Many-body dynamics of a Bose system with attractive interactions on a ring

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PACS numbers: 03.75.Kk, 05.30.Jp, 67.40.Db, 05.45.Pq

I. INTRODUCTION

Bose-Einstein condensates (BECs) in quasi-one-dimensional systems have attracted much more attention in recent years because it, depending on the repulsive or attractive nature of two-body interaction, can form dark- or bright-soliton states, respectively. It is believed that atomic matter bright solitons are of primary importance for developing concrete applications of BECs in the future, so of particular interests have been focused on the study of effectively attractive BECs. Using the Feshbach resonances, the coupling constant was tuned to a negative value and both bright-soliton trains and a single bright-soliton were realized experimentally.

The Gross-Pitaevskii mean-field theory (MFT) is a powerful tool in the study of BECs, which has successfully described an impressive set of experiments. However it breaks down if quantum many-body (QMB) correlations are important, such as in quantum phase transitions. For one-dimensional Bose system with attractive interaction, MFT predicts the transition from a uniform state to a soliton state at a critical point when increasing the absolute value of the strength of the interaction. Recently, Kamamoto et.al. and Kavoulakis carefully checked the ground state and low-lying excitation spectra near and above the transition. They revealed that quantum correlations become crucial. The resulting properties of QMB correlations, such as the singularity in the transition is replaced by a crossover region, the condensate fraction begins to decrease, etc. suggest that the MFT should qualitatively be modified and QMB effects have to be considered. On the other hand, most of the previous studies regard QMB effects in the stationary situations, which attempt to beyond MFT by using the diagonalization schemes. Therefore it is desirable to explore the quantum dynamics of this system. In the present paper, we numerically study the quantum dynamics of N interacting Bosons in a toroidal trap, which plays an important role in the physics of trapped dilute gases. As a function of interaction strength, we calculate the Shannon entropy of the wave packets to characterize the quantum dynamics. A modulation instability at critical point is discovered, which is reflected by a rapid increase of entropy. As a direct result, the interference fringes, which occur after switching off the confinement and letting the particles spread in a ballistic way, vanish near and above the critical point.

The toroidal one-dimensional regime can be realized when the transverse dimensions are on the order of the healing length, and the longitudinal dimension is much longer than the transverse ones. Experimentally, this geometry may be achieved by an optical trap with Laguerre-Gaussian beams. The system is tightly confined in the radial direction so that the energy-level spacings in this direction greatly exceed the interaction energy. Thus N Bosons are confined on a ring of radius R and cross section S = πr² with r ≪ R. The Hamiltonian for this system reads

\[ H = \int_0^{2\pi} d\theta \left[ -\Psi^\dagger(\theta) \frac{\partial^2}{\partial \theta^2} \Psi(\theta) + \frac{U}{2} \Psi^\dagger(\theta) \Psi^\dagger(\theta) \Psi(\theta) \Psi(\theta) \right] \]

where \( \Psi(\theta) \) is the field operator, \( U = 8\pi a R/S \), with \( a \) being the s-wave scattering length, and \( \theta \) denotes the azimuthal angle. Here the length and the energy are measured in units of \( R \) and \( \hbar^2/2mR^2 \). The normalization of \( \Psi(\theta) \) is \( \int |\Psi(\theta)|^2 d\theta = N \). The Hamiltonian is characterized by a single dimensionless parameter \( \gamma \)

\[ \gamma \equiv \frac{U N}{2\pi} \]

which is essential the ratio between the interaction energy and the kinetic energy. The transition occurs when \( \gamma < \gamma_c \equiv -1/2 \).
This paper is organized as follows. In sec.II, after numerically obtaining the eigenenergies and eigenstates of the system, we study the dynamical evolutions of Shannon entropy of the system. The evolutions show quasiperiodical and irregular behaviors below and above critical point correspondingly. The irregular one is interpreted as much more number of virtually excited particles involved in the dynamics due to strong interaction. In sec.III, we discuss the momentum distribution of single-particle states. Above critical point the system exhibits a statistical relaxation to a steady distribution. The interference fringes of momentum, which can be obtained experimentally by releasing the trap, are studied too. We conclude in sec.IV

II. SHANNON ENTROPY OF THE SYSTEM

In cylindrical coordinate expanding the field operator $\Psi(\theta, t)$ in terms of plane waves as

$$\Psi(\theta, t) = \frac{1}{\sqrt{2\pi}} \sum_l \hat{c}_l(t) e^{il\theta}$$

(3)

with $l$ integer. Where $\hat{c}_l$ is annihilation operator of a boson with angular momentum $l$. Then Hamiltonian (1) can be rewritten as

$$\hat{H} = \sum_l l^2 \hat{c}_l^\dagger \hat{c}_l + \frac{\gamma}{2N} \sum_{kl:mn} \hat{c}_k^\dagger \hat{c}_l \hat{c}_m \hat{c}_n$$

(4)

Expressing the single-particle states $|l\rangle$ according to their angular momentum $l = 0, \pm 1, \pm 2, \ldots$, a many-body basis state $|k\rangle$ can be represented as $|n_{-L}, \ldots, n_{-1}, n_0, n_1, \ldots, n_L\rangle$ restricted to the conservations of the total number of particles and the total angular momentum

$$\sum_{l=-L}^L n_l = N, \sum_{l=-L}^L ln_l = 0$$

(5)

where $n_l$ denotes the number of bosons occupying the single-particle states $|l\rangle$, and $L$ is the cutoff of the angular momentum. Since $N$ bosons define the smallest spacing on a ring, it is practical to make $L \approx N$. The system is occupied by $m = 2L + 1$ single-particle states. With these assumptions, eigenvalues and eigenstates of Hamiltonian (4) can be solved directly by a smart method (see Appendix).

At first we introduce Shannon entropy to characterize the dynamical behavior of the system, which has been used to manifest the many-body dynamics of Tonks-Girardeau gas. The Shannon entropy is defined

$$S(t) = -\sum_k |\Psi_k(t)|^2 \ln |\Psi_k(t)|^2$$

(6)

FIG. 1: Entropy versus rescaled time $t$ for $L = 8, N = 8$ and different $\gamma$. Dotted curve describes $\gamma = -0.14$. Dashed curve corresponds to $\gamma = -0.54$. Thin solid curve represents $\gamma = -1.62$ and thick solid curve represents $\gamma = -6.48$.

FIG. 2: Time dependence of principal components for same parameters as fig.1. The correspondences between line styles and $\gamma$ are same as fig.1 too.

Where $\Psi_k(t) = \langle k | \Psi(t) \rangle$ is the projection of the wave function onto the many-body basis state $|k\rangle$. And the time-dependent wave function reads

$$|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle$$

(7)

Where $|\Psi(0)\rangle$ is state of the system at $t = 0$.

Then, by varying $a$ via the Feshbach technology, let us consider an interesting case for which initially the system is condensed on the ground state without interaction. After switch on the interaction, the wave function spreads over all the basis.
The time dependence of entropy is reported in Fig.1. For $0 > \gamma > \gamma_c$, the entropy oscillates quasiperiodically in time. For $\gamma < \gamma_c$, the small oscillation is disappeared and the entropy increases linearly to a saturation for a strong enough interaction. For weak interactions the system is condensed on the ground state. The small deviations from ground state are suppressed with time increasing $\gamma < \gamma_c$, which leads to the periodic oscillations of the entropy. It also ensures the validity of MFT with only a few modes $19, 20$. When $\gamma$ is lowered ($|\gamma|$ is increased), the number of virtually excited particles due to stronger interaction is increased $19, 20$. This indicates that much more basis states are involved in the dynamics. As a result, the entropy is evolved with a linear increase followed by a saturation. It implies that the initial ground state decreases exponentially while the basis excited in the dynamics increase exponentially in time (see Fig.2). It can not be explained in the MFT framework. To address the number of basis obviously, one may calculate the principal components for wave packets in the basis representation $\hat{H}$, which is defined by the entropy $N_{eff}(t) = \exp[S(t)]$, which is regarded as the effective number of unperturbed many-body states in the dynamics. A recent paper about the two-body random interaction model $25$ shows that in a finite system, the number of basis states excited in the system is finite too. Although Hilbert space is same for the system independent of strength of interaction, the dynamics will depend upon it. Stronger the strength of interaction, more basis involved in the dynamics. So the finity of basis leads to the saturation of the entropy. And the linear part of the time entropy may be interpreted as the onset of the chaotic superpositions of the basis states $25$.

Although the complicity of the quantum correlations near and above the critical point $14$, the Shannon entropy is appropriate to characterize the dynamical properties of the system. To compare the results for different $\gamma$, we calculate the rescaled entropy $S_{nor} = S_{ave}/N_H$, where $S_{ave}$ is the mean value of the entropy averaged over time after the saturation and $N_H$ is the dimension of the Hilbert space, i.e. the total number of basis. Fig 3 presents the results for $L = 6, 7, 8$ and $N = 8$. Here, different $L$ can make us find the dependence of entropy on $L$. We can see that the data are sensitive to the modulation instability. For $0 > \gamma > \gamma_c$, $S_{nor}$ is stagnated to zero. This is same to the MFT results, where bosons are almost condensed to the ground state. Near the critical point, the virtual high-momentum single-particle states are excited and the system is influenced significantly by the quantum fluctuations $13$. In this region, slight variations of $\gamma$ alter the renormalized entropy obviously for given $L$. For $\gamma < \gamma_c$, $S_{nor}$ reaches to a plateau with altitude $\sim 0.85$. We guess that for the limitation case $L \to \infty$, the renormalized entropy will reach to unity corresponding to the situation that all of the basis states have the same probabilities to be occupied.

III. MOMENTUM DISTRIBUTION AND FRINGE VISIBILITY

Quantum dynamics characterized by the Shannon entropy is investigated for different $\gamma$ in previous section. Similarly to MFT, a modulation instability is founded at a critical point. But near and above critical point, the irregular behaviors of quantum dynamics can not be obtained by MFT due to quantum correlations.

QMB effects influence not only the entropy, but also the momentum distribution, which can be observed in experiment. What we do in the following is to address the influence.

Using the formal Fourier transforming, we obtain the momentum function

$$\hat{\chi}(\vec{p}) = \int dr \hat{\Psi}(\theta)e^{-i\vec{p} \cdot \vec{r}}$$

(8)

Let us consider momentum along the $y$-axis $\vec{p} = p \cdot \hat{e}_y$. We get

$$\hat{\chi}(\vec{p}) = A \sum_k \hat{c}_k \int_0^{2\pi} d\theta e^{i k \theta} e^{-ipR\sin^ \theta}$$

(9)

Where $A$ is a normalization constant. Note that the integration is the Bessel function $23$ and substitute it in the equation. Then

$$\hat{\chi}(p) = A \sum_n \hat{c}_n J_n(pR)$$

(10)

Where $J_n(x)$ is the first kind Bessel function of order $n$. The occupation number distribution is

$$n(p, t) = \langle \Psi(t) | \hat{\chi}^\dagger(p) \hat{\chi}(p) | \Psi(t) \rangle$$

(11)
Where $C$ is a normalization constant.

Steady momentum distribution means the collapses of the condensates. This suggests that we can use the fringe visibility to address dynamical evolutions. The visibility is defined as:

$$v(t) = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}}$$  \hspace{1cm} (14)$$

where $I_{\text{max}} = n(p = 0, t), I_{\text{min}} = n(p_0, t)$ and $p_0$ is the first zero of $J_0$. The results are reported in Fig.5. The periodical and irregular behaviors of the dynamics are showed for some $\gamma$. For $0 > \gamma > \gamma_c$, quantum dynamics study reveals that the visibility of a uniform ground state oscillates with time (see inset of Fig.5), contrasting with the GP framework where the visibility is unity $[8, 19]$. For strong interaction, the visibility is described by a fast decay followed by irregular fluctuations. It can be used to distinguish the modulation instability. So entropy, momentum distribution and visibility are strictly consistent to characterize the dynamical behaviors of the one-dimensional Bose system.

IV. CONCLUSION

We studied the dynamical properties of attractive one-dimensional Bose system with the QMB dynamics. Our main result is the onset of a modulation stability at $\gamma_c$. We found that the quantum dynamics of the system is distinguished by the periodical and irregular behaviors for $\gamma$ below and above critical point respectively.

Within the MFT, a phase transition between uniform state and bright soliton is expected at a critical point $[8, 19]$. We investigate the dynamical behaviors by varying the interacting strength. Far below the critical point, the system is confined to the ground state corresponding to

FIG. 4: Particle density distributions in the momentum representation for four parameters $\gamma$ at different times. Thicker curves describe latter times. Dotted, Dashed, Thin solid, thick solid curves correspond $t = 0, 0.1, 0.6, 1.$

Using Eq.(7) and Eq.(10), we obtain

$$n(p, t) = \sum_{kk'} \Psi_k^\dagger \Psi_{k'} \sum_{l,l'} J_l(pR) J_{l'}(pR) \langle k| \hat{c}^\dagger_l \hat{c}_{l'} |k' \rangle$$  \hspace{1cm} (12)$$

With $\langle k| \hat{c}^\dagger_l \hat{c}_{l'} |k' \rangle = n_l^k \delta_{kk'} \delta_{ll'}$ where $n_l^k$ denotes the number of particles with angular momentum $l$ in the basis $|k\rangle$, we get

$$n(p, t) = C \sum_k |\Psi_k|^2 \sum_l J_l^2(pR) a_l^k$$  \hspace{1cm} (13)$$

Where $C$ is a normalization constant.

The momentum distributions for several $\gamma$ are shown in Fig.4. One can see that for weakly interacting system the occupation number distribution will not be changed obviously in time. But small perturbations lead to quasi-periodical oscillation of the entropy (Fig.1). When the two-body interaction strength exceeds the critical point, many high-momentum single-particle states are excited. With time increasing, the momentum distribution becomes flat and the condensate can not be retrieval. A steady momentum distribution means the collapses of the condensates. This suggests that we can use the fringe visibility to address dynamical evolutions. The visibility is defined as $[22]$

$$v(t) = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}}$$  \hspace{1cm} (14)$$

We investigated the dynamical behaviors by varying the interacting strength. Far below the critical point, the system is confined to the ground state corresponding to
a condensate. The entropy oscillates quasiperiodically in this situation, which is connected with the collapses and revivals of the condensate fraction (Fig.1 and Fig.2). And the system is well described by MFT. In the crossover region the behaviors of entropy change gradually depending on $\gamma$. For strong interaction regime, evolution of the entropy is characterized by a fast linear increase followed by a saturation. This indicates that the ground state is relaxed to steady states distribution and the condensate is destroyed. The results of momentum distributions and the fringe visibility confirm the modulation instability of the system. They are of particular interest because they can be studied experimentally.

The QMB effects, such as entropy, principal component, momentum distribution and visibility have important meaning in the study of the finite Bose system\[22]. The investigation in the context demonstrates that QMB effects significantly influence the dynamical behaviors and can help us to comprehend both the quantum evolution of the system and the mechanism of the transition beyond MFT.

Acknowledgments

The work is supported in part by the NSF of China (Grant Nos. 60478029 and 10125419) and by the National Fundamental Research Program of China, Grant No.2001CB309310. W.L. thanks Dr. Y. Wu for a stimulating discussion.

APPENDIX: ENERGY SPECTRUM AND EIGENSTATES OF A BOSE SYSTEM ON A RING

In the following, we give an introduction about the algebra method to calculate the eigenvalues of Hamiltonian\[11] based on Ref.\[24]. One can see Ref.\[24] for a detail.

A many-body basis can be represented as

$$|n_L, \cdots, n_1, n_0, \cdots, n_L\rangle = \sqrt{\prod_l n! l!} \prod_l (\hat{c}_l^\dagger)^{n_l} |\text{vac}\rangle \quad \text{(A.1)}$$

where $|\text{vac}\rangle$ is the vacuum state satisfying $\hat{c}|\text{vac}\rangle = 0$. Hence, we denote the eigenstate of Hamiltonian\[11] as

$$|\Psi\rangle = F(\hat{c}_L^\dagger, \cdots, \hat{c}_1^\dagger, \hat{c}_0^\dagger, \cdots, \hat{c}_L^\dagger)|\text{vac}\rangle \quad \text{(A.2)}$$

where $F$ is the linear combination of the terms $\prod_l (\hat{c}_l^\dagger)^{n_l}$. The eigenvalue equation $H|\Psi\rangle = E|\Psi\rangle$ can be rewritten as $EF|\text{vac}\rangle = HF|\text{vac}\rangle = [H, F]|\text{vac}\rangle$ due to $H|\text{vac}\rangle = 0$. Using $[\hat{c}_l, F] = \partial F/\partial \hat{c}_l^\dagger$, $[\hat{c}_l \hat{c}_m, F] = \partial^2 F/\partial \hat{c}_l^\dagger \partial \hat{c}_m^\dagger$, the polynomial function $F$ satisfies the following differential equation

$$\sum_l \hat{c}_l^2 x_l \partial F/\partial x_l + \sum_{l\neq m} x_l x_m \partial^2 F/\partial x_m \partial x_n = EF \quad \text{(A.3)}$$

where $x_l = \hat{c}_l^\dagger$. And the polynomial function $F = F(x_L, x_{L-1}, \cdots, x_1, x_0)$ is the linear combinations of the terms $\prod_l x_l^{n_l}$. Now, the eigenvalues problem is reduced to seek polynomial solutions to a second order linear differential equation. Utilizing this method, it is not a hard work to use a MATHEMATICA program to calculate all the eigenvalues and eigenstates of Hamiltonian\[11] for finite $N$ and $L$.
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