A note on DSR-like approach to space-time

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In this note we discuss the possibility to define a space-time with a DSR based approach. We show that the strategy of defining a non linear realization of the Lorentz symmetry with a consistent vector composition law cannot be reconciled with the extra request of an invariant length (time) scale. The latter request forces to abandon the group structure of the translations and leaves a space-time structure where points with relative distances smaller or equal to the invariant scale can not be unambiguously defined.

I. INTRODUCTION

It is widely believed that the space-time, where physical process and measurements take place, might have a structure different from a continuous and differentiable manifold, when it is probed at the Planck length \( \ell_p \). For example, the space-time could have a foamy structure \(^1\), or it could be non-commutative in a sense inspired by string theory results \(^2\) or in the sense of \(\kappa\)-Minkowski approach \(^3\).

If this happens in the space-time, in the momentum space there must also be a scale, let say \( p_p \), that signs this change of structure of the space-time, even if the interplay between length and momentum (\( p \sim c\hbar \lambda^{-1} \)) will presumably change when we approach such high energy scales.

One could argue that, if the Planck length gives a limit at which one expects that quantum gravity effects become relevant, then it would be independent from observers, and one should look for symmetries that reflect this property. Such argument gave rise to the so called DSR proposals, that is, a deformation of the Lorentz symmetry (in the momentum space) with two invariant scales: the speed of light \(c\) and \( p_p \) (or \( E_p \)) \(^4\) \(^5\) \(^6\).

In this note, we will discuss this class of deformations of the Lorentz symmetry and its realization in the space-time. Approaches to the problem inspired by the momentum space formulation have been presented \(^3\) \(^4\), but our approach is quite different from these because we demand the existence of an invariant measurable physical scale compatible with the deformation of the composition law of space-time coordinates induced by the non linear transformation.

It has also been claimed that \(\kappa\)-Minkowski \(^8\) gives a possible realization of the DSR principles in space-time \(^9\), however the construction is still not satisfactory since the former is only compatible with momentum composition law non symmetric under the exchange of particles labels (see discussions in \(^10\)). In this work we are dealing with non linear realizations of the Lorentz algebra which induce symmetric composition law and therefore it is not compatible with the \(\kappa\)-Minkowski approach.

The main results of our studies are: i) the strategy of defining a non linear realization of the Lorentz symmetry with a consistent vector composition law cannot be reconciled with the extra request of an invariant length (time) scale; ii) the request of an invariant length forces to abandon the group structure of the translations and leaves a space-time structure where points with relative distances smaller or equal to the invariant scale can not be unambiguously defined.

In the next section we will explore the approach to DSR in the momentum space and will implement these ideas in the space-time sector. In the final section conclusion and discussions are presented.

II. NON LINEAR REALIZATION OF LORENTZ GROUP APPROACH TO SPACE-TIME

In this section we will first review the approach to DSR as a non linear realization of the Lorentz transformations in the energy-momentum space, and then try to apply these ideas to the space-time. For a more general review of DSR see for example \(^11\) and references therein.

DSR principles are realized in the energy-momentum
space by means of a non-linear action of the Lorentz group \( \mathbb{R} \). More precisely, if the coordinates of the physical space \( P \) are \( p_\mu = \{p_0, \mathbf{p}\} \), we can define a non-linear function \( F : \mathbb{P} \to \mathbb{P} \), where \( \mathbb{P} \) is the space with coordinates \( \pi_\mu = \{\alpha, \pi\} \), on which the Lorentz group acts linearly. We will refer to \( \mathbb{P} \) as a classical space-time.

In terms of the previous variables, a boost of a single particle with momentum \(^1 p \) to another reference frame, where the momentum of the particle is \( p' \), is given by

\[
p' = F^{-1} \circ \Lambda \circ F [p] \equiv \mathcal{B}[p].
\]

Finally, an addition law \((+\)\) for momenta, which is covariant under the action of \( \mathcal{B} \), is

\[
p_a + p_b = F^{-1} [F[p_a] + F[p_b]],
\]

and satisfies \( \mathcal{B}[p_a + p_b] = \mathcal{B}[p_a] + \mathcal{B}[p_b] \).

In this formulation, the requirement of having an invariant scale fixes the action of \( F \) on some points of the real space \( P \). Indeed, since the Lorentz transformation leaves the points \( \pi = 0 \) and \( \pi = \infty \) invariant, one sees that if we demand invariance of Planck momentum \( (p_\mathbf{p}) \) (or energy \( (E_\mathbf{p}) \)) then \( F[p_\mathbf{p}] \) (or \( F[E_\mathbf{p}] \)) only can be 0 or \( \infty \). A general discussion on the possible deformations is given in \( [5, 12] \).

In the following we explore the possibility to extend the above discussion generalizing it to define a non-linear realization of Lorentz symmetry in space-time. In analogy with the momentum space, we will consider a real space-time \( X \) with coordinates \( x^\mu = \{x^0, \ldots, x^3\} \) and will assume: a) the existence of an auxiliary space-time \( \mathcal{X} \) with coordinates \( \xi^\mu = \{\xi^0, \ldots, \xi^3\} \) (called classical space-time), where the Lorentz group acts linearly and b) the existence of an invertible map \( G[x] \) such that \( G[x] : X \to \mathcal{X} \).

Boosts in the space-time will be defined in the same way as DSR boosts in the momentum space, that is

\[
\mathcal{B} = G^{-1} \circ \Lambda \circ G,
\]

where \( \Lambda \) is the Lorentz boost, which acts linearly on \( \mathcal{X} \).

As was done in the energy-momentum space, we want to define a space-time vector composition law covariant under the action of deformed boost:

\[
x_a + x_b = G^{-1}[G[x_a] + G[x_b]].
\]

This definition implies that a vector can always be written as the sum of two (or more) vectors and this decomposition is covariant under boosts:

\[
\delta_{(ab)} = G^{-1}[G[x_a] - G[x_b]] = \delta_{(ac)} + \delta_{(cb)},
\]

for any \( x_c \). With relation (4) we can define the operation of translation of all vectors of our space by a fixed vector \( \alpha \):

\[
\hat{T}_\alpha(x) \equiv x + \alpha.
\]

With the above definition the translations behave as in the standard case under the action of boosts:

\[
\mathcal{B}\left(\hat{T}_\alpha(x)\right) = \hat{T}_{\mathcal{B}(\alpha)}(\mathcal{B}(x)).
\]

It is easy to check that these transformations form a group. We will call \( e \) the neutral element of the group \( i.e. \) the neutral element for the composition law. From the relations (1),(6) we read that the neutral element is such that \( G[e] = 0 \) \( i.e. \) its image corresponds to the origin of the classical space.

I. Invariant scales

With the previous definitions we want to add the condition that, under the deformed boosts, an invariant measurable physical scale (both a time or length scale) has to exist. In doing that we have in mind that, eventually, this scale is related with the Planck length (or time). Let us call \( \delta_p \) the vector which defines the invariant scale. By this we mean that \( \hat{\delta}_p \) is any vector of the form

\[
\begin{pmatrix}
T_p \\
\mathbf{x}
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
t \\
\mathbf{x}_p
\end{pmatrix},
\]

where \( \mathbf{x}_p \) is any spatial vector with modulus equal to the invariant scale and \( T_p \) is the invariant time scale. The invariance condition we impose is that a time (length) equal to the invariant scale is not affected by boosts. This corresponds to the physical intuition that any Planck-length segment (whatever his time position) or Planck-time interval (whatever his space position) should remain unaffected by a change in reference frame: \( i.e. \)

\[
\mathcal{B}\left(\begin{pmatrix} T_p \\ \mathbf{x}\end{pmatrix}\right) = \begin{pmatrix} T_p \\ \mathbf{x}'\end{pmatrix} \quad \text{and} \quad \mathcal{B}\left(\begin{pmatrix} t \\ \mathbf{x}_p\end{pmatrix}\right) = \begin{pmatrix} t' \\ \mathbf{x}_p\end{pmatrix}.
\]

In the classical space we have the invariant \((\xi^0)^2 - (\xi)^2\) whose image in the physical space is obviously invariant under the action of deformed boost. This, together with the requirement of invariance under rotations, allows to demonstrate that the above relations have to satisfy \( t = t' \) and \( \mathbf{x} = \mathbf{x}' \).

The above result is valid if we assume the existence of \( i) \) only a temporal invariant scale, \( ii) \) only a spatial invariant scale, \( iii) \) both a temporal and a spatial invariant scale.

For example, if we assume an invariant time scale \( T_p \), the vector \( (T_p, \mathbf{x}) \) (with arbitrary \( \mathbf{x} \)) under the action of a boost transformation has to be modified in such a way to keep \( T_p \) fixed as well as to leave the Casimir

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1 From here we will omit all the indexes. When necessary we will use the convention \( a^\mu = (a, a) \)
\( C(T_p, x) = (\xi^0)^2 - (\xi)^2 = (G^0(T_p, x))^2 - (G(T_p, x))^2 \)

The invariant points of standard Lorentz transformations are 0 and \( \infty \) (i.e., vectors where all the components are zero or infinity) and we can only use one of this two points to ensure the condition \( |x| = |\xi| \). In DSR in the momentum space we have a similar situation: in that case the fixed point at infinity is used to guarantee invariance of (high) energy or momentum scales. Since we demand our model to be equivalent to usual space-time when the distances (times) are much bigger than the invariant scale we expect \( G \) to approach the identity in this conditions and this implies \( G[\infty] = \infty \). Then the only possibility left is that the image of the invariant vectors under \( G \) must be zero. This condition is equivalent to the condition we used to define the neutral element of translations \( e \). This result indicates that at this point we are getting in troubles: infinitely many different physical vectors are mapped to the same vector in the classical space i.e. the function \( G \) can not be any more invertible when we approach the invariant scale. For example if we consider two distinct points \( x_a, x_b \) and assume that their distance (in space and/or time) equals the invariant quantity we have to write

\[
x_b - x_a = \delta_p
\]

and, after applying the \( G \) function to previous relation, we get \( G[x_a] = G[x_b] \) in contrast with our assumption of \( G \) invertible. We conclude that, to satisfy the invariance condition, we loose the uniqueness of the neutral element of translations and therefore its group structure.

In figure 1 we give an explicit realization of the temporal part of the \( G \) function for a vector with a fixed spatial part and a varying temporal component in order to represent it as an ordinary one-dimensional function. As required, we recover the standard Lorentz transformation for large times i.e. the \( G \) function becomes the identity for \( t >> T_p \) while goes to 0 once the invariant scale is reached. It is clear that both the images of \( T_p \) and \(-T_p \) are zero; we loose the uniqueness (\( G \) is not invertible at this points) and an invertible extension of \( G \) can not be defined on points in the interval \([-T_p, T_p] \).

The final result is that our invariant vector(s) has to satisfy the following relation

\[
B\delta_p = \delta_p,
\]

or, equivalently,

\[
\Lambda G[\delta_p] = G[\delta_p].
\]

Two comments are in order here. We have demonstrated the impossibility of implementing a minimal scale (in space, time or both) in the space-time by means of a DSR-like approach. However, it is clear that –as occurs in DSR in the momentum space– it is always possible to map the invariant scale (space or time) to infinity in the classical space time and to map the zero of the physical
space time to the zero of the classical space-time. The resulting space-time will differ from the Lorentz invariant one at large scales but it will not suffer the problems we discussed above: it will have a maximal scale (and possibly a minimum momentum) and will mirror the usual DSR in the momentum space.

Finally, let us say that the statement that the approach with a minimal scale is not possible, but the one with a maximal scale is allowed, can be understood by a dimensional argument. If we assume: i) a continuous differentiable manifold structure for the space time, ii) the existence of a length scale $\ell$, it is always possible to express any quantity depending on the coordinates as a series containing only negative powers of $\ell$. If we put the extra condition that iii) it should exist a smooth limit toward the undeformed space time, it is clear the small $\ell$ limit can not be accepted. The limit of large $\ell$, instead, is well defined and the interpretation of the scale as a maximal scale becomes clear. These arguments were already discussed in [14].

### III. CONCLUSIONS

In this note we have explored a possible scenario for the a space-time with an invariant scale in a DSR based approach. We started constructing a non linear realization of Lorentz transformations defining a non linear, invertible map between the physical space and an auxiliary space where the Lorentz group acts linearly. In doing that we introduce a deformed composition law for vectors in the physical space to guarantee its invariance under boosts. Up to this point we can still define a translation operator compatible with the deformed action of boosts, and this translations define a group.

Then we try to impose the physical condition that some (small) measurable physical length (in space or time) should remain invariant under boosts. To define the space length we use the standard expression for the invariant scale using the standard Lorentz fixed point at infinity. In present case instead, we are forced to map the invariant scale to the first Lorentz fixed point: the origin of vector space (recall that both in coordinate and momentum space the Lorentz transformations are linear and the only two fixed point are zero and infinity). This procedure unavoidably leaves all vectors with spatial or temporal length smaller or equal to the invariant length without counterpart in the classical space.

We understand why we encounter differences respect to the DSR approach in momentum space. For the latter the invariant momentum is realized mapping the physical momentum space up to the maximum momentum (energy) to the entire classical space (i.e. we obtain the invariant scale using the standard Lorentz fixed point at infinity). In present case instead, we are forced to map the invariant scale to the first Lorentz fixed point: the origin of vector space (recall that both in coordinate and momentum space the Lorentz transformations are linear and the only two fixed point are zero and infinity). This procedure unavoidably leaves all vectors with spatial or temporal length smaller or equal to the invariant length without counterpart in the classical space.

The main difference of our result with other approaches in the literature [22], resides in the definition of the composition law [11], which has been introduced in order to extend the notion of covariance. We think that the assumption of the standard composition law for vectors in the physical space is not correct. To assume [13] is an unavoidable step if we want to consistently construct a non linear realization of Lorentz invariance.

At the end of our construction we arrive to inconsistencies. A first possibility is, of course, to reject the idea that a DSR-like transformation can be defined in a coordinate space-time with an invariant scale. Indeed the connection of DSR models (defined in momentum space) with coordinate space is unclear [15] and may be this connection will be realized only via a completely different approach.

In any case, since the non-linear realizations of Lorentz symmetry have been very successful in constructing new versions of particle’s momentum space, we think it is worthwhile to explore the same technique further in the attempt to understand the possible structure of space-time.

A possible way out is to accept that translations do not form any more a group and interpret the peculiar behavior of the non linear mapping $G$ as a clue for a fundamental indetermination at the scale of the invariant length. This interpretation suggests a sort of (space-time) uncertainty, something that resembles what is expected to happen in a non commutative space-time. Alternatively we can speculate that defects (as in a worm-hole QG vacuum [16]) might be present in space-time.

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