An Improved Splitting Function for Small $x$ Evolution

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Abstract

We summarize our recent result for a splitting function for small $x$ evolution which includes resummed small $x$ logarithms deduced from the leading order BFKL equation with the inclusion of running coupling effects. We compare this improved splitting function with alternative approaches.

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In recent years the theory of scaling violations for deep inelastic structure functions at small $x$ has attracted considerable interest, prompted by the experimental information coming from HERA. New effects beyond the low–order perturbative approximation to anomalous dimensions or splitting functions should become important at small-$x$. However, no major deviation of the data from a standard next–to–leading order perturbative treatment of scaling violations has been found. By now the origin of this situation has been mostly understood [1,2].

The BFKL kernel $\chi(\alpha_s, M)$ has been computed to next-to-leading accuracy (NLO):

$$\chi(M, \alpha_s) = \alpha_s \chi_0(M) + \alpha_s^2 \chi_1(M) + \ldots$$

The problem is how to use the information contained in $\chi_0$ and $\chi_1$ in order to improve the splitting function derived from the perturbative leading singlet anomalous dimension function $\gamma(\alpha_s, N)$ which is known up to NLO in $\alpha_s$:

$$\gamma(N, \alpha_s) = \alpha_s \gamma_0(N) + \alpha_s^2 \gamma_1(N) + \ldots$$

in such a way that the improved splitting function remains a good approximation down to small values of $x$. This can be accomplished [1] by exploiting the fact that the solutions of the BFKL and GLAP equations coincide at leading twist if their respective evolution kernels are related by a “duality” relation. In the fixed coupling limit the duality relation is simply given by:

$$\chi(\gamma(N, \alpha_s), \alpha_s) = N.$$  

The splitting function then will contain all relative corrections of order $(\alpha_s \log 1/x)^n$, derived from $\chi_0(M)$, and of order $\alpha_s^2 (\alpha_s \log 1/x)^n$, derived from $\chi_1(M)$.

The early wisdom on how to implement the information from $\chi_0$ was completely shaken by the computation of $\chi_1$, which showed that the naive expansion for the improved anomalous dimension had a hopelessly bad behaviour. However, as a consequence of the physical requirement of momentum conservation, this problem is cured if the small-$x$ resummation is combined with the standard resummation of collinear singularities, by constructing a ‘double-leading’ perturbative expansion.

However, higher order corrections to the kernel qualitatively change the asymptotic small-$x$ behaviour of structure functions by changing it from $x^{-\lambda_0}$ to $x^{-\lambda} = x^{-\lambda_0} e^{\Delta \lambda \xi} \approx x^{-\lambda_0}[1 + \Delta \lambda \xi + \ldots]$, with:

$$\lambda = \lambda_0 + \Delta \lambda, \quad \Delta \lambda = \alpha_s^2 \chi_1(\frac{1}{2}) + \ldots$$

The computed correction $\chi_1(\frac{1}{2})$ to the leading result is quite large, and this suggests that the correct asymptotic exponent is not reliably determined by the two known terms. Therefore in our previous work we have treated $\lambda$ as a parameter to be fitted from the data. Good agreement with the HERA data was found, but at the price of having to sharply fine-tune this parameter.

In our recent work [1] we have shown that also this problem can be solved by a full treatment of running coupling corrections. Indeed, it has been known for some time that running coupling effects can be included perturbatively order by order at small $x$ by...
adding effective subleading $\Delta \chi_i$ contributions to the BFKL kernels $\chi_1, \chi_2, \ldots$. However, these additional terms turn out to have singularities at $M = 1/2$, which correspond to an enhancement by powers of $\ln 1/x$ of the associated splitting functions, which may offset the perturbative suppression by powers of $\alpha_s$. In our approach, we have shown that these contributions can be resummed to all orders at the level of splitting functions, in a way compatible with factorization and a smooth behaviour in the small-$x$ limit.

Based on these results, we now know the way the information contained in $\chi_0(M)$ should be used in order to construct a better first approximation for the improved anomalous dimension. Indeed, we find that, once running coupling effects are properly included in the improved anomalous dimension, the asymptotic behavior near $x = 0$ is much softened with respect to the naive Lipatov exponent. Hence, the corresponding dramatic rise of structure functions at small $x$, which is phenomenologically ruled out, is replaced by a milder rise. The effective exponent corresponding to this rise turns out to be in surprisingly good agreement with the fine-tuned value of $\lambda$ indicated by the fit, which in turn led to a splitting function which closely followed the NLO GLAP in the region of the data. This suggests that additional higher order corrections are small, and that a leading–order approximation based on the standard BFKL kernel $\chi_0$ is phenomenologically viable.

We discuss now explicitly our proposed improved splitting function. Assuming that one only knows $\gamma_0(N)$, $\gamma_1(N)$ and $\chi_0(M)$, the improved anomalous dimension has the following expression:

$$
\gamma_I^{NL}(\alpha_s, N) = [\alpha_s \gamma_0(N) + \alpha_s^2 \gamma_1(N) + \gamma_s(\frac{\alpha_s^2}{N}) - \frac{n_c \alpha_s}{2 \pi N}] + \\
\gamma_A(c_0, \alpha_s, N) - \frac{1}{2} + \sqrt{\frac{2}{\kappa_0 \alpha_s}} [N - \alpha_s c_0] + \frac{1}{4} \beta_0 \alpha_s (1 + \frac{\alpha_s}{N} c_0) - \text{mom. sub.} \quad (5)
$$

The first line on the right-hand side, within square brackets, is the double-leading expression for the improved anomalous dimension at this level of accuracy, made up of the NLO perturbative term $\alpha_s \gamma_0(N) + \alpha_s^2 \gamma_1(N)$ plus the power series of terms $(\alpha_s/N)^n$ in $\gamma_s(\frac{\alpha_s^2}{N})$, obtained from $\chi_0$ using eq. (3), with subtraction of the order $\alpha_s$ term to avoid double counting ($c_A = n_c = 3$). In the second line, the “Airy” anomalous dimension $\gamma_A(c_0, \alpha_s, N)$ contains the running coupling resummation, and the remaining terms subtract the contributions to $\gamma_A(c_0, \alpha_s, N)$ which are already included in $\gamma_s$, $\gamma_0$ and $\gamma_1$. The Airy anomalous dimension $\gamma_A(c_0, \alpha_s, N)$ is the exact solution of the running coupling BFKL equation corresponding to a quadratic approximation of $\chi_0$ near $M = 1/2$: $\chi_0 \approx [c_0 + \frac{1}{2} \kappa_0 (M - \frac{1}{2})^2]$. Finally “mom. sub.” is a subleading subtraction that ensures momentum conservation $\gamma_I(\alpha_s, N = 1) = 0$.

We now summarize the properties of the improved anomalous dimension in this approximation. In the limit $\alpha_s \to 0$ with arbitrary $N$, $\gamma_I(\alpha_s, N)$ reduces to $\alpha_s \gamma_0(N) + \alpha_s^2 \gamma_1(N) + O(\alpha_s^3)$. For $\alpha_s \to 0$ with $\alpha_s/N$ fixed, $\gamma_I(\alpha_s, N)$ reduces to $\gamma_{\text{DL–LO}} = \alpha_s \gamma_0(N) + \gamma_s(\frac{\alpha_s^2}{N}) - \frac{n_c \alpha_s}{2 \pi N}$, i.e. the leading term of the double-leading expansion. Thus the Airy term is subleading in both limits. In spite of this, its role is very significant because of the singularity structure of the different terms in eq. (5). In fact, $\gamma_0(N)$ has a pole at $N = 0$, $\gamma_s$ has a branch cut at $N = \alpha_s c_0$, and $\gamma_A$ has a pole at $N = N_0 < \alpha_s c_0$, where $N_0$ is the position of the rightmost zero of the Airy function. The importance of the Airy term is that the square root term subtracted from $\gamma_A$ cancels, within the relevant accuracy,
the branch cut of $\gamma_s$ at $N = \alpha_s c_0$ and replaces the corresponding asymptotic behaviour at small $x$ with the much softer one from $\gamma_A$. Note that the quadratic approximation is sufficient to give the correct asymptotic behaviour up to terms which are of higher order in comparison to those included in the double-leading expression in eq. (5).

While we refer the interested reader to ref. [1] for further details, we show here two new figures for the splitting function which illustrate the success of this approach and compare it both to our older results and to those of ref. [2]. In figure 1 we show (for $\alpha_s = 0.2$) the singlet splitting function obtained from eq. (5) compared with the NLO GLAP kernel and with the DL-LO approximation, which displays the sharp small-$x$ rise characteristic of the BFKL resummation. In the region of the HERA data, our improved splitting function, with no free parameters, closely follows the NLO GLAP evolution with a behaviour at small $x$ which is much softer than that of BFKL. It is interesting to note that the agreement between GLAP and resummed results is significantly improved by the inclusion in eq. (5) of $\gamma_1$ and the corresponding double-counting subtraction. This improvement was already shown in ref. [1] in the anomalous dimension, and is even more apparent in the splitting function shown here.

In figure 2 the improved splitting function is compared with those of our earlier ap-
approach to this problem and with that recently obtained in ref. [2]. It is seen that the correct inclusion of running coupling corrections is by itself sufficient to produce the softening of the behaviour at small $x$ required in order to reproduce the data. Thus it is no longer necessary to introduce $\lambda$ by hand as a free parameter. In fact, the fine-tuned value $\lambda \approx 0.21$ (for $\alpha_s \sim 0.2$) obtained from fitting the data is in remarkable agreement with the value $N_0 = 0.2112$ determined from the Airy anomalous dimension with $\alpha_s = 0.2$. This improved splitting function is so good that subleading corrections to it are presumably very small.

In the same figure, we also show the splitting function from ref. [2]. In comparison to the approach adopted there, we share the general physical framework, but there are a number of differences. Specifically, the authors of ref. [2] determine the asymptotic small-$x$ behaviour by making some assumptions on the way to implement a symmetrisation $M \rightarrow (1 - M)$ of the BFKL kernel $\chi(\alpha_s, M)$, which we prefer not to do because of the ambiguities it introduces. Also, our resummed curve in figure 2 includes the effects of $\gamma_1$ but not fully those of $\chi_1$. This is because we find that after running coupling resummation, the impact of the left-over terms in $\chi_1$ is comparable to the resummation ambiguities.
By contrast, their curve includes the effects of $\chi_1$ but not those of $\gamma_1$. Furthermore, in ref. [2] the running coupling evolution equation is solved explicitly but numerically to the relevant order. Instead, through the Airy expansion, we obtain an analytic expression which holds to the same accuracy. This opens up the possibility to fit the data and match to usual evolution equations in a straightforward way. The resummed splitting functions shown in figure 2 all rise asymptotically with approximately the same power, but differ in the preasymptotic region. These differences can be taken as a measure of the remaining resummation uncertainties.

References

[1] G. Altarelli, R. D. Ball and S. Forte, hep-ph/0306156 and ref. therein.
[2] M. Ciafaloni, D. Colferai, G. P. Salam and A. M. Stasto, hep-ph/0307188 and ref. therein; D. Colferai, these Proceedings