TURBULENT CONVECTION MODEL IN THE OVERSHOOTING REGION. II. THEORETICAL ANALYSIS

Q. S. Zhang1,2,3 and Y. Li1,2

1 National Astronomical Observatories/Yunnan Observatory, Chinese Academy of Sciences, P.O. Box 110, Kunming 650011, China; zqs@ynao.ac.cn, ly@ynao.ac.cn
2 Laboratory for the Structure and Evolution of Celestial Objects, Chinese Academy of Sciences, Kunming 650011, China
3 Graduate School of Chinese Academy of Sciences, Beijing 100039, China

Received 2011 February 23; accepted 2012 February 19; published 2012 April 10

ABSTRACT

Turbulent convection models (TCMs) are thought to be good tools to deal with the convective overshooting in the stellar interior. However, they are too complex to be applied to calculations of stellar structure and evolution. In order to understand the physical processes of the convective overshooting and to simplify the application of TCMs, a semi-analytic solution is necessary. We obtain the approximate solution and asymptotic solution of the TCM in the overshooting region, and find some important properties of the convective overshooting. (1) The overshooting region can be partitioned into three parts: a thin region just outside the convective boundary with high efficiency of turbulent heat transfer, a power-law dissipation region of turbulent kinetic energy in the middle, and a thermal dissipation area with rapidly decreasing turbulent kinetic energy. The decaying indices of the turbulent correlations $k$, $u'_T T'$, and $T'T'$ are only determined by the parameters of the TCM, and there is an equilibrium value of the anisotropic degree $\omega$. (2) The overshooting length of the turbulent heat flux $u'_T T'$ is about $1H_k$ ($H_k = |dr/d \ln k|$). (3) The value of the turbulent kinetic energy at the convective boundary $k_c$ can be estimated by a method called the maximum of diffusion. Turbulent correlations in the overshooting region can be estimated by using $k_c$ and exponentially decreasing functions with the decaying indices.

Key words: convection – diffusion – turbulence

1. INTRODUCTION

Convective overshooting is an important physical process in the stellar structure and evolution. Phenomenologically, the acceleration of a fluid element is zero at the convective boundary, but its speed is not zero. It is able to go across the convective boundary into the dynamically stable zone. This phenomenon is called convective overshooting. Convective overshooting transports heat and matter, and affects the structure and evolution of stars. A phenomenological theory of overshooting was developed by Zahn (1991), which predicts an adiabatic overshooting region. However, Xiong & Deng (2001) pointed out that turbulent velocity and temperature are strongly correlated in Zahn’s theory. Recently, Christensen-Dalsgaard et al. (2011) found that convective overshooting only described by the turbulent convection models (TCMs) could be in agreement with the helioseismic data.

The TCMs are based on fully hydrodynamic moment equations, and applied to the investigation of convective overshooting (Xiong 1981, 1985, 1989; Xiong & Deng 2001; Canuto 1997, 1998, 1999; Canuto & Dubovikov 1998; Marik & Petrovay 2002; Deng & Xiong 2006, 2008; Li & Yang 2007; Zhang & Li 2009). There are two main difficulties restricting the applications of the TCMs. One is to solve the equations of the TCMs, which are highly nonlinear and unstable in numerical calculations. The other is to incorporate the TCMs into a stellar evolution code. In general, to solve the TCMs, the parameters of the stellar structure (e.g., temperature $T$, density $\rho$, pressure $P$, radius $r$, luminosity $L$, and elements abundance vector) are needed, and solving the equations of stellar structure requires the temperature gradient $\nabla$, which is determined by the TCMs. Thus, in order to apply the TCMs, one must solve both the TCMs and the equations of stellar structure, which are enormously difficult.

Although developing a numerical technique is very important, getting an approximate solution of the TCMs is more interesting because an approximate solution helps us understand the physical processes and may significantly simplify the application of the TCMs. Xiong (1989) found the asymptotic solution of his TCM in the overshooting region, the turbulent correlations exponentially decreasing in the overshooting region. However, his solution for the heat flux $u'_T T'$ is not suitable near the convective boundary, and the initial turbulent kinetic energy $k_0$ is unknown so that the value of the turbulent correlations in the overshooting region actually cannot be determined without numerical calculations.

In this paper, we investigate the properties of convective overshooting by analyzing Li & Yang’s TCM (Li & Yang 2007), which was tested in the solar convective zone (Li & Yang 2007; Yang & Li 2007). We try to get a semi-analytical solution of the TCM in the overshooting region. We introduce the TCM in Section 2, investigate the properties of the overshooting in Section 3, and summarize the conclusions in Section 4.

2. TURBULENT CONVECTION MODEL

The closure assumptions of Li & Yang’s TCM are (Li & Yang 2001, 2007) as follows: the three-order moment terms are modeled with a gradient-type scheme; the dissipation rate $\epsilon$ of the turbulent kinetic energy $k$ is assumed to be local; the dissipation rates of the turbulent heat flux $u'_T T'$ and turbulent fluctuation of temperature $T'T'$ are assumed to be determined by both the convective timescale of the turbulent dissipation, $\tau_1^{-1} = \epsilon/k$, and the thermal dissipation one, $\tau_2^{-1} = (\lambda/\rho c_P)(\epsilon^2/k^3)$. According to those closure assumptions, fully hydrodynamic moment equations on the quasi-steady approximation result in the complete equations of two-order moment terms.
(Li & Yang 2007):

\[
\frac{1}{\rho r^2} \frac{\partial}{\partial r} \left( C_s \rho r^2 \frac{k}{\epsilon} \frac{\partial u'_{\perp} u'_r}{\partial r} \right) = \frac{2}{3} \epsilon - \frac{2 \beta g_r}{T} (u'_{\perp} u'_r - \frac{2}{3} k) \quad (1)
\]

\[
\frac{1}{\rho r^2} \frac{\partial}{\partial r} \left( C_s \rho r^2 \frac{k}{\epsilon} \frac{\partial k}{\partial r} \right) = \epsilon + \frac{\beta g_r}{T} u'_{\perp} T' \quad (2)
\]

\[
\frac{2}{\rho r^2} \frac{\partial}{\partial r} \left( C_t \rho r^2 \frac{k}{\epsilon} \frac{\partial u'_{\perp} T'}{\partial r} \right) = - \frac{T}{H_P} (\nabla - \nabla_{ad}) u'_{\perp} T' + \frac{\beta g_r}{T} T' T' + C_t \left( \frac{\epsilon}{k} + \frac{\lambda \epsilon^2}{\rho c_P k^3} \right) u'_{\perp} T' \quad (3)
\]

\[
\frac{1}{\rho r^2} \frac{\partial}{\partial r} \left( C_t \rho r^2 \frac{k}{\epsilon} \frac{\partial u'_{\perp} u'_r}{\partial r} \right) = - \frac{2T}{H_P} (\nabla - \nabla_{ad}) u'_{\perp} T' + 2C_t \left( \frac{\epsilon}{k} + \frac{\lambda \epsilon^2}{\rho c_P k^3} \right) T' T'. \quad (4)
\]

The temperature gradient is calculated as

\[
\nabla = \nabla_R - \frac{H_P}{T} \frac{\rho c_P}{\lambda} u'_{\perp} T'. \quad (5)
\]

The meaning of these equations and each term in them have been described in previous works (Li & Yang 2007; Zhang & Li 2009) in detail. We simply re-introduce them here.

Equations (1)–(4) describe the equilibrium (time-independent) structure of the radial kinetic energy \( u'_{\perp} u'_r \), the turbulent kinetic energy \( k \), the turbulent heat flux \( u'_{\perp} T' \), and the turbulent fluctuation of temperature \( T' T' \). On the left side of those equations, there is the non-local term (i.e., the diffusion term) of each turbulent correlation. On the right side are the local terms which describe the generation and the dissipation of each turbulent correlation.

In Equations (1) and (2), \( \epsilon \) is the turbulent dissipation rate of \( k \) and \( \epsilon = \langle k^{3/2} \rangle / l \), where \( l = \alpha H_P \), and the second term on the right side is the generation rate of the kinetic energy due to the contribution of the buoyancy. The last term in Equation (1) is the return to isotropy term which attempts to make the turbulent motion isotropic. In Equation (3), the first two terms on the right side is the generation rate of the turbulent heat flux \( u'_{\perp} T' \), and the last one is the dissipation rate that comprises the turbulent dissipation and the thermal dissipation. In Equation (4), the first term on the right side is the generation rate of the turbulent fluctuation of temperature \( T' T' \), and the last one is the dissipation rate. The meanings of the other symbols are as follows: \( H_P = -(dr/d\ln P) \) is the local pressure scale height, \( \beta = -(d \ln \rho / d \ln T)_P \) is the expansion coefficient, \( g_r = -GM/r^2 \) is the radial component of gravity acceleration, \( V = (d \ln T / d \ln P) \) is the temperature gradient in the stellar interior, \( \nabla_{ad} = (d \ln T / d \ln P)_S \) is the adiabatic temperature gradient, \( \lambda = (4acT^3/3k\rho) \) is the thermal conduction coefficient, \( c_T = (d H / d T)_P \) is the specific heat, \( C_t \) is the parameter of the return to isotropy term, \( (C_t, C_{t1}, C_{t2}) \) are the diffusion parameters, and \( \alpha, C_t, C_e \) are the dissipation parameters of turbulent variations \( k, u'_{\perp} T', T' T' \).

In Equations (1)–(4), overbars are only used in three turbulent correlations, \( \bar{u}'_{\perp} \bar{u}'_r, \bar{u}'_{\perp} T', \) and \( T' T' \). The other variations (density \( \rho \) and the temperature \( T, \) etc.) are all mean state quantities that should use overbars but we ignore them for convenience.

Equation (5) describes the energy transport in the stellar interior by both turbulent motions (i.e., convection and overshooting) and radiation. \( \nabla_R \) is the radiative temperature gradient.

3. THEORETICAL ANALYSIS OF TCM IN THE OVERSHOOTING REGION

In our previous work (Zhang & Li 2009), we applied the TCM to the solar overshooting region and found some properties of the overshooting region: \( u'_{\perp} T' < 0, \nabla_R < \nabla < \nabla_{ad} \), and the peak of \( T' T' \), which are similar to Xiong’s (1985) and Xiong & Deng’s (2001) works. In this section, we attempt to get semi-analytical solutions of the TCM.

Some approximations are adopted to simplify Equations (1)–(5) in the overshooting region.

Approximation I. Péclet number \( P_e \gg 1 \), where \( P_e = (\rho C_T \sqrt{k}) / \lambda \). That is, \( \epsilon/k > (\lambda / \rho c_P) (\epsilon^2/k^3) \), which means the turbulent dissipation is much stronger than the thermal dissipation. This assumption is reasonable in most cases except for the region near the surface of a star or with very small \( k \).

Approximation II. All variations, except the turbulent fluctuations, are thought to be constant because the turbulent fluctuations change much faster than others in the overshooting region.

Approximation III. Far away from the convective boundary, \( \nabla \approx \nabla_R \). This assumption is acceptable if the heat flux \( u'_{\perp} T' \) is small.

3.1. Turbulent Heat Transport in the Overshooting Region

Defining the anisotropic degree \( \omega = \langle u'_{\perp} u'_r \rangle / 2k \), which is the ratio of radial kinetic energy to total kinetic energy, and applying Approximation II and Equation (5), we can rewrite Equation (3) as

\[
\frac{\partial}{\partial r} \left( 4C_{t1} \alpha^2 \omega \frac{\partial u'_{\perp} T'}{\partial r} \right) = - \frac{T}{H_P} (\nabla_R - \nabla_{ad}) u'_{\perp} T' + \frac{\beta g_r}{T} T' T' + [2 \omega P_e + C_t(1 + P_e^{-1})] \frac{\sqrt{k}}{T} u'_{\perp} T'. \quad (6)
\]

In the last bracket in Equation (6), Approximation I \( (P_e \gg 1) \) makes the dissipation term \( C_t(1 + P_e^{-1}) \langle \sqrt{k} / T \rangle u'_{\perp} T' \) negligible. And, by using Equation (5) and Approximation II, it is easy to find that the diffusion term is on the same order as the negligible dissipation term:

\[
\frac{\partial}{\partial r} \left( 4C_{t1} \alpha^2 \omega \frac{\partial u'_{\perp} T'}{\partial r} \right) \approx 2C_t \alpha^2 \omega \frac{d \ln k}{d \ln P} \cdot \frac{d \ln (\nabla_R - \nabla)}{d \ln P} \times \left( \frac{\sqrt{k}}{T} u'_{\perp} T' \right) \sim P e^{\omega} \left( \frac{\sqrt{k}}{T} u'_{\perp} T' \right). \quad (7)
\]

Therefore, the diffusion term is also negligible. Equation (3) is in local equilibrium:

\[
- \frac{T}{H_P} (\nabla_R - \nabla_{ad}) u'_{\perp} T' + \frac{\beta g_r}{T} T' T' \approx 0. \quad (8)
\]
In the overshooting region, the most important process is the diffusion of kinetic energy. Thus, we ignore the diffusion of \( \nabla T T \) (i.e., setting \( C_{a1} = 0 \)). The solution of the TCM with \( C_{a1} = 0 \) can be thought of as the zero-order solution of the TCM.

Ignoring the diffusion of \( \nabla T T \) and the dissipation terms of \( \dot{\omega} T \), using Approximations I and II, one can rewrite Equations (1)–(4) as

\[
\frac{2C_{a} l}{k} \frac{\partial}{\partial r} \left( \omega k \frac{\partial \omega}{\partial r} \right) = \left( C_{k} - 1 \right) \left( \omega - \frac{1}{3} \right) \frac{k^{2}}{l} + \frac{\beta g r}{T} u_{r} T (1 - \omega)
\]

\[
2C_{a} l \frac{\partial}{\partial r} \left( \omega k \frac{\partial k}{\partial r} \right) = \frac{k^{2}}{l} + \frac{\beta g r}{T} u_{r} T^2
\]

\[
0 = -\frac{2T}{H_{P}} (\nabla - \nabla_{ad}) \omega k + \frac{\beta g r}{T} u_{r} T^2
\]

\[
0 = -\frac{2T}{H_{P}} (\nabla - \nabla_{ad}) u_{r} T + 2C_{c} \beta g \kappa^{3}\frac{1}{k} T^2.
\]

Equation (9), resulting from Equations (1) and (2), describes the equilibrium structure of the anisotropic degree \( \omega \). The left side is the diffusion of \( \omega \). The first term on the right side is the dissipation rate due to the return to isotropy term in Equation (1). The last term is the generation rate of \( \omega \) due to the buoyancy.

Equations (11) and (12) show

\[
0 = (\nabla - \nabla_{ad}) \left( u_{r} T^2 + 2C_{c} \beta g \kappa^{3} \frac{1}{k} T^2 \right).
\]

The solution is \( u_{r} T^2 = -2C_{c} \beta g (T/\beta g) \), or \( \nabla = \nabla_{ad} \). The latter is equivalent to \( u_{r} T^2 = -(T/H_{P}) (\lambda/\rho c_{P}) (\nabla_{ad} - \nabla_{R}) \). Because \( u_{r} T^2 \) is close to zero near the convective boundary and gradually decreases far away from the convective boundary (Xiong 1989; Xiong & Deng 2001; Zhang & Li 2009), the physically acceptable result is

\[
u_{r} T^2 = \text{Max} \left\{ -\frac{T}{H_{P}} \frac{\lambda}{\rho c_{P}} (\nabla_{ad} - \nabla_{R}), -2C_{c} \beta g \kappa^{3} \frac{1}{k} T^2 \right\}.
\]

Equation (14) shows that there is an adiabatic stratification zone in the overshooting region in the case of \( C_{a1} = 0 \). In order to investigate the property of heat transport in the overshooting region, we must know the length of the adiabatic stratification zone. From Equation (14), the boundary of the adiabatic stratification is the location where \( (T/H_{P}) (\lambda/\rho c_{P}) (\nabla_{ad} - \nabla_{R}) = 2C_{c} \beta g (k^{3/2}/l) \). Solving the equation for \( \omega \) is not easy because it is nonlinear. However, this problem is avoidable. Turbulent motions are isotropic when \( \omega = 1/3 \). In the convection zone, \( \omega > 1/3 \) because the buoyancy boosts radial turbulent motion. In most of the overshooting region, \( \omega \) should be less than 1/3 because buoyancy prevents radial turbulent motion. Therefore, \( \omega \) should not be far from 1/3 near the convective boundary. Furthermore, taking \( \omega \) as a constant, one can rewrite Equation (10) as

\[
2C_{a} l \omega \frac{\partial}{\partial r} \left( k \frac{\partial k}{\partial r} \right) = \frac{k^{2}}{l} + \frac{\beta g c}{T} u_{r} T^2.
\]

Substituting Equation (14) into the above equation, one can get the approximate solution:

\[
k^{2} \approx k_{C}^{2} \exp \left( -\frac{3}{4C_{c} \beta g} \left| \frac{r - r_{C}}{l} \right| \right)
\]

if \( \frac{T}{H_{P}} \frac{\lambda}{\rho c_{P}} (\nabla_{ad} - \nabla_{R}) \leq 2C_{c} \beta g (k^{3/2}/l) \), and

\[
k^{2} = k_{A}^{2} \exp \left( -\frac{3(1 + 2C_{c} \beta g)}{4C_{c} \beta g} \left| \frac{r - r_{A}}{l} \right| \right)
\]

if \( (T/H_{P}) (\lambda/\rho c_{P}) (\nabla_{ad} - \nabla_{R}) > 2C_{c} \beta g (k^{3/2}/l) \).

In Equation (16), point \( C \), which is the convective boundary where \( \nabla_{ad} = \nabla_{R} \), is set to be the initial point, \( k_{C} \) and \( r_{C} \) being \( k \) and \( r \). The contribution of the buoyancy term (i.e., the last term in Equation (15)) is ignored in obtaining the solution Equation (16). In the deep convection zone, turbulent motions are almost in local equilibrium, thus the ratio of \( -\beta g r (T/\beta g) u_{r} T^{2} \) to \( k^{3/2}/l \) is about 1. However, near the convective boundary, buoyancy is about zero; meanwhile, the diffusion of \( k \) dominates. Those make the ratio to be much less than 1. Therefore the buoyancy term is negligible.

In Equation (17), point \( A \), where \( k = k_{A} \) and \( r = r_{A} \), is the boundary of the adiabatic overshooting region. In the region beyond point \( A \), the ratio of \( -\beta g r (T/\beta g) u_{r} T^{2} \) to \( k^{3/2}/l \) is \( 2C_{c} \beta g \), which is on the order of one, and thus the buoyancy term remains.

The exponentially decreasing function of \( k \) is due to the fact that there is no generation in the overshooting region. Contrary to the situation in the convection zone, buoyancy dissipates \( k \) because it prevents the radial motion of fluid elements in the overshooting region. The distribution of \( k \) results from equilibrium between the diffusion and the dissipation. \( k \) should decrease faster if the buoyancy is as effective as the turbulent dissipation, which is found by comparing the exponential indices of Equations (16) and (17).

The location of point \( A \) is determined by \( (T/H_{P}) (\lambda/\rho c_{P}) (\nabla_{ad} - \nabla_{R}) = 2C_{c} \beta g (k^{3/2}/l) \). Using Equation (16), we get a property of point \( A \):

\[
k_{A}^{2} \exp \left( -\frac{3}{4C_{c} \beta g} \left| \frac{r_{A} - r_{R}}{l} \right| \right) = \frac{1}{2C_{c} \beta g \rho c_{P} \nabla_{ad}} \left| \nabla_{ad} - \nabla_{R,A} \right|
\]

The relation between \( r_{A} \) and \( \nabla_{R,A} \) is needed in order to solve this equation and to locate point \( A \). Near the convective boundary,

\[
|\nabla_{ad} - \nabla_{R,A}| \approx |\nabla_{ad}| \chi (\ln P_{A} - \ln P_{C}) = |\nabla_{ad}| \chi |l_{ad}/H_{P}|
\]

where \( l_{ad} = |r_{A} - r_{C}| \) is the length of the adiabatic overshooting region, \( P_{A} \) and \( P_{C} \) are the pressures at points \( A \) and \( C \), and \( \chi = (d \ln \nabla_{R})/(d \ln P) \), which is approximately a constant.

Substituting Equation (19) into Equation (18), one finds

\[
k_{C}^{2} \exp \left( -\frac{1}{\alpha} \sqrt{\frac{3}{4C_{c} \beta g H_{P}}} \frac{l_{ad}}{H_{P}} \right) = \frac{1}{2C_{c} \beta g \rho c_{P} \nabla_{ad}} \left| \nabla_{ad} \right| \frac{l_{ad}}{H_{P}}
\]
In the area $|r - r_C| \leq |r_A - r_C|$ in the overshooting region, the temperature gradient $\nabla$ is almost equal to the adiabatic one. In the area $|r - r_C| > |r_A - r_C|$, however, according to Equations (14), (17), and (5), the temperature gradient $\nabla$ is gradually close to $\nabla_R$:

$$\nabla - \nabla_R = (\nabla_{ad} - \nabla_{R,A}) \cdot \exp \left[ -\frac{3(1 + 2C_e\omega)}{4C_e\omega} \frac{|r - r_A|}{l} \right].$$

(25)

Although $\omega$ in Equations (24) and (25) is still unknown, we can estimate it roughly. Equations (24) and (25) describe the turbulent motion near the convective boundary, thus we can use $\omega \approx \omega_C$, where $\omega_C$ is $\omega$ at the convective boundary. In the deep convection zone, $\omega$ is almost equal to the equilibrium value $\omega_C = (2)/(3C_1) + 1/3$, which is derived from the localized TCM (see Appendix A). $\omega_C < \omega_{Cz}$ because the buoyancy is zero at the boundary, and $\omega_{Cz} > 1/3$ because of the diffusion of $\omega$. Therefore, the typical value of $\omega_C$ can be taken as the average, i.e., $\omega_C \approx 1/2(\omega_{Cz} + 1/3)$. If Equation (25) is used in the region far away from the convective boundary (beyond the peak of $T^T$), $\omega \approx \omega_C$ is not appropriate. One can use $\omega = \omega_0$, where $\omega_0$ is the equilibrium value of $\omega$ in the overshooting region which is introduced in the next subsection.

Another turbulent correlation is $T^T\overline{T}$, which can be worked out by using Equation (11):

$$\overline{T^T} \approx 0, \quad (|r - r_C| \leq |r_A - r_C|),$$

(26)

and

$$T^T\overline{T} = \frac{2T^T}{\overline{T^T}} \frac{T}{H_P} \beta g (\nabla_{ad} - \nabla)\omega_k, \quad (|r - r_C| > |r_A - r_C|).$$

(27)

Equation (26) seems to go against Cauchy’s theorem $u'_r u'_r T^T \overline{T} \geq u'_r T^T$. Actually, $T^T\overline{T} \approx 0$ is only an approximate solution on the order of $(Pe^4(\sqrt{k}/\omega r_0^T))$ because Equation (8) is an approximation on that order. Numerical calculations show no conflict.

Results obtained above are based on $C_{e1} = 0$. Numerical results of $\nabla$ with both $C_{e1} = 0$ and $C_{e1} \neq 0$ are shown in Figure 2. It is found that the effects of the diffusion of $T^T\overline{T}$ are only making $\nabla$ smoother. However, there is no adiabatic overshooting region when the diffusion of $T^T\overline{T}$ is present, because $T^T\overline{T}$ increases near the convective boundary due to the turbulent diffusion, thus $\nabla$ decreases according to Equation (8). Numerical results of the turbulent correlations in both $C_{e1} = 0$ and $C_{e1} \neq 0$ with different TCM parameters and for different stellar models are shown in Figures 3–5. It is found that the theoretical solutions fit the numerical solutions well in the case of $C_{e1} = 0$. This also validates that the boundary value $k_C$ derived from the maximum of diffusion is a good approximation. The diffusion of $T^T\overline{T}$ modifies and smoothes the profile of $T^T\overline{T}$ and $u'_r T^T$. However, $k$ is insensitive to the diffusion of $T^T\overline{T}$ because $k$ is mainly dominated by its diffusion. The diffusion of $T^T\overline{T}$ does not significantly change the integral value of $T^T\overline{T}$. According to Equation (8), the integral value of $\nabla$ or $u'_r T^T$ is also insensitive to the diffusion of $T^T\overline{T}$, which is found in Figures 2–5.

The distribution of $T^T\overline{T}$ reveals an important property of overshooting. In the nonadiabatic overshooting region, using
Our theoretical result shows Equation (27). This result indicates a maximum of $\pm C_e$ boundary in both $\nabla \approx \nabla$ Figure 3. Numerical results of the temperature gradient near the convective boundary in both $C_{e1} = 0$ and $C_{e1} \neq 0$, $V_0$ being the temperature gradient of the model with $C_{e1} = 0$, and $V_1$ corresponding to $C_{e1} = 0.01$. Dotted line $V_T$, which is almost identical to $V_0$, shows the theoretical solution of the temperature gradient with $C_{e1} = 0$. The stellar model and other TCM parameters are the same as Figure 1. Point A is the boundary of the adiabatic overshooting region. Our theoretical result shows $l_{ad} \approx 0.015H_P$ in that TCM parameter set, the numerical calculation being 0.015$H_P$.

$\nabla \approx \nabla$, one finds $T^{TT'} \propto T(\nabla_{ad} - \nabla_R)k$ according to Equation (27). This result indicates a maximum of $T^{TT'}$ (Xiong 1985; Zhang & Li 2009) which is shown in Figures 3–5. Beyond the location of the maximum of $T^{TT'}$, the temperature of a turbulent element is gradually close to the temperature of the environment, and the efficiency of heat transport significantly decreases. Therefore, the area between the convective boundary and the location of the maximum of $T^{TT'}$ can be thought of as the overshooting region of $u^T$. It is found from Figures 3–5 that the width of the valley of $\overline{u^T}$ is approximately equal to the distance from the convective boundary to the location of the maximum of $T^{TT'}$. In order to get the overshooting length of heat transport, we need to locate the maximum of $T^{TT'}$.

Using Equation (17), defining $\theta_0 = (d \ln k/d \ln P) = \pm(1/\alpha)\sqrt{(1 + 2C_{e\omega})/(2C_{e\omega})}$ as the decaying index of $k$ (in the case of $C_{e1} = 0$), we get

$$T^{TT'} \propto T(\nabla_{ad} - \nabla_R)P^{\theta_0}.$$  \hspace{1cm} (28)

The derivative of $T^{TT'}$ is zero at the peak of $\overline{u^T}$. We get $\nabla_R$ there (denoted as $\nabla_R^*$):

$$(\nabla_R^* + \theta_0)(\nabla_{ad} - \nabla_R^*) - \chi \nabla_R^* \approx 0.$$ \hspace{1cm} (29)

$\nabla_R^*$ is determined by only one turbulent parameter $\theta_0$.

The typical overshooting length of $u^T$ (or $V$) can be estimated with $\nabla_R^*$:

$$\chi = \left| \frac{d \ln \nabla_R}{d \ln P} \right| \approx \left| \frac{\ln \nabla_{R,C} - \ln \nabla_R^*}{\ln P_C - \ln P^*} \right|$$

$$= \left| \frac{\ln \nabla_{ad} - \ln \nabla_R^*}{\ln P_C - \ln P^*} \right| = \frac{\ln \nabla_{ad}^*}{H_P}.$$ \hspace{1cm} (30)
where $\nabla_{R,C}$ is $\nabla_R$ at the convective boundary, and $l_\nabla$ is the distance from the convective boundary to the location of the maximum of $T''T'$ and also the typical overshooting length of $\nabla$. $l_\nabla$ works out to be

$$l_\nabla \approx \frac{1}{|\chi|} \ln \frac{\nabla_{ad}}{\nabla_R} H_p. \quad (31)$$

Usually, $|\theta_0|$ is much larger than $|\chi|$ and $\nabla_{ad}$, and $\nabla_R^*$ can be approximately solved from Equation (29):

$$\nabla_R^* \approx \left(1 - \frac{\chi}{\theta_0}\right) \nabla_{ad}. \quad (32)$$

Finally, we find

$$l_\nabla \approx \frac{H_p}{|\theta_0|} = H_k, \quad (33)$$

where $H_k$ is the scale height of turbulent kinetic energy $k$ defined by $H_k = (dr/d \ln k)$. The result indicates that $\nabla$ is remarkably modified by overshooting only in about 1H_k. It is found from Figure 3 that $l_\nabla = \ln(k_c/k_A)H_k \approx 0.8H_k$, which is in agreement with Equation (33). It is shown in Figure 2 that $\nabla$ is remarkably modified only in $1H_k$.

### 3.2. Asymptotic Analysis

In the preceding subsection, we have discussed turbulent heat transport and the solution of turbulent correlations in the overshooting region near the convective boundary based on the assumption $C_{e1} = 0$. The diffusion of $T''T'$ only modifies turbulent correlations to be smoother near the convective boundary. However, it has more effects on turbulent motions in the overshooting region farther than $1H_k$ away from the convective boundary. In this subsection, we investigate the turbulence properties in the outer overshooting region (beyond $1H_k$).

In the numerical calculations of the TCM, we found that the anisotropic degree $\omega$ always showed an equilibrium value in the overshooting region. A typical numerical result is shown in Figure 6. In order to understand it, we discuss the behavior of the anisotropic degree $\omega$ in both convection zone and overshooting region. $\omega$ should be larger than 1/3 in the convection zone because buoyancy boosts the radial movement of turbulent elements. Actually, $\omega$ is almost equal to the equilibrium value in the convection zone $\omega_{cz} = (2/3C_e) + 1/3$ (see Appendix A) due to the equilibrium between the buoyancy and the return to isotropy term. When turbulent elements go across the convective boundary into the overshooting region, buoyancy prevents convective elements from moving, thus $\omega$ decreases to less than 1/3 near the convective boundary. However, as $u''T'$ exponentially decreases, the equilibrium of $\omega$ is established again in the overshooting region. This results in an asymptotic property of the overshooting region: there is an equilibrium value of $\omega$ in the overshooting region, $\omega \approx \omega_o$.

By using the asymptotic property $\omega \approx \omega_o$ and Approximations I, II, and III, it is easy to get the asymptotic solution of the TCM in the overshooting region (see Appendix B):

$$u''T' = \frac{(C_k - 1)(\omega_o - \frac{1}{3})}{1 - \omega_o} \frac{T}{k^2} \frac{P}{\beta g l}, \quad (34)$$

$$T''T' = 2\omega_o (\nabla_{ad} - \nabla_R) \frac{T^2}{\beta g H_p} k \quad (35)$$

where $\omega_o$ is defined at $|\chi| = 0$.

In the numerical calculations of the TCM, we found that $\omega_o$ is remarkably modified by overshooting only in about 1H_k. It is found from Figure 3 that $l_\nabla = \ln(k_c/k_A)H_k \approx 0.8H_k$, which is in agreement with Equation (33). It is shown in Figure 2 that $\nabla$ is remarkably modified only in $1H_k$.

### Figure 6.

Numerical result of the structure of $\omega$ in overshooting region. The stellar model is the solar model at present age, $C_{e1} = 0.01$. The other TCM parameters are the same as Figure 1, except $\alpha = 0.2$ in order to enlarge $\theta$ to show the thermal dissipation region in which $P_e < 1$. With those parameters, the equilibrium value in convection zone is $\omega_{cz} = 0.6$, and the equilibrium value in the overshooting region is $\omega_o = 0.293$ which is denoted by the dotted line.

$$k = k_0 \left(\frac{P}{P_0}\right)^\theta, \quad (36)$$

where $\theta$ is the asymptotic solution of $d \ln k/d \ln P$,

$$\theta = \pm \frac{1}{3} \left[1 - \frac{1}{3C_e(1/C_k + 1/3)} \right]. \quad (37)$$

where $k$ takes the decreasing expression in the overshooting region, which means “+” is adopted in the upward overshooting region and “−” in the downward one. The equilibrium value $\omega_o$ is determined by

$$(2C_eC_{e1} - C_{e1}C_k)\omega_o^2 + \left[1 + \frac{1}{3C_e(C_k + 1)} \right] \times \omega_o + \frac{1}{3C_e(C_k - 1)} = 0. \quad (38)$$

The equilibrium value $\omega_o$ is only a function of turbulent parameters ($C_e, C_{e1}, C_s, C_k$). The fact that buoyancy prevents the radial movement of turbulent elements in the overshooting region restricts the turbulent parameters to ensure $\omega_o < 1/3$.

Where $\omega$ reaches its equilibrium value $\omega_o$, important. According to Equation (9), the equilibrium of $\omega$ can be realized only if the buoyancy term synchronically decreases with decreasing $k$. Therefore, $\omega$ starts to reach its equilibrium value $\omega_o$ beyond the peak of $T''T'$ due to $|u''T'|$ decreasing.

Setting $C_{e1} = 0$ in Equation (38), we find that the asymptotic solution is the same as the results in the overshooting region with $|r - r_c| \geq |r_A - r_c|$ by setting $\omega = \omega_o$ in Equations (14), (17), and (27). Because Equation (8) is correct whether $C_{e1} = 0$ or not, the conclusion that the maximum of $T''T'$ is located at about $1H_k$ is also correct in both cases.

It must be mentioned that we have used Approximation I (i.e., $P_e \gg 1$), which means that the turbulent dissipation is much larger than the thermal dissipation. If $k$ decreases enough to satisfy $P_e < 1$, the thermal dissipation should become
significant thus $\overline{T T'}$ and the turbulent kinetic energy $k$ should rapidly decrease to zero. Then, $\omega$ also rapidly decreases as shown in Figure 6. In other words, turbulent movement can hardly overshoot into the thermal dissipation zone where $P_e < 1$.

According to discussions above, we can separate the overshooting region into three parts as shown in Figure 7: the overshooting region of $\overline{u'_rT'}$ or $\nabla$ with the length of about $1H_k$, the turbulent dissipation region in which the asymptotic solution holds, and the thermal dissipation region in which the turbulent movement quickly vanishes. The boundaries among those parts are the peak of $\overline{T T'}$ and the location of $P_e = 1$.

4. CONCLUSIONS AND DISCUSSIONS

TCMs are better tools in dealing with the convective overshooting than non-local mixing length theories. However, they are too complex to be applied to the calculations of stellar structure and evolution. In order to investigate the property of convective overshooting and to make it easy to apply TCMs, we have analyzed the TCM developed by Li & Yang (Li & Yang 2007) and obtained approximate and asymptotic solutions of the TCM in the overshooting region with $P_e \gg 1$. The main conclusions and corresponding discussions are listed as follows.

1. The overshooting region can be partitioned into three parts: a thin turbulent heat flux overshooting region, a power-law dissipation region of turbulent kinetic energy, and a thermal dissipation area with rapidly decreasing $k$. The turbulent fluctuations $k$, $\overline{u'_rT'}$, and $\overline{T T'}$ exponentially decrease in the overshooting region as Equations (34)–(36). The equilibrium value of the anisotropic degree $\alpha_o$ and the exponential indices of the turbulent fluctuations are only determined by the parameters of the TCM. The decaying behaviors of the turbulent fluctuations are similar to Xiong & Deng’s results (Xiong 1989; Xiong & Deng 2001).

2. The peak of $\overline{T T'}$ in the overshooting region is located about $1H_k$ away from the convective boundary. In this distance, the modification of $\nabla$ caused by the overshooting is remarkable. An approximate profile of $\nabla$ comprises an adiabatic overshooting region with the length of $l_{ad}$ and an exponentially decreasing function, as described in Equations (24) and (25). Beyond $1H_k$, the modification of $\nabla$ is negligible and $\nabla \approx \nabla_g$. It should be noted that the result of the $1H_k$ overshooting distance of turbulent heat transfer is independent of the parameters of TCM, so it may be a general property of overshooting. Our result is similar to Marik & Petrovay (2002) whose result shows that the length between the peak of $\overline{T T'}$ and the convective boundary is about $1.2H_k$. Meakin & Arnett (2010) simulated the turbulent convection of a $23M_\odot$ star, with the data of the turbulent kinetic energy and the convective flux in the overshooting region shown in Figure 8. It is found that the overshooting length of the convective flux $\overline{u'_rT'}$ is about $0.5 \sim 2H_k$, which is in agreement with our result.

3. The value of the turbulent kinetic energy at the convective boundary $k_C$ can be estimated by a method called the maximum of diffusion. The value of turbulent fluctuations in the overshooting region can be estimated by using exponentially decreasing functions and the initial value $k_C$. This may significantly simplify the application of the TCM in calculations of the stellar structure and evolution.

There is a distinction between the non-local model of Zahn (1991) and our results, i.e., the temperature gradient jumps from nearly adiabatic to radiative in Zahn’s model but continuously changes in our results (see Figure 2). This is caused by the assumption in Zahn’s model that the turbulent velocity and temperature fluctuation are strongly correlated (Xiong & Deng 2001). In our results, the correlativity of turbulent velocity and temperature fluctuation $R_{VT} = (\overline{u'_rT'})/(\sqrt{2\kappa T'T'})$ quickly decreases to zero then turns to be negative near the convective boundary (see Figure 9), and the asymptotic solution shows that $R_{VT} \propto \sqrt{k}$ and exponentially decreases in the turbulent dissipation overshooting region. Our results are in agreement with three-dimensional simulations such as Figure 6 in Singh et al. (1995) and Figure 15 in Meakin & Arnett (2007).
then the diffusion significantly enlarges the overshooting region. In the convection zone near the convective boundary, temperature $RVT$ shows the equilibrium value of $RVT$ is $RVT, e = \sqrt{2\omega/\omega_{a}C_{e} + 1/2\omega_{a}C_{f}}$ (see Appendix A). The TCM parameters show $RVT, e = 0.384$.

We thank the anonymous referee for valuable comments which helped improve the paper. We thank C. A. Meakin for providing the numerical data of Figure 8. Fruitful discussions with J. Su, X. J. Lai, and C. Y. Ding are highly appreciated. This work is co-sponsored by the National Natural Science Foundation of China through grant nos. 10673030 and 10973035 and Science Foundation of Yunnan Observatory through grant No. Y0ZX011009.

APPENDIX A

THE LOCALIZED TCM IN THE CONVECTION ZONE

The localized TCM results from Equations (1)–(4) by ignoring the diffusion terms. It is a good approximation of the TCM in the convection zone (Li & Yang 2001). We attempt to work out the solution in this appendix.

Some symbols are defined for convenience: $U = \dot{w}_{i}T'$, $V = T'T'$, $W = \sqrt{\omega}$, $A = (T/H_{P})(\nabla_{R} - \nabla_{ad})$, $B = -(\beta g_{r})/(T)$, $D = \lambda/\rho C_{P}$, and $f = (\nabla - \nabla_{ad})/2(\nabla_{R} - \nabla_{ad})$.

Ignoring the diffusion terms of Equations (1)–(4), we get the localized TCM:

$$0 = 2\frac{W^{3}}{l} - 2BU + 2C_{e}\left(\omega - \frac{1}{3}\right)\frac{W^{3}}{l}$$

(A1)

$$0 = \frac{W^{3}}{l} - BU$$

(A2)

$$0 = -2\omega f AW^{2} - BV + C_{e}(1 + P_{e}^{-1})\frac{BU}{l}$$

(A3)

$$0 = -2f AU + 2C_{e}(1 + P_{e}^{-1})\frac{WV}{l}$$

(A4)

$$U = AD(1 - f).$$

(A5)

Equations (A1) and (A2) show

$$\omega = \frac{2}{3C_{k}} + \frac{1}{3}. \quad (A6)$$

This is the equilibrium value $\omega_{a}$ in the convection zone.

Describing $W$, $V$, $U$ by $f$ and $P_{e} (= lW/D)$, we find

$$f = \frac{C_{e}C_{f}P_{e}^{-1}(1 + P_{e}^{-1})^{2}}{C_{e}C_{f}P_{e}^{-1}(1 + P_{e}^{-1})^{3} + 2C_{e}w(1 + P_{e}^{-1}) + 1}.$$  (A7)

$W$, $V$ can be worked out as

$$W^{3} = ADl(1 - f), \quad (A8)$$

$$V = \frac{A_{f}W^{2}}{C_{f}(1 + P_{e}^{-1})}.$$  (A9)

According to $P_{e} = lW/D$, Equations (A8) and (A7), we get the equation of $P_{e}$:

$$aP_{e}^{4} + (b + 1)P_{e}^{3} + 2bP_{e}^{2} + (b - at)P_{e} - t = 0, \quad (A10)$$

where $a = 1 + 1/2\omega_{a}C_{e}$, $b = C_{e}/2\omega$, $t = AB^{4}/D^{2}$. $f$ is determined by $f = 1 - P_{e}^{2}/a$ according to Equation (A8).

Solving Equation (A10), we can obtain all turbulent fluctuations of the localized TCM by using Equations (A5), (A8), (A9), and (A11).

An important case is $t \gg 1$, thus $P_{e} \gg 1$ according to Equation (A10). In that case, Equation (A7) shows

$$f = \frac{C_{e}C_{f}P_{e}^{-1}}{2C_{e}\omega + 1}. \quad (A11)$$

which corresponds to adiabatic convection.

Finally, we obtain the turbulent fluctuations according to Equations (A8), (A5), and (A9),

$$W^{3} \approx ADl \quad (A12)$$

$$V \approx \frac{C_{e}}{2C_{e}\omega + 1} \quad (A13)$$

$$U \approx AD, \quad (A14)$$

and the correlativity of turbulent velocity and temperature $RVT$,

$$RVT = \frac{U}{\sqrt{2\omega W^{2}V}} \approx \frac{2C_{e}\omega + 1}{2C_{e}\omega}. \quad (A15)$$

APPENDIX B

DETAILS OF DERIVING THE ASYMPTOTIC SOLUTION OF THE TCM IN THE OVERSHOOTING REGION

These are the details of obtaining the asymptotic solution of the TCM in the overshooting region.

Some symbols are defined for convenience: $U = \dot{u}_{i}T'$, $V = T'T'$, $W = \sqrt{\omega}$, $A = -(T/H_{P})(\nabla_{R} - \nabla_{ad}) \approx -(T/H_{P})(\nabla_{R} - \nabla_{ad})$ (Approximation III is used), and $B = -(\beta g_{r})/(T)$.

Applying the asymptotic property $\omega = \omega_{a}$ and Approximations I, II, and III, one can rewrite TCM as

$$0 = (C_{k} - 1)\left(\omega_{a} - \frac{1}{3}\right)\frac{W^{3}}{l} - BU(1 - \omega_{a}) \quad (B1)$$

$$IC_{e}\omega_{a}\frac{\partial}{\partial r}\left(W\frac{\partial W}{\partial r}\right) = \frac{W^{3}}{l} - BU \quad (B2)$$
\[ 0 = -BV + 2A\omega_0W^2 \]  \hspace{1cm} (B3)

\[ lC_{e1}\omega_0 \frac{\partial}{\partial r} \left( W \frac{\partial V}{\partial r} \right) = AU + \frac{Ce}{T} W V. \]  \hspace{1cm} (B4)

Equation (B1) is equivalent to

\[ U = \frac{(C_k - 1) (\omega_0 - \frac{1}{3})}{(1 - \omega_0)} \frac{W^3}{Bl}. \]  \hspace{1cm} (B5)

Putting it into Equation (B2), one gets the equation of \( W \):

\[ \frac{\partial^2 W^3}{\partial r^2} = \frac{3}{4C_{e1}\omega_0 l^2} \left[ 1 - \frac{(C_k - 1) (\omega_0 - \frac{1}{3})}{(1 - \omega_0)} \right] W^3. \]  \hspace{1cm} (B6)

Equation (B3) is equivalent to

\[ V = \frac{2A\omega_0}{B} W^2. \]  \hspace{1cm} (B7)

From Equations (B4), (B5), and (B7), we get another equation of \( W \):

\[ \frac{\partial^2 W^3}{\partial r^2} = \frac{3}{4C_{e1}\omega_0 l^2} \left[ \frac{(C_k - 1) (\omega_0 - \frac{1}{3})}{(1 - \omega_0)} + 2C_{e1}\omega_0 \right] W^3. \]  \hspace{1cm} (B8)

Comparing Equation (B6) with Equation (B8), one finds

\[ \frac{3}{4C_{e1}\omega_0 l^2} \left[ \frac{(C_k - 1) (\omega_0 - \frac{1}{3})}{(1 - \omega_0)} + 2C_{e1}\omega_0 \right] = \frac{3}{4C_{e1}\omega_0 l^2} \left[ 1 - \frac{(C_k - 1) (\omega_0 - \frac{1}{3})}{(1 - \omega_0)} \right]. \]  \hspace{1cm} (B9)

Therefore, the equation of \( \omega_0 \) is

\[ (2C_{e1} Ce - C_{e1} C_k)\omega_0^2 + \left[ \frac{1}{3} C_{e1} (C_k + 2) - C_k (C_k + 2C_{e1} - 1) \right] \]  \hspace{1cm} (B10)

\[ \times \omega_0 + \frac{1}{3} C_k (C_k - 1) = 0. \]

The asymptotic solution of \( W \) is derived from Equation (B6):

\[ W = W_0 \exp \left\{ \pm \frac{1}{2\alpha} \sqrt{\frac{1}{3C_{e1}\omega_0} \left[ 1 - \frac{(C_k - 1) (\omega_0 - \frac{1}{3})}{(1 - \omega_0)} \right] \ln \left( \frac{P}{P_0} \right)} \right\} \]  \hspace{1cm} (B11)

\( W \) takes the decreasing expression in the overshooting region: “+” is adopted in the upward overshooting region and “-” in the downward one.

REFERENCES

Canuto, V. M. 1997, ApJ, 489, L71
Canuto, V. M. 1998, ApJ, 508, 767
Canuto, V. M. 1999, ApJ, 524, 311
Canuto, V. M., & Dubovikov, M. 1998, ApJ, 493, 834
Christensen-Dalsgaard, J., Monteiro, M. J. F. G., Rempel, M., & Thompson, M. J. 2011, MNRAS, 414, 1158
Deng, L., & Xiong, D. R. 2006, ApJ, 643, 426
Deng, L., & Xiong, D. R. 2008, MNRAS, 386, 1979
Li, Y., & Yang, J. Y. 2001, Chin. J. Astron. Astrophys., 1, 66
Li, Y., & Yang, J. Y. 2007, MNRAS, 375, 388
Marik, D., & Petrovay, K. 2002, A&A, 396, 1011
Meakin, C. A., & Arnett, D. 2007, ApJ, 667, 448
Meakin, C. A., & Arnett, W. D. 2010, Ap&SS, 328, 221
Singh, H. P., Roxburgh, I. W., & Chen, K. L. 1995, A&A, 295, 703
Xiong, D. R. 1981, Sci. Sin., 24, 1406
Xiong, D. R. 1985, A&A, 150, 133
Xiong, D. R. 1989, A&A, 213, 176
Xiong, D. R., & Deng, L. 2001, MNRAS, 327, 1137
Yang, J. Y., & Li, Y. 2007, MNRAS, 375, 403
Zahn, J. P. 1991, A&A, 252, 179
Zhang, Q. S., & Li, Y. 2009, Res. Astron. Astrophys., 9, 585