ESSENTIAL DIMENSION, PRO-FINITE GROUP SCHEMES AND ANABELIAN GEOMETRY

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ABSTRACT. In the present paper we mix ideas from the fields of anabelian geometry and essential dimension, obtaining results in both.

With regards to anabelian geometry, we formulate a dimensional version of Grothendieck’s section conjecture. Grothendieck’s conjecture implies the dimensional conjecture. We prove that the dimensional version holds for abelian varieties, unconditionally.

With regards to essential dimension, we prove two general criteria showing that the essential dimension of pro-finite, non-finite group schemes is almost always infinite. We thus propose a new definition of essential dimension, the fce dimension fced \( G \) of a group scheme \( G \), which coincides with the classical one if \( G \) is of finite type but has a better behaviour for pro-finite groups. Over any field, we compute \( fced \mathbb{Z} = \dim G \) where \( TG \) is the Tate module of a torus \( G \), in particular \( fced \mathbb{Z}(1) = 1 \). Over fields finitely generated over \( \mathbb{Q} \), we compute \( fced \mathbb{Z}_p = 0 \) and \( fced TA = \dim A \) where \( TA \) is the Tate module of an abelian variety \( A \).

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1. INTRODUCTION

Essential dimension has been introduced by Buhler and Reichstein in [BR97] as a measure of the complexity of torsors under a group scheme \( G \): it is the minimal number of parameters necessary in order to define a generic \( G \)-torsor. Up to now, essential dimension has only been studied for group schemes of finite type, for which a number of tools has been developed: most notably, the existence of the so-called versal torsors, which among other things ensures that every affine group scheme of finite type has finite essential dimension.

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1.1. Essential dimension of pro-finite group schemes. The situation is completely different for pro-finite group schemes, but in some sense much simpler: essential dimension is almost always infinite. We prove the following two criteria.

**Theorem 3.9.** Let $G$ be a pro-finite group scheme over a field $k$, and suppose there exists an extension $k'/k$ and a non-trivial morphism $G_{k'} \to \mathbb{Z}_p$. Then $\text{ed}_k G = \infty$.

**Theorem 3.10.** Let $G$ be a pro-finite étale abelian group scheme over a field $k$. Then $\text{ed}_k G < \infty$ if and only if $G$ is finite.

Even if plain essential dimension is almost always infinite for pro-finite group schemes, we think that its formalism and ideas may still give non-trivial information about them, in particular with respect to Grothendieck’s section conjecture.

1.2. The section conjecture. In 1983, Grothendieck proposed in a famous letter to Faltings [Gro97] a series of ideas and conjectures that described how the geometry of a particular class of varieties over fields finitely generated over $\mathbb{Q}$, called abelian varieties, should be reflected completely in their étale fundamental group. Among these conjectures, one remains largely open and is a major problem in number theory: the section conjecture.

Recall that the étale fundamental group scheme $\pi_1(X)$ of a variety $X$ is a pro-finite étale group scheme which carries the same information of the classical étale fundamental group $\pi_1(X)$ plus its projection $\pi_1(X) \to \text{Gal}(\bar{k}/k)$ to the absolute Galois group. In characteristic 0, it coincides with Nori’s fundamental group scheme. There is a natural morphism of functors $X(\_ ) \to H^1(\_ , \pi_1(X))$.

The section conjecture, as reformulated by Borne and Vistoli in [BV15, §9], says that if $X$ is a smooth, projective, hyperbolic curve over a field $k$ finitely generated over $\mathbb{Q}$, then the natural map $X(L) \to H^1(L, \pi_1(X))$ is a bijection for fields $L$ finitely generated over $k$.

Consider now the following observation of Vistoli. Assume that Grothendieck’s section conjecture holds: then for $X$ as above $\pi_1(X)$ should somehow have essential dimension 1, since its functor of torsors coincides with the points of the curve. Because of Theorem 3.9 this cannot be true in a naïve sense.

In order to obtain a meaningful theory of essential dimension for pro-finite group schemes and formalize Vistoli’s observation, we define two variants of essential dimension: finite type essential dimension and continuous essential dimension. These two variants operate in orthogonal directions, and both coincide with classical essential dimension for algebraic group schemes. We merge them in what we call the fce dimension (finite type, continuous, essential dimension) which may be thought as the right extension of essential dimension to pro-finite group schemes.

1.3. Finite type essential dimension. In the first variant of essential dimension we consider only torsors defined over fields finitely generated over the base field, and call the resulting invariant the finite type essential dimension $\text{fed}_k G$ of a pro-finite group scheme $G$. This was also the original definition used by Z. Reichstein, see [Rei00, §3, §12]: later, the distinction was overlooked since it is not relevant for groups of finite type. We compute the finite type essential dimension for $\mathbb{Z}_p(1) = \lim_{\leftarrow n} \mu_p^n$ and $\mathbb{Z}_p$. 
Theorem 5.12. Over any field \( k \), \( \text{fed}_k \mathbb{Z}_p(1) = \infty \).

Theorem 5.19. Let \( k \) be finitely generated over \( \mathbb{Q} \). Then \( \text{fed}_k \mathbb{Z}_p = 0 \).

Theorem 5.19 has a purely Galois-theoretic interpretation: \( \mathbb{Z}_p \)-extensions of fields finitely generated over \( \mathbb{Q} \) are defined over number fields.

Corollary 5.20. Let \( k \) be finitely generated over \( \mathbb{Q} \), and let \( k = \mathbb{Q}^K \) be the algebraic closure of \( \mathbb{Q} \) in \( K \). If \( H/K \) is a \( \mathbb{Z}_p \)-extension, there exists a \( \mathbb{Z}_p \)-extension \( h/K \) such that \( H = hK \).

As Theorem 5.12 and Theorem 5.19 show, finite type essential dimension depends heavily on the arithmetic of the base field: for instance, \( \text{fed}_k \mathbb{Z}_p = \infty \) if \( k \) is a number field but \( \text{fed}_k \mathbb{Z}_p = 0 \) if \( k = \mathbb{Q} \), because \( \mathbb{Z}_p \simeq \mathbb{Z}_p(1) \) over \( \mathbb{Q} \). In fact, the proof of Theorem 5.19 relies on both the Mordell-Weil and Faltings’ theorems.

We also emphasize that Theorem 5.12 is completely different from the two criteria for infinite essential dimension given above: while the criteria follow from very general constructions for pro-finite group schemes, the fact that \( \text{fed}_k \mathbb{Z}_p(1) = \infty \) really depends on the structure of \( \mathbb{Z}_p(1) \), and in fact the proof does not generalize to \( \mathbb{Z}_p \).

These two results are quite surprising. A theorem of Florence [Flo08, Theorem 4.1] implies that \( \lim_n \text{ed}_k \mathbb{Z}/p^n = \infty \) over fields finitely generated over \( \mathbb{Q} \), but still we have \( \text{fed}_k \mathbb{Z}_p = 0 \). This paradox is explained simply by the fact that torsors of positive essential dimension do not extend.

For \( \mathbb{Z}_p(1) \), we have that \( \lim_n \text{ed}_k \mu_{p^n} = 1 \) but \( \text{fed}_k \mathbb{Z}_p(1) = \infty \). This other paradox is more subtle: it relies on a pathological phenomenon that appears when we pass to the limit, the same phenomenon by which the \( \mathbb{Z} \)-module \( \mathbb{Z}_p \) has rank equal to \( \infty \) but topological rank equal to \( 1 \).

1.4. Continuous essential dimension. The second variant of essential dimension we introduce, i.e. the continuous essential dimension \( \text{ced}_k G \) of a pro-algebraic group scheme \( G \), corrects this pathology.

If \( M \) is a pro-finite abelian group, it is possible to define the topological rank of \( M \) as the limit of the ranks of its finite quotients. If \( T \) is a torsor for a pro-finite group scheme \( G = \varprojlim G_i \), we define the continuous essential dimension \( \text{ced}_k T \) of \( T \) as the limit of the essential dimensions of its finite pushforwards \( \text{ed}_k T \times^G G_i \). The continuous essential dimension of \( G \) is the supremum of the continuous essential dimensions of \( G \)-torsors.

It is possible to merge in an obvious way the finite type and continuous variants, and we obtain the fce dimension \( \text{fced}_k G \). All these variants of essential dimension coincide with the classical one for group schemes of finite type.

If \( A \) is an algebraic torus and \( T_p A = \varprojlim A[p^n], TA = \prod_p T_p A \) are its local and global Tate modules, we prove the following.

Theorem 6.10. Let \( k \) be a field and \( A \) an algebraic torus over \( k \). Then

\[
\text{ced}_k T_p A = \text{fced}_k T_p A = \text{ced}_k TA = \text{fced}_k TA = \dim A.
\]

In particular, for \( A = \mathbb{G}_m^d \) we have

\[
\text{ced}_k \mathbb{Z}_p(1)^d = \text{fced}_k \mathbb{Z}_p(1)^d = \text{ced}_k \mathbb{Z}(1)^d = \text{fced}_k \mathbb{Z}(1)^d = d.
\]
For a general pro-algebraic group scheme $G = \lim_{\leftarrow} G_i$ we have $\text{ced}_k G \leq \lim\inf_i \text{ed}_k G_i$, and the inequality may be strict. In 6.4 we give a counterexample found by F. Scavia: a 1-dimensional torus $A$ on $\mathbb{Q}$ for which $\text{ed}_\mathbb{Q} A[2^n] \geq 2$ for $n > 1$. Still, $\text{ced}_\mathbb{Q} T_2 A = 1$ thanks to Theorem 6.10. See Remark 6.5 for a discussion on why we don’t simply study the asymptotic behaviour of $\text{ed}_k G_i$.

1.5. Fce dimension and anabelian geometry. We can now formalize Vistoli’s observation.

**Dimensional section conjecture.** Let $k$ be a field finitely generated over $\mathbb{Q}$, and $X$ a smooth, geometrically connected hyperbolic curve. Then $\text{fced}_k \pi_1(X) = 1$ and, if $X$ is proper, $\text{fed}_k \pi_1(X) = 1$.

In Proposition 7.1 we show that if $X$ is an affine curve (except $X = \mathbb{A}^1$) then $\text{fed} \pi_1(X) = \infty$, thus asking $\text{fed}_k \pi_1(X) = 1$ only makes sense for proper curves.

**Proposition 7.2.** Grothendieck’s section conjecture implies the dimensional section conjecture.

We prove that the dimensional section conjecture holds for abelian varieties. If $A$ is an abelian variety and $T_p A = \lim_{\leftarrow} A[2^n]$, $T_A = \prod T_p A$ are its local and global Tate modules, we prove the following.

**Theorem 7.6.** Let $A$ be an abelian variety over a field $k$ finitely generated over $\mathbb{Q}$, and $p$ a prime number. Then $\text{fced}_k TA = \text{fced}_k T_p A = \dim A$.

Compare Theorem 7.6 with Theorem 6.10: the Tate module of a torus is a free $\mathbb{Z}$-module of rank equal to the dimension, while the Tate module of an abelian variety is a free $\mathbb{Z}$-module of rank equal 2 times the dimension. Still, their fce dimension is equal to the dimension in both cases. While for a torus this holds over any field, for abelian varieties the fact that the base field is finitely generated over $\mathbb{Q}$ gives a crucial arithmetic input.

Finally, we show using the example of abelian varieties that neither finite type essential dimension nor continuous essential dimension alone are enough in order to study questions arising from anabelian geometry.

**Proposition 7.7.** If $A$ is an abelian variety over any field $k$ of characteristic different from $p$, then $\text{ced}_k TA \geq \text{ced}_k T_p A \geq 2 \dim A$.

**Theorem 7.10.** Over any field $k$ of characteristic 0 and for any prime number $p$, if $A$ is a positive dimensional abelian variety then $\text{fed}_k TA = \text{fed}_k T_p A = \infty$.

Étale fundamental group schemes provide a natural source of pro-finite group schemes. We plan to study their fce dimension in a future paper, with a focus on the exceptional properties implied by Grothendieck’s section conjecture.

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2. TORSORS FOR PRO-ALGEBRAIC GROUP SCHEMES

If \( G = \lim_{\rightarrow} G_i \) is a pro-algebraic group scheme over \( k \) (every affine group scheme is pro-algebraic) and \( L/k \) is an extension, we have to clarify what we mean by \( H^1(L, G) \): there are at least three possibilities:

- \( \lim_i H^1(L, G_i) \).
- The set \( \text{Tors}(L, G) \) of \( G \)-torsors over \( L \), where by \( G \)-torsor we mean a scheme \( T \) over \( L \) with an action of \( G \) such that \( G_L \times_L T \to T \times_L T \) is an isomorphism. Observe that these are torsors for the fpqc topology, but not for the fppf one: it may happen that, since \( G \) is not of finite type, such a \( T \) is not trivialized by any finite extension of \( k \).
- If \( G \) is abelian, the fppf continuous cohomology in the sense of Jannsen, see [Jan88].

We have a natural map

\[
\tau_L : H^1_{\text{fpqc}}(L, G) = \text{Tors}(L, G) \to \lim_i H^1(L, G_i) = \lim_i \text{Tors}(L, G_i).
\]

We are going to prove that, if \( G \) is pro-finite, the map \( \tau_L \) is an isomorphism.

**Lemma 2.1.** Let \( G = \lim_i G_i \) a pro-finite group scheme over a field \( k \), and let \( G'_i \) be the scheme theoretic image of \( G \to G_i \). Then \( G_j \to G_i \) factorizes as \( G_j \to G'_i \\to G_i \) for every \( j \gg i \) great enough.

**Proof.** If \( G_i \) was finite étale, this would reduce to the analogous fact for finite groups, which in turn follows from the fact that a projective system of finite, non-empty sets is non-empty.

For finite group schemes, write \( G_i = \text{Spec} A_i \), then \( G \) is the spectrum of

\[
A = \lim_i A_i = \bigsqcup_i A_i / \sim
\]

where \( \sim \) is the equivalence relation that identifies an element \( a \in A_i \) with its image in \( A_j \) for every \( i \leq j \). Call \( K_{ij} \subseteq A_i \) the kernel of \( A_i \to A_j \) for every \( i \leq j \), \( K_{ij} \) increases with \( j \) and since \( A_i \) is a finite dimensional vector space over \( k \) we have that \( K_{ij} \) is eventually stable, call \( K_i \) the stable kernel. It is immediate to check that \( K_i \) is the kernel of \( A_i \to \bigsqcup_i A_i / \sim \), thus \( G'_i = \text{Spec} A_i / K_i \) and the thesis follows. \( \square \)

**Lemma 2.2.** Let \( G = \lim_i G_i \) a pro-finite group scheme over a field \( k \), and let \( G'_i \) be the scheme theoretic image of \( G \to G_i \). \( \{G'_i\}_i \) defines a second projective system of finite group schemes. It is possible to define a third projective system \( \{G_i\}_i \sqcup \{G'_i\}_i \) such that \( \{G_i\}_i \) and \( \{G'_i\}_i \) are both cofinal sub-systems.

**Proof.** Let \( I \) be the poset of indexes of \( \{G_i\}_i \), and consider a copy of it \( I' = I \). On the disjoint union \( I \sqcup I' \), define an order in the following way.

The restriction of the order to each component is just the order on \( I \). If \( i \in I \) and \( j' \in I' \) (corresponding to \( j \in I \)), then \( j' \geq i \) if \( j \geq i \). If \( i' \in I' \) and \( j \in I \), then \( j \geq i' \) if \( j \geq i \) and the morphism \( G_j \to G_i \) factorizes as \( G_j \to G'_i \to G_i \). It is obvious that \( I' \) is cofinal in \( I \sqcup I' \). For every \( i \) and for every \( j >> i \) great enough, we have that \( G_j \to G_i \) factorizes as \( G_j \to G'_i \to G_i \) thanks to Lemma 2.1: this tells us that \( I \) is cofinal in \( I \sqcup I' \), too. \( \square \)
The proof of the following is due to A. Vistoli.

**Lemma 2.3.** Let $G = \varprojlim_i G_i$, a pro-finite group. Then

$$\tau : \text{Tors}(L, G) \to \varprojlim_i H^1(L, G_i) = \varprojlim_i \text{Tors}(L, G_i)$$

is an isomorphism.

**Proof.** Thanks to Lemma 2.2, we may suppose that $G \to G_i$ is surjective for every $i$, by which we mean that the associated morphism of Hopf algebras is injective.

First, let us prove surjectivity of $\tau$. Consider an element $(T_i)_i \in \varprojlim_i \text{Tors}(L, G_i)$, for every $i \leq j$ let $H_{ij}$ be the set of $G_j \to G_i$-equivariant morphisms $\sigma_{ij} : T_j \to T_i$. By hypothesis $H_{ij}$ is nonempty, we want to show that we can choose one $\sigma_{ij} \in H_{ij}$ for every $i \leq j$ such that $\sigma_{ij} \circ \sigma_{jk} = \sigma_{ik}$ for every $i \leq j \leq k$. Consider $H = \prod_{i \geq j} H_{ij}$, we have that $H_{ij}$ is finite and thus if we consider $H$ with the product topology, it is compact. For $a \leq b \leq c$, let $C_{abc} \subseteq H$ be subset of $(\sigma_{ij})_{ij} \in H$ such that $\sigma_{ab} \circ \sigma_{bc} = \sigma_{ac}$, it is a closed subset. The thesis is equivalent to showing that $\bigcap_{a \leq b \leq c} C_{abc}$ is non-empty: since $H$ is compact, it is enough to check that the intersection of a finite number of them is non-empty.

Let $S$ be a finite set of triplets $a \geq b \geq c$, and choose an index $l$ such that $l \geq a$ for every triplet in $S$. For every $i \leq l$, choose any $\sigma_i : T_i \to T_i$, and for every $i \leq j \leq l$ define $\sigma_{ij} : T_j \to T_i$ as the only equivariant morphism such that $\sigma_{ij} \circ \sigma_j = \sigma_i$: the unicity follows from the fact that $G_l \to G_j$ is surjective. Then we have that

$$\sigma_{ij} \circ \sigma_{jk} \circ \sigma_k = \sigma_{ij} \circ \sigma_j = \sigma_i,$$

and thus $\sigma_{ij} \circ \sigma_{jk} = \sigma_{ik}$ as desired.

For injectivity, let $T, T'$ two G-torsors such that $T_i \simeq T'_i$ for every $i$. Let $H_i$ be the set of $G_i$-equivariant isomorphisms $T_i \simeq T'_i$, $H_i$ is finite for every $i$. This makes $(H_i)$ into a projective system of finite, non-empty sets, thus its limit is non-empty, and this allows us to define an isomorphism $T \simeq T'$.

**Lemma 2.3** clarifies the situation for pro-finite groups schemes. For more general pro-algebraic group schemes, the situation is much more complicated, but something can be said if the projective system is countable. Under this hypothesis, it is easy to find a cofinal subsystem isomorphic to $\mathbb{N}$, thus from now on we suppose that the set of indexes $I$ is just $\mathbb{N}$.

If the set of indexes is $\mathbb{N}$, the map

$$\tau_L : \text{Tors}(L, G) \to \varprojlim_{n \in \mathbb{N}} H^1(L, G_n) = \varprojlim_{n \in \mathbb{N}} \text{Tors}(L, G_n)$$

is easily seen to be surjective: if we have a system $(T_n)_n$ of $(G_n)_n$ torsors, just choose any equivariant morphism $\sigma_n : T_{n+1} \to T_n$ for every $n$, these fit into a tower whose limit is the desired $G$-torsor.

We want now to look at fibers of $\tau_L$. If $T \to \text{Spec} L$ is a $G$-torsor, call $G' = \text{Aut}_G(T)$ the group of automorphisms of $T$, it is an inner form of $G_L$ and we have a natural bijection $\text{Tors}(L, G) = \text{Tors}(L, G')$ sending $T$ to the trivial torsor: this means that in order to study fibers of $\tau_L$, it is enough to study the fiber of the image of the trivial $G'$ torsor for every inner form $G'$ of $G_L$.

Hence, we want to understand $G'$-torsors $T \to \text{Spec} L$ such that $T \times^{G'}(G'_n)$ is trivial for every $n$: a sufficient condition for them to be trivial is that the projective
system of sets \((G'_n(L))_n\) satisfies the Mittag-Leffler condition. Hence, a sufficient condition for \(\tau_i\) to be injective is that the projective system of sets \((G'_n(L))_n\) satisfies the Mittag-Leffler condition for every inner form \(G'\) of \(G_L\).

In the abelian case, suppose we have

\[ 0 \to A \to B \to C \to 0 \]

a short exact sequence of pro-algebraic abelian group schemes, where right exactness means that \(B \to C\) is surjective in the fpqc topology. It is easy to see that this is equivalent to asking that the short exact sequence is a projective limit of short exact sequences of algebraic groups in the fppf topology. Then we have a long exact sequence

\[ 0 \to H^0(L, A) \to H^0(L, B) \to H^0(L, C) \to \text{Tors}(L, A) \to \text{Tors}(L, B) \to \text{Tors}(L, C) \]

functorial in \(A, B, C\) and thus \(\text{Tors}(L, ) = H^1_{\text{fppf}}(L, )\) corresponds to continuous \(H^1_{\text{fppf}}\) in the sense of Jannsen.

Let us now look more closely to the injectivity of \(\tau_i\) in the abelian case. By taking the obvious spectral sequence (see [Jan88, eq. 0.2]) we can say more precisely that we have an obstruction

\[ 0 \to \lim_{\leftarrow i}^1 G_n(L) \to H^1_{\text{fppf}, \text{cont}}(L, G) \to \lim_{\leftarrow i} H^1(L, G_n) \to 0 \]

to injectivity, which again vanishes if \((G_n(L))_n\) satisfies the Mittag-Leffler condition.

3. ESSENTIAL DIMENSION OF PRO-FINITE GROUP SCHEMES

We want now to prove two criteria showing that the essential dimension of pro-finite group schemes is infinite very often, up to the point that it is natural to ask whether it exists a pro-finite étale, non-finite group scheme with finite essential dimension. The following is the key lemma: even if it has very restrictive hypotheses, it is general enough to prove all the necessary cases.

**Lemma 3.1.** Let \(G = \lim_{\leftarrow n} G_n\) be a pro-finite abelian group scheme over an infinite field \(k\), and let \(H_n = \ker(G_n \to G_{n-1})\). Suppose that for every extension \(L/k\) and every \(n\)

\[ H^1(L, G_{n+1}) \to H^1(L, G_n) \]

is surjective and that for every \(n\) there exists an \(H_n\)-torsor on \(k(t)\) of essential dimension 1. Then there exists a \(G\)-torsor \(T \to \text{Spec} k(t_1, t_2, \ldots)\) such that \(\text{ed}_k T = \infty\).

**Proof.** For every \(n\), let \(S_n \to \text{Spec} k(t_n)\) be an \(H_n\)-torsor with \(\text{ed}_k S_n = 1\), set \(S_{n,G_n} = T_n \times_{H_n} G_n\).

Define by recursion a \(G_n\)-torsor \(T_n \to \text{Spec} k(t_1, \ldots, t_n)\) in the following way. For \(n = 1\), set

\[ T_1 = S_{1,G_1}. \]

For \(n \geq 2\), choose \(T'_n \to \text{Spec} k(t_1, \ldots, t_{n-1})\) as any lifting of \(T_{n-1}\) (which exists by hypothesis) and define

\[ T_n = T'_{n,k(t_1,\ldots,t_n)} + S_{n,G_n,k(t_1,\ldots,t_n)}. \]

By passing to the limit, we get a \(G\)-torsor \(T \to \text{Spec} k(t_1, t_2, \ldots)\).
Suppose by contradiction that \( \text{ed}_k T < \infty \), then there exists an integer \( m \) and a \( G \)-torsor \( Q \to \text{Spec} k(t_1, \ldots, t_m) \) which extends to \( T \). Consider now the \( G_{m+1} \) torsor \( Q_{G_{m+1}} = Q \times^G G_{m+1} \). By construction,
\[
Q_{G_{m+1}}(t_1, t_2, \ldots) = T_{m+1} k(t_1, t_2, \ldots).
\]
Consider the difference \( G_{m+1} \)-torsor on \( k(t_1, \ldots, t_{m+1}) \)
\[
D = Q_{G_{m+1}} k(t_1, \ldots, t_{m+1}) - T_{m+1},
\]
we have that \( D_k(t_1, t_2, \ldots) \) is trivial. But \( G_{m+1} \) is finite and \( k \) is algebraically closed in \( k(t_1, t_2, \ldots) \), hence \( D \) is already trivial on \( k(t_1, \ldots, t_{m+1}) \), i.e.
\[
Q_{G_{m+1}} k(t_1, \ldots, t_{m+1}) = T_{m+1}
\]
which means that \( T_{m+1} \) is defined on \( k(t_1, \ldots, t_{m+1}) \).

Now recall that \( T_{m+1} \) is by definition the sum of \( T'_{m+1, k(t_1, \ldots, t_{m+1})} \) a torsor which is defined on \( k(t_1, \ldots, t_m) \), and \( S_{m+1, k(t_1, \ldots, t_{m+1})} \) a torsor which is defined on \( k(t_{m+1}) \). Now define
\[
R_{m+1} = Q_{G_{m+1}} - T'_{m+1, k(t_1, \ldots, t_m)} \to \text{Spec} k(t_1, \ldots, t_m)
\]
Thus we have a \( G_{m+1} \)-torsor \( R_{m+1} \) on \( k(t_1, \ldots, t_m) \) and a \( G_{m+1} \)-torsor \( S_{m+1} \) on \( k(t_{m+1}) \) such that their extensions to \( k(t_1, \ldots, t_{m+1}) \) are equal, and this gives a contradiction. In fact, let \( \bar{R}_{m+1} \) and \( \bar{S}_{m+1} \) be \( G_{m+1} \)-torsors which are spreading outs of \( R_{m+1} \), \( S_{m+1} \) on open subsets \( U, V \) of \( \mathbb{A}^n, \mathbb{A}^1 \). Up to shrinking \( U, V \), we may suppose that
\[
U \times \bar{S}_{m+1} \simeq \bar{R}_{m+1} \times V \to U \times V \subseteq \mathbb{A}^{m+1}
\]
since this is true generically. Since \( k \) is infinite, we can choose a rational point \( u \in U(k) \). If we restrict the equality above to \( u \times \text{Spec} k(t_m) \), we get
\[
S_{m+1} \simeq \bar{R}_{m+1, u} \times \text{Spec} k(t_{m+1}),
\]
but \( \text{ed}_k S_{m+1} = 1 \) by hypothesis and hence we have a contradiction. \( \square \)

**Lemma 3.2.** Over any field \( k \), for every prime \( p \) and for every \( n \) there exists a \( \mu_{p^n} \)-torsor on \( k(t) \) of essential dimension 1.

**Proof.** We have
\[
H^1(k(t), \mu_{p^n}) = k(t)^*/(k(t)^{p^n}).
\]
Let \( t \in k(t)^*/(k(t)^{p^n}) \), then \( t \) is obviously not defined over \( k \). Since \( k \) is algebraically closed in \( k(t) \), this tells us that \( \text{ed}_k t = 1 \). \( \square \)

**Lemma 3.3.** If \( \text{char } k = p \), then \( H^0(k, \mathbb{Z}/p^n) = 0 \) for every \( n \) and every \( q \geq 2 \).

**Proof.** This follows from the more general fact that fields of characteristic \( p \) have cohomological dimension less than or equal to 1, see [Ser65, Ch. 2, Proposition 3]. Alternatively, one can give a direct proof by induction on \( n \) using the Artin-Schreier exact sequence. \( \square \)

**Lemma 3.4.** If \( \text{char } k = p \) and for every \( n \), there exists a \( \mathbb{Z}/p^n \) torsor on \( k(t) \) of essential dimension 1.
Proof. Let us first do this for \( n = 1 \). Let \( \Phi : k(t) \to k(t) \) the homomorphism \( x \mapsto x^p - x \). Using Artin-Schreier theory,

\[
H^1(k(t), \mathbb{Z}/p) \simeq k(t)/\Phi(k(t)).
\]

We have that \( t \in k(t)/\Phi(k(t)) \) has essential dimension 1. In fact, if \( t \) is defined on \( k \) (the only algebraic sub-extension of \( k(t)/k \)) we have

\[
t = \lambda + q^p - q
\]

for some \( \lambda \in k \) and \( q \in k(t) \). But then \( k(t) \subseteq k(q) \subseteq k(t) \) and \( [k(q) : k(t)] = p \), which is absurd.

Now let \( T \in H^1(k(t), \mathbb{Z}/p^n) \) a \( \mathbb{Z}/p^n \)-torsor of essential dimension 1. Thanks to Lemma 3.3, we can lift it to a \( \mathbb{Z}/p^{n+1} \)-torsor \( T' \). We have

\[
1 \geq \text{ed}_k T' \geq \text{ed}_k T = 1
\]

\( \square \)

Corollary 3.5. Over any field \( k \), \( \text{ed}_k \mathbb{Z}_p(1) = \infty \).

Proof. Since essential dimension decreases along extensions, we may suppose that \( k \) is infinite. Write \( \mathbb{Z}_p(1) = \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/p^n \). Thanks to Lemma 3.3 and Lemma 3.4, the hypotheses of Lemma 3.1 are satisfied.

\( \square \)

Corollary 3.6. Over any field \( k \), \( \text{ed}_k \mathbb{Z}_p = \infty \).

Proof. Thanks to Corollary 3.5, it only remains to prove the case \( \text{char} k = p \). As before, since essential dimension decreases along extensions we may suppose \( k \) to be infinite. Write \( \mathbb{Z}_p = \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/p^n \), let us check the hypothesis of Lemma 3.1.

The surjectivity of \( H^1(L, \mathbb{Z}/p^{n+1}) \to H^1(L, \mathbb{Z}/p^n) \) comes from Lemma 3.3. The kernel of \( \mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n \) is just \( \mathbb{Z}/p \), thanks to Lemma 3.4 there exists a \( \mathbb{Z}/p \)-torsor on \( k(t) \) of essential dimension 1.

\( \square \)

Corollary 3.7. Let \( k \) be a field, \( p_1, p_2, \ldots \) a sequence of not necessarily distinct prime numbers and \( n_1, n_2, \ldots \) positive integers. Then

\[
\text{ed}_k \prod_{i=1}^{\infty} \mathbb{Z}/p_i^{n_i} = \infty
\]

Proof. We may extend \( k \) and suppose \( k = \bar{k} \), and hence \( \mu_{p^n} = \mathbb{Z}/p^n \) if \( \text{char} k \neq p \). Write

\[
\prod_{i=1}^{\infty} \mathbb{Z}/p_i^{n_i} = \varprojlim_{n \in \mathbb{N}} \prod_{i=1}^{n} \mathbb{Z}/p_i^{n_i},
\]

and let us check the hypotheses of Lemma 3.1.

The surjectivity of

\[
H^1 \left( L, \prod_{i=1}^{n} \mathbb{Z}/p_i^{n_i} \right) \to H^1 \left( L, \prod_{i=1}^{n-1} \mathbb{Z}/p_i^{n_i} \right)
\]

is obvious, since cohomology commutes with direct product. The fact that there exists a \( \mathbb{Z}/p_i^{n_i} \)-torsor of essential dimension 1 on \( k(t) \) comes either from Lemma 3.2 if \( \text{char} k \neq p_i \) or from Lemma 3.4 if \( \text{char} k = p_i \).

\( \square \)
**Lemma 3.8.** Let $G = \varprojlim G_i$ be a pro-finite group scheme over a field $k$, and suppose that there exists a non-trivial morphism $f : G \to \mathbb{Z}_p$ for some $p$. Then there exists a possibly different surjective morphism $G \to \mathbb{Z}_p$ with a section $\mathbb{Z}_p \to \overline{G}_k$ over the algebraic closure $\overline{k}$.

**Proof.** Since $\cap_n p^n\mathbb{Z}_p = \{0\}$ and $f$ is non-trivial, there exists a maximum $n$ such that $f$ has image contained in $p^n\mathbb{Z}_p$. Since $p^n\mathbb{Z}_p \cong \mathbb{Z}_p$, we may suppose $n = 0$, i.e. there exists an invertible element $u \in \mathbb{Z}_p^*$ in the image of $f$.

Up to composing $f$ with $\cdot u^{-1} : \mathbb{Z}_p \to \mathbb{Z}_p$, we may suppose that 1 is in the image of $f$.

Now replace $k$ with $\overline{k}$, we may suppose that 1 = $f(g)$ for some rational point $g \in G(k)$. Since $G$ is pro-finite, we can write $G = \varprojlim G_i$ with $G_i$ finite. Let $g_i \in G_i(k)$ be the projection of $g \in G(k)$, since $G_i$ is finite $g_i$ has finite order $n_i$. This allows us to define a homomorphism $\hat{\mathbb{Z}} \to G_i$ which maps 1 to $g_i$, and these morphisms fit into a tower giving us a continuous homomorphism $\hat{\mathbb{Z}} \to G(k)$, where $G(k)$ has the pro-discrete topology. Since the pro-discrete topologies on $\hat{\mathbb{Z}}$ and $G(k)$ coincide with the topologies induced by the scheme structure, we get an homomorphism of group schemes $\hat{\mathbb{Z}} \to G$.

By construction, the composition $\hat{\mathbb{Z}} \to G \to \mathbb{Z}_p$

is the canonical projection $\hat{\mathbb{Z}} = \prod_q \mathbb{Z}_q \to \mathbb{Z}_p$, hence if we compose with the embedding $\mathbb{Z}_p \hookrightarrow \hat{\mathbb{Z}}$ we get a section $\mathbb{Z}_p \to G$ of $G \to \mathbb{Z}_p$. Observe that $\mathbb{Z}_p \to G$ does not, in general, send 1 to $g$: we have used the embedding $\mathbb{Z}_p \hookrightarrow \hat{\mathbb{Z}}$ as a shortcut for the chinese remainder theorem. 

**Theorem 3.9.** Let $G$ be a pro-finite group scheme over a field $k$, and suppose there exists an extension $k' / k$ and a non-trivial morphism $G_{k'} \to \mathbb{Z}_p$. Then $\text{ed}_k G = \infty$.

**Proof.** Thanks to Lemma 3.8, up to extending the base field we may suppose we have a surjective morphism $G \to \mathbb{Z}_p$ with a section $\mathbb{Z}_p \to G$. Then thesis follows from Corollary 3.6. 

**Theorem 3.10.** Let $G$ be a pro-finite étale abelian group scheme over a field $k$. Then $\text{ed}_k G \leq \infty$ if and only if $G$ is finite.

**Proof.** If $G$ is finite, then it is a classical fact that $\text{ed}_k G < \infty$, it follows from the existence of versal torsors.

Now suppose that $G$ is not finite and write $G = \varprojlim G_i$ with $G_i$ abelian, finite étale group scheme and $G \to G_i$ surjective. We want to prove that $\text{ed}_k G = \infty$, hence we can extend the base field freely.

$G_i$ is finite étale, up to enlarging $k$ enough we may suppose that it is discrete, i.e. it is just a finite abelian group. For every prime $p$, let $G_{i,p} \subseteq G_i$ be the subgroup of $p$-torsion. We identify three cases.

**Case 1:** $\varprojlim |G_{i,p}| < \infty$ for every prime $p$. 

Claim 1: there exists an infinite sequence \( p_1 < p_2 < \ldots \) of primes and positive integers \( n_1, n_2, \ldots \) with morphisms

\[
\pi : G \to \prod_{i=1}^{\infty} \mathbb{Z}/p_i^{n_i},
\]

\[
s : \prod_{i=1}^{\infty} \mathbb{Z}/p_i^{n_i} \to G,
\]

whose composition \( \pi \circ s \) is the identity of \( \prod_{i=1}^{\infty} \mathbb{Z}/p_i^{n_i} \). Case 1 then follows from the claim thanks to Corollary 3.7.

Since \( \lim_{i} |G_{i,p}| < \infty \) for every \( p \), for \( i \) great enough and \( j \geq i \) we have that \( G_j \to G_i \) induces an isomorphism on \( p \)-torsion. If \( |G_{i,p}| > 1 \) for \( i \) great enough, it is then easy to construct homomorphisms \( \mathbb{Z}/p^n \to G \) and \( G \to \mathbb{Z}/p^n \) whose composition is the identity of \( \mathbb{Z}/p^n \) for some integer \( n \geq 1 \). Since \( G \) is not finite and \( |G_{i,p}| \) is bounded for every \( i \), there are infinite primes \( p \) for which \( |G_{i,p}| > 1 \).

Case 2: \( \lim_{i} |G_{i,p}| = \infty \) for some prime \( p \) and there exists no non-trivial homomorphism \( G \to \mathbb{Z}_p \).

Claim 2: There are positive integers \( n_1, n_2, \ldots \) with morphisms

\[
\pi : G \to \prod_{i=1}^{\infty} \mathbb{Z}/p_i^{n_i},
\]

\[
s : \prod_{i=1}^{\infty} \mathbb{Z}/p_i^{n_i} \to G,
\]

whose composition \( \pi \circ s \) is the identity of \( \prod_{i=1}^{\infty} \mathbb{Z}/p_i^{n_i} \). Case 2 then follows from the claim thanks to Corollary 3.7.

Call a surjective morphism \( G \to \mathbb{Z}/p^n \) maxed if there exists no extension \( G \to \mathbb{Z}/p^{n'} \to \mathbb{Z}/p^n \) with \( n' > n \). Since there exists no non-trivial homomorphism \( G \to \mathbb{Z}_p \), every surjective morphism \( G \to \mathbb{Z}/p^n \) extends to a maxed one.

Since \( G_p \) is not trivial, and we can extend morphisms to maxed morphisms, there exists a maxed morphism \( G \to \mathbb{Z}/p^n \). Let \( i \) be an index such that we have a factorization \( G \to G_{i,p} \to \mathbb{Z}/p^n \): I claim that there exists a section \( \mathbb{Z}/p^n \to G_{i,p} \). Write \( G_{i,p} = \mathbb{Z}/p^{n_1} \oplus \cdots \oplus \mathbb{Z}/p^{n_k} \). There exists a \( j \) such that \( \mathbb{Z}/p^{n_j} \to \mathbb{Z}/p^n \) is surjective, in particular \( n_j \geq n \). Since we have a lifting \( G \to \mathbb{Z}/p^{n_j} \to \mathbb{Z}/p^n \) and \( G \to \mathbb{Z}/p^n \) is maxed, we have \( n_j = n \), and thus we get the desired section \( \mathbb{Z}/p^n \to G_{i,p} \).

Since the set of sections \( \mathbb{Z}/p^n \to G_{i,p} \subseteq G_i \) is finite and projective limits of finite, non-empty sets are non-empty, we get a section \( \mathbb{Z}/p^n \to G \) and thus a splitting \( G = G' \oplus \mathbb{Z}/p^n \). It is obvious that \( G' \) still satisfies the hypotheses of case 2, i.e. infinite \( p \)-part and no non-trivial morphism \( G' \to \mathbb{Z}_p \). By recursion, this allows us to construct a morphism as in claim 2.

Case 3: \( \lim_{i} |G_{i,p}| = \infty \) and there exists a non-trivial morphism \( G \to \mathbb{Z}_p \). Case 3 then follows from Theorem 3.9. \( \square \)

Question. In view of Theorem 3.9 and Theorem 3.10, it is natural to ask if there exists a pro-finite étale group scheme which is not finite but has finite essential dimension.

We have thus proved that essential dimension is infinite very often for pro-finite group schemes, up to the point that we have no examples of non-finite, pro-finite étale group schemes with finite essential dimension: thus, essential dimension is not a very interesting invariant for pro-finite group schemes. One may be content with this, and be done with it. However, we think that the ideas and formalism of essential dimension may still give non-trivial information, at the cost of modifying the basic definition of essential dimension.
We are going to define two new variants of essential dimension: finite type essential dimension and continuous essential dimension. These two variants operate in orthogonal directions, and both coincide with classical essential dimension for algebraic group schemes. We can then combine them in what we call the fce dimension (finite type, continuous, essential dimension) which may be thought as the right extension of essential dimension to pro-finite group schemes.

4. EXTENSION OF TORSORS

The variants of essential dimension that we are going to define focus on torsors defined over fields finitely generated over the base field. In this context, an often meaningful question is the following: if we have a torsor defined on the generic point of a variety, does it extend to the whole variety? In this section, which is completely independent from the concept of essential dimension, we prove some results that we need regarding this question.

We are going to use extensively the notion of gerbes and in particular we are going to replace étale fundamental groups with étale fundamental gerbes, see [BV15]. This is not strictly necessary, but it allows us to handle better various situations where fixing a base point is troublesome. Every result about gerbes can be translated into a result about torsors by considering the gerbe $BG$ if $G$ is a group scheme.

**Lemma 4.1.** Let $G$ be a pro-finite étale group scheme over a field $k$, $V$ a variety and $T = \text{Spec} \, A \to \text{Spec} \, k(V)$ a torsor defined on the generic point. Let $\tilde{A} \subseteq A$ be the normalization of $\mathcal{O}_V$ in $A$, and $\tilde{T} \to V$ the relative spectrum.

- The action of $G$ on $T$ extends to $\tilde{T}$.
- If $\tilde{T} \to V$ is étale, then $V$ is normal and $\tilde{T} \to V$ is a $G$-torsor.
- On the other hand, if $V$ is normal and an extension of $T$ to $V$ exists, then it is unique and it coincides with $\tilde{T}$.

**Proof.** The problem is local, we may suppose $V = \text{Spec} \, \tilde{B}$ an integral $k$ algebra of finite type with fraction field $k(V) = B$. Moreover, it is straightforward to get the general case from the one in which $G$ is finite étale, hence we make this assumption.

By definition, $\tilde{A}$ is the integral closure of $\tilde{B} \subseteq B$ in $A$. Using the Yoneda lemma, $G$ acts by ring homomorphisms on $A$ and fixes the elements of $\tilde{B} \subseteq B \subseteq A$, hence elements integral over $A$ are sent to elements integral over $A$ by the action, i.e. the action of $G$ restricts to $\tilde{A}$: in doing so, we are subtly using the fact that $G$ is étale and integral closure commutes with étale base change, see [EGAIV-4, Proposition 18.12.15]. For a more down-to-earth proof, write $G = \text{Spec} \, R$ and consider the comodule structure

$$A \to A \otimes R.$$ 

The elements of $\tilde{A}$ are integral over $\tilde{B}$ and thus they are mapped to elements of $A \otimes R$ integral over $\tilde{B} \otimes R$. Since integral closure commutes with étale base change, these are exactly $\tilde{A} \otimes R$.

Now suppose that $\tilde{T} \to V$ is étale. In particular, if $\tilde{V}$ is the normalization of $V$ in $k(V)$, we have a factorization $\tilde{T} \to \tilde{V} \to V$, hence the normalization $\tilde{V} \to V$ is étale and thus an isomorphism.
Now, let $\rho : G \times \bar{T} \to \bar{T}$ the action. We have a natural morphism $\rho \times p_2 : G \times \bar{T} \to \bar{T} \times V \bar{T}$; the fact that $\bar{T}$ is a $G$-torsor is equivalent to the fact that $\rho \times p_2$ is an isomorphism. But $G \times \bar{T}$ and $\bar{T} \times V \bar{T}$ are both finite étale covers of $V$ and $\rho \times p_2$ is a morphism of covers, thus in order to prove that it is an isomorphism it is enough to show that it is generically an isomorphism, and this is true by hypothesis since $T$ is a torsor.

On the other hand, suppose that $\bar{T}' \to V$ is some extension of $T$ and that $V$ is normal. Then $\bar{T}' \to V$ is finite étale (because $G$ is finite étale), hence $\bar{T}'$ is normal and finite over $V$: the former says that $\mathcal{O}_{T'} \subseteq A$ contains the normalization $\tilde{A}$ of $\tilde{B}$ (since $\mathcal{O}_V \supseteq \tilde{B}$ and is normal), the latter that it is contained in it.

**Corollary 4.2.** Let $\Phi$ be a finite étale gerbe over a field $k$, $V$ a smooth variety and $s : \text{Spec} k(V) \to \Phi$ a morphism. Call $\xi$ the generic point. Then there exists an open subset $U_{\text{max}} \subseteq V$ with a morphism $u_{\text{max}} : U_{\text{max}} \to \Phi$ and an isomorphism

$$\varphi_{\text{max}} : s \to u_{\text{max}, \xi}$$

in $\Phi(k(V))$ such that

- for every other $u : U \to \Phi$, $\varphi : s \to u_\xi$ as above, we have $U \subseteq U_{\text{max}}$ and there exists a unique isomorphism $\psi : u_{\text{max}}|_U \to u$ such that $\varphi \circ \varphi_{\text{max}} = \varphi$,
- $V \setminus U_{\text{max}} \subseteq V$ has pure codimension 1,
- if $k'/k$ is a separable extension, then $U_{\text{max}} \times k' = (U \times k')_{\text{max}}$ and $u_{\text{max}} \times k' = (u \times k')_{\text{max}}$.

**Proof.** If $\Phi$ has a section $\text{Spec} k \to \Phi$, the first point is a direct consequence of Lemma 4.1: there exists the greatest open subset where a torsor extends and the extension is unique. Since $U_{\text{max}}$ is the étale locus of a finite cover, by purity of branch locus we get the second point. Since integral closure commutes with étale base change (see [EGAIV-4, Proposition 18.12.15]), we get the third.

Otherwise, since $\Phi$ is finite étale there exists a finite Galois extension $k'/k$ with a section $\text{Spec} k'/V \to \Phi$, this let us identify $\Phi_{\text{max}} = BG'$ for some finite étale group scheme $G'$ over $k'$. Write $V' = V \times \text{Spec} k'$. We have a $G'$-torsor $T' \to \text{Spec} k(V')$. For every element $\sigma \in \text{Gal}(k'/k)$ we have a $\sigma$-equivariant isomorphism of schemes $\sigma^*: T' \to T'$, thus the étale loci of $T'$ and $\sigma^* T'$ coincide i.e. the étale locus $U'_{\text{max}} \subseteq V'$ of $T'$ is Galois-invariant, let $U_{\text{max}} \subseteq V$ be its image in $V$. We have that $T'|_{U'_{\text{max}}}$ defines a morphism $U'_{\text{max}} \to \Phi_{k'} = BG'$ which is Galois invariant. To check that it descends to a morphism $U_{\text{max}} \to \Phi$ we only have to check the cocycle condition: but this can be checked on the generic point, where it is obviously satisfied.

It is immediate to check that $U_{\text{max}} \to \Phi$ satisfies the requested conditions since $U'_{\text{max}} \to \Phi_{k'}$ satisfies them. $\Box$

**Corollary 4.3.** Let $\Phi = \lim_{\leftarrow i} \Phi_i$ be a pro-finite étale gerbe over a field $k$, $V$ a smooth variety and $s : \text{Spec} k(V) \to \Phi$ a morphism. Write $U_{\text{max}, j} \subseteq V$ for the open subset given by Corollary 4.2 with respect to Spec $k(V) \to \Phi_i$, we have $U_{\text{max}, j} \subseteq U_{\text{max}, i}$ if $j \geq i$.

Suppose that there exists an index $i_0$ such that, for every index $i \geq i_0$, $U_{\text{max}, j} = U_{\text{max}, i_0}$. Then the conclusions of Corollary 4.2 hold for $\Phi$, and $U_{\text{max}} = U_{\text{max}, i_0}$. $\Box$

Let $V$ be a smooth variety over a field $k$ of characteristic 0, $D \subseteq V$ an irreducible codimension 1 subvariety. Let $\sqrt{(V, D)}$ be the infinite root stack of $V$ at $D$, see
There are natural morphisms

\[ V \setminus D \leftrightarrow \sqrt[n]{(V,D)} \rightarrow V, \]

and \( V \setminus D \rightarrow \sqrt[n]{(V,D)} \) induces an isomorphism of fundamental gerbes

\[ \Pi_{V\setminus D/k} \simeq \Pi_{\sqrt[n]{(V,D)}/k}. \]

This last fact follows directly from the analogous fact for fundamental groups, which is proved in \([\text{Bor09}, \text{Proposition 3.2.2}]\). If \( d \in D \) is a smooth closed point, the fiber of \( \sqrt[n]{(V,D)} \) over \( d \in V \) is a gerbe over \( k(d) \) non-canonically isomorphic to \( B_{k(d)}\hat{\mathbb{Z}}(1) \). Hence, we get a morphism

\[ B_{k(d)}\hat{\mathbb{Z}}(1) \rightarrow \sqrt[n]{(V,D)} \rightarrow \Pi_{\sqrt[n]{(V,D)}/k} = \Pi_{V\setminus D/k}. \]

\textbf{Definition 4.4.} Let \( V \) be a smooth variety, \( D \subseteq V \) a codimension 1 subvariety, \( d \in D \) a closed, smooth point. We call the morphism

\[ B_{k(d)}\hat{\mathbb{Z}}(1) \rightarrow \Pi_{V\setminus D/k} \]

constructed above the hole at \( d \).

\textbf{Lemma 4.5.} Let \( V \) be a smooth variety over a field of characteristic 0, \( D \subseteq V \) a codimension 1 subvariety, \( d \in D \) a closed, smooth point.

Suppose we have a pro-finite gerbe \( \Phi \) and a morphism \( V \setminus D \rightarrow \Phi \) with a 2-commutative diagram

\[ B_{k(d)}\hat{\mathbb{Z}}(1) \rightarrow \Pi_{V\setminus D/k} \]

\[ \downarrow \quad \downarrow \]

\[ \text{Spec} k(d) \rightarrow \Phi \]

where \( B_{k(d)}\hat{\mathbb{Z}}(1) \rightarrow \Pi_{V\setminus D/k} \) is the hole at \( d \) and \( \text{Spec} k(d) \rightarrow \Phi \) is some section. Then \( V \setminus D \rightarrow \Phi \) extends uniquely to a morphism \( V \rightarrow \Phi \).

\textbf{Proof.} Since we are in characteristic 0, with standard arguments we can reduce to the case \( k = \mathbb{C} \), in particular \( k(d) = \mathbb{C} \). With an abuse of notation, we call \( d \) not only the point \( d \in D \) but also the tautological section \( \text{Spec} \mathbb{C} \rightarrow B\hat{\mathbb{Z}}(1) \) and its images in \( \sqrt[n]{(V,D)} \), \( \Pi_{\sqrt[n]{(V,D)}/k} \), \( \Pi_{V\setminus D/k} \Pi_{V/k} \). By taking the image of \( d \in \Phi \), we can identify \( \Phi \) with \( BG \) for some pro-finite group \( G \) and reduce everything to a purely group-theoretic problem: we want to show that the homomorphism of étale fundamental groups

\[ \pi_1(V \setminus D,d) \rightarrow G \]

extends uniquely to a morphism

\[ \pi_1(V,d) \rightarrow G. \]

Now, since we are over \( \mathbb{C} \) étale fundamental groups are just profinite completions of topological fundamental groups. We have an induced homomorphism from the topological fundamental group

\[ \pi_1(V \setminus D^{an},d) \rightarrow G, \]

and \( \pi_1(V \setminus D^{an},d) \rightarrow \pi_1(V^{an},d) \) is surjective with kernel normally generated by a simple loop around \( D \) near \( d \).
Our hypothesis on the morphism $B\hat{Z}(1) \to \Phi$ precisely says that this loop maps to 0 in $G$, hence we have a unique extension $\pi_1(V^{an}, d) \to G$. Since $G$ is pro-finite, this extends uniquely to an homomorphism $\pi_1(V, d) \to G$, as desired. 

**Remark 4.6.** In the proof of Lemma 4.5 we have reduced ourselves to topological fundamental groups in order to use the fact that the kernel of $\pi_1(V \setminus D^{an}, d) \to \pi_1(V^{an}, d)$ is normally generated by a loop around $d$. Observe that this is not in general guaranteed for étale fundamental groups, even if we consider the smallest closed, normal subgroup containing the loop. This would be guaranteed to be the kernel if $\pi_1(V^{an}, d)$ was good in the sense of Serre (see [Ser65, §I.2.6 Exercises 1, 2]), but in general this is false.

**Corollary 4.7.** Let $k$ be a field of characteristic 0, $V$ a smooth variety over $k$, $\Phi$ a pro-finite étale gerbe over $k$.

Suppose that for every finite extension $k'/k$ and for every section $s : \text{Spec } k' \to \Phi$ there are no non-trivial morphisms $\hat{Z}(1) \to \text{Aut}_{k'}(s)$ of group schemes over $k'$.

Then every section $\text{Spec } k(V) \to \Phi$ extends uniquely to a morphism $V \to \Phi$.

**Proof.** Let $s : \text{Spec } k(V) \to \Phi$ be a section, write $\Phi = \varprojlim \Phi_i$ with $\Phi_i$ finite étale and let $s_i : \text{Spec } k(V) \to \Phi \to \Phi_i$ be the composition. Thanks to Corollary 4.2, $s_i$ extends to an open subset $U$ of the form $U = V \setminus (D_1 \cup \cdots \cup D_n)$ with $D_j$ irreducible codimension 1 subvariety, and $U \neq V$ if and only if $n > 0$.

Let us show that $n = 0$. Otherwise, call $V' = V \setminus (D_2 \cup \cdots \cup D_n)$, we have $U = V' \setminus D_1$. Since $s_i$ extends to $U$, we have an induced morphism $U \to \Phi_i$, we want to apply Lemma 4.5 in order to show that this morphism extends to $V' \to \Phi_i$ thus giving a contradiction (since $U$ is the greatest open subset where $s_i$ extends).

Choose a smooth point $d \in D_1$, we have morphisms $B_{k(d)}\hat{Z}(1) \to \Phi \to \Phi_i$ and hence $B_{k(d)}\hat{Z}(1) \to \Phi_{k(d)} \to \Phi_{i,k(d)}$. By abuse of notation, we also call $d$ the tautological section of $B_{k(d)}\hat{Z}(1)$ and $\Phi_{k(d)}$: thanks to the hypothesis, the homomorphism $\hat{Z}(1) \to \text{Aut}_{k(d)}(d)$ is trivial and thus we have a factorization $B_{k(d)}\hat{Z}(1) \to \text{Spec } k(d) \to \Phi$.

In particular, $B_{k(d)}\hat{Z}(1) \to \Phi_i$ factorizes through a section $\text{Spec } k(d) \to \Phi_i$, thus we may apply Lemma 4.5 and $s_i$ extends to $V'$, which is absurd.

Since $s_i$ extends to a morphism $\tilde{s}_i : V \to \Phi_i$ for every $i$, to check that these give a morphism $\tilde{s} : V \to \Phi$ we have to check a cocycle condition. This condition can be checked on the generic point, where it is obvious, thus we get the thesis.

**Remark 4.8.** The hypothesis of Corollary 4.7 seems to be very restrictive. However, applications exist more often than one might think in arithmetic. For instance, we are going to show that the hypothesis holds for Tate modules of abelian varieties over fields finitely generated over $Q$.

**Proposition 4.9.** Let $A$ be an abelian variety over a field finitely generated over $Q$. Then there are no non-trivial homomorphisms of group schemes $T_p A \to Z_p$.

**Proof.** Suppose that $T_p A \to Z_p$ is such a non-trivial homomorphism, up to replacing $Z_p$ with a closed subgroup we may suppose that $T_p A \to Z_p$ is surjective (all non-trivial closed subgroups of $Z_p$ are isomorphic to $Z_p$).

The surjective morphism $T_p A \to Z_p$ induces a tower

$$
\cdots \to A_{n+1} \to A_n \to \cdots \to A_0 = A
$$
where $\mathcal{A}_n$ is an abelian variety and $\pi_n : \mathcal{A}_n \to \mathcal{A}_0$ is a $\mathbb{Z}/p^n$-torsor which is an homomorphism of abelian varieties. In particular, if $a_0 \in \mathcal{A}_0(k)$ is the origin, $\pi_n^{-1}(a_0) \subseteq \mathcal{A}_n$ is a trivial $\mathbb{Z}/p^n$ torsor, and hence the $p$-torsion of $\mathcal{A}_n(k)$ has at least $p^n$ elements.

But $\mathcal{A}_n$ is isogenous to $\mathcal{A}_0$, hence by Faltings' isogeny theorem there is only a finite number of isomorphism classes of abelian varieties in the tower, and thus we get a contradiction using the Mordell-Weil theorem.

From now on, we are going to generalize all of our statements about abelian varieties to torsors under abelian varieties. This is a necessary step for applications since one does not always have a rational point to get the Albanese variety, while the Albanese torsor exists in general.

**Lemma 4.10.** If $\mathcal{A}^1$ is a torsor for an abelian variety $\mathcal{A}$, then $\Pi_{\mathcal{A}/k}$ is banded by $\mathcal{T}_{\mathcal{A}} = \prod_q \mathcal{T}_q \mathcal{A}$.

**Proof.** The band $G$ of an abelian gerbe $\Phi$ is characterized by the fact that the inertia of $\Phi$ is isomorphic to $G \times \Phi$. Since $\mathcal{A}^1$ is an $\mathcal{A}$-torsor, we have the usual isomorphism $\mathcal{A} \times \mathcal{A}^1 \to \mathcal{A}^1 \times \mathcal{A}^1$ which induces an isomorphism $\Pi_{\mathcal{A}/k} \times \Pi_{\mathcal{A}^1/k} \to \Pi_{\mathcal{A}^1/k} \times \Pi_{\mathcal{A}^1/k}$. We have thus a 2-cartesian diagram

$$
\begin{array}{ccc}
\Pi_{\mathcal{A}/k} = \mathcal{T}_{\mathcal{A}} \times \Pi_{\mathcal{A}^1/k} & \rightarrow & \Pi_{\mathcal{A}^1/k} \\
\downarrow & & \downarrow \\
\Pi_{\mathcal{A}^1/k} \rightarrow \Pi_{\mathcal{A}^1/k} \times \Pi_{\mathcal{A}^1/k} = \Pi_{\mathcal{A}/k} \times \Pi_{\mathcal{A}^1/k}
\end{array}
$$

which is the base change along $\Pi_{\mathcal{A}^1/k} \to \text{Spec } k$ of the 2-cartesian diagram

$$
\begin{array}{ccc}
\mathcal{T}_{\mathcal{A}} & \rightarrow & \text{Spec } k \\
\downarrow & & \downarrow \\
\text{Spec } k & \rightarrow & \Pi_{\mathcal{A}/k}
\end{array}
$$

\[ \square \]

**Corollary 4.11.** Let $\mathcal{A}^1$ be a torsor for an abelian variety $\mathcal{A}$ over a field $k$ finitely generated over $\mathbb{Q}$. Then every morphism $\Pi_{\mathcal{A}^1/k} \to B\mathbb{Z}_p$ factorizes through a section $\text{Spec } k \to B\mathbb{Z}_p$.

**Proof.** It is enough to show that $\Pi_{\mathcal{A}^1/k} \to B\mathbb{Z}_p$ induces a trivial morphism of bands. Since the band of $\Pi_{\mathcal{A}^1/k}$ is $\mathcal{T}_{\mathcal{A}} = \prod_q \mathcal{T}_q \mathcal{A}$, this follows from Proposition 4.9.

\[ \square \]

The following is essentially the Weil pairing.

**Lemma 4.12.** Let $\mathcal{A}$ be an abelian variety over $k$ with dual abelian variety $\hat{\mathcal{A}}$, and $T_p \mathcal{A}$ its $p$-adic Tate module. Then $\text{Hom}(T_p \mathcal{A}, \mathbb{Z}_p(1))$ is naturally isomorphic to $T_p \hat{\mathcal{A}}$.

**Proof.** We prove this using the Yoneda lemma, at the level of functors. The subgroup $\text{Hom}(T_p \mathcal{A}, \mu_n) \subseteq \text{Hom}(T_p \mathcal{A}, \mathbb{G}_m)$ represents the group of $n$-torsion line...
bundles \( L \) over \( A \) with a trivialization of the restriction of \( L \) to the identity, i.e. \( \text{Hom}(T_p A, \mu_n) = \hat{A}[n] \). Passing to the limit,
\[
\text{Hom}(T_p A, \mathbb{Z}_p(1)) \cong \varprojlim_n \text{Hom}(T_p A, \mu_{p^n}) = \varprojlim_n \hat{A}[p^n] = T_p \hat{A}.
\]
\( \square \)

**Corollary 4.13.** If \( k \) is finitely generated over \( \mathbb{Q} \) and \( A \) is an abelian variety over \( k \), there are no non-trivial morphisms \( \mathbb{Z}_p(1) \to T_p A \).

**Proof.** Consider the category \( \text{TwFr}_p \) of group schemes over \( k \) which are twisted forms of \( \mathbb{Z}_p^N \) for some \( N \), \( T_p A \) is such a group scheme. The functor
\[
F : \text{TwFr}_p^\text{op} \longrightarrow \text{TwFr}_p
\]
\[
M \longmapsto \text{Hom}(M, \mathbb{Z}_p(1))
\]
is a contravariant equivalence of \( \text{TwFr}_p \) with itself: the natural morphism \( M \to \text{Hom}(\text{Hom}(M, \mathbb{Z}_p(1))) \) is an isomorphism since this can be checked after base changing to \( \bar{k} \), where it is obvious.

Hence, by Lemma 4.12 we have
\[
\text{Hom}(\mathbb{Z}_p(1), T_p A) = \text{Hom}(F(T_p(A)), F(\mathbb{Z}_p(1))) = \text{Hom}(T_p \hat{A}, \mathbb{Z}_p)
\]
which is trivial by Proposition 4.9. \( \square \)

**Corollary 4.14.** Let \( A^1 \) be a torsor for an abelian variety \( A \) over a field \( k \) finitely generated over \( \mathbb{Q} \). Then every morphism \( B \to \Pi_{A^1/k} \) factorizes through a section \( \text{Spec} k \to \Pi_{A^1/k} \).

**Proof.** It is enough to show that \( B \to \Pi_{A^1/k} \) induces a trivial morphism of bands. Since the band of \( \Pi_{A^1/k} \) is \( TA = \prod_q T_q A \), this follows from Corollary 4.13. \( \square \)

**Corollary 4.15.** Let \( k \) be finitely generated over \( \mathbb{Q} \), \( A^1 \) a torsor for an abelian variety and \( V \) a smooth variety over \( k \). Every morphism \( \text{Spec} k(V) \to \Pi_{A^1/k} \) extends uniquely to \( V \).

**Proof.** The uniqueness follows from Corollary 4.3. For existence, apply Corollary 4.14 and Corollary 4.7. \( \square \)

**Lemma 4.16.** Let \( W \to V \) be a surjective smooth morphism of normal varieties, \( \Phi \) a pro-finite étale gerbe, \( s_V : \text{Spec} k(V) \to \Phi \) a section and \( s_W \) its composition with \( \text{Spec} k(W) \to \text{Spec} k(V) \).

Suppose that \( s_W \) extends to a morphism \( W \to \Phi \). Then \( s_V \) extends to a morphism \( V \to \Phi \).

**Proof.** Let \( k'/k \) a separable extension with a section \( \text{Spec} k' \to \Phi \). Over \( k' \), the thesis follows directly from Lemma 4.1 plus the fact that integral closure commutes with smooth base change, see [Stacks, Tag 03GG]. Since the maximal locus of definition commutes with base change along separable extensions (see Corollary 4.3) and \( \text{Spec} k(V \times k') \to \Phi_{k'} \) extends to \( V \times k' \to \Phi'_{k'} \) we get the thesis. \( \square \)

If we assume that \( \Phi \) is torsion free, we may drop the smoothness assumption.
Lemma 4.17. Let \( W \to V \) a surjective morphism of smooth varieties over a field of characteristic 0, \( \Phi \) a torsion-free pro-finite étale gerbe, \( s_V : \text{Spec} \, k(V) \to \Phi \) a section and \( s_W \) its composition with \( \text{Spec} \, k(W) \to \text{Spec} \, k(V) \).

Suppose that \( s_W \) extends to a morphism \( W \to \Phi \). Then \( s_V \) extends to a morphism \( V \to \Phi \).

Proof. By generic smoothness and thanks to Lemma 4.16, \( s_V \) extends to an open subset of \( U \subseteq V \), and we may restrict it to have the form \( U = V \setminus (D_1 \cup \ldots \cup D_n) \) with \( D_i \subseteq V \) codimension 1 subvarieties. We want to extend \( U \to \Phi \) to a morphism \( V \to \Phi \), and in order to do this we may suppose that \( n = 1 \) and then conclude by induction on \( n \).

Hence, \( D = D_1 \) and \( s_V \) extends to \( V \setminus D \). We want to apply Lemma 4.5. Let \( C \subseteq W \) be one irreducible component of the inverse image of \( D \), choose a smooth closed point \( c \in C \) where \( C \to D \) is smooth, and let \( d \in D \) be its image. Let \( r \) be the ramification index of \( W \to V \) at the generic point of \( C \). As in Lemma 4.5, we have a morphism
\[
B_{k(d)} \hat{Z}(1) \to \Pi_{V \setminus D} \to \Phi
\]
and we want to show that it factorizes through some section \( \text{Spec} \, k(d) \to \Phi \). Call \( \varphi \) the image of the tautological section \( \text{Spec} \, k(d) \to B_{k(d)} \hat{Z}(1) \) in \( \Phi \). We want to show that the induced homomorphism
\[
\sigma_{V,D} : \hat{Z}(1) \to \text{Aut}_{\Phi_{k(d)}}(\varphi)
\]
is trivial. By hypothesis, this is true over \( W \), i.e. the analogous homomorphism
\[
\sigma_{W,C} : \hat{Z}(1) \to \text{Aut}_{\Phi_{k(c)}}(\varphi)
\]
is trivial. But now we have a commutative diagram
\[
\begin{array}{ccc}
\hat{Z}(1) & \xrightarrow{r} & \hat{Z}(1) \\
\sigma_{W,C} = 0 & \downarrow & \sigma_{W,C} \\
\text{Aut}_{\Phi_{k(c)}}(\varphi) & \nearrow & \sigma_{V,D}
\end{array}
\]
where the horizontal arrow \( \hat{Z}(1) \to \hat{Z}(1) \) is just multiplication by the ramification index \( r \) of \( W \to V \) at \( C \). Hence, since \( \sigma_{W,C} \) is trivial and \( \text{Aut}_{\Phi_{k(c)}}(\varphi) \) is torsion free by hypothesis, we get that \( \sigma_{V,D} \) is trivial too, as desired. \( \square \)

Lemma 4.18. Let \( W \dashrightarrow V \) a rational map of smooth projective varieties over a field of characteristic 0, \( \Phi \) a torsion-free pro-finite étale gerbe, \( s_V : \text{Spec} \, k(V) \to \Phi \) a section and \( s_W \) its composition with \( \text{Spec} \, k(W) \to \text{Spec} \, k(V) \).

Suppose that \( s_W \) extends to a morphism \( W \to \Phi \). Then \( s_V \) extends to a morphism \( V \to \Phi \).

Proof. There exists a smooth projective variety \( W' \) with morphisms \( W' \to W \), \( W' \to V \) which commute with the given rational map \( W \dashrightarrow V \): up to replacing \( W \) with \( W' \), we may suppose that the rational map \( W \dashrightarrow V \) is a projective dominant morphism, hence surjective. Now apply Lemma 4.17. \( \square \)
5. Finite type essential dimension

Observe that if $G$ is a group scheme of finite type over $k$ and $L/k$ is an extension, every $G$-torsor over $L$ is defined over a finitely generated extension of $k$. If $G$ is not of finite type, extensions which are not finitely generated make a difference.

In fact, all the proofs of section 3 are based on Lemma 3.1, where we construct a single torsor of infinite essential dimension. Observe that the proof of Lemma 3.1 does not adapt to the construction of torsors with finite and arbitrarily large essential dimension: we really use the "gap" between finite and infinite.

Then we define finite type essential dimension by focusing only on finitely generated extension, thus avoiding this pathology.

**Definition 5.1.** Let $F : \text{Fields}_k \to \text{Set}$ be a functor from the category of extensions of $k$ to Set. The *finite type essential dimension* $\text{fed}_k F$ is the supremum of the essential dimensions $\text{ed}_k(a)$ where $a$ varies among objects $a \in F(L)$ with $L$ a finitely generated extension of $k$.

To remain in Merkurjev’s general framework for essential dimension one can give the following alternative definition, for which I thank Z. Reichstein.

Given a functor $F : \text{Fields}_k \to \text{Set}$, we may define the functor $F^{\text{fin}} : \text{Fields}_k \to \text{Set}$ as

$$F^{\text{fin}}(L) = \begin{cases} F(L) & \text{if } L/k \text{ is finitely generated} \\ \{\ast\} & \text{otherwise} \end{cases}$$

Then we have $\text{fed}_k F = \text{ed}_k F^{\text{fin}}$.

We point out that in the original work of Z. Reichstein only finitely generated extensions where considered, see [Rei00, §3, §12]. In fact, he had a different perspectives on essential dimension: rather than something associated to an object of a functor $\text{Fields}_k \to \text{Set}$, it was associated to varieties with group actions, see [Rei00, Definition 3.1]. In this perspective, the fields considered where automatically finitely generated: this subtlety was later overlooked since for groups of finite type it is not crucial.

**Remark 5.2.** If $F$ is the functor of points of an algebraic stack locally of finite type, every point is defined on a finitely generated extension of $k$, and then $\text{fed}_k F = \text{ed}_k F$. Since the vast majority of functors for which essential dimension is studied are algebraic stacks locally of finite type, we can think of finite type essential dimension as a generalization of essential dimension rather than as a variant of it.

**Example 5.3.** It is easy to come up with examples of functors for which essential dimension and finite type essential dimension are different. For example, define $F(L) = \{\ast\}$ if $\text{trdeg}_k L < \infty$, and $F(L) = \{\bullet, \ast\}$ if $\text{trdeg}_k L = \infty$. Then $\text{fed}_k F = 0$ and $\text{ed}_k F = \infty$.

For a less trivial example, consider the group $\mathbb{Z}_p$ over a field $k$ finitely generated over $\mathbb{Q}$. As we will see in Theorem 5.19, $\text{fed}_k \mathbb{Z}_p = 0$ even if $\text{ed}_k \mathbb{Z}_p = \infty$ thanks to Theorem 3.10.

If Grothendieck’s section conjecture is true, étale fundamental group schemes of smooth, proper, hyperbolic curves provide another example. Let $X$ be such a curve with a rational point $x \in X(k)$ and $\pi_1(X, x)$ its étale fundamental group scheme, with $k$ finitely generated over $\mathbb{Q}$. If Grothendieck’s section conjecture is true, then

$$X(k') = H^1(k', \pi_1(X, x))$$
for every finitely generated extension \( k'/k \), and thus
\[
\text{fed}_k \pi_1(X, x) = \text{fed}_k X = 1.
\]
However, we have that \( \text{ed}_k \pi_1(X, x) = \infty \) since \( \pi_1(X, x) \) clearly satisfies the hypothesis of Theorem 3.9.

5.1. Fin. type essential dim. of \( \mathbb{Z}_p(1) \). In this subsection we are going to compute the finite type essential dimension of \( \mathbb{Z}_p(1) \). As we will see, this is still infinite, but the reason is far more subtle than for plain essential dimension. In order to prove this, we need to develop the theory of higher discrete valuations.

5.1.1. Higher discrete valuations.

**Definition 5.4.** A rank \( n \) valuation on a field \( L \) is a valuation \( v : L^* \to \mathbb{Z}^n \) where the value group \( \mathbb{Z}^n \) has the lexicographic order (we want \( v \) to be surjective, \( \mathbb{Z}^n \) must really be the value group).

If \( v \) is a rank \( n \) valuation on \( L \), its determinant \( \det(v) \) is the composition
\[
\det(v) : \bigoplus_n L^* \xrightarrow{v^n} \mathbb{Z}^n \xrightarrow{M_n(\mathbb{Z})} \mathbb{Z}.
\]

For \( n = 1 \), rank 1 valuations are just the usual discrete valuations. The idea of using rank \( n \) valuations in order to study essential dimension has been already explored in [Mey12].

**Definition 5.5.** A chain of discrete valuations \( (v_1, \ldots, v_n) \) of length \( n \) on a field \( L \) is the following data.

Set \( L_0 = L \). For every \( i = 1, \ldots, n \), let \( v_i \) be a discrete valuation \( v_i : L_{i-1} \to \mathbb{Z} \), and define \( L_i \) as the residue field \( L_{i-1}/v_i \). We define \( L_n \) as the residue field of the chain.

If \( k \subseteq L_0 \) is a subfield and the restriction of \( v_1 \) to \( k \) is trivial, then \( k \) embeds naturally in \( L_1 \). Then we say that the chain \( (v_1, \ldots, v_n) \) is trivial on \( k \) if we have recursively that \( v_i \) is trivial on \( k \) for every \( i \).

Recall that a lower triangular matrix with only ones on the diagonal is called a lower unitriangular matrix, and these are precisely the automorphisms of \( \mathbb{Z}^n \) as an ordered group.

**Lemma 5.6.**

i) Let \( v \) be a rank \( n \) valuation on \( L \), and write \( L_0 = L, w_0 = v \). Define \( v_1 : L_0 \to \mathbb{Z} \) as the first coordinate of the rank \( n \) valuation \( w_0 = v \). The function \( v_1 \) is a rank 1 valuation on \( L_0 \). Set \( L_1 = L_0/v_1 \), \( w_0 \) induces a rank \( n - 1 \) valuation \( w_1 \) on \( L_1 \).

Repeating the process, we may construct a chain of discrete valuations \( (v_1, \ldots, v_n) \) associated to \( v \), and the residue field of the chain coincides with the residue field of \( v \).

ii) Given a chain of discrete valuations \( (v_1, \ldots, v_n) \) on \( L \), it is possible to construct a rank \( n \) valuation \( v : L \to \mathbb{Z}^n \) such that \( (v_1, \ldots, v_n) \) is associated to \( v \).

iii) Let \( v, v' \) be two rank \( n \) valuations \( L^* \to \mathbb{Z}^n \). The following are equivalent:
   - \( v, v' \) are isomorphic valuations,
   - \( v, v' \) differ by a lower unitriangular matrix,
   - \( v \) and \( v' \) have the same associated chain.

iv) All the preceding points remain true if we restrict everything to valuations and chains of valuations trivial on a base field \( k \).
Proof. i) Let us check that the first coordinate $v_1$ is a valuation. It is clearly an homomorphism with respect to multiplication. If
$$v_1(a + b) < \min\{v_1(a), v_1(b)\},$$
since $v_1$ is the first coordinate of $v$ we get that
$$v(a + b) < \min\{v(a), v(b)\},$$
hence we have a contradiction.

Now take $a, a' \in L^*$ with $v_1(a) = v_1(a') = 0$. If $a$ and $a'$ map to the same element of $L^*_v$, we have that $a' - a$ maps to 0. Hence, $v_1(a' - a) > 0 = v_1(a)$ and thus $v(a' - a) > v(a)$. This implies that $v(a') = v((a' - a) + a) = v(a)$, i.e. $v$ defines a map $L^*_v \to \mathbb{Z}^n$. Since the first coordinate is 0, we may ignore it, thus getting a map $L^*_v \to \mathbb{Z}^{n-1}$. It can be checked that this is a rank $n - 1$ valuation, thus we conclude by induction.

ii) Let $(v_1, \ldots, v_n)$ be a chain of discrete valuations on $L$, we want to construct $v$. For $n = 1$ this is obvious. By induction, we have a rank $n - 1$ valuation $w : L^*_v \to \mathbb{Z}^{n-1}$: we want to put together $v_1$ and $w$ to construct $v$. Fix $\pi \in L^*$ an uniformizing parameter for $v_1$. Now for any $a \in L^*$ define
$$v(a) = \left( v_1(a), w\left( a \cdot \pi^{-v_1(a)} \right) \right) \in \mathbb{Z} \oplus \mathbb{Z}^{n-1} = \mathbb{Z}^n.$$  

It can be easily checked that $v$ satisfies the properties of a rank $n$ valuation, and that its associated discrete valuations are $v_1, \ldots, v_n$. Observe that the construction of $v$ depends on the choices of the uniformizing parameter $\pi$ and of $w$.

iii) Since the ordered automorphisms of $\mathbb{Z}^n$ are given by lower unitriangular matrices, $v$ and $v'$ are isomorphic as abstract valuations if and only if they differ by such a matrix.

If two rank $n$ valuations $v, v'$ differ by a lower unitriangular matrix, it is obvious that they have the same associated chain.

On the other hand, suppose that $v$ and $v'$ have the same associated chain $(v_1, \ldots, v_n)$. Let $\pi_i \in L^*$ be such that $v_i(\pi_j) = 0$ for every $j < i$, and $v_i(\pi_i) = 1$. Let $c_i, c'_i$ be the coordinates of $v$ and $v'$: these are in general different from $v_i, v'_i$, but still we have $c_i(\pi_i) = c'_i(\pi_i) = 0$ for $j < i$ and $c_i(\pi_i) = c'_i(\pi_i) = 1$ (see the construction of point (1)).

This tells us that the square matrices
$$(v(\pi_1)| \ldots |v(\pi_n)), (v'(\pi_1)| \ldots |v'(\pi_n))$$
are both lower unitriangular. Hence, up to multiplying $v'$ by a lower unitriangular matrix we may suppose that $v(\pi_i) = v'(\pi_i)$ for every $i = 1, \ldots, n$. But now, given any $a \in L^*$, it is easy to write by recursion
$$a = a_1 = \pi_1^{v_1(a_1)} \cdot a_2 = \pi_1^{v_1(a_1)} \cdot \pi_2^{v_2(a_2)} \cdot a_3 = \cdots = \prod_i \pi_i^{v_i(a)} \cdot a_{n+1}$$

with $v_i(a_j) = 0$ for $i < j$. In particular, $v(a_{n+1}) = v'(a_{n+1}) = 0$ and
$$v(a) = \sum_i v_i(a_i) \cdot v(\pi_i) = \sum_i v_i(a_i) \cdot v'(\pi_i) = v'(a).$$

iv) Obvious. $\square$
Recall that if $M$ is a proper variety over $k$ and $v$ is a valuation on $k(M)$ trivial on $k$, we can define the center of the valuation: if $A$ is the value ring with fraction field $L$, then the valuative criterion of properness gives us a morphism $\text{Spec } A \to M$ and the center of the valuation is the image of the closed point. If $A$ is a DVR and the valuation is associated to an hypersurface $V \subseteq M$, then the center of the valuation is the generic point of $V$.

**Corollary 5.7.** Let $M$ be an integral scheme and $n$ a positive integer. Consider a chain $M_0, \ldots, M_n$ where $M_0 = M$ and $M_{i+1}$ is a codimension 1 integral locally closed subscheme of the normalization $\overline{M}_i$ of $M_i$:

$$M_n \subseteq \overline{M}_{n-1} \to M_{n-1} \subseteq \cdots \to M_1 \subseteq \overline{M}_0 \to M_0 = M$$

There exists a rank $n$ valuation $v : k(M)^* \to \mathbb{Z}^n$ with associated chain $(v_1, \ldots, v_n)$ such that $v_i$ corresponds to the codimension 1 sub-variety $M_i \subseteq \overline{M}_{i-1}$.

**5.1.2. Algebraic dependence and higher valuations.** In the following, we fix a base field $k$. All valuations and chains are tacitly assumed to be trivial on $k$.

**Proposition 5.8.** Given a finitely generated extension $L/k$ and $n$ elements $x_1, \ldots, x_n \in L^*$, they are algebraically independent over $k$ if and only if there exists a rank $n$ valuation $v$ trivial on $k$ such that $\det(v)(x_1, \ldots, x_n) \neq 0$.

**Proof.** One implication, i.e. $\det(v)(x_1, \ldots, x_n) \neq 0$ implies algebraic independence, is classical and holds without assuming $L/k$ finitely generated, see for example [ZS60, ch.VI, §10, rmk.B].

The other implication can be done by induction. For $n = 0$, the empty set is algebraically independent and the empty matrix has determinant 1, hence the unique 0-valuation works.

Let now $n > 0$ be a positive integer, and suppose we have proven the lemma for $n - 1$. Choose $x_1, \ldots, x_n \in L^*$ which are algebraically independent. Consider the discrete valuation $v : k[x_1, \ldots, x_n] \to \mathbb{Z}$ such that $v(x_n) = 1$ and $v(p) = 0$ if $p \in k[x_1, \ldots, x_n]$ is prime with $x_n$. We can extend $v$ to a valuation $v' : L \to \mathbb{Z}$ in the sense that, if $t \in k(x_1, \ldots, x_n)$, $v(t) > 0$ if and only if $v'(t) > 0$ (there might be ramification, but this is finite since $L/k$ is finitely generated). Since the restriction of $v'$ to $k(x_1, \ldots, x_{n-1})$ is trivial, we have an immersion

$$k(x_1, \ldots, x_{n-1}) \hookrightarrow L_{v'}$$

and hence $x_1, \ldots, x_{n-1}$ are algebraically independent also in $L_{v'}$. By induction hypothesis, there exists a rank $n - 1$ valuation $u : L_{v'} \to \mathbb{Z}^{n-1}$ such that

$$\det(u)(x_1, \ldots, x_{n-1}) \neq 0.$$

Now, combining $u$ with $v'$, we obtain a rank $n$ valuation $v : L^* \to \mathbb{Z}^n$ such that

$$\det(v)(x_1, \ldots, x_n) = v'(x_n) \det(u)(x_1, \ldots, x_{n-1})$$

since $v'$ is zero when restricted to $k(x_1, \ldots, x_{n-1})$. Now,

$$\det(u)(x_1, \ldots, x_{n-1}) \neq 0$$

by inductive hypothesis and $v'(x_n) \neq 0$ because $v(x_n) = 1$, hence we have $\det(v)(x_1, \ldots, x_{n}) \neq 0$ too. \hfill $\square$
Example 5.9. To see why we need the hypothesis $L/k$ finitely generated in Proposition 5.8, consider $k = \mathbb{Q}$ and $L$ the algebraic closure of $\mathbb{Q}(t)$. Clearly $t \in L$ is transcendental over $k$, but $t$ has an $n$-th root in $L$ for every $n$ and thus every discrete valuation is trivial on $t$.

Lemma 5.10. Let $M$ be a variety over $k$ of dimension $m$. Let $k(M)/L/k$ be a sub-extension of transcendence degree $n \leq m$. Then there exist a transcendence basis $x_1, \ldots, x_n \in L$ and a rank $n$ valuation $v : k(M)^* \to \mathbb{Z}^n$ such that

- $\det(v)(x_1, \ldots, x_n) \neq 0$,
- the center of $v$ is the generic point of a codimension $n$ sub-variety of $M$.

Proof. Choose any $t_1, \ldots, t_n \in L$ transcendental over $k$, we have a rational morphism $f : M \to \mathbb{A}^n$.

For the sake of clarity, suppose first that we are in characteristic 0. Thanks to generic smoothness, we can choose a closed point $q \in M$ where $f$ is defined and smooth. Choose hypersurfaces $H_1, \ldots, H_n$ regular at $f(q)$ cutting it transversally. Since $f$ is smooth at $q$, the irreducible components $V_i \subseteq M$ of $f^{-1}(H_i)$ containing $q$ are hypersurfaces which meet transversally. Set $M_i = \cap_{j=1}^i V_j$ and use them to define a valuation $v$ as in Corollary 5.7. If $x_j \in k[t_1, \ldots, t_n]$ is the equation defining $H_j$, then by construction the matrix $v(x_1, \ldots, x_n)$ is lower unitriangular, and thus it has determinant 1, and the center of the valuation is the generic point of $M_n \subseteq M$.

We can generalize this idea to positive characteristic by replacing generic smoothness with generic flatness, but the process is more complex, and we have to allow ramification.

Set $N_0 = \mathbb{A}^n$, it is a regular affine scheme. Thanks to generic flatness there exists an open subset $M_0 \subseteq M$ over which $(t_1, \ldots, t_n)$ defines a flat dominant morphism $f_0 : M_0 \to N_0$ of pure codimension $m - n$. For $i = 1, \ldots, n$ apply the following recursive process:

1. Choose $p_{i-1} \in N_{i-1}$ the image of a closed point of $M_{i-1}$ through $f_{i-1} : M_{i-1} \to N_{i-1}$.
2. Choose $x_i \in m_{p_{i-1}}^2 \setminus m_{p_{i-1}}^1$ where $m_{p_{i-1}}$ is the local ideal of $p_{i-1}$ in $N_{i-1}$.
3. Up to shrinking $N_{i-1}$ (and $M_{i-1}$) around $p_{i-1}$ we may suppose that $x_i$ is defined globally and $N_i = V(x_{i-1}) \subseteq N_{i-1}$ is a regular closed sub-scheme of dimension $n - i$.
4. Choose one irreducible component $M_i$ of the normalization of $f_{i-1}^{-1}(N_i) \subseteq M_{i-1}$, which is nonempty because we have chosen $p_{i-1} \in N_{i-1}$ in the image of $f_{i-1}$. Since $f_{i-1}$ is flat of pure codimension $m - n$, $M_i$ has dimension $n - i + m - n = m - i$.
5. The induced morphism $f_i : M_i \to N_i$ is dominant because $M_i$ dominates one irreducible component of $f_{i-1}^{-1}(N_i)$, and $f_{i-1}$ is flat. Thanks to generic flatness, up to shrinking $N_i$ and $M_i$ we may suppose that $f_i$ is flat of pure codimension $m - n$.

By Corollary 5.7, the sequence $M_1, \ldots, M_n$ defines a rank $n$ valuation $v$ on $k(M)$ whose center is the image of the generic point of the generically finite morphism $M_n \to M$.

We have chosen $x_i$ as a function on $N_{i-1} \subseteq \mathbb{A}^n$, with a small abuse of notation we also denote by $x_i$ one lifting to $k(\mathbb{A}^n) = k(t_1, \ldots, t_n)$. Since $x_i \in k(\mathbb{A}^n)$ restricts to a nonzero rational function on $N_j$ for $j < i$, $f^* x_i \in k(M)$ restricts to a nonzero
rational function on $M_j$ for $j < i$. This implies that the $j$-th coordinate of $v(x_i)$ is 0, i.e. the $n \times n$ matrix $v(x_1, \ldots, x_n)$ is triangular.

We have now to show that $\det(v)(x_1, \ldots, x_n) \neq 0$. The $i$-th diagonal entry is

$$v_i \left( f_i^{s_i}(x_i | N_i) \right)$$

where $v_i : k(M_{i-1})^* \to \mathbb{Z}$ is the rank 1 valuation associated to the generic point of $M_i$. But this is exactly the ramification index of $f_i$ at the generic point of $M_i$, since $x_i | N_i$ is a local equation for $N_i = f_i(M_i)$, and thus it is a nonzero integer.

5.1.3. Computation of $\text{fed}_k \mathbb{Z}_p(1)$. If $A$ is an abelian group, write $\land_p A$ for the projective limit $\varprojlim_n A/p^n A$. If $L$ is a field, we have $H^1(L, \mu_{p^n}) = L^*/(L^*)^{p^n}$ and thus

$$H^1(L, \mathbb{Z}_p(1)) = \varprojlim_n L^*/(L^*)^{p^n} = \land_p L^*.$$

If $v : L^* \to \mathbb{Z}^n$ is a rank $n$ valuation (or any homomorphism of groups), observe that it makes sense to evaluate $v$ on elements of $\land_p L^*$, i.e. we have an induced homomorphism

$$\land_p L^* \to \land_p \mathbb{Z}^n = \mathbb{Z}_p^n$$

which, by a small abuse of notation, we still call $v$. This extension is not a valuation anymore: $\land_p L^*$ is not a field and the order of $\mathbb{Z}$ does not extend to $\mathbb{Z}_p$. Still, it is an interesting homomorphism, and it allows us to generalize Proposition 5.8 to elements of $\land_p L^*$.

**Lemma 5.11.** Consider $x_1, \ldots, x_n \in H^1(L, \mathbb{Z}_p(1)) = \land_p L^*$ and suppose that for some rank $n$ valuation $v : L^* \to \mathbb{Z}^n$ we have

$$\det(v)(x_1, \ldots, x_n) \neq 0 \in \mathbb{Z}_p.$$

Then there exists $s \in \mathbb{N}$ such that the image of $(x_1, \ldots, x_n)$ in $L^*/L^* \oplus \cdots \oplus L^*/L^*$ has essential dimension $n$. In particular, $\text{ed}_k(x_1, \ldots, x_n) \geq n$.

**Proof.** The reduction modulo $p^s$ of $\det(v)(x_1, \ldots, x_n)$ is nonzero for some $s$ large enough. This implies that the image of $(x_1, \ldots, x_n)$ in $(L^*/L^*)^n = H^1(L, \mu_{p^n})$ has essential dimension $n$.

In fact, for any choice of $x_{1,s}, \ldots, x_{n,s} \in L^*$ such that $x_i \equiv x_{j,s} (\text{mod } L^{p^s})$ we have that $x_{1,s}, \ldots, x_{n,s}$ are algebraically independent thanks to Proposition 5.8. □

**Theorem 5.12.** Let $a_1, \ldots, a_n \in \mathbb{Z}_p$ be linearly independent over $\mathbb{Q}$. Then

$$\prod_{i=1}^n t_i^{e_i} \in \land_p k(t_1, \ldots, t_n)^* = H^1(k(t_1, \ldots, t_n), \mathbb{Z}_p(1))$$

has essential dimension $n$. In particular, $\text{fed}_k \mathbb{Z}_p(1) = \infty$.

**Proof.** Suppose that $\text{ed}_k t_1^{a_1} \cdots t_n^{a_n} < n$, this means that there exists a subfield $k' \subseteq k(t_1, \ldots, t_n)$ of transcendence degree $n - 1$ such that $t_1^{a_1} \cdots t_n^{a_n}$ is in the image of $\land_p k^{s_n} \to \land_p k(t_1, \ldots, t_n)^*$.

Identify $k(t_1, \ldots, t_n)$ with the function field of $\mathbb{P}^n$, and choose a transcendence basis $x_2, \ldots, x_n$ of $k'$ as in Lemma 5.10. We have then a rank $n - 1$ valuation $v'$ whose center is the generic point of an irreducible curve $C \subseteq \mathbb{P}^n$ and such that $\det(v')(x_2, \ldots, x_n) \neq 0$. □
There is at least one of the coordinate hyperplanes not containing $C$, say $H_1 = \{t_1 = 0\}$. Choose a point $p$ in the normalization $\overline{C}$ of $C$ mapping to a point of $C \cap H_1$. Then we may use $p$ to extend $v'$ to a rank $n$ valuation $v : k(t_1, \ldots, t_n) \to \mathbb{Z}^n$ whose first $n-1$ coordinates are just $v'$ (for details on how to construct this extension, see Lemma 5.6).

Write

$$v(t_1^{a_1} \cdots t_n^{a_n}) = \left( \sum_{i=1}^{n} r_{j,i} \alpha_i \right)$$

with $r_{j,i} = v(t_j)_i \in \mathbb{Z}$. By construction, $r_{j,1} = v(t_1)_j = 0$ for $j < n$ and $r_{n,1} = v(t_1)_n \neq 0$ since $C \not\subseteq H_1$ but $p \in \overline{C}$ maps to a point of $H_1$. Consider the following determinant

$$\det(v(t_1^{a_1} \cdots t_n^{a_n})|v(x_2)| \ldots |v(x_n)) = \sum_{i=1}^{n} \alpha_i \det((r_{j,i})_j|v(x_2)| \ldots |v(x_n)) =$$

$$= \alpha_1 \cdot r_{n,1} \det(v') (x_2, \ldots, x_n) + \sum_{i=2}^{n} \alpha_i \cdot s_i$$

where

$$s_i = \det((r_{j,i})_j|v(x_2)| \ldots |v(x_n)) \in \mathbb{Z}, i = 2, \ldots, n.$$ 

Since $r_{n,1} \neq 0$, $\det(v') (x_2, \ldots, x_n) \neq 0$, $s_i$ are integers and $\alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$, this determinant is different from 0. Using Lemma 5.11, this implies that

$$\text{ed}_k(t_1^{a_1} \cdots t_n^{a_n}, x_2, \ldots, x_n) \geq n.$$ 

On the other hand, both $t_1^{a_1} \cdots t_n^{a_n}$ and $x_2, \ldots, x_n$ are defined on $k'$ which has transcendence degree $n-1$ over $k$, hence we get a contradiction. 

If $G$ is an abelian group scheme and $p$ is a prime number, we write

$$T_pG = \lim_{\overline{n}} G[p^n], \quad TG = \lim_{\overline{n}} G[n] = \prod_p T_pG$$

for the $p$-local and global Tate modules, which are pro-finite group schemes.

**Corollary 5.13.** Let $G$ be an algebraic torus over a field $k$, then $\text{fed}_k T_pG = \text{fed}_k TG = \infty$.

**Proof.** Since $TG = \prod_p T_pG$, it is enough to prove $\text{fed}_k T_pG = \infty$. If the torus is split of dimension $d$, $T_pG = \mathbb{Z}_p(1)^d$ and then the thesis follows from Theorem 5.12. In general, there exists a splitting field $k'/k$ for $G$ finite over $k$, and finite type essential dimension decreases along finite extensions of the base field (this is analogous to the same fact for plain essential dimension).

### 5.2. Fin. type essential dim. of $\mathbb{Z}_p$
5.2.1. **Characteristic p.** In this subsection, char $k = p$.

A. Ledet in [Led04a] and [Led04b] obtained the main known results about essential dimension of cyclic $p$-groups in characteristic $p$. For every $n$, there is a short exact sequence of group schemes

$$0 \to \mathbb{Z}/p^n \to W_n \to W_n \to 0$$

where $W_n$ is the group scheme of Witt vectors. We have that $H^1(k, W_n) = 0$ and $\dim W_n = n$, thus we obtain that $\text{ed}_k \mathbb{Z}/p^n \leq n$.

**Conjecture (Ledet).** If char $k = p$, then $\text{ed}_k \mathbb{Z}/p^n = n$.

The conjecture is known for $n = 1, 2$.

Since $H^2(k, \mathbb{Z}/p) = 0$ by Lemma 3.3, we have a short exact sequence

$$0 \to H^1(k, \mathbb{Z}/p) \to H^1(k, \mathbb{Z}/p^{n+1}) \to H^1(k, \mathbb{Z}/p^n) \to 0$$

and hence

$$\text{ed}_k \mathbb{Z}/p^n \leq \text{ed}_k \mathbb{Z}/p^{n+1} \leq \text{ed}_k \mathbb{Z}/p^n + 1$$

since $\text{ed}_k \mathbb{Z}/p = 1$ and thanks to [BF03, Corollary 1.15]. Ledet’s conjecture is thus equivalent to saying that

$$\text{ed}_k \mathbb{Z}/p^{n+1} = \text{ed}_k \mathbb{Z}/p^n + 1.$$ 

A much weaker statement than Ledet’s conjecture asks that the essential dimension keeps increasing with $n$.

**Weak Ledet’s conjecture.** If char $k = p$, then $\lim_{n \to \infty} \text{ed}_k \mathbb{Z}/p^n = \infty$.

**Proposition 5.14.** If the weak Ledet’s conjecture holds, then $\text{fed}_k \mathbb{Z}/p = \infty$.

**Proof.** Since $H^2(L, \mathbb{Z}/p) = 0$ for every extension $L/k$, every $\mathbb{Z}/p^n$-torsor lifts to a $\mathbb{Z}/p^{n+1}$-torsor, and by recursion to a $\mathbb{Z}_p$-torsor. If $\lim_{n \to \infty} \mathbb{Z}/p^n = \infty$, we have thus $\mathbb{Z}_p$-torsors of arbitrarily large essential dimension defined over finitely generated extensions of $k$. \qed

5.2.2. **Characteristic different from p.** Let char $k \neq p$, and for any $n$ choose $\bar{\zeta}_p^n \in \bar{k}$ a primitive $p^n$th root of the unity such that $\bar{\zeta}_p^n = \bar{\zeta}_p^n$. Call

$$k(\zeta_{p^n}) = k(\bar{\zeta}_p, \bar{\zeta}_p^2, \ldots)$$

the field where we add all $p$-adic roots of the unity.

**Definition 5.15.** We say that $k$ has almost all $p$-adic roots of the unity if $[k(\zeta_{p^n}) : k]$ is finite, and that it has all $p$-adic roots of unity if $k = k(\zeta_{p^n})$.

**Lemma 5.16.** If $\bar{\zeta}_p \in k (\zeta_4 \in k$ if $p = 2)$ and $k$ has almost all $p$-adic roots of unity, then $k$ has all $p$-adic roots of unity. In other words, if $k$ has almost all $p$-adic roots of unity, then $k(\zeta_{p^n}) = k(\zeta_p) (or k(\zeta_{2^n}) = k(\zeta_4))$.

**Proof.** The Galois group $\Gamma = \text{Gal}(k(\zeta_{p^n})/k)$ acts continuously and faithfully on $\mathbb{Z}_p(1)(k(\zeta_{p^n})) = \mathbb{Z}_p$, thus $\Gamma$ is a closed subgroup of $\mathbb{Z}_p^\ast$. We have $\mathbb{Z}_p^\ast \cong \mathbb{Z}_p \times \mathbb{Z}/p - 1$ for $p \neq 2$ and $\mathbb{Z}_2^\ast \cong \mathbb{Z}_2 \times \mathbb{Z}/2$. Thus, if $\Gamma$ is finite, it is a subgroup of $\mathbb{Z}/p - 1 (or \mathbb{Z}/2)$, and $\mathbb{Z}/p - 1 (or \mathbb{Z}/2)$ acts faithfully on $\mu_p \subseteq k(\zeta_{p^n}) (or \mu_4)$. If $\bar{\zeta}_p \in k (or \zeta_4)$, the action of $\Gamma$ is trivial on $\mu_p (or \mu_4)$, thus $\Gamma$ must be trivial. \qed

**Corollary 5.17.** If $k$ is finitely generated over $\mathbb{Q}$, then it does not have almost all $p$-adic roots of unity.
Lemma 5.18. A field $k$ of characteristic different from $p$ does not have almost all $p$-adic roots of the unity if and only if every homomorphism of group schemes $\mathbb{Z}_p(m) \to \mathbb{Z}_p(n)$ is trivial for $m \neq n$.

Proof. Since $\text{Hom}(\mathbb{Z}_p(m), \mathbb{Z}_p(n)) = \text{Hom}(\mathbb{Z}_p(m + d), \mathbb{Z}_p(n + d))$ for every $d$, we may suppose $m = 0$. Suppose by contradiction that such a non-trivial homomorphism exists, since every closed subgroup of $\mathbb{Z}_p(n)$ is of the form $p^n\mathbb{Z}_p(n) \simeq \mathbb{Z}_p(n)$ we may suppose the homomorphism is surjective. In particular, $\mathbb{Z}_p(n)(k(\zeta_{p^n}))$ is a trivial Galois module.

The Galois group $\Gamma$ of $k(\zeta_{p^n})/k$ acts faithfully on $\mathbb{Z}_p(1)(k(\zeta_{p^n})) \simeq \mathbb{Z}_p$, thus it is a closed subgroup of $\text{Aut}(\mathbb{Z}_p) = \mathbb{Z}_p^* = \mathbb{Z}_p \times \mathbb{Z}/p - 1$ if $p \neq 2$, and $\mathbb{Z}_2^2 = \mathbb{Z}_2 \times \mathbb{Z}/2$ if $p = 2$. If $\Gamma$ is infinite, thanks to the description above it is easy to check that its action on $\mathbb{Z}_p(n)(k(\zeta_{p^n}))$ is non-trivial.

If $k$ has almost all $p$-adic roots of the unity then $\Gamma \subseteq \mathbb{Z}/p - 1 \subseteq \mathbb{Z}_p^*$ (or $\Gamma \subseteq \mathbb{Z}/2 \subseteq \mathbb{Z}_2^2$), hence the action of $\Gamma$ is trivial on $\mathbb{Z}_p(p - 1)$ (or $\mathbb{Z}_2(2)$). In particular, $\mathbb{Z}_p \simeq \mathbb{Z}_p(p - 1)$ (or $\mathbb{Z}_2 \simeq \mathbb{Z}_2(2)$). \qed

If $k$ has almost all $p$-adic roots of the unity, then

$$\text{fed}_k \mathbb{Z}_p \geq \text{fed}_{k(\zeta_{p^n})} \mathbb{Z}_p, k(\zeta_{p^n}) = \text{fed}_{k(\zeta_{p^n})} \mathbb{Z}_p, k(\zeta_{p^n})(1) = \infty.$$ 

If $k$ does not have almost all $p$-adic roots of the unity, then it is an easy consequence of a theorem of Florence [Flo08, Theorem 4.1] that

$$\lim_{n} \text{ed}_k \mathbb{Z}/p^n \mathbb{Z} = \infty.$$ 

In view of this, the following is rather surprising.

Theorem 5.19. Let $k$ be finitely generated over $\mathbb{Q}$. Then

$$\text{fed}_k \mathbb{Z}_p = 0.$$ 

Proof. Let $T \to \text{Spec} L$ be a $\mathbb{Z}_p$-torsor with $L/k$ finitely generated: we are going to show that $T$ is defined on the algebraic closure $\overline{k}$ of $k$ in $L$. Up to replacing $k$ with $\overline{k}$ we may suppose that $k$ is algebraically closed in $L$, and by induction we may suppose that $\text{trdeg}_k L = 1$, i.e. $L = k(X)$ is the function field of a smooth, projective, geometrically connected curve $X$ over $k$.

Thanks to Corollary 4.7 and to Lemma 5.18, the generic morphism $\text{Spec} k(X) \to B\mathbb{Z}_p$ extends to a morphism $X \to B\mathbb{Z}_p$. Since $\mathbb{Z}_p$ is abelian and the abelianization of $\Pi_X/k$ is $\Pi_{\text{Pic}^0_X}/k$, we have a factorization $\Pi_X/k \to \Pi_{\text{Pic}^0_X}/k \to B\mathbb{Z}_p$. But now the morphism $\Pi_{\text{Pic}^0_X}/k \to B\mathbb{Z}_p$ factorizes through a section $\text{Spec} k \to B\mathbb{Z}_p$ thanks to Corollary 4.11: in particular, the generic $\mathbb{Z}_p$-torsor $T \to \text{Spec} L$ is defined on $k$. \qed

From Theorem 5.19, we obtain as a corollary the fact that every $\mathbb{Z}_p$-extension over a field finitely generated over $\mathbb{Q}$ is defined over a number field.
Corollary 5.20. Let $K$ be finitely generated over $\mathbb{Q}$, and let $k = \overline{\mathbb{Q}}^K$ be the algebraic closure of $\mathbb{Q}$ in $K$. If $H/K$ is a $\mathbb{Z}_p$-extension, there exists a $\mathbb{Z}_p$-extension $h/k$ such that $H = hK$.

Proof. We have that $\text{Spec } H \to \text{Spec } K$ is a $\mathbb{Z}_p$ torsor, by Theorem 5.19 we have a $\mathbb{Z}_p$ torsor $\text{Spec } h \to \text{Spec } k$ such that $\text{Spec } H = \text{Spec } h \times_k \text{Spec } K$. In particular, $\text{Spec } h \to \text{Spec } k$ is connected and pro-étale, thus $h/k$ is $\mathbb{Z}_p$-Galois extension. The isomorphism $h \otimes_k K \simeq H$ allows us to fix an embedding $h \subseteq H$ such that $hK = H$. □

5.3. Why asking finite transcendence degree is not enough. We have defined finite type essential dimension by focusing on finitely generated extension of the base field in order to avoid the fact that pro-finite group schemes almost always have infinite essential dimension as shown in section 3.

Excluding the category of fields finitely generated over the base field, there is another category of extensions of the base field that might have been a good candidate, i.e. the category of fields of finite transcendence degree over the base field. Here we show that this category is still not small enough in order to get a meaningful variant of essential dimension.

Proposition 5.21. Let $G$ be a pro-finite group scheme over a field $k$ of characteristic different from $p$, and suppose that there exists an extension $k'/k$ and a non-trivial homomorphism $G_k \to \mathbb{Z}_p$.

For every $n$, there exists an extension $L/k$ of transcendence degree $n$ and a $G$-torsor over $L$ of essential dimension $n$.

Proof. Since $G$ is pro-finite, $\text{Hom}(G_{\bar{k}}, \mathbb{Z}_p) = \text{Hom}(G_k, \mathbb{Z}_p)$, and thus there exists a non-trivial homomorphism $G_k \to \mathbb{Z}_p$ defined on the algebraic closure of $k$. Then we can use Lemma 3.8, and up to modifying $G_k \to \mathbb{Z}_p$ we may suppose it has a section $\mathbb{Z}_p \to G_k$.

Since $k$ has characteristic different from $p$, over $\bar{k}$ we have $\mathbb{Z}_p \simeq \mathbb{Z}_p(1)$. Consider the $\mathbb{Z}_p(1)$-torsor over $L = \bar{k}(t_1, \ldots, t_n)$ given in Theorem 5.12: its pushforward to $G$ has essential dimension $n$, since we can push it forward to $\mathbb{Z}_p$ again where it has essential dimension $n$. □

6. Continuous essential dimension

The second variant of essential dimension, continuous essential dimension, is more subtle. It is not defined at the level of functors but at the level of single objects.

If $G$ is a pro-affine group scheme (every affine group scheme is pro-affine) and $T$ a $G$-torsor, consider the projective system of torsors $(T \times^G H)_{G \to H}$ where $H$ is a group scheme of finite type and $G \to H$ is an homomorphism. If we have two homomorphisms $G \to H, G \to H'$ with a third homomorphism $H \to H'$ that makes the diagram commute, a basic property of essential dimension tells us that $\text{ed}_k T \times^G H \geq \text{ed}_k T \times^G H'$, i.e. essential dimension increases along the projective system.

If we think the torsors $T \times^G H$ as increasingly better approximations of $T$ (this is particularly convenient if $G$ is pro-finite thanks to Lemma 2.3), then it makes sense to consider the limit of the essential dimensions of $T \times^G H$. Thanks to the
argument above, this limit exists and is just the supremum of \( ed_k T \times^G H \) where \( G \to H \) varies as above. Hence, we give the following definition.

**Definition 6.1.** Let \( \Phi \) be a pro-algebraic gerbe (i.e., a projective limit of gerbes of finite type) over a field \( k \), and \( s : \text{Spec} \ L \to \Phi \) a section.

The **continuous essential dimension** \( ced_k(s) \) is the supremum of the essential dimensions of \( \psi(s) : \text{Spec} \ L \to \Phi \xrightarrow{\psi} \Psi \), where \( \psi : \Phi \to \Psi \) varies among all morphisms from \( \Phi \) to a gerbe of finite type \( \Psi \).

The continuous essential dimension \( ced_k(\Phi) \) of \( \Phi \) is the supremum of \( ced_k(s) \), where \( s \) varies among all sections \( \text{Spec} \ K \to \Phi \) and all field extensions \( K/k \). If \( G \) is a pro-algebraic group scheme, we write \( ced_k G \) for \( ced_k BG \).

**Remark 6.2.** If \( G \) is a group scheme of finite type and \( T \) is a torsor, then \( T \times^G G \) with respect to the identity \( G \to G \) is an initial object of the projective system described above. Hence, we have \( ed_k T = ced_k T \) and \( ced_k G = ed_k G \). More generally, \( ced_k \Phi = ed_k \Phi \) for a gerbe \( \Phi \) of finite type over \( k \). Again, this tells us that we can think of continuous essential dimension as a generalization of essential dimension rather than a variant.

If \( \Phi = \lim_{i \in I} \Phi_i \) is a pro-algebraic gerbe, since every morphism \( \Phi \to \Psi \) to an algebraic gerbe \( \Psi \) factors as \( \Phi \to \Phi_i \to \Psi \) for some \( i \) then for every section \( \text{Spec} \ L \to \Phi \) with images \( s_i : \text{Spec} \ L \to \Phi_i \) we have

\[
ced_k(s) = \sup_i ed_k(s_i) = \lim_i ed_k(s_i).
\]

Since for every section \( s \) we have \( ed_k(s_i) \leq ed_k \Phi_i \), we get that

\[
ced_k \Phi \leq \lim_i \inf ed_k \Phi_i,
\]

where we have used the nonstandard, but obvious, notion of \( \lim \inf \) along the projective system \( I \). In general, there is no reason why equality should hold.

**Example 6.3.** Let \( G = \lim_{i < n} G_i \) be a pro-finite group scheme with transition morphisms \( \phi_i : G_i \to G_i \) and suppose that for every \( i \) there exists a group scheme \( H_i \) such that \( ed_k G_i \times H_i \geq ed_k G_i + 1 \). For every \( j > i \), define an homomorphism

\[
G_j \times H_j \to G_i \times H_i
\]

\[
(g, h) \mapsto (\phi_j(g), 0).
\]

It is immediate to check that \( \lim_{i \to n} G_i \times H_i = G \), and thus we get a strict inequality

\[
ced_k G \leq \lim_i \inf ed_k G_i < \lim_i \inf ed_k G_i + 1 \leq \lim_i \inf ed_k G_i \times H_i
\]

The preceding example is not very satisfying, since with the groups \( H_i \) we have added a lot of useless information in the presentation of \( G \) as a projective limit. A reasonable guess is that equality might hold if the homomorphisms \( G \to G_i \) in the presentation are surjective. The following counterexample of F. Scavia shows that this is not the case.

**Example 6.4.** Let \( G \) be the 1-dimensional torus \( x^2 + y^2 = 1 \) over \( Q \), it splits over \( Q(i) \). Let \( T_2 G = \lim_{n \to 1} G[2^n] \). As we will prove in Theorem 6.10, we have \( ced_Q T_2 G = \dim G = 1 \). We are now going to show that \( ed_Q G[2^n] \geq 2 \) for \( n > 1 \) using a result from [LMMR13].
Let $\Gamma = \text{Gal}(\mathbb{Q}(i)/\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z}$, the character module $M$ of $G$ is $\mathbb{Z}$ where $\Gamma$ acts by $x \mapsto -x$.

The character module of $T[2^n]$ is $M/2^nM$. A permutation module $P$ is a $\Gamma$-module which is free as $\mathbb{Z}$-module, and such that $\Gamma$ acts by permutations of a basis. A 2-presentation of $M/2^nM$ is a morphism $\varphi : P \rightarrow M/2^nM$ such that $P$ is a permutation module and the cokernel is finite of odd order, in our case this is equivalent to surjectivity since $M/2^nM$ is finite of even order.

Thanks to [LMMR13, Corollary 5.1], we have that $\text{ed}_Q(G[2^n]) \geq \text{ed}_Q(G[2^n]; 2) = \min \text{rk ker } \varphi$

where $\varphi$ ranges among all 2-presentations of $M/2^nM$.

Let $\varphi : P \rightarrow M/2^nM$ be a 2-presentation, since $M/2^nM$ is finite we have $\text{rk ker } \varphi = \text{rk } P$. If $n > 1$, the action of $\Gamma$ is non-trivial on $M/2^nM$, thus $\Gamma$ must act non-trivially on $P$ too. But then $\text{rk } P > 1$, because a rank 1 permutation module is a trivial Galois module.

**Remark 6.5.** A natural question is why we should take the limit defining the continuous essential dimension at the level of torsors and not at the level of groups, i.e. why not define $\text{ced}_k G$ as $\lim_i \text{ed}_k G_i$ for a pro-finite group scheme $G = \lim_i G_i$.

Obviously, this depends on taste. From our point of view, there are three reasons.

- It may happen that the limit $\lim_i \text{ed}_k G_i$ does not exists, and we don’t see any particular reason to prefer $\lim \inf_i \text{ed}_k G_i$ or $\lim \sup_i \text{ed}_k G_i$. On the other hand, the limit always exists at the level of torsors.
- The limit $\lim_i \text{ed}_k G_i$ depends on the presentation of $G = \lim_i G_i$ as a projective limit, while our definition depends only on $G$.
- Most importantly, we are interested in studying $G$-torsors, and $\lim_i \text{ed}_k G_i$ depends on $G_i$-torsors that do not extend to $G$.

Finally, we can merge in an obvious way the finite type and continuous essential dimensions and define the **fee dimension** $\text{fced}_k \Phi$ of a pro-algebraic gerbe $\Phi$.

**Definition 6.6.** If $\Phi$ is a pro-algebraic gerbe, the **fee dimension** $\text{fced}_k \Phi$ of $\Phi$ is the supremum of the continuous essential dimensions $\text{ced}_k s$ where $\text{Spec } L \rightarrow \Phi$ is a section over a field $L$ finitely generated over $k$.

If $G$ is a pro-algebraic group scheme, we write $\text{fced}_k G$ for $\text{fced}_k BG$.

**Lemma 6.7.** If $\Phi$ is a gerbe of finite type over $k$, then $\text{ed}_k \Phi = \text{fed}_k \Phi = \text{ced}_k \Phi = \text{fced}_k \Phi$. \hfill $\Box$

**Lemma 6.8.** Let $\Phi$ be a pro-algebraic gerbe over a field $k$.

(i) For every section $s : \text{Spec } L \rightarrow \Phi$, we have $\text{ced}_k s \leq \text{ed}_k s$.

(ii)

$$
\begin{align*}
\text{ced}_k \Phi & \leq \text{fced}_k \Phi \\
\text{fced}_k \Phi & \leq \text{fed}_k \Phi \\
\text{fed}_k \Phi & \leq \text{ed}_k \Phi
\end{align*}
$$

(iii) There are examples for which $\text{fed}_k \Phi > \text{ced}_k \Phi$ and others for which $\text{fed}_k \Phi < \text{ced}_k \Phi$.

(iv) If $k'/k$ is an extension, $\text{ced}_{k'} \Phi_{k'} \leq \text{ced}_k \Phi$. If $k'/k$ is finitely generated, the inequality holds for $\text{fed}$ and $\text{fced}$ too.
Proof. The proof of (i) follows directly from the definition, (ii) follows from (i) and (iv) is identical to the analogous fact for classical essential dimension.

The only non-trivial one is (iii). Thanks to Theorem 5.12 and the following Theorem 6.10, we have

\[ \text{fed}_k \mathbb{Z}_p(1) = \infty > \text{ced}_k \mathbb{Z}_p(1) = 1. \]

On the other hand, thanks to Theorem 5.19 and the following Proposition 6.11, if \( k \) is finitely generated over \( \mathbb{Q} \) then

\[ \text{fed}_k \mathbb{Z}_p = 0 < 1 \leq \text{ced}_k \mathbb{Z}_p. \]

\[ \square \]

In dimension 0, essential dimension and continuous essential dimension coincide for pro-finite gerbes.

**Proposition 6.9.** Let \( \Phi \) be a pro-finite gerbe over \( k \), and let \( s : \text{Spec} L \to \Phi \) be a section where \( L/k \) is an extension of fields. Then \( \text{ced}_k s = 0 \) if and only if \( \text{ed}_k s = 0 \).

**Proof.** Since \( \text{ced}_k s \leq \text{ed}_k s \), one implication is obvious. Let us suppose now that \( \text{ced}_k s = 0 \). Up to replacing \( k \) with \( k^L \), we may suppose that \( k \) is algebraically closed in \( L \).

Write \( \Phi = \lim_{\leftarrow i} \Phi_i \) with \( \Phi_i \) finite gerbes. By hypothesis, \( s_i : \text{Spec} L \to \Phi \to \Phi_i \) is defined over \( k \), i.e. there exist sections \( r_i : \text{Spec} k \to \Phi_i \) with 2-commutative diagrams

\[
\begin{array}{ccc}
\text{Spec} L & \xrightarrow{s} & \Phi \\
\downarrow & & \downarrow \\
\text{Spec} k & \xrightarrow{r_i} & \Phi_i
\end{array}
\]

We want to show that the \( r_i \) form a projective system whose limit is a section \( r : \text{Spec} k \to \Phi \) such that \( r_L = s \).

Let \( j \geq i \) in the projective system, and define \( r_{j,i} \in \Phi_i(k) \) the image of \( r_j \) in \( \Phi_i \). We want to give isomorphisms \( r_{j,i} \simeq r_i \) for every \( j \geq i \). Now, \( \text{Isom}_\Phi(r_{j,i}, r_i) \) is a finite scheme with an \( L \)-rational point, because we have isomorphisms

\[ r_{j,i,L} \simeq s_i \simeq r_{i,L}. \]

Since \( k \) is algebraically closed in \( L \) and \( \text{Isom}_\Phi(r_{j,i}, r_i) \) is finite, the isomorphism \( r_{j,i,L} \simeq r_{i,L} \) given above is defined over \( k \), i.e. it is the base change of an isomorphism \( \alpha_{ij} : r_{ji} \simeq r_i \).

These isomorphisms respect the cocycle condition: if \( j \geq i \geq h \) and we write \( \varphi_{h,i} : \Phi_i \to \Phi_h \), we have

\[ \alpha_{h,i} \circ \varphi_{h,i}(\alpha_{ij}) = \alpha_{h,j}. \]
In fact, this equality can be checked after base change to \( L \), and over \( L \) it amounts to the commutativity of the following diagram:

\[
\begin{array}{ccc}
r_{i,j,L} & \\ & \phi_{k,j}(a_{i,j}) & \downarrow \\
& l & \\
rs_{h} & & \alpha_{k,j} \\
r_{i,j,L} & \downarrow a_{i,j} & \rightarrow \alpha_{k,j} \\
& l & \\
& k & \\
\end{array}
\]

which is obvious. Hence \( r = \lim_{i \downarrow s} r_i : \text{Spec} k \rightarrow \Phi \) is a section, and clearly \( r_L \simeq s \).

As we have seen in Corollary 5.13, if \( G \) is a torus the finite type essential dimension is infinite for \( T_pG \) and \( TG \). With continuous essential dimension we get a much more interesting result.

**Theorem 6.10.** Let \( k \) be a field and \( G \) an algebraic torus over \( k \) of dimension \( d \). Then
\[
ced_k T_pG = \text{fced}_k T_pG = \text{ced}_k TG = \text{fced}_k TG = d.
\]

In particular, for \( G = G^d_m \), we have
\[
\text{ced}_k \mathbb{Z}_p(1)^d = \text{fced}_k \mathbb{Z}_p(1)^d = \text{ced}_k \hat{\mathbb{Z}}(1)^d = \text{fced}_k \hat{\mathbb{Z}}(1)^d = d.
\]

**Proof.** Since \( TG = \prod_p T_pG \) and \( \text{fced}_k \leq \text{ced}_k \), it is enough to prove \( \text{fced}_k T_pG \geq d \) and \( \text{ced}_k TG \leq d \).

Let us first prove the lower bound. Let \( k' / k \) a finite splitting field for \( G \), thanks to Lemma 6.8.\( \text{iv} \) we may suppose that \( G = G^d_m \). We have
\[
H^1(L, T_pG^d_m) = H^1(L, \mathbb{Z}_p(1))^d = (\wedge_p L^*)^d.
\]

The element
\[
(t_1, \ldots, t_d) \in (\wedge_p k(t_1, \ldots, t_d)^*)^d
\]
has continuous essential dimension \( d \) thanks to Lemma 5.11 (use the rank \( n \) valuation associated to the chain of discrete valuations given by coordinate hyperplanes), thus we have that \( \text{fced}_k T_pG \geq d \).

We establish now the upper bound. Let \( k' / k \) be a finite splitting field of \( G \), and \( r = [k' : k] \) its degree. Now let \( L/k \) be any field extension, there exists a finite extension \( L'/L \) such that \( [L' : L] \leq r \) and \( k'/k \) is a subextension of \( L/k \). Since \( G^d_m \) has trivial cohomology, every \( G \)-torsor over \( L \) is splitted by \( L' \). Since the period of a torsor divides the index (see [Cla04, Proposition 9]), the period of any \( G \)-torsors divides \( r! \), i.e. \( H^1(L, G) \) is \( r! \)-torsion for every extension \( L/k \).

Now let \( L/k \) be an extension \( T \rightarrow \text{Spec} L \) be a \( TG \)-torsor. For every \( n \), write \( T_n = T \times TG \mathbb{Z}[n] \) for the induced \( G[n] \)-torsor. We have to show that \( \text{ced}_k T_n \leq d \), i.e. that \( \text{ed}_k T_n \leq d \) for every \( n \). If \( m \) divides \( n \), the usual Kummer exact sequence gives us a commutative diagram of long exact sequences in cohomology

\[
\begin{array}{ccc}
G(L) & \rightarrow & H^1(L, G[n]) \\
\downarrow \text{id} & & \downarrow \alpha/m \\
G(L) & \rightarrow & H^1(L, G[m]) \\
\end{array}
\]
In particular, this tells us that $T_n \in H^1(L, G[n])$ maps to a divisible element in $H^1(L, G)$. Since $H^1(L, G)$ is $r!$-torsion, it has no non-trivial divisible elements, and hence it $T_n$ comes from a point $g \in G(L)$. But then $T_n$ is defined on the residue field $k(g)$ of $g$, and since $G$ has dimension $d$ we get the desired upper bound.

For completeness, we summarize what we know about the various types of essential dimension for $\mathbb{Z}_p$. This does not require any additional effort.

**Proposition 6.11.** Let $k$ be a field of characteristic different from $p$. If $k$ contains all $p$-adic roots of unity, then $\text{fced}_k \mathbb{Z}_p = \text{ced}_k \mathbb{Z}_p = 1$. In general, $\text{ced}_k \mathbb{Z}_p \geq 1$. If $k$ is a finitely generated extension of $\mathbb{Q}$, then $\text{fced}_k \mathbb{Z}_p = 0$.

**Proof.** If $k$ contains all $p$-adic roots of unity, $\mathbb{Z}_p = \mathbb{Z}_p(1)$ and thus this is Theorem 6.10. Continuous essential dimension decreases along extension of the base field (this follows from the analogous fact for plain essential dimension), hence we can see that $\text{ced}_k \mathbb{Z}_p \geq 1$ by adding to $k$ the $p$-adic roots. If $k$ is finitely generated over $\mathbb{Q}$, we have $\text{fced}_k \mathbb{Z}_p \leq \text{fced}_k \mathbb{Z}_p = 0$ thanks to Theorem 5.19. 

7. **FCE dimension and anabelian geometry**

We have already defined the fce dimension in the previous section as the merging of finite type and continuous essential dimension. Here, we give some results that show how these two variants work in synergy, in particular in anabelian geometry.

A. Vistoli observed that, if Grothendieck’s section conjecture is true, then the étale fundamental group scheme $\pi'_1(X)$ of an hyperbolic curve over a field finitely generated over $\mathbb{Q}$ should somehow have essential dimension 1. Using the fce dimension we can make his observation formal.

First, let us give a negative result: finite type essential dimension is not refined enough for affine curves.

**Proposition 7.1.** Let $X$ be a smooth, affine curve over any field. If $\text{char } k = 0$, assume $X \neq \mathbb{A}^1$. Then $\text{fctd} \Pi_{X/k} = \infty$.

**Proof.** Suppose first that $\deg \tilde{X} \setminus X \geq 2$, where $\tilde{X}$ is the smooth completion. Up to a finite extension of the base field we may suppose that $\tilde{X} \setminus X$ has two rational points. Choose any prime $l \neq \text{char } k$. Using the explicit description of the abelianized fundamental group of a curve and the holes described in Definition 4.4, we have morphisms $B\mathbb{Z}_l(1) \to \Pi_{X/k} \to B\mathbb{Z}_l(1)$ whose composition is the identity, thus $\text{fctd} \Pi_{X/k} \geq \text{fctd} \mathbb{Z}_l(1) = \infty$ thanks to Theorem 5.12.

If $\deg \tilde{X} \setminus X = 1$, $\tilde{X}$ is not a Brauer-Severi variety because it has a divisor of degree 1. If $X \neq \mathbb{A}^1$, then $g(X) \geq 1$ and there exists $X' \to X$ a non-trivial, connected finite étale cover. If $X = \mathbb{A}^1$, then $\text{char } k = p \neq 0$ by hypothesis and thus we have the Abhyankar cover $G_m \to \mathbb{A}^1, x \mapsto x^p + 1/x$, see [Abh57, Theorem 1].

In any case, we have a non-trivial finite étale cover $X' \to X$ and $\deg \tilde{X'} \setminus X' \geq 2$, thus $\text{fctd} \Pi_{X'/k} = \infty$. Now apply [BRV07, Proposition 2.17] to $\Pi_{X'/k} \to \Pi_{X/k}$ and get $\text{fctd} \Pi_{X/k} \geq \text{fctd} \Pi_{X'/k} = \infty$. Observe that [BRV07, Proposition 2.17] is stated for plain essential dimension, but it is clear from the proof that it works for finite type essential dimension, too.
**Dimensional section conjecture.** Let $k$ be a field finitely generated over $\mathbb{Q}$, and $X$ a smooth, geometrically connected hyperbolic curve. Then $\text{cend}_k \Pi_{X/k} = 1$ and, if $X$ is proper, $\text{cend}_k \Pi_{X/k} = 1$.

Observe that because of Proposition 7.1 it makes sense to use finite type essential dimension only for proper curves.

**Proposition 7.2.** Grothendieck’s section conjecture implies the dimensional section conjecture.

*Proof.* If Grothendieck’s section conjecture is true and $X$ is a smooth, proper, geometrically connected hyperbolic curve over a field $k$ finitely generated over $\mathbb{Q}$, then $\Pi_{X/k}(L) = X(L)$ for every $L$ finitely generated over $\mathbb{Q}$, hence $\text{cend}_k \Pi_{X/k} = \text{cend}_k X = 1$.

Let now $X$ be as above, but we drop the properness assumption. Then the fact that $\text{cend} \Pi_{X/k} = 1$ follows from the description of Grothendieck’s section conjecture for affine curves given in [Bre18, §8]. Let us recall it briefly.

It is possible to construct the so-called infinite root stack $\hat{X}$ of $\bar{X}$ at $\bar{X} \setminus X$: this is a projective limit $\hat{X} = \varprojlim_n X_n$ where $X_n$ are smooth, proper orbicurves with a morphism $X_n \to \hat{X}$ which has ramification index $n$ at the points over $\bar{X} \setminus X$ and is an isomorphism outside of $\bar{X} \setminus X$.

It turns out that $\Pi_{X/k} = \Pi_{\hat{X}/k}$, hence we may replace $X$ with $\hat{X}$ in order to study $\Pi_{X/k}$: this has the advantage that $\hat{X}$ is somewhat "proper", being the projective limit of proper objects. For every rational point $p \in \hat{X} \setminus X(k)$ at infinite, we have a morphism $B\mathbb{Z}(1) \to \hat{X}$ and thus $B\mathbb{Z}(1) \to \Pi_{X/k}$, hence a functor $H^1(\_ , \hat{X}(1)) \to \Pi_{X/k}(\_ )$: its image is the so called "packet of tangential sections" at $p$.

Using $\hat{X}$, Grothendieck’s section conjecture for affine curves says that $\Pi_{X/k}(L) = \hat{X}(L)$ for every $L$ finitely generated over $\mathbb{Q}$. Since $\hat{X}(\_ )$ is the disjoint union of $X(\_ )$ and the “packets” $H^1(\_ , \hat{X}(1))$, thanks to Theorem 6.10 we have that $\text{cend}_k \Pi_{X/k} = 1$. □

We prove that the dimensional section conjecture holds for torsors under abelian varieties.

In order to do this, we have first to establish a base-point free version of Faltings’ theorem for torsors under abelian varieties (we need this more general version already to prove the conjecture for abelian varieties, not only for torsors). Tate modules are replaced by étale fundamental gerbes. The formulation is more involved than Faltings’ theorem because hom-sets are not groups if we don’t fix base points, and we need to avoid base points if we want to work with torsors rather than abelian varieties.

**Lemma 7.3 (Faltings’ theorem).** Let $k$ be a field finitely generated over $\mathbb{Q}$, and $E, F \to \text{Spec } k$ torsors for abelian varieties $A, B$ over $k$ and $p$. Let $\Pi_{E/k}, \Pi_{F/k}$ be the étale fundamental gerbes of $E, F$, and $\rho : \Pi_{E/k} \to \Pi_{F/k}$ a morphism.
For every finite gerbe $\Phi$ and every morphism $\varphi : \Pi_{F/k} \to \Phi$ there exists a $B$-torsor $F'$ and a morphism $f : E \to F'$ such that the following diagram 2-commutes

\[
\begin{array}{ccc}
E & \xrightarrow{\pi_E} & \Pi_{E/k} \\
\downarrow{f} & & \downarrow{\pi(f)} \\
F' & \xrightarrow{\pi_{F'}} & \Pi_{F'/k} \\
\end{array}
\xrightarrow{\varphi} \begin{array}{ccc}
\Pi_{F'/k} & \xrightarrow{\varphi} & \Phi \\
\end{array}
\]

The same statement holds if we replace étale fundamental gerbes $\Pi_{\_k}$ with their $p$-parts $\Pi_{\_//k,p}$ at any prime $p$.

**Proof.** Thanks to [BV15, Lemma 5.12] we may suppose that $\Pi_{F/k} \to \Phi$ is Nori-reduced (this essentially amounts to the fact that the induced homomorphisms between automorphism groups are surjective), and thus we may suppose that $\Phi$ is abelian since $\Pi_{F/k}$ is abelian. Let us first prove the statement at a prime $p$.

Thanks to Lemma 4.10, the bands of $\Pi_{E/k,p}$ and $\Pi_{F/k,p}$ are respectively $T_{pA}$ and $T_{pB}$, call $G$ the band of $\Phi$. We have induced morphisms of bands $T_{pA} \xrightarrow{\rho} T_{pB} \xrightarrow{\varphi} G$. Since $G$ is finite, by Faltings’ theorem we have a morphism of abelian varieties $f : A \to B$ such that $\varphi \circ T_p f = \varphi \circ \rho$.

Write

$$F' = E \times^A B$$

the induced torsor along $f : A \to B$, we have a natural equivariant morphism $E \to F'$. By construction, we have a natural isomorphism

$$\Pi_{F'/k,p} \simeq \Pi_{E/k,p} \times^{T_p f} T_p B$$

between the fundamental gerbe of $F'$ and the induced gerbe along $T_p f$. Since $\varphi \circ T_p F = \varphi \circ \rho$ the diagram

\[
\begin{array}{ccc}
T_p A & \xrightarrow{\rho} & T_p B \\
\downarrow{T_p f} & & \downarrow{\varphi} \\
T_p B & \xrightarrow{\varphi} & G \\
\end{array}
\]

commutes. If we push $\Pi_{E/k,p}$ along this diagram we get the commutative diagram

\[
\begin{array}{ccc}
\Pi_{E/k,p} & \xrightarrow{\rho} & \Pi_{F/k,p} \\
\downarrow{\pi(f)} & & \downarrow{\varphi} \\
\Pi_{F'/k,p} & \xrightarrow{\varphi} & \Phi \\
\end{array}
\]

as desired.

Now write $|G| = p_1^{a_1} \cdots p_n^{a_n}$ with primes $p_i \neq p_j$ if $i \neq j$, and using Bezout’s theorem we write

$$1 = b_1 \cdot \prod_{i \neq 1} p_i^{a_j} + \cdots + b_n \cdot \prod_{i \neq n} p_i^{a_1}.$$
Suppose that for every $i = 1, \ldots, n$ we have constructed an homomorphism $f_i : A \to B$ as above. Write

$$f'_j = \left( b_j \cdot \prod_{i \neq j} p_i^{a_i} \right) \cdot f_j$$

and consider the homomorphism

$$f' = f'_1 + \cdots + f'_n.$$

Let $F'$ be the $B$-torsor induced by $E$ along $f'$. We have to check the commutativity of

$$\begin{array}{ccc}
\Pi_{E/k} & \xrightarrow{\rho} & \Pi_{F/k} \\
\downarrow_{\pi(f')} & & \downarrow_{\varphi} \\
\Pi_{F'/k} & \xrightarrow{\varphi} & \Phi
\end{array}$$

which amounts to the commutativity of

$$\begin{array}{ccc}
\Pi_{E/k, p_j} & \xrightarrow{\rho_{p_j}} & \Pi_{F/k, p_j} \\
\downarrow_{\pi(f')} & & \downarrow_{\varphi_{p_j}} \\
\Pi_{F'/k, p_j} & \xrightarrow{\varphi_{p_j}} & \Phi_{p_j}
\end{array}$$

for every $j = 1, \ldots, n$. As before, this is equivalent to the commutativity of the diagram

$$\begin{array}{ccc}
T_{p_j} A & \xrightarrow{\rho_{p_j}} & T_{p_j} B \\
\downarrow_{T_{p_j} f'} & & \downarrow_{\varphi_{p_j}} \\
T_{p_j} B & \xrightarrow{\varphi_{p_j}} & G_{p_j}
\end{array}$$

Now, $f' = f'_1 + \cdots + f'_n$ and $\varphi_{p_j} \circ T_{p_j} f'_i : T_p A \to G_{p_j}$ is 0 for $i \neq j$ because $f'_i$ is multiple of $p_j^{a_j}$ and $|G_{p_j}| = p_j^{a_j}$, hence

$$\varphi_{p_j} \circ T_{p_j} f' = \varphi_{p_j} \circ T_{p_j} f'_j.$$ 

Moreover, we have

$$\varphi_{p_j} \circ T_{p_j} f'_j = \left( b_j \cdot \prod_{i \neq j} p_i^{a_i} \right) \cdot \varphi_{p_j} \circ T_{p_j} f'_j = \varphi_{p_j} \circ T_{p_j} f'_j$$

because

$$b_j \cdot \left( \prod_{i \neq j} p_i^{a_i} \right) \equiv 1 \pmod{p_j^{a_j}}$$

hence we get

$$\varphi_{p_j} \circ T_{p_j} f' = \varphi_{p_j} \circ T_{p_j} f'_j = \varphi_{p_j} \circ T_{p_j} f'_j$$

as desired. \qed
Lemma 7.4. Let $V$ be a smooth variety over a field $k$ of characteristic 0 with Albanese torsor $V \to A^1$, and let $\Pi^{ab}_{V/k}$ be the abelianized fundamental gerbe of $V$. The natural morphism

$$\Pi^{ab}_{V/k} \to \Pi_{A^1/k}$$

is a quotient of gerbes, and the kernel is torsion.

Proof. This is classical for $k$ algebraically closed and étale fundamental groups, see [Sza09, Corollary 5.8.10]. The general case follows from the fact that the étale fundamental gerbe behaves well under base change, see [BV15, Proposition 6.1]. □

Corollary 7.5. If $V$ is a smooth projective variety over a field $k$ of characteristic 0 with Albanese torsor $V \to A^1$ and $V \to \Phi$ is a morphism to an abelian, torsion-free gerbe $\Phi$ then we have a factorization $V \to A^1 \to \Phi$. □

Theorem 7.6. Let $A^1$ be an $A$-torsor for an abelian variety $A$ over a field $k$ finitely generated over $\mathbb{Q}$, and $p$ a prime number. Then $\text{fcd}_k \Pi_{A^1/k} = \text{fcd}_k \Pi_{A^1/k,p} = \dim A^1$. In particular, for $A^1 = A$ we get $\text{fcd}_k T A = \text{fcd}_k T_p A = \dim A$.

Proof. We prove this for $\Pi_{A^1/k}$, the argument for $\Pi_{A^1/k,p}$ is analogous.

Let $k'/k$ be a field finitely generated over $k$, and $\text{Spec } k' \to \Pi_{A^1/k}$ a section. Up to replacing $k$ with $\overline{k}$, we may suppose that $k$ is algebraically closed in $k'$. By resolution of singularities there exists a smooth, geometrically connected projective variety $V$ with $k(V) = k'$. Thanks to Corollary 4.13 and Corollary 4.7, $\text{Spec } k' \to \Pi_{A^1/k}$ extends uniquely to a morphism $V \to \Pi_{A^1/k}$.

Let $V \to B^1$ be the Albanese torsor of $V$, it is a torsor for the Albanese variety $B$. Since $\Pi_{A^1/k}$ is abelian and torsion free, by Corollary 7.5 we have a factorization

$$V \to B^1 \to \Pi_{B^1/k} \to \Pi_{A^1/k}.$$ 

Let us suppose we have a morphism $\phi : \Pi_{A^1/k} \to \Phi$ with $\Phi$ a finite gerbe, we have to show that the composition $\text{Spec } k' \to V \to \Pi_{A^1/k} \to \Phi$ factorizes through a field of transcendence degree less than or equal to $\dim A$.

By Lemma 7.3 there exists a morphism $f : B \to A$ such that, if $A' = B^1 \times_B A$ is the induced $A$-torsor, the following diagram commutes:

$$\begin{array}{ccc}
V & \longrightarrow & B^1 \\
\downarrow f & & \downarrow \pi(f) & \downarrow \phi \\
A' & \longrightarrow & \Pi_{A'/k} & \longrightarrow & \Phi
\end{array}$$

In particular, this tells us that the composed morphism $\text{Spec } k' \to \Phi$ factorizes through the residue field of a point of $A'$, which has transcendence degree less than or equal to $\dim A$, as desired. □

Abelian varieties show that the fce dimension is the right definition in order to study questions arising from anabelian geometry: if $A$ is an abelian variety over any field of characteristic 0, we prove that $\text{fcd}_k T A = \infty$ and $\text{fcd}_k T_p A \geq 2 \dim A$.

Proposition 7.7. If $A$ is an abelian variety over any field $k$ of characteristic different from $p$, then $\text{fcd}_k T A \geq \text{fcd}_k T_p A \geq 2 \dim A$. 

Proof. We may base change to \(k_0\), then \(T_pA \simeq \mathbb{Z}_p^{2 \dim A} \simeq \mathbb{Z}_p(1)^{2 \dim A}\), thus we may apply Theorem 6.10.

**Lemma 7.8.** Let \(\phi_m \in H^1(A, A[m])\) be the \(A[m]\)-torsor \(A \to A\) given by multiplication by \(m\), and \(n\) an integer. Then
\[
n^*\phi_m = n\phi_m,
\]
i.e. the pullback of \(\phi_m\) along \(n : A \to A\) is \(n\phi_m\).

**Proof.** We have a \(n : A[m] \to A[m]\)-equivariant diagram of torsors
\[
\begin{array}{ccc}
A & \longrightarrow & \frac{n^*\phi_m}{A} \\
\downarrow & & \downarrow \\
A & \longrightarrow & A
\end{array}
\]

**Corollary 7.9.** Let \(n \in \mathbb{Z}_p\) be a \(p\)-adic integer and \(A\) an abelian variety over a field \(k\) and \(\psi \in H^1(A, T_pA)\) the \(T_pA\)-torsor over \(A\) given by the tower
\[
\ldots A \xrightarrow{p} A \xrightarrow{p} A \xrightarrow{p} A.
\]
The \(T_pA\)-torsor over \(A\) induced on \(A\) by \(\text{id} \otimes n \in \text{Hom}(A, A) \otimes \mathbb{Z}_p\) (the “pullback along multiplication by \(n\))") is \(n\psi \in H^1(A, T_pA)\).

**Theorem 7.10.** Over any field \(k\) of characteristic 0 and for any prime number \(p\), if \(A\) is a positive dimensional abelian variety then \(\text{fed}_k TA = \text{fed}_k T_pA = \infty\).

**Proof.** Since the global Tate module is the product of all the local Tate modules it is enough to prove the second statement. Let \(g\) be the dimension of \(A\), and fix any integer \(d\), we are going to construct a \(T_pA\) torsor of essential dimension \(dg\).

Let \(\phi\) be structure \(T_pA\)-torsor over \(A\) as in Corollary 7.9. Choose \(d\) \(p\)-adic integers \(n_1, \ldots, n_d \in \mathbb{Z}_p^*\) which are linearly independent over \(\mathbb{Q}\). This gives us an element
\[
\text{id} \otimes n_1 + \ldots + \text{id} \otimes n_d \\
\in \text{Hom}(A, A) \otimes \mathbb{Z}_p + \ldots + \text{Hom}(A, A) \otimes \mathbb{Z}_p = \text{Hom}(A^d, A) \otimes \mathbb{Z}_p
\]
which in turn induces a \(T_pA\)-torsor over \(A^d\) (the “pullback of \(\phi\) along \(n_1 \otimes \cdots \otimes n_d\)”), with an abuse of notation we denote it by \((n_1 \otimes \cdots \otimes n_d)^*\). If \(S\) is a scheme and \((f_1, \ldots, f_d) : S \to A^d\) is a morphism, then
\[
(f_1, \ldots, f_d)^* (n_1 \otimes \cdots \otimes n_d)^* \phi = f_1^* n_1^* \phi + \cdots + f_d^* n_d^* \phi = n_1 f_1^* \phi + \cdots + n_d f_d^* \phi
\]
by Corollary 7.9.

Now, consider the restriction of \((n_1 \otimes \cdots \otimes n_d)^*\phi\) to the generic point of \(A^d\), and suppose by absurd that it is defined on a subextension \(k(A^d)/k'/k\) of strictly smaller transcendence degree, we have thus a \(T_pA\)-torsor on \(\text{Spec} k'\). By resolution of singularities, let \(V\) be a smooth projective variety with \(k(V) = k'\).

Thanks to Lemma 4.18, the generic \(T_pA\)-torsor extends to \(V\) and hence we have a factorization
\[
V \to B^1 \to BT_pA
\]
where $V \to B^1$ is the Albanese torsor thanks to Corollary 7.5. Since morphisms between torsors for abelian varieties extend, we have a factorization

$$A^d \to B^1 \to B \psi .$$

In particular, since $\dim V < \dim A^d$, there is a positive-dimensional sub-abelian variety $K \subseteq A^d$ where the torsor $(n_1 \otimes \cdots \otimes n_d)^* \psi$ is trivial. We want to show that this gives a contradiction.

Let $f_1, \ldots, f_d$ be the coordinates $f_j : K \to A$: at least one of these is not trivial. By Corollary 7.9, the restriction of $(n_1 \otimes \cdots \otimes n_d)^* \psi$ to $K$ is

$$n_1 f_1^* \psi + \cdots + n_d f_d^* \psi .$$

For every $j = 1, \ldots, d$, the pullback $f_j^* \psi$ is an element of the $\mathbb{Q}$-vector space $\text{Hom}(K, A) \otimes \mathbb{Z} Q$. We have a $\mathbb{Q} \subseteq \mathbb{Q}_p$-sub-vector space

$$\text{Hom}(K, A) \otimes \mathbb{Z} Q \subseteq H^1(K, T_p A) \otimes \mathbb{Z}_p Q_p ,$$

and at least one of the elements $f_j^* \psi$ is not trivial. Since $n_1, \ldots, n_s \in \mathbb{Q}_p$ are linearly independent over $\mathbb{Q}$ the linear combination

$$n_1 f_1^* \psi + \cdots + n_d f_d^* \psi \in H^1(K, T_p A) \otimes \mathbb{Z}_p Q_p$$

cannot be 0, and this gives a contradiction. $\square$

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