Magnetohydrodynamics effect on convective boundary layer flow and heat transfer of viscoelastic micropolar fluid past a sphere

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Abstract. The main interest of this study is to investigate the effect of MHD on the boundary layer flow and heat transfer of viscoelastic micropolar fluid. Governing equations are transformed into dimensionless form in order to reduce their complexity. Then, the stream function is applied to the dimensionless equations to produce partial differential equations which are then solved numerically using the Keller-box method in Fortran programming. The numerical results are compared to published study to ensure the reliability of present results. The effects of selected physical parameters such as the viscoelastic parameter, $K$, micropolar parameter, $K_1$ and magnetic parameter, $M$ on the flow and heat transfer are discussed and presented in tabular and graphical form. The findings from this study will be of critical importance in the fields of medicine, chemical as well as industrial processes where magnetic field is involved.

1. Introduction

Fluid can either be classified as Newtonian or non-Newtonian fluid with Newtonian fluid defined as fluid that retained its viscosity despite the amount of shear stress applied while non-Newtonian is the opposite. The study of non-Newtonian fluid is huge among researchers due to the practical importance especially in industrial processes since in reality most fluids are non-Newtonian. Countless studies of non-Newtonian fluid has been published including the study of micropolar fluid by [1], [2] and [3] and the investigation on boundary layer flow of Jeffrey fluid conducted by [4] and [5]. Casson fluid and viscoelastic fluid are also among the non-Newtonian fluid which has garnered the interest of researchers. The boundary layer flow of viscoelastic fluid embedded in porous medium has been studied in [6] while [7] explores the topic of drag detection past a sphere.

However, recent researches on non-Newtonian fluid are inclined towards the study of fluid that displays the characteristics of two different fluid family. Instead of choosing one or the other, the combination of both fluid model would be a better fit. This study for instance, will propose a viscoelastic micropolar model that would accommodate fluid that is viscoelastic and at the same time contain microstructures. Among other researches that followed the current trend of the complex non-Newtonian fluid is the study of micropolar nanofluid in [8], Jeffrey micropolar in [9] and also Casson nanofluid [10].
The aligned MHD effect will also be explored in this study. The MHD effect on non-Newtonian fluid has been highlighted in several studies including [11] and [12]. However, to the best of author’s knowledge, none of them have taken interest in the effect of aligned MHD as the fluid flows past a solid sphere. The flow of viscoelastic micropolar fluid past a sphere has also never been addressed in any study.

2. Mathematical Formulation
Consider the convective boundary layer flow of a viscoelastic micropolar fluid with MHD effect as it passes a solid sphere with radius $a$. Figure 1 is the graphical representation of the physical model and coordinate system of the problem.

![Figure 1. Physical model and coordinate system.](image)

It is assumed that the constant wall temperature of the sphere surface is $T_w$ and the ambient temperature is $T_\infty$. Suppose $u$ and $v$ are the velocity components along the $x$ and $y$ axes, $T$ is the temperature of the fluid and $H$ is the microrotation components orthogonal to the $x$-$y$ plane, then according to [13] and [14], the governing equations of the boundary layer flow for this problem can be written as:

Continuity equation:

$$\frac{\partial}{\partial x}(\bar{\rho} \bar{u}) + \frac{\partial}{\partial y}(\bar{\rho} \bar{v}) = 0$$  \hspace{1cm} (1)

Momentum equation:

$$\frac{\bar{u}}{\rho} \frac{\partial \bar{u}}{\partial x} + \frac{\bar{v}}{\rho} \frac{\partial \bar{u}}{\partial y} = \bar{u} \frac{d \bar{u}}{d x} + \left( \frac{\mu + \kappa}{\rho} \right) \frac{\partial^2 \bar{u}}{\partial y^2} + \kappa \frac{\partial}{\partial x} \left( \frac{\bar{u}}{\rho} \frac{\partial \bar{u}}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{\bar{v}}{\rho} \frac{\partial \bar{u}}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{\bar{v}}{\rho} \frac{\partial \bar{v}}{\partial y} \right) + g \beta (T - T_\infty) \sin \left( \frac{x}{a} \right)$$

$$+ \frac{\kappa}{\rho} \frac{\partial \bar{H}}{\partial y} - \frac{\sigma}{\rho} (\bar{u} - \bar{u}_w) B^2 \sin^2 \alpha$$  \hspace{1cm} (2)

Micropolar Equation:

$$\rho \left( \bar{u} \frac{\partial \bar{H}}{\partial x} + \bar{v} \frac{\partial \bar{H}}{\partial y} \right) = -\kappa \left( 2 \bar{H} + \frac{\partial \bar{H}}{\partial x} \right) + \frac{\rho}{\mu} \frac{\partial \bar{H}}{\partial y}$$  \hspace{1cm} (3)

Energy equation:
subject to the boundary conditions
\[ \bar{u} = \bar{v} = 0, \quad T = T_w, \quad \bar{H} = -\frac{1}{2} \frac{\partial \bar{u}}{\partial y} \text{ on } \bar{y} = 0, \]
\[ \bar{u} = \bar{u}_e(\bar{x}), \quad \frac{\partial \bar{u}}{\partial y} = 0, \quad T = T_e, \quad \bar{H} = 0 \quad \text{as } \bar{y} \to \infty \]
where \( \bar{u}_e \) is velocity outside the boundary layer and \( \bar{r}(\bar{x}) \) is the radial distance from the symmetrical axis to the sphere surface defined as
\[ \bar{u}_e(\bar{x}) = \frac{3}{2} U_x \sin \left( \frac{\bar{x}}{a} \right), \quad \bar{r}(\bar{x}) = a \sin \left( \frac{\bar{x}}{a} \right) \]

In the equations, \( \mu, \kappa, \rho, \kappa, g, \beta, \sigma \) and \( B \) represents the dynamic viscosity, micropolar vortex viscosity, density, viscoelastic vortex viscosity, gravitational acceleration, coefficient of thermal expansion, electrical conductivity and magnetic field, respectively. The microinertia per unit mass, \( J \) and the spin gradient viscosity, \( \gamma \) are given by
\[ J = \frac{a v}{U_x}, \quad \gamma = \left( \mu + \frac{\kappa}{2} \right) J. \]

In order to reduce the complexity of the equation and the number of parameters involved, the following non-dimensional variables are introduced.
\[ x = \bar{x}/a, \quad y = \bar{r}/U_x, \quad u = \bar{u}/U_x, \quad v = \bar{v}/U_x, \]
\[ u_e(x) = \bar{u}_e(\bar{x})/U_x, \quad r(x) = \bar{r}(\bar{x})/a, \quad H = (a/U_x) \bar{H} \]
\[ \theta = \frac{T - T_e}{T_w - T_e} \]

As a result, equations (1) to (5) are transformed into a set of non-dimensional equations as follows.

Continuity equation:
\[ \frac{\partial}{\partial x} (ru) + \frac{\partial}{\partial y} (rv) = 0 \]

Momentum equation:
\[ \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = u \frac{dx}{dt} + (1 + K_t) \frac{\partial^2 u}{\partial y^2} + K \left[ \frac{\partial}{\partial x} \left( u \frac{\partial^2 u}{\partial y^2} \right) + v \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} \frac{\partial^2 v}{\partial y^2} \right] + \lambda \theta \sin(x) + K_1 \frac{\partial H}{\partial y} \]
\[ -M (u - u_e) \sin^2 \alpha \]

Micropolar equation:
\[ u \frac{\partial H}{\partial x} + v \frac{\partial H}{\partial y} = -K_1 \left( 2H + \frac{\partial u}{\partial y} \right) + \left( 1 + K_i \right) \frac{\partial^2 H}{\partial y^2} \]

Energy equation:
\[ u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \frac{1}{\Pr} \frac{\partial^2 \theta}{\partial y^2} \]

New parameters are introduced in the new dimensionless equations where \( \Pr \) is the Prandtl number, \( K \) is the viscoelastic parameter defined by \( K = \frac{k_U}{a \rho \nu} \) and \( K_i = \frac{\kappa}{\mu} \) is the material parameter. The mixed
convection parameter, $\lambda$, is given by $\lambda = \frac{Gr}{Re}$ where the Grashoff number, $Gr = \frac{g \beta (T - T_0) a^3}{\nu^2}$ and the Reynolds number, $Re = \frac{U_a a}{\nu}$. The boundary conditions is also transformed to

$$u = v = 0, \quad \theta = 1, \quad H = -\frac{1}{2} \frac{\partial u}{\partial y} \text{ on } y = 0,$$

$$u_y = \frac{3}{2} \sin x, \quad \frac{\partial u}{\partial y} = 0, \quad \theta = 0, \quad H = 0 \quad \text{as } y \to \infty$$

(11)

In order to solve equations (7) to (10) with subject to the boundary condition in (11), similarity method is applied to the equations using the transformation

$$\psi = xr(x)f(x,y), \quad H = xh(x,y), \quad \theta = \theta(x,y)$$

(12)

and the stream function, $\psi$ is defined as

$$u = \frac{1}{r} \frac{\partial \psi}{\partial y}, \quad v = -\frac{1}{r} \frac{\partial \psi}{\partial x}$$

(13)

Substituting equation (12) into equations (8) to (10), they are transformed into a set of partial differential equations given by

Momentum equation:

$$(1 + K_1) \frac{\partial^3 f}{\partial y^3} \left( \frac{\partial f}{\partial y} \right)^2 + \frac{9 \sin x \cos x}{x} + \frac{3 \sin x}{x} \theta + K_1 \frac{\partial h}{\partial y} - M \left( \frac{\partial f}{\partial y} \frac{3 \sin x}{2} \right) \sin^2 \alpha$$

$$+ \left( 1 + \frac{x \cos x}{\sin x} \right) f \frac{\partial^2 f}{\partial y^2} + K_1 \left[ 2 \frac{\partial f}{\partial y} \frac{\partial^2 f}{\partial y^2} + \left( 1 + 2 \frac{x \cos x}{\sin x} \right) \left( f \frac{\partial^4 f}{\partial y^4} + \left( \frac{\partial^2 f}{\partial y^2} \right)^2 \right) \right]$$

$$= x \left( \frac{\partial f}{\partial y} \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial f}{\partial x} \frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial x} \frac{\partial^2 f}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial y} \frac{\partial^2 f}{\partial x \partial y} \right)$$

(14)

Micropolar equation

$$\left( 1 + \frac{K_1}{2} \right) \frac{\partial^2 h}{\partial y^2} + \left( 1 + \frac{x \cos x}{\sin x} \right) f \frac{\partial h}{\partial y} - K_1 \left( 2 h + \frac{\partial^2 f}{\partial y} \right) - \frac{\partial f}{\partial y} \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial y} \frac{\partial^2 f}{\partial x \partial y} \right)$$

(15)

Energy equation:

$$\frac{1}{P} \frac{\partial^2 \theta}{\partial y^2} + \left( 1 + \frac{x \cos x}{\sin x} \right) f \frac{\partial \theta}{\partial y} - x \left( \frac{\partial f}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial \theta}{\partial y} \right)$$

(16)

Boundary condition:

$$f(0) = f'(0) = 0, \quad \theta(0) = 1, \quad h(0) = -\frac{1}{2} f''(0) \quad \text{on } y = 0,$$

$$f' = \frac{3 \sin x}{2}, \quad f'' = 0, \quad \theta = 0, \quad h = 0 \quad \text{as } y \to \infty$$

(17)

At the lower stagnation point of the sphere, i.e. $x \approx 0$ equations (14) to (17) will reduce to the following ordinary differential equations:
Momentum equation:

\[
(1 + K_i) f''' - f''^2 + \frac{9}{4} + \lambda \vartheta + K_i h' - M \left( f' - \frac{3}{2} \right) \sin^2 \alpha + 2ff'' \\
+ 2K \left( f' f''' - f f'' - (f')^2 \right) = 0
\]  

(18)

Micropolar equation:

\[
\left( 1 + \frac{K_i}{2} \right) h'' + 2f h' - f' h - K_i (2h + f'') = 0
\]

(19)

Energy equation:

\[
\frac{1}{\text{Pr}} \theta'' + 2f \theta' = 0
\]

(20)

and the boundary condition becomes

\[
f(0) = f'(0) = 0, \quad \theta(0) = 1, \quad h(0) = -\frac{1}{2} f''(0) \quad \text{at} \quad y = 0
\]

\[
f'(\infty) = \frac{3}{2}, \quad f''(\infty) = 0, \quad \theta(\infty) = 0, \quad h(\infty) = 0 \quad \text{as} \quad y \to \infty
\]

(21)

where prime denotes differentiation with respect to the variable y.

3. Results and Discussion

Equations (18) to (20) with respect to the boundary conditions in equation (21) are solved numerically in Fortran program using the Keller-box method, a finite difference method that has been explained in [15]. For validation purpose, current results are compared to the limiting cases from [14] (i.e. \(K = K_i = M = 0\)) at Pr=0.7. Since both results shows excellent agreement, our finding is deemed reliable.

| \(\lambda\) | \(f''(0)\) | \(\vartheta'(0)\) |
|----------|----------|----------|
| Nazar et al. (2003) | Present values | Nazar et al. (2003) | Present values |
| 0        | 2.4151   | 2.532705 | 0.8162 | 0.81985 |
| 1        | 2.8064   | 2.801357 | 0.8463 | 0.840638 |
| 2        | 3.1804   | 3.174699 | 0.8648 | 0.863636 |
| 3        | 3.5401   | 3.533803 | 0.8857 | 0.884575 |
| 4        | 3.888    | 3.880975 | 0.905  | 0.903855 |
| 5        | 4.2257   | 4.217931 | 0.923  | 0.921762 |
| 6        | 4.5546   | 4.545991 | 0.9397 | 0.938511 |
| 7        | 4.8756   | 4.866201 | 0.9555 | 0.954268 |
| 8        | 5.1896   | 5.179403 | 0.9704 | 0.969164 |
| 9        | 5.4974   | 5.486292 | 0.9846 | 0.983305 |
| 10       | 5.7995   | 5.78745  | 0.9981 | 0.996776 |
| 20       | 8.5876   | 8.565406 | 1.1077 | 1.106159 |
Table 2. Values of \( f'''(0) \) and \(-\theta'(0)\) for various values of \( \lambda \) at \( K = 0.5, K_1 = 1, M = 0.5, \text{Pr} = 25, \alpha = \frac{\pi}{6} \).

| \( \lambda \) | \( f'''(0) \) | \(-\theta'(0)\) |
|---|---|---|
| 1 | 1.349332 | 2.444838 |
| 2 | 1.434752 | 2.477711 |
| 3 | 1.517521 | 2.508733 |
| 4 | 1.597913 | 2.538130 |
| 5 | 1.676158 | 2.566088 |
| 6 | 1.752448 | 2.592758 |
| 7 | 1.826949 | 2.618270 |
| 8 | 1.899801 | 2.642734 |
| 9 | 1.971129 | 2.666244 |
| 10 | 2.041041 | 2.688881 |
| 20 | 2.679030 | 2.879437 |

Table 3. Values of \( f'''(0) \) and \(-\theta'(0)\) for various values of \( K \) at \( \lambda = 0.5, K_1 = 1, M = 0.5, \text{Pr} = 25, \alpha = \frac{\pi}{6} \).

| \( K \) | \( f'''(0) \) | \(-\theta'(0)\) |
|---|---|---|
| 0.1 | 1.797243 | 2.686245 |
| 0.5 | 1.305525 | 2.427623 |
| 1.0 | 1.032302 | 2.252837 |
| 5.0 | 0.508221 | 1.791748 |
| 10.0 | 0.362553 | 1.602926 |
| 50.0 | 0.163508 | 1.223746 |
| 100.0 | 0.116075 | 1.082783 |

From Table 2, it can be observed that the values of the skin friction and heat transfer rate increases as the mixed convection parameter, \( \lambda \) increases and the opposite effect occurs to the shear stress coefficient as the viscoelastic parameter, \( K \) increases in Table 3. Figure 2, 3 and 4 shows the velocity, temperature and microrotation profile for various values of \( K \) when \( \lambda = 0.5, M = 0.5, K_1 = 1, \text{Pr} = 25 \) and \( \alpha = \frac{\pi}{6} \). At ascending values of \( K \), the velocity and temperature profiles decrease and increase, respectively. For microrotation profile, the boundary layer thickness seems to have an effect on the profile. As \( K \) increases, the profile increases when \( y < 1 \), and decreases as \( y \) gets bigger.

![Figure 2. Velocity profile for various values of \( K \).](image-url)
4. Conclusion

The convective boundary layer flow of viscoelastic micropolar fluid with aligned MHD effect is investigated in this study. The viscoelastic micropolar model will be a generalised model that is applicable for viscoelastic fluid, micropolar fluid and the combination of both. This model will be a better representation of fluid that are simultaneously viscoelastic and micropolar. The partial differential equations transformed from the governing equations are solved numerically using the Keller-box scheme in Fortran program and results are found promising upon validation with previous work. From this study, holding other parameters constant, the following results are obtained:

- As the value of the viscoelastic parameter gets bigger, the temperature profile increases while the velocity profile decreases.
- As the mixed convection parameter increases, the shear stress coefficient also increases.
- The skin friction and heat transfer rate descends as the viscoelastic parameter increases.
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