Mathematical model of rod oscillations with account of material relaxation behaviour

I V Kudinov, V A Kudinov, A V Eremin and V V Zhukov

Samara State Technical University, 244, Molodogvardeyskaya St., Samara, 443100, Russia
E-mail: totoig@yandex.ru

Abstract. Taking into account the bounded velocity of strains and deformations propagation in the formula given in the Hooke’s law, the authors have obtained the differential equation of rod damped oscillations that includes the first and the third time derivatives of displacement as well as the mixed derivative (with respect to space and time variables). Study of its precise analytical solution found by means of separation of variables has shown that rod recovery after being disturbed is accompanied by low-amplitude damped oscillations that occur at the start time and only within the range of positive displacement values. The oscillations amplitude decreases with increase of relaxation factor. Rod is recovered virtually without an oscillating process both in the limit and with any high values of the relaxation factor.

1. Introduction
Precise analytical solutions of hyperbolic equations of resilient bodies damped oscillations that occur under some force have been obtained only in individual cases with exactly specified laws of variation of disturbing load [1 – 6]. The Hooke’s law and the second Newton’s law are used for obtaining these equations. It is known that the formula given in the Hooke’s law implies the infinite velocity of disturbances propagation. That means that strain caused by some displacement immediately (abruptly) reaches the value relevant to this displacement. This happens due to the fact that strain and displacement are not time-spaced in the Hooke’s law formula (there is no temporary variable in the formula).

Yet propagation velocities of any physical values cannot be infinite and, therefore, both strains and deformations cannot immediately reach some exact values. In actual physical environment they change with some time delay according to material relaxation behaviour expressed as a relaxation factor.

2. Statement of the problem
In order to define the material relaxation behaviour, velocities of strain and deformation change in time in the Hooke’s law formula \( \sigma = E \varepsilon \) will be represented as \( \sigma + \tau, \partial \sigma / \partial t \) and \( \varepsilon + \tau, \partial \varepsilon / \partial t \), where \( \sigma \) – normal strain; \( \varepsilon = \partial u / \partial x \) – deformation; \( u \) – displacement; \( x \) – coordinate; \( t \) – time; \( E \) – resilience modulus; \( \tau \) – relaxation factor [7 – 10].

Then the Hooke’s law formula looks as follows:

\[
\sigma = E(\varepsilon + \tau, \partial \varepsilon / \partial t) - \tau, \partial \sigma / \partial t .
\]
Let us insert the relation for $\sigma$ (1) into the dynamic equation for displacement $\rho \partial \varepsilon / \partial t = \partial \sigma / \partial x$ and after some transformations one obtains the following:

$$\tau, \frac{\partial^3 u}{\partial t^3} + \frac{\partial^2 u}{\partial x^2} = e^2 \frac{\partial^2 u}{\partial x^2} + e^2 \tau, \frac{\partial^3 u}{\partial x^2 \partial t},$$

where $e = \sqrt{E / \rho}$ is velocity of disturbance propagation.

The equation (2) defines displacement change with account of material relaxation behaviour. Yet it does not contain the summand expressing the internal resistance of the environment with load applied. In order express it, let us suppose that resistance force is proportional to velocity of change of displacement in time [2].

$$-r \tau, \partial \sigma / \partial t - \partial u / \partial t,$$

where $r$ – resistance factor, 1/s. The minus sign means that the direction of the resistance force $\sigma$ is opposite to that of the displacement change velocity.

Let us insert the relation for $\sigma$ into the right hand part of the equation (2):

$$\tau, \frac{\partial^3 u}{\partial t^3} + \frac{\partial^2 u}{\partial x^2} = e^2 \left( \frac{\partial^2 u}{\partial x^2} + \tau, \frac{\partial^3 u}{\partial x^2 \partial t} \right) - r \frac{\partial u}{\partial t},$$

so that the second Newton’s law is valid.

It is obvious that if $\tau, = 0$ and $r = 0$, the equation (3) is reduced to the equation of rod undamped oscillations.

In the following mathematical arrangement, let us solve the boundary value problem regarding oscillations of a rod that is rigidly fixed at one end and has been deformed at the start time in linear fashion:

$$\tau, \frac{\partial^3 u(x,t)}{\partial t^3} + \frac{\partial^2 u(x,t)}{\partial x^2} = e^2 \left[ \frac{\partial^2 u(x,t)}{\partial x^2} + \tau, \frac{\partial^3 u(x,t)}{\partial x^2 \partial t} \right] - r \frac{\partial u(x,t)}{\partial t} ; \quad (t > 0; \quad 0 < x < \delta)$$

$$u(x,0) = b(\delta - x); \quad (5) \quad \partial u(x,0) / \partial t = 0; \quad (6) \quad \partial^2 u(x,0) / \partial t^2 = 0; \quad (7)$$

$$\partial u(0,t) / \partial x = 0; \quad (8) \quad u(\delta,t) = 0,$$

where $b$ – factor; $\delta$ – rod length.

Problem (4) – (9) in dimensionless form will look as follows:

$$F_0, \frac{\partial \Theta(\xi,F_0)}{\partial F_0} + F_0, \frac{\partial^3 \Theta(\xi,F_0)}{\partial F_0^3} + \frac{\partial^2 \Theta(\xi,F_0)}{\partial F_0^2} = \frac{\partial^2 \Theta(\xi,F_0)}{\partial \xi^2}$$

$$(F_0 > 0; \quad 0 < \xi < 1)$$

$$\partial \Theta(\xi,0) / \partial F_0 = 0; \quad (11) \quad \partial \Theta(\xi,0) / \partial \xi = 0; \quad (12) \quad \partial^2 \Theta(\xi,0) / \partial F_0^2 = 0; \quad (13)$$

$$\partial \Theta(0,F_0) / \partial \xi = 0; \quad (14) \quad \Theta(1,F_0) = 0,$$

where $\Theta = u / u_0; \quad \xi = x / \delta; \quad F_0 = e t / \delta; \quad F_0, = e t, / \delta; \quad F_0, = \delta r / e; \quad u_0 = b \delta$; dimensionless displacement, coordinate, time, relaxation factor, resistance factor respectively.

3. Method of Solving the Problem

Solution of problem (10) – (15) is taken to be as follows:

$$\Theta(\xi,F_0) = \varphi(F_0)R(\xi),$$

where $R(\xi) = \cos(r \pi \xi / 2)$ (r = 1, 3, 5, ...).
Let us insert the relation for $\Theta(\zeta, F_0)$ (16) into equation (10), then divide the obtained relation by $R(\zeta)$ and find the following:

$$
F_0 \frac{\partial^3 \Phi}{\partial F_0^3} + \frac{\partial^2 \Phi}{\partial F_0^2} + (F_0 + v_k F_0) \frac{\partial \Phi}{\partial F_0} + v_k \Phi = 0,
$$

(17)

where $v_k = r^2 \pi^2 / 4$. ($r = 2k - 1; \ k = \Gamma_{\infty}$) — eigenvalues of the Sturm-Liouville boundary value problem.

Let us find particular solutions of the equation (17) by means of taking integral:

$$
\Phi_k (F_0) = [C_{1k} \exp(i\beta F_0) + C_{2k} \exp(-i\beta F_0)] \exp(y F_0) + C_{3k} \exp(z_{3k} F_0),
$$

(18)

where $C_{\beta} (j = 1, 2, 3; \ k = \Gamma_{\infty})$ — constants of integration; $\beta = \sqrt{3}(A/2 + 2B/A)/(6F_0)$; $\gamma = (B/A - A/4 - 1)/(3F_0)$; $i = \sqrt{-1}$;

$$
A = \{36F_0, (v_k F_0, + F_0) - 108v_k F_0^2 - 8 + 12\sqrt{3} \left[2v_k m_1 \left(2v_k^2 F_0^2 + 6v_k m_2 + 6F_0^2 - 9v_k \right) - v_k F_0 \left(2F_0^2 + 27v_k^2 F_0^2 + 4v_k \right) \right]^{1/3} - v_k F_0, (v_k F_0 + 2F_0) + 2m_2 \left(2F_0^2 - 9v_k \right) - F_0^2 + 27v_k F_0^2 + 4v_k \right]^{1/3};
$$

$$
B = 3F_0, (v_k F_0 + F_0) - 1; \ m_1 = F_0^2; \ m_2 = F_0; \ m_3 = F_0; F_0.
$$

Let us use Euler’s formula:

$$
\exp(i s) = \cos s + i \sin s; \ \exp(-i s) = \cos s - i \sin s,
$$

to reduce relation (18) to:

$$
\Phi_k (F_0) = \exp(y F_0) \left[B_{1k} \cos(\beta F_0) - B_{2k} \sin(\beta F_0) \right] + C_{3k} \exp(z_{3k} F_0),
$$

(19)

where $B_{1k} = C_{1k} + C_{2k}; \ B_{2k} = i(C_{2k} - C_{1k})$.

Let us insert relation (19) into relation (16) and obtain the following:

$$
\Theta(\zeta, F_0) = \{\exp(y F_0) \left[B_{1k} \cos(\beta F_0) - B_{2k} \sin(\beta F_0) \right] + C_{3k} \exp(z_{3k} F_0) \} \cos(r\pi \zeta / 2) \ (r = 2k - 1; \ k = \Gamma_{\infty}).
$$

(20)

Each particular solution (20) satisfies the equation (10) and boundary conditions (14), (15) yet none of them satisfies the initial conditions (11) – (13). To satisfy them, let us compose a sum of particular solutions:

$$
\Theta(\zeta, F_0) = \sum_{k=1}^{\infty} \{\exp(y F_0) \left[B_{1k} \cos(\beta F_0) - B_{2k} \sin(\beta F_0) \right] + C_{3k} \exp(z_{3k} F_0) \} \cos(r\pi \zeta / 2), \ (r = 2k - 1)
$$

(21)

Constants of integration $B_{1k}, B_{2k}, C_{3k}$ are obtained from initial conditions (11) – (13). Let us insert relation (21) into initial conditions (12), (13) and obtain the following:

$$
B_{1k} = (2\beta - C_{3k} z_{3k}^2) / \gamma; \ B_{2k} = \left[1 - \beta^2 \right] B_{1k} + C_{3k} z_{3k}^2 / (2 \gamma \beta).
$$

If one inserts relation (21) into initial conditions (11), one will obtain the following:

$$
\sum_{k=1}^{\infty} (B_{1k} + C_{3k}) \cos(r\pi \zeta / 2) = 1 - \zeta. \ (r = 2k - 1)
$$

(22)
Relation (22) represents Fourier transformation of function $1 - \zeta$ in eigenfunctions of the Sturm-Liouville boundary value problem within the interval $[0; 1]$. In order to obtain the unknown factor $C_{3k}$, let us multiply relation (22) by $\cos(j\pi\zeta/2)$ and take integral of the obtained expression within the range from $\zeta = 0$ to $\zeta = 1$:

$$\int_0^1 \sum_{k=1}^\infty (B_{lk} + C_{3k}) \cos(r\pi\zeta/2) \cos(j\pi\zeta/2) d\zeta = \int_0^1 (1 - \zeta) \cos(j\pi\zeta/2) d\zeta. \quad (r = j = 2k - 1) \quad (23)$$

Due to cosines orthogonality, relation (23) will look as follows:

$$\int_0^1 (B_{lk} + C_{3k}) \cos^2(r\pi\zeta/2) d\zeta = \int_0^1 (1 - \zeta) \cos(r\pi\zeta/2) d\zeta. \quad (24)$$

Let us define the integrals in relation (24) and obtain the following:

$$C_{3k} = -\left[(D + r^2\pi^2)B_{lk} + 8 \cos(r\pi/2) - 8\right]/(D + r^2\pi^2), \quad (r = 2k - 1; \ k = \Gamma, \infty) \quad (25)$$

where $D = 2r\pi\cos(r\pi/2)\sin(r\pi/2)$.

After constants $B_{lk}, B_{3k}$ and $C_{3k}$ have been defined, the precise analytical solution of problem (10) – (15) in a closed form is obtained from relation (21).

4. Analysis of the results obtained

Analysis of the results of displacement calculation according to the formula (21) allows us to come to conclusion that undamped oscillations virtually occur with $F_0 = F_0_1 \geq 10^{-4}$. See Figure 1a for calculation variant with damped oscillations. Its analysis suggests that the oscillations amplitude decreases with time in compliance with some exponential relationship. Symmetry of rod deviation regarding the zero value $\Theta$ can be observed with small values of $F_0_1$ ($F_0_1 < 0,1$). Oscillations asymmetry occurs with increase of $F_0_1$ (see $F_0_1 = 10$ in Figure 1b). Furthermore, if $F_0_1 = 0$ the oscillations amplitude first decreases at some start time, then it becomes stable and subsequently the oscillations become undamped.

If $F_0_1 = F_0_1 = 5$ the number of oscillations within the positive displacement range increases (Figure 1c). E.g., if $F_0_1 = 5$, oscillations within this range occur up to $F_0 = 19$, therefore, for rods having high values of $F_0_1$ their recovery from some deformed starting condition to the ignition condition is accompanied by low-amplitude oscillations occurring at some start time within the positive displacement range only. The rod is recovered virtually without oscillating process as the dimensionless resistance factor $F_0_1$ increases (Figure 1d).
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