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THE CONSTRUCTION OF NORMAL BASES FOR THE SPACE OF
CONTINUOUS FUNCTIONS ON $V_q$, WITH THE AID OF OPERATORS

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Abstract. Let $a$ and $q$ be two units of $\mathbb{Z}_p$, $q$ not a root of unity, and let $V_q$ be the closure of the set $\{aq^n \mid n = 0, 1, 2, \ldots \}$. $K$ is a non-archimedean valued field, $K$ contains $\mathbb{Q}_p$, and $K$ is complete for the valuation $| \cdot |$, which extends the $p$-adic valuation. $C(V_q \to K)$ is the Banach space of continuous functions from $V_q$ to $K$, equipped with the supremum norm.

Let $\mathcal{E}$ and $D_q$ be the operators on $C(V_q \to K)$ defined by $(\mathcal{E}f)(x) = f(qx)$ and $(D_qf)(x) = (f(qx) - f(x))/(x(q-1))$. We will find all linear and continuous operators that commute with $\mathcal{E}$ (resp. with $D_q$), and we use these operators to find normal bases $(r_n(x))$ for $C(V_q \to K)$. If $f$ is an element of $C(V_q \to K)$, then there exist elements $\alpha_n$ of $K$ such that $f(x) = \sum_{n=0}^{\infty} \alpha_n r_n(x)$ where the series on the right-hand-side is uniformly convergent. In some cases it is possible to give an expression for the coefficients $\alpha_n$.

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1. Introduction

Let $p$ be a prime, $\mathbb{Z}_p$ the ring of the $p$-adic integers, $\mathbb{Q}_p$ the field of the $p$-adic numbers. $K$ is a non-archimedean valued field, $K \supset \mathbb{Q}_p$, and we suppose that $K$ is complete for the valuation $| \cdot |$, which extends the $p$-adic valuation. Let $a$ and $q$ be two units of $\mathbb{Z}_p$ (i.e. $|a| = |q| = 1$), $q$ not a root of unity. Let $V_q$ be the closure of the set $\{aq^n \mid n = 0, 1, 2, \ldots \}$.

We denote by $C(V_q \to K)$ (resp. $C(\mathbb{Z}_p \to K)$) the set of all continuous functions $f : V_q \to K$ (resp. $f : \mathbb{Z}_p \to K$) equipped with the supremum norm. If $f$ is an element of $C(V_q \to K)$ then we define the operators $\mathcal{E}$ and $D_q$ as follows:

$(\mathcal{E}f)(x) = f(qx)$
(\text{D}_q f)(x) = \frac{f(qx) - f(x)}{x(q - 1)}

We remark that the operator $E$ does not commute with $D_q$. Furthermore, the operator $D_q$ lowers the degree of a polynomial with one, whereas the operator $E$ does not.

If $\mathcal{L}$ is a non-archimedean Banach space over a non-archimedean valued field $\mathbb{L}$, and $e_1, e_2, \ldots$ is a finite or infinite sequence of elements of $\mathcal{L}$, then we say that this sequence is orthogonal if $||e_1 + \cdots + e_k|| = \max\{||e_i|| : i = 1, \ldots, k\}$ for all $k$ in $\mathbb{N}$ (or for all $k$ that do not exceed the length of the sequence) and for all $e_1, \ldots, e_k$ in $\mathcal{L}$. An orthogonal sequence $e_1, e_2, \ldots$ is called orthonormal if $||e_i|| = 1$ for all $i$. A family $(e_i)$ of elements of $\mathcal{L}$ forms a (orthonormal) basis of $\mathcal{L}$ if the family $(e_i)$ is orthonormal and also a basis. We will call a sequence of polynomials $(p_n(x))$ a polynomial sequence if $p_n$ is exactly of degree $n$ for all natural numbers $n$.

The aim here is to find normal bases for $C(V_q \rightarrow K)$, which consist of polynomial sequences. Therefore we will use linear, continuous operators which commute with $D_q$ or with $E$. If $(r_n(x))$ is such a polynomial sequence, and if $f$ is an element of $C(V_q \rightarrow K)$, there exist coefficients $\alpha_n$ in $K$ such that $f(x) = \sum_{n=0}^{\infty} \alpha_n r_n(x)$ where the series on the right-hand-side is uniformly convergent. In some cases it is possible to give an expression for the coefficients $\alpha_n$.

We remark that all the results (with proofs) in this paper can be found in [5], except for theorem 5.

2. Notations.

Let $V_q$, $K$ and $C(V_q \rightarrow K)$ be as in the introduction. The supremum norm on $C(V_q \rightarrow K)$ will be denoted by $|| \cdot ||$. We introduce the following:

$A_0(x) = 1$, $A_n(x) = (x - aq^{n-1})A_{n-1}(x)$ ($n \geq 1$),

$B_n(x) = A_n(x)/A_n(aq^n)$, $C_n(x) = a^{-n}q^{n(n-1)/2}(q - 1)^n B_n(x)$

It is clear that $(A_n(x)), (B_n(x))$ and $(C_n(x))$ are polynomial sequences. The sequence $(C_n(x))$ forms a basis for $C(V_q \rightarrow K)$ and the sequence $(B_n(x))$ forms a normal basis for $C(V_q \rightarrow K)$. From this it follows that $||B_n|| = 1$ and $||C_n|| = |(q - 1)^n|$. Let $E$ and $D_q$ be as in the introduction. Then we introduce the following:

Definition. Let $f$ be a function from $V_q$ to $K$. We define the following operators:

$(D_q^n f)(x) = (D_q(D_q^{n-1} f))(x)$

$(E^n f)(x) = f(q^n x)$

$D f(x) = D^{(1)} f(x) = f(qx) - f(x) = ((E - 1)f)(x)$

$D^{(n)} f(x) = ((E - 1) . . . (E - q^{n-1})f)(x)$, $D^{(0)} f(x) = f(x)$

The operator $D_q$ does not commute with $D$. The following properties are easily verified:

$D_q j C_k(x) = C_{k-j}(x)$ if $k \geq j$, $D_q j C_k(x) = 0$ if $j > k$. So $D_q j$ lowers the degree of a polynomial with $j$.\n
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\[ D^{(j)}B_k(x) = (x/a)^jq^{(j-k)}B_{k-j}(x) \] if \( j \leq k \), \[ D^{(j)}B_k(x) = 0 \] if \( j > k \)

If \( p(x) \) is a polynomial of degree \( n \), then \( (D^{(j)}p)(x) \) is a polynomial of degree \( n \) if \( n \) is at least \( j \), and \( (D^{(j)}p)(x) \) is the zero-polynomial if \( n \) is strictly smaller than \( j \).

If \( f \) is an element of \( C(V_q \rightarrow K) \), then we also have

i) \( (D_n f)(x) = x^n q^{n(n-1)/2}(q-1)^n(D_q^n f)(x) \)

ii) \( (q-1)^n D_q^n f(x) \rightarrow 0 \) uniformly

iii) \( D_n f(x) \rightarrow 0 \) uniformly

(i) can be found in [1], p. 60, ii) can be found in [3], p. 124-125, iii) follows from i) and ii).

3. Linear Continuous Operators which Commute with \( E \) or with \( D_q \)

Let us start this section with the following known result:

If \( f \) is an element of \( C(\mathbb{Z}_p \rightarrow K) \), then the translation operator \( E \) on \( C(\mathbb{Z}_p \rightarrow K) \) is the operator defined by \( Ef(x) = f(x+1) \).

If we put \( G_n(x) = \binom{x}{n} \) (the binomial polynomials), then L. Van Hamme ([4]) proved the following theorem:

A linear, continuous operator \( Q \) on \( C(\mathbb{Z}_p \rightarrow K) \) commutes with the translation operator \( E \) if and only if the sequence \( (g_n) \) is bounded, where \( g_n = QG_n(0) \).

Such an operator \( Q \) can be written in the following way:

\[ Q = \sum_{i=0}^{\infty} g_i \Delta^i \]

where \( \Delta \) is the operator defined as follows:

\( (\Delta f)(x) = f(x+1) - f(x) \)

We can prove analogous theorems for the operators \( E \) and \( D_q \) on \( C(V_q \rightarrow K) \):

**Theorem 1** An operator \( Q \) on \( C(V_q \rightarrow K) \) is continuous, linear and commutes with \( E \) if and only if the sequence \( (b_n) \) is bounded, where \( b_n = (QB_n)(a) \).

From the proof of the theorem it follows that \( Q \) can be written in the form \( Q = \sum_{i=0}^{\infty} b_i D^{(i)} \).

If \( f \) is an element of \( C(V_q \rightarrow K) \), then \( (Qf)(x) = \sum_{i=0}^{\infty} b_i (D^{(i)}f)(x) \) and the series on the right-hand-side is uniformly convergent (since \( D^{(n)}f(x) \rightarrow 0 \) uniformly). Clearly we have

\( b_n = (QB_n)(a) \), since \( (QB_n)(a) = \sum_{i=0}^{\infty} b_i D^{(i)}B_n)(a) = \sum_{i=0}^{n} b_i (x/a)^i q^{(i-n)}B_{n-i}(a) = b_n \).

Furthermore, \( Qx^n \) is a \( K \)-multiple of \( x^n \).

If \( b_0 = \ldots = b_{N-1} = 0 \), \( b_N \neq 0 \), and if \( p(x) \) is a polynomial, then \( x^N \) divides \( (Qp)(x) \).

**Some examples**

1) For the operator \( E \) we have: \( (EB_n)(x) = B_n(qx) \), so \( (EB_0)(a) = 1 \), \( (EB_1)(a) = 1 \), and \( (EB_n)(a) = 0 \) if \( n \geq 2 \). This gives us \( E = \sum_{i=0}^{n} D^{(i)} \).
2) The operator $\mathcal{E} \circ D = \mathcal{E}D$ clearly commutes with $\mathcal{E}$. We have $((\mathcal{E}D)B_0)(a) = 0$, and since $(n > 1) ((\mathcal{E}D)B_n)(x) = (\mathcal{E} (q^{1-n}B_{n-1}))(x) = \frac{q^n}{a}q^{1-n}B_{n-1}(qx)$, we find $((\mathcal{E}D)B_1)(a) = q$, $(\mathcal{E}D)B_2)(a) = 1$ and $((\mathcal{E}D)B_n)(a) = 0$ if $n \geq 3$. We conclude that $\mathcal{E}D = q^{D(1)} + D(2)$.

Analogous to theorem 1 we have:

**Theorem 2** An operator $Q$ on $C(V_q \rightarrow K)$ is continuous, linear and commutes with $D_q$ if and only if the sequence $(c_n/(q-1)^n)$ is bounded, where $c_n = (QC_n)(a)$.

Such an operator $Q$ can be written in the form $Q = \sum_{i=0}^{\infty} c_i D_q^i$, and if $f$ is an element of $C(V_q \rightarrow K)$ it follows that $(Qf)(x) = \sum_{i=0}^{\infty} c_i (D_q^i f)(x)$, where the series on the right-hand-side converges uniformly (since $(q-1)^n D_q^n f(x) \rightarrow 0$ uniformly). Furthermore, we have $c_n = (QC_n)(a)$ since

$$ (QC_n)(a) = (\sum_{i=0}^{\infty} c_i D_q^i C_n)(a) = \sum_{i=0}^{n} c_i C_{n-i}(a) = c_n. $$

**Remarks**

1) Let $R$ and $Q$ be linear, continuous operators on $C(V_q \rightarrow K)$, with $R$ of the form $R = \sum_{i=1}^{\infty} b_i D_q^{(i)}$ (i.e. $R$ commutes with $\mathcal{E}$, $b_0 = 0$), and $Q$ of the form $Q = \sum_{i=1}^{\infty} c_i D_q^i$ (i.e. $Q$ commutes with $D_q$, $c_n - 0$). The main difference between the operators $Q$ and $R$ is that $Q$ lowers the degree of each polynomial with at least one, where $R$ does not necessarily lowers the degree of a polynomial.

2) If $Q_1$ and $Q_2$ both commute with $D_q$ and if $Q_1 = \sum_{i=0}^{\infty} c_{1;i} D_q^i$, then $Q_2 = \sum_{i=0}^{\infty} c_{2;i} D_q^i$, then $(Q_1 o Q_2)(f) = (Q_2 o Q_1)(f) = \sum_{k=0}^{\infty} D_q^k f \left( \sum_{j=0}^{k} c_{1;j} c_{2;k-j} \right)$.

If we take two formal power series $q_1(t) = \sum_{i=0}^{\infty} c_{1;i} t^i$, $q_2(t) = \sum_{i=0}^{\infty} c_{2;i} t^i$, then

$$ q_1(t) \cdot q_2(t) = \sum_{k=0}^{\infty} t^k \left( \sum_{j=0}^{k} c_{1;j} c_{2;k-j} \right), $$

so the composition of two operators which commute with $D_q$, corresponds with multiplication of power series.
This is not the case if we take two operators which commute with $\mathcal{E}$: Take e.g. $\mathcal{E} = D^{(0)} + D^{(1)}$ and $D^{(1)}$, then $\mathcal{E} o D^{(1)} = \mathcal{E} D^{(1)} = q D^{(1)} + D^{(2)}$, whereas for power series this gives $q_1(t) = 1 + t$, $q_2(t) = t$ and $q_1(t) \cdot q_2(t) = t + t^2$.

4. Normal bases for $C(V_q \rightarrow K)$

We use the operators of theorems 1 and 2 to make polynomials sequences $(p_n(x))$ which form normal bases for $C(V_q \rightarrow K)$. If $Q$ is an operator as found in theorem 1, with $b_0$ equal to zero, we associate a (unique) polynomial sequence $(p_n(x))$ with $Q$. We remark that the operator $R = \sum_{i=0}^{\infty} b_i D(i)$ does not necessarily lowers the degree of a polynomial.

**Proposition 1** Let $Q = \sum_{i=N}^{\infty} b_i D(i) \ (N \geq 1)$ with $|b_N| > |b_n|$ if $n > N$. There exists a unique polynomial sequence $(p_n(x))$ such that $(Qp_n)(x) = x^N p_{n-N}(x)$ if $n \geq N$, $p_n(aq^i) = 0$ if $n \geq N$, $0 \leq i < N$ and $p_n(x) = B_n(x)$ if $n < N$.

In the same way as in proposition 1 we have.

**Proposition 2** Let $Q = \sum_{i=N}^{\infty} c_i D_q^i \ (N \geq 1)$, $c_N \neq 0$, $(c_n/(q-1)^n)$ bounded. Then there exists a unique polynomial sequence $(p_n(x))$ such that $(Qp_n)(x) = p_{n-N}(x)$ if $n \geq N$, $p_n(aq^i) = 0$ if $n \geq N$, $0 \leq i < N$ and $p_n(x) = B_n(x)$ if $n < N$.

We use the operators of theorems 1 and 2 to make polynomials sequences $(p_n(x))$ which form normal bases for $C(V_q \rightarrow K)$. If $f$ is an element of $C(V_q \rightarrow K)$, there exist coefficients $\alpha_n$ such that $f(x) = \sum_{n=0}^{\infty} \alpha_n p_n(x)$ where the series on the right-hand-side is uniformly convergent. In some cases, it is also possible to give an expression for the coefficients $\alpha_n$.

**Theorem 3** Let $Q = \sum_{i=N}^{\infty} b_i D(i) \ (N \geq 1)$ with $|b_n| < |b_N| = 1$ if $n > N$

1) There exists a unique polynomial sequence $(p_n(x))$ such that $(Qp_n)(x) = x^N p_{n-N}(x)$ if $n \geq N$, $p_n(aq^i) = 0$ if $n \geq N$, $0 \leq i < N$ and $p_n(x) = B_n(x)$ if $n < N$. This sequence forms a normal basis for $C(V_q \rightarrow K)$ and the norm of $Q$ equals one.

2) If $f$ is an element of $C(V_q \rightarrow K)$, then $f$ can be written as a uniformly convergent series $f(x) = \sum_{n=0}^{\infty} \beta_n p_n(x)$, $\beta_n = ((D(i)(x^{-N}Q)^k)f)(a)$ if $n = i + kN \ (0 \leq i < N)$, with $\|f\| = \max_{0 \leq k, 0 \leq i < N} \|((D(i)(x^{-N}Q)^k)f)(a)\|$, where $x^{-N}Q$ is a linear continuous operator with norm equal to one.
And analogous to theorem 3 we have

**Theorem 4** Let \( Q = \sum_{i=N}^{\infty} c_i D_i^q \) \((N \geq 1)\) with \( |c_N| = \left| (q-1)^N \right| \), \( |c_n| \leq \left| (q-1)^n \right| \) if \( n > N \).

1) There exists a unique polynomial sequence \((p_n(x))\) such that \((Q p_n)(x) = p_{n-N}(x)\) if \( n \geq N \), \( p_n(a q^i) = 0 \) if \( n \geq N \), \( 0 \leq i < N \) and \( p_n(x) = B_n(x) \) if \( n < N \). This sequence forms a normal basis for \( C(V_q \to K) \) and the norm of \( Q \) equals one.

2) If \( f \) is an element of \( C(V_q \to K) \), there exists a unique uniformly convergent expansion of the form \( f(x) = \sum_{n=0}^{\infty} \gamma_n p_n(x) \), where \( \gamma_n = a^i (q-1)^i q^{i(i-1)/2} D_i^q f(a) \) if \( n = i + kN \) \((0 \leq i < N)\), with \( \|f\| = \max_{0 \leq k, 0 \leq i < N} \{ \| (q-1)^i D_i^q f(a) \| \} \).

**Remark.** Here we have \( |c_n| \leq |c_N| \), in contrast with theorem 3, where we need \( |b_n| < |b_N| \) \((n > N)\).

**An example**
Let us consider the following operator \( Q = (q-1) D_q \). Then \( c_1 = (q-1) \) and \( c_k = 0 \) if \( k \neq 1 \).
The polynomials \( p_k(x) \) are given by \( p_k(x) = C_k(x)/(q-1)^k \), and they form a normal basis for \( C(V_q \to K) \). The expansion \( f(x) = \sum_{k=0}^{\infty} ((q-1)^k D_k^q f(a)) p_k(x) = \sum_{k=0}^{\infty} (D_k^q f(a)) C_k(x) \) is known as Jackson's interpolation formula \((2),(3)\).

If \( Q \) is an operator as found in theorem 4, with \( N \) equal to one, then we can prove a theorem analogous to theorem 2:

**Theorem 5** Let \( Q \) be an operator such that \( Q = \sum_{i=1}^{\infty} c_i D_i^q \), with \( |c_1| = \left| (q-1) \right| \), \( |c_n| \leq \left| (q-1)^n \right| \) if \( n > 1 \), and let \( p_n(x) \) be the polynomial sequence as found in theorem 4.

An operator \( T \) on \( C(V_q \to K) \) is continuous, linear and commutes with \( D_q \) if and only if \( T \) is of the form \( T = \sum_{i=0}^{\infty} d_i Q^i \), where the sequence \((d_n)\) is bounded, where \( d_n = (Tp_n)(a) \).

**Remark.** In theorem 2 the sequence \((c_n/(q-1)^n)\) must be bounded, whereas here the sequence \((d_n)\) must be bounded. This follows from the fact that the norm of the operator \( D_q \) equals \( |q-1|^{-1} \), whereas the norm of the operator \( Q \) equals 1.

**5. More Normal Bases**
We want to make more normal bases, using the ones we found in theorems 3 and 4. For operators which commute with \( E \) we can prove the following theorem:
Theorem 6 \[ \text{Let } (p_n(x)) \text{ be a polynomial sequence which forms a normal basis for } C(V_q \rightarrow K), \]
and let \( Q = \sum_{i=N}^{\infty} b_i D^{(i)} \) \((N \geq 0)\) with \( 1 = |b_N| > |b_k| \) if \( k > N \). If \( Qp_n(x) = x^N r_{n-N}(x) \) \((n \geq N)\), then the polynomial sequence \((r_k(x))\) forms a normal basis for \( C(V_q \rightarrow K) \).

And analogous for operators which commute with the operator \( D_q \) we have:

Theorem 7 \[ \text{Let } (p_n(x)) \text{ be a polynomial sequence which forms a normal basis for } C(V_q \rightarrow K), \]
and let \( Q = \sum_{i=N}^{\infty} c_i D_q^i \) \((N \geq 0)\) with \( |c_N| = |(q-1)^N| \), \( |c_n| \leq |(q-1)^n| \) if \( n > N \).
If \( (Qp_n)(x) = r_{n-N}(x) \) \((n \geq N)\), then the polynomial sequence \((r_k(x))\) forms a normal basis for \( C(V_q \rightarrow K) \).

We remark that analogous results can be found on the space \( C(\mathbb{Z}_p \rightarrow K) \) for linear continuous operators which commute with the translation operator \( E \). The result analogous to theorems 3 and 4 for the case \( N \) equal to one, was found by L. Van Hamme (see [4]), and the extensive version of theorems 3 and 4, and the analogons of theorems 5, 6 and 7 can be found with proofs similar to the proofs of the theorems in this paper.

REFERENCES

[1] F.H. Jackson, \textit{Generalization of the Differential Operative Symbol with an Extended Form of Boole's Equation}, Messenger of Mathematics, vol. 38, 1909, p. 57-61.
[2] F.H. Jackson, \textit{q-form of Taylor's Theorem}, Messenger of Mathematics, vol 38, (1909) p. 62-64.
[3] L. Van Hamme, \textit{Jackson's Interpolation Formula in p-adic Analysis} Proceedings of the Conference on p-adic Analysis, report nr. 7806, Nijmegen, June 1978, p. 119-125.
[4] L. Van Hamme, \textit{Continuous Operators which commute with Translations, on the Space of Continuous Functions on } \( \mathbb{Z}_p \text{, in "p-adic Functional Analysis"}, \text{ Bayod / Martinez-Maurica / De Grande - De Kimpe (Editors)}, \text{ p. 75-88, Marcel Dekker, 1992.}
[5] A. Verdoodt, \textit{The Use of Operators for the Construction of Normal Bases for the Space of Continuous Functions on } \( V_q \), Bulletin of the Belgian Mathematical Society - Simon Stevin, vol 1, 1994, p.685-699.