Structural descriptions of limits of the parabolic Ginzburg-Landau equation on closed manifolds

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Abstract

In the setting of a compact Riemannian manifold of dimension \( N \geq 3 \) we provide a structural description of the limiting behaviour of the energy measures of solutions to the parabolic Ginzburg-Landau equation. In particular, we provide a decomposition of the limiting energy measure into a diffuse part, which is absolutely continuous with respect to the volume measure, and a concentrated part supported on a codimension 2 rectifiable subset. We also demonstrate that the time evolution of the diffuse part is determined by the heat equation while the concentrated part evolves according to a Brakke flow. This paper extends the work of Bethuel, Orlandi, and Smets from [8].

1 Introduction

In this paper we extend the work of Bethuel, Orlandi, and Smets on the parabolic Ginzburg-Landau equation from [8] to the setting of a compact Riemannian manifold \((M, g)\) of dimension \( N \geq 3 \). More specifically, we are interested in providing a detailed description of the limiting behaviour as \( \varepsilon \to 0^+ \) of solutions of the PDE initial value problem

\[
\begin{align*}
\partial_t u_\varepsilon &= \Delta u_\varepsilon + \frac{1}{\varepsilon} u_\varepsilon (1 - |u_\varepsilon|^2) \quad \forall x \in M \text{ and } \forall t > 0 \\
u_\varepsilon(x, 0) &= u_0^\varepsilon(x) \quad \forall x \in M
\end{align*}
\]

(PGL)\(\varepsilon\)

for a given \( u_0^\varepsilon \) which, throughout this paper, we assume satisfies

\[\mathcal{E}_\varepsilon(u_0^\varepsilon) \leq M_0 |\log(\varepsilon)| \quad \text{where } M_0 \text{ is a fixed positive constant } (H_0)\]

and where

\[\mathcal{E}_\varepsilon(u) := \int_M e_\varepsilon(u) \, d\text{vol}_g, \quad e_\varepsilon(u) := \frac{1}{2} |\nabla u|^2 + V_\varepsilon(u) \quad (1.1)\]
\[
V_\varepsilon(x) := \frac{1}{4\varepsilon^2}(1 - |x|^2)^2.
\]

The asymptotics of solutions to the equation \((PGL)_\varepsilon\) has been extensively studied in the setting of Euclidean space. For \(N \geq 3\) it was shown in \([20, 24]\), in a variety of settings, including \(\mathbb{R}^N\) and bounded open subsets of \(\mathbb{R}^N\), that for well-prepared initial data, the energy of solutions to \((PGL)_\varepsilon\) concentrates around a codimension 2 mean curvature flow, as long as that flow remains smooth. It was then shown in \([3]\) that if the limiting energy measure satisfies a lower density bound, then this result may be extended past the formation of singularities, thereby giving a conditional proof of convergence of rescaled energy measures, globally in time, to a codimension 2 Brakke flow – a measure-theoretic weak solution of the mean curvature flow.

Following this work results were obtained in \([26]\), for \(N = 3\) and on a bounded domain, relating a local energy condition with the local absence of vortex behaviour and using this to demonstrate energy concentration on a rectifiable 1-varifold. The relationship between the local energy condition and the absence of vortex behaviour was shown to hold for \(\mathbb{R}^4\) in \([33]\) where the energy is weighted by a Gaussian function.

Finally, this line of research concluded with \([8]\) which, among other improvements, removed the lower density bound imposed in \([3]\), giving an unconditional proof that the concentrated part of a limiting energy measure evolves via a Brakke flow in \(\mathbb{R}^N\), globally in \(t\), for every \(N \geq 3\), and without requiring well-prepared initial data.

The description of the dynamics of the limiting energy measure over \(\mathbb{R}^N\) in \([8]\) raised the question of possible extensions to other settings. One such extension is found in \([27]\) who demonstrated the conclusions of \([8]\) for the parabolic Ginzburg-Landau equation with magnetic potential in \(\mathbb{R}^3\). Related work in the Riemannian setting includes \([31]\) and \([30]\) for the Allen-Cahn equation over a compact Riemannian manifold without boundary as well as \([32]\) which extends the Monotonicity formula to a suitably restricted class of compact Riemannian manifolds, possibly with boundary. Despite these efforts, an extension of the results of \([8]\) to the case of a compact Riemannian manifold without boundary has not been shown.

The main result of this paper is, in the setting of a compact smooth Riemannian manifold \((M, g)\) without boundary, a careful study of the family of energy measures

\[
\mu^t_\varepsilon(x) := \frac{e_\varepsilon(u_\varepsilon(x,t))}{|\log(\varepsilon)|}d\text{vol}_g(x)
\]

for \(t > 0\) as \(\varepsilon \to 0^+\). Of particular note is that we impose no topological or curvature restrictions on \(M\) beyond what is guaranteed by compactness. As a result, our analysis applies to compact manifolds with possibly non-trivial topology. The result of this analysis, stated in Theorem 1.1, is
that the limiting energy decomposes into a diffuse energy and a concentrated vortex energy which do not interact. The evolution of the diffuse energy will be governed by the heat equation while the vortex energy evolves according to a Brakke flow, a measure theoretic formulation of mean curvature flow. More specifically, we have:

**Theorem 1.1.** Let $M$ be of dimension $N \geq 3$ and suppose that $\{u_\varepsilon\}_{\varepsilon \in (0,1)}$ are a family of solutions to (PGL), for corresponding $\varepsilon$ and with respective initial data $\{u^0_\varepsilon\}_{\varepsilon \in (0,1)}$. Let $\mu^t_\varepsilon$ be, for each $t > 0$, the measure on $M$ defined by

$$
\mu^t_\varepsilon := \frac{e^\varepsilon(u_\varepsilon(\cdot,t))}{|\log(\varepsilon)|} \text{dvol}_g.
$$

Then, after perhaps passing to a subsequence $\{u_{\varepsilon_n}\}_{n \in \mathbb{N}}$, there exists a family of limiting measures $\{\mu^t_\ast\}_{t > 0}$ and subsets $\{\Sigma^t_\ast\}_{t > 0}$ in $M$, as well as a function $\Phi_\ast : M \times (0, \infty) \to \mathbb{R}/2\pi\mathbb{Z}$ such that the following properties hold:

1. $\mu^t_{\varepsilon_n} \to \mu^t_\ast$ in $\mathcal{M}(M)$ for each $t > 0$.
2. $\Phi_\ast$ satisfies the heat equation on $M \times (0, \infty)$.
3. For each $t > 0$, the measure $\mu^t_\ast$ can be exactly decomposed as

$$
\mu^t_\ast = \frac{|\nabla \Phi_\ast|^2}{2} \mathcal{H}^N + \nu^t_\ast
$$

where

$$
\nu^t_\ast = \Theta_\ast(x,t) \mathcal{H}^{N-2} \mathbb{L} \Sigma^t_\ast
$$

and where $\Theta_\ast(\cdot,t)$ is a bounded measurable function.
4. There exists a positive function $\eta$ defined on $(0, \infty)$ such that, for $\mathcal{L}^1$-almost every $t > 0$, the set $\Sigma^t_\ast$ is $(N-2)$-rectifiable and

$$
\Theta_\ast(x,t) = \Theta_{N-2}(\mu^t_\ast, x) = \lim_{r \to 0^+} \frac{\mu^t_\ast(B_r(x))}{\omega_{N-2} r^{N-2}} \geq \eta(t),
$$

for $\mathcal{H}^{N-2}$-almost every $x \in \Sigma^t_\ast$.
5. The family of measures $t \mapsto \Theta_\ast(x,t) \mathcal{H}^{N-2} \mathbb{L} \Sigma^t_\ast$ forms a Brakke flow.

These conclusions were first demonstrated in [8] for the, non-compact, smooth manifold $\mathbb{R}^N$ paired with the standard metric.

In general we follow the strategy developed in [8]. However, a number of details need to be adapted in order for the strategy to extend to the more general setting.

- When defining the weighted energy, which is used to establish a monotonicity formula, we use an approximation to the heat kernel as a weight. The form of the alteration that we employ differs from the earlier works [31], [30] and is designed to facilitate a comparison of the weighted energy at distinct points in space-time, see Lemma 3.7.
A consequence of modifying the weighted energy is that new error terms $\Phi$ and $\Psi$ arise, see (3.7) and (3.8) for definitions. The error term $\Phi$, as seen in Theorem 1.1 of [17], corresponds to the fact that we are not working over Euclidean space while $\Psi$, as seen in [32], reflects the fact that we have replaced the heat kernel on $M$ with an approximation. These error terms are handled by appealing to the Hessian Comparison Theorem which is discussed in (2.5).

When following the Hodge de Rham decomposition strategy from Subsection 3.6 of [8] we need to solve a Poisson problem over $M$. Since we do not impose any topological restrictions on $M$ some care is needed to ensure that a solution exists. Specifically, we needed to modify the argument from [8] to account for the harmonic part of the data as well as provide additional estimates for the resultant error terms.

When decomposing the solution to $(\text{PGL})_\varepsilon$, as in Theorem 3 of [8], we now have to account for the fact that no topological restrictions were placed on $M$. This, in particular, has the effect of adding an additional term, $u_{h,\varepsilon} : M \times (0, \infty) \to \mathbb{S}^1$, which corresponds to the harmonic part of the Hodge de Rham decomposition of $u_\varepsilon \times d u_\varepsilon$ at time $t = 0$. The presence of this additional term also has consequences on how we are able to express the limiting energy density in Theorem 1.1.

The use of the Hessian Comparison Theorem gives rise to curvature-dependent constants in many of our estimates. In our arguments, it is often convenient to rescale the metric $g$ to a dilated metric $g/\alpha$ with $\alpha \in (0, 1)$. All estimates that we need continue to hold with the same, often better, constants after such rescaling. Indeed, such a rescaling decreases bounds on the curvature and hence improves all curvature-dependent constants.

Inevitably, there are numerous arguments in the proof of Theorem 1.1, that are very similar to corresponding points in [8]. We omit discussions that would essentially duplicate prior arguments. However, we have taken a couple of steps to explain these points and to document their correctness. First, we attempt to sketch these proofs well enough to make it clear that no significant new subtleties arise in the Riemannian case. As a result, in places our exposition resembles a sort of reader’s guide to parts of [8]. This seems to us necessary for a reasonably complete account of the proof of Theorem 1.1. Second, the author’s Ph.D. thesis [12] contains an expanded version of this paper, and it includes an appendix in which we discuss in detail a number of the points omitted here. These are arguments that involve few novel ingredients, but for which some documentation may be useful. We refer to this appendix often.

While Theorem 1.1 is interesting in its own right it is worth noting that this result is a key ingredient in demonstrating the existence of solutions to the elliptic Ginzburg-Landau equation over $(M, g)$, when $N = 3$, for which the energy and a quantity associated to vorticity concentrate about a non-length minimizing geodesic as $\varepsilon \to 0^+$. This is shown in [13] which improves on earlier work such as [22] and [29].
We conclude this introduction by describing some issues in the proof of Theorem 1.1. First, as in [8], an important intermediate result is the following “clearing out” theorem. It involves a weighted energy, $\tilde{E}_\varepsilon$, whose definition is provided in (2.11).

**Theorem 1.2.** For any $\sigma \in (0,1)$ and $T > 0$ there exists positive numbers $\varepsilon_0$, $R(\sigma)$, and $\eta(\sigma)$ such that if $u_\varepsilon$ is a solution to (PGL)$_\varepsilon$ on $M \times (0,T)$ satisfying (H$_0$) for $0 < \varepsilon < \varepsilon_0$, $R$ satisfies $\sqrt{2\varepsilon} < R < \min\{R(\sigma), \sqrt{T}\}$, and $x_T$ is a point such that

$$\tilde{E}_\varepsilon(u_\varepsilon, (x_T,T), R) \leq \eta(\sigma)|\log(\varepsilon)|,$$

then

$$|u_\varepsilon(x_T,T)| \geq 1 - \sigma.$$ 

The overall strategy of the proof follows that of Theorem 1 in [8], on which Theorem 1.2 is modelled. We start by presenting an overview in Section 4, drawing on the work of [8]. In this overview we highlight elements of the proof in which substantial new considerations arise. All such points are treated in detail in Section 5. The overview of Section 4 also identifies many aspects of the proof that carry over to the Riemannian setting with only superficial changes. Detailed verification of these points can be found in Appendix A of [12]. In addition, for such points we attempt in Section 4 to describe the underlying ideas in sufficient detail to explain why the arguments of [8] do not involve any substantial changes in the Riemannian context.

The next result is an adaptation of Theorem 3 from [8]. New issues arise from the possibly non-trivial topology of $M$. This is reflected in the presence of the $S^1$-valued map $u_{h,\varepsilon}$. We refer the reader to (2.12) for the definition of $ju$, where $u : M \to \C$, which is used in the statement of the next theorem.

**Theorem 1.3.** Suppose $u_\varepsilon$ satisfies (PGL)$_\varepsilon$ and (H$_0$). Then there exists a $S^1$-valued function $u_{h,\varepsilon}$ depending only on the initial data of $u_\varepsilon$ such that, for any compact set $K \subset M \times (0,\infty)$ and $\varepsilon$ sufficiently small, there is a real-valued function $\phi_\varepsilon$ and a complex-valued function $w_\varepsilon$ defined on a neighbourhood of $K$, such that

1. $u_\varepsilon = w_\varepsilon e^{i\phi_\varepsilon}u_{h,\varepsilon}$ on $K$,
2. $\phi_\varepsilon$ verifies the heat equation on $K$,
3. $|\nabla \phi_\varepsilon(x,t)| \leq C(K)\sqrt{(M_0+1)|\log(\varepsilon)|}$ for all $(x,t) \in K$,
4. $\|\nabla w_\varepsilon\|_{L^p(K)} \leq C(p,K)$, for any $1 \leq p < \frac{N+1}{N+1}$,
5. $u_{h,\varepsilon}$ does not depend on $t$, $ju_{h,\varepsilon}$ is a harmonic 1-form on $M$, and

$$|\nabla u_{h,\varepsilon}(x,t)| \leq K_M\sqrt{M_0|\log(\varepsilon)|}$$

for all $(x,t) \in K$.

Here, $C(K)$ and $C(p,K)$ are constants depending only on $K$ and $K,p$ (and $M_0$) respectively and $K_M$ is a constant depending only on $M$.

This is proved in Section 6. Finally, the proof of Theorem 1.1 is completed in Section 7.
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2 Preliminaries

In this section we record some of the specialized notation and definitions used throughout this paper.

At each $x \in M$ we use $\langle \cdot, \cdot \rangle_g$ and $|\cdot|_g$ to denote, respectively, the inner product and norm on $T_xM$ given by $g$. For $x, y \in M$ we use $d_g(x, y)$ to denote the distance between $x$ and $y$ induced by the metric $g$. For $p \in M$ and $r > 0$ we use the notation $B_{r,g}(p)$ to denote the geodesic ball about $p$ of radius $r$ in the metric $g$ which is defined by

$$B_{r,g}(p) := \{ x \in M : d_g(x, p) < r \}.$$  

We will write $\text{vol}_g$ to denote the unique Radon measure on $M$ with the property that $\text{vol}_g(A)$ is the Riemannian volume of $A$ for all sufficiently regular $A$, and for non-negative $f \in L^1(M; \text{vol}_g)$, we write $f \text{vol}_g$ to denote the measure defined by

$$f \text{vol}_g(A) := \int_A f \text{dvol}_g.$$  

We define the injectivity radius of $M$ according to the metric $g$, denoted $\text{inj}_g(M)$, by

$$\text{inj}_g(M) := \sup \left\{ r > 0 : \exp_x : T_xM \to M \text{ is a diffeomorphism onto } B_{r,g}(x) \text{ for all } x \in M \right\}. \quad (2.1)$$

We define the diameter of $M$ according to the metric $g$, denoted $\text{diam}_g(M)$, by

$$\text{diam}_g(M) := \sup \{ d_g(x, y) : \forall x, y \in M \}. \quad (2.2)$$

In the above notation we may, for convenience, remove the subscript $g$.

We note for $p \in M$ and $0 < s < \text{inj}_g(M)$ that the function

$$r(x) := \frac{1}{2}(d_g(x, p))^2 \quad (2.3)$$

satisfies

$$\nabla r(x) = -\exp^{-1}_s(p) \quad (2.4)$$

on $B_s(p)$, see Theorem 6.6.1 of [23]. Also, if the sectional curvature, $K$, of $M$ satisfies

$$\lambda \leq K \leq \mu, \quad \text{with } \lambda \leq 0 \leq \mu$$

then, for $0 < \rho < \min \left\{ \frac{\text{inj}_g(M)}{\sqrt{\mu}}, \text{inj}_g(M) \right\}$ if $\mu > 0$ and $0 < \rho < \text{inj}_g(M)$ otherwise, we have

$$\sqrt{\mu}d(x, p) \cot(\sqrt{\mu}d(x, p))|v|^2 \leq \text{Hess}(r)(v, v) \leq \sqrt{\lambda}d(x, p) \coth(\sqrt{\lambda}d(x, p))|v|^2 \quad (2.5)$$

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for \( x \in B_\rho(p) \) and \( v \in T_xM \), see Theorem 6.6.1 of [23]. This is referred to as the Hessian Comparison Theorem.

We use the notation \( \Lambda_\alpha(x_0, T, R, \Delta T) \) for \( 0 < \alpha \leq 1, \ x_0 \in M, \ T \geq 0, \ \Delta T > 0, \) and \( R > 0 \) to refer to

\[
\Lambda_\alpha(x_0, T, R, \Delta T) := B_\alpha R(x_0) \times [T + (1 - \alpha^2)\Delta T, T + \Delta T].
\]  

We also use the abbreviations \( \Lambda_\alpha \) for (2.6) and \( \Lambda := \Lambda_1(x_0, T, R, \Delta T) \) when the other parameters are understood.

For \( y \in M \) we define the approximate heat kernel about \( y \) evaluated at \( (x, t) \in M \times (0, \infty) \), denoted \( K_{ap}(x, t; y) \), by

\[
K_{ap}(x, t; y) := \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp \left[ -\frac{(d_{+, g}(x, y))^2}{4t} \right]
\]  

where \( d_{+, g} : M \times M \to [0, \infty) \) is a smooth function defined so that

\[
d_{+, g}(x, y) = \text{inj}_g(M) f \left( \frac{d_g(x, y)}{\text{diam}_g(M)} \right)
\]  

where \( f : [0, \infty) \to [0, \infty) \) is a smooth function chosen so that

1. \( f(s) = s \) for \( s \in [0, \frac{1}{2}] \),
2. \( f(s) = 1 \) for \( s \geq 1 \),
3. \( f(s) \geq s \) for \( 0 \leq s \leq 1 \),
4. \( f \) is non-decreasing,
5. \( \|f'\|_{L^\infty(\mathbb{R})} < \sqrt{2} \).

We note that \( d_{+, g} \) satisfies

\[
c_*d_g(x, y) \leq d_{+, g}(x, y) \leq 2d_g(x, y)
\]  

where

\[
c_* = \frac{\text{inj}_g(M)}{\text{diam}_g(M)}.
\]

We will use the notation \( K_{ap, g}(x, t; x_\ast) \) when we wish to explicitly indicate the dependence of \( K_{ap} \) on the metric \( g \). Also, for a fixed point \( p \in M \) we use the notation \( r_+ \) to denote

\[
r_+(x) := \frac{1}{2}(d_+(x, p))^2.
\]  

Next we introduce notation for energy weighted by the approximate heat kernel on \( M \). For \( z_\ast = (x_\ast, t_\ast) \in M \times (0, \infty) \) and \( 0 < R \leq \sqrt{r_\ast} \), we use the notation

\[
\tilde{E}_s(z_\ast, R) := R^2 \int_M e_\ast(u(x, t_\ast - R^2))K_{ap}(x, R^2; x_\ast) d\text{vol}_g(x).
\]  

We may also use variations of this notation which include \( g \) in the subscript to emphasize particular dependence on the metric.
For a given $u: M \to \mathbb{C}$ we introduce the notation $ju$ for the 1-form
\[ ju := u \times du \] (2.12)
which in coordinates can be expressed as
\[ u \times du := \sum_{i=1}^{N} u \times \frac{\partial u}{\partial x_i} dx^i. \]

Now we provide a series of definitions related to Brakke flows.

**Definition 2.1.** A Radon measure $\nu$ on $M$ is said to be $k$-rectifiable if there exists a $k$-rectifiable set $\Sigma$, and a density function $\Theta \in L^1_{\text{loc}}(\mathcal{H}^k \llcorner \Sigma)$ such that
\[ \nu = \Theta(\cdot) \mathcal{H}^k \llcorner \Sigma. \] (2.13)

Next, we define the distributional first variation of a rectifiable Radon measure. To do this, we remark that if $\Sigma$ is $k$-rectifiable then at $H^k$-almost every point $x \in \Sigma$ there is a unique tangent space $T_x \Sigma$ belonging to the Grassmannian $G_{N,k,x}$. Similar to [8] we associate $G_{N,k,x}$ to projection operators onto $k$-dimensional subspaces of $T_x M$.

**Definition 2.2.** Let $\nu$ be a $k$-rectifiable Radon measure. Then we define the distributional first variation of $\nu$ to be the distribution, $\delta \nu$, defined by
\[ \delta \nu(X) := \int_{\Sigma} \text{div}_{T_x \Sigma}(X) d\nu \] (2.14)
for all $X \in \chi(M)$, where $\chi(M)$ denotes the space of smooth vector fields over $M$ and, following Section 2 of [30], we define
\[ \text{div}_{T_x \Sigma}(X) := \sum_{k=1}^{N-2} \langle De_k X(x), e_i \rangle \] (2.15)
where $\{e_1, e_2, \ldots, e_{N-2}\}$ denote any orthonormal basis of $T_x \Sigma$ and $D_{e_k} X$ denote the associated covariant derivatives. When $|\delta \nu|$ is absolutely continuous with respect to $\nu$, we say that $\nu$ has a first variation and we may write
\[ \delta \nu = H \nu \]
where $H$ is the Radon-Nikodym derivative of $\delta \nu$ with respect to $\nu$. In this case, (2.14) becomes
\[ \int_S \text{div}_{T_x \Sigma}(X) d\nu = \int_S \langle H, X \rangle d\nu. \] (2.16)

Next, we let $\nu_t^i \geq 0$ be a family of Radon measures on $M$. For $\chi \in C^2(M; [0, \infty))$, we define
\[ T_t \nu_0^i(\chi) := \limsup_{t \to t_0} \frac{\nu_t^i(\chi) - \nu_0^i(\chi)}{t - t_0}. \]
If $\nu^i \llcorner \{ \chi > 0 \}$ is a $k$-rectifiable measure which has a first variation verifying $\chi|H|^2 \in L^1(\nu^i)$, then we set
\[ B(\nu^i, \chi) := -\int \chi|H|^2 d\nu^i + \int \langle \nabla \chi, P(\nu^i) \rangle d\nu^i, \]
where \( P \), as in Section 2 of [30] and consistent with our identification of the Grassmannian with projections, denotes \( \mathcal{H}^k \)-almost everywhere the orthogonal projection onto the tangent space to \( \nu^t \), otherwise, we set \( B(\nu^t, \chi) = -\infty \).

We are now in a position to give the definition of a Brakke flow.

**Definition 2.3.** Let \( \{\nu^t\}_{t \geq 0} \) be a family of Radon measures on \( M \). We say that \( \{\nu^t\}_{t \geq 0} \) is a \( k \)-dimensional Brakke flow if and only if

\[
D_t \nu^t(\chi) \leq B(\nu^t, \chi),
\]

for every \( \chi \in C^\infty(M; (0, \infty)) \) and for all \( t \geq 0 \).

### 3 Toolbox

We record a few helpful results that will be needed for the proof of Theorem 1.1. These are generalizations of corresponding results found in [8].

**Lemma 3.1.** Let \( \chi \) be a Lipschitz function on \( M \). Then, for any \( T \geq 0 \),

\[
\frac{d}{dt} \int_{M \times \{t\}} e_\varepsilon(u_\varepsilon) \chi(x) = - \int_{M \times \{T\}} |\partial_t u_\varepsilon|^2 \chi(x) - \int_{M \times \{T\}} \partial_t u_\varepsilon \cdot \langle \nabla u_\varepsilon, \nabla \chi \rangle.
\]

and

\[
\frac{1}{2} \int_{M \times \{t\}} |\partial_t u_\varepsilon|^2 \chi^2 + \frac{d}{dt} \int_{M \times \{t\}} e_\varepsilon(u_\varepsilon) \chi^2 \leq 4 \|\nabla \chi\|_{L^\infty}^2 \int_{\text{supp}(\chi)} e_\varepsilon(u_\varepsilon).
\]

In particular, for any \( 0 \leq T_1 \leq T_2 \),

\[
\int_{M \times \{T_2\}} e_\varepsilon(u_\varepsilon) \chi(x) - \int_{M \times \{T_1\}} e_\varepsilon(u_\varepsilon) \chi(x) = - \int_{M \times [T_1, T_2]} |\partial_t u_\varepsilon|^2 \chi(x) - \int_{M \times [T_1, T_2]} \partial_t u_\varepsilon \cdot \langle \nabla u_\varepsilon, \nabla \chi \rangle.
\]

**Proof.** The proof of (3.1) follows from differentiation under the integral while (3.3) follows by integrating (3.1) in \( t \). To see (3.2) we replace \( \chi \) with \( \chi^2 \) in Lemma 3.1 and use standard estimates.

The next result, the basis for a monotonicity formula, will play a fundamental role in the proof of Theorem 1.2.

**Lemma 3.2.** Suppose \((M, g)\) is an \( N \)-dimensional compact Riemannian manifold without boundary and suppose that \( u_\varepsilon \) solves \( (\text{PGL})_\varepsilon \) on \( M \). Let \( K_{ap} \) be the approximate heat kernel as in (2.7). Then for \( 0 < R < \sqrt{T} \) and \( y \in M \): 

\[
Z'(R) = 2R \int_{M} \left[ V_\varepsilon(u_\varepsilon(x, T - R^2)) + \Xi(u_\varepsilon(y, T))(x, T - R^2) \right] (x, T - R^2) \]

\[ + 2R \int_{M} \Psi(u_\varepsilon(y, T))(x, T - R^2) \]

\[ + 2R \int_{M} \Phi(u_\varepsilon(y, T))(x, T - R^2) \]

\[ = 2R \int_{M} \left[ V_\varepsilon(u_\varepsilon(x, T - R^2)) + \Xi(u_\varepsilon(y, T))(x, T - R^2) \right] (x, T - R^2) \]

\[ + 2R \int_{M} \Psi(u_\varepsilon(y, T))(x, T - R^2) \]

\[ + 2R \int_{M} \Phi(u_\varepsilon(y, T))(x, T - R^2) \]

\[ = 2R \int_{M} \left[ V_\varepsilon(u_\varepsilon(x, T - R^2)) + \Xi(u_\varepsilon(y, T))(x, T - R^2) \right] (x, T - R^2) \]

\[ + 2R \int_{M} \Psi(u_\varepsilon(y, T))(x, T - R^2) \]

\[ + 2R \int_{M} \Phi(u_\varepsilon(y, T))(x, T - R^2) \]
where

$$Z(R) := R^2 \int_M e_\varepsilon(u_\varepsilon(x, T - R^2)) K_{ap}(x, R^2; y) d\text{vol}_g(x)$$  \hspace{1cm} (3.5)$$

and where, for $0 < t < T$, we have

$$\Xi(u_\varepsilon, (y, T)) (x, t) := (T - t) \left| \partial_t u_\varepsilon(x, t) + \frac{\langle \nabla u_\varepsilon(x, t), \nabla K_{ap}(x, T - t; y) \rangle}{K_{ap}(x, T - t; y)} \right|^2,$$  \hspace{1cm} (3.6)$$

$$\Phi(u_\varepsilon, (y, T)) (x, t) := (T - t) \left[ \text{Hess}(K_{ap}(x, T - t; y)) (\nabla u_\varepsilon(x, t), \nabla u_\varepsilon(x, t)), \right. \hspace{1cm} (3.7)$$

$$\left. - \frac{|\langle \nabla u_\varepsilon(x, t); \nabla K_{ap}(x, T - t; y) \rangle|^2}{2K_{ap}(x, T - t; y)} + \frac{| \nabla u_\varepsilon|^2 K_{ap}(x, T - t; y)}{2(T - t)} \right].$$

$$\Psi(u_\varepsilon, (y, T)) (x, t) := (T - t) e_\varepsilon(u_\varepsilon(x, t)) \left[ \partial_t K_{ap}(x, T - t; y) - (\Delta K_{ap})(x, T - t; y) \right].$$  \hspace{1cm} (3.8)$$

We also have, for any $z_T = (x_T, T) \in M \times (0, \infty)$ and $R_* = \sqrt{T}$, that

$$\bar{E}_\varepsilon(z_T, R_*) = \int_{M \times [0, T]} (V_\varepsilon(u_\varepsilon) + \Xi(u_\varepsilon, z_T)) K_{ap}(x, T - t; x_T) d\text{vol}_g(x) dt$$

$$+ \int_{M \times [0, T]} \Phi(u_\varepsilon, z_T) d\text{vol}_g(x) dt + \int_{M \times [0, T]} \Psi(u_\varepsilon, z_T) d\text{vol}_g(x) dt.$$  \hspace{1cm} (3.9)$$

Proof. Computations like (3.4) are quite standard, and very similar ones can be found for example in the proof of Theorem 2.1 of [32]. Then (3.9) follows by integrating (3.4) from $R = 0$ to $R = \sqrt{T}$ and changing variables. For a detailed exposition see A.3.1.1 of [12].

As remarked in the introduction, the terms $\Phi$ and $\Psi$ reflect the non-Euclidean character of the metric and the use of the approximate, rather than exact, heat kernel. They are estimated using arguments that ultimately rely on the Hessian Comparison Theorem. We illustrate this first for $\Psi$.

**Lemma 3.3.** Let $(M, g)$ be an $N$-dimensional compact Riemannian manifold and suppose $y \in M$. Let $K_{ap}$ be the approximate heat kernel from (2.7) and $\Psi$ be as in (3.8). Then there is $c_0 > 0$ such that for all $0 < t < T$ we have

$$\int_M \Psi(u, (y, T))(x, T - t) \geq - \frac{N \mu t^\frac{1}{2}}{4} \int_M e_\varepsilon(u) K_{ap} - c_0 \int_M e_\varepsilon(u)$$  \hspace{1cm} (3.10)$$

where the constants remain bounded when dividing the metric by $0 < a \leq 1$ and we have used the abbreviations $K_{ap}$ for $K_{ap}(x, t; y)$ and $u$ for $u(x, T - t)$. Similarly, there is $c_1 > 0$ such that for all $0 < t < T$ we have

$$\int_M \Psi(u, (y, T))(x, T - t) \leq \frac{N |\lambda| t^\frac{1}{2}}{6} \int_M e_\varepsilon(u) K_{ap} + c_1 \int_M e_\varepsilon(u).$$  \hspace{1cm} (3.11)$$
where the constants remain bounded when dividing the metric by $0 < a \leq 1$.

It is worth noting that we also have

\[ \int_M \Psi(u, (y,T))(x,T-t) \leq \frac{N|\lambda|}{6} \int_M (d_+(x,y))^2 e_\varepsilon(u)K_{ap} + C_M \int_M e_\varepsilon(u)K_{ap} + C_0 E_0 t \]

(3.12)

where $C_M, C_0$ remain bounded when dividing the metric by $0 < a \leq 1$.

Proof. By computing $\partial_t K_{ap} - \Delta K_{ap}$ we obtain, using the notation from (2.10), that

\[ \partial_t K_{ap} - \Delta K_{ap} = \frac{[\Delta r_+(x) - N]}{2t}K_{ap} + \frac{r_+(x) - \frac{1}{2}\nabla r_+(x)^2}{2t^2}K_{ap}. \]

First observe that if $s := \min\left\{ \frac{s}{\pi r_p}, \frac{\min(M)}{2} \right\}$ and $x \in B_s(y)$ then the right-most term is zero and by using the notation (2.3) as well as (2.5) we obtain

\[ \frac{[\Delta r_+(x) - N]}{2t}K_{ap} = \frac{[\Delta r(x) - N]}{2t}K_{ap} \geq -\frac{N\mu(d(x,y))^2}{4t}K_{ap}. \]

(3.13)

Next, observe that for $x \in M \setminus B_s(y)$ we have

\[ \partial_t K_{ap} - \Delta K_{ap} \geq -\frac{C_M \max\{t,1\}}{t^2}K_{ap}. \]

(3.14)

Using (3.13) and (3.14) leads to

\[ \int_M e_\varepsilon(u)[\partial_t K_{ap} - \Delta K_{ap}] \geq -\frac{N\mu}{4t} \int_{B_s(y)} (d(x,y))^2 e_\varepsilon(u)K_{ap} - \frac{C_M \max\{t,1\}}{t^2} \int_{M \setminus B_s(y)} e_\varepsilon(u)K_{ap}. \]

Note that, since $d_+(\cdot, y)$ is a function of distance from $y$, we have

\[ \frac{C_M \max\{t,1\}}{t^2} \int_{M \setminus B_s(y)} e_\varepsilon(u)K_{ap} \leq \frac{C_M \max\{t,1\}e^{\frac{s^2}{2}}e^{\frac{t^2}{4}}}{t^2(4rt)} \int_{M \setminus B_s(y)} e_\varepsilon(u) \]

\[ \leq C_M e^{\frac{s^2}{2}} \int_M e_\varepsilon(u) \]

\[ \leq C_M \int_M e_\varepsilon(u). \]

Note that if we rescale the metric by dividing by $0 < a \leq 1$ then the constant $C'_M$ only becomes smaller. Observe that we either have $t^\# \geq s$ or $0 < t^\# < s$. If $t^\# \geq s$ then

\[ -\frac{N\mu}{4t} \int_{B_s(y)} (d(x,y))^2 e_\varepsilon(u)K_{ap} \geq -\frac{N\mu}{4t^2} \int_{B_s(y)} e_\varepsilon(u)K_{ap}. \]

If $0 < t^\# < s$ then we have, using the notation $A_{t^\#}^s(y) := B_s(y) \setminus B_{t^\#}^s(y)$
for \( y \in M \), that

\[
- \frac{N \mu}{4t} \int_{B_{\epsilon}(y)} (d(x, y))^2 e_\epsilon(u) K_{ap} \\
= - \frac{N \mu}{4t} \int_{A_{\epsilon}^{\frac{1}{4}}(y)} (d(x, y))^2 e_\epsilon(u) K_{ap} - \frac{N \mu}{4t} \int_{A_{\epsilon}^{\frac{1}{4}}(y)} (d(x, y))^2 e_\epsilon(u) K_{ap} \\
\geq - \frac{N \mu}{4t} \int_{B_{\epsilon}^{\frac{1}{4}}(y)} e_\epsilon(u) K_{ap} - \frac{N \mu (\text{inj}(M))^2}{16t} \int_{A_{\epsilon}^{\frac{1}{4}}(y)} e_\epsilon(u),
\]

where the constants remain bounded when dividing the metric by \( a \) becomes smaller if we divide the metric by \( a \). Setting \( t > 0 \) and multiplying by \( e^{-\epsilon \frac{|x|^2}{4t}} \) gives

\[
\geq - \frac{N \mu}{4t^{\frac{1}{2}}} \int_{M} e_\epsilon(u) K_{ap} - \frac{N \mu (\text{inj}(M))^2}{16t} \sup_{t > 0} \left\{ \frac{e^{-\epsilon \frac{|x|^2}{4t}}}{t(4t)^{\frac{N}{2}}} \right\} \int_{M} e_\epsilon(u).
\]

Notice that \( C''_M \) is invariant under rescaling in the metric and \( \mu \) only becomes smaller if we divide the metric by \( a \). Setting this altogether gives

\[
\int_{M} e_\epsilon(u) [\partial_t K_{ap} - \Delta K_{ap}] \geq -\frac{N \mu}{4t^{\frac{1}{2}}} \int_{M} e_\epsilon(u) K_{ap} - 2 \max\{C'_M, C''_M\} \int_{M} e_\epsilon(u).
\]

Setting

\[
c_0 := 2 \max\{C'_M, C''_M\}
\]

and multiplying by \( t \) gives the desired result. Observe that a similar proof holds for (3.11) and that (3.12) is demonstrated through the proof of the upper bound.

We next record estimates of a similar character for \( \Phi \).

**Lemma 3.4.** Suppose \( (M, g) \) is an \( N \)-dimensional compact Riemannian manifold without boundary. Let \( K_{ap} \) be the approximate heat kernel from (2.7). Then there is \( c_2 > 0 \) such that for all \( 0 < t < T \) that

\[
\int_{M} \Phi(u, (y, T))(x, T - t) \geq -\frac{|\lambda| t^{\frac{N}{2}}}{3} \int_{M} e_\epsilon(u) K_{ap} - c_2 \int_{M} e_\epsilon(u) \tag{3.15}
\]

where the constants remain bounded when dividing the metric by \( 0 < a \leq 1 \) and where we have used the abbreviations \( K_{ap} \) for \( K_{ap}(x, t; y) \) and \( u \) for \( u(x, T - t) \). Similarly, there is \( c_3 > 0 \) such that for all \( 0 < t < T \) that

\[
\int_{M} \Phi(u, (y, T))(x, T - t) \leq \frac{\mu t^{\frac{N}{2}}}{2} \int_{M} e_\epsilon(u) K_{ap} + c_3 \int_{M} e_\epsilon(u) \tag{3.16}
\]
where the constants remain bounded when dividing the metric by \(0 < a \leq 1\). It is worth noting that we have

\[
\int_M \Phi(u,T)(x,T-t) \leq \mu \int_M \frac{(d(x,y))^2}{4} e_\epsilon(u)K_{ap} + D_M \int_M e_\epsilon(u)K_{ap}
\]

where \(D_M\) remains bounded when dividing the metric by \(0 < a \leq 1\).

\[
\text{(3.17)}
\]

Proof. The proof is similar to that of Lemma 3.3. More discussion is provided in A.3.1.2 of [12].

We now prove a monotonicity formula for solutions to \((PGL)_\epsilon\). As noted before, this result will be instrumental to demonstrating many of the estimates needed in the proof of Theorem 1.2.

**Proposition 3.5.** Let \(K_{ap}\) be the approximate heat kernel and suppose that \(y \in M\) and \(T > 0\). Then there exists positive constants \(C_1 \geq 1\) and \(C_2\) such that if \(0 \leq R_1 \leq R_2 \leq \min\{\sqrt{T}, 1\}\) then

\[
C_1 E_0 R_1 + \exp[C_2 R_1] Z(R_1) \leq C_1 E_0 R_2 + \exp[C_2 R_2] Z(R_2)
\]

(3.18)

where

\[
E_0 := \int_M e_\epsilon(u_0^0(x))d\text{vol}_g(x).
\]

That is, the function \(r \mapsto C_1 E_0 r + \exp[C_2 r] Z(r)\) is non-decreasing on \([0, \min\{\sqrt{T}, 1\}]\).

\[
\text{(3.19)}
\]

Proof. Combining (3.10) and (3.15) for \(u = u_\epsilon\) with the expression for \(Z'(R)\) from Lemma 3.2 gives an inequality of the form

\[
Z'(R) \geq -\tilde{C}Z(R) - \tilde{D}E_0
\]

where \(\tilde{C}\) and \(\tilde{D}\) are positive constants that remain bounded when dividing the metric by \(0 < a \leq 1\). Setting \(C_2 := \tilde{C}\) and \(C_1 := \tilde{D}e^{\tilde{C}}\) as well as using that \(R \leq 1\) leads to (3.18). More details are provided in A.3.1.3 of [12].

\[
\text{(3.18)}
\]

**Remark 3.6.** As one might guess from the appeal to the Hessian Comparison Theorem, the constants \(C_1\) and \(C_2\) from the above proposition can all be estimated in terms of upper and lower bounds on the sectional curvature. As a result, all such constants are preserved by dividing the metric \(g\) by factors smaller than one. As noted in the introduction, this is generally the case for all curvature-dependent constants appearing in this paper.

The next result facilitates comparison of the weighted energy centred about two different points in space-time.

**Lemma 3.7.** Let \(0 < t_\ast < T\), and \(z_\ast = (x_\ast, t_\ast) \in M \times (0, \infty)\). Then,

\[
\tilde{E}_\epsilon(z_\ast, \sqrt{T}) \leq \left(\frac{T}{t_\ast}\right)^{N-1} \exp\left[\frac{C_f(d_\epsilon(x_\ast, x))^2}{T-t_\ast}\right] \tilde{E}_\epsilon(u_\epsilon, (x_\ast, T), \sqrt{T})
\]

where the constants remain bounded when dividing the metric by \(0 < a \leq 1\). It is worth noting that we have

\[
\int_M \Phi(u,T)(x,T-t) \leq \mu \int_M \frac{(d(x,y))^2}{4} e_\epsilon(u)K_{ap} + D_M \int_M e_\epsilon(u)K_{ap}
\]

where \(D_M\) remains bounded when dividing the metric by \(0 < a \leq 1\).
for all \( x_T \in M \) where \( C_f := \max \{ 1, \| f' \|_{L^\infty([0,\infty))} \} \) and \( f \) is as defined below (2.8). In particular,

\[
\tilde{E}_\varepsilon(z_*, \sqrt{T}) \leq \left( \frac{T}{t_*} \right)^{N-1} \frac{2C_f (d(x_T, x_*)^2)}{4T - d(x_T, x_*)^2} \tilde{E}_\varepsilon(u_\varepsilon, (x_T, T), \sqrt{T})
\]

for all \( x \in M \) where \( C_f \) is as above.

**Proof.** The proof proceeds in the same way as the proof of Lemma 2.3 of [8] except a careful estimate of the supremum of the function

\[
x \mapsto \exp \left( \frac{(d_+(x, x_T))^2}{4T} - \frac{(d_+(x, x_*)^2}{4t_*} \right)
\]

is required. The corresponding estimate in [8] is done completely explicitly. Here it is carried out by considering several cases, depending on the relative size of \( d_+(x, x_T) \), \( d_+(x, x_*) \), and \( \text{inj}_g(M) \). Details can be found in A.3.2.1 of [12].

The next proposition is an important localization method that converts information about the energy density on a small ball to information about the weighted energy. This will be helpful when analyzing the structure of the energy density measure in the proof of Theorem 1.1.

**Proposition 3.8.** Suppose \( T > 0 \) and \( \sqrt{2} < R < 1 \). Then for any \( \lambda > 0 \) and \( x_T \in M \) the following inequality holds

\[
\int_M \epsilon_\varepsilon(u_\varepsilon(\cdot, T)) e^{-\frac{(d_+(\cdot, x_T))^2}{4T^2}} \leq \int_{B_\lambda R(x_T)} \epsilon_\varepsilon(u_\varepsilon(\cdot, T))
\]

\[
+ M_0 e^{-\frac{\lambda^2}{4T}} \frac{2}{\sqrt{2}} \left[ e_+ \left( \frac{2R^2}{T + 2R^2} \right)^{N-2} C_1(4\pi) \frac{1}{\sqrt{2}} (\sqrt{2}R)^{N-2} \sqrt{T} \right] \log(\varepsilon).
\]

**Proof.** The proof is essentially the same as that of Proposition 2.3 of [8]. The point is to estimate

\[
\int_{M \setminus B_\lambda R(x_T)} \epsilon_\varepsilon(u_\varepsilon) e^{-\frac{(d_+(x, x_T))^2}{4T^2}}.
\]

This is achieved by applying the monotonicity formula, see Proposition 3.5, in addition to the properties of \( d_+ \), as in (2.9). See A.3.5.1 of [12] for details.

In the next proposition we exploit the monotonicity formula to obtain good estimates of the solution of a nonhomogeneous heat equation when the right-hand side is dominated by \( V_\varepsilon(u_\varepsilon) \).

**Proposition 3.9.** If \( 0 < T < 1 \), \( x_T \in M \), and \( \omega: M \times (0, \infty) \to \wedge^2 M \) solves

\[
\begin{cases}
\partial_\tau \omega - \Delta \omega = h & \text{on } M \times (0, \infty) \\
\omega(x, 0) = 0 & x \in M
\end{cases}
\]

where \( h \in L^\infty(M \times [0, T]; \wedge^2 M) \) satisfies

\[
|h(x, t)| \leq V_\varepsilon(u_\varepsilon(x, t)) & \text{for } (x, t) \in M \times [0, T]
\]

(3.20)
then for any \( z = (x, t) \in M \times [0, T] \), the following estimate holds:

\[
|\omega(z)| \leq C_3(T + 1) \left( \frac{T}{t} \right)^{\frac{N-1}{2}} e^{\frac{\gamma_f(d(x, x_T))^2}{8(T-t)}} \left( \bar{E}_\epsilon(u_\epsilon, (x_T, T), \sqrt{T}) + C_1 E_0 T \right) \\
\leq C_3(T + 1) \left( \frac{T}{t} \right)^{\frac{N-1}{2}} e^{\frac{4 \gamma_f(d(x, x_T))^2}{8(T-t)}} \left( \bar{E}_\epsilon(u_\epsilon, (x_T, T), \sqrt{T}) + C_1 E_0 T \right)
\]

where \( C_f \) is as in Lemma 3.7 and \( C_3 \) depends on \( M \).

**Proof.** The proof proceeds in the same way as Proposition 2.2 of [8], the idea being to represent \( \omega(z) \) by Duhamel’s formula and then exploit the fact that, since \( |f| \leq V_\gamma(u_\epsilon) \), the right-hand side of Duhamel’s formula is controlled by the weighted energy. In our setting we must estimate the heat kernel for 2-forms, appearing in Duhamel’s formula, by the approximate heat kernel \( K_{ap} \), appearing in \( \bar{E}_\epsilon \). This may be done using estimates on the heat kernel for differential forms which are provided in [28]. Details can be found in A.3.3.1 and Proposition A.3.2 of [12].

The next proposition is a localization method that originated from [25] and was used in [8]. As in [8] this result is vital to our proof of Theorem 1.2 as it permits us to localize our estimate of the weighted energy to a small coordinate ball. It is based on a Pohozaev type inequality.

**Proposition 3.10.** Let \((M, g)\) be an \( N \)-dimensional compact Riemannian manifold without boundary and suppose \( 0 < t < T \) is chosen so that \((T-t)\) is small enough that

\[
1 - C_5(T-t) \geq \frac{1}{2}
\]

where \( C_5 > 0 \) depends linearly on the sectional curvature of \( M \). Then, there is a constant \( C_0 > 0 \) invariant under dilations of the metric \( g \) by factors larger than one and \( D_f > 0 \) dependent only on \( f \) such that if \( z_T = (x_T, T) \in M \times (0, \infty) \) then

\[
\int_{M \times \{t\}} e_\epsilon(u_\epsilon) \frac{(d_+(x, x_T))^2}{4(T-t)} e^{-\frac{(d_+(x, x_T))^2}{8(T-t)}} \leq (4\pi)^{\frac{N}{2}} C_0(T-t)^{\frac{N}{2}+1} E_0 + 2D_f C_0(4\pi)^{\frac{N}{2}} (T-t)^{\frac{N}{2}+1} E_0 + 2D_f [4\pi(T-t)]^{\frac{N}{2}} \int_{M \times \{t\}} [V_\epsilon(u_\epsilon) + \Xi(u_\epsilon, z_T)] K_{ap}(x, \sqrt{T-t}; x_T)
\]

and consequently

\[
\int_{M \times \{t\}} e_\epsilon(u_\epsilon) e^{-\frac{(d_+(x, x_T))^2}{4(T-t)}} \leq 2 \int_{A \times \{t\}} e_\epsilon(u_\epsilon) e^{-\frac{(d_+(x, x_T))^2}{4(T-t)}} + (4\pi)^{\frac{N}{2}} C_0(T-t)^{\frac{N}{2}+1} E_0 + \frac{2[4\pi(T-t)]^{\frac{N}{2}}}{N} \int_{M \times \{t\}} [V_\epsilon(u_\epsilon) + \Xi(u_\epsilon, z_T)] K_{ap}(x, T-t; x_T)
\]

where

\[
A := \left\{ x \in M : \frac{(d_+(x, x_T))^2}{8(T-t)} \leq C_6 \right\}.
\]
Proof. The proof is mostly similar to the one found in Proposition 2.4 of [8], the only exception being we replace usage of the distance function $d$ with $d_\epsilon$ and use properties relating to the definition of $d_\epsilon$. We refer the reader to Lemma A.3.4 and A.3.5.2 of [12] for more details. First, for $0 < T_1 \leq T_2 < T$ and $x_T \in M$, we establish the inequality

$$
\int_{T_1}^{T_2} \int_M \frac{(d_\epsilon(x, x_T))^2}{4(T-t)} c_\epsilon(u_e) e^{-\frac{(d_\epsilon(x, x_T))^2}{4(T-t)}} \leq [4\pi(T-T_1)]^{\frac{N}{2}} D_f E_{\epsilon}(z_T, \sqrt{T-T_1}) - [4\pi(T-T_2)]^{\frac{N}{2}} D_f E_{\epsilon}(z_T, \sqrt{T-T_2})
$$

where

$$
D_f := (2 - \|f'\|_{L^\infty(B_1)})^{-1}.
$$

To do this, we take the dot product of $(PGL)_\epsilon$ with $2(T-t)\partial_t u_e e^{-\frac{(d_\epsilon(x, x_T))^2}{4(T-t)}}$, integrate by parts in time, and apply elementary inequalities. The only difference from Lemma 2.6 of [8] involves using properties of $d_\epsilon$, such as that

$$
|\nabla r_+(x)| \leq \|f'\|_{L^\infty(B_1)} d_+(x, y)
$$

where $r_+$ is as in (2.10) and $y \in M$.

Then, setting $T_1 = t$, letting $T_2 \searrow t$, and using (3.9) leads to

$$
\int_{M \times \{t\}} c_\epsilon(u_e) \frac{(d_\epsilon(x, x_T))^2}{4(T-t)} e^{-\frac{(d_\epsilon(x, x_T))^2}{4(T-t)}} \leq \frac{N D_f (4\pi)^{\frac{N}{2}}}{2} (T-t)^{\frac{N}{2} - 1} E_{\epsilon}(z_T, \sqrt{T-t})
$$

(3.24)

+ $(4\pi)^{\frac{N}{2}} D_f (T-t)^{\frac{N}{2}} \int_{M \times \{t\}} [V_\epsilon(u_e) + \Xi(u_e, z_T)] K_{\epsilon p}(x, T-t; x_T)$

+ $(4\pi)^{\frac{N}{2}} D_f (T-t)^{\frac{N}{2}} \int_{M \times \{t\}} [\Phi(u_e, z_T)(x, t) + \Psi(u_e, z_T)(x, t)] \text{dvol}(x)$.

Combining (3.12) and (3.16) with (3.24) leads to

$$
\left[1 - (4\pi)^{\frac{N}{2}} D_f \left[\mu - \frac{2N\lambda}{3}\right](T-t) \right] \int_{M \times \{t\}} c_\epsilon(u_e) \frac{(d_\epsilon(x, x_T))^2}{4(T-t)} e^{-\frac{(d_\epsilon(x, x_T))^2}{4(T-t)}} \leq (4\pi)^{\frac{N}{2}} D_f [N + C_M + D_M](T-t)^{\frac{N}{2} - 1} E_{\epsilon}(z_T, \sqrt{T-t})
$$

+ $(4\pi)^{\frac{N}{2}} D_f (T-t)^{\frac{N}{2}} \int_{M \times \{t\}} [V_\epsilon(u_e) + \Xi(u_e, z_T)] K_{\epsilon p}(x, T-t; x_T)$

+ $C_0 (4\pi)^{\frac{N}{2}} D_f (T-t)^{\frac{N}{2} - 1} E_0$.

Defining $C_5 := (4\pi)^{\frac{N}{2}} D_f \left[\mu - \frac{2N\lambda}{3}\right]$, using that we assume $1 - C_5 (T-t) \geq \frac{1}{2}$, and rearranging gives (3.22). Setting $C_6 := 2D_f [N + C_M + D_M]$ and arguing as in [8] then gives (3.23).

The following permits us to find a good bound for the weighted energy on a scale $R_1$ by the weighted energy on a smaller scale $\delta_0 R_1$.
Proposition 3.11. Fix $0 < \delta_0 < \frac{1}{T}$, $T > 0$, and let $0 < R \leq \min\{\sqrt{T}, 1\}$. There exists a constant $\varepsilon_1 > 0$ depending only on $R$, $T$, and $\delta_0$, such that for $0 < \varepsilon \leq \varepsilon_1$, there exists $R_1 > 0$, satisfying

$$R_1 \in (\sqrt[6]{\varepsilon}, R)$$

such that

$$\begin{align*}
\{C_7E_0R_1 + Z(R_1)\} - \{C_7E_0(\delta_0R_1) + Z(\delta_0R_1)\} \\
\leq \frac{4C_7e^{C_8}\log(\delta_0)}{\log(\varepsilon)}[RE_0 + Z(R)]
\end{align*}$$

(3.25)

where $C_7 := C_1e^{2C_2}$ and $C_8 := 2C_2$. We also have

$$\int_{T - R_1^2}^{T - (\delta_0R_1)^2} \int_M \left[\left(\Xi(u_\varepsilon, z_T)\right)(x, t) + V_\varepsilon(u_\varepsilon(x, t))\right]K_{ap}(x, T - t; xT)d\text{vol}_g(x)dt$$

$$\leq \frac{4C_7e^{C_8}\log(\delta_0)}{\log(\varepsilon)}[RE_0 + Z(R)].$$

(3.26)

Proof. The proof is essentially the same as in Proposition 2.6 of [8]. The idea in proving (3.25) is to average increments of $r \mapsto C_7e^{C_8}E_0r + e^{C_8r}Z(r)$ over time intervals $[\delta_0^jR, \delta_0^{j+1}R]$ for $j = 2, 3, \ldots, k_0$ where

$$k_0 \approx \left\lfloor \frac{\log(\varepsilon)}{2\log(\delta_0)} \right\rfloor$$

and to find an interval, $[\delta_0^{k_1}R, \delta_0^{k_1+1}R]$, for which the increment is small. This is achieved by repeatedly making use of Proposition 3.5. The inequality (3.26) then follows from (3.9), additional estimates on the error terms $\Phi$ and $\Psi$ in terms of the weighted energy and the initial energy due to (3.10), (3.15), as well as our choice of constants $C_7$ and $C_8$. For more details we refer the reader to A.3.6.1 of [12].

4 Overview of the proof of Theorem 1.2

We start by presenting a detailed outline of the proof of Theorem 1.2. We closely follow the proof presented in [8], so much so that this section may be used as a reader’s guide to the arguments found there. As we proceed, we will distinguish between

1. arguments that can be adapted from the Euclidean to the Riemannian setting with only cosmetic changes; we will describe these but not present them in detail; and

2. places where more effort is needed in order to adjust earlier arguments to the present setting. These points will be discussed at greater length in Section 5.

Note that, unless otherwise specified, all metric related quantities will be associated to $g_{R_1}$ and the metric will be suppressed from the notation.
Reduction via rescaling:

Throughout the proof of the theorem, \(0 < \delta_0 < \frac{1}{16}\) will denote a fixed parameter whose precise value will not be specified until a late stage of the proof. Applying Proposition 3.11 with this choice of \(\delta_0\) and \(R, T\) as in the statement of Theorem 1.2 we find a suitable time interval, \([\delta_0 R_1, R_1]\), in which the weighted energy of \(u_\varepsilon\) satisfies (3.25) and (3.26). Next we define \(v_\varepsilon : M \times (0, \infty) \to \mathbb{C}\), a rescaling of \(u_\varepsilon\) in \(\varepsilon\) and in time, by

\[
v_\varepsilon(x, t) := u_\varepsilon(x, T + R_1^2 |t - 1|)
\]

where \(\varepsilon := \frac{\varepsilon_0}{R_1^2}\). We also introduce the rescaled metric

\[
g_{R_1} := \frac{g}{R_1^2}.
\]

It follows from standard parabolic estimates that there is \(K > 0\) such that for \(x \in M\) and \(t > 0\) that

\[
|v_\varepsilon(x, t)| \leq 3, \quad |\nabla v_\varepsilon(x, t)| \leq \frac{K}{\varepsilon}, \quad |\partial_t v_\varepsilon(x, t)| \leq \frac{K}{\varepsilon^2}.
\]

Rescaling (3.25) to be written in terms of \(v_\varepsilon\) as well as applying \((H_0)\) and the assumptions of Theorem 1.2 we obtain

\[
\{ \tilde{E}_{\varepsilon, g_{R_1}}(v_\varepsilon, (x, T, 1), 1) + C_7 E_0 R_1 \} - \{ \tilde{E}_{\varepsilon, g_{R_1}}(v_\varepsilon, (x, T, 1), \delta_0) + C_7 E_0 (\delta_0 R_1) \} 
\leq 4 C_7 \varepsilon e^{C_8} |\log(\delta_0)| |RM_0 + \eta|.
\]

Finally, a change of variables applied to (3.26) in addition to an application \((H_0)\) and the assumptions of Theorem 1.2 leads to

\[
\int_{M \times [0, 1 - \delta_0]} [V_i(v_\varepsilon) + \Xi(v_\varepsilon, (x, T, 1))] K_{ap, g_{R_1}}(x, 1 - t; x, T) \leq 4 C_7 \varepsilon e^{C_8} |\log(\delta_0)| |RM_0 + \eta|.
\]

From here Theorem 1.2 is reduced to demonstrating the following result.

**Proposition 4.1.** Let \(T > 0\) and \(x_T \in M\). Then there exists constants \(0 < \delta_0 < \frac{1}{16}\), \(0 < \varepsilon_0 < \frac{1}{2} \min\{1, \text{inj}_g(M)\}\), \(0 < R_0 < 1\), and \(\eta_0 > 0\) such that for \(0 < \eta \leq \eta_0\), \(0 < \varepsilon < \varepsilon_0\), and \(0 < R < \min\{\sqrt{T}, R_0\}\) the following inequality holds:

\[
\tilde{E}_{\varepsilon, g_{R_1}}(v_\varepsilon, (x, T, 1), \delta_0) \leq \frac{1}{2} \left( \tilde{E}_{\varepsilon, g_{R_1}}(v_\varepsilon, (x, T, 1), 1) + C_7 E_0 R_1 \right) + \mathcal{R}(\eta, R)
\]

where \(\mathcal{R}(\eta, R)\) tends to zero as \(\eta, R \to 0^+\) and \(R_1\) is as in Proposition 3.11.

The proof of Theorem 1.2 using Proposition 4.1 proceeds with only minor differences to the argument in the Euclidean setting from [8]. We refer the reader to Section 5 for additional details. We then reduce proving Proposition 4.1 to demonstrating the existence of \(0 < \delta_0 < \frac{1}{16}\) for which there is some \(\delta \in [\delta_0, 2\delta_0]\) for which

\[
\tilde{E}_{\varepsilon, g_{R_1}}(v_\varepsilon, (x, T, 1), \delta) \leq \frac{e^{C_2}}{8} \left( \tilde{E}_{\varepsilon, g_{R_1}}(v_\varepsilon, (x, T, 1), 1) + C_7 E_0 R_1 \right) + \mathcal{R}(\eta, R)
\]
where $\mathcal{R}(\eta, R) \rightarrow 0^+$ as $\eta, R \rightarrow 0^+$. The argument reducing the proof of Proposition 4.1 to (4.7) is essentially the same as in [8] and is sketched in Subsection 5.1.

**Preliminary choice of good time slice:**

By Chebyshev’s inequality applied to (4.5) in time over the interval $[1 - 4\delta_0^2, 1 - 2\delta_0^2]$ we see that there are a large number of time slices $t = 1 - \delta^2$, where $\delta \in [\delta_0, 2\delta_0]$, for which

$$\int_M V_{\epsilon}(v_\epsilon) K_{ap,GR_1}(x, 1 - t; xT) \leq C(\delta_0)[RM_0 + \eta] \quad (4.8)$$

$$\int_M \Xi(v_\epsilon, (xT, 1)) K_{ap,GR_1}(x, 1 - t; xT) \leq C(\delta_0)[RM_0 + \eta]. \quad (4.9)$$

The inequalities (4.8) and (4.9) will be used to determine $\mathcal{R}(\eta, R)$ from (4.7) as well as obtain the coefficient of $E_{x, GR_1}(v_\epsilon, (xT, 1), 1) + C_7 E_0 R_1$ from (4.7). We use the notation $\Theta_1$ to denote

$$\Theta_1 := \{ t \in [1 - 4\delta_0^2, 1 - \delta_0^2] : (4.8) \text{ and } (4.9) \text{ both hold at } t \}. \quad (4.10)$$

In particular, we provide a more explicit estimate on the number of slices in Lemma A.4.1 of [12]. The strategy in proving (4.7) will be to decompose $E_{x, GR_1}(v_\epsilon, (xT, 1), \delta)$ into suitable components and estimate the resultant terms using PDE techniques by showing that the data can be controlled by (4.8) and (4.9).

**Localization and decomposition:**

We make use of Proposition 3.10, applied through $u_\epsilon$, in combination with (4.8) and (4.9) to obtain

$$\tilde{E}_{x, GR_1}(v_\epsilon, (xT, 1), \delta) \leq \frac{2}{(4\pi)^{N/2}} \delta^{N - 2} \int_{B_1(\sqrt{\|v_\epsilon\|}(xT))} e_\epsilon(v_\epsilon) \text{dvol} \quad (4.11)$$

$$+ K_0 \delta_0^2 R^2 \tilde{E}_{x, GR_1}(v_\epsilon, (xT, 1), 1) + C_7 E_0 R_1 + C(\delta_0)[RM_0 + \eta].$$

Since $C(\delta_0)[RM_0 + \eta]$ can be included in $\mathcal{R}(\eta, R)$ and $\delta_0^2 R^2$ can be chosen suitably small it suffices to estimate the remaining term from (4.11). To do this we decompose $e_\epsilon(u_\epsilon)$. We first observe that there is a constant $K > 0$ such that

$$e_\epsilon(v_\epsilon) \leq K \left( |v_\epsilon \times dv_\epsilon|^2 + |v_\epsilon|^2 |\nabla|v_\epsilon| |v_\epsilon|^2 + V_\epsilon(v_\epsilon) \right). \quad (4.12)$$

We further decompose $v_\epsilon \times dv_\epsilon$ by using a Hodge de Rham decomposition. To do this we first define $H(\omega)$ to be the harmonic part of a 2-form $\omega$. Explicitly,

$$H(\omega) := \sum_{i=1}^{2(M)} \left< \omega, \gamma_{i, GR_1} \right>_{L^2} \gamma_{i, GR_1} \quad (4.13)$$

where $\{ \gamma_{i, GR_1} \}_{i=1}^{2(M)}$ is an $L^2$-orthonormal basis for the space of harmonic 2-forms on $M$ in the metric $g_{R_1}$, obtained by rescaling an $L^2$-orthonormal
basis \( \{ \gamma_{i,j} \}^{\beta_2(M)} \) for the space of harmonic 2-forms on \( M \) in the metric \( g \), and \( \beta_2(M) \) denotes the 2nd Betti number of \( M \). We may then use a Hodge de Rham decomposition to find \( \varphi_t, \psi_t, \) and \( \xi_t \) satisfying

\[
v_i \times dv_i = d\varphi_t + d^*\psi_t + \xi_t \quad \text{on } B_{3r/2}(x_T) \times \{ t \}\]

(4.14)

\[
d^*\xi_t = 0 \quad \text{on } B_{3r/2}(x_T) \times \{ t \}\]

(4.15)

\[
d\xi_t = d^*d\psi_t + H(d[v_i \times dv_i]\chi) \quad \text{on } B_{3r/2}(x_T) \times \{ t \}\]

(4.16)

\[-\Delta\psi_t = d[v_i \times dv_i]\chi - H(d[v_i \times dv_i]\chi), \quad \text{on } M \times \{ t \}\]

(4.17)

where \( r > 0 \) is chosen sufficiently small toward the end of the argument and \( \chi \) is a smooth cutoff function supported in \( B_r(x_T) \) which is identically 1 on \( B_{2r}(x_T) \), \( 0 \leq \chi \leq 1 \), and \( \|\nabla\chi\|_{L_\infty} \leq \frac{2}{r} \). To make the notation more compact we set

\[
H^-(\omega) := \omega - H(\omega)
\]

(4.18)

where \( \omega \) is a 2-form over \( M \times \{ t \} \). We note that \( H \) is a new consideration for the Hodge de Rham decomposition that does not appear in the corresponding identities from \([8]\). This term arises because we impose no topological restrictions on \( M \).

From (4.11), (4.12), and (4.14) it will suffice to estimate each of the following:

\[
\int_{B_{3r/2}(x_T)} \left[ |v_i|^2 |\nabla v_i|^2 + V_c(v_i) \right] \]

(4.19)

\[
\int_{B_{3r/2}(x_T)} |d\varphi_t|^2 \]

(4.20)

\[
\int_{B_{3r/2}(x_T)} \left\{ |d\psi_t|^2 + |d^*\psi_t|^2 \right\} \]

(4.21)

\[
\int_{B_{3r/2}(x_T)} |\xi_t|^2. \]

(4.22)

Since \( H(d[v_i \times dv_i]\chi) \) is present in both (4.16) and (4.17) then to achieve our goal we will also need to provide estimates related to \( H \). Below we outline the strategy for estimating (4.19)–(4.22). Where necessary, we will provide additional details in Section 5.

**Modulus Estimate:**

As in \([8]\) the goal is to demonstrate that (4.19) satisfies an estimate of the form

\[
\int_{B_r(x_T)} \left\{ |v_i|^2 |\nabla v_i|^2 + V_c(v_i) \right\} \leq C(\delta_0, r)[RM_0 + \eta]^2 \left[ E_e, g_{H_1} (v, (x_T, 1), \delta) + CRE_0 + 1 \right].
\]

(4.23)

The proof of (4.23) follows the same procedure as Section 3.5 of \([8]\). As such, we will describe it briefly and refer the reader to Subsection A.4.2 of \([12]\) additional details. We first define \( \sigma_c := 1 - |v_i|^2 \) and observe that \( \sigma_c \) solves the PDE

\[
\partial_t \sigma_c - \Delta \sigma_c = 2|\nabla v_i|^2 - \frac{2}{\varepsilon^2} \sigma_c (1 - \sigma_c).
\]

(4.24)
By moving $\partial_t \sigma_\epsilon$ to the right-hand side we can treat this as a Poisson problem so that elliptic techniques can be applied to obtain an interior estimate for $\nabla \sigma_\epsilon$. We then apply various algebraic manipulations to estimate the terms on the right-hand side of the Poisson problem by quantities involving $\Xi$ and $V_\epsilon$. From there we make use of the assumption that $t \in \Theta_1$.

**Estimate of $\xi_t$:**

As in [8] we use that $\xi_t$ solves (4.15) and (4.16) in addition to elliptic estimates, see Lemma 5.2 of [1] and note the correction found in [2], to obtain

$$\|\xi_t\|_{L^2(B_{3\epsilon}(x_T))} \leq K \left[ \|d\psi_t\|_{L^2(B_{2\epsilon}(x_T))} + \|H(d[v_\epsilon \times d\nu_\epsilon])\|_{L^2(M)} \right].$$

(4.25)

The argument for (4.25) is the same as Lemma 3.4 of [8].

**Estimate of $\phi_t$:**

As in [8] the goal in estimating $\phi_t$ is to obtain an inequality of the form

$$\int_{B_s(x_T) \times \{t\}} (|d^* \psi_t|^2 + |\xi_t|^2) \leq \frac{K_M \delta_0^N}{r} \left[ \tilde{E}_{\epsilon, \delta R_1}(v_\epsilon, (x_T, 1), 1) + C_7 R_1 E_0 \right]$$

$$+ C(\delta_0, r) \left[ (RM_0 + \eta) + (RM_0 + \eta) \frac{\delta^2 R_1}{2} \left( \tilde{E}_{\epsilon, \delta R_1}(v_\epsilon, (x_T, 1), 1) + C_7 R_1 E_0 \right) + R_2(t) \right]$$

(4.26)

where

$$R_2(t) := \int_{B_{2\epsilon}(x_T)} (|d^* \psi_t|^2 + |\xi_t|^2)$$

$$+ \left( \int_{B_{2\epsilon}(x_T)} (|d^* \psi_t|^2 + |\xi_t|^2) \right)^{\frac{1}{2}} \left( \tilde{E}_{\epsilon, \delta R_1}(v_\epsilon, (x_T, 1), 1) + C_7 R_1 E_0 \right)^{\frac{1}{2}}.$$

Much of the proof extends with little change to the Riemannian setting with the exception of a computation done in coordinates. We present this new ingredient in Section 5.2 and provide a brief outline of the general argument below.

To achieve (4.26), we introduce an elliptic PDE that $\phi_t$ solves over $B_s(x_T)$ for $s \in [r, \frac{3r}{\epsilon}]$ where $s$ will be carefully chosen to ensure good properties. As shown in Section 5.2 we have that for each $s \in [r, \frac{3r}{\epsilon}]$, $\phi_t$ solves

$$\begin{cases}
L_\delta \varphi = h & \text{on } B_s(x_T) \times \{t\} \\
\frac{\partial \varphi}{\partial r} = g & \text{on } \partial B_s(x_T) \times \{t\},
\end{cases}$$

(4.27)

where $L_\delta$ is defined, using the abbreviation $K_{ap} := K_{ap, \delta R_1}(x; \delta^2; x_T)$, by

$$L_\delta := -\frac{1}{K_{ap}} \text{div}[K_{ap} \nabla].$$

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and
\[ h := v_\epsilon \times \left( \frac{\langle \nabla K_{ap}, \nabla v_\epsilon \rangle}{K_{ap}} - \partial_t v_\epsilon \right) + \left\langle \frac{dK_{ap}}{K_{ap}}, d^* \psi_t + \xi_t \right\rangle \quad \text{on } B_s(x_T) \times \{t\} \]
\[ g := v_\epsilon \times \frac{\partial v_\epsilon}{\partial r} - (d^* \psi_t + \xi_t) \quad \text{on } \partial B_s(x_T) \times \{t\}, \]
where $\omega_N$ denotes the normal part of $\omega$ and we recall that $t = 1 - \delta^2$ is assumed to be an element of $\Theta_1$. Next, we make use of elliptic estimates for (4.27) to obtain
\[ \int_{B_s(x_T)} |\nabla \phi|^2 e^{-\frac{(d(x,x_T))^2}{4\delta^2}} \leq C(\delta, r) \left[ \int_{B_s(x_T)} h^2 e^{-\frac{(d(x,x_T))^2}{4\delta^2}} + \left( \int_{B_s(x_T)} h^2 e^{-\frac{(d(x,x_T))^2}{4\delta^2}} \right)^{\frac{1}{2}} \right] + K_M r \int_{\partial B_s(x_T)} g^2 e^{-\frac{(d(x,x_T))^2}{4\delta^2}}. \]

Finally, to obtain (4.26) we estimate the data, $h$ and $g$, in terms of $\Xi$, $V_\epsilon(v_\epsilon)$, $d^* \psi_t$, and $\xi_t$ with one exception in which we obtain an estimate in terms of $v_\epsilon$. In particular, in estimating $g$ an averaging process is used for $s \in [r, \frac{3r}{2}]$ to estimate integrals over $\partial B_s(x_T)$ in terms of integrals over $B_{\frac{3r}{2}}(x_T) \setminus B_r(x_T)$. We note that the described exception results in the first term on the right-hand side of (4.26). This is the only term where $\delta$ will be needed to manufacture the leading coefficient of (4.7). We also note that the procedure for estimating $g$ is the same as in [8] except an application of (2.5) is needed. We refer the reader to Section 5 for more details.

**Estimate of $\psi_t$:**

Estimating $\psi_t$ is more involved and will be outlined through a number of steps. The goal of each step will be to successively decompose $\psi_t$ into terms with more specific information that can be utilized. Unlike previous estimates, subterms of $\psi_t$ are not always estimated by appealing to PDE techniques. Instead we may make use of detailed pointwise information as well as the the work of [21].

**Step 1: Decomposition of $\psi_t$**

Before proceeding with the decomposition we introduce some notation as well as some useful pointwise estimates. We introduce a real-valued function defined on $M \times (0, \infty)$ in terms of $|v_\epsilon|$ so that if $\tilde{v}_\epsilon = \tau v_\epsilon$ then
\[ |1 - \tau^2(x, t)| \leq K |1 - |v_\epsilon(x, t)||^2 \]  \hspace{1cm} (4.29)
\[ \tilde{v}_\epsilon = v_\epsilon \quad \text{if } |v_\epsilon| \leq \frac{1}{4}. \]  \hspace{1cm} (4.30)
\[ |\tilde{v}_\epsilon| = 1 \quad \text{if } |v_\epsilon| \geq \frac{1}{2}. \]  \hspace{1cm} (4.31)
To prove this, note that

$\Delta \psi_{1,t} = H^+ (d[\tilde{v}_t \times d\tilde{v}_t] \chi)$ on $M \times \{t\}$ (4.32)

$\Delta \psi_{2,t} = H^+ ([1 - \tau^2] v_t \times d\psi_t \chi)$ on $M \times \{t\}$. (4.33)

The smallness (4.29) of $1 - \tau^2$ will aid in estimates of $\psi_{2,t}$. A key point in estimates of $\psi_{1,t}$ is that

$$|d(\tilde{v}_t \times d\tilde{v}_t)| \leq K \frac{(1 - |v_t|^2)^2}{4\epsilon^2} = KV_r(v_t) \quad \text{on} \quad M \times (0, \infty). \quad (4.34)$$

To prove this, note that

$$d[\tilde{v}_t \times d\tilde{v}_t] = \sum_{i \neq j} \frac{\partial \tilde{v}_t}{\partial x_i} \times \frac{\partial \tilde{v}_t}{\partial x_j} dx^i \wedge dx^j. \quad (4.35)$$

By (4.3), the right-hand side is always bounded by $K/\epsilon^2$ and vanishes when $|\tilde{v}_t| = 1$, that is, when $|v_t| \geq 1/2$, so (4.34) follows from the definition of $V_r$. We also see from (4.35) that $d(\tilde{v}_t \times d\tilde{v}_t)$ has a Jacobian structure which we will exploit to apply [21].

**Step 2: Decomposition and estimate of $\psi_{2,t}$**

We further decompose $\psi_{2,t}$ as $\psi_{2,t} = \psi_{2,1,t} + \psi_{2,2,t}$ where $\psi_{2,1,t}$, $\psi_{2,2,t}$ solve

$$-\Delta \psi_{2,1,t} = d(1 - \tau^2)(v_t \times d\psi_t) \chi) \quad \text{on} \quad M \times \{t\} \quad (4.36)$$

$$-\Delta \psi_{2,2,t} = H^+ ((1 - \tau^2)[v_t \times d\psi_t] \wedge d\chi) \quad \text{on} \quad M \times \{t\} \quad (4.37)$$

where in (4.36) we have used that $H^+$ is the identity on exact forms. The argument to estimate $\psi_{2,1,t}$ and $\psi_{2,2,t}$ is similar in style to that of Lemma 3.8 of [8] though executed differently. In addition, the data of (4.37) requires additional estimates due to the harmonic projection term. We provide more details in Section 5.

Appealing to, among other things, elliptic regularity, the pointwise estimates (4.29), (4.3), and that $t \in \Theta_1$ we can estimate both terms in the decomposition to find that

$$\int_{M \times \{t\}} \left\{|d\psi_{2,t}|^2 + |d^* \psi_{2,t}|^2\right\} \leq C(\delta_0, r)[RM_0 + \eta]. \quad (4.38)$$

**Step 3: Decomposition of $\psi_{1,t}$**

The decomposition and estimates presented here represent an additional step required to extend the argument of [8] to the manifold setting. This arises due to the presence of the harmonic part in (4.32). More details are provided in Section 5. Since $\psi_{1,t}$ solves (4.32) then we may write

$$\psi_{1,t}(x) = \int_M \bigg\langle G(x, y), H^+ (d[\tilde{v}_t \times d\tilde{v}_t] \chi) \bigg\rangle$$

$$= \int_M \bigg\langle G(x, y), d[\tilde{v}_t \times d\tilde{v}_t] \chi \bigg\rangle - \int_M \bigg\langle G(x, y), H(d[\tilde{v}_t \times d\tilde{v}_t] \chi) \bigg\rangle$$

$$=: \psi_{1,t}(x) - (G \ast H(d[\tilde{v}_t \times d\tilde{v}_t] \chi))(x)$$

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where $G$ is the Dirichlet kernel for 2-forms on $M$. In addition, we can use (4.32) to establish
\[ \int_M \{ |d\psi_{1,t}|^2 + |d^\ast \psi_{1,t}|^2 \} = \int_M \left\langle \psi_{1,t}, H^+ (d[\tilde{\nu}_x \times d\tilde{\nu}_z] \chi) \right\rangle. \] (4.40)

Combining (4.39), (4.40), and estimates on the harmonic projection gives
\[ \int_M \{ |d\psi_{1,t}|^2 + |d^\ast \psi_{1,t}|^2 \} \leq \int_M \left\langle \tilde{\psi}_{1,t}, d[\tilde{\nu}_x \times d\tilde{\nu}_z] \chi \right\rangle + C(\delta_0, r)[RM_0 + \eta]^2 \] (4.41)
and so it suffices to estimate
\[ \int_M \left\langle \tilde{\psi}_{1,t}, d[\tilde{\nu}_x \times d\tilde{\nu}_z] \chi \right\rangle. \] (4.42)

The advantage of (4.42) is that both the “convolution” defining $\tilde{\psi}$ as well as the integral defining (4.42) can be taken over small geodesic ball. This facilitates the use of normal coordinates and allows for the use of a detailed coordinate description of the Green’s function.

**Step 4: Decomposition of $\tilde{\psi}_{1,t}$**

Here we follow the decomposition technique presented in [8] which adapts to the setting of a Riemannian manifold with only minor differences. A more complete discussion can be found in Section 5.

Following [8] we choose an appropriate $\alpha > 0$ and carefully construct a Lipschitz function $l: [0, \infty) \to [0, \infty)$ supported on $[e^\alpha r, 32r]$ such that $l \equiv 1$ on $[2e^\alpha r, 16r]$. From this we define $m: [0, \infty) \to [0, \infty)$ by
\[ m(s) := \begin{cases} 1 - l(s) & \text{for } s \in [0, 16r] \\ 0 & \text{for } s \in (16r, \infty) \end{cases} \] (4.43)
and set
\[ G(x, y) = m(d(x, y))G(x, y) + (1 - m(d(x, y)))G(x, y) =: G^i(x, y) + G^e(x, y). \]
This decomposition allows us to define
\[ \tilde{\psi}^i_{1,t} := \int_{B_{2r}(x,T)} \left\langle G^i(x, y), d[\tilde{\nu}_x \times d\tilde{\nu}_z] \chi \right\rangle \] (4.44)
\[ \tilde{\psi}^e_{1,t} := \int_{B_{32r}(x)} \left\langle G^e(x, y), d[\tilde{\nu}_x \times d\tilde{\nu}_z] \chi \right\rangle. \] (4.45)
Using (4.44) and (4.45) we reduce estimating (4.42) to estimating
\[ \int_M \left\langle \tilde{\psi}^i_{1,t}, d[\tilde{\nu}_x \times d\tilde{\nu}_z] \chi \right\rangle \] (4.46)
\[ \int_M \left\langle \tilde{\psi}^e_{1,t}, d[\tilde{\nu}_x \times d\tilde{\nu}_z] \chi \right\rangle. \] (4.47)
Step 5: Estimating $\tilde{\psi}_{1,t}$

Extending the argument to the setting of a Riemannian manifold retains much of the style of the corresponding one from [8]. The main difference is that we need to make use of detailed local coordinate expressions for the Green’s function on 2-forms, $G$, as well as of the integrand of (4.46). We highlight the main ideas below and refer the reader to Section 5 for a more complete account.

Here, working in normal coordinates, we exploit the Jacobian structure of $d[\tilde{v}_e \times d\tilde{v}_e]$, see (4.35), together with estimates from [21] to obtain an $L^\infty$ estimate on $\tilde{\psi}_{1,t}$ which can be paired with pointwise estimates of $d[\tilde{v}_e \times d\tilde{v}_e]\chi$ to show

$$\int_M \langle \tilde{\psi}_{1,t}, d[\tilde{v}_e \times d\tilde{v}_e]\chi \rangle \leq C(\delta_0, r)(\tilde{E}_e, g)_{R_1}(v_e, (x_T, 1), 1) + C_7 R_1 E_0 + 1)[RM_0 + \eta].$$

These computations use detailed information on the form of the Green’s function in normal coordinates.

Step 6: Auxiliary parabolic problem

Unfortunately, the obtainable estimates for $\tilde{\psi}_{1,t}$ are insufficient to make use of duality in (4.47). As a result we, following [8], introduce $\psi^*_1$ the solution to the parabolic problem

$$\begin{aligned} \partial_t \psi^*_1 + \Delta \psi^*_1 &= d[\tilde{v}_e \times d\tilde{v}_e]\chi \quad &\text{on } M \times (0, \infty) \setminus \{0\}, \\ \psi^*_1(\cdot, 0) &= 0 &\text{on } M \times \{0\}, \end{aligned}$$

and use this to replace $d[\tilde{v}_e \times d\tilde{v}_e]\chi$ with $\partial_t \psi^*_1 - \Delta \psi^*_1$ in (4.47). Thus, it suffices to estimate each of

$$\int_M \langle \tilde{\psi}_{1,t}, \partial_t \psi^*_1 \rangle \quad \text{(4.50)}$$

$$\int_M \langle \tilde{\psi}_{1,t}, -\Delta \psi^*_1 \rangle \quad \text{(4.51)}$$

The arguments to estimate (4.50) and (4.51) extend to the Riemannian setting with a bit of additional work. For $\tilde{\psi}_{1,t}$ the main obstacle is the need for local coordinate expressions of the Green’s function and direct computations of the distributional Laplacian of $G$. The techniques for estimating $\psi^*_1$ extend to the Riemannian setting but a bit of care is needed. We provide more details in Section 5.

Using (4.49) it is possible to find a time slice for which we have $L^2$ estimates on $\partial_t \psi^*_1$ as well as be a member of $\Theta_1$. For such $t$ we can estimate (4.50) by Cauchy-Schwarz. We estimate (4.51) by making use of information about the distributional laplacian of $G$, the integral kernel defining $\tilde{\psi}_{1,t}$. These arguments closely follow [8], but some adjustments are needed to adapt them to the Riemannian setting.
5 Clearing Out Proof

In this section we present the details omitted from the outline presented in Section 4. As in Section 4, unless otherwise specified, all metric related quantities will be associated to \( g_{R_1} \) and the metric will be suppressed from the notation.

5.1 Reduction via rescaling

Following [8] we first reduce the proof of Theorem 1.2 to that of Proposition 4.1, which is stated in Section 4. We refer the reader to Section 4 for the relevant definitions surrounding the rescaled solution \( v_\epsilon \) as well as to A.4.5.1 of [12] for a detailed account of this reduction.

Proof of Theorem 1.2, assuming Proposition 4.1. Using the conclusion of Proposition 4.1 as well as (4.4) leads to

\[
\tilde{E}_{\epsilon, g_{R_1}}(v_\epsilon(x_T, 1), 1) + C_7 E_0 R_1 \leq R_1(\eta, R)
\]

where \( R_1(\eta, R) \to 0^+ \) as \( \eta, R \to 0^+ \). It is then possible to choose \( T_\epsilon \), also dependent on \( \sigma \), such that \( T_\epsilon = 1 + O_\sigma(\epsilon^2) \) and, by an extension of Lemma (III.3) of [6] to our setting,

\[
1 - |v_\epsilon(x_T, T_\epsilon)| \leq C_M \left[ \frac{1}{\epsilon^N} \int_{B_\epsilon(x_T)} (1 - |v_\epsilon(x, T_\epsilon)|^2)^2 \right]^{\frac{1}{N+1}} \leq D_M R_1(\eta, R)
\]

(5.1)

where \( C_M \) and \( D_M \) are constants that depend on \( M \). Next, using the time derivative estimate from (4.3) and our choice of \( T_\epsilon \), we have

\[
|v_\epsilon(x_T, T_\epsilon) - v_\epsilon(x_T, 1)| \leq \frac{\sigma}{2}.
\]

(5.2)

Hence, after combining (5.1) and (5.2), as well as choosing \( R_0 \) and \( \eta_0 \) small enough that we can ensure \( D_M R_1(\eta, R) \leq \frac{\sigma}{2} \), we will have the desired result. \( \square \)

As remarked in Section 4, to prove Proposition 4.1 it suffices to demonstrate that for some \( 0 < \delta_0 < \frac{1}{16} \) there is \( \delta \in [\delta_0, 2\delta_0] \) such that

\[
\tilde{E}_{\epsilon, g_{R_1}}(v_\epsilon(x_T, 1), \delta) \leq e^{-C_2} \tilde{E}_{\epsilon, g_{R_1}}(v_\epsilon(x_T, 1), 1) + C_7 E_0 R_1 + R(\eta, R)
\]

where \( R(\eta, R) \) tends to zero as \( \eta, R \to 0^+ \) and \( R_1 \) is as in Proposition 3.11. To see this, observe that by applying Proposition 3.5 through \( u_\epsilon \), using (4.7), as well as that \( C_1 \leq C_7 \) and \( \delta < 2\delta_0 < \frac{1}{8} \) we have

\[
\tilde{E}_{\epsilon, g_{R_1}}(v_\epsilon(x_T, 1), \delta_0) \leq e^{C_2} \tilde{E}_{\epsilon, g_{R_1}}(v_\epsilon(x_T, 1), \delta) + \delta[C_1 E_0 R_1]
\]

\[
\leq e^{C_2} \left( \tilde{E}_{\epsilon, g_{R_1}}(v_\epsilon(x_T, 1), 1) + C_7 E_0 R_1 \right) + \delta[C_1 E_0 R_1] + e^{C_2} R(\eta, R)
\]

\[
\leq \frac{1}{4} \left( \tilde{E}_{\epsilon, g_{R_1}}(v_\epsilon(x_T, 1), 1) + C_7 E_0 R_1 \right) + e^{C_2} R(\eta, R).
\]
5.2 Estimate of $\varphi_t$

To obtain the PDE (4.27) we begin by taking the cross product of (PGL), with $v_t$ we obtain

$$0 = v_t \times \partial_t v_t + d^*(v_t \times dv_t) \quad \text{on } M \times (0, \infty).$$

(5.3)

By applying $d^*$ to (4.14) we see that

$$d^*(v_t \times dv_t) = -\Delta \varphi_t \quad \text{on } B_{2p}(x_T) \times \{t\}.$$  

(5.4)

Rewriting $v_t \times \partial_t v_t$ as

$$v_t \times \partial_t v_t = v_t \times \left( \frac{\langle \nabla K_{ap}, \nabla v_t \rangle}{K_{ap}} + \partial_t v_t \right) - \left( \frac{dK_{ap}}{K_{ap}} d\varphi_t + d^* \psi_t + \xi_t \right)$$

and then applying (5.3) and (5.4) leads to

$$-\Delta \varphi_t - \left( \frac{\nabla K_{ap}}{K_{ap}}, \nabla \varphi_t \right) = v_t \times \left( \frac{-\langle \nabla K_{ap}, \nabla v_t \rangle}{K_{ap}} - \partial_t v_t \right) + \left( \frac{dK_{ap}}{K_{ap}} d\varphi_t + d^* \psi_t + \xi_t \right).$$

(5.5)

Finally, noting that

$$-\Delta \varphi_t - \left( \frac{\nabla K_{ap}}{K_{ap}}, \nabla \varphi_t \right) = \frac{-1}{K_{ap}} \text{div}(K_{ap} \nabla \varphi_t)$$

we may rewrite (5.5) as well as introduce boundary conditions coming from (4.14) to obtain (4.27). We use elliptic PDE techniques to estimate the $L^2$ norm of $\nabla \varphi_t$ on $B_s(x_T)$, where $s \in \left[ r, \frac{3}{2} \right]$ will be chosen later, in terms of the data from (4.27). First, we multiply the PDE from (4.27) by $-2\delta^2 \langle \nabla v, \nabla K_{ap} \rangle$ and integrate by parts to obtain

$$-2\delta^2 \int_{B_s(x_T)} h(\nabla \varphi, \nabla K_{ap}) = \frac{2\delta^2}{B_s(x_T)} \text{div}(K_{ap} \nabla \varphi) \frac{\langle \nabla \varphi, \nabla K_{ap} \rangle}{K_{ap}}$$

$$\quad = -2\delta^2 \int_{B_s(x_T)} K_{ap} \left( \nabla \varphi, \nabla \left( \frac{\langle \nabla \varphi, \nabla K_{ap} \rangle}{K_{ap}} \right) \right)$$

$$\quad \quad + 2\delta^2 \int_{\partial B_s(x_T)} g(\nabla \varphi, \nabla K_{ap}).$$

(5.6)

Then one can verify, for example by a pointwise computation in normal coordinates, that

$$K_{ap} \left( \nabla \varphi, \nabla \left( \frac{\langle \nabla \varphi, \nabla K_{ap} \rangle}{K_{ap}} \right) \right) = \text{Hess}(K_{ap})(\nabla \varphi, \nabla \varphi) - \frac{\langle \nabla \varphi, \nabla K_{ap} \rangle^2}{K_{ap}} + \frac{1}{2} \langle \nabla (|\nabla \varphi|^2), \nabla K_{ap} \rangle.$$ 

(5.7)

Using (5.7) in (5.6) followed by integrating by parts leads to

$$2\delta^2 \int_{B_s(x_T)} \left[ -\frac{\Delta K_{ap}}{2} |\nabla \varphi|^2 + \text{Hess}(K_{ap})(\nabla \varphi, \nabla \varphi) - \frac{\langle \nabla \varphi, \nabla K_{ap} \rangle^2}{K_{ap}} \right] = -2\delta^2 \int_{B_s(x_T)} h(\nabla \varphi, \nabla G)$$

$$\quad \quad \quad = -\delta^2 \int_{\partial B_s(x_T)} |\nabla \varphi|^2 \frac{\partial K_{ap}}{\partial r} + 2\delta^2 \int_{\partial B_s(x_T)} g(\nabla \varphi, \nabla K_{ap}).$$

(5.8)
We then compute the integrand of the braced term of (5.8) more explicitly in terms of the function (2.3) and apply (2.5) to obtain
\[
\int_{B_s(x_T)} \frac{(N-2)}{2} |\nabla \varphi|^2 K_{ap} - \int_{B_s(x_T)} \frac{(d(x,x_T))^2}{4\delta^2} \left[ 1 + \frac{2N\lambda \delta^2}{3} - 2\mu \delta^2 \right] |\nabla \varphi|^2 K_{ap} - 2\delta^2 \int_{B_s(x_T)} h(\nabla \varphi, \nabla K_{ap})
\]
(5.9)
\[
\geq -\delta^2 \int_{\partial B_s(x_T)} |\nabla \varphi|^2 \frac{\partial K_{ap}}{\partial r} + 2\delta^2 \int_{\partial B_s(x_T)} g(\nabla \varphi, \nabla K_{ap}).
\]

More details can be found in Lemma A.4.4 of [12]. From there we appeal to standard estimates and choose \(d_0\) such that \(0 < 2d_0 \leq \frac{1}{2(\lambda + \frac{1}{2}) + \frac{1}{2}}\) to obtain
\[
-2\delta^2 \int_{B_s(x_T)} h(\nabla \varphi, \nabla K_{ap}) - \int_{B_s(x_T)} \frac{(d(x,x_T))^2}{4\delta^2} \left[ 1 + \frac{2N\lambda \delta^2}{3} - 2\mu \delta^2 \right] |\nabla \varphi|^2 K_{ap}
\]
\[
\leq \frac{1}{2} \int_{B_s(x_T)} h^2 K_{ap} - \frac{1}{2} \int_{B_s(x_T)} \frac{(N-2)}{2} |\nabla \varphi|^2 K_{ap} + \frac{1}{2} \int_{B_s(x_T)} h^2 K_{ap}
\]
(5.10)

which, when combined with (5.9), leads to
\[
\int_{B_s(x_T)} \frac{(N-2)}{2} |\nabla \varphi|^2 K_{ap} + \frac{1}{2} \int_{B_s(x_T)} h^2 K_{ap}
\]
\[
\geq -\delta^2 \int_{\partial B_s(x_T)} |\nabla \varphi|^2 \frac{\partial K_{ap}}{\partial r} + 2\delta^2 \int_{\partial B_s(x_T)} g(\nabla \varphi, \nabla K_{ap}).
\]

Then an explicit computation of \(\frac{\partial K_{ap}}{\partial r}\) and \((\nabla \varphi, \nabla K_{ap})\), analogous to [8], give
\[
\int_{\partial B_s(x_T)} |\nabla \varphi|^2 K_{ap} \leq \int_{B_s(x_T)} \frac{(N-2)}{2s} |\nabla \varphi|^2 K_{ap} + \frac{1}{s} \int_{B_s(x_T)} h^2 K_{ap}
\]
(5.11)
\[
+ \int_{\partial B_s(x_T)} g^2 K_{ap}
\]
for each \(s \in [r, 3r/2]\) where
\[
\nabla \varphi := \nabla \varphi - \frac{\partial \varphi}{\partial r} \frac{\partial}{\partial r}
\]

More details can be found in Corollary A.4.5 of [12]. This estimate will permit us to obtain control over the \(L^2\) norm of \(\varphi\) on \(\partial B_s(x_T)\) in terms of \(h\) and \(g\). Next if we multiply the PDE of (4.27) by \(\varphi\), integrate by parts, and make use of (5.11), the Poincaré-Wirtinger, and Young’s inequality we obtain
\[
\int_{B_s(x_T)} |\nabla \varphi|^2 e^{-\frac{(d(x,x_T))^2}{4\delta^2}}
\]
(5.12)
\[
\leq C(\delta,r) \left[ \int_{B_s(x_T)} h^2 e^{-\frac{(d(x,x_T))^2}{4\delta^2}} + \left( \int_{B_s(x_T)} h^2 e^{-\frac{(d(x,x_T))^2}{4\delta^2}} \right)^{\frac{1}{2}} \left( \int_{\partial B_s(x_T)} g^2 e^{-\frac{(d(x,x_T))^2}{4\delta^2}} \right)^{\frac{1}{2}} \right]
\]
+ \(K_M\) \int_{\partial B_s(x_T)} g^2 e^{-\frac{(d(x,x_T))^2}{4\delta^2}}
\]
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for each \( s \in [r, 3r/2] \). We notice that when \( h \) is the data from (4.27) then we have the pointwise estimate
\[
h^2 \leq C(\delta_0, r) \left[ \Xi(v_s, (x_T, 1) + |d^* \psi_t|^2 + |\xi_t|^2 \right]. \tag{5.13}
\]

As a result of (5.13) and the assumption that \( t \in \Theta_1 \) we have
\[
\int_{B_s(x_T)} h^2 e^{-\frac{(d(x, x_T))^2}{4r^2}} \leq C(\delta_0, r) \left[ RM_0 + \eta + \int_{B_{\delta_0}(x_T)} (|d^* \psi_t|^2 + |\xi_t|^2) \right]. \tag{5.14}
\]

A similar pointwise estimate for the data \( g \) from (4.27) leads to
\[
\int_{\partial B_s(x_T)} g^2 e^{-\frac{(d(x, x_T))^2}{4r^2}} \leq K_M \int_{\partial B_s(x_T)} \left[ |\nabla v_s|^2 + |d^* \psi_t|^2 + |\xi_t|^2 \right] e^{-\frac{(d(x, x_T))^2}{4r^2}}. \tag{5.15}
\]

Averaging to find a suitable \( s \in [r, 3r/2] \) and manipulating Gaussian functions gives
\[
\int_{\partial B_s(x_T)} g^2 e^{-\frac{(d(x, x_T))^2}{4r^2}} \leq \frac{K_M}{r} \int_{\partial B_s(x_T)} |\nabla v_s|^2 e^{-\frac{(d(x, x_T))^2}{4r^2}} + \frac{K_M}{r} \int_{B_s(x_T)} \left[ |d^* \psi_t|^2 + |\xi_t|^2 \right] \tag{5.16}
\]
\[
= \frac{(4\pi)^N K_M\delta_0^N}{r} \int_{B_s(x_T)} |\nabla v_s|^2 K_{ap,g_{B_s}} + \frac{K_M}{r} \int_{B_s(x_T)} \left[ |d^* \psi_t|^2 + |\xi_t|^2 \right].
\]

More details are provided in A.4.3.1 of [12]. Combining (5.12) with (5.14) and (5.16) gives (4.26) if we choose \( \delta_0 \) small enough that \( 2\delta_0 \sqrt{SC} \leq r \).

### 5.3 Estimate of \( \psi_t \)

The strategy from [8] to estimate \( \psi_t \) extends to our setting with a few modifications mostly caused by the use of coordinates and the possibility of non-trivial homology. In particular, some additional work is due to the presence of the harmonic projection \( H \).

We first observe that since \( \psi_t \) solves (4.17) then we can represent \( \psi_t \) as
\[
\psi_t(x) = \int_M \left( G(x, y), H^\perp (d[v_s(y) \times dv_s(y)] \chi(y) \right) d\text{vol}(y) \tag{5.17}
\]
where \( G \) is the Dirichlet kernel for 2-forms on \( M \) with the metric \( g_{B_1} \), which can be constructed in coordinates using the results of [4] and [9]. In particular, for each \( x, y \in M \) we have \( G(\cdot, y), G(x, \cdot) \in W^{1,1}(M; \wedge^{(2,2)} M), -\Delta_s G(\cdot, y) \in L^1(M; \wedge^2 M) \), and when \( x, y \) are elements of a normal coordinate neighbourhood then
\[
|G(x, y)| \leq K(d(x, y))^{2-N}, \quad |DG(x, y)| \leq K(d(x, y))^{1-N} \tag{5.18}
\]
\[
|\Delta_s G(x, y)| \leq K(d(x, y))^{2-N}. \tag{5.19}
\]

In particular, if we were to rescale the metric by a factor \( a^2 \) for \( a > 0 \) we would find
\[
G_{a^2 g} = a^{6-N} G_g \tag{5.20}
\]
where $G_g$ denotes the Green’s function constructed using the metric $g$.

Next we introduce a smooth function $\rho$ such that
\[
\rho(s) = \begin{cases} 
1 & \text{for } s \in [0, 1/4], \\
\frac{1}{s} & \text{for } s \geq 1/2,
\end{cases} \quad \|\rho^\prime\|_{L^{\infty}(\mathbb{R})} \leq 4
\] (5.21)
as well as the localized version of $v_{\epsilon}$
\[
\tilde{v}_{\epsilon}(x, t) := \tau(x, t)v_{\epsilon}(x, t), \quad \tau(x, t) := \rho(|v_{\epsilon}(x, t)|).
\] (5.22)

Noting that $\tilde{v}_{\epsilon} \times d\tilde{v}_{\epsilon} = \tau^2 v_{\epsilon} \times dv_{\epsilon}$, we split (5.17) into
\[
\psi_t(x) = \int_M \left\langle G(x, y), H^\perp(d[\tilde{v}_{\epsilon} \times d\tilde{v}_{\epsilon}]\chi) \right\rangle + \int_M \left\langle G(x, y), H^\perp(d(1 - \tau^2)v_{\epsilon} \times dv_{\epsilon}) \right\rangle
\] (5.23)
\[
=: \psi_{1,t} + \psi_{2,t}.
\]
From the definitions of $\psi_{1,t}$ and $\psi_{2,t}$ we see that
\[
-\Delta \psi_{1,t} = H^\perp(d[\tilde{v}_{\epsilon} \times d\tilde{v}_{\epsilon}]\chi) \quad \text{on } M \times \{t\}
\] (5.24)
\[
-\Delta \psi_{2,t} = H^\perp(d(1 - \tau^2)v_{\epsilon} \times dv_{\epsilon}) \chi) \quad \text{on } M \times \{t\}.
\] (5.25)

We observe that there is $K > 0$ such that
\[
|1 - \tau^2(x, t)| \leq K|1 - |v_{\epsilon}(x, t)|^2|
\] (5.26)
\[
|d[\tilde{v}_{\epsilon} \times d\tilde{v}_{\epsilon}]\chi| \leq KV_{\epsilon}(v_{\epsilon})
\] (5.27)
over $M \times \{t\}$.

### 5.3.1 Estimate of Harmonic Projection

Before we begin estimating $\psi_{1,t}$ and $\psi_{2,t}$ we first obtain estimates for $H(d[v_{\epsilon} \times dv_{\epsilon}]\chi)$. Note that
\[
d[v_{\epsilon} \times dv_{\epsilon}]\chi = d[\tilde{v}_{\epsilon} \times d\tilde{v}_{\epsilon}]\chi + d(1 - \tau^2)\chi (v_{\epsilon} \times dv_{\epsilon})\chi) + (1 - \tau^2)[v_{\epsilon} \times dv_{\epsilon}] \wedge d\chi
\] (5.28)
and so the definition of $H$ implies that
\[
H(d[v_{\epsilon} \times dv_{\epsilon}]\chi) = H(d[\tilde{v}_{\epsilon} \times d\tilde{v}_{\epsilon}]\chi) + H((1 - \tau^2)\chi (v_{\epsilon} \times dv_{\epsilon}) \wedge d\chi).
\] (5.29)

We now estimate each of the terms on the right-hand side of (5.29). By straightforward estimates using (4.13), the definition of $H$, we may find a constant $K$ such that
\[
\|H(\omega)\|_{L^{2}(\mathbb{R}^2 \times M)} \leq KR_1^{\frac{N}{2}} \|\omega\|_{L^1(\mathbb{R}^2 \times M)}
\] (5.30)
\[
\|H(\omega)\|_{L^{\infty}(\mathbb{R}^2 \times M)} \leq KR_1^{N} \|\omega\|_{L^1(\mathbb{R}^2 \times M)}
\] (5.31)
for all 2-forms $\omega$, where the exponents on $R_1$ are due to the scaling properties of the basis $\{\gamma_{\theta R_1}g^{-1}M\}_{i=1}^{\frac{N}{2}}$ appearing in (4.13). Next observe that by (5.27) and manipulations with Gaussian functions using that $\chi$ is supported on $B_{\delta_0}(x_T)$, we have
\[
\|d[\tilde{v}_{\epsilon} \times d\tilde{v}_{\epsilon}]\chi\|_{L^1(\mathbb{R}^2 \times M)} \leq C(\delta_0, r) \int_M V_{\epsilon}(v_{\epsilon}) K_{\theta R_1}.
\] (5.32)
Now we consider \((1 - \tau^2)[v_\epsilon \times d\nu_\epsilon] \wedge d\chi\). Using (4.3), Cauchy-Schwarz, (5.26), and manipulations with Gaussian functions using that \(\chi\) is supported on \(B_\epsilon(x_T)\) leads to

\[
\| (1 - \tau^2)[v_\epsilon \times d\nu_\epsilon] \wedge d\chi \|_{L^1(\Lambda^2 M)} \leq C(\delta_0, r) \left( \int_M V_\epsilon(v_\epsilon) K_{ap, g} \right)^\frac{1}{2}.
\] (5.33)

We will make use of various combinations of (5.30) and (5.31) with (5.32) and (5.33).

### 5.3.2 Estimate of \(\psi_{2, t}\)

The aim of this subsection is to establish the following estimate:

\[
\int_{M \times \{t\}} \left\{ |d\psi_{2, t}|^2 + |d^*\psi_{2, t}|^2 \right\} \leq C(\delta_0, r)[RM_0 + \eta].
\] (5.34)

This will be achieved by making use of the Poisson problem (5.25), appealing to elliptic regularity, and applying our assumption that \(t \in \Theta_1\). We note that, as in previous estimates, the goal is to show that the data from the PDE (5.25) can be estimated in terms of (4.8) and (4.9).

We notice that \(\psi_{2, t}\) can be further decomposed as \(\psi_{2, t} = \psi_{2, t}^1 + \psi_{2, t}^2\) where \(\psi_{2, t}^1, \psi_{2, t}^2\) satisfy

\[
-\Delta \psi_{2, t}^1 = d[(1 - \tau^2)(v_\epsilon \times d\nu_\epsilon)\chi] \text{ on } M \times \{t\}
\] (5.35)

\[
-\Delta \psi_{2, t}^2 = H^\perp(1 - \tau^2)[v_\epsilon \times d\nu_\epsilon] \wedge d\chi \text{ on } M \times \{t\}
\] (5.36)

where in (5.35) we have used that \(H^\perp\) is the identity on exact forms. Since \(\psi_{2, t}^1\) solves (5.35) elliptic regularity gives

\[
\int_{M \times \{t\}} \left\{ |d\psi_{2, t}^1|^2 + |d^*\psi_{2, t}^1|^2 \right\} \leq K \| (1 - \tau^2)(v_\epsilon \times d\nu_\epsilon)\chi \|^2_{L^2(\Lambda^2 M)}.
\] (5.37)

Using (4.3), (5.26), and manipulations with Gaussian functions using that the support of \(\chi\) is \(B_\epsilon(x_T)\) gives

\[
\int_{M \times \{t\}} |(1 - \tau^2)(v_\epsilon \times d\nu_\epsilon)\chi|^2 \leq C(\delta_0, r) \int_{M \times \{t\}} V_\epsilon(v_\epsilon) K_{ap, g}.
\] (5.38)

Combining (5.37), (5.38), and using that \(t \in \Theta_1\) leads to

\[
\int_{M \times \{t\}} \left\{ |d\psi_{2, t}^1|^2 + |d^*\psi_{2, t}^1|^2 \right\} \leq C(\delta_0, r)[RM_0 + \eta].
\] (5.39)

Next, since \(\psi_{2, t}^2\) solves (5.36) then elliptic regularity gives

\[
\int_{M \times \{t\}} \left\{ |d\psi_{2, t}^2|^2 + |d^*\psi_{2, t}^2|^2 \right\} \leq K \left\| H^\perp((1 - \tau^2)[v_\epsilon \times d\nu_\epsilon] \wedge d\chi) \right\|_{L^2(\Lambda^2 M)}^2.
\] (5.40)
It then follows from (5.30), (5.33), and similar considerations as in (5.38) that we have
\[
\left\| H^\perp ((1 - \tau^2)\nabla v_v \wedge d\chi) \right\|_{L^2(M)}^2 \leq C(\delta_0, r) \int_{M \times \{t\}} V_r(v_v) K_{ap, 9R_1}.
\]
(5.41)
From (5.40), (5.41), and the assumption that \( t \in \Theta_1 \) it follows that
\[
\left\{ |d\psi^2_{2,t}|^2 + |d^* \psi^2_{2,t}|^2 \right\} \leq C(\delta_0, r)[R\delta_0 + \eta].
\]
(5.42)
Finally, combining (5.39) and (5.42) gives (5.34).

5.3.3 Estimate of \( \psi_{1,t} \)
Next we estimate \( \psi_{1,t} \). We proceed with the a slightly modified version of the strategy applied in [8]. The main difference is the need to estimate terms relating to the harmonic projection \( H \). In addition, some computations need to be done in coordinates, for example the estimate of the low frequency term \( \tilde{\psi}^1_{1,t} \).

Taking the inner product of (5.24) with \( \psi_{1,t} \) and integrating by parts we obtain
\[
\int_{M \times \{t\}} \left[ |d\psi_{1,t}|^2 + |d^* \psi_{1,t}|^2 \right] = \int_{M \times \{t\}} \left\langle \psi_{1,t}, H^\perp (d[\tilde{v}_v \times d\tilde{v}_v] \chi) \right\rangle.
\]
(5.43)
Thus, to obtain control of the \( L^2 \) norms of the differential and codifferential of \( \psi_{1,t} \) it suffices to estimate
\[
\int_{M \times \{t\}} \left\langle \psi_{1,t}, H^\perp (d[\tilde{v}_v \times d\tilde{v}_v] \chi) \right\rangle.
\]
(5.44)
As a result, we focus on obtaining an upper bound of (5.44). We proceed through a series of steps.

**Step 1: Localization**
Due to the presence of the harmonic projection we will need a few additional estimates not needed in [8]. We begin by noting that \( \psi_{1,t} \) can be decomposed as
\[
\psi_{1,t} = \int_M \left\langle G, H^\perp (d[\tilde{v}_v \times d\tilde{v}_v] \chi) \right\rangle
\]
(5.45)
\[
= \int_M \left( G, d[\tilde{v}_v \times d\tilde{v}_v] \chi \right) - \int_M \left\langle G, H(d[\tilde{v}_v \times d\tilde{v}_v] \chi) \right\rangle
\]
\[
= : \tilde{\psi}_{1,t} - G \ast H (d[\tilde{v}_v \times d\tilde{v}_v] \chi).
\]
We then use (5.45) to rewrite (5.44) as
\[
\int_{M \times \{t\}} \left\langle \psi_{1,t}, H^\perp (d[\tilde{v}_v \times d\tilde{v}_v] \chi) \right\rangle = \int_M \left\langle \tilde{\psi}_{1,t}, d[\tilde{v}_v \times d\tilde{v}_v] \chi \right\rangle - \int_M \left\langle G \ast H (d[\tilde{v}_v \times d\tilde{v}_v] \chi), d[\tilde{v}_v \times d\tilde{v}_v] \chi \right\rangle
\]
(5.46)
\[
- \int_M \left\langle \psi_{1,t}, H (d[\tilde{v}_v \times d\tilde{v}_v] \chi) \right\rangle.
\]
We now estimate the last two terms of (5.46). We will start with

\[ \int_{M \times \{t\}} \langle \psi_{1,t}, H(d[\tilde{v}_x \times d\tilde{v}_z] \chi) \rangle. \]

Observe that by the integral representation of \( \psi_{1,t} \), standard integral kernel estimates, as well as \( W^{1,1} \) estimates on \( G \) we obtain

\[ \int_{M \times \{t\}} |\psi_{1,t}| \leq KR_1^{-2} \left\{ \int_M |d[\tilde{v}_x \times d\tilde{v}_z] | \chi | + \| H(d[\tilde{v}_x \times d\tilde{v}_z] \chi) \|_{L^\infty(\alpha^2 M)} \text{ vol}(M) \right\}. \]

(5.47)

Combining (5.31), (5.32), (5.27), manipulations of Gaussians that use that \( \chi \) is supported on \( B_4(x_T) \), as well as the assumption that \( t \in \Theta_1 \) with (5.47) we obtain

\[ \int_M |\psi_{1,t}| \leq C(\delta_0, r) R_1^{-2} [RM_0 + \eta]. \]

(5.48)

Finally, by (5.48), (5.31), and (5.32) we have

\[ \left| \int_M \langle \psi_{1,t}, H(d[\tilde{v}_x \times d\tilde{v}_z] \chi) \rangle \right| \leq C(\delta_0, r) R_1^{-2} [RM_0 + \eta]^2. \]

(5.49)

Next we estimate \( G * H(d[\tilde{v}_x \times d\tilde{v}_z] \chi) \). Observe that by (5.31), (5.32), and manipulations with Gaussian functions which use that \( \chi \) is supported on \( B_{4r}(x_T) \) we have

\[ \| G * H(d[\tilde{v}_x \times d\tilde{v}_z] \chi) \|_{L^\infty(\alpha^2 M)} \leq C(\delta_0, r) R_1^{-2} \int_{M \times \{t\}} V_\epsilon(\psi) K_{ap, g R_1}. \]

Thus, combining this with (5.27), similar Gaussian function manipulations to those in (5.48), and the assumption that \( t \in \Theta_1 \), we have

\[ \left| \int_M \langle G * H(d[\tilde{v}_x \times d\tilde{v}_z] \chi), d[\tilde{v}_x \times d\tilde{v}_z] \chi \rangle \right| \leq C(\delta_0, r) R_1^{-2} [RM_0 + \eta]^2. \]

(5.50)

**Step 2: Decomposition of \( \tilde{\psi}_{1,t} \)**

As a result of the previous step we focus on estimating

\[ \int_{M \times \{t\}} \langle \tilde{\psi}_{1,t}, d[\tilde{v}_x \times d\tilde{v}_z] \chi \rangle. \]

(5.51)

To achieve this, we proceed as in [8] and decompose \( \tilde{\psi}_{1,t} \) by splitting \( G \) into its the high and low frequency parts. For the low frequency term, \( \tilde{\psi}_{1t}^l \), we will be interested in establishing an \( L^\infty \) estimate by appealing to the work of [21]. For the high frequency term, \( \tilde{\psi}_{1t}^h \), we will be interested in \( L^2 \) estimates in addition to an operator norm bound on the distributional Laplacian of \( G^p \), the integral kernel of \( \psi_{1,t}^p \).
Given $\alpha \in (2/3, 3/4)$ and assuming that $36r < \frac{\ln(M)}{2}$ we consider the function $l: [0, \infty) \to [0, \infty)$ defined by

$$l(s) := \begin{cases} 0 & \text{if } s \leq e^r \\ \left(\frac{r}{36r}\right)^{N-1} \left(2^{N-1} - 1\right)^{N-1} & \text{if } e^r s \leq 2e^r \\ \left(2^{N-1} - \left(\frac{r}{36r}\right)\right)^{N-1} \left(2^{N-1} - 1\right)^{N-1} & \text{if } 2e^r \leq s \leq 16r \\ 0 & \text{if } s \geq 32r. \end{cases}$$

From this we define $m: [0, \infty) \to [0, \infty)$ by

$$m(s) := \begin{cases} 1 - l(s) & \text{for } s \in [0, 16r] \\ 0 & \text{for } s \in (16r, \infty) \end{cases}$$

and note that $m$ satisfies

$$\begin{cases} m(s) \equiv 1 & \text{for } s \in (0, e^r) \\ m(s) \equiv 0 & \text{for } s \in (2e^r, \infty) \\ |m'(s)| \leq \frac{\alpha}{15r}. \end{cases}$$

Then we set

$$G(x, y) = m(d(x, y))G(x, y) + (1 - m(d(x, y)))G(x, y) =: G^i(x, y) + G^o(x, y).$$

This decomposition allows us to define

$$\tilde{\psi}^i_{1,t} := \int_{B_{2e^r}(x)} \langle G^i(x, y), d[v_x \times d\tilde{v}_t]\rangle \chi$$

(5.54)

$$\tilde{\psi}^e_{1,t} := \int_{B_{2e^r}(x)} \langle G^o(x, y), d[v_x \times d\tilde{v}_t]\rangle \chi.$$  

(5.55)

In the above $\tilde{\psi}^i_{1,t}$ represents the high frequencies of $\tilde{\psi}_{1,t}$ while $\tilde{\psi}^e_{1,t}$ represents the low frequencies of $\tilde{\psi}_{1,t}$.

Next, we note that by (5.18), the definition of (5.53), and computations in normal coordinates we obtain

$$\left\| G^i \right\|_{L^1_\alpha(M)} \| G^o \|_{L^\infty_\alpha(M)} \leq K_M e^r$$

(5.56)

$$\left\| DG^i \right\|_{L^1_\alpha(M)} \| DG^o \|_{L^\infty_\alpha(M)} \leq K_M e^r,$$  

(5.57)

where $K_M$ is a constant depending only on $M$. Similar computations in normal coordinates for the more delicate estimate of $\tilde{\psi}^i_{1,t}$ are presented in detail below, see for example (5.63). We refer the reader to Lemma A.4.9 of [12] for more details.

In addition, through direct computations related to the distributional Laplacian of $G^i$ we obtain

$$\left\| \langle G^i(\cdot, y), -\Delta h \rangle \right\| \leq K_M \| h \|_{L^\infty_\alpha(M)}$$  

(5.58)

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for all $h \in C^2(M; \wedge^2 M)$ and each $y \in M$. We refer the reader to Lemma A.4.9 of [12] for more details. Estimates (5.56) and (5.57) along with the integral kernel expression for $\tilde{\psi}_{1,t}$ allow us to obtain

$$\int_M |\tilde{\psi}_{1,t}|^2 \leq KC(\delta_0, r)2^{2\alpha} \left( \tilde{E}_{\epsilon,g_{R_1}}(\nu, (x_T, 1), 1) + C\gamma_1 E_0 \right)$$  (5.59)

for $t \in \Theta_1$. We refer the reader to Lemma A.4.11 of [12] for more details.

Finally, we use the decomposition of $\tilde{\psi}_{1,t}$ in (5.51) to conclude that it suffices to estimate

$$\int_M \left\langle \tilde{\psi}_{1,t}, d[\tilde{v}_\epsilon \times d\tilde{v}_\epsilon] \right\rangle \chi$$  (5.60)

$$\int_M \left\langle \tilde{\psi}_{1,t}, d[\tilde{v}_\epsilon \times d\tilde{v}_\epsilon] \right\rangle.$$  (5.61)

**Step 3: Estimate of $\tilde{\psi}_{1,t}$**

We now focus on estimating the $L^\infty$ norm of $\tilde{\psi}_{1,t}$ in (5.51) to conclude that it suffices to estimate

$$\int_M \left\langle \tilde{\psi}_{1,t}, d[\tilde{v}_\epsilon \times d\tilde{v}_\epsilon] \right\rangle.$$  (5.60)

$$\int_M \left\langle \tilde{\psi}_{1,t}, d[\tilde{v}_\epsilon \times d\tilde{v}_\epsilon] \right\rangle.$$  (5.61)

Doing this will permit us to provide an upper bound on (5.60). We proceed in the same way as in [8] except we need to make use of normal coordinates in order to have an explicit expression for the integrand. The idea is to rewrite $\tilde{\psi}_{1,t}$ into a distributional pairing of the coordinate components of $d[\tilde{v}_\epsilon \times d\tilde{v}_\epsilon]$ and a Lipschitz function. Then, the work of [21] is able to provide an $O(1)$ estimate for the Lipschitz dual norm of the coordinate components of $d[\tilde{v}_\epsilon \times d\tilde{v}_\epsilon]$.

We will estimate $\tilde{\psi}_{1,t}$ at a fixed $x \in B_4(x_T)$. We let $y$ denote normal coordinates centered at $x$. In these coordinates, of course $x$ corresponds to zero, and if $p \in M$ is the point corresponding to the coordinate $y$, then $d(x,p) = |y|$. In the coordinates, $y$, we will write a 2-form as

$$\omega = \sum_{j_1 \neq j_2} \omega_{j_1,j_2}(y)dy^{j_1} \wedge dy^{j_2} = \sum_j \omega_j(y)dy^j.$$  (5.62)

We recall that the Green’s function $G$ is a tensor of type $(2,2)$ such that, if the first and second components $z$ and $p$ are written respectively in normal coordinates $\tilde{y}$ and $y$ centred on $x$, then $G$ acts on 2-forms $\omega$ via

$$\langle G(z, p), \omega(p) \rangle = \sum_l \left( \sum_j G^l_j(\tilde{y}, y)\omega_j(y) \right)dy^l.$$  

In particular,  

$$\langle G(x, p), \omega(p) \rangle = \sum_l \left( \sum_j G^l_j(0, y)\omega_j(y) \right)dy^l.$$  

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In these coordinates, the Green’s function for 2-forms, evaluated with one argument fixed at \(x\), has components

\[
G^2_I(0, y) = |y|^{2-N} H^2_I(y), \quad I = (i_1, i_2), \quad J = (j_1, j_2)
\]

where \(H^2_I\) is a Lipschitz function in \(y\) for each \(I\) and \(J\), see [9] and Proposition 4.12 from [4]. \((g_{R_1})_{ij}\) and \(\tilde{g}_{R_1}^j\) denote, respectively, the metric tensor and its inverse with respect to these coordinates. Due to our choice of \(r\), \(G^c\) has support in the domain of this coordinate system, and so the above discussion gives

\[
\tilde{\psi}_{1,t}(x,t) = \sum_I \left[ \sum_J \int_{B_{2\sigma_t}(0)} l(|y|)|y|^{2-N} H^2_I(\chi) \frac{\partial \tilde{\psi}_r}{\partial x_{j_1}} \times \frac{\partial \tilde{\psi}_r}{\partial x_{j_2}} \sqrt{|g_{R_1}(y)|} dy \right] dy^l
\]

where we have set \(|g_{R_1}(y)| = \det((g_{R_1})_{ij}(y))\). We now estimate each of the summands from (5.63). Following the proof of Lemma 3.12 of [8] and applying Fubini’s Theorem we obtain, for all \(k \in L^1(M)\), that

\[
\int_{B_{2\sigma_t}(0)} l(|y|)|y|^{2-N} k(y) dy = \int_{c_t}^{16r} s^{-1} \mathcal{J}^k_s ds + \frac{1}{N+2} \left[ \mathcal{J}^k_{16r} - \mathcal{J}^k_{c_t} \right]
\]

(5.64)

and

\[
h(u,s) := \frac{(N-1)(N-2)}{2^{N-1} - 1} \begin{cases} 1 & \text{if } 0 \leq u \leq s, \\ 2 - \frac{u}{s} & \text{if } s \leq u \leq 2s, \\ 0 & \text{if } u \geq 2s. \end{cases}
\]

We refer the reader to A.4.4.1 of [12] for more details. We then use (5.64) with \(k = a_I\). As in [8], we exploit the Jacobian structure of \(\frac{\partial \tilde{\psi}_r}{\partial x_{j_1}} \times \frac{\partial \tilde{\psi}_r}{\partial x_{j_2}}\) by applying Theorem 2.1 of [21] to \(\mathcal{J}^a\) to obtain

\[
\sup_{s \in [c_t, 16r]} \mathcal{J}^a_s \leq C(\delta_0, r) \left( \frac{\tilde{E}_{c, gR_1}(v_c, (x_T, 1), 1) + C_7 R_1 E_0}{\log(\epsilon)} + \epsilon^{\beta} \right)
\]

(5.65)

for some \(\beta > 0\). We refer the reader to Lemma A.4.10 of [12] for more details. Combining (5.63), (5.64), and (5.65) leads to

\[
\left\| \tilde{\psi}_{1,t} \right\|_{L^\infty(\mathcal{M})} \leq C(\delta_0, r) \left( \tilde{E}_{c, gR_1}(v_c, (x_T, 1), 1) + C_7 R E_0 + 1 \right).
\]

(5.66)

Observe that by (5.66), (5.27), manipulations of Gaussian functions that use that the support of \(\chi\) is \(B_r(x_T)\), in addition to using the assumption that \(t \in \Theta_1\) we have

\[
\left\| \tilde{\psi}_{1,t} \right\|_{L^\infty(\mathcal{M})} \leq C(\delta_0, r) \left( \tilde{E}_{c, gR_1}(v_c, (x_T, 1), 1) + C_7 R E_0 + 1 \right)[RM_0 + \eta].
\]
Step 4: Auxiliary parabolic problem

Since we only have control over the $L^1$-norm of $d[\tilde{v}_t \times d\tilde{v}_t] \chi$ then we fall slightly short of using Hölder’s inequality since the estimates (5.56), (5.57) only permit us to obtain the $L^2$ estimate (5.59). As a result, as in [8], we introduce $\psi^*_1$ solving a parabolic PDE to obtain better regularity through parabolic estimates.

We introduce $\psi^*_1$ solving

$$\begin{cases}
\partial_t \psi^*_1 - \Delta \psi^*_1 = d[\tilde{v}_t \times d\tilde{v}_t] \chi & \text{on } M \times (0, \infty) \\
\psi^*_1(\cdot, 0) = 0 & \text{on } M \times \{0\}.
\end{cases} \tag{5.67}$$

By appealing to standard parabolic techniques, the monotonicity formula, Gaffney’s inequality, Lemma 3.7, as well as Proposition 3.9 and Proposition 3.5 we can show that

$$\|\psi^*_1(\cdot, 1 - \delta^2)\|_{L^\infty(M)} \leq C(\delta_0, r) \left( \bar{E}_{r,j,R_1}(v_\epsilon, (x_\epsilon, 1, 1) + C \tau R_1 E_0 \right) \tag{5.68}$$

$$\|D\psi^*_1\|_{L^2(M)} \leq C(\delta_0, r) \left( \bar{E}_{r,j,R_1}(v_\epsilon, (x_\epsilon, 1, 1) + C \tau R_1 E_0 \right). \tag{5.69}$$

We refer the reader to Lemma A.4.12 of [12] for more details. Next, by using (3.22), Proposition 3.11, and Proposition 3.5 as well as its proof we obtain

$$\int_{M \times \{t\}} |\partial_t v_\epsilon|^2 e^{-\frac{(d_\epsilon(x_\epsilon, x_T))^2}{4(1-t)}} \leq K_M \left( \bar{E}_{r,j,R_1}(v_\epsilon, (x_\epsilon, 1, 1) + C \tau R_1 E_0 \right). \tag{5.70}$$

We refer the reader to A.4.4.2 of [12] for additional details. Finally, we argue that we can find $t \in [1 - 4\delta_0^2, 1 - \delta_0^2]$ for which

$$\int_{M \times \{t\}} |\partial_t \psi^*_1|^2 \leq C(\delta_0, r) \epsilon^{-1} \left( \bar{E}_{r,j,R_1}(v_\epsilon, (x_\epsilon, 1, 1) + C \tau R_1 E_0 \right). \tag{5.71}$$

We introduce the notation $\Theta_2 := \{ t \in [1 - 4\delta_0^2, 1 - \delta_0^2] : (5.71) \text{ holds at } t \}. \tag{5.72}$

To show (5.71) we proceed as in [8]. Taking the inner product of (5.67) with $\partial_t \psi^*_1$, integrating over $M \times [0, 1 - \delta_0^2]$, and integrating by parts leads to

$$\int_{M \times [0, 1 - \delta_0^2]} |\partial_t \psi^*_1|^2 = -\frac{1}{2} \int_{M \times (1 - \delta_0^2)} \left\{ |d\psi^*_1|^2 + |d^* \psi^*_1|^2 \right\} + \int_{M \times [0, 1 - \delta_0^2]} (\partial_t \psi^*_1, d[\tilde{v}_t \times d\tilde{v}_t] \chi). \tag{5.73}$$

Next introducing normal coordinates, $y$, centred at $x_T$ and expressing $\psi^*_1$ in these coordinates, similar to (5.62), as

$$\psi^*_1 = \sum_I \psi^*_{1,I}(y)dy$$

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we may write \((d\tilde{v}_t \times d\tilde{u}_t)\chi, \partial_t \psi_1^*\) as 
\[
\sum_{I} \sum_{j < j} g_{ij}^{(I)}(y) \partial v_{i+1}^I(y) \left[ \partial_y (\tilde{v}_t \times \partial_j \tilde{v}_t(y)) - \partial_j (\tilde{v}_t \times \partial y \tilde{v}_t(y)) \right] \chi(y) \sqrt{|g_{ij}(y)|} 
\]
where \(I = (i_1, i_2)\), \(J = (j_1, j_2)\), \(|g_{ij}(y)|\) is as in (5.63), and we have set 
\[
g_{ij}^{(I)}(y) = \begin{pmatrix} g_{ij}^{(I_1)} & g_{ij}^{(I_2)} \\ g_{ij}^{(I_1)} & g_{ij}^{(I_2)} \end{pmatrix} 
\]
where \(g_{ij}^{(I)}\) denotes the \(i, j\) component of the inverse of metric tensor, \(g_{ij}(y)\), in these coordinates. By our choice of \(\chi\) and \(r > 0\) we see that the support of \(\chi\) is contained in the domain of this coordinate system. Setting \(\chi_{g_{ij}}^{(I)} = \chi_{g_{ij}}^{(I)} \sqrt{|g_{ij}|}\), integrating by parts repeatedly as in \([5]\), and using that \(\chi_{g_{ij}}^{(I)}\) is supported on \(B_{4r}(0)\) we can write 
\[
\int_{M \times [0, 1 - \delta^2_0]} (\partial_t \psi_1^*, d[\tilde{v}_t \times d\tilde{u}_t])\chi = T_1 + T_2 + T_3 + T_4
\]
where 
\[
T_1 = 2 \sum_{I} \sum_{j < j} \int_{B_{4r}(0) \times [0, 1 - \delta^2_0]} \left[ \partial_j \psi_1^* \left( \partial_y \tilde{v}_t \times \partial_j \tilde{v}_t \right) - \partial_j \psi_1^* \left( \partial_y \tilde{v}_t \times \partial_j \tilde{v}_t \right) \right] \chi_{g_{ij}}^{IJ}
\]
\[
T_2 = - \sum_{I} \sum_{j < j} \int_{B_{4r}(0) \times [0, 1 - \delta^2_0]} \left( \tilde{v}_t \times \partial_j \tilde{v}_t \right) \left[ \partial_j \psi_1^* \left( \partial_y \tilde{v}_t \times \partial_j \tilde{v}_t \right) - \partial_j \psi_1^* \left( \partial_y \tilde{v}_t \times \partial_j \tilde{v}_t \right) \right] \chi_{g_{ij}}^{IJ}
\]
\[
T_3 = - \sum_{I} \sum_{j < j} \int_{B_{4r}(0) \times [0, 1 - \delta^2_0]} \partial_j \psi_1^* \left( \partial_y \tilde{v}_t \times \partial_j \tilde{v}_t \right) \chi_{g_{ij}}^{IJ}
\]
\[
T_4 = - \sum_{I} \sum_{j < j} \int_{B_{4r}(0) \times [0, 1 - \delta^2_0]} \left[ \partial_j \psi_1^* \left( \tilde{v}_t \times \partial_y \tilde{v}_t \right) - \partial_j \psi_1^* \left( \tilde{v}_t \times \partial_y \tilde{v}_t \right) \right] \chi_{g_{ij}}^{IJ}.
\]
We estimate \(T_1, T_2, T_3, \) and \(T_4\) as in \([5]\) by using (5.68), (5.69), (4.3), (5.70), and Proposition 3.5. The only change required is in the estimate of \(T_4\) in which an additional appeal to Gaffney’s inequality and \(L^2\) estimates of \(\psi_1^*\) obtained from the proof of (5.69) are applied. Proceeding in this way we obtain 
\[
\int_{M \times [0, 1 - \delta^2_0]} |\partial_t \psi_1^*|^2 \leq C(\delta_0, r) \epsilon^{-1} \left( \tilde{E}_c : g_{ij} \left( v_t, (xT, 1) \right) + C_T R_1 E_0 \right). \tag{5.74}
\]
An application of Chebyshev’s inequality then allows us to find \(t \in [1 - 4\delta_0^2, 1 - \delta^2_0]\) for which (5.71) holds.

**Step 5: Estimate of \(\tilde{\psi}_{1, t}\)**

We assume that \(t \in \Theta_1 \cap \Theta_2\). Using (5.67) we can write 
\[
\int_{M \times (t)} \langle \tilde{\psi}_{1, t}, d[\tilde{v}_t \times d\tilde{u}_t] \rangle \chi = \int_M \langle \tilde{\psi}_{1, t}, \partial_t \psi_1^* \rangle + \int_M \langle \tilde{\psi}_{1, t}, -\Delta \psi_1^* \rangle.
\]

The first term can be estimated using (5.59) and (5.71) to obtain
\[
\left| \int_{M \times \{t\}} \left\langle \tilde{\psi}_{1,t}, \partial_1 \psi^*_1 \right\rangle \right| \leq C(\delta_0, r) \epsilon^{\alpha - \frac{1}{2}} \left( \tilde{E}_{\epsilon, g R_1}(v_\epsilon, (x_T, 1), 1) + C_7 R_1 E_0 \right).
\] (5.75)

The second term can be estimated using (5.58), the proof of Proposition 3.9, and (4.8) to obtain
\[
\left| \int_{M \times \{t\}} \left\langle \tilde{\psi}_{1,t}, -\Delta \psi^*_1 \right\rangle \right| \leq C(\delta_0, r) \left( \tilde{E}_{\epsilon, g R_1}(v_\epsilon, (x_T, 1), 1) + C_7 R_1 E_0 \right)[R M_0 + \eta].
\] (5.76)

Combining estimates (5.34), (5.45), (5.50), (5.75), and (5.76) with (5.43) gives
\[
\int_{M \times \{t\}} \left\{ |d\psi_{1,t}|^2 + |d^* \psi_{1,t}|^2 \right\} \leq C(\delta_0, r) \epsilon^{\alpha - \frac{1}{2}} \left( \tilde{E}_{\epsilon, g R_1}(v_\epsilon, (x_T, 1), 1) + C_7 R_1 E_0 \right)
+ C(\delta_0, r) \left( \tilde{E}_{\epsilon, g R_1}(v_\epsilon, (x_T, 1), 1) + C_7 R_1 E_0 + 1 \right)(R M_0 + \eta + [R M_0 + \eta]^2). 
\] (5.77)

Finally, combining (4.25) (4.26), (5.34), (4.41), the estimate of (5.60), and (5.77) and choosing the parameters sufficiently small completes the proof of Proposition 4.1.

6 Energy Decompositions

In this section we present the proof of Theorem 1.3. Compared to [8], there are new considerations related to the homology of $M$. More specifically, when applying the Hodge de Rham decomposition we must, since we impose no homological restrictions on $M$, consider the harmonic part. In particular, these considerations are responsible for the presence of $u_{h, \epsilon}$ in the conclusions of Theorem 1.3.

We start by stating a local energy decomposition for solutions of (PGL)$_\epsilon$, valid in a region where the modulus is bounded away from zero.

**Theorem 6.1.** Suppose that $0 < R < \text{inj}_g(M)$, $T > 0$, and $\Delta T > 0$ are given. Consider the cylinder
\[
\Lambda := B_R(x_0) \times [T, T + \Delta T].
\]
There exists a constant $0 < \sigma \leq \frac{1}{4}$ and $\beta > 0$ depending only on $N$, such that if
\[
|u_\epsilon| \geq 1 - \sigma \text{ on } \Lambda,
\] (6.1)
then
\[
e_\epsilon(u_\epsilon)(x, t) \leq C(\Lambda) \int^t_\Lambda e_\epsilon(u_\epsilon),
\] (6.2)
for any $(x, t) \in \Lambda_{\frac{1}{2}}$. Moreover,
\[
e_\epsilon(u_\epsilon) = \frac{|
abla \Phi_\epsilon|^2}{2} + \kappa_\epsilon \text{ in } \Lambda_{\frac{1}{2}},
\] (6.3)
where the functions $\Phi_\varepsilon$ and $\kappa_\varepsilon$ are defined on $\Lambda_{\frac{1}{2}}$ and verify
\[
\partial_t \Phi_\varepsilon - \Delta \Phi_\varepsilon = 0 \quad \text{in} \quad \Lambda_{\frac{1}{2}},
\]
\[
\|\kappa_\varepsilon\|_{L^\infty(\Lambda_{\frac{1}{2}})} \leq C(\Lambda)\varepsilon^\beta, \quad \|\nabla \Phi_\varepsilon\|^2_{L^\infty(\Lambda_{\frac{1}{2}})} \leq C(\Lambda)M_0|\log(\varepsilon)|.
\]
In addition, it follows from our choice of $\Phi_\varepsilon$ that if $u_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}$ on $\Lambda$ then
\[
\|\nabla \Phi_\varepsilon - \nabla \varphi_\varepsilon\|_{L^\infty(\Lambda_{\frac{1}{2}})} \leq C(\Lambda)\varepsilon^\beta.
\]
This is an adaptation to the present setting of Theorem 2 in [8]. Since the analysis is entirely local, and because it does not involve any delicate properties of test functions adapted to the metric, the proof ends up being essentially identical in the Riemannian case. This being the case, we omit all details here. An interested reader can consult [8], or A.6.0.1 of [12], where it is verified in detail that the arguments of [8] remain valid on a manifold.

As was done in [8], we record a straightforward consequence obtained by combining the results of Theorem 1.2 with Proposition 3.8.

**Proposition 6.2.** Let $u_\varepsilon$ be a solution of (PGL)$_\varepsilon$ verifying assumption (H$_0$) and $\sigma > 0$ be given. Let $x_T \in M, \; T > 0$, and $0 < 2\varepsilon < R^2 < R(\sigma)$ where $R(\sigma)$ is as in Theorem 1.2. There exists a positive continuous function $\lambda$ defined on $(0, \infty)$ such that, if
\[
\hat{\eta}(x_T, T, R) := \frac{1}{(4\pi)^{\frac{N}{2}}R^{N-2}|\log(\varepsilon)|} \int_{B_{\lambda(T)R}(x_T)} e_\varepsilon(u_\varepsilon(\cdot, T)) \leq \frac{\eta_1(\sigma)}{2}
\]
then
\[
|u_\varepsilon(x, t)| \geq 1 - \sigma \quad \text{for} \; t \in [T + T_0, T + T_1] \; \text{and} \; x \in B_{\frac{R}{2}}(x_T).
\]
Here $T_0$ and $T_1$ are defined by
\[
T_0 := \left(\frac{2\hat{\eta}}{\eta_1}\right)^{-\frac{N-2}{2}}R^2, \quad T_1 := R^2.
\]
In particular, a more precise estimate shows that we can find $\lambda$ defined on $(0, \infty) \times (0, \infty)$ satisfying
\[
\lambda(T, R) \sim \sqrt{\frac{8}{c^2} \log \left(\frac{(4\pi)^\frac{N}{2}}{M_0e^{c^2}} \left[\frac{2}{T + 2R^2}\right]^{\frac{N-2}{2}}\right)}
\]
as $(T, R) \to (0, 0)$. In particular, $\lambda(T, R)R$ is bounded as $R \to 0^+$ for any $T > 0$.

Following [8] we also record the following consequence of Proposition 6.2 combined with Theorems 1.2 and 6.1 for future use.
Proposition 6.3. For each \( \sigma > 0 \) there exists positive constants \( \eta_2(\sigma) \) and \( R(\sigma) \) as well as a positive function \( \lambda \) defined on \((0, \infty)\) such that if, for \( x \in M, t > 0, \) and \( \sqrt{2}r < r < R(\sigma) \) we have

\[
\int_{B_{\lambda(t)r}(x)} e_{\varepsilon}(u_{\varepsilon}) \leq \eta_2 r^{N-2} \log(\varepsilon),
\]

then

\[
e_{\varepsilon}(u_{\varepsilon}) = \frac{\left| \nabla \Phi_{\varepsilon} \right|^2}{2} + \kappa_{\varepsilon}
\]
in \( \Lambda_{\frac{1}{4}}(x,t,r) := B_{\frac{1}{4}}(x) \times \left[ t + \frac{1}{16}r^2, t + \frac{3}{16}r^2 \right], \) where \( \Phi_{\varepsilon} \) and \( \kappa_{\varepsilon} \) are as in Theorem 6.1. In particular,

\[
\mu_{\varepsilon} = \frac{e_{\varepsilon}(u_{\varepsilon})}{\log(\varepsilon)} \leq C(t,r) \quad \text{on} \quad \Lambda_{\frac{1}{4}}(x,t,r).
\]

We use the remainder of this section to prove Theorem 1.3. We begin by introducing some notation. We let \( \Omega := M \times (0, \infty) \) and use \( \delta \) to denote the exterior derivative on \( M \times (0, \infty) \). In addition we let \( \delta^* \) denote its formal adjoint with respect to the natural product metric. If \( \eta \) is a \( k \)-form on \( M \times (0, \infty) \) and \( \Sigma \) is a smooth hypersurface, we will write \( \eta_T \) to denote the \( k \)-form on \( \Sigma \) defined by

\[
\eta_T := i^* \eta = \text{the tangential part of} \quad \eta \quad \text{on} \quad \Sigma,
\]
where \( i : \Sigma \to M \times (0, \infty) \) is the inclusion map. We also write

\[
\eta_N := \eta - \eta_T = \text{the normal part of} \quad \eta \quad \text{on} \quad \Sigma.
\]

We note that if \( u_{\varepsilon} \) solves \((PGL)_{\varepsilon}\) and satisfies \((H_0)\) then standard parabolic estimates give, for sufficiently small \( \varepsilon \), that

\[
\int_{M \times (t)} e_{\varepsilon}(u_{\varepsilon}) \leq M_0 |\log(\varepsilon)| \quad \forall t > 0, \tag{6.7}
\]

\[
|u_{\varepsilon}(x, t)| \leq 3 \quad \forall (x, t) \in \Omega. \tag{6.8}
\]

In particular, (6.7) allows us to conclude that

\[
\int_{M \times [0, t_2]} e_{\varepsilon}(u_{\varepsilon}) \leq C(\Omega)M_0 |\log(\varepsilon)| \tag{6.9}
\]

\[
\int_{\partial \Omega} e_{\varepsilon}(u_{\varepsilon}) \leq 2M_0 |\log(\varepsilon)|. \tag{6.10}
\]

The next result is the main decomposition tool used in the proof of Theorem 1.3.

Proposition 6.4. Assume that \( u_{\varepsilon} \) is a solution to \((PGL)_{\varepsilon}\) on \( M \times (0, \infty) \) that satisfies \((H_0)\). Then there is a smooth 1-form \( \gamma \) dependent only on the initial data of \( u_{\varepsilon} \) such that, on \( \Omega \), there exists a smooth function \( \Phi \), a smooth 1-form \( \zeta \), and a smooth 2-form \( \Psi \) for which

\[
u_{\varepsilon} \times \delta u_{\varepsilon} = \delta \Phi + \delta^* \Psi + \gamma + \zeta, \quad \delta \Psi = 0 \quad \text{in} \quad \Omega, \quad \Psi_T = 0 \quad \text{on} \quad \partial \Omega.\tag{6.11}
\]
and

\[ \| \Phi \|_{W^{1,2}(\Omega)} + \| D\Phi \|_{L^2(\Omega)} + \| \gamma \|_{L^2(\Sigma)} \leq C(\Omega) \sqrt{(M_0 + 1) \log(\varepsilon)}. \] (6.12)

In addition, we have that \( \gamma \) is constant in time, independent of \( dt \), a harmonic 1-form on \( M \) for all \( t > 0 \), and there is a time independent \( S^1 \)-valued function \( u_{h,\varepsilon} \) such that \( ju_{h,\varepsilon} = \gamma \). Moreover, for any \( 1 \leq p < \frac{N+1}{N-1} 
\[ \left\{ \begin{array}{l}
\| D\Phi \|_{L^p(\Omega)} \leq C(p,\Omega)(M_0 + 1), \\
\| \xi \|_{L^p(\Omega)} \leq C(p,\Omega)(M_0 + 1)\varepsilon^\frac{1}{p},
\end{array} \right. \] (6.13)

where \( C(p,\Omega) \) is a constant depending only on \( p \) and \( \Omega \).

**Proof.** As in [8] we split the proof into two steps. We begin by dealing with \( \Sigma := \partial \Omega \). Notice that \( \Sigma = (M \times \{t_1\}) \cup (M \times \{t_2\}) \).

**Step 1: Hdr decompositions on \( \Sigma \).** Since \( \partial \Sigma = \emptyset \) then a standard Hodge-de Rham decomposition applied to the tangential part of \( u_{h,\varepsilon} \times \delta u_{\varepsilon} \) allows us to write

\[ (u_{\varepsilon} \times \delta u_{\varepsilon})_T = u_{\varepsilon} \times du_{\varepsilon} = d\Phi^i + d^* \Psi^i + \gamma^i \] on \( M \times \{t_i\} \), (6.14)

for \( i = 1, 2 \), with

\[ d\Psi^i = 0 \quad \text{on} \quad M \times \{t_i\} \quad \text{for} \quad i = 1, 2, \] (6.15)

\[ d\gamma^i = 0 = d^* \gamma^i \quad \text{on} \quad M \times \{t_i\} \quad \text{for} \quad i = 1, 2, \] (6.16)

\[ \int_{M \times \{t_i\}} \Phi^i d\text{vol}_g = 0 \quad \text{for} \quad i = 1, 2. \] (6.17)

See for example Theorem 5 of Section 5.2.5 of [16], which also shows that

\[ \left\| \Phi^i \right\|_{W^{1,2}(M \times \{t_i\})}^2 + \left\| \Psi^i \right\|_{W^{1,2}(M \times \{t_i\})}^2 + \left\| \gamma^i \right\|_{L^2(M \times \{t_i\})}^2 \leq KM_0 \log(\varepsilon) \] (6.18)

for \( i = 1, 2 \). Next observe that by applying \( d \) to (6.14) at \( t = t_i \) for \( i = 1, 2 \) and using (6.15) we obtain

\[ -\Delta_M \Psi^i = J_M u_{\varepsilon} \quad \text{on} \quad M \times \{t_i\} \] (6.19)

for \( i = 1, 2 \) where \( J_M u_{\varepsilon} := \frac{1}{2} du_{\varepsilon} \times \delta u_{\varepsilon} \) and \( \Delta_M \) is the Laplacian on \( M \). Thus, by Theorem 2.1 of [21], the Sobolev Embedding Theorem, duality, and elliptic regularity, see Lemma 2.9 of [6] and Propositions 5.17 and 6.5 of [19], we have that for all \( q > N, \ p := \frac{N}{q-1}, \) and \( \alpha := 1 - \frac{N}{q} \) that

\[ \left\| \Psi^i \right\|_{W^{1,p}(M \times \{t_i\})} \leq C(p, M) \left\| J_M u_{\varepsilon} \right\|_{W^{-1,p}(M \times \{t_i\})} \] (6.20)

\[ \leq C(p, M) \left\| J_M u_{\varepsilon} \right\|_{[C^{0,\alpha}(M \times \{t_i\})]^*} \leq C(p, M)(M_0 + 1) \]

for \( i = 1, 2 \).

Next we provide an approximation to the harmonic parts from (6.14) that stores most of the energy. This is a new ingredient needed to extend
the corresponding result of [8] to our setting. As a result, we go over the
associated estimates in more detail.

We consider a collection of closed curves, \( \{c_j\}_{j=1}^{\beta_1(M)} \) where \( \beta_1(M) \) is
the first Betti number of \( M \), generating the first homology group \( H_1(M) \).
It follows from item (ii) of Theorem 4 of Section 5.3.2 and Theorem 6 of
Section 5.2.5 of [16] that, associated to these curves, we can find a basis
\( \{c^k\}_{k=1}^{\beta_1(M)} \) for \( H^1(M) \), the space of harmonic 1-forms on \( M \), satisfying

\[
\int_{c_j} c^k = 2\pi \delta_{jk}.
\]

Using this basis we can express \( \gamma_\varepsilon^0 \), the harmonic part of \( u_\varepsilon \times du_\varepsilon \) at \( t = 0 \), as

\[
\gamma_\varepsilon^0 = \sum_{k=1}^{\beta_1(M)} a^0_k(\varepsilon)c^k \tag{6.21}
\]

where \( a^0_k(\varepsilon) \) may depend on \( \varepsilon \). From the representation (6.21) we may define

\[
[\gamma_\varepsilon^0] := \sum_{k=1}^{\beta_1(M)} [a^0_k(\varepsilon)]c^k \tag{6.22}
\]

which is a harmonic 1-form on \( M \). Notice that we may extend (6.22) to
\( M \times (0, \infty) \), in particular to \( \Omega \), by being constant in time to obtain

\[
\gamma_\varepsilon(x, t) := [\gamma_\varepsilon^0](x). \tag{6.23}
\]

We observe that this extension has no term corresponding to \( dt \). We also
note that by construction we have

\[
\|\gamma_\varepsilon - [\gamma_\varepsilon^0]\|_{L^2(\Lambda^1 M)} \leq \left\| \sum_{k=1}^{\beta_1(M)} \left( a^0_k(\varepsilon) - [a^0_k(\varepsilon)] \right)c^k \right\|_{L^2(\Lambda^1 M)} \leq \sum_{k=1}^{\beta_1(M)} \|c^k\|_{L^2(\Lambda^1 M)} \tag{6.24}
\]

where the rightmost quantity is not dependent on \( \varepsilon \). Next we establish
that \( \gamma_\varepsilon^0 \) and \( \gamma_\varepsilon \) are not too far from \( [\gamma_\varepsilon^0] \). We only demonstrate this for
\( \gamma_\varepsilon^0 \) as the proof is similar for \( \gamma_\varepsilon \). To do this we first extend \( \gamma_\varepsilon^0, \gamma_\varepsilon \) to
\( M \times (0, \infty) \), in particular \( \Omega \), by being constant in time and, respectively,
use \( \Gamma_\varepsilon^0, \Gamma_\varepsilon \) to denote this. Next we define the 2-form \( \eta \) by

\[
\eta := (\Gamma_\varepsilon^0 - \Gamma_\varepsilon^0) \wedge dt. \tag{6.25}
\]

Observe that

\[
\delta^* \eta = -[d^*(\Gamma_\varepsilon^0 - \Gamma_\varepsilon^0) \wedge dt = 0
\]

where we have identified \( \Gamma_\varepsilon^0 - \Gamma_\varepsilon^0 \) with an element of \( H^1(M) \) since this
1-form is independent of \( t \) and \( dt \). From this computation we can now see that, after integrating by parts, we obtain

\[
2 \int_{M \times [0, t_1]} (J u_\varepsilon, \eta)_{M \times [0, t_1]} = \int_{M \times [0, t_1]} \langle \delta (u_\varepsilon \times \delta u_\varepsilon), \eta \rangle_{M \times [0, t_1]} \tag{6.26}
\]

\[
= \int_{M \times \{t_1\}} (u_\varepsilon \times du_\varepsilon) \wedge \eta_N + \int_{M \times \{0\}} (u_\varepsilon \times du_\varepsilon) \wedge \eta_N
\]
where \( Ju_c := \frac{1}{2} \delta [u_c \times \delta u_c] \). By noting that \( \eta_N = (\gamma_\epsilon^1 - \gamma_\epsilon^0) \wedge \delta t \) at \( t \times \{ t \} \) and \( \eta_N = - (\gamma_\epsilon^1 - \gamma_\epsilon^0) \wedge \delta t \) at \( t \times \{ 0 \} \) we can rewrite this last expression as

\[
\int_{t \times \{ t \}} (u_c \times du_c) \wedge \ast \eta_N + \int_{t \times \{ 0 \}} (u_c \times du_c) \wedge \ast \eta_N
\]

(6.27)

\[
= (-1)^{N-1} \left[ \int_{t \times \{ t \}} \langle u_c \times du_c, \gamma_\epsilon^1 - \gamma_\epsilon^0 \rangle_M - \int_{t \times \{ 0 \}} \langle u_c \times du_c, \gamma_\epsilon^1 - \gamma_\epsilon^0 \rangle_M \right]
\]

(6.28)

\[
= (-1)^{N-1} \int_M \langle \gamma_\epsilon^1, \gamma_\epsilon^1 - \gamma_\epsilon^0 \rangle_M - \int_M \langle \gamma_\epsilon^0, \gamma_\epsilon^1 - \gamma_\epsilon^0 \rangle_M
\]

(6.29)

\[
= (-1)^{N-1} \int_M |\gamma_\epsilon^1 - \gamma_\epsilon^0|^2_M
\]

where we have used (6.14) in the third line. Putting (6.26) and (6.27) together and using the Jerrard-Soner estimate, Theorem 2.1 of [21], together with equivalence of norms on \( H^1(M) \) gives

\[
\| \gamma_\epsilon^1 - \gamma_\epsilon^0 \|^2_{L^2(\Lambda, M)} = 2 \left( \int_{t \times \{ t \}} \langle Ju_c, \eta \rangle_M \right) \leq 2 \| Ju_c \|_{C^{0,\alpha}(\Lambda^2 M \times [0,t_1])} \| \eta \|_{C^{0,\alpha}(\Lambda^2 M \times [0,t_1])}
\]

(6.30)

\[
\leq C(\alpha, \Omega) \| \eta \|_{L^2(\Lambda^2 M \times [0,t_1])} \left[ \frac{\int_{t \times \{ t \}} e_\epsilon(u_c)}{\log(\epsilon)} + 1 \right]
\]

(6.31)

\[
\leq C(\alpha, \Omega)(M_0 + 1) \| \gamma_\epsilon^1 - \gamma_\epsilon^0 \|_{L^2(\Lambda, M)}
\]

Thus, we obtain

\[
\| \gamma_\epsilon^1 - \gamma_\epsilon^0 \|^2_{L^2(\Lambda, M)} \leq C(\alpha, \Omega)(M_0 + 1).
\]

(6.32)

It then follows from (6.24) and (6.28) that for \( i = 1, 2 \)

\[
\| \gamma_\epsilon^i - [\gamma_\epsilon^0] \|^2_{L^2(\Lambda, M)} \leq C(\alpha, \Omega)(M_0 + 1)
\]

(6.33)

Next we notice that since the integral of \( \gamma_\epsilon^i \) over every closed loop in \( M \) has a value in \( 2\pi \mathbb{Z} \) there exists \( u_{h,\epsilon} : M \rightarrow \mathbb{S}^1 \) such that

\[
u_{h,\epsilon} \times du_{h,\epsilon} = \gamma_\epsilon^i.
\]

(6.34)

We may extend \( u_{h,\epsilon} \) to be constant in time to obtain \( u_{h,\epsilon} : \Omega \rightarrow \mathbb{S}^1 \) such that

\[
u_{h,\epsilon} \times du_{h,\epsilon} = \gamma_\epsilon^i,
\]

(6.35)

\( u_{h,\epsilon} \) is independent of \( t \) and \( u_{h,\epsilon} \times du_{h,\epsilon} \) is independent of \( dt \). We also consider the linear extension \( \Phi_{h,\epsilon}^i \) of \( \Phi_{\epsilon}^i \) to \( \Phi_2^i \) in \( \Omega \) defined by

\[
\Phi_{h,\epsilon}^i(x,t) := \left( \frac{t_2 - t}{t_2 - t_1} \right) \Phi_{\epsilon}^i(x) + \left( \frac{t - t_1}{t_2 - t_1} \right) \Phi_2^i(x).
\]

(6.36)

Note that by (6.18) this extension satisfies

\[
\| \Phi_{h,\epsilon}^i \|_{W^{1,2}(\Omega)} \leq K(\Omega) \sqrt{M_0} |\log(\epsilon)|.
\]

(6.37)
Step 2: “Gauge transformation” of \(u_\varepsilon\). On \(\Omega\) we consider the map \(w_\varepsilon\) defined by
\[
w_\varepsilon := u_\varepsilon e^{-\varepsilon \Phi_1^{1/2}} \text{ in } \Omega.
\]
Notice that \(|w_\varepsilon| = |u_\varepsilon|\). Moreover, one can show
\[
w_\varepsilon \times \delta w_\varepsilon = u_\varepsilon \times \delta u_\varepsilon - \delta \Phi_1^{1/2} - \gamma_\varepsilon + (1 - |u_\varepsilon|^2)(\delta \Phi_1^{1/2} + \gamma_\varepsilon).
\] (6.33)
Since \(|u_\varepsilon| \leq 3\) then
\[
|\nabla_x w_\varepsilon| \leq |\nabla_x u_\varepsilon| + 3|\nabla_x \Phi_1^{1/2}| + 3|\gamma_\varepsilon|
\] (6.34)
and hence
\[
\|\nabla w_\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon^{-2} \|1 - |w_\varepsilon|^2\|_{L^2(\Omega)}^2 \leq K M_0 |\log(\varepsilon)|.
\] (6.35)
By Hölder’s inequality, (6.8), (6.9), (6.18), and (6.32) we have that for \(1 \leq p < 2\)
\[
\| (1 - |u_\varepsilon|^2) \delta \Phi_1^{1/2} \|_{L^p(\Omega)}^p \leq K(\Omega) M_0 \varepsilon^{2-p} |\log(\varepsilon)|
\] (6.36)
\[
\| (1 - |u_\varepsilon|^2) \gamma_\varepsilon \|_{L^p(\Omega)}^p \leq K(\Omega) M_0 \varepsilon^{2-p} |\log(\varepsilon)|
\] (6.37)
and similarly
\[
\| (1 - |u_\varepsilon|^2) \delta \Phi_1^{1/2} \|_{L^p(M \times (t_1, 1))}^p \leq K(\Omega) M_0 \varepsilon^{2-p} |\log(\varepsilon)|
\] (6.38)
\[
\| (1 - |u_\varepsilon|^2) \gamma_\varepsilon \|_{L^p(M \times (t_1, 1))}^p \leq K(\Omega) M_0 \varepsilon^{2-p} |\log(\varepsilon)|
\] (6.39)
for \(i = 1, 2\). Next, by Corollary 5.6 of [19], we have the following Hodge decomposition of \(w_\varepsilon \times \delta w_\varepsilon\) on \(\Omega\):
\[
\begin{cases}
  w_\varepsilon \times \delta w_\varepsilon = \delta \Phi_\varepsilon + \delta^* \Psi_\varepsilon + \eta & \text{ in } \Omega, \\
  \delta \Psi_\varepsilon = 0 & \text{ in } \Omega, \\
  (\Phi_\varepsilon)_T = 0, (\Psi_\varepsilon)_T = 0, \eta_T = 0 & \text{ on } \Sigma \\
  \delta \eta = \delta^* \eta = 0 & \text{ on } \Omega
\end{cases}
\] (6.40)
also satisfying
\[
\| \Phi_\varepsilon \|_{W^{1,2}(\Omega)} + \| \Psi_\varepsilon \|_{W^{1,2}(\Omega)} \leq K(\Omega) \sqrt{M_0 |\log(\varepsilon)|}.
\] (6.41)
Next we include another new estimate needed to extend the argument of [8] to the setting of a Riemannian manifold. As this represents an addition to the argument from [8] we provide a more detailed discussion. We will write
\[
H^1_\varepsilon(\Omega) := \{ \text{ 1-forms } \eta \text{ on } \Omega : \delta \eta = \delta^* \eta = 0 \text{ in } \Omega, \eta_T = 0 \text{ on } \partial \Omega \}.
\]
It follows from the discussion in Lemma 10 of Section 5.3 of [5] that \(H^1_\varepsilon(\Omega)\) is a real vector space of dimension \((\# \text{ of components of } \partial \Omega) - 1 = 1\). Since \(H^1_\varepsilon(\Omega)\) clearly includes all 1-forms of the form \(\eta = \text{const} \, dt\), we deduce that
\[
\eta = a_\varepsilon \, dt
\] (6.42)
where \(a_\varepsilon \in \mathbb{R}\) that may depend on \(\varepsilon\). Next, we observe that
\[
a_\varepsilon \cdot \text{vol}_n(M)(t_2 - t_1) = \int_{\Omega} (j_\Omega w_\varepsilon, dt)_{\Omega} \quad (6.43)
\]
where we have used the abbreviation \(j_\Omega w_\varepsilon := w_\varepsilon \times \delta w_\varepsilon\). By (6.33) and the fact that \(\gamma_\varepsilon\) is independent of \(dt\) we can rewrite (6.43) as
\[
\int_{\Omega} (j_\Omega w_\varepsilon, dt)_{\Omega} = \int_{\Omega} (j_\Omega u_\varepsilon, dt)_{\Omega} - \int_{\Omega} \langle \delta \Phi_\varepsilon^{1,2}, dt \rangle_{\Omega} + \int_{\Omega} \langle (1 - |u_\varepsilon|^2) \delta \Phi_\varepsilon^{1,2}, dt \rangle_{\Omega}
\]
\[
=: (A) + (B) + (C).
\]
Observe that since \(u_\varepsilon\) solves \((PGL)_\varepsilon\) then we have
\[
(A) = \int_{t_1}^{t_2} \int_{M} \nabla u_\varepsilon \cdot \nabla \nabla fraction{\partial_i u_\varepsilon} dv_{\Omega}(x) dt = \int_{t_1}^{t_2} \int_{M} \nabla u_\varepsilon \cdot \Delta_M u_\varepsilon dv_{\Omega}(x) dt
\]
\[
= -\int_{t_1}^{t_2} \int_{M} \langle \nabla^\ast (u_\varepsilon \times d u_\varepsilon), 1 \rangle_M \rangle dt = -\int_{t_1}^{t_2} \int_{M} \langle u_\varepsilon \times d u_\varepsilon, d(1) \rangle_M \rangle dt = 0
\]
where we integrated by parts over \(M\). Next, observe that by (6.17)
\[
(B) = \int_{M} \int_{t_1}^{t_2} \nabla \Phi_\varepsilon^{1,2} \cdot dv_{\Omega}(x) = \int_{M} \nabla \Phi_\varepsilon^{1,2} \cdot dv_{\Omega}(x) = 0.
\]
Finally, observe that by Cauchy-Schwarz, (6.9), and (6.32)
\[
|| (C) || \leq \int_{t_1}^{t_2} \int_{M} |1 - |u_\varepsilon|^2| || \nabla \Phi_\varepsilon^{1,2} || \leq C(\Omega) M \varepsilon |\log(\varepsilon)|.
\]
Combining (6.44), (6.45), and (6.46) with (6.43) we obtain
\[
|a_\varepsilon| \leq C(\Omega) M \varepsilon |\log(\varepsilon)|.
\]
Next, note that \(\Psi_\varepsilon\) satisfies
\[
\begin{align*}
-\Delta_\Omega \Psi_\varepsilon = \omega_\varepsilon :&= Jw_\varepsilon, & \text{in } \Omega, \\
(\Psi_\varepsilon)_{T} = &0, & \text{on } \partial \Omega \\
(\delta^\ast \Psi_\varepsilon)_{T} = &A_\varepsilon := d^\ast \Psi_\varepsilon^{1} + (\gamma_\varepsilon^{1} - \gamma_\varepsilon^{0}) + (1 - |u_\varepsilon|^2) \langle d \Phi_\varepsilon^{1} + |\gamma_\varepsilon^{0}| \rangle & \text{on } M \times \{t_i\}
\end{align*}
\]
for \(i = 1, 2\) where \(\Delta_\Omega\) is the Laplacian on \(\Omega\). By (6.20), (6.28), (6.29), (6.38), and (6.39) we have, for \(i = 1, 2, 1 \leq p < \frac{N+1}{N-1}\), and \(q = \frac{p}{p-1}\) that
\[
\| A_\varepsilon \|_{L^p((\partial \Omega))} \cdot \| \Psi_\varepsilon \|_{L^q((\partial \Omega))} = \| A_\varepsilon \|_{L^p((\partial \Omega))} \leq C(p, \Omega)(M_0 + 1).
\]
Arguing as in (6.20) we also have
\[
\| \omega_\varepsilon \|_{L^p((\partial \Omega))} \cdot \| \omega_\varepsilon \|_{C^{0,1}((\partial \Omega))} \cdot \| \omega_\varepsilon \|_{C^{0,1}((\partial \Omega))} \cdot \| \Psi_\varepsilon \|_{L^p((\partial \Omega))} \leq C(\alpha, \Omega)(M_0 + 1).
\]
Thus, by elliptic regularity, obtained by a Stampacchia duality argument obtained by combining Proposition A.2 of [6] and Corollary 5.6 of [19], we have
\[
\| \Psi_\varepsilon \|_{W^{1,p}((\Omega))} \leq C(p, \Omega)(M_0 + 1).
\]
We refer the reader to [8] as well as Proposition 6.4 for more details regarding this estimate. We set
\[
\Psi = \Psi_{\varepsilon}, \quad \Phi = \Phi_{1,2}^{\varepsilon} + \Phi_{\varepsilon}, \quad \gamma = \gamma_{\varepsilon}, \quad \zeta = -(1 - \|u_{\varepsilon}\|^2)(\delta \Phi_{1,2}^{\varepsilon} + \gamma_{\varepsilon}) + \eta.
\]
Then
\[
u_{\varepsilon} \times \delta u_{\varepsilon} = w_{\varepsilon} \times \delta w_{\varepsilon} + |u_{\varepsilon}|^2(\delta \Phi_{1,2}^{\varepsilon} + \gamma_{\varepsilon})
= \delta \Phi_{\varepsilon} + \delta^* \Psi_{\varepsilon} + \gamma_{\varepsilon} + \zeta.
\]
The conclusion follows from (6.40), (6.32), (6.36), (6.37), (6.41), (6.47), and (6.50).

Next we demonstrate, following [8], that the phase portion, \(\Phi\), of \(u_{\varepsilon}\) is close to satisfying the heat equation.

**Lemma 6.5.** Suppose \(u_{\varepsilon}\) satisfies \((PGL)\), on \(M \times (0, \infty)\) and \((H_0)\), and suppose, for \(0 < t_1 < t_2 < \infty\), we set \(\Omega = M \times (t_1, t_2)\). For \(\varepsilon > 0\) sufficiently small we let \(\Phi, \Psi, \gamma, \) and \(\zeta\) satisfy the conclusions of Proposition 6.4. Then the function \(\Phi\) verifies the equation
\[
\partial_t \Phi - \Delta \Phi = -d^*(\delta^* \Psi_{\varepsilon} + \gamma_{\varepsilon} - P_t(\delta^* \Psi_{\varepsilon} + \gamma_{\varepsilon} - \zeta)dt) - P_t(\delta^* \Psi_{\varepsilon} + \gamma_{\varepsilon} - \zeta)\text{ in } \Omega.
\]
(6.51)
Here, for a 1-form \(\omega\) on \(\Omega\), \(P_t(\omega)\) denotes its \(dt\) component.

**Proof.** By Proposition 6.4 we have
\[
u_{\varepsilon} \times \delta u_{\varepsilon} = \delta \Phi + \delta^* \Psi + \gamma + \zeta,
\]
(6.52)
where \(\Phi, \Psi, \gamma,\) and \(\zeta\) verify the conclusions of Proposition 6.4. Taking the cross product of \((PGL)\) with \(u_{\varepsilon}\) leads to
\[
u_{\varepsilon} \times \partial_t u_{\varepsilon} = -d^*(u_{\varepsilon} \times du_{\varepsilon})\text{ in } \Omega.
\]
(6.53)
On the other hand, we also have by (6.52)
\[
\begin{align*}
\nu_{\varepsilon} \times du_{\varepsilon} &= d\Phi + \gamma + (\delta^* \Psi + \zeta) - P_t(\delta^* \Psi + \zeta)dt, \\
u_{\varepsilon} \times \partial_t u_{\varepsilon} &= \Phi_t + P_t(\delta^* \Psi + \zeta).
\end{align*}
\]
(6.54)
Notice that \(d^* \gamma = 0\) since \(\gamma\) is a harmonic 1-form on \(M \times \{t\}\) for all \(t\). As a result of this last observation along with (6.53) and (6.54) we obtain the conclusion.

With the above ingredients in hand, the proof of Theorem 1.3 exactly follows arguments in [8]. We recall some details for the convenience of the reader.

**Proof of Theorem 1.3**

Let \(u_{\varepsilon}\) be a solution of \((PGL)\) verifying \((H_0)\) on \(M \times (0, \infty)\). Let \(K\) be a compact subset of \(M \times (0, \infty)\). Choose \(0 < t_1 < t_2 < \infty\) so that \(K \subset M \times (t_1, t_2)\). Let \(\Omega = M \times (t_1, t_2)\) and suppose that \(\Phi, \Psi, \gamma,\) and \(\zeta\) be as in Proposition 6.4 and Lemma 6.5. We choose \(t_3\) and \(t_4\) such
that \( t_1 < t_3 < t_4 < t_2 \) and so that \( K \subset M \times (t_3, t_4) =: \Lambda. \) By perhaps perturbing \( t_3 \) and \( t_4 \) we may assume

\[
\int_{\partial \Lambda} |\Phi|^2 + \int_{\partial \Lambda} |\nabla_{x,t} \Phi|^2 \leq C(K)(M_0 + 1)|\log(\varepsilon)|. \tag{6.55}
\]

This is possible because of (6.12). We split the proof into two steps.

**Step 1: Defining \( \varphi_\varepsilon \).** Let \( \varphi_\varepsilon \) verify the homogeneous heat equation

\[
\begin{cases}
\partial_t \varphi_\varepsilon - \Delta \varphi_\varepsilon = 0 & \text{in } \Lambda \\
\varphi_\varepsilon = \Phi & \text{on } \mathcal{O}_1
\end{cases}
\]

and define

\[ w_\varepsilon := u_\varepsilon e^{-i\varphi_\varepsilon} \tilde{u}_{h, \varepsilon} \]

where \( u_{h, \varepsilon} \) is the \( S^1 \)-valued function described in Proposition 6.4 satisfying

\[ j u_{h, \varepsilon} = \gamma. \]

From the standard regularity theory for the heat equation, see Theorems 8 and 9 of [14], in addition to (6.12) we have

\[
\|\nabla \varphi_\varepsilon\|_{L^\infty(K)}^2 \leq C(K)\|\Phi\|_{W^{1,2}(\mathcal{O}_0)}^2 \leq C(K)(M_0 + 1)|\log(\varepsilon)|. \tag{6.57}
\]

Next, for later use we set \( \Phi_1 = \Phi - \varphi_\varepsilon \). Then \( \Phi_1 \) solves

\[
\begin{cases}
\partial_t \Phi_1 - \Delta \Phi_1 = -d^*(\delta^* \Psi + \zeta - P_t(\delta^* \Psi + \zeta)dt) - P_t(\delta^* \Psi + \zeta) & \text{in } \Lambda, \\
\Phi_1 = 0 & \text{on } \mathcal{O}_1.
\end{cases}
\]

Since by (6.13) we have

\[
\|\delta^* \Psi + \zeta - P_t(\delta^* \Psi + \zeta)dt\|_{L^p(\Lambda)} + \|P_t(\delta^* \Psi + \zeta)\|_{L^p(\Lambda)} \leq C(p, K)(M_0 + 1)
\]

it follows from standard estimates for the non-homogeneous heat equation that

\[
\|\nabla \Phi_1\|_{L^p(\Lambda)} \leq C(p, K)(M_0 + 1). \tag{6.59}
\]

**Step 2: \( W^{1,p} \) estimates for \( w_\varepsilon \).** First observe that

\[
|w_\varepsilon|^2 |\nabla w_\varepsilon|^2 = \left|w_\varepsilon|^2 \right| |\nabla |w_\varepsilon||^2 + |w_\varepsilon \times \nabla w_\varepsilon|^2,
\]

and hence

\[
\int_{K \cap (|u_\varepsilon| \geq \frac{1}{2})} |\nabla w_\varepsilon|^p \leq C(p) \left[ \int_K |w_\varepsilon \times \delta w_\varepsilon|^p + \int_K |\nabla |w_\varepsilon||^p \right]. \tag{6.60}
\]

On the other hand, by standard estimates for (PGL), (6.57), (6.12), and equivalence of norms for \( \gamma \) we have

\[
|\nabla w_\varepsilon| \leq |\nabla u_\varepsilon| + 3|\nabla \varphi_\varepsilon| + 3|\gamma| \leq C(K)M_0 \varepsilon^{-1},
\]

where we have used that since \( |u_{h, \varepsilon}| = 1 \) then \( |\nabla u_{h, \varepsilon}| = |ju_{h, \varepsilon}| = |\gamma| \). As a result, we have

\[
\int_{K \cap (|u_\varepsilon| \leq \frac{1}{2})} |\nabla w_\varepsilon|^p \leq C(p, K)M_0^p \varepsilon^{2-p} \int_K V_\varepsilon(u_\varepsilon) \leq C(p, K)M_0^{p+1}. \tag{6.61}
\]
By the definition of $w_\varepsilon$ and Proposition 6.4 we have

$$w_\varepsilon \times \delta w_\varepsilon = \delta^* \Psi + \delta \Phi_1 + \zeta + (1 - |u_\varepsilon|^2)(\delta \varphi + \gamma).$$  \hfill (6.62)

By Hölder’s inequality we have

$$\| (1 - |u_\varepsilon|^2) \delta \varphi \|_{L^p(K)} + \| (1 - |u_\varepsilon|^2) \gamma \|_{L^p(K)} \leq C(p, K)M_0 \varepsilon^{\frac{2-p}{p}} \| \log(\varepsilon) \|,$$

and hence, when combined with (6.62), Proposition 6.4, and (6.59), we have

$$\int_K |w_\varepsilon \times \delta w_\varepsilon|^p \leq C(p, K)(M_0 + 1).$$  \hfill (6.63)

The proof for the gradient of the modulus remains the same as in [8] except we use a cutoff function in time, $\chi_K$, and work over a set $K' := M \times [t_4, t_5] \subset \Omega$ containing $K$. Following this procedure we have, for $1 \leq p < 2$, that

$$\int_{B_K} |\nabla |w_\varepsilon||^p \leq C(K)(M_0 + 1)\varepsilon^{1 - \frac{p}{2}} \| \log(\varepsilon) \|.$$

We refer the reader to [8] for additional details. Combining (6.64) with (6.60), (6.61), and (6.63) completes the proof.

\section{Analysis of Limiting Measures}

In this section we complete the proof of Theorem 1.1. To do this we will, as in [8], combine the results of Theorems 1.2, 6.1, 1.3 as well as their consequences and apply a detailed analysis of the limiting energy measure. Much of the corresponding proof used in [8] carries over to the general setting with minor variations. However, new ingredients are needed in the globalization of $\Phi_*$ due to the presence of $u_{h, \varepsilon}$ from Theorem 1.3. We refer the reader to Section A.7 of [12] for more detail.

We fix solutions $\{u_\varepsilon\}_{0 < \varepsilon < 1}$ of (PGL)$_\varepsilon$ satisfying assumption (H$_0$), and we define Radon measures over $M \times [0, \infty)$ and its time slices by

$$\mu_\varepsilon(x, t) := \frac{e_\varepsilon(u_\varepsilon(x, t))}{|\log(\varepsilon)|} \text{dvol}_g(x) \text{d}t$$

$$\mu^t_\varepsilon(x) := \frac{e_\varepsilon(u_\varepsilon(x, t))}{|\log(\varepsilon)|} \text{dvol}_g(x).$$

As a result of assumption (H$_0$) and standard estimates for (PGL)$_\varepsilon$, together with well-known arguments from [10, 18], there is a subsequence $\varepsilon_n \rightarrow 0^+$ and Radon measures $\mu_*$ and $\mu_\varepsilon^t$, defined on $M \times [0, \infty)$ and on $M$ respectively, such that

$$\mu_{\varepsilon_n} \rightarrow \mu_* \text{ as measures,}$$

$$\mu_{\varepsilon_n}^t \rightarrow \mu_*^t \text{ as measures for all } t > 0, \text{ where } \mu_* = \mu_*^t \text{d}t.$$

We will write $\varepsilon$ instead of $\varepsilon_n$ when this is not misleading. We also identify the measure $\mu_*^t$ with a measure on $M \times \{t\}$, and we will sometimes identify $M$ with $M \times \{t\}$. We record a consequence of the monotonicity formula on the limit measures.
Lemma 7.1. For each $t > 0$ and $x \in M$, the function $r \mapsto G((x,t), \cdot)$ defined on $(0, \infty)$ by

$$G((x,t), r) := e^{C_2 r^2 / (4\pi t)} \int_{M} e^{-(d(x,y))^2 / 4r^2} \, d\mu^*_t(y) + C_1 M_0 r,$$

where $C_1$ and $C_2$ are determined by Proposition 3.5, is non-decreasing for $0 < r < \min\{\sqrt{t}, 1\}$.

Next, we record an important consequence of the previous analysis.

Theorem 7.2. There exists an absolute constant $\eta_2 > 0$, and a positive continuous function $\lambda$ defined on $(0, \infty)$ such that if, for $x \in M$, $t > 0$, and $r > 0$ sufficiently small, and

$$\mu^*_t(B_{\lambda(t)r}(x)) < \eta_2 r^{N-2},$$

then for every $s \in [t + \frac{5}{3} r^2, t + r^2]$, $\mu^*_s$ is absolutely continuous with respect to the volume measure on the ball $B_{\frac{s}{2}}(x)$. More precisely,

$$\mu^*_s = \frac{|\nabla \Phi^*|^2}{2} \, d\text{vol}_g(x) \text{ on } B_{\frac{s}{2}}(x),$$

where $\Phi^*$ satisfies the heat equation in $\Lambda^*_1 = \Lambda^*_1(x, t, r, r^2)$.

Proof. The proof is the same as in [8] and is a straightforward consequence of Proposition 6.3. \qed

7.1 Densities and the concentration set

We begin by introducing some notation for measure densities.

Definition 7.3. Let $\nu$ be a Radon measure on $M$. For $m \in \mathbb{N}$, the $m$-dimensional lower and upper densities of $\nu$ at the point $x$, denoted $\Theta^*_m(\nu, x)$ and $\Theta^*_m(\nu, x)$ respectively, are defined by

$$\Theta^*_m(\nu, x) := \liminf_{r \to 0^+} \frac{\nu(B_r(x))}{\omega_m r^m} \quad \text{and} \quad \Theta^*_m(\nu, x) := \limsup_{r \to 0^+} \frac{\nu(B_r(x))}{\omega_m r^m},$$

where $\omega_m$ denotes the volume of the $m$-dimensional Euclidean unit ball in the standard metric. When both quantities coincide, $\nu$ admits an $m$-dimensional density $\Theta_m(\nu, x)$ at the point $x$, defined as the common value.

Next, following [8], we record a lemma regarding upper bounds on measure densities.

Lemma 7.4. For all $x \in M$ and for all $t > 0$,

$$\Theta^*_{m-2}(\mu^*_t, x) \leq \Theta^*_{m-2}(\mu^*_t, x) \leq M_0 e^{\frac{1}{\omega_{m-2}} \left[ e^{C_2 t^2 \frac{\omega_m}{2}} + (4\pi)^{\frac{m}{2}} C_1 \sqrt{t} \right]},$$

where $\omega_m$ denotes the volume of the $m$-dimensional Euclidean unit ball in the standard metric. When both quantities coincide, $\nu$ admits an $m$-dimensional density $\Theta_m(\nu, x)$ at the point $x$, defined as the common value.

Proof. The first inequality follows from the definition of lower and upper densities while the second inequality follows from the fact that $d$ agrees with $d_*$ on $B_t(x)$ for each $x \in M$ and $0 < r < \min\{1, \frac{\text{inj}(M)}{2}\}$ combined with Lemma 7.1 and (H0). \qed
Proceeding as in [8] we introduce a suitable notion (not the usual one in this context) of parabolic $m$-dimensional lower density of a Radon measure $\nu$.

**Definition 7.5.** Let $\nu$ be a Radon measure on $M \times [0, \infty)$ such that $\nu = \nu' \, dt$. For $t > 0$ and $m \in \mathbb{N}$, the parabolic $m$-dimensional density of $\nu$ at the point $(x, t)$ is defined by

$$\Theta_m^P(\nu, (x, t)) := \lim_{r \to 0^+} \frac{1}{(4\pi)^{\frac{m}{2}} r^m} \int_M e^{-\frac{\langle d(x, y) \rangle^2}{4r^2}} \, d\nu_{t-r}^m(y)$$

when it exists.

Observe that since $r \mapsto \mathcal{G}_m((x, t), r)$ is non-decreasing then $\Theta_{m-2}^P(\mu^*, (x, t))$ is defined everywhere in $M \times (0, \infty)$. Next, analogously to [8], we will relate the parabolic density to the lower $(N - 2)$-dimensional measure density.

**Lemma 7.6.** Suppose $x \in M$ and $t > 0$. Then, there exists $K_M > 0$, depending on $M$, such that

$$\Theta_{N-2}^P(\mu^*, (x, t)) \geq K_M \Theta_{N-2}(\mu^*, x). \tag{7.3}$$

**Proof.** The proof is the same as found in Subsection 6.2 of [8]. We refer the reader to [77] for additional details.

Let $(x, t) \in M \times (0, \infty)$ be given. Let $0 < r < \min \{ t, 1, \frac{\inj(M)}{2} \}$ be fixed. Similar to the proof of Lemma 7.4, we conclude from Lemma 7.1 that

$$\mu^*_r(B_r(x)) \leq \frac{e^{\frac{1}{4} + C_2r}}{r^{N-2}} \int_M e^{-\frac{\langle d(x, y) \rangle^2}{4r^2}} \, d\mu_*^{t-r}(y) + (4\pi)^{\frac{N}{2}} e^4 C_1 M_0 \sqrt{r^2 + r}.$$ 

Observe that on $B_{\inj(M)}(x)$ that

$$e^{-\frac{\langle d(x, y) \rangle^2}{4(r^2 + r)}} = e^{-\frac{\langle d(x, y) \rangle^2}{4r^2}} e^{\frac{\langle d(x, y) \rangle^2}{4(r^2 + r)}} \leq e^{-\frac{\langle d(x, y) \rangle^2}{16}} e^{-\frac{\langle d(x, y) \rangle^2}{4r^2}}.$$ 

On $M \setminus B_{\inj(M)}(x)$ we have

$$\int_{M \setminus B_{\inj(M)}(x)} e^{-\frac{\langle d(x, y) \rangle^2}{4(r^2 + r)}} \, d\mu_*^{t-r}(y) \leq e^{-\frac{\inj(M)^2}{16(r^2 + r)}} M_0.$$ 

Putting these together we obtain

$$\mu^*_r(B_r(x)) \leq \frac{e^{\frac{1}{4} + \frac{\inj(M)^2}{16}} + C_2r}{r^{N-2}} \int_M e^{-\frac{\langle d(x, y) \rangle^2}{4r^2}} \, d\mu_*^{t-r}$$

$$+ \frac{e^{\frac{1}{4} + C_2r - \frac{\inj(M)^2}{16(r^2 + r)}}}{r^{2}} M_0 + (4\pi)^{\frac{N}{2}} e^4 C_1 M_0 \sqrt{r^2 + r}.$$ 

Letting $r \to 0^+$ gives the conclusion. \qed
Just as in [8] we define
\[ \Sigma := \{ (x, t) \in M \times (0, \infty) : \Theta_{N-2}^{\mu}(\mu_*, (x, t)) > 0 \}, \] (7.4)
\[ \Sigma_t := \Sigma_t \cap (M \times \{ t \}) \text{ for } t > 0. \] (7.5)

A consequence of Lemma 7.6 is
\[ \Theta_{*, N-2}^{\mu}(\mu^t_*, x) \equiv 0 \text{ on } M \setminus \Sigma^t. \] (7.6)

Next we record, for future use, that the function \((x, t) \mapsto \Theta_{N-2}^{\mu}(\mu_*, (x, t))\) is upper semi-continuous on \(M \times (0, \infty)\). We note that the proof of this is the same as in [8]. More detail regarding its extension can be found in A.7.1.1 of [12].

**Lemma 7.7.** The map \((x, t) \mapsto \Theta_{N-2}^{\mu}(\mu_*, (x, t))\) is upper semi-continuous on \(M \times (0, \infty)\).

**7.2 First properties of \(\Sigma_{\mu}\)**

We begin this subsection by demonstrating a lower bound estimate on the \((N-2)\)-dimensional lower density over the set \(\Sigma_{\mu}\). The proof follows [8] closely so we refer the reader to A.7.2.1 of [12] for more details.

**Lemma 7.8.** Suppose \(0 < r < \sqrt{t} \) and \(x \in M\). Then, if \((x, t) \in \Sigma_{\mu}\) it follows that
\[ r^{2-N} \mu_\star^{t-r^2}(B_{\lambda(t-r^2)r}(x)) > \eta_2, \]
where \(\eta_2\) is the constant in Theorem 7.2.

**Proof.** We proceed by proving the contrapositive statement. Suppose there is \((x, t) \in M \times (0, \infty)\) and \(0 < r < \sqrt{t}\) for which
\[ r^{2-N} \mu_\star^{t-r^2}(B_{\lambda(t-r^2)r}(x)) \leq \eta_2. \]
By Theorem 7.2, for all \(\tau \in \left[ t - \frac{r^2}{2}, t \right]\) we have
\[ \mu_\star^\tau = \frac{|\nabla \Phi_\star|^2}{2} \text{dvol}_g(x) \quad B_{\tau}(x) \]
where \(\Phi_\star\) is smooth. Straightforward computations then show that \(\Theta_{N-2}^{\mu}(\mu_*, (x, t)) = 0\). \(\Box\)

Next we prove a clearing out lemma related to the set \(\Sigma_{\mu}\).

**Theorem 7.9.** There exists a positive continuous function \(\eta_3\) defined on \((0, \infty)\), such that for any \((x, t) \in M \times (0, \infty)\) and any \(0 < r < \sqrt{t}\), if
\[ \mathcal{F}_\mu((x, t), r) := \frac{1}{r^{N-2}} \int_M e^{-\frac{(d_\mu(x, y))^2}{r^2}} \text{d}\mu_\star^{t-r^2}(y) \leq \eta_3(t - r^2) \]
then \((x, t) \not\in \Sigma_{\mu}\).

**Proof.** The proof extends to our setting without change to the argument from the proof of Theorem 6 of [8]. We refer the reader to [8] or A.7.2.2 of [12] for additional details. \(\Box\)
Following [8] we note that a consequence of Theorem 7.9, for which details can be found in A.7.2.3 of [12], is the following:

**Corollary 7.10.** For any \((x,t) \in \Sigma_\mu\),
\[
\Theta^P_{N-2}(\mu_*, (x,t)) \geq \eta_3(t).
\]

Next we provide a decomposition for \(\mu_*^t\) and demonstrate a few properties of \(\Sigma_\mu^t\) and \(\Sigma_\mu\). The proof of this proposition is the same as in [8] with the exception that we rescale the metric instead of the function in the argument for (2). More details can be found in A.7.2.4 of [12].

**Proposition 7.11.**

1. The set \(\Sigma_\mu\) is closed in \(M \times (0, \infty)\).
2. For any \(t > 0\) we have
\[
\mathcal{H}^{N-2}(\Sigma_\mu^t) \leq K M_0 < \infty.
\]
3. For any \(t > 0\), the measure \(\mu_*^t\) can be decomposed as
\[
\mu_*^t = g(x,t)\mathcal{H}^N + \Theta_*(x,t)\mathcal{H}^{N-2} \mathcal{L}_{\Sigma_\mu^t},
\]
where \(g\) is some smooth function defined on \([M \times (0,\infty)] \setminus \Sigma_\mu\) and \(\Theta_*\) verifies the bound \(\Theta_*(x,t) \leq K M M_0 [e^{C_M t} \frac{N-2}{2} + D_M \sqrt{t}]\) for \(C_M, D_M, K_M > 0\) depending on \(M\).

### 7.3 Regularity of \(\Sigma_\mu^t\)

Next we record that the \((N-2)\)-dimensional parabolic density of \(\mu_*^t\) is controlled by \(\Theta_{*,N-2}(\mu_*^t, x)\) for most \(t\) and \(x\). This gives the reverse relationship illustrated in Lemma 7.6. The proof is very similar to the corresponding one from [8] the only exceptions are that we invoke the Besicovitch-Federer Covering Theorem, see Theorem 2.8.14 of [15], and we do not restrict our analysis to a finite region of time. As a result, we refer the reader to A.7.3.1 of [12] for more details.

**Proposition 7.12.** For \(\mathcal{L}^1\)-almost every \(t > 0\), the following inequality holds:
\[
\Theta_{*,N-2}(\mu_*^t, x) \geq K \Theta^P_{N-2}(\mu_*, (x,t)) \tag{7.7}
\]
for \(\mathcal{H}^{N-2}\)-almost every \(x \in M\).

Next we show that a lower density bound holds on \(\Sigma_\mu^t\) for most points.

**Corollary 7.13.** For \(\mathcal{L}^1\)-almost every \(t \geq 0\)
\[
\Theta_{*,N-2}(\mu_*^t, x) \geq K \eta_3(t) \tag{7.8}
\]
for \(\mathcal{H}^{N-2}\)-almost every \(x \in \Sigma_\mu^t\).

**Proof.** The corollary follows from Corollary 7.10 and Proposition 7.12. Details can be found in A.7.3.2 of [12].

Finally, we show that for \(\mathcal{L}^1\)-almost every \(t > 0\) and \(\mathcal{H}^{N-2}\)-almost every \(x \in \Sigma_\mu^t\) the upper and lower densities of \(\mu_*^t\) agree. As a result, for \(\mathcal{L}^1\)-almost every \(t > 0\) the set \(\Sigma_\mu^t\) is \((N-2)\)-rectifiable.
Proposition 7.14. For $\mathcal{L}^{1}$-almost every $t > 0$,
\[
\Theta_{*,N-2}(\mu^{t}, x) = \Theta_{N-2}^{\ast}(\mu^{t}, x) \geq K\eta_{3}(t)
\]
for $\mathcal{H}^{N-2}$-almost every $x \in \Sigma^{t}_{\mu}$. Consequently, for $\mathcal{L}^{1}$-almost every $t > 0$ the set $\Sigma^{t}_{\mu}$ is $(N - 2)$-rectifiable.

Proof. The proof essentially follows ideas from [8]. One begins by defining the vector space, $F$, for a fixed $(x, t) \in \Omega_{\omega}$ by
\[
F := \left\{ g \in L^{\infty}((0, \infty); \mathbb{R}) : I(g) := \lim_{r \to 0^{+}} I_{r}(g) \text{ exists and is finite} \right\}
\]
where for $r > 0$,
\[
I_{r}(y) := \frac{1}{r^{N-2}} \int_{M} g \left( \frac{d_{+}(x, y)}{r} \right) d\mu_{t}^{\ast}(y).
\]
The same definition appears in [8], with the Euclidean distance in place of $d_{+}$.

To prove the proposition, it suffices to show that $\chi_{[0, 1]} \in F$. The starting point is the fact that, if we write $e_{s}(t) = e^{-st^{2}}$, then $e_{1/4} \in F$; this is established in the proof of Proposition 7.12. One can then proceed using the same technique as in [8], which involves a number of steps which we now outline.

It is now shown that if $g \in F$ then for $s > 0$ the rescaling $g_{s}(\ell) = g(\ell^{s})$ belongs to $F$ as well. Since $e_{1/4} \in F$ this shows that $e_{s} \in F$ for all $s$. Next, we proceed to inductively demonstrate that functions of the form $\ell \mapsto \ell^{2k} e^{-\ell^{2}}$ for $k \in \mathbb{N} \cup \{0\}$ are member of $F$. We then show that $g \in C^{2}_{c}((0, \infty))$ satisfying $g'(0) = 0$ are also members of $F$ by appealing to Hermite polynomials and an approximation argument. Finally, we use members of $C^{2}_{c}((0, \infty))$ to approximate $\chi_{[0, 1]}$ and show that $\chi_{[0, 1]} \in F$.

We refer the reader to the proof of Proposition 8 and simply note that the proof presented there only depends on functions over the real line.

7.4 Globalizing $\Phi^{*}$

In this subsection we demonstrate that the function $\Phi^{*}$ has a globally defined differential and partial derivative in $t$ even though its construction was merely local.

Lemma 7.15. The locally defined function $\Phi^{*}$ from Theorem 7.2 extends to a function $\Phi^{*} : M \times (0, \infty) \to \mathbb{R}/2\pi\mathbb{Z}$. In particular, $\Phi^{*}$ has a differential, $d\Phi^{*}$, that is globally defined and satisfies $d\Phi^{*} = d\phi^{*} + \gamma^{*}$ where $\phi^{*}, \gamma^{*}$ are globally defined so that $\phi^{*}$ solves the heat equation on $M \times (0, \infty)$ and $\gamma^{*}$ is a harmonic 1-form on $M \times (0, \infty)$ that is only a function of $x$ and has no term corresponding to $dt$. In addition, $\partial_{t}\Phi^{*}$ is globally defined and equal to $\partial_{t}\phi^{*}$.
Proof. For $m \in \mathbb{N} \setminus \{1\}$ we set $\mathcal{K}_m = M \times [\frac{1}{m}, m]$, so that $\bigcup_{m \geq 2} \mathcal{K}_m = M \times (0, \infty)$. Applying Theorem 1.3 to $\mathcal{K} = \mathcal{K}_m$ we may write, for $\varepsilon$ sufficiently small,

$$u_\varepsilon = e^{i\phi_\varepsilon^m} u_{h, \varepsilon} \quad \text{on } \mathcal{K}_m,$$

(7.9)

where $\phi_\varepsilon^m$ solves the heat equation on $\mathcal{K}_m$, $u_{h, \varepsilon} \times du_{h, \varepsilon} = \gamma_\varepsilon$ is a harmonic 1-form on $\mathcal{K}_m$ not dependent, as a function, on $t$ and $m$ and has no component corresponding to $dt$. Theorem 1.3 yields the estimates

$$\|\nabla \phi_\varepsilon^m\|_{L^\infty(\mathcal{K}_m)} + \|\nabla u_{h, \varepsilon}\|_{L^\infty(\mathcal{K}_m)} \leq C(m) \sqrt{(M_0 + 1)\log(\varepsilon)}$$

(7.10)

$$\|\nabla u_\varepsilon^m\|_{L^p(\mathcal{K}_m)} \leq C(m, p) \quad \text{for any } 1 \leq p < \frac{N + 1}{N}. $$

(7.11)

For fixed $m$, we may pass to a further subsequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ such that

$$\frac{\phi_\varepsilon^m}{\sqrt{\log(\varepsilon)}} \to \phi_\varepsilon^m \quad \text{in } C^2(\mathcal{K}_{m-1})$$

(7.12)

$$\frac{\gamma_\varepsilon}{\sqrt{\log(\varepsilon)}} \to \gamma_\varepsilon \quad \text{in } C^2(\mathcal{K}_{m-1})$$

(7.13)

where $\phi_\varepsilon^m$ also satisfies the heat equation on $\mathcal{K}_{m-1}$, and $\gamma_\varepsilon$ is a harmonic 1-form. We have used the fact that the space of harmonic forms is finite dimensional. Note also that $\gamma_\varepsilon$ does not depend on $m$ or $t$ and has no component corresponding to $dt$.

Next, let $x_0 \in \Omega_\mu := (M \times (0, \infty)) \setminus \Sigma_\mu$. By (1) of Proposition 7.11 we have that $\Omega_\mu$ is open. Thus, we can find a set $\Lambda_{x_0} = B_R(x_0) \times [t_0, t_1]$ contained in $\Omega_\mu$. For $m_0$ large enough we will have, for $m \geq m_0$, that $\Lambda_{x_0} \subset \mathcal{K}_m$. For $\varepsilon$ sufficiently small we have

$$|u_\varepsilon| \geq 1 - \sigma \geq \frac{1}{2} \quad \text{on } \Lambda_{x_0}$$

(7.14)

where $\sigma$ is the constant in Theorem 6.1. This lower bound on the norm allows us to write

$$u_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}$$

(7.15)

for some real-valued function $\varphi_\varepsilon : M \times (0, \infty) \to \mathbb{R}/2\pi\mathbb{Z}$. By (7.14) we may apply (6.6) of Theorem 6.1 to demonstrate that there exists a solution $\Phi_\varepsilon$ of the heat equation on $\Lambda_{x_0}$ such that

$$\|\nabla \Phi_\varepsilon - \nabla \varphi_\varepsilon\|_{L^\infty(\Lambda_{x_0})} \leq C\varepsilon^\beta.$$  

(7.16)

On the other hand, since $|w_\varepsilon^m| = |u_\varepsilon|$ we may write, for $m \geq m_0$

$$w_\varepsilon^m = \rho_\varepsilon e^{i\psi_\varepsilon^m} \quad \text{on } \Lambda_{x_0}$$

(7.17)

where $\psi_\varepsilon^m : M \times (0, \infty) \to \mathbb{R}/2\pi\mathbb{Z}$. Combining (7.9), (7.15), and (7.17) we obtain

$$d\varphi_\varepsilon = d\phi_\varepsilon^m + \gamma_\varepsilon + d\psi_\varepsilon^m.$$  

(7.18)

By (7.16) for fixed $m$ we have

$$\left|\frac{d\phi_\varepsilon^m + \gamma_\varepsilon - d\Phi_\varepsilon}{\sqrt{\log(\varepsilon)}}\right| \leq \left|\frac{d\psi_\varepsilon^m}{\sqrt{\log(\varepsilon)}}\right| + C\varepsilon^\beta \quad \text{on } (\Lambda_{x_0})_\varepsilon.$$  

(55)
By (7.11) we obtain
\[ \left\| \frac{d\phi_m^\varepsilon}{\sqrt{|\log(\varepsilon)|}} + \frac{\gamma_\varepsilon}{\sqrt{|\log(\varepsilon)|}} - \frac{d\Phi_\varepsilon}{\sqrt{|\log(\varepsilon)|}} \right\|_{L^p((\Lambda_{x_0})_{1/2})} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0^+. \]

Since \( \frac{d\phi_m^\varepsilon}{\sqrt{|\log(\varepsilon)|}} \rightarrow \phi_m^* \) and \( \frac{d\Phi_\varepsilon}{\sqrt{|\log(\varepsilon)|}} \rightarrow \gamma^* \) from (7.12) and (7.10) then we deduce that \( \frac{d\Phi_\varepsilon}{\sqrt{|\log(\varepsilon)|}} \rightarrow d\Phi^* \) on \((\Lambda_{x_0})_{1/2}^\perp\)

\[ d\Phi^* = \frac{d\phi_m^*}{\sqrt{|\log(\varepsilon)|}} + \gamma^* \text{ on } (\Lambda_{x_0})_{1/2}. \]

Observe that since \( \gamma^* \) and \( \Phi^* \) are independent of \( m \) then by changing \( \phi_m^* \) by a constant we may assume that all \( \phi_m^* \) coincide on \((\Lambda_{x_0})_{1/2}^\perp\). By analyticity, for each \( n \geq m_0 \) the functions \( \{\phi_m^*\}_{m \geq n} \) coincide on \( K_m \).

Letting \( n \) go to infinity, we define their common value \( \phi^* \) on \( M \times (0, \infty) \). We then have
\[ d\Phi^* = d\phi^* + \gamma^* \quad (7.19) \]
on \((\Lambda_{x_0})_{1/2}^\perp\). Since the right-hand-side is globally defined we can then extend \( \Phi^* \). We also note that \( \partial_t \Phi^* \) is globally defined and equal to \( \partial_t \phi^* \).

\[ 7.5 \quad \text{Mean Curvature Flows} \]

The goal of this subsection is to prove (5) from Theorem 1.1. In particular, we focus on studying the properties of the singular parts of \( \{\nu_t^\varepsilon\}_{t > 0} \), denoted \( \{\nu_t^\varepsilon\}_{t > 0} \), which for each \( t > 0 \) satisfy
\[ \nu_t^\varepsilon = \Theta^*(x,t)\mathcal{H}^{N-2} \downharpoonright \Sigma_{\mu}^t \quad (7.20) \]
where \( \Theta^* \) and \( \Sigma_{\mu}^t \) are as in (1.2). As in [8], and following the same proof, we will study limiting behaviour of
\[ \omega^t_\varepsilon := \frac{|\partial_t u_\varepsilon|^2}{|\log(\varepsilon)|} \text{dvol}_\rho(x) \quad (7.21) \]
and
\[ \sigma^t_\varepsilon := -\frac{\partial_t u_\varepsilon \cdot \nabla u_\varepsilon}{|\log(\varepsilon)|} \text{dvol}_\rho(x). \quad (7.22) \]

\[ 7.5.1 \quad \text{Convergence of } \sigma^t_\varepsilon \]

By the Cauchy-Schwarz inequality \( \sigma_\varepsilon \) is uniformly bounded on \( M \times [0, T] \) for every \( T > 0 \). By perhaps passing to a further subsequence, we may assume that \( \sigma_\varepsilon \rightarrow \sigma^* \) as measures. The Radon-Nikodym derivative of \( |\sigma_\varepsilon| \) with respect to \( \mu_\varepsilon \) verifies
\[ \frac{d|\sigma_\varepsilon|}{d\mu_\varepsilon} = \frac{|\partial_t u_\varepsilon \cdot \nabla u_\varepsilon|}{e_\varepsilon(u_\varepsilon)} \leq \frac{\sqrt{2} |\partial_t u_\varepsilon| e_\varepsilon(u_\varepsilon)}{e_\varepsilon(u_\varepsilon)} = \sqrt{2} \frac{|\partial_t u_\varepsilon|}{e_\varepsilon(u_\varepsilon)}. \]
On the other hand,

\[
\left\| \frac{\partial u_\varepsilon}{\sqrt{e_\varepsilon(u_\varepsilon)}} \right\|_{L^2(M \times [0, T], d\mu_\varepsilon)}^2 = \int_{M \times [0, T]} \frac{\left| \partial_t u_\varepsilon \right|^2}{e_\varepsilon(u_\varepsilon)} d\mu_\varepsilon dt
\]

\[
= \int_{M \times [0, T]} \frac{\left| \partial_t u_\varepsilon \right|^2}{e_\varepsilon(u_\varepsilon)} \frac{e_\varepsilon(u_\varepsilon)}{|\log(\varepsilon)|} d\text{vol}_g(x) dt
\]

\[
= \int_{M \times [0, T]} \frac{\left| \partial_t u_\varepsilon \right|^2}{|\log(\varepsilon)|} d\text{vol}_g(x) dt
\]

\[
\leq M_0
\]

where we used standard energy estimates for \((\text{PGL})_\varepsilon\) and assumption \((H_0)\) for the last inequality. We conclude that

\[
\sigma_* = h \mu_t^* dt
\]

where \(h \in L^2(M \times [0, T], \mu_t^* dt)\). We use \((1.2)\) from Theorem 1.1 to decompose \(\mu_t^*\) into its absolutely continuous part with respect to \(d\text{vol}_g\) and its singular part \(\nu_t^*\) satisfying \((7.20)\). Arguing as in Proposition 3.1 of \([7]\) combined with Theorem 6.1 and Lemma 7.15 we see that the part of \(\sigma_t^*\) absolutely continuous with respect to \(d\text{vol}_g\) has density \(-\partial_t \Phi_* \cdot \nabla \Phi_*\). We now have

**Lemma 7.16.** The measure \(\sigma_*\) decomposes as \(\sigma_* = \sigma_*^t dt\), where for \(L^1\)-almost every \(t \geq 0\),

\[
\sigma_*^t = -\partial_t \Phi_* \cdot \nabla \Phi_* d\text{vol}_g(x) + h \nu_*^t.
\]

Next we observe that for every \(t \geq 0\), by appealing to the ideas found in Lemmas 7.5 and 7.6 of \([30]\), we have for all smooth vector fields, \(X\), that

\[
\frac{1}{|\log(\varepsilon)|} \int_{M \times \{t\}} [e_\varepsilon(u_\varepsilon)I - \nabla u_\varepsilon \otimes du_\varepsilon] : DX d\text{vol}_g(x) = \int_M \left\langle X, \frac{\partial_t u_\varepsilon \cdot \nabla u_\varepsilon}{|\log(\varepsilon)|} \right\rangle d\text{vol}_g(x)
\]

\[
= -\int_M \left\langle X, \sigma_*^t \right\rangle
\]

where \(I\) is the identity operator, \(\nabla u_\varepsilon \otimes du_\varepsilon = \nabla u_\varepsilon^1 \otimes du_\varepsilon^1 + \nabla u_\varepsilon^2 \otimes du_\varepsilon^2\), \(DX\) is the \((1,1)\)-tensor field defined at a point \(p \in M\) by

\[
DX_p: v \in T_p M \rightarrow D_v X,
\]

and we use the notation \(A : B\) to denote the inner product of \((1,1)\)-tensor fields on \(T_x M\) defined by

\[
A : B := \sum_{i=1}^N \sum_{j=1}^N \langle A(e_i), e_j \rangle \langle B(e_i), e_j \rangle
\]

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where \( \{e_1, e_2, \ldots, e_N\} \) is any orthonormal basis for \( T_x M \). Following [8] we use (7.23) as motivation to analyze the weak limit of
\[
\alpha^t_\varepsilon = \left( I - \frac{\nabla u_\varepsilon \otimes du_\varepsilon}{\varepsilon (u_\varepsilon)} \right) d\mu^t_\varepsilon.
\]
Since \( |\alpha^t_\varepsilon| \leq KN \mu^t_\varepsilon \) then we may assume that, by perhaps passing to a subsequence, that
\[
\alpha^t_\varepsilon \rightharpoonup \alpha^t \equiv A \cdot \mu^t
\]
where \( A \) is a symmetric \((1,1)\)-tensor field and where a \((1,1)\)-tensor field is symmetric if for each \( x \in M \) and each \( u, v \in T_x M \) we have
\[
\langle A_x(u), v \rangle = \langle u, A_x(v) \rangle.
\]
We also recall that a symmetric \((1,1)\)-tensor is referred to as positive semi-definite if for each \( x \in M \) and each \( u \in T_x M \) we have
\[
\langle A_x(u), u \rangle \geq 0.
\]
Finally, notice that if \( A, B \) are symmetric \((1,1)\)-tensor fields then we write \( A \leq B \) if \( B_x - A_x \) is positive semi-definite for each \( x \in M \). We now notice that since \( \nabla u_\varepsilon \otimes du_\varepsilon \) is a positive semi-definite \((1,1)\)-tensor field then
\[
A \leq I. \quad (7.26)
\]
On the other hand, computing in normal coordinates about a point \( x \in M \), we have, at \( x \), that
\[
\operatorname{tr}_g [\varepsilon(u_\varepsilon)I - \nabla u_\varepsilon \otimes du_\varepsilon] = (N - 2) \varepsilon_c(u_\varepsilon) + 2 \varepsilon V_\varepsilon(u_\varepsilon).
\]
Therefore, since the trace is a linear operation, passing to the limit we obtain
\[
\operatorname{tr}_g(A) = (N - 2) + 2 \frac{dV_*}{d\mu_*} \quad (7.27)
\]
where \( \frac{dV_*}{d\mu_*} \) is the non-negative limiting measure, obtained after passing to a subsequence, of \( \frac{\varepsilon_c(u_\varepsilon)}{\varepsilon (u_\varepsilon)} \). Taking the limit \( \varepsilon \to 0^+ \) in (7.23), decomposing \( \mu^t_\varepsilon \) using (1.2) of Theorem 1.1, and using the pointwise estimates provided by Theorem 6.1 we obtain for \( \mathcal{L}^1 \)-almost every \( t \geq 0 \)

\[
\int_M A : DXd\nu^t_* + \int_M \left[ \frac{\langle \nabla \Phi_* \rangle^2}{2} I - \nabla \Phi_* \otimes d\Phi_* \right] : DXd\nu^t_\mu \quad (7.28)
\]

\[
= - \int_M \langle X, h \rangle d\nu^t_* - \int_M \langle X, \partial_t \Phi_* \nabla \Phi_* \rangle d\nu^t_\mu. 
\]

Since \( \Phi_* \) solves the heat equation then we also have, by multiplying the heat equation by \( \langle X, \nabla \Phi_* \rangle \) and arguing in coordinates similar to Lemmas 7.5 and 7.6 of [30], that

\[
\int_M \left[ \frac{\langle \nabla \Phi_* \rangle^2}{2} I - \nabla \Phi_* \otimes d\Phi_* \right] : DXd\nu^t_\mu = - \int_M \langle X, \partial_t \Phi_* \cdot \nabla \Phi_* \rangle d\nu^t_\mu. \quad (7.29)
\]

Combining (7.28) and (7.29) now gives the following
**Lemma 7.17.** For $\mathcal{L}^1$-almost every $t \geq 0$ and for every smooth vector field $X$ we have

$$
\int_M A : DX d\nu^*_t = - \int_M (X, h) d\nu^*_t. \quad (7.30)
$$

We see that the conclusion of Lemma 7.17 is close to (2.16). Thus, if we can show that $A$ is the orthogonal projection operator from $T_x M$ onto $T_x \Sigma^t_\mu$, then we will have shown that $\nu^*_t$ has first variation with mean curvature $h$. Following [8] we proceed in this direction by first demonstrating that $A$ is perpendicular to normal vectors to $T_x \Sigma^{t, \mu}$.

**Lemma 7.18.** For $\mathcal{L}^1$-almost every $t \geq 0$ and $\mathcal{H}^{N-2}$-almost every $x \in \Sigma^t_\mu$ we have

$$
A_x \left[ \int_{T_x \Sigma^{t, \mu}} \nabla \chi(y) d\mathcal{H}^{N-2}(y) \right] = 0 \quad (7.31)
$$

where $\chi$ is a compactly supported smooth function on $T_x M$ where we use the exponential map to identify a neighbourhood of zero in $T_x M$ with subsets of $M$.

**Proof.** As in the corresponding proof from [8] we choose $t \geq 0$ for which (7.30) holds and $x \in \Sigma^t_\mu$ such that $T_x \Sigma^{t, \mu}$ exists and such that $x$ is a Lebesgue point for $\Theta_*$ with respect to $\mathcal{H}^{N-2}$, and of $A$ with respect to $\nu^*_t$. We now consider a smooth function $\chi$ with support contained in a normal coordinate neighbourhood centred at $x$. We then consider, written in normal coordinates centred at $x$, the vector field defined by

$$
X_{r,l}(y) := \chi \left( \frac{y}{r} \right) \frac{\partial}{\partial x^l} \quad \text{for } l \in \{1, 2, \ldots, N\}.
$$

Inserting $X_{r,l}$ into (7.30), taking the limit $r \to 0^+$, and appealing to the difference of homogeneity of the right-hand side as in Theorem 3.8 of [3], we conclude that

$$
A_x \left[ \int_{T_x \Sigma^{t, \mu}} \nabla \chi(y) d\mathcal{H}^{N-2}(y) \right] = 0.
$$

We have, due to the arguments of Section 6 of [3], the following consequence:

**Corollary 7.19.** For $t$ and $x$ as in Lemma 7.18,

$$(T_x \Sigma^{t, \mu})^\perp \subset \text{ker}(A_x).$$

We now show that $A_x = P$ where $P$ is the orthogonal projection of $T_x M$ onto $T_x \Sigma^{t, \mu}$.

**Corollary 7.20.** For $t$ and $x$ as in Lemma 7.18, $A_x = P$ is the orthogonal projection onto the tangent space $T_x \Sigma^{t, \mu}$.

**Proof.** By (7.26) we have $A_x \leq I_x$ for each $x \in M$, and therefore all the eigenvalues of $A_x$ are less than or equal to 1. By (7.27), $\text{tr}_g(A_x) \geq N - 2$ so that the sum of the eigenvalues of $A_x$ is at least $N - 2$. By Corollary 7.19 and our choice of $x$ and $t$ we know that $A_x$ has at least two zero eigenvalues. Combining the above information allows us to conclude that $A_x$ has precisely two zero eigenvalues and $(N - 2)$ eigenvalues equal to 1. In particular, since the kernel is $(T_x \Sigma^{t, \mu})^\perp$ then $A_x$ is the orthogonal projection onto $T_x \Sigma^{t, \mu}$.\qed
Combining Lemma 7.17 and Corollary 7.20 we obtain:

**Proposition 7.21.** For $L^1$-almost every $t \geq 0$, $\nu_t^\epsilon$ has a first variation and
\[
\delta\nu_t^\epsilon = h\nu_t^\epsilon.
\]
That is, $h$ is the mean curvature of $\nu_t^\epsilon$.

Next, following [8], we demonstrate the semi-continuity of $\omega_t^\epsilon$ defined in (7.21). First, we introduce the bundle $B$ whose fiber over $x \in M$ is the space of linear maps $T_x M \to \mathbb{R}^2$, which we identify with $(T_x M)^2$. On $B$ we define the measure
\[
\tilde{\omega}_t^\epsilon := \delta_{p_\epsilon(x)} \frac{|\partial_t u_\epsilon - p_\epsilon|^2}{\log(\epsilon)} d\nu_0(x)
\]
where $p_\epsilon := \frac{\nabla \omega_t^\epsilon}{|\nabla \omega_t^\epsilon|}$. By perhaps passing to a further subsequence, we may assume that $\tilde{\omega}_t^\epsilon dt \rightharpoonup \omega_*$ as measures. We deduce from the decomposition provided by Theorem 6.1 and the Portmanteau Theorem that:

**Lemma 7.22.** The measure $\tilde{\omega}_* \rightharpoonup \omega_*$ decomposes as $\tilde{\omega}_* = \tilde{\omega}_t^\epsilon dt$, and for $L^1$-almost every $t \geq 0$
\[
\tilde{\omega}^\epsilon_t = \Pi^t_{x,p}(p)|\partial_t \Phi_\epsilon|^2 d\nu_0(x) + \mathcal{M}^t_\epsilon,
\]
where $\Pi^t_{x,p}$ is a probability measure on $(T_x M)^2$ with support on the unit ball and $\mathcal{M}^t_\epsilon = \tilde{\omega}^\epsilon_t \Sigma_\mu$.

We borrow the following proposition, after adapting it to the case of a manifold, from Section 6 of [3]

**Proposition 7.23.** For $L^1$-almost every $t \geq 0$ and every smooth function $\chi$ we have
\[
\int_B \chi(x) \mathcal{M}^t_\epsilon(x,p) \geq \int_M \chi |h|^2 d\nu_\epsilon(t).
\]

We are now ready to prove (5) of Theorem 1.1.

**Proof.** We begin by using Lemma 3.1, integrating over $[T_0, T_1]$, and dividing by $|\log(\epsilon)|$. Next we let $\epsilon \to 0^+$. Then by combining Lemma 7.16, Proposition 7.21, Lemma 7.22, and Theorem 6.1 we obtain
\[
\nu_{T_1} - \nu_{T_0} + \int_{M \times (T_1)} \chi \frac{|\nabla \Phi_*|^2}{2} d\nu_0(x) - \int_{M \times \{T_0\}} \chi \frac{|\nabla \Phi_*|^2}{2} d\nu_0(x) = \int_{M \times [T_0, T_1]} \chi |h|^2 d\nu_\epsilon + \int_{M \times [T_0, T_1]} (\nabla \chi, P(h)) d\nu_\epsilon + \int_{M \times [T_0, T_1]} (\partial_t \Phi_\epsilon \nabla \Phi_\epsilon).
\]

Since $\Phi_\epsilon$ solves the heat equation, we have the identity
\[
\int_{M \times (T_1)} \chi \frac{|\nabla \Phi_*|^2}{2} d\nu_0(x) - \int_{M \times \{T_0\}} \chi \frac{|\nabla \Phi_*|^2}{2} d\nu_0(x) = \int_{M \times [T_0, T_1]} \chi |\partial_t \Phi_*|^2 d\nu_0(x) dt + \int_{M \times [T_0, T_1]} (\partial_t \Phi_* \cdot \nabla \Phi_*, \nabla \chi) d\nu_0(x) dt.
\]
Combining (7.32) and (7.33) gives

\[ \nu^T_1 - \nu^T_1 \leq - \int_{M \times [T_0, T_1]} \chi |h|^2 d\nu_* + \int_{M \times [T_0, T_1]} (\nabla \chi, P(h)) d\nu_* . \]

Applying Theorem 4.4 of [3], whose proof extends to the case of a compact Riemannian manifold, completes the proof of (5) of Theorem 1.1.

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