A spectrahedral representation of the first derivative relaxation of the positive semidefinite cone

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Abstract If $X$ is an $n \times n$ symmetric matrix, then the directional derivative of $X \mapsto \det(X)$ in the direction $I$ is the elementary symmetric polynomial of degree $n - 1$ in the eigenvalues of $X$. This is a polynomial in the entries of $X$ with the property that it is hyperbolic with respect to the direction $I$. The corresponding hyperbolicity cone is a relaxation of the positive semidefinite (PSD) cone known as the first derivative relaxation (or Renegar derivative) of the PSD cone. A spectrahedral cone is a convex cone that has a representation as the intersection of a subspace with the cone of PSD matrices in some dimension. We show that the first derivative relaxation of the PSD cone is a spectrahedral cone, and give an explicit spectrahedral description of size $(n+1)/2 - 1$. The construction provides a new explicit example of a hyperbolicity cone that is also a spectrahedron. This is consistent with the generalized Lax conjecture, which conjectures that every hyperbolicity cone is a spectrahedron.

Keywords Spectrahedron · Hyperbolic polynomial · Hyperbolicity cone · Elementary symmetric polynomial

1 Introduction

1.1 Preliminaries

Hyperbolic polynomials, hyperbolicity cones, and spectrahedra A multivariate polynomial $p$, homogeneous of degree $d$ in $n$ variables, is hyperbolic with respect to $e \in \mathbb{R}^n$ if...
if \( p(e) \neq 0 \) and for all \( x \), the univariate polynomial \( t \mapsto p(x - te) \) has only real roots. Associated with such a polynomial is a cone

\[
\Lambda_+(p, e) = \{ x \in \mathbb{R}^n : \text{all roots of } t \mapsto p(x - te) \text{ are non-negative} \}.
\]

A foundational result of Gårding [8] is that \( \Lambda_+(p, e) \) is actually a convex cone, called the \textit{closed hyperbolicity cone associated with } \( p \) \text{ and } \( e \).

For example \( p(x) = \prod_{i=1}^n x_i \) is hyperbolic with respect to \( 1_n \), the vector of all ones, and the corresponding closed hyperbolicity cone is the non-negative orthant, \( \mathbb{R}_+^n \). Similarly \( p(X) = \det(X) \) (where \( X \) is a symmetric \( n \times n \) matrix), is hyperbolic with respect to the identity matrix \( I \), and the corresponding closed hyperbolicity cone is the positive semidefinite cone \( \mathcal{S}_+^n \).

If a polynomial \( p \) has a representation of the form

\[ p(x) = \det \left( \sum_{i=1}^n A_i x_i \right) \quad (1) \]

for symmetric matrices \( A_1, \ldots, A_n \), and there exists \( e \in \mathbb{R}^n \) such that \( \sum_{i=1}^n A_i e_i \) is positive definite, we say that \( p \) has a \textit{definite determinantal representation}. In this case \( p \) is hyperbolic with respect to \( e \). The associated closed hyperbolicity cone is

\[ K = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n A_i x_i \succeq 0 \right\} \quad (2) \]

where we write \( X \succeq 0 \) to indicate that \( X \) is positive semidefinite (and \( X \succ 0 \) to indicate that \( X \) is positive definite). Such convex cones are called \textit{spectrahedral cones}. If the matrices \( A_1, A_2, \ldots, A_n \) are \( d \times d \) we call (2) a \textit{spectrahedral representation of size } \( d \).

\textbf{Derivative relaxations} One way to produce new hyperbolic polynomials is to take directional derivatives of hyperbolic polynomials in directions of hyperbolicity [2, Section 3.10], a construction emphasized in the context of optimization by Renegar [13]. If \( p \) has degree \( d \) and is hyperbolic with respect to \( e \), then for \( k = 0, 1, \ldots, d \), the \( k \)-th directional derivative in the direction \( e \), i.e.,

\[
D_e^{(k)} p(x) = \frac{d^k}{dt^k} p(x + te) \bigg|_{t=0},
\]

is also hyperbolic with respect to \( e \). Moreover

\[
\Lambda_+(D_e^{(k)} p, e) \supseteq \Lambda_+(D_e^{(k-1)} p, e) \supseteq \cdots \supseteq \Lambda_+(p, e)
\]

so the hyperbolicity cones of the directional derivatives form a sequence of \textit{relaxations} of the original hyperbolicity cone.
Suppose \( p(x) = \prod_{i=1}^{n} x_i \) and \( e = 1_n \). Then, for \( k = 0, 1, \ldots, n \),
\[
D_{i_n}^{(k)} p(x) = k! e_{n-k}(x)
\]
where \( e_{n-k} \) is the elementary symmetric polynomial of degree \( n - k \) in \( n \) variables. We use the notation \( \mathbb{R}_+^{n,(k)} \) for \( \Lambda_+(e_{n-k}, I_n) \), the closed hyperbolicity cone corresponding to \( e_{n-k} \).

Suppose \( p(X) = \det(X) \) is the determinant restricted to \( n \times n \) symmetric matrices, and \( e = I_n \) is the \( n \times n \) identity matrix. Then, for \( k = 0, 1, \ldots, n \),
\[
D_{i_n}^{(k)} p(x) = k! E_{n-k}(X) = k! e_{n-k}(\lambda(X))
\]
where \( E_{n-k}(X) \) is the elementary symmetric polynomial of degree \( n - k \) in the eigenvalues of \( X \) or, equivalently, the coefficient of \( t^k \) in \( \det(X + t I_n) \). We use the notation \( S_+^{n,(k)} \) for \( \Lambda_+(E_{n-k}, I_n) \), the closed hyperbolicity cone corresponding to \( E_{n-k} \). We use the notation \( \lambda(X) \) for the eigenvalues of a symmetric matrix \( X \) ordered so that \( |\lambda_1(X)| \geq |\lambda_2(X)| \geq \cdots \geq |\lambda_n(X)| \). We use this order so that \( \lambda_i(X^2) = \lambda_i(X)^2 \) for all \( i \).

The focus of this paper is the cone \( S_+^{n,(1)} \), the hyperbolicity cone associated with \( E_{n-1} \). In particular, we consider whether \( S_+^{n,(1)} \) can be expressed as a ‘slice’ of some higher dimensional positive semidefinite cone. Such a description allows one to reformulate hyperbolic programs with respect to \( S_+^{n,(1)} \) (linear optimization over affine ‘slices’ of \( S_+^{n,(1)} \)) as semidefinite programs.

**Generalized Lax conjecture** We have seen that every spectrahedral cone is a closed hyperbolicity cone. The **generalized Lax conjecture** asks whether the converse holds, i.e., whether every closed hyperbolicity cone is also a spectrahedral cone. The original Lax conjecture, now a theorem due to Helton and Vinnikov [9] (see also [11]), states that if \( p \) is a trivariate polynomial, homogeneous of degree \( d \), and hyperbolic with respect to \( e \in \mathbb{R}^2 \), then \( p \) has a definite determinantal representation. While a direct generalization of this algebraic result does not hold in higher dimensions [3], the following geometric conjecture remains open.

**Conjecture 1** (Generalized Lax Conjecture (geometric version)) *Every closed hyperbolicity cone is spectrahedral.*

An equivalent algebraic formulation of this conjecture is as follows.

**Conjecture 2** (Generalized Lax Conjecture (algebraic version)) *If \( p \) is hyperbolic with respect to \( e \in \mathbb{R}^n \), then there exists a polynomial \( q \), hyperbolic with respect to \( e \in \mathbb{R}^n \), such that \( qp \) has a definite determinantal representation and \( \Lambda_+(q, e) \supseteq \Lambda_+(p, e) \).

The algebraic version of the conjecture implies the geometric version because it implies the existence of a multiplier \( q \) such that the hyperbolic cone associated with \( qp \) is spectrahedral and \( \Lambda_+(qp, e) = \Lambda_+(p, e) \cap \Lambda_+(q, e) = \Lambda_+(p, e) \). To see that the geometric version implies the algebraic version requires more algebraic machinery, and is discussed, for instance, in [17, Section 2].
1.2 Main result: a spectrahedral representation of $S_{+}^{n,(1)}$

In this paper, we show that $S_{+}^{n,(1)}$, the first derivative relaxation of the positive semidefinite cone, is spectrahedral. We give an explicit spectrahedral representation of $S_{+}^{n,(1)}$ (see Theorem 1 to follow). Moreover, in Theorem 3 in Section 2 we find an explicit hyperbolic polynomial $q$ such that $q(X)E_{n−1}(X)$ has a definite determinantal representation and $A_{+}(q, I) \supseteq S_{+}^{n,(1)}$.

Theorem 1 Let $d = \binom{n+1}{2} − 1$ and let $B_1, \ldots, B_d$ be any basis for the $d$-dimensional space of real symmetric $n \times n$ matrices with trace zero. If $B(X)$ is the $d \times d$ symmetric matrix with $i,j$ entry equal to $\text{tr}(B_i X B_j)$ then

$$S_{+}^{n,(1)} = \{X \in S^n : B(X) \succeq 0\}.$$  

Section 2 is devoted to the proof of this result. At this stage we make a few remarks about the statement and some of its consequences.

- The spectrahedral representation of $S_{+}^{n,(1)}$ in Theorem 1 has size $d = \binom{n+1}{2} − 1 = \frac{1}{2}(n + 2)(n − 1)$. This is about half the size of the smallest previously known projected spectrahedral representation of $S_{+}^{n,(1)}$, i.e., representation as the image of a spectrahedral cone under a linear map [15].
- A straightforward extension of this result shows that if $p$ has a definite determinantal representation and $e$ is a direction of hyperbolicity for $p$, then the hyperbolicity cone associated with the directional derivative $D_ep$ is spectrahedral. We discuss this in Sect. 3.1.
- It also follows from Theorem 1 that $\mathbb{R}_{+}^{n,(2)}$, the second derivative relaxation of the orthant in the direction $1_n$, has a spectrahedral representation of size $(\binom{n}{2}) − 1$. We discuss this in Sect. 3.1. This representation is significantly smaller than the size $O(n^{n−3})$ representation constructed by Brändén [4], and about half the size of the smallest previously known projected spectrahedral representation of $\mathbb{R}_{+}^{n,(2)}$ [15].

1.3 Related work

We briefly summarize related work on spectrahedral and projected spectrahedral representations of the hyperbolicity cones $\mathbb{R}_{+}^{n,(k)}$ and $S_{+}^{n,(k)}$. Sanyal [14] showed that $\mathbb{R}_{+}^{n,(1)}$ is spectrahedral by giving the following explicit definite determinantal representation of $e_{n−1}(x)$, which we use repeatedly in the paper.

Proposition 1 If $1_{n}^\perp = \{x \in \mathbb{R}^n : 1^T_n x = 0\}$, and $V_n$ is a $n \times (n − 1)$ matrix with columns spanning $1_{n}^\perp$, then there is a positive constant $c$ such that

$$c e_{n−1}(x) = \det(V_n^T \text{diag}(x)V_n) \quad \text{and so} \quad \mathbb{R}_{+}^{n,(1)} = \{x \in \mathbb{R}^n : V_n^T \text{diag}(x)V_n \succeq 0\}.$$  

This representation is also implicit in the work of Choe et al. [5]. Zinchenko [18], gave a projected spectrahedral representation of $\mathbb{R}_{+}^{n,(1)}$. Brändén [4], established that each
of the cones \( \mathbb{R}^{n,(k)}_+ \) are spectrahedral by constructing graphs \( G \) with edges weighted by linear forms in \( x \), such that the edge weighted Laplacian \( L_G(x) \) is positive semidefinite if and only if \( x \in \mathbb{R}^{n,(k)}_+ \). Amini showed that the hyperbolicity cones associated with certain multivariate matching polynomials are spectrahedral [1], and used these to find new spectrahedral representations of the cones \( \mathbb{R}^{n,(k)}_+ \) of size \( \frac{(n-1)!}{(k-1)!} + 1 \).

Explicit projected spectrahedral representations of the cones \( S^{n,(1)}_+ \) of size \( O(n^2 \min\{k, n-k\}) \) were given by Saunderson and Parrilo [15], leaving open (except in the cases \( k = n-2, n-1 \)) the question of whether these cones are spectrahedra. The main result of this paper is that \( S^{n,(1)}_+ \) is a spectrahedron.

2 Proof of Theorem 1

In this section we give two proofs of Theorem 1. The first proof is convex geometric in nature whereas the second is algebraic in nature. Both arguments are self-contained. We present the geometric argument first because it suggests the choice of multiplier \( q \) for the algebraic argument.

Both arguments take advantage of the fact that the cone \( S^{n,(1)}_+ \) satisfies \( QS^{n,(1)}_+ Q^T = S^{n,(1)}_+ \) for all \( Q \in O(n) \). One way to see this is to observe that the hyperbolic polynomial \( E_{n-1} X \) that determines the cone satisfies \( E_{n-1}(QXQ^T) = E_{n-1}(X) \) for all \( Q \in O(n) \) and the direction of hyperbolicity (the identity) is also invariant under this group action.

2.1 Geometric argument

We begin by stating a slight reformulation of Sanyal’s spectrahedral representation (Proposition 1).

**Proposition 2** Let \( 1_{n}^\perp = \{ y \in \mathbb{R}^n : 1_n^T y = 0 \} \) be the subspace of \( \mathbb{R}^n \) orthogonal to \( 1_n \). Then

\[
\mathbb{R}_+^{n,(1)} = \{ x \in \mathbb{R}^n : y^T \text{diag}(x)y \geq 0 \text{ for all } y \in 1_n^\perp \}.
\]

**Proof** This follows from Proposition 1 since \( V_n^T \text{diag}(x)V_n \geq 0 \) holds if and only if \( u^TV_n^T \text{diag}(x)V_nu \geq 0 \) for all \( u \in \mathbb{R}^{n-1} \) which holds if and only if \( y^T \text{diag}(x)y \geq 0 \) for all \( y \in 1_n^\perp \). \( \square \)

In this section we establish a ‘matrix’ analogue of Proposition 2.

**Theorem 2** Let \( I_n^\perp = \{ Y \in S^n : \text{tr}(Y) = 0 \} \) be the subspace of \( n \times n \) symmetric matrices with trace zero. Then

\[
S^{n,(1)}_+ = \{ X \in S^n : \text{tr}(XY) \geq 0, \text{ for all } Y \in I_n^\perp \}.
\] (4)

The concrete spectrahedral description given in Theorem 1 follows immediately from Theorem 2. Indeed if \( B_1, B_2, \ldots, B_d \) are a basis for \( I_n^\perp \) then an arbitrary \( Y \in I_n^\perp \)
can be written as $Y = \sum_{i=1}^{d} y_i B_i$. The condition $\text{tr}(YXY) \geq 0$ for all $Y \in I_n^{\perp}$ is equivalent to

$$\sum_{i,j=1}^{d} y_i y_j \text{tr}(B_i X B_j) \geq 0 \text{ for all } y \in \mathbb{R}^d \quad \text{which holds if and only if } B(X) \geq 0.$$ 

**Proof of Theorem 2** The convex cone $\mathcal{S}_n^{(1)}$ is invariant under the action of the orthogonal group on $n \times n$ symmetric matrices by congruence transformations. Similarly, the convex cone

$$\{ X \in S^n : \text{tr}(YXY) \geq 0 \text{ for all } Y \in I_n^{\perp} \}$$

is invariant under the same action of the orthogonal group. This is because $X \in I_n^{\perp}$ if and only if $QXQ^T \in I_n^{\perp}$ for any orthogonal matrix $Q$.

Because of these invariance properties, the following (straightforward) result tells us that we can establish Theorem 2 by showing that the diagonal ‘slices’ of these two convex cones agree.

**Lemma 1** Let $K_1, K_2 \subset S^n$ be such that $QK_1Q^T = K_1$ for all $Q \in O(n)$ and $QK_2Q^T = K_2$ for all $Q \in O(n)$. If $\{ x \in \mathbb{R}^n : \text{diag}(x) \in K_1 \} = \{ x \in \mathbb{R}^n : \text{diag}(x) \in K_2 \}$ then $K_1 = K_2$.

**Proof** Assume that $X \in K_1$. Then there exists $Q$ such that $QXQ^T = \text{diag}(\lambda(X))$. Since $K_1$ is invariant under orthogonal congruence, $\text{diag}(\lambda(X)) \in K_1$. By assumption, it follows that $\text{diag}(\lambda(X)) \in K_2$. Since $K_2$ is invariant under orthogonal congruence, $X = Q^T \text{diag}(\lambda(X))Q \in K_2$. This establishes that $K_1 \subseteq K_2$. Reversing the roles of $K_1$ and $K_2$ completes the argument. \qed

**Relating the diagonal slices** To complete the proof of Theorem 2, it suffices (by Lemma 1) to show that the diagonal slices of the left- and right-hand sides of (4) are equal. Since the diagonal slice of $\mathcal{S}_n^{(1)}$ is $\mathbb{R}_n^{(1)}$, it is enough (by Proposition 2) to establish the following result.

**Lemma 2**

$$\{ x \in \mathbb{R}^n : \text{tr}(Y \text{diag}(x)Y) \geq 0 \text{ for all } Y \in I_n^{\perp} \}
= \{ x \in \mathbb{R}^n : y^T \text{diag}(x)y \geq 0 \text{ for all } y \in I_n^{\perp} \}.$$

**Proof** Suppose that $\text{tr}(Y \text{diag}(x)Y) \geq 0$ for all $Y \in I_n^{\perp}$. Let $y \in I_n^{\perp}$. Then $\text{diag}(y) \in I_n^{\perp}$ and so it follows that $\text{tr}(\text{diag}(y) \text{diag}(x) \text{diag}(y)) = y^T \text{diag}(x)y \geq 0$. This shows that the left hand side is a subset of the right hand side.

For the reverse inclusion suppose that $y^T \text{diag}(x)y \geq 0$ for all $y \in I_n^{\perp}$. Let $Y \in I_n^{\perp}$. Suppose the symmetric group on $n$ symbols, $S_n$, acts on $\mathbb{R}^n$ by permutations. Then for every $\sigma \in S_n$, we have that $\sigma \cdot \lambda(Y) \in 1_n^{\perp}$ and thus

\[ \text{tr}(\text{diag}(\sigma \cdot \lambda(Y^2)) \text{diag}(x)) = (\sigma \cdot \lambda(Y))^T \text{diag}(x)(\sigma \cdot \lambda(Y)) \geq 0. \]
The diagonal of a symmetric matrix is a convex combination of permutations of its eigenvalues, a result due to Schur [16] (see also, e.g., [12]). Hence \( \text{diag}(Y^2) \) is a convex combination of permutations of \( \lambda(Y^2) \), i.e.,

\[
\text{diag}(Y^2) = \sum_{\sigma \in S_n} \eta_\sigma \left( \sigma \cdot \lambda(Y^2) \right)
\]

where the \( \eta_\sigma \) satisfy \( \eta_\sigma \geq 0 \) and \( \sum_{\sigma \in S_n} \eta_\sigma = 1 \). It then follows that

\[
\text{tr}(Y \text{diag}(x) Y) = \text{tr}(\text{diag}(Y^2) \text{diag}(x)) = \sum_{\sigma \in S_n} \eta_\sigma \text{tr}(\text{diag}(\sigma \cdot \lambda(Y^2)) \text{diag}(x)) \geq 0.
\]

This shows that the right hand side is a subset of the left hand side.

This completes the proof of Theorem 2.

\[ \square \]

### 2.2 Algebraic argument

In this section, we establish the following algebraic version of Theorem 1.

**Theorem 3** Let \( n \geq 2 \) and \( B_1, \ldots, B_d \) be a basis for \( I_n^\perp \), the subspace of \( n \times n \) symmetric matrices with trace zero. Then there is a positive constant \( c \) (depending on the choice of basis) such that

1. \( q(X) = \prod_{1 \leq i < j \leq n} (\lambda_i(X) + \lambda_j(X)) \) is hyperbolic with respect to \( I_n \);
2. the hyperbolicity cone associated with \( q \) satisfies

\[
\Lambda^+(q, I_n) = \{ X \in S^n : \lambda_i(X) + \lambda_j(X) \geq 0 \text{ for all } 1 \leq i < j \leq n \} \supseteq S_n^{n,1} ;
\]

3. \( q(X) E_{n-1}(X) \) has a definite determinantal representation as

\[
\text{det}(L_2(X)) = \det(\mathcal{B}(X)).
\]

We remark that \( q(X) \) is defined as a symmetric polynomial in the eigenvalues of \( X \), and so can be expressed as a polynomial in the entries of \( X \). Although our argument does not use this fact, it can be shown that \( q(X) = \text{det}(L_2(X)) \) where \( L_2(X) \) is the second additive compound matrix of \( X \) [7]. This means that \( q \) is not only hyperbolic with respect to \( I_n \), but also has a definite determinantal representation.

**Proof of Theorem 3** The three items in the statement of Theorem 3 are established in the following three Lemmas (Lemmas 3, 4, and 5).

**Lemma 3** If \( q(X) = \prod_{1 \leq i < j \leq n} (\lambda_i(X) + \lambda_j(X)) \) then \( q \) is hyperbolic with respect to \( I_n \).
First observe that $q(I_n) = 2^{\binom{n}{2}} \neq 0$. Moreover, for any real $t$,

$$q(X - t I_n) = \prod_{1 \leq i < j \leq n} (\lambda_i(X - t I_n) + \lambda_j(X - t I_n)) = \prod_{1 \leq i < j \leq n} (\lambda_i(X) + \lambda_j(X) - 2t)$$

which has $\binom{n}{2}$ real roots given by $\frac{1}{2}(\lambda_i(X) + \lambda_j(X))$ for $1 \leq i < j \leq n$. Hence $q$ is hyperbolic with respect to $I_n$.

**Lemma 4** If $n \geq 2$ then

$$\Lambda_+(q, I_n) = \{X \in S^n : \lambda_i(X) + \lambda_j(X) \geq 0 \text{ for all } 1 \leq i < j \leq n \} \supseteq S^{n, (1)}_+.$$ 

**Proof** Since the roots of $t \mapsto q(X - t I_n)$ are $\frac{1}{2}(\lambda_i(X) + \lambda_j(X))$, the description of $\Lambda_+(q, I_n)$ is immediate. Both sides of the inclusion are invariant under congruence by orthogonal matrices. By Lemma 1 it is enough to show that the inclusion holds for the diagonal slices of both sides. Note that

$$\{x \in \mathbb{R}^n : \text{diag}(x) \in \Lambda_+(q, I_n)\} = \{x \in \mathbb{R}^n : x_i + x_j \geq 0 \text{ for all } 1 \leq i < j \leq n\}.$$ 

Hence it is enough to establish that

$$\{x \in \mathbb{R}^n : x_i + x_j \geq 0 \text{ for all } 1 \leq i < j \leq n\} \supseteq \mathbb{R}^{n, (1)}_+.$$ 

To do so, we use the characterization of $\mathbb{R}^{n, (1)}_+$ from Proposition 2. This tells us that if $x \in \mathbb{R}^{n, (1)}_+$ then $v^T \text{diag}(x)v = \sum_{\ell=1}^n x_\ell v_\ell^2 \geq 0$ for all $v \in 1^n_\perp$. In particular, let $v$ be the element of $1^n_\perp$ with $v_i = 1$ and $v_j = -1$ and $v_k = 0$ for $k \not\in \{i, j\}$. Then, if $x \in \mathbb{R}^{n, (1)}_+$ it follows that $\sum_{\ell=1}^n x_\ell v_\ell^2 = x_i + x_j \geq 0$. This completes the proof. □

**Lemma 5** If $B_1, \ldots, B_d$ is a basis for $I^n_\perp$, then there is a positive constant $c$ (depending on the choice of basis) such that

$$c q(X) E_{n-1}(X) = \det(B(X)).$$

**Proof** Since both sides are invariant under orthogonal congruence, it is enough to show that the identity holds for diagonal matrices. In other words, it is enough to show that

$$c \prod_{1 \leq i < j \leq n} (x_i + x_j) e_{n-1}(x) = \det(B(\text{diag}(x))).$$

Since a change of basis for the subspace of symmetric matrices with trace zero only changes $\det(B(X))$ by a positive constant (which is one if the change of basis is orthogonal with respect to the trace inner product), it is enough to choose a particular basis for the subspace of symmetric matrices with trace zero, and show that the identity holds for a particular constant.
Let $v_1, v_2, \ldots, v_{n-1}$ be a basis for $1_n = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 0\}$. Let $M_{ij}$ be the $n \times n$ matrix with a one in the $(i, j)$ and the $(j, i)$ entry, and zeros elsewhere. Clearly the $M_{ij}$ for $1 \leq i < j \leq n$ form a basis for the subspace of symmetric matrices with zero diagonal. Together $\text{diag}(v_1), \text{diag}(v_2), \ldots, \text{diag}(v_{n-1})$ and $M_{ij}$ for $1 \leq i < j \leq n$ form a basis for the subspace of symmetric matrices with trace zero. Using this basis we evaluate the matrix $B(\text{diag}(x))$. We note that

$$\text{tr}(\text{diag}(v_i) \text{diag}(x) \text{diag}(v_j)) = v_i^T \text{diag}(x) v_j \text{ for } 1 \leq i, j \leq n$$

$$\text{tr}(\text{diag}(v_i) \text{diag}(x) M_{jk}) = 0 \text{ for all } 1 \leq i \leq n \text{ and } 1 \leq j < k \leq n$$

since $M_{jk}$ has zero diagonal, and that

$$\text{tr}(M_{ij} \text{diag}(x) M_{k\ell}) = \begin{cases} x_i + x_j & \text{if } i = k \text{ and } j = \ell \\ 0 & \text{otherwise} \end{cases}$$

for all $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq n$. This means that $B(\text{diag}(x))$ is block diagonal, and so

$$\det(B(\text{diag}(x))) = \prod_{1 \leq i < j \leq n} (x_i + x_j) \det(\text{diag}(x) V_n)$$

(6)

where $V_n$ is the $n \times (n - 1)$ matrix with columns $v_1, v_2, \ldots, v_n$. By Proposition 1, there is a positive constant $c$ such that

$$\det(V_n^T \text{diag}(x) V_n) = c e_{n-1}(x).$$

(7)

Combining (6) and (7) gives the stated result. □

This completes the proof of Theorem 3. □

3 Discussion

3.1 Consequences of Theorem 1

A straightforward consequence of Theorem 1 is that if $p$ has a definite determinantal representation, and $e$ is a direction of hyperbolicity for $p$, then the hyperbolicity cone associated with the directional derivative $D_e p$ is spectrahedral.

Corollary 1 If $p(x) = \det(\sum_{i=1}^n A_i x_i)$ for symmetric $\ell \times \ell$ matrices $A_1, \ldots, A_n$, and $A_0 = \sum_{i=1}^m A_i e_i$ is positive definite, then $\Lambda_+(D_e p, e)$ has a spectrahedral representation of size $\binom{\ell + 1}{2} - 1$.

Proof The hyperbolicity cone $\Lambda_+(D_e p, e)$ can be expressed as

$$\Lambda_+(D_e p, e) = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n A_0^{-1/2} A_i A_0^{-1/2} x_i \in S_{+}^n(1) \right\}.$$
(see, e.g., [15, Proposition 4]). Applying Theorem 1 then gives

$$\Lambda_+(D_e p, e) = \left\{ x \in \mathbb{R}^n : B \left( \sum_{i=1}^{n} A_0^{-1/2} A_i A_0^{-1/2} x_i \right) \succeq 0 \right\}.$$ 

Our main result also yields a spectrahedral representation of $\mathbb{R}^{n,(2)}$, the second derivative relaxation of the non-negative orthant, of size $\binom{n}{2} - 1$. This is, in fact, a special case of Corollary 1. In the statement below, $V_n$ is any $n \times (n-1)$ matrix with columns that span $1_n^\perp$.

**Corollary 2** The hyperbolicity cone $\mathbb{R}^{n,(2)}$ has a spectrahedral representation of size $\binom{n}{2} - 1$ given by

$$\mathbb{R}^{n,(2)}_+ = \left\{ x \in \mathbb{R}^n : B(V_n^T \text{diag}(x)V_n) \succeq 0 \right\}$$

**Proof** First, we use the fact that $\mathbb{R}^{n,(2)}_+ = \Lambda_+(D_1 e_{n-1}, 1_n)$. Then, by Sanyal’s result (Proposition 1), we know that $e_{n-1}(x)$ has a definite determinantal representation. The stated result then follows directly from Corollary 1 with polynomial $p = e_{n-1}$ and direction $e = 1_n$. □

### 3.2 Questions

**Constructing spectrahedral representations** It is natural to ask for which values of $k$ the cones $\mathcal{S}^{n,(k)}$ are spectrahedral. Our main result shows that $\mathcal{S}^{n,(1)}_+$ has a spectrahedral representation of size $d = \binom{n+1}{2} - 1$. The only other cases for which spectrahedral representations are known are the straightforward cases $k = n-1$ and $k = n-2$. If $k = n-1$ then

$$\mathcal{S}^{n,(n-1)}_+ = \{ X \in \mathcal{S}^n : \text{tr}(X) \geq 0 \}$$

is a spectrahedron (with a representation of size 1). Since $\mathcal{S}^{n,(n-2)}_+$ is a quadratic cone, it is a spectrahedron. To give an explicit representation, let $d = \binom{n+1}{2} - 1$ and $B_1, B_2, \ldots, B_d$ be an orthonormal basis (with respect to the trace inner product) for the subspace $I_{n-1}^\perp$. Now $X \in \mathcal{S}^{n,(n-2)}_+$ if and only if (see, e.g., [15, Section 5.1])

$$\text{tr}(X) \geq 0 \quad \text{and} \quad \text{tr}(X)^2 - \text{tr}(X^2) = \left[ \sqrt{\frac{n-1}{n} \text{tr}(X)} \right]^2 - \sum_{i=1}^{d} \text{tr}(B_i X)^2 \geq 0.$$  (8)
By a well-known spectrahedral representation of the second-order cone, (8) holds if and only if
\[
\sqrt{\frac{n - 2}{n}} \text{tr}(X) I_d + \begin{bmatrix}
\text{tr}(B_1 X) & \text{tr}(B_2 X) & \text{tr}(B_3 X) & \cdots & \text{tr}(B_d X) \\
\text{tr}(B_2 X) & -\text{tr}(B_1 X) & 0 & \cdots & 0 \\
\text{tr}(B_3 X) & 0 & -\text{tr}(B_1 X) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\text{tr}(B_d X) & 0 & 0 & \cdots & -\text{tr}(B_1 X)
\end{bmatrix} \succeq 0.
\] (9)

So we see that \( S_+^{n,(n-2)} \) has a spectrahedral representation of size \( d = \left(\frac{n+1}{2}\right) - 1 \). At this stage, it is unclear how to extend the approach in this paper to the remaining cases.

**Question 1** Are the cones \( S_+^{n,(k)} \) spectrahedral for \( k = 2, 3, \ldots, n-3 \)?

At first glance, it may seem that Corollary 1 allows us to construct a spectrahedral representation for \( S_+^{n,(2)} \) from a spectrahedral representation for \( S_+^{n,(1)} \). However, this is not the case. To apply Corollary 1 to this situation, we would need a definite determinantal representation of \( E_{n-2}(X) \), which our main result (Theorem 1) does not provide.

**Lower bounds on size** Another natural question concerns the size of spectrahedral representations of hyperbolicity cones. Given a hyperbolicity cone \( K \), there is a unique (up to scaling) hyperbolic polynomial \( p \) of smallest degree \( d \) that vanishes on the boundary of \( K \) (see, e.g., [10]). Clearly any spectrahedral representation must have size at least \( d \), but it seems that in some cases the smallest spectrahedral representation (if it exists at all) must have larger size.

**Question 2** Is there a spectrahedral representation of \( S_+^{n,(1)} \) with size smaller than \( \left(\frac{n+1}{2}\right) - 1 \)?

Recently, there has been considerable interest in developing methods for producing lower bounds on the size of projected spectrahedral descriptions of convex sets (see, e.g., [6]). There has been much less development in the case of lower bounds on the size of spectrahedral descriptions. The main work in this direction is due to Kummer [10]. For instance it follows from [10, Theorem 1] that any spectrahedral representation of the quadratic cone \( S_+^{n,(n-2)} \) must have size at least \( \frac{1}{2} \left(\frac{n+1}{2}\right) - 1 \). Furthermore, in the special case that \( \left(\frac{n+1}{2}\right) - 1 = 2^k + 1 \) for some \( k \) (which occurs if \( n = 3 \) and \( k = 2 \) or \( n = 4 \) and \( k = 3 \)) then Kummer’s work shows that any spectrahedral representation of \( S_+^{n,(n-2)} \) must have size at least \( \left(\frac{n+1}{2}\right) - 1 \). This establishes that the construction in (9) is optimal when \( n = 3 \) and \( n = 4 \). Furthermore, in the case \( n = 3 \) we have that \( S_+^{n,(1)} = S_+^{n,(n-2)} \). Hence our spectrahedral representation for \( S_+^{n,(1)} \) is also optimal if \( n = 3 \).

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