Superdualities: 
Below and beyond U-duality $^{1,2}$

B. Julia

Laboratoire de Physique Théorique de l’École Normale Supérieure$^{3}$
24 Rue Lhomond - 75231 Paris CEDEX 05

ABSTRACT

Hidden symmetries are the backbone of Integrable two-dimensional theories. They provide classical solutions of higher dimensional models as well, they seem to survive partially quantisation and their discrete remnants in M-theory called U-dualities, would provide a way to control infinities and nonperturbative effects in Supergravities and String theories. Starting from Einstein gravity we discuss the building blocks of these large groups of internal symmetries, and embed them in superalgebras of dynamical symmetries. The classical field equations for all bosonic matter fields of all toroidally compactified supergravities are invariant under such “superdualities”. Possible extensions are briefly discussed.

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1 Introduction

Since the discovery of discrete U-dualities, of their role in the control of the divergences of string theories \cite{1, 2, 3} and of the duality between Large N super-Yang-Mills theory and AdS compactification of eleven dimensional supergravity \cite{4} the need for a better conceptual understanding of the latter has become rather urgent. The superspace approach is notoriously hard but component approaches are rather cumbersome, this is unsatisfactory as more miracles are being discovered \cite{5}. These dualities are important also to clarify the constraints on allowed counterterms and nonperturbative effects in Supergravities see \cite{3, 7} and references therein.

In section 2 we shall review the relation between the background spacetime geometry and the duality symmetries. This is a long and pedestrian approach towards the background independent stucture of M-theory. It turns out that all massless bosonic matter fields of the toroidally compactified theory obey one rather simple universal self-duality classical equation of motion. The self-duality involution is the product of Hodge duality on all forms (the bosonic matter fields) by an internal twist which in particular compensates for the noninvolutive character of Hodge duality in some dimensions of spacetime. More conceptually the twist exchanges generators and their conjugates in a doubled superalgebra (still in the bosonic sector) that captures all the nonlinearities. In section 3 we shall recall previous instances of this self-duality in even dimensions $D = 2k$ for the k-forms, and proceed to generalise it to all forms and all dimensions following \cite{5}. The superalgebra of dualities contains as subalgebra the symmetry of a supertorus with one fermionic dimension on top of the compactification torus one is assuming. In the next section we shall discuss the importance of deformation theory for the construction of SUGRAS and discuss the deformation that leads to M-self-duality. Finally we shall comment on possible extensions.

2 Ehlers’ and other symmetry enhancements

2.1 From stationary gravity to Kaehler moduli

Let us start our discussion with the realistic (low energy) model of four dimensional gravity. It is of course invariant under diffeomorphisms and if needed under local Lorentz transformations. This possibility reflects the fact that gauge restoration may increase the symmetry, it is well known that gauge unfixing often makes other symmetries manifest but we shall see that changes of
our choice of fundamental fields also modify the faithful symmetry group. If one considers the space of solutions admitting one non-lightlike Killing vector, it turns out that diffeomorphisms of the cyclic coordinate become gauge transformations of the abelian connection defined by the orthogonal hyperplanes to the Killing orbits: local domains become fibered by the orbits and inherit a principal connection. Actually the global (or rigid) rescalings of the cyclic coordinate imply also a scaling symmetry $\mathbb{R}$. (More generally dimensional reduction on $T^k$ implies an internal symmetry $GL(k, \mathbb{R})$). If one however dualises the vector potential in the remaining three dimensions to a scalar field $B$ defined up to a gauge freedom, namely the addition of a constant, then the two degrees of freedom of the graviton conspire to parameterise the Poincaré upper half-plane and the abelian gauge invariance of the connection disappears to leave room for a rigid $SL(2, \mathbb{R})$ of internal symmetries. This group I called the Ehlers group although the original name was given to its maximal compact subgroup $SO(2)$. The latter is the only true surprise as the other two generators of $SL(2, \mathbb{R})$ are the rescalings and the constant shifts of the dual $B$ field. In fact its action is highly nontrivial (if somewhat singular) as it for example transforms an asymptotically flat Schwarzschild black hole into a Taub-NUT spacetime. These transformations act nonlocally on the original four-dimensional fields.

More instances of this miracle occur in other theories, Einstein-Maxwell theory reduced from 4 to 3 dimensions has the structure of the nonlinear sigma model $SU(2,1)/S(U(2) \times U(1))$ with four freedoms whereas one rescaling and two shifts are predicted $[3]$. Similarly compactification of pure gravity from $d$ dimensions to 3 leads to an internal symmetry $SL(d-2, \mathbb{R})$ whereas one expects $GL(d-3, \mathbb{R})$. Supergravities representing the effective low energy theories of type I or heterotic strings (compactified on $T^6$) have been conjectured in 1990 to exhibit also a non-perturbative (in the string-string coupling constant $g_{\text{string}}$) so-called S-duality, namely $SL(2, \mathbb{Z})$ inside the Lie group $SL(2, \mathbb{R})$. The latter has been known in the classical supergravity context since the construction of the $N = 4$ SUGRA in 4 dimensions and the analysis by Chamseddine of the reduction of type I 10d SUGRA on $T^6$. In this last case a form (here a 2-form) can again be dualised to a scalar (axion or Kalb-Ramond) field to parameterise the Poincaré upper half-plane. We refer the reader to some reviews for more references: see $[3, 4, 11]$ for instance. Let us recall also that in $[12]$ the same S-duality in four dimensions is conjugated to a subgroup of Ehlers’ type namely the $SL(2, \mathbb{Z})$ associated to a seventh Killing vector by the T-duality corresponding to its direction. T-duality is a discrete symmetry of string theories associated to an internal isometry. It does act on
the string interactions but in perturbative way and it acts nonperturbatively on the geometry by inverting the radius of the compactification circle in string units.

Typically the Lie group (over the real numbers) is a symmetry of the low energy effective SUGRA type action and the discrete arithmetic group over \( \mathbb{Z} \) is its quantum remnant and it is believed to be the (gauge) symmetry of the full theory. More generally S-dualities are those dualities that exchange weak and strong string-string couplings. The discreteness of S-dualities is the non-perturbative result of string-string interactions: for instance in the IIB theory non-holomorphic S-modular Eisenstein series appear as coefficients of the (10 dimensional) gravitational coupling expansion [13] where the string length appears as a cut-off. It has been shown that S-duality of the heterotic string on \( T^6 \) is “string-string dual” to T-duality of type IIA on \( K3 \times T^2 \), in fact this correspondence between string theories exchanges perturbative and “non-perturbative” dualities.

Yet another enhancement to \( SL(2, \mathbb{R}) \) occurs by \( T^2 \) compactification of string theories, the complex structure modulus of the torus is as expected a coordinate on the Poincaré upper half-plane, it corresponds to a “geometrical” \( SL(2, \mathbb{R}) \). At the string level, i.e. with all massive states included, the continuous geometric Lie group acts on the moduli and only its discrete subgroup \( SL(2, \mathbb{Z}) \) is a quantum gauge (but perturbative) symmetry. But the background 2-form flux or integral over the torus and its volume combine to form its Kaehler modulus. The latter parameterises another \( SL(2, \mathbb{Z})/SO(2) \) [14]. For instance in the type I case reduced to 8 dimensions on top of the geometric \( GL(2, \mathbb{R}) \) invariance of the tangent space to the compactification torus which is relevant in the zero mass sector there appears another \( SL(2, \mathbb{R}) \), in the type IIA case the same group emerges. From the M-theory point of view the flux of the three form and the volume of \( T^3 \) (\( SL(3, \mathbb{R}) \) scalars) make a similar complex valued modulus and in the type IIB case the 2-form fluxes enhance the \( SL(2, \mathbb{R}) \) already present in 10 dimensions to \( SL(3, \mathbb{R}) \) commuting with the diffeomorphism invariance induced \( SL(2, \mathbb{R}) \). In these examples double T-duality of the torus \( T^2 \) is the nontrivial generator of the extra \( SL(2, \mathbb{Z}) \) that commutes with the obvious (geometric) \( SL(2, \mathbb{R}) \).

Let us note that the T-duality relating type IIA and type IIB exchanges 2 rather different dimensions, not only are the lengths of the dual circles inversely proportional but in 8 dimensions the \( SL(3, \mathbb{R}) \) coming from unimodular diffeomorphisms of the M-theory compactified on \( T^3 \) commute with the \( SL(2, \mathbb{R}) \) coming from unimodular diffeomorphisms of type IIB compactified on \( T^2 \), although one dualises only one direction. this means that our classi-
cal approximation of spacetime and even its number of dimensions are model dependent, an important issue is to determine the domain of validity of each of these “complementary” (in the sense of Bohr) classical approximations, in that connection see [15].

For completeness let us recall that the $SL(2, \mathbb{R})$ symmetry of IIB SUGRA in 10 dimensions may be traced back to its 12 dimensional origin, namely F-theory compactified on a torus $T^2$ with frozen Kaehler modulus, in particular this means that in 12 dimensions if there is diffeomorphism invariance it is only for the volume preserving subgroup, I have emphasized this point in my study of the group disintegration of $E_8$ in for instance [15]. We note here that the group of unimodular diffeomorphisms is precisely the invariance group of the action of isentropic perfect fluids (compressible or not) expressed in Lagrangian coordinates. In fact if we fix the volume of $T^2$ the large radius limit of one of its directions corresponds to a small radius limit for the other one and hence effectively 11 dimensions not 12. Another peculiar compactification along a torus $T^2$ of null radii produces IIB superstring theory from M theory, this was analysed by Aspinwall and leads to the interpretation of the $SL(2, \mathbb{Z})$ invariance of IIB string theory in 10 dimensions as the geometric invariance of the torus. One of the null radii is actually infinite on the IIB side after T-duality, so the limit corresponds to a ten dimensional theory.

2.2 More “accidents”

Let us now review quickly the build-up of the large U-duality symmetry groups when one increases the dimension of the compactification torus. The geometrical symmetries grow regularly as expected but beyond the Ehlers type accidents listed above even more dramatic symmetries drop out of the low energy analysis. For instance the M-theory (11d supergravity) symmetries are the Lie groups of E type suitably defined for low rank or rank 9 and maybe rank 10(?). After reduction on $T^n$ one obtains the split real form of $E_n$ the algebra is non simple for $1 \leq n \leq 3$ and beyond that it becomes simple by glueing of the Ehlers type factor to the geometric symmetry. Specifically $A_2 \times A_1 = E_3$ becomes $A_4 = E_4$. Other glueings occur for type I SUGRA on $T^3$ where $D_2$ becomes $D_3$ and on $T_7$ where $D_6 \times A_1$ becomes $D_8$. In that family the algebras of type D appear also in their split form. In the string context they also appear [15].

It is a classical result that $SO(n, n + k)$ symmetries occur on $T^n$ if one starts from type I SUGRA coupled to k vector multiplets in 10 dimensions. The groups $SO(n, n + 16)$ correspond to heterotic strings where 16 is the rank
of the internal gauge group.

In all cases the symmetry extends to the affine (Kac-Moody) Lie algebras corresponding to the three dimensional theory upon further reduction to two dimensions. At the classical level, the action has been successively found to be non symplectic (see for instance the discussion and references in [13]) and Lie-Poisson [17, 18] leading at the quantum level to the proposal to use a quantum group, the stringy version remains unknown. The interplay between arithmetic, affine and quantum groups deserves more study.

Conversely one might ask oneself whether a curved space sigma model in three dimensions, namely (topological) gravity coupled to a symmetric space sigma model (notwithstanding any supersymmetry) is actually the result of a toroidal dimensional reduction. This has been studied extensively in the last century [19]. The result is strikingly simple: the rule of group disintegration (also called oxidation) [20] requires that the affine Dynkin diagram of the 3d group contains a geometric $SL(D-3) \times \mathbb{R}$ ending at the affine root for an ancestor theory to exist in dimension D. This had been used to predict a new SUGRA in 6 dimensions that was actually constructed recently [21]. If one considers the scalar sectors of the $E_n$ family, but one puts them now in 3 curved dimensions (we stick to the split forms, this ruins most supersymmetries) then one obtains a surprising magic triangle of higher dimensional ancestors. The geometry of that triangle still escapes mathematicians [22].

3 Self-duality equations

3.1 k-forms in 2k dimensions

It is well known that self-duality requires $d = 4k + 2$ on a spacetime of Minkowskian signature, or $d = 4k$ for Riemannian spaces. The discovery of instantons in 1975 has launched a search for exact classical solutions of non-linear equations including Einstein and Yang-Mills equations. It followed the discovery of regular magnetic monopole (and dyon) solitons and was concomitant with that of the corresponding self-dual solutions in the so-called Prasad-Sommerfield-Bogomolny (BPS) limit, (actually the pseudoparticle paper appeared precisely between the PS and the B papers and the latter emphasized the stability aspect). More recently the BPS conditions gained importance because of their realisation as conditions for unbroken supersymmetry. As a toy example, the scalar wave equation in two dimensions admits self-dual solutions, the left and right moving modes. The (i)-self-duality equation on a Riemann surface $df = i \ast df$ implies harmonicity.
Conversely the Cauchy-Riemann equations are related to the real solutions of the harmonic equation, they can be seen as a twisted self-duality equation for a pair of functions \((Re f = a \text{ and } Im f = b)\): \(da = \ast db\) \(a\) and \(b\) are conjugate harmonic functions, this fact has been used in \cite{23} to render the action of the \(SL(2, \mathbb{R})\) subgroup of the conformal group manifest: \(a\) and \(b\) are coordinates on the Poincare half-plane again. Let us introduce the doublet \(C = (a, b)\), the above equation can be rewritten

\[
dC = \ast dCS.
\] (3.1)

Note that we chose euclidean signature and preserved the reality of \(C\) nevertheless, the twisting matrix \(S := i\tau_2\) has square \(-1\) precisely to compensate the annoying \(\ast\ast = -1\). To summarize the procedure, we replaced one second order equation for one unknown, here the harmonic equation for \(a\), by two first order equations for two unknowns which are equivalent to the original problem. The first order system is then shown to possess a rigid symmetry, the duality \(S\) that acts locally on the pair of field strengths but nonlocally on the original function \(a\). This despairingly simple example is actually the prototype of our final result. Strictly speaking there are topological and gauge subtleties because the data of \(C\) is not equivalent to that of \(a\) there is an integration constant to be handled by a “normalisation” condition.

The Geroch group action on Einstein plane waves is realised on an infinite set of potentials related also by duality equations called Baecklund transformations and usually combined into a linear system depending on one parameter. The consistency conditions are the original equations and the corresponding equations for the dual fields. But let us stay in two effective (“active”) dimensions and consider the principal sigma model for a semi-simple group \(G\). We shall identify the Lie algebra \(\mathcal{G}\) and its dual by the Killing form (which appears in the action). The equations read:

\[
A := dgg^{-1} = A_c T^c
\] (3.2)
\[
dA - A \wedge A \equiv 0 \tag{3.3}
\]
\[
d \ast A - A \wedge A \ast A + A \wedge A = 0 \tag{3.4}
\]

In order to exhibit the symmetry between the Bianchi identity and the equation of motion one is led to define dual generators \(\tilde{T}^c = S(T^c)\) and to impose \((S \ast)^2 = \text{Id}\.\) by defining suitably the action of the linear “involution” \(S\) on the dual generators. The equations now read

\[
dC - C \wedge C \equiv 0 \tag{3.5}
\]
\[
C = S(\ast C) \tag{3.6}
\]
provided $T^c$ and $\tilde{T}_c$ form the Lie algebra $\mathcal{G} \ltimes \mathcal{G}^*$. Note that the Killing form is only used for the definition of $S$. We see the second example of a universal twisted self-duality equation that encodes the full original second order system.

Our next example comes from abelian vector fields in maximal supergravity in 4 dimensions. This has been analysed first with gauge fixed coset in $N \leq 4$ SUGRA [24, 25] and then extended to include the maximal SUGRA with the general gauge invariant structure under the maximal compact subgroup [24, 26]. Shortly thereafter this technology has been transferred to statistical mechanics. The duality symmetry $E_7$ can only act on the doubled set of the fundamental 28 potentials plus their 28 dual potentials that together form the 56 representation of $E_7$: the abelian one forms $C$. They obey again an analogous system of equations:

$$dC \equiv 0 \quad (3.7)$$

$$VC = *SVC \quad (3.8)$$

where $S^2 = -\text{Id.}$ and $V$ is the scalar matrix representing $E_7$ in the fundamental representation 56. This structure extends to higher dimension $d = 2k$ for k-forms which now fall into representations of the groups $E_5 = D_5$ and $E_3$ [27].

### 3.2 11d SUGRA has a twelfth fermionic dimension

Dualisations of all forms is possible in the toroidal compactifications of 11d SUGRA [28, 27, 5] or at least the equations of motion for the doubled set take a nice form. The dualisability of the 3 form was discovered in 10 dimensions long ago, but for the eleven dimensional theory although one can double the set of fields one cannot dualise the three form, see [29, 30] and references therein. In fact independently of our group doubled Lagrangians were proposed [31] but they do not exhibit the nonabelian structure that generalises the semi-direct product algebra of sigma models presented above.

The bosonic action we shall consider in 11d reads:

$$\mathcal{L}_{11} = \kappa^{-2} R * 1 - \frac{1}{2} * F_{(4)} \wedge F_{(4)} - \kappa \frac{1}{6} F_{(4)} \wedge F_{(4)} \wedge A_{(3)} \quad , \quad (3.9)$$

the matter equations can be rewritten in our favorite form by introducing the dual 6-form $\hat{A}_{(6)}$:

$$*G = SG \quad (3.10)$$

$$dG - G \wedge G \equiv 0 \quad (3.11)$$

provided $G = dV V^{-1}$ and

$$V = e^{A_{(3)}} V \ e^{\hat{A}_{(6)}} \hat{V} \ . \quad (3.12)$$
In these expressions the forms should be expanded and treated as odd Grassmann parameters if of odd degree, then the full content of the nonlinear Chern-Simons term (dictated by supersymmetry and proportional, with a specific coefficient, to the gravitational coupling constant $\kappa$) follows from the Clifford-type superalgebra structure:

$$\{V, V\} = -\kappa \tilde{V}, \quad [V, \tilde{V}] = 0, \quad [\tilde{V}, \tilde{V}] = 0.$$

Equation 3.12 is a generalisation of ordinary sigma models, the rigid symmetry has become a gauge symmetry in the generalised sense of [32] namely:

$$V' = V e^{\Lambda_{(3)}} V e^{\tilde{\Lambda}_{(6)}} \tilde{V},$$

where $\Lambda_{(3)}$ and $\tilde{\Lambda}_{(6)}$ are closed forms. The reader may wonder what has been gained by the replacement of a p-form by a closed (p+1) form, it turns out that the (p+1)th de Rham cohomology leads to conserved generalised charges [33] and the superalgebra of gauge symmetries implies nonlinear relations of the type

$$t_k t_l = t_{k+l}$$

where the $t_k$ are the renormalised tensions of the $(k-1)$ branes (fundamental ones or their duals) $t_k := T_{k-1}/2\pi$. In the present situation one finds $(t_3)^2 = t_6$. One recovers one of the relations found in [34]. More generally the known relations between tensions are of several types: they are either Dirac-Nepomechie-Teitelboim (DNT) quantisation conditions or they express global well-definiteness of the action classically or absence of anomalies, finally they may also be obtained from T-duality as shown first by Polchinski. We would like to stress that by the present classical considerations and the “single” valuedness of the action [34] one recovers the DNT quantisation condition. In general eqs. 3.15 are quite powerful.

What is rewarding is that after toroidal compactifications all the equations of motion of the bosonic matter fields of maximal SUGRA’s are reproduced (some of them really derived for the first time ab initio) and encoded by a relatively simple superalgebra. Note that some degrees of freedom of the graviton become progressively matter fields, eventually all of them when the theory has been reduced to 3 dimensions. All the matter field equations do follow our universal pattern, namely 3.10 and 3.11. The necessary superalgebra is a deformation described compactly in [5] of the following finite dimensional superalgebra:

$$\mathcal{G} := \mathcal{A} \ltimes \mathcal{A}^*$$

$$\mathcal{A} := \mathfrak{sl}_+(n|1) \ltimes (\wedge w)^3$$
where \( n \) is the dimension of the torus, the twelfth fermionic dimension appears in the classical superalgebra \( sl(n|1) \), \( w \) is its fundamental representation which decomposes as \( w = v + 1 \) (with \( v \) an \( n \)-component vector) as representation of \( sl(n) \). Only the Borel (triangular) subsuperalgebra appeared yet (this is the meaning of the + index), and its semidirect product is with the (super)antisymmetric tensor of third order. The latter unifies the three form and its descendent 2-forms etc. in a single representation. The deformation we alluded to reflects the Clifford structure above and is proportional to \( \kappa \). It is my conjecture that we shall find an even larger (“simple”) structure whose solvable part will be \( G \).

As promised we have encoded all equations for the bosons (but for the graviton) and extended the U-duality symmetry. Strictly speaking we have only recovered the Borel subgroup of \( E_n \) as follows. The superalgebra involved is \( \mathbb{Z} \)-graded (with nonpositive degrees) and its coefficients are forms whose degrees compensate those of the generators. In degree zero the coefficients are scalar fields and their dual (D-2)-forms are coefficients of the generators of degree -(D-2) (note these are odd in odd spacetime dimensions). The degree zero sector of the above superalgebra is \( b_0 = sl(n) \ltimes (\wedge v)^3 \), this gives precisely the Lie algebra of the scalar Borel manifold \( B \) (as Iwasawa told us: for a noncompact symmetric space \( B \approx K \backslash G \)), see [26, 35] and references therein.

4 Towards M-theory

4.1 Low energy theory

Clearly the other massless fields appropriate to the low energy approximation should be included, firstly the graviton and then all the fermionic partners. As far as the metric is concerned we would like to give two reasons for hope. The first one is an old result [36] on a BPS type condition for a fourth order gravity theory which leads to the Einstein spaces vacuum equations. These equations are of second order and allow for an arbitrary cosmological constant in four dimensions. The (twisted) self-duality equation reads again:

\[
R = *SR
\]

where \( R \) is the Lorentz algebra valued curvature 2-form (torsion is set equal to zero) and \( S \) is the Lorentz “Hodge” dualisation. It is not exactly what we are looking for but in four dimensions and up to the cosmological constant problem it comes close!
Another encouraging sign is the fact that the deformation proportional to the gravitational coupling constant is the only mysterious feature in the matter sector, the undeformed superalgebras listed above are quite natural. The fact that the semidirect product $\mathcal{A} \ltimes \mathcal{A}^*$ occurs everywhere is quite reminiscent of the orbit method in group theory with the cotangent bundle to the group as the basic symplectic object. At the quantum level it could be related to some “Quantum doubles”. Another hint along the same line is what we called the “Jade rule”, namely the property that if the commutator of two generators $V$ and $V'$ is equal to $V''$ the commutator of $V$ and $S(V')$ is up to a sign $S(V'')$, this is clearly true for semidirect products of the type $\mathcal{A} \ltimes \mathcal{A}^*$ but it is significant that the Jade rule is preserved by deformation! Furthermore if we could characterise our deformation we could envisage to include the graviton which in fact is also deformed from a linear (free) spin 2 field to the nonlinear metric by the requirement of (super)diffeomorphism invariance. We shall return to this observation in the next section.

Let us now return to the fermionic fields. They quite generally transform under the local gauge groups ($K$ the maximal compact U-duality subgroup or the Lorentz group $L$ for the spacetime symmetries) and are inert under U-dualities once the corresponding gauge invariances ($K \times L$) have been restored. One obstacle to progress has been the absence of appropriate simple finite dimensional or even affine superalgebra. In three dimensional maximal SUGRA the Borel subgroup of $E_8$ appears as the degree zero part of our symmetry superalgebra, the rest is the odd dual but as the superalgebra is far from simple, it has only nonpositive degrees, it does not appear in the tables of simple ones. This remark has consequences for our program of restoration of a larger supergroup beyond the present Borel part: it is unlikely to be finite dimensional if simple.

Finally one should include the extended objects and their massive excitations, this should reduce the symmetry to some kind of arithmetic subgroup.

4.2 The Noether method and GDA’s

The cohomology of infinite Lie algebras is at the heart of the so-called Noether method of construction of gauge theories like SUGRAS, it is in fact the only method available to derive 11d supergravity. In [28] we resummed the gravitational infinite series leading to Einstein’s action by using our knowledge of differential geometry. Then the local supersymmetry invariance was implemented order by order in $\kappa$ and in the absence of scalar fields this terminates after a finite number of steps. The result is the trilinear term in the three form
if one starts from the flat space linearised theory. Very much like in Yang- 
Mills theory one starts with abelian gauge invariances and a rigid non-abelian 
symmetry and one deforms the gauge invariance and its invariant action pre-
serving the rigid symmetry. What we have found here is similar, indeed the 
superalgebra of dualities is also deformed into another more nonlinear one and 
the deformation parameter is again the gravitational coupling constant. The 
equations of motion have been shown to be invariant.

Graded Differential Algebras (GDA’s) have been studied in recent times 
in Differential Topology [37], but with a restriction to simply connected man-
ifolds, they have been also considered in SUGRA theories [38] but with the 
same restriction (freeness to be lifted for sure) and finally in the analysis of 
BRS cohomology. Here the nonlinearities are different from Yang-Mills non 
linearities instead of expressions like $F = dA - A \wedge A$ one encounters

$$F = dA - dA' \wedge A''...$$ (4.2)

The (super)group is obtained by exponentiating (degree zero) combinations of 
dergree $n$ ($n \geq 0$ for the time being) differential forms and degree $-n$ generators 
of a $\mathbb{Z}$-graded superalgebra. Nonlinearities of the type (4.2) have been encoun-
tered before also in the coupling of type I SUGRA to abelian gauge fields, they 
were once more dictated by the requirement of local supersymmetry. In the 
nonabelian case the Chapline-Manton coupling in a sense combines the present 
nonlinearities and those of Yang-Mills theory.

5 Conclusion

The selfdual superalgebras have other applications under study, they repro-
duce, simplify and extend many known results but small Mysteries have con-
verged to a big one. Let us add that the Green-Schwarz term that cancels the 
perturbative anomalies of naive SUGRA is also of a (mixed) Chern-Simons 
nature. In [39] the modified Green-Schwarz mechanism involves distributions 
defined by the boundary cycles but it should fit into this circle of ideas. If so 
the relation Horava and Witten found between the Yang-Mills and the gravita-
tional coupling constants $\lambda_{YM} \propto \kappa^4$ strongly suggests a deformation analysis 
of their anomaly considerations but in terms of a parameter $\mu$ such that $\lambda \propto \mu^2$ 
and $\kappa \propto \mu^3$, this suggests a role for triality in the gravitational sector at least 
in 10/11 dimensions.
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References

[1] C. Hull and P. Townsend, Nucl. Phys. B438 (1995) 109.
[2] P. Townsend, Phys. Lett. 350B (1995) 184.
[3] E. Witten, Nucl. Phys. B443 (1995) 85.
[4] J. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231.
[5] E. Cremmer, B. Julia, H. Lü and C. N. Pope, Nucl. Phys. B535 (1998) 242, hep-th/9806106.
[6] M. B. Green and P. Vanhove, Phys.Lett. 408B (1997) 122.
[7] D. Dunbar, B. Julia, D. Seminara and M. Trigiante Counterterms in type I Supergravities, to appear in JHEP, hep-th/9911158.
[8] B.K. Harrison, J. Math. Phys. 9 (1968) 1744
W. Kinnersley, J. Math. Phys. 14 (1973) 651.
[9] B. Julia, Kac-Moody symmetry of gravitation and supergravity theories, Proc. AMS-SIAM Summer Seminar on Applications of Group Theory in Physics and Mathematical Physics, Chicago 1982, LPTENS preprint 82/22, eds. M. Flato, P. Sally and G. Zuckerman, Lectures in Applied Mathematics, 21 (1985) 335.
[10] J. Schwarz, Talk at Strings 1993, hep-th/9307121.
[11] J. Louis and B. de Wit, in Comptes-rendus de l’Ecole de Cargèse Mai 1997, “Strings, Branes and Dualities”, eds. P. Windey et al. Kluwer (1999) ASI C520, hep-th/9801132.
[12] I. Bakas, Phys. Lett B343 (1995) 103, hep-th/9410104.
[13] M. Green, hep-th/9903124 and Talk at Strings 99.

[14] R. Dijkgraaf, E. Verlinde and H. Verlinde, in Perspectives in String theory, P. Di Vecchia and J.L. Petersen eds. (World Scientific Singapore 1988).

[15] B. Julia, Dualities in the classical supergravity limits, hep-th/9805083. Comptes-rendus de l’Ecole de Cargèse de Mai 1997, “Strings, Branes and Dualities”, eds. P. Windey et al. Kluwer (1999) ASI C520.

[16] A. Hanany and B. Julia, in preparation.

[17] D. Korotkin and H. Samtleben, Nucl.Phys. B527 (1998) 657-689, hep-th/9710210.

[18] K. Koepsell, H. Nicolai, H. Samtleben, JHEP 9904 (1999) 023, hep-th/9903111.

[19] E. Cremmer, B. Julia, H. Lü and C. Pope, Higher-dimensional Origin of D=3 Coset Symmetries, hep-th/9909099.

[20] B. Julia, Group disintegrations; in Superspace and Supergravity, Eds. S.W. Hawking and M. Rocek (Cambridge Univ. Press, 1981) 331.

[21] Riccardo D’Auria, Sergio Ferrara, Costas Kounnas, Phys.Lett. 420B (1998) 289, hep-th/9711048.

[22] B. Julia, in preparation.

[23] B. Julia and H. Nicolai, Nucl. Phys. B482 (1996) 431, hep-th/9608082.

[24] S. Ferrara, J. Scherk and B. Zumino, Nucl. Phys. B121 (1977) 393.

[25] E. Cremmer, S. Ferrara and J. Scherk, Phys. Lett. 74B (1978) 61.

[26] E. Cremmer and B. Julia, Phys. Lett. 80B (1978) 48; Nucl. Phys. B156 (1979) 141.

[27] E. Cremmer, B. Julia, H. Lü and C.N. Pope, Nucl. Phys. B523 (1998) 73, hep-th/9710119.

[28] E. Cremmer, B. Julia and J. Scherk, Phys. Lett. 76B (1978) 409.

[29] H. Nishino and S.J. Gates Jr, Nucl. Phys. B268 (1986), 532.
[30] G. Dall’Agata, K. Lechner and M. Tonin, JHEP 9807 (1998) 017, hep-th/9806140.

[31] I. Bandos, N. Berkovits and D. Sorokin, Nucl. Phys. B522 (1998) 214, hep-th/9711055.

[32] B. Julia, Effective gauge fields, in Proceedings of the 4th Johns Hopkins Workshop on “Current problems in particle theory,” Bonn (1980) ed. R. Casalbuoni et al.

[33] I.V. Lavrinenko, H. Lü, C.N. Pope and K.S. Stelle, Nucl.Phys. B555 (1999) 201, hep-th/990357.

[34] M. Duff, J. Liu and R. Minasian, Nucl. Phys. B452 (1995) 261, hep-th/9506126.

[35] H. Lü, C.N. Pope and K.S. Stelle, Nucl. Phys. B476 (1996) 89, hep-th/9602140.

[36] A. Belavin and D. B Burlankov, Phys. Lett. 58A (1976) 7.

[37] D. Sullivan, Infinitesimal computations in topology, in Publications Math. de l’IHES, 47 (1977) 269.

[38] R. D’Auria and P. Fré, in Proceedings of the 1982 Trieste Summer School, eds. S. Ferrara et al., (1983) World Scientific.

[39] P. Horava, E. Witten, Nuclear Phys. B475 (1996) 94.