SOME PATHOLOGIES OF FANO MANIFOLDS IN POSITIVE CHARACTERISTIC

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Abstract. We construct examples of Fano manifolds, which are defined over a field of positive characteristic, but not over $\mathbb{C}$.

1. Introduction

1.1. Let $X$ be a Fano variety defined over a ground field $k = \overline{k}$ (see [12] or [16, Ch. V] for the basic notions and facts). We will also assume $X$ to be smooth.

The main concern of the present note is the following:

Question. Suppose $p := \text{char } k > 0$. Does there exist $X$ non-liftable to char = 0 (hence, in particular, $H^2(X, T_X) \neq 0$), i.e. is it true that there is always some variety $X_0$, defined over a $\mathbb{Z}$-subalgebra $R \subset \mathbb{C}$ of finite type, so that $X = X_0 \otimes_R k$ for $k := R/p$ and a prime ideal $p \subset R$ with $p \cap \mathbb{Z} = (p)$?

The problem is trivial for $\dim X = 1$, a bit trickier when $\dim X = 2$ (but still one does not obtain any new del Pezzo surfaces in positive characteristic due to e.g. [19, Ch. IV, §2]), and does not seem to get much attention in the case of $\dim X \geq 3$.

Remark 1.2. In [5], many (both non- and algebraic) Calabi-Yau 3-folds, defined over $\mathbb{F}_p$, have been constructed. The idea was to take a rigid (i.e. with $H^1(T) = 0$) nodal CY variety $\mathcal{X}$, defined over $\mathbb{Z}$, in such a way that its mod $p$ reduction $X_p := \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{F}_p$ acquires an additional node. Then the needed CY 3-fold $X$ is constructed via a small resolution $X \to X_p$ (see [5, Theorem 4.3]). We will discuss other (general type) examples in Remark 1.5 below.

Recall that partial classification of Fano 3-folds $X$, subject to the “numerical” constraints $\text{rk Pic} = 1$ and either $p > 5$ or $D \cdot c_2(X) > 0$ for any ample divisor $D$ on $X$, was obtained in [30]. As it turns out, all these $X$ do admit a lifting to $\mathbb{C}$, which is insufficient for us (cf. [1], [7], [20], [29]).

Thus, in order to find $X$ with an interesting behavior in positive characteristic one should turn to the “qualitative” properties of Fano varieties, as the following example suggests (compare with Remark 1.2).

Example 1.3. The hypersurface $Y_p := (\sum x_i^p y_i = 0) \subset \mathbb{P}^n \times \mathbb{P}^n$ (see [16, 1.4.3.2]]) is a Fano variety iff $p \leq n$. Furthermore, $Y_p$ is an (obvious) $SL(n+1)$-homogeneous space, which can deform to a non-homogeneous variety. This violates a similar property of homogeneous spaces defined over $k \subseteq \mathbb{C}$. On the other hand, one

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1) Actually, in the light of Corollary 1.7 below, for the arguments in [30] to carry on (cf. [30, Theorem 1.4]) one has to assume in addition that $H^1(X, \mathcal{O}_X(-D)) = 0$ for any (or at least those considered in [29]) ample divisors $D$ on $X$. 

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may consider quotient schemes $G/P$, where $G$ is a semi-simple algebraic group and $P \subset G$ is its non-reduced parabolic subgroup. (These $G/P$ are rational for $p > 3$ due to [35].) Yet, unfortunately, both of the examples lift to characteristic 0, at least when $\text{rk Pic} = 1$ (cf. the discussion in [30]). Finally, let us mention $p$-coverings $X$ of Fano hypersurfaces in $\mathbb{P}^{n+1}$, for which $\wedge^{n-1}\Omega_X^1$ contains a positive line subbundle (see [15]). But again, even though this property of $\wedge^{n-1}\Omega_X^1$ is specific to positive characteristic (see e.g. [3]), $X$ is (obviously) liftable to $\mathbb{C}$.

We answer our Question via the next

**Theorem 1.4.** For any $p \geq 3$ and $d, N \gg 1$, there exists a flat family $X \rightarrow B_d$ of Fano 3-folds, parameterized by an algebraic variety $B_d$ of dimension $2d$, such that

- the fiber $X := X_b$ is non-liftable to char = 0 for generic $b \in B_d$;
- the fibers $X_b$ and $X_{b'}$ are non-isomorphic for generic $b \in B_d, b' \in B_d$, with $d \neq d'$;
- $(-K_X^3) \geq N$ for the anticanonical degree of $X = X_0$.

**Remark 1.5.** Examples of manifolds (in arbitrary dimension) of general type violating Kodaira vanishing – more precisely, having $H^1(L^{-1}) \neq 0$ for an ample line bundle $L$, – have been constructed in [22]. This was a development of the method from [23], where a surface $S$ of general type (with $H^1(S, L^{-1}) \neq 0$) is constructed together with a morphism $S \rightarrow C$ onto a curve, so that the $k(C)$-curve $S$ violates the Mordell-Weil property. We follow these trends starting from 2.6 below, for the 3-fold $X$ we need is constructed by finding a $\mathbb{P}^1$-bundle $W$ over $S$ first, with $-K_W$ nef, and then taking a $p$-covering $X \rightarrow W$ à la [9].

Next corollary complements the results mentioned in Remarks 1.2, 1.5:

**Corollary 1.6.** For generic $X$ as in Theorem 1.4 and $m \gg 1$, there is a cyclic $m$-covering $\pi_m : X_m \rightarrow X$ such that $X_m$ is smooth, $K_{X_m}$ is nef, and $X_m$ is non-liftable to char = 0.

The proof of Theorem 1.4 also yields (compare with [31])

**Corollary 1.7.** In the notations from Corollary 1.6 irregularity $q(X) \neq 0$, as well as $q(X_m) \neq 0$.

(Both of Corollaries 1.6, 1.7 are proved at the end of Section 2.)

**1.8.** Theorem 1.4 and Corollaries 1.6, 1.7 seem to be the “minimal” illustrations of pathological behavior of Fano varieties in positive characteristic (cf. the discussion after Remark 1.2). Note also that Theorem 1.4 provides an unbounded family of (smooth) Fano 3-folds (cf. [17], [26]). Furthermore, it follows easily from the arguments after Lemma 2.13 below that the cones $\text{NE}(X_b)$, with varying $b \in B_d$, admit infinitely many “jumps” (compare with [31], [6], [36]).

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2) Another possible approach to answer Question affirmatively is via the wild conic bundle structures (for $p = 2$) on Fano manifolds (see [30], Theorem 5.5). Yet again one does not get anything new in characteristic 2 due to [21], Theorem 7 (see also [33] for some discussion and results on (birational) geometry of wild conic bundles).
Thus, one can see that essentially every “standard” property of Fano manifolds breaks in positive characteristic, the fact which is due (in our opinion) to the following principle behind (compare with Remark 2.11 below):

*every object over $k$, $\text{char } k > 0$, is defined only up to the Frobenius twist.*

(This was probably first exploited in [32], while the ultimate reading on the subject are [23, 24] and [25] of course.)

Finally, it would be interesting to work out the constructions in [2.8] taking a (supersingular) Kummer surface in place of $S$ (cf. [4]).

2. The construction

2.1. Fix a smooth projective surface $S$. We will denote by $\mathcal{E}$ (resp. $\mathcal{F}$) a vector bundle (resp. a coherent sheaf) on $S$. Let us recall some standard notions and facts about $\mathcal{E}$ and $\mathcal{F}$ (see e.g. [8, 10, 27]).

First of all, one defines the Chern classes $c_i := c_i(\mathcal{E}) \in A^i(S)$ for $\mathcal{E}$ (and similarly for $\mathcal{F}$, using a locally free resolution), with $c_1 = \det \mathcal{E}$. In fact, letting $W := \mathbb{P}(\mathcal{E})$ there is a natural inclusion $A^*(S) \hookrightarrow A^*(W)$ of groups of cycles induced by the projection $\pi : W \to S$, and the following identity (called the *Hirsch formula*) holds:

$$H^2 + H \cdot c_1 + c_2 = 0.$$ 

Here $H := \mathcal{O}(1) \in A^1(W)$ is the Serre line bundle on $W$ and $c_i$ are identified with $\pi^*(c_i)$ (cf. Remark 2.2). In particular, for $r := \text{rk} \, \mathcal{E} = 2$ we have

$$H^3 = -c_1^2 - c_2,$$

which together with the Euler’s formula

$$K_W = -rH + \pi^*(K_S + c_1)$$

gives

$$(-K^3_W) = 6K^2_S + 10c_1^2 + 24K_S \cdot c_1 - 8c_2.$$

*Remark 2.2.* When $k \subseteq \mathbb{C}$ (so that $c_1 \in H^{2i}(S, \mathbb{Z})$) and the structure group of $\mathcal{E}$ is not $SU(2)$ (or $H^1(S, \mathbb{Z}) \neq 0$), the previous formulae (except for the Euler’s one and with suitably corrected $(-K^3_W)$) should be read with all the $c_i$ up to $\pm$. In fact, when $\mathcal{E} \simeq \mathcal{E}^*$ (the dual of $\mathcal{E}$) and $\simeq$ is non-canonical, there is no preference in choosing $\pi^*(c_i)$ or $-\pi^*(c_i)$ as one may change the orientation in the fibers of $\mathcal{E}$. In particular, the Hirsch formula turns into $H^2 \pm H \cdot c_1 \pm c_2 = 0$, with both $c_i$ having a definite sign at once.

Further, let $Z \subset S$ be a 0-dimensional subscheme supported at a finite number of points $p_1, \ldots, p_m$. Put $\ell_i := \text{dim } \mathcal{O}_{S,p_i}/I_{Z,p_i}$ and $\ell(Z) := \sum_i \ell_i$ (the length of $Z$) for the defining ideal $I_Z$ of $Z$. One can show that $c_2(j_*\mathcal{O}_Z) = -\ell(Z) \in A^2(S)$, where $j : Z \to S$ is the inclusion map. In particular, if $\mathcal{E}$ admits a splitting

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}' \otimes I_Z \to 0$$

for some line bundles $\mathcal{L}, \mathcal{L}' \in \text{Pic}(S)$, we obtain (using the Whitney’s formula)

$$c_1 = c_1(\mathcal{L}) + c_1(\mathcal{L}'), \quad c_2 = c_1(\mathcal{L}) \cdot c_1(\mathcal{L}') + \ell(Z).$$

As, for example, the hyperelliptic curve $y^2 = x^p - a, a \in k$, of genus $(p - 1)/2$, covered (after Frobenius twist) by a rational curve (see [32 Corollary 1]).
Example 2.7. Let $s \in H^0(S, \mathcal{E})$ be a section. Assume for simplicity that the zero locus $(s)_0 \subset S$ of $s$ has codimension $\geq 2$. Then one gets an exact sequence of sheaves $0 \to \mathcal{O}_S \to \mathcal{E} \xrightarrow{\lambda_s} \mathcal{L}(s)_0 \to 0$ (cf. (2.3)).

Suppose now that $r = 2$. Let $\mathcal{L} \subset \mathcal{E}$ be a line subbundle such that the sheaf $\mathcal{E}/\mathcal{L}$ is torsion-free. Then by working locally it is easy to obtain an exact sequence (2.3). All such sequences (extensions of $\mathcal{E}$) are classified by the group $\operatorname{Ext}^1(\mathcal{L} \otimes I_2, \mathcal{L})$, which in the case when $Z$ is a locally complete intersection coincides with $\mathcal{O}_Z$. (More precisely, since $\mathcal{L}, \mathcal{L}'$, etc. are defined up to the $k^*$-action, the classifying space for the extensions is the projectivization $\mathbb{P}(\operatorname{Ext}^1(\mathcal{L}' \otimes I_2, \mathcal{L})) = \mathbb{P}(\mathcal{O}_Z)$.) Note also that (2.3) provides a locally free extension when $H^2(S, \mathcal{L}' \otimes \mathcal{L}) = 0$ (Serre’s criterion).

2.6. Let $Y$ be a smooth projective variety with a line bundle $L \in \operatorname{Pic}(Y)$. Frobenius $F : Y \to Y$ induces a homomorphism

$$F^* : H^1(Y, L^{-1}) \to H^1(Y, L^{-p})$$

and one has

$$\operatorname{Ker} F^* \simeq \{ df \in \Omega_{k(Y)} \mid f \in k(Y), (df) \geq pD \}$$

once $L = \mathcal{O}_Y(D)$ for $D \geq 0$ (see [22, Theorem 1]). We may assume that $\operatorname{Ker} F^* \neq 0$ (see [22, Theorem 2]), $H^1(Y, \mathcal{O}_Y(-pD)) = 0$ for an ample $D$ (cf. the proof of [30, Lemma 1.4]), which yields a non-split extension

$$0 \to \mathcal{O}_Y(-D) \to \mathcal{E}_{L,Y} \to \mathcal{O}_Y \to 0$$

such that the corresponding extension for $F^* \mathcal{E}_{L,Y}$ splits. This defines a $\mathbb{G}_a$-torsor over $Y$ (w.r.t. the line bundle $L^{-1}$) and a subscheme $X \subset \mathbb{P}(\mathcal{E}_{L,Y})$ which projects “$p$-to-1” onto $Y$. More precisely, the morphism $\phi : X \to Y$ is purely inseparable of degree $p$, so that $X$ is regular and $K_X = \phi^*(K_Y - (p-1)D)$ (see [22, Proposition 1.7, (16)])

Example 2.7. Let $P(y)$ be a polynomial of degree $e$ and $C \subset \mathbb{P}^2$ a plane curve given by the equation $P(y^p) - y = z^pe^{-1}$ (see [22, Example 1.3]). One easily checks that $(dz) = pe(3e - 3)(\infty)$ for $C \subset \mathbb{P}^2$ identified with its projectivization and $\infty \in C$. Then, taking $y^dz$ (with $d(y^dz) = y^pdz$) and $e \gg 1$, we obtain $\dim \operatorname{Ker} F^* = h^1(C, (3 - pe)(\infty)) \geq 2$ for $D := (pe - 3)(\infty)$. Moreover, [22, Propositions 2.3, 2.6] shows that instead of $C$ we may take $Y$ as above, with $h^1(Y, L^{-1}) \geq 2$, ample $K_Y$ (for $p \geq 3$) and arbitrary $\dim Y \geq 2$.

2.8. Now let $S := Y$ be the Raynaud surface. Recall that $S$ can be realized as a double cover of the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E}_{L,C})$ over the curve $C$ from Example 2.7 ramified along a section and the $p$-cover $X$ of $C$ as above (run the constructions in [22, 2.1] with $k := 2$). Let $\psi : S \to C$ be the induced rational curve fibration.

Proposition 2.9. In the previous setting, there exists a vector bundle $\mathcal{E}$ on $S$ such that

- $r = 2$, $c_2 = 0$ and $c_1 = -nK_S$ for an arbitrary $n \gg 1$;
- the divisor $-K_W$ is nef on $W = \mathbb{P}(\mathcal{E})$ (cf. 2.1).

Proof. In the exact sequence (2.3), take $\mathcal{L}$ an arbitrary ample, $\mathcal{L}' = -\mathcal{L} - nK_S$ and $Z$ any finite with $\ell(Z) = -c_1(\mathcal{L}) \cdot c_1(\mathcal{L}')$. Then from (2.3) we get $c_1 = -nK_S, c_2 = 0,$
Remark 2.11. \( \Omega^2_{\Sigma_l} \) is typical to char \( > \) char.

\[ \text{Hom}(\Omega^2_{\Sigma_l},\mathcal{L}^{-1} \otimes \mathcal{L}') \]

There is a natural morphism

\[ \text{sheaf } \Omega^1_{E} \text{ identified with the zero-section of restrictions} \]

\( \text{(Here } \xi \text{ for some } \Omega^2_{\Sigma_l} \text{)} \)

It follows from Lemma 2.10 that the line bundle \( \Omega^2_{\Sigma_l} \) is isomorphic to the dual of \( \Omega^2_{\Sigma_l} \).

**Proof.** Note that \( l \) is a rational curve with unique singular point \( o := \text{Sing}(l) \) such that \( l \) is given by the equation \( y^2 = z^p \) (affine) locally near \( o \). This shows that the sheaf \( \Omega^1_l \) has torsion, with support in \( o \), for \( 2ydy = pz^{p-1}dz = 0 \). In particular, we get \( (\Omega^1_l)^** = \mathcal{O}_l \cdot dz \) near \( o \), with \( dy \) being the second (torsion) generator of \( \Omega^1_l \).

This easily yields

\[ h^0(l,(\Omega^1_l)^**) \geq \frac{(p-1)(p-2)}{2} - \frac{(p-2)(p-3)}{2} - 1 \geq 0. \]

Indeed, if \( \tilde{l} \subseteq \mathbb{P}^2 \) is the closure of the curve \( (y^2 = z^p) \), then \( \text{Sing}(\tilde{l}) = \{ o, \infty \} \) and there is a natural morphism \( l \rightarrow \tilde{l} \). The above estimate \( h^0(l,(\Omega^1_l)^**) \geq 0 \) now follows because \( l \rightarrow \tilde{l} \) is the normalization near \( \infty \) and \( \text{mult}_\infty(l) = p - 2 \).

Further, we have

\[ \mathbb{P}(\mathcal{E}|_l) = \mathbb{P}(\mathcal{L}|_l \otimes \mathcal{L}'|_l) \]

with a local fiber coordinate \( t \), so that \( (\text{Kähler}) \) \( dt \) induces a section \( \tau \) of \( (\Omega^2_{\Sigma_l})^** \).

Namely, given the indicated properties of \( \mathcal{E}|_l \) and \( \Omega^1_l \), we obtain

\[ dt \wedge \xi = \tau|_{U \cap U'} = f dt \wedge \xi|_{U \cap U'} \]

for some \( \xi \in H^0(l,(\Omega^1_l)^**) \) and \( f \in \mathcal{O}(U \cap U' \cap l) \) generating a cycle in \( H^1(l,\mathcal{O}_l^*) \).

(Here \( U,U' \subseteq \Sigma_l \) are arbitrary open charts pulled back from \( l \), and the latter is identified with the zero-section of restrictions \( \mathcal{E}|_{U'}, \mathcal{E}|_{U''} \)).

More precisely, we get that \( \tau \) lifts to a section of \( (\mathcal{L}^{-1} \otimes \mathcal{L}')|_l \), which gives \( (\Omega^2_{\Sigma_l})^** \simeq \omega_{\Sigma_l/\mathbb{P}_1} \otimes \mathcal{L}^{-1} \otimes \mathcal{L}' \).

\[ \Box \]

**Remark 2.11.** The proof of Lemma 2.10 illustrates the following phenomenon typical to \( \text{char} > 0 \): the sheaf \( \Omega^1_X \) may be rank 1 torsion, thus not equal to \( \text{Hom}(T_X,\mathcal{O}_X) \), where \( \text{dim}T_{X,x} \geq 2 \) for some \( x \in \text{Sing}(X) \). Let us also point out that the interrelation between \( \Omega^2_{\Sigma_l} \) and canonical bundle of the normalization of \( \Sigma_l \) is not clear (meaning the conductor cycle need not be effective) because \( \Sigma_l \) violates the \( S_2 \)-property (for the curve \( l \) obviously does, – the rational function \( y^{1/p} : l \rightarrow \mathbb{P}^1 \) is defined at every point near \( o \), but is not regular at \( o \)).

**Lemma 2.12.** The linear system \( |-K_W|_{\Sigma_l} \) is basepoint-free.

**Proof.** It follows from Lemma 2.10 that the line bundle \( (\Omega^2_{\Sigma_l})^* \) satisfies

\[ H^0(\Sigma_l,(\Omega^2_{\Sigma_l})^*) \supseteq H^0(\Sigma_l,\mathcal{L} \otimes \mathcal{L}'^{-1}). \]

Indeed, since \( \mathbb{P}^1 \) normalizes \( l \), we get \( \Omega^1_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1} \cdot dz \) near (the preimage of \( o \) (cf. the proof of Lemma 2.10). Hence the sheaf \( \omega_{\Sigma_l/\mathbb{P}^1} \) pulls back to \( \omega_{\Sigma_l/\mathbb{P}^1} \) for the ruled surface \( \Sigma := \mathbb{P}(\mathcal{E}|_{\mathbb{P}^1}) \). It remains to notice that (the class of) \( (\omega_{\Sigma_l/\mathbb{P}^1})^* = 2[\text{minimal section on } \Sigma] + 2[\text{fiber}] \) is ample.

\[ ^4 \text{Note also that } H^2(S,\mathcal{L}'^{-1} \otimes \mathcal{L}) = H^2(nKS) = 0 \text{ and so } \mathcal{E} \in \text{Ext}^1(\mathcal{L}' \otimes I_Z,\mathcal{L}) \text{ is locally free.} \]
On the other hand, for \( s \in H^0(\Sigma_l, (\Omega^2_{\Sigma_l})^*) \) we get
\[-K_W|_{\Sigma_l} = (s = 0) + \Sigma_l|_{\Sigma_l} = (s = 0)\]
by adjunction, and the claim follows. \(\square\)

**Lemma 2.13.** \( H^1(W, -K_W - \Sigma_l) = 0. \)

**Proof.** Notice that
\[ H^1(W, \pi_* (2H + \pi^* (n-1)K_S - \Sigma_l)) = H^1(S, S^2E \otimes O_S((n-1)K_S - l)) = 0 \]
by Serre vanishing (for \( K_S \) is ample). Furthermore, we have
\[ R^1 \pi_* (2H + \pi^* (n-1)K_S - \Sigma_l) = 0 \]
by Grothendieck comparison, since the divisor \( 2H + \pi^* (n-1)K_S - \Sigma_l \) is \( \pi \)-ample. The assertion now follows from the Leray spectral sequence. \(\square\)

From the defining exact sequence for \( \Sigma_l \) and Lemma 2.13 we get a surjection
\[ H^0(W, -K_W) \twoheadrightarrow H^0(\Sigma_l, -K_W|_{\Sigma_l}). \]
Then by Lemma 2.12 the base locus of \(-K_W\) is contained in the fibers of \( \pi \). This shows that \(-K_W\) is nef and finishes the proof of Proposition 2.9. \(\square\)

Recall that \( H^1(S, L^{-1}) \neq 0 \) for \( L = O_S(D) \) (cf. the beginning of 2.8). One can also observe that \( H^1(W, \pi^* L^{-1}) \simeq H^1(S, L^{-1}) \). Hence, setting \( Y := W \) (resp. \( L := \pi^* L \)) in the notations of 2.6 we can pass to the corresponding \( p \)-cover \( \phi : X = \mathbb{P}(E_{\pi^* L, W}) \rightarrow W \). The divisor \(-K_X\) is ample by Proposition 2.9 and Kleiman’s criterion (see [11] Ch. I, Theorem 8.1)).

Finally, since \( W = \mathbb{P}(E) \) for \( E \) constructed as an extension (cf. 2.1), to complete the proof of Theorem 1.4 it suffices to vary \( L \) and \( Z \) as above in such a way that both \( c_1(E) = -nK_S, c_2(E) = 0 \) remain fixed, while \( \ell(Z) \rightarrow \infty \) (cf. 2.3) and the properties of \( E \) in Proposition 2.9. The assertion then follows from the formula for \((−K^3_W)\) in 2.1 and the fact that the cone \( \overline{NE}(X) \) is finite polyhedral (see e.g. [14], [13]). More precisely, we have obtained that \( X = X_b \) surjects onto \( W = \mathbb{P}(E) \), with \( E \in \text{Ext}^1(L' \otimes I_Z, E) \) varying together with \( Z \) (or \( b \in B_d \)). Then it follows from the discussion after Example 2.5 that for different \( E \) the 3-folds \( W \) are non-isomorphic. This gives \( X_b \neq X_{b'} \) for generic \( b' \in B_{d'}, d' \neq d \), and proves the second claim of Theorem 1.4 whereas the other two are evident from the construction of \( X \).

**Remark 2.14.** Alternatively, one could show that already \( W \) is Fano, by applying the results in [2], [37] (and the formula for \((−K^3_W)\) of course).

We now pass on to the proof of Corollaries:

**Proof of Corollary 1.1.** \( X_m \) is obtained by the cyclic covering of \( X \) with ramification at a smooth surface from \( |−mK_X| \) (see e.g. [18]). This gives \( \mu_m \in \text{Aut}(X_m) \) (with \( X = X_m/\mu_m \)) and \( (m, p) = 1 \). We may assume the \( \mu_m \)-action on \( X_m \) comes from a cyclic projective action on some \( \mathbb{P}^N \subset X_m \).

Now, if \( X_m \) were liftable to \( \mathbb{C} \), the \( \mu_m \)-action on \( X_m \subset \mathbb{P}^N \) must also lift. Then we would get that \( X = X_m/\mu_m \) is liftable to \( \mathbb{C} \), a contradiction. \(\square\)
Proof of Corollary 1.7. Suppose that \( q(X) = 0 \). We are going to find an ample line bundle \( \mathcal{L} \) on \( X \) such that \( H^1(X, \pi_m^* \mathcal{L}^{-1}) \neq 0 \) and \( (p-1)\pi_m^* \mathcal{L} - K_{X_m} \) is ample. This will contradict [16, Ch. II, Corollary 6.3]

The description of \( \text{Ker} F^* \) in [2,6] and the fact that \( h^1(W, \pi^* L^{-1}) = h^1(S, L^{-1}) \geq 2 \) (cf. Example 2.7) yield \( h^1(X, (\phi \circ \pi)^* L^{-1}) \geq 2 \) – for not every element in \( H^1(W, \pi^* L^{-1}) \) is a \( p \)-power (when lifted to \( X \)). Furthermore, the same argument shows that \( H^1(X_m, \pi_m^* \mathcal{L}^{-1}) \neq 0 \), provided we have found \( \mathcal{L} \) on \( X \) as needed.

Now set \( \mathcal{L} := \mathcal{O}_X(-K_X + (\pi \circ \phi)^* D) \). This is obviously ample. Moreover, by the vanishing in [30, Proposition 6.1] and results from [30, §§3,4] we may assume generic surface \( S_X \in |-K_X| \) to be smooth (hence \( K3-like \)), which leads to an exact sequence

\[
0 \to H^1(X, -S_X - (\pi \circ \phi)^* D) \to H^1(X, -(\pi \circ \phi)^* D) \to H^1(S, -(\pi \circ \phi)^* D |_S). \]

Here \( H^1(S, -(\pi \circ \phi)^* D |_S) = 0 \) by [22, Proposition 3.4], \( H^1(X, -(\pi \circ \phi)^* D) \neq 0 \) by the previous considerations, \( \mathcal{O}_X(K_X - (\pi \circ \phi)^* D) = \mathcal{L}^{-1} \), and thus we are done.

Hence we get \( q(X) \neq 0 \). Then also \( q(X_m) \neq 0 \) for the flat morphism \( \pi_m \).

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\[ \text{Appendix} \]

\textbf{A.1.} Let me elaborate further on the footnote 1) on page 1 of the present paper. The second sentence in the proof of [30, Theorem 1.4] claims that “By Serre vanishing, we may assume that \( H^1(X, \mathcal{O}(-pD)) = 0 \ldots \)” This phrase is actually the cornerstone for the whole [30, Section 1] and is totally misleading (in fact false) – it was never explained in [30] why one can apply Serre vanishing to the \textit{fixed} \( p \) and an arbitrary ample \( D \) (whereas vanishing requires (a priori arbitrary) \textit{sufficiently large} \( p \)). There are more confusing assertions in [30, Section 1] (such as e.g. the name \textit{irregularity} (and the notation \( q(X) \)) for the number \( h^0(X, \Omega_X^1) \)). Amazingly, the author of [30] has presented same results at a recent conference in Moscow (2017), with apparently a fare success.

\textbf{A.2.} Initially the present paper was accepted for publishing in Manuscripta Mathematica. However, some time later I have received a message from the Editors, informing me that there were several anonymous emails which claimed that the constructions of my paper are incorrect. Basically, there were two mathematically fruitful extra-reports, expressing the concerns, whose content and my responses to

\[ ^5 \text{Note that } (p-1)\pi_m^* \mathcal{L} - K_{X_m} \equiv \pi_m^* (- (p-1 + m \frac{1}{m})K_X + (\pi \circ \phi)^* D) \text{ is ample.} \]
them are available at request. Let me stress though that neither of the reports contained a satisfactory support for their authors’ strong acquisitions. Nevertheless, despite this fact the Editors of Manuscripta initiated a retraction process for the paper, and the manuscript had actually been retracted recently. My strong disagreement and the evidence in its favor were completely ignored. Nor I was allowed to support my paper with the corresponding Addendum.

A.3. The present situation around my paper is a result of personal conflict and is a part of continuous manhunt on me (initiated unfortunately by my former advisor Cheltsov). It is quite sad that such mechanisms as scientific publishing are used, on the one hand, in order to push forward wrong papers (cf. [30]), and suppress ones mathematical personality on the other. There is no trace here of objective delivering the scientific results to the community.

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