Boundedness of Rough Singular Integral Operators on Homogeneous Herz Spaces with Variable Exponents

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Abstract. We establish the boundedness of rough singular integral operators on homogeneous Herz spaces with variable exponents. As an application, we obtain the boundedness of related commutators with BMO functions on homogeneous Herz spaces with variable exponents.

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1 Introduction

Let $S^{n-1}$ be the unit sphere in $\mathbb{R}^n (n \geq 2)$ and $\Omega$ be a measurable function defined on $S^{n-1}$. The rough singular integral operator $T_\Omega$ is defined by

$$T_\Omega f(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy.$$ 

In 1952, Calderón and Zygmund first studied the operator $T_\Omega$ in [3] and proved that $T_\Omega$ is bounded in $L^p(\mathbb{R}^n)$ for $p \in (1,\infty)$ if $\Omega \in C^\infty(S^{n-1})$ is a homogeneous function of degree zero and

$$\int_{S^{n-1}} \Omega(x) d\sigma(x) = 0.$$  (1.1)

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The condition on $\Omega$ can be weakened or revised; see [4, 8, 27]. Later, the results were further extended to the weight Lebesgue spaces by Duoandikoetxea [14]. Further development of the topic in other function spaces with constant exponents can be found in [5–7, 15, 16, 23] and the references therein.

Variable exponent function spaces received considerable attentions in recent decades [37]. They are important not only in theory as generalizations of classical function spaces, but also for their wide applications in the fields of fluid dynamics, elasticity dynamics, the differential equations with nonstandard growth. We refer to [1, 2, 12, 28] for the details. The rich development can be found in many research works of the theory of variable exponent function spaces. For example, Lebesgue spaces with variable exponent were studied in [10, 12, 19], Herz spaces with variable exponent were studied in [17, 26, 29], Morrey spaces with variable exponent were studied in [20, 32, 34], and some other type of function spaces with variable exponent can be found in [12, 13, 21, 25, 35, 36, 38–41].

Along with the development of the theory of variable exponent function spaces, the theories of the rough singular integral operators and their commutators on these function spaces with variable exponents have attracted many researchers’ attentions. In the variable exponent Lebesgue spaces, Cruz-Uribe et al obtained the boundedness of the rough singular integral operator [10]. The related works were generalized to Herz spaces with variable exponents. For example, Wang proved the rough singular integral operator $T_\Omega$ and its commutators are bounded from the variable exponent Herz space $H_{q(\cdot)}^{a,p_1}(\mathbb{R}^n)$ to the variable exponent Herz space $K_{q(\cdot)}^{a,p_2}(\mathbb{R}^n)$ [31]. Besides it, Wang et al considered the parameterized Littlewood-Paley operators and their commutators on Herz spaces with variable exponents $K_{q(\cdot),p(\cdot)}^a(\mathbb{R}^n)$ [33].

Motivated by the above works, in the paper, we devote to solve the boundedness of the rough singular integral operators and their commutators on the homogeneous Herz spaces with variable exponents $K_{q(\cdot),p(\cdot)}^a(\mathbb{R}^n)$. The rest of this paper is arranged as follows. In Section 2 we recall the definition of the homogeneous Herz spaces with variable exponents $K_{q(\cdot),p(\cdot)}^a(\mathbb{R}^n)$ and state our main results. The proofs of the main theorems will be proved in Sections 3 and 4, respectively.

Finally, some conventions should be explained. $C$ is denoted by a positive constant whose value may be different from line to line. The symbol $A \lesssim B$ stands for the inequality $A \leq CB$. Other notations will be explained when we meet it.

## 2 Preliminary and main results

Let $\lambda \in (0,\infty)$ and $p(\cdot) : \mathbb{R}^n \to [1,\infty)$ be a measurable function. The Lebesgue space with variable exponent $p(\cdot)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) := \left\{ f \text{ is measurable} : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, dx < \infty \right\},$$
for some $\lambda > 0$. Then $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space when it is equipped with the norm

$$
\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, dx \leq 1 \right\}.
$$

The boundedness of the Hardy-Littlewood maximal operator in the variable exponent Lebesgue spaces plays an essential role to extend results in classical harmonic analysis and function theory to the variable exponent case; see [9, 10] and the references therein. Let $L^1_{\text{loc}}(\mathbb{R}^n)$ be the collection of locally integrable functions on $\mathbb{R}^n$. Given a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator $M$ is defined by

$$
Mf(x) := \sup_{r > 0} r^{-n} \int_{B(x,r)} |f(y)| \, dy, \quad \forall x \in \mathbb{R}^n,
$$

where $B(x,r) := \{ y \in \mathbb{R}^n : |x - y| < r \}$.

Let

$$
p_- := \text{essinf} \left\{ p(x) : x \in \mathbb{R}^n \right\} \quad \text{and} \quad p_+ := \text{esssup} \left\{ p(x) : x \in \mathbb{R}^n \right\}.
$$

Then the notations $\mathcal{P}(\mathbb{R}^n)$ and $\mathcal{P}^0(\mathbb{R}^n)$ are defined respectively by

$$
\mathcal{P}(\mathbb{R}^n) := \{ p(x) \text{ is measurable} : 1 < p_- \leq p(x) \leq p_+ < \infty \},
$$

$$
\mathcal{P}^0(\mathbb{R}^n) := \{ p(x) \text{ is measurable} : 0 < p_- \leq p(x) \leq p_+ < \infty \}.
$$

We say $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. In particular, if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and satisfies the following two conditions:

$$
|p(x) - p(y)| \leq \frac{C}{-\log(|x - y|)}, \quad \text{if } |x - y| \leq 1/2,
$$

and

$$
|p(x) - p(y)| \leq \frac{C}{\log(|y| + \epsilon)}, \quad \text{if } |y| \geq x,
$$

then $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. In the sequel, the conjugate exponent of $p(x)$ is denoted by $p'(x)$, i.e.,

$$
p'(x) = p(x) / (p(x) - 1).
$$

Let $\Omega \in L^s(S^{n-1})$ with $s \in [1, \infty)$, $\rho$ be a rotation in $\mathbb{R}^n$ and $|\rho| = \|\rho - I\|$. The integral modulus of continuity of order $s$ of the function $\Omega$ is defined by

$$
W_s(\delta) := \sup_{|\rho| < \delta} \left( \int_{S^{n-1}} |\Omega(\rho x') - \Omega(x')|^s \, dx' \right)^{1/s}.
$$
We say $\Omega$ satisfies the $L^s$-Dini condition if it is homogeneous of degree zero on $\mathbb{R}^n$ and
\[
\int_0^1 \frac{W_s(\delta)}{\delta} d\delta < \infty.
\]

Let $p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. The mixed Lebesgue-sequence space $L^{p(\cdot)}(L^{q(\cdot)})$ is defined on sequences of $L^{q(\cdot)}$-functions by the modular
\[
\rho_{L^{p(\cdot)}(L^{q(\cdot)})} \left( \{ f_\nu \}_{\nu = -\infty}^{+\infty} \right) := \sum_{\nu = -\infty}^{+\infty} \inf \left\{ \lambda_\nu > 0 : \rho_{q(\cdot)} \left( \frac{f_\nu}{\lambda_\nu^{1/p(\cdot)}} \right) \leq 1 \right\}.
\]
The quasi-norm is defined by
\[
\left\| \{ f_\nu \}_{\nu = -\infty}^{+\infty} \right\|_{L^{p(\cdot)}(L^{q(\cdot)})} := \inf \left\{ \mu > 0 : \rho_{L^{p(\cdot)}(L^{q(\cdot)})} \left( \{ f_\nu \}_{\nu = -\infty}^{+\infty} \right) \leq 1 \right\}.
\]  

(2.1)

If $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, then the Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ can be defined by
\[
L^{p(\cdot)}(\mathbb{R}^n) := \left\{ f \text{ is measurable: } |f|^p \in L^{q(\cdot)}(\mathbb{R}^n) \text{ for some } 0 < p_0 < p_\cdot \text{ and } q(x) = \frac{p(x)}{p_0} \right\}
\]
and its quasinorm is given by
\[
\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \|f|^{p_0}\|^{1/p_0}_{L^{q(\cdot)}(\mathbb{R}^n)}.
\]
So, we can replace (2.1) by a simple expression

\[
\rho_{L^{p(\cdot)}(L^{q(\cdot)})} \left( \{ f_\nu \}_{\nu = -\infty}^{+\infty} \right) := \sum_{\nu = -\infty}^{+\infty} \left\| f_\nu \right\|_{L^{p(\cdot)}(L^{q(\cdot)})}^{q(\cdot)}.
\]

We now recall the variable exponent Herz space $\dot{K}^{s_{\eta(\cdot),p(\cdot)}}_{q(\cdot)}(\mathbb{R}^n)$. Let $B_k := \{ x \in \mathbb{R}^n : |x| \leq 2^k \}$, $A_k := B_k \setminus B_{k-1}$, and $\chi_k := \chi_{A_k}$ for $k \in \mathbb{Z}$. Denote $\mathbb{Z}_+$ as the sets of non-negative integers, $\chi_k := \chi_k$ if $k \in \mathbb{Z}_+$ and $\chi_0 := \chi_{B_0}$.

**Definition 2.1** ([33]). Let $\alpha \in \mathbb{R}$, $p(\cdot)$ and $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. The homogeneous Herz space with variable exponents $\dot{K}^{s_{\eta(\cdot),p(\cdot)}}_{q(\cdot)}(\mathbb{R}^n)$ is defined by
\[
\dot{K}^{s_{\eta(\cdot),p(\cdot)}}_{q(\cdot)}(\mathbb{R}^n) := \left\{ f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus 0) : \|f\|_{\dot{K}^{s_{\eta(\cdot),p(\cdot)}}_{q(\cdot)}(\mathbb{R}^n)} < \infty \right\},
\]
where
\[
\|f\|_{\dot{K}^{s_{\eta(\cdot),p(\cdot)}}_{q(\cdot)}(\mathbb{R}^n)} := \left\{ \left\| 2^{k\alpha} |f| \chi_k \right\|_{L^{p(\cdot)}(L^{q(\cdot)})} \right\}_{k = -\infty}^{+\infty} := \inf \left\{ \eta > 0 : \sum_{k = -\infty}^{+\infty} \left\| \left( 2^{k\alpha} |f| \chi_k \right) \right\|_{L^{p(\cdot)}(L^{q(\cdot)})}^{q(\cdot)} \leq 1 \right\}.
\]
Lemma 2.1 ([18]). Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant $C$ such that for all balls $B \subset \mathbb{R}^n$ and all measurable subsets $S \subset B$, then

\begin{enumerate}[(i)]
  \item $\frac{1}{|B|} \| \chi_B \|_{L^{p(\cdot)}(\mathbb{R}^n)} \| \chi_B \|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C$;
  \item $\frac{\| \chi_S \|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\| \chi_S \|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \leq \frac{|S|}{|B|}$;
  \item $\frac{\| \chi_S \|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\| \chi_S \|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \leq \left( \frac{|S|}{|B|} \right)^{\delta_1}$ and $\frac{\| \chi_S \|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\| \chi_S \|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \leq \left( \frac{|S|}{|B|} \right)^{\delta_2}$, where $\delta_1$ and $\delta_2$ are positive constants less than one.
\end{enumerate}

Now it is the position to state our main results. Theorem 2.1 is about the boundedness of the rough singular integral operator $T_\Omega$ on homogeneous Herz spaces with variable exponents.

Theorem 2.1. Let $p_1(\cdot)$ and $p_2(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ with $(p_1)_+ \leq (p_2)_- \cdot q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, and $\Omega$ satisfy the $L^s$-Dini condition $(s > \max\{q_+, q_-\})$. Suppose that $\alpha \in ((n-1)/s - n\delta_1, n\delta_2)$, where $\delta_1$ and $\delta_2$ are defined in Lemma 2.1 below. Then $T_\Omega$ is bounded from $\dot{K}^a_{q(\cdot), p_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^a_{q(\cdot), p_2(\cdot)}(\mathbb{R}^n)$.

Let $b$ be a locally integrable function on $\mathbb{R}^n$. The commutator $[b, T_\Omega]$ is defined by

$$[b, T_\Omega]f(x) := p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} [b(x) - b(y)] f(y) dy.$$  

The boundedness of the commutator $[b, T_\Omega]$ on homogeneous Herz spaces with variable exponents will be obtained in Theorem 2.2.

Recall that the $\text{BMO}(\mathbb{R}^n)$ space consists of all locally integrable functions $f$ such that

$$\|f\|_{\text{BMO}} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx < \infty,$$

where $f_Q = |Q|^{-1} \int_Q f(y) \, dy$, the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes and $|Q|$ denotes the Lebesgue measure of $Q$.

Theorem 2.2. Let $p_1(\cdot)$ and $p_2(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ with $(p_1)_+ \leq (p_2)_- \cdot q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, and $\Omega$ satisfy the $L^s$-Dini condition $(s > \max\{q_+, q_-\})$. Suppose that $b \in \text{BMO}(\mathbb{R}^n)$ and $\alpha \in ((n-1)/s - n\delta_1, n\delta_2)$, where $\delta_1$ and $\delta_2$ are defined in Lemma 2.1. Then $[b, T_\Omega]$ is bounded from $\dot{K}^a_{q(\cdot), p_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^a_{q(\cdot), p_2(\cdot)}(\mathbb{R}^n)$.

3 Proof of Theorem 2.1

To give the proof of Theorem 2.1, we need the following lemmas.
Lemma 3.1 ([22]). Let $x \in \mathbb{R}^n$, $s \in (1, \infty)$ and $\Omega$ satisfy the $L^s$-Dini condition. Then there is a positive constant $C$ for any $|y| \in (0, R/2)$ with $R \in (0, \infty)$ such that
\[
\left( \int_{R <|x| < 2R} \left| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n} \right|^s \right)^{1/s} \leq R^{\frac{n}{s} - n} \left\{ \frac{|y|}{R} + \int_{|y|/2R < |y|/|\Omega|/\delta} \frac{W_s(d)}{\delta} \right\}.
\]

Lemma 3.2 ([9]). Let $E \subset \mathbb{R}^n$, $p(\cdot) \in \mathcal{P}(E)$, and $f : E \times E \to \mathbb{R}$ be a measurable function (with respect to product measure) such that for almost every $y \in E$, $f(\cdot, y) \in L^{p(\cdot)}(E)$. Then
\[
\left\| \int_E f(\cdot, y) \, dy \right\|_{L^{p(\cdot)}(E)} \leq \int_E \left\| f(\cdot, y) \right\|_{L^{p(\cdot)}(E)} \, dy.
\]

Lemma 3.3 ([25]). Let $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$, $p_+ < q \leq \infty$ and $1 \leq q$. Denote $\tilde{q}(\cdot)$ be $\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{1}{q}$ $(x \in \mathbb{R}^n)$. Then
\[
\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \|f\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{q(\cdot)}(\mathbb{R}^n)}
\]
for all measurable functions $f$ and $g$.

Lemma 3.4 ([33]). Let $p(\cdot), q(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$. Assume that $f \in L^{(\cdot, \cdot)}(\mathbb{R}^n)$, then
\[
\min \left( \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right) \leq \left\| \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right\|_{L^1(\mathbb{R}^n)} \leq \max \left( \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right).
\]

Lemma 3.5. Let $k \in \mathbb{Z}$, $a_k \in [0, +\infty)$ and $1 \leq p_k \leq \sup_{k \in \mathbb{Z}} p_k < \infty$. Then
\[
\sum_{k \in \mathbb{Z}} a_k^{p_k} \leq 2 \left( \sum_{k \in \mathbb{Z}} a_k \right)^{p_*},
\]
where
\[
p_* = \begin{cases} \inf_{k \in \mathbb{Z}} p_k, & \sum_{k \in \mathbb{Z}} a_k \leq 1, \\ \sup_{k \in \mathbb{Z}} p_k, & \sum_{k \in \mathbb{Z}} a_k > 1. \end{cases}
\]

Proof. If $\sum_{k \in \mathbb{Z}} a_k \leq 1$, we have
\[
\sum_{k \in \mathbb{Z}} a_k^{p_k} \leq \sum_{k \in \mathbb{Z}} a_k^{\inf_{k \in \mathbb{Z}} p_k} \leq \left( \sum_{k \in \mathbb{Z}} a_k \right)^{\inf_{k \in \mathbb{Z}} p_k}.
\]
On the other hand, if $\sum_{k \in \mathbb{Z}} a_k > 1$, we have
\[
\sum_{k \in \mathbb{Z}} a_k^{p_k} = \sum_{a_k > 1, k \in \mathbb{Z}} a_k^{p_k} + \sum_{a_k \leq 1, k \in \mathbb{Z}} a_k^{p_k}.
\]
where $\in$ the conjugate exponent of $s$. Then for any $f$, Lemma 3.7 (10)

Let $\Omega \in L^s(S^{n-1})$. Assume that $d \in (0,s]$ and $\mu \in (-n+(n-1)d/s,\infty)$, then

$$\left( \int_{|x| \leq |y|} |\Omega(x-y)|^d |x|^\mu \, dx \right)^{\frac{1}{d}} \lesssim |y|^\frac{\mu}{d} \|\Omega\|_{L^s(S^{n-1})}.$$

Lemma 3.7 (10). Let $\Omega \in L^s(S^{n-1})$ satisfies (1.1), $q(\cdot) \in B(R^n)$ and $q_\cdot \in (s',\infty)$, where $s'$ is the conjugate exponent of $s$. Then for any $f \in L^{q}(\mathbb{R}^n)$, $T_\Omega$ is bounded on $L^{q}(\mathbb{R}^n)$.

By using the above lemmas we are able to prove Theorem 2.1 now.

Proof of Theorem 2.1. Let $f \in K^a_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$. We can write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x) \chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

By the definition of $K^a_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$, we write

$$\|T_\Omega(f)\|_{K^a_{q(\cdot),p(\cdot)}(\mathbb{R}^n)} \lesssim \eta_{1,1} + \eta_{1,2} + \eta_{1,3},$$

where

$$\eta_{1,1} = \left\{ \left\| 2^{ka} \left| \sum_{j=-\infty}^{k-2} T_\Omega(f_j) \chi_k \right| \right\|_{p_2(1)(L^q(\cdot))}^\infty, \right\}$$

$$\eta_{1,2} = \left\{ \left\| 2^{ka} \left| \sum_{j=k-1}^{k+1} T_\Omega(f_j) \chi_k \right| \right\|_{p_2(1)(L^q(\cdot))}^\infty, \right\}$$
First, we consider \( \eta \) and \( \eta_0 \). Denote \( \eta_0 = \| f \|_{K_p^{a q_1(p_1)}(\mathbb{R}^n)} \). To prove the theorem, it suffices to prove \( \eta_{1,1}, \eta_{1,2}, \eta_{1,3} \lesssim \eta_0 \). First, we consider \( \eta_{1,2} \). Applying Lemmas 3.4 and 3.7, using Lemma 3.2.5 in [11], we have

\[
\sum_{k=-\infty}^{\infty} \left\| \left( \frac{2^{ka} \sum_{j=k+1}^{\infty} T_{\alpha} (f_j) \chi_k}{\eta_0} \right)^{p_2} \right\|_{L^{q_1(p_1)}(\mathbb{R}^n)} \leq \sum_{k=-\infty}^{\infty} \left( \sum_{j=k+1}^{\infty} \left\| \frac{2^{ka} T_{\alpha} (f_j) \chi_k}{\eta_0} \right\|_{L^{q_1(p_1)}(\mathbb{R}^n)} \right)^{p_2},
\]

where

\[
(p_2^1)_k = \begin{cases} 
(p_2)_-, & \left\| \frac{2^{ka} \sum_{j=k+1}^{\infty} T_{\alpha} (f_j) \chi_k}{\eta_0} \right\|_{L^{q_1(p_1)}(\mathbb{R}^n)} \leq 1, \\
(p_2)_+, & \left\| \frac{2^{ka} \sum_{j=k+1}^{\infty} T_{\alpha} (f_j) \chi_k}{\eta_0} \right\|_{L^{q_1(p_1)}(\mathbb{R}^n)} > 1,
\end{cases}
\]

and \( C = \max \left\{ C_1 (p_2^1)_-, C_1 (p_2)_+ \right\} \). Since \( f \in K_p^{a q_1(p_1)}(\mathbb{R}^n) \), it is easy to see that

\[
\sum_{k=-\infty}^{\infty} \left( \frac{2^{ka} |f \chi_k|}{\eta_0} \right)^{q_1(p_1)} \leq 1, \quad \left\| \frac{2^{ka} |f \chi_k|}{\eta_0} \right\|_{L^{q_1(p_1)}(\mathbb{R}^n)} \leq 1.
\]
Hence, by Lemmas 3.4 and 3.5, we obtain

\[
\sum_{k=-\infty}^{\infty} \left\| \frac{\left(2^{ka} \sum_{j=k-1}^{k+1} T_{\Omega} (f_j) \chi_k \right)}{\eta_0} \right\|_{L^{p_2} (\mathbb{R}^n)}^2 \leq C,
\]

where \((p_1)^+ \leq (p_2)_- \leq (p_2^1)_k\) and \(p_2^1 = \inf_{k \in \mathbb{Z}} \frac{(p_1)_k}{(p_2)_k}\). Consequently, we have \(\eta_{1,2} \leq \eta_0 = \|f\|_{K^{\alpha_{\Omega, \Omega}} (\mathbb{R}^n)}\).

Now let us turn to \(\eta_{1,1}\). By Lemma 3.4 we have

\[
\sum_{k=-\infty}^{\infty} \left\| \frac{\left(2^{ka} \sum_{j=-\infty}^{k-2} T_{\Omega} (f_j) \chi_k \right)}{\eta_0} \right\|_{L^{p_2} (\mathbb{R}^n)}^2 \leq C,
\]

where \((p_2)_- \leq (p_2^2)_k \leq (p_2^+_2)\),

\[
(p_2^2)_k = \begin{cases} 
(p_2)_-, & \text{if } \left\| \frac{\left(2^{ka} \sum_{j=-\infty}^{k-2} T_{\Omega} (f_j) \chi_k \right)}{\eta_0} \right\|_{L^{p_2} (\mathbb{R}^n)} \leq 1, \\
(p_2^+_2), & \text{if } \left\| \frac{\left(2^{ka} \sum_{j=-\infty}^{k-2} T_{\Omega} (f_j) \chi_k \right)}{\eta_0} \right\|_{L^{p_2} (\mathbb{R}^n)} > 1.
\end{cases}
\]
To estimate \( \| T_\Omega (f_j) \chi_k \|_{L^q (\mathbb{R}^n)} \), we use Lemma 3.2 and obtain
\[
\| T_\Omega (f_j) \chi_k \|_{L^q (\mathbb{R}^n)} \lesssim \int_{B_j} \left| \frac{\Omega (-y)}{|y|^n} \right| \left| \frac{\chi_k (\cdot)}{|\cdot|^n} \right| \left| f_j (y) \right| dy.
\]

For each \( k \in \mathbb{Z}, j \leq k - 2, x \in A_k, y \in A_j, s > q_+ \), denote \( \tilde{q} (\cdot) > 1 \) and \( \frac{1}{q (x)} = \frac{1}{q (x)} + \frac{1}{s} \). By Lemmas 3.1 and 3.3, we have
\[
\| \frac{\Omega (-y)}{|y|^n} \frac{\chi_k (\cdot)}{|\cdot|^n} \|_{L^q (\mathbb{R}^n)} \lesssim 2^{(k-1)(\frac{q_+}{q} - n)} \left\{ \frac{|y|}{2^{k-1}} + \int_{|y|/2^k} W_s (\delta) \frac{d\delta}{\delta} \right\} \| \chi B_k \|_{L^q (\mathbb{R}^n)}
\]
\[
\lesssim 2^{(k-1)(\frac{q_+}{q} - n)} \left\{ 2^j - k + 1 + \int_0^1 W_s (\delta) \frac{d\delta}{\delta} \right\} \| \chi B_k \|_{L^q (\mathbb{R}^n)} \| B_k \|^{-\frac{1}{q}}
\]
\[
\lesssim 2^{-kn} \| \chi B_k \|_{L^q (\mathbb{R}^n)} \] (3.1)

and
\[
\| \frac{\Omega (\cdot)}{|\cdot|^n} \chi_k (\cdot) \|_{L^q (\mathbb{R}^n)} \lesssim 2^{-kn} \| \Omega (\cdot) \chi_k (\cdot) \|_{L^q (\mathbb{R}^n)} \| \chi B_k \|_{L^q (\mathbb{R}^n)}
\]
\[
= 2^{-kn} \left( \int_{2^{k-1}}^{2^k} r^{n-1} dr \int_{S^{n-1}} |\Omega (x')|^s d\nu \right)^{\frac{1}{s}} \| \chi B_k \|_{L^q (\mathbb{R}^n)}
\]
\[
\lesssim 2^{-kn} 2^{m_2} \| \Omega \|_{L^1 (S^{n-1})} \| \chi B_k \|_{L^q (\mathbb{R}^n)}
\]
\[
\lesssim 2^{-kn} 2^{m_2} \| \Omega \|_{L^1 (S^{n-1})} \| \chi B_k \|_{L^q (\mathbb{R}^n)} \| B_k \|^{-\frac{1}{q}}
\]
\[
\lesssim 2^{-kn} \| \chi B_k \|_{L^q (\mathbb{R}^n)} \] (3.2)

where the last inequality is based on the fact that \( \| \chi B_k \|_{L^q (\mathbb{R}^n)} \approx \| \chi B_k \|_{L^q (\mathbb{R}^n)} \| B_k \|^{-\frac{1}{q}} \); see
Thus, we discuss it in two cases.

\[
\|T_\Omega(f_j)\chi_k\|_{L^p(\mathbb{R}^n)} \lesssim 2^{-kn} \|\chi_{B_k}\|_{L^{q'}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q'}(\mathbb{R}^n)} \|f_j\|_{L^q(\mathbb{R}^n)} \\
\lesssim 2^{-kn} |B_k| \|\chi_{B_k}\|_{L^{q'}(\mathbb{R}^n)} \|f_j\|_{L^q(\mathbb{R}^n)} \\
\lesssim 2^{(j-k)n\delta_2} \|f_j\|_{L^q(\mathbb{R}^n)}.
\]

Thus,

\[
\sum_{k=-\infty}^{2} \left\| \left( \frac{2^{ka} \sum_{j=-\infty}^{k-2} T_\Omega(f_j)\chi_k}{\eta_0} \right) \right\|_{L^{p_1}(\mathbb{R}^n)}^{p_2(-)} \frac{\|f_j\|_{L^q(\mathbb{R}^n)}}{(p^2_2)_k} \\
\lesssim \sum_{k=-\infty}^{2} \left( \sum_{j=-\infty}^{k-2} \frac{2^{ka}}{\eta_0} \|f_j\|_{L^q(\mathbb{R}^n)} \right) \left( \frac{2^{j\alpha} f_j}{\eta_0} \right)_{L^q(\mathbb{R}^n)}^{p_1(-)} \frac{1}{\|f_j\|_{L^q(\mathbb{R}^n)}} (p^2_2)_k \\
= \sum_{k=-\infty}^{2} \left( \sum_{j=-\infty}^{k-2} \frac{2^{j\alpha} f_j}{\eta_0} \right)_{L^q(\mathbb{R}^n)}^{p_1(-)} \frac{1}{\|f_j\|_{L^q(\mathbb{R}^n)}} (p^2_2)_k.
\]

We discuss it in two cases.

\textbf{Case 1:} let \(0 < (p_1)_+ \leq 1\),

\[
\sum_{k=-\infty}^{2} \left\| \left( \frac{2^{ka} \sum_{j=-\infty}^{k-2} T_\Omega(f_j)\chi_k}{\eta_0} \right) \right\|_{L^{p_1}(\mathbb{R}^n)}^{p_2(-)} \frac{1}{\|f_j\|_{L^q(\mathbb{R}^n)}} (p^2_2)_k \\
\lesssim \sum_{k=-\infty}^{2} \left( \sum_{j=-\infty}^{k-2} 2^{(j-k)(n\delta_2-a)} (p_1)_+ \right) \left( \frac{2^{j\alpha} f_j}{\eta_0} \right)_{L^q(\mathbb{R}^n)}^{p_1(-)} \frac{1}{\|f_j\|_{L^q(\mathbb{R}^n)}} (p^2_2)_k \\
\lesssim \left\{ \sum_{j=-\infty}^{2} \left( \frac{2^{j\alpha} f_j}{\eta_0} \right)_{L^q(\mathbb{R}^n)}^{p_1(-)} \frac{1}{\|f_j\|_{L^q(\mathbb{R}^n)}} \sum_{k=j+2}^{\infty} 2^{(j-k)(n\delta_2-a)} (p_1)_+ \right\} p^2 \\
\lesssim C,
\]

[31]
where \( p^*_2 = \inf_{k \in \mathbb{Z}(p_1)_+} \frac{\langle p \rangle_k}{k} \).

Case 2: let \((p_1)_+ > 1,
\[
\sum_{k = -\infty}^{\infty} \left\| \frac{2^k |\sum_{j = -\infty}^{k-2} T_\Omega(f_j) \chi_k|}{\eta_0} \right\|_{L_{\frac{p_1}{p_1'}}(\mathbb{R}^n)}^{p_2(\cdot)} \approx \sum_{j = -\infty}^{\infty} \left\| \frac{2^j a f_j}{\eta_0} \right\|_{L_{\frac{p_1}{p_1'}}(\mathbb{R}^n)}^{p_2(\cdot)} \sum_{k = j+2}^{\infty} 2^{j-k} (n\delta_2 - a)(p_1)_+ / 2
\leq C,
\]
where \( p^*_2 = \inf_{k \in \mathbb{Z}(p_1)_+} \frac{\langle p \rangle_k}{k} \). This implies that
\[
\eta_{1,1} \lessapprox \eta_0 = \|f\|_{L^p(\mathbb{R}^n)}. \]

Finally we estimate \( \eta_{1,3} \). For each \( k \in \mathbb{Z}, j \geq k+2 \), when \( x \in A_k, y \in A_j \), there hold \( |x| \leq |y| / 2 \) and \( |x - y| \geq |y| - |x| \geq 2^{j-2} \). Noting \( s > q_+, \tilde{q}(\cdot) > 1 \) with \( \frac{1}{q(x)} = \frac{1}{\tilde{q}(x)} + \frac{\alpha}{p} \), since \( \alpha > \frac{\alpha - 1}{q} - n\delta_1 \), we can choose \( v \) such that \( \alpha > v + \frac{\alpha - 1}{q} - n\delta_1 > \frac{\alpha - 1}{p} - n\delta_1 \), and applying Lemmas 3.2, 3.3, 3.6 and 2.1 to obtain
\[
\| T_\Omega(f_j) \chi_k \|_{L^s(\mathbb{R}^n)} \lesssim \int_{B_j} \left\| \frac{\Omega(-y)}{|-y|^n} \right\|_{L^s(\mathbb{R}^n)} \| \chi_k(\cdot) \|_{L^v(\mathbb{R}^n)} \| f_j(y) \|_{L^v(\mathbb{R}^n)} dy
\lesssim \int_{B_j} 2^{-jn} \left\| \Omega(-y) \right\|_{L^s(\mathbb{R}^n)} \| \chi_k(\cdot) \|_{L^v(\mathbb{R}^n)} \| f_j(y) \|_{L^v(\mathbb{R}^n)} dy
\lesssim \int_{B_j} 2^{-jn} \left\| \Omega(-y) \right\|_{L^s(\mathbb{R}^n)} \| \chi_k(\cdot) \|_{L^v(\mathbb{R}^n)} \| f_j(y) \|_{L^v(\mathbb{R}^n)} dy
= \int_{B_j} 2^{-jn} \left( \int_{A_k} \| \Omega(x-y) \|_{L^s(\mathbb{R}^n)} \| \chi_k(\cdot) \|_{L^v(\mathbb{R}^n)} \| f_j(y) \|_{L^v(\mathbb{R}^n)} dx \right) \| f_j(y) \|_{L^v(\mathbb{R}^n)} dy
\lesssim \int_{B_j} 2^{-jn} 2^{-kv} \left( \int_{|x| \leq |y| / 2} |\Omega(x-y)|^s |x|^{sv} dx \right)^{\frac{1}{s}} \| \chi_{B_k} \|_{L^v(\mathbb{R}^n)} \| f_j(y) \|_{L^v(\mathbb{R}^n)} dy
\]
\[
\begin{align*}
& \lesssim \int_{B_j} 2^{-jn_2(j-k)\nu} |y|^{\frac{n_2}{2n-2\nu}} \| \Omega \|_{L^q(S^{n-1})} \| \chi B_k \|_{L^q(S^n)} \| B_k \|^{-\frac{1}{2}} \left| f_j(y) \right| \, dy \\
& \lesssim \int_{B_j} 2^{-jn_2(j-k)(v+\frac{3}{2})} \| \Omega \|_{L^q(S^{n-1})} \| \chi B_k \|_{L^q(S^n)} \| f_j(y) \| \, dy \\
& \lesssim 2^{-jn_2(j-k)(v+\frac{3}{2})} \| \chi B_k \|_{L^q(S^n)} \| \chi B_j \|_{L^q(S^n)} \| f_j(y) \| \| L^q(S^n) \\
& \lesssim 2^{-jn_2(j-k)(v+\frac{3}{2})} \| \chi B_k \|_{L^q(S^n)} \| \chi B_j \|_{L^q(S^n)} \| f_j(y) \| \| L^q(S^n) \\
& \lesssim 2 \left(k-j\right) \left(n\delta_1 - v - \frac{3}{2}\right) \| f_j(y) \| \| L^q(S^n) \\
\end{align*}
\]

Similar to $\eta_{1,1}$, we have

\[
\begin{align*}
& \lesssim \sum_{k=-\infty}^{\infty} \left\| \left( \frac{2^{ka} \sum_{j=k+2}^{\infty} T_{\Omega}(f_j) \chi_k}{\eta_0} \right) \right\|_{L^2_{\Omega}(\mathbb{R}^n)} \\
& \lesssim \sum_{k=-\infty}^{\infty} \left( \sum_{j=k+2}^{\infty} \frac{2^{ka}}{\eta_0} \left\| T_{\Omega}(f_j) \chi_k \right\|_{L^q(S^n)} \right) \\
& \lesssim \sum_{k=-\infty}^{\infty} \left( \sum_{j=k+2}^{\infty} \frac{2^{ka}}{\eta_0} 2^{\left(k-j\right)(n\delta_1 - v - \frac{3}{2})} \left\| f_j \right\|_{L^q(S^n)} \right) \\
& \lesssim \sum_{k=-\infty}^{\infty} \left( \sum_{j=k+2}^{\infty} 2^{\left(k-j\right)(a+n\delta_1 - v - \frac{3}{2})} \left\| \left( \frac{2^{ja} f_j}{\eta_0} \right) \right\|_{L^q(S^n)} \right) \\
& \lesssim \left\{ \begin{array}{ll}
\left( p_2 \right)_-, & \text{if } \left\| \frac{2^{ka} \sum_{j=k+2}^{\infty} T_{\Omega}(f_j) \chi_k}{\eta_0} \right\|_{L^q(S^n)} \leq 1, \\
\left( p_2 \right)_+, & \text{if } \left\| \frac{2^{ka} \sum_{j=k+2}^{\infty} T_{\Omega}(f_j) \chi_k}{\eta_0} \right\|_{L^q(S^n)} > 1.
\end{array} \right.
\end{align*}
\]

where

\[
\left( p_2 \right)_k = \begin{cases} 
\left( p_2 \right)_-, & \text{if } \left\| \frac{2^{ka} \sum_{j=k+2}^{\infty} T_{\Omega}(f_j) \chi_k}{\eta_0} \right\|_{L^q(S^n)} \leq 1, \\
\left( p_2 \right)_+, & \text{if } \left\| \frac{2^{ka} \sum_{j=k+2}^{\infty} T_{\Omega}(f_j) \chi_k}{\eta_0} \right\|_{L^q(S^n)} > 1.
\end{cases}
\]

When $0 < \left( p_1 \right)_+ \leq 1,$

\[
\begin{align*}
& \lesssim \sum_{k=-\infty}^{\infty} \left\| \left( \frac{2^{ka} \sum_{j=k+2}^{\infty} T_{\Omega}(f_j) \chi_k}{\eta_0} \right) \right\|_{L^2_{\Omega}(\mathbb{R}^n)} \\
& \lesssim \sum_{k=-\infty}^{\infty} \left( \sum_{j=k+2}^{\infty} 2^{\left(k-j\right)(a+n\delta_1 - v - \frac{3}{2})} \left( \frac{2^{ja} f_j}{\eta_0} \right) \right) \left\| \left( \frac{2^{ja} f_j}{\eta_0} \right) \right\|_{L^q(S^n)} \\
& \lesssim \left( \frac{p_2}{p_1} \right)_k \\
\end{align*}
\]
\[ \sum_{j=-\infty}^{\infty} \left( \frac{2^{ja_j} f_j}{\eta_0} \right)^{p_1(\cdot)} \left( \frac{\mathfrak{g}(\cdot)}{L^P(\mathbb{R}^n)} \right)^{\frac{1}{p_1(\cdot)}} \leq C, \]

where \( p_3^* = \inf_{k \in \mathbb{Z}} \left( \frac{p_3(\cdot)}{p_1(\cdot)} \right)_+ \). When \( (p_1)_+ > 1, \)

\[ \sum_{k=-\infty}^{\infty} \left( \sum_{j=k+2}^{\infty} 2^{(k-j)(a+n\delta_1-v-\frac{\alpha}{p})} \right) \left( \frac{2^{ja_j} f_j}{\eta_0} \right)^{p_1(\cdot)} \left( \frac{\mathfrak{g}(\cdot)}{L^P(\mathbb{R}^n)} \right)^{\frac{1}{p_1(\cdot)}} \]

\[ \times \sum_{j=k+2}^{\infty} 2^{(k-j)(a+n\delta_1-v-\frac{\alpha}{p})} \left( \frac{2^{ja_j} f_j}{\eta_0} \right)^{p_1(\cdot)} \left( \frac{\mathfrak{g}(\cdot)}{L^P(\mathbb{R}^n)} \right)^{\frac{1}{p_1(\cdot)}} \]

\[ \leq C, \]

where \( p_3^* = \inf_{k \in \mathbb{Z}} \left( \frac{p_3(\cdot)}{p_1(\cdot)} \right)_+ \). So we have \( \eta_1, 3 \lesssim \eta_0 = \| f \|_{k^*_1, (p_1)_+}^{k^*_1} \). This completes the proof of Theorem 2.1.

\section{Proof of Theorem 2.2}

The following lemma will play an important role in our proof.

\textbf{Lemma 4.1 ([17])}. Let \( p(\cdot) \in \mathcal{B}(\mathbb{R}^n), k \) be a positive integer and \( b \in BMO(\mathbb{R}^n) \), then there is a positive \( C \) such that for all balls \( B \) in \( \mathbb{R}^n \) and all \( j, i \in \mathbb{Z} \) with \( j > i \),

\[ \frac{1}{C} \| b \|^k \leq \sup_B \left( \frac{1}{\| \chi_B \|_{L^p(\mathbb{R}^n)}} \right) \| (b-b_B)^k \chi_B \|_{L^p(\mathbb{R}^n)} \leq C \| b \|^k, \]

\[ \| (b-b_B)^k \chi_B \|_{L^p(\mathbb{R}^n)} \leq C (j-i)^k \| b \|^k \| \chi_B \|_{L^p(\mathbb{R}^n)}. \]
Similar to Lemma 3.7, we can also obtain the boundedness of $[b, T_{\Omega}]$ on $L^{q(\cdot)}(\mathbb{R}^n)$; see [10] for the details.

\textbf{Proof of Theorem 2.2.} Let $b \in \text{BMO}$, $f \in K^a_{q(\cdot), p(\cdot)}(\mathbb{R}^n)$. Similar to the proof of Theorem 2.2, we divide the estimation into three parts. Write

$$\begin{aligned}
\left\| \frac{2^{ka} \left[ [b, T_{\Omega}] (f) \chi_k \right]}{\eta} \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(\cdot)} \lesssim \eta_{2,1} + \eta_{2,2} + \eta_{2,3},
\end{aligned}$$

where $\eta_{2,i}$ is defined similarly to $\eta_{1,i} (i=1,2,3)$ in the proof in Theorem 2.2, except that the operator $T_{\Omega}$ in $\eta_{1,i}$ has to be changed to the commutator $[b, T_{\Omega}]$. Analogously, it suffices to prove $\eta_{2,i} \lesssim \|f\|_{K^a_{q(\cdot), p(\cdot)}(\mathbb{R}^n)}$.

First we consider $\eta_{2,2}$. By the boundedness of $[b, T_{\Omega}]$ on $L^{p(\cdot)}$ [10], as discussed about $\eta_{1,2}$ in the proof of Theorem 2.1, we will get

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} \left\| \frac{2^{ka} \left[ \sum_{j=k-1}^{k+1} [b, T_{\Omega}] (f_j) \chi_k \right]}{\eta_0 \|b\|_*} \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(\cdot)} \leq C.
\end{aligned}$$

Thus

$$\eta_{2,2} \lesssim \|b\|_* \eta_0 = \|b\|_* \|f\|_{K^a_{q(\cdot), p(\cdot)}(\mathbb{R}^n)}.$$

We turn to estimate $\eta_{2,1}$. For each $k \in \mathbb{Z}$, $j \leq k-2$, $x \in A_k$, $y \in A_j$, using Lemma 3.2, we have

$$\begin{aligned}
\| [b, T_{\Omega}] (f_j) \chi_k \|_{L^{q(\cdot)}(\mathbb{R}^n)} &\leq \int_{B_j} \left\| \frac{\Omega(\cdot - y)}{|y|^n} \right\| \left[ b(\cdot) - b(y) \right] \chi_k(\cdot) \|f_j(y)\| \, dy \\
&\lesssim \int_{B_j} \left\| \frac{\Omega(\cdot - y)}{|y|^n} \left[ \frac{\Omega(\cdot)}{|\cdot|^n} \right] \left[ b(\cdot) - b(y) \right] \chi_k(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|f_j(y)\| \, dy \\
&\quad + \int_{B_j} \left\| \frac{\Omega(\cdot - y)}{|y|^n} \left[ \frac{\Omega(\cdot)}{|\cdot|^n} \right] \chi_k(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \left| b_j - b(y) \right| |f_j(y)| \, dy \\
&\quad + \int_{B_j} \left\| \frac{\Omega(\cdot)}{|\cdot|^n} \chi_k(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \left| b_j - b(y) \right| |f_j(y)| \, dy \\
&\quad + \int_{B_j} \left\| \frac{\Omega(\cdot)}{|\cdot|^n} \chi_k(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \left| b_j - b(y) \right| |f_j(y)| \, dy \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$
Noting \( s > q_+ \), we denote \( \tilde{q} ( \cdot ) > 1 \) and \( \frac{1}{q ( \cdot )} = \frac{1}{q ( \cdot )} + \frac{1}{s} \). By Lemmas 3.1 and 3.3, as well as Lemma 4.1, we have

\[
\left\| \frac{\Omega ( \cdot - y )}{\cdot - y \cdot - y} \mathbf{b} ( \cdot - b_B ) \chi_k ( \cdot ) \right\|_{L^q ( \mathbb{R}^n )} \lesssim \left\| \frac{\Omega ( \cdot - y )}{\cdot - y \cdot - y} \chi_k ( \cdot ) \right\|_{L^q ( \mathbb{R}^n )} \left\| \mathbf{b} ( \cdot - b_B ) \right\|_{L^q ( \mathbb{R}^n )} \lesssim 2^{(k-1) \left( \frac{n}{q} - n \right)} \left\{ 2^{j-k+1} + \int_0^1 \frac{W_\epsilon ( \delta )}{\delta} d\delta \right\} (k-j) \left\| \mathbf{b} \right\|^s \left\| \chi B_k \right\|_{L^{q \left( \mathbb{R}^n \right)}},
\]

and

\[
\left\| \frac{\chi_k ( \cdot )}{\cdot - y \cdot - y} \chi_k ( \cdot ) \right\|_{L^q ( \mathbb{R}^n )} \lesssim \left\| \frac{\chi_k ( \cdot )}{\cdot - y \cdot - y} \chi_k ( \cdot ) \right\|_{L^q ( \mathbb{R}^n )} \left\| \mathbf{b} ( \cdot - b_B ) \chi_k ( \cdot ) \right\|_{L^q ( \mathbb{R}^n )} \lesssim 2^{-k} 2^{sn} \left\| \Omega \right\|_{L^q ( \mathbb{R}^n )} \left( k-j \right) \left\| \mathbf{b} \right\|^s \left\| \chi B_k \right\|_{L^{q \left( \mathbb{R}^n \right)}},
\]

From (3.1), (3.2), (4.1) and (4.2), for \( I_1 \) and \( I_3 \), we obtain

\[
I_i \lesssim 2^{(j-k)n2^s} (k-j) \left\| \mathbf{b} \right\|^s \left\| f_i \right\|_{L^q ( \mathbb{R}^n )} \quad (i = 1,3).
\]

For \( I_2 \) and \( I_4 \),

\[
I_i \lesssim 2^{(j-k)n2^s} \left\| \mathbf{b} \right\|^s \left\| f_i \right\|_{L^q ( \mathbb{R}^n )} \quad (i = 2,4).
\]

Similar to the estimation of \( \eta_{1,3} \) in the proof of Theorem 2.1, we can find

\[
\sum_{k=-\infty}^{\infty} \left\| \left( \frac{2^k \sum_{j=-\infty}^{k-2} \left[ b \mathbf{T} \Omega ( f ) \chi_k \right]}{\eta_0 \left\| \mathbf{b} \right\|^s} \right) \right\|_{L^{p_2 \left( \mathbb{R}^n \right)}} \leq C.
\]

Therefore, \( \eta_{2,1} \lesssim \left\| \mathbf{b} \right\|^s \eta_0 \).
Finally we estimate $\eta_{23}$. For each $k \in \mathbb{Z}$, $j \geq k+2$, when $x \in A_k$, $y \in A_j$, there hold $|x| \leq |y|/2$, $|x-y| \geq |y|-|x| \geq 2^{-2}$. By Lemma 3.2, we have
\[
\left\| [b, T_\alpha] (f_j) \chi_k \right\|_{L^q(R^n)} \leq 2^{-jn} \int_{B_j} \left\| \Omega(\cdot - y) \left[ b(\cdot - b_{B_k}) \chi_k(\cdot) \right] f_j(y) \right\| \, dy
\]
\[
+ 2^{-jn} \int_{B_j} \left\| \Omega(\cdot - y) \chi_k(\cdot) \left| b_{B_k} - b(y) \right| f_j(y) \right\| \, dy
\]
\[
= : J_1 + J_2.
\]
Noting $s > q_+$, we denote $\tilde{q}(\cdot) > 1$ and $\frac{1}{q(x)} = \frac{1}{q(\cdot) + \frac{1}{2}}$. Since $\alpha > \frac{n+1}{s} - n\delta_1$, we can choose $\nu$ such that $\alpha > \nu + \frac{\mu}{s} - n\delta_1 > \frac{n+1}{s} - n\delta_1$. By Lemmas 3.3, 3.6 and 4.1, we have
\[
\left\| \Omega(\cdot - y) \left[ b(\cdot - b_{B_k}) \chi_k(\cdot) \right] f_j(y) \right\|_{L^q(R^n)} \leq 2^{-k\nu} \left\| \Omega \right\|_{L^q(S^{n-1})} \left\| b \right\|_{\ast} \left\| \chi_{B_k} \right\|_{L^{q(\cdot)}(R^n)},
\]
and
\[
\left\| \Omega(\cdot - y) \chi_k(\cdot) \left| b_{B_k} - b(y) \right| f_j(y) \right\|_{L^q(R^n)} \leq 2^{(j-k)(\nu + \frac{1}{s})} \left\| b \right\|_{\ast} \left\| \chi_{B_k} \right\|_{L^{q(\cdot)}(R^n)}.
\]
Consequently,
\[
J_1 \lesssim 2^{(k-j)(n\delta_1 - \nu - \frac{1}{s})} \left\| b \right\|_{\ast} \left\| f_j \right\|_{L^q(R^n)},
\]
\[
J_2 \lesssim 2^{(k-j)(n\delta_1 - \nu - \frac{1}{s})(j-k)} \left\| b \right\|_{\ast} \left\| f_j \right\|_{L^q(R^n)}.
\]
Therefore,
\[
\sum_{k=-\infty}^{\infty} \left\| \frac{2^{ka} \sum_{j=k+2}^{\infty} [b, T_\alpha] (f_j) \chi_k}{\eta_0 \left\| b \right\|_{\ast}} \right|_{L^{q(\cdot)}(R^n)} \lesssim \frac{\tilde{q}(\cdot)}{L^{p(\cdot)}(R^n)} \leq C.
\]
Thus, $\eta_{2,3} \lesssim \left\| b \right\|_{\ast} \eta_0$. This completes the proof of Theorem 2.2.

\[\square\]

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