An application of metric cotype to quasisymmetric embeddings

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Abstract

We apply the notion of metric cotype to show that $L^p$ admits a quasisymmetric embedding into $L^q$ if and only if $p \leq q$ or $q \leq p \leq 2$.

This note is a companion to [4]. After the final version of [4] was sent to the journal for publication I learned from Juha Heinonen and Leonid Kovalev of a long-standing open problem in the theory of quasisymmetric embeddings, and it turns out that this problem can be resolved using the methods of [4]. The argument is explained below. I thank Juha Heinonen and Leonid Kovalev for bringing this problem to my attention.

Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. An embedding $f : X \to Y$ is said to be a quasisymmetric embedding with modulus $\eta : (0, \infty) \to (0, \infty)$ if $\eta$ is increasing, $\lim_{t \to 0} \eta(t) = 0$, and for every distinct $x, y, z \in X$ we have

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right).$$

We refer to [1] and the references therein for a discussion of this notion.

It was not known whether every two separable Banach spaces are quasisymetrically equivalent. This is asked in [6] (see problem 8.3.1 there). We will show here that the answer to this question is negative. Moreover, it turns out that under mild assumptions the cotype of a Banach space is preserved under quasisymmetric embeddings. Thus, in particular, our results imply that $L^p$ does not embed quasisymetrically into $L^q$ if $p > 2$ and $q < p$. The question of determining when $L^p$ is quasisymetrically equivalent to $L^q$ was asked in [6] (see problem 8.3.3 there). We also deduce, for example, that the separable space $c_0$ does not embed quasisymetrically into any Banach space which has an equivalent uniformly convex norm.

We recall some definitions. A Banach space $X$ is said to have (Rademacher) type $p > 0$ if there exists a constant $T < \infty$ such that for every $n$ and every $x_1, \ldots, x_n \in X$,

$$E_{\varepsilon}\left\| \sum_{j=1}^{n} \varepsilon_j x_j \right\|_X^p \leq T^p \sum_{j=1}^{n} \|x_j\|_X^p,$$

where the expectation $E_{\varepsilon}$ is with respect to a uniform choice of signs $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$. $X$ is said to have (Rademacher) cotype $q > 0$ if there exists a constant $C < \infty$ such that for every $n$ and every $x_1, \ldots, x_n \in X$,

$$E_{\varepsilon}\left\| \sum_{j=1}^{n} \varepsilon_j x_j \right\|_X^q \geq \frac{1}{C^q} \sum_{j=1}^{n} \|x_j\|_X^q.$$

We also write

$$p_X = \sup\{p \geq 1 : X \text{ has type } p\} \quad \text{and} \quad q_X = \inf\{q \geq 2 : X \text{ has cotype } q\}.$$
X is said to have non-trivial type if \( p_X > 1 \), and X is said to have non-trivial cotype if \( q_X < \infty \). For example, \( L_p \) has type \( \min\{p, 2\} \) and cotype \( \max\{p, 2\} \) (see for example [5]).

**Theorem 1.** Let X be a Banach space with non-trivial type. Assume that Y is a Banach space which embeds quasisymmetrically into X. Then \( q_Y \leq q_X \).

**Proof.** Let \( f : Y \to X \) be a quasisymmetric embedding with modulus \( \eta \). Assume for the sake of contradiction that X has cotype \( q \) and that \( p := q_Y > q \). By the Maurey-Pisier theorem [2] for every \( n \in \mathbb{N} \) there is a linear operator \( T : \ell^n_p \to Y \) such that for all \( x \in \ell^n_p \) we have \( \|x\|_p \leq \|T(x)\|_Y \leq 2\|x\|_p \). For every integer \( m \in \mathbb{N} \) consider the mapping \( g : \mathbb{Z}_m^n \to X \) given by

\[
g(x_1, \ldots, x_n) = f \circ T \left( \frac{2^n}{m}, \ldots, \frac{2^n}{m} \right).
\]

By Theorem 4.1 in [4] there exist constants \( A, B > 0 \) which depend only on the type and cotype constants of X such that for every integer \( m \geq An^{1/q} \) which is divisible by 4 and every \( h : \mathbb{Z}_m^n \to X \) we have

\[
\sum_{j=1}^n \mathbb{E}_x \left[ \left\| h \left( x + \frac{m}{2} e_j \right) - h(x) \right\|_X^q \right] \leq B^q m^q \mathbb{E}_{x,x} \left[ \|h(x + \varepsilon) - h(x)\|_X^q \right], \tag{1}
\]

where the expectations above are taken with respect to uniformly chosen \( x \in \mathbb{Z}_m^n \) and \( \varepsilon \in \{-1, 0, 1\}^n \) (here, and in what follows we denote by \( \{e_j\}_{j=1}^n \) the standard basis of \( \mathbb{R}^n \).

From now on we fix \( m \) to be the smallest integer which is divisible by 4 and \( m \geq An^{1/q} \). Thus \( m \leq 8An^{1/q} \). For every \( x \in \mathbb{Z}_m^n \), \( j \in \{1, \ldots, n\} \) and \( \varepsilon \in \{-1, 0, 1\}^n \) we have

\[
\left\| \frac{g(x + \varepsilon) - g(x)}{g \left( x + \frac{m}{2} e_j \right) - g(x)} \right\|_X \leq \eta \left( \frac{n^{1/p}}{m} \right) \leq \eta \left( \frac{n^{1/p}}{m} \right).
\]

Thus, using (1) for \( g = h \) we see that

\[
n \mathbb{E}_{x,x} \|g(x + \varepsilon) - g(x)\|_X^q \leq \eta \left( \frac{n^{1/p}}{A} \right)^q \sum_{j=1}^n \mathbb{E}_x \left[ \left\| g \left( x + \frac{m}{2} e_j \right) - g(x) \right\|_X^q \right] \leq \eta \left( \frac{n^{1/p}}{A} \right)^q (8AB)^q n \mathbb{E}_{x,x} \|g(x + \varepsilon) - g(x)\|_X^q.
\]

Canceling the term \( \mathbb{E}_{x,x} \|g(x + \varepsilon) - g(x)\|_X^q \) we deduce that

\[
\eta \left( \frac{n^{1/p}}{A} \right)^q \geq \frac{1}{8AB}.
\]

Since \( p > q \) this contradicts the fact that \( \lim_{t \to 0} \eta(t) = 0 \). \( \square \)

Using the same argument as in [4] (and noting that the snowflake embedding from [3] is a quasisymmetric embedding), we obtain the following complete answer to the question when \( L_p \) embeds quasisymmetrically into \( L_q \).

**Corollary 2.** For \( p, q > 0 \), \( L_p \) embeds quasisymmetrically into \( L_q \) if and only if \( p \leq q \) or \( q \leq p \leq 2 \).
References

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