An Exploration into why Output Regularization Mitigates Label Noise

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Abstract

Label noise presents a real challenge for supervised learning algorithms. Consequently, mitigating label noise has attracted immense research in recent years. Noise robust losses is one of the more promising approaches for dealing with label noise, as these methods only require changing the loss function and do not require changing the design of the classifier itself, which can be expensive in terms of development time. In this work we focus on losses that use output regularization (such as label smoothing and entropy). Although these losses perform well in practice, their ability to mitigate label noise lack mathematical rigor. In this work we aim at closing this gap by showing that losses, which incorporate an output regularization term, become symmetric as the regularization coefficient goes to infinity. We argue that the regularization coefficient can be seen as a hyper-parameter controlling the symmetricity, and thus, the noise robustness of the loss function.

1. Introduction

Tagging of unlabeled data is an expensive and time consuming effort. The desire to cut costs and speedup the tagging process often results in data that suffers from label noise. The ability to learn under label noise is thus one of the most common machine learning problems, when it comes to real life scenarios. As such, mitigation of label noise has attracted a vast amount of machine learning research [5] with a recent surge in the field of deep learning [1, 15]. In this work we only consider uniform (sometimes called symmetric) label noise.

The Robust Loss Approach: A branch of methods promote learning with robust loss functions [6, 16, 17, 18, 13, 12, 3, 8]. A loss function is said to be robust if the same test accuracy is achieved when a model is trained with noisy or clean data. The robust loss approach is appealing since it only requires replacing of one loss with another. Recent works show that this “simplicity” does not come on the expense of accuracy [7, 12]. Moreover, robust loss functions, which are based on symmetric loss functions have the advantage of being based on strong theoretical foundations [6, 16, 4].

Output Regularization and Confidence Penalty: Adding an output regularization is another approach that can be considered for label noise mitigation. Output regularization, when applied to the output of a softmax layer, can also be thought of as a confidence penalty. Intuitively, that prevents the training process from overfitting on falsely labeled data. For example, label smoothing, that can be seen as form of confidence penalty [14], was shown experimentally to mitigate label noise [11]. However, a full theoretical explanation of the merits of output regularization in the context of label noise is still missing.

The main contribution of this work is a novel theoretical explanation for the success of output regularization in the mitigation of label noise. This explanation is based on a novel discovery that in many cases the loss function becomes asymptotically symmetric, and thus robust, when the coefficient of the output regularization goes to infinity.

2. Notations and Preliminaries

Let $X \in \mathcal{X}$, $Y \in [C] := \{1, \ldots, C\}$ be jointly distributed random variables and let $\bar{Y}$ be a noisy version of $Y$ such that for some $\rho < \frac{1}{C-1}$.

$$P \left( \bar{Y} = i \mid Y, X \right) = \begin{cases} \frac{\rho}{1-\rho} & i \neq Y, \\ 1-\rho & i = Y. \end{cases}$$

Let $\mathcal{F} \subseteq [C]^\mathcal{X}$ be a family of measurable functions and define a set of random variable $Z = \{f(X) \mid f \in \mathcal{F}\}$. For a loss function $l : \mathbb{R}^C \times [C] \rightarrow \mathbb{R}$ and $Z \in \mathcal{Z}$ we define $L_l(Z) = E[l(Z, Y)], \bar{L}_l(Z) = E\left[l(\bar{Z}, \bar{Y})\right]$. 

We say that a loss function \( l \) is robust if
\[
\bar{Z}^* \in \arg \min_{Z \in \mathcal{Z}} \bar{L}_l (Z) \quad \implies \quad \exists Z^* \in \arg \min_{Z \in \mathcal{Z}} L_l (Z),
\]
\[
\text{pred} (\bar{Z}^*) = \text{pred} (Z^*),
\]
where \( \text{pred} (z) = \arg \max_{\ell \in [C]} (z_\ell) \). Note that
\[
\arg \min_{\bar{Z} \in \mathcal{Z}} \bar{L}_l (Z) \subseteq \arg \min_{Z \in \mathcal{Z}} L_l (Z)
\]
is sufficient for \( l \) to be robust.

A regularizer \( g \) is any function from \( \mathbb{R}^C \) to \( \mathbb{R} \) that has a single minima at \( 0 \in \mathbb{R}^C \). ¹For a regularizer \( g \) and \( Z \in \mathcal{Z} \) we use the notation: \( G_g (Z) = \mathbb{E} [g (Z)] \).

When \( \mathcal{F} = \{ f_\theta \}_{\theta \in \Theta} \) we use the following notations:
\[
L_l (\theta) = L_l (Z (\theta)) \quad \text{and} \quad G_g (\theta) = G_g (Z (\theta)), \quad \text{where} \quad Z (\theta) = f_\theta (X).
\]

Finally, for a random variable \( Z \) we define the \( L^2 \) norm
\[
||Z||_L^2 = \sqrt{\mathbb{E} [||Z||^2]}.
\]

3. Related Work

Symmetric Loss Functions: A sufficient property for a loss function \( l : \mathbb{R}^C \times [C] \rightarrow \mathbb{R} \) to be robust is that \( \sum_{\ell \in [C]} l (z, \ell) \) is constant (does not depend on \( z \)) [6]. A loss function that has this property is called a symmetric loss. For a softmax output \( p \), Ghosh et al. [6] proposed the Mean Absolute Error (MAE):
\[
l (p, y) = \| p - \text{onehot} (y) \|_1 = 2 (1 - p_y)
\]
as a symmetric loss function.

Asymptotically Symmetric Losses: Recent works showed an inefficiency of the MAE loss from a gradient decent perspective [18, 17]. To solve this issue Zhang and Sabuncu [18] proposed the Generalized Cross Entropy (GCE) loss function:
\[
l_q (p, y) = \frac{1 - p_y^q}{q}, \quad q \in (0, 1]
\]
as a sort of compromise between MAE and Cross Entropy (CE). Note that when \( q \to 0 \) GCE converges to CE, while it is proportional to MAE when \( q \to 1 \). Later, motivated by the idea of Reversed Cross Entropy (RCE), Wang et al. [17] suggested the Symmetric Cross Entropy (SCE). This loss can be practically written as
\[
l_\lambda (p, y) = - \log (p_y) + \lambda (1 - p_y).
\]
Again, when \( \lambda = 0 \) it is just equal to the standard CE, and when \( \lambda = 1 \) it is proportional to MAE.

¹We note that confidence penalty regularizers, like label smoothing and entropy are not compiled with this definition. At the end of this article we have a special treatment for these regularizers.

Mitigation of Label Noise with Regularization: Hu et al. [7] gave proven generalization guarantees for deep learning using two special regularization techniques. However, they assumed an additive noise and a linear model, where the connection to deep learning is achieved only through the notion of Neural Tangent Kernel (NTK) [9, 10, 2] and requires a very wide network. Lukasik et al. [11] showed empirically, that confidence penalty (in the form of label smoothing) can mitigate label noise. They further showed that label smoothing can be translated to parameter regularization. The question of how that leads to a label noise robustness was left open, however.

Asymptotical Robustness of Losses with an \( l_2 \) Regularized Parameter: Asymptotical symmetricity of convex binary losses, combined with \( l_2 \) regularization on the parameter was discussed in Van Rooyen et al. [16]. However, it seems that this concept has not yet been developed further to multi-class classification, and to output regularization.

4. The Multi-Category Unhinged loss

Let us define a new symmetric loss function, the Multi Category Unhinged (MUH) loss:
\[
l_{\text{MUH}} (z, y) = \frac{1}{C} \sum_{\ell \in [C]} z_{\ell} - y_{\ell} = - (z, \text{onehot}^* (y))
\]
where,
\[
\text{onehot}^* (y) = \text{onehot} (y) - \frac{1}{C}.
\]
This loss can be seen as a multi-categorial extension of the binary unhinged loss,
\[
l (z, y) = 1 - yz, \quad y \in \{-1, 1\}
\]
presented in Van Rooyen et al. [16], that in turn was inspired by the hinge loss (made famous by its use in SVM):
\[
l (z, y) = \max (1 - yz, 0) \quad y \in \{-1, 1\}.
\]

Our intent in proposing this new loss, is to later use it to derive conditions under which regularized losses are asymptotically robust to label noise.

Since the MUH loss is unbounded from bellow, it makes sense to add a regularization term to it. Intuitively, adding a regularization term to a symmetric loss should not harm robustness by much, as the regularization term does not depend on the labels at all. This is made clear by the following lemma which is true for any symmetric loss:

Lemma 1. Let \( l \) be a symmetric loss function and let \( g \) be a regularizer. Let
\[
\lambda = \frac{1}{a}, \quad \text{where} \quad a = 1 - \frac{p^C}{C - 1} > 0 \quad (1)
\]
Then
\[
\arg\min_{Z \in \mathbb{Z}} \bar{L}_t(Z) + G_g(Z) \subseteq \arg\min_{Z \in \mathbb{Z}} L_t(Z) + \lambda G_g(Z)
\]

The following theorem shows that in the special case where \( l = l^{\text{Muh}} \) and \( g \) is quadratic with a positive definite Hessian, the regularized loss \( l + g \) is robust under a wide family of models which includes neural networks:

**Theorem 1.** Assume that for all \( t > 0 \)
\[
Z \in \mathbb{Z} \iff tZ \in \mathbb{Z}.
\]

If \( g(z) = z^T A z \) for some positive definite matrix \( A \in \mathbb{R}^{C \times C} \), then \( l^{\text{Muh}} + g \) is robust.

**Proof.** Assume that \( Z^* \) minimizes:
\[
\bar{L}_t(Z) + \tilde{G}_g(Z) = E \left[ -Z^T \text{onehot}^* (\tilde{Y}) + Z^T A Z \right] \quad \text{s.t.} Z \in \mathbb{Z}
\]

Define \( \lambda \) as in eq 1, then by theorem 1 \( Z^* \) also minimizes:
\[
L_t(Z) + \lambda G_g(Z) = E \left[ -Z^T \text{onehot}^* (Y) + \lambda Z^T A Z \right] = E \left[ -\frac{1}{2\lambda} \left\| \sqrt{2\lambda} A^{1/2} Z - A^{-1/2} \text{onehot}^* (Y) \right\|^2 \right] + \text{const} \quad \text{s.t.} Z \in \mathbb{Z}
\]

From this and from our assumption that implies
\[
Z = \{ \lambda Z \mid Z \in \mathbb{Z} \},
\]

we have that \( \tilde{Z} = \lambda Z^* \) minimizes:
\[
E \left[ -\frac{1}{2\lambda} \left\| \sqrt{2\lambda} A^{1/2} Z - A^{-1/2} \text{onehot}^* (Y) \right\|^2 \right] = \frac{1}{\lambda} E \left[ -Z^T \text{onehot}^* (Y) + Z^T A Z \right] + \text{const} \quad \text{s.t.} Z \in \mathbb{Z}
\]

Thus, \( \tilde{Z} \) also minimizes \( L_t(Z) + G_g(Z) \), and since \( \text{pred}(Z) = \text{pred}(\tilde{Z}) \), we have what we need. \( \square \)

Using the last theorem we can now generalize a result by Manwani and Sastry [13] which showed that the binary square loss is robust for the linear classifiers family.

**Corollary 1.** If \( Z \) is such that \( Z \in \mathbb{Z} \iff tZ \in \mathbb{Z} \) for all \( t > 0 \), then the square loss \( l(z, y) = \| z - y \|^2 \) is robust to uniform label noise.

**Proof.** Just substitute \( A = 1/2 I \) and use the fact that
\[
\frac{1}{2} \| z - \text{onehot}^* (y) \|^2 = l^{\text{Muh}}(z, y) + \frac{1}{2} \| z \|^2 + \text{const}.
\]

Note that we generalize the result of Manwani and Sastry [13] in both: the number of classes and the variety of models. In addition we note that our result gives a novel justification for using \( \text{onehot}^* (y) \) instead of \( \text{onehot} (y) \) when using the square loss for multi-category classification.

**5. Asymptotical Robustness**

In the previous section we established the robustness of the MUH loss with quadratic output regularization. In this section we use this fact to show that an important class of regularized loss functions become asymptotically robust when the regularization coefficient goes to infinity. This class is comprised of loss functions of the form \( l_\lambda(z, y) = l(z, y) + \lambda g(z) \), where \( \nabla_z l(0, y) = \nabla_z l^{\text{MUH}}(0, y) \) and \( g \) is twice differentiable.

If \( Z^*_\lambda \) minimizes
\[
L_{l_\lambda}(Z) = L_t(Z) + \lambda G_g(Z)
\]

Then it also minimizes
\[
L_t(Z) \quad \text{s.t.} G_g(Z) \leq a_\lambda := G(Z^*_\lambda)
\]

Thus, intuitively we expect that when \( \lambda \to \infty \) (and \( a_\lambda \to 0 \)) \( Z \) will converge towards zero, where by Taylor approximation
\[
l(z, y) \approx l^{\text{lin}}(z, y) := \nabla_z l(0, y) z = l^{\text{MUH}}(z, y) \quad \text{(2)}
\]

and \( g(z) \approx g^{\text{q}}(z) := z^T \nabla^2 g(0) z \).

Using Theorem 1 we can now conclude that \( l_\lambda \) is robust when \( \lambda \to \infty \).

We will now give a formal definition of asymptotical robustness. It will be convenient to work with losses of the form \( l_\alpha = \alpha l + g \) and let \( \alpha \to 0 \). The asymptotical results are equivalent to those achieved with \( l_\lambda = l + \lambda g \) and \( \lambda \to \infty \). We will also assume from now that \( \mathcal{F} = \{ f_\theta \mid \theta \in \mathbb{R}^m \} \).

When \( \mathcal{L}_{l_\alpha}(\theta) \) has a single minimizer (for example, when \( \mathcal{F} \) is the family of linear classifiers and \( l_\alpha \) is strongly convex) the following definition is useful:

**Definition 1.** Let \( l_\alpha \) be a loss function such that \( \mathcal{L}_{l_\alpha}(\theta) \) has a unique minimizer \( \theta_\alpha \). We say that \( l_\alpha \) is asymptotically robust if there exists a robust loss function \( l_\alpha \) such that \( \mathcal{L}_{l_\alpha}(\theta) \) has a unique minimizer \( \theta_\alpha \) and
\[
\frac{\left\| Z(\theta_\alpha) \right\|_{L^2} - \left\| Z(\tilde{\theta}_\alpha) \right\|_{L^2}}{\left\| Z(\tilde{\theta}_\alpha) \right\|_{L^2}} \xrightarrow[\alpha \to 0 \text{ } a_\lambda = l + \lambda g]{} 0, \quad \text{(3)}
\]

In the general case, where \( \mathcal{L}_{l_\alpha} \) doesn’t have a unique minimizer we need to use a weaker definition:
Definition 2. Let \( l_\alpha \) be a loss function. We say that \( l_\alpha \) is asymptotically locally robust at \( \theta_0 \) if there exists a robust loss function \( l_\beta \) and there exists \( 0 < \beta_\alpha \to 0, \theta_\alpha \to \theta_0 \) a local minimizer of \( L_{l_\beta} \), such that for any \( 0 < \alpha_\alpha \to 0 \) and \( \theta_\alpha \to \theta_0 \) a minimizer of \( L_{l_\alpha} \), it holds that:

\[
\frac{\|Z(\theta_\alpha)\|_{L^2}}{\|Z(\theta_0)\|_{L^2}} \to 0 \quad (4)
\]

We are now ready to present our main theorem:

Theorem 2. Let \( l \) be a twice continuously differentiable loss function such that \( \nabla l(0, y) = \nabla l_{\text{MUH}}(0, y) \) and let \( g \) be a twice continuously differentiable regularizer. Let \( \theta_0 \in \mathbb{R}^m \) be such that \( Z(\theta_0) = 0 \). Assume that \( Z(\theta) \) is almost surely twice continuously differentiable on the closure of an open neighborhood \( \Omega \) of \( \theta_0 \). Also assume that \( \text{E}[\nabla^T Z(\theta) \nabla Z(\theta)] \in \mathbb{R}^{m \times m} \) is positive definite on \( \Omega \). If in addition \( \nabla Z(\theta_0) \left[ \nabla^2 G(\theta_0) \right]^{-1} \nabla L(\theta_0) \neq 0 \), it holds that \( l_\alpha = \alpha l + g \) is asymptotically locally robust at \( \theta_0 \).

Equipped with Theorem 2, it is now easy to prove a (global) asymptotical robustness when \( \mathcal{F} \) is the family of linear classifiers and \( \nabla Z(0, y) = \nabla Z_{\text{MUH}}(0, y) \):

Theorem 3. Let \( l \) be a twice continuously differentiable convex loss function such that \( \nabla l(0, y) = \nabla l_{\text{MUH}}(0, y) \) and let \( g \) be a twice continuously differentiable regularizer. Assume that \( \mathcal{F} = \{ x \mapsto (\theta_1^T \phi(x), \ldots, \theta_k^T \phi(x)) \mid \theta_i \in \mathbb{R}^{\mathbb{R}^p} \}_{i=1}^k \), for some \( \phi : \mathcal{X} \to \mathbb{R}^p \) such that \( \text{E}[\phi(X) \phi(X)^T] \) is positive definite, then \( l \) is asymptotically robust to uniform label noise.

6. Softmax Cross Entropy

CE is a very common loss function in modern machine learning. Naturally, we would like to know if adding regularization to CE makes it asymptotically robust. Usually CE is preceded by a softmax layer. We can thus consider them together as one loss function:

\[
l(z, y) = -z_g + \log \left( \sum_{i \in [C]} e^{z_i} \right).
\]

In this case we have that

\[
\nabla l(0, y) = -\text{onehot}^*(y) = \nabla Z_{\text{MUH}}(0, y),
\]

which is exactly what we need for asymptotical robustness.

If we only consider the pure CE (without softmax), \( l(p, y) = -\log(p_y) \), we need to take into account that its domain is the simplex \( \{ p \succ 0 \in \mathbb{R}^C \mid \|p\|_1 = 1 \} \). Currently, our theory only supports losses which have the full \( \mathbb{R}^C \) as a domain. In addition, in this case, common output regularizers pushes the \( p \) towards \( (1/c, \ldots, 1/c)^T \) and not towards 0. However, we can still check if the first order Taylor approximation is symmetric near \( p = (1/c, \ldots, 1/c)^T \) which hints at the existence of possible asymptotical robustness, and indeed:

\[
\sum_{i \in [C]} l_{\text{lin}}(p, i) = \sum_{i \in [C]} \nabla l \left( \left( \frac{1}{C}, \ldots, \frac{1}{C} \right)^T, i \right) p = -C p_i - C
\]

Handling Confidence Penalty Regularizers: When a regularizer has the form of \( g(z) = h(\text{softmax}(z)) \) it can be considered as a confidence penalty. Common confidence penalties are, for example, entropy: \( h(p) = \sum_{i \in [C]} \log(p_i) \) and label-smoothing: \( h(p) = \sum_{i \in [C]} \log(p_i) \) [14]. These output regularizers do not have a unique minimum at 0, but rather they are minimized on the line

\[
\mathcal{A} = \{ z \in \mathbb{R}^C \mid z_1 = \cdots = z_C \}.
\]

Thus, even in the linear case, there is no unique solution \( Z_\alpha \to 0 \) to the problem

\[
\min_{Z \in \mathcal{A}} L_{l+\alpha g}(Z).
\]

Even if \( l \) is convex.

Roughly speaking, a possible way to overcome this difficulty, is to apply the whole theory of Section 5 on an alternative output space \( \{ z \in \mathbb{R}^C \mid z_C = 0 \} \) which is generated by subtracting the last coordinate of each output from the rest of its coordinates. This operation does not change the result of the softmax function and the new output space is isomorphic to \( \mathbb{R}^{C-1} \) which allows us to apply the theory.

7. Conclusion

In this work we proposed a new concept of asymptotical robustness to uniform label noise. We showed that asymptotical robustness exists, and suggested the idea that this is what stands behind the success of output regularization methods to mitigate label noise. The cornerstone of our theory is the robustness of the MUH loss with a quadratic output regularizer. As a byproduct of this robustness we also proved that the square loss, with a modified onehot function, is robust to uniform label noise under a classifiers family that can be represented by a neural network. At the end we showed that the softmax-CE loss can be asymptotically robust if is equipped with a confidence penalty regularizer.

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A. Proofs

Lemma 2. Let \( l \) be a loss function. Assume that \( Y \) is corrupted with uniform label noise of level \( p \) then for any \( Z \in \mathcal{Z} \)
\[
\bar{L}_l(Z) = \frac{\rho K}{C - 1} + \left(1 - \frac{\rho C}{C - 1}\right) L_l(Z)
\]

Proof. This Lemma is proved in Ghosh et al. \cite{6} as a part of their main theorem.

Lemma 3. Let \( h, g : \Theta \subseteq \mathbb{R}^m \rightarrow \mathbb{R} \). Assume that \( \theta_0 \) is a minimizer of \( g \) and that \( g \) is strictly convex and twice continuously differentiable at \( \theta_0 \). Assume that \( h \) is twice continuously differentiable at \( \theta_0 \) such that \( \nabla h(\theta_0) \neq 0 \). Let \( \alpha_n \rightarrow 0 \), and assume that \( \theta_n \in \arg\min \alpha_n h + g \) such that \( f \) and \( g \) are differentiable at \( \theta_n \), then
\[
\frac{\delta_n - \theta_0}{\|\delta_n - \theta_0\|} \rightarrow \frac{v}{\|v\|}
\]
where \( v := -\left[\nabla^2 g(\theta_0)\right]^{-1} \nabla h(\theta_0) \).

Proof. Let \( \delta_n = \theta_n - \theta_0 \). Since \( h \) and \( g \) are differentiable at \( \theta_n \):
\[
\alpha_n \nabla h(\theta_n) = -\nabla g(\theta_n)
\]
From the continues second differentiability of \( h \) and \( g \) at \( \theta_0 \) and form the fact that \( \nabla g(\theta_0) = 0 \) we have:
\[
\alpha_n \left[\nabla h(\theta_n) + \nabla^2 h(\theta_n) \delta_n + o(\delta_n)\right]
\]
\[
= -\nabla^2 g(\theta_n) \delta_n + O(\|\delta_n\|) \delta_n
\]
and since \( g \) is strictly convex and \( h \) has a bounded Hessian on \( \Theta \) it holds that:
\[
\frac{\delta_n}{\alpha_n} \rightarrow v = -\left[\nabla^2 g(\theta_0)\right]^{-1} \nabla h(\theta_0).
\]
Now, since \( \frac{\delta_n}{\alpha_n} \rightarrow \|v\| \) and \( \frac{\|\delta_n\|}{\alpha_n} \frac{\delta_n}{\|\delta_n\|} = \frac{\delta_n}{\alpha_n} \rightarrow v \) it holds that:
\[
\frac{\delta_n}{\alpha_n} \rightarrow \frac{v}{\|v\|}.
\]

Lemma 4. Let \( l \) be a twice continuously differentiable loss function and let \( g \) be a twice continuously differentiable regularizer. Assume \( Z(\theta) = f_\theta(X), \theta \in \mathbb{R}^m \). For some \( \theta_0 \in \mathbb{R}^m \) assume that \( Z(\theta_0) = 0, Z(\theta) \) is almost surely twice continuously differentiable on an open neighborhood \( \Omega \) of \( \theta_0 \) and that \( E[\nabla Z^T(\theta_0) \nabla Z(\theta_0)] \in \mathbb{R}^{m \times m} \) is positive definite. Then, there is an open neighborhood \( \Omega_0 \subseteq \Omega \) of \( \theta_0 \) such that for small enough \( \alpha \geq 0 \) it holds that \( \alpha L_l(\theta) + G_g(\theta) \) is strictly convex on \( \Omega_0 \).

Proof. It holds that:
\[
\nabla^2 [\alpha L_l + G_g](\theta_0) = \alpha \nabla^2 L(\theta_0) + E \sum_{L \in \mathcal{C}} \frac{\partial g}{\partial z_i}(0) \nabla^2 Z_i
\]
\[
+ E \left[\nabla^T Z(\theta_0) \nabla^2 g(0) \nabla Z(\theta_0)\right]
\]
\[
= \alpha \nabla^2 L(\theta_0) + E \left[\nabla^T Z(\theta_0) \nabla^2 g(0) \nabla Z(\theta_0)\right]
\]
Now, since \( E \left[\nabla^T Z(\theta_0) \nabla Z(\theta_0)\right] \succ 0 \) and \( \nabla^2 g(0) \succ 0 \) it holds that:
\[
E \left[\nabla^T Z(\theta_0) \nabla^2 g(0) \nabla Z(\theta_0)\right] \succ 0,
\]
and thus there exists \( \alpha_0 > 0 \) such that:
\[
\nabla^2 [\alpha_0 L_l + G_g](\theta_0), \nabla^2 [-\alpha_0 L_l + G_g](\theta_0) \succ 0
\]
Thus, from the fact that \( \nabla^2 L_l \) and \( \nabla^2 G_g \) are continues on \( \Omega \) there is \( \Omega_0 \subseteq \Omega \), an open neighborhood of \( \theta_0 \) such that for all \( \theta \in \Omega_0 \) it holds that:
\[
\nabla^2 [\alpha_0 L_l + G_g](\theta), \nabla^2 [-\alpha_0 L_l + G_g](\theta) \succ 0
\]
and thus for all \( \alpha > 0 \) it holds that:
\[
\nabla^2 [\alpha L_l + G_g](\theta) \succ 0,
\]
which is what we need.

Proof of Lemma 1

Proof. Let \( Z^* \in \arg\min_{Z \in \mathcal{Z}} \bar{L}_l(Z) + G_g(Z) \) and define \( \alpha = \left(1 - \rho \frac{C}{C - 1}\right) > 0 \). From Lemma 2 it holds that for any \( Z \in \mathcal{Z} \)
\[
[\bar{L}(Z^*) + G_g(Z^*)] - [\bar{L}(Z) + G_g(Z)]
\]
\[
= [\alpha L_l(Z^*) + G_g(Z^*)] - [\alpha L_l(Z) + G_g(Z)].
\]
From the optimality of \( f^* \) it holds that the left hand side is smaller then 0 and after dividing by \( \alpha > 0 \) we have what we need.

Proof of Theorem 2

Proof. Let \( 0 < \alpha_n \rightarrow 0 \) and \( \Omega \ni R_n \rightarrow 0 \) and that \( R_n \) minimizes \( \mathcal{L}_{\alpha_l + g} = \alpha L_l + G_g \). From the fact that \( g \) is strictly convex and the assumption that \( E \left[\nabla^T Z(\theta_0) \nabla^2 Z(\theta_0)\right] \succ 0 \) implies that \( \nabla L_l(\theta_0) \neq 0 \), and thus by Lemma 3, it holds for \( \alpha_n \geq 0 \) that:
\[
\frac{\delta_n}{\alpha_n} \rightarrow \delta := -\left[\nabla^2 G_g(\theta_0)\right]^{-1} \nabla L_l(\theta_0) \neq 0.
\]
By that and by our assumption that
\[
\nabla Z(\theta_0) \delta = \nabla Z(\theta_0) \left[\nabla^2 G_g(\theta_0)\right]^{-1} \nabla L_l(\theta_0) \neq 0.
\]
using a Taylor approximation of $Z_n$ we have that
\begin{equation}
\frac{Z_n}{\|Z_n\|_{L^2}} \xrightarrow{a.s.} \frac{\nabla Z (\theta_0) \delta_n + O (\|\delta_n\|) \|\delta_n\|}{\|\nabla Z (\theta_0) \frac{\delta_n}{\|\delta_n\|} + O (\|\delta_n\|)\|L^2\|}
\end{equation}

\begin{align}
\xrightarrow{a.s.} Z := \frac{\nabla Z (Z_0) \delta}{\|\nabla Z (Z_0) \delta\|_{L^2}}.
\end{align}

Let
\begin{equation}
\hat{l}_\beta = \beta l_{\text{lin}} + g^{sq},
\end{equation}
where
\begin{align}
l_{\text{lin}} (z, y) = \nabla l (0, y) z \text{ and } g^{sq} (z) = z^T \nabla^2 g (z) z.
\end{align}

By 1 (setting $A = 1/\beta z^T \nabla^2 g (z) z$) we have that $\hat{l}_\beta$ is robust to uniform label noise and it is enough to show that there exists $\beta_n \to 0$ and $\hat{\theta}_n \to \theta_0$ such that $\hat{\theta}_n$ is a local minimizer of $\mathcal{L}_{\hat{l}_\beta}$ and
\begin{align}
\mathbb{E} \left[ \phi (X) \phi (X)^T \right] > 0 \text{ it holds by Lemma 4 that for some } \beta_0 > 0 \text{ if } \beta < \beta_0 \mathcal{L}_{\hat{l}_\beta} = \beta \mathcal{L}_{l_{\text{lin}}} + \mathcal{G}_{g^{sq}} \text{ is strictly convex on } \Omega_0 \subseteq \Omega, \text{ a neighborhood of } \theta_0, \text{ and thus its minimizer converges to } \theta_0. \text{ We thus can choose some } \beta_n > 0, \beta_n \to 0 \text{ and letting } \hat{\theta}_n \text{ be the minimizer of } \mathcal{L}_{\hat{l}_\beta} \text{ on } \Omega_0 \text{ for big enough } n, \text{ when it is attained. By repeating the argument in the beginning of the proof and substituting } \mathcal{L}_{l_{\text{lin}}}, \mathcal{G}_{g^{sq}}, \beta_n \text{ and } \hat{\theta}_n \text{ in the places of } \mathcal{L}_l, \mathcal{G}_{g^\alpha}, \alpha_n \text{ and } \theta_n \text{ we get what we need.}
\end{align}

Proof of Theorem 3.

Proof. Let $\hat{\ell}_\alpha = \alpha l_{\text{lin}} + g^{sq}$ be as defined in eq. 6. Since
\begin{align}
\mathbb{E} \left[ \phi (X) \phi (X)^T \right] \text{ is positive definite } l_{\alpha} \text{ are } \hat{\ell}_\alpha \text{ uniquely minimized at } \theta_\alpha \text{ and } \theta_\alpha . \text{ From the fact that the limit in eq. 3 holds what we need is a direct consequences of the proof of Theorem 2.}