We present an overview of the intimate relationship between string and D-brane dynamics, and the dynamics of gauge and gravitational fields in three spacetime dimensions. The successes, prospects and open problems in describing both perturbative and non-perturbative aspects of string theory in terms of three-dimensional quantum field theory are highlighted.

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1. Introduction

In this article we will describe a very simple example of a holographic correspondence, that between two-dimensional conformal field theories and three-dimensional topological quantum field theories. Generally, this duality is described as an isomorphism between the space of conformal blocks on the conformal field theory side and the space of physical states on the topological field theory side (where by “physical” we mean, for instance, gauge invariant). The goal is to compute correlators of two-dimensional conformal field theory, with open or closed worldsheets, from such “bulk” three-dimensional quantum field theories. The best known example of this holography is the equivalence between a Chern-Simons gauge theory defined on a three-manifold $M_3$ with boundary and a Wess-Zumino-Novikov-Witten (WZNW) model on $\partial M_3$ [1, 2].

On quite general grounds, any conformal field theory can be shown to give rise to a topological quantum field theory by extracting a modular tensor category from the conformal field theory chiral vertex operator algebra [3]. For example, the Moore-Seiberg data of a rational conformal field theory encode the basic braiding, fusing and $S$-matrices, along with the appropriate pentagon and hexagon identities [4]. They give rise to a topological quantum field theory in three-dimensions which can be used to compute invariants of knots and links in three-manifolds. They also give rise to a modular tensor category $\mathcal{C}$, the category of representations of the rational chiral vertex operator algebra, which may be thought of as a basis-independent formulation of the Moore-Seiberg data. $\mathcal{C}$ generalizes the well-known category of finite-dimensional vector spaces. With this one can develop a powerful graphical calculus in terms of ribbon graphs which correspond to framed Wilson lines in three-dimensions [5].

One problem with this correspondence is that the Hilbert space $\mathcal{H}$ of a topological field theory is only isomorphic to the space of holomorphic conformal blocks of the associated conformal field theory. A chiral correlator in the conformal field theory is completely determined by a choice of vector in $\mathcal{H}$. Thus the equivalence just described is not exactly an example of a holographic correspondence, in which the full conformal field theory correlation functions, comprising both holomorphic and antiholomorphic sectors, on the boundary would be reproduced by some three-dimensional theory in the bulk. The goal is then to find some three-dimensional theory that corresponds to the full conformal field theory.

Once this correspondence is achieved, we can try to use it to reformulate perturbative string theory in the simpler language of quantum field theory.
But in order to implement the various nonperturbative ingredients of string theory, we need to somehow add additional structure to this formalism. In this article we shall describe how to do this in the simplest instance of WZNW models, describing string propagation in group manifolds, which can always be constructed by using Chern-Simons theories [2]. We will thereby derive string theory from a theory of higher-dimensional extended objects which we might dub “Worldsheet M-Theory”, because all string theories will originate from a single bulk theory in three dimensions.

### 1.1. Topological Membranes

The higher-dimensional object mentioned above will be referred to as a “topological membrane” [6,7], and it is obtained by filling in the string worldsheet and viewing it as the boundary of a three-manifold. Despite some similarities which we describe below, these membranes are fundamentally different from the membrane degrees of freedom in 11-dimensional M-Theory. There are various ways to regard the induced string theory from three-dimensions. For instance, we may identify the string worldsheet $\Sigma$ with the boundary of the three-manifold $M_3$ in question, $\Sigma = \partial M_3$, but as pointed out above the induced field theory carries only chiral degrees of freedom. Instead, we shall proceed with the observation [1] that locally the three-manifold can always be subjected to a Heegaard splitting $M_3 = M \#_\Sigma M'$, as depicted in Figure 1 and the holomorphic and antiholomorphic sectors of the two-dimensional theory can be identified with the induced degrees of freedom on the two boundaries $\partial M$ and $\partial M'$. In a neighbourhood of this slice, $M_3$ may be identified with the three-manifold $\Sigma \times [0,1]$ depicted in Figure 2. The two boundary components $\Sigma_0 = \Sigma \times \{0\}$ and $\Sigma_1 = \Sigma \times \{1\}$ then respectively contain the holomorphic and antiholomorphic degrees of freedom of the induced two-dimensional conformal field theory [8]. This is the picture that we shall always have in mind throughout all our string constructions.

A crucial point will be that in order to see this program through it will be necessary to destroy the topological invariance of the original bulk quantum field theory. The dynamical ingredients which will go into the construction of the topological membrane can be summarized as follows. In addition to the Chern-Simons term, we will add the Yang-Mills action and consider a topologically massive gauge theory in the bulk [8]–[10]. The presence of propagating gluon degrees of freedom will enable us to control boundary conditions simultaneously on the left-moving and right-moving worldsheets $\Sigma_0$ and $\Sigma_1$, and to thereby induce the full non-chiral conformal field theory on $\Sigma$ [11]. We will then couple this gauge theory to topologically massive gravity [10],
Figure 1. A Heegaard splitting of a three-manifold $M_3$. The three-manifold is cut along a Riemann surface $\Sigma$ into two three-manifolds $M$ and $M'$, whose boundaries are identified by an orientation reversing homeomorphism as $\Sigma = \partial M$, $-\Sigma = \partial M'$. The closed curve represents a Wilson loop in the bulk.

Figure 2. The three-manifold $M_3 = \Sigma \times [0, 1]$. The Wilson lines propagate between the two boundaries $\Sigma_0$ and $\Sigma_1$, which are each copies of $\Sigma$ but carry opposite orientations.

Consisting of the Einstein-Hilbert and gravitational Chern-Simons actions in three dimensions, which will have the effect of inducing two-dimensional quantum gravity on the boundary $\Sigma$ [12,13]. A further conformal coupling to a three-dimensional scalar field theory will then produce the dilaton field [14], and hence the string coupling $g_s$. Finally, minimal couplings of the gauge theory to charged matter fields in the bulk will yield deformations of the induced conformal field theory [15] enabling us to construct vertex operators and, ultimately, states corresponding to D-branes [16].

Let us conclude this introduction with some of the primary motivations for rewriting string theory this way in terms of topological membranes:

1. Many aspects of string dynamics have natural interpretations in terms of the dynamics of gauge and gravitational fields in the bulk.
2. Various algebraic properties of two-dimensional conformal field theories can be understood geometrically and dynamically in the three-dimensional picture. In this sense, important dynamical effects are responsible for fundamental properties of the induced conformal field theory. This yields new dynamical perspectives on string construc-
tions, which are simpler in the language of three-dimensional quantum field theories.

(3) As a “Worldsheet M-Theory”, it has the highly desirable feature that the basic principles inherent to three-dimensional gauge theory are far fewer than those to all of the existing string theories.

(4) There are various pieces of evidence that the topological membrane framework gives a potential dynamical origin for the eleventh extra dimension of M-Theory from fundamental string fields. First of all, the conformal scalar coupling mentioned above can be adjusted to induce a dynamical string coupling $g_s$ [14], analogously to the way that the string coupling emerges as a dynamical variable in M-Theory [17]. Secondly, the topological membrane construction is precisely a lower-dimensional version of the Hořava-Witten mechanism [18,19], in which the cancellation of gauge and gravitational anomalies on the boundaries of an open cylindrical membrane stretched between the two boundaries of an 11-dimensional spacetime $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$ requires there to be one $E_8$ gauge group on each spacetime boundary, and it thereby induces the $E_8 \times E_8$ heterotic string theory. However, this latter approach relies crucially on spacetime structures such as 11-dimensional supergravity, whereas in the present case the emphasis is on the worldsheet properties of the open topological membrane. Finally, the supermembrane in the spectrum of 11-dimensional supergravity and the topological membrane, although not identical, do share some amusing similarities. The supermembrane couples to a three-form supergravity potential whose action is the 11-dimensional Chern-Simons term [20], while its worldvolume description when it wraps degenerate three-cycles of a Calabi-Yau manifold is a three-dimensional Chern-Simons theory [21]. All of these similarities hint that the extra dimension of topological membrane theory could potentially be embedded in 11-dimensional M-Theory.

2. Chern-Simons Gauge Theory

We will begin with an investigation of pure Chern-Simons gauge theory and hence describe the basic holographic duality with chiral conformal field theories.

2.1. The Chern-Simons Action

Let $M_3$ be a compact oriented three-manifold, possibly with boundary, and let $G$ be a compact connected Lie group equipped with an invariant bilinear
The Chern-Simons action is defined by the functional \[ S_{\text{CS}}[A] = \frac{k}{4\pi} \int_{M_3} \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) + \frac{k}{4\pi} \int_{\partial M_3} \text{Tr} \left( A^{1,0} \wedge A^{0,1} \right) \] (2.1)

where \( A \) is a connection one-form on a principal \( G \)-bundle \( P \to M_3 \). The first term defines a three-dimensional topological quantum field theory, containing no gluons or explicit dependence on any metric of \( M_3 \). The second term ensures that the action has classical extrema when \( M_3 \) has a non-empty boundary and it depends on the choice of a complex structure on \( \partial M_3 \). \( A^{1,0} \) (resp. \( A^{0,1} \)) denotes the holomorphic (resp. antiholomorphic) component of the gauge connection on \( \partial M_3 \) with respect to this auxiliary complex structure.

When the gauge group \( G \) is simple, it has non-trivial homotopy group \( \pi_3(G) = \mathbb{Z} \). Then the action (2.1) is not invariant under homotopically non-trivial (large) gauge transformations but instead changes by \( 2\pi k w \), where \( w \in \mathbb{Z} \) is the winding number of the map \( M_3 \to G \) [10]. Consistency of the quantum field theory, in a path integral formalism to be described below, then requires the quantization condition \( k \in \mathbb{Z} \) on the Chern-Simons coefficient in (2.1).

### 2.2. Path Integral Formulation

We will study the Chern-Simons path integral with prescribed boundary conditions for the gauge field \( A \) on the Riemann surface \( \Sigma = \partial M_3 \), and hence define the Hartle-Hawking type state

\[ \mathcal{F}[A] = \int_{A^{1,0}=A} DA \ e^{iS_{\text{CS}}[A]} . \] (2.2)

This state defines a vector in the Chern-Simons Hilbert space \( \mathcal{H}^{\text{CS}}_{\Sigma} \) which depends only on the topology and framing of the three-manifold \( M_3 \) [1]. It determines a modular functor \( \mathcal{F} : \mathcal{H}^{\text{CS}}_{\Sigma} \to \mathbb{C} \) defined by

\[ \mathcal{F}(\Psi) = \langle \mathcal{F} | \Psi \rangle = \frac{1}{\text{vol} \mathcal{G}_{\Sigma}} \int DA \ \overline{DA} \ \mathcal{F}[A] \ \Psi[A] , \] (2.3)

where \( \mathcal{G}_{\Sigma} = C^\infty(\Sigma, G) \) is the group of gauge transformations on the boundary \( \Sigma \). The formula (2.3) utilizes the natural inner product on the Hilbert space \( \mathcal{H}^{\text{CS}}_{\Sigma} \) usually defined in holomorphic (coherent state) quantization [24, 25].
Given a connection one-form $A$, let us now parametrize its gauge orbit as

$$A = g^{-1} \bar{A} g + g^{-1} \, d g ,$$

(2.4)

where $g : \Sigma \to G$ is a gauge transformation on the boundary and $\bar{A}$ is a fixed representative of the gauge equivalence class on $M_3$ of the gauge field $A$. With standard gauge-fixing techniques, one then finds that the path integral (2.2) factorizes as $[2, 26]

$$\mathcal{F}[A] = \int_{A^{1.0}=A} D \bar{A} \, \delta (F_{\bar{A}}) \, \Delta_{FP} (\bar{A}) \, e^{i S_{\text{CS}}[\bar{A}]} \int Dg \, e^{i k S^+_W[g,A]} ,$$

(2.5)

where $F_{\bar{A}} = d \bar{A} + \bar{A} \wedge A$ is the curvature of $\bar{A}$, $\Delta_{FP}$ denotes the usual Faddeev-Popov determinant, and the delta-function in (2.5) restricts the functional integration to classical gauge field configurations (with the specified boundary conditions on $\Sigma$) which extremize the action (2.1). The functional

$$S^+_W[g,A] = \frac{1}{4\pi} \int_{\Sigma} \text{Tr} \left( |g^{-1} \, \partial g|^2 - 2 g^{-1} \, \partial g \wedge A \right)$$

$$+ \frac{1}{12\pi} \int_{M_3} \text{Tr} \left( (g^{-1} \, d g)^3 \right)$$

(2.6)

is the action of the chiral gauged WZNW model for the group field $g$ coupled to an external gauge field $A$ on the Riemann surface $\Sigma = \partial M_3$ $[2, 8, 27]$.

The induced boundary conformal field theory is chiral precisely because of the background field coupling, which leaves only one conserved current $g^{-1} \, \partial g$. After setting $A = 0$ by an appropriate choice of boundary conditions, the Chern-Simons path integral exhibits a complete decoupling into bulk and surface degrees of freedom. In other words, on a three-manifold with boundary the gauge symmetry of the field theory defined by the action (2.1) becomes anomalous, and we absorb this anomaly by identifying new degrees of freedom on the boundary.

### 2.3. Phase Space

Near the boundary $\partial M_3$, the three-manifold may be regarded topologically as the product $M_3 = \Sigma \times \mathbb{R}$ and we may analyse the Chern-Simons gauge theory using a standard Hamiltonian formalism $[1]$. The canonical variables are the two components of the gauge field $A$ on $\Sigma$, and the Gauss law of (2.1) shows that the phase space $\mathcal{P}_{\text{CS}}$ of Chern-Simons theory is the moduli space of flat $G$-connections, $F_A = 0$, on $\Sigma$ modulo gauge transformations. If $\Sigma$ is a compact oriented Riemann surface of genus $g \geq 0$ with $n > 2 - 2g$
punctures, then the phase space has the explicit presentation
\[
P_{\Sigma}^{\text{CS}} = \text{Hom}(\pi_1(\Sigma), G) / G \tag{2.7}
\]
reflecting the fact that flat connections are determined entirely by their holonomies (Wilson lines) around loops on \(\Sigma\). This is a symplectic manifold of finite dimension
\[
\dim P_{\Sigma}^{\text{CS}} = (2g + n - 2) \dim G , \tag{2.8}
\]
with symplectic leaves obtained by restricting the holonomies of \(A\) around the punctures to lie in conjugacy classes of the gauge group \(G\). As the volume of the leaves is finite, quantization of (2.7) will produce a finite dimensional Hilbert space \(H_{\Sigma}^{\text{CS}}\). As we will discuss in more detail later on, with appropriate parabolic conditions at the punctures in the path integral formalism, \(H_{\Sigma}^{\text{CS}}\) coincides with the finite dimensional vector space of conformal blocks of a chiral WZNW model on a genus \(g\) surface \(\Sigma\) with \(n\) primary vertex operator insertions [1]. The dimension of this space is computed by the Verlinde formula [28].

2.4. Conformal Field Theory Constructions

Let us now summarize the list of (chiral) conformal field theories that can be constructed in this way from pure Chern-Simons gauge theory:

(1) One can build a multitude of rational conformal field theories in the Chern-Simons formalism using GKO coset constructions [29]. Let \(H \subset G\) be a subgroup such that the Virasoro algebra over \(G\) can be decomposed into the orthogonal direct sum of the Virasoro algebra over \(H\) and the Virasoro algebra over the coset \(G/H\). Then one can construct the coset current algebra based on \(G/H\) with action \(k S_{W}^{+}[g] - l S_{W}^{+}[h]\) for group fields \(g : \Sigma \to G\) and \(h : \Sigma \to H\). This theory is holographically dual to three-dimensional Chern-Simons gauge theory with action \(k S_{\text{CS}}[A] - l S_{\text{CS}}[B]\), where \(A\) is a \(G\)-connection and \(B\) an \(H\)-connection on \(M_3\) [2]. For example, the standard minimal models can be obtained through the coset constructions
\[
\mathcal{M}_k = SU(2)_k \times SU(2)_{1} / SU(2)_{k+1} \tag{2.9}
\]
where the quotient is by the diagonal \(SU(2)\) action. In this way it is possible to characterize the zoo of rational conformal field theories on the basis of a single gauge theory in three-dimensions.
(2) Supersymmetric extensions of all of these constructions are also possible and straightforward to carry out, giving the standard $N = 1$ [30] and $N = 2$ [31, 32] superconformal field theories in two dimensions. They will not be discussed in this article.

(3) The holographic correspondence can also be extended to more general conformal field theories using model-independent topological field theory formulations [3,5]. These more general scenarios are axiomatic and no action formalism is possible for them in the manner described above. They also will not be dealt with in this article.

3. Topologically Massive Gauge Theory

We will now modify the Chern-Simons action (2.1) by adding to it a Yang-Mills term. If it is not present initially, then it will be induced in any case by quantum radiative corrections when the gauge theory is minimally coupled to charged matter, as we will do later on. As we explain, the inclusion of this term enables us to significantly expand the list of conformal field theory constructions of the previous section and to begin working our way towards building up complete string theories.

3.1. Classical Aspects

Let us fix a Lorentzian metric on $M_3$ of signature $(2,1)$ and denote the corresponding Hodge duality operator by $*: \Omega^p(M_3, \text{ad} P) \rightarrow \Omega^{3-p}(M_3, \text{ad} P)$. The action of topologically massive gauge theory is defined by the functional [9, 10]

$$S_{\text{TMGT}}[A] = \int_{M_3} \text{Tr} \left[ -\frac{1}{e^2} F_A \wedge * F_A + \frac{k}{4\pi} A \wedge (dA + \frac{2}{3} A \wedge A) \right] + \int_{\partial M_3} \text{Tr} \left( A^{1,0} \wedge \Pi^{0,1} \right), \quad (3.1)$$

where the Yang-Mills coupling constant $e^2$ has dimensions of mass and

$$\Pi = \frac{k}{4\pi} A - \frac{2}{e^2} * F_A \quad (3.2)$$

is the canonical momentum conjugate to the gauge field $A$. The second term in (3.1) imposes conformal Dirichlet boundary conditions fixing the connection components $A^{1,0}$ on $\Sigma = \partial M_3$. The Yang-Mills term in (3.1) is gauge invariant even when $\partial M_3 \neq \emptyset$, and so the same chiral WZNW model as in the last section is induced on the boundary within the path integral framework for the action (3.1). The
crucial consequence of its addition is that the bulk field theory is no longer topological, because it depends explicitly on the metric of $M_3$ through the Hodge operator. In contrast to the pure Chern-Simons gauge theory, there are now propagating gluon degrees of freedom and, as we show in the next subsection, the Hilbert space of physical states becomes infinite-dimensional. To see this, we let $D_A = d + A$ denote the usual gauge covariant derivative, and note that one can write the Euler-Lagrange equations arising from the action (3.1) in the form

$$ (D_A^2 - \mu^2) * F_A = * (F_A \wedge * F_A) . \quad (3.3) $$

This is the equation of motion for a single propagating degree of freedom with topological mass

$$ \mu = \frac{e^2 |k|}{4\pi} . \quad (3.4) $$

Because of the presence of a massive gluon, the Yang-Mills term serves as an infrared regularization of the pure Chern-Simons theory, which is recovered in the infrared limit $e^2 \to \infty$ wherein the energies of all states containing the gluon decouple from the rest of the spectrum. We may thereby regard pure Chern-Simons theory as the ground state of topologically massive gauge theory. A particularly important consequence of the presence of this new degree of freedom is the enlargement of the phase space of the gauge theory. In the pure Chern-Simons case of the previous section, there are only two canonical variables $A^{1,0}$ and $A^{0,1}$ which are conjugate to each other, and so it is not possible to fix both components of the gauge field at the same time on $\partial M_3$, as this would violate the canonical Poisson brackets. In contrast, the action (3.1) involves four independent canonical variables $\Pi^{1,0}$, $\Pi^{0,1}$, $A^{1,0}$ and $A^{0,1}$. Now $A^{1,0}$ and $A^{0,1}$ Poisson commute, and it is consistent to fix both gauge field components on the boundary.

Furthermore, one can now vary the choice of worldsheet complex structure in the induced conformal field theory on $\Sigma$ via its coupling to the metric of $M_3$ [7]. This means that we can now generate both holomorphic and antiholomorphic degrees of freedom on the connected components of $\partial M_3 = \Sigma_0 \sqcup \Sigma_1$ holding different induced conformal field theories, one in each boundary component as illustrated in Figure 2. Alternatively, we may choose to kill all degrees of freedom in one boundary component by fixing both $A^{1,0}$ and $A^{0,1}$ there, leading us into various string constructions [11]. For example, the heterotic string construction starts with a topological membrane that has a semi-simple gauge group $G_L \times G_R$, where $G_L$ is the gauge group for the ordinary bosonic topologically massive gauge theory, and $G_R$
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is the gauge group for the \( N = 1 \) supersymmetric extension of topologically massive gauge theory \([7, 30]\). The heterotic worldsheet is constructed by placing conformal boundary conditions on \( \Sigma_0 \) and killing all degrees of freedom on \( \Sigma_1 \) for the theory based on \( G_L \), thereby inducing bosonic left-moving worldsheet degrees of freedom, while killing all modes on \( \Sigma_0 \) and selecting conformal boundary conditions on \( \Sigma_1 \) for the \( G_R \) theory, leading to supersymmetric right-moving degrees of freedom on the string worldsheet. Thus the seemingly hybrid nature of the heterotic string construction becomes more natural in the topological membrane framework, wherein the different sectors have a common geometric origin in the choice of membrane boundary conditions.

3.2. Hilbert Space

Let us now analyse in detail the structure of the Hilbert space of physical states of topologically massive gauge theory, focusing for simplicity on the abelian case with gauge group \( G = U(1) \), corresponding to the \( c = 1 \) conformal field theory of a single boson \([24]\). Consider a canonical split of the gauge connection \( A = A + A_0 \, dt \) on a spacetime of the form \( M_3 = \Sigma \times \mathbb{R} \), where \( \Sigma \) is a compact oriented surface of genus \( g \). The metric on \( M_3 \) is of the form \( ds_3^2 = -dt^2 \) while its orientation is given by the three-form \( dvol_{\Sigma} \wedge dt \).

Varying the action (3.1) with respect to \( A_0 \) yields the Gauss law, which in the \( A_0 = 0 \) gauge reads

\[
\frac{1}{e^2} \, d \ast_2 \dot{A} - \frac{k}{4\pi} F_A = 0 \tag{3.5}
\]

where \( \ast_2 \) is the Hodge operator on \( \Sigma \) and a dot denotes differentiation with respect to the time coordinate \( t \in \mathbb{R} \). At each fixed time slice \( t \in \mathbb{R} \), we write the one-form \( A \in \Omega^1(\Sigma) \) using its Hodge decomposition as

\[
A = d\xi + \ast_2 d\chi + \alpha, \tag{3.6}
\]

where \( \xi \) and \( \chi \) are scalar fields on \( \Sigma \) while \( \alpha = (a_1^{(\alpha)}, a_2^{(\alpha)})_{\alpha=1,...,g} \in H^1(\Sigma, \mathbb{R}) = \mathbb{R}^{2g} \) are the harmonic degrees of freedom of the gauge field.

Substituting (3.6) into the abelian version of the action (3.1) and applying the Gauss law constraint (3.5) shows that the resulting action after diagonalization decouples into \( g + 1 \) independent pieces as

\[
S_{\text{TMGT}}[A] = S_I[\varphi] + \sum_{\alpha=1}^g S_L \left[ a_1^{(\alpha)}, a_2^{(\alpha)} \right]. \tag{3.7}
\]
The first term is the free particle action for the non-local scalar field $\varphi = \sqrt{\nabla^2/\epsilon^2} \chi$ of mass $\mu$ given by [10]

$$S_f[\varphi] = \frac{1}{2} \int_{M_3} \left( d\varphi \wedge * d\varphi - \mu^2 * \varphi^2 \right),$$

(3.8)

describing the dynamics of the single propagating gluon mode. The second set of $g$ terms are topological and consist of identical copies of the quantum mechanical action [34]–[38]

$$S_L[a_1, a_2] = \int dt \left( \frac{1}{2e^2} \dot{a}_i^2 - \frac{k}{4\pi} \epsilon^{ij} a_i \dot{a}_j \right)$$

(3.9)

where $\epsilon^{12} = +1$. This is the Landau action describing propagation of a charged particle of mass $m = 1/\epsilon^2$ on the plane $(a_1, a_2)$ in a uniform magnetic field $B = k/4\pi$.

After diagonalization, the quantum Hilbert space of topologically massive gauge theory is therefore given by

$$H_{TMGT}^\Sigma = H_f[\varphi] \otimes (H_L)^\otimes g,$$

(3.10)

where the first factor is the infinite dimensional Hilbert space of the free massive scalar field $\varphi$, while the second factor is the Hilbert space for $g$ copies of the Landau problem on the plane. $H_L$ is thus composed of infinitely many Landau levels, with the mass gap between consecutive levels being equal to $\Delta = |B|/m = \mu$, the mass of the gauge boson. This calculation ignores large gauge transformations, which are parametrized by elements of the lattice $H^1(\Sigma, \mathbb{Z})$ of rank $2g$. Demanding invariance of the quantum gauge theory under them restricts the harmonic modes to the torus $a \in H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z})$ and yields the Hilbert space for the Landau problem on $\Sigma$. The Hilbert space (3.10) then contains a dependence on the moduli of the Riemann surface $\Sigma$ arising from the non-topological nature of the gauge theory, but the mass gap $\Delta$ is always independent of these moduli [39]. From this result we may also extract the pure Chern-Simons Hilbert space $H_{CS}^\Sigma$ as the projection of (3.10) onto the lowest Landau level in the infrared limit $\mu \to \infty$ (equivalently the strong coupling limit $\epsilon^2 \to \infty$).

4. Hamiltonian Quantization

In the previous sections we have used path integral quantization to describe howboundary degrees of freedom are induced in order to cancel the gauge anomaly of topologically massive gauge theory on a three-manifold with
boundary. Drawing from the analysis of Section 3.2, we shall now investigate in detail how these modes alternatively appear within the formalism of canonical quantization. This will provide the basic building blocks for the construction of quantum states of the topological membrane, which will in turn be used to construct string amplitudes and to describe non-perturbative string excitations of the membrane.

4.1. Hamiltonian Formalism for Topologically Massive Gauge Theory

Let us consider a canonical split $A = A + A_0$ $dt = A_i$ $d x^i + A_0$ $dt$ of the generic (non-abelian) topologically massive gauge theory (3.1) on the three-manifold $M_3 = \Sigma \times \mathbb{R}$. We define the electric and magnetic fields by

\begin{equation}
E = \Pi - \frac{k}{8\pi} *_2 A ,
\end{equation}

\begin{equation}
B = *_2 F_A .
\end{equation}

The electric field (4.1) is analogous to the velocity operator in the Landau problem. The Hamiltonian of topologically massive gauge theory can then be written as

\begin{equation}
H_{TMGT} = \int_\Sigma \text{Tr} \left( e^2 E \wedge *_2 E + \frac{1}{e^2} B \wedge *_2 B \right) ,
\end{equation}

while the Gauss law reads

\begin{equation}
*_2 G = d *_2 E - i A \wedge E + \frac{k}{4\pi} *_2 B \sim 0 .
\end{equation}

The scalar field $G$ on $\Sigma$ is the generator of time-independent gauge transformations, and one easily checks that it commutes with the gauge-invariant Hamiltonian (4.3). The condition $G \sim 0$ is a weak equality which will be imposed as a physical state condition in the quantum field theory, rather than as a relation among quantum operators.

To proceed with canonical quantization, we decompose form components as $A_i = A^a_i T^a$ and so on, where $T^a$, $a = 1, \ldots, \text{dim} G$ are the generators of the gauge group $G$ which we take to be normalized as $\text{Tr} (T^a T^b) = \frac{1}{2} \delta^{ab}$. The equal time canonical quantum commutators are then given by

\begin{equation}
\left[ A^a_i (x) , \Pi^b_j (y) \right] = i \delta_{ij} \delta^{ab} \delta^{(2)} (x - y) \end{equation}

with all other commutators vanishing. These imply the commutation rela-
tions of the electric and magnetic field operators given by
\[
\begin{align*}
\left[ E_a^i(x), E_b^j(y) \right] &= -\frac{i}{4\pi} \epsilon_{ij} \delta^{ab} \delta^{(2)}(x - y), \\
\left[ E_a^i(x), B_b^j(y) \right] &= -i \delta^{ab} \epsilon_{ij} \partial^j \delta^{(2)}(x - y), \\
\left[ B_a^i(x), B_b^j(y) \right] &= 0.
\end{align*}
\] (4.6) (4.7) (4.8)

4.2. Functional Schrödinger Picture

We shall work in a functional Schrödinger polarization wherein the physical states are the wavefunctionals \( \Psi[A] \) [36]. The quantum commutators (4.5) are then satisfied by representing the canonical momenta as the functional derivative operators
\[
\Pi_a^i = -i \frac{\delta}{\delta A_a^i}.
\] (4.9)

Stationary states obey the functional Schrödinger equation \( H_{\text{TMGT}} \Psi[A] = \mathcal{E} \Psi[A] \) giving
\[
\int_{\Sigma} \text{dvol}_{\Sigma} \left[ \frac{e^2}{2} \sum_{i=1,2} \sum_{a=1}^{\text{dim}G} \left( -\frac{i}{\delta A_a^i} - \frac{k}{8\pi} \epsilon^{ij} A_j^a \right)^2 + \frac{1}{2e^2} \sum_{a=1}^{\text{dim}G} (B_a^a)^2 \right] \Psi[A] = \mathcal{E} \Psi[A],
\] (4.10)

where \( \mathcal{E} \) is the energy of the state. Physical states, respecting the gauge symmetry, must further be annihilated by the generator (4.4) of infinitesimal gauge transformations, \( G \Psi[A] = 0 \), which gives the constraint equations
\[
\left[ \sum_{i=1,2} \sum_{a=1}^{\text{dim}G} (D A_i^a)^b \left( -i \delta A_b^i - \frac{k}{4\pi} \epsilon^{ij} A_j^a \right) + \frac{k}{4\pi} B_a^a \right] \Psi[A] = 0 \] (4.11)

for each \( a = 1, \ldots, \text{dim}G \).

The Gauss law constraint has an immediate consequence for the structure of the physical state wavefunctionals [40]. Let us integrate (4.11) over the Riemann surface \( \Sigma \) and substitute in the gauge orbit parametrization (2.4). Because of the magnetic field term, we then find that the gauge symmetry is represented projectively as
\[
\Psi[A] = e^{i\alpha[A,g]} \Psi[\bar{A}],
\] (4.12)
where the projective phase is a cocycle

\[
\alpha [\mathcal{A}, g] = -\frac{i k}{4\pi} \int_\Sigma \text{Tr} (\mathcal{A} \wedge dg g^{-1}) - \frac{k}{12\pi} \int_{M^3} \text{Tr} (g^{-1} dg)^3
\]

in the group cohomology of the gauge group \( G \). Large gauge transformations again change the second term in (4.13) by the winding number of the map \( \Sigma \to G \), and so the projective phase factor in (4.12) is well-defined only if \( k \in \mathbb{Z} \). This exhibits the quantization of the Chern-Simons coefficient solely within the Hamiltonian formalism, without recourse to any path integral description of the quantum gauge theory. The projective cocycle is related to the chiral anomaly in two dimensions.

Let us now solve for the physical states in the infrared limit \( e^2 \to \infty \) of the topologically massive gauge theory, or more precisely in the energy regime in which all momenta are much smaller than the topological gluon mass \( (3.4) \) \([41]\). The normal ordered Hamiltonian operator (4.3) is given in this regime by

\[
H_{\text{TMGT}} = e^2 \int_\Sigma \text{Tr} (E^{0,1} \wedge E^{1,0})
\]

with respect to a chosen complex structure on \( \Sigma \). From (4.6) it follows that the electric field creation and annihilation operators satisfy the commutation relations

\[
\left[ E^a_z(z), E^b_w(w) \right] = \frac{k}{2\pi} \delta^{ab} \delta^{(2)}(z - w) .
\]

The problem of finding the physical states in the infrared limit of topologically massive gauge theory is thereby formally a field theoretic version of the Landau problem, exactly as we anticipated in the previous section.

Since the Hamiltonian (4.14) is non-negative, the vacuum state has zero energy and is destroyed by all of the annihilation operators,

\[
E^a_z \Psi_vac[A] = \left( -2i \frac{\delta}{\delta A^a_z} - i A^a_z \right) \Psi_vac[A] = 0 .
\]

The Gauss law (4.11) can be written in this complex polarization as

\[
\sum_{b=1}^{\dim G} \left( (D_A)^a_{\overline{\tau}} \frac{\delta}{\delta A^a_{\overline{\tau}}} + \frac{k}{4\pi} (D_A)^a_{\overline{\tau}} A^b_z - \frac{k}{2\pi} B^a \right) \Psi_vac[A] = 0 .
\]

This equation is related to the anomaly equation for \(|k|\) flavours of chiral fermions in two-dimensional Euclidean space \([41]\).
The equations (4.16) and (4.17) are simultaneously solved by a Euclidean path integral over two-dimensional group fields as [16, 42, 44]

\[
\Psi_{\text{vac}}^A = \int Dg \ e^{-|k| S^g_{\text{WZW}}[g,A]} \ e^{-\frac{|k|}{4\pi} \int_S \text{Tr} (A^{1,0} \wedge A^{0,1}).}
\] (4.18)

In the first factor we recognize the action (2.6) of the chiral gauged WZNW model on \(\Sigma\), with \(A\) identified as either \(A^{1,0}\) or \(A^{0,1}\) depending on the sign \(k > 0\) or \(k < 0\) of the Chern-Simons coefficient. The second factor automatically produces the normalization factor usually required in holomorphic quantization. It corresponds precisely to the second term in (2.1) that was inserted by hand to ensure that the quantum gauge theory has a well-defined classical limit. In this way we have exhibited the appearance of induced chiral WZNW degrees of freedom on the boundary of the membrane directly in the Hamiltonian formalism of topologically massive gauge theory. The state \(\Psi_{\text{vac}}^A\) obtained in this way is completely analogous to the Hartle-Hawking state \(F^A\) that was constructed in Section 2.2.

### 4.3. Geometrical Interpretation

The construction presented in the previous subsection is closely related to the geometric quantization of Chern-Simons gauge theory [43], which gives an alternative characterization of the equivalence between the strong coupling limit of topologically massive gauge theory and the WZNW model. Let \(\mathcal{A}_\Sigma\) denote the space of \(G\)-connections on the compact oriented Riemann surface \(\Sigma\) which we assume is equipped with a fixed complex structure. Then the commutation relations (4.15) mean that the electric field operators \(E^{1,0} + E^{0,1}\) define a connection of a unitary complex line bundle \(\mathcal{L} \otimes k\) over \(\mathcal{A}_\Sigma\) of constant curvature \(k \in \mathbb{Z}\), with \(\mathcal{L}\) the basic prequantum line bundle of geometric quantization. The Gauss law constraint, in the form (4.12), thereby implies that the vacuum wavefunctional \(\Psi_{\text{vac}}^A\) is a gauge-invariant section of \(\mathcal{L} \otimes k\), while the ground state condition (4.16) implies that it is holomorphic with respect to the canonical connection on \(\mathcal{L} \otimes k\).

The vector space \(\mathcal{H}^\text{CS}_\Sigma\) of holomorphic gauge invariant sections of \(\mathcal{L} \otimes k\) may be presented as the cohomology group

\[
\mathcal{H}^\text{CS}_\Sigma = \Pi^0(\mathcal{P}^\text{CS}_\Sigma, \mathcal{L} \otimes k)
\] (4.19)

where \(\mathcal{P}^\text{CS}_\Sigma\), as in Section 2.3, is the moduli space of flat \(G\)-connections on the Riemann surface \(\Sigma\). The line bundles \(\mathcal{L} \otimes k\) are closely related to the Friedan-Shenker bundles of conformal field theory, and in this way one may establish a natural isomorphism between the vector space (4.19) and the correspond-
ing finite-dimensional vector space of WZNW conformal blocks [1, 42]. The choice of vacuum state (4.18) in the space (4.19) thereby characterizes a chiral WZNW correlation function on Σ.

Given this correspondence, we may now proceed to compute the amplitude for propagation on the three-geometry $M^3 = \Sigma \times [0, 1]$ depicted in Figure 2 [44]. We insert an initial state described by a vacuum wavefunctional $\Psi_0^{\text{vac}}[A]$ of the form (4.18) on the lower surface $\Sigma_0$ at time $t = 0$. We then allow it to evolve in time through the bulk of the membrane, according to the dynamics of topologically massive gauge theory, until it reaches a final state at time $t = 1$ described by a vacuum wavefunctional $\Psi_1^{\text{vac}}[A]$ on the upper surface $\Sigma_1$ of opposite chirality to (4.18). The partition function of topologically massive gauge theory is thereby determined after diagonalization as the inner product

$$Z_{\Sigma}^{\text{TMGT}} = \langle \Psi_0^{\text{vac}} \mid \Psi_1^{\text{vac}} \rangle$$

$$= \frac{1}{\text{vol} G} \int \mathcal{D}A \mathcal{D}A' \ e^{i S_{\text{TMGT}}[A]} \bar{\Psi}_0^{\text{vac}}[A] \Psi_1^{\text{vac}}[A]$$

$$= \sum_{\lambda, \lambda' \in (\Lambda_{G_k}^*/\Lambda_{G_k})^g} \zeta_{\lambda \lambda'}^{\lambda'\lambda} \bar{\psi}_\lambda(m) \bar{\psi}_{\lambda'}(\overline{m}).$$

(4.20)

Here $\Lambda_{G_k}$ denotes the root lattice of the affine Lie algebra at level $k$ based on the gauge group $G$, so that the sums in (4.20) run over $g$-tuples of particular irreducible representations of $G$ (precisely, the integrable highest weight representations of the current algebra at level $k$). The topological wavefunctions $\bar{\psi}_\lambda(m)$ determine holomorphic conformal blocks and they span the Hilbert space $\mathcal{H}_{\Sigma}^{\text{CS}}$ [24, 25]. They depend on the complex moduli $m \in \mathbb{C}^{3g-3}$ of the Riemann surface $\Sigma$ and can be computed from the lowest Landau level wavefunctions for the Landau problem on $\Sigma$ [39], as described in Section 3.2. They are succinctly expressed in this way in terms of holomorphic genus $g$ Jacobi theta-functions at level $k$ [44].

The coupling coefficients $\zeta_{\lambda \lambda'}^{\lambda'\lambda} \in \mathbb{N}_0$ ensure that the bilinear form (4.20) is a modular invariant of the string worldsheet $\Sigma$ [45]. In most cases the partition function will be given by a diagonal sum, $\zeta_{\lambda \lambda'}^{\lambda'\lambda} = \delta_{\lambda \lambda'}$, but there can be examples of three-manifolds $M_3$ for which the inner product assumes an entangled form. The complete three-dimensional version of the standard ADE classification of rational conformal field theories [46] is not known, and it presumably involves the gravitational sector of the topological membrane, to be described in the next section. As is evident from the second line of (4.20), the modular invariant statistical sum for the topological membrane contains boundary degrees of freedom, appearing as in (4.18), which ensure
bulk gauge invariance of the inner product and also that classical extrema contribute to the path integral governing the quantum theory of the membrane [44].

However, in spite of what is written here, there is not quite a complete holomorphic factorization into left-moving and right-moving worldsheet degrees of freedom. There are subtleties in arriving at such a factorization for generic gauged WZNW models which are avoided by constructing vacuum wavefunctionals involving two gauge connections on $\Sigma$ [42]. We will not enter any further into this discussion here.

5. Topologically Massive Gravity

Because the Yang-Mills term in the topologically massive gauge theory action $\mathcal{L}$ couples to the spacetime metric of $M_3$, radiative corrections will generate three-dimensional gravitational terms. The proper formulation of the membrane quantum theory must thereby include a sum over all metrics weighted by the appropriate gravity actions. In this section we will briefly describe the gravitational sector of the topological membrane theory and the ensuing emergence of the string dilaton field. For the most part, in subsequent sections we will ignore gravitational contributions, but here we include a quick description for completeness.

5.1. Conformal Coupling and the Dilaton

To incorporate gravitational terms into the action $\mathcal{L}$, we will work in the first order formalism of general relativity. The fibers of the frame bundle over the three-manifold $M_3$ are spanned by local triad fields $e^a \in \Omega^1(M_3)$, $a = 1, 2, 3$ which together transform as a vector under local $SO(2,1)$ Lorentz transformations. The frame bundle carries a canonical spin-connection $\omega$ transforming as a gauge connection under the local $SO(2,1)$ group, which is torsion-free, compatible with the metric of $M_3$, and has curvature

$$R^a(\omega) = d\omega^a + \epsilon^{abc} \omega^b \wedge \omega^c$$

with $\epsilon^{123} = +1$. The compatibility condition on $\omega$ means that the triads $e^a$ are not independent variables. Let us further introduce a dimensionless scalar field $D$ on $M_3$.

The action for the conformal coupling of topologically massive gauge the-
ory to topologically massive gravity is defined by the non-local functional [14]
\[ S_{\text{CTMGT}}[A, \omega; D] = \int_{M_3} \left[ \kappa D^2 e^a \wedge R^a(\omega) + 8\kappa \, dD \wedge *dD 
- \frac{1}{e^2} \text{Tr} (F_A \wedge *F_A) \right] 
- 8\kappa \int_{\partial M_3} D *_2 i_{\partial_{\perp}} dD + S_{\text{CS}}[A, \omega], \] (5.2)
where
\[ S_{\text{CS}}[A, \omega] = \int_{M_3} \left[ \frac{k}{4\pi} \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) 
+ \frac{k'}{8\pi} \left( \omega^a \wedge d\omega^a + \frac{2}{3} \epsilon^{abc} \omega^a \wedge \omega^b \wedge \omega^c \right) \right] \] (5.3)
is the sum of the gauge and gravitational Chern-Simons actions [1, 10]. In (5.2), \( \kappa \) is the three-dimensional Planck mass and the first term is a modification of the usual Einstein-Hilbert action. \( i_{\partial_{\perp}} : \Omega^p(M_3) \to \Omega^{p-1}(M_3) \) denotes interior multiplication by the vector field \( \partial_{\perp} \) which locally spans the one-dimensional fibers of the normal bundle \( N\Sigma \) to the boundary \( \Sigma = \partial M_3 \) in \( M_3 \). Since the non-compact Lie group \( \text{SO}(2,1) \) is contractible, there is no quantization condition that needs to be imposed on the gravitational Chern-Simons coefficient in (5.3) and one may choose any \( k' \in \mathbb{R} \).

The boundary term in (5.2) ensures that the action is invariant under three-dimensional conformal transformations when \( M_3 \) has a non-empty boundary [14]. We have suppressed for simplicity the boundary terms required to give the corresponding quantum field theory a well-defined classical limit. In the phase with a non-vanishing vacuum expectation value \( \langle D^2 \rangle \neq 0 \), corresponding to a vacuum with spontaneously broken conformal symmetry, one can gauge the scalar field \( D \) away by a Weyl transformation of the metric with conformal factor \( \Omega \) given by \( \Omega^2 = \langle D^2 \rangle / D^2 \). Then the resulting action (5.2, 5.3) describes topologically massive gravity coupled minimally to topologically massive gauge theory. This phase contains a propagating graviton mode of topological mass
\[ \mu' = \frac{8\pi \kappa \langle D^2 \rangle}{|k'|}, \] (5.4)
along with a propagating gluon of topological mass
\[ \mu = \frac{e^2 |k| \langle D^2 \rangle}{4\pi}. \] (5.5)
The quantum fluctuations of the scalar field $D$ thereby set the mass scales of the bulk theory.

Let us now turn off the gauge sector of the theory and set $A = 0$. Following the analogous procedures to those used before, the bulk quantum field theory defined by (5.2,5.3) can be shown to induce a two-dimensional gravity action on the boundary $\Sigma = \partial M$ [12]– [14], [47]. We refer to this two-dimensional quantum gravity as a “deformed” Liouville theory, and it is described by the action functional [14]

$$S_L[D, \phi] = \int_\Sigma \left[ -\frac{1}{4\pi} \left( \ln D^4 + \phi \right) R^{(2)} - 2\kappa D \ast_2 i\partial_\perp dD + \frac{1}{16\pi} d\phi \wedge \ast_2 d\phi + \Lambda \ast_2 e^{-\phi} \right],$$

(5.6)

where $\phi$ is the Liouville field and $R^{(2)}$ the curvature two-form of $\Sigma$. The cosmological constant is determined by the topological graviton mass (5.4) as [13]

$$\Lambda = \left( \mu' \right)^2 .$$

(5.7)

The appearence of a Liouville field theory from topologically massive gravity is not entirely surprising, given that the boundary theory can be formulated in terms of an $SL(2,\mathbb{R})$ WZNW model [48], while the bulk theory is formulated in terms of an $SO(2,1)$ Chern-Simons gauge theory. The natural identifications of the two models can now be heuristically deduced from the group isomorphism

$$SO(2,1) = PSL(2,\mathbb{C}) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) ,$$

(5.8)

inducing both chiralities of the $SL(2,\mathbb{R})$ worldsheet theory as in (5.6). The gravitational dressing of conformal field theories by topologically massive gravity is studied in [49, 50].

In addition to inducing the gravitational sector of the string theory, we see from (5.6) that the scalar field $D$ can be naturally identified as the three-dimensional version of the string dilaton field. In particular, the string coupling is given by

$$g_s = \langle D^4 \rangle .$$

(5.9)

Let us summarize a few of the generic features of the identification of the dilaton in this way [14]:

1. In string theory, the target space tachyon operator usually depends on the dilaton, which can be used to change $g_s$ and thus it controls the
scale transformation properties of the strings. This is consistent with the manner in which the field $D$ sets the bulk mass scales in (5.4) and (5.5), and this feature is important for correctly formulating T-duality within the framework of topological membranes.

(2) The dilaton in the topological membrane approach has a nice dynamical origin in terms of the fluctuating geometry and propagating bosons in the bulk three-manifold.

(3) With suitable boundary conditions for the field $D$ on $\partial M_3$, one can also generate a dynamical string coupling $g_s$ through the identification (5.9). This is reminiscent of the relationship between M-Theory and Type IIA superstring theory, and suggests that it could provide a possible worldsheet origin for the eleventh spacetime dimension of M-Theory.

5.2. Hamiltonian Quantization

Let us now analyse the Schrödinger wavefunctionals for pure topologically massive gravity ($A = D = 0$ in (5.2,5.3)) on the three-manifold $M_3 = \Sigma \times [0,1]$. We begin with the infrared limit of the theory, $k' \to 0$, in which the massive graviton decouples from the spectrum and the theory reduces to pure Einstein gravity in three-dimensions which is a topological field theory [51]. Let us consider the string worldsheet which is a torus $\Sigma = T^2$ of modulus $\tau = \tau_1 + i \tau_2$, $\tau_2 > 0$. With $J_a$, $a = 1, 2, 3$ denoting the generators of $SO(2,1)$, the topological wavefunctions then depend on the mean extrinsic curvature $K$ of $\Sigma$ in $M_3$ and two commuting $SO(2,1)$ holonomies $e^{\lambda_1 J_2}$, $e^{\lambda_2 J_2}$ as [52]

$$\Psi_{\text{grav}}(\lambda_1, \lambda_2; K) = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \frac{\lambda_1 - \tau \lambda_2}{\pi \tau_2^{1/2} K} e^{-i|\lambda_1 - \tau \lambda_2|^2/\tau_2 K} \chi(\tau, \bar{\tau}), \quad (5.10)$$

where $\mathcal{F}$ is a fundamental modular domain of the upper complex half-plane, and the Schrödinger equation is equivalent to the requirement that the function $\chi(\tau, \bar{\tau})$ be an automorphic Maass form of modular weight $\frac{1}{2}$.

The ground state corresponds to the choice

$$\chi^{(0)}(\tau, \bar{\tau}) = \tau_2^{1/2} \eta(\tau)^2 \quad (5.11)$$

in (5.10), where

$$\eta(\tau) = e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) \quad (5.12)$$

is the Dedekind function. We can now work out the gravitational analog of
the membrane inner product which we defined in Section 4.3, one finds
\[ \langle \Psi^{(0)}_{\text{grav}} | \Psi^{(0)}_{\text{grav}} \rangle = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} |\eta(\tau)|^4 . \] (5.13)
This result coincides exactly with the diffeomorphism ghost contribution to the string theory torus partition function.

Reinstating the gauge sector of the bulk theory, the full transition amplitude for the propagation of states between the boundaries \( \Sigma_0 \) and \( \Sigma_1 \) is then given using the result of Section 4.3 as
\[ Z_{\text{grav}} = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} |\eta(\tau)|^4 \sum_{\lambda, \lambda' \in (\Lambda^* G_k / \Lambda G_k)^g} \zeta^{\lambda \lambda'} \overline{\psi}_\lambda(\mathbf{r}) \psi_{\lambda'}(\tau) . \] (5.14)
This amplitude contains the correct integration over moduli space, which is induced by the integration over the holonomy parameters \( \lambda_1 \) and \( \lambda_2 \) in the overlap (5.13) of initial and final gravitational wavefunctions. Note that the two dimensional ghosts do not emerge from gauge fixing the gravitational diffeomorphisms in three-dimensions, as these bulk ghost fields completely decouple on the three-manifold \( M_3 = \Sigma \times [0, 1] \) \([53]\). The proper description of string ghost fields in three-dimensional language is not presently known.

A related problem is the expression of the worldsheet BRST formalism in the topological membrane. The BRST conditions are presumably related to the loop equations of the bulk gauge theory.

This is the only detailed result concerning the gravitational wavefunctionals that is presently available. However, from the structure of the gravitational action written in a suitable parametrization, it is possible to show, in a manner analogous to what we did in Section 3.2, that the physical Hilbert space of the full topologically massive gravity theory (\( \mu' = 8\pi \kappa' / |k'| < \infty \)) on \( \Sigma \times [0, 1] \) is the product \([33, 54]\)
\[ \mathcal{H}^{\text{TMG}} = \mathcal{H}_1[\varphi] \otimes (\mathcal{H}_m)^{\otimes (3g-3)} . \] (5.15)
The first factor is the Hilbert space of the propagating graviton of mass \( \mu' \), while the second factors are topological contributions from quantum mechanical degrees of freedom induced by the moduli \( m \in \mathbb{C}^{3g-3} \) of the Riemann surface \( \Sigma \).

6. Wilson Lines

In this section we will further deform the topological Chern-Simons gauge theory by coupling it to charged matter fields in the bulk. The consequences of this will be many-fold. Adding charged matter corresponds to deforming
the corresponding conformal field theory and allows us to describe the simplest vertex operators that create primary string states. Later on, the careful incorporation of such a deformation will be used to describe the boundary states that correspond to D-branes.

6.1. Deformations of Conformal Field Theories

Suppose that we deform our worldsheet conformal field theory, described generically by an action $S_{\text{CFT}}$, by some collection of operators $V_I$, $I \in \mathcal{I}$ to produce an action [55, 56]

$$S_\Lambda = S_{\text{CFT}} + \sum_{I \in \mathcal{I}} \lambda^I(\Lambda) \int_\Sigma d^2z \ V_I(z, \bar{z}),$$

(6.1)

where the couplings $\lambda^I(\Lambda)$ generally depend on some worldsheet scale $\Lambda$. For a relevant deformation, the original conformal field theory with action $S_{\text{CFT}}$ is described as an ultraviolet fixed point of the corresponding renormalization group flows, while for an irrelevant deformation it corresponds to an infrared fixed point. It is possible to have renormalization group flows which connect two different conformal field theories.

In the topological membrane approach, let us now consider the addition of charged matter in the bulk, so that even the pure Chern-Simons theory is no longer topological, as there are local propagating degrees of freedom. This will induce a deformed two-dimensional conformal field theory on the boundary. To see this, consider for simplicity the case of a single-charge deformation of the worldsheet theory, whose fields we denote collectively by $\Phi$. The (suitably normalized) partition function is given by

$$Z_{\text{ws}} = \int D\Phi \ e^{-S_{\text{CFT}} + \lambda \int_\Sigma d^2z \ V(z, \bar{z})}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \int_\Sigma d^2z_1 \cdots \int_\Sigma d^2z_n \ \langle V(z_1, \bar{z}_1) \cdots V(z_n, \bar{z}_n) \rangle$$

(6.2)

where the averages denote $n$-point correlation functions in the unperturbed conformal field theory. In WZNW models, correlators of primary vertex operators generically exhibit a holomorphic factorization into chiral and anti-chiral conformal blocks as

$$\langle \prod_{i=1}^{n} V_{\lambda_i}(z_i, \bar{z}_i) \rangle = \langle \prod_{i=1}^{n} V_{\lambda_i}^L(z_i) \rangle \langle \prod_{i=1}^{n} V_{\lambda_i}^R(\bar{z}_i) \rangle$$

(6.3)

where $\lambda_i$ label irreducible representations of the group $G$ of the WZNW model. More precisely, the left-hand side of (6.3) is actually a sum over left-
right conformal blocks, but to avoid clutter we simply write the factorization as displayed.

Let us now consider the membrane geometry $M_3 = \Sigma \times [0, \beta]$, where we give the time direction an arbitrary length $\beta \in \mathbb{R}$. Then the conformal field theory correlation functions are induced by open (gauge non-invariant) Wilson lines

$$W^{(\lambda_i)}[A] = W_{\bigcup_i C_i}(\lambda_i) [A] = \prod_{i=1}^{n} \text{Tr}_{\lambda_i} P \exp \left( i \int_{C_i} A \right), \quad (6.4)$$

where $C_i$ for each $i = 1, \ldots, n$ is an oriented, vertical open contour in $M_3$ with endpoints $z_i \in \Sigma_0$ and $\overline{z}_i \in \Sigma_1$ (see Figure 2). The insertion of the operator (6.4) in the bulk gives the Aharonov-Bohm phase factor (holonomy) for propagating charged particles, minimally coupled to the gauge field $A$, which move from left to right worldsheets. In other words, a gas of (open) Wilson lines describes charged matter in the bulk. More precisely, the operators (6.4) are really Polyakov lines at inverse finite temperature $\beta$, the internal size of the topological membrane. We have thereby derived a remarkable new aspect of our underlying holographic correspondence, that charged three-dimensional matter at finite temperature is equivalent to a deformed two-dimensional conformal field theory. The bulk parameter $\beta$ plays the role of the fugacity $\lambda$ of (6.2) in three dimensions [15].

### 6.2. Polyakov Loops in Topologically Massive Gauge Theory

Given the correspondence we arrived at in the previous subsection, let us now study in more detail the Polyakov loop operators of topologically massive gauge theory. We will use them to study chiral aspects of the induced boundary conformal field theory [16]. For this, we choose a gluing automorphism which identifies the left and right worldsheets $\Sigma_0 = \Sigma_1$, and thereby study gauge dynamics on the membrane geometry $M_3 = \Sigma \times S^1$. The Polyakov loop operators (6.4) then correspond to characters of the pure gauge parts $g$ of the gauge field $A$ (see (2.4)) in representations $\lambda_i$ defined by

$$\chi_{\lambda_i}(z_i) = \text{Tr}_{\lambda_i}(g(z_i)) = \text{Tr} \left[ g(z_i) \right]_{\lambda_i}. \quad (6.5)$$

We will consider again the infrared limit $e^2 \to \infty$ whereby the vacuum amplitude of the gauge theory gives the pure Chern-Simons theory partition function at finite temperature, which may be computed as the thermal average

$$Z_{\Sigma}^{CS} = \text{Tr}_{\mathcal{H}_{\Sigma}^{CS}} \left( e^{-\beta H_{CS}} \right). \quad (6.6)$$
Because Chern-Simons theory is a topological field theory, it has a vanishing Hamiltonian, \( H_{\text{CS}} = 0 \), and the partition function (6.6) thereby computes the dimension of the corresponding Hilbert space \( \mathcal{H}_{\Sigma}^{\text{CS}} \),

\[
Z_{\Sigma}^{\text{CS}} = \dim \mathcal{H}_{\Sigma}^{\text{CS}},
\]

(6.7)

independently of the internal membrane size \( \beta \) [57].

The Chern-Simons Hilbert spaces over punctured Riemann surfaces correspond to non-dynamical external charged particles in the bulk, or equivalently to Polyakov loop insertions in the path integral [1]. The associated vacuum wavefunctionals generalizing (4.18) are given by [16, 41]

\[
\Xi^{(\lambda_i)}[\{z_i\}; \mathcal{A}] = e^{-\frac{|k|}{4\pi} \int_{\Sigma} \text{Tr}(A^{1,0} \wedge A^{0,1})}
\times \int \mathcal{D}g \bigotimes_{i=1}^{n} [g(z_i)]_{\lambda_i} e^{-|k|S_{\text{V}}[g, \mathcal{A}]}. \quad (6.8)
\]

The representations \( \lambda_i \) act on complex vector spaces \( V_i, i = 1, \ldots, n \). Thus the wavefunctional (6.8) may be regarded as an operator on the product of the corresponding representation spaces \( V_i \), and one has

\[
\Xi^{(\lambda_i)}[\{z_i\}; \mathcal{A}] \in \mathbf{V} = \bigotimes_{i=1}^{n} (V_i^* \otimes V_i). \quad (6.9)
\]

Under a local gauge transformation (2.5), these states transform under a projective representation of the gauge group as

\[
\Xi^{(\lambda_i)}[\{z_i\}; \mathcal{A}] = e^{i\alpha[A, g]} \Xi^{(\lambda_i)}[\{z_i\}; \mathcal{A}] \otimes \bigotimes_{i=1}^{n} [g(z_i)]_{\lambda_i}, \quad (6.10)
\]

where the projective phase \( \alpha[A, g] \) is again the group cocycle given by (4.13). Thus the wavefunctionals (6.8) behave analogously to gauge theory correlators of the open Wilson lines (6.4). Furthermore, the property of being annihilated by the electric field operators \( E^{1,0} \) of Section 4 is clearly insensitive to insertions of the group-valued field \( g \) on the string worldsheet \( \Sigma \). Thus the wavefunctionals (6.8) still define vacuum states of the infrared limit of topologically massive gauge theory for the membrane geometry \( M_3 = \Sigma \times S^1 \).

### 6.3. The Verlinde Formula

The computation of the dimensions (6.7) proceeds by introducing an inner product, as in Section 4.3, appropriate to the states (6.8). Under the holographic correspondence, this produces conformal field theory correlators of the corresponding character insertions (6.5) on \( \Sigma \), and a careful treatment
of the resulting gauge theory path integral \[16, 57\] reproduces exactly the anticipated Verlinde formula for the dimensions of the spaces of conformal blocks on an \(n\)-punctured Riemann surface \(\Sigma\) of genus \(g\) \[28\]

\[
\langle \prod_{i=1}^{n} \chi_{\lambda_i}(z_i) \rangle = \frac{1}{\text{vol } G_{\Sigma}} \text{Tr } \mathbb{V} \langle \Xi^{\{\lambda_i\}}\{z_i\} \mid \Xi^{\{\lambda_i\}}\{z_i\} \rangle
\]

\[
= \sum_{\lambda \in \Lambda_k^*/\Lambda_k} (S_{0\lambda})^{2g-2} \prod_{i=1}^{n} S_{\lambda_i \lambda} . \quad (6.11)
\]

Here \(S_{\lambda\lambda'}\) is the modular \(S\)-matrix of the gauge group \(G\), which we assume is compact, connected and simple, given by the formula

\[
S_{\lambda\lambda'} = \sqrt{\frac{(-1)^{\Delta_+}}{(k + c_v)^r}} \frac{\text{vol } \Lambda_{G_k}^*}{\text{vol } \Lambda_{G_k}} \sum_{w \in \mathcal{W}_{G_k}} \text{sgn}(w) e^{-\frac{2\pi i}{k+c_v} (\lambda + \rho, w(\lambda' + \rho))} \quad (6.12)
\]

with \(|\Delta_+|\) the number of positive roots, \(r\) the rank, \(\mathcal{W}_{G_k}\) the Weyl group, \(c_v\) the quadratic Casimir in the adjoint representation, and \(\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha\) the Weyl vector of the current algebra based on \(G\) at level \(k\). The quadratic form in \((6.12)\) is inherited from the inner product on the root lattice. When \(\Sigma = \mathbb{T}^2 \) \((g = 1, n = 0)\), the formula \((6.11)\) also computes the number of integrable highest weight representations of the affine Lie algebra associated to \(G\) at level \(k\). The independence of this correlation function on the puncture positions \(z_i \in \Sigma\) owes to the topological invariance of the quantum gauge theory in the infrared limit.

The \(S\)-matrix has a very explicit description in three-dimensional terms via the formalism of surgery constructions on three-manifolds \([1]\). It can be represented in terms of the vacuum expectation value of the Wilson line operator \((6.4)\) taken around the components of the Hopf link (Figure 3) carrying charges \(\lambda, \lambda'\), with respect to the Chern-Simons path integral defined on the three-sphere \(S^3\), as

\[
S_{\lambda\lambda'} = \langle W_{\text{Hopf}(\lambda,\lambda')} \rangle_{S^3} . \quad (6.13)
\]

This result is derived by exploiting the fact that the path integral defines a modular functor to cut out a tubular neighbourhood surrounding the link, giving the partition function of Chern-Simons gauge theory on a solid torus, and then gluing it back after performing a modular transformation along its boundary \(\mathbb{T}^2\). With it one arrives at a purely geometrical and dynamical derivation of the Verlinde diagonalization formula. However, the proper way to encode the fusion rules of the underlying rational conformal field theory,
upon which the Verlinde formula is based [28], within the Chern-Simons framework is as yet an unsolved problem in the general case. This problem has been addressed from the point of view of the Chern-Simons inner product in [58]–[60]. Some progress has been made in the axiomatic approaches to three-dimensional topological quantum field theories [5].

\[ \text{Hopf}(\lambda, \lambda') = \]  

\[ \lambda \]  

\[ \lambda' \]  

Figure 3. The Hopf linking of two charges \( \lambda \) and \( \lambda' \).

It is instructive to consider the simplest instance of gauge group \( G = U(1) \), in which case the description of the fusion rules in three-dimensional terms is completely understood [16]. When the Chern-Simons coefficient is a rational number \( k = \frac{4p}{q} \), with \( p \) and \( q \) relatively prime positive integers, the holographic dual of the three-dimensional theory is the \( c = 1 \) conformal field theory of the rational circle, with its extended chiral algebra [16, 24, 44]. The modular \( S \)-matrix in this case is given by

\[
S_{\lambda\lambda'} = \frac{1}{\sqrt{pq}} \ e^{-\frac{2\pi i}{pq} \lambda \lambda' / p q},
\]

where \( \lambda, \lambda' \in \mathbb{Z}_{pq} \) generate a Narain lattice of charges [61, 62]. These matrix elements can be thought of in three-dimensional terms entirely by computing the correlation function on the three-sphere for a generic Wilson loop (6.4) consisting of \( n \) connected components \( C_i \) corresponding to the periodic worldlines of particles with charges \( \lambda_i \). In this simple case the result can be expressed very explicitly as [63]

\[
\langle W(\lambda_i) \rangle_{S^3} = \prod_{i,j=1}^{n} e^{-\frac{2\pi i}{pq} \lambda_i \lambda_j \#(C_i, C_j)},
\]

where \( \#(C_i, C_j) \) is the linking number which counts the number of signed intersections of \( C_i \) with the surface spanned by \( C_j \). The phase factors in (6.15) can be interpreted as Aharonov-Bohm phases arising from the circulation of one charged particle about another. In particular, this formula expresses the equivalence between the anomalous spins

\[
\Delta = \frac{\lambda^2}{k}
\]

in the dual two-dimensional and three-dimensional theories [49].
7. Compactification

In this section we will briefly describe how the Narain lattice containing the allowed spectrum of string charges arises in the topological membrane framework. The construction presented in the following quickly generalizes to any toroidal compactification of the target space, but for simplicity we will only describe the case of a spacetime compactified on a circle $S^1$. The proper description in the case of strings propagating in generic group manifolds is not presently understood.

7.1. Monopole-Instanton Operators

To incorporate the Narain lattice of charges, we need to construct string winding modes, which in turn requires some process that leads to charge non-conservation in topological membrane theory. Such processes arise due to the presence of monopole-instantons in compact $U(1)$ Chern-Simons gauge theory [64]–[66], which are remnants of ’t Hooft-Polyakov monopoles in a spontaneously broken $SU(2)$ Chern-Simons-Higgs model and lead to the possibility of tunneling between states of different magnetic flux or charge. These topological defects also arise in compact $U(1)$ topologically massive gauge theory and lead to a BKT phase transition on the string worldsheet [67]–[69].

In $U(1)$ topologically massive gauge theory on the three-geometry $M_3 = \Sigma \times [0,1]$, the Chern-Simons coefficient $k$ is related to the compactification radius $R$ of the target space through

$$ |k| = \frac{4R^2}{\alpha'}, $$

(7.1)

where $\alpha'$ is the string slope. Restricting to a compact $U(1)$ gauge group means that the pure gauge part $g$ of the gauge connection $A$ in (2.4) is a compact, $S^1$-valued field on the string worldsheet $\Sigma$. Thus the compactification of the gauge group yields the desired target space compactification, and also the appropriate topological field configurations that generate the Narain lattice, as we now proceed to demonstrate.

From the Gauss law (4.4) it follows that the elements of the gauge group are the operators

$$ U = \exp \left[ -i \int_{\Sigma} \theta \star_2 \left( \star_2 d \star_2 E - \frac{k}{4\pi} B - \rho \right) \right], $$

(7.2)

where $\rho$ is the charge density of external minimally coupled bulk matter and the quantity in the second set of brackets is the gauge group generator $G$. 
They generate the infinitesimal time-independent gauge transformations

\[ U \mathbf{A} U^{-1} = \mathbf{A} + d\theta , \quad (7.3) \]

so that the physical states \(|\Psi\rangle\) of the gauge theory are those which are invariant under their action on the quantum Hilbert space, \(U|\Psi\rangle = |\Psi\rangle\). In the presence of topologically non-trivial gauge field configurations we must also take into account large gauge transformations. They can be incorporated by taking the gauge function \(\theta(z)\) in (7.2) to be the multi-valued angle function of the Riemann surface \(\Sigma\) \[70\]

\[ \theta_{z_0}(z) = \text{Im} \ln \left( \frac{E_{z_0}(z)}{E_{z'}(z) E_{z_0}(z')} \right), \quad (7.4) \]

giving the angle between \(z\) and some reference point \(z_0\) on \(\Sigma\). Here \(z'\) is an arbitrary but fixed reference point on \(\Sigma\), while \(E_{z_0}(z)\) denotes the prime form of \(\Sigma\).

Via an integration by parts in (7.2), we thereby discover the extra local operators \[66, 71, 72\]

\[ V(z_0) = \exp \left[ -i \int_{\Sigma} \left( E + \frac{k}{4\pi} \ast_2 A \right) \wedge \ast_2 d \ln E_{z_0} \right. \]
\[ + \left. i \int_{\Sigma} \theta_{z_0} \ast_2 \rho \right]. \quad (7.5) \]

These operators commute with non-compact gauge transformations, and the physical state conditions \(V(z_0)|\Psi\rangle = |\Psi\rangle\) can be imposed simultaneously for all \(z_0 \in \Sigma\). From the canonical commutation relations \(4.6) - 4.8)\) it follows that

\[ [B(z), V^n(z_0)] = 2\pi n \delta^{(2)}(z-z_0) \quad V^n(z_0) , \quad (7.6) \]

and so the operator \(V^n(z_0)\) creates a point-like magnetic vortex at \(z_0 \in \Sigma\) of flux

\[ \frac{1}{2\pi} \int_{\Sigma} F_A = n , \quad (7.7) \]

which is the monopole number of the topologically non-trivial gauge field configuration \(\mathbf{A}\).

Furthermore, we can integrate the Gauss law \[124\] and use the exponential decay of the electric field \(E\) at large distance scales, owing to the
topological mass of the photon in the theory. It then follows that the operator $V^n(z_0)$ also carries an electric charge

$$\Delta Q = -\frac{n k}{2}.$$  (7.8)

This charge is unobservable in the long wavelength limit far from the vortex because of the exponential fall-offs, and the Aharonov-Bohm linking phases are all equal to 1 [72]. We conclude that $V^n(z_0)$ is a monopole-instanton operator, creating a dyon that interpolates between topologically inequivalent vacua of the topologically massive gauge theory.

### 7.2. The Narain Lattice

The monopole-instantons also shift the allowed charge spectrum of the quantum field theory. From (4.12) it follows that, in the functional Schrödinger picture, the physical states acquire a projective phase under gauge transformations as

$$V(z_0)\Psi[A] = e^{i \int_{\Sigma} \theta_{z_0} \star_2 \left( \frac{k}{8\pi} B - \rho \right)} \, \Psi[A + d\theta_{z_0}] .$$  (7.9)

Now single-valuedness of the gauge cocycle in (7.9) under periodic shifts $\theta_{z_0} \to \theta_{z_0} + 2\pi$ of the angle function requires the quantization condition

$$Q = \int_{\Sigma} \star_2 \rho = m + \frac{k}{8\pi} \int_{\Sigma} F_A ,$$  (7.10)

where the integer $m$ can be interpreted as a particle winding number around the monopole-instanton at $z_0 \in \Sigma$. The condition (7.10) generalizes the usual Dirac charge quantization of compact quantum electrodynamics (recovered in the limit $k = 0$). From the flux quantization condition (7.7) it follows that the spectrum of electric charges is thereby given as

$$Q = m + \frac{n k}{4} .$$  (7.11)

It is evident from (7.11) that the monopole-instantons shift the spectrum of allowed string momenta, and the magnetic charges $n \in \mathbb{Z}$ correspond to string winding modes [72]. As depicted in Figure 4, the presence of monopole-instantons in the bulk of the membrane, carrying electric charge (7.8), now shifts the initial charge of a particle from $Q_0$ on the right-moving worldsheet $\Sigma_0$ to a final charge $Q_1 = Q_0 + \Delta Q$ on the left-moving worldsheet $\Sigma_1$. The collection of charges $(Q_0, Q_1)$ live in the Narain lattice $R \cdot H^{1,1} \subset \mathbb{R}^{1,1}$ with the usual hyperbolic inner product [61, 62]. When $k = 4p/q$ is a rational number with $\gcd(p, q) = 1$, the sublattices $Q_0 = 0$ and $Q_1 = 0$ are
of finite index $pq$ in $R \cdot I^{1,1}$. Using this inner product and the identification (7.1) we may compute the spectrum of induced spins (6.16) as

$$2 (\Delta + \overline{\Delta}) = \frac{2Q_0^2}{k} + \frac{2Q_1^2}{k}$$

$$= \frac{2 (m + \frac{n k}{4})^2}{k} + \frac{2 (m - \frac{n k}{4})^2}{k}$$

$$= m^2 \frac{\alpha'}{R^2} + n^2 \frac{R^2}{\alpha'} ,$$

(7.12)

and the induced angular momentum from propagation between left and right worldsheets as

$$\Delta - \overline{\Delta} = m n .$$

(7.13)

The relations (7.12) and (7.13) reproduce the mass-shell relation and level-matching condition for bosonic strings compactified on a circle of radius $R$. In particular, T-duality in this picture has the interpretation of interchanging monopole charges and Dirac charges [32].

![Figure 4.](image)

Figure 4. The propagation of a charged particle along a Wilson line between left and right worldsheets in the presence of a monopole-instanton in the bulk of the membrane. The interaction of the particle with the vortex, indicated by X, shifts its charge from $Q_0$ to $Q_1 = Q_0 + \Delta Q$.

Because of the relation (7.13), any charge non-conserving process such as that depicted in Figure 4 must be accompanied by photon emission in the bulk such that the total angular momentum of the bulk theory is conserved. With suitable boundary conditions, and by taking into account all linking and monopole-instanton processes for charged particle propagation in the bulk, it is possible to show that all full (non-chiral) three-dimensional amplitudes for which the initial and final charges $(Q_0, Q_1) \notin R \cdot I^{1,1}$ vanish identically [73]. Thus the non-perturbative dynamics of the topological membrane naturally singles out the Narain lattice as the one describing the
appropriate compactified string spectrum. In this way, when \( k \in 2\mathbb{Q} \), the \( U(1) \) topologically massive gauge theory is dual to the \( c = 1 \) conformal field theory of the extended current algebra of the rational circle. It is not clear though at this stage how the enhanced, non-abelian \( SU(2) \) Kac-Moody symmetry at level \( k = 1 \) of the conformal field theory at the self-dual radius \( R = \sqrt{\alpha'} \) manifests itself in the bulk \( U(1) \) gauge theory.

Going back to our construction of the heterotic string theory in Section 3.1, we now discover a remarkable prediction of topological membrane theory [72]. The only way to induce the required change in spectrum between the left-movers and right-movers is through non-perturbative charge non-conserving processes in topologically massive gauge theory, and these in turn can only be induced by topologically non-trivial gauge field configurations. Thus the asymmetry between left-moving and right-moving modes is generated by a compact gauge group in the bulk, which translates into a compact target space in the dual string theory. Thus in the topological membrane construction of the heterotic string, all spacetime dimensions are required to be compact.

8. Open Strings

We will now show how to describe open strings in terms of topological membranes, with an eye towards constructing membrane states corresponding to D-branes. On a first thought, this does not seem possible to do. The string worldsheet is the boundary of a three-manifold, \( \Sigma = \partial M_3 \), and so it is necessarily closed, \( \partial \Sigma = \emptyset \), because the boundary of a boundary is always empty. This is certainly true for smooth spaces \( M_3 \). But if we allow our membrane geometry to contain singularities, such as those arising from orbifold constructions, then worldsheet boundaries can appear at bulk singular points. In this section we will show how to generate open string worldsheet degrees of freedom by taking suitable orbifolds of the membrane geometries described in the previous sections. With the appropriate modifications of these orbifold operations, everything we say here can also be used to generate unoriented (Type I) strings, but we will stick to the open string description with the ultimate goal of establishing the appearance of D-branes in topological membrane theory.

8.1. Worldsheet Orbifolds

Our construction will be based on the standard description of open strings as worldsheet orbifolds of closed strings [74]–[76]. In conformal field theory, a correlation function on a worldsheet \( \Sigma \) is completely determined by a choice
of vector in the space of conformal blocks associated to a double cover \( \hat{\Sigma} \) of \( \Sigma \), obeying factorization constraints and modular invariance \([77]\). The double \( \hat{\Sigma} \) is a closed oriented Riemann surface which generates the worldsheet \( \Sigma \) via the orbifold

\[
\Sigma = \hat{\Sigma} / \sigma
\]  

(8.1)

with respect to an anti-conformal involution \( \sigma : \hat{\Sigma} \to \hat{\Sigma}, \sigma \circ \sigma = \text{id} \) which generates the worldsheet parity symmetry group \( \mathbb{Z}_{w_2} \). The fixed points of \( \hat{\Sigma} \) under the involution \( \sigma \) correspond to boundary points of \( \Sigma \). For example, if \( \Sigma \) is a disk \( \mathbb{D}^2 \), then its double \( \hat{\Sigma} \) is a sphere \( S^2 \) and \( \sigma : S^2 \to S^2 \) is reflection about the equatorial plane. If the worldsheet \( \Sigma \) is oriented and closed, then its double \( \hat{\Sigma} = \Sigma \cap -\Sigma = \Sigma_0 \cup \Sigma_1 \) is also oriented and closed, and supports a full non-chiral conformal field theory.

Generally, the worldsheet orbifold group \( G_{\text{orb}} \) combines \( \mathbb{Z}_{w_2} \) together with the target space symmetry group \( G \) such that \( G_{\text{orb}} \subset G \times \mathbb{Z}_{w_2} \). We will usually deal with “standard” worldsheet orbifolds in which it is a direct product \( G_{\text{orb}} = G_0 \times \mathbb{Z}_{w_2} \), \( G_0 \subset G \). An open string may be viewed as an orbifold \( O_{\text{str}} = S^1/\mathbb{Z}_2 \) of a closed string, with the cyclic group \( \mathbb{Z}_2 \) generated by the reflection \( S^1 \to S^1 \) through the equatorial line of the circle. Its fundamental group is the infinite dihedral group

\[
\pi_1(O_{\text{str}}) = \mathcal{D}_{\infty} = \mathbb{Z}_2 \star \mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 ,
\]  

(8.2)

where \( \star \) denotes the free product of discrete abelian groups.

The monodromies of fields in the open string sector \( O_{\text{str}} \) correspond to the representations of (8.2) in the orbifold group such that the triangle diagram

\[
\begin{array}{ccc}
\mathbb{Z}_2 \star \mathbb{Z}_2 & \longrightarrow & G_{\text{orb}} \\
\downarrow & & \searrow \\
\mathbb{Z}_{w_2} & & \\
\end{array}
\]  

(8.3)

is commutative. The partition function of the corresponding open worldsheet field theory is then a sum over all such monodromies of the form

\[
Z_{\Sigma}(m) = \frac{1}{|G_{\text{orb}}|^g} \sum_{\alpha : \pi_1(\Sigma) \to G_{\text{orb}}} Z_{\Sigma}(\alpha; m) .
\]  

(8.4)

For example, with this prescription the cylinder amplitude takes the form

\[
Z_{\mathbb{R} \times S^1}(t) = \frac{1}{|G_{\text{orb}}|} \sum_{g_1, g_2, h \atop g_i^2 = \mathbb{1}, [g, h] = \mathbb{1}} Z_{\mathbb{R} \times S^1}(g_1, g_2, h; t) ,
\]  

(8.5)
where the modulus $t \in \mathbb{R}$ of the cylinder corresponds to the circumference of the non-contractible cycle of $\mathbb{R} \times S^1$, and the monodromy elements in the sum assume the standard forms $g_i = (\tilde{g}_i, \sigma)$, $h = (\tilde{h}, \text{id})$ with $\tilde{g}_i, h \in G_0 \subset G$.

### 8.2. Membrane Orbifolds

We will now cast the orbifold constructions of the previous subsection into the framework of topological membranes [3, 5, 16, 44, 78, 79]. The chiral sector is induced by introducing the connecting three-manifold

\[
M_\Sigma = \left( \hat{\Sigma} \times [0,1] \right) / \mathbb{Z}_2 ,
\]

where the cyclic group $\mathbb{Z}_2$ combines the worldsheet involution $\sigma$ on the double $\hat{\Sigma}$ with time reversal $T : t \mapsto 1 - t$ on the interval $[0,1]$. This is a three-manifold with boundary $\partial M_\Sigma = \hat{\Sigma}$. For example, for $\Sigma = \mathbb{D}^2$, $\hat{\Sigma} = S^2$, the connecting three-manifold is the ball $M_\Sigma = \mathbb{D}^3$ with boundary $\partial \mathbb{D}^3 = S^2$. However, for the construction of orbifold membrane amplitudes to be carried out in subsequent sections, it is more convenient to use a slightly different version of the $\mathbb{Z}_2$ orbifold in (8.6) [16, 44, 78, 79], in which the time reversal involution $T$ acts only to create a new open surface at its fixed point $t = \frac{1}{2}$ on $[0,1]$. This operation is depicted in Figure 5.

\[\begin{array}{c}
\hat{\Sigma}_0 \\
\hat{\Sigma}_1
\end{array}\]

\[\begin{array}{c}
t = 0 \\
t = 1/2 \\
t = 1
\end{array}\]

\[\begin{array}{c}
\hat{\Sigma}_0 = \hat{\Sigma}_1
\end{array}\]

\[\hat{\Sigma}_{1/2} / \mathbb{Z}_2 = \Sigma\]

Figure 5. The orbifold of the topological membrane creates an open string worldsheet $\Sigma$ at the time reversal fixed point $t = \frac{1}{2}$ and identifies the holomorphic and antiholomorphic sectors $\hat{\Sigma}_0 = \hat{\Sigma}_1$ of the associated closed string double.

Let us consider the allowed orbifold operations on the membrane which are discrete symmetries of topologically massive gauge theory defined on the three-geometry $\hat{\Sigma} \times [0,1]$ [79]. The Chern-Simons action is odd under both time-reversal $T$ and the standard worldsheet parity $\sigma = P$. Because of the presence of the additional Yang-Mills kinetic term in the topologically
massive gauge theory action, it is not difficult to see that the only discrete three-dimensional spacetime symmetries compatible with bulk gauge invariance are the combinations $\text{PT}$ and $\text{PCT}$, where $C$ is the usual charge conjugation operation in three dimensions. These are therefore the only $\mathbb{Z}_2$ actions that we are allowed to take in defining our gauge theory on the orbifold \((\mathcal{S},\sigma)\). The action functional on the connecting three-manifold is then defined in terms of the previous gauge theory action as

$$2S_{\text{TMGT}}^{M_\Sigma}[A] = S_{\text{TMGT}}^\widehat{\Sigma \times [0,1]}[\widehat{A}], \quad (8.7)$$

where $\widehat{A}$ is the extension of the gauge field $A$ from $M_\Sigma$ to the covering cylinder $\widehat{\Sigma} \times [0,1]$.

For an induced string theory which is compactified on a circle as in the previous section, it is also straightforward to work out how the allowed $\text{PT}$ and $\text{PCT}$ automorphisms of the topological membrane act on the charge spectrum \((8.8)\) and appropriately truncate it according to the standard boundary conditions in open string theory \([79]\). For the $\text{PT}$ orbifold of topologically massive gauge theory one finds

$$\text{PT} : Q \mapsto -Q. \quad (8.8)$$

After performing the orbifold operation which identifies left and right worldsheets, the identification $Q_1 = -Q_0$ leaves a spectrum of pure winding modes $Q = nk/4$ and corresponds to Dirichlet boundary conditions on the open string embedding fields. Similarly, one has

$$\text{PCT} : Q \mapsto Q \quad (8.9)$$

and the orbifold identification $Q_1 = Q_0$ truncates the charge spectrum to pure KK-modes $Q = m$, corresponding to Neumann boundary conditions on the open strings.

### 8.3. Singleton Orbifolds

The final truncation to open strings that we need to make is the appropriate specification of how the harmonic modes of the gauge connections map under the orbifold operations in \((8.6)\) \([5]\). The worldsheet involution $\sigma : \widehat{\Sigma} \to \widehat{\Sigma}$ induces an involutive isomorphism on homology $\sigma_* : H_1(\widehat{\Sigma}, \mathbb{R}) \to H_1(\widehat{\Sigma}, \mathbb{R})$. The homology group $H_1(\widehat{\Sigma}, \mathbb{R})$ is a symplectic vector space over $\mathbb{R}$ with respect to the canonical intersection form on the closed oriented Riemann surface $\widehat{\Sigma}$. The subspaces $L_{\pm}(\widehat{\Sigma}) \subset H_1(\widehat{\Sigma}, \mathbb{R})$ defined as the $\pm 1$ eigenspaces of the involution, $\sigma_*L_{\pm}(\widehat{\Sigma}) = \pm L_{\pm}(\widehat{\Sigma})$, are Lagrangian subspaces with
respect to this symplectic structure and the homology group decomposes into the orthogonal direct sum
\[ H_1(\hat{\Sigma}, \mathbb{R}) = L_+ (\hat{\Sigma}) \oplus L_- (\hat{\Sigma}) . \] (8.10)

We then identify the homology of the worldsheet orbifold obtained from the double \( \hat{\Sigma} \) as the Lagrangian subspace
\[ H_1(\Sigma, \mathbb{R}) = L_- (\hat{\Sigma}) . \] (8.11)

The identification (8.11) has a very natural geometrical interpretation from the topological membrane perspective. Given the connecting three-manifold (8.6), the canonical inclusion \( \bar{\iota} : \hat{\Sigma} = \partial M_\Sigma \to M_\Sigma \) induces a homomorphism \( \bar{\iota}_* : H_1(\hat{\Sigma}, \mathbb{R}) \to H_1(M_\Sigma, \mathbb{R}) \) with null space
\[ \ker(\bar{\iota}_*) = L_- (\hat{\Sigma}) . \] (8.12)

In other words, the homology group (8.11) of the open string worldsheet \( \Sigma \) consists of those homology cycles of its double \( \hat{\Sigma} \) which are contractible in the corresponding connecting three-manifold \( M_\Sigma \). These truncated topological degrees of freedom can also be related more directly to the membrane geometry \( \hat{\Sigma} \times [0,1] \) by using a \( \mathbb{Z}_2 \)-equivariant version of this homological construction [16]. Equivalently, since \( M_\Sigma \) retracts to \( \Sigma = \hat{\Sigma}/\mathbb{Z}_2 \), we can pick an arbitrary imbedding \( j : \Sigma \hookrightarrow M_\Sigma \) to carry out the appropriate truncation [3, 5].

9. Orbifold Amplitudes

The remainder of this article is devoted to explaining how to use the set-up of the previous section to describe states corresponding to D-branes in three-dimensional terms. In this section we will describe how open string amplitudes easily arise from orbifolds of topologically massive gauge theory, and how they can be used to systematically derive open string vertex operators in the induced two-dimensional boundary conformal field theory. We shall see that the vertex operators for both Neumann and Dirichlet branes arise very naturally in this formalism.

9.1. Open String Amplitudes

We will begin with a heuristic explanation of how three-dimensional orbifolds of the previously derived closed string amplitudes of topologically massive gauge theory naturally induce the expected open string amplitudes [44].
Recall the membrane inner product (4.20) which we write in the form

$$Z_{\Sigma}^{TMGT} = \sum_{\lambda, \lambda' \in (\Lambda^*_G / \Lambda_G)^g} \zeta^{\lambda \lambda'} \psi_{\lambda}(0) \otimes \bar{\psi}_{\lambda'}(1),$$

(9.1)

where the non-negative integers $\zeta^{\lambda \lambda'}$ ensure modular invariance of the world-sheet amplitude. We have emphasized its interpretation as the propagation amplitude for the evolution of an initial state $\psi_{\lambda}(0)$, inserted at time $t = 0$ on the holomorphic boundary of the membrane geometry depicted on the left-hand side of Figure 5, through the bulk of the membrane to a final state $\psi_{\lambda'}(1)$ at time $t = 1$ on the anti-holomorphic sector of the closed string double $\hat{\Sigma}$. It is regarded as a vector in the Chern-Simons Hilbert space

$$Z_{\Sigma}^{TMGT} \in \mathcal{H}_{\Sigma}^{CS} \otimes \mathcal{H}_{\Sigma}^{CS}$$

(9.2)

of non-chiral conformal blocks.

We can use this bilinear form to induce an amplitude corresponding to propagation of states between times $t = 0$ and $t = \frac{1}{2}$ in the membrane geometry depicted on the right-hand side of Figure 5 by taking a membrane orbifold. For this, we insert a complete set of states $\psi_{\rho}(\frac{1}{2})$ into (9.1) at $t = \frac{1}{2}$ to write

$$Z_{\Sigma}^{TMGT} = \sum_{\lambda, \lambda' \in (\Lambda^*_G / \Lambda_G)^g} \zeta^{\lambda \lambda'} \psi_{\lambda}(0) \otimes \sum_{\rho \in (\Lambda^*_G / \Lambda_G)^g} \bar{\psi}_{\rho}(\frac{1}{2}) \psi_{\rho}(\frac{1}{2}) \otimes \bar{\psi}_{\lambda'}(1).$$

(9.3)

Let us now consider the orbifold involutions of Section 8.2. At the orbifold fixed point $t = \frac{1}{2}$, Wilson lines transform as $W_{C(\rho)}[A] \mapsto W_{-C(\rho)}[A]$ under both the PT and PCT orbifolds whose overall effect is to change the orientation of the worldline $C$ of the charge $\rho$. Making the desired orbifold identifications thereby leads to the requirement

$$W_{C(\rho)}[A] = W_{-C(\rho)}[A]$$

(9.4)

for arbitrary oriented contours $C \subset M_3$. If $C \subset \hat{\Sigma}_{1/2}$, then this is only possible in the trivial charge sector $\rho = 0$. Since the conformal field theory wavefunctions are generated by Wilson lines lying entirely in $\hat{\Sigma}$ (via the appropriate monopole-instanton processes), it follows that the only membrane state possible at $\hat{\Sigma}_{1/2}$ is the identity character state $\psi_{\rho=0}(\frac{1}{2}) = 1$.

Thus, after orbifolding, the insertion of a complete set of states in (9.3) leaves simply the orthogonal projection, acting on the Chern-Simons Hilbert
space, onto those membrane states which are invariant under interchange of left and right moving worldsheet modes. We thereby find the orbifold amplitude

$$Z_{\Sigma}^{\text{orb}} = \sum_{\lambda, \lambda' \in \{\Lambda^{*}\Lambda G_k / \Lambda G_k\}_{\text{orb}}} \zeta^{\lambda \lambda'} \psi_{\lambda}(0) \otimes \frac{1}{2} \left( \mathbb{1} + \sigma_{\#}^{-1} \right) \overline{\psi}_{\lambda'}(1),$$

(9.5)

where $\sigma_{\#} : \mathcal{H}_{\Sigma}^\text{CS} \to \mathcal{H}_{\Sigma}^\text{CS}$ is the induced representation of the $\text{PT}$ or $\text{PCT}$ involution on the three-dimensional states of the membrane and the subscript $\text{orb}$ denotes the appropriate truncation of the charge lattice under the orbifold operation. After taking into account the induced identifications $\psi_{\lambda}(0) = \psi_{\lambda}(1) = \psi_{\lambda}^{\text{orb}}$, the amplitude (9.5) acquires the form [44]

$$Z_{\Sigma}^{\text{orb}} = \sum_{\lambda \in \{\Lambda^{*}\Lambda G_k / \Lambda G_k\}_{\text{orb}}} h^{\lambda} \psi_{\lambda}^{\text{orb}},$$

(9.6)

and it is regarded as a vector in the Hilbert space

$$Z_{\Sigma}^{\text{orb}} \in \mathcal{H}_{\Sigma}^\text{CS}.$$  

(9.7)

Thus we easily derive the fact that for closed oriented worldsheets $\tilde{\Sigma}$, the partition function is a bilinear form in the characters of the induced conformal field theory, while for open (or unoriented) worldsheets $\Sigma$ it is simply linear in the characters [44, 78]. Moreover, these same arguments serve to show that open string membrane amplitudes can be regarded as square roots of the corresponding closed string amplitudes [16]

$$\left| \langle \Psi_0 \mid \Psi_{1/2} \rangle_{\text{orb}} \right| = \sqrt{\left| \langle \Psi_0 \mid \Psi_1 \rangle \right|}.$$  

(9.8)

In other words, closed strings come in a double volume of open strings.

### 9.2. Brane Vertex Operators

We can also carry out the construction of orbifold amplitudes more precisely, using our previous path integral formalism for membrane amplitudes and the appropriate Schrödinger wavefunctionals [16]. For this, we will consider the simplest instance of $U(1)$ topologically massive gauge theory minimally coupled to a conserved current represented by a (dual) one-form $J = J + J_0 \, dt$ on $M_3 = \tilde{\Sigma} \times [0, 1]$. The action is

$$S_J[A] = S_{\text{TMGT}}[A] + \int_{M_3} A \wedge \ast J$$

(9.9)
while the continuity equation for the source is given by

\[ d \ast J = 0 \]

(9.10)

The appropriate boundary conditions which are compatible with the PT and PCT orbifold involutions of the membrane require us to fix the sources in terms of an auxiliary one-form \( \tilde{Y} = \tilde{Y}^{1,0} + \tilde{Y}^{0,1} \) on the closed string double \( \hat{\Sigma} \) as

\[ \begin{align*}
J_0 \, d\text{vol}_{\hat{\Sigma}} &= d\tilde{Y}, \\
J^{1,0} &= \frac{\mu}{2} *_2 \tilde{Y}^{0,1}, \\
J^{0,1} &= \frac{\mu}{2} *_2 \tilde{Y}^{1,0},
\end{align*} \]

(9.11)

with \( \mu \) the topological photon mass (3.4). This restriction further guarantees that the amplitudes, defined originally in terms of wavefunctionals living on the boundaries \( \hat{\Sigma}_0 \) and \( \hat{\Sigma}_1 \), can be systematically extended into the bulk of the membrane. It is also the key to constructing vacuum wavefunctionals of the matter-coupled gauge theory which respect gauge invariance as before [16].

It is now straightforward to incorporate the addition of charged matter in this way for arbitrary source configurations to the construction of vacuum Schrödinger wavefunctionals, extending the constructions we carried out earlier for charged particle Wilson lines. Doing so, and then carefully computing the orbifold of the corresponding membrane inner product, we arrive at the orbifold partition function [16]

\[ Z_{\Sigma}^{\text{orb}} = \langle \Psi_{\text{vac}}^1 | \Psi_{\text{vac}}^{1/2} \rangle_{\text{orb}} \]

(9.12)

with the vacuum wavefunctionals

\[ \begin{align*}
\Psi_{\text{vac}}^0 \left[ A, \tilde{Y} \right] &= e^{-i \int_{\Sigma} A^{1,0} \wedge \left( \frac{|k|}{4\pi} A^{0,1} + \tilde{Y}^{0,1} \right)} \\
&\times \int D\phi \ e^{-i \frac{|k|}{8\pi} \int_{\Sigma} (d\phi \wedge *_2 d\phi - 2 \partial\phi \wedge A^{0,1})}, \\
\Psi_{\text{vac}}^{1/2} \left[ A, \tilde{Y} \right] &= \int D\phi \ e^{-i \frac{|k|}{4\pi} \int_{\Sigma} \phi (\tilde{Y}^{1/2} - \frac{|k|}{4\pi} A^{1/2})} \\
&\times e^{-i \frac{|k|}{8\pi} \int_{\Sigma} [A^{1,0} \wedge (\partial\phi - \frac{8\pi}{|k|} \tilde{Y}^{0,1}) - A^{0,1} \wedge (\partial\phi - \frac{8\pi}{|k|} \tilde{Y}^{1,0})]}.
\end{align*} \]

(9.13, 9.14)

Here \( \phi \) is the boson field of the induced \( c = 1 \) conformal field theory in this case, and in (9.14) the superscript \( || \) denotes the projections of the one-forms onto the cotangent bundle over the boundary \( \partial\Sigma \) in the open string worldsheet \( \Sigma \). A simple field redefinition in the path integrals removes the source terms from the bulk of \( \Sigma \), and thus in this case a change of conformal
background due to the insertion of bulk charged matter is induced solely by boundary deformations.

Let us now insert the local Hodge decomposition on \( \hat{\Sigma} \) of the charged matter deformation given by

\[
\tilde{Y} = \frac{|k|}{4\pi} (dY_D + *_2 dY_N)
\]

with \( Y_D, Y_N \in \Omega^0(\hat{\Sigma}) \). The absence of harmonic modes in (9.15) is imposed by the requirement that the sources be non-dynamical in \( M_3 \). In addition, we use the Hodge decomposition (3.6) of the topologically massive gauge field \( A \) on the closed string double \( \Sigma \). The functional integrations over the harmonic degrees of freedom \( a \) in (9.12, 9.14) for the PT and PCT orbifolds of the topological membrane then respectively yield the delta-function constraints [16]

\[
\text{PT} : \delta \left( \nabla^2 (\phi - 2Y_D) \right) \big|_{\Sigma} = \delta \left( (\phi - Y_D) \right) \big|_{\partial \Sigma}, \quad (9.16)
\]

\[
\text{PCT} : \delta \left( \nabla^2 (\phi - 2Y_D) \right) \big|_{\Sigma} = \delta \left( i_{\partial} d\phi \right) \big|_{\partial \Sigma}. \quad (9.17)
\]

The first delta-function in both (9.16) and (9.17) imposes (after a simple field redefinition) the bulk equation of motion for the free scalar field \( \phi \) on the open string worldsheet \( \Sigma \). For the PT orbifold the second delta-function imposes Dirichlet boundary conditions for \( \phi \) on \( \partial \Sigma \), while for the PCT orbifold it selects Neumann boundary conditions. Thus the only roles played by the wavefunctional (9.14) at the orbifold branch point are to enforce the worldsheet equations of motion and to select the appropriate boundary conditions of open string theory. Otherwise they simply correspond to inserting the identity character state into the membrane inner product, consistently with what we argued in the previous subsection.

What is particular interesting about the orbifold vacuum wavefunctional (9.14) is the boundary exponential term. After functional integration, for the PT and PCT orbifolds of the topological membrane it yields a boundary deformation of the usual bulk \( \sigma \)-model action given respectively by [16]

\[
\text{PT} : V_D = e^{-\frac{|k|}{4\pi} \int_{\partial \Sigma} Y_D} \int d\phi \quad (9.18)
\]

\[
\text{PCT} : V_N = e^{-\frac{|k|}{4\pi} \int_{\partial \Sigma} Y_N} \int d\phi \quad (9.19)
\]

The insertion (9.18) into the conformal field theory path integral is the vertex operator for a D-brane described by the collective coordinate \( Y_D \) in the target space direction \( \phi \). Using the identification (7.1), one sees that this correspondence is exact. Similarly, the operator (9.19) is the open string
photon Wilson line for the $U(1)$ gauge field component $Y_N/2\pi \alpha'$ in direction $\phi$. Thus D-branes in topological membranes correspond to charged matter on an orbifold line in three dimensions. In this picture, the collective coordinate of the D-brane is controlled by the bulk charge distribution. This is in perfect harmony with our description of deformed conformal field theories through the addition of charged matter fields in the bulk that was given in Section 6.1.

10. Boundary States

The results of Section 9.2 provide very encouraging evidence that the wavefunctionals (9.14) describe three-dimensional states corresponding to D-branes. Of course, the fact that they induce the correct vertex operators in the effective $\sigma$-model action is not a proof of this fact, and one now needs to carefully explore to what extent the dynamics of the topological membrane truly captures the physics of D-branes. One way to proceed towards this goal is to analyse to what extent the orbifolds of topologically massive gauge theory induce boundary conformal field theories. In this section we will show how to construct standard closed string boundary states in three-dimensional terms and also how the bulk-boundary correspondence of conformal field theory manifests itself in the topological membrane approach.

10.1. Ishibashi States

Let us recall the vacuum wavefunctionals $\Xi^{\{\lambda_i\}}[\{z_i\}; A]$ in (6.8) describing states of chiral, non-dynamical charged particles in the topological membrane. They are elements of the complex vector space $V$ as given in (6.9), and they correspond to primary chiral field insertions on the closed Riemann surface $\hat{\Sigma}$. Later on, we will see how to also incorporate string descendant fields into the membrane wavefunctionals and amplitudes. But for the time-being, we assume that this has been done and extend the representation spaces $V_i$ to the appropriate Virasoro-Sugarawa modules by application of the corresponding Virasoro descendant fields. We will also denote these modules by $V_i$.

We shall now analyse the structure of these operators in the case $n = 1$, corresponding to a single particle insertion of charge $\lambda$. Consider a vertical Wilson line operator $W^\lambda[A]$ in the instance when the bulk topologically massive gauge theory is assumed to possess its full discrete PCT invariance. Then the orbifold involution acts on this Wilson line as

$$\text{PCT} : W^\lambda[A] \mapsto W^\lambda[A]$$

(10.1)
and thus the charge non-conserving monopole-instanton induced processes are suppressed. Moreover, the corresponding wavefunctional \( \Xi^\lambda[z; A] \) is an operator

\[
\Xi^\lambda[z; A] : V_\lambda \rightarrow V_\lambda .
\] (10.2)

By gluing together left and right moving worldsheet sectors as before, one finds that the PCT-invariant membrane states describing propagation between the boundaries \( \hat{\Sigma}_0 \) and \( \hat{\Sigma}_1 \) are given by the vacuum wavefunctionals

\[
\Phi^\lambda[z, \overline{z}; A] = \Xi^\lambda[z; A] \otimes \overline{\Xi}^{\lambda=\overline{\lambda}}[\overline{z}; A] .
\] (10.3)

The identification of left and right moving charges \( \overline{\lambda} = \lambda \) arises from the identification under worldsheet parity \( P : \hat{\Sigma}_0 \equiv \hat{\Sigma}_1 \). The operator (10.3) acts only in the diagonal, left-right symmetric product \( V_\lambda \otimes V_{\overline{\lambda}=\lambda} \subset V \). It is therefore proportional to the orthogonal projection

\[
P_\lambda = \sum_{I \in I} |\lambda, I\rangle \langle \lambda = \lambda, I| ,
\] (10.4)

onto this subspace.

By choosing the proportionality constant to be 1, and using the natural inner product on the Hilbert space \( V_\lambda \), the homomorphism \( \Phi^\lambda : V_\lambda \rightarrow V_\lambda \) is in a one-to-one correspondence with the closed string Ishibashi state [80,81] given by

\[
|\lambda\rangle^D = \sum_{I \in I} |\lambda, I\rangle \otimes U_p U_c |\lambda, I\rangle ,
\] (10.5)

where the anti-unitary operators \( U_p \) and \( U_c \) implement the actions of the parity and charge conjugation automorphisms on the right-moving Hilbert space. The vector (10.5) is just a Dirichlet boundary state of the induced closed string theory. In a completely analogous manner, by assuming only the PT sub-invariance of the bulk quantum field theory, one can readily construct Neumann Ishibashi boundary states [16]. In this case one must account for non-trivial monopole-instanton transitions in the bulk. Again it is not clear though how, for the free boson at the self-dual radius, the family of branes parametrized by the \( SU(2) \) group manifold at level \( k = 1 \) [82] appears here, with the Dirichlet brane corresponding to the identity element of \( SU(2) \). A possible clue may lie in boundary state of [83] which is a global \( SU(2) \) rotation of the usual Neumann boundary state.
10.2. The Bulk-Boundary Correspondence

Let us now see how the usual bulk-boundary correspondence of two-dimensional conformal field theory manifests itself in the three dimensional framework. According to this principle, the \( n \)-point correlators on an open surface \( \Sigma \) are in one-to-one correspondence with chiral \( 2n \)-point correlation functions on the double \( \hat{\Sigma} \) [77]. The interaction of a local field with \( \partial \Sigma \), in the form of boundary conditions, is then simulated by the interaction between mirror images of the same holomorphic field on \( \hat{\Sigma} \), carrying conjugate primary charges \( \lambda, \bar{\lambda} \).

To each conformal boundary condition \( \alpha \), we insert a Wilson loop \( W^{\lambda_\alpha} [A] \) in the bulk corresponding to a prescribed representation \( \lambda_\alpha \) of the gauge group. More precisely, let us suppose that the boundary of the open string worldsheet consists of \( B \) connected components given by the disjoint union

\[
\partial \Sigma = \bigcup_{\alpha=1}^{B} C_\alpha . \tag{10.6}
\]

In the double \( \hat{\Sigma} \), the pre-image of each loop \( C_\alpha, \alpha = 1, \ldots, B \) is a \( \mathbb{Z}_2 \)-invariant equatorial circle corresponding to a Wilson loop in the covering cylinder \( \hat{\Sigma} \times [0, 1] \), which becomes a circle of singular points in the three-dimensional orbifold (Figure 6). Any connected component \( C_\alpha \) of the singular locus of the orbifold \( \Sigma \) can be represented as a sum over Wilson loops with the topology of \( C_\alpha \) [5, 78].

![Figure 6](image)

Figure 6. Three-dimensional representation of the bulk-boundary computation of \( n \)-point conformal correlation functions. The interactions of local field insertions with \( \partial \Sigma \) are represented by the interactions with Wilson loops \( W^{\lambda_\alpha} [A] \) corresponding to the open string boundaries \( C_\alpha \). The chiral correlator on the right is computed using the finite temperature prescription of Section 6.2 along with the linking rules described in Section 6.3.

These facts can all be derived systematically by examining orbifold membrane amplitudes with the prescribed boundary conditions on \( \partial \Sigma \) [16]. The
boundary conditions are charges of external matter inside the topological membrane, and are thereby naturally labelled by primary charges, just as in conformal field theory. These are the boundary conditions which preserve all bulk symmetries. This three-dimensional prescription thus provides a very effective computation of the correlation functions in boundary conformal field theory. An open problem in this context is the description of symmetry-breaking boundary conditions in the three-dimensional framework, and hence of D-branes which are not maximally symmetric.

11. The Cardy Condition

We can now describe one of the fundamental results of boundary conformal field theory, the celebrated Cardy condition. Just like in the case of the Verlinde formula, it has a very natural dynamical origin within the framework of open topological membranes. This analysis allows one to select the totality of three-dimensional states corresponding to D-branes based on the single guiding principle of bulk gauge invariance.

11.1. Fundamental Brane States

The Cardy condition of boundary conformal field theory [84] follows from the equality of the closed string cylinder amplitude, computed as a matrix element between two boundary states, and the open string annulus amplitude with the corresponding boundary conditions. In the language of topological membranes, it arises as a compatibility condition between conservation of orbifold charges and bulk gauge invariance in $M_3$ [16]. To understand this point, let us consider the “fundamental” wavefunctionals

$$
\Upsilon^\lambda[z;A] = \sum_{\lambda' \in (\Lambda^* G_k / \Lambda G_k)^a} \beta_{\lambda\lambda'} \Phi^{\lambda'}[z;A] \quad (11.1)
$$

given as linear superpositions of the Ishibashi wavefunctionals describing D-brane states. The coupling coefficients may be fixed by demanding that membrane inner products with the wavefunctionals (11.1) be determined as overlaps with the trivial $\lambda = 0$ Wilson line (corresponding to the identity character state as described earlier) in the orbifold topologically massive gauge theory. An elementary computation gives [16]

$$
\beta_{\lambda\lambda'} = \frac{S_{\lambda\lambda'}}{S_{\lambda0}}, \quad (11.2)
$$

and thereby yields a remarkably simple derivation of the Cardy solution (up to normalization) of the sewing constraints in boundary conformal field
theory [84].

The coupling coefficients (11.2) are described dynamically by the Hopf linking amplitudes (6.13). With them we may compute the matrix elements [16]

\[ \langle 1 \mid W^{\lambda'} \Upsilon^\lambda \rangle = S_{-\lambda,\lambda'} \langle 1 \mid \Upsilon^\lambda \rangle. \]  

(11.3)

We interpret (11.3) to mean that as a charged particle propagates through the bulk it interacts with a soliton-like defect described by the boundary state (11.1,11.2), producing the usual Hopf linking factors in \( S^3 \). In the worldsheet picture, this defect clearly corresponds to a D-brane, producing the correct charge deformation of the boundary field.

11.2. Surgery Calculation

The relationship (11.1,11.2) between brane states and Ishibashi wavefunctionals can be derived in an alternative setting by using surgery techniques on three-manifolds [5]. Let us compute the one-point conformal correlation function on the disk \( \Sigma = \mathbb{D}^2 \) of a primary field of charge \( \lambda \) with boundary condition \( \lambda' \). In the membrane picture, we compute this correlator by inserting a vertical Wilson line of charge \( \lambda \) through the connecting three-manifold \( M_\Sigma = \mathbb{D}^3 \), linked with the unknotted Wilson loop of charge \( \lambda' \) representing the boundary interaction as prescribed in the previous section. This membrane state is depicted on the left in Figure 7. Since the corresponding Chern-Simons Hilbert space is one-dimensional, this state is proportional to the closed string state given by the Ishibashi boundary state on the two-sphere \( S^2 \). This basis vector for the physical Hilbert space is represented on the right in Figure 7.

Figure 7. Three-dimensional version of the representation of the correlator of a bulk field on the disk in terms of the standard two-point conformal block on the sphere.

We can determine the coupling coefficients in this setting by gluing another three-ball \( \mathbb{D}^3 \) to each of the balls in Figure 7 along their common (but oppositely oriented) \( S^2 \) boundaries. The left-hand side then gives the expec-
tation value of the Hopf link in \( S^3 \) with component charges \( \lambda \) and \( \lambda' \), while the right-hand side gives the gauge theory correlator on \( S^3 \) of a single unknot of charge \( \lambda \). Using the functorial property of the quantum gauge theory, this results in the equality

\[
\langle W_{\text{Hopf}(\lambda,\lambda')} \rangle_{S^3} = \beta_N \langle W_{\text{unknot}(\lambda)} \rangle_{S^3},
\]

(11.4)

and from (6.13) the Cardy solution (11.2) again unambiguously follows. With the canonical normalization of boundary conformal field theory [5], this gives the usual expression for the one-point correlation function of a bulk field on the disk \( D^2 \) as \( S_{\lambda\lambda'}/\sqrt{S_{\lambda0}} \) times the standard two-point conformal block on the sphere \( S^2 \). Furthermore, gauge invariance of the bulk theory implies that the branes obtained in this way correspond to all relevant branes of the topological membrane [16].

11.3. The Annulus Amplitude

As a final check of the consistency of our membrane identifications, let us now look at the genus 1 annulus amplitude (Figure 8). By a direct calculation in the \( \text{PCT} \) orbifold of topologically massive gauge theory, it is given by the Neumann amplitude [16, 44]

\[
Z_{\lambda\lambda'}^{\text{PCT}}(t) = \sum_{\rho \in [\Lambda_{G_k}/\Lambda_{G_k}]_{\text{PCT}}} N_{\lambda\lambda'}^\rho \psi^{\text{PCT}}_\rho(t)
\]

(11.5)

where \( N_{\lambda\lambda'}^\rho \) are the fusion coefficients for the Kac-Moody algebra based on the gauge group \( G \) at level \( k \). One can compare this result with the Dirichlet cylinder amplitude computed in a completely analogous way in the \( \text{PT} \) orbifold. The two results are related after a modular transformation and a Poisson resummation [44], and immediately lead to the Verlinde formula (6.11) for the three-punctured sphere \( S^2 \).

Figure 8. The annulus with prescribed conformal boundary conditions \( \lambda \) and \( \lambda' \). The inner radius is parametrized as \( a = e^{-t} \) where \( t \in \mathbb{R} \) is the modulus of the annulus.

One can also arrive at this result via a surgery prescription [5]. The connecting three-manifold in this case is a solid torus, with an annulus Wilson
graph representing the boundary interactions. Again, one immediately has the expression (11.5) for the annulus amplitude. Let us now take the connected sum of the connecting three-manifold with another solid torus. This produces the Chern-Simons invariant of $S^3$ with three unknots, and functoriality then leads once again to the Verlinde formula for the three-punctured two-sphere. This is completely analogous to the original derivation [1] of the Verlinde diagonalization formula in terms of closed string amplitudes in three dimensions. Thus from the point of view of topological membranes, the open string derivation of the Verlinde formula [84] is completely equivalent to the closed string derivation [1].

12. Descendent States

As our final piece of evidence for the appearance of D-brane states, we will now show how the dynamics of the topological membrane naturally induces the correct tension of a D-brane. To derive the tension formula, we shall need to further extend our construction of vacuum Schrödinger wavefunctionals to describe excited membrane eigenstates of the topologically massive gauge theory Hamiltonian. At the same time, this will solve another problem of topological membrane theory that we have not yet addressed, namely the proper description of gauge invariant states in three dimensions which correspond to string descendent fields.

12.1. Landau Levels and Excited Membrane States

We will begin by constructing excited wavefunctionals of topologically massive gauge theory in a fixed, non-trivial dilaton background of the conformally coupled topologically massive gravity described in Section 5.1 [16]. For simplicity we will restrict our attention to a $U(1)$ gauge group, or equivalently to the $c = 1$ conformal field theory of a single boson. Using the dilaton equation of motion arising from variation of the action (5.2) with respect to the scalar field $D$, the Hamiltonian of the gauge sector can be written as

$$H_D = \frac{1}{2} \int_{\Sigma} \left( \frac{e^2}{D^2} E^{0,1} \wedge E^{1,0} + 2\kappa \, \text{dvol}_\Sigma \, V(D, \omega) \right),$$

where the dilaton potential $V(D, \omega)$ defined by

$$\text{dvol}_\Sigma \, V(D, \omega) = 8D \ast_2 \nabla^2 D - D^2 \, R^{(2)}(\omega)$$

is essentially the three-dimensional energy density of the Liouville field theory (5.0). The dilaton background shifts the ground state energy $\mathcal{E} = 0$ to
the non-vanishing value
\[ E_0 = \kappa \langle V(D, \omega) \rangle . \] (12.3)

As observed previously, Hamiltonian quantization of topologically massive gauge theory is equivalent to a field theoretic version of the Landau problem. Using this observation, one can build higher membrane states. We start from the vacuum wavefunctionals (6.8) which are destroyed by the electric field annihilation operators as in (4.16),
\[ E_z \Xi^{(\lambda_i)}[\{z_i\}; A] = 0 . \] (12.4)

A natural set of gauge-invariant excited states of the topological membrane is then obtained via successive insertions of powers of the electric field creation operators at the insertion points \( z_i \in \Sigma \) to give
\[
\Psi_n^{(\lambda_i)}[\{z_i\}, \{n_i\}; A] = \prod_{i=1}^n \frac{1}{\sqrt{n_i!}} \left( \frac{4\pi i}{k} E_\Xi(z_i) \right)^{n_i} \Xi^{(\lambda_i)}[\{z_i\}; A] \\
= e^{-\frac{|k|}{8\pi} \int_\Sigma A^{1,0} \wedge A^{0,1} \int D\phi} e^{-\frac{|k|}{8\pi} \int_\Sigma (\partial \phi - 2A^{1,0}) \wedge \partial \phi} \\
\times \prod_{i=1}^n \frac{1}{\sqrt{n_i!}} (A_2(z_i) - \partial_2 \phi(z_i))^{n_i} e^{i\lambda_i \phi(z_i)} \] (12.5)

with \( n_i \in \mathbb{N}_0 \). As in (6.9), these wavefunctionals are regarded as operators on products of the corresponding representation spaces,
\[
\Psi_n^{(\lambda_i)}[\{z_i\}, \{n_i\}; A] \in V , \] (12.6)

and they are eigenstates of the Hamiltonian (12.1),
\[
H_D \Psi_n^{(\lambda_i)}[\{z_i\}, \{n_i\}; A] = \mathcal{E}_N \Psi_n^{(\lambda_i)}[\{z_i\}, \{n_i\}; A] , \] (12.7)

where the excited state energies are those of Landau levels
\[
\mathcal{E}_N = \mathcal{E}_0 + \frac{\mu}{\langle D^4 \rangle} N \] (12.8)

with
\[
N = \sum_{i=1}^n n_i . \] (12.9)

The wavefunctionals (12.5) evidently describe gauge invariant membrane excitations that correspond to string descendent states at level \( N \) given by (12.9). They correspond to \( n \) gauge invariant combinations of external
charged particles and photons situated at the points \( z_i \in \Sigma \). Strictly speaking, the infrared limit \( \mu \to \infty \) projects these states onto the lowest Landau level \( N = 0 \), i.e. \( n_i = 0 \ \forall i = 1, \ldots, n \), but we shall soon describe how appropriate field theoretic renormalizations can be employed such that the higher Landau levels contribute to quantities in the ground state of the topologically massive gauge theory. In this way the topological photon mass (3.4) will naturally set the mass scales for both perturbative and non-perturbative states of the induced string theory.

12.2. D-Brane Tension

To derive a formula for the tension of a D-brane in the membrane formalism, we will use the observation that the tension in open string theory can be computed as the appropriate regulated dimension of the conformal field theory state space corresponding to the one-graviton vertex operators [85]. In three dimensional language, this means that we need to compute the dimension of the Hilbert space \( \mathcal{H}_k(\lambda) \) spanned by the wavefunctionals (12.5) for \( n = 1 \) and for a fixed \( U(1) \) charge \( \lambda \), incorporating all Landau levels \( N \geq 0 \). Of course, this Hilbert space is infinite dimensional, but we can compute its dimension as in (6.6,6.7) by treating the membrane size \( \beta \) as a regulator, performing an appropriate renormalization, and then taking the high-temperature limit \( \beta \to 0 \). The calculation is much different than the derivation of the Verlinde formula described in Section 6.3, because the Hamiltonian of topologically massive gauge theory in higher Landau levels does not vanish and the finite temperature amplitudes now depend explicitly on both \( \beta \) and the insertion points \( z_i \in \Sigma \).

However, the limit \( \beta \to 0 \) shrinks the size of the membrane and so does not properly induce string dynamics. Instead, one should formulate the theory in the dual finite temperature formalism by compactifying the Euclidean time direction on a circle of circumference \( \tilde{\beta} = 1/\mu^2 \beta \), and then take the equivalent limit \( \tilde{\beta} \to \infty \) which decompactifies the membrane [16]. The regulated dimension of the physical state space then computes a modified version of the Verlinde formula defined by

\[
\text{reg dim } \mathcal{H}_k(\lambda) = \lim_{\tilde{\beta} \to \infty} \text{Tr} \mathcal{H}_k(\lambda) \left( e^{-\tilde{\beta} H_D} \right).
\]

The trace in (12.10) can be represented as a membrane inner product of \( n = 1 \) wavefunctionals (12.5) analogous to those used before, which now however requires regularization by an ultraviolet cutoff \( \Lambda \). In the limit \( \tilde{\beta} \to \infty \), \( \Lambda \to \infty \) with \( \ln(\Lambda)/\tilde{\beta} \) held fixed, the topological graviton mass undergoes a finite
renormalization and only the lowest Landau level $N = 0$ contributes to the amplitude. Corresponding to the PT orbifold of the topological membrane, this calculation thereby results in the expression [16]

\[
\text{reg dim } \mathcal{H}_k(\lambda) = \langle \Psi_1^{\lambda}(z) | \Psi_1^{\lambda}(z) \rangle_{\text{ren}} = \frac{|k|}{4\pi}.
\] (12.11)

By using the identification (7.1), the dimension mass formulæ [85]

\[
\mathcal{M}^2 = \alpha' (g_s) \chi(\Sigma) \sqrt{\text{reg dim } \mathcal{H}_k(\lambda)}
\] (12.12)

agrees exactly with the formula for the tension of a D-brane wrapping the target space circle $S^1$ of radius $R$, with $\chi(\Sigma)$ the Euler characteristic of the Riemann surface $\Sigma$. Because of bulk charge conservation, this result is independent of $\lambda$. It also agrees with a direct calculation of orbifold inner products that yield the correct Born-Infeld effective action for the target space string dynamics within the membrane formalism [16].

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References

1. E. Witten, *Comm. Math. Phys.* **121**, 351 (1989).
2. G. Moore and N. Seiberg, *Phys. Lett.* **B220**, 422 (1989).
3. J. Fuchs, I. Runkel and C. Schweigert, *Nucl. Phys.* **B646**, 353 (2002) [hep-th/0204148].
4. G. Moore and N. Seiberg, *Comm. Math. Phys.* **123**, 177 (1989).
5. G. Felder, J. Fröhlich, J. Fuchs and C. Schweigert, *Compos. Math.* **131**, 452 (2002) [hep-th/9912239].
6. I. I. Kogan, *Phys. Lett.* **B231**, 377 (1989).
7. S. Carlip and I. I. Kogan, *Mod. Phys. Lett.* **A6**, 171 (1991).
8. S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, *Nucl. Phys.* **B326**, 108 (1989).
9. J. F. Schonfeld, *Nucl. Phys.* **B185**, 157 (1981).
10. S. Deser, R. Jackiw and S. Templeton, *Ann. Phys.* **140**, 372 (1982).
11. L. Cooper and I. I. Kogan, *Phys. Lett.* **B383**, 271 (1996) [hep-th/9602062].
12. S. Carlip, *Nucl. Phys.* **B362**, 111 (1991).
13. I. I. Kogan, *Nucl. Phys.* **B375**, 362 (1992).
14. I. I. Kogan, A. Momen and R. J. Szabo, *JHEP* **9812**, 013 (1998) [hep-th/9811006].
15. I. I. Kogan, *Phys. Lett.* **B390**, 189 (1997) [hep-th/9608031].
16. P. Castelo Ferreira, I. I. Kogan and R. J. Szabo, *Nucl. Phys.* **B676**, 243 (2004) [hep-th/0308101].
17. E. Witten, *Nucl. Phys.* **B443**, 85 (1995) [hep-th/9503124].
18. P. Hořava and E. Witten, *Nucl. Phys.* **B460**, 506 (1996) [hep-th/9510209].
19. P. Hora and E. Witten, *Nucl. Phys.* B475, 94 (1996) [hep-th/9603142].
20. E. Cremmer, B. Julia and J. Scherk, *Phys. Lett.* B76, 409 (1978).
21. M. O’Loughlin, *Phys. Lett.* B385, 103 (1996) [hep-th/9601179].
22. S. S. Chern and J. Simons, *Ann. Math.* 99, 48 (1974).
23. A. Schwarz, *Lett. Math. Phys.* 2, 247 (1978).
24. M. O’Loughlin, *Phys. Lett.* B385, 103 (1996) [hep-th/9601179].
25. M. O’Loughlin, *Phys. Lett.* B385, 103 (1996) [hep-th/9601179].
26. W. Ogura, *Phys. Lett.* B229, 61 (1989).
27. E. Witten, *Comm. Math. Phys.* 92, 455 (1984).
28. E. Verlinde, *Nucl. Phys.* B300, 360 (1988).
29. P. Goddard, A. Kent and D. Olive, *Comm. Math. Phys.* 103, 105 (1986).
30. N. Sakai and Y. Tani, *Prog. Theor. Phys.* 83, 968 (1990).
31. L. Cooper, I. I. Kogan and R. J. Szabo, *Nucl. Phys.* B498, 492 (1997) [hep-th/9702088].
32. L. Cooper, I. I. Kogan and R. J. Szabo, *Ann. Phys.* 268, 61 (1998) [hep-th/9710179].
33. S. Carlip and I. I. Kogan, *Phys. Rev. Lett.* 64, 1487 (1990).
34. I. I. Kogan and A. Yu. Morozov, *Sov. Phys. JETP* 61, 1 (1985).
35. I. I. Kogan, *Comm. Nucl. Part. Phys.* 19, 305 (1990).
36. G. Dunne, R. Jackiw and C. A. Trugenberger, *Phys. Rev.* D41, 661 (1990).
37. A. P. Polychronakos, *Ann. Phys.* 203, 231 (1990).
38. X. G. Wen, *Int. J. Mod. Phys.* B4, 239 (1990).
39. I. I. Kogan, *Int. J. Mod. Phys.* A9, 3887 (1994) [hep-th/9401093].
40. M. Asorey, F. Falceto and S. Carlip, *Phys. Lett.* B312, 477 (1993) [hep-th/9304081].
41. G. Grignani, P. Sodano, G. W. Semenoff and O. Tirkkonen, *Nucl. Phys.* B489, 360 (1997) [hep-th/9609228].
42. E. Witten, *Comm. Math. Phys.* 144, 189 (1992).
43. S. Axelrod, S. Della Pietra and E. Witten, *J. Diff. Geom.* 33, 787 (1991).
44. P. Castelo Ferreira, I. I. Kogan and R. J. Szabo, *JHEP* 0204, 035 (2002) [hep-th/0112104].
45. A. Cappelli, C. Itzykson and J.-B. Zuber, *Nucl. Phys.* B280, 445 (1987).
46. A. Cappelli, C. Itzykson and J.-B. Zuber, *Comm. Math. Phys.* 113, 1 (1987).
47. M. C. Ashworth, *Mod. Phys. Lett.* A10, 2749 (1995) [hep-th/9510192].
48. A. Alekseev and S. Shatashvili, *Nucl. Phys.* B323, 719 (1989).
49. G. Amelino-Camelia, I. I. Kogan and R. J. Szabo, *Nucl. Phys.* B489, 413 (1996) [hep-th/9607037].
50. I. I. Kogan and R. J. Szabo, *Nucl. Phys.* B502, 383 (1997) [hep-th/9703071].
51. E. Witten, *Nucl. Phys.* B311, 46 (1988/9).
52. S. Carlip, *Phys. Rev.* D45, 3584 (1992) [hep-th/9109006].
53. S. Carlip and I. I. Kogan, *Phys. Rev. Lett.* 67, 3647 (1991) [hep-th/9110005].
54. I. I. Kogan, *Phys. Lett.* B256, 369 (1991).
55. A. B. Zamolodchikov, *Sov. J. Nucl. Phys.* 46, 1090 (1987).
56. A. A. Migdal and J. L. Cardy, *Nucl. Phys.* B285, 687 (1987).
57. M. Blau and G. Thompson, *Nucl. Phys.* B408, 345 (1993) [hep-th/9305010].
58. F. Falceto and K. Gawedzki, *Lett. Math. Phys.* 38, 155 (1996) [hep-th/9502181].
59. F. Falceto and K. Gawedzki, *Comm. Math. Phys.* 183, 267 (1997) [hep-th/9604094].
60. K. Gawedzki and P. Tran-Ngoc-Bich, *J. Math. Phys.* 41, 4695 (2000) [hep-th/9803101].
61. K. S. Narain, *Phys. Lett.* B169, 41 (1986).
62. K. S. Narain, M. H. Sarmadi and E. Witten, *Nucl. Phys.* B279, 369 (1987).
63. A. M. Polyakov, *Mod. Phys. Lett.* A3, 325 (1988).
64. M. Lüscher, *Nucl. Phys.* **B326**, 557 (1989).
65. K. Lee, *Nucl. Phys.* **B373**, 735 (1992).
66. I. I. Kogan and A. Kovner, *Phys. Rev.* **D51**, 1948 (1995) [hep-th/9410067].
67. B. Sathiapalan, *Phys. Rev.* **D35**, 3277 (1987).
68. I. I. Kogan, *JETP Lett.* **45**, 709 (1987).
69. A. A. Abrikosov, Jr. and I. I. Kogan, *Sov. Phys. JETP* **96**, 418 (1989).
70. M. Bergeron and G. W. Semenoff, *Ann. Phys.* **245**, 1 (1996) [hep-th/9306050].
71. E. C. Marino, *Phys. Rev.* **D38**, 3194 (1988).
72. L. Cooper, I. I. Kogan and K.-M. Lee, *Phys. Lett.* **B394**, 67 (1997) [hep-th/9611107].
73. P. Castelo Ferreira, I. I. Kogan and B. Tekin, *Nucl. Phys.* **B589**, 167 (2000) [hep-th/0004078].
74. J. A. Harvey and J. A. Minahan, *Phys. Lett.* **B188**, 44 (1987).
75. G. Pradisi and A. Sagnotti, *Phys. Lett.* **B216**, 59 (1989).
76. P. Hořava, *Nucl. Phys.* **B327**, 461 (1989).
77. J. L. Cardy and D. C. Lewellen, *Phys. Lett.* **B259**, 274 (1991).
78. P. Hořava, *J. Geom. Phys.* **21**, 1 (1996) [hep-th/9404101].
79. P. Castelo Ferreira and I. I. Kogan, *JHEP* **0106**, 056 (2001) [hep-th/0012188].
80. N. Ishibashi, *Mod. Phys. Lett.* **A4**, 251 (1989).
81. R. E. Behrend, P. A. Pearce, V. B. Petkova and J.-B. Zuber, *Nucl. Phys.* **B579**, 707 (2000) [hep-th/9908036].
82. M. R. Gaberdiel, A. Recknagel and G. M. T. Watts, *Nucl. Phys.* **B626**, 344 (2002) [hep-th/0108102].
83. C. G. Callan, Jr., I. R. Klebanov, A. W. W. Ludwig and J. M. Maldacena, *Nucl. Phys.* **B422**, 417 (1994) [hep-th/9402113].
84. J. L. Cardy, *Nucl. Phys.* **B324**, 581 (1989).
85. J. A. Harvey, S. Kachru, G. Moore and E. Silverstein, *JHEP* **0003**, 001 (2000) [hep-th/9909072].