CANONICAL DUALITY-TRIALITY THEORY FOR SOLVING GENERAL GLOBAL OPTIMIZATION PROBLEMS IN COMPLEX SYSTEMS

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Abstract. General nonconvex optimization problems are studied by using the canonical duality-triality theory. The triality theory is proved for sums of exponentials and quartic polynomials, which solved an open problem left in 2003. This theory can be used to find the global minimum and local extrema, which bridges a gap between global optimization and nonconvex mechanics. Detailed applications are illustrated by several examples.

1. Introduction and Motivation

This paper intends to solve the following nonconvex optimization problem (\(\mathcal{P}\)) in short):

\[
\begin{align*}
\mathcal{P} : \quad & \max \left\{ \Pi(x) = W(x) + \frac{1}{2} x^T A x - f^T x \mid x \in \mathbb{R}^n \right\}, \\
\end{align*}
\]

where \(\max\{\ast\}\) denotes finding extremum points of a function given in \(\{\ast\}\), \(f \in \mathbb{R}^n\) is a given (input) vector, \(A \in \mathbb{R}^{n \times n}\) is a given symmetric matrix, and \(W : \mathbb{R}^n \rightarrow \mathbb{R}\) is a combination of fourth order polynomials (double-well functions) and quadratic-exponential functions, namely:

\[
W(x) := \sum_{i \in I_m} \exp \left( \frac{1}{2} x^T B_i x - \alpha_i \right) + \sum_{j \in I_p} \frac{b_j}{2} \left( \frac{1}{2} x^T C_j x - \theta_j \right)^2,
\]

where \(I_m = \{1, \ldots, m\}\), \(I_p = \{1, \ldots, p\}\) are two integer sets with \(m, p\) which are fixed integers; all the coefficients \(b_j\) with \(j \in I_p\) are positive constants, and \(\alpha_i, \theta_j \in \mathbb{R}\ \forall i \in I_m, j \in I_p\) are given parameters; the matrices \(\{B_i\}_{i \in I_m}\) and \(\{C_j\}_{j \in I_p}\) are assumed to be symmetric, positive semi-definite such that the cone generated by them contains a positive definite matrix.

The nonconvex optimization problem \(\mathcal{P}\) arises naturally in complex systems with a wide range of applications, including chaotic dynamical systems \([11,14,16]\), computational biology \([39]\), chemical database analysis \([38]\), large deformation computational mechanics \([6,32]\), population growing \([29]\), location/ allocation, network communication \([17]\), and phase transitions of solids \([14,15,21]\), etc.

For example, the popular sensor network location problem is to solve the following system of nonlinear equations (see \([1,25]\)):

\[
\|u_i - u_j\|^2 = d_{ij}^2, \quad \forall (i, j) \in \mathcal{I}_p, \quad u_k = a_k, \quad \forall k \in \mathcal{I}_b
\]
where the vectors $u_i = \{a_i^p\} \in \mathbb{R}^d$ ($i = 1, \ldots, p$) represent the locations of the unknown sensors, $I_p = \{(i, j) : i < j, d_{ij} \text{ is specified}\}$ and $I_b = \{k : u_k = a_k \text{ is specified}\}$ are two given index sets, $d_{ij}$ are given distances for $(i, j) \in I_p$, the given vectors $a_1, a_2, \ldots, a_q \in \mathbb{R}^d$ are the so-called anchors. The notation $\|u_i - u_j\|_2$ denotes the Euclidian distance between $u_i$ and $u_j$, i.e.,

$$\|u_i - u_j\|_2 = \sqrt{\sum_{a=1}^{d} (a_i^a - a_j^a)^2}.$$

By using the least squares method, the quadratic equations (2) of the sensor localization problem can be reformulated as an optimization problem:

$$\min \left\{ P(u) = \sum_{(i,j) \in I_p} \frac{1}{2} (\|u_i - u_j\|_2^2 - d_{ij}^2)^2 : u_i \in U_a \right\},$$

where $U_a = \{u \in \mathbb{R}^{d \times p} | u_i = a_k \forall k \in I_b\}$ is a feasible space. Let $x = \{\{u_1^1, \ldots, u_1^d\}, \ldots, \{u_q^1, \ldots, u_q^d\}\} \in \mathbb{R}^n$ ($n = d \times p$) denote an extended vector. By using Lagrange multiplier method to relax the boundary conditions in $U_a$, the least squares method for the sensor localization problem (3) can be written in the problem (1) for certain properly defined matrices $\{C_j\}$, which is the so-called deformation matrix in structural mechanics. The sensor network localization type problems also appear in computational biology, Euclidean ball packing, molecular confirmation, and recently, wireless network communication, etc [30, 39]. Due to the nonconvexity, the sensor network localization problem is considered to be NP-hard even for the simplest case $d = 1$ [25, 33]. Recent result of Aspnes et al [11] shows that the problem of computing a realization of the sensors on the plane is NP-complete in general.

Mathematics and mechanics have been two complementary partners since the Newton times. Many fundamental ideas, concepts, and mathematical methods extensively used in calculus of variations and optimization are originated from mechanics. For examples, the Lagrange multiplier method was first proposed by Lagrange from the classical analytic mechanics; while the concepts of super-potential and sub-differential in modern convex analysis were introduced by Moreau from frictional mechanics [26, 27]. From the point view of computational large deformation mechanics, both the fourth-order polynomial minimization problem ($P$) and the sensor localization problem (3) are actually two special cases of discretized finite deformation problems [6]. It is known that in continuum mechanics and differential geometry, the deformation $u(x) : \Omega \rightarrow \mathbb{R}^r$ is a vector field over an open domain $\Omega \subset \mathbb{R}^r$, and the minimal potential variational problem is defined by

$$\min \left\{ P(u) = \int_{\Omega} [W(\nabla u) - u^T f] \, d\Omega \mid u \in U_a \right\},$$

where $W(F)$ is the so-called stored strain energy, which is usually a nonconvex function of the deformation gradient $F = \nabla u$, the feasible set $U_a$ in this nonconvex variational problem is called the kinematically admissible space, where certain boundary conditions are prescribed. According to the hyper-elasticity law (see Chapter 6.1.2 [9] or [23]), the stored strain energy should be an objective function of the deformation gradient $F$, i.e., there exists an objective strain measure $E(F)$
and a convex function $V(E)$ such that

$$W(\nabla u) = V(E(\nabla u)).$$

One of the most simple objective strain measures is the well-known Green-St. Venant strain tensor $E = \frac{1}{2}(F^TF - I)$. Clearly, this strain measure satisfies the objectivity condition, i.e. $E(QF) = E(F)$ for any given orthonormal (rotation) matrix $Q$. For the most simple St. Venant-Kirchhoff material, $V(E)$ is a quadratic function of $E$, i.e.

$$V(E) = \frac{1}{2} \lambda (\text{tr} E)^2 + \mu \text{tr} (E)^2,$$

where, $\lambda, \mu > 0$ are the classical Lamé constants, $\text{tr} E$ represents the trace of $E$. Therefore, the stored energy $W(F)$ is a fourth-order polynomial tensor function of $F = \nabla u$. While for bio-materials, the stored energy could be the combination of the polynomial and exponential functions of the Cauchy-Green strain tensor. By using finite difference method (FDM), the deformation gradient $\nabla u$ can be directly approximated by the difference $Du = u(x_i) - u(x_j) = u_i - u_j$. While in finite element method (FEM), the domain $\Omega = \bigcup_m \Omega^m$ is discretized by a finite number of elements $\Omega^e \subset \Omega$ and in each element, the deformation field $u(x) = \sum_i N_i(x) u_i$ is numerically represented by the nodal vectors $u_i$ via piecewise interpolation (polynomial) function $N_i(x)$ (cf. [6]). Therefore, by either FDM or FEM, the minimal potential variational problem (4) can be eventually reduced to a very complicated large-scale fourth-order polynomial/exponential minimization problem with the problems $(P)$ as its the most simple case. In the contact mechanics and elasto-plastic design of large deformed structures, the nonconvex problems are usually subjected to inequality constraints. In these cases, the global optimal solution could be local minima (see [2]) and to solve such problems is fundamentally difficult by using traditional direct methods.

Canonical duality theory was developed originally from Gao and Strang’s work in 1989 [19] for solving general variational problem (4) in finite deformation theory, where the stored energy $W(F)$ is nonconvex and even nonsmooth. By introducing a so-called complementary gap function, they recovered the complementary energy principle in large deformation (geometrically nonlinear) systems. They proved that the nonnegative gap function can be used to identify the global minimizer of the nonconvex potential variational problems. Seven years later, it was discovered that the negative gap function can be used to identify the largest local minimum and maximum. Therefore, a so-called triality theory was first proposed in nonconvex mechanics [5], and then generalized to global optimization [10]. This triality theory is composed of a canonical min-max duality and two pairs of double-min, double-max dualities, which reveals an intrinsic duality pattern in complex systems and has been used successfully for solving a wide class of challenging problems in complex systems [7, 8, 13, 18]. However, it was realized in 2003 [11, 12] that the double-min duality holds under certain additional conditions. Recently, this problem is partly solved for a class of fourth order polynomial optimization problems [20]. Based on these results, this paper intends to solve the more challenging problem $(P)$. We will show that by the canonical dual transformation, all critical solutions of $(P)$ can be analytically presented in terms of the canonical dual solutions. The extremality of these solutions can be identified by the triality theory. Several solved examples are
listed in the last section.

2. Canonical Dual Problem and Analytical Solutions

Following the standard procedure of the canonical dual transformation (cf. e.g., [12]), first we need to choose a geometric operator \( \Lambda = (\Lambda_1(x), \Lambda_2(x)) : \mathbb{R}^n \to \mathbb{R}^{m+p} \), where

\[
\Lambda_1(x) = \left\{ \frac{1}{2} x^t B_i x - \alpha_i \right\} : \mathbb{R}^n \to \mathbb{R}^m,
\]

\[
\Lambda_2(x) = \left\{ \frac{1}{2} x^t C_j x - \theta_j \right\} : \mathbb{R}^n \to \mathbb{R}^p.
\]

Therefore, the nonconvex function \( W(x) \) can be written in the following canonical form

\[
W(x) = V(\Lambda(x)) = V_1(\Lambda_1(x)) + V_2(\Lambda_2(x))
\]

with

\[
V_1(\epsilon) = \sum_{i \in I_m} \exp(\epsilon_i), \quad V_2(\gamma) = \sum_{j \in I_p} \frac{1}{2} b_j \gamma_j^2.
\]

Clearly, the canonical function \( V(\epsilon) \) is convex on

\[
\mathcal{V}_a = \{ \epsilon = (\epsilon, \gamma) \in \mathbb{R}^{m+p} \mid \epsilon_i \in [-\alpha_i, +\infty), \gamma_j \in [-\theta_j, +\infty), \forall i \in I_m, j \in I_p \}
\]

such that the canonical dual variable \( \varsigma = (\tau, \sigma) \) of \( \epsilon = (\epsilon, \gamma) \) can be uniquely defined by

\[
\varsigma = \nabla V(\epsilon) \Rightarrow \tau = \nabla V_1(\epsilon) = \{\exp(\epsilon_i)\}, \quad \sigma = \nabla V_2(\gamma) = \{b_j \gamma_j\},
\]

and on the canonical dual space

\[
\mathcal{V}_a^* = \{ \varsigma = (\tau, \sigma) \in \mathbb{R}^{m+p} \mid \tau_i \in [\exp(-\alpha_i), \infty), \sigma_j \in [-b_j \theta_j, \infty), \forall i \in I_m, j \in I_p \},
\]

the Legendre conjugate of \( V(\epsilon) \) can be defined by

\[
V^*(\varsigma) = \text{sta}\{\epsilon^t \varsigma - V(\epsilon) \mid \epsilon \in \mathcal{V}_a\} = V_1^*(\tau) + V_2^*(\sigma)
\]

where \( \text{sta}\{\} \) denotes finding stationary points of the function given in \{\} and

\[
V_1^*(\tau) = \sum_{i \in I_m} \left( \tau_i \ln \tau_i - \tau_i \right), \quad V_2^*(\sigma) = \sum_{j \in I_p} \frac{1}{2} b_j \sigma_j^2.
\]

By using the canonical dual transformation \( W(x) = V(\Lambda(x)) = \Lambda(x)^t \varsigma - V^*(\varsigma) \), the Gao-Strang total complementary function \( \Xi : \mathbb{R}^n \times \mathcal{V}_a^* \to \mathbb{R} \) associated with the problem \((P)\) can be given by

\[
\Xi(x, \varsigma) = \langle \Lambda(x), \varsigma \rangle - V^*(\varsigma) + \frac{1}{2} x^t A x - f^t x
\]

\[
= \frac{1}{2} x^t G(\varsigma) x - \alpha^t \tau - \theta^t \sigma - V_1^*(\tau) - V_2^*(\sigma) - f^t x,
\]

where

\[
G(\varsigma) = A + \sum_{i \in I_m} \tau_i B_i + \sum_{j \in I_p} \sigma_j C_j.
\]
Via this $\Xi(x, \varsigma)$, the canonical dual function $\Pi^d : V_n^* \to \mathbb{R}$ can be defined by

$$\Pi^d(\varsigma) := \text{sta} \{ \Xi(x, \varsigma) | x \in \mathbb{R}^n \} = \{ \Xi(x(\varsigma), \varsigma) : \nabla_x \Xi(x(\varsigma), \varsigma) = 0 \}. $$

Notice that $\nabla_x \Xi(x, \varsigma) = G(\varsigma)x - f = 0$ if and only if

$$G(\varsigma)x = f. $$

Let $C_{ad}(G(\varsigma))$ be the space generated by the columns of the matrix $G(\varsigma)$. Then, on the dual feasible space

$$S_a = \{ \varsigma \in V_a^* : f \in C_{ad}(G(\varsigma)) \},$$

the primal solution $x = (G(\varsigma))^{-1}f$ is well defined (if $G(\varsigma)$ is singular, $(G(\varsigma))^{-1}$ denotes its pseudo-inverse, see [3], [28] and references therein) and we have $\Pi^d : S_a \to \mathbb{R}$

$$\Pi^d(\varsigma) = -\frac{1}{2}(G(\varsigma))^{-1}f - V_1^c(\tau) - V_2^c(\sigma) - \alpha^t \tau - \theta^t \sigma. $$

Therefore, the canonical dual problem is proposed in the following form:

$$(P^d) : \text{ext}\{ \Pi^d(\varsigma) : \varsigma \in S_a \}. $$

By the canonical duality theory, it is not difficult to show that

$$\Pi(x) = \text{sta}\{ \Xi(x, \varsigma) : \varsigma \in S_a \} = \Xi(x, \varsigma(x)), $$

where $\varsigma(x) = (\tau(x), \sigma(x))$ and

$$\begin{align*}
(\tau(x))_i &= \exp((\Lambda_1(x))_i), \ i \in I_m, \\
(\sigma(x))_j &= b_j(\Lambda_2(x))_j, \ j \in I_p.
\end{align*}$$

According to the general theory presented in [12], we have the following result.

**Theorem 1 (Analytical Solutions).** Suppose that for a given $f \in \mathbb{R}^n$ the canonical dual space $S_a$ is not empty. If $\varsigma \in S_a$ is a stationary point of $\Pi^d$, then

$$\overline{x} = (G(\varsigma))^{-1}f$$

is a stationary point of $\Pi$ and

$$\Pi(\overline{x}) = \Pi^d(\varsigma). $$

**Proof:** Let us calculate $\nabla \Pi^d(\varsigma)$ and $\nabla^2 \Pi^d(\varsigma)$. We know that

$$\nabla \Pi^d(\varsigma) = \begin{bmatrix} \nabla_{\tau} \Pi^d(\varsigma) \\ \nabla_{\sigma} \Pi^d(\varsigma) \end{bmatrix} \in \mathbb{R}^{m+p},$$

then

$$\begin{align*}
(\nabla_{\tau} \Pi^d(\varsigma))_i &= \frac{1}{2}f^t (G(\varsigma))^{-1}B_i(G(\varsigma))^{-1}f - \ln \tau_i - \alpha_i, \ i \in I_m; \\
(\nabla_{\sigma} \Pi^d(\varsigma))_j &= \frac{1}{2}f^t (G(\varsigma))^{-1}C_j(G(\varsigma))^{-1}f - \frac{\sigma_j}{b_j} - \theta_j, \ j \in I_p.
\end{align*}$$

On the other hand,

$$\nabla^2 \Pi^d(\varsigma) = \begin{bmatrix} \nabla^2_{\tau \tau} \Pi^d(\varsigma) & \nabla^2_{\tau \sigma} \Pi^d(\varsigma) \\ \nabla^2_{\sigma \tau} \Pi^d(\varsigma) & \nabla^2_{\sigma \sigma} \Pi^d(\varsigma) \end{bmatrix} \in \mathbb{R}^{m+p} \times \mathbb{R}^{m+p},$$

where $\nabla^2_{\tau \sigma} \Pi^d(\varsigma) := (\nabla_{\tau}(\nabla_{\sigma} \Pi^d(\varsigma))^t)$). Let $\delta_{ij}$ be the Kronecker’s delta. Then
\[(\nabla^2_{\tau\tau}\Pi^d(\varsigma))_{ij} = -f'(G(\varsigma))^{-1}B_i(G(\varsigma))^{-1}B_j(G(\varsigma))^{-1}f - \frac{\delta_{ij}}{\tau_j}, \quad i, j \in I_m.\]

\[(\nabla^2_{\tau\sigma}\Pi^d(\varsigma))_{ij} = -f'(G(\varsigma))^{-1}B_i(G(\varsigma))^{-1}C_j(G(\varsigma))^{-1}f \quad i \in I_m; j \in I_p.\]

\[(\nabla^2_{\sigma\tau}\Pi^d(\varsigma))_{ij} = -f'(G(\varsigma))^{-1}C_i(G(\varsigma))^{-1}B_j(G(\varsigma))^{-1}f \quad i \in I_m; j \in I_p.\]

\[(\nabla^2_{\sigma\sigma}\Pi^d(\varsigma))_{ij} = -f'(G(\varsigma))^{-1}C_i(G(\varsigma))^{-1}C_j(G(\varsigma))^{-1}f - \frac{\delta_{ij}}{b_j} \quad i, j \in I_p.\]

By making \(x = (G(\varsigma))^{-1}f\) and \(F(x) \in \mathbb{R}^{n \times (m+p)}\) be \(F(x) = [B_1x, \ldots, B_mx, C_1x, \ldots, C_px]\), we have

\[(\nabla^2\Pi^d(\varsigma)) = -F(x)^t(G(\varsigma))^{-1}F(x) - \text{Diag} \left( \frac{1}{\tau_1}, \ldots, \frac{1}{\tau_m}, \frac{1}{b_1}, \ldots, \frac{1}{b_p} \right).\]

Let \(D = \text{Diag} \left( \tau_1, \ldots, \tau_m, b_1, \ldots, b_p \right)\), then \(\nabla^2\Pi^d(\varsigma)\) can be written as

\[(\nabla^2\Pi^d(\varsigma)) = -F(x)^t(G(\varsigma))^{-1}F(x) - D^{-1}.\]

Calculating \(\nabla\Pi(x)\) and \(\nabla^2\Pi(x)\), we have respectively

\[\nabla\Pi(x) = \sum_{i \in I_m} \exp \left( \frac{1}{2}x^tB_ix - \alpha_i \right) B_i x + \sum_{j \in I_p} b_j \left( \frac{1}{2}x^tC_jx - \theta_j \right) C_j x + Ax - f.\]

\[\nabla^2\Pi(x) = A + \sum_{i \in I_m} \exp \left( \frac{1}{2}x^tB_ix - \alpha_i \right) (B_i x (B_i x)^t + B_i)\]

\[+ \sum_{j \in I_p} b_j \left( C_j x (C_j x)^t + \frac{1}{2}x^tC_jx - \theta_j \right) C_j.\]

Since \(\mathbf{\zeta} = (\mathbf{\tau}, \mathbf{\sigma})\) is a stationary point of \(\Pi^d\) then by Equations \(21\) and \(22\) we have that

\[(27) \quad (A_1(\mathbf{\zeta}))_i = \ln \tau_i, \quad i \in I_m;\]

\[(28) \quad (A_2(\mathbf{\zeta}))_j = \frac{\sigma_j}{b_j}, \quad j \in I_p.\]

Using Equations \(27\) and \(28\) in Equation \(23\), we obtain

\[\nabla\Pi(\mathbf{\zeta}) = G(\mathbf{\zeta})\mathbf{\zeta} - f = G(\mathbf{\zeta})(G(\mathbf{\zeta}))^{-1}f - f = 0.\]

Notice that Equations \(27\) and \(28\) together with Equations \(16\) and \(18\) imply that

\[(29) \quad \Pi(\mathbf{\zeta}) = \Xi(\mathbf{\zeta}, \mathbf{\zeta}) = \Xi((G(\mathbf{\zeta}))^{-1}f, \mathbf{\zeta}) = \Pi^d(\mathbf{\zeta}).\]

And this finishes the proof. ■
Remark 1. This theorem shows that the problem \((P^d)\) is canonical dual to the nonconvex primal problem \((P)\) in the sense that \(\Pi(\boldsymbol{\tau}) = \Pi^d(\boldsymbol{\tau})\) at each critical point of \(\Xi(\boldsymbol{x}, \xi)\). By the criticality condition \(15\), we know that if \(G(\xi)\) is singular at \(\boldsymbol{\tau}\), the canonical equilibrium equation \(16\) may have infinite number of solutions: \(\boldsymbol{\tau} = G(\boldsymbol{\tau})^* \mathbf{f} + \mathbf{N}\xi^o\), where \(G^*\) represents the Moore-Penrose generalized inverse, \(\mathbf{N}\) is a basis matrix of the null space of \(G(\boldsymbol{\tau})\), and \(\xi^o\) is a free vector. In this case, Theorem 1 still holds, but the canonical dual function \(\Pi^d\) will have additional parametrical vector \(\xi^o\). In order to avoid this case, a quadratic perturbation method is introduced in \(30\), i.e. in the case that \(G(\boldsymbol{\tau})\) is singular, replace it by the following perturbed form

\[
(30) \quad G_\alpha(\boldsymbol{\tau}) = G(\boldsymbol{\tau}) + \alpha \mathbf{D}
\]

where \(\alpha > 0\) is a perturbation parameter and \(\mathbf{D}\) is a given positive-definite matrix.

Very often, \(\mathbf{D} = \mathbf{I}\). Detailed study on this quadratic perturbation method is given in \(30\).

In the next section, we will show that the extremality of some of these solutions can be identified by a refined triality theory.

3. Triality Theory

Before presenting the refined triality theory, we need the following sets

\[
\mathcal{S}_a^+ := \{\xi \in \mathcal{S}_a : G(\xi) \succeq 0\}, \quad \mathcal{S}_a^- := \{\xi \in \mathcal{S}_a : G(\xi) \prec 0\}.
\]

Lemma 1. Suppose that \(m + p < n\), \(\boldsymbol{\tau} \in \mathcal{S}_a^-\) is a stationary point and a local minimizer of \(\Pi^d\) and \(\mathbf{y} = (G(\boldsymbol{\tau}))^{-1} \mathbf{f}\). Then, there exists a matrix \(\mathbf{L} \in \mathbb{R}^{n \times (m+p)}\) with \(\text{Rank}(\mathbf{L}) = m + p\) such that

\[
(31) \quad \mathbf{L}^t \nabla^2 \Pi(\boldsymbol{\tau}) \mathbf{L} \succeq 0.
\]

Proof: Since \(\boldsymbol{\tau} \in \mathcal{S}_a^-\) is a local minimizer of \(\Pi^d\), we have that \(\nabla^2 \Pi^d(\boldsymbol{\tau}) \succeq 0\). It follows from Equation \(24\) that

\[
-F(\chi)^t (G(\boldsymbol{\tau}))^{-1} F(\chi) \succeq \mathbf{D}^{-1} > 0.
\]

Thus, \(\text{Rank}(\mathbf{F}(\chi)) = m + p\). Since \(\boldsymbol{\tau} \in \mathcal{S}_a^-\) and \(\mathbf{F}(\chi)\mathbf{D}(\mathbf{F}(\chi))^t \succeq 0\) there exists a nonsingular matrix \(\mathbf{T} \in \mathbb{R}^{n \times n}\) such that

\[
(32) \quad \mathbf{T}^t G(\boldsymbol{\tau}) \mathbf{T} = \text{Diag}(-\lambda_1, \ldots, -\lambda_n)
\]

and

\[
(33) \quad \mathbf{T}^t \mathbf{F}(\chi) \mathbf{D}(\mathbf{F}(\chi))^t \mathbf{T} = \text{Diag}(a_1, \ldots, a_{m_1 + m_2}, 0, \ldots, 0),
\]

where \(\lambda_i > 0\) for every \(i = 1, \ldots, n\) and \(a_j > 0\) for every \(j = 1, \ldots, m + p\) (see \(4\), \(22\) and references therein). According to Lemma 3 in the Appendix, we know that there exists orthogonal matrices \(\mathbf{U} \in \mathbb{R}^{n \times n}\) and \(\mathbf{E} \in \mathbb{R}^{(m+p) \times (m+p)}\) such that

\[
(34) \quad \mathbf{T}^t \mathbf{F}(\chi) \mathbf{D}^2 = \mathbf{URE},
\]

where \(\mathbf{R} \in \mathbb{R}^{n \times (m+p)}\) and

\[
R_{ij} = \left\{ \begin{array}{ll}
\sqrt{\alpha_i}, & i = j \text{ and } i = 1, \ldots, m + p \\
0, & \text{otherwise}.
\end{array} \right.
\]
According to the singular value decomposition theory, we know that $U$ is the identity matrix. Then
\[
\nabla^2 \Pi^q(\bar{\varphi}) = -F'(\bar{x}) (G(\bar{\varphi}))^{-1} F(\bar{x}) - D^{-1}
\]
\[= - (F'(\bar{x})^T) T^T G(\bar{\varphi}) T^{-1} (T^T F(\bar{x})) - D^{-1}
\]
\[= - D^{-\frac{1}{2}} E' R' \text{Diag} \left( \frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n} \right) R E - I_{(m+p)\times(m+p)} \succeq 0.
\]

Multiplying by $D^\frac{1}{2}$ from the left and the right
\[(35) \quad D^\frac{1}{2} \nabla^2 \Pi^q(\bar{\varphi}) D^\frac{1}{2} = -E' R' \text{Diag} \left( \frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n} \right) R E - I_{(m+p)\times(m+p)} \succeq 0.
\]

If we multiply the right side of the last equation by $E$ from the left and $E'$ from the right, we have
\[
0 \preceq -R' \text{Diag} \left( \frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n} \right) R - I_{(m+p)\times(m+p)}
\]
\[
\preceq \text{Diag} \left( \frac{a_1}{\lambda_1} - 1, \ldots, \frac{a_{m+p}}{\lambda_{m+p}} - 1 \right).
\]

thus $a_i \geq \lambda_i$, for every $i = 1, \ldots, m + p$. On the other hand
\[
T^T \nabla^2 \Pi(\bar{x}) T = T^T G(\bar{\varphi}) T + T^T F(\bar{x}) D F(\bar{x})' T
\]
\[= \text{Diag} (-\lambda_1, \ldots, -\lambda_n) + \text{Diag} (a_1, \ldots, a_{m+p}, 0, \ldots, 0)
\]
\[= \text{Diag} (a_1 - \lambda_1, \ldots, a_{m+p} - \lambda_{m+p}, -\lambda_{m+p+1}, \ldots, -\lambda_n).
\]

Let $J \in \mathbb{R}^{n \times n}$ be defined by
\[
J_{ij} = \begin{cases} 
1, & i = j \text{ and } i = 1, \ldots, m + p \\
0, & \text{otherwise}.
\end{cases}
\]

Then we have
\[(36) \quad J^T T^T \nabla^2 \Pi(\bar{x}) T J = \text{Diag} (a_1 - \lambda_1, \ldots, a_{m+p} - \lambda_{m+p}) \succeq 0.
\]

Let $L = TJ$, clearly Rank ($L$) = $m + p$ and $L^T \nabla^2 \Pi(\bar{x}) L \succeq 0$, this completes the proof.

In a similar way, we can prove the following lemma.

**Lemma 2.** Suppose that $m + p > n$, $\bar{\varphi} \in S^*_q$ is a stationary point $\Pi^q$ and $\bar{x} = (G(\bar{\varphi}))^{-1} f$ is a local minimizer of $\Pi$. Then, there exists a matrix $Q \in \mathbb{R}^{(m+p) \times n}$ with Rank ($Q$) = $n$ such that
\[(37) \quad Q^T \nabla^2 \Pi^q(\bar{\varphi}) Q \succeq 0.
\]

Let the $m + p$ column vectors of $L$ be respectively as $l_1, \ldots, l_{m+p}$ and the $n$ column vectors of $Q$ be respectively as $q_1, \ldots, q_n$. Clearly, $l_1, \ldots, l_{m+p}$ are $m + p$ independent vectors and $q_1, \ldots, q_n$ are $n$ independent vectors. Now the subspaces $X_b$ and $S_b$ are defined as follows:

\[(38) \quad X_b = \left\{ x \in \mathbb{R}^n : x = \bar{x} + \sum_{i=1}^{m+p} v_i l_i, \{v_i\}_{i=1}^{m+p} \subset \mathbb{R} \right\},
\]
\[(39) \quad S_b = \left\{ \varsigma \in \mathbb{R}^{m+p} : \varsigma = \bar{x} + \sum_{j=1}^{n} \vartheta_j q_j, \{\vartheta_j\}_{j=1}^{n} \subset \mathbb{R} \right\}.
\]
Now we are ready to present the Refined Triality Theory.

**Theorem 2** (Triality Theory). Let $\mathbf{v}$ be a stationary point of $\Pi^d$ and $\mathbf{x} = (G(\mathbf{v}))^{-1} \mathbf{f}$. Assume that $\det(\nabla^2 \Pi(\mathbf{x})) \neq 0$.

(i) If $\mathbf{v} \in S^+_a$, then $\mathbf{v}$ is the only global maximizer of $\Pi^d$ in $S^+_a$ and $\mathbf{x}$ is the only global minimizer of $\Pi$.

(ii) If $\mathbf{v} \in S^-_a$, then $\mathbf{v}$ is a local maximizer of $\Pi^d$ in $S^-_a$ if and only if $\mathbf{x}$ is a local maximizer of $\Pi$.

(iii) If $\mathbf{v} \in S^-_a$ and
   a) if $n = m + p$, then $\mathbf{v}$ is a local minimizer of $\Pi^d$ if and only if $\mathbf{x}$ is a local minimizer of $\Pi$, i.e., there exists respectively neighborhoods $\mathcal{X}, S \subset \mathbb{R}^n$ of $\mathbf{x}$ and $\mathbf{v}$ such that
   
   \[ \Pi(\mathbf{x}) = \min_{\mathbf{x} \in \mathcal{X}} \Pi(\mathbf{x}) = \min_{\mathbf{v} \in S} \Pi^d(\mathbf{v}) = \Pi^d(\mathbf{v}); \]

   b) if $m + p < n$ and $\mathbf{v}$ is a local minimizer of $\Pi^d$, then $\mathbf{x}$ is a saddle point of $\Pi$ and there exists respectively neighborhoods $\mathcal{X}, S \subset \mathbb{R}^n$ of $\mathbf{x}$ and $\mathbf{v}$, such that
   
   \[ \Pi(\mathbf{x}) = \min_{\mathbf{x} \in \mathcal{X} \cap S} \Pi(\mathbf{x}) = \min_{\mathbf{v} \in S} \Pi^d(\mathbf{v}) = \Pi^d(\mathbf{v}); \]

   c) if $n < m + p$ and $\mathbf{x}$ is a local minimizer of $\Pi$, then $\mathbf{v}$ is a saddle point of $\Pi^d$ and there exists respectively neighborhoods $\mathcal{X}, S \subset \mathbb{R}^n$ of $\mathbf{x}$ and $\mathbf{v}$ such that
   
   \[ \Pi(\mathbf{x}) = \min_{\mathbf{x} \in \mathcal{X}} \Pi(\mathbf{x}) = \min_{\mathbf{v} \in S \cap \mathcal{X}} \Pi^d(\mathbf{v}) = \Pi^d(\mathbf{v}). \]

**Proof:**

(i) Since $\mathbf{v} \in S^+_a$, from Equation (24) it is not difficult to show that $\Pi^d$ is strictly concave in $S^+_a$ and $\Xi(\cdot, \mathbf{v})$ is strictly convex in $\mathbb{R}^n$ and therefore $\mathbf{x}$ must be the only global maximizer of $\Pi^d$ in $S^+_a$ and $\mathbf{x}$ is the only global minimizer of $\Xi(\cdot, \mathbf{v})$. By the definition of $\Xi$ given in Equation (13) and the convexity of $V$, the Fenchel inequality leads to

\[ \Xi(\mathbf{x}, \mathbf{v}) \leq \Pi(\mathbf{x}), \ \forall (\mathbf{x}, \mathbf{v}) \in \mathbb{R}^n \times S^+_a. \]

Let us assume now that there exists a vector $\mathbf{x}' \in \mathbb{R}^n \setminus \{\mathbf{x}\}$ such that $\Pi(\mathbf{x}') \leq \Pi(\mathbf{x})$, then

\[ \Pi(\mathbf{x}) \geq \Pi(\mathbf{x}') \geq \Xi(\mathbf{x}', \mathbf{v}) > \Xi(\mathbf{x}, \mathbf{v}) = \Pi(\mathbf{x}), \]

where the last equality comes from Equation (24). This contradiction proves that $\mathbf{x}$ must be the only global minimizer of $\Pi$.

(ii) Notice first that using Equations (27) and (28) in Equation (26) we have

\[ \nabla^2 \Pi(\mathbf{x}) = G(\mathbf{v}) + F(\mathbf{x}) D F(\mathbf{x})^t, \]

where $F(\mathbf{x})$ and $D$ are defined in Equation (24). If $\mathbf{v}$ is a local maximizer of $\Pi^d$ in $S^-_a$ we must have that $\nabla^2 \Pi^d(\mathbf{v}) \preceq 0$, from Equation (24) which is equivalent to

\[ \mathbf{D}^{-1} + F(\mathbf{x})^t (G(\mathbf{v}))^{-1} F(\mathbf{x}) \succeq 0. \]

* If $m + p = n$ and $F$ is invertible, multiplying Equation (44) by $(F(\mathbf{x}))^{-1}$ from the left and $(F(\mathbf{x}))^{-1}$ from the right, we have:

\[ (F(\mathbf{x})^{-1})^{-1} F(\mathbf{x})^{-1} (G(\mathbf{v}))^{-1} F(\mathbf{x})^{-1} \preceq 0. \]
this is equivalent to
\[
(F(\mathbf{x})')^{-1}D^{-1}(F(\mathbf{x}))^{-1} \succeq -(G(\mathbf{z}))^{-1} > 0,
\]
which in turn is equivalent to (Lemma 3 in the Appendix)
\[
-G(\mathbf{z}) \succeq F(\mathbf{z})DF(\mathbf{z})' \iff \nabla^2 \Pi(\mathbf{x}) \preceq 0.
\]
By assumption det(\nabla^2 \Pi(\mathbf{z})) \neq 0, then \mathbf{z} is a local maximum of \Pi.

- If \( m+p \neq n \) or \( F \) is not invertible, then by Lemma 2 there exists orthogonal matrices \( E \in \mathbb{R}^{n \times n}, K \in \mathbb{R}^{(m+p) \times (m+p)} \) and a matrix \( R \in \mathbb{R}^{n \times (m+p)} \) such that
\[
R_{ij} = \begin{cases} 
s_i, & i = j \text{ and } i = 1, \ldots, r \\
0, & \text{otherwise}
\end{cases}
\]
where \( s_i > 0 \) for every \( i, \) \( r = \text{Rank } (F(\mathbf{z})) \) and
\[
F(\mathbf{z})D^{\frac{1}{2}} = ERK.
\]
Using Equation (40), Equation (41) can be rewritten as:
\[
D^{-1} + D^{-\frac{1}{2}}K'R'E'(G(\mathbf{z}))^{-1}ERKD^{-\frac{1}{2}} \succeq 0
\]
after multiplying this equation by \( KD^{\frac{1}{2}} \) from the left and \( D^{\frac{1}{2}}K' \) from the right, we have
\[
I_{(m+p)\times(m+p)} + R'(E'G(\mathbf{z})E)^{-1}R \succeq 0.
\]
This equation is equivalent to
\[
-I_{(m+p)\times(m+p)} - R'(E'G(\mathbf{z})E)^{-1}R \preceq 0.
\]
By Lemma 5 in the Appendix, the last equation is equivalent to
\[
0 \succeq E'G(\mathbf{z})E + RR' = E'G(\mathbf{z})E + R(KD^{-\frac{1}{2}}DD^{-\frac{1}{2}}K')R'
\]
multiplying by \( E \) from the left and \( E' \) from the right, we can obtain that
\[
0 \succeq G(\mathbf{z}) + (ERKD^{-\frac{1}{2}})D(D^{-\frac{1}{2}}K'R'E') = G(\mathbf{z}) + F(\mathbf{z})DF(\mathbf{z})' = \nabla^2 \Pi(\mathbf{z}).
\]
By the assumption det(\nabla^2 \Pi(\mathbf{z})) \neq 0, \mathbf{z} is a local maximum of \Pi.

Notice that every step of the proof is equivalent, so if \( \mathbf{z} \) is a local maximum of \( \Pi \)
then \( \mathbf{z} \) must be a local maximum of \( \Pi' \).

(iii) Let us consider the three cases:

a) \( n = m + p \): if \( \mathbf{z} \) is a local minimizer of \( \Pi' \) then
\[
\nabla^2 \Pi'(\mathbf{z}) = -F(\mathbf{z})'(G(\mathbf{z}))^{-1}F(\mathbf{z}) - D^{-1} \succeq 0
\]
\[
\iff -F(\mathbf{z})'(G(\mathbf{z}))^{-1}F(\mathbf{z}) \succeq D^{-1}.
\]
This implies that \( \text{Rank } (F(\mathbf{z})) = n \). By multiplying the last inequality by \( (F(\mathbf{z})')^{-1} \) from the left and by \( (F(\mathbf{z}))^{-1} \) from the right, we have
\[
-(G(\mathbf{z}))^{-1} \succeq (F(\mathbf{z})')^{-1}D^{-1}(F(\mathbf{z}))^{-1}.
\]
By Lemma 3 this is equivalent to
\[
-G(\mathbf{z}) \succeq F(\mathbf{z})DF(\mathbf{z})' \iff \nabla^2 \Pi(\mathbf{z}) \succeq 0.
\]
And since \( \text{det}(\nabla^2 \Pi(\mathbf{z})) \neq 0 \), \( \mathbf{z} \) is a local minimizer of \( \Pi \). In a similar way we can prove the converse.
b) From Equation (24) we know that
\[-F(x)^t(G(\varsigma))^{-1}F(x) \succeq D^{-1},\]
then \(-F(x)^t(G(\varsigma))^{-1}F(x)\) is a nonsingular matrix and Rank \((F(x)) = m + p < n\). We claim now that \(x\) is not a local minimizer of \(\Pi\). This is because if \(x\) is also a local minimizer, we would have
\[\nabla^2\Pi(x) = G(\varsigma) + F(x)DF(x)^t \succeq 0,\]
thus
\[F(x)DF(x)^t \succeq -G(\varsigma).\]
This implies that
\[n = \text{Rank } (-G(\varsigma)) = \text{Rank } (F(x)DF(x)^t) = m + p,\]
which is a contradiction. Therefore, \(x\) is a saddle point of \(\Pi\).

To prove Equation (41), we let \(L\) be the matrix as given in Lemma 1 and \(\{l_i\}_{i=1}^{m+p}\) be the column vectors of \(L\). Define
\[\varphi(t_1, \ldots, t_{m+p}) := \Pi(x + t_1l_1 + \ldots + t_{m+p}l_{m+p}).\]
We need to show that \((0, \ldots, 0) \in \mathbb{R}^{m+p}\) is a local minimizer of the function \(\varphi\). Notice that
\[\nabla \varphi(0, \ldots, 0) = L^t\nabla \Pi(x) = 0\]
and
\[\nabla^2 \varphi(0, \ldots, 0) = L^t\nabla^2 \Pi(x)L \succeq 0,\]
which is a consequence of Lemma 1. Furthermore, from Equation (36) we have that
\[\nabla^2 \varphi(0, \ldots, 0) = \text{Diag } (a_1 - \lambda_1, \ldots, a_{m+p} - \lambda_{m+p}),\]
and since \(\text{det}(\nabla^2 \Pi(x)) \neq 0\) it can be proven that \(a_i > \lambda_i\) for every \(i\). The proof is complete.
c) The proof is similar with item b).

Remark 2. Theorem 3 shows that in order to solve the problem \((P)\) by means of the canonical duality theory, a necessary condition is that the problem \((P)\) should have a unique solution. It was indicated in [30] that if the nonconvex minimization problem has more than one global minimizer, it could be NP-hard. In order to solve this type of problems, the perturbation methods should be used.

Remark 3. The triality theory states precisely that if \(\varsigma\) is a global maximizer of \(\Pi^d\) on a certain set, then \(x\) is a global minimizer for \(\Pi\). This is known from the general result by Gao and Strang in [19]. If \(\varsigma\) is a local maximizer for \(\Pi^d\) then \(x\) is also a local maximizer for \(\Pi\). This is the so-called double-max duality statement. If \(\varsigma\) is a local minimizer for \(\Pi^d\), then \(x\) is also a local minimizer for \(\Pi\) in certain directions. This is so-called double-min duality in the standard triality form proposed in [9]. The triality theory was first discovered in nonconvex mechanics [5]. It was realized in 2003 that the double-min duality holds under certain additional condition, which was left as an open problem (see [11, 12]). Recently, this open problem is solved for quartic polynomial optimization problem [20]. This result is now generalized to the general nonconvex problem \((P)\). Part (iii) of Theorem 4 shows that if \(m + p = n\),
then $\varsigma$ is a local minimizer if and only if $x$ is also a local minimizer. In other cases either $x$ is a saddle point of $\Pi$ or $\varsigma$ is a saddle point of $\Pi^d$.

**Remark 4.** The canonical duality-triality theory has been challenged recently by C. Zălinescu and his co-workers R. Strugariu, M. D. Voisei in several papers (see [35]). By list some simple “counterexamples”, they claimed that this theory is false. Unfortunately, most of these counterexamples are not new, which were first discovered by Gao in 2003 [11, 12]. However, [11, 12] never been cited in their papers. Some of their “counterexamples” are fundamentally wrong, i.e. they oppositely choose linear functions as the stored energy and nonlinear functions as external energy (see [36]). These conceptual mistakes show a big gap between mathematics and mechanics.

4. Numerical Examples

In the following examples, $m = p = 1$ and $b_1 = 1$. The graphs provided and the numerical results were obtained using Maxima [24].

4.1. One stationary point in $S^+_a$. First, we consider the case that the primal function has a unique solution. We let $\alpha_1 = \theta_1 = 1$ and

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, f = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$  

Clearly, the function $\Pi : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$\Pi(x, y) = \exp\left(\frac{1}{2}(x^2 + 2y^2) - 1\right) + \frac{1}{2}\left(\frac{1}{2}(x^2 + y^2) - 1\right)^2 + \frac{1}{2}(x^2 - y^2) - x - y,$$

and the dual function has the form of

$$\Pi^d(\tau, \sigma) = -\frac{1}{2}\left(\frac{1}{1 + \tau + \sigma} + \frac{1}{2\tau + \sigma - 1}\right) - \tau \cdot \ln(\tau) - \frac{1}{2}\sigma^2 - \sigma.$$ 

![Figure 1. $\Pi$ function of Example 1](image)

It can be shown that $\Pi^d$ has only one critical point in $S^+_a$ and it is given (approximately) by

$$\bar{\varsigma} = (1.171057661103504, -0.34599084656216).$$

By the triality theory, the vector

$$\bar{x} = G(\bar{\varsigma})^{-1}f = (0.54792514555217, 1.003890602479819)$$

is the only global minimizer of the primal problem.
4.2. **One stationary point in** \(S^+_a\) **and one in** \(S^-_a\). Let \(\alpha_1 = 1\), \(\theta_1 = 50\), and 
\[
A = \begin{bmatrix} 1 & 0 \\ 0 & -16 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, f = \begin{bmatrix} -25 \\ 9 \end{bmatrix}.
\]

The primal function \(\Pi : \mathbb{R}^2 \rightarrow \mathbb{R}\) is then given by
\[
\Pi(x, y) = \exp\left(\frac{1}{2}(x^2 + y^2) - 1\right) + \frac{1}{2}\left(\frac{1}{2}(x^2 + 2y^2) - 50\right)^2 + \frac{1}{2}(x^2 - 16y^2) + 25x - 9y
\]
and its canonical dual is
\[
\Pi^d(\tau, \sigma) = -\frac{1}{2}\left(\frac{81}{-16 + \tau + 2\sigma} + \frac{625}{1 + \tau + \sigma}\right) - \tau \cdot \ln(\tau) - \frac{1}{2}\sigma^2 - 50\sigma,
\]
which has two critical points:
\[
\varsigma_1 = (96.61711963278241, -38.94928057661689) \in S^+_a,
\varsigma_2 = (0.42157060067968, -49.86072154366873) \in S^-_a.
\]

Therefore, by the triality theory, the associated vector
\[
\mathbf{x}_1 = G(\varsigma_1)^{-1}f = (-0.42612784793499, 3.310578038951848)
\]
is the only global minimizer of \(\Pi(x)\) and
\[
\mathbf{x}_2 = (0.51611144112381, -0.078057328303129)
\]
is a local maximizer (see Figure 3) since \(\mathbf{x}_2\) is a local maximum of \(\Pi^d\) in \(S^-_a\) (see Figure 5).
4.3. One stationary point in $S_a^+$ and two in $S_a^-$. In order to illustrate the triality theory, we let $\alpha_1 = \theta_1 = 2$, and

$$A = \begin{bmatrix} -16 & 0 \\ 0 & -4 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, C_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, f = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$ 

Accordingly, we have

$$\Pi(x, y) = \exp\left(\frac{1}{2}x^2 - 2\right) + \frac{1}{2}\left(\frac{1}{2}y^2 - 2\right)^2 + \frac{1}{2}\left(-16x^2 - 4y^2\right) - 2x - 2y,$$

$$\Pi^d(\tau, \sigma) = -\frac{1}{2}\left(\frac{4}{\sigma - 4} + \frac{4}{\tau - 16}\right) - \tau \cdot \ln(\tau) - \tau - \frac{1}{2}\sigma^2 - 2\sigma.$$

In this case, $\Pi^d$ has in total six critical points but only one

$$\underline{\tau} = (16.64468576727409, 4.552474610531074) \in S_a^+,$$

(see Figure 7) and two

$$\underline{\tau}_2 = (0.1364153779858, -1.943380912562619) \in S_a^-,$$

$$\underline{\tau}_3 = (15.34981976568548, 3.39006302031545) \in S_a^-.$$

From Figures 8 we can see that $\underline{\tau}_2$ is a local maximizer and $\underline{\tau}_3$ is a local minimizer of $\Pi^d$. Therefore, by the triality theory, we know that

$$\underline{x}_\tau = G(\underline{\tau})^{-1} f = (3.102286573591542, 3.620075858467906).$$
is the only global minimizer:

$$\overline{x}_2 = (-0.12607490787063, -0.33650880356205)$$

is a local maximizer and

$$\overline{x}_3 = (-3.076070133243102, -3.283567054905852)$$

is a local minimizer of $\Pi(x)$ (see Figure 8).
4.4. Non-unique global minima. In the case that no stationary point can be found in \( S^+_a \), the primal problem could have more than one global minima. To see this, we let \( f \equiv 0, \alpha_1 = \theta_1 = 2, \) and

\[
A \equiv 0, B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, C_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\]

In this case, the primal function

\[
\Pi(x, y) = \exp \left( \frac{1}{2} x^2 - 2 \right) + \frac{1}{2} \left( \frac{1}{2} y^2 - 2 \right)^2
\]

has 2 global minimums at \((0, -2), (0, 2)\) and a local maximum at \((0, 0)\). While the dual function

\[
\Pi^d(\tau, \sigma) = -\tau \ln \tau - \tau - \frac{1}{2} \sigma^2 - 2\sigma
\]

does not have a stationary point in \( S^+_a \). There is however a critical point in the boundary of \( S^+_a \), namely, \( \varpi = (\exp(-2), 0) \). By defining \( \mathbf{x} = G(\varpi)^{-1}f \), we have that \( \mathbf{x} = (0, 0) \).

In order to find a global minimum of \( \Pi \), we need to introduce the following perturbations:

\[
A_n = \begin{bmatrix} -\frac{16}{n} & 0 \\ 0 & -\frac{4}{n} \end{bmatrix} \quad \text{and} \quad f_n = \begin{bmatrix} \frac{2}{n} \end{bmatrix}, \quad \text{for every } n \in \mathbb{N}.
\]

Then, the associated primal and dual functions are

\[
\Pi_n(x, y) = \exp \left( \frac{1}{2} x^2 - 2 \right) + \frac{1}{2} \left( \frac{1}{2} y^2 - 2 \right)^2 + \frac{1}{2} \left( -\frac{16}{n} x^2 - \frac{4}{n} y^2 \right) - \frac{2}{n} x - \frac{2}{n} y,
\]

\[
\Pi^d_n(\tau, \sigma) = -\frac{1}{2} \left( \frac{4}{n^2 (\tau - \frac{16}{n}) + \frac{4}{n^2 (\sigma - \frac{4}{n})}} \right) - \tau \ln \tau - \tau - \frac{1}{2} \sigma^2 - 2\tau - 2\sigma.
\]

Notice that if \( n = 1 \) we are in the case presented in Example 3. Let us show that for sufficiently large values of \( n \) we can find a stationary point for \( \Pi^d_n \) in \( S^+_a \), namely \( \varpi_n \). Furthermore, by defining \( \mathbf{x}_n = G(\varpi)^{-1}f \), we will have a convergent sequence.

Let us calculate the gradient of \( \Pi^d_n \):

\[
\nabla\Pi^d_n(\tau, \sigma) = \begin{bmatrix} -2 - \ln \tau + \frac{2}{(n\tau - 16)^2} \\ -\sigma - 2 + \frac{2}{(n\sigma - 4)^2} \end{bmatrix}.
\]

Let \( h(\tau) = -2 - \ln \tau + \frac{2}{(n\tau - 16)^2} \) and \( g(\sigma) = -\sigma - 2 + \frac{2}{(n\sigma - 4)^2} \). It is not difficult to show that there exists a sufficiently large \( N \in \mathbb{N} \), such that if \( n > N \), the following are true:

a) \( n \cdot \exp\left(-2 + \frac{1}{n}\right) - 16 \) and \( n \cdot \exp(-2) - 16 \) are positive numbers.

b) \( h\left(\exp\left(-2 + \frac{1}{n}\right)\right) = \frac{2}{(n \cdot \exp(-2 + \frac{1}{n}) - 16)^2} - \frac{1}{n} \)

\( < 0 < h(\exp(-2)) = \frac{2}{(n \cdot \exp(-2) - 16)^2} \).

\[
c) \ g\left(\frac{5.1}{n}\right) \approx -\frac{5.1}{n} - 0.34710743801 < 0 < g\left(\frac{4.9}{n}\right) \approx 0.46913580247 - \frac{4.9}{n}.
\]
Based on these results, we know that for every \( n > N \), \( \nabla \Pi^d \) has a stationary point \( \bar{\mathbf{s}}_n = (\mathbf{\tau}_n, \mathbf{\sigma}_n) \in [\exp(-2), \exp(-2 + \frac{1}{n})] \times [\frac{4n}{\exp(-2)}, \frac{\exp(-2 + 1/n)}{n}] \). Moreover, by the fact that \( g(\bar{\mathbf{s}}_n) = 0 \), it is easy to obtain \( \lim_{n \to +\infty} n \cdot \mathbf{\sigma}_n = 5 \).

Notice also that

\[
G(\bar{\mathbf{s}}_n) = \begin{bmatrix} \mathbf{\tau}_n - \frac{16}{n} & 0 \\ 0 & \mathbf{\sigma}_n - \frac{4}{n} \end{bmatrix}
\]

is positive definite. Therefore, the perturbed solution can be obtained as

\[
\bar{\mathbf{x}}_n = G(\bar{\mathbf{s}}_n)^{-1} \mathbf{f}_n = \begin{bmatrix} \frac{2}{n \exp(-2) - 16} \\ \frac{n \exp(-2) - 16}{n \exp(-2) - 1} \end{bmatrix}.
\]

Since \( \mathbf{\tau}_n \in [\exp(-2), \exp(-2 + \frac{1}{n})] \) then \( \lim_{n \to +\infty} \mathbf{\tau}_n = \exp(-2) \). By the fact that \( \lim_{n \to +\infty} n \cdot \mathbf{\sigma}_n = 5 \), we have

\[
\lim_{n \to +\infty} \bar{\mathbf{x}}_n = \begin{bmatrix} 0 \\ 2 \end{bmatrix},
\]

which is a solution of \( \Pi \).

Canonical perturbation method was originally introduced in [31] for solving non-convex polynomial minimization problems. This method has been used successfully in integer programming and network communication (see [17, 37]).

5. Future Research

Some open questions that will be studied in the future are the following:

- As stated in Remark 1, in order to use the canonical dual transformation a necessary condition is that \((\mathcal{P})\) has a unique solution. Is this also a sufficient condition? In other words, giving \((\mathcal{P})\) such that it has a unique solution, can we find a stationary point of \(\Pi^d\) in \(\mathcal{S}^+_n\)?
- Example 4 shows an interesting perturbation method that allows us to solve a problem when the necessary condition of Remark 1 is not satisfied. Can we generalize this method and develop an algorithm?

6. Appendix: Some Lemmas in Matrix Analysis

The following results are needed in the proofs of Section 2.

**Lemma 3.** (Singular value decomposition [22]) For any given matrix \( \mathbf{M} \subset \mathbb{R}^{m \times n} \) with \( \text{Rank} \ (\mathbf{M}) = r \), there exists \( \mathbf{U} \subset \mathbb{R}^{m \times m}, \mathbf{R} \subset \mathbb{R}^{m \times n} \) and \( \mathbf{E} \subset \mathbb{R}^{n \times n} \) such that

\[
\mathbf{M} = \mathbf{U} \mathbf{R} \mathbf{E};
\]

where \( \mathbf{U} \) and \( \mathbf{E} \) are orthogonal matrices, and

\[
R_{ij} = \begin{cases} s_i, & i = j, \ i = 1, \ldots, r \\ 0, & i \neq j \end{cases}
\]

where \( s_i > 0 \) for every \( i = 1, \ldots, r \).

**Lemma 4.** [22] If \( \mathbf{G} \) and \( \mathbf{U} \) are positive definite matrices in \( \mathbb{R}^{n \times n} \), then \( \mathbf{G} \succeq \mathbf{U} \) if and only if \( \mathbf{U}^{-1} \succeq \mathbf{G}^{-1} \).
Lemma 5. \[20\] Suppose \( P, U \) and \( D \) are three matrices in \( \mathbb{R}^{n \times n} \) such that

\[
D = \begin{bmatrix}
D_{11} & 0_{m \times (n-m)} \\
0_{(n-m) \times n} & 0_{(n-m) \times (n-m)}
\end{bmatrix},
\]

where \( D_{11} \in \mathbb{R}^{m \times m} \) is nonsingular and

\[
P = \begin{bmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix} \prec 0, \quad U = \begin{bmatrix}
U_{11} & 0_{m \times (n-m)} \\
0_{(n-m) \times m} & U_{22}
\end{bmatrix} \succ 0,
\]

\( P_{ij} \) and \( U_{ij} \) are appropriate dimensional matrices for \( i, j = 1, 2 \). Then,

\[
P + DUD^t \preceq 0 \iff -D^tP^{-1}D - U^{-1} \preceq 0.
\]

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