MATROID BASE POLYTOPE DECOMPOSITION II : SEQUENCES OF HYPERPLANE SPLITS

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Abstract. This is a continuation of an early paper [Adv. Appl. Math. 47(2011), 158-172] about matroid base polytope decomposition. We will present sufficient conditions on a matroid $M$ so its base polytope $P(M)$ has a sequence of hyperplane splits. These yields to decompositions of $P(M)$ with two or more pieces for infinitely many matroids $M$. We also present necessary conditions on the Euclidean representation of rank three matroids $M$ for the existence of decompositions of $P(M)$ into 2 or 3 pieces. Finally, we prove that $P(M_1 \oplus M_2)$ has a sequence of hyperplane splits if either $P(M_1)$ or $P(M_2)$ also has a sequence of hyperplane splits.

Keywords: Matroid base polytope, polytope decomposition

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1. Introduction

This paper is a continuation of the paper by the two present authors. For general background in matroid theory we refer the reader to [13],[16]. A matroid $M = (E, \mathcal{B})$ of rank $r = r(M)$ is a finite set $E = \{1, \ldots, n\}$ together with a nonempty collection $\mathcal{B} = \mathcal{B}(M)$ of $r$-subsets of $E$ (called the bases of $M$) satisfying the following basis exchange axiom:

if $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \setminus B_2$, then there exists $f \in B_2 \setminus B_1$ such that $(B_1 - e) + f \in \mathcal{B}$.

We denote by $\mathcal{I}(M)$ the family of independent sets of $M$ (consisting of all subsets of bases of $M$). For a matroid $M = (E, \mathcal{B})$, the matroid base polytope $P(M)$ of $M$ is defined as the convex hull of the incidence vectors of bases of $M$, that is,

$$ P(M) := \text{conv} \left\{ \sum_{i \in B} e_i : B \text{ a base of } M \right\}, $$

where $e_i$ is the $i^{th}$ standard basis vector in $\mathbb{R}^n$. $P(M)$ is a polytope of dimension at most $n - 1$.

A matroid base polytope decomposition of $P(M)$ is a decomposition

$$ P(M) = \bigcup_{i=1}^t P(M_i) $$

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where each $P(M_i)$ is a matroid base polytope for some matroid $M_i$ and, for each $1 \leq i \neq j \leq t$, the intersection $P(M_i) \cap P(M_j)$ is a face of both $P(M_i)$ and $P(M_j)$. It is known that nonempty faces of matroid base polytope are matroid base polytopes [5, Theorem 2]. So, the common face $P(M_i) \cap P(M_j)$ (whose vertices correspond to elements of $B(M_i) \cap B(M_j)$) must also be a matroid base polytope. $P(M)$ is said to be decomposable if it admits a matroid base polytope decomposition with $t \geq 2$ and indecomposable otherwise. A decomposition is called hyperplane split when $t = 2$.

Matroid base polytope decomposition were introduced by Lafforgue [10, 11] and have appeared in many different contexts: quasisymmetric functions [11, 12], compactification of the moduli space of hyperplane arrangements [6, 8], tropical linear spaces [14, 15], etc. In [3], we have studied the existence (and nonexistence) of such decompositions. Among other results, we presented sufficient conditions on a matroid $M$ so $P(M)$ admits a hyperplane split. This yielded us to different hyperplane splits for infinitely many matroids. A natural question is the following one: given a matroid base polytope $P(M)$, is it possible to find a sequence of hyperplane splits providing a decomposition of $P(M)$? In other words, is there a hyperplane split of $P(M)$ such that one of the two obtained pieces has a hyperplane split such that, in turn, one of the two new obtained pieces has a hyperplane split, and so on, giving a decomposition of $P(M)$?

In [7, Section 1.3], Kapranov showed that all decompositions of a (appropriately parametrized) rank-2 matroid can be achieved by a sequence of hyperplane splits. However, this is not the case in general. Billera, Jia and Reiner [2] provided a decomposition into three indecomposable pieces of $P(W)$ where $W$ is the rank three matroid on $\{1, \ldots, 6\}$ with $B(W) = \binom{[6]}{2} \setminus \{(1,2,3), (1,4,5), (3,5,6)\}$. They proved that this decomposition cannot be obtained via hyperplane splits. However, we notice that $P(W)$ may admits other decompositions into three pieces that can be obtained via hyperplane splits; this is illustrated in Example 3.

A difficulty arising when we apply successive hyperplane splits is that the intersection $P(M_i) \cap P(M_j)$ also must be a matroid base polytope. For instance, consider a first hyperplane split $P(M) = P(M_1) \cup P(M'_1)$ and suppose that $P(M'_1)$ admits a hyperplane split, say $P(M'_1) = P(M_2) \cup P(M'_2)$. This sequence of 2 hyperplane splits would give the decomposition $P(M) = P(M_1) \cup P(M_2) \cup P(M'_2)$ if $P(M_1) \cap P(M_2), P(M_1) \cap P(M'_2)$ were matroid base polytopes. By definition of hyperplane split, $P(M_2) \cap P(M'_2)$ is the base polytope of a matroid, however the other two intersections might not be matroid base polytopes. Recall that the intersection of two matroids is not necessarily a matroid (for instance, $B(M_1) = \{\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}\}$ and $B(M_2) = \{\{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}\}$ are matroids while $B(M_1) \cap B(M_2) = \{\{1,3\}, \{2,3\}, \{2,4\}\}$ is not).
In the next section, we give sufficient conditions on $M$ so that $P(M)$ admits a sequence of $t \geq 2$ hyperplane splits. This allows us to provide decompositions of $P(M)$ with $t + 1$ pieces for infinitely many matroids. We say that two decompositions $P(M) = \bigcup_{i=1}^{t} P(M_i)$ and $P(M) = \bigcup_{i=1}^{t} P(M'_i)$ are equivalent if there exists a permutation $\sigma$ of $\{1, \ldots, t\}$ such that $P(M_i)$ is combinatorially equivalent to $P(M'_{\sigma(i)})$. They are different otherwise. We present a lower bound for the number of different decompositions of $P(U_{n,r})$ into $t$ pieces.

In Section 3, we present necessary geometric conditions (on the Euclidean representation) of rank three matroids $M$ for the existence of decompositions of $P(M)$ into 2 or 3 pieces. Finally, in Section 4, we show that the direct sum $P(M_1 \oplus M_2)$ has a sequence of hyperplane splits if either $P(M_1)$ or $P(M_2)$ also has a sequence of hyperplane splits.

2. Sequence of hyperplane splits

Let $M = (E, B)$ be a matroid of rank $r$ and let $A \subseteq E$. We recall that the independent sets of the restriction of matroid $M$ to $A$, denoted by $M|_A$, are given by $I(M|_A) = \{I \subseteq A : I \in I(M)\}$.

Let $t \geq 2$ be an integer with $r \geq t$. Let $E = \bigcup_{i=1}^{t} E_i$ be a $t$-partition of $E = \{1, \ldots, n\}$ and let $r_i = r(M|_{E_i}) > 1$, $i = 1, \ldots, t$. We say that $\bigcup_{i=1}^{t} E_i$ is a good $t$-partition if there exist integers $0 < a_i < r_i$ with the following properties:

(P1) $r = \sum_{i=1}^{t} a_i$.

(P2) (a) For any $j$ with $1 \leq j \leq t - 1$
if $X \in I(M|_{E_1 \cup \cdots \cup E_j})$ with $|X| \leq a_1$ and $Y \in I(M|_{E_{j+1} \cup \cdots \cup E_t})$ with $|Y| \leq a_2$, then $X \cup Y \in I(M)$.

(b) For any pair $j, k$ with $1 \leq j < k \leq t - 1$
if $X \in I(M|_{E_1 \cup \cdots \cup E_j})$ with $|X| \leq \sum_{i=1}^{j} a_i$,

$Y \in I(M|_{E_{j+1} \cup \cdots \cup E_k})$ with $|Y| \leq \sum_{i=j+1}^{k} a_i$,

$Z \in I(M|_{E_{k+1} \cup \cdots \cup E_t})$ with $|Z| \leq \sum_{i=k+1}^{t} a_i$,
then $X \cup Y \cup Z \in I(M)$.

Notice that the good 2-partitions provided by (P2) case (a) with $t = 2$ are the good partitions defined in [3]. Good partitions were used to give sufficient conditions for the existence of hyperplane splits. The latter was a consequence of the following two results:
Lemma 1. [3] Lemma 1 | Let $M = (E, \mathcal{B})$ be a matroid of rank $r$ and let $E = E_1 \cup E_2$ be a good 2-partition with integers $0 < a_i < r(M|_{E_i})$, $i = 1, 2$. Then,
\[
\mathcal{B}(M_1) = \{ B \in \mathcal{B}(M) : |B \cap E_1| \leq a_1 \} \quad \text{and} \quad \mathcal{B}(M_2) = \{ B \in \mathcal{B}(M) : |B \cap E_2| \leq a_2 \}
\]
are the collections of bases of matroids.

Theorem 1. [3] Theorem 1 | Let $M = (E, \mathcal{B})$ be a matroid of rank $r$ and let $E = E_1 \cup E_2$ be a good 2-partition with integers $0 < a_i < r(M|_{E_i})$, $i = 1, 2$. Then, $P(M) = P(M_1) \cup P(M_2)$ is a hyperplane split, where $M_1$ and $M_2$ are the matroids given by Lemma [1].

We shall use these two results as the initial step in our construction of a sequence of $t \geq 2$ hyperplane splits.

Lemma 2. Let $t \geq 2$ be an integer and let $E = \bigcup_{i=1}^{t} E_i$ be a good $t$-partition with integers $0 < a_i < r(M|_{E_i})$, $i = 1, \ldots, t$. Let
\[
\mathcal{B}(M_1) = \{ B \in \mathcal{B}(M) : |B \cap E_1| \leq a_1 \}
\]
and, for each $j = 1, \ldots, t$, let
\[
\mathcal{B}(M_j) = \{ B \in \mathcal{B}(M) : |B \cap E_1| \geq a_1, \ldots, |B \cap \bigcup_{i=1}^{j-1} E_i| \geq \sum_{i=1}^{j-1} a_i, |B \cap \bigcup_{i=1}^{j} E_i| \leq \sum_{i=1}^{j} a_i \}.
\]
Then $\mathcal{B}(M_i)$ is the collection of bases of a matroid for each $i = 1, \ldots, t$.

Proof. By Properties (P1) en (P2) we have that

if $X \in I(M|_{E_1})$ with $|X| \leq a_1$ and $Y \in I(M|_{E_2 \cup \cdots \cup E_t})$ with $|Y| \leq \sum_{i=2}^{t} a_i$,

then $X \cup Y \in I(M)$. So, by Lemma [1] $\mathcal{B}(M_1)$ is the collection of bases of a matroid. Now, notice that $\mathcal{B}(\overline{M_1}) = \{ B \in \mathcal{B}(M) : |B \cap E_1| \geq a_1 \}$ is also the collection of bases of a matroid on $E$. We claim that $P(\overline{M_1}) = P(M_2) \cup P(\overline{M_2})$ is a hyperplane split where
\[
\mathcal{B}(M_2) = \{ B \in \mathcal{B}(M) : |B \cap E_1| \geq a_1 \quad \text{and} \quad |B \cap (E_1 \cup E_2)| \leq a_1 + a_2 \}
\]
and
\[
\mathcal{B}(\overline{M_2}) = \{ B \in \mathcal{B}(M) : |B \cap E_1| \geq a_1 \quad \text{and} \quad |B \cap (E_1 \cup E_2)| \geq a_1 + a_2 \}.
\]
Indeed, since $\mathcal{B}(\overline{M_1})$ is the collection of bases of a matroid on $E$, then, by properties (P1) and (P2) (a),

if $X \in I(\overline{M}|_{E_1 \cup E_2})$ with $|X| \leq a_1 + a_2$ and $Y \in I(\overline{M}|_{E_3 \cup \cdots \cup E_t})$ with $|Y| \leq \sum_{i=3}^{t} a_i$,

then $X \cup Y \in I(\overline{M})$. So, by Lemma [1] $\mathcal{B}(M_2)$ is the collection of bases of a matroid (and thus $\mathcal{B}(\overline{M_2})$ also is). Inductively applying the above argument to $\overline{M_j}$, it can be easily checked that for all $j$ $\mathcal{B}(M_j)$ is the collection of bases of a matroid. \qed
**Theorem 2.** Let $t \geq 2$ be an integer and let $M = (E, \mathcal{B})$ be a matroid of rank $r$. Let $E = \bigcup_{i=1}^{t} E_i$ be a good $t$-partition with integers $0 < a_i < r(M|_{E_i})$, $i = 1, \ldots, t$. Then $P(M)$ has a sequence of $t$ hyperplane splits yielding the decomposition

$$P(M) = \bigcup_{i=1}^{t} P(M_i),$$

where $M_i$, $1 \leq i \leq t$, are the matroids defined in Lemma 2.

**Proof.** By Theorem 1, the result holds for $t = 2$. Moreover, by the inductive construction of Lemma 2, we clearly have that $P(M) = \bigcup_{i=1}^{t} P(M_i)$ with $\mathcal{B}(M) = \bigcup_{i=1}^{t} \mathcal{B}(M_i)$. We only need to show that $\mathcal{B}(M_j) \cap \mathcal{B}(M_k)$ is the collection of bases of a matroid for any $1 \leq j < k \leq t$. For, by definition of $\mathcal{B}(M_i)$, we have

$$\mathcal{B}(M_j) \cap \mathcal{B}(M_k) = \{ B \in \mathcal{B}(M) : \text{the condition } C_h(B) \text{ is satisfied for all } 1 \leq h \leq k \}$$

where for $A \subseteq E$:

- $C_h(A)$ is satisfied if $|A \cap \bigcup_{i=1}^{h} E_i| \geq \sum_{i=1}^{h} a_i$ and $1 \leq h \leq k$, $h \neq j, k$,

- $C_j(A)$ is satisfied if $|A \cap \bigcup_{i=1}^{j} E_i| = \sum_{i=1}^{j} a_i$,

and

- $C_k(A)$ is satisfied if $|A \cap \bigcup_{i=1}^{k} E_i| \leq \sum_{i=1}^{k} a_i$.

We will check the exchange axiom for any $X, Y \in \mathcal{B}(M_j) \cap \mathcal{B}(M_k)$. Since $X, Y \in \mathcal{B}(M)$ for any $e \in X \setminus Y$ there exists $f \in Y \setminus X$ such that $X - e + f \in \mathcal{B}(M)$. We will verify that $X - e + f \in \mathcal{B}(M_j) \cap \mathcal{B}(M_k)$. We distinguish three cases (depending which of the conditions $C_i(X - e)$ is satisfied).

**Case 1.** There exists $1 \leq l \leq j$ such that $C_l(X - e)$ is not satisfied. We suppose that $l$ is minimal with this property. Since, by definition of $\mathcal{B}(M_j) \cap \mathcal{B}(M_k), l \leq j \leq k$, $C_l(X)$ is satisfied, and $C_l(X - e)$ is not satisfied, we obtain

(a) $|X \cap \bigcup_{i=1}^{l} E_i| = \sum_{i=1}^{l} a_i$,

(b) $e \in \bigcup_{i=1}^{l} E_i$,

(c) $|(X - e) \cap \bigcup_{i=1}^{l} E_i| = \sum_{i=1}^{l} a_i - 1$. 


Since $Y \in \mathcal{B}(M_j) \cap \mathcal{B}(M_k)$, then $|Y \cap \bigcup_{i=1}^{l} E_i| \geq \sum_{i=1}^{l} a_i$.

Therefore, by using (c), $I_1, I_2 \in \mathcal{I}(M|_{E_1 \cup \cdots \cup E_l}) \subseteq \mathcal{I}(M)$ with $|I_1| < |I_2|$. So, there exists $f \in I_2 \setminus I_1 \subset Y \setminus X$ with $I_1 \cup f \in \mathcal{I}(M|_{E_1 \cup \cdots \cup E_l})$. Thus, $f \in \bigcup_{i=1}^{l} E_i$ and

$$|I_1 \cup f \cap \bigcup_{i=1}^{l} E_i| = \sum_{i=1}^{l} a_i - 1. \quad (1)$$

Moreover, since $X$ is a base, $|X| = r = \sum_{i=1}^{l} a_i$ and, by (a), we have

$$|(X - e + f) \cap \bigcup_{i=1}^{l} E_i| = |X \cap \bigcup_{i=1}^{l} E_i| = \sum_{i=1}^{l} a_i - \sum_{i=1}^{l} a_i = \sum_{i=1}^{l} a_i.$$  

We also have $I_3 \in \mathcal{I}(M|_{E_{l+1} \cup \cdots \cup E_t})$, thus, by (P2) (b),

$I_1 \cup f \cup I_3 \in \mathcal{I}(M)$ with $|I_1 \cup f \cup I_3| = \sum_{i=1}^{l} a_i + 1 + \sum_{i=l+1}^{t} a_i = r$

and so $I_1 \cup f \cup I_3 = X - e + f \in \mathcal{B}(M)$.

Finally we need to show that $X - e + f \in \mathcal{B}_j \cap \mathcal{B}_k$, that is $C_h(X - e + f)$ holds for each $1 \leq h \leq k$.

(i) $h < l$: Since $l$ is the minimum for which $C_l(X - e)$ is not verified, $C_h(X - e)$ is satisfied for each $1 \leq h < l$ and thus $C_h(X - e + f)$ is also satisfied (we just added a new element).

(ii) $h = l$: By equation (1), $C_l(X - e + f)$ is satisfied.

(iii) $h > l$: Since $e, f \in \bigcup_{i=1}^{l} E_i$,

$$|X - e + f \cap \bigcup_{i=1}^{h} E_i| = |X \cap \bigcup_{i=1}^{h} E_i|,$$

thus $C_h(X - e + f)$ is satisfied if and only if $C_h(X)$ is satisfied, which is the case since $h > l$.

Case 2. $C_{l'}(X - e)$ is satisfied for all $1 \leq l' \leq j$ and there exists $j + 1 \leq l \leq k - 1$ such that $C_l(X - e)$ is not satisfied. We suppose that $l$ is minimal with this property. Since $C_l(X)$ is satisfied and $C_l(X - e)$ is not,

(a) $|X \cap \bigcup_{i=1}^{l} E_i| = \sum_{i=1}^{l} a_i,$

(b) $e \in \bigcup_{i=j+1}^{l} E_i$ (since $C_j(X - e)$ is satisfied),
(c) \(|(X - e) \cap \bigcup_{i=1}^{l} E_i| = \sum_{i=1}^{l} a_i - 1.\)

Since \(C_j(X - e)\) is satisfied,

\[
| (X - e) \cap \bigcup_{i=1}^{l} E_i | = |(X - e) \cap \bigcup_{i=1}^{l} E_i| - |(X - e) \cap \bigcup_{i=1}^{j} E_i |
\]

\[
= \sum_{i=1}^{l} a_i - 1 - \sum_{i=1}^{j} a_i = \sum_{i=j+1}^{l} a_i - 1.
\]  \(\text{(2)}\)

Let \(Y \in \mathcal{B}(M_j) \cap \mathcal{B}(M_k)\). Since \(C_j(Y)\) and \(C_l(Y)\) are satisfied,

\[
|Y \cap \bigcup_{i=1}^{l} E_i | = |Y \cap \bigcup_{i=1}^{l} E_i| - |Y \cap \bigcup_{i=1}^{j} E_i |
\]

\[
\geq \sum_{i=1}^{l} a_i - \sum_{i=1}^{j} a_i = \sum_{i=j+1}^{l} a_i.
\]

Since \(|I_1| < |I_2|\), there exists \(f \in I_2 \setminus I_1\) such that \(I_1 + f \in \mathcal{I}(M|_{E_{j+1} \cup \cdots \cup E_t})\). So, \(f \in \bigcup_{i=j+1}^{l} E_i\) and, by (b), we have

\[(X - e + f) \cap \bigcup_{i=1}^{j} E_i = X \cap \bigcup_{i=1}^{j} E_i.\]

Since \(X\) is a base, \((X - e + f) \cap \bigcup_{i=1}^{j} E_i \in \mathcal{I}(M|_{E_{j+1} \cup \cdots \cup E_t})\) (also notice that \((X - e + f) \cap \bigcup_{i=1}^{l} E_i \in \mathcal{I}(M|_{E_{j+1} \cup \cdots \cup E_t})\)). Moreover, since \(X \in \mathcal{B}_j \cap \mathcal{B}_k\), \(C_j(X)\) is satisfied and thus

\[(X - e + f) \cap \bigcup_{i=1}^{j} E_i| = \sum_{i=1}^{j} a_i\]  \(\text{(3)}\)

and, by equation (2), we have

\[(X - e + f) \cap \bigcup_{i=1}^{l} E_i| = \sum_{i=j+1}^{l} a_i\]  \(\text{(4)}\)

obtaining that

\[(X - e + f) \cap \bigcup_{i=1}^{l} E_i| = r - \sum_{i=1}^{j} a_i - \sum_{i=j+1}^{l} a_i = \sum_{i=l+1}^{t} a_i.\]
Now, by (P2) (b), we have
\[
\left( (X - e + f) \cap \bigcup_{i=1}^{j} E_i \right) \cup \left( (X - e + f) \cap \bigcup_{i=j+1}^{l} E_i \right) \cup \left( (X - e + f) \cap \bigcup_{i=l+1}^{t} E_i \right) = X - e + f \in \mathcal{I}(M).
\]

Since \(|X - e + f| = r, X - e + f \in \mathcal{B}(M)|

Finally we need to show that \(X - e + f \in \mathcal{B}(M)\), that is, that \(C_h(X - e + f)\) is verified for each \(1 \leq h \leq k\).

(i) \(h < l\) and \(h \neq j\): Since \(C_h(X - e)\) is satisfied, by the minimality of \(l\), \(C_h(X - e + f)\) is also satisfied.

(ii) \(h = j\): By equation (2), \(C_j(X - e + f)\) is satisfied.

(iii) \(h = l\): By equations (3) and (4), \(C_l(X - e + f)\) is satisfied.

(iv) \(h > l\): Since \(e, f \in \bigcup_{i=1}^{l} E_i, |X - e + f \cap \bigcup_{i=1}^{h} E_i| = |X \cap \bigcup_{i=1}^{h} E_i|\), thus \(C_h(X - e + f)\) is satisfied if and only if \(C_h(X)\) is satisfied, which is the case because \(h > l\).

Case 3. \(C_i(X - e)\) is satisfied for every \(1 \leq i \leq k\).

Subcase (a) \(|(X - e) \cap \bigcup_{i=1}^{k} E_i| = \sum_{i=1}^{k} a_i.\) We first notice that \(e \in \bigcup_{i=k+1}^{t} E_i\) (otherwise \(|X - e \cap \bigcup_{i=1}^{k} E_i| < |X \cap \bigcup_{i=1}^{k} E_i|\) which is impossible since \(C_k(X)\) holds). Now,

\[
\sum_{i=1}^{k} a_i = \sum_{i=k+1}^{t} a_i - 1. \tag{5}
\]

Let \(Y \in \mathcal{B}(M_j) \cap \mathcal{B}(M_k)\). Since \(C_j(Y)\) and \(C_l(Y)\) are satisfied, \(|Y \cap \bigcup_{i=1}^{k} E_i| \leq \sum_{i=1}^{k} a_i,\) and

so \(|Y \cap \bigcup_{i=k+1}^{t} E_i| \geq \sum_{i=k+1}^{t} a_i.\)

Since \(|I_2| < |I_2|\) there exists \(f \in I_2 \setminus I_1\) such that \(I_1 + f \in \mathcal{I}(M|_{E_{k+1} \cup \ldots \cup E_t}).\) So,

\(f \in \bigcup_{i=k+1}^{t} E_i\) and since \(e \in \bigcup_{i=k+1}^{t} E_i,\)

\[
(X - e + f) \cap \bigcup_{i=1}^{k} E_i = X \cap \bigcup_{i=1}^{k} E_i \in \mathcal{I}(M|_{E_{k+1} \cup \ldots \cup E_t}).
\]

Also, since \((X - e + f) \cap \bigcup_{i=k+1}^{t} E_i \in \mathcal{I}(M|_{E_{k+1} \cup \ldots \cup E_t}),\) by (P2)(b) we have

\[
X - e + f = \left( X - e + f \cap \bigcup_{i=1}^{k} E_i \right) \cup \left( X - e + f \cap \bigcup_{i=k+1}^{t} E_i \right) \in \mathcal{I}(M).
\]
Moreover, by using equation (5) and the fact that \( f \in \bigcup_{i=k+1}^{t} E_i \) we obtain that
\[
|(X - e + f) \cap \bigcup_{i=k+1}^{t} E_i| = \sum_{i=k+1}^{t} a_i.
\]
Since \( |(X - e) \cap \bigcup_{i=1}^{k} E_i| = \sum_{i=1}^{k} a_i \),
\[
|(X - e + f) \cap \bigcup_{i=1}^{k} E_i| = \sum_{i=1}^{k} a_i.
\]
Therefore,
\[
|(X - e + f) \cap \bigcup_{i=1}^{t} E_i| = (X - e + f) \cap \bigcup_{i=1}^{k} E_i + |(X - e + f) \cap \bigcup_{i=k+1}^{t} E_i| = \sum_{i=1}^{t} a_i = r
\]
and so \( X - e + f \in \mathcal{B}(M) \).

Finally we need to show that \( X - e + f \in \mathcal{B}_j \cap \mathcal{B}_k \), that is, that \( C_h(X - e + f) \) is verified for each \( 1 \leq h \leq k \). Since \( e, f \in \bigcup_{i=k+1}^{t} E_i \), \( C_h(X - e + f) \) becomes \( C_h(X) \) for all \( 1 \leq h \leq k \), which is satisfied.

**Subcase (b)** If \( |(X - e) \cap \bigcup_{i=1}^{k} E_i| < \sum_{i=1}^{k} a_i \), then \( e \in \bigcup_{i=k+1}^{t} E_i \) (otherwise \( |(X - e) \cap \bigcup_{i=1}^{j} E_i| < |X \cap \bigcup_{i=1}^{j} E_i| \) which is impossible since \( C_j(X) \) holds). Now, since \( C_j(X - e) \) is satisfied,
\[
|(X - e) \cap \bigcup_{i=1}^{j} E_i| = \sum_{i=1}^{j} a_i,
\]
and thus
\[
|(X - e) \cap \bigcup_{i=j+1}^{t} E_i| = \sum_{i=j+1}^{t} a_i - 1.
\]
Let \( Y \in \mathcal{B}(M_j) \cap \mathcal{B}(M_k) \). Since \( C_j(Y) \) and \( C_l(Y) \) are satisfied,
\[
|Y \cap \bigcup_{i=1}^{j} E_i| = \sum_{i=1}^{j} a_i,
\]
and thus
\[
|Y \cap \bigcup_{i=j+1}^{t} E_i| = \sum_{i=j+1}^{t} a_i.
\]
Since \( |I_1| < |I_2| \), there exists \( f \in I_2 \setminus I_1 \) such that \( I_1 + f \in \mathcal{I}(M | E_{j+1} \cup \cdots \cup E_t) \). So,
\[
\bigcap_{i=1}^{j} E_i = X \cap \bigcap_{i=1}^{j} E_i \in \mathcal{I}(M | E_{j+1} \cup \cdots \cup E_t). \tag{6}
\]
and, by (P2) (b), we have
\[
\left( X - e + f \cap \bigcup_{i=1}^{j} E_i \right) \cup \left( X - e + f \cap \bigcup_{i=j+1}^{t} E_i \right) \in \mathcal{I}(M)
\]
Therefore, \( X - e + f \in B(M) \).

Finally, we need to show that \( X - e + f \in B \cap B_k \), that is, \( C_h(X - e + f) \) is verified for each \( 1 \leq h \leq k \).

(i) \( h < j \): Since \( C_h(X - e) \) is satisfied, \( C_h(X - e + f) \) is also satisfied.

(ii) \( h = j \): \( C_j(X - e + f) \) is satisfied by equation (6).

(iii) \( j + 1 \leq h \leq k - 1 \): Since \( C_h(X - e) \) is satisfied then \( C_h(X - e + f) \) is also satisfied.

(iv) \( h = k \): Since \( |X - e \cap \bigcup_{i=1}^{k} E_i| < \sum_{i=1}^{k} a_i \) then \( |X - e + f \cap \bigcup_{i=1}^{k} E_i| \leq \sum_{i=1}^{k} a_i \) and thus \( C_h(X - e + f) \) is satisfied. \( \square \)

2.1. Uniform matroids.

**Corollary 1.** Let \( n, r, t \geq 2 \) be integers with \( n \geq r + t \) and \( r \geq t \). Let \( p_t(n) \) be the number of different decompositions of the integer \( n \) of the form \( n = \sum_{i=1}^{t} p_i \) with \( p_i \geq 2 \) and let \( h_t(U_{n,r}) \) be the number of decompositions of \( P(U_{n,r}) \) into \( t \) pieces. Then,
\[
h_t(U_{n,r}) \geq p_t(n).
\]

**Proof.** We consider the partition \( E = \{1, \ldots, n\} = \bigcup_{i=1}^{t} E_i \), where
\[
E_1 = \{1, \ldots, p_1\}, \\
E_2 = \{p_1 + 1, \ldots, p_1 + p_2\}, \\
\vdots \\
E_t = \{\sum_{i=1}^{t-1} p_i + 1, \ldots, \sum_{i=1}^{t} p_i\}.
\]

We claim that \( \bigcup_{i=1}^{t} E_i \) is a good \( t \)-partition. For, we first notice that \( M|_{E_i} \) is isomorphic to \( U_{p_i, \min\{p_i, r\}} \) for each \( i = 1, \ldots, t \). Let \( r_i = r(M|_{E_i}) = \min\{p_i, r\} \). We now show that
\[
\sum_{i=1}^{t} r_i \geq r + t.
\] (7)

For, we note that
\[
\sum_{i=1}^{t} r_i = \sum_{i=1}^{t} r(M|_{E_i}) = \sum_{i \in T \subseteq \{1, \ldots, t\}} p_i + (t - |T|)r.
\]
We distinguish three cases.

1) If \( t = |T| \), then \( \sum_{i=1}^{t} r_i = \sum_{i=1}^{t} p_i = n \geq r + t \).

2) If \( t = |T| + 1 \), then \( \sum_{i=1}^{t} r_i = \sum_{i=1}^{t-1} p_i + r \geq 2(t-1) + r \geq t + t - 2 + r \geq t + r \).

3) If \( t = |T| + k \), with \( k \geq 2 \), then \( \sum_{i=1}^{t} r_i \geq kr \geq 2r \geq r + t \).

So, by equation (1), we can find integers \( a'_i \geq 1 \) such that \( \sum_{i=1}^{t} r_i = r + \sum_{i=1}^{t} a'_i \). Therefore, there exist integers \( a_i = r(M|E_i) - a'_i \) with \( 0 < a_i < r(M|E_i) \) such that \( r = \sum_{i=1}^{t} a_i \). Moreover, if \( X \in \mathcal{I}(M|E_1 \cup \cdots \cup E_t) \) with \( |X| \leq \sum_{i=1}^{t} a_i \), \( Y \in \mathcal{I}(M|E_{j+1} \cup \cdots \cup E_k) \) with \( |Y| \leq \sum_{i=j+1}^{k} a_i \), and \( Z \in \mathcal{I}(M|E_{k+1} \cup \cdots \cup E_t) \) with \( |Z| \leq \sum_{i=k+1}^{t} a_i \) for \( 1 \leq j < k \leq t-1 \), then \( |X \cup Y \cup Z| \leq \sum_{i=1}^{t} a_i = r \) and so \( X \cup Y \cup Z \) is always a subset of one of the bases of \( U_{n,r} \). Thus, \( X \cup Y \cup Z \in \mathcal{I}(U_{n,r}) \) and \((P2)\) is also verified.

Notice that there might be several choices for the values of \( a_i \) (each providing a good \( t \)-partition). However, it is not clear if these choices give different sequences of \( t \) hyperplane splits.

**Example 1:** Let us consider the uniform matroid \( U_{8,4} \). We take the partition \( E_1 = \{1, 2\} \), \( E_2 = \{3, 4\} \), \( E_3 = \{5, 6\} \), and \( E_4 = \{7, 8\} \). Then \( r(M|E_i) = 2 \), \( i = 1, \ldots, 4 \). It is easy to check that if we set \( a_i = 1 \) for each \( i \) then \( E_1 \cup E_2 \cup E_3 \cup E_4 \) is a good \( 4 \)-partition and thus \( \mathcal{P}(U_{8,3}) = \mathcal{P}(M_1) \cup \mathcal{P}(M_2) \cup \mathcal{P}(M_3) \cup \mathcal{P}(M_4) \) is a decomposition where

- \( \mathcal{B}(M_1) = \{B \in \mathcal{B}(U_{8,4}) : |B \cap \{1, 2\}| \leq 1\} \),
- \( \mathcal{B}(M_2) = \{B \in \mathcal{B}(U_{8,4}) : |B \cap \{1, 2\}| \geq 1, |B \cap \{3, 4\}| \leq 1\} \),
- \( \mathcal{B}(M_3) = \{B \in \mathcal{B}(U_{8,4}) : |B \cap \{1, 2\}| \geq 1, |B \cap \{3, 4\}| \geq 1, |B \cap \{5, 6\}| \leq 1\} \),
- \( \mathcal{B}(M_4) = \{B \in \mathcal{B}(U_{8,4}) : |B \cap \{1, 2\}| \geq 1, |B \cap \{3, 4\}| \geq 1, |B \cap \{5, 6\}| \geq 1\} \).

2.2. **Relaxations.** Let \( M = (E, \mathcal{B}) \) be a matroid of rank \( r \) and let \( X \subset E \) be both a circuit and a hyperplane of \( M \) (recall that a *hyperplane* is a flat, that is \( X = c\ell(X) = \{e \in E : r(X \cup e) = r(X)\} \), of rank \( r - 1 \)). It is known [13 Proposition 1.5.13] that \( \mathcal{B}(M') = \mathcal{B}(M) \cup \{X\} \) is the collection of bases of a matroid \( M' \) (called, relaxation of \( M \)).

**Corollary 2.** Let \( M = (E, \mathcal{B}) \) be a matroid and let \( E = \bigcup_{i=1}^{t} E_i \) be a good \( t \)-partition. Then, \( \mathcal{P}(M') \) has a sequence of \( t \) hyperplane splits where \( M' \) is a relaxation of \( M \).
Proof. It can be checked that the desired sequence of \( t \) hyperplane splits of \( P(M') \) can be obtained by using the same given good \( t \) partition \( E = \bigcup_{i=1}^{t} E_i \).

We notice that the above result is not the only way to define a sequence of hyperplane splits for relaxations. Indeed it is proved in [3] that binary matroids (and thus graphic matroids) do not have hyperplane splits, however there is a sequence of hyperplane splits for relaxations of graphic matroids as it is shown in Example 3 below.

3. Rank-three matroids: geometric point of view

We recall that a matroid of rank three on \( n \) elements can be represented geometrically by placing \( n \) points on the plane such that if three elements form a circuit, then the corresponding points are collinear (in such diagram the lines need not be straight). Then the bases of \( M \) are all subsets of points of cardinal 3 which are not collinear in this diagram. Conversely, any diagram of points and lines in the plane in which a pair of lines meet in at most one point represents a unique matroid whose bases are those 3-subsets of points which are not collinear in this diagram.

The combinatorial conditions \((P1)\) and \((P2)\) can be translated into geometric conditions when \( M \) is of rank three. The latter is given by the following two corollaries.

**Corollary 3.** Let \( M \) be a matroid of rank 3 on \( E \) and let \( E = E_1 \cup E_2 \) be a partition of the points of the geometric representation of \( M \) such that

1) \( r(M|E_1) \geq 2 \) and \( r(M|E_2) = 3 \);
2) for each line \( l \) of \( M \), if \( |l \cap E_1| \neq 0 \), then \( |l \cap E_2| \leq 1 \).

Then, \( E = E_1 \cup E_2 \) is a 2-good partition.

**Proof.** \((P2)(a)\) can be easily checked with \( a_1 = 1 \) and \( a_2 = 2 \).

**Example 2.** Let \( M \) be the rank-3 matroid arising from the configuration of points given in Figure 1. It can be easily checked that \( E_1 = \{1, 2\} \) and \( E_2 = \{3, 4, 5, 6\} \) verify the conditions of Corollary 3. Thus, \( E_1 \cup E_2 \) is a 2-good partition.

**Corollary 4.** Let \( M \) be a matroid of rank 3 on \( E \) and let \( E = E_1 \cup E_2 \cup E_3 \) be a partition of the points of the geometric representation of \( M \) such that

1) \( r(M|E_i) \geq 2 \) for each \( i = 1, 2, 3 \),
2) for each line \( l \) with at least 3 points of \( M \),
   a) if \( |l \cap E_1| \neq 0 \) then \( |l \cap (E_2 \cup E_3)| \leq 1 \),
   b) if \( |l \cap E_3| \neq 0 \) then \( |l \cap (E_1 \cup E_2)| \leq 1 \).

Then, \( E = E_1 \cup E_2 \cup E_3 \) is a 3-good partition.

**Proof.** \((P2)\) can be easily checked with \( a_1 = a_2 = a_3 = 1 \).
Example 3. Let $W^3$ be the 3-whirl on $E = \{1, \ldots, 6\}$ shown in Figure 2. $W^3$ is the example given by Billera et al. [2] that we mentioned by the end of the introduction. $W^3$ is a relaxation of $M(K_4)$ (by relaxing circuit $\{2, 4, 6\}$) and it is not graphic.

It can be checked that $E_1 = \{1, 6\}$, $E_2 = \{2, 5\}$, and $E_3 = \{1, 4\}$ verify the conditions of Corollary 4. Thus, $E_1 \cup E_2 \cup E_3$ is a good 3-partition.

We finally notice that given the 2-good partition $E_1 \cup E_2$ of the matroid $M$ in Example 2, we can apply a hyperplane split to the matroid $M|_{E_2}$ induced by the set of points in $E_2 = \{3, 4, 5, 6\}$. Indeed, it can be checked that $E_2^1 = \{3, 4\}$ and $E_2^2 = \{5, 6\}$ verify conditions in Corollary 4 and thus it is a good 2-partition of $M|_{E_2}$. Moreover, it can be checked that $E_1 = \{1, 2\}$, $E_2^1 = \{3, 4\}$, and $E_2^2 = \{5, 6\}$ verify the conditions of Corollary 4 and thus $E_1 \cup E_2^1 \cup E_2^2$ is a good 3-partition for $M$. 
4. Direct sum

Let \( M_1 = (E_1, \mathcal{B}) \) and \( M_2 = (E_2, \mathcal{B}) \) be matroids of rank \( r_1 \) and \( r_2 \) respectively where \( E_1 \cap E_2 = \emptyset \). The direct sum, denoted by \( M_1 \oplus M_2 \), of matroids \( M_1 \) and \( M_2 \) has as ground set the disjoint union \( E(M_1) \oplus E(M_2) = E(M_1) \cup E(M_2) \) and as set of bases \( \mathcal{B}(M_1 \oplus M_2) = \{ B_1 \cup B_2 | B_1 \in \mathcal{B}(M_1), B_2 \in \mathcal{B}(M_2) \} \). Further, the rank of \( M_1 \oplus M_2 \) is \( r_1 + r_2 \).

In \cite{3}, we proved the following result.

**Theorem 3.** \cite{3} Let \( M_1 = (E_1, \mathcal{B}) \) and \( M_2 = (E_2, \mathcal{B}) \) be matroids of rank \( r_1 \) and \( r_2 \) respectively where \( E_1 \cap E_2 = \emptyset \). Then, \( P(M_1 \oplus M_2) \) has a hyperplane split if and only if either \( P(M_1) \) or \( P(M_2) \) has a hyperplane split.

Our main result in this section is the following.

**Theorem 4.** Let \( M_1 = (E_1, \mathcal{B}) \) and \( M_2 = (E_2, \mathcal{B}) \) be matroids of rank \( r_1 \) and \( r_2 \) respectively where \( E_1 \cap E_2 = \emptyset \). Then, \( P(M_1 \oplus M_2) \) admits a sequence of hyperplane splits if either \( P(M_1) \) or \( P(M_2) \) admits a sequence of hyperplane splits.

**Proof.** Without loss of generality, we suppose that \( P(M_1) \) has a sequence of hyperplane splits yielding to the decomposition \( P(M_1) = \bigcup_{i=1}^{t} P(N_i) \). For each \( i = 1, \ldots, t \), we let

\[
L_i = \{ X \cup Y : X \in \mathcal{B}(N_i), Y \in \mathcal{B}(M_2) \}.
\]

Since \( N_i \) and \( M_2 \) are matroids, \( L_i \) is also the matroid given by \( N_i \oplus M_2 \).

Now for all \( 1 \leq i, j \leq t, \ i \neq j \) we have

\[
L_i \cap L_j = \{ X \cup Y : X \in \mathcal{B}(N_i) \cap \mathcal{B}(N_j), Y \in \mathcal{B}(M_2) \}
\]

Since \( \mathcal{B}(N_i) \cap \mathcal{B}(N_j) = \mathcal{B}(N_i \cap N_j) \) and \( M_2 \) are matroids, \( L_i \cap L_j \) is also a matroid given by \( (N_i \cap N_j) \oplus M_2 \). Moreover, \( P(M_1) = \bigcup_{i=1}^{t} P(N_i) \) so \( \mathcal{B}(M_1) = \bigcup_{i=1}^{t} \mathcal{B}(N_i) \) and thus

\[
\bigcup_{i=1}^{t} L_i = \{ X \cup Y : X \in \bigcup_{i=1}^{t} \mathcal{B}(N_i), Y \in \mathcal{B}(M_2) \}
\]

\[
= \{ X \cup Y : X \in \mathcal{B}(M_1), Y \in \mathcal{B}(M_2) \}
\]

\[
= \mathcal{B}(M_1 \oplus M_2).
\]

We now show that this matroid base decomposition induces a \( t \)-decomposition of \( P(M_1 \oplus M_2) \). Indeed, we claim that \( P(M_1 \oplus M_2) = \bigcup_{i=1}^{t} P(L_i) \). For, we proceed by induction on \( t \). The case \( t = 2 \) is true since, in the proof of Theorem \cite{3} was showed that \( P(M_1 \oplus M_2) = P(L_1) \cup P(L_2) \). We suppose that the result is true for \( t \) and let

\[
P(M_1) = \bigcup_{i=1}^{t-1} P(N_i) \cup P(N_i^1) \cup P(N_i^2),
\]

(8)
where \( N_i, i = 1, \ldots, t - 1, N_1^2, N_2^2 \) are matroids. Moreover, we suppose that throughout the sequence of hyperplane splits of \( P(M_1) \) we had \( P(M_i) = \bigcup_{i=1}^{t} P(N_i) \) and that the last hyperplane split was applied to \( P(N_t) \) (obtaining \( P(N_t) = P(N_1^t) \cup P(N_2^t) \)) and yielding to equation (8).

Now, by the inductive hypothesis, the decomposition \( P(M_1) = \bigcup_{i=1}^{t} P(N_i) \) implies the decomposition \( P(M_1 \oplus M_2) = \bigcup_{i=1}^{t} P(L_i) \). But, by the case \( t = 2 \), \( P(N_t) = P(N_1^t) \cup P(N_2^t) \) implying the decomposition \( P(N_t \oplus M_2) = P(L_1^t) \cup P(L_2^t) \) where

\[
L_1^t = \{ X \cup Y : X \in \mathcal{B}(N_1^t), Y \in \mathcal{B}(M_2) \}, \quad \text{and} \quad L_2^t = \{ X \cup Y : X \in \mathcal{B}(N_2^t), Y \in \mathcal{B}(M_2) \}
\]

Therefore,

\[
P(M_1 \oplus M_2) = \bigcup_{i=1}^{t} P(L_i) = \bigcup_{i=1}^{t-1} P(L_i) \cup P(L_1^t) \cup P(L_2^t).
\]

\[\square\]

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