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Fractional coloring of triangle-free planar graphs\footnote{This research was supported by the Czech-French Laboratory LEA STRUCO.}

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Abstract

We prove that every planar triangle-free graph on $n$ vertices has fractional chromatic number at most $3 - \frac{1}{n+1/3}$.

1 Introduction

Coloring of triangle-free planar graphs is an attractive topic. It started with Grötzsch’s theorem \cite{7}, stating that such graphs are 3-colorable. Since then, several simpler proofs have been given, e.g., by Thomassen \cite{13, 14}. Algorithmic questions have also been addressed: while most proofs readily yield quadratic algorithms to 3-color such graphs, it takes considerably more effort to obtain asymptotically faster algorithms. Kowalik \cite{11} proposed an algorithm running in time $O(n \log n)$, which relies on the design of an advanced data structure. More recently, Dvořák et al. \cite{2} managed to obtain a linear-time algorithm, yielding at the same time a yet simpler proof of Grötzsch’s theorem.

The fact that all triangle-free planar graphs admit a 3-coloring implies that all such graphs have an independent set containing at least one third of the vertices. Albertson et al. \cite{1} had conjectured that there is always a larger independent set, which was confirmed by Steinberg and Tovey \cite{12} even in a stronger sense: all triangle-free planar $n$-vertex graphs admit a 3-coloring...
where not all color classes have the same size, and thus at least one of them forms an independent set of size at least \( \frac{n+1}{3} \). This bound turns out to be tight for infinitely many triangle-free graphs, as Jones [9] showed. As an aside, let us mention that the graphs built by Jones have maximum degree 4: this is no coincidence as Heckman and Thomas later established that all triangle-free planar \( n \)-vertex graphs with maximum degree at most 3 have an independent set of order at least \( \frac{3n}{4} \), which again is a tight bound—actually attained by planar graphs of girth 5.

All these considerations naturally lead to investigate the fractional chromatic number of triangle-free planar graphs. Indeed, as we shall later see, this invariant actually corresponds to a weighted version of the independence ratio. In addition, since \( \chi_f(G) \leq \chi(G) \) for every graph \( G \), Grötzsch’s theorem implies that \( \chi_f(G) \leq 3 \) whenever \( G \) is triangle-free and planar. On the other hand, Jones’s construction shows the existence of triangle-free planar graphs with fractional chromatic number arbitrarily close to 3. Thus one wonders whether there exists a triangle-free planar graph with fractional chromatic number exactly 3. Let us note that this happens for the circular chromatic number \( \chi_c \), which is a different relaxation of ordinary chromatic number such that \( \chi_f(G) \leq \chi_c(G) \leq \chi(G) \).

The purpose of this work is to answer this question. We do so by establishing the following upper bound on the fractional chromatic number of triangle-free planar \( n \)-vertex graphs, which depends on \( n \).

**Theorem 1.** Every planar triangle-free graph on \( n \) vertices has fractional chromatic number at most \( 3 - \frac{1}{n+1/3} \).

Consequently, no (finite) triangle-free planar graph has fractional chromatic number equal to 3. We also note that the bound provided by Theorem 1 is tight up to the multiplicative factor. Indeed, the aforementioned construction of Jones [9] yields, for each \( n \geq 2 \) such that \( n \equiv 2 \pmod{3} \), a triangle-free planar graph \( G_n \) with \( \alpha(G_n) = \frac{n+1}{3} \). Consequently, \( \chi_f(G_n) \geq \frac{3n}{n+1} = 3 - \frac{3}{n+1} \).

Our result can be improved for triangle-free planar graphs with maximum degree at most four, giving an exact bound for such graphs.

**Theorem 2.** Every planar triangle-free \( n \)-vertex graph of maximum degree at most four has fractional chromatic number at most \( \frac{3n}{n+1} \).

Furthermore, the graphs of Jones’s construction contain a large number of separating 4-cycles (actually, all their faces have length five). We show that planar triangle-free graphs of maximum degree 4 and without separating 4-cycles cannot have fractional number arbitrarily close to 3.
Theorem 3. There exists $\delta > 0$ such that every planar triangle-free graph of maximum degree at most four and without separating 4-cycles has fractional chromatic number at most $3 - \delta$.

Dvořák and Mnich [5] proved that there exists $\beta > 0$ such that all planar triangle-free $n$-vertex graphs without separating 4-cycles contain an independent set of size at least $n/(3 - \beta)$. This gives an evidence that the restriction on the maximum degree in Theorem 3 might not be necessary.

Conjecture 1. There exists $\delta > 0$ such that every planar triangle-free graph without separating 4-cycles has fractional chromatic number at most $3 - \delta$.

Faces of length four are usually easy to deal with in the proofs by collapsing; thus the following seemingly simpler variant of Conjecture 1 is likely to be equivalent to it.

Conjecture 2 (Dvořák and Mnich [5]). There exists $\delta > 0$ such that every planar graph of girth at least five has fractional chromatic number at most $3 - \delta$.

2 Notation and auxiliary results

Let $\mu$ be the Lebesgue measure on real numbers. Let $G$ be a graph. If a function $\varphi$ assigns to each vertex of $G$ a measurable subset of $[0,1]$ and $\varphi(u) \cap \varphi(v) = \emptyset$ for all edges $uv$ of $G$, we say that $\varphi$ is a fractional coloring of $G$. Let $f : V(G) \to Q \cap [0,1]$ be a function with rational values. If the fractional coloring $\phi$ satisfies $\mu(\varphi(v)) \geq f(v)$ for every $v \in V(G)$, then we say that $\varphi$ is an $f$-coloring of $G$. If $\mu(\varphi(v)) = f(v)$ for every $v \in V(G)$, then we say that $\varphi$ is a tight $f$-coloring. Note that if $G$ has an $f$-coloring, then it also has a tight one. For $x \in Q \cap [0,1]$, let $c_x$ denote the constant function assigning the value $x$ to each vertex of $G$. The fractional chromatic number of $G$ is defined as

$$\chi_f(G) = \frac{1}{\sup \{ x \in Q \cap [0,1] : G \text{ has a } c_x\text{-coloring} \}}.$$  

Let $w : V(G) \to \mathbb{R}^+$ be an arbitrary function. For a set $X \subseteq V(G)$, by $w(X)$ we mean $\sum_{v \in X} w(v)$. Let $w(f) = \sum_{v \in V(G)} f(v)w(v)$. An integer $N \geq 1$ is a common denominator of $f$ if $Nf(v)$ is an integer for every $v \in V(G)$. Setting $[N] = \{1, \ldots, N\}$, a function $\psi : V(G) \to \mathcal{P}([N])$ is an $(f,N)$-coloring of $G$ if $\psi(u) \cap \psi(v) = \emptyset$ for every $uv \in E(G)$ and $|\psi(v)| \geq N f(v)$ for every $v \in V(G)$. The $(f,N)$-coloring is tight if $|\psi(v)| = N f(v)$ for every $v \in V(G)$.
The fractional chromatic number of a graph can be expressed in various equivalent ways, based on its well known linear programming formulation and duality. The proof of the following lemma can be found e.g. in Dvořák et al. [6, Theorem 2.1].

**Lemma 4.** Let $G$ be a graph and $f : V(G) \to Q \cap [0,1]$ a function. The following statements are equivalent.

- The graph $G$ has an $f$-coloring.
- There exists a common denominator $N$ of $f$ such that $G$ has an $(f,N)$-coloring.
- For every $w : V(G) \to \mathbb{R}^+$, there exists an independent set $X \subseteq V(G)$ with $w(X) \geq w(f)$.

We need several results related to Grötzsch’s theorem. The following lemma was proved for vertices of degree at most three by Steinberg and Tovey [12]. The proof for vertices of degree four follows from the results of Dvořák and Lidický [4], as observed by Dvořák et al. [3].

**Lemma 5.** If $G$ is a triangle-free planar graph and $v$ is a vertex of $G$ of degree at most four, then there exists a 3-coloring of $G$ such that all neighbors of $v$ have the same color.

In fact, Dvořák et al. [3] proved the following stronger statement.

**Lemma 6.** There exists an integer $D \geq 4$ with the following property. Let $G$ be a triangle-free planar graph without separating 4-cycles and let $X$ be a set of vertices of $G$ of degree at most four. If the distance between every two vertices in $X$ is at least $D$, then there exists a 3-coloring of $G$ such that all neighbors of vertices of $X$ have the same color.

Let $G$ be a triangle-free plane graph. A 5-face $f = v_1v_2v_3v_4v_5$ of $G$ is safe if $v_1$, $v_2$, $v_3$ and $v_4$ have degree exactly three, their neighbors $x_1, \ldots, x_4$ (respectively) not incident with $f$ are pairwise distinct and non-adjacent, and

- the distance between $x_2$ and $v_5$ in $G - \{v_1, v_2, v_3, v_4\}$ is at least four, and
- $G - \{v_1, v_2, v_3, v_4\}$ contains no path of length exactly three between $x_3$ and $x_4$.
Lemma 7 (Dvořák et al. [2, Lemma 2.2]). If $G$ is a plane triangle-free graph of minimum degree at least three and all faces of $G$ have length five, then $G$ has a safe face.

Finally, let us recall the folding lemma, which is frequently used in the coloring theory of planar graphs.

Lemma 8 (Klostermeyer and Zhang [10]). Let $G$ be a planar graph with odd-girth $g > 3$. If $C = v_0v_1 \ldots v_{r-1}$ is a facial circuit of $G$ with $r \neq g$, then there is an integer $i \in \{0, \ldots, r-1\}$ such that the graph $G'$ obtained from $G$ by identifying $v_{i-1}$ and $v_{i+1}$ (where indices are taken modulo $r$) is also of odd-girth $g$.

3 Proofs

First, let us show a lemma based on the idea of Hilton et. al. [8].

Lemma 9. Let $G$ be a planar triangle-free graph and let $w : V(G) \rightarrow \mathbb{R}^+$ be an arbitrary function. If $v \in V(G)$ has degree at most 4, then $G$ contains an independent set $X$ such that $w(X) \geq w(V(G)) + \frac{w(v)}{3}$.

Proof. Lemma 5 implies that there exists a 3-coloring of $G$ such that all neighbors of $v$ have the same color. Consequently, $G$ has an $f_v$-coloring for the function $f_v$ such that $f_v(z) = 1/3$ for $z \in V(G) \setminus \{v\}$ and $f_v(v) = 2/3$. By Lemma 4, there exists an independent set $X \subseteq V(G)$ such that $w(X) \geq w(f_v) = \frac{w(V(G)) + w(v)}{3}$.

Theorem 2 now readily follows.

Proof of Theorem 2. Let $G$ be a planar triangle-free $n$-vertex graph of maximum degree at most four. Consider any function $w : V(G) \rightarrow \mathbb{R}^+$, and let $v$ be the vertex to which $w$ assigns the maximum value. We have $w(v) \geq w(V(G))/n$. By Lemma 9, there exists an independent set $X$ such that $w(X) \geq \frac{w(V(G)) + w(v)}{3} \geq \frac{n+1}{3n} w(V(G))$. Therefore, for every $w : V(G) \rightarrow \mathbb{R}^+$, there exists an independent set $X$ with $w(X) \geq w(c_{(n+1)/(3n)})$. By Lemma 4, it follows that the fractional chromatic number of $G$ is at most $\frac{3n}{n+1}$.

Similarly, Lemma 6 implies Theorem 3.

Proof of Theorem 3. Let $D$ be the constant of Lemma 6, let $\delta_0 = \frac{1}{34n}$ and $\delta = \frac{9\delta_0}{3\delta_0 + 1} = \frac{3}{4n+1}$. Let $G$ be a planar triangle-free graph of maximum degree
of vertices at distance at most \( D \) length exactly 5 arbitrary, Lemma 4 implies that \( G \) has a \( c_{1/3+\delta_0} \)-coloring.

Let \( G' \) be the graph obtained from \( G \) by adding edges between all pairs of vertices at distance at most \( D - 1 \). The maximum degree of \( G' \) is less than \( 4D \), and thus \( G' \) has a coloring by at most \( 4D \) colors. Let \( C_1, \ldots, C_{4D} \) be the color classes of this coloring. For \( i \in \{1, \ldots, 4D\} \), let \( f_i \) be the function defined by \( f_i(v) = 2/3 \) for \( v \in C_i \) and \( f_i(v) = 1/3 \) for \( v \in V(G) \setminus C_i \).

Lemma 6 ensures that \( G \) has an \( f_i \)-coloring.

Consider any function \( w : V(G) \to R^+ \). There exists \( i \in \{1, \ldots, 4D\} \) such that \( w(C_i) \geq w(V(G))/4D \). By Lemma 4 applied for \( f_i \), we conclude that \( G \) contains an independent set \( X \) such that \( w(X) \geq w(f_i) = \frac{w(V(G))+w(C_i)}{4D} \geq (1/3 + \delta_0)w(V(G)) = w(c_{1/3+\delta_0}) \). Since the choice of \( w \) was arbitrary, Lemma 4 implies that \( G \) has a \( c_{1/3+\delta_0} \)-coloring.

\( \square \)

The proof of Theorem 1 is somewhat more involved. Let \( \varepsilon = 1/9 \) and for \( n \geq 1 \), let \( b(n) = 1/3 + \varepsilon/n \). Let \( G \) be a plane triangle-free graph. We say that \( G \) is a counterexample if \( G \) does not have a \( c_{b(|V(G)|)} \)-coloring. We say that \( G \) is a minimal counterexample if \( G \) is a counterexample and no plane triangle-free graph with fewer than \( |V(G)| \) vertices is a counterexample. Since \( b \) is a decreasing function, every minimal counterexample is connected.

**Lemma 10.** If \( G \) is a minimal counterexample, then \( G \) is 2-connected. Consequently, the minimum degree of \( G \) is at least two.

**Proof.** Since \( b(n) \leq 1/2 \), every counterexample has at least three vertices; hence, it suffices to prove that \( G \) is 2-connected, and the bound on the minimum degree will follow. Let \( n \) be the number of vertices of \( G \).

Suppose that \( G \) is not 2-connected, and let \( G_1 \) and \( G_2 \) be subgraphs of \( G \) such that \( G = G_1 \cup G_2 \), the graph \( G_1 \) intersects \( G_2 \) in exactly one vertex \( v \), and both \( n_1 = |V(G_1)| \) and \( n_2 = |V(G_2)| \) are greater than 1. Since \( n = n_1 + n_2 - 1 \), we have \( n_1, n_2 < n \), and thus neither \( G_1 \) nor \( G_2 \) is a counterexample. Consequently, \( G_i \) has a \( c_{b(n_i)} \)-coloring for \( i \in \{1, 2\} \). Since \( b \) is a decreasing function, we deduce that \( G_i \) has a \( c_{b(n)} \)-coloring and hence, by Lemma 4, there exists \( N \geq 1 \) such that \( G_i \) has a \( (c_{b(n)}, N) \)-coloring \( \varphi_i \). By permuting the colors if necessary, we can assume that \( \varphi_1(v) = \varphi_2(v) \), and thus \( \varphi_1 \cup \varphi_2 \) is a \( (c_{b(n)}, N) \)-coloring of \( G \). This contradicts the assumption that \( G \) is a counterexample.

\( \square \)

**Lemma 11.** If \( G \) is a minimal counterexample, then every face of \( G \) has length exactly 5.
Proof. Let $n$ be the number of vertices of $G$. Suppose that $G$ has a face $f$ of length other than 5. Since $G$ is triangle-free, it has odd girth at least five, and by Lemma 8, there exists a path $v_1v_2v_3$ in the boundary of $f$ such that the graph $G'$ obtained by identifying $v_1$ with $v_3$ to a single vertex $z$ has odd girth at least five as well. It follows that $G'$ is triangle-free. Since $G$ is a minimal counterexample, $G'$ has a $c_{b(n-1)}$-coloring, and by giving both $v_1$ and $v_3$ the color of $z$, we obtain a $c_{b(n-1)}$-coloring of $G$. Since $b(n) < b(n-1)$, this contradicts the assumption that $G$ is a counterexample.

Given a counterexample $G$ on $n$ vertices, a function $w: V(G) \to \mathbb{R}^+$ is a witness if $G$ has no independent set $X$ satisfying $w(X) \geq w(c_{b(n)})$. By Lemma 4, every counterexample has a witness. Let us now state a useful special case of Lemma 9.

**Lemma 12.** If $G$ is a counterexample on $n$ vertices, $w$ is a witness and $v \in V(G)$ has degree at most three, then $w(v) < 3\varepsilon w(V(G))/n$.

**Proof.** Let $n$ be the number of vertices of $G$. By Lemma 9, there exists an independent set $X \subseteq V(G)$ with $w(X) \geq \frac{w(V(G)) + w(v)}{3}$. On the other hand, since $w$ is a witness, we have $w(X) < w(c_{b(n)}) = \frac{w(V(G))}{3} + \frac{2}{n} w(V(G))$. The claim of this lemma follows.

**Lemma 13.** If $G$ is a minimal counterexample, then $G$ has minimum degree at least three.

**Proof.** Let $n$ be the number of vertices of $G$ and let $w: V(G) \to \mathbb{R}^+$ be a witness for $G$. By Lemma 10, the graph $G$ has minimum degree at least two. Suppose that $v \in V(G)$ has degree two. By Lemma 12, we have $w(v) < 3\varepsilon w(V(G))/n$.

Since $G$ is a minimal counterexample, there exists $N \geq 1$ and a tight $(c_{b(n-1)}, N)$-coloring $\psi$ of $G - v$. Let $f(x) = b(n-1)$ for $x \in V(G - v)$ and $f(v) = 1 - 2b(n-1)$. Clearly, $\psi$ extends to an $(f, N)$-coloring of $G$. By
Lemma 4, there exists an independent set \( X \subseteq V(G) \) such that

\[
 w(X) \geq w(f) \\
= b(n-1)w(V(G)) - (3b(n-1) - 1)w(v) \\
> b(n-1)w(V(G)) - \frac{3(3b(n-1) - 1)\varepsilon}{n}w(V(G)) \\
= \left[ b(n-1) - \frac{9\varepsilon^2}{n(n-1)} \right]w(V(G)) \\
= \left[ b(n) + \frac{\varepsilon}{n(n-1)} - \frac{9\varepsilon^2}{n(n-1)} \right]w(V(G)) \\
= b(n)w(V(G)) = w(v_{b(n)}).
\]

This contradicts that \( w \) is a witness for \( G \).

\[ \square \]

**Lemma 14.** No minimal counterexample contains a safe 5-face.

**Proof.** Let \( G \) be a minimal counterexample containing a safe 5-face \( f = v_1v_2v_3v_4v_5 \), and let \( x_1, \ldots, x_4 \) be the neighbors of \( v_1, \ldots, v_4 \), respectively, that are not incident with \( f \). Let \( n \) be the number of vertices of \( G \) and let \( w: V(G) \to \mathbb{R}^+ \) be a witness for \( G \). By Lemma 12, we have \( w(v_i) < 3\varepsilon w(V(G))/n \) for \( 1 \leq i \leq 4 \).

Let \( G' \) be the graph obtained from \( G - \{v_1, v_2, v_3, v_4\} \) by identifying \( x_2 \) with \( v_5 \) into a new vertex \( u_1 \), and \( x_3 \) with \( x_4 \) into a new vertex \( u_2 \). Since \( f \) is safe, \( G' \) is triangle-free. Since \( G \) is a minimal counterexample, there exists \( N \geq 1 \) and a tight \((v_{b(n-6)}, N)\)-coloring \( \psi \) of \( G' \). Let \( f(x) = b(n-6) \) for \( x \in V(G - \{v_1, v_2, v_3, v_4\}) \) and \( f(v_i) = 1 - 2b(n-6) \) for \( 1 \leq i \leq 4 \). We use \( \psi \) to design an \((f, N)\)-coloring of \( G \).

Let \( \psi(x_2) = \psi(v_5) = \psi(u_1) \) and \( \psi(x_3) = \psi(x_4) = \psi(u_2) \). Let \( \psi(v_1) \) be a subset of \( [N] \setminus (\psi(x_1) \cup \psi(v_3)) \) of size \( f(v_1)N \), and let \( \psi(v_2) \) be a subset of \( [N] \setminus (\psi(x_2) \cup \psi(v_1)) \) of size \( f(v_2)N \). Let \( M_3 = [N] \setminus (\psi(x_2) \cup \psi(v_3)) \) and \( M_4 = [N] \setminus (\psi(x_3) \cup \psi(x_4)) \). Note that \( |M_3| \geq f(v_3)N \) and \( |M_4| \geq f(v_4)N \). Furthermore, since \( \psi(x_3) = \psi(x_4) \) and \( \psi(v_2) \cap \psi(v_3) = \emptyset \), we have \( |M_3 \cup M_4| = 1 - |\psi(x_3)| = 1 - b(n-6) \geq f(v_3) + f(v_4) \). Therefore, we can choose disjoint sets \( \psi(v_3) \subseteq M_3 \) and \( \psi(v_4) \subseteq M_4 \) of size \( f(v_3)N = f(v_4)N \). This gives an \((f, N)\)-coloring of \( G \).
By Lemma 4, there exists an independent set $X \subseteq V(G)$ such that

$$w(X) \geq w(f)$$

$$= b(n - 6)w(V(G)) - (3b(n - 6) - 1)\sum_{i=1}^{4} w(v_i)$$

$$> b(n - 6)w(V(G)) - \frac{12(3b(n - 6) - 1)\varepsilon}{n}w(V(G))$$

$$= \left[ b(n - 6) - \frac{36\varepsilon^2}{n(n - 6)} \right] w(V(G))$$

$$= \left[ b(n) + \frac{6\varepsilon}{n(n - 6)} - \frac{36\varepsilon^2}{n(n - 6)} \right] w(V(G))$$

$$\geq b(n)w(V(G)) = w(c_{b(n)}).$$

This contradicts that $w$ is a witness for $G$. \qed

We can now establish Theorem 1.

**Proof of Theorem 1.** Note that $\frac{1}{3 - n+1/3} = b(n)$. Suppose that there exists a planar triangle-free graph $G$ on $n$ vertices with fractional chromatic number greater than $3 - \frac{1}{n+1/3}$. Then $G$ has no $c_{b(n)}$-coloring, and thus $G$ is a counterexample. Therefore, there exists a minimal counterexample $G_0$. Lemmas 13, 11 and 7 imply that $G_0$ has a safe 5-face. However, that contradicts Lemma 14. \qed

**References**

[1] M. Albertson, B. Bollobás, and S. Tucker, *The independence ration and the maximum degree of a graph*, Congr. Numer., 17 (1976), pp. 43–50.

[2] Z. Dvořák, K. Kawarabayashi, and R. Thomas, *Three-coloring triangle-free planar graphs in linear time*, Trans. on Algorithms, 7 (2011), p. article no. 41.

[3] Z. Dvořák, D. Král’, and R. Thomas, *Three-coloring triangle-free graphs on surfaces V. Coloring planar graphs with distant anomalies*. Manuscript.

[4] Z. Dvořák and B. Lidický, *3-coloring triangle-free planar graphs with a precolored 8-cycle*, ArXiv e-prints, 1305.2467 (2013).
[5] Z. Dvořák and M. Mnich, *Large Independent Sets in Triangle-Free Planar Graphs*, ArXiv e-prints, 1311.2749 (2013).

[6] Z. Dvořák, J.-S. Sereni, and J. Volec, *Subcubic triangle-free graphs have fractional chromatic number at most 14/5*, ArXiv e-prints, 1301.5296 (2013).

[7] H. Grötzsch, *Ein Dreifarbenzatz für Dreikreisfreie Netze auf der Kugel*, Math.-Natur. Reihe, 8 (1959), pp. 109–120.

[8] A. Hilton, R. Rado, and S. Scott, *A (< 5)-colour theorem for planar graphs*, Bull. London Math. Soc., 5 (1973), pp. 302–306.

[9] K. F. Jones, *Minimum independence graphs with maximum degree four*, in Graphs and applications (Boulder, Colo., 1982), Wiley-Intersci. Publ., Wiley, 1985, pp. 221–230.

[10] W. Klostermeyer and C. Q. Zhang, *$(2 + \epsilon)$-coloring of planar graphs with large odd-girth*, J. Graph Theory, 33 (2000), pp. 109–119.

[11] L. Kowalik, *Fast 3-coloring triangle-free planar graphs*, in ESA, S. Albers and T. Radzik, eds., vol. 3221 of Lecture Notes in Computer Science, Springer, 2004, pp. 436–447.

[12] R. Steinberg and C. A. Tovey, *Planar Ramsey numbers*, J. Combin. Theory, Ser. B, 59 (1993), pp. 288–296.

[13] C. Thomassen, *Grötzsch’s 3-color theorem and its counterparts for the torus and the projective plane*, J. Combin. Theory, Ser. B, 62 (1994), pp. 268–279.

[14] ——*, *A short list color proof of Grotzsch’s theorem*, J. Combin. Theory, Ser. B, 88 (2003), pp. 189–192.