Simple derivation from postulates of generalized vacuum Maxwell equations

Chun Wa Wong

Department of Physics and Astronomy, University of California, Los Angeles, CA 90095-1547, USA

(Dated: May 15, 2013)

The two postulates of special relativity plus the postulates of conserved charges, both electric and magnetic, and a resulting linear system are sufficient for the derivation of the generalized vacuum Maxwell equations with both charges. The derivative admits another set of Maxwell equations for charges that are the opposite-parity partners of the usual electric and magnetic charges. These new charges and their photons are parts of the parallel universe of dark matter.

I. INTRODUCTION

The purpose of this paper is to display the conceptual basis of classical electromagnetism by deriving the usual generalized vacuum (gv) Maxwell equations with both electric and magnetic charges from the following four postulates:

P1. Principle of relativity:
Physical laws are the same in all inertial frames.

P2. Constancy of light speed:
Light propagates in vacuum with the same speed c in all inertial frames.

P3. Charge conservation:
Electric and other charges have the same values in all inertial frames.

P4. Linearity:
Classical electromagnetic (EM) theory is described by differential equations linear in the EM fields.

P1 and P2 are Einstein’s postulates for Special Relativity (SR) needed to ensure that the resulting dynamical equations have the same form in all inertial frames. P3 defines the essential attribute of the source charges of the EM fields. P4 defines classical EM theory as a linear approximation to more complicated physics. Finally, Maxwell equations have both retarded and advanced wave solutions. So causality is not one of the required postulates. By not using causality, the present derivation differs significantly from that given by Heras, although charge conservation is required in both derivations, as it should.

The derivation given in Section 2 is simple and elementary and is readily accessible to readers familiar with the basic properties of 4-vectors in spacetime.

In Section 3, the space-time symmetries of the 4-currents and EM fields in the gvMaxwell equations are described. It then becomes clear that the given derivation admits another set of gvMaxwell equations that are the opposite-parity partners of the first set. The new equations involve opposite-parity charges and photons that are parts of the parallel universe of dark matter.

II. DERIVING THE USUAL GVMAXWELL EQUATIONS

The present derivation is based on the following idea. The derivation has to be relativistic to comply with the two SR postulates. In the very intuitive Minkowski notation, all vectors and tensors carry only subscripts with time treated as a space-like component, but made imaginary to display its unique role:

\[ x = (x, x_4 = ict), \]

\[ \partial = \frac{\partial}{\partial x_\mu} = \left( \nabla, \frac{\partial}{ic\partial t} \right). \]  (1)

The two inhomogeneous Maxwell equations to be derived can then be written as one equation

\[ s_G J_\mu = \partial_\nu F_{\mu \nu}(x), \quad s_G = 4\pi/c, \]  (2)

where \( J_\mu = (J, ic\rho) \) is the electric 4-current and \( F_{\mu \nu}(x) = -F_{\nu \mu}(x) \) is the antisymmetric rank-2 Maxwell field tensor containing the EM fields \( E \) and \( B \) as their six nontrivial elements that can be nonzero. The factor \( 1/c \) in \( s_G \) identifies the dimension of \( F \) as that of a dipole moment of the charge density \( \rho \), namely charge per unit area. The appearance of \( 4\pi \), the total solid angle in 3D space, shows that we are using Gaussian (G) units.

Differentiation of Eq. (2) gives

\[ s_G \partial_\mu J_\mu = \partial_\nu \partial_\mu F_{\mu \nu} = -\partial_\mu \partial_\nu F_{\nu \mu} = 0. \]  (3)

This is the (Lorentz-\-) invariant continuity equations that extends electric charge conservation (\( dp/dt = 0 \)) for the charge density \( \rho \) in the laboratory frame to all inertial frames. Thus Maxwell equations imply charge conservation, a well-known result obtained in many textbooks. See, for example, Sakurai.

In deriving the gvMaxwell equations from postulates, we need to invert the process by going from Eq. (3) to Eq. (2). There are logical gaps big and small that have to be bridged. We shall proceed in two steps: (i) Derive the inhomogeneous (electric source) equation, and (ii) derive the inhomogeneous (magnetic source) equation when magnetic sources are also present. The standard Maxwell equations without magnetic sources are then obtained as the special case of zero magnetic 4-current.

At the risk of some repetition, let us start from the beginning. In one inertial frame, charge conservation can be expressed conveniently in the differential form

\[ \frac{dp}{dt} = 0 = \partial_t \rho + \nabla \cdot J, \quad \partial_t = \frac{\partial}{\partial t}, \quad J = \nabla \rho. \]  (4)
This continuity equation becomes explicitly frame-independent when written in the invariant form $\partial_{\mu} J_\mu(x) = 0$.

Any conserved 4-current $J_\mu$ has the structure of Eq. (2), $s_G J_\mu = \partial_\nu F_{\mu\nu}(x)$, where $F_{\mu\nu}(x) = -F_{\nu\mu}(x)$ is an antisymmetric rank-2 tensor. This is because Eq. (3) is then guaranteed.

A rank-2 spacetime tensor can be displayed as a 4D matrix. If the matrix $F$ is antisymmetric, it has zero diagonal elements and only 6 independent off-diagonal elements that can be nonzero. Eq. (3) thus contains 12 terms in 6 canceling pairs.

A 4D spacetime matrix contains a 3D spatial submatrix at its “core”, bordered by an extra row and column involving time. In the Minkowskian notation, the 2-sided border contains the last row and the last column. It is convenient to make all purely spatial quantities real in physics, because many of them are measurable experimentally in terms of real numbers. So the antisymmetric field tensor/matrix $F$ (or $F_{\mu\nu}$ in the so-called index notation) can be defined to have a real spatial part $F_{jk}$, $j, k = 1, 2, 3$. This means that a border element is imaginary in the Minkowskian notation, because it carries only one time index. It is timelike if the spatial elements in the 3D core are spacelike. All elements of matrix $F$ have the same dimension.

The three nontrivial elements residing in the border row defines a 3D vector

$$F_{4j} = iE_j,$$  \hspace{1cm} (5)

where $E$ has been taken to be real. By antisymmetry, the border column elements are $F_{j4} = -iE_j$.

The remaining 3 nontrivial, off-diagonal elements reside in the spatial core, and are antisymmetric in two spatial indices $jk$. They too can be made into a 3D vector by using the 3D permutation (Levi-Civita) symbol:

$$F_{jk} = \epsilon_{jkl} B_l,$$  \hspace{1cm} (6)

This spatial antisymmetry gives rise to a rotation that is responsible for the Lorentz force and the Hall effect. $B$ is in general different from $E$, as we shall explain in Section III.

We now apply the linearity postulate P4 by identifying the vectors $E$ and $B$ as the EM fields themselves. It is useful to display separately the time and space parts of the dynamical Eq. (2).

$$s_G J_4 = \partial_4 F_{4j}(x),$$

$$s_G J_j = \partial_j F_{4j}(x).$$  \hspace{1cm} (7)

In terms of the two 3-vectors $E$ and $B$ contained in $F$, these equations read

$$4\pi \rho(x) = \nabla \cdot E(x),$$

$$4\pi J_j = c(\nabla \times B)_j - \partial_j E_j.$$  \hspace{1cm} (8)

These are linear, inhomogeneous dynamical equations for the EM fields driven by an arbitrary conserved current $J_\mu$. For electric charges in particular, they state the Gauss and Ampère–Maxwell EM laws (summaries of experimental facts), respectively.

Thus charge conservation implies the two inhomogeneous dynamical (Maxwell) equations if the EM fields $E$ and $B$ appear only linearly. These are the dynamical equations when the field tensor $F$ has the structure $\{E, B\}$ specified by the vectors appearing in the border and in the spatial core, respectively.

Are there other solutions? First note that it is the border vector, here $E$, that betrays the presence of the source electric charge density $\rho$. Since $\rho$ can contain both positive and negative charges, $E$ can always be used without an extra negative sign. This leaves only one other tensor structure $\{E, -B\}$ for the electric 4-current. The resulting solution is mathematically and physically the same as the first solution but uses a left-handed convention for all rotations (curls and cross products). Hence the first solution for electric 4-currents is physically unique when $E$ is in the border.

However, $F$ is not the only EM field tensor that can be constructed from the given EM fields. The second and only remaining possibility is to relocate $B$ to the border and $E$ to the core. With $B$ in the border, we are now concerned with magnetic 4-currents driving the same EM fields. Two sign choices are possible for the core vector: $[E_m = B, B_m = \pm E]$. They differ in their Poynting vectors giving the directions of their EM waves and energy transports:

$$s_G S_m = E_m \times B_m = \pm B \times E = \mp s_G S_e.$$  \hspace{1cm} (9)

The solution with waves moving in the same direction as waves generated by the partner electric 4-current comes from the tensor structure $\{B, -E\}$. The fields and currents of this magnetic solution, called the dual partner of the electric solution, are marked by left subscripts $.$. The duality transformation of the original EM fields that makes explicit the magnetic 4-current contribution is then:

$$\{E, B\} \rightarrow \{., E, .B\} \equiv \{B, -E\},$$

$$J \rightarrow \pm J.$$  \hspace{1cm} (10)

where both currents can be chosen arbitrarily. The resulting dual dynamical equation is just

$$s_G J_\mu = \partial_\nu F_{\mu\nu}(x).$$  \hspace{1cm} (11)

Its time and space parts can be read off from Eq. (8) with the duality substitution made on the fly:

$$4\pi \rho = \nabla \cdot B,$$

$$4\pi J_j = -c(\nabla \times E)_j - \partial_j E_j.$$  \hspace{1cm} (12)

This completes the derivation of the gvMaxwell equations from the given four postulates.

When magnetic charges are absent, Eq. (12) gives the standard homogeneous equations describing the absence of magnetic charges and the Faraday law, respectively.

The gvMaxwell equations have an additional duality property not shown in Eq. (10). On taking its dual, we have

$$\{., E, .B\} \rightarrow \{., B, -.E\} = \{E, -.B\},$$

$$J \rightarrow -(J).$$  \hspace{1cm} (13)
The \( (J_\nu) = -J_\nu \) relation is useful for sign checking when working with \( \text{gvMaxwell} \) equations. The dual \( J_\nu \) is just a Hodge dual\(^\ast\)\(^\ast\) in the mathematics of differential forms. Our derivation of the duality transformation is completely self-contained, however, and does not require a knowledge of differential forms.

What about the first sign choice in Eq. (9) for the field tensor with the structure \( [\mathbf{B}, \mathbf{E}] \)? Its EM wave moves in a direction opposite to that in the original solution for electric currents. If interpreted as a left-handed way of referring to a wave moving in the same direction, it is the dual of the left-handed vector set \( \{ \mathbf{E}, -\mathbf{B} \} \). It is actually the same physical solution, and it also satisfies the same duality transformation (10).

### III. SPACE-TIME SYMMETRIES

All terms of a Maxwell equation have the same symmetries under the space-time transformations of parity \( \mathcal{P} \): \( r \rightarrow -r \), and time reversal \( \mathcal{P} \): \( t \rightarrow -t \), respectively. Each term can be separated into four parts of different space-time symmetries (±,±). A classical electric charge is a point charge without any space-time extension. It therefore has the intrinsic symmetries of (even, even) = (+,+) under \( \mathcal{P} \) and \( \mathcal{T} \), respectively. These intrinsic symmetries are those of the vacuum. Thus the space-time symmetries of \( \rho \) are purely external or extrinsic. Consider, for example, that part of \( \rho \) with symmetries (+,+). Its associated \( J \) is (−,−) in space-time symmetries, its \( \mathbf{E} \) is (−,+), its \( \mathbf{B} \) is (+,−), its \( \mathbf{J} \) is (−,+), and its \( \mathbf{J} \) is (−,+).

This symmetry analysis is particularly useful in visualizing the structure of the field tensor and field equations. On going from the border vector \( \mathbf{E} \) of the field tensor \( \mathbf{F} \) to its core vector \( \mathbf{B} \), a time index is changed into a space index. Hence both space-time symmetries of \( \mathbf{B} \) must be opposite in sign to those of \( \mathbf{E} \) if one is dealing with fields of definite symmetries. This explains why in Eq. (8), a spatial current \( \mathbf{J} \) can be generated by both the time variation of the border vector \( \mathbf{E} \) and the spatial variation of the core vector \( \mathbf{B} \). Conversely, each EM field can have contributions from both \( \mathbf{J} \) and \( \mathbf{J} \).

The \( \mathbf{B} \) field of symmetries (+,−) in the above example can have a contribution from the magnetostatic field of a magnetic charge at rest at the origin. The external part of such a field has the functional form \( \mathbf{F} / \mathcal{P} \), where \( |\mathcal{P}| = 1 \), and the space-time symmetries (−,−). The intrinsic symmetries of the magnetic charge must then be (−,−). Hence a magnetic charge cannot be a classical point charge with no internal spacetime structure. It is at best semi-classical and pointlike.

From the viewpoint of internal symmetries, two other such semi-classical and pointlike charges of symmetries (−,+ and (+,−) are admissible. They satisfy a set of \( \text{gvMaxwell} \) equations that are the opposite-parity partners of the usual set, namely

\[
\begin{align*}
\mathbf{E}^\ast \mathbf{F}^\ast & = \mathbf{E}^\ast \mathbf{F}^\ast(x), \\
\mathbf{B}^\ast \mathbf{F}^\ast & = \mathbf{B}^\ast \mathbf{F}^\ast(x),
\end{align*}
\]

where \( J^\ast \) are the opposite-parity partners of \( J \), respectively. This completes our derivation of both new and old sets of \( \text{gvMaxwell} \) equations.

The two new charges are similar to magnetic charges in that they have intrinsic symmetries different from those of the vacuum. All three charges necessarily have internal structures. For a more complete description of the structure and other aspects of magnetic charges, see the excellent review by Goldhaber and Trower.

The new set of \( \text{gvMaxwell} \) equations is mathematically disconnected from the usual set. In the quantum theory, their new photons have positive intrinsic parity, opposite in sign to the negative intrinsic parity of our own visible photons. Since the new photons can be emitted or absorbed only by the new opposite-parity charges, both new photons and new charges exist only in the parallel universe of dark matter. To inhabitants of this dark universe, however, it is our own universe that appears dark.

---

1. Electronic address: cwong@physics.ucla.edu
2. J. J. Sakurai, *Advanced Quantum Mechanics* (Addison-Wesley, Reading MA, 1967), pp. 5–6.
3. J. J. Sakurai, *Advanced Quantum Mechanics* (Knopf, New York, 2005), pp. 442–450.
4. J. A. Heras, “How to obtain the covariant form of Maxwell’s equations from the continuity equation”, *Eur. J. Phys.* 30, 845–854 (2009).
5. E. Kapuscik, “Comment on “Can Maxwell’s equations be obtained from the continuity equations?”” by J. A. Heras [Am. J. Phys. 75, 652–657 (2007)], *Am. J. Phys.* 77, 754 (2009).
6. J.D. Jackson, *Classical electrodynamics* (Wiley, New York, 1999), 3rd ed., p. 556.
7. P. Penrose, *The road to reality* (Knopf, New York, 2005), pp. 442–446.
8. J. D. Jackson [3], pp. 269–273.
9. A. S. Goldhaber and W. E. Trower, “Resource Letter MM-1: Magnetic monopoles”, *Am. J. Phys.* 58, 429–439 (1990).