Γ-CONVERGENCE FOR FREE-DISCONTINUITY PROBLEMS IN LINEAR ELASTICITY: HOMOGENIZATION AND RELAXATION

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ABSTRACT. We analyze the Γ-convergence of sequences of free-discontinuity functionals arising in the modeling of linear elastic solids with surface discontinuities, including phenomena as fracture, damage, or material voids. We prove compactness with respect to Γ-convergence and represent the Γ-limit in an integral form defined on the space of generalized special functions of bounded deformation (GSBD). We identify the integrands in terms of asymptotic cell formulas and prove a non-interaction property between bulk and surface contributions. Eventually, we investigate sequences of corresponding boundary value problems and show convergence of minimum values and minimizers. In particular, our techniques allow to characterize relaxations of functionals on GSBD, and cover the classical case of periodic homogenization.

1. Introduction

This paper deals with the Γ-convergence of sequences of free-discontinuity functionals (E_n)_n of the form

\[ E_n(u) = \int_{\Omega} f_n(x, e(u)(x)) \, dx + \int_{J_u \cap \Omega} g_n(x, u^+(x), u^-(x), \nu_u(x)) \, dH^{d-1}(x), \tag{1.1} \]

where Ω ⊂ R^d denotes the reference configuration, e(u) is the symmetric part of the gradient of a vector-valued displacement u : Ω → R^d, and J_u denotes the set of discontinuities of u, oriented by a normal vector ν_u with one-sided traces u^+ and u^-. (By H^{d-1} we indicate the (d-1)-dimensional Hausdorff measure.) Such functionals are prototypes for many variational models of fracture mechanics in a small strain setting. In this framework, the bulk density f_n depending only on the linearized stress tensor accounts for elastic bulk terms for the unfractured region of the body, while the surface integrand g_n represents the energy spent to produce a crack. Usually, g_n is assumed to be bounded and accounts for both brittle [47] and cohesive [8] fracture, where in the latter case g_n depends explicitly on the crack opening \[ |u| := u^+ - u^- \].

Minimization problems for (1.1) are usually complemented with Dirichlet data. Their well-posedness in the space of generalized functions of bounded deformation (GSBD) [35] has been a challenging task in very recent years, see [7, 24, 26, 27, 28, 30, 43]. We also refer to [1, 20, 22, 25, 32, 38, 45, 49] for some recent applications. In the present paper, we are interested in the effective behavior of a sequence of functionals (E_n)_n and corresponding minimization problems. The parameter n may have different meanings: it may account for a regularization of the energy, represent the size of a microstructure, or model the different mechanical responses of a composite material in each of its components. Identifying the limit of E_n in the sense of Γ-convergence [10, 34] arises as a natural problem with various possible applications: let us mention, for instance, the case of homogenization, i.e., f_n(x, ξ) = f(x/ε_n, ξ) and g_n(x, η_1, η_2, ν) = g(x/ε_n, η_1 - η_2, ν) for f and g.

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$g$ being 1-periodic in the first variable. Here, the limiting functional corresponds to the effective energy of the homogenized material.

**State-of-the-art:** Due to both its theoretical interest and its relevance for applications, the $\Gamma$-convergence analysis of free-discontinuity problems has been the subject of many contributions over the last three decades. Most of the attention, however, has been focused on the related but different context of functionals of the form

$$\mathcal{E}^{BV}_n(u) = \int_{\Omega} f_n(x, \nabla u(x)) \, dx + \int_{\partial_\Omega} g_n(x, [u](x), \nu_n(x)) \, d\mathcal{H}^{d-1}(x) \quad (1.2)$$

involving the **full deformation gradient** $\nabla u$. (With a slight abuse of notation, we still use the notation $g_n$ although the density depends on $u^+$ and $u^-$ only in terms of $[u] = u^+ - u^-$. The first result in this direction is the seminal work [11] addressing the case of periodic homogenization. By assuming a linear growth of $g_n$, the authors derive a limiting homogenized functional with $x$-independent densities $f_{\text{hom}}$ and $g_{\text{hom}}$ on the natural energy space of **special functions of bounded variation** (SBV) [6, Section 4]. Unfortunately, the growth assumptions on $g_n$ do not comply with standard models for brittle and cohesive fracture where surface densities are assumed to be bounded.

The result has been extended in [46] to the non-periodic case and more natural growth conditions. There, however, the authors study a slightly simplified setting motivated by applications to quasistatic crack growth: first, competitors for (1.2) are **scalar-valued**. Secondly, the surface energy densities $g_n$ are independent of the jump height $[u]$. The scalar nature of the problem allows to use truncation techniques (at least for bounded Dirichlet data) such that SBV endowed with the $L^1$-topology is still a natural setting for the $\Gamma$-convergence result. In both contributions [11, 46], a remarkable property is that the limiting densities $f_{\text{hom}}$ and $g_{\text{hom}}$ are completely determined by $f$ and $g$, respectively. This **non-interaction** of the bulk and the surface part of the energy is due to the exponent $p > 1$ in the growth assumption for $f_n$. In contrast, for $p = 1$, it is indeed known by other examples that interaction effects in the limit are possible, see, e.g., [9, 13, 19].

A comprehensive treatment of the $\Gamma$-convergence analysis for functionals of the form (1.2) has been achieved recently in the paper [17], which includes the vector-valued setting, assumes no periodicity, and complies with weaker coercivity conditions. More precisely, the densities are assumed to satisfy

$$\alpha|\xi|^p \leq f_n(x, \xi) \leq \beta(1 + |\xi|^p), \quad \alpha \leq g_n(x, \zeta, \nu) \leq \beta(1 + |\zeta|) \quad (1.3)$$

for $x \in \Omega$, $\xi \in \mathbb{R}^{d\times d}$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{R}^d$ with $|\nu| = 1$, where $0 < \alpha \leq \beta$ are positive constants. Due to the weaker growth assumptions on $g_n$ compared to [11] and the vector-valued nature, the model is more relevant for applications in fracture mechanics. This, however, comes at the expense of considering a weaker functional setting. Indeed, compactness of competitors with bounded energy can now only be expected with respect to the convergence in measure in the larger space of generalized special functions of bounded variation $GSBV$ [6, Section 4]. Eventually, based on a general compactness result in $GSBV^p$ (the subspace of $GSBV$ of functions with $p$-integrable gradient), [42] complements the $\Gamma$-convergence analysis in [17] by investigating corresponding boundary value problems and showing convergence of minimizers.

The authors in [17] make use of an abstract viewpoint: they first show that, under assumptions (1.3), the $\Gamma$-limit of (1.2) can still be represented as an integral functional on $GSBV^p$. To this aim, they use the localization method for $\Gamma$-convergence and the global method for relaxation developed in [12, 13]. This method compares asymptotic Dirichlet problems on small balls with different boundary data depending on the local properties of the functions, and provides a characterization of the energy densities in terms of cell formulas. A technical difficulty lies in the fact that the
procedure needs linear growth of \( g_n \) in \([u]\) which is not available due to (1.3). This can be overcome, however, by means of a perturbation trick: a small perturbation of the functionals, depending on the jump height, is considered, which can be represented as an integral functional on \( SBV^p \). Then, by letting the perturbation parameter vanish and by truncating functions suitably, the representation is extended to \( GSBV^p \). Similar truncation techniques are also employed for the localization method in connection with the fundamental estimate [11 Proposition 3.1] to pass from the \( L^p \)-topology to the topology of measure convergence. We already remark that such a tool is not available for functionals of the form (1.1). In fact, given a control only on the symmetrized gradient, it is in general not possible to use smooth truncations to decrease the energy up to a small error.

After establishing the abstract representation result, the authors in [17] show that the bulk and surface parts of the energy do not interact in the limit. This allows to derive the results of [11, 16] as a corollary of their approach. For further discussion below, we point out that a key step for the identification of the surface density relies on the coarea formula to approximate \( GSBV^p \) functions by piecewise constant functions. A similar method has been used for the characterization of lower semicontinuity in \( SBV \) [2, 8], for the so-called jump transfer [37] Theorem 2.1, and it is also at the core of the compactness result [12] needed to treat boundary value problems. This is a delicate issue for functionals of the form (1.1) since this technique is not available if only symmetrized gradients are controlled.

The present paper: Summarizing, a rather complete picture of \( \Gamma \)-convergence for functionals given in (1.2) has been developed over the last years, extending also to the case of stochastic homogenization [15]. By way of contrast, the understanding of the counterpart in the linearized setting is scarce. This leads us to the purpose of our paper, which exactly aims at extending the results in [17, 12] to the more general framework of free-discontinuity functionals of the form (1.1).

Our first main result (Theorem 2.1) provides a general compactness and representation result for \( \Gamma \)-limits of sequences \((\mathcal{E}_n)\) on the space \( GSBD^p \) (the subspace of \( GSBD \) with \( e(u) \in L^p \)), endowed with the topology of measure convergence. More precisely, we assume that the energy densities \( f_n \) and \( g_n \) satisfy

\[
\alpha |(\xi^T + \xi)/2|^p \leq f_n(x, \xi) \leq \beta (1 + |(\xi^T + \xi)/2|^p), \quad \alpha \leq g_n(x, a, b, \nu) \leq \beta
\]

for \( x \in \Omega, \xi \in \mathbb{R}^{d \times d}, a, b \in \mathbb{R}^d \), and \( \nu \in \mathbb{R}^d \) with \(|\nu| = 1\), where \( p > 1 \) and \( 0 < \alpha \leq \beta \). We prove that, up to a subsequence, the functionals \( \mathcal{E}_n \) converge to a \( \Gamma \)-limit \( \mathcal{E} \) which is still an integral functional of the form (1.1).

Our second main result (Theorem 2.4) deals with the identification of the \( \Gamma \)-limit \( \mathcal{E} \) by analyzing the relation of the densities \((f_n)\) and \((g_n)\), with the densities \( f_\infty \) and \( g_\infty \) of \( \mathcal{E} \). In particular, we investigate under which conditions bulk and surface effects decouple in the limit. For the bulk density, we obtain

\[
f_\infty(x, \xi) = \lim_{\rho \to 0^+} \limsup_{n \to \infty} \frac{1}{\rho^d} \int_{Q_\rho(x)} f_n(x, e(u)(x)) \, dx,
\]

where we denote by \( Q_\rho(x) \) the cube centered at \( x \) with sidelength \( \rho \), and the infimum is taken among all functions \( v \in W^{1,p}(Q_\rho(x); \mathbb{R}^d) \) satisfying \( v(y) = \xi y \) near \( \partial Q_\rho(x) \). When it comes to the surface energy instead, we consider some additional restrictions: we focus on the case where \( g_n(x, \nu) \) is independent of the traces at the jump, and for \( d > 2 \) we further assume that \( g_n = h \) is independent of \( n \). We prove that

\[
g_\infty(x, \xi) = \lim_{\rho \to 0^+} \limsup_{n \to \infty} \frac{1}{\rho^{d-1}} \int_{Q_\rho(x)} g_n(x, \nu_\alpha(x)) \, d\mathcal{H}^{d-1},
\]
where $Q_p^\rho(x)$ is a cube of sidelength $\rho$ oriented by $\nu$, and the infimum is taken over all piecewise rigid functions $v \in PR(Q_p^\rho(x))$ which near $\partial Q_p^\rho(x)$ agree with the jump function

$$
\bar{u}_{x,\nu}(y) = \begin{cases} 
0 & \text{if } (y-x) \cdot \nu \geq 0, \\
(1,0,\ldots,0) & \text{if } (y-x) \cdot \nu < 0.
\end{cases}
$$

Here, $PR$ is the subset of $\text{GSBD}^p$ consisting of functions $u$ with $e(u) \equiv 0$. (Due to independence of $g_n$ on the jump height, we point out that equivalently piecewise constant functions could be considered, i.e., $u$ satisfies $\nabla u \equiv 0$.) The reason for considering $d = 2$ for general sequences $(g_n)_n$ lies in a technique for approximating $\text{GSBD}^p$ functions by piecewise rigid functions which is only available in the planar setting. As a corollary, again restricted to $d = 2$, we are also able to deduce a (periodic) homogenization result (Corollary 2.5).

In general dimensions, we can treat the case of constant sequences $g_n = h$. Here, it turns out that the limiting density $g_\infty$ is the so-called $\text{BV}$-elliptic envelope introduced in [5] as a condition for lower semicontinuity of functionals defined on partitions. As $h$ is independent of the jump height, $g_\infty$ also coincides with the $BD$-elliptic envelope introduced in [43] in the more general context of variational problems in spaces of Caccioppoli-affine functions. As a special case (Corollary 2.6), we deduce in any space dimensions that the relaxation with respect to measure convergence of integral functionals on $\text{GSBD}^p$ of the form

$$
\int_{\Omega} f(x, e(u)(x)) \, dx + \int_{J_u \cap \Omega} g(x, \nu_u(x)) \, d\mathcal{H}^{d-1}
$$

has the same structure with densities $\bar{f}$ and $\bar{g}$, where $\bar{f}$ denotes the quasiconvex envelope of $f$ and $\bar{g}$ is the $\text{BV}$-elliptic envelope of $g$. In particular, it is simply given by the superposition of the relaxation of the bulk energy in $W^{1,p}$ and of the surface energy in the space of piecewise constant functions.

In our third main result, we eventually incorporate Dirichlet boundary data (Proposition 2.9), and show the convergence of (almost) minimizers for a sequence $(\mathcal{E}_n)_n$ with the given conditions to minimizers of $\mathcal{E}$ (Theorem 2.10).

**Proof techniques and challenges:** In the sequel, we highlight some of the proof techniques focusing on the additional challenges with respect to models (1.2) in $\text{GSBV}^p$. For the compactness of $\Gamma$-convergence (Theorem 2.1), we specify the localization technique already used in [17] to the setting at hand. The key ingredient is a construction for joining two functions $u, v \in \text{GSBD}^p(\Omega)$, which is usually called the fundamental estimate (Proposition 1.1). In doing this, one must ensure that the energy spent in a transition layer is small, when the two functions are close in the considered topology. Typically, this is achieved by means of a cut-off construction of the form $w := u\varphi + (1-\varphi)v$ for some smooth $\varphi$ with $0 \leq \varphi \leq 1$. This, however, requires $L^p$-integrability of the functions $u$ and $v$, which is not a priori given in our context. In contrast to the $\text{GSBD}^p$ setting, this cannot be recovered with truncation arguments, and we need a considerably more involved strategy to overcome this issue.

The main novel tool is a Korn-type inequality for functions with small jump, established recently by Cagnetti, Chambolle, and Scardia [21], which generalizes a two-dimensional result in [29] (see also [39]) to arbitrary dimension. It provides a control of the full gradient in terms of the symmetrized gradient, up to an exceptional set whose perimeter has a surface measure comparable to that of the discontinuity set. We combine this tool with a covering technique of the transition layer by means of small cubes, which enables us to cut out an exceptional set with small volume and perimeter such that, in the residual set, the $L^p$ norm of $u - v$ is controlled in terms of $\|e(u-v)\|_{L^p}$ (up to a small rest). This finally allows to perform the usual cut-off construction. Concerning the
representation of the limit, we profit of a very recent integral representation result proved in [31, Theorem 2.1], tailored for energies of the form (1.1).

For the identification of the limiting densities \( f_\infty \) and \( g_\infty \) (Theorem 2.4), the essential point is to show that the minimization in the asymptotic cell formulas (1.4)–(1.5) can indeed be restricted from \( GSBD^p \) to Sobolev and piecewise constant functions, respectively. For the bulk density, this is achieved by using the Korn inequality for functions with small jump set [21] to approximate \( GSBD^p \) functions with asymptotically vanishing jump set by Sobolev functions. Afterwards, we can follow the lines of the \( SBV \) proof [17, 46], in particular involving truncation methods to obtain a sequence of equiintegrable Sobolev functions, see Lemma 5.1. For the surface density instead, the challenge lies in approximating a sequence in \( GSBD^p \) with vanishing symmetrized gradient by a sequence of characteristic functions of sets with finite perimeter, see Lemma 5.2. The nonavailability of the coarea formula in our setting makes this a very delicate task which can be overcome by the application of a piecewise Korn-Poincaré inequality, see Proposition 3.4 and [45], which has been derived only for \( d = 2 \). For the relaxation result in general space dimensions, this technique is not at our disposal, and we use directly a recent lower semicontinuity result for surface integrals in \( GSBD^p \) [43] for so-called symmetric jointly convex functions, see Definition 3.9.

Finally, the extension to this case of Dirichlet boundary conditions (Proposition 2.9 and Theorem 2.10) is not straightforward because of two main reasons. First, the construction of recovery sequences complying with the given data requires the usage of our novel fundamental estimate (Proposition 4.1) and a recent extension result [16]. Secondly, according to the compactness results of [26, 28, 45], sequences with equibounded energy in the Dirichlet setting converge in a slightly weaker sense compared to convergence in measure.

Summarizing, we believe that the present paper gives a thorough analysis of the \( \Gamma \)-convergence and relaxation problem for free-discontinuity problems in linearized elasticity. We fix a convenient setting, develop a number of technical tools (all arising from very recent advances in the topic), and illustrate the main issues to be overcome in order to remove the restrictions on the surface density that we need to impose in Theorem 2.4. This will hopefully be the object of forthcoming achievements and contributions to the problem.

The paper is organized as follows. In Section 2 we introduce the model and state our three main results. In Section 3 we collect basic properties of the function space \( GSBD^p \), and we recall integral representation formulas for functionals defined on Sobolev functions and piecewise constant functions. Section 4 is devoted to the proof of Theorem 2.1, in particular we formulate and prove a fundamental estimate in \( GSBD^p \) (Proposition 4.1). In Section 5 we present two approximation results of \( GSBD^p \) functions by Sobolev and characteristic functions, respectively. These are fundamental ingredients for the proof of Theorem 2.4 in Section 6 but also of independent interest. In Section 7 finally we address the minimization problems with Dirichlet boundary data.

2. Setting of the problem and main results

In this section we present our main results. We start with some basic notation. In the sequel, \( \Omega \subset \mathbb{R}^d \) always denotes an open set. Let \( A(\Omega) \) be the family of open and bounded subsets of \( \Omega \). We write \( \chi_A \) for the characteristic function of any \( A \subset \mathbb{R}^d \), which is 1 on \( A \) and 0 otherwise. If \( A \) is a set of finite perimeter, we denote its essential boundary by \( \partial^* E \), see [6, Definition 3.60]. For two sets \( A, B \subset \mathbb{R}^d \), we denote by \( A \triangle B \) their symmetric difference and by \( \text{dist}(A, B) \) their Hausdorff distance. Moreover, we write \( B \Subset \subset A \) if \( B \subset \subset \Omega \).

For \( x, y \in \mathbb{R}^d \), we use the notation \( x \cdot y \) for the scalar product and \( |x| \) for the Euclidean norm. Moreover, we set \( S^{d-1} := \{ x \in \mathbb{R}^d : |x| = 1 \} \). We denote by \( \mathbb{R}^{d \times d} \) the set of \( d \times d \) matrices. By
In particular, for $\xi \in \mathbb{R}^{d \times d}$, we indicate the subsets of symmetric and skew-symmetric matrices, respectively. In particular, for $\xi \in \mathbb{R}^{d \times d}$, we define $\text{sym}(\xi) = (\xi + \xi^T)/2$. The Frobenius norm of a matrix $\xi$ is denoted by $|\xi|$.

For every $x \in \mathbb{R}^d$ and $\rho > 0$ we indicate by $B_\rho(x) \subset \mathbb{R}^d$ the open ball with center $x$ and radius $\rho$. Additionally, for $\nu \in \mathbb{S}^{d-1}$, we denote by $Q_\rho^\nu(x)$ the cube centered at $x$ with sidelength $\rho$ and two faces normal to $\nu$. For $x = 0$, we simply write $Q_\rho^\nu$. We let $e_1 = (1,0,\ldots,0) \in \mathbb{R}^d$ and set $Q_\rho(x) = Q_{\rho}^{e_1}(x)$ for all $x \in \mathbb{R}^d$ and $\rho > 0$. We denote by $\mathcal{L}^d$ and $\mathcal{H}^k$ the $d$-dimensional Lebesgue measure and the $k$-dimensional Hausdorff measure, respectively. We set $\mathbb{R}_+ = [0,\infty)$.

For definition and properties of the space $\text{GSBD}^p(\Omega)$, $1 < p < \infty$, we refer the reader to [35]. Some relevant properties are collected in Subsection 3.1 below. In particular, the approximate $\Gamma$-convergence result for functionals of this form. (For an exhaustive treatment of $\Gamma$-convergence we refer the reader to [10, 34].) To formulate the result, we introduce some further notation: for each $u \in \text{GSBD}^p(\Omega)$ and $\xi \in \mathbb{R}^{d \times d}$, we have

$$\alpha|\text{sym}(\xi)|^p \leq f(x,\xi) \leq \beta(1 + |\text{sym}(\xi)|^p). \quad (2.1)$$

Moreover, let $g : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,\infty)$ be a Borel function satisfying

$$\alpha \leq g(x,a,b,\nu) \leq \beta \quad (2.2)$$

for $\mathcal{H}^{d-1}$-a.e. $x \in \Omega$ and for all $a,b \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$. In what follows, we consider energy functionals $\mathcal{E} : \text{GSBD}^p(\Omega) \times \mathcal{A}(\Omega) \to [0,\infty)$ of the form

$$\mathcal{E}(u,A) = \int_A f(x,e(u)(x)) \, dx + \int_{J_u \cap A} g(x,u^+(x),u^-(x),\nu_u(x)) \, d\mathcal{H}^{d-1}(x) \quad (2.3)$$

for each $u \in \text{GSBD}^p(\Omega)$ and each $A \in \mathcal{A}(\Omega)$. In this subsection, we present a general $\Gamma$-convergence result for functionals of this form. (For an exhaustive treatment of $\Gamma$-convergence we refer the reader to [10, 34].) To formulate the result, we introduce some further notation: for every $u \in \text{GSBD}^p(\Omega)$ and $A \in \mathcal{A}(\Omega)$ we define

$$\textbf{m}_\mathcal{E}(u,A) = \inf \{ \mathcal{E}(v,A) : v = u \text{ in a neighborhood of } \partial A \}. \quad (2.4)$$

For $x_0 \in \Omega$, $u_0 \in \mathbb{R}^d$, and $\xi \in \mathbb{R}^{d \times d}$ we introduce the functions $\ell_{x_0,u_0,\xi} : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\ell_{x_0,u_0,\xi}(x) = u_0 + \xi(x-x_0). \quad (2.5)$$

Moreover, for $x_0 \in \Omega$, $a,b \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$ we introduce $u_{x_0,a,b,\nu} : \mathbb{R}^d \to \mathbb{R}^d$ by

$$u_{x_0,a,b,\nu}(x) = \begin{cases} a & \text{if } (x-x_0) \cdot \nu \geq 0, \\ b & \text{if } (x-x_0) \cdot \nu < 0. \end{cases} \quad (2.6)$$

We now proceed with our first main result.

**Theorem 2.1 (\Gamma-convergence).** Let $\Omega \subset \mathbb{R}^d$ be open. Let $(f_n)_n$ and $(g_n)_n$ be sequences of functions satisfying 2.1 and 2.2, respectively. Let $\mathcal{E}_n : \text{GSBD}^p(\Omega) \times \mathcal{A}(\Omega) \to [0,\infty)$ be the corresponding sequence of functionals given in 2.3. Then, there exists $\mathcal{E} : \text{GSBD}^p(\Omega) \times \mathcal{A}(\Omega) \to [0,\infty)$ and a subsequence (not relabeled) such that

$$\mathcal{E}(.A) = \Gamma- \lim_{n \to \infty} \mathcal{E}_n(.,A) \quad \text{with respect to convergence in measure on } A$$
for all $A \in \mathcal{A}(\Omega)$. Moreover, for every $u \in GSBD^p(\Omega)$ and $A \in \mathcal{A}(\Omega)$ we have that

$$
\mathcal{E}(u, A) = \int_A f_\infty(x, u(x), \nabla u(x)) \, dx + \int_{J_u \cap A} g_\infty(x, u^+(x), u^-(x), \nu_u(x)) \, d\mathcal{H}^{d-1}(x),
$$

(2.7)

where $f_\infty$ is given by

$$
f_\infty(x_0, u_0, \xi) := \limsup_{\rho \to 0^+} \frac{m_\mathcal{E}(f_{x_0, u_0, \xi}, Q_{\rho}(x_0))}{\rho^d},
$$

(2.8)

for all $x_0 \in \Omega$, $u_0 \in \mathbb{R}^d$, $\xi \in \mathbb{R}^{d \times d}$, and $g_\infty$ is given by

$$
g_\infty(x_0, a, b, \nu) := \limsup_{\rho \to 0^+} \frac{m_\mathcal{E}(u_{x_0, a, b, \nu}, Q_{\rho}'(x_0))}{\rho^{d-1}},
$$

(2.9)

for all $x_0 \in \Omega$, $a, b \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$.

The compactness of $\Gamma$-convergence is proved via the localization technique for $\Gamma$-convergence, see Section 4. Here, the main ingredient is a novel fundamental estimate in the space $GSBD^p$, see Proposition 4.1. Afterwards, the representation (2.7) in terms of the densities $f_\infty$ and $g_\infty$ follows by the recent integral representation result [31].

**Remark 2.2** (Invariance under rigid motions, cell formulas). (i) Suppose that each $\mathcal{E}_\alpha$ satisfies $\mathcal{E}_\alpha(u + a, A) = \mathcal{E}_\alpha(u, A)$ for all affine functions $a: \mathbb{R}^d \to \mathbb{R}^d$ with $e(a) = 0$ and all $A \in \mathcal{A}(\Omega)$. Then, as $\Gamma$-limit, $\mathcal{E}$ satisfies the same property. Thus, [31] Remark 2.2(iii) implies that $\mathcal{E}$ has the form

$$
\mathcal{E}(u, A) = \int_A f_\infty(x, e(u)(x)) \, dx + \int_{J_u \cap A} g_\infty(x, [u](x), \nu_u(x)) \, d\mathcal{H}^{d-1}(x),
$$

(2.10)

where $[u](x) := u^+(x) - u^-(x)$, and the densities $f_\infty$, $g_\infty$ are given by

$$
f_\infty(x_0, \text{sym}(\xi)) = \limsup_{\rho \to 0^+} \frac{m_\mathcal{E}(f_{x_0, \text{sym}(\xi)}, Q_{\rho}(x_0))}{\rho^d}, \quad g_\infty(x_0, \xi, \nu) = \limsup_{\rho \to 0^+} \frac{m_\mathcal{E}(u_{x_0, \xi, 0, \nu}, Q_{\rho}'(x_0))}{\rho^{d-1}}
$$

(2.11)

for all $x_0 \in \Omega$, $\xi \in \mathbb{R}^{d \times d}$, $\xi \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$.

(ii) A variant of the proof shows that, in the minimization problems (2.8)–(2.9), one may replace the cubes by balls $B_\rho(x_0)$ with radius $\rho$, centered at $x_0$.

### 2.2. Identification of the $\Gamma$-limit: homogenization and relaxation.

We now address the structure of the $\Gamma$-limit by showing that there is no interaction between the bulk and surface densities, i.e., $f_\infty$ is only determined by $(f_n)_n$ and $g_\infty$ is only determined by $(g_n)_n$. As applications, we discuss homogenization and relaxation results. The statements announced in this subsection are proved in Section 6. In this part, we restrict our assumptions to a more specific setting, namely to surface densities $g$ of the form $g: \Omega \times \mathbb{S}^{d-1} \to [0, +\infty)$, still being Borel functions and satisfying

$$
\alpha \leq g(x, \nu) \leq \beta
$$

(2.12)

for $\mathcal{H}^{d-1}$-a.e. $x \in \Omega$ and for all $\nu \in \mathbb{S}^{d-1}$, where $0 < \alpha \leq \beta < +\infty$ as before. Moreover, for some parts we will further restrict our analysis to the planar setting $d = 2$ and to exponents $p \geq 2$. We refer to Remark 2.7 at the end of the subsection for comments on these restrictions.

To formulate the non-interaction between bulk and surface density, we need to restrict functionals $\mathcal{E}$ of the form (2.3) to Sobolev functions $W^{1,p}(\Omega; \mathbb{R}^d)$ and to piecewise rigid functions $PR(\Omega)$,
respectively. Here, we set $PR(\Omega) := \{u \in \text{GSBD}^p(\Omega): e(u) \equiv 0\}$, see \cite{[11]}. Then, similarly to (2.4), for every $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$ we define
\[
m^{1,p}_\xi(u, A) = \inf_{v \in W^{1,p}(\Omega; \mathbb{R}^d)} \{\mathcal{E}(v, A): \ v = u \ \text{in a neighborhood of } \partial A\},
\] (2.13)
and, for every $u \in PR(\Omega)$, we let
\[m^{PR}_\xi(u, A) = \inf_{v \in PR(\Omega)} \{\mathcal{E}(v, A): \ v = u \ \text{in a neighborhood of } \partial A\}.
\] (2.14)
Due to the fact that the surface integral vanishes on $W^{1,p}$, and by the definition of $PR(\Omega)$ as well as the upper control in (2.1), we find for all $A \in \mathcal{A}(\Omega)$ that
\[
\mathcal{E}(u, A) = \int_A f(x, e(u)(x)) \, dx \quad \text{for all } u \in W^{1,p}(\Omega; \mathbb{R}^d),
\] (2.15)
\[
|\mathcal{E}(u, A) - \int_{J_u \cap A} g(x, \nu_u(x)) \, d\mathcal{H}^{d-1}| \leq \beta \mathcal{L}^d(A) \quad \text{for all } u \in PR(\Omega).
\] (2.16)
For convenience, we specify the notation in (2.5)–(2.6) and write
\[
\ell_\xi = \ell_{e_0, \xi} \quad \text{for } \xi \in \mathbb{R}^{d \times d} \quad \text{and} \quad \hat{u}_{x,\nu} = u_{x_0, e_1, 0, \nu} \quad \text{for } e_1 = (1,0,\ldots,0).
\] (2.17)
In particular, we have $\bar{\ell}_\xi(y) = \xi y$ for all $y \in \mathbb{R}^d$.

**Proposition 2.3.** Let $\Omega \subset \mathbb{R}^d$ be open. Let $(f_n)$ and $(g_n)$ be sequences of functions satisfying (2.1) and (2.12), respectively. Correspondingly, define $(\mathcal{E}_n)$ as in (2.3). Then, by passing to a subsequence (not relabeled) the following holds: for all $x \in \Omega$ and all $\xi \in \mathbb{R}^{d \times d}$ we have
\[
\limsup_{\rho \to 0^+} \liminf_{n \to \infty} \frac{m^{1,p}_{\mathcal{E}_n}(\ell_{\xi}, Q_\rho(x))}{\rho^d} = \limsup_{\rho \to 0^+} \limsup_{n \to \infty} \frac{m^{1,p}_{\mathcal{E}_n}(\bar{\ell}_{\xi}, Q_\rho(x))}{\rho^d} =: f(x, \xi),
\] (2.18)
and it holds that $f(x, \xi) = f(x, \text{sym}(\xi))$. For all $x \in \Omega$ and all $\nu \in \mathbb{S}^{d-1}$ we have
\[
\limsup_{\rho \to 0^+} \liminf_{n \to \infty} \frac{m^{PR}_{\mathcal{E}_n}(\hat{u}_{x,\nu}, Q_\rho(x))}{\rho^{d-1}} = \limsup_{\rho \to 0^+} \limsup_{n \to \infty} \frac{m^{PR}_{\mathcal{E}_n}(\bar{u}_{x,\nu}, Q_\rho(x))}{\rho^{d-1}} =: g(x, \nu).
\] (2.19)

In the above formulas, we intend that $\rho$ is always chosen sufficiently small such that the cubes $Q_\rho(x)$ and $Q_\rho'(x)$ are contained in $\Omega$. In view of (2.15)–(2.16), the density $f$ is completely determined by $(f_n)$, whereas $g$ is completely determined by $(g_n)$. This motivates the definition of (2.13)–(2.14). The proof of Proposition 2.3 essentially relies on $\Gamma$-convergence results for Sobolev functions and piecewise constant functions, see Subsection 3.2.

By Theorem 2.1 we get that up to subsequence (not relabeled), the functionals $\mathcal{E}_n(\cdot, A)$ given in (2.3) with densities $f_n$ and $g_n$, $\Gamma$-converge with respect to the convergence in measure to a functional $\mathcal{E}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$. As each $\mathcal{E}_n$ satisfies $\mathcal{E}_n(u + a, A) = \mathcal{E}_n(u, A)$ for all affine functions $a$ with $e(a) = 0$ and all $A \in \mathcal{A}(\Omega)$, Remark 2.2(i) shows that the densities $f_\infty$ and $g_\infty$ of the $\Gamma$-limit can be represented by (2.11). We now proceed with our second main result. We show that the density $f_\infty$ coincides with the function $f$ provided by Proposition 2.3. Hence, the surface energies are not contributing to the bulk part of the limiting functional. We also prove the analogous property $g_\infty = g$ for the surface densities in two specific situations: (a) in the planar case $d = 2$ and (b) for $d > 2$, whenever the surface densities $g_n$ are independent of $n$.

**Theorem 2.4 (Identification of the $\Gamma$-limit).** Let $\Omega \subset \mathbb{R}^d$ be open. Let $(f_n)$ and $(g_n)$ be sequences of functions satisfying (2.1) and (2.12), respectively. Suppose that (2.18)–(2.19) hold, and define $f$ and $g$ accordingly. Let $f_\infty$ and $g_\infty$ be defined by (2.11). Then, the following holds:
Corollary 2.5 (Homogenization). Let \( p \geq 2 \), \( \Omega = \mathbb{R}^2 \), and consider \( f \) and \( g \) satisfying (2.12), respectively. Suppose that for every \( x \in \mathbb{R}^2 \), \( \xi \in \mathbb{R}^{2 \times 2} \), and \( \nu \in S^1 \) we have the limits

\[
 f_{\text{hom}}(\text{sym}(\xi)) := \lim_{r \to \infty} \frac{m_\xi^p(\tilde{u}_r, Q_r(x))}{r^2}, \quad g_{\text{hom}}(\nu) := \lim_{r \to \infty} \frac{m_\nu^{PR}(\tilde{u}_{rx, \nu}, Q_r(x))}{r}
\]

exist and are independent of \( x \). Let \((\varepsilon_n)_{n \in \mathbb{N}} \subset (0, \infty)\) be a sequence with \( \varepsilon_n \to 0 \), and for \( n \in \mathbb{N} \) let

\[
 f_n(x, \xi) := f(x/\varepsilon_n, \xi), \quad g_n(x, \nu) = g(x/\varepsilon_n, \nu)
\]

for \( x \in \mathbb{R}^2 \), \( \xi \in \mathbb{R}^{2 \times 2} \), and \( \nu \in S^1 \). Then, for all \( A \in \mathcal{A}(\Omega) \) the functionals \( \mathcal{E}_n(\cdot, A) \), given by (2.3), with densities \( f_n \) and \( g_n \), \( \Gamma \)-converge with respect to the convergence in measure to \( \mathcal{E}_{\text{hom}}(\cdot, A) \), where \( \mathcal{E} \) is defined in (2.20) with densities \( f_{\text{hom}} \) and \( g_{\text{hom}} \).

In view of [11] Propositions 2.1 and 2.2, (2.24) can always be verified, whenever \( f \) and \( g \) are periodic of period 1 with respect to the coordinates \( e_1 \) and \( e_2 \). In general dimension, using part (iii) of Theorem 2.4 we obtain the following relaxation result.
Corollary 2.6 (Relaxation). Let $\Omega \subset \mathbb{R}^d$ be open for $d \geq 2$. Suppose that $f$ and $g$ satisfy (2.1) and (2.12), respectively. Denote by $\tilde{E}$ the relaxation of $E$ given by (2.3) corresponding to $f$ and $g$, i.e.,

$$\tilde{E}(u, A) = \inf \left\{ \liminf_{n \to \infty} E(u_n, A) : u_n \to u \text{ in measure on } A \right\}$$

for all $u \in GSBD^p(\Omega)$ and $A \in \mathcal{A}(\Omega)$. Then, $\tilde{E}$ is characterized by

$$\tilde{E}(u, A) = \int_A \tilde{f}(x, e(u(x))) \, dx + \int_{J_u \cap A} \tilde{g}(x, \nu_u(x)) \, dH^{d-1}(x)$$

for all $u \in GSBD^p(\Omega)$ and $A \in \mathcal{A}(\Omega)$, where $\tilde{f}$ denotes the quasiconvex envelope (with respect to the second variable) of $f$, and $\tilde{g}$ is the BD-elliptic envelope of $g$ defined in (2.23).

Remark 2.7 (Discussion on assumptions). Theorem 2.4(ii) only holds in dimension $d = 2$ since for the identification of the surface density we apply a piecewise Korn-Poincaré inequality [40] which is only available in the planar setting. For similar reasons, we need to restrict ourselves to exponents $p \geq 2$. We refer to Remark 5.3 for more details in that direction. In the statement of Theorem 2.4(iii) we need to assume continuity of $h$ in order to apply relaxation results for piecewise constant functions [41], see Proposition 3.16. Eventually, the assumption that $h$ is even turns out to be instrumental to apply lower semicontinuity results in $GSBD^p$, see Theorem 3.10 and Proposition 3.11 below. Let us also mention that our strategy exploits explicitly the fact that the surface densities are of the form (2.12), i.e., do not depend on $u^+(x)$ and $u^-(x)$.

Remark 2.8 (Continuity of $h$ in Theorem 2.4(iii)). As a final remark, we record that the continuity assumption on $h$ in Theorem 2.4(iii) can be slightly altered in the following sense: suppose that there exists $D \in \mathcal{A}(\Omega)$ with Lipschitz boundary such that $h$ is uniformly continuous on $D$ and

$$\sup_{(x, \nu) \in D \times S^{d-1}} h(x, \nu) \leq \inf_{(x, \nu) \in (\Omega \setminus D) \times S^{d-1}} h(x, \nu).$$

(2.25)

Then (2.22) holds for all $u \in GSBD^p(\Omega)$ with $J_u \subset \overline{D} \cap \Omega$. We refer to Subsection 6.3 for details.

2.3. Minimization problems for given boundary data. We complement the Γ-convergence results of the previous subsection by convergence results for minimizers of certain boundary value problems, as it is customary in many applications. We impose Dirichlet data on $\partial_D \Omega := \Omega' \cap \partial \Omega$, where $\Omega$ denotes a bounded Lipschitz domain and $\Omega' \supset \Omega$ denotes another bounded Lipschitz domain such that also $\Omega' \setminus \overline{\Omega}$ has Lipschitz boundary. This will be achieved by requiring $u = u^0$ on $\Omega' \setminus \overline{\Omega}$ for some datum $u^0 \in W^{1,p}(\Omega'; \mathbb{R}^d)$, i.e., we will treat the non-attainment of the boundary data (in the sense of traces) as internal jumps. To this end, we introduce energy functionals defined on $\Omega'$. Consider sequences of densities $(f_n)_n$ and $(g_n)_n$ as in (2.1) and (2.12), respectively. We define

$$f'_n(x, \xi) := \begin{cases} f_n(x, \xi) & \text{if } x \in \Omega, \\ \alpha(\text{sym(}\xi))^{\beta} & \text{otherwise.} \end{cases}$$

(2.26)

and

$$g'_n(x, \nu) := \begin{cases} g_n(x, \nu) & \text{if } x \in \Omega, \\ \beta + 1 & \text{otherwise.} \end{cases}$$

(2.27)

We assume that, both for $E_n$ and $E'_n$, (2.18)–(2.19) hold, which is always true up to taking a subsequence. Accordingly, we define $f$, $g$ (for $E_n$), and $f'$, $g'$ (for $E'_n$). We remark that, in this setting, one can show that $f'(x, \xi) = f(x, \xi)$ for $x \in \Omega$ and $f'(x, \xi) = \alpha(\text{sym(}\xi))^{\beta}$ else, as well as $g'(x, \nu) = g(x, \nu)$ for $x \in \Omega$. Finally, for $x \in \partial_D \Omega$ the value of $g'(x, \nu)$ is completely determined by $(g_n)_n$, and is independent of the choice of $\Omega'$, see [42, Remark 4.4] for details.
By Theorem \(\text{2.1(i)}\) and Remark \(\text{2.2(i)}\) the functionals \(\mathcal{E}_n\), with densities \(f_n\) and \(g_n\), \(\Gamma\)-converge with respect to the convergence in measure (up to a subsequence) to a limiting functional \(\mathcal{E}\) with densities \(f_\infty\) and \(g_\infty\). By the results in the previous subsection, we know that \(f_\infty\) agrees with the function \(f\), see Theorem \(\text{2.4(i)}\). In the sequel, we suppose that \(g_\infty = g\). (For instance, such a property holds in the setting of Theorem \(\text{2.4(ii),(iii)}\).) In a similar fashion, the functionals \(\mathcal{E}'_n\) with densities \(f'_n\) and \(g'_n\), \(\Gamma\)-converge to some \(\mathcal{E}'\). Again, we know that the bulk density of \(\mathcal{E}'\) is the function \(f'\) in \(\text{(2.18)}\) and we assume that the surface density is given by \(g'\) in \(\text{(2.19)}\). As before, this characterization can be ensured in the setting of Theorem \(\text{2.4(ii)}\) or in the setting of Theorem \(\text{2.4(iii)}\) under the assumption that \(h\) is uniformly continuous on \(\Omega\). For the latter case, we need to resort to Remark \(\text{2.8}\) (with \(\Omega'\) in place of \(\Omega\) and \(D = \Omega\)) since the continuity of the density in \(\text{(2.27)}\) gets lost through the extension. (Note that indeed \(J_u \subset \mathcal{D} \cap \Omega'\) holds since we require \(u = u^0\) on \(\Omega' \setminus \overline{\Omega}\).)

We now present the following version of the \(\Gamma\)-convergence result which takes boundary data into account. We remark that the statement takes a more general point of view than assuming the setting of Theorem \(\text{2.4(ii),(iii)}\): the result below is true \textit{whenever} the limiting surface density \(g\) is determined solely by the functions \(g_n\) through the asymptotic minimization problems discussed in the previous subsection.

**Proposition 2.9 (\(\Gamma\)-convergence with boundary data).** Let \((f_n)\) and \((g_n)\) be sequences of functions satisfying \(\text{(2.1)}\) and \(\text{(2.12)}\), respectively. Consider the sequence of functionals \(\mathcal{E}'_{n}^{\prime}\) with densities \((f'_n)\), \((g'_n)\) defined as in \(\text{(2.26)}\)–\(\text{(2.27)}\). Assume that, both for \(\mathcal{E}_n\) and \(\mathcal{E}'_{n}\), \(\text{(2.18)}\)–\(\text{(2.19)}\) hold, and accordingly define \(f\), \(g\) (for \(\mathcal{E}_n\)), and \(f'\), \(g'\) (for \(\mathcal{E}'_{n}\)). Consider the \(\Gamma\)-limit \(\mathcal{E}'\) of \(\mathcal{E}'_{n}\) with densities \(f'_n\) and \(g'_n\), and assume that \(g'_\infty = g'\). Suppose that \((u^0_n) \subset W^{1,p}(\Omega'; \mathbb{R}^d)\) converges strongly to \(u^0\) in \(W^{1,p}(\Omega'; \mathbb{R}^d)\). Then the sequence of functionals

\[
\mathcal{E}'_{n}(u) = \begin{cases} 
\mathcal{E}'_{n}(u) & \text{if } u = u^0_n \text{ on } \Omega' \setminus \overline{\Omega}, \\
+\infty & \text{otherwise},
\end{cases}
\]

\(\Gamma\)-converges with respect to the convergence in measure to

\[
\mathcal{E}'(u) = \begin{cases} 
\mathcal{E}'(u) & \text{if } u = u^0 \text{ on } \Omega' \setminus \overline{\Omega}, \\
+\infty & \text{otherwise}.
\end{cases}
\]

We emphasize once more that the assumption \(g'_\infty = g'\) on the surface density covers the setting of Theorem \(\text{2.4(ii),(iii)}\). However, it is not limited to that since above we have no restriction on the dimension. Instead, for our main result about convergence of minimizers, we focus again on the setting of Theorem \(\text{2.4(ii)}\).

**Theorem 2.10 (Convergence of minima and minimizers).** Let \((f_n)\) and \((g_n)\) be sequences of functions satisfying \(\text{(2.1)}\) and \(\text{(2.12)}\), respectively. Suppose either that

(i) \(d = p = 2\),
(ii) \(d \geq 2\) and \(g_n = \tilde{g}\) for \(n \in \mathbb{N}\), where \(\tilde{g}\) denotes a uniformly continuous density with \(\nu \mapsto \tilde{g}(x, \nu)\) being even for all \(x \in \Omega\).

Consider the sequence of functionals \(\mathcal{E}'_{n}\) and the limiting energy \(\mathcal{E}'\) given by Proposition \(\text{2.9}\) for boundary data \((u^0_n) \subset W^{1,p}(\Omega'; \mathbb{R}^d)\) which converge strongly in \(W^{1,p}(\Omega'; \mathbb{R}^d)\) to \(u^0\). Then

\[
\inf_{v \in GSBD^p(\Omega')} \mathcal{E}'_{n}(v) \to \min_{v \in GSBD^p(\Omega')} \mathcal{E}'(v)
\]

for \(n \to \infty\). Moreover, for each sequence \((u_n)\) with

\[
\mathcal{E}'_{n}(u_n) \leq \inf_{v \in GSBD^p(\Omega')} \mathcal{E}'_{n}(v) + \varepsilon_n
\]
for a sequence $\varepsilon_n \to 0$, there exist a subsequence (not relabeled), modifications $(y_n)_n$ satisfying $\mathcal{L}^d(\{e(y_n) \neq e(u_n)\}) \to 0$ as $n \to \infty$, and $u \in \text{GSBD}^p(\Omega')$ with $y_n \to u$ in measure on $\Omega'$ such that
\[
\lim_{n \to \infty} \mathcal{E}_n'(u_n) = \mathcal{E}'(u) = \min_{v \in \text{GSBD}^p(\Omega')} \mathcal{E}'(v).
\]
In case (i), we additionally have $\lim_{n \to \infty} \mathcal{E}_n'(y_n) = \mathcal{E}'(u)$, i.e., $(y_n)_n$ is a minimizing sequence converging to the minimizer $u$.

For the proofs of the results we refer to Section 7. We point out that in case (i) we obtain a slightly stronger statement. This is due to compactness properties of $\text{GSBD}^p$ functions and the construction of certain modifications, see Theorem 3.8 below.

3. Preliminaries

In this section, we collect basic properties of the function space $\text{GSBD}^p$ and we recall integral representation formulas for functionals defined on Sobolev functions and piecewise constant functions.

3.1. Generalized special functions of bounded deformation. In this subsection, we collect fundamental properties of the function space $\text{GSBD}^p$.

**GSBD-functions, basic properties:** The space $\text{GSBD}(\Omega)$ of generalized special functions of bounded deformation has been introduced in [35, Definitions 4.1 and 4.2]. We recall that every $u \in \text{GSBD}(\Omega)$ has an approximate symmetric gradient $e(u) \in L^1(\Omega; \mathbb{R}^{d \times d}_{\text{sym}})$ and an approximate jump set $J_u$. For $x \in J_u$ there exist $u^+(x), u^-(x) \in \mathbb{R}^d$ and $\nu_u(x) \in \mathbb{S}^{d-1}$ such that
\[
\lim_{\rho \to 0} \rho^{-d} \mathcal{L}^d(\{y \in B_{\rho}(x) : \pm (y-x) \cdot \nu_u(x) > 0\} \cap \{|u-u^\pm(x)| > \varepsilon\}) = 0
\]
for every $\varepsilon > 0$, and the function $[u] := u^+ - u^- : J_u \to \mathbb{R}^d$ is measurable. For $1 < p < +\infty$, the space $\text{GSBD}^p(\Omega)$ is given by
\[
\text{GSBD}^p(\Omega) := \{u \in \text{GSBD}(\Omega) : e(u) \in L^p(\Omega; \mathbb{R}^{d \times d}_{\text{sym}}), \mathcal{H}^{d-1}(J_u) < \infty\}.
\]
For $u \in \text{GSBD}^p(\Omega)$, the approximate gradient $\nabla u$ exists $\mathcal{L}^d$-a.e. in $\Omega$, see [21, Corollary 5.2]:

**Lemma 3.1** (Approximate gradient). Let $\Omega \subset \mathbb{R}^d$ be open, let $1 < p < +\infty$, and $u \in \text{GSBD}^p(\Omega)$. Then for $\mathcal{L}^d$-a.e. $x_0 \in \Omega$ there exists a matrix in $\mathbb{R}^{d \times d}$, denoted by $\nabla u(x_0)$, such that
\[
\lim_{\rho \to 0} \rho^{-d} \mathcal{L}^d\left(\left\{x \in B_{\rho}(x_0) : \frac{|u(x) - u(x_0) - \nabla u(x_0)(x-x_0)|}{|x-x_0|} > \varepsilon\right\}\right) = 0
\]
for all $\varepsilon > 0$, and $\text{sym}(\nabla u(x_0)) = e(u)(x_0)$, where $e(u)(x_0)$ denotes the approximate symmetric gradient.

We point out that the result in Lemma 3.1 has already been obtained in [10] for $p = 2$, as a consequence of the embedding $\text{GSBD}^2(\Omega) \subset (\text{GBV}(\Omega))^d$, see [10, Theorem 2.9].

**Korn inequalities in $\text{GSBD}^p$:** We recall Korn and Poincaré inequalities in $\text{GSBD}^p$. In what follows, we say that $a : \mathbb{R}^d \to \mathbb{R}^d$ is a rigid motion if $a$ is affine with $e(a) = \frac{1}{2}(\nabla a + (\nabla a)^\top) = 0$. We start by Korn and Korn-Poincaré inequalities for functions with small jump set, see [21, Theorem 1.1, Theorem 1.2].

**Theorem 3.2** (Korn inequality for functions with small jump set). Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $1 < p < +\infty$. Then there exists a constant $c = c(\Omega, p) > 0$ such that for all $u \in \text{GSBD}^p(\Omega)$ there exists a set of finite perimeter $\omega \subset \Omega$ with
\[
\mathcal{H}^{d-1}(\partial^* \omega) \leq c \mathcal{H}^{d-1}(J_u), \quad \mathcal{L}^d(\omega) \leq c(\mathcal{H}^{d-1}(J_u))^{d/(d-1)}
\]
and a rigid motion $a$ such that
\[
\|u - a\|_{L^p(\Omega)} + \|\nabla u - \nabla a\|_{L^p(\Omega)} \leq c\|e(u)\|_{L^p(\Omega)}.
\] (3.2)
Moreover, there exists $v \in W^{1,p}(\Omega; \mathbb{R}^d)$ such that $v = u$ on $\Omega \setminus \omega$ and
\[
\|e(v)\|_{L^p(\Omega)} \leq c\|e(u)\|_{L^p(\Omega)}.
\]

Note that in [21, Theorem 1.1] $L^d(\omega) \leq c(\mathcal{H}^{d-1}(J_u))^{d/(d-1)}$ has not been stated explicitly, but it readily follows from $\mathcal{H}^{d-1}(\partial^* \omega) \leq c\mathcal{H}^{d-1}(J_u)$ by the isoperimetric inequality.

**Remark 3.3** (Scaling invariance on cubes). If $\Omega = Q_\rho$ for $\rho > 0$, then we find $\omega \subset Q_\rho$ and a rigid motion $a$ such that
\[
\mathcal{H}^{d-1}(\partial \omega) \leq \bar{c}\mathcal{H}^{d-1}(J_u), \quad \mathcal{L}^d(\omega) \leq \bar{c}(\mathcal{H}^{d-1}(J_u))^{d/(d-1)}
\]
and
\[
\|u - a\|_{L^p(Q_\rho \setminus \omega)} \leq \bar{c} \rho^d \|e(u)\|_{L^p(Q_\rho)},
\]
where $\bar{c} = \bar{c}(p) > 0$ is independent of the sidelength $\rho$. This follows by a standard rescaling argument.

Note that the above result is indeed only relevant for functions with sufficiently small jump set, as otherwise one can choose $\omega = \Omega$, and (3.2) trivially holds. In other words, for functions with jump set whose measure is comparable to the size of the domain, Theorem 3.2 might not give any information. A finer result, yet restricted to the two-dimensional setting, is provided by the following piecewise Korn-Poincaré inequality. (For the definition and properties of Caccioppoli partitions we refer to [25, Section 4.4].)

**Proposition 3.4** (Piecewise Korn-Poincaré inequality). Let $\Omega \subset \mathbb{R}^2$ be an open, bounded set with Lipschitz boundary, and let $0 < \theta \leq \theta_0$ for some $\theta_0$ sufficiently small. Then, there is some $C_\theta = C_\theta(\theta) > 0$ such that the following holds: for each $u \in GSBD^2(\Omega)$ we find a (finite) Caccioppoli partition $\Omega = R \cup \bigcup_{j=1}^J P_j$, and corresponding rigid motions $(a_j)_{j=1}^J$ such that
\[
\begin{align*}
&\quad (i) \quad \sum_{j=1}^J \mathcal{H}^1((\partial^* P_j \cap \Omega) \setminus J_u) + \mathcal{H}^1((\partial^* R \cap \Omega) \setminus J_u) \leq \theta(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial \Omega)), \\
&\quad (ii) \quad \mathcal{L}^2(R) \leq \theta(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial \Omega))^2, \quad \mathcal{L}^2(P_j) \geq \theta^3 \quad \text{for all } j = 1, \ldots, J, \\
&\quad (iii) \quad \|u - a_j\|_{L^\infty(P_j)} \leq C_\theta \|e(u)\|_{L^2(\Omega)} \quad \text{for all } j = 1, \ldots, J.
\end{align*}
\] (3.3)

**Proof.** The statement is a slightly simplified version of [44, Theorem 4.1]. We briefly explain how the result can be obtained therefrom. We define $\theta_0 \leq 1/c$, where $c$ is the constant from [45, Theorem 4.1] and apply [44, Theorem 4.1] for $\theta/c$ in place of $\theta$. Then, (3.3)(i) follows from [44, (18)(i)], where we denote the component $P_0$ by $R$. Item (3.3)(ii) follows from [44, (17)(i), (18)(ii)], choosing $\theta_0$ sufficiently small such that $C_\theta \geq \theta_0$. Finally, (3.3)(iii) follows from [44, (18)(iii)], where also a corresponding Korn-type estimate has been proved. 

To control the affine mappings appearing in the above results, we will also make use of the following elementary lemma (see [44, Lemma 3.4]).

**Lemma 3.5.** Let $G \in \mathbb{R}^{d \times d}$, $b \in \mathbb{R}^d$. Let $\delta > 0$, $R > 0$, and let $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous, strictly increasing function with $\psi(0) = 0$. Consider a measurable, bounded set $E \subset \mathbb{R}^d$ with $E \subset B_R(0)$ and $\mathcal{L}^d(E) \geq \delta$. Then there exists a continuous, strictly increasing function $\tau_\psi: \psi(\mathbb{R}_+) \to \mathbb{R}_+$ with $\tau_\psi(0) = 0$ only depending on $\delta$, $R$, and $\psi$ such that
\[
|G| + |b| \leq \tau_\psi \left( \int_E \psi(|G x + b|) \, dx \right).
\]
Moreover, if \( p \in [1, \infty) \), then \( \tau_p \) can be chosen as \( \tau_p(t) = ct^{1/p} \) for \( c = c(p, \delta, R) > 0 \). Moreover, there exists \( c_0 > 0 \) only depending on \( \delta, d, \) and \( R \) such that

\[
\|Gx + b\|_{L^\infty(B_R(0))} \leq c_0\|Gx + b\|_{L^1(E)}.
\]  

(3.4)

**Approximation:** The following result is a special version of [21, Theorem 5.1]. (For the definition and properties of GSBD functions we refer to [6, Section 4.5].)

**Theorem 3.6** (Approximation). Let \( \Omega \subset \mathbb{R}^d \) be a bounded Lipschitz domain, and let \( 1 < p < +\infty \).

Let \( u \in GSBD^p(\Omega) \). Then there exists a sequence \( (u_n)_n \subset GSBD^p(\Omega; \mathbb{R}^d) \cap L^p(\Omega; \mathbb{R}^d) \) such that

\begin{itemize}
  \item[(i)] \( u_n \rightarrow u \) in measure on \( \Omega \),
  \item[(ii)] \( \|e(u_n) - e(u)\|_{L^p(\Omega)} \rightarrow 0 \),
  \item[(iii)] \( \mathcal{H}^{d-1}(J_{u_n} \Delta J_u) \rightarrow 0 \).
\end{itemize}

Moreover, each \( u_n \) lies in \( W^{1,p}(\Omega \setminus (\Gamma_n \cup \bar{\Omega}_n)) \), where \( \Gamma_n \) is closed and the finite union of \( C^1 \)-manifolds, and \( \dot{\omega}_n \) is a finite union of cubes.

**Compactness:** We recall the following compactness result in GSBD\(^p\) (see [26, Theorem 1.1]).

**Theorem 3.7** (Compactness). Let \( \Omega \subset \mathbb{R}^d \) be open and bounded. Let \( (u_n)_n \) be a sequence in GSBD\(^p\)(\( \Omega \)) such that

\[
\sup_{n \in \mathbb{N}} \left( \|e(u_n)\|_{L^p(\Omega)} + \mathcal{H}^{d-1}(J_{u_n}) \right) < +\infty.
\]

Then, there exists a subsequence, still denoted \( (u_n)_n \), such that \( G_\infty := \{ x \in \Omega : |u_n(x)| \rightarrow \infty \} \) has finite perimeter, and a function \( u \in GSBD^p(\Omega) \) with \( u = 0 \) in \( G_\infty \) such that

\begin{itemize}
  \item[(i)] \( u_n \rightarrow u \) \( \mathcal{L}^d \)-a.e. on \( \Omega \setminus G_\infty \),
  \item[(ii)] \( e(u_n) \rightharpoonup e(u) \) in \( L^p(\Omega \setminus G_\infty; \mathbb{R}^{d \times d}) \),
  \item[(iii)] \( \liminf_{n \rightarrow \infty} \mathcal{H}^{d-1}(J_{u_n}) \geq \mathcal{H}^{d-1}(J_u \cup (\partial^* G_\infty \cap \Omega)) \).
\end{itemize}

(3.6)

A control on (3.5) does in general not imply that the sequence converges in measure which is reflected by the presence of the set \( G_\infty \). The latter can be understood as the parts of the domain which are (almost) completely disconnected by the jump set \( (J_{u_n})_n \) such that the functions \( (u_n)_n \) can take arbitrarily large values on these pieces. To ensure measure convergence on the entire domain, one needs to pass to modifications of \( (u_n)_n \).

**Theorem 3.8** (Compactness for modifications). Let \( \Omega \subset \Omega' \subset \mathbb{R}^d \) be bounded Lipschitz domains. Let \( (\mathcal{E}_n)_n \) be a sequence of functionals of the form (2.3) with densities satisfying (2.1) and (2.12).

Let \( (u_n^0)_n \subset W^{1,p}(\Omega'; \mathbb{R}^d) \) be converging in \( L^p(\Omega'; \mathbb{R}^d) \) to some \( u^0 \in W^{1,p}(\Omega'; \mathbb{R}^d) \). Consider \( (u_n)_n \subset GSBD^p(\Omega') \) with \( u_n = u_n^0 \) on \( \Omega' \setminus \overline{\Omega} \) and \( \sup_{n \in \mathbb{N}} \mathcal{E}_n(u_n, \Omega') < +\infty \).

Then, we find a subsequence (not relabeled), modifications \( (y_n)_n \subset GSBD^p(\Omega') \) satisfying

\[
y_n = u_n^0 \text{ on } \Omega' \setminus \overline{\Omega}, \quad \mathcal{L}^d\left( \{ e(y_n) \neq e(u_n) \} \right) \leq \frac{1}{n} \text{ for all } n \in \mathbb{N},
\]

(3.7)

and a limiting function \( u \in GSBD^p(\Omega') \) with \( u = u^0 \) on \( \Omega' \setminus \overline{\Omega} \) such that \( y_n \rightarrow u \) in measure on \( \Omega' \) and \( e(y_n) \rightharpoonup e(u) \) weakly in \( L^p(\Omega'; \mathbb{R}^{d \times d}) \).

Moreover, if \( p = d = 2 \) and \( (|\nabla u_n^0|^2)_n \) are equiintegrable, then the modifications \( (y_n)_n \) can be chosen in such a way that we also have

\[
\mathcal{E}_n(y_n, \Omega') \leq \mathcal{E}_n(u_n, \Omega') + \frac{1}{n} \text{ for all } n \in \mathbb{N}.
\]

(3.8)
Proof. Up to small adaptations, the case \( p = d = 2 \) has been addressed in [45, Theorem 6.1, Remark 6.3]. In [45], only a single energy and a single boundary datum was considered, but an inspection of the proof shows that the statement can be extended to the above setting. In fact, the crucial point is that the growth conditions (2.1) and (2.12) hold uniformly in \( n \in \mathbb{N} \). Moreover, the property \( L^2(\{e(y_n) \neq e(u_n)\}) \leq \frac{1}{n} \) was not noted explicitly in [45], but follows from the construction, see [43, (65)-(66)]. We also refer to [42, Theorem 3.1] for an analogous statement in GSBD\( p \).

The statement for \( d \geq 2 \) and \( 1 < p < +\infty \) can be found in [28, Theorem 1.1]. The result is weaker than the one in [45] in the sense that (3.8) cannot be guaranteed. We briefly explain that (3.7) is satisfied. Indeed, the modifications \((y_n)_n\) are obtained from \((u_n)_n\) by subtracting piecewise rigid motions \((a^n_j)_j\) associated to a fixed Caccioppoli partition \((P_j)_j\), i.e., \( y_n = u_n - \sum_j a^n_j \chi_{P_j} \), see [28, (1.4)]. This even yields

\[
L^d(\{e(y_n) \neq e(u_n)\}) = 0 \quad \text{for all } n \in \mathbb{N}.
\]

As \( u_n = u^0_n \) on \( \Omega' \setminus \overline{\Gamma} \) and \( u^0_n \to u^0 \), [28, (1.5b)] allows us to choose \( a^n_j = 0 \) for all components \( P_j \) intersecting \( \Omega' \setminus \overline{\Gamma} \). This ensures \( y_n = u_n = u^0_n \) on \( \Omega' \setminus \overline{\Gamma} \).

Lower semicontinuity: We start with a definition from [43].

**Definition 3.9** (Symmetric joint convexity). We say that \( \tau: \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty) \) is a symmetric jointly convex function if

\[
\tau(i, j, \nu) = \sup_{h \in \mathbb{N}} (g_h(i) - g_h(j)) \cdot \nu \quad \text{for all } (i, j, \nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \text{ with } i \neq j,
\]

where \( g_h: \mathbb{R}^d \to \mathbb{R}^d \) is a uniformly continuous, bounded, and conservative vector field for every \( h \in \mathbb{N} \).

The following result can be found in [43, Theorem 5.1].

**Theorem 3.10** (Lower semicontinuity of surface integrals in GSBD\( p \)). Let \( \tau: \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty) \) be a symmetric jointly convex function. Then, for every sequence \( (u_n)_n \subset \text{GSBD}^p(\Omega) \), \( p > 1 \), converging in measure to \( u \in \text{GSBD}^p(\Omega) \), and satisfying the condition

\[
\sup_{n \in \mathbb{N}} (\|e(u_n)\|_{L^p(\Omega)} + \mathcal{H}^{d-1}(J_{u_n})) < +\infty,
\]

we have that

\[
\int_{J_u} \tau(u^+, u^-, \nu_u) \, d\mathcal{H}^{d-1} \leq \liminf_{n \to \infty} \int_{J_{u_n}} \tau(u^+_n, u^-_n, \nu_{u_n}) \, d\mathcal{H}^{d-1}.
\]

In the present paper, we will use that certain densities depending only on the normal are symmetric jointly convex. More precisely, we have the following result (see [43, Proposition 4.11]).

**Proposition 3.11.** A function \( \tau: \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to [c, +\infty) \), \( c > 0 \), of the form \( \tau(i, j, \nu) = \psi(\nu) \) for all \( (i, j, \nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \), \( i \neq j \), is symmetric jointly convex if \( \psi: \mathbb{R}^d \to [0, +\infty) \) is even, positively 1-homogeneous, and convex.

We close with a lower semicontinuity result for modifications in the setting of Theorem 3.8. It is a special case of [28, Theorem 1.2] for integrands not depending on the jump height.

**Lemma 3.12** (Lower semicontinuity of surface integrals for modifications). Consider the setting of Theorem 3.8 for a sequence \( (u_n)_n \) with corresponding modifications \( (y_n)_n \) and limiting function \( u \). Furthermore, let \( h: \Omega' \times \mathbb{S}^{d-1} \to [0, +\infty) \) be a density satisfying (2.12) such that \( \nu \mapsto h(x, \nu) \) is even and symmetric jointly convex for all \( x \in \Omega \). Moreover, suppose that \( h \) is uniformly continuous...
on $\Omega \times \mathbb{S}^{d-1}$ and that for $\mathcal{H}^{d-1}$-a.e. $x \in \partial \Omega \cap \Omega'$ it holds that $h(x, \nu_{\Omega}(x)) = \lim_{n \to \infty} h(x_n, \nu_n)$ for sequences $(x_n)_{n} \subset \Omega$ and $(\nu_n)_{n} \subset \mathbb{S}^{d-1}$ with $x_n \to x$ and $\nu_n \to \nu_{\Omega}$, where $\nu_{\Omega}(x)$ denotes the outer normal at $x \in \partial \Omega \cap \Omega'$. Then, it holds that

$$\int_{J_u} h(x, \nu_n) \, d\mathcal{H}^{d-1} \leq \liminf_{n \to \infty} \int_{J_{\nu_n}} h(x, \nu_{u_n}) \, d\mathcal{H}^{d-1}. \quad (3.10)$$

**Proof.** First, we reduce the problem to the case that $h$ is continuous on the *entire* set $\Omega' \times \mathbb{S}^{d-1}$.

We can construct an extension of $h$ to $\Omega'$, called $\tilde{h}$, with $\tilde{h} = h$ on $\Omega \times \mathbb{S}^{d-1}$, which still satisfies that $\nu \to \tilde{h}(x, \nu)$ is even and symmetric jointly convex for all $x \in \Omega'$. This can be done by a local construction, first extending to the boundary and then reflecting with respect to $x$ across $\partial \Omega$: clearly, the two properties which hold for fixed $x$ with respect to $\nu$ are preserved. Moreover, we have $h(x, \nu_{\Omega}(x)) = \tilde{h}(x, \nu_{\Omega}(x))$ for $\mathcal{H}^{d-1}$-a.e. $x \in \partial \Omega \cap \Omega'$. Now it suffices to check (3.10) for $\tilde{h}$ in place of $h$. In fact, as $J_{\nu_n}, J_u \subset \Omega' \cap \Omega'$, see Theorem 3.8 it holds that $\nu_u(x) = \nu_{\Omega}(x)$ for $\mathcal{H}^{d-1}$-a.e. $x \in J_{\nu_n} \cap \partial \Omega$ (and likewise for $u_n$).

Now, (3.10) for $\tilde{h}$ can be deduced from [28, Theorem 1.2]: first, observe that $\tilde{h}$ satisfies $(g_1)$–$(g_5)$ therein, where in particular $(g_5)$ does not depend on the jump height, and $(g_4)$ follows from Theorem 3.10. Then, the lower semicontinuity of the surface term follows from [28, Equation (4.3)], once we clarify the role played by the Caccioppoli partition $(P_j)_j$. To this end, recall that the modifications $(y_n)_{n}$ are defined as $y_n = u_n - \sum_j \tilde{a}^n_j \chi_{P_j}$, see [28, (1.4)]. By a suitable choice of $(\tilde{a}^n_j)$, see [28, below equation (4.52)], one can ensure that $\bigcup_j \partial^r P_j \cap \Omega \subset J_u$ up to an $\mathcal{H}^{d-1}$-negligible set. Therefore, in our setting, [28, Equation (4.3)] can be simplified to

$$\frac{d\mu}{d\mathcal{H}^{d-1}}(x_0) \geq \tilde{h}(x_0, \nu_u(x_0))$$

for $\mathcal{H}^{d-1}$-a.e. $x_0 \in J_u$. This implies (3.10). \(\square\)

### 3.2. $\Gamma$-convergence and integral representation on Sobolev functions and piecewise constant functions.

This subsection is devoted to integral representation formulas for bulk and surface integrals, respectively.

**Bulk integrals:** Let $1 < p < +\infty$, and let $f_n : \Omega \times \mathbb{R}^{d \times d} \to [0, +\infty)$ be a sequence of Carathéodory functions satisfying (2.1) for some $\alpha, \beta > 0$. Let us consider the functionals $\mathcal{F}_n : L^1(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0, +\infty]$ defined by

$$\mathcal{F}_n(u, A) := \int_A f_n(x, e(u)(x)) \, dx$$

$$+ \infty, \quad \text{otherwise.} \quad (3.11)$$

**Proposition 3.13.** There exists $\mathcal{F} : L^1(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0, +\infty)$ such that, up to subsequence (not relabeled), the functionals $\mathcal{F}_n(\cdot, A)$ $\Gamma$-converge in the strong topology of $L^1(\Omega; \mathbb{R}^d)$ to $\mathcal{F}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$. Moreover, we have that

$$\mathcal{F}(u, A) = \int_A f_0(x, e(u)(x)) \, dx,$$

where for all $x \in \Omega$ and $\xi \in \mathbb{R}^{d \times d}$ the density $f_0$ is given by

$$f_0(x, \xi) = f_0(x, \text{sym}(\xi)) = \limsup_{\rho \to 0^+} \frac{m^{1,p}_x (\ell_\xi, Q_\rho(x))}{\rho^d}. \quad (3.12)$$

Here, $\ell_\xi$ is defined in (2.17) and $m^{1,p}_x$ in (2.13). Moreover, $f_0$ is a Carathéodory function satisfying (2.1) and it holds that

$$f_0(x, \xi) = \limsup_{\rho \to 0^+} \liminf_{n \to \infty} \frac{m^{1,p}_x (\ell_\xi, Q_\rho(x))}{\rho^d} = \limsup_{\rho \to 0^+} \limsup_{n \to \infty} \frac{m^{1,p}_x (\ell_\xi, Q_\rho(x))}{\rho^d}. \quad (3.13)$$
Proof. The proof is standard (see e.g. [12, 15]) and we only provide the main steps. The proof of
the \( \Gamma \)-convergence part is based on standard localization techniques, see e.g. [51, Sections 18, 19].
The integral representation result and (3.12) follow by adapting the global method of relaxation
(see [12, Theorem 2]) to our setting with a weaker growth condition from below in contrast to
[12], as we only control the symmetric part of the gradients. In fact, by using a Korn-Poincaré
inequality instead of the classical Poincaré inequality, one can follow the arguments in the proof of
[12, Theorem 2].

Finally, (3.13) is also well-known and obtained as a consequence of \( \Gamma \)-convergence: given \( x \in \Omega \)
and \( \xi \in \mathbb{R}^d \), by the \( \Gamma \)-liminf inequality and the coercivity of the functionals we get
\[
\begin{align*}
m_{\mathcal{E}}^{1,p} (\bar{\ell}_\xi, Q_\rho(x)) & \leq \sup_{\rho' \in (0, \rho)} \liminf_{n \to 0} m_{\mathcal{E}}^{1,p} (\bar{\ell}_\xi, Q_{\rho'}(x)) \\
\end{align*}
\] (3.14)
for all \( \rho > 0 \). (The passage to smaller cubes \( Q_{\rho'}(x) \) is necessary to ensure that boundary values
at \( \partial Q_\rho(x) \) are preserved under convergence.) On the other hand, the \( \Gamma \)-limsup inequality and the
fact that boundary values can be adjusted by the fundamental estimate show
\[
\begin{align*}
\limsup_{n \to 0} m_{\mathcal{E}}^{1,p} (\bar{\ell}_\xi, Q_\rho(x)) & \leq m_{\mathcal{E}}^{1,p} (\bar{\ell}_\xi, Q_\rho(x)) \\
\end{align*}
\] (3.15)
for all \( \rho > 0 \). By combining (3.14)–(3.15) and using (3.12) we get (3.13). \( \square \)

Surface integrals: We now address the surface part of functionals defined in (2.3). Consider
the representation formula (2.14) for functions \( u = u_x, \zeta, 0, \nu \) given in (2.6) for \( x \in \Omega \), \( \nu \in \mathbb{S}^{d-1} \),
and \( \zeta \in \mathbb{R}^d \). First, it is instrumental to simplify (2.14) whenever the density \( g \)
satisfies (2.12). Indeed, by [23, Theorem A.1], each \( v \in PR(\Omega) \) can be represented as \( v(x) = \sum_{k \in \mathbb{N}} (A_k x + b_k) \chi_{P_k}(x) \) for
\( x \in \Omega \), where \( (A_k)_k \subset \mathbb{R}^{d \times d} \), \( (b_k)_k \subset \mathbb{R}^d \), and \( (P_k)_k \) denotes a Caccioppoli partition of \( \Omega \) (see
[6, Section 4.4]). Therefore, in view of the fact that \( g \) does not depend on the jump height (see
(2.12)), one can check that the minimization problem (2.14) can be restricted to functions where
the partition consists of exactly two sets. More precisely, for \( x \in \Omega \), \( \nu \in \mathbb{S}^{d-1} \), and \( \zeta \in \mathbb{R}^d \), (2.14)
can be rewritten as
\[
\begin{align*}
m_{\mathcal{E}}^{R \mathcal{E}} (u_x, \zeta, 0, \nu, A) = m_{\mathcal{E}}^{PC (\tilde{u}_x, \nu), A} := \inf_{v \in PC(\Omega)} \{ \mathcal{E}(v, A) : v = \tilde{u}_x, \nu \text{ in a neighborhood of } \partial A \},
\end{align*}
\] (3.16)
where \( PC(\Omega) = \{ u \in L^1(\Omega; \mathbb{R}^d) : u = e_1 \chi_T : T \subset \Omega \text{ with } T \text{ set of finite perimeter} \} \) denotes the space of piecewise constant functions attaining only the values 0 and \( e_1 \), and \( \tilde{u}_x, \nu \) is defined in
(2.17).

We now address \( \Gamma \)-convergence and integral representation of functionals \( \mathcal{G}_n : PC(\Omega) \times A(\Omega) \to [0, +\infty) \) defined by
\[
\begin{align*}
\mathcal{G}_n(u, A) := \int_{J_n \cap \partial A} g_n(x, \nu_n(x)) \, d\mathcal{H}^{d-1}(x)
\end{align*}
\] (3.17)
for all \( A \in A(\Omega), u \in PC(\Omega) \), where \( g_n : \Omega \times \mathbb{S}^{d-1} \to [0, +\infty) \) is a Borel function satisfying (2.12).

Proposition 3.14. There exists \( \mathcal{G} : PC(\Omega) \times A(\Omega) \to [0, +\infty) \) such that, up to subsequence (not relabeled), \( \mathcal{G}_n(\cdot, A) \) \( \Gamma \)-converges with respect to the strong \( L^1(\Omega; \mathbb{R}^d) \)-convergence to \( \mathcal{G}(\cdot, A) \) for all
\( A \in A(\Omega) \). Moreover, for all \( u \in PC(\Omega) \) we have that
\[
\begin{align*}
\mathcal{G}(u, A) = \int_{\partial \Omega \cap \partial A} g_0(x, \nu_0(x)) \, d\mathcal{H}^{d-1}(x),
\end{align*}
\] (3.18)
where for all \( x \in \Omega \) and \( \nu \in \mathbb{S}^{d-1} \) the density \( g_0 \) is given by
\[
g_0(x, \nu) := \limsup_{\rho \to 0^+} \frac{m_{\rho}^{PC}(\tilde{u}_{x,\nu}, Q_\rho'(x))}{\rho^{d-1}}. \tag{3.19}
\]
Moreover, \( g_0 \) satisfies \( (2.12) \) and it holds that
\[
g_0(x, \nu) = \limsup_{\rho \to 0^+} \liminf_{n \to \infty} \frac{m_{\rho,n}^{PC}(\tilde{u}_{x,\nu}, Q_\rho'(x))}{\rho^{d-1}} = \limsup_{\rho \to 0^+} \limsup_{n \to \infty} \frac{m_{\rho,n}^{PC}(\tilde{u}_{x,\nu}, Q_\rho'(x))}{\rho^{d-1}}. \tag{3.20}
\]

**Proof.** We apply [4, Theorem 3.2] and [12, Theorem 3] to obtain the representation \( (3.18) \)–\( (3.19) \).

Finally, \( (3.20) \) can be derived by \( \Gamma \)-convergence as explained in \( (3.14) \)–\( (3.15) \), where we employ the fundamental estimate on \( PC(\Omega) \) for the inequality analogous to \( (3.15) \), see [4, Lemma 4.4]. (We also refer to [44, Lemmas 6.3, 7.5] for similar arguments in \( PR(\Omega) \).) \( \square \)

**Proof of Proposition 2.3.** Consider a sequence of functionals \( (\mathcal{E}_n)_n \) of the form \( (2.3) \) for densities \( (f_n) \) and \( (g_n) \), respectively. Then, with the notation in \( (3.11) \) and \( (3.17) \), thanks to \( (2.15) \)–\( (2.16) \), for all \( A \in \mathcal{A}(\Omega) \) it holds that \( \mathcal{E}_n(u, A) = \mathcal{F}_n(u, A) \) for all \( u \in W^{1,p}(\Omega, \mathbb{R}^2) \) and \( |\mathcal{E}_n(u, A) - \mathcal{F}_n(u, A)| \leq \beta C^d(A) \) for all \( u \in PR(\Omega) \). Now, the statement of Proposition 2.3 follows immediately from \( (3.13) \), \( (3.16) \), and, \( (3.20) \). \( \square \)

**Remark 3.15.** For later reference, we point out that we have shown that \( f = f_0 \) and \( g = g_0 \), where the densities \( f, g \) are given in Proposition 2.3 and \( f_0, g_0 \) are defined in \( (3.12) \) and \( (3.19) \), respectively.

The following result can be found in [5, Theorem 3.1].

**Proposition 3.16** (Relaxation). Consider a continuous density \( h \) satisfying \( (2.12) \) and denote the corresponding functional in \( (3.17) \) by \( \mathcal{S} \). Then, the relaxed functional
\[
\tilde{S}(u, A) := \inf \left\{ \liminf_{n \to \infty} \mathcal{S}(u_n, A) : u_n \to u \text{ in measure on } \Omega \right\},
\]
for all \( u \in PC(\Omega) \) and \( A \in \mathcal{A}(\Omega) \) admits an integral representation
\[
\tilde{S}(u, A) = \int_{J_u \cap A} \tilde{h}(y, \nu_u(y)) \, d\mathcal{H}^{d-1}(y),
\]
where for each \( x \in \Omega \), the density \( \tilde{h} \) is the BV-elliptic envelope of \( h \), i.e.,
\[
\tilde{h}(x, \nu) := \inf_{v \in PC(\mathcal{Q}^r_1)} \left\{ \int_{J_u \cap \mathcal{Q}^r_1} h(x, \nu_v(y)) \, d\mathcal{H}^{d-1}(y) : v = \bar{u}_{0, \nu} \text{ in a neighborhood of } \partial \mathcal{Q}^r_1 \right\} \tag{3.21}
\]
for all \( \nu \in \mathbb{S}^{d-1} \), where \( \bar{u}_{0, \nu} \) is defined in \( (2.17) \).

**Corollary 3.17.** Suppose that \( h \) is given as in Proposition 3.16 and suppose that \( \nu \mapsto h(x, \nu) \) is even for all \( x \in \Omega \). Then, it holds that
\[
\tilde{h}(x, \nu) := \limsup_{\rho \to 0^+} \frac{m_{\rho}^{PC}(\bar{u}_{x,\nu}, Q_\rho'(x))}{\rho^{d-1}} \tag{3.22}
\]
for all \( x \in \Omega \) and \( \nu \in \mathbb{S}^{d-1} \). Moreover, for each \( x \in \Omega \), the function \( \tilde{h}(x, \cdot) \) is even and symmetric jointly convex, as defined in Definition 3.9 (as a function independent of the variables \( i \) and \( j \)).
Proof. First, as $\bar{S}$ is the $\Gamma$-limit of the constant sequence $S$, (3.19)–(3.20) imply (3.22). We fix $x \in \Omega$, and show that $\bar{h}(x, \cdot)$ is symmetric jointly convex. First, since the functional $\bar{S}$ is lower semicontinuous in $PC(\Omega)$, we get that $\bar{h}(x, \cdot)$ is convex by [5, Theorem 5.11]. It is elementary to check that $\bar{h}(x, \cdot)$ is still even. Eventually, $\nu \mapsto |\nu| \bar{h}(x, \nu)/|\nu|$ can be understood as a positively 1-homogeneous function for $\nu \in \mathbb{R}^d$. Then Proposition 3.11 implies that $\bar{h}(x, \cdot)$ is symmetric jointly convex (as a function independent of the variables $i$ and $j$).

Remark 3.18 (BV- and BD-elliptic envelope). In view of (2.23) and (3.21), we observe that the characterizations given in (2.23) and (3.21) coincide, i.e., in the present setting the BV- and BD-elliptic envelope are the same.

4. Compactness of $\Gamma$-convergence in $GSBD^p$

This section is devoted to the proof of Theorem 2.1. The result is based on the localization method of $\Gamma$-convergence along with the recent integral representation result [31]. For the first part, the main ingredient is a fundamental estimate in $GSBD$.

4.1. Fundamental estimate in $GSBD^p$. We use the following convention in the whole subsection: given $A \in \mathcal{A}(\Omega)$, we may regard every $u \in GSBD^p(A)$ as a measurable function on $\Omega$, extended by $u = 0$ on $\Omega \setminus A$. We start by formulating the fundamental estimate.

Proposition 4.1 (Fundamental estimate in $GSBD^p$). Let $\Omega \subset \mathbb{R}^d$ be open, and let $1 < p < +\infty$. Let $\eta > 0$ and let $A', A, B \in \mathcal{A}(\Omega)$ with $A' \subset \subset A$. Assume that $A \setminus A' \subset B$, or that $B$ has Lipschitz boundary. Then, there exists a function $A': GSBD^p(A) \times GSBD^p(B) \to [0, +\infty]$ which is lower semicontinuous with respect to convergence in measure and satisfies

$$A(z_1, z_2) \to 0 \quad \text{whenever} \quad z_1 - z_2 \to 0 \quad \text{in measure on} \ (A \setminus A') \cap B \ (4.1)$$

such that the following holds: for every functional $E$ in (2.3) with densities $f$, $g$ satisfying (2.1)–(2.2) and for every $u \in GSBD^p(A)$, $v \in GSBD^p(B)$ there exists a function $w \in GSBD^p(A' \cup B)$ such that

(i) $E(w, A' \cup B) \leq (1 + \eta)(E(u, A) + E(v, B)) + A(u, v) + \eta$,

(ii) $\|\min\{|w - u|, |w - v|\}\|_{L^p(A' \cup B)} \leq A(u, v) + \eta(E(u, A) + E(v, B)) + \eta$,

(iii) $w = u$ on $A'$ and $w = v$ on $B \setminus A$. \hspace{1cm} (4.2)

Remark 4.2. Let us start with some comments on the result:

(i) A main technique of the proof is the Korn inequality for $GSBD^p$-functions with small jump sets, see Theorem 3.2. This allows us to establish an $L^p$-control on $min\{|w - u|, |w - v|\}$ in (4.2)(ii). In contrast, we point out that each function $u, v, w$ itself might not even be integrable.

(ii) The statement is much easier to prove when (4.1) is replaced by

$$A(z_1, z_2) \to 0 \quad \text{whenever} \quad \|z_1 - z_2\|_{L^p(A' \cup B)} \to 0.$$ 

This corresponds to a fundamental estimate in $GSBD^p(\Omega) \cap L^p(\Omega; \mathbb{R}^d)$, see [31, Lemma 3.7]. The latter in turn is inspired by the original statement in $SBV^p$ formulated in [11, Proposition 3.1]. The arguments there basically rely on a suitable cut-off construction between the functions $u$ and $v$. This special case is not enough for our purposes as it requires $L^p$-integrability of the functions which is not available in our setting. We also point out that a truncation argument as [11, Lemma 3.5] is not applicable. As a remedy, we use an alternative technique, based on Theorem 3.2.
Proof of Proposition

We begin with a short outline of the proof. We start by partitioning the set \( A \setminus A' \) into ‘layers’ where we will eventually ‘join’ \( u \) and \( v \) by a cut-off construction. These layers are additionally covered by a collection of small cubes (Step 1). In each of these cubes, we apply a Korn inequality in \( GSBD^p \) (Theorem 3.2) on the function \( u - v \) (Step 2), and we analyze the corresponding exponential sets (Step 3) and rigid motions (Step 4). Based on this, we introduce modifications of \( u \) and \( v \) such that their difference lies in \( L^p \) (Step 5). Then, we apply a cut-off construction similar to [31, Lemma 3.7] or [11, Proposition 3.1] (Step 6) and obtain the desired function \( w \) satisfying (4.2) (Step 7).

We will focus on the case where \( A \setminus A' \subset B \). At the end, we will indicate the minor changes to be done when this is not assumed, but instead \( B \) has Lipschitz boundary. To account for this alternative assumption, along the proof we will write \( (A \setminus A') \cap B \) in place of \( A \setminus A' \) several times, although the intersection with \( B \) is redundant under condition \( A \setminus A' \subset B \).

**Step 1: Preliminaries.** Let \( \eta > 0 \), and let the sets \( A', A, B \in \mathcal{A}(\Omega) \) with \( A' \subset A \) and \( A \setminus A' \subset B \) be given. In this step, we introduce several parameters and coverings that we will use throughout the proof. We fix \( k \in \mathbb{N} \) sufficiently large such that

\[
\frac{12\beta^2 d \bar{c}}{k \alpha^2 (1 + L^d(A \setminus A'))} \leq \eta, \tag{4.3}
\]

where \( \bar{c} \geq 1 \) denotes the constant from Remark 3.3, \( d \) the dimension, and \( \alpha, \beta \) are defined in (2.1).

Let \( A_1, \ldots, A_{k+1} \) be open subsets of \( \mathbb{R}^d \) with \( A' \subset A_1 \subset \cdots \subset A_{k+1} \subset A \). We also define further open sets \( A_i \subset A_i^+ \subset \cdots \subset A_{i+1} \subset A_{i+1}^+ \) for \( i = 1, \ldots, k \), and let

\[
S_i = (A_{i+1} \setminus A_i^+) \cap B, \quad T_i = (A_{i+1}^+ \setminus A_i^+) \cap B \tag{4.4}
\]

for \( i = 1, \ldots, k \). As \( A \setminus A' \subset B \), we get \( T_i \subset S_i \subset A \cap B \) for \( i = 1, \ldots, k \). (The intersection in (4.4) with \( B \) is redundant, but added in order to highlight that the sets are contained in \( A \cap B \).) Moreover, let \( \varphi_i \in C^\infty(\mathbb{R}^d, [0, 1]) \) with \( \varphi_i = 1 \) on \( A_i^+ \) and \( \varphi_i = 0 \) on \( \mathbb{R}^d \setminus A_{i+1} \), i.e.,

\[
\{0 < \varphi_i < 1\} \subset T_i. \tag{4.5}
\]

Define \( \psi: \mathbb{R}_+ \to [0, 1) \) by \( \psi(t) := \frac{t}{1 + t} \) for \( t \geq 0 \) and observe that

\[
u_n \to u \text{ in measure on } U \in \mathcal{A}(\Omega) \text{ if and only if } \int_U \psi(\|u_n - u\|) \, dx \to 0. \tag{4.6}
\]

We apply Lemma 3.3 for \( \delta = 1/2 \) and \( R = \sqrt{d} \), and let \( \tau_\psi: [0, 1) \to \mathbb{R}_+ \) be the continuous, strictly increasing function with \( \tau_\psi(0) = 0 \). As \( \tau_\psi \) is uniformly continuous on \( [0, 1/2] \), we can choose \( \lambda \in (0, +\infty) \) such that

(i) \( 2\lambda < 1/2 \),

(ii) \( 12\beta^2 (1 + \max_{i=1,\ldots,k} \|\nabla \varphi_i\|_\infty) L^d(A \setminus A') \max_{t \in [0,1/2]} |\tau_\psi^p(t + 2\lambda) - \tau_\psi^p(t)| \leq \eta/2 \), \tag{4.7}

where here and the following \( \tau_\psi^p(\cdot) := (\tau_\psi(\cdot))^p \). The constant \( \lambda \) will become relevant later for the definition of \( A \), see in particular (4.16) and (4.31). Recalling \( T_i \subset S_i \), we pick a further constant \( \rho \in (0, 1) \) sufficiently small such that

(i) \( 2^p \lambda^{-p} \bar{c} \rho^{p-1} \leq \lambda \),

(ii) \( \rho \leq \min_{i=1,\ldots,k} \{ \text{dist}(T_i, S_i), \|\nabla \varphi_i\|_\infty^{-1} \} \). \tag{4.8}
For each \( i = 1 \ldots k \), we cover \( T_i \) up to set of \( \mathcal{L}^d \)-negligible measure with a finite number of pairwise disjoint open cubes \( \Omega^i := (Q^i_j)_j \) with centers \( (x^i_j)_j \subset \rho \mathbb{Z}^d \cap T_i \) and sidelength \( \rho \). In view of (4.12)(i) and (4.8)(ii), we get
\[
T_i \subset \bigcup_j \overline{Q^i_j} \subset \subset S_i \subset A \cap B.
\]
We now show that for each cube \( Q^i_j \),
\[
\text{Step 3: Korn's inequality and exceptional sets.}
\]
For each \( Q^i_j \), we define \( \Lambda^i_j \) as in (4.10). For each \( Q^i_j \), we define \( \Lambda^i_j \) as in (4.10). We also let \( \Omega^i_{\text{good}} := \Omega^i \setminus \Omega^i_{\text{bad}} \). For each \( Q^i_j \in \Omega^i_{\text{good}} \), we apply Theorem 3.2 for \( u - v \) to obtain exceptional sets \( \omega^i_j \) and rigid motions \( a^i_j \) such that
\[
\begin{align*}
(\text{i}) & \quad \mathcal{H}^{d-1}(\partial^* \omega^i_j) \leq \bar{c} \mathcal{H}^{d-1}(\bar{\Omega}^i_j) + \mathcal{L}^d(\omega^i_j), \\
(\text{ii}) & \quad \|u - v - a^i_j\|_{L^p(\omega^i_j)} \leq \tilde{c} \rho^d \|e(u - v)\|_{L^p(\Omega^i_j)}.
\end{align*}
\]
(See also Remark 3.3.) For each \( Q^i_j \in \Omega^i_{\text{bad}} \), we define the exceptional set \( \omega^i_j \) simply by
\[
\omega^i_j := Q^i_j.
\]
\[
\text{Step 3: Korn's inequality and exceptional sets.}
\]
We now show that for each cube \( Q^i_j \) we have
\[
\mathcal{H}^{d-1}(\partial^* \omega^i_j) \leq C \left( \mathcal{H}^{d-1}(\bar{\Omega}^i_j) + \mathcal{L}^d(\omega^i_j) \right),
\]
where \( C := 4\bar{c} \tilde{c} \) and \( \bar{c} \geq 1 \) denotes again the constant from Remark 3.3. Indeed, for \( Q^i_j \in \Omega^i_{\text{good}} \), this follows directly from (4.12)(i). For \( Q^i_j \in \Omega^i_{\text{bad}} \), we first observe that \( \mathcal{H}^{d-1}(\partial^* \omega^i_j) = 2d \rho^{d-1} \), see (4.13). Then, by (4.11) we obtain
\[
\frac{1}{2d} \mathcal{H}^{d-1}(\partial^* \omega^i_j) = \rho^{d-1} \leq (2\bar{c} \bar{c}^2) \mathcal{H}^{d-1}(\bar{\Omega}^i_j) + \mathcal{L}^d(\omega^i_j) \leq \frac{1}{2} \rho^d = \frac{1}{2} \mathcal{L}^d(\Omega^i_j),
\]
from which (4.14) follows. Moreover, for \( Q^i_j \in \Omega^i_{\text{good}} \), by (4.11) and (4.12)(i) we get
\[
\mathcal{L}^d(\omega^i_j) \leq \tilde{c} \mathcal{H}^{d-1}(\bar{\Omega}^i_j) \leq \frac{1}{2} \mathcal{L}^d(\Omega^i_j).
\]
\[
\text{Step 4: Korn's inequality and rigid motions.}
\]
Recall that \( \tau_u : [0, 1] \to \mathbb{R}_+ \) is the function obtained by Lemma 3.3 for \( \delta = 1/2 \) and \( R = \sqrt{d} \). For each \( z_1 \in GSBD^p(A) \) and \( z_2 \in GSBD^p(B) \) we define
\[
\Lambda_\epsilon(z_1, z_2) = 2^p \tau_u^p \left( 2 \rho^d \int_{(A \setminus A') \cap B} \psi(|z_1 - z_2|) \, dx + 2 \lambda \right),
\]
whenever \( 2 \rho^d \int_{(A \setminus A') \cap B} \psi(|u - v|) \, dx \leq 1/2 \) and \( \Lambda_\epsilon(z_1, z_2) = +\infty \) else. (Note that this is well defined by (4.31) and it will lead to a definition of \( A \) in (4.31) that it is consistent with the definition below (4.10).)
The goal of this step is to prove the estimate
\[ \|a_j^i\|_{L^p(Q_j^i \setminus \omega_j^i)}^p \leq C(Q_j^i) A_j(u,v) \quad \text{for all } Q_j^i \subset Q^i. \] (4.17)
By definition of \( \omega_j^i \), see (4.13), it is clear that this needs to be checked only for cubes in \( Q^i_{\text{good}} \). To this end, we first note by (4.15) that
\[ \mathcal{L}^d(Q_j^i \setminus \omega_j^i) \geq \frac{1}{2} \mathcal{L}^d(Q_j^i) = \frac{1}{2} \rho^d. \] (4.18)
We write the rigid motions \( a_j^i \) as \( a_j^i(x) = A_j^i \cdot x + b_j^i \), and denote by \( x_j^i \) the center of the cube \( Q_j^i \). We can apply Lemma 3.5 for \( \delta = 1/2 \), \( R = \sqrt{d} \), \( \rho \to A_j^i \), \( b = b_j^i + A_j^i x_j^i \), and \( E = \rho^{-1}(Q_j^i \setminus \omega_j^i - x_j^i) \) to find
\[ \rho |A_j^i| + |b_j^i + A_j^i x_j^i| \leq \tau_\psi \left( \int_{Q_j^i \setminus \omega_j^i} \psi(|Gx + b|) \, dx \right) = \tau_\psi \left( \int_{Q_j^i \setminus \omega_j^i} \psi(|a_j^i|) \, dx \right), \] (4.19)
where in the second step we used a change of variables. We now estimate the integral on the right hand side of (4.19). By the triangle inequality, the monotonicity of \( \psi \), and the subadditivity of \( \psi \) we get
\[ \int_{Q_j^i \setminus \omega_j^i} \psi(|a_j^i|) \, dx \leq \int_{Q_j^i \setminus \omega_j^i} \psi(|u - v - a_j^i|) \, dx + \int_{Q_j^i \setminus \omega_j^i} \psi(|u - v|) \, dx. \]
Note that \( \psi(t) = \frac{1}{1 + t^2} \leq \lambda + \lambda^{-p} \rho^p \) for all \( t \geq 0 \). Therefore, we get
\[ \int_{Q_j^i \setminus \omega_j^i} \psi(|a_j^i|) \, dx \leq \lambda^{-p} \|u - v - a_j^i\|_{L^p(Q_j^i \setminus \omega_j^i)}^p + \lambda \mathcal{L}^d(Q_j^i \setminus \omega_j^i) + \int_{Q_j^i \setminus \omega_j^i} \psi(|u - v|) \, dx. \] (4.20)
For the first addend, we further compute by (4.11) and (4.12)(ii) that
\[ \|u - v - a_j^i\|_{L^p(Q_j^i \setminus \omega_j^i)}^p \leq C_1 \rho^p \|e(u) - e(v)\|_{L^p(Q_j^i)}^p \leq \frac{1}{p^2} \int_{Q_j^i \setminus \omega_j^i} \psi(|u - v|) \, dx \]
where we used that \( Q_j^i \subset Q^i_{\text{good}} \). This along with (4.3)(i), (4.18), and (4.20) yields
\[ \int_{Q_j^i \setminus \omega_j^i} \psi(|a_j^i|) \leq \frac{1}{\rho^d} \lambda^{-1} + \frac{1}{\rho^d} \int_{Q_j^i \setminus \omega_j^i} \psi(|u - v|) \leq 2 \lambda + \frac{1}{2 \rho^d} \int_{Q_j^i \setminus \omega_j^i} \psi(|u - v|). \] (4.21)
A simple calculation also yields
\[ \|a_j^i\|_{L^p(Q_j^i \setminus \omega_j^i)}^p \leq \mathcal{L}^d(Q_j^i) \sup_{x \in Q_j^i} |A_j^i x + b_j^i| = \mathcal{L}^d(Q_j^i) \sup_{x \in Q_j^i} |A_j^i (x + x_j^i) + b_j^i| \]
\[ \leq \mathcal{L}^d(Q_j^i) 2^{p-1} \left( (|A_j^i| \rho)^p + |b_j^i + A_j^i x_j^i| \right). \] (4.22)
Now we obtain (4.17) by using that \( \tau_\psi \) is increasing and by combining (4.19), (4.21), and (4.22).

**Step 5: Modifications of \( u \).** In this step of the proof, we will modify the function \( u \) on \( S_i \) (recall (4.4)) such that its difference to \( v \) restricted to \( T_i \) lies in \( L^p \). For each \( i = 1, \ldots, k \), we define \( \omega_i = \bigcup_j \omega_j^i \), and we note that \( \omega_i \subset \bigcup_j Q_j^i \subset S_i \) by (4.3). We introduce the function
\[ u_i = u x_i \setminus \omega_i + v x_i \in GSB\mathcal{D}^p(A). \] (4.23)
We now prove the estimates
\[ \begin{array}{ll}
(i) & \mathcal{E}(u_i, S_i) \leq (1 + C\beta \alpha^{-1}) (\mathcal{E}(u, S_i) + \mathcal{E}(v, S_i)), \\
(ii) & \|u_i - v\|_{L^p(T_i)}^p \leq C4^{p-1} \alpha^{-1} \left( \mathcal{E}(u, S_i) + \mathcal{E}(v, S_i) \right) + 2^{p-1} \mathcal{L}^d(S_i) A_j(u,v),
\end{array} \] (4.24)
where $C = 4d\bar{c}$ is the constant of (4.14). To prove (i), we first use (2.2) to get
\[ E(u_i, S_i) \leq E(u, S_i) + E(v, S_i) + \beta H^{d-1}(\partial^* \omega^i). \] (4.25)

By (4.9) and (4.14) we then compute
\[
\beta H^{d-1}(\partial^* \omega^i) \leq \beta \sum_j H^{d-1}(\partial^* \omega^j) \leq C \beta \sum_j \left( H^{d-1}(J_u \cap Q^*_j) + H^{d-1}(J_v \cap Q^*_j) + \|e(u)\|_{L^p(Q^*_j)}^p + \|e(v)\|_{L^p(Q^*_j)}^p \right) \leq C \beta \left( H^{d-1}(J_u \cap S_i) + H^{d-1}(J_v \cap S_i) + \|e(u)\|_{L^p(S_i)}^p + \|e(v)\|_{L^p(S_i)}^p \right).
\]
where we used that the cubes are pairwise disjoint. Then, (i) follows from (4.25) and the lower bound in (2.1)–(2.2). We now address (4.24)(ii). To this end, for each cube $Q^*_j$, by using (4.12)(ii) and (4.17) we get
\[
\left\| \hat{w} - w \right\|_{L^p(T_j)} \leq \sum_j \left\| \hat{w} - w \right\|_{L^p(Q^*_j \setminus \omega^j)} \leq 2^{p-1} \|u - v\|_{L^p(Q^*_j \setminus \omega^j)} + 2^{p-1} \|a^j\|_{L^p(Q^*_j \setminus \omega^j)} \leq 2^{p-1} c \rho \|e(v - u)\|_{L^p(Q^*_j)} + 2^{p-1} \|a^j\|_{L^p(Q^*_j \setminus \omega^j)} \leq 4^{p-1} c \rho \left( \|e(u)\|_{L^p(S_i)}^p + \|e(v)\|_{L^p(S_i)}^p \right) + 2^{p-1} L^d(Q^*_j) A_1(u, v). \]

Then, summing over all cubes and using (4.9) as well as (4.23) we derive
\[
\left\| u - v \right\|_{L^p(T_i)} \leq \sum_j \left\| u - v \right\|_{L^p(Q^*_j \setminus \omega^j)} \leq 4^{p-1} c \rho \left( \|e(u)\|_{L^p(S_i)}^p + \|e(v)\|_{L^p(S_i)}^p \right) + 2^{p-1} L^d(S_i) A_1(u, v). \]

In view of (2.1) and $C \geq \bar{c}$, this shows (4.24)(ii), and concludes this step of the proof.

**Step 6: Cuff-off construction.** We now perform a cut-off to join the functions $u_i$ and $v$. Recalling $u_i$ defined in (4.21) and the functions $\varphi_i$ introduced before (1.5), we define the functions $w_i := \varphi_i u_i + (1 - \varphi_i) v \in GSBD^p(A' \cup B)$ for $i = 1, \ldots, k$. By (4.4), we get
\[
E(w_i, A' \cup B) \leq E(u_i, (A' \cup B) \cap A_i^+ \cap B \cap A_i^-) + E(v, B \setminus A_i^-) + E(w_i, S_i) \leq E(u, A) + E(u_i, S_i) + E(v, B) + E(w_i, S_i). \] (4.26)
The second term has already been estimated in (4.24)(i). We now address the last term. By using the upper bounds in (2.1)–(2.2) we compute (by $\circ$ we denote the symmetrized vector product)
\[
E(w_i, S_i) \leq \int_{S_i} \beta (1 + |e(u_i)|^p) \, dx + \beta H^{d-1}(J_{u_i} \cap S_i) \leq \beta L^d(S_i) + \beta \int_{S_i} |\varphi_i e(u_i) + (1 - \varphi_i) e(v) + \nabla \varphi_i \circ (u_i - v)|^p \, dx + \beta H^{d-1}(J_{u_i} \cap S_i) \leq \beta L^d(S_i) + 3^{p-1} \beta \int_{S_i} (|e(u_i)|^p + |e(v)|^p + |\nabla \varphi_i|^p |u_i - v|^p) \, dx + \beta (H^{d-1}(J_{u_i} \cap S_i) + H^{d-1}(J_{v} \cap S_i)) + \beta L^d(S_i). \]

Using the lower bounds (2.1)–(2.2), (4.5), and (4.24)(i) we then get
\[
E(w_i, S_i) \leq 3^{p-1} \beta \alpha^{-1} \left( E(u_i, S_i) + E(v, S_i) \right) + 3^{p-1} \beta L^d(S_i) + \beta L^d(S_i) \leq 3^{p-1} \beta \alpha^{-1} (2 + C \beta \alpha^{-1}) (E(u, S_i) + E(v, S_i)) + 3^{p-1} \beta L^d(S_i). \]
By (4.8) (ii), (4.24), (4.26), and the fact that $C\beta\alpha^{-1} \geq \beta\alpha^{-1} \geq 1$ we thus derive after some computation
\[
\mathcal{E}(v_i, A') \leq \mathcal{E}(u, A) + \mathcal{E}(v, B) + 2C\beta\alpha^{-1}(\mathcal{E}(u, S_i) + \mathcal{E}(v, S_i)) + \mathcal{E}(w_i, S_i) \\
\leq \mathcal{E}(u, A) + \mathcal{E}(v, B) + (2C + C3\rho + C12\rho^{-1})(\beta\alpha^{-1})^2(\mathcal{E}(u, S_i) + \mathcal{E}(v, S_i)) \\
+ 6\rho^{-1}\beta\|\nabla\varphi\|_{L^\infty}(S_i)\Lambda_s(u, v) + \beta\mathcal{L}^d(S_i).
\] (4.27)

**Step 7: Definition of $w$ and $A$.** We finally define $w$ and $A$, and we show estimate (4.2). Recalling that the sets $(S_i)_{i=1}^k$ are pairwise disjoint and contained in $(A \setminus A') \cap B \subset A \cap B$, see (4.4), we can choose $i_0 \in \{1, \ldots, k\}$ such that
\[
\mathcal{E}(u, S_{i_0}) + \mathcal{E}(v, S_{i_0}) + \frac{1}{2}\mathcal{L}^d(S_{i_0}) \leq \frac{1}{k} \sum_{i=1}^k \left( \mathcal{E}(u, S_i) + \mathcal{E}(v, S_i) + \frac{1}{2}\mathcal{L}^d(S_i) \right) \\
\leq \frac{1}{k} \left( \mathcal{E}(u, A) + \mathcal{E}(v, B) + \frac{1}{2}\mathcal{L}^d(A \setminus A') \right).
\] (4.28)

Recall $C = 4d\delta$. As $\alpha \leq 1$, we have $12\rho^{k+1}d\delta\beta/(2\alpha^2) \geq 1$. Then, by (4.3), (4.27), and (4.28) the function $w := w_{i_0}$ satisfies
\[
\mathcal{E}(w, A' \cup B) \leq \mathcal{E}(v, A) + \mathcal{E}(v, B) + \frac{12\rho^{k+1}d\delta\beta^2}{\alpha^2}(\mathcal{E}(u, S_{i_0}) + \mathcal{E}(v, S_{i_0}) + \frac{1}{2}\mathcal{L}^d(S_{i_0})) + M\Lambda_s(u, v) \\
\leq (1 + \eta)\mathcal{E}(v, A) + \mathcal{E}(v, B) + M\Lambda_s(u, v) + \eta/2
\] (4.29)
where for shorthand we have defined $M := 6\rho^{-1}\beta(1 + \max_{i=1, \ldots, k}\|\nabla\varphi_i\|_{L^\infty}(S_i \setminus A'))$. In a similar fashion, as $\beta \geq 1$, by analogous estimates, taking (4.24) (ii), (4.3) and (4.28) into account, we get
\[
\|u_{i_0} - v\|_{L^p(T_{i_0})} \leq \eta\mathcal{E}(v, A) + \mathcal{E}(v, B) + M\Lambda_s(u, v) + \eta/2.
\] (4.30)

We let
\[
\Lambda(z_1, z_2) = M2^p\tau_\psi\left(2\rho^{-d}\int_{(A \setminus A') \cap B} \psi(|z_1 - z_2|)\,dx\right)
\] (4.31)
whenever $2\rho^{-d}\int_{(A \setminus A') \cap B} \psi(|u - v|)\,dx \leq 1/2$ and $A(z_1, z_2) = +\infty$ else. (Note that this is consistent with the definition below (4.10).) Then, $A$ is lower semicontinuous by Fatou’s lemma and the fact that $\tau_\psi$ is continuous and increasing. Moreover, in view of (4.6), we easily check that (4.1) is satisfied since $\tau_\psi(0) = 0$, see Lemma 4.5. Eventually, by (4.7) (ii) and (4.16) we find
\[
|A(u, v) - M\Lambda_s(u, v)| \leq \eta/2.
\] (4.32)

This along with (4.29) yields (4.2) (i). Recalling that $w = \varphi_{i_0}u_{i_0} + (1 - \varphi_{i_0})v$ and that $\{u \neq u_{i_0}\} \subset S_{i_0} \subset A \setminus A' \setminus B$ as well as $\varphi_{i_0} = 1$ on $A'$ and $\varphi_{i_0} = 0$ outside $A$, we get (4.2) (iii). Moreover, in view of (4.23), $w = v$ on $\omega^u$, and the fact that $\{0 < \varphi_{i_0} < 1\} \subset T_{i_0} \subset S_{i_0}$ (see (4.5)), we compute
\[
\|\min\{|u - w|, |w - v|\}\|_{L^p(A \cup B)} = \|\min\{|w - u_{i_0}|, |w - v|\}\|_{L^p(S_{i_0} \setminus \omega^u)} \leq \|u_{i_0} - v\|_{L^p(T_{i_0})}.
\]
Finally, by (4.30) and (4.32) we get that (4.2) (ii) holds true. This concludes the proof whenever $A \setminus A' \subset B$.

If $B$ has Lipschitz boundary, the condition $A \setminus A' \subset B$ is dispensable. In fact, we can still cover each set $T_i$ defined in (4.4) with cubes of sidelength $\rho$, see Step 1. These cubes, however, are not necessarily contained in $B$. Still, we can apply Korn’s inequality on the cubes in Step 2 by extending $v = 0$ outside $B$, at the expense of an additional term $C\beta L^{d-1}(\partial B \cap Q_j^0)$ on the right hand side of the estimate on $H^{d-1}(\partial w_j^0)$, see (4.14). This implies an additional addend $C\beta L^{d-1}(\partial B \cap S_i)$ on the right hand side of (4.24) (i). Eventually, in (4.28), this leads to an additional addend on the right hand side of the form $\frac{1}{k}H^{d-1}(\partial B \setminus (A \setminus A'))$. This can be made arbitrarily small for $k$ sufficiently large. □
4.2. Proof of Theorem 2.1. We consider a sequence of functionals $(\mathcal{E}_n)_n$ of the form (2.3). We start by proving some properties of the $\Gamma$-liminf and $\Gamma$-limsup with respect to the topology of the convergence in measure. To this end, we define

$$\mathcal{E}'(u, A) := \Gamma - \liminf_{n \to \infty} \mathcal{E}_n(u, A) = \inf \{ \liminf_{n \to \infty} \mathcal{E}_n(u_n, A) : u_n \to u \text{ in measure on } A \},$$

$$\mathcal{E}''(u, A) := \Gamma - \limsup_{n \to \infty} \mathcal{E}_n(u, A) = \inf \{ \limsup_{n \to \infty} \mathcal{E}_n(u_n, A) : u_n \to u \text{ in measure on } A \} \quad (4.33)$$

for all $u \in \text{GSBD}^p(\Omega)$ and $A \in \mathcal{A}(\Omega)$.

Lemma 4.3 (Properties of $\Gamma$-liminf and $\Gamma$-limsup). Let $\Omega \subset \mathbb{R}^d$ be open. Let $\mathcal{E}_n : \text{GSBD}^p(\Omega) \times \mathcal{A}(\Omega) \to [0, \infty)$ be a sequence of functionals as in (2.3), where we assume that $f_n$ and $g_n$ satisfy (2.1) and (2.2), respectively, for all $n \in \mathbb{N}$. Define $\mathcal{E}'$ and $\mathcal{E}''$ as in (4.33). For brevity, we write $I(u, A) := \|e(u)\|_{L^p(A)}^p + \mathcal{H}^{d-1}(J_u \cap A)$. Then we have

(i) $\mathcal{E}'(u, A) \leq \mathcal{E}'(u, B)$, $\mathcal{E}''(u, A) \leq \mathcal{E}''(u, B)$ whenever $A \subset B$,

(ii) $\alpha I(u, A) \leq \mathcal{E}'(u, A) \leq \beta I(u, A) + \beta \mathcal{L}^d(A)$,

(iii) $\mathcal{E}'(u, A) = \sup_{B \subset A} \mathcal{E}'(u, B)$, $\mathcal{E}''(u, A) = \sup_{B \subset A} \mathcal{E}''(u, B)$ whenever $A \in \mathcal{A}(\Omega)$,

(iv) $\mathcal{E}'(u, A \cup B) \leq \mathcal{E}'(u, A) + \mathcal{E}'(u, B)$, $\mathcal{E}''(u, A \cup B) \leq \mathcal{E}''(u, A) + \mathcal{E}''(u, B)$ whenever $A, B \in \mathcal{A}(\Omega)$, \hspace{1cm} (4.34)

where $\alpha$, $\beta$ appear in (2.1) and (2.2).

Proof of Lemma 4.3. Apart from (iii), the proof is standard. For convenience of the reader, however, we describe the arguments here to some extent. First, property (i) follows from the fact that all $\mathcal{E}_n(u, \cdot)$ are measures. The upper bound in (ii) follows by (2.1), (2.2) and by taking the constant sequence $u_n = u$ in (4.33). For the lower bound in (ii), let us consider an (almost) optimal sequence $(w_n)_n$ in (4.33). By the growth conditions (2.1) and (2.2) we get that (3.5) is satisfied, i.e., we can apply Theorem 3.7. Since the sequence $(w_n)_n$ converges in measure to $u$, the set $G_\infty$ has $\mathcal{L}^d$-negligible measure. Then, (2.1)–(2.2) along with (3.6)(ii),(iii) imply the lower bound.

As a preparation for (iii) and (iv), we show the following: for all sets $D, E, F \in \mathcal{A}(\Omega)$, $E \subset \subset F \subset \subset D$, we have

$$\mathcal{E}'(u, D) \leq \mathcal{E}'(u, F) + \mathcal{E}''(u, D \setminus \overline{E}), \quad \mathcal{E}''(u, D) \leq \mathcal{E}''(u, F) + \mathcal{E}''(u, D \setminus \overline{E}). \quad (4.35)$$

Indeed, let $(u_n)_n, (v_n)_n \subset \text{GSBD}^p(\Omega)$ be sequences converging in measure to $u$ on $F$ and $D \setminus \overline{E}$, respectively, such that

$$\mathcal{E}''(u, F) = \limsup_{n \to \infty} \mathcal{E}_n(u_n, F), \quad \mathcal{E}''(u, D \setminus \overline{E}) = \limsup_{n \to \infty} \mathcal{E}_n(v_n, D \setminus \overline{E}). \quad (4.36)$$

Fix $\eta > 0$. We apply Proposition 4.1 for $A = F$, $B = D \setminus \overline{E}$, and some $A' \in \mathcal{A}(\Omega)$ with $E \subset \subset A' \subset \subset F$. Note that we indeed have $A' \subset \subset A$ and $A \nearrow A' \subset B$. We get a function $w_n^\eta \in \text{GSBD}^p(D)$ satisfying (see (4.2)(i))

$$\mathcal{E}_n(w_n^\eta, D) \leq (1 + \eta)(\mathcal{E}_n(u_n, F) + \mathcal{E}_n(v_n, D \setminus \overline{E})) + A_\eta(u_n, v_n) + \eta, \quad (4.37)$$

where $A_\eta$ is the function given in (4.1). (We include $\eta$ in the notation to highlight the fact that the definition of $A_\eta$ depends on $\eta$.) We observe that $u_n - v_n$ tends to $0$ in measure on $F \setminus \overline{E}$. Hence, we get $A_\eta(u_n, v_n) \to 0$ by (4.1). By a diagonal argument we can find a sequence $(\eta_n)_n \subset (0, \infty)$ such that $\eta_n \to 0$ and $A_{\eta_n}(u_n, v_n) \to 0$ as $n \to \infty$. We now define $\tilde{w}_n := w_n^{\eta_n}$ for $n \in \mathbb{N}$. Recall that $(u_n)_n$ and $(v_n)_n$ converge in measure to $u$ on $F$ and $D \setminus \overline{E}$, respectively. In view of (4.2)(ii),(iii),
we get that \( \bar{w}_n \) converges in measure to \( u \) on \( D \). Then, by using \([4.33],[4.36]–[4.37]\), and \( \eta_n \to 0 \) we obtain
\[
\mathcal{E}''(u, D) \leq \limsup_{n \to \infty} \mathcal{E}_n(\bar{w}_n, D) \leq \mathcal{E}''(u, F) + \mathcal{E}''(u, D \setminus E).
\]
This implies the second estimate in \([4.35]\). A similar argument yields the first one.

Let us now show (iii), i.e., the inner regularity of \( \mathcal{E}' \) and \( \mathcal{E}'' \). By \([4.34]\)(ii) and \([4.35]\) we get
\[
\mathcal{E}''(u, D) \leq \mathcal{E}''(u, F) + \beta \mathcal{I}(u, D \setminus E) + \beta \mathcal{L}^d(D \setminus E).
\]
Since we can choose \( D \) and \( E \) in such a way that \( \mathcal{I}(u, D \setminus E) \) and \( \mathcal{L}^d(D \setminus E) \) can be taken arbitrarily small, and \( \mathcal{E}''(u, \cdot) \) is an increasing set function, we obtain \( \mathcal{E}''(u, D) \leq \sup_{F \subset D} \mathcal{E}''(u, F) \). This shows (iii) for \( \mathcal{E}'' \). The proof for \( \mathcal{E}' \) is similar.

Finally, we show (iv). Observe that the inequalities are clear if \( A \cap B = \emptyset \). Let \( A, B \in \mathcal{A}(\Omega) \) with nonempty intersection. Given \( \varepsilon > 0 \), one can choose \( M \subset M' \subset A \) and \( N \subset N' \subset B \) such that \( M, M', N, N' \in \mathcal{A}(\Omega), M' \cap N' = \emptyset \), and \( \mathcal{I}(u, (A \cup B) \setminus (M \cup N')) + \mathcal{L}^d((A \cup B) \setminus (M \cup N')) \leq \varepsilon \), see \([1]\) Proof of Lemma 5.2 for details. Then using, \([4.34]\)(i),(ii) and \([4.35]\) we get
\[
\mathcal{E}''(u, A \cup B) \leq \mathcal{E}''(u, M' \cup N') + \mathcal{E}''(u, (A \cup B) \setminus (M \cup N')) \leq \mathcal{E}''(u, M') + \mathcal{E}''(u, N') + \beta \varepsilon,
\]
where we also used \( \mathcal{E}''(u, M' \cup N') \leq \mathcal{E}''(u, M') + \mathcal{E}''(u, N') \) which holds due to \( M' \cap N' = \emptyset \). Since \( \varepsilon \) was arbitrary, the statement follows. The proof for \( \mathcal{E}' \) is again similar.

We can now prove Theorem 2.1.

**Proof of Theorem 2.1.** We apply a compactness result for \( \bar{\Gamma} \)-convergence, see \([34]\) Theorem 16.9, to find an increasing sequence of integers \((n_k)_k\) such that the objects \( \mathcal{E}' \) and \( \mathcal{E}'' \) defined in \([4.33]\) with respect to \((n_k)_k\) satisfy
\[
(\mathcal{E}')_-(u, A) = (\mathcal{E}'')_-(u, A)
\]
for all \( u \in GSBD^p(\Omega) \) and \( A \in \mathcal{A}(\Omega) \), where \( (\mathcal{E}')_- \) and \( (\mathcal{E}'')_- \) denote the inner regular envelope. By \([4.34]\)(iii) we know that \( \mathcal{E}' \) and \( \mathcal{E}'' \) are inner regular, and thus they both coincide with their respective inner regular envelopes. This shows that the \( \Gamma \)-limit, denoted by \( \bar{\mathcal{E}} := \mathcal{E}' = \mathcal{E}'' \), exists for all \( u \in GSBD^p(\Omega) \) and all \( A \in \mathcal{A}(\Omega) \).

We now check that \( \mathcal{E} \) satisfies the assumptions of the integral representation result \([31]\) Theorem 2.1. First, the definition in \([4.33]\) and the locality of each \( \mathcal{E}_n \) show that \( \mathcal{E}(\cdot, A) \) is local for any \( A \in \mathcal{A}(\Omega) \), i.e., \( \mathcal{E}(u, A) = \mathcal{E}(v, A) \) for all \( u, v \in GSBD^p(\Omega) \) satisfying \( u = v \ \mathcal{L}^d \text{-a.e. in } A \). Moreover, in view of \([34]\) Remark 16.3, \( \mathcal{E}(\cdot, A) \) is lower semicontinuous with respect to convergence in measure on \( \Omega \) for any \( A \in \mathcal{A}(\Omega) \). Next, we check that \( \mathcal{E}(u, \cdot) \) can be extended to a Borel measure for any \( u \in GSBD^p(\Omega) \). Indeed, \( \mathcal{E} \) is increasing, superadditive, and inner regular, see \([34]\) Proposition 16.12 and Remark 16.3. Moreover, by \([4.34]\)(iv) we find that \( \mathcal{E} \) is subadditive. Then, by De Giorgi-Letta (see \([34]\) Theorem 14.23), \( \mathcal{E}(u, \cdot) \) can be extended to a Borel measure. Eventually, by \([4.34]\)(ii) we get that
\[
\alpha \left( \int_B |e(u)|^p \, dx + \mathcal{H}^{d-1}(J_u \cap B) \right) \leq \mathcal{E}(u, B) \leq \beta \left( \int_B (1 + |e(u)|^p) \, dx + \mathcal{H}^{d-1}(J_u \cap B) \right).
\]
for every \( u \in GSBD^p(\Omega) \) and every Borel set \( B \subset \Omega \). Therefore, \( \mathcal{E} \) satisfies the assumptions of \([31]\) Theorem 2.1, and we conclude that \( \bar{\mathcal{E}} \) admits a representation of the form \([2.7]–[2.9]\). (The minimization problems \([2.8]–[2.9]\) can be formulated both in terms of balls and cubes, see \([31]\) Theorem 2.1, Remark 2.2(iv)].)
5. Two approximation results

In this section we present two approximation results for GSBD\(_p\) functions which are instrumental for the proof of Theorem 2.4 but also of independent interest.

5.1. Approximation of GSBD\(_p\) functions by \(W^{1,p}\). In the first result we show that a sequence of GSBD\(_p\) functions with asymptotically vanishing jump sets can be approximated by a sequence of equiintegrable Sobolev functions. The result is a variant in GSBD of similar results for Sobolev functions \([36]\) and BV-functions \([48]\). It crucially relies on the recent Korn inequality stated in Theorem 3.2 Later we will apply this result in the blow-up around points with approximate gradient.

Lemma 5.1 (Approximation of GSBD\(_p\) functions by \(W^{1,p}\)). Let \(\Omega \subset \mathbb{R}^d\) be an open, bounded set with Lipschitz boundary, and let 1 < \(p < +\infty\). Let \(u: \Omega \to \mathbb{R}^d\) be a measurable function, and let \((u_n)_n \subset GSBD(\Omega)\) be a sequence satisfying

\[
\begin{align*}
\text{(i)} & \quad \sup_{n \in \mathbb{N}} \|e(u_n)\|_{L^p(\Omega)} < \infty, \\
\text{(ii)} & \quad \lim_{n \to \infty} \mathcal{H}^{d-1}(J_{u_n}) = 0, \\
\text{(iii)} & \quad u_n \to u \text{ in measure on } \Omega \text{ as } n \to +\infty. 
\end{align*}
\]

Then, there exists a sequence \((w_n)_n \subset W^{1,p}(\Omega; \mathbb{R}^d)\) such that \(|\nabla w_n|^p_n\) is equiintegrable, and

\[
\begin{align*}
\text{(i)} & \quad \lim_{n \to \infty} \mathcal{L}^d(\{|w_n \neq u_n\} \cup \{e(w_n) \neq e(u_n)\}) = 0, \\
\text{(ii)} & \quad \lim_{n \to \infty} \|w_n - u\|_{L^p(\Omega)} = 0.
\end{align*}
\]

Proof. We first introduce the sequence \((w_n)_n\), and afterwards we study its properties.

Step 1: Existence of a bounded sequence in \(W^{1,p}\). Let us prove that there exists a bounded sequence \((v_n)_n \subset W^{1,p}(\Omega; \mathbb{R}^d)\) such that

\[
\mathcal{L}^d(\{|v_n \neq u_n\} \cup \{e(v_n) \neq e(u_n)\}) \to 0. 
\]

By Theorem 3.2 for every \(n \in \mathbb{N}\) there exists \(v_n \in W^{1,p}(\Omega; \mathbb{R}^d)\) and a set of finite perimeter \(\omega_n \subset \Omega\) such that \(u_n = v_n\) in \(\Omega \setminus \omega_n\), and \(\mathcal{L}^d(\omega_n) \leq c(\mathcal{H}^{d-1}(J_{u_n}))^{d/(d-1)}\), where the constant \(c\) depends only on \(p, d,\) and \(\Omega\). Thus, by construction and (5.1)(ii), (5.3) holds true. We now show that the sequence \((v_n)_n\) is bounded in \(W^{1,p}(\Omega; \mathbb{R}^d)\). To this end, we first apply Korn’s and Poincaré’s inequality for Sobolev functions to get

\[
\|v_n - a_n\|_{L^p(\Omega)} + \|\nabla v_n - A_n\|_{L^p(\Omega)} \leq C\|e(v_n)\|_{L^p(\Omega)} \leq C\|e(u_n)\|_{L^p(\Omega)},
\]

where each \(a_n\) is a rigid motions, i.e., \(a_n(x) := A_n x + b_n\) for \(A_n \in \mathbb{R}^{d \times d}_{\text{skew}}\) and \(b_n \in \mathbb{R}^d\). It now suffices to prove

\[
\sup_{n \in \mathbb{N}} (\|A_n\| + |b_n|) < +\infty.
\]

Indeed, this also shows \(\sup_{n \in \mathbb{N}} \|a_n\|_{L^p(\Omega)} < \infty\), and then along with (5.1)(i), (5.4), and the triangle inequality we obtain the boundedness of \((v_n)_n\) in \(W^{1,p}(\Omega; \mathbb{R}^d)\).

Let us show (5.5). Since \((v_n)_n\) converges in measure to \(u\) (see (5.1)(iii)), by Remark 2.2 there exists a strictly increasing concave function \(\psi: \mathbb{R}_+ \to \mathbb{R}_+\) with \(\psi(0) = 0\) and \(\psi(\mathbb{R}_+) = \mathbb{R}_+\), such that, up to passing to a subsequence \((v_{n_k})_k\), we have

\[
\sup_{k \in \mathbb{N}} \int_{\Omega} \psi(|v_{n_k}|) \, dx \leq 1.
\]
As $\psi \geq 0$ is increasing and concave, it is also subadditive. Therefore, we get by \((5.6)\) and the triangle inequality that
\[
\int_{\Omega} \psi(|a_{n_k}|) \, dx \leq \int_{\Omega} \psi(|v_{n_k} - a_{n_k}|) \, dx + \int_{\Omega} \psi(|v_{n_k}|) \, dx \leq \int_{\Omega} \psi(|v_{n_k} - a_{n_k}|) \, dx + 1.
\]
Then, by using Jensen’s inequality for concave functions, H"older’s inequality, and the monotonicity of $\psi$ we obtain
\[
\int_{\Omega} \psi(|a_{n_k}|) \, dx \leq \psi \left( \int_{\Omega} |v_{n_k} - a_{n_k}| \, dx \right) + \frac{1}{\mathcal{L}^d(\Omega)} \leq \psi \left( \left( \mathcal{L}^d(\Omega) \right)^{-\frac{1}{p}} \|v_{n_k} - a_{n_k}\|_{L^p(\Omega)} \right) + \frac{1}{\mathcal{L}^d(\Omega)}.
\]
This along with \((5.1)(i)\) and \((5.4)\) shows that $\sup_{n \in \mathbb{N}} \int_{\Omega} \psi(|a_{n_k}|) \, dx$ is bounded. By Rellich’s theorem, \((5.1)(i)\) and \((5.4)\) we get that a further subsequence converges strongly in $L^p(\Omega; \mathbb{R}^d)$ to $u$. By Urysohn’s property we deduce that the whole sequence $(v_n)_n$ converges to $u$ in $L^p(\Omega; \mathbb{R}^d)$. In particular, $\sup_{n \in \mathbb{N}} \|v_{n_k}\|_{L^p(\Omega)} < +\infty$ which along with \((5.1)(i)\) and \((5.4)\) shows $\sup_{n \in \mathbb{N}} \|a_{n_k}\|_{L^p(\Omega)} < +\infty$. Then Lemma 3.5 applied for $\psi(t) = t^p$ yields \((5.5)\) for the whole sequence. This concludes Step 1 of the proof.

**Step 2: Equiintegrability and \((5.2)\)**. Thanks to Step 1, the sequence $(v_n)_n$ is bounded in $W^{1,p}(\Omega; \mathbb{R}^d)$. Therefore, we can apply \[\text{Lemma 3.1} (\text{see also Lemma 2.1}),\] and we deduce that there exists a sequence $(w_n)_n \subset W^{1,p}(\Omega; \mathbb{R}^d)$ such that $(|\nabla w_n|^p)_n$ is equiintegrable, and
\[
\mathcal{L}^d \left( \{w_n \neq v_n\} \cup \{e(w_n) \neq e(v_n)\} \right) \to 0.
\]
By \((5.3)\) we get that \((5.2)(i)\) holds true. We recall that $\sup_{n \in \mathbb{N}} \|\nabla w_n\|_{L^p(\Omega)} < +\infty$ and $v_n \to u$ in measure on $\Omega$. This along with Poincaré’s inequality, Rellich’s theorem, and \((5.7)\) shows that \((5.2)(ii)\) holds true. This concludes the proof of the lemma. \(\square\)

### 5.2. Approximation of jump sets by boundary of partitions

The goal of the subsection is to prove the following result which allows to approximate the jump sets of $\text{GSBD}^p$ functions by the boundary of partitions. To this end, we define the sets
\[
Q_\nu^{\pm} = Q_\nu^+ \cap \{x \in \mathbb{R}^d : \pm x \cdot \nu \geq 0\}, \quad (5.8)
\]
where here and in the following $\pm$ is a placeholder for both $+$ and $-$.

**Lemma 5.2** (Approximation of jump sets by boundary of partitions). Let $d = 2$ and $p \geq 2$. Let $\zeta \in \mathbb{R}^2 \setminus \{0\}$ and $\nu \in \mathbb{S}^1$. Let $(u_n)_n \subset \text{GSBD}^p(Q_\nu^+)$ be a sequence satisfying
\[
\begin{align*}
(i) & \quad \lim_{n \to \infty} \|e(u_n)\|_{L^p(Q_\nu^+)} = 0, \\
(ii) & \quad \sup_{n \in \mathbb{N}} \mathcal{H}^1(J_n) < +\infty, \\
(iii) & \quad u_n \to u_{0,\zeta,0,\nu} \text{ in measure on } Q_\nu^+ \text{ as } n \to +\infty.
\end{align*}
\]
Then, there exists a sequence of neighborhoods $N_n \subset Q_\nu^+$ of $\partial Q_\nu^+$ and pairwise disjoint sets $S_n^+$ and $S_n^-$ with $\mathcal{L}^2(\overline{Q_\nu^+} \setminus (S_n^+ \cup S_n^-)) = 0$ for all $n \in \mathbb{N}$ such that
\[
\begin{align*}
(i) & \quad \lim_{n \to \infty} \mathcal{L}^2(S_n^\pm \Delta Q_\nu^{\pm}) = 0, \\
(ii) & \quad N_n \cap Q_\nu^{\nu,\pm} \subset S_n^\pm \text{ for all } n \in \mathbb{N}, \\
(iii) & \quad \lim_{n \to \infty} \mathcal{H}^1((\partial^* S_n^+ \cap \partial^* S_n^-) \setminus J_n) = 0.
\end{align*}
\]
\[\text{Lemma 5.2}\]
Later we will apply this result in the blow-up around jump points. Indeed, assumptions (5.9) are satisfied in the blow-up around \( H^1 \) a.e. jump point. The result states that, up to an asymptotically small set, the jump set covers the boundary of a partition which consists of two sets approximating the upper and lower half cubes \( Q_1^{+} \) and \( Q_1^{-} \), see (5.10)(i),(iii). Condition (5.10)(ii) will ensure that the piecewise constant functions \( v_n := \zeta \chi_{S_n^+} \) satisfy \( v_n = u_{0,\zeta,0,\nu} \) in a neighborhood of \( \partial Q_1^+ \).

**Remark 5.3** (Dimension \( d = 2 \), exponent \( p \)). (i) We emphasize that the result holds in dimension two only due to the application of a piecewise Korn-Poincaré inequality, see Proposition 3.4 and [45], which has been derived only for \( d = 2 \). The result, crucially based on similar results of this type [41, 40], heavily relies on combining different components of the jump set by segments via an explicit construction: a technique whose extension to higher dimension seems to be very difficult.

(ii) Let us also mention that, for simplicity, Proposition 3.4 has been derived in \( GSBD^2 \) and not in \( GSBD^p \) for general \( 1 < p < +\infty \). Therefore, we can derive Lemma 5.2 only for \( p \geq 2 \). (For \( p > 2 \) we can resort to \( GSBD^2 \) via Hölder’s inequality.) Although a generalization of Proposition 3.4 to general \( 1 < p < +\infty \) would be possible without significant changes in the proof, we refrain from entering into such minor issues and prefer to address the problem in this slightly less general fashion.

Besides the actual proof of Lemma 5.2, we also give a simplified proof in Appendix A which works under the assumption that each \( J_{u_n} \) consists of a bounded number of closed, continuous curves. Our motivation to present this simpler version of the proof is twofold. On the one hand, it provides elementary self-contained arguments avoiding the deep and complicated result from [45]. On the other hand, the construction of combining different parts of the jump set represents (in a simplified way) a main technique used in [40, 41, 45] being essential in the proof of Proposition 3.4.

We now proceed with the proof of Lemma 5.2, and refer the reader to Appendix A for the simplified proof.

**Proof of Lemma 5.2.** Let \( (u_n)_n \subset GSBD^p(Q_1^+) \) for \( p \geq 2 \) be given. By Hölder’s inequality we clearly have \( (u_n)_n \subset GSBD^2(Q_1^+) \) with \( \lim_{n \to \infty} \|e(u_n)\|_{L^2(Q_1^+)} = 0 \) due to (5.9)(i). We define

\[
C_0 := \sup_{n \in \mathbb{N}} H^1(J_{u_n}) + H^1(\partial Q_1^+) < +\infty. \tag{5.12}
\]

Our strategy relies on applying Proposition 3.4 for \( u_n \) and for fixed \( 0 < \theta < \min\{\theta_0, \frac{1}{2}\} \), where \( \theta_0 \) is the constant from Proposition 3.4. At the end of the proof, we pass to the limit \( \theta \to 0 \) and perform a diagonal argument. For simplicity, we do not indicate the \( \theta \)-dependence of the objects explicitly in the notation. We start by using (5.9)(iii) to find a sequence \( (\eta_n)_n \subset (0, +\infty) \) with \( \eta_n \to 0 \) and some \( n_\theta \in \mathbb{N} \) depending on \( \theta \) such that the sets

\[
B_n^- = \{ x \in Q_1^{\nu^-} : |u_n(x)| < \eta_n \}, \quad B_n^+ = \{ x \in Q_1^{\nu^+} : |u_n(x) - \zeta| < \eta_n \}, \tag{5.13}
\]

satisfy for all \( n \geq n_\theta \) that

\[
\mathcal{L}^2(Q_1^{\nu^\pm} \setminus B_n^{\pm}) \leq \theta^8/4. \tag{5.14}
\]
By Proposition 3.4 for $\theta^2$ in place of $\theta$ and by (5.12) we obtain partitions $Q^*_\theta = R_n \cup \bigcup_{j=1}^{J_n} P^n_j$ and corresponding rigid motions $(a^n_j)_{j=1}^{J_n}$ such that

(i) $\mathcal{H}^1((\partial^* R_n \cap Q^*_\theta) \setminus J_{u_n}) + \sum_{j=1}^{J_n} \mathcal{H}^1((\partial^* P^n_j \cap Q^*_\theta) \setminus J_{u_n}) \leq C_0 \theta^2$,

(ii) $\mathcal{L}^2(R_n) \leq C_0^2 \theta^2$, $\mathcal{L}^2(P^n_j) \geq \theta^6$ for all $j = 1, \ldots, J_n$.

(iii) $\max_{1 \leq j \leq J_n} \|u_n - a^n_j\|_{L^\infty(P^n_j)} \leq C_0 \|e(u_n)\|_{L^2(Q^*_\theta)}$.

(5.15)

Step 1: Components essentially contained in half-cubes. In this step we show that, by possibly passing to a larger $n_\theta \in \mathbb{N}$ only depending on $\theta$, for $n \geq n_\theta$ we have for all $j = 1, \ldots, J_n$ that

$$\mathcal{L}^2(P^n_j \cap B_n^-) = 0 \quad \text{or} \quad \mathcal{L}^2(P^n_j \cap B_n^+) = 0.$$  

(5.16)

To see this, by (5.14) and (5.15)(ii), for each $j = 1, \ldots, J_n$ we get

$$\max \left\{ \mathcal{L}^2(B_n^+ \cap P^n_j), \mathcal{L}^2(B_n^- \cap P^n_j) \right\} \geq \theta^6/4.$$  

(5.17)

We now first assume that the maximum is attained for $B_n^-$ and show $\mathcal{L}^2(P^n_j \cap B_n^+) = 0$. Afterwards, we briefly indicate the changes if the maximum is attained for $B_n^+$. Writing $a^n_j$ as $a^n_j(x) = A^n_j x + b^n_j$, we get by Lemma 3.3 for $\delta = \theta^6/4$ and $R = \sqrt{2}/2$, see (3.4), that

$$\|A^n_j x + b^n_j\|_{L^\infty(Q^*_\theta)} \leq c_\theta \|A^n_j x + b^n_j\|_{L^1(B_n^\cap P^n_j)}.$$  

(5.18)

where $c_\theta > 0$ is a constant depending on $\theta$. In view of (5.13), (5.15)(ii), and (5.18), this implies

$$\|a^n_j\|_{L^\infty(Q^*_\theta)} \leq c_\theta \|a^n_j\|_{L^\infty(B_n^- \cap P^n_j)} \leq c_\theta (\|A^n_j - u_n\|_{L^\infty(P^n_j)} + \|u_n\|_{L^\infty(B_n^-)}) \leq c_\theta (C_0^2 \|e(u_n)\|_{L^2(Q^*_\theta)} + \eta_n).$$  

(5.19)

Now, suppose by contradiction that $\mathcal{L}^2(P^n_j \cap B_n^+) > 0$. Then (5.13), (5.15)(iii), and (5.19) give

$$\|\zeta - \eta_n\|_{L^\infty(B_n^+ \cap P^n_j)} \leq \|u_n - a^n_j\|_{L^\infty(P^n_j)} + \|a^n_j\|_{L^\infty(Q^*_\theta)} \leq (c_\theta + 1) (C_0^2 \|e(u_n)\|_{L^2(Q^*_\theta)} + \eta_n).$$

By (5.11) and the fact that $\eta_n \to 0$ this yields a contradiction for $n$ sufficiently large only depending on $\theta$. If the maximum in (5.17) is attained for $B_n^+$ instead, we can show $\mathcal{L}^2(P^n_j \cap B_n^-) = 0$ by a similar argument, where we repeat the argument in (5.17)–(5.19) for $A^n_j x + (b^n_j - \zeta)$ instead of $A^n_j x + b^n_j$. We omit the details.

Step 2: Cutting of components. We define the set

$$V_\theta = \{ x \in Q^*_\theta : |x \cdot \nu| \leq \theta \}.$$  

(5.20)

Let $n \geq n_\theta$. In this step, up to sets of negligible $\mathcal{L}^2$-measure, we cut the components $R_n$ and $(P^n_j)_{j=1}^{J_n}$ into sets

$$R_n = R_n^+ \cup R_n^-; \quad P^n_j = P^n_j^+ \cup P^n_j^- \quad \text{for all } j = 1, \ldots, J_n,$n such that

(i) $R_n^+ \cup \bigcup_{j=1}^{J_n} P^n_j^\pm \subset Q^*_\theta \pm \cup V_\theta$,

(ii) $\sum_{j=1}^{J_n} \mathcal{H}^1((\partial^* P^n_j^\pm \setminus \partial^* P^n_j^\pm) + \mathcal{H}^1((\partial^* R_n^\pm \setminus \partial^* R_n)) \leq (C_0^2 + 1) \theta.$

(5.21)

To this end, let us fix $P^n_j$. We can assume without restriction that $\mathcal{L}^2(P^n_j \cap B_n^+) = 0$ by Step 1, see (5.16). (The other case can be treated in a similar fashion.) By Fubini’s theorem, $\theta \leq \frac{1}{2}$, and
we find that
\[
\int_0^1 \mathcal{H}^1(P_n^+ \cap L(s)) \, ds \leq \int_0^{1/2} \mathcal{H}^1((Q_1^{i+} \setminus B_n^+) \cap L(s)) \, ds \leq \mathcal{L}^2(Q_1^{i+} \setminus B_n^+) \leq \theta^8 / 4,
\]
where \( L(s) := \{ x \in Q_1^i : x \cdot \nu = s \} \). Therefore, we find \( s_j^+ \in (0, \theta) \) such that the sets \( P_j^+ = P_n^+ \cap \{ x \in Q_1^i : x \cdot \nu > s_j^+ \} \) and \( P_j^- = P_n^+ \cap \{ x \in Q_1^i : x \cdot \nu < s_j^+ \} \) satisfy
\[
\mathcal{H}^1(\partial^* P_j^+ \setminus \partial^* P_j^-) = \mathcal{H}^1(P_n^+ \cap L(s_j^+)) \leq \theta^{-1} \theta^8 / 4 \leq \theta^7.
\]
Clearly, by construction we also have \( P_j^+ \subset Q_1^{i+} \cup V_0 \). We repeat this construction for each \( P_n^+ \).
As \( J_n \leq \theta^{-6} \), see (5.15), we then get
\[
\sum_{j=1}^{J_n} \mathcal{H}^1(\partial^* P_j^+ \setminus \partial^* P_j^-) \leq \# J_n \theta^7 \leq \theta.
\]
By a similar construction we can define \( R_n = R_n^+ \cup R_n^- \) such that \( R_n^+ \subset Q_1^{i+} \cup V_0 \) and
\[
\mathcal{H}^1(\partial R_n^+ \setminus \partial R_n^-) \leq \theta^{-1} \mathcal{L}^2(R_n) \leq C_0^2 \theta,
\]
where the last step follows from (5.15)(ii). This shows (5.21) and concludes the proof of Step 2.

**Step 3: Definition of \( S_n^+ \) and \( S_n^- \).** We now define the sets \( S_n^+ \) and \( S_n^- \) and establish (5.10). For each \( 0 < \theta < \min\{\theta_0, \frac{1}{2}\} \) and each \( n \geq n_0 \), we first define the sets
\[
\tilde{S}_{n,\theta}^- = \bigcup_{j=1}^{J_n} P_j^+ \setminus R_n^-, \quad \tilde{S}_{n,\theta}^+ = \bigcup_{j=1}^{J_n} P_j^+ \setminus R_n^+.
\]
(We include \( \theta \) in the notation to highlight the dependence of the definition on \( \theta \).) By (5.15)(i) and (5.21)(ii) we note that
\[
\mathcal{H}^1((\partial^* \tilde{S}_{n,\theta}^- \cap \partial^* \tilde{S}_{n,\theta}^+) \setminus J_{u_n}) \leq \sum_{j=1}^{J_n} \mathcal{H}^1(\partial^* P_j^+ \setminus \partial^* P_j^-) + \mathcal{H}^1(\partial^* R_n^+ \setminus \partial^* R_n^-)
+ \mathcal{H}^1((\partial^* R_n \cap Q_1^i) \setminus J_{u_n}) + \sum_{j=1}^{J_n} \mathcal{H}^1((\partial^* P_j^+ \cap Q_1^i) \setminus J_{u_n}) \leq (C_0^2 + 1) \theta + C_0 \theta^2 \leq (C_0 + 1)^2 \theta.
\]
Define also the neighborhood \( N_\theta = Q_1^i \setminus Q_1^{i-} \). The sets \( \tilde{S}_{n,\theta}^- \) and \( \tilde{S}_{n,\theta}^+ \) do possibly not satisfy (5.10)(ii), and therefore we introduce the sets
\[
S_{n,\theta}^+ = (\tilde{S}_{n,\theta}^+ \cup (N_\theta \cap Q_1^{i+} \cup V_0)) \setminus (N_\theta \cap Q_1^{i-} \cap V_0),
S_{n,\theta}^- = (\tilde{S}_{n,\theta}^- \cup (N_\theta \cap Q_1^{i+} \cap V_0)) \setminus (N_\theta \cap Q_1^{i-} \cap V_0).
\]
Clearly, we have \( \mathcal{L}^2(Q_1^i \setminus S_{n,\theta}^+ \cap S_{n,\theta}^-) = 0 \) and we can check that
\[
\begin{align*}
(i) \quad & \mathcal{L}^2(S_{n,\theta}^+ \Delta Q_1^{i+}) \leq 2 \theta, \\
(ii) \quad & N_\theta \cap Q_1^{i+} \subset S_{n,\theta}^+, \\
(iii) \quad & \mathcal{H}^1((\partial S_{n,\theta}^+ \cap \partial S_{n,\theta}^-) \setminus J_{u_n}) \leq c \theta,
\end{align*}
\]
for a constant \( c > 0 \) independent of \( \theta \) and \( n \). In fact, (i) and (ii) follow from (5.20), (5.21)(i), and (5.23). To see (iii), we observe that (5.23) implies
\[
\mathcal{H}^1((\partial S_{n,\theta}^+ \cap \partial S_{n,\theta}^-) \setminus J_{u_n}) \leq \mathcal{H}^1((\partial S_{n,\theta}^+ \cap \partial S_{n,\theta}^-) \setminus J_{u_n}) + \sum_{j=1}^{J_n} \mathcal{H}^1(\partial (N_\theta \cap Q_1^{i+} \cap V_0)),
\]
Since \( \mathcal{H}^1(\partial (N_\theta \cap Q_1^{i+} \cap V_0)) \leq \theta^2 \) for a universal \( c > 0 \), the estimate then follows from (5.22).

Finally, we obtain the sets \( S_n^+ \) and \( S_n^- \) satisfying (5.10) from (5.24) by a diagonal argument. \( \square \)
6. Identification of the $\Gamma$-limit

This section is devoted to the proof of the results announced Subsection 2.2. To prove Theorem 2.4 we need to recover the estimates (2.20)–(2.22) (the implications about $\Gamma$-convergence then follow immediately by Theorem 2.1). This we will do in Subsections 6.1–6.3 below. Eventually, in Subsection 6.4 we prove the corollaries of Theorem 2.4.

Remark 6.1. In the proof of (2.20)–(2.22), we will use the following general property of recovery sequences: if $(u_n)_n$ is a recovering sequence for $u$ with respect to $\mathcal{E}_n(\cdot,\Omega)$, then $(u_n)_n$ is optimal for $u$ with respect to $\mathcal{E}(\cdot,\Omega)$ for every $A \in \mathcal{A}(\Omega)$ such that $\mathcal{E}(u,\partial A) = 0$. This follows from the fact that $\mathcal{E}(u,\cdot)$ is a Radon measure for each $u \in \text{GSBD}^p(\Omega)$.

6.1. Bulk part: Proof of (2.20). We start with the bulk density by showing two inequalities.

Step 1: $f_\infty(x,e(u)(x)) \leq f(x,e(u)(x))$ for $\mathcal{L}^d$-a.e. $x \in \Omega$. First, in view of (2.4) and (2.13), we get $\mathbf{m}_\epsilon(\tilde{\ell}_\xi,Q_p(x)) \leq \mathbf{m}_\epsilon^{1,p}(\tilde{\ell}_\xi,Q_p(x))$ for all $\xi \in \mathbb{R}^{d \times d}$, where we recall the notation $\tilde{\ell}_\xi(y) = \xi y$ for $y \in \mathbb{R}^d$ in (2.17). Then (2.11) implies

$$f_\infty(x,\text{sym}(\xi)) = \limsup_{\rho \to 0^+} \frac{\mathbf{m}_\epsilon(\tilde{\ell}_\xi,Q_p(x))}{\rho^d} \leq \limsup_{\rho \to 0^+} \frac{\mathbf{m}_\epsilon^{1,p}(\tilde{\ell}_\xi,Q_p(x))}{\rho^d}.$$  \hspace{1cm} (6.1)

On the other hand, by Remark 3.15 we get that the density $f$ in (2.18) coincides with $f_0$ given in (3.12). Thus, by (3.12) and (2.15) we find

$$f(x,\text{sym}(\xi)) = \limsup_{\rho \to 0^+} \frac{\mathbf{m}_\epsilon^{1,p}(\tilde{\ell}_\xi,Q_p(x))}{\rho^d}.$$  \hspace{1cm} (6.2)

By combining (6.1)–(6.2) we obtain $f_\infty(x,\text{sym}(\xi)) \leq f(x,\text{sym}(\xi))$. This concludes Step 1.

Step 2: $f_\infty(x,e(u)(x)) \geq f(x,e(u)(x))$ for $\mathcal{L}^d$-a.e. $x \in \Omega$. By Remark 2.2(i) and the Radon-Nikodým Theorem we have for $\mathcal{L}^d$-a.e. $x \in \Omega$ that

$$f_\infty(x,e(u)(x)) = \lim_{\rho \to 0^+} \frac{\mathcal{E}(u,Q_p(x))}{\rho^d} < \infty.$$  \hspace{1cm} (6.3)

Let $(u_n)_n$ be a recovering sequence for $\mathcal{E}(u,\Omega)$. This along with the growth condition (2.12) yields that the sequence $(\mathcal{H}^{d-1}(J_{u_n}))_n$ is uniformly bounded. Thus, up to a subsequence (not relabeled), there exists a finite positive Radon measure $\mu$ such that

$$\mu_n := \mathcal{H}^{d-1}\llcorner J_{u_n} \rightharpoonup \mu \quad \text{weakly}^* \text{ in the sense of measures.}$$  \hspace{1cm} (6.4)

Let us notice that for $\mathcal{L}^d$-a.e. $x \in \Omega$ we have that

$$\limsup_{\rho \to 0^+} \frac{\mu(Q_p(x))}{\rho^{d-1}} = 0.$$  \hspace{1cm} (6.5)

Indeed, by contradiction we suppose that there exists a Borel set $B \subset \Omega$ with $\mathcal{L}^d(B) > 0$ and $t > 0$ such that

$$\limsup_{\rho \to 0^+} \frac{\mu(Q_p(x))}{\rho^{d-1}} > t,$$

for all $x \in B$. Then, as a consequence of [\text{[Theorem 2.56]}] we would get $\mu \llcorner B \geq t\mathcal{H}^{d-1}\llcorner B$ implying that $\mu(B) = \infty$. But this is a contradiction since $\mu$ is finite. Since $u \in \text{GSBD}^p(\Omega)$, the approximate gradient $\nabla u(x)$ exists for $\mathcal{L}^d$-a.e. $x \in \Omega$, see Lemma 3.1. As the main inequality needs to hold for $\mathcal{L}^d$-a.e. $x \in \Omega$, we can suppose that (6.3) and (6.5) are satisfied at $x$ and that the approximate gradient $\nabla u(x)$ exists. Since $\mathcal{E}(u,\cdot)$ is a Radon measure, there exists a subsequence
By (2.3), (6.7), and by the change of variables 

\[ f(x, e(u)(x)) := \lim_{i \to \infty} \limsup_{n \to \infty} \frac{1}{\rho_i^d} m_{\mathcal{E}_n}(\tilde{\ell}_{u_n}(x), Q_{\rho_i}(x)). \]  

(6.6)

By Remark 6.1 for every \( i \in \mathbb{N} \) there exists \( n_i \in \mathbb{N} \) such that for every \( n \geq n_i \), we get

\[ \frac{\mathcal{E}(u, Q_{\rho_i}(x))}{\rho_i^d} \geq \frac{\mathcal{E}_n(u_n, Q_{\rho_i}(x))}{\rho_i^d} - \frac{1}{i}. \]  

(6.7)

We define the functions

\[ \tilde{u}_i(y) := \frac{u_n(x + \rho_i y) - u_n(x)}{\rho_i} \quad \text{and} \quad \tilde{u}(y) := \frac{u(x + \rho_i y) - u(x)}{\rho_i} \]  

for \( y \in Q_1 \), and note that \( \tilde{u}_n \to \tilde{u} \) in measure on \( Q_1 \) as \( n \to \infty \) since \( u_n \to u \) in measure on \( \Omega \). We now show by a diagonal argument that, up to passing to larger \( n_i \in \mathbb{N} \), the sequence \( v_i := \tilde{u}_n \), satisfies

\[ v_i \to \tilde{\ell}_{u_n}(x) \]  

in measure on \( Q_1 \),

(recall that \( \tilde{\ell}_{u_n}(x) = \nabla u(x)y \) for \( y \in \mathbb{R}^d \), see (2.17)) and

\[ \lim_{i \to \infty} \mathcal{H}^{d-1}(J_{v_i}) = 0. \]  

(6.8)

Indeed, recall that \( \tilde{u}_n \to \tilde{u} \) in measure on \( Q_1 \) as \( n \to \infty \). Since \( u \) is approximately differentiable at \( x \), we also have \( \tilde{u} \to \tilde{\ell}_{u_n}(x) \) in measure on \( Q_1 \) as \( i \to \infty \), cf. Lemma 3.1. Consequently, (6.8) can be achieved. Moreover, by a change of variables and by recalling (6.4) we get

\[ \limsup_{n \to \infty} \mathcal{H}^{d-1}(J_{u_n}) = \limsup_{n \to \infty} \frac{\mathcal{H}^{d-1}(J_{u_n} \cap Q_{\rho_i}(x))}{\rho_i^{d-1}} \leq \limsup_{n \to \infty} \frac{\mu_n(Q_{\rho_i}(x))}{\rho_i^{d-1}} \leq \frac{\mu(Q_{\rho_i}(x))}{\rho_i^{d-1}}. \]

Thus, by (6.9) we get

\[ \limsup_{i \to \infty} \limsup_{n \to \infty} \mathcal{H}^{d-1}(J_{\tilde{u}_n}) = 0. \]  

(6.10)

Then, by a diagonal argument, (6.9) can be ensured. Note by (6.6) that we can choose \( (n_i)_i \) such that additionally we have

\[ f(x, e(u)(x)) := \lim_{i \to \infty} \frac{1}{\rho_i^d} m_{\mathcal{E}_n}(\tilde{\ell}_{u_n}(x), Q_{\rho_i}(x)). \]  

(6.11)

By (2.3), (6.7), and by the change of variables \( y' = x + \rho_i y \), we get

\[ \int_{Q_1} f_n(x + \rho_i y, e(v_i)(y)) \, dy = \int_{Q_{\rho_i}(x)} f_n(y', e(u_n)(y')) \, dy' \leq \frac{\mathcal{E}(u, Q_{\rho_i}(x))}{\rho_i^d} + \frac{1}{i}. \]  

(6.12)

In addition, taking into consideration (6.3) we get

\[ \limsup_{i \to \infty} \int_{Q_1} f_n(x + \rho_i y, e(v_i)(y)) \, dy \leq \lim_{i \to \infty} \frac{\mathcal{E}(u, Q_{\rho_i}(x))}{\rho_i^d} = f_{\infty}(x, e(u)(x)). \]  

(6.13)

Let us observe that the sequence \( (v_i)_i \subset GSBD^p(Q_1) \) satisfies the assumptions of Lemma 5.1. Indeed, by (6.13) and the growth condition (2.1) we have that (5.1)(i) holds true. Thanks to (6.9), we get (5.1)(ii), while condition (5.1)(iii) is a consequence of (6.8). Thus, by Lemma 5.1 applied to the sequence \( (v_i)_i \subset GSBD^p(Q_1) \) there exists a sequence \( (w_i)_i \subset W^{1,p}(Q_1; \mathbb{R}^d) \) such that \( 5.2 \).
holds true, and \(|\nabla w_i|^p|\) is equiintegrable. In particular, from this latter fact, (5.2)(i), and (2.1) we get
\[
\limsup_{i \to \infty} \int_{Q_1} f_{n_i}(x + \rho_i y, e(v_i)(y)) \, dy = \limsup_{i \to \infty} \int_{Q_1} f_{n_i}(x + \rho_i y, e(w_i)(y)) \, dy.
\] (6.14)
Moreover, by (5.2) (ii) and (6.8) we get that
\[
\lim_{i \to \infty} \|w_i - \bar{\ell}_{\mathcal{V}u(x)}\|_{L^p(Q_1)} = 0.
\] (6.15)
We now modify the sequence \((w_i)\) in such a way that it will attain the boundary datum of the function \(\bar{\ell}_{\mathcal{V}u(x)}\) in a neighborhood of \(\partial Q_1\) (see [17, Proof of Theorem 5.2(b), Step 2] for a similar argument). By [34, Theorem 19.4] we know that the family of functionals \(\tilde{F}_i\) defined by
\[
\tilde{F}_i(u, A) = \int_A f_{n_i}(x + \rho_i y, e(u)(y)) \, dy
\] (6.16)
for \(A \in \mathcal{A}(Q_1)\) and \(u \in W^{1,p}(A; \mathbb{R}^d)\) satisfies uniformly the Fundamental Estimate (see [34, Chapter 18]): for fixed \(0 < \varepsilon < 1\) there exists a constant \(C(\varepsilon) > 0\), and a sequence \((z_i) \subset W^{1,p}(Q_1; \mathbb{R}^d)\) with \(z_i = \tilde{\ell}_{\mathcal{V}u(x)}\) in a neighborhood of \(\partial Q_1\) for all \(i \in \mathbb{N}\) such that
\[
\tilde{F}_i(z_i, Q_1) \leq (1 + \varepsilon) \left( \tilde{F}_i(w_i, Q_1) + \tilde{F}_i(\tilde{\ell}_{\mathcal{V}u(x)}, Q_1 \setminus Q_{1-\varepsilon}) \right) + C(\varepsilon) \|w_i - \tilde{\ell}_{\mathcal{V}u(x)}\|_{L^p(Q_1)} + \varepsilon
\] (more precisely, \(z_i := \varphi w_i + (1 - \varphi)\tilde{\ell}_{\mathcal{V}u(x)}\), where \(\varphi\) lies in a finite collection of cut-off functions \(\varphi \in C_0^\infty(Q_1; [0, 1])\), with \(\varphi = 1\) in \(Q_{1-\varepsilon}\). Since \(\mathcal{L}^d(Q_1 \setminus Q_{1-\varepsilon}) \leq \varepsilon\), thanks to the growth condition (2.1) and (6.15), we get that
\[
\limsup_{i \to \infty} \tilde{F}_i(z_i, Q_1) \leq (1 + \varepsilon) \limsup_{i \to \infty} \tilde{F}_i(w_i, Q_1) + d\varepsilon(1 + \varepsilon)\beta(1 + |e(u)(x)|^p) + \varepsilon.\] (6.17)
Then, by (6.13), (6.14), (6.16), and (6.17) we derive
\[
\limsup_{i \to \infty} \tilde{F}_i(z_i, Q_1) \leq (1 + \varepsilon) f_\infty(x, e(u)(x)) + d\varepsilon(1 + \varepsilon)\beta(1 + |e(u)(x)|^p) + \varepsilon.\] (6.18)
On the other hand, by a change of variables we have that
\[
\tilde{F}_i(z_i, Q_1) = \int_{Q_1} f_{n_i}(x + \rho_i y', e(z_i)(y')) \, dy' = \frac{1}{\rho_i^d} \int_{Q_{n_i}(x)} f_{n_i}(y, e(\check{z}_i)(y)) \, dy,
\] (6.19)
where \(\check{z}_i \in W^{1,p}(Q_{\rho_i}(x); \mathbb{R}^d)\) is defined by \(\check{z}_i(y) := \rho_i z_i((y - x)/\rho_i) + \tilde{\ell}_{\mathcal{V}u(x)}x\) for \(y \in Q_{\rho_i}(x)\). Since \(z_i = \tilde{\ell}_{\mathcal{V}u(x)}\) in a neighborhood of \(\partial Q_1\), we get \(\check{z}_i = \tilde{\ell}_{\mathcal{V}u(x)}\) in a neighborhood of \(\partial Q_{\rho_i}(x)\). Thus, (2.13), (2.15), and (6.19) imply
\[
\frac{1}{\rho_i^d} m_{\mathcal{E}_{n_i}}^1(\tilde{\ell}_{\mathcal{V}u(x)}, Q_{\rho_i}(x)) \leq \frac{1}{\rho_i^d} \int_{Q_{n_i}(x)} f_{n_i}(y, e(\check{z}_i)(y)) \, dy = \tilde{F}_i(z_i, Q_1).
\]
This along with (6.18) and the arbitrariness of \(\varepsilon\) yields
\[
\limsup_{i \to \infty} \frac{1}{\rho_i^d} m_{\mathcal{E}_{n_i}}^1(\tilde{\ell}_{\mathcal{V}u(x)}, Q_{\rho_i}(x)) \leq f_\infty(x, e(u)(x)).
\]
Thanks to (6.11), we obtain the desired inequality \(f(x, e(u)(x)) \leq f_\infty(x, e(u)(x))\). This concludes the proof.
6.2. Surface part in Theorem 2.4. Proof of (2.21). The proof is again split into two inequalities. The first one is obtained similarly as in Subsection 6.1. The other inequality is the only point where we need to restrict ourselves to dimension \(d = 2\), due to the application of Lemma 5.2.

Step 1: \(g_\infty(x, |u|(x), \nu_{\rho}(x)) \leq g(x, \nu_{\rho}(x))\) for \(H^{d-1}\)-a.e. \(x \in J_u\). With the notation in (2.4), we set \(\tilde{u} := u, |u|, 0, \nu_{\rho}(x)\) for brevity. First, in view of (2.4) and (2.14), we get \(m^{\rho}_{\xi}(\tilde{u}, Q^{\nu_{\rho}}_{\rho}(x)) \leq m^{\rho R}_{\xi}(\tilde{u}, Q^{\nu_{\rho}}_{\rho}(x))\). Then (2.11) entails

\[
g_\infty(x, |u|(x), \nu_{\rho}(x)) = \limsup_{\rho \to 0^+} \frac{m^{\rho}_{\xi}(\tilde{u}, Q^{\nu_{\rho}}_{\rho}(x))}{\rho^{d-1}} \leq \limsup_{\rho \to 0^+} \frac{m^{\rho R}_{\xi}(\tilde{u}, Q^{\nu_{\rho}}_{\rho}(x))}{\rho^{d-1}}. \tag{6.20}
\]

On the other hand, by Remark 3.15 we get that the density \(g\) in (2.19) coincides with \(g_0\) given in (3.19). Therefore, (2.16), (3.16), and (3.19) yield

\[
g(x, \nu_{\rho}(x)) = \limsup_{\rho \to 0^+} \frac{m^{\rho PC}_{\xi}(\tilde{u}, Q^{\nu_{\rho}}_{\rho}(x))}{\rho^{d-1}} = \limsup_{\rho \to 0^+} \frac{m^{\rho R}_{\xi}(\tilde{u}, Q^{\nu_{\rho}}_{\rho}(x))}{\rho^{d-1}}, \tag{6.21}
\]

where we used the notation in (2.17). By (6.20)–(6.21) we obtain \(g_\infty(x, |u|(x), \nu_{\rho}(x)) \leq g(x, \nu_{\rho}(x))\).

Step 2: \(g_\infty(x, |u|(x), \nu_{\rho}(x)) \geq g(x, \nu_{\rho}(x))\) for \(H^{d-1}\)-a.e. \(x \in J_u\). We recall that in this step we explicitly use that \(d = 2\). Still, for later purposes in Subsection 6.3 we write \(d\) instead of 2 whenever the arguments hold in every dimension. In view of (2.10) and the Radon-Nikodým Theorem, we get for \(H^{d-1}\)-a.e. \(x \in J_u\) that

\[
g_\infty(x, |u|(x), \nu_{\rho}(x)) = \lim_{\rho \to 0^+} \frac{1}{\rho^{d-1}} \mathcal{E}(u, Q^{\nu_{\rho}}_{\rho}(x)) < +\infty. \tag{6.22}
\]

Moreover, we choose \((\rho_i)_i \subset (0, \infty)\) such that \(\rho_i \searrow 0\), \(\mathcal{E}(u, \partial Q^{\nu_{\rho_i}}_{\rho_i}(x)) = 0\) for every \(i \in \mathbb{N}\), and such that the representation formula (2.19) holds along \((\rho_i)_i\), i.e.,

\[
g(x, \nu_{\rho_i}(x)) := \lim_{i \to \infty} \limsup_{n \to \infty} \frac{1}{\rho_i^{d-1}} m^{\rho R}_{\xi_{\rho_i}}(\tilde{u}, Q^{\nu_{\rho_i}}_{\rho_i}(x)). \tag{6.23}
\]

In what follows, since the main inequality needs to hold for \(H^{d-1}\)-a.e. \(x \in J_u\), we can fix \(x \in J_u\) such that the above property holds. We write \(\nu\) in place of \(\nu_{\rho_i}\) for notational simplicity.

Let \((u_n)_n \subset GSB\mathcal{D}^p(\Omega)\) be a recovery sequence for \(\mathcal{E}(u, \Omega)\). As \(\mathcal{E}(u, \partial Q^{\nu_{\rho_i}}_{\rho_i}(x)) = 0\), by Remark 6.1 we get that for every \(i \in \mathbb{N}\) there exists \(n_i \in \mathbb{N}\) such that for all \(n \geq n_i\) it holds that

\[
\frac{1}{\rho_i^{d-1}} \mathcal{E}(u_n, Q^{\nu_{\rho_i}}_{\rho_i}(x)) \geq \frac{1}{\rho_i^{d-1}} \mathcal{E}(u_n, Q^{\nu_{\rho_i}}_{\rho_i}(x)) - \frac{1}{i}. \tag{6.24}
\]

We define \(\tilde{u}^i_n(y) = u_n(x + \rho_i y)\) and \(\tilde{u}^i(y) = u(x + \rho_i y)\) for \(y \in Q^\gamma \) and note that \(\tilde{u}^i_n \to \tilde{u}^i\) in measure on \(Q^\gamma\) since \(u_n \to u\) in measure on \(\Omega\). Since \(x \in J_u\) is an approximate jump point, we find \(\tilde{u}^i \to u_{0, n^i(x), u^i(x), \nu(x), \nu^i(x)}\) in measure for \(i \to \infty\), see (3.1) and recall notation (2.6). Thus, by a diagonal argument, up to passing to larger \(n_i \in \mathbb{N}\), we can suppose that \(n_i \to +\infty\) as \(i \to \infty\) and

\[
v_i \to u_{0, n^i(x), u^i(x), \nu(x), \nu^i(x)} \quad \text{in measure on } Q^\gamma, \tag{6.25}
\]

where we define \(v_i \in GSB\mathcal{D}^p(Q^\gamma)\) by \(v_i = \tilde{u}^i_n\). Note that by (6.23) we can choose \((n_i)_i\) such that additionally

\[
g(x, \nu) := \lim_{i \to \infty} \frac{1}{\rho_i^{d-1}} m^{\rho R}_{\xi_{\rho_i}}(\tilde{u}, Q^{\nu_{\rho_i}}_{\rho_i}(x)) \tag{6.26}
\]
holds. By a change of variables, and by using (2.3) and (6.24) we get
\[
\int_{J_{\omega} \cap Q_{\rho_i}^0} g_{n_i}(x + \rho_i y, \nu_{n_i}(y)) \, dH^d(y) = \frac{1}{\rho_i^{d-1}} \int_{J_{\omega_i} \cap Q_{\rho_i}^0(x)} g_{n_i}(z, \nu_{u_{n_i}}(z)) \, dH^d(z)
\leq \frac{1}{\rho_i^{d-1}} E(u, Q_{\rho_i}^0(x)) + \frac{1}{i}. \tag{6.27}
\]

We also define \( \tilde{v}_i(y) := v_i(y) - u^-(x) \) for \( y \in Q_{\rho_i}^0 \). We check that \( \tilde{v}_i \), satisfies the assumptions of Lemma 5.2. First, to see (5.9)(ii), we use (2.12), (6.22), and (6.27) to find
\[
\sup_{i \in \mathbb{N}} H^d(J_{E_i} \cap Q_{\rho_i}^0) = \sup_{i \in \mathbb{N}} H^d(J_{E_i} \cap Q_{\rho_i}^0) \leq \frac{1}{\rho_i^{d-1}} \sup_{i \in \mathbb{N}} \left( \frac{1}{\rho_i^{d-1}} E(u, Q_{\rho_i}^0(x)) + \frac{1}{i} \right) < +\infty. \tag{6.28}
\]

Now we show (5.9)(i), namely \( \lim_{i \to \infty} \|e(\tilde{v}_i)\|_{L^p(Q_{\rho_i}^0)} = 0 \). Indeed, using the growth condition (2.1), (2.3), and a change of variables we get
\[
\int_{Q_{\rho_i}^0} |e(\tilde{v}_i(y))|^p \, dy = \frac{\rho_i^p}{\rho_i^d} \int_{Q_{\rho_i}^0(x)} |e(u_{n_i}(z))|^p \, dz \leq \frac{\rho_i^{p-1}}{\rho_i^{d-1}} \frac{1}{\rho_i^{d-1}} E(u, Q_{\rho_i}^0(x)).
\]

Then, by (6.22), (6.24), and the fact that \( \rho_i \to 0 \) as \( i \to +\infty \) we conclude \( \lim_{i \to \infty} \|e(\tilde{v}_i)\|_{L^p(Q_{\rho_i}^0)} = 0 \). Finally, (6.25) holds for \( \zeta = u^+(x) - u^-(x) \) by the definition of \( \tilde{v}_i \) and (6.25). Thus, thanks to Lemma 5.2 we can define a sequence of piecewise constant functions \( (w_i)_i \subset PC(Q_{\rho_i}^0) \) by
\[
w_i(y) := \begin{cases} 0 & y \in S_i^{-}, \\ e_1 & y \in S_i^{+}, \end{cases}
\]
where \( S_i^{\pm} \) are given in Lemma 5.2 and \( e_1 = (1,0) \). (From now on, we explicitly set \( d = 2 \).) Let us observe some properties of the sequence \( (w_i)_i \). First of all, by (5.10)(i) we get that \( w_i \to \vec{u}_{0,\nu} \) strongly in \( L^1(Q_{\rho_i}^0; \mathbb{R}^2) \). Thanks to property (5.10)(ii), we have that \( \vec{w}_i = \vec{u}_{0,\nu} \) in a neighborhood of \( \partial Q_{\rho_i}^0 \). Finally, by property (5.10)(iii) it holds that
\[
H^1(J_{\omega_i} \setminus J_{E_i}) = H^1(J_{\omega_i} \setminus J_{E_i}) \leq \eta_i,
\]
with \( \lim_{i \to \infty} \eta_i = 0 \). This along with (2.12) implies
\[
\int_{J_{\omega_i} \cap Q_{\rho_i}^0} g_{n_i}(x + \rho_i y, \nu_{n_i}(y)) \, dH^1(y) \leq \int_{J_{\omega_i} \cap Q_{\rho_i}^0} g_{n_i}(x + \rho_i y, \nu_{n_i}(y)) \, dH^1(y) + \beta \eta_i. \tag{6.29}
\]
Due to (6.22), (6.27), and (6.29) we get
\[
\limsup_{i \to \infty} \int_{J_{\omega_i} \cap Q_{\rho_i}^0} g_{n_i}(x + \rho_i y, \nu_{n_i}(y)) \, dH^1(y) \leq \limsup_{i \to \infty} \frac{1}{\rho_i} E(u, Q_{\rho_i}^0(x)) = g_{\infty}(x, [u](x), \nu). \tag{6.30}
\]
By defining \( \tilde{w}_i(z) = w_i((z - x)/\rho_i) \) for \( z \in Q_{\rho_i}^0(x) \) and rescaling back to \( Q_{\rho_i}^0(x) \) we obtain
\[
\int_{J_{\omega_i} \cap Q_{\rho_i}^0} g_{n_i}(x + \rho_i y, \nu_{n_i}(y)) \, dH^1(y) = \frac{1}{\rho_i} \int_{J_{\omega_i} \cap Q_{\rho_i}^0(x)} g_{n_i}(z, \nu_{\tilde{w}_i}(z)) \, dH^1(z).
\]
Then (6.30) yields
\[
\limsup_{i \to \infty} \frac{1}{\rho_i} \int_{J_{\omega_i} \cap Q_{\rho_i}^0(x)} g_{n_i}(z, \nu_{\tilde{w}_i}(z)) \, dH^1(z) \leq g_{\infty}(x, [u](x), \nu). \tag{6.31}
\]
Observe that by construction \( \tilde{w}_i = u_{x,\nu} \) in a neighbourhood of \( \partial Q_{\rho_i}^0(x) \). Therefore, by (2.14) and (2.16) we find
\[
\mathbf{m}^{P_R}_{E_{\rho_i}}(u_{x,\nu}, Q_{\rho_i}^0(x)) \leq \int_{J_{\omega_i} \cap Q_{\rho_i}^0(x)} g_{n_i}(z, \nu_{\tilde{w}_i}(z)) \, dH^1(z) + \beta \rho_i^2.
\]
This along with \((6.31)\) shows
\[
\limsup_{i \to \infty} \frac{1}{\rho_i} \mathcal{E}^{PR}_{\varepsilon_i}(u_{x,\nu}, Q_{\rho_i}^\nu(x)) \leq g_\infty(x, [u](x), \nu).
\]
By \((6.26)\), writing again \(\nu_a(x)\) in place of \(\nu\), we conclude \(g(x, \nu_a(x)) \leq g_\infty(x, [u](x), \nu_a(x))\). \(\square\)

6.3. **Surface part in Theorem 2.4. Proof of (2.22).** We proceed with the proof of \((2.22)\). Recall that \(g_n = \hat{g}\) for all \(n \in \mathbb{N}\), and therefore \(E_n\) takes the form
\[
E_n(u, A) = \int_A f_n(x, e(u)(x)) \, dx + \int_{J_n \cap A} h(x, u_n(x)) \, dH^{d-1}.
\]
We argue along the lines of the proof in Subsection 6.2 and replace the application of Lemma 3.2 by Proposition 3.16 and the lower semicontinuity result in Theorem 3.10.

**Step 1:** \(g_\infty(x, [u](x), \nu_a(x)) \leq \tilde{h}(x, \nu_a(x))\) for \(\mathcal{H}^{d-1}\)-a.e. \(x \in J_\nu\). We proceed exactly as in Step 1 in Subsection 6.2 with the only difference that in place of \((2.19)\) and Proposition 3.14 we use \((3.22)\).

**Step 2:** \(g_\infty(x, [u](x), \nu_a(x)) \geq \hat{h}(x, \nu_a(x))\) for \(\mathcal{H}^{d-1}\)-a.e. \(x \in J_\nu\). We first follow the lines of Step 2 in Subsection 6.2. We may suppose that \(x \in J_\nu\) satisfies
\[
g_\infty(x, [u](x), \nu_a(x)) = \lim_{\rho \to 0^+} \frac{1}{\rho^{d-1}} \mathcal{E}(u, Q_{\rho}^\nu(x)) < +\infty. \tag{6.32}
\]
By \((u_n)_n\) we denote a recovery sequence for \(\mathcal{E}(u, \Omega)\). In a similar fashion to \((6.24)-(6.25)\), we find sequences \((\rho_i)_i\) and \((n_i)_i\) with \(\rho_i \to 0\) and \(n_i \to +\infty\) as \(i \to \infty\) such that
\[
\frac{1}{\rho_i^{d-1}} \mathcal{E}(u, Q_{\rho_i}^\nu(x)) \geq \frac{1}{\rho_i^{d-1}} \mathcal{E}_{n_i}(u_{n_i}, Q_{\rho_i}^\nu(x)) - \frac{1}{i}, \tag{6.33}
\]
where we again use the shorthand \(\nu = \nu_a(x)\), and
\[
v_i \to u_{0, u^+(x), u^-(x), \nu} \quad \text{in measure on } Q_1^\nu, \tag{6.34}
\]
where \(v_i(y) := u_{n_i}(x + \rho_i y)\) for \(y \in Q_1^\nu\). By \((6.33)\) and a change of variables we get
\[
\frac{1}{\rho_i^{d-1}} \mathcal{E}(u, Q_{\rho_i}^\nu(x)) \geq \frac{1}{\rho_i^{d-1}} \int_{J_{n_i} \cap Q_1^\nu(x)} h(z, \nu_{u_{n_i}(z)}) \, d\mathcal{H}^{d-1}(z) - \frac{1}{i}
\]
\[
= \int_{J_{n_i} \cap Q_1^\nu(x)} h(x + \rho_i y, \nu_{v_i}(y)) \, d\mathcal{H}^{d-1}(y) - \frac{1}{i}.
\]
As in \((6.28)\), by using that \(h\) satisfies \((2.12)\), we have \(\sup_{i \in \mathbb{N}} \mathcal{H}^{d-1}(J_{n_i} \cap Q_1^\nu) < +\infty\). By continuity of the function \(h\), we thus find a sequence \((\eta_i)_i\) \(\subset (0, +\infty)\) with \(\eta_i \to 0\) such that
\[
\frac{1}{\rho_i^{d-1}} \mathcal{E}(u, Q_{\rho_i}^\nu(x)) \geq \int_{J_{n_i} \cap Q_1^\nu(x)} h(x, \nu_{v_i}(y)) \, d\mathcal{H}^{d-1}(y) - \eta_i. \tag{6.35}
\]
As \(\tilde{h}(x, \cdot) \leq h(x, \cdot)\) by \((3.21)\), we further get
\[
\frac{1}{\rho_i^{d-1}} \mathcal{E}(u, Q_{\rho_i}^\nu(x)) \geq \int_{J_{n_i} \cap Q_1^\nu(x)} \tilde{h}(x, \nu_{v_i}(y)) \, d\mathcal{H}^{d-1}(y) - \eta_i. \tag{6.36}
\]
Since \(v_i\) converges in measure to \(\tilde{u} := u_{0, u^+(x), u^-(x), \nu}\) on \(Q_1^\nu\) by \((6.34)\) and the density \(\tilde{h}(x, \cdot)\) is symmetric jointly convex (see Corollary 3.17, Theorem 3.10 along with \((6.36)\) and the fact that \(\eta_i \to 0\) yields
\[
\liminf_{i \to \infty} \frac{1}{\rho_i^{d-1}} \mathcal{E}(u, Q_{\rho_i}^\nu(x)) \geq \liminf_{i \to \infty} \int_{J_{n_i} \cap Q_1^\nu(x)} \tilde{h}(x, \nu_{v_i}(y)) \, d\mathcal{H}^{d-1}(y) \geq \int_{J \cap Q_1^\nu} \tilde{h}(x, \nu_{v}(y)) \, d\mathcal{H}^{d-1}(y).
\]
The definition of \( \hat{u} \) implies \( \int_{J_u \cap Q_1^0} \hat{h}(x, \nu_u(y)) \, dH^{d-1}(y) = \bar{h}(x, \nu) \). Now, by recalling the notation \( \nu = \nu_u(x) \) and by using (6.32) we conclude \( g_\infty(x, [u](x), \nu_u(x)) \geq \hat{h}(x, \nu_u(x)) \).

With the results of Subsections 6.1–6.3 Theorem 2.4 is now completely proved. We close this subsection with the proof of Remark 2.8.

**Proof of Remark 2.8.** First, as \( h \) is continuous on \( D \), identity (2.22) clearly holds for \( H^{d-1}\)-a.e. \( x \in J_u \cap D \). Since \( J_u \subset D \cap \Omega \), it remains to address the case \( x \in \partial D \cap \Omega \). As \( D \) has Lipschitz boundary, the outer normal \( \nu_D(x) \) exists for \( H^{d-1}\)-a.e. \( x \in \partial D \). Then, as \( J_u \subset \bar{D} \), we deduce that \( \nu_u(x) = \nu_D(x) =: \nu_x \) for \( H^{d-1}\)-a.e. \( x \in J_u \cap \partial D \), i.e., we need to show

\[
g_\infty(x, [u](x), \nu_x) = \hat{h}(x, \nu_x) \quad \text{for } H^{d-1}\text{-a.e. } x \in J_u \cap \partial D.
\]

As before, we split the proof into two inequalities.

**Step 1:** \( g_\infty(x, [u](x), \nu_x) \leq \hat{h}(x, \nu_x) \). We first set \( \hat{u}^x := u_{x, [u](x), 0, \nu_x} \) (recall the notation in (2.6)). In view of (2.4) and (2.14), we get \( m_x(\hat{u}^x, Q^{\nu_x}_\rho(x)) \leq m_x^{PR}(\hat{u}^x, Q^{\nu_x}_\rho(x)) \). Then (2.11) gives

\[
g_\infty(x, [u](x), \nu_x) = \limsup_{\rho \to 0^+} \frac{m_x(\hat{u}^x, Q^{\nu_x}_\rho(x))}{\rho^{d-1}} \leq \limsup_{\rho \to 0^+} \frac{m_x^{PR}(\hat{u}^x, Q^{\nu_x}_\rho(x))}{\rho^{d-1}}.
\]

At the end of the step we will show that, for given \( \varepsilon > 0 \), we can find \( \bar{x} \in D \) such that

\[
\begin{align}
(i) & \quad \limsup_{\rho \to 0^+} \frac{m_x^{PR}(\hat{u}^x, Q^{\nu_x}_\rho(x))}{\rho^{d-1}} - \liminf_{\rho \to 0^+} \frac{m_x^{PR}(\hat{u}^x, Q^{\nu_x}(\bar{x}))}{\rho^{d-1}} \leq C\varepsilon \alpha^{-1} \|h\|_\infty, \\
(ii) & \quad |h(x, \nu_x) - \hat{h}(\bar{x}, \nu_x)| \leq C\varepsilon \alpha^{-1} \|h\|_\infty
\end{align}
\]

for some universal \( C > 0 \). Then we conclude as follows: by (2.16), (3.16), (3.22), and the fact that \( \bar{x} \in D \) we have

\[
\begin{align}
&h(\bar{x}, \nu_x) = \limsup_{\rho \to 0^+} \frac{m_x^{PC}(\bar{u}^x, Q^{\nu_x}_\rho(\bar{x}))}{\rho^{d-1}} = \limsup_{\rho \to 0^+} \frac{m_x^{PR}(\bar{u}^x, Q^{\nu_x}(\bar{x}))}{\rho^{d-1}}. \\
&\text{By (6.37), (6.38) (i), and (6.39) we obtain } g_\infty(x, [u](x), \nu_x) \leq h(\bar{x}, \nu_x) + C\varepsilon \alpha^{-1} \|h\|_\infty. \text{ Then (6.38) (ii) yields } g_\infty(x, [u](x), \nu_x) \leq h(x, \nu_x) + 2C\varepsilon \alpha^{-1} \|h\|_\infty. \text{ Since } \varepsilon \text{ was arbitrary, we obtain the desired inequality.}
\end{align}
\]

To conclude this step, we need to show (6.38). We only prove (6.38) (i) as (6.38) (ii) follows along similar lines. For convenience, as in Proposition 3.10 we denote the surface integral with density \( h \) by \( S \). Let \( \varepsilon > 0 \) and suppose without restriction that \( 1/\varepsilon \in \mathbb{N} \). By uniform continuity of \( h \) in \( D \) we can choose \( \bar{x} \in D \) and \( \delta = \delta(\varepsilon) > 0 \) such that

\[
|h(y, \cdot) - h(\bar{y}, \cdot)|_\infty \leq \varepsilon \quad \text{for all } y \in Q^{\nu_x}_\rho(\bar{x}) \cap D, \ y \in Q^{\nu_x}_\rho(\bar{x}) \cap D.
\]

Pick \( 0 < \rho_0 \leq \delta \) sufficiently small such that \( Q^{\nu_x}_\rho(\bar{x}) \subset D \) and choose \( v \in PC(Q^{\nu_x}_\rho(\bar{x})) \) with \( v = \bar{u}_{\bar{x}, \nu_x} \) in a neighborhood of \( \partial Q^{\nu_x}_\rho(\bar{x}) \) such that

\[
S(v, Q^{\nu_x}_\rho(\bar{x})) \leq m_x^{PC}(\bar{u}_{\bar{x}, \nu_x}, Q^{\nu_x}_\rho(\bar{x})) + \rho_0^d.
\]

Let us observe that

\[
H^{d-1}(J_v) \leq \alpha^{-1} \|h\|_\infty \rho_0^{d-1}.
\]

Indeed, using \( \bar{u}_{\bar{x}, \nu_x} \) as a competitor, we get \( m_x^{PC}(\bar{u}_{\bar{x}, \nu_x}, Q^{\nu_x}_\rho(\bar{x})) \leq \|h\|_\infty \rho_0^{d-1} \). This along with the lower bound in (2.12) shows (6.42).
We now construct a competitor for \( m^{PC}_{S}(\bar{u}_{x,v_{x}}, Q^{\nu}_{\rho}(x)) \) for all \( \rho \leq \rho' \), where \( 0 < \rho' \leq \delta \) is chosen sufficiently small such that
\[
D \cap Q^{\nu}_{\rho}(x) \supset \{ y \in Q^{\nu}_{\rho}(x) : (y-x) \cdot \nu_{x} \leq -\varepsilon \rho \}.
\] (6.43)

Now, we define \( v^\rho \in PC(Q^{\nu}_{\rho}(x)) \) by \( v^\rho = \bar{u}_{x,v_{x}} \) on \( Q^{\nu}_{\rho}(x) \setminus Q^{\nu}_{\rho(1-\varepsilon)}(x) \) and on \( Q^{\nu}_{\rho(1-\varepsilon)}(x) \) we set
\[
v^\rho(y) = \begin{cases} 
\varepsilon & \text{if } (y-x) \cdot \nu_{x} > -\varepsilon \rho, \\
v(\bar{x} + \rho_0(\varepsilon \rho)^{-1}(y-x_n)) & \text{if } -2\varepsilon \rho < (y-x) \cdot \nu_{x} < -\varepsilon \rho \text{ and } y \in Q_n, \\
0 & \text{if } (y-x) \cdot \nu_{x} < -2\varepsilon \rho,
\end{cases}
\]
where \( (Q_n) \) denotes a partition of the set \( \{ y \in Q^{\nu}_{\rho(1-\varepsilon)}(x) : -2\varepsilon \rho < (y-x) \cdot \nu_{x} < -\varepsilon \rho \} \) consisting of \( \varepsilon^{-d} \) cubes with side length \( \varepsilon \rho \), and \( x_n \) indicates the center of \( Q_n \). Since \( v = \bar{u}_{x,v_{x}} \) in a neighborhood of \( \partial Q^{\nu}_{\rho_0}(\bar{x}) \), the functions \( v^\rho \) have \( \mathcal{H}^{d-1} \)-negligible jump on \( \partial Q_n \). Hence, we get
\[
S(v^\rho, Q^{\nu}_{\rho}(x)) \leq \sum_{n=1}^{v^{-d}} S(v^\rho, Q_n) + \| h \|_{\infty} \mathcal{H}^{d-1} (J_{v^\rho} \cap (Q^{\nu}_{\rho}(x) \setminus Q^{\nu}_{\rho(1-\varepsilon)}(x))).
\]

Then, due to \( J_{v^\rho} \cap Q^{\nu}_{\rho(1-\varepsilon)}(x) \subset D \) (see (6.43)), a scaling argument and (6.40) imply
\[
S(v^\rho, Q^{\nu}_{\rho}(x)) \leq \rho^{d-1} \rho_0^{-1} (d-1) S(v, Q^{\nu}_{\rho_0}(\bar{x})) + \varepsilon \mathcal{H}^{d-1}(J_{v}) + C \| h \|_{\infty} \varepsilon \rho^{d-1},
\]
where \( C > 0 \) is a universal constant. This along with (6.42) shows
\[
S(v^\rho, Q^{\nu}_{\rho}(x)) \leq \rho^{d-1} \rho_0^{-1} (d-1) S(v, Q^{\nu}_{\rho_0}(\bar{x})) + C \varepsilon \rho^{d-1}.
\]
By (6.41) and the fact that \( v^\rho = \bar{u}_{x,v_{x}} \) in a neighborhood of \( \partial Q^{\nu}_{\rho}(x) \) we conclude
\[
\limsup_{\rho \to 0^+} \frac{m^{PC}_{S}(\bar{u}_{x,v_{x}}, Q^{\nu}_{\rho}(x))}{\rho^{d-1}} \leq \limsup_{\rho \to 0^+} \frac{m^{PC}_{S}(\bar{u}_{x,v_{x}}, Q^{\nu}_{\rho}(\bar{x}))}{\rho^{d-1}} + C \varepsilon \rho^{d-1} \| h \|_{\infty}.
\]
On the other hand, (2.25) and (6.40) immediately yield the converse inequality
\[
\limsup_{\rho \to 0^+} \frac{m^{PC}_{S}(\bar{u}_{x,v_{x}}, Q^{\nu}_{\rho}(x))}{\rho^{d-1}} \geq \limsup_{\rho \to 0^+} \frac{m^{PC}_{S}(\bar{u}_{x,v_{x}}, Q^{\nu}_{\rho}(\bar{x}))}{\rho^{d-1}} - C \varepsilon \rho^{d-1} \| h \|_{\infty}.
\]

This along with (2.16) and (3.16) shows (6.38)(i). In view of the characterization (3.21), the proof of (6.38)(ii) can be obtained along similar lines. (Indeed, it is even easier since instead of (6.40) we only need \( \| h(x,\cdot) - h(\bar{x},\cdot) \|_{\infty} \leq \varepsilon \).

Step 2: \( g_{\infty}(x, (u)(x), (\nu_{x})) \geq \hat{h}(x, (\nu_{x})) \). We follow Step 2 of the previous proof by employing a slightly different continuity argument in (6.35). More precisely, by the uniform continuity of \( h \) in \( D \) and property (2.25), for given \( \varepsilon > 0 \), we can choose \( \bar{x} \in D \) such that (6.38)(ii) holds and (6.35) is replaced by
\[
\liminf_{i \to \infty} \frac{1}{\rho_{i}^{d-1}} E(u, Q^{\nu_{i}}(x)) \geq \liminf_{i \to \infty} \int_{J_{v_{i}} \cap Q^{\nu_{i}}(x)} \hat{h}(\bar{x}, (\nu_{x})(y)) d\mathcal{H}^{d-1}(y) - \varepsilon.
\]

Since \( \bar{x} \in D \), we can proceed as in the previous proof to find, \( g_{\infty}(x, (u)(x), (\nu_{x})) \geq \hat{h}(x, (\nu_{x})) - \varepsilon \). This along with (6.38)(ii) and the arbitrariness of \( \varepsilon \) shows \( g_{\infty}(x, (u)(x), (\nu_{x})) \geq \hat{h}(x, (\nu_{x})) \).

\[ \square \]

Remark 6.2. The proof shows that \( \hat{h} \) is uniformly continuous on \( D \) and that for \( \mathcal{H}^{d-1} \)-a.e. \( x \in \partial D \cap \Omega \) it holds that \( \hat{h}(x, (\nu_{D})(x)) = \lim_{n \to \infty} \hat{h}(x_{n}, (\nu_{n})) \) for sequences \( (x_{n})_{n} \subset D \) and \( (\nu_{n})_{n} \subset \mathcal{S}^{d-1} \) with \( x_{n} \to x \) and \( \nu_{n} \to (\nu_{D}) \).
6.4. Proof of Corollaries 2.5 and 2.6. We deduce the announced corollaries of Theorem 2.4

Proof of Corollary 2.5. We only need to show that the limits in (2.19) coincide with $f$ and $g$ in (2.18). Then the result follows from Theorem 2.4. The argument used to prove this property is standard and for this reason we omit it. For instance, we can follow closely the proof of [17, Theorem 3.11]. □

Proof of Corollary 2.6. Thanks to Theorem 2.4(iii), applied to the constant sequence of functionals $\mathcal{E}_n = \mathcal{E}$ given by (2.3) corresponding to $f$ and $g$, we get that

$$\tilde{\mathcal{E}}(u, A) = \int_A \tilde{f}(x, e(u)(x)) \, dx + \int_{J_u \cap A} \tilde{g}(x, \nu_u(x)) \, d\mathcal{H}^{d-1}(x)$$

for all $u \in GSBD^p(\Omega)$ and $A \in \mathcal{A}(\Omega)$, where $\tilde{f}$ is defined as in (2.18), and $\tilde{g}$ is the BD-elliptic envelope of $g$ defined in (2.23). By Remark 3.15 we get that $\tilde{f}$ coincides with the density $f_0$ in Proposition 3.13. Since in the case of a constant sequence the $\Gamma$-limit $F$ is simply the relaxation of the integral functional with density $f$, it turns out that $f_0 = \tilde{f}$ is the quasiconvex envelope (with respect to the second variable) of $f$, see [33, Theorem 9.8]. □

7. Minimization problems for given boundary data

This section is devoted to the proofs of the results announced in Subsection 2.3. Before coming to the proof of Proposition 2.9 we state an auxiliary lower semicontinuity result.

Lemma 7.1 (Lower semicontinuity). In the setting of Proposition 2.9, consider a sequence $(v_n)_n \subset GSBD^p(\Omega')$ with $\sup_{n \in \mathbb{N}}(\|e(v_n)\|_{L^p(\Omega')} + \mathcal{H}^{d-1}(J_{v_n})) < +\infty$ such that $v_n \to v$ in measure on $\Omega'$ for some $v \in GSBD^p(\Omega')$. Then, for all $A \in \mathcal{A}(\Omega')$ it holds that

\begin{align}
\tag{7.1}
& (i) \quad \int_A f'(x, e(v)(x)) \, dx \leq \liminf_{n \to \infty} \int_A f'_n(x, e(v_n)(x)) \, dx, \\
& (ii) \quad \int_{J_n \cap A} g' (x, \nu_n) \, d\mathcal{H}^{d-1} \leq \liminf_{n \to \infty} \int_{J_n \cap A} g'_n (x, \nu_n) \, d\mathcal{H}^{d-1}.
\end{align}

Proof. The statement follows by repeating the argument in [40, Proposition 4.3] which we detail here for the convenience of the reader. First, we notice that by Theorem 2.4(i) the bulk density $f'_\infty$ of $\mathcal{E}'$ agrees with the function $f'$ given by (2.18). Moreover, the surface density $g'_\infty$ agrees with $g'$ given in (2.19) by assumption in Proposition 2.9. As $f'$ and $g'$ are characterized by (2.18) and (2.19), respectively, we find that the energies $\mathcal{E}_n^k$ with densities $k f'_n$ and $l g'_n$ for $k, l \in \mathbb{N}$ converge to $\mathcal{E}^{k,l}$ with densities $k f'$ and $l g'$. Therefore, we obtain

$$\int_A k f'(x, e(v)(x)) \, dx + \int_{J_n \cap A} l g' (x, \nu_n) \, d\mathcal{H}^{d-1}$$

$$\leq \liminf_{n \to \infty} \left( \int_A k f'_n(x, e(v_n)(x)) \, dx + \int_{J_n \cap A} l g'_n (x, \nu_n) \, d\mathcal{H}^{d-1} \right).$$

In view of (2.1), (2.12), and $\sup_{n \in \mathbb{N}}(\|e(v_n)\|_{L^p(\Omega')} + \mathcal{H}^{d-1}(J_{v_n})) < +\infty$, by dividing the estimate by $k$ and sending $k \to +\infty$, we obtain (7.1)(i). In a similar fashion, (7.1)(ii) follows by dividing first by $l$ and by letting $l \to +\infty$. □

Proof of Proposition 2.9. We follow the lines of similar results in the $GSBV^p$ setting, see [46, Lemma 7.1] and [42, Lemma 4.3]. Our focus lies on the adaptations necessary to our $GSBD^p$
setting, including more delicate constructions for extensions and fundamental estimates. (In particular, besides Proposition 4.14, we use the recent extension result [16] and approximations from [21].) To keep the exposition self-contained, however, we provide all details of the proof.

We start by noticing that the \( \Gamma \)-liminf is immediate from the \( \Gamma \)-convergence of \( E_n' \) to \( E' \) and the fact that the boundary condition on \( \Omega' \setminus \overline{\Omega} \) is preserved under the convergence in measure. We now address the \( \Gamma \)-limsup. As \( E_n' \) \( \Gamma \)-converges to \( E' \), there exists a recovery sequence \( (u_n)_n \) for \( u \), i.e., \( u_n \to u \) in measure on \( \Omega' \) and \( \lim_{n \to \infty} E_n'(u_n) = E'(u) \). By Remark 6.1 we get that \( E_n'(u_n, A) \to E'(u, A) \) for each \( A \in A(\Omega') \) with \( E'(u, \partial A) = 0 \). This along with (7.1) shows

\[
\begin{align*}
(\text{i}) & \quad \int_A f'(x, e(u_n(x))) \, dx = \lim_{n \to \infty} \int_A f'_n(x, e(u_n(x))) \, dx, \\
(\text{ii}) & \quad \int_{J_n \cap A} g'(x, \nu_n) \, d\mathcal{H}^{d-1} = \lim_{n \to \infty} \int_{J_n \cap A} g'_n(x, \nu_{u_n}) \, d\mathcal{H}^{d-1},
\end{align*}
\]

whenever \( \mathcal{E}'(u, \partial A) = 0 \).

**Step 1: Definition of the recovery sequence.** We need to modify the sequence \( (u_n)_n \) to ensure that the boundary conditions are satisfied on \( \Omega' \setminus \overline{\Omega} \). This is subject of this step. In Step 2 we will estimate the energy of the modified sequence to see that it is indeed a recovery sequence. At the end of the proof in Step 3, we will check the following auxiliary properties:

\[
\begin{align*}
&\text{(i) } u_n - u_n^0 \to 0 \text{ in measure on } \Omega' \setminus \overline{\Omega}, \\
&\text{(ii) } e(u_n) - e(u_n^0) \to 0 \text{ strongly in } L^p(\Omega' \setminus \overline{\Omega}; \mathbb{R}^{d \times d}), \\
&\text{(iii) } J_n \cap (\Omega' \setminus \Omega) \to 0.
\end{align*}
\]

For the moment, we suppose that (7.3) holds. We can find a Lipschitz neighborhood \( U \supset \Omega' \setminus \overline{\Omega} \) and an extension \( (y_n)_n \subset GSBD^p(\Omega) \) satisfying \( y_n = u_n - u_n^0 \) on \( \Omega' \setminus \overline{\Omega} \) such that for \( n \to \infty \)

\[
\begin{align*}
&\text{(i) } \|e(y_n)\|_{L^p(U)} + \mathcal{H}^{d-1}(J_n \cap U) \to 0, \\
&\text{(ii) } y_n \to 0 \text{ in measure on } U.
\end{align*}
\]

Indeed, by the extension result [16, Theorem 1.1] we can choose \( (y_n)_n \subset GSBD^p(\Omega) \) with \( y_n|_{\Omega \setminus \Omega'} \in W^{1,p}(U \cap \Omega; \mathbb{R}^d) \) such that \( y_n = u_n - u_n^0 \) on \( \Omega' \setminus \overline{\Omega} \) and

\[
\|e(y_n)\|_{L^p(U)} + \mathcal{H}^{d-1}(J_n \cap U) \leq C\|e(u_n - u_n^0)\|_{L^p(U \setminus \Omega)} + C\mathcal{H}^{d-1}(J_n \cap (\Omega' \setminus \overline{\Omega})),
\]

where \( C > 0 \) depends only on \( \Omega, \Omega' \) and \( p \). Then, (7.4)(i) follows from (7.3)(ii),(iii). We now show (7.4)(ii). To this end, we use (7.4)(i) and apply Theorem 3.2 on \( (y_n)_n \) to find sets \( (\omega_n)_n \subset \subset \subset \Omega \) with \( \mathcal{L}^d(\omega_n) \to 0 \) and rigid motions \( (a_n)_n \subset \subset \subset \Omega \) such that \( \|y_n - a_n\|_{L^p(U \setminus \omega_n)} \to 0 \). By (7.3)(i) and \( y_n = u_n - u_n^0 \) on \( \Omega' \setminus \overline{\Omega} \) we get \( a_n \to 0 \) in measure on \( \Omega' \setminus \overline{\Omega} \). As \( a_n \) is affine, this also yields \( a_n \to 0 \) in measure on \( U \). This along with \( \|y_n - a_n\|_{L^p(U \setminus \omega_n)} \to 0 \) and \( \mathcal{L}^d(\omega_n) \to 0 \) shows (7.4)(ii).

Let \( \varepsilon > 0 \) and choose \( V \) open with \( V \supset \partial_D \Omega, V \subset U, \mathcal{E}'(u, \partial(V \cap \Omega')) = 0, \mathcal{L}^d(V) \leq \varepsilon, \) and \( \int_{V \cap \Omega'} f'(x, e(u(x))) \, dx < \varepsilon \). (Here, \( \partial_D \Omega = \Omega' \cap \partial \Omega \).) Then by (7.2)(ii) we also obtain

\[
\lim_{n \to \infty} \int_{V \cap \Omega'} f'_n(x, e(u_n(x))) \, dx < \varepsilon.
\]

Our goal is to apply the fundamental estimate. To this end, choose \( A \subset \mathbb{R}^d \) such that \( A \cap \Omega' \subset \Omega \) and \( A \supset \supset \supset \Omega \setminus V \), and choose \( A' \subset \subset \subset A \) such that \( \Omega \setminus V \subset A' \subset \subset \subset \Omega \) (see for instance Picture 1). Moreover, we let \( A = U \cap \Omega' \), and we note that \( A' \cup B = \Omega' \) since \( V \subset U \). Define the functional \( \mathcal{I} \) by \( \mathcal{I}(w, A) = \|e(w)\|_{L^p(\Omega')} + \mathcal{H}^{d-1}(J_n \cap A) \) for all \( w \in GSBD^p(\Omega') \) and \( A \in A(\Omega') \). We apply Proposition 4.1 and Remark 4.2(iii) for \( \eta > 0 \) and \( \mathcal{I}, \) for the functions \( u \equiv 0 \) and \( v = y_n \), to find a
sequence \((\varphi^n_n)_n \subset \text{GSBD}^p(\Omega')\) with \(\varphi^n_n = y_n\) on \(\Omega' \setminus \overline{B} \setminus A\) and \(\varphi^n_n = 0\) on \(\Omega \setminus V \subset A'\) (see \(4.2\)(iii)) such that by \(4.2\)(i) we have

\[
\mathcal{I}(\varphi^n_n, \Omega') \leq (1 + \eta) \left( \mathcal{I}(0, A) + \mathcal{I}(y_n, B) \right) + A(u, y_n) + \eta.
\]

As \(A(0, y_n) \to 0\) for \(n \to \infty\) by \(7.4\)(ii) and \(4.1\), we find \(\limsup_{n \to \infty} \mathcal{I}(\varphi^n_n, \Omega') \leq \eta\). By sending \(\eta \to 0\) we can choose a suitable diagonal sequence \((\varphi_n)_n\), still satisfying \(\varphi_n = y_n\) on \(\Omega' \setminus \overline{B}\) and \(\varphi_n = 0\) on \(\Omega \setminus V\) such that \(I(\varphi_n, \Omega') \to 0\). This in turn implies

\[
\|e(\varphi_n)\|_{L^p(\Omega')} + \mathcal{H}^{d-1}(J_{\varphi_n} \cap \Omega') \to 0.
\]

\[\text{(7.6)}\]

In a similar fashion, by \(4.2\)(ii) and \(7.4\)(ii) we find

\[
\varphi_n \rightarrow 0 \text{ in measure on } \Omega'.
\]

\[\text{(7.7)}\]

Now, we let \(\tilde{u}_n := u_n - \varphi_n\). Then \(\tilde{u}_n = u_n - y_n\) on \(\Omega' \setminus \overline{B}\), and therefore \(\tilde{u}_n = u^n_0\) on \(\Omega' \setminus \overline{B}\) as \(y_n = u_n - u^n_0\) on \(\Omega' \setminus \overline{B}\). Moreover, \(\tilde{u}_n = u_n\) on \(\Omega \setminus V\) as \(\varphi_n = 0\) on \(\Omega \setminus V\). We also observe that \(\tilde{u}_n \to u\) in measure on \(\Omega'\) by \(7.7\) and the fact that \(u_n \to u\) in measure on \(\Omega'\).

**Step 2: Estimate on the energy.** To conclude that \((\tilde{u}_n)_n\) is a recovery sequence, it remains to estimate the energy \(\hat{E}^0_n(\tilde{u}_n)\). As \(\mathcal{H}^{d-1}(J_{\varphi_n} \cap \Omega') \to 0\) by \(7.6\), we find by \(2.26\) that

\[
\limsup_{n \to \infty} \int_{J_u} g'_{n}(x, \nu_{u_n}) \, d\mathcal{H}^{d-1} \leq \limsup_{n \to \infty} \int_{J_{\tilde{u}_n}} g'_{n}(x, \nu_{u_n}) \, d\mathcal{H}^{d-1}.
\]

\[\text{(7.8)}\]

By \(2.26\) and \(\tilde{u}_n = u_n\) on \(\Omega \setminus V\) we further get

\[
\int_{\Omega} |f'_n(x, e(u_n)) - f'_n(x, e(\tilde{u}_n))| \, dx \leq \int_{\Omega \setminus V} (f_n(x, e(u_n)) + f_n(x, e(\tilde{u}_n))) + \alpha \int_{\Omega \setminus V} |e(u_n)|^p - |e(\tilde{u}_n)|^p.
\]

The rightmost term converges to zero for \(n \to \infty\) by \(7.3\)(ii). By \(2.1\), \(7.5\)–\(7.6\), \(\mathcal{L}^d(V) \leq \varepsilon\), and the definition \(\tilde{u}_n := u_n - \varphi_n\) we derive

\[
\limsup_{n \to \infty} \int_{\Omega} (f_n(x, e(u_n)) + f_n(x, e(\tilde{u}_n))) \, dx \leq \beta \mathcal{L}^d(V) + 2^{p-1} \beta \limsup_{n \to \infty} \int_{\Omega} |e(\varphi_n)|^p \, dx
\]

\[
+ (1 + 2^{p-1} \beta \alpha^{-1}) \limsup_{n \to \infty} \int_{\Omega} f_n(x, e(u_n)) \, dx
\]

\[
\leq \beta \varepsilon + (1 + 2^{p-1} \beta \alpha^{-1}) \varepsilon.
\]
Thus, by (7.3) and the fact that \( \tilde{\mathcal{E}}'_n(u_n) = \mathcal{E}'_n(u_n) \to \mathcal{E}'(u) = \tilde{\mathcal{E}}'(u) \), we then conclude
\[
\limsup_{n \to \infty} \tilde{\mathcal{E}}'_n(\tilde{u}_n) \leq \limsup_{n \to \infty} \mathcal{E}'_n(u_n) + \beta \varepsilon + (1 + 2^{p-1} \beta \alpha^{-1}) \varepsilon \leq \tilde{\mathcal{E}}'(u) + \beta \varepsilon + (1 + 2^{p-1} \beta \alpha^{-1}) \varepsilon.
\]

Since \( \varepsilon \) was arbitrary, we obtain the \( \Gamma \)-limsup inequality by using a diagonal argument.

**Step 3: Proof of (7.3).** To conclude, it remains to show (7.3). First, as \((u_n)_n\) is a recovery sequence, we find \( u_n \to u^0 \) in measure on \( \Omega' \setminus \overline{\Omega} \). This along with \( u^0_n \to u^0 \) in \( L^p(\Omega'; \mathbb{R}^d) \) shows (i). To see (ii), we consider \( A \in \mathcal{A}(\Omega') \), with \( \overline{A} \subset \Omega' \setminus \overline{\Omega} \) and \( \mathcal{E}'(u, \partial A) = 0 \). Then (2.26) and (7.2) yield
\[
e(u_n) \to e(u^0) \quad \text{in} \quad L^p(A; \mathbb{R}^{d \times d}).
\]

(9.7)

For \( \varepsilon > 0 \), we consider \( V \) open with \( V \supset \partial D \overline{\Omega} \) such that \( \mathcal{E}'(u, \partial(V \cap \Omega')) = 0 \), \( \mathcal{L}^d(V) < \varepsilon \), and
\[
\int_{V \cap \Omega'} f'(x, e(u_n))(x) \, dx < \varepsilon, \quad \int_{V \cap \Omega'} f'(x, e(u^0_n))(x) \, dx < \varepsilon \quad \text{for all } n \in \mathbb{N}.
\]

(7.10)

(The second estimate is achieved by (2.1) and the fact that \( e(u^0_n) \to e(u^0) \) strongly in \( L^p(\Omega'; \mathbb{R}^{d \times d}) \). For \( n \) large enough, we also get \( \int_{V \cap \Omega'} f'_n(x, e(u_n))(x) \, dx < \varepsilon \) by (7.2)(i). Then, by (2.1) we obtain
\[
\int_{\Omega' \setminus \overline{\Omega}} |e(u_n) - e(u^0_n)|^p \, dx = \int_{\Omega' \setminus (\Omega \cup V)} |e(u_n) - e(u^0_n)|^p \, dx + \int_{V \cap (\Omega' \setminus \overline{\Omega})} |e(u_n) - e(u^0_n)|^p \, dx
\]
\[
\leq \int_{\Omega' \setminus (\Omega \cup V)} |e(u_n) - e(u^0_n)|^p \, dx + \frac{2^{p-1} \alpha}{\alpha} \int_{V \cap \Omega'} (f'_n(x, e(u_n)) + f'(x, e(u^0_n))).
\]

By (9.7), (7.10), and the fact that \( \|e(u^0_n) - e(u^0)\|_{L^p(\Omega')} \to 0 \) we conclude
\[
\limsup_{n \to \infty} \int_{\Omega' \setminus \overline{\Omega}} |e(u_n) - e(u^0_n)|^p \, dx \leq 2^p \alpha^{-1} \varepsilon.
\]

Since \( \varepsilon \) was arbitrary, we obtain (ii). We finally prove (iii). Up to a subsequence we have
\[
\mu_n := \mathcal{H}^{d-1}|_{J_{u_n \cap (\Omega' \setminus \Omega)}} \rightharpoonup ^* \mu \quad \text{weakly}^* \text{ in the sense of measures.}
\]

By (2.12) and (7.2)(ii) we get \( \mathcal{H}^{d-1}(J_{u_n \cap U}) \to 0 \) for all \( U \in \mathcal{A}(\Omega') \) with \( \overline{U} \subset \Omega' \setminus \overline{\Omega} \) and \( \mathcal{E}'(u, \partial U) = 0 \). Consequently, to prove the conclusion of (iii), it suffices to show \( \mu(\partial_D \Omega) = 0 \). We argue by contradiction. Let us assume that \( \mu(\partial_D \Omega) > 0 \). Then there exists a cube \( Q_{2p}(x) \) with \( x \in \partial_D \Omega \) such that \( Q_{2p}(x) \subset \Omega' \), \( \mathcal{E}'(u, \partial Q_{2p}(x)) = 0 \), and \( \mu(Q_{2p}(x)) > \sigma > 0 \). For notational simplicity, we write \( Q'' = Q_{2p}(x) \) and also let \( \tilde{Q}'' = Q_{8p}(x) \). We may also assume that \( \tilde{Q}'' \subset \Omega' \). For \( n \) large enough we get
\[
\mathcal{H}^{d-1}(J_{u_n \cap (Q'' \setminus \Omega)}) = \mu_n(Q'') > \sigma > 0.
\]

(7.11)

Our goal is now to modify the sequence \((u_n)_n\) by a reflection method and to move the jump set inside \( \Omega \). This will lead to a contradiction as we assumed that \((u_n)_n\) is a recovery sequence, but inside \( \Omega \) the surface energy is much less than in \( \Omega' \setminus \Omega \), cf. (2.27). The reflection method is a bit more delicate compared to [10,12] since we deal with \( GSBD^p \) instead of \( GSBV^p \). Possibly after passing to a smaller \( \rho \) (not relabeled), we can assume that in a suitable coordinate system
\[
\Omega \cap \tilde{Q}'' = \{(x', y) : x' \in (-4\rho, 4\rho)^{d-1}, \ y \in (-4\rho, \tau(x'))\}
\]
for a Lipschitz function \( \tau \) with \( \|\tau\|_{\infty} \leq \rho \). We choose \( \eta \in (2\rho, 3\rho) \) such that
\[
V_{\rho} := \{(x', y) : x' \in (-\rho, \rho)^{d-1}, \ y \in (\tau(x') - \eta, \tau(x') + \eta)\}
\]
satisfies $E'(u, \partial V) = 0$. Note that $Q^\rho \subset V_\rho \subset \tilde{Q}^\rho \subset \Omega'$ since $\eta \in (2\rho, 3\rho)$. Let $\hat{u}$ be the function defined on $V_\rho$ by reflecting $u$ at $\tau(x')$, $x' \in (-\rho, \rho)^d - 1$, i.e.,

$$\hat{u}(x', y) = \begin{cases} 
  u(x', y) & \text{if } y > \tau(x'), \\
  u(x', 2\tau(x') - y) & \text{if } y < \tau(x'). 
\end{cases}$$

Clearly $\hat{u} \in W^{1,p}(V_\rho; \mathbb{R}^d)$ as $u \in W^{1,p}(\Omega' \setminus \overline{\Omega}; \mathbb{R}^d)$. In a similar fashion, we need to reflect the sequence $(u_n)_n$. As a preliminary step, we apply Theorem [3.6] for $\Omega = \tilde{Q}^\rho$ to obtain a sequence $(v_n)_n \subset GSBV^p(\tilde{Q}^\rho; \mathbb{R}^d)$ such that

$$v_n - u_n \to 0 \quad \text{in measure on } \tilde{Q}^\rho \text{ for } n \to \infty, \quad \mathcal{H}^{d-1}( (J_{u_n} \cap J_{v_n}) \cap \tilde{Q}^\rho ) \leq \frac{1}{n} \text{ for all } n \in \mathbb{N}.$$  

(7.12)

We define $\hat{u}_n \in GSBV^p(V_\rho; \mathbb{R}^d)$ by

$$\hat{u}_n(x', y) = \begin{cases} 
  v_n(x', y) & \text{if } y > \tau(x') - \lambda_n, \\
  v_n(x', 2(\tau(x') - \lambda_n) - y) & \text{if } y < \tau(x') - \lambda_n, 
\end{cases}$$

where $0 < \lambda_n \leq 1/n$ is chosen such that

$$\mathcal{H}^{d-1}( \{ (x', y) \in J_{v_n} : x' \in (-\rho, \rho)^d - 1, \, y \in (\tau(x') - \lambda_n, \tau(x')) \} ) \leq \frac{1}{n}.$$  

(7.13)

Observe that the functions are well defined since $\|\tau\|_\infty \leq \rho$ and $\eta < 3\rho$. We now introduce the sequence

$$w_n := v_n + \hat{u} - \hat{u}_n \in GSBV^p(V_\rho; \mathbb{R}^d).$$

By (7.12), $\lambda_n \to 0$, and the fact that $u_n \to u$ in measure on $\Omega'$, we get that $w_n \to u$ in measure on $V_\rho$. By letting $\Gamma_n := J_{w_n} \cap J_{v_n}$, we further find

(i) $\mathcal{H}^{d-1}( J_{w_n} \cap (V_\rho \setminus \Omega) ) = 0,$

(ii) $\mathcal{H}^{d-1}( J_{w_n} \setminus \Gamma_n ) \leq \mathcal{H}^{d-1}( \{ (x', y) \in V_\rho \setminus J_{v_n} : y > \tau(x') - \lambda_n \} ).$  

(7.14)

Here, the essential point is that the jump of $w_n$ lies completely inside $\Omega$. By (2.12) and (7.14(i)) we now get

$$G(w_n) := \int_{J_{w_n} \cap V_\rho} g_n(x, \nu_{v_n}) \, d\mathcal{H}^{d-1} \leq \int_{J_{w_n} \cap \Gamma_n} g_n(x, \nu_{v_n}) \, d\mathcal{H}^{d-1} + \beta \mathcal{H}^{d-1}( J_{w_n} \setminus \Gamma_n ).$$

Then by (2.12), (7.12), (7.13), and (7.14(ii)) we derive

$$G(w_n) \leq \int_{J_{v_n} \cap \Gamma_n} g_n'(x, \nu_{v_n}) \, d\mathcal{H}^{d-1} + \beta \mathcal{H}^{d-1}( J_{v_n} \cap (V_\rho \setminus \Omega) ) + \beta/n + \beta \mathcal{H}^{d-1}( J_{w_n} \setminus (V_\rho \setminus \Omega) ) + 3\beta/n.$$  

In the second step, we also used $\Gamma_n \subset V_\rho \cap \Omega$ by (7.14(i)). Now, by (2.27), (7.11), and $Q^\rho \subset V_\rho$ we get

$$G(w_n) \leq \int_{J_{w_n} \cap (V_\rho \setminus \Omega)} g_n'(x, \nu_{u_n}) \, d\mathcal{H}^{d-1} + 3\beta/n + (\beta + 1) \mathcal{H}^{d-1}( J_{u_n} \cap (V_\rho \setminus \Omega) ) - \sigma$$

$$\leq \int_{J_{u_n} \cap V_\rho} g_n'(x, \nu_{u_n}) \, d\mathcal{H}^{d-1} + 3\beta/n - \sigma.$$  

(7.15)
On the other hand, recalling that $w_n \to u$ in measure on $V_\rho$, we get by (7.1)(ii)
\[
\int_{J_n \cap V_\rho} g'_x(x, \nu_x) \, d\mathcal{H}^{-1} \leq \liminf_{n \to \infty} \int_{J_{w_n} \cap V_\rho} g'_x(x, \nu_{w_n}) \, d\mathcal{H}^{-1} = \liminf_{n \to \infty} G(w_n).
\]
Moreover, as $(u_n)_n$ is a recovery sequence for $u$ and $\mathcal{E}'(u, \partial V_\rho) = 0$, (7.2)(ii) yields
\[
\int_{J_n \cap V_\rho} g'_x(x, \nu_x) \, d\mathcal{H}^{-1} = \lim_{n \to \infty} \int_{J_{w_n} \cap V_\rho} g'_x(x, \nu_{w_n}) \, d\mathcal{H}^{-1}.
\]

The previous two equations contradict (7.15). This concludes the proof of (iii).

**Proof of Theorem 2.16** The statement follows in the spirit of the fundamental theorem of $\Gamma$-convergence, see [14, Theorem 1.21]. As we employ nonstandard compactness results, we indicate the details for both cases (i) and (ii).

(i) Given $(u_n)_n \subset GSBD^2(\Omega')$ satisfying (2.29), we apply Theorem 3.8 on the functionals $(\mathcal{E}'_n)_n$ and find a subsequence (not relabeled), $(y_n)_n \subset GSBD^2(\Omega')$ with $\mathcal{L}^d(\{e(y_n) \neq e(u_n)\}) \to 0$ and
\[
\liminf_{n \to \infty} \mathcal{E}'_n(y_n) = \liminf_{n \to \infty} \mathcal{E}'_n(u_n) \leq \liminf_{n \to \infty} \mathcal{E}'_n(u_n) = \liminf_{n \to \infty} \mathcal{E}'_n(u_n) = \liminf_{n \to \infty} \inf_{v \in GSBD^2(\Omega')} \mathcal{E}'_n(v).
\]

Here, the first equality holds as $y_n = u_n^0$ on $\Omega \setminus \overline{\Omega}$, and the first inequality follows from (3.8). By Theorem 3.8 we also find $u \in GSBD^2(\Omega')$ with $u = u^0$ on $\Omega \setminus \overline{\Omega}$ such that $y_n \to u$ in measure on $\Omega'$. Therefore, by the $\Gamma$-liminf inequality in Lemma 2.9 we derive
\[
\mathcal{E}'(u) \leq \liminf_{n \to \infty} \mathcal{E}'_n(y_n) \leq \liminf_{n \to \infty} \mathcal{E}'_n(u_n) \leq \liminf_{n \to \infty} \inf_{v \in GSBD^2(\Omega')} \mathcal{E}'_n(v). \tag{7.16}
\]

By using Lemma 2.9 once more, for each $w \in GSBD^2(\Omega')$ with $w = u^0$ on $\Omega \setminus \overline{\Omega}$ we find a recovery sequence $(w_n)_n$ converging to $w$ in measure satisfying $\lim_{n \to \infty} \mathcal{E}'_n(w_n) = \mathcal{E}'(w)$. This yields
\[
\limsup_{n \to \infty} \inf_{v \in GSBD^2(\Omega')} \mathcal{E}'_n(v) \leq \lim_{n \to \infty} \mathcal{E}'_n(w_n) = \mathcal{E}'(w). \tag{7.17}
\]

By combining (7.16) and (7.17) we derive
\[
\mathcal{E}'(u) \leq \liminf_{n \to \infty} \inf_{v \in GSBD^2(\Omega')} \mathcal{E}'_n(v) \leq \limsup_{n \to \infty} \inf_{v \in GSBD^2(\Omega')} \mathcal{E}'(v) \leq \mathcal{E}'(w). \tag{7.18}
\]

Since $w$ was arbitrary, we get that $u$ is a minimizer of $\mathcal{E}'$. The statement follows from (7.16) and (7.18) with $w = u$. In particular, the limit in (2.28) does not depend on the specific choice of the subsequence and thus (2.28) holds for the whole sequence.

(ii) Given $(u_n)_n \subset GSBD^p(\Omega')$ satisfying (2.29), we apply Theorem 3.8 on the functionals $(\mathcal{E}'_n)_n$ and find a subsequence (not relabeled), $(y_n)_n \subset GSBD^p(\Omega')$ with $\mathcal{L}^d(\{e(y_n) \neq e(u_n)\}) = 0$ (see (3.9)) and $u \in GSBD^p(\Omega')$ with $u = u^0$ on $\Omega' \setminus \overline{\Omega}$ such that $y_n \to u$ in measure on $\Omega'$ and $e(y_n) \to e(u)$ weakly in $L^p(\Omega'; \mathbb{R}^{d \times d})$. Now, by (7.1)(i) and the fact that $e(y_n) = e(u_n)$ $\mathcal{L}^d$-a.e. in $\Omega'$ we find
\[
\int_{\Omega'} f'(x, e(u)(x)) \, dx \leq \liminf_{n \to \infty} \int_{\Omega'} f'(x, e(y_n)(x)) \, dx = \liminf_{n \to \infty} \int_{\Omega'} f'(x, e(u_n)(x)) \, dx. \tag{7.19}
\]

Denote the extension of $\hat{g}$ defined in (2.27) by $\hat{g}'$. The surface density $g'$ of the $\Gamma$-limit $\mathcal{E}'$ satisfies $g' \leq \hat{g}'$. Moreover, as $\hat{g}$ is continuous on $\Omega \times \mathbb{S}^{d-1}$, we get that $g'(x_0, \cdot)$ is even and symmetric jointly convex for all $x_0 \in \Omega$ by Corollary 3.17. Note that $g'$ satisfies the properties stated in Remark 6.2 (with $\Omega'$ in place of $\Omega$ and $D = \Omega$). Therefore, we can apply Lemma 3.12 on $g'$. Hence, we obtain
\[
\int_{J_u} g'(x, \nu_x) \, d\mathcal{H}^{-1} \leq \liminf_{n \to \infty} \int_{J_{u_n}} g'(x, \nu_{u_n}) \, d\mathcal{H}^{-1} \leq \liminf_{n \to \infty} \int_{J_{u_n}} \hat{g}'(x, \nu_{u_n}) \, d\mathcal{H}^{-1}. \tag{7.20}
\]
By combining (7.19)–(7.20) we find \( \mathcal{E}'(u) \leq \liminf_{n \to \infty} \mathcal{E}'(u_n) \). Clearly, as \( u = u^0 \) on \( \Omega' \setminus \overline{\Omega} \), this also yields \( \mathcal{E}'(u) \leq \liminf_{n \to \infty} \mathcal{E}'(u_n) \). Now we can proceed as in (i) below (7.16) to conclude the proof (with the only difference that we do not get \( \lim_{n \to \infty} \mathcal{E}'(y_n) = \mathcal{E}'(u) \)). □

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**Appendix A. Proof of Lemma 5.2 under simplifying assumption**

*Proof.* Here, we present a simplified proof where we assume that each \( J_{u_n} \) consists of a bounded number of closed, continuous curves, denoted by \( \gamma_1^n, \ldots, \gamma_{G_n}^n \), where \( \sup_{n \in \mathbb{N}} G_n < +\infty \). To simplify notation, we suppose that \( \nu = e_2 \). This is not restrictive since we can first prove the statement for the functions \( u_n(x) := R^T u_n(Rx) \) for \( x \in Q_1^2 = Q_1 \), where \( R \) is a rotation with \( Re_2 = \nu \), and then rotate back the obtained partition onto \( Q_1^2 \). In view of (5.9)(ii), we define

\[
C_0 := \max_{n \in \mathbb{N}} G_n + \sup_{n \in \mathbb{N}} \mathcal{H}^1(J_{u_n}) < +\infty.
\]

(A.1)

For \( k \in \mathbb{N} \) fixed, we define

\[
U_k = Q_1 \cap \{ |x \cdot e_2| < \frac{1}{2}k^{-1} \}, \quad V_k = Q_1 \cap \{ |x \cdot e_2| \leq k^{-1/4} \}.
\]

(A.2)

For each \( k \in \mathbb{N} \), our strategy will lie in combining different components of the jump set \( J_{u_n} \) inside \( V_k \). Then, we will obtain the sets \( S_n^+ \) and \( S_n^- \) in the statement by a diagonal argument. We introduce some further notation: we cover \( U_k \) up to a set of negligible \( L^2 \)-measure by \( k \) pairwise disjoint cubes of sidelength \( 1/k \), denoted by \( Q_1^k, \ldots, Q_k^k \) with corresponding centers \( x_1^k, \ldots, x_k^k \).

We will first show that each of these cubes necessarily intersects the jump set \( J_{u_n} \) (Step 1). Based on this, we will combine different connected components of the jump set \( J_{u_n} \) with small segments of length at most \( \sqrt{5}/k \) (Step 2). This will eventually allow us to define the sets \( S_n^+ \) and \( S_n^- \) satisfying (Step 3).

**Step 1:** \( J_{u_n} \cap Q_j^k \neq \emptyset \) for all \( j = 1, \ldots, k \) and each \( n \geq n_k \), where \( n_k \in \mathbb{N} \) depends on \( k \). We suppose by contradiction that the statement were wrong, i.e., we find \( Q_j^k \) with \( J_{u_n} \cap Q_j^k = \emptyset \) for some \( n \geq n_k \), where \( n_k \in \mathbb{N} \) depending on \( k \) is specified below (see (A.5) and (A.8)). First, we observe that then \( u_n \) would be a Sobolev function when restricted to \( Q_j^k \). Thus, the Korn-Poincaré inequality implies

\[
\|u_n - a_n\|_{L^1(Q_j^k)} \leq C_k \|e(u_n)\|_{L^p(Q_j^k)} \leq C_k \|e(u_n)\|_{L^p(Q_1^2)}
\]

for a constant \( C_k \) only depending on \( k \), where \( a_n \) defined by \( a_n(x) = A_n x + b_n \) for \( x \in \mathbb{R}^2 \) is a suitable rigid motion. As \( u_n \to u_{0,\xi,0,e_2} \) in measure on \( Q_1 \), see (5.9)(iii), we find a sequence \( (\eta_n)_n \subset (0, +\infty) \) with \( \eta_n \to 0 \) and \( n_k \in \mathbb{N} \) sufficiently large such that the sets (recall (5.8))

\[
B_n^- = \{ x \in Q_1^{1-2} : |u_n(x)| < \eta_n \}, \quad B_n^+ = \{ x \in Q_1^{1+2} : |u_n(x) - \xi| < \eta_n \}
\]

(A.4)

fulfill for each \( n \geq n_k \) that

\[
L^2(Q_1^{1\pm} \setminus B_n^\pm) \leq \frac{1}{4}k^{-2}.
\]

(A.5)
As $L^2(Q_{1}^{\pm}\cap Q_{j}^{\pm}) = \frac{1}{2}k^{-2}$, this implies

$$L^2(B_{n}^{\pm}\cap Q_{j}^{\pm}) \geq \frac{1}{4}k^{-2}. \tag{A.6}$$

Now, we define $b_{n}^{-} = b_{n}$ and $b_{n}^{+} = b_{n} - \zeta$. By Lemma 3.5 for $\psi(t) = t$, $R = \sqrt{2}/2$, $\delta = 1/4$, $G = A_{n}/k$, $b = b_{n}^{\pm} + A_{n}x_{j}^{k}$, and $E = k((B_{n}^{\pm}\cap Q_{j}^{\pm}) - x_{j}^{k})$, and by a change of variables we get

$$|A_{n}|/k + |b_{n}^{\pm} + A_{n}x_{j}^{k}| \leq c \int_{E} |G \cdot x + b| \, dx = \frac{c}{L^{2}(B_{n}^{\pm}\cap Q_{j}^{\pm})} \int_{B_{n}^{\pm}\cap Q_{j}^{\pm}} |A_{n}x + b_{n}^{\pm}| \, dx, \tag{A.7}$$

where $c > 0$ is a universal constant. (Here, we used that $L^{2}(E) \geq 1/4$.) As $b_{n}^{+} - b_{n} = -\zeta$ and $b_{n}^{-} - b_{n} = 0$, we derive by (A.3) and (A.4) that

$$\|A_{n}x + b_{n}^{\pm}\|_{L^{1}(B_{n}^{\pm}\cap Q_{j}^{\pm})} = \|a_{n} + b_{n}^{\pm} - b_{n}\|_{L^{1}(B_{n}^{\pm}\cap Q_{j}^{\pm})} \leq \|a_{n} - u_{n}\|_{L^{1}(Q_{j}^{\pm})} + \|u_{n} + b_{n}^{\pm} - b_{n}\|_{L^{1}(B_{n}^{\pm}\cap Q_{j}^{\pm})} \leq Ck\|e(u_{n})\|_{L^{p}(Q_{j}^{\pm})} + \eta_{n}L^{2}(B_{n}^{\pm}\cap Q_{j}^{\pm}).$$

This along with (A.6)–(A.7) yields $|b_{n}^{+} + A_{n}x_{j}^{k}| \leq 4ek^{2}Ck\|e(u_{n})\|_{L^{p}(Q_{j}^{\pm})} + \eta_{n}. Therefore, by (5.9) (i) and $\eta_{n} \to 0$ we find

$$|b_{n}^{+} + A_{n}x_{j}^{k}| \leq \zeta/3 \quad \text{for all } n \geq n_{k}. \tag{A.8}$$

where $n_{k} \in N$ depends on $k$. Now, however, (A.8) contradicts the fact that $|b_{n}^{+} - b_{n}^{-}| = |\zeta|$. This shows that $Q_{j}^{\pm}$ necessarily intersects $J_{n_{k}}$, and the first step of the proof is concluded.

Step 2: Combining components of the jump set with segments. Recall the definition of $U_{k}$ and $V_{k}$ in (A.2). Let $k \in N$ and $n \geq n_{k}$. The goal of this step is to construct a closed set $\Gamma_{n_{k}}^{k} \subset V_{k}$ such that $\Gamma_{n_{k}}^{k}$ contains the points $(-\frac{1}{2}, 0)$ and $(\frac{1}{2}, 0)$, is path connected, and satisfies

$$H_{1}(\Gamma_{n_{k}}^{k} \setminus J_{n_{k}}) \leq c/k^{1/2}, \tag{A.9}$$

where $c$ depends only on $C_{0}$ in (A.1).

Let us construct $\Gamma_{n_{k}}^{k}$. We denote by $\tau_{1}^{n}, \ldots, \tau_{n_{k}}^{n}$ the connected components of $J_{n_{k}} \cap V_{k}$, which intersect $U_{k}$. Note that each of these components is a closed, continuous curve, and that we have

$$J_{n_{k}} \cap U_{k} = \bigcup_{j=1}^{n_{k}} \tau_{j}^{n} \cap U_{k}. \tag{A.10}$$

We now estimate the number $T_{n}$ of these connected components. To this end, we decompose the components of the original jump set $J_{n_{k}}$ into the index sets

$$T_{n}^{\text{small}} = \{j = 1, \ldots, G_{n} : H_{1}(\tau_{j}^{n}) \leq \frac{1}{2}k^{-1/4}\} \quad \text{and} \quad T_{n}^{\text{large}} = \{1, \ldots, G_{n}\} \setminus T_{n}^{\text{small}}.$$

Then, as each component in $(\tau_{j}^{n})_{j=1}^{T_{n}}$ intersects $U_{k}$ and each curve in $T_{n}^{\text{large}}$ is split into different curves of $(\tau_{j}^{n})_{j=1}^{T_{n}}$, we obtain by (A.1)

$$T_{n} \leq \#T_{n}^{\text{small}} + \sum_{j \in T_{n}^{\text{large}}} \frac{H_{1}(\tau_{j}^{n})}{\text{dist}(U_{k}, \partial V_{k} \cap Q_{1})} \leq G_{n} + 2k^{1/4} \sum_{j=1}^{G_{n}} H_{1}(\tau_{j}^{n}) \leq 2C_{0}k^{1/4}. \tag{A.11}$$

Here, in the second step we used that $\text{dist}(U_{k}, \partial V_{k} \cap Q_{1}) = k^{-1/4} - \frac{1}{2}k^{-1} \geq \frac{1}{2}k^{-1/4}$.

We now add two additional elements to the family of curves, namely the segments

$$\tau_{T_{n} + 1}^{n} = [-\frac{1}{2}, \frac{1}{2} + 1/k] \times \{0\} \quad \text{and} \quad \tau_{T_{n} + 2}^{n} = [\frac{1}{2} - 1/k, \frac{1}{2}] \times \{0\}. \tag{A.12}$$
We connect the different components \((\tau^n_j)_{j=1}^{n+2}\) with segments: for each pair \((\tau^n_{j_1}, \tau^n_{j_2}), j_1 \neq j_2\), with \(\text{dist}(\tau^n_{j_1}, \tau^n_{j_2}) \leq \sqrt{5}/k\), we choose a closed segment of length at most \(\sqrt{5}/k\) contained in \(V_k\) which connects \(\tau^n_{j_1}\) with \(\tau^n_{j_2}\). Denote the union of the components \((\tau^n_j)_{j=1}^{n+2}\) with these segments by \(\Gamma_n^k\).

We show that \(\Gamma_n^k\) has the desired properties. First, \(\Gamma_n^k \subset V_k\) by definition. By construction, \(\Gamma_n^k\) contains the points \((-\frac{1}{2}, 0)\) and \((\frac{1}{2}, 0)\), see (A.12). To see that \(\Gamma_n^k\) is path connected, we first note that \(\Gamma_n^k\) intersects each cube \(Q^j_k\), \(j = 1, \ldots, k\), by Step 1 and by (A.10). Then, each component in \((\tau^n_j)_{j=1}^{n+2}\) intersecting a cube \(Q^j_k\) is connecting to all components intersecting the cube \(Q^j_k\) or the adjacent cubes \(Q_{k-1}^j\) and \(Q_{k+1}^j\) (if existent), for the maximal distance of two points in adjacent cubes is \(\sqrt{5}/k\). This shows that \(\Gamma_n^k\) is path connected. Finally, since \(\bigcup_{j=1}^{n} \tau^n_j \subset J_{u_n}\), we get by (A.11) that

\[
H^1(\Gamma_n^k \setminus (J_{u_n} \cup \tau^n_{n+1} \cup \tau^n_{n+2})) \leq \sqrt{5}k^{-1}(T_n + 2)(T_n + 1)/2 \leq c\kappa^{-1}(C_0 k^{-1/4})^2 \leq c k^{-1/2},
\]

where \(c\) depends on \(C_0\). As \(H^1(\tau^n_{T_{n+j}}) \leq k^{-1}\) for \(j = 1, 2\), this shows (A.9) and concludes Step 2 of the proof.

**Step 3: Definition of \(S^+_n\) and \(S^-_n\).** We now define the sets \(S^+_n\) and \(S^-_n\) and establish (5.10). For each \(n \in \mathbb{N}\) and each \(n \geq n_k\), we denote the (possibly countably many) connected components of the open set \(Q_1 \setminus \Gamma_n^k\) by \((P^k_n)^J_{j=1}, J \in \mathbb{N} \cup \{+\infty\}\). As \(\Gamma_n^k\) is path connected and contained in \(V_k\), as well as \((-\frac{1}{2}, 0), (\frac{1}{2}, 0) \in \Gamma_n^k\), we observe that each set \(P^k_n\) intersects at most one of the sets \(Q_{1,2}^+ \setminus V_k\) and \(Q_{1,2}^- \setminus V_k\). We now define \(\hat{S}^-_{n,k}\) as the union of components \((P^k_{n,j})^J_{j=1}\) which do not intersect \(Q_{1,2}^+ \setminus V_k\). We also let \(\hat{S}^+_{n,k} := (Q_1 \setminus \Gamma_n^k) \setminus \hat{S}^-_{n,k}\), and note that \(\hat{S}^+_{n,k}\) does not intersect \(Q_{1,2}^- \setminus V_k\). We define the neighborhoods \(N_k = Q_1 \setminus Q_{1-k}^-\) and observe that the sets \(\hat{S}^-_{n,k}\) will possibly not satisfy (5.10)(ii). Therefore, we introduce the sets

\[
\begin{align*}
S^+_n &= (\hat{S}^+_{n,k} \cup (N_k \cap Q_{1,2}^+ \cap V_k)) \setminus (N_k \cap Q_{1,2}^- \cap V_k), \\
S^-_n &= (\hat{S}^-_{n,k} \cup (N_k \cap Q_{1,2}^- \cap V_k)) \setminus (N_k \cap Q_{1,2}^+ \cap V_k). 
\end{align*}
\]

(A.13)

Clearly, by definition we have \(L^2(\hat{S}_{n,k} \setminus (S^+_n \cup S^-_n)) = 0\) for all \(k \in \mathbb{N}\) and \(n \geq n_k\), and

\[
Q_{1,2}^+ \setminus V_k \subset S^+_n \subset Q_{1,2}^+ \cup V_k.
\]

(A.14)

We can now check that

\[
\begin{align*}
(i) & \quad L^2(S^+_n \cup Q_{1,2}^+) \leq 2k^{-1/4}, \\
(ii) & \quad N_k \cap Q_{1,2}^+ \subset S^+_n, \\
(iii) & \quad H^1((\partial S^+_n \cap \partial S^-_n) \setminus J_{u_n}) \leq c k^{-1/4}, 
\end{align*}
\]

(A.15)

for \(c > 0\) depending on \(C_0\). Indeed, (i) is a consequence of (A.2) and (A.14). By (A.13) and (A.14) we get (ii). Finally, we show (iii). First, (A.13) and the definition of \(N_k\) imply

\[
H^1((\partial S^+_n \cap \partial S^-_n) \setminus J_{u_n}) \leq H^1((\partial S^+_n \cap \partial S^-_n) \setminus J_{u_n}) + \sum_{j=\pm} H^1(\partial (N_k \cap Q_{1,2}^j \cap V_k)).
\]

Then, as \(\partial S^+_n \cap \partial S^-_n \subset \Gamma_n^k\) and \(H^1(\partial (N_k \cap Q_{1,2}^j \cap V_k)) \leq c k^{-1/4}\), (iii) follows from (A.9).

Finally, we obtain the desired sets \(S^+_n\) and \(S^-_n\) satisfying (5.10) from (A.15) by a suitable diagonal argument. This concludes the proof.
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