Algebraic approach to renormalization

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I. INTRODUCTION

The idea of eliminating irrelevant modes in order to focus on the dynamics of few selected degrees of freedom has a long history. The Bloch-Feshbach formalism, developed in the late 50’s to describe selected features of nuclear dynamics, employs projection operators in Hilbert space, thereby discarding dynamical information that pertains to the irrelevant modes. The irrelevant modes no longer appear explicitly in the effective theory, yet their residual influence on the dynamics of the remaining modes is taken into account through adjustments of the effective interaction.

In a similar spirit, Wilson’s renormalization group, developed in the early 70’s to better understand critical phenomena, is a mathematical tool that allows one to iteratively eliminate short-wavelength modes and thus to arrive at effective (“renormalized”) theories which describe the dynamics on successively larger length scales. In the original context of second-order phase transitions the renormalization group mainly served to calculate critical exponents, and to provide a satisfactory theoretical explanation for their universality. More recently, it has been pointed out that the renormalization group rationale also affords a natural framework for the physics of interacting fermion systems, and that it helps to understand such diverse phenomena as Landau’s fermi liquid theory, charge-density waves, BCS instability, or screening.

The two calculational tools for systematic mode elimination – on the one hand the Bloch-Feshbach projection technique in Hilbert space, based on algebraic concepts such as linear subspaces and operators, and on the other hand Wilson’s renormalization group, commonly formulated in terms of functional integrals and Feynman diagrams – both derive their power from the fact that they allow one to study accurately selected features of the dynamics (for instance its infrared limit) without ever having to solve the full underlying microscopic theory. Beyond this common root for their success, mounting evidence like the success of Anderson’s “poor man’s scaling” approach to the Kondo problem, Seke’s projection method treatment of the nonrelativistic Lamb shift, or a calculation of the 1-loop renormalization of $\phi^4$ theory with the help of Bloch-Feshbach techniques suggests that the two methods are very closely related, and that in some cases the latter (Wilson) can even be regarded as a special case of the former (Bloch-Feshbach).

It is the purpose of the present paper to propose a slightly modified version of the Bloch-Feshbach formalism as an efficient tool to do renormalization (Sec. I). In the literature it has been put forth that the practical use of such a projection operator approach was limited to simple, essentially one-dimensional problems, and that the study of more complicated physical systems like a Fermi liquid would instead require the use of functional integral methods. Contrary to this assertion, I wish to show how algebraic techniques in Hilbert space can be successfully employed to tackle a variety of non-trivial phenomena in more than one dimension. I shall focus on the application to interacting quantum gases (Sec. II), both Bose and Fermi, and illustrate the versatility of the algebraic approach in deriving four well-known results: the flow of the coupling constant for bosons with point interaction; closely related, the one-loop $\beta$ function of $\phi^4$ theory; the screening of fermion-fermion interactions; and the BCS instability (Sec. III).

Despite the obvious parallels between the Bloch-Feshbach formalism and Wilson’s renormalization, both in spirit and – as I will show – in practical calculations, there remains however an important difference: beyond the elimination of short-wavelength modes, which is common to both approaches, the renormalization program
generally requires the additional step of rescaling. How this additional step, too, can be implemented within the algebraic framework, and under which circumstances it is called for, will be discussed in a separate section (Sec. \[\text{V}\]) at the end of the paper.

II. PROJECTION TECHNIQUE

We consider a macroscopic system (e. g., an interacting quantum gas) whose possible microstates span a \(\sim\)typically large– Hilbert space (e. g., boson or fermion Fock space), and whose dynamics in this Hilbert space is governed by some Hamiltonian \(H\). In equilibrium the state of the system is described by a canonical statistical operator

\[
\rho(b) = \exp[-H/b - \ln Z(b)] \quad b \geq 0
\]

with partition function \(Z(b)\), where \(b\) may represent the temperature \(T\), Boltzmann’s constant \(k_B\), their product \(k_BT\), or possibly some other quantity which we choose to keep explicitly as an external parameter; it will be specified later. All other parameters, such as masses, coupling constants, the chemical potential or – in case \(b \notin \{T, k_BT\}\) – the temperature, are absorbed into the definition of the Hamiltonian. Given the statistical operator as a function of \(b\), the underlying Hamiltonian can be extracted via

\[
b^2 \frac{\partial}{\partial b} \rho(b) = [H + c(b)]\rho(b) \quad (2)
\]

Here \(c(b)\) is a \(c\)-number which stems from the derivative of the partition function. Its specific value need not concern us, as long as we disregard simple shifts of the ground state energy. The partial derivative with respect to the external parameter is taken at fixed masses, couplings, chemical potential, and other parameters in the Hamiltonian.

We presume that we are only interested in certain selected features of the macroscopic system (e. g., its long-wavelength properties), and that these selected features are represented by observables which merely act in some subspace of the original Hilbert space (e. g., in the subspace spanned by many-particle states with all momenta below some cutoff). Let the operator which projects the original Hilbert space onto this selected subspace be denoted by \(P\), and its complement by \(Q = 1 - P\). Information pertaining to the selected degrees of freedom is then entirely encoded in the reduced statistical operator

\[
\rho_{\text{eff}}(b) := P\rho(b)P \quad (3)
\]

the other parts of the original density matrix, \(P\rho Q, Q\rho P\) and \(Q\rho Q\), only carry information which to us is irrelevant.\[\text{II}\]

We would now like to cast the reduced statistical operator again into the canonical form \([\text{I}]\), with some modified, effective Hamiltonian \(H_{\text{eff}}\) which acts in the smaller subspace only (\(H_{\text{eff}} \equiv PH_{\text{eff}}P\)). This effective Hamiltonian will in general acquire a dependence on the parameter \(b\), so we must write

\[
\rho_{\text{eff}}(b) = \exp[-H_{\text{eff}}(b)/b - \ln Z_{\text{eff}}(b)] \quad (4)
\]

We assume, however, that the effective Hamiltonian is a smooth function of \(b\) which can be Taylor expanded around \(b = 0\). For small parameter values we may then set, as a first approximation,\[\text{III}\]

\[
H_{\text{eff}}(b) \approx H_{\text{eff}}(0) \quad (5)
\]

This approximate effective Hamiltonian can be determined in analogy to Eq. \([\text{II}]\), via

\[
b^2 \frac{\partial}{\partial b} \rho_{\text{eff}}(b) \bigg|_{b=0} = [H_{\text{eff}}(0) + c']\rho_{\text{eff}}(0) \quad (6)
\]

again up to some (generally different) \(c\)-number.

In order to evaluate the left-hand side of the equation \([\text{III}]\), we first use \([\text{II}]\) to obtain

\[
b^2 \frac{\partial}{\partial b} \rho_{\text{eff}}(b) = [P\rho + c]\rho_{\text{eff}}(b) + PH\rho(b)P \quad (7)
\]

which includes a new term \(PH\rho(b)P\) that accounts for the overlap between the \(P\)- and \(Q\)-sectors. Assuming that the original Hamiltonian can be decomposed into a free part and an interaction part,

\[
H = H^{(0)} + V \quad (8)
\]

where the free part \(H^{(0)}\) commutes with the projection, this overlap term is \(O(V^2)\) and hence expected to be small for a weak interaction. With the help of the identity

\[
Q\rho(b)P = -Q \int_0^{1/b} d\lambda \frac{d}{d\lambda} \left[ e^{-\lambda H} Qe^{\lambda H} \right] \rho(b)P = -Q \int_0^{1/b} d\lambda e^{-\lambda H}[Q, V]e^{\lambda H} \rho(b)P \quad (9)
\]

we find, to lowest nontrivial order in perturbation theory,

\[
Q\rho(b)P = Q \left[ \exp(-L^{(0)}/b) - 1 \right](QVP) \rho_{\text{eff}}(b) \quad (10)
\]

Here \(L^{(0)}\) denotes the Liouville “super” operator which, when acting on an arbitrary Hilbert space operator \(A\), takes the commutator with the free Hamiltonian:

\[
L^{(0)}A := [H^{(0)}, A] \quad (11)
\]

Since

\[
L^{(0)}|E_i^{(0)}\rangle\langle E_j^{(0)}| = (E_i^{(0)} - E_j^{(0)})|E_i^{(0)}\rangle\langle E_j^{(0)}| \quad (12)
\]

for free energy eigenstates \(|E_i^{(0)}\rangle\}, it may be viewed as probing the energy difference between in- and out-states.
Provided the states in the Q-sector have higher energies than those in the P-sector, the equation \( \mathcal{L}(0) \) has a well-defined limit \( b \to 0 \): simply, \( \exp(-\mathcal{L}(0)/b) \to 0 \). Inserting this limit into Eq. (\ref{eq:hamiltonian}) then yields

\[
H_{\text{eff}}(0) = PHP + \Sigma + \epsilon'' , \quad (13)
\]

with \( \Sigma \) given to second order perturbation theory by

\[
\Sigma^{(2)} = -PVQ \left\{ \frac{1}{\mathcal{L}(0)} \right\} VQPH . \quad (14)
\]

Clearly, the effective Hamiltonian is not just the projection \( PHP \) of the original Hamiltonian, but contains also the extra term \( \Sigma \) to account for the residual influence of the eliminated \( Q \)-modes.

III. APPLICATION TO INTERACTING QUANTUM GASES

A. Mode Elimination

We are interested in the low-temperature properties of interacting quantum gases, and hence in the effective dynamics of low-energy excitations above the many-particle ground state. For noninteracting bosons the ground state has all particles in the lowest energy, zero momentum single-particle mode; while for noninteracting fermions the ground state consists of a filled Fermi sea, with all momentum modes occupied up to some Fermi momentum \( K_F \). (For simplicity, the Fermi surface will always be taken to be spherical.) We will assume that, at least to a good approximation, these essential features of the ground state survive even in the presence of interaction. More specifically, we will assume that in the case of interacting bosons the ground state still has most particles in modes with zero, or at least very small, momentum; and that in the case of interacting fermions there still exists a well-defined Fermi surface. Low-energy excitations then correspond to the promotion of bosons from small to some slightly higher momentum, or of fermions from just below the Fermi surface to just above it. \[ \Lambda(s + \Delta s) := \exp(-\Delta s)\Lambda(s) , \quad \Delta s \geq 0 \] with \( \Delta s \) infinitesimal, thereby discarding from the theory momentum modes pertaining to an infinitesimal shell (in the fermionic case: two shells) of thickness \( \Delta s = \Delta s \Delta s \); determine the effective dynamics of the remaining modes; then eliminate the next infinitesimal shell, again determine the effective dynamics of the remaining modes, and so on. After each infinitesimal step we obtain a new effective Hamiltonian with slightly modified coupling constants. These may also include novel couplings which had not been present in the original theory; in fact, the mode elimination procedure will typically generate an infinite number of such novel couplings. But in many cases only a few coupling constants will change appreciably and thus suffice to study the physical system at hand. How these coupling constants evolve as the flow parameter \( s \) increases and hence the cutoff \( \Lambda(s) \) approaches zero, can then be described by a small set of coupled differential equations. Modulo rescaling, which I shall discuss separately in Sec. III, these are the renormalization group equations of the theory.

Whether or not the temperature is among the quantities that flow as a function of \( s \), depends on whether it has been included in the definition of the external parameter \( b \). If \( b \) is chosen to be \( k_B \) or some other temperature-independent quantity then the temperature may flow, like all variables which have been absorbed into the definition of the Hamiltonian. If \( b = T \) or \( b = k_B T \), on the other hand, then the temperature is regarded as a parameter that is controlled externally and hence fixed: it does not change upon mode elimination. While the former scenario applies to the study of isolated systems which, depending on the modes selected, may exhibit effective dynamics at varying apparent temperatures, the latter scenario is adapted to the study of systems which are coupled to a heat bath of prescribed temperature. It is this latter case which we shall consider, as we wish to study the effective dynamics in the low-temperature limit \( T \to 0 \) and thus explicitly control the temperature. Consequently, we choose \( b = k_B T \).

Our aim is now to illustrate in a few prominent cases—bosons with point interaction, \( \phi^4 \) theory, screening of fermion-fermion interactions, BCS instability—how the applicable renormalization group equations can be derived efficiently with the help of our projection technique. To this end we must specify the appropriate many-particle Hilbert space (Fock space) and the appropriate projection operator for each infinitesimal elimination step.

At a given cutoff \( \Lambda \) the boson Fock space is spanned by the particle-free vacuum \( |0_b\rangle \) and all \( n \)-particle states \((n = 1 \ldots \infty)\)

\[
|k_1 \ldots k_n\rangle \propto \prod_{i=1}^{n} a^\dagger(k_i) |0_b\rangle , \quad |k_i| \leq \Lambda , \quad (16)
\]

where the \( \{k_i\} \) denote the particle momenta and \( \{a^\dagger(k_i)\} \) the associated bosonic creation operators.
The fermion Fock space, on the other hand, is spanned by the filled Fermi sea (fermionic vacuum) $|0\rangle$ and all its excitations which have particles above the Fermi surface and/or missing particles (“holes”) below it, all within a shell of thickness $2\Lambda$. In order to cast this into a mathematical formulation it is convenient to change coordinates, from the true particle momenta $\{K_i\}$ to little (“quasiparticle”) momenta

$$k_i := (|K_i| - K_F)\hat{K}_i$$

and additional discrete labels

$$\sigma_i := \text{sign}(|K_i| - K_F) \ .$$

This coordinate transformation $K \rightarrow (k, \sigma)$ is invertible except for modes which lie exactly on the Fermi surface. States above the Fermi surface are labeled $\sigma = 1$, while those below are labeled $\sigma = -1$. The allowed excitations in fermion Fock space then have the form $(n = 1 \ldots \infty)$

$$|k_1^\pm \ldots k_n^\pm \rangle \propto \prod_{i=1}^n \theta(\sigma_i) a^\dagger(k_i, \sigma_i) + \theta(-\sigma_i) a(-k_i, \sigma_i) |0\rangle, \quad |k_i^\pm| \leq \Lambda \ ,$$

where the $\{k_i^\pm\}$ denote the momenta of particles ($+$) or holes ($-$), respectively, and $\{a^\dagger\}, \{a\}$ the associated fermionic creation and annihilation operators. For simplicity, we have omitted any spin quantum numbers. It is now obvious which form the projection operator will have that is associated with the infinitesimal cutoff reduction (14): if applied to any of the excitations (16) or (17) it will simply multiply the respective state by a product of $\theta$ functions, $\prod_i \theta(\Lambda - e^{\Delta s}[k_i^\pm])$, to enforce the new cutoff constraint.

B. Hamiltonian

We consider interacting quantum gases whose dynamics in the original, full Hilbert space is governed by a Hamiltonian of the form

$$H = \sum_k \varepsilon_k :a_k^\dagger a_k:+ \frac{1}{4} \sum_{ijkl} \langle lk|V|ji\rangle_{\pm} :a_i^\dagger a_k^\dagger a_j a_l:$$

$$= H^{(0)} + V \ ,$$

with kinetic energy $H^{(0)}$ and a two-body interaction $V$. By definition the single-particle energies $\varepsilon_k$ include the chemical potential. The annihilation and creation operators obey

$$[a_i, a_j^\dagger]_{\pm} = \delta_{ij} \quad \text{for bosons (upper sign) or fermions (lower sign), respectively.}$$

Each term in the Hamiltonian is normal ordered ($:\ldots:) \text{ with respect to the many-particle ground state.}$

In the bosonic case this just coincides with the usual normal ordering: all annihilation operators to the right, all creation operators to the left. The explicit normal ordering of the Hamiltonian is then redundant. In the fermionic case, on the other hand, normal ordering means shuffling all operators which annihilate the fermionic vacuum ($a_i$ with $\sigma_i = 1$ or $a_i^\dagger$ with $\sigma_i = -1$) to the right, all others ($a_i^\dagger$ with $\sigma_i = 1$ or $a_i$ with $\sigma_i = -1$) to the left, thereby changing sign depending on the degree of the permutation. The explicit normal ordering of the Hamiltonian then becomes nontrivial. In both cases the normal ordering ensures that the energy of the (bosonic or fermionic) vacuum is set to zero,

$$(0_{b,f} | H | 0_{b,f} ) = 0 \ .$$

C. Modification of the 2-Body Interaction

As we discussed earlier, each mode elimination will yield an effective Hamiltonian that will generally contain a slightly altered mass, chemical potential, two-body interaction, etc., and possibly new interactions such as an effective three-body interaction. Here we shall restrict our attention to the modification of the two-body interaction. In order to determine this modification we must consider the general formula (13) for the effective Hamiltonian. As we have seen, the effective Hamiltonian is the sum of the projected original Hamiltonian, $PHP$, and an extra term $\Sigma$ that accounts for the residual influence of the eliminated modes. For our purposes the projection $PHP$ of the original Hamiltonian need not concern us, as it will not lead to any modification of the two-body interaction. Rather, we must investigate the ramifications of the extra term $\Sigma$.

Application of the perturbative result (14) yields:

$$\Sigma^{(2)} = \frac{-1}{16} \sum_{abcdijkl} \langle lk|V|ji\rangle_{\pm} \langle dc|V|ba\rangle_{\pm} P :a_i^\dagger a_k^\dagger a_j a_l: + Q 1_{L(0)} Q :a_i^\dagger a_k^\dagger a_l a_j: P \ .$$

The two projectors $P$ at both ends of the operator product ensure that all external momenta lie below the new, reduced cutoff, whereas the projectors $Q$ in the center force at least one internal momentum to lie in the infinitesimal shell which has just been eliminated. Therefore, at least one pair of field operators must pertain to the eliminated $Q$-modes, and hence be contracted: $a^{(i)} a^{(l)} \rightarrow (0_{b,f}) a^{(i)} a^{(l)} |0_{b,f}\rangle$. The product of the remaining six field operators can then be rearranged with the help of Wick’s theorem, to yield a decomposition

$$\Sigma^{(2)} = \Sigma_{0}^{(2)} + \Sigma_{1}^{(2)} + \Sigma_{2}^{(2)} + \Sigma_{3}^{(2)} \quad \text{where each } \Sigma^{(2)} \text{ is normal ordered and contains } n \text{ field operators.}$$

The various terms shift the ground state energy ($n = 0$), modify the mass, the chemical potential,
or more generally the form of the single-particle dispersion relation \((n = 2)\), modify the two-body interaction \((n = 4)\), and generate a new effective three-body interaction \((n = 6)\).

Since we want to focus on the modification of the two-body interaction we consider only the term with \(n = 4\). Calculating this term from Eq. \((23)\) involves two contractions, at least one of which must pertain to the eliminated Q-modes (see above), and both of which must go across the central \(Q(1/L(0))Q\) (due to momentum conservation). Neglecting the energy of the external modes, we find for bosons

\[
\Delta \langle k|V|ji\rangle_+ = -\Delta \left[ \sum_{a,b} \frac{1}{2(\epsilon_a + \epsilon_b)} \langle lk|V|ba\rangle_+ \langle ab|V|ji\rangle_+ \right]
\]

and for fermions

\[
\Delta \langle k|V|ji\rangle_- = \Delta_{klji}^{(ZS)} + \Delta_{klji}^{(ZS')} + \Delta_{klji}^{(BCS)}
\]

with

\[
\Delta_{klji}^{(ZS)} = -\Delta \left[ \sum_{a,b} \frac{\theta(\sigma_a)\theta(-\sigma_b) - \theta(-\sigma_a)\theta(\sigma_b)}{\epsilon_a - \epsilon_b} \right] \langle la|V|bi\rangle_- \langle bk|V|ja\rangle_- ,
\]

its cross term

\[
\Delta_{klji}^{(ZS')} = -\Delta_{klji}^{(ZS)}
\]

and

\[
\Delta_{klji}^{(BCS)} = -\Delta \sum_{a,b} \frac{\theta(\sigma_a)\theta(\sigma_b) - \theta(-\sigma_a)\theta(-\sigma_b)}{2(\epsilon_a + \epsilon_b)} \langle lk|V|ba\rangle_- \langle ab|V|ji\rangle_- .
\]

The \(\Delta\) in front of the sums signifies that at least one of the internal modes \((a, b)\) must lie in the eliminated shell. In the bosonic case the modification of the two-body interaction can be associated with a 1-loop “ladder” diagram; in the fermionic case, on the other hand, there are three distinct contributions which, with hindsight, can be identified with “zero sound” (ZS,ZS’) and BCS diagrams.\(^{11}\) The ZS contribution and its cross term ZS’ account for particle-hole excitations \((\sigma_a = \pm 1, \sigma_b = \mp 1)\), while the BCS term describes 2-particle \((\sigma_a = \sigma_b = +1)\) or 2-hole \((\sigma_a = \sigma_b = -1)\) excitations.

IV. EXAMPLES

A. Bosons with point interaction

For spinless bosons with point interaction (\(\delta\) function potential in real space) it is

\[
\langle lk|V|ji\rangle_+ = \frac{2U}{\Omega} \delta_{kl+ki,k_j+k_l},
\]

with the Kronecker symbol enforcing momentum conservation, \(\Omega\) being the spatial volume, and \(U\) the coupling constant. Provided the magnitude of the external momenta \(k_j, k_l\) is negligible compared to the cutoff \(\Lambda\), momentum conservation implies that the internal modes \(a, b\) must both lie in the eliminated shell, and hence \(\epsilon_a = \epsilon_b = \epsilon_\Lambda\). Application of the general formula \((25)\) then yields

\[
\Delta U = -\frac{U^2}{2\Omega\epsilon_\Lambda} \Delta \left[ \sum_{|k_a|,|k_b|\leq \Lambda} \delta_{k_a+k_b,k_j+k_l} \right],
\]

where the sum

\[
\Delta \left[ \sum_{|k_a|,|k_b|\leq \Lambda} \delta_{k_a+k_b,k_j+k_l} \right] \approx \sum_{|k_a|\geq |\Lambda-\Delta\Lambda,\Lambda|} 1
\]

simply counts the number of eliminated states. For a spherical cut in momentum space this number of states is given by

\[
\sum_{|k_a|\geq |\Lambda-\Delta\Lambda,\Lambda|} 1 = \rho(\epsilon_\Lambda) \frac{d\Lambda}{d\Delta\Lambda} \Delta\Lambda ,
\]

with \(\rho(\epsilon_\Lambda)\) denoting the density of states at the cutoff. With \(\Delta\Lambda = \Lambda \Delta s\) we thus obtain the flow equation

\[
\Delta U = -\frac{d\ln \rho(\epsilon_\Lambda)}{d\ln \Lambda} \frac{\rho(\epsilon_\Lambda)}{2\Omega} U^2 \cdot \Delta s .
\]

For a dilute gas of nonrelativistic bosons in three spatial dimensions, with mass \(m\), dispersion relation \(\epsilon_\Lambda = \Lambda^2/2m\) and density of states \(\rho(\epsilon_\Lambda) = \Omega m\Lambda/2\pi^2\) the flow equation reduces to

\[
\Delta U = -\frac{m\Lambda}{2\pi^2} U^2 \cdot \Delta s .
\]

By its very definition the sequence of effective theories retains complete information about the system’s low-energy dynamics. Observables pertaining to this low-energy dynamics are therefore unaffected by the successive mode elimination, and hence independent of \(s\). For example, the scattering length\(^{12}\)

\[
a = \frac{m}{4\pi} \left[ U(s) - U(s)^2 \int_{|p|\leq \Lambda(s)} \frac{d^3p}{(2\pi)^3} \frac{m}{p^2} \right]
\]

stays constant under the flow \((28)\), as the \(s\)-dependence of the parameters \(U\) and \(\Lambda\) just cancels out (up to third order corrections).
B. The link to $\phi^4$ theory

There is an interesting relationship between the result [24] and the 1-loop $\beta$ function for real $\phi^4$ theory. The $\phi^4$ Hamiltonian describes the dynamics of coupled anharmonic oscillators. It reads, in three spatial dimensions, $H = H^{(0)} + V$ with kinetic energy

$$H^{(0)} = \frac{1}{2} \int d^3x \left[ \pi(x)^2 + |\nabla \phi(x)|^2 + m^2 \phi(x)^2 \right] = \sum_k \epsilon_k \phi_k^\dagger \phi_k$$

and interaction

$$V = \frac{g}{4} \int d^3x \phi(x)^4 = \frac{g}{4!} \sum_k \prod_{i=1}^4 \frac{1}{\sqrt{2\epsilon_k}} (\phi_k + \phi_k^\dagger) \delta \sum_{k,0} \epsilon_k .$$

Here $m$ denotes the mass, $\Omega$ the spatial volume, $g$ the coupling constant, and $\epsilon_k$ the single-particle energy

$$\epsilon_k = \sqrt{k^2 + m^2} .$$

The field $\phi$ and its conjugate momentum $\pi$ are time-independent (Schrödinger picture) operators which satisfy the commutation relations for bosons, and $\phi, \phi^\dagger$ are the associated annihilation and creation operators. While the kinetic part of the Hamiltonian is normal ordered (:. . .), the interaction is not.

When expressed in terms of annihilation and creation operators the Hamiltonian takes on a form which is very similar to that of the quantum gas Hamiltonian [20]. More precisely, the $\phi^4$ Hamiltonian contains a Bose gas Hamiltonian with two-body interaction matrix element

$$\langle lk|V|ji \rangle = \frac{g}{4!} \delta_{k_i, k_j} \delta_{k_l, k_k} \delta_{k_l, k_k} \delta_{k_l, k_k} \delta_{k_l, k_k} .$$

The derivation of a flow equation for $g$ can now proceed in the same vein as that for $U$, again starting from Eq. [23]. Now, however, apart from 2 → 2 particle scattering the $\phi^4$ Hamiltonian with its novel interactions $a^\dagger a^\dagger a^\dagger a^\dagger$ etc. also permits 2 → 4 and 2 → 6 scattering. Therefore, in Eq. [23] the intermediate state may be not just $|ab\rangle$, but also $|ab\rangle$, $|ahl\rangle$, $|abj\rangle$, $|abjl\rangle$. As long as the magnitude of the external momenta is negligible compared to the cutoff, it is in all six cases $\epsilon_a = \epsilon_b = \epsilon_A$ and

$$\langle lk|V|ji \rangle = \frac{g}{4! \epsilon_A^2} \delta_{k_l, k_j} \delta_{k_i, k_k} \delta_{k_l, k_i} \delta_{k_i, k_k} ,$$

Hence, in order to account for the larger set of allowed intermediate states we merely have to introduce an extra factor 6, and obtain thus

$$\Delta g = - \frac{d \ln \epsilon_A}{d \ln \Lambda} \frac{3 \rho(\epsilon_A)}{8 \Omega \epsilon_A^2} g^2 \Delta s .$$

For $\Lambda \gg m$ it is $\epsilon_A = \Lambda$, $\rho(\epsilon_A) = \Omega \epsilon_A^2 / 2\pi^2$, and the flow equation reduces to

$$\Delta g = - \frac{3g^2}{16\pi^2} \Delta s ,$$

in agreement with the well-known 1-loop result for the $\beta$ function of $\phi^4$ theory [26].

C. Screening of fermion-fermion interactions

We consider nonrelativistic fermions in spatial dimension $d$ ($d \geq 2$) which interact through a two-body interaction

$$\langle lk|V|ji \rangle = [V(q)\delta_{s_l,s_i} \delta_{s_k,s_j} - V(q')\delta_{s_k,s_i} \delta_{s_l,s_j}] \times \delta_{k_i, k_j} + k_j + k_i .$$

duly antisymmetrized to account for Fermi statistics, and with $\{s_a\}$ denoting the spin quantum numbers and $q, q'$ the respective momentum transfers

$$q := K_l - K_i = K_j - K_k ,$$

$$q' := K_j - K_i = K_k - K_l .$$

We investigate scattering processes for which

$$0 < |q|, \Lambda < |q'|, |K_i + K_j|, K_F .$$

In this regime only the ZS contribution [27] can significantly modify the two-body interaction; its cross term ZS’ (Eq. [28]), as well as the BCS contribution [28], are suppressed by a factor $\Lambda / K_F$. This can be seen directly from the geometry of the Fermi surface. The three constraints on the intermediate state: (i) both $K_a$ and $K_b$ lie in the cutoff shell of thickness $2\Lambda$; (ii) more stringently, one of them lie in the infinitesimal shell to be eliminated; and (iii) $K_a - K_b = -q' (ZS')$ or $K_a + K_b = K_i + K_j$ (BCS), respectively — reduce the momentum space volume available to the internal momentum $K_a$ to $O(K_F^{-d-2} \Delta \Lambda)$. In contrast, for $|q| \sim \Lambda$ the ZS contribution with its condition $K_a - K_b = -q$ allows a momentum space volume of the order $K_F^{-d-1} \Delta \Lambda$.

To evaluate the ZS contribution at some given momentum transfer $q$, we first define the angle $\theta$ between $-q$ and the internal momentum $K_a$,

$$\cos \theta := - \frac{q \cdot K_a}{|q||K_a|} ,$$

change coordinates from original $(K)$ to little $(k)$ momenta, and write, up to corrections of order $|q|/K_F$,

$$\epsilon_a - \epsilon_b = v_F(|K_a| - |K_b|) = v_F(\sigma_a|K_a| - \sigma_b|K_b|) = v_F|q|z$$

(48)
with $v_F$ denoting the Fermi velocity. Next we note that the term with $\theta(\sigma_a)\theta(-\sigma_b)$ and the term with $\theta(-\sigma_a)\theta(\sigma_b)$ yield identical contributions; therefore it suffices to consider only the first term and then multiply it by two. Finally, assuming that in the interaction matrix element (44) the cross term is negligible,

$$|V(q')| \ll |V(q)|,$$

(49)

the two matrix elements in Eq. (27) can simply be replaced by $V(q)^2$ modulo Kronecker symbols for spin and momentum conservation. By virtue of these Kronecker symbols one of the two summations over internal modes collapses trivially, leaving

$$\Delta V(q) = -\frac{2}{v_F} \Delta \sum_a \theta(\Lambda - |k_a|) \theta(|k_a| - |q|z + \Lambda) \theta(\sigma_a) \theta(|q|z - |k_a|) \cdot V(q)^2 .$$

(50)

Here the first two $\theta$ functions explicitly enforce the sharp cutoff constraint for both $k_a$ and $k_b$ ($|k_a|, |k_b| \leq \Lambda$), while the latter two $\theta$ functions enforce $\sigma_a = 1$ and $\sigma_b = -1$, respectively. Under these constraints it is always $\theta \in [0, \pi/2)$ and hence $z > 0$.

The above equation can be immediately integrated from cutoff $\Lambda \gg |q|, |k_a|$ (symbolically, $\Lambda \to \infty$) down to $\Lambda \ll |q|, |k_a|$ (symbolically, $\Lambda \to 0$), to yield the total modification of the two-body interaction:

$$\frac{1}{V_{\text{eff}}(q)} - \frac{1}{V_{\text{bare}}(q)} = \frac{2}{v_F} \sum_a \theta(\Lambda - |k_a|) \theta(|k_a| - |q|z + \Lambda) \theta(\sigma_a) \theta(|q|z - |k_a|) \int_0^\infty \frac{\delta(\Lambda - |q|z - |k_a|) \cdot V(q)^2}{|q|z} .$$

(51)

At the lower bound ($\Lambda \to 0$) the various conditions imposed by the $\theta$ functions cannot all be satisfied simultaneously, and therefore the product of $\theta$ functions vanishes. At the upper bound ($\Lambda \to \infty$), on the other hand, the cutoff constraints imposed by the first two $\theta$ functions are trivially satisfied and thus can be omitted. In this case the sum over $a$ is evaluated by turning it into two integrals, one over a radial variable such as $|k_a|$ or $\epsilon_a$, the other over the solid angle. At a given solid angle and hence given $z$, the fourth $\theta$ function restricts the radial integration to the range $|k_a| \in [0, |q|z]$ or, equivalently, $\epsilon_a \in [0, v_F|q|z]$. This energy interval in turn corresponds to a number $|\rho(\epsilon_F)v_F| |q|z]$ of states, $\rho(\epsilon_F)$ being the density of states at the Fermi surface. (It includes a factor to account for the spin degeneracy.) The integration over the solid angle is constrained to a semi-sphere, due to $\theta \in [0, \pi/2)$, and hence reduced by a factor 1/2 as compared to a full-sphere integration. Altogether we obtain

$$\frac{1}{V_{\text{eff}}(q)} - \frac{1}{V_{\text{bare}}(q)} = \frac{2}{v_F} \frac{1}{\Delta} \rho(\epsilon_F) v_F |q|z \int_0^\infty \frac{\delta(\Lambda - |q|z - |k_a|) \cdot V(q)^2}{|q|z} = \rho(\epsilon_F)$$

(52)

and thus

$$V_{\text{eff}}(q) = \left[ \frac{1}{V_{\text{bare}}(q)} + \rho(\epsilon_F) \right]^{-1} .$$

(53)

This result describes the well-known screening of fermion-fermion interactions.

D. BCS instability

Our last example pertains to fermions with an attractive pairing interaction

$$\langle lk|V|ji\rangle_\sigma = -V \delta_{K_j,-K_i} \delta_{K_i,-K_k} \delta_{\epsilon_{K_i}s_i,\epsilon_{K_k}s_j} - \delta_{\epsilon_{K_i}s_i,\epsilon_{K_k}s_j} ,$$

(54)

which is the simplest form of BCS theory. Due to the pairing condition $K_j = -K_i$, $K_i = -K_k$ it is impossible to satisfy in the ZS and ZS’ terms the requirement that at least one of the internal modes be in the eliminated shell. Hence only the BCS term (29) can modify the coupling constant. In the BCS term there are contributions with $\theta(\sigma_a)\theta(\sigma_b)$ and $\theta(-\sigma_a)\theta(-\sigma_b)$, respectively, which yield identical results; therefore it suffices to consider only the first contribution and then multiply it by two. The pairing condition implies $\epsilon_a = \epsilon_b = \epsilon_A$, which for modes in the upper $\sigma = +1$ eliminated shell is given by $\epsilon_A = v_F\Lambda$. The eliminated shell itself covers an infinitesimal energy interval of width $[v_F\Lambda \Delta s]$, which in turn corresponds to a number $|\rho(\epsilon_F)v_F\Lambda \Delta s|$ of states. Of the two summations over internal modes one collapses trivially due to momentum and spin conservation, leaving

$$\Delta V = \frac{V^2}{2v_F\Lambda} \Delta \sum_a \theta(\sigma_a)$$

$$= \frac{V^2}{2v_F\Lambda} \rho(\epsilon_F) v_F \Lambda \Delta s = \frac{\rho(\epsilon_F)}{2} V^2 \Delta s .$$

(55)

From this flow equation for the BCS coupling $V(s)$ we immediately conclude that as long as the initial coupling $V(0)$ is positive, $V(s)$ diverges as $s \to \infty$. This indicates the occurrence of binding (“Cooper pairs”) at very low temperatures. Furthermore, we can again convince ourselves that the sequence of effective theories retains complete information about the system’s low-energy dynamics: low-energy observables such as the zero-temperature gap $\Delta_{0}$ do not depend on the flow parameter $s$ and are thus unaffected by the successive mode elimination.
V. RESCALING

So far we have only considered the systematic elimination of modes. Yet there is a second operation, namely rescaling, which in the context of renormalization plays an equally important role. In this section we shall first discuss the rescaling operation by itself, and later its interplay with the elimination of modes.

Again we would like to couch the rescaling procedure into the language of linear operators in Hilbert space. To this end we define a unitary dilatation operator $D(s)$ which, if acting on a (box-normalized) boson excitation $|0\rangle$ or fermion excitation $|\Psi\rangle$, simply rescales all momenta by $e^s$.

$$D(s)|k_1^{(\pm)} \ldots k_n^{(\pm)}\rangle := |e^s k_1^{(\pm)} \ldots e^s k_n^{(\pm)}\rangle \quad \forall s \geq 0.$$  (57)

This operation can be viewed in two different ways, analogous to the active and passive interpretations of coordinate transformations: either as an enlargement of the physical momenta at fixed length unit (active interpretation), or as an enlargement of the length unit at fixed physical momenta (passive interpretation). The latter view might be pictured as the observer with a television camera “zooming out” away from the probe, thereby watching on the camera’s monitor a larger section of the probe and excitations with apparently smaller wavelengths.

Let the state of a physical system be described by a canonical statistical operator

$$\rho(b) \propto \exp[-H(\bar{\Omega})/b] \quad ,$$  (58)

where we have written explicitly the possible dependence of the Hamiltonian on a finite volume $\Omega$. After “zooming out” the canonical distribution will appear distorted, namely as

$$\rho_{zoom}(b, s) := D(s)\rho(b)D^\dagger(s) \quad .$$  (59)

We wish to cast this zoomed distribution again into a canonical form,

$$\rho_{zoom}(b, s) \propto \exp[-H_{zoom}(e^{-d+s}\bar{\Omega}, s)/b(s)] \quad ,$$  (60)

with a modified external parameter $b(s)$ and some modified, “zoomed” Hamiltonian $H_{zoom}$. Since the rescaling makes the system appear smaller on the observer’s (imaginary) monitor, the modified Hamiltonian must now depend on the rescaled volume $e^{-d+s}\bar{\Omega}$ ($d$ being the spatial dimension) rather than on $\bar{\Omega}$. The thus defined zoomed Hamiltonian is easily determined:

$$H_{zoom}(\Omega, s) = \frac{b(s)}{b} D(s)H(e^{+d+s}\Omega)D^\dagger(s) \quad .$$  (61)

In general, the $s$-dependence of the external parameter $b$ is not a priori known and must be fixed by additional constraints. One such constraint might be the requirement that some given reference Hamiltonian $H_{ref}$ be scale invariant, $H_{zoom}^{ref} = H_{ref}$.

Just like the effective Hamiltonian obtained via mode elimination, the zoomed Hamiltonian generally contains modified masses, chemical potential and coupling constants; however, it does not contain any qualitatively new interactions. How the various parameters evolve as the flow parameter $s$ increases, is again governed by a set of differential equations, the so-called scaling laws.

To illustrate these general ideas we re-consider our first example, bosons with point interaction. Making use of the identity

$$D(s)a^{(l)}(k)D^\dagger(s) = a^{(l)}(e^s k)$$  (62)

for annihilation and creation operators, we find for the free part of the Hamiltonian

$$H_{zoom}^{(0)}(\Omega, s) = \sum_k \left[ \frac{b(s)}{b} e(-s k) \right] a^\dagger(k)a(k) \quad ,$$  (63)

and for the interaction

$$V_{zoom}(\Omega, s) = \frac{e(-s) b(s)}{b} V(\Omega) \quad .$$  (64)

Taking $b = k_B T$, assuming a relativistic single-particle energy

$$\epsilon(k) = \sqrt{k^2 + m^2} - \mu$$  (65)

with mass $m$ and chemical potential $\mu$, and requiring that in the limit $m = \mu = 0$ the free dynamics become scale invariant ($H_{zoom}^{(0)} = H^{(0)}$), we obtain the scaling relations

$$T(s) = e^{s} T \quad , \quad m(s) = e^s m \quad , \quad \mu(s) = e^s \mu$$  (66)

which in turn imply

$$U(s) = e^{-1/(d+1)s} U \quad .$$  (67)

This last result immediately translates into a scaling law for the coupling constant $g$ of $\phi^4$ theory. The coupling $g$ differs from $U$ essentially by a factor $\sqrt{\epsilon_{ij}^l \epsilon_{kj}^l}$, which just leads to an extra factor $e^{2s}$ in the scaling:

$$g(s) = e^{(4/(d+1))s} g \quad .$$  (68)

As expected, the number $\epsilon := 4 - (d + 1)$ in the exponent is the deviation of the space-time dimension from the upper critical dimension four $d_c$.

The combination of rescaling and mode elimination constitutes a complete renormalization group transformation. In the by now familiar spirit we define the renormalized statistical operator

$$\rho_{ren}(b, s) := D(s)P(s)\rho(b)P(s)D^\dagger(s) \quad ,$$  (69)
which is obtained by first eliminating high momentum modes in the shell \([e^{-\Lambda} \Lambda, \Lambda]\) and then rescaling all momenta so as to recover the original cutoff. In our intuitive picture this transformation corresponds to first coarse-graining the physical spatial resolution and then “zooming out,” thereby rescaling the length unit such that the apparent spatial resolution, defined in rescaled rather than original length units, stays constant. (One might visualize the apparent spatial resolution as the fixed pixel size of the observer’s monitor, which, as the camera zooms out, corresponds to ever worse physical resolutions.) As in the previous cases we wish to cast the renormalized statistical operator into the canonical form

\[
\rho_{\text{ren}}(b, s) \propto \exp\left[-H_{\text{ren}}(e^{-d_s \tilde{\Omega}}, b(s), s)/b(s)\right] 
\]

with modified external parameter \(b(s)\) and renormalized Hamiltonian \(H_{\text{ren}}\). How the various parameters in the renormalized Hamiltonian (masses, couplings, etc.) evolve under the combined action of mode elimination and rescaling, is governed by the full renormalization group equations. These can be obtained by adding the respective flows due to mode elimination and rescaling.

The rescaling part of the full renormalization group equations is specifically adapted to the study of critical phenomena. Critical phenomena are distinguished by the lack of an intrinsic length scale: as the observer is “zooming out,” at fixed apparent resolution, he keeps seeing the long-wavelength dynamics governed by the same masses and coupling constants. In other words, the observer has no means of inferring from these parameters the scale at which the dynamics is taking place. Critical phenomena are therefore naturally associated with a fixed point—the “critical” fixed point—of the full renormalization group equations.

If one studies effects other than critical phenomena, such as screening or the BCS instability, then there is generally no reason for a repeated rescaling.

VI. CONCLUSION

In this paper we have formulated the renormalization program exclusively in terms of algebraic operations in Hilbert space. This new formulation has given us the opportunity to expose the close relationship between Wilson’s renormalization and the projection technique à la Bloch and Feshbach, and to consider some of the ideas underlying renormalization from a slightly different conceptual perspective. Aside from these formal developments we have also explored the potential for practical uses of the algebraic approach, and found that in four examples—bosons with point interaction, \(\phi^4\) theory, screening of fermion-fermion interactions, and BCS instability—the flow equations for the respective couplings could be derived in an efficient and straightforward manner.

The algebraic approach carries the potential for interesting future developments. First, it is easily generalized since projection operators permit the elimination not just of short-wavelength modes, but also of other kinds of information deemed irrelevant, such as high angular momenta, spin degrees of freedom, or entire particle species. Secondly, the algebraic approach builds a bridge between renormalization and macroscopic transport theories, as also the transition from micro- to macrodynamics can be effected by means of very similar projection techniques. It may thus provide a natural language for the study of issues such as the renormalization of macroscopic transport theories, or effective kinetic theories.

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