Quantum information masking of an arbitrary unknown state can be realized in the multipartite lower-dimensional systems

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Abstract
Quantum information masking is a protocol that hides the original quantum information from subsystems and spreads it over quantum correlation, which is available to multipartite except bipartite systems. In this work, we explicitly study the quantum information masking in multipartite scenario and prove that there exists an universal quantum masker can mask the information encoding in the unknown states into a multipartite system with lower-dimensional local systems, namely, all the \(k\)-level quantum states can be masked into a \(m\)-qudit systems \((m \geq 4)\) whose local dimension \(d \leq k\). In order to observe the masking process intuitively, explicitly controlled operations are provided. Our scheme well demonstrates the abundance of quantum correlation between multipartite quantum system and has potential application in the security of quantum information processing.

1. Introduction

In quantum information theory, since the information processing process adheres to the unitary evolution and linear superposition principle, several operations that can be complemented in the classical information process are prohibited in a closed physical system. The notion that reveals these phenomena is called the ‘no-go’ theorem. For instance, there exists no universal cloning machine that can replicate an arbitrary unknown pure quantum state, which is known as the no-cloning theorem [1–3]. A contrary version of the no-cloning theorem states that it is impossible to delete one of two copied arbitrary unknown quantum states without affecting the other in a closed physical system resulting in no-deleting theorem [4]. With the in-depth study of quantum information theory, more and more no-go theorems have been mooted like no-broadcasting theorem [5, 6], no-superposition theory [7–9], no-hiding theory [10]. These theorems shed light on the discrepancies between quantum mechanics and classical physics from the perspective of information theory, and also root in the security of quantum information processing tasks such as quantum secret sharing [11–13], quantum key distribution [14, 15] and quantum teleportation [16–18], \textit{et al}.

In 2018, a new no-go theorem named no-masking theorem was proposed by Kavan Modi \textit{et al} which states that it’s impossible to hide the original arbitrary unknown quantum states into the quantum correlation between bipartite quantum systems and make it inaccessible to marginal systems [19]. Moreover, this result not only provides a broader version of the quantum bit commitment named quantum qubit commitment [20, 21] but also demonstrates its potential application in quantum secret sharing.

This topic attracted wide attention and a series of significant discussions were published. When the masked set of states is all the states in a quantum system, there is evidence that the no-masking theorem is robust, that is, even if replacing unitary transformations with general linear transformations, it’s impossible probabilistically to mask arbitrary unknown quantum state into a bipartite quantum system [22]. Interesting, no-masking theorem is available even the set of states is reduced to the set of nonzero measure [23]. This finding further enriches the no-masking theorem. Similar to the no-cloning theorem, when the masked set consists of mutual orthogonal
(or linearly independent) states, it can be deterministically (or probabilistically) masked into bipartite systems [24]. Different from the no-cloning theorem, the maskable set of states is abundant. In geometry, the maskable set of states in high dimensional systems is described by the one on two or more hyperdisks [25]. This result could be simplified to qubit case, where the maskable states are located on a spherical circle on the Bloch sphere which has been verified experimentally [26, 27]. In addition, there exists a masker for the quantum information encoding into the states described by real destiny matrices [28, 29].

The above conclusions well describe the situation of quantum information masking in a bipartite system. How does it behave in multipartite systems? Li and Wang proved that it’s possible to mask all the states of a quantum system into multipartite systems and observe no information in each local system [30]. Meanwhile, several schemes were proposed, namely, $H^d \rightarrow (H^d)^{\otimes 2d}$, $H^d \rightarrow H^d \otimes H^d \otimes H^d$ when $d$ is odd and $H^d \rightarrow (H^d)^{\otimes d+1} \otimes H^d$ when $d$ is even respectively. An operation like this that can mask all quantum states of a system is called a universal masker. This finding opens up a new direction for quantum information masking. Recently, its upgraded version named $k$-uniform quantum information masking was proposed which reveals the strong relationship between quantum information masking and quantum error-correcting codes (QECCs) in heterogeneous systems [31]. A similar topics also were discussed [32, 33]. In addition, the close relationship between quantum multipartite masker and quantum teleportation was studied and a new masking scheme ($H^d \rightarrow H^d \otimes H^d \otimes H^d \otimes H^d$) was provided [34]. These conclusions all convey a signal that quantum information masking has wide prospect in the security of quantum information process.

So far, the states in the maskable set have been masked into multipartite systems whose local dimension is not lower than that of the system in which they are located. Generally, the operations allowed are simpler in lower dimensional quantum systems especially in experiments. In this paper, first, we show that it’s possible to mask arbitrary unknown quantum states into a multipartite qubit system with lower-dimensional local system. To facilitate the understanding of the masking process, then, the vivid controlled operations in four-partite scenario, by means of the tight relation of $k$-uniform states and QECC, it has been proved that, the reductions to each party are identity. As a consequence, a necessary definition can be seen as the 1-uniform version of the $k$-uniform quantum information masking in multipartite systems which proposed in the [31]. Moreover, in the $k$-uniform masking scenario, by means of the tight relation of $k$-uniform states and QECC, it has been proved that, the reductions to any $k$ bodies are identity. As a consequence, a necessary definition is given as follows.

**Definition 2.** The set of pure states $\{|\psi\rangle\}$ is called a ‘maximum entangled basis’ (MEB) of $N$-qudit systems $(H^d)^{\otimes N}$, if it satisfies the conditions as follows

(i) For an arbitrary pure state $|\psi\rangle$, it’s reductions to each party are

$$\rho_{A_i} = Tr_{A_j} |\psi\rangle \langle \psi| = \frac{I}{d};$$

(ii) The set of pure states $\{|\psi\rangle\}$ forms an orthonormal basis for the Hilbert space $(H^d)^{\otimes N}$.

In addition, to prove the existence of the universal multipartite masker which described by a unitary operator $U$, we notice the following fact [35].
Lemma 1. If two sets of states \( \{ \mid \psi_i \rangle \}_{i=1}^n \) and \( \{ \mid \Psi_i \rangle \}_{i=1}^n \) satisfy the condition
\[
\langle \psi_i | \psi_j \rangle = \langle \Psi_i | \Psi_j \rangle ,
\]
for \( i, j = 1, 2, \cdots, n \), there always exists a unitary operator \( U \) to complete the following transformation
\[
U | \psi_i \rangle = | \Psi_i \rangle ,
\]
where \( i = 1, 2, \cdots, n \).

3. Multi-qubit masker

As is shown in [30], arbitrary unknown qubit states \( | \psi \rangle = a_0 | 0 \rangle + a_1 | 1 \rangle \) can be deterministically masked into 4-qubit systems which reads as
\[
| 0 \rangle \rightarrow \frac{1}{\sqrt{2}} (| 00 \rangle + | 11 \rangle) \otimes (| 00 \rangle + | 11 \rangle),
\]
\[
| 1 \rangle \rightarrow \frac{1}{\sqrt{2}} (| 00 \rangle - | 11 \rangle) \otimes (| 00 \rangle - | 11 \rangle).
\]

Nevertheless, we find the masking capacity of 4-qubit systems is not limited to this. Next, we give an example to illustrate that it is possible to mask all the states of a 4-level quantum system into 4-qubit systems.

Example 1. For an arbitrary unknown quantum state \( | \phi \rangle \) in the system \( H^4 \), namely
\[
| \phi \rangle = a_0 | 0 \rangle + a_1 | 1 \rangle + a_2 | 2 \rangle + a_3 | 3 \rangle ,
\]
where \( |a_0|^2 + |a_1|^2 + |a_2|^2 + |a_3|^2 = 1 \). According to the lemma 1, there exists a masking processing defined by
\[
| 0 \rangle \rightarrow \frac{1}{\sqrt{2}} (| 00 \rangle + | 11 \rangle) \otimes (| 00 \rangle + | 11 \rangle),
\]
\[
| 1 \rangle \rightarrow \frac{1}{\sqrt{2}} (| 00 \rangle - | 11 \rangle) \otimes (| 00 \rangle - | 11 \rangle),
\]
\[
| 2 \rangle \rightarrow \frac{1}{\sqrt{2}} (| 01 \rangle + | 10 \rangle) \otimes (| 01 \rangle + | 10 \rangle),
\]
\[
| 3 \rangle \rightarrow \frac{1}{\sqrt{2}} (| 01 \rangle - | 10 \rangle) \otimes (| 01 \rangle - | 10 \rangle).\]

Proof. The total final state in 4-qubit system is
\[
| \Phi \rangle = \frac{1}{4} (a_0 + a_1) (| 0000 \rangle + | 1111 \rangle + (a_0 - a_1) (| 0011 \rangle + | 1100 \rangle) + (a_2 + a_3) (| 0101 \rangle + | 1010 \rangle) + (a_2 - a_3) (| 0110 \rangle + | 1001 \rangle).
\]
After calculation, the reduction to each local system is derived as follows
\[
\rho_i = \frac{1}{4} (|a_0 + a_1|^2 + |a_0 - a_1|^2 + |a_2 + a_3|^2 + |a_2 - a_3|^2) I
\]
\[
= I_2 ,
\]
for \( i = 1, 2, 3, 4 \). That is, we observe no information from each local systems of 4-qubit systems which reads that there exists a universal masker for \( H^4 \to H^2 \otimes H^2 \otimes H^2 \otimes H^2 \).

Next, in order to reveal the masking details of the processing above, the universal masker described by controlled operation is given as follows.

For arbitrary unknown quantum states \( | \phi \rangle \) in the system \( H^4 \) given as (6), according to lemma 1, we can complete the unitary transformation reads as
\[
| 0 \rangle \rightarrow | 00 \rangle_{12} ,
\]
\[
| 1 \rangle \rightarrow | 10 \rangle_{12} ,
\]
\[
| 2 \rangle \rightarrow | 01 \rangle_{12} ,
\]
\[
| 3 \rangle \rightarrow | 11 \rangle_{12} .
\]
Then the total state in 2-qubit systems is obtained as
\[ |\Phi_0\rangle = (a_0|00\rangle + a_1|10\rangle + a_2|01\rangle + a_3|11\rangle)_{12}. \] (11)

The marginal states of \(|\Phi_1\rangle\) are
\[ \rho_1 = \left( \begin{array}{cc} |a_0|^2 + |a_2|^2 & a_0a_1^* + a_2a_3^* \\ a_0^*a_1 + a_2^*a_3 & |a_1|^2 + |a_3|^2 \end{array} \right), \]
\[ \rho_2 = \left( \begin{array}{cc} |a_0|^2 + |a_1|^2 & a_0a_2^* + a_1a_3^* \\ a_0^*a_2 + a_1^*a_3 & |a_2|^2 + |a_3|^2 \end{array} \right). \] (12)

Now, it's obvious that each marginal system contains the original quantum information. Then we add two auxiliary systems and proceed in four steps.

First, we do the C-Not \(C_{13}\) on \(|\Phi_1\rangle\), then the total state in 4-qubit systems is
\[ |\Phi_1\rangle = (a_0|0000\rangle + a_1|1010\rangle + a_2|0101\rangle + a_3|1111\rangle)_{1234}. \] (13)

Now, the reduced density matrices of the first three local systems are derived as follows
\[ \rho_1 = \rho_3 = \left( \begin{array}{cc} |a_0|^2 + |a_2|^2 & 0 \\ 0 & |a_1|^2 + |a_3|^2 \end{array} \right), \]
\[ \rho_2 = \left( \begin{array}{cc} |a_0|^2 + |a_1|^2 & a_0a_2^* + a_1a_3^* \\ a_0^*a_2 + a_1^*a_3 & |a_2|^2 + |a_3|^2 \end{array} \right). \] (14)

It is apparent that the original information contained in the off-diagonal elements of the reduced density matrices of 1 and 3 systems is masked.

Second, we do the C-Not operation \(C_{24}\) on the total state \(|\Phi_2\rangle\) and obtain the state as follows
\[ |\Phi_2\rangle_{1234} = a_0|0000\rangle + a_1|1010\rangle + a_2|0101\rangle + a_3|1111\rangle. \] (15)

After calculation, we can have the reduction to each local system as follows
\[ \rho_1 = \rho_3 = \left( \begin{array}{cc} |a_0|^2 + |a_2|^2 & 0 \\ 0 & |a_1|^2 + |a_3|^2 \end{array} \right), \]
\[ \rho_2 = \rho_4 = \left( \begin{array}{cc} |a_0|^2 + |a_1|^2 & 0 \\ 0 & |a_2|^2 + |a_3|^2 \end{array} \right). \] (16)

It can be seen that all the original information contained in the off-diagonal elements of the reduced density matrices of four local systems vanish.

Third, we send the first qubit through Hadamard gate, then do the C-Not \(C_{12}\) and new total state becomes
\[
|\Phi_3\rangle_{1234} = \frac{1}{\sqrt{2}}[a_0(|0000\rangle + |1100\rangle) + a_1(|0010\rangle - |1110\rangle) \\
+ a_2(|0101\rangle + |1001\rangle) + a_3(|0111\rangle - |1011\rangle)].
\] (17)

Now, four reduced density matrices can be obtained as
\[ \rho_1 = \rho_2 = \frac{1}{2}I, \]
\[ \rho_3 = \left( \begin{array}{cc} |a_0|^2 + |a_2|^2 & 0 \\ 0 & |a_1|^2 + |a_3|^2 \end{array} \right), \]
\[ \rho_4 = \left( \begin{array}{cc} |a_0|^2 + |a_1|^2 & 0 \\ 0 & |a_2|^2 + |a_3|^2 \end{array} \right). \] (18)

It is distinct that we have no information about the original states in both 1, 2 systems and there is still some information leaked on the diagonal elements of 3, 4 system.
Finally, we send the third qubit through Hadamard gate, then perform the C-Not operation $C_{34}$ on $|\Phi_4\rangle$ and obtain the final total state

$$
|\Phi_6\rangle_{1234} = \frac{1}{2}[a_0(|0000\rangle + |1111\rangle + |0011\rangle + |1100\rangle) \\
+ a_1(|0000\rangle + |1111\rangle - |0011\rangle - |1100\rangle) \\
+ a_2(|0101\rangle + |1010\rangle + |0110\rangle + |1001\rangle) \\
+ a_3(|0101\rangle + |1010\rangle - |0110\rangle - |1001\rangle)].
$$

(19)

It is easy to verify that $\rho_i = \frac{I}{2}$, $i = 1, 2, 3, 4$, which means we complete the masking process $H^4 \rightarrow H^2 \otimes H^2 \otimes H^2 \otimes H^2$. All the C-Not operations above have the form

$$
C_{st} = |0\rangle \langle 0| \otimes I + |1\rangle \langle 1| \otimes \sigma_s,
$$

(20)

where $s = 1, 3$, $t = 2, 4$. It should note that (19) is equivalent to the state given as

$$
|\Phi_7\rangle_{1234} = \frac{1}{2}[a_0(|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle) \\
+ a_1(|00\rangle - |11\rangle) \otimes (|00\rangle - |11\rangle) \\
+ a_2(|01\rangle + |10\rangle) \otimes (|01\rangle + |10\rangle) \\
+ a_3(|01\rangle - |10\rangle) \otimes (|01\rangle - |10\rangle)].
$$

(21)

In order to further explore the masking capacity of multi-qubit systems we give another example

**Example 2.** For an arbitrary unknown quantum state $|\phi\rangle$ in the system $H^4$

$$
|\phi\rangle = \sum_{k=0}^{3} a_k|k\rangle.
$$

(22)

where $\sum_{k=0}^{3}|a_k|^2 = 1$.

According to the lemma 1, we can complete the deterministic masking process as follows

$$
|0\rangle \rightarrow \frac{1}{2}(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle),
$$

$$
|1\rangle \rightarrow \frac{1}{2}(|001\rangle + |110\rangle) \otimes (|001\rangle + |110\rangle),
$$

$$
|2\rangle \rightarrow \frac{1}{2}(|010\rangle + |101\rangle) \otimes (|010\rangle + |101\rangle),
$$

$$
|3\rangle \rightarrow \frac{1}{2}(|100\rangle + |011\rangle) \otimes (|100\rangle + |011\rangle),
$$

$$
|4\rangle \rightarrow \frac{1}{2}(|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle),
$$

$$
|5\rangle \rightarrow \frac{1}{2}(|001\rangle - |110\rangle) \otimes (|001\rangle - |110\rangle),
$$

$$
|6\rangle \rightarrow \frac{1}{2}(|010\rangle - |101\rangle) \otimes (|010\rangle - |101\rangle),
$$

$$
|7\rangle \rightarrow \frac{1}{2}(|100\rangle - |011\rangle) \otimes (|100\rangle - |011\rangle).
$$

(23)

**Proof.** After the unitary transformation, the total state becomes

$$
|\Phi\rangle = \frac{1}{2}[(a_0 + a_4)(|000000\rangle + |111111\rangle) \\
+ (a_0 - a_4)(|000111\rangle + |111000\rangle) \\
+ (a_1 + a_5)(|001001\rangle + |110110\rangle) \\
+ (a_1 - a_5)(|001101\rangle + |110011\rangle) \\
+ (a_2 + a_6)(|010010\rangle + |101101\rangle) \\
+ (a_2 - a_6)(|010101\rangle + |101010\rangle) \\
+ (a_3 + a_7)(|100100\rangle + |011101\rangle) \\
+ (a_3 - a_7)(|100011\rangle + |011110\rangle)].
$$

(24)
We can verify the reduced density matrices of each local system by calculation as
\[
\rho_i = \frac{1}{4} (|a_0 + a_d|^2 + |a_0 - a_d|^2 + |a_1 + a_d|^2 + |a_1 - a_d|^2
+ |a_2 + a_d|^2 + |a_2 - a_d|^2 + |a_3 + a_d|^2 + |a_3 - a_d|^2) I
= \frac{I}{2}.
\] (25)

where \(i = 1, 2, 3, 4, 5, 6\).

Obviously, we can observe no information about the original states \(|\phi\rangle\) which means we complete the masking process \(H^d \rightarrow (H^2)^{\otimes 6}\).

The above two examples exhibit that it’s possible to mask an arbitrary unknown quantum state located on the systems \(H^4\) and \(H^8\) into the 4-qubit and 6-qubit systems respectively. From the equations (7) and (23), we can notice that the MEBs of the systems \(H^4\) and \(H^8\) ascertain the range of the set of the unknown quantum states that can be masked. Thus we can derive the following corollary about the masking capacity of the 2n-qubit systems naturally.

**Corollary 1.** For an arbitrary unknown quantum state \(|\phi\rangle\) in the system \(H^d\)
\[
|\phi\rangle = \sum_{k=0}^{d-1} a_k |k\rangle,
\] (26)

If the dimension of the system \(H^d\) satisfies
\[
d \leq 2^n,
\] (27)

there always exists a universal masker, which can mask all the states of \(H^d\) into 2n-qubit systems deterministically, namely \(H^d \rightarrow (H^2)^{\otimes 2^n}\).

### 4. Multi-qudit masker

With the increase of the dimensions of the local systems involved in the masking process, the range of maskable states expands. To move forward, we propose the first general conclusion.

**Theorem 1.** For any positive integer \(2 \leq h \leq d^2\), it’s possible to mask all the states of \(h\)-level system \(H^h\) into 4-qudit systems, i.e., \(H^h \rightarrow H^d \otimes H^d \otimes H^d \otimes H^d\).

**Proof.** For an arbitrary unknown quantum state of the \(d^2\)-level system \(H^{d^2}\) namely
\[
|\psi\rangle = \sum_{k=0}^{d^2-1} a_k |k\rangle,
\] (28)

where \(\sum_{k=0}^{d^2-1} |a_k|^2 = 1\). According to lemma 1, there exists a unitary can complete the following transformation
\[
|k\rangle \rightarrow |\psi_k\rangle \otimes |\psi_k\rangle.
\] (29)

where
\[
|\psi_k\rangle = \frac{1}{d} \sum_{j=0}^{d-1} \omega^{j(k \bmod d)} |j\rangle |(j + [k/d]) \bmod d\rangle.
\] (30)

for \(\omega = e^{i\pi/d}\) and \([k/d]\) denotes the quotient of \(k\) divided by \(d\). Thus the total state in systems \(H^d \otimes H^d \otimes H^d \otimes H^d\) can be written as
\[
|\Psi\rangle = \sum_{k=0}^{d^2-1} a_k |\psi_k\rangle |\psi_k\rangle.
\] (31)
It’s distinct that \( \{ |\psi_k\rangle \} \) is a MEB of the 2-qudit systems. The reductions to all local systems are derived as
\[
\rho_{A_i} = \text{Tr}_{A_j}(|\Psi\rangle \langle \Psi|) = \sum_{k=0}^{d^2-1} \sum_{l=0}^{d^2-1} |a_k|^2 \text{Tr}_{A_j}(|\psi_k\rangle \langle \psi_k|)(|\psi_l\rangle \langle \psi_l|) \delta_{kl}
\]
\[
= \sum_{k=0}^{d^2-1} |a_k|^2 \text{Tr}_{A_j}(|\psi_k\rangle \langle \psi_k|)(|\psi_k\rangle \langle \psi_k|)
\]
\[
= \sum_{k=0}^{d^2-1} |a_k|^2 \frac{I}{d}
\]
\[
= \frac{I}{d}. \tag{32}
\]

Then, we provide the corresponding controlled operation. For an arbitrary unknown state \( |\psi_k\rangle \in H^{d^2} \)
\[
|\psi_k\rangle = \sum_{k=0}^{d^2-1} a_k |k\rangle.
\tag{33}
\]
First, the unitary evolution goes as
\[
|k\rangle \rightarrow |k \mod d\rangle |[k/d]\rangle. \tag{34}
\]
Next, by adding two auxiliary systems, the total state holds the form as follows
\[
|\psi_1\rangle = \sum_{k=0}^{d^2-1} a_k |k \mod d\rangle |[k/d]\rangle |00\rangle. \tag{35}
\]
Second, we obtain the new total state by performing the controlled operation \( C_{13} \)
\[
|\psi_2\rangle = \sum_{k=0}^{d^2-1} a_k |k \mod d\rangle |[k/d]\rangle |k \mod d\rangle |0\rangle, \tag{36}
\]
where
\[
C_{13} = \sum_{k=0}^{d^2-1} |k \mod d\rangle \langle k \mod d| \otimes U^{(k \mod d)}, \tag{37}
\]
with
\[
U = \sum_{j=0}^{d-1} |(j+1) \mod d\rangle \langle j|. \tag{38}
\]
Third, we perform the controlled operation \( C_{24} \) on \( |\psi_2\rangle \) and obtain the states as follows
\[
|\psi_3\rangle = \sum_{k=0}^{d^2-1} a_k |k \mod d\rangle |[k/d]\rangle |k \mod d\rangle |[k/d]\rangle, \tag{39}
\]
where
\[
C_{24} = \sum_{k=0}^{d^2-1} |[k/d]\rangle \langle [k/d]| \otimes U^{[k/d]}, \tag{40}
\]
We send the first and third qudits through the general Hadamard gate, namely
\[
|j\rangle \rightarrow \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \omega^j |j\rangle. \tag{41}
\]
Finally, two controlled not gates given as follows are performed on \( |\psi_3\rangle \) subsequently
\[
C_{22} = \sum_{j=0}^{d^2-1} |j\rangle \langle j| \otimes U^j
\]
\[
C_{34} = \sum_{i=0}^{d^2-1} |i\rangle \langle i| \otimes U^i. \tag{42}
\]
The final total state becomes
\[ |\psi_f\rangle = \frac{1}{d^{d-1}d^{d-1}} \sum_{i=0}^{d} \sum_{j=0}^{d} a_k \omega^{(i+j)(k \text{ mod } d)} |j \rangle |i \rangle |(i + [k/d]) \text{ mod } d \rangle. \tag{43} \]

It should be noted that $|\psi_f\rangle$ is equivalent to (31).

It can be deduced from the quantum masking in multi-qubit systems that as the number of local systems participating in masking tasks increases, all quantum states in the higher-dimensional system will be deterministically masked. Thus a more general conclusion can be obtained.

**Theorem 2.** For all the states of the $w$-level system $H^w$,
\[ |\phi\rangle = \sum_{k=0}^{w-1} a_k |k\rangle, \tag{44} \]
If we mask all the states into $m$-qudit ($m \geq 4$) systems, the supremum of the level of the system in which all the masked states located can be derived as
\[ w \leq d \left\lfloor \frac{\pi}{2} \right\rfloor. \tag{45} \]

**Proof.** Without loss of generality, we divide the $m$-qudit system into two parts. Let $\{ |\psi_w\rangle \}$, $\{ |\psi_w\rangle \}$ be the subsets of the MEBs of $(H^d)^{\left\lfloor \frac{\pi}{2} \right\rfloor}$ and $(H^d)^{\left\lfloor \frac{\pi}{2} \right\rfloor}$ respectively. According to lemma 1, the masking process can be defined by
\[ |k\rangle \rightarrow |\psi_k\rangle |v_k\rangle. \tag{46} \]
where $k = 0, 1, 2, \cdots, w$. That is,
\[ U: |\phi\rangle = \sum_{k=0}^{w-1} a_k |k\rangle \rightarrow |\Phi\rangle = \sum_{k=0}^{w-1} a_k |\psi_k\rangle |v_k\rangle \tag{47} \]
For the final state $|\Phi\rangle$, the reductions to the local systems of the first $\left\lfloor \frac{m}{2} \right\rfloor$ systems are
\[ \rho_{A_i} = \text{Tr}_{\widehat{A}_i}(|\Phi\rangle \langle \Phi|) = \sum_{k=0}^{w-1} \sum_{l=0}^{w-1} |a_k|^2 \text{Tr}_{\widehat{A}_i} |\psi_k\rangle \langle \psi_l| (|v_k\rangle \langle v_l|) \delta_{kl} \]
\[ = \sum_{k=0}^{w-1} |a_k|^2 \text{Tr}_{\widehat{A}_i} |\psi_k\rangle \langle \psi_k| (|v_k\rangle \langle v_k|) \]
\[ = \sum_{k=0}^{w-1} |a_k|^2 \frac{l}{d} \]
\[ = \frac{l}{d}. \tag{48} \]
Where $l \in \{1, 2, \cdots, \left\lfloor \frac{m}{2} \right\rfloor \}$ and $\widehat{A}_i$ denotes the set $\{ A_1, A_2, \cdots, A_l \} \setminus \{ A_1 \}$ and $\lceil \cdot \rceil (\lfloor \cdot \rceil)$ denotes the integer part.

The same result can be obtained for the last $\left\lfloor \frac{m}{2} \right\rfloor$-qudit systems namely $\rho_{A_i} = \frac{l}{d}$ for $l \in \left( \left\lfloor \frac{m}{2} \right\rfloor , \left\lfloor \frac{m}{2} \right\rfloor + 1, \cdots, m \right)$. Thus we observe no information about $|\phi\rangle$ from all the local systems of the $m$-qudit systems. \[ \square \]

This conclusion implies that, in the quantum information masking process, when the dimension of each local system $d$ and the level $w$ of the system that all the masked states located in are determined and $w > d$, it can be seen that at least $2t$ qudit systems need to be employed for $t = \left\lfloor \log_d w \right\rfloor$.

**5. Conclusion**

In this paper, we study the structure of the quantum multipartite masker. Noticing all the known masking schemes, each local system’s dimension is not less than the level of the system which all the states to be masked located in. Here we present some schemes to show it’s possible masking $k$-level states into a $m$-partite system with local dimension $d$ for $k \geq d, m \geq 4$. We also provide the controlled operation in four-partite systems to unveil the mystery of quantum information masking.
Our conclusion provides hands for the question ‘can all quantum states of level \(d\) be hidden into a tripartite quantum system \(H^d \otimes H^d \otimes H^d\) with \(n < d\) or not?’ raised in [30]. Moreover, our schemes further enriched the \(k\)-uniform quantum information masking in multipartite systems. Undoubtedly, as a generalized concept, \(k\)-uniform quantum information masking has a broader significance. As a consequence, a new question is raised: can our conclusion be generalized to \(k\)-uniform scenario?

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**Data availability statement**

The data that support the findings of this study are available upon reasonable request from the authors.

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