A Note on Single Soft Scalar Emission of $\mathcal{N} = 8$ SUGRA and $E_7(7)$ Symmetry

Song He* and Hua Xing Zhu†

School of Physics, Peking University, Beijing, 100871, China

Abstract

We study single soft scalar emission amplitudes of $\mathcal{N} = 8$ supergravity (SUGRA) at the one-loop level using an explicit formula for one-loop amplitudes in terms of tree amplitudes, which in turn are evaluated using supersymmetric BCFW recursion relations. It turns out that the infrared-subtracted amplitudes vanish in the soft momentum limit, which supports the conjecture that $E_7(7)$ symmetry has no anomalies at the one-loop level.

*Electronic address: songhe@aei.mpg.de
†Electronic address: hxzhu@pku.edu.cn
I. INTRODUCTION

There has been a renewed interest in $\mathcal{N} = 8$ supergravity (SUGRA) in recent years. In [1], the authors made the bold conjecture that the theory may be ultraviolet (UV) finite up to all orders in perturbation theory. To support the conjecture, they provided all-loop evidence for the finiteness by promoting the “no-triangle hypothesis”\(^1\) to higher loops and using string duality arguments in [2]. Since then, various higher loop calculations have confirmed explicitly that $\mathcal{N} = 8$ SUGRA in four dimensional spacetime is UV finite at three loops [4], and very recently at four loops [5]. There are also string theory arguments in favor of the finiteness of $\mathcal{N} = 8$ SUGRA [3, 6].

On the other hand, it has long been known that on-shell classical $\mathcal{N} = 8$ SUGRA has a local $SU(8)$ symmetry and a hidden global $E_{7(7)}$ symmetry [7, 8]. Before gauge-fixing, $E_{7(7)}$ is linearly realized and acts on 133 scalars as well as vectors present in the classical action, independent from the local $SU(8)$ symmetry. The 63 local parameters of $SU(8)$ can be made used of to remove 63 non-physical scalars, leaving 70 massless scalars, which leads to an non-linearly realization of $E_{7(7)}$ on the remaining scalars. The action of the non-linearly realized $E_{7(7)}$ on $\mathcal{N} = 8$ SUGRA fields was only revealed recently [9], exact to all orders in gravitational coupling constant. It is possible but still not clear that this hidden $E_{7(7)}$ is relevant to the conjectured finiteness of $\mathcal{N} = 8$ SUGRA.

Recently, the emission of a single soft scalar in $\mathcal{N} = 8$ SUGRA tree amplitudes was examined in [10], in order to find the imprint of $E_{7(7)}$ symmetry, as expected from low energy theorem associated with soft Goldstone boson emission. The amplitudes for a single soft scalar emission were found to vanish generally, and it should be noted that the result is beyond the expectation from low energy theorem in pion physics, where a single soft pion emission is generally non-vanishing and can be obtained from the sum of Feynman diagrams in which the soft pion is attached to other external lines [11]. This is due to the fact that in the diagrams where the soft Goldstone boson is attached to external particles, taking the soft limit could lead to propagator singularities when the external particles are on-shell [12]. In $\mathcal{N} = 8$ SUGRA case there are cubic vertices in the Lagrangian through which a soft scalar could attach to external particles. However, these vertices vanish by themselves and

\(^1\) Note that for one-loop $\mathcal{N} = 8$ SUGRA amplitudes, the absence of triangle, bubble and rational terms has been proven, and the use of this terminology is merely a convention.
overcompensate the propagator singularities. The same result has been obtained in \cite{13}. By
generalizing the BCFW recursion relations \cite{17} to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory
and $\mathcal{N} = 8$ SUGRA, the authors of \cite{13} related general tree amplitudes to three-particle
amplitudes, which vanish fast enough to overcompensate the propagator singularities in the
soft limit, thus established that amplitudes with single soft scalar emission vanish generally.
They also found that double soft scalar emission amplitudes can be related to commutator
of two “broken generators” which label the soft scalars. In \cite{14}, the footprint of $E_{7(7)}$
symmetry on tree amplitudes was examined from a very different perspective. Considering
the consequences of Noether current conservation associated with $E_{7(7)}$ symmetry, it turns
out that single soft scalar emission amplitudes can be related to amplitudes without soft
scalar, but with extra “axial” charge attached to external particles. The result shows that
such “axial” charge vanishes at the tree level. As was explained in \cite{14}, when combining
with the results of \cite{10,13}, this establishes the result of low energy theorem for an $E_{7(7)}$
symmetry at the tree level.

Given the low energy theorem of $E_{7(7)}$ symmetry at the tree level, it is very natural and
interesting to see if this symmetry persists at the higher-order level. From the fact that chiral
$SU(8)$ one-loop triangle anomalies vanish \cite{15}, it is expected that $E_{7(7)}$ is not anomalous at
least at the one-loop level. The authors of \cite{16} have established the low energy theorem for
one-loop $n$-point amplitudes, by assuming the $E_{7(7)}$ symmetry at the one-loop level. They
found that the soft limit of the bosonic 4-point amplitudes vanish for complex momenta and
the “axial” charge vanishes for all one-loop amplitudes. It remains to examine the soft limit
of one loop $n$-point ($n \geq 5$) amplitudes, in order to confirm the low energy theorem of $E_{7(7)}$
symmetry at the one-loop level.

In this note we make a first attempt towards this goal. Our perspective is different from
that of \cite{16} but closer to that of \cite{13}. We shall use a simple formula for general one-loop
amplitude in terms of tree amplitudes \cite{13} follows from “no triangle hypothesis” \cite{2}. In
section II we review supersymmetric BCFW recursion relations and the vanishing result of
single soft scalar emission at the tree level. Then the result is generalized to the one-loop
level for the infrared-subtracted amplitudes in section III, which can be viewed as a strong
evidence for the absence of $E_{7(7)}$ anomalies. Conclusion and Discussions are presented in
the end.
II. SINGLE SOFT SCALAR EMISSION OF $\mathcal{N} = 8$ SUGRA AT THE TREE LEVEL

It has been argued from different perspectives [10, 13, 14, 16] that tree amplitudes with a single soft scalar emission in $\mathcal{N} = 8$ SUGRA vanish, which can be viewed as hints of a hidden $E_7(7)$ symmetry. Besides, double soft scalar emission has been calculated in [13] to reveal the non-trivial structure of $E_7(7)$. The key point leads to this result in [13] is the generalization of BCFW recursion relations [17] to maximally supersymmetric theories which we review below, and we refer to [13, 18] for details.

One beautiful insight in [13, 19] is that amplitudes of particles with higher spin have better large $z$ scaling, where $z$ is associated with the BCFW deformation

$$\lambda_1(z) = \lambda_1 + z\lambda_2, \quad \tilde{\lambda}_2(z) = \tilde{\lambda}_2 - z\tilde{\lambda}_1.$$  

(2.1)

In addition, maximal SUSY can relate all the helicity states in a CPT invariant supermultiplet to each other, which allows one to label the external states in a natural, continuous way. It’s realized as follows. The external states are represented by Grassmann coherent states $|\eta\rangle$ or $|\bar{\eta}\rangle$, which diagonalize not only the momentum but also the supercharge $Q_I$ or $\bar{Q}^I$, respectively. For a massless particle with momentum ($\sigma^\mu p_\mu$)$_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$, the Grassmann coherent states are defined as

$$|\bar{\eta}, \lambda, \tilde{\lambda}\rangle = e^{Q_{I\alpha} \bar{w}_I \alpha | + s, \lambda, \tilde{\lambda}\rangle}, \quad |\eta, \lambda, \tilde{\lambda}\rangle = e^{\bar{Q}^{I\dot{\alpha}} \bar{w}^{I\dot{\alpha}} | - s, \lambda, \tilde{\lambda}\rangle}$$  

(2.2)

where $w_\alpha$ and $\bar{w}_{\dot{\alpha}}$ are spinors such that $\langle w, \lambda \rangle = 1$ and $[\bar{w}, \tilde{\lambda}] = 1$. Note that $w_\alpha$ and $\bar{w}_{\dot{\alpha}}$ are not uniquely defined, but up to an additive shift, e.g. $w_\alpha \sim w_\alpha + c\lambda_\alpha$. The Grassmann coherent states defined in Eq. (2.2) are built from the highest spin states $| + s, \lambda, \tilde{\lambda}\rangle$ and $| - s, \lambda, \tilde{\lambda}\rangle$, respectively, where $Q| + s\rangle = \bar{Q}| - s\rangle = 0$.

Note that $\eta$ and $\bar{\eta}$ are equally valid descriptions of the complete supermultiplet; they are related to each other by a Grassmann Fourier transformation

$$|\eta\rangle = \int d^N \eta e^{\eta \eta\eta}|\eta\rangle, \quad \eta\rangle = \int d^N \bar{\eta} e^{\bar{\eta}\eta\eta}|\bar{\eta}\rangle.$$  

(2.3)

Now all the amplitudes can be expressed as smooth functions of $\eta$ and $\bar{\eta}$

$$\mathcal{M}(|\eta_h, \lambda_i, \tilde{\lambda}_{i}\rangle; |\bar{\eta}_h, \lambda_i, \tilde{\lambda}_{i}\rangle)$$  

(2.4)

In terms of amplitudes in $\eta$ representation, the BCFW recursion relations are generalized
\[ \mathcal{M}\{\{\eta_1(z), \lambda_1(z), \bar{\lambda}_1\}, \{\eta_2, \lambda_2, \bar{\lambda}_2(z)\}, \eta_i\} = \sum_{L,R} \int d^8\eta \mathcal{M}_L(\{\eta_1(z_P), \lambda_1(z_P), \bar{\lambda}_1\}, \eta, \eta_L) \frac{1}{p^2(z)} \mathcal{M}_R(\{\eta_2, \lambda_2, \bar{\lambda}_2(z_P)\}, \eta, \eta_R) \]  

(2.5)

where \( \eta_1(z_P) = \eta_1 + z_P \eta_2 \) is the supersymmetric counterpart of BCFW deformation Eq. (2.1).

Now the vanishing result of single soft scalar emission in \( \mathcal{N} = 8 \) SUGRA can be derived from supersymmetric BCFW recursion relations. Starting with any amplitude containing a particle with momentum \( p \) in \( \eta \) or \( \bar{\eta} \) representation, denoted as \( \mathcal{M}(\eta, ...) \) or \( \mathcal{M}(\bar{\eta}, ...) \), the single soft scalar emission is obtained by

\[ \lim_{p \to 0} \int d^8\eta^{abcd} \mathcal{M}(\eta, ...), \quad \text{or} \quad \lim_{p \to 0} \int d^8\bar{\eta}\bar{\eta}^{abcd} \mathcal{M}(\bar{\eta}, ...), \]  

(2.6)

where we first multiply the amplitude by \( \eta^{abcd} \equiv \eta^a\eta^b\eta^c\eta^d \) or \( \bar{\eta}^{abcd} \equiv \bar{\eta}^a\bar{\eta}^b\bar{\eta}^c\bar{\eta}^d \) with (sub)superscripts in (anti-)fundamental representations of \( SU(8) \), and integrate it over \( \eta \) or \( \bar{\eta} \), forcing the corresponding particle to be a scalar, then take the soft limit \( p \to 0 \). In terms of spinors, the soft limit is not uniquely defined. If we want to use \( p \sim \delta \) and then take \( \delta \to 0 \), we can take \( \lambda \sim \delta^\alpha \) and \( \bar{\lambda} \sim \delta^{1-\alpha} \) for any \( \alpha \in [0, 1] \), and the soft limit should be given by the largest contribution with a certain \( \alpha \). In the following we use \( f(p) \sim O(g(\delta)) \) to express that when \( p \sim \delta \to 0 \), \( f(p) \) is of the same order as or higher order than \( g(p) \) in \( \delta \), i.e.

\[ \lim_{p \to 0} \frac{f(p)}{g(p)} = c, \]  

(2.7)

where it is understood as the soft limit if \( f(p) \) is expressed as a function of \( \lambda \) and \( \bar{\lambda} \). Besides, \( c \) is a finite constant which can be zero.

To begin our discussion, we recall that the three particle amplitude can be either holomorphic or anti-holomorphic,

\[ \mathcal{M}_3(\eta_1, \eta_2, \eta_3) = \frac{\delta^{16}(\sum_{i=1}^{3} \bar{\lambda}_i \eta_i)}{([1 2] [2 3] [3 1])^2}, \quad \text{or} \quad \prod_{i=1}^{3} \int d^8\bar{\eta}_i \exp(\eta_i\bar{\eta}_i) \frac{\delta^{16}(\sum_{i=1}^{3} \lambda_i \bar{\eta}_i)}{([1 2] [2 3] [3 1])^2}, \]  

(2.8)

It is instructive to take a close look on the effect of taking soft limit in different ways. For a general three particle amplitude \( \mathcal{M}_3(1, 2, 3) \), we take the momentum of the first particle to be soft \( p_1 \sim \delta \) by taking \( \lambda_1 \sim \delta^\alpha \) and \( \bar{\lambda}_1 \sim \delta^{1-\alpha} \) for any \( \alpha \in [0, 1] \). The anti-holomorphic part of \( \mathcal{M}_3 \) is

\[ \mathcal{M}_3^{ah} = \frac{\delta^{16}(\bar{\lambda}_i \eta_i)}{([1 2] [2 3] [3 1])^2} \]  

(2.9)
By momentum conservation $\lambda_2 = -\delta^a \lambda_1 - \lambda_3$, we have

$$M^{ah}_3 = \delta^{2-4\alpha}[1 3]^2 \delta^8(\eta_1 - \delta^a \eta_2)\delta^8(\eta_3 - \eta_2),$$

(2.10)

where we have rescaled the spinor of particle 1 as $\lambda_1 = \delta^a \tilde{\lambda}_1$ and $\tilde{\lambda}_1 = \delta^{1-a} \hat{\lambda}_1$, where $\hat{\lambda}_1$ and $\tilde{\lambda}_1$ are hard spinors. The holomorphic part of $M_3$ is

$$M^h_3 = \prod_i \int d^8 \eta_i e^{\hat{\eta}_i} \frac{\delta^{16}(\lambda_i \bar{\eta}_i)}{(\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 1 \rangle)^2}$$

(2.11)

$$= \prod_i \int d^8 \eta_i e^{\hat{\eta}_i} \delta^{-2+4\alpha} \langle 1 3 \rangle^2 \delta^8(\bar{\eta}_1 - \delta^{1-a} \bar{\eta}_2)\delta^8(\bar{\eta}_3 - \bar{\eta}_2)$$

Therefore, for a soft particle other than scalar, the soft limit of three particle amplitude depends on $\alpha$, i.e. it depends on the way to take soft limit. However, for soft scalar limit which needs the delta function to provide exactly four components of $\eta_1$ or $\bar{\eta}_1$, the delta function contributes $\delta^{4\alpha}$ for anti-holomorphic part, and $\delta^{1-4\alpha}$ for holomorphic part, which yields a total contribution of the order $\delta^2$ for whatever $\alpha$ and either holomorphic or anti-holomorphic case,

$$\int d^8 \eta_1 \eta_1^{abcd} M_3(\eta_1, \eta_2, \eta_3) \sim O(\delta^2) \rightarrow 0 \text{ as } p_1 \sim \delta \rightarrow 0.$$  

(2.12)

For $n + 1 (n \geq 3)$-point amplitude with a single soft scalar and $n$ hard particles, it is straightforward to iteratively use supersymmetric BCFW recursion relations by deforming any two hard particles in the same (sub-)amplitude containing the soft scalar in every step, until there is a three particle amplitude with the single soft scalar in the factorization, which is of the order $\sim O(\delta^2)$. However, this is accompanied by a propagator pole since the propagator attached to this three amplitude is

$$\frac{1}{(p_1 + p'_i)^2} = \frac{1}{2p_1 \cdot p'_i} \sim O(\delta^{-1}) \text{ as } p_1 \sim \delta \rightarrow 0,$$

(2.13)

where $p'_i (2 \leq i \leq n + 1)$ is the (possibly deformed in previous steps of decomposition) momentum of the hard external particle in this three particle amplitude and $p'_i \sim O(1)$. Therefore, we have

$$\int d^8 \eta_1 \eta_1^{abcd} M_{n+1}(\eta_1, \eta_2, \ldots, \eta_{n+1}) \sim O(\delta) \rightarrow 0 \text{ as } p_1 \sim \delta \rightarrow 0$$

(2.14)

for $n \geq 3$. 

\[ \text{Page 6} \]
III. GENERALIZATION TO ONE-LOOP AMPLITUDES

The “no-triangle hypothesis” for one-loop amplitudes in $\mathcal{N} = 8$ SUGRA has been discovered and proved in [2]. It was also proved from a different perspective in [13]. The absence of triangle, bubble coefficients and rational terms leads to the simple formula for any one-loop amplitude in $\mathcal{N} = 8$ SUGRA, which is a sum of box integrals with coefficients purely given by products of tree amplitudes,

$$
\mathcal{M}_n^{1-\text{loop}} = \sum_{A,B,C,D \subseteq \{n\}} \sum_{l^* \in \{AB,BC,CD,DA\}} \prod_{a=AB,BC,CD,DA} \int d^8 \eta_a
$$

$$
\mathcal{M}_A(\eta_{DA}, -l^*_{DA}; A; \eta_{AB}, l^*_{AB}) \mathcal{M}_B(\eta_{AB}, -l^*_{AB}; B; \eta_{BC}, l^*_{BC}) \times \mathcal{M}_C(\eta_{BC}, -l^*_{BC}; C; \eta_{CD}, l^*_{CD}) \mathcal{M}_D(\eta_{CD}, -l^*_{CD}; D; \eta_{DA}, l^*_{DA})
$$

$$
\times I_4(P_A, P_B, P_C, P_D).
$$

(3.1)

Here the first summation is over all non-empty, non-intersecting subsets $A, B, C$ and $D$(corners) of $n$ particles, $A \cup B \cup C \cup D = \{n\} \equiv \{1, \ldots, n\}$, and the second is over (generally two) solutions of equations

$$
l^2 = (l - P_B)^2 = (l - P_B - P_C)^2 = (l + P_A)^2 = 0,
$$

(3.2)

where $P_A = \sum_{i \in A} p_i$ and similarly for $B, C$ and $D$. The $4 \times 8$ fold Grassmann integrations include those over 8 Grassmann variables for the internal line between two corners $D$ and $A$, $\eta_{DA}^I$ with $I = 1, \ldots, 8$, and similarly for $AB, BC$ and $CD$. In addition, the corresponding momenta are denoted by $l^*_{DA} = l^* + P_A$, $l^*_{AB} = l^*$, $l^*_{BC} = l^* - P_B$ and $l^*_{CD} = l^* - P_B - P_C$, as presented in the four tree amplitudes $\mathcal{M}_A, \mathcal{M}_B, \mathcal{M}_C$ and $\mathcal{M}_D$. $A$ is short for $\{\eta_i, p_i\}$ or $\{\eta_i, \lambda_i, \tilde{\lambda}_i\}$ with $i \in A$, and similarly for $B, C$ and $D$. The product of these four tree amplitudes is the coefficient of the box integral $I_4(P_A, P_B, P_C, P_D)$ which is given by [20]

$$
I_4(K_1, K_2, K_3, K_4) = -\frac{r_\Gamma}{2 \sqrt{\det S}} F_4,
$$

(3.3)

where $r_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma^2(1-2\epsilon)}{\Gamma(1-2\epsilon)}$, the symmetric $4 \times 4$ matrix $S$ is,

$$
S_{ij} = -\frac{1}{2} (K_i + \ldots + K_{j-1})^2 \text{ for } i \neq j, \quad S_{ii} = 0,
$$

(3.4)
and box functions are given by,

\[ F^{4m}(K_1, K_2, K_3, K_4) = \frac{1}{2} (-\text{Li}_2((1 - \lambda_1 + \lambda_2 + \rho)/2) + \text{Li}_2((1 - \lambda_1 + \lambda_2 - \rho)/2) \\
-\text{Li}_2(-(1 - \lambda_1 - \lambda_2 - \rho)/(2\lambda_1)) + \text{Li}_2(-(1 - \lambda_1 - \lambda_2 + \rho)/(2\lambda_1)) - \frac{1}{2} \ln \left( \frac{\lambda_1}{\lambda_2^2} \right) \ln \left( \frac{1 + \lambda_1 - \lambda_2 + \rho}{1 + \lambda_1 - \lambda_2 - \rho} \right), \]

\[ F^{3m}(k_1, K_2, K_3, K_4) = -\frac{1}{2\epsilon^2} \left( (-s)^{-\epsilon} + (-t)^{-\epsilon} - (-K_2^2)^{-\epsilon} - (-K_4^2)^{-\epsilon} \right) \\
+\text{Li}_2 \left( 1 - \frac{K_2^2}{s} \right) + \text{Li}_2 \left( 1 - \frac{K_3^2}{t} \right) - \text{Li}_2 \left( 1 - \frac{K_2^2K_4^2}{st} \right) \\
+\frac{1}{2} \ln^2 \left( \frac{s}{t} \right) - \frac{1}{2} \ln \left( \frac{K_3^2}{s} \right) \ln \left( \frac{K_2^2}{t} \right) - \frac{1}{2} \ln \left( \frac{K_3^2}{t} \right) \ln \left( \frac{K_2^2}{t} \right), \]

\[ F^{2m}(k_1, K_2, k_3, K_4) = -\frac{1}{\epsilon^2} \left( (-s)^{-\epsilon} + (-t)^{-\epsilon} - (-K_2^2)^{-\epsilon} - (-K_4^2)^{-\epsilon} \right) \\
+\text{Li}_2 \left( 1 - \frac{K_2^2}{s} \right) + \text{Li}_2 \left( 1 - \frac{K_3^2}{t} \right) + \text{Li}_2 \left( 1 - \frac{K_2^2}{s} \right) + \text{Li}_2 \left( 1 - \frac{K_2^2}{t} \right) \\
-\text{Li}_2 \left( 1 - \frac{K_2^2K_4^2}{st} \right) - \frac{1}{2} \ln^2 \left( \frac{s}{t} \right), \]

\[ F^{2m}(k_1, k_2, K_3, K_4) = -\frac{1}{2\epsilon^2} \left( (-s)^{-\epsilon} + 2(-t)^{-\epsilon} - (-K_3^2)^{-\epsilon} - (-K_4^2)^{-\epsilon} \right) \\
+\text{Li}_2 \left( 1 - \frac{K_3^2}{s} \right) + \text{Li}_2 \left( 1 - \frac{K_3^2}{t} \right) + \frac{1}{2} \ln^2 \left( \frac{s}{t} \right) - \frac{1}{2} \ln \left( \frac{K_3^2}{s} \right) \ln \left( \frac{K_3^2}{s} \right), \]

\[ F^{1m}(k_1, k_2, k_3, K_4) = -\frac{1}{\epsilon^2} \left( (-s)^{-\epsilon} + (-t)^{-\epsilon} - (-K_4^2)^{-\epsilon} \right) \\
+\text{Li}_2 \left( 1 - \frac{K_4^2}{s} \right) + \text{Li}_2 \left( 1 - \frac{K_4^2}{t} \right) + \frac{1}{2} \ln^2 \left( \frac{s}{t} \right) + \frac{\pi^2}{6}, \]

\[ F^{0m}(k_1, k_2, k_3, k_4) = -\frac{1}{\epsilon^2} \left( (-s)^{-\epsilon} + (-t)^{-\epsilon} \right) + \frac{1}{2} \ln^2 \left( \frac{s}{t} \right) + \frac{\pi^2}{2}. \tag{3.5} \]

for four, three, ..., zero-mass case respectively, where \( \epsilon \) is the infrared cutoff in dimensional regularization. \( k_i \) are on-shell momenta for the case when there is only one external leg in a corner and \( K_i \) off-shell momenta for more generic case. The Mandelstam variables are \( s = (k_1 + k_2)^2 \) and \( t = (k_1 + k_4)^2 \) for zero-mass case and similarly for other cases with possible off-shell momenta. The dilogarithm function is defined as

\[ \text{Li}_2(z) = -\int_0^z \frac{dx}{x} \ln(1 - x), \tag{3.6} \]

and in four-mass case \( \rho \) is defined as,

\[ \rho = \sqrt{1 - 2(\lambda_1 + \lambda_2) + (\lambda_1 - \lambda_2)^2} \text{ with } \lambda_1 = \frac{K_2^2K_3^2}{st} \text{ and } \lambda_2 = \frac{K_2^2K_4^2}{st}. \tag{3.7} \]
Before proceeding, an important remark on infrared divergences is needed. These box functions, except the four-mass one, all possess infrared divergences, which are regularized by computing in $D = 4 - 2\varepsilon$ dimension. A general one-loop amplitude then can be expanded in powers of $\varepsilon$,

$$M^{1\text{-loop}}(\varepsilon) = \frac{C_2}{\varepsilon^2} + \frac{C_1}{\varepsilon} + C_0 + \mathcal{O}(\varepsilon),$$

(3.8)

where $C_2, C_1$ and $C_0$ are functions of kinematic invariants. The result we shall present for one-loop single soft scalar emission has two folds of meanings. First, as it stands, we shall prove that for any one-loop amplitude with a single soft scalar,

$$\lim_{\delta \to 0} C_i(\delta) = 0,$$

(3.9)

for $i = 0, 1, 2$. The same conclusion holds for the $\mathcal{O}(\varepsilon)$ terms but they can be neglected when we take the $\varepsilon \to 0$ limit. On the other hand, it is well known that as long as one is concerning about proper “infrared safe” observables, infrared divergences do not show up in the final result. For example, this can be done by subtracting the IR divergences from the 1-loop amplitudes via dipole subtracting scheme [21], and since $C_0, C_1$ and $C_2$ all vanish in the soft limit, our result is independent of subtraction schemes. Therefore, we will always refer to our result by stating its physical implication: for whatever scheme one uses to subtract the infrared divergences, the one-loop infrared-subtracted amplitudes for single soft scalar emission always vanish.

FIG. 1: Generic terms of one-loop amplitude with single soft scalar emission.
FIG. 2: Special terms of one-loop amplitude with single soft scalar emission.

Now we want to calculate the single soft scalar emission in one-loop amplitudes, which by Eq. (3.1) is given by

\[
\lim_{p_1\to 0} \int d^8\eta_1 \eta_1^{abcd} M_{n+1}^{1\text{-loop}} = \\
\lim_{p_1\to 0} \int d^8\eta_1 \eta_1^{abcd} \sum_{A,B,C,D \subset \{2,\ldots,n+1\}} \sum_{l^*} \prod_{a=AB,BC,CD,DA} \int d^8\eta_a \\
M_A(\eta_{DA}, -l^*_{DA}; 1; A; \eta_{AB}, l^*_{AB}) M_B(\eta_{AB}, -l^*_{AB}; B; \eta_{BC}, l^*_{BC}) \\
\times M_C(\eta_{BC}, -l^*_{BC}; C; \eta_{CD}, l^*_{CD}) M_D(\eta_{CD}, -l^*_{CD}; D; \eta_{DA}, l^*_{DA}) \\
\times I_4(P_A + p_1, P_B, P_C, P_D) + \text{similar terms}
\]

\[
+ \lim_{p_1\to 0} \int d^8\eta_1 \eta_1^{abcd} \sum_{A=\emptyset, B,C,D \subset \{2,\ldots,n+1\}} \sum_{l^*} \prod_{a=AB,BC,CD,DA} \int d^8\eta_a \\
M_A(\eta_{DA}, -l^*_{DA}; 1; \eta_{AB}, l^*_{AB}) M_B(\eta_{AB}, -l^*_{AB}; B; \eta_{BC}, l^*_{BC}) \\
\times M_C(\eta_{BC}, -l^*_{BC}; C; \eta_{CD}, l^*_{CD}) M_D(\eta_{CD}, -l^*_{CD}; D; \eta_{DA}, l^*_{DA}) \\
\times I_4(p_1, P_B, P_C, P_D) + \text{similar terms},
\]

(3.10)

where the first collected term, denoted by \( M_{\text{gen}} \) (Fig. 1), is the generic case where every corner has hard external particles and the second one, \( M_{\text{spe}} \) (Fig. 2), is the special case where one certain corner has only a single soft external particle. Besides, similar terms are
those with the soft scalar in corner $B, C$ and $D$. By Eq. (2.14), we have

$$
\mathcal{M}_{\text{gen}} = \sum_{A,B,C,D \in \{2, \ldots, n+1\}} \sum_{I^*} \prod_{a=AB,BC,CD,DA} \int d^8 \eta_a

\left[ \lim_{p_1 \to 0} \int d^8 \eta_1 \eta_1^{abcd} \mathcal{M}_A(\eta_{DA}, -l^*_DA; 1; \eta_{AB}, l^*_AB) \right] \mathcal{M}_B(\eta_{AB}, -l^*_AB; B; \eta_{BC}, l^*_BC)

\times \mathcal{M}_C(\eta_{BC}, -l^*_BC; C; \eta_{CD}, l^*_CD) \mathcal{M}_D(\eta_{CD}, -l^*_CD; D; \eta_{DA}, l^*_DA)

\times \lim_{p_1 \to 0} I_4(P_A + p_1, P_B, P_C, P_D) + \text{similar terms.}

(3.11)

where the soft limit in square parenthesis vanishes ($\mathcal{O}(\delta)$ as $\delta \to 0$) and other tree amplitudes are regular since they generally do not depend on the soft limit (Notice the frozen momenta are generally hard in this case).

It is easy to see that $\det S$ remains regular in the soft limit, but there are soft momentum divergences arising in the box functions. Here we only need to consider in $\mathcal{M}_{\text{gen}}$ box functions with at least one mass and check their soft limits with a massive corner $P + p_1$ goes to $P$ as $p_1 \to 0$. If $P$ is massive, it is easy to see that any of these box functions is regular when $p_1 \to 0$. For $P$ massless, that is when the corner has only one soft scalar and a single hard particle, we need to check potential discontinuities of transitions between $m$-mass functions and $(m-1)$-mass functions for $m = 1, 2, 3, 4$ when taking the soft limit. As discussed in details in [20], one-mass and two-mass-easy functions smoothly goes to zero-mass and one-mass functions in the soft limit, respectively, but two-mass-hard, three-mass and four-mass functions can have discontinuities proportional to $1/\epsilon$ in the soft limit. Nevertheless, the coefficients, which contribute to $C_1$ only diverge as $\mathcal{O}(\ln(\delta))$ as $p_1 \sim \mathcal{O}(\delta) \to 0$, which are overcompensated by the $\mathcal{O}(\delta)$ vanishing behavior of the product of tree amplitudes.

Now the only non-trivial thing one needs to check is $\mathcal{M}_{\text{spe}}$

$$
\mathcal{M}_{\text{spe}} = \sum_{A=\emptyset, B,C,D \in \{2, \ldots, n+1\}} \sum_{I^*} \prod_{a=AB,BC,CD,DA} \int d^8 \eta_a

\left[ \lim_{p_1 \to 0} \int d^8 \eta_1 \eta_1^{abcd} \mathcal{M}_A(\eta_{DA}, -l^*_DA; 1; \eta_{AB}, l^*_AB) \right] \mathcal{M}_B(\eta_{AB}, -l^*_AB; B; \eta_{BC}, l^*_BC)

\times \mathcal{M}_C(\eta_{BC}, -l^*_BC; C; \eta_{CD}, l^*_CD) \mathcal{M}_D(\eta_{CD}, -l^*_CD; D; \eta_{DA}, l^*_DA)

\times \lim_{p_1 \to 0} I_4(p_1, P_B, P_C, P_D) + \text{similar terms.}

(3.12)

It is enough to consider the three-, two-, one-mass cases for $n > 3$ and we leave the special
case $n = 3$ corresponding to the zero-mass case later for an explicit estimate. For $n > 3$, the solution to Eq. (3.2) is generally hard ($O(1)$), thus we can use Eq. (2.12) to obtain the soft limit in square parenthesis, which vanishes as $O(\delta^2)$, and we now explicitly estimate the soft limit of box integral when the momentum of a corner goes to zero.

By Eq. (3.4), the soft limit of $\det S^{-1/2}$ depends on whether these momenta are on-shell, and it is straightforward to obtain $\det S^{-1/2} \sim O(1)$ for three-mass and two-mass-easy cases, while $\det S^{-1/2} \sim O(\delta^{-1})$ for two-mass-hard, and one-mass cases. For the box functions defined in Eq. (3.5), we have the following results of their soft limits.

For three-mass case, the box function is $O(1)$ since $s$ goes to a finite value $K^2_2$ when $k_1 \to 0$ and there is no singular contribution in this soft limit, so does two-mass-easy case since $s \to K^2_2$ and $t \to K^2_2$ in any soft limit. For two-mass-hard case, there can be singular terms from $-\frac{1}{\epsilon^2}(-s)^{-\epsilon}$, and $\frac{1}{2} \ln^2 \left(\frac{s}{\delta}\right) - \frac{1}{2} \ln \left(\frac{K^2_2}{\delta}\right)$ which give $O(\ln^2 \delta)$. Similarly, there can be singular terms in one-mass case which are $O(\ln^2 \delta)$.

Therefore, for $n > 3$, the soft limit of the box integrals $\lim_{p_1 \to 0} I_4(p_1, P_B, P_C, P_D)$ can have soft momentum divergences, but these are overcompensated by the three particle amplitude which vanishes as $O(\delta^2)$ in the soft limit, and we obtain that

$$\lim_{p_1 \to 0} \int d^8 q_1 \eta_1^{abcd} M_{n+1}^{1\text{-loop}}(1, 2, ..., n+1) = 0 \quad (3.13)$$

for $n > 3$.

The case $n = 3$ is more subtle and we treat it separately here (Fig. 3). Take the term with $p_1 = P_A$ in the corner A as an example and relabel $P_A = k_1 \sim \delta, P_B = k_2, P_C = k_3$ and $P_D = k_4$. Since $k_1, k_2, k_3$ and $k_4$ are all on-shell, in the soft limit $k_1 \sim \delta \to 0$, we have not only $s = 2k_1 \cdot k_2 \sim \delta$, but also $t = 2k_1 \cdot k_4 \sim \delta$. Therefore, the pre-factor $\det S^{-1/2} \sim O(\delta^{-2})$ and the corresponding zero-mass box function is $O(\ln^2 \delta)$ in this limit. In addition, we can not just take the soft limit inside square parenthesis of Eq. (3.12) because in this case some internal (fixed) momenta $l^*$ become soft!

To see this, we shall use Eq. (3.2) which gives,

$$l^* \cdot k_1 = l^* \cdot k_2 = l^* k_4 - k_2 \cdot k_3 = 0, \quad (3.14)$$

where in the last equality $l^* \cdot k_2 = k_2 \cdot k_3 = -k_1 \cdot k_4 \sim O(\delta)$ implies that $l^* \sim O(\delta)$ and further $(l^* + k_1) \sim O(\delta)$, thus three momenta $-l^*_{DA}, p_4$ and $l^*_{AB}$ in $M_A$, $-l^*_{AB}$ in $M_B$ and
\[ l_{DA}^{*} \text{in } M_D \text{ are all } \mathcal{O}(\delta). \]

Adopting a simpler notation, we have

\[
\int d^8 \eta_1 \eta_1^{abcd} M_4(1, 2, 3, 4) = \\
\int d^8 \eta_1 \eta_1^{abcd} \prod_{a=1}^{4} \int d^8 \eta_a \eta_a M_A(-4, 1, 1') M_B(-1', 2, 2') M_C(-2', 3, 3') M_D(-3', 4, 4') \\
\times I_4(1, 2, 3, 4) + \text{similar terms.} \tag{3.15}
\]

![FIG. 3: One-loop 4-point amplitudes with one soft particle and three hard particle.](image)

First we assume that the sub-amplitude \( M_A \) is anti-holomorphic. The similar argument can be applied to making the opposite choice of \( M_A \) be holomorphic. The sub-amplitudes \( M_B, M_C \) and \( M_D \) can be either holomorphic or anti-holomorphic. But it will be clear later that the case where all of the sub-amplitudes are anti-holomorphic is irrelevant for single soft scalar emission. For all the sub-amplitudes, we choose to work in \( \eta \) representation. In this representation three particle amplitude is

\[
M_3^{ah}(\eta_i) = \frac{\delta^{16}(\bar{\lambda}_i \eta_i)}{([1 2] [2 3] [3 4])^2} \tag{3.16}
\]

for \( M_3 \) anti-holomorphic and

\[
M_3^h(\eta_i) = \int \prod_{i=1,2,3} d^8 \bar{\eta}_i e^{\eta_i \eta_i} \frac{\delta^{16}(\lambda_i \bar{\eta}_i)}{([1 2] \langle 2 3 \rangle \langle 3 1 \rangle)^2} \tag{3.17}
\]

for \( M_3 \) holomorphic. Note that in \( \eta \) representation, the power of \( \eta \) is 16 for an anti-holomorphic amplitude and 8 for a holomorphic amplitude. We define an useful quantity
\(\Delta(M_n(\eta))\) to be the power of \(\eta\) minus the power of \(d\eta\) in \(M_n\). Obviously, \(0 \leq \Delta(M_n) \leq 8n\) for any amplitude to be non-vanishing. It is easy to see that for the amplitude defined in Eq. (3.15), we have

\[
\Delta(M_4) = \begin{cases} 
32, & \text{if the number of holomorphic amplitude in } \{M_B, M_C, M_D\} \text{ is } 0, \\
24, & \text{if the number of holomorphic amplitude in } \{M_B, M_C, M_D\} \text{ is } 1, \\
16, & \text{if the number of holomorphic amplitude in } \{M_B, M_C, M_D\} \text{ is } 2, \\
8, & \text{if the number of holomorphic amplitude in } \{M_B, M_C, M_D\} \text{ is } 3.
\end{cases}
\]

(3.18)

Now we can understand why the case when all of the sub-amplitudes are anti-holomorphic is irrelevant. In this case \(\Delta(M_4) = 32\), when taking into account the \(4 \times 8\) fold \(d\eta\) integral, the only choice for the external particles species is 4 gravitons! If two of the sub-amplitudes are anti-holomorphic, \(\Delta = 16\) and there are \(4 \times 8\) fold \(d\eta\) external particle integral, left 8 power of \(\eta\) assignment for external particle species. For the bosonic case, it can correspond to cases studied in [16], a 4-point amplitude of two scalars and two vectors, or that of four scalars.

Since \(M_A\) is anti-holomorphic, we can set \(\lambda_4', \lambda_1\) and \(\lambda_1'\) to be parallel and take the soft limit as

\[
\lambda_4' = \lambda_1' = \lambda_1 = \delta^\alpha \hat{\lambda}_1
\]

and

\[
\hat{\lambda}_4' = \delta^{1-\alpha} \hat{\lambda}_4', \quad \hat{\lambda}_1' = \delta^{1-\alpha} \hat{\lambda}_1', \quad \hat{\lambda}_1 = \delta^{1-\alpha} \hat{\lambda}_1.
\]

By momentum conservation we have \(\hat{\lambda}_1 = \hat{\lambda}_4' - \hat{\lambda}_1'\). Then \(M_A\) is given by

\[
M_A = \delta^{4-4\alpha} \left[4' \hat{1}'\right]^2 \delta^8(\eta_1' - \eta_1) \delta^8(\eta_1' + \eta_1) \sim O(\delta^{4-4\alpha}).
\]

(3.19)

The power of \(\eta_1\) in \(M_A\) is 4 in order to match the pre-factor \(\eta^{abcd}\) when considering a scalar emission. Thus in total \(M_A\) provides 12 power of \(\eta_1'\) plus \(\eta_1\). In order to match the internal \(d\eta_1'\) and \(d\eta_1\) integral, \(M_B\) and \(M_D\) must provide 4 power of \(\eta_1'\) plus \(\eta_1\).

The holomorphic part of \(M_B\) is given by

\[
M_B^h = \prod_i \int d^8 \bar{\eta}_i e^{\bar{\eta}_i \eta_i} \delta^{-2+4\alpha} \left[4' \hat{1}'\right]^2 \delta^8(\eta_1' + \delta^{1-\alpha} \bar{\eta}_2) \delta^8(\eta_2' - \bar{\eta}_2),
\]

(3.20)

and the anti-holomorphic part is

\[
M_B^{\bar{h}} = \delta^{2-4\alpha} \left[4' \hat{1}'\right]^2 \delta^8(\eta_1' + \delta^{\alpha} \bar{\eta}_2) \delta^8(\eta_2' + \bar{\eta}_2).
\]

(3.21)
Besides, $\mathcal{M}_D^{h(\text{ah})} = \mathcal{M}_B^{h(\text{ah})}(1' \to 3', 2' \to 4', 2 \to 4)$. There is no soft momentum in the other holomorphic sub-amplitude $\mathcal{M}_D$, thus it is always $O(1)$.

We discuss several choices for $\mathcal{M}_B$ and $\mathcal{M}_D$. If both $\mathcal{M}_B$ and $\mathcal{M}_D$ are holomorphic, we have

\[ \mathcal{M}_4 \sim \int d^8 \bar{\eta} \eta^{abcd} d^8 \bar{\eta}' d^8 \eta' \delta^{4-4\alpha}[\hat{2}' \hat{1}']^2 \delta^8(\eta' - \eta) \delta^8(\eta' + \eta) \times \prod_i \int d^8 \bar{\eta}_i e^{\bar{\eta}_i, \eta_i} \delta^{-2+4\alpha} \langle \hat{\eta}' \hat{2}' \rangle \delta^8(\bar{\eta}_v + \delta^{1-\alpha} \bar{\eta}_2) \times \delta^2 \log^2 \delta. \tag{3.22} \]

As mentioned before, $\mathcal{M}_B$ and $\mathcal{M}_D$ must provide 4 power of $\eta'$ plus $\eta_v$. This can only come from the exponential of second and third line in Eq. (3.22). At the same time it brings down 4 power of $\bar{\eta}_v$ plus $\bar{\eta}_v$ from the exponential. In order to match the $d\eta_v$ and $d\bar{\eta}_v$ integral, there must be 12 power of $\bar{\eta}_v$ plus $\bar{\eta}_v$ from the delta functions in the second and third line in Eq. (3.22). Thus the same delta functions provide $\delta^{4-4\alpha}$,

\[ \mathcal{M}_4 \sim \delta^{4-4\alpha} \cdot \delta^{-2+4\alpha} \cdot \delta^{-2+4\alpha} \cdot \delta^{4-4\alpha} \cdot \delta^{-2} \log^2 \delta \sim \delta^2 \log^2 \delta. \tag{3.23} \]

This establishes that $\mathcal{M}_4$ vanishes as $\delta \to 0$.

Next we consider the case where $\mathcal{M}_D$ is holomorphic and $\mathcal{M}_B$ is anti-holomorphic. The opposite choice is similar.

\[ \mathcal{M}_4 \sim \int d^8 \eta \eta^{abcd} d^8 \bar{\eta} d^8 \eta \delta^{4-4\alpha}[\hat{2}' \hat{1}']^2 \delta^8(\eta' - \eta) \delta^8(\eta' + \eta) \times \delta^{2-4\alpha}[\hat{2}' \hat{1}']^2 \delta^8(\eta_v + \delta^{1-\alpha} \eta_2) \delta^8(\bar{\eta}_v - \eta_2) \times \prod_i \int d^8 \bar{\eta}_i e^{\bar{\eta}_i, \eta_i} \delta^{-2+4\alpha} \langle \hat{\eta}' \hat{2}' \rangle \delta^8(\bar{\eta}_v - \delta^{1-\alpha} \bar{\eta}_4) \delta^8(\bar{\eta}_v + \bar{\eta}_4) \times \delta^{-2} \log^2 \delta. \tag{3.24} \]

The $\delta$ power counting reads

\[ \mathcal{M}_4 \sim \delta^{4-4\alpha} \cdot \delta^{2-4\alpha} \cdot \delta^{-2+4\alpha} \cdot \delta^{8-2\alpha} \cdot \delta^{4-4\alpha} \cdot \delta^{-2} \cdot \log^2 \delta \sim \delta^{6-j} \log^2 \delta. \tag{3.26} \]
where $j$ is the number of $\eta_i'$ in Eq. (3.24). The dominant contribution comes from taking $j = 4$, and $M_4$ scales as $\delta^2 \log \delta$.

If both $M_B$ and $M_D$ are anti-holomorphic, we have

\[
M_4 \sim \int d^8 \eta_1 \eta_4^{abcd} d^8 \eta_1 d^8 \eta_4 \delta^{4-4\alpha} [\hat{\eta}^{[1]}_1 \hat{\eta}^{[1]'}_1] \delta^8 (\eta_1' - \eta_1) \delta^8 (\eta_4' + \eta_1) \\
\times \delta^{2-4\alpha} [\hat{\eta}^{[2]}_1 \hat{\eta}^{[2]'}_1] \delta^8 (\eta_1' + \delta^\alpha \eta_2) \delta^8 (\eta_2' - \eta_2) \\
\times \delta^{2-4\alpha} [\hat{\eta}^{[3]}_1 \hat{\eta}^{[3]'}_1] \delta^8 (\eta_1' - \delta^\alpha \eta_4) \delta^8 (\eta_4' + \eta_4) \\
\times \delta^{-2} \log^2 \delta \\
\sim \delta^6 \log^2 \delta \to 0.
\] 

(3.27)

It is clear that our result is in agreement with the result derived in [16] for $\Delta(M_4) = 16$, although it is more general because it is applicable directly to cases with fermions. In addition, our result holds for cases with $\Delta(M_4) = 8, 24$ because we did not assume whether $M_C$ is holomorphic or anti-holomorphic (except the case when both $M_B$ and $M_D$ are anti-holomorphic for which it must be holomorphic), thus our result covers all possible arrangement of four external particles with at least one scalar and it shows that one-loop four-point amplitudes with single soft scalar emission vanish in all cases. Together with Eq. (3.13), the conclusion is, for the infrared-subtracted amplitude,

\[
\lim_{p_1 \to 0} \int d^8 \eta_1 \eta_1^{abcd} M_{n+1}^{1-loop} (1, 2, ..., n + 1) = 0
\] 

(3.28)

for $n \geq 3$.

Naively we should be able to generalize our result to one-loop double soft scalar emission. At the tree level, authors of [13] have obtained a finite result which reveals the non-trivial structure of $E_7(7)$ group. It is expected that the same result at the one-loop level should directly follows from Eq. (3.1) and the vanishing result of the tree level single soft emission. However, the discontinuities between different box functions make the problem non-trivial since we must explicitly take into account discontinuities to check if the same finite result can be obtained at the one-loop level. This work is in progress [22].

IV. CONCLUSION AND DISCUSSIONS

In this note we have studied single soft scalar emission of $\mathcal{N} = 8$ SUGRA. As investigated from different perspectives in [10, 13, 14], at the tree level, the single soft scalar emission
vanishes which indicates a hidden $E_{7(7)}$ symmetry in addition to the $SU(8)$ symmetry. Here we generalize the result to the one-loop level using supersymmetric BCFW construction and a simple formula for one-loop amplitude of $\mathcal{N} = 8$ SUGRA in terms of tree amplitudes, due to the absence of triangle, bubble and rational terms. It turns out that for the one-loop infrared-subtracted amplitude, the single soft scalar emission vanishes, which implies that there may be no anomalies of the $E_{7(7)}$ symmetry at the one loop level.

Our result is in agreement with that of [16] for special cases studied there, i.e. four-scalar and two-scalar-two-vector amplitudes. Although we have not obtained the explicit expression for general amplitudes as for special cases in [16], the vanishing result for infrared finite parts of general amplitudes is obtained for the first time. As argued in [16], this should directly imply the “axial” charge vanishes, which by the low energy theorem implies the conservation of the corresponding Noether current.

Clearly more works are needed to reveal the role of $E_{7(7)}$ and possible enlarged symmetry in $\mathcal{N} = 8$ SUGRA. First, by analyzing the subtraction of infrared divergences properly, it is straightforward to study the double soft scalar emission at the one-loop level to further confirm the non-trivial structure of $E_{7(7)}$ group obtained by double emission at the tree level [13]. Furthermore, it would be very interesting to study the soft emission of arbitrary numbers of scalars at both the tree and loop level, and the results should indicate the exponentiation and the full finite action of $E_{7(7)}$ group on the Hilbert space. Besides, as discussed in [13], the single soft emission of graviphoton can go to a constant, which may indicate further enlarged symmetry of the theory, and further investigations for such emissions at both tree and loop level are desirable. We hope that the result of soft scalar and graviphoton emissions, which reveals $E_{7(7)}$ and possible enlarged symmetry, can shed some light on the possible UV finiteness of $\mathcal{N} = 8$ SUGRA.

Acknowledgement

We are grateful to J. Kaplan for helpful discussions. S.H. thanks N. Arkani-Hamed and F. Cachazo for encouragement on working along this direction. H.Z. is grateful to Chong Sheng Li for his support on this work. S.H.’s work is supported by the National Natural Science Foundation (NFS) of China under grant No. 10721063, No. 10675005 and No. 10835002. H.Z. is supported by National Natural Science Foundation of China, under
Grants No.10721063, No.10575001 and No.10635030.

[1] Z. Bern, L. J. Dixon and R. Roiban, “Is N = 8 SUGRA Ultraviolet Finite?,” Phys. Lett. B 644, 265 (2007) [arXiv:hep-th/0611086];

[2] Z. Bern, L. J. Dixon, M. Perelstein and J. S. Rozowsky, “Multi-leg one-loop gravity amplitudes from gauge theory,” Nucl. Phys. B 546, 423 (1999) [arXiv:hep-th/9811140]; Z. Bern, N. E. J. Bjerrum-Bohr and D. C. Dunbar, “Inherited twistor-space structure of gravity loop amplitudes,” JHEP 0505, 056 (2005) [arXiv:hep-th/0501137]; N. E. J. Bjerrum-Bohr, D. C. Dunbar and H. Ita, “Six-point one-loop N = 8 supergravity NMHV amplitudes and their IR behaviour,” Phys. Lett. B 621, 183 (2005) [arXiv:hep-th/0503102]; N. E. J. Bjerrum-Bohr, D. C. Dunbar, H. Ita, W. B. Perkins and K. Risager, “The no-triangle hypothesis for N = 8 supergravity,” JHEP 0612, 072 (2006) [arXiv:hep-th/0610042]; N. E. J. Bjerrum-Bohr and P. Vanhove, “Explicit Cancellation of Triangles in One-loop Gravity Amplitudes,” JHEP 0804, 065 (2008) [arXiv:0802.0868 [hep-th]]; N. E. J. Bjerrum-Bohr and P. Vanhove, “Absence of Triangles in Maximal Supergravity Amplitudes,” JHEP 0810, 006 (2008) [arXiv:0805.3682 [hep-th]].

[3] M. B. Green, J. G. Russo and P. Vanhove, “Non-renormalisation conditions in type II string theory and maximal supergravity,” JHEP 0702, 099 (2007) [arXiv:hep-th/0610299];

[4] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, D. A. Kosower and R. Roiban, “Three-Loop Superfiniteness of N=8 SUGRA,” Phys. Rev. Lett. 98, 161303 (2007) [arXiv:hep-th/0702112]; Z. Bern, J. J. Carrasco, D. Forde, H. Ita and H. Johansson, “Unexpected Cancellations in Gravity Theories,” Phys. Rev. D 77, 025010 (2008) [arXiv:0707.1035 [hep-th]]; Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, “Manifest Ultraviolet Behavior for the Three-Loop Four-Point Amplitude of N=8 Supergravity,” Phys. Rev. D 78, 105019 (2008) [arXiv:0808.4112 [hep-th]];

[5] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, “The Ultraviolet Behavior of N=8 Supergravity at Four Loops,” Phys. Rev. Lett. 103, 081301 (2009) [arXiv:0905.2326 [hep-th]].

[6] N. Berkovits, “New higher-derivative R**4 theorems,” Phys. Rev. Lett. 98, 211601 (2007) [arXiv:hep-th/0609006]; B. Green, J. G. Russo and P. Vanhove, “Ultraviolet properties of
maximal SUGRA,” Phys. Rev. Lett. 98, 131602 (2007) [arXiv:hep-th/0611273].

[7] E. Cremmer and B. Julia, “The $\mathcal{N} = 8$ SUGRA Theory. 1. The Lagrangian,” Phys. Lett. B 80, 48(1978); E. Cremmer, B. Julia and J. Scherk, “SUGRA theory in 11 dimensions”; Phys. Lett. B 76, 409(1978); “The SO(8) SUGRA,” Nucl. Phys. B 159,141(1979).

[8] B. de Wit and H. Nicolai, “$\mathcal{N} = 8$ SUGRA”, Nucl.Phys.B 208,323(1982).

[9] L. Brink, S. S. Kim and P. Ramond, “$E_7(7)$ on the Light Cone,” JHEP 0806, 034 (2008) [AIP Conf. Proc. 1078, 447 (2009)] [arXiv:0801.2993 [hep-th]]; R. Kallosh and M. Soroush, “Explicit Action of $E_7(7)$ on N=8 SUGRA Fields,” Nucl. Phys. B 801, 25 (2008) [arXiv:0802.4106 [hep-th]].

[10] M. Bianchi, H. Elvang and D. Z. Freedman, “Generating Tree Amplitudes in N=4 SYM and N = 8 SG,” JHEP 0809, 063 (2008) [arXiv:0805.0757 [hep-th]].

[11] M. Bando, T. Kugo and K. Yamawaki, “Nonlinear Realization and Hidden Local Symmetries,” Phys. Rept. 164, 217 (1988).

[12] S. Weinberg, “The quantum theory of fields. Vol. 2: Modern applications,” Cambridge, UK: Univ. Pr. (1996) 489 p

[13] N. Arkani-Hamed, F. Cachazo and J. Kaplan, “What is the Simplest Quantum Field Theory?” arXiv:0808.1446 [hep-th].

[14] R. Kallosh, T. Kugo, “The footprint of $E_{7(7)}$ in amplitudes of $\mathcal{N} = 8$ SUGRA”, arXiv:0811.3414 [hep-th].

[15] N. Marcus, “Composite Anomalies In SUGRA,” Phys. Lett. B 157, 383 (1985); P. di Vecchia, S. Ferrara and L. Girardello, “Anomalies Of Hidden Local Chiral Symmetries In Sigma Models And Extended Supergravities,” Phys. Lett. B 151, 199 (1985).

[16] R. Kallosh, C. Lee and T. Rube, “$\mathcal{N} = 8$ SUGRA 4-point Amplitudes”, arXiv:0811.3417 [hep-th].

[17] R. Britto, F. Cachazo and B. Feng, “New recursion relations for tree amplitudes of gluons,” Nucl. Phys. B 715, 499 (2005) [arXiv:hep-th/0412308]. R. Britto, F. Cachazo, B. Feng and E. Witten, “Direct proof of tree-level recursion relation in Yang-Mills theory,” Phys. Rev. Lett. 94, 181602 (2005) [arXiv:hep-th/0501052].

[18] A. Brandhuber, P. Heslop and G. Travaglini, “A note on dual superconformal symmetry of the N=4 super Yang-Mills S-matrix,” arXiv:0807.4097 [hep-th].

[19] N. Arkani-Hamed and J. Kaplan, “On Tree Amplitudes in Gauge Theory and Gravity,” JHEP
[20] Z. Bern and G. Chalmers, “Factorization in one loop gauge theory,” Nucl. Phys. B 447, 465 (1995) [arXiv:hep-ph/9503236].

[21] S. Catani and M. H. Seymour, in NLO QCD,” Nucl. Phys. B 485, 291 (1997) [Erratum-ibid. B 510, 503 (1998)] [arXiv:hep-ph/9605323].

[22] S. He, H. Zhu, in preparation.