Two Flows Kowalevski Top as the Full Genus Two Jacobi’s Inversion Problem and $\text{Sp}(4, \mathbb{R})$ Lie Group Structure

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Abstract

By using the first and second flows of the Kowalevski top, we can recreate the Kowalevski top into two-flows Kowalevski top, which has two-time variables. Then, we demonstrate that equations of the two-flows Kowalevski top become those of the full genus two Jacobi inversion problem. In addition to the Lax pair for the first flow, we construct a Lax pair for the second flow. Using the first and second flows, we demonstrate that the Lie group structure of these two Lax pairs is $\text{Sp}(4, \mathbb{R})/\mathbb{Z}_2 \cong \text{SO}(3,2)$. With the two-flows Kowalevski top, we can conclude that the Lie group structure of the genus two hyperelliptic function is $\text{Sp}(4, \mathbb{R})/\mathbb{Z}_2 \cong \text{SO}(3,2)$.

1 Introduction

We are interested in the reason why exact solutions for some special non-linear differential equations exist, as well as a series of infinitely many solutions in some cases. Soliton equations are examples of such equations, such that various methods for studying soliton systems are beneficial to our objective. Starting from the inverse scattering method [1–3], the soliton theory has several interesting developments, such as the AKNS formulation [4], geometrical approach [5–8], Bäcklund transformation [9–11], Hirota equation [12, 13], Sato theory [14], vertex construction of the soliton solution [15–17], and Schwarzian type mKdV/KdV equation [18].

We have a dogma that the reason why some special non-linear differential equations have a series of infinitely many solutions is that such non-linear systems have the Lie group structure. Owing to an addition formula of the Lie group, we obtain a series of infinitely many solutions. As a representation of the addition formula of the Lie group, algebraic functions will emerge, such as trigonometric/elliptic/hyperelliptic functions, as the solutions of such special non-linear differential equations. A product of Lie group elements is given by an addition of exponential arguments of corresponding Lie algebra, and such multiplication formula is usually called an addition formula of the Lie group. In the KdV case, the Bäcklund formula plays a role of an addition formula to provide new soliton solutions.

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In our previous papers, using the Lie group structure \( \text{SL}(2, \mathbb{R})/\mathbb{Z}_2 \cong \text{SO}(2,1) \cong \text{SU}(1,1)/\mathbb{Z}_2 \cong \text{Sp}(2, \mathbb{R})/\mathbb{Z}_2 \) as the guiding principle, we revisited and studied why two dimensional integrable models, such as \( \text{KdV}/\text{mKdV}/\text{sinh-Gordon} \), have quite optimal properties such as having a series of infinitely many solutions from the perspective of the Lie group’s structure [19–24].

Here, we would like to revisit the famous Kowalevski top [25, 26]. It is quite surprising that the Kowalevski top was first solved more than one hundred years ago, yet it remains in progress [27–37]. The original first flow (time-dependent) Kowalevski top can be formulated into the special genus two hyperelliptic function has the SO(3,2) structure.

For the Kowalevski top, we take \( A = B = 2C \), \( \zeta_0 = 0 \), which provides four conserved quantities. In this case, the center of mass is on the \( \xi\eta \) plane. Owing to the fact that \( A = B \), we can rotate the \( \xi \) and \( \eta \) axes around \( \zeta \) axis. Hence, we can put \( \eta_0 = 0 \) without losing generality. Then, the set of equations for the Kowalevski top becomes

\[
2 \frac{d\omega_1}{dt} = \omega_2 \omega_3, \tag{2.1}
\]

In this research, we study the generalized two-flows Kowalevski top by using the second flow of the Kowalevski top [29,30]. We will demonstrate that equations of the two-flows Kowalevski top become those of the full genus two Jacobi’s inversion problem. Next, in addition to the Lax pair of the original Kowalevski top [28,31], we will provide the Lax pair for the second flow of the Kowalevski top. Using the Lax pairs of the first and second flows, we will demonstrate that Lax pairs have SO(3,2)\( \cong \text{Sp}(4, \mathbb{R})/\mathbb{Z}_2 \) Lie group structure. Combining these two results, we conclude that the genus two hyperelliptic function has the SO(3,2)\( \cong \text{Sp}(4, \mathbb{R})/\mathbb{Z}_2 \) Lie group structure.

2 Two flows Kowalevski top as the full genus two Jacobi’s inversion problem

2.1 Review of Kowalevski’s work

We first review Kowalevski’s work [25, 26]. A set of equations that describe the rotation of a rigid body around a fixed point under uniform gravitational force with a magnitude of \( Mg \) is given in the form

\[
\frac{dX_1}{\sqrt{f_5(X_1)}} + \frac{dX_2}{\sqrt{f_5(X_2)}} = 0, \quad \frac{X_1dX_1}{\sqrt{f_5(X_1)}} + \frac{X_2dX_2}{\sqrt{f_5(X_2)}} = idt,
\]

where \( f_5(X) \) denotes some fifth-degree polynomial function of \( X \).

In this research, we study the generalized two-flows Kowalevski top by using the second flow of the Kowalevski top [29,30]. We will demonstrate that equations of the two-flows Kowalevski top become those of the full genus two Jacobi’s inversion problem. Next, in addition to the Lax pair of the original Kowalevski top [28,31], we will provide the Lax pair for the second flow of the Kowalevski top. Using the Lax pairs of the first and second flows, we will demonstrate that Lax pairs have SO(3,2)\( \cong \text{Sp}(4, \mathbb{R})/\mathbb{Z}_2 \) Lie group structure. Combining these two results, we conclude that the genus two hyperelliptic function has the SO(3,2)\( \cong \text{Sp}(4, \mathbb{R})/\mathbb{Z}_2 \) Lie group structure.
Here, we have introduced the parameters $\ell$ respectively. Note that $|x|$. We also consider $x$.

We regard $\xi$ a convenient complex valuable defined as a quantity, which were used in the original Kowalevski’s work [25]. In the following, we adopt $x$ in the form

$$2\frac{d\omega_2}{dt} = -\omega_3\omega_1 - c_0\gamma_3,$$  
(2.2)

$$\frac{d\omega_2}{dt} = c_0\gamma_2,$$  
(2.3)

$$\frac{d\gamma_1}{dt} = \omega_3\gamma_2 - \omega_2\gamma_3,$$  
(2.4)

$$\frac{d\gamma_2}{dt} = \omega_1\gamma_3 - \omega_3\gamma_1,$$  
(2.5)

$$\frac{d\gamma_3}{dt} = \omega_2\gamma_1 - \omega_1\gamma_2.$$  
(2.6)

Here, we have used $c_0 = -Mg\xi_0/C$ as a convenient notation.

In this Kowalevski’s case, we define new variables as

$$\gamma_1 = \gamma_1 + \frac{\omega_1^2 - \omega_2^2}{c_0},$$  
(2.7)

$$\gamma_2 = \gamma_2 + \frac{2\omega_1\omega_2}{c_0}.$$  
(2.8)

These variables satisfy the following equations

$$\frac{d\gamma_1}{dt} = \omega_3\gamma_2, \quad \frac{d\gamma_2}{dt} = -\omega_3\gamma_1.$$  
(2.9)

These equations demonstrate that $\gamma_1^2 + \gamma_2^2$ is conserved, which is the Kowalevski’s conserved quantity.

Now, we summarize four conserved quantities in the Kowalevski top.

$$\begin{align*}
\text{a) Energy:} & \quad 2\omega_1^2 + 2\omega_2^2 + \omega_3^2 - 2c_0\gamma_1 = 6\ell_1 \\
\text{b) Vertical component of angular momentum:} & \quad 2\omega_1\gamma_1 + 2\omega_2\gamma_2 + \omega_3\gamma_3 = 2\ell \\
\text{c) Direction cosines:} & \quad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \\
\text{d) Kowalevski’s conserved quantity:} & \quad (c_0\gamma_1)^2 + (c_0\gamma_2)^2 = k^2
\end{align*}$$  
(2.10) - (2.13)

Here, we have introduced the parameters $\ell_1$, $\ell$, and $k$ that represent values of each conserved quantity, which were used in the original Kowalevski’s work [25]. In the following, we adopt a convenient complex valuable defined as

$$\xi = c_0(\gamma_1 + i\gamma_2) = c_0(\gamma_1 + i\gamma_2) + (\omega_1 + i\omega_2)^2.$$  
(2.14)

We regard $\xi$ and its complex conjugate $\tilde{\xi}$ as independent valuables and denote $\xi_1$ and $\xi_2$, respectively. Note that $|\xi|^2 = \xi_1\xi_2 = k^2$. In addition, we introduce other complex variables $x_1$ and $x_2$ as

$$x_1 = \omega_1 + i\omega_2, \quad x_2 = \omega_1 - i\omega_2.$$  
(2.15)

We also consider $x_1$ and $x_2$ to be independent. In the following, we derive differential equations satisfied by $x_1$ and $x_2$, which will lead to the Jacobi’s inversion problem.

By using these conserved quantities, $\omega_3$ and $\gamma_3$ are solved with variables of Eqs. (2.14) and (2.15) in the form

$$\omega_3^2 = 6\ell_1 - (x_1 + x_2)^2 + \xi_1 + \xi_2 = A + \xi_1 + \xi_2,$$  
(2.16)

$$c_0\omega_3\gamma_3 = 2c_0 + x_1x_2(x_1 + x_2) - x_1\xi_2 - x_2\xi_1 = B - x_1\xi_2 - x_2\xi_1,$$  
(2.17)

$$c_0^2\gamma_3^2 = c_0^2 - k^2 - x_1^2x_2^2 + x_1^2\xi_2 + x_2^2\xi_1 = C + x_1^2\xi_2 + x_2^2\xi_1,$$  
(2.18)
where we used $x_1^2 x_2^2 = (\omega_1^2 + \omega_2^2)^2$, $x_1^2 \xi_2 + x_2^2 \xi_1 = 2(\omega_1^2 + \omega_2^2)^2 + 2c_0 ((\omega_1^2 - \omega_2^2)\gamma_1 + 2\omega_1 \omega_2 \gamma_2)$. In the right-hand sides of Eqs.(2.16)–(2.18), we have separated the expressions into two parts, i.e., a part which depends on $\xi_1, \xi_2$ and others with

\[ A = 6\ell_1 - (x_1 + x_2)^2, \quad B = 2c_0 + x_1 x_2(x_1 + x_2), \quad C = c_0^2 - k^2 - x_1^2 x_2^2. \]

From an identity $\omega_3^2 \times c_0^2 \gamma_3^2 = (c_0 \omega_3 \gamma_3)^2$, we have

\[ 0 = (A + \xi_1 + \xi_2)(C + x_1^2 \xi_2 + x_2^2 \xi_1) - (B - x_1 \xi_2 - x_2 \xi_1)^2 \\
= AC - B^2 + k^2(x_1 - x_2)^2 + (Ax_1^2 + 2Bx_1 + C)\xi_2 + (A\xi_1^2 + 2B\xi_1 + C)\xi_1 \\
= R_1(x_1, x_2) + k^2(x_1 - x_2)^2 + R(x_1)\xi_2 + R(x_2)\xi_1, \quad \text{(2.19)} \]

where

\[ R(x) = Ax^2 + 2Bx + C = -x^4 + 6\ell_1 x^2 + 4c_0 x + c_0^2 - k^2, \quad \text{(2.20)} \]

\[ R_1(x_1, x_2) = AC - B^2 \\
= -6\ell_1 x_1^2 x_2^2 - 4c_0(x_1 + x_2)x_1 x_2 - (c_0^2 - k^2)(x_1 + x_2)^2 + 6\ell_1(c_0^2 - k^2) - 4\ell_1^2 c_0^2. \quad \text{(2.21)} \]

To this point, we have adopted six dynamical variables $x_1, x_2, \xi_1, \xi_2, \omega_3, \gamma_3$ so far. Assuming that $x_1$ and $x_2$ have been solved, $\xi_1$ and $\xi_2$ can be obtained in principle from the relationship

\[ (\xi_1 - x_1^2) \cdot (\xi_2 - x_2^2) = c_0^2(\gamma_1^2 + \gamma_2^2) = c_0^2(1 - \gamma_3^2) = c_0^2 - (C + x_1^2 \xi_2 + x_2^2 \xi_1), \quad \text{(2.22)} \]

and Eq.(2.19). In addition, $\omega_3$ and $\gamma_3$ could be obtained from Eqs.(2.16) and (2.18), respectively. Accordingly, we can reduce the number of dynamical variables from six to two, i.e., $x_1$ and $x_2$ with four conserved quantities. In the calculation process below, there appears the following combination $R(x_1)R(x_2) - (x_1 - x_2)^2 R_1(x_1, x_2)$, which gives

\[ R(x_1)R(x_2) - (x_1 - x_2)^2 R_1(x_1, x_2) = R(x_1, x_2)^2, \quad \text{(2.23)} \]

where

\[ R(x_1, x_2) = Ax_1 x_2 + B(x_1 + x_2) + C \\
= -x_1^2 x_2^2 + 6\ell_1 x_1 x_2 + 2c_0(x_1 + x_2) + c_0^2 - k^2. \quad \text{(2.24)} \]

We note that $R(x_1, x_1) = R(x_1)$ and $R(x_2, x_2) = R(x_2)$.

Let us consider differential equations for two dynamical variables $x_1$ and $x_2$:

\[ 2\frac{dx_1}{dt} = 2 \left( \frac{d\omega_1}{dt} + i \frac{d\omega_2}{dt} \right) = -i(\omega_3(x_1 + i\omega_2) + c_0 \gamma_3) = -i(\omega_3 x_1 + c_0 \gamma_3), \quad \text{(2.25)} \]

\[ 2\frac{dx_2}{dt} = 2 \left( \frac{d\omega_1}{dt} - i \frac{d\omega_2}{dt} \right) = i(\omega_3(x_1 - i\omega_2) + c_0 \gamma_3) = i(\omega_3 x_2 + c_0 \gamma_3). \quad \text{(2.26)} \]

By using Eqs.(2.16)–(2.18), we have

\[ -4 \left( \frac{dx_1}{dt} \right)^2 = \omega_3^2 x_1^2 + 2c_0 \omega_3 \gamma_3 x_1 + c_0^2 \gamma_3^2 \\
= (A + \xi_1 + \xi_2)x_1^2 + 2(B - x_1 \xi_2 - x_2 \xi_1)x_1 + (C + x_1^2 \xi_2 + x_2^2 \xi_1) \\
= (Ax_1^2 + 2Bx_1 + C) + (x_1 - x_2)^2 \xi_1 \\
= R(x_1) + (x_1 - x_2)^2 \xi_1, \quad \text{(2.27)} \]

\[ -4 \left( \frac{dx_2}{dt} \right)^2 = \omega_3^2 x_2^2 + 2c_0 \omega_3 \gamma_3 x_2 + c_0^2 \gamma_3^2 \]
Then, we have

\[ \frac{4}{R(x_1)} \left( \frac{dx_1}{dt} \right)^2 = 1 + \frac{(x_1 - x_2)^2}{R(x_1)} \xi_1, \]  

(2.30)

\[ \frac{4}{R(x_2)} \left( \frac{dx_2}{dt} \right)^2 = 1 + \frac{(x_1 - x_2)^2}{R(x_2)} \xi_2, \]  

(2.31)

\[ \frac{8}{\sqrt{R(x_1)R(x_2)}} \frac{dx_1}{dt} \frac{dx_2}{dt} = \frac{2R(x_1) + 2R(x_1)R(x_2) - k^2(x_1 - x_2)^4}{R(x_1)R(x_2)}. \]  

(2.32)

Owing to the identity of Eq. (2.19), we can eliminate \( \xi_1, \xi_2 \) by considering the following combination of Eqs. (2.30)–(2.32)

\[ \frac{4}{R(x_1)} \left( \frac{dx_1}{dt} \right)^2 - \frac{4}{R(x_2)} \left( \frac{dx_2}{dt} \right)^2 + \frac{8}{\sqrt{R(x_1)R(x_2)}} \frac{dx_1}{dt} \frac{dx_2}{dt} = 2R(x_1)R(x_2) + (x_1 - x_2)^2 \left( \frac{R(x_1)}{R(x_2)} \xi_2 + \frac{R(x_2)}{R(x_1)} \xi_1 \right) \]  

\[ + 2R(x_1) \frac{R(x_2)}{R(x_1)} - k^2(x_1 - x_2)^4 \]  

\[ = 4 \left( \frac{R(x_1)}{R(x_2)} \frac{dx_1}{dt} + \frac{dx_2}{dt} \right)^2 = 4 \left( \frac{R(x_1)}{R(x_2)} \left( \frac{R(x_1)}{R(x_2)} + \frac{dx_1}{dt} \right)^2 \right) - \frac{k^2}{4}. \]  

(2.33)

where we have used Eq. (2.23). Then, we have

\[ \left( \frac{dx_1}{dt} + \frac{dx_2}{dt} \right)^2 = \frac{(x_1 - x_2)^4}{R(x_1)R(x_2)} \left( \frac{R(x_1)}{R(x_2)} + \frac{R(x_1)R(x_2)}{2(x_1 - x_2)^2} \right)^2 - \frac{k^2}{4}. \]  

(2.34)

Next, we introduce Kowalevski variables \( s_1, s_2 \) in the form

\[ s_1 = \frac{R(x_1, x_2) - \sqrt{R(x_1)R(x_2)}}{2(x_1 - x_2)^2}, \quad s_2 = \frac{R(x_1, x_2) + \sqrt{R(x_1)R(x_2)}}{2(x_1 - x_2)^2}. \]  

(2.35)

In terms of the Kowalevski variables, we have

\[ \frac{\sqrt{R(x_1)R(x_2)}}{(x_1 - x_2)^2} = s_2 - s_1, \]  

(2.36)

while Eq. (2.34) gives

\[ \left( \frac{dx_1}{dt} + \frac{dx_2}{dt} \right)^2 = \frac{1}{(s_2 - s_1)^2} \left( s_1^2 - \frac{k^2}{4} \right). \]  

(2.37)
\[
\left( \frac{dx_1/dt}{\sqrt{R(x_1)}} - \frac{dx_2/dt}{\sqrt{R(x_2)}} \right)^2 = -\frac{1}{(s_2 - s_1)^2} \left( s_2^2 - \frac{k^2}{4} \right). \tag{2.38}
\]

We now assume that the following \( \varphi(s) \) exists

\[
\frac{dx_1/dt}{\sqrt{R(x_1)}} + \frac{dx_2/dt}{\sqrt{R(x_2)}} = i\sqrt{\frac{s_1^2 - k^2/4}{s_1 - s_2}} = \frac{ds_1}{\sqrt{f_5(s_1)}}, \tag{2.39}
\]
\[
-\frac{dx_1/dt}{\sqrt{R(x_1)}} + \frac{dx_2/dt}{\sqrt{R(x_2)}} = i\sqrt{\frac{s_2^2 - k^2/4}{s_2 - s_1}} = \frac{ds_2}{\sqrt{f_5(s_2)}}. \tag{2.40}
\]

Later, we will explicitly prove that \( \varphi(s) \) is the third-degree polynomial function of \( s \), which implies that the above assumption is true. Considering the square root of Eqs.(2.37) and (2.38), various signs emerge. After obtaining the explicit form of \( \varphi(s) \), we return to Eqs.(2.39) and (2.40) ; in addition, via numerical calculations, we have checked that the above signs are correct. From the second and third terms of Eqs.(2.39) and (2.40), we have

\[
\frac{ds_1}{\sqrt{f_5(s_1)}} = i\frac{dt}{s_1 - s_2}, \quad \frac{ds_2}{\sqrt{f_5(s_2)}} = -i\frac{dt}{s_1 - s_2}, \tag{2.41}
\]

where

\[
f_5(s) = \varphi(s)(s^2 - k^2/4), \tag{2.42}
\]

which gives a special genus two Jacobi’s inversion problem of the form

\[
\frac{ds_1}{\sqrt{f_5(s_1)}} + \frac{ds_2}{\sqrt{f_5(s_2)}} = 0, \quad \frac{s_1 ds_1}{\sqrt{f_5(s_1)}} + \frac{s_2 ds_2}{\sqrt{f_5(s_2)}} = i dt. \tag{2.43}
\]

A general genus two Jacobi’s inversion problem is written in the form [38]

\[
du_1 = \frac{dX_1}{Y_1} + \frac{dX_2}{Y_2}, \quad du_2 = \frac{X_1 dX_1}{Y_1} + \frac{X_2 dX_2}{Y_2}, \quad Y = \sqrt{f_5(X)}, \tag{2.44}
\]

where \( f_5(X) \) is a fifth-degree polynomial function of \( X \). We will use the term “full” Jacobi’s inversion problem for the case where \( du_1 \neq 0 \) and \( du_2 \neq 0 \) and “special” for \( du_1 = 0 \) and \( du_2 = 0 \). From Eq.(2.44), we have

\[
\frac{\partial X_1}{\partial u_2} = \frac{Y_1}{X_1 - X_2}, \quad \frac{\partial X_2}{\partial u_2} = -\frac{Y_2}{X_1 - X_2}, \tag{2.45}
\]
\[
\frac{\partial X_1}{\partial u_1} = -\frac{X_2 Y_1}{X_1 - X_2}, \quad \frac{\partial X_2}{\partial u_1} = \frac{X_1 Y_2}{X_1 - X_2}. \tag{2.46}
\]

By making the correspondence of \( s_1 \leftrightarrow X_1, s_2 \leftrightarrow X_2, 0 \leftrightarrow du_1, i dt \leftrightarrow du_2, \) Eq.(2.44) gives the special genus two Jacobi’s inversion problem Eq.(2.43).

If we notice that Eq.(2.35) is the change of variables \( x_1, x_2 \rightarrow s_1(x_1, x_2), s_2(x_1, x_2) \), we can find \( \varphi(s) \) in Eqs.(2.39) and (2.40) from the following dynamics independent identical relationships:

\[
\frac{dx_1}{\sqrt{R(x_1)}} + \frac{dx_2}{\sqrt{R(x_2)}} = \frac{ds_1}{\sqrt{\varphi(s_1)}}, \quad -\frac{dx_1}{\sqrt{R(x_1)}} + \frac{dx_2}{\sqrt{R(x_2)}} = \frac{ds_2}{\sqrt{\varphi(s_2)}}. \tag{2.47}
\]

From Eqs.(2.47), we have

\[
\frac{\partial s_1}{\partial x_1} = \sqrt{\frac{\varphi(s_1)}{R(x_1)}}, \quad \frac{\partial s_1}{\partial x_2} = \sqrt{\frac{\varphi(s_1)}{R(x_2)}}, \quad \frac{\partial s_2}{\partial x_1} = -\sqrt{\frac{\varphi(s_2)}{R(x_1)}}, \quad \frac{\partial s_2}{\partial x_2} = \sqrt{\frac{\varphi(s_2)}{R(x_2)}}. \tag{2.48}
\]
The existence of \( \varphi(s) \) that satisfies Eqs.(2.48) implies that the original Kowalevski top can be expressed in the genus two Jacobi’s inversion problem.

First, we consider the limit \( x_2 \to x_1 \) and take the most singular term. In this limit, \( R(x_1, x_2) \to R(x_1), R(x_2) \to R(x_1) \). Then, we have

\[
\begin{align*}
    s_2 & \sim \frac{R(x_1)}{(x_1 - x_2)^2}, \\
    \frac{\partial s_2}{\partial x_1} & \sim -\frac{2R(x_1)}{(x_1 - x_2)^3}, \\
    \frac{\partial s_2}{\partial x_2} & \sim \frac{2R(x_1)}{(x_1 - x_2)^3}, \\
\end{align*}
\]

(2.49)

from Eq.(2.35). By adopting Eqs.(2.48) and (2.49), we obtain

\[
\frac{\partial s_2}{\partial x_1} \frac{\partial s_2}{\partial x_2} \sim \varphi(s_2) - \frac{4R(x_1)^2}{(x_1 - x_2)^6} \sim -\frac{4s_2^3}{R(x_1)}.
\]

(2.50)

This implies that \( \varphi(s_2) = 4s_2^3 + (\text{lower order}) \), which is the third-degree polynomial function of \( s \), i.e.,

\[
\varphi(s) = 4s^3 + k_2 s^2 + k_1 s + k_0.
\]

(2.51)

The coefficients \( k_2, k_1, \) and \( k_0 \) are determined by using the relationship

\[
\varphi(s_1) + \varphi(s_2) = \sqrt{R(x_1)R(x_2)} \left( \frac{\partial s_1}{\partial x_1} \frac{\partial s_1}{\partial x_2} - \frac{\partial s_2}{\partial x_1} \frac{\partial s_2}{\partial x_2} \right).
\]

(2.52)

\((x_1 - x_2)^6 \times (\text{l.h.s of Eq.}(2.52)) \) represents the polynomial of \( x_1 \) and \( x_2 \) that involves \( k_2, k_1, \) and \( k_0 \). By using the definitions of \( s_1 \) and \( s_2 \) given in Eqs.(2.35), \((x_1 - x_2)^6 \times (\text{r.h.s of Eq.}(2.52)) \) is proven to be the polynomial of \( x_1 \) and \( x_2 \). Hence, from Eq.(2.52), the coefficients \( k_2, k_1, \) and \( k_0 \) are determined. Accordingly, we obtained \( \varphi(s) \) in the form

\[
\varphi(s) = 4s^3 + 6\ell_1 s^2 - (k^2 - c_0^2)s - \frac{3}{2} \ell_1 (k^2 - c_0^2) - \ell^2 c_0^2.
\]

(2.53)

By using Eq.(2.53), we have checked Eqs.(2.48) to ensure that it is actually satisfied. We conclude that the original Kowalevski top can be expressed in the special genus two Jacobi’s inversion problem Eq.(2.43).

### 2.2 Second flow of the Kowalevski Top

We generalize the original Kowalevski top to two-flows Kowalevski top, and we replace \( \omega_i(t) \to \omega_i(r, t), \gamma_i(t) \to \gamma_i(r, t) \) \((i = 1, 2, 3) \) by introducing another time \( r \), such that \( x_1(t) \to x_1(r, t), x_2(t) \to x_2(r, t), s_1(t) \to s_1(r, t), s_2(t) \to s_2(r, t) \) and \( d/dt \to d/dr \). We call the set of Eqs.(2.1)–(2.6) as the first flow equations. The second flow equations are given in the form [29,30]

\[
\begin{align*}
    2\frac{\partial \omega_1}{\partial r} & = -c_0 \gamma_2 \gamma_3 + \omega_3(\gamma_1 \omega_2 - \gamma_2 \omega_1) - 2\gamma_3 \omega_1 \omega_2 - \frac{1}{c_0}(\omega_1^2 + \omega_2^2)\omega_2 \omega_3, \\
    2\frac{\partial \omega_2}{\partial r} & = c_0 \gamma_3 \gamma_1 + \omega_3(\gamma_1 \omega_1 + \gamma_2 \omega_2) + \gamma_3(\omega_1^2 - \omega_2^2) + \frac{1}{c_0}(\omega_1^2 + \omega_2^2)\omega_1 \omega_3, \\
    \frac{\partial \omega_3}{\partial r} & = \gamma_2(\omega_1^2 - \omega_2^2) - 2\gamma_1 \omega_1 \omega_2, \\
    \frac{\partial \gamma_1}{\partial r} & = \gamma_3 \left( \gamma_1 \omega_2 - \gamma_2 \omega_1 - \frac{1}{c_0}(\omega_1^2 + \omega_2^2)\omega_2 \right), \\
    \frac{\partial \gamma_2}{\partial r} & = \gamma_3 \left( \gamma_1 \omega_1 + \gamma_2 \omega_2 + \frac{1}{c_0}(\omega_1^2 + \omega_2^2)\omega_1 \right), \\
    \frac{\partial \gamma_3}{\partial r} & = -\omega_2(\gamma_1^2 + \gamma_2^2) + \frac{1}{c_0}(\gamma_1 \omega_2 - \gamma_2 \omega_1)(\omega_1^2 + \omega_2^2).
\end{align*}
\]

(2.54)–(2.59)
Under these second flow equations, Eqs.(2.10)–(2.13) are also conserved. Furthermore, the first and second flows are integrable, i.e., the integrability conditions of $\omega_i, \gamma_i$ ($i = 1, 2, 3$) are satisfied. For example, the integrability condition

$$\partial_r(\omega_2 \omega_3) = \partial_t \left( -c_0 \gamma_2 \gamma_3 + \omega_3 (\gamma_1 \omega_2 - \gamma_2 \omega_1) - 2 \gamma_3 \omega_1 \omega_2 - \frac{1}{c_0} (\omega_1^2 + \omega_2^2) \omega_3 \right). \quad (2.60)$$

It can be observed that Eq.(2.60) holds by using the first flow equations Eqs.(2.1)–(2.6) and the second flow equations Eqs.(2.54)–(2.59).

Let us denote

$$\varphi_{22} = s_1 + s_2 = \frac{R(x_1, x_2)}{(x_1 - x_2)^2}, \quad (2.61)$$

$$\varphi_{21} = -s_1 s_2 = \frac{R(x_1) R(x_2) - R(x_1, x_2)^2}{4(x_1 - x_2)^3} = \frac{R_1(x_1, x_2)}{4(x_1 - x_2)^2}. \quad (2.62)$$

By using the first and second flow equations, we have checked

$$\frac{c_0}{2} \frac{\partial \varphi_{22}}{\partial r} = \frac{\partial \varphi_{21}}{\partial t}, \quad \text{i.e.,} \quad \frac{c_0}{2} \frac{\partial}{\partial r}(s_1 + s_2) = -\frac{\partial}{\partial t}(s_1 s_2).$$

It can be considered as the integrability condition by identifying $\varphi_{22}, \varphi_{21}$ as the genus two hyperelliptic $\varphi$ functions with $du_1 : du_2 = 2dr/c_0 : dt$. This fact suggests that we have a full Jacobi inversion problem in the form

$$\frac{ds_1}{\sqrt{f_5(s_1)}} + \frac{ds_2}{\sqrt{f_5(s_2)}} = du_1 = \frac{2}{c_0} dr, \quad (2.63)$$

$$\frac{s_1 ds_1}{\sqrt{f_5(s_1)}} + \frac{s_2 ds_2}{\sqrt{f_5(s_2)}} = du_2 = idt. \quad (2.64)$$

Comparing Eqs.(2.63) and (2.64) with Eq.(2.44), we have achieved the correspondence $s_1 \leftrightarrow X_1, s_2 \leftrightarrow X_2, \sqrt{f_5(s_1)} \leftrightarrow Y_1, \sqrt{f_5(s_2)} \leftrightarrow Y_2, du_1 \leftrightarrow 2idr/c_0, du_2 \leftrightarrow idt$. Then, from Eqs.(2.45) and (2.46), we have

$$\frac{1}{i} \frac{\partial \varphi_{21}}{\partial t} = \frac{\sqrt{f_5(s_1)}}{s_1 - s_2}, \quad \frac{1}{i} \frac{\partial \varphi_{22}}{\partial t} = \frac{\sqrt{f_5(s_2)}}{s_1 - s_2}, \quad (2.65)$$

$$\frac{c_0}{2i} \frac{\partial \varphi_{21}}{\partial r} = \frac{s_2 \sqrt{f_5(s_1)}}{s_1 - s_2}, \quad \frac{c_0}{2i} \frac{\partial \varphi_{22}}{\partial r} = \frac{s_1 \sqrt{f_5(s_2)}}{s_1 - s_2}. \quad (2.66)$$

If two-flows Kowalevski top can be written in a full genus two Jacobi’s inversion problem, Eqs.(2.65) and (2.66) must be satisfied. From Eq.(2.41), it can be observed that Eq.(2.65) is definitely satisfied. Then, we will check to confirm that Eq.(2.66) is completely satisfied. The relative sign of Eqs.(2.65) and (2.66) is determined in the above value from the integrability conditions of $\varphi_{22}$ and $\varphi_{21}$. Then, we take the square and check the relationship. The first term of Eq.(2.66) gives

$$-\frac{c_0^2}{4} \left( \frac{\partial \varphi_{21}}{\partial r} \right)^2 = \frac{s_2^2 f_5(s_1)}{(s_1 - s_2)^2} = \frac{s_2^2 \varphi(s_1)(s_1^2 - k^2/4)}{(s_1 - s_2)^2}$$

$$\Leftrightarrow \frac{(\partial \varphi_{21}/\partial r)^2}{\varphi(s_1)} = -\frac{4 s_2^2 (s_1^2 - k^2/4)}{c_0^2 (s_1 - s_2)^2}. \quad (2.67)$$

The second term of Eq.(2.66) gives

$$-\frac{c_0^2}{4} \left( \frac{\partial \varphi_{22}}{\partial r} \right)^2 = \frac{s_1^2 f_5(s_2)}{(s_1 - s_2)^2} = \frac{s_1^2 \varphi(s_2)(s_2^2 - k^2/4)}{(s_1 - s_2)^2}.$$
\[ (\frac{\partial s_2}{\partial t})^2 = -\frac{4 s_1^2 (s_2^2 - k^2/4)}{c_0^2 (s_1 - s_2)^2}. \] (2.68)

However, by using Eq. (2.47), we have
\[
\left( \frac{\partial x_1}{\partial t} + \frac{\partial x_2}{\partial t} \right)^2 = \left( \frac{\partial s_1}{\partial t} \right)^2 = -\frac{4 s_1^2 (s_2^2 - k^2/4)}{c_0^2 (s_1 - s_2)^2},
\]
\[
\left( \frac{\partial x_1}{\partial t} + \frac{\partial x_2}{\partial t} \right)^2 = \left( \frac{\partial s_2}{\partial t} \right)^2 = -\frac{4 s_1^2 (s_2^2 - k^2/4)}{c_0^2 (s_1 - s_2)^2}.
\]
(2.69) \quad (2.70)

The following equations are derived from the sum and difference between Eqs. (2.69) and
\[(\frac{\partial x_1}{\partial t})^2 + \frac{\partial x_2}{\partial t} = \frac{1}{c_0^2} \left( \frac{4(s_1 s_2^2 - k^2 ((s_1 + s_2)^2 - 2s_1 s_2)}{2} \right)
\]
\[= -\frac{1}{4c_0^2} \frac{R(x_1, x_2)^2 + k^2 R(x_1, x_2)^2 + (x_1 - x_2)^2 R_1(x_1, x_2)^2}{R(x_1) R(x_2)},
\]
\[= \frac{k^2 s_2 + s_1}{2c_0^2} \sqrt{R(x_1) R(x_2)} = \frac{k^2}{4c_0^2} R(x_1, x_2).
\]
(2.71) \quad (2.72)

Using Eqs. (2.54)–(2.59), we have verified that Eqs. (2.71) and (2.72) are completely satisfied.
Then, we can conclude that the first and second flows, which constitute the integrable two-flows Kowalevski top, provide the full genus two Jacobi inversion problem of Eqs. (2.63) and (2.64).

3 Lax pairs for two flows Kowalevski top and Sp(4, \mathbb{R})

Lie group structure

We usually formulate integrable models with a Lax pair to explicitly demonstrate an integrability of models and systematically obtain various conserved quantities. Let us consider the KdV equation of the form
\[ u_t - u_{xxx} + 6uu_x = 0. \]
From one Lax pair \( L, B \) of the form
\[ L = -\partial_x^2 + u, \quad B = 4\partial_x^3 - 6u\partial_x - 3u_x, \]
(3.1)
\[ \partial_t L = [B, L] \]
gives the KdV equation. From another Lax pair \( L, B \), which is known as the AKNS formalism, of the form
\[ L = \begin{pmatrix} \lambda & -u \\ -1 & -\lambda \end{pmatrix}, \quad B = \begin{pmatrix} 4\lambda^3 - 2\lambda u - u_x & -u_{xx} - 2\lambda u_x - 4\lambda^2 u + 2u^2 \\ -4\lambda^2 + 2u & -4\lambda^3 + 2\lambda u + u_x \end{pmatrix}, \]
(3.2)
where \( \lambda \) represents a constant spectral parameter, \([\partial_x - L, \partial_t - B] = \partial_t L - \partial_x B - [B, L] = 0 \) gives the KdV equation. We construct a Lax pair to examine the Lie group structure of an integrable model. As an infinitesimal Lie group transformation of an integrable model, we formulate a Lax pair using Lie algebra elements with only linear partial differential operators. The Lax pair of Eq. (3.2) is appropriate for this purpose because it contains only the linear differential operators as \( \partial_x \) and \( \partial_t \). The matrices \( L \) and \( B \) are constructed from the SL(2, \mathbb{R})/\mathbb{Z}_2 \cong SO(2,1)
Lie algebra elements
\[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \]
Then, we can infer that the KdV equation has the SL(2, \mathbb{R})/\mathbb{Z}_2 \cong SO(2,1) Lie group structure.

In this study, we construct the Lax pairs of the first and second flows for two-flows Kowalevski top, to examine its Lie group structure. Reyman et al. have constructed the Lax pair for the first flow \([28, 31]\) by using the \( 4 \times 4 \) spinor representation of \( SO(3,2) \) Lie algebra. Because \( SO(3,2) \cong Sp(4, \mathbb{R})/\mathbb{Z}_2 \), we formulate Lax pairs with the \( Sp(4, \mathbb{R}) \) Lie algebra.
3.1 The bases for the $\text{Sp}(4, \mathbb{R})$ Lie algebra

We first construct the bases of the $\text{Sp}(4, \mathbb{R})$ Lie algebra by using an almost complex structure $J$, which is a skew symmetric real matrix with $J^2 = -1$. For the $\text{Sp}(4, \mathbb{R})$ case, we take the representation

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$  \hfill (3.3)

Then, the Lie algebra of $\text{Sp}(4, \mathbb{R})$ is represented by the spaces of matrices $A = (a_{ij}), (1 \leq i, j \leq 4)$, which satisfy $JA + A^TJ = 0$. It is a ten-dimensional representation spanned by the following basis elements

$$I_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad I_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$I_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad I_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$I_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad I_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$I_9 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad I_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

3.2 Lax pairs of the first and second flows

The Lax equation of the first flow is given by

$$\frac{\partial L}{\partial t} = [B_2, L],$$ \hfill (3.4)

where the operators $L$ and $B_2$ have the following form

$$L = \frac{c_0}{2\lambda} \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & 0 \\ \gamma_2 & -\gamma_1 & 0 & -\gamma_3 \\ \gamma_3 & 0 & -\gamma_1 & \gamma_2 \\ 0 & -\gamma_3 & \gamma_2 & \gamma_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\omega_2 & -\omega_1 \\ 0 & 0 & \omega_1 & -\omega_2 \\ \omega_2 & -\omega_1 & -2\lambda & -\omega_3 \\ \omega_1 & \omega_2 & \omega_3 & 2\lambda \end{pmatrix},$$ \hfill (3.5)

$$= \frac{c_0}{2\lambda} \left( \gamma_1(I_3 - I_{10}) + \gamma_2(I_1 + I_2 + I_8 + I_9) + \gamma_3(I_4 - I_5) \right)$$

$$+ \omega_1(-I_6 + I_7) + \omega_2(-I_4 - I_5) + \omega_3(-I_8 + I_9) - 2\lambda I_{10},$$ \hfill (3.6)

$$B_2 = \frac{1}{2} \begin{pmatrix} 0 & \omega_3 & -\omega_2 & -\omega_1 \\ -\omega_3 & 0 & \omega_1 & -\omega_2 \\ \omega_2 & -\omega_1 & -2\lambda & -\omega_3 \\ \omega_1 & \omega_2 & \omega_3 & 2\lambda \end{pmatrix}$$ \hfill (3.7)

$$= \frac{1}{2}\omega_1(-I_6 + I_7) + \frac{1}{2}\omega_2(-I_4 - I_5) + \frac{1}{2}\omega_3(I_1 - I_2 - I_8 + I_9) - \lambda I_{10},$$ \hfill (3.8)

where $\lambda$ is a constant spectral parameter. For the first flow, we denote the time-evolving operator as $B_2$ instead of $B_1$, because $\partial t$ is proportional to the second argument $du_2$ of the
Jacobi’s inversion relationship of Eq.(2.44). Eq.(3.4) gives Eqs.(2.1)–(2.6). In constructing a Lax pair of $L$ and $B_2$, eight Lie algebra elements

$$(I_1 + I_2 + I_3 + I_4), \quad (I_1 - I_2 - I_3 + I_4), \quad (I_1 - I_3), \quad (I_2 - I_4),$$

$$(I_5 - I_6), \quad (I_5 + I_6), \quad (I_7 - I_8), \quad (I_8 - I_9), \quad I_{10},$$

emerge, i.e., $(I_1 + I_9), (I_2 + I_8), (I_6 - I_7), (I_8 - I_9), I_3, I_4, I_5,$ and $I_{10}$ are necessary. Then, in the case of the first flow, we cannot conclude that all ten bases of $Sp(4, R)$ Lie algebra are necessary.

Next, we construct the Lax pair of the second flow. The operator $L$ is the same as Eq.(3.5). Assuming that the second flow can be written by the basis of the $Sp(4, R)$ Lie algebra, we rearrange $B_1$ in the form

$$B_1 = \begin{pmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & -a_{11} & a_{23} & a_{24} \\
  -a_{24} & a_{14} & a_{33} & a_{34} \\
  a_{23} & -a_{13} & a_{43} & -a_{33}
\end{pmatrix}
$$

(3.9)

$$= a_{11}I_3 + a_{12}I_1 + a_{13}I_4 + a_{14}I_6 + a_{21}I_2$$

$$+ a_{23}I_7 + a_{24}I_5 + a_{33}I_{10} + a_{34}I_8 + a_{43}I_9.
$$

(3.10)

The Lax equation for the second flow is given by

$$\frac{\partial L}{\partial t} = [B_1, L].
$$

(3.11)

To ensure that this Lax equation gives Eqs.(2.54)–(2.59), ten independent linear relationships for ten variables $a_{11}, a_{12}, a_{13}, a_{14}, a_{21}, a_{23}, a_{24}, a_{33}, a_{34},$ and $a_{43}$ must be satisfied. These ten equations to be satisfied are provided in Appendix A. However, the expressions of the solutions are too long, and the entire expressions cannot be presented in this paper; hence, we present only the first few terms in Appendix A. To achieve our, it is not necessary to know the explicit form of $B_1$; however it is sufficient that there really exist solutions. By using these solutions $a_{ij}$’s, we have confirmed that the Lax equation $\partial_t L = [B_1, L]$ is completely satisfied by using the second flow equations Eqs.(2.54)–(2.59). As $a_{23} = -1, a_{24} = -2$, we have $a_{24} = 2a_{23};$ however other $a_{ij}$’s are independent, such that we need nine bases

$$I_1, \ I_2, \ I_3, \ I_4, \ I_6, \ I_8, \ I_9, \ I_{10}, \ (I_7 + 2I_5),$$

(3.12)

for the Lax pair of the second flow. However, for the first flow, $I_5$ is required. Then $I_7$ itself becomes necessary in the combination of both flows. This fact implies that all of the ten $Sp(4, R)/Z_2 \cong SO(3, 2)$ Lie algebra bases are necessary and sufficient to construct Lax pairs of the first and second flows for two-flows Kowalevski top. It is important that coefficients of all ten $Sp(4, R)/Z_2$ Lie algebra bases are independent. If some coefficients of the Lie algebra bases are not independent, there is a possibility that the structure of the Lie algebra of two flow Kowalevski top reduces to some Lie subalgebra of the $Sp(4, R)/Z_2$ Lie algebra. In the Lax formalism, it is quite non-trivial and is quite difficult to find the necessary and sufficient Lie algebra to obtain the two-flows Kowalevski top. In such a situation, we say that the two-flows Kowalevski top has the Lie group structure.

On one hand, we demonstrated that the two-flows Kowalevski top is equivalent to the full genus two Jacobi’s inversion problem in Section 2. On the other hand, we proved that the Lie group structure of Lax pairs of two-flows Kowalevski top is $Sp(4, R)/Z_2 \cong SO(3, 2)$ in Section 3. Combining these results, we conclude that the genus two hyperelliptic function has the $Sp(4, R)/Z_2 \cong SO(3, 2)$ structure.
4 Summary and Discussions

First, we reviewed the Kowalevski top to explain the formulation and notation, as it is quite prevalent, yet was first introduced more than one hundred years ago. The equations of the original Kowalevski top, or the first flow, can be formulated into the special genus two Jacobi’s inversion problem. In addition to the first flow, we have adopted the second flow, i.e., other equations of the Kowalevski top by introducing another time \( r \). The first and second flows satisfy the integrability condition. In addition, we demonstrated that the two-flows Kowalevski top is formulated into the full genus two Jacobi’s inversion problem.

Next, in addition to the Lax pair of the first flow for the Kowalevski top, we constructed the Lax pair of the second flow for the Kowalevski top. Then, we deduced the \( \text{Sp}(4, \mathbb{R})/\mathbb{Z}_2 \cong \text{SO}(3,2) \) Lie group structure of Lax pairs of the first and second flows. Combining these results, using the two-flows Kowalevski top, we infer that the genus two hyperelliptic function, which is the solution of the full genus two Jacobi’s inversion problem has the \( \text{Sp}(4, \mathbb{R})/\mathbb{Z}_2 \cong \text{SO}(3,2) \) Lie group structure.

The Lie group structure is \( \text{Sp}(2, \mathbb{R}) \) for the genus one elliptic function and \( \text{Sp}(4, \mathbb{R}) \) for the genus two hyperelliptic function, which resembles the Siegel’s discrete \( \text{Sp}(2g, \mathbb{Z}) \) modular transformation for the genus \( g \) hyperelliptic theta functions [39].

Appendix A Ten equations for \( a_{ij} \)'s in the second flow Lax equations and expression of the solution

Ten equations for \( a_{ij} \)'s in the second flow Lax equations are given in the form

1) \[
\begin{align*}
\gamma_2 c_0 + a_{13}(\gamma_3 c_0 + 2\omega_2 \lambda) + 2a_{14}\omega_1 \lambda - a_{21}\gamma_2 c_0 + 2a_{23}\omega_1 \lambda + a_{24}(\omega_3 c_0 - 2\omega_2 \lambda) & \\
- \gamma_3 c_0(\gamma_1 \omega_2 - \gamma_2 \omega_1) + \gamma_3 \omega_3(\omega_1^2 + \omega_2^2) & = 0,
\end{align*}
\] (A.1)

2) \[
\begin{align*}
2a_{11}\gamma_2 c_0 - 2a_{12}\gamma_1 c_0 - 4a_{13}\omega_1 \lambda - 2a_{14}(\gamma_3 c_0 - 2\omega_2 \lambda) & - \gamma_3 c_0(\gamma_1 \omega_1 + 2\omega_2) & \\
- \gamma_3 \omega_1(\omega_1^2 + \omega_2^2) & = 0,
\end{align*}
\] (A.2)

3) \[
\begin{align*}
a_{11}c_0(\gamma_3 c_0 - 2\omega_2 \lambda) + 2a_{12}\omega_1 c_0 - 2a_{13}c_0(\gamma_1 c_0 + 2\lambda^2) & + a_{14}c_0(\gamma_2 c_0 + 2\lambda) & \\
- a_{23}\gamma_2 c_0^2 & - a_{33}c_0(\omega_3 c_0 - 2\omega_2 \lambda) + 2a_{34}\omega_1 c_0 \lambda & - \omega_2 c_0(\gamma_1^2 + \gamma_2^2) & + \omega_3 c_0(\gamma_1 \omega_1 + 2\omega_2) & \\
- (\gamma_1 \omega_2 c_0 - 2\gamma_2 c_0 - \omega_1 \omega_3 \lambda)(\omega_1^2 + \omega_2^2) & + \gamma_3 c_0(\gamma_1 c_0 + \omega_1^2 - \omega_2^2) & = 0,
\end{align*}
\] (A.3)

4) \[
\begin{align*}
2a_{11}\omega_1 c_0 - a_{12}c_0(\gamma_3 c_0 + 2\omega_2 \lambda) + a_{13}c_0(\gamma_2 c_0 - 2\omega_3 \lambda) & + 4a_{14}c_0 \lambda^2 & - a_{21}\gamma_2 c_0 & \\
- 2a_{33}\omega_1 c_0 \lambda & - a_{34}c_0(\gamma_3 c_0 - 2\omega_2 \lambda) & - \omega_3 c_0(\gamma_1 \omega_2 - \gamma_2 \omega_1) & - \gamma_3 c_0(\gamma_2 c_0 + 2\omega_1 \omega_2) & \\
- \omega_2 \omega_3 \lambda(\omega_1^2 + \omega_2^2) & = 0,
\end{align*}
\] (A.4)

5) \[
\begin{align*}
- 2a_{11}\gamma_2 c_0 + 2a_{21}\gamma_1 c_0 + 2a_{23}(\omega_3 c_0 + 2\omega_2 \lambda) & + 4a_{24}\omega_1 \lambda - \gamma_3 c_0(\gamma_1 \omega_1 + 2\omega_2) & \\
- \gamma_3 \omega_1(\omega_1^2 + \omega_2^2) & = 0,
\end{align*}
\] (A.5)

6) \[
\begin{align*}
- 2a_{11}\omega_1 c_0 - a_{13}\gamma_2 c_0^2 & + a_{23}c_0(\gamma_3 c_0 - 2\omega_2 \lambda) & - 4a_{23}c_0 \lambda^2 & + a_{24}c_0(\gamma_2 c_0 + 2\omega_3 \lambda) & \\
- 2a_{33}\omega_1 c_0 \lambda & + a_{34}c_0(\gamma_3 c_0 + 2\omega_2 \lambda) & - \omega_3 c_0(\gamma_1 \omega_2 - \gamma_2 \omega_1) & + \gamma_3 c_0(\gamma_2 c_0 + 2\omega_1 \omega_2) & \\
+ \omega_2 \omega_3 \lambda(\omega_1^2 + \omega_2^2) & = 0,
\end{align*}
\] (A.6)

7) \[
\begin{align*}
a_{11}c_0(\gamma_3 c_0 + 2\omega_2 \lambda) - a_{14}\gamma_2 c_0^2 & - 2a_{21}\omega_1 c_0 \lambda & + a_{23}c_0(\gamma_2 c_0 - 2\omega_3 \lambda) & + a_{24}(\gamma_1 c_0 + 2\lambda) & \\
- a_{33}c_0(\omega_3 c_0 + 2\omega_2 \lambda) & - 2a_{34}\omega_1 c_0 \lambda & - \omega_2 c_0(\gamma_1^2 + \gamma_2^2) & + \omega_3 c_0(\gamma_1 \omega_1 + 2\omega_2) & \\
+ (\gamma_1 \omega_2 c_0 - 2\gamma_2 c_0 - \omega_1 \omega_3 \lambda)(\omega_1^2 + \omega_2^2) & + \gamma_3 c_0(\gamma_1 c_0 + \omega_1^2 - \omega_2^2) & = 0,
\end{align*}
\] (A.7)

8) \[
\begin{align*}
- a_{13}(\omega_3 c_0 + 2\omega_2 \lambda) & + 2a_{14}\omega_1 \lambda & + 2a_{23}\omega_1 \lambda - a_{24}(\omega_3 c_0 - 2\omega_2 \lambda) & + a_{34}(\gamma_2 c_0 + 2\lambda) & \\
- a_{43}(\gamma_2 c_0 - 2\omega_3 \lambda) & + \gamma_3 c_0(\gamma_1 \omega_2 - \gamma_2 \omega_1) & - \gamma_3 \omega_2(\omega_1^2 + \omega_2^2) & = 0,
\end{align*}
\] (A.8)

9) \[
\begin{align*}
- 2a_{14}(\gamma_3 c_0 + 2\omega_2 \lambda) & + 4a_{24}\omega_1 \lambda & + 2a_{33}(\gamma_2 c_0 - 2\omega_3 \lambda) & + 2a_{34}(\gamma_1 c_0 + 4\lambda^2)
\end{align*}
\]
\[-(\gamma_3 c_0 + 2\omega_2 \lambda)(\gamma_1 \omega_1 + \gamma_2 \omega_2) - 2\omega_1 \lambda(\gamma_1 \omega_2 - \gamma_2 \omega_1) - \gamma_3 \omega_1 (\omega_1^2 + \omega_2^2) = 0, \quad (A.9)\]

\[-4a_{13} \omega_1 \lambda + 2a_{23}(\gamma_3 c_0 - 2\omega_2 \lambda) - 2a_{33}(\gamma_2 c_0 + 2\omega_3 \lambda) - 2a_{43}(\gamma_1 c_0 + 4\lambda^2)\]
\[-(\gamma_3 c_0 - 2\omega_2 \lambda)(\gamma_1 \omega_1 + \gamma_2 \omega_2) + 2\omega_1 \lambda(\gamma_1 \omega_2 - \gamma_2 \omega_1) - \gamma_3 \omega_1 (\omega_1^2 + \omega_2^2) = 0. \quad (A.10)\]

The first few terms of \(a_{ij}\)'s are given by

1) \[a_{11} = \frac{1}{2c_0 D}(16\gamma_1^5 \gamma_3 \omega_1^2 c_0^5 + 32\gamma_1^5 \omega_1^2 \omega_2 c_0^4 \lambda + 32\gamma_1^4 \gamma_2 \gamma_3 \omega_1 \omega_2 c_0^5 + \cdots), \quad (A.11)\]

2) \[a_{12} = \frac{1}{2c_0 D}(16\gamma_1^4 \gamma_2 \gamma_3 \omega_1^2 c_0^5 + 32\gamma_1^4 \omega_1^2 \omega_2 c_0^4 \lambda + 32\gamma_1^3 \gamma_2^2 \gamma_3 \omega_1 \omega_2 c_0^5 + \cdots), \quad (A.12)\]

3) \[a_{13} = \frac{1}{D}(8\gamma_1^4 \gamma_3 \omega_1^2 c_0^4 - 32\gamma_1^4 \omega_1^2 \omega_2 c_0^2 \lambda^2 + 16\gamma_1^3 \gamma_2 \gamma_3^2 \omega_1 \omega_2 c_0^4 + \cdots), \quad (A.13)\]

4) \[a_{14} = \frac{1}{D}(-32\gamma_1^3 \gamma_3 \omega_1^2 \omega_2 \omega_3 c_0^3 \lambda^2 - 64\gamma_1^2 \omega_1^2 \omega_2^2 \omega_3 c_0^3 \lambda^3 + 32\gamma_1^2 \gamma_2^2 \gamma_3^2 \omega_1^2 \omega_2 c_0^3 + \cdots), \quad (A.14)\]

5) \[a_{21} = \frac{1}{2c_0 D}(16\gamma_1^4 \gamma_2 \gamma_3 \omega_1^2 c_0^5 + 32\gamma_1^4 \gamma_2 \omega_1^2 \omega_2 c_0^4 \lambda + 32\gamma_1^3 \gamma_2^2 \gamma_3 \omega_1 \omega_2 c_0^5 + \cdots), \quad (A.15)\]

6) \[a_{23} = -1, \quad (A.16)\]

7) \[a_{24} = -2, \quad (A.17)\]

8) \[a_{33} = \frac{1}{2c_0 D}(-16\gamma_1^5 \gamma_3 \omega_1^2 c_0^5 - 32\gamma_1^5 \omega_1^2 \omega_2 c_0^4 \lambda - 32\gamma_1^4 \gamma_2 \gamma_3 \omega_1 \omega_2 c_0^5 + \cdots), \quad (A.18)\]

9) \[a_{34} = \frac{1}{2c_0 D}(16\gamma_1^4 \gamma_2 \gamma_3 \omega_1^2 c_0^5 + 32\gamma_1^4 \gamma_2 \omega_1^2 \omega_2 c_0^4 \lambda - 32\gamma_1^3 \gamma_2^2 \gamma_3 \omega_1 \omega_2 c_0^5 + \cdots), \quad (A.19)\]

10) \[a_{43} = \frac{1}{2c_0 D}(16\gamma_1^4 \gamma_2 \gamma_3 \omega_1^2 c_0^5 + 32\gamma_1^4 \gamma_2 \omega_1^2 \omega_2 c_0^4 \lambda + 32\gamma_1^3 \gamma_2^2 \gamma_3 \omega_1 \omega_2 c_0^5 + \cdots), \quad (A.20)\]

with

\[D = 4\gamma_1^3 \gamma_3^2 \omega_1^2 c_0^4 + 16\gamma_1^2 \gamma_3 \omega_1^2 \omega_2 c_0^3 \lambda + 16\gamma_1^2 \omega_1^2 \omega_2 c_0^2 \lambda^2 + \cdots. \quad (A.21)\]

It appears that \(a_{12} = a_{21}\) and \(a_{33} = -a_{11}\) hold at first glance; however, they are actually different. Namely, \(a_{12} \neq a_{21}\) and \(a_{33} \neq -a_{11}\).
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