Anisotropic Sunyaev-Zel’dovich Effect

Sujan Sengupta
Indian Institute of Astrophysics, Koramangala, Bangalore 560 034, India

(November 18, 2018)

The spectral distortion in the Cosmic Microwave Background Radiation (CMBR) caused from the inverse Compton scattering of low energy CMB photons by thermally distributed hot electrons in massive clusters of galaxies - better known as the thermal Sunyaev-Zel’dovich effect (TSZE) [1,2] results in a systematic transfer of photons from the Rayleigh-Jeans to the Wien side of the spectrum causing a distortion in the Planckian nature of the spectrum. The measurements of the effect yield directly the properties of the hot intra-cluster gas, the total dynamical mass of the cluster as well as the indirect information on the cosmological evolution of the clusters. The effect is also used as an important tool to determine the Hubble constant $H_0$ and the density parameter $\Omega_0$ of the Universe [3-9]. Recent interferometric imaging of the TSZE [10,11] has been used to estimate the mass of various galaxy clusters which could constrain the cosmological parameters of structure formation models. Measurements of TSZE and the kinetic Sunyaev Zel’dovich effect (KSZE) from the Sunyaev-Zel’dovich Infrared Experiment and OVRO/BIMA determine the central Compton y parameter and constrain the radial peculiar velocity of the clusters. In near future sensitive observations of the effect with ground based and balloon-borne telescopes, equipped with bolometric multi-frequency arrays, are expected to yield high quality measurements. However, in modeling the observational data, the analytical solution of Kompaneets equation [12] with relativistic corrections are used in general. The Kompaneets equation describes the isotropic intensity of the radiation field after scattering. This means the expression used so far does not describe the anisotropy or the angle-dependence of the radiation after scattering. Hence the angle dependence of the spectral distortion in CMBR is completely ignored in any theoretical discussion or interpretation of the observed data in spite of the fact that scattering of an initially isotropic radiation should become anisotropic. In this letter I show that the thermal Sunyaev-Zel’dovich effect is strongly angle dependent.

Chandrasekhar [13] has provided the radiative transfer equations for the anisotropic Compton scattering by assuming that the electron energy is much less than the photon energy. On the other hand, for the scattering of CMBR, the photon energy is much less than the electron energy. In fact, in the TSZE, the electrons are considered to have relativistic motion described by relativistic Maxwellian distribution. The relevant radiative transfer equations that describe the anisotropic Compton scattering of low energy photons can be written as [14-16]

$$\frac{\partial I(\mu, \nu, z)}{\partial \tau(z)} = -I(\mu, \nu, z) + \frac{\omega_0}{2} \int_{-1}^{1} \left[ P_0 - \frac{2kT_e}{m_e c^2} P_1 + \frac{2kT_e}{m_e c^2} \left( \nu^2 \frac{\partial^2}{\partial \nu^2} - 2\nu \frac{\partial}{\partial \nu} \right) P_2 \right] I(\mu, \nu, z) d\mu' . \tag{1}$$

Here $\omega_0$ is the albedo for single scattering, $\mu = \cos \theta$ where $\theta$ is the angle between the axis of symmetry ($z$ axis) in the rest frame of electron gas and the ray path, $\tau(z)$ is the total optical depth along the axis of symmetry given by $d\tau(z) = \int \sigma_T n_e(z) dz$, $\sigma_T$ and $n_e$ being the Thomson scattering cross section and the electron number density respectively. $k$, $c$, $m_e$, $\nu$ and $T_e$ are Boltzmann constant, velocity of light, electron rest mass, frequency of the photon and the temperature of the electron respectively. $\omega_0 = 1$ for a purely scattering medium. The phase functions $P_0$, $P_1$ and $P_2$ are given as :

$$P_0(\mu, \mu') = \frac{3}{8} \left[ 3 - \mu^2 - \mu'^2 (1 - 3\mu^2) \right] , \tag{2}$$

$$P_1(\mu, \mu') = -\frac{3}{4} \left[ 1 - \mu^2 \right] \frac{\partial P_0}{\partial \mu} \tag{3}$$

$$P_2(\mu, \mu') = -\frac{3}{16} \left[ 1 - 2\mu^2 \right] \frac{\partial P_0}{\partial \mu} \tag{4}$$

Inverse Compton scattering of the Cosmic Microwave Background Radiation (CMBR) by hot intra-cluster gas - better known as the thermal Sunyaev-Zel’dovich effect (TSZE) [1,2] results in a systematic transfer of photons from the Rayleigh-Jeans to the Wien side of the spectrum causing a distortion in the Planckian nature of the spectrum. The measurements of the effect yield directly the properties of the hot intra-cluster gas, the total dynamical mass of the cluster as well as the indirect information on the cosmological evolution of the clusters. The effect is also used as an important tool to determine the Hubble constant $H_0$ and the density parameter $\Omega_0$ of the Universe [3-9]. Recent interferometric imaging of the TSZE [10,11] has been used to estimate the mass of various galaxy clusters which could constrain the cosmological parameters of structure formation models. Measurements of TSZE and the kinetic Sunyaev Zel’dovich effect (KSZE) from the Sunyaev-Zel’dovich Infrared Experiment and OVRO/BIMA determine the central Compton y parameter and constrain the radial peculiar velocity of the clusters. In near future sensitive observations of the effect with ground based and balloon-borne telescopes, equipped with bolometric multi-frequency arrays, are expected to yield high quality measurements. However, in modeling the observational data, the analytical solution of Kompaneets equation [12] with relativistic corrections are used in general. The Kompaneets equation describes the isotropic intensity of the radiation field after scattering. This means the expression used so far does not describe the anisotropy or the angle-dependence of the radiation after scattering. Hence the angle dependence of the spectral distortion in CMBR is completely ignored in any theoretical discussion or interpretation of the observed data in spite of the fact that scattering of an initially isotropic radiation should become anisotropic. In this letter I show that the thermal Sunyaev-Zel’dovich effect is strongly angle dependent.

Chandrasekhar [13] has provided the radiative transfer equations for the anisotropic Compton scattering by assuming that the electron energy is much less than the photon energy. On the other hand, for the scattering of CMBR, the photon energy is much less than the electron energy. In fact, in the TSZE, the electrons are considered to have relativistic motion described by relativistic Maxwellian distribution. The relevant radiative transfer equations that describe the anisotropic Compton scattering of low energy photons can be written as [14-16]

$$\frac{\partial I(\mu, \nu, z)}{\partial \tau(z)} = -I(\mu, \nu, z) + \frac{\omega_0}{2} \int_{-1}^{1} \left[ P_0 - \frac{2kT_e}{m_e c^2} P_1 + \frac{2kT_e}{m_e c^2} \left( \nu^2 \frac{\partial^2}{\partial \nu^2} - 2\nu \frac{\partial}{\partial \nu} \right) P_2 \right] I(\mu, \nu, z) d\mu' . \tag{1}$$

Here $\omega_0$ is the albedo for single scattering, $\mu = \cos \theta$ where $\theta$ is the angle between the axis of symmetry ($z$ axis) in the rest frame of electron gas and the ray path, $\tau(z)$ is the total optical depth along the axis of symmetry given by $d\tau(z) = \int \sigma_T n_e(z) dz$, $\sigma_T$ and $n_e$ being the Thomson scattering cross section and the electron number density respectively. $k$, $c$, $m_e$, $\nu$ and $T_e$ are Boltzmann constant, velocity of light, electron rest mass, frequency of the photon and the temperature of the electron respectively. $\omega_0 = 1$ for a purely scattering medium. The phase functions $P_0$, $P_1$ and $P_2$ are given as :

$$P_0(\mu, \mu') = \frac{3}{8} \left[ 3 - \mu^2 - \mu'^2 (1 - 3\mu^2) \right] , \tag{2}$$

$$P_1(\mu, \mu') = -\frac{3}{4} \left[ 1 - \mu^2 \right] \frac{\partial P_0}{\partial \mu} \tag{3}$$

$$P_2(\mu, \mu') = -\frac{3}{16} \left[ 1 - 2\mu^2 \right] \frac{\partial P_0}{\partial \mu} \tag{4}$$

Inverse Compton scattering of the Cosmic Microwave Background Radiation (CMBR) by hot intra-cluster gas - better known as the thermal Sunyaev-Zel’dovich effect (TSZE) [1,2] results in a systematic transfer of photons from the Rayleigh-Jeans to the Wien side of the spectrum causing a distortion in the Planckian nature of the spectrum. The measurements of the effect yield directly the properties of the hot intra-cluster gas, the total dynamical mass of the cluster as well as the indirect information on the cosmological evolution of the clusters. The effect is also used as an important tool to determine the Hubble constant $H_0$ and the density parameter $\Omega_0$ of the Universe [3-9]. Recent interferometric imaging of the TSZE [10,11] has been used to estimate the mass of various galaxy clusters which could constrain the cosmological parameters of structure formation models. Measurements of TSZE and the kinetic Sunyaev Zel’dovich effect (KSZE) from the Sunyaev-Zel’dovich Infrared Experiment and OVRO/BIMA determine the central Compton y parameter and constrain the radial peculiar velocity of the clusters. In near future sensitive observations of the effect with ground based and balloon-borne telescopes, equipped with bolometric multi-frequency arrays, are expected to yield high quality measurements. However, in modeling the observational data, the analytical solution of Kompaneets equation [12] with relativistic corrections are used in general. The Kompaneets equation describes the isotropic intensity of the radiation field after scattering. This means the expression used so far does not describe the anisotropy or the angle-dependence of the radiation after scattering. Hence the angle dependence of the spectral distortion in CMBR is completely ignored in any theoretical discussion or interpretation of the observed data in spite of the fact that scattering of an initially isotropic radiation should become anisotropic. In this letter I show that the thermal Sunyaev-Zel’dovich effect is strongly angle dependent.
\[ P_1(\mu, \mu') = \frac{3}{8} \left[ 1 - 3\mu'^2 - 3\mu^2(1 - 3\mu'^2) + 2\mu^3\mu'(3 - 5\mu'^2) + 2\mu\mu'(3\mu^2 - 1) \right] , \tag{3} \]

and

\[ P_2(\mu, \mu') = \frac{3}{8} \left[ 3 - \mu^2 - \mu'^2 + \mu\mu'(3\mu'_2 - 5 + 3\mu^2 + 3\mu^2 - 5\mu^2\mu'^2) \right] . \tag{4} \]

For an isotropic radiation field \( \frac{1}{2} \int_{-1}^{1} I(\mu', \nu) d\mu' = I(\nu), \frac{1}{2} \int_{-1}^{1} I(\mu', \nu) \mu'^2 d\mu' = \frac{1}{4} I(\nu) \) and \( \int_{-1}^{1} I(\mu', \nu) \mu'^3 d\mu' = \int_{-1}^{1} I(\mu, \nu) \mu'^3 d\mu' = 0 \) Therefore, for isotropic scattering with \( s \) as the ray path that does not change after scattering, equation (1) reduces to

\[ \frac{\partial I(\nu)}{\partial \tau(s)} = \frac{2kT_e}{m_e c^2} \left( \nu^2 \frac{\partial^2 I(\nu)}{\partial \nu^2} - 2\nu \frac{\partial I(\nu)}{\partial \nu} \right) \tag{5} \]

which is well known as the Kompaneets equation [12] for low frequency. Note that for plane parallel medium, the operator \( \partial / \partial s \) becomes \( \mu \partial / \partial z \) [13] which incorporates the change in the ray path with respect to the \( z \) axis after scattering.

The analytical solution of equation (5) provides the thermal component of the distortion \( \Delta I \) and is written as

\[ \Delta I(x) = I_0 y \frac{x^4 e^x}{(e^x - 1)^2} \left[ \frac{x(e^x + 1)}{e^x - 1} - 4 \right] (1 + \delta \tau) \tag{6} \]

where \( x = \frac{h\nu}{kT_{CMBR}}, I_0 = 2(kT_{CMBR})^3/(hc)^2 \). The term \( y = \frac{2kT_e}{m_e c^2} \int \sigma_T n_e ds \) is usually referred to as the Compton \( y \) parameter and \( \delta \tau \) is a relativistic correction to the thermal effect [17] significant if \( kT_e > 10 \text{KeV} \).

The above expression is usually used in order to estimate the TSZE. Clearly, the angle dependence of the emergent intensity and hence the distortion is neglected as the radiation field after scattering is assumed to remain isotropic.

Now equation (1) is a coupled integro partial differential equation and so it cannot be solved analytically. I solve it numerically by discretization method. In this method the medium is divided into several shells and the integration is performed over two dimensional grids of angular and radial points. For the angle integration I have adopted an eight point Gauss-Legendre roots and weights. I have taken eighty frequency points with equal spacing. For the initial condition, I have provided equal amount of intensity corresponding to \( T_{CMBR} = 2.728 \text{K} \) at \( \tau = 0 \) along all directions i.e., \( I(\mu, \nu, \tau = 0) = \frac{2kT_e}{c^2 \sigma_T n_e} (e^x - 1)^{-1} \). The code is thoroughly tested for stability and flux conservation. The numerical results coincide with the analytical solution given by equation (6) when the radiation is made isotropic. On the other hand, if \( P_1 = P_2 = 0 \), the results are well matched with that presented in [13] for Thomson scattering.

Usually the isothermal \( \beta \) models [18] is considered for the density distribution of the clusters. The spherical isothermal model density is described by

\[ n_e(r) = n_0 \left( 1 + \frac{r^2}{r_c^2} \right)^{-3\beta/2} \tag{7} \]

where the core radius \( r_c \) and \( \beta \) are shape parameters, \( n_0 \) is the central electron density. However, for the present purpose it is sufficient to consider an isothermal, homogeneous and plane parallel medium with a constant optical depth. In the present work I have taken a constant value of \( n_e = 10^{-3} \) and the size of the cluster is 4 Mpc. The electron temperature \( T_e \) is taken to be 1 KeV and 10 KeV. The results are presented graphically in Fig. 1 and in Fig. 2.

The spectral distortion for the isotropic case is characterized by three distinct frequencies: the crossover frequency \( x_0 = 3.83 \) where the TSZE vanishes; \( x_{\text{min}} = 2.26 \) which gives the minimum decrement of the CMB intensity and \( x_{\text{max}} = 6.51 \) which gives the maximum distortion due to this effect. The value of \( x_0 \) is however pushed to higher values of \( x \) with the increase in \( T_e \) for the relativistic case.

First of all there is no change in the values of \( x_0, x_{\text{min}} \) and \( x_{\text{max}} \) in any direction even if the radiation field is anisotropic. In order to show this, the result for the isotropic scattering is also presented in Fig. 1 and in Fig. 2. I have taken the ray path for the isotropic case along the axis of symmetry. The values of the three characteristic frequency points remain unchanged because of the fact that the frequency dependence of the distortion is independent of the angle dependence of the distortion as can be seen from equation (1).

\[ ^1 \text{In Ref. [15] } I_0 = \frac{1}{2} I \text{ should be } I = \frac{1}{2} I_0. \]
FIG. 1. Anisotropic Sunyaev-Zel’dovich effect with $kT_e = 1 \text{ KeV}$. From top to bottom the solid lines represent the spectral distortion due to anisotropic scattering along the angular directions (1) $\mu = 0.1$, (2) $\mu = 0.24$, (3) $\mu = 0.4$, (4) $\mu = 0.6$, (5) $\mu = 0.76$, (6) $\mu = 0.9$, (7) $\mu = 0.98$. The dashed line represent the distortion due to isotropic scattering with the ray path being along the axis of symmetry.

FIG. 2. Same as Fig.1 but with $kT_e = 10 \text{ KeV}$. 

3
It is worth mentioning at this point that the photon changes its direction each time it gets scattered. With respect to the z axis, the new ray path becomes $s = z/\mu$. As a consequence, the optical depth through the new ray path becomes $d\tau(z)/\mu$ where $d\tau(z)$ corresponds to the optical depth along the z axis. Therefore, for anisotropic scattering it is incorrect to take $\partial I/\partial \tau(s)$ instead of $\mu \partial I/\partial \tau(z)$ in the left hand side of equation (1) because $\partial I/\partial \tau(s)$ describes the change in the intensity along the same ray path $s$ before and after scattering. Flux conservation confirms this fact.

The Compton $y$ parameter, for a single anisotropic scattering in a plane-parallel medium, therefore, can be defined as

$$y = \frac{1}{\mu} \frac{2kT_e}{m_ec^2} \int \sigma_T n_e(z) dz$$

where $\mu$ is now the angle between the incident direction to the emergent angle of the radiation. In reality, however, the photon would suffer multiple scattering depending on the density of the medium and hence would change its direction several times before it emerges out. If the medium is spherically symmetric then the situation becomes more complicated. In plane parallel geometry the ray makes a constant angle $\theta$ with the normal while in the spherically symmetric geometry the angle made by the ray direction and the radius vector changes constantly.

Now, the most interesting and important message conveyed by Fig.1 and Fig. 2 is that the spectral distortion is strongly angle dependent. The degree of distortion in the CMBR due to TSZE changes drastically with the change in the angular direction of the photons. This clearly demonstrates the fact that the TSZE produces anisotropic distortion in the CMBR spectrum.

The degree of distortion increases as the angle between the emergent intensity and the axis of symmetry increases. Therefore the minimum decrement and maximum distortion of CMBR occur at $x = 2.26$ and $x = 6.51$ respectively for the radiation emerging almost perpendicular to the axis of symmetry. For a single scattering this can be explained easily. With the increase in $\theta$ and hence with the decrease in $\mu$ the Compton $y$ parameter increases. As a result the distortion increases when $\theta$ increases. For a plane parallel medium the Compton $y$ parameter becomes infinite when $\theta = \pi/2$ and hence the distortion becomes infinite at $\theta = \pi/2$. It should be mentioned here that the results presented in Fig. 1 and in Fig. 2 incorporate multiple scattering of photons incident and emergent at any angular direction before and after scattering. The radiation traversing at the opposite direction, i.e., the backscattered radiation is also taken into care while solving equation (1) numerically. The qualitative nature of the result will not alter if we consider spherical symmetry except the fact that the ray would peak with the radius vector towards the outer boundary of the sphere. The results with spherical symmetry will be published in a forthcoming paper.

Fig. 1 and Fig. 2 show that the spectral distortion due to isotropic scattering matches with that due to anisotropic scattering when $\mu = 0.76$. Therefore the distortion due to anisotropic scattering would be less as compared to that due to isotropic scattering if $\mu < 0.76$. Now, the spectral distortion along large angular direction should be irrelevant from the observational point of view. But the important point to be noted is that- if the effect is measured along the line of sight ( which can be assumed as the axis of symmetry) then for the same value of the Compton $y$ parameter, the distortion due to anisotropic scattering would be much less than that calculated under isotropic assumption. In other words, the spectral distortion along the line of sight would result into underestimation of the Compton $y$ parameter if isotropic scattering is assumed.

Therefore it is extremely important to know the direction of the observed intensity with respect to the axis of symmetry in the rest frame of the electron gas while estimating the density of the clusters. The isotropic assumption would either underestimate or overestimate it depending on the angular direction of the emergent intensity of CMBR.

In conclusion I would like to emphasize that the anisotropy in the CMBR, induced by Compton scattering from hot electrons in the intra-cluster, not only yields anisotropic distortion in the Planckian spectrum but also would results into polarization in the CMBR.

I am thankful to P. Bhattacharjee for bringing my attention on this problem and for discussions.
[7] J. P. Hughes and M. Birkinshaw, Astrophys. J. 501, 1 (1998).
[8] E. D. Reese, et al., Astrophys. J. 533, 38 (2000).
[9] T. Herbig, C. R. Lawrence, A. C. S. Readhead and S. Guikus, Astrophys. J. 449, L5 (1995).
[10] M. Joy, et al., Astrophys. J. 551, L1 (2001).
[11] S. J. LaRoque, J. E. Carlstrom, E. D. Reese, G. P. Holder, W. L. Holzapfel, M. Joy and L. Grego, astro-ph/0204124 (2002).
[12] A. S. Kompaneets, Soviet Phys.- JETP, 4, 730 (1957).
[13] S. Chandrasekhar, Radiative Transfer (Dover, New York, 1960).
[14] J. P. Babuel-Peyrissac and G. Rouvillois, J. Quantum Spectroscopy and Radiative Transfer, 10, 1277 (1970).
[15] R. F. Stark, MNRAS, 195, 115 (1981).
[16] F. K. Hansen and P. B. Lilje, MNRAS, 306, 153 (1999).
[17] N. Itoh, Y. Kohyama and S. Nozawa, Astrophys. J. 502, 7 (1998).
[18] A. Cavaliere, and R. Fusco-Femiano, Astron. & Astrophys. 49, 137 (1996); Astron & Astrphys. 100, 194 (1981).