New Branching Rules: Improvements on Independent Set and Vertex Cover in Sparse Graphs

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Abstract. We present an \(O^*(1.0919^n)\)-time algorithm for finding a maximum independent set in an \(n\)-vertex graph with degree bounded by 3, which improves the previously known algorithm of running time \(O^*(1.0977^n)\) by Bourgeois, Escoffier and Paschos [IWPEC 2008]. We also present an \(O^*(1.1923^k)\)-time algorithm to decide if a graph with degree bounded by 3 has a vertex cover of size \(k\), which improves the previously known algorithm of running time \(O^*(1.1939^k)\) by Chen, Kanj and Xia [ISAAC 2003].

Two new branching techniques, branching on a bottle and branching on a 4-cycle, are introduced, which help us to design simple and fast algorithms for the maximum independent set and minimum vertex cover problems and avoid tedious branching rules.

Key words. Graph Algorithm, Independent Set, Vertex Cover, Sparse Graph

1 Introduction

The maximum independent set problem (MIS), to find a maximum set of vertices in a graph such that there is no edge between any two vertices in the set, is one of the basic NP-hard optimization problems and has been well studied in the literature, in particular in the line of research on worst-case analysis of algorithms for NP-hard optimization problems. In 1977, Tarjan and Trojanowski [1] published the first algorithm for this problem, which runs in \(O^*(2^{n/3})\) time and polynomial space. Later, the running time was improved to \(O^*(2^{0.304n})\) by Jian [2]. Robson [3] obtained an \(O^*(2^{0.296n})\)-time polynomial-space algorithm and an \(O^*(2^{0.276n})\)-time exponential-space algorithm. In a technical report [4], Robson also claimed better running times. Recently, Fomin et al. [5] got a simple \(O^*(2^{0.288n})\)-time polynomial-space algorithm by using the “Measure and Conquer” method. There is also a considerable amount of contributions to

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the maximum independent set problem in sparse graphs, especially in degree-3 graphs \([6],[7],[8],[9]\). We summarize the results on low-degree graphs as well as general graphs in Table 1.

| Authors               | Running times           | References | Notes                                      |
|-----------------------|-------------------------|------------|--------------------------------------------|
| Tarjan & Trojanowski  | \(O^*(1.2600^n)\) for MIS | 1977 [1]   |                                            |
| Jian                  | \(O^*(1.2346^n)\) for MIS | 1986 [2]   |                                            |
| Robson               | \(O^*(1.2109^n)\) for MIS | 1986 [3]   | Exponential space                          |
| Beigel               | \(O^*(1.0823^n)\) for MIS | 1999 [4]   |                                            |
|                      | \(O^*(1.1259^n)\) for 3-MIS |            |                                            |
| Robson               | \(O^*(1.1893^n)\) for MIS | 2001 [4]   |                                            |
| Chen et al.          | \(O^*(1.1254^n)\) for 3-MIS | 2003 [4]   |                                            |
| Xiao et al.          | \(O^*(1.1034^n)\) for 3-MIS | 2005 [8]   | Published in Chinese                       |
| Fomin et al.         | \(O^*(1.2210^n)\) for MIS | 2006 [5]   |                                            |
| Fomin & Høie         | \(O^*(1.1225^n)\) for 3-MIS | 2006 [10]  |                                            |
| Fürer                | \(O^*(1.1120^n)\) for 3-MIS | 2006 [11]  |                                            |
| Razgon               | \(O^*(1.1034^n)\) for 3-MIS | 2006 [12]  |                                            |
| Bourgeois et al.     | \(O^*(1.0977^n)\) for 3-MIS | 2008 [9]   |                                            |
| Xiao                 | \(O^*(1.0919^n)\) for 3-MIS | This paper |                                            |

Table 1. Exact algorithms for the maximum independent set problem

In the literature, there are several methods of designing algorithms for finding maximum independent sets in graphs. One method is to find a minimum vertex cover (a set of vertices such that each edge in the graph has at least one endpoint in the set), and then to get a maximum independent set by taking all the remaining vertices, such as the algorithms presented in \([7],[13]\). In this kind of algorithms, the dominating part of the running time is the running time for finding a minimum vertex cover. Another method is based on the search tree method. We will use a branch-and-reduce paradigm. We choose a parameter, such as the number of vertices or edges or others, as a measure of the size of the problem. When the parameter is zero or a negative number, the problem can be solved in polynomial time. We branch on the current graph \(G\) into serval graphs \(G_1, G_2, \ldots, G_l\) such that the parameter \(r_i\) of graph \(G_i\) is less than the parameter \(r\) of graph \(G\) \((i = 1, 2, \ldots, l)\), and a maximum independent set in \(G\) can be found in polynomial time if a maximum independent set in each of the \(l\) graphs \(G_1, G_2, \ldots, G_l\) is known. By this method, we can build up a search tree, and the exponential part of the running time of the algorithm is corresponding to the size of the search tree. The running time analysis leads to a linear recurrence for each node in the search tree that can be solved by using standard techniques. Let \(C(r)\) denote the worst-case size of the search tree when the parameter of graph \(G\) is \(r\), then we get recurrence relation \(C(r) \leq \sum_{i=1}^l C(r_i)\). Solving the recurrence, we get \(C(r) = [\alpha(r, r_1, r_2, \ldots, r_l)]^r\), where \(\alpha(r, r_1, r_2, \ldots, r_l)\) is the largest root of the function \(f(x) = 1 - \sum_{i=1}^l x^{r_i-r}\). As for the measure (the parameter...
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A natural one is the number of vertices or edges in the graph. Most previous algorithms for the maximum independent set problem are analyzed by using the number of vertices as a measure \[1,2,3,5\]. The number of edges is considered in Beigel’s algorithm \[6\]. There are also some other measures. Xiao et al. \[8\] used the number of degree-3 vertices as a measure to analyze algorithms and got an \(O^*(1.1034^n)\)-time algorithm for MIS in degree-3 graphs. Unfortunately, that paper was published in Chinese. Recently, Razgon \[12\] also got an \(O^*(1.1034^n)\)-time algorithm for MIS in degree-3 graphs by measuring the number of degree-3 vertices. But the two algorithms are totally different. F"urier \[11\] designed an algorithm for MIS in degree-3 graphs by tackling \(m-n\), where \(m\) is the number of edges and \(n\) the number of vertices. Based upon a refined branching with respect to F"urier’s algorithm, Bourgeois et al. \[9\] got the current best algorithm for MIS in degree-3 graphs with running time \(O^*(1.0977^n)\). In this paper, we still use the number of degree-3 vertices as a measure to analyze our algorithm. Based on two new branching rules, branching on a bottle and branching on a 4-cycle, we design an even faster algorithm for MIS in degree-3 graphs, which runs in \(O^*(1.0919^n)\) time. Our algorithm is simple and does not contain many branching rules. Furthermore, it can be used to solve the \(k\)-vertex cover problem (to decide if the graph has a vertex cover of size \(k\)) in degree-3 graphs in \(O^*(1.1923^k)\) time, which improves the previously known result of \(O^*(1.1939^k)\) by Chen et al. \[7\].

2 Preliminaries

We shall try to be consistent in using the following notation. The number of vertices in a graph will be denoted by \(n\) and the number of degree-3 vertices (vertices of degree \(\geq 4\) will also be counted with a weight) by \(r\). For a vertex \(v\) in a graph, 
\(d(v)\) is the degree of \(v\), \(N(v)\) the set of all neighbors of \(v\), \(N[v] = N(v) \cup \{v\}\) the set of vertices with distance at most 1 from \(v\), and \(N_2(v)\) the set of vertices with distance exactly 2 from \(v\). We say edge \(e\) is incident on a vertex set \(V'\), if at least one endpoint of \(e\) is in \(V'\). In our algorithm, when we remove a set of vertices, we also remove all the edges that are incident on it. Throughout the paper we use a modified \(O\) notation that suppresses all polynomially bounded factors. For two functions \(f\) and \(g\), we write \(f(n) = O^*(g(n))\) if \(f(n) = O(g(n)poly(n))\), where \(poly(n)\) is a polynomial.

Our algorithms are based on the branch-and-reduce paradigm. We will first apply some reduction rules to reduce the size of instances of the problem. Then we apply some branching rules to branch on the graph by including some vertices in the independent set or excluding some vertices from the independent set. In each branch, we will get a maximum independent set problem in a graph with a smaller measure. Next, we introduce the reduction rules and branching rules that will be used in our algorithms.

2.1 Reduction Rules

There are several standard preprocesses to reduce the size of instances of the problem. Folding a degree-1 or degree-2 vertex and removing a dominated vertex
are frequently used rules. Besides these reduction rules, we still need to reduce some other local structures called 2-3 structure, 3-3 structure and 3-4 structure.

**Folding a degree-1 vertex**

**Folding a degree-1 vertex** \( v \) means removing \( v \) and \( u \) from the graph, where \( u \) is the unique neighbor of \( v \).

**Folding a degree-2 vertex**

**Folding a degree-2 vertex** \( v \) (with two neighbors \( u \) and \( w \)) means

(a) removing \( v \), \( u \) and \( w \) from the graph, when \( u \) and \( w \) are adjacent.

(b) removing \( v \), \( u \) and \( w \) from the graph and introducing a new vertex \( s \) that is adjacent to all neighbors of \( u \) and \( w \) in \( G \) (except the removed vertex \( v \)), when \( u \) and \( w \) are nonadjacent.

Please refer to Figure 1 for an illustration of the operation in case (b) of folding a degree-2 vertex. Let \( \alpha(G) \) denote the size of a maximum independent set of graph \( G \) and \( G^*(v) \) the graph after folding a degree-1 or degree-2 vertex \( v \). Then we have the following lemma.

**Lemma 1.** For any degree-1 or degree-2 vertex \( v \) in graph \( G \),

\[
\alpha(G) = 1 + \alpha(G^*(v)).
\]

![Fig. 1. Illustrations of folding operations](image)

The correctness of folding a degree-1 or degree-2 vertex has been discussed in many previous papers. In fact, general folding rules are known in the literature,
which can deal with a vertex of degree $\geq 3$ or a set of independent vertices $\text{[13],[5]}$. In this paper, we still need to fold the following three local structures called 2-3 structure, 3-3 structure and 3-4 structure.

Let $u$ and $v$ be two independent degree-3 vertices, if they have three common neighbors $a, b$ and $c$, then we say that the five vertices compose a 2-3 structure (see Figure 1), and denote it by $\{u, v\}-\{a, b, c\}$. Let $v$ be a degree-3 vertex, and $u$ and $w$ two adjacent vertices of degree $\geq 3$. If $N(u) \cup N(w) - \{u, w\} = N(v)$, then we say that the six vertices $\{u, v, w\} \cup N(v)$ compose a 3-3 structure (see Figure 1), and denote it by $\{v, u, w\}-\{a, b, c\}$. Let $u, v$ and $w$ be three independent vertices of degree $\geq 3$, if they have exact four neighbors $a, b, c$ and $d$, then we say that the seven vertices compose a 3-4 structure, and denote it by $\{u, v, w\}$-$\{a, b, c, d\}$.

**Folding a 2-3 structure, 3-3 structure or 3-4 structure**

Let $A-B$ be a 2-3 structure or 3-3 structure or 3-4 structure. Folding $A-B$ means

(a) removing $A \cup B$ from the graph, when $B$ is not an independent set.

(b) removing $A \cup B$ from the graph and introducing a new vertex $s$ that is adjacent to all neighbors of vertices in $B$ (except the removed vertices), when $B$ is an independent set.

**Lemma 2.** If graph $G$ has a 2-3 structure or 3-3 structure, then

$$\alpha(G) = 2 + \alpha(G'_{2}),$$

where $G'_{2}$ is the graph after folding a 2-3 structure or 3-3 structure in $G$.

If graph $G$ has a 3-4 structure, then

$$\alpha(G) = 3 + \alpha(G'_{3}),$$

where $G'_{3}$ is the graph after folding a 3-4 structure in $G$.

A degree-2 vertex can be regarded as a 1-2 structure according to our definitions. In fact, a degree-2 vertex, 2-3 structure and 3-4 structure are special cases described in Lemma 2.4 in [13]. The 3-3 structure is for the first time being introduced. The correctness of folding an $A-B$ structure (a 1-2 structure, 2-3 structure, 3-3 structure or 3-4 structure) follows from this observation: When $B$ is not an independent set, there is a maximum independent set that contains $A$ (or two independent vertices in $A$, when $A-B$ is a 3-3 structure). When $B$ is an independent set, there is a maximum independent set that contains either $B$ or $A$ (or two independent vertices in $A$, when $A-B$ is a 3-3 structure). We ignore the detailed proof here.

**Dominance**

If there are two vertices $v$ and $u$ such that $N[u] \subseteq N[v]$, we say $u$ dominates $v$.

**Lemma 3.** If vertex $v$ is dominated by any other vertex in graph $G$, then

$$\alpha(G) = \alpha(G - \{v\}).$$

**Definition 1.** A graph is called a reduced graph, if it has no degree-1 vertex, degree-2 vertex, dominated vertex, 2-3 structure, 3-3 structure or 3-4 structure.
2.2 Branching Rules

Next we introduce two branching techniques, branching on a bottle and branching on a 4-cycle, which are simple and obvious, but can avoid tedious branching rules in the description of the algorithms.

Let $a$ be a degree-3 vertex, and $b, c, d$ the three neighbors of $a$. If two neighbors of $a$, say $c$ and $d$, are adjacent, then we say that the four vertices compose a bottle and denote it by $b-a\{-c,d\}$.

**Lemma 4.** Let $b-a\{-c,d\}$ be a bottle in graph $G$, then there is a maximum independent set $S$ in $G$ such that either $a \in S$ or $b \in S$.

**Proof.** If $b$ is not in a maximum independent set, we can directly remove $b$ from the graph. In the remaining graph $a$ becomes a degree-2 vertex and the two neighbors of it are adjacent. In this case, there is a maximum independent set that contains $a$.

Based on Lemma 4, we get the following branching rule.

**Branching on a bottle**

Branching on a bottle $b-a\{-c,d\}$ means branching by either including $a$ in the independent set or including $b$ in the independent set.

**Note.** In fact, we can fold a bottle by using the general folding rule mentioned in [5] (also in [6]), but this folding rule is helpless for our analysis, especially when the three neighbors of the degree-3 vertex are high-degree vertices.

Let $a, b, c$ and $d$ be four vertices in graph $G$, if $G$ has four edges $ab$, $bc$, $cd$ and $da$, then we say that $abcd$ is a 4-cycle in $G$.

**Lemma 5.** Let $abcd$ be a 4-cycle in graph $G$, then for any independent set $S$ in $G$, either $a, c \not\in S$ or $b, d \not\in S$.

**Proof.** Since any independent set contains at most 2 vertices in a 4-cycle and the two vertices can not be adjacent, we know the lemma holds.

Based on Lemma 5, we get the following branching rule.

**Branching on a 4-cycle**

Branching on a 4-cycle $abcd$ means branching by either excluding $a$ and $c$ from the independent set or excluding $b$ and $d$ from the independent set.

3 A Simple Algorithm

Our algorithm for the maximum independent set problem is described in Figure 2. It works as follows. If the graph has a component of at most 15 vertices, we find a maximum independent set in this component directly (Step 1). If the graph has a degree-1 or degree-2 vertex, we fold it in Step 2. If the graph has a dominated vertex, we remove it in Step 3. If the graph has a 2-3 structure or 3-3 structure or 3-4 structure, we fold it in Step 4 and Step 5. When the graph can not be reduced, we apply our branching rules. If there is a bottle, we branch
on a bottle (Step 6). Else if there is a 4-cycle, we branch on a 4-cycle (Step 7). Else in Step 8, we greedily select a vertex of maximum degree and branch on it by including it in the independent set or excluding it from the independent set.

**Fig. 2. The Algorithm** $MIS(G)$

| Input: A graph $G$. |
| Output: The size of a maximum independent set in $G$. |

1. **If** $\{G$ has a component $P$ of at most 15 vertices$, \textbf{return} t + MIS(G - P)$, where $t$ is the size of a minimum independent set in $P$.
2. **Else if** $\{\exists v \in V: d(v) = 1 \text{ or } 2\}$, return $1 + MIS(G'\cdot(v))$.
3. **Else if** $\{\exists v, u \in V: N[u] \subseteq N[v]\}$, return $MIS(G - \{v\})$.
4. **Else if** $\{$there is a 2-3 structure or 3-3 structure$\}$, return $2 + MIS(G_3^2)$.  
5. **Else if** $\{$there is a 3-4 structure$\}$, return $3 + MIS(G_3^d)$.  
6. **Else if** $\{$there is a bottle $b$-$a$-$\{c, d\}\}$, return $\max\{1 + MIS(G - N[a]), 1 + MIS(G - N[b])\}$.
7. **Else if** $\{$there is a 4-cycle $abcd\}$, return $\max\{MIS(G - \{a, c\}), MIS(G - \{b, d\})\}$.  
8. **Else**, pick up a vertex $v$ of maximum degree, and return $\max\{MIS(G - \{v\}), 1 + MIS(G - N[v])\}$. 

**Note:** With a few modifications, the algorithm can provide a maximum independent set.

4 The Analysis

To analyze the time complexity of our algorithm, we will consider recurrence relations related to parameter $r$, the number of degree-3 vertices (vertices of degree $\geq 4$ will also be counted with a weight) in the corresponding graph. When $r = 0$, the graph has only degree-0, degree-1 and degree-2 vertices and the maximum independent set problem can be solved in linear time. We use $C(r)$ to denote the worst-case size of the search tree in our algorithm when the parameter of the graph is $r$. In our algorithm, it is possible to create a vertex of degree $\geq 4$ when folding. We will regard a degree-$d$ ($d \geq 3$) vertex as a combination of $d - 2$ degree-3 vertices and count $d - 2$ in parameter $r$. Then when a degree-$d$ vertex is removed, parameter $r$ will be reduced by $d - 2$. When an edge incident on a degree-$d$ vertex is removed, parameter $r$ will be reduced by 1. In the remaining of the paper, when we say a graph has $x$ degree-3 vertices, it does not mean that the graph really has exactly $x$ vertices of degree 3. In fact,
all the vertices of degree $\geq 3$ are counted. Next, we analyze how much $r$ can be reduced in each step of our algorithm.

**Lemma 6.** After folding a degree-1 or degree-2 vertex, parameter $r$ will not increase.

**Lemma 7.** Let $G$ be a graph having no degree-1 or degree-2 vertex, then after folding a 2-3 structure or 3-3 structure or 3-4 structure, or removing a dominated vertex in $G$, parameter $r$ will be reduced by at least 4.

**Proof.** In each case, a degree-3 vertex is removed (or an even better case occurs), and then we can further reduce $r$ by 3 from 3 neighbors of the vertex. Totally $r$ will be reduced by at least 4.

**Lemma 8.** Let $G$ be a connected graph. If $G$ has at least $x$ degree-1 vertices and $x$ vertices of degree $\geq 3$ (a degree-$d$ $(d \geq 3)$ vertex will be regarded as $d - 2$ degree-3 vertices), then after iteratively folding degree-1 vertices until the graph has no degree-1 vertex, parameter $r$ will be reduced by at least $x$.

**Proof.** Let $V' \neq \emptyset$ be the set of vertices of degree $\geq 2$ in the remaining graph after iteratively folding degree-1 vertices (The lemma obviously holds, when $V' = \emptyset$). Assume there are $y$ edges between $V'' = V - V'$ and $V'$. After removing $V''$, we can reduce $y$ degree-3 vertices from $V'$. We will prove that there are at least $x - y$ degree-3 vertices in $V''$. To prove that, we first construct a new graph $G'$ from $G$ by contracting $V'$ into a single vertex $v$ and remove all self-loops incident on it (keeping parallel edges). Then we only need to prove that except vertex $v$, $G'$ has at least $x - y$ degree-3 vertices.

Since all the $x$ degree-1 vertices of $G$ are in $V''$, $G'$ has at least $x'$ degree-1 vertices, where $x' = x + 1$ when $v$ is a degree-1 vertex and $x' = x$ when $v$ is not a degree-1 vertex. Note that a tree with $x'$ degree-1 vertices has at least $x' - 2$ degree-3 vertices. We know that $G'$ has least $x' - 2$ degree-3 vertices ($G'$ is a connected graph). We consider the following three cases. Case 1: $y = 1$. For this case, $v$ is a degree-1 vertex and $x' = x + 1$. Then $G'$ has at least $x' - 2 = x - 1$ degree-3 vertices. Case 2: $y = 2$. For this case, $v$ is a degree-2 vertex and $x' = x$, and $G'$ still has at least $x - 2$ degree-3 vertices. Case 3: $y \geq 3$. For this case, $v$ is a degree-$y$ vertex and $x' = x$. Excepting $y - 2$ degree-3 vertices counted from $v$, there are still $x - 2 - (y - 2) = x - y$ degree-3 vertices.

Therefore, after removing $V''$, $r$ will be reduced by at least $x$.

**Corollary 1.** Let $G$ be a graph having not any component of a path. If $G$ has any degree-1 vertex, then we can reduce $r$ by at least 1 by iteratively folding degree-1 vertices. If $G$ has exactly 2 degree-1 vertices, then we can reduce $r$ by at least 2 by iteratively folding degree-1 vertices.

**Lemma 9.** Let $G$ be a reduced graph and $v$ a degree-3 vertex in $G$. Then not any degree-0 vertex or component of a 1-path or component of a 2-path is created after removing $N[v]$.
Proof. If a degree-0 vertex $u$ is created, then $G$ has a 2-3 structure $\{v, u\} \cup N(v)$. If a 1-path $ab$ is created, then there is a 3-3 structure $\{v, a, b\} \cup N(v)$. If a 2-path $abc$ is created, then there is a 3-4 structure $\{a, c, v\} \cup N(v) \cup \{b\}$.

**Lemma 10.** Let $G$ be a connected reduced graph of more than 7 vertices and $v$ a degree-3 vertex in $G$. Then after removing $N[v]$, parameter $r$ will be reduced by at least 8. Furthermore, if each 3-cycle in $G$ contains at least one vertex of degree $\geq 4$, then after removing $N[v]$, parameter $r$ will be reduced by at least 10.

Proof. There is at most one edge with both endpoints in $N(v)$, otherwise $v$ will dominate a neighbor of it. Therefore, there are at least four edges between $N(v)$ and $N_2(v)$. If $|N_2(v)| \geq 4$, $r$ will be reduced by $4 + 4 = 8$ directly after removing $N[v]$. If $|N_2(v)| \leq 3$, it is impossible to create a component of a 1-path ($l \geq 3$) after removing $N[v]$. By Lemma 8 and Corollary 1 and Lemma 9, we know that eventually $r$ will be reduced by at least 8.

Next, we assume that in each 3-cycle in $G$ there is a vertex of degree $\geq 4$. We distinguish the following two cases. **Case 1:** All vertices in $N(v)$ are degree-3 vertices. In this case, none pair of vertices in $N(v)$ are adjacent and there are exactly six edges between $N(v)$ and $N_2(v)$, which means at most 3 degree-1 vertices will be created after removing $N[v]$. It is impossible to create a component of a path after removing $N[v]$ (Obviously, no path of length $\geq 4$ will be created. Lemma 9 shows no path of length $\leq 2$ will be created. If a 3-path is created, then the graph $G$ has only 7 vertices). So by Corollary 1 if a component with 1 or 2 degree-1 vertices is created after removing $N[v]$, we can further reduce $r$ by 1 or 2 by further reducing degree-1 vertices in the component. If a component with 3 degree-1 vertices is created, then the component also contains at least 3 degree-3 vertices, otherwise the only possibility of the component is that it has 4 vertices: a degree-3 vertex adjacent with three degree-1 vertices, which also implies a contradiction—the graph $G$ has only 7 vertices. By Lemma 8 we still can further reduce $r$ by at least 3. In any case, totally we can reduce $r$ by at least $4 + 6 = 10$. **Case 2:** There is a vertex of degree $\geq 4$ in $N(v)$. Then there are at least five edges between $N(v)$ and $N_2(v)$ (Note that there is at least one edge with both endpoints in $N(v)$). By Lemma 8 and Lemma 9, we know that $r$ will be reduced by at least $5 + 5 = 10$.

**Lemma 11.** Let $G$ be a connected reduced graph of more than 8 vertices and $v$ a vertex of degree $\geq 4$ in $G$. Then after removing $N[v]$, parameter $r$ will be reduced by at least 10.

Proof. The lemma obviously holds when $v$ is a vertex of degree $\geq 5$ or a degree-4 vertex with $|N_2(v)| \geq 4$. Now we assume $v$ is a degree-4 vertex and $|N_2(v)| \leq 3$. **Case 1:** $|N_2(v)| = 1$. In this case, after removing $N[v]$, $r$ is reduced by at least $6 + 4 = 10$, or $r$ is reduced by at least $6 + 3$ and the only vertex in $N_2(v)$ becomes a degree-1 vertex (Note that there are at least $|N(v)| = 4$ edges between $N(v)$ and $N_2(v)$). In the later case, we can reduce $r$ by at least 1 by folding degree-1 vertices. **Case 2:** $|N_2(v)| = 2$. The two vertices $a, b \in N_2(v)$ are adjacent, otherwise there is a 3-4 structure $\{v\} \cup N_2(v) \cup N(v)$. Then after removing $N[v]$,
at most one of \(a\) and \(b\) becomes a degree-1 vertex, otherwise \(G\) has only 7 vertices. Therefore, we also can reduce \(r\) by at least 4 from \(V - N[v]\). **Case 3:** \(|N_2(v)| = 3\).

If one vertex in \(N(v)\) is a vertex of degree \(\geq 4\), then the lemma holds. Otherwise, all vertices in \(N(v)\) are degree-3 vertices, and then the number of edges between \(N(v)\) and \(N_2(v)\) is 4 or 6 or 8. If no degree-1 vertex is created after removing \(N[v]\), then \(r\) will be reduced by at least 6+4 directly (Note that it is impossible to create two degree-0 vertices, and when one degree-0 vertex is created, there are at least three edges between \(N(v)\) and \(N_2(v)\) that are incident on the other two vertices in \(N_2(v)\)). If some degree-1 vertices are created but no path component is created, then we can further reduce \(r\) by at least 1 by Lemma 8. The difficult case occurs when a path component is created. The path can only be a 1-path or 2-path. If it is a 2-path, then the graph has only 8 vertices. Therefore the path is a 1-path. If one vertex in the path is a vertex of degree \(\geq 4\) in \(G\), then after removing \(N[v]\), \(r\) is reduced by at least 6 + 4 = 10 directly. If the two vertices in the path are degree-3 vertices in \(G\), then there are at least two edges between \(N(v)\) and \(N_2(v)\) that are incident on the third vertex \(u\) in \(N_2(v)\). So after removing \(N[v]\), \(r\) is reduced by at least 6 + 4 = 10, or \(r\) is reduced by at least 6 + 3 and \(u\) becomes a degree-1 vertex, folding which will further reduce \(r\) by at least 1. We have checked all the cases and then finished the proof.

**Lemma 12.** Let \(G\) be a connected reduced graph of more than 7 vertices. If \(G\) has a bottle, then algorithm MIS\((G)\) will branch on a bottle with recurrence relation

\[
C(r) \leq 2C(r - 8),
\]

where \(C(r)\) is the worst-case size of the search tree in our algorithm.

Moreover, if each 3-cycle in \(G\) contains at least one vertex of degree \(\geq 4\), then MIS\((G)\) will branch on a bottle with recurrence relation

\[
C(r) \leq 2C(r - 10).
\]

**Proof.** Let the bottle called by our algorithm be \(b-a-\{c, d\}\). Our algorithm will branch by either removing \(N[a]\) or \(N[b]\). By Lemma 10 and Lemma 11 we get (1) and (2) directly.

**Lemma 13.** Let \(G\) be a connected bottle-free reduced graph of more than 7 vertices. If \(G\) has a 4-cycle, then algorithm MIS\((G)\) will branch on a 4-cycle with recurrence relation

\[
C(r) \leq 2C(r - 8).
\]

Moreover, if each 3-cycle or 4-cycle in \(G\) contains at least one vertex of degree \(\geq 4\), then MIS\((G)\) will branch on a 4-cycle with recurrence relation

\[
C(r) \leq 2C(r - 10).
\]
Proof. Let the 4-cycle called by our algorithm be \( a b c d \). Our algorithm will branch by removing either \( \{a, c\} \) or \( \{b, d\} \) from the graph. We look at the branch where \( \{a, c\} \) is removed (It is the same to \( \{b, d\} \)). Since none of the four vertices is dominated by others, each of the four vertices will be adjacent to a vertex different from the four vertices. If after removing \( \{a, c\} \), no degree-1 vertex is created, then we can reduce \( r \) by 8 directly in this branch. If some degree-1 vertices are created, then our algorithm will fold one, say \( x \), in the next step. Obviously, \( x \) is a degree-3 vertex in the original graph. The operation of removing \( \{a, c\} \) and then folding \( x \) is equivalent to the removing of \( N[x] \). We can reduce \( r \) by at least 8 by Lemma 10. Therefore, we get (3).

Next, we prove (4). There is at least one vertex of degree \( \geq 4 \), say \( a \), in the 4-cycle. We distinguish the following three cases. Case 1: There is only one vertex of degree \( \geq 4 \) in the 4-cycle. No matter we remove \( \{a, c\} \) or \( \{b, d\} \), at least one degree-1 will be created. As discussed above, after further folding a degree-1 vertex, we can reduce \( r \) by at least 10 in each branch by Lemma 10. Then we get (4). Case 2: There are exactly two vertices of degree \( \geq 4 \) in the 4-cycle and the two vertices are adjacent to each other in the 4-cycle (the two vertices are not \( \{a, c\} \) or \( \{b, d\} \)). Without loss of generality, we can assume the two vertices of degree \( \geq 4 \) are \( a \) and \( b \). In the branch where \( \{a, c\} \) is removed, \( d \) becomes a degree-1 vertex. In the branch where \( \{b, d\} \) is removed, \( c \) becomes a degree-1 vertex. Then in each branch we will remove \( N[v] \) for some degree-3 vertex \( v \) in \( G \). We still can get (4). Case 3: There are exactly two vertices of degree \( \geq 4 \) in the 4-cycle and the two vertices are a pair of opposite vertices in the 4-cycle. Then the two vertices of degree \( \geq 4 \) are \( a \) and \( c \) (We have assumed that \( a \) is a vertex of degree \( \geq 4 \)). It is easy to see that after removing \( \{b, d\} \), \( r \) will be reduced by at least 8. In the branch where \( \{a, c\} \) is removed, some degree-1 vertices are created (at least \( b \) and \( d \)). Then in this branch we will remove \( N[v] \) for some degree-3 vertex \( v \) in \( G \), where \( v \) has two vertices of degree \( \geq 4 \) \( a \) and \( c \). Since \( G \) has no bottle, there is not any edge with both endpoints in \( N(v) \). Therefore, there are at least 8 edges between \( N(v) \) and \( N_2(v) \). If no degree-1 vertex is created after removing \( N[v] \) (Lemma 9 also shows that no degree-0 vertex will be created), then \( r \) is reduced by \( 6 + 8 = 14 \) directly. If some degree-1 vertices are created, the case becomes complicated. In fact, as we do in the proof of Lemma 10 we can prove that no component of less than 3 degree-3 vertices will be created. By Lemma 8 we know that \( r \) will also be reduced by at least \( 6 + 7 = 13 \) in this branch. We get

\[ C(r) \leq C(r - 8) + C(r - 13). \tag{5} \]

Case 4: There are exactly three vertices of degree \( \geq 4 \) in the 4-cycle. Without loss of generality, we assume the remaining degree-3 vertex is \( c \). After removing \( \{a, c\} \), \( r \) will be reduced by at least 10. After removing \( \{b, d\} \), \( r \) will be reduced by at least 12. We get

\[ C(r) \leq C(r - 10) + C(r - 12). \tag{6} \]
Case 5: All the four vertices in the cycle are vertices of degree \( \geq 4 \). It is clear that \( r \) will be reduced by at least 10 in each branch. We also get (4). Since (4) covers (5) and (6), we know that the lemma holds.

**Lemma 14.** Let \( G \) be a connected reduced graph of more than 15 vertices that has no bottle or 4-cycle. If \( G \) has a vertex of degree \( \geq 4 \), then algorithm \( MIS(G) \) will branch on a vertex of maximum degree with recurrence relation

\[
C(r) \leq C(r - 6) + C(r - 14).
\]

(7)

**Proof.** Our algorithm will select a vertex \( v \) of maximum degree and branch on it by excluding it from the independent set or including it in the independent set. In the former branch, \( v \) is removed and \( r \) decreases by at least \( 2 + 4 = 6 \). In the latter branch, \( N[v] \) is removed. Since \( G \) has no bottle or 4-cycle, there are at least 8 vertices in \( N_2(v) \). Then in this branch, \( r \) will be reduced by at least \( 6 + 8 = 14 \). Therefore, we get (7).

**Lemma 15.** Let \( G \) be a connected reduced graph of more than 15 vertices that has no bottle or 4-cycle. If \( G \) is also a 3-regular graph, then algorithm \( MIS(G) \) can branch with recurrence relation

\[
C(r) \leq C(r - 10) + 2C(r - 14).
\]

(8)

**Proof.** Our algorithm will select a degree-3 vertex and branch on it. Since \( G \) is 3-regular graph that has no 3-cycle or 4-cycle, there are exactly 8 vertices in \( N_2(v) \). In the branch where \( N[v] \) is removed, 10 degree-3 vertices are reduced. So we can branch with recurrence relation

\[
C(r) \leq C(r - 10) + Q(r - 4),
\]

where \( Q \leq C \) is some function corresponding to the size of the branch where \( v \) is removed. Next, we focus on refining analysis of \( Q \).

In the branch where \( v \) is removed, 3 nonadjacent degree-2 vertices are created. Our algorithm will fold the three degree-2 vertices in the next step. Let \( G' \) be the resulted graph. Then \( G' \) has exactly 3 degree-4 vertices (Note that the original graph has no 3-cycle or 4-cycle. It is impossible to create a degree-3 vertex after folding a degree-2 vertex), and each 3-cycle or 4-cycle in the current graph contains at least one degree-4 vertex. If \( G' \) has a bottle or 4-cycle, we can branch with \( Q(r) \leq 2C(r - 10) \) by Lemma 12 and Lemma 13. If \( G' \) has no bottle or 4-cycle, we will branch on a degree-4 vertex \( v' \). We further distinguish three different cases. **Case 1:** The other two degree-4 vertices are adjacent to \( v' \). In this case, we have \( |N_2(v')| \geq 8 \) (the three degree-4 vertices may form a triangle). In the branch where \( v' \) is removed, \( r \) is reduced by at least 6, and in the branch where \( N[v'] \) is removed, \( r \) is reduced by at least \( 8 + 8 = 16 \). We get \( Q(r) \leq C(r - 6) + C(r - 16) \). **Case 2:** There is only one degree-4 vertex adjacent to \( v' \). Since there is no bottle and 4-cycle, we get \( |N_2(v')| \geq 9 \). In the branch where \( N[v'] \) is removed, \( r \) is reduced by \( 7 + 9 = 16 \). We also get \( Q(r) \leq C(r - 6) + C(r - 16) \). **Case 3:** There is no degree-4 vertex adjacent to \( v' \). We will
branch on $v'$ with (7) directly, and in the branch where $v'$ is removed, some other degree-4 vertices are left. We can further branch with (7) at least. Then we get
$$Q(r) \leq C(r-14) + C(r-6-6) + C(r-6-14) = C(r-12) + C(r-14) + C(r-20).$$

The worst case is that after branching with (9) we branch with (2) or (4), in which we get
$$C(r) \leq C(r-10) + Q(r-4) \leq C(r-10) + 2C(r-14),$$
as claimed in the lemma.

Among all the cases in our algorithm, the worst running time corresponds to recurrence relation (8). Since $C(r) = O(1.0919^r)$ satisfies (8), we get

**Theorem 1.** Algorithm MIS($G$) can find a minimum independent set in a degree-3 graph in $O^*(1.0919^n)$ time.

5 Improvement on $k$-Vertex Cover

Given a graph $G$ and a parameter $k$, the $k$-vertex cover problem is to decide if $G$ has a vertex cover of size at most $k$. The $k$-vertex cover problem is one of the most extensively studied problems in the area of Parameterized Algorithms. In this section, we show that the $k$-vertex cover problem can be solved in $O^*(1.1923^k)$ time, which improves the previously known result of $O^*(1.1939^k)$ by Chen et al. [7].

Nemhauser and Trotter [14] proved the following theorem:

**Proposition 1.** For a graph $G = (V,E)$ with $n$ vertices and $m$ edges, we can compute two disjoint vertex sets $C_0, V_0 \subset V$ in $O(\sqrt{mn})$ time, such that

1. Every minimum vertex cover in induced subgraph $G(V_0)$ plus $C_0$ forms a minimum vertex cover of $G$.
2. A minimum vertex cover of $G(V_0)$ contains at least $|V_0|/2$ vertices.

Our simple algorithm works as follows. Given an instance ($G, k$) of the $k$-vertex cover problem in degree-3 graphs, we first use Nemhauser and Trotter’s algorithm to construct $C_0$ and $V_0$. If $|V_0| > 2k$, then $G$ does not have a vertex cover of size at most $k$. Else we use our algorithm presented in Section 3 to find a maximum independent set $S$ in $G(V_0)$ in $O^*(1.0919^{|V_0|}) = O^*(1.1923^k)$ time. Then $C_0 + V_0 - S$ is a minimum vertex cover of $G$. If $k > |C_0 + V_0 - S|$, then $G$ does not have a vertex cover of size at most $k$. Else $C_0 + V_0 - S$ is satisfied vertex cover.

**Theorem 2.** The $k$-vertex cover problem in degree-3 graphs can be solved in $O^*(1.1923^k)$ time.

6 Concluding Remarks

In this paper, we have presented a simple $O^*(1.0919^n)$-time algorithm for the minimum independent set problem in degree-3 graphs and a simple $O^*(1.1923^k)$-time algorithm for the $k$-vertex cover problem in degree-3 graphs. Both algorithms improve previously known algorithms.
Unlike most previous algorithms, our algorithms do not contain many branching rules. We use two new branching techniques, called branching on a bottle and branching on a 4-cycle, to avoid tedious examinations of the local structures. In fact, new branching rules catch the structural properties of small cycles in graphs, which make our algorithms simple and practical. It is easy to see that many previous algorithms can apply these two new branching rules to simplify the description and analysis.

Our algorithm for the maximum independent set problem is analyzed by measuring the number of degree-3 vertices. We have checked that our algorithm \( \text{MIS}(G) \) can also be analyzed by measuring parameter \( m - n + t \) to get the same running time bound, where \( m \) is the number of edges, \( n \) the number of vertices, and \( t \) the number of tree components in the graph. Readers may note that the algorithms presented by F"urer [11] and Bourgeois et al. [9] are analyzed by measuring \( m - n \). In fact, their algorithms also need to consider the tree components created in the graphs and they have a separate section to analyze them. We guess that considering the tree components in the parameter may lead to a clearer analysis.

References

1. Tarjan, R., Trojanowski, A.: Finding a maximum independent set. SIAM Journal on Computing 6(3) (1977) 537–546
2. T.Jian: An \( O(2^{0.304n}) \) algorithm for solving maximum independent set problem. IEEE Transactions on Computers 35(9) (1986) 847–851
3. Roboson, J.: Algorithms for maximum independent sets. Journal of Algorithms 7(3) (1986) 425–440
4. Roboson, J.: Finding a maximum independent set in time \( O(2^{n/4}) \). Technical Report 1251-01, LaBRI, Universite Bordeaux I (2001)
5. Fomin, F.V., Grandoni, F., Kratsch, D.: Measure and conquer: a simple \( O(2^{0.288n}) \) independent set algorithm. In: SODA, ACM Press (2006) 18–25
6. Beigel, R.: Finding maximum independent sets in sparse and general graphs. In: Proceedings of the 10th annual ACM-SIAM symposium on discrete algorithms (SODA 1999). (1999) 856–857
7. Chen, J., Kanj, I.A., Xia, G.: Labeled search trees and amortized analysis: Improved upper bounds for NP-hard problems. Algorithmica 43(4) (2005) 245–273 A preliminary version appeared in ISAAC 2003.
8. Xiao, M.Y., Chen, J.E., Han, X.L.: Improvement on vertex cover and independent set problems for low-degree graphs. Chinese Journal of Computers 28(2) (2005) 153–160
9. Bourgeois, N., Escoffier, B., Paschos, V.T.: An \( O^\ast(1.0977^n) \) exact algorithm for max independed6nt set in sparse graphs. In Grohe, M., Niedermeier, R., eds.: IWPEC. Volume 5018 of Lecture Notes in Computer Science., Springer (2008) 55–65
10. Fomin, F.V., Hsie, K.: Pathwidth of cubic graphs and exact algorithms. Inf. Process. Lett. 97(5) (2006) 191–196
11. F"urer, M.: A faster algorithm for finding maximum independent sets in sparse graphs. In Correa, J.R., Hevia, A., Kiwi, M.A., eds.: LATIN. Volume 3887 of Lecture Notes in Computer Science., Springer (2006) 491–501
12. Razgon, I.: A faster solving of the maximum independent set problem for graphs with maximal degree 3. In Broersma, H., Dantchev, S.S., 0002, M.J., Szeider, S., eds.: ACiD. Volume 7 of Texts in Algorithmics., King’s College, London (2006) 131–142

13. Chen, J., Kanj, I., Xia, G.: Simplicity is beauty: Improved upper bounds for vertex cover. Technical Report TR05-008, School of CTI, DePaul University (2005)

14. Nemhauser, G.L., Trotter, L.E.: Vertex packings: Structural properties and algorithms. Mathematical Programming 8(1) (1975) 232–248