COUNIVERSAL SPACES WHICH ARE EQUIVARIANTLY COMMUTATIVE RING SPECTRA

J.P.C. GREENLEES

Abstract. We identify which couniversal spaces have suspension spectra equivalent to commutative orthogonal ring $G$-spectra for a compact Lie group $G$. These are precisely those whose cofamily is closed under passage to finite index subgroups. Equivalently these are the couniversal spaces admitting an action of an $E_\infty^G$-operad.

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1. Introduction

For a compact Lie group $G$, Theorem 4.7 shows that a number of simple $G$-equivariant homotopy types have suspension spectra which are commutative orthogonal ring $G$-spectra. Because equivariant commutativity implies a large amount of additional structure, including norm maps, this has significant implications.

These homotopy types are naturally used for isotropic decompositions of the sphere, and as such they play a significant role in understanding the structure of $G$-equivariant spectra where $G$ is a torus in [3]. That analysis involves constructing the model category of rational $G$-spectra for a torus $G$ from a diagram of much simpler model categories. The simplest way to do this is to construct the simpler model categories as categories of modules over commutative ring $G$-spectra, and for this diagram to arise from a diagram of commutative ring $G$-spectra.

The homotopy types of the ring $G$-spectra are apparent from the construction, and it remains to show that they are indeed commutative ring $G$-spectra. If the ambient category of $G$-spectra is the category of orthogonal spectra, the commutative monoids admit multiplicative norm maps, which is a substantial restriction on the homotopy type. Accordingly, [3] works instead with the Blumberg-Hill category of orthogonal $L$-spectra [1], where many more $G$-spectra admit the structure of commutative rings.

The motivating application of the present note is to show that in fact the ring spectra required in the construction of [3] can be represented by commutative rings in the category

[1] I am grateful to M.Hill and M.Kedziorek for the conversation at EuroTalbot17 when we observed that we knew of no obstruction to Corollary [4.8]
orthogonal $G$-spectra. It follows that the the argument of [3] can be conducted directly in the category of orthogonal $G$-spectra rather than in the more elaborate category of spectra with an $L$-action.

2. Operadic preliminaries

There are rare examples of spectra which are obviously strictly commutative rings, but it is much more usual to show that a spectrum admits the action of a suitable operad, and then use general results to show this means the homotopy type is represented by a ring spectrum.

2.A. $N_\infty$-operads. In the equivariant world there is a range of essentially different operads governing commutative ring spectra: these are the $N_\infty$-operads of Blumberg-Hill [2]. These are operads $O$ in $G$-spaces whose $n$-th term $O(n)$ is a universal space for a family $\mathcal{F}O(n)$ of subgroups of $G \times \Sigma_n$: it is essential that $O(n)$ is $G$-fixed and $\Sigma_n$-free, but within that class there is a wide range of options. We need only discuss the two extreme types of $N_\infty$-$G$-operads.

At one extreme we have the non-equivariant $E_\infty$-operads, which are as free as possible whilst being $G$-fixed. Equivalently, the $n$-th term is the universal space for the family

$$\mathcal{F}(n) = \{H \times 1 \subseteq G \times \Sigma_n \mid H \subseteq G\}.$$

There are of course many $E_\infty$-operads, and we write $E_\infty$ for a chosen one. For example we might use the linear isometries operad on a $G$-fixed universe, but we will use no special properties of the operad.

At the other extreme we have the $E_{G\infty}$-operads which are as fixed as possible whilst their $n$-th term is $\Sigma_n$-free, so their $n$-th term is a universal space for the family

$$\mathcal{F}_{G}(n) = \{\Gamma \mid \Gamma \cap \Sigma_n = 1\}.$$

There are of course many $E_{G\infty}$-operads, and we write $E_{G\infty}$ for a chosen one. For example we might use the linear isometries operad on a complete $G$-universe, but we will use no special properties of the operad. We pause to recall that if $\Gamma \cap \Sigma_n = 1$ then $\Gamma$ is a ‘graph subgroup’ in the sense that we have $\Gamma = \Gamma(L, \alpha)$ for some subgroup $L$ of $G$ and some homomorphism $\alpha : L \rightarrow \Sigma_n$, where $\Gamma(L, \alpha) = \{(x, \alpha(x)) \mid x \in L\}$.

2.B. Commutative monoids and $E_{G\infty}$-operads. The relevance of $E_{G\infty}$-operads is the connection to the standard symmetric monoidal product of spectra.

Lemma 2.1. The commutative monoids in the category of orthogonal $G$-spectra are the $E_{G\infty}$-algebras.

Proof: This uses the traditional argument of [6, 15.5], using [4, B.117], which in turn corrects [3, III.8.4]. We note that the statement in [4] is only given for finite groups, but the argument applies as written to arbitrary compact Lie groups, giving the full replacement for the statement in [3]. □
2.C. **Endomorphism operads.** The other piece of standard material is to consider the endomorphism operad $\mathcal{E}_Y$ on a based space $Y$, defined by

$$\mathcal{E}_Y(n) = \text{Map}_*(Y^\wedge n, Y).$$

We automatically find $Y$ is an $\mathcal{E}_Y$-algebra. Equally, if $Y$ is a based $G$-space $\mathcal{E}_Y$ is an operad in $G$-spaces and $Y$ is an algebra over it.

3. McClure’s argument

McClure [7] argued as follows to construct an $E_\infty$-operad acting on $\tilde{E}G$.

First we consider the endomorphism operad $E_{\tilde{E}G}$, and then note that passage to fixed points gives a map

$$\phi(n) : \mathcal{E}_{\tilde{E}G}(n)^G = \text{Map}_*^G(\tilde{E}G^\wedge n, \tilde{E}G) \longrightarrow \text{Map}_*(S^0, S^0).$$

We write

$$D_{M_C}(n) = \phi(n)^{-1}(id),$$

and note that this is also an operad acting on $\tilde{E}G$. Because $\phi(n)$ is a weak equivalence $D_{M_C}(n)$ is contractible, so that $E_\infty \times D_{M_C}$ is an $E_\infty$-operad acting on $\tilde{E}G$ as required.

4. Generalizing McClure’s argument

4.A. **Couniversal spaces.** Given a group $G$ and a family $\mathcal{F}$ of subgroups of $G$, we say that $\tilde{E}\mathcal{F} = S^0 \ast E\mathcal{F}$ is the **couniversal space** for the complementary cofamily $\text{All} \setminus \mathcal{F}$. Simplifying notation, for a cofamily $\mathcal{C}$, we write simply

$$EC = \tilde{E}(C^c).$$

This has two essential features: it has geometric isotropy $C$, and $(EC)^H = S^0$ whenever $H \in \mathcal{C}$.

4.B. **The endomorphism operad of a cofamily.** We consider the endomorphism operad of $EC$:

$$\mathcal{E}_{EC}(n) = \text{Map}_*(EC^\wedge n, EC).$$

The following partial information about the homotopy type of this space will be useful later.

**Lemma 4.1.** Given cofamilies $\mathcal{C}$ and $\mathcal{D}$ the space

$$\text{map}_*(EC, ED)$$

has the following properties

- It is $H$-contractible if $H \not\in \mathcal{C} \cap \mathcal{D}$
- It is $H$-couniversal if no subgroup of $H$ lies in $\mathcal{D} \setminus \mathcal{C}$

**Proof:** It is clear that if $H$ is not in $\mathcal{C} \cap \mathcal{D}$ then $\text{map}_*(EC, ED)$ is $H$-contractible, since one or other of the spaces is.

If $H \in \mathcal{C} \cap \mathcal{D}$ we wish to argue that the map

$$\text{map}_*(EC, ED)^H \longrightarrow \text{map}_*(S^0, S^0) = S^0$$

is an equivalence. In other words, that any $H$-map $f : EC \longrightarrow ED$ is determined by the map from $S^0 \longrightarrow ED$. The obstruction to extension and uniqueness lie in $[EC_+^c \wedge S^k, ED]^H$, which vanishes unless $H$ has a subgroup $K \in \mathcal{D} \setminus \mathcal{C}$.

□
4.C. **The couniversal operad of a cofamily.** There is a $G \times \Sigma_n$-map $i_n : S^0 = (S^0)^\wedge n \to (EC)^\wedge n$ inducing a $G \times \Sigma_n$-map

$$i_n^* : \mathcal{E}_{EC}(n) = Map_\ast(EC^\wedge n, EC) \to Map_\ast(S^0, EC) = EC.$$ 

We take

$$DC(n) = (i_n^*)^{-1}(1).$$ 

We note that when $C = \mathcal{N}T$ consists of the non-trivial subgroups the fixed point set $\mathcal{D}\mathcal{N}T^G = D_{\mathcal{M}C}$ is McClure’s operad.

**Lemma 4.2.** $DC$ is an operad acting on $EC$. □

Using this, we will show that for suitable cofamilies $C$, the space $EC$ is an algebra over an $N_\infty$-operad with more highly structured algebras than $E_\infty$.

4.D. **Permutation powers and cofamilies.** Let us think of the symmetric group $\Sigma_n$ as the permutations of $\{1, 2, \ldots, n\}$. We consider the group $G \times \Sigma_n$ and let $p : G \times \Sigma_n \to \Sigma_n$ and $\pi : G \times \Sigma_n \to G$ be the projections.

If $C$ is a cofamily of subgroups of $G$, we view $EC$ as a trivial $\Sigma_{n-1}$-space and form the $n$th smash power $(EC)^\wedge n$ and view it as a $G \times \Sigma_n$-space.

**Lemma 4.3.** The $G \times \Sigma_n$-space $EC^\wedge n$ is couniversal.

**Proof:** Consider any $G$-space $X$ and form the $G \times \Sigma_n$-space $X^\wedge n$. We will consider fixed points under a subgroup $\Delta \subseteq G \times \Sigma_n$.

Consider the orbits $o_1, \ldots, o_s$ of $\{1, \ldots, n\}$ under $p(\Delta)$, and choose orbit representatives $d_i \in o_i$. Now write $\Delta_i = p^{-1}((\Sigma_n)_{d_i}) \cap \Delta$ for the subgroup of $\Delta$ fixing $d_i$.

We then see that there is a homeomorphism

$$h : \bigwedge_{i=1}^s X^{\pi(\Delta_i)} \cong (X^\wedge n)^\Delta.$$ 

The $i$th factor in the domain gives the $d_i$th coordinate in $X^\wedge n$ and hence determines the coordinates in $o_i$. More precisely, if $m \in o_i$ we may choose $\delta \in \Delta$ with $p(\delta)(d_i) = m$, and then

$$h(x_1 \wedge \ldots \wedge x_s)_m = \pi(\delta)x_i.$$ 

Since $x_i$ is fixed by $\Delta_i$ this is independent of the choice of $\delta$. The verification that $h$ is a homeomorphism is straightforward.

Applying this to $X = EC$ we see that $X^\Delta$ is always either $S^0$ or contractible. The collection of subgroups for which it is $S^0$ is obviously a cofamily.

□

If we write $C(C, n)$ for the geometric isotropy of $EC^\wedge n$, then by the lemma $EC^\wedge n \simeq EC(C, n)$.

**Lemma 4.4.**

$$C(C, n) \subseteq \pi^*C$$
Proof: We show that if $\Delta$ is not in the right hand side it is not in the left hand side.

If $\pi(\Delta)$ does not lie in $\mathcal{C}$, then $(EC)^{\pi(\Delta)} \neq S^0$. Suppose then that $x = x(1) \in EC \setminus S^0$ is a non-trivial element of $\mathcal{C}$ fixed by $\pi(\Delta)$. Now write $x(i) = \sigma^i x(1)$ where $\sigma = (123 \cdots n)$. We then have $x = x(1) \wedge w(2) \wedge \cdots \wedge w(n)$ fixed by $\pi(\Delta) \times \Sigma_n$ and hence by its subgroup $\Delta$. Hence $\Delta \not\in C(\mathcal{C}, n)$.

\medskip

Lemma 4.5. If $\mathcal{C}$ is closed under passage to finite index subgroups then

$$\pi^* \mathcal{C} \cap \mathcal{F}_G(n) \subseteq C(\mathcal{C}, n)$$

Proof: Suppose $\Delta \subseteq G \times \Sigma_n$ lies in the intersection, which is to say $L := \pi(\Delta) \in \mathcal{C}$, and $\Delta = \Gamma(L, \alpha)$ is a graph subgroup. We will show that $\Delta \in C(\mathcal{C}, n)$. Since $C(\mathcal{C}, n)$ is a cofamily, it suffices to show that the subgroup $\Delta' = \Gamma(L_e, \alpha|_{L_e})$ lies in $C(\mathcal{C}, n)$, where $L_e$ is the identity component of $L$.

However, since $\Sigma_n$ is discrete $\alpha|_{L_e}$ is trivial, so that $\Delta' = \Gamma(L_e, \text{const}) = L_e$. However $L = \pi(\Delta)$ lies in $\mathcal{C}$, so its finite index subgroup $L_e$ also lies in $\mathcal{C}$ by hypothesis:

$$(EC^{\wedge n})^\Delta \subseteq (EC^{\wedge n})^{\Delta'} = (EC^{L_e})^{\wedge n} = (S^0)^{\wedge n} = S^0.$$

Hence $(EC^{\wedge n})^\Delta = S^0$ as required.

\medskip

Lemma 4.6. The map

$$i^*_n : \mathcal{E}_{EC}(n) = \text{map}_*(EC^n, EC) \longrightarrow \text{map}_*(S^0, EC) = EC$$

is an $\mathcal{F}_G(n)$-equivalence.

Proof: We observe that by Lemmas 4.4 and 4.5 if $H \in \mathcal{F}_G(n)$ then $C(\mathcal{C}, n)|_H = \pi_* \mathcal{C}|_H$. The result follows from Lemma 4.1.

\medskip

4.E. McClure's argument extended. We now apply the above to the operad $DC$ of Subsection 4.B.

Theorem 4.7. If $\mathcal{C}$ is a cofamily then the space $EC$ is an $E^G_\infty$-algebra if and only if $\mathcal{C}$ is closed under passage to finite index subgroups.

Proof: If there is a finite index inclusion $K \subseteq H$ of subgroups with $H \in \mathcal{C}$ and $K \not\in \mathcal{C}$, then the assumption that $EC$ is $E^G_\infty$ leads to a contradiction. Indeed $\pi^H_0(EC) = 0$ so that $1 = 0$ in that ring. On the other hand, by Segal-tom Dieck splitting, $\pi^H_0(EC) \neq 0$ so that $1 \neq 0$ in $\pi^H_0(EC)$. The existence of a norm map then gives a contradiction since norm$_H^H(1) = 1$.

Now suppose $\mathcal{C}$ is closed under passage to finite index subgroups. By Lemma 4.2 there is an action of $DC$ on $EC$, and hence also an action of $E^G_\infty \times DC$. It remains to show that the nth term in this operad is universal for $\mathcal{F}_G(n)$. In other words, we need to show that if $\Gamma \in \mathcal{F}_G(n)$ is a graph subgroup then $DC(n)^\Gamma \simeq *$.

Now by Lemma 4.6 the map

$$i^*_n : \mathcal{E}_{EC}(n) = \text{map}_*(EC^n, EC) \longrightarrow \text{map}_*(S^0, EC) = EC$$

is an $\mathcal{F}_G(n)$-equivalence, and hence $DC(n)$ is $\mathcal{F}_G(n)$-contractible as required.
Corollary 4.8. If $G$ is a torus and $K$ is a connected subgroup then $S^{\infty V(K)} = \bigcup_{V^K = 0} S^V$ is an $E^G_\infty$-algebra.

Proof: The space $S^{\infty V(K)}$ is couniversal for the cofamily $V(K) = \{H \mid H \supseteq K\}$ of subgroups containing $K$. Since $K$ is connected $V(K)$ is closed under passage to finite index subgroups. \qed

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School of Mathematics and Statistics, Hicks Building, Sheffield S3 7RH. UK.
E-mail address: j.greenlees@sheffield.ac.uk