Alice without quaternions: another look at the mad tea-party

ANNE VAN WEERDEN

Utrecht University Library, PO Box 80124, 3508 TC Utrecht, The Netherlands

Ever since the publication of Lewis Carroll’s *Alice’s adventures in Wonderland* (1866), interpretations of its apparent nonsense have been given. In 2009, Melanie Bayley added new interpretations, for instance, that the chapter about the mad tea-party mocked the quaternions of Sir William Rowan Hamilton. In 2017, Francine Abeles argued against Bayley’s quaternion interpretation of the tea-party, and these arguments will be supported and extended by showing that Bayley’s interpretation is based on erroneous assumptions about quaternions. It can be concluded that it is indeed very unlikely that Dodgson had the quaternions in mind when writing the tea-party chapter.

1. Introduction

In 1866, the mathematician Charles Lutwidge Dodgson (1832–98), alias Lewis Carroll, published the famous fantasy novel *Alice’s adventures in Wonderland*, followed in 1872 by *Through the looking glass, and what Alice found there*. Many interpretations have been given of the apparent nonsense in the *Alice* books, to which in the 1980s Helena Pycior added mathematical ones (Pycior 1984).

Algebra used to obey the laws of arithmetic, but by Dodgson’s time mathematicians had been stirring up the absolute truths of mathematics. Already before the start of the nineteenth century, imaginary and negative numbers had been introduced, and in 1830 George Peacock (1791–1858) published *A treatise on algebra* (Peacock 1830), about a symbolical approach to algebra which admitted undefined entities. In 1843, Sir William Rowan Hamilton (1805–65) found the quaternions, which were non-commutative; it was the first time that one of the laws of arithmetic appeared not to be universally true.

Dodgson could not accept some of these new developments. Showing that, in early nineteenth-century England, criticisms of mathematics had been given humorously, Pycior (1984, 157) argued that through his *Alice* books Dodgson also used humour to ridicule these developments. She further suggested that symbolic algebra in particular had inspired Dodgson to write the *Alice* books:

it was with visions of lines multiplied by lines, quantities less than nothing, and symbolic algebra dancing in his head that Dodgson first set about constructing...
an other-world (the underground) in which meaninglessness and arbitrariness prevailed (Pycior 1984, 163).

In 2009, Melanie Bayley published an article in *New Scientist*, ‘Alice’s adventures in algebra: Wonderland solved’ (Bayley 2009). One of Bayley’s suggestions was that in the chapter about the mad tea-party (Carroll 1866, 95–111) Dodgson mocked Hamilton’s quaternions. In 2017, Francine Abeles cast doubt on this interpretation, in an article called ‘On the truth of some new mathematical ideas in *Alice’s adventures in Wonderland*’ (Abeles 2017).

I asked [Bayley] why she invoked Hamilton in these interpretations of *AAIW* despite the knowledge that there was no evidence that Carroll knew anything about his work. She gave as her reason that ‘During Carroll’s time, Hamilton was in the air.’ However, from an historiographic point of view that is not a valid reason to make such a connection (Abeles 2017, 14).

Abeles’ objection will be extended here by showing why Bayley’s quaternion interpretation was erroneous.

2. The tea-party and erroneous assumptions

In chapter VII of *Alice’s adventures in Wonderland*, Alice meets the Hatter, the March Hare and the Dormouse, sitting at a table set for tea-time at six o’clock (Carroll 1866, 95–111) (Figure 1). The character Time is not present, but because he stopped the clock at six it is always tea-time, and the characters at the table do not have time to
‘wash the things between whiles’; they ‘keep moving round’ ‘as the things get used up’ (Carroll 1866, 104).

Bayley’s quaternion interpretation of the tea-party chapter was motivated by non-commutativity in one of the conversations, and by the rotations of the characters at the table. It will be shown here, however, that non-commutativity does not necessarily imply knowing about quaternions, and that Bayley’s view on quaternions and rotations was flawed.

**Non-commutativity** Hamilton had been searching, intermittently, for an extension of the imaginary or complex numbers to describe three-dimensional space. With \( a \) and \( b \) real numbers and \( i = \sqrt{-1} \), the complex numbers \( a + ib \) consist of one real and one imaginary term. They can be used to describe a plane, and what was expected for three-dimensional space were triplets, consisting of one real and two imaginary terms: \( a + ib + jc \). But Hamilton could not find a suitable multiplication for the triplets, and in 1843 he discovered that he had to add yet another imaginary term yielding four-term sets, \( a + ib + jc + kd \), which he called quaternions.\(^2\) The quaternions formed an algebra, but one without commutativity.

Bayley argued that this non-commutativity can be recognized in the conversations at the table:

Alice’s answers are equally non-commutative. When the Hare tells her to ‘say what she means,’ she replies that she does, ‘at least I mean what I say – that’s the same thing.’ ‘Not the same thing a bit!’ says the Hatter. ‘Why, you might just as well say that ‘I see what I eat’ is the same thing as ‘I eat what I see’!’ (Bayley 2009), (Carroll 1866, 98).

Yet in 1858, fifteen years after the discovery of the quaternions, Arthur Cayley (1821–95) had published a ‘A memoir on the theory of matrices’ in which he treated square matrices and their determinants. Showing the ‘rule for the multiplication or composition of two matrices’ he remarked, ‘it is to be observed, that the operation is not a commutative one’ (Cayley 1858, 21).

Clearly, in the early 1860s, when Dodgson was writing his first *Alice* book, non-commutativity was no longer the exclusive preserve of the quaternions. Moreover, in 1867 Dodgson himself published an *Elementary treatise on determinants* (Dodgson 1867). Only once did he mention the word ‘matrix’, because he preferred his own word ‘Blocks’ (Dodgson 1867, iv); yet, obviously having been very familiar with matrices, he doubtless knew about their non-commutative multiplication.

**Rotations and added terms** Bayley further suggested that quaternions can be recognized in the rotations at the table of the mad tea-party (Bayley 2009).

Just as complex numbers work with two terms, quaternions belong to a number system based on four terms [...] Alice is now at a table with three strange characters: the Hatter, the March Hare and the Dormouse. The character Time [...] is absent [...]. Reading this scene with Hamilton’s maths in mind, the members of the Hatter’s tea party represent three terms of a quaternion, in which the

\(^2\)That Hamilton initially saw it as adding a third imaginary term to the triplets appears from a letter he wrote on the day after the discovery of the quaternions to his friend John Graves (1806–70) (Hamilton 1844, 491).
all-important fourth term, time, is missing. Without Time, we are told, the characters are stuck at the tea table, constantly moving round to find clean cups and saucers. Their movement around the table is reminiscent of Hamilton’s early attempts to calculate motion, which was limited to rotations in a plane before he added time to the mix (Bayley 2009).

It is not immediately obvious what Bayley assumed to be the quaternions’ ‘time’ term. Hamilton called the real term of a quaternion its ‘scalar’ part and the three imaginary terms together its ‘vector’ part, and it can be inferred that Bayley had the scalar part in mind; in his Lectures, Hamilton called the scalar part of a quaternion an ‘extra-spatial unit’ (Hamilton 1853, preface 60), and Bayley suggests that that was the ‘extra’ time term he added:

when he added the fourth, he got the three-dimensional rotation he was looking for, but he had trouble conceptualising what this extra term meant. Like most Victorians, he assumed this term had to mean something, so in the preface to his Lectures on Quaternions of 1853 he added a footnote: ‘It seemed (and still seems) to me natural to connect this extra-spatial unit with the conception of time’ (Bayley 2009).

Thus associating the scalar term with time, Bayley’s view on quaternions and rotations appears to be based on two misunderstandings.

The first is that triplets cannot describe rotations because that would need triplet multiplication. Hamilton, indeed, could not find a suitable multiplication for triplets, and his ‘early attempts’ with three terms therefore did not lead to ‘rotations in a plane.’ Moreover, only in 1844, the year after the discovery of the quaternions, Hamilton made the connection with rotations (Pujol 2014).

The second is that the term Hamilton added to the triplets as the fourth one was not the scalar part, which Bayley assumed was connected with time, but the third imaginary part; Hamilton did not ‘add time to the mix’. The complex numbers $a + ib$, the triplets $a + ib + jc$ and the quaternions $a + ib + jc + kd$ all have a scalar part, and if the scalar part were personified in the character Time, his absence from the table would end any correspondence with quaternions, triplets or complex numbers.

That Dodgson knew about complex numbers can be inferred from his 1867 Treatise; he referred to Peacock’s Algebra which treated complex numbers (Dodgson 1867, preface v). Therefore, even if Dodgson also knew about quaternions, he would not have made such a basic mistake as removing Time from the table.

3. Quaternions and time

Abeles’ second objection was precisely against this ‘time’ interpretation of the ‘extra-spatial unit’. It appears that the sentence Bayley gave from the footnote of the Lectures is only a small part of what Hamilton wrote:

it seemed (and still seems) to me natural to connect this extra-spatial unit with the conception […] of time, regarded here merely as an axis of continuous and unidimensional progression. But whether we thus consider jointly time and space, or conceive generally any system of four independent axes, or scales of progression
(u, i, j, k), I am disposed to infer from the above investigation [the] Law Of The
Four Scales (Hamilton 1853, preface 60).

From this Abeles concluded that ‘time is not an element of quaternion algebra’
(Abeles 2017, 13). And not only did Hamilton discuss his conception of ‘time’ in
this footnote; to show where the quaternions came from he dedicated a large part
of the preface to the Lectures to a discussion of his previous ideas, such as his
‘Algebra as the science of pure time’ which he had communicated in 1835 (Hamilton
1837).

About this conception of algebra Bayley remarked:

where geometry allowed the exploration of space, Hamilton believed, algebra
allowed the investigation of ‘pure time’, a rather esoteric concept he had
derived from Immanuel Kant that was meant to be a kind of Platonic ideal of
time, distinct from the real time we humans experience (Bayley 2009).

Or, as Hamilton expressed it, he had considered ‘time’ ‘to be abstract, ideal, or pure, like
that “space” which is the object of geometry’ (Hamilton 1853, preface 2).

Hamilton started to write his Lectures in 1848, but already in 1845 or 1846 he had
revised his view about algebra as the science of pure time. The main reason was the
limitation it posed on algebra; a consequence had been that multiplication was
always associative (Øhrstrøm 1985, 46, 49). This limitation had been acceptable as
long as algebras obeyed the laws of arithmetic, but in December 1843, two months
after Hamilton had found the non-commutative quaternions, John Graves found
the octaves, or octonions, extensions of the quaternions by four more imaginary
terms, and they were neither commutative nor associative.3 Even though Hamilton
felt uncomfortable about an algebra without associativity, its restriction on algebra
as the science of pure time became untenable, and he ‘converted’ to Peacock’s competing
view, that algebra is a ‘system of signs and of their combinations’ (Øhrstrøm 1985,
52–53).

Therefore, instead of time being a part of quaternion algebra, their extension, the
octaves, even ended the connection Hamilton had made between algebra and pure
time.

Time and space But from the footnote in the preface to the Lectures it is also
obvious that Hamilton did not completely abandon his idea of ‘considering jointly
time and space’ in a philosophical way. In 1852 Hamilton wrote that his view on
algebra as a science of pure time had been ‘the parent of the quaternions’ and that
knowing about these views would be useful for students of his later work on geometry
(Øhrstrøm 1985, 54).

Moreover, in 1846 he had referred to it in the poem ‘The Tetractys’, a Greek term
signifying a four-term or quaternion (Graves 1882–89, 524),

And how the One of Time, of Space the Three,
     Might in the Chain of Symbol girdled be.
     […]

3 Apparently while investigating Graves’ new system in 1844, Hamilton found, or recognized, the what he
called ‘associative’ law of arithmetic (Øhrstrøm 1985, 50).
A dimly traced Pythagorean lore;
A westward floating, mystic dream of four.

This leads to another contemplation. Suppose that Dodgson did know about quaternions, not in depth but through this poem; Hamilton sent his poems to many friends. Then even if Dodgson erroneously assumed that time was a part of quaternion algebra, the problem remains that the character Time is absent from the table, and without the scalar term there are no quaternions, triplets or complex numbers.

4. Victorian etiquette

Still, it might be objected that the erroneous mathematical description was Bayley’s mistake and not Dodgson’s. It can be wondered then whether the mad tea-party scene would be consistent if it is assumed that Dodgson meant to set the scalar part of the quaternion to zero. Although it would still be illogical to remove Time from the table completely, the four characters then would personify a pure quaternion $0 + ib + jc + kd$, and these quaternions do represent rotations in real space.

Yet this idea is undermined by the last part of the chapter; the last time Alice saw them, the Hatter and the March Hare were trying to put the Dormouse into the teapot (Carroll 1866, 110; Figure 2).

Figure 2. Trying to put the Dormouse into the teapot (Carroll 1866, 110).
This led Bayley to suggest that

this could be their route to freedom. If they could only lose him, they could exist
independently, as a complex number with two terms. Still mad, according to
Dodgson, but free from an endless rotation around the table.

But that is not what would happen. As the vector part of a quaternion, \( ib + jc + kd \),
losing the Dormouse they would go back to being the vector part of a triplet, \( ib + jc \),
or remain the vector part of a quaternion of which now both the Time and the Dor-
mouse term are set to zero: \( 0 + ib + 0 + kd \). There does not seem to be much math-
ematical humour in a triplet with a zero scalar term, or a quaternion with two zero
terms; it is easier to suggest that the humour is that they tried to put the Dormouse
into the teapot.

Indeed, Dodgson had more to poke fun at than mathematics; in 2014 Philip
Ardagh suggested that in the tea-party chapter Dodgson was ridiculing the very
strict Victorian rules of behaviour. In 1855 Dodgson had published a humorous
short article, ‘Hints for etiquette; or, dining out made easy’ (Dodgson 1855, 33), in
which he satirized the Victorian ‘Dos and Don’ts at the dinner table’ (Ardagh 2014).

Living in such socially strict times, for Victorians it may simply have been hilarious
to read about the characters messing around with food, and to see the Hatter and the
March Hare standing on their chairs and even on the table, while trying to put the
Dormouse into the teapot.

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ORCID
Anne van Weerden ⓒ http://orcid.org/0000-0003-3272-8007

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