Abstract. The fundamental theorem of complex interpolation is

**Boundedness Theorem:** If, for \( j = 0, 1 \), a linear operator \( T \) is a bounded map from the Banach space \( X_j \) to the Banach space \( Y_j \) then, for each \( \theta, 0 < \theta < 1 \), \( T \) is a bounded map between the complex interpolation spaces \([X_0, X_1]_\theta \) and \([Y_0, Y_1]_\theta \).

Alberto Calderón, in his foundational presentation of this material fifty-one years ago [5], also proved the following companion result:

**Compactness Theorem/Question:** Furthermore in some cases, if \( T \) is also a compact map from \( X_0 \) to \( Y_0 \), then, for each \( \theta \), \( T \) is a compact map from \([X_0, X_1]_\theta \) to \([Y_0, Y_1]_\theta \).

The fundamental question of exactly which cases could be covered by such a result was not resolved then, and is still not resolved.

The paper [20], which is the focus of this commentary, is a contribution to that question. We will not summarize in any detail here the contents of [20] or of related works. (Some of that may be done later in [27].) Rather we will take this opportunity to sketch the mathematical world, historical and current, in which that paper lives. We will see that there have been many very talented contributors and many fine contributions; however the core problem remains open. We will at least be able to announce some small new partial results.

This paper is the fourth (and, together with its technical sequel [27], probably the last) in a series [24, 25, 26] which we have recently written (the earlier ones in collaboration with Mario Milman) to survey and discuss some small parts of the very extensive and very impressive body of research created by our brilliant colleague and dear friend Nigel Kalton. We have also submitted material from these papers to possibly appear in a forthcoming “Selecta” volume to be published in memory of Nigel [1].

The topic which we are about to discuss here has an extensive and varied bibliography. We will certainly not be able to do full justice to the contributions of all who have worked on it and on other closely related topics. We are quite possibly not aware of some of these contributions. We apologize in advance for all omissions.

We are going to be dealing with lots of different Banach spaces, including \( L^p \) spaces. Let us specify at the outset that all of them will be over the complex field. Thus, also, the terminology Banach couple (see below), will always refer here to a couple of complex Banach spaces. To avoid any ambiguity, we should also mention that, for any Banach space \( X \),
we will be using the notation $\ell^p(X)$ or $c_0(X)$ to denote the Banach space, equipped with its natural norm, of all two-sided $X$ valued sequences $\{x_n\}_{n \in \mathbb{Z}}$ for which the two-sided numerical sequence $\{\|x_n\|\}_{n \in \mathbb{Z}}$ is in $\ell^p$ or $c_0$ respectively.

We have begun writing a companion paper or set of lecture notes [27], which is intended to be a sequel to this commentary. We plan to post a preliminary version of it on the arXiv concurrently with this document. It will deal with issues which constraints of space and time have prevented us from including here. It will contain some minor new results, and discussions of some technical aspects of [20], including, for example, some more explicit details for the proof of the result of that paper dealing with arbitrary couples of Banach lattices.

1. The history of the problem before the writing of [20]

Our story begins with the classical Riesz-Thorin Theorem. According to that theorem, any linear operator $T : L^{p_0} + L^{p_1} \to L^{q_0} + L^{q_1}$ which is a bounded map from $L^{p_0}$ into $L^{q_0}$ and also from $L^{p_1}$ into $L^{q_1}$ must also be a bounded map from $L^p$ into $L^q$ whenever the two exponents $p$ and $q$ are related by the convexity formulae

$$1/p = (1 - \theta)/p_0 + \theta/p_1 \quad \text{and} \quad 1/q = (1 - \theta)/q_0 + \theta/q_1$$

for some $\theta \in (0, 1)$. In 1926 Marcel Riesz gave the first proof of a somewhat limited version of this theorem. (One could claim that he was more or less “forced” to obtain this theorem after he found a clever trick to show that the Hilbert transform is bounded on $L^p$, but only when $p$ is an even integer.) In 1939 Olof Thorin published a quite different new proof of the complete version, using holomorphic $L^p$ valued functions, a device which John Edensor Littlewood once described as “the most impudent idea in mathematics”. (Even those of us who have the greatest admiration for this wonderful proof might feel obliged to concede that Littlewood was perhaps just a little too carried away when he wrote that.)

In 1960 Mark Krasnosel’skii investigated the interplay of compactness with the Riesz-Thorin Theorem. He showed [33] that if the above-mentioned operator $T$ has the additional property that it maps $L^{p_0}$ compactly into $L^{q_0}$, then it also maps $L^p$ compactly into $L^q$.

In the mid 1960’s Selim Krein, Jacques-Louis Lions and Alberto Calderón independently found ways to upgrade Thorin’s “impudence” from the setting of $L^p$ spaces into the setting of Banach couples $^2$, i.e., general pairs $(X_0, X_1)$ of Banach spaces $X_0$ and $X_1$ which are both continuously embedded in some Hausdorff topological vector space.

The theory developed by Alberto Calderón had a somewhat wider scope than those of Krein and Lions. As described in [5] (and also in many other papers and monographs, including [31, 33, 32, etc.] for each Banach couple $(X_0, X_1)$, Calderón’s construction uses certain holomorphic $X_0 + X_1$ valued functions to obtain, for each $\theta \in (0, 1)$, a Banach space, usually denoted by $[X_0, X_1]_\theta$ and referred to as a complex interpolation space. This space almost automatically has the following Riesz-Thorin-like interpolation property: Given any two Banach couples $(X_0, X_1)$ and $(Y_0, Y_1)$, any linear operator $T : X_0 + X_1 \to Y_0 + Y_1$ whose restriction to $X_j$ is a bounded map $T : X_j \to Y_j$ for $j = 0, 1$ is also a bounded map from $[X_0, X_1]_\theta$ to $[Y_0, Y_1]_\theta$ for each $\theta \in (0, 1)$.

$^2$Some papers use the terminology interpolation pair or Banach pair, instead of Banach couple.
The first example of a Banach couple that comes to mind is a pair of $L^p$ spaces, say $(L^{p_0}, L^{p_1})$. (Of course they must both have the same underlying measure space.) Calderón showed that

\[(L^{p_0}, L^{p_1})_{\theta} = L^p,\]

where $p$, $p_0$, $p_1$ and $\theta$ are connected by the first of the above two convexity formulae (1.1). We refer e.g. to [3] pp. 106–107 for a quick direct proof of (1.2) which is essentially a rewriting of Thorin’s proof of the Riesz-Thorin theorem. In Calderón’s original presentation of these matters [5], this result emerged as a particular example (using Section 13.3 of [5]) of his simple formula identifying $[X_0, X_1]_{\theta}$ as the “product” $X_0^{1-\theta}X_1^\theta$ (Section 13.6 pp. 124–125) in the case where $(X_0, X_1)$ is a lattice couple, i.e., when $X_0$ and $X_1$ are Banach lattices of complex valued measurable functions on the same measure space. Some of the main ideas for the proof of this characterization formula (Section 33.6 pp. 171–180), also came naturally from Thorin’s proof of the Riesz-Thorin Theorem.

Although it is of course explicitly defined, the process for obtaining $[X_0, X_1]_{\theta}$ from $X_0$ and $X_1$ when these two spaces are not $L^p$ spaces, or do not have some kind of compatible lattice structure, works in ways which seem very mysterious. These ways bewildered even Alberto Calderón himself, as he once “confessed” to Mitchell Taibleson. Some partial compensation for this mystery would be found some decades later when Svante Janson, motivated by some ideas of Jaak Peetre, revealed, among many other interesting results, that the complex interpolation method is a special concrete case, involving Fourier series, of an abstract interpolation method going back to Aronszajn and Gagliardo.

Calderón’s remarkable exposition [5] of his complex interpolation method contains many “gems”. They include a beautiful theorem (in Sections 12.1, pp 121 and 32.1, pp. 148–156) characterizing the dual space of $[X_0, X_1]_{\theta}$ (cf. a slightly different way of presenting it in [13]) and, in its deeper depths (in Sections 34.1 and 34.2, pp. 180–188), is the development and application of the formula still referred to as the “Calderón Reproducing Formula” which is one of the forerunners of modern wavelet analysis, where the formula also survives under the name “continuous wavelet transform”.

But among those “gems”, the one that mainly concerns us here is the following generalization (implicit in Sections 9.4 p. 118 and 29.4, pp. 137–138) of Krasnosel’skii’s compactness theorem: If the Banach couple $(Y_0, Y_1)$ satisfies a certain “natural” condition, and if the above-mentioned linear operator $T$ which maps the space $X_j$ to $Y_j$ boundedly for $j = 0$ and $j = 1$ is also compact for at least one of these values of $j$, say $j = 0$, then $T$ is also compact as a map from $[X_0, X_1]_{\theta}$ into $[Y_0, Y_1]_{\theta}$ for each $\theta \in (0, 1)$. The “natural” condition

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3In fact, for the formula $[X_0, X_1]_{\theta} = X_0^{1-\theta}X_1^\theta$ to hold, the lattices $X_0$ and $X_1$ have to satisfy a certain mild condition which we shall not describe here. In fact Calderón obtains a rather more general “vector-valued” version of this formula where the measurable functions take values in Banach spaces.

4In November 1999, immediately after one of us gave a lecture at Washington University about our forthcoming joint paper [21] with Nigel and Mario Milman, Mitch spontaneously rose to his feet and gave us all a touching account of a stroll he had taken in Hyde Park, Chicago with Alberto Calderón in the 1960s, just after Alberto had given a lecture about the exciting new interpolation method that he had just developed. This was the occasion on which Alberto made the “confession” mentioned above.

5Here, and also elsewhere, we need to refer to two different sections of [5] because the format of that paper is usually that a result is announced in Section $N$ and its proof is given in Section $N + 20$. 

which Calderón had to impose in this theorem is, roughly speaking, that $Y_0$ can be “ap-
proximated” by some sequence or net of its finite dimensional subspaces in a way which is
suitably consistent with $Y_1$.

To streamline the rest of this discussion, we will generally use the following notation
(cf. [12] p. 22): Given two Banach couples $(X_0, X_1)$ and $(Y_0, Y_1)$, we may write
\[(1.3) \quad (X_0, X_1) \triangleright (Y_0, Y_1)\]
to mean that $T : [X_0, X_1]_\theta \to [Y_0, Y_1]_\theta$ is compact for every $\theta \in (0, 1)$ whenever $T : X_0 + X_1 \to Y_0 + Y_1$ is a linear operator for which $T : X_0 \to Y_0$ compactly and $T : X_1 \to Y_1$ boundedly. Thus, for example, Krasnosel’skii’s theorem can be expressed by writing $(L^{p_0}, L^{p_1}) \triangleright (L^{p_0}, L^{p_1})$.

We may then also use the notation
\[(\ast, \ast) \triangleright (Y_0, Y_1)\]
to mean that $(Y_0, Y_1)$ is a Banach couple with the property that $(X_0, X_1) \triangleright (Y_0, Y_1)$ for all Banach couples $(X_0, X_1)$. Thus, Calderón’s result in [5] can be expressed by stating that $(\ast, \ast) \triangleright (Y_0, Y_1)$ for every Banach couple $(Y_0, Y_1)$ which has the particular approximation property mentioned above.

Analogously it will sometimes be convenient to write $(X_0, X_1) \triangleright (\ast, \ast)$ to mean that $(X_0, X_1)$ is a Banach couple with the property that $(X_0, X_1) \triangleright (Y_0, Y_1)$ for all Banach couples $(Y_0, Y_1)$. Finally, we can use the notation $(\ast, \ast) \triangleright (\ast, \ast)$ as shorthand for the (thus far unconfirmed) statement that the following question has an affirmative answer.

**Question CIC:** Does the property $(X_0, X_1) \triangleright (Y_0, Y_1)$ hold for all Banach couples $(X_0, X_1)$ and $(Y_0, Y_1)$?  

Calderón was evidently not able to answer this question. But then no one else has been able to do this either, in the more than half a century since Calderón submitted his paper [5] to Studia Mathematica. Quite a number of us have made valiant efforts to do so, and some of the world’s best mathematicians, (including Alberto Calderón himself\(^6\)) have been invited to consider (or reconsider!) this problem. Several of them, including Nigel Kalton, have expressed the opinion that the answer to Question CIC is “no”. But where is the counterexample?

**Remark 1.1.** One would expect it to be easier to obtain an affirmative answer to the variant of Question CIC for “two-sided compactness”, i.e., where instead of showing that $(X_0, X_1) \triangleright (Y_0, Y_1)$ holds, one only seeks to show that a weaker property holds, in which $T : [X_0, X_1]_\theta \to [Y_0, Y_1]_\theta$ is required to be compact, only for those operators $T$ which satisfy the additional condition that $T : X_1 \to Y_1$ is also compact. But this supposedly easier question is also still open.

Quite a number of partial results have been obtained during this said half century. Let us here briefly mention a few\(^7\) of them which preceded the particular paper which we have been asked to discuss. (Some other more recent results will be mentioned later.)

\(^6\)CIC=Complex interpolation of compactness.
\(^7\)He discussed it briefly with one of us during a visit to the Technion in 1989.
\(^8\)Please keep in mind the apology at the beginning of this document.
Arne Persson [37] proved a variant of Calderón’s result, namely that \((*,*) \triangleright (Y_0, Y_1)\) holds also when \((Y_0, Y_1)\) satisfies a kind of approximation property which is somewhat different from that imposed by Calderón. A remarkable paper [11] by Fernando Cobos and Jaak Peetre, even though it did not directly deal with this question, injected some powerful new ideas into the game. Helped by those ideas and by others in [30], one of us [11] was able to show, essentially, that the problem of answering Question CIC is equivalent to determining whether or not \((X_0, X_1) \triangleright (Y_0, Y_1)\) holds for just one special particular choice of the couples \((X_0, X_1)\) and \((Y_0, Y_1)\).

In order to prepare for a precise formulation of this result, we first need to recall the definitions of the Banach sequence spaces \(FL_p\) and their weighted counterparts \(FL^p_\alpha\) (sometimes also denoted by \(FL_p(e^{\alpha n})\)). For each \(p \in [1, \infty]\), the space \(FL_p\) consists of all sequences \(\{\lambda_n\}_{n \in \mathbb{Z}}\) of complex numbers which arise as the Fourier coefficients of some function \(f : \mathbb{T} \to \mathbb{C}\) in \(L^p(\mathbb{T})\). Then, for each \(\alpha \in \mathbb{R}\), \(FL^\alpha_p\) consists of all sequences \(\{\lambda_n\}_{n \in \mathbb{Z}}\) for which \(\{e^{\alpha n}\lambda_n\}_{n \in \mathbb{Z}}\) is an element of \(FL_p\). Your first guess as to how these spaces are normed will be correct. These spaces, for \(p = 1\) and \(p = \infty\), play crucial roles in Svante Janson’s important alternative characterisations [30] of complex interpolation spaces, which we mentioned above. For later purposes we should also mention the subspaces \(FC\) and \(FC^\alpha\) of \(FL_\alpha\) and \(FL^\alpha_\infty\) which are obtained when \(L_\infty\) is replaced by its subspace of continuous functions.

We can now state the above-mentioned result of [11] explicitly:

The answer to Question CIC is affirmative if and only if the property

\[
(\ell^1(FL^1_\alpha), \ell^1(FL^1_\alpha)) \triangleright (\ell^\infty(FL^\infty_\alpha), \ell^\infty(FL^\infty_\alpha))
\]

holds.

(A more explicit and detailed proof of this equivalence would appear later in [22], pp. 355–358.)

Although [11] and Question CIC remained unresolved in [11], the interplay between them led to another result: Without imposing any approximation assumption on \((Y_0, Y_1)\), even if compactness perhaps does not “interpolate”, then at least it “extrapolates”. I.e., if \(T : X_j \to Y_j\) is bounded for \(j = 0, 1\) and if \(T : [X_0, X_1]_\theta \to [Y_0, Y_1]_\theta\) is compact for just one value of \(\theta\) in \((0, 1)\), then it is compact for all \(\theta \in (0, 1)\). A different proof of this result was immediately obtained by Fernando Cobos, Thomas Kühn and Tomas Schonbek [11]. They also proved that \((X_0, X_1) \triangleright (Y_0, Y_1)\) holds, without the requirement of Calderón’s or Persson’s approximation hypotheses, in the case where \((X_0, X_1)\) and \((Y_0, Y_1)\) are both lattice couples (subject to some mild additional conditions).

Of course, more or less in parallel with the development of the complex interpolation method, another important method, the so-called real interpolation method, was also developed in the 1960’s, one of the main steps in that process being the work [35] of Jacques–Louis Lions and Jaak Peetre. It was natural to ask a question analogous to Question CIC, where, instead of the spaces \([X_0, X_1]_\theta\) and \([Y_0, Y_1]_\theta\) one considers the Lions–Peetre spaces generated by this method, and which are usually denoted by \((X_0, X_1)_{\theta, p}\) and \((Y_0, Y_1)_{\theta, p}\). The answer to this question was also not immediately evident. Affirmative answers for some of its special cases finally led the way to an affirmative answer for its general form. This can in fact be found in the same papers [9] and [11] which we mentioned just above in

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9This is implicit in Step 1 of the proof of Theorem 2.1 on pp. 339-340 of [11].
connection with Question CIC. Some sequels to that answer will be briefly discussed below in Section 4.

2. The paper [20] itself

The second named author of this document once good-naturedly claimed that the first named author is afflicted with a “disease” of wrestling with Question CIC, and from time to time he “infests” others with the same “disease”. The first named author does not deny this claim, even if in his older years the “disease” has become somewhat milder. Perhaps his greatest success was that he “infected” Nigel Kalton, at least temporarily, and this led Nigel to make very substantial contributions in [20]. Of course Nigel’s extremely broad and deep and intensive interests and activities in so many topics would guarantee him immunity from succumbing to this disease for any period significantly beyond the time during which he worked with the first named author on [20].

Here would be the natural place in this document for a summary of the main results and techniques of [20], including a list of the various (quite large) classes of Banach couples \((X_0, X_1)\) and \((Y_0, Y_1)\), not previously investigated, for which it is shown there that \((X_0, X_1) \triangleright (Y_0, Y_1)\) holds. However the paper speaks quite well for itself in these matters, and exactly that summary can be seen on page 262 of [20] (or on page 2 of the preliminary arXiv version).

Let us mention that one of the results of [20], namely that

\[
(X_0, X_1) \triangleright (\ast, \ast) \quad \text{for every lattice couple} \quad (X_0, X_1),
\]

is obtained via an interplay (see pages 269–270 of [20]) with a different interpolation method which probably deserves to be better and more widely known. It is often called the “plus-minus” method\(^{10}\). It comes in two “flavors”. In its original version this method was invented by Jaak Peetre\(^{10}\) and, in a modified version which was motivated by applications to Orlicz spaces, it first appeared in a joint paper [28] of Jan Gustavsson and Jaak. We intend to mention some more details about this method and the way it is used in [20] and beyond in a forthcoming second more expanded version of [27].

As already mentioned, Nigel believed that ultimately a counterexample will show that \((X_0, X_1) \triangleright (Y_0, Y_1)\) does not hold in general. In parallel with his gentle and modest ways, he was very ambitious, maybe even fiercely ambitious, and hoped very much for the discovery of that counterexample. One of his wishes was that it would show that the results of [20] were just about as strong as one could hope for. Well, in the almost 20 years since the appearance of [20], as far as we know, only a small number of additional partial answers to Question CIC have emerged. Some of them will be described below. One could interpret this as hinting that perhaps Nigel’s wish may yet be granted, more or less.

It is quite difficult by now to recall very much of the conversations and ideas that were in the air during the writing of [20], beyond those which appear explicitly in the paper. A pity, because perhaps they could yet be useful. Nigel liked to refer to some steps of some arguments as “the method of gliding humps”. Could that mental picture be refined to guide us further?

\(^{10}\)The reason for this name comes from a property of (weakly) unconditionally convergent series which is mentioned, for example, on line 20 of [30, p. 58].
It certainly seems worth going back to look again at many details of [20]. This is another one of the things we plan to do in the future expanded version of [27]. There is considerably more to be commented upon than could reasonably fit into the limited framework of this document.

3. Some developments since then

3.1. Some subsequently discovered additional cases where \((X_0, X_1) \succ (Y_0, Y_1)\) holds. Here, as far as we know, are all the examples of couples \((X_0, X_1)\) and \((Y_0, Y_1)\) which have been shown to satisfy \((X_0, X_1) \succ (Y_0, Y_1)\) since the appearance of [20].

In 2007 it was shown in [19] that \((*,*) \succ (Fl_\infty, Fl_\infty^1)\). Perhaps the main interest of this result lies in the hope that it could maybe be some kind of first step towards showing that (1.4) holds, and thus resolving Question CIC completely.

It was particularly pleasing in that paper (and also in [18]) to see Svante Janson returning to this field, when we recall that his impressive work from some decades earlier (notably in the above-mentioned paper [30]) is exceptionally important in interpolation theory. We are also grateful to him for several very helpful conversations in the intervening years.

In 2010 it was shown [12] that \((*,*) \succ (Y_0, Y_1)\) holds whenever \((Y_0, Y_1)\) is a lattice couple for which the underlying measure space is \(\sigma\)-finite and for which some other mild condition is satisfied. This result was obtained, roughly speaking, by taking a suitable variant of an adjoint operator and applying a suitable variant of Schauder’s classical theorem about compact operators to the above-mentioned result (2.1) of [20] (for which (as we intend to explain in more detail in the future expanded version of [27]) there are no extra conditions required on the lattice couple).

In the light of this result and of Theorem 11 of [20, pp. 274–275], one is tempted to think that taking adjoints and using Schauder’s theorem provide an immediate and obvious way of obtaining a few more partial new results. I.e., it seems reasonable to conjecture that \((X_0, X_1) \succ (Y_0, Y_1)\) if and only if their dual couples satisfy \((Y'_0, Y'_1) \succ (X'_0, X'_1)\). At this stage we only know how to show half of this conjecture, namely that

\[(3.1) \quad (Y'_0, Y'_1) \succ (X'_0, X'_1) \implies (X_0, X_1) \succ (Y_0, Y_1)\]

for any two regular couples \((X_0, X_1)\) and \((Y_0, Y_1)\). (See Section 2 of the current preliminary version of [27] for details.) This general fact seems to have been somehow overlooked till now, though it was shown in one special setting in the proof of Theorem 11 of [20, pp. 274–275]. The reverse implication of (3.1) eludes us so far because, although all bounded linear operators have adjoints, they do not always have “pre-adjoints”. In fact, whatever difficulty exists here is no smaller than the difficulty of completely answering Question CIC. Why? Recalling the definitions of the spaces \(Fl^1, Fl^1_1, FC\) and \(FC_1\) above, a few lines before (1.1), let us note that Calderón’s original result is sufficient to establish that \((l^1(FL^1), l^1(FL^1_1)) \succ (c_0(FC), c_0(FC_1))\). If the reverse implication of (3.1) holds, then we can almost immediately deduce that (1.4) holds, which is equivalent to answering “yes” to Question CIC.

One immediate consequence of (3.1) is a new “sister” result for Theorem 9 of [20, p. 271]. That theorem told us that \((X_0, X_1) \succ (Y_0, Y_1)\) holds whenever \(X_0\) is a UMD-space. Now
we know that this holds also whenever $Y_0$ is a UMD-space. (Again we refer to Section 2 of the current version of [27] for more details.)

Some as yet unpublished results show that $(\ast, \ast) \triangleright (E_{n,0}, E_{n,1})$ for every $n \in \mathbb{N}$ for some special particular couples $(E_{n,0}, E_{n,1})$, which form a sequence which “converges” in some rather weak sense to the couple which we would so like to have in place of them, namely $(\ell^\infty (FL^\infty), \ell^\infty (FL_1^\infty))$. The elements in dense subclasses of all the spaces in all of these couples can be conveniently visualized as sequences of holomorphic functions of a complex variable on the “unit annulus” $\{ z \in \mathbb{C} : 1 \leq |z| \leq 1 \}$. Complex interpolation requires us to consider holomorphic functions of these functions. Thus we are dealing with sequences of holomorphic functions of two complex variables.

3.2. Other “post-[20]” results related to Question CIC. Soon after the appearance of [20], since Question CIC had been defying attempts to answer it for so long, one of us, in collaboration with Natan Kruglyak and Mieczysław Mastlo found it to be reasonable to publish a “toolbox” paper [22] identifying various special cases of Question CIC, whose resolution would suffice to give a complete answer. An informal sequel to that paper was also posted on the internet ([23]). Subsequently, a remarkable calculation by Fedja Nazarov (in more formal settings we should address him as Fedor) the details of which are available at [15], hinted that perhaps there might be a counterexample to a question which is closely related to Question CIC, which is formulated as Question 2 on p. 362 of [22]. (See also [23] or [18, p. 168].) The result of Fedja’s calculation could also be interpreted as a first vague hint that, even if the answer to Question CIC is affirmative, there perhaps cannot be a generally valid quantitative version, in terms of entropy numbers or covering numbers, of the property $(\ast, \ast) \triangleright (\ast, \ast)$.

It was remarked on p. 358 of [22] that one can readily show that

\[
(\ell^\infty (FL^\infty), \ell^\infty (FL_1^\infty) \triangleright (\ell^1 (FL^1), \ell^1 (FL_1^1)))
\]

via some results of Bill Johnson and Grothendieck. But (3.2) also follows from Theorem 11 of [20, pp. 273–274]. (Note that (3.2) is a tantalizing “mirror image” of the result (1.4) that would completely answer Question CIC.)

Some years later, some other authors also provided some additional tools which might ultimately be helpful in further investigations of Question CIC:

In 2000, Tomasz Schonbek offered several interesting insights in his paper [40]. He provided a more unified treatment of results of [20] and of other papers, clarifying the central role which the property of equicontinuity plays in many of them. For example, his Theorem 3.5 on p. 1241 of [40] gave an attractive alternative way of obtaining the result of Theorem 10 of [20, p. 272]. In Section 2 of his paper he indicated a possible role for a variant of the Radon-Nikodým Theorem. We remark that, related to this, his Theorem 2.2 (p. 1234) might perhaps turn out, in some contexts, to be able to replace or reinforce the role played by Lemma 1 of [20, p. 264].

In 2004, in their Theorem 3.2 of [8, p. 71], Fernando Cobos, Luz M. Fernández-Cabrera and Antón Martínez gave necessary and sufficient conditions, somewhat in the style of

\[\text{Note, however, that at least one alternative strategy for trying to answer Question CIC explicitly dispenses with using equicontinuity. See Proposition 4 and the discussion preceding it on p. 359 of [22].}\]
Lemma 5 of [20, p. 268] for a bounded operator $T : [A_0, A_1]_\theta \to [B_0, B_1]_\theta$ to be compact. (Their paper also dealt extensively with related issues for other interpolation methods.)

In 2006, Svante Janson and one of us [18] briefly reviewed some old and new possible strategies for dealing with Question CIC. In particular, Section 3, pp. 166–168 of that paper contained some observations about possible refinements, for this purpose, of the method of “infinite matrices of infinite matrices” that was used in [11] Section 2, pp. 339–343 to prove the “extrapolation of compactness” result mentioned here above, near the end of Section 1.

One year later Nigel, collaborating with Svitlana Mayboroda and Marius Mitrea, obtained some interesting variants of that same earlier “extrapolation of compactness” result. In Section 10 of [31, pp. 160-163] they ventured, as Nigel had so significantly done on so many other occasions, beyond Banach spaces, into the much less “comfortable” setting of quasi-Banach spaces. They could deal with the cases where these spaces are Besov or Triebel-Lizorkin spaces, and they applied their results to partial differential equations.

Finally we mention a much more recent paper of Jürgen Voigt [43]. He considered the cases where the couples $(X_0, X_1)$ and $(Y_0, Y_1)$ satisfy either $X_0 = X_1$ or $Y_0 = Y_1$. In such cases one immediately has $(X_0, X_1) \succ (Y_0, Y_1)$ because of some classical results of Lions-Peetre. However the novelty of the results in [43] is that, instead of dealing with a single operator $T : X_0 + X_1 \to Y_0 + Y_1$, they treat an operator valued holomorphic function $T(z)$ defined for all $z$ in the strip $\{z \in \mathbb{C} : 0 \leq \text{Re} z \leq 1\}$ for which $T(it) : X_0 \to Y_0$ is compact for all $t$ in some subset of $\mathbb{R}$ having positive measure. (Of course operator valued holomorphic functions, often also referred to as analytic families of operators, have had many interactions with complex interpolation theory, ever since they appeared in various special cases in [29] and in [11].)

4. SOME RELATED RESULTS, ALSO FOR OTHER INTERPOLATION METHODS, AND ALSO FOR NONLINEAR OPERATORS

It seems appropriate to say at least a few words about the large body of results dealing with compact operators in the parallel contexts of other interpolation methods. Fortunately we can refer to an extensive and detailed survey [7] of many such results, written by Fernando Cobos. (We are also grateful to Fernando for his many other energetic initiatives to encourage research and interaction in interpolation theory including but also well beyond issues of compactness, via various conferences and meetings, and of course via his own numerous results.)

We recall that, as already mentioned above, the papers [9] and [11] provide an affirmative answer for the analogue of Question CIC for the Lions-Peetre real method of interpolation, and that some special cases of this result had been known long before. Once such qualitative compactness results have been obtained, it becomes natural to seek quantitative refinements of them. Typically these are expressed in terms of entropy numbers, or the measure of non-compactness. Such results are amply described in [7]. Of course there is a qualitative compactness result for the complex method too, namely Calderón’s seminal result, provided we remain within the setting where the range couple $(Y_0, Y_1)$ satisfies Calderón’s approximation condition. Very recently, Radosław Szwedek [39] has obtained quantitative results in this setting.
It is sometimes possible to obtain interpolation of compactness results which apply simultaneously to a wide class of different interpolation methods. Evgeniy Pustynik [38] has done this when the range couple \((Y_0, Y_1)\) is a lattice couple having certain extra properties. His results have some overlap with [12]. Similar issues are addressed by Yuri Brudnyi in [4].

Finally let us briefly discuss interpolation of compact nonlinear operators by the real method. We confess to doing this mainly to give us an opportunity to recall a special moment shared with Nigel. An early paper [6] about this topic was written by Fernando Cobos. It showed that, in some special cases, the affirmative answer to the analogue of Question CIC for the real method of interpolation (then already known from a preprint) could also be extended to the case of nonlinear operators satisfying suitable properties, such as Lipschitz conditions. We make no attempt to survey other research on this topic, but only mention that when one of us met Nigel in Adelaide in 2008 we casually started telling him about some joint work [17] extending the results of [6] and taking account of some counterexamples we had found in [16]. Nigel, being Nigel, instantaneously understood all factors in play, and immediately guessed what condition we had needed to impose in order to obtain our desired result.

5. Anyway, why should we care about Question CIC?

Of course we are attracted to Question CIC, as we are to other open problems, in large part because of the fun and challenge of trying to solve them, just as others are attracted to climbing Mount Everest. However there seem to be some other good reasons for working on Question CIC.

Allowing ourselves to be a little “carried away”, we can cautiously claim that even the partial results obtained by Nigel and his coauthor in [20] might ultimately be relevant for confronting energy crises. As evidence for this, we can consider the research by Sören Bartels, Max Jensen and Rüdiger Müller, which is presented in [11] and then more formally in [2]. It deals with numerical solutions within a mathematical model relevant for processes which extract oil from those underground reservoirs where it is mixed with other fluids, and hints at the economic importance of better understanding of such processes. Suddenly, in the middle of both of these documents, there is an unexpected need for a tool from interpolation theory, and Theorem 11 of [20, p. 273–274] fulfills that need exactly. Nigel’s coauthor saw fit to inform these researchers that, in their particular setting, the seminal theorem of Alberto Calderón, mentioned above ([5] Sections 9.4 p. 118 and 29.4, pp. 137–138) would also suffice for their purposes. One could also use results from the real interpolation method.

Returning to the realm of “pure” or “purer” mathematics, we can remark that there are references to [20] in papers dealing with such topics as compact Weyl operators, sub-Laplacians of holomorphic \(L^p\) type on exponential solvable groups, and boundary value problems for Waldenfels operators.

It could be argued that by now the answer to Question CIC is known for all cases where the spaces \(Y_0\) and \(Y_1\) are “reasonable”. But mathematics and also its applications, sometime have a tendency to move beyond what was previously considered “reasonable”.

\[12\]But note that Fernando does not discuss nonlinear operators in [7].
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