Mechanical Systems in the Generalized Lie Algebroids Framework

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Abstract

Mechanical systems called by use, mechanical \((\rho, \eta)\)-systems, Lagrange mechanical \((\rho, \eta)\)-systems or Finsler mechanical \((\rho, \eta)\)-systems are presented. The canonical \((\rho, \eta)\)-semi(spray) associated to a mechanical \((\rho, \eta)\)-system is obtained. New and important results are obtained in the particular case of Lie algebroids. The Lagrange mechanical \((\rho, \eta)\)-systems are the spaces necessary to develop a new Lagrangian formalism. We obtain the \((\rho, \eta)\)-semispray associated to a regular Lagrangian \(L\) and external force \(F_e\) and we derive the equations of Euler-Lagrange type. So, a new solution for the Weinstein’s Problem in the general framework of generalized Lie algebroids is presented.

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1 Introduction

In general, if \(C\) is a category, then we denote \(|C|\) the class of objects and for any \(A, B \in |C|\), we denote \(\mathcal{C}(A, B)\) the set of morphisms of \(A\) source and \(B\) target and \(\text{Iso}_C(A, B)\) the set of \(C\)-isomorphisms of \(A\) source and \(B\) target. Let \(\text{LieAlg}, \text{Mod}, \text{Man} \) and \(\mathcal{B}^\gamma\) be the category of Lie algebras, modules, manifolds and vector bundles respectively.

We know that if
\[
(E, \pi, M) \in |\mathcal{B}^\gamma|,
\]
\[
\Gamma(E, \pi, M) = \{u \in \text{Man}(M, E) : u \circ \pi = \text{Id}_M\}
\]
and
\[
\mathcal{F}(M) = \text{Man}(M, \mathbb{R}),
\]
then \((\Gamma(E, \pi, M), +, \cdot)\) is a \(\mathcal{F}(M)\)-module.

If \((\varphi, \varphi_0) \in \mathcal{B}^\gamma((E, \pi, M), (E', \pi', M'))\) such that \(\varphi_0 \in \text{Iso}_{\text{Man}}(M, M')\), then, using the operation
\[
\mathcal{F}(M) \times \Gamma(E', \pi', M') \longrightarrow \Gamma(E', \pi', M') \quad (f, u') \mapsto f \circ \varphi_0^{-1} \cdot u'
\]
it results that \((\Gamma(E', \pi', M'), +, \cdot)\) is a \(\mathcal{F}(M)\)-module and we obtain the \(\text{Mod}\)-morphism
\[
\Gamma(E, \pi, M) \xrightarrow{\Gamma(\varphi, \varphi_0)} \Gamma(E', \pi', M')
\]
defined by
\[
\Gamma(\varphi, \varphi_0) u(y) = \varphi \left(u_{\varphi_0^{-1}(y)}\right) = (\varphi \circ u \circ \varphi_0^{-1})(y),
\]
for any \( y \in M' \).

If \( (F, \nu, M) \in |\mathcal{B}| \) so that there exists

\[
(\rho, \text{Id}_M) \in \mathcal{B} ((F, \nu, M), (TM, \tau_M, M))
\]

and an operation

\[
\Gamma (F, \nu, M) \times \Gamma (F, \nu, M) \xrightarrow{[\cdot, \cdot]} \Gamma (F, \nu, M)
\]

\[
(u, v) \mapsto [u, v]_F
\]

with the following properties:

**LA1.** the equality holds good

\[
[u, f \cdot v]_F = f [u, v]_F + \Gamma (\rho, \text{Id}_M) (u) f \cdot v,
\]

for all \( u, v \in \Gamma (F, \nu, M) \) and \( f \in \mathcal{F} (M) \).

**LA2.** the 4-tuple \( (\Gamma (F, \nu, M), +, \cdot, [\cdot, \cdot]_F) \) is a Lie \( \mathcal{F} (M) \)-algebra,

**LA3.** the \textbf{Mod}-morphism \( \Gamma (\rho, \text{Id}_M) \) is a \textbf{LieAlg}-morphism of

\[
(\Gamma (F, \nu, M), +, \cdot, [\cdot, \cdot]_F)
\]

source and

\[
(\Gamma (TN, \tau_N, M), +, \cdot, [\cdot]_{TM})
\]

target, then the triple

\[
((F, \nu, M), [\cdot, \cdot]_F, (\rho, \text{Id}_M))
\]

is an object of the category \( \textbf{LA} \) of Lie algebroids. The couple \( ([\cdot, \cdot]_F, (\rho, \text{Id}_M)) \) is called Lie algebroid structure.

Used earlier by many authors, the study of Lie algebroids were considerably improved by J. Pradines in [30], who noticed that the Lie algebroids are infinitesimal versions of Lie groupoids in a functorial manner.

A generalization of a Lie algebroid is the \textit{Lie bialgebroid} defined in [24] by K.C.H. Mackenzie and P. Xu.

A Lie algebroid \( (\Gamma (F, \nu, M), [\cdot, \cdot]_F, (\rho, \text{Id}_M)) \) is called \textit{Lie bialgebroid} if there exists a Lie algebroid structure \( ([\cdot]_F, (\hat{\rho}, \text{Id}_M)) \) for its dual \( (\hat{F}, \hat{\nu}, M) \) such that

\[
d_{\hat{F}} ([u, v]_F) = \left[[u, d_{\hat{F}} v]_F - [v, d_{\hat{F}} u]_F, \forall u, v \in \Gamma (F, \nu, M)\right.
\]

where \( d_{\hat{F}} \) is the exterior differentiation operator for the exterior differential algebra of the Lie algebroid \( \left((\hat{F}, \hat{\nu}, M), [\cdot]_{\hat{F}}, (\hat{\rho}, \text{Id}_M)\right) \).

The Courant algebroids defined in [23] by Z.J. Liu, A. Weinstein and P. Xu are new generalizations of Lie algebroids. This notion is the result of an effort to unify the Courant bracket and the Manin bracket. (see: [24]).
If \((F, \nu, M) \in |B^v|\) so that there exists
\[(\rho, I_dM) \in B^v ((F, \nu, M), (TM, \tau_M, M)),\]
a skewsymmetric and \(\mathbb{R}\)-linear bracket
\[
\Gamma (F, \nu, M) \times \Gamma (F, \nu, M) \xrightarrow{[\cdot, \cdot]_F} \Gamma (F, \nu, M)
\]
a nondegenerate symmetric and bilinear form
\[
\Gamma (F, \nu, M) \times \Gamma (F, \nu, M) \xrightarrow{\langle \cdot, \cdot \rangle_F} F (M)
\]
and we have the section application
\[
\mathcal{F} (M) \xrightarrow{\mathcal{S}} \Gamma (F, \nu, M)
\]
defined by
\[
\langle \mathcal{S} (f), u \rangle_F = \frac{1}{2} \cdot \Gamma (\rho, I_dM) u (f), \ \forall u \in \Gamma (F, \nu, M)
\]
such that the following properties are satisfied:

\(CA_1\). the equality holds good
\[
[u, f \cdot v]_F = f [u, v]_F + \Gamma (\rho, I_dM) u (f) \cdot v - \langle u, v \rangle_F \cdot \mathcal{S} (f),
\]
for all \(u, v \in \Gamma (F, \nu, M)\) and \(f \in \mathcal{F} (M)\).

\(CA_2\). \(\sum_{\text{cyclic}} [[u, v]_F, z]_F = \frac{1}{6} \cdot \mathcal{S} \left( \sum_{\text{cyclic}} \langle [u, v]_F, z \rangle_F \right),\)

\(CA_3\). \(\Gamma (\rho, I_dM) [u, v]_F = [\Gamma (\rho, I_dM) u, \Gamma (\rho, I_dM) v]_{TM}, \) for all \(u, v \in \Gamma (F, \nu, M)\),

\(CA_4\). \(\langle \mathcal{S} (f), \mathcal{S} (g) \rangle_F = 0, \) for all \(f, g \in \mathcal{F} (M)\),

\(CA_5\). \(\Gamma (\rho, I_dM) u (\langle v, z \rangle_F) = \langle [u, v]_F + \mathcal{S} ([u, v]_F), z \rangle_F + \langle u, [v, z]_F + \mathcal{S} ([v, z]_F) \rangle_F,\)
for all \(u, v, z \in \Gamma (F, \nu, M)\),

then the 5-tuple
\[\left((F, \nu, M), [\cdot, \cdot]_F, (\rho, I_dM), \langle \cdot, \cdot \rangle_F, \mathcal{S}\right)\] (2)
is a Courant algebroid. The 4-tuple \(((\cdot, \cdot)_F, (\rho, I_dM), \langle \cdot, \cdot \rangle_F, \mathcal{S})\) is called Courant algebroid structure.

Z.J. Liu, A. Weinstein and P. Xu \[23\] proved that any Lie bialgebroid is a Courant algebroid and some general new problems are proposed. Trying to give some possible proofs for these problems, a new class of generalized algebroids are discovered by Paul Popescu in \[35\].

If \((F, \nu, M) \in |B^v|\) so that there exists
\[(\rho, I_dM) \in B^v ((F, \nu, M), (TM, \tau_M, M)),\]
a skewsymmetric and $\mathbb{R}$-linear bracket

$$\Gamma (F,\nu,M) \times \Gamma (F,\nu,M) \xrightarrow{[\cdot,\cdot]_F} \Gamma (F,\nu,M)$$

and a submodul $(\mathcal{M},+\cdot)$ of $(\Gamma (F,\nu,M),+\cdot)$ such that $\Gamma (\rho,\text{Id}_M)u = 0$, for all $u \in \mathcal{M}$ and the following properties are satisfied:

1. \( GA_1 \): $[u, f \cdot v]_F - f \cdot [u,v]_F - \Gamma (\rho,\text{Id}_M)u(f) \cdot v \in \mathcal{M}$ and $[f \cdot u, v]_F - f \cdot [u,v]_F + \Gamma (\rho,\text{Id}_M)u(f) \cdot v \in \mathcal{M}$, for all $u,v \in \Gamma (F,\nu,M)$ and $f \in F(M)$.

2. \( GA_2 \): $\sum_{\text{cyc}}[[u,v]_F,z]_F \in \mathcal{M}$, for all $u,v,z \in \Gamma (F,\nu,M)$.

3. \( GA_3 \): $\Gamma (\rho,\text{Id}_M)[u,v]_F = [\Gamma (\rho,\text{Id}_M)u,\Gamma (\rho,\text{Id}_M)v]_{\mathcal{T}M}$, for all $u,v \in \Gamma (F,\nu,M)$.

4. \( GA_4 \): $[u,v]_F \in \mathcal{M}$, whenever $u$ or $v$ are in $\mathcal{M}$,

then the 4-tuple

$$(F,\nu,M),[\cdot,\cdot]_F, (\rho,\text{Id}_M), \mathcal{M}$$

(3)

is a generalized algebroid. The triple $(\mathcal{M},[\cdot,\cdot]_F, (\rho,\text{Id}_M), \mathcal{M})$ is called generalized algebroid structure.

We know that the secret of the Ehresmann connection is given by the diagrams

$$ \begin{array}{cccc} E & \xrightarrow{\rho} & (T\mathcal{M},[\cdot]_{\mathcal{T}M}) & \xrightarrow{\text{Id}_{\mathcal{T}M}} & (T\mathcal{M},[\cdot]_{\mathcal{T}M}) \\ \downarrow \pi & & \downarrow \tau_M & & \downarrow \tau_M \\ \mathcal{M} & \xrightarrow{\text{Id}_M} & \mathcal{M} & \xrightarrow{\text{Id}_M} & \mathcal{M} \end{array} $$

(4)

where $(E,\pi,\mathcal{M})$ is a fiber bundle and $((T\mathcal{M},\tau_M,\mathcal{M}),[\cdot]_{\mathcal{T}M},(\text{Id}_{\mathcal{T}M},\text{Id}_M))$ is the standard Lie algebroid.

First, there appeared the idea of changing the standard Lie algebroid with an arbitrary Lie algebroid as in the diagrams

$$ \begin{array}{cccc} E & \xrightarrow{\rho} & (F,\cdot) & \xrightarrow{\rho} & (T\mathcal{M},[\cdot]_{\mathcal{T}M}) \\ \downarrow \pi & & \downarrow \nu & & \downarrow \tau_M \\ \mathcal{M} & \xrightarrow{\text{Id}_M} & \mathcal{M} & \xrightarrow{\text{Id}_M} & \mathcal{M} \end{array} $$

(5)

Secondly, there appeared the idea of changing in the previous diagrams the identities morphisms with arbitrary $\text{Man}$-isomorphisms $h$ and $\eta$ as in the diagrams

$$ \begin{array}{cccc} E & \xrightarrow{\rho} & (F,\cdot) & \xrightarrow{\rho} & (T\mathcal{M},[\cdot]_{\mathcal{T}M}) \\ \downarrow \pi & & \downarrow \nu & & \downarrow \tau_M \\ \mathcal{M} & \xrightarrow{h} & \mathcal{N} & \xrightarrow{\eta} & \mathcal{M} & \xrightarrow{h} & \mathcal{N} \end{array} $$

(6)

where

$$(\rho,\eta) \in \mathcal{B}^Y((F,\nu,M),(T\mathcal{M},\tau_M,\mathcal{M}))$$
and
\[ \Gamma(F,\nu,N) \times \Gamma(F,\nu,N) \xrightarrow{[\cdot],F,h} \Gamma(F,\nu,N) \]
is an operation with the following properties:

GLA\(_1\). the equality holds good
\[ [u,f \cdot v]_{F,h} = f [u,v]_{F,h} + \Gamma(Th \circ \rho, h \circ \eta)(u) f \cdot v, \]
for all \( u, v \in \Gamma(F,\nu,N) \) and \( f \in F(N) \).

GLA\(_2\). the 4-tuple \( (\Gamma(F,\nu,N), +, \cdot, [,],_{F,h}) \) is a Lie \( F(N) \)-algebra,

GLA\(_3\). the \( \text{Mod} \)-morphism \( \Gamma(Th \circ \rho, h \circ \eta) \) is a \( \text{LieAlg} \)-morphism of
\[ \left( \Gamma(F,\nu,N), +, \cdot, [,],_{F,h} \right) \]
source and
\[ \left( \Gamma(TN,\tau,N), +, \cdot, [,],_{TN} \right) \]
target.

We say that the triple
\[ \left( (F,\nu,N), [,],_{F,h}, (\rho, \eta) \right) \]
is a generalized Lie algebroid. The couple \( ([\cdot],_{F,h}, (\rho, \eta)) \) is called generalized Lie algebroid structure.

So, we extend the notion of Lie algebroid from one base manifold to a pair of diffeomorphic base manifolds and we obtain the notion of generalized Lie algebroid via Ehresmann connections theory, independent of the previous generalizations of Lie algebroids appeared in literature.

In the following particular case, \( (\eta, h) = (Id_M, Id_M) \), we obtain the definition of Lie algebroid. (see also [1])

We can define the set of morphisms of
\[ \left( (F,\nu,N), [,],_{F,h}, (\rho, \eta) \right) \]
source and
\[ \left( (F',\nu',N'), [,],_{F',h'}, (\rho', \eta') \right) \]
target as being the set
\[ \{ (\varphi, \varphi_0) \in B^Y((F,\nu,N),(F',\nu',N')) \} \]
such that \( \varphi_0 \in Iso_{\text{Man}}(N,N') \) and the \( \text{Mod} \)-morphism \( \Gamma(\varphi, \varphi_0) \) is a \( \text{LieAlg} \)-morphism of
\[ \left( \Gamma(F,\nu,N), +, \cdot, [,],_{F,h} \right) \]
source and 
\( \left( \Gamma (F', \nu', N'), \tau', [\cdot]_{F', \nu'} \right) \)

target.

So, we can discuss about the category \textbf{GLA} of generalized Lie algebroids. We remark that \textbf{GLA} is a subcategory of the category \textbf{B}^v.

**Example 1** Let \( M, N \in |\text{Man}|, h \in \text{Iso}_{\text{Man}}(M, N) \) and \( \eta \in \text{Iso}_{\text{Man}}(N, M) \) be. Using the tangent \textbf{B}^v-morphism \((T \eta, \eta)\) and the operation

\[
\Gamma (TN, \tau_N, N) \times \Gamma (TN, \tau_N, N) \xrightarrow{[\cdot]_{TN,h}} \Gamma (TN, \tau_N, N)
\]

where

\[
[u, v]_{TN,h} = \Gamma \left( T (h \circ \eta)^{-1}, (h \circ \eta)^{-1} \right) \left( [\Gamma (T (h \circ \eta), h \circ \eta) u, \Gamma (T (h \circ \eta), h \circ \eta) v]_{TN} \right),
\]

for any \( u, v \in \Gamma (TN, \tau_N, N) \), we obtain that

\[
\left( (TN, \tau_N, N), (T \eta, \eta), [\cdot]_{TN,h} \right)
\]

is a generalized Lie algebroid. (see [1])

For any \textbf{Man}-isomorphisms \( \eta \) and \( h \), new and interesting generalized Lie algebroid structures for the tangent vector bundle \((TN, \tau_N, N)\) are obtained. For any base \( \{t_\alpha, \alpha \in \overline{1, m}\} \) of the module of sections \((\Gamma (TN, \tau_N, N), +, \cdot)\) we obtain the structure functions

\[
L^\gamma_{\alpha \beta} = \left( \theta^i_\alpha \frac{\partial \theta^i_\beta}{\partial x^j} - \theta^i_\beta \frac{\partial \theta^i_\alpha}{\partial x^j} \right) \tilde{\theta}^\gamma_j, \quad \alpha, \beta, \gamma \in \overline{1, m}
\]

where

\[
\theta^i_\alpha, \quad i, \alpha \in \overline{1, m}
\]

are real local functions so that

\[
\Gamma (T (h \circ \eta), h \circ \eta) t_\alpha = \theta^i_\alpha \frac{\partial}{\partial x^i}
\]

and

\[
\tilde{\theta}^\gamma_j, \quad i, \gamma \in \overline{1, m}
\]

are real local functions so that

\[
\Gamma \left( T (h \circ \eta)^{-1}, (h \circ \eta)^{-1} \right) \left( \frac{\partial}{\partial x^j} \right) = \tilde{\theta}^\gamma_j t_\gamma.
\]

In particular, using arbitrary isometries (symmetries, translations, rotations,...) for the Euclidean 3-dimensional space \( \Sigma \), and arbitrary basis for the module of sections we obtain a lot of generalized Lie algebroid structures for the tangent vector bundle \((T \Sigma, \tau_\Sigma, \Sigma)\).
The problem to develop a Lagrangian formalism directly on a Lie algebroid similar to Klein’s formalism for ordinary Lagrangian Mechanics \[19\] was proposed by Alan Weinstein in his work \[40\].

In that work, the author gave a theory of Lagrangian systems on Lie algebroids and obtained the Euler-Lagrange equations using the dual of a Lie algebroid and the Legendre transformation defined by a regular Lagrangian.

Paulette Liberman \[22\] showed later that such a formalism is not possible if one considers the tangent bundle of a Lie algebroid as space for developing the theory. Eduardo Martinez \[25\] gave a full description using the notion of prolongation of a Lie algebroid.

Recently, there has been a lot of activity around mechanical systems on Lie algebroids (due to its unifying view many different problems) and some extensions to more general structures \[7, 8, 21, 26, 33, 34, 38, 39\].

Recently, a unified approach of constraining implicit Lagrangian and Hamiltonian systems on Dirac algebroid was presented in \[17\]. A Dirac algebroid on a vector bundle \((E, \pi, M)\) has been viewed as a double vector bundle morphism

\[\varepsilon : T^*E \longrightarrow T^*E\]
covering the identity on \( (E, \pi, M) \). (see \[15, 16\])

The double vector bundles presented in \[20, 31, 32\] has been applied to geometric formalism of Analytical Mechanics, including nonholonomic constrains in \[13, 14\].

There are obtained \(\varepsilon\) as the composition of the canonical isomorphism of double vector bundles

\[R_\pi : T^*E \longrightarrow T^*E\]

and

\[\Pi_\varepsilon : T^*E \longrightarrow T^*E.\]

An application of this approach to Analytical Mechanics so that \((E, \pi, M)\) plays the role of kinematic configurations, is based on some ideas of Tulczyjew presented in \[36, 37\].

We know that the use of connections for the geometry of systems of second order differential equations has been proposed by M. Crampin \[10\] and C. Griffone \[18\].

Using the generalized Lie algebroids, a new class of (linear) connections in Ehresmann sense on the base of the Lie algebroid generalized tangent bundle is presented in Section 2. (see also \[1\]) Using this \((\rho, \eta)\)-connection theory, the adapted basis are presented.

Using the generalized Lie algebroids, a new class of (linear) connections in Ehresmann sense on the base of the Lie algebroid generalized tangent bundle is presented in Section 2. (see also \[1\]) Using this \((\rho, \eta)\)-connection theory, the adapted basis are presented.

The lift \(I \xrightarrow{c} E\) of a curve \(I \xrightarrow{c} M\) is presented and studied in the Section 3. Section 4 studies remarkable endomorphisms of the Lie algebroid sections of the Lie algebroid generalized tangent bundle and in Section 5 a new class of linear connections, called by use distinguished linear \((\rho, \eta)\)-connections, is presented.
We know that the geometry of Lagrange space is the geometry of its canonical semispray and the associated nonlinear connection. It has been developed by R. Miron and M. Anastasiei [28]. Using techniques that are specific to Lagrange geometry, R. Miron [27] introduces and investigates some geometric aspects of nonconservative mechanical systems by means of the corresponding semispray and nonlinear connection.

When the external force field depends on both position and velocity, the geometry of nonconservative mechanical systems was rigorously investigated by Klein [19] and Godbillon [12]. In paper [19] Klein introduces a second rank skew symmetric tensor as external force tensor of a nonconservative mechanical system.

Some aspects regarding first integrals for nonconservative mechanical systems were investigated by Djukic and Vicianovic [11] and Cantrijn [9]. I. Bucătaru and R. Miron [6] extend the geometric investigations of nonconservative mechanical systems using the evolution nonlinear connection and the almost symplectic structure of the nonconservative mechanical system.

Finding a new solution for the Weinstein’s Problem in the general framework of generalized Lie algebroids, a new class of mechanical systems called by use, mechanical \((\rho, \eta)\)-systems, Lagrange mechanical \((\rho, \eta)\)-systems or Finsler mechanical \((\rho, \eta)\)-systems is presented in Section 6. In Section 7 we study the canonical \((\rho, \eta)\)-semispray associated to a mechanical \((\rho, \eta)\)-system. Finally, an important result about the canonical \((\rho, \eta)\)-spray associated to a mechanical \((\rho, \eta)\)-system is presented. In particular, we obtain similar results with I. Bucătaru and R. Miron [5, 6].

The Section 8 of this paper is dedicated to study the geometry of Lagrange mechanical \((\rho, \eta)\)-systems. We determine and we study the \((\rho, \eta)\)-semispray associated to a regular Lagrangian \(L\) and external force \(F_e\) which are applied on the total space of a generalized Lie algebroid.

The equations of Euler-Lagrange type are derived. In particular, using the Lie algebroid generalized tangent bundle of a Lie algebroid, we obtain a new solution for the Weinstein’s Problem, different by the Martínez’s solution [25]. (see also [3, 4])

Finally, we obtain that the integral curves of the canonical \((\rho, \eta)\)-semispray associated to Lagrange mechanical \((\rho, \eta)\)-system \(((E, \pi, M), F_e, L)\) and from locally invertible \(B^*\)-morphism \((g, h)\) are the \((g, h)\)-lifts solutions of the equations of Euler-Lagrange type (107).

In particular, if \(F\) is a Finsler fundamental function, then the geodesics on the manifold \(M\) are the curves such that the components of their \((g, h)\)-lifts are solutions for equations of Euler-Lagrange type (107).

Notice that our theory of mechanical systems is a progress because, in particular, we obtain all previous mechanical systems presented in literature.

As there exists (see: [1]) a construction of the generalized tangent bundle of a dual vector bundle and a connection theory in the dual case, we ask:

- Can we develop an alternative approach for the dual case and can we find a transformation of Legendre type with whom to show the equivalence
between these Mechanical Systems?

The answer, in the next paper.

2 \((\rho, \eta)\)-connections and adapted basis

We consider the diagram:

\[
\begin{array}{ccc}
E & \xrightarrow{\pi} & \left(F, [\cdot, F]_{\rho, \eta}, (\rho, \eta)\right) \\
\downarrow \pi & \downarrow \nu & \downarrow \nu \\
M & \xrightarrow{h} & N
\end{array}
\]

(8)

where \((E, \pi, M) \in [B^v]\) and \(\left((F, \nu, N), [\cdot, F]_{\rho, \eta}, (\rho, \eta)\right) \in [GLA]\).

We obtain the \(B^v\)-morphism

\[
\begin{array}{ccc}
\pi^* (h^* F) & \hookrightarrow & F \\
\downarrow \pi^* (h^* \nu) & \downarrow \nu \\
E & \xrightarrow{h \circ \pi} & M
\end{array}
\]

(9)

We take \((x^i, y^a)\) as canonical local coordinates on \((E, \pi, M)\), where \(i \in \overline{1, m}\) and \(a \in \overline{1, r}\). Let

\[
(x^i, y^a) \rightarrow (x^i (x^i), y^a (x^i, y^a))
\]

be a change of coordinates on \((E, \pi, M)\). Then the coordinates \(y^a\) change to \(y'^a\) according to the rule:

\[
y'^a = M^a_n y^a.
\]

(10)

**Theorem 2** Let \(\left(\pi^* (h^* F), Id_E\right)\) be the \(B^v\)-morphism of \((\pi^* (h^* F), \pi^* (h^* \nu), E)\) source and \((TE, \tau_E, E)\) target, where

\[
\begin{array}{ccc}
\pi^* (h^* F) & \xrightarrow{\rho} & TE \\
Z^\alpha T_\alpha (u_x) & \mapsto & \left(Z'^\alpha \cdot \rho^i_\alpha \circ h \circ \pi \frac{\partial}{\partial x^i} (u_x)\right)
\end{array}
\]

(11)

Using the operation

\[
\Gamma \left(\pi^* (h^* F), \pi^* (h^* \nu), E\right)^2 \xrightarrow{[\cdot]_\pi^* (h^* F)} \Gamma \left(\pi^* (h^* F), \pi^* (h^* \nu), E\right)
\]

defined by

\[
\begin{align*}
[T_\alpha, T_\beta]_{\pi^* (h^* F)} &= L^\gamma_{\alpha \beta} \circ h \circ \pi \cdot T_\gamma, \\
[T_\alpha, f T_\beta]_{\pi^* (h^* F)} &= f L^\gamma_{\alpha \beta} \circ h \circ \pi T_\gamma + \rho^\gamma_\alpha \circ h \circ \pi \frac{\partial f}{\partial x^\gamma} T_\beta, \\
[f T_\alpha, T_\beta]_{\pi^* (h^* F)} &= -[T_\beta, f T_\alpha]_{\pi^* (h^* F)},
\end{align*}
\]

(12)
for any \( f \in \mathcal{F}(E) \), it results that
\[
\left( (\pi^* (h^* F), \pi^* (h^* \nu), E), [\cdot, \cdot]_{\pi^* (h^* F)}, \left( \pi^* (h^* F), \rho, Id_E \right) \right)
\]
is a Lie algebroid which is called the pull-back Lie algebroid of the generalized Lie algebroid \( (F, \nu, N), [\cdot, \cdot]_F, (\rho, \eta) \).

If \( z = z^\alpha t_\alpha \in \Gamma (F, \nu, N) \), then we obtain the section
\[
Z = (z^\alpha \circ h \circ \pi) T_\alpha \in \Gamma (\pi^* (h^* F), \pi^* (h^* \nu), E)
\]
so that \( Z (u_x) = z (h (x)) \), for any \( u_x \in \pi^{-1} (U \cap h^{-1} V) \).

Let
\[
\left( \partial_i, \hat{\partial}_a \right) \overset{\text{put}}{=} \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a} \right)
\]
be the base sections for the Lie \( \mathcal{F}(E) \)-algebra
\[
(\Gamma (TE, \tau_E, E), + , [\cdot, \cdot]_{TE}).
\]

For any sections
\[
Z^\alpha T_\alpha \in \Gamma (\pi^* (h^* F), \pi^* (h^* F), E)
\]
and
\[
Y^a \hat{\partial}_a \in \Gamma (VTE, \tau_E, E)
\]
we obtain the section
\[
Z^\alpha \hat{\partial}_a + Y^a \hat{\partial}_a =: Z^\alpha \left( T_\alpha \oplus (\rho^a \circ h \circ \pi) \partial_i \right) + Y^a \left( 0_{\pi^* (h^* F)} \oplus \hat{\partial}_a \right)
\]
\[
= Z^\alpha T_\alpha \oplus \left( Z^\alpha \left( \rho^a \circ h \circ \pi \right) \partial_i + Y^a \hat{\partial}_a \right) \in \Gamma \left( \pi^* (h^* F) \oplus TE, \frac{\partial}{\partial x^i}, E \right).
\]

Since we have
\[
Z^\alpha \hat{\partial}_a + Y^a \hat{\partial}_a = 0
\]
\[
\Downarrow
\]
\[
Z^\alpha T_\alpha = 0 \wedge Z^\alpha \left( \rho^a \circ h \circ \pi \right) \partial_i + Y^a \hat{\partial}_a = 0,
\]
it implies \( Z^\alpha = 0, \ \alpha \in \overline{1, p} \) and \( Y^a = 0, \ \alpha \in \overline{1, r} \).

Therefore, the sections \( \hat{\partial}_1, ..., \hat{\partial}_p, \hat{\partial}_1, ..., \hat{\partial}_r \) are linearly independent.

We consider the vector subbundle \((\rho, \eta) TE, (\rho, \eta) \tau_E, E\) of the vector bundle
\( (\pi^* (h^* F) \oplus TE, \frac{\partial}{\partial x^i}, E) \), for which the \( \mathcal{F}(E) \)-module of sections is the \( \mathcal{F}(E) \)-submodule of \( \left( \Gamma \left( \pi^* (h^* F) \oplus TE, \frac{\partial}{\partial x^i}, E \right), + , [\cdot, \cdot] \right) \), generated by the set of sections
\( \left( \hat{\partial}_a, \hat{\partial}_a \right) \).
The base sections $\left( \hat{\partial}_a, \partial_a \right)$ will be called the natural $(\rho, \eta)$-base.

The matrix of coordinate transformation on $((\rho, \eta) T E, (\rho, \eta) \tau_E, E)$ at a change of fibred charts is

$$\begin{pmatrix}
L^\alpha_{\alpha'} \circ h \circ \pi & 0 \\
\left( \rho^a_i \circ h \circ \pi \right) \frac{\partial M^a_{\alpha'} \circ \pi}{\partial x^i} Y^b_a \circ \pi \\
\end{pmatrix}.$$  (13)

Easily we obtain:

**Theorem 3** Let $(\tilde{\rho}, Id_E)$ be the $B^\nu$-morphism of $((\rho, \eta) T E, (\rho, \eta) \tau_E, E)$ source and $(T E, \tau_E, E)$ target, where

$$\begin{pmatrix}
Z^\alpha \hat{\partial}_a + Y^a \hat{\partial}_a \\
\end{pmatrix}(u_x) \mapsto \begin{pmatrix}
Z^\alpha \left( \rho^a_i \circ h \circ \pi \right) \partial_i + Y^a \hat{\partial}_a \\
\end{pmatrix}(u_x).$$  (14)

Using the operation

$$\Gamma (\rho, \eta) T E \overset{\tilde{\rho}}{\rightarrow} T E$$

$$\Gamma \left( (\rho, \eta) T E, (\rho, \eta) \tau_E, E \right)^2 \overset{\left[ \cdot \right]_{(\rho, \eta) T E}}{\rightarrow} \Gamma \left( (\rho, \eta) T E, (\rho, \eta) \tau_E, E \right)$$

defined by

$$\left[ Z^\alpha \hat{\partial}_a + Y^a \hat{\partial}_a, Z^\beta \hat{\partial}_b + Y^b \hat{\partial}_b \right]_{(\rho, \eta) T E}$$

$$\quad = \left[ Z^\alpha_{\alpha'} T_\alpha, Z^\beta_{\beta'} T_\beta \right]_{\pi^* (h^* E)} \otimes \left[ Z^\alpha_{\alpha'} \left( \rho^a_i \circ h \circ \pi \right) \partial_i + Y^a \hat{\partial}_a, Z^\beta_{\beta'} \left( \rho^b_j \circ h \circ \pi \right) \partial_j + Y^b \hat{\partial}_b \right]_{\tau_E},$$  (15)

for any $Z^\alpha \hat{\partial}_a + Y^a \hat{\partial}_a$ and $Z^\beta \hat{\partial}_b + Y^b \hat{\partial}_b$, we obtain that the couple

$$\left[ \cdot \right]_{(\rho, \eta) T E}, (\tilde{\rho}, Id_E)$$

is a Lie algebroid structure for the vector bundle $((\rho, \eta) T E, (\rho, \eta) \tau_E, E)$.

**Remark 4** In particular, if $h = Id_M$, then the Lie algebroid

$$\left( \left( Id_{TM}, Id_M \right) T E, \left( Id_{TM}, Id_M \right) \tau_E, E \right), \left[ \cdot \right]_{\left( Id_{TM}, Id_M \right) T E}, \left( \tilde{Id}_{TM}, Id_E \right)$$

is isomorphic with the usual Lie algebroid

$$\left( (T E, \tau_E, E), \left[ \cdot \right]_{T E}, (Id_{TE}, Id_E) \right).$$
This is a reason for which the Lie algebroid
\[
\left(\left((\rho, \eta) \tau E, (\rho, \eta) \tau E, E, \left[\cdot, \cdot\right]_{(\rho, \eta) \tau E}, (\hat{\rho}, Id_E)\right),\right)
\]
will be called the Lie algebroid generalized tangent bundle. (see also [1])

We consider the \(B_v\)-morphism \(((\rho, \eta) \pi!, Id_E)\) given by the commutative diagram

\[
\begin{array}{ccc}
(\rho, \eta) \tau E & \xrightarrow{\pi^* (h^* F)} & (\rho, \eta) \tau E, E \\
\downarrow & & \downarrow \pi^* (h^* \nu) \\
E & \xrightarrow{Id_E} & E \\
\end{array}
\]

This is defined as:
\[
(\rho, \eta) \pi! \left(\left(Z^\alpha \tilde{\partial}_\alpha + Y^a \tilde{\partial}_a\right) (u_x)\right) = (Z^\alpha T_\alpha) (u_x),
\]
for any \(Z^\alpha \tilde{\partial}_\alpha + Y^a \tilde{\partial}_a \in \Gamma ((\rho, \eta) \tau E, (\rho, \eta) \tau E, E).\)

Using the \(B_v\)-morphism \(((\rho, \eta) \pi!, Id_E)\) we obtain the tangent \((\rho, \eta)\)-application \(((\rho, \eta) \tau E, (\rho, \eta) \tau E, E)\) source and \((F, \nu, N)\) target.

**Definition 5** The kernel of the tangent \((\rho, \eta)\)-application is written
\[
(V (\rho, \eta) \tau E, (\rho, \eta) \tau E, E)
\]
and it is called the vertical interior differential system. (see [2])

We remark that the set \(\{ \tilde{\partial}_a, a \in \Gamma, \tau \}\) is a base of the \(F (E)\)-module
\[
(\Gamma (V (\rho, \eta) \tau E, (\rho, \eta) \tau E, E), +, \cdot).
\]

**Proposition 6** The short sequence of vector bundles

\[
\begin{array}{cccc}
0 & \hookrightarrow & V (\rho, \eta) \tau E & \xrightarrow{\pi^* (h^* F)} & 0 \\
\downarrow & & \downarrow Id_E & \downarrow & \downarrow \\
E & \xrightarrow{Id_E} & E & \xrightarrow{Id_E} & E \\
\end{array}
\]

is exact.

**Definition 7** A Man\-morphism \((\rho, \eta) \Gamma\) of \((\rho, \eta) \tau E\) source and \(V (\rho, \eta) \tau E\) target defined by
\[
(\rho, \eta) \Gamma \left(Z^\gamma \tilde{\partial}_\gamma + Y^a \tilde{\partial}_a\right) (u_x) = (Y^a + (\rho, \eta) \Gamma^a Z^\gamma) \tilde{\partial}_a (u_x),
\]
so that the \(B_v\)-morphism \(((\rho, \eta) \Gamma, Id_E)\) is a split to the left in the previous exact sequence, will be called \((\rho, \eta)\)-connection for the vector bundle \((E, \pi, M)\).
The \((\rho, Id_M)\)-connection is called \(\rho\)-connection and is denoted \(\rho \Gamma\) and the \((Id_T M, Id_M)\)-connection is called connection and is denoted \(\Gamma\).

**Definition 8** If \((\rho, \eta) \Gamma\) is a \((\rho, \eta)\)-connection for the vector bundle \((E, \pi, M)\), then the kernel of the \(B^\nu\)-morphism \((\rho, \eta) \Gamma, Id_E\) is written

\[
(H (\rho, \eta) TE, (\rho, \eta) \tau_E, E)
\]

and is called the horizontal interior differential system. (see \[2\])

**Definition 9** If \((E, \pi, M) \in |B^\nu|\) and \(\{s_a, a \in \Gamma \tau^r\}\) is a base of the \(F(M)\)-module of sections \((\Gamma (E, \pi, M), +, \cdot)\), then we obtain the \(B^\nu\)-morphism \((\Pi, \pi)\) defined by the commutative diagram

\[
\begin{array}{ccc}
V (\rho, \eta) TE & \xrightarrow{\Pi} & E \\
(\rho, \eta) \tau_E & \downarrow \pi & \downarrow \pi \\
E & \xrightarrow{\pi} & M
\end{array}
\]

so that

\[
\Pi \left( Y^a \partial \hat{s}_a (u_x) \right) = Y^a (u_x) s_a (x).
\]

**Theorem 10** (see \[1\]) If \((\rho, \eta) \Gamma\) is a \((\rho, \eta)\)-connection for the vector bundle \((E, \pi, M)\), then its components satisfy the law of transformation

\[
(\rho, \eta) \Gamma^a_k \gamma = M^a_k \circ \pi \left[ \rho_k \circ h \circ \pi \frac{\partial M^a_k \circ \pi}{\partial x^k} y^k + (\rho, \eta) \Gamma^a_k \right] \Lambda^\gamma \circ h \circ \pi.
\]

In the particular case of Lie algebroids (see also \[3\]), \((\eta, h) = (Id_T M, Id_M)\), the relations \[22\] become

\[
\rho \Gamma^a_k = M^a_k \circ \pi \left[ \rho_k \circ h \circ \pi \frac{\partial M^a_k \circ \pi}{\partial x^k} y^k + \rho \Gamma^a_k \right] \Lambda^\gamma \circ h \circ \pi.
\]

In the classical case, \((\rho, \eta, h) = (Id_T M, Id_M, Id_M)\), the relations \[23\] become

\[
\Gamma^a_k = \frac{\partial \gamma^a}{\partial x^k} \circ \tau_M \left[ \frac{\partial}{\partial x^k} \left( \frac{\partial \gamma^a}{\partial x^k} \circ \tau_M \right) y^k + \Gamma^a_k \right] \frac{\partial x^k}{\partial x^m} \circ \tau_M.
\]

**Remark 11** If we have a set of real local functions \((\rho, \eta) \Gamma^a_k\) which satisfies the relations of passing \[22\], then we have a \((\rho, \eta)\)-connection \((\rho, \eta) \Gamma\) for the vector bundle \((E, \pi, M)\).

**Example 12** If \(\Gamma\) is an Ehresmann connection for the vector bundle \((E, \pi, M)\) on components \(\Gamma^a_k\), then the differentiable real local functions

\[
(\rho, \eta) \Gamma^a_k = (\rho^a_k \circ h \circ \pi) \Gamma^a_k
\]

are the components of a \((\rho, \eta)\)-connection \((\rho, \eta) \Gamma\) for the vector bundle \((E, \pi, M)\). This \((\rho, \eta)\)-connection will be called the \((\rho, \eta)\)-connection associated to the connection \(\Gamma\).
We put the problem of finding a base for the $F(E)$-module

$$(\Gamma (H (\rho, \eta) TE, (\rho, \eta) \tau E), +, \cdot)$$

of the type

$$\tilde{\delta}_\alpha = Z_\alpha^\beta \hat{\partial}_\beta + Y_\alpha^a \hat{\partial}_a, \alpha \in \Gamma, r$$

which satisfies the following conditions:

$$\Gamma ((\rho, \eta) \pi!, Id E) \left( \tilde{\delta}_\alpha \right) = T_\alpha, \tag{25}$$

$$\Gamma ((\rho, \eta) \Gamma, Id E) \left( \tilde{\delta}_\alpha \right) = 0. \tag{26}$$

Then we obtain the sections

$$\delta_\alpha \tilde{\delta}_\alpha = \tilde{\partial}_\alpha \ominus \left( (\rho^i \circ h \circ \pi) \partial_i - (\rho, \eta) \Gamma^a \hat{\partial}_a \right). \tag{27}$$

such that their law of change is a tensorial law under a change of vector fiber charts.

The base $\left( \tilde{\delta}_\alpha, \hat{\partial}_a \right)$ will be called the adapted $(\rho, \eta)$-base.

**Remark 13** The following equality holds good

$$\Gamma (\tilde{\rho}, Id E) \left( \tilde{\delta}_\alpha \right) = (\rho^i \circ h \circ \pi) \partial_i - (\rho, \eta) \Gamma^a \hat{\partial}_a. \tag{28}$$

Moreover, if $(\rho, \eta) \Gamma$ is the $(\rho, \eta)$-connection associated to a connection $\Gamma$, then we obtain

$$\Gamma (\tilde{\rho}, Id E) \left( \tilde{\delta}_\alpha \right) = (\rho^i \circ h \circ \pi) \delta_i, \tag{29}$$

where $\left( \delta_i, \hat{\partial}_a \right)$ is the adapted base for the $F(E)$-module $(\Gamma (TE, \tau E), +, \cdot)$.

**Theorem 14** The following equality holds good

$$\left[ \tilde{\delta}_\alpha, \tilde{\delta}_\beta \right]_{(\rho, \eta)TE} = L^\gamma_{\alpha\beta} \circ (h \circ \pi) \tilde{\delta}_\gamma + (\rho, \eta, h) R^a_{\alpha\beta} \tilde{\partial}_a, \tag{30}$$

where

$$(\rho, \eta, h) R^a_{\alpha\beta} = \Gamma (\tilde{\rho}, Id E) \left( \tilde{\delta}_\beta \right) \left( (\rho, \eta) \Gamma^a \right)$$

$$- \Gamma (\tilde{\rho}, Id E) \left( \tilde{\delta}_\alpha \right) \left( (\rho, \eta) \Gamma^a_\beta \right) + \left( L^\gamma_{\alpha\beta} \circ h \circ \pi \right) (\rho, \eta) \Gamma^a_{\gamma}. \tag{31}$$

Moreover, we have:

$$\left[ \tilde{\delta}_\alpha, \hat{\partial}_b \right]_{(\rho, \eta)TE} = \Gamma (\tilde{\rho}, Id E) \left( \hat{\partial}_b \right) \left( (\rho, \eta) \Gamma^a_\alpha \right) \hat{\partial}_a, \tag{32}$$
and

\[ \Gamma (\hat{\rho}, Id_E) \left[ \delta_\alpha, \delta_\beta \right]_{(\rho,\eta)TE} = \left[ \Gamma (\hat{\rho}, Id_E) \left( \delta_\alpha \right), \Gamma (\hat{\rho}, Id_E) \left( \delta_\beta \right) \right]_{TE}. \quad (32) \]

Let \((d\tilde{z}^\alpha, d\tilde{y}^\beta)\) be the natural dual \((\rho, \eta)\)-base of natural \((\rho, \eta)\)-base \(\left( \tilde{\partial}_\alpha, \tilde{\partial}_\alpha \right)\).

This is determined by the equations

\[
\begin{align*}
\left( d\tilde{z}^\alpha, \tilde{\partial}_\beta \right) & = \delta_\alpha^\beta, \\
\left( d\tilde{z}^\alpha, \tilde{\partial}_\alpha \right) & = 0, \\
\left( d\tilde{y}^a, \tilde{\partial}_\beta \right) & = 0, \\
\left( d\tilde{y}^a, \tilde{\partial}_\alpha \right) & = \delta_a^\alpha.
\end{align*}
\]

We consider the problem of finding a base for the \(\mathcal{F}(E)\)-module

\[
\left( \Gamma \left( (V (\rho, \eta) TE)^{\ast}, ((\rho, \eta) \tau E)^{\ast}, E \right)^{\ast}, +, \right)
\]

of the type

\[
d\tilde{y}^a = \theta_a^a d\tilde{z}^\alpha + \omega_a^b d\tilde{y}^b, \quad a \in \overline{1,n}
\]

which satisfies the following conditions:

\[
\left\{ \begin{array}{ll}
\left( \tilde{\partial}_a, \tilde{\partial}_\alpha \right) & = \theta_a^\alpha \\
\left( \tilde{\partial}_a, \tilde{\partial}_\beta \right) & = 0
\end{array} \right.
\]

We obtain the sections

\[
d\tilde{y}^a = (\rho, \eta) \Gamma_a^\alpha d\tilde{z}^\alpha + \tilde{y}^a, \quad a \in \overline{1,n}.
\]

such that their changing rule is tensorial under a change of vector fiber charts. The base \((d\tilde{z}^\alpha, d\tilde{y}^a)\) will be called the adapted dual \((\rho, \eta)\)-base.

3 The \((g, h)\)-lift of a differentiable curve

We consider the following diagram:

\[
\begin{array}{ccc}
E & \xrightarrow{[\cdot, \cdot]_{F, h}} & \left( \mathcal{F}(\nu), (\rho, \eta) \right) \\
\pi \downarrow & & \downarrow \nu \\
M & \xrightarrow{h} & N
\end{array}
\]

where \((E, \pi, M) \in \mathcal{B}Y\) and \(\left( \mathcal{F}(\nu), N \right) \xrightarrow{[\cdot, \cdot]_{F, h}}, (\rho, \eta)\) is a generalized Lie algebroid.

We admit that \((\rho, \eta) \Gamma\) is a \((\rho, \eta)\)-connection for the vector bundle \((E, \pi, M)\) and \(I \hookrightarrow M\) is a differentiable curve. We know that

\[
\left( E|_{\text{Im}(\eta \circ h \circ c)}, \pi|_{\text{Im}(\eta \circ h \circ c)}, \text{Im} (\eta \circ h \circ c) \right)
\]

is a vector subbundle of the vector bundle \((E, \pi, M)\).
Definition 15 If
\[ I \xrightarrow{\dot{c}} E|_{\text{Im}(\eta \circ h \circ c)} \]
\[ t \mapsto y^a(t) s_a(\eta \circ h \circ c(t)) \]
is a differentiable curve such that there exists \( g \in \text{Man}(E, F) \) such that the following conditions are satisfied:

1. \((g, h) \in B^v((E, \pi, M), (F, \nu, N))\) and

2. \( \rho \circ g \circ \dot{c}(t) = \frac{d}{dt}(\eta \circ h \circ c)(t) \), for any \( t \in I \),

then we will say that \( \dot{c} \) is the \((g, h)\)-lift of the differentiable curve \( c \).

Remark 16 The second condition is equivalent with the following affirmation:
\[ \rho^i_\alpha(\eta \circ h \circ c(t)) \cdot g^a_\alpha(h \circ c(t)) \cdot y^a(t) = \frac{d(\eta \circ h \circ c)(t)}{dt}, \quad i \in \overline{1, m}. \]  

Definition 17 If \( I \xrightarrow{\dot{c}} E|_{\text{Im}(\eta \circ h \circ c)} \) is a differentiable \((g, h)\)-lift of the differentiable curve \( c \), then the section
\[ \eta \circ h \circ c(t) \mapsto \dot{c}(t) \]
will be called the canonical section associated to the couple \((c, \dot{c})\).

Definition 18 If \((g, h) \in B^v((E, \pi, M), (F, \nu, N))\) has the components
\[ g^a_\alpha; a \in \overline{1, r}, \quad \alpha \in \overline{1, p} \]
such that for any vector local \((n + p)\)-chart \((V, t_V)\) of \((F, \nu, N)\) there exists the real functions
\[ V \xrightarrow{\mathbb{Z}^a_\alpha} \mathbb{R}; \quad a \in \overline{1, r}, \quad \alpha \in \overline{1, p} \]
such that
\[ \mathbb{Z}^b_\alpha(\varkappa) \cdot g^a_\alpha(\varkappa) = \delta^b_a, \]  
for any \( \varkappa \in V \), then we will say that the \(B^v\)-morphism \((g, h)\) is locally invertible.

Remark 19 In particular, if \((\text{Id}_{TM}, \text{Id}_M, \text{Id}_M) = (\rho, \eta, h)\) and the \(B^v\) morphism \((g, \text{Id}_M)\) is locally invertible, then we have the differentiable \((g, \text{Id}_M)\)-lift
\[ I \xrightarrow{\dot{c}} TM \]
\[ t \mapsto \tilde{g}^i_j(c(t)) \frac{dc^j(t)}{dt} \frac{\partial}{\partial x^i}(c(t)). \]  
Moreover, if \( g = \text{Id}_{TM} \), then we obtain the usual lift of tangent vectors
\[ I \xrightarrow{\dot{c}} TM \]
\[ t \mapsto \frac{dc^i(t)}{dt} \frac{\partial}{\partial x^i}(c(t)). \]
Definition 20  If \( I \xrightarrow{\hat{c}} E|_{\text{Im}(\psi_{\text{inh}})} \) is a differentiable \((g, h)\)-lift of differentiable curve \( c \), such that its components functions \((y^a, a \in \overline{1,n})\) are solutions for the differentiable system of equations:

\[
\frac{du^a}{dt} + (\rho, \eta)^a \circ (c, \dot{c}) \circ (\eta \circ h \circ c) \cdot g^a_b \circ h \circ c \cdot u^b = 0,
\]

then we will say that the \((g, h)\)-lift \( \dot{c} \) is parallel with respect to the \((\rho, \eta)\)-connection \((\rho, \eta) \Gamma \).

Remark 21  In particular, if \((\rho, \eta, h) = (\text{Id}_{TM}, \text{Id}_M, \text{Id}_M)\) and the \( B^y \)-morphism \((g, Id_M)\) is locally invertible, then the differentiable \((g, Id_{TM})\)-lift

\[
I \xrightarrow{\hat{c}} TM
\]

\[
t \mapsto \left( \tilde{g}^i_j \circ c \cdot \frac{dc^j}{dt} \right) \frac{\partial}{\partial x^i} (c(t)),
\]

is parallel with respect to the connection \( \Gamma \) if the component functions

\[
\left( \tilde{g}^i_j \circ c \cdot \frac{dc^j}{dt}, i \in \overline{1,n} \right)
\]

are solutions for the differentiable system of equations

\[
\frac{du^i}{dt} + \Gamma^i_k \circ u (c, \dot{c}) \circ c \cdot g^k_i \circ c \cdot u^k = 0,
\]

namely

\[
\frac{d}{dt} \left( \tilde{g}^i_j (c(t)) \cdot \frac{dc^j}{dt} \right) + \Gamma^i_k \left( \tilde{g}^i_j (c(t)) \cdot \frac{dc^j}{dt} \right) \cdot \frac{\partial}{\partial x^k} (c(t)) \cdot \frac{dc^k}{dt} = 0.
\]

Moreover, if \( g = \text{Id}_{TM} \), then the usual lift of tangent vectors \((41)\) is parallel with respect to the connection \( \Gamma \) if the component functions \( \left( \frac{dc^j}{dt}, j \in \overline{1,n} \right) \) are solutions for the differentiable system of equations

\[
\frac{du^i}{dt} + \Gamma^i_k \circ u (c, \dot{c}) \circ c \cdot u^k = 0,
\]

namely

\[
\frac{d}{dt} \left( \frac{dc^i}{dt} \right) + \Gamma^i_k \left( \frac{dc^j}{dt} \cdot \frac{\partial}{\partial x^j} (c(t)) \right) \cdot \frac{dc^k}{dt} = 0.
\]
4 Remarkable Mod-endomorphisms

We consider the following diagram:

\[
\begin{array}{ccc}
E & \rightarrow & \left(F,\left[,\right]_{F,h},(\rho,\eta)\right) \\
\pi \downarrow & & \downarrow \nu \\
M & \rightarrow & N
\end{array}
\] (48)

where \((E,\pi,M)\in |B^Y|\) and \(\left(F,\nu,N,\left[,\right]_{F,h},(\rho,\eta)\right)\) is a generalized Lie algebroid.

**Definition 22** For any Mod-endomorphism \(e\) of \(\Gamma\left(\left(\rho,\eta\right)_{TE},\left(\rho,\eta\right)_{\tau E},E\right)\) we define the application of Nijenhuis type defined by

\[
N_e(X,Y) = \left[eX,eY\right]_{\left(\rho,\eta\right)_{TE}} + e^2\left[X,Y\right]_{\left(\rho,\eta\right)_{TE}} - e\left[eX,Y\right]_{\left(\rho,\eta\right)_{TE}} - e\left[X,eY\right]_{\left(\rho,\eta\right)_{\rho TE}}
\]

for any \(X,Y \in \Gamma\left(\left(\rho,\eta\right)_{TE},\left(\rho,\eta\right)_{\tau E},E\right)\).

4.1 Projectors

**Definition 23** Any Mod-endomorphism \(e\) of \(\Gamma\left(\left(\rho,\eta\right)_{TE},\left(\rho,\eta\right)_{\tau E},E\right)\) with the property

\[
e^2 = e
\] (49)

will be called projector.

**Example 24** The Mod-endomorphism

\[
\Gamma\left(\left(\rho,\eta\right)_{TE},\left(\rho,\eta\right)_{\tau E},E\right) \xrightarrow{\nu} \Gamma\left(\left(\rho,\eta\right)_{TE},\left(\rho,\eta\right)_{\tau E},E\right)
\]

\[
Z^a\hat{\delta}_a + Y^a\hat{\gamma}_a \quad \mapsto \quad Y^a\hat{\gamma}_a
\]

is a projector which will be called vertical projector.

**Remark 25** We have \(V\left(\hat{\delta}_a\right) = 0\) and \(V\left(\hat{\gamma}_a\right) = \hat{\gamma}_a\). Therefore, it follows

\[
V\left(\hat{\delta}_a\right) = \left(\rho,\eta\right)\Gamma^a\hat{\gamma}_a.
\]

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In addition, we obtain the equality
\[
\Gamma((\rho, \eta) \Gamma, \text{Id}_E) \left( Z^\alpha \partial_\alpha + Y^a \partial_a \right) = \mathcal{V} \left( Z^\alpha \partial_\alpha + Y^a \partial_a \right),
\]  
for any \( Z^\alpha \partial_\alpha + Y^a \partial_a \in \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E) \).

**Theorem 26** A \((\rho, \eta)\)-connection for the vector bundle \((E, \pi, M)\) is characterized by the existence of a \textbf{Mod}-endomorphism \( \mathcal{V} \) of \( \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E) \) with the properties:
\[
\mathcal{V}(\Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)) \subset \Gamma(\mathcal{V}(\rho, \eta) TE, (\rho, \eta) \tau_E, E)
\]
\[
\mathcal{V}(X) = X \iff X \in \Gamma(\mathcal{V}(\rho, \eta) TE, (\rho, \eta) \tau_E, E).
\]

**Example 27** The \textbf{Mod}-endomorphism
\[
\Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E) \xrightarrow{\mathcal{H}} \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)
\]
\[
Z^\alpha \partial_\alpha + Y^a \partial_a \mapsto Z^\alpha \partial_\alpha
\]
is a projector which will be called horizontal projector.

**Remark 28** We have \( \mathcal{H}(\partial_\alpha) = \partial_\alpha \) and \( \mathcal{H}(\partial_a) = 0 \). Therefore, we obtain \( \mathcal{H}(\partial_\alpha) = \partial_\alpha \).

**Theorem 29** A \((\rho, \eta)\)-connection for the vector bundle \((E, \pi, M)\) is characterized by the existence of a \textbf{Mod}-endomorphism \( \mathcal{H} \) of \( \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E) \) with the properties:
\[
\mathcal{H}(\Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)) \subset \Gamma(\mathcal{H}(\rho, \eta) TE, (\rho, \eta) \tau_E, E)
\]
\[
\mathcal{H}(X) = X \iff X \in \Gamma(\mathcal{H}(\rho, \eta) TE, (\rho, \eta) \tau_E, E).
\]

**Corollary 30** A \((\rho, \eta)\)-connection for the vector bundle \((E, \pi, M)\) is characterized by the existence of a \textbf{Mod}-endomorphism \( \mathcal{H} \) of \( \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E) \) with the properties:
\[
\mathcal{H}^2 = \mathcal{H},
\]
\[
\text{Ker} \, \mathcal{H} = (\Gamma(\mathcal{V}(\rho, \eta) TE, (\rho, \eta) \tau_E, E), +, \cdot).
\]

**Remark 31** For any \( X \in \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E) \) we obtain the unique decomposition
\[
X = \mathcal{H}X + \mathcal{V}X.
\]

**Proposition 32** After some calculations we obtain
\[
N_{\mathcal{V}}(X, Y) = \mathcal{V}[\mathcal{H}X, \mathcal{H}Y]_{(\rho, \eta) TE} = N_{\mathcal{H}}(X, Y),
\]
for any \( X, Y \in \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E) \).

**Corollary 33** The horizontal interior differential system
\[
(H(\rho, \eta) TE, (\rho, \eta) \tau_E, E)
\]
is involutive if and only if \( N_{\mathcal{V}} = 0 \) or \( N_{\mathcal{H}} = 0 \).
4.2 The almost product structure

**Definition 34** Any \textbf{Mod}-endomorphism \(e\) of \(\Gamma (\rho, \eta)TE, (\rho, \eta\tau_E, E)\) with the property
\[
e^2 = Id
\] (55)
will be called the almost product structure.

**Example 35** The \textbf{Mod}-endomorphism
\[
\Gamma (\rho, \eta)TE, (\rho, \eta\tau_E, E) \xrightarrow{P} \Gamma (\rho, \eta)TE, (\rho, \eta\tau_E, E)
\]
\[
Z^a\delta_a + Y^a\tilde{\delta}_a \mapsto Z^a\delta_a - Y^a\tilde{\delta}_a
\]
is an almost product structure.

**Remark 36** The previous almost product structure has the properties:
\[
\mathcal{P} = (2\mathcal{H} - Id); \\
\mathcal{P} = (Id - 2\mathcal{V}); \\
\mathcal{P} = (\mathcal{H} - \mathcal{V}).
\] (56)

**Remark 37** We obtain that \(P(\delta_a) = \delta_a\) and \(P(\tilde{\delta}_a) = -\tilde{\delta}_a\). Therefore, it follows
\[
\mathcal{P}(\tilde{\delta}_a) = \delta_a - \rho^a\delta_a.
\]

**Theorem 38** A \((\rho, \eta)\)-connection for the vector bundle \((E, \pi, M)\) is characterized by the existence of a \textbf{Mod}-endomorphism \(\mathcal{P}\) of \(\Gamma (\rho, \eta)TE, (\rho, \eta\tau_E, E)\) with the following property:
\[
\mathcal{P}(X) = -X \iff X \in \Gamma (V(\rho, \eta)TE, (\rho, \eta\tau_E, E)).
\] (57)

**Proposition 39** After some calculations, we obtain
\[
N_{\mathcal{P}}(X, Y) = 4\mathcal{V}[\mathcal{H}X, \mathcal{H}Y],
\]
for any \(X, Y \in \Gamma ((\rho, \eta)TE, (\rho, \eta\tau_E, E))\).

**Corollary 40** The horizontal interior differential system \((H\rho, \eta)TE, (\rho, \eta\tau_E, E)\) is involutive if and only if \(N_{\mathcal{P}} = 0\).

4.3 The almost tangent structure
Definition 41 Any \( \text{Mod-endomorphism} e \) of \( (\Gamma (\rho, \eta) T E, (\rho, \eta) \tau E, E) \) with the property
\[
e^2 = 0i
will be called the almost tangent structure.

Example 42 If \((E, \pi, M) = (F, \nu, N), g \in \text{Man}(E, E)\) such that \((g, h)\) is a locally invertible \(B^v\)-morphism, then the \(\text{Mod-endomorphism}
\[
\Gamma ((\rho, \eta) T E, (\rho, \eta) \tau E, E) \overset{J(g, h)}{\longrightarrow} (\tilde{g}^h_a \circ h \circ \pi) Z_a \delta_a + Y^b \partial_b(58)
\]
is an almost tangent structure which will be called the almost tangent structure associated to the \(B^v\)-morphism \((g, h)\). (see: [18])

Example 43 We obtain that
\[
J_{(g, h)} (\tilde{\delta}_a) = J_{(g, h)} (\tilde{\partial}_a) = (\tilde{g}^h_a \circ h \circ \pi) \tilde{\partial}_a \text{ and } J_{(g, h)} (\tilde{\partial}_b) = 0.
\]
and we have the following properties:
\[
\begin{align*}
J_{(g, h)} \circ P &= J_{(g, h)}; \\
P \circ J_{(g, h)} &= -J_{(g, h)}; \\
J_{(g, h)} \circ H &= J_{(g, h)}; \\
H \circ J_{(g, h)} &= 0; \\
J_{(g, h)} \circ V &= 0; \\
V \circ J_{(g, h)} &= J_{(g, h)}; \\
N J_{(g, h)} &= 0.
\end{align*}
\]

5 Distinguished linear \((\rho, \eta)\)-connections

We consider the following diagram:
\[
\begin{array}{ccc}
E & \xrightarrow{F \cdot [\cdot, \cdot]_{F, h}, (\rho, \eta)} & \left( F, [\cdot, \cdot]_{F, h}, (\rho, \eta) \right) \\
\pi \downarrow & & \downarrow \nu \\
M & \xrightarrow{h} & N
\end{array}
\]
where \((E, \pi, M) \in [B^\gamma] \) and \( \left( (F, \nu, N), [\cdot, \cdot]_{F, h}, (\rho, \eta) \right) \) is a generalized Lie algebroid.

Let
\[
(T^{\rho, \eta}_{q, \nu} ((\rho, \eta) T E, (\rho, \eta) \tau E, +, +))
\]
be the $\mathcal{F}(E)$-module of tensor fields by $(p,r)$-type from the generalized tangent bundle

$$(H(\rho,\eta)TE, (\rho,\eta)\tau_E, E) \oplus (V(\rho,\eta)TE, (\rho,\eta)\tau_E, E).$$

An arbitrary tensor field $T$ is written as

$$T = T^{a_1...a_r}_{\alpha_1...\alpha_p} \delta_{\alpha_1} \otimes \ldots \otimes \delta_{\alpha_p} \otimes \delta^{b_1} \otimes \ldots \otimes \delta^{b_s} \otimes \partial_{\beta_1} \otimes \ldots \otimes \partial_{\beta_q} \otimes \delta_{\gamma_1} \otimes \ldots \otimes \delta_{\gamma_s}.$$ 

Let

$$(\mathcal{T} ((\rho,\eta)TE, (\rho,\eta)\tau_E, E), +, \cdot, \otimes)$$

be the tensor fields algebra of generalized tangent bundle $((\rho,\eta)TE, (\rho,\eta)\tau_E, E)$.

**Definition 44** Let $(E, \pi, M)$ be a vector bundle endowed with a $(\rho,\eta)$-connection $(\rho,\eta)\Gamma$ and let

$$(X, T) \xrightarrow{(\rho,\eta)D} (\rho,\eta)DXT$$

be a covariant $(\rho,\eta)$-derivative for the tensor algebra

$$(\mathcal{T} ((\rho,\eta)TE, (\rho,\eta)\tau_E, E), +, \cdot, \otimes)$$

of the generalized tangent bundle

$$(\rho,\eta)TE, (\rho,\eta)\tau_E, E)$$

which preserves the horizontal and vertical interior differential systems by parallelism. (see [2])

The real local functions

$$((\rho,\eta)H^\alpha_{\beta\gamma}, (\rho,\eta)H^a_{b\gamma}, (\rho,\eta)V^\alpha_{\beta\gamma}, (\rho,\eta)V^a_{b\gamma})$$

defined by the following equalities:

$$(\rho,\eta)D_{\delta_\gamma} \delta_\beta = (\rho,\eta)H^\alpha_{\beta\gamma} \delta_\alpha, \quad (\rho,\eta)D_{\delta_\gamma} \partial_b = (\rho,\eta)H^a_{b\gamma} \partial_a$$

$$(\rho,\eta)D_{\delta_\gamma} \delta_\beta = (\rho,\eta)V^\alpha_{\beta\gamma} \delta_\alpha, \quad (\rho,\eta)D_{\delta_\gamma} \partial_b = (\rho,\eta)V^a_{b\gamma} \partial_a$$

(61)

are the components of a linear $(\rho,\eta)$-connection $((\rho,\eta)H, (\rho,\eta)V)$ for the generalized tangent bundle $((\rho,\eta)TE, (\rho,\eta)\tau_E, E)$ which will be called the distinguished linear $(\rho,\eta)$-connection.

If $h = Id_M$, then the distinguished linear $(Id_{TM}, Id_M)$-connection is the classical distinguished linear connection.

The components of a distinguished linear connection $(H, V)$ will be denoted

$$(H^i_{jk}, H^a_{bk}, V^i_{jc}, V^a_{bc}).$$
Theorem 45 If \((\rho, \eta, H, (\rho, \eta)V)\) is a distinguished linear \((\rho, \eta)\)-connection for the generalized tangent bundle \(((\rho, \eta)TE, (\rho, \eta)\tau_E, E)\), then its components satisfy the change relations:

\[
(\rho, \eta) H_{\beta\gamma} = \Lambda^a_{\beta\gamma} \circ \pi \cdot \left(\Gamma(\tilde{\rho}, Id_E) \left(\delta_{\gamma} \right) \left(\Lambda^a_{\beta\gamma} \circ \pi\right) + \left(\rho, \eta\right) H_{\beta\gamma}^a \cdot \Lambda^b_{\beta\gamma} \circ \pi\right) \cdot \Lambda^c_{\gamma} \circ \pi, \\
(\rho, \eta) H_{\beta\gamma}^a = M_{\beta\gamma}^a \circ \pi \cdot \left(\Gamma(\tilde{\rho}, Id_E) \left(\delta_{\gamma} \right) \left(M_{\beta\gamma}^a \circ \pi\right) + (\rho, \eta) H_{\beta\gamma}^a \cdot M_{\beta\gamma}^b \circ \pi\right) \cdot \Lambda^c_{\gamma} \circ \pi, \\
(\rho, \eta) V_{\beta\gamma}^a = \Lambda^a_{\beta\gamma} \circ \pi \cdot (\rho, \eta) V_{\beta\gamma}^a \cdot \Lambda^b_{\beta\gamma} \circ \pi \cdot M_{\beta\gamma}^c \circ \pi, \\
(\rho, \eta) V_{\beta\gamma}^c = M_{\beta\gamma}^a \circ \pi \cdot (\rho, \eta) V_{\beta\gamma}^a \cdot M_{\beta\gamma}^b \circ \pi \cdot M_{\beta\gamma}^c \circ \pi.
\]

Corollary 46 In the particular case of Lie algebroids (see [3]), \((\eta, h) = (Id_M, Id_M)\), we obtain

\[
\rho H_{\beta\gamma} = \Lambda^a_{\beta\gamma} \circ \pi \cdot \left[\Gamma(\tilde{\rho}, Id_E) \left(\delta_{\gamma} \right) \left(\Lambda^a_{\beta\gamma} \circ \pi\right) + \rho H_{\beta\gamma}^a \cdot \Lambda^b_{\beta\gamma} \circ \pi\right] \cdot \Lambda^c_{\gamma} \circ \pi, \\
\rho H_{\beta\gamma}^a = M_{\beta\gamma}^a \circ \pi \cdot \left[\Gamma(\tilde{\rho}, Id_E) \left(\delta_{\gamma} \right) \left(M_{\beta\gamma}^a \circ \pi\right) + \rho H_{\beta\gamma}^a \cdot M_{\beta\gamma}^b \circ \pi\right] \cdot \Lambda^c_{\gamma} \circ \pi, \\
\rho V_{\beta\gamma}^a = \Lambda^a_{\beta\gamma} \circ \pi \cdot \rho V_{\beta\gamma}^a \cdot \Lambda^b_{\beta\gamma} \circ \pi \cdot M_{\beta\gamma}^c \circ \pi, \\
\rho V_{\beta\gamma}^c = M_{\beta\gamma}^a \circ \pi \cdot \rho V_{\beta\gamma}^a \cdot M_{\beta\gamma}^b \circ \pi \cdot M_{\beta\gamma}^c \circ \pi.
\]

In the classical case, \((\rho, \eta, h) = (Id_{TE}, Id_M, Id_M)\), we obtain that the components of a distinguished linear connection \((H, V)\) verify the change relations:

\[
H_{jk}^i = \frac{\partial x^i}{\partial y^k} \circ \pi \cdot \left[\delta_{jk} \left(\frac{\partial x^i}{\partial y^k} \circ \pi\right) + H_{jk}^i \cdot \frac{\partial x^i}{\partial y^k} \circ \pi\right] \cdot \frac{\partial x^k}{\partial y^k} \circ \pi, \\
H_{jk} = \frac{\partial x^c}{\partial y^k} \circ \pi \cdot \frac{\partial x^j}{\partial y^k} \circ \pi \cdot \frac{\partial x^i}{\partial y^c} \circ \pi \cdot \frac{\partial x^k}{\partial y^i} \circ \pi, \\
V_{jk}^c = \frac{\partial x^c}{\partial y^k} \circ \pi \cdot V_{jk} \circ \pi \cdot \frac{\partial x^i}{\partial y^c} \circ \pi \cdot \frac{\partial x^k}{\partial y^i} \circ \pi, \\
V_{jk} = \frac{\partial x^c}{\partial y^k} \circ \pi \cdot V_{jk} \circ \pi \cdot \frac{\partial x^i}{\partial y^c} \circ \pi \cdot \frac{\partial x^k}{\partial y^i} \circ \pi.
\]

Example 47 If \((E, \pi, M) = (F, \nu, N)\) is a vector bundle endowed with the \((\rho, \eta)\)-connection \(((\rho, \eta)\Gamma, \eta)\), then the local real functions

\[
\left(\frac{\partial (\rho, \eta)\Gamma^a}{\partial y^k}, \frac{\partial (\rho, \eta)\Gamma^a}{\partial y^k}, 0, 0\right)
\]

are the components of a distinguished linear \((\rho, \eta)\)-connection for the generalized tangent bundle \(((\rho, \eta)TE, (\rho, \eta)\tau_E, E)\), which will by called the Berwald linear \((\rho, \eta)\)-connection.

The Berwald linear \((Id_{TE}, Id_M)\)-connection are the usual Berwald linear connection.
Corollary 49 In the particular case of Lie algebroids (see [3]), $(\rho, \eta) = (Id_M, Id_M)$,
we obtain
\[
T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s} = \Gamma(\tilde{\rho}, I_M) \left( \tilde{\partial}_C \right) T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s}
+ \rho H_{\alpha} T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s} + \ldots + \rho H_{\alpha}^r T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s}
- \rho H_{\beta} T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s} + \ldots - \rho H_{\beta}^r T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s}
+ \rho H_{b} T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s} + \ldots + \rho H_{b}^r T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s}
- \rho \rho H_{\beta} T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s} + \ldots - \rho \rho H_{\beta}^r T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s}
+ \rho \rho H_{b} T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s} + \ldots + \rho \rho H_{b}^r T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s}
(69)
\]
and
\[
T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s} | c = \Gamma(\tilde{\rho}, I_M) \left( \tilde{\partial}_c \right) T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s}
+ \rho V_{\alpha} T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s} + \ldots + \rho V_{\alpha}^r T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s}
- \rho V_{\beta} T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s} + \ldots - \rho V_{\beta}^r T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s}
+ \rho V_{b} T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s} + \ldots + \rho V_{b}^r T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s}
- \rho V_{\beta} T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s} + \ldots - \rho V_{b} T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s}
+ \rho V_{b}^r T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s} + \ldots + \rho V_{b} T_{\beta_1, \ldots, \beta_{p-1}, \beta_p b_1, \ldots, b_s}
(70)
\]
In the classical case, \((\rho, \eta, h) = (I_{TE}, I_M, I_M)\), we obtain
\[
T_{j_1, \ldots, j_{p-1}, j_p b_1, \ldots, b_s} = \delta_k \left( T_{j_1, \ldots, j_{p-1}, j_p b_1, \ldots, b_s} \right)
+ H_{jk} T_{j_1, \ldots, j_{p-1}, j_p b_1, \ldots, b_s} + \ldots + H_{jk}^r T_{j_1, \ldots, j_{p-1}, j_p b_1, \ldots, b_s}
- H_{jk} T_{j_1, \ldots, j_{p-1}, j_p b_1, \ldots, b_s} + \ldots - H_{jk}^r T_{j_1, \ldots, j_{p-1}, j_p b_1, \ldots, b_s}
+ H_{jk} T_{j_1, \ldots, j_{p-1}, j_p b_1, \ldots, b_s} + \ldots + H_{jk}^r T_{j_1, \ldots, j_{p-1}, j_p b_1, \ldots, b_s}
- H_{jk} T_{j_1, \ldots, j_{p-1}, j_p b_1, \ldots, b_s} + \ldots - H_{jk}^r T_{j_1, \ldots, j_{p-1}, j_p b_1, \ldots, b_s}
(71)
\]
and
\[
T_{i_1, \ldots, i_{p-1}, i_p b_1, \ldots, b_s} | c = \delta_k \left( T_{i_1, \ldots, i_{p-1}, i_p b_1, \ldots, b_s} \right)
+ V_{i_1} T_{i_1, \ldots, i_{p-1}, i_p b_1, \ldots, b_s} + \ldots + V_{i_1}^r T_{i_1, \ldots, i_{p-1}, i_p b_1, \ldots, b_s}
- V_{i_1} T_{i_1, \ldots, i_{p-1}, i_p b_1, \ldots, b_s} + \ldots - V_{i_1}^r T_{i_1, \ldots, i_{p-1}, i_p b_1, \ldots, b_s}
+ V_{i_1} T_{i_1, \ldots, i_{p-1}, i_p b_1, \ldots, b_s} + \ldots + V_{i_1}^r T_{i_1, \ldots, i_{p-1}, i_p b_1, \ldots, b_s}
- V_{i_1} T_{i_1, \ldots, i_{p-1}, i_p b_1, \ldots, b_s} + \ldots - V_{i_1}^r T_{i_1, \ldots, i_{p-1}, i_p b_1, \ldots, b_s}
(72)
\]
Definition 50 If \((E, \pi, M) = (F, \nu, N)\), \((\rho, \eta)\) \(\Gamma\) is a \((\rho, \eta)\)-connection for the vector bundle \((E, \pi, M)\) and
\[
\left( \rho, \eta \right) H_{bc}^a, \left( \rho, \eta \right) \bar{H}_{bc}^a, \left( \rho, \eta \right) V_{bc}^a, \left( \rho, \eta \right) \bar{V}_{bc}^a
\]
26
are the components of a distinguished linear \((\rho, \eta)\)-connection for the generalized tangent bundle \(((\rho, \eta) TE, (\rho, \eta) \tau_{E, E})\) such that

\[
(\rho, \eta) H^{\alpha}_{bc} = (\rho, \eta) \tilde{H}^{\alpha}_{bc} \quad \text{and} \quad (\rho, \eta) V^{\alpha}_{bc} = (\rho, \eta) \tilde{V}^{\alpha}_{bc},
\]

then we will say that the generalized tangent bundle \(((\rho, \eta) TE, (\rho, \eta) \tau_{E, E})\) is endowed with a normal distinguished linear \((\rho, \eta)\)-connection \(((\rho, \eta) H, (\rho, \eta) V)\) on components \(((\rho, \eta) H^{\alpha}_{bc}, (\rho, \eta) V^{\alpha}_{bc})\).

In the particular case of Lie algebroids, \((\eta, h) = (\text{Id}_M, \text{Id}_M)\), the components of a normal distinguished linear \((\rho, \text{Id}_M)\)-connection \((\rho H, \rho V)\) will be denoted \((\rho H^{\alpha}_{bc}, \rho V^{\alpha}_{bc})\).

In the classical case, \((\rho, \eta, h) = (\text{Id}_T E, \text{Id}_M, \text{Id}_M)\), the components of a normal distinguished linear \((\text{Id}_T M, \text{Id}_M)\)-connection \((H, V)\) will be denoted \((H^{i}_{jk}, V^{i}_{jk})\).

### 6 Mechanical systems

We consider the following diagram:

\[
\begin{array}{ccc}
E & \xrightarrow{\left(E, [], E_h, (\rho, \eta)\right)} & \left(E, [], E_h, (\rho, \eta)\right) \\
\pi \downarrow & & \downarrow \pi \\
M & \xrightarrow{h} & M
\end{array}
\]

where \(\left(E, \pi, M\right), [\cdot], (\rho, \eta)\) is a generalized Lie algebroid.

**Definition 51** The triple

\[
\left(E, \pi, M\right), F_e, (\rho, \eta) \Gamma,
\]

where

\[
F_e = F^a \frac{\partial}{\partial y^a} \in \Gamma(V(\rho, \eta) TE, (\rho, \eta) \tau_{E, E}, E)
\]

is an external force and \((\rho, \eta) \Gamma\) is a \((\rho, \eta)\)-connection for the vector bundle \((E, \pi, M)\), will be called mechanical \((\rho, \eta)\)-system.

**Definition 52** A smooth Lagrange fundamental function on the vector bundle \((E, \pi, M)\) is a mapping \(E \xrightarrow{L} \mathbb{R}\) which satisfies the following conditions:

1. \(L \circ u \in C^\infty(M)\), for any \(u \in \Gamma(E, \pi, M) \setminus \{0\}\);
2. \(L \circ 0 \in C^0(M)\), where 0 means the null section of \((E, \pi, M)\).

Let \(L\) be a Lagrangian defined on the total space of the vector bundle \((E, \pi, M)\).
If \((U, s_U)\) is a local vector \((m + r)\)-chart for \((E, \pi, M)\), then we obtain the following real functions defined on \(\pi^{-1}(U)\):

\[
\begin{align*}
L_i &= \frac{\partial L}{\partial x^i}, \\
L_{ab} &= \frac{\partial^2 L}{\partial x^a \partial y^b}.
\end{align*}
\]

(76)

**Definition 53** If for any vector local \(m+r\)-chart \((U, s_U)\) of \((E, \pi, M)\), we have:

\[
\text{rank} \|L_{ab}(u_x)\| = r,
\]

(77)

for any \(u_x \in \pi^{-1}(U) \setminus \{0_x\}\), then we will say that the Lagrangian \(L\) is regular.

**Proposition 54** If the Lagrangian \(L\) is regular, then for any vector local \(m+r\)-chart \((U, s_U)\) of \((E, \pi, M)\), we obtain the real functions \(\tilde{L}_{ab}\) locally defined by

\[
\begin{array}{c}
\pi^{-1}(U) \\
u_x
\end{array} \xrightarrow{\tilde{L}_{ab}} \mathbb{R} \\
\xrightarrow{\tilde{L}_{ab}(u_x)}
\]

(78)

where \(\|\tilde{L}_{ab}(u_x)\| = \|L_{ba}(u_x)\|^{-1}\) for any \(u_x \in \pi^{-1}(U) \setminus \{0_x\}\).

**Definition 55** A smooth Finsler fundamental function on the vector bundle \((E, \pi, M)\) is a mapping \(E \rightarrow \mathbb{R}_+\) which satisfies the following conditions:

1. \(F \circ u \in C^\infty (M)\), for any \(u \in \Gamma (E, \pi, M) \setminus \{0\}\);
2. \(F \circ \tilde{0} \in C^0 (M)\), where \(0\) means the null section of \((E, \pi, M)\);
3. \(F\) is positively 1-homogenous on the fibres of vector bundle \((E, \pi, M)\);
4. For any vector local \(m+r\)-chart \((U, s_U)\) of \((E, \pi, M)\), the hessian:

\[
\|F_{a'b'}(u_x)\|
\]

(79)

is positively define for any \(u_x \in \pi^{-1}(U) \setminus \{0_x\}\).

**Definition 56** If \(L\) respectively \(F\) is a smooth Lagrange respectively Finsler function, then the triple

\[
((E, \pi, M), F_e, L)
\]

(80)

respectively

\[
((E, \pi, M), F_e, F)
\]

(81)

where \(F_e = F^a \frac{\partial}{\partial y^a} \in \Gamma (V (\rho, \eta) TE, (\rho, \eta) \tau_{E}, E)\) is an external force, is called Lagrange mechanical \((\rho, \eta)\)-system and Finsler mechanical \((\rho, \eta)\)-system, respectively.
Definition 57  Any Lagrange mechanical \((\rho, \text{Id}_M)\)-system and any Finsler mechanical \((\rho, \text{Id}_M)\)-system will be called Lagrange mechanical \(\rho\)-system and Finsler mechanical \(\rho\)-system, respectively.

Any Lagrange mechanical \((\text{Id}_{TM}, \text{Id}_M)\)-system and any Finsler mechanical \((\text{Id}_{TM}, \text{Id}_M)\)-system will be called Lagrange mechanical system and Finsler mechanical system, respectively.

7  \((\rho, \eta)\)-semisprays and \((\rho, \eta)\)-sprays for mechanical \((\rho, \eta)\)-systems

Let \(((E, \pi, M), F, (\rho, \eta)\Gamma)\) be an arbitrary mechanical \((\rho, \eta)\)-system.

Definition 58  The vertical section \(C = y^a \partial_a\) will be called the Liouville section.

A section \(S \in \Gamma ((\rho, \eta)TE, (\rho, \eta)\tau E, E)\) will be called \((\rho, \eta)\)-semispray if there exists an almost tangent structure \(e\) such that \(e(S) = C\).

Let \(g \in \text{Man} (E, E)\) be such that \((g, h)\) is a locally invertible \(\mathbf{B}^v\)-morphism of \((E, \pi, M)\) source and target.

Theorem 59  The section

\[
S = (g^a_b \circ h \circ \pi) y^b \partial_a - 2 (G^a - \frac{1}{4} F^a) \partial_a
\]  

(82)

is a \((\rho, \eta)\)-semispray such that the real local functions \(G^a, a \in 1, n, \) satisfy the following conditions

\[
(\rho, \eta) \Gamma^a_c = (g^b_c \circ h \circ \pi) \frac{\partial(G^a - \frac{1}{4} F^a)}{\partial y^b} - \frac{1}{2} (g^d_c \circ h \circ \pi) y^f L^f_{dc} (g^a_d \circ h \circ \pi) \\
+ \frac{1}{2} (\rho^j_c \circ h \circ \pi) \frac{\partial(g^a_j \circ h \circ \pi)}{\partial x^j} y^e (g^a_c \circ h \circ \pi) \\
- \frac{1}{2} (g^b_c \circ h \circ \pi) y^e (\rho^j_c \circ h \circ \pi) \frac{\partial(g^a_j \circ h \circ \pi)}{\partial x^j}
\]

(83)

In addition, we remark that the local real functions

\[
(\rho, \eta) \hat{\Gamma}^a_c = (g^b_c \circ h \circ \pi) \frac{\partial G^a}{\partial y^b} - \frac{1}{2} (g^d_c \circ h \circ \pi) y^f L^f_{dc} (g^a_d \circ h \circ \pi) \\
+ \frac{1}{2} (\rho^j_c \circ h \circ \pi) \frac{\partial(g^a_j \circ h \circ \pi)}{\partial x^j} y^e (g^a_c \circ h \circ \pi) \\
- \frac{1}{2} (g^b_c \circ h \circ \pi) y^e (\rho^j_c \circ h \circ \pi) \frac{\partial(g^a_j \circ h \circ \pi)}{\partial x^j}
\]

(84)

are the components of a \((\rho, \eta)\)-connection \((\rho, \eta) \hat{\Gamma}\) for the vector bundle \((E, \pi, M)\).

The \((\rho, \eta)\)-semispray \(S\) will be called the canonical \((\rho, \eta)\)-semispray associated to mechanical \((\rho, \eta)\)-system \(((E, \pi, M), F, (\rho, \eta)\Gamma)\) and from locally invertible \(\mathbf{B}^v\)-morphism \((g, h)\).
Proof. We consider the Mod-endomorphism

\[ \Gamma((\rho, \eta) TE, (\rho, \eta) e, E) \xrightarrow{p} \Gamma((\rho, \eta) TE, (\rho, \eta) e, E) \]

\[ X \quad \longmapsto \quad \mathcal{J}(g, h) [S, X]|_{(\rho, \eta) TE} - [S, \mathcal{J}(g, h) X]|_{(\rho, \eta) TE}. \]

Let \( X = Z^a \partial_a + Y^a \partial_a \) be an arbitrary section. Since

\[ [S, X]|_{(\rho, \eta) TE} = \left[ (g^a \circ h \circ \pi \cdot y^c) \partial_a, Z^b \partial_b \right]|_{(\rho, \eta) TE} + \left[ (g^a \circ h \circ \pi \cdot y^c) \partial_a, Y^b \partial_b \right]|_{(\rho, \eta) TE} \]

\[ - \left[ 2 \left( G^a - \frac{1}{4} F^a \right) \partial_a, Z^b \partial_b \right]|_{(\rho, \eta) TE} - \left[ 2 \left( G^a - \frac{1}{4} F^a \right) \partial_a, Y^b \partial_b \right]|_{(\rho, \eta) TE} \]

and

\[ \left[ (g^a \circ h \circ \pi \cdot y^c) \partial_a, Z^b \partial_b \right]|_{(\rho, \eta) TE} = (g^a \circ h \circ \pi) y^c \left( \rho^i_a \circ h \circ \pi \right) \frac{\partial Z^c}{\partial x^i} \partial_c \]

\[ - Z^b \left( \rho^i_a \circ h \circ \pi \right) \frac{\partial (g^c \circ h \circ \pi)}{\partial x^j} y^c \partial_c \]

\[ + (g^a \circ h \circ \pi) y^c Z^b L_{ab} \partial_c, \]

\[ \left[ (g^a \circ h \circ \pi \cdot y^c) \partial_a, Y^b \partial_b \right]|_{(\rho, \eta) TE} = (g^a \circ h \circ \pi) y^c \left( \rho^i_a \circ h \circ \pi \right) \frac{\partial Y^c \partial_c}{\partial x^i} \partial_c \]

\[ - Y^b \rho^i_a \circ h \circ \pi \frac{\partial (G^a - \frac{1}{4} F^a)}{\partial x^i} \partial_c, \]

\[ \left[ 2 \left( G^a - \frac{1}{4} F^a \right) \partial_a, Z^b \partial_b \right]|_{(\rho, \eta) TE} = 2 \left( G^a - \frac{1}{4} F^a \right) \frac{\partial Z^c}{\partial y^a} \partial_c \]

\[ - 2 Z^b \rho^i_a \circ h \circ \pi \frac{\partial (G^a - \frac{1}{4} F^a)}{\partial x^i} \partial_c, \]

\[ \left[ 2 \left( G^a - \frac{1}{4} F^a \right) \partial_a, Y^b \partial_b \right]|_{(\rho, \eta) TE} = 2 \left( G^a - \frac{1}{4} F^a \right) \frac{\partial V^c \partial_c}{\partial y^a} \partial_c - 2 Y^b \frac{\partial (G^a - \frac{1}{4} F^a)}{\partial y^b} \partial_c, \]

it results that

\[ \mathcal{J}(g, h) [S, X]|_{(\rho, \eta) TE} = (g^a \circ h \circ \pi) y^c \left( \rho^i_a \circ h \circ \pi \right) \frac{\partial Z^c}{\partial x^i} \left( \tilde{g}^d \circ h \circ \pi \right) \tilde{d} \]

\[ - Z^b \left( \rho^i_a \circ h \circ \pi \right) \frac{\partial (g^c \circ h \circ \pi \cdot y^c)}{\partial x^j} \left( \tilde{g}^d \circ h \circ \pi \right) \tilde{d} \]

\[ + (g^a \circ h \circ \pi) y^c Z^b L_{ab} \left( \tilde{g}^d \circ h \circ \pi \right) \tilde{d} - Y^d \tilde{d} \]

\[ - 2 \left( G^a - \frac{1}{4} F^a \right) \frac{\partial Z^c}{\partial y^a} \left( \tilde{g}^d \circ h \circ \pi \right) ^\partial \tilde{d}. \]
Since

\[ \left[ S, J_{(g,h)} X \right]_{(\rho, \eta)TE} = \left[ (g_a^c \circ h \circ \pi) y^c \partial_a, Z^b (\hat{g}_d^b \circ h \circ \pi) \hat{d}_c \right]_{(\rho, \eta)TE} \]

and

\[ \left[ (g_a^c \circ h \circ \pi) y^c \partial_a, Z^b (\hat{g}_d^b \circ h \circ \pi) \hat{d}_c \right]_{(\rho, \eta)TE} = -Z^d \hat{d}_d + (g_a^c \circ h \circ \pi) y^c \left( \rho_a^i \circ h \circ \pi \right) \frac{\partial Z^b}{\partial x^i} \left( \hat{g}_d^b \circ h \circ \pi \right) \hat{d}_d \]

\[ - (g_a^c \circ h \circ \pi) y^c \left( \rho_a^i \circ h \circ \pi \right) Z^b \frac{\partial \left( \hat{g}_d^b \circ h \circ \pi \right)}{\partial y^c} \hat{d}_d \]

\[ 2 \left( G^a - \frac{1}{4} F^a \right) \partial_a, Z^b \left( \hat{g}_d^b \circ h \circ \pi \right) \hat{d}_c \]

\[ = 2 \left( G^a - \frac{1}{4} F^a \right) \frac{\partial Z^b}{\partial y^c} \left( \hat{g}_d^b \circ h \circ \pi \right) \hat{d}_d - Z^b \left( \hat{g}_d^b \circ h \circ \pi \right) \frac{\partial 2 \left( G^d - \frac{1}{4} F^d \right)}{\partial y^c} \hat{d}_d \]

it results that

\[ \left[ S, J_{(g,h)} X \right]_{(\rho, \eta)TE} = -Z^d \hat{d}_d + (g_a^c \circ h \circ \pi) y^c \left( \rho_a^i \circ h \circ \pi \right) \frac{\partial Z^b}{\partial x^i} \left( \hat{g}_d^b \circ h \circ \pi \right) \hat{d}_d \]

\[- (g_a^c \circ h \circ \pi) y^c \left( \rho_a^i \circ h \circ \pi \right) Z^b \frac{\partial \left( \hat{g}_d^b \circ h \circ \pi \right)}{\partial y^c} \hat{d}_d \]

\[-2 \left( G^a - \frac{1}{4} F^a \right) \frac{\partial Z^b}{\partial y^c} \left( \hat{g}_d^b \circ h \circ \pi \right) \hat{d}_d \]

\[+ Z^b \left( \hat{g}_d^b \circ h \circ \pi \right) \frac{\partial 2 \left( G^d - \frac{1}{4} F^d \right)}{\partial y^c} \hat{d}_d. \]

(P2)

Using equalities (P1) and (P2), we obtain:

\[ P \left( Z^a \partial_a + Y^a \partial_a \right) = Z^a \partial_a - Y^a \partial_a + (g_a^c \circ h \circ \pi) y^c Z^b \left( L_{ab}^c \circ h \circ \pi \right) \left( \hat{g}_d^c \circ h \circ \pi \right) \hat{d}_a \]

\[ - Z^b \left( \rho_a^i \circ h \circ \pi \right) \frac{\partial \left( g_a^c \circ h \circ \pi \right)}{\partial x^i} y^c \left( \hat{g}_d^c \circ h \circ \pi \right) \hat{d}_a \]

\[+ (g_a^c \circ h \circ \pi) y^c \left( \rho_a^i \circ h \circ \pi \right) Z^b \frac{\partial \left( \hat{g}_d^c \circ h \circ \pi \right)}{\partial x^i} \hat{d}_a \]

\[ - Z^b \left( \hat{g}_d^c \circ h \circ \pi \right) \frac{\partial 2 \left( G^d - \frac{1}{4} F^d \right)}{\partial y^c} \hat{d}_a \]

After some calculations, it results that P is an almost product structure.

Using the equalities (51) and (52), it results that

\[ P \left( Z^a \dot{\partial}_a + Y^a \dot{\partial}_a \right) = (Id - 2 (\rho, \eta) \Gamma) \left( Z^a \dot{\partial}_a + Y^a \dot{\partial}_a \right), \]
for any $Z^a\dot{\partial}_a + Y^a\dot{\partial}_a \in \Gamma((\rho, \eta) T E, (\rho, \eta) \tau_E, E)$ and we obtain

$$(\rho, \eta) \Gamma \left( Z^a\dot{\partial}_a + Y^a\dot{\partial}_a \right) = Y^d\dot{\partial}_d - \frac{1}{2} (g^b_c \circ h \circ \pi) y^e Z^b \left( L^b_{ac} \circ h \circ \pi \right) \left( \dot{\gamma}^d_c \circ h \circ \pi \right) \dot{\partial}_d$$

$$+ \frac{1}{2} Z^b \left( \rho^d_b \circ h \circ \pi \right) \frac{\partial (g^c_e \circ h \circ \pi)}{\partial x^i} \frac{\partial (\dot{\gamma}^d_c \circ h \circ \pi)}{\partial y^i} \dot{\partial}_d$$

$$- \frac{1}{2} (g^a_c \circ h \circ \pi) y^e \left( \rho^d_a \circ h \circ \pi \right) Z^b \frac{\partial (g^c_e \circ h \circ \pi)}{\partial y^e} \dot{\partial}_d$$

$$+ Z^b \left( \dot{\gamma}^c_i \circ h \circ \pi \right) \frac{\partial (\dot{\gamma}^d_c \circ h \circ \pi)}{\partial y^e} \dot{\partial}_d.$$

Since

$$(\rho, \eta) \Gamma \left( Z^a\dot{\partial}_a + Y^a\dot{\partial}_a \right) = \left( Y^d + (\rho, \eta) \Gamma^d_b Z^b \right) \dot{\partial}_d$$

it results the relations (83). In addition, since

$$(\rho, \eta) \Gamma^a_c = (\rho, \eta) \Gamma^a_c + \frac{1}{4} \dot{\gamma}^d_c \circ h \circ \pi \frac{\partial F^a_c}{\partial y^e}$$

and

$$(\rho, \eta) \Gamma^a_c = (\rho, \eta) \Gamma^a_c + \frac{1}{2} \dot{\gamma}^d_c \circ h \circ \pi \frac{\partial F^a_c}{\partial y^e}$$

$$= M^a_c \circ \pi \left( \rho^d_c \circ h \circ \pi \cdot \frac{\partial M^a_c}{\partial x^i} y^b + (\rho, \eta) \Gamma^a_c \right) M^c_b \circ h \circ \pi$$

$$+ M^a_c \circ \pi \left( \frac{1}{4} \dot{\gamma}^d_c \circ h \circ \pi \cdot \frac{\partial F^a_c}{\partial y^b} \right) M^c_b \circ h \circ \pi$$

$$= M^a_c \circ \pi \left( \rho^d_c \circ h \circ \pi \cdot \frac{\partial M^a_c}{\partial x^i} y^b + (\rho, \eta) \Gamma^a_c + \frac{1}{4} \dot{\gamma}^d_c \circ h \circ \pi \cdot \frac{\partial F^a_c}{\partial y^b} \right) M^c_b \circ h \circ \pi$$

$$= M^a_c \circ \pi \left( \rho^d_c \circ h \circ \pi \cdot \frac{\partial M^a_c}{\partial x^i} y^b + (\rho, \eta) \Gamma^a_c \right) M^c_b \circ h \circ \pi$$

it results the conclusion of the theorem. ■

**Remark 60** If $\eta = Id_M$, $(g, h) = (Id_E, Id_M)$ and $F_e = 0$, then we obtain the canonical semispray associated to $\rho$-connection $\rho \Gamma$ presented in [3], pp131-132 and [4].

If $(\rho, \eta) = (Id_{TM}, Id_M)$, $(g, h) = (Id_E, Id_M)$ and $F_e \neq 0$, then we obtain the canonical semispray associated to connection $\Gamma$ presented in [5].

In particular, if $(\rho, \eta) = (Id_{TM}, Id_M)$, $(g, h) = (Id_E, Id_M)$, and $F_e = 0$, then we obtain the classical canonical semispray associated to connection $\Gamma$.

Using Theorem 59 we obtain the following:

**Theorem 61** The following properties hold good:
follows:

The tensor of integrability of the $(\rho, \eta)$-connection $(\tilde{\Gamma})$ is as follows:

\[
(\rho, \eta, h) \tilde{R}^{a}_{cd} = (\rho, \eta) \tilde{R}^{a}_{cd} + \frac{1}{4} \left( \tilde{g}^{a}_{\delta} \circ h \circ \pi \frac{\partial F^{a}}{\partial y^{f}} \tilde{g}^{f}_{\delta} \circ h \circ \pi \frac{\partial F^{a}}{\partial y^{f}} |_{c} - \tilde{g}^{a}_{\delta} \circ h \circ \pi \frac{\partial F^{a}}{\partial y^{f}} \tilde{g}^{f}_{\delta} \circ h \circ \pi \frac{\partial F^{a}}{\partial y^{f}} |_{d} \right) + \frac{1}{16} \left( \tilde{g}^{a}_{\delta} \circ h \circ \pi \frac{\partial F^{b}}{\partial y^{f}} \tilde{g}^{f}_{\delta} \circ h \circ \pi \frac{\partial^{2} F^{a}}{\partial y^{f} \partial y^{f}} - \tilde{g}^{a}_{\delta} \circ h \circ \pi \frac{\partial F^{b}}{\partial y^{f}} \tilde{g}^{f}_{\delta} \circ h \circ \pi \frac{\partial^{2} F^{a}}{\partial y^{f} \partial y^{f}} \right) + \frac{1}{4} \left( L^{f}_{cd} \circ h \circ \pi \right) \left( \tilde{g}^{a}_{\delta} \circ h \circ \pi \right) \frac{\partial F^{a}}{\partial y^{f}},
\]

where $|_{c}$ is the $h$-covariant derivation with respect to the normal Berwald linear $(\rho, \eta)$-connection $\tilde{\Gamma}$.

Proof. Since

\[
(\rho, \eta, h) \tilde{R}^{a}_{cd} = \Gamma (\tilde{\rho}, Id_{E}) \left( \frac{\partial}{\partial x^{c}} \right) \left( (\rho, \eta) \tilde{\Gamma}^{a}_{d} \right) - \Gamma (\tilde{\rho}, Id_{E}) \left( \frac{\partial}{\partial x^{c}} \right) \left( (\rho, \eta) \tilde{\Gamma}^{a}_{d} \right) + L^{c}_{d} \circ h \circ \pi (\rho, \eta) \tilde{\Gamma}^{a}_{c},
\]

and

\[
\Gamma (\tilde{\rho}, Id_{E}) \left( \frac{\partial}{\partial x^{c}} \right) \left( (\rho, \eta) \tilde{\Gamma}^{a}_{d} \right) = \Gamma (\tilde{\rho}, Id_{E}) \left( \frac{\partial}{\partial x^{c}} \right) \left( (\rho, \eta) \tilde{\Gamma}^{a}_{d} \right) + \frac{1}{4} \Gamma (\tilde{\rho}, Id_{E}) \left( \frac{\partial}{\partial x^{c}} \right) \left( \tilde{g}^{a}_{\delta} \circ h \circ \pi \frac{\partial F^{a}}{\partial y^{f}} \tilde{g}^{f}_{\delta} \circ h \circ \pi \frac{\partial F^{a}}{\partial y^{f}} \right) - \frac{1}{4} \tilde{g}^{a}_{\delta} \circ h \circ \pi \frac{\partial F^{a}}{\partial y^{f}} \frac{\partial}{\partial y^{f}} \left( (\rho, \eta) \tilde{\Gamma}^{a}_{d} \right) - \frac{1}{16} \tilde{g}^{a}_{\delta} \circ h \circ \pi \frac{\partial F^{a}}{\partial y^{f}} \frac{\partial}{\partial y^{f}} \left( \tilde{g}^{a}_{\delta} \circ h \circ \pi \frac{\partial F^{a}}{\partial y^{f}} \right),
\]

33
\[
\Gamma (\tilde{\rho}, Id_E) \left( \tilde{\delta}_d \right) \left( (\rho, \eta) \tilde{\Gamma}_c \right) = \Gamma (\tilde{\rho}, Id_E) \left( \tilde{\delta}_d \right) \left( (\rho, \eta) \Gamma_c^a \right) \\
+ \frac{1}{4} \Gamma (\tilde{\rho}, Id_E) \left( \tilde{\delta}_d \right) \left( g^a_c \circ h \circ \pi \frac{\partial F^a}{\partial y^f} \right) \\
- \frac{1}{4} \tilde{g}_d^c \circ h \circ \pi \frac{\partial F^i}{\partial y^f} \frac{\partial}{\partial y^f} ((\rho, \eta) \Gamma_c^a) \\
- \frac{1}{16} \tilde{g}_d^c \circ h \circ \pi \frac{\partial F^i}{\partial y^f} \frac{\partial}{\partial y^f} \left( g^c_\circ h \circ \pi \frac{\partial F^a}{\partial y^f} \right),
\]

\[
L_{c,d}^a \circ h \circ \pi \cdot (\rho, \eta) \tilde{\Gamma}_c^a = L_{c,d}^a \circ h \circ \pi \cdot (\rho, \eta) \Gamma_c^a \\
+ L_{c,d}^a \circ h \circ \pi \cdot \left( \tilde{g}_c^a \circ h \circ \pi \frac{\partial F^a}{\partial y^f} \right)
\]

It results the conclusion of the theorem. ■

**Proposition 64** If \(S\) is the canonical \((\rho, \eta)\)-semispray associated to the mechanical \((\rho, \eta)\)-system \(((E, \pi, M), F_\epsilon, (\rho, \eta) \Gamma)\) and from \(B^\eta\)-morphism \((g, h)\), then

\[
2G^a = 2G^a M^a_a \circ h \circ \pi - (g^a_b \circ h \circ \pi) y^b \left( \rho^a_a \circ h \circ \pi \right) \frac{\partial F^a}{\partial x^i}.
\]

**Proof.** Since the Jacobian matrix of coordinates transformation is

\[
\begin{pmatrix}
M^a_{\circ h \circ \pi} & 0 \\
\rho^a_a \circ (h \circ \pi) \frac{\partial M^a_{\circ h \circ \pi}}{\partial x^i} y^a & M^a_{\circ h \circ \pi}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
M^a_{\circ h \circ \pi} & 0 \\
\rho^a_a \circ (h \circ \pi) \frac{\partial y^a}{\partial x^i} M^a_{\circ h \circ \pi}
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
M^a_{\circ h \circ \pi} & 0 \\
\rho^a_a \circ (h \circ \pi) \frac{\partial y^a}{\partial x^i} M^a_{\circ h \circ \pi}
\end{pmatrix}
\cdot \begin{pmatrix}
(g^a_b \circ h \circ \pi) y^b \\
-2 \left( G^a - \frac{1}{4} F^a \right)
\end{pmatrix}
\]

\[= \begin{pmatrix}
(g^a_b \circ h \circ \pi) y^b \\
-2 \left( G^a - \frac{1}{4} F^a \right)
\end{pmatrix},
\]

the conclusion results immediately. ■

In the following, we consider a differentiable curve \(I \xrightarrow{\phi} M\) and its \((g, h)\)-lift \(\tilde{\phi}\).

**Definition 65** If it is verifies the following equality:

\[
\frac{d\phi(t)}{dt} = \Gamma (\tilde{\rho}, Id_E) S (\circ \phi(t)),
\]

then we say that the curve \(\phi\) is an integral curve of the \((\rho, \eta)\)-semispray \(S\) of the mechanical \((\rho, \eta)\)-system \(((E, \pi, M), F_\epsilon, (\rho, \eta) \Gamma)\).

**Theorem 66** All \((g, h)\)-lifts solutions of the equations:

\[
\frac{dG^a}{dt} + 2G^a \circ u (c, \phi) (x (t)) = \frac{1}{2} F^a \circ u (c, \phi) (x (t)), \ a \in \Gamma, \rho,
\]

where \(x (t) = (\eta \circ h \circ c) (t)\), are integral curves of the canonical \((\rho, \eta)\)-semispray associated to mechanical \((\rho, \eta)\)-system \(((E, \pi, M), F_\epsilon, (\rho, \eta) \Gamma)\) and from locally invertible \(B^\eta\)-morphism \((g, h)\).
Proof. Since the equality
\[ \frac{d\dot{c}(t)}{dt} = \Gamma(\tilde{p}, Id_E) S(\dot{c}(t)) \]
is equivalent to
\[ \frac{d}{dt}((\eta \circ h \circ c)^{\dot{c}}(t), y^{\dot{a}}(t)) = \left(\rho_o^{\dot{a}} \circ \eta \circ h \circ c(t) g^\alpha_h \circ h \circ c(t) y^\beta(t), -2 \left(G^a - \frac{1}{4} F^a\right) ((\eta \circ h \circ c)^{\dot{c}}(t), y^{\dot{a}}(t))\right), \]
it results
\[ \frac{dy^{\dot{a}}(t)}{dt} + 2 G^a (x^i(t), y^{\dot{a}}(t)) = \frac{1}{2} F^a (x^i(t), y^{\dot{a}}(t)), \quad a \in \overline{1,n}, \]
\[ \frac{dx^i(t)}{dt} = \rho_o^{\dot{a}} \circ \eta \circ h \circ c(t) g^\alpha_h \circ h \circ c(t) y^\beta(t), \]
where \(x^i(t) = (\eta \circ h \circ c)^{\dot{c}}(t)\). □

**Definition 67** If \( S \) is a \((\rho, \eta)\)-semispray, then the vector field
\[ [C, S]_{(\rho, \eta)T_E} - S \]
will be called the derivation of \((\rho, \eta)\)-semispray \( S \).

The \((\rho, \eta)\)-semispray \( S \) will be called \((\rho, \eta)\)-spray if the following conditions are verified:

1. \( S \circ 0 \) is differentiable of class \( C^1 \) where \( 0 \) is the null section;
2. Its derivation is the null vector field.

The \((\rho, \eta)\)-semispray \( S \) will be called quadratic \((\rho, \eta)\)-spray if there are verified the following conditions:

1. \( S \circ 0 \) is differentiable of class \( C^2 \), where \( 0 \) is the null section;
2. Its derivation is the null vector field.

In particular, if \((\rho, \eta) = (id_{TM}, Id_M)\) and \((g, h) = (Id_E, Id_M)\), then we obtain the spray and the quadratic spray which is similar with the classical spray and quadratic spray.

**Theorem 68** If \( S \) is the canonical \((\rho, \eta)\)-spray associated to mechanical \((\rho, \eta)\)-system \(((E, \pi, M), F_e, (\rho, \eta) \Gamma)\) and from locally invertible \( B^\ast \)-morphism \((g, h)\), then
\[ 2 \left(G^a - \frac{1}{4} F^a\right) = (\rho, \eta) \Gamma^a_c \left(g^c_j \circ h \circ \pi \right) y^f \]
\[ + \frac{1}{2} \left(g^d_l \circ h \circ \pi \right) y^c \left(L^a_{dc} \circ h \circ \pi \right) \left(g^e_h \circ h \circ \pi \right) \left(g^c_j \circ h \circ \pi \right) y^f \]
\[ - \frac{1}{2} \left(\rho_o^{\dot{a}} \circ h \circ \pi \right) \frac{\partial (g^c_j \circ h \circ \pi)}{\partial x^c} y^f \left(g^e_h \circ h \circ \pi \right) \left(g^c_j \circ h \circ \pi \right) y^f \]
\[ + \frac{1}{2} \left(g^e_h \circ h \circ \pi \right) y^c \left(\rho_o^{\dot{a}} \circ h \circ \pi \right) \frac{\partial (g^c_j \circ h \circ \pi)}{\partial x^c} \left(g^c_j \circ h \circ \pi \right) y^f \]
(94)
We obtain the spray

\[
S = (g^b \circ h \circ \pi) y^b \dot{\partial}_a - (\rho, \eta) \Gamma^a_c \left( g^b_j \circ h \circ \pi \right) y^j \dot{\partial}_a
- \frac{1}{2} \left( g^c_e \circ h \circ \pi \right) y^c \left( L^b_{de} \circ h \circ \pi \right) \left( \ddot{g}^a_e \circ h \circ \pi \right) \left( g^f_j \circ h \circ \pi \right) y^f \dot{\partial}_a
+ \frac{1}{2} \left( \rho^c_e \circ h \circ \pi \right) \frac{\partial \left( g^a_{\text{cho}} \pi \right)}{\partial x^j} \left( \ddot{g}^a_e \circ h \circ \pi \right) \left( g^f_j \circ h \circ \pi \right) y^f \dot{\partial}_a
- \frac{1}{2} \left( g^c_e \circ h \circ \pi \right) y^c \left( \rho^b_e \circ h \circ \pi \right) \frac{\partial \left( g^a_{\text{cho}} \pi \right)}{\partial x^j} \left( g^f_j \circ h \circ \pi \right) y^f \dot{\partial}_a
\]

This spray will be called the canonical \((\rho, \eta)\)-spray associated to mechanical system \(((E, \pi, M), F_c, (\rho, \eta) \Gamma)\) and from locally invertible \(B^\pi\)-morphism \((g, h)\).

In particular, if \((\rho, \eta) = (id_{TM}, Id_M)\) and \((g, h) = (Id_E, Id_M)\), then we get the canonical spray associated to connection \(\Gamma\) which is similar with the classical canonical spray associated to connection \(\Gamma\).

**Proof.** Since

\[
[C, S]_{(\rho, \eta)TE} = \left[ y^a \dot{\partial}_a, \left( g^b_c \circ h \circ \pi \cdot y^c \right) \dot{\partial}_b \right]_{(\rho, \eta)TE} - 2 \left[ y^a \dot{\partial}_a, \left( G^b - \frac{1}{4} F^b \right) \dot{\partial}_b \right]_{(\rho, \eta)TE},
\]

\[
\left[ y^a \dot{\partial}_a, \left( g^b_c \circ h \circ \pi \cdot y^c \right) \dot{\partial}_b \right]_{(\rho, \eta)TE} = \left( g^b_e \circ h \circ \pi \cdot y^c \right) \dot{\partial}_b
\]

and

\[
\left[ y^a \dot{\partial}_a, \left( G^b - \frac{1}{4} F^b \right) \dot{\partial}_b \right]_{(\rho, \eta)TE} = y^a \frac{\partial \left( G^b - \frac{1}{4} F^b \right)}{\partial y^a} \dot{\partial}_b - \left( G^b - \frac{1}{4} F^b \right) \dot{\partial}_b
\]

it results that

\[
[C, S]_{(\rho, \eta)TE} - S = 2 \left( -y^j \frac{\partial \left( G^a - \frac{1}{4} F^a \right)}{\partial y^j} + 2 \left( G^a - \frac{1}{4} F^a \right) \right) \dot{\partial}_a
\]

Using equality \(S3\), it results that

\[
\frac{\partial \left( G^a - \frac{1}{4} F^a \right)}{\partial y^j} = (\rho, \eta) \Gamma^a_c \left( g^b_j \circ h \circ \pi \right)
+ \frac{1}{2} \left( g^c_e \circ h \circ \pi \right) y^c \left( L^b_{de} \circ h \circ \pi \right) \left( \ddot{g}^a_e \circ h \circ \pi \right) \left( g^f_j \circ h \circ \pi \right)
- \frac{1}{2} \left( \rho^c_e \circ h \circ \pi \right) \frac{\partial \left( g^a_{\text{cho}} \pi \right)}{\partial x^j} \left( \ddot{g}^a_e \circ h \circ \pi \right) \left( g^f_j \circ h \circ \pi \right)
+ \frac{1}{2} \left( g^c_e \circ h \circ \pi \right) y^c \left( \rho^b_e \circ h \circ \pi \right) \frac{\partial \left( g^a_{\text{cho}} \pi \right)}{\partial x^j} \left( g^f_j \circ h \circ \pi \right)
\]

Using equalities \((S1)\) and \((S2)\), it results the conclusion of the theorem. 

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Theorem 69 All \((g, h)\)-lifts solutions of the following system of equations:

\[
\frac{dy^a}{dt} + (\rho, \eta) \Gamma^a_c (g^c_j \circ h \circ \pi) y^j \\
+ \frac{1}{2} (g^d_c \circ h \circ \pi) y^e (L^{bd}_{ce} \circ h \circ \pi) (\tilde{g}^a_e \circ h \circ \pi) (g^f_i \circ h \circ \pi) y^f \\
- \frac{1}{2} (\rho^e_i \circ h \circ \pi) \frac{\partial (\tilde{g}^a_e \circ h \circ \pi)}{\partial x^i} y^e (\tilde{g}^c_j \circ h \circ \pi) (g^f_i \circ h \circ \pi) y^f \\
+ \frac{1}{2} (g^b_c \circ h \circ \pi) y^e (\rho^b_e \circ h \circ \pi) \frac{\partial (\tilde{g}^a_e \circ h \circ \pi)}{\partial x^e} (g^c_j \circ h \circ \pi) y^f = 0,
\]

are the integral curves of canonical \((\rho, \eta)\)-spray associated to mechanical \((\rho, \eta)\)-system \(((E, \pi, M), F_e, (\rho, \eta) \Gamma)\) and from locally invertible \(B^\Sigma\)-morphism \((g, h)\).

8 A Lagrangian formalism for Lagrange mechanical \((\rho, \eta)\)-systems

Let \(((E, \pi, M), F_e, L)\) be an arbitrary Lagrange mechanical \((\rho, \eta)\)-system.

Let \((d\tilde{z}^a, d\tilde{y}^a)\) be the natural dual \((\rho, \eta)\)-base of the natural \((\rho, \eta)\)-base \((\tilde{\partial}_a, \tilde{\partial}_a)\).

It is very important to remark that the 1-forms \(d\tilde{z}^a, d\tilde{y}^a, a \in \overline{1, p}\) are not the differentials of coordinates functions as in the classical case, but we will use the same notations. In this case

\((d\tilde{z}^a) \neq d^{(\rho, \eta)TE} (\tilde{z}^a)\),

where \(d^{(\rho, \eta)TE}\) is the exterior differentiation operator associated to exterior differential \(F(E)\)-algebra

\((\Lambda ((\rho, \eta) TE, (\rho, \eta) \tau E, E), +, \cdot, \wedge)\).

Let \(L\) be a regular Lagrangian and let \((g, h)\) be a locally invertible \(B^\Sigma\)-morphism of \((E, \pi, M)\) source and target.

Definition 70 The 1-form

\[
\theta_L = (\tilde{g}^e_a \circ h \circ \pi \cdot L_e) d\tilde{z}^a
\]

will be called the 1-form of Poincaré-Cartan type associated to the Lagrangian \(L\) and to the locally invertible \(B^\Sigma\)-morphism \((g, h)\).

Easily, we obtain:

\[
\theta_L (\tilde{\partial}_a) = \tilde{g}^e_a \circ h \circ \pi \cdot L_e, \quad \theta_L (\tilde{\partial}_b) = 0.
\]
**Definition 71** The 2-form

\[ \omega_L = d^{(\rho, \eta)TE} \theta_L \]

will be called the 2-form of Poincaré-Cartan type associated to the Lagrangian \( L \) and to the locally invertible \( B^\nu \)-morphism \((g, h)\).

By the definition of \( d^{(\rho, \eta)TE} \), we obtain:

\[ \omega_L (U, V) = \Gamma (\tilde{\rho}, \text{Id}_E) (U) (\theta_L (V)) \]

\[ - \Gamma (\tilde{\rho}, \text{Id}_E) (V) (\theta_L (U)) - \theta_L ([U, V]_{(\rho, \eta)TE}), \]

for any \( U, V \in \Gamma ((\rho, \eta)TE, (\rho, \eta)\tau_E, E) \).

**Definition 72** The real function

\[ E_L = y^a L_a - L \]

will be called the energy of regular Lagrangian \( L \).

**Theorem 73** The equation

\[ i_S (\omega_L) = -d^{(\rho, \eta)TE} (E_L), \quad S \in \Gamma ((\rho, \eta)TE, (\rho, \eta)\tau_E, E), \]

has an unique solution \( S_L (g, h) \) of the type:

\[ (g^a_c \circ h \circ \pi) y^f \tilde{\partial}_a - 2 \left( G^a - \frac{1}{4} F^a \right) \tilde{\partial}_a, \]

where

\[ -2 \left( G^a - \frac{1}{4} F^a \right) = E_b (L, \rho, g, h) \tilde{L}^ae \]

\[ (g^a_c \circ h \circ \pi) y^f \tilde{\partial}_a, \]

and

\[ E_b (L, \rho, g, h) = (\rho^b_c \circ h \circ \pi) L_1 - (\rho^b_c \circ h \circ \pi) y^a L_{1a} \]

\[ - \left( g^d_f \circ h \circ \pi \right) y^f \left( \rho^d_c \circ h \circ \pi \right) \frac{\partial (y^e_c \circ h \circ \pi) L_e}{\partial x^i} \]

\[ + \left( g^d_f \circ h \circ \pi \right) y^f \left( \rho^d_c \circ h \circ \pi \right) \frac{\partial (y^e_c \circ h \circ \pi) L_e}{\partial x^i} \]

\[ + \left( g^d_f \circ h \circ \pi \right) y^f \left( L_{db}^c \circ h \circ \pi \right) \left( \tilde{g}^d_c \circ h \circ \pi \right) L_e \]

\( S_L (g, h) \) will be called the canonical \((\rho, \eta)\)-semispray associated to Lagrange mechanical \((\rho, \eta)\)-system \((E, \pi, M), F_c, L)\) and from locally invertible \( B^\nu \)-morphism \((g, h)\).

**Proof.** We obtain that

\[ i_S (\omega_L) = -d^{(\rho, \eta)TE} (E_L) \]

if and only if

\[ \omega_L (S, X) = -\Gamma (\tilde{\rho}, \text{Id}_E) (X) (E_L), \]
for any $X \in \Gamma \left( (\rho, \eta) TE, (\rho, \eta) \tau_E, E \right)$. Particularly, we obtain:

$$\omega_L \left( S, \hat{\partial}_b \right) = -\Gamma \left( \hat{\partial}, \text{Id}_E \right) \left( \hat{\partial}_b \right) (E_L).$$

If we expand this equality, we obtain

$$\left( g^d_j \circ h \circ \pi \right) y^l \left[ \left( \rho^d_a \circ \pi \right) \frac{\partial (L_b \circ h \circ \pi)}{\partial x^l} - \left( \rho^d_b \circ \pi \right) \frac{\partial (L_c \circ h \circ \pi)}{\partial x^l} \right] - \left( L_{ab} \circ h \circ \pi \right) \left( \tilde{g}^c_e \circ h \circ \pi \right) \left( \frac{1}{4} F^a \right) \left( \tilde{g}^c_a \circ h \circ \pi \right) \cdot L_{eb}$$

$$= \rho^d_b \circ h \circ \pi \cdot L_i - \left( \rho^d_b \circ h \circ \pi \right) \frac{\partial (y^a L_i)}{\partial x^i}.$$

After some calculations, we obtain the conclusion of the theorem. □

**Remark 74** If $F_c = 0$ and $\eta = \text{Id}_M$, then

$$E_b \left( L, \rho, \text{Id}_E, \text{Id}_M \right) = \left( \rho^d_i \circ \pi \right) L_i - \left( \rho^d_b \circ \pi \right) y^d L_{id} + y^d \left( L_{ab} \circ \pi \right) L_{eb}$$

and $S_L \left( \text{Id}_E, \text{Id}_M \right)$ is the canonical $\rho$-semispray associated to regular Lagrangian $L$ which is similar with the semispray presented in [25] by E. Martínez. (see also [27], [26]) The canonical $\rho$-semispray $S_L \left( \text{Id}_E, \text{Id}_M \right)$ is the same $\rho$-semispray presented in [3], [4].

In addition, if $F_c \neq 0$ and $(\rho, \eta) = (\text{Id}_TM, \text{Id}_M)$, then $S_L \left( \text{Id}_E, \text{Id}_M \right)$ will be the canonical semispray presented in [5], [6] by I. Bucătaru and R. Miron.

**Theorem 75** If $S_L \left( g, h \right)$ is the canonical $(\rho, \eta)$-semispray associated to Lagrange mechanical $(\rho, \eta)$-system $((E, \pi, M), F_c, L)$ and from locally invertible $B^r$-morphism $(g, h)$, then the real local functions

$$\left( \rho, \eta \right) \Gamma^a_c = -\frac{1}{2} \left( \tilde{g}^d_e \circ h \circ \pi \right) \frac{\partial (E_b (L, \rho, g, \tau) \tilde{L}^{ab} \circ \pi)}{\partial y^d}$$

$$- \frac{1}{2} \left( g^d_e \circ h \circ \pi \right) \frac{\partial (L_{bc} \circ h \circ \pi)}{\partial x^d} \left( \tilde{g}^d_f \circ h \circ \pi \right)$$

$$+ \frac{1}{2} \left( \rho^d_a \circ h \circ \pi \right) \frac{\partial \left( \tilde{g}^d_e \circ h \circ \pi \right)}{\partial x^d} \frac{\partial \left( \tilde{g}^d_f \circ h \circ \pi \right)}{\partial x^d}$$

(105)

are the components of a $(\rho, \eta)$-connection $(\rho, \eta) \Gamma$ for the vector bundle $(E, \pi, M)$ which will be called the $(\rho, \eta)$-connection associated to Lagrange mechanical $(\rho, \eta)$-system $((E, \pi, M), F_c, L)$ and from locally invertible $B^r$-morphism $(g, h)$.

In the particular case of Lie algebroids, $\eta = h = \text{Id}_M$ and $g = \text{Id}_E$, we obtain

$$\rho \Gamma^a_c = -\frac{1}{2} \frac{\partial \left( E_b \left( L, \rho, \text{Id}_E, \text{Id}_M \right) \tilde{L}^{ab} \right)}{\partial y^c} - \frac{1}{2} y^b \tilde{L}^{ab} \circ \pi.$$

(106)

**Theorem 76** The parallel $(g, h)$-lifts with respect to $(\rho, \eta)$-connection $(\rho, \eta) \Gamma$ are the integral curves of the canonical $(\rho, \eta)$-semispray associated to mechanical $(\rho, \eta)$-system $((E, \pi, M), F_c, L)$ and from locally invertible $B^r$-morphism $(g, h)$.
Definition 77  The equations

\[ \frac{dy^a}{dt}(t) - \left( E_b(L, \rho, g, h) \tilde{L}^{ae}_{\rho} \circ h \circ \pi \right) \circ u(c, \dot{c})(x(t)) = 0, \quad (107) \]

where \( x(t) = \eta \circ h \circ c(t) \), will be called the equations of Euler-Lagrange type associated to Lagrange mechanical \((\rho, \eta)\)-system \(((E, \pi, M), F_e, L)\) and from locally invertible \(B^\nu\)-morphism \((g, h)\).

The equations

\[ \frac{dy^a}{dt}(t) - \left( E_b(L, \rho, Id_{E}, Id_{M}) \tilde{L}^{ab}_{\rho} \right) \circ u(c, \dot{c})(x(t)) = 0, \quad (108) \]

where \( x(t) = c(t) \), will be called the equations of Euler-Lagrange type associated to Lagrange mechanical \(\rho\)-system \(((E, \pi, M), F_e, L)\).

Remark 78  The integral curves of the canonical \((\rho, \eta)\)-semispray associated to mechanical \((\rho, \eta)\)-system \(((E, \pi, M), F_e, L)\) and from locally invertible \(B^\nu\)-morphism \((g, h)\) are the \((g, h)\)-lifts solutions for the equations of Euler-Lagrange type \((107)\).

Using our theory, we obtain the following

Theorem 79  If \( F \) is a Finsler fundamental function, then the geodesics on the manifold \( M \) are the curves such that the components of their \((g, h)\)-lifts are solutions for the equations of Euler-Lagrange type \((107)\).

Therefore, it is natural to propose to extend the study of the Finsler geometry from the usual Lie algebroid \(((TM, \tau_M, M), [\cdot, \cdot]_{TM}, (Id_{TM}, Id_M))\), to an arbitrary (generalized) Lie algebroid \(((E, \pi, M), [\cdot, \cdot]_{E,h}, (\rho, \eta))\).

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