SOME LEVIN-STEČKIN’S TYPE INEQUALITIES FOR OPERATOR CONVEX FUNCTIONS ON HILBERT SPACES

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Abstract. Let $f$ be an operator convex function on $I$ and $A, B \in \mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in $I$. Assume that $p : [0, 1] \to \mathbb{R}$ is symmetric and non-decreasing on $[0, 1/2]$. In this paper we obtained, among others, that

$$0 \leq \int_0^1 p(t) \, dt \int_0^1 f(tA + (1-t)B) \, dt - \int_0^1 p(t) \, f(tA + (1-t)B) \, dt$$

$$\leq \frac{1}{4} \left[ p\left(\frac{1}{2}\right) - p(0) \right] \left[ \frac{f(A) + f(B)}{2} - f\left(\frac{A + B}{2}\right) \right]$$

in the operator order.

Several other similar inequalities for either $p$ or $f$ is differentiable, are also provided. Applications for power function and logarithm are given as well.

1. Introduction

A real valued continuous function $f$ on an interval $I$ is said to be operator convex (operator concave) on $I$ if

$$f((1 - \lambda) A + \lambda B) \leq (\geq) (1 - \lambda) f(A) + \lambda f(B)$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator $A$ and $B$ on a Hilbert space $H$ whose spectra are contained in $I$. Notice that a function $f$ is operator concave if $-f$ is operator convex.

A real valued continuous function $f$ on an interval $I$ is said to be operator monotone if it is monotone with respect to the operator order, i.e., $A \leq B$ with $\text{Sp}(A), \text{Sp}(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [12] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0, \infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0, \infty)$. The entropy function $f(t) = -t \ln t$ is operator concave on $(0, \infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

In [8] we obtained among others the following Hermite-Hadamard type inequalities for operator convex functions $f : I \to \mathbb{R}$

$$f\left(\frac{A + B}{2}\right) \leq \int_0^1 f((1 - s) A + sB) \, ds \leq f\left(\frac{A}{2}\right) + f\left(\frac{B}{2}\right).$$

1991 Mathematics Subject Classification. 47A63, 26D15, 26D10.

Key words and phrases. Operator convex functions, Integral inequalities, Hermite-Hadamard inequality, Féjer’s inequalities, Levin-Stečkin’s inequality.
where $A, B$ are selfadjoint operators with spectra included in $I$.

From the operator convexity of the function $f$ we have

$$f\left(\frac{A + B}{2}\right) \leq \frac{1}{2} \left[f((1 - s)A + sB) + f(sA + (1 - s)B)\right]$$

for all $s \in [0, 1]$ and $A, B$ selfadjoint operators with spectra included in $I$.

If $p : [0, 1] \to [0, \infty)$ is Lebesgue integrable and symmetric in the sense that $p(1 - s) = p(s)$ for all $s \in [0, 1]$, then by multiplying (1.3) with $p(s)$, integrating on $[0, 1]$ and taking into account that

$$\int_0^1 p(s) f((1 - s)A + sB) ds = \int_0^1 p(s) f(sA + (1 - s)B) ds,$$

we get the weighted version of (1.2) for $A, B$ selfadjoint operators with spectra included in $I$

$$\left(\int_0^1 p(s) ds\right) f\left(\frac{A + B}{2}\right) \leq \int_0^1 p(s) f(sA + (1 - s)B) ds$$

which are the operator version of the well known Féjer’s inequalities for scalar convex functions.

For recent inequalities for operator convex functions see [1]-[2], [4], [6]-[14], and [21]-[26].

The following result is known in the literature as Levin-Stečkin’s inequality [16]:

**Theorem 1.** If the function $p : [0, 1] \to \mathbb{R}$ is symmetric, namely $p(1 - t) = p(t)$ for $t \in [0, 1]$ and non-decreasing (non-increasing) on $[0, 1/2]$ , then for every convex function $g$ on $[0, 1]$,

$$(LS) \quad \int_0^1 p(t) g(t) dt \leq \geq \int_0^1 p(t) dt \int_0^1 g(t) dt.$$

If the function $g$ is concave on $[0, 1]$, then the signs of inequalities reverse in (LS).

For some recent results related to Levin-Stečkin’s inequality, see [18], [19] and [27].

Motivated by the above operator inequalities, we provide in this paper the operator version of Levin-Stečkin’s inequality as well as several reverses. Applications for power function and logarithm are also given.

## 2. Operator Inequalities

Let $f$ be an operator convex function on $I$. For $A, B \in \mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in $I$, we consider the auxiliary function $\varphi_{(A,B)} : [0, 1] \to \mathcal{SA}(H)$, the class of all selfadjoint operators on $H$, defined by

$$\varphi_{(A,B)}(t) := f((1 - t)A + tB).$$

For $x \in H$ we can also consider the auxiliary function $\varphi_{(A,B);x} : [0, 1] \to \mathbb{R}$ defined by

$$\varphi_{(A,B);x}(t) := \langle \varphi_{(A,B)}(t)x, x \rangle = (f((1 - t)A + tB)x, x).$$
We have the following basic fact [10]:

**Lemma 1.** Let \( f \) be an operator convex function on \( I \). For any \( A, B \in S\mathcal{A}_I(H) \), \( \varphi_{(A,B)} \) is well defined and convex in the operator order. For any \( (A,B) \in S\mathcal{A}_I(H) \) and \( x \in H \) the function \( \varphi_{(A,B);x} \) is convex in the usual sense on \([0,1]\).

A continuous function \( g : S\mathcal{A}_I(H) \to \mathcal{B}(H) \) is said to be *Gâteaux differentiable* in \( A \in S\mathcal{A}_I(H) \) along the direction \( B \in \mathcal{B}(H) \) if the following limit exists in the strong topology of \( \mathcal{B}(H) \)

\[
\nabla g_A(B) := \lim_{s \to 0} \frac{g(A + sB) - g(A)}{s} \in \mathcal{B}(H).
\]

If the limit \( (2.3) \) exists for all \( B \in \mathcal{B}(H) \), then we say that \( f \) is *Gâteaux differentiable* in \( A \) and we can write \( g \in G(A) \). If this is true for any \( A \) in an open set \( S \) from \( S\mathcal{A}_I(H) \) we write that \( g \in G(S) \).

If \( g \) is a continuous function on \( I \), by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators \( A, B \in S\mathcal{A}_I(H) \) we consider the segment of selfadjoint operators

\[
[A, B] := \{(1 - t)A + tB \mid t \in [0, 1]\}.
\]

We observe that \( A, B \in [A, B] \) and \( [A, B] \subset S\mathcal{A}_I(H) \).

We also have [10]:

**Lemma 2.** Let \( f \) be an operator convex function on \( I \) and \( A, B \in S\mathcal{A}_I(H) \), with \( A \neq B \). If \( f \in G([A,B]) \), then the auxiliary function \( \varphi_{(A,B)} \) is differentiable on \((0,1)\) and

\[
\varphi_{(A,B)}' (t) = \nabla f_{(1-t)A+tB} (B-A).
\]

Also we have for the lateral derivative that

\[
\varphi_{(A,B)}' (0+) = \nabla f_A (B-A)
\]

and

\[
\varphi_{(A,B)}' (1-) = \nabla f_B (B-A).
\]

and

**Lemma 3.** Let \( f \) be an operator convex function on \( I \) and \( A, B \in S\mathcal{A}_I(H) \), with \( A \neq B \). If \( f \in G([A,B]) \), then for \( 0 < t_1 < t_2 < 1 \) we have

\[
\nabla g_{(1-t_1)A+t_1B} (B-A) \leq \nabla g_{(1-t_2)A+t_2B} (B-A)
\]

in the operator order.

We also have

\[
\nabla f_A (B-A) \leq \nabla g_{(1-t_1)A+t_1B} (B-A)
\]

and

\[
\nabla g_{(1-t_2)A+t_2B} (B-A) \leq \nabla f_B (B-A).
\]

In particular, we observe that:

**Corollary 1.** Let \( f \) be an operator convex function on \( I \) and \( A, B \in S\mathcal{A}_I(H) \), with \( A \neq B \). If \( f \in G([A,B]) \), then for all \( t \in (0,1) \) we have

\[
\nabla f_A (B-A) \leq \nabla f_{(1-t)A+tB} (B-A) \leq \nabla f_B (B-A).
\]
For two Lebesgue integrable functions \( h, g : [a, b] \to \mathbb{R} \), consider the Čebyšev functional:

\[
(2.11) \quad C(h, g) := \frac{1}{b-a} \int_a^b h(t)g(t) \, dt - \frac{1}{(b-a)^2} \int_a^b h(t) \, dt \int_a^b g(t) \, dt.
\]

In 1935, Grüss [15] showed that

\[
(2.12) \quad |C(h, g)| \leq \frac{1}{4} (M - m) (N - n),
\]

provided that there exists the real numbers \( m, M, n, N \) such that

\[
(2.13) \quad m \leq h(t) \leq M \quad \text{and} \quad n \leq g(t) \leq N \quad \text{for a.e. } t \in [a, b].
\]

The constant \( \frac{1}{4} \) is best possible in (2.11) in the sense that it cannot be replaced by a smaller quantity.

We have the following operator inequalities:

**Theorem 2.** Let \( f \) be an operator convex function on \( I \) and \( A, B \in \mathcal{SA}_I(H) \).

Assume that \( p : [0, 1] \to \mathbb{R} \) is symmetric and non-decreasing on \([0, 1/2], \) then we have the operator inequality

\[
(2.14) \quad 0 \leq \int_0^1 p(t) \, dt \int_0^1 f(tA + (1-t)B) \, dt - \int_0^1 p(t) f(tA + (1-t)B) \, dt\leq \frac{1}{4} \left[p\left(\frac{1}{2}\right) - p(0)\right]\left[\frac{f(A) + f(B)}{2} - f\left(\frac{A + B}{2}\right)\right].
\]

If \( p : [0, 1] \to \mathbb{R} \) is symmetric and non-increasing on \([0, 1/2], \) then

\[
(2.15) \quad 0 \leq \int_0^1 p(t) \, dt \int_0^1 f(tA + (1-t)B) \, dt - \int_0^1 p(t) f(tA + (1-t)B) \, dt\leq \frac{1}{4} \left[p(0) - p\left(\frac{1}{2}\right)\right]\left[\frac{f(A) + f(B)}{2} - f\left(\frac{A + B}{2}\right)\right].
\]

**Proof.** For \( x \in H \) we consider the auxiliary function \( \varphi_{(A,B);x} : [0, 1] \to \mathbb{R} \) defined by

\[
\varphi_{(A,B);x} (t) := \left< \varphi_{(A,B)}(t) x, x \right> = \left< (1-t) A + tB, x, x \right>.
\]

Since \( p \) is symmetric on \([0, 1]\), then

\[
\int_0^1 p(t) \frac{\varphi_{(A,B);x} (t) + \varphi_{(A,B);x} (1-t)}{2} \, dt
= \frac{1}{2} \left[ \int_0^1 p(t) \varphi_{(A,B);x} (t) \, dt + \int_0^1 p(t) \varphi_{(A,B);x} (1-t) \, dt \right]
= \frac{1}{2} \left[ \int_0^1 p(t) \varphi_{(A,B);x} (t) \, dt + \int_0^1 p(t) \varphi_{(A,B);x} (1-t) \, dt \right].
\]

By changing the variable \( 1-t = s, \ s \in [0, 1] \) we have

\[
\int_0^1 p(1-t) \varphi_{(A,B);x} (1-t) \, dt = \int_0^1 p(s) \varphi_{(A,B);x} (s) \, ds
\]

and then

\[
\int_0^1 p(t) \frac{\varphi_{(A,B);x} (t) + \varphi_{(A,B);x} (1-t)}{2} \, dt = \int_0^1 p(t) \varphi_{(A,B);x} (t) \, dt.
\]
Also
\[
\int_0^1 \frac{\varphi(A,B;x)(t) + \varphi(A,B;x)(1-t)}{2} \, dt = \int_0^1 \varphi(A,B;x)(t) \, dt.
\]

Therefore
\[
\int_0^1 p(t) \, dt \int_0^1 \varphi(A,B;x)(t) \, dt - \int_0^1 p(t) \varphi(A,B;x)(t) \, dt = \int_0^1 p(t) \, dt \int_0^1 \varphi(A,B;x)(t) \, dt - \int_0^1 p(t) \varphi(A,B;x)(t) \, dt,
\]
where
\[
\hat{\varphi}(A,B;x)(t) := \frac{\varphi(A,B;x)(t) + \varphi(A,B;x)(1-t)}{2}, \quad t \in [0, 1]
\]
is the symmetrical transform of \(\varphi(A,B;x)\) on the interval \([0, 1]\).

Now, if we use the Levin-Stečkin's inequality for the symmetric function \(p\) and the convex function \(g = \varphi(A,B;x)^r\), then we obtain
\[
0 \leq \int_0^1 p(t) \, dt \int_0^1 \varphi(A,B;x)(t) \, dt - \int_0^1 p(t) \varphi(A,B;x)(t) \, dt,
\]
for all \(x \in H\).

Since, by Lemma 1, \(\varphi(A,B;x)\) is convex, then \(\hat{\varphi}(A,B;x)\) is symmetric and convex, which implies that
\[
\varphi(A,B;x) \left( \frac{1}{2} \right) = \hat{\varphi}(A,B;x) \left( \frac{1}{2} \right) \leq \hat{\varphi}(A,B;x)(t)
\]
\[
\leq \hat{\varphi}(A,B;x)(1) = \frac{\varphi(A,B;x)(0) + \varphi(A,B;x)(1)}{2}, \quad t \in [0, 1],
\]
for all \(x \in H\).

Also \(p(0) \leq p(t) \leq p \left( \frac{1}{2} \right), \quad t \in [0, 1]\) and by Grüss' inequality for \(h = p\) and \(g = \hat{\varphi}(A,B;x)\) we get
\[
0 \leq \int_0^1 p(t) \, dt \int_0^1 \hat{\varphi}(A,B;x)(t) \, dt - \int_0^1 p(t) \hat{\varphi}(A,B;x)(t) \, dt
\]
\[
\leq \frac{1}{4} \left[ p \left( \frac{1}{2} \right) - p(0) \right] \left[ \frac{\varphi(A,B;x)(0) + \varphi(A,B;x)(1)}{2} - \varphi(A,B;x) \left( \frac{1}{2} \right) \right]
\]

namely, by (2.16) and (2.17)
\[
0 \leq \int_0^1 p(t) \, dt \int_0^1 \varphi(A,B;x)(t) \, dt - \int_0^1 p(t) \varphi(A,B;x)(t) \, dt
\]
\[
\leq \frac{1}{4} \left[ p \left( \frac{1}{2} \right) - p(0) \right] \left[ \frac{\varphi(A,B;x)(0) + \varphi(A,B;x)(1)}{2} - \varphi(A,B;x) \left( \frac{1}{2} \right) \right]
\]
for all \(x \in H\).
The inequality (2.18) can be written in terms of inner product as
\[
0 \leq \left\langle \left( \int_0^1 p(t) dt \int_0^1 f((1-t)A + tB) dt \right) x, x \right\rangle \\
- \left\langle \left( \int_0^1 p(t) f((1-t)A + tB) dt \right) x, x \right\rangle \\
\leq \frac{1}{4} \left[ p \left( \frac{1}{2} \right) - p(0) \right] \left[ f \left( \frac{A + B}{2} \right) - \frac{f(A) + f(B)}{2} \right]
\]
for all \( x \in H \), which is equivalent to the operator inequality (2.14).

\[ \square \]

Remark 1. If \( f \) is an operator concave function on \( I \) and \( A, B \in \mathcal{SA}_I(H) \), while \( p : [0,1] \rightarrow \mathbb{R} \) is symmetric and non-decreasing on \([0,1/2] \), then

\[
0 \leq \int_0^1 p(t) f((1-t)A + tB) dt - \int_0^1 p(t) dt \int_0^1 f((1-t)A + tB) dt dt \\
\leq \frac{1}{4} \left[ p \left( \frac{1}{2} \right) - p(0) \right] \left[ f \left( \frac{A + B}{2} \right) - \frac{f(A) + f(B)}{2} \right].
\]

Also, in this case, if \( p : [0,1] \rightarrow \mathbb{R} \) is symmetric and non-increasing on \([0,1/2] \), then

\[
0 \leq \int_0^1 p(t) f((1-t)A + tB) dt - \int_0^1 p(t) dt \int_0^1 f((1-t)A + tB) dt dt \\
\leq \frac{1}{4} \left[ p(0) - p \left( \frac{1}{2} \right) \right] \left[ f \left( \frac{A + B}{2} \right) - \frac{f(A) + f(B)}{2} \right].
\]

The following inequality obtained by Ostrowski in 1970, [20] also holds

\[
|C(h, g)| \leq \frac{1}{8} (b-a)(M-m)\|g'\|_{\infty},
\]
provided that \( h \) is Lebesgue integrable and satisfies (2.13) while \( g \) is absolutely continuous and \( g' \in L_{\infty}[a,b] \). The constant \( \frac{1}{8} \) is best possible in (2.21).

We have the following operator inequalities when some differentiability conditions are imposed.

Theorem 3. Let \( f \) be an operator convex function on \( I \) and \( A, B \in \mathcal{SA}_I(H) \) while \( p : [0,1] \rightarrow \mathbb{R} \) is symmetric and non-decreasing on \([0,1/2] \).

(i) If \( p \) is differentiable on \((0,1) \), then

\[
0 \leq \int_0^1 p(t) dt \int_0^1 f((1-t)A + tB) dt - \int_0^1 p(t) f((1-t)A + tB) dt \\
\leq \frac{1}{8} \|p'\|_{\infty} \left[ f \left( \frac{A + B}{2} \right) - \frac{f(A) + f(B)}{2} \right].
\]

(ii) If \( f \in \mathcal{G}([A,B]) \), then

\[
0 \leq \int_0^1 p(t) dt \int_0^1 f((1-t)A + tB) dt - \int_0^1 p(t) f((1-t)A + tB) dt \\
\leq \frac{1}{16} \left[ p \left( \frac{1}{2} \right) - p(0) \right] \|\nabla f_B(B - A) - \nabla f_A(B - A)\|.
\]
Proof: The inequality (2.22) follows by (2.21) for \( g = p \) and \( h = \varphi_{(A,B);x}, \ x \in H \) and proceed like in the proof of Theorem 2.

Now, by Lemma 2

\[
(2.24) \quad \left( \varphi_{(A,B);x} (t) \right)' = \left( \varphi_{(A,B);x} (t) \right)' + \left( \varphi_{(A,B);x} (1-t) \right)'
\]

\[
= \left( \varphi'_{(A,B)} (t) x, x \right) \frac{2}{2} - \left( \varphi'_{(A,B)} (1-t) x, x \right)
\]

\[
= \left( \nabla f_{(1-t)A+tB} (B-A) x, x \right) - \left( \nabla f_{tA+(1-t)B} (B-A) x, x \right)
\]

\[
= \left( \left[ \nabla f_{(1-t)A+tB} (B-A) - \nabla f_{tA+(1-t)B} (B-A) \right] x, x \right)
\]

for all \( t \in (0,1) \) and any \( x \in H \),

Since \( \varphi_{(A,B);x} \) is convex on \( (0,1) \), then

\[
\left( \varphi_{(A,B);x} (t) \right)'_{t=0+} \leq \left( \varphi_{(A,B);x} (t) \right)'_{t=1-}, \ t \in (0,1)
\]

namely, by Lemma 3

\[
\left\langle \left[ \frac{\nabla f_A (B-A) - \nabla f_B (B-A)}{2} \right] x, x \right\rangle
\]

\[
\leq \left\langle \left[ \frac{\nabla f_{(1-t)A+tB} (B-A) - \nabla f_{tA+(1-t)B} (B-A)}{2} \right] x, x \right\rangle
\]

\[
\leq \left\langle \left[ \frac{\nabla f_B (B-A) - \nabla f_A (B-A)}{2} \right] x, x \right\rangle
\]

for all \( t \in (0,1) \) and any \( x \in H \).

Therefore

\[
\left| \left( \varphi_{(A,B);x} (t) \right)' \right| \leq \left| \left\langle \left[ \frac{\nabla f_B (B-A) - \nabla f_A (B-A)}{2} \right] x, x \right\rangle \right|
\]

for all \( t \in (0,1) \) and any \( x \in H \), which implies that

\[
\sup_{t \in (0,1)} \left| \left( \varphi_{(A,B);x} (t) \right)' \right| \leq \left| \left\langle \left[ \frac{\nabla f_B (B-A) - \nabla f_A (B-A)}{2} \right] x, x \right\rangle \right|
\]

\[
= \left| \left\langle \left[ \frac{\nabla f_B (B-A) - \nabla f_A (B-A)}{2} \right] x, x \right\rangle \right|
\]

for any \( x \in H \), since by Corollary 1, we have \( f_B (B-A) \geq \nabla f_A (B-A) \).
If we use Ostrowski’s inequality (2.21) for $h = p$ and $g = \hat{\varphi}(A,B)x$, then we obtain

$$0 \leq \int_0^1 p(t) \, dt \int_0^1 \hat{\varphi}(A,B) : x \, (t) \, dt - \int_0^1 p(t) \, \hat{\varphi}(A,B) : x \, (t) \, dt$$

$$\leq \frac{1}{8} \left[ p \left( \frac{1}{2} \right) - p(0) \right] \sup_{t \in (0,1)} \left| \left( \hat{\varphi}(A,B) : x \right)' \right|$$

$$\leq \frac{1}{8} \left[ p \left( \frac{1}{2} \right) - p(0) \right] \left\langle \left[ \frac{\nabla f_B (B - A) - \nabla f_A (B - A)}{2} \right] x, x \right\rangle,$$

namely

$$0 \leq \left( \int_0^1 p(t) \, dt \int_0^1 f ((1 - t) A + tB) \, dt - \int_0^1 p(t) \, f ((1 - t) A + tB) \, dt \right) x, x \rangle$$

$$\leq \frac{1}{8} \left[ p \left( \frac{1}{2} \right) - p(0) \right] \left\langle \left[ \frac{\nabla f_B (B - A) - \nabla f_A (B - A)}{2} \right] x, x \right\rangle,$$

which is equivalent to the operator inequality (2.23).

Another, however less known result, even though it was obtained by Čebyšev in 1882, [3], states that

$$|C(h,g)| \leq \frac{1}{12} \|h'\|_\infty \|g'\|_\infty (b - a)^2,$$

provided that $h'$, $g'$ exist and are continuous on $[a,b]$ and $\|h'\|_\infty = \sup_{t \in [a,b]} |h'(t)|$.

The case of euclidean norms of the derivative was considered by A. Lupaş in [17] in which he proved that

$$|C(h,g)| \leq \frac{1}{\pi^2} \|h'\|_2 \|g'\|_2 (b - a),$$

provided that $h$, $g$ are absolutely continuous and $h'$, $g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Further, we have:

**Theorem 4.** Let $f$ be an operator convex function on $I$ and $A, B \in S A_I (H)$ while $p : [0,1] \rightarrow \mathbb{R}$ is symmetric and non-decreasing on $[0,1/2]$.

(i) If $p$ is differentiable on $(0,1)$ and $f \in G([A,B])$, then

$$0 \leq \int_0^1 p(t) \, dt \int_0^1 f (tA + (1 - t) B) \, dt - \int_0^1 p(t) \, f (tA + (1 - t) B) \, dt$$

$$\leq \frac{1}{24} \|p'\|_\infty \left[ \nabla f_B (B - A) - \nabla f_A (B - A) \right].$$
(ii) If $p$ is differentiable on $(0, 1)$ with $p' \in L^2[0, 1]$ and $f \in G([A, B])$, then

$$
0 \leq \int_0^1 p(t) \, dt \int_0^1 f(tA + (1 - t)B) \, dt - \int_0^1 p(t) f(tA + (1 - t)B) \, dt
$$

$$
\leq \frac{1}{2\pi^2} \|p'\|_2
\times \left( \int_0^1 \left\| \nabla f_{(1-t)A+tB}(B-A) - \nabla f_{tA+(1-t)B}(B-A) \right\|^2 \, dt \right)^{1/2}
$$

$$
\leq \frac{1}{\pi^2} \|p'\|_2 \left( \int_0^1 \left\| \nabla f_{(1-t)A+tB}(B-A) \right\|^2 \, dt \right)^{1/2}
$$

provided the last integral is finite.

**Proof.** The inequality (2.27) follows by (2.25) for $h = p$ and $g = \varphi_{(A,B),x}$, $x \in H$ and proceed like in the proof of Theorem 2.

From (2.24) we have

$$
\int_0^1 \left[ \left( \varphi_{(A,B),x}(t) \right) \right]^2 \, dt
$$

$$
= \int_0^1 \left\| \left[ \frac{\nabla f_{(1-t)A+tB}(B-A) - \nabla f_{tA+(1-t)B}(B-A)}{2} \right] x, x \right\|^2 \, dt
$$

$$
\leq \int_0^1 \left\| \nabla f_{(1-t)A+tB}(B-A) - \nabla f_{tA+(1-t)B}(B-A) \right\|^2 \, dt
$$

$$
\leq \frac{1}{4} \|x\|^2 \int_0^1 \left\| \nabla f_{(1-t)A+(1-t)B}(B-A) \right\|^2 \, dt
$$

for all $x \in H$, implying that

$$
\left( \int_0^1 \left[ \left( \varphi_{(A,B),x}(t) \right) \right]^2 \, dt \right)^{1/2}
$$

$$
\leq \frac{1}{2} \|x\|^2 \left( \int_0^1 \left\| \nabla f_{(1-t)A+tB}(B-A) - \nabla f_{tA+(1-t)B}(B-A) \right\|^2 \, dt \right)^{1/2}
$$

for all $x \in H$.

By using (2.26) for $h = p$ and $g = \varphi_{(A,B),x}$, $x \in H$, we derive

$$
0 \leq \left\langle \left( \int_0^1 p(t) \, dt \int_0^1 f((1-t)A+tB) \, dt - \int_0^1 p(t) f((1-t)A+tB) \, dt \right) x, x \right\rangle
$$

$$
\leq \frac{1}{2\pi^2} \|p'\|_2 \|x\|^2 \left( \int_0^1 \left\| \nabla f_{(1-t)A+tB}(B-A) - \nabla f_{tA+(1-t)B}(B-A) \right\|^2 \, dt \right)^{1/2},
$$

which is equivalent to the first inequality in (2.28).
By the triangle inequality, we have
\[
\left( \int_0^1 \| \nabla f_{(1-t)A+tB} (B - A) - \nabla f_{tA+(1-t)B} (B - A) \|^2 \, dt \right)^{1/2} \\
\leq \left( \int_0^1 \| \nabla f_{(1-t)A+tB} (B - A) \|^2 \, dt \right)^{1/2} + \left( \int_0^1 \| \nabla f_{tA+(1-t)B} (B - A) \|^2 \, dt \right)^{1/2} \\
= 2 \left( \int_0^1 \| \nabla f_{(1-t)A+tB} (B - A) \|^2 \, dt \right)^{1/2},
\]
which proves the last part of (2.28).

**Remark 2.** If either \( p \) is non-increasing on \([0, 1/2]\) or \( f \) is an operator concave function on \( I \), then the interested reader may state similar results to the ones in Theorem 3 and Theorem 4. We omit the details.

3. **Some Examples**

The function \( f (t) = t^r \) is operator convex on \((0, \infty)\) if either \( 1 \leq r \leq 2 \) or \(-1 \leq r \leq 0\). Assume that \( p : [0, 1] \to \mathbb{R} \) is symmetric and non-decreasing on \([0, 1/2]\), then we have by (2.14) the operator inequality
\[
0 \leq \int_0^1 p(t) \, dt \int_0^1 (tA + (1-t)B)^r \, dt - \int_0^1 p(t) \, dt \int_0^1 (tA + (1-t)B)^r \, dt \\
\leq \frac{1}{4} \left[ p \left( \frac{1}{2} \right) - p(0) \right] \left[ \frac{A^r + B^r}{2} - \left( \frac{A + B}{2} \right)^r \right]
\]
for all \( A, B > 0 \).

Moreover, if \( p \) is differentiable on \((0, 1)\), then by (2.22)
\[
0 \leq \int_0^1 p(t) \, dt \int_0^1 (tA + (1-t)B)^r \, dt - \int_0^1 p(t) \, dt \int_0^1 (tA + (1-t)B)^r \, dt \\
\leq \frac{1}{8} \| p' \|_{\infty} \left[ \frac{A^r + B^r}{2} - \left( \frac{A + B}{2} \right)^r \right]
\]
for all \( A, B > 0 \).

The function \( f (x) = x^{-1} \) is operator convex on \((0, \infty)\), operator Gâteaux differentiable and
\[
\nabla f_T (S) = -T^{-1}ST^{-1}
\]
for \( T, S > 0 \).

If we use (2.23), then we get the inequality
\[
0 \leq \int_0^1 p(t) \, dt \int_0^1 (tA + (1-t)B)^{-1} \, dt - \int_0^1 p(t) \, dt \int_0^1 (tA + (1-t)B)^{-1} \, dt \\
\leq \frac{1}{16} \left[ p \left( \frac{1}{2} \right) - p(0) \right] \left[ \frac{1}{2} A^{-1} (B - A) A^{-1} - B^{-1} (B - A) B^{-1} \right]
\]
provided that \( p : [0, 1] \to \mathbb{R} \) is symmetric and non-decreasing on \([0, 1/2]\) and \( A, B > 0 \).
Moreover, if $p$ is differentiable on $(0, 1)$ then by (2.27) we derive

\begin{equation}
0 \leq \int_0^1 p(t) \, dt \int_0^1 (tA + (1 - t) B)^{-1} \, dt - \int_0^1 p(t) \, dt \int_0^1 (tA + (1 - t) B)^{-1} \, dt
\leq \frac{1}{24} \|p'\|_{\infty} \left[ A^{-1} (B - A) A^{-1} - B^{-1} (B - A) B^{-1} \right]
\end{equation}

for $A, B > 0$.

If we use the first and last term in (2.28), then we also have

\begin{equation}
0 \leq \int_0^1 p(t) \, dt \int_0^1 (tA + (1 - t) B)^{-1} \, dt - \int_0^1 p(t) \, dt \int_0^1 (tA + (1 - t) B)^{-1} \, dt
\leq \frac{1}{2} \|p'\|_2
\times \left( \int_0^1 \left\| \left( (1 - t) A + tB \right)^{-1} (B - A) \left( (1 - t) A + tB \right)^{-1} \right\|^2 \, dt \right)^{1/2} \frac{1}{16},
\end{equation}

provided that $p' \in L^2 [0, 1]$ and $A, B > 0$.

The logarithmic function $f(t) = \ln t$ is operator concave on $(0, \infty)$. Assume that $p : [0, 1] \to \mathbb{R}$ is symmetric and non-decreasing on $[0, 1/2]$, then we have by (2.14) the operator inequality

\begin{equation}
0 \leq \int_0^1 p(t) \ln (tA + (1 - t) B) \, dt - \int_0^1 p(t) \, dt \int_0^1 \ln (tA + (1 - t) B) \, dt
\leq \frac{1}{4} \left[ p \left( \frac{1}{2} \right) - p (0) \right] \left[ \ln \left( \frac{A + B}{2} \right) - \frac{\ln A + \ln B}{2} \right],
\end{equation}

for all $A, B > 0$.

Moreover, if $p$ is differentiable on $(0, 1)$, then by (2.22)

\begin{equation}
0 \leq \int_0^1 p(t) \ln (tA + (1 - t) B) \, dt - \int_0^1 p(t) \, dt \int_0^1 \ln (tA + (1 - t) B) \, dt
\leq \frac{1}{8} \|p'\|_{\infty} \left[ \ln \left( \frac{A + B}{2} \right) - \frac{\ln A + \ln B}{2} \right],
\end{equation}

for all $A, B > 0$.

We note that the function $f(x) = \ln x$ is operator concave on $(0, \infty)$. The ln function is operator Gâteaux differentiable with the following explicit formula for the derivative (cf. Pedersen [21, p. 155]):

\begin{equation}
\nabla \ln_T (S) = \int_0^\infty (s1_H + T)^{-1} S (s1_H + T)^{-1} \, ds
\end{equation}

for $T, S > 0$.

If we use inequality (2.23) for $\ln$ we get for $A, B > 0$,

\begin{equation}
0 \leq \int_0^1 p(t) \ln (tA + (1 - t) B) \, dt - \int_0^1 p(t) \, dt \int_0^1 \ln (tA + (1 - t) B) \, dt
\leq \frac{1}{16} \left[ p \left( \frac{1}{2} \right) - p (0) \right] \left[ \int_0^\infty (s1_H + B)^{-1} (B - A) (s1_H + B)^{-1} \, ds
\right.
- \left. \int_0^\infty (s1_H + A)^{-1} (B - A) (s1_H + A)^{-1} \, ds \right],
\end{equation}

provided that $p : [0, 1] \to \mathbb{R}$ is symmetric and non-decreasing on $[0, 1/2]$. 

If $p$ is differentiable, then by (2.27) we

\begin{align}
0 & \leq \int_0^1 p(t) \ln(tA + (1-t)B) \, dt - \int_0^1 p(t) \, dt \int_0^1 \ln(tA + (1-t)B) \, dt \\
& \leq \frac{1}{24} \|p'\|_\infty \left[ \int_0^\infty (s1_H + B)^{-1} (B - A) (s1_H + B)^{-1} \, ds \\
& \quad - \int_0^\infty (s1_H + A)^{-1} (B - A) (s1_H + A)^{-1} \, ds \right],
\end{align}

for $A, B > 0$.

A similar inequality can be derive from (2.28), however the details are omitted. The interested author can also state the corresponding operator inequalities for $f(t) = t \ln t$ that is operator convex on $(0, \infty)$.

Finally, if we take $p(t) = t (1 - t)$, then we observe that $p$ is symmetric and non-decreasing on $[0, 1/2]$ and by (3.1) we obtain

\begin{align}
0 & \leq \frac{1}{6} \int_0^1 (tA + (1-t)B)^r \, dt - \int_0^1 t (1-t) (tA + (1-t)B)^r \, dt \\
& \leq \frac{1}{16} \left[ A^r + B^r - \left( \frac{A + B}{2} \right)^r \right]
\end{align}

if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and $A, B > 0$.

From (3.3) we derive

\begin{align}
0 & \leq \frac{1}{6} \int_0^1 (tA + (1-t)B)^{-1} \, dt - \int_0^1 t (1-t) (tA + (1-t)B)^{-1} \, dt \\
& \leq \frac{1}{64} \left[ A^{-1} (B - A) A^{-1} - B^{-1} (B - A) B^{-1} \right]
\end{align}

for $A, B > 0$.

From (3.6) we obtain the logarithmic inequality

\begin{align}
0 & \leq \int_0^1 t (1-t) \ln(tA + (1-t)B) \, dt - \frac{1}{6} \int_0^1 \ln(tA + (1-t)B) \, dt \\
& \leq \frac{1}{16} \left[ \ln \left( \frac{A + B}{2} \right) - \ln A + \ln B \right],
\end{align}

while from (3.9), the inequality

\begin{align}
0 & \leq \int_0^1 t (1-t) \ln(tA + (1-t)B) \, dt - \frac{1}{6} \int_0^1 \ln(tA + (1-t)B) \, dt \\
& \leq \frac{1}{64} \left[ \int_0^\infty (s1_H + B)^{-1} (B - A) (s1_H + B)^{-1} \, ds \\
& \quad - \int_0^\infty (s1_H + A)^{-1} (B - A) (s1_H + A)^{-1} \, ds \right]
\end{align}

for $A, B > 0$.

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