Green Functions For Wave Propagation on a 5D manifold and the Associated Gauge Fields Generated by a Uniformly Moving Point Source

I. Aharonovich$^a$ and L. P. Horwitz$^{abc}$

$^a$ Bar-Ilan University, Department of Physics, Ramat Gan, Israel.
$^b$ Tel-Aviv University, School of Physics, Ramat Aviv, Israel.
$^c$ College of Judea and Samaria, Ariel, Israel.

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Abstract

Gauge fields associated with the manifestly covariant dynamics of particles in (3, 1) spacetime are five-dimensional. We provide solutions of the classical 5D gauge field equations in both (4, 1) and (3, 2) flat spacetime metrics for the simple example of a uniformly moving point source. Green functions for the 5D field equations are obtained, which are consistent with the solutions for uniform motion obtained directly from the field equations with free asymptotic conditions.

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1 Introduction

Maxwell electrodynamics arises in a natural way in the study of quantum dynamical evolution of particles in 3D space. The non-relativistic Schrödinger equation has a form which invariant under the transformation

$$\Psi_t(x) \rightarrow e^{i\Lambda(x,t)}\Psi_t(x)$$

when the so-called gauge compensation fields are added to the space and time derivation. One finds in this method how the 3D dynamics associated with non-relativistic theory results in a 4D gauge field, which has an $O(3,1)$ invariance for the homogeneous field equations.

In a similar way, the manifestly covariant Stueckelberg Schrödinger equation (discussed below in Section 2) induces five gauge fields. Here, we study
mathematical and physical properties of these 5D gauge fields. The work is divided as follows.

In Section 2, an overview of 5D off-mass-shell gauge field theory based on the framework of Stueckelberg \cite{1,2,3} is given. 5D gauge theories also arise in other studies, such as a special case of higher dimensional relativistic dynamics and electrodynamics (cf. \cite{15,16}), or modern Kaluza-Klein type theories (cf. \cite{21} and references therein). In this paper, we concentrate on the construction which from the Stueckelberg framework (cf. \cite{2,3}).

Previous studies of the fields have been conducted (cf. \cite{10,11,12}) using certain types of Green functions (GF’s). Since the field equation in higher dimension admits many types of GF’s, in order to gain some insight into criteria for selecting useful ones, we study here a direct solution by Fourier transform, for the special case of a uniformly moving point source (UMS).

In Section 3, a derivation of 5D gauge fields generated by a uniformly moving point source is given, for both (4,1) and (3,2) metrics, followed by classification to regions of source motion, namely, spacelike and timelike. The wave equations are solved with asymptotically free conditions, in which the boundary value of the fields at infinity vanish pointwise.

In Section 4, a derivation is given of the principal part GF’s consistent with the fields generated by a uniformly moving source of Section 3.

The GF’s obtained agree with a particular form of fundamental solutions of 5D wave equations, found in, for example ref. \cite{13,14}, i.e.,

\begin{equation}
g(x, \tau) = \lim_{\epsilon \to 0^+} \frac{1}{4\pi^2} G_\epsilon(x^2 + \sigma_5 \tau^2) \quad (2)
\end{equation}

\begin{equation}
G_\epsilon(y) = \frac{\partial}{\partial \epsilon} \frac{\theta(-\sigma_5 y + \epsilon)}{\sqrt{-\sigma_5 y + \epsilon}} \quad (3)
\end{equation}

where \(\sigma_5\) determines the metric signature, \(\pm 1\) for (4,1) and (3,2) metrics, respectively.

Our present study differs from the previous literature in the following:

- The GF’s carry the group symmetry in all coordinates, whereas normally, only the \(t\) retarded solutions are considered.
- The GF’s are treated in a unified manner in both (4,1) and (3,2) metrics.
- We shall show that the derivative present in (3) is useful in regularizing the fields, whereas non derivative forms (cf. \cite{15} and references therein) have an additional infinite part, which may be removed by other methods such as Hadamard’s finite part (cf. \cite{17}).
- We study, in particular, the properties of the gauge fields, derived both from the GF’s and a more direct method, generated by a uniformly moving point source.
2 Fundamentals

An offshell classical and quantum electrodynamics has been constructed \[3\] from a fundamental theory of relativistic dynamics of 4D particles, termed events, in a framework first derived by Stueckelberg \[1, 2\].

Stueckelberg defined a Lorentz invariant Hamiltonian-like generator of evolution, over 8D phase space, parameterized by a Lorentz invariant $\tau$, in both classical and quantum relativistic mechanics. Solutions of the relativistic quantum two body bound state problem agree, up to relativistic corrections \[4\], with solutions of the non-relativistic Schrödinger equation. The experiments of Lindner, et. al. \[23\], moreover, showing quantum interference in time can be explained in a simple and consistent way in the framework of this theory \[22\], and provides strong evidence that the time $i$ is a quantum observable, as required in this framework.

In the classical manifestly covariant theory, the Hamiltonian of a free particle is given by

$$K = \frac{p_\mu p^\mu}{2M}$$

where $x^\mu = [ct, \mathbf{x}]$ and $p^\mu = [E/c, \mathbf{p}]$. A simple model for an interacting system is provided by the potential model

$$K = \frac{p_\mu p^\mu}{2M} + V(x)$$

The equation are

$$\dot{x}^\mu = \frac{\partial K}{\partial p^\mu} = \frac{1}{M}p^\mu, \quad \dot{p}^\mu = -\frac{\partial K}{\partial x^\mu} = -\frac{\partial V}{\partial x^\mu}$$

It follows from (4) that

$$v = \frac{dx}{dt} = \frac{\dot{x}}{\dot{t}} = \frac{p}{E}$$

which is the standard formula obtained for velocity in special relativity (we take $c = 1$ in the following).

Horwitz and Piron \[2\] generalized the framework to many-body systems, and gave $\tau$ the physical meaning of a universal historical time, correlating events in spacetime.

The general many-body, $\tau$ invariant, classical evolution function is defined as

$$K = \sum_{n=1}^{N} \frac{1}{2M_n} \eta_{\mu\nu} p^\mu_n p^\nu_n + V(x_1, x_2, ..., x_N)$$

where $\eta_{\mu\nu} = diag(-, +, +, +)$ and $n$ sums over all particles of the system, and, in this case, we have taken the potential function $V$ not to be a function of momenta.
The classical equations of motion, for a single particle system in an external potential $V(x)$, are similar to the non-relativistic Hamilton equations, with, in addition, motion and "momentum" along the $t$ axis:

$$\dot{x}_n = \frac{\partial K}{\partial p_{n \mu}} = \frac{1}{M_n} p_n^\mu \quad \quad \dot{p}_{n \mu} = -\frac{\partial K}{\partial x_n^\mu} = -\frac{\partial V}{\partial x_n^\mu}$$  \quad \quad \quad (8)

In the usual formulation of relativistic dynamics (cf. [24]), the energy-momentum is constrained to a mass-shell defined as

$$p_\mu p^\mu = p^2 - E^2 = -m^2$$  \quad \quad \quad (9)

where $m$ is a given fixed quantity, a property of the particle. In the Stueckelberg formulation, however, the event mass is generally unconstrained. Since in (8), the value of $K$ is absolutely conserved, $p_\mu p^\mu = -m^2$ is constant only in the special case where

$$\frac{d}{d\tau} V(x) = \left[ \dot{x}_\mu \frac{\partial}{\partial x^{\mu}} + \frac{\partial}{\partial \tau} \right] V(x) = 0$$

Thus, the proper time $d\tau$, and universal time $d\tau$, are related through the ratio between the dynamical Lorentz invariant mass $m$, and the Galilean target mass $M$. If $V(x)$ goes to zero asymptotically, then it becomes constant. Since this asymptotic value is usually what is measured in experiment, we may assume that it takes on the value of the Galilean target mass. Although there are no detailed models at present, one assumes that there is a stabilizing mechanism (for example, self-interaction or, in terms of statistical mechanics and condensation phenomenon [25]) which brings the particle, at least to a good approximation, to a defined mass value [25], such that

$$K = \frac{1}{2M} p_\mu p^\mu = -\frac{m^2}{2M} = \frac{M}{2}$$

For the quantum case, for which $P^\mu$ is represented by $-i\partial/\partial x_\mu$, the Stueckelber Schrödinger equation is taken to be (we take $\hbar = 1$ in the following)

$$i \frac{\partial \Psi_\tau(x)}{\partial \tau} = K \Psi_\tau(x)$$  \quad \quad \quad (11)

\[1\] In the non-relativistic limit, the mass distribution converges to a single point; one may choose the parameter $M$ to have this Galilean target mass value [25]. We shall assume that $M$ has this value in the following.
The Stueckelberg classical and quantum relativistic dynamics have been studied for various systems in some detail, including the classical relativistic Kepler problem \cite{2} and the quantum two body problem for central potential \cite{4}.

2.1 Off-Shell Electrodynamics

Pre-Maxwell off-shell electrodynamics is constructed in a similar fashion to the construction of Maxwell electrodynamics from the Schr"odinger equation \cite{3}.

Under the local gauge transformation

\[ \Psi'_{\tau}(x) = e^{-ie_0\chi(x,\tau)}\Psi_{\tau}(x) \] (12)

5 compensation fields \( a_\alpha(x, \tau) \) \( (\alpha \in \{0, 1, 2, 3, 5\}) \) are implied, such that with the transformation \( a'_\alpha(x, \tau) = a_\alpha(x, \tau) - \partial_\alpha \chi(x, \tau) \) the following modified Stueckelberg-Schr"odinger equation remains form invariant

\[ \left[ i \frac{\partial}{\partial \tau} + e_0 a_5(x, \tau) \right] \Psi_{\tau}(x) = \frac{1}{2M} \left[ (p^\mu - e_0 a^\mu)(p_\mu - e_0 a_\mu) \right] \Psi_{\tau}(x) \] (13)

under the transformation (12).

We can see this by observing the following relations:

\[ [p_\mu - e_0 a_\mu'] \Psi' = \left[ -i \frac{\partial}{\partial x^\mu} - e_0 \left( a_\mu - \frac{\partial}{\partial x^\mu} \chi \right) \right] e^{-ie_0\chi} \Psi = \]
\[ = \left[ e_0 \frac{\partial}{\partial x^\mu} \chi - i \frac{\partial \Psi}{\partial x^\mu} - e_0 \left( a_\mu - \frac{\partial}{\partial x^\mu} \chi \right) \right] e^{-ie_0\chi} \Psi = \]
\[ = e^{-ie_0\chi} [p_\mu - e_0 a_\mu] \Psi \]

\[ [i \frac{\partial}{\partial \tau} + e_0 (a_5 - \frac{\partial \chi}{\partial \tau})] e^{-ie_0\chi} \Psi = \]
\[ = e^{-ie_0\chi} [i \frac{\partial}{\partial \tau} + e_0 a_5] \Psi \]

The result is then, of the same form as for the usual \( U(1) \) gauge compensation argument for the non-relativistic Schrödinger equation. Thus, the classical (and quantum) evolution function for a particle, under an external field, assumed to be given by

\[ K = \frac{1}{2M} [p - e_0 a(x, \tau)]^2 - e_0 a^5(x, \tau) \] (14)

5
(where we have used the shorthand notation of $x^2 = x_\mu x^\mu$) and the corresponding Hamilton equations are

$$\dot{x}^\mu(\tau) = \frac{\partial K}{\partial p_\mu} = \frac{1}{M} [p^\mu - e_0 a^\mu] \tag{15}$$

$$\dot{p}_\mu(\tau) = -\frac{\partial K}{\partial x_\mu} = \frac{e_0}{M} (p - e_0 a(x, \tau))_\mu \partial^\nu a^\nu(x, \tau) + e_0 \partial^\nu a^\nu(x, \tau) \tag{16}$$

Here, $e_0$ is proportional to the Maxwell charge $e$ through a dimensional constant, which is derived below. Second order equations of motion for $x^\mu(\tau)$, a generalization of the usual Lorentz force, follow from the Hamilton equations (15) and (16) (17)

$$M \ddot{x}^\mu = e_0 \dot{x}^\nu f^\mu_\nu + e_0 f^\mu_5$$

where for $\alpha, \beta = 0, 1, 2, 3, 5$ the antisymmetric tensor

$$f^{\alpha \beta} \equiv \partial^\alpha a^\beta - \partial^\beta a^\alpha$$

is the 5D field tensor. Moreover, second order wave equation for the fields $f^{\alpha \beta}$ can be derived from a Lagrangian density as follows (18):

$$\mathcal{L} = -\frac{\lambda}{4} f^{\alpha \beta} f_{\alpha \beta} - e_0 a_\alpha j^\alpha$$

which produces the wave equation

$$\lambda \partial_\alpha f^{\alpha \beta} = e_0 j^\beta$$

$\lambda$ is a dimensional constant, which will be shown below to have dimensions of length. The sources $j^\beta(x, \tau)$ depend both on spacetime and on $\tau$, and obey the continuity equation

$$\partial_\alpha j^\alpha = \partial_\mu j^\mu + \partial_\tau \rho = 0 \tag{21}$$

where $j^5 \equiv \rho$ is a Lorentz invariant spacetime density of events. This equation follows from (13) for $\rho_\tau(x) = \Psi^*_\tau(x) \Psi_\tau(x)$

$$j^\mu_\tau(x) = -\frac{i}{2M} [\Psi^*_\tau(x) (i\partial^\mu - e_0 a^\mu (x, \tau)) \Psi_\tau(x) + c.c.]$$

as we discuss below, and also the classical from the argument given below.

2.1.1 Currents of point events

Maxwell current conservation, for point charges, can be derived (cf. [5]) by defining the current of a point charge as

$$J^\mu(x) = e \int_{-\infty}^{+\infty} ds \, \dot{z}^\mu(s) \delta^4[x - z(s)] \tag{22}$$
In that case, \( s \) is the proper time, and \( z^\mu(s) \) the world-line of the point charge (for free motion, \( s \) may coincide with \( \tau \)), and 
\[
\frac{dz^\mu(s)}{ds} = \frac{dz^\mu(s)}{d\tau}
\]
Then,
\[
\partial_\mu J^\mu = -e \int_{-\infty}^{+\infty} ds \frac{d}{ds} \delta^4[x - z(s)] = -e \lim_{L \to \infty} \frac{+L}{-L} \delta^4[x - z(s)]
\]
which vanishes if \( z^\mu(s) \) (or, for example, just the time component \( z^0(s) \)) becomes infinite for \( s \to \pm \infty \), and the observation point \( x^\mu \) is restricted to a bounded region of spacetime, e.g., the laboratory. We therefore, with Jackson [5], identify \( J^\mu \) as the Maxwell current. We see that this current is a functional on the world line, and the usual notion of a "particle" corresponds to this functional on the world line.

If we identify \( \delta^4[x - z(s)] \) with a density \( \rho_s(x) \) and the local (in \( \tau \)) current 
\[
\rho_s(x) = \delta^4[x - z(s)] \quad j^\mu(x, s) = \frac{dz^\mu(s)}{ds} \delta^4[x - z(s)]
\]
then the relation
\[
\frac{d}{ds} \delta^4[x - z(s)] = -\frac{dz^\mu(s)}{ds} \partial_\mu \delta^4[x - z(s)]
\]
used in the above demonstration in fact corresponds to the conservation law (reverting to the more general parameter \( \tau \) in place of the proper time \( s \))
\[
\partial_\mu j^\mu(x, \tau) + \partial_\tau \rho(x, \tau) = 0
\]
What we call the pre-Maxwell current of a point event is then defined as
\[
j^\alpha(x, \tau) = \frac{dz^\alpha(\tau)}{d\tau} \delta^4[x - z(\tau)]
\]
where \( j^5(x, \tau) \equiv \rho(x, \tau) \) and \( z^5(\tau) \equiv 1 \) (since \( z^5(\tau) \equiv \tau \)). Integrating over \( \tau \), we recover the standard Maxwell equations for Maxwell fields defined by
\[
A^\mu(x) = \int a^\mu(x, \tau)d\tau
\]
We therefore call the fields \( a^\mu(x, \tau) \) pre-Maxwell fields. Thus, Maxwell theory is properly contained in the more general pre-Maxwell theory.

For the quantum theory, a real positive definite density function \( \rho_\tau(x) \) can be derived from the Stueckelberg-Schrödinger equation
\[
\rho_\tau(x) = |\Psi_\tau|^2 = \Psi_\tau^*(x)\Psi_\tau(x)
\]
which can be identified with the \( \rho(x, \tau) = \delta^4[x - z(\tau)] \) in the classical (relativistic) limit. The continuity equation is then satisfied for the gauge invariant currents
\[
j_5^\mu(x) = -\frac{1}{2M} \left[ \Psi_\tau^*(x)(i\partial^\mu - e_0 a^\mu(x, \tau))\Psi_\tau(x) + c.c. \right]
\]
Combining (22) with (24), we obtain

$$J^\mu(x) = e \int_{-\infty}^{+\infty} j^\mu(x, \tau) \, d\tau$$

(30)

This is a restatement of the 5D continuity equation (25) and provides a connection between pre-Maxwell OSE and Maxwell electrodynamics. It also follows from (11) that

$$\frac{\partial \rho}{\partial t} + \partial_\mu j^\mu = 0$$

(31)

where \( \rho = |\Psi_\tau(x)|^2 \), as for the classical case.

2.1.2 The wave equation

From equations (20) and (18) one can derive the wave equation for the potentials \( a^\alpha(x, \tau) \):

$$\lambda \partial_\beta \partial^\beta a^\alpha - \lambda \partial^\alpha (\partial_\beta a^\beta) = e_0 j^\alpha$$

(32)

Under the generalized Lorentz gauge \( \partial_\beta a^\beta = 0 \), the wave equation takes the simpler form

$$\lambda \partial_\beta \partial^\beta a^\alpha = \lambda \left[ \Box^2 a^\alpha + \sigma_5 \frac{\partial^2 a^\alpha}{\partial \tau^2} \right] = e_0 j^\alpha(x, \tau)$$

(33)

where a 5th diagonal metric component can take either signs \( \sigma_5 = \pm 1 \), corresponding to \( O(4,1) \) and \( O(3,2) \) symmetries of the homogeneous field equations, respectively.

Integrating (33) with respect to \( \tau \), and assuming that \( \lim_{\tau \to \pm \infty} \partial_\tau a^\alpha(x, \tau) = 0 \), we obtain

$$\lambda \int_{-\infty}^{+\infty} \, d\tau \left[ \Box^2 a^\alpha + \sigma_5 \frac{\partial^2 a^\alpha}{\partial \tau^2} \right] = \frac{e_0}{e} j^\alpha(x)$$

Identifying

$$A^\mu(x) = \int_{-\infty}^{+\infty} \, d\tau \, a^\mu(x, \tau)$$

(34)

we obtain

$$\lambda \Box^2 A^\mu(x) = \frac{e_0}{e} J^\mu(x)$$

(35)

(where we’ve restricted our attention to \( \mu = 0, 1, 2, 3 \)), from which a relation between \( e_0, \lambda \) and the Maxwell charge \( e \) can be obtained:

$$e = \frac{e_0}{\lambda}$$

Therefore, the Maxwell electrodynamics is properly contained in the 5D electromagnetism.
2.1.3 A note about units

In natural units ($\hbar = c = 1$), the Maxwell potentials $A^\mu$ have units of $1/L$. Therefore, the pre-Maxwell OSE potentials $a^\alpha$ have units of $1/L^2$, and in order to maintain the action integral

$$S = \int_{-\infty}^{+\infty} \mathcal{L} \, d\tau \, d^4 x$$

(36)

dimensionless, the coefficient $\lambda$ in (19) must have units of $L$, forcing $e_0$ to have units of $L$ as well.

The Fourier transform of the pre-Maxwell OSE fields

$$\tilde{a}^\mu(x, s) = \int_{-\infty}^{+\infty} e^{isx} a^\mu(x, \tau) \, d\tau$$

(37)

and equation (34) suggest that the Maxwell potentials and fields are the zero mode of the pre-maxwell OSE fields, with respect to the $\tau$ axis, i.e.,

$$A^\mu(x) = \tilde{a}^\mu(x, s)|_{s=0}$$

(38)

2.2 Solutions of the wave equation

The classical 5D wave-equation (33) can be solved by the method of Green-functions. Such GF’s have been found [7, 10] through a 5-fold Fourier transform of the wave equation. The GF obeys the wave-equation of a point source

$$\partial_\beta \partial^\beta g(x, \tau) = \delta^4(x) \delta(\tau)$$

(39)

After transformation to momentum space, (39) becomes ($k^2 = k_\mu k^\mu$)

$$(k^2 + \sigma_5 k_5^2) \tilde{g}(k, k_5) = 1$$

(40)

i.e., in terms of the inverse transform

$$g(x, \tau) = \frac{1}{(2\pi)^5} \int d^4k \, \frac{1}{k^2 + \sigma_5 k_5^2} e^{i[k \cdot x + \sigma_5 k_5 \tau]}$$

(41)

Although (41) is not a well defined integral, there are, as for GF’s in 4D, several ways of defining the integral, which result in GF’s all of which satisfy (39). These different forms of solutions have physical consequences and it is part of the motivation of our work to obtain criteria which could determine this choice.

For example, Land and Horwitz [7] found what they called the Principal-Part GF to be

$$g_{PP}(x, \tau) = -\frac{1}{4\pi} \delta(x^2) \delta(\tau) - \frac{1}{2\pi^2} \frac{\partial}{\partial x^2} \frac{\theta(-\sigma x^2 - \tau^2)}{\sqrt{-\sigma x^2 - \tau^2}}$$

(42)
where \( \sigma = \sigma_5 = \pm 1 \) is the signature of the fifth dimension, \( \tau \).

A later work by Oron and Horwitz \[10\] found another, \( \tau \)-retarded GF of the form

\[
g(x, \tau) = \begin{cases} 
\frac{1}{2\pi^3} \ln \left| \frac{\tau - \sqrt{\tau^2 + x^2}}{\tau + \sqrt{\tau^2 + x^2}} \right| - \frac{\tau}{x^2(x^2 + \tau^2)} & x^2 + \tau^2 < 0 \\
\frac{1}{\pi} \left( \frac{d}{d\tau} \frac{1}{\pi dt^2} \right)^{(n-2)/2} \theta(t - |x|) \frac{\sqrt{t^2 - x^2}}{\sqrt{t^2 - x^2}} & x^2 + \tau^2 > 0 
\end{cases}
\]

(43)

where only the \((4,1)\) case was explicitly given.

GF's of \((n,1)\) wave equations are well known in the mathematical literature (this is taken from \[13\], cf. also \[14\], \[15\], \[16\]):

\[
g(x, t) = \begin{cases} 
\frac{\theta(t)}{2\pi} \left( \frac{1}{\pi dt^2} \right)^{(n-3)/2} \delta(t^2 - x^2) & n = 3, 5, 7, \cdots \\
\frac{1}{2\pi} \left( \frac{1}{\pi dt^2} \right)^{(n-2)/2} \theta(t - |x|) \frac{\sqrt{t^2 - x^2}}{\sqrt{t^2 - x^2}} & n = 4, 6, 8, \cdots 
\end{cases}
\]

(44)

where \( n = 4 \) is the case of \((4,1)\) metric. These well known GF's, which are retarded in \( t \), can be made symmetric by proper choice of contour of integration on the original Fourier integral representation.

Once the GF's have been found, the general field generated by a given source can then be found by integration on its support

\[
a^\alpha(x, \tau) = e_0 \int d^4x' \, d\tau' \, g(x - x', \tau - \tau') \, j^\alpha(x', \tau')
\]

(45)

and applying it on a point particle given by \[21\]. The potentials of point events can then be found

\[
a^\alpha(x, \tau) = e_0 \int_{-\infty}^{+\infty} d\tau' \, g(x - z(\tau'), \tau - \tau') \, \dot{z}^\alpha(\tau')
\]

(46)

In order to get some insight into the criteria for choosing appropriate GF's, we study solutions of the differential equation \[63\] (for the particular choice of uniformly moving point source) without using the GF, i.e., we compute directly

\[
a^\alpha(x, \tau) = \frac{e}{(2\pi)^3} \int_{-\infty}^{+\infty} d\tau' \, \dot{z}^\alpha(\tau') \int d^4k \, dk_5 \, \frac{e^{i[k \cdot (x - z(\tau')) + \sigma_5 k_5 (\tau - \tau')]}}{k^2 + \sigma_5 k_5^2}
\]

(47)

Solutions of the integral \[64\] are the subject of the next section.
3 Fields of a uniformly moving point charge

3.1 Solutions of the wave equation

Let us seek a solution to the field equation generated by a uniformly moving point source. Such a source has a general worldline description

\[ z^\alpha(\tau) = z_0^\alpha + b^\alpha(\tau - \tau_0) \equiv D^\alpha + b^\alpha \tau \quad \dot{z}^\alpha(\tau) = b^\alpha \quad \alpha \in \{0, 1, 2, 3, 5\} \]

(48)

where for \( z^5 \equiv \tau \) we have \( b^5 = 1 \). However, we leave \( b^5 \) unspecified, leaving the possibility for a 5D symmetry of the solution to emerge, as indeed we find.

Without loss of generality, we can eliminate \( D^\alpha \) by choosing a coordinate system in which \( D^\alpha = 0 \). The current of such a source is then given by

\[ j^\alpha(x, \tau) = b^\alpha \delta^4 [x - b\tau] \]

(49)

Substituting (48) into (47) we obtain an integral representation of the uniform motion fields, which could be called pre-Coulomb fields:

\[ a^\alpha(x, \tau) = e \frac{1}{(2\pi)^5} \int_{-\infty}^{+\infty} d\tau' b^\alpha \int d^4k \int_{-\infty}^{+\infty} d\tau' \frac{e^{ik_5(x - b\tau') + \sigma_5 k_5(\tau - b\tau')}}{k^2 + \sigma_5 k_5^2} = \]

\[ = e \frac{b^\alpha}{(2\pi)^5} \int_{-\infty}^{+\infty} d\tau' \int d^3k \frac{e^{ik_\beta(x^\beta - b^\beta \tau')}}{k_\beta k_\beta} = \]

\[ = e \frac{b^\alpha}{(2\pi)^5} \int d^3k \int_{-\infty}^{+\infty} d\tau' \frac{e^{ik_\beta(x^\beta - b^\beta \tau')}}{k_\beta k_\beta} = \]

\[ = e \frac{b^\alpha}{(2\pi)^5} \int d^3k \frac{e^{ik_\beta x^\beta}}{k_\beta k_\beta} \int_{-\infty}^{+\infty} d\tau' e^{-ik_\beta b^\beta \tau'} = \]

\[ = e \frac{b^\alpha}{(2\pi)^4} \int d^3k \frac{e^{ik_\beta x^\beta}}{k_\beta k_\beta} \delta (k_\beta b^\beta) \]

(50)

The argument \( k_\beta b^\beta \) of the \( \delta \)-function causes the 5-fold integration to be constrained to a 5D hyperplane,

\[ S[b] = \{ k \in \mathbb{R}^5 | k_\alpha b^\alpha = 0 \} \]

whose normal is just \( b^\beta \).

In order to proceed, we select a pivot axis, for which integration would put the remaining 4-fold integral to be in that hyperplane. Naturally, we select \( k^5 \),
since we will take $b^5 > 0$:
\[
k_\beta b^\beta = k \cdot b + \sigma_5 k^5 b^5 = b^5 \left[ k \cdot b' + \sigma_5 k^5 \right]
\]
where $\mu \in \{0, 1, 2, 3\}$ and
\[
b'_{\mu} = \frac{b_{\mu}}{b^5}
\]
$b'_{\mu}$ is the $(3,1)$ velocity of the source relative to its motion in the $\tau$ direction.

We then have
\[
\delta (k_\beta b^\beta) = \frac{1}{|b^5|} \delta (k \cdot b' + \sigma_5 k^5) \implies k^5 = -\frac{1}{\sigma_5} k \cdot b'
\]
\[
k_\beta k^\beta = k^2 + \sigma_5 (k^5)^2 = k^2 + \sigma_5 \left[ -\frac{1}{\sigma_5} (k \cdot b') \right]^2 = k^2 + \sigma_5 (k \cdot b')^2
\]
\[
k_\beta x^\beta = k \cdot x + \sigma_5 k^5 \tau = k \cdot x + \sigma_5 -\frac{1}{\sigma_5} (k \cdot b') \tau = k \cdot (x - b' \tau)
\]
(51)

And thus, we obtain:
\[
a^\alpha (x, \tau) = \frac{e^{ib^\alpha}}{2\pi^4 |b^5|} \int \frac{e^{ik \cdot (x - b' \tau)}}{k^2 + \sigma_5 (k \cdot b')^2} \quad (52)
\]
The integral can be solved by introducing a rotation in $k$ space in which $b'$ takes a particularly simple form, namely, along one of the axes. Aside from the special case of $b'^2 = 0$, $b'$ can be rotated to be along one of the axes by an $SO(3,1)$ transformation. We shall now divide our discussion to the spacelike case where $b'^2 > 0$ and the timelike case where $b'^2 < 0$, and to avoid complications, we shall solve the zero measure case of $b'^2 = 0$ by a limiting procedure.

3.2 $a$-fields due to a (3,1) timelike source
Since $b'^2 < 0$, we can find $\Lambda \in SO(3,1)$ such that
\[
b' = \Lambda b'' \quad \text{such that} \quad b'' = [b'^0, 0]
\]
\[
b'^0 \equiv s = \epsilon(b^0) \sqrt{-b'^2}
\]
and since $|\Lambda| = 1$, we have $d^4k'' = d^4k$. Replacing $b'$ with $b''$ and $k$ with $k''$, we obtain:

$$a^\alpha(x, \tau) = \frac{e b^\alpha}{2\pi^4|b^5|} \int d^4k'' e^{i[k'' \cdot x'' - k''(x'' - s\tau)]} = \frac{e b^\alpha}{2\pi^4|b^5|} \int d^4k'' e^{i[k'' \cdot x'' - k''(x'' - s\tau)]} \frac{1}{k''^2 + (k''0)^2(\sigma_5 s^2 - 1)}$$

(53)

where $x'' = \Lambda^{-1}x$. We follow the convention of boldface corresponding to the space part of a four-vector. Since $b''0$ is a 4-vector along the time axis, we can find simple closed form expressions for $x''$ as follows:

$$x''0 = \frac{x''0}{s} = -x'' \cdot b'' \sqrt{-b''^2}$$

$$(x''0)^2 = x''^2 + (x''0)^2 = x^2 + \frac{(b' \cdot x)^2}{-b''^2} = x^2 - \frac{(b' \cdot x)^2}{b''^2}$$

Integral (53) depends on the value of the denominator along the path of integration, where 2 types of source motion emerge. The types are given as follows:

| Source motion $\sigma_5 b^2$ | Description |
|-------------------------------|-------------|
| $\sigma_5 b^2 > 1$            | Supershell case, where the integral has a well defined solution, essentially the Laplace GF in 4D. |
| $\sigma_5 b^2 < 1$            | Undershell case, where the integral is not well defined. The integral is essentially the Maxwell GF. |

In the following we describe the properties of these cases.

### 3.2.1 Undershell timelike $\alpha$-fields $\sigma_5 b^2 > -1$

As mentioned, the denominator of integral (53) is not positive definite. Nevertheless, we shall proceed first by absorbing the coefficient $1 - \sigma_5 s^2$ into $k^0$.

$$k^0 \rightarrow \frac{k^0}{\sqrt{1 - \sigma_5 s^2}}$$

Equation (53) obtains the form

$$a^\alpha(x, \tau) = \frac{e b^\alpha}{2\pi^4|b^5|\sqrt{1 - \sigma_5 s^2}} \int d^4k \frac{1}{k^2 - (k^0)^2} \left\{ 1 \left( \frac{1}{2\pi^4} \int d^4k \frac{e^{ik \cdot y}}{k^2} \right) \right.$$

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where

\[ y^\mu = \left[ x''' - \frac{s \tau}{\sqrt{1 - \sigma_5 s^2}} ; x'' \right] \]

The last integral inside the braces is the well known Fourier integral form for Maxwell wave-equations’s GF in 4 dimensions, and though it is ill-defined, it has many well known solutions, corresponding to different limits of the integration contour chosen. We shall choose the Principal Part solution for our present study.

\[ G_P(x) = \frac{1}{(2\pi)^4} P \int_{\mathbb{R}^4} d^4 k \frac{e^{i k \cdot x}}{k^2} = \frac{\delta(x^2)}{4\pi} \]  

(54)

corresponding the sum of retarded and advanced GF’s. Using the \( G_P \) above, we arrive at the undershell \( a^- \)-fields:

\[ a^\alpha(x, \tau) = e^{b^\alpha} \frac{4\pi |b^5|}{\sqrt{1 + \sigma_5 b^2}} \delta \left[ x''^2 - \frac{(x''' - s \tau)^2}{1 - \sigma_5 s^2} \right] = \frac{e^{b^\alpha}}{4\pi |b^5| \sqrt{1 + \sigma_5 b^2}} \delta \left[ x^2 - \frac{(b' \cdot x)^2}{b'^2} + \frac{(-b' \cdot x + b'^2 \tau)^2}{b'^2 (1 + \sigma_5 b'^2)} \right] \]  

(55)

We call these undershell solutions because they correspond to the offshell mass of the source below its Galilean target mass. Equation (55) has \( O(3, 1) \) symmetry, with respect to the lower 4 coordinates of \( x^\alpha \). It can be further broken to a sum of \( \delta \)-functions with linear arguments in \( \tau \), as follows:

\[ p(\tau_{1,2}) = x^2 - \frac{(b' \cdot x)^2}{b'^2} + \frac{(-b' \cdot x + b'^2 \tau_{1,2})^2}{b'^2 (1 + \sigma_5 b'^2)} = 0 \]

\[ \tau_{1,2} = \frac{b' \cdot x}{b'^2} \pm \frac{1}{b'^2} \sqrt{1 + \sigma_5 b'^2} \sqrt{(b' \cdot x)^2 - b'^2 x^2} \]

Using the linearity of the \( \delta \)-function

\[ \delta(p(\tau)) = \frac{b'^2 (1 + \sigma_5 b'^2)}{(b'^2)^2} \delta \left[ (\tau - \tau_1)(\tau - \tau_2) \right] \delta(\tau - \tau_1) + \delta(\tau - \tau_2) \]

we then have

\[ a^\alpha(x, \tau) = \frac{e^{b^\alpha} \Delta_+}{8\pi |b^5| \sqrt{(b' \cdot x)^2 - b'^2 x^2}} \]

where

\[ \Delta_+(x, \tau) = \delta(\tau - \tau_1) + \delta(\tau - \tau_2) \]

The \( a^- \)-field depends on the the fifth metric component, \( \sigma_5 \), only through \( \tau_{1,2} \), i.e., the coefficient is independent of the signature of the 5D space. However,
the values \( \tau = \tau_{1,2} \) correspond to a 4D surface in 5D space where the \( a \)-fields are non-zero, and therefore, the metric appears, to some extent, in the geometry of the non-zero surfaces.

After some algebra, the \( a \)-fields can gain yet another, 5D covariant form. We start by rewriting the \( \delta \)-function argument in 5D form:

\[
p(\tau) = x^2 - \frac{(b' \cdot x)^2}{b'^2} + \frac{(-b' \cdot x + b'^2 \tau_{1,2})^2}{b'^2(1 + \sigma_5 b'^2)} = \]

\[
x_\beta x^\beta - \frac{b_\beta x^\beta}{b_\beta b^\beta} \tag{56}
\]

where the metric signature of \((4,1)\) or \((3,2)\) is used in the contraction products, e.g.:

\[
b_\beta x^\beta = b \cdot x + \sigma_5 b^5 x^5 = b \cdot x + \sigma_5 b^5 \tau
\]

For \((4,1)\) case, we have:

\[
b_\alpha b^\alpha = b^2 + (b^5)^2 = 1 \left[ (b^5)^2 + 1 \right] > 0
\]

since \( \sigma_5 b^2 > -1 \) in the present case. Thus, in the \((4,1)\) metric, the undershell motion corresponds to the 5D \textit{spacelike} region in the \( b^\alpha \) velocity space. As shall be observed later, this region is not limited to 4D timelike source motion \( b'^2 < 0 \), and it includes \( b'^2 \geq 0 \) as well. For the \((3,2)\) metric, on the other hand, the motion is in the 5D timelike region \( b_\alpha b^\alpha < 0 \).

Furthermore, one can define

\[
n^\alpha = \frac{b^\alpha}{\sqrt{\sigma_5 b_\beta b^\beta}} \tag{57}
\]

Substituting equations \(56,57\) into \(55\), we arrive at the 5D covariant form:

\[
a^\alpha(x, \tau) = \frac{en^\alpha}{4\pi} \delta \left[ x_\beta x^\beta - \sigma_5 (n_\beta x^\beta)^2 \right] \tag{58}
\]

The term \textit{undershell source motion} stems from the mass shell equation

\[
P_\alpha P^\alpha = M^2 \dot{x}_\alpha \dot{x}^\alpha = M^2 \left[ -\frac{m^2}{M^2} + \sigma_5 \right] = \sigma_5 M^2 \left[ -\sigma_5 \frac{m^2}{M^2} + 1 \right] = \]

\[
M^2 b_\alpha b^\alpha \tag{59}
\]

or

\[
b_\alpha b^\alpha = \sigma_5 \left[ -\sigma_5 \frac{m^2}{M^2} + 1 \right] \tag{60}
\]

For \((4,1)\) metric, \( \sigma_5 = 1 \), where undershell timelike motion leads to \( |m| < M \). Hence, undershell motion refers to the \textit{mass-shell of the source} being \textit{less} than its non-relativistic mass-shell \( M \).
3.2.2 Supershell timelike $a$-fields $\sigma_5 b^2 < -1$

As the name suggests, in the (4,1) metric of the source motion, the supershell case is determined by $|m| > M$, i.e., the relativistic mass $|m|$ being greater than the its non-relativistic Galilean target mass $M$. In this case, however, only $\sigma_5 = 1$ is applicable, since there is no timelike motion $b^2 < 0$ such that $(-1)b^2 + 1 < 0$, unless $b^2 > 1$, which is no longer timelike. Such a case will be investigated later. In this case, the integral is well defined, as the zeros in the denominator are no longer real. By following a similar procedure of absorbing the coefficient of $k^0$ in the denominator into $k^0$, we obtain ($\sigma_5 = 1$):

$$a^\alpha(x, \tau) = \frac{e^{b^\alpha}}{2\pi^4 |b^5| \sqrt{s^2 - 1}} \int d^4k \frac{e^{i[k \cdot x'' - k^0(x''\cdot \tau)/\sqrt{s^2 - 1}]} k^2 + (k^0)^2}{k^2 + (k^0)^2}\int_0^+ dk \sin(k|x''|) \int_0^+ dk^0 e^{-ik^0(x''\cdot \tau)/\sqrt{s^2 - 1}} =$$

$$= \frac{4\pi e^{b^\alpha}}{(2\pi)^4 |b^5| |x''| \sqrt{s^2 - 1}} \int_0^+ k dk \sin(k|x''|) \int_0^+ dk^0 \frac{e^{-ik^0(x''\cdot \tau)/\sqrt{s^2 - 1}}}{k^2 + (k^0)^2} =$$

$$= \frac{e^{b^\alpha}}{4\pi^3 |b^5| |x''| \sqrt{s^2 - 1}} \int_0^+ k dk \sin(k|x''|)(-1)\frac{\pi}{k} e^{-ik(x''\cdot \tau)/\sqrt{s^2 - 1}} =$$

$$= \frac{e^{b^\alpha}}{4\pi^2 |b^5| |x''| \sqrt{s^2 - 1} - 2i} \left[ \frac{0}{-|x''\cdot \tau|/\sqrt{s^2 - 1} + i|x''|} - \frac{0 - 1}{-|x''\cdot \tau|/\sqrt{s^2 - 1} - i|x''|} \right] =$$

$$= \frac{e^{b^\alpha}}{4\pi^2 |b^5| \sqrt{-b^2 - 1}} \left[ x^2 - \frac{(b' \cdot x)^2}{b'^2} + (-b' \cdot x + b'^2 \tau)^2}{b'^2(b'^2 + 1)} \right] =$$

$$= \frac{e n^\alpha}{4\pi^2 [x_\beta x^\beta + (n_\beta x^\beta)^2]}$$

where we have defined $n^\alpha \equiv b^\alpha / \sqrt{-b_\beta b^\beta}$.

The supershell $a$-field is found to be a smooth function on 5D spacetime; it is the 5D analogue of the well-known (cf. on-shell 4D Maxwell $a$-field of a uniformly moving charge

$$A^\mu(x) = \frac{en^\mu}{4\pi \sqrt{x^2 + (n \cdot x)^2}},$$

where $n^\mu = \frac{dz^\mu(s)}{ds}$ is the constant 4-velocity obeying the mass-shell constraint $n^2 = -1$. Clearly, the 5D $a$-field, proportional to $r^{-2}$, as opposed to the Maxwell $A$-field being proportional to $r^{-1}$, is a consequence of the additional dimension.

The supershell $a$-field has the same form as the GF for the 4D Laplace operator:

$$\nabla^2 G_L(x) = \delta^4(x) \quad (62)$$

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where \( x \in \mathbb{R}^4 \), \( r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 \)

and the GF \( G_L(x) \) is given by (cf. [13]):

\[
G_L = \frac{1}{(2\pi)^4} \int \frac{e^{ik_c x_c}}{k_c k_c} = \frac{1}{2\pi^2 r^2} \tag{63}
\]

where in this case

\[
k_c x_c = k^1 x^1 + k^2 x^2 + k^3 x^3 + k^4 x^4
\]

This is far from being coincidental, as the 5D scalar \( x_\beta x^\beta + (n_\beta x^\beta)^2 \) reduces to \( r^2 \) in 4D, when the source’s uniform velocity is purely temporal \( n_\beta = [1; 0, 0, 0] \).

### 3.3 \( a \)-fields due to a spacelike source

We now solve the integral (52) for spacelike source motion, \( b'^2 > 0 \), which may not be regarded as a possible physical source, since it implies faster than light motion of the source particle. Once again, we choose to integrate in a \( k \)-frame such that \( b''^3 = [0; 0, 0, b''^3] \) is along one of the spatial axes, e.g., the \( z \)-axis, and we now define

\[
b'^3 \equiv s = \sqrt{b'^2} = \sqrt{b'^2}, \quad b'^2 > 0 \tag{64}
\]

The current in the \( b'^3 \) frame can be expressed by

\[
j'^\mu(x'', \tau) = b'^{\mu} \delta^4[x-z''^\mu(\tau)] = b'^3 \delta^3(t) \delta(y'') \delta(z'' - b'^3 \tau) \tag{65}
\]

Returning to pre-Maxwell \( a \)-field integral (52),

\[
a'^\alpha(x, \tau) = \frac{e b'^\alpha}{|b'^3|(2\pi)^4} \int d^4 k'' e^{i[k''^\mu x''^\mu + k''^3 (z'' - s\tau) - k''^0 t']} \frac{e^{i[k''^1 x''^1 + k''^2 y''^2 + k''^3 (z'' - s\tau) - k''^0 t']} \delta^4(x'' - z''^\mu(\tau))}{(k''^1)^2 + (k''^2)^2 + (k''^3)^2 - (k''^0)^2 + \sigma_5 (sk'')^2}
\]

where we have renamed \( k'' \) as \( k \) and \( x'' \) as \( x \).

The coefficient of \( (k'^3)^2 \) changes sign when \( 1 + \sigma_5 s^2 = 1 + \sigma b'^2 = 0 \) which can only occur when \( \sigma_5 = -1 \) (since for spacelike motion, \( b'^2 > 0 \)) and \( |s| \geq 1 \) \( (s = e(b'^0)\sqrt{b'^2}) \), causing the denominator to obtain a \((2, 2)\) quadratic form.

Once again, the form of the fields are characterized by the types of source motion. In the following, we shall treat both possible types of source motion separately.
3.3.1 Under Spacelike motion $1 + \sigma_5 b'^2 > 0$

Rescaling $k^3 \rightarrow k^3/\sqrt{1 + \sigma_5 s^2}$, the spacelike a-field integral can be expressed by

$$a^\alpha(x, \tau) = \frac{e^{b^\alpha}}{|b^\alpha|(2\pi)^4 \sqrt{1 + \sigma_5 s^2}} \int d^4k \frac{e^{ik\cdot y}}{k^2} =$$

$$= \frac{e^{b^\alpha} \delta(y^2)}{4\pi |b^\alpha| \sqrt{1 + \sigma_5 s^2}}$$

where $y^\mu = \left[ x^0; x^1, x^2, \frac{x^3 - s\tau}{\sqrt{1 + \sigma_5 s^2}} \right]$, $k^2 = k_\mu k^\mu$, and we have chosen the Principal Part contour.

However:

$$y^2 = (x^1)^2 + (x^2)^2 + \frac{(x^3 - s\tau)^2}{1 + \sigma_5 s^2} - (x^0)^2 =$$

$$= x_\mu x^\mu - (x^3)^2 + \frac{(x^3 - \sqrt{1 + \sigma_5 s^2} \tau)^2}{1 + \sigma_5 s^2}$$

We can furthermore put $x^3$ into an invariant form:

$$x^3 = \frac{x^3 s}{s} = \frac{x \cdot b'}{b'^2}$$

$$y^2 = x^2 - \frac{(b' \cdot x)^2}{b'^2} + \frac{(-b' \cdot x + b'^2 \tau)^2}{b'^2(1 + \sigma_5 s^2)}$$

The spacelike $a$-fields then obtain a 4D covariant form:

$$a^\alpha(x, \tau) = \frac{e^{b^\alpha}}{8\pi |b^\alpha| \sqrt{(b' \cdot x)^2 - b'^2 x^2}} \delta \frac{1}{b'^2(1 + \sigma_5 s^2)} \left[ x^2 - \frac{(b' \cdot x)^2}{b'^2} + \frac{(-b' \cdot x + b'^2 \tau)^2}{b'^2(1 + \sigma_5 s^2)} \right] =$$

$$= \frac{e^{b^\alpha} \Delta_+}{8\pi |b^\alpha| \sqrt{(b' \cdot x)^2 - b'^2 x^2}}$$

where we have, as before:

$$\Delta_+ = \delta(\tau - \tau_1) + \delta(\tau - \tau_2)$$

$$\tau_{1,2} = \frac{b' \cdot x}{b'^2} \pm \frac{1}{b'^2} \sqrt{1 + \sigma_5 b'^2 \sqrt{(b' \cdot x)^2 + b'^2 x^2}}$$

Thus, the under spacelike motion fields are of the same form as their timelike under-shell counterparts.

3.3.2 Super Spacelike motion $1 + \sigma_5 b'^2 > 0$

As mentioned above, in this case we have $b'^2 > 1$ and the choice $\sigma_5 = -1$ is therefore necessary. Therefore, the integral takes the form:

$$a^\alpha(x, \tau) = \frac{e^{b^\alpha}}{|b^\alpha|(2\pi)^4} \int d^4k \frac{e^{i(k^4 x + k^7 x + k^3 (z^3 - s \tau) - k^0 \tau)}}{(k^1)^2 + (k^2)^2 + (s^2 - 1)(k^3)^2 - (k^0)^2}$$

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We shall solve this integral by transforming the integrand to a Gaussian form

\[ \int_{-\infty}^{+\infty} \frac{e^{iax}}{\pi^{2}} \, dx = \frac{1}{2i} \int_{-\infty}^{+\infty} \varepsilon(q) \, dq \int_{-\infty}^{+\infty} e^{iax + iqx^{2}} \, dx \]

where \( \varepsilon(q) \) is the sign function. Using this relation in (68) we obtain

\[ a^{\alpha}(x, \tau) = \frac{e^{b^{\alpha}}}{(2\pi)^{4} |b^{5}| \sqrt{s^{2} - 1}} \frac{1}{2i} \int_{-\infty}^{+\infty} dq \varepsilon(q) \int d^{4}k \exp \left[ i k \cdot y + i q \left( (k^{1})^{2} + (k^{2})^{2} - (k^{3})^{2} - (k^{0})^{2} \right) \right] = \]

\[ = \frac{e^{b^{\alpha}}}{(2\pi)^{4} |b^{5}| \sqrt{s^{2} - 1}} \frac{1}{2i} \int_{-\infty}^{+\infty} dq \varepsilon(q) \left( \frac{\pi}{q} \right)^{4} \exp \left[ \frac{i}{4q} \left( (y^{1})^{2} + (y^{2})^{2} - (y^{3})^{2} - (y^{0})^{2} \right) \right] = \]

\[ = \frac{e^{b^{\alpha}} |b^{5}| \pi^{2}}{(2\pi)^{4} \sqrt{s^{2} - 1}} \frac{1}{2i} \int_{0}^{+\infty} du \left\{ \exp \left[ \frac{iu}{4} \left( (y^{1})^{2} + (y^{2})^{2} - (y^{3})^{2} - (y^{0})^{2} \right) \right] - \exp \left[ -\frac{iu}{4} \left( (y^{1})^{2} + (y^{2})^{2} - (y^{3})^{2} - (y^{0})^{2} \right) \right] \right\} \]

Where we have put \( u = 1/q \). Here, the singularity at \( q = 0 \) is controlled by the oscillation in the exponent (although one can find the same result by other methods).

Using

\[ \int_{0}^{+\infty} \exp [iax] \, dx = P \left[ \frac{i}{a} \right] + \pi \delta(a) \]

we find the fields to be:

\[ a^{\alpha}(x, \tau) = -\frac{e^{b^{\alpha}}}{(2\pi)^{4} |b^{5}| \sqrt{s^{2} - 1}} \frac{1}{2} \left[ (y^{1})^{2} + (y^{2})^{2} - (y^{3})^{2} - (y^{0})^{2} \right] = \]

\[ = \frac{e^{b^{\alpha}}}{4\pi^{2} |b^{5}| \sqrt{s^{2} - 1}} \frac{1}{2} \left[ (y^{1})^{2} + (y^{2})^{2} - (y^{3})^{2} - (y^{0})^{2} \right] \]

We have

\[ y^{\mu} = \left[ x^{0}, x^{1}, x^{2}, \frac{x^{3} - s \tau}{\sqrt{s^{2} - 1}} \right] \]

\[ (y^{1})^{2} + (y^{2})^{2} - (y^{3})^{2} - (y^{0})^{2} = \left( x^{1} \right)^{2} + \left( x^{2} \right)^{2} - \frac{\left( x^{3} - s \tau \right)^{2}}{s^{2} - 1} - \left( x^{0} \right)^{2} = \]

\[ = x^{2} - \frac{(b' \cdot x)^{2}}{b'^{2}} - \frac{(b \cdot x - b^{2} \tau)^{2}}{(b'^{2} - 1)b'^{2}} \]
Thus we obtain the final form

\[ a^\alpha(x, \tau) = -\frac{e b^\alpha}{4\pi^2|b^5|\sqrt{|b^2| - 1}} \left( \frac{1}{x^2 - \frac{(b \cdot x)^2}{b^2} - \frac{(b \cdot x - b^2\tau)^2}{b^2(b^2 - 1)}} \right) \]  

(69)

The corresponding 5D covariant form is then

\[ a^\alpha(x, \tau) = \frac{en^\alpha}{4\pi^2} \frac{1}{\left( (n_\alpha x^\alpha)^2 - x_\alpha x^\alpha \right)} \]  

(70)

where \( n^\alpha = \frac{b^\alpha}{\sqrt{b_\beta b^\beta}} \)

3.4 Summary of fields generated by a Uniformly Moving Source

We present a short summary of the results obtained in this section:

\[
 a^\alpha(x, \tau) = \begin{cases} 
\frac{en^\alpha}{4\pi} \delta \left[ x_\beta x^\beta - (n_\beta x^\beta)^2 \right] & \text{Undershell } \sigma_5 = +1, n_\alpha n^\alpha = +1 \\
\frac{en^\alpha}{4\pi^2} \frac{1}{x_\beta x^\beta + (n_\beta x^\beta)^2} & \text{Supershell } \sigma_5 = +1, n_\alpha n^\alpha = -1 \\
\frac{en^\alpha}{4\pi} \delta \left[ x_\beta x^\beta + (n_\beta x^\beta)^2 \right] & \text{Under-spacelike } \sigma_5 = -1, n_\alpha n^\alpha = -1 \\
\frac{en^\alpha}{4\pi^2} \left( (n_\beta x^\beta)^2 - x_\beta x^\beta \right) & \text{Super-spacelike } \sigma_5 = -1, n_\alpha n^\alpha = +1 
\end{cases} 
\]  

(71)

In a more general compact representation, we have

\[
 a^\alpha(x, \tau) = \begin{cases} 
\frac{en^\alpha}{4\pi} \delta \left[ (n_\beta x^\beta)^2 - \sigma_5 x_\beta x^\beta \right] & \zeta = +1 \\
\frac{en^\alpha}{4\pi^2} \frac{1}{((n_\beta x^\beta)^2 + \sigma_5 x_\beta x^\beta)} & \zeta = -1 
\end{cases} 
\]  

(72)

where:

\[ \zeta = \sigma_5 \cdot \varepsilon [b_\alpha b^\alpha] = \sigma_5 \cdot \varepsilon [b^2 + \sigma_5 (b^5)^2] \]

\[ n^\alpha = \frac{b^\alpha}{|b_\beta b^\beta|} \]

The various values for \( \sigma_5 \) and \( \zeta \) are given in table I.
| Metric | $\sigma_5$ | Velocity region | Mass shell | $\varepsilon(b_\alpha b^\alpha)$ | $\zeta$ |
|--------|----------|-----------------|------------|-------------------------------|-------|
| (4, 1) | +1       | Undershell      | $m^2 < M^2$| +1                            | +1    |
| (4, 1) | +1       | Supershell      | $m^2 > M^2$| −1                            | −1    |
| (3, 2) | −1       | Under-spacelike | $-m^2 < M^2$| −1                            | +1    |
| (3, 2) | −1       | Super-spacelike | $-m^2 > M^2$| +1                            | −1    |

Table 1: Regions of source velocity summary.

### 3.5 Concatenation

As we have pointed out above, the pre-Maxwell theory can be contracted to Maxwell form by integration (as in (34)) over $\tau$ (called concatenation) Applying this procedure to the 5D pre-Maxwell fields which we have obtained above, we find

$$A^\mu(x) = \int_{-\infty}^{+\infty} a^\mu(x, \tau) d\tau = \frac{e n^\mu}{4\pi} \frac{\theta([n \cdot x]^2 - n^2 x^2)}{\sqrt{[n \cdot x]^2 - n^2 x^2}}$$

(73)

where

$$n^2 = n_\mu n^\mu = n_\alpha n^\alpha - \sigma_5 (n^5)^2$$

and $n_\alpha n^\alpha = \pm 1$ according to the velocity regions of source motion (see table 1). It should be emphasized that (73) is a general Maxwell field, for all values of $n^2$. The solutions can be put into a more specific form for the 3 regions of (3, 1) spacetime:

$$A^\mu(x) = \begin{cases} 
\frac{e n^\mu}{4\pi \sqrt{[n' \cdot x]^2 + x^2}} & n'^2 = -1 \\
\frac{e n^\mu}{4\pi} \frac{\theta([n' \cdot x]^2 - x^2)}{\sqrt{[n' \cdot x]^2 - x^2}} & n'^2 = +1 \\
\frac{e n^\mu}{4\pi |n' \cdot x|} & n'^2 = 0
\end{cases}$$

(74)

where

$$n'^\mu = \frac{n^\mu}{|n_\alpha n^\alpha|}$$

We will discuss the possibility of integrating on a smaller interval of $\tau$ (Land regularization [12]) adequate in some cases to reproduce the results of ordinary Maxwell scattering.
4 Green functions

In this section, Green functions for both (4,1) and (3,2) wave equations are given. Green functions for equations of this type have been discussed \[13, 14, 18\]. In particular, two distinct versions of the fundamental solution for (4,1) wave-equation have been given:

\[
G_{4,1}(x, t) = -\frac{1}{4\pi^2} \frac{\theta(t - |x|)}{|t^2 - x^2|^{3/2}} \quad \text{cf. [13], based on [19]} \quad (75)
\]

\[
H_{4,1}(x, t) = \frac{1}{2\pi^2} \frac{d}{dt} \frac{\theta(t - |x|)}{\sqrt{t^2 - x^2}} \quad \text{cf. [13]} \quad (76)
\]

The difference expression \((H_{4,1} - G_{4,1})(x, t)\) is a distribution

\[
(H_{4,1} - G_{4,1})(x, t) = \frac{\delta(t - |x|)}{2\sqrt{t^2 - x^2}}
\]

In the analysis below, on the other hand, we shall provide a distinct derivation of the GF’s for both (4,1) and (3,2) which are similar to \(H_{4,1}(x, t)\) as follows:

\[
g_{\sigma_5}(x, \tau) = \lim_{\epsilon \to 0^+} \frac{\sigma_5}{4\pi^2} \frac{\partial}{\partial \epsilon} \theta[-\sigma_5(x^2 + \sigma_5 \tau^2) + \epsilon] \quad \text{for} \quad \sigma_5 = +1
\]

\[
g_{\sigma_5}(x, \tau) \text{ contains a singular distribution term as well:}
\]

\[
\Delta_{\sigma_5} = \lim_{\epsilon \to 0^+} \frac{\sigma_5}{4\pi^2} \frac{\delta[-\sigma_5(x^2 + \sigma_5 \tau^2) + \epsilon]}{\sqrt{-\sigma_5(x^2 + \sigma_5 \tau^2) + \epsilon}}
\]

In the following sections, GF’s are derived for the (4,1) and (3,2) wave equations, which are symmetric in \(t\). Then, we apply the GF’s to the current of a uniformly moving point source, and re-derive the results of section 3. We shall describe the importance of the form of \(\Delta_{\sigma_5}\) in the derivation of the fields.

4.1 (4,1) Green function

With \(\sigma_5 = +1\) in (41), we have

\[
g_+(x, \tau) = \frac{1}{(2\pi)^5} \int d^4k \, dk_5 \, \frac{1}{k^2 + k_5^2} e^{ik \cdot x + ik_5 \tau} = \quad (77)
\]

\[
= \frac{1}{(2\pi)^5} \int d^3k \, dk_5 \, e^{ik \cdot x + ik_5 \tau} \int_{-\infty}^{+\infty} \frac{dk_0}{k^2 + k_5^2 - k_0^2} e^{-ik_0 t}
\]
The Cauchy Principal Part of the $k_0$ integral is

\[
P \int_{-\infty}^{+\infty} \frac{dk_0}{k^2 + k_5^2 - k_0^2} e^{-i k_0 t} = \frac{\pi i \varepsilon(-t)}{2 \sqrt{k^2 + k_5^2}} \left[ e^{+it\sqrt{k^2 + k_5^2}} - e^{-it\sqrt{k^2 + k_5^2}} \right] = \pi \sin \left( t \sqrt{k^2 + k_5^2} \right) \sqrt{k^2 + k_5^2}
\]

We then have

\[
g_+(x, \tau) = \frac{\pi \varepsilon(t)}{(2\pi)^5} \int d^3 k \int d^3 k_5 \frac{\sin \left( t \sqrt{k^2 + k_5^2} \right)}{\sqrt{k^2 + k_5^2}} e^{ik \cdot x + k_5 \tau}
\]

We now orient the $k, k_5$ space such the 4D "observation" vector $(x, \tau)$ is along $k_3$ (one observes at time $\tau$ on the laboratory clock at the point $x$). Defining $l = \sqrt{k^2 + k_5^2}$ and $R = \sqrt{x^2 + \tau^2}$, and using $\alpha, \theta$ and $\phi$ as the 4D polar angles we find:

\[
\begin{align*}
k_3 &= l \cos \alpha \\
k_5 &= l \sin \alpha \cos \theta \\
k \cdot x + k_5 \tau &= R k_3 = R l \sin \alpha \cos \theta
\end{align*}
\]

In terms of these variables,

\[
g_+(x, \tau) = \frac{\pi \varepsilon(t)}{(2\pi)^5} \int_0^{+\infty} l^2 \int_0^{\pi} \sin^2 \alpha \, d\alpha \int_0^{\pi} \sin \theta \, d\theta \int_0^{2\pi} \sin(tl) \, d\phi \frac{\sin(lR \sin \alpha \cos \theta)}{l} e^{ilR \sin \alpha \cos \theta} = \]

\[
= \frac{\varepsilon(t)4\pi^2}{R(2\pi)^5} \int_0^{+\infty} l^3 \int_0^{\pi} \sin^2 \alpha \, d\alpha \int_0^{\pi} \sin(lR \sin \alpha) \sin(tl) \, d\alpha \int_0^{\pi} \sin(lR \sin \alpha) \, d\alpha = \]

\[
= \frac{\varepsilon(t)4\pi^2}{R(2\pi)^5} \int_0^{+\infty} l^3 \int_0^{\pi} \sin(lt) \sin(lR \sin \alpha) \sin(lR \sin \alpha) \, d\alpha = \]

\[
= \left. \frac{-\varepsilon(t)4\pi^2}{R(2\pi)^5} \right| \frac{\partial}{\partial R} \int_0^{+\infty} l^3 \int_0^{\pi} \sin(lt) \cos(lR \sin \alpha) \, d\alpha = \]

The choice of orientation in $k$-space resulted in a first order derivative with respect to the "4D observation point" $R$. This form is important in convergence of the UMS solution. The $\alpha$ integral is simply $\pi J_0(lR)$ and using the sine transform of $J_0(x)$ (cf. [20])}

\[
J_0(x) = \frac{2}{\pi} \int_1^{+\infty} \frac{\sin(xs)}{\sqrt{s^2 - 1}} \, ds
\]

(78)
we find:
\[ g_+(x, \tau) = -\frac{8 \pi^2 \varepsilon(t)}{(2\pi)^5} \frac{1}{R} \frac{1}{\partial R} \int_0^{+\infty} dl \int_1^{+\infty} ds \sin(lt) \frac{\sin(lRs)}{\sqrt{s^2 - 1}} \]

Changing the order of integration and noting that the \( l \) integrand is symmetric under \( l \rightarrow -l \) we have
\[ g_+(x, \tau) = -\frac{8 \pi^2 \varepsilon(t)}{(2\pi)^5} \frac{1}{R} \frac{1}{\partial R} \int_1^{+\infty} ds \frac{1}{\sqrt{s^2 - 1}} \int_{-\infty}^{+\infty} dl \sin(lt) \sin(lRs) = \]
\[ = -\frac{4 \pi^2 \varepsilon(t)}{(2\pi)^5} \frac{1}{R} \frac{1}{\partial R} \int_1^{+\infty} ds \frac{2\pi}{\sqrt{s^2 - 1}} \frac{\delta(t + Rs) - 2\delta(t - Rs)}{(-4)} \]

Since \( Rs > 0 \), then using
\[ \delta(t + Rs) - \delta(t - Rs) = -\varepsilon(t) [\delta(t + Rs) + \delta(t - Rs)] \]

we find:
\[ g_+(x, \tau) = \frac{4 \pi^3 \varepsilon^2(t)}{(2\pi)^5} \frac{1}{R} \frac{1}{\partial R} \int_1^{+\infty} ds \frac{1}{\sqrt{s^2 - 1}} [\delta(t + Rs) + \delta(t - Rs)] = \]
\[ = \frac{4 \pi^3 \varepsilon^2(t)}{(2\pi)^5} \frac{1}{R} \frac{1}{\partial R} \int_1^{+\infty} ds \frac{1}{\sqrt{s^2 - 1}} \theta(-t/R - 1) + \theta(+t/R - 1) = \]
\[ = \frac{1}{8\pi^2} \frac{1}{R} \frac{1}{\partial R} \frac{\theta(t^2 - R^2)}{\sqrt{t^2 - R^2}} \]

We have \( t^2 - R^2 = t^2 - x^2 - \tau^2 = -x_\alpha x^\alpha \), and since \( R > 0 \), we can use \( 1/R\partial/\partial R = 2\theta/\partial R^2 \), which is linear in \( x_\alpha x^\alpha \), thus:
\[ g_+(x, \tau) = -\lim_{\epsilon \rightarrow 0^+} \frac{1}{\pi^2} \frac{1}{\partial \epsilon} \frac{\theta(-x_\alpha x^\alpha + \epsilon)}{\sqrt{-x_\alpha x^\alpha + \epsilon}} \]
(79)

### 4.2 (3, 2) Green function

We shall repeat the procedure for \( \sigma_5 = -1 \), as follows:
\[ g_-(x, \tau) = \frac{1}{(2\pi)^5} \int d^3k \frac{1}{k_2 - k_5^2} e^{i[\mathbf{k} \cdot \mathbf{x} - k_5 \tau]} = \]
\[ = \frac{1}{(2\pi)^5} \int_{R^3} d^3k e^{i[\mathbf{k} \cdot \mathbf{x}]} \int_{R^2} \frac{dk_5}{k_2 - k_5^2} \frac{dk_0}{k_0} e^{-i[k_0 t + k_5 \tau]} \]

The integration is separated into the two subspaces \( R^3 \) for the spatial coordinates, and \( R^2 \) for the temporal coordinates. We shall use polar coordinates in both spaces, using the following substitutions:
\[ k^2 = k_1^2 + k_2^2 + k_3^2 \]
\[ r^2 = x^2 + y^2 + z^2 \]
\[ l^2 = k_0^2 + k_5^2 \]
\[ s^2 = l^2 + r^2 \]
\[ k \cdot x = kr \cos \theta \]
\[ dk_0 \, dk_5 = l \, dl \, d\alpha \]
\[ k_0 t + k_5 \tau = sl \cos \alpha \]
The integral then takes the form

\[ g_-(x, \tau) = \frac{1}{(2\pi)^5} \int_0^{+\infty} k^2 dk \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi e^{ikr \cos \theta} \int_{+\infty}^{l} dl \int_0^{2\pi} d\alpha \frac{e^{ils \cos \alpha}}{k^2 - l^2} \]

We can integrate immediately on \( \phi, \theta \) and \( \alpha \) as follows:

\[ \int_0^{2\pi} d\phi = 2\pi, \quad \int_0^{\pi} d\theta = \frac{2 \sin(kr)}{kr}, \quad \int_0^{2\pi} d\alpha = 2\pi J_0(ls) \]

and thus:

\[ g_-(x, \tau) = \frac{1}{(2\pi)^5} \frac{8\pi^2}{r} \int_0^{+\infty} k \, dk \sin(kr) \int_0^{+\infty} l \, dl \frac{J_0(ls)}{k^2 - l^2} = \]

\[ = -\frac{1}{(2\pi)^5} \frac{2 \partial}{r \partial r} \int_0^{+\infty} l \, dl J_0(ls) \int_0^{+\infty} dk \frac{\cos(kr)}{k^2 - l^2} = \]

\[ = -\frac{1}{(2\pi)^5} \frac{1 \partial}{r \partial r} \int_0^{+\infty} l \, dl J_0(ls) \int_{-\infty}^{+\infty} dk \frac{\cos(kr)}{k^2 - l^2} \]

The Principal Part value of the \( k \) integral is:

\[ P \int_{-\infty}^{+\infty} \cos(kr) \, dk = P.P. \mathfrak{R} \left[ \int_{-\infty}^{+\infty} \frac{e^{ikr}}{k^2 - l^2} \, dk \right] = \]

\[ = \mathfrak{R} \left\{ \frac{i\pi}{2l} \left[ e^{ilr} - e^{-ilr} \right] \right\} = -\frac{\pi}{l} \sin(lr) \]

Thus, the expression for \( g_-(x, \tau) \) one obtains the form

\[ g_-(x, \tau) = \frac{\pi}{(2\pi)^5} \frac{1 \partial}{r \partial r} \int_0^{+\infty} l \, dl J_0(ls) \frac{\sin(lr)}{l} = \]

\[ = \frac{\pi}{(2\pi)^5} \frac{1 \partial}{r \partial r} \int_0^{+\infty} dl J_0(ls) \sin(lr) \]

Once again, using the sine transform of \( J_0(x) \) (see (78)), we have

\[ g_-(x, \tau) = \frac{\pi}{(2\pi)^5} \frac{2 \partial}{r \partial r} \int_1^{+\infty} \frac{du}{\sqrt{u^2 - 1}} \frac{1}{2} \int_{-\infty}^{+\infty} dl \sin(isu) \sin(lr) = \]

\[ = \frac{1}{(2\pi)^5} \frac{1 \partial}{r \partial r} \int_1^{+\infty} \frac{du}{\sqrt{u^2 - 1}} \frac{2\pi}{(-4)} [\delta(su + r) - 2\delta(su - r)] = \]

\[ = -\frac{\pi}{(2\pi)^5} \frac{1 \partial}{r \partial r} \int_1^{+\infty} \frac{du}{\sqrt{u^2 - 1}} e(-s) [\delta(su + r) + \delta(su - r)] \]

However, \( su \geq 0 \) and \( r \geq 0 \), and thus, the first term \( \delta(su+r) \) cancels identically, leaving:

\[ g_-(x, \tau) = \frac{1}{2(2\pi)^2} \frac{\partial}{r \partial r} \frac{\theta(r/s - 1)}{s / \sqrt{r^2 / s^2 - 1}} = \]

\[ = \frac{1}{2(2\pi)^2} \frac{\partial}{r \partial r} \frac{\theta(r - s)}{\sqrt{r^2 - s^2}} \]

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Using the same arguments that were made for the \((4,1)\) case, we have:

\[
g_{-}(x, \tau) = \lim_{\epsilon \to 0^+} \frac{1}{4\pi^2} \frac{\partial}{\partial \epsilon} \frac{\theta(x_{\alpha} x^\alpha + \epsilon)}{\sqrt{x_{\alpha} x^\alpha + \epsilon}} \tag{81}
\]

where \(x_{\alpha} x^\alpha = x^2 - \tau^2\) in this case.

Combining both \((4,1)\) and \((3,2)\) cases, we obtain:

\[
g_{\sigma_5}(x, \tau) = \lim_{\epsilon \to 0^+} \frac{\sigma_5 \partial}{4\pi^2} \frac{\partial}{\partial \epsilon} \frac{\theta[-\sigma_5(x^2 + \sigma_5 \tau^2) + \epsilon]}{\sqrt{-\sigma_5(x^2 + \sigma_5 \tau^2) + \epsilon}} = \lim_{\epsilon \to 0^+} \frac{\sigma_5 \partial}{4\pi^2} \frac{\partial}{\partial \epsilon} \frac{\theta[-\sigma_5 x_{\alpha} x^\alpha + \epsilon]}{\sqrt{-\sigma_5 x_{\alpha} x^\alpha + \epsilon}} \tag{82}
\]

The factor 2 between \((82)\) and \((76)\) stems from the fact that both retarded and advanced \(t\) are used, picking up additional contribution from the future of \(t\).

An attempt to provide GF’s which are \(\tau\) retarded in \((4,1)\) and \((3,2)\) is currently under study.

### 4.3 Fields solution through GF’s

We shall now apply the GF’s \((82)\) to the current generated by a uniformly moving point source. Recalling the UMS path \((48)\), generating the current \((20)\), we shall use \((15)\) to find the fields:

\[
a_{\sigma_5}^\alpha(x, \tau) = \lim_{\epsilon \to 0^+} \frac{e\sigma_5 \partial}{4\pi^2} \frac{\partial}{\partial \epsilon} \int_{-\infty}^{+\infty} d\tau' b^\alpha \frac{\theta[-\sigma_5((x - b\tau') + \sigma_5(\tau - b^5 \tau' + \epsilon)]}{\sqrt{-\sigma_5((x - b\tau') + \sigma_5(\tau - b^5 \tau') + \epsilon}} \tag{83}
\]

Clearly, the limits of integration depend on the coefficients of the quadratic argument:

\[
p(\tau') = -\sigma_5((x - b\tau') + \sigma_5(\tau - b^5 \tau') + \epsilon) = -\sigma_5 \left[(b^2 + \sigma_5(b^5))^2 - 2b \cdot x + 2\sigma_5 b^5 \tau' + x^2 + \sigma_5 \tau'^2\right] + \epsilon = -\sigma_5 b_\alpha b^\alpha \tau' + 2\sigma_5(b_\alpha x^\alpha) \tau' - \sigma_5 [x_\alpha x^\alpha - \sigma_5 \epsilon]
\]

Thus, the polarity of the quadratic form \(p(\tau')\) depends on the sign of \(\zeta = \sigma_5 \epsilon (b_\alpha b^\alpha)\), which we shall treat individually.

### 4.4 UMS fields for \(\zeta = +1\)

We have \(-\sigma_5 b_\alpha b^\alpha < 0\). The condition \(p(\tau') > 0\) is then limited to \(\tau_1 < \tau < \tau_2\) where \(\tau_{1,2}\) are the roots of \(p(\tau')\):

\[
\tau_{1,2} = \frac{-b_\alpha x^\alpha \pm \sqrt{(b_\alpha x^\alpha)^2 - b_\alpha b^\alpha (x_\alpha x^\alpha - \sigma_5 \epsilon)}}{-b_\alpha b^\alpha} \tag{84}
\]

the fields become

\[
a_{\sigma_5}^\alpha(x, \tau) = \lim_{\epsilon \to 0^+} \frac{e\sigma_5 b^\alpha \partial}{4\pi^2} \frac{\partial}{\partial \epsilon} \int_{\tau_1}^{\tau_2} d\tau' \frac{\theta[p(\tau')]}{\sqrt{p(\tau')}} \tag{85}
\]
Clearly, the integral would be become zero \textit{identically} if $\tau_{1,2}$ are complex, thus:

$$(\tau_1' - \tau_2')^2 = \frac{4}{(b_\alpha b^{\alpha})^2} \left[ (b_\alpha x^{\alpha})^2 - b_\alpha b^{\alpha} (x_\alpha x^{\alpha} - \sigma_5 \epsilon) \right] > 0$$

We can now write $p(\tau')$ as follows:

$$p(\tau') = \frac{\sigma_5}{b_\alpha b^{\alpha}} R^2 - A^2 (\tau' - B)^2$$

where

$$R^2 = (b_\alpha x^{\alpha})^2 - b_\alpha b^{\alpha} (x_\alpha x^{\alpha} - \sigma_5 \epsilon) \quad A = \sqrt{\sigma_5 b_\alpha b^{\alpha}} \quad B = \frac{b_\alpha x^{\alpha}}{b_\alpha b^{\alpha}}$$

where $R^2 > 0$ is a requirement for the integral to be non-zero, and $A^2 > 0$ in this case. After making the substitution

$$\sqrt{\frac{\sigma_5}{b_\alpha b^{\alpha}}} R \tanh \beta = A(\tau' - B)$$

we have

$$a_{\sigma_5}^\alpha (x, \tau) = \lim_{\epsilon \to 0^+} \sqrt{\frac{\epsilon \sigma_5}{b_\alpha b^{\alpha}}} \frac{\partial}{\partial \epsilon} \theta(R^2) \int_{-\infty}^{+\infty} \frac{R d\beta}{A \cosh^2 \beta \sqrt{(\sigma_5/b_\alpha b^{\alpha}) R^2 [1 - \tanh^2 \beta]}} = \lim_{\epsilon \to 0^+} \frac{1}{\sqrt{\sigma_5 b_\alpha b^{\alpha}}} \frac{\partial}{\partial \epsilon} \theta(R^2) \int_{-\infty}^{+\infty} \frac{d\beta}{\cosh \beta}$$

The remaining $\beta$ integral is a constant and equal to $\pi$ (easily verified by substituting $u = e^\beta$). Thus

$$a_{\sigma_5}^\alpha (x, \tau) = \lim_{\epsilon \to 0^+} \frac{\epsilon \sigma_5 b^{\alpha}}{4\pi \sqrt{\sigma_5 b_\alpha b^{\alpha}}} \frac{\partial}{\partial \epsilon} \theta(R^2)$$

where $\partial \theta(R^2) = \sigma_5 b_\alpha b^{\alpha} \delta(R^2)$, which gives the final form for $a_{\sigma_5}^\alpha (x, \tau)$ for this case:

$$a_{\sigma_5}^\alpha (x, \tau) = \frac{\epsilon b^{\alpha} (\sigma_5 b_\beta b^{\beta})}{4\pi \sqrt{\sigma_5 b_\alpha b^{\alpha}}} \delta \left[ ((b_\alpha x^{\alpha})^2 - b_\alpha b^{\alpha} x_\alpha x^{\alpha}) \right]$$

where the limit of $\epsilon \to 0^+$ was taken explicitly. Defining the normalized 5D velocity $n^{\alpha} = b^{\alpha} / \sqrt{\sigma_5 b_\beta b^{\beta}}$, we obtain the solution consistent with $\zeta = 1$ of (72):

$$a_{\sigma_5}^\alpha (x, \tau) = \frac{\epsilon n^{\alpha}}{4\pi} \delta \left[ (n_\alpha x^{\alpha})^2 - \sigma_5 x_\alpha x^{\alpha} \right]$$

where it is stressed again that $\sigma_5$ appears implicitly in the scalar products such as $n_\alpha x^{\alpha}$.
4.5 UMS fields for $\zeta = +1$

We shall repeat the analysis of the last section for the case of $-\sigma_5 b_\alpha > 0$. The roots $[3][4]$ are applicable in the present case as well. However, the range of integration of $[5][6]$ now reveals that $p(\tau') > 0$ for the exterior region $\tau' \in (-\infty, \tau'_1) \cup (\tau'_2, +\infty)$. Therefore, $[5][6]$ now becomes:

$$
p(\tau') = (-\sigma_5 b_\alpha) \left( \tau' - \frac{b_\alpha x^\alpha}{b_\alpha b_\alpha^\alpha} \right)^2 - \left( -\frac{\sigma_5}{b_\alpha b_\alpha^\alpha} \right) \left[ (b_\alpha x^\alpha)^2 - b_\alpha b_\alpha^\alpha (x_\alpha x^\alpha - \sigma_5) \right] =
\quad = A^2 (\tau' - B)^2 - \left( -\frac{\sigma_5}{b_\alpha b_\alpha^\alpha} \right) R^2
$$

(90)

where in this case

$$
A = \sqrt{-\sigma_5 b_\alpha b_\alpha^\alpha}
$$

However, in order that the field integral $[7][8]$ converges, we shall define $p_\lambda(\tau')$ as follows:

$$
p_\rho(\tau') = p(\tau') + \rho^2
$$

(91)

The field integral $[7][8]$ obtains the form

$$
a_{\sigma_5}^\alpha (x, \tau) = \frac{e\sigma_5 b_\alpha}{4\pi} \lim_{\rho \to 0^+} \frac{\partial}{\partial \epsilon} \int_{-\infty}^{+\infty} \frac{\theta(p(\tau'))}{\sqrt{p_\rho(\tau')}} d\tau' =
\quad = \frac{e\sigma_5 b_\alpha}{4\pi} \lim_{\rho \to 0^+} \frac{\partial}{\partial \epsilon} \int_{-\infty}^{+\infty} \left[ \frac{\delta(p(\tau'))}{\sqrt{p_\rho(\tau')}} - \frac{1}{2} \frac{\theta(p(\tau'))}{[p_\rho(\tau')]^{3/2}} \right] \frac{\partial p(\tau')}{\partial \epsilon} d\tau'
$$

(92)

where we used that fact that $\partial_\epsilon p(\tau') = \partial_\epsilon p_\rho(\tau')$. The first $\delta(p(\tau'))$ term breaks up over the roots of $p(\tau')$:

$$
\frac{\delta(p(\tau'))}{\sqrt{p_\rho(\tau')}} = \frac{1}{A^2} \frac{\delta(\tau - \tau'_1) + \delta(\tau - \tau'_2)}{\sqrt{p_\rho(\tau')}}
$$

(93)

We also have

$$
\partial_\epsilon p(\tau') = \partial_\epsilon p_\rho(\tau') = 1
$$

(94)

Integrating the first $\delta$-term, and combining $[9][10]$ with $[7][8]$ and $[9][11]$, we obtain

$$
\int_{-\infty}^{+\infty} \frac{\delta(p(\tau'))}{\sqrt{p_\rho(\tau')}} \frac{\partial p(\tau')}{\partial \epsilon} d\tau' = \frac{1}{A^2} \frac{[b_\alpha b_\alpha^\alpha]}{2R} \left[ \frac{1}{\sqrt{p_\rho(\tau')}} \right]_{\tau' = \tau'_1}^{\tau' = \tau'_2} + \frac{1}{\sqrt{p_\rho(\tau')}}
\quad = \frac{1}{R \rho}
$$

(95)
which diverges as $1/\rho$. The second term can be integrated with the substitution

$$C \cosh \beta = A(\tau' - R) \quad \text{where} \quad C = \left[ \frac{-\sigma_5}{b_\alpha b^\alpha} R^2 - \lambda^2 \right]$$

Thus

$$\int_{-\infty}^{+\infty} \frac{\theta(p(\tau'))}{[p_p(\tau')]^{1/2}} d\tau' = \frac{2C}{A} \theta(R^2) \int_{\beta_0}^{+\infty} \sinh \beta \frac{d\beta}{\sinh^2 \beta} \left[ \frac{1}{[C^2(\cosh^2 \beta - 1)]^{1/2}} \right] = (96)$$

$$= 2 \frac{\theta(R^2)}{AC^2} \int_{\beta_0}^{+\infty} \frac{d\beta}{\sinh^2 \beta} = 2 \frac{\theta(R^2)}{AC^2} \left( -1 \right) \coth \beta \bigg|_{\beta_0}^{+\infty}$$

$$= 2 \frac{\theta(R^2)}{AC^2} \left[ \coth \beta_0 - 1 \right]$$

The $\beta_0$ lower bound is given by

$$\sinh^2 \beta_0 = \frac{\rho^2}{C^2} \implies \coth \beta_0 = \sqrt{1 + \frac{1}{\sinh^2 \beta_0}} = \frac{1}{\rho} \sqrt{C^2 + \rho^2} = C + O(\rho)$$

which provides the complete solution for the second term

$$\int_{-\infty}^{+\infty} \frac{\theta(p(\tau'))}{[p_p(\tau')]^{1/2}} d\tau' = 2 \frac{\theta(R^2)}{AC^2} \left[ \frac{C}{\rho} - 1 \right] = (97)$$

$$= 2 \theta(R^2) \left[ \frac{1}{\sqrt{R^2 + \sigma_5 \rho^2 b_\alpha b^\alpha}} - \frac{1}{\sqrt{-\sigma_5 b_\alpha b^\alpha(-\sigma_5 R^2/b_\alpha b^\alpha - \rho^2)}} \right] (98)$$

The sum of the $\delta$-term and the smooth term becomes

$$a^\alpha_{\delta}(x, \tau) = \frac{e \sigma_5 b^\alpha}{4\pi^2} \lim_{\rho \to 0^+} \lim_{\epsilon \to 0^+} \theta(R^2) \left[ \frac{1}{R \rho} - \frac{1}{\sqrt{R^2 + \sigma_5 \rho^2 b_\alpha b^\alpha}} - \frac{1}{\sqrt{-\sigma_5 b_\alpha b^\alpha(-\sigma_5 R^2/b_\alpha b^\alpha - \rho^2)}} \right]$$

$$= \frac{eb^\alpha}{4\pi^2 \sqrt{-\sigma_5 b_\alpha b^\alpha}} \left[ (b_\alpha x^\alpha)^2 - (b_\alpha b^\alpha)(x_\alpha x^\alpha) \right]$$

Once again, defining $n^\alpha = b^\alpha/b_\alpha$, we obtain the final result for $\zeta = -1$:

$$a^\alpha_{\delta}(x, \tau) = \frac{en^\alpha}{4\pi^2} \theta((n_\alpha x^\alpha)^2 + \sigma_5 x_\alpha x^\alpha) (99)$$

5 Conclusions

The $a$-fields (see (72)) generated by a uniformly moving point source in (4, 1) and (3, 2) offshell electrodynamics clearly resemble the expected UMS fields in a 5D Maxwell electrodynamics. However, the latter, generally in a framework
of relativistic dynamics, normally regarded as producing fields from timelike sources only. Stueckelberg based offshell electrodynamics, on the other hand, that are based on a 4D dynamics parameterized along an invariant parameter, have no apparent limit on the region of the source velocity, though normally, the equations are set with constant $z^5(\tau) \equiv \tau \Rightarrow b^5 \equiv 1$. Moreover, the 4D dynamics place no a priori restriction on $b^{\mu} \equiv \frac{b^5}{\tau}$, and all cases of $b^{\mu} < 0$ or $b^{\mu} > 0$ were shown. However, the two forms of the fields given in (72) differ dramatically, and in particular, one is required to explain the $\delta$-functions fields found for the 5D spacelike and timelike regions of source velocity for the $(4,1)$ and $(3,2)$ flat metric equations, respectively.

The $\delta$-function fields ($\zeta = -1$) have support on a 4D null surface given by $x_{\alpha}x^\alpha - \sigma_5(n_{\alpha}x^\alpha)^2 = 0$, which is orthogonal to the direction of motion of the source $n^\alpha$. Thus, this null surface is actually the $(3,1)$ light-cone, as can easily be observed in the frame $n_{\alpha} = \{0; 0, 0, 0, 1\}$, in which case, both $(4,1)$ and $(3,2)$ fields reduce to the Maxwell time-symmetric GF $\delta(x^2)$. This reflects the choice of Principal Part which was taken in the derivation of those fields. These singular fields are in fact the analog for a 4D UMS field of a spacelike moving source.

In a subsequent study, we plan to show that when a Lorentz force derived from those fields is applied to a test particle, it produces a finite force in an infinitesimally short $\tau$ interval, and thus, has no noticeable effect on test particles. The reason for this is that the field tensor $f_{\alpha\beta} = \partial_{[\alpha}a_{\beta]} - \partial_{[\beta}a_{\alpha]}$ contains derivatives of the $\delta$-functions, and when integrated by parts, a coupling of the $\delta$ to acceleration terms is obtained, causing a large mass renormalization effect when the test particle hits the surface of singular support. This effect actually reduces the impact to a finite value, causing it to behave as a zero-measure force.

The smooth fields, on the other hand, obey a $1/r^2$ decay power-law. However, the $(4,1)$ and $(3,2)$ fields differ dramatically in this case. For the $(4,1)$ metric case, the denominator is positive definite, which can easily be observed when a non-physical frame of $n^\alpha = \{1; 0, 0, 0, 0\}$ is taken ($b^5 = 0$ in this case, contradicting the theory). In fact, for any $n^5 \neq 0$, it causes the field to be a transient phenomenon, decaying as $1/\tau^2$ for large $\tau \gg \sqrt{x^2 + t^2}$.

On the other hand, For the $(3,2)$ case, the fields follow an $O(2,2)$ symmetry as well, which can be seen when $n^\alpha = \delta_5^i$ for one of $i \in \{1, 2, 3\}$. When this field is integrated over $\tau$, it produces the $O(2,1)$ GF (proportional to $[t^2 - x^2 - y^2]^{-1/2}$), which is also the Maxwell field produced by a uniformly moving 3D point source in spacelike motion.

In [12], M.C. Land studied the equations of motion of a test particle in a field with similar singular support behavior. In particular, the scattering problem in the non-relativistic limit was derived, in which he noted a failure in matching the well known Rutherford scattering formula. Land then used the mass-$\tau$ uncertainty relations, similar to the time-energy uncertainty in non-relativistic QM, to argue that a true point-wise 4D particle is insufficient to describe a physical source, and thus defined a distribution of events along the $\tau$ parameter,
acting coherently as a single particle. He chose the following distribution

\[ j^\alpha(x, \tau) = \frac{b\alpha}{2\lambda} \int_{-\infty}^{+\infty} d\tau' e^{-|\tau-\tau'|/\lambda} \delta^4[x-b\tau'] \]  

(100)

which approaches the point-wise distribution for \( \lambda \to 0^+ \), and the Maxwell worldline (see eq. (22)) for \( \lambda \to +\infty \). Since the fields are linear, the cumulative contribution smoothed out the \( \delta \)-function fields. Using numerical computation, Land found a constraint on \( \lambda \).

We shall show in a subsequent study how this method of regularization applies to the type of fields we have found here, and make comparison with observed phenomena.

It was found that the GF’s (52) are consistent with the UMS fields. \( \epsilon \) derivative is used to indicate derivation with respect to the argument, which is maintained even once the fields are applied on a test particle. Although the derivative seems to contain a strong distribution \( \delta(y)/\sqrt{y} \), this term has proved essential in the derivation of \( \zeta = -1 \) smooth fields (59), in which it counter balanced an infinite contribution from the bounds \( \tau \to \tau_2^+, \tau_1^- \). Geometrically, it regularized the singular support at the 5D light-cone.

The GF’s obtained would be used for subsequent studies of radiation-reaction, 2 particle systems and various models of regularizations.

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