Bifurcation analysis and chaos control for a plant–herbivore model with weak predator functional response

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ABSTRACT
The interaction between plants and herbivores is one of the most fundamental processes in ecology. Discrete-time models are frequently used for describing the dynamics of plants and herbivores interaction with non-overlapping generations, such that a new generation replaces the old at regular time intervals. Keeping in view the interaction of the apple twig borer and the grape vine, the qualitative behaviour of a discrete-time plant–herbivore model is investigated with weak predator functional response. The topological classification of equilibria is investigated. It is proved that the boundary equilibrium undergoes transcritical bifurcation, whereas unique positive steady-state of discrete-time plant–herbivore model undergoes Neimark–Sacker bifurcation. Numerical simulation is provided to strengthen our theoretical discussion.

1. Introduction

In mathematical biology, plant–herbivore models are basically modifications of prey–predator models [8]. The interaction between plants and herbivores has been investigated by many researchers both in differential and difference equations. Kartal [31] investigated the dynamical behaviour of a plant–herbivore model including both differential and difference equations. In [30] Kang et al. discussed bistability, bifurcation and chaos control in a discrete-time plant–herbivore model. In [38] Liu et al. investigated stability, limit cycle, Neimark–Sacker bifurcations and homoclinic bifurcation for a plant–herbivore model with toxin-determined functional response. Li et al. [37] discussed period-doubling and Neimark–Sacker bifurcations for a plant–herbivore model incorporating plant toxicity in the functional response of plant–herbivore interactions. Similarly, for some other discussions related to qualitative behaviour of plant–herbivore models, the interested reader is referred to [2,10,21,25–27,29,33,45,47,52] and references are therein.

Taking into account the interaction of the apple twig borer and the grape vine [3], a mathematical model for plant–herbivore interaction based on weak predator functional
response is given by
\begin{align}
x_{n+1} &= \frac{x_n}{\alpha(1 + y_n^2) + \beta x_n}, \\
y_{n+1} &= \gamma y_n(1 + x_n),
\end{align}
(1)
where $x_n$ and $y_n$ are population densities for grapevine and apple twig borer, respectively. Moreover, $\alpha$, $\beta$ and $\gamma$ are positive parameters. For some earlier investigations related to model (1), we refer to [4–7,34,35]. Moreover, taking into account strong predator functional response, Din [12] discussed global stability of system (1), and Khan et al. [32] investigated Neimark–Sacker bifurcation for system (1) with strong predator functional response.

The main findings of this paper are summarized as follows:

- Existence of equilibria and their stability analysis is investigated.
- It is proved that model (1) undergoes transcritical bifurcation at its boundary equilibrium by implementing bifurcation theory of normal forms and centre manifold theorem.
- Direction and existence criteria for Neimark–Sacker bifurcation are investigated at positive steady-state of plant–herbivore model (1).
- Pole-placement and hybrid control methods are implemented in order to discuss chaos control in system (1).

Moreover, in Section 2 the existence of steady-states and their local asymptotic behaviour is analysed. Section 3 is dedicated to investigate transcritical bifurcation about boundary steady-state of system (1). In Section 4, we discuss Neimark–Sacker bifurcation at positive steady-state of model (1). We study pole-placement chaos control and hybrid control strategies in Section 5. Finally, in Section 6, numerical simulations are carried out to authenticate the theoretical discussion.

2. Existence of equilibria and stability

In this section, first we investigate fixed points for system (1). For this, these fixed points must solve the following algebraic system:
\begin{align}
x &= \frac{x}{\alpha(1 + y^2) + \beta x}, \\
y &= \gamma (1 + x)y.
\end{align}
(2)
Simple computation yields the following steady-states for the plant–herbivore model (1):
\[
P_0 = (0, 0), \quad P_1 = \left( \frac{1 - \alpha}{\beta}, 0 \right), \quad P_* = \left( \frac{1 - \gamma}{\gamma}, \sqrt{\gamma(1 + \beta) - (\alpha \gamma + \beta)} \right) \frac{1 - \gamma}{\alpha \gamma}.
\]
Assume that $0 < \alpha < 1$, $\beta > 0$, $\beta/(1 - \alpha + \beta) < \gamma < 1$, then $P_*$ is unique positive equilibrium point of the system (1). For $\beta = 0.2$, the region in $\alpha \gamma$-plane where inequalities $0 < \alpha < 1$, $\beta/(1 - \alpha + \beta) < \gamma < 1$ are satisfied is depicted in Figure 1 by red shaded region. In order to see the dynamical behaviour of system (1) at equilibria, we first compute
Figure 1. Existence region (red) for $P_*$ at $\beta = 0.2$.

Jacobian matrix of system (1) at $(x, y)$ as follows:

$$
J(x, y) := \begin{pmatrix}
\frac{\alpha (1 + y^2)}{(\alpha(1 + y^2) + \beta x)^2} \gamma y & -\frac{2\alpha xy}{(\alpha(1 + y^2) + \beta x)^2} \\
\gamma (1 + x) & 0
\end{pmatrix}.
$$

(3)

Then, at $(x, y) = (0, 0)$ the Jacobian matrix $J(x, y)$ given in (3) reduces to

$$
J(0, 0) := \begin{pmatrix}
\frac{1}{\alpha} & 0 \\
0 & \gamma
\end{pmatrix}.
$$

Then topological classification of $P_0$ is given as follows:

- $P_0$ is a sink if and only if $\alpha > 1$ and $0 < \gamma < 1$.
- $P_0$ is a saddle point if and only if $0 < \alpha < 1$ and $0 < \gamma < 1$, or $\alpha > 1$ and $\gamma > 1$.
- $P_0$ is a source if and only if $0 < \alpha < 1$ and $\gamma > 1$.
- $P_0$ is a non-hyperbolic point if and only if $\alpha = 1$, or $\gamma = 1$.

Next, for $\alpha \in [0, 2]$ and $\gamma \in [0, 2]$ the aforementioned topological classification for $P_0$ is depicted in Figure 2. Moreover, for the existence of boundary equilibrium $P_1$, we assume
that $0 < \alpha < 1$, then the Jacobian matrix $J(x, y)$ at $P_1$ is computed as follows:

$$J(P_1) := \begin{pmatrix} \alpha & 0 \\ 0 & \frac{\gamma(1 - \alpha + \beta)}{\beta} \end{pmatrix}.$$

Furthermore, topological classification for boundary equilibrium $P_1$ is itemized as follows:

- $P_1$ is a sink if and only if $0 < \alpha < 1$ and $0 < \gamma < \beta/(1 - \alpha + \beta)$.
- $P_1$ is a saddle point if and only if $0 < \alpha < 1$ and $\gamma > \beta/(1 - \alpha + \beta)$.
- $P_1$ is a non-hyperbolic point if and only if $\gamma = \beta/(1 - \alpha + \beta)$.

Furthermore, for $\alpha \in [0, 1]$, $\beta \in [0, 200]$ and $\gamma = 0.995$ the topological classification of $P_1$ is depicted in Figure 3. At the end of this section, we explore dynamics of model (1) at its unique positive steady-state $P_\ast$. For this, first Jacobian matrix at this equilibrium is computed as follows:

$$J(P_\ast) := \begin{pmatrix} 1 + \beta - \frac{\beta}{\gamma} & 2(\gamma - 1)\sqrt{\alpha(\beta(\gamma - 1) + \gamma - \alpha\gamma)} \\ \frac{\gamma}{\alpha} & \frac{\gamma^{3/2}}{\sqrt{\gamma(\beta(\gamma - 1)+\gamma-\alpha\gamma)}} \end{pmatrix}.$$
Moreover, characteristic polynomial for $J(P_*)$ is computed as follows:

$$ F(\lambda) = \lambda^2 - \left(2 - \beta \left(\frac{1}{\gamma} - 1\right)\right) \lambda + 3 + 5\beta - 2\alpha(1 - \gamma) - \frac{3\beta}{\gamma} - 2\gamma(1 + \beta). \quad (4) $$

From (4) we have

$$ F(1) = \frac{2(1 - \gamma)(\beta(\gamma - 1) + \gamma - \alpha\gamma)}{\gamma} $$

and

$$ F(-1) = 6 + 6\beta - 2\alpha(1 - \gamma) - \frac{4\beta}{\gamma} - 2(1 + \beta)\gamma. $$

Assume that $0 < \alpha < 1$, $\beta > 0$, and $\beta/(1 - \alpha + \beta) < \gamma < 1$, then from first part of last inequality we have $\beta < \gamma(1 - \alpha + \beta)$, or equivalently $\beta(\gamma - 1) + \gamma - \alpha\gamma > 0$. Hence, it follows that $F(1) > 0$. Similarly, one can prove that $F(-1) > 0$ under the existence conditions for positive steady-state $P_*$ of system (1). That is, $F(-1) > 0$ for $0 < \alpha < 1$, $\beta > 0$, and $\beta/(1 - \alpha + \beta) < \gamma < 1$. Indeed, it follows that

$$ F(-1) = 6 + 6\beta - 2\alpha(1 - \gamma) - \frac{4\beta}{\gamma} - 2(1 + \beta)\gamma $$

$$ = 2 + 6\beta - 2\alpha(1 - \gamma) - \frac{4\beta}{\gamma} - 2(1 + \beta)\gamma + 4 $$

$$ = 2(1 - \gamma)(1 - \alpha + \beta) - 4(1 - \gamma) \left(\frac{\beta}{\gamma}\right) + 4 $$

$$ > 2(1 - \gamma) \left(\frac{\beta}{\gamma}\right) - 4(1 - \gamma) \left(\frac{\beta}{\gamma}\right) + 4 $$

$$ = 2 \left(2 + \beta - \frac{\beta}{\gamma}\right) > 2(1 + \alpha) > 0. $$

Therefore, according to Jury condition roots of $F(\lambda) = 0$ inside the unit open disk if and only if $F(0) < 1$, which on simplification gives that $2\alpha + \beta > 2$, or $\gamma < 3\beta/(2 - 2\alpha + 2\beta)$ and $2\alpha + \beta \leq 2$. Similarly, $F(0) > 1$ if and only if $2\alpha + \beta < 2$ and $3\beta/(2 - 2\alpha + 2\beta) < \gamma$. Moreover, $F(0) = 1$ if and only if $\gamma = 3\beta/(2 - 2\alpha + 2\beta)$ and $2\alpha + \beta < 2$. Due to aforementioned computations, we have the following results:

**Lemma 2.1:** Assume that $0 < \alpha < 1$, $\beta > 0$, and $\beta/(1 - \alpha + \beta) < \gamma < 1$, then the following conditions hold true:

- $P_*$ is a sink if and only if $2\alpha + \beta > 2$, or $\gamma < 3\beta/(2 - 2\alpha + 2\beta)$ and $2\alpha + \beta \leq 2$.
- $P_*$ is a source if and only if $2\alpha + \beta < 2$ and $3\beta/(2 - 2\alpha + 2\beta) < \gamma$.
- $P_*$ is a non-hyperbolic fixed point if and only if $\gamma = 3\beta/(2 - 2\alpha + 2\beta)$ and $2\alpha + \beta < 2$.

In Figure 4, topological classification of $P_*$ is depicted for $0 < \alpha < 1$, $0 < \gamma < 1$ and $\beta = 0.1$.
Figure 3. Topological classification of \( P_1 \) for \( \alpha \in [0, 1], \beta \in [0, 200] \) and \( \gamma = 0.995 \).

3. Transcritical bifurcation

In this section, we investigate that boundary equilibrium \( P_1 \) undergoes transcritical bifurcation. For this, we assume that

\[ \gamma \equiv \gamma_0 := \frac{\beta}{1 - \alpha + \beta}. \]

Consider the set \( \Omega_{TB} \) given by

\[ \Omega_{TB} := \left\{ (\alpha, \beta, \gamma_0) \in \mathbb{R}_+^3 : \gamma_0 = \frac{\beta}{1 - \alpha + \beta}, \ 0 < \alpha < 1, \ \beta > 0 \right\}. \]

Assume that \( (\alpha, \beta, \gamma_0) \in \Omega_{TB} \), then system (1) is equivalently represented by the following 2-dimensional map:

\[
\begin{pmatrix}
u \\
u
\end{pmatrix} \rightarrow \begin{pmatrix} u \\ (\gamma_0 + \gamma) v(1 + u) \end{pmatrix},
\]

(5)
Figure 4. Topological classification of $P_\ast$ for $0 < \alpha < 1$, $0 < \gamma < 1$ and $\beta = 0.1$.

where $\bar{\gamma}$ is very small perturbation in $\gamma_0$. Furthermore, if we consider $x = u - (1 - \alpha)/\beta$ and $y = v$, then map (5) is transformed into the following map:

$$
\begin{pmatrix}
    x \\
    y
\end{pmatrix}
\rightarrow
\begin{pmatrix}
    x + \frac{1-\alpha}{\beta} \\
    \alpha(1+y^2) + \beta \left( x + \frac{1-\alpha}{\beta} \right) \\
    (\gamma_0 + \bar{\gamma})y \left( 1 + x + \frac{1-\alpha}{\beta} \right)
\end{pmatrix}.
$$

(6)

An application of Taylor series expansion about $(x, y, \bar{\gamma}) = (0, 0, 0)$ yields that

$$
\begin{pmatrix}
    x \\
    y
\end{pmatrix}
\rightarrow
\begin{pmatrix}
    \alpha \\
    0
\end{pmatrix}
\begin{pmatrix}
    0 \\
    \gamma_0(1 - \alpha + \beta)
\end{pmatrix}
\begin{pmatrix}
    x \\
    y
\end{pmatrix}
+ \begin{pmatrix}
    f(x, y) \\
    g(x, y, \bar{\gamma})
\end{pmatrix},
$$

(7)

where

$$
f(x, y) = -\alpha \beta x^2 - \frac{\alpha(1-\alpha)}{\beta} y^2 + \alpha \beta^2 x^3 + \alpha(1-2\alpha)xy^2 + O\left( (|x| + |y|)^4 \right)
$$

and

$$
g(x, y, \bar{\gamma}) = \gamma_0 xy + \left( \frac{1-\alpha + \beta}{\beta} \right) y \bar{\gamma} + xy \bar{\gamma}.
$$
Since at $\gamma_0 = \beta / (1 - \alpha + \beta)$ the linear part of map (7) is already in canonical form. Therefore, in order to implement the centre manifold theorem [9], let $W^C(0, 0, 0)$ denotes the centre manifold for the map (7) which is evaluated at $(0, 0)$ in a small neighbourhood of $\bar{\gamma} = 0$. Then, $W^C(0, 0, 0)$ is computed as follows:

$$W^C(0, 0, 0) = \left\{ (x, y, \bar{\gamma}) \in \mathbb{R}^3 : y = m_1 x^2 + m_2 x \bar{\gamma} + m_3 \bar{\gamma}^2 + O\left((|x| + |\bar{\gamma}|)^3\right) \right\},$$

where

$$m_1 = m_2 = m_3 = 0.$$ 

Moreover, we define the following mapping which is restricted to the centre manifold $W^C(0, 0, 0)$:

$$F : x \rightarrow x + \bar{\gamma} + s_1 x^2 + s_2 x \bar{\gamma} + s_3 \bar{\gamma}^2 + O\left((|x| + |\bar{\gamma}|)^3\right),$$

where

$$s_1 = -\alpha \beta, \quad s_2 = s_3 = 0.$$ 

Furthermore, it follows that

$$F(0, 0) = 0, \quad F_x(0, 0) = 1, \quad F_{\bar{\gamma}}(0, 0) = 1, \quad F_{xx}(0, 0) = -2 \alpha \beta < 0.$$ 

Then, the following theorem gives the parametric conditions for existence and direction of transcritical bifurcation for system (1) at its boundary fixed point $P_1$.

**Theorem 3.1:** Suppose that $\gamma = \beta / (1 - \alpha + \beta)$ and $0 < \alpha < 1$, then system (1) undergoes transcritical bifurcation at its boundary steady-state $P_1$ when parameter $\gamma$ varies in a small neighbourhood of $\gamma_0 = \beta / (1 - \alpha + \beta)$. Furthermore, two steady-states bifurcate from $P_1$ for $\gamma < \gamma_0$, and merge as the steady-state $P_1$ at $\gamma = \gamma_0$ and disappear at $\gamma > \gamma_0$.

### 4. Neimark–Sacker bifurcation

In this section, we study that unique positive steady-state of system (1) undergoes Neimark–Sacker bifurcation. For this, necessary conditions for existence of Neimark–Sacker bifurcation at $P_*$ are given as follows:

$$\gamma = \frac{3 \beta}{2 - 2 \alpha + 2 \beta}, \quad 2 \alpha + \beta < 2.$$ 

Next, we consider the following set:

$$S_{NB} := \left\{ (\alpha, \beta, \gamma_1) \in \mathbb{R}^3_+ : \gamma_1 = \frac{3 \beta}{2 - 2 \alpha + 2 \beta}, \quad 2 \alpha + \beta < 2, \quad 0 < \alpha < 1, \quad \beta > 0 \right\}.$$ 

Assume that $(\alpha, \beta, \gamma_1) \in S_{NB}$, and $\tilde{\gamma}$ be small perturbation in $\gamma_1$, then system (1) can be expressed by the following two-dimensional perturbed map:

$$\begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} u \\ \alpha (1 + v^2) + \beta u \\ (\gamma_1 + \tilde{\gamma}) v (1 + u) \end{pmatrix}. \quad (8)$$
In order to translate the unique positive fixed point

\[
\begin{pmatrix}
1 - (\gamma_1 + \gamma) \\
\gamma_1 + \gamma
\end{pmatrix},
\sqrt{\frac{(\gamma_1 + \gamma)(1 + \beta) - (\alpha(\gamma_1 + \gamma) + \beta)}{\alpha(\gamma_1 + \gamma)}}
\]

of the perturbed map (8) at origin, we consider the following translations:

\[
x = u - \frac{1 - (\gamma_1 + \gamma)}{\gamma_1 + \gamma},
y = v - \sqrt{\frac{(\gamma_1 + \gamma)(1 + \beta) - (\alpha(\gamma_1 + \gamma) + \beta)}{\alpha(\gamma_1 + \gamma)}}.
\]

Then, from (8) and (9) it follows that

\[
\begin{pmatrix}
x \\
y
\end{pmatrix}
\to
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
+ \begin{pmatrix}
f_2(x, y) \\
g_2(x, y)
\end{pmatrix},
\]

where

\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
= \left(\begin{array}{c}
1 + \beta - \frac{\beta}{\gamma_1 + \gamma} \\
\sqrt{\frac{(\gamma_1 + \gamma)(\beta(\gamma_1 + \gamma) - \alpha(\gamma_1 + \gamma))}{\alpha}}
\end{array}\right)
\times
\left(\begin{array}{c}
\frac{2((\gamma_1 + \gamma)-1)\sqrt{\alpha(\beta(\gamma_1 + \gamma)-1)+\alpha(\gamma_1 + \gamma)}}{(\gamma_1 + \gamma)^{3/2}}
\end{array}\right),
\]

\[
f_2(x, y) = -\left(\frac{\alpha \beta + \alpha \beta Y^2}{(\alpha + \beta X + \alpha Y^2)^3}\right) x^2
- \left(\frac{2\alpha Y (\alpha - \beta X + \alpha Y^2)}{(\alpha + \beta X + \alpha Y^2)^3}\right) xy
- \left(\frac{X (\alpha^2 + \alpha \beta X - 3 \alpha^2 Y^2)}{(\alpha + \beta X + \alpha Y^2)^3}\right) y^2
+ \left(\frac{\alpha \beta^2 + \alpha \beta^2 Y^2}{(\alpha + \beta X + \alpha Y^2)^4}\right) x^3
+ \left(\frac{2\alpha Y (2 \alpha \beta - \beta^2 X + 2 \alpha \beta Y^2)}{(\alpha + \beta X + \alpha Y^2)^4}\right) x^2 y
+ \left(\frac{\alpha \beta^2 X^2 - \alpha^3 - 8 \alpha^2 \beta X Y^2 + 3 \alpha^3 Y^2 + 2 \alpha^3 Y^2}{(\alpha + \beta X + \alpha Y^2)^4}\right) x y^2
+ \left(\frac{4\alpha (\alpha^2 \beta X Y - \alpha^2 Y^3 + \alpha^3 Y)}{(\alpha + \beta X + \alpha Y^2)^4}\right) y^3
+ O\left((|x| + |y|)^4\right),
\]

\[
g_2(x, y) = (\gamma_1 + \gamma) x y,
X = \frac{1 - (\gamma_1 + \gamma)}{\gamma_1 + \gamma},
\]

\[
Y = \sqrt{\frac{(\gamma_1 + \gamma)(1 + \beta) - (\alpha(\gamma_1 + \gamma) + \beta)}{\alpha(\gamma_1 + \gamma)}}.
\]
Moreover, the characteristic polynomial for \((a_{11} \ a_{12})\) is computed as follows:

\[
P(\rho) = \rho^2 - \left(2 - \beta \left(\frac{1}{\gamma_1 + \tilde{\gamma}} - 1\right)\right) \rho + 3 + 5\beta - 2\alpha(1 - (\gamma_1 + \tilde{\gamma}))
\]

\[
- \frac{3\beta}{\gamma_1 + \tilde{\gamma}} - 2(\gamma_1 + \tilde{\gamma})(1 + \beta).
\]

(11)

Assume that \(\rho(\tilde{\gamma})\) and \(\bar{\rho}(\tilde{\gamma})\) be complex conjugate roots of (11), then a simple computation yields that

\[
|\rho(\tilde{\gamma})| = \sqrt{3 - 2\alpha + 2\beta + 2(\alpha - 1 - \beta)\tilde{\gamma} - \frac{6\beta(1 - \alpha + \beta)}{3\beta + 2(1 - \alpha + \beta)\tilde{\gamma}}}
\]

Furthermore, it follows that \(|\rho(0)| = 1\), but \(\rho^i(0) \neq 1\) for all \(i = 1, 2, 3, 4\) if and only if

\[
- \frac{1}{3}(4 + 2\alpha + \beta) \neq -2, 0, 1, 2.
\]

(12)

Since \(-\frac{1}{3}(4 + 2\alpha + \beta) < 0\) and \((\alpha, \beta, \gamma_1) \in S_{NB}\), therefore \(-\frac{1}{3}(4 + 2\alpha + \beta) \neq -2\). Hence, condition (12) is satisfied. Moreover, it follows that

\[
\left(\frac{d|\rho(\tilde{\gamma})|}{d\tilde{\gamma}}\right)_{\tilde{\gamma}=0} = \frac{(1 - \alpha + \beta)(2 - 2\alpha - \beta)}{3\beta} > 0.
\]

The following similarity transformation is considered in order to convert linear part of (10) into canonical form at \(\tilde{\gamma} = 0\):

\[
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{12} & 0 \\ \vartheta - a_{11} & -\varphi \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix},
\]

(13)

where

\[
\vartheta := \frac{4 + 2\alpha + \beta}{6}, \quad \varphi := \frac{\sqrt{(2 - 2\alpha - \beta)(10 + 2\alpha + \beta)}}{6}
\]

and

\[
a_{11} = \frac{1}{3}(1 + 2\alpha + \beta), \quad a_{12} = \frac{2\alpha\sqrt{\frac{1 - \alpha + \beta}{\alpha}}(2\alpha - 2 + \beta)}{3\sqrt{3}\beta}.
\]

Then, from transformation (13) it follows that

\[
\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{\vartheta - a_{11}} & 0 \\ \varphi a_{12} & \frac{-1}{\varphi} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\]

(14)

Assuming \(\tilde{\gamma} = 0\), then from (10), (13) and (14) we obtain the following normal form of map (10):

\[
\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} \vartheta & -\varphi \\ \varphi & \vartheta \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} + \begin{pmatrix} F(w, z) \\ G(w, z) \end{pmatrix},
\]

(15)
where
\[ F(w, z) := -\left( \frac{3\sqrt{3}\beta \sqrt{\frac{\beta}{1-\alpha+\beta}}}{2\sqrt{\alpha\beta(2-2\alpha-\beta)}} \right) f_2(a_{12}w, (\vartheta - a_{11})w - \varphi z) \]
and
\[ G(w, z) := -\left( \frac{3\left(4\sqrt{\alpha\beta} g_2(a_{12}w, (\vartheta - a_{11})w - \varphi z) \right.}{2\sqrt{\alpha\beta\sqrt{(2-2\alpha-\beta)(10+2\alpha+\beta)}}} \right) \]

Taking into account the bifurcation theory of normal forms \([28,36,43,48,49]\), the first Lyapunov exponent at \((w, z) = (0, 0)\) is computed as follows:
\[ L = -\text{Re} \left( \frac{(1-2\rho(0))\bar{\rho}(0)^2}{1-\rho(0)} \tau_{20}\tau_{11} \right) - \frac{1}{2} |\tau_{11}|^2 - |\tau_{02}|^2 + \text{Re}(\bar{\rho}(0)\tau_{21}), \]
where
\[ \tau_{20} = \frac{1}{8} \left[ F_{ww} - F_{zz} + 2G_{wz} + i(G_{ww} - G_{zz} - 2F_{wz}) \right], \]
\[ \tau_{11} = \frac{1}{4} \left[ F_{ww} + F_{zz} + i(G_{ww} + G_{zz}) \right], \]
\[ \tau_{02} = \frac{1}{8} \left[ F_{ww} - F_{zz} - 2G_{wz} + i(G_{ww} - G_{zz} + 2F_{wz}) \right], \]
and
\[ \tau_{21} = \frac{1}{16} \left[ F_{www} + F_{wwz} + G_{wwz} + G_{zzz} + i(G_{www} + G_{wzz} - F_{wwz} - F_{zzz}) \right]. \]

Due to aforementioned computations, we have the following theorem:

**Theorem 4.1:** Suppose that \( L \neq 0, 0 < \alpha < 1, \beta/(1-\alpha+\beta) < \gamma < 1 \) and \( 2\alpha + \beta < 2 \), then unique positive equilibrium point \( P_\ast \) of system (1) undergoes Neimark–Sacker bifurcation when the bifurcation parameter \( \gamma \) varies in a small neighbourhood of \( \gamma_1 = \frac{3\beta}{(2-2\alpha+2\beta)} \). Moreover, if \( L < 0 \), then an attracting invariant closed curve bifurcates from the equilibrium point for \( \gamma > \gamma_1 \), and if \( L > 0 \), then a repelling invariant closed curve bifurcates from the equilibrium point for \( \gamma < \gamma_1 \).

## 5. Chaos control

Chaos control is a method of stabilization with the help of small perturbations which are applied to unstable periodic orbits for a given system. The purpose of chaos control is to make chaotic behaviour more predictable and stable. For this, a small perturbation is applied as compare to the original size of the system under consideration, and in a result the natural dynamics of the system is prevented from any major modification. Recently, chaos control for irregular complex dynamics has developed as one of the leading topics in nonlinear applied science. The pioneer articles related to chaos control were published in
1990 and after that their number has been steadily increasing [46]. Moreover, the practical methods related to chaos control can be implemented in various areas such as communications, physics laboratories, biochemistry, turbulence, and cardiology [40]. Furthermore, in the case of dynamical systems related to biological breeding of species, chaos control is considered to be an important feature for investigation. On the other hand, population models with non-overlapping generations have more irregular complex behaviour. For some recent investigation related to chaos control in discrete-time models we refer to [1,13–20,22–24] and references are therein.

In this section, first we discuss pole-placement chaos control method based on state feedback control which was introduced by Romeiras et al. [44] (also see [41]). This method is also known as generalized or modified OGY method proposed by Ott et al. [42]. For the application of pole-placement technique to model (1), one can rewrite this system as follows:

\[
\begin{align*}
    x_{n+1} &= \frac{x_n}{\alpha(1 + y_n^2) + \beta x_n} = f(x_n, y_n, \gamma), \\
    y_{n+1} &= \gamma y_n(1 + x_n) = g(x_n, y_n, \gamma),
\end{align*}
\]

where \(\gamma\) denotes parameter for chaos control. Furthermore, it is assumed that \(\gamma\) lies in a small interval of type \(|\gamma - \gamma_0| < \delta\), where \(\delta > 0\) and \(\gamma_0\) represents the nominal parameter lies in the chaotic region. One can apply pole-placement method in order to move unstable trajectory towards desired stable orbit. For this, suppose that \((x^*, y^*)\) be an unstable fixed point for model (1) lies in a chaotic region under the influence of transcritical or Neimark–Sacker bifurcations. Keeping in view the method of linearization, one can approximate system (16) in the neighbourhood of the unstable fixed point \((x^*, y^*)\) as follows:

\[
\begin{bmatrix}
    x_{n+1} - x^* \\
    y_{n+1} - y^*
\end{bmatrix} \approx A \begin{bmatrix}
    x_n - x^* \\
    y_n - y^*
\end{bmatrix} + B[\gamma - \gamma_0],
\]

where

\[
A = \begin{bmatrix}
    \frac{\partial f(x^*, y^*, \gamma_0)}{\partial x_n} & \frac{\partial f(x^*, y^*, \gamma_0)}{\partial y_n} \\
    \frac{\partial g(x^*, y^*, \gamma_0)}{\partial y_n} & \frac{\partial g(x^*, y^*, \gamma_0)}{\partial y_n}
\end{bmatrix}, \quad B = \begin{bmatrix}
    \frac{\partial f(x^*, y^*, \gamma_0)}{\partial \gamma} \\
    \frac{\partial g(x^*, y^*, \gamma_0)}{\partial \gamma}
\end{bmatrix} = \begin{bmatrix}
    0 \\
    y^*(1 + x^*)
\end{bmatrix}.
\]

In order to check that system (16) is controllable, the following matrix is computed:

\[
C = [B : AB] = \begin{bmatrix}
    0 & \left(\frac{\partial f(x^*, y^*, \gamma_0)}{\partial y_n}\right) (y^*(1 + x^*)) \\
    y^*(1 + x^*) & \left(\frac{\partial g(x^*, y^*, \gamma_0)}{\partial y_n}\right) (y^*(1 + x^*))
\end{bmatrix}.
\]

Assume that \((x^*, y^*)\) is positive fixed point, then it follows that \(y^*(1 + x^*) \neq 0\). Therefore, rank of \(C\) is 2 at positive equilibrium point. Next, we assume that \([\gamma - \gamma_0] = -K \begin{bmatrix}
    x_n - x^* \\
    y_n - y^*
\end{bmatrix}\).
where \( K = [k_1 \ k_2] \), then system (17) can be written as
\[
\begin{bmatrix}
x_{n+1} - x^* \\
y_{n+1} - y^*
\end{bmatrix} \approx [A - BK] \begin{bmatrix}
x_n - x^* \\
y_n - y^*
\end{bmatrix}.
\tag{19}
\]
Moreover, equilibrium point \((x^*, y^*)\) is a sink if and only if both eigenvalues \(\mu_1\) and \(\mu_2\) for the matrix \(A - BK\) lie in an open unit disk. Moreover, \(\mu_1\) and \(\mu_2\) are called regulator poles. Placing these eigenvalues at appropriate position is called pole-placement method. Furthermore, pole-placement problem has unique solution because rank of matrix \(C\) is two. Denote \(\lambda^2 + \alpha_1 \lambda + \alpha_2 = 0\) as characteristic equation for matrix \(A\) and \(\mu^2 + \beta_1 \mu + \beta_2 = 0\) is characteristic equation of \(A - BK\), then unique solution of the pole-placement problem is given as follows:
\[
K = \begin{bmatrix}
\beta_2 - \alpha_2 & \beta_1 - \alpha_1
\end{bmatrix} T^{-1},
\tag{20}
\]
where \(T = CM\) and \(M = \begin{bmatrix}
\alpha_1 & 1 \\
1 & 0
\end{bmatrix}\). Then it follows that
\[
T = CM = \begin{bmatrix}
\frac{\partial f}{\partial y_n} \frac{\partial g}{\partial y} & 0 \\
\alpha_1 \frac{\partial g}{\partial \gamma} + \frac{\partial g}{\partial y_n} \frac{\partial g}{\partial \gamma} & \frac{\partial g}{\partial \gamma}
\end{bmatrix},
\tag{21}
\]
where all partial derivatives in (21) are evaluated at \((x^*, y^*, \gamma_0)\). From (20) and (21), we have the following unique solution of pole-placement problem:
\[
k_1 = \frac{\beta_2 - \alpha_2}{\left(\frac{\partial f(x^*, y^*, \gamma_0)}{\partial y_n}\right) \left(\frac{\partial g(x^*, y^*, \gamma_0)}{\partial c}\right)} - \frac{\beta_1 - \alpha_1}{\left(\frac{\partial f(x^*, y^*, \gamma_0)}{\partial y_n}\right) \left(\frac{\partial g(x^*, y^*, \gamma_0)}{\partial \gamma}\right)}, \quad k_2 = \frac{\beta_1 - \alpha_1}{\frac{\partial g(x^*, y^*, \gamma_0)}{\partial \gamma}}.
\]
Secondly, we apply another comparatively simple chaos control method known as hybrid control strategy based on parameter perturbation and state feedback control [39,50]. An application of hybrid control strategy to model (1) yields the following controlled system:
\[
\begin{align*}
x_{n+1} &= \kappa \left(\frac{x_n}{\alpha(1 + y_n^2) + \beta x_n}\right) + (1 - \kappa) x_n, \\
y_{n+1} &= \kappa \left(\gamma y_n(1 + x_n)\right) + (1 - \kappa) y_n,
\end{align*}
\tag{22}
\]
where \(0 < \kappa < 1\) is control parameter. Assume that \((x^*, y^*)\) be an equilibrium point of system (22), then Jacobian matrix for system (22) computed at \((x^*, y^*)\) is given as follows:
\[
J(x^*, y^*) := \begin{bmatrix}
\frac{\kappa \alpha(1 + (y^*)^2)}{(\alpha(1 + (y^*)^2) + \beta x^*)^2} + 1 - \kappa & -\frac{2\kappa \alpha x^* y^*}{\gamma \kappa (1 + x^*) + 1 - \kappa} \\
\frac{\gamma \kappa (1 + x^*) + 1 - \kappa}{\gamma \kappa (1 + x^*) + 1 - \kappa}
\end{bmatrix}.
\tag{23}
\]
It is easy to observe that system (22) is controllable as long as the steady-state \((x^*, y^*)\) of system (22) is locally asymptotically stable. Keeping in view this fact, one has the following result for controllability of system (22).

**Lemma 5.1:** The equilibrium point \((x^*, y^*)\) of system (22) is locally asymptotically stable if the following holds true:
\[
|\text{Tr} J(x^*, y^*)| < 1 + \det J(x^*, y^*) < 2.
\]
6. Numerical simulation and discussion

First, we choose $\alpha = 0.99, \beta = 0.008, \gamma \in [0.2, 0.7]$ and $(x_0, y_0) = (1.25, 0.001)$. Then, system (1) undergoes transcritical bifurcation at its boundary equilibrium point $(1.25, 0)$ as bifurcation parameter $\gamma$ passes through $\gamma \approx 0.4444$. Furthermore, at $\gamma \approx 0.666667$ the positive steady-state $(0.499999, 0.0778499)$ exists and undergoes Neimark–Sacker bifurcation. At $(\alpha, \beta, \gamma) = (0.99, 0.008, 0.666667)$ the characteristic equation for Jacobian matrix of model (1) is computed as follows:

$$\lambda^2 - 1.996\lambda + 1 = 0.$$ 

Obviously, $\lambda_1 = 0.998 - 0.0632139i$ and $\lambda_2 = 0.998 + 0.0632139i$ be roots for aforementioned characteristic equation with $|\lambda_{1,2}| = 1$. The maximum Lyapunov exponents (MLE) and bifurcation diagrams are depicted in Figure 5. Secondly, we select $\alpha = 0.2, \beta = 0.5, \gamma \in [0.45, 0.95]$ and $(x_0, y_0) = (0.78, 1.426)$, then model (1) undergoes Neimark–Sacker bifurcation at its positive steady-state $(0.733333, 1.47196)$ as bifurcation parameter $\gamma$ varies in a small neighbourhood of $\gamma = 0.576923$. For the parametric values $(\alpha, \beta, \gamma) = (0.2, 0.5, 0.576923)$ the characteristic equation of the Jacobian matrix of system (1) is computed as follows:

$$\lambda^2 - 1.63333\lambda + 1 = 0.$$
Moreover, we have $\lambda_1 = 0.816667 - 0.57711i$ and $\lambda_2 = 0.816667 + 0.57711i$ as roots for aforementioned characteristic equation with $|\lambda_{1,2}| = 1$. MLE and bifurcation diagrams are depicted in Figure 6. On the other hand, some phase portraits for $\gamma = 0.573, 0.576923, 0.58, 0.65$ are depicted in Figure 7.

Finally, we take $\alpha = 0.2$, $\beta = 0.5$ and $\gamma = 0.95$ to verify the effectiveness of chaos control strategies introduced in Section 5. Under this selection of parameters, system (1) has unique positive steady-state given by $(0.0526316, 1.96683)$. An application of pole-placement method gives the following controlled system:

$$
\begin{align*}
    x_{n+1} &= \frac{x_n}{0.2(1 + y_n^2) + 0.5x_n}, \\
y_{n+1} &= (0.95 - k_1(x_n - 0.0526316) - k_2(y_n - 1.96683)) y_n(1 + x_n).
\end{align*}
$$

Then, variational matrix for system (24) is computed as follows:

$$
\begin{pmatrix}
    0.973684 & -0.041407 \\
    1.86849 - 2.07035k_1 & 1 - 2.07035k_2
\end{pmatrix}.
$$
Figure 7. Phase portraits of system (1) for $\alpha = 0.2$, $\beta = 0.5$, $x_0 = 0.78$, $y_0 = 1.426$ and with different values of $\gamma$: (a) phase portrait for $\gamma = 0.573$, (b) phase portrait for $\gamma = 0.576923$, (c) phase portrait for $\gamma = 0.58$ and (d) phase portrait for $\gamma = 0.65$.

Figure 8. Bounded stability region for system (24).
Figure 9. Plots for system (25) with \( \kappa = 0.33 \) and \((x_0, y_0) = (0.0526316, 1.96683)\): (a) plot for \( x_n \), (b) plot for \( y_n \) and (c) phase portrait.

Furthermore, the lines for marginal stability are computed as follows:

\[
L_1 : k_2 = 0.984955 - 0.0209795k_1,
\]
\[
L_2 : k_2 = -1.42005 + 1.57346k_1,
\]

and

\[
L_3 : k_2 = 0.0253254 - 0.0425261k_1.
\]

The triangular stability region bounded by these marginal stability lines is depicted in Figure 8.

Next, for same parametric values we apply hybrid control method to system (1). In this case system (22) is written as follows:

\[
x_{n+1} = \kappa \left( \frac{x_n}{0.2(1 + y_n^2) + 0.5x_n} \right) + (1 - \kappa)x_n,
\]
\[
y_{n+1} = \kappa (0.95y_n(1 + x_n)) + (1 - \kappa)y_n.
\]
Then, Jacobian matrix of system (25) and its characteristic equation are computed as follows:

$$
\begin{bmatrix}
1 - 0.0263158\kappa & -0.041407\kappa \\
1.86849\kappa & 1
\end{bmatrix}
$$

and

$$
\lambda^2 - (2 - 0.0263158\kappa)\lambda + 1 - 0.0263158\kappa + 0.0773684\kappa^2 = 0.
$$

Keeping in view Jury condition and aforementioned characteristic equation, $|2 - 0.0263158\kappa| < 2 - 0.0263158\kappa + 0.0773684\kappa^2 < 2$ if and only if $0 < \kappa < 0.340136$. At $\kappa = 0.33$ the plots for system (25) are depicted in Figure 9.

7. Concluding remarks

Local asymptotic stability, bifurcation analysis and chaos control are investigated for the apple twig borer and the grape vine type plant–herbivore model. The discrete-time model is obtained by taking into account the weak predator functional response. Furthermore, existence of equilibria and their stability analysis is investigated. Taking into account the bifurcating behaviour for biological populations, it must be noted that these phenomena are essential for competition between plants and herbivores [11]. It is proved that plant–herbivore model undergoes transcritical bifurcation at its boundary equilibrium by implementing bifurcation theory of normal forms and centre manifold theorem. Direction and existence criteria for Neimark–Sacker bifurcation are investigated at positive steady-state of plant–herbivore model. Bifurcating behaviour and chaos have always been considered as disadvantageous phenomena in biology. On the other hand, due to chaotic behaviour population can undergo a higher risk of extinction due to the unpredictability. Therefore, these are catastrophic for the breeding of biological population [51]. In order to prevent biological populations from this ruinous situation, one might think about applications of chaos and bifurcation control. Pole-placement and hybrid control methods are implemented in order to discuss chaos control in the system. Furthermore, our investigation reveals that the pole-placement method is more effective as compared to the hybrid control method.

8. Future work

It must be noted that the weak predator functional response in system (1) is similar to Holling type-III function 1. Therefore, it is more interesting to apply Holling type-II functional response. With such implementation of Holling type-II functional response, plant–herbivore model (1) can be rewritten as follows:

$$
\begin{align*}
x_{n+1} &= \frac{x_n}{\alpha(1 + y_n) + \beta x_n}, \\
y_{n+1} &= \gamma y_n(1 + x_n).
\end{align*}
$$

(26)

Stability, bifurcation analysis and chaos control for model (26) is our future work for investigation.
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