On spaces in countable web

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Abstract. Improving a result from the paper “Spaces in countable web” by
Yoshikazu Yasui and Zhi-min Gao we show that Tychonoff spaces in countable
discrete web may contain closed discrete subsets of arbitrarily big cardinality.

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web, star-Lindelöf.

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1 Results and discussion

In [19], Yoshikazu Yasui and Zhi-min Gao define a space $X$ to be in countable
web provided for every open cover $\mathcal{U}$ of $X$ there is a countable subset $A \subset X$
such that $St(A, \mathcal{U}) = X$. Actually, this property was known, under several
different names, long before [19]. Thus, it was called $\omega$-star in [3],
star-Lindelöfness in [2, 4] and other papers, strong star-Lindelöfness in [3]
and other papers, *Lindelöfness in [13, 16] and other papers, countability of weak
extent in [5]. A number of results, such as Theorem 3.1 are not new, too.

Further, Yasui and Gao define a space $X$ to be in countable discrete web
provided for every open cover $\mathcal{U}$ of $X$ there is a countable, closed and discrete
subset $A \subset X$ such that $St(A, \mathcal{U}) = X$. This property seems to be new and
interesting. It is easily seen (and pointed out in [19]) that the property of
being in countable discrete web is between being in countable web (i.e. star-
Lindelöf) and having countable extent (i.e. all closed discrete subspaces are
at most countable). Moreover, it is strictly between and, so to say, more close to countable extent than to countable web. Indeed, the examples of Tychonoff spaces in countable web which are not in countable discrete web are easy to find: the square of the Sorgenfrey line is like this \[19\], every pseudocompact \(\Psi\)-space is like this, etc. However, the examples of spaces in countable discrete web but with uncountable extent seem not so easy to be found. In \[19\], only a \(T_1\) example (similar to example 2 in \[3\]) is presented. Here we present a ZFC Tychonoff example and a consistent normal example. Below, \(D = \{0, 1\}\) is the two-point discrete space, \(c\) stands for the cardinality of continuum; the definition of the cardinal \(p\) can be found in \[4\].

**Example 1** For every cardinal \(\tau\) there is a Tychonoff space \(X\) in countable discrete web with \(e(X) = \tau\).

For every \(\alpha < \tau\) denote by \(z_\alpha\) the point in \(D^\tau\) with the \(\alpha\)-th coordinate equal to 1 and the rest of the coordinates equal to 0. Put \(Z = \{z_\alpha : \alpha < \tau\}\) and

\[
X = (D^\tau \times (\omega + 1)) \setminus ((D^\tau \setminus Z) \times \{\omega\}).
\]

Then \(\tilde{Z} = Z \times \{\omega\}\) is a closed discrete subspace of \(X\) of cardinality \(\tau\); the proof that \(X\) is in countable discrete web is not so straightforward; it is presented in section 2.

**Remarks.** 1. A Tychonoff space in countable discrete web with \(e(X) = c\) was constructed, also by Yan-Kui Song ([13], Example 3.1). However, Song’s construction can not be extended to \(\tau > c\). Another improvement, as compared with the Song’s construction, is that our space is separable if \(\tau = c\).

2. The proof of \(X\) in our Example 1 being in countable discrete web, presented in section 2, is similar to the proof of Theorem 1 in [12] where, for every cardinal \(\tau\), a pseudocompact Tychonoff space \(X\) countable web and with \(e(X) \geq \tau\) is constructed. However, the space \(X\) in Example 1 is not pseudocompact. So the following question remains open.

**Question 1** How big can be the extent of a pseudocompact Tychonoff space in countable discrete web?
Example 2  \((\omega_1 < p)\) A normal space \(S\) in countable discrete web with 
\(e(S) = c\).

A space \(X\) is called an \((a)\)-space \([11]\) provided for every open cover \(U\) 
of \(X\) and every dense subspace \(Y \subset X\) there is a closed in \(X\) and discrete 
\(A \subset Y\) such that \(St(A, U) = X\). It is clear that a separable \((a)\) space 
is in countable discrete web. Now let \(X \subset \mathbb{R}\) and let \(Y_X\) denote the space 
\((X \times \{0\}) \cup (\mathbb{R} \times (0, 1))\) endowed with the subspace topology inherited from the 
Moore-Niemytzki plane. By Theorem 5 from \([10]\), proved by Paul Szeptycki, 
\(Y_X\) has property \((a)\) as soon as \(|X| < p\). So take \(X \subset \mathbb{R}\) of cardinality \(\omega_1\) 
and put \(S = Y_X\).

Or, alternatively, take as \(S\) a \(\Psi\)-space of cardinality \(\omega_1\). Since \(|S| < p\), 
by \([17]\), \(S\) is an \((a)\)-space and thus it is in countable discrete web.

Question 2 Is there a ZFC example of a normal space in countable discrete 
web with uncountable extent?

Question 3 Is there a normal space in countable web which is not in count-
able discrete web?

A space \(X\) is called \(\delta\theta\)-refinable (see e.g. \([3]\)) if every open cover of \(X\) has 
an open refinement of the form \(\bigcup \{V_n : n \in \omega\}\) where each \(V_n\) covers \(X\) and 
for every \(x \in X\) there is \(n \in \omega\) such that \(|\{U \in V_n : x \in U\}| < \omega\). Aull has 
proved in \([1]\) that every \(\delta\theta\)-refinable space of countable extent is Lindelöf. So 
it is worth to note here that the spaces from Examples 1 and 2 above are in 
countable discrete web, \(\delta\theta\)-refinable and non-Lindelöf.

We conclude this section with one correction to \([19]\). In the cited theorem 
from \([4]\) (the first lines in the section “Preliminaries” in \([19]\) starcompactness 
is equivalent to countable compactness not only in regular \(T_1\) spaces but in 
all Hausdorff spaces as well (in \([4]\) this fact was stated without proof; a proof 
can be found in \([5]\)); here, a space \(X\) is called starcompact provided for every 
open cover \(U\) of \(X\) there is a finite subset \(A \subset X\) such that \(St(A, U) = X\).

2 The proof

Here we prove that the space \(X\) from example \([4]\) is in countable discrete web. 
All notation \((X, Z, \tilde{Z}, z_\alpha, \text{ etc.})\) is like above. We need several lemmas.
The first lemma is a weaker form of a Theorem of Fodor ([7], see also [18, Theorem 3.1.5]). Let $A$ be a set and $\lambda$ a cardinal. A set mapping of order $\lambda$ is a mapping that assigns to each $s \in A$ a subset $f(s) \subset A$ such that $|f(s)| < \lambda$ and $s \not\in f(s)$. A subset $T \subset A$ is called $f$-free if $f(t) \cap T = \emptyset$ for every $t \in T$.

**Lemma 1** Let $A$ be a set and $f$ a set mapping on $A$ of order $\omega$. Then there is a countable family $H$ of $f$-free subsets of $A$ such that $\bigcup H = A$.

**Lemma 2** For every assignment to the points $z_\alpha$ their neighbourhoods $U_\alpha$ in $D^\tau$ ($0 \leq \alpha < \tau$) there is an at most countable $S \subset D^\tau$ such that $S \cap U_\alpha \neq \emptyset$ for all $\alpha < \tau$ and $S \cap Z = \emptyset$.

**Proof:** Without loss of generality we assume that the sets $U_\alpha$ take the form $U_\alpha = \{f \in D^\tau : f(\alpha) = 1 \text{ and } f(\beta) = 0 \ \forall \beta \in f(\alpha)\}$ where $f(\alpha)$ is some finite subset of $c \setminus \{\alpha\}$. So $f$ is a set mapping on $\tau$ of order $\omega$. By Lemma 1 there is a countable family $H$ of $f$-free subsets of $\tau$ such that $\bigcup H = \tau$. Let $H = \{H_n : n \in \omega\}$. Without loss of generality we assume that $H_n \cap H_m = \emptyset$ whenever $n \neq m$ and that $|H_n| > 1$ for every $n$. Denote by $p_n$ the point in $D^\tau$ such that

$$p_n(\alpha) = \begin{cases} 1 & \text{if } \alpha \in H_n, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $p_n \in U_\alpha$ for every $\alpha \in H_n$. If, for some $n$ and $\alpha$, $|H_n| = \{\alpha\}$ then exactly one, namely the $\alpha$s, coordinate of $p_n$ equals one. In that case, redefine $p_n$ so that one more coordinate of $p_n$ equals one and still $p_n \in U_\alpha$. After having done this we have $p_n \not\in Z$ for all $n$.

Further, put $S = \{p_n : n \in \omega\}$. Then $S \cap U_\alpha \neq \emptyset$ for all $\alpha < \tau$. Further, $S \cap Z = \emptyset$ and $S$ is in fact a sequence converging to the point with all coordinates equal to zero. Therefore $S \cap Z = \emptyset$. $\square$

**Lemma 3** For every countable family $U$ of nonempty open sets in $D^\tau$ there is a way to choose points $p_U \in U$ for all $U \in U$ so that $P \cap Z = \emptyset$ where $P = \{p_U : U \in U\}$.

**Proof:** It is easy to see that in every nonempty open set $U \subset D^\tau$ one can pick a point, $p$, such that all but finitely many coordinates of $p$ are equal to 1. Pick such a point $p_U$ in every $U \in U$. Put $A_U = \{\gamma < c : p_U(\gamma) = 0\}$ and $A = \bigcup \{A_U : U \in U\}$. Let $\alpha < \tau$. Pick $\beta \in \tau \setminus (A \cup \{\alpha\})$. Put
\[ O_\alpha = \{ f \in D^c : f(\beta) = 0 \text{ and } f(\alpha) = 1 \}. \] Then \( O_\alpha \) is a neighbourhood of \( z_\alpha \) in \( D^r \) and \( O_\alpha \cap \mathcal{P} = \emptyset \). So \( \mathcal{P} \cap Z = \emptyset \). \( \square \)

Now let \( \mathcal{V} \) be an open cover of the space \( X \) from example \([1]\). For every \( n \in \omega \) put \( \mathcal{V}_n = \{ V \cap (D^r \times \{n\}) : V \in \mathcal{V} \} \). Then \( \mathcal{V}_n \) is an open cover of \( D^r \times \{n\} \) and thus it has a finite subcover consisting of nonempty sets, say \( \mathcal{V}_n \).

Put \( \widetilde{\mathcal{V}}_n = \{ U : U \times \{n\} \in \mathcal{V}_n \} \) and \( \mathcal{U} = \cup\{ \mathcal{V}_n : n \in \omega \} \). Then \( \mathcal{U} \) is a countable family of nonempty open sets in \( D^r \). So let \( \{p_U : U \in \mathcal{U}\} \) be like in Lemma \([3]\). For each \( n \in \omega \) put \( Q_n = \{ (p_U, n) : U \times \{n\} \in \mathcal{V}_n \} \) and put \( Q = \cup\{ Q_n : n \in \omega \} \). It is clear that \( \pi_1(Q) = P \) where \( \pi_1 : D^r \times (\omega+1) \to D^r \) is the projection of the product onto the first factor. Now, since \( P \cap Z = \emptyset \) we have \( \overline{Q} \cap \overline{Z} = \emptyset \). Since \( |Q \cap (D^r \times \{n\})| < \omega \) for every \( n \in \omega \) it follows that \( Q \) is discrete and closed in \( X \). On the other hand, \( Q \) is countable and \( St(Q, \mathcal{V}) \supset D^r \times \omega \). It remains to find another countable closed and discrete set \( R \subset X \) such that \( \overline{R} \cap \overline{Z} = \emptyset \) and \( St(R, \mathcal{V}) \supset \overline{Z} \). For every \( \alpha \in \tau \) choose \( O_\alpha \in \mathcal{V} \) such that \( (z_\alpha, \omega) \in O_\alpha \). Also choose \( U_\alpha \) open in \( D^r \) and \( n_\alpha \in \omega \) so that \( U_\alpha \times [n_\alpha, \omega] \subset O_\alpha \). Then the sets \( U_\alpha \) are like in Lemma \([2]\) so let \( S \subset D^r \) also be like in Lemma \([3]\). Enumerate \( S \) as \( S = \{ s_k : k \in \omega \} \). For every \( n \in \omega \) put \( R_n = \{ (s_k, n) : k \leq n \} \). Last, put \( R = \cup\{ R_n : n \in \omega \} \). It is clear that \( \pi_1(R) = S \), so \( \overline{R} \cap \overline{Z} = \emptyset \). Again, since \( |R \cap (D^r \times \{n\})| < \omega \) for every \( n \in \omega \) it follows that \( R \) is discrete and closed in \( X \). Let \( \alpha \in \tau \). Then \( s_k \in U_\alpha \) for some \( k \in \omega \). Put \( n = \max\{k, n_\alpha\} \). Then \( (s_k, n) \in O_\alpha \). On the other hand, \( (s_k, n) \in R \), so \( (z_\alpha, \omega) \in St(R, \mathcal{V}) \). \( \square \)

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