The $Z$-boson hadronic decay width up to $\mathcal{O}(\alpha_s^4)$-order QCD corrections using the principle of maximum conformality

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In the paper, we study the properties of the $Z$-boson hadronic decay width by using the $\mathcal{O}(\alpha_s^4)$-order QCD corrections with the help of the principle of maximum conformality (PMC). By using the PMC, we obtain an accurate renormalization scale-and-scheme independent pQCD correction for the $Z$-boson hadronic decay width. More explicitly, after applying the PMC, a more convergent pQCD series has been obtained; and the contributions from the unknown $\mathcal{O}(\alpha_s^4)$-order and higher-order terms are highly suppressed, e.g. conservatively, we have $\Delta\Gamma_{\text{had}}^{\text{High order}} \simeq 0.004 \text{ MeV}$. In combination with the known electro-weak (EW) corrections, the QED corrections, the EW-QCD mixed corrections, and the QED-QCD mixed corrections, our final prediction of the hadronic $Z$ decay width is $\Gamma_{\text{had}}^{\text{Z}} = 1744.43 \pm 1.39 \text{ MeV}$, which agrees well with the PDG global fit experimental value, $1744.4 \pm 2.0 \text{ MeV}$. Thus, one obtains optimal fixed-order predictions for the $Z$-boson hadronic decay width, enabling high precision test of the Standard Model.

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I. INTRODUCTION

In quantum chromodynamics (QCD), the $Z$-boson hadronic decay width plays an important role in determining the strong coupling constant ($\alpha_s$). The $Z$-boson hadronic decay width has been measured by various collaborations at the electron-positron colliders LEP and SLC [1, 2], which could also be precisely measured in future high luminosity colliders such as the super $Z$ factory [3] or CEPC [4]. Theoretically, the one-loop electro-weak (EW) contributions and the mixed EW-QCD contributions to the $Z$-boson hadronic decay have been investigated in Refs.[5–8], and the two-loop EW contribution has been given in Refs.[9–11]. In large-$m_t$ limit, the higher-loop corrections have been calculated up to $\mathcal{O}(\alpha_s\alpha_s^2)$ [12, 13], $\mathcal{O}(\alpha_s^2\alpha_s)$, $\mathcal{O}(\alpha_s^3)$ [14, 15], and $\mathcal{O}(\alpha_s\alpha_s^2)\mathcal{O}(\alpha_s^4)$ [16–18], respectively, where $\alpha_s \equiv g_s^2/(4\pi)$ with $g_s$ being the top-quark Yukawa coupling constant. The final-state QED radiations have been computed up to $\mathcal{O}(\alpha_s)$, $\mathcal{O}(\alpha_s^2)$, $\mathcal{O}(\alpha_s^3)$ in Ref.[19]. The non-factorizable QCD corrections have been estimated in Refs.[8, 20]. The pure pQCD corrections up to $\mathcal{O}(\alpha_s^4)$ [21, 22], $\mathcal{O}(\alpha_s^3)$ [23–27], $\mathcal{O}(\alpha_s^4)$ [28–30] have also been performed in the literature. Moreover, the mass corrections to both the vector and axial vector correlators can be found in Refs.[21, 22, 31–35]. Those achievements give us good opportunities for precise determining of $\alpha_s(M_Z)$. A determination of $\alpha_s$ from the $Z$-boson hadronic decay width has been tried in Ref.[36].

Following the standard renormalization group invariance, the physical observable is independent to theoretical conventions, such as the choices of the renormalization scale and scheme, which is ensured by mutual cancelation of the scale and scheme dependence among different orders for an infinite-order pQCD prediction. However, for the fixed-order pQCD predictions, if the perturbative coefficients and the corresponding $\alpha_s$ do not match properly, the pQCD series shall have scale and scheme ambiguities. Conventionally, the renormalization scale is usually taken as the guessed momentum flow of the process, as well as the one to eliminate the large logs or to minimize the contributions from high-order terms or to achieve the prediction in agreement with the experimental data; This naive treatment directly causes the mismatching between the strong coupling constant and its coefficients, breaking the renormalization group invariance [37] and resulting in conventional renormalization scale and scheme ambiguities [38, 39]. Such guessing treatment decreases the predictive power of pQCD. In fact, predictions based on conventional scale setting are even incorrect for Abelian theory – Quantum Electrodynamics (QED); the renormalization scale of the QED coupling constant can be set unambiguously by using the Gell-Mann-Low method [40].

A correct scale-setting approach is thus important. Many ways have been suggested in the literature, most of them such as the renormalization group improved effective coupling method (FAC) [41, 42] and the principle of minimum sensitivity (PMS) [43–45] are designed to find an optimal renormalization scale of the process; on the other hand, the principle of maximum conformality (PMC) [46–50] provides a systematic and rigorous scale-
setting approach, whose purpose is to determine the effective \(\alpha_s\) of a fixed-order pQCD series with the help of renormalization group equation (RGE). The determined effective \(\alpha_s\) is independent to the choice of renormalization scale, thus the conventional renormalization scale ambiguity is eliminated. Moreover, since all the scheme-dependent non-conformal \(\{\beta_i\}\)-terms have been eliminated, the resultant pQCD series becomes scheme independent conformal series; thus the conventional renormalization scheme ambiguity can be eliminated simultaneously. A recent demonstration of renormalization scheme independence can be found in Refs.[51, 52], where by using the \(C\)-scheme coupling [53], it has been proven that the PMC prediction is independent of the choice of renormalization scale and scheme up to any fixed order. The convergence of the pQCD series can be naturally improved, since the divergent renomalon terms [54–56] is eliminated. The accurate renormalization scale-and-scheme independent conformal series is helpful not only for achieving precise pQCD predictions but also for a reliable prediction of the contributions of unknown higher-orders; some applications can be found in Refs.[57–61] which are estimated by using the Padé resummation approach [62–64]. In the present paper, we shall adopt the PMC single-scale approach [65] to analyze the \(Z\)-boson hadronic decay width. The PMC multi-scale approach and single-scale approach are equivalent to each other in sense of perturbative theory, but the residual scale dependence emerged in PMC multi-scale method can be greatly suppressed by applying the single-scale approach.

The remaining parts of the paper are organized as follows. In Sec.II, we will give the detailed PMC treatment for a precise determination of the \(Z\)-boson hadronic decay width. In Sec.III, we will give the numerical results. Sec.VI is reserved for a summary.

II. THE \(Z\)-BOSON HADRONIC DECAY WIDTH USING THE PMC

The hadronic decay width of the \(Z\)-boson can be expressed as
\[
\Gamma^\text{had}_Z = \Gamma_0 R^{\text{mc}} + \Delta\Gamma^\text{Extra}_Z,
\]
where the first term stands for the pure pQCD correction with the leading-order (LO) decay width \(\Gamma_0 = \frac{G_F M^2_Z}{24\pi}\), and the Fermi constant, \(G_F = 1.166378 \times 10^{-5}\). The second term \(\Delta\Gamma^\text{Extra}_Z\) contains four parts, i.e.,
\[
\Delta\Gamma^\text{Extra}_Z = \Delta\Gamma_1 + \Delta\Gamma_2 + \Delta\Gamma_3 + \Delta\Gamma_4
\]
\[
= -1.577^{+0.183}_{-0.233} + 0.695^{+0.061}_{-0.061} + 6.577^{+0.560}_{-0.560}
+ 0.609^{+0.061}_{-0.061}
= 6.304^{+0.804}_{-0.847} \text{ (MeV)},
\]
where \(\Delta\Gamma_1\) is the \(b\)- and \(t\)-quark mass correction to the vector and axial vector correlators [31–35], \(\Delta\Gamma_2\) is the quark final-state QED radiation and the mixed QED-QCD correction [19], \(\Delta\Gamma_3\) is the electroweak two-loop corrections and the higher-loop corrections in the large-\(m_t\) limit [11], \(\Delta\Gamma_4\) is the mixed EW-QCD correction and nonfactorizable QCD correction [6–8, 20]. Here the central values are for \(\mu_r = M_Z\), and the errors are for \(\mu_r \in [M_Z/2, 2M_Z]\). The perturbative QCQ corrections to the dominant correlator of the neutral current can be divided as the following four contributions,
\[
R^{\text{mc}} = 3\sum_f v_f^2 r_{\text{NS}}^V + \left( \sum_f v_f \right)^2 r_S^V + \sum_f a^2 r_{\text{NS}}^A + r_S^A \]
where \(v_f \equiv 2I_f - 4q_f s_f^V\), \(a_f \equiv 2I_f, q_f\) is the \(f\)-quark electric charge, \(s_f^V\) is the effective weak mixing angle, and \(I_f\) is the third component of weak isospin of the left-handed component of \(f\), \(r_{\text{NS}}^V = r_{\text{NS}}^A = r_{\text{NS}}^V = r_{\text{NS}}^V = r_{\text{NS}}^A = r_{\text{NS}}^A\) for the non-singlet, vector-singlet, and axial-singlet part, respectively. Those contributions can be further expressed as
\[
r_{\text{NS}} = 1 + \sum_{i=1}^C C_i^n a_i, a_i = 3
\]
where \(a_i = \alpha_s/(4\pi)\), and the coefficients of \(r_{\text{NS}}, r_{\text{NS}}^V\), and \(r_{\text{NS}}^A\) can be obtained from Refs.[28–30, 67, 68]. As for \(r_{\text{NS}}^A\), we adopt conventional scale setting approach to perform our analysis [2], and numerically, we obtain \(r_{\text{NS}}^A = [-1.725, -1.685]\) MeV for \(\mu_r \in [M_Z/2, 2M_Z]\) by using the formulas given by Ref.[28], whose magnitude is quite small in comparison to that of \(r_{\text{NS}}, r_{\text{NS}}^V\); fortunately, this approximate treatment will not affect our final conclusions.

The R-ratio can be rewritten as the following perturbative form by using the degeneracy relations [49, 50, 69], i.e.,
\[
R^{\text{mc}} = r_0 + r_{1,0} a_0(\mu_r) + (r_{2,0} + \beta_0 r_{2,1}) a_0^2(\mu_r)
+ (r_{3,0} + \beta_1 r_{3,1} + 2\beta_0 r_{3,1} + \beta_2^2 r_{3,2}) a_0^3(\mu_r)
+ (r_{4,0} + \beta_2 r_{4,1} + 2\beta_1 r_{4,1} + 5\beta_0 r_{4,3}) a_0^4(\mu_r) + O(\alpha_s^5),
\]
\[
\nonumber 2\text{ From the known } O(\alpha_s^4)\text{-order expressions, we cannot derive the exact RG-dependent } \mu_r\text{-series for } r_{\text{NS}}^V, \text{ which is however very important for using the PMC scale-setting; so we have to take this approximation.}
where $r_0 = 3 \left( \sum_j c_j^2 + \sum_i a_i^2 \right)$, and the coefficients $r_{i,j}$ can be obtained from the known coefficients $C^{NS}$, $C^{VS}$, and $C^{AS}$ of $r_{NS}$, $r_{VS}^V$, and $r_{AS}^V$. The coefficients $r_{i,0}$ are $\{ \beta_i \}$-independent conformal coefficients, and the $\{ \beta_i \}$-dependent non-conformal coefficients $r_{i,j}$ ($j \neq 0$) are generally functions of $\ln \mu^2_r / M_Z^2$, i.e.,

$$r_{i,j} = \sum_{k=0}^j C^k_{r_{i-k,j-k}} \ln^k \left( \mu^2_r / M_Z^2 \right),$$

(5)

where the reduced coefficients $\tilde{r}_{i,j} = r_{i,j} |_{\mu_r = M_Z}$, the combination coefficients $C^k_{r_{i-k,j-k}} = j! / [k! (j-k)!]$. We put the known coefficients $\tilde{r}_{i,j}$ up to $\mathcal{O}(\alpha^4_s)$-level in the Appendix.

Following the standard PMC single-scale procedures as described in detail in Ref.\[65\], with the help of RGE, one can determine an effective coupling $\alpha_s(Q_s)$ by absorbing all the non-conformal $\{ \beta_i \}$-terms into the running coupling, and the resultant pQCD series becomes the following conformal series,

$$R^{\text{PMC}}_{\text{conv}} = r_0 + r_{1,0} \alpha_s(Q_s) + r_{2,0} \alpha_s^2(Q_s) + r_{3,0} \alpha_s^3(Q_s) + \mathcal{O}(\alpha^4_s),$$

(6)

where $Q_s$ is the PMC scale, which corresponds to the overall effective momentum flow of the process and can be determined up to next-to-next-to-leading log (NNLL) accuracy by using the present known $\mathcal{O}(\alpha^4_s)$-order pQCD series; i.e., the $\ln Q_s^2 / M_Z^2$ can be expanded as the following perturbative series,

$$\ln \frac{Q^2}{M_Z^2} = T_0 + T_1 \alpha_s(M_Z) + T_2 \alpha_s^2(M_Z) + \mathcal{O}(\alpha^3_s),$$

(7)

where

$$T_0 = \frac{\tilde{r}_{2,1}}{r_{1,0}},$$

(8)

$$T_1 = \frac{\tilde{r}_{3,1}}{r_{1,0}} + \frac{2(\tilde{r}_{2,0} \tilde{r}_{3,1} - \tilde{r}_{1,0} \tilde{r}_{3,2})}{r_{1,0}^2},$$

(9)

and

$$T_2 = \frac{3 \beta_1 (\frac{\tilde{r}_{2,1}}{r_{1,0}} - \frac{\tilde{r}_{1,0} \tilde{r}_{3,2}}{2 r_{1,0}^2})}{r_{1,0}^2} + \frac{4(\tilde{r}_{1,0} \tilde{r}_{2,0} \tilde{r}_{3,1} - \tilde{r}_{2,0}^2 \tilde{r}_{2,1}) + 3(\tilde{r}_{1,0} \tilde{r}_{2,0} \tilde{r}_{3,1} - \tilde{r}_{2,0}^2 \tilde{r}_{2,1})}{r_{1,0}^3} + \frac{\beta_0 (4 \tilde{r}_{2,0} \tilde{r}_{3,1} \tilde{r}_{1,0} - 3 \tilde{r}_{4,2} \tilde{r}_{2,1} + 2 \tilde{r}_{2,0} \tilde{r}_{3,2} \tilde{r}_{1,0} - 3 \tilde{r}_{2,0} \tilde{r}_{2,1})}{r_{1,0}^4} + \frac{\beta_0^2 (2 \tilde{r}_{1,0} \tilde{r}_{3,2} \tilde{r}_{2,1} - \tilde{r}_{3,1}^2 - \tilde{r}_{2,1} \tilde{r}_{4,1})}{r_{1,0}^5}.$$  

(10)

It can be found that $Q_s$ is exactly free of $\mu_r$, and together with the $\mu_r$-independent conformal coefficients $r_{i,0}$, the conventional renormalization scale ambiguity is eliminated. Therefore, the precision of $R^{\text{PMC}}$ can be greatly improved by using the PMC. Moreover, the precision of the predictions depend on the perturbative nature of both the $R^{\text{PMC}}$ and the $\ln Q^2_s / M_Z^2$, which shall be numerically analyzed in the following paragraphs.

### III. NUMERICAL RESULTS

To do the numerical calculation, we adopt the Z-boson mass $M_Z = 91.1876 \pm 0.0021$ GeV and top-quark pole mass $M_t = 172.9$ GeV \[70\]. Taking $\alpha_s(M_Z) = 0.1181$ \[70\], we obtain $\Lambda^{\text{QCD}}_{\text{NNLO}} = 209.5$ GeV for the four-loop $\alpha_s$-running.

First, by setting all input parameters to be their central values, we present the Z-boson hadronic decay width $\Gamma^{\text{had}}_Z$ under the conventional scale-setting up to $\mathcal{O}(\alpha_s), \mathcal{O}(\alpha_s^2), \mathcal{O}(\alpha_s^3), \mathcal{O}(\alpha_s^4)$ orders versus the renormalization scale $\mu_r$, respectively.

![FIG. 1. The Z-boson hadronic decay width $\Gamma^{\text{had}}_Z$ under the conventional scale-setting approach in Fig. 1. It shows that in agreement of the conventional wisdom, the renormalization scale dependence becomes small when we have known more loop terms. For examples, we obtain $\Gamma^{\text{had}}_Z|_{\text{conv.}} = [1744.378, 1744.587]$ MeV for $\mu_r \in [M_Z/2, 2M_Z]$, and $\Gamma^{\text{had}}_Z|_{\text{conv.}} = [1744.378, 1745.008]$ MeV for $\mu_r \in [M_Z/3, 3M_Z]$, e.g., the net scale errors are only $0.01\%$, and $0.04\%$, respectively. We should point out that as has been mentioned in the Introduction, such small net scale dependence for the $\mathcal{O}(\alpha_s^4)$-order prediction is due to good convergence of the perturbative series, e.g., the relative magnitudes of the $\alpha_s$-terms: $\alpha_s^2$-terms: $\alpha_s^3$-terms: $\alpha_s^4$-terms: $1: 2.9\%: -2.2\%: -0.4\%$ for the case of $\mu_r = M_Z$; and also due to the cancellation of the scale dependence among different orders. The scale errors for each order term are changed and large. For example, the $\Gamma^{\text{had}}_Z$ has the following perturbative series up to $\mathcal{O}(\alpha_s^4)$-order:]

$$\Gamma^{\text{had}}_Z|_{\text{conv.}} = 1681.262 + 62.960 \beta_1^{0.925} + 1.802 \beta_2^{4.838} - 1.382 \beta_3^{1.131} - 0.230 \beta_4^{0.555} - 0.169 \beta_5^{0.040} \text{ (MeV)},$$  

(11)

\[\beta_n = 1744.418 \beta_{n+1}^{0.040} \text{ (MeV)}.

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where the central values are for $\mu_r = M_Z$, and the errors are obtained by varying $\mu_r \in [M_Z/2, 2M_Z]$. It shows that the absolute scale errors are 16%, 495%, 131%, and 143% for the $\alpha_s$-terms, $\alpha_s^2$-terms, $\alpha_s^3$-terms, and $\alpha_s^4$-terms, respectively; and there do have large scale cancellations among different orders.

One may observe that the relative magnitudes of each $\alpha_s$-terms, $\alpha_s^2$-terms, $\alpha_s^3$-terms, and $\alpha_s^4$-terms, respectively; and there do have large scale cancellations among different orders.

Third, it is helpful to predict the magnitude of the “unknown” higher-order pQCD corrections. The renormalization scale independent PMC series is helpful for such purpose. Because the PMC series has a good perturbative convergence, e.g., the magnitude of $\mathcal{O}(\alpha_s^4)$-order term is only 0.01% of $\mathcal{O}(\alpha_s)$-order term, it is reasonable to take the magnitude of the last known term $\pm |r_{4,0}^a(Q_s)|$ as a conservative prediction of the uncalculated higher-order terms [39]. By further taking the variation of $\Delta Q_s \simeq \pm 1.9$ GeV which is obtained by using the last known term of Eq. (13) as the magnitude of its unknown NNNLL term into consideration, we obtain

$$
\Delta \Gamma_Z^{\text{had}}|_{\text{PMC}, \text{High order}} \simeq \pm 0.004 \text{ (MeV)}.
$$

Finally, after eliminating the renormalization scale uncertainty by applying the PMC, we still have uncertainties from the $\alpha_s$ fixed-point error $\Delta \alpha_s(M_Z)$ and the Z-boson mass error $\Delta M_Z$. As for the $\alpha_s$ fixed-point error, by using $\Delta \alpha_s(M_Z) = 0.0011$ [70] together with the four-loop $\alpha_s$-running behavior, we obtain $A_{QCD}^{p_n=5} = 209.5_{-12.6}^{+13.2}$ MeV and

$$
\Delta \Gamma_Z^{\text{had}}|_{\text{PMC}, \Delta M_Z} = \pm 0.574 \text{ (MeV)}.
$$

And for the error of Z-boson mass $\Delta M_Z = \pm 0.0021$ GeV, we obtain

$$
\Delta \Gamma_Z^{\text{had}}|_{\text{PMC}, \Delta M_Z} = \pm 0.120 \text{ (MeV)}.
$$

Here, when discussing one uncertainty, the other input parameters shall be set as their central values. The squared average of those three errors results in a net error of $\pm 0.586$ MeV to the PMC prediction of decay width, in which $\Delta \alpha_s(M_Z)$ dominates the error sources. Thus the exact value of the reference point $\alpha_s(M_Z)$ is important for precise pQCD prediction.

**IV. SUMMARY**

In the present paper, we have presented a more accurate prediction of the Z-boson hadronic decay width by applying the PMC to eliminate the conventional renormalization scale ambiguity.

As a summary, we obtain

$$
\Gamma_Z^{\text{had}}|_{\text{PMC}} = 1744.439_{-1.390}^{+1.433} \text{ (MeV)}.
$$

where the errors are the sum of two parts, one is the squared average of those from $\Delta \alpha_s(M_Z)$, $\Delta M_Z$, and the uncalculated higher-order terms, another is the error from $\Delta \Gamma_Z^{\text{extra}}$ as given in Eq.(2). We present the PMC prediction of the Z-boson hadronic decay width in Fig. 3, where the experimental data are presented as a comparison. One may observe that the present PMC prediction agrees well with the global fit of the experimental measurements, 1744.3 $\pm$ 2 MeV [70].

Under conventional scale-setting approach, the scale-setting ambiguity could be softened by including enough

FIG. 2. The Z-boson hadronic decay width $\Gamma_Z^{\text{had}}$ under the PMC scale-setting up to $\mathcal{O}(\alpha_s^4)$-terms versus the renormalization scale $\mu_r$, respectively.

Second, we present the Z-boson hadronic decay width $\Gamma_Z^{\text{had}}$ under the PMC scale-setting approach in Fig. 2. It shows that after applying the PMC, the pQCD convergence is greatly improved, e.g., the relative magnitudes of the $\alpha_s$-terms: $\alpha_s^2$-terms: $\alpha_s^3$-terms: $\alpha_s^4$-terms of the pQCD series changes to 1: 4.33%: −0.49%: 0.01%. And there is no renormalization scale dependence for $\Gamma_Z^{\text{had}}$ at any fixed order, i.e.,

$$
\Gamma_Z^{\text{had}}|_{\text{PMC}} = 1681.262 + 60.838 + 2.634 - 0.299 + 0.004
$$

where each perturbative terms and the net total decay width are unchanged for any choice of $\mu_r$. This behavior is consistent with that of the previous PMC multi-scale approach analysis on $R^{\text{PC}}$ [71]. The PMC single scale is an effective scale which effectively replaces the individual PMC scales introduced in the PMC multi-scale approach in the sense of a mean value theorem, which can be regarded as the overall effective momentum flow of the process; it shows stability and convergence with increasing order in pQCD via the pQCD approximates. More explicitly, we obtain that the PMC scale $Q_s = 114.9$ GeV $\sim 1.3M_Z$, which can be fixed up to NNLL accuracy by using the present known $\mathcal{O}(\alpha_s^4)$-order pQCD series, i.e.,

$$
\ln \frac{Q^2}{M_Z^2} = 0.2249 + 21.7363a_s(M_Z) + 376.287a_s^2(M_Z)
$$

One may observe that the relative magnitudes of each order terms in $Q_s$, perturbative series are $1 : 91% : 15\%$, which also shows a good convergence behavior.

In the present paper, we have presented a more accurate prediction of the Z-boson hadronic decay width by applying the PMC to eliminate the conventional renormalization scale ambiguity.

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higher-order loop terms due to large cancelation among different orders; for the present considered decay width up to $\mathcal{O}(\alpha_s^5)$-order, the scale uncertainty is $(\pm 0.10^\circ_0)$ MeV for $\mu_r \in [M_Z/2, 2M_Z]$; and by further including the mentioned other error sources, we have $T_{Z}^{\text{had. conv.}} = 1744.418^{+1.595}_{-1.621}$ (MeV). After applying the PMC scale-setting approach, the pQCD series becomes scale independent and more convergent, thus a more reliable pQCD prediction can be achieved, enabling high precision test of the Standard Model.

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APPENDIX: THE PMC REDUCED PERTURBATIVE COEFFICIENTS $\hat{r}_{i,j}$

In this appendix, we give the required PMC reduced coefficients $\hat{r}_{i,j}$ for the perturbative series of the $Z$-boson hadronic decay width, which can be obtained from Refs.[28–30, 67, 68] with proper transformations, i.e.,

$$\hat{r}_{1,0} = 9\gamma_{1}^{NS}(\sum_{f} a_{f}^{2} + \sum_{f} v_{f}^{2}),$$  \hspace{1cm} (18)

$$\hat{r}_{2,0} = 4 \left[ 9\gamma_{2}^{NS}(\sum_{f} a_{f}^{2} + \sum_{f} v_{f}^{2}) - 37 + 12 \ln \frac{M_{Z}^{2}}{M_{t}^{2}} \right],$$  \hspace{1cm} (19)

$$\hat{r}_{1,1} = 9\Pi_{1}^{NS}(\sum_{f} a_{f}^{2} + \sum_{f} v_{f}^{2}),$$  \hspace{1cm} (20)

$$\hat{r}_{2,1} = 9\Pi_{2}^{NS}(\sum_{f} a_{f}^{2} + \sum_{f} v_{f}^{2}),$$  \hspace{1cm} (21)

$$\hat{r}_{3,0} = 144 \left[ \gamma_{3}^{NS} + \frac{\gamma_{3}^{S}(\sum_{f} q_{f})^{2}}{\sum_{f} q_{f}^{2}} \right] (\sum_{f} a_{f}^{2} + \sum_{f} v_{f}^{2})$$

$$+ (\sum_{f} v_{f})^{2} \left( \frac{440 - 320 \xi_{3}}{3} \right) + \frac{2144}{3} \ln \frac{M_{Z}^{2}}{M_{t}^{2}}$$

$$+ 368 \ln^{2} \frac{M_{Z}^{2}}{M_{t}^{2}} + 192 \xi_{3} + \frac{368 \pi^{2}}{3} - \frac{40600}{9},$$  \hspace{1cm} (22)

$$\hat{r}_{3,1} = 36\Pi_{2}^{NS}(\sum_{f} a_{f}^{2} + \sum_{f} v_{f}^{2}),$$  \hspace{1cm} (23)

$$\hat{r}_{3,2} = -3\pi^{2}\gamma_{i}^{NS}(\sum_{f} a_{f}^{2} + \sum_{f} v_{f}^{2}),$$  \hspace{1cm} (24)

$$\hat{r}_{4,0} = 576 \left[ \gamma_{4}^{NS} + \frac{\gamma_{4}^{S}(\sum_{f} q_{f})^{2}}{\sum_{f} q_{f}^{2}} \right] (\sum_{f} a_{f}^{2} + \sum_{f} v_{f}^{2})$$

$$+ (\sum_{f} v_{f})^{2} \left( \frac{3980 - 28960 \xi_{3}}{9} + \frac{11200 \xi_{5}}{9} \right)$$

$$+ \left( \frac{356000}{9} - 44192 \xi_{3} \right) \ln \frac{M_{Z}^{2}}{M_{t}^{2}} + \frac{33776}{3} \ln^{2} \frac{M_{Z}^{2}}{M_{t}^{2}}$$

$$+ \frac{8464}{3} \ln^{3} \frac{M_{Z}^{2}}{M_{t}^{2}} - \frac{13083735979}{56700} + \frac{35934343}{525}$$

$$+ \frac{12328 \xi_{5}}{9} - \frac{170272 \ln^{4} 2}{405} + \frac{512 \ln^{5} 2}{45}$$

$$+ \left( \frac{11748 + 512 \ln 2}{3} + \frac{170272 \ln^{2} 2}{405} - \frac{512 \ln^{3} 2}{27} \right) \pi^{2},$$  \hspace{1cm} (25)

$$\hat{r}_{4,1} = 144 \left[ \Pi_{3}^{NS} + \frac{\Pi_{3}^{S}(\sum_{f} q_{f})^{2}}{\sum_{f} q_{f}^{2}} \right] (\sum_{f} a_{f}^{2} + \sum_{f} v_{f}^{2})$$

$$+ (\sum_{f} v_{f})^{2} \left( \frac{59600}{27} - \frac{1040 \xi_{3}}{3} - \frac{320 \xi_{5}^{2}}{3} + \frac{800 \xi_{5}}{3} \right)$$

$$+ 12 \pi^{2}\gamma_{i}^{NS}(\sum_{f} a_{f}^{2} + \sum_{f} v_{f}^{2}),$$  \hspace{1cm} (26)

$$\hat{r}_{4,2} = -9\pi^{2}\Pi_{1}^{NS}(\sum_{f} a_{f}^{2} + \sum_{f} v_{f}^{2}),$$  \hspace{1cm} (27)

where the expressions for the coefficients $\gamma_{i}$ and $\Pi_{i}$ can be found in Refs.[68, 76].

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