The constancy of principal curvatures of curvature-adapted submanifolds in symmetric spaces

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Abstract

In this paper, we investigate complete curvature-adapted submanifolds with maximal flat section and trivial normal holonomy group in symmetric spaces of compact type or non-compact type under certain condition, and derive the constancy of the principal curvatures of such submanifolds. As its result, we can derive that such submanifolds are isoparametric.

1 Introduction

Let $G/K$ be a symmetric space of compact type or non-compact type, and $M$ a complete (embedded oriented) Riemannian submanifold in $G/K$. Denote by $R$ the curvature tensor of $G/K$. Also, denote by $T^\perp_x M$ the normal space of $M$ at $x \in M$, $A$ the shape tensor of $M$, $\nabla^\perp$ the normal connection of $M$ and $\exp^\perp$ the normal exponential map of $M$. If, for each $x \in M$ and each $v \in T^\perp_x M$, the normal Jacobi operator $R(v) := R(\cdot, v)v$ preserves $T_x M$ invariantly and $R(v)|_{T_x M}$ commutes with $A_v$, then $M$ is said to be curvature-adapted. This notion was introduced by Berndt-Vanhecke ([3]). Curvature-adapted hypersurfaces (in some cases with constant principal curvatures) in rank one symmetric spaces were studied by some geometers (see [1,2,5,28] for example). If, for each $x \in M$, the normal umbrella $\Sigma_x := \exp^\perp(T^\perp_x M)$ is totally geodesic, then $M$ is called a submanifold with section. Furthermore, if, for each $x \in M$, the induced metric on $\Sigma_x$ is flat, then $M$ is called a submanifold with flat section. Furthermore, if the codimension of $M$ is equal to the rank of $G/K$, then we call $M$ a submanifold with maximal flat section. Assume that $M$ is a complete curvature-adapted submanifold with maximal flat section and trivial normal holonomy group. Then, since $M$ has flat section, $R(v)|_{T_x M}$’s ($v \in T^\perp_x M$) commute to one another for each $x \in M$. Hence they have the common eigenspace decomposition. It is shown that there exist the smooth distributions $D^R_i (i = 1, \cdots, m_R)$ on $M$ such that, for each $x \in M$, $T_x M = \oplus_{i=1}^{m_R} (D^R_i)_x$ holds and that this decomposition is the common eigenspace decomposition of $R(v)|_{T_x M}$’s ($v \in T^\perp_x M$). Note that $m_R = \max_{v \in T^\perp_x M} \# \Spec R(v)$, where $x$ is an arbitrary point of $M$, $\Spec(\cdot)$ is
the spectrum of $\cdot$ and $\sharp(\cdot)$ is the cardinal number of $(\cdot)$. Let $\pi : \hat{M} \to M$ be the universal covering of $M$. Then there exist smooth sections $\alpha_i$ of $(\pi^*T^\perp M)^*$ such that, for each $x \in \hat{M}$ and each $v \in T^\perp_{\pi(x)} M$, $R(v)|_{(D^R_i)^*_{\pi(x)}} = \varepsilon ((\alpha_i)_\pi(v))^2 \text{id}$, where $\pi^*T^\perp M$ is the induced bundle of the normal bundle $T^\perp M$ by $\pi$, $(\pi^*T^\perp M)^*$ is its dual bundle, id is the identity transformation of $(D^R_i)^*_{\pi(x)}$ and $\varepsilon = 1$ (resp. $\varepsilon = -1$) in the case where $G/K$ is of compact type (resp. of non-compact type). Note that each $\alpha_i$ is unique up to the $(\pm 1)$-multiple. Set $\mathcal{R}_M := \{\pm \alpha_1, \cdots, \pm \alpha_{m_R}\}$ and, for each $x \in M$, define a subset $\mathcal{R}_{xM}^\perp$ of $(T^\perp_x M)^*$ by

$$\mathcal{R}_{xM}^\perp := \{\pm (\alpha_1)_\pi, \cdots, \pm (\alpha_{m_R})_\pi\},$$

where $\hat{x}$ is an arbitrary point of $\pi^{-1}(x)$. Note that $\mathcal{R}_{xM}^\perp$ is independent of the choice of $\hat{x} \in \pi^{-1}(x)$. The system $\mathcal{R}_{xM}^\perp$ gives a root system and that it is isomorphic to the (restricted) root system of the symmetric pair $(G, K)$. Hence, if $\alpha, \beta \in \mathcal{R}_M$ and if $\beta = F\alpha$ for some $F \in C^\infty(M)$, then $F = \pm 1$ or $\pm 2$. For convenience, we denote $D^R_i$ by $D^R_{\alpha_i}$. For each $\alpha \in \mathcal{R}_M$ and each parallel normal vector field $\tilde{v}$ of $M$, we define a function $\alpha(\tilde{v})^2$ over $M$ by

$$\alpha(\tilde{v})^2(x) := \alpha_{\hat{x}}(\tilde{v}_x)^2 \quad (x \in M),$$

where $\hat{x}$ is an arbitrary point of $\pi^{-1}(x)$.

On the other hand, since $M$ has flat section and trivial normal holonomy group, it follows from the Ricci equation that $A_v$’s $(v \in T^\perp_x M)$ commute to one another for each $x \in M$. Hence they have the common eigenspace decomposition. Set

$$m_A := \max_{x \in M} \max_{v \in T^\perp_x M} \sharp \text{Spec } A_v$$

and

$$U_A := \{x \in M \mid \max_{v \in T^\perp_x M} \sharp \text{Spec } A_v = m_A\}.$$

It is clear that $U_A$ is open in $M$. Let $U^0_A$ be one of components of $U_A$. It is shown that there exist smooth distributions $D^A_i$ $(i = 1, \cdots, m_A)$ on $U^0_A$ such that, for each $x \in U^0_A$, $T^\perp_x M = \bigoplus_{i=1}^{m_A} (D^A_i)_x$ holds and that this decomposition is the common eigenspace decomposition of $A_v$’s $(v \in T^\perp_x M)$. Also, there exist smooth sections $\lambda_i$ of $(T^\perp M)^*$ $(i = 1, \cdots, m_A)$ such that, for each $x \in U^0_A$ and each $v \in T^\perp_x M$, $A_v|_{(D^A_i)_x} = (\lambda_i)_x(v) \text{id}$ holds. Set $\mathcal{A}_M := \{\lambda_1, \cdots, \lambda_{m_A}\}$. For convenience, we denote $D^A_i$ by $D^A_{\lambda_i}$.

In this paper, we first prove the following results.

**Theorem A.** Let $M$ be a complete curvature-adapted submanifold with maximal flat section and trivial normal holonomy group in a symmetric space $G/K$ of compact type or non-compact type. Then the following statements (i) and (ii) hold:

(i) For each parallel normal vector field $\tilde{v}$ of $M$, the eigenvalues $\varepsilon\alpha(\tilde{v})^2$’s $(\alpha \in \mathcal{R}_M)$ of $R(\tilde{v})$ are constant over $M$.  

\[2\]
Let $\lambda \in A_M$. Assume that, for any $\alpha \in R_M$ with $D^R_\alpha \cap D^A_\lambda \neq \{0\}$, $\dim(D^R_\alpha \cap D^A_\lambda) \geq 2$. Then, for each parallel normal vector field $\bar{v}$ of $U^0_A$, the principal curvature $\lambda(\bar{v})$ of $U^0_A$ for $\bar{v}$ is constant along any curve tangent to $D^A_\lambda$.

In 2006, Heintze-Liu-Olmos ([10]) defined the notion of an isoparametric submanifold in a general Riemannian manifold as a (properly embedded) complete submanifold $M$ with section and trivial normal holonomy group satisfying the following condition:

(Is) Sufficiently close parallel submanifolds of $M$ have constant mean curvature with respect to the radial direction.

In the sequel, we assume that all isoparametric submanifolds have flat section. Next we prove the following result.

**Theorem B.** Under the hypothesis of Theorem A, assume that, for each $\alpha \in R_M$, there exists $\lambda \in A_M$ such that $D^R_\alpha \subset D^A_\lambda$ holds over $U^0_A$ and that $\dim D^R_\alpha \geq 2$. Then the following statements (i)-(iii) hold:

(i) The set $U^0_A$ is equal to $M$ and, for each parallel normal vector field $\bar{v}$ of $M$, the principal curvatures $\lambda(\bar{v})$’s ($\lambda \in A_M$) of $M$ for $\bar{v}$ are constant over $M$.

(ii) If $G/K$ is of compact type, then $M$ is isoparametric.

(iii) If $G/K$ is of non-compact type and if $M$ is real analytic, then $M$ is isoparametric.

**Remark 1.1.** (i) Principal orbits of a Hermann action $H \acts G/K$ with cohom $H = \text{rank} G/K$ are curvature-adapted isoparametric submanifolds with maximal flat section and trivial normal holonomy group (see [8,18]).

(ii) Principal orbits of the isotropy action of any symmetric space of compact type (or non-compact type) satisfy the condition that, for each $\alpha \in R_M$, there exists $\lambda \in A_M$ with $D^R_\alpha \subset D^A_\lambda$ (see [8, Theorem 5.3]).

(iii) The proofs of Theorem A and Theorem B(i) do not require strong results proved in other papers, whereas Theorem B(ii)-(iii) make use of several strong results in different papers.

By using Theorem B and several strong results in different papers, we derive the following result.

**Theorem C.** Under the hypothesis of Theorem B, assume that $G/K$ is a simply connected and irreducible symmetric space of compact type and rank greater than one. Then $M$ is congruent to a principal orbit of the isotropy action of $G/K$.

**Theorem D.** Under the hypothesis of Theorem B, assume that $G/K$ is an irreducible symmetric space of non-compact type and rank greater than one, and that $M$ is real analytic and has no non-Euclidean type focal point on the ideal boundary of $G/K$. Then $M$ is a principal orbit of a Hermann action on $G/K$. 

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The notion of a non-Euclidean type focal point on the ideal boundary of $G/K$ was introduced in [19]. See the next section about the definition of this notion.

**Remark 1.2.** (i) The principal orbits of any Hermann action on a symmetric space $G/K$ of non-compact type have no non-Euclidean type focal point on the ideal boundary of $G/K$.

(ii) Let $G/K$ be an irreducible symmetric space of non-compact type and rank greater than one, $N$ the nilpotent part in the Iwasawa’s decomposition $G = KAN$ of $G$ and $M$ a principal orbit of the $N$-action on $G/K$. Assume that the multiplicity of each root of the (restricted) root system of the symmetric pair $(G, K)$ is greater than one. Then $M$ satisfies all the hypothesis of Theorem B (see [20]) and it is real analytic. However it has a non-Euclidean type focal point on the ideal boundary of $G/K$. On the other hand, we can show that $M$ does not occur as a principal orbit of a Hermann action on $G/K$. Thus, in Theorem D, is indispensable the condition that $M$ has no non-Euclidean type focal point on the ideal boundary of $G/K$.

(iii) There exists a Hermann action on an irreducible symmetric space $G/K$ of non-compact type and rank greater than one such that its principal orbits satisfy all the hypothesis in Theorem D but that it is not conjugate to the isotropy actions of $G/K$. For example, see Table 1 in Section 4 about all of such Hermann actions on irreducible rank two symmetric spaces of non-compact type.

### 2 Basic notions and facts

In this section, we shall recall the basic notions and facts in a symmetric space. See [7, Pages 94-95] or [30, Page 177] about the algebraic structure of a symmetric space and the Jacobi field on the space. We use the notations in Introduction. Let $M$ be an embedded submanifold in a Riemannian manifold $N$. Denote by $\nabla$, $\bar{\nabla}$ and $A$ the Riemannian connection of $M$, that of $N$ and the shape tensor of $M$, respectively. Take a unit normal vector $v$ of $M$ at $x$ and denote by $\gamma_v$ the geodesic in $N$ with $\gamma_v'(0) = v$, where $\gamma_v'(0)$ is the velocity vector of $\gamma_v$ at 0. If there exists an $M$-Jacobi field $J$ along $\gamma_v$ satisfying $J(0) \neq 0$ and $J(s_0) = 0$, then the real number $s_0$ is called a focal radius along $\gamma_v$. We consider the case where $N$ is a symmetric space $G/K$. In this case, the strongly $M$-Jacobi field $J$ along $\gamma_v$ with $J(0) = X$ (hence $\frac{\partial}{\partial s} J|_{s=0} = -A_v X$) is given by

$$J(s) = \left(P_{\gamma_v[0,s]} \circ \left(\cos(s\sqrt{R(v)}) - \frac{\sin(s\sqrt{R(v)})}{\sqrt{R(v)}} \circ A_v\right)\right)(X),$$

where $\nabla_v$ is the covariant derivative along $\gamma_v$ with respect to $\bar{\nabla}$ and $P_{\gamma_v[0,s]}$ is the parallel translation along $\gamma_v[0,s]$. In the case where $M$ has flat section, any focal radius of $M$ along $\gamma_v$ is given as a zero point of a strongly $M$-Jacobi field along $\gamma_v$. Hence the set of all focal radii of $M$ along $\gamma_v$ coincides with the zero point set of the
real-valued function $F_v$ over $\mathbb{R}$ defined by

$$F_v(s) := \det \left( \cos(s\sqrt{R(v)}) - \frac{\sin(s\sqrt{R(v)})}{\sqrt{R(v)}} \circ A_v \right).$$

In 1995, Terng-Thorbergsson ([32]) defined the notion of an *equifocal submanifold* as a compact submanifold with flat section and trivial normal holonomy group satisfying the following condition:

(PF) $M$ has parallel focal structure, that is, for any parallel normal vector field $\tilde{v}$ of $M$, the focal radii along $\gamma_{\tilde{v}}$ are independent of the choice of $x \in M$ (with considering their multiplicities).

Let $H$ be a closed subgroup of $G$. The $H$-action on $G/K$ is called a *polar action* if $H$ is compact and if there exists a complete embedded submanifold $\Sigma$ meeting all principal $H$-orbits orthogonally. The submanifold $\Sigma$ is called a *section* of this action. Furthermore, if the induced metric on $\Sigma$ is flat, then the $H$-action is called a *hyperpolar action*. It is known that principal orbits of a hyperpolar action are equifocal. On the other hand, in 1995, E. Heintze, R.S. Palais, C.L. Terng and G. Thorbergsson ([11]) proved that, in the case where $G/K$ is a simply connected symmetric space of compact type, any homogeneous submanifold with flat section in $G/K$ is a principal orbit of a hyperpolar action. If $G/K$ is of compact type and if there exists an involution $\sigma$ of $G$ with $(\text{Fix} \sigma)_0 \subset H \subset \text{Fix} \sigma$, then the $H$-action on $G/K$ is called a *Hermann action*. It is known that Hermann actions are hyperpolar ([11]) and that the principal orbits of a Hermann action are curvature-adapted (see [8]). In 2001, A. Kollross ([24]) proved that, in the case where $G/K$ is an irreducible simply connected symmetric space of compact type, hyperpolar actions of cohomogeneity greater than one on an irreducible simply connected symmetric space of compact type are orbit equivalent to Hermann actions. In 2002, U. Christ ([6]) proved that, in the case where $G/K$ is an irreducible simply connected symmetric space of compact type, any irreducible equifocal submanifold of codimension greater than one in $G/K$ is homogeneous. Note that there was a gap in his proof but, in 2012, C. Gorodski and E. Heintze ([9]) closed the gap.

Therefore we obtain the following fact.

**Fact 2.1.** Any equifocal submanifold of codimension greater than one in any irreducible simply connected symmetric space of compact type is a principal orbit of a Hermann action.

Heintze-Liu-Olmos ([10]) showed the following fact.

**Fact 2.2.** For a compact submanifold with flat section in a symmetric space of compact type, it is equifocal if and only if it is isoparametric.

**Remark 2.1.** In more general, it is shown that, for a (not necessarily compact)
submanifold with flat section and trivial normal holonomy group in a symmetric space of compact type, it has parallel focal structure if and only if it is isoparametric.

When a non-compact submanifold $M$ in a symmetric space $G/K$ of non-compact type deforms as its principal curvatures approach to zero, its focal set vanishes beyond the ideal boundary $(G/K)(\infty)$ of $G/K$. For example, when an open potion of a totally umbilic sphere in a hyperbolic space of constant curvature $c(<0)$ deforms as its principal curvatures approach to $\sqrt{-c}$, its focal point approach to $(G/K)(\infty)$ and, when it furthermore deforms as its principal curvatures approach to a positive value smaller than $\sqrt{-c}$, the focal point vanishes beyond $(G/K)(\infty)$. On the base of this fact, we recognized that, for a non-compact submanifold in a symmetric space of non-compact type, the parallelity of the focal structure is not an essential condition. So, in 2004, we [16] introduced the notion of a complex focal radius along the normal geodesic $\gamma_v$ of such a submanifold as a general notion of a focal radius along $\gamma_v$. This notion was defined as the zero points of the complex-valued function $F^C_v$ over $\mathbb{C}$ defined by

$$F^C_v(z) := \det \left( \cos(z\sqrt{R(v)^C}) - \frac{\sin(z\sqrt{R(v)^C})}{\sqrt{R(v)^C}} \circ A^C_v \right),$$

where $R(v)^C$ (resp. $A^C_v$) is the complexification of $R(v)$ (resp. $A_v$). In the case where $M$ is real analytic, we [17] showed that complex focal radii along $\gamma_v$ indicate the positions of focal points of the extrinsic complexification $M^C(\hookrightarrow G^C/K^C)$ of $M$ along the complexified geodesic $\gamma^C_{\iota^*v}$, where $G^C/K^C$ is the anti-Kaehlerian symmetric space associated with $G/K$ and $\iota$ is the natural immersion of $G/K$ into $G^C/K^C$.

We ([16]) defined the notion of a complex equifocal submanifold as a (properly embedded) complete submanifold with flat section and trivial normal holonomy group satisfying the following condition:

(\text{PCF}) $M$ has parallel complex focal structure, that is, for any parallel normal vector field $\tilde{v}$ of $M$, the complex focal radii along $\gamma^c_{\tilde{v}}$ are independent of the choice of $x \in M$ (considering their multiplicities).

We should call this submanifold a equi-complex-focal submanifold but called a complex equifocal submanifold for simplicity. We ([17]) showed the following fact.

**Fact 2.3.** Let $M$ be a complete submanifold with flat section in a symmetric space $G/K$ of non-compact type. If $M$ is isoparametric, then it is complex equifocal. Conversely, if $M$ is real analytic, complex equifocal and curvature-adapted, then it is isoparametric.

Let $G/K$ be a symmetric space of non-compact type and $H$ a closed subgroup of $G$. The $H$-action on $G/K$ is called a polar action if there exists a complete embedded submanifold $\Sigma$ meeting all principal $H$-orbits orthogonally. Furthermore, if the induced metric on $\Sigma$ is flat, then the $H$-action is called a hyperpolar action.
Note that a polar action (resp. a hyperpolar action) on a symmetric space of non-compact type was called a complex polar action (resp. complex hyperpolar action) in [17,18,20]. In [17], it was proved that principal orbits of a hyperpolar action on \(G/K\) are complex equifocal and that any homogeneous submanifold with flat section in \(G/K\) is a principal orbit of a hyperpolar action. If there exists an involution \(\sigma\) of \(G\) with \((\text{Fix}\,\sigma)_0 \subset H \subset \text{Fix}\,\sigma\), then the \(H\)-action on \(G/K\) is called a Hermann type action. For simplicity, we call this action a Hermann action in this paper. It is easy to show that Hermann actions are hyperpolar.

At the end of this section, we recall the notion of a non-Euclidean type focal point on the ideal boundary of a Hadamard manifold introduced in [19]. Let \(N\) be a Hadamard manifold, \(N(\infty)\) the ideal boundary of \(N\) and \(M\) a submanifold in \(N\). Take \(v \in T_x^\perp M\). Let \(\gamma_v : [0, \infty) \to N\) be the normal geodesic of \(M\) of direction \(v\). If there exists an \(M\)-Jacobi field \(J\) along \(\gamma_v\) satisfying \(\lim_{s \to \infty} \frac{||J(s)||}{s} = 0\), then we call \(\gamma_v(\infty) (\in N(\infty))\) a focal point of \(M\) on the ideal boundary \(N(\infty)\) along \(\gamma_v\), where \(\gamma_v(\infty)\) is the asymptotic class of \(\gamma_v\). In fact, if such an \(M\)-Jacobi field \(J\) exists, then for the standard geodesic variation \(\delta : [0, \infty) \times (-\epsilon, \epsilon) \to N\) having \(J\) as the variational vector field, the asymptotic classes \(\gamma_s(\infty)\)'s of \(\gamma_s : t \mapsto \delta(t,s) (-\epsilon < s < \epsilon)\) coincide with \(\gamma_v(\infty)\), where \(\epsilon\) is a positive number. Hence \(\gamma_v(\infty)\) should be interpreted as a focal point of \(M\). Also, if there exists an \(M\)-Jacobi field \(J\) along \(\gamma_v\) satisfying \(\lim_{s \to \infty} \frac{||J(s)||}{s} = 0\) and \(\text{Sec}(v,J(0)) \neq 0\), then we call \(\gamma_v(\infty)\) a non-Euclidean type focal point \(M\) on \(N(\infty)\) along \(\gamma_v\), where \(\text{Sec}(v,J(0))\) is the sectional curvature for the 2-plane spanned by \(v\) and \(J(0)\). If, for any nonzero normal vector \(v\) of \(M\), \(\gamma_v(\infty)\) is not a non-Euclidean type focal point of \(M\) on \(N(\infty)\) along \(\gamma_v\), then we say that \(M\) has no non-Euclidean type focal point on the ideal boundary.

3 Proofs of Theorems A, B and C

Let \(M\) be a complete curvature-adapted submanifold in \(G/K\) as in the statement of Theorem A. We shall use the notations in Introduction. Denote by \(\nabla\) and \(\tilde{\nabla}\) the Riemannian connections of \(M\) and \(G/K\), respectively. In the sequel, for each \(\alpha \in \mathcal{R}_M\) with \(2\alpha \notin \mathcal{R}_M\), \(D^R_{2\alpha}\) implies the zero distribution. First we note that the following relations hold:

\[
R(D^R_{\alpha}, T^\perp M)T^\perp M \subset D^R_{\alpha}, \quad R(D^R_{\alpha}, D^R_{\beta})T^\perp M \subset D^R_{\alpha + \beta} \oplus D^R_{\alpha - \beta},
\]

\[
R(D^R_{\alpha}, D^R_{\beta})D^R_{\alpha + \beta} \subset D^R_{\alpha + \beta} \oplus D^R_{\alpha - \beta} \oplus T^\perp M.
\]

(3.1)

because \(M\) has maximal flat section.

We shall prove Theorem A.

Proof of Theorem A. First we shall show the statement (i). Take a parallel normal vector field \(\tilde{\gamma}\) of \(M\). Let \(V\) be a sufficiently small open set of \(M\). Take a local unit section \(\tilde{X}\) of \(TM \oplus D^R_{\alpha}\) defined over \(V\) and a local unit section \(\tilde{Y}\) of \(D^R_{\alpha}\) defined
over $V$, where $TM \oplus D_{2\alpha}^R$ means $TM \cap (D_{2\alpha}^R)^\perp$. Then we have

$$\tilde{\nabla}_\tilde{X} \left( R(\tilde{Y}, \tilde{v})\tilde{v} \right) = \varepsilon \tilde{X}(\alpha(\tilde{v})^2)\tilde{Y} + \varepsilon \alpha(\tilde{v})^2 \tilde{\nabla}_\tilde{X}\tilde{Y}. \quad (3.2)$$

On the other hand, we have

$$\tilde{\nabla}_\tilde{X} \left( R(\tilde{Y}, \tilde{v})\tilde{v} \right) = R(\nabla_{\tilde{X}}\tilde{Y}, \tilde{v})\tilde{v} - R(\tilde{Y}, A_{\tilde{v}}\tilde{X})\tilde{v} - R(\tilde{Y}, \tilde{v})A_{\tilde{v}}\tilde{X}, \quad (3.3)$$

where we use $R(\tilde{v})|_{T_{\perp}M} = 0$. From $\langle \nabla_{\tilde{X}}\tilde{Y}, \tilde{Y} \rangle = 0$, we have $\langle R(\nabla_{\tilde{X}}\tilde{Y}, \tilde{v})\tilde{v}, \tilde{Y} \rangle = 0$. Hence, by taking the inner product of $(3.2)$ and $(3.3)$ with $\tilde{Y}$, we obtain

$$\varepsilon \tilde{X}(\alpha(\tilde{v})^2) = -\langle R(\tilde{Y}, A_{\tilde{v}}\tilde{X})\tilde{v}, \tilde{Y} \rangle - \langle R(\tilde{Y}, \tilde{v})A_{\tilde{v}}\tilde{X}, \tilde{Y} \rangle. \quad (3.4)$$

Also, since $M$ is curvature-adapted, $A_{\tilde{v}}\tilde{X}$ is a local section of $TM \oplus D_{2\alpha}^R$. Hence it follows from $(3.1)$ that

$$R(\tilde{Y}, A_{\tilde{v}}\tilde{X})\tilde{v} \in (D_{\alpha}^R)^\perp \quad \text{and} \quad R(\tilde{Y}, \tilde{v})A_{\tilde{v}}\tilde{X} \in (D_{\alpha}^R)^\perp. \quad (3.5)$$

From $(3.4)$ and $(3.5)$, we obtain

$$\tilde{X}(\alpha(\tilde{v})^2) = 0. \quad (3.6)$$

Take local unit sections $\tilde{Z}_i \ (i = 1, 2)$ of $D_{2\alpha}^R$ defined over $V$. Then, in similar to $(3.4)$, we can show

$$\varepsilon \tilde{Z}_1(2\alpha(\tilde{v})^2) = -\langle R(\tilde{Z}_2, A_{\tilde{v}}\tilde{Z}_1)\tilde{v}, \tilde{Z}_2 \rangle - \langle R(\tilde{Z}_2, \tilde{v})A_{\tilde{v}}\tilde{Z}_1, \tilde{Z}_2 \rangle. \quad (3.7)$$

Also, it follows from $(3.1)$ that

$$R(\tilde{Z}_2, A_{\tilde{v}}\tilde{Z}_1)\tilde{v} = 0 \quad \text{and} \quad R(\tilde{Z}_2, \tilde{v})A_{\tilde{v}}\tilde{Z}_1 \in T_{\perp}M. \quad (3.8)$$

Hence we obtain

$$\tilde{Z}_1(\alpha(\tilde{v})^2) = 0. \quad (3.8)$$

From $(3.6)$, $(3.8)$, the arbitrariness of $\tilde{X}, \tilde{Z}_1$ and $V$, it follows that $\alpha(\tilde{v})^2$ is constant over $M$.

Next we shall show the statement (ii). Take $x \in U^{0}_\alpha$ and $\alpha \in R_M$ with $D_{\alpha}^R \cap D_{\alpha}^A \neq \{0\}$. Take local sections $\tilde{X}_i \ (i = 1, 2)$ of $D_{\alpha}^R \cap D_{\alpha}^A$ over a neighborhood $V$ of $x$ in $U^{0}_\alpha$ which give an orthonormal system at each point of $V$, where we use $\dim(D_{\alpha}^R \cap D_{\alpha}^A) \geq 2$. Then we have

$$R((\tilde{X}_1)_x, (\tilde{X}_2)_x)\tilde{v}_x = \tilde{\nabla}_{(\tilde{X}_1)}(\tilde{X}_2)_x \tilde{v} - \tilde{\nabla}_{(\tilde{X}_2)}(\tilde{X}_1)_x \tilde{v} - \tilde{\nabla}_{[\tilde{X}_1, \tilde{X}_2]}(\tilde{v}_x) \quad (3.9)$$

$$= -\langle (\tilde{X}_1)_x(\lambda(\tilde{v}))((\tilde{X}_2))_x + ((\tilde{X}_2)_x(\lambda(\tilde{v})))((\tilde{X}_1))_x - (\lambda)_x(\tilde{v}_x)[(\tilde{X}_1, \tilde{X}_2)]_x + A_{\tilde{v}_x}[(\tilde{X}_1, \tilde{X}_2)]_x. \)
On the other hand, since \((\widetilde{X}_i)_x\) \((i = 1, 2)\) belong to \((D^R_\alpha)_x\) and \(\widetilde{v}_x\) belongs to \(T^\perp_x M\), it follows from (3.1) that
\[
R((\widetilde{X}_1)_x, (\widetilde{X}_2)_x)\widetilde{v}_x \in (D^R_{2\alpha})_x.
\]
From (3.9) and (3.10), we have
\[
((\widetilde{X}_1)_x(\lambda(\widetilde{v})))(\widetilde{X}_2)_x - ((\widetilde{X}_2)_x(\lambda(\widetilde{v}))(\widetilde{X}_1)_x
\begin{equation}
\equiv (A_{\widetilde{v}_x} - (\lambda)_x(\widetilde{v}_x))(\widetilde{X}_1, \widetilde{X}_2)_x \pmod{(D^R_{2\alpha})_x}.
\end{equation}
\]

The left-hand side of this relation belongs to \((D^A_\lambda)_x\) but the right-hand side of this relation is orthogonal to \((D^A_\lambda)_x\). Hence, it follows that the left-hand side of (3.11) vanishes. Therefore, it follows from the linear independency of \((\widetilde{X}_1)_x\) and \((\widetilde{X}_2)_x\) that \((\widetilde{X}_1)_x(\lambda(\widetilde{v})) = 0\). From the arbitrariness of \(x\) and \(\widetilde{X}_1\), it follows that \(\lambda(\widetilde{v})\) is constant along any curve tangent to \(D^R_{\alpha} \cap D^A_{\lambda}\). Furthermore, the statement (ii) of Theorem A follows from the arbitrariness of \(\alpha\), where we use the curvature-adaptedness of \(M\). q.e.d.

Next we shall prove the statement (i) of Theorem B.

Proof of (i) of Theorem B. Let \(D^R_\alpha \subset D^A_\lambda\) and set \(D^R_{\alpha, 2\alpha} := D^R_\alpha \oplus D^R_{2\alpha}\).

(Step I) First we shall show that \(D^R_{\alpha, 2\alpha}\) is a totally geodesic distribution on \(U^0_{\lambda, A}\). Take a parallel normal vector field \(\widetilde{v}\) of \(U^0_{\lambda, A}\). Fix \(x_0 \in U^0_{\lambda, A}\). Take local sections \(\widetilde{X}_1\) and \(\widetilde{X}_2\) of \(D^R_\alpha\) defined over a neighborhood \(V\) of \(x_0\) in \(U^0_{\lambda, A}\). Since \(\alpha(\widetilde{v})^2\) is constant over \(M\) by the statement (i) of Theorem A, we have
\[
\nabla_{(\widetilde{X}_1)_x}(\nabla(\widetilde{X}_2, \widetilde{v})\widetilde{v}) = \varepsilon \alpha(\widetilde{v})^2(x)\nabla_{(\widetilde{X}_1)_x}\widetilde{X}_2
\]

On the other hand, since \(M\) has flat section, we obtain
\[
\nabla_{(\widetilde{X}_1)_x}(R(\widetilde{X}_2, \widetilde{v})\widetilde{v}) = R(\nabla_{(\widetilde{X}_1)_x}\widetilde{X}_2, \widetilde{v}_x)\widetilde{v}_x
\begin{equation}
- \lambda_x(\widetilde{v}_x)R((\widetilde{X}_2)_x, (\widetilde{X}_1)_x)\widetilde{v}_x
\end{equation}
- \lambda_x(\widetilde{v}_x)R((\widetilde{X}_2)_x, \widetilde{v}_x((\widetilde{X}_1)_x).
\]

It follows from (3.1) that
\[
R((\widetilde{X}_2)_x, (\widetilde{X}_1)_x)\widetilde{v}_x \in (D^R_{2\alpha})_x
\]
and
\[
R((\widetilde{X}_2)_x, \widetilde{v}_x((\widetilde{X}_1)_x) \subset T^\perp_x M \oplus (D^R_{2\alpha})_x.
\]

Also, since \(M\) has flat section, we have
\[
R(\nabla_{(\widetilde{X}_1)_x}\widetilde{X}_2, \widetilde{v}_x)\widetilde{v}_x \in T_x M.
\]

From (3.12) – (3.16), we obtain
\[
R(\nabla_{(\widetilde{X}_1)_x}\widetilde{X}_2, \widetilde{v}_x)\widetilde{v}_x \equiv \varepsilon \alpha(\widetilde{v})^2(x)\nabla_{(\widetilde{X}_1)_x}\widetilde{X}_2 \pmod{(D^R_{2\alpha})_x}.
\]
Therefore, from the arbitrariness of $\tilde{v}$, we obtain $\nabla_{\tilde{X}_1} X_2 \in \langle D^{R}_{\alpha,2\alpha} \rangle_x$. Furthermore, from the arbitrariness of $\tilde{X}_1, \tilde{X}_2$ and $x$, it follows that $\nabla_{D^R_{\alpha}} D^R_{\alpha} \subset D^R_{\alpha,2\alpha}$ holds on $V$. Furthermore, it follows from the arbitrariness of $x_0$ that $\nabla_{D^R_{\alpha}} D^R_{\alpha} \subset D^R_{\alpha,2\alpha}$ holds on $U^0_\alpha$. Similarly, we can show that $\nabla_{D^{R}_{2\alpha}} D^{R}_{\alpha} \subset D^{R}_{2\alpha, \alpha}$, $\nabla_{D^{R}_{2\alpha}} D^{R}_{\alpha} \subset D^{R}_{\alpha,2\alpha}$ and $\nabla_{D^{R}_{\alpha}} D^{R}_{\alpha} \subset D^{R}_{\alpha,2\alpha}$ hold on $U^0_\alpha$. Therefore $D^{R}_{\alpha,2\alpha}$ is a totally geodesic distribution on $U^0_\alpha$.

(Step II) Since $D^{R}_{\alpha,2\alpha}|_{U^0_\alpha}$ is totally geodesic, it is integrable. Denote by $\mathcal{F}_{\alpha,2\alpha}$ the foliation on $U^0_\alpha$ whose leaves are the integral manifolds of $D^{R}_{\alpha,2\alpha}|_{U^0_\alpha}$ and $L^{\alpha,2\alpha}_x$ the leaf of $\mathcal{F}_{\alpha,2\alpha}$ through $x \in U^0_\alpha$. Let $\xi_{\alpha,2}$ be the element of $T^*_M x$ defined by $\langle \xi_{\alpha,2}, \cdot \rangle = \alpha_x(\cdot)$. Set $E^x_{\alpha} := (D^{R}_{\alpha,2\alpha})(x) \oplus \text{Span}\{\xi_{\alpha,2}\}$ and $\Sigma^0_{\alpha} := \exp_x(E^x_{\alpha})$, where $\exp_x$ is the exponential map of $G/K$ at $x$. In this step, we shall show that $L^{\alpha,2\alpha}_x = \Sigma^0_{\alpha} \cap U^0_\alpha$. Take $x \in \pi^{-1}(x)$. Let $K_x$ be the isotropy subgroup of $G$ at $x$ and $\theta_x$ an involution of $G$ with $(\text{Fix } \theta_x) \subset K_x \subset \text{Fix } \theta_x$, where $\text{Fix } \theta_x$ is the fixed point group of $\theta_x$ and $(\text{Fix } \theta_x)$ is its identity component. Denote by the same symbol $\theta_x$ the involution of $\mathfrak{g}$ induced from $\theta_x$. Let $\mathfrak{g} = \mathfrak{t}_x + \mathfrak{p}_x$ be the eigenspace decomposition of $\theta_x$. The subspace $\mathfrak{p}_x$ is identified with $T_x(G/K)$. It is easy to show that $R(E^x_{\alpha}, E^x_{\alpha}) E^x_{\alpha} \subset E^x_{\alpha}$, which implies that $E^x_{\alpha}$ is a Lie triple system of $\mathfrak{g}$. Therefore $\Sigma^0_{\alpha}$ is a totally geodesic in $G/K$ (see P. 224 of [12]). In case of $2\alpha \in R_M$, $\Sigma^0_{\alpha}$ is a totally geodesic in $G/K$. Hence $\nabla_{\tilde{X}_1} \tilde{X}_2$ is tangent to $\Sigma^0_{\alpha} \cap U^0_\alpha$. This implies that $\Sigma^0_{\alpha} \cap U^0_\alpha$ is totally geodesic in $M$. From these facts, we can derive $L^{\alpha,2\alpha}_x = \Sigma^0_{\alpha} \cap U^0_\alpha$.

(Step III) Take any two points $x$ and $y$ of $U^0_\alpha$. Since $L^{\alpha,2\alpha}_x = \Sigma^0_{\alpha} \cap U^0_\alpha$ and $L^{\alpha,2\alpha}_y = \Sigma^0_{\alpha} \cap U^0_\alpha$, $L^{\alpha,2\alpha}_x$ (resp. $L^{\alpha,2\alpha}_y$) is a hypersurface in $\Sigma^0_{\alpha}$ (resp. $\Sigma^0_{\alpha}$). We shall compare these hypersurfaces. Let $\xi_\alpha$ be a unit normal vector field of $\tilde{M}$ such that $(\xi_\alpha)_\tilde{z} = \xi_{\alpha,2}$ for any $\tilde{z} \in \tilde{M}$. According to (i) of Theorem A, $\alpha$ is parallel with respect to the normal connection of $\tilde{M}$. Hence so is also $\xi_\alpha$. According to the assumption, $D^{R}_{2\alpha} \subset D^A_{\lambda}$ hold over $U^0_\alpha$ for some $\lambda \in A_M$. Since $\Sigma^0_{\alpha}$ and $\Sigma^0_{\alpha}$ are totally geodesic in $G/K$, it follows from $D^{R}_{\alpha} \subset D^A_{\lambda}$ and $D^{R}_{2\alpha} \subset D^A_{\lambda}$ that $\lambda(\tilde{w}) = \lambda(\tilde{w}) = 0$ for any parallel normal vector field $\tilde{w}$ of $M$ orthogonal to $\xi_\alpha$. On the other hand, since $\Sigma^0_{\alpha}$ is totally geodesic in $G/K$, the shape operator of the hypersurface $L^{\alpha,2\alpha}_x$ in $\Sigma^0_{\alpha}$ coincides with $A_{\xi_\alpha}|_{T_{L^{\alpha,2\alpha}_x}}$. Hence $L^{\alpha,2\alpha}_x$ is a curvature-adapted hypersurface in $\Sigma^0_{\alpha}$. Also, it is clear that the hypersurface $L^{\alpha,2\alpha}_x$ has at most two principal curvatures $\lambda(\xi_\alpha)$ and $\lambda(\xi_\alpha)$, and, furthermore, it follows from (ii) of Theorem A that they are constant along leaves of the corresponding eigendistributions. Since $\Sigma^0_{\alpha}$ is a rank one symmetric space and $L^{\alpha,2\alpha}_x$ is a complete curvature adapted hypersurface with two distinct
principal curvatures satisfying a special condition that the eigendistributions of the normal Jacobi operator coincide with those of the shape operator, this hypersurface \( L_\alpha \) has constant principal curvatures (see [14,15,26,27]). Hence \( \lambda(\xi_\alpha) \) and \( \bar{\lambda}(\xi_\alpha) \) are constant along \( L_\alpha \). Similarly, \( \lambda(\xi_\alpha) \) and \( \bar{\lambda}(\xi_\alpha) \) are constant along \( L_{y,2}\alpha \). Since \( \alpha(\xi_\alpha)^2 \) is constant over \( M \), it follows that \( \Sigma_{x,2}\alpha \) and \( \Sigma_y \) are isometric to each other. Therefore it follows that \( L_{x,2}\alpha \) and \( L_{y,2}\alpha \) are locally isometric to each other. Furthermore, since \( M \) is complete, the orthogonal complementary distribution of \( F_{x,2}\alpha \) is a totally geodesic foliation, it follows from the result in [4] that the element of holonomy along any curve orthogonal to leaves of \( F_{x,2}\alpha \) consists of local isometries between the leaves. See [4] about the definition of an element of holonomy. Furthermore, from this fact, \( U_{x,2}\alpha = M \) is derived directly. This completes the proof. q.e.d.

Next we shall prove the statements (ii) and (iii) of Theorem B.

Proof of (ii) and (iii) of Theorem B. The focal radii and the complex focal radii along the normal geodesic \( \gamma_v \) of \( M \) are given as the zero points of the functions \( F_v \) and \( F^c_v \) as in Section 2. Hence, it follows from the additional assumption in Theorem B that, in the case where \( G/K \) is of compact type, the set of all the focal radii along \( \gamma_v \) is equal to

\[
\left\{ \frac{\arctan(\alpha_{\xi}(v)/\lambda_{\xi}(v)) + j\pi}{\alpha_{\xi}(v)} \right\} \quad (\alpha, \lambda) \in \mathcal{R}_M \times \mathcal{A}_M \text{ s.t. } D_{\alpha} \subset D_{\lambda}, \quad j \in \mathbb{Z}
\]

and, in the case where \( G/K \) is of non-compact type, the set of all the complex focal radii along \( \gamma_v \) is equal to

\[
\left\{ \frac{\arctanh(\alpha_{\xi}(v)/\lambda_{\xi}(v)) + j\pi \sqrt{-1}}{\alpha_{\xi}(v)} \right\} \quad (\alpha, \lambda) \in \mathcal{R}_M \times \mathcal{A}_M \text{ s.t.}

\begin{align*}
&\text{"} D_{\alpha} \subset D_{\lambda} \text{ and } |\lambda_{\xi}(v)| > |\alpha_{\xi}(v)|", \quad j \in \mathbb{Z} \\
&\bigcup \left\{ \frac{\arctanh(\lambda_{\xi}(v)/\alpha_{\xi}(v)) + (j + \frac{1}{2}) \pi \sqrt{-1}}{\alpha_{\xi}(v)} \right\} \quad (\alpha, \lambda) \in \mathcal{R}_M \times \mathcal{A}_M \text{ s.t.}

&\text{"} D_{\alpha} \subset D_{\lambda} \text{ and } |\lambda_{\xi}(v)| < |\alpha_{\xi}(v)|", \quad j \in \mathbb{Z} \biggr\}
\end{align*}

Hence it follows from (i) of Theorem A and (i) of Theorem B that \( M \) has parallel focal structure (resp. complex equifocal) in the case where \( G/K \) is of compact type (resp. non-compact type). Therefore the statements (ii) and (iii) of Theorem B follow from Remark 2.1 and Fact 2.3. q.e.d.
Next we shall prove Theorem C.

*Proof of Theorem C.* According to the proof of (ii) of Theorem B, $M$ is equifocal. Hence, according to Fact 2.1, it follows from the additional conditions in Theorem C that $M$ is a principal orbit of a Hermann action on $G/K$. Denote by $H$ this Hermann action. Furthermore, it follows from the additional assumption in Theorem B that the $H$-action satisfies the condition $\Delta^V_+ \cap \Delta^H_+ = \emptyset$, where $\Delta^V_+$ and $\Delta^H_+$ are the systems determined by the triple $(H, G, K)$ as in [21]. Hence, according to Proposition 4.39 in [13], the $H$-action is conjugate to the isotropy action of $G/K$. That is, $M$ is congruent to a principal orbit of the isotropy action. q.e.d.

Finally we shall prove Theorem D.

*Proof of Theorem D.* According to (iii) of Theorem B, $M$ is isoparametric. According to the result in [23], it follows from this fact and the additional assumptions in Theorem D that $M$ is a principal orbit of a Hermann action on $G/K$. q.e.d.

### 4 Examples

Let $G/K$ be a symmetric space of non-compact type such that the multiplicity of each root of the (restricted) root system of the symmetric pair $(G, K)$ is greater than one. Then the principal orbits of the isotropy actions of $G/K$ are complete curvature-adapted submanifolds as in Theorem B. Also, the principal orbits of Hermann actions $H \curvearrowright G/K$'s (of cohomogeneity two) in Table 1 and their dual actions $H^d \curvearrowright G^d/K$'s are complete curvature-adapted submanifolds as in Theorem B (see [8,18,25]), where $G^d/K$ is the compact dual of $G/K$. Note that the dual actions are conjugate to the isotropy actions of $G^d/K$ by Proposition 4.39 of [13]. The Hermann actions in
Table 1 are all of Hermann actions on irreducible rank two symmetric spaces of non-compact type such that their principal orbits satisfy all the hypothesis in Theorem B and that they are not conjugate to the isotropy actions (see [21,24]).

| $H \curvearrowright G/K$ | $\dim M$ | $\sharp A_M$ |
|--------------------------|---------|---------|
| $Sp(1,2) \curvearrowright SU^*(6)/Sp(3)$ | 12 | 3 |
| $SO_0(2,3) \curvearrowright SO(5,\mathbb{C})/SO(5)$ | 8 | 4 |
| $Sp(2,\mathbb{R}) \curvearrowright Sp(2,\mathbb{C})/Sp(2)$ | 8 | 4 |
| $Sp(1,1) \curvearrowright Sp(2,\mathbb{C})/Sp(2)$ | 8 | 4 |
| $F_4^{-20} \curvearrowright E_6^{-26}/F_4$ | 24 | 3 |
| $G_2^2 \curvearrowright G_2^7/G_2$ | 12 | 6 |

$(M : \text{a principal orbit of } H \curvearrowright G/K)$

| Table 1. |

References

[1] J. Berndt, Real hypersurfaces with constant principal curvatures in complex hyperbolic space, J. Reine Angew. Math. 395 132-141 (1989).

[2] J. Berndt, Real hypersurfaces in quaternionic space forms, J. Reine Angew. Math. 419 9-26 (1991).

[3] J. Berndt and L. Vanhecke, Curvature adapted submanifolds, Nihonkai Math. J. 3 177-185 (1992).

[4] R.A. Blumenthal and J.J. Hebda, Complementary distributions which preserve the leaf geometry and applications to totally geodesic foliations, Quart. J. Math. Oxford. Ser. (2) 35 383-392 (1984).

[5] T.E. Cecil and P.J. Ryan, Focal sets and real hypersurfaces in complex projective space, Trans. Amer. Math. Soc. 269 481-499 (1982).

[6] U. Christ, Homogeneity of equifocal submanifolds, J. Differential Geometry 62 1-15 (2002).

[7] P.B. Eberlein, Geometry of nonpositively curved manifolds, Chicago Lectures in Mathematics, Univerity of Chicago Press, Chicago, IL, 1996.

[8] O. Goertsches and G. Thorbergsson, On the Geometry of the orbits of Hermann actions, Geom. Dedicata 129 101-118 (2007).

[9] C. Gorodski and E. Heintze, Homogeneous structures and rigidity of isoparametric submanifolds in Hilbert space, J. Fixed Point Theory Appl. 11 (2012) 93-136.

[10] E. Heintze, X. Liu and C. Olmos, Isoparametric submanifolds and a Chevalley type restriction theorem, Integrable systems, geometry, and topology, 151-190, AMS/IP Stud. Adv. Math. 36, Amer. Math. Soc., Providence, RI, 2006.

[11] E. Heintze, R.S. Palais, C.L. Terng and G. Thorbergsson, Hyperpolar acti-
ons on symmetric spaces, Geometry, topology and physics for Raoul Bott (ed. S. T. Yau), Conf. Proc. Lecture Notes Geom. Topology 4, Internat. Press, Cambridge, MA, 1995 pp214-245.

[12] S. Helgason, Differential geometry, Lie groups and symmetric spaces, Academic Press, New York, 1978.

[13] O. Ikawa, The geometry of symmetric triad and orbit spaces of Hermann actions, J. Math. Soc. Japan 63 79-139 (2011).

[14] N. Innami, Y. Mashiko and K. Shiohama, Metric spheres in the projective spaces with constant holomorphic sectional curvature, Tsukuba J. Math. 35 79-90 (2011).

[15] N. Innami, Y. Itokawa and K. Shiohama, Complete real hypersurfaces and special K-line bundles in K-hyperbolic spaces, Intern. J. Math. 24 1350082, 14 pp. (2013).

[16] N. Koike, Submanifold geometries in a symmetric space of non-compact type and a pseudo-Hilbert space, Kyushu J. Math. 58 167-202 (2004).

[17] N. Koike, Complex equifocal submanifolds and infinite dimensional anti-Kaehlerian isoparametric submanifolds, Tokyo J. Math. 28 201-247 (2005).

[18] N. Koike, Actions of Hermann type and proper complex equifocal submanifolds, Osaka J. Math. 42 599-611 (2005).

[19] N. Koike, On curvature-adapted and proper complex equifocal submanifolds, Kyungpook Math. J. 50 509-536 (2010).

[20] N. Koike, Examples of a complex hyperpolar action without singular orbit, Cubo A Math. J. 12 131-147 (2010).

[21] N. Koike, Collapse of the mean curvature flow for equifocal submanifolds, Asian J. Math. 15 101-128. (2011)

[22] N. Koike, An Cartan type identity for isoparametric hypersurfaces in symmetric spaces, Tohoku Math. J. (to appear) (arXiv:math. DG/1010.1652 v3).

[23] N. Koike, The classifications of certain kind of isoparametric submanifolds in non-compact symmetric spaces, arXiv:math.DG/1209.1933v1.

[24] A. Kollross, A Classification of hyperpolar and cohomogeneity one actions, Trans. Amer. Math. Soc. 354 571-612 (2001).

[25] A. Kollross, Duality of symmetric spaces and polar actions, J. Lie Theory 21 961-986 (2011).

[26] S. Montiel, Real hypersurfaces of a complex hyperbolic space, J. Math. Soc. Japan 37 515-535 (1985).

[27] M. Ortega and J.D. Pérez, On the Ricci tensor of a real hypersurface of quaternionic hyperbolic space, manuscripta math. 93 49-57 (1997).

[28] T. Murphy, Curvature-adapted submanifolds of symmetric spaces, Indiana Univ. Math. J. 61 831-847 (2012).

[29] R.S. Palais and C.L. Terng, Critical point theory and submanifold geometry, Lecture Notes in Math. 1353, Springer, Berlin, 1988.

[30] T. Sakai, Riemannian Geometry, Translations of Mathematical Monographs, 149. American Math. Soc., Providence, RI,1996.
[31] C.L. Terng, Isoparametric submanifolds and their Coxeter groups, J. Differential Geometry 21 79-107 (1985).
[32] C.L. Terng and G. Thorbergsson, Submanifold geometry in symmetric spaces, J. Differential Geometry 42 665-718 (1995).

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