ON THE TOPOLOGICAL RANK OF THE VARIETY OF RIGHT ALTERNATIVE METABELIAN LIE-NILPOTENT ALGEBRAS

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In 1981, S. V. Pchelintsev introduced the notion of topological rank for Spechtian varieties of algebras as a certain tool for studying the structure of non-nilpotent subvarieties in a given variety. We provide a variety of right alternative algebras of arbitrary given finite topological rank. Namely, we prove that the topological rank of the variety of right alternative metabelian (solvable of index two) algebras that are Lie-nilpotent of step not more than $s$ over a field of characteristic distinct from two and three is equal to $s$.

Keywords: right alternative algebra, metabelian algebra, Lie-nilpotent algebra, superalgebra, variety of algebras, free algebra of variety, polynomial identity, Spechtian variety, Specht property of variety, topological rank of variety.

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1. Introduction

In 1986, A. R. Kemer [1,2] solved affirmatively the famous Specht problem [3] by proving that an arbitrary variety of associative algebras over a field of characteristic zero is finitely based. It is also known [4–6] that there are non-finitely based varieties of associative algebras over an arbitrary field of prime characteristic.

Recall that a variety of algebras is said to be Spechtian (or to have the Specht property) if its every subvariety is finitely based. The Kemer’s theorem has certain analogs in the cases of Jordan, alternative, and Lie algebras over a field of characteristic 0. Namely, A. Ya. Vais, E. I. Zel’manov [7] proved the Specht property of the variety generated by Jordan PI-algebra on a finite set of generators. A. V. Iltyakov [8] obtained the similar result for alternative PI-algebras and also proved in [9] that

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the variety generated by a finite dimensional Lie algebra is Spechtian. U. U. Umirbaev [10] proved the Specht property of every variety of solvable alternative algebras over a field of characteristic distinct from two and three. The questions about the Specht property for the varieties of all alternative, Lie, and Jordan algebras over a field of characteristic zero are still open problems.

Since 1976, it is known [11] that the variety of all right alternative metabelian algebras over an arbitrary field is not Spechtian. In 1985, I. M. Isaev [12] proved that non-finitely based varieties of right alternative metabelian algebras can even be generated by finite-dimensional algebras. The Specht property for certain varieties of right alternative algebras were also studied in [13]–[16].

Recall [17] the notion of topological rank for Spechtian varieties of algebras. Let $V$ be a Spechtian variety of algebras and $M$ be a proper subvariety of $V$. By a system distinguishing $M$ from $V$ we mean a set $S = \{f_1, \ldots, f_n\}$ of nonzero homogeneous polynomials of the free $V$-algebra such that $M$ can be defined by the union of the identities $f_1 = 0, \ldots, f_n = 0$ with the defining identities of $V$. A degree of the system $S$ is the maximal degree of its polynomials. A dimension $\dim V_M$ of $M$ with respect to $V$ is the minimal possible degree of a system distinguishing $M$ from $V$. Let $\mathcal{P}(V)$ be the set of all subvarieties of $V$. For every $\mathcal{R} \in \mathcal{P}(V)$ and $n \in \mathbb{N}$ by $\mathcal{U}_n(\mathcal{R})$ we denote the set of all proper subvarieties $M \subset \mathcal{R}$ of dimensions $\dim \mathcal{R} \geq n$ and put $\mathcal{U}_n(\mathcal{R}) = \mathcal{U}_n(\mathcal{R}) \cup \{\mathcal{R}\}$. By definition, it is clear that $\mathcal{U}_n(\mathcal{R}) \cap \mathcal{U}_{n'}(\mathcal{R}) = \mathcal{U}_{\max(n,n')}(\mathcal{R})$ and, for every $\mathcal{M} \in \mathcal{U}_n(\mathcal{R})$, we have $\mathcal{U}_n(\mathcal{M}) \subset \mathcal{U}_n(\mathcal{R})$. Therefore considering a set $\mathcal{B} = \{\mathcal{U}_n(\mathcal{R}) \mid \mathcal{R} \in \mathcal{P}(V), n \in \mathbb{N}\}$ as a base for the neighborhoods, we endow $\mathcal{P}(V)$ with a topology. Every subset $\mathcal{P}(V)$ gains a structure of the topological space with respect to the induced topology. For every $\Omega \subseteq \mathcal{P}(V)$ by $\Omega'$ denote a subspace of $\Omega$ obtained by the exclusion of all its isolated points. Note that by virtue of the Specht property of $V$, every descending chain of varieties in $\mathcal{P}(V)$ stabilizes and, consequently, every topological subspace of $\mathcal{P}(V)$ contains isolated points. Therefore one can consider the strictly descending chain $\Omega \supset \Omega' \supset \Omega'' \supset \cdots \supset \Omega^{(n)} \supset \cdots$.

The topological rank $r_t(\Omega)$ of the space $\Omega$ is the minimal $n$ such that $\Omega^{(n)} = \emptyset$ or $\aleph_0$ if such an $n$ doesn’t exist. The value $r_t(\mathcal{P}(V))$ is called the topological rank of the variety $V$ and is denoted shortly by $r_t(V)$. For example, if $V$ is nilpotent, then every subvariety of $V$ turns out to be an isolated point of $\Omega = \mathcal{P}(V)$ and, consequently, $r_t(V) = 1$. Otherwise, $\Omega'$ consists of all non-nilpotent subvarieties of $V$ and its isolated points are the varieties that were limit points in $\Omega$ for the sequences of nilpotent varieties only. Further, if $\Omega''$ is not empty, then its isolated points are the varieties that were limit points in $\Omega'$ for the sequences containing only isolated points of $\Omega'$, etc.
The structures of a set of non-nilpotent subvarieties for various varieties of nearly associative metabelian algebras were studied in [17]–[19]. A. V. Badeev [20] provided a chain $V_1 \subset \cdots \subset V_n \subset \cdots \subset V$ of varieties of commutative alternative nil-algebras over a field of characteristic three such that $r_t(V_n)$ is a linear function on $n$ and $r_t(V) = \aleph_0$. In 2007, S. V. Pchelintsev [16] constructed a variety $M$ of right alternative metabelian algebras of almost finite topological rank, i.e., a variety $M$ such that $r_t(M) = \aleph_0$ and $r_t(M')$ is finite for every proper subvariety $M' \subset M$.

Formulation of the result

Let $F$ be a field of characteristic $\text{char}(F) \neq 2, 3$ and $\text{RA}_2$ be the variety of right alternative metabelian algebras over $F$ defined by the identities

\begin{align*}
(x, y, z) + (x, z, y) &= 0 \quad \text{(the right alternative identity),} \\
(xy)(zt) &= 0 \quad \text{(the metabelian identity),}
\end{align*}

where $(x, y, z) = (xy)z - x(yz)$ is the associator of the variables $x, y, z$. By $\text{RA}_2^{(s)}$ we denote the subvariety of $\text{RA}_2$ distinguished by the identity

$$
\left[\ldots [x_1, x_2], \ldots, x_s, x_{s+1}\right] = 0
$$

of Lie-nilpotency of step $s$, where $[x, y] = xy - yx$ is the commutator of $x, y$.

The Specht property of $\text{RA}_2^{(s)}$ is proved by the author in [15]. By virtue of nilpotency of every commutative subvariety of $\text{RA}_2$, we have $r_t(\text{RA}_2^{(2)}) = 1$. S. V. Pchelintsev established in [17] that $r_t(\text{RA}_2^{(2)}) = 2$. In the present paper, we prove the following

**Theorem.** The topological rank of the variety $\text{RA}_2^{(s)}$ is equal to $s$ for all natural $s$.

The paper is organized as follows. In Sec. 2 we provide some preliminary results about the free $\text{RA}_2$-algebra $F_{\text{RA}_2}[X]$ on a countable set $X$ of generators over $F$. Sec. 3 is devoted to the studying of relations of the free algebra $F_{\text{RA}_2^{(s)}}[X]$. In Sec. 4 we construct a system of linear generators for the space of multilinear polynomials in $F_{\text{RA}_2^{(s)}}[X]$ of sufficiently high degree and obtain the upper bound $r_t(\text{RA}_2^{(s)}) \leq s$ by estimating the values of topological ranks of some subvarieties in $\text{RA}_2^{(s)}$ of special type. Finally, in Sec. 5 we construct an auxiliary $\text{RA}_2^{(s)}$-superalgebra and considering the identities of their Grassmann envelopes obtain the low bound $r_t(\text{RA}_2^{(s)}) \geq s$.

2. Preliminaries

Throughout the paper, $F$ is a field of characteristic $\text{char}(F) \neq 2, 3$; all vector spaces (algebras, superalgebras) are considered over $F$; $X = \{x_1, x_2, \ldots\}$ is a countable set; $\mathfrak{A} = F_{\text{RA}_2}[X]$ is a free $\text{RA}_2$-algebra on the set $X$ of generators; $R_x$ and $L_x$ are, respectively, the operators of right and left multiplication by the element $x$; $H_x = R_x - L_x$; $\mathfrak{A}^*$ is the associative algebra generated by all the operators $R_x$ and
$L_x$, for $x \in \mathfrak{A}$, acting on $\mathfrak{A}^2$ and by the identical mapping id; $\text{Var} A$ is the variety generated by an algebra $A$.

Recall \cite{15,16} that $\mathfrak{A}^*$ satisfies the relations

\begin{align}
R_x^2 &= 0, \\
[R_x R_y, L_z] &= 0, \\
[R_x, L_y] &= -L_x L_y.
\end{align}

Relations (2.1), (2.2) imply immediately the following

**Lemma 2.1.** The operator $R_x R_y$ lies in the center of $\mathfrak{A}^*$.

**Proposition 2.1.** The algebra $\mathfrak{A}^*$ satisfies the relation

\[3R_x R_y + H_x H_y = 2[R_x, H_y] + H_y R_y + H_y R_x.\]  \hspace{1cm} (2.4)

**Proof.** Using (2.3), we have

\[H_x H_y = (R_x - L_x) (R_y - L_y) = R_x R_y + L_x L_y - L_x R_y - R_x L_y =
\]
\[= R_x R_y - [R_x, L_y] - L_x R_y - L_y R_x - [R_x, L_y] =
\]
\[= R_x R_y - 2[R_x, L_y] - L_x R_y - L_y R_x.
\]

Combining the obtained relation with (2.1) and (2.3), we get

\[3R_x R_y + H_x H_y = 4R_x R_y - 2[R_x, L_y] - L_x R_y - L_y R_x =
\]
\[= 2[R_x, R_y] - 2[R_x, L_y] + (H_x - R_x) R_y + (H_y - R_y) R_x =
\]
\[= 2[R_x, H_y] + H_x R_y + H_y R_x. \quad \Box
\]

In what follows, we use the symbol $T$ as a common notation for the operator symbols $R$ and $H$. The notation $w = T_x \ldots T_y$ means that each operator symbol of the word $w$ can be equal to $R$ or $H$ independently. In the case when all operator symbols in some word are assumed to be equal to each other, we use the notation

\[T(i_1, \ldots, i_n) = \begin{cases} R_{i_1} \ldots R_{i_n}, & \text{if } T = R, \\
H_{i_1} \ldots H_{i_n}, & \text{if } T = H
\end{cases}
\]

and set $T(\emptyset) = \text{id}$.

**Lemma 2.2.** The algebra $\mathfrak{A}^*$ is spanned by the operators

\[H(i_1, \ldots, i_n) R(j_1, \ldots, j_m).
\]

**Proof.** Let $I$ be a linear span of all operators $H(i_1, \ldots, i_n) R(j_1, \ldots, j_m)$. It suffices to prove the inclusions $R(k) I \subseteq I$ and $I H(k) \subseteq I$. Note that (2.1) yields $R(i) H(j) \in I$. Hence the inclusion $R(k) I \subseteq I$ can be easily proved by induction on the length of the operator $H(i_1, \ldots, i_n)$. At the same time, Lemma 2.1 implies $I H(k) \subseteq I$. \hfill $\Box$
Let $L$ be a linear span in $A^*$ of all operators of the form

$$L_x, w, \ w \in A^*.$$ 

By virtue of (2.3), $L$ forms an ideal of $A^*$ and by induction on $n$ one can prove the congruence

$$H(1, \ldots, n) \equiv R(1, \ldots, n) \pmod{L}, \ n \in \mathbb{N}. \quad (2.5)$$

3. Relations of the free $\text{RA}_2(s)$-algebra

Let $A_s = F_{\text{RA}_2(s)}[X]$ be the free $\text{RA}_2(s)$-algebra on the set $X$ of generators. Lemma 2.2 implies immediately the following

**Lemma 3.1.** The linear span of all operators of degree $d \geq s$ in $A_s^*$ is spanned by the operators

$$H(i_1, \ldots, i_n) R(j_1, \ldots, j_{d-n}), \ n < s.$$ 

In what follows, the term "polynomial" means a homogeneous polynomial of degree not less than two.

**Definition 3.1.** Let $\approx$ be a symmetric relation on the set of polynomials of $A^*$ such that $f_0 \approx f_1$ if $f_i = f_{1-i} R(j_1, \ldots, j_{2k}), i \in \{0, 1\}$, and $f_{1-i}$ doesn’t depend on the variables $x_{j_1}, \ldots, x_{j_{2k}}$. By the same symbol $\approx$ we denote the induced relation on $A^*$: $\xi \approx \eta$ for $\xi, \eta \in A^*$ if $(x_i x_j) \xi \approx (x_i x_j) \eta$ and $\xi, \eta$ do not depend on $x_i, x_j$.

**Proposition 3.1.** The algebra $A_s$ satisfies the relation

$$x^3 \approx 0. \quad (3.1)$$

**Proof.** Using (1.1) and (1.2), we have

$$2yx^3 = y(x \circ x^2) = (yx^2) x = ((yx) x)x = (yx)x^2 = 0.$$ 

Hence, $x^3L = 0$. Therefore applying (2.5), for even $n \geq s$, and taking into account (1.3), we obtain

$$x^3 \approx x^3 R(1, \ldots, n) = x^3 H(1, \ldots, n) = 0. \quad \Box$$

We say that almost all polynomials of $A_s$ (operators of $A_s^*$) satisfy some condition $\vartheta$ if there is a natural $n$ such that $\vartheta$ holds for all polynomials (operators) of degree more then $n$.

**Lemma 3.2.** If $f \approx 0$ for $f \in A_s$, then almost all operators of $A_s^*$ annihilate $f$.

**Proof.** Assume that $f R(j_1, \ldots, j_{2k}) = 0$, where $f$ doesn’t depend on $x_{j_1}, \ldots, x_{j_{2k}}$.

In view of Lemma 3.1, every operator word $\xi \in A_s^*$ of the degree $d \geq s + 2k$ can be represented as

$$\xi = \eta R(j_1, \ldots, j_{2k}), \ \eta \in A_s^*.$$
Hence by Lemma 2.1, we have
\[ f_x = f_R(j_1, \ldots, j_{2k}) \eta = 0. \]

**Lemma 3.3.** Almost all operators of \( A_s^* \) are skew-symmetric with respect to all their variables.

**Proof.** We set \( w \in A_2^* \). By virtue of (1.2) and (2.1), the partial linearization (see [21, Chap. 1]) of (3.1) has the form
\[ (wx) x + (xw) x + x^2w = (xw) x \approx 0, \]
whence, \( H_xR_x = -L_xR_x \approx 0 \). Thus in view of Lemma 3.1, it remains to calculate
\[ H_xH_x \approx -H_xH_xR_yR_z \approx -H_xH_xR_yR_z \approx 0. \]

**Proposition 3.2.** The algebra \( A_s \) satisfies the relation
\[ (xy)T_yT_x \approx 0. \] (3.2)

**Proof.** By virtue of Lemmas 2.1, 3.3, it suffices to verify that \( (xy)R_xR_y \approx 0 \). Using (2.1), (1.1), and Lemma 3.3, we have
\[ (xy)R_xR_y = -(xy)R_yR_x = y^2L_xR_x \approx 0. \]

**Proposition 3.3.** The algebra \( A_s^* \) satisfies the relations
\[ 3R_xR_y - 2[R_x, H_y] + H_xH_y \approx 0, \] (3.3)
\[ [R_x, H_yH_z] \approx 0. \] (3.4)

**Proof.** By Lemma 3.3, relation (3.3) follows from (2.1). Using (3.3) and combining Lemmas 2.1, 3.3 with the Jacobian identity, we obtain
\[ [R_x, H_yH_z] \approx 2[R_x, [R_y, H_z]] \approx [R_x, [R_y, H_z]] - [R_y, [R_x, H_z]] = [H_z, [R_y, R_x]] = 0. \]

**Definition 3.2.** Let \( \mathcal{I} \) be an ideal of \( \mathfrak{A}_s^* \). For \( \xi, \eta \in \mathfrak{A}_s^* \) we write \( \xi \equiv \eta \pmod{\mathcal{I}} \) if there is a \( \theta \in \mathcal{I} \) such that \( \xi - \eta \approx \theta \).

Let \( \mathcal{H}_n \) be the ideal of \( \mathfrak{A}_s^* \) generated by all the elements \( H(i_1, \ldots, i_n) \).

**Proposition 3.4.** The algebra \( \mathfrak{A}_s^* \) satisfies the relation
\[ H(1, \ldots, 2t) \equiv 0 \pmod{\mathcal{H}_{2t+1}}. \] (3.5)

**Proof.** We set \( \eta = H(1, \ldots, 2t) \). Applying (3.3) and (3.4), we have
\[ 3\eta \approx 3\eta R_xR_y \approx 2\eta R_xH_y \approx 2R_x\eta H_y \approx 0 \pmod{\mathcal{H}_{2t+1}}. \]
4. Upper bound for the topological rank of $\text{RA}_2^{(s)}$

**Definition 4.1.** An $n$-allotted variety $(1 \leq n \leq s)$ is a subvariety $V$ of $\text{RA}_2^{(s)}$ such that the free $V$-algebra on the set $X$ of generators satisfies the relation

$$\varphi(x_1, \ldots, x_{n+1}) \approx 0,$$

(4.1)

where

$$\varphi(x_1, \ldots, x_{n+1}) = \begin{cases} 
\ldots[x_1, x_2], \ldots, x_n, x_{n+1}], & \text{if } n \text{ is even}, \\
\ldots[x_1 x_2, x_3], \ldots, x_n, x_{n+1}], & \text{if } n \text{ is odd}.
\end{cases}$$

By definition, every 1-allotted variety $M$ is right nilpotent. Moreover, applying Lemma 3.1, it is not hard to prove that $M$ is nilpotent and, consequently, $r_t(M) = 1$.

We also stress that the variety $\text{RA}_2^{(s)}$ is $s$-allotted: for even $s$, it is clear by definition and, for odd $s$, it follows from (3.5).

Let $V$ be an $n$-allotted variety $(n \geq 2)$, $A$ be the free $V$-algebra on the set $X$ of generators, and $\mathcal{P}_{d,n}$ $(d \geq 3)$ be the subspace of multilinear polynomials in $A$ on the variables $x_1, \ldots, x_d$. In order to avoid complicated formulas while writing down the polynomials of $\mathcal{P}_{d,n}$ we omit the indices of variables at the operator symbols and assume them to be arranged at the ascending order. For example, the notation $w = (x_2 x_5) H^2 R^{3}$ means the monomial

$$w = (x_2 x_5) H (1, 3) R (4, 6, 7).$$

**Definition 4.2.** Regular words are the polynomials of $\mathcal{P}_{d,n}$ of the following types:

1) $(x_i \circ x_j) H^2 R^{d-2j-2},$
2) $[x_i, x_j] H^{2j} R^{d-2j-2},$
3) $[x_2, x_3] H^{2j} R^{d-2j-2},$
4) $[x_1, x_2] H^{2k-1} R^{d-2k-1},$

where $i = 2, 3, \ldots, d; j = 0, 1, \ldots, t-1; k = 1, 2, \ldots, n-t-1; t = \lceil \frac{n}{2} \rceil$.

**Lemma 4.1.** Almost all polynomials of $\bigcup_{d=3}^{\infty} \mathcal{P}_{d,n}$ are linear combinations of regular words.

**Proof.** By Lemma 3.3 there is a degree $d$ such that every monomial

$$(x_1 x_2) T_3 \ldots T_d \in \mathcal{P}_{d,n}$$

is skew-symmetric w.r.t. $x_3, \ldots, x_d$. Consequently, in view of Lemma 3.1 and relation 3.5, $\mathcal{P}_{d,n}$ can be spanned by the polynomials

$$(x_i \circ x_j) H^k R^{d-k-2}, [x_i, x_j] H^k R^{d-k-2},$$

where $a \circ b = ab + ba, 1 \leq i < j \leq d$, and $k = 0, 1, \ldots, 2t-1.$
Linearizing (3.1) and (3.2), we have

\[(x \circ y) T_x + (y \circ z) T_x + (z \circ x) T_y \approx 0,\]
\[[x, y] T_x T_t + [x, t] T_x T_y + [z, y] T_x T_t + [z, t] T_x T_y \approx 0.\]

Applying these relations, it is not hard to prove that \(P_{d,n}\) can be spanned by the polynomials:

\[1') \quad (x_1 \circ x_i) H^k R^{d-k-2},\]
\[2') \quad [x_1, x_i] H^k R^{d-k-2},\]
\[3') \quad [x_2, x_3] H^k R^{d-k-2},\]

where \(i = 2, \ldots, d\) and \(k = 0, 1, \ldots, 2t - 1\).

By \(W\) denote the linear span of all regular words of types 1)–3). Note that the polynomials of types 1')–3') lie in \(W\) for even \(k\). Let us verify that the polynomials of types 1')–3') for odd \(k\) can be represented as linear combinations of regular words.

By virtue of (3.1), we have

\[(x \circ y) H_z = (x \circ y) z - z (x \circ y) = (x \circ y) z - (zx) y - (zy) x.\]

Hence, in view of (3.3), every polynomial of type 1') lie in \(W\). Further, using Lemmas 2.1, 3.3, the partial linearization

\[(xy) y + (yx) y + y^2 x \approx 0\]

of (3.3), identity (3.1) and relation (3.3), we get

\[\begin{align*}
[x, y] H_y & \approx [x, y] H_y R_z R_u \approx [x, y] R_z R_u H_y = [x, y] R_u R_z H_u = \\
& = ((xy) y - (yx) y) R_z H_u \approx (2 (xy) y + y^2 x) R_z H_u = y^2 (2 L_x + R_z) R_z H_u = \\
& = y^2 (3 R_x - 2 H_x) R_z H_u \approx y^2 (2 [R_x, H_z] - H_x H_z - 2 H_x R_z) H_u \approx \\
& \approx y^2 (2 R_x - H_x) H_z H_u = y^2 (R_x + L_x) H_z H_u = (y^2 R_x + (xy) R_y) H_z H_u.
\end{align*}\]

In view of (3.3), the obtained relation implies that the polynomials of types 2'), 3') for odd \(k\) are skew-symmetric modulo \(W\) with respect to all their variables. Therefore every polynomial of type 2'), 3') is proportional modulo \(W\) to a regular word of type 4).

Lemma 4.2. For every \(n\)-allotted variety \(V\) \((n \geq 2)\) there is a punctured neighborhood \(\mathcal{U}_d(V)\) such that every variety of \(\mathcal{U}_d(V)\) is \((n - 1)\)-allotted.

Proof. By virtue of the restriction \(\text{char}(F) \neq 2\), Lemma 3.3 and relation (3.1) imply that in some punctured neighborhood of \(V\) every variety can be defined by a system of identities where all polynomials starting from some sufficiently high degree are multilinear. Consequently by Lemma 4.1 we can choose a punctured neighborhood \(\mathcal{U}_d(V)\) such that every variety \(M \in \mathcal{U}_d(V)\) satisfies an identity \(f = 0\), where \(f\) is a nontrivial linear combination of regular words of \(P_{d,n}\). Let \(A\) be the
free $\mathfrak{M}$-algebra on the set $X$ of generators. We write down relation $[4,1]$ shortly as $\mathcal{A}^2H^{2t} \approx 0$ if $n = 2t + 1$ and as $\mathcal{A}H^{2t} \approx 0$ if $n = 2t$.

First consider the case $n = 2t + 1$. We prove that relation $\mathcal{A}^2H^{2t} \approx 0$ and identity $f = 0$ imply $\mathcal{A}H^{2t} \approx 0$. By Lemma $[4,1]$ $f$ can be presented in the form

$$f \equiv \sum_{i=2}^{d} \sum_{j=0}^{t-1} \left( \alpha_{2j}^{(i)} (x_1 \circ x_i) H^{2j} R^{d-2j-2} + \alpha_{2j+1}^{(i)} [x_1, x_i] H^{2j} R^{d-2j-2} \right) \pmod{\mathcal{W}_{3,4}},$$

where $\mathcal{W}_{3,4}$ is the linear span of regular words of types 3), 4). We fix $i \geq 4$ and a minimal index $\ell$ such that $\alpha_{\ell}^{(i)} \neq 0$. Then by the substitution $x_i := aH^{2t-\ell}$, for $a \in \mathcal{A}$, using the equality $R_y + L_y = 2R_y - H_y$ and relation $[3,4]$, we obtain $aH^{2t} \approx 0$. Otherwise, we can rewrite $f$ in the form

$$f = \sum_{k=0}^{t-1} g_k + \sum_{k=1}^{t} h_k,$$

where

$$g_0 = \left( \alpha_0 [x_1, x_2] x_3 + \beta_0 [x_3, x_1] x_2 + \gamma_0 [x_2, x_3] x_1 \right) R^{d-3},$$

$$h_t = \zeta_t [x_1, x_2] H^{2t-1} R^{d-2k-1},$$

and

$$g_k = \left( \alpha_k [x_1, x_2] x_3 + \beta_k [x_3, x_1] x_2 + \gamma_k [x_2, x_3] x_1 \right) H^{2k-1} R^{d-2k-2},$$

$$h_k = \delta_k (x_1 \circ x_2) H^{2k} R^{d-2k-2} + \varepsilon_k (x_1 \circ x_3) H^{2k} R^{d-2k-2} + \zeta_k [x_1, x_2] H^{2k-1} R^{d-2k-1},$$

for $k = 1, \ldots, t - 1$. If at least one of the coefficients $\alpha_0, \beta_0, \gamma_0$ is not zero, then by three successive substitutions $x_i := aH^{2t-1} (i = 1, 2, 3)$, we have

$$\begin{align*}
(\alpha_0 + \beta_0) aH^{2t} & \approx 0, \\
(\alpha_0 + \gamma_0) aH^{2t} & \approx 0, \\
(\beta_0 + \gamma_0) aH^{2t} & \approx 0.
\end{align*}$$

Hence in view of the restriction char($F$) $\neq 2$, we obtain either $aH^{2t} \approx 0$ or $g_0 = 0$. Further, if $\varepsilon_1 \neq 0$, then by the substitution $x_3 := aH^{2t-2}$, we have $aH^{2t} \approx 0$. Otherwise, if at least one of the coefficients $\delta_1$ or $\zeta_1$ is not zero, by two successive substitutions $x_i := aH^{2t-2} (i = 1, 2)$, we obtain

$$\begin{align*}
(2\delta_1 + \zeta_1) aH^{2t} & \approx 0, \\
(2\delta_1 - \zeta_1) aH^{2t} & \approx 0.
\end{align*}$$

Thus we have either $aH^{2t} \approx 0$ or $h_1 = 0$ and, consequently, $f$ gets the form

$$f = \sum_{k=1}^{t-1} g_k + \sum_{k=2}^{t} h_k.$$
Therefore by the same arguments as above, we obtain either \( aH^{2t} \equiv 0 \) or
\[
g_1 = h_2 = \cdots = g_{t-2} = h_{t-1} = g_{t-1} = 0
\]
and
\[
f = h_t = \zeta_t [x_1, x_2] H^{2t-1} P_t^{d-2t-1}.
\]
But in this case, the assumption of the lemma implies \( \zeta_t \neq 0 \) and, consequently, \( A H^{2t} \approx 0 \).

Now consider the case \( n = 2t \). We need to prove that relation \( A H^{2t} \approx 0 \) and identity \( f = 0 \) imply \( A^2 H^{2t-2} \approx 0 \). Unlike the case \( n = 2t + 1 \), the regular word of type 4) corresponding to the index \( k = t \) vanishes. All the other regular words are the same. Therefore by the similar arguments as above, reducing per unit the power \( p(t) \) for every substitution \( x_i := aH^{p(t)} \) and assuming \( a \in A^2 \), one can prove that \( A^2 H^{2t-1} \approx 0 \). By (3.5), the obtained relation yields \( A^2 H^{2t-2} \approx 0 \).

As it was stressed above, every 1-allotted variety has the topological rank 1. Consequently Lemma 4.2 implies that the topological rank of every \( n \)-allotted variety is not more than \( n \). In particular, \( r_t (\mathcal{R}A_2^{(s)}) \leq s \).

5. Low bound for the topological rank of \( \mathcal{R}A_2^{(s)} \)

Let \( \mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \) be a superalgebra (\( \mathbb{Z}_2 \)-graded algebra) with the even part \( \mathcal{A}_0 \) and the odd part \( \mathcal{A}_1 \), i.e. \( \mathcal{A}_i \mathcal{A}_j \subseteq \mathcal{A}_{i+j \bmod 2} \) for \( i, j \in \{0, 1\} \); \( G \) be the Grassmann algebra on a countable set of anticommuting generators \( \{e_1, e_2, \ldots \} \) with the natural \( \mathbb{Z}_2 \)-grading (\( G_0 \) and \( G_1 \) are spanned by the words of even and, respectively, odd length on \( \{e_i\} \)). The Grassmann envelope \( G(\mathcal{A}) \) of \( \mathcal{A} \) is the subalgebra \( G_0 \otimes \mathcal{A}_0 + G_1 \otimes \mathcal{A}_1 \) of the tensor product \( G \otimes \mathcal{A} \). It is well known that \( G(\mathcal{A}) \) satisfies a multilinear identity \( f = 0 \) iff \( \mathcal{A} \) satisfies the certain graded identity \( \tilde{f} = 0 \). Here \( \tilde{f} \) denotes the so-called superpolynomial corresponding to \( f \) and we also say that \( \mathcal{A} \) satisfies the superidentity \( \tilde{f} = 0 \).

The descriptions of the process of constructing of superpolynomials (the superizing process) can be found in [22–25]. For a given variety \( \mathcal{V} \) of algebras, \( \mathcal{A} \) is said to be a \( \mathcal{V} \)-superalgebra if \( G(\mathcal{A}) \in \mathcal{V} \), i.e. if \( \mathcal{A} \) satisfies all the superizations of the defining identities of \( \mathcal{V} \).

Let \( \varepsilon \) be one of the elements 0, 1 \( \in F \) and \( \mathcal{A}^{(\varepsilon)} = \mathcal{A}_0^{(\varepsilon)} \oplus \mathcal{A}_1^{(\varepsilon)} \) be a superalgebra with the countable basis \( x, a_{i,j} \ (i, j = 0, 1, \ldots) \) such that \( x \in \mathcal{A}_1^{(\varepsilon)} \) and \( a_{i,j} \in \mathcal{A}_0^{(\varepsilon)} \) iff \( i \equiv j \pmod{2} \). We introduce the multiplication of the basis elements of \( \mathcal{A}^{(\varepsilon)} \) as follows:
\[
x^2 = \frac{\varepsilon}{2} a_{0,0}, \quad a_{i,j} \cdot x = a_{i,j+1}, \quad x \cdot a_{i,2j} = (-1)^j (a_{i,2j+1} - a_{i+1, 2j}), \quad x \cdot a_{i,2j+1} = \frac{(-1)^j}{2} (a_{i,2j+2} - 2a_{i+1, 2j+1} + a_{i+2, 2j}),
\]
Lemma 5.1. The algebra $A^{(c)}$ is an RA$_2$-superalgebra.

Proof. We check that $A^{(c)}$ satisfies the superization of (1.1):

$$ (a, b, c) + (-1)^{|b|c}(a, c, b) = 0, $$

where $a, b, c$ are homogeneous basis elements of $A^{(c)}$ and $|a|$ denotes the parity of $a$, i.e. $|a| = k$ for $a \in A^{(c)}_k$ ($k = 0, 1$). In view of metability of $A^{(c)}$ and the odd parity of $x$, it suffices to verify the relation

$$ a_{i,j}([L_x, R_x] - (-1)^{i+j} L_x^2) = 0. \tag{5.1} $$

First for even $j$, we calculate

$$ a_{i,j} [L_x, R_x] = (-1)^i (a_{i, j+1} - a_{i+1,j}) R_x - a_{i,j+1} L_x = $$

$$ = (-1)^i (a_{i,j+2} - a_{i+1,j+1}) - \frac{(-1)^i}{2} (a_{i,j+2} - 2a_{i+1,j+1} + a_{i+2,j}) = $$

$$ = \frac{(-1)^i}{2} (a_{i,j+2} - a_{i+2,j}); $$

$$ a_{i,j} L_x^2 = (-1)^i (a_{i, j+1} - a_{i+1,j}) L_x = $$

$$ = \frac{1}{2} (a_{i,j+2} - 2a_{i+1,j+1} + a_{i+2,j}) + a_{i+1,j+1} - a_{i+2,j} = $$

$$ = \frac{1}{2} (a_{i,j+2} - a_{i+2,j}). $$

To conclude the proof it remains to make the similar calculations for odd $j$:

$$ a_{i,j} [L_x, R_x] = \frac{(-1)^i}{2} (a_{i,j+1} - 2a_{i+1,j} + a_{i+2,j-1}) R_x - a_{i,j+1} L_x = $$

$$ = \frac{(-1)^i}{2} (a_{i,j+2} - 2a_{i+1,j+1} + a_{i+2,j}) + (-1)^{i+1} (a_{i,j+2} - a_{i+1,j+1}) = $$

$$ = \frac{(-1)^{i+1}}{2} (a_{i,j+2} - a_{i+2,j}); $$

$$ a_{i,j} L_x^2 = \frac{(-1)^i}{2} (a_{i,j+1} - 2a_{i+1,j} + a_{i+2,j-1}) L_x = $$

$$ = \frac{1}{2} (a_{i,j+2} - a_{i+1,j+1} + a_{i+1,j+1} - 2a_{i+2,j} + a_{i+3,j-1} + a_{i+2,j} - a_{i+3,j-1}) = $$

$$ = \frac{1}{2} (a_{i,j+2} - a_{i+2,j}). \quad \Box $$

Proposition 5.1. The algebra $A^{(c)}$ satisfies the relations

$$ [a_{i,j}, x]_a = a_{i+1,2j}, \tag{5.2} $$

$$ [[a_{i,j}, x]_a, x]_a = a_{i+2,j}. \tag{5.3} $$
where \([a, b]_s = ab - (-1)^{|a||b|}ba\) is a supercommutator of the elements \(a, b\).

**Proof.** First we calculate

\[
[a_{i,2j}, x]_s = a_{i,2j} R_x - (-1)^i a_{i,2j} L_x = a_{i,2j+1} - (a_{i,2j+1} - a_{i+1,2j}) = a_{i+1,2j}.
\]

Thus (5.2) and, consequently, (5.3), for even \(j\), are proved. Further, for odd \(j\), we have

\[
[a_{i,j}, x]_s = a_{i,j} R_x + (-1)^i a_{i,j} L_x = a_{i,j+1} + \frac{1}{2} (a_{i,j+1} - 2a_{i+1,j} + a_{i+2,j-1}) = \frac{3}{2} a_{i,j+1} - a_{i+1,j} + \frac{1}{2} a_{i+2,j-1}.
\]

Finally, combining the obtained relation with (5.2), we get

\[
[(a_{i,j}, x)_s]_s = \frac{1}{2} \left(3a_{i,j+1} - 2a_{i+1,j} + a_{i+2,j-1}\right), x]_s = \frac{1}{2} \left(3a_{i,j+1} - 3a_{i+1,j+1} + 2a_{i+2,j} - a_{i+3,j-1} + a_{i+3,j-1}\right) = a_{i+2,j}. \quad \Box
\]

Let \(\mathcal{I}^{(k)} (k \in \mathbb{N})\) be the span of all elements \(a_{i,j} \in \mathcal{A}^{(c)}\) such that \(i \geq 2k\).

By the definition of multiplication in \(\mathcal{A}^{(c)}\), every nonzero product \(a_{i,j} \mathcal{I}_x\) is a linear combination of elements \(a_{i',j'}\) such that \(i' \geq i\). Consequently, \(\mathcal{I}^{(k)}\) is an ideal of \(\mathcal{A}^{(c)}\).

For every natural \(n \geq 2\), we introduce the quotient superalgebra \(\mathcal{A}^{(n)}\) as follows:

\[
\mathcal{A}^{(2k)} = \mathcal{A}^{(0)} \mathcal{I}^{(k)}, \quad \mathcal{A}^{(2k+1)} = \mathcal{A}^{(1)} \mathcal{I}^{(k)}.
\]

**Lemma 5.2.** The algebra \(\mathcal{A}^{(n)}\) is an \(\mathcal{R}A_2^{(n)}\)-superalgebra.

**Proof.** Taking into account Lemma 5.1 it suffices to prove that \(\mathcal{A}^{(n)}\) satisfies the superization of (1.3). By virtue of metability of \(\mathcal{A}^{(n)}\), we need to verify the relation

\[
\underbrace{\ldots \left[\left[\left[\mathcal{A}^{(n)}, x\right]_s, x\right]_s, \ldots, x\right]_s}_n = 0.
\]

By the definition of \(\mathcal{A}^{(n)}\), using (5.3), for \(k = \frac{n}{2}\), we have

\[
\underbrace{\ldots \left[\left[\left[a_{i,j}, x\right]_s, x\right]_s, \ldots, x\right]_s}_{2k} = a_{i+2k, j} = 0.
\]

Thus the required relation is proved for even \(n\). In the case of odd \(n\), it remains to check the following:

\[
\underbrace{\ldots \left[\left[\left[x, x\right]_s, x\right]_s, \ldots, x\right]_s}_{2k+1} = \underbrace{\ldots \left[\left[\left[a_{0,0}, x\right]_s, x\right]_s, \ldots, x\right]_s}_{2k} = a_{i+2k, j} = 0. \quad \Box
\]

**Lemma 5.3.** The variety \(\text{Var} \mathcal{G}\left(\mathcal{A}^{(n)}\right)\) is not \((n - 1)\)-allotted.
Proof. In the case $n = 2k + 1$, we need to verify that

$$\cdots \left[ \left[ A^{(n)}, x \right]_a, x \right]_a, \ldots, x \right]_a R^j_x \neq 0, \quad j \in \mathbb{N}.$$  

Applying (5.2), we calculate

$$\cdots \left[ x, x \right]_a, x \right]_a, \ldots, x \right]_a R^j_x = \cdots \left[ a_{0,0}, x \right]_a, x \right]_a, \ldots, x \right]_a R^j_x = a_{2k-1,j} \neq 0.$$  

For $n = 2k$, we need to check that

$$\cdots \left[ \left[ A^{(n)} \right]_2, x \right]_a, x \right]_a, \ldots, x \right]_a R^j_x \neq 0, \quad j \in \mathbb{N}.$$  

Using (5.3), we get

$$\cdots \left[ a_{0,1}, x \right]_a, x \right]_a, \ldots, x \right]_a R^j_x = a_{2k-2,j+1} \neq 0.$$  

In view of (3.5), Lemma 5.2 implies that the variety $\text{Var} G \left( A^{(n)} \right)$ is $n$-allotted. Consequently by Lemma 5.3 we have $r_1(\text{RA}_{2}^{(n)}) \geq n$ for $n = 2, \ldots, s$. Finally, comparing this estimate with the result of Sec. 4 we obtain $r_1(\text{RA}_{2}^{(s)}) = s$.

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