Improved bounds for box dimensions of potential singular points to the Navier–Stokes equations

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Abstract
In this paper, we study the potential singular points of interior and boundary suitable weak solutions to the 3D Navier–Stokes equations. It is shown that the upper box dimensions of interior singular points and boundary singular points are bounded by 7/6 and 10/9, respectively. Both proofs rely on the recent progress of $\varepsilon$-regularity criteria at one scale.

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1. Introduction
We study the following 3D incompressible Navier–Stokes equations

\[
\begin{cases}
    u_t - \Delta u + u \cdot \nabla u + \nabla \Pi = 0, \\
    \text{div} u = 0, \\
    u|_{t=0} = u_0,
\end{cases}
\]

where $u$ is the velocity field, $\Pi$ is the scalar pressure, and the initial velocity $u_0$ satisfies the condition $\text{div} u_0 = 0$. We study the regularity problem of the Navier–Stokes equations. Our main objective of this paper is to further lower the box dimension of potential singular points of suitable weak solutions to the Navier–Stokes equations.

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A point \( z = (x, t) \) in (1.1) is said to be regular if \( u \) belongs to \( L^\infty \) in a neighborhood of \( z \). Otherwise, it is called singular. In a celebrated work [1], Caffarelli et al obtained two \( \varepsilon \)-regularity criteria to the suitable weak solutions of (1.1): \( z = 0 \) is a regular point provided that one of the following conditions holds for an absolute positive constant \( \varepsilon \),
\[
\|u\|_{L^2(Q(1))} + \|\Pi\|_{L^2(Q(1))} + \|\Pi\|_{L^{2/3}(Q(1))} < \varepsilon
\]
and
\[
\limsup_{\mu \to 0} \frac{1}{\mu} \|\nabla u\|^2_{L^2(\mu)} < \varepsilon.
\]
Lin [14], Ladyzhenskaya and Seregin [13] gave an alternative condition
\[
\|u\|_{L^{7/5}(Q(1))} + \|\Pi\|_{L^{7/3}(Q(1))} < \varepsilon,
\]
instead of (1.2). Recently, Guevara and Phuc [7] improved (1.2) and (1.4) to
\[
\|u\|_{L^{20/7}(Q(1))} + \|\Pi\|_{L^{10/3}(Q(1))} < \varepsilon,
\]
for some \( p, q \) satisfying \( 2/p + 3/q = 7/2 \) and \( 1 \leq p \leq 2 \). Subsequently, the authors in [9] found that (1.5) can be replaced by
\[
\|u\|_{L^{2p}(Q(1))} + \|\Pi\|_{L^{2q}(Q(1))} < \varepsilon
\]
for some \( p, q \) satisfying \( 1 \leq 2/p + 3/q < 2 \) and \( 1 \leq p, q \leq \infty \). Other interior regularity criteria similar to (1.2) can be found in [3, 21, 23]. Since the gradient of the pressure appears in (1.1), one can replace \( \Pi \) by \( \Pi - \Pi_B(1) \) by subtracting an average in (1.2), (1.4)–(1.6). Let \( S_i \) denote the possible interior singular points of suitable weak solutions to the 3D Navier–Stokes equations. One can use the condition (1.2) and (1.3) to obtain that
\[
\dim_H(S_i) \leq 1 \quad \text{and} \quad \dim_B(S_i) \leq 5/3,
\]
where \( \dim_H(S) \) and \( \dim_B(S) \) denote the Hausdorff dimension and box dimension of a set \( S \), respectively. For the background of fractal dimension, we refer the reader to [4]. The relationship between the Hausdorff dimension and box dimension is that the former is less than or equal to the latter.

In the past decade, starting from Robinson and Sadowski’s work [17], several authors have tried to lower the box dimension of potential interior singular points for suitable weak solutions to the 3D Navier–Stokes equations to 1 (Robinson and Sadowski [17] \( 5/3 \approx 1.67 \); Kukavica [11] \( 135/82 \approx 1.65 \); Kukavica and Pei [12] \( 45/29 \approx 1.55 \); Koh and Yang [10] 95/63 (1.51); Wang and Wu (360/277 \( \approx 1.30 \); Ren et al [16] (975/758 \( \approx 1.29 \); He et al [9] (2400/1903 \( \approx 1.261 \)).

In this paper, we also consider the potential boundary singular points of boundary suitable weak solutions to the Navier–Stokes equations. In particular, we consider the Navier–Stokes equations with no-slip conditions, which are given by
\[
\begin{align*}
  u_t - \Delta u + u \cdot \nabla u + \nabla \Pi &= 0, \quad \mathbb{R}^3_+ \times (0, T) \\
  \text{div} u &= 0, \quad u|_{\partial \mathbb{R}^3_+} = 0 \\
  u|_{t=0} &= u_0.
\end{align*}
\]

For the boundary regularity problem of the Navier–Stokes equation (1.7), Seregin [20] obtained the following two \( \varepsilon \)-regularity conditions...
\[
\limsup_{\mu \to 0} \frac{1}{\mu} \|
abla u\|_{L^2(Q^+(\mu))}^2 < \varepsilon
\]  
(1.8)

and

\[
\|u\|_{L^1(Q^+(1))} + \|\Pi\|_{L^{3/2}(Q^+(1))} < \varepsilon.
\]  
(1.9)

After that, Gustafson et al. [8] obtained the \(\varepsilon\)-regularity condition

\[
\|u\|_{L^1(Q^+(1))} + \|\Pi\|_{L^{3/2}(Q^+(1))} < \varepsilon,
\]  
(1.10)

where \(\lambda\) and \(\kappa^*\) are defined in (3.4). Let \(S_b\) denote the possible boundary singular points of boundary suitable weak solutions to the 3D Navier–Stokes equations. Here, a point \(z = (x, t)\) in (1.7) is said to be a singular point if \(u\) is not Hölder continuous in any neighborhood of \(z\) (see [8, 20]). Using these \(\varepsilon\)-regularity conditions, Choe and Yang [2] recently obtained a result for the upper box dimension \(\dim_B(S_b) \leq \frac{3}{2}\). In this paper we improve all the previous bounds for the interior and boundary singular points.

The rest of this paper is organized as follows. In section 2, we give a precise statement of our main results. In section 3, we introduce relevant notation and definitions of suitable weak solutions to (1.1) and (1.7), respectively. In section 4, we present a few auxiliary lemmas which are interpolation inequalities and decay estimates for the pressure. In section 5, we prove proposition 2.2 and deduce theorem 2.1, which is an estimation of the box dimension of \(S_i\) in (1.1). In section 6, we prove theorem 2.6 and proposition 2.4 and deduce theorem 2.3, which is an estimation of the box dimension of \(S_b\) in (1.7). Concluding remarks are given in section 7.

2. Main results

In this section we state the main results of this paper.

**Theorem 2.1.** The upper box dimension of \(S_i\) in (1.1) is at most \(7/6\).

**Remark 2.1.** The bound \(7/6\) is better than the previous results obtained in [9–12, 16–18, 22]. It is a direct consequence of the following regularity criterion.

**Proposition 2.2.** Suppose that \((u, \Pi)\) is a suitable weak solution to (1.1). Then for each \(\gamma < 1/2\) there exist positive numbers \(\varepsilon_1\) and \(r_0 < 1\) such that \(z = (x, t)\) is a regular point if for some \(r < r_0\)

\[
\int_{Q(z,r)} \|\nabla u\|^2 + |u|^{10/3} + |\Pi - \Pi_{B(x,t)}|^{5/3} + |\nabla \Pi|^{5/4} \, dx \, ds < r^{5/3 - \gamma} \varepsilon_1
\]  
(2.1)

where \(Q(z, r)\) denotes a parabolic cylinder (see the next section for notations and definitions).

The proof of proposition 2.2 is different from recent arguments in [9, 10, 16, 22]. The key point is to apply the following \(\varepsilon\)-regularity criteria, for any \(\delta > 0\),

\[
\|u\|_{L^p(Q(1))} + \|\Pi\|_{L^{3/2}(Q(1))} < \varepsilon \quad \text{and} \quad p > \frac{5}{2} + \delta,
\]  
(2.2)

to establish an iteration scheme. Our starting point is the following \(\varepsilon\)-regularity criterion

\[
\|\nabla u\|_{L^2(Q(1))} + \|u\|_{L^2(Q(1))} + \|\nabla \Pi\|_{L^{3/2}(Q(1))} < \varepsilon,
\]  
(2.3)
which is derived from
\[ \|u\|_{L^{2+}(Q(1))} + \|\nabla u\|_{L^{3/2}(Q(1))} < \varepsilon. \] (2.4)

It is worth noting that we get (2.4) by (1.6) and Poincaré–Sobolev inequality. Next, we explain the reason for the choice of (2.3). First, recall the interpolation inequality recently obtained in [9, 16], for \( p \geq 2, \)
\[ \frac{1}{r^2-p} \|u\|_{L^{r}(Q(\rho))}^p \leq \left( \frac{p}{r} \right)^{p/2} \left( \frac{1}{\rho} \|u\|_{L^{2+}(Q(\rho))}^p + \frac{1}{\rho} \|\nabla u\|_{L^{3/2}(Q(\rho))}^p \right)^{p/2}. \] (2.5)

All the proof in [9, 10, 16, 22] is based on (2.2) and (2.5), which leads to the restriction \( p > \frac{5}{2} \) in (2.5). However, as was observed in [22, remark 1.4, p 1762], it seems that it is useful to use the quantity \( \|\nabla u\|_{L^{2}} \) instead of \( \|u\|_{L^{\infty}} \) in estimating the box dimension of the singular set in (1.1). When we take \( p = 2 \) in (2.5), the quantity \( \|u\|_{L^{\infty}} \) disappears in the first part of the estimate (2.5). This is the reason why we chose \( \|u\|_{L^{2}} \) in (2.3). From (1.5) and (1.6), \( \|u\|_{L^{2}} \) is not enough to ensure the regularity. Fortunately, even though we add the \( \|\nabla u\|_{L^{2}} \) on the left-hand side of (2.5), the inequality still holds. Second, corresponding to \( \|u\|_{L^{2}} \), for pressure \( \Pi \), a natural candidate is \( \|\Pi\|_{L^{1}} \), but the Calderón–Zygmund theorem breaks down in \( L^1 \). Hence, the quantity \( \|\Pi\|_{L^{1}} \) is not appropriate. Motivated by the recent work of Choe and Yang [2], we note that there holds a decay estimate [8, lemma 17, p 617](see also lemma 4.3 in section 4) in terms of \( \nabla \Pi \) like (2.5). The borderline case of this decay estimate is parallel to (2.5) with \( p = 2 \). The endpoint case is new and corresponding to the quantity \( \|\nabla \Pi\|_{L^{1/2}} \). We will present the proof of this case. Indeed, we can slightly improve the known result [8, lemma 17, p 617]. Third, in the spirit of [12, 22], \( \|\nabla \Pi\|_{L^{1/2}} \) allows us to directly use the quantity \( \|\nabla \Pi\|_{L^{1/2}} \) on the right-hand side of the decay estimate involving the pressure instead of an interpolation between \( \|\nabla \Pi\|_{L^{5/4}} \) and \( \|\Pi\|_{L^{5/4}} \), in [9, 10, 16, 22]. Based on this, (2.3) helps us to make full use of \( \|\nabla u\|_{L^{2}} \) and \( \|\nabla \Pi\|_{L^{1/2}} \) in our proof.

We briefly illustrate the key points of the proof of proposition 2.2; see section 5 for its detailed proof. We bound
\[ (\theta \rho)^{-3/2} \|u\|_{L^{2}(Q(\theta \rho))} + (\theta \rho)^{-1} \|\nabla u\|_{L^{2}(Q(\theta \rho))} \]
via interpolation inequality (2.5) and hypothesis (2.1). Then, making use of the divergence-free condition, we establish a pressure decay estimate (4.3) in terms of \( \nabla \Pi \). Using the same scaled quantities, we get the smallness of \( (\theta \rho)^{-3/2} \|u\|_{L^{2}(Q(\theta \rho))} + (\theta \rho)^{-1} \|\nabla u\|_{L^{2}(Q(\theta \rho))} \) as well as
\[ (\theta \rho)^{-1} \|\nabla \Pi\|_{L^{1/2}(Q(\theta \rho))}. \]

This together with (2.3) completes the proof of proposition 2.2. Roughly speaking, we finish the proof by using the decay estimate once rather than many times as in [9, 10, 16, 22]. In addition, one can show that \( \dim_{H}(S_{\varepsilon}) \leq 37/30 \) via a combination of the proof described above and the \( \varepsilon \)-regularity criterion
\[ \|u\|_{L^{5/2+}(Q(1))} + \|\nabla u\|_{L^{5/4}(Q(1))} < \varepsilon. \] (2.6)

Remark 2.2. In [15], Ożański and Robinson studied partial regularity of a surface growth model below
\[ u_t + u_{xxxx} + \partial_x u_x^2 = 0. \] (2.7)
More precisely, they proved that the singular points set in (2.7) has 1D parabolic Hausdorff measure zero and box dimension no larger than 7/6. This yields the interesting fact that the fractal dimension of the singular points set in system (1.1) and in equation (2.7) is completely the same.

For the boundary case (1.7) we have the following theorem.

**Theorem 2.3.** The upper box dimension of $S_b$ in (1.7) is at most $10/9$.

**Remark 2.3.** The bound $10/9$ is better than the previous result obtained in [2]. It is a direct consequence of the following regularity criterion.

**Proposition 2.4.** Suppose that $(u, \Pi)$ is a suitable weak solution to (1.7). Then there exist positive numbers $\varepsilon_2$ and $r_0 < 1$ such that $z$ is a regular point if for some $0 < r < r_0$

$$
\int_{Q^+((x, r))} |\nabla u|^2 + |u|^{10/3} + |\Pi - \Pi_{B^+((x, r))}|^{5/3} + |\nabla \Pi|^{5/4} \, dx < r^{10/9} \varepsilon_2.
$$

Compared with the proof of proposition 2.2, the proof of proposition 2.4 includes a new ingredient of an application of new $\varepsilon$-regularity criteria below.

**Theorem 2.5.** Suppose that $(u, \Pi)$ is a suitable weak solution to the 3D Navier–Stokes equation (1.7) in $Q^+(1)$. Then there exists an absolute positive constant $\varepsilon$ such that $z = 0$ is a regular point if

$$
\|\nabla u\|_{L^p(\lambda^+(Q^+(1))} + \|\Pi - \Pi_{B^+_{\varepsilon}}(x)\|_{L^{\kappa^*}(Q^+(1))} < \varepsilon
$$

(2.9)

for some $p, q$ satisfying $2 \leq 2/q + 3/p < 3$ and $1 \leq p, q \leq \infty$.

**Remark 2.4.** A special case is

$$
\|\nabla u\|_{L^2(Q^+(1))} + \|\nabla \Pi\|_{L^{5/4}(Q^+(1))} < \varepsilon,
$$

which can be used to show that $\text{dim}_B(S_b) \leq 5/4$.

The Poincaré–Sobolev inequality guarantees theorem 2.5 follows from the next theorem, which is of independent interest.

**Theorem 2.6.** Suppose that $(u, \Pi)$ is a suitable weak solution to the 3D Navier–Stokes equation (1.7) in $Q^+(1)$. Then there exists an absolute positive constant $\varepsilon$ such that $z = 0$ is a regular point if

$$
\|u\|_{L^p(Q^+(1))} + \|\Pi - \Pi_{B^+_{\varepsilon}}(x)\|_{L^{\kappa^*}(Q^+(1))} < \varepsilon
$$

(2.10)

for some $p, q$ satisfying $1 \leq 2/q + 3/p < 2$ and $1 \leq p, q \leq \infty$.

**Remark 2.5.** Theorem 2.6 is an improvement of (1.9) and (1.10). The proof is in part inspired by recent results (1.5) and (1.6).
3. Notations and definitions

We denote by $L^q(-T,0;X)$, $1 \leq q \leq \infty$, the set of measurable functions on the interval $(-T,0)$ with values in $X$ and $\|f(t,\cdot)\|_X \in L^q(-T,0)$. We denote by $z = (x,t) \in \mathbb{R}^3 \times (-T,0)$ a space-time point and denote balls and cylinders by

$$B(x,r) = \{y \in \mathbb{R}^3 : |x-y| \leq r\},$$

$$Q(z,r) = B(x,r) \times (t-r^2,t).$$

We recall the definition of suitable weak solutions to (1.1).

**Definition 3.1.** A pair $(u, \Pi)$ is called a suitable weak solution to the Navier–Stokes equation (1.1) provided the following conditions are satisfied:

1. $u \in L^\infty(-T,0; L^2(\mathbb{R}^3)) \cap L^2(-T,0; \dot{H}^1(\mathbb{R}^3))$, $\Pi \in L^{3/2}(-T,0;L^{3/2}(\mathbb{R}^3))$.
2. $(u, \Pi)$ solves (1.1) in $\mathbb{R}^3 \times (-T,0)$ in the sense of distributions.
3. $(u, \Pi)$ satisfies the following inequality, for a.e. $t \in (-T,0)$,

$$\int_{\mathbb{R}^3} |u(x,t)|^2 \phi(x,t) \, dx + 2 \int_{-T}^t \int_{\mathbb{R}^3} |\nabla u|^2 \phi \, dx \, ds 
\leq \int_{-T}^t \int_{\mathbb{R}^3} |u|^2 (\partial_t \phi + \Delta \phi) \, dx \, ds + \int_{-T}^t \int_{\mathbb{R}^3} u \cdot \nabla \phi (|u|^2 + 2\Pi) \, dx \, ds,$$

where $\phi(x,s) \in C_0^\infty(\mathbb{R}^3 \times (-T,0))$ is non-negative.

**Remark 3.1.** From Giga and Sohr’s work [6], we know that the pressure $\Pi$ in the above definition also meets the regularity

$$\Pi \in L^{3/3}(-T,0;L^{3/3}(\mathbb{R}^3)), \nabla \Pi \in L^{5/4}(-T,0;L^{5/4}(\mathbb{R}^3)).$$

We would like to mention that the regularity (3.2) is required in the derivation from proposition 2.2 to theorem 2.1. See [22, p 1765–6] and [8, p 2095–6].

In the light of the natural scaling property of the Navier–Stokes equations, we introduce the following scaling-invariant quantities:

$$A(r) = \sup_{t-r^2 \leq s < t} r^{-1} \int_{B(x,r)} |u(y,s)|^2 \, dy,$$

$$E(r) = r^{-1} \int_{Q(x,r)} |\nabla u(y,s)|^2 \, dy \, ds,$$

$$E_p(r) = r^{p-5} \int_{Q(x,r)} |u(y,s)|^p \, dy \, ds,$$

$$P_{\frac{5}{4}}(\nabla \Pi, r) = r^{-1} \left( \int_{Q(x,r)} |\nabla \Pi|^5 \, dy \, ds \right)^{4/5},$$

$$P_{\frac{1}{5}, \frac{5}{4}}(\nabla \Pi, r) = r^{-1} \int_{t-r^2}^t \left( \int_{B(x,r)} |\nabla \Pi|^5 \, dy \right)^{4/5} \, ds.$$  

(3.3)
Before going further, we introduce the triplet \((\lambda, \kappa, \kappa^*)\) satisfying
\[
1 < \lambda < 2, \quad \frac{2}{\lambda} + \frac{3}{\kappa} = 4, \quad \frac{1}{\kappa^*} = \frac{1}{\kappa} - \frac{1}{3}.
\] (3.4)

In addition, we need the following scaling-invariant quantities involving the pressure
\[
\hat{D}(r) = r^{-1} \left( \int_{B(x,r)} \left( \int_{B(x,rs)} |\Pi(y,s) - \Pi_B(y,s)|^{\kappa^*} \, dy \right) \frac{ds}{s} \right)^{\frac{1}{\kappa^*}},
\] (3.5)
\[
\hat{D}_1(r) = r^{-1} \left( \int_{B(x,r)} |\nabla \Pi(y,s)|^{\kappa} \, dy \right)^{\frac{1}{\kappa}},
\] (3.6)
where we have used the notation \(\hat{f}_E\) which is the average of \(f\) over the set \(E\).

For any boundary point \(z = (x, t) = (x_1, x_2, 0, t) \in \partial \mathbb{R}^{1+3} \times (0, T)\), we denote half-balls and half-cylinders by
\[
B^+(x, r) = \{ y \in B(x, r) : y_3 > 0 \},
\]
\[
Q^+(z, r) = \{ (y, t) \in Q(z, r) : y_3 > 0 \}.
\]

We recall the definition of boundary suitable weak solutions to (1.7).

**Definition 3.2.** Let \(\Omega = \mathbb{R}^3_+\) and \(Q_T = \mathbb{R}^{1+3}_+ \times (-T, 0)\). The pair of \((u, \Pi)\) is a suitable weak solution to (1.7) if the following conditions are satisfied:

1. The functions \(u : Q_T \to \mathbb{R}^3\) and \(\Pi : Q_T \to \mathbb{R}\) satisfy
   \[
   u \in L^{\infty} (I; L^2(\Omega)) \cap L^2 (I; W^{1,2}(\Omega)), \quad \Pi \in L^\lambda (I; L^\kappa(\Omega)),
   \]
   \[
   \nabla^2 u \in L^\lambda (I; L^\kappa(\Omega)), \quad \nabla \Pi \in L^\lambda (I; L^\kappa(\Omega)),
   \]
   where \(\lambda\), \(\kappa\) and \(\kappa^*\) are fixed numbers satisfying (3.4).

2. \((u, \Pi)\) solves the Navier–Stokes equations in \(Q_T\) in the sense of distributions and \(u\) satisfies the boundary conditions in the sense of traces.

3. \(u\) and \(\Pi\) satisfy the local energy inequality
   \[
   \int_{B^+(x,t)} |u(x, t)|^2 \phi(x, t) \, dx + 2 \int_0^t \int_{B^+(x,s)} |\nabla u(x, s)|^2 \phi(x, s) \, dx \, ds \leq \int_0^t \int_{B^+(x,s)} |u|^2 (\partial_t \phi + \Delta \phi) \, dx \, ds + \int_0^t \int_{B^+(x,s)} (|u|^2 + 2\Pi) u \cdot \nabla \phi \, dx \, ds
   \] (3.7)
   for all \(t \in I = (0, T)\) and for all non-negative functions \(\phi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})\).

We shall use the same scaling-invariant quantities (3.3) for the boundary case replacing \(B(x, r)\) and \(Q(z, r)\) by \(B^+(x, r)\) and \(Q^+(z, r)\) in (3.3). Readers can tell them in the context.

For \(\kappa^*\) and \(\lambda\) in (3.4), we denote their Hölder conjugates \(i\) and \(j\) through the relations
\[
\frac{1}{j} = 1 - \frac{1}{\lambda}, \quad \frac{1}{i} = 1 - \frac{1}{\kappa^*}.
\]
Hence, it follows from (3.4) that
\[
\frac{2}{i} + \frac{3}{j} = 2, \quad 2 < i < \infty.
\] (3.8)

We end this section by giving a few more notations. When the center of a ball or a cylinder is located at the origin, we put them simply as
\[
B(r) = B(0, r) \quad \text{and} \quad Q(r) = Q(0, r).
\]
\[
B^+(r) = B^+(0, r) \quad \text{and} \quad Q^+(r) = Q^+(0, r).
\]

For simplicity, we also write
\[
\tilde{f}_r = \tilde{f}_{B(r)}
\]
\[
\|f\|_{L^p(Q(r))} \leq \|f\|_{L^p(-\rho/2, 0; L^Q(B(r)))}.
\]
\[
\|f\|_{L^p(Q(r))} \leq \|f\|_{L^p(B(r))}.
\]

We shall use the summation convention on repeated indices. We shall use the notation
\[
A \lessapprox B
\]
if there is a generic positive constant \(C\) such that
\[
|A| \leq C|B|.
\]
The generic positive constant \(C\) may be different from line to line unless otherwise stated.

4. Auxiliary lemmas

In this section, we present interpolation inequalities and decay estimates for the pressure.

Lemma 4.1 ([9, 16]). For \(0 < r \leq \rho/2\) and \(2 \leq p \leq 10/3\),
\[
E_p(r) \lesssim \left(\frac{\rho}{r}\right)^{p/2} A^{(p-2)/2}(\rho) E(\rho) + \left(\frac{r}{\rho}\right)^p A^{p/2}(\rho)
\]
for the interior case and
\[
E_p(r) \lesssim \left(\frac{\rho}{r}\right)^{p/2} A^{(p-2)/2}(\rho) E(\rho)
\]
for the boundary case, where the implied positive constants do not depend on \(r\) and \(\rho\).

We derive decay estimates in terms of \(\nabla \Pi\). See [12] for a different version.

Lemma 4.2. For \(0 < r \leq \rho/8\),
\[
P_{1,3/2}(\nabla \Pi, r) \lesssim \left(\frac{\rho}{r}\right) E(\rho) + \left(\frac{L}{\rho}\right) P_{5/4}(\nabla \Pi, \rho)
\]
for the interior case, where the implied positive constant does not depend on \(r\) and \(\rho\).

Proof. Without loss of generality, we assume that \(z = 0\). We consider the usual cut-off function \(\phi \in C_0^\infty(B(\rho/2))\) such that \(\phi \equiv 1\) on \(B(3\rho/8)\) with \(0 \leq \phi \leq 1\) and
\[
|\nabla \phi|^2 + |\nabla^2 \phi| \leq C\rho^{-2}.
\]

Due to the incompressible condition, we infer that
\[
\partial_i \partial_j [(\partial_k \Pi) \phi] = -\phi \partial_i \partial_j [\partial_k U_{i,j}] + 2(\partial_i \phi) \partial_j \partial_k \Pi + (\partial_k \Pi) \partial_i \partial_j \phi
\]
where \( U_{ij} = (u_i - \overline{u}_{i(2/3)})(u_j - \overline{u}_{j(2/3)}) \). Let \( K \) denote the normalized fundamental solution of the Laplace equation. Then for \( x \in B(3\rho/8) \)

\[
\partial_k \Pi(x) = K \ast \{ \phi \partial_k \partial_j [\partial_k U_{ij}] - 2(\partial_i \phi) \partial_k \partial_j \Pi - (\partial_k \Pi) \partial_k \partial_j \phi \}
\]

\[
= \partial_k \partial_j K \ast \{ \phi \partial_k U_{ij} \}
\]

\[
= 2\partial_k \partial_j K \ast \{ (\partial_i \phi) \partial_k U_{ij} \} + K \ast \{ (\partial_i \partial_j \phi) (\partial_k U_{ij}) \}
\]

\[
- 2\partial_k \partial_j K \ast \{ (\partial_i \partial_j \phi) \Pi \} - 2\partial_k K \ast \{ (\partial_i \partial_j \phi) \Pi \} - K \ast \{ (\partial_i \partial_j \partial_k \phi) \Pi \}
\]

\[
=: \partial_k P_1(x) + \partial_k P_2(x) + \partial_k P_3(x).
\]

(4.5)

Since \( \phi(x) = 1 \), where \( x \in B(r) (0 < r \leq \rho/4) \), we know that there is no singularity in \( \partial_k P_2(x) \) and \( \partial_k P_3(x) \). This together with (4.4) allows us to obtain

\[
|\partial_k P_2(x)| \leq \rho^{-3} \int_{B(\rho/2)} |\partial_k U_{ij}| \, dx,
\]

\[
|\partial_k P_3(x)| \leq \rho^{-4} \int_{B(\rho/2)} |\Pi| \, dx,
\]

and by Hölder’s inequality

\[
\| \partial_k P_2(x) \|_{L^{2,2}(B(\rho))} \lesssim \left( \frac{r}{\rho} \right)^2 \| \partial_k U_{ij} \|_{L^{2,2}(B(\rho/4))},
\]

(4.6)

\[
\| \partial_k P_3(x) \|_{L^{2,2}(B(\rho))} \lesssim \left( \frac{r}{\rho} \right)^2 \| \Pi \|_{L^{2,2}(B(\rho/4))}.
\]

(4.7)

Using the Hölder inequality and the Poincaré–Sobolev inequality, we see that

\[
\| \partial_k U_{ij} \|_{L^{2,2}(Q(\rho/2))} \lesssim \| u - \overline{u}_{i(2/3)} \|_{L^{2,5}(Q(\rho/2))} \| \nabla u \|_{L^2(Q(\rho/2))} \lesssim \| \nabla u \|_{L^2(Q(\rho/2))}^2.
\]

(4.8)

It follows from (4.6) and (4.8) that

\[
\| \partial_k P_2(x) \|_{L^{2,2}(Q(\rho))} \lesssim \left( \frac{r}{\rho} \right)^2 \| \nabla u \|_{L^2(Q(\rho/2))}^2.
\]

(4.9)

Notice that \( \Pi - \overline{\Pi} \) also satisfies (4.5), and we derive from (4.7) that

\[
\| \partial_k P_3(x) \|_{L^{2,2}(Q(\rho))} \lesssim \left( \frac{r}{\rho} \right)^2 \| \Pi - \overline{\Pi} \|_{L^{2,4,2}(Q(\rho/2))} \| \nabla \Pi \|_{L^{4,2}(Q(\rho/2))}.
\]

(4.10)

According to the Calderón–Zygmund theorem and (4.8), we get

\[
\| \partial_k P_1(x) \|_{L^{2,2}(Q(\rho))} \lesssim \| \partial_k U_{ij} \|_{L^{2,2}(Q(\rho/2))} \lesssim \| \nabla u \|_{L^2(Q(\rho))}^2.
\]

(4.11)
Combining the estimates (4.9)–(4.11), we get
\[ \|\partial_\kappa\Pi\|_{L^{3/2}(Q(r))} \lesssim \|\partial_\kappa P_1(x)\|_{L^{3/2}(Q(r))} + \|\partial_\kappa P_2(x)\|_{L^{3/2}(Q(r))} + \|\partial_\kappa P_3(x)\|_{L^{3/2}(Q(r))} \]
\[ \lesssim \|\nabla u\|_{L^2(Q(\rho))}^2 + \left(\frac{r}{\rho}\right)^2 \|\partial_\kappa\Pi\|_{L^{3/2}(Q(\rho))}. \]
(4.12)

Hence
\[ P_{1,3/2}(\nabla\Pi, r) \lesssim \left(\frac{\rho}{r}\right) E(\rho) + \left(\frac{r}{\rho}\right) P_{5/4}(\nabla\Pi, \rho). \]

**Lemma 4.3.** For \(0 < r \leq \rho/8\),
\[ D_1(r) \lesssim \left(\frac{\rho}{r}\right) E^{1/\lambda}(\rho)A^{1-1/\lambda}(\rho) + \left(\frac{r}{\rho}\right) D_1(r), \]
where \(\lambda\) is defined in (3.4) and the implied positive constant does not depend on \(r\) and \(\rho\). In addition, this lemma remains valid for \(\kappa = 1\).

**Proof.** We get the result by replacing (4.5) with
\[ \partial_\kappa\Pi(x) = K \ast \{\phi_{\partial_\kappa\partial_\kappa}[\partial_\kappa U_{i,j}] - 2\partial_\kappa\phi\partial_\kappa\partial_\kappa\Pi - \partial_\kappa\Pi\partial_\kappa\partial_\kappa\phi\}
= \partial_\kappa\partial_\kappa(K \ast \{\phi_{\partial_\kappa U_{i,j}}\})
+ 2\partial_\kappa K \ast (\partial_\kappa\phi[\partial_\kappa U_{i,j}])
+ 2\partial_\kappa K \ast (\partial_\kappa\partial_\kappa\phi\partial_\kappa\Pi)
+ 2\partial_\kappa K \ast (\partial_\kappa\partial_\kappa\partial_\kappa\Pi)
= : \partial_\kappa P_1(x) + \partial_\kappa P_2(x) + \partial_\kappa P_3(x) \]
and by making a slight variation to the proof of lemma 4.2.

**Remark 4.1.** This lemma gives an improvement of [8, lemma 17, p 617]. Since the standard parabolic regularization theory is not applicable to the case \(L^1\), a particularly interesting case in this lemma is \(\kappa = 1\).

In the spirit of the above lemma 4.2 and [2, lemma 4, p 11], we obtain the following result for the boundary case.

**Lemma 4.4.** If \(z = (x, t), x \in \partial\mathbb{R}^+_{+}, t - \rho^2 > 0, \) and \(t < T\), then for \(0 \leq r \leq \rho/4\) and \(\lambda \leq 5/4\)
\[ \tilde{D}_1(r) \lesssim \left(\frac{\rho}{r}\right) E^{1/\lambda}(\rho)A^{1-1/\lambda}(\rho) + \left(\frac{r}{\rho}\right) \left(E^{1/2}(\rho) + P_{5/4}(\nabla\Pi, \rho)\right), \]
where \(\lambda\) is defined in (3.4) and the implied positive constant does not depend on \(r\) and \(\rho\).

**Proof.** A slight variation of the proof [8, lemma 11, p 608] provides the estimates
\[ \tilde{D}_1(r) \lesssim \left(\frac{\rho}{r}\right) E^{1/\lambda}(\rho)A^{1-1/\lambda}(\rho) + \left(\frac{r}{\rho}\right) \left(E^{1/2}(\rho) + \rho^{-2}\|\Pi - \nabla\Pi\|_{L^{3/2}(Q(\rho))}\right). \]
From $\lambda \leq 5/4$ and $\kappa \leq 15/7$ and Hölder’s inequality, we see that
\[
\rho^{-2}||\Pi - \Pi_{\rho}||_{L^{\lambda,3}(Q^{+}(\rho))} \lesssim \rho^{-1}||\Pi - \Pi_{\rho}||_{L^{5/3,1/(5/3)}(Q^{+}(\rho))} \lesssim \rho^{-1}||\nabla\Pi||_{L^{5/4}(Q^{+}(\rho))}.
\]
Combining the two estimates yields the result.

5. Proof of theorem 2.1

In this section, we consider the upper box dimension of potential interior singular points for suitable weak solutions to the Navier–Stokes equation (1.1). Actually, theorem 2.1 is a direct consequence of proposition 2.2. The proof is based on the contradiction argument, which is very well known (see, for example, [2, 12, 22]). Thus, it suffices to prove proposition 2.2. We divide its proof into a few steps.

Step (1) We may prove the theorem assuming that $z = 0$. We shall show that the quantities on the left of (5.6) can be controlled by the following assumption (5.1), which was used in [22, p 1762, inequality (3.2)]. Assume that for some fixed $\rho \leq \rho_{0} < 1$
\[
\int_{Q(2\rho)} |\nabla u|^{2} + |u|^{10/3} + ||\Pi - \Pi_{2\rho}||^{5/3} + ||\nabla\Pi||^{5/4} \, dx \, dt \lesssim (2\rho)^{5/3 - \gamma/1}.
\]
(5.1)
Consider the usual cut-off positive function $\phi(x, t)$ supported in $Q(2\rho)$ and with value 1 on the ball $Q(\rho)$ such that
\[
\rho|\nabla \phi| + \rho^{2}|\nabla^{2} \phi| + \rho^{2}||\partial_{t} \phi(x, t)|| \leq C.
\]
(5.2)
This and the local energy inequality (3.1) give
\[
\sup_{-\rho^{2} \leq t < 0} \int_{B(\rho)} |u|^{2} \, dx + \int_{Q(\rho)} |\nabla u|^{2} \, dx \, dt \lesssim \left( \int_{Q(2\rho)} |u|^{10/3} \, dx \, dt \right)^{3/5} + \frac{C}{\rho} \int_{Q(2\rho)} ||u|^{2} - |\overline{u}|^{2} \rho|| \, dx \, dt
\]
\[
+ \frac{C}{\rho} \int_{Q(2\rho)} ||\Pi - \Pi_{2\rho}||u| \, dx \, dt
\]
(5.3)
where the divergence-free condition and integration by parts are used.

Step (2) By Hölder’s inequality and the Gagliardo–Nirenberg inequality, we estimate
\[
\frac{C}{\rho} \int_{Q(2\rho)} ||u|^{2} - |\overline{u}|^{2} \rho|| \, dx \, dt \lesssim \rho^{-1}||u|^{2} - |\overline{u}|^{2} \rho||_{L^{5/3,1/(5/3)}(Q(2\rho))} ||u||_{L^{5/3,1/(5/3)}(Q(2\rho))}
\]
\[
\lesssim \rho^{-1}||u|^{2}||_{L^{5/3}(Q(\rho))}^{1/2} ||u \nabla u||_{L^{5/(3\gamma)}(Q(\rho))}^{1/2} ||u||_{L^{5/3,1/(5/3)}(Q(\rho))}
\]
\[
\lesssim \rho^{-1/2}||u||_{L^{5/3}(Q(\rho))}^{1/2} ||\nabla u||_{L^{5/(3\gamma)}(Q(\rho))}^{1/2}
\]
(5.4)
and
\[
\frac{C}{\rho} \int_{Q(2\rho)} |\Pi - \Pi_{2\rho}| |u| \, dx \, dr \\
\lesssim \rho^{-1} \|\Pi - \Pi_{2\rho}\|_{L^{10/3,1}(Q(2\rho))} \|u\|_{L^{10/3,1}(Q(2\rho))} \\
\lesssim \rho^{-1/2} \|\Pi - \Pi_{2\rho}\|_{L^{5/3,1}(Q(2\rho))} \|\Pi\|_{L^{5/3,1}(Q(2\rho))} \|u\|_{L^{10/3,1}(Q(2\rho))} \\
\lesssim \rho^{-1/2} \|\Pi - \Pi_{2\rho}\|_{L^{5/3,1}(Q(2\rho))} \|\nabla \Pi\|_{L^{5/3,1}(Q(2\rho))} \|u\|_{L^{10/3,1}(Q(2\rho))}. \tag{5.5}
\]

Plugging (5.4) and (5.5) into (5.3) and using the assumption (5.1), we get
\[
\frac{C}{\rho} \int_{Q(2\rho)} |\Pi - \Pi_{2\rho}| |u| \, dx \, dr \\
\lesssim \|u\|_{L^{10/3,1}(Q(2\rho))} + \rho^{-1/2} \|u\|_{L^{10/3,1}(Q(2\rho))} \|\nabla u\|_{L^{5/3,1}(Q(2\rho))} \|\Pi - \Pi_{2\rho}\|_{L^{5/3,1}(Q(2\rho))} \\
+ \rho^{-1/2} \|u\|_{L^{10/3,1}(Q(2\rho))} \|\Pi\|_{L^{5/3,1}(Q(2\rho))} \|\nabla \Pi\|_{L^{5/3,1}(Q(2\rho))} \\
\lesssim \varepsilon_1^{3/5} \rho^{1 - 3\gamma/5} + \varepsilon_1 \rho^{7/6 - \gamma} \\
\lesssim \varepsilon_1^{3/5} \rho^{7/6 - \gamma}, \tag{5.6}
\]

where we have used \(\gamma \geq 5/12\). Hence we get
\[
A(u, \rho) \lesssim \varepsilon_1^{3/5} \rho^{1/6 - \gamma}. \tag{5.7}
\]

**Step (3)** If we take
\[
\beta = 1/6 \quad \text{and} \quad \theta = \rho^{3} < 1/8,
\]
then from (4.1), (5.1) and (5.7) we get
\[
E_2(\theta \rho) + E(\theta \rho) \lesssim \theta^{-1} E(\rho) + \theta^2 A(\rho) \\
\lesssim \varepsilon_1^{3/5} \rho^{2/3 - \gamma} + \varepsilon_1^{3/5} \theta^2 \rho^{1/6 - \gamma} \\
\lesssim \varepsilon_1^{3/5} \rho^{-\beta + 2/3 - \gamma} + \varepsilon_1^{3/5} \rho^{3\beta + 1/6 - \gamma} \\
\lesssim \varepsilon_1^{3/5} \rho^{1/2 - \gamma}. \tag{5.8}
\]

If \(\gamma \leq 1/2\), then
\[
E_2(\theta \rho) + E(\theta \rho) \lesssim \varepsilon_1^{3/5}. \tag{5.8}
\]

**Step (4)** From lemma 4.2 and (5.1) we have
\[
P_{1,3/2}(\nabla \Pi, \theta \rho) \\
\lesssim \theta^{-1} E(\rho) + \theta P_{3/4}(\nabla \Pi, \rho) \\
\lesssim \varepsilon_1^{1/5} \rho^{2/3 - \gamma} + \varepsilon_1^{1/5} \rho^{1/3 - 4\gamma/5} \\
\lesssim \varepsilon_1^{4/5} \rho^{-1/6 + 2/3 - \gamma} + \varepsilon_1^{4/5} \rho^{1/6 + 1/3 - 4\gamma/5} \\
\lesssim \varepsilon_1^{4/5} \rho^{1/2 - \gamma}. \tag{5.9}
\]
Combining (5.8) and (5.9), we get
\[ E(\theta\rho) + E_2(\theta\rho) + P_{1,3/2}(\nabla\Pi, \theta\rho) \lesssim \varepsilon^{3/5}. \]
This together with (2.3) yields that \( z = 0 \) is a regular point. This means that \( u \in L^\infty(Q(r)) \) for some \( r > 0 \). This completes the proof of proposition 2.2.

6. Proof of theorem 2.3

In the first place, we prove proposition 6.1. Then theorems 2.5 and 2.6 follow from proposition 6.1 and (1.10). After that, we shall give the proof of proposition 2.4. Then theorem 2.3 is a direct consequence of proposition 2.4 by the contradiction argument.

**Proposition 6.1.** Let \( p, q \) be defined in theorem 2.6 and denote \( \alpha = \frac{2}{1 + \frac{2}{p} + \frac{1}{q}} \). Suppose that \( (u, \Pi) \) is a suitable weak solution to the Navier–Stokes equation (1.7) in \( Q^+(R) \). Then there holds, for any \( R > 0 \),
\[ \|u\|_{L^2(Q^+(R/2))}^2 + \|\nabla u\|_{L^2(Q^+(R/2))}^2 \lesssim R^{(3\alpha-4)/\alpha}\|u\|_{L^{2\alpha}(Q^+(R))}^2 + R^{(3\alpha-5)/(\alpha-1)}\|u\|_{L^{(\alpha-1)}(Q^+(R))}^{2\alpha/(\alpha-1)} + R^{-1}\|\Pi - \Pi_{B^+(R)}\|_{L^{\alpha-1}(Q^+(R))}^2 \]
where the implied positive constants do not depend on \( R \).

We divide its proof into a few steps.

**Proof.**

**Step (1)** Fix \( r \) and \( \rho \) satisfying
\[ 0 < R/2 \leq r < \rho \leq R. \]
Let \( \phi(x, t) \) be a non-negative smooth function supported in \( Q(\rho) \) such that \( \phi(x, t) = 1 \) on \( Q(r) \) and
\[ |\partial_t \phi| + |\nabla \phi|^2 + |\nabla^2 \phi| \lesssim (\rho - r)^{-2}. \]
From the local energy inequality (3.1) we have
\[ \int_{B^+(\rho)} |u(x, t)|^2 \phi(x, t) dx + 2 \int_{Q^+(\rho)} |\nabla u|^2 \phi dx ds \lesssim L_1 + L_2 + L_3 \]
where
\[ L_1 = \frac{1}{(\rho-r)^2} \int_{Q^+(\rho)} |u|^2 dx ds, \]
\[ L_2 = \frac{1}{(\rho-r)} \int_{Q^+(\rho)} |u|^3 \phi dx ds, \]
\[ L_3 = \frac{1}{(\rho-r)} \int_{Q^+(\rho)} u(\Pi - \Pi_{B^+(\rho)}) dx ds. \]
In order to estimate $L_1, L_2$ and $L_3$, we shall use the following interpolation inequality. If $k$ and $l$ satisfy
\[
\frac{2}{l} + \frac{3}{k} = \frac{3}{2} \quad \text{and} \quad 2 \leq l \leq \infty,
\]
then
\[
\|u\|_{L^2(Q(\rho))} \lesssim \|u\|_{L^\infty(Q(\rho))}^{1-2/l} \|u\|_{L^{2k}(Q(\rho))}^{2/l} \lesssim \|u\|_{L^\infty(Q(\rho))}^{1-2/l} \|\nabla u\|_{L^2(Q(\rho))}^{1/2} \lesssim \|u\|_{L^\infty(Q(\rho))} + \|\nabla u\|_{L^2(Q(\rho))},
\]
(6.3)

**Step (2)** By Hölder’s inequality and (6.3), we have
\[
L_1 \lesssim \frac{\rho^{5/3}}{(\rho-r)^{2/3}} \left( \int_{Q^+(\rho)} |u|^3 \, dx \, dt \right)^{2/3} \lesssim \frac{\rho^{5/3}}{(\rho-r)^{2/3}} \|u\|_{L^{2\alpha+3\rho}(Q^+(\rho))}^{2\alpha/3} \left( \|u\|_{L^\infty(Q^+(\rho))}^{2/3} + \|\nabla u\|_{L^2(Q^+(\rho))}^{2/3} \right)^{(3-\alpha)/3}.
\]

By Young’s inequality, there is a positive constant $C$ such that
\[
L_1 \leq \frac{1}{16} \left( \|u\|_{L^\infty(Q^+(\rho))}^2 + \|\nabla u\|_{L^2(Q^+(\rho))}^2 \right) + \frac{C\rho^{3\alpha/2}}{(\rho-r)^{\alpha/2}} \|u\|_{L^{2\alpha+3\rho}(Q^+(\rho))}^{2\alpha/3}.
\]
(6.4)

Similarly, by Hölder’s inequality and (6.3), we have
\[
L_2 = \left( \frac{1}{\rho-r} \right) \int_{Q^+(\rho)} |u|^\alpha |u|^{3-\alpha} \, dx \, dt \lesssim \frac{1}{(\rho-r)^{\alpha/2}} \|u\|_{L^{2\alpha+3\rho}(Q^+(\rho))} \left( \|u\|_{L^{\infty}(Q^+(\rho))}^{2\alpha/3} + \|\nabla u\|_{L^2(Q^+(\rho))}^{2\alpha/3} \right)^{(3-\alpha)/2}.
\]

By Young’s inequality, there is a positive constant $C$ such that
\[
L_2 \leq \frac{1}{16} \left( \|u\|_{L^\infty(Q^+(\rho))}^2 + \|\nabla u\|_{L^2(Q^+(\rho))}^2 \right) + \frac{C\rho^3}{(\rho-r)^{\alpha/2}} \|u\|_{L^{2\alpha+3\rho}(Q^+(\rho))}^{2\alpha/3},
\]
(6.5)

Finally, by Hölder’s inequality and (6.3), we have
\[
L_3 \lesssim \frac{1}{(\rho-r)^{\alpha/2}} \|\Pi - \overline{\Pi}_{B^+(\rho)}\|_{L^{\infty}(Q^+(\rho))} \|\nabla u\|_{L^2(Q^+(\rho))} \lesssim \rho^{1/2} \left( \|\Pi - \overline{\Pi}_{B^+(\rho)}\|_{L^{\infty}(Q^+(\rho))} \left( \|u\|_{L^\infty(Q^+(\rho))}^2 + \|\nabla u\|_{L^2(Q^+(\rho))}^2 \right)^{1/2} \right)^{1/2}
\]
where $\kappa^*, \lambda, i$ and $j$ are numbers in (3.4) and (3.8). By Young’s inequality, there is a positive constant $C$ such that
\[ L_3 \leq \frac{1}{16} \left( \|u\|_{L^\infty(Q^+(\rho))}^2 + \|\nabla u\|_{L^2(Q^+(\rho))}^2 \right) + \frac{C_\rho}{(\rho - r)^2} \|\Pi - \Pi_{B^+(\rho)}\|_{L^\infty(Q^+(\rho))}^2. \]  

(6.6)

**Step (3)** Substituting the inequalities (6.4)–(6.6) into (6.2), we obtain that

\[
\sup_{-\rho^2 \leq t < 0} \int_{B(t)} |u|^2 \, dx + \int_{Q^+(\rho)} |\nabla u|^2 \, dxdt \\
\leq \frac{1}{4} \left( \|u\|_{L^\infty(Q^+(\rho))}^2 + \|\nabla u\|_{L^2(Q^+(\rho))}^2 \right) + \frac{C_\rho^5}{(\rho - r)^{6/\alpha}} \|\Pi - \Pi_{B^+(\rho)}\|_{L^\infty(Q^+(\rho))}^2 \\
+ \frac{C_\rho^3}{(\rho - r)^{2/(\alpha - 1)}} \|u\|_{L^{2\alpha/(\alpha - 1)}(Q^+(\rho))}^2 + \frac{C_\rho}{(\rho - r)^2} \|\Pi - \Pi_{B(\rho)}\|_{L^\infty(Q^+(\rho))}^2.
\]

Applying the standard iteration argument, lemma V.3.1 in [5], we get the result. □

We end this section by giving the proof of proposition 2.4.

**Proof of proposition 2.4.** We may assume \( z = 0 \). First we assume that

\[
\int_{Q^+(2\rho)} |\nabla u|^2 + |u|^{10/3} + \|\Pi - \Pi_{2\rho}\|^{5/3} + |\nabla \Pi|^{5/4} \, dxdt \leq (2\rho)^{5/3 - \epsilon} \epsilon_2. \]  

(6.7)

Following the argument in [10] we shall determine a suitable parameter \( \gamma \). From (6.7), we see that

\[
\rho^{-1} \|\nabla \Pi\|_{L^{5/4}(Q^+(\rho))} \lesssim \epsilon_2^{4/5} \rho^{1/3 - 4\gamma/5}.
\]  

(6.8)

By the Poincaré inequality, we know that

\[
\int_{Q^+(2\rho)} |u|^2 (\partial_t \phi + \Delta \phi) \lesssim \int_{Q^+(2\rho)} |\nabla u|^2 \lesssim \epsilon_2 \rho^{5/3 - \gamma}.
\]  

(6.9)

Plugging (6.9) into the first term on the right-hand side of local energy inequality (3.7) and controlling other terms by the same derivation in (5.4) and (5.5), we arrive at

\[
\sup_{-\rho^2 \leq t < 0} \int_{B^+(\rho)} |u|^2 \, dx + \int_{Q^+(\rho)} |\nabla u|^2 \, dxdt \lesssim \rho^{5/3 - \gamma} + \rho^{7/6 - \gamma} \lesssim \rho^{5/6 - \gamma}.
\]  

(6.10)

Consequently,

\[
A(u, \rho) \lesssim \epsilon_2 \rho^{1/6 - \gamma}.
\]  

(6.11)

If we take

\[
\beta = \frac{1}{4\lambda} - \frac{1}{12} - \frac{\gamma}{10} \quad \text{and} \quad \theta = \rho^\beta < \frac{1}{4},
\]
then from (4.2), (4.15) and (6.8) we conclude that

\[ E(\theta \rho) + \tilde{D}_1(\theta \rho) \]
\[ \lesssim \theta^{-1} E(\rho) + \theta^{-1} E^{1/\lambda}(\rho) A^{1-1/\lambda}(\rho) + \theta E^{1/2}(\rho) + \theta P_{3/4}(\nabla \Pi, \rho) \]
\[ \lesssim \varepsilon_2^2 (2/3 - \gamma) + \varepsilon_2^2 (2/3 - \gamma)/\lambda + \varepsilon_2^2 (2/3 - \gamma)/2 + \varepsilon_2^2 (2/3 - \gamma)/5 \]
\[ \lesssim \varepsilon_2^2 \rho^{-\beta + 2/3 - \gamma} + \varepsilon_2^2 \rho^{-\beta + (2/3 - \gamma)/\lambda} + \varepsilon_2^2 \rho^{2/3 - \gamma} + \varepsilon_2^2 \rho^{1/3 - 4\gamma/5} \]

(6.12)

since our choice of \( \beta \) satisfies

\[-\beta + (1 - 1/\lambda)(1/6 - \gamma) + (2/3 - \gamma)/\lambda = \beta + 1/3 - 4\gamma/5.\]

All the exponents of \( \rho \) on the right of (6.12) are non-negative if

\[ \gamma \leq \min \left\{ \frac{5(3\lambda - 1)}{18\lambda}, \frac{5(1 + \lambda)}{12\lambda}, \frac{5(1 + \lambda)}{18\lambda} \right\}. \]

Thus, we can choose \( \lambda \) sufficiently close to 1 to obtain \( \gamma \leq 5/9 \) and

\[ E(\theta \rho) + \tilde{D}_1(\theta \rho) \lesssim \varepsilon_2^{1/2}. \]

Hence theorem 2.5 implies that \( \varepsilon = 0 \) is a regular point, namely, \( u \) is Hölder-continuous in a neighborhood of \((0, 0)\).

\[ \square \]

7. Concluding remarks

Making use of recent \( \varepsilon \)-regularity criteria [9] and previous observations in [2, 10, 12, 16, 22], we further derive that the upper box dimension of the possible interior singular points \( S_i \) in the 3D Navier–Stokes equations is at most 7/6. Until now, the gap between Kukavica’s issue [11] (\( \dim_B(S_i) \leq 1 \)) and known result (\( \dim_B(S_i) \leq 7/6 \)) is 1/6 through the effort of researchers in the past decade [9, 10, 12, 16, 22]. We believe that a new iteration technique, new quantities such as the vorticity curl \( u \) introduced in the proof, new \( \varepsilon \)-regularity criteria at one scale and appropriate interpolation inequality may be helpful to fill this gap.

On the other hand, we would like to mention that Scheffer [19] proved that \( \dim_H(S_i) \leq 1 \) is optimal if one just utilizes the local energy inequality and not the Navier–Stokes system. Therefore, a natural question is what is the optimal upper box dimension of \( S_i \).

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