Massive Three-Dimensional Supergravity From $R + R^2$
Action in Six Dimensions

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ABSTRACT

We obtain a three-parameter family of massive $\mathcal{N} = 1$ supergravities in three dimensions from the 3-sphere reduction of an off-shell $\mathcal{N} = (1,0)$ six-dimensional Poincaré supergravity that includes a curvature squared invariant. The three-dimensional theory contains an off-shell supergravity multiplet and an on-shell scalar matter multiplet. We then generalise this in three dimensions to an eight-parameter family of supergravities. We also find a duality relationship between the six-dimensional theory and the $\mathcal{N} = (1,0)$ six-dimensional theory obtained through a $T^4$ reduction of the heterotic string effective action that includes the higher-order terms associated with the supersymmetrisation of the anomaly-cancelling $\text{tr}(R \wedge R)$ term.
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1 Introduction

Three-dimensional topologically massive gravity (TMG) was introduced long ago as a toy model for studying quantum gravity [1, 2]. The topological mass term was introduced in order to give dynamics to the gravitational field, but in order for the graviton to have positive mass it was necessary to reverse the usual sign of the Einstein-Hilbert term. This had the unfortunate effect that the BTZ black hole solution then has negative mass. Recently, it was observed that if the coupling constant of the topological mass term takes a certain critical value, the bulk massive graviton decouples, one can revert to the usual sign for the Einstein-Hilbert term, and thus one obtains a unitary theory where the BTZ black hole has positive mass [3]. The theory is dual to a two-dimensional chiral conformal field theory on the boundary.

In subsequent developments in three-dimensional gravity, further generalisations were introduced involving the addition of higher-order curvature terms [4, 5]. These multi-parameter theories allow families of parameter choices that again exhibit critical behaviour, with the massive graviton decoupling in the bulk. Supersymmetric extensions of these theories [6, 7], and also the original TMG theory [2, 8, 9, 6], have been constructed.

Even though the original motivation for considering three-dimensional gravity theories was that they might stand in their own right as consistent toy models for quantum gravity, it is nevertheless natural to ask whether these theories can be embedded in string theory. If this could be done, it could, as a toy model, help to shed light on the quantum description of gravity within string theory in higher dimensions. A first step was taken in [10], where the 2-sphere reduction of a five-dimensional supergravity with curvature-squared terms [11] was considered. After the reduction in [10], non-local and higher-derivative field redefinitions were performed in order to remove the resulting curvature-squared terms in three dimensions, giving precisely the usual TMG theory. The use of such field redefinitions is somewhat questionable, since the solution space of the redefined theory is different from that of the original one [10]. The string theory origin of the five-dimensional theory, which is itself complete without requiring any terms beyond the quadratic order in curvature, is unclear.

More recently, it was shown that theories of gravity with topological terms (of the

\footnote{By the same token, such non-local higher-derivative field redefinitions could ostensibly be used to transform the “new massive gravity” theory of [4] into pure Einstein-Hilbert gravity. Again, the non-local, and non-invertible, nature of the transformations implies that the solution spaces in the two sets of variables are inequivalent.}
form $\text{tr}(R \wedge R \wedge \cdots R \wedge \omega)$, etc.) in $4k + 3$ dimensions can be recursively embedded in higher-dimensional such theories via consistent reductions on homogeneous Einstein spaces of dimension $4n$ \cite{12}. A special case of such reductions yields three-dimensional TMG from a seven-dimensional starting point \cite{13}.

In this paper, we shall study the 3-sphere reduction of a six-dimensional off-shell Poincaré supergravity with $R + \alpha \text{Riem}^2$ type action, where $\alpha$ is an arbitrary constant, which was constructed in \cite{14, 15}. We find that this reduction, upon a suitable and consistent truncation, yields a class of massive $\mathcal{N} = 1$ supergravities in three dimensions which include a Lorentz Chern-Simons term. These are disjoint from a larger class of such models that were recently constructed directly in three dimensions \cite{7}. We shall not carry out any non-local field redefinitions of the kind described in \cite{10}, which could ostensibly allow us to remove all the higher-order curvature terms except the Lorentz Chern-Simons term, for the reasons we discussed above.

In section 2, we present the action, supersymmetry transformation rules and bosonic equations of motion for the six-dimensional $R + \alpha \text{Riem}^2$ supergravity model constructed in \cite{14, 15}. In section 3, we perform a consistent reduction of the six-dimensional theory on a 3-sphere, and obtain a three-parameter family of massive supergravities in three dimensions. (There are two continuous parameters and a discrete parameter, which is the coefficient of the Einstein-Hilbert term, and which takes the values 1, 0 or $-1$.) These theories comprise a mixing of an off-shell $\mathcal{N} = 1$ supergravity multiplet coupled to an on-shell scalar multiplet. We give the full supersymmetric completion of the bosonic action, including those terms which have not previously been studied in \cite{7}. In section 4 we generalise our three-dimensional theory to one containing eight parameters, thus extending previous results in \cite{7}. We then study the critical points of these three-dimensional theories, i.e. the hypersurfaces in parameter space where the propagating gravity modes decouple.

Having successfully embedded a generalised three-dimensional topologically massive supergravity in six dimensions, we can address the further question of whether the embedding can be lifted to string theory. The off-shell nature of the model ensures that the $R$ and $\text{Riem}^2$ actions are separately supersymmetric. By contrast, adding higher derivative terms in the on-shell supergravities, such as the ten-dimensional heterotic supergravity, requires a derivative expansion which is supersymmetric only order by order in $\alpha'$. In attempting to embed the six-dimensional model we are studying here into ten-dimensional heterotic supergravity, we find, as discussed in section 5, that the 4-torus reduction of the latter, followed by a consistent truncation, is related to the six-dimensional model in a dual formu-
lation. More precisely, it appears that the $T^4$ dimensional reduction of the (infinite number of) terms in the heterotic $\alpha'$ expansion that are directly associated with the supersymmetric completion of the $\text{tr}(R \wedge R)$ anomaly-cancelling term seem to be reproduced simply by dualising the 2-form potential of the exact $R + \alpha \text{Riem}^2$ model. We provide evidence for this phenomenon by computing the leading terms in the correspondence between these two models connected by duality transformation.

Our conclusions appear in section 6. Appendix A sets out our notation and conventions, and appendix B contains some formulae useful for the calculations in section 4. In appendix C, we present the six-dimensional off-shell $\mathcal{N} = (1, 0)$ supergravity model, which was given in the string frame in section 2, in the Einstein frame.

2 The Six-Dimensional Theory

2.1 The off-shell Poincaré multiplet and field redefinitions

Our starting point is the six-dimensional $\mathcal{N} = (1, 0)$ supergravity constructed in [14], using the $48 + 48$ component off-shell Poincaré supermultiplet consisting of the fields

$$\{ e^a_\mu, \psi^i_\mu, V^{ij}_\mu, B_{\mu\nu}, C_{\mu\nu\rho\sigma}, \phi, \psi^i \} ,$$

(2.1)

where $e^a_\mu$ is the vielbein, $\psi^i_\mu$, ($i = 1, 2$), is the symplectic Majorana-Weyl gravitino, $V^{ij}_\mu = V^{ji}_\mu$ is a triplet of $SU(2)$ gauge fields, $B_{\mu\nu}$ is a real 2-form potential with 1-form gauge invariance, $C_{\mu\nu\rho\sigma}$ is a real 4-form potential with 3-form gauge invariance, $\phi$ is a real scalar, and $\psi^i$ is a symplectic Majorana-Weyl spinor. We shall work with the field strengths $H = dB$ and $G = dC$.

The full Lagrangian we shall study is a sum of the off-shell Poincaré supergravity [14] and an off-shell supersymmetrisation of $R^2_{\mu\nu\alpha\beta}$ [15]. The sum can be schematically written as

$$e^{-1} \mathcal{L} = \sigma R(e) + \cdots + \frac{1}{4} \alpha e^\phi (R_{\mu\nu\alpha\beta}(e)^2 + \cdots) ,$$

(2.2)

where $\alpha$ is a constant, $\sigma = \pm 1, 0$, and we have set the six-dimensional gravitational coupling
constant to unity. It was shown in [15] that the field redefinitions
\begin{align*}
\tilde{e}_\mu^a &= e^{\phi/2} e_\mu^a, \\
\tilde{\psi}_\mu &= e^{\phi/4} \psi_\mu - e^{-3\phi/4} \Gamma_\mu \psi, \\
\tilde{\epsilon} &= e^{\phi/4} \epsilon.
\end{align*}
(2.3)
lead to considerable simplifications in the supersymmetry transformation rules (detailed in the next subsection and appendix C) that facilitates the construction of the $R_{\mu\nu ab} R^{\mu\nu ab}$ invariant by analogy with the super Yang-Mills system. In particular, the dilaton dependence disappears in this invariant and the result now takes the form
\[ e^{-1} \mathcal{L} = \sigma e^{-2\phi} R(\hat{e}) + \cdots + \frac{1}{4} \alpha (R_{\mu\nu ab}(\hat{e})^2 + \cdots) = e^{-1} \left( \sigma \mathcal{L}_R + \frac{1}{4} \alpha \mathcal{L}_{\text{Riem}^2} \right). \]
(2.4)

2.2 The bosonic part of the off-shell Poincaré action and supersymmetry

In more detail, and dropping the hats for simplicity, the bosonic part of the full action is given by
\[ e^{-1} \mathcal{L}_R = e^{-2\phi} \left( R + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{2} V_{ij}^\mu V_{ij}^\mu \right) \\
- \frac{1}{5! \sqrt{2}} e^{\mu_1 \nu_1 \cdots \nu_5} G_{\nu_1 \cdots \nu_5} V_{\mu_1}^i \delta^i + \frac{1}{2 \times 5!} e^{2\phi} G_{\mu_1 \cdots \mu_5} G^{\mu_1 \cdots \mu_5}, \]
(2.5)
and
\[ e^{-1} \mathcal{L}_{\text{Riem}^2} = R_{\mu\nu ab}(\omega_+) R^{\mu\nu ab}(\omega_+) - G_{\mu_1 \mu_2}^i G_{ij}^\mu + \frac{1}{4} e^{\mu_\rho \sigma \lambda \tau} R_{\mu\nu ab}(\omega_+) R_{\rho\sigma}^{ab}(\omega_+) B_{\lambda \tau}, \]
(2.6)
where $H_{\mu\nu\rho} e^\nu_a e^\rho_b = H_{\mu ab}$ enters the spin connection as a torsion,
\[ \omega_{\mu\pm}^{ab} = \omega_{\mu}^{ab} \pm \frac{1}{2} H_{\mu}^{ab}. \]
(2.7)
The spin connection $\omega_{\mu ab}$ is the standard one that follows from $de^a + \omega^a_b \wedge e_b = 0$. The curvatures are defined as
\[ R_{\mu\nu}^{ab}(\omega_\pm) = 2 \partial_{[\mu} \omega_{\nu]}^{ab} + \omega_{[\mu}^{ac} \omega_{\nu]}^{cb}, \]
\begin{align*}
G_{ij}^{\mu\nu} &= 2 \partial_{[\mu} V_{ij}^{\nu]} + V_{[\mu}^{k(i} V_{ij}^{j)k}. \end{align*}
(2.8)

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2The redefinition of $V_{ij}^{\mu}$ given in [15] is quadratic in fermions, which is not relevant at the order in fermions we are working in this paper.

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6
The off-shell supersymmetry transformations, up to cubic fermion terms, are given by [14, 15] (see appendix C for further details).

\[ \delta e^a_{\mu} = \frac{1}{2} \bar{\epsilon} \Gamma^a \psi_{\mu}, \]

\[ \delta \psi_{\mu} = D_{\mu}(\omega_{-}) \epsilon, \]

\[ \delta B_{\mu\nu} = \bar{\epsilon} \Gamma_{[\mu} \psi_{\nu]}, \]

\[ \delta V^{ij}_{\mu} = \bar{\epsilon} (i \Gamma^\lambda \psi^{ij}_\lambda (\omega_{-}) + \frac{1}{6} \bar{\epsilon} (i \Gamma \cdot H \psi^{ij}_\mu), \]

\[ \delta C_{\mu\nu\rho\sigma} = 2\sqrt{2} e^{-\phi} \bar{\epsilon} \Gamma_{[\mu \nu \rho} \psi^{ij]}_{\sigma]} \delta_{ij} + 2\sqrt{2} e^{-13\phi/4} \bar{\epsilon} \Gamma_{\mu \nu \rho \sigma} \psi^{ij} \delta_{ij}, \]

\[ \delta \psi = \frac{1}{4} e^{5\phi/4} \Gamma^\mu \partial_{\mu} \phi \epsilon - \frac{1}{48} e^{5\phi/4} \Gamma \cdot H \epsilon - e^\phi \eta \]

\[ \delta \phi = e^{-5\phi/4} \bar{\epsilon} \psi, \] (2.9)

where \( \Gamma \cdot H = \Gamma^{\mu\nu} H_{\mu\nu} \). The parameter \( \eta \), up to cubic fermions, is defined by

\[ \eta_i = \left( - \frac{1}{4 \sqrt{2}} e^{\phi/4} \Gamma^\mu V_{\mu}^{(j)} k \delta^{jk} \epsilon_j + \frac{1}{5! \times 8 \sqrt{2}} e^{9\phi/4} \Gamma^{\mu_1 \ldots \mu_5} G_{\mu_1 \ldots \mu_5} \epsilon^i \right) \delta \epsilon_i, \] (2.10)

and

\[ D_{\mu}(\omega_{-}) \epsilon^i = \partial_{\mu} \epsilon^i + \frac{1}{4} \omega_{\mu ab}(\omega_{-}) \Gamma^{ab} \epsilon^i + \frac{1}{2} V^{ij}_{\mu} \epsilon_j, \]

\[ \psi^i_{\mu\nu}(\omega_{-}) = 2D_{[\mu}(\omega_{-}) \psi^i_{\nu]} = \left( \partial_{\mu} + \frac{1}{4} \omega_{\mu - ab} \Gamma_{ab} \right) \psi^i_{\nu} + \frac{1}{2} V^{ij}_{\mu} \psi_{\nu j} - \mu \leftrightarrow \nu. \] (2.11)

The transformation rules for \( C_{\mu\nu\rho\sigma} \) and \( \psi, \phi \) in the form above were not given in [15] because they were not needed for the construction of the \( R^{\mu\nu}_{ab} R^{\mu\nu} \) invariant which involves only the fields \( (e^a_{\mu}, \psi_{\mu}, B_{\mu\nu}, V^i_{\mu}) \). We shall need these transformation rules, however, in the off-shell Poincaré supergravity sector.

### 2.3 The \( R^{\mu\nu}_{ab} R^{\mu\nu} \) Invariant

The supersymmetrization of \( R^{\mu\nu}_{ab} R^{\mu\nu} \) in six dimensions has been accomplished in [15] by using the Noether procedure, and in [16] by exploiting a map between the Yang-Mills supermultiplet and a set of fields in the off-shell Poincaré supermultiplet in [16]:

\[ \left( A^I_{\mu}, Y^{ij}_I, \lambda^{ij}_I \right) \rightarrow \left( 2\tilde{\omega}^{ab}_{\mu +}, -\tilde{G}^{ij}_{ab}, \psi^{ab}_{\mu\nu}(\omega_{-}) \right), \] (2.12)

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3 In the results of [15], we have multiplied the action by 2 and let \( H_{\mu\nu} \rightarrow -H_{\mu\nu} \). Furthermore, in eqn. (4.29) of [13], \( 1/8 \rightarrow 1/4 \) and \( -3/4 \rightarrow -3/8 \), thereby correcting the typos there. See appendix A for further information on conventions.
where
\[ \tilde{G}_{ij}^{\mu
u} = G_{ij}^{\mu
u} + 2\tilde{\psi}_{[\mu}^{(i} \Gamma^{\nu]} \psi_{j]}^{(j)} (\omega_-) + \frac{1}{6} \psi_{[\mu}^{(i} \Gamma \cdot H \psi_{j]}^{j)} , \] (2.13)
up to quartic fermions, and
\[ \tilde{\omega}^{ab}_{\mu +} = \omega^{ab}_{\mu +} + \frac{1}{4} \tilde{\psi}_{\mu}^{[a} \Gamma^{b]} + \frac{1}{4} \tilde{\psi}^a \Gamma_\mu \psi^b + \frac{1}{2} H^a_b \] . (2.14)

As explained in [16], applying this map to the super Yang-Mills multiplet coupled to Poincaré supergravity given in [14] leads to the action
\[ e^{-1} \mathcal{L}_{\text{Riem}^2} = R_{\mu \nu ab} (\tilde{\omega}^+_{+}) R^{\mu \nu ab} (\tilde{\omega}^+_{+}) + 2 \tilde{\psi}^{ab} (\omega^+) \mathcal{D} (\omega, \omega^+) \psi_{ab} (\omega_) - \tilde{G}_{ij}^{\mu
u} \tilde{G}_{ij}^{\mu
u} \]
\[ - R^{\mu \nu ab} (\omega_-) \tilde{\psi}_\lambda \Gamma_{ab} \Gamma^\lambda \psi_{\mu \nu} \psi_\lambda \omega_- + \frac{1}{12} \tilde{\psi}^{ab} (\omega_-) \Gamma \cdot H \psi_{ab} (\omega_-) \]
\[ + \frac{1}{4} \epsilon^{\mu \nu \rho \sigma \lambda \tau} R_{\mu \nu}^{ab} (\tilde{\omega}^+_{+}) R_{\rho \sigma ab} (\tilde{\omega}^+_{+}) B_{\lambda \tau} , \] (2.15)
where, up to quartic fermions,
\[ \mathcal{D}_\mu (\omega, \omega^+) \psi_{ab}^i = \left( \partial_\mu + \frac{1}{4} \omega_{\mu}^{cd} \Gamma_{cd} \right) \psi_{ab}^i - 2 \omega_{\mu + [a} \psi_{b]}^{\lambda i} + \frac{1}{2} V_{\lambda i} \psi_{ab}^i . \] (2.16)

The action is given modulo quartic fermions, and so in addition to the quartic fermion terms that have been suppressed, those which arise from the term $R_{\mu \nu ab} (\tilde{\omega}^+_{+}) R^{\mu \nu ab} (\tilde{\omega}^+_{+})$ and $\tilde{G}_{ij}^{\mu
u} \tilde{G}_{ij}^{\mu
u}$ can be dropped. Note in particular that there will be terms bilinear in fermions coming from the fermionic torsion in the first term in the action. We have also used the fact that $R_{\mu \nu ab} (\omega_-) = R_{ab\mu \nu} (\omega^+) + \text{fermion bilinears}$ [15].

It will prove to be convenient to rewrite $\tilde{\psi}^{ab} \mathcal{D} (\omega, \omega^+) \psi_{ab}$ in terms of the torsionful spin connection $\omega_{\mu - ab}$. This leads to the action [15] [16]
\[ e^{-1} \mathcal{L}_{\text{Riem}^2} = R_{\mu \nu ab} (\tilde{\omega}^+_{+}) R^{\mu \nu ab} (\tilde{\omega}^+_{+}) + 2 \tilde{\psi}^{ab} (\omega_-) \mathcal{D} (\omega, \omega^+) \psi_{ab} (\omega_-) - \tilde{G}_{ij}^{\mu
u} \tilde{G}_{ij}^{\mu
u} \]
\[ - R^{\mu \nu ab} (\omega_-) \psi_\lambda \Gamma_{ab} \Gamma^\lambda \psi_{\mu \nu} \psi_\lambda \omega_- + \frac{1}{12} \tilde{\psi}^{ab} (\omega_-) \Gamma \cdot H \psi_{ab} (\omega_-) \]
\[ + 4 H^{\mu \nu} \tilde{\psi}_\mu (\omega_-) \Gamma_\rho \psi_\nu \psi_\lambda \omega_- + \frac{1}{4} \epsilon^{\mu \nu \rho \sigma \lambda \tau} R_{\mu \nu}^{ab} (\omega^+) R_{\rho \sigma ab} (\omega^+) B_{\lambda \tau} , \] (2.17)
where
\[ \mathcal{D}_\mu (\omega_-) \psi_{ab}^i = \left( \partial_\mu + \frac{1}{4} \omega_{\mu - [a}^{cd} \Gamma_{cd} \right) \psi_{ab}^i - 2 \omega_{\mu - [a} \psi_{b]}^{\lambda i} + \frac{1}{2} V_{\lambda i} \psi_{ab}^i . \] (2.18)
A few notational clarifications are in order. Firstly, $\psi_{ab} (\omega_-) = e_{a}^{\nu} e_{b}^{\mu} \psi_{\mu \nu} (\omega_-)$, with $\psi_{\mu \nu} (\omega_-)$ as defined in [2.11]. Using this definition in the second term in the above action gives
\[ 2 \tilde{\psi}^{\mu \nu} (\omega_-) \mathcal{D} (\omega, \Gamma^-) \psi_{\mu \nu} (\omega_-) \] where
\[ \Gamma_{\pm \mu \nu} = \Gamma_{\mu \nu}^{\rho} \pm \frac{1}{2} H_{\mu \nu}^{\rho} , \] (2.19)
satisfying the vielbein postulate
\[ \partial_{\mu} e^a_{\nu} + \omega_{\mu \pm a b} e_{\nu b} - \Gamma^\rho_{\pm \mu \nu} e^a_{\rho} = 0. \] (2.20)

and
\[ \mathcal{D}_\lambda(\omega_-, \Gamma_-) \psi^i_{\mu \nu} = \left( \partial_\lambda + \frac{1}{4} \omega_{\lambda \pm a b} \Gamma_{ab} \right) \psi^i_{\mu \nu} + 2 \Gamma^{\sigma}_{\lambda \mu} \psi^j_{\nu \sigma} + \frac{1}{2} V^{ij}_{\lambda} \psi_{\mu \nu j}. \] (2.21)

### 2.4 The bosonic equations of motion

After performing a consistent truncation of the auxiliary fields, by setting
\[ V^{ij}_{\mu} = 0, \quad C_{\mu \rho \sigma} = 0, \] (2.22)

the six-dimensional action is given by
\[ \mathcal{L} = \sqrt{-g} e^{-2\phi} \left[ R + 4(\partial \phi)^2 - \frac{1}{12} H^{\mu \rho \sigma} H_{\mu \rho \sigma} \right] + \frac{1}{4} \alpha \sqrt{-g} \tilde{R}^{\mu \rho \sigma \lambda} \tilde{R}_{\mu \rho \sigma \lambda} + \frac{1}{4} \beta \mathcal{L}_{CS}, \] (2.23)

where \( H = dB \) and
\[ \mathcal{L}_{CS} = \frac{1}{4} \epsilon^{\mu \rho \sigma \lambda \tau} \tilde{R}^{\alpha \beta \mu \nu} \tilde{R}_{\alpha \beta \rho \sigma} B_{\lambda \tau}. \] (2.24)

Here we have introduced the additional parameter \( \beta \) for convenience, so that we can distinguish between terms coming from curvature squared, versus terms coming from the Chern-Simons term. In what follows we should keep in mind that six-dimensional supersymmetry will require \( \beta = \pm \alpha \), and in fact with the fermion conventions we shall be using, supersymmetry requires
\[ \beta = +\alpha. \] (2.25)

(For simplicity, we have not included the discrete parameter \( \sigma \), introduced in (2.2) and (2.4) here. It can easily be introduced if desired by making the field redefinition \( \phi \rightarrow \phi - \frac{1}{2} \log \sigma \), with \( \sigma \) initially allowed to be any constant and then taken to be \( \pm 1 \) or 0 after the substitution.)

In the remainder of this section, we shall use the notation
\[ \Gamma^\rho_{\mu \nu} \equiv \tilde{\Gamma}^\rho_{\mu \nu} = \Gamma^\rho_{\mu \nu} - \frac{1}{2} H^\rho_{\mu \nu}. \] (2.26)
The Riemann tensor for this connection is defined by
\[ \tilde{R}^{\alpha}_{\beta \mu \nu} = \partial_\mu \tilde{\Gamma}^\alpha_{\nu \beta} - \partial_\nu \tilde{\Gamma}^\alpha_{\mu \beta} + \tilde{\Gamma}^\alpha_{\mu \gamma} \tilde{\Gamma}^\gamma_{\nu \beta} - \tilde{\Gamma}^\alpha_{\nu \gamma} \tilde{\Gamma}^\gamma_{\mu \beta}. \] (2.27)

Thus we have
\[ \tilde{R}^{\alpha}_{\beta \mu \nu} = R^{\alpha}_{\beta \mu \nu} + \nabla_\mu [H_\nu]^{\alpha}_{\beta} - \frac{1}{2} H^{\alpha}_{\lambda \mu} H^{\beta \lambda}_{\nu}. \] (2.28)

The Chern-Simons term \( \mathcal{L}_{CS} \) can be written as the 6-form
\[ \mathcal{L}_{CS} = -2\tilde{\Theta}^{\alpha \beta} \wedge \tilde{\Theta}_{\alpha \beta} \wedge B, \] (2.29)
where \( \tilde{\Theta}^{\alpha \beta} = \frac{1}{2} \tilde{R}^{\alpha}_{\beta \mu \nu} dx^\mu \wedge dx^\nu \) is the curvature 2-form with torsion. Up to a total derivative, it may also be written as
\[ \mathcal{L}_{CS} = 2\tilde{I}_3 \wedge H, \] (2.30)
where
\[ d\tilde{I}_3 = \text{tr}(\tilde{\Theta} \wedge \tilde{\Theta}) = -\tilde{\Theta}^{\alpha \beta} \wedge \tilde{\Theta}_{\alpha \beta}. \] (2.31)

The Chern-Simons form \( \tilde{I}_3 \) is given by
\[ \tilde{I}_3 = (\tilde{\Gamma}^\alpha_{\mu \beta} \partial_\nu \tilde{\Gamma}^\beta_{\rho \alpha} + \frac{2}{3} \tilde{\Gamma}^\alpha_{\mu \beta} \tilde{\Gamma}^\beta_{\nu \gamma} \tilde{\Gamma}^\gamma_{\rho \alpha}) dx^\mu \wedge dx^\nu \wedge dx^\rho \]
\[ = 2\tilde{\mathcal{L}}_{LCS} d^3 x, \] (2.32)
where we have used \( dx^\mu \wedge dx^\nu \wedge dx^\rho = -\epsilon^{\mu \nu \rho} \tilde{I}_3 \) and defined the Lorentz-Chern-Simons Lagrangian
\[ \tilde{\mathcal{L}}_{LCS} = -\frac{1}{2} \epsilon^{\mu \nu \rho} \left( \tilde{\Gamma}^\alpha_{\mu \beta} \partial_\nu \tilde{\Gamma}^\beta_{\rho \alpha} + \frac{2}{3} \tilde{\Gamma}^\alpha_{\mu \beta} \tilde{\Gamma}^\beta_{\nu \gamma} \tilde{\Gamma}^\gamma_{\rho \alpha} \right). \] (2.33)

Under a variation of the connection, we have
\[ \delta \tilde{I}_3 = \delta \tilde{\Gamma}^\alpha_{\mu \beta} \tilde{R}^{\beta \alpha \rho} dx^\mu \wedge dx^\nu \wedge dx^\rho + d\delta \nu, \] (2.34)
where
\[ \delta \nu = \delta \tilde{\Gamma}^\alpha_{\mu \beta} \tilde{\Gamma}^\beta_{\nu \alpha} dx^\mu \wedge dx^\nu. \] (2.35)

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4From the vielbein postulate (2.20), it follows that \( R_{\mu \nu}^{ab}(\omega_\nu) \epsilon_a^\alpha \epsilon_b^\beta = \tilde{R}^{\alpha \beta}_{\mu \nu} \). Note that when writing the Riemann tensor with all coordinate indices, we follow the original and standard general relativity convention of putting the manifestly-antisymmetric index pair to the right, \( R_{\mu \nu}^{\rho \sigma} = 2\partial_\mu \Gamma^{\rho \sigma}_{\nu \sigma} + \cdots \), whereas when the Riemann tensor is written with two coordinate and two local Lorentz indices, we follow the supergravity convention of putting the manifestly-antisymmetric coordinate-index pair to the left, \( R_{\mu \nu}^{ab} = 2\partial_\mu \omega^{ab}_{\nu} + \cdots \). Since in the former case all the indices on the Riemann tensor are greek, whereas in the latter case there are two greek and two latin indices, there should be no confusion.
Thus in terms of components, we have

$$\delta \tilde{I}_{\mu\nu} = 6 \tilde{R}^\alpha_{\alpha[\mu} \delta \tilde{\Gamma}^{\alpha}_{\rho]\beta} + 3 \partial_{[\mu} \delta \nu_{\nu]} . \quad (2.36)$$

We find that the equations of motion are given by

$$e^{-2\phi} \left( R_{\mu\nu} + 2 \nabla_\mu \nabla_\nu \phi - \frac{1}{4} H_{\mu\rho\sigma} H^{\rho\sigma}_{\nu} \right) + \frac{1}{4} E_{\mu\nu} = 0 , \quad (2.37)$$

$$\nabla_\mu \left( e^{-2\phi} H^{\mu\nu\rho} \right) - 12 \alpha \nabla_\mu \nabla_\nu \phi \tilde{R}^{[\mu\nu\rho]}_{\sigma} + 6 \alpha \nabla_\mu \left( \tilde{R}^{[\mu\nu}_{\sigma\lambda} H^\rho_{\sigma\lambda} \right) - \frac{1}{2} \beta \tilde{R}^{\alpha}_{\beta\mu\sigma} \tilde{R}^{\beta}_{\alpha\lambda\tau} \varepsilon^{\mu\sigma\lambda\tau\nu\rho} - 6 \alpha \nabla_\mu \left( \tilde{R}^{[\mu\nu}_{\sigma\lambda} * H^{\rho\sigma\lambda} \right) = 0 , \quad (2.38)$$

$$R - 4(\nabla \phi)^2 + 4 \Box \phi - \frac{1}{12} H^2 = 0 , \quad (2.39)$$

where

$$E_{\mu\nu} = 2 \alpha \tilde{R}^{\alpha}_{\mu\rho} \tilde{R}^{\rho}_{\alpha\beta\nu} - \frac{1}{2} \alpha \tilde{R}^{\alpha}_{\mu\beta\rho} \tilde{R}^{\rho}_{\alpha\beta\mu} g_{\mu\nu} - 4 \alpha \nabla_\alpha \nabla_\lambda \tilde{R}^{\alpha}_{\mu\nu} \lambda$$

$$- 2 \alpha \left( \nabla_\lambda \tilde{R}^{\alpha}_{\mu(\lambda} H^{\nu)\sigma} \right) H_{\nu\sigma\alpha} + 2 \alpha \nabla_\beta \left( \tilde{R}^{\beta}_{\mu(\alpha} H^{\nu)\sigma} \right) - \alpha \tilde{R}^{\beta}_{\mu(\rho\lambda} H^{\nu)\gamma} H^{\gamma}_{\rho\lambda}$$

$$- 2 \beta \nabla_\beta \left( \tilde{R}^{\beta}_{\mu(\gamma} * H^{\nu)\alpha} \right) + \beta \tilde{R}^{\beta}_{\mu(\rho\lambda} H^{\nu)\gamma} H^{\gamma}_{\rho\lambda} \quad (2.40)$$

and $* H^{\mu\nu\rho} = \frac{1}{6} \varepsilon^{\mu\nu\rho\sigma\lambda\tau} H_{\sigma\lambda\tau}$. (Note that the occurrence of some covariant derivatives with torsion, and others without torsion, is intended, and is not a misprint.)

3 3-Sphere Reduction to Three Dimensions

3.1 The full bosonic action in three dimensions

We now consider the 3-sphere reduction, with the ansatz given by

$$ds^2 = dS^2 + d\Sigma^2 , \quad H_3 = 2S \epsilon_{(3)} + 2m \Sigma_{(3)} , \quad (3.1)$$

where $m$ is a constant, and $d\Sigma^2$ is the metric of the round $S^3$ with $R_{ij} = 2m g_{ij}$. Substituting the ansatz into the six-dimensional equations of motion (with the parameter $\sigma$ introduced by sending $\phi \rightarrow \phi - \frac{1}{2} \log \sigma$, as discussed previously), we obtain equations of motion for
the three-dimensional fields:

\[ 0 = \alpha S + \sigma e^{-2\phi}S + m - \frac{1}{2}(\beta m - \alpha S)(R + 6S^2), \tag{3.2} \]

\[ 0 = \sigma (4\phi - 4(\partial \phi)^2 + R + 2S^2 + 4m^2), \tag{3.3} \]

\[ 0 = \sigma e^{-2\phi} [R_{\mu \nu} + 2\nabla_{\mu} \nabla_{\nu} \phi] - 2mSg_{\mu \nu} \]

\[ + \alpha \left[ \Box R_{\mu \nu} - \frac{1}{2} \nabla_{\mu} \nabla_{\nu} R - 4R_{\mu}^{\ \lambda} R_{\lambda \nu} + \frac{5}{2} RR_{\mu \nu} + \frac{3}{2} g_{\mu \nu} (R_{\rho \sigma} R^{\rho \sigma} - \frac{7}{12} R^2) \right. \]

\[ \left. - \frac{3}{2} S^4 g_{\mu \nu} + G_{\mu \nu} S^2 - (\nabla_{\mu} \nabla_{\nu} - g_{\mu \nu} \Box) S^2 - 2\partial_{\mu} S \partial_{\nu} S + (\partial S)^2 g_{\mu \nu} \right] \]

\[ + 2\beta m \left[ S^3 g_{\mu \nu} - G_{\mu \nu} S + (\nabla_{\mu} \nabla_{\nu} - g_{\mu \nu} \Box) S - C_{\mu \nu} \right], \tag{3.4} \]

where \( G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} \), and \( C_{\mu \nu} \) is the Cotton tensor defined by

\[ C_{\mu \nu} = \varepsilon_{\mu \rho \sigma} \nabla_{\rho} (R_{\sigma \nu} - \frac{1}{4} g_{\sigma \nu} R), \tag{3.5} \]

and the \( \phi \) field equation has been used to simplify the Einstein equation. Note from (3.2) that although \( S \) is an auxiliary field in the lowest-order theory (where \( \alpha = 0 \) and \( \beta = 0 \)), it becomes dynamical when \( \alpha \neq 0 \).

It is useful to note, when performing the dimensional reduction, that the connection with torsion in the 3-sphere directions is flat, and so the curvature \( \tilde{R}^{ij}_{\ k\ell} \) on \( S^3 \) vanishes.

We find that the three-dimensional equations can be derived from the Lagrangian

\[ e^{-1} \mathcal{L} = \sigma e^{-2\phi} \left[ R + 4(\partial \phi)^2 + 4m^2 + 2S^2 \right] + 4mS - 2\beta m (RS + 2S^3 - e^{-1} \mathcal{L}_{LCS}^{\text{Bos}}) \]

\[ + \frac{1}{4} \alpha \left[ 4R_{\mu \nu} R^{\mu \nu} - R^2 - 8(\partial S)^2 + 12S^4 + 4RS^2 \right]. \tag{3.6} \]

For later purposes, we shall write this as

\[ \mathcal{L} = \sigma \mathcal{L}_{EH}^{\text{Bos}} + 4m \mathcal{L}_C^{\text{Bos}} + \frac{1}{4} \alpha \mathcal{L}_{Riem}^{\text{Bos}} + 2\beta m (\mathcal{L}_{LCS}^{\text{Bos}} - \mathcal{L}_{S^3}^{\text{Bos}}), \tag{3.7} \]

where

\[ e^{-1} \mathcal{L}_{EH}^{\text{Bos}} = e^{-2\phi} \left[ R + 2S^2 + 4(\partial \phi)^2 + 4m^2 \right], \tag{3.8} \]

\[ e^{-1} \mathcal{L}_C^{\text{Bos}} = S, \tag{3.9} \]

\[ e^{-1} \mathcal{L}_{Riem}^{\text{Bos}} = 4R_{\mu \nu} R^{\mu \nu} - R^2 - 8(\partial S)^2 + 12S^4 + 4RS^2, \tag{3.10} \]

\[ e^{-1} \mathcal{L}_{LCS}^{\text{Bos}} = \frac{1}{4} \varepsilon_{\mu \nu \rho \sigma} \left( R_{\mu \nu}^{\ ab} \omega_{\rho a b} + \frac{2}{3} \omega_{\mu}^{\ a b} \omega_{\nu}^{\ c} \omega_{\rho}^{\ d} e^{c} \right), \tag{3.11} \]

\[ e^{-1} \mathcal{L}_{S^3}^{\text{Bos}} = RS + 2S^3, \tag{3.12} \]
where the spin connections and curvatures are both fermionic and bosonic torsion-free. (We write $\omega$ for $\omega(e)$ for simplicity in notation.) It is worth remarking that if we had simply substituted the ansatz \[ \text{(3.1)} \] into the six-dimensional Lagrangian we would have obtained the three-dimensional Lagrangian \[ \text{(3.6)} \] but without the $4mS$ term, and of course it would therefore not have given rise to the correct three-dimensional equations of motion. It is well known that substituting a field-strength ansatz such as in \[ \text{(3.1)} \] into a higher-dimensional Lagrangian typically fails to give the correct lower-dimensional Lagrangian. It is interesting that the correct Lagrangian is obtained, even when $H_{\mu\nu\rho}$ enters in higher-order terms too, simply by adding the term $4mS$.

The higher-order terms in the Lagrangian, proportional to $\alpha$ and $\beta$, can simply be written as
\[ \frac{1}{4\alpha} R^{\mu\nu\rho\sigma} \tilde{R}_{\mu\nu\rho\sigma} - 2\beta m L^{\text{LCS}}_{\text{Bos}}, \tag{3.13} \]
where the tildes indicate, as usual the curvatures and connections are those involving the bosonic torsion, as in \[ \text{(2.26)} \] with $H_{\mu\nu\rho} = 2S\varepsilon_{\mu\nu\rho}$. However, $S$ cannot simply be absorbed as a torsion in the full theory, as can be seen even in the leading-order bosonic terms $e^{-1} L^{\text{EH}}_{\text{Bos}}$ and $e^{-1} L^{\text{Bos}}_{\text{C}}$. Furthermore, as we shall discuss below, the supersymmetric completions of the bosonic terms in \[ \text{(3.13)} \] involve $S$-dependent terms that cannot be absorbed as a torsion.

The three-dimensional model we have obtained is an intriguing mix of the off-shell supergravity multiplet and an on-shell dilatonic scalar multiplet. The combination of invariants in the supergravity multiplet is special case of the more general massive supergravity obtained in \[ \text{(3.3)} \]. However, our theory should not be viewed as trivial generalization of the more general massive supergravity by adding a matter coupling. Truncating out the scalar multiplet in our theory will not lead to the more general massive supergravity, but rather to the trivial Einstein-Hilbert term with a cosmological constant. This can be seen from the three-dimensional supersymmetry transformation rules, which will be given in \[ \text{(3.25)} \] below. Truncating out $(\phi, \psi)$ requires us to take $S = -m$, and it follows from \[ \text{(3.3)} \] that the Ricci scalar becomes a constant. Thus the matter coupling in our model is more closely related to the supergravity multiplet than a typical matter multiplet and the scalar multiplet should be viewed as an integral part of the theory.

The scalar $\phi$ in the three-dimensional multiplet has its origin in a mixing of the six-dimensional dilaton and the breathing mode of the reduction ansatz. Turning off the higher-order derivative terms, the relevant Lagrangian is given by (we set $\sigma = 1$ in the remainder of this subsection for simplicity)
\[ e^{-1} \mathcal{L}_3 = e^{-2\phi}(R + 2S^2 + 4(\partial\phi)^2 + 4m^2) + 4\xi mS. \tag{3.14} \]
Here, we have added a parameter $\xi$, which takes the value 0 or 1, since $S$ is an independent invariant. Integrating out the auxiliary field $S$, we have

$$e^{-1} \mathcal{L}_3 = e^{-2\phi}(R + 4(\partial \phi)^2 + 4m^2 - 2\xi m^2 e^{4\phi}).$$  \hspace{1cm} (3.15)

To see how this scalar $\phi$ arises as a mixing of the six-dimensional dilaton and the breathing mode, let us examine the six-dimensional Lagrangian in the Einstein frame,

$$\mathcal{L}_6 = \frac{1}{2} \sqrt{-g} \left( \mathcal{R} - \frac{1}{2}(\partial \phi)^2 - \frac{1}{12} e^{-\sqrt{2} \phi} \mathcal{H}^2 \right).$$  \hspace{1cm} (3.16)

The reduction ansatz including the breathing mode is given by

$$ds_6^2 = e^{2a\phi} ds_3^2 + e^{2b\phi} d\Sigma_3^2, \quad \mathcal{H} = 2m(\epsilon + \xi \Sigma),$$  \hspace{1cm} (3.17)

where $a^2 = \frac{3}{8}$ and $b = -\frac{1}{3}a$. Thus we have

$$\mathcal{L}_3 = \sqrt{-g} \left( R - \frac{1}{2}(\partial \phi)^2 - \frac{1}{2}(\partial \phi)^2 - V \right),$$

$$V = 2m^2(\xi e^{\sqrt{2} \phi} + e^{-\sqrt{2} \phi})e^{4a\phi} - 6m^2 e^{8a\phi}. \hspace{1cm} (3.18)$$

It turns out that we can make the consistent truncation,

$$\hat{\phi} = \frac{1}{2}\phi, \quad \varphi = \sqrt{2} a\phi,$$  \hspace{1cm} (3.19)

so that the resulting Lagrangian is given by

$$\mathcal{L} = \sqrt{-g} \left( R - \frac{1}{2}(\partial \phi)^2 - V \right),$$

$$V = -2m^2(2e^{\sqrt{2} \phi} - \xi e^{2\sqrt{2} \phi}). \hspace{1cm} (3.20)$$

The potential $V$ can be expressed in terms of a superpotential as

$$V = \left( \frac{dW}{d\phi} \right)^2 - W^2, \quad W = \sqrt{2} m(2e^{\sqrt{2} \phi} - \xi e^{2\sqrt{2} \phi}).$$  \hspace{1cm} (3.21)

The reduction ansatz now becomes

$$ds_6^2 = e^{-\frac{1}{\sqrt{3}} \phi} \left( e^{\sqrt{2} \phi} ds_3^2 + d\Sigma_3^2 \right) = e^{-\frac{1}{\sqrt{3}} \phi} \left( ds_{\text{str}}^2 + d\Sigma_3^2 \right). \hspace{1cm} (3.22)$$

It is clear that the vacuum solution $\text{AdS}_3 \times S^3$ with $\xi = 1$ is the decoupling limit of the self-dual string. For $\xi = 0$, the metric of the vacuum solution is a domain wall, which is the decoupling limit of the electric string.
3.2 The three-dimensional supersymmetry transformations

Upon reduction to three dimensions, the supersymmetry parameter $\epsilon^i$, which is a symplectic Majorana spinor, turns into a spinor $\epsilon^{iA}$ where the $SO(2,1)$ spinor index as well as the spinor index on which the $\Sigma$ matrices act are suppressed, while the $SO(3)$ spinor index $A$ is exhibited. This spinor has 8 real components, and therefore it is associated with $N = 4$ supersymmetry in three dimensions. We shall truncate the theory to $N = 1$ by setting

$$\epsilon^{iA} = \frac{1}{\sqrt{2}} \epsilon \Omega^A .$$

The six-dimensional chirality condition now translates into $\tau_3 \epsilon = \epsilon$, and the six-dimensional symplectic Majorana condition becomes $\epsilon^* = -i \epsilon$.

In the reduction to three dimensions, we shall let $\mu \to (\mu, \mu')$ and $a \to (a, a')$ where the primes are used in labeling the internal coordinate world and Lorentz vectors. In the bosonic sector we truncate as in (2.22) and use ansatz (3.1), while in the fermionic sector we set

$$\psi^{iA} = \frac{1}{\sqrt{2}} \psi \Omega^A , \quad \bar{\psi} = \frac{1}{\sqrt{2}} \sum_2 \psi \Omega^A , \quad \psi^{iA} = 0 .$$

As a consequence, we are left with the three-dimensional fields $(e^a_{\mu}, \psi_\mu, S)$ and $(\psi, \phi)$. We find their supersymmetry transformations to be

$$\delta e^a_{\mu} = \frac{1}{2} \bar{\epsilon} \gamma^a \psi_\mu ,$$
$$\delta \psi_\mu = D_\mu (\omega_-) \epsilon = D_\mu \epsilon + \frac{1}{2} \gamma_\mu \epsilon S ,$$
$$\delta S = \frac{1}{8} \bar{\epsilon} \gamma^{\mu\nu} \psi_{\mu\nu} (\omega_-) = \frac{1}{8} \bar{\epsilon} \gamma^{\mu\nu} \psi_{\mu\nu} - \frac{1}{16} \epsilon S ,$$
$$\delta \psi = \frac{1}{4} \epsilon S^{5/4} (\gamma^\mu \partial_\mu \phi + S + m) \epsilon ,$$
$$\delta \phi = \epsilon S^{5/4} \bar{\psi} ,$$

where $\psi_{\mu\nu} (\omega_-) = 2 D_{[\mu} (\omega_-) \psi_{\nu]}$ and

$$\omega^a_{\mu} \pm = \omega^a_{\mu} \pm \epsilon^a_{\mu} S ,$$

Using (3.1), we find that $\omega^a_{\mu'} \psi_{\mu'} = 0$ which simplifies the reduction formulae considerably. For example, $\psi_{\mu'} = 0$ and $\psi_{\mu'} = 0$.

Our results for the transformation rules for $(e^a_{\mu}, \psi_\mu, S)$ agree precisely with the known off-shell $N = 1$ supergravity multiplet transformations in three dimensions. The field $S$ admits a torsion interpretation [6].
In section 3.1, we explained why the scalar field $S$ (and its fermionic partner) cannot be truncated away in presence of the higher derivative couplings, by considering the field equations. The supersymmetry transformation rules above provide another simple explanation of this phenomenon as follows. Setting $\phi = 0$ implies that $S = -m$. But then the supersymmetry variation of $S$ implies the gravitino field equation without higher derivative terms. Hence, the higher derivative couplings must be absent altogether if we are to be able to truncate out the scalar multiplet.

### 3.3 The supersymmetric completion of $L_{EH}^\text{Bos}$ and $L_C^\text{Bos}$

The supersymmetric completion of $L_{EH}^\text{Bos}$ and $L_C^\text{Bos}$ can be obtained by performing the 3-sphere reduction of the off-shell Poincaré sector of the six-dimensional theory. Since the fermionic sector of this theory has not been provided until now, we construct the supersymmetric completion directly in three dimensions, by starting from the bosonic sector and supersymmetry transformation rules we obtained from the 3-sphere reduction. We find, up to quartic fermion terms,

\[
e^{-1}L_{EH} = e^{-2\phi} \left[ R + 2S^2 + 4(\partial\phi)^2 + 4m^2 \right] + e^{-2\phi} \left[ -\bar{\psi}_\mu R^\mu + 2\bar{\psi}^{\mu\nu}\gamma_\nu \psi_\mu \partial_\mu\phi + m\bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu \right] - 8e^{-\frac{13}{4}\phi} \left[ \bar{\psi}\gamma_\mu R^\mu + \bar{\psi}\gamma_\mu \gamma_\nu \psi_\mu \partial_\nu\phi + m\bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu \right] + 8e^{-\frac{9}{2}\phi} \left[ \bar{\psi}\psi S + 2\bar{\psi}^{\mu} D_\mu \psi - 2m\bar{\psi}\psi \right], \tag{3.27}
\]

\[
e^{-1}L_C = S + \frac{1}{8}\bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu. \tag{3.28}
\]

### 3.4 The supersymmetric completion of $L_{Riem^2}^\text{Bos}$, $L_{LCS}^\text{Bos}$ and $L_S^3$

Using (2.22), (3.1) and (3.24), we find that the 3-sphere reduction of the six-dimensional Lagrangian $L_{Riem^2}$ given in (2.15) yields\(^5\)

\(^5\)The reduction of the second term in (2.15) gives rise to $\bar{\psi}^{ab} D(\omega_+^2)\psi_{ab}$ with the covariant derivative defined in (2.18) for an $Sp(1)$ singlet. We convert that to $\bar{\psi}^{ab} D(\omega_-^2)\psi_{ab}$ with the covariant derivative defined in (3.30) by adding and subtracting the required terms. Our result corrects that of [3] for the Riemann$^2$ invariant, where $\bar{\psi}^{ab} D(\omega)\psi_{ab}$ is used, instead of $\bar{\psi}^{ab} D(\omega_+^2)\psi_{ab}$.  

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We have grouped the terms in (3.29) as a sum of three terms, each enclosed in square brackets. Each bracketed set is separately invariant under three-dimensional $\mathcal{N} = 1$ supersymmetry. The terms in the first bracket furnish a supersymmetrization of the Riemann tensor squared term (with bosonic torsion). The second set of bracketed terms agrees with the topologically massive supergravity action $\mathcal{L}_{LCS}$ that has been known for some time [2]. The third bracket provides a superextension of the combination $RS + 2S^3$. Although the existence of such a super-invariant had been noted in [16], its explicit form has not previously been given.

Up to quartic fermion terms, the Lagrangians for the three super-invariants can be
written more explicitly as

\[
ee^{-1}L_{Riem} = 4R_{\mu\nu}R^{\mu\nu} - R^2 - 8\partial^\mu S\partial_\mu S + 4RS^2 + 12S^4
+ 4\bar{\psi}\gamma^\nu\psi\nabla_\mu R_{\mu\nu} + \psi\gamma^\nu\psi R_{\mu\nu} - 4\nabla_\mu \partial_\nu S - 2\bar{\psi}\gamma^\nu\psi\partial_\mu S^2

+ \frac{1}{8}\bar{\psi}\gamma^\mu\psi_\nu (3RS + 8\Box S + 16S^3) + 2\bar{\psi}_{ab}(\omega_-)\bar{D}(\omega_-)\psi_{ab}(\omega_-)

-R_{\mu\nu ab}(\omega_-)\bar{\psi}_\lambda\Gamma^{ab\lambda}\psi_\mu(\omega_-) + 2S\bar{\psi}_\mu(\omega_-)\gamma^\mu\gamma^\nu(\omega_-),
\]

(3.34)

\[
ee^{-1}L_{LCS} = \frac{1}{4}e^{\mu \nu \rho} \left( R_{\mu \nu}^{ab} \omega_{p a b} + \frac{2}{3}\omega_{\mu}^{a} b_{\omega_{\nu}^{b} c_{\omega_{\rho}^{e}}} a \right) + \frac{1}{2}\bar{R}^{\mu \nu} \gamma_\mu R^{\nu}

+ \frac{1}{2}e^{\mu \nu \rho} \left( R_{\rho \sigma} - \frac{1}{4}g_{\rho \sigma} R \right) \bar{\psi}_\gamma \gamma^\sigma \psi_\rho,
\]

(3.35)

\[
ee^{-1}L_{S^3} = RS + 2S^3 - \frac{1}{2}\bar{R}^{\mu \nu} \gamma_\mu R^{\nu} - \bar{\psi}_\mu \gamma^\mu \psi_\nu \partial_\nu S

+ \frac{1}{2}\bar{\psi}_\mu \gamma^{\mu \nu} R_{\nu} S - \frac{1}{2}\bar{\psi}_\mu \psi_\mu S^2,
\]

(3.36)

where all curvatures in which the arguments are not indicated are understood to be torsion-free.

4 Generalization of the Model in Three Dimensions and its Critical Points

The three dimensional model we have obtained through the 3-sphere reduction from six dimensions has two continuous parameters, namely the cosmological constant \( m^2 \), and the coupling constant \( \alpha \) in front of the Riemann squared action, which are both dimensionless if measured in units of \( \kappa \) (which we set to unity, for convenience). We can also include the discrete parameter \( \sigma \), the coefficient of the Einstein-Hilbert term, taking the values 1, 0 or \(-1\). Since the three-dimensional \( \mathcal{N} = 1 \) supersymmetry is less restrictive than the original six-dimensional \( \mathcal{N} = (1, 0) \) supersymmetry, we can generalise the three-dimensional \( \mathcal{N} = 1 \) theory to include eight parameters, with the Lagrangian in the bosonic sector given by

\[
\mathcal{L} = \sigma e^{-2\phi} \left[ R + 2S^2 + 4(\partial\phi)^2 + 4m^2 \right] + M \mathcal{L}_{\text{Bos}}^{C}

+ \frac{1}{4} \alpha \mathcal{L}_{Riem}^{\text{Bos}} + 2\beta m \left( \mathcal{L}_{\text{LCS}}^{\text{Bos}} - a\mathcal{L}_{S^3}^{\text{Bos}} \right) + b\mathcal{L}_{R^2}^{\text{Bos}} + c\mathcal{L}_{S^4}^{\text{Bos}},
\]

(4.1)
where $\mathcal{L}^{\text{Bos}}_C$, $\mathcal{L}^{\text{Bos}}_{\text{Riem}}$, $\mathcal{L}^{\text{Bos}}_{\text{LCS}}$ and $\mathcal{L}^{\text{Bos}}_S$ are as defined in (3.8), (3.9), (3.10) and (3.11), while the last two terms in the Lagrangian are given by [15, 16]

$$e^{-1}\mathcal{L}^{\text{Bos}}_{R^2} = R^2 - 16(\partial S)^2 + 12RS^2 + 36S^4, \quad (4.2)$$

$$e^{-1}\mathcal{L}^{\text{Bos}}_{S^4} = 3RS^2 + 10S^4. \quad (4.3)$$

Note that we have set $\kappa^2 = 1$ and introduced the new positive or negative real parameters $M, a, b, c$. Thus the count of eight parameters comprises seven real dimensionless parameters (measured in units of $\kappa$), and the discrete parameter $\sigma = \pm 1, 0$.\footnote{One could take the view that $\sigma$ should not strictly speaking be thought of as a non-trivial parameter in the theory, since, as noted in section 2, any value of $\sigma$ can be obtained from $\sigma = 1$ by means of the field transformation $\phi \rightarrow \phi - \frac{1}{2} \log \sigma$. However, it is useful to include it explicitly in the Lagrangian since the cases where $\sigma$ is negative and $\sigma = 0$ have properties that are physically distinct from the case when $\sigma$ is positive. Thus perhaps the most useful viewpoint is that there are three distinct seven-parameter theories, corresponding to $\sigma = +1$, $\sigma = -1$ and $\sigma = 0$.}

Compared to the seven-parameter model of [16] our extra parameter is $m$.\footnote{The counting of seven parameters in [16] also includes the discrete constant $\sigma$, and again, one could perhaps most appropriately view the model as comprising three distinct six-parameter theories. Although there is no $\phi$ field in the theory in [16], $\sigma$ is again in a sense a “redundant” parameter, since it can be introduced, starting from the case $\sigma = 1$, by means of appropriate scaling transformations of the fields and the other coupling constants.}

We note that

$$D = 6 \text{ supersymmetry } \implies M = 4m, \quad \beta = \alpha, \quad a = 1, \quad b = c = 0. \quad (4.4)$$

It should be emphasised that the case with $\beta = -\alpha$ can also be lifted to six-dimensional supergravity, provided that the supersymmetry transformation rules and spinor chiralities are modified appropriately.

Turning to the generalized massive supergravity model, the bosonic part of the full Lagrangian takes the form

$$e^{-1}\mathcal{L} = \sigma e^{-2\phi} \left( R + 4(\partial \phi)^2 + 2S^2 + 4m^2 \right) + MS + \alpha R_{\mu\nu}R^{\mu\nu} + \frac{1}{4}(4b - \alpha)R^2$$

$$-2(\alpha + 8b)(\partial S)^2 + (3\alpha + 36b + 10c)S^4 + (\alpha + 12b + 3c)RS^2$$

$$-2\beta a(RS + 2S^3) + 2\beta me^{-1}\mathcal{L}_{\text{LCS}}. \quad (4.5)$$

For generic values of these parameters, the fluctuations around the AdS$_3$ vacuum solution is expected to describe two helicity $|\nu| = 2$ states and three scalars, the latter coming from the trace of the metric, the auxiliary field $S$ and the dilatonic scalar $\phi$. For special
values of the coupling constants, however, some or all of the helicity $|\nu| = 2$ states may become singletonic in the sense that they become confined to propagate on the boundary of AdS$_3$. Additionally, it may be possible that the trace of the metric can be gauged away by residual coordinate transformations. The massive supergravity model with bosonic sector given in (4.1) differs from that studied recently in [16] owing to the replacement of $R - 2S^2$, which is the bosonic sector of the simple supergravity, by the Lagrangian (3.8). One of the consequences of doing so is that we have one extra parameter in the full Lagrangian.

We expand the metric around the AdS$_3$ background as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, and impose the gauge condition

$$\nabla^\mu H_{\mu\nu} = 0$$

(4.6)

where $H_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{3}\bar{g}_{\mu\nu}h$, $h \equiv h_{\mu\nu}\bar{g}^{\mu\nu}$. We expand the scalar fields $S$ and $\phi$ around the supersymmetric vacuum solution $\bar{S} = -m$ and $\bar{\phi} = 0$, and denote the fluctuation fields by $s$ and $\phi$, respectively. The requirement that the $S$ field equation be satisfied by the vacuum solution implies that

$$M = 4m(\sigma + m^2c).$$

(4.7)

We shall use this relation in subsequent calculations to eliminate $M$. Setting $m = 1$ from here on for simplicity, the resulting field equations for the scalar fields $\phi$, $s$, and the trace of the Einstein equation, respectively, take the form

$$6\sigma \Box \phi - \sigma(\Box - 3)h - 6\sigma s = 0,$$

(4.8)

$$3(\alpha + 8b)\Box s + 3(\sigma + 3c + 6\gamma)s + \gamma(\Box - 3)h + 6\sigma \phi = 0,$$

(4.9)

$$(\Box - 3)Y = 0,$$

(4.10)

where $\Box$ is defined in the AdS$_3$ background, and

$$Y = (\alpha + 8b)(\Box - 3)h - \gamma h + 12(\rho s + \sigma \phi),$$

$$\gamma \equiv \sigma + 3c - \alpha + 2\beta a,$$

$$\rho \equiv 12b + 3c + \alpha + \beta a.$$ 

(4.11)

The $s$ field equation differs from [16] only in presence of the scalar field $\phi$ and the sign of $\sigma$. As commented upon earlier, the scalar $\phi$ couples to the supergravity multiplet in a non-trivial way. It follows that, unless $\sigma = 0$, there is no choice of parameters for which all three scalars can be eliminated by using their equations of motion. Consequently, the discussion
of the unitarity of the theory is more complicated than the discussion in [7], where, for a certain choice of parameters, s and h could be eliminated allowing a unitarity condition to be obtained. If $\sigma = 0$, the partial results of [16] on perturbative unitarity can be used. With the parameters required by $D = 6$ supersymmetry, however, one does not obtain a model for which a conclusion about perturbative unitarity can be drawn, based on the results of [16] alone.

There remains the traceless part of the Einstein equation, which can be expressed as

$$\mathcal{D}(1)\mathcal{D}(-1)\mathcal{D}(\eta_+)\mathcal{D}(\eta_-)H_{\mu\nu} = \gamma^{-1} J_{\mu\nu}, \quad (4.12)$$

where we have defined the differential operator

$$[\mathcal{D}(\eta)]\nu = \delta^\nu_\nu + \eta \varepsilon^{\alpha\nu} \nabla_\alpha,$$  

for a constant $\eta$, and

$$\eta_\pm = \gamma^{-1} \left( -\beta \pm \sqrt{\beta^2 - \gamma\alpha} \right). \quad (4.14)$$

Note that this result is independent of the parameter $b$ that occurs in front of $\mathcal{L}_{\mathcal{R}^2}$ in the total Lagrangian. We have assumed that $\gamma \neq 0$. The source term is given by

$$J_{\mu\nu} = -\frac{1}{3} \left( \nabla_\mu \nabla_\nu - \frac{1}{3} g_{\mu\nu} \Box \right) Y. \quad (4.15)$$

These satisfy $\eta_+ \eta_- = \alpha \gamma^{-1}$. The integrability condition $\nabla^\mu J_{\mu\nu} = 0$ is satisfied by virtue of the field equation (4.10). Provided that $\gamma \neq 0$, the source term $J_{\mu\nu}$ can be absorbed into the definition of $H_{\mu\nu}$ in such a way that it maintains the traceless and transverse properties of $H_{\mu\nu}$.

For appropriate boundary conditions, the vanishing of the left-hand side of (4.12) implies that $H_{\mu\nu}$ is annihilated by one or another of the four commuting $\mathcal{D}$ factors. In general, the helicity $\nu$ and lowest energy $E_0$ of an excitation satisfying $\mathcal{D}(\eta)H_{\mu\nu} = 0$ are given by [7]

$$\nu = \frac{2|\eta|}{|\eta|}, \quad E_0 = 1 + \frac{1}{|\eta|}, \quad (4.16)$$

and the mode furnishes a unitary irreducible representation of the AdS$_3$ group if $E_0 \geq |\nu|$, which means $|\eta| \leq 1$. If $\eta = \pm 1$, the mode decouples in the bulk, and just describes an excitation in the boundary theory. Thus generically, when $|\eta| \neq 1$, there are two bulk graviton modes with $|\nu| = 1$ and two boundary modes.

Note that by definition $\mathcal{D}(\eta)H_{\mu\nu}$ means $[\mathcal{D}]\nu^\epsilon H_{\mu\epsilon}$, and that, as can easily be verified, this preserves the transversality and tracelessness of $H_{\mu\nu}$. Furthermore, the operators commute on $H_{\mu\nu}$, in the sense that $[\mathcal{D}(\eta_1), \mathcal{D}(\eta_2)]H_{\mu\nu} = 0$ for any $\eta_1$ and $\eta_2$. Another useful identity satisfied by these operators is that $\mathcal{D}(\eta_1)\mathcal{D}(\eta_2)H_{\mu\nu} = \eta_1 \eta_2 (\Box + 3) H_{\mu\nu} + \mathcal{D}(\eta_1 + \eta_2)H_{\mu\nu}$.
The expression (4.14), for $\gamma \neq 0$, agrees with that found for the seven-parameter action in [16]. The additional parameter that we have in our action and the coupling of the scalar multiplet does not change this result, because upon expansion around the vacuum solution we have employed, the linearization of $\sqrt{-g} \left[ \sigma e^{-2\phi}(R + 2S^2 + 4m^2) + MS \right]$ gives the same result as $\sqrt{-g} \left[ \sigma (R - 2S^2) + MS \right]$ in [16]. As we saw earlier, however, the field equations for the scalar fields do differ in the two cases.

For $\gamma = 0$, a straightforward limit of (4.12) gives

$$D(1)D(-1)D(\eta)\varepsilon_{\mu}^{\alpha \beta} \nabla_\alpha H_{\beta \nu} = -\frac{1}{2\beta} J_{\mu \nu}, \quad (4.17)$$

where $\eta = -\alpha/(2\beta)$. The source term cannot be absorbed into a redefinition of $H_{\mu \nu}$ in this case. Upon acting with $\varepsilon_{\lambda \tau \mu} \nabla_\tau$, the source term drops out, yielding [7]

$$D(1)D(-1)D(\eta)\left[ \Box + 3 \right]H_{\mu \nu} = 0. \quad (4.18)$$

In addition to a single helicity 2 massive graviton, this equation also describes a partially-massless graviton [17, 18, 19].

The critical points where the massive graviton decouples arise when either $|\eta_+| = 1$ and/or $|\eta_-| = 1$ and/or $\eta_+ = \eta_-$, with $\eta_\pm$ given in (4.14). The criticality condition in our eight-parameter model coincides with that in [7], where an extensive list of critical points was given. For our three-parameter theory that can be lifted to six dimensions, $a = 1$, $b = 0 = c$ and the critical points are given by

$$\sigma^2 = 1 : \quad \text{Case 1:} \quad \beta = +\alpha : \quad \sigma \alpha = -\frac{1}{4}, \quad \eta_+ = 1,$$

$$\text{Case 2:} \quad \beta = -\alpha : \quad \sigma \alpha = +\frac{1}{4}, \quad \eta_+ = \eta_- = 1,$$

$$\sigma = 0 : \quad \text{Case 3:} \quad \text{Any } \alpha \neq \beta, \quad \eta_- = -1,$$

$$\text{Case 4:} \quad \text{Any } \alpha = \beta, \quad \eta_+ = \eta_- = -1. \quad (4.19)$$

In Case 1 and Case 3, there are only single helicity $-2$ bulk states with AdS energies $E_0 = 4$ and $E_0 = 1 + |(2\beta - \alpha)/\alpha|$, respectively. In Case 3, taking $\beta = -\alpha$ gives $E_0 = 4$ as well. In Case 2 and Case 4, there are no propagating bulk gravitons at all.

Finally, we may evaluate the central charges for the right-handed and left-handed Virasoro algebras of the boundary CFT, following the procedure described in [20, 21, 22, 7], finding

$$C_L = \frac{3}{2G} \left( \sigma + 3\alpha + 2\beta(a + 1) \right), \quad C_R = \frac{3}{2G} \left( \sigma + 3\alpha + 2\beta(a - 1) \right). \quad (4.20)$$
(We have restored Newton’s constant in order to simplify comparison with previous results.)

For our three-parameter theory coming from the reduction from six dimensions, we have $a = 1$ and $c = 0$. Thus for $\sigma^2 = 1$, we have $c_L = 0$ if $\sigma \beta = -\frac{1}{4}$, while $C_R = 3\sigma/(2G)$. This corresponds to the critical points listed as Case 1 and Case 2 in (4.19). If instead $\sigma = 0$, then $C_R = 0$ for any $\beta$, and $C_L = 6\beta/G$. In this case, $\beta$ can be chosen so that $C_L$ has any desired value. This may have interesting consequences for the corresponding CFT. This case leads to the critical points listed as Case 2 and Case 4 in (4.19).

In summary, Case 2 can be viewed as the higher-derivative generalization of chiral gravity proposed in [3], and Case 4 is an alternative higher-derivative version in which the Hilbert-Einstein term is omitted, with the parameters chosen in each case so that no massive gravity modes arise. Moreover, this phenomenon occurs in the alternative theory for any value of the parameters with $\alpha = \beta$.

5 Dualisation and the Heterotic String

The six-dimensional supergravity whose bosonic Lagrangian is given by (2.23) is closely related to the dimensional reduction of the effective theory of the heterotic string. To be more precise, we may consider the ten-dimensional supergravity constructed in [23], where the supersymmetrisation of the anomaly-canceling $\text{tr}(R \wedge R)$ term in the Bianchi identity for the 3-form $H_{(3)}$ was studied. The goal in [23] was to consider only those terms that are necessary in order to obtain a Lagrangian that remains supersymmetric when the the Bianchi identity $dF_{(3)} = \text{tr}(F \wedge F)$ is modified to $dF_{(3)} = \text{tr}(F \wedge F) - \alpha' \text{tr}(R \wedge R)$. It was shown that this requires introducing higher-order curvature terms in the Lagrangian, starting with $\frac{1}{4} \alpha e^{-2\phi} R^\mu_{\nu\rho\sigma} R_{\nu\rho\sigma}$, and that furthermore the curvature in these terms is built from the connection $\tilde{\Gamma}^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} - \frac{1}{2} F^\mu_{\nu\rho}$ with bosonic torsion. Supersymmetry requires that the Lagrangian with the anomaly-canceling $\text{tr}(R \wedge R)$ term have corrections to arbitrarily high order in $\alpha'$ (and hence arbitrarily high powers of the curvature). In [23], these corrections were studied up to and including order $\alpha'^2$.

If the theory of [23] is reduced on $T^4$ to six dimensions (setting the Yang-Mills fields to zero, and consistently truncating to keep only the relevant fields), it can be compared with the theory of [14, 15], given in (2.23). The reduced heterotic effective action gives

$$L_6 = \sqrt{-g} e^{-2\phi} \left[ R + 4(\partial \phi)^2 - \frac{1}{12} F^\mu_{\nu\rho} F_{\mu\nu\rho} + \frac{1}{4} \alpha \tilde{R}^\mu_{\nu\rho\sigma} \tilde{R}_{\mu\nu\rho\sigma} \right] + \mathcal{O}(\alpha^2),$$

where here the curvature in the terms at order $\alpha$ is calculated using the connection $\tilde{\Gamma}^\mu_{\nu\rho} =$
$\Gamma^{\mu\nu\rho} - \frac{1}{2} F^{\mu\nu\rho}$, and $F^{(3)}$ is given by

$$F^{(3)} = dA^{(2)} - \frac{1}{2} \alpha \tilde{I}_3. \tag{5.2}$$

If we neglect for a moment the torsion contributions to the higher-order curvature terms in the two six-dimensional theories, it is easy to see that (2.23) and (5.1) are related by dualisation, with the dilaton and metric of the theory (2.23) transformed according to

$$\phi \rightarrow -\phi, \quad g_{\mu\nu} \rightarrow e^{-2\phi} g_{\mu\nu}. \tag{5.3}$$

(All fields on the right-hand sides are the transformed fields.) Then we define (using the dualised dilaton and metric)

$$F_{\mu\nu\rho} = \frac{1}{6} \varepsilon_{\mu\nu\rho\alpha\beta\gamma} e^{2\phi} H^{\alpha\beta\gamma}. \tag{5.4}$$

Note that under this dualisation, the Chern-Simons term $L_{CS}$ in (2.23) gives rise to the anomaly-canceling $\text{tr}(R \wedge R)$ term in the theory described by (5.1).

If we include the torsion contributions, the duality between (2.23) and (5.1) is harder to see, but we conjecture that it does still exist, and it implemented by exactly the same transformations (5.3) and (5.4). It should be emphasised that in particular, this conjectured duality relates the exact, closed-form, theory given by (2.23) (which has no curvature corrections beyond $O(\alpha)$) to the theory described by (5.1) with its curvature corrections to all orders in $\alpha$.

The strongest reason for believing this duality conjecture is that the theory described by (2.23) (together with its fermionic terms as given in [14, 15]) is exactly supersymmetric, with $\mathcal{N} = (1,0)$ supersymmetry. On the other hand, the theory described by (5.1) (together with its fermionic terms) is the dimensional reduction of the supersymmetrisation of the anomaly-canceling $\text{tr}(R \wedge R)$ term in ten dimensions, and this theory (after the reduction and truncation we have performed) also has $\mathcal{N} = (1,0)$ supersymmetry in six dimensions. In each case, the supersymmetrisation procedure gave a unique result. Since the dualisation of (2.23) will certainly give rise to some six-dimensional theory with an anomaly-canceling $\text{tr}(R \wedge R)$ term, the uniqueness of the constructions implies that it can only be giving rise to the theory described by (5.1).

A remarkable consequence of this duality is that the infinite set of correction terms to all orders in $\alpha$ that are needed in order to supersymmetrise the anomaly-canceling $\text{tr}(R \wedge R)$ term in the theory studied in (23) can be deduced (modulo terms that vanish in the $T^4$ reduction) simply by performing the dualisation of the theory constructed in (2.23), which is exactly supersymmetric without the need for any corrections beyond the order $\alpha$. 

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Providing a complete proof of the duality would be quite involved, and we shall not attempt it here. Instead, we shall just focus on a sub-calculation that is already non-trivial, and that elucidates a seemingly puzzling aspect of the dualisation. The puzzle is that in both the formulation in (2.23) and the formulation in (5.1), the quadratic curvature terms are constructed from a connection with bosonic torsion; $-\frac{1}{2}H_{\mu\nu\rho}$ in the case of (2.23) and $-\frac{1}{2}F_{\mu\nu\rho}$ in the case of (5.1). However, these fields are related by (5.4), and so one might have expected that if the (2.23) theory had torsion proportional to $H_{\mu\nu\rho}$ then the dual theory would have torsion proportional to $\epsilon_{\mu\nu\rho\alpha\beta\gamma}F_{\alpha\beta\gamma}$ rather than $F_{\mu\nu\rho}$. Here, we shall look specifically at the contributions at linear order in $F_{\mu\nu\rho}$, associated with the anomaly-canceling term $\text{tr}(\tilde{R} \wedge \tilde{R})$, in the expression for the dualised field-strength. This calculation shows how the torsion is indeed proportional to $F_{\mu\nu\rho}$, and not $\epsilon_{\mu\nu\rho\alpha\beta\gamma}F_{\alpha\beta\gamma}$. A further simplification in our calculation will be to neglect terms where derivatives land on the dilaton field.

To begin, we expand the the terms in the Lagrangian (2.23) in powers of $H_{\mu\nu\rho}$, keeping only those of quadratic or lower order. From (2.28) we have, up to quadratic order in $H_{\mu\nu\rho}$,

\[ \tilde{R}^{\mu\nu\rho\sigma} \tilde{R}_{\mu\nu\rho\sigma} = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - R_{\alpha\beta\mu\nu} H_{\alpha\gamma\mu} H_{\beta\nu} + \frac{1}{2} (\nabla_\mu H_{\nu}^{\alpha\beta} - \nabla_\nu H_{\mu}^{\alpha\beta}) \nabla^\mu H^\nu_{\alpha\beta}, \]

which, upon use of the Bianchi identity $dH_{(3)} = 0$ and the zeroth-order equation of motion $\nabla^\mu H_{\mu\nu\rho} = 0$ (recall we are neglecting terms involving derivatives of $\phi$ here) gives

\[ \tilde{R}^{\mu\nu\rho\sigma} \tilde{R}_{\mu\nu\rho\sigma} = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 3 R^{\alpha\beta\mu\nu} H_{\alpha\gamma\mu} H_{\beta\nu} + O(H^3). \]

The Chern-Simons term in (2.23) has the expansion

\[ L_{CS} = -\frac{1}{12} \beta \sqrt{-g} \epsilon^{\mu\nu\rho\alpha\beta\gamma} \tilde{I}_{\mu\nu\rho} H_{\alpha\beta\gamma}, \]

\[ = -\frac{1}{12} \beta \sqrt{-g} \epsilon^{\mu\nu\rho\alpha\beta\gamma} \left( I_{\mu\nu\rho} H_{\alpha\beta\gamma} + 3 H_{\mu\sigma\lambda} R_{\lambda\nu\rho} H_{\alpha\beta\gamma} \right) + O(H^3). \]

Using the Schoutens identity $R_{\sigma}^{\alpha\beta\gamma\nu} \epsilon^\mu_{\nu\rho\alpha\beta\gamma} = 0$, we find, after some manipulations, that the second term in (5.6) is proportional to the Ricci tensor and hence by using the zeroth-order Einstein equation (with derivatives of $\phi$ neglected), it becomes of higher than quadratic order in $H$. To the order we are working, Lagrangian (2.23) can therefore be expanded as

\[ L = \sqrt{-g} \left[ e^{-2\phi} (R - \frac{1}{12} H^2) - \frac{1}{12} \beta I_{\mu\nu\rho} H_{\alpha\beta\gamma} \epsilon^{\mu\nu\rho\alpha\beta\gamma} - \frac{3}{4} \alpha R^{\alpha\beta\mu\nu} H_{\alpha\lambda\mu} H_{\beta\lambda\nu} \right]. \]

To perform the dualisation, we next add a Lagrange multiplier term

\[ L_{LM} = -\frac{1}{36} \sqrt{-g} \tilde{F}_{\mu\nu\rho} H_{\alpha\beta\gamma} \epsilon^{\mu\nu\rho\alpha\beta\gamma}, \]
to the Lagrangian, where $\tilde{F}^{(3)} \equiv dA_{(2)}$, make the changes of variables given in (5.3) and (5.4), and then vary with respect to $H_{\alpha \beta \gamma}$. This gives the result that

$$F_{\mu \nu \rho} = 3 \partial_{[\mu} A_{\nu \rho]} - \frac{1}{2} \beta I_{\mu \nu \rho} - \frac{3}{2} \alpha R^{\alpha \beta}_{\mu \nu \rho} F_{\rho |\alpha \beta} .$$

(5.9)

With $\alpha = \beta$ we see that indeed this is the correct expansion, up to linear order in $F_{(3)}$, of the expression

$$F_{(3)} = dA_{(2)} - \frac{1}{2} \alpha \tilde{I}_3 ,$$

(5.10)

where

$$\tilde{I}_3 = (\tilde{\Gamma}^\alpha_{\mu \beta} \partial_{\nu} \tilde{\Gamma}^\beta_{\rho \alpha} + \frac{2}{3} \tilde{\Gamma}^\alpha_{\mu \beta} \tilde{\Gamma}^\beta_{\nu \gamma} \tilde{\Gamma}^\gamma_{\rho \alpha}) dx^\mu \wedge du^\nu \wedge dx^\rho ,$$

(5.11)

with $\tilde{\Gamma}^\mu_{\nu \rho} = \Gamma^\mu_{\nu \rho} - \frac{1}{2} F^\mu_{\nu \rho}$. Thus at the order to which we have worked here, we have seen how the two six-dimensional theories are related by duality.

It should be emphasised that from (5.3), the relationship between the six-dimensional theory constructed in [14, 15], and the dimensional reduction of the heterotic string, is non-perturbative in nature, since the sign of the dilaton is reversed. Consequently, the embedding of the three-dimensional massive supergravity in string theory is also of a non-perturbative nature. This is consistent with the fact that both the three-dimensional and the six-dimensional theories are complete, whereas the higher-order terms in the heterotic theory require infinite sequences of higher curvature terms.

6 Conclusions

In this paper we have obtained a new type of massive three-dimensional gravity, by performing a 3-sphere reduction of the off-shell six-dimensional $\mathcal{N} = (1, 0)$ supergravity that was constructed in [14, 15]. This six-dimensional starting point is of particular interest because it is fully supersymmetric with just quadratic curvature terms added to the basic Poincaré supergravity, and hence we can obtain a closed-form result in three dimensions. The three-dimensional theory comprises an $\mathcal{N} = 1$ off-shell supergravity multiplet coupled to an on-shell scalar multiplet which cannot be non-trivially decoupled. The theory has three parameters (two continuous plus one discrete).

Because the constraints of $\mathcal{N} = 1$ supersymmetry in three dimensions are weaker than those of $\mathcal{N} = (1, 0)$ supersymmetry in six dimensions, we can actually relax the relations between the coefficients of the terms in the dimensionally-reduced Lagrangian, whilst still maintaining $\mathcal{N} = 1$ supersymmetry. In this way, we then generalised our original three-
dimensional theory to one containing eight parameters (of which seven are continuous and one is discrete).

We also considered the possible relation between our six-dimensional starting point in \[14, 15\], and the heterotic string in ten dimensions. More specifically, we focussed on the ten-dimensional supergravity considered in \[23\], in which the anomaly-cancelling tr\((R \wedge R)\) term of the heterotic theory was supersymmetrised. This required the introduction of curvature-squared terms, and in fact an infinite sequence of higher-order curvature terms (which were not constructed in \[23\]). If this theory is reduced on \(T^{4}\) and truncated to \(\mathcal{N} = (1, 0)\), the resulting six-dimensional theory must evidently be the dual of the theory constructed in \[14, 15\] that served as our starting point in this paper. This is rather remarkable, since the reduction of \[23\] would yield an infinite sequence of higher-order curvature terms, whereas the theory in \[14, 15\] is exactly supersymmetric with no curvature terms beyond the quadratic order.

As is known from recent work, there are quite large classes of higher-order superinvariants in three dimensions that each involve only a finite number of terms. So far, in six dimensions, the only known example is the curvature-squared invariant studied in \[14, 15\]. It would be interesting to investigate whether further such invariants might exist in six dimensions.

The 3-sphere reduction that we performed in this paper was a very simple one that did not involve non-singlets under the action of the isometry group of the sphere. It is unclear whether a consistent reduction that retained all the \(SO(4) = SU(2)_L \times SU(2)_R\) Kaluza-Klein gauge fields is possible, but it would certainly be possible to perform a consistent DeWitt group-manifold reduction, retaining the gauge fields of \(SU(2)_R\) and all other singlets under \(SU(2)_L\). Such reductions of six-dimensional supergravity, in the absence of higher-order curvature terms, have been considered in the past \[24, 25, 26\]. It would be interesting to carry out analogous reductions including the higher-order terms.

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A Notation and conventions

The six-dimensional $\Gamma$-matrices obey the Clifford algebra \{\(\Gamma_a, \Gamma_b\)\} = 2\(\eta_{ab}\) with $\eta_{ab} = \text{diag}(-,+,\ldots,+)$\footnote{This convention differs from that in \cite{14} where $\eta_{ab} = \text{diag}(+,+\ldots,+)$. Accordingly, we let $\varepsilon^{a_1\ldots a_6} \rightarrow -i\varepsilon^{a_1\ldots a_6}$ in \cite{14}. Our definition of the Ricci tensor $R_{\mu\nu} = g^{\lambda\tau}R_{\mu\lambda\nu\tau}$ also differs from that of \cite{14} where $R_{\mu\nu} = g^{\lambda\tau}R_{\mu\lambda\nu\tau}$.}. The spinors in six dimensions are symplectic Majorana-Weyl, obeying the reality condition
\[
(\psi_i)^* = B\Omega^{ij}\psi_j , \quad i = 1, 2 , \tag{A.1}
\]
where $\Omega^{ij} = -\Omega_{ji}$ is the $Sp(1)$ invariant constant tensor, and $B$ is constructed from six-dimensional $\Gamma$-matrices such that
\[
\Gamma_a^* = -B \Gamma_a B^{-1} , \quad B^T = -B , \quad B^* = -B . \tag{A.2}
\]

The Dirac conjugate is defined as $\bar{\psi}_i = i (\psi_i)^* \gamma_0$. The $Sp(1)$ indices are raised and lowered using $\Omega$:
\[
\psi_i = \Omega_{ij} \psi_j , \quad \bar{\psi}_i = \bar{\psi}_j \Omega^{ji} , \quad \Omega_{ij} \Omega^{jk} = -\delta^k_i . \tag{A.3}
\]
The contraction of the $Sp(1)$ indices is such that $\bar{\varepsilon} \psi_i = \bar{\varepsilon}_i \psi_i$. The fermionic bilinears have the property $\bar{\varepsilon}_i \Gamma_{a_1\ldots a_6} \psi_j = -(-1)^n \bar{\psi}_j \Gamma_{a_n\ldots a_1} \varepsilon_i$.

Under $SO(5,1) \rightarrow SO(2,1) \times SO(3)$, we let $a \rightarrow (a,a')$, where now $a = 0,1,2$ and $a' = 1,2,3$, and the $\Gamma$-matrices decompose as
\[
\Gamma_a = \gamma_a \times 1 \times \Sigma_1 , \quad \Gamma_{a'} = 1 \times \sigma_{a'} \times \Sigma_2 , \\
B = 1 \times \sigma_2 \times \Sigma_3 , \quad \Gamma_7 = -\Gamma_0 \Gamma_1 \ldots \Gamma_6 = 1 \times 1 \times \Sigma_3 , \\
\Gamma_{\mu_1\ldots\mu_6} = -\varepsilon_{\mu_1\ldots\mu_6} \Gamma_7 . \tag{A.4}
\]

where \{\(\gamma_a, \gamma_b\)\} = 2\(\eta_{ab}\) with $\eta_{ab} = \text{diag}(-,+,\ldots,+)$, and $\sigma_{a'}$ as well as $(\Sigma_1, \Sigma_2, \Sigma_3)$ are are the standard Pauli matrices. In our conventions $\gamma_{012} = 1$. The supersymmetry parameter has positive chirality $\Gamma_7 \varepsilon^i = \varepsilon^i$, which implies $\Sigma_3 \varepsilon^i = \varepsilon^i$. Our conventions for the Levi-Civita tensor densities in six and three dimensions are that $\epsilon_{012345} = +1$ and $\epsilon_{012} = +1$.

B Expansion formula

Given the action
\[
I = \int d^3x \sqrt{-g} \left( c_1 R_{\mu\nu}^2 + c_2 R^2 \right) , \tag{B.1}
\]
its variation with respect to the metric is
\[ \delta I = \int d^3 x \delta g^{\mu \nu} \sqrt{-g} L_{\mu \nu}, \]  
where
\[ L_{\mu \nu} = -(2c_2 + c_1) \nabla_\mu \nabla_\nu R + c_1 \Box R_{\mu \nu} + \frac{1}{2} (4c_2 + c_1) g_{\mu \nu} \Box R \]
\[ - \frac{1}{2} (2c_2 + 2c_1) g_{\mu \nu} R^2 + (2c_2 + 3c_1) R R_{\mu \nu} - \frac{1}{2} c_1 \left( 8 R_{\mu \lambda} R^{\lambda \nu} - 3 g_{\mu \nu} R_{\sigma \tau}^2 \right). \]

The linearizations of various quantities about AdS_3 with \( \bar{R} = 6\Lambda \), in the gauge \( \nabla_\mu H_{\mu \nu} = 0 \) with \( H_{\mu \nu} = h_{\mu \nu} - \frac{1}{3} \bar{g}_{\mu \nu} h \), and with total derivative terms discarded, take the form:
\[ R^{(1)}_{\mu \nu} = - \frac{1}{2} \left( \Box - 6\Lambda \right) H_{\mu \nu} - \frac{1}{6} \left( \nabla_\mu \nabla_\nu + g_{\mu \nu} \Box \right) h, \]
\[ R^{(1)} = \left( g^{\mu \nu} R^{(1)}_{\mu \nu} \right) = - \frac{2}{3} \left( \Box + 3\Lambda \right) h, \]
\[ G^{(1)}_{\mu \nu} = - \frac{1}{2} \Box H_{\mu \nu} - \frac{1}{6} \left( \nabla_\mu \nabla_\nu - \bar{g}_{\mu \nu} \Box \right) h, \]
\[ C^{(1)}_{\mu \nu} = - \frac{1}{2} \left( \Box - 2\Lambda \right) \varepsilon^{\alpha \beta} \nabla_\alpha H_{\beta \nu}. \]

(A superscript \((1)\) indicates that the quantity on which it has been placed is linearized around the AdS_3 background. After doing this, we then drop the bars from derivative operators in the background.) At the higher derivative level, the following expansion formula is useful:
\[ \left( \Box R^{(1)}_{\mu \nu} \right) = \Box \left( R^{(1)}_{\mu \nu} - 2\Lambda h_{\mu \nu} \right) \]
\[ = - \frac{1}{2} \Box \left( \Box - 2\Lambda \right) H_{\mu \nu} - \frac{1}{6} \nabla_\mu \nabla_\nu \left( \Box + 6\Lambda \right) h - \frac{1}{6} g_{\mu \nu} \Box \left( \Box + 2\Lambda \right) h. \]

Note also that \( \left( \Box R \right)^{(1)} = \Box R^{(1)} \) and \( \left( \nabla_\mu \nabla_\nu R \right)^{(1)} = \nabla_\mu \nabla_\nu R^{(1)} \). Other useful formulae include the arbitrary variation of the Einstein-Hilbert Lagrangian
\[ \delta \left( \sqrt{-g} R \right) = \sqrt{-g} \left( G_{\mu \nu} - \nabla_\mu \nabla_\nu + g_{\mu \nu} \Box \right) \delta g^{\mu \nu}, \]
and the commutators
\[ [\Box, \nabla_\mu] h = \nabla_\mu (\Box + \Lambda) h, \]
\[ [\Box, \nabla_\mu \nabla_\nu] h = 6\Lambda \left( \nabla_\mu \nabla_\nu - \frac{1}{3} \bar{g}_{\mu \nu} \Box \right) h. \]

Finally, we record the three-dimensional identity
\[ R_{\mu \nu}^{ab} = 4 R_{[\mu}^{[a} e_{\nu]} b]} - R e_{[\mu}^{[a} e_{\nu]} b] . \]
C Off-shell $\mathcal{N} = (1, 0)$ supergravity in 6D in Einstein frame

The bosonic part of the Lagrangian is given by [14]

$$
e^{-1} \mathcal{L} = R + 4 V^i V^j - \partial_\mu \phi \partial_\mu \phi - \frac{1}{12} e^{-2\phi} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{2 \times 5!} G_{\mu_1...\mu_6} G^{\mu_1...\mu_6} - \frac{1}{5! \sqrt{2}} \varepsilon_{\mu_1...\nu_5} G_{\nu_1...\nu_5} V^i \delta_{ij}. \quad (C.1)$$

The action is invariant under off-shell supersymmetry transformations, which, up to cubic fermions, take the form [14]

$$
\delta e^a_\mu = \frac{1}{2} \bar{\epsilon} \Gamma^a \psi_\mu, \\
\delta \psi_\mu = D_\mu (\omega) \epsilon - \frac{1}{16} e^{-\phi} \Gamma \cdot H \Gamma \mu \epsilon + \Gamma \mu \eta, \\
\delta B_{\mu\nu} = -\bar{\epsilon} \Gamma_{[\mu} \psi_{\nu]} e^\phi - \bar{\epsilon} \Gamma_{\mu\nu} \psi, \\
\delta V^{ij} = -4 e^{(i} \phi \rho \Gamma \cdot H \psi^{j)} - 2 e^{-\phi} e^{(i} \Gamma \mu \psi^{j)} - 4 \eta (i \psi \mu), \\
\delta C_{\mu\nu\rho\sigma} = 2 \sqrt{2} \bar{\epsilon} e^{[i} \Gamma_{[\mu\nu} \psi_{\rho\sigma]} \delta_{ij}, \\
\delta \psi = \frac{1}{4} e^\phi \partial_\mu \phi \epsilon - \frac{1}{48} \Gamma \cdot H \epsilon - \epsilon^\phi \eta, \\
\delta \phi = e^{-\phi} \bar{\epsilon} \psi, \quad (C.2)
$$

where [14]

$$
\phi_\mu = -\frac{1}{16} (\Gamma^\rho \Gamma_\mu - \frac{2}{3} \Gamma_\mu \Gamma^\rho) \psi^\rho - \frac{1}{48} \Gamma \mu \chi, \\
\chi = 6 e^{-\phi} \Gamma^\mu \tilde{D}_\mu \psi + \frac{1}{4} e^{-2\phi} \Gamma \cdot H \psi, \\
\eta_i = \left( -\frac{1}{4 \sqrt{2}} \Gamma^\rho \Gamma^{(i} V^j \Gamma_\rho \delta_{i}^{j)} \epsilon_j + \frac{1}{5 \times 8 \sqrt{2}} \Gamma^\rho \Gamma^{(i} G_{[\mu_1...\mu_5} \Gamma_\rho \Gamma_{\mu_6...\mu_5} \epsilon^j \delta_{i}^{j)} \right), \quad (C.3)
$$

and

$$
\tilde{D}_\mu \psi = D_\mu (\omega) \psi + \frac{1}{48} \Gamma \cdot H \psi_\mu - \frac{1}{16} e^\phi \Gamma_\mu \phi \psi_\mu + e^\phi \phi_\mu, \quad (C.4) \\
\psi'_{\mu\nu} = 2 D_{[\mu} (\omega) \psi_{\nu]} + \frac{1}{4} e^{-\phi} \Gamma \cdot H \Gamma_{[\mu} \psi_{\nu]} \cdot (C.5)
$$

It was observed in [15] that the fields $\phi$ and $\psi$ can be eliminated from the transformation rules of the multiplet of fields $(e_\mu, \psi_\mu, V^i_\mu, B_{\mu\nu})$ by means of the the field redefinitions [23]. These redefinitions lead to drastic simplifications and, dropping the hats on $\hat{e}_\mu, \hat{\psi}_\mu$ and $\hat{\epsilon}$ for simplicity, yield the transformation rules [23].
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