THE NORM OF A TRUNCATED TOEPLITZ OPERATOR

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Abstract. We prove several lower bounds for the norm of a truncated Toeplitz operator and obtain a curious relationship between the $H^2$ and $H^\infty$ norms of functions in model spaces.

1. Introduction

In this paper, we continue the discussion initiated in [6] concerning the norm of a truncated Toeplitz operator. In the following, let $H^2$ denote the classical Hardy space of the open unit disk $\mathbb{D}$ and $K_\Theta := H^2 \cap (\Theta H^2)^\perp$, where $\Theta$ is an inner function, denote one of the so-called Jordan model spaces [2, 4, 7]. If $H^\infty$ is the set of all bounded analytic functions on $\mathbb{D}$, the space $K_\Theta^\infty := H^\infty \cap K_\Theta$ is norm dense in $K_\Theta$ (see [2, p. 83] or [9, Lem. 2.3]). If $P_\Theta$ is the orthogonal projection from $L^2 := L^2(\partial\mathbb{D}, \frac{|d\zeta|}{2\pi})$ onto $K_\Theta$ and $\varphi \in L^2$, then the operator

$$A_\varphi f := P_\Theta(\varphi f), \quad f \in K_\Theta^\infty,$$

is densely defined on $K_\Theta$ and is called a truncated Toeplitz operator. Various aspects of these operators were studied in [3, 5, 6, 9, 10].

If $\| \cdot \|$ is the norm on $L^2$, we let

$$\|A_\varphi\| := \sup\{\|A_\varphi f\| : f \in K_\Theta^\infty, \|f\| = 1\}$$

and note that this quantity is finite if and only if $A_\varphi$ extends to a bounded operator on $K_\Theta$. When $\varphi \in L^\infty$, the set of bounded measurable functions on $\partial\mathbb{D}$, we have the basic estimates

$$0 \leq \|A_\varphi\| \leq \|\varphi\|_\infty.$$

However, it is known that equality can hold, in nontrivial ways, in either of the inequalities above and hence finding the norm of a truncated Toeplitz operator can be difficult. Furthermore, it turns out that there are many unbounded symbols $\varphi \in L^2$ which yield bounded operators $A_\varphi$. Unlike the situation for classical Toeplitz operators on $H^2$, for a given $\varphi \in L^2$, there many $\psi \in L^2$ for which $A_\varphi = A_\psi$ [9 Thm. 3.1].

For a given symbol $\varphi \in L^2$ and inner function $\Theta$, lower bounds on the quantity (1) are useful in answering the following nontrivial questions:

(i) is $A_\varphi$ unbounded?

(ii) if $\varphi \in L^\infty$, is $A_\varphi$ norm-attaining (i.e., is $\|A_\varphi\| = \|\varphi\|_\infty$)?

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When Θ is a finite Blaschke product and ϕ ∈ H∞, the paper [6] computes ∥Aϕ∥ and gives necessary and sufficient conditions as to when ∥Aϕ∥ = ∥ϕ∥∞. The purpose of this short note is to give a few lower bounds on ∥Aϕ∥ for general inner functions Θ and general ϕ ∈ L². Along the way, we obtain a curious relationship (Corollary 5) between the H² and H∞ norms of functions in K∞Θ.

2. LOWER BOUNDS DERIVED FROM POISSON’S FORMULA

For ϕ ∈ L², let

\[(\mathcal{P}\varphi)(z) := \int_{\partial D} \frac{1 - |z|^2}{|\zeta - z|^2} \varphi(\zeta) \frac{|d\zeta|}{2\pi}, \quad z \in \mathbb{D},\]

be the standard Poisson extension of ϕ to D. For ϕ ∈ C(∂D), the continuous functions on ∂D, recall that \(\mathcal{P}\varphi\) solves the classical Dirichlet problem with boundary data ϕ. Also note that

\[k_\lambda(z) := \frac{1 - \Theta(\lambda)\overline{\Theta(z)}}{1 - |\lambda|^2}, \quad \lambda, z \in \mathbb{D},\]

is the reproducing kernel for \(K_\Theta\) [9].

Our first result provides a general lower bound for ∥Aϕ∥ which yields a number of useful corollaries:

**Theorem 1.** If ϕ ∈ L², then

\[\sup_{\lambda \in \mathbb{D}} \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} \left| \int_{\partial D} \varphi(z) \left| \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda} \right|^2 |dz| \right| \frac{1 - |\lambda|^2}{2\pi} \leq \|A\varphi\|.\]  

In other words,

\[\sup_{\lambda \in \mathbb{D}} \int_{\partial D} \varphi(z) d\nu_\lambda(z) \leq \|A\varphi\|\]

where

\[d\nu_\lambda(z) := \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} \left| \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda} \right|^2 |dz| \frac{1 - |\lambda|^2}{2\pi}\]

is a family of probability measures on ∂D indexed by \(\lambda \in \mathbb{D}\).

**Proof.** For \(\lambda \in \mathbb{D}\) we have

\[\|k_\lambda\| = \sqrt{\frac{1 - |\Theta(\lambda)|^2}{1 - |\lambda|^2}},\]

from which it follows that

\[\|A\varphi\| \geq \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} \|\langle A\varphi k_\lambda, k_\lambda \rangle\| \]

\[= \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} \|\langle P\varphi k_\lambda, k_\lambda \rangle\| \]

\[= \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} \|\langle \varphi k_\lambda, k_\lambda \rangle\| \]

\[= \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} \left| \int_{\partial D} \varphi(z) \left| \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda} \right|^2 |dz| \right|.\]

That the measures \(d\nu_\lambda\) are indeed probability measures follows from (4). \(\square\)
Now observe that if \( \Theta(\lambda) = 0 \), the argument in the supremum on the left hand side of (3) becomes the absolute value of the expression in (2). This immediately yields the following corollary:

**Corollary 1.** If \( \varphi \in L^2 \), then

\[
\sup_{\lambda \in \Theta^{-1}(\{0\})} |(\mathcal{P}\varphi)(\lambda)| \leq \|A_{\varphi}\|,
\]

where the supremum is to be regarded as 0 if \( \Theta^{-1}(\{0\}) = \emptyset \).

Under the right circumstances, the preceding corollary can be used to prove that certain truncated Toeplitz operators are norm-attaining:

**Corollary 2.** Let \( \Theta \) be an inner function having zeros which accumulate at every point of \( \partial \mathbb{D} \). If \( \varphi \in C(\partial \mathbb{D}) \) then \( \|A_{\varphi}\| = \|\varphi\|_\infty \).

**Proof.** Let \( \zeta \in \partial \mathbb{D} \) be such that \( |\varphi(\zeta)| = \|\varphi\|_\infty \). By hypothesis, there exists a sequence \( \lambda_n \) of zeros of \( \Theta \) which converge to \( \zeta \). By continuity, we conclude that

\[
\|\varphi\|_\infty = \lim_{n \to \infty} |(\mathcal{P}\varphi)(\lambda_n)| \leq \|A_{\varphi}\| \leq \|\varphi\|_\infty
\]

whence \( \|A_{\varphi}\| = \|\varphi\|_\infty \).

The preceding corollary stands in contrast to the finite Blaschke product setting. Indeed, if \( \Theta \) is a finite Blaschke product and \( \varphi \in H^\infty \), then it is known that \( \|A_{\varphi}\| = \|\varphi\|_\infty \) if and only if \( \varphi \) is the scalar multiple of the inner factor of some function from \( K_\Theta \) [3, Thm. 2].

At the expense of wordiness, the hypothesis of Corollary 2 can be considerably weakened. A cursory examination of the proof indicates that we only need \( \zeta \) to be a limit point of the zeros of \( \Theta \), \( \varphi \in L^\infty \) to be continuous on an open arc containing \( \zeta \), and \( |\varphi(\zeta)| = \|\varphi\|_\infty \).

Theorem 1 yields yet another lower bound for \( \|A_{\varphi}\| \). Recall that an inner function \( \Theta \) has a finite angular derivative at \( \zeta \in \partial \mathbb{D} \) if \( \Theta \) has a non-tangential limit \( \Theta(\zeta) \) of modulus one at \( \zeta \) and \( \Theta' \) has a finite non-tangential limit \( \Theta'(\zeta) \) at \( \zeta \). This is equivalent to asserting that

\[
\frac{\Theta(z) - \Theta(\zeta)}{z - \zeta}
\]

has the non-tangential limit \( \Theta'(\zeta) \) at \( \zeta \). If \( \Theta \) has a finite angular derivative at \( \zeta \), then the function in (6) belongs to \( H^2 \) and

\[
\lim_{r \to 1^-} \int_{\partial \mathbb{D}} \left| \frac{\Theta(z) - \Theta(r\zeta)}{z - r\zeta} \right|^2 \frac{|dz|}{2\pi} = \int_{\partial \mathbb{D}} \left| \frac{\Theta(z) - \Theta(\zeta)}{z - \zeta} \right|^2 \frac{|dz|}{2\pi}.
\]

Furthermore, the above is equal to

\[
\lim_{r \to 1^-} \frac{1 - |\Theta(r\zeta)|^2}{1 - r^2} = |\Theta'(\zeta)| > 0.
\]

See [1] [8] for further details on angular derivatives. Theorem 1 along with the preceding discussion and Fatou’s lemma yield the following lower estimate for \( \|A_{\varphi}\| \).

**Corollary 3.** For an inner function \( \Theta \), let \( D_\Theta \) be the set of \( \zeta \in \partial \mathbb{D} \) for which \( \Theta \) has a finite angular derivative \( \Theta'(\zeta) \) at \( \zeta \). If \( \varphi \in L^\infty \) or if \( \varphi \in L^2 \) with \( \varphi \geq 0 \), then

\[
\sup_{\zeta \in D_\Theta} \frac{1}{|\Theta'(\zeta)|} \left| \int_{\partial \mathbb{D}} \varphi(z) \left| \frac{\Theta(z) - \Theta(\zeta)}{z - \zeta} \right|^2 \frac{|dz|}{2\pi} \right| \leq \|A_{\varphi}\|.
\]
In other words,
\[
\sup_{\zeta \in \mathcal{D}_\Theta} \left| \int_{\partial \mathbb{D}} \varphi(z) d\nu_\lambda(z) \right| \leq \| A \varphi \|,
\]
where
\[
d\nu_\lambda(z) := \frac{1}{|\Theta'(\zeta)|} \left| \frac{\Theta(z) - \Theta(\zeta)}{z - \zeta} \right|^2 \frac{|dz|}{2\pi}
\]
is a family of probability measures on \( \partial \mathbb{D} \) indexed by \( \zeta \in \mathcal{D}_\Theta \).

### 3. Lower Bounds and Projections

Our next several results concern lower bounds on \( \| A \varphi \| \) involving the orthogonal projection \( P_\Theta : L^2 \to K_\Theta \).

**Theorem 2.** If \( \Theta \) is an inner function and \( \varphi \in L^2 \), then
\[
\left\| P_\Theta(\varphi) - \overline{\Theta(0)} P_\Theta(\Theta \varphi) \right\| \leq \| A \varphi \|.
\]

**Proof.** First observe that \( \| k_0 \| = (1 - |\Theta(0)|^2)^{1/2} \). Next we see that if \( \varphi \in L^2 \) and \( g \in K_\Theta \) is any unit vector, then
\[
(1 - |\Theta(0)|^2)^{1/2} \| A \varphi \| \geq |\langle A \varphi, k_0 \rangle| = |\langle P_\Theta(\varphi), g \rangle| = |\langle P_\Theta(\varphi) - \overline{\Theta(0)} P_\Theta(\Theta \varphi), g \rangle|.
\]
Setting
\[
g = \frac{P_\Theta(\varphi) - \overline{\Theta(0)} P_\Theta(\Theta \varphi)}{\left\| P_\Theta(\varphi) - \overline{\Theta(0)} P_\Theta(\Theta \varphi) \right\|}
\]
yields the desired inequality. \( \square \)

In light of the fact that \( P_\Theta(\Theta \varphi) = 0 \) whenever \( \varphi \in H^2 \), Theorem 2 leads us immediately to the following corollary:

**Corollary 4.** If \( \Theta \) is inner and \( \varphi \in H^2 \), then
\[
\frac{\| P_\Theta(\varphi) \|}{(1 - |\Theta(0)|^2)^{1/2}} \leq \| A \varphi \|. \tag{7}
\]

It turns out that (7) has a rather interesting function-theoretic implication. Let us first note that for \( \varphi \in H^\infty \), we can expect no better inequality than
\[
\| \varphi \| \leq \| \varphi \|_{\infty}
\]
(with equality holding if and only if \( \varphi \) is a scalar multiple of an inner function). However, if \( \varphi \) belongs to \( K^*_\Theta \), then a stronger inequality holds.

**Corollary 5.** If \( \Theta \) is an inner function, then
\[
\| \varphi \| \leq (1 - |\Theta(0)|^2)^{1/2} \| \varphi \|_{\infty} \tag{8}
\]
holds for all \( \varphi \in K^*_\Theta \). If \( \Theta \) is a finite Blaschke product, then equality holds if and only if \( \varphi \) is a scalar multiple of an inner function from \( K_\Theta \).
Proof. First observe that the inequality
\[ \| \varphi \| \leq (1 - |\Theta(0)|^2)^{1/2} \| \varphi \|_\infty \]
follows from Corollary 4 and the fact that \( P_\Theta \varphi = \varphi \) whenever \( \varphi \in K_\Theta \). Now suppose that \( \Theta \) is a finite Blaschke product and assume that equality holds in the preceding for some \( \varphi \in K_\Theta^\infty \). In light of (7), it follows that 
\[ \| A \varphi \| = \| \varphi \|_\infty. \]
From [6, Thm. 2] we see that \( \varphi \) must be a scalar multiple of the inner part of a function from \( K_\Theta \). But since \( \varphi \in K_\Theta^\infty \), then \( \varphi \) must be a scalar multiple of an inner function from \( K_\Theta \). □

When \( \Theta \) is a finite Blaschke product, then \( K_\Theta \) is a finite dimensional subspace of \( H^2 \) consisting of bounded functions [3, 5, 9]. By elementary functional analysis, there are \( c_1, c_2 > 0 \) so that
\[ c_1 \| \varphi \| \leq \| \varphi \|_\infty \leq c_2 \| \varphi \| \]
for all \( \varphi \in K_\Theta \). This prompts the following question:

Question. What are the optimal constants \( c_1, c_2 \) in the above inequality?

4. LOWER BOUNDS FROM THE DECOMPOSITION OF \( K_\Theta \)

A result of Sarason [9, Thm. 3.1] says, for \( \varphi \in L^2 \), that
\[ A_\varphi \equiv 0 \iff \varphi \in \Theta H^2 + \Theta H^2. \]
It follows that the most general truncated Toeplitz operator on \( K_\Theta \) is of the form 
\( A_\psi, \chi \) where \( \psi, \chi \in K_\Theta \). We can refine this observation a bit further and provide another canonical decomposition for the symbol of a truncated Toeplitz operator.

Lemma 1. Each bounded truncated Toeplitz operator on \( K_\Theta \) is generated by a symbol of the form
\[ \varphi = \frac{\psi}{\in H^2} + \frac{\chi \Theta}{\in \Theta H^2} \] (10)
where \( \psi, \chi \in K_\Theta \).

Before getting to the proof, we should remind the reader of a technical detail. It follows from the identity \( K_\Theta = H^2 \cap \Theta \Theta H^2 \) (see [2, p. 82]) that
\[ C : K_\Theta \to K_\Theta, \quad Cf := z \Theta, \]
is an isometric, conjugate-linear, involution. In fact, when \( A_\varphi \) is a bounded operator we have the identity \( CA_\varphi C = A_\varphi^* [9, Lemma 2.1]. \)

Proof of Lemma 7 If \( T \) is a bounded truncated Toeplitz operator on \( K_\Theta \), then there exists some \( \varphi \in L^2 \) such that \( T = A_\varphi \). We claim that this \( \varphi \) can be chosen to have the special form (10). First let us write \( \varphi = f + zg \) where \( f, g \in H^2 \). Using the orthogonal decomposition \( H^2 = K_\Theta \oplus \Theta H^2 \), it follows that \( \varphi \) may be further decomposed as
\[ \varphi = (f_1 + \Theta f_2) + zg \]
where \( f_1, g \in K_\Theta \) and \( f_2, g \in H^2 \). By [4], the symbols \( \Theta f_2 \) and \( z \Theta g \) yield the zero truncated Toeplitz operator on \( K_\Theta \). Therefore we may assume that
\[ \varphi = f + zg \]
for some \( f, g \in K_\Theta \). Since \( Cg = \bar{g} \Theta \), we have \( \bar{z} g = (Cg) \Theta \) and hence (10) holds with \( \psi = f \) and \( \chi = Cg \). □
Corollary 6. Let $\Theta$ be an inner function. If $\psi_1, \psi_2 \in K_{\Theta}$ and $\varphi = \psi_1 + \psi_2 \Theta$, then
\[
\| \psi_1 - \Theta(0) \psi_2 \| \leq \left( 1 - |\Theta(0)|^2 \right)^{\frac{1}{2}} \| A \varphi \|.
\]

Proof. If $\varphi = \psi_1 + \psi_2 \Theta$, then, since $\psi_1, \psi_2 \in K_{\Theta}$ and $\psi_2 \Theta \in \overline{zH^2}$, we have
\[
P_{\Theta}(\varphi) - \Theta(0)P_{\Theta}(\Theta \varphi) = \psi_1 - \Theta(0)\psi_2.
\]
The result now follows from Theorem 2. □

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