A Lower Bound to the Receiver Operating Characteristic of a Cognitive Radio Network

(submitted to the IEEE Transactions on Information Theory, July 2010)

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Abstract

Cooperative cognitive radio networks are investigated by using an information-theoretic approach. This approach consists of interpreting the decision process carried out at the fusion center as a binary (asymmetric) channel, whose input is the presence of a primary signal and output is the fusion center decision itself. The error probabilities of this channel are the false-alarm and missed-detection probabilities. After calculating the mutual information between the binary random variable representing the primary signal presence and the set of sensor (or secondary user) output samples, we apply the data-processing inequality to derive a lower bound to the receiver operating characteristic. This basic idea is developed through the paper in order to consider the cases of full channel and signal knowledge and of knowledge in probability distribution. The advantage of this approach is that the ROC lower bound derived is independent of the particular type of spectrum detection algorithm and fusion rule considered. Then, it can be used as a benchmark for existing practical systems.

Index Terms

Cognitive radio networks, Data-Processing inequality, Spectrum sensing, Sensor networks, Receiver Operating Characteristic.

I. INTRODUCTION

Cognitive Radio (CR) technologies have gained considerable interest in the last few years because of two factors: i) the increasing demand for wireless spectrum from a large number of applications;
and \( ii \) the fact that many portions of licensed spectrum are neglected or underutilized by the regular licensees [1]–[6].

The concept of CR depends considerably on the application context [7]. Nevertheless, an official definition has been given by the Global Standards Collaboration (GSC) group within the ITU [8]: “A radio or system that senses its operational electromagnetic environment and can dynamically and autonomously adjust its radio operating parameters to modify system operation, such as maximize throughput, mitigate interference, facilitate interoperability, access secondary markets.” According to this definition, a CR device should be able to autonomously exploit unused portions spectrum to increase its own signalling rate without limiting the use of the radio spectrum from licensed users. Thus, the most important feature of a CR device is the ability to detect the availability of spectrum holes [7], which can be accomplished by suitable spectrum sensing techniques. The key role of spectrum sensing has been recognized in the technical literature as the enabling technique for CR systems. Different strategies have been envisaged to effectively implement this feature and a comprehensive taxonomy can be found in [7].

A simple statement of the CR detection problem can be given as follows. In a CR network there are two classes of users: \( i \) primary users, i.e., those users who have license rights of some other form of priority with respect to the radio channel access; \( ii \) secondary users, i.e., those users who have no licence rights or have more limited priority to the channel access than the primary users. Secondary users are those who need CR capabilities, such as spectrum sensing, in order to avoid causing interference to primary users. Thus, secondary users have to estimate the radio channel condition before attempting a transmission, i.e., they need to assess whether the channel is idle or busy (hypotheses \( H_0 \) and \( H_1 \), respectively). This estimation is usually affected by error and characterized by two error probabilities:

- The false-alarm probability \( P_{fa} \), corresponding to the detection of hypothesis \( H_1 \) when \( H_0 \) is true.
- The missed-detection probability \( P_{md} \), corresponding to the detection of hypothesis \( H_0 \) when \( H_1 \) is true.

Ideally, secondary users should operate toward reaching the goal of having \( P_{fa} = P_{md} = 0 \). However, radio channel impairments prevent to attain this operating level, and a suitable tradeoff has to be sought. Typically, secondary users are allowed a maximum level of interference to the primary users, which translates into a maximum probability of missed detection. Then, the CR users can maximize their throughput by maximizing the false-alarm probability \( P_{fa} \) under the constraint of a given \( P_{md} \). Typical values of these probabilities have been set to \( P_{md} = P_{fa} = 0.1 \) in the contest of the developing standard IEEE 802.22 [6]. A complete picture of the performance of a CR system is provided by the receiver
operating characteristic (ROC) plot. The ROC is a plot of the missed-detection probability $P_{md}$ versus the false-alarm probability $P_{fa}$. Its derivation depends on the radio channel parameters (fading, noise power) and on the type of decision process implemented to detect the presence of a primary signal.

It has been widely recognized in the literature (see, e.g., [7] and references therein) that user cooperation enhances the performance of a CR system, both in terms of ROC, and of avoiding the hidden primary user problem. This problem is considered one of the major challenges to the implementation of a CR system, and is similar to the hidden node problem experienced in Carrier Sense Multiple Accessing (CSMA) [7]. The hidden primary user problem derives from the shadowing of secondary users, occurring while sensing the primary signal transmission. More precisely, a secondary user can be in the range of a primary user receiver but out of the range of another primary user transmitter. Then, the secondary user senses the channel idle, because it cannot capture the primary user signal, and then starts its transmission. However, since it is in the range of the other primary user receiver, it eventually interferes with the reception of the primary signal. Having multiple secondary users sensing the channel reduces the chances of falling into this situation.

Fig. 1 illustrates the block diagram of a CR system based on user cooperation. We can see that the primary signal is present if $\xi = 1$. This signal is received by a set of $K$ secondary users (or sensors) which sample it during a certain observation window. Secondary users can exploit individually this information in order to make a decision on the spectrum availability. Otherwise, they can share it by sending a suitable signal through a control channel to a central processing unit (i.e., implementing user cooperation). This unit provides for the fusion of the user information and is then called fusion center (FC) [7].

The goal of this paper is to analyze, by using information-theoretic results, the behavior of a cooperative CR system. Our approach is based on the observation that appending the decision process implemented at the FC to the primary signal transmission channel yields an equivalent binary channel with input $\xi$, the random variable indicating the signal presence, and output $\hat{\xi}$, the FC decision. According to this interpretation, the false-alarm and missed-detection probabilities correspond to the two error probabilities of this binary channel (conditioned to $\xi = 0$ and $\xi = 1$). In general, this channel turns out to be an asymmetric binary channel because the error probabilities are different.

Then, by using the data-processing inequality, we can calculate an upper bound to the channel capacity, which translates into a lower bound to the ROC. This basic idea is developed through the paper in order to derive the lower bound of a cooperative CR system ROC, which is independent of the spectrum detection and fusion strategy used. This lower bound can be applied to assess the validity of specific combinations of spectrum sensing and fusion strategy.
The remainder of the paper is organized as follows. The system model is illustrated in Section II, where the key concept of applying the data-processing inequality to the cooperative CR system is introduced and analyzed in detail. Section III deals with the derivation of the mutual information of the cooperative CR system without the FC channel and detection. This section considers the case of known channel gains and signal, as a baseline, and the case of known channel gain and signal distribution, as a further development. Relevant asymptotic cases are also studied, in order to mitigate the numerical difficulties in the derivation of the results. Section IV illustrates the analytic results through numerical examples including. Lower bounds to the ROC are reported in this section along with a comparison of these results with an energy detection estimator. Finally, our conclusions are collected in Section V.

II. SYSTEM MODEL AND OPTIMUM ROC

We consider a CR system (illustrated in Fig. 1) equipped with $K$ sensors sensing the wireless spectrum over $N$ sampling times in order to provide information about the channel availability to secondary users. Intentionally, the diagram does not show the terminal part consisting in the collection of the sensor measurements, their compacting, their transmission to the FC through a control channel, and the FC processing block providing the output decision about the signal presence.

We assume a block fading channel where the $n$th sampled signal received by sensor $k$ is given by

$$y_k(n) = \begin{cases} z_k(n) & \xi = 0 \\ h_k s(n) + z_k(n) & \xi = 1 \end{cases}$$

(1)
for \(k = 1, \ldots, K\) and \(n = 1, \ldots, N\). Here, \(z_k(n) \sim \mathcal{N}_c(0, \sigma_k^2)\) are the iid received noise samples, \(h_k\) are the block fading gain coefficients, \(s(n)\) are the primary user’s symbols, and \(\xi\) denotes the the random variable indicating that the primary signal is present (\(\xi = 1\)) or absent (\(\xi = 0\)). The variances \(\sigma_k^2\) are known parameters. We can interpret \(\xi\) as the imponderable primary user decision to convey information through the channel at the time the CR system is trying to check the existence of a spectrum hole.

In the following we assume that the random variable \(\xi\) is not necessarily equiprobable but rather we have \(P(\xi = 0) = \alpha\). Then, \(\alpha\) represents the a priori probability of primary signal absence.

As already mentioned, in this framework we do not consider the remaining part of the communication system beyond the block diagram of Fig. 1. This part consists of a distributed algorithm at the \(K\) sensors and at the fusion center (FC) aimed at condensing the available channel sensing information (at the sensors), sending it to the FC, and jointly processing in order to make a reliable decision on the presence of a primary transmitted signal.

On the contrary, we regard the block diagram in Fig. 1 as a binary input-continuous output vector channel, which we study in order to derive the mutual information

\[
\mathcal{I} \triangleq I(\xi; \{y_k(n)\}_{k=1,n=1}^{K,N}).
\]

(2)

Using the data processing inequality (DPI) \([9]\), we can see that the mutual information \(\mathcal{I}\) upper bounds the mutual information of the channel corresponding to the completion of the transmission chain to the FC by any conceivable distributed algorithm.

Completing the transmission chain up to the FC’s output yields a binary-input binary-output channel. Denoting the FC’s output by \(\hat{\xi}\), we have from the DPI:

\[
I(\xi; \hat{\xi}) \leq \mathcal{I}.
\]

(3)

Now, by the definition of the false-alarm and missed-detection probabilities (denoted by \(P_{fa}\) and by \(P_{md}\), respectively), we have:

\[
\begin{align*}
P_{fa} &= P(\hat{\xi} = 1 | \xi = 0) \\
P_{md} &= P(\hat{\xi} = 0 | \xi = 1)
\end{align*}
\]

In general, we have a binary asymmetric channel whose transition probability matrix can be written as

\[
P = \begin{pmatrix}
1 - P_{fa} & P_{fa} \\
P_{md} & 1 - P_{md}
\end{pmatrix}.
\]

Notation \(z \sim \mathcal{N}_c(\mu, \Sigma)\) denotes a circularly symmetric complex Gaussian distributed vector with mean \(\mu\), covariance matrix \(\Sigma = \mathbb{E}[zz^H] - \mu\mu^H\), and pdf \(\det(\pi\Sigma)^{-1} \exp\left[-\frac{1}{2}(z - \mu)^H\Sigma^{-1}(z - \mu)\right]\).
Fig. 2. ROC lower bound curves corresponding to $\alpha = 0.5$ and mutual information $I$ indicated by the labels.

The mutual information, assuming $P(\xi = 0) = \alpha$, is given by [9]:

$$I(\xi, \hat{\xi}) = H_b(\alpha(1 - P_{fa}) + \bar{\alpha}P_{md})$$

where $\bar{\alpha} \triangleq 1 - \alpha$ and $H_b(p) \triangleq -p \log_2 p - (1 - p) \log_2 (1 - p)$ is the binary entropy function [9].

Finally, inserting (4) into inequality (3), we obtain a relationship between the false-alarm and missed-detection probabilities, which represents a lower bound to the ROC for the given CR system.

The parametric dependence of the ROC lower bound on the mutual information is illustrated in Fig. 2. As expected, as $I \uparrow 1$, the ROC lower bounds decrease monotonically to $P_{fa} = P_{md} = 0$. 

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III. Calculation of $\mathcal{I}$

Let us define for convenience the following matrices and vectors:

$$
\begin{align*}
Y & \triangleq (y_k(n))_{k=1,n=1}^{K,N} \\
Z & \triangleq (z_k(n))_{k=1,n=1}^{K,N} \\
h & \triangleq (h_1, \ldots, h_K)^T \\
s & \triangleq (s(1), \ldots, s(N))^H.
\end{align*}
$$

Then, we can simplify (1) by writing it as follows:

$$Y = \xi hs^H + Z,$$

and hence the mutual information (2) becomes

$$\mathcal{I} = h(Y) - h(Y \mid \xi),$$

where $h(\cdot)$ denotes the differential entropy [9]. First, it is plain to see that

$$h(Y \mid \xi) = h(Z) = N \sum_{k=1}^{K} \log_2(\pi e \sigma_k^2).$$

The evaluation of $h(Y)$ is more difficult. We distinguish among different assumptions concerning the distribution of the secondary channel gain vector $h$ and the signal vector $s$. In the following we consider the cases of $i)$ known gains and signal at the receiver, and of $ii)$ known gain and signal distribution at the receiver.

A. Known gains and signal at the receiver

In order to equalize the noise variances, we transform the channel equation (5) by pre-multiplying by the inverse of the square root of the noise covariance matrix

$$\Sigma_z \triangleq \text{diag}(\sigma_1^2, \ldots, \sigma_K^2).$$

We obtain

$$Y = \xi A + Z,$$

where $A \triangleq \Sigma_z^{-1/2} hs^H$ and the entries of $Z$ are then iid as $\mathcal{N}(0,1)$. This linear transformation is invertible and does not change the mutual information $\mathcal{I}$. In order to calculate the mutual information, we resort to Theorem A.1 (Appendix A). Since $\xi = 0, 1$, in order to use this result we can subtract $A/2$
and obtain symmetric input. Theorem A.1 tells us that the channel is equivalent to a binary-input real
additive Gaussian channel with SNR \( \|A\|^2/2 \). We obtain

\[
I = H_b(\alpha) - \alpha \mathbb{E} \left[ \log_2 \left( 1 + \frac{\bar{\alpha}}{\alpha} e^{Z - \|A\|^2} \right) \right] \\
- \bar{\alpha} \mathbb{E} \left[ \log_2 \left( 1 + \frac{\alpha}{\bar{\alpha}} e^{Z - \|A\|^2} \right) \right],
\]

where \( Z \sim \mathcal{N}(0, 2\|A\|^2) \).

**Remark III.1** It is plain to see that (7) is invariant to the mapping \( \alpha \mapsto 1 - \alpha \), i.e., to exchanging the a priori probabilities of primary signal presence and absence. The ROC performance improves as these probabilities get closer to 0 or to 1, as illustrated in the following. The symmetry of the resulting ROC lower bound suggests to define an equilibrium point corresponding to \( P_{fa} = P_{md} \), which is referred to as equilibrium probability and denoted by \( P_{eq} \) in the sequel. Under these operating conditions, the binary channel \( \xi \rightarrow \hat{\xi} \) is symmetric.

**Remark III.2** It is worth noting that the mutual information \( I \), and hence the lower bound to the ROC, depend only on \( \|A\|^2 \) (in this case). This parameter can be written as

\[
\text{SNR} \triangleq \|A\|^2 = \sum_{k=1}^{K} \frac{|h_k|^2 \|s\|^2}{\sigma_k^2},
\]

which corresponds to the sum of the secondary users’ receive SNR’s. For this reason, we refer to it in the following by the term additive SNR.

1) **Limiting behavior for \( \alpha \rightarrow 0 \):** Expanding (7) for \( \alpha \rightarrow 0 \) we obtain:

\[
I = \mathbb{E} \left[ 1 - Z + \text{SNR} - e^{Z-\text{SNR}} \right] \frac{\alpha \log_2 e}{2 \log(2)} + O(\alpha^3)
\]

We can see that the first- and second-order approximations represent upper and lower bounds, respectively, to the mutual information \( I \). These lower bounds are illustrated in Fig. [3] plotting the ratio \( I/\alpha \) versus \( \alpha \) and its second-order approximation (lower bound) for SNR = 1 (0 dB).

\[\text{SNR} \triangleq \mathcal{N}(\mu, \sigma^2)\]

represents a real Gaussian random variable with mean \( \mu \) and variance \( \sigma^2 \).

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Fig. 3. Plot of the ratio $I/\alpha$ and of its second-order approximation $[\text{SNR} - 0.5(e^{2\text{SNR}} - 1)\alpha]\log_2 e$ versus $\alpha$ for $\text{SNR} = 1$ (0 dB).

2) Limiting behavior for $\text{SNR} \to \infty$: Applying Theorem B.1 (Appendix B), we obtain the following bounds:

$$H_b(\alpha) - 2\frac{\sqrt{\pi\alpha}}{\ln 2} \frac{1}{\text{SNR}^{1/2}} e^{-\text{SNR}/4} \leq I$$

$$\leq H_b(\alpha) - 2\frac{\sqrt{\pi\alpha}}{\ln 2} \frac{1}{\text{SNR}^{1/2}} e^{-\text{SNR}/4} + \frac{\sqrt{\pi\alpha}}{2\ln 2} \frac{1}{\text{SNR}^{3/2}} e^{-\text{SNR}/4}.$$

These bounds yield, for $\text{SNR} \to \infty$, the following asymptotic approximation:

$$I \sim H_b(\alpha) - 2\frac{\sqrt{\pi\alpha}}{\ln 2} \frac{1}{\text{SNR}^{1/2}} e^{-\text{SNR}/4}.$$  \hspace{1cm} (9)
3) Limiting behavior for $\alpha \to 0$ and $\text{SNR} \to \infty$: Finally, we can also expand (4) for $\alpha \to 0$ at the equilibrium point of the ROC, and obtain

$$I(\xi; \hat{c}) = (1 - 2P_{eq}) \log_2 \frac{1 - P_{eq}}{P_{eq}} \alpha - \frac{(1 - 2P_{eq})^2}{2P_{eq}(1 - P_{eq})} \alpha^2 + O(\alpha^3).$$

When both $\alpha$ and $P_{eq} \to 0$, we obtain the approximation

$$P_{eq} \approx e^{-\text{SNR}}.$$  \hfill (10)

B. Known gain and signal distribution at the receiver

The case of known gain and signal distribution at the receiver can be handled by exploiting the results derived in Section III-A. First, we notice that $A = \Sigma^{-1/2}_c h s^H$ is a random matrix whose joint pdf of the entries depends on the distributions of the channel gain vector $h$ and of the signal vector $s$. Then, starting from (7), we can apply the chain rule for the mutual information and the independence between $\xi$ and the vectors $h, s$ in order to obtain the following result:

$$I = I(\xi; A) + I(\xi; Y | A) = I(\xi; Y | A) = H_b(\alpha) - \mathbb{E} \left[ \alpha \log_2 \left( 1 + \frac{\alpha}{\bar{\alpha}} e^{\sqrt{2T}Z_1 - \Gamma} \right) \right] + \bar{\alpha} \log_2 \left( 1 + \frac{\alpha}{\bar{\alpha}} e^{\sqrt{2T}Z_1 - \Gamma} \right).$$  \hfill (11)

where $Z_1 \sim \mathcal{N}_c(0, 1)$ is independent of $\Gamma \triangleq \|A\|^2$, and the average is with respect to both $Z_1$ and $\Gamma$. In accordance with eq. (8), we have

$$\Gamma = \sum_{k=1}^{K} \frac{|h_k|^2 \|s\|^2}{\sigma_k^2},$$

but in this framework $\Gamma$ is a random variable whose mean value is defined as the additive SNR, i.e.,

$$\text{SNR} \triangleq \mathbb{E} [\Gamma].$$

The lower bound to the ROC depends on the distribution of $\Gamma$. Some examples illustrate this dependence in Section IV.

IV. NUMERICAL EXAMPLES

In this section we illustrate the ROC bound obtained by numerical examples in order to compare the lower bounds obtained with some real estimation scheme.
Fig. 4. ROC lower bound in the case of known channel gains and signal for different values of the additive SNR (reported on the plot) and a priori probability of primary signal absence \( \alpha = 0.5 \).

A. Known gains and signal at the receiver

The first example reported in Fig. 4 which consider the case of known channel gains and signal (Section III-A) with a priori probability of primary signal absence \( \alpha = 0.5 \). It can be noticed that the curves are symmetric with respect to exchanging the probabilities \( P_{fa} \) and \( P_{md} \). We can also notice a threshold behavior with respect to the SNR, which is better illustrated in Fig. 5. The SNR threshold lies between 5 and 10 dB: below the threshold, the equilibrium probability decreases slowly; above the threshold, the decrease rate becomes faster. The curves in Fig. 5 are lower bounds to the equilibrium probability versus the additive SNR for different values of the probability of signal absence (or presence) and in the asymptotic case of \( \alpha \rightarrow 0 \), which is given by eq. (10).
Fig. 5. Lower bound to the equilibrium probability $P_{eq}$ versus the additive SNR with a priori probability of primary signal absence (or presence) $\alpha = 0.5, 0.1, 10^{-2}, 10^{-3}$ (solid curve). The dashed curve corresponds to $\alpha \to 0$ (asymptotic case).

B. Comparison with an energy detection scheme

A simple spectrum sensing scheme based on energy detection corresponds to the following estimation rule:

$$\hat{\xi} = \begin{cases} 
0, & \|Y\|^2 < \theta + \ln \alpha \\
1, & \|Y\|^2 > \theta + \ln \bar{\alpha} 
\end{cases}$$

(12)

From the equivalent channel equation (6), the resulting false-alarm and missed-detection probabilities are given by:

$$\hat{\xi} = \begin{cases} 
P_{fa} = P(\|Z\|^2 > \theta + \ln \bar{\alpha}) \\
P_{md} = P(\|A + Z\|^2 < \theta + \ln \alpha) 
\end{cases}$$

(13)

Since $\|Z\|^2$ and $\|A + Z\|^2$ are central and noncentral $\chi^2$-distributed random variables, we can find explicit expressions of the two probabilities. In fact, the cdf’s can be found in standard textbooks, such
as \[10\]. We have:

\[ P(\|Z\|^2 < u) = \gamma(KN, u), \]

where \(\gamma(n, x) \triangleq \Gamma(n) - \int_x^\infty u^{n-1}e^{-u}du\) is the normalized upper incomplete Gamma function, and

\[ P(\|A + Z\|^2 < u) = 1 - Q_{KN}(\sqrt{2}\|A\|^2, \sqrt{2u}), \]

where \(Q_m(a, b) \triangleq \int_b^\infty x(a/b)^m e^\left(-\frac{x^2 + a^2}{2}\right)I_{m-1}(ax)dx\) is the generalized Marcum’s Q function defined in \[10\].

Figures 6 and 7 show the ROC corresponding to an energy detector spectrum sensing scheme for two values of the product \(KN\) and \(\text{SNR} = 0\) and \(10\) dB, respectively. The diagrams also report the information-theoretic lower bound derived in Section III-A. We can see that increasing the product \(KN\) for a fixed \(\text{SNR}\) degrades the resulting ROC. This can be understood by observing that the variances of \(\|A + Z\|^2\) and \(\|Z\|^2\) are proportional to \(KN\) (they are \((1 + 2\text{SNR})KN\) and \(KN\), respectively), while the mean value difference is equal to \(\text{SNR}\). Therefore, as the \(KN\) increases, the overlapping of the two pdf’s increases, and hence the probabilities of false-alarm and missed-detection.

\textbf{Remark IV.1} It is worth noting that the previous results hold for fixed additive SNR. Then, increasing either \(K\) or \(N\) implies that the individual secondary user SNR’s \(\|h_k\|^2\|s\|^2/\sigma_k^2\) must decrease to keep the overall additive SNR constant. On the contrary, if one fixes the individual SNR’s, the additive SNR increases and both the lower bound and the energy-detector ROC improve. Thus, the fact that the ROC curves decrease as \(KN\) increases shall be interpreted by saying that the energy detector performance would improve if we could concentrate all the available sensors in a single one by keeping the total additive SNR constant.

\textbf{C. Known gain and signal distribution at the receiver}

Here we consider the case of iid Rayleigh fading gains, where \(\gamma_k \triangleq \mathbb{E}[|h_k|^2]/\sigma_k^2\), and \(\|s\|^2\) has probability distribution \(P(\|s\|^2 = S_m) = p_m\) for \(m = 1, \ldots, M\). If we assume that all the \(\gamma_k\) are different, the pdf of \(\Gamma\) can be derived as follows:

\[ p_\Gamma(G) = \sum_{m=1}^M p_m \sum_{k=1}^K \frac{\exp(-G/(\gamma_k S_m))}{\gamma_k S_m} \prod_{\ell \neq k} \frac{1}{1 - \gamma_\ell/\gamma_k}. \]
We can use this result to calculate the double integral

\[
\mathcal{I} = H_b(\alpha) - \int_{-\infty}^{\infty} \frac{\exp(-z^2/2)}{\sqrt{2\pi}} \int_{0}^{\infty} p_{\Gamma}(G) \left[ \alpha \log_2 \left( 1 + \frac{\alpha}{\tilde{\alpha}} e^{\sqrt{2Gz-G}} \right) + \tilde{\alpha} \log_2 \left( 1 + \frac{\alpha}{\tilde{\alpha}} e^{\sqrt{2Gz-G}} \right) \right] dG dz.
\]

(14)

As an illustrative example, we consider the following scenario:

- \(K = 4\) secondary users.
- \(\gamma_k = (4 + k)\) dB for \(k = 1, \ldots, K\).
- \(\|s\|^2 = 1\).
The ROC curves are reported in Fig. 8. The lowest curve corresponds to the lower bound calculated by using (14). The other curves correspond to the implementation of a spectrum sensing algorithm based on energy detection for different combinations of the number of secondary users $K$ and sampling times $N$. In all cases, the same additive SNR is assumed, $\text{SNR} = \sum_{k=1}^{K} \gamma_k$, namely,

$$10 \log_{10}(10^{0.5} + 10^{0.6} + 10^{0.7} + 10^{0.8}) = 12.66 \text{ dB}.$$ 

As already noticed in Remark IV.1, the best operating condition for the energy detector corresponds to the case of $K = N = 1$ (at fixed additive SNR).

V. CONCLUSIONS

In this work we proposed an information-theoretic method to derive a lower bound to the receiver operating characteristic of a cognitive radio network based on cooperative sensors. The bound stems from the application of the data-processing inequality to the binary asymmetric channel arising by considering
the primary signal presence as a binary input and the fusion center decision on the signal presence as a binary output. The bound takes into account the possible knowledge of the \textit{a priori} probability of primary signal presence and applies to every kind of single-input multiple-output channel connecting the primary signal to the multiple cooperative sensors (i.e., the secondary users of the cognitive radio system).

Key advantages of this approach are: \( i \) independence from the implementation of the connection between the sensors and the fusion center; and \( ii \) independence from the fusion rule. Both features derive from the information-theoretic method we have followed, based on the equivalence between the actual channel model (connecting the primary transmitter to the sensors and then to the fusion center) and the binary asymmetric channel with error probabilities corresponding to the false-alarm and missed-detection events.

In order to illustrate this basic idea, we considered two scenarios of interest for cognitive radio networks:
1) The case of full channel gain and primary signal information at the fusion center.

2) The case of full distribution information about the channel gain and the primary signal at the fusion center.

The first case has been investigated in full detail by deriving the mutual information $I$ between the primary signal presence variable $\xi$ and the set of sensor observations $Y$. This expression has been analyzed asymptotically, both for large additive SNR and for vanishing $\alpha$ (probability of signal absence). The asymptotic expression for large additive SNR, eq. (9), is based on an integral which is an extension of the one calculated in [11, Prob. 4.12]. In essence, this expression can lead to an asymptotic expansion of the mutual information of the binary symmetric channel for large SNR. Full details on the derivation are reported in Appendix B.

In the second case, a general expression of the mutual information $I$ required to obtained the receiver operating characteristic lower bound has been derived as the average value of an expression depending on two random variables (the random additive SNR $\Gamma$ and the auxiliary Gaussian random variable $Z_1$). This average leads to a double integral, which has been expanded in detail in the case of iid Rayleigh distributed channel gains. In the numerical results section, the distribution of $\Gamma$ is reported in a fairly general case, and numerical results are included for illustration purposes.

Finally, it is worth mentioning that the series expansion of the mutual information of a binary input additive Gaussian channel (Theorem B.1) is also a novel contribution of this paper. It extends the series expansion of the capacity of the same channel, which can be found in [11, Prob. 4.12]. In the present context, this series expansion is needed to account for the possible knowledge of the a priori probability of primary signal presence, which is available in several cognitive radio system and worth being used to improve the quality of the decision rule at the fusion center.

APPENDIX A

EQUIVALENCE OF BINARY INPUT GAUSSIAN CHANNELS

In this appendix we show that a binary-input, vector-output additive Gaussian channel is equivalent, as far as concerns the mutual information, to a binary-input additive Gaussian channel with scalar output.

Let the binary input be $X = \pm 1$ with $\alpha \triangleq P(X = -1)$ and $\bar{\alpha} \triangleq 1 - \alpha = P(X = +1)$.

Let the channel equation be

$$y = X a + z,$$  \hspace{1cm} (15)

where $a \in \mathbb{R}^{n \times 1}$ is a given constant vector and $z$ is a vector of iid Gaussian random variables distributed as $\mathcal{N}(0, 1)$. 

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The mutual information is given by
\[ I(X; y) = h(y) - h(y|X) = h(y) - h(z). \]

We know that \( h(z) = \left(\frac{n}{2}\right) \log_2(2\pi e) \). To calculate \( h(y) \), we note that
\[
p_y(y) = \frac{\alpha \exp(-\|y + a\|^2/2) + \bar{\alpha} \exp(-\|y - a\|^2/2)}{(2\pi)^{n/2}}.
\]

Hence,
\[
I(X; y) = -\mathbb{E} \left[ \log_2(\alpha e^{-(y+a)^2/2} + \bar{\alpha} e^{-(y-a)^2/2}) \right] + \frac{\log_2 e}{2}
\]
\[
= H_b(\alpha) - \alpha \mathbb{E} \left[ \log_2 \left( 1 + \frac{\bar{\alpha}}{\alpha} e^{2a^T z - 2\|a\|^2} \right) \right]
\]
\[
- \bar{\alpha} \mathbb{E} \left[ \log_2 \left( 1 + \frac{\alpha}{\bar{\alpha}} e^{2a^T z - 2\|a\|^2} \right) \right].
\]

(16)

The scalar product \( a^T z \) is a real Gaussian random variable with zero mean and variance \( \|a\|^2 \).

Similarly, we can find the mutual information of the channel
\[ Y = aX + Z, \]
where \( a \in \mathbb{R} \) and \( Z \sim \mathcal{N}(0, 1) \). In this case,
\[
I(X; Y) = -\mathbb{E} \left[ \log_2(\alpha e^{-(Y+a)^2/2} + \bar{\alpha} e^{-(Y-a)^2/2}) \right] + \frac{\log_2 e}{2}
\]
\[
= H_b(\alpha) - \alpha \mathbb{E} \left[ \log_2 \left( 1 + \frac{\bar{\alpha}}{\alpha} e^{2az - 2a^2} \right) \right]
\]
\[
- \bar{\alpha} \mathbb{E} \left[ \log_2 \left( 1 + \frac{\alpha}{\bar{\alpha}} e^{2az - 2a^2} \right) \right].
\]

(18)

Then, the mutual information eqs. (16) and (18) coincide provided that \( a^2 = \|a\|^2 \). These results are summarized by the following theorem.

**Theorem A.1** A binary-input vector-output additive Gaussian channel \( y = Xa + z \), where the entries of \( z \) are iid and distributed as \( \mathcal{N}(0, 1) \), is equivalent, in terms of mutual information, to a binary-input real additive Gaussian channel with SNR \( \gamma = \|a\|^2 \). The mutual information is:
\[
I(X; Y) = H_b(\alpha) - \alpha \mathbb{E} \left[ \log_2 \left( 1 + \frac{\bar{\alpha}}{\alpha} e^{2\sqrt{\gamma}Z_1 - 2\gamma} \right) \right]
\]
\[
- \bar{\alpha} \mathbb{E} \left[ \log_2 \left( 1 + \frac{\alpha}{\bar{\alpha}} e^{2\sqrt{\gamma}Z_1 - 2\gamma} \right) \right],
\]

(19)
where \( Z_1 \sim \mathcal{N}(0, 1) \).

**Remark A.1** The equivalence between the scalar and \( n \)-vector channels in terms of mutual information can be predicted by observing that the vector channel is a combination of \( n \) parallel Gaussian channels. Then, provided the receiver knows the vector \( \mathbf{a} \), the useful part of the signal can be combined coherently at the receiver while the noise is combined incoherently. This implies an \( n \)-fold increase of the SNR, for a given mutual information, when passing from the scalar to the \( n \)-vector channel.

**APPENDIX B**

**ASYMPTOTIC APPROXIMATION OF (19)**

In this appendix we derive an asymptotic approximation of the mutual information (19). This result extends [11, Prob. 4.12], corresponding to the equiprobable input case.

**Theorem B.1** The asymptotic expansion of (19) for \( \gamma \to \infty \) is given by

\[
I(X; Y) = H_b(\alpha) - \sum_{n=0}^{\infty} \frac{(-1)^n k_n(\alpha)}{\gamma^{n+1/2}} e^{-\gamma/2},
\]

where

\[
k_n(\alpha) = \frac{2\sqrt{\pi \alpha}}{n! \ln 2} \sum_{k=0}^{n} \frac{(2n)! 2^{(n-k)}|E_2(n-k)|}{k^{2(n-k)} \pi (n-k)!} \times \sum_{m=0}^{k} 2^{2m} (2k)^{(2m)} (\ln(\alpha/\bar{\alpha}))^{2(k-m)}.
\]

Consecutive partial sums in (20) are lower and upper bounds to \( I(X; Y) \).

**Proof:** We start by considering the integral

\[
\int_{-\infty}^{\infty} \ln(1 + \rho e^{z-2\gamma}) e^{-z^2/(8\gamma)} \frac{dz}{\sqrt{8\pi \gamma}}
\]

\[=
\int_{-\infty}^{\infty} \ln(1 + \rho e^{z}) e^{-z^2/(8\gamma)} \frac{dz}{\sqrt{8\pi \gamma}}
\]

\[=
\int_{-\infty}^{\infty} \ln(1 + \rho e^{z}) e^{-(z+2\gamma)^2/(8\gamma)} \frac{dz}{\sqrt{8\pi \gamma}}
\]

\[=\]

\[2\sqrt{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n c_n(\rho)}{(8\gamma)^{n+1/2}} e^{-\gamma/2}
\]

where

\[
c_n(\rho) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \ln(1 + \rho e^{z}) \frac{z^{2n}}{n!} e^{-z^2/2} dz.
\]
Since consecutive partial sums of the series expansion of \( e^{-x} \) are lower and upper bound of the limit, also consecutive partial sums of (22) are lower and upper bounds of the lhs.

To calculate the coefficients \( c_n(\rho) \), we notice that

\[
c'_n(\rho) = \frac{1}{4\pi n!\sqrt{\rho}} \int_{-\infty}^{\infty} \frac{z^{2n}}{\cosh((z + \ln \rho)/2)} dz
\]

\[
= \frac{1}{2\pi n!\sqrt{\rho}} \int_{-\infty}^{\infty} \frac{(2u - \ln \rho)^{2n}}{\cosh u} du
\]

\[
= \frac{1}{2\pi n!\sqrt{\rho}} \sum_{k=0}^{n} \binom{2n}{2k} 2^{2k} (\ln \rho)^{2(n-k)} \int_{-\infty}^{\infty} \frac{u^{2k}}{\cosh u} du
\]

\[
= \frac{1}{2\pi n!} \sum_{k=0}^{n} \binom{2n}{2k} \pi^{2k} (\ln \rho)^{2(n-k)} \left| E_{2k} \right|
\]

by using the integral [12, 3.523-4]

\[
\int_{-\infty}^{\infty} \frac{u^{2k}}{\cosh u} du = 2 \left( \frac{\pi}{2} \right)^{2k+1} |E_{2k}|,
\]

for every integer \( k \geq 0 \), where \( E_n \) is the \( n \)th Euler number (\( E_0 = 1, E_2 = -1, E_4 = 5, E_6 = -61, \ldots \)).

Since we have:

\[
\int_{0}^{x} \frac{(\ln u)^n}{\sqrt{u}} du = 2 \sqrt{x} \sum_{k=0}^{n} (-1)^k 2^k n^{(k)} (\ln x)^{n-k},
\]

for every integer \( n \geq 0 \) and with \( n^{(m)} \triangleq (n!/(n - m)!) \), we can integrate (24) term by term and obtain the following final expression:

\[
c_n(\rho) = \frac{1}{2\pi n!} \sum_{k=0}^{n} \binom{2n}{2k} \pi^{2k} |E_{2k}| \int_{0}^{\rho} \frac{(\ln u)^{2(n-k)}}{\sqrt{u}} du
\]

\[
= \frac{\sqrt{\rho}}{n!} \sum_{k=0}^{n} \binom{2n}{2k} \pi^{2(n-k)} |E_{2(n-k)}| |E_{2k}|
\]

\[
\times \sum_{m=0}^{2k} (-1)^m 2^m (2k)^{(m)} (\ln \rho)^{2k-m}.
\]

(25)

In the special case of \( \rho = 1 \), we have:

\[
c_n(1) = \frac{1}{n!} \sum_{k=0}^{n} 2^{2k} \binom{2n}{2k} \pi^{2(n-k)} |E_{2(n-k)}|.
\]
Then, in accordance with [11, Prob. 4.12], where $c_n$ corresponds to $c_n(1)/2^{2n}$, we have:

\[
\begin{align*}
  c_0(1) &= 1 \\
  c_1(1) &= 8 + \pi^2 \\
  c_2(1) &= 384 + 48\pi^2 + 5\pi^4 \\
  c_3(1) &= 46080 + 5760\pi^2 + 600\pi^4 + 61\pi^6 \\
  &\vdots
\end{align*}
\]

Finally, we apply (25) to derive the asymptotic expansion (20) and the coefficients (21).

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