Quantum Scaling Approach to Nonequilibrium Models

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Abstract

Stochastic nonequilibrium exclusion models are treated using a real space scaling approach. The method exploits the mapping between nonequilibrium and quantum systems, and it is developed to accommodate conservation laws and duality symmetries, yielding exact fixed points for a variety of exclusion models. In addition, it is shown how the asymmetric simple exclusion process in one dimension can be written in terms of a classical Hamiltonian in two dimensions using a Suzuki-Trotter decomposition.

1 Introduction

Stochastic models of lattice gas dynamics provide insight into the nonequilibrium behaviour of a variety of physical processes, such as surface reactions and growth, catalysis and transport phenomena. These models are systems of many interacting particles — the dynamics of the particles are prescribed in the model definition — and the evolution is typically governed by a Master equation. They exhibit steady state phase transitions and very rich dynamics, but there exists no general framework in which to analyse nonequilibrium models. Exact treatments are scarce (see e.g. [1, 2, 3]) and so approximate

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techniques are required. To this end we present a scaling treatment designed
to capture universal and non-universal critical properties of nonequilibrium
systems.

The scaling method was developed in detail in [4]. It exploits the well
known equivalence between the Master equation and the Schrödinger equa-
tion in imaginary time [1] — the stochastic lattice gas model is written as
a quantum spin model. The scaling is achieved using a real space blocking
procedure [5] in order to thin out the number of degrees of freedom. It has
been applied to the contact process [6, 7], where very accurate results for the
critical point and certain critical exponents were obtained. Here, we show
how to adapt the method to models which possess a conservation law (e.g.
conserved particle number) in order to obtain exact fixed points. Further,
from a stability analysis of the fixed points, we infer the role of bias in the
model dynamics.

2 Quantum Scaling for Exclusion Models

In the following, we consider exclusion models — models where sites on a lat-
tice are either occupied by a single particle or vacant. In the quantum formul-
ation, these models are spin-1/2 quantum chains. The mapping is achieved
by interpreting configurations of particles and vacancies in the nonequilib-
rium model as a configuration of quantum spins, where particles are replaced
by an up-spin and vacancies are replaced by a down-spin, say. Since the dy-
namics in the nonequilibrium model become processes involving spin flips,
they can be expressed in terms of a quantum Hamiltonian. Hence, for a
lattice containing \( L \) sites, the configuration is written \(|\{\sigma_l\}\rangle = \prod_{l=1}^{L} \otimes|\sigma_l\rangle\),
where \( \{\sigma_l\} = \sigma_1, \ldots, \sigma_L \). We use the notation \( \sigma_l = +_l \) to represent an up-
spin at site \( l \) (i.e. a particle in the nonequilibrium system), and \( \sigma_l = -_l \) to
represent a down-spin (i.e. a vacancy in the nonequilibrium system). The
steady state of the nonequilibrium system is equivalent to the ground state
of the corresponding quantum problem.

2.1 Quantum Renormalisation Group Scheme

We now outline the renormalisation group scheme for quantum systems [4].
The first step is to divide the lattice into adjacent blocks, each containing \( b \)
sites, as indicated in figure \( \text{1} \). The lattice contains \( L' = L/b \) blocks, labelled
\[ H = \sum_{\nu=1}^{L'} [H_{\nu} + H_{\nu,\nu+1}] . \]  

The renormalisation is achieved as follows. We treat the intrablock Hamiltonian \( H_{\nu} \) exactly, and regard the interblock part \( H_{\nu,\nu+1} \) as a perturbation. Thus, we find the eigenvectors of \( H_{\nu} \) and use only those of lowest energy to form a truncated basis of states — in a spin-1/2 system we aim to keep the two lowest lying eigenstates. The configuration of the renormalised lattice is written as a direct product over blocks:

\[ |\{\sigma_{\nu}\}\rangle = \prod_{\nu=1}^{L'} \otimes |\sigma_{\nu}\rangle , \]  

where \( \{\sigma_{\nu}\} = \sigma_1, \ldots, \sigma_{L'} \) and the block spin states \( |\sigma_{\nu}\rangle \) are the renormalised basis of states obtained from the lowest lying eigenstates of \( H_{\nu} \). Then the renormalised Hamiltonian \( H' \) is obtained by writing \( H' = \sum_{\nu}[H'_{\nu} + H'_{\nu,\nu+1}] \). The matrix elements of \( H'_{\nu} \) are given by

\[ \langle \sigma_{\nu}'|H'_{\nu}|\sigma_{\nu}'\rangle = \langle \sigma_{\nu}|H_{\nu}|\sigma_{\nu}\rangle , \]  

where on the r.h.s., we use the fact that we have assigned eigenvectors of \( H_{\nu} \) to the block spin states (since these eigenvectors are orthogonal, i.e.
\[ \langle \sigma | \sigma' \rangle = \delta_{\sigma, \sigma'} \text{ although } H_\nu \text{ is not Hermitian, } H'_\nu \text{ is diagonal}. \] Similarly, the matrix elements of \( H'_{\nu, \nu+1} \) are given by

\[ \langle \sigma_\nu | H'_{\nu, \nu+1} | \sigma'_\nu \rangle = \langle \sigma_\nu | H_{\nu, \nu+1} | \sigma'_\nu \rangle , \] (4)

and thus \( H'_{\nu, \nu+1} \) contributes the interaction terms in the renormalised Hamiltonian \( H' \).

Thus we have a prescription whereby we retain only the lowest lying eigenstates of a block Hamiltonian in order to thin out the number of degrees of freedom and still retain the features important for criticality — the renormalisation is carried out near the ground state of the quantum system, equivalent to the steady state of the nonequilibrium system. By assigning this truncated basis of states to a block spin variable in an appropriate way, one hopes to obtain a rescaled Hamiltonian \( H' \) of the same form as the original Hamiltonian \( H \) but with rescaled parameters.

### 2.2 Scaling for the Asymmetric Simple Exclusion Process

In this section, we apply the above transformation to the asymmetric simple exclusion process (ASEP) in one dimension with periodic boundary conditions. In the ASEP, particles hop to the right (left) with rate \( p \) (\( q \)) provided that the target site is empty. These dynamics can be expressed in terms of a quantum Hamiltonian given by

\[ H = \sum_{l=1}^{L} \left[ p \left( P_l^+ P_{l+1}^- - \sigma_l^- \sigma_{l+1}^+ \right) + q \left( P_l^- P_{l+1}^+ - \sigma_l^+ \sigma_{l+1}^- \right) \right] . \] (5)

where \( P_l^\pm = \frac{1}{2} (1 \pm \sigma_l^z) \) are projection operators and \( \sigma_l^\pm \) (\( \sigma_l^- \)) creates (annihilates) a particle at site \( l \). Thus the terms \( \sigma_l^- \sigma_{l+1}^+ \) and \( \sigma_l^+ \sigma_{l+1}^- \) generate particle hopping to the right and left respectively — they derive from the gain terms of the original Master equation (i.e. the terms due to particle hopping which increase the probability of finding the system in a particular configuration). The terms involving projection operators arise due to the loss terms in the Master equation (i.e. the terms which contribute to the probability that the system is in a particular configuration provided that no particle performs a hop). Therefore probability and particle number (which is related to the \( z \)-component of the spin) are both conserved. This model also possesses a
particle-hole symmetry under interchange of $p \leftrightarrow q$. The aim is to maintain this duality at all stages of scaling.

We begin by dividing the lattice into blocks of size $b = 2$. Then we split $H$ into an intrablock Hamiltonian $H_\nu$ given by

$$H_\nu = p \left( P_{\nu,1}^+ P_{\nu,2}^- - \sigma_{\nu,1}^- \sigma_{\nu,2}^+ \right) + q \left( P_{\nu,1}^- P_{\nu,2}^+ - \sigma_{\nu,1}^+ \sigma_{\nu,2}^- \right),$$  \hspace{1cm} (6)

where the suffix $\nu, i$ indicates that the operator acts at site $i$ in block $\nu$, and an interblock Hamiltonian $H_{\nu,\nu+1}$ given by

$$H_{\nu,\nu+1} = p \left( P_{\nu,2}^+ P_{\nu+1,1}^- - \sigma_{\nu,2}^- \sigma_{\nu+1,1}^+ \right) + q \left( P_{\nu,2}^- P_{\nu+1,1}^+ - \sigma_{\nu,2}^+ \sigma_{\nu+1,1}^- \right),$$  \hspace{1cm} (7)

such that both $H_\nu$ and $H_{\nu,\nu+1}$ possess the duality under interchange of $p \leftrightarrow q$ when $+ \leftrightarrow -$.

The next step is to find the lowest lying eigenstates of $H_\nu$ and use them to form the renormalised basis of spin states. The ground state of $H_\nu$ is three-fold degenerate — since the dynamics conserve particle number $H_\nu$ decomposes into $b+1$ disconnected sectors. This particle conservation can be respected in the renormalised Hamiltonian if it is present in the renormalised basis of states, therefore the ground eigenstates of $H_\nu$ are organised according to their eigenvalue of the block spin operator $\sum_{i=1}^{b} \sigma_{\nu,i}^z$. This leads us to define block spin states

$$| + 1_\nu \rangle = | +_{\nu,1}, +_{\nu,2} \rangle,$$

$$| 0_\nu \rangle = \frac{1}{p+q} \left[ q | +_{\nu,1}, -_{\nu,2} \rangle + p | -_{\nu,1}, +_{\nu,2} \rangle \right],$$

$$| - 1_\nu \rangle = | -_{\nu,1}, -_{\nu,2} \rangle,$$

for each block $\nu$. The corresponding left eigenstates of $H_\nu$ are assigned to the left block spin states

$$\langle +_{\nu} | = \langle +_{\nu,1}, +_{\nu,2} |,$$

$$\langle 0_{\nu} | = \langle +_{\nu,1}, -_{\nu,2} | + \langle -_{\nu,1}, +_{\nu,2} |,$$

$$\langle -_{\nu} | = \langle -_{\nu,1}, -_{\nu,2} |.$$

The left ground eigenstates of a quantum Hamiltonian describing a nonequilibrium process are always given by such sums over vectors (where the coefficient of each vector is equal to one) due to conservation of probability. By defining the block states in this way, we aim to maintain particle conservation (which we cannot maintain by arranging these states in two linear
superpositions forming a spin-1/2 basis) and the particle-hole duality of the model. Thus the renormalised Hamiltonian will describe a spin-1 quantum chain.

We are now able to calculate the matrix elements of $\cal H'_\nu$ and $\cal H'_{\nu,\nu+1}$. Firstly, we note that because our basis states are degenerate eigenstates of $\cal H_\nu$, the contribution due to $\cal H'_\nu$ is a constant and moreover, because the groundstate eigenvalue is zero (which is always the case for the ground state eigenvalue of quantum systems representing nonequilibrium models) this constant is zero. The task then is to evaluate the matrix elements of $\cal H'_{\nu,\nu+1}$, as prescribed by equation (14). For example, the term in $\cal H_{\nu,\nu+1}$ given by $\sigma^-_\nu \sigma^+_{\nu+1}$ contributes the matrix elements

$$\langle \sigma_\nu, \sigma_{\nu+1} | \cal H'_{\nu,\nu+1} | \sigma'_\nu, \sigma'_{\nu+1} \rangle = \langle \sigma_\nu | \sigma^-_\nu | \sigma'_\nu \rangle \langle \sigma_{\nu+1} | \sigma^+_{\nu+1} | \sigma'_{\nu+1} \rangle.$$  \hspace{1cm} (14)

Thus the operator $\sigma^-_\nu$ is replaced by a renormalised operator given by

$$\sigma^-_\nu = \begin{pmatrix}
+1_\nu | \sigma^-_\nu, 1_\nu > & +1_\nu | \sigma^-_\nu, 0_\nu > \\
0_\nu | \sigma^-_\nu, 1_\nu > & 0_\nu | \sigma^-_\nu, 0_\nu > \\
-1_\nu | \sigma^-_\nu, 1_\nu > & -1_\nu | \sigma^-_\nu, 0_\nu >
\end{pmatrix},$$

where the $\nu$ suffix on the matrix represents an operator acting on the truncated basis of states in block $\nu$. Evaluating such matrix elements for all the operators appearing in $\cal H_{\nu,\nu+1}$ yields

$$P^+_{\nu,\nu} = P^+_{\nu,\nu} + \frac{a}{p+q} P^0_{\nu} \quad P^+_{\nu,\nu+1,1} = P^+_{\nu,\nu+1,1} + \frac{a}{p+q} P^0_{\nu+1},$$
$$P^-_{\nu,\nu} = \frac{a}{p+q} P^0_{\nu} + P^-_{\nu} \quad P^-_{\nu,\nu+1,1} = \frac{a}{p+q} P^0_{\nu+1} + P^-_{\nu+1},$$
$$\sigma^+_{\nu,\nu} = \frac{a}{p+q} \sigma^+_{\nu} + \sigma^+_{\nu}, \quad \sigma^+_{\nu,\nu+1,1} = \frac{a}{p+q} \sigma^+_{\nu+1} + \sigma^+_{\nu+1,1},$$
$$\sigma^-_{\nu,\nu} = \sigma^-_{\nu} + \frac{a}{p+q} \sigma^-_{\nu} \quad \sigma^-_{\nu,\nu+1,1} = \sigma^-_{\nu+1} + \frac{a}{p+q} \sigma^-_{\nu+1,1},$$

where $P^j_{\nu}$ projects into the block spin state $i$ and $\sigma^j_{\nu}$ raises or lowers the spin from block spin state $i$ to block spin state $j$. These expressions are substituted into $\cal H_{\nu,\nu+1}$ leading to a renormalised Hamiltonian given by $\cal H' = \sum_{\nu} \cal H'_{\nu,\nu+1}$.

Thus the renormalised Hamiltonian describes a three-state stochastic process (probability is still conserved) where the z-component of the spin is still conserved. In order to obtain scaling equations for the rates $p$ and $q$ this Hamiltonian has to be projected onto a basis of spin-1/2 states. To do this, we note that raising (lowering) operators in the spin-1/2 system are written only in terms of operators which raise (lower) the spin in the spin-1 system.
Further, each of the dynamical processes, generated by the raising and lowering operators in the spin-1 system, can be identified with a corresponding term involving projection operators due to no transition. Hence we rewrite

\[ P_{\nu,2}^+ = (1 + \frac{p}{p+q}) P_{\nu}^+, \quad P_{\nu+1,1}^+ = (1 + \frac{p}{p+q}) P_{\nu+1}^+; \]
\[ P_{\nu,2}^- = (1 + \frac{q}{p+q}) P_{\nu}^-, \quad P_{\nu+1,1}^- = (1 + \frac{q}{p+q}) P_{\nu+1}^-; \]
\[ \sigma_{\nu,2}^+ = (1 + \frac{q}{p+q}) \sigma_{\nu}^+, \quad \sigma_{\nu+1,1}^+ = (1 + \frac{p}{p+q}) \sigma_{\nu+1}^+; \]
\[ \sigma_{\nu,2}^- = (1 + \frac{p}{p+q}) \sigma_{\nu}^-, \quad \sigma_{\nu+1,1}^- = (1 + \frac{q}{p+q}) \sigma_{\nu+1}^-; \]

whence the renormalised Hamiltonian assumes the same form as the original:

\[ H' = \sum_{\nu} \left[ p' \left(P_{\nu}^+ P_{\nu+1}^- - \sigma_{\nu}^- \sigma_{\nu+1}^+\right) + q' \left(P_{\nu}^- P_{\nu+1}^+ - \sigma_{\nu}^+ \sigma_{\nu+1}^-\right)\right], \quad (15) \]

where the rescaled rates \( p' \) and \( q' \) are given by

\[ p' = p(1 + \frac{p}{p+q})^2, \quad q' = q(1 + \frac{q}{p+q})^2. \quad (16) \]

In order to exploit this renormalisation, we consider the ratio \( \gamma = p/q \). Stable fixed points for \( \gamma \) are found at \( \gamma^* = 0 \) and \( \infty \), and these are separated by an unstable fixed point at \( \gamma^* = 1 \). Hence symmetric diffusion is unstable with respect to bias. At the symmetric fixed point the Hamiltonian describes a spin-1/2 Heisenberg chain whose dynamics are governed by a dynamic exponent \( z = 2 \). This behaviour therefore is unstable and in the presence of any bias the dynamics are described by a new exponent. A Bethe ansatz calculation for \( q = 0 \) shows that this exponent is \( z = \frac{3}{2} \). That we find the exact value for the unstable fixed point is a consequence of our preservation of duality at all stages in the blocking.

### 2.3 Scaling for the Pair Evaporation and Deposition Model

Another model possessing a duality is a process whereby pairs of particles evaporate from adjacent lattice sites with a rate \( \epsilon \) or are deposited onto adjacent vacancies with a rate \( \delta \). The quantum Hamiltonian representing these processes takes the form

\[ H = \sum_{l} \left[ \delta \left(P_{l}^- P_{l+1}^- - \sigma_{l}^- \sigma_{l+1}^+\right) + \epsilon \left(P_{l}^+ P_{l+1}^+ - \sigma_{l}^+ \sigma_{l+1}^-\right)\right]. \quad (17) \]
The duality in this model is again a particle-hole symmetry under the interchange of $\delta \leftrightarrow \epsilon$. There is also a conservation law similar to the particle conservation in the ASEP: if we label the sublattice of odd (even) sites by A (B), then the density of particles on sublattice A (B) is $\rho_A$ ($\rho_B$). The conserved quantity is the difference in the sublattice densities, $\rho_A - \rho_B$.

The duality of this Hamiltonian is preserved if we divide it into an intra-block part given by

$$H_\nu = \delta \left(P_{\nu,1}^- P_{\nu,2}^- - \sigma_{\nu,1}^+ \sigma_{\nu,2}^+\right) + \epsilon \left(P_{\nu,1}^+ P_{\nu,2}^+ - \sigma_{\nu,1}^- \sigma_{\nu,2}^-\right),$$

and an interblock part given by

$$H_{\nu,\nu+1} = \delta \left(P_{\nu,2}^- P_{\nu+1,1}^- - \sigma_{\nu,2}^+ \sigma_{\nu+1,1}^+\right) + \epsilon \left(P_{\nu,2}^+ P_{\nu+1,1}^+ - \sigma_{\nu,2}^- \sigma_{\nu+1,1}^-\right).$$

Again, the ground state of $H_\nu$ is three-fold degenerate with eigenvectors given by

$$|1_\nu\rangle = |-\nu,1, +\nu,2\rangle,$$

$$|2_\nu\rangle = \frac{1}{\delta + \epsilon}[\delta|+\nu,1, +\nu,2\rangle + \epsilon|-\nu,1, -\nu,2\rangle],$$

$$|3_\nu\rangle = |+\nu,1, -\nu,2\rangle.$$

We can maintain the conservation of the sublattice densities with respect to both the intrablock and the interblock Hamiltonians if we assign spin-1 block states in the following way:

$$| + 1_\nu\rangle = |1_\nu\rangle, \quad | + 1_{\nu+1}\rangle = |3_{\nu+1}\rangle,$$

$$|0_\nu\rangle = |2_\nu\rangle, \quad |0_{\nu+1}\rangle = |2_{\nu+1}\rangle,$$

$$|- 1_\nu\rangle = |3_\nu\rangle, \quad |- 1_{\nu+1}\rangle = |1_{\nu+1}\rangle.$$
of bias. Again, the Hamiltonian (17) of the symmetric problem is given by
that of the spin-1/2 Heisenberg chain. Thus the dynamic exponent is \( z = 2 \)
in the biased and unbiased cases. This result is consistent with a mean field
result which predicts that the diffusion constant is independent of bias, and
it is supported by numerical simulation [9].

2.4 Scaling for the Pair Evaporation and Deposition
Process with Diffusion in One Dimension

In this section, we apply the renormalisation group transformation to a model
incorporating both the dynamics of the ASEP and of the pair evaporation
and deposition process. These dynamics, on a chain with periodic boundary
conditions, are represented by a quantum Hamiltonian given by

\[
H = \sum_l \left[ \delta \left( P_{l}^- P_{l+1}^- - \sigma_l^+ \sigma_{l+1}^+ \right) + \epsilon \left( P_{l}^+ P_{l+1}^+ - \sigma_l^- \sigma_{l+1}^- \right) \right. \\
\left. + p \left( P_{l}^+ P_{l+1}^- - \sigma_l^- \sigma_{l+1}^+ \right) + q \left( P_{l}^- P_{l+1}^+ - \sigma_l^+ \sigma_{l+1}^- \right) \right].
\] (25)

Exact results have been obtained when \( \delta = \epsilon \) and \( p = q \) in which case the
model is equivalent to a spin-1/2 XXZ ferromagnet, and also for \( \delta + \epsilon = p + q \)
which is the condition that the evolution operator can be written as a free-
fermion Hamiltonian [9, 2].

A blocking, with a dilation factor \( b = 2 \), is implemented by dividing \( H \)
into an intrablock part \( H_\nu \) given by

\[
H_\nu = \delta \left( P_{\nu,1}^- P_{\nu,2}^- - \sigma_{\nu,1}^+ \sigma_{\nu,2}^+ \right) + \epsilon \left( P_{\nu,1}^+ P_{\nu,2}^+ - \sigma_{\nu,1}^- \sigma_{\nu,2}^- \right) \\
+ p \left( P_{\nu,1}^+ P_{\nu,2}^- - \sigma_{\nu,1}^- \sigma_{\nu,2}^+ \right) + q \left( P_{\nu,1}^- P_{\nu,2}^+ - \sigma_{\nu,1}^+ \sigma_{\nu,2}^- \right),
\] (26)

and an interblock part given by

\[
H_{\nu,\nu+1} = \delta \left( P_{\nu,2}^- P_{\nu+1,1}^- - \sigma_{\nu,2}^+ \sigma_{\nu+1,1}^+ \right) + \epsilon \left( P_{\nu,2}^+ P_{\nu+1,1}^+ - \sigma_{\nu,2}^- \sigma_{\nu+1,1}^- \right) \\
+ p \left( P_{\nu,2}^+ P_{\nu+1,1}^- - \sigma_{\nu,2}^- \sigma_{\nu+1,1}^+ \right) + q \left( P_{\nu,2}^- P_{\nu+1,1}^+ - \sigma_{\nu,2}^+ \sigma_{\nu+1,1}^- \right).
\] (27)

The blocking proceeds as in the previous two sections. The ground state of
\( H_\nu \) is two-fold degenerate with eigenvectors denoted

\[
|1_\nu\rangle = \frac{1}{\delta + \epsilon} \left[ \delta \left( \sigma_{\nu,1}^+ + \sigma_{\nu,2}^+ \right) + \epsilon \left( \sigma_{\nu,2}^- - \sigma_{\nu,1}^- \right) \right],
\] (28)

\[
|2_\nu\rangle = \frac{1}{p + q} \left[ q \left( \sigma_{\nu,1}^- + \sigma_{\nu,2}^- \right) + p \left( \sigma_{\nu,1}^+ + \sigma_{\nu,2}^+ \right) \right].
\] (29)
Effective spin states for the block $\nu$ are identified by taking the block up-spin $|+\nu\rangle = |1,\nu\rangle$ and the block down-spin $|-\nu\rangle = |2,\nu\rangle$ for all $\nu$. Thus the block spin states observe two symmetries of the model: the particle-hole symmetry under interchange of rates $\delta \leftrightarrow \epsilon$ and $p \leftrightarrow q$, and a symmetry whereby the spins on the even sublattice (say) of sites are flipped and the pair evaporation and deposition processes are transformed into the hopping processes and vice versa. This assignment of block spin states yields a renormalised Hamiltonian $H'$ of the same form as (25), but with rescaled rates $\delta', \epsilon', p'$ and $q'$ given by

$$\delta' = \frac{p^3 + q^3 + pq(\delta + \epsilon)}{(p + q)^2},$$

$$\epsilon' = \frac{\delta \epsilon(\delta + \epsilon + p + q)}{(\delta + \epsilon)^2},$$

$$p' = \frac{\delta \epsilon(p + q) + \delta p^2 + \epsilon q^2}{(\delta + \epsilon)(p + q)},$$

$$q' = \frac{\delta \epsilon(p + q) + \epsilon p^2 + \delta q^2}{(\delta + \epsilon)(p + q)}.$$

A flow diagram is obtained where the symmetric fixed point (i.e. where all rates are equal) is fully stable. This suggests that the dynamic transition in the ASEP is removed when the pair evaporation and deposition processes are included — the dynamics in this model are described by the exponent $z = 2$ for all choices of $\delta$, $\epsilon$, $p$ and $q$ (provided $\delta$ and $\epsilon$ are not both equal to zero). Also, all the fixed points satisfy the free-fermion condition $\delta + \epsilon = p + q$ for which exact results are available \[2, 9\]. Moreover, this condition is stable (after iterating the transformation an infinite number of times the rescaled rates always satisfy the free-fermion condition, regardless of the original choice of rates), therefore we do not expect to observe any new macroscopic behaviour in the regions of parameter-space to which the exact solution does not apply.

### 3 Trotter Decomposition for the Asymmetric Simple Exclusion Process

One difficulty that arises when constructing renormalisation group transformations for quantum systems is associated with the non-commutation of operators appearing in the quantum Hamiltonian. In this section, we remove
this problem by exploiting the Suzuki-Trotter decomposition \cite{10, 11, 12, 13} to rewrite the quantum Hamiltonian representing the ASEP as a classical Hamiltonian for Ising spin variables in two dimensions. As a byproduct, direct contact is made between stochastic nonequilibrium models in one dimension and vertex models in two dimensions \cite{14}.

The general scheme for the mapping is to split the Hamiltonian into a set of operators \( \{ H_i \} \) such that

\[
H = \sum_{i=0}^{j} H_i, \tag{34}
\]

where each term in \( H_i \) commutes with every other (or at least, where any non-commutation can be easily dealt with), but \([H_l, H_m] \neq 0 \) for \( l \neq m \). The exponential of a sum of operators is then expanded as a product of exponentials by dealing with the non-commutation through the Trotter formula \cite{10}

\[
e^{\sum_{i=0}^{j} H_i} = \lim_{n \to \infty} \left[ e^{H_0} \cdots e^{H_j} \right]^n. \tag{35}
\]

For the ASEP, we begin with the quantum Hamiltonian \cite{25}, and divide it up in the following way:

\[
H_0 &= \sum_{l \text{ odd}} \left[ p P_l^+ P_{l+1}^- + q P_l^- P_{l+1}^+ \right], \tag{36}
\]
\[
H_1 &= -\sum_{l \text{ odd}} \left[ p \sigma_l^- \sigma_{l+1}^+ + q \sigma_l^+ \sigma_{l+1}^- \right], \tag{37}
\]
\[
H_2 &= \sum_{l \text{ even}} \left[ p P_l^+ P_{l+1}^- + q P_l^- P_{l+1}^+ \right], \tag{38}
\]
\[
H_3 &= -\sum_{l \text{ even}} \left[ p \sigma_l^+ \sigma_{l+1}^- + q \sigma_l^- \sigma_{l+1}^+ \right]. \tag{39}
\]

Ultimately, this choice must be made to reflect the update mechanism — here we consider a parallel sublattice update. According to equation \cite{25}, the partition function for the quantum system is written

\[
Z = \lim_{n \to \infty} Z(n), \tag{40}
\]

where we have defined \( Z(n) \) by

\[
Z(n) \equiv \text{Tr} \left[ e^{-\frac{\beta H_0}{n}} e^{-\frac{\beta H_1}{n}} e^{-\frac{\beta H_2}{n}} e^{-\frac{\beta H_3}{n}} \right]^n. \tag{41}
\]

The next step is to insert complete sets of basis states into \cite{41}. We choose the basis \( |s\rangle = |\sigma_1, \sigma_2, \ldots, \sigma_L\rangle \), where \( \sigma_i \) is the eigenvalue of the Pauli matrix
\[\Phi_{l,\tau} = \frac{1}{2}(1 \pm \sigma_{l,\tau}) \] and \(\sigma_{l,\tau}\) is an Ising spin variable associated with the site \((l, \tau)\), and where

\[ h_{l,\tau} = \Delta \Phi_{l,\tau} - \ln \left( \frac{\beta p}{n} \right) P_{l,\tau}^{+} P_{l+1,\tau}^{-} P_{l+1,\tau+1}^{+} P_{l+1,\tau+1}^{-} \]
The parameter $\Delta$ has been introduced in order to project away unwanted configurations, i.e. those which do not represent processes allowed under the original stochastic dynamics; for instance, the final term in equation (45) corresponds to a particle hopping to the left, which is forbidden for $q = 0$.

To obtain the full effective Hamiltonian for the two-dimensional classical system, $Z(n) = \lim_{\Delta \to \infty} \text{Tr} e^{-H^{(\text{eff})}}$, we must include the contributions from the Hamiltonians $H_0$, $H_2$ and $H_3$. Then the effective Hamiltonian can be written

$$H^{(\text{eff})} = \sum_{l, \tau \text{ odd}} h^{(\text{eff})}_{l, \tau} + \sum_{l, \tau \text{ even}} h^{(\text{eff})}_{l, \tau},$$  

(46)

with

$$h^{(\text{eff})}_{l, \tau} = \Delta \Phi_{l, \tau} - \ln \left( \frac{\beta}{\pi} \right) P_{l, \tau} P_{l, \tau+1} P^{-}_{l+1, \tau} P^{+}_{l+1, \tau+1}$$

$$- \ln \left( 1 - \frac{\beta}{\pi} \right) P^{+}_{l, \tau} P^{+}_{l, \tau+1} P^{-}_{l+1, \tau} P^{-}_{l+1, \tau+1},$$  

(47)

where the final term here corresponds to the ‘stay-put’ probability that a particle, with a vacancy to its right, does not perform a hop.

The quantum Hamiltonian (5) has now been rewritten as a classical Hamiltonian (46) for an Ising system in two dimensions. The nonequilibrium steady state of the original model is characterised by the zero temperature behaviour of the quantum model. This limit is recovered in the Ising system by taking the extent of the Trotter axis to be infinite. The temperature in the quantum system is not the same as the temperature in the classical Ising system. Instead, the inverse temperature of the quantum model translates into the extent of the classical system in the Trotter direction. Thus the critical behaviour in the quantum ground state is expressed through the critical behaviour of the finite temperature Ising system in equilibrium. Zero temperature transitions in the quantum model, which occur as a function of the couplings, are caused in the infinite Ising system by varying the temperature.

### 3.1 ‘Plaquette’ Structure of Ising Hamiltonian

The Ising Hamiltonian $H^{(\text{eff})}$ representing the ASEP contains a parameter $\Delta$ which is infinite. This constraint can be incorporated in a compact fashion into a vertex model description. In the representation we have chosen, $H^{(\text{eff})}$ contains only four-spin interactions. These are shown in Figure 2 as the five possible arrangements of spins around a plaquette.
Figure 2: The allowed plaquette configurations with their weights

\[ \omega_1, \omega_2, \omega_3, \omega_4, \omega_5 \]

Figure 3: Lattice on which the 2d classical Ising model is defined. The original spatial axis is labelled by \( l \) and \( \tau \) labels the Trotter axis. The configurations around the shaded squares are determined by the allowed plaquette configurations shown in Figure 2.

The weights \( \omega_1 \) to \( \omega_5 \) of the allowed vertices are

\[ \omega_1 = \omega_2 = \omega_4 = 1, \omega_3 = 1 - \frac{\beta p}{n}, \omega_5 = \frac{\beta p}{n}. \tag{48} \]

All other vertices have zero weight. A lattice configuration is then specified by placing the allowed plaquette configurations on the shaded squares of the lattice shown in Figure 3. The open plaquettes in this diagram play the role of passive plaquettes, whose configurations are determined only by the surrounding configurations shown in Figure 2. Every configuration is allowed for an open plaquette and each occurs with weight 1. The partition function for this system is

\[ Z = \sum_{\text{allowed configurations}} \prod_{\text{plaquettes } i} \omega_i, \tag{49} \]

As mentioned previously, the division of the quantum Hamiltonian into sums over commuting operators must be chosen with a specific update mechanism.
in mind. From Figure 3, we see that the division (36) to (39) combined with
the insertion of basis states in equation (42) describes a parallel sublattice
update mechanism — the shaded plaquettes in a row of the diagram represent
the bonds chosen for an update in one time-step. Alternatively, we may
have chosen to describe sequential update. This is achieved by dividing
the quantum Hamiltonian into $L$ pairs of local bond Hamiltonians. One of
each pair contains only diagonal terms, the other contains only non-diagonal
terms, and there is one pair for every bond in the system. Then, inserting
basis states between each pair of local Hamiltonians leads to a sequential
update mechanism.

We also note that in the chosen basis, we can write a transfer matrix for
the evolution of configurations along the Trotter axis. Its matrix elements
are identical to those appearing in the matrix representation of the discrete
time Master equation (for a specified update mechanism) with a hopping
rate given by $\beta p/n$.

3.2 Scaling

By rewriting the quantum Hamiltonian as a classical Hamiltonian, the prob-
lem of non-commutation of operators in the quantum Hamiltonian has been
removed and the classical Hamiltonian provides Boltzmann weights for phys-
ical processes (in the sense that each plaquette configuration has a direct
physical interpretation in terms of the original stochastic dynamics). The
restricted number of plaquette configurations and the way they are placed
on the lattice suggest that we can devise direct and straightforward scaling
procedures. Configurations (and their weights) are built up by piecing pla-
quettes together in an allowed way. Then one can coarse-grain by matching
configurations under a change of length scale. This dilation of scale may be
applied to either the time axis or the spatial axis individually, or to both at
once. One can also explore how changing the update mechanism effects the
scaling.

To illustrate these ideas, consider a putative decimation eliminating sites
along the time axis for a parallel sublattice update mechanism. This is
shown, for a particular matching of active and inactive plaquettes, in Figure
4. The weight for the unrenormalised system is obtained by tracing over
all allowed plaquettes consistent with a given configuration $\{\sigma_i\}$. A scaling
equation may be obtained, for example, by matching this weight with the
renormalised weight for a plaquette with spins given by $\sigma_1, \sigma_2, \sigma_4$ and $\sigma_5$. 

15
Figure 4: Decimation of the Trotter lattice along the time axis. Sites labelled by a circle are traced over.

4 Conclusion

Scaling techniques which are simple to implement have been described within the quantum formulation of nonequilibrium exclusion models. In particular, the exact fixed points are obtained for the ASEP in which the known dependence of the dynamics on asymmetry is recovered. For a model involving the evaporation and deposition of adjacent pairs exact fixed points are obtained; the resulting flow diagram indicates that the dynamics are independent of the ratio of evaporation and deposition rates. Further, the stability of the free-fermion condition in the model combining the dynamics of the ASEP with pair evaporation and deposition indicates that the exact solutions provide a complete account of the large scale behaviour. Again, the dynamics are found to be independent of bias.

We note that the projections of spin-1 operators onto a basis of spin-1/2 operators in Sections 2.2 and 2.3 are not necessary. Indeed, one could continue scaling the system to renormalised bases of higher and higher spin. An approach to yield the continuum limit of the ASEP in this way [15] is under investigation.

We have also shown how to map a quantum Hamiltonian representing a nonequilibrium exclusion process onto a classical Hamiltonian in one higher dimension. In this way, we have shown how to write the steady state probabilities for configurations of the nonequilibrium system in terms of classical Boltzmann weights. This should enable one to borrow real-space renormalisation group techniques for classical Hamiltonian systems and apply them to nonequilibrium systems.
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