PURE SUBRINGS OF REGULAR LOCAL RINGS, 
ENDOMORPHISM RINGS AND FROBENIUS MORPHISMS 

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Abstract. The aim of this paper is threefold: first, to prove that the endomorphism ring associated to a pure subring of a regular local ring is a noncommutative crepant resolution if it is maximal Cohen-Macaulay; second, to see that in that situation, a different, but Morita equivalent, noncommutative crepant resolution can be constructed by using Frobenius morphisms; finally, to study the relation between Frobenius morphisms of noncommutative rings and the finiteness of global dimension. As a byproduct, we will obtain a result on wild quotient singularities: If the smooth cover of a wild quotient singularity is unramified in codimension one, then the singularity is not strongly F-regular.

1. Introduction

1.1. Pure subrings of regular local rings and NCCRs (Section 2). After Van den Bergh introduced the notion of NCCR (noncommutative crepant resolution), its importance is now well recognized. A nice reference of the theory is Leuschke’s recent survey article. The most basic example is the following:

Example 1.1. Let \( S := k[[x_1, \ldots, x_d]] \), a power series ring over a field \( k \), \( G \subset GL_d(k) \) a small finite subgroup with \( \text{char}(k) \nmid \#G \), and \( R := S^G \) the invariant subring. Then the endomorphism ring of the \( R \)-module \( S \), \( \text{End}_R(S) \), is isomorphic to the skew group ring \( S * G \) and so a NCCR. Namely it has finite global dimension and is a MCM (maximal Cohen-Macaulay) \( R \)-module. (Unlike the original definition of NCCR by Van den Bergh, we do not assume that the base ring \( R \) is Gorenstein.)

We are interested in generalizing this example. In particular, we would like to investigate the following problem:

Problem 1.2. Let \( R \subset S \) be a module-finite and pure extension of commutative domains such that they are both Noetherian, complete, local and normal, and moreover \( S \) is regular. Then is \( \text{End}_R(S) \) a NCCR?

Our main motivation for this problem is the situation in positive characteristic. Then the extension \( R \subset S \) can be purely inseparable and we have no Galois group \( G \) and cannot use the isomorphism \( \text{End}_R(S) \cong S * G \) to show that \( \text{End}_R(S) \) is a NCCR.

The purity condition in the problem does not matter in characteristic zero, because every module-finite extension of normal domains is pure. As Corollary 3.3 shows, the condition is necessary in positive characteristic.

Apart from Example 1.1, the answer to the problem is known to be positive in dimension two. Indeed, in this case, \( \text{End}_R(S) \) is MCM, as every reflexive module

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is MCM. Moreover $S$ contains every indecomposable MCM $R$-module as a direct summand [Ha]. Then $\text{End}_R(S)$ has finite global dimension from the following theorem:

**Theorem 1.3** ([Le1]). Let $S$ be a commutative Noetherian complete local CM ring and $M$ a finitely generated MCM $S$-module which includes every indecomposable MCM $S$-module as a direct summand. Then $\text{End}_S(M)$ has finite global dimension.

In dimension $\geq 3$, one can hardly expect that a ring $R$ with a regular covering $S$ has only finitely many indecomposable MCM modules. However we will generalize Theorem 1.3 as follows:

**Theorem 1.4** (Theorem 2.6). Let $S$ and $M$ be as in Theorem 1.3. Suppose that $R \subset S$ is a pure subring such that $S$ is a finitely generated $R$-module. Suppose also that $\text{Hom}_R(S, M)$ is MCM. Then $\text{End}_R(M)$ has finite global dimension.

As a corollary we obtain a partial answer of Problem 1.2:

**Corollary 1.5** (Corollary 2.11). Let $R$ and $S$ be as in Problem 1.2. If $\text{End}_R(S)$ is MCM, then it is a NCCR.

Thus the remaining problem is:

**Problem 1.6.** With the assumption as in Problem 1.2 is $\text{End}_R(S)$ always MCM? Or, when is it?

### 1.2. A result on wild quotient singularities (Section 3).

Using the above result and Yi's theorem of the global dimension of a skew group ring [Yi], we will prove the following result on wild quotient singularities:

**Corollary 1.7** (Corollary 3.3). Let $S := k[[x_1, \ldots, x_d]]$, $G \subset \text{Aut}_k(S)$ a finite group of order divisible by the characteristic of $k$ and $R := S^G$. Suppose that $S$ is unramified over $R = S^G$ in codimension one. Then $R$ is not a pure subring of $S$ or strongly F-regular.

**Remark 1.8.** Several cases are known where the invariant ring $S^G$ is not Cohen-Macaulay, hence nor F-regular. For instance, it is the case if $G \cong \mathbb{Z}/p^n\mathbb{Z}$, the action is linear and the fixed point locus has codimension $\geq 3$ [ES].

### 1.3. NCCRs via Frobenius (Section 4).

We are interested also in the role of Frobenius morphism in the theory of noncommutative (crepant) resolution. (By a NCR (noncommutative resolution), we mean an endomorphism ring $\text{End}_R(M)$ having finite global dimension.) Firstly an interesting problem similar to Problem 1.2 is:

**Problem 1.9.** Let $R$ be a commutative ring of characteristic $p > 0$ such that its Frobenius map $R \hookrightarrow R^{1/p}$ is finite. Then one can consider the endomorphism ring $\text{End}_R(R^{1/p^e})$ associated to the $e$-iterated Frobenius $R \hookrightarrow R^{1/p^e}$. When is it a NC(C)R?

This is of particular interest, because $\text{End}_R(R^{1/p^e}) \cong \text{End}_{R^{p^e}}(R)$ and

$$D(R) := \bigcup_{e \geq 0} \text{End}_{R^{p^e}}(R)$$

is the ring of differential operators on $R$. If for sufficiently large $e$’s, $\text{End}_R(R^{1/p^e})$ have global dimensions bounded from above, then from [Ko], $D(R)$ also has finite
global dimension. The ring \( \text{End}_R(R^{1/p^e}) \) is also closely related to the F-blowup \([TY]\).

The following is our partial answer to Problem 1.9:

**Theorem 1.10 (Corollary 4.2).** Let \( S \) and \( R \) be as in Problem 1.2. Suppose that \( \text{End}_R(S) \) is MCM. Then for sufficiently large \( e \), \( \text{End}_R(R^{1/p^e}) \) is a NCCR and Morita equivalent to \( \text{End}_R(S) \).

In fact, the core of our proof is due to Hara \([Ha]\), who proved the corresponding result in dimension two. Also Toda and the author \([TY]\) previously proved the theorem in the case of tame quotient singularities. Their proof uses a result from the representation theory, while Hara’s argument replaces it with an elegant ring-theoretic argument.

We should notice that \( \text{End}_R(R^{1/p^e}) \) is not a NCCR for a general singularity \( R \). Indeed Dao proved \([Da]\) that if \( R \) is a local complete intersection which is regular in codimension two, then \( \text{End}_R(R^{1/p^e}) \) is not a NCCR for any \( e \).

1.4. **Noncommutative Frobeniuses and global dimensions (Sections 5 and 6).** For commutative rings or schemes, the Frobenius morphism has been exploited as a tool to study singularities: Kunz’s characterization of regularity (smoothness) \([Ku]\), the study of F-singularities (see \([Sm]\)) among others. In \([Ya]\), the author defined the Frobenius morphism of the endomorphism ring of a module over a commutative ring. In Section 6 we will also define the Frobenius morphism of the skew group ring associated to a commutative ring and a finite group in an obvious way. Then we will study relation between the finiteness of global dimension and the flatness of Frobenius morphism. For commutative rings, both properties are characterizations of regularity and Herzog \([He]\) gave a direct proof that the latter implies the former. Koh and Lee \([KL]\) refined his result and proved certain constraint which the minimal resolution of every module satisfies. We will prove the noncommutative version of Koh and Lee’s result (Theorem 5.2).

However, in order to obtain finite global dimension using this, the flatness of Frobenius morphism is not sufficient unlike the commutative case. Then we will axiomize the properties of Frobenius morphism which are necessary to deduce finite global dimension. Moreover if the relevant noncommutative ring is \( \text{End}_R(M) \), then we will relate the properties with those of \( M \) in terms of the Frobenius of \( R \).

1.5. **Convention.** Throughout the paper, \( R \) denotes some commutative Noetherian local complete domain of Krull dimension \( d \). Unless otherwise noted, a *ring* means a (commutative or noncommutative) \( R \)-algebra which is a finitely generated torsion-free \( R \)-module. Thus every commutative ring has Krull dimension \( d \).

A *module* means a finitely generated left module unless otherwise noted. Then the category of modules over a ring has the Krull-Schmidt property: Every module uniquely decomposes into the direct sum of indecomposable modules. A ring or module is called *maximal Cohen-Macaulay* or simply *Cohen-Macaulay* (for short, CM) if they are maximal Cohen-Macaulay \( R \)-modules. Similarly a ring or module is called *reflexive* if it is a reflexive \( R \)-module. The notation \( _RM \) (resp. \( _M \text{CM} \)) means that \( M \) is a left (resp. right) \( R \)-module.

We denote the category of \( R \)-modules by \( \text{mod}(R) \) and subcategories of projective (resp. CM, reflexive) \( R \)-modules by \( \text{proj}(R) \) (resp. \( \text{CM}(R) \), \( \text{ref}(R) \)). A sequence of modules in such a subcategory is said to be *exact* if it is exact in the ambient
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category \text{mod}(R). A functor between such subcategories is said to be \textit{exact} if it preserves exact sequences.

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2. NCRs of pure subrings

2.1. Preliminaries. First of all we recall some basic notions.

\textbf{Definition 2.1.} For a module \(M\) over a ring \(S\), the \textit{additive closure} of \(M\), denoted \(\text{add}(M)\), is the category consisting of modules isomorphic to a direct summand of \(M^\oplus l, l \geq 0.\) We say that \(M\) \textit{is an additive generator} of the category \(\text{add}(M)\).

\textbf{Definition 2.2.} We say that a ring \(S\) \textit{is of finite CM type} if there are, up to isomorphism, only finitely many indecomposable CM \(S\)-modules. Then there is an additive generator \(M\) of \(\text{CM}(S)\), which is a CM module containing every indecomposable CM module as a direct summand. Such \(M\) is called a \textit{CM generator} (over \(S\)).

\textbf{Definition 2.3.} A (necessarily module-finite) extension \(R \subset S\) of commutative rings is said to be \textit{pure} if the inclusion map \(R \hookrightarrow S\) splits as an \(R\)-module map. If it is the case, we also say that \(R\) is a \textit{pure subring} of \(S\).

\textbf{Definition 2.4.} Let \(\Lambda\) be a ring and \(M\) a \(\Lambda\)-module. The \textit{reflexive hull} of \(M\) is defined as the \(\Lambda\)-module

\[ M^{\text{ref}} := \text{Hom}_R(\text{Hom}_R(M, R), R). \]

Then \(M\) is reflexive if and only if the natural map \(M \rightarrow M^{\text{ref}}\) is an isomorphism.

\textbf{Definition 2.5 \cite{BO}, \cite{VdB}.} Let \(M\) be a torsion-free \(R\)-module.

1. \(\text{End}_R(M)\) is called a \textit{NCR} if it has finite global dimension.

2. \(\text{End}_R(M)\) is called a \textit{NCCR} if it is CM and has finite global dimension.

2.2. NCRs of pure subrings.

\textbf{Theorem 2.6.} Let \(S\) be a commutative CM ring of finite CM type with a CM generator \(M\). Suppose that \(R \subset S\) is pure and that \(\text{Hom}_R(S, M)\) is CM. Then

\[ \text{gl. dim } \text{End}_R(M) \leq \max\{d, 2\}. \]

In particular, \(\text{End}_R(M)\) is a NCR. Moreover if \(d \geq 2\), then

\[ \text{gl. dim } \text{End}_R(M) = d. \]

\textit{Proof.} Let \(\Lambda\) be the opposite ring of \(\text{End}_R(M)\). Being Noetherian, \(\Lambda\) and \(\text{End}_R(M)\) have equal global dimension. Therefore we may show the theorem for \(\Lambda\) instead of \(\text{End}_R(M)\).

Notice that since \(S\) is CM, we have \(S \in \text{CM}(S) = \text{add}(S)\) and \(RS \in \text{add}(RM)\). Since \(R\) is a direct summand of \(S\), we have \(R \in \text{add}(RM)\). Then we define functors

\[ \alpha := \text{Hom}_R(M, -), \quad \text{proj}(\Lambda) \]

\[ \alpha^{-1} := \text{Hom}_\Lambda(\alpha(R), -). \]
These are equivalences which are inverses to each other. (The equivalence is known as the projectivization \cite{ARS}.) Let $e := \max\{d, 2\}$, $A$ an arbitrary \Lambda-module and

$$P_\bullet : P_{e-1} \to \cdots \to P_1 \to P_0$$

the first $e$-step of a projective resolution of $A$. Set

$L_\bullet := \alpha^{-1}(P_\bullet) : L_{e-1} \to \cdots \to L_1 \to L_0$

and $L_e := \text{Ker}(L_{e-1} \to L_{e-2})$.

**Claim 2.7.** We have that $L_e \in \text{add}(R M)$.

If the claim is true, then applying $\alpha$ to

$$0 \to L_e \to L_{e-1} \to \cdots \to L_0,$$

we obtain an exact sequence

$$(1) \quad 0 \to \alpha(L_e) \to P_{e-1} \to P_{e-2} \to \cdots \to P_0.$$ 

Here the exactness follows from the left exactness of $\alpha$ and the exactness of $P_\bullet$. Sequence (1) is a projective resolution of $A$. Hence $\Lambda$ has global dimension $\leq e$. On the other hand, from \cite{Ra} (see also \cite{Le2}), it has global dimension $\geq d$. Hence if $d \geq 2$, then the equality in the theorem holds.

The proof up to this point is basically the same as the one in \cite{Le1}. The difference lies in the proof of Claim 2.7.

**Proof of Claim 2.7.** By assumption, $\text{Hom}_R(S, N)$ is CM for every $N \in \text{add}(R M)$. Hence

$$S \text{Hom}_R(S, N) \in \text{add}(S M) = \text{CM}(S).$$

Thus we have the functor

$$\psi := \text{Hom}_R(S, -) : \text{add}(R M) \to \text{CM}(S).$$

Set $\Gamma := \text{End}_S(M)^{\text{op}}$ and define equivalences

$$\beta := \text{Hom}_S(M, -), \qquad \beta^{-1} := \text{Hom}_\Gamma(\beta(S), -),$$

which are inverses to each other. Since $\Gamma$ is a subring of $\Lambda$, we have the forgetting functor

$$\phi : \text{mod}(\Lambda) \to \text{mod}(\Gamma), \ A \mapsto \gamma A,$$

which is obviously exact. For $N \in \text{add}(R M)$, we have isomorphisms of $\Gamma$-modules,

$$\phi \circ \alpha(N) \cong \text{Hom}_R(M, N)$$

$$\cong \text{Hom}_R(S \otimes_S M, N)$$

$$\cong \text{Hom}_S(M, \text{Hom}_R(S, N))$$

$$\cong \beta \circ \psi(N).$$
Therefore the following diagram is (2-)commutative:
\[
\begin{array}{ccc}
\text{proj}(\Lambda) & \xrightarrow{\phi_{\text{proj}(\Lambda)}} & \text{proj}(\Gamma) \\
\downarrow{\alpha} & & \downarrow{\gamma} \\
\text{add}(R M) & \xrightarrow{\psi} & \text{CM}(S)
\end{array}
\]

Put \( K_\bullet := \psi(L_\bullet) \). Since
\[ K_\bullet \cong \beta^{-1}(\_P_\bullet), \]
and \( \beta^{-1} \) is exact, \( K_\bullet \) is exact. Let
\[ K_e := \ker(K_{e-1} \to K_{e-2}) \cong \psi(L_e). \]
From the depth lemma, \( K_e \) is CM and belongs to \( \text{add}(S M) = \text{CM}(S) \). Hence \( R K_e \in \text{add}(R M) \). Since \( R \subset S \) is pure, \( L_e \cong \text{Hom}_R(R, L_e) \) is a direct summand of \( R K_e \cong \text{Hom}_R(S, L_e) \). Hence the claim holds. \( \square \)

We have completed the proof of Theorem 2.6. \( \square \)

Remark 2.8. From Lemma 2.9, if \( S \) is normal, then we can replace the condition in Theorem 2.6 that \( \text{Hom}_R(S, M) \) is CM with the condition that \( (S \otimes_R M)^{\text{ref}} \) is CM.

Lemma 2.9. Let \( S \) be a commutative normal CM ring of finite CM type with a CM generator \( M \). Then for any \( R \)-module \( N \), \( (N \otimes_R M)^{\text{ref}} \) is MCM if and only if \( \text{so is } \text{Hom}_R(N, M) \).

Proof. Let \( K_S \) denote the canonical module of \( S \). We have
\[ \text{Hom}_S((N \otimes_R M)^{\text{ref}}, K_S) \cong \text{Hom}_S(N \otimes_R M, K_S) \cong \text{Hom}_R(N, \text{Hom}_S(M, K_S)). \]
Here the left isomorphism follows from the fact \( K_S \) is reflexive. Thus \( (N \otimes_R M)^{\text{ref}} \) and \( \text{Hom}_R(N, \text{Hom}_S(M, K_S)) \) are reflexive modules which are the canonical duals to each other. Since \( \text{Hom}_S(\_K_S) \) is an auto-equivalence of \( \text{CM}(S) \) (see [Yo]), \( \text{Hom}_S(M, K_S) \) is also a CM generator over \( S \). Therefore \( \text{Hom}_R(N, M) \) is CM if and only if \( \text{so is } \text{Hom}_R(N, \text{Hom}_S(M, K_S)) \) is CM if and only if \( (N \otimes_R M)^{\text{ref}} \) is CM. \( \square \)

Example 2.10. Let \( S \) be a commutative normal CM ring of finite CM type with a CM generator \( M \), \( G \) a finite group acting on \( S \) and \( R := S^G \). Suppose that the ring extension \( R \subset S \) is pure and unramified in codimension one. Then \( S * G \cong \text{End}_R(S) \) (see [Au, page 118]). Hence we have isomorphisms of \( S \)-modules,
\[ M^{\otimes_{SG}} \cong (S * G) \otimes_S M \cong \text{End}_R(S) \otimes_S M \cong \text{Hom}_R(S, M). \]
(Here the last isomorphism holds, since the both hand sides are reflexive modules and the isomorphism is valid in codimension one.) In particular, \( \text{Hom}_R(S, M) \) is CM. From Theorem 2.6 \( \text{End}_R(M) \) has global dimension \( d \) and is a NCR.

Corollary 2.11. Let \( S \) be a commutative regular local ring with the extension \( R \subset S \) pure. Then if \( \text{End}_R(S) \) is CM, then it is a NCCR.

Proof. Since \( S \) is regular, \( \_S \) is a CM generator. Now the corollary is a direct consequence of Theorem 2.6. \( \square \)
3. A result on wild quotient singularities

In this section, we suppose that the base ring $R$ has characteristic $p > 0$. We will always denote by $q$ a power of $p$; $q = p^e$ $(e \in \mathbb{Z}_{\geq 0})$. We define $R^{1/q} := \{ f^{1/q} | f \in R \}$ in the algebraic closure of the quotient field of $R$. Then $R$ is a subring of $R^{1/q}$ and $R^{1/q}$ has a natural $R$-module structure. The Frobenius map, $F : R \to R$, $f \mapsto f^p$, is isomorphic to the inclusion map $R \hookrightarrow R^{1/p}$. We will make also the assumption that $R$ is F-finite. Namely the $R$-module $R \subset R^{1/p}$ is finitely generated.

**Definition 3.1** ([HH]). We say that $R$ is strongly F-regular if for every $0 \neq c \in R$, there exists $q = p^e$ such that the $R$-linear map $R \to R^{1/q}$ with $1 \mapsto c^{1/q}$ splits.

**Definition 3.2.** Let $S$ be a ring and $G$ a group acting on it. Then the skew group ring $S \ast G$ is defined as follows: It is a free $S$-module, $\bigoplus_{g \in G} S \cdot g$, with the multiplication defined by $(sg)(s'g') = (sg')(gg')$.

**Corollary 3.3.** Let $S$ be a commutative regular ring and $G \subset \text{Aut}(S)$ a finite group of automorphisms of $S$. Suppose that the induced $G$-action on the residue field of $S$ is trivial and that $S$ is unramified over $R := S^G$ in codimension one. Let $\Lambda := S \ast G$, which is by assumption isomorphic to $\text{End}_R(S)$. Then the following are equivalent:

1. Either $p = 0$ or $p \nmid \#G$.
2. $\Lambda$ has finite global dimension.
3. $\Lambda$ has global dimension $d$.
4. The ring extension $R \subset S$ is pure.
5. $R$ is strongly F-regular.

**Proof.** $(1) \Leftrightarrow (2)$: [Yi, Theorem 5.2].

$(1) \Rightarrow (3)$: [MR, 7.5.6].

$(3) \Rightarrow (2)$: Obvious.

$(1) \Rightarrow (4)$: Well-known.

$(4) \Rightarrow (1)$: This follows from Corollary 2.11.

$(4) \Rightarrow (5)$: [HH].

$(5) \Rightarrow (4)$: A strongly F-regular ring is a splinter, that is, every module-finite extension of it is pure [Hu].

See Corollary 6.18 for two more equivalent conditions.

4. NCCRs via Frobenius

In this section, we will continue to suppose that the base ring $R$ has characteristic $p > 0$.

The following is a straightforward generalization of Hara’s similar result in dimension two [Ha].

**Proposition 4.1.** Let $S$ be a commutative CM ring of finite CM type with a CM generator $M$. We suppose that $\text{End}_R(M)$ is CM (see Remark 2.3), that $R \subset S$ is pure and that $R$ is strongly F-regular. Then for sufficiently large $e$, $F^e_* R = R(R^{1/p^e})$ is an additive generator of $\text{add}(R^M)$.

**Proof.** Our proof is essentially the same as Hara’s one. Firstly, for every $q$, since $S^{1/q} \subset \text{CM}(S) = \text{add}(S)$, we have $R^{1/q} \subset \text{add}(R^M)$. Being a direct summand of $R^{1/q}$, $R^{1/q}$ is also in $\text{add}(R^M)$.
Indeed every indecomposable \( L \in \text{add}(R) \) is embedded in the free module of the same rank, as it is torsion-free. Then for sufficiently factorial \( c_L \in R \), the image of the induced map \( c_L : R^\oplus \text{rank } L \to R^\oplus \text{rank } L \) is contained in \( L \). We can choose \( \prod L c_L \) as the desired \( c \).

If we put \( M^* := \text{Hom}_R(M, R) \), it is by assumption CM. Hence \( _SM^* \in \text{add}(S) \) and \( _RM^* \in \text{add}(R) \). Let \( M^{1/q} \) be the \( S^{1/q} \)-module corresponding to \( M \) by the obvious isomorphism \( S \cong S^{1/q} \). Since \( \text{Hom}_R(M^*, S^{1/q}) \) is CM, \( S^{1/q} \text{Hom}_R(M^*, S^{1/q}) \in \text{add}(S^{1/q}M^{1/q}) \). Therefore \( R^{1/q} \text{Hom}_R(M^*, R^{1/q}) \in \text{add}(R^{1/q}M^{1/q}) \). Hence the \( R^{1/q} \)-linear map
\[
c^{1/q} : \text{Hom}_R(M^*, R^{1/q}) \to \text{Hom}_R(M^*, R^{1/q})
\]
factors as
\[
\text{Hom}_R(M^*, R^{1/q}) \to (R^{1/q})^\oplus m \to \text{Hom}_R(M^*, R^{1/q}),
\]
where \( m \) is the rank of \( M \).

On the other hand, since \( R \) is strongly F-regular, there exists \( q = p^e \) such that the \( R \)-linear map \( R \to R^{1/q} \), \( 1 \mapsto c^{1/q} \) splits. Applying \( \text{Hom}_R(M^*, -) \) to it, we obtain a splitting \( R \)-linear map
\[
(2) \quad M = \text{Hom}_R(M^*, R) \to \text{Hom}_R(M^*, R^{1/q}).
\]
This factors as
\[
M \to \text{Hom}_R(M^*, R^{1/q}) \xrightarrow{c^{1/q}} \text{Hom}_R(M^*, R^{1/q}),
\]
and furthermore as
\[
M \to \text{Hom}_R(M^*, R^{1/q}) \to (R^{1/q})^\oplus m \to \text{Hom}_R(M^*, R^{1/q}).
\]
The splitting of (2) yields that of \( M \to (R^{1/q})^\oplus m \). Hence \( R M \) is a direct summand of \( (R^{1/q})^\oplus m \). In consequence, \( R^{1/q} \) is an additive generator of \( \text{add}(R M) \).

**Corollary 4.2.** With the assumption as in Proposition 4.1, for sufficiently large \( e \), \( \text{End}_R(R^{1/p^e}) \) is Morita equivalent to \( \text{End}_R(M) \) and a NCCR as well.

**Proof.** As is well-known, the equality \( \text{add}(R^{1/q}) = \text{add}(R M) \) induces the Morita equivalence of \( \text{End}_R(R^{1/q}) \) and \( \text{End}_R(M) \). \( \square \)

The ring of differential operators on \( R \) is expressed as follows [Ye]:
\[
D(R) = \bigcup_{e \geq 0} \text{End}_{R^{p^e}}(R).
\]

**Corollary 4.3.** With the assumption as in Proposition 4.1, \( D(R) \) has global dimension \( \leq d + 1 \).

**Proof.** We have obvious isomorphisms \( \text{End}_{R^{p^e}}(R) \cong \text{End}_R(R^{1/p^e}) \). Hence \( \text{End}_{R^{p^e}}(R) \) has global dimension \( d \) for \( e \gg 0 \). Since \( D(R) \) is a direct limit of them, the corollary follows from a general result on the global dimension of direct limits by Bernstein [Be]. \( \square \)
5. Noncommutative Herzog-Koh-Lee and Exact Order-Raising Endofunctors

5.1. Noncommutative Herzog-Koh-Lee. Koh and Lee [KL] proved a result on the minimal resolution of a module over a commutative local ring, which is a refinement of Herzog’s result [He]. We will generalize their result to the noncommutative setting along the lines of Koh and Lee.

Let \( \Lambda \) be a ring. From assumption (see [1.5] and [La2 (23.3)], \( \Lambda \) is semiperfect and every (left or right) finitely generated \( \Lambda \) module admits the minimal projective resolution. Let \( e_1, \ldots, e_l \) be a basic set of primitive idempotents for \( \Lambda \) so that \( Q_i := \Lambda e_i, i = 1, \ldots, l \), are the irredundant set of the indecomposable projective left \( \Lambda \) modules (see [AE Proposition 27.10]).

Let \( j \subset \Lambda \) be the Jacobson radical. The socle of a right \( \Lambda \) module \( V \) is defined to be its largest semisimple submodule and denoted by \( \text{Soc}(V) \). Since \( \Lambda \) is semiperfect, \( \text{Soc}(V) \) is equal to the annihilator of \( j \) (see [AE pages 118 and 171]):

\[
\text{Soc}(V) = \{ v \in V | \forall i = 0 \} \subset V.
\]

From now on, we simply write \( \text{Soc}(V) = \text{Soc}(V_\Lambda) \). For some \( 1 \leq i \leq l \), we have \( \text{Soc}(V) \otimes_\Lambda Q_i \neq 0 \), say \( \text{Soc}(V) \otimes_\Lambda Q_1 \neq 0 \). Then set

\[
s_\Lambda := \inf \{ t \geq 1 | \text{Soc}(V) \otimes_\Lambda Q_1 \not\subset Vm^t \otimes_\Lambda Q_1 \}.
\]

Let

\[
P_\bullet : \cdots \rightarrow P_{j+1} \xrightarrow{\delta_{j+1}} P_j \xrightarrow{\delta_j} P_{j-1} \rightarrow \cdots \rightarrow P_0
\]

be the minimal projective resolution of a left \( \Lambda \) module \( U \). Here for each \( j \), we can write \( P_j = \bigoplus_{i=1}^l Q_i^{\otimes n_{j,i}} \).

**Lemma 5.1.** For \( j > r \), in particular, for \( j > d \), we have either that \( n_{j,1} = 0 \) or that \( \delta_{j+1} \) is nonzero modulo \( m^{s_\Lambda} \) (that is, \( \text{Im} \delta_{j+1} \not\subset m^{s_\Lambda} P_j \)).

**Proof.** Let \( j > r \). The right \( \Lambda \) module \( V \) has projective dimension \( r \) [La1 (5.32)]. Hence \( \text{Tor}_j(V, U) = 0 \). Hence

\[
V \otimes_\Lambda P_\bullet : \cdots \rightarrow V \otimes_\Lambda P_{j+1} \xrightarrow{\delta_{j+1}} V \otimes_\Lambda P_j \xrightarrow{\delta_j} V \otimes_\Lambda P_{j-1} \rightarrow \cdots
\]

is exact in the middle. Since \( P_\bullet \) is minimal, we have

\[
\text{Soc}(V) \otimes_\Lambda P_j \subset \text{Ker} \delta_j = \text{Im} \delta_{j+1}.
\]

To obtain a contradiction, we make the assumptions that \( n_{j,1} > 0 \) and that \( \delta_{j+1} \) were zero modulo \( m^{s_\Lambda} \). From the latter,

\[
\text{Soc}(V) \otimes_\Lambda P_j \subset \text{Im} \delta_{j+1} \subset Vm^{s_\Lambda} \otimes_\Lambda P_j.
\]

From the former, this implies that \( \text{Soc}(V) \otimes_\Lambda Q_1 \subset Vm^{s_\Lambda} \otimes_\Lambda Q_1 \), which contradicts the definition of \( s_\Lambda \). We have proved the proposition. \( \square \)
For a nonempty subset \( I \subset \{1, 2, \ldots, l\} \), we define a ring \( \Lambda_I := (\sum_{i \in I} e_i)\Lambda(\sum_{i \in I} e_i) \) and a number \( s_I = s_{\Lambda_I} \) in the same way as \( s_{\Lambda} \). Then we put
\[
s := \max\{s_I | \emptyset \neq I \subset \{1, 2, \ldots, l\}\}.
\]

**Theorem 5.2** (Noncommutative Herzog-Koh-Lee). Suppose that for every \( j \leq l(d + 1) \), \( \delta_{j+1} \) is zero modulo \( m^s \). Then \( P_{l(d+1)} = 0 \). Equivalently proj. dim \( U < l(d + 1) \).

**Proof.** The proof is by induction on \( l \). If \( l = 1 \), then the proposition is a direct consequence of Lemma 5.1. We now turn to the general case. For \( d < j \leq l(d + 1) \), since \( \delta_{j+1} \) is zero modulo \( m^s \), again from Lemma 5.1 we have \( n_{j, 1} = 0 \). Put \( e := \sum_{i \geq 2} e_i \) and \( \Lambda' := e\Lambda e \). Then consider the complex \( P_*' := (eP_*)(d+1) \) of \( B \)-modules. Namely the complex \( P_0' := eP_{d+1} \) with the obvious differentials. For \( -1 < j \leq (l - 1)(d + 1) \), since \( n_{j+d+1, 1} = 0 \), \( P_j' = eP_{j+d+1} \) is a projective \( \Lambda' \)-module. Hence \( P_0' \) is the minimal projective resolution of \( \text{Coker}(P_{d+1} \to P_0) \) at least in degree \( \leq (l - 1)(d + 1) \) such that for \( j \leq (l - 1)(d + 1) \), the differential \( \delta_{j+1} : P_{j+1} \to P_j \) is zero modulo \( m^s \). From the induction hypothesis, we have
\[
eP_{l(d+1)} = P_{(l-1)(d+1)} = 0 \quad \text{and} \quad P_{l(d+1)} = 0.
\]

\( \Box \)

When \( \Lambda \) is CM, we do not need the inductive argument and have a better result.
In this case, we have \( r := \text{depth}(\sigma \Lambda) = d \) and for every \( i \), \( \text{Soc}(V) \otimes_{\Lambda} Q_i \neq 0 \). Set
\[
s_0 := \max\{t \geq 1 | \text{Soc}(V) \otimes_{\Lambda} Q_i \not\subset V^t \otimes_{\Lambda} Q_i\}.
\]

Then we can see the following by an argument similar to that of Lemma 5.1.

**Theorem 5.3.** Suppose that \( \Lambda \) is CM. For \( j > d \), if \( \delta_{j+1} \) is zero modulo \( m^s \), then \( P_j = 0 \) and proj. dim \( U \leq d \).

### 5.2. Exact order-raising endofunctors and the finiteness of global dimension

We keep the notation of the preceding subsection.

**Definition 5.4.** Let \( \Phi : \text{proj}(\Lambda) \to \text{proj}(\Lambda) \) be an endofunctor. We say that \( \Phi \) is order-raising if for every \( i > 0 \), there exists \( e_0 > 0 \) such that for every \( e \geq e_0 \) and for every morphism \( f : P \to Q \) in \( \text{proj}(\Lambda) \) which factors through \( jQ \), \( \Phi^e(f) : \Phi^e(P) \to \Phi^e(Q) \) factors through \( j^e\Phi(Q) \).

**Definition 5.5.** A functor \( \Phi \) between subcategories of abelian categories is said to have zero kernel if \( \Phi(N) \cong 0 \Rightarrow N \cong 0 \).

**Corollary 5.6.** Suppose that there exists an exact and order-raising endofunctor \( \Phi : \text{proj}(\Lambda) \to \text{proj}(\Lambda) \) which has zero kernel. Then \( \Lambda \) has finite global dimension. Moreover if \( \Lambda \) is CM, then \( \Lambda \) has global dimension \( d \).

**Proof.** Since for some \( n \), \( j^n \subset m\Lambda \), we may replace \( j^e\Phi(Q) \) in Definition 5.3 with \( m^e\Phi(Q) \). Let \( U \) be an arbitrary finitely generated \( \Lambda \)-module and
\[
P_0 \to P_1 \to \cdots \to P_{j+1} \to P_j \to P_{j-1} \to \cdots \to P_0
\]
its minimal projective resolution. Since \( \Phi \) is exact, for every \( e \), \( \Phi^e(P_0) \) is an exact sequence. Since \( \Phi \) is order-raising, if \( e \) is sufficiently large, then
\[
\Phi^e(P_{l(d+1)}) \to \Phi^e(P_{l(d+1)-1}) \to \cdots \to \Phi^e(P_0)
\]
is the first steps of the minimal projective resolution of \( \Phi^e(U) \), whose differentials are zero modulo \( m^e \). From Theorem 5.2, \( \Phi^e(P_{l(d+1)}) = 0 \). Since \( \Phi \) has zero kernel, \( P_{l(d+1)} = 0 \). Therefore \( \text{proj. dim } U \leq l(d + 1) \) and hence \( \text{gl. dim } \Lambda < \infty \).

For the second assertion, using Theorem 5.3 instead, we can similarly show that \( \text{gl. dim } \Lambda \leq d \). On the other hand, from [Ra] (see also [Lo2]), \( \text{gl. dim } \Lambda \geq d \), and the corollary follows. \( \square \)

6. Noncommutative Frobeniuses and the finiteness of global dimension

In this section we will define the Frobenius morphism for two classes of noncommutative rings, endomorphism rings of modules and skew group rings. Then we study when the Frobenius pullback functor for such a ring satisfies the conditions in the last section for that the ring has finite global dimension.

We now suppose that \( R \) is normal and of characteristic \( p > 0 \).

6.1. Frobenius of endomorphism rings. Let \( M \) be a nonzero reflexive \( R \)-module and \( M_1/p \) the corresponding \( R_1/p \)-module under the isomorphism \( R_1/p \cong R \), \( f_1/p \leftrightarrow f \). Put \( E := \text{End}_R(M) \) and \( E_1/p := \text{End}_{R_1/p}(M_1/p) \). The Hom set
\[
H := \text{Hom}_R(M, M_1/p)
\]
has a natural \((E_1/p, E)\)-bimodule structure.

Definition 6.1 ([Ya]). We define the Frobenius pullback of \( E \) as
\[
F^* := H \otimes_E - : \text{mod}(E) \rightarrow \text{mod}(E_1/p),
\]
and the Frobenius pushforward as its right adjoint
\[
F_* := \text{Hom}_{E_1/p}(H, -) : \text{mod}(E_1/p) \rightarrow \text{mod}(E).
\]
We call the pair \((F^*, F_*)\) the Frobenius morphism of \( E \).

Note that \( E \) and \( E_1/p \) are canonically isomorphic and one may regard the above functors as endofunctors.

Example 6.2. If \( M = R \), then \( H = R_1/p \), \( E = R \) and \( E_1/p = R_1/p \). Therefore \( F^* \) and \( F_* \) are respectively the pullback and pushforward of the ordinary relative Frobenius \( R \hookrightarrow R_1/p \).

Definition 6.3. Let \( F : R \rightarrow R \), \( f \mapsto f^p \) be the absolute Frobenius map of \( R \). We define the reflexive pullback of an \( R \)-module \( N \) by \( F \) as \( F^*N := (F^*N)_{\text{ref}} \).

Definition 6.4. Given an \( R \)-module \( N \), we define as follows:

1. \( N \) is \( F^* \)-closed if \( F^*N \in \text{add}(N) \).
2. \( N \) is \( F_* \)-closed if \( F_*N \in \text{add}(N) \).
3. \( N \) is strongly \( F_* \)-closed if \( F_*N \) is an additive generator of \( \text{add}(N) \).

Lemma 6.5. Let \( N \) be a reflexive \( R \)-module. Consider the following conditions:

1. \( N \in \text{add}(M) \).
2. \( \text{Hom}_R(M, N) \) is a projective right \( E \)-module.
3. \( \text{Hom}_R(N, M) \) is a projective left \( E \)-module.

Then (1) \( \iff \) (2) \( \Rightarrow \) (3).
Proof. We have an equivalence of categories of reflexive modules
\[ \text{Hom}_R(M, -) : \text{ref}(R) \xrightarrow{\sim} \text{ref}(E^{\text{op}}), \]
which restricts to an equivalence \( \text{add}(M) \xrightarrow{\sim} \text{proj}(E^{\text{op}}) \) (see [IR]). This shows (1) \( \iff \) (2).

If \( N \in \text{add}(M) \), then \( E\text{Hom}_R(N, M) \) is a direct summand of a free module, and projective. Hence (1) \( \Rightarrow \) (3). \( \square \)

Proposition 6.6. The following are equivalent:

1. \( F^* \) is exact.
2. \( H \) is a projective right \( E \)-module.
3. \( M \) is \( F^* \)-closed.

Proof. (1)\( \iff \) (2): Since \( E \) is Noetherian, \( H_E \) is flat if and only if it is projective, which shows this equivalence.

(2)\( \iff \) (3): From the preceding lemma, \( H \) is a projective right \( E \)-module if and only if \( RM^{1/p} = F_*M \in \text{add}(M) \), that is, \( M \) is \( F_* \)-closed. \( \square \)

Proposition 6.7. If \( M \) is strongly \( F_* \)-closed, then for every nonzero \( E \)-module \( A \), \( F^*A \) is nonzero. Namely \( F^* \) has zero kernel.

Proof. If \( M \) is strongly \( F_* \)-closed, then \( H \) contains every indecomposable projective \( E \)-module as a direct summand. Hence for some \( l \), \( H \oplus l \) contains \( E \) as a direct summand. Hence \( H \oplus l \otimes_E A \neq 0 \) and \( H \otimes_E A \neq 0 \). \( \square \)

Proposition 6.8. Consider the following conditions:

1. \( F^* \) preserves projective modules.
2. \( F_* \) is exact.
3. \( H \) is a projective left \( E^{1/p} \)-module.
4. \( M \) is \( F^* \)-closed.

Then (1) \( \iff \) (2) \( \iff \) (3) \( \iff \) (4).

Proof. (1)\( \iff \) (2)\( \iff \) (3): Obvious.

(3)\( \iff \) (4): Let \( F_\text{rel} \) denote the relative Frobenius map, \( R \rightarrow R^{1/p} \). Define the reflexive pullback \( F_\text{rel}^* \) similarly. We have isomorphisms of left \( E^{1/p} \)-modules,
\[ H \cong \text{Hom}_{R^{1/p}}(F_\text{rel}^* M, M^{1/p}) \cong \text{Hom}_{R^{1/p}}(F_\text{rel} X, M^{1/p}). \]
Here the left isomorphism follows from the adjunction and the right from the fact that \( M^{1/p} \) is reflexive. Now (3)\( \iff \) (4) follows from Lemma 5.5. \( \square \)

Corollary 6.9. If \( M \) is strongly \( F_* \)-closed and \( F^* \)-closed, and if \( F^* |_{\text{proj}(E)} \) is order-raising (regarded as an endofunctor), then \( E \) has finite global dimension. Moreover if \( E \) is CM, then its global dimension is \( d \).

Proof. The functor \( F^* \) preserves projectives and its restriction \( F^* |_{\text{proj}(E)} \) is exact and order-raising and has zero kernel. Now the corollary follows from Corollary 5.6. \( \square \)
6.2. The case where $M$ is a commutative regular local ring. In this subsection, we additionally suppose that the $R$-module $M$ is also a commutative regular local ring such that the $R$-module structure on $M$ is the one induced from the ring extension $R \subset M$.

**Lemma 6.10.** $R M$ is strongly $F_\ast$-closed and $F^\ast$ is exact and has zero kernel.

**Proof.** Since $M$ is regular, $F_\ast M = M^{1/p}$ is a free $M$-module. Hence $R M$ is strongly $F_\ast$-closed. The rest assertions follow from Propositions 6.6 and 6.7.

**Proposition 6.11.** Suppose that $E = \text{End}_R(M)$ is CM and that the ring extension $R \subset M$ is pure. Then $M$ is $F^\ast$-closed and $F^\ast$ preserves projectives.

**Proof.** From Lemma 2.9, $(M \otimes_R M)^{\text{ref}}$ is also CM. Since $M^{1/p}$ is a free $M$-module, $(M^{1/p} \otimes_R M)^{\text{ref}}$ is also CM and a free $M^{1/p}$-module. Therefore $R_{1/p}(M^{1/p} \otimes_R M)^{\text{ref}} \in \text{add}(R_{1/p}(M^{1/p})^{\text{ref}})$. Since $R_{1/p}(R^{1/p} \otimes_R M)^{\text{ref}}$ is a direct summand of $R_{1/p}(M^{1/p} \otimes_R M)^{\text{ref}}$, we have $R_{1/p}(R^{1/p} \otimes_R M)^{\text{ref}} \in \text{add}(R_{1/p}(M^{1/p})^{\text{ref}})$, which saids that $M$ is $F^\ast$-closed. From Proposition 6.8, $F^\ast$ preserves projectives.

Since $M$ is naturally regarded as a subring of $E = \text{End}_R(M)$, there are forgetting functors $\text{mod}(E) \to \text{mod}(M)$ and similarly $\text{mod}(E^{1/p}) \to \text{mod}(M^{1/p})$.

**Proposition 6.12 (Ya).** The diagram

\[
\begin{array}{ccc}
\text{mod}(M) & \longrightarrow & \text{mod}(M^{1/p}) \\
\uparrow & & \uparrow \\
\text{mod}(E) & \longrightarrow & \text{mod}(E^{1/p})
\end{array}
\]

is commutative. Here $F_M : M \to M^{1/p}$ is the relative Frobenius map of $M$.

**Proof.** The two composite functors from $\text{mod}(E)$ to $\text{mod}(M^{1/p})$ in the diagram send $E$ to $M^{1/p} \otimes_M E$ and $M^{1/p} H$ respectively. It suffices to show that there is a natural isomorphism between the two modules. Since $M$ is regular and $M^{1/p}$ is a free $M$-module, we have

\[M^{1/p} \otimes_M E \cong \text{Hom}_R(M, M^{1/p}) = H.\]

**Proposition 6.13.** Let $j$ be the Jacobson radical of $E$ and $m_M$ the maximal ideal of $M$. Suppose that $F^\ast$ preserves projectives. Then $F^\ast|_{\text{proj}(E)}$ is order-raising if and only if $j \subset m_M E$.

**Proof.** The “if” part: Let $\phi : P \to Q$ be a morphism in $\text{proj}(E)$ which factors through $j Q$, and hence through $m_M Q$. From Proposition 6.12, $(F^\ast)^e(\phi)$ factors through $m_M^{[p]}(F^\ast)^e(Q)$. Here $m_M^{[p]}$ is the ideal of $M$ generated by $f^p$, $f \in m_M$. Let $m_R$ be the maximal ideal of $R$. For every $i > 0$, there exists $n > 0$ such that $j^i \supset m_R^n E$. Then for every $n > 0$, there exists $e > 0$ such that $m_M^{[p]} E \supset m_M^{[p]} E$. Therefore for every $i > 0$, if $e$ is sufficiently large, then $(F^\ast)^e(\phi)$ factors through $j^i(F^\ast)^e(Q)$. Thus $F^\ast|_{\text{proj}(E)}$ is order-raising.

The “only if” part: Suppose $j \not\subset m_M E$. Then choose an element $f \in j \setminus m_M E$ and let $\phi : E \to E$ be the map sending 1 to $f$. From Proposition 6.12, for every
e, \((F^*)^\ast(\phi)\) does not factor through \(m_M E\). However if \(F^*\) were order-raising, for \(e \gg 0\), \((F^*)^\ast(\phi)\) would factor through \(m_M E\). Therefore \(F^*\) is not order-raising. □

We now prove Corollary 2.11 in a different way under the additional assumption on the Jacobson radical:

**Corollary 6.14.** If \(E\) is CM, the extension \(R \subset M\) is pure and \(j \subset m_M E\), then \(E\) has global dimension \(d\) and is a NCCR.

**Proof.** From Proposition 6.11, \(F^*\) preserves projectives. From Lemma 6.10 and Proposition 6.13, the restricted endofunctor \(F^*|_{\text{proj}(E)}\) is exact and order-raising and has zero kernel. Now the assertion follows from Corollary 5.6. □

**Problem 6.15.** If \(E\) is CM and the extension \(R \subset M\) is pure, then is the Jacobson radical \(j\) of \(E\) included in \(m_M E\)?

### 6.3. Frobeniuses of skew group rings.

Let \(S\) be a commutative regular local ring and \(G\) a finite group acting on it. Let \(F_S : S \hookrightarrow S^{1/p}\) be the Frobenius map of \(S\). Since \(S\) is regular, \(F_S\) is flat and \(S^{1/p}\) is a free \(S\)-module. We note that \(S^{1/p}\) has a natural \(G\)-action such that \(F_S\) is \(G\)-equivariant. Therefore the skew group ring \(S^{1/p} \ast G\) is also defined.

**Definition 6.16.** We define the Frobenius map of \(S \ast G\) just as the inclusion map

\[
F : S \ast G \hookrightarrow S^{1/p} \ast G,
\]

by which \(S \ast G\) becomes a subring of \(S^{1/p} \ast G\). Accordingly we define the Frobenius pullback and pushforward functors

\[
F^* : \text{mod}(S \ast G) \to \text{mod}(S^{1/p} \ast G), \ A \mapsto S^{1/p} \ast G \otimes_{S \ast G} A,
\]

\[
F_* : \text{mod}(S^{1/p} \ast G) \to \text{mod}(S \ast G), \ S^{1/p} \ast G A \mapsto S \ast G A.
\]

The following proposition is a direct consequence of the definition:

**Proposition 6.17.** \(F^*\) is exact, preserves projectives and has zero kernel. Also \(F_*\) is exact. Furthermore the diagram

\[
\begin{array}{ccc}
\text{mod}(S) & \xrightarrow{F^*} & \text{mod}(S^{1/p}) \\
\uparrow & & \uparrow \\
\text{mod}(S \ast G) & \xrightarrow{F_*} & \text{mod}(S^{1/p} \ast G)
\end{array}
\]

is (2-)commutative.

**Corollary 6.18.** Let \(n\) be the maximal ideal of \(S\). Suppose that \(G \subset \text{Aut}(S)\), that \(G\) acts trivially on the residue field \(S/n\) and that \(S\) is unramified over \(R = S^G\) in codimension one. Then the following are equivalent:

1. The five equivalent conditions in Corollary 3.3 hold.
2. The Jacobson radical of \(S \ast G\) is \(n \ast G\).
3. \(F^*\) is order-raising.

**Proof.** (1)⇒(2): Villamayor’s theorem [Vi] (see also [Pa]).

(2)⇒(3): Similar to Proposition 6.13.

(3)⇒(1): This follows from Proposition 6.17 and Corollary 5.6. □
6.4. Agreement of the two definitions of noncommutative Frobenius.

**Proposition 6.19.** Let $S$ be a commutative regular local ring and $G \subset \text{Aut}(S)$ a finite group of automorphisms of $S$. Suppose that $S$ is unramified over $R = S^G$ in codimension one. Then with identifications $S \ast G = \text{End}_R(S)$ and $S^{1/p} \ast G = \text{End}_{R^{1/p}}(S^{1/p})$, the Frobenius morphisms of $S \ast G$ and $\text{End}_R(S)$ in Definitions 6.1 and 6.16 coincide.

**Proof.** Frobenius morphisms are respectively given by the bimodules $S^{1/p} \ast G(S^{1/p} \ast G) S \ast G$ and $\text{End}_{R^{1/p}}(S^{1/p}) \text{Hom}_R(S, S^{1/p}) \text{End}_R(S)$. Therefore it suffices to show the isomorphism of these bimodules under the mentioned identifications of rings. By assumption, we have an isomorphism of $(S, R^{1/p})$-bimodules, $S^{1/p} \cong (S \otimes_R R^{1/p})\text{ref}$. Therefore we have isomorphisms of bimodules,

$$S^{1/p} \ast G \cong \text{Hom}_{R^{1/p}}(S^{1/p}, S^{1/p}) \cong \text{Hom}_{R^{1/p}}((S \otimes_R R^{1/p})\text{ref}, S^{1/p}) \cong \text{Hom}_{R^{1/p}}(S \otimes_R R^{1/p}, S^{1/p}) \cong \text{Hom}_R(S, S^{1/p})$$

We have completed the proof. □

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