An example of a covering surface with movable natural boundaries

Claudi Meneghin

March 29, 2022

Abstract

We exhibit a covering surface of the punctured complex plane (with no points over 0) whose natural projection mapping fails to be a topological covering, due to the existence of branches with natural boundaries, projecting over different slits in $\mathbb{C}^*$.

In [1], A.F.Beardon shows an example of a 'covering surface' over a region in the unit disc which is not a topological covering (We recall that a 'covering surface' of a region $D \subset \mathbb{C}$ is a Riemann surface $S$ admitting a surjective conformal mapping onto $D$, among which there are the analytical continuations of holomorphic germs; a continuous mapping $p$ of a topological space $Y$ onto another one $X$ is a 'topological covering' provided that each point $x \in X$ admits an open neighbourhood $U$ such that the restriction of $p$ to each connected component $V_i$ of $p^{-1}(U)$ is a homeomorphism of $V_i$ onto $U$, see [2, 3]).

Beardon’s example takes origin by the analytical continuation of a holomorphic germ, namely a branch of the inverse of the Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \left( \frac{z - a_n}{1 - \overline{a_n}z} \right),$$

with suitable hypotheses about the $a_n$’s. It is shown that there is a sequence of inverse branches $f_k$ of $B$ at 0 such that the radii of convergence of their Taylor developments tend to 0 as $k \to \infty$. This of course prevents the natural projection (taking a germ at $z_0$ to the point $z_0$) of the analytical continuation of any branch of $B^{-1}$ to be a topological covering.
Quoting Beardon, such an example shows that there is a significant difference between the definition of a covering surface used by complex analysts and that used by topologists.

We remark that, by another point of view, it is easy to construct examples of covering surfaces having some kind of isolated 'missing points' (i.e. singular points for the analytical continuation), preventing their projection mappings from being topological coverings.

For instance, consider the entire function defined by $g(z) := \exp \exp(\cdot)$. We can easily define infinitely many branches of its 'inverse' in a neighbourhood of any point in $\mathbb{C}^*$ by choosing suitable branches of the 'logarithm function'. Let $\gamma : I \rightarrow \mathbb{C}^*$ be defined by $\gamma(t) = t + (1 - t) \cdot e$.

Now let us construct the following branches of $\log$: let $\ell_1$ be the branch taking the value $1$ at $w = e$, and $\ell_2$ be the branch defined on $D(1, 1)$ by extending the Taylor development of the real logarithm function, centered at $1$ (thus $\ell_2(1) = 0$) and $\ell_3 := \ell_1 + 2\pi i$. Now both $\ell_2 \circ \ell_1$ and $\ell_2 \circ \ell_3$ are inverse branches of $g$ and both admit analytical continuation across $\gamma|_{[0,1]}$.

Clearly $\ell_2 \circ \ell_1$ does not admit analytical continuation up to $1 = \gamma(1)$, whereas $\ell_2 \circ \ell_3$ does; thus, again, the natural projection is not a topological covering.

One could ask if we can construct such examples dealing with nonisolated singularities. In this note, we aim at showing that the answer is yes: in fact we construct a covering surface of $\mathbb{C}^*$ (once more originating from the analytical continuation of a holomorphic germ), admitting singular-point arcs over the slits $E_n := D(0, e^n) \cup [e^{n-1}, e^n]$, $n \in \mathbb{Z}$.

By imagery, we could thought of these singularities as 'movable natural boundaries' (not to be confused with those arising in the theory of complex ordinary differential equations, see e.g. [?]). These singularities will appear (or 'disappear') at $E_n$, after a quantity connected to the winding number around $0$ of the path which we carry out analytical continuation along (see definition [ ]).

Of course, this prevents the natural projection of the analytical continuation from being a topological covering (see figure aside). To start constructing our example, let us consider the power series $h(z) := \sum_{\nu=0}^{\infty} z^{2^\nu} = 1 + z^2 + z^4 + z^8 + ..., $ defining a holomorphic function in $\mathbb{D}$ and having a natural boundary at $\partial \mathbb{D}$.

Now, let $E_n := \{x + iy \in \mathbb{C}: n \leq x \leq n + 1 \text{ and } y > 2\pi n\}$ $n \in \mathbb{Z}$ and
\[ E := \bigcup_{n=-\infty}^{\infty} E_n. \] Since \( E \) is simply connected, by Riemann mapping theorem, there exists a biholomorphic map \( \psi: E \to \mathbb{D} \), thus \( h \circ \psi \) has a natural boundary at \( \partial E \).

Now we proceed to define a slight generalisation of the notion of winding number, aimed at coping with nonclosed paths:

**Definition 1** Let \( \gamma: [0, 1] \to \mathbb{C} \setminus \{0\} \) be a path: we say that \( \gamma \) winds \( n \) times around \( 0 \) if
\[
\left\lfloor \frac{\Im (\ell_1 \circ \gamma(1) - \ell_0 \circ \gamma(0))}{2\pi} \right\rfloor = n,
\]
where \((U_0, \ell_0)\) is any branch of the logarithm in a neighbourhood of \( \gamma(1) \), \((U_1, \ell_1)\) the branch got by analytical continuation across \( \gamma \) and \([\ ]\) denotes the integer-part operator. We shall write \( W(\gamma) = n \).

In the following theorem, the main construction of this note will be fully depicted: let \( \lambda \) be the branch of the complex 'logarithm function' taking the value \( \ln(1/2) \) at \( 1/2 \) (defined by its Taylor development in \( \mathbb{D}(1/2, 1/2) \)) and \( f := h \circ \psi \circ \lambda \). Let \( \gamma \) be a path as in definition [1]. We have:

**Theorem 2** (a) - conditioned everywhere continuability: for every \( \omega \in \mathbb{C}^* \) there exists a path \( \beta \) from \( 1/2 \) to \( \omega \) along which \( f \) can be continued;
(b) - movable boundaries: let \( M, N \in \mathbb{Z} \); for every \( \omega \in \partial \mathbb{D}(0, e^M) \cup [e^{M-1}, e^M] \) let \( \varphi \) be a path joining \( 1/2 \) and \( \omega \) in \( \mathbb{C}^* \) such that \( W(\gamma) = N \) and \( f \) admits analytical continuation along \( \varphi|_{[0, 1]} \). If \( N > M \), then \( f \) admits analytical continuation along \( \varphi \) up to \( \omega = \varphi(1) \); if \( N = M \), then \( f \) does not.

**Proof**
(a) let \( \omega \in \mathbb{C}^* \): since the exponential function admits the period \( 2\pi i \), we can find \( \zeta \in E \) such that \( e^{\zeta} = \omega \). Let \( \alpha \) be a curve in \( E \) joining \( \ln(1/2) \) and \( \zeta \); the thesis follows now by setting \( \beta := \exp(\alpha) \cdot \lambda \) can of course be continued along \( \beta \) in such a way to get a finite chain of holomorphic function elements \((U_i, \Lambda_i)_{i=0...M} \) such that \( \bigcup_{i=0}^{N} \Lambda_i(U_i) \subset E \); the analytical continuation of \( f \) along \( \beta \) now follows by composing the \( \Lambda_i \)'s with \( h \circ \psi \) on the left.

(b): Now, \( \lambda \) can be certainly continued across \( \varphi \); let \( \ell \) be the branch of the
logarithm obtained in this way at $\varphi(1)$. Since $f$ admits analytical continuation along $\varphi|_{[0,1]}$, we have $\ell \circ \varphi|_{[0,1]} \subset \mathcal{E}$. Now $\mathcal{W}(\varphi) = N$, so if $\omega \in \partial \mathbb{D}(0,e^M)$, then $\Re(\ell(\omega)) = M$ and $2\pi(N - 1) \leq \Im(\ell(\omega)) < 2\pi N$; if $\omega \in [e^{M-1},e^M]$, then $M - 1 \leq \Re(\ell(\omega)) \leq M$ and $\Im(\ell(\omega)) = 2\pi(N - 1)$. Hence, if $N > M$, then $\ell(\omega) \in \mathcal{E}$, implying that $h \circ \psi \circ \ell$ admits analytical continuation up to $\omega = \varphi(1)$; by consequence, so does $f$; if $N = M$, then $\ell(\omega) \in \partial \mathcal{E}$, therefore we cannot carry out analytical continuation of $h \circ \psi$ at $\ell \circ \varphi(1)$, which means that $f$ could not be continued at $\omega$. ■

The picture in theorem 2 is clear: there are some kind of ‘natural boundaries’ for the analytical continuation of $f$ which can be ‘pushed farther’ by winding the analytical-continuation path a suitable number of times around 0; as already stated, this prevents the natural projection of the analytical continuation of $f$ from being a topological covering.

References

[1] A.F. Beardon *A remark on analytic continuation* Proceedings of the AMS, 128, 5

[2] Otto Forster *Lectures on Riemann surfaces* Springer Verlag, 1981

[3] Klaus Jänich *Topology* Springer Verlag, 1994