ON THE MULTIGRADED HILBERT AND POINCARÉ-BETTI SERIES AND THE GOLOD PROPERTY OF MONOMIAL RINGS

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Abstract. In this paper we study the multigraded Hilbert and Poincaré-Betti series of $A = S/\mathfrak{a}$, where $S$ is the ring of polynomials in $n$ indeterminates divided by the monomial ideal $\mathfrak{a}$. There is a conjecture about the multigraded Poincaré-Betti series by Charalambous and Reeves which they proved in the case, where the Taylor resolution is minimal. We introduce a conjecture about the minimal $A$-free resolution of the residue class field and show that this conjecture implies the conjecture of Charalambous and Reeves and, in addition, gives a formula for the Hilbert series. Using Algebraic Discrete Morse theory, we prove that the homology of the Koszul complex of $A$ with respect to $x_1, \ldots, x_n$ is isomorphic to a graded commutative ring of polynomials over certain sets in the Taylor resolution divided by an ideal $\mathfrak{r}$ of relations. This leads to a proof of our conjecture for some classes of algebras $A$. We also give an approach for the proof of our conjecture via Algebraic Discrete Morse theory in the general case.

The conjecture implies that $A$ is Golod if and only if the product (i.e. the first Massey operation) on the Koszul homology is trivial. Under the assumption of the conjecture we finally prove that a very simple purely combinatorial condition on the minimal monomial generating system of $\mathfrak{a}$ implies Golodness for $A$.

1. Introduction

In this note, we study the multigraded Hilbert and Poincaré-Betti series of algebras $A = S/\mathfrak{a}$, where $S$ is the commutative polynomial ring in $n$ indeterminates and $\mathfrak{a}$ is a monomial ideal with minimal monomial generating system $\text{MinGen}(\mathfrak{a}) := \{m_1, \ldots, m_l\}$.

Recall that the multigraded Poincaré-Betti series $P^A_k(\underline{x}, t)$ and $\text{Hilb}_A(\underline{x}, t)$ of $A$ are defined as

$$P^A_k(\underline{x}, t) := \sum_{i=0}^{\infty} \sum_{\alpha \in \mathbb{N}^n} \dim_k(\text{Tor}^A_i(k,k)^{\alpha}) \underline{x}^\alpha t^i,$$

$$\text{Hilb}_A(\underline{x}, t) := \sum_{i=0}^{\infty} \sum_{\alpha \in \mathbb{N}^n \mid |\alpha| = i} \dim_k(A^{\alpha}) \underline{x}^\alpha t^i.$$

In [5] Charalambous and Reeves proved that in the case where the Taylor resolution of $\mathfrak{a}$ over $S$ is minimal the Poincaré-Betti series takes the following form:

$$P^A_k(\underline{x}, t) = \prod_{i=1}^{n} (1 + x_i t) \frac{1}{1 + \sum_{I \subset \{1, \ldots, i\}} (-1)^{cl(I)} m_I t^{cl(I) + |I|}};$$

where $cl(I)$ is the number of equivalence classes of $I$ with respect to the relation defined as the transitive closure of $i \sim j :\Leftrightarrow \gcd(m_i, m_j) \neq 1$ and $m_I := \text{lcm}(m_i \mid
\( i \in I \) is the least common multiple.

In the general case, they conjecture that

\[
P_k^A(x, t) = \frac{\prod_{i=1}^{n}(1 + x_i t)}{1 + \sum_{I \subset [l]} \sum_{I \in U} (-1)^{cl(I)} m_I t^{cl(I)+|I|}},
\]

where \([l] = \{1, \ldots, l\}\) and \(U \subset 2^{[l]}\) is the “basis”-set. However, the conjecture does not include a description of the basis-set \(U\).

Using Algebraic Discrete Morse theory (see [10]), we are able to specify the basis-set \(U\) and prove the conjecture in several cases. In fact, we give a general conjecture about the multigraded minimal \(A\)-free resolution of \(k\) over \(A\). This conjecture implies in these cases an explicit description of the multigraded Hilbert and Poincaré-Betti series, hence it implies the conjecture by Charalambous and Reeves.

Section 2 recalls Algebraic Discrete Morse theory. For more details and a proof see [10].

In Section 3 we apply Algebraic Discrete Morse theory to the Taylor resolution. We define a standard matching which we need for the formulation of our conjecture, and we define special acyclic matchings for ideals generated in degree two. In particular, we define matchings (not necessarily acyclic) for Stanley Reisner ideals of order complexes of a partially ordered set.

In Section 4 we formulate our conjecture on the multigraded minimal resolution of \(k\) as an \(A\)-module and we show that our conjecture gives an explicit form of the multigraded Hilbert and Poincaré-Betti series. This generalizes the conjecture by Charalambous and Reeves. We say that an algebra \(A\) has property (P) (resp. (H)) if the multigraded Poincaré-Betti series (resp. multigraded Hilbert series) has the conjectured form.

In Section 5 we give a description of the Koszul homology \(H_\bullet(K^A)\) of the Koszul complex over \(A\) with respect to the sequence \(x_1, \ldots, x_n\) in terms of a standard matching on the Taylor resolution. We need this description later in the proof of our conjecture.

In Section 6 we prove that the Stanley Reisner ring \(A = k[\Delta]\), where \(\Delta = \Delta(P)\) is the order complex of a partially ordered set \(P\), satisfies property (P) and property (H).

In the first subsection of Section 7 we prove our conjecture for algebras for which \(H_\bullet(K^A)\) is an \(M\)-ring, a notion introduced by Fröberg [3]. Using a theorem of Fröberg, we also prove property (P) for algebras \(A = S/\mathfrak{a}\) for which in addition the minimal free resolution of \(\mathfrak{a}\) carries the structure of a differential-graded algebra. In the second part we prove our conjecture for all Koszul algebras \(A\). Note that this, as a particular case, gives another proof that \(A = k[\Delta]\) satisfies property (P) and (H).

Finally, we explain why our conjecture makes sense in general. We generalize the Massey operation in order to get an explicit description of the Eagon complex. On this complex we define an acyclic matching. If the resulting Morse complex is minimal, one has to find an isomorphism to the conjectured complex. We give some ideas on how to construct this isomorphism. This construction justifies our conjecture.
Since an algebra is Golod if and only if

\[ P^A_k(x, t) = \frac{\prod_{i=1}^{n}(1 + x_i t)}{1 - t \sum_{\beta_{\alpha,i} \neq 0} \beta_{\alpha,i} x^\alpha t^i}, \]

where \( \beta_{i,\alpha} := \dim_k (\text{Tor}^S_i(A, k)_{\alpha}) \), we can give some applications to the Golod property of monomial rings in the last section of this note. We prove, under the assumption of property (P), that \( A \) is Golod if and only if the first Massey operation is trivial. In addition we give, again under the assumption of property (P), a very simple, purely combinatorial condition on the minimal monomial generating system \( \text{MinGen}(a) \) which implies Golodness. We conjecture that this is an equivalence. This would imply that, in the monomial case, Golodness is independent of the characteristic of the residue class field \( k \).

Recently, Charalambous proved in [4] that if

\[ P^A_k(x, t) = \frac{\prod_{i=1}^{n}(1 + x_i t)}{Q_R(x, t)} \quad \text{with} \quad Q_R(x, t) = \sum \left( \sum c_{\alpha} x^\alpha \right) t^i, \]

then \( x^\alpha \) equals to a least common multiple of a subset of the minimal monomial generating system \( \text{MinGen}(a) \). However an explicit form of \( Q_R(x, t) \) in terms of subsets of \( \text{MinGen}(a) \) is still not known.

In addition, Charalambous proves a new criterion for generic ideals to be Golod. In Section 8 we reprove this criterion using our approach.

In another recent paper, Berglund gives an explicit form of the denominator \( Q_R(x, t) \) in terms of the homology of certain simplicial complexes. Since there seems to be no obvious connection of the approach taken in [2] and our approach, it is an interesting problem to link these two methods.

2. Algebraic Discrete Morse Theory

In this section we recall Algebraic Discrete Morse theory from [10].

Let \( R \) be a ring and \( C_* = (C_i, \partial_i)_{i \geq 0} \) be a chain complex of free \( R \)-modules \( C_i \).

We choose a basis \( X = \bigcup_{i=0}^{n} X_i \) such that \( C_i \simeq \bigoplus_{c \in X_i} R \cdot c \). From now on we write the differentials \( \partial_i \) with respect to the basis \( X \) in the following form:

\[ \partial_i : \begin{cases} C_i & \rightarrow & C_{i-1} \\ c & \mapsto & \partial_i(c) = \sum_{c' \in X_{i-1}} [c : c'] \cdot c'. \end{cases} \]

Given the complex \( C_* \) and the basis \( X \), we construct a directed, weighted graph \( G(C_*) = (V, E) \). The set of vertices \( V \) of \( G(C_*) \) is the basis \( V = X \) and the set \( E \) of (weighted) edges is given by the rule

\[(c, c', [c : c']) \in E \iff c \in X_i, c' \in X_{i-1}, \text{ and } [c : c'] \neq 0.\]

We often omit the weight and write \( e \rightarrow c' \) to denote an edge in \( E \). Also by abuse of notation we write \( e \in G(C_*) \) to indicate that \( e \) is an edge in \( E \).

**Definition 2.1.** A subset \( \mathcal{M} \subset E \) of the set of edges is called an acyclic matching if it satisfies the following three conditions:

1. (Matching) Each vertex \( v \in V \) lies in at most one edge \( e \in \mathcal{M} \).
2. (Invertibility) For all edges \( (c, c', [c : c']) \in \mathcal{M} \) the weight \( [c : c'] \) lies in the center of \( R \) and is a unit in \( R \).
(3) (Acyclicity) The graph $G_M(V,E_M)$ has no directed cycles, where $E_M$ is given by

$$E_M := (E \setminus M) \cup \left\{ (c', c, \frac{-1}{c : c'}) \mid (c, c', [c : c']) \in M \right\}.$$ 

For an acyclic matching $M$ on the graph $G(C) = (V,E)$ we introduce the following notation.

1. We call a vertex $c \in V$ critical with respect to $M$ if $c$ does not lie in an edge $e \in M$; we write

$$X_i^M := \{ c \in X_i \mid c \text{ critical} \}$$

for the set of all critical vertices of homological degree $i$.

2. We write $c' \leq c$ if $c \in X_i$, $c' \in X_{i-1}$, and $[c : c'] \neq 0$.

3. $\text{Path}(c, c')$ is the set of paths from $c$ to $c'$ in the graph $G_M(C)$.

4. The weight $w(p)$ of a path $p = c_1 \to \cdots \to c_r \in \text{Path}(c_1, c_r)$ is given by

$$w(c_1 \to \cdots \to c_r) := \prod_{i=1}^{r-1} w(c_i \to c_{i+1}),$$

$$w(c \to c') := \begin{cases} -\frac{1}{[c : c']} & , \quad c \leq c', \\ [c : c'] & , \quad c' \leq c. \end{cases}$$

5. We write $\Gamma(c, c') = \sum_{p \in \text{Path}(c, c')} w(p)$ for the sum of weights of all paths from $c$ to $c'$.

Now we are in position to define a new complex $C^M_*$, which we call the Morse complex of $C_*$ with respect to $M$. The complex $C^M_* = (C^M_i, \partial^M_i)_{i \geq 0}$ is defined by

$$C^M_i := \bigoplus_{c \in X_i^M} Rc,$$

$$\partial^M_i : \left\{ \begin{array}{c} C^M_i \to C^M_{i-1} \\ c \mapsto \sum_{c' \in X_{i-1}^M} \Gamma(c, c')c', \end{array} \right.$$ 

**Theorem 2.2.** $C^M_*$ is a complex of free $R$-modules and is homotopy-equivalent to the complex $C_*$; in particular, for all $i \geq 0$

$$H_i(C_*) \cong H_i(C^M_*).$$

The maps defined below give a chain homotopy between $C_*$ and $C^M_*$:

$$f : \left\{ \begin{array}{c} C_* \to C^M_* \\ c \in X_i \mapsto f(c) := \sum_{c' \in X_i^M} \Gamma(c, c')c', \end{array} \right.$$ 

$$g : \left\{ \begin{array}{c} C^M_* \to C_* \\ c \in X_i^M \mapsto g_i(c) := \sum_{c' \in X_i} \Gamma(c, c')c'. \end{array} \right.$$ 

Sometimes we consider the same construction for matchings which are not acyclic. Clearly, Theorem 2.2 does not hold anymore for $C^M_*$ if $M$ is not acyclic. In general, there is not even a good definition of the differentials $\partial^M_i$. But for calculating invariants it is sometimes useful to consider $C^M_*$ for matchings that are not acyclic. In these cases we consider just the vector space $C^M_i$. 
3. Algebraic Discrete Morse Theory on the Taylor Resolution

In this section we consider acyclic matchings on the Taylor resolution. First, we introduce a standard matching, which we use in later order to formulate and prove our conjecture. Then Section 3.2 considers the Taylor resolution for monomial ideals which are generated in degree two. The resolutions of those ideals are important for the proof of our conjecture in the case where $A$ is Koszul (see Section 7). Next, we give a matching on the Taylor resolution of Stanley Reisner ideals of the order complex of a partially ordered set, which we use in Section 6 in order to prove property (P) and property (H) for this type of ideal. Finally, we introduce the (strong) gcd-condition for monomial ideals and give a special acyclic matching on the Taylor resolution for this type of ideals, which are in connection with the Golod property of monomial rings (see Section 5).

3.1. Standard Matching on the Taylor Resolution. Let $S = k[x_1, \ldots, x_n]$ be the commutative polynomial ring over a field $k$ of arbitrary characteristic and $a \subseteq S$ a monomial ideal.

The basis of the Taylor resolution is given by the subsets $I \subseteq \text{MinGen}(a)$ of the minimal monomial generating system $\text{MinGen}(a)$ of the ideal $a$. For a subset $I \subseteq \text{MinGen}(a)$ we denote by $m_I$ the least common multiple of the monomials in $I$, $m_I : = \text{lcm}(m \in I)$.

On this basis we introduce an equivalence relation: We say that two monomials $m, n \in I$ with $I \subseteq \text{MinGen}(a)$ are equivalent if $\text{gcd}(m, n) \neq 1$ and write $m \sim n$. The transitive closure of $\sim$ gives us an equivalence relation on each subset $I$. We denote by $cl(I) : = \# I/ \sim$ the number of equivalence classes of $I$.

Based on the Taylor resolution, we define a product by

$$I \cdot J = \begin{cases} 0 & \text{gcd}(m_I, m_J) \neq 1 \\ I \cup J & \text{gcd}(m_I, m_J) = 1. \end{cases}$$

Then the number $cl(I)$ counts the factors of $I$ with respect to the product defined above.

The aim of this section is to introduce an acyclic matching on the Taylor resolution which preserves this product.

We call two subset $I, J \subseteq \text{MinGen}(a)$ a matchable pair and write $I \to J$ if $|J| + 1 = |I|$, $m_J = m_I$, and the differential of the Taylor complex maps $I$ to $J$ with coefficient $[I, J] \neq 0$.

Let $I \to J$ be a matchable pair in the Taylor resolution with $cl(I) = cl(J) = 1$ such that no subset of $J$ is matchable. Then define

$$M_{11} : = \{ I \cup K \to J \cup K \text{ for each } K \text{ with } \text{gcd}(m_K, m_I) = \text{gcd}(m_K, m_J) = 1 \}.$$  

For simplification we write $I \in M_{11}$ if there exists a subset $J$ with $I \to J \in M_{11}$ or $J \to I \in M_{11}$. It is clear that this is an acyclic matching. Furthermore, the differential changes in each homological degree in the same way and for two subsets $I, K$ with $\text{gcd}(m_I, m_K) = 1$ we have $I \cup K \in M_{11} \iff I \in M_{11}$ or $K \in M_{11}$. Because of these facts, we can repeat this matching $M_{11}$ on the resulting Morse complex. This gives us a sequence of acyclic matchings, which we denote by $M_1 : = \bigcup_{i \geq 1} M_{1i}$.

If no repetition is possible, we reach a resolution with basis given by some subsets $I \subseteq \text{MinGen}(a)$ with the following property: If we have a matchable pair $I \to J$ where $J$ has a higher homological degree than $J$, then $cl(I) \geq 1$ and $cl(J) \geq 2$. We now construct the second sequence:

Let $I \to J$ be a matchable pair in the resulting Morse complex with $cl(I) = 1, cl(J) = 2$ such that no subset of $J$ is matchable. Then define

$$M_2 : = \{ I \cup K \to J \cup K \text{ for each } K \text{ with } \text{gcd}(m_K, m_I) = \text{gcd}(m_K, m_J) = 1 \}.$$
With the same arguments as before this defines an acyclic matching, and a repetition
is possible. The third sequence starts if no repetition of $M_2$ is possible and is given
by a matchable pair $I \rightarrow J$ in the resulting Morse complex with $cl(I) = 1, cl(J) = 3$
such that no subset of $J$ is matchable. Then define

$$M_3 := \{ I \cup K \rightarrow J \cup K \text{ for each } K \text{ with } \gcd(m_K, m_I) = \gcd(m_K, m_J) = 1 \}.$$ 

Since every matchable pair is of the form $I \cup K \rightarrow J \cup K$ with $m_I = m_J,$
$\gcd(m_I, m_K) = 1,$ and $cl(I) = 1, cl(J) \geq 1,$ we finally reach with this procedure a
minimal resolution of the ideal $a$ as $S$-module. Let $M$ be the union of all matchings.
As before we write $I \in M$ if there exists a subset $J$ with $I \rightarrow J \in M$ or $J \rightarrow I \in M.$
Then the minimal resolution has a basis given by $\text{MinGen}(a) \setminus M.$
We give a matching of this type a special name:

**Definition 3.1** (standard matching). A sequence of matchings $M := \bigcup_{i \geq 1} M_i$ is
called a standard matching on the Taylor resolution if all the following holds:

1. $M$ is graded, i.e. for all edges $I \rightarrow J$ in $M$ we have $m_I = m_J,$
2. $T^{\bullet,M}$ is minimal, i.e. for all edges $I \rightarrow J$ in $T^{\bullet,M}$ we have $m_I \neq m_J,$
3. $M_i$ is a sequence of acyclic matchings on the Morse complex $T^{M_{i+1}}$ ($M_{i+1} :=
\bigcup_{j=1}^{i+1} M_j, T^{M_{i+1}} = T_3$),
4. for all $I \rightarrow J \in M_i$ we have

$$cl(J) - cl(I) = i - 1,$$
$$|J| + 1 = |I|,$$
5. there exists a set $B_i \subset M_i$ such that

(a) $M_i = B_i \cup \{ I \cup K \rightarrow J \cup K \mid K \text{ with } \gcd(m_K, m_J) = 1 \text{ and } I \rightarrow J \in B_i \}$
(b) for all $I \rightarrow J \in B_i$ we have $cl(I) = 1$ and $cl(J) = i.$

The construction above shows that a standard matching always exists. For a
standard matching we have two easy properties, which we will need in Section 5.

**Lemma 3.2.** Let $M$ and $M'$ be two different standard matchings. Then

1. for all $i \geq 1$ we have

$$1 + \sum_{I \in M_{i+1}} (-1)^{cl(I)} m_I t^{cl(I)+|I|} = 1 + \sum_{I \in M_{i+1}} (-1)^{cl(I)} m_I t^{cl(I)+|I|},$$
2. if $I, J \notin M,$ $\gcd(m_I, m_J) = 1,$ and $I \cup J \in M,$ then there exists a set $K$
with $|K| = |I| + |J| + 1, cl(K) = 1,$ and $(I \cup J \rightarrow K) \in M.$

**Proof.** The result follows directly from the definition of a standard matching.

If the ideal is generated in degree two, every standard matching ends after the
second sequence: Assume that we have a matchable pair $I \rightarrow J$ such that $cl(I) = 1$
and $cl(J) \geq 3.$ Then $J$ has at least three subsets $J = J_1 \cup J_2 \cup J_3$ such that
$\gcd(m_{J_1}, m_{J_2}) = 1, i, i' = 1, 2, 3.$ Since $I$ and $J$ have the same multidegree and
$cl(I) = 1,$ there would exist a generator $u \in \text{MinGen}(a)$ such that $\gcd(m_{J_1}, u) \neq 1$
for $i = 1, 2, 3.$ But $u$ is a monomial of degree two, which makes such a situation
impossible.

In this case we have

**Lemma 3.3.** If every standard matching ends after the second sequence, i.e. $M =
M_1 \cup M_2,$ then

$$\sum_{I \in M_1} (-1)^{cl(I)} m_I t^{cl(I)+|I|} = \sum_{I \in M} (-1)^{cl(I)} m_I t^{cl(I)+|I|}.$$
Proof. By definition an edge $I \to J$ matched by the second sequence has the property $|I| = |J| + 1$ and $cl(I) = cl(J) - 1$ and $m_I = m_J$. Therefore,

$$(-1)^{cl(I)}m_Ie^{cl(I)+|I|} = - \left((-1)^{cl(J)}m_Je^{cl(J)+|J|}\right),$$

which proves the assertion. \qed

3.2. Resolutions of Monomial Ideals Generated in Degree Two. Let $a \subseteq S$ be a monomial ideal with minimal monomial generating system MinGen($a$) such that for all monomials $m \in$ MinGen($a$) we have $deg(m) = 2$. We assume, in addition, that $a$ is squarefree. This is no restriction since via polarization we get similar results for the general case.

First we fix a monomial order $\prec$. We introduce the following notation: To each subset $I \subset$ MinGen($a$) we associate an undirected graph $G_I = (V, E)$ on the ground set $V = [n]$, by setting $\{i, j\} \in E$ if the monomial $x_i x_j$ lies in $I$. We call a subset $I$ an nbc-set if the associated graph $G_I = (V, E)$ contains no broken circuit, i.e. there exists no edge $\{i, j\}$ such that

1. $E \cup \{\{i, j\}\}$ contains a circuit $c$ and
2. $x_i x_j = \max_\prec \{x_i' x_j' \mid \{i', j'\} \in c\}$.

Proposition 3.4. There exists an acyclic matching $M_1$ on the Taylor resolution such that

1. $M_1$ is the first sequence of a standard matching,
2. the resulting Morse complex $T_*^{M_1}$ is a subcomplex of the Taylor resolution and
3. $T_*^{M_1}$ has a basis indexed by the nbc-sets.

Proof. Let $Z$ be a circuit in $T_*$ of maximal cardinality. Let $x_i x_j := \max_\prec \{Z\}$. We then define

$$M_{1,0} := \left\{ (Z \cup I) \to ((Z \setminus \{x_i x_j\}) \cup I) \mid I \in T_* \text{ with } Z \cap I = \emptyset \right\}.$$ 

It is clear that $I$ is an acyclic matching and the resulting Morse complex $T^{M_{1,0}}$ is a subcomplex of the Taylor resolution. Now let $Z_1$ be a maximal circuit in $T^{M_{1,0}}$ and let $x_{\nu} x_l := \max_\prec \{Z_1\}$. We then define

$$M_{1,1} := \left\{ (Z_1 \cup I) \to ((Z_1 \setminus \{x_{\nu} x_l\}) \cup I) \mid I \in T^{M_{1,0}} \text{ with } Z_1 \cap I = \emptyset \right\}.$$ 

We only have to guarantee that $(Z_1 \cup I) \not\in M_{1,0}$. Assume $(Z_1 \cup I) \in M_{1,0}$. Since $(Z_1 \setminus \{x_{\nu} x_l\}) \cup I \not\in M_{1,0}$, we see that $x_{\nu} x_l \not= x_i x_j$ and $x_{\nu} x_l \in Z$. But then $W := Z \cup (Z_1 \setminus \{x_{\nu} x_l\})$ is a circuit, which is a contradiction to the maximality of $Z$. Therefore, $M_{1,1}$ is a well defined acyclic matching and the resulting Morse complex is a subcomplex of the Taylor resolution.

If we continue this process, we reach a subcomplex $T^{M_1}$ of the Taylor resolution with a basis indexed by all nbc-sets. It is clear that $M_1 := \bigcup_i M_{1,i}$ satisfies all conditions of the first sequence of a standard matching. Furthermore, if $I$ is an nbc-set and $m_I = m_{I \setminus \{m\}}$, then it follows that $cl(I) = cl(I \setminus \{m\}) - 1$ (otherwise we would have a circuit). This implies that $M_1$ is exactly the first sequence of a standard matching. \qed

We denote by $T_{nbc}$ the resulting Morse complex.

Corollary 3.5. Let $a \subseteq S$ be a monomial ideal generated in degree two. We denote with $nbc_i$ the number of nbc-sets of cardinality $i - 1$. Then for the Betti number of $a$ we have the inequality $\beta_i \leq nbc_i$. 

3.3. Resolution of Stanley Reisner Ideals of a Partially Ordered Set. In this subsection we give a (not acyclic) matching on the subcomplex $T_{\text{nbc}}$ in the case where $\mathfrak{a} = J_{\mathcal{A}(P)}$ is the Stanley Reisner ideal of the order complex of a partially ordered set $(P, \prec)$. In this case $\mathfrak{a}$ is generated in degree two by monomials $x_ix_j$ where $\{i, j\}$ is an antichain in $P$. For simplification we assume that $P = [p] = \{1, \ldots, p\}$ and the order $\prec$ preserves the natural order, i.e. $i < j \Rightarrow i < j$, where $<$ is the natural order on the natural numbers $\mathbb{N}$. Then the minimal monomial generating system $\text{MinGen}(\mathfrak{a})$ of the Stanley Reisner ideal is given by

$$\text{MinGen}(\mathfrak{a}) := \left\{ x_i x_j \mid i < j \text{ and } i \neq j \right\}.$$  

Since $\text{MinGen}(\mathfrak{a})$ consists of monomials of degree two, we can work on the subcomplex $T_{\text{nbc}}$ of the Taylor resolution, where $T_{\text{nbc}}$ is constructed with respect to the lexicographic order such that $x_1 > x_2 > \ldots > x_n$.

First we introduce some notation:

**Definition 3.6.** A subset $I \subset \text{MinGen}(\mathfrak{a})$ is called a sting-chain if there exists a sequence of monomials $x_{i_1}x_{j_2}x_{i_2}x_{j_3}, \ldots, x_{i_{\nu-1}}x_{i_{\nu}} \in I$ with

1. $1 \leq i_1 < \ldots < i_{\nu} \leq n$,
2. $i_j = \min\{j \mid x_j \text{ divides } \text{lcm}(m_{I_j})\}$,
3. $i_j = \max\{j \mid x_j \text{ divides } \text{lcm}(m_{I_j})\}$,
4. for all monomials $x_r x_s \in I$ with $r < s$ exists an index $1 \leq j \leq \nu - 1$ such that either
   a. $x_r x_s = x_{i_j}x_{i_{j+1}}$ or
   b. $r = i_j$, $s < i_{j+1}$, and $x_s x_{i_{j+1}} \notin I$ or
   c. $r > i_j$, $s = i_{j+1}$, and $i_j \prec s$ (i.e. $x_i x_r \notin \mathfrak{a}$).

Let $\mathcal{B}$ be the set of all chains of sting-chains:

$$\mathcal{B} := \left\{ (I_1, \ldots, I_l) \left| \begin{array}{c} I_j \text{ sting-chain for all } j = 1, \ldots, l \text{ and } \\
\max(I_j) < \min(I_{j+1}) \text{ for all } j = 1, \ldots, l - 1 \end{array} \right. \right\},$$

where

$$\max(I) := \max\{i \mid x_i \text{ divides } \text{lcm}(m_{I})\}$$

$$\min(I) := \min\{i \mid x_i \text{ divides } \text{lcm}(m_{I})\}.$$  

Note that a sting-chain is not necessarily an nbc-set. For example, the set $\{x_i x_{i+1}, x_{i+1} x_{i+2}, x_{i+2} x_{i+3}\}$ with $i < j < l$ is a sting-chain, if $x_i x_{i+1}, x_{i+1} x_{i+2} \notin \mathfrak{a}$, but it contains a broken circuit if $x_i x_j \notin \mathfrak{a}$. But with an identification of those sets we get the following Proposition:

**Proposition 3.7.** There exists a matching $\mathcal{M}_2$ (not necessary acyclic) on the complex $T_{\text{nbc}}$ such that

1. there exists a bijection between the sets $I \in T_{\text{nbc}}^{\mathcal{M}_2}$ and the chains of sting-chains $I \in \mathcal{B}$,
2. for $I \rightarrow I' \in \mathcal{M}_2$ we have
   a. $\text{lcm}(m_I) = \text{lcm}(m_{I'})$ and
   b. $\ell(I) = \ell(I') - 1$ and $|I| = |I'| + 1$.

**Proof.** For a set $I \in T_{\text{nbc}} \setminus \mathcal{B}$ let $x_i x_{i+1} x_{i+2}$ be the maximal monomial with respect to the lexicographic order such that $i < \nu < j < l$ and at least one of the following conditions is satisfied:

1. $x_i x_j, x_{i+1} x_{j+2} \in I$ and $x_{i+1} x_{j+2} \notin I$,
2. $x_i x_{i+1}, x_{i+2} x_{j+2} \in I$.

Case $x_i x_j, x_{i+1} x_{i+2} \in I$: Because of the transitivity of the order $\prec$ on $P$ we have either $x_i x_{\nu} \in \mathfrak{a}$ or $x_{\nu} x_j \in \mathfrak{a}$. 
Corollary 3.8. Let \( \mathcal{M} \) be a monomial ideal generated in degree two and \( \mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \) a standard matching on the Taylor resolution. With the notation above we get:

\[
\sum_{t \in \mathcal{M}_1} (-1)^{d(I)} m_t t^{d(I)+|I|} = \sum_{t \notin \mathcal{M}} (-1)^{d(I)} m_t t^{d(I)+|I|} = \sum_{t \text{ nbc-set}} (-1)^{d(I)} m_t t^{d(I)+|I|}.
\]
If $a$ is the Stanley Reisner ideal of the order complex of a partially ordered set $P$, then
\begin{equation}
(3.2) \quad (3.7) = \sum_{I \in B} (-1)^{c(I)} m_I t^{c(I)+|I|}.
\end{equation}

**Proof.** Lemma 3.3 implies the first equality and the second equality follows by Proposition 3.4. If $a$ is the Stanley Reisner ideal of the order complex of a partially ordered set $P$, then Proposition 3.7 together with the proof of Lemma 3.3 imply Equation (3.2). □

3.4. The gcd-Condition. In this subsection we introduce the gcd-condition. Let $a \leq S$ be a monomial ideal in the commutative polynomial ring and $\text{MinGen}(a)$ a minimal monomial generating system.

**Definition 3.9 (gcd-condition).**

1. We say that $a$ satisfies the gcd-condition, if for any two monomials $m, n \in \text{MinGen}(a)$ with $\text{gcd}(m, n) = 1$ there exists a monomial $m, n \neq u \in \text{MinGen}(a)$ with $u | \text{lcm}(m, n)$;

2. We say that $a$ satisfies the strong gcd-condition if there exists a linear order $\prec$ on $\text{MinGen}(a)$ such that for any two monomials $m \prec n \in \text{MinGen}(a)$ with $\text{gcd}(m, n) = 1$ there exists a monomial $m, n \neq u \in \text{MinGen}(a)$ with $m \prec u$ and $u | \text{lcm}(m, n)$.

**Example 3.10.** Let $a = \langle x_1 x_2, x_2 x_3, x_3 x_4, x_4 x_5, x_1 x_5 \rangle$ be the Stanley Reisner ideal of the triangulation of the 5-gon. Then $a$ satisfies the gcd-condition, but not the strong gcd-condition.

**Proposition 3.11.** Let $a$ be a monomial ideal which satisfies the strong gcd-condition. Then there exists an acyclic matching $\mathcal{M}$ on the Taylor resolution such that for all $\text{MinGen}(a) \ni I \notin \mathcal{M}$ we have $cl(I) = 1$. We call the resulting Morse complex $T_{\text{gcd}}$.

**Proof.** Assume $\text{MinGen}(a) = \{ m_1 \prec m_2 \prec \ldots \prec m_i \}$. We start with $m_1$. Let $m_{i_0} \in \text{MinGen}(a)$ be the smallest monomial such that $\text{gcd}(m_1, m_{i_0}) = 1$. Then there exists a monomial $m_1 \prec u_0 \in \text{MinGen}(a)$ with $u_0 | \text{lcm}(m_1, m_{i_0})$. Then we define
\[
\mathcal{M}_0 := \left\{ \left( \{ m_1, m_{i_0}, u_0 \} \cup I \right) \rightarrow \left( \{ m_1, m_{i_0} \} \cup I \right) \mid I \subset \text{MinGen}(a) \right\}.
\]

It is clear that this is an acyclic matching and that the Morse complex $T_{\mathcal{M}_0}$ is a subcomplex of the Taylor resolution. Now let $m_{i_1}$ be the smallest monomial $\neq m_{i_0}$ such that $\text{gcd}(m_1, m_{i_1}) = 1$. Then there exists a monomial $m_1 \prec u_1 \in \text{MinGen}(a)$ with $u_1 | \text{lcm}(m_1, m_{i_1})$ and we define
\[
\mathcal{M}_1 := \left\{ \left( \{ m_1, m_{i_1}, u_1 \} \cup I \right) \rightarrow \left( \{ m_1, m_{i_1} \} \cup I \right) \mid I \subset \text{MinGen}(a) \right\}.
\]

Again, it is straightforward to prove that $\mathcal{M}_1$ is an acyclic matching on $T_{\mathcal{M}_0}$ and that the Morse complex is a subcomplex of the Taylor resolution. We repeat this process for all $m_1 \prec m_i$ with $\text{gcd}(m_1, m_i) = 1$ and we reach a subcomplex $T_{\mathcal{M}_m}$, $\mathcal{M}_m = \bigcup_i \mathcal{M}_i$, of the Taylor resolution which satisfies the following condition: For all remaining subsets $I \subset \text{MinGen}(a) \setminus \mathcal{M}_m$, we have:

1. $m_1 \in I \Rightarrow cl(I) = 1$,

2. $m_1 \notin I \Rightarrow cl(I) \geq 1$.

We repeat now this process with the monomial $m_2$. Here we have to guarantee that for a set $\{ m_2, m_i \} \cup I$ the corresponding set $\{ m_2, m_i, u_i \} \cup I$, with $\text{gcd}(m_2, m_i) = 1$ and $m_2 \prec u_i$ and $u_i | \text{lcm}(m_2, m_i)$, is not matched by the first sequence $\mathcal{M}_m$. Since all sets $J \in \mathcal{M}_m$ satisfy $m_1 \in J$, this would be the case if either $u_i = m_1$ or...
m_1 \in I$. The first case is impossible since $m_1 < m_2 < u_i$. In second case we have $cl((m_2, m_i) \cup I) = 1$. We define:

$$
M_2 := \left\{ \left( \{m_2, m_i, u_2 \} \cup I \right) \rightarrow \left( \{m_2, m_i \} \cup I \right) \mid I \subset \text{MinGen}(a) \setminus M_{m_1} \text{ and } cl((m_2, m_i) \cup I) \geq 2 \right\}.
$$

Condition (1) implies then that $M_2$ is a well defined sequence of acyclic matchings. Since we make this restriction, the resulting Morse complex is not anymore a subcomplex of the Taylor resolution, but we have still the following fact: For all remaining subsets $I \subset \text{MinGen}(a) \setminus (M_{m_1} \cup M_{m_2})$ we have:

1. $m_1 \in I \Rightarrow cl(I) = 1$,
2. $m_2 \in I \Rightarrow cl(I) = 1$,
3. $m_1, m_2 \not\in I \Rightarrow cl(I) \geq 1$.

We apply this process to all monomials. Then we finally reach a complex with the desired properties. □

4. The Multigraded Hilbert and Poincaré–Betti Series

Let $a \subseteq S$ be a monomial ideal and $\mathcal{M} = M_1 \cup \bigcup_{i \geq 2} \mathcal{M}_i$ a standard matching on the Taylor resolution. We introduce a new non-commutative polynomial ring $\tilde{R}$, defined by

$$
\tilde{R} := k\{Y_I \text{ for MinGen}(a) \supset I \notin M_1 \text{ and cl}(I) = 1\}.
$$

On this ring, we define three gradings:

$$
|Y_I| := |I| + 1, \quad \deg(Y_I) := \alpha, \text{ with } \underline{x}^\alpha = m_I, \quad \deg_i(Y_I) := ||\alpha||, \text{ with } \underline{x}^\alpha = m_I,
$$

where $||\alpha|| = \sum_i\alpha_i$ is the absolute value of $\alpha$. This makes $\tilde{R}$ into a multigraded ring:

$$
\tilde{R} = \bigoplus_{\alpha \in \mathbb{N}^n} \bigoplus_{i \geq 0} \tilde{R}_{i,\alpha}
$$

with $\tilde{R}_{i,\alpha} := \{u \in \tilde{R} \mid \deg(u) = \alpha \text{ and } |u| = i\}$.

Let $[Y_I, Y_J] := Y_IY_J - (-1)^{|Y_I||Y_J|}Y_JY_I$ be the graded commutator of $Y_I$ and $Y_J$.

We define the following multigraded two-side ideal

$$
\mathfrak{r} := ([Y_I, Y_J] \text{ for gcd}(m_I, m_J) = 1),
$$

and set

$$
R := \tilde{R}/\mathfrak{r}.
$$

Let $\text{Hilb}_R(\underline{x}, t, z) := \sum_{\alpha \in \mathbb{N}^n} \sum_{i \geq 0} \dim_k(R_{i,\alpha}) \underline{x}^\alpha t^{||\alpha||} z^i$ be the multigraded Hilbert series of $R$. We have the following fact:

**Proposition 4.1.** The multigraded Hilbert series $\text{Hilb}_R(\underline{x}, t, z)$ of $R$ is given by

$$
\text{Hilb}_R(\underline{x}, t, z) = \frac{1}{1 + \sum_{I \subset \text{MinGen}(a) \setminus \{m_i \mid I \notin M_{m_1} \cup M_{m_2}\}} (-1)^{cl(I)} m_I t^{m_i} z^{cl(I)+|I|},
$$

where $t^{m_i} := t^\alpha$ with $\underline{x}^\alpha = m_I$. 
Proof. In [3], Cartier and Foata prove that the Hilbert series of an arbitrary noncommutative polynomial ring divided by an ideal, which is generated by some (graded) commutators, is given by

\[
\text{Hilb}_R(x, t, z) := \frac{1}{1 + \sum_F (-1)^{|F|} \prod_{y \in F}^{\deg(y)} x^{|y|} t^{deg(y)} z^{|y|}},
\]

where \( F \subset \{ Y_I \text{ with } I \notin \mathcal{M}_1, cl(I) = 1 \} \) is a commutative part (i.e. \( Y_I Y_J = (-1)^{|J||I|} Y_J Y_I \) for all \( Y_I, Y_J \in F \)) and \( Y_F = \prod_{Y_I \in F} Y_I \).

Therefore, we only have to calculate the commutative parts. Since \( r \) is generated by the relations \( Y_I Y_J = (-1)^{|J||I|} Y_J Y_I \), if \( gcd(m_I, m_J) = 1 \), we see that the commutative parts are given by

\[
F := \left\{ Y_I \mid \text{gcd}(m_{I_j}, m_{I_{j'}}) = 1 \text{ for all } j \neq j' \right\}.
\]

But the fact that \( Y_{I_{t_1}} \cdots Y_{I_{t_r}} \) is a commutative part is equivalent to \( I_{t_1} \cup \cdots \cup I_{t_r} \notin \mathcal{M}_1 \). Therefore, we can identify the commutative parts \( F \) with the elements \( I \notin \mathcal{M}_1 \) and sum over all \( I \notin \mathcal{M}_1 \). It is clear that the cardinality of a commutative part equals to the number \( cl(I) \). If \( I = I_{t_1} \cup \cdots \cup I_{t_r} \) with \( cl(I_j) = 1 \) is a commutative part, it follows that \( Y_I = Y_{I_{t_1}} \cdots Y_{I_{t_r}} \), which implies the exponents of \( t, z, x \). □

We formulate the following conjecture:

**Conjecture 4.2.** Let \( F_\bullet \) be a multigraded minimal \( A \)-free resolution of \( k \) as an \( A \)-module with \( F_i = \bigoplus_{\alpha \in \mathbb{N}^n} A(-\alpha)^{\beta_\alpha} \) for \( i \geq 0 \). Then we have the following isomorphism as \( k \)-vectorspaces:

\[
F_i \cong \bigoplus_{|J| = 1} \bigoplus_{\alpha_J \in \mathbb{Z}^{|J|}} A(-\alpha_J + \deg(u)),
\]

where \( \alpha_J \) is the characteristic vector of \( J \), defined by

\[
(\alpha_J)_i = \begin{cases} 0, & i \notin J, \\ 1, & i \in J \end{cases}
\]

This conjecture gives a precise formulation of the conjecture by Charalambous and Reeves on the multigraded Poincaré-Betti series. In addition, we get an explicit form of the multigraded Hilbert series of \( S/a \) for monomial ideals \( a \).

**Proposition 4.3.** Let \( A = S/a \) be the quotient of the commutative polynomial ring by a monomial ideal \( a \), and let \( \mathcal{M} := \mathcal{M}_1 \cup \bigcup_{i \geq 2} \mathcal{M}_i \) be a standard matching on the Taylor resolution. If Conjecture 4.2 holds, then the multigraded Poincaré-Betti and Hilbert series have the following form:

\[
P_k^A(x, t) = \prod_{i=1}^n (1 + x_i t) \text{ Hilb}_R(x, 1, t) = \frac{1}{1 + \sum_{I \subseteq \text{MinGen}(a)} \left( -1 \right)^{cl(I)} m_I t^{cl(I) + |I|}},
\]
\( \text{Hilb}_A(x, t) = \left( \prod_{i=1}^{n} (1 - x_i t) \text{ Hilb}_R(x, t, -1) \right)^{-1} \)

\[
1 + \sum_{I \subseteq \text{MinGen}(a) \atop I \notin M_1} (-1)^{|I|} m_I t^{m_I} \prod_{i=1}^n (1 - x_i t).
\]

Note that Equation (4.1) is a reformulation of the conjecture by Charalambous and Reeves.

**Proof.** The form of the Poincaré-Betti series follows directly from the conjecture, by counting basis elements of \( F_i \).

For the Hilbert series we consider the complex \( F_* \rightarrow k \rightarrow 0 \), which is exact since \( F_* \) is a minimal free resolution of \( k \). Since the Hilbert series of \( k \) is 1, the Euler characteristic implies:

\[
\sum_{i \geq 0} (-1)^i \text{Hilb}_{F_i}(x, t) = 1.
\]

Conjecture 4.2 implies

\[
\text{Hilb}_{F_i}(x, t) = \sum_{J \subseteq \{1, \ldots, n\}} \sum_{u \in R \atop |u| = i-l} x^{\alpha_j} t^{|J|} x^{\deg(u)} t^{\deg_t(u)} \text{Hilb}_A(x, t).
\]

The Cauchy product finally implies:

\[
\sum_{i \geq 0} (-1)^i \text{Hilb}_{F_i}(x, t) = \text{Hilb}_A(x, t) \sum_{i \geq 0} \sum_{J \subseteq \{1, \ldots, n\}} (-1)^i x^{\alpha_j} t^{|J|}
\]

\[
\sum_{u \in R \atop |u|=i-l} (-1)^{i-l} x^{\deg(u)} t^{\deg_t(u)}
\]

\[
= \text{Hilb}_A(x, t) \left( \sum_{J \subseteq \{1, \ldots, n\}} x^{\alpha_j} (-t)^{|J|} \right)
\]

\[
\left( \sum_{u \in R} x^{\deg(u)} t^{\deg_t(u)} (-1)^{|u|} \right)
\]

\[
= \text{Hilb}_A(x, t) \prod_{i=1}^n (1 - t x_i) \text{ Hilb}_R(x, t, -1).
\]

It is known that if \( A \) is Koszul, then \( \text{Hilb}_A(x, t) = 1/P_k A(x, -t) \). In our case, this means:

**Proposition 4.4.** If \( A \) is Koszul, then \( \text{Hilb}_R(x, t, -1) = \text{Hilb}_R(x, 1, -t) \).

**Proof.** In the monomial case, the Koszul property is equivalent to the fact that \( a \) is generated in degree two. We prove that a subset \( I \in \text{MinGen}(a) \) which is not matched by \( M_1 \) satisfies \( d(I) + |I| = \deg_t(Y_I) \). It is clear that this proves the assertion.

It is enough to prove it for subsets \( I \subseteq \text{MinGen}(a) \) with \( d(I) = 1 \). Let \( m_I = x^\alpha \) be the least common multiple of the generators in \( I \). Since all generators have degree two, it follows \( ||\alpha|| \leq 2 + |I| - 1 = |I| + 1 = |I| + d(I) \). Since Tor\( _1 \) \( (S/a, k)_I = 0 \), we get \( ||\alpha|| = |I| + 1 = |I| + d(I) \). 

We introduce some notation for rings \( A \) satisfying the consequences of Conjecture 4.2.
Definition 4.5. We say that $A$ has property (P) if $P^A_k(x, t) = \prod_{i=1}^{n}(1 + x_i t) \text{Hilb}_R(x_i, 1, t)$ and has property (H) if $\text{Hilb}_A^k(x, t) = \left( \prod_{i=1}^{n}(1 - x_i t) \text{Hilb}_R(x_i, t, -1) \right)^{-1}$.

5. The Homology of the Koszul Complex $K^A$

Let $\mathcal{M}$ be a standard matching on the Taylor resolution of $a$. The basis of the $k$-vectorspace $T^M_\bullet \otimes_S k$ is then given by the sets $I \subset \text{MinGen}(a)$ with $I \notin \mathcal{M}$.

We denote with $K^A_\bullet$ the Koszul complex of $A$ with respect to the sequence $x_1, \ldots, x_n$, i.e.

$$K_i := \bigoplus_{\{j_1 < \ldots < j_i\}} A e_{\{j_1 < \ldots < j_i\}}$$

with differential

$$\partial_i : \left\{ \begin{array}{ll}
K_i & \rightarrow K_{i-1} \\
e_{\{j_1 < \ldots < j_i\}} & \rightarrow \sum_{l=1}^{i+1} \sum_{\{j_1 < \ldots < j_l-1 < j_l+1 < \ldots < j_i\}} x_j e_{\{j_1 < \ldots < j_l-1 < j_l+1 < \ldots < j_i\}}
\end{array} \right.$$  

We denote further by $Z(K_\bullet)$ (resp. $B(K_\bullet)$) the set of cycles (resp. boundaries) of the complex $K_\bullet$. Finally, we denote with $H(K_\bullet)$ the homology of the Koszul complex.

Proposition 5.1. If $\mathcal{M}$ is a standard matching, then there exists a homogeneous homomorphism

$$\phi : \left\{ \begin{array}{ll}
T^M_\bullet \otimes_S k & \rightarrow K^A_\bullet \\
I & \mapsto \phi(I)
\end{array} \right.$$  

such that for all $I, J \notin \mathcal{M}$ with $\gcd(m_I, m_J) = 1$ we have

1. $\phi(I)$ is a cycle,
2. $\phi(I) \phi(J) = \phi(I \cup J)$ if $I \cup J \notin \mathcal{M}$,
3. if $I \cup J \in \mathcal{M}$,

$$\phi(I) \phi(J) = \partial(c) + \sum_{L \notin \mathcal{M} : cl(L) \geq cl(I) + cl(J)} a_L \phi(L) \quad \text{for some } a_L \in k,$$

for some $c \in K^A_\bullet$.

Note that $\phi(I) \phi(J) \in B(K_\bullet)$ might happen if all coefficients $a_L$ are zero.

Proof. We consider the following double complex:

$$\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & T^M_\bullet \otimes_S k & \rightarrow & \ldots & \rightarrow & T^M_0 \otimes_S k & \rightarrow & S/I \otimes_S k & \rightarrow 0 \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & T^M_\bullet \otimes_S S_0^k & \rightarrow & \ldots & \rightarrow & T^M_0 \otimes_S S_0^k & \rightarrow & S/I \otimes_S S_0^k & \rightarrow 0 \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & \vdots & \rightarrow & \ldots & \rightarrow & \vdots & \rightarrow & \vdots & \rightarrow 0 \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & T^M_\bullet \otimes_S S_0^k & \rightarrow & \ldots & \rightarrow & T^M_0 \otimes_S S_0^k & \rightarrow & S/I \otimes_S S_0^k & \rightarrow 0 \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}$$

Since every row and every column, except the first row and the right column, are exact, we get by diagram chasing a homogeneous homomorphism

$$\phi : \left\{ \begin{array}{ll}
T^M_\bullet \otimes_S k & \rightarrow K_\bullet \\
I & \mapsto \phi(I).
\end{array} \right.$$
By construction it is clear that $\phi(I)$ is a cycle. The second condition of a standard matching is: if $(I \to J) \in \mathcal{M}$, then $(I \cup K \to J \cup K) \in \mathcal{M}$ for all $K$ with $\gcd(m_K, m_I) = 1$. This condition implies that one can chose the homomorphism $\phi$ such that $\phi(I)\phi(J) = \phi(I \cup J)$ if $I \cup J \notin \mathcal{M}$.

Now let $I \cup J \in \mathcal{M}$. Since $I, J \notin \mathcal{M}$, it follows from the standard matching that $I \cup J$ is matched with a set $\hat{I}$ of higher homological degree. We now consider $\mathcal{M} := \mathcal{M} \setminus \{I \to I \cup J\}$. We then have

$$0 = \partial^{\mathcal{M}'} \partial^{\mathcal{M}}(\hat{I}).$$

Hence we get:

$$\partial^{\mathcal{M}'}(I \cup J) = \sum_{L \notin \mathcal{M}} a_L \partial^{\mathcal{M}}(L).$$

Since we take the tensor product $\otimes_k k$ with $k$, all summands with $a_L \notin k$ cancel out. Hence $\phi(I)\phi(J) \in B(K_*)$ or, again with diagram chasing:

$$\phi(I)\phi(J) - \sum_{L \notin \mathcal{M}, cL = c(I)+c(J)} a_L \phi(L) \in B(K_*).$$

If from the construction of the standard matching it follows, in addition, that $cl(L) \geq cl(I) + cl(J)$ (otherwise $L$ would have been matched before). \hfill $\Box$

We define the following new $k$-algebra:

For each $I \notin \mathcal{M}$ with $cl(I) = 1$ we define one indeterminate $Y_I$ with total degree $deg_t(Y_I) := |I|$ and multidegree $deg_m(Y_I) := x^n$, if $x^n = m_I$. Let $R' := k(Y_I, I \notin \mathcal{M}, cl(I) = 1)/'r'$ be the quotient algebra of the graded commutative polynomial ring $k(Y_I, I \notin \mathcal{M}, cl(I) = 1)$ (i.e. $Y_I Y_J = (-1)^{|I||J|} Y_J Y_I$) and the multigraded ideal $'r'$ that is generated by the relations given by Proposition 5.1, i.e.:

1. $Y_I Y_J = 0$ if $\gcd(m_I, m_J) \neq 1$,
2. $Y_{I_1} \cdots Y_{I_r} = \sum a_L Y_L$ if $\phi(I_{i_1}) \cdots \phi(I_{i_r}) = \sum a_L \phi(L) + \text{boundary}$,
3. $Y_{I_1} \cdots Y_{I_r} = 0$ if $[\phi(I_{i_1}) \cdots \phi(I_{i_r})] = 0$.

**Theorem 5.2.** If $\mathcal{M}$ is a standard matching, then $R'$ is isomorphic to $H(K_*)$.

**Proof.** The isomorphism is given by Proposition 5.1. We only have to prove that $[\phi(I)][\phi(J)] = 0$ if $\gcd(m_I, m_J) \neq 1$. This follows from the next lemma and the next corollary. \hfill $\Box$

**Lemma 5.3.** Let $c = \sum_I \alpha_I \frac{m_I}{x_I} e_I$ be a homogeneous cycle with multidegree $\text{deg}(c) = m$. We fix an $x_0 \mid m$. Then there exists a cycle $c' = \sum_{I'} \alpha_{I'} \frac{m_{I'}}{x_{I'}} e_{I'}$, homologous to $c$, such that $x_0 \mid x_{I'}$ for all $I'$.

**Proof.** Let $I$ be an index set such that $\alpha_I \neq 0$ in the expansion of $c$ with $x_0 \nmid x_I$. Then

$$\frac{m_I}{x_I} e_I = \sum_{i \in I} (-1)^{pos(i)+1} \frac{m_{x_I}}{x_0 x_I} e_{x_0} \wedge e_{I\setminus\{i\}} + \partial \left( \frac{m_I}{x_0 x_I} e_{x_0} \wedge e_I \right).$$

If we replace each index set $I$ with respect to (5.1), we finally reach a cycle $c'$ with the desired property. By construction there exists an element $d$ with $c - c' = \partial(d) \in B(K_*)$. \hfill $\Box$

**Corollary 5.4.** Let $c_1, c_2$ be two homogeneous cycles with multidegrees $\text{deg}(c_1) = m$ and $\text{deg}(c_2) = n$. If $\gcd(m, n) \neq 1$, we have $[c_1][c_2] = 0$.

**Proof.** Let $c_1 := \sum_I \alpha_I \frac{m_I}{x_I} e_I$ and $c_2 := \sum_J \beta_J \frac{n_J}{x_J} e_J$ with $\gcd(m, n) \neq 1$. We fix a $j \in \text{supp}(\gcd(m, n))$. By Lemma 5.3, we can assume that $j \in I \cup J$ for all $I, J$. This implies $[c_1][c_2] = 0$. \hfill $\Box$

**Corollary 5.5.** $H(K_*)$ is generated by $I \notin \mathcal{M}$ with $cl(I) = 1$.
6. HILBERT AND POINCARÉ-BETTI SERIES OF THE ALGEBRA $A = k[\Delta]$

In this section we prove property (P) and (H) for $A = S/a$ where $a = I_{\Delta(P)}$ is the Stanley Reisner ideal of the order complex $\Delta(P)$ of a partially ordered set $P$.

Let $P := (\{1, \ldots, n\}, \prec)$ be a partially ordered set, where $i \prec j$ implies $i < j$.

The Stanley Reisner ring of the order complex $\Delta = \Delta(P)$ is given by

$$A := k[\Delta] = k[x_i, i \in P]/\langle x_ix_j \text{ with } i < j \text{ and } i \neq j \rangle.$$ 

We now define a sequence of regular languages $L_i$ over the alphabet $\Gamma_i := \{x_1, \ldots, x_n\}$:

1. $x_ix_j \in L_i$ for all $i < j$ and $i \neq j$,
2. $x_ix_{j_1} \cdots x_{j_l} \in L_i$ if $x_ix_{j_1} \cdots x_{j_{l-1}} \in L_i$ and $i < j_r$ for all $r = 1, \ldots, l$ and either
   a. $j_{l-1} \neq j_l$ or
   b. $x_ix_{j_1} \cdots x_{j_{l-2}}x_{j_l} \in L_i$ and $j_l < j_{l-1}$.

Let $f_i(x,t) := \sum_{w \in L_i} t^{|w|} w$ be the word counting function of $L_i$.

Corollary 3.8 and Corollary 3.9 of [10] imply the following theorem:

Theorem 6.1. The Poincaré-Betti series of $A$ is given by:

$$P^{A}(x,t) := \prod_{i=1}^{n} (1 + t \cdot x_i) \prod_{i=1}^{n} (1 + f_i(x,t)) = \prod_{i=1}^{n} \frac{1 + t \cdot x_i}{1 - f_i(x,t)}.$$

where $f_i(x,t) := \frac{1}{1 - f_i(x,t)}$.

We only have to calculate the word counting functions $f_i$. Since the language $L_n$ is empty, it follows that $f_n := 0$. We construct recursively non-deterministic finite automata $A_i$ such that the language $L(A_i)$ accepted by $A_i$ is $L_i$ (for the basic facts on deterministic finite automata we use here [9]). We assume that $A_j$ is defined for all $j > i$. Let $A^+_j$ be the automaton which accepts the language $L^+_j \cup \{w \cdot x_j \mid w \in L^+_j\}$, where

$$L^+_j := \{w_1 \circ \cdots \circ w_i \mid i \in \mathbb{N} \setminus \{0\} \text{ and } w_j \in L, j = 1, \ldots, i\},$$

$$L^+_i := L^+_j \cup \{\varepsilon\} = \{w_1 \circ \cdots \circ w_i \mid i \in \mathbb{N} \text{ and } w_j \in L, j = 1, \ldots, i\},$$

where $\circ$ denotes the concatenation and $\varepsilon$ is the empty word. It follows that the word counting function of $L(A^+_j)$ is given by $\frac{L^+_j}{1-f_j}$.

We now construct $A_i$:

- From the starting state we go to the state $i$ if we read the letter $x_i$, otherwise we reject the input word.
- From the state $i$ we can switch by reading the empty word to the state $j$, which represents the automaton $A^+_j$, if $i < j$ and $i \neq j$. We then accept if $A^+_j$ accepts.
- Now assume we have the transitions $i \to j_1$ and $i \to j_2$ with $j_1 < j_2$. Because of condition 26 we can switch by reading the empty word from state $j_2$ to state $j_1$.
- Assume that we have the transition $i \to j_2$ and do not have the transition $i \to j_1$, with $j_1 < j_2$. This means $i < j_1$ and $i \neq j_2$. Therefore, we must have $j_1 < j_2$, otherwise we get a contradiction to the transitivity of the order in $P$. It follows by condition 11 that we can switch by reading the empty word from state $j_2$ to $j_1$.

It is clear that $A_i$ accepts the language $L_i$. Since the state $j$ represents the automaton $A^+_j$, we get a recursion for the word counting functions:
Lemma 6.2. For the word counting functions $f_i$ we get the following recursion:

$$
\begin{align*}
& f_n := 0, \\
& f_i := t x_i \sum_{i < j \in \mathfrak{a}} \frac{f_j + t x_j}{1 - f_j} \prod_{r=i+1}^{j-1} \frac{1 + t x_r}{1 - f_r}.
\end{align*}
$$

Proof. The state $j$ represents the automaton $A_j^+$ with word counting function $\frac{f_j + t x_j}{1 - f_j}$. By the argumentation above we have $j \rightarrow \nu$ for all $\nu = i+1, \ldots, j-1$ if we have $i \rightarrow j$. Since we accept when the automaton $A_j^+$ accepts, we get the desired recursion. □

By standard facts on regular languages the functions $f_i$ are rational functions, but we want to have an expression of the Poincaré-Betti series by polynomials:

Lemma 6.3. For the rational functions $f_i$ we have:

$$
\begin{align*}
& f_i := \frac{w_i}{1 - \sum_{r=i+1}^{n} w_r},
\end{align*}
$$

where $w_i$ are polynomials and $w_n = 0$.

Proof. We prove it by induction: $w_n$ is a polynomial and we have $f_n = \frac{w_n}{1 - 0}$.

We now assume that $f_j$ satisfies the desired condition for all $j > i$. Then

$$
\begin{align*}
& f_i = t x_i \sum_{i < j \in \mathfrak{a}} \frac{t x_j f_j}{1 - f_j} \prod_{r=i+1}^{j-1} \frac{1 + t x_r}{1 - f_r} \\
= & \quad t x_i \sum_{i < j \in \mathfrak{a}} \left(1 - \sum_{r > j} w_r \right) \prod_{r=i+1}^{j-1} \frac{1 + t x_r}{1 - \sum_{r > j} w_r} \\
= & \quad t x_i \sum_{i < j \in \mathfrak{a}} \left(1 - \sum_{r > j} w_r \right) \prod_{r=i+1}^{j-1} \frac{1 + t x_r}{1 - \sum_{r > j} w_r} \\
= & \quad t x_i \sum_{i < j \in \mathfrak{a}} \left(w_j + t x_j - t x_j \sum_{r > j} w_r \right) \prod_{r=i+1}^{j-1} \frac{1 + t x_r}{1 - \sum_{r > j} w_r} \\
= & \quad \frac{w_i}{1 - \sum_{l \geq i+1} w_l}
\end{align*}
$$

with

$$
\begin{align*}
& w_i := t x_i \sum_{i < j \in \mathfrak{a}} \left(w_j + t x_j - t x_j \sum_{r > j} w_r \right) \prod_{r=i+1}^{j-1} \frac{1 + t x_r}{1 - \sum_{r > j} w_r}.
\end{align*}
$$
By induction, \( w_r \) is a polynomial and therefore \( w_i \) is a polynomial. \( \square \)

**Corollary 6.4.** The Poincaré-Betti series of \( A \) is given by:

\[
P_k^A(x, t) := \prod_{i=1}^n \frac{1}{(1 + tx_i) - w_1 - \ldots - w_n}
\]

with

\[
w_n := 0,
\]

\[
w_i := tx_i \sum_{1 \leq j < \nu, x_j \leq a} \left( w_j + tx_j - t \sum_{r \geq j} w_r \right) \left( \prod_{r'=1}^{j-1} (1 + tx_r) \right).
\]

**Proof.** The result is a direct consequence of Lemma 6.3 and Theorem 6.1. \( \square \)

We now solve the recursion of \( w_i \)’s. For this, we introduce a directed graph \( G = (V, E) \) with vertex set \( V = \{1, \ldots, n\} \) and two vertices \( i, j \) are joined (i.e. \( i \to j \)) if \( i < j \) and \( i \neq j \). We write \( G|_{t_1, \ldots, t_{i_\nu}} \) for the induced subgraph on the vertices \( t_1, \ldots, t_{i_\nu} \).

For a sequence \( 1 \leq i_1 < \ldots < i_{i_\nu} \leq n \) we define

\[
d(i_1, \ldots, i_{i_\nu}) := \# \{ \text{paths from } i_1 \text{ to } i_{i_\nu} \text{ in } G|_{t_1, \ldots, t_{i_\nu}} \},
\]

\[
c(i_1, \ldots, i_{i_\nu}) := \sum_{0=a_0 < a_1 < \ldots < a_{i_\nu} = i_{i_\nu}} \cdots \sum_{r \geq 1} (-1)^r d(i_{a_0+1}, \ldots, a_1) \cdots d(i_{a_{r-1}+1}, \ldots, i_{a_r}).
\]

Note that a path counted by \( d(i_1, \ldots, i_{i_\nu}) \) needs not to pass through all vertices \( t_1, \ldots, t_{i_\nu} \).

With this notation we get

**Corollary 6.5.** The Poincaré-Betti series of \( A \) is given by:

\[
P_k^A(x, t) := \prod_{i=1}^n \frac{1}{(1 + tx_i) - W(t, x)}
\]

with

\[
W(t, x) = 1 + \sum_{1 \leq i_1 < \ldots < i_{i_\nu} \leq n} c(i_1, \ldots, i_{i_\nu}) t^{i_{i_\nu}} x_{i_1} \cdots x_{i_{i_\nu}}.
\]

**Proof.** The result follows if one solves the recursion of the \( w_i \)’s and collects the coefficients of the monomials \( x_{i_1} \cdots x_{i_{i_\nu}} \). \( \square \)

In order to prove property (P), we give a bijection between the paths in \( G|_{t_1, \ldots, t_{i_\nu}} \) and the sting-chains:

**Lemma 6.6.** For any sequence \( 1 \leq i_1 < \ldots < i_{i_\nu} \leq n \) there exists a bijection between the paths from \( i_1 \) to \( i_{i_\nu} \) in \( G|_{t_1, \ldots, t_{i_\nu}} \) and the sting-chains \( I \) with \( \text{lcm}(I) = x_{i_1} \cdots x_{i_{i_\nu}} \).

**Proof.** We consider the path \( i_1 \to j_2 \to j_3 \to \ldots \to j_r \to i_{i_\nu} \). To this path, we associate the set \( I := \{x_{i_1}x_{j_2}, x_{j_2}x_{j_3}, \ldots, x_{j_r}x_{i_{i_\nu}}\} \). Now we define the stings:

Assume \( j_r < i_{i_0}, \ldots, j_r < j_{r+1} \). Then we must have either \( j_r \neq i_s \) or \( i_s \neq j_{r+1} \) for all \( s = i_0, \ldots, l_1 \) (otherwise we would have a contradiction to \( j_r \neq j_{r+1} \)). This implies

\[
\{x_{j_r}x_{i_1}, x_{i_1}x_{j_{r+1}}\} \cap a \neq \emptyset \text{ for all } s = i_0, \ldots, l_1.
\]

If \( x_{j_r}x_{i_1} \in \{x_{j_r}x_{i_1}, x_{i_1}x_{j_{r+1}}\} \cap a \), we choose \( x_{j_r}x_{i_1} \), otherwise we choose \( x_{i_1}x_{j_{r+1}} \).

With this choice we get that \( I \) satisfies condition (11) and (13) of Definition 3.6.

By construction we have \( \text{lcm}(I) = x_{i_1} \cdots x_{i_{i_\nu}} \).
If we start with a sting-chain $I$ with $\text{lcm}(I) = x_{i_1} \cdots x_{i_\nu}$, then by definition there exist monomials $x_{i_1}x_{j_2}, x_{j_2}x_{j_3}, \ldots, x_{j_r}x_{i_\nu} \in I$. This sequence defines a path $i_1 \Rightarrow j_2 \Rightarrow \ldots \Rightarrow j_r \Rightarrow i_\nu$. Since both constructions are inverse to each other, the assertion follows.

It follows:

$$W(t, x) = 1 + \sum_{I \in B} (-1)^{cl(I)} m_I t^{cl(I)+|I|},$$

where $B$ is the set of chains of sting-chains, defined in Section 3.

We now can prove property (P) and (H) for the ring $A = k[\Delta]$.

**Theorem 6.7.** Let $P$ be a partially ordered set and $\Delta$ the order complex of $P$. The multigraded Poincaré-Betti and Hilbert series of the Stanley Reisner ring $A = k[\Delta] = S/a$ are given by:

$$P_A^k(x, t) := \prod_{i \in P} \frac{(1 + tx_i)}{W(t, x)}$$

$$\text{Hilb}_A(x, t) := \frac{W(-t, x)}{\prod_{i \in P} (1 - tx_i)},$$

where

$$W(t, x) = 1 + \sum_{I \notin M} (-1)^{cl(I)} m_I t^{cl(I)+|I|}$$

$$= 1 + \sum_{I \notin M_1} (-1)^{cl(I)} m_I t^{cl(I)+|I|}$$

$$= 1 + \sum_{I \in B} (-1)^{cl(I)} m_I t^{cl(I)+|I|}$$

$$= 1 + \sum_{I \in nbc-set} (-1)^{cl(I)} m_I t^{cl(I)+|I|}$$

with $M = M_1 \cup M_2$ a standard matching on the Taylor resolution $T_\bullet$ of $a$.

**Proof.** The assertion is a direct consequence of Corollary 3.8, Corollary 6.5, and Equation (6.1). □

**7. Proof of Conjecture 4.2 for Several Classes of Algebras $A$**

In this section we prove Conjecture 4.2 in some special cases. In the first subsection, we prove the conjecture for algebras $A$ for which the Koszul homology is an $M$-ring - a notion introduced by Fröberg [6]. If in addition the minimal resolution of $a$ has the structure of a differential-graded algebra, we prove property (P) for $A$.

In the second subsection, we prove Conjecture 4.2 for all Koszul algebras. Note that this gives another proof that for a partially ordered set $P$ the Stanley Reisner ring $A = k[\Delta(P)]$ satisfies property (P) and (H).

In the last subsection, we outline an idea for a proof of Conjecture 4.2 in general.

**7.1. Proof for Algebras $A$, with $H_\bullet(K^A)$ is an M-ring.**

The first class for which we can prove Conjecture 4.2 uses a theorem by Fröberg [6]. We use the notation of Fröberg:

**Definition 7.1.** A $k$-algebra $R$ isomorphic to a (non-commutative) polynomial ring $k(X_1, \ldots, X_r)$ divided by an ideal $\mathfrak{r}$ of relations is called

1. a weak M-ring if $\mathfrak{r}$ is generated by relations of the following types:

   a) the (graded) commutator $[X_i, X_j] = 0$,
(b) \( m = 0 \), where \( m \) is a monomial in \( X_i \).

(2) an M-ring if \( r \) is generated by relations of the following types:

(a) the (graded) commutator \([X_i, X_j] = 0\).

(b) \( m = 0 \) with \( m \) a quadratic-monomial in \( X_i \).

Now we assume that \( H(K_\bullet) \) is an M-ring and \( \mathcal{M} \) is a standard matching. Let \( R'' := k(Y_1, I \notin \mathcal{M}, cl(I) = 1)/\tau'' \) be the non-commutative polynomial ring divided by an ideal \( \tau'' \), where \( \tau'' \) is generated by the following relations:

\[
Y_1Y_j = (-1)^{deg(Y_j)Y_j}Y_jY_1, \quad \text{if } \gcd(m_i, m_j) = 1 \text{ and } I \cup J \notin \mathcal{M} \text{ for all } I, J \notin \mathcal{M} \text{ with } cl(I) = cl(J) = 1.
\]

In the notion of Fröberg, \( R'' \otimes R' \) is the MM-ring belonging to the M-ring \( R' \cong H(K_\bullet) \). Each literal \( Y_1 \) has two degrees: the total degree \( |Y_1| := |I| + 1 \) and the multidegree \( \deg(Y_1) := \alpha \), with \( x^\alpha = m_1 \).

We define \( F_\bullet := R'' \otimes_k K^{\bullet \bullet}_\bullet \). Since \( K^{\bullet \bullet}_\bullet \) is an A-module, \( F_\bullet \) is a free graded A-module with \( \deg(m \cdot n) := \deg_{k^n}^k(m) + \deg_{k^n}^{\bullet \bullet}(n) \). Let \( F_i \) be the homogeneous part of degree \( i \). The next theorem proves Conjecture 4.2 in our situation.

**Theorem 7.2.** Let \( \mathcal{M} \) be a standard matching. Assume \( H(K_\bullet) \) an M-ring. If there exists a homomorphism \( s : H_\bullet(K^{\bullet \bullet}_\bullet) \to Z_\bullet(K^{\bullet \bullet}_\bullet) \), such that \( \pi \circ s = id_{H_\bullet(K^{\bullet \bullet}_\bullet)} \), then \( A \) satisfies Conjecture 4.2.

**Corollary 7.3.** Under the assumptions of Theorem 7.2, the algebra \( A \) has properties (P) and (H). \( \square \)

**Proof of Theorem 7.2.** Theorem 5.2 verifies the conditions for Theorem 3 in [6]. In the proof of this theorem, Fröberg shows that \( F_\bullet \) defines a minimal free resolution of \( k \) as an \( A \)-module. By Theorem 5.2, the homology of the Koszul complex is isomorphic to the ring \( R'/\tau' \). Since \( H_\bullet(K^{\bullet \bullet}_\bullet) \) is an M-ring, it follows that the ideal \( \tau' \) is generated in degree two. The construction of the ideal \( \tau' \) implies that every standard matching ends after the second sequence. In the second sequence of \( \mathcal{M} \), we have that \( I \to J \in \mathcal{M}_2 \) satisfies \( cl(I) = cl(J) - 1 \) and \( |I| = |J| + 1 \). Now let \( I \to J \in \mathcal{M}_2 \) with \( cl(I) = 1 \) and \( cl(J) = cl(J_1) + cl(J_2) = 2 \). The difference between the ring \( R'' \) and the ring \( R \) is that in \( R \) we have a variable \( Y_1 \) and the variables \( Y_1, Y_2 \) commute in the ring \( R'' \). The variables \( Y_1, Y_2 \) do not commute and the variable \( Y_1 \) is omitted. Identifying \( Y_1, Y_2 \in R'' \) with \( Y_1, Y_2 \in R \) and \( Y_1, Y_2 \in R'' \) with \( Y_1 \in R \) gives an isomorphism as \( k \)-vector spaces of \( R \) and \( R'' \). The property \( cl(I) = cl(J) - 1 \) and \( |I| = |J| + 1 \) proves that this isomorphism preserves the degrees, and we are done. \( \square \)

The theorem includes the theorem by Charalambous and Reeves since in their case every standard matching is empty and Charalambous and Reeves proved the existence of the map \( s : H_\bullet(K^{\bullet \bullet}_\bullet) \to Z_\bullet(K^{\bullet \bullet}_\bullet) \):

**Corollary 7.4.** If the Taylor resolution of \( a \) is minimal, then \( A = S/\mathfrak{a} \) satisfies Conjecture 4.2. \( \square \)

Note that \( H_\bullet(K^{\bullet \bullet}_\bullet) \cong R' \) carries three gradings. Let \( u \in R' \) with \( u = Y_1 \cdots Y_{l_r} \). Then we have \( \gcd(m_{l_1}, m_{l_j}) = 1 \), for \( j \neq j' \), and \( I_1 \cup \ldots \cup I_r \notin \mathcal{M} \) (otherwise \( u \in \tau' \)). We set

\[
\deg(u) = \alpha \text{ if } \tau'' = m_{l_1} \cdots m_{l_r} = m_{l_1} \cup \ldots \cup m_{l_r},
\]

\[
\deg_k(u) = r = cl(I_1 \cup \ldots \cup I_r),
\]

\[
|u| = |I_1| + \ldots + |I_r| = |I_1 \cup \ldots \cup I_r|.
\]
It follows:

\[ H_\bullet(K^A) \cong R' = \bigoplus_{\alpha \in \mathbb{N}^{n}} R'_{\alpha,i,j} = \bigoplus_{I \notin \mathcal{M}} k Y_I, \]

where \( Y_I = Y_{I_1} \cdot \cdots Y_{I_r} \) if \( \text{cl}(I) = r \) and \( \text{gcd}(m_{I_j}, m_{I_j'}) = 1 \), for \( j \neq j' \).

Fröberg proved that in the case where \( H_\bullet(K^A) \) is an M-ring and the minimal resolution of \( a \) has the structure of a differential-graded algebra we have:

\[ P_k^A(x,t) = \frac{\text{Hilb}_{K^A \otimes_k} k(x,t)}{\text{Hilb}_{H_\bullet(K^A)}(x,-t,t)} = \prod_{i=1}^{n}(1 + tx_i) \frac{1}{\text{Hilb}_{H_\bullet(K^A)}(x,-t,t)}. \]

Therefore, we only have to calculate the Hilbert series \( \text{Hilb}_{H_\bullet(K^A)}(x,-t,t) \):

\[ \text{Hilb}_{H_\bullet(K^A)}(x,-t,t) = \sum_{\alpha \in \mathbb{N}^{n}} \dim_k(R'_{\alpha,i,j}) x^{\alpha} (-t)^i t^j = \sum_{I \notin \mathcal{M}} m_I (-t)^{\text{cl}(I)} t^{|I|} = 1 / \text{Hilb}_R(x,1,t). \]

The last equation follows from Lemma since if \( H_\bullet(K^A) \) is an M-ring, every standard matching ends after the second sequence. It follows:

**Corollary 7.5.** If \( H_\bullet(K^A) \) is an M-ring and the minimal resolution of \( a \) has the structure of a differential-graded algebra, then \( A \) has property (P). □

### 7.2. Proof for Koszul Algebras

In this subsection we give the proof of Conjecture for Koszul algebras \( A = S/\mathfrak{a} \). Note that since \( \mathfrak{a} \) is monomial, this is equivalent to the fact that \( \mathfrak{a} \) is generated in degree two. We assume in addition that \( \mathfrak{a} \) is squarefree. This is no restriction since via polarization we can reduce the calculation of the Hilbert and Poincaré-Betti series of \( S/\mathfrak{a} \) to the calculation of the series for \( S/\mathfrak{b} \) for a squarefree ideal \( \mathfrak{b} \leq S \).

**Theorem 7.6.** Let \( A = S/\mathfrak{a} \) be the quotient algebra of the polynomial ring and a squarefree monomial ideal \( \mathfrak{a} \) generated by monomials of degree two and \( \mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \) a standard matching of \( \mathfrak{a} \). Then \( A \) satisfies Conjecture.

**Corollary 7.7.** The multigraded Poincaré-Betti and Hilbert series of Koszul algebras \( A = S/\mathfrak{a} \) for a squarefree monomial ideal \( \mathfrak{a} \leq S \) are given by:

\[ P_k^A(x,t) := \prod_{i \in P} \frac{(1 + tx_i)}{W(t, x)}, \]

\[ \text{Hilb}_A(x,t) := \prod_{i \in P} \frac{W(-t, x)}{(1 - tx_i)}, \]

where

\[ W(t, x) = 1 + \sum_{I \notin \mathcal{M}} (-1)^{\text{cl}(I)} m_I t^{\text{cl}(I)} + |I| = 1 + \sum_{I \notin \mathcal{M}_1} (-1)^{\text{cl}(I)} m_I t^{\text{cl}(I)} + |I| = 1 + \sum_{I \in \text{nc-set}} (-1)^{\text{cl}(I)} m_I t^{\text{cl}(I)} + |I|. \]
Proof. The assertion follows directly from Theorem 7.6 the standard matching for ideals generated in degree two given in Section 3 and the fact that, in this case, every standard matching ends after the second sequence. □

Note that if $\mathfrak{a} \leq S$ is any ideal with a quadratic Gröbner basis, this corollary gives a form of the multigraded Hilbert and Poincaré-Betti series of $A = S/\mathfrak{a}$ since, in this case, the series coincide with the series of $S/\in_{-\infty}(\mathfrak{a})$.

Proof of Theorem 7.6 In this proof we sometimes consider the variables $x_1, \ldots, x_n$ as elements of the polynomial ring $S$ and sometimes as letters. In the second case the variables do not commute and we consider words over the alphabet $\Gamma := \{x_1, \ldots, x_n\}$. It will be clear from the context if we consider $w$ as a monomial in $S$ or as a word over $\Gamma$. For example, if we write $w \in \mathfrak{a}$ or $x_i \mid w$, we see $w$ as a monomial.

For $j = 1, \ldots, n$, let $L_j$ be the sets of words $x_1 x_2 \cdots x_r$, $r \geq 2$, over the alphabet $\{x_1, \ldots, x_n\}$, such that

1. $i_1 = j < i_2, \ldots, i_r$,
2. for all $2 \leq l \leq r$ there exists an $1 \leq l' < l$ such that $x_{i_l} x_{i_l'} \in \mathfrak{a}$ and $i_l > i_{l'}$ for all $l' < l$.

We define

$$L := \left\{ w_{i_1} \cdots w_{i_r} \mid i_1 > \cdots > i_r \right\}.$$ 

Note that here the variables $x_i$ are considered as letters and do not commute. In [10] we construct for Koszul algebras $A$ a minimal free resolution of $k$. The basis in homological degree $i$ in this resolution is given by the following set (see Corollary 3.9 of [10]):

$$\mathcal{B}_i = \left\{ e_I w \mid I \subseteq \{1, \ldots, n\} \quad \text{ and } \quad |J| + |w| = i \right\},$$

where $|w|$ is the length of the word $w$.

Thus in order to prove the theorem, we have to find a bijection between the words $w \in L$ of length $i$ and the monomials $u \in R$ with degree $|u| = i$. Remember that in our case the subsets $I \not\subseteq M_1$ are exactly the $\text{nbc}$-sets (see Section 3.2) and therefore the ring $R$ has the following form:

$$R = \frac{k(Y_1, I \text{ is an \text{nbc}-set } \cup \text{cl}(I) = 1)}{(\langle Y_I, Y_J \mid \gcd(m_I, m_J) = 1 \rangle)}.$$ 

We assume that the monomials $u \in R$ are ordered, i.e. if $u = Y_{I_1} \cdots Y_{I_r}$ and $Y_{I_j}$ commute with $Y_{I_{j+1}}$, then $\min(I_j) > \min(I_{j+1})$.

Clearly, it is enough to construct a bijection between the sets $L_j$ and the ordered monomials $u = Y_{I_1} \cdots Y_{I_r}$, with $\text{cl}(I_1 \cup \cdots \cup I_r) = 1$ and $j = \min(I_1) < \min(I_i)$, for $i = 2, \ldots, r$.

For a word $w$ over the alphabet $\{x_1, \ldots, x_n\}$ we denote by $x_{f(w)}$ (resp. $x_{l(w)}$) the first (resp. the last) letter of $w$, i.e. $w = x_{f(w)} w'$ (resp. $w = w' x_{l(w)}$).

We call a word $w$ over the alphabet $\{x_1, \ldots, x_n\}$ an $\text{nbc}$-word if there exists an index $j$ such that $w \in L_j$ and each variable $x_i$, $i = 1, \ldots, n$, appears at most once in the word $w$.

The existence of the bijection follows from the following four claims.

Claim 1: For each $j$ and each word $w \in L_j$ which is not an $\text{nbc}$-word there exists a unique subdivision of the word $w$,

$$\phi_1(w) := u_1 | v_1 | u_2 | v_2 | \cdots | u_r | v_r,$$

such that

(i) $u_1 v_1 \cdots u_r v_r = w$. 

(ii) The subword $u_i$ is either a variable or an $\text{ncb}$-word in the language $L_{f(u_i)}$.
(iii) The words $v_i$ are either the empty word $\varepsilon$ or a descending chain of variables, i.e. $v_i = x_{j_1} \cdots x_{j_n}$ with $j_1 > \ldots > j_n$.
(iv) If $v_i \neq \varepsilon$ and $u_i$ is an $\text{ncb}$-word, then
\[ f(u_i) \geq f(v_i) > l(v_i) > f(u_{i+1}). \]
(v) If $v_i \neq \varepsilon$ and $u_i$ is a variable, then
\[ f(u_i) < f(v_i) > l(v_i) > f(u_{i+1}). \]
(vi) If $v_i = \varepsilon$ and $u_i$ is a variable, then
\[ f(u_i) \geq f(u_{i+1}). \]
(vii) If $v_i = \varepsilon$ and $u_i$ is a variable, then
\[ f(u_i) < f(u_{i+1}). \]

**Claim 2:** There exists an injective map $\phi_2$ on the subdivisions of Claim 1 such that
\[ \phi_2(\phi_1(w)) := w_1 || w_2 || \ldots || w_s \]
and for each $w_i$, $i = 1, \ldots, s$, we have the following properties:
(i) If $w_i = x_{j_1} \cdots x_{j_n}$, then for all $1 \leq l \leq t$ there exists an index $0 \leq l' < l$ with $x_{j_l}x_{j_{l'}} \in a$ and $j_{l'} > j_l$ for all $l' < l < l$.
(ii) In each word $w_i$, each variable $x_1, \ldots, x_n$ appears at most once.
(iii) $w_i$ is not a variable.
(iv) There exists an index $t$ such that $x_{t} \mid w_1 \cdots w_{i-1}$ and $x_{t}x_{f(w_i)} \in a$ and either $x_{f(w_i)} \mid w_1 \cdots w_{i-1}$ or $t > f(w_i)$.
(v) For all $x_{t} \mid w_i$, $j < f(w_i)$, and $x_{t} \mid w_1 \cdots w_{i-1}$ with $x_{t}x_{j} \in a$, we have $t < j$.
(vi) If $\gcd(w_i, w_{i+1}) = 1$, then $f(w_i) > f(w_{i+1})$.

**Claim 3:** There exists an injection $\phi_3$ between the sequences $\phi_2\phi_1(L_j)$ from Claim 2 and the sequences $w_1 || w_2 || \ldots || w_s$, satisfying, in addition to the conditions from Claim 2, the following properties:
(i) There exists an $j < i$ such that $\gcd(w_i, w_j) \neq 1$.

**Claim 4:** For each $f$ there is a bijection
\[
\phi_4 : \phi_3\phi_2\phi_1(L_j) \to \left\{ \begin{array}{c}
Y_{I_1} \cdots Y_{I_r} \\
Y_{I_1} \cdots Y_{I_r} \quad \text{ordered}
\end{array} \right\}
\]
\[ j = \min(I_i) < \min(I_j), \quad \text{for } i = 2, \ldots, r \]

Since $\phi_1, \ldots, \phi_3$ are injections and $\phi_4$ is a bijection, the composition $\phi_4\phi_3\phi_2\phi_1$ is the desired map.

**Proof of Claim 1.** Let $x_{j_1} \cdots x_{j_r} \in L_j$, for some $j$, which is not an $\text{ncb}$-word. Then we have the following uniquely defined subdivision:
\[
\begin{array}{c|c|c|c}
\hline
x_{i_1}x_{i_2} \cdots x_{i_{j_1}-1} & x_{i_{j_1}} \cdots x_{i_{j_2}-1} & x_{i_{j_2}} \cdots x_{i_{j_3}-1} & \cdots \\
_{i_2 > \ldots > i_{j_1-1}} & _{i_{j_1} > \ldots > i_{j_2-1}} & _{i_{j_2} > \ldots > i_{j_3-1}} & \\
\hline
\end{array}
\]

The first part $x_{i_1}x_{i_2} \cdots x_{i_{j_0}-1}$ we split again into
\[ u_1 || v_1 := x_{i_1} || x_{i_2} \cdots x_{i_{j_0}-1}. \]
Thus, we get the subdivision
\[ u_1 || v_1 || u_2 || v_2 || \ldots || u_s || v_s, \]
where $u_1$ is a variable, $v_i$ are the monomials of the descending chains of variables (note that $v_i = \varepsilon$ is possible) and the words $u_i$, $i \geq 2$, are words in $L_{f(u_i)}$. If all $u_i$ are $\text{ncb}$-words, we are done. But in general, it is not the case. Therefore, we
define the following map $\varphi$: For an \textbf{nbc}-word $w$ we set $\varphi(w) := w$. If $w$ is not an \textbf{nbc}-word, we construct the above subdivision and set

$$\varphi(w) := u_1 \parallel v_1 \parallel \varphi(u_2) \parallel v_2 \parallel \ldots \parallel \varphi(u_s) \parallel v_s.$$ 

Since the word $w$ is of finite length the recursion, is finite and $\varphi(w)$ produces a subdivision of the word $w$.

Since each $\varphi(w)$ ends with a word $v$, which is possibly the empty word $\varepsilon$, the $u$'s and $v$'s do not always alternate in $\varphi(w)$. In order to define the desired subdivision, we therefore have to modify $\varphi(w)$:

- If we have the situation $v_i \parallel u_{i+1}$ such that $v_i, v_{i+1}$ are descending chains of variables, possibly $\varepsilon$, then by construction we have that the word $v_i v_{i+1}$ is a descending chain of variables. We replace the subdivision $v_i \parallel u_{i+1}$ by the word $v_i v_{i+1}$.

The construction implies that the resulting subdivision fulfills all desired properties. Let $\phi_1$ be the map which associates to each word $w$ the corresponding subdivision.

Clearly, this subdivision is unique and therefore $\phi_1$ is an injection.

**Proof of Claim 2.** Let $\phi_1(w) = u_1 \parallel v_1 \parallel u_2 \parallel v_2 \parallel \ldots \parallel u_s \parallel v_s$ be a subdivision of Claim 1. We construct the image under $\phi_2$ by induction.

**(R)** If $f(u_s) \leq f(u_a)$ and there exists a variable $x$ with $u_1 v_1 \cdots u_{s-1} v_{s-1}$ with $x f(u_s) \in a$, we replace $v_{s-1}$ by $v_{s-1} := v_{s-1} x f(u_s)$, else we replace $u_s$ by $u_s := u_s x f(u_s)$. Finally, we replace $v_s$ by the $v_s'$ such that $v_s = x f(u_s) v_s'$.

We repeat this process until $v_s' = \varepsilon$. We get a word

$$u_1 \parallel v_1 \parallel \ldots \parallel u_{s-1} \parallel v_{s-1}' \parallel u'_s,'$$

such that $u_i, v_i$, for $i = 1, \ldots, s - 2$, and $u_s, v_s$ are as before, $v_{s-1}'$ is a descending chain of variables and for $u'_s$ we have:

- (i) If there exist variables $x_i \parallel u'_s$ with $i < f(u'_s)$ and $x_j \parallel u_1 v_1 \cdots u_{s-1} v_{s-1}'$ such that $x_i x_j \in a$, then $j < i$.

Now we repeat the same process for $u_{s-1} \parallel v_{s-1}'$. We get a word

$$u_1 \parallel v_1 \parallel \ldots \parallel u_{s-2} \parallel v_{s-2}' \parallel u_{s-1}' \parallel u'_s,'$$

such that $u_i, v_i$ are from the original decomposition and $u_s', v_{s-1}'$ have property (i).

We repeat this process for all words $u_i \parallel v_i$ and we reach a sequence of words

$$\phi_{2,1}(\phi_1(w)) := u'_1 \parallel u'_2 \parallel \ldots \parallel u'_{s-1} \parallel u'_s.$$

By construction this sequence satisfies the conditions (i), (ii), and (v).

Note that our construction implies that each word $u'_s$ has a unique decomposition $u'_s = u''_i v''_i$ such that $u''_i$ is either a variable or an \textbf{nbc}-word in $L(f(u''_i))$ and $v''_i$ is a descending chain of variables. Now we begin with $v''_1$ and permute the variables with respect to the rule (R) to the right, if necessary, and go on by induction. It is clear that these two algorithms are inverse to each other and therefore $\phi_{2,1}$ is an injection onto its image.

In order to satisfy conditions (iii), (iv), and (vi), we define an injective map $\phi_{2,2}$ on the image of $\phi_{2,1}$. The composition $\phi_2 := \phi_{2,2} \phi_{2,1}$ gives then the desired map.

Let $\phi_{2,1}(\phi_1(w)) = u_1 \parallel u_2 \parallel \ldots \parallel u_s \parallel v_s$. Let $i$ be the smallest index such that $\gcd(u_i, u_{i+1}) = 1$ and $f(u_i) < f(u_{i+1})$. By construction the word $u_i = u_i v_i$ has a decomposition such that $v_i$ is a descending chain of variables and $f(u_i) < f(u_{i+1})$ ($v_i$ was constructed by the map $\phi_{2,1}$). The word $u_{i+1}$ has a decomposition $u_{i+1} = u'_{i+1} v_{i+1}$ such that $u'_{i+1}$ is either a variable or an \textbf{nbc}-word and $v_{i+1}$ a descending chain of variables. We replace $u_i \parallel u_{i+1}$ by the new word $\varphi(u_i \parallel u_{i+1}) := u'_{i+1} c(v, v_{i+1})$ where $c(v, v_{i+1})$ is the descending chain of variables consisting of the variables of $v_i$ and $v_{i+1}$.

We repeat this procedure until there are no words $u_i, u_{i+1}$ with $\gcd(u_i, u_{i+1}) = 1$.
and \( f(u_i) < f(u_{i+1}) \).

It is straightforward to check that the resulting sequence

\[
\phi_{2,2}\phi_{2,1}(\phi_1(w)) := \tilde{u}_1|\tilde{u}_2|\ldots|\tilde{u}_{s-1}|u_{\tilde{s}}
\]

satisfies all desired conditions.

To reverse the map \( \phi_{2,2} \), we apply to each word \( u_i \) the maps \( \phi_1 \) and \( \phi_{2,1} \). Then it is easy to see that the sequence

\[
\phi_{2,1}\phi_1(u_1)||\phi_{2,1}\phi_1(u_2)||\ldots||\phi_{2,1}\phi_1(u_{s-1})||\phi_{2,1}\phi_1(u_s)
\]

is the preimage of \( \phi_{2,2} \). Therefore, \( \phi_{2,2} \) is an injection and the map \( \phi_2 := \phi_{2,2}\phi_{2,1} \) is the desired injection.

**Proof of Claim 3:** Let \( \phi_2\phi_1(w) = u_1||u_2||\ldots||u_{s-1}||u_s \) be a sequence from Claim 2. In order to satisfy the desired condition, we construct a map \( \phi_3 \) similar to \( \phi_{2,2} \).

Let \( i \) be the largest index such that \( \gcd(\lcm(u_1, \ldots, u_i), u_{i+1}) = 1 \). Then it follows from Claim 2 that \( f(u_i) > f(u_{i+1}) \). If we replace \( u_i||u_{i+1} \) by a new word which is constructed in a similar way as in the map \( \phi_{2,2} \), we risk to violate condition (v) from Claim 2. Therefore, we first have to permute the word \( u_{i+1} \) in the correct position. Let \( l < i + 1 \) be the smallest index such that there exists an index \( t > f(u_{i+1}) \) with \( x_l | u_i \) and \( x_l x_{f(u_{i+1})} \in a \). By Condition (iv) from Claim 2, such an index always exists. We replace the sequence \( u_1||u_2||\ldots||u_{s-1}||u_s \) by the sequence

\[
\varphi(u_1)||\varphi(u_1)||\varphi(u_{i+1})||\varphi(u_{i+1})||\ldots||u_i||u_{i+2}||\ldots||u_s,
\]

where \( \varphi(u_i||u_{i+1}) \) is the map from the construction of \( \phi_{2,2} \) of Claim 2. Now the construction implies that all conditions of Claim 2 are still satisfied.

We repeat this procedure until the sequence satisfies the desired condition.

To reverse this procedure we reverse the map \( \varphi \) with the maps \( \phi_1 \) and \( \phi_2 \) and permute the words to the right until Condition (vi) from Claim 2 is satisfied. It follows that \( \phi_3 \) is an injection onto its image.

**Proof of Claim 4.** Let \( \phi_2\phi_1(w) = w_1||w_2||\ldots||w_s \) be a sequence from Claim 3. We now construct a bijection between these sequences of words and the ordered monomials \( Y_{I_1} \cdots Y_{I_r} \) with \( c(I_1 \cup \ldots \cup I_r) = 1 \) and \( \min(I_1) < \min(I_j) \) for all \( j = 2, \ldots, r \). We now assume:

**Assumption A:**

(a) For each \( \text{nbc-set } I \) and each index \( i \) with \( x_i \mid m_I = \lcm(I) \), there exists a unique word \( \psi(I) := w \) such that \( w = x_iw' \) and \( w \) satisfies conditions (i) - (iii) from Claim 2.

(b) For each word \( w \) satisfying conditions (i) - (iii) from Claim 2, there exists a unique \( \text{nbc-set } \varphi(w) := I \).

In addition, the maps \( \psi \) and \( \varphi \) are inverse to each other.

We now prove Claim 4:

Let \( Y_{I_1} \cdots Y_{I_s} \) be an ordered monomial with \( c(I_1 \cup \ldots \cup I_s) = 1 \) and \( \min(I_1) < \min(I_j) \) for \( j = 2, \ldots, s \). Let \( j_t \) be the smallest index \( i \) such that \( x_i \mid \lcm(I_t) \) and either

- there exists a variable \( x_t \mid w_1w_2\cdots w_{t-1} \) with \( t > i \) and \( x_i x_t \in a \)
- or \( x_i \mid \lcm(I_t, I_{t+1}, \ldots, I_{i-1}) \).

Such an index always exists since \( \gcd(m_{I_1 \cup I_2 \cup \ldots \cup I_{i-1}}, m_{I_t}) \neq 1 \). By definition the variables \( Y_t, Y_s \) commute if \( \gcd(m_t, m_s) = 1 \). It is easy to see that one can reorder the monomial \( Y_{I_1} \cdots Y_{I_s} \) such that if \( \gcd(m_{I_t}, m_{I_{t+1}}) = 1 \), we have \( j_t > j_{t+1} \). We now construct a bijection between monomials \( Y_{I_1} \cdots Y_{I_s} \) ordered in that way and the sequences of Claim 3.
Let \( \phi_1 \phi_2 \phi_1 (w) = w_1 || w_2 || \ldots || w_s \) be a sequence of Claim 3 and \( I_j \) be the \( \text{nbc} \)-sets corresponding to the words \( w_j \). Then we associate to the sequence the following monomial
\[
\phi_4 (w_1 || w_2 || \ldots || w_s) := Y_1 \cdots Y_s.
\]
Condition (i) from Claim 3 and Condition (vi) from Claim 2 imply that we get an ordered monomial.

On the other hand, consider an ordered monomial \( Y_{l_1} \cdots Y_{l_s} \). We associate to \( Y_{l_1} \) the corresponding \( \text{nbc} \)-word \( w_1 \) whose front letter is \( x_{\min(l_1)} \).

For \( l = 2, \ldots, s \) let \( w_l \) be the word corresponding to \( I_l \) whose front letter is \( x_{l}\).

It follows directly from the construction that the sequence \( w_1 || w_2 || \ldots || w_s \) satisfies all desired conditions.

Conditions (iv) and (v) of Claim 2 imply that both constructions are inverse to each other and therefore \( \phi_4 \) is a bijection.

In order to finish our proof, we have to verify Assumption A.

To a word \( w = x_{j_1} \cdots x_{j_r} \) satisfying Conditions (i) - (iii) we associate a graph on the vertex set \( V = [n] \). The edges are constructed in the following way: We set \( E := \{\{j_1, j_2\}\} \). For \( j_s \) there exists an index \( 0 \leq l < s \) such that \( x_{j_s} x_{j_s} \in \mathfrak{a} \). Let \( P_{j_s} \) be the set of those indices. Now let \( l_2 \) be the maximum of \( P_{j_2} \). If \( E \cup \{\{j_2, j_2\}\} \) contains no broken circuit (with respect to the lexicographic order), we set \( E := E \cup \{\{j_2, j_2\}\} \).

Else we set \( P_{j_2} := P_{j_2} \setminus \{l_2\} \) and repeat the process. It is clear that there exists at least one index in \( P_{j_2} \) such that the constructed graph contains no broken circuit. We repeat this for \( P_{j_3}, P_{j_4}, \ldots, P_{j_r} \). By construction we obtain a graph which contains no broken circuit. Now graphs without broken circuits are in bijection with the \( \text{nbc} \)-sets (define \( I := \{x_i x_j \mid \{i,j\} \in E\} \)).

Given an \( \text{nbc} \)-graph and a vertex \( i \) such that there exist \( j \in V \) with \( \{i, j\} \in E \), we construct a word \( w \) satisfying Conditions (i) - (iii) by induction: Assume we can construct to each graph of length \( \nu \) and each vertex \( i \) a word \( w \) which satisfies the desired conditions.

Given a graph of length \( \nu + 1 \) and a vertex \( i \). Let \( P_i := \{i < j \mid \{i, j\} \in E\} \) and \( E_1 := E \setminus \{\{i, j\} \in E \mid j \in P_i\} \). Then \( E \setminus E_1 \) decomposes in \( |P_i| + 1 \) connected components. One component is the vertex \( i \) and for each \( j > i \) we have exactly one component \( G_j \) with \( j \in G_j \). By induction we can construct words \( w_j \) corresponding to \( G_j \). Now assume \( P_i = \{j_1 < \ldots < j_r\} \). We set \( w := i w_{j_1} \cdots w_{j_r} \). Finally, we permute \( x_t \in w_j \), with \( t < j_{t+1} \) to the right until it is in the correct position.

Let \( w \) be a word constructed from a graph. Assume there is \( x_t \in w_j \) which was permuted to the right in the word \( w_{j'} \), \( j < j' \). If there exists an index \( l \) such that \( x_l x_t \in \mathfrak{a} \), and \( l > t \), then we would add an edge \( \{l, t\} \). But since \( x_t \in w_j \) and the original graph was connected, this leads to a broken circuit for the constructed graph. Therefore, the edge for the vertex \( t \) has to be constructed with the corresponding index in \( w_j \). This proves that both constructions are inverse to each other. \( \square \)

7.3. Idea for a Proof in the General Case. In this section we outline a program which we expect to yield a proof of Conjecture \ref{conj:main} in general.

The only way to prove the conjecture is to find a minimal \( A \)-free resolution of the field \( k \), which in general is a very hard problem. With the Algebraic Discrete Morse theory one can minimize a given free resolution, but one still needs a free resolution to start. The next problem is the connection to the minimized Taylor resolution of the ideal \( \mathfrak{a} \).

The Eagon complex is an \( A \)-free resolution of the field \( k \) which has a natural connection to the Taylor resolution of the \( \mathfrak{a} \) since the modules in this complex are tensor products of \( H_s(K^A) \simeq T^M \otimes_{S, k} T \). The problem with the Eagon complex is that the differential is defined recursively.
In the first part of this section, we define a generalization of the Massey operations which gives us an explicit description of the differential of the Eagon complex. We apply Algebraic Discrete Morse theory to the Eagon complex. The resulting Morse complex is not minimal in general, but it is minimal if for example $H_*(K^A)$ is an M-ring. In order to prove our conjecture in general, one has to find an isomorphism between the minimized Eagon complex and the conjectured minimal resolution. We can not give this isomorphism in general, but with this Morse complex we can explain our conjecture.

For the general case, we think that one way to prove the conjecture is the following:

- calculate the Eagon complex.
- minimize it with the given acyclic matching.
- find a degree-preserving $k$-vectorspaces-isomorphism to the ring $K_\bullet \otimes_k R$.

As before we fix one standard matching $M$ on the Taylor resolution of $a$. The set of cycles $\{\phi(I) \mid I \notin M\}$ is a system of representatives for the Koszul homology. With the product on the homology, we can define the following operation:

For two sets $J, I \notin M$ we define:

$$I \wedge J := \begin{cases} 0 & \text{gcd}(m_I, m_J) \neq 1 \\ 0 & \text{gcd}(m_I, m_J) = 1, I \cup J \in M \text{ and } [\phi(I)][\phi(J)] = 0 \\ \sum_{L \notin \mathcal{M}} a_L L & [\phi(I)][\phi(J)] = [\phi(I \cup J)] \text{ and } I \cup J \notin M \end{cases}$$

Now we can define the function $(I, J) \mapsto g(I, J) \in K^A_*$ such that

$$\partial(g(I, J)) := \phi(I) \phi(J) - \frac{m_I m_J}{m_{I \cup J}} \phi(I \wedge J).$$

By Proposition 5.1 this function is well defined.

We now define a function for three sets $\gamma(I_1, I_2, I_3)$ by:

$$\gamma(I_1, I_2, I_3) := \phi(I_1) g(I_2, I_3) + (-1)^{|I_1|+1} g(I_1, I_2) \phi(I_3) + (-1)^{|I_1|+1} \frac{m_{I_1 \cup I_2}}{m_{I_1 \cup I_2} m_{I_2 \cup I_3}} g(I_1 \wedge I_2, I_3).$$

It is straightforward to prove that $\partial(\gamma(I_1, I_2, I_3)) = 0$. If $\gamma(I_1, I_2, I_3)$ is a boundary for all sets $I_1, I_2, I_3$, we can define $g(I_1, I_2, I_3)$ such that $\partial(g(I_1, I_2, I_3)) = \gamma(I_1, I_2, I_3)$.

Similar to the Massey-operation we go on by induction:

Assume $\gamma(I_1, \ldots , I_l)$ vanishes for all $l$-tuples $I_1, \ldots , I_l$, with $l \geq \nu - 1$. Then there exist cycles $g(I_1, \ldots , I_l)$ such that $\partial(g(I_1, \ldots , I_l)) = \gamma(I_1, \ldots , I_l)$. We then define:

$$\gamma(I_1, \ldots , I_\nu) := \phi(I_1) g(I_2, \ldots , I_\nu) + (-1)^{\sum_{j=1}^{\nu-1} |I_j|+1} g(I_1, \ldots , I_{\nu-1}) \phi(I_\nu)$$

$$+ \sum_{i=2}^{\nu-2} (-1)^{\sum_{j=1}^{i} |I_j|+1} g(I_1, \ldots , I_i) g(I_{i+1}, \ldots , I_\nu)$$

$$+ \sum_{i=1}^{\nu-1} (-1)^{\sum_{j=1}^{i} |I_j|+1} \frac{m_{I_i \cup I_i+1}}{m_{I_i \cup I_i+1} m_{I_i \cup I_i+2}} g(I_1, \ldots , I_{i-1}, I_i \wedge I_{i+1}, I_{i+2}, \ldots , I_\nu)$$

$$- (-1)^{\sum_{j=1}^{\nu-2} |I_j|+1} \frac{m_{I_{\nu-1} \cup I_\nu}}{m_{I_{\nu-1} \cup I_\nu} m_{I_{\nu-2} \cup I_\nu}} g(I_1, \ldots , I_{\nu-2}, I_{\nu-1} \wedge I_\nu).$$

It is straightforward to prove that $\gamma(I_1, \ldots , I_\nu)$ is a cycle. Therefore, we get an induced operation on the Koszul homology. Since the first three summands are exactly the summands of the Massey operations, we call $\gamma(I_1, \ldots , I_\nu)$ the $\nu$-th generalized Massey operations.

From now on we assume that all generalized Massey operations vanish. We then can give an explicit description of the Eagon complex:

We define free modules $X_i$ to be the free $A$-modules over $I \notin M$ with $|I| = i$. It is
clear that we have $X_i \otimes_A k \simeq H_i(K_A)$. The Eagon complex is defined by a sequence of complexes $Y^i$, with $Y^0 = K^n$ and $Y^n$ is defined by

\[
Y_i^{n+1} := Y_{i+1}^n \oplus Y_0^n \otimes X_i, \quad i > 0,
\]

\[
Y_0^{n+1} = Y_1^n.
\]

Let $Z_i(Y^*)$ and $B_i(Y^*)$ denote cycles and boundaries, respectively. The differentials $d^s$ on $Y^s$ are defined by induction. $d^0$ is the differential on the Koszul complex. Assume $d^{s-1}$ is defined. One has to find a map $\alpha$ that makes the diagram in Figure 1 commutative: One can then define $d^s := (d^{s-1}, \alpha)$. 

![Diagram](image)

**Figure 1.**

The map $d^s$ satisfies $H_i(Y^s) = H_0(Y^s) \otimes X_i$ and $B_{i-1}(Y^s) = d^s(Y_1^s) = Z_i(Y^{s-1})$. The first property allows us to continue this procedure for $s + 1$ and the second gives us exactness of the following complex:

\[
F_s : \cdots \rightarrow Y_0^{s+1} \xrightarrow{d^s} Y_0^s \xrightarrow{d^{s-1}} Y_0^{s-1} \rightarrow \cdots \rightarrow Y_0^1 \rightarrow k.
\]

Note that to make the diagram commutative, it is enough to define $\alpha(n \otimes f)$ for all generators $n \otimes f$ of $Y_0^s \otimes X_i$ such that $\alpha(n \otimes f) = (m, d^{s-1}(n) \otimes f)$, with $m \in Y_i^{s+1}$ and the property that $d^{s-1}(m) + d^{s-1}(d^{s-1}(n) \otimes f) = 0$.

The $\nu$-th module of the complex $Y^s$ is given by $Y^s = K_i \otimes X_i \oplus \cdots \otimes X_n$, with $j + r + \sum_{j=1}^r i_j = \nu + s$. We fix an $R$-basis of $Y^s$ by $e_L \otimes I_1 \otimes \cdots \otimes I_r$ with $I_j \notin M$ and $e_L = e_{i_1} \land \cdots \land e_{i_r}$. We are now able to define the maps $\alpha$: Since all generalized Massey operations vanish, there exists elements $g(I_1, \ldots, I_r)$ such that $\partial(g(I_1, \ldots, I_r)) = (1 \otimes I_1, \ldots, 1 \otimes I_r)$

**Lemma 7.8.** Suppose that $d^{s-1} : Y_i^{s-1} \rightarrow Y_i^{s-1}$ is such that

\[
d^{s-1}(e_L \otimes I_1 \otimes \cdots \otimes I_r) = \partial^K(e_L) \otimes I_1 \otimes \cdots \otimes I_r
\]

\[
+ (-1)^{|L|}e_L \phi(I_1) \otimes I_2 \otimes \cdots \otimes I_r,
\]

\[
+ (-1)^{|L|} \sum_{j=1}^{r-1} (-1)^{|j|+1} |I_j|+1 \frac{m_{i_1} \cdots m_{i_{j+1}}}{m_{I_1 \cup I_{j+1}}} e_L \otimes I_1 \otimes \cdots \otimes I_j \land I_{j+1} \otimes \cdots \otimes I_r,
\]

\[
+ (-1)^{|L|} \sum_{j=1}^{r-1} (-1)^{|j|+1} e_L g(I_1, \ldots, I_{j+1}) \otimes I_{j+2} \otimes \cdots \otimes I_r.
\]

If $n := e_L \otimes I_1 \otimes \cdots \otimes I_r \in Y^s$ and $J$ is a generator of $X_i$, we define $\alpha(n \otimes J)$ to be the map that sends $n \otimes J$ to $(m, d^{s-1}(n) \otimes J)$ with

\[
m = (-1)^{|L|}(-1)^{|j|+1} |I_j|+1 \frac{m_{I_1 \cup I_{j+1}}}{m_{I_j \cup J}} e_L \otimes I_1 \otimes \cdots \otimes I_{r-1} \otimes I_r \land J
\]

\[
+ (-1)^{|L|}(-1)^{|j|+1} e_L g(I_1, \ldots, I_r, J).
\]

Then $\alpha$ makes the diagram in Figure 1 commutative.

**Proof.** We only have to check that $d^{s-1}(m) + d^{s-1}(d^{s-1}(n) \otimes f) = 0$. This is a straightforward calculation and is left to the reader. \qed
Corollary 7.9. The map $d^a$ can be defined as follows:

$$d^a(e_L \otimes I_1 \otimes \ldots \otimes I_r) = \partial^K(e_L) \otimes I_1 \otimes \ldots \otimes I_r$$

$$+ (-1)^{|I|} e_L \otimes (I_1 \otimes I_2 \otimes \ldots \otimes I_r)$$

$$+ (-1)^{|I|} \sum_{j=1}^{r-1} (-1)^{\sum_{i=1}^{j} |I_j| + 1} \frac{m_{I_j} m_{I_{j+1}}}{m_{I_j} m_{I_{j+1}}} e_L \otimes I_1 \otimes \ldots \otimes I_j \otimes I_{j+1} \otimes \ldots \otimes I_r$$

$$+ (-1)^{|I|} \sum_{j=1}^{r-1} (-1)^{\sum_{i=1}^{j} |I_j| + 1} e_L \otimes g(I_1, \ldots, I_{j+1}) \otimes I_{j+2} \otimes \ldots \otimes I_r.$$ 

With this corollary we get an explicit description of the Eagon resolution of $k$ over $A$.

In order to define the acyclic matching, we first use Theorem 5.2 to define the Eagon complex with the ring $H_*(K^A) \cong R' = k[Y_I \mid cl(I) = 1, I \notin M]/e'$ instead of $H_*. The operation $I \wedge J$ then is nothing but the multiplication $Y_I Y_J$ in $R'$. We write $y_I$ for the class of $Y_I$ in $R'$.

It is clear that this complex is not minimal in general. The idea now is to minimize this complex via Algebraic Discrete Morse theory. It is easy to see, that the only invertible coefficient occurs by mapping $\ldots \otimes y_I \otimes y_J \otimes \ldots$ to the element $\ldots \otimes y_I y_J \otimes \ldots$, with $gcd(m_{I_1}, m_{J_1}) = 1$. The idea is to match all such basis elements, with $I \wedge J = I \cup J$ and $I \cup J \notin M$. In order to do this, we have to define an order on the variables $y_I$ with $I \notin M$: We order the sets $I$ by cardinality and if two sets have the same cardinality by the lexicographic order on the multidegrees $m_{I_1}, m_{J_1}$. The monomials in $R'$ are ordered by the degree-lexicographic order. The acyclic matching is similar to the Morse matching on the normalized Bar resolution (see [10]). Since $M$ is a standard matching on the Taylor resolution, we know that if $I_1 \cup I_2 \cup \ldots \cup I_r \notin M$ with $cl(I_j) = 1$ and $gcd(m_{I_j}, m_{I_{j+1}}) = 1$ for all $j \neq j'$, then it follows that $I_2 \cup \ldots \cup I_r \notin M$. Therefore, the following matching is well defined:

$$e_L \otimes y_{I_1} \otimes y_{I_2} \cdots y_I \otimes \ldots \to e_L \otimes y_{I_1} y_{I_2} \cdots y_I \otimes \ldots,$$

where $I_1 < I_2 < \ldots < I_r$ and $I_1 \cup I_2 \cup \ldots \cup I_r \notin M$ and $cl(I_j) = 1$ and $gcd(m_{I_j}, m_{I_{j+1}}) = 1$ for all $j \neq j'$. On the remaining basis elements we do the same matching on the second coordinate, and so on. The exact definition of the acyclic matching and the proof is given in Definition 3.1 of [10].

We describe the remaining basis elements, as in [10], by induction. $[y_I | u_1]$ with $u_1 = y_{I_1} \cdots y_{I_r}$ is called fully attached (see Definition 3.3 of [10]) if one of the following conditions is satisfied:

1. $r = 1$ and $gcd(m_{I_1}, m_{J_1}) \neq 1$ or $y_I > y_{J_1}$,
2. $gcd(m_{I_1} m_{J_1}) = 1$ for all $i$ and $I \cup J \subseteq M$, and for all $1 \leq i \leq r$ we have $I \cup J_1 \cup \ldots \cup J_i \notin M$.

A tuple $[y_I | u_1] \ldots | u_r]$ is called fully attached if $[y_I | u_1] \ldots | u_{r-1}]$ is fully attached, one of the following properties is satisfied and $u_r$ is minimal in the sense that there is no proper divisor $v_r | u_r$ satisfying one of the conditions below:

1. $u_r$ is a variable and $gcd(m_{u_{r-1}}, m_{u_r}) \neq 1$,
2. $u_r, u_{r-1}$ are both variables and $u_{r-1} > u_r$,
3. $[y_I | u_1] \ldots | u_{r-2} | u_r]$ is a fully attached tuple and $u_{r-1} > u_r$,
4. $u_{r-1} = y_{I_1} \cdots y_{I_r} u_r = y_{J_1} \cdots y_{J_s} \otimes \text{such that } gcd(m_{u_{r-1}, m_{u_r}}) = 1$ and $I_1 \cup \ldots \cup I_l \cup J_1 \cup \ldots \cup J_s \in M$.

Here $m_u := \text{lcm}(I_1 \cup \ldots \cup I_r)$ if $u = y_{I_1} \cdots y_{I_r}$.

The basis of the Morse complex is given by elements $e_L | w$, where $w$ is fully attached tuple. If $H_*(K^A)$ is an $M$-ring, the Morse complex is minimal since in this case the fully attached tuple has the form $[y_I | y_{I_2} \cdots | y_{I_r}]$. In order to prove
Conjecture 8.2 one has to find an isomorphism between the fully attached tuples and the monomials in \( R \).

We cannot give this isomorphism in general, but we think that this Morse complex helps for the understanding of our conjecture:

Let \([y_1, y_2, \ldots, y_l]\) be a fully attached tuple, with \(y_1 > \ldots > y_l\). We map such a tuple to the monomial \(Y_1 \cdots Y_l \in R\). Clearly, this map preserves the degree. We get a problem if \([y_1|u_1]\ldots[u_r]\) is a fully attached tuple and \(u_1 = I_1 \cup \ldots \cup I_r\) with \(r > 1\). For example, assume \(J \mapsto I_1 \cup \ldots \cup I_r \in \mathcal{M}_r\), with \(\text{cl}(J) = \text{cl}(I_1) = \ldots = \text{cl}(I_r) = 1\) and \(\gcd(m_{I_j}, m_{I_{j'}}) = 1\) for \(j \neq j'\), is matched. Assume further \(y_1 < \ldots < y_l\). Then \([y_1|y_2|\ldots|y_l]\) is a fully attached tuple. We cannot map \([y_1, y_2, \ldots, y_l]\) to \(Y_1 Y_2 \cdots Y_l\), since in \(R\) the variables commute, i.e. \(Y_1 Y_2 \cdots Y_l = Y_1 Y_{l-1} \cdots Y_2\) and the tuple \([y_1, y_{l-1}, \ldots, y_1]\) maps already to this element. But we can define

\[ [y_1|y_2|\ldots|y_l] \mapsto Y \in R. \]

The degree of \(Y \in R\) is \(|J| + 1\) and the homological degree of \([y_1|y_2|\ldots|y_l]\) is

\[ |I_1| + 1 + |I_2| + \ldots + |I_r| + 1 = |I_1| + \ldots + |I_r| + 1 + 1 = |J| + 1, \]

therefore this map preserves the degree.

These facts demonstrate that the variables \(Y_1\), for which \(I \in \mathcal{M}\), \(\text{cl}(I) = 1\), and \(I \notin \mathcal{M}_1\), are necessary. We consider this as a justification of our conjecture.

8. Applications to the Golod Property of Monomial Rings

In this section we give some applications to the Golod property. Remember that a ring \(A\) is Golod if and only if one of the following conditions is satisfied (see [27]):

\begin{equation}
P_k^A(x, t) = \frac{\prod_{i=1}^{n} (1 + x_i t)}{1 - t \sum_{\alpha \in \mathbb{N}^n, i \geq 0} \dim_k(\text{Tor}^S_k(A, k)x^\alpha t^i)}.
\end{equation}

(8.1)

All Massey operations on the Koszul homology vanish.

If an algebra satisfies property (P), then we get in the monomial case the following equivalence:

**Theorem 8.1.** If \(A = S/\mathfrak{a}\) satisfies property (P), then \(A\) is Golod if and only if one of the following conditions is satisfied:

1. For all subsets \(I \subset \text{MinGen}(\mathfrak{a})\) with \(\text{cl}(I) \geq 2\) we have \(I \in \mathcal{M}\) for any standard matching \(\mathcal{M}\).
2. The product (i.e. the first Massey operation) on the Koszul homology is trivial.

**Proof.** Property (P) implies the equivalence of (8.1) and the first condition. Theorem 5.2 implies the equivalence of the first and the second condition. \(\square\)

**Corollary 8.2.** If \(A = S/\mathfrak{a}\) satisfies one of the following conditions, then \(A\) is Golod if and only if the first Massey operation vanishes.

1. \(\mathfrak{a}\) is generated in degree two,
2. \(H_*(K^A)\) is an \(M\)-ring and either there is a homomorphism \(s : H_*(K^A) \to \mathbb{Z}_*(K^A)\) such that \(\pi \circ s = \text{id}_{H_*(K^A)}\) or the minimal resolution of \(\mathfrak{a}\) has the structure of a differential graded algebra.

**Proof.** In the previous section we proved property (P) in these cases, therefore the result follows from the theorem above. \(\square\)
Recently, Charalambous proved in [4] a criterion for generic ideals to be Golod. Remember that a monomial ideal \( \mathfrak{a} \) is generic if the multidegree of two minimal monomial generators of \( \mathfrak{a} \) are equal for some variable, then there is a third monomial generator of \( \mathfrak{a} \) whose multidegree is strictly smaller than the multidegree of the least common multiple of the other two. It is known that for generic ideals \( \mathfrak{a} \) the Scarf resolution is minimal. Charalambous proved the following proposition:

**Proposition 8.3** ([4]). Let \( \mathfrak{a} \subseteq S \) be a generic ideal. \( A = S/\mathfrak{a} \) is Golod if and only if \( m_im_J \neq m_{i\cup J} \) whenever \( I \cup J \in \Delta_S \) for \( I, J \subset \text{MinGen}(\mathfrak{a}) \).

Here \( \Delta_S \) denotes the Scarf resolution.

Assuming property \((P)\), our Theorem 5.1 gives a second proof of this fact:

**Proof.** It is easy to see that the condition

\[
m_im_J \neq m_{i\cup J} \text{ whenever } I \cup J \in \Delta_S
\]

is equivalent to fact that the product on the Koszul homology is trivial. Thus, Theorem 8.1 implies the assertion. \( \square \)

We have the following criterion:

**Lemma 8.4.** Let \( A = S/\mathfrak{a} \) with \( \mathfrak{a} = \langle m_1, \ldots, m_t \rangle \).

1. If \( \gcd(m_i, m_j) \neq 1 \) for all \( i \neq j \), then \( A \) is Golod (see [5], [8]).
2. If \( A = S/\mathfrak{a} \) is Golod, then \( \mathfrak{a} \) satisfies the gcd-condition.

**Proof.** If a ring \( A \) is Golod, then the product on \( H_*(K^A) \) is trivial. This implies \( Y_iY_j = 0 \) if \( \gcd(m_i, m_j) = 1 \). With Theorem 8.2 it follows that all sets \( I \cup J \) with \( \gcd(m_i, m_j) = 1 \) are matched. In particular, all sets \( \{m_i, m_j\} \) with \( \gcd(m_i, m_j) = 1 \). Such a set can only be matched with a set \( \{m_i, m_j, m_k\} \) with the same lcm. But this implies that there must exist a third generator \( m_r \) with \( m_r|m_i m_j \). \( \square \)

The following counterexample shows that the converse of the second statement is false: Let \( \mathfrak{a} := \langle xy, yz, zw, wt, xt \rangle \) be the Stanley Reisner ideal of the triangulation of the 5-gon. It is easy to see that \( \mathfrak{a} \) satisfies the gcd-condition. But \( \mathfrak{a} \) is Gorenstein and therefore not Golod. But we have:

**Theorem 8.5.** If \( A = S/\mathfrak{a} \) has property \((P)\) and \( \mathfrak{a} \) satisfies the strong gcd-condition, then \( A \) is Golod.

**Proof.** We prove that \( H_*(K^A) \) is an \( M \)-ring and isomorphic as an algebra to the ring

\[
R := k(Y_i \mid I \notin \mathcal{M}, c(I) = 1) / \langle Y_iY_j \text{ for all } I, J \notin \mathcal{M}_0 \cup \mathcal{M} \rangle,
\]

where \( \mathcal{M}_0 \) is the sequence of matchings constructed in Proposition 3.1 in order to obtain the complex \( \mathcal{T}_{\gcd} \) and \( \mathcal{M} \) is a standard matching on the complex \( \mathcal{T}_{\gcd} \). It follows that the first Massey operation is trivial and then Theorem 5.1 implies the assertion.

The idea is to make the same process as in Section 5 with the complex \( \mathcal{T}_{\gcd} \) from Proposition 3.1 instead of the Taylor resolution \( \mathcal{T}_* \). Since all sets \( I \) in \( \mathcal{T}_{\gcd} \) satisfy \( c(I) = 1 \), the result follows directly from property \((P)\).

Note that \( \mathcal{M}_0 \) satisfies all conditions required in the proof of Proposition 5.1 except the following: Assume \( I \cup J \in \mathcal{M}_0 \) with \( \gcd(m_i, m_j) = 1 \) and \( I, J \notin \mathcal{M}_0 \). Then there exists a set \( \tilde{I} \) such that \( \tilde{I} \to I \cup J \in \mathcal{M}_0 \). It follows

\[
0 = \partial^2(\tilde{I}) = \partial(I \cup J) + \sum_{L \notin \mathcal{M}_0} a_L L
\]

and therefore as in the proof of Proposition 5.1

\[
\phi(I \cup J) = \sum_{L \notin \mathcal{M}_0} a_L \phi(L) \quad \text{for some } a_L \in k.
\]
In the case of Proposition 8.5 we could guarantee that \( cl(L) \geq cl(I \cup J) \). We can not deduce this fact here, but this is the only difference between \( \mathcal{M}_0 \cup \mathcal{M} \) and a standard matching on the Taylor resolution. Since all sets \( L \) with \( cl(L) \geq 2 \) are matched, we only could have

\[
\phi(I \cup J) = \sum_{\substack{L \in \mathcal{M}_0 \cup \mathcal{M} \colon cl(L) = 1 \; \text{for some } a_L \in k.}}
\]

We prove that this cannot happen. If \( I \cup J \) is matched, then there exists a monomial \( m \) with \( I \cup J \cup \{m\} \rightarrow I \cup J \in \mathcal{M}_0 \). But then, since \( cl(I \cup J \setminus \{n\}) \geq cl(I \cup J) \geq 2 \), by the definition of \( \mathcal{M}_0 \) any image \( I \cup J \cup \{m\} \setminus \{n\} \) is also matched:

\[
I \cup J \cup \{m\} \setminus \{n\} \rightarrow I \cup J \setminus \{n\} \in \mathcal{M}_0.
\]

This proves that the situation above is not possible and we are done. \( \square \)

**Corollary 8.6.** Suppose that \( A = S/\mathfrak{a} \) has property (P). Then \( A \) is Golod if

1. \( \mathfrak{a} \) is shellable (for the definition see [1]),
2. \( \text{MinGen}(\mathfrak{a}) \) is a monomial ordered family (for the definition see [11]),
3. \( \mathfrak{a} \) is stable and \( \# \text{supp}(m) \geq 2 \) for all \( m \in \text{MinGen}(\mathfrak{a}) \),
4. \( \mathfrak{a} \) is \( p \)-Borel fixed and \( \# \text{supp}(m) \geq 2 \) for all \( m \in \text{MinGen}(\mathfrak{a}) \).

Here \( \text{supp}(m) := \{ 1 \leq i \leq n \mid x_i \text{ divides } m \} \).

**Proof.** We order MinGen(\( \mathfrak{a} \)) with the lexicographic order. Then it follows directly from the definitions of the ideals that \( \mathfrak{a} \) satisfies the strong gcd-condition. The assertion follows then from Theorem 8.5. \( \square \)

Theorem 8.5 and the preceding Lemma give rise to the following conjecture:

**Conjecture 8.7.** Let \( \mathfrak{a} = (m_1, \ldots, m_l) \subset S \) be a monomial ideal and \( A = S/\mathfrak{a} \). Then \( A \) is Golod if and only if \( \mathfrak{a} \) satisfies the strong gcd-condition.

In particular: Golodness is independent of the characteristic of \( k \).

It is known that if \( \mathfrak{a} \) is componentwise linear, then \( A \) is Golod (see [5]). One can generalize this result to the following:

**Corollary 8.8.** Let \( \mathfrak{a} \) be generated by monomials with degree 1.

1. If \( \dim_k(\text{Tor}_i^S(S/\mathfrak{a}, k)_{i+j}) = 0 \) for all \( j \geq 2(l-1) \), then \( A = S/\mathfrak{a} \) is Golod,
2. if \( A \) is Golod, then \( \dim_k(\text{Tor}_i^S(S/\mathfrak{a}, k)_{i+j}) = 0 \) for all \( j \geq i(l-2)+2 \).

In particular: If \( A \) is Koszul, then \( A \) is Golod if and only if the minimal free resolution of \( \mathfrak{a} \) is linear.

**Proof.** Let \( I \subset \{m_1, \ldots, m_l\} \) with \( cl(I) = 1 \) and \( \text{lcm}(I) \neq \text{lcm}(I \setminus \{m\}) \) for all \( m \in I \). Then \( l+|I|-1 \leq \deg(I) \leq (l-1)|I| + 1 \). Now assume that \( L = I \cup J \notin \mathcal{M} \) with \( \gcd(m_l, m_J) = 1 \), then \( \deg(L) \geq 2l - 2 + |I \cup J| \), which is a contradiction to \( \dim_k(\text{Tor}_i^S(S/\mathfrak{a}, k)_{i+j}) = 0 \) for all \( j \geq 2l-2 \). Therefore, the product on the Koszul homology is trivial. By the same multidegree reasons it follows that all Massey operations have to vanish, hence \( A \) is Golod.

If \( A \) is Golod, then the product on \( H^*(K^A) \) is trivial, hence (by theorem 8.5) \( J \notin \mathcal{M} \) implies \( cl(I) = 1 \). But for those subsets we have \( l+|I|-1 \leq \deg(I) \leq (l-1)|I| + 1 \). Therefore, it follows that \( \dim_k(\text{Tor}_i^S(S/\mathfrak{a}, k)_{i+j}) = 0 \) for all \( j \geq i(l-2)+2 \). \( \square \)

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References

[1] E. Batzies, *Discrete Morse theory for cellular resolutions*, PhD thesis. Philipps-Universität Marburg (2002).

[2] A. Berglund, *Poincaré Series of Monomial Rings*, arXiv:math.AC/0412282 v1 (2004).

[3] A. Cartier, D. Foata, *Problèmes combinatoires de commutation et réarrangements*, Lecture Notes in Mathematics, Springer (1969).

[4] H. Charalambous, *On the Denominator of the Poincaré Series of Monomial Quotient Rings*, arXiv:math.AC/0412295 v1 (2004).

[5] H. Charalambous, A. Reeves, *Poincare series and resolutions of the residue field over monomial rings*, Comm. in Alg. 23 (1995), 2389-2399.

[6] R. Fröberg *Some complex constructions with applications to Poincaré series*, Semin. d’Algèbre Paul Dubreil, Proc., Paris 1977/78, 31ème Année, Lect. Notes Math. 740 (1979), 272-284.

[7] T.H. Gulliksen, G. Levin, *Homology of local rings*, Queen’s Papers in Pure and Applied Mathematics, 20, Kingston, Ontario: Queen’s University. X (1969), p. 192

[8] J. Herzog, V. Reiner, V. Welker, *Componentwise linear ideals and Golod rings*, Mich. Math. J. 46 (1999), 211-223.

[9] J. E. Hopcroft, R. Motwani, Rotwani, J. D. Ullman, *Introduction to Automata Theory, Languages and Computability*, 2nd edition, Addison-Wesley Longman Publishing Co., Inc. Boston, MA, USA (2000).

[10] M. Jöllenbeck, V. Welker *Resolution of the Residue Class Field via Algebraic Discrete Morse Theory*, arXiv:math.AC/0501179 (2005).

[11] A. Postnikov, B. Shapiro, *Trees, Parking functions, Syzygies, and Deformations of Monomial Ideals*, arXiv:math.CO/0301110 v2 (2003).

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