ \frac{1}{3} \| \nabla \cdot e_h^n \|_0^2 + \frac{T}{6} \| \nabla \cdot e_h^{n-1} \|_0^2 + \frac{\| e_h^{n+1} - e_h^n \|_0^2}{4 \Delta t}.
\end{align}
(50)

The last term on the right-hand side of (50) will be absorbed with the left-hand side of (47).

In the sequel, we assume that, for $0 \leq n \leq N - 1$,
\begin{align}
4 c_{inv}^2 \| \hat{\mathbf{u}}_h^n \|_{C_{inv}}^2 \frac{\Delta t}{h^2} \leq \frac{\kappa}{T}.
\end{align}
(51)

At the end of the proof we will show that (51) always holds for $h$ sufficiently small.

Inserting (50) into (47) we reach
\begin{align}
\frac{1}{4} \frac{1}{\Delta t} \| e_h^{n+1} \|^2_0 + \frac{1}{4} \| e_h^{n+1} \|^2_0 - \frac{1}{4} \| e_h^n \|^2_0 - \frac{1}{4} \| e_h^n \|^2_0 \\
- \nu \| \nabla e_h^{n+1} \|_0^2 + \frac{17}{18} \| \nabla \cdot e_h^{n+1} \|_0^2 \\
\leq \left( \| \nabla \hat{s}_h^n \|_\infty + \frac{9}{2 \mu} \| \hat{s}_h^n \|_\infty^2 + \frac{\Delta t}{T} \right) \| e_h^{n+1} \|_0^2 + 3 \| \nabla \hat{s}_h^n \|_\infty \| e_h^n \|_0 \| e_h^{n+1} \|_0^2 \\
+ \frac{3}{2} \| \nabla \hat{s}_h^n \|_\infty \| e_h^{n-1} \|_0^2 \\
+ \frac{1}{3} \| \nabla \cdot e_h^n \|_0^2 + \frac{T}{6} \| \nabla \cdot e_h^{n-1} \|_0^2 + \frac{\| e_h^{n+1} - e_h^n \|_0^2}{4 \Delta t}.
\end{align}
(52)

Taking the sum of terms in (52) we get
\begin{align}
\frac{1}{4} \| e_h^{n+1} \|^2_0 + \frac{1}{4} \| e_h^{n+1} \|^2_0 + \sum_{k=2}^{n+1} \nu \Delta t \| \nabla e_h^n \|_0^2 + \frac{4 \mu}{9} \sum_{k=2}^{n+1} \Delta t \| \nabla \cdot e_h^n \|_0^2 \\
\leq \frac{1}{4} \| e_h^n \|_0^2 + \frac{1}{4} \| e_h^n \|^2_0 \\
+ \frac{\mu}{6} (\Delta t) \| \nabla \cdot e_h^n \|_0^2 + \frac{\mu}{6} (\Delta t) \| \nabla \cdot e_h^n \|_0^2 + \frac{3 \mu}{2} (\Delta t) \| \nabla \hat{s}_h^n \|_\infty \| e_h^n \|_0^2 \\
+ (\Delta t) \left( \frac{3}{2} \| \nabla \hat{s}_h^n \|_\infty + 3 \| \nabla \hat{s}_h^n \|_\infty \right) e_h^n \|_0^2 + \sum_{k=2}^{n+1} \Delta t \left( \frac{T}{2} \| \tau_1^{n+1} \|_0^2 + \frac{T}{2} \| \tau_2^{n+1} \|_0^2 + \frac{9}{2 \mu} \| \tau_3^{n+1} \|_0^2 \right) \\
+ \sum_{k=2}^{n+1} \Delta t \left( \frac{\| \nabla \hat{s}_h^n \|_\infty}{2} + 3 \| \nabla \hat{s}_h^n \|_\infty + \frac{\| e_h^{n-1} \|_0^2}{T} + \frac{\| e_h^{n-1} \|_0^2}{T} + \frac{3}{2} \| \nabla \hat{s}_h^n \|_\infty + \frac{3}{2} \| \nabla \hat{s}_h^n \|_\infty \right) \| e_h^n \|_0^2,
\end{align}
(53)
where in the last term of (52) to simplify writing we assume \( \|\nabla \hat{s}_h^{k+1}\|_\infty = 0 \) for \( k = n + 1 \). In the sequel, to simplify notation we define

\[
M^k = 4 \left( \frac{\|\nabla \hat{s}_h^{k+1}\|_\infty}{2} + \frac{9}{2\mu} \|\hat{s}_h^{k-1}\|_\infty^2 + 3\|\nabla \hat{s}_h^k\|_\infty + \frac{3}{2} \|\nabla \hat{s}_h^{k+1}\|_\infty \right)
\]

and

\[
E^0 = 4 \left( \frac{1}{4} \|\epsilon_h^0\|_0^2 + \frac{1}{2} \|\epsilon_h^0\|_0^2 + \frac{H_n}{4} (\Delta t) \|\nabla \cdot \epsilon_h^0\|_0^2 + \frac{\mu}{4} (\Delta t) \|\nabla \cdot \epsilon_h^0\|_0^2 + \frac{3}{2} (\Delta t) \|\nabla \hat{s}_h^0\|_\infty \|\epsilon_h^0\|_0^2 + \frac{3}{2} (\Delta t) \|\nabla \hat{s}_h^1\|_\infty \|\epsilon_h^0\|_0^2 \right) + (\Delta t) \left( \frac{3}{2} \|\nabla \hat{s}_h^1\|_\infty + 3\|\nabla \hat{s}_h^1\|_\infty \right) \|\epsilon_h^1\|_0^2 .
\]

Let us observe that all the terms in (54) can be bounded in terms of \( \|u\|_{L^\infty(L^\infty)} \) and \( \|\nabla u\|_{L^\infty(L^\infty)} \) applying (15) and (16).

With the above simplifications we can write equation (53) in the form

\[
\|\epsilon_h^{n+1}\|_0^2 + \sum_{k=2}^{n+1} \nu (\Delta t) \|\nabla \epsilon_h^k\|_0^2 + \mu \sum_{k=2}^{n+1} \Delta t \|\nabla \cdot \epsilon_h^k\|_0^2 \leq E^0 + \sum_{k=2}^{n+1} (\Delta t \left( \frac{4K+1}{\mu} + M^k \right)) \|\epsilon_h^k\|_0^2 + 2 \sum_{k=2}^{n+1} \Delta t \left( T(\|\tau_k^0\|_0^2 + \|\tau_k^1\|_0^2) + \frac{9}{\mu} \|\tau_k^2\|_0^2 \right).
\]

Assuming (45) we can apply Gronwall Lemma 4 to obtain

\[
\|\epsilon_h^{n+1}\|_0^2 + \sum_{k=2}^{n+1} \nu \Delta t \|\nabla \epsilon_h^k\|_0^2 + \mu \sum_{k=2}^{n+1} \Delta t \|\nabla \cdot \epsilon_h^k\|_0^2 \leq e^{8(n+1) + 2T \max_j M^j} \left( E^0 + 2 \sum_{k=2}^{n+1} \Delta t \left( T(\|\tau_k^0\|_0^2 + \|\tau_k^1\|_0^2) + \frac{9}{\mu} \|\tau_k^2\|_0^2 \right) \right).
\]

In view of (35) and (26)-(27)-(39) from Lemma 5 and taking into account that \( \omega_n = 1 \), \( n = 1, \ldots, N = T/\Delta t \), we have

\[
\Delta t \sum_{k=2}^{n} \|\tau_k^0\|_0^2 \leq C \left( \|u_t\|_{L^2(H^l)}^2 h^{2k} + \|u_t\|_{L^2(L^2)}^2 (\Delta t)^4 \right),
\]

\[
\Delta t \sum_{k=2}^{n} \|\tau_k^1\|_0^2 \leq C \left( \mu^2 \|u_t\|_{L^\infty(H^{l+1})}^2 h^{2k} + \|p\|_{L^\infty(H^{l+1})}^2 h^{2(l+1)} \right),
\]

and

\[
\Delta t \sum_{k=2}^{n} \|\tau_k^0\|_0^2 \leq C \left( \|u\|_{L^\infty(W^{l+1})}^2 \left( T^{l/2} \|u\|_{L^\infty(H^{l+1})}^2 h^{2k} + \|u_t\|_{L^2(L^2)}^2 (\Delta t)^4 \right) + C \|u\|_{L^\infty(L^\infty)}^2 \|u_t\|_{L^2(H^l)}^2 (\Delta t)^4 \right).
\]

Consequently,

\[
\sum_{k=2}^{n+1} \Delta t \left( T(\|\tau_k^0\|_0^2 + \|\tau_k^1\|_0^2) + \frac{9}{\mu} \|\tau_k^2\|_0^2 \right) \leq T \left( C_1^2 h^{2k} + C_2^2 h^{2l+2} + C_3^2 (\Delta t)^4 \right),
\]

where

\[
C_1^2 = C \left( \|u_t\|_{L^2(H^l)}^2 + \left( \mu + T \|\Omega\|^{2/d} \|u\|_{L^\infty(W^{l+1})}^2 \right) \|u\|_{L^\infty(H^{l+1})}^2 \right),
\]

\[
C_2^2 = C \left( \frac{1}{\mu} \|u\|_{L^\infty(H^{l+1})}^2 \right),
\]

\[
C_3^2 = C \left( \|u_t\|_{L^2(L^2)}^2 + \|u\|_{L^\infty(W^{l+1})}^2 \|u_t\|_{L^2(L^2)}^2 + \|u\|_{L^\infty(L^\infty)}^2 \|u_t\|_{L^2(H^l)}^2 \right).
\]
Finally, since we have assumed \( u^n_k = s^n_k \) to bound \( E_0 \), we only need to bound \( e^n_k \). To this end, let us observe that after one step using Euler method with the explicit form of the nonlinear term, and arguing as before, it is easy to get
\[
\frac{1}{\Delta t_0} \| e^n_k \|_0 + \nu \| \nabla e^n_k \|_0 + \mu \| \nabla \cdot e^n_k \|_0 = (\tau^n_1 + \tau^n_2, e^n_k) + (\nabla \cdot e^n_k).
\]  
From (60) we get
\[
\frac{1}{2} \| e^n_k \|_0^2 + \mu \frac{\Delta t_0}{2} \| \nabla \cdot e^n_k \|_0^2 \leq \frac{(\Delta t_0)^2}{2} \| \tau^n_1 \|_0^2 + \tau^n_1 \|_0^2 + \frac{\Delta t_0}{\mu} \| \tau^n_2 \|_0^2.
\]
Now, in view of (42) and taking into account that \( \Delta \), from (60) we get
\[
(\Delta t)^2 \| \tau^n_1 \|_0^2 \leq C \| u_{et} \|_{L_\infty(L^2)} (\Delta t)^4 + TC_1 h^2,
\]  
where \( C_1 \) is the constant in (59). Also in view of (43) we have
\[
(\Delta t)^2 \| \tau^n_2 \|_0^2 \leq C \| u_{et} \|_{L_\infty(W^{1,\infty})} \| u_l \|_{L_\infty(L^2)} + \| u \|_{L_\infty(L^\infty)} \| u \|_{L_\infty(H^1)} + TC_1 h^2.
\]
Thus, we finally obtain
\[
\| e^n_k \|_0^2 + \mu \Delta t \| \nabla \cdot e^n_k \|_0^2 \leq C_0^2 (\Delta t)^4 + T \left( C_1 h^2 + C_2 h^{2 + 2} \right),
\]  
where
\[
C_0^2 = C \| u_{et} \|_{L_\infty(L^2)} + \| u \|_{L_\infty(W^{1,\infty})} \| u_l \|_{L_\infty(L^2)} + \| u \|_{L_\infty(L^\infty)} \| u \|_{L_\infty(H^1)}.
\]
Going back to the definition of \( E_0 \), (50), we reach
\[
E_0 \leq (5 + \Delta t \left[ 6 \| \nabla s^n_k \|_\infty + 12 \| \nabla s^n_1 \|_\infty) \right] \| e^n_k \|_0^2 + 2 \mu (\Delta t) \| \nabla \cdot e^n_k \|_0^2,
\]
so that applying (62) and (10) we finally obtain
\[
E_0 \leq C \left( 1 + \Delta t \| u \|_{L_\infty(W^{1,\infty})} \right) \left( C_0^2 (\Delta t)^4 + T \left( C_1 h^2 + C_2 h^{2 + 2} \right) \right).
\]  
Inserting (58), (64) into (67), we conclude (10) as long as condition (61) holds. Thus, in order to finish the proof, we have to check that this is the case if \( h \) and \( \Delta t \) are taken sufficiently small. To this end we devote the rest of the proof.

We will show that for \( h \) sufficiently small it holds
\[
\| \tilde{u}^n_k \|_\infty \leq 4 \| u \|_{L_\infty(L^\infty)},
\]  
for \( 0 \leq n \leq N - 1 \), so that (51) will be a consequence of (44). For this purpose, we start by taking \( h_0 \) such that the right-hand side of (15) is bounded by \( \| u \|_{L_\infty(L^\infty)} / 6 \), so that by writing \( s_h = s_h - u + u \), and in view of (15), we have that, if \( h \leq h_{0,1} \), then, \( \| s_h \|_{L_\infty(L^\infty)} \leq (7/6) \| u \|_{L_\infty(L^\infty)} \), and, consequently,
\[
\| s^n_h \| \leq (7/2) \| u \|_{L_\infty(L^\infty)}, \quad 0 \leq t_n \leq T.
\]  
For \( \| \tilde{u}^n_k \|_\infty \), adding and subtracting \( \tilde{s}^n_h \) and using inverse inequality (3) we get
\[
\| \tilde{u}^n_k \|_\infty \leq \| \tilde{u}^n_k - s^n_h + s^n_h \|_\infty \leq c_{inv} h^{d-2} \| \tilde{u}^n_k - s^n_h \|_0 + \| s^n_h \|_\infty \leq c_{inv} h^{d-2} \| \tilde{u}^n_k - s^n_h \|_0 + c_{inv} h^{d-2} \| u^n_{et} - s^n_h \|_0 + (7/2) \| u \|_{L_\infty(L^\infty)},
\]  
where, in the last inequality, we have also applied (66). We now take \( h_0 \leq h_{0,1} \) such that if \( h \leq h_{0,2} \) the right-hand sides of (40) and (62) are smaller that
\[
r(h) = \frac{h^d}{36 \gamma_{inv}^d}.
\]  
(68)
Notice that this is possible since, on the one hand, due to (43), \( \Delta t \) can be bounded in terms of \( h^2 \), and, on the other hand, the right-hand sides of (46) and (62), being at least \( O(h^4) \), decay faster with \( h \) than \( r(h) \) in (68), which is, at most, \( O(h^3) \).

For \( h \leq h_{0,2} \), and since we are taking \( u_n^h = s_n^h \), in view of (67), we have that (65) holds for \( n = 0, 1 \). Assuming that (65) holds for \( n \leq m \), we will now show that it also holds for \( n = m + 1 \), and this will finish the proof. Indeed, if (65) holds for \( n \leq m \), then, as argued above, (46) holds for \( n = m \). But, since \( h \leq h_{0,2} \), the right-hand side of (46) is smaller that \( r(h) \) in (68), and, consequently, for \( n = m + 1 \), the right-hand side of (67) is smaller that \( (7/2 + 1/2) \|u\|_{L^{\infty}(L^\infty)} \), that is, (65) also holds for \( n = m + 1 \). □

**Remark 3** In view of (44), we observe that as \( \Delta t, h \to 0 \) the ratio \( \Delta t / h^2 \) must be bounded. One can allow the upper bound to be larger or smaller, but its size reflects in the error bound (46) through the factor \( \exp(8(\kappa + 1)) \) on the right-hand side of the error bound (46).

As pointed out in the introduction, arguing as in [9], it is possible to get a different CFL condition in which the quantity that has to be bounded is \( (\Delta t)h^{-1} \nu^{-1} \) instead of \( (\Delta t)h^{-2} \) and that involves a weaker norm for \( u \) in (44). However, since we focus on getting bounds that hold for high Reynolds numbers we will develop this line of research in future works.

For the semi-implicit method (22) we get the following result.

**Theorem 2** Let \( \Delta t \) satisfy the following bound for \( k = 2, \ldots, n + 1 \),

\[
\Delta t \left( \frac{1}{T} + N_k \right) \leq \frac{1}{2}, \quad k = 2, \ldots, n + 1,
\]

where

\[
N_k = 4 \left( \frac{\|\nabla s_h^k\|_{\infty}}{2} + \frac{9}{2\mu} \|s_h^k\|_{2}^2 + 3\|\nabla s_h^{k+1}\|_{\infty} + \frac{3}{2} \|\nabla s_h^{k+1}\|_{\infty} \right).
\]

Then, there exist a positive constant \( h_0 \), such that for \( h < h_0 \) the error \( e_n^{n+1} = s_n^{n+1} - u_n^{n+1} \) of the semi-implicit method (22), satisfies the following bound for \( 1 \leq n \leq N - 1 = T / \Delta t - 1 \),

\[
\|e_n^{n+1}\|_{0}^2 + \sum_{k=2}^{n+1} \nu \Delta t \|e_h^k\|_{2}^2 + \mu \sum_{k=2}^{n+1} \Delta t \|e_h^k\|_{L}^2 \leq e^{8 + 2T \max_{1 \leq n \leq N} N_j} \left( C_0^2 + T C_3^2 \right) (\Delta t)^4 + T \left( C_0^2 h^{2k} + C_2^2 h^{2l+2} \right),
\]

where the constants \( C_2^r \) are defined in (59) and (65).

**Proof** Arguing as in the case of method (29), we have that (47) also holds for the semi-implicit method but with the first term on the right-hand side of (47) replaced by

\[
|b(s_h^n, s_h^{n+1}, e_h^{n+1}) - b(u_h^n, u_h^{n+1}, e_h^{n+1})|,
\]

and \( \tau_h^n \) replaced by \( \tau_h^n \). Then, recalling Lemma 3 and taking into account Remark 2 we have that (62) also holds true for method (22) with \( \kappa = 0 \), \( s_h^n \) replaced by \( s_h^{n+1} \) and \( \tau_h^n \) replaced by \( \tau_h^n \). Then the proof follows the steps of the proof of Theorem 1. In the present case, there is no need to argue about the boundedness of \( u_h^n \) since it is not necessary that condition (51) is satisfied. □

**Remark 4** From Theorems 1 and 2 using triangle inequality and applying (13), we reach

\[
\max_{1 \leq n \leq N} \|u_h^n - u^n\|_{0} = O((\Delta t)^2 + h^{k} + h^{l+1}),
\]

which means that the rate of convergence is of optimal order 2 in time. Choosing for example Hood-Taylor elements with \( l = k - 1 \) we get a rate of convergence of order \( k \) in space, which matches previously results in [7], as stated in the introduction.
3.2 Variable stepsize

In this section we carry out the error analysis for the variable stepsize case.

We start with a technical lemma that is needed to prove the main result.

Lemma 7  Assuming condition (19) and denoting by
\[ G_n = \frac{\omega_n}{2(1 + \omega_n - 1)} \| e_n^+ \|_0^2 + \frac{1}{4} \| e_n^+ + (e_n^+ - e_n^-) \|_0^2, \quad n = 1, 2, \ldots, N, \] (69)
the following inequality holds, for a constant \( K_0 \leq 75/14, \)
\[ \Delta t_n(D e_n^{n+1}, e_n^{n+1}) \geq G_{n+1} - G_n + \frac{\omega_n}{2(1 + \omega_n)} \| e_n^{n+1} - \hat{e}_n^n \|_0^2 \] (70)
\[ - \frac{K_0}{\gamma} (1 + \Gamma)^2 (|\omega_{n-1} - 1| + |\omega_n - 1|) G_n. \]

Proof  We first rewrite adequately the term \((D e_n^{n+1}, e_n^{n+1})\). Using the identity
\[ (v - w, v) = \frac{1}{2}(\| v \|_0^2 - \| w \|_0^2 + \| v - w \|_0^2), \] (71)
we have that
\[ \Delta t_n(D e_h^{n+1}, e_h^{n+1}) = \frac{1}{2}(\| e_h^{n+1} \|_0^2 - \| e_h^n \|_0^2 + \| e_h^{n+1} - e_h^n \|_0^2) \] (72)
\[ + \frac{\omega_n}{2(1 + \omega_n)} (\| e_h^{n+1} \|_0^2 - \| \hat{e}_h^n \|_0^2 + \| e_h^{n+1} - \hat{e}_h^n \|_0^2). \]

For the first term on the right-hand side above, using the identity (71), we write
\[ \frac{1}{2}(\| e_h^{n+1} \|_0^2 - \| e_h^n \|_0^2 + \| e_h^{n+1} - e_h^n \|_0^2) = \frac{1}{4}(\| e_h^{n+1} \|_0^2 - \| e_h^n \|_0^2 + \| e_h^{n+1} - e_h^n \|_0^2) \]
\[ + \frac{1}{2}(e_h^{n+1}, e_h^{n+1} - e_h^n) \]
\[ = \frac{1}{4} \| e_h^{n+1} + (e_h^{n+1} - e_h^n) \|_0^2 - \frac{1}{4} \| e_h^n \|_0^2. \]

Then
\[ \Delta t_n(D e_h^{n+1}, e_h^{n+1}) = G_{n+1} - G_n + R_n + \frac{\omega_n}{2(1 + \omega_n)} \| e_h^{n+1} - \hat{e}_h^n \|_0^2, \] (73)
where
\[ R_n = G_n - \left( \frac{1}{4} \| e_h^n \|_0^2 + \frac{\omega_n}{2(1 + \omega_n)} \| \hat{e}_h^n \|_0^2 \right). \]

Noticing that
\[ \| e_h^n + \omega_n(e_h^n - e_h^{n-1}) \|_0^2 = (1 + \omega_n)^2 \| e_h^n \|_0^2 - 2\omega_n(1 + \omega_n)(e_h^n, e_h^{n-1}) + \omega_n^2 \| e_h^{n-1} \|_0^2, \]
a straightforward computation shows
\[ R_n = c_{1,n} \| e_h^n \|_0^2 - c_{2,n}(e_h^n, e_h^{n-1}) + c_{3,n} \| e_h^{n-1} \|_0^2, \]
where
\[ c_{1,n} = \frac{\omega_n}{2(1 + \omega_n - 1)} - \frac{1}{4} + 1 - \frac{\omega_n}{2}, \] (74)
\[ c_{2,n} = 1 - \omega_n^2, \] (75)
\[ c_{3,n} = \frac{1}{4} - \frac{\omega_n^3}{2(1 + \omega_n)}. \] (76)
We now express these three coefficients in terms of $\omega_n - 1$ and $\omega_{n-1} - 1$. For $c_{1,n}$, we have

$$c_{1,n} = \frac{\omega_{n-1} - 1}{4(1 + \omega_{n-1})} + \frac{2 - \omega_n - \omega_n^2}{2} = \frac{\omega_{n-1} - 1}{4(1 + \omega_{n-1})} + \frac{2 + \omega_n}{2} (1 - \omega_n).$$

For $c_{3,n}$, we have

$$c_{3,n} = \frac{1 + \omega_n - 2\omega_n^3}{4(1 + \omega_n)} = \frac{1 + 2\omega_n + 2\omega_n^2}{4(1 + \omega_n)} (1 - \omega_n).$$

From [19] it is easy to check that

$$|c_{1,n}| \leq \frac{1}{4} \left| \omega_{n-1} - 1 \right| + (1 + \Gamma/2) \left| \omega_n - 1 \right|, \quad (77)$$

$$|c_{2,n}| \leq (1 + \Gamma) \left| \omega_{n-1} - 1 \right|, \quad (78)$$

so that

$$\max_{1 \leq j \leq 3} |c_{j,n}| \leq \frac{1}{4} \left| \omega_{n-1} - 1 \right| + (1 + \Gamma) \left| \omega_n - 1 \right|. \quad (79)$$

Furthermore, since

$$G_n = \left( 1 + \frac{\omega_{n-1}}{2(1 + \omega_{n-1})} \right) \|e_n^0\|^2_0 - (e_n^0, e_n^{n-1}) + \frac{1}{4} \|e_n^{n-1}\|^2_0,$$

and the smallest eigenvalue of matrix

$$\begin{bmatrix} 1 + x & -1/2 \\ -1/2 & 1/4 \end{bmatrix}$$

is $\lambda = \frac{(5 + 4x) - \sqrt{(5 + 4x)^2 - 16x}}{8}$, which can be seen to be $\lambda \geq 0.14x$, for $0 \leq x \leq 1/2$ (see Fig. 1). Then, if follows that

$$G_n \geq \frac{0.07}{(1 + \Gamma)^2} \omega_{n-1} (\|e_n^{n-1}\|^2_0 + \|e_n^0\|^2_0). \quad (81)$$

Thus, if [19] holds, from [79] and [81] it follows that

$$|R_n| \leq K_0 \frac{1 + \Gamma}{4\gamma} \left( |\omega_{n-1} - 1| + 4(1 + \Gamma) |\omega_n - 1| \right) G_n$$

$$\leq K_0 \frac{1 + \Gamma}{\gamma} \left( |\omega_{n-1} - 1| + |\omega_n - 1| \right) G_n, \quad (82)$$

with $K_0 \leq (3/8)100/7 = 75/14$. Inserting [82] into [75] we finally obtain [70]. □
In the sequel, we define
\[ \Delta t = \max_{i \leq n \leq N} \Delta t_n. \]

Also, for \( \alpha \in [1, 10] \), we consider the polynomial
\[ \frac{x(1 + 2x)}{\alpha} x^2 + \frac{(1 + 2x)(1 + x)}{\alpha} x - 17. \]

One can check that it has two complex roots and two real ones, one negative and one positive. Let \( \Gamma^*(\alpha) \) be this positive real root. It is possible to check that \( \Gamma^*(\alpha) \) is an increasing function of \( \alpha \) whose values for \( \alpha = 1, 5, 10 \) are larger than, 1.25, 2.12 and 2.61, respectively. Fixed \( \alpha \in [1, 10] \), we restrict ourselves to meshes satisfying
\[ \Gamma \leq \Gamma^*(\alpha). \]

We also assume
\[ \sum_{n=2}^{N-1} (|\omega_{n-1} - 1| + |\omega_n - 1|) \leq \Lambda, \]
for some positive \( \Lambda \).

We now state and prove the convergence result for the IMEX method (20).

**Theorem 3** Fix \( \kappa > 0 \), and \( \sigma > 1 \), and let \( \Delta t_n \) and \( h \) satisfy, the following CFL-type condition
\[ 3C_{\text{inv}}^2 \|u\|_{L^\infty(L^\infty)}^2 (1 + 2\Gamma)^2 \frac{(1 + \Gamma)}{\gamma} \frac{\Delta t_n}{h^2} \leq \frac{\kappa}{\tau}, \quad n \geq 1, \]
and
\[ 1 - \Delta t_n \frac{(1 + \Gamma) L_n^2}{0.07\omega_n} > \frac{1}{\sigma}, \quad n \geq 1. \]

where \( L_n^2 \) is defined in (95). Let \( G_n \) be as defined in (99) for the error \( e^n_h \) of method (20). Then, the following bound holds with \( m_\kappa \) defined in (99)
\[ G_{n+1} + \nu \frac{n+1}{k=2} \Delta t_{k-1} \|\nabla e_k^n\|_0^2 + \mu \frac{n+1}{k=2} \Delta t_k m_k \|\nabla \cdot e_k^n\|_0^2 \leq \]
\[ C_n \left((C_0^2 + TC_3^2)(\Delta t)^4 + T \left(C_1^2 h^{2k} + C_2^2 h^{2l+2}\right)\right), \]
where
\[ C_n = \exp \left(\sigma_{n+1} \Delta t_n \frac{(1 + \Gamma) L_n^2}{0.07\omega_n} + \frac{n}{k=2} \left(\sigma_k \Delta t_{k-1} \frac{(1 + \Gamma) L_k^{k-1}}{0.07\omega_{k-1}} + \omega_k f_k\right)\right), \]
and the constants \( f_k \) and \( \sigma_k \) are defined in (97), (102), respectively, and the constants \( C_0^2 \) are defined in (99) and (103).

**Proof** Taking \( u = e_h^{n+1} \) in (20) and applying (70) from Lemma 7 we reach
\[ G_{n+1} - G_n + \frac{\omega_n}{2(1 + \omega_n)} \|e_h^{n+1} - e_h^n\|_0^2 + \nu \Delta t_n \|\nabla e_h^{n+1}\|_0^2 + \mu \Delta t_n \|\nabla \cdot e_h^{n+1}\|_0^2 \leq \]
\[ \Delta t_n |b(s_h^n, s_h^n, e_h^{n+1}) - b(u_h^n, u_h^n, e_h^{n+1})| + \Delta t_n (\tau_1^{n+1} + \tau_3^{n+1} - c_h^{n+1}) + \Delta t_n (\tau_2^{n+1} - \nabla \cdot e_h^{n+1}) + \frac{K_2}{\tau}(1 + \Gamma)^2 (|\omega_{n-1} - 1| + |\omega_n - 1|) G_n. \]
As before, we now apply Lemma 3 to estimate the nonlinear term and, further, we bound

\[ 2\|\tilde{u}_h^n\|_{\text{inv}} h^{-1} \|e_h^{n+1} - \tilde{e}_h^n\|_0 \|e_h^{n+1}\|_0 \leq 2c_{\text{inv}}^2 \|\tilde{u}_h^n\|_2 \left( \frac{(1 + \omega_n) \Delta t_n}{h^2} \|e_h^{n+1}\|_0^2 \right) + \frac{(1 + \omega_n)}{2(1 + \omega_n)} \frac{\|e_h^{n+1} - \tilde{e}_h^n\|_0^2}{\Delta t_n}. \]  

(91)

Instead of (86), we will now assume

\[ 2c_{\text{inv}}^2 \|\tilde{u}_h^n\|_2 \left( \frac{(1 + \omega_n) \Delta t_n}{h^2} \right) \leq \kappa \frac{T}{\tau}. \]

(92)

As before, we will see at the end of the proof that (92) holds if (86) holds and \( h \) is sufficiently small. Thus, from (30) in Lemma 6 and assuming (92) we obtain

\[ \Delta t_n \|b(\hat{s}_h^n, s_h^n, e_h) - b(\hat{u}_h^n, u_h^n, e_h)\| \leq \Delta t_n \left( \left( \frac{\|\nabla \hat{s}_h^n\|_{\infty}}{2} + \frac{9\alpha}{2\mu} \|\hat{s}_h^n\|_2 + \frac{\kappa}{T} \right) \|e_h^{n+1}\|_0^2 \right) + \Delta t_n \left( \frac{(1 + 2\Gamma)(1 + \Gamma)}{2} \|\nabla \hat{s}_h^n\|_{\infty} \|e_h^n\|_0^2 + \Delta t_n \frac{\Gamma(1 + 2\Gamma)}{2} \|\nabla \hat{s}_h^n\|_{\infty} \|e_h^{n-1}\|_0^2 \right) + \Delta t_n \frac{\mu(1 + 2\Gamma)(1 + \Gamma)}{18\alpha} \|\nabla \cdot e_h^{n-1}\|_0^2 + \frac{(1 + 2\Gamma)(1 + \Gamma)}{2} \|\nabla \cdot e_h^{n-1}\|_0^2 \right) + \Delta t_n \frac{\mu(1 + 2\Gamma)(1 + \Gamma)}{18\alpha} \|\nabla \cdot e_h^{n-1}\|_0^2 
\]

(93)

Inserting (93) into (60) and, arguing as before with the terms involving \( \tau_1^{n+1} + \tau_3^{n+1} \) and \( \tau_2^{n+1} \), we get

\[ G_{n+1} - G_n + \nu \Delta t_n \|\nabla e_h^{n+1}\|_0^2 + \frac{17}{18} \mu \Delta t_n \|\nabla \cdot e_h^{n+1}\|_0^2 \leq \]

\[ \Delta t_n \left( \left( \frac{\|\nabla \hat{s}_h^n\|_{\infty}}{2} + \frac{9\alpha}{2\mu} \|\hat{s}_h^n\|_2 + \frac{\kappa + 1}{T} \right) \|e_h^{n+1}\|_0^2 + \frac{(1 + 2\Gamma)(1 + \Gamma)}{2} \Delta t_n \|\nabla \hat{s}_h^n\|_{\infty} \|e_h^n\|_0^2 \right) + \Delta t_n \frac{\Gamma(1 + 2\Gamma)}{2} \|\nabla \hat{s}_h^n\|_{\infty} \|e_h^{n-1}\|_0^2 + \Delta t_n \frac{\mu(1 + 2\Gamma)(1 + \Gamma)}{18\alpha} \|\nabla \cdot e_h^{n-1}\|_0^2 
\]

(94)

Let us denote by

\[ L_h^n = \frac{\|\nabla \hat{s}_h^n\|_{\infty}}{2} + \frac{9\alpha}{2\mu} \|\hat{s}_h^n\|_2 + \frac{\kappa + 1}{T}, \]

(95)

so that, using (81), the first term on the right-hand side of (94) can be written as

\[ \Delta t_n L_h^n \|e_h^{n+1}\|_0^2 \leq \Delta t_n \frac{(1 + \Gamma)}{0.07\tau_n} G_{n+1}. \]

Similarly, for the terms involving \( \|e_h^n\|_0^2 \) and \( \|e_h^{n-1}\|_0^2 \) on the right-hand side in (94), we have

\[ \frac{(1 + 2\Gamma)(1 + \Gamma)}{2} \Delta t_n \|\nabla \hat{s}_h^n\|_{\infty} \|e_h^n\|_0^2 + \frac{\Gamma(1 + 2\Gamma)}{2} \Delta t_n \|\nabla \hat{s}_h^n\|_{\infty} \|e_h^{n-1}\|_0^2 \leq \frac{(1 + \Gamma)(1 + 2\Gamma)^2}{0.14\tau_n} \Delta t_n \|\nabla \hat{s}_h^n\|_{\infty} G_n. \]
Thus, going back to (94) we reach

\[
G_{n+1} - G_n + \nu \Delta t_n \|
\nabla e_{h,k}^{n+1}\|^2_0 + \frac{17}{18} \mu \Delta t_n \|
\nabla \cdot e_{h,k}^{n+1}\|^2_0 \leq \Delta t_n \left( \frac{1 + \Gamma}{0.07 \omega_n} \right) G_{n+1}
\]

\[
+ \left( (1 + \Gamma)(1 + 2\Gamma)^2 \Delta t_n \|
\nabla s_h^n\|^\infty + \frac{K_0}{\gamma} (1 + \Gamma) (|\omega_{n-1} - 1| + |\omega_n - 1|) \right) G_n
\]

\[
+ \Delta t_n \left( \frac{\mu(1 + 2\Gamma)(1 + \Gamma)}{18\alpha} \|
\nabla \cdot e_{h,k}^{n}\|^2_0 + \Delta t_n \left( \frac{\mu(1 + 2\Gamma)}{18\alpha} \|
\nabla \cdot e_{h,k}^{n-1}\|^2_0 \right) \right) (96)
\]

Denoting by

\[
f_n = \frac{(1 + \Gamma)(1 + 2\Gamma)^2}{0.14 \omega_n} \Delta t_n \|
\nabla s_h^n\|^\infty + \frac{K_0}{\gamma} (1 + \Gamma) (|\omega_{n-1} - 1| + |\omega_n - 1|), (97)
\]

and adding terms we get

\[
G_{n+1} + \nu \sum_{k=2}^{n+1} \Delta t_{k-1} \|
\nabla e_{h,k}^{n}\|^2_0
\]

\[
+ \mu \sum_{k=2}^{n+1} \Delta t_k \left( \frac{17}{18} \omega_k - \frac{(1 + 2\Gamma)(1 + \Gamma)}{18\alpha} \right) \|
\nabla \cdot e_{h,k}^{n}\|^2_0 \leq
\]

\[
G_1 + \mu \left( \frac{(1 + 2\Gamma)(1 + \Gamma) \Delta t_1}{18\alpha} + \frac{\Gamma(1 + 2\Gamma) \Delta t_2}{18\alpha} \right) \|
\nabla \cdot e_{h,k}^{n}\|^2_0 + \mu \left( \frac{(1 + 2\Gamma) \Delta t_1}{18\alpha} \|
\nabla \cdot e_{h,k}^{n}\|^2_0 \right)
\]

\[
+ \sum_{k=2}^{n+1} \Delta t_{k-1} \left( \frac{(1 + \Gamma) L_{h,k-1}^-}{0.07 \omega_{k-1}} + \sum_{k=2}^{n+1} f_{k-1} G_{k-1} \right)
\]

\[
+ \frac{1}{2} \sum_{k=2}^{n+1} \Delta t_{k-1} \left( T(\|\tau_1^n\|^2_0 + \|\tau_3^n\|^2_0) + \frac{9}{\mu} \|\tau_2^n\|^2_0 \right). \quad (98)
\]

As mentioned before, for any fixed value of \(\alpha\), the number \(\Gamma^*(\alpha)\) in (84) is chosen to be the the positive real root of (83). Since \(\omega_k \leq \Gamma^*, \ k = 1, \ldots, N - 1\), it follows that

\[
m_k := \frac{17}{18} \omega_k - \frac{(1 + 2\Gamma)(1 + \Gamma)}{18\alpha} \Gamma(1 + 2\Gamma) \omega_{k+1} \geq 0. \quad (99)
\]

Let us also denote by

\[
E_0^n = (f_1 G_1 + \mu \left( \frac{(1 + 2\Gamma)(1 + \Gamma) \Delta t_1}{18\alpha} + \frac{\Gamma(1 + 2\Gamma) \Delta t_2}{18\alpha} \right) \|
\nabla \cdot e_{h,k}^{n}\|^2_0
\]

\[
+ \mu \left( \frac{(1 + 2\Gamma)(1 + \Gamma) \Delta t_1}{18\alpha} \|
\nabla \cdot e_{h,k}^{n}\|^2_0 \right). \quad (100)
\]

Taking into account (100), (98) can be written as

\[
G_{n+1} + \nu \sum_{k=2}^{n+1} \Delta t_{k-1} \|
\nabla e_{h,k}^{n}\|^2_0 + \mu \sum_{k=2}^{n+1} \Delta t_k m_k \|
\nabla \cdot e_{h,k}^{n}\|^2_0 \leq \Delta t_n \left( \frac{(1 + \Gamma)L_{h,k}^-}{0.07 \omega_n} G_{n+1} \right)
\]

\[
+ \sum_{k=2}^{n+1} \Delta t_{k-1} \left( T(\|\tau_1^n\|^2_0 + \|\tau_3^n\|^2_0) + \frac{9}{\mu} \|\tau_2^n\|^2_0 \right) + E_0^n. \quad (101)
\]
Now assumption (87) implies that \( \sigma_{n+1} \) defined by
\[
\sigma_{n+1} = \left( 1 - \Delta t \frac{(1 + \Gamma)L_{c}^{n}}{0.07 \omega_{n}} \right)^{-1},
\] (102)
is positive. Thus, applying Lemma 2 we have
\[
(104), \quad \text{and arguing as we did with (64) we get}
\]
\[
C_{n+1} + \frac{1}{2} \sum_{k=2}^{n+1} \Delta t_{k-1} \| \nabla e_{k}^{h} \|_{0}^{2} + \mu \sum_{k=2}^{n+1} \Delta t_{k} m_{k} \| \nabla \cdot e_{k}^{h} \|_{0}^{2} \leq \frac{1}{2} \sum_{k=2}^{n+1} \Delta t_{k-1} \left( T(\| r_{1}^{h} \|_{0}^{2} + \| r_{3}^{h} \|_{0}^{2}) + \frac{9}{\mu} \| r_{2}^{h} \|_{0}^{2} \right),
\] (103)
where \( C_{n} \) is defined in (89).

To conclude we bound the initial and truncation errors. For the initial errors, we argue as in the fixed stepsize case. Since we assume \( u_{h}^{0} = \hat{s}_{0}^{h} \) so that \( e_{0}^{h} = 0 \). Consequently,
\[
G_{1} = \frac{\omega_{0}}{2(1 + \omega_{0})} \| e_{1}^{h} \|_{0}^{2} + \frac{1}{4} \| 2e_{1}^{h} \|_{0}^{2} \leq \frac{3}{2} \| e_{1}^{h} \|_{0}^{2},
\]
and, in view of (100),
\[
E_{0}^{0} \leq (1 + f_{1}) \frac{3}{2} \| e_{1}^{h} \|_{0}^{2} + \omega_{0} \left( \frac{(1 + 2\Gamma)(1 + \Gamma)}{18 \alpha} + \frac{\Gamma(1 + 2\Gamma)\omega_{1}}{18 \alpha} \right) \mu \Delta t_{0} \| \nabla \cdot e_{1}^{h} \|_{0}^{2}. \] (104)

The first step is carried out with the implicit-explicit Euler method and then, in view of (104), and arguing as we did with (64) we get
\[
E_{0}^{0} \leq C \left( 1 + \Delta t \| u \|_{L_{\infty}(W^{1,\infty})} \right) \left( C_{0}^{2}(\Delta t)^{4} + T \left( C_{0}^{2} h^{2k} + C_{0}^{2} h^{2.12} \right) \right), \] (105)
where \( C \) depends on \( \Gamma, 1/\gamma, \omega_{0} \) and \( \omega_{1} \).

For the truncation errors we also argue as in the fixed stepsize case. Then, arguing exactly as in (68) we get
\[
\frac{1}{2} \sum_{k=2}^{n+1} \Delta t_{k-1} \left( T(\| r_{1}^{h} \|_{0}^{2} + \| r_{3}^{h} \|_{0}^{2}) + \frac{9}{\mu} \| r_{2}^{h} \|_{0}^{2} \right) \leq T \left( C_{0}^{2} h^{2k} + C_{0}^{2} h^{2.12} + C_{3}^{2}(\Delta t)^{4} \right), \] (106)
with the constants \( C_{3}^{2} \) defined in (59) and avoiding writing the explicit dependency on \( \omega_{n} \).

Inserting (105) and (106) into (103) we finally reach (88).

To conclude the proof, as in the fixed step size, we now show that taking \( h \) sufficiently small, the CFL type condition (86) implies (92). Arguing as in the fixed stepsize, it is possible to choose \( h_{0,1} \) such that \( \| \hat{s}_{n}^{h} - u^{n} \|_{\infty} \leq \| u \|_{L_{\infty}(L_{\infty})} / 12 \) for \( h \leq h_{0,1} \), which implies
\[
\| \hat{s}_{n}^{h} \|_{\infty} \leq \frac{13}{12} \| u \|_{L_{\infty}(L_{\infty})},
\] (107)
and, since \( \| \hat{s}_{n}^{h} \| \leq (1 + 2\Gamma) \| s_{n}^{h} \|_{L_{\infty}(L_{\infty})} \), by writing \( \hat{u}_{n}^{h} = (\hat{u}_{n}^{h} - \hat{s}_{n}^{h}) + \hat{s}_{n}^{h} \), it also holds
\[
\| \hat{u}_{n}^{h} \|_{\infty} \leq c_{inv} h^{-d/2} (1 + \Gamma) \| s_{n}^{h} - u_{n}^{h} \|_{0} + \Gamma \| s_{n-1}^{h} - u_{n-1}^{h} \|_{0} + (1 + 2\Gamma) \frac{13}{12} \| u \|_{L_{\infty}(L_{\infty})}. \] (108)

If an \( h_{0,2} > 0 \) exists such that for \( h \leq h_{0,2} \) both the right-hand side of (62) and the right-hand side of (88) multiplied by 0.07(1 + \Gamma)/\gamma are smaller than \( h^{d} \| u \|_{L_{\infty}(L_{\infty})} / (12c_{inv})^{2} \) it follows that
\[
\| s_{k}^{h} - u_{k}^{h} \|_{0} \leq \frac{h^{d/2}}{12c_{inv}} \| u \|_{L_{\infty}(L_{\infty})}, \quad k = 1, \ldots, n.
\]
Thus, in view of (108), it follows that
\[
\| \mathbf{u}_h^n \|_\infty \leq (1 + 2\Gamma)^\frac{7}{6} \| \mathbf{u} \|_{L^\infty(L^\infty)},
\]
so that (102) holds as a consequence of assumption (56). We now argue why \( h_{0,2} \) exists, and this will finish the proof. The exponents of the powers of \( h \) on right-hand sides of (102) and (88) are larger than \( d \), so that they will decay with \( h \) faster than \( h^d \) if the exponential term in (89) remains bounded as \( h \to 0 \). To show that this is the case, we first notice that, in view of the expression of \( L_h^n \) in (97) and recalling (10) and (107), we have
\[
L_h^n \leq (1 + 2\Gamma) \left( \frac{(1 + C_\infty)}{2} \| \mathbf{u} \|_{L^\infty(W^{1,\infty})} + \frac{9\alpha}{2\mu} \left( \frac{13}{12} \right)^2 \| \mathbf{u} \|_{L^\infty(L^\infty)}^2 \right) + \frac{\kappa + 1}{T}
\]
and in view of the expression of \( f_k \) in (97), we also have
\[
f_k \leq \frac{(1 + \Gamma)(1 + 2\Gamma)^3}{0.14\gamma} \Delta t_n (1 + C_\infty) \| \mathbf{u} \|_{L^\infty(W^{1,\infty})} + \frac{K_\alpha}{\gamma}(1 + \Gamma)^2 (|\omega_n - 1| + |\omega_n - 1|).
\]
Finally, due to assumption (87), \( \sigma_n \) defined in (102) satisfies \( \sigma_n < \sigma \) and noticing that the sum of the increments \( \Delta t_n \) is always bounded by \( T \) and recalling assumption (85), it is clear that the constants \( C_n \) in (89) remain bounded as \( h \to 0 \). □

**Remark 5** In view of (86) we notice that the ratios \( \Delta t_n / h^2 \) must remain bounded. As stated in Remark 3, the size of the constant \( \kappa \) in the CFL condition (86) reflects in the error bound (85) through an exponential factor. More precisely, considering also the size of the constant \( \sigma \) in (87), taking into account \( \sigma_n < \sigma \) (with \( \sigma_n \) defined in (102)) and in view of the expression of \( L_h^n \) in (95), the influence of the constants \( \kappa \) and \( \sigma \) affect the constant \( C_n \) in (103) by a factor of size
\[
\exp \left( \kappa \sigma \frac{1 + \Gamma}{0.07 T} \sum_{k=1}^n \Delta t_k \omega_k \right) = \exp \left( \kappa \sigma \frac{1 + \Gamma}{0.07 T} \sum_{k=1}^n \Delta t_{k-1} \omega_{k-1} \right) \leq \exp \left( \kappa \sigma \frac{1 + \Gamma}{0.07} \right).
\]
On the other hand, the value of \( \Gamma^*(\alpha) \) in (84) increases with the value of the free parameter \( \alpha \). However, arguing as above, we can observe that the effect of increasing \( \alpha \) (to allow for larger increases in step length) also affects the constant \( C_n \) in (89) since the term \( \frac{9\alpha}{2\mu} \| \mathbf{u}_h^n \|_\infty^2 \) is present in the definition of \( L_h^n \) in (97).

**Theorem 4** Fix \( \sigma > 1 \), and let \( \Delta t_n \) and \( h \) satisfy
\[
1 - \frac{(1 + \Gamma)\mathcal{L}_h^n}{0.07 \omega_n} > \frac{1}{\sigma}, \quad n \geq 1.
\]
where
\[
\mathcal{L}_h^n = \frac{\| \nabla s_h^{n+1} \|_\infty^2}{2} + \frac{9\alpha}{2\mu} \| s_h^{n+1} \|_\infty^2 + \frac{1}{T},
\]
Let \( G_n \) be as defined in (108) for the error \( e_h^n \) of the semi-implicit method (22). Then, the following bound holds with \( m_k \) defined in (99)
\[
G_{n+1} + \nu \sum_{k=2}^{n+1} \Delta t_{k-1} \| \nabla e_h^k \|_0^2 + \mu \sum_{k=2}^{n+1} \Delta t_k m_k \| \nabla \cdot e_h^k \|_0^2 \leq C_n \left( (C_0^2 + TCG_0^2)(\Delta t)^4 + T \left( C_1^2 + C_2^2 h^{2k} + C_3^2 h^{2l+2} \right) \right),
\]
where
\[
C_n = \exp \left( \sigma_n \Delta t_n \frac{(1 + \Gamma)\mathcal{L}_h^n}{0.07 \omega_n} + \sum_{k=2}^n \left( \sigma_k \Delta t_{k-1} \frac{(1 + \Gamma)\mathcal{L}_h^{k-1}}{0.07 \omega_{k-1}} + \omega_k f_k \right) \right),
\]
22
and the constants $f_k$ and $\sigma_k$ are defined in (97), (102), respectively, and the constants $C^2_i$ are defined in (59) and (63).

**Proof** The proof follows that of Theorem 3 with the changes that we now comment on. Indeed, it is easy to see that (90) also holds for the semi-implicit method (22) with the first term on the right-hand side of (90) replaced by

$$b(s^n_h, s^{n+1}_h, e^{n+1}_h) - b(u^n_h, u^{n+1}_h, e^{n+1}_h)$$

and $\tau_3$ by $\tau_4$. Then, recalling Lemma 4 and Remark 2, we have that (93) holds with $\kappa = 0$, $s^n_h$ replaced by $s^{n+1}_h$ and $\tau_3$ by $\tau_4$. The proof is concluded following the steps of the proof of Theorem 3 but without having to prove that the CFL condition (92) holds. \qed

**Remark 6** Analogous comments to those in Remark 4 apply in this case taking into account the definition of $G_n$ in (69).

## 4 Numerical Experiments

We test methods (20) and (22) with variable step size. We now describe the algorithm used, which follows standard procedures in the numerical integration of ordinary differential equations (ODEs). For the implementation of the method, we used a variable formula strategy as described in [14, §III.5] for the fully nonlinear BDF formulae, but with explicit treatment of the convective term as specified in (20). In particular we notice that the local error estimate of the method of order $k$ is given by

$$\text{EST}_n = \frac{\Delta t_n}{t_{n+1} - t_{n-k}} \left\| U_h^{n,k} \right\|_0,$$

where

$$U_h^{n,k} = \prod_{i=0}^{k-1} (t_{n+1} - t_{n-i}) u_h[t_{n+1}, \ldots, t_{n-k}],$$

where $u_h[t_{n+1}, \ldots, t_{n-k}]$ is the standard $(k+1)$-th divided difference based on $t_{n+1}, \ldots, t_{n-k}$ and $u_h^{n+1}, \ldots, u_h^{n-k}$. This corresponds to the fully nonlinear BDF, but, nevertheless, it was used with the methods (20) and (22). At each time level $t_n$, the estimation $\text{EST}_n$ is compared with the quantity

$$\text{TOL}_n = TOL_f \left( \max \left\{ \left\| u_h^{n+1} \right\|_0, \left\| u_h^n \right\|_0 \right\} + 0.001 \right),$$

where $TOL_f$ is a given tolerance. If $\text{EST}_n > \text{TOL}_n$, $u_h^{n+1}$ is not considered sufficiently accurate and is rejected. It is then recomputed from $t_n$ with a new step length given by

$$\Delta t_n^{\text{new}} = 0.9 \Delta t^n_{k+1} \sqrt{\frac{TOL_n}{\text{EST}_n}}. \quad (110)$$

On the contrary, if $\text{EST}_n < \text{TOL}_n$, then, $u_h^{n+1}$ is accepted and the algorithm proceeds to compute $u_h^{n+2}$ with a step length $\Delta t_{n+1}$ given by the right-hand side of (110).

The algorithm starts with $k = 1$ and $\Delta t_0 = \Delta t_1 = \sqrt{TOL_f}/100$. At $t_1$ the first local error estimation $\text{EST}_1$ is computed. If $\text{EST}_1 > \text{TOL}_f$ the computation is restarted again from $t_0$ with the step size changed according to (110). From $t_2$ onwards, estimates corresponding to the two-step BDF can be estimated. When these estimates are smaller than those corresponding to $k = 1$, then $k$ is switched to $k = 2$. In the experiments below, no more than eight or ten steps were performed with $k = 1$.

We now comment on the solution of the linear systems to be solved at each step, where, again, we follow standard procedures in the numerical integration of ODEs. If we denote by $y^+$ the vector containing the coefficients of the velocity $u_h^{n+1}$ and the pressure $p_h^{n+1}$ in
the corresponding nodal basis of the finite element spaces, then, since \(20\) and \(22\) are linear in \(u_h^{n+1}\) and \(p_h^{n+1}\), to find \(y^n\) one has to solve a linear system

\[
A_n y^n = b^n.
\]

(111)

This was solved by iterative refinement. Starting with an approximation \(y^{n,[0]} = \hat{y}^n\), extrapolated from the previous \(k\) values \((k = 1, 2)\), the approximation is updated as \(y^{n,[j+1]} = y^{n,[j]} + e^{n,[j]}\), where \(e^{n,[j]}\) is obtained by solving the system \(A_n e^{n,[j]} = b^n - A_n y^{n,[j]}\), where, we notice, \(A_n\) has been factored on time level \(t_n \leq t_m\). The iteration is stopped either when the velocity in two consecutive iterations satisfies

\[
\|u_h^{n,[j+1]} - u_h^{n,[j]}\|_0 \leq \min \left\{ 10^{-8}, \frac{TOLr_{\text{rel}}}{100} \right\} \left(\|u_h^{n-1}\|_0 + 0.001\right),
\]

(112)
in which case we set \(u_h^n = u_h^{n,[j+1]}\), or if five iterations are performed without ever \([112]\) being satisfied, in which case, \(A_n\) is factored, replaces \(A_m\) and system \([111]\) is solved. As we will see below, the number of factorizations never exceeded 5\% of the total number of steps.

The code was programmed in MATLAB and for the factorizations in linear systems we used command \texttt{decomposition} with no extra arguments. The set of equations obtained from \([20]\) when \(v_h = 0\) was rewritten as

\[-(\nabla \cdot u_h^{n+1}, q_h) = 0,
\]

so that matrices \(A_n\) were symmetric (although indefinite).

### 4.1 Problem with known solution

We consider the Navier-Stokes equations in the domain \(\Omega = [0,1]^2\) with \(T = 4\), and the forcing term \(f\) chosen so that the solution \(u\) and \(p\) are given by

\[
u(x, y, t) = \frac{6 + 4 \cos(4t)}{10} \left[ \frac{8 \sin^2(\pi x)(2y(1-y)(1-2y)}{-8 \pi \sin(2 \pi x \pi y((1-y)^2)} \right] \]

(113)

\[
p(x, y, t) = \frac{6 + 4 \cos(4t)}{10} \sin(\pi x) \cos(\pi y).
\]

(114)

We present results with the \(P_2/P_1\) pair of mixed finite-elements on a regular triangulation with SW-NE diagonals. The meshes have \(N = 6, 12, 24\) and \(48\) subdivisions in each coordinate directions, and the tolerances for the BDF2 on these meshes are, respectively, \(10^{-4}, 10^{-5}, 10^{-6}\) and \(10^{-7}\). We checked that they are sufficiently small so that the error arising from time discretization is negligible when compared to that arising form spatial discretization. In the experiments in this section, the value of the grad-div parameter is \(\mu = 0.05\).

In Fig.\(2\) for the final time \(t = 4\), we show the \(L^2\) errors in velocity

\[
\|u^n_h - I_h(u^n)\|_0.
\]

(115)

where \(I_h\) is the standard Lagrange interpolant. In Fig.\(2\), as in the rest of the pictures in this section, results of IMEX method \([20]\) are joined by continuous lines, while those of the semi-implicit method \([22]\) by discontinuous lines. It can be seen that continuous and discontinuous lines are super imposed, reflecting the fact that the errors shown in Fig.\(2\) correspond to spatial discretization and are independent of the method chosen for time integration. It can be seen that, as \(h\) decreases, the errors also decrease, but with a higher rate for the two largest values of the viscosity, \(\nu = 10^{-2}\) and \(\nu = 10^{-4}\). The results corresponding to \(\nu = 10^{-6}\), \(\nu = 10^{-8}\) and \(\nu = 10^{-10}\) coincide and are on top of one another. We also show the slopes of a least squares fit to the results corresponding to \(\nu = 10^{-2}\) (in blue) and


\[ \nu = 10^{-10}, \text{ confirming the analysis that the rate of decay is } O(h^2) \text{ independent of } \nu \text{ in the convection dominated regime, i.e., for } \nu \text{ small enough. For } \nu = 10^{-2}, \text{ and the last three results of } \nu = 10^{-4}, \text{ the slope close to 4 reflects that the numerical approximation } u_h \text{ is supraconvergent, in the sense that the error (115) decays faster that } \|u^n - I_h(u^n)\|_0, \text{ which is } O(h^3). \]

We check that our implementation uses variable step sizes in Fig. 3, where we show the step lengths of method (20) (the ones corresponding to method (22) being hardly distinguishable). They correspond to \( \nu = 10^{-6}, TOL_r = 10^{-5} \) and the mesh with \( N = 12 \) subdivisions in each coordinate directions. After the initial steps corresponding to the first order method, the step lengths grow quickly, and then vary between 0.001 and 0.005 approximately. In Fig. 3 we marked with a red cross the locations where a step was rejected. We see that the number of rejections is really low. Most of the results corresponding to different values of \( \nu, TOL_r \) and \( N \) showed similar variations.

In Fig. 4 we compare the number of factorizations with the number of steps for the results shown in Fig. 2 (recall each factorization is reused in subsequent steps for iterative refinement until failure to converge, where it is updated). We can see that the number of factorizations hardly grows as \( h \) (and the corresponding tolerances) are decreased, and that at worst (large \( h \), small \( \nu \)) they are hardly more than 5% of the number of steps. Some differences can be seen between methods (20) and (22), for large values of \( \nu \). We do not have an explanation for them. We notice, however, that the differences between the methods become less significant for smaller values of \( \nu \).

Next we check if the CFL-type condition (86) is necessary in practice for the IMEX method (20), whereas this is not the case of the semi-implicit method (22). For that purpose, we repeat the above computations (and including also meshes with \( N = 96 \) subdivisions in each coordinate direction) but with tolerance \( TOL_r = 10^{-4} \) for all meshes. Notice that in principle the number of steps depends on the tolerance, so, if the tolerance is kept constant while the spatial mesh size \( h \) is decreased, the number of steps should not change much. On the other hand, if stability worsens as \( h \) is decreased, errors arising from time discretization increase with every step, and will become larger than the tolerance more easily, forcing the algorithm to take smaller step sizes, this resulting in a larger number of steps. In Fig. 5 we show the number of steps (left plot). For the IMEX method (continuous lines), we see that although they remain fairly constant for \( \nu = 10^{-2} \), this is not the case of the rest of the values of the viscosity. Indeed, for the three smallest values of the viscosity, the number of steps...
Figure 3: Step lengths corresponding to $\nu = 10^{-6}$, $TOL_r = 10^{-5}$ and $N = 12$. Rejections marked with a red cross.

Figure 4: Number of factorizations (left) and steps (right) for results shown in Fig. 2.
steps roughly doubles from $h = 1/24$ to $h = 1/48$, and it is multiplied by three from $h = 1/48$ to $h = 1/96$, approaching the factor 4 predicted by the CFL-type condition \( \text{(86)} \). On the other hand, for the semi-implicit method (discontinuous lines) we see that, contrary to the IMEX method, the number of steps fairly constant as $h$ is refined. As we have shown, this method does not require the CFL-type condition \( \text{(86)} \) to converge. On the contrary, in view of Fig. 5, we may conclude that method with explicit convection \( \text{(20)} \) indeed suffers from restriction \( \text{(86)} \).

In Fig 5 we can also see that the number of factorizations remains fairly independent of the number of steps, as it was the case in the previous example where tolerances decreased with $h$.

4.2 Flow past a cylinder

We now present results for the well-known benchmark problem defined in [19]. The domain is given by

$$\Omega = (0, 2.2) \times (0, 0.41)/ \{(x, y) \mid (x - 0.2)^2 + (y - 0.2)^2 \leq 0.0025\}$$

and the time interval is \([0, 8]\). The velocity is identical on both vertical sides, and is given by

$$u(0, y) = u(2.2, y) = \frac{6}{0.41^2} \sin\left(\frac{\pi t}{8}\right) \left(\begin{array}{c} y(0.41 - y) \\ 0 \end{array}\right).$$

In the rest of the boundary the velocity is set $u = 0$. At $t = 0$, the initial velocity is $u = 0$. The viscosity is set to $\nu = 10^{-3}$ and the forcing term is $f = 0$.

It is well-known that around $t = 4$ a vortex sheet develops behind the cylinder, as it can be seen in Fig. 6 where we show the speed and velocity fields for $t = 5$ to $t = 8$. We use the same scale in the four plots of the velocity fields, and the results we obtain are virtually identical to those in [16, Fig. 2].

In Fig 7 we show the mesh used in all the results below, which has 6624 elements whose diameters range from $5.53 \times 10^{-3}$ to $3.38 \times 10^{-2}$. The number of degrees of freedom for the velocity on this mesh is 27168, and for the pressure, 3480. Following experience in [8], the grad-div parameter for the experiments in this section is set to $\mu = 0.01$. 

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In Fig 8 we show the step sizes used with tolerances $10^{-4}$, $10^{-6}$, and $10^{-8}$. We notice that, for each tolerance, after the very small steps near $t = 0$, step sizes vary by a factor of at least ten, so that, as in the previous section, the algorithm uses variable steps sizes. It can also be seen that the smaller the tolerance, the smaller the step sizes, except for the IMEX method (20) with tolerances $10^{-4}$ and $10^{-6}$ on the interval $[1.7, 3.7]$ where step sizes are the same for both tolerances and clearly smaller than those of the semi-implicit method (22) for the same tolerances. This suggest that, for these two tolerances, the step size in method (20) is limited by the CFL-type condition (86) rather than by the roughness of the solution, suggesting, once again, that restriction (86) is not a consequence of the techniques used in the proof of Theorem 3.

From the practical point of view, we may mention that the number of steps for the three tolerances, $10^{-4}$, $10^{-6}$, and $10^{-8}$, were 11221, 21187 and 93671, respectively, for the IMEX method, and 4384, 19885 and 92086 for the semi-implicit method, but the number of factorizations were, respectively, 67, 53, and 92 for the IMEX method and, 72, 84 and 89 for the semi-implicit method, low figures for both methods, in line with those of the previous section.

Quantities of practical interest in this example are the drag and lift coefficients, $c_d$ and $c_l$, respectively. In Fig. 9 for tolerances $10^{-4}$ and $10^{-6}$, we show the evolution of the drag and lift coefficients obtained with the IMEX method, which were computed with the following formulae, [16], [17]:

\[
c_d(t_n) = -20 \left( \frac{1}{\Delta t_{n-1}} \nabla u_h^n, v_d \right) + \nu (\nabla u_h^n, \nabla v_d) + b(u_h^n, u_h^n, v_d) - (p_h^n(t), \nabla \cdot v_d),
\]

\[
c_l(t_n) = -20 \left( \frac{1}{\Delta t_{n-1}} \nabla u_h^n, v_l \right) + \nu (\nabla u_h^n, \nabla v_l) + b(u_h^n, u_h^n, v_l) - (p_h^n(t), \nabla \cdot v_l),
\]
Figure 8: Flow around a cylinder: step lengths for $TOL_r = 10^{-4}$, $10^{-6}$, and $10^{-8}$. Top: IMEX method (20) bottom; semi-implicit method (22).

Figure 9: Flow around a cylinder: drag and lift coefficients for $TOL_r = 10^{-4}$, and $10^{-6}$. Reference values for the maximum of the lift and drag coefficients marked with an asterisk.
where \(v_d\) and \(v_l\) are piecewise quadratic functions taking values \(v_d = [1, 0]^T\) and \(v_l = [0, 1]^T\) on those nodes on the circumference \(c \equiv (x - 0.2)^2 + (y - 0.2)^2 = 0.0025\), and vanishing on triangles without vertices on \(c\).

The results of both tolerances are indistinguishable, suggesting sufficient accuracy already for \(TOL_r = 10^{-4}\). We also show the reference values of the maximum values, \(c_{d,max} = 2.950921575\) and \(c_{l,max} = 0.47795\), plotted with a red asterisk at their corresponding times \(t_{d,max} = 3.93625\) and \(t_{l,max} = 5.693125\) (all taken from [16]) showing a very good coincidence. In fact, the relative errors for the times where the maximum values are reached are in both cases below 0.0006, and for the drag and lift coefficients are 0.012 and 0.043, respectively. Furthermore, there is good agreement between the plots in Fig. 9 and those in [16, Fig. 4]. Finally, also of interest is the pressure difference between the front and the back of the cylinder, whose reference value for \(t = 8\) is, according to [10], \(\Delta p_b = -0.1116\). For the tolerance \(10^{-4}\), the relative error in \(\Delta p_b\) is below 0.002.

The results of these two sections suggest that, as predicted by the theory, the IMEX method (20) is stable and convergent, but indeed suffers from the CFL-type restriction (86). On the other hand, the computational cost of the semi-implicit method (22) is not much larger than the IMEX one, see Figure 5, and has the advantage of having no step restriction.

5 Conclusions

The numerical simulation of the Navier-Stokes equations is still a challenge in which there are several questions that deserve some research. Important aspects of the present paper are the following:

- We study the possibility of optimizing the cost of the temporal integration. To this end we consider two IMEX schemes for the incompressible Navier-Stokes equations, which reduce the cost of every time step, compared with an implicit method. In the first method the nonlinear term is explicit and in the second one is semi-implicit.
- We prove that using an explicit form for the nonlinear term a CFL type condition is required. This condition is stronger than the usual one in which \((\Delta t)h^{-1}\) has to be bounded (we required \((\Delta t)h^{-2}\) to be bounded). Our numerical experiments confirm experimentally that the CFL-type restriction cannot be weakened.
- Adaptive time stepping is an important tool in Computational Fluid Dynamics. In the present paper, we include the error analysis of a second order backward differentiation formulae (BDF2) scheme with variable time step both for an explicit and semi-implicit form of the nonlinear term. Concerning BDF2 method with variable time step we did not found in the literature any reference with the error analysis for the Navier-Stokes equations.
- Methods for whom robust estimates can be derived enable stable flow simulations for small viscosity coefficients on comparatively coarse grids. Adding grad-div stabilization to the standard mixed finite element formulation we are able to prove error bounds with constants independent of the Reynolds number for the two methods we analyzed.
- In summary, to our knowledge, this is the first time in the literature variable step BDF2 methods are analyzed for the Navier-Stokes equations and also the first time in the literature robust bounds (independent of viscosity coefficient) are proved for Navier-Stokes with fully discrete methods using variable time-step.

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