Cosmological Quantum Metrology

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We investigate the cosmological quantum metrology of Dirac fields in a two-dimensional asymptotically flat Robertson-Walker spacetime, which describes the universe well. We show that the expansion of the universe generates quantum Fisher information (QFI) between modes of the Dirac particles. Meanwhile, the QFI gives optimal bounds to the error of the quantum parameter estimations under the influence of universe expansion. We demonstrate the possibility for high precision estimation of parameters that appear in the expanding universe including the volume and expansion velocity. We find that the precision of the estimation can be improved by choosing special measurements that are marked by mass of the Dirac particles.

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Introduction.—Quantum metrology is mainly to study the ultimate limit of precision in estimating a physical quantity if quantum physics is considered [1]. Classically, for a given measurement scheme, the effect of statistical errors can be reduced by repeating the same measurement and averaging the outcomes. Explicitly, by repeating the measurements $N$ times separately, the uncertainty about the measured parameter $\Theta$ will be reduced to the standard quantum limit $\Delta \Theta \simeq 1/\sqrt{N}$. However, if the quantum properties are exploited such as the squeezed or entangled states are used as probe system which undergoes an evolution depending on the parameters to be estimated, the precision can be enhanced such that the uncertainty will approach the Heisenberg limit $\Delta \Theta \simeq 1/N$ which beats the standard quantum limit [2]. This has been realized in gravitational wave observation [3]. Recently, the study of quantum metrology has been applied to the area of relativistic quantum information [4–14]. It is shown that quantum metrology methods can be exploited to improve probing technologies of Unruh-Hawking effect [15, 16] and gravitation [17]. Most recently, it is found that quantum metrology can be employed to estimate the Unruh temperature of a moving cavity at experimental reachable acceleration [18, 19]. These preliminary results are of great importance for the observation of relativistic effects in a laboratory [20] and pace-based quantum information processing tasks [21, 22].

It is remarkable that the universe expands acceleratingly, which is discovered through the observations of Type Ia supernovae [24, 25]. The accurate measurement of the expansion velocity of the universe is significant not only for estimating the age of our universe but also for the study of dark energy. Currently, the expansion velocity is mainly measured through distance measurement [26, 27], which is hardly be performed and easily be interfered. In this work, we propose a quantum metrology technique to estimate the expansion velocity of the universe in Robertson-Walker (RW) metric [28–32], which describes an isotropic and homogeneous expanding universe. The RW universe is widely regarded as the Standard Model of the present-day cosmology [33]. We study a system of a vacuum state for Dirac fields in the asymptotically past but evolves to an entangled fermion and antifermion state as the expanding of the universe. By applying quantum metrology strategy on the final states, the precision of the estimated parameter can be related to the quantum Fisher information (QFI) [34]. We analytically calculate the QFI in terms of the mass $m$ of the Dirac particles and module $k$ of the wave vector. Such process corresponds to finding the set of measurements that allows us to estimate the expansion velocity [17] with the highest precision. Our proposal is independent of the measurement on the velocities and distances of the galaxies. We demonstrate that the dynamic spacetime background actually creates QFI. At the same time, information about the history of the expanding universe can be extracted from measurements on the states in the asymptotic future region, while the ultimate precision limit is related with QFI. We then find that the precision of the estimation can be improved by choosing the proper measurements that marked by the mass of the Dirac particles. As far as we know, this is the first time that the quantum metrology is studied in the cosmological background. Our work opens a new avenue for the development of quantum detection and estimation technologies for cosmological parameters in the expanding universe.

Quantum Metrology and Fisher information.—The aim of quantum metrology is to determine the value of a parameter with higher precision than classical approaches, by using entanglement and other quantum resources [1]. A key concept for the metrology is Fisher information [34], which relates with the result $\xi$ of a positive operator valued measurement

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(POVM) \{ \hat{E}(\xi) \} and takes the form
\begin{equation}
\mathcal{F}_\xi(\Theta) = \sum_\xi P(\xi|\Theta) \left( \frac{\partial \ln P(\xi|\Theta)}{\partial \Theta} \right)^2 ,
\end{equation}
where \( P(\xi|\Theta) \) is the probability with respect to a chosen POVM and the corresponding measurement result \( \xi \) and \( \Theta \) is the parameter to be estimated. According to the classical Cramér-Rao inequality [35], the mean square fluctuation of the unbiased error for \( \Theta \) is
\begin{equation}
\Delta \Theta \geq \frac{1}{\sqrt{\mathcal{F}_\xi(\Theta)}}.
\end{equation}
Furthermore, the Fisher information is \( n\mathcal{F}_\xi(\Theta) \) for \( n \) repeated trials of measurement, which leads to Cramér-Rao Bound (CRB) on the phase uncertainty, \( n(\Delta \Theta)^2 \geq \frac{1}{\mathcal{F}_\xi(\Theta)} \) which gives the maximum precision in \( \Theta \) for a particular measurement scheme. Optimizing over all the possible quantum measurements provides an lower bound [2], i.e.,
\begin{equation}
n(\Delta \Theta)^2 \geq \frac{1}{\mathcal{F}_\xi(\Theta)^2} \geq \frac{1}{\mathcal{F}_{Q}(\Theta)} ,
\end{equation}
where \( \mathcal{F}_Q(\Theta) \) is the quantum Fisher information (QFI). Usually, the QFI can be calculated from the density matrix of the state by the maximum \( \mathcal{F}_Q(\Theta) := \max_\xi \mathcal{F}_\xi(\Theta) \) which is saturated by a particular POVM and defined in terms of the symmetric logarithmic derivative (SLD) Hermitian operator \( \mathcal{L}_\Theta \)
\begin{equation}
\mathcal{F}_Q(\Theta) = \text{Tr} \left( \rho_\Theta \mathcal{L}_\Theta^2 \right) = \text{Tr} \left[ (\partial_\Theta \rho_\Theta) \mathcal{L}_\Theta \right] ,
\end{equation}
where
\begin{equation}
\partial_\Theta \rho_\Theta = \frac{1}{2} \{ \rho_\Theta , \mathcal{L}_\Theta \} ,
\end{equation}
where \( \partial_\Theta \equiv \frac{\partial}{\partial \Theta} \) and \( \{ , \} \) denotes the anticommutator.

Quantization of Dirac fields in the RW universe.— In quantum field theory, particles are excitations of quantum fields [31]. To study the quantum metrology for Dirac fields (or particles) in the RW spacetime [28–31], we introduce the local tetrad field
\begin{equation}
g^{\mu\nu} = e^\mu_a(x)e^\nu_b(x)\eta^{ab} ,
\end{equation}
where \( g^{\mu\nu} \) and \( \eta_{ab} \) are metrics of the RW spacetime and Minkowski spacetime, respectively. The covariant Dirac equation for field \( \Psi \) with mass \( m \) in a curved background reads [36]
\begin{equation}
i\gamma^\mu(x) \left( \frac{\partial}{\partial x^\mu} - \Gamma^\mu_\rho \right) \Psi = m \Psi .
\end{equation}
We solve the Dirac equation in a two-dimensional RW spacetime, which is proved to embody all the fundamental features of the higher dimensional counterparts [29]. The corresponding line element reads
\begin{equation}
ds^2 = [a(\eta)]^2(\ d\eta^2 - dx^2) ,
\end{equation}
where the dimensionless conformal time \( \eta \) is related to cosmological time \( \tau \) by \( a(\eta) = \int a^{-1}(\tau)\ d\tau \). We consider a special case, as presented in [29 31 32 37], the conformal factor \( a(\eta) \) can be written as
\begin{equation}
a(\eta) = 1 + \epsilon (1 + \tanh \rho \eta) ,
\end{equation}
where \( \epsilon \) and \( \eta \) are positive real parameters, corresponding to the total volume and the expansion velocity of the universe [29 31], respectively. Note that \( a(\eta) \) is constant in the far past \( \eta \to -\infty \) and far future \( \eta \to +\infty \), which means that the spacetime tends to a flat Minkowskian spacetime asymptotically. In the asymptotic past (in)- and future (out)-regions the vacuum states and one-particle states may be well defined. In this special spacetime background the spin connections read
\begin{equation}
\Gamma_\mu = \frac{1}{4} \frac{\partial a(\eta)}{a(\eta)} [\gamma_\mu , \gamma_0] ,
\end{equation}
where \( \partial a(\eta) = \frac{\partial a(\eta)}{a(\eta)} \). To solve the Dirac equation Eq. (7) in the metric Eq. (8), we re-scale the field as
\begin{equation}
\Psi = a^{-3/2} \gamma^\nu \partial_\nu - m a(\eta) )\psi ,
\end{equation}
and the dynamic equation evolves to
\begin{equation}
\eta^{\mu\nu} \partial_\mu \partial_\nu \psi - \frac{\partial}{\partial \eta} m a(\eta) \psi - [m a(\eta)]^2 \psi = 0 .
\end{equation}
Moreover, by introducing the flat spinors \( U_d \) and \( V_d \) that satisfy the relations \( \bar{\eta}^0 U_d = -iU_d \) and \( \bar{\eta}^0 V_d = iV_d \), we can represent the re-scaled solution \( \psi \) into a radial and temporal separable form, that is \( \psi := N^{(+)\psi} (\eta) U_d e^{i k^x x} \) (or \( \psi := N^{(+)\psi} (\eta) V_d e^{i k^x x} \)). Here the functions \( \psi^\pm_k \) obey the following coupled differential equation
\begin{equation}
\psi^+_k (\pm) + \left[ k^2 + m^2 a(\eta)^2 + im\tilde{a}(\eta) \right] \psi^\pm_k = 0 .
\end{equation}
Actually \( \psi^+_k (+) \) and \( \psi^+_k (-) \) are positive and negative frequency modes with respect to conformal time \( \eta \) near the asymptotic past and future, i.e. \( \psi^+_k (\pm) (\eta) \approx \mp \omega_{in/out} \psi^\pm_k (\eta) \) with \( \omega_{in/out} = \sqrt{|k|^2 + m^2 a^2 (\eta \to \mp\infty)} \) and \( |k|^2 = k^2 \). In the asymptotic past region, the positive and negative frequency solutions [29 32 38] of Eq. (13) are
\begin{equation}
\psi^\pm_{in} = \exp \left[-i \omega(\pm) \eta - \frac{i \omega(\mp)}{\rho} \ln(2 \cosh \rho \eta) \right] \times \sum_{n=0}^{\infty} \left[ \rho + i \zeta_{in}^\pm \right] \Lambda^\pm_n \frac{\Lambda^\pm_n}{2^n \eta} ,
\end{equation}
where \( [n] = n(n+1) \ldots (q+n-1) \), \( \Lambda^\pm_n = [1 + \tanh(\rho \eta)]_n \), \( \zeta_{in}^\pm = \omega_{in} (\pm) \pm m \epsilon \) and \( \omega_{in} = (\omega_{out} + \omega_{in})/2 \). Similarly,
one may obtain the modes of the Dirac fields that behaving as positive and negative frequency modes in the asymptotic future region

\[ \psi_{\text{out}}^{(\pm)} = \exp \left[ -i\omega_{(+)} \eta - \frac{i\omega_{(-)}}{\rho} \ln(2 \cosh \rho \eta) \right] \times \sum_{n=0}^{\infty} \frac{[\rho + i\omega_{(n)}]_{\eta}}{[\rho + i\omega_{(n)}]_{\eta}} \Lambda_{n}^{n} 2 \eta \rho^{n}, \] (15)

where \( \Lambda_{n}^{n} = [1 - \tanh(\rho \eta)]^{n} \). It is now possible to quantize the field and find the relation between the vacuum state in the asymptotic future region and that in asymptotic past. Such relation can be described by a certain set of Bogoliubov transformations \([3]\) between in and out modes

\[ \psi_{\text{in}}^{(\pm)}(k) = A_{k}^{\pm} \psi_{\text{out}}^{(\pm)}(k) + B_{k}^{\pm} \psi_{\text{out}}^{(\mp)}(k), \] (16)

where \( A_{k}^{\pm} \) and \( B_{k}^{\pm} \) are Bogoliubov coefficients that take the form,

\[ A_{k}^{\pm} = \sqrt{\frac{\omega_{\text{out}}}{\omega_{\text{in}}}} \frac{(1 - \frac{\omega_{\text{in}}}{\rho}) \Gamma(-i\omega_{(n)}/\rho)}{(1 - \frac{\omega_{\text{out}}}{\rho}) \Gamma(-i\omega_{(n)}/\rho)}, \]

\[ B_{k}^{\pm} = \sqrt{\frac{\omega_{\text{out}}}{\omega_{\text{in}}}} \frac{(1 + i\omega_{(n)}/\rho) \Gamma(i\omega_{(n)}/\rho)}{(1 + i\omega_{(n)}/\rho) \Gamma(i\omega_{(n)}/\rho)}. \] (17)

We can see that the Dirac field undergoes a \( \Theta_{i} \)-dependent Bogoliubov transformation, where \( \Theta_{i} = \epsilon, \rho \) are the parameters that we want to estimate.

To find the relation between the asymptotic past and future vacuum states, we use the relationship between the operators

\[ b_{\text{in}}(k) = \left[ A_{k}^{\ast} b_{\text{out}}(k) + B_{k}^{\ast} \chi(k) a_{\text{out}}^{\dagger}(-k) \right], \] (18)

where \( \chi(k) = \frac{\omega_{\text{out}}}{\omega_{\text{in}}} \frac{\omega_{\text{in}}}{\rho} - \frac{\omega_{\text{in}}}{\rho} \). Imposing \( b_{\text{in}}(k)|0\rangle_{\text{in}} = 0 \) and the vacuum normalization \( \langle 0 | 0 \rangle_{\text{in}} = 1 \), we obtain the asymptotically past vacuum state in terms of the asymptotic future Fock basis

\[ |0\rangle_{\text{in}} = \sqrt{1 + |\gamma_{k} F\rangle_{\text{out}}^{2}} |1\rangle_{\text{out}}^{\text{out}} - \frac{|\gamma_{k} F\rangle_{\text{out}}}{1 + |\gamma_{k} F\rangle_{\text{out}}^{2}} |1\rangle_{\text{out}}^{\text{out}} \] (19)

where

\[ |\gamma_{F}\rangle_{\text{out}}^{2} = \frac{\zeta_{(-)}^{+} \sinh \left( \frac{\pi \zeta_{(-)}^{+}}{\rho} \right)}{\zeta_{(-)}^{\mp} \sinh \left( \frac{\pi \zeta_{(-)}^{\mp}}{\rho} \right)} \frac{\sinh \left( \frac{\pi \zeta_{(+)}^{+}}{\rho} \right)}{\sinh \left( \frac{\pi \zeta_{(+)}^{\mp}}{\rho} \right)}. \] (20)

We can see that the system which was a natural vacuum state \( |0\rangle_{\text{in}} \) of Dirac field in the asymptotically past region evolves to an entangled particle and antiparticle state as the expansion of the universe. Particles and antiparticles were produced and entangled by gravity during the expanding and the information about the velocity of the expansion is codified in the final states.

**Quantum Metrology in the expanding universe.**—Our aim is to study how precisely one can in principle estimate the cosmological parameters that appear in the expansion of the universe. Specifically, we will estimate the volume \( \epsilon \) and the expansion velocity \( \rho \) of the universe based on the quantum Cramer–Rao inequality. Thus we should at first calculate the operational QFI in terms of the Bogoliubov coefficients which encode in the estimated parameters. The reduced density matrix corresponding to particle modes is found to be

\[ \phi_{k}^{\rho} = \frac{1}{(1 + |\gamma_{k} F\rangle_{\text{out}}^{2})} \langle 0 | 0 \rangle + |\gamma_{k} F\rangle_{\text{out}}^{2} |\gamma_{k} F\rangle_{\text{out}}^{2} \langle 1 | 1 \rangle, \] (21)

which is obtained after tracing out the antiparticle modes \( n_{-} \) and all other modes with different \( k \) since the modes of the out states are unique for all inertial particle (or antiparticle) detectors \([23]\). Assuming that now we live in the universe that corresponds to the asymptotic future of the spacetime, we can estimate the volume and the expansion velocity of the universe by taking measurements on the particle state. In this system, the states of the Dirac fields act as the probes, which is prior available. Now our main task is to find the optimal estimation strategy, i.e., finding an optimal measurement realizing the QFI that allows us to estimate the parameters with the highest precision. The symmetric logarithmic derivative operator \( L_{\Theta_{i}} \) is given by \( \mathcal{L}_{\Theta_{i}} = \frac{1}{2} (j \partial_{\Theta_{i}} (\rho_{k}^{F})) \| j \rangle \langle j \| \), where \( j(\neq l) = (0, 1) \) and the QFI is found to be

\[ \mathcal{F}_{Q}(\Theta_{i}) = \frac{1}{(1 + |\gamma_{k} F\rangle_{\text{out}}^{2})} \langle \partial_{\Theta_{i}} \rho_{k}^{F} (\rho_{k}^{F})^{\dagger} \rangle^{2} \]

\[ + \frac{1}{(1 + |\gamma_{k} F\rangle_{\text{out}}^{2})} \langle \partial_{\Theta_{i}} (\rho_{k}^{F})^{2} \rangle \] (22)

where the expanding parameters \( \rho \) and \( \epsilon \) are encoded.

**FIG. 1:** (Color online) The QFI of the Dirac fields vs the expansion velocity \( \rho \) for different \( \epsilon \). The values of \( \epsilon \) for each line with increasing gap length are 0.2, 0.4, 0.6, 0.8, and 1.0, corresponding to the color sequence red, orange, green, blue and purple, respectively. Both \( m \) and \( k \) are fixed with \( m = k = 1 \).

In Fig. 1 we plot the QFI of the Dirac fields in an expanding universe as a function of expansion velocity \( \rho \) for different...
It is shown that the expansion generates QFI between certain modes of the Dirac particles that are only influenced by the expansion of the universe. The QFI first increases and then decreases as the increase of both ρ and ε. We can see that the QFI of the final states depends sensitively on the cosmological parameters, which means that the expansion of the universe has a significant effect on the precision of quantum metrology. Conversely, information about the history of the expanding universe can be extracted from the measurements on the final states in the asymptotic future region, in which the QFI shows the limit on the precision of the measurements.

For a further study, we calculate the optimal bound for the error in the estimation process by using Eqs. (3) and (22). In Fig. 2 we plot the optimal bounds (Δρ)² and (Δε)² (dashed line) in the estimation of the volume ε and the expansion velocity ρ of the universe over the mass m of the Dirac particles. It is shown that the optimal bounds behave almost the same when we estimate ε or ρ. It is found that the bounds decrease rapidly as the increase of mass for the light Dirac particles (for example neutrinos). In the standard cosmology model, relic neutrinos act as witnesses and participants for landmark events in the history of the universe from the era of big-bang nucleosynthesis to the era of large scale structure formation [39, 40]. It is shown that the relic neutrinos are crucial for estimating the age of our universe and its fate. It is also shown that the bounds become very small (the order of magnitude 10⁻¹²) as the measurements repeated a number of times for the medium weight Dirac particles (for example the strange and charm quarks).

In Fig. 3 we plot the optimal bounds (Δθ)² in the estimation of the volume ε (dashed line) and the expansion velocity ρ of the universe over k. The parameters are fixed with ρ = ε = k = 1. The measurements repeated n = 10¹¹ times.

Discussion and summary.— We studied the cosmological quantum metrology by incorporating the effect of universe expansion in quantum parameter estimation. The expanding spacetime background has a significant effect on the value of QFI. The expansion velocity of the universe can be estimated from the measurements on the final states of the Dirac particles (for example the neutrinos) in the asymptotic future region. This cosmologic quantum metrology goes beyond the classical method though may not achieve the Heisenberg limit. As mentioned before, the expansion velocity is currently measured through the distance measurement. Such a method requires measuring both the velocities and distances of the supernovae [26, 27] and is hardly be performed and easily be interfered for the following three reasons. First, the supernovae explosions are very rare events, so the probability of seeing one nearby is very low. Second, the supernovae interact gravitationally with their neighbours and the velocities become perturbed, inducing “proper” motions that are superimposed onto the overall expansion. Third, performing an accurate extragalactic distance measurement is far more difficult because the distances depend on the expansion velocity itself [26, 27]. Our proposal is based on the local measurements on the final states of Dirac particles which are detectable in cosmic rays [41] or cosmic neutrino background [42]. According to the standard cosmology, the relic neutrinos are abundant particles ranking only second to the cosmic microwave background in the present-day universe [40, 43]. One of the most important task of the neutrino detectors, for example the IceCube in the Antarctica, the Super-Kamiokande and Sudbury Neutrino Observatory (SNO) located underground, are exploring the background of neutrinos produced in the Big Bang. Thus, the measurements can be performed once the relic neutrinos has been detected in the observatories.
sive understanding of quantum metrology in the background of cosmology will enable us to make the necessary estimation technologies that affected by the expansion the universe.

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