Discontinuous BPS spectra in $N = 2$ gauge theory

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Abstract

We consider the spectrum of BPS saturated states in $N = 2$ gauge theories in four dimensions. This spectrum may be discontinuous across real codimension one submanifolds of marginal stability in the moduli space of vacua. An example, which can be treated with semiclassical methods in the weak coupling limit, is the decay of quark-soliton bound states. For a quark and a soliton of electric-magnetic charge vectors $Q$ and $Q'$ respectively, we find that as the manifold of marginal stability is crossed, the number of soliton states changes by a factor of $2^{Q \cdot Q'}$, where the dot denotes the symplectic product.

October 1995
1. Introduction

An important feature of $N = 2$ supersymmetric theories in four space-time dimensions is the existence of a central charge $Z$ in the supersymmetry algebra \( [1] \):

\[
\{ Q^A_\alpha, Q^B_\beta \} = \epsilon_{\alpha\beta} \epsilon^{AB} Z. \tag{1.1}
\]

The central charge is in general a linear combination of conserved abelian charges with coefficients that depend on the vacuum moduli and the parameters of the Lagrangian. In a unitary representation of supersymmetry, the mass $m$ of a state is bounded \( [2] \) by

\[
m \geq |Z|. \tag{1.2}
\]

If we limit ourselves to maximum spin 1 (i.e. we do not consider supergravity theories) we have the following representations of the supersymmetry algebra \( [1] \): When the inequality \( [1.2] \) is not saturated, an irreducible representation contains five scalar, four spinor and one vector particle, i.e. a total of sixteen helicity states. For a so called BPS representation, which saturates \( [1.2] \), there are two possibilities; either two scalar and one spinor particle, or one scalar, two spinor and one vector particle, for a total of four or eight helicity states respectively.

Particles in a BPS representation enjoy the following stability property: For a BPS state with central charge $Z$ and mass $m$ to decay into states with central charges $Z_1$ and $Z_2$ and masses $m_1$ and $m_2$, we must have $Z = Z_1 + Z_2$ by charge conservation. But this means that $m = |Z| \leq |Z_1| + |Z_2| \leq m_1 + m_2$ by the triangle inequality, so this decay cannot take place. These inequalities are saturated exactly when the would-be decay products are also BPS states and the central charges $Z_1$ and $Z_2$ are aligned in the complex plane. This will happen on a real codimension one subvariety of ‘marginal stability’ in the moduli space of vacua. As explained in \( [3] \), the spectrum of BPS one-particle states must sometimes be discontinuous across such a boundary. This phenomenon is well understood in two space-time dimensions \( [4] \), where the number of particles appearing or disappearing in this way can be calculated from Picard-Lefschetz theory in the case of Landau-Ginzburg theories, or from the Tr$(\mathcal{L}^F \mathcal{F} \exp(-\beta H)$ index \( [3] \) and the $tt^*$ topological-anti-topological fusion equations \( [8] \) for general $N = 2$ theories.

The purpose of this article is to study such discontinuities in the context of $N = 2$ gauge theories in four dimensions. In the simplest example, the $N = 2$ $SU(2)$ gauge theory without extra matter \( [8] \), there are only two conserved abelian charges, namely the
electric and magnetic charge for the unbroken $U(1)$ gauge group. Consequently, there is a
single curve of marginal stability where their coefficients in the central charge are aligned.
Unfortunately, this curve lies entirely in the strong coupling regime, where it is difficult
to exhibit the discontinuity of the spectrum explicitly. In more complicated examples, i.e.
with larger gauge groups and/or extra matter, there are more conserved abelian charges
and the manifold of marginal stability depends on the quantum numbers of the particles
involved in the decay. (The set of all such curves, regardless of which particles actually
exist in the spectrum, is in fact dense in the moduli space.) More important for our
purposes is that models with extra matter have dimensionful parameters, namely the bare
masses of the matter fields. When such a parameter is large compared to the dynamically
generated scale of the theory, marginal stability might occur also at weak coupling, where
semiclassical methods are applicable.

This paper is organized as follows: In section two, we quickly review some aspects of
$N = 2$ gauge theories, and in section three, we discuss their BPS spectra at weak coupling.
In section four, we calculate the number of quark-soliton bound states that disappear as
a curve of marginal stability is crossed. In section five, we consider the case of an $SU(2)$
gauge group in somewhat more detail.

2. $N = 2$ supersymmetric gauge theories

The field content of our model consists of an $N = 2$ gluon vector multiplet and an
$N = 2$ quark hypermultiplet. In terms of $N = 1$ superfields, the vector multiplet may
be decomposed as an $N = 1$ vector multiplet $V$ and an $N = 1$ chiral multiplet $\Phi$, both
transforming in the adjoint of the gauge group $G$. The hypermultiplet consists of two
$N = 1$ chiral multiplets $Q$ and $\tilde{Q}$ transforming in complex conjugate (in general reducible)
representations $R$ and $\bar{R}$ under $G$. To have an asymptotically free or scale invariant
model, the index of the representation $R$ must be less than or equal to the index of the
adjoint representation of $G$. The action of the model is completely determined by $N = 2$
supersymmetry to be of the form

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \left( \text{Im} \, \tau \Phi^\dagger e^V \Phi + Q^\dagger e^V Q + \tilde{Q}^\dagger e^V \tilde{Q} \right) + \int d^2\theta \left( \tau W^\alpha W_\alpha + \tilde{Q}(\Phi + m)Q \right) + \text{c.c.},$$

where $m$ gives the bare masses of the irreducible components of the hypermultiplet, and
the gauge coupling constant and the theta-parameter are combined as $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$. (We
use the conventions of \[8\] for \(N = 1\) superspace. Note that our normalization of \(m\) differs from that of \[9\] by a factor of \(\sqrt{2}\). The classical action (2.1) with vanishing bare masses is invariant under an \(U(1)_R\) symmetry such that the \(R\)-characters of \(\Phi\), \(W_\alpha\), \(Q\), and \(\tilde{Q}\) are 1, 1/2, 0, and 0 respectively. The masses \(m\) and the dynamically generated scale \(\Lambda\) in an asymptotically free theory break this symmetry, but we may restore it by assigning \(R\)-character 1 to \(m\) and \(\Lambda\).

The theory has a moduli space of inequivalent vacuum states. We will consider the Coulomb branch of this moduli space in which only the lowest component \(\phi\) of the chiral superfield \(\Phi\) has a non-vanishing vacuum expectation value, subject to the constraint \([\phi, \phi^\dagger] = 0\). The theory will be weakly coupled when \(\langle \phi \rangle\) is large compared to the scale \(\Lambda\). At a generic point on this branch, the gauge symmetry is spontaneously broken to its maximal abelian subgroup \(U(1)_r\), where \(r = \text{rank}(G)\), and we have a vector \(q\) of corresponding electric charges. As usual in spontaneously broken gauge theories where the exact symmetry group contains abelian factors, there will also be a vector of magnetic charges \(g\) in the theory. In general, these charges are not well-defined over the moduli space of vacua; if we assemble \(q\) and \(g\) to an electric-magnetic charge vector \(Q = (q, g)\), this ambiguity amounts to the action of an element of \(Sp(2r, \mathbb{Z})\) on \(Q\) as we encircle a singularity \([3]\). Finally, there is a set of conserved quark number charges \(S\), one for each irreducible component of the representation \(R\). These charges can pick up contributions proportional to \(q\) and \(g\) as we encircle a singularity of the moduli space \([9]\).

3. The BPS spectrum

The central charge \(Z\) for a state is a linear combination of the electric, magnetic and quark number charges:

\[
Z = a \cdot q + a_D \cdot g + m \cdot S, \tag{3.1}
\]

where classically \(a\) is given by the eigenvalues of \(\phi\) and \(a_D = \tau a\), but these coefficients get modified at the quantum level. The formula (3.1) is derived by explicitly constructing the supercharges and calculating their Poisson bracket \([2] [3] [9]\). (The contributions from electric and quark number charges are most easily checked by calculating the anticommutator of two supersymmetry transformations on the component fields of the vector and hypermultiplets.)

We will now determine the spectrum of BPS states in the theory, at least at weak coupling (i.e. for \(\langle \phi \rangle\) large compared to \(\Lambda\) in asymptotically free theories or \(\text{Im} \tau\) large)
where semiclassical methods can be trusted. We start with the elementary field quanta. The fields of the \(N = 2\) vector multiplet fill out a BPS representation of maximum spin 1. Indeed, the bound (1.2) is saturated by the usual Higgs formula for the mass of gauge bosons in a situation with spontaneous symmetry breaking \([10]\). The \(N = 2\) hypermultiplet must also give rise to BPS representations, since it only contains fields of maximum spin 1/2. The central charges of the different particles are given by the eigenvalues of the matrix \(\phi + m\), and the masses, which can be read off from the action (2.1), again saturate (1.2).

The theory also contains magnetically charged solitonic excitations with non-vanishing expectation values of \(\phi\) and the gauge field \(A_\mu\). The bound (1.2) is saturated by the Bogomolny-Prasad-Sommerfield limit \([11, 12]\) of monopoles and dyons \([13, 14, 15]\). Such a configuration breaks half of the supersymmetries \([16]\). Acting with the broken supersymmetry generators on a state of spin 0, we can create two helicity states transforming as spin 1/2 and one more spin 0 state, i.e. a BPS multiplet of maximum spin 1/2. We should also include the CPT conjugate states of opposite electric and magnetic charges for a total of eight helicity states. After an electric-magnetic duality transformation, these states could be described by a hypermultiplet coupled to the dual photons in a \(U(1)^r\) gauge theory \([3]\).

Another type of BPS states can be thought of as bound states in a quark-soliton system. The fermionic fields in the \((Q, \bar{Q})\) hypermultiplet make up a Dirac spinor \(\psi\) and a conjugate spinor \(\bar{\psi} = \psi^\dagger \gamma^0\) transforming in the \(R\) and \(\bar{R}\) representations respectively. The equation of motion for these fields derived from (2.1) in a fixed \(A_\mu\) and \(\phi\) background reads

\[
(i\gamma^\mu D_\mu - \text{Re}(\phi + m) + \gamma^5 \text{Im}(\phi + m)) \psi = 0, \tag{3.2}
\]

where we have decomposed the matrix \(\phi + m\) in its Hermitian and anti-Hermitian parts. A convenient basis for the Dirac matrices is

\[
\gamma^0 = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^i = i \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \quad i = 1, 2, 3, \quad \gamma^5 = i \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{3.3}
\]

where the \(\sigma^i\) are the Pauli matrices. We decompose the Dirac spinor accordingly as \(\psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}\). Choosing the gauge \(A_0 = 0\), replacing \(-i\partial_0\) by the energy eigenvalue \(E\) and multiplying equation (3.2) by \(\gamma^0\) from the left, we get

\[
\begin{pmatrix} \text{Im}(\phi + m) - E & L \\ L^\dagger & -\text{Im}(\phi + m) - E \end{pmatrix} \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = 0, \tag{3.4}
\]
where the operator $L$ is given by

$$L = i\sigma^i D_i - i\text{Re}(\phi + m).$$  \hspace{1cm} (3.5)

Each normalizable solution to (3.4) doubles the number of BPS solitons, since a fermionic state can be either empty or occupied.

4. The discontinuity

It might seem difficult to find solutions to (3.4). However, we know that the spectrum of BPS states cannot change unless we cross a curve of marginal stability. To determine the number of bound states that appear or disappear across such a curve, we choose a specific point on the curve where $\phi$ is Hermitian up to a phase, which we can rotate away by a $U(1)_R$ transformation so that $\text{Im} \phi = 0$. If we now have a solution to $L\psi^- = 0$, a solution of (3.4) would be to take $\psi^+ = 0$ and $E = -\text{Im} m$. Similarly, a solution to $L^\dagger \psi^+ = 0$ gives a solution to (3.4) with $\psi^- = 0$ and $E = \text{Im} m$. Note that since the central charges of the quark components are given by the eigenvalues of $\phi + m$ and the central charge $Z$ of a magnetically charged soliton configuration in the weak coupling limit is a large imaginary number, the mass difference between a quark-soliton bound state and the soliton state should indeed equal $|\text{Im} m|$.

We have thus found a space of solutions to (3.4) of dimension $\dim \text{Ker}(L) + \dim \text{Ker}(L^\dagger)$. In general, this quantity is difficult to determine, but a lower bound is given by the absolute value of $\text{index}(L) = \dim \text{Ker}(L) - \dim \text{Ker}(L^\dagger)$. Now we can use the three-dimensional version of the Callias index theorem [17]:

$$\text{index}(L) = \frac{-1}{16\pi i} \int \text{Tr} \left( UdU \wedge dU \right),$$  \hspace{1cm} (4.1)

where $U = -|\text{Re}(\phi + m)|^{-1}\text{Re}(\phi + m)$ and the integral is taken over a two-sphere at spatial infinity. The matrix $U$ is only well-defined if the eigenvalues of $\text{Re}(\phi + m)$ are non-zero; this is in fact a necessary condition for $L$ to be Fredholm so that $\text{index}(L)$ is well-defined. As long as we stay away from zero eigenvalues of $\text{Re}(\phi + m)$, $\text{index}(L)$ depends continuously on $\phi + m$ and is therefore a constant.

However, the index can change when an eigenvalue changes sign. Indeed, the central charges of the quark components are given by the eigenvalues of $\phi + m$. With $\text{Im} \phi = 0$, there will be a purely imaginary central charge whenever $\text{Re}(\phi + m)$ has a zero eigenvalue.
This aligns with the central charge of a magnetically charged soliton in the weak coupling limit, and a discontinuity in the BPS spectrum is therefore possible. For \( \text{Re } m \) large compared to the scale \( \Lambda \), the discontinuity will take place in the weak coupling regime, thus justifying our reliance on semiclassical methods. We can decompose \( \text{Re}(\phi + m) \) as

\[
\text{Re}(\phi + m) = \sum_A \lambda_A \chi_A \chi_A^\dagger,
\]

where the \( \lambda_A \) are the real eigenvalues of \( \text{Re}(\phi + m) \) and the \( \chi_A \) the corresponding eigenvectors normalized up to an arbitrary phase by \( \chi_A^\dagger \chi_B = \delta_{AB} \) and fulfilling the completeness relation \( \sum_A \chi_A \chi_A^\dagger = 1 \). The matrix \( U \) in (4.1) is then given by

\[
U = \sum_A \text{sign}(\lambda_A) \chi_A \chi_A^\dagger.
\]

We consider a situation where one of the eigenvalues, \( \lambda_0 \) say, changes sign. A short calculation gives

\[
\lim_{\lambda_0 \to 0^+} \text{Tr } (U dU \wedge dU) - \lim_{\lambda_0 \to 0^-} \text{Tr } (U dU \wedge dU) = 4d\chi_0^\dagger \wedge d\chi_0.
\]

An important point is now that \( \chi_0 \) is in general not a globally defined function over \( S^2 \) but a section of the line-bundle \( V \) of eigenvectors of \( \phi \) with eigenvalue \( -\text{Re } m \). The quantity on the right-hand side of (4.4) is globally well-defined, though, and we can calculate the jump in the index as

\[
\Delta \text{index}(L) = -\frac{1}{2\pi i} \int d\chi_0^\dagger \wedge d\chi_0 = c_1(V)[S^2].
\]

This is of course an integer as it should be, since the first Chern class of \( V \) is an element of \( H^2(S^2, \mathbb{Z}) \). In physical terms, this amounts to the Dirac quantization condition: The wave function of a particle with electric-magnetic charge vector \( Q \) in the presence of charges \( Q' \) at the origin is a section of a line-bundle over \( \mathbb{R}^3 - \{0\} \), the first Chern number of which equals the integer symplectic product \( Q \cdot Q' \). The number of solitonic states thus changes by a factor of \( 2^Q Q' \) as a curve of marginal stability is crossed. This number is invariant under the action of \( Sp(2r, \mathbb{Z}) \) on \( Q \) and \( Q' \) and thus well-defined on the moduli space of vacua. One should note that although we have considered fundamental quarks in a soliton background in the weak coupling regime, the result has a wider validity. For example, as the mass parameters of the model are turned off, the elementary quarks may be continuously changed to magnetically charged solitons, and curves of marginal stability may move from the weak coupling to the strong coupling regime \[9\]. The discontinuities of the BPS spectrum are unchanged, though. It would be interesting to check the validity of this result in cases which can not be treated at weak coupling, and also for dyon-dyon bound states.
5. The SU(2) case

The case of an SU(2) gauge group is of particular importance, since embedded SU(2) monopoles and dyons seem to be the fundamental one-particle solitons also for larger gauge groups [18].

A gauge invariant parametrization of the moduli space of vacua is given by the vacuum expectation value of \( u = \frac{1}{2} \text{Tr}(\phi^2) \). By using the \( U(1)_R \) symmetry, we can take \( u \) to be real and positive. The monopole configuration [13] [14] is of the form

\[
\phi = \hat{x} \cdot \sigma \phi(r) \quad A_0 = 0 \quad A_i = \epsilon_{ijk} \hat{x}^j A^k(r), \tag{5.1}
\]

where \( r = (x \cdot x)^{1/2} \), \( \hat{x}^i = r^{-1} x^i \) and \( \phi(r) \) is real and positive. We have \( \phi(r) \to \sqrt{u} \) and \( A(r) \to 0 \) as \( r \to \infty \). This background is invariant under rotations generated by \( J_i = L_i + S_i + T_i \), where \( L_i \), \( S_i \) and \( T_i \) are the generators of orbital angular momentum, spin and isospin respectively.

With an SU(2) gauge group, asymptotic freedom or scale invariance permits us to consider quarks in (up to four copies of) the doublet representation or (a single copy of) the triplet representation. The Callias formula (4.1) then predicts one or two normalizable modes respectively, which have been explicitly constructed in the case of zero bare quark mass [19]. The two modes for a triplet quark transform as spin \( 1/2 \) under \( J_i \), leading to solitons in BPS representations of maximum spin 1. Doublets and triplets of SU(2) are the only representations that arise also when we embed a monopole in an asymptotically free or scale invariant theory with a larger gauge group. Triplets arise when the quarks are in the adjoint representation of the gauge group (i.e. in \( N = 4 \) theories), but also in SU(\( n \)) theories with quarks in the symmetric product of two fundamental representations.

Here we will consider the case of isodoublet quarks with non-zero bare mass in somewhat more detail. The lowest mode should share the symmetry of the monopole background under rotations generated by \( J_i \). This leads to the Ansatz

\[
(\psi^{-})_\alpha^\kappa = \delta_\alpha^\kappa \psi^{-}_s(r) + (\hat{x} \cdot \sigma)_\alpha^\kappa \psi^{-}_v(r), \tag{5.2}
\]

where \( \alpha = 1, 2 \) and \( \kappa = 1, 2 \) are spinor and SU(2) doublet indices respectively. With \( L \) as in (3.3), the equation \( L \psi^{-} = 0 \) then amounts to

\[
\frac{d}{dr} \begin{pmatrix} \psi^{-}_s \\ \psi^{-}_v \end{pmatrix} = \begin{pmatrix} -2A - \phi & -\text{Re} m \\ -\text{Re} m & -2r^{-1} + 2A - \phi \end{pmatrix} \begin{pmatrix} \psi^{-}_s \\ \psi^{-}_v \end{pmatrix}. \tag{5.3}
\]
This system of coupled linear differential equations has two linearly independent solutions. However, to have a regular solution at the origin, we must impose the boundary condition \( \psi_u^- (0) = 0 \). For large \( r \), this solution will generically have components along both eigenvectors of the matrix in (5.3), so to have a normalizable solution we must require that the eigenvalues \( \lambda_1 = -\sqrt{u} - \text{Re} \, m \) and \( \lambda_2 = -\sqrt{u} + \text{Re} \, m \) in the \( r \to \infty \) limit are both negative. This will be the case unless

\[
\text{Re} \left( \frac{m}{\sqrt{u}} \right) > 1,
\]

where we have stated the condition in a \( U(1)_R \) invariant way. For fixed \( m \), this inequality determines a heart-shaped region in the finite \( u \)-plane where the quark-soliton bound state is absent.

Finally, we briefly comment on the quantum numbers of the soliton states. In the region of moduli space where we have two such states, their quark number charges \( S \) differ by the quark number charge of the elementary charge, i.e. by one unit. Furthermore, for \( m = 0 \), the theory is invariant under quark number conjugation, which fixes the quark numbers of the soliton states to \( S = -1/2 \) and \( S = 1/2 \) [19]. For non-zero \( m \), the quark number of the lowest soliton state has been calculated in [20], and \( S \to 0 \) as \( m \to \infty \). At weak coupling, the decrease of \( S \) with increasing \( m \) is rapid.

I am grateful to the Aspen Center for Physics for its hospitality and to G. Moore for discussions. This research was supported by DOE under grant DE-FG02-92ER40704.
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