SPACE OF ONE CYCLES AND CONIVEAU FILTRATIONS

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Abstract. We prove a structural result about the space of one cycles of a separably rationally connected variety or a separably rationally connected fibration over a curve, either as a topological group or as an h-sheaf. This has the following consequences: a proof that the strong coniveau filtration agrees with the coniveau filtration on degree 3 homology, and a result on the integral Tate conjecture for homologically trivial one cycles.

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1. Introduction

1.1. Space of one cycles and Lawson homology. In this paper we undertake the study of the space of one cycles on a separably rationally connected variety or a separably rationally connected fibration over a curve, with a view towards understanding the relation between Lawson homology/higher Chow groups and homologies for such varieties.

Given a complex projective variety $X$, one can give the space of $r$-dimensional cycles a natural topology such that it becomes a topological group, denoted by $Z_r(X)$ [Law89]. There are many equivalent ways to define the topology. See Section 4 for some of them.

Lawson [Law89] was the first to realize the importance of understanding the homotopy type of this topological group.

Definition 1.1. Let $X$ be a reduced complex projective scheme. Define the Lawson homology $L_r H_{n+2r}(X)$ as the homotopy group $\pi_n(Z_r(X))$. 

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In a similar fashion, but more in the spirit of motivic complex developed by Voevodsky and his collaborators, one can consider the presheaf that assigns to each variety a family of \( r \)-dimensional cycles in \( X \). Homology of the associated Suslin complex computes Bloch’s higher Chow groups.

Even though we will not use this later in the paper, we would like to mention that Lawson homology can be seen as the semi-topological analogue of higher Chow groups. Moreover, for finite coefficients, Suslin-Voevodsky [SV96, Section 8] proves that Lawson homology agrees with higher Chow groups. This explains why our main results applies to Lawson homology for complex varieties and higher Chow groups about varieties defined over other algebraically closed fields in this paper.

Understanding this topological group or the sheaf is important for understanding algebraic cycles on \( X \). In both cases, the space/sheaf has a description as a colimit of topological spaces/sheaves, taken over all smooth projective varieties and families of cycles over these varieties. A fundamental problem is to understand more precisely about the structure of this colimit. For example, for several applications discussed later in the paper, one would like to know whether or not it is a filtered colimit or a homotopy colimit. While this may not be the case in general, for the space of one cycles on a separably rationally connected varieties, we prove some results about this colimit in the positive direction. The precise statement is a little technical and involved to be included here. See Section 3, in particular Theorem 3.5, for the precise statement. Roughly speaking, this theorem suggests that we might have something close to a homotopy colimit.

Without going into the details about this structural result about the space of one cycles, we describe one of its consequences and some of its applications.

**Theorem 1.2 (=Theorem 4.8).** Let \( X \) be a complex smooth projective variety. Assume that either \( X \) is rationally connected or \( X \) is a rationally connected fibration over a curve. Then for any loop \( L \) in \( Z_1(X) \), there is a smooth projective variety \( Y \) with a family of 1 dimensional cycles in \( X \) over \( Y \) such that the map 
\[
\Phi : L_0 H_1(Y) = \pi_1(Z_0(Y)) \to L_1 H_3(X) = \pi_1(Z_1(X))
\]
induced by the family of one cycles contains the class \([L]\) in \( L_1 H_3(X) \).

There is also a version for varieties defined over any algebraically closed field, in which one use higher Chow groups with finite coefficients (Theorem 5.7).

In the following, we describe some applications of this theorem and its higher Chow group counterpart.

### 1.2. Coniveau and Strong Coniveau

In [BO21] [Voi20], Benoist-Ottem and Voisin introduced several a priori different notions of coniveau filtrations on the cohomology of a variety. Let us first review the definitions.

**Definition 1.3.** Let \( X \) be a smooth projective variety of dimension \( n \) defined over an algebraically closed field. Given an abelian group \( A \) that is one of \( \mathbb{Z}/n\mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}_\ell \), we simply write \( H^k(X, A) \) as either the étale cohomology with coefficient \( A \) or the singular cohomology with coefficient \( A \) (if \( X \) is a complex variety), we have the following closely related filtrations on the cohomology \( H^k(X, A) \).

1. **The coniveau filtration:**
   \[
   N^c H^k(X, A) := \bigoplus_{f: Y \to X} f_* (H_{2n-k}(Y, A)) \subset H_{2n-k}(X, A) \cong H^k(X, A),
   \]
where the sum is taken over all morphisms from projective algebraic sets $f : Y \to X, \dim Y \leq n - c$.

(2) The strong coniveau filtration:
\[ \tilde{N}^c H^k(X, A) := \sum_{f: Y \to X} f_*(H_{2n-k}(Y, A)) \subset H_{2n-k}(X, A) \cong H^k(X, A), \]
where the sum is taken over all morphisms from *smooth* projective varieties $f : Y \to X, \dim Y \leq n - c$.

(3) The strong cylindrical filtration:
\[ \tilde{N}_{c, cyl} H^k(X, A) := \sum_\Gamma \Gamma_*(H_{2n-k-2c}(Z, A)) \subset H_{2n-k}(X, A) \cong H^k(X, A) \]
where the sum is taken over all smooth projective varieties $Z$ and correspondences $\Gamma \subset Z \times X$ of relative dimension $c$ over $Z$.

(4) The strong equidimensional cylindrical filtration:
\[ \tilde{N}_{c, cyl, eq} H^k(X, A) := \sum_\Gamma \Gamma_*(H_{2n-k-2c}(Z, A)) \subset H_{2n-k}(X, A) \cong H^k(X, A) \]
where the sum is taken over all smooth projective varieties $Z$ and correspondences $\Gamma \subset Z \times X$ that is equidimensional of relative dimension $c$ over $Z$.

(5) The semi-stable filtration: $N_{c, st, cyl} H^k(X, Z)$ as the group generated by the cylinder homomorphisms
\[ f_* \circ p^* : H_{2n-k-2c}(Z, Z) \to H_{2n-k}(X, Z) \cong H^k(X, Z), \]
for all morphisms $f : Y \to X$, and flat projective morphisms $p : Y \to Z$ of relative dimension $c$ with simple normal crossing fibers, where $\dim Z \leq 2n - k - 2c$.

(6) We use the notations $N^c H_k$ etc. to denote the filtrations on Borel-Moore or singular homology $H_k$. Since $X$ is smooth, this is the same as the filtrations $N^c H^{2d-k}$.

A general relation between these filtrations is the following.

**Lemma 1.4.** [Voi20, Proposition 1.3] We have the following inclusions:
\[ \tilde{N}_{n-c, cyl, eq} H^{2c-1}(X, A) \subset \tilde{N}_{n-c, cyl} H^{2c-1}(X, A) = \tilde{N}^c H^{2c-1}(X, A) \subset N^c H^{2c-1}(X, A). \]

The only non-obvious part, the equality in the middle, is proved by Voisin [Voi20, Proposition 1.3].

A natural question is whether or not these filtrations agree with each other. If we use $\mathbb{Q}$ or $\mathbb{Q}_l$ coefficients, the theory of weights shows that the strong coniveau and coniveau filtrations are equivalent. Since the difference between some of these filtrations also gives stable birational invariants, one wonders if this could be used to prove non-stable-rationality for some rationally connected varieties.

Examples with $\mathbb{Z}$-coefficients where the strong coniveau filtration and coniveau filtration differ are constructed in [BO21]. More precisely, they prove the following.

**Theorem 1.5.** [BO21, Theorem 1.1] For all $c \geq 1$ and $k \geq 2c + 1$, there is a smooth projective complex variety $X$ such that the inclusion
\[ N^c H^k(X, Z) \subset N^c H^k(X, Z) \]
is strict. One may choose $X$ to have torsion canonical bundle. If $c \geq 2$, one may choose $X$ to be rational.
The examples as above usually have large dimension especially when $c$ is large. For lower dimensional examples, Benoist-Ottem proved the following.

**Theorem 1.6.** [BO21, Theorem 1.2] For $k \in \{3, 4\}$, there is a smooth projective complex variety $X$ of dimension $k + 1$ with torsion canonical bundle such that the inclusion

$$\tilde{N}^1 H^k(X, \mathbb{Z}) \subset N^1 H^k(X, \mathbb{Z})$$

is strict.

These examples leave the case of $c = 1$ open for threefolds and for rationally connected varieties. Voisin studied the strong coniveau filtrations on $H^{2d-3}$ [Voi20].

**Theorem 1.7.** [Voi20, Theorem 2.6, Corollary 2.7, Theorem 2.17] Let $X$ be a smooth projective variety of dimension $d$ defined over $\mathbb{C}$.

1. Assume the Walker Abel-Jacobi map [Wal07]
   $$\phi : CH_1(X)_{alg} \to J(N^1 H^{2d-3}(X, \mathbb{Z}))$$
   is injective on torsions. Then we have
   $$N_{1, st, cyl} H^{2d-3}(X, \mathbb{Z}) / \text{Tor} = N^1 H^{2d-3}(X, \mathbb{Z}) / \text{Tor}.$$

2. If $\dim X$ is 3, we have
   $$N_{1, cyl, st} H^3(X, \mathbb{Z}) / \text{Tor} = N^1 H^3(X, \mathbb{Z}) / \text{Tor}.$$

3. If $X$ is rationally connected, we have
   $$N_{1, cyl, st} H^{2d-3}(X, \mathbb{Z}) = \tilde{N}_{1, st} H^{2d-3}(X, \mathbb{Z}).$$

As a consequence,
   $$N_{1, cyl, st} H^{2d-3}(X, \mathbb{Z}) / \text{Tor} = \tilde{N}^1 H^{2d-3}(X, \mathbb{Z}) / \text{Tor} = N^1 H^{2d-3}(X, \mathbb{Z}) / \text{Tor}.$$

The first application of our general result about the space of cycles is an improvement of Voisin’s result for (separably) rationally connected varieties.

**Theorem 1.8.** Let $X$ be a complex smooth projective rationally connected variety or a rationally connected fibration over a curve of dimension $d$. There is a smooth projective curve $C$ with a family of 1-dimensional cycles $\Gamma \subset C \times X$ such that

$$\Gamma_* : H_1(C, \mathbb{Z}) \to H_3(X, \mathbb{Z})$$

has the same image as the $s$-map $s : L_1 H_3(X) \to H_3(X)$, which is the same as the $N_1 H_3(X, \mathbb{Z})$. In particular, the following filtrations on $H^{2d-3}(X, \mathbb{Z})$ introduced in Definition 1.3 are the same:

$$\tilde{N}_{1, cyl, eq} H^{2d-3}(X, \mathbb{Z}) = \tilde{N}_{1, cyl} H^{2d-3}(X, \mathbb{Z}) = \tilde{N}^{d-1} H^{2d-3}(X, \mathbb{Z}) = N^{d-1} H^{2d-3}(X, \mathbb{Z}).$$

An immediate corollary is the following.

**Theorem 1.9.** Let $X$ be a complex smooth projective rationally connected variety or a rationally connected fibration over a curve of dimension 3. Then the following filtrations on $H^3(X, \mathbb{Z})$ introduced in Definition 1.3 coincide with the whole cohomology group:

$$\tilde{N}_{1, cyl, eq} H^3(X, \mathbb{Z}) = \tilde{N}_{1, cyl} H^3(X, \mathbb{Z}) = \tilde{N}^1 H^3(X, \mathbb{Z}) = N^1 H^3(X, \mathbb{Z}) = H^3(X, \mathbb{Z}).$$
Remark 1.10. Using the decomposition of the diagonal argument, one can show that when \( X \) is rationally connected or a rationally connected fibration, the kernel and cokernel of \( \Gamma : H_k(Y) \to H_{k+2}(X) \) is \( N \)-torsion for a fixed \( N \). So we may consider the \( s \)-map from \( \mathbb{Z}/N \) Lawson homology (defined as the homotopy group of the topological group \( \mathbb{Z}_r(X) \otimes \mathbb{Z}/N \) to \( H_2(X, \mathbb{Z}/N) \). We have long exact sequences

\[
L_1H_k(X, \mathbb{Z}) \to L_1H_k(X, \mathbb{Z}/N) \to L_1H_{k-1}(X, \mathbb{Z}) \to 0
\]

By results of Suslin-Voevodsky [SV96] and the Bloch-Kato conjecture proved by Voevodsky, there is an isomorphism

\[
L_1H_{2+k}(X, \mathbb{Z}/N) \cong CH_1(X, k, \mathbb{Z}/N) \cong \mathbb{H}^1(X, \tau^{\leq \dim X-1} R\pi_*(\mathbb{Z}/N))
\]

between torsion Lawson homology, Bloch’s higher Chow group, and certain Zariski cohomology group, where \( \pi : X_{cl} \to X_{zar} \) is the continuous map from \( X(\mathbb{C}) \) with the analytic topology to \( X \) with the Zariski topology.

We also have a long exact sequence:

\[
\ldots \to L_1H_k(X, \mathbb{Z}/N) \to H_k(X, \mathbb{Z}/N) \to KH_k(X, \mathbb{Z}/N) \to L_1H_{k-1}(X, \mathbb{Z}/N) \to \ldots
\]

where \( KH_k(X, \mathbb{Z}/N) = H^{\dim X-k}(X, \mathbb{R}^{\dim X} \pi_* \mathbb{Z}/N) \) is the so-called Kato homology. The author has made a number of conjectures about Kato homologies of a rationally connected fibration in [Tian20]. Special cases of these conjectures imply that there is an isomorphism \( L_1H_k(X, \mathbb{Z}/N) \cong H_k(X, \mathbb{Z}/N) \) for all \( k \) and all rationally connected varieties and rationally connected fibrations over a curve defined over the complex numbers. This in turn would imply the \( s \)-maps \( L_1H_k(X, \mathbb{Z}) \to H_k(X, \mathbb{Z}) \) are isomorphisms.

We have a similar result that applies to fields of positive characteristic.

Theorem 1.11 (=Theorem 5.13). Let \( X \) be a \( d \)-dimensional smooth projective separably rationally connected variety or a separably rationally connected fibration over a curve defined over an algebraically closed field. There is a smooth projective curve \( C \) with a family of 1-dimensional cycles \( \Gamma \subset C \times X \) such that

\[
\Gamma_* : H_1^{BM}(C, \mathbb{Z}_\ell) \to H_3^{BM}(X, \mathbb{Z}_\ell) \cong H^{2d-3}(X, \mathbb{Z}_\ell)
\]

surjects onto \( N^1H^{2d-3}(X, \mathbb{Z}_\ell) \).

Theorem 1.12 (=Theorem 5.14). Let \( X \) be a smooth projective separably rationally connected variety or a separably rationally connected fibration over a curve defined over an algebraically closed field. Assume \( X \) is a 3-fold. Then the following filtrations on \( H^3(X, \mathbb{Z}_\ell) \) introduced in Definition 1.3 equal the whole cohomology group:

\[
\tilde{N}_{1, cl} \mathcal{H}^3(X, \mathbb{Z}_\ell) = N_{1, cl} \mathcal{H}^3(X, \mathbb{Z}_\ell) = N^1 \mathcal{H}^3(X, \mathbb{Z}_\ell) = N^1H^3(X, \mathbb{Z}_\ell) = H^3(X, \mathbb{Z}_\ell).
\]

1.3. Integral Tate conjecture for one cycles. Given a smooth projective variety defined over a finite field \( \mathbb{F}_q \), we have the cycle class map

\[
CH_r(X, \mathbb{Z}_\ell) \to H^{2d-2r}(X, \mathbb{Z}_\ell(d-r))
\]

The integral Tate conjecture asks when is this cycle class map surjective.
Thus the integral Tate conjecture consists of a geometric part, i.e. surjectivity of $CH_1(X) \otimes \mathbb{Z}_\ell \to H^{2d-2r}_\text{et}(\bar{X}, \mathbb{Z}_\ell(d-r))_G$, and an arithmetic part, i.e. surjectivity of $CH_r(X) \otimes \mathbb{Z}_\ell \to H^{2d-2r}_\text{et}(\bar{X}, \mathbb{Z}_\ell(d-r))_G$.

In a recent preprint [SS22], Scavia and Suzuki systematically investigated the question of the surjectivity in the arithmetic part and relate them to the strong coniveau filtration. For codimension 2 cycles, they obtain the following results.

**Theorem 1.13.** [SS22] Theorem 1.3] Let $F$ be a finite field and $\ell$ be a prime number invertible in $F$, and suppose that $F$ contains a primitive $\ell^2$-th root of unity. There exists a smooth projective geometrically connected $F$-variety $X$ of dimension $2\ell + 2$ such that the map

$$CH^2(X) \otimes \mathbb{Z}_\ell \to H^4_{\text{et}}(X, \mathbb{Z}_\ell(4))_G$$

is surjective whereas the map

$$CH^2(X)_\text{hom} \otimes \mathbb{Z}_\ell \to H^1(F, H^3_{\text{et}}(\bar{X}, \mathbb{Z}_\ell(2)))$$

is not.

**Theorem 1.14.** [SS22] Theorem 1.4] Let $p$ be an odd prime. There exist a finite field $\mathbb{F}$ of characteristic $p$ and a smooth projective geometrically connected fourfold $X$ over $\mathbb{F}$ for which the image of the composition

$$H^1(\mathbb{F}, H^3_{\text{et}}(X, \mathbb{Z}_2(2)))_\text{tors} \to H^1(\mathbb{F}, H^3_{\text{et}}(X, \mathbb{Z}_2(2))) \to H^4(\mathbb{F}, \mathbb{Z}_2(2))$$

contains a non-algebraic torsion class.

Taking products with projective spaces of the example in the above theorem gives higher dimensional examples.

The failure of the surjectivity is related to the discrepancy of the strong coniveau and coniveau filtration by the following.

**Theorem 1.15.** [SS22] Theorem 1.5, Proposition 7.6] Let $X$ be a smooth projective geometrically connected variety over a finite field $\mathbb{F}$ and $\ell$ be a prime number invertible in $\mathbb{F}$. Suppose that the coniveau and strong coniveau on $H^3_{\text{et}}(X, \mathbb{Z}_\ell(2))$ coincide: $N^1H^3(X, \mathbb{Z}_\ell(2)) = \hat{N}^1H^3_{\text{et}}(X, \mathbb{Z}_\ell(2))$. Then the $\ell$-adic algebraic Abel-Jacobi map is an isomorphism:

$$CH^2(X)_\text{alg} \otimes \mathbb{Z}_\ell \to H^1(\mathbb{F}, N^1H^3(X, \mathbb{Z}_\ell(2))).$$

In general, we have a surjection

$$CH^r(X)_{F\text{-alg}} \otimes \mathbb{Z}_\ell \to \text{Image}(H^1(\mathbb{F}, \hat{N}^{r-1}H^{2r-1}(X, \mathbb{Z}_\ell(r))) \to H^1(\mathbb{F}, N^{r-1}H^{2r-1}(X, \mathbb{Z}_\ell(r))).$$
In the paper of Scavia-Suzuki [SS22], they use \( CH^r(X)_{F\text{-alg}} \) to denote cycles algebraically equivalent to zero over \( F \), and \( CH^r(X)_{\text{alg}} \) to denote cycles algebraically equivalent to zero over \( \bar{F} \). However, for codimension 2 cycles on varieties defined over \( F \), there is no known example where \( CH^2(X)_{F\text{-alg}} \) and \( CH^2(X)_{\text{alg}} \) differ (Question 8.2, [SS22]), and \( CH^2(X)_{F\text{-alg}} \otimes \mathbb{Z}_\ell \) and \( CH^2(X)_{\text{alg}} \otimes \mathbb{Z}_\ell \) are isomorphic if the strong coniveau coincides with the coniveau filtration on \( H^3 \) ([SS22, Proposition 7.10, 7.11]). Moreover, for one cycles on a separably rationally connected variety or a separably rationally connected fibration over a curve, we know \( CH^1(X)_{F\text{-alg}} \) and \( CH^1(X)_{\text{alg}} \) are the same [Tia22, Theorem 1.13].

While the surjectivity in the arithmetic part is not true in general, we do expect this to be true for separably rationally connected varieties and separably rationally connected fibrations over a curve.

As a corollary of Theorem 5.7 and the work of Scavia-Suzuki, we get the following results regarding the arithmetic part of the cycle class map.

**Corollary 1.16** (=Corollary 5.15). *Let \( X \) be a smooth projective variety of dimension \( d \) defined over a finite field \( \mathbb{F}_q \), that is either separably rationally connected or a separably rationally connected fibration over a curve. Then we have a surjection*

\[
CH^1(X)_{\text{alg}} \to H^1(\mathbb{F}_q, N^1 H^{2d-3}_\text{et}(\bar{X}, \mathbb{Z}_\ell(d-1))).
\]

Furthermore, assume one of the followings

1. \( N^1 H^{2d-3}_\text{et}(\bar{X}, \mathbb{Z}_\ell(d-1)) = H^{2d-3}_\text{et}(\bar{X}, \mathbb{Z}_\ell(d-1)) \).

2. The cycle class map

\[
cl : \lim_{\ell \to \infty} CH^1(\bar{X}, 1, \mathbb{Z}/\ell^n) \to H^{2d-3}_\text{et}(\bar{X}, \mathbb{Z}_\ell(d-1))
\]

is surjective.

Then every class in \( H^1(\mathbb{F}_q, H^3(\bar{X}, \mathbb{Z}_\ell(d-1))) \) is the class of an algebraic cycle. In particular, this holds if \( X \) has dimension 3.

1.4. **Structure of the paper.** The key technical part consists of Section 2 and 3, where we develop the relevant deformation techniques to prove the structural result about the space of 1-cycles. Section 2 is a brief recap of some of the results contained in [Tia22]. The most important concept is that of \( m \)-free curves and \( n \)-connecting curves. We systematically use these curves later. This section also contains a very useful lemma, the sliding lemma 2.9. Section 3 proves the main technical result, Theorem 3.5. While the idea is simple and geometric, the actual proof is a little complicated. We illustrate the basic idea and constructions in Example 3.12 and 3.13.

The applications are contained in Section 4 and 5. The complex varieties case and the general case are similar in spirit, but differ slightly in the statement and proof due to some technicality. We present the more geometric argument in the complex case first, hoping to show the readers the flavor of the argument.

2. **Deformation of stable maps**

In this section we recall some results proved in [Tia22] for deformations of stable maps. For the proof and more details for the following results, see [Tia22, Section 2].

We start with some definitions.
Definition 2.1. A *comb* is a stable map \( f : C \to X \) such that the dual graph of the domain (a connected nodal curve) has vertices \( v_0, v_1, \ldots, v_n \) and edges \( e_i, 1 \leq i \leq n \) connecting \( v_0 \) and \( v_i \). The irreducible component corresponding to \( v_0 \) is called the *handle*. The other irreducible components are called *teeth*.

Definition 2.2. Let \( X \) be a smooth projective variety. A curve \( f : C \to X \) is \( m \)-free, if it is embedded in the case \( \dim X \geq 3 \), immersion in the case \( \dim X = 2 \), and for any effective divisor \( D \) of degree \( m \) on \( C \), \( \mathbb{H}^2(C, \Omega_C^* (-D)) = 0 \).

A family of curves \( F : C \to X, p : C \to S \) is \( n \)-connecting (resp. generically \( n \)-connecting), if for any distinct \( n \) points (resp. any general \( n \)-points) in \( X \), the subfamily of curves in \( C \to S \) passing through these points is non-empty and geometrically irreducible.

Remark 2.3. We can take the product of \( X \) with some \( \mathbb{P}^N \) and choose a morphism \( C \to \mathbb{P}^N \). Deformations in the product gives deformations in \( X \). For almost all of the applications, we can replace \( X \) by the product and assume \( \dim \geq 3 \).

First, we show the existence of \( m \)-free \( n \)-connecting curves on a separably rationally connected fibration.

Lemma 2.4. Let \( X \) be a smooth projective variety that is a separably rationally connected fibration over a curve or a separably rationally connected variety defined over any field. Then there exists a family of \( m \)-free \( n \)-connecting curves defined over the same field for any positive integers \( m, n \).

Lemma 2.5. Let \( X \) be a smooth projective variety that is either a separably rationally connected fibration over a smooth projective curve \( B \) or a separably rationally connected variety. For any stable map \( f : C \to X \) from a connected prestable curve, and any effective divisor \( D \) contained in the smooth locus of \( C \), one can attach \( 2 \)-free curves to \( C \) at general points along general directions, and the resulting stable map \( F : \bar{C} \to X \) satisfies the vanishing
\[
\mathbb{H}^2(\bar{C}, \Omega_{\bar{C}}^* \otimes \mathcal{O}_{\bar{C}}(-D)) = 0,
\]
and a general deformation has smooth irreducible domain. Moreover, a general deformation of the stable map is \( \deg D \)-free.

It turns out that \( 2 \)-free \( 2 \)-connecting curves gives higher connecting curves.

Lemma 2.6. Let \( X \) be a smooth projective variety that is a separably rationally connected fibration over a curve or a separably rationally connected variety. Assume that \( X \) has dimension at least 3. Let \( f : C \to X \) be a morphism from a smooth projective curve. Fix positive integers \( m, n \). Fix a family of \( 2 \)-free \( 2 \)-connecting curves and let \( g : D \to X \) be the comb with handle \( C \) and with sufficiently many teeth (depending on \( m, n \)) at general points along general directions as in Lemma 2.5. Then the family of curves coming from general deformations of \( g \) is \( m \)-free \( n \)-connecting.

Corollary 2.7. Let \( X \) be a smooth projective variety that is a separably rationally connected fibration over a curve or a separably rationally connected variety. Assume that \( X \) has dimension at least 3. Let \( f : \mathbb{P}^1 \to X \) be a constant morphism. Fix a family of \( 2 \)-free \( 2 \)-connecting curves and let \( g : D \to X \) be the comb with handle \( C \) and with teeth that belong to this family. Then the family of curves coming from general deformations of \( g \) is \( 2 \)-free \( 2 \)-connecting.
The next lemma explains the advantage of having \( n \)-connecting curves.

**Lemma 2.8.** Let \( X \) be a smooth projective variety that is a separably rationally connected fibration over a curve or a separably rationally connected variety. Let \( f : \Sigma \to X, p : \Sigma \to T \) be a family of pre-stable maps parameterized by a smooth curve \( T \) with geometrically irreducible generic fiber. Fix a positive integer \( n \). Fix \( n \)-free \( n \)-connecting curves \( C \to S, C \to X \). Let \( t_1, t_2 \) be two points in the curve \( T \). Choose any \( n \)-tuple of points in the smooth locus of \( \Sigma_{t_1} \) (resp. \( \Sigma_{t_2} \)) whose image in \( X \) consisting of \( n \) distinct points, and any curve \( C_1 \) (resp. \( C_2 \)) in the family of \( m \)-free \( n \)-connecting curves containing the image of \( n \)-tuple of points in \( \Sigma_{t_1} \) (resp. \( \Sigma_{t_2} \)), one can construct two new stable maps by taking the union \( \Sigma_1 = \Sigma_{t_1} \cup C_1 \) (resp. \( \Sigma_2 = \Sigma_{t_2} \cup C_2 \)) along the \( n \)-tuple of points and there is a deformation parameterized by an irreducible curve \( T' \) from \( \Sigma_1 \) to \( \Sigma_2 \).

Next we introduce some useful type of deformations. Recall that the dual graph of a stable map is a graph whose vertices correspond to irreducible components and edges correspond to nodes. For a schematic illustration of the deformations in the following lemma, see Figure 1.

1. Deform the curve \( D \) along the curve \( C \) to the node.
2. A new \( \mathbb{P}^1 \) component with three nodes connecting \( B, C, D \) is created.
3. Move \( D \) away from the node along the curve \( B \).

**Figure 1.** Sliding the curve \( D \) from \( C \) to \( B \)

**Lemma 2.9** (The sliding lemma). Let \( f : \Gamma \to V \) be a stable map with dual graph \( G \). Let \( B, C, D \) be three irreducible components of \( \Gamma \) corresponding to a chain \( V_B \to V_C \to V_D \) of adjacent vertices in \( G \). Assume that every irreducible component is \( m \)-free (in particular, every irreducible component is an embedded smooth curve by definition of \( m \)-freeness) for \( m \) larger than the number of nodes on the component. Then there is a deformation parameterized by an irreducible curve from \( f \) to a stable map \( f' : \Gamma' \to V \), whose irreducible components are unions of deformations of \( B, C, D \) and all the other irreducible components of \( \Gamma \) and whose dual graph \( G' \) is obtained from \( G \) by changing the chain \( V_B \to V_C \to V_D \) in \( G \) to a \( V \)-shape \( V_D \leftarrow V_B \to V_C \) and keeping the other edges unchanged (here we use the same symbol to denote the vertex corresponding to deformations of irreducible components).

Moreover, if the \( D \) belong to a family of \( n \) connecting curves (where \( n \) is the number of nodes connecting \( D \) and other irreducible components in \( \Gamma \)), then we may choose the deformations of \( B \) and \( C \) to form the stable map \( \Gamma' \) to be trivial.

We finish this section with two particular type of deformations.

**Lemma 2.10** (The break lemma). Let \( f : C \to D \) be a generically étale morphism of degree \( d \) between smooth projective connected curves. There is a family of curves \( W \to \mathbb{P}^1, F : W \to D \) with the following properties.
(1) \(W_0\) is a connected nodal curve, which is the union of a copy of \(C\) and \(\mathbb{P}^1\)'s.
(2) Each copy of \(\mathbb{P}^1\) connects \(C\) at two points.
(3) The morphism \(F\) maps each copy of \(\mathbb{P}^1\) to a point in \(D\), and \(F\) restricted to the copy of \(C\) is the same as \(f\).
(4) \(W_\infty\) is the union of \(d\) copies of \(D\) (with identity morphism \(D \to D\)), and a number of \(\mathbb{P}^1\)'s that map to points in \(D\).

**Lemma 2.11.** Let \(C\) be a connected projective nodal curve and \(D\) a smooth projective curve, and let \(f : C \to D\) be a morphism. There is a family of curves \(W \to \mathbb{P}^1, F : W \to D\) with the following properties.

1. \(W_0\) is a connected nodal curve, which is the union of a copy of \(C\) and a smooth irreducible projective curve \(C'\).
2. \(W_\infty\) is a smooth irreducible curve.
3. \(F|_{C'} : C' \to D\) and \(F|_{W_\infty} : W_\infty \to D\) are generically étale.

### 3. Space of cycles

In this section, we fix an algebraically closed field \(k\) of any characteristic. We remind the readers that a variety is always assumed to be irreducible through out the paper, and thus connected. Sometimes we add the word irreducible just to emphasis this.

**Definition 3.1.** Let \(X, Y\) be finite type reduced separated \(k\)-schemes. A family of relative cycles of equi-dimension \(r\) over \(Y\) is a formal linear combination of integral subschemes \(Z = \sum m_i Z_i, Z_i \subset Y \times X, m_i \in \mathbb{Z}\) such that

1. Each \(Z_i\) dominates one irreducible component of \(Y\).
2. Each fiber of \(Z_i \to Y\) has dimension \(r\).
3. A fat point condition is satisfied (see [Kol96, Chapter I, 3.10.4] or [SV00, Definition 3.1.3]).
4. A field of definition condition is satisfied. Namely, for any point \(y \in Y\), there are integral subschemes \(\gamma_i\) of \(X\) defined over the residue field \(\kappa(y)\) such that the cycle theoretic fiber ([Kol96, Chapter I, 3.10.4]) of \(Z\) over \(y\) is \(\sum m_i \gamma_i\).

We say this family has proper support if \(Z_i \to Y\) are proper for every \(i\).

We refer the interested readers to [Kol96] Section I.3, I.4 and [SV00] Section 3 for details about the last two conditions and the subtle points about these definitions. We only remark that with this definition, one can pull-back families of cycles.

We decided to adopt Kollár’s [Kol96] convention of only considering families of cycles over a reduced base. Suslin and Voevodsky [SV00] consider more general base.

For our purpose, it is perfectly fine to always work over a reduced base.

**Remark 3.2.** We would also like to mention that for a normal variety, condition 3 is automatic. So for our purpose, one can safely ignore this condition later in the discussion. Condition 4 is always satisfied in characteristic 0 or if all the \(m_i\)'s are invertible in the field \(k\). It is introduced to make sure that pulling-back of families of cycles is always defined and one has a presheaf of relative cycles.

**Definition 3.3.** Let \(I^{eq}\) be the category, whose objects are pairs \((S, \Gamma)\) consisting of a normal \(k\)-variety \(S\) with a family of relative 1-cycles in \(S \times X\), and whose
morphism set between \((S_1, \Gamma_1)\) and \((S_2, \Gamma_2)\) is the set of morphisms \(f : S_1 \to S_2\) such that \(\Gamma_1\) is the pull-back of \(\Gamma_2\).

**Definition 3.4.** Let \(I^{st}\) be the category, whose objects are pairs \((S, \Gamma)\) consisting of a normal \(k\)-variety \(S\) with families of stable maps (with possibly disconnected domains) \(F_i : C_i \to X, C_i \to S\) and a family of cycles \(\Gamma = \sum_i m_i[C_i]\) over \(S\), and whose morphism set between \((S_1, \Gamma_1)\) and \((S_2, \Gamma_2)\) is the set of morphisms \(f : S_1 \to S_2\) such that \(\Gamma_1\) is the pull-back of \(\Gamma_2\) (i.e. the families of stable maps are pull-back families).

There is a functor \(\Phi : I^{st} \to I^{eq}\), the Kontsevich-Chow functor, that associates to a family of stable maps the corresponding cycle class of the images.

Now we can state the main technical result of the paper.

**Theorem 3.5.** Let \(X\) be a smooth projective separably rationally connected variety or a separably rationally connected fibration over a curve defined over an algebraically closed field \(k\).

1. Let \((S_1, \Gamma_1)\) and \((S_2, \Gamma_2)\) be two objects in \(I^{st}\) or \(I^{eq}\). Assume that cycles in the families \(\Gamma_1\) and \(\Gamma_2\) are algebraically equivalent. There is an object \((S, \Gamma)\) in \(I^{st}\) or \(I^{eq}\), and morphisms \(f_1 : (S_1, \Gamma_1) \to (S, \Gamma), f_2 : (S_2, \Gamma_2) \to (S, \Gamma)\). Moreover, if both \(S_1\) and \(S_2\) are smooth/projective, we may choose \(S\) to be smooth/projective.

2. Let \((S_1, \Gamma_1)\) be an object in \(I^{eq}\) and let \(f, g : S_0 = \text{Spec} k \to S_1\) be two morphisms between schemes such that \((S_0, f^*\Gamma_1) = (S_0, g^*\Gamma_1) = (S_0, \Gamma_0) \in I^{eq}\).

Then there is an object \((\tilde{S}_1, \tilde{\Gamma}_1)\) in \(I^{eq}\) with a morphism \((\tilde{S}_1, \tilde{\Gamma}_1) \to (S_1, \Gamma_1)\), and two liftings \(\tilde{f}, \tilde{g} : S_0 \to \tilde{S}_1\) of \(f, g\), such that

(a) The morphism \(\tilde{S}_1 \to S_1\) is projective, dominant. The variety \(\tilde{S}_1\) may be chosen to be normal.

(b) There are objects \((\tilde{S}_2, \tilde{\Gamma}_2), (T_1, \gamma_1), \ldots, (T_n, \gamma_n)\) in \(I^{eq}\) and two commutative diagrams

\[
\begin{array}{ccc}
(S_0, \Gamma_0) & \xrightarrow{f} & \coprod_{i=1}^n (T_i, \gamma_i) \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
(S_1, \Gamma_1) & \xrightarrow{\tilde{f}} & (\tilde{S}_1, \tilde{\Gamma}_1) \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
(S_0, \Gamma_0) & \xrightarrow{g} & \coprod_{i=1}^n (T_i, \gamma_i) \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
(S_1, \Gamma_1) & \xrightarrow{\tilde{g}} & (\tilde{S}_1, \tilde{\Gamma}_1) \\
\end{array}
\]

where, \(\coprod_{i=1}^n (T_i, \gamma_i)\) is a chain of constant family of cycles in the sense that for any \(k\)-point of \(T_i\), the cycle \(\gamma_i\) over that point is the same as \(\Gamma_0\) and there are points \(t_i, t'_i \in T_i\) such that \(f'(S_0) = t_i, g'(S_0) = t'_i\) and that \(t'_i\) and \(t_{i+1}\) are mapped to the same point in \(\tilde{S}_1\) for \(i = 1, \ldots, n-1\). We may choose every variety to be normal.

(c) Fix a general point \(s \in S_1\). For any pair of points \(s', s''\) in the inverse image of \(s\) in \(\tilde{S}_1\), there are objects \((C_1, \gamma_{C_1}), \ldots, (C_n, \gamma_{C_n})\) in \(I^{eq}\) such that \(\coprod_{i=1}^n (C_i, \gamma_{C_i})\) is a chain of constant family of cycles connecting \(s'\)
and $s''$. We may choose every variety to be normal and that $(C_i, \gamma_{C_i})$ admits a morphism to $(\tilde{S}_2, \tilde{\Gamma}_2)$.

(d) In characteristic 0, we may assume $S_2$ is smooth and projective.

**Remark 3.6.** We remark that in the second part of the theorem, the positive and negative part of the families of cycles $\gamma_i$ may vary along $T_i$. The second part of the theorem only states that the difference remains constant.

**Remark 3.7.** Even if we start with a family of effective cycles, for the statements to be true, we have to use non-effective cycles. Moreover, neither of the statement is a simple corollary of the existence of the universal family over the Chow variety (true only in characteristic 0). This is because we require that the family is parameterized by a normal variety, while the Chow variety is only semi-normal in [Kol96] by definition or satisfies no such normality condition at all in some other references such as [Fri91]. In the first statement, even if we take two effective cycles, it is not true in general that the Chow points parameterizing them lie in the same irreducible component. As for the second statement, it is possible that the morphism from the normalization of the Chow variety to the Chow variety maps two points to the same point $s$. If this happens, we take $S_0$ to be a point with the cycle over $s$, $S_1$ to be the normalization of the Chow variety, $f, g$ to be the two morphisms to the inverse image of $s$ in the normalization, the existence of $\tilde{S}_1$ in this case cannot be deduced from the existence of the universal family over the Chow variety.

A key ingredient in the proof of Theorem 3.5 is the following results of the author in [Tia22].

**Theorem 3.8** (Theorem 3.3 in [Tia22]). Let $V$ be a smooth projective variety defined over an algebraically closed field of dimension at least 3. Assume that either $V$ is separably rationally connected or a separably rationally connected fibration over a curve. Fix a family $C \to M$ of $n$-connecting $m$-free curves ($m, n \geq 2$). Let $\Gamma_1 \to V$ and $\Gamma_2 \to V$ be two morphisms from smooth connected projective curves of the same geometric genus to $V$ that are algebraically equivalent as cycles. Then there are two combs such that

1. The handles are $\Gamma_1$ and $\Gamma_2$.
2. There is a one-to-one correspondence between the teeth in the two combs. Each pair of teeth corresponds to two smooth points in the same irreducible component of the moduli space of stable maps.
3. The two combs correspond to two smooth points in the same irreducible component of the moduli space of stable maps, whose general member is a smooth embedded curve.

Moreover, we may assume that each tooth is $n'$-free and belongs to an $n'$-connecting family of curves for any positive integers $n', n'$.

**Corollary 3.9** (Corollary 3.4 in [Tia22]). Theorem 3.8 is true in the case $f_1 : \Gamma_1 \to X, f_2 : \Gamma_2 \to X$ are stable maps with connected but possibly reducible domains.

We first prove special cases of Theorem 3.5.

**Proposition 3.10.** The first statement in Theorem 3.5 is true if $S_1, S_2$ are Spec $k$. In this case, we may take $S$ to be a smooth projective curve.
Proof. We write $\Gamma_{S_1} - \Gamma_{S_2} = \Gamma^+ - \Gamma^-$, where $\Gamma^+$ (resp. $\Gamma^-$) is the positive (resp. negative) part of the cycle. By assumption, $\Gamma^+ , \Gamma^-$ are algebraically equivalent.

To prove the proposition, we only need to construct a family of stable maps over a smooth base $B$ so that

1. Some linear combinations of the cycle classes of the fibers over a point $b^+$ give $\Gamma^+$ and (similarly for $\Gamma^-$ over point $b^-$).
2. General members of the family are embedded, unobstructed smooth connected curves.

Given this family, we can produce the desired family by adding a constant family of cycles $\Gamma_{S_2} - \Gamma^-$ over $B$. Clearly we can restrict the family to a smooth curve. We may even assume that this curve is projective by applying stable reduction to the family of stable maps.

Now we explain the construction. Fix a family of 2-free curves. For each irreducible component of $\Gamma^+$, we take its normalization. After adding enough 2-free curves to make a comb $C$, we can make sure that the whole stable map has unobstructed deformation. If this irreducible component appears with multiplicity in the cycle $\Gamma^+$, we take copies of the comb.

We do this for each irreducible component and get combs $C_i, i = 1, \ldots$. We add one more curve $C_0$ that is embedded, sufficiently unobstructed, and connects all combs. Denote by $f^+: C^+ \to X$ the union of the stable maps.

We do the same thing for $\Gamma^-$ and have a stable map $f^- : C^- \to X$. We also arrange that we added the same number of 2-free curves in the chosen family in $C^+$ and $C^-$. We also choose the curve that connects all the combs to lie in the same irreducible component.

Now both $C^+$ and $C^-$ have connected domains. But they may have different arithmetic genus.

Finally, we add a stable map $g^+: D^+ \to X$ (resp. $g^- : D^- \to X$) to $C^+$ (resp. $C^-$). We choose the curves $D^+$ and $D^-$ to be smooth embedded, sufficient unobstructed and lie in the same irreducible component. Moreover, we make the arithmetic genus of $C^+ \cup D^+$ and $C^- \cup D^-$ equal. To achieve this, we simply normalize the curves at a suitable number of nodes between $C^+, D^+$ and $C^-, D^-$. This is possible since we may choose $D^+, D^-$ to pass through a large number of chosen points.

Now we can apply Corollary 3.9 to finish the proof.

Proposition 3.11. The second statement of Theorem 3.5 holds if $(S_1, \Gamma_1)$ comes from a family of stable maps.

Before we begin the long and complicated proof, we would like to give some examples first. The proof of Theorem 3.5 boils down to answer the following basic question: suppose that we are given two stable maps such that the push forward of the fundamental classes gives the same cycle class. Can we deform them from one to the other? We illustrate some of the difficulties and solutions in the following examples.

Example 3.12. Let $f_1 : C_1 \to D, f_2 : C_2 \to D$ be two morphisms of degree $d$ between smooth projective curves. Suppose that we have an embedding $D \subset X$. As a cycle in $X$, the image of $f_1, f_2$ are the same.

The question is: can one find a deformation of cycles from $f_1$ to $f_2$?
A first try might be to deform them as stable maps. Unfortunately, in this case the answer in general is no. There are two extreme cases of a degree $d$ covers. Namely, étale covers and purely inseparable covers. Both are rigid and cannot be deformed to other covers.

The solution is also very simple. We add and substract the same cycle, but represented by different stable maps. More precisely, we add $C_2$ to $f_1$ and add $d$ copies of the curve $D$ to $f_2$. Then we add $C_1$ to $f_2$ and add $d$ copies of $D$ to $f_1$. We call the resulting stable maps $f_1^+$ and $f_2^+$. These are the $+$-side stable maps. This has the effect of adding the cycles class $2dD$. So we take $2d$ copies of $D$ and denote them by $f_1^-, f_2^-$. These are the $-$-side stable maps. Now we have more than $d$ copies of $D$ in the $+$-side. We make an arbitrary choice of $d$-copies of $D$ and think of them as the stable maps that give the cycle class $dD$. These components are to be called $\Gamma$ curves later in the proof.

Then we clearly have a deformation of the $+$-side maps and the $-$-side stable maps, which leaves the $\Gamma$ curves fixed, and gives the desired deformation of cycles.

In the actual proof, we need to be able to add curves to a family of stable maps, which adds some extra difficulties. Also there are additional difficulties as explained in the next example.

**Example 3.13.** Suppose that we have two stable maps $f_1, f_2$ such that there is a one-to-one correspondence between irreducible components, and that the restriction of $f_1, f_2$ to each corresponding irreducible component gives the same morphism. But the dual graph of the domain curves are different. This can happen if we have multiple covers of the image curve.

How do we get the deformation from $f_1$ to $f_2$? The solution is to insert a $\mathbb{P}^1$ at every node and add 2-free curves to $\mathbb{P}^1$. Then we can smooth these combs. Let us call the smoothings connecting curves. We say that these curves are in the $+$-side. Since only the dual graph differs, we may use Lemma 2.9 to move these connecting curves. It is possible to deform the two stable maps to have the same dual graph in the $+$-side. To make the cycle class remain the same, we need to have families of curve in the $-$-side. This is easily achieved. We simply add all the connecting curves in the $-$-side. Whatever deformations of connecting curves we have in the $+$-side, we can follow these deformations in the $-$-side. This produces the desired deformation of cycles.

The above two examples explain the basic ideas and constructions in the proof. Now we have to carefully carry them out.

**Proof of Proposition 3.11.**

**Step 0.** Setup.

Denote by $\gamma$ the cycle given by the fiber over $f(S_0)$ and $g(S_0)$. There is a sequence of blow-ups at points $\tilde{X} = X_n \to \ldots \to X_0 = X$ such that the strict transforms of each irreducible component of $\gamma$ is smooth [SS10 Appendix Theorem A.2].

Then we take the strict transform of the families of stable maps over $S_1$ along these blow-ups such that we have a morphism over $S_1 | \Gamma'_1 | \to S_1 \times \tilde{X}$. The strict transforms may not be families of stable maps anymore. But we can make a base change to turn them into families of stable maps to $\tilde{X}$, and we may assume that each irreducible component of the inverse images of $f(S_0), g(S_0)$ is a smooth divisor.

We choose a lifting $\tilde{f}$ (resp. $\tilde{g}$) of $f$ (resp. $g$) at a general point in one of the irreducible components.
For a pre-stable map to $X$, we divide the irreducible components of the domain into several types.

**Type I** The pre-stable map to $X$ restricted to this component is finite but not generically étale.

**Type II** The pre-stable map to $X$ restricted to this component is generically étale of degree larger than 1.

**Type III** The pre-stable map to $X$ restricted to this component is birational.

**Type IV** The pre-stable map to $X$ restricted to this component is constant, and this component is not isomorphic to $\mathbb{P}^1$.

**Type V** The pre-stable map to $X$ restricted to this component is constant, and this component is isomorphic to $\mathbb{P}^1$. There are three subtypes depending on the number of nodes on this $\mathbb{P}^1$. Type V.1, there is only one node. Type V.2, there are only two nodes. Type V.3, there are at least 3 nodes.

Note that by our construction, for the lifting $\tilde{f}, \tilde{g}$, the strict transform of the image curve is smooth in $\tilde{X}$ in the first three cases. Thus in the type III case, this irreducible component has to be smooth and the stable map restricted to this component is an isomorphism to its image in $\tilde{X}$. Moreover, in the first three cases, the stable map restricted to the irreducible component factors through a smooth curve (i.e. its image in $\tilde{X}$).

For each irreducible component of $\tilde{f}$ (resp. $\tilde{g}$) of type I, II, and IV, we construct a fibered surface in the following way. Let us denote this irreducible component by $C$ and in the case of type I, II components, its image in $V_n$ by $D$.

**Type I**: There is a finite but non-generically-étale morphism $C \to D$. Lemma 2.11 gives a fibration $W \to \mathbb{P}^1, F : W \to D$ such that

1. $W_0$ is a connected nodal curve, which is the union of a copy of $C$ and a smooth irreducible projective curve $C'$.
2. $W_\infty$ is a smooth irreducible curve.
3. $F|_{C'} : C' \to D$ and $F|_{W_\infty} : W_\infty \to D$ are generically étale.

We can construct a fibration over $\Sigma_C \to \mathbb{P}^1$. Moreover, we may choose the fibration in such a way that if the irreducible component $C$ is over $f(S_0)$ (resp. $g(S_0)$), the fiber of $\Sigma_C$ over $\infty$ (resp. 0) is $C + C'$ and the fiber of $\Sigma_C$ over 0 (resp. $\infty$) is $C''$.

Via the inclusion of $D$ in $X$, we have a family of stable maps to $\tilde{X}$. Other than $C$, we only introduce new type II irreducible components.

**Type II**: We apply Lemma 2.10. This gives a family $W \to \mathbb{P}^1, F : W \to D$, which gives a deformation from $C$ with a number of $\mathbb{P}^1$s to a union of copies of $D$ with a number of $\mathbb{P}^1$s. Moreover, we may choose the fibration in such a way that if the irreducible component $C$ is over $f(S_0)$ (resp. $g(S_0)$), the fiber of $W \times_{\mathbb{P}^1} B$ over $\infty$ (resp. 0) is $C$ union a bunch of rational curves that are maps to a point in $D$. Compose this family with $D \to V$ gives a family of stable maps.

Similarly, via the inclusion of $D$ in $X$, we have a family of stable maps to $\tilde{X}$. We only introduce new type III and V irreducible components.

**Type IV**: The map $C \to V$ factors through a constant map to a point $x$. So there is a deformation from the curve $C$ to a stable curve whose irreducible components are all isomorphic to $\mathbb{P}^1$. We take the fibred surface given by this family over a base curve $B$ with two marked points $b_1, b_2$. Moreover, we may choose the fibration in such a way that if the original irreducible component $C$ is over $f(S_0)$ (resp. $g(S_0)$),
the new constructed surface has fiber $C$ over $b_2$ (resp. $b_1$). This new family of curves gives a constant pre-stable map to $X$ by mapping every thing to $x$.

Via the inclusion of the point in $X$, we have a family of constant stable maps to $X$. Other than $C$, we only introduce new type V irreducible components.

Up to a further generically finite projective base change, we may assume that $S_1$ admits morphisms to $\mathbb{P}^1$ and $B$ that map the inverse image of $f(S_0)$ (resp. $g(S_0)$) to 0 and $b_1$ (resp. $\infty$ and $b_2$).

We construct new families of cycles coming from stable maps in the following steps.

We say an irreducible component is in the $f^+$ or $f^-$ (resp. $g^+$ or $g^-$) side, if it appears as an irreducible component of the stable map in $\tilde{f}^+$ or $\tilde{f}^-$ (resp. $\tilde{g}^+$ or $\tilde{g}^-$).

**Step 1.** For each irreducible component $C$ (with image $D$ in $\tilde{X}$) of type I, II, IV that is in the $f^+$ or $g^+$ side, we use the above construction to add a new family of stable maps over $\tilde{S}_1$. When we apply these constructions, we first deal with type I components, which only introduces new type II irreducible components. Then we deal with all the type II components, both old and new, which only introduces new type III and V components. Finally we deal with type IV components, which also introduces new type V irreducible components.

To make the cycle class remain the same as $\gamma$, we subtract families of stable maps, that is, we add families of stable maps over $\tilde{S}_1$ on the $-$ side, which are suitable number of copies of constant families given by the inclusion $D \subset \tilde{X}$.

Here we adopt the convention that if a family of stable maps comes with multiplicity $n$, we take $n$ disjoint unions of the family.

With this construction, we achieve that all the irreducible components of type I, II, IV in the $f^+$ side also appear in $g^+$ side, and vice versa.

**Step 2.** We perform a similar construction as in Step 1, applied to the irreducible components in the $f^-, g^-$ side, to achieve that all the irreducible components of type I, II, IV in the $f^-$ also appear in $g^-$ side, and vice versa.

**Step 3.** Denote by $\Gamma^+_f, \Gamma^-_f$ (resp. $\Gamma^+_g, \Gamma^-_g$) the positive part and negative part of the cycles over $S_0$ obtained by pulling back via $\tilde{f}$ (resp. $\tilde{g}$), and $\Gamma^+_0, \Gamma^-_0$ the positive part and negative part of the cycles over $S_0$.

Since we have added and subtracted the same cycles in the cycle over $\tilde{f}(S_0)$ and $\tilde{g}(S_0)$ in Step 1 and Step 2, and since we assume that the corresponding cycle over $f(S_0)$ and $g(S_0)$ are the same, we know that there are effective cycles $\Delta_f, \Delta_g$ such that

$$\Gamma^+_f = \Gamma^+_0 + \Delta_f, \Gamma^-_f = \Gamma^-_0 + \Delta_f,$$

$$\Gamma^+_g = \Gamma^+_0 + \Delta_g, \Gamma^-_g = \Gamma^-_0 + \Delta_g.$$

For an irreducible component in $\Gamma^{+/−}_0$ with multiplicity $m$, we may assume that we have $m$ irreducible components of type III (i.e. birational onto image) in the family of stable maps. If this is not the case, we simply add more families of constant stable maps over $\tilde{S}_1$ from the normalization of this irreducible component to $\tilde{X}$ and $X$ to both the $+$ and the $−$ side.

We call these $m$-tuple type III components $\Gamma$ curves, and all the other components $\Delta$ curves. The sum of the cycle class of these $\Gamma$ curves gives $\Gamma^+_0$ and $\Gamma^-_0$. 
Step 4. Recall that after Step 2, any component $C$ of type I, II, and IV that appears in the $f+$ side (resp. $f-$ side) also appears in the $g+$ side (resp. $g-$ side). So it makes sense to say a component appears in the $+/−$ side. For any such component $C$ in the $+$ side that is a $\Delta$ curve, we add a constant family of pre-stable maps given by this component in the $−$ side, and if this component is not of type IV or V, to make the cycle class remain the same, we also add a suitable number of constant families of pre-stable maps in the $+$ side that are families of normalizations of the image $D$. We perform the same operation for components in the $−$ side.

Step 5. In this step, we use suitable blow-ups to adjust the number of components of type V for $f+, f-, g+, g-$ sides.

Up to a generically finite projective base change over $S_1$, we may assume that we have families of stable maps with many marked points on every irreducible component. We may add a V-type component by blowing up the locus of the marked points over the irreducible component of the inverse image of $f(S_0)$ (resp. $g(S_0)$) containing $\tilde{f}(S_0)$, and produce a family of pre-stable maps to $X$ so that type V components increase by 1 in one of the $f+, f-, g+, g-$ sides.

We note that at this point, we have a one-to-one correspondence between irreducible components of fibers of $f\mapsto \Gamma^0$ and that there are families over $\tilde{f}(S_0)$ and $\tilde{g}(S_0)$ of these two families of $m$-free curves are the same stable maps.

We choose a one-to-one correspondence between the type III components in the $f-$ and $g-$ side. The same argument shows that we also have the same number of irreducible components of type III in the $f+$ and $g+$ side.

Finally, for the same reason, we have a one-to-one correspondence between the $\Delta$ curves in the $+-$ and $−-$ side.

Step 6. Up to replacing $X$ with $X \times \mathbb{P}^N$, we may find two families of $m$-free curves over $\tilde{S}_0$ that connects the every irreducible component in the $+$ side and $−$ side. We may even assume that the fibers over $\tilde{f}(S_0)$ and $\tilde{g}(S_0)$ of these two families of $m$-free curves are the same stable maps.

These families of $m$-free curves are called handles in the following.

To summarize, we have constructed two families of stable maps with connected domains $f^+:C^+\to X, f^-:C^-\to X$ over $S_1$ such that

1. There is a one to one correspondence between the irreducible components of fibers of $C^+$ (resp. $C^-$) over $\tilde{f}(S_0), \tilde{g}(S_0)$.
2. The restriction of the morphism $F^+$ (resp. $F^-$) to each pair of corresponding irreducible components of fibers of $C^+$ (resp. $C^-$) over $\tilde{f}(S_0), \tilde{g}(S_0)$ gives the same morphism.
3. There is a one-to-one correspondence between the $\Delta$ curves in the $f^+, f^-, g^+, g^−$ side. The restriction of the morphism $F^+$ (resp. $F^-$) to the corresponding irreducible components gives the same morphism.
4. The cycle classes of $F^+(C^+|_{f(S_0)}) − F^-(C^-|_{f(S_0)})$ and $F^+(C^+|_\tilde{g}(S_0)) − F^-(C^-|_\tilde{g}(S_0))$ are the same as $\Gamma_0$.

Step 7. Up to making a further base change, we may assume that we can insert a $\mathbb{P}^1$ to every nodes of the fiber over $\tilde{f}(S_0), \tilde{g}(S_0)$, and that there are families of marked points over $\tilde{S}_1$ such that there is a marked point in every inserted $\mathbb{P}^1$ component over $\tilde{f}(S_0), \tilde{g}(S_0)$. 

Step 8. Up to a generically finite projective base change over $S_1$, we may assume that we can...
Let us still denote the two families of stable maps by $F^+: C^+ \to X, F^- : C^- \to X$.

We call such $\mathbb{P}^1$'s ghost components.

Note that by construction, for every irreducible component, there is at least one ghost component connecting it to the handle.

We claim that the number of ghost components is the same in the $f^+$ side and $g^+$ side (resp. $f^-$ side and $g^-$ side). So in particular, it makes sense to talk about the number of ghost components in the $+$ side. To prove the claim, we look at the dual graph of the curve before we add the ghost components, and consider the genus of the curve and betti numbers of the dual graph. For the genus, we have

$$g = \sum_{\text{vertex } V} g_V + b_1,$$

where $b_1$ is the first betti number of the graph. For the Euler characteristic of the dual graph $G$,

$$\chi(G) = 1 - b_1 = v - e,$$

where $v$ (resp. $e$) is the number of vertices (resp. edges).

Since there is a one-to-one correspondence between the components, and since the genus of the whole curve is constant in the family, we know that the number of edges, i.e. the number of nodes, is the same for $f^+$ and $g^+$ side (resp. $f^-$ and $g^-$ side).

**Step 8.** We add families of $m$-free $n$-connecting curves (with $m, n$ large) to the curves in the $+/−$-sides to kill the obstructions. This requires some explanation.

First of all, up to making a base change, we may assume that we have two families of stable maps (in the $+/−-side$) that have sufficiently many marked points. The number of marked points is determined by the following condition. For each irreducible component over $\tilde{f}(S^0), \tilde{g}(S^0)$, there is a number $N$ such that if we add $N$ 2-free curves at general points along general directions, the resulting comb is $M$-free for $M$ larger than the number of nodes it has. We may choose this number uniform for all irreducible components over $\tilde{f}(S^0), \tilde{g}(S^0)$. We assume that there are sufficiently many marked points for this family and that for each irreducible component over $\tilde{f}(S^0), \tilde{g}(S^0)$, there are exactly $N$ marked points in general position. Since the number of components of type I to V are in one-to-one correspondence, the total number of marked points adds up the same for the fibers over $\tilde{f}(S^0), \tilde{g}(S^0)$. Note that we single out the ghost components and do not consider them as type V components. Then we add $m$-free $n$-connecting curves at these marked points. This is possible if we make further not necessarily surjective base changes. We may not have surjectivity since we do not make an effort to make the curves added for every fiber over $\tilde{S}_1$ to be $m$-free $n$-connecting. Rather, we only care about the fiber over $\tilde{f}(S^0), \tilde{g}(S^0)$ and nearby fibers. In the following steps, we implicitly only work with neighborhoods of $\tilde{f}(S^0), \tilde{g}(S^0)$. We will make the morphism $\tilde{S}_1 \to \tilde{S}_0$ projective at the end of the proof.

There are lots of freedom in choosing the curves to add. We assume that the curves added to irreducible components over $\tilde{f}(S^0), \tilde{g}(S^0)$ are along general directions and that for each corresponding irreducible component, we add the same set of $m$-free $n$-connecting curves. This is possible since the stable maps restricted to the corresponding components are the same.
Now the stable maps over \( \tilde{f}(S_0), \tilde{g}(S_0) \) are unobstructed. In fact, they consist of combs that are \( M \)-free for \( M \) larger than the number of ghost components connected to it.

After adding these curves, the stable maps in the \( f^+ \) side and \( g^+ \) side consist of the same set of stable maps to \( X \) plus the set of ghost components connecting these stable maps.

For each family of curves added in the \( + \)-side, we add the corresponding family of curves in the \( - \)-side. These newly added families are disjoint from the family \( C^{+/-} \). The cycle class remains the same.

Finally we assume that there are families of marked points whose fiber over \( \tilde{f}(S_0), \tilde{g}(S_0) \) is a marked point in one of the ghost components. For every such family, we construct a family of bouquets and glue these families of bouquets to \( C^{+/-} \) along these families of marked points.

**Step 9.** In this step, we construct a one-to-one correspondence between the ghost components in the \( + \)-side (resp. \( - \)-side).

There are deformations from the stable map in the \( f^+ \) side to the \( g^+ \) side, which only deform the ghost components while keeping all the other components fixed. This is done in the following way. We first take a deformation of the union of ghost components and the bouquets attached to them, which turns this union into a comb with \( m \)-free \( n \)-connecting curves as teeth. This can be done with the two connecting nodes in the ghost components fixed. Then we take general deformations of these combs with again the connecting nodes fixed. These general deformations are 2-free and 2-connecting by Corollary 2.7. For simplicity, we still call these general deformations the ghost components.

So we can apply the sliding lemma 2.9 to move the ghost components to a standard position: one ghost component connecting the handle to a single irreducible component and other ghost components connecting the handle at two points. Since all the ghost components lie in the same 2-connecting family, Lemma 2.8 implies that we can get a deformation from the stable map in the \( f^+ \) side to the \( g^+ \) side while keeping all the other components fixed.

There are many ways to perform this deformation. We make one choice.

Since ghost components deforms to ghost components during this deformation, our choice of the deformation gives a one-to-one correspondence between the ghost components in the \( + \)-side.

Similarly, we choose a deformation of the stable maps in the \( - \)-side by deforming the ghost components. This also gives a one-to-one correspondence between the ghost components in the \( - \)-side.

**Step 10.** We introduce free components in this step.

For each family of marked points in the \( + \)-side (resp. \( - \)-side) whose fiber over \( \tilde{f}(S_0), \tilde{g}(S_0) \) lie in the ghost component, we have a family of bouquets. This family of bouquets naturally comes with a family of marked points in the distinguished \( \mathbb{P}^1 \) component. Choose a smooth divisor \( H \) in \( S_0 \) containing \( f(S_0), g(S_0) \). Given a family of bouquets, we blow-up the total space of the family along the family of marked points over \( H \). We then add this new family of pre-stable maps to the \( - \) (resp. \( + \)) side. Note that the fiber of this family of pre-stable maps over \( \tilde{f}(S_0), \tilde{g}(S_0) \) is the same as the union of the ghost component over \( \tilde{f}(S_0), \tilde{g}(S_0) \) and the bouquet attached to it.
In particular, after adding these, the cycle class over \( \tilde{f}(S_0) \), \( \tilde{g}(S_0) \) remains to be \( \gamma \).

We call these newly added components the free components.

**Step 12.** We are now ready to construct the deformation of cycles required by the theorem.

In step 9, we already see that we can deform the \( f^+ \) (resp. \( f^- \)) side to \( g^+ \) (resp. \( g^- \)) side by only deforming the ghost components.

Now the desired deformation is the following. While we deform the ghost components on the + side, we can deform the corresponding free components in the − side the same way. As a result, the resulting cycle remain the same.

Similarly we take further deformations of the ghost components in the − side and deform the free components in the + side following these deformations. This gives the required deformation of constant family of cycles connecting \( \tilde{f}(S_0) \) and \( \tilde{g}(S_0) \).

Now we construct the required deformation for inverse images of a general point in \( S_0 \). First note that when pulling back families of stable maps from \( \mathbb{P}^1 \) and the curve \( B \), we may assume that the morphism descends to a rational map over a Zariski open subset of \( S_1 \) containing this general point. In other words, we add the same stable maps for all the inverse images. Then observe that in the next steps we just add families of \( m\)-free \( n\)-connecting curves to the same stable maps. Thus it suffices to deform the added \( m\)-free \( n\)-connecting curves, which is possible by Lemma 2.8. Then on the opposite side, we may deform the corresponding \( m\)-free \( n\)-connecting curve in the same way so that the cycle class remain the same during the deformation.

Now we have constructed the \( \tilde{S}^0_1 \) (which will be show to be an open subset of the desired \( S_1 \) and the curves \( T_1, \ldots, C_1, \ldots \)). We note that we may choose the deformations over irreducible curves \( T_1, \ldots, C_1, \ldots \) such that every stable map parameterized by the curve is unobstructed. We do not assume the curves are projective. The important thing is that the stable map at the nodes are unobstructed so that there is a unique irreducible component containing all the curves. Choose an embedding of these families of stable maps in to \( X \times \mathbb{P}^n \). So we may think of these stable maps as induced by an embedding of \( \tilde{S}_1 \cup T_1 \ldots \cup C_1 \ldots \) in the Hilbert scheme of \( X \times \mathbb{P}^n \). Furthermore, the Hilbert scheme is smooth near \( \tilde{S}_1 \cup T_1 \ldots \cup C_1 \ldots \). In particular, the images lie in a unique irreducible component. So we may take \( S_2 \) to be the normalization of the reduced closed subscheme of the irreducible component containing \( \tilde{S}_1 \cup T_1 \ldots \cup C_1 \ldots \). Now we take \( \tilde{S}_1 \) to be the normalization of the closure of the image of \( \tilde{S}^0_1 \) in \( S_1 \times S_2 \).

This family of subschemes in \( X \times \mathbb{P}^n \) gives a family of cycles over \( \tilde{S}_2 \) in \( X \times \mathbb{P}^n \). Then projection to \( X \) gives a family of cyles in \( X \) over \( \tilde{S}_2 \).

In characteristic 0, we may take a resolution of singularities of \( \tilde{S}_2 \) that is an isomorphism near the images of \( \tilde{S}_1 \cup T_1 \ldots \cup C_1 \ldots \). □

The proof also shows the following.

**Corollary 3.14.** Let \( X \) be a smooth projective separably rationally connected variety or a separably rationally connected fibration over a curve. Let \( f : \Gamma \to X, f' : \Gamma' \to X \) be two stable maps to \( X \) such that the push-forward of the fundamental class of \( \Gamma \) and \( \Gamma' \) are the same as cycles. Denote this cycle by \( \gamma \). Fix a family of \( m\)-free \( n\)-connecting curves for \( m, n \geq 2 \). Then there are two families of stable
maps \( F_i : (B_i, x_i, y_i) \mapsto C_i^{+/-} \to X, i = 0, \ldots, m \) over smooth curves \( B_i \) with two points \( x_i, y_i \) such that for each point \( b \in \cup B_i \), the cycle \( C_i^{+/-}_b \) is \( \gamma \), and that the fiber over \( x_0 \) of the family \( F_0 : C_0^+ \to X \) (resp. the fiber over \( y_m \) of the family \( C_m^+ \) contains \( f \) (resp. \( g \)) as a sub-stable map, and that \( F_i|_{y_i} = F_{i+1}|_{x_i} \) as stable maps for \( i = 0, \ldots, m - 1 \). Moreover, the other irreducible components of the stable map over \( x_0 \) (resp. \( y_m \)) are of the following types:

1. A contracted component isomorphic to \( \mathbb{P}^1 \).
2. A member of the family of \( m \)-free \( n \)-connecting curves.
3. A multiple cover of an irreducible component of the cycle \( \gamma \).

Now we are ready to prove Theorem 3.5.

**Proof of Theorem 3.5.** We prove the first statement for the equidimensional case. The semistable case is the same. One simply replace the word cycle with semistable families. In fact, the key argument in the proof is to work with stable maps.

Choose general point \( s_1 \in S_1, s_2 \in S_2 \) and denote the cycles over \( s_1 \) (resp. \( s_2 \)) by \( \Gamma_{s_1} \) (resp. \( \Gamma_{s_2} \)). We write \( \Gamma_{s_1} - \Gamma_{s_2} = \Gamma^+ - \Gamma^- \), where \( \Gamma^+/\Gamma^- \) are the the positive parts and negative parts.

By Proposition 3.10, there is a family of cycles \( \Gamma \subset S_3 \times X \) over a smooth projective curve \( S_3 \) such that \( \Gamma^+ \) and \( \Gamma^- \) appear as cycles in some fiber \( s_+, s_- \). We simply take \( S = S_1 \times S_2 \times S_3 \), with a family of cycles \( p_1^* \Gamma_1 + p_2^* \Gamma_2 + p_3^* (\Gamma_3) - S \times \Gamma^- - S \times \Gamma_{s_1} \), and inclusions \( S_1 \to (S_1, s_+, s_-), S_2 \to (s_1, S_2, s_-) \). We use the notation \( S \times \Gamma_{s_1} \) (resp. \( S \times \Gamma^- \)) to mean the constant family of cycles \( \Gamma_{s_1} \) (resp. \( \Gamma^- \)) over \( S \). Since \( S_3 \) is a smooth projective curve, if both \( S_1, S_2 \) are smooth or projective, so is \( S \).

For the second statement, First of all, if \((S_1, \Gamma_1)\) is a family of equidimensional family of cycles, up to taking some purely inseparable base change and normalization, we may assume that the generic fiber of each family of cycles over \( S_1 \) is smooth. In particular, we already have a family of stable maps over a Zariski open subset.

Then by the stable reduction theorem, we may perform a generically finite, generically separable projective base change to produce a family of stable maps. Thus Proposition 3.11 proves the general case.

\[ \square \]

4. **Complex varieties: Lawson homology**

Let \( X \) be a complex projective variety and we fix a very ample line bundle \( \mathcal{O}(1) \).

All the degree’s are taken with respect to this line bundle. Let \( \text{Chow}_{r,d}(X) \) be the Chow variety parameterizing degree \( d \), \( r \)-dimensional cycles of \( X \) and \( \text{Chow}(X) = \cup_d \text{Chow}_{r,d}(X) \). We give the set \( \text{Chow}(X)(\mathbb{C}) \) the structure of a topological monoid, where the topological structure comes from the analytic topology on \( \text{Chow}(X)(\mathbb{C}) \) and the monoid structure is the sum of cycles. Define \( \text{Z}_r(X) \) to be the group completion of \( \text{Chow}(X)(\mathbb{C}) \). It has a topological group structure. The topology can be defined in several equivalent ways. These are studied by Lima-Filho [LF94].

**Definition 4.1.** Define the topological space \( \text{Z}_r(X)^{eq} \) as the colimit of all the topological spaces \( S(\mathbb{C}) \) over the category \( I^{eq} \) (Definition 3.3).
Lemma 4.2. In the definition of \( Z_r(X)^{eq} \), we may take a subset consisting of family of equidimensional cycles over normal projective varieties (or smooth projective varieties).

Proof. Given any family of equidimensional cycles \( \Gamma \to S \), we may find a normal projective variety (resp. smooth projective variety) \( T \), a family \( \Gamma_T \to T \), and an open subset \( T^0 \) of \( T \) such that there is a surjective proper map \( p : T^0 \to S \) and \( \Gamma_T|_{T^0} = \Gamma \times_S T^0 \).

Note that we have a factorization \( T^0(\mathbb{C}) \to S(\mathbb{C}) \to Z_r(X) \). A set in \( S(\mathbb{C}) \) is closed if and only if its inverse image under \( p^{-1} \) in \( T^0(\mathbb{C}) \) is closed.

On the other hand, if we have an equidimensional family over a normal variety \( S \), we may pull back this family to a resolution \( S' \). The topology of \( S(\mathbb{C}) \) is the quotient topology coming from \( S'(\mathbb{C}) \to S(\mathbb{C}) \).

Thus the topology on \( Z_r(X)^{eq} \) is determined by families over normal varieties (resp. smooth varieties) such that the family has an extension over a normal (resp. smooth) projective compactification.

Therefore, when defining \( Z_r(X)^{eq} \) as a colimit, we may take only normal (resp. smooth) projective varieties. \( \square \)

Definition 4.3. Define the topological space \( Z_r(X)^{Chow} \) as the quotient of \( Ch_r(X)(\mathbb{C}) \times Ch_r(X)(\mathbb{C}) \) by \( Ch_r(X)(\mathbb{C}) \), where the action is \( (a,b) \mapsto (a+c,b+c) \) for \( c \in C_r(X)(\mathbb{C}) \).

Theorem 4.4 ([LF94], Theorem 3.1, Theorem 5.2, Corollary 5.4). The identity map induces homeomorphisms

\[ Z_r(X)^{eq} \cong Z_r(X)^{Chow}. \]

Recall the definition of Lawson homology.

Definition 4.5. Let \( X \) be a complex projective variety. Define the Lawson homology \( L_nH_{n+2r}(X) \) as the homotopy group \( \pi_n(Z_r(X)) \).

Example 4.6 (Dold-Thom isomorphism). Consider \( Z_0(X) \), the group of zero cycles on \( X \). The classical Dold-Thom theorem implies that there is an isomorphism

\[ L_0H_n(X) \cong H_n(X, \mathbb{Z}). \]

Example 4.7 (Hurewitz map). The Hurewitz map is induced by the inclusion \( X \to Z_0(X) \):

\[ \pi_k(X) \to \pi_k(Z_0(X)) \cong H_k(X, \mathbb{Z}). \]

Theorem 4.8. Let \( X \) be a complex smooth projective variety. Assume that either \( X \) is rationally connected or \( X \) is a rationally connected fibration over a curve. Then for any loop \( L \) in \( Z_1(X) \), there is a smooth projective variety \( Y \) with a family of 1 dimensional cycles in \( X \) over \( Y \) such that the map

\[ \Phi : L_0H_1(Y) = \pi_1(Z_0(Y)) \to L_1H_3(X) = \pi_1(Z_1(X)) \]

induced by the family of one cycles contains the class \([L]\) in \( L_1H_3(X) \).

We first introduce some notations. Given a normal projective variety \( S \) parameterizing a family of 1 dimensional cycles of \( X \), there is an induced continuous map between topological groups:

\[ Z_0(S) \to Z_1(X). \]
We denote by $I(S)$ the image of this map, i.e. the closed subgroup of $Z_1(X)$ generated by the cycles over $S$, and $K(S)$ the kernel of this map.

The first observation in the proof of Theorem 4.8 is the following.

**Lemma 4.9.** Let $X$ be a smooth projective variety defined over an algebraically closed field $k$ of characteristic $p$. Assume that $X$ is either a separably rationally connected variety or a separably rationally connected fibration over a curve. Fix an integer $N$ that is invertible in $k$. For any class $[L]$ in $L_1H_3(X) = \pi_1(Z_1(X))$, there is a normal projective variety $U$ and a family of equidimensional one cycles $\gamma_U$ over $U$ such that $[L]$ is represented by a continuous map

$$I = [0,1] \to U \to Z_1(X).$$

Note that $I \to U$ is not a loop in general.

**Proof.** By the first part of Theorem 3.5 and Lemma 4.2, the topological group $Z_1(X)$ is a filtered colimit over closed subgroups generated by one-dimensional cycles parameterized by smooth/normal projective varieties.

Homotopy groups commutes with filtered colimits. Thus there is an irreducible smooth projective variety $S$ with a family of one dimensional cycles over $S$ such that the induced map

$$\pi_1(I(S)) \to \pi_1(Z_1(X)) \cong L_1H_3(X)$$

contains the class $[L]$ in $\pi_1(Z_1(X))$.

The fibration $K(S) \to Z_0(S) \to I(S)$ gives a long exact sequence of homotopy groups:

$$\ldots \to \pi_1(Z_0(S)) \to \pi_1(I(S)) \to \pi_0(K(S)) \to \ldots .$$

A loop in $I(S)$ lifts to a continuous map from the unit interval $I = [0,1]$ to $Z_0(S)$, such that 0, 1 maps to two points in $Z_0(S)$ that parameterize the same cycle in $X$.

We may assume that the family over $I$ is the difference of two families of effective 0-cycles of degree $d$ in $S$. That is, it corresponds to the difference of two continuous maps $f^+ : I \to S^{(d^+)}$, $f^- : I \to S^{(d^-)}$, which is the same as a continuous map $f : I \to S^{(d^+) \times S^{(d^-)}}$ with 0 mapping to a point $x = (x^+,x^-)$ and 1 mapping to a point $y = (y^+,y^-)$.

A family of one cycles over $S$ induces a family of cycles over $S^{(d^+)}$ and $S^{(d^-)}$. Let us denote them by $\Gamma_{d^+}, \Gamma_{d^-}$.

The loop is the composition $I \to S^{(d^+) \times S^{(d^-)}} \to Z_0(S) \to Z_1(X)$, where the middle map is taking the difference.

Let us use a different family of cycles $\pi^+_d \Gamma_{d^+} - \pi^-_d \Gamma_{d^-}$ on the product $S^{d^+} \times S^{d^-}$, where $\pi^+_d$ is the projection to $S^{d^+/d^-}$. This family of cycles induces a continuous map $S^{d^+} \times S^{d^-} \to Z_1(X)$ such that the composition $I \to S^{d^+} \times S^{d^-} \to Z_1(X)$ is the loop $L$.

We take $U$ to be $S^{d^+} \times S^{d^-}$ and $\gamma_U$ to be $\pi^+_d \Gamma_{d^+} - \pi^-_d \Gamma_{d^-}$. □

**Proof of Theorem 4.8.** By Lemma 4.9, there is a normal projective variety $U$ and a family of equidimensional one cycles $\gamma_U$ over $U$ such that $[L]$ is represented by a continuous map

$$f : I = [0,1] \to U \to Z_1(X).$$

Denote by $x, y \in U$ the image of 0, 1 by $f$. The cycle over $x, y$ are the same by assumption. Now we are in the set-up of the second part of Theorem 3.5. Thus there is a smooth projective variety $T$ with a generically finite surjective morphism
generated by cycles of the form $aT \to 24T$. 

Note that $I$ projective variety $Y$ all these families come from pulling back from a family of cycles over a smooth projective variety $Y$.

The morphism $T \to U$ induces a continuous map between topological groups $Z_0(T) \to Z_0(U)$. Denote by $K$ the kernel topological group. As a group, $K$ is generated by cycles of the form $a - b$, where $a, b$ are points in a fiber of $T \to U$. Note that $I(T) = I(U)$. Thus we have a fibration sequence of topological groups:

$$0 \to K \to K(T) \to K(U) \to 0.$$ 

We have commutative diagrams:

$$
\begin{array}{ccc}
\pi_1(Z_0(Y)) & \longrightarrow & \pi_1(I(Y)) \\
\uparrow & & \uparrow \\
\pi_1(Z_0(T)) & \longrightarrow & \pi_1(I(T)) \\
\downarrow & & \downarrow \\
\pi_1(Z_0(U)) & \longrightarrow & \pi_1(I(U)) \\
\end{array}
$$

The obstruction of lifting the class $[L]$ in $\pi_1(I(T))$ is in $\pi_0(K(T))$ and maps to $[x - y]$ in $\pi_1(K(U))$. The class $[x_T - y_T]$ differs from the obstruction class by an element in $\pi_0(K)$.

We take the Stein factorization $T \to T' \to U$, where $T \to T'$ has connected fibers (hence birational) and $T' \to U$ is finite. Therefore $\pi_0(K)$ is finitely generated by classes of the form $[a - b]$, where $a, b$ are points in the fiber over a general point in $U$.

The class $[L]$ maps to $\pi_1(I(Y))$, with obstruction class the push-forward of $[x_T - y_T]$ modulo classes in $\pi_0(K)$.

By the existence of the families of constant cycles in the second part of Theorem 3.5 we have

1. The composition $\pi_0(K) \to \pi_0(K(T)) \to \pi_0(K(Y))$ is the zero map.

2. The push-forward of the class $[x_T - y_T]$ vanishes in $\pi_0(K(Y))$.

Thus the class of the loop $L$ in $\pi_1(I(Y))$ is contained in $\pi_1(Z_0(Y))$. \hfill \Box

Now we introduce another ingredient.

**Lemma 4.10.** [FM94, Page 709, 1.2.1] There is a continuous map, the s-map: $Z_r(X) \wedge \mathbb{P}^1 \to Z_{r-1}(X)$ inducing the s-map on Lawson homologies $s : L_{r}H_k(X) \to L_{r-1}H_k(X)$.

**Remark 4.11.** The construction of the s-map depends on a deep result: Lawson’s algebraic suspension theorem. A geometric way of describing the s-map is the following. Given a cycle $\Gamma$, take a general pencil of divisors $D_t(t \in \mathbb{P}^1)$ that intersect $\Gamma$ properly, and the s-map sends $\Gamma$ to the cycle $\Gamma \cdot D_\infty - \Gamma \cdot D_0$.

**Definition 4.12.** Let $Y$ be a semi-normal variety. Let $Z \subset Y \times X$ be a family of $r$-dimensional cycle over $Y$ corresponding to a morphism $f : Y \to Z_r(X)$. We
define the correspondence homomorphism
\[ \Phi_f : H_k(Y, Z) \to H_{k+2r}(X, Z) \]
as the composition
\[
H_k(Y, Z) \cong \pi_k(Z_0(Y)) \to \pi_k(Z_r(X)) \xrightarrow{s^k} \pi_{k+2r}(Z_0(X)) \cong H_{k+2r}(X, Z),
\]
where the map \( \pi_k(Z_r(X)) \to \pi_{k+2r}(Z_0(X)) \) is induced by \( k \)-th iterations of the \( s \)-map.

**Theorem 4.13** ([FM94] Theorem 3.4). Let \( Y \) be a smooth projective variety and \( \Gamma \subset Y \times X \) be a family of \( r \)-dimensional cycle over \( Y \) corresponding to a morphism \( f : Y \to Z_r(X) \). We have
\[
\Phi_f = \Gamma_* : H_k(Y, Z) \to H_{k+2r}(X, Z),
\]
where \( \Gamma_* \) is the map defined using \( Z \) as a correspondence.

With this result, we can prove the main results over complex numbers.

**Theorem 4.14.** Let \( X \) be a smooth projective rationally connected variety or a rationally connected fibration over a curve. For each \( k \geq 1 \), there is a smooth projective curve \( C \) with a family of \( 1 \)-dimensional cycles \( \Gamma \subset C \times X \) induced by \( f : C \to Z_1(X) \) such that
\[
\Phi_f = \Gamma_* : H_1(C, Z) \to H_3(X, Z)
\]
has the same image as the \( s \)-map \( s : L_1H_3(X) \to H_3(X) \), which is the same as the coniveau filtration \( N_1H_3(X, Z) \).

**Proof.** Recall that there is an isomorphism \( L_0H_k(S) \cong H_k(S) \) for any projective variety \( S \) by the Dold-Thom theorem. We have a commutative diagram:
\[
\begin{array}{ccc}
L_0H_1(Y) & \xrightarrow{\Gamma_*} & L_1H_3(X) \\
\cong \downarrow & & \downarrow s \\
H_1(Y) & \xrightarrow{\Gamma_*} & H_3(X)
\end{array}
\]
Therefore the image of \( \Phi_f = \Gamma_* \) is contained in the image of the \( s \)-map for any smooth projective variety \( Y \).

The image of the \( s \)-map is finitely generated. Theorem 4.8, combined with the first part of theorem 3.5, shows that there is smooth projective variety \( Y \) parameterizing a family of one-dimensional cycles in \( X \), whose corresponding correspondence homomorphism \( \Phi_f \) contains the image of the \( s \)-map.

By taking general hyperplane sections, we may find a smooth projective curve \( C \subset Y \) such that \( \pi_1(C) \to \pi_1(Y) \) is surjective. Then we simply restrict the family of cycles to \( C \).

Finally, we note that the image of the \( s \)-map
\[
s : L_1H_3(X) \to H_3(X)
\]
is \( N^1H_3(X, Z) \) by [Wal07, Proposition 2.8].

The immediate consequence is the following.
Theorem 4.15. Let $X$ be a smooth projective rationally connected variety or a rationally connected fibration over a curve. Assume $X$ is a 3-fold. Then all the filtrations on $H^3(X, \mathbb{Z})$ introduced in Definition 1.3 equal the whole cohomology group:

$$\tilde{N}_{1,\text{cyl,eq}}H^3(X, \mathbb{Z}) = \tilde{N}_{1,\text{cyl}}H^3(X, \mathbb{Z}) = N^1H^3(X, \mathbb{Z}) = H^3(X, \mathbb{Z}).$$

Proof. By the decomposition of the diagonal argument, $L_1H_k(X) \otimes \mathbb{Q} \cong H_k(X, \mathbb{Q}) \cong H^k(X, \mathbb{Q})$. Thus we know that $s : L_1H_3(X) \to H_3(X, \mathbb{Z}) \cong H^3(X, \mathbb{Z})$ is surjective by [Vois08 Corollary 3.1]. □

5. General case: Chow sheaves

In this section we discuss the general case of a smooth projective separably rationally connected variety or a smooth projective separably rationally connected fibration over a curve defined over an algebraically closed field $k$.

Sometimes we may “invert $p$”, by taking the tensor product of a sheaf with $\mathbb{Z}[\frac{1}{p}]$. In this scenario, we understand that $p$ is 1 if the base field has characteristic 0 and equal the characteristic otherwise. We use $\mathbb{Z}_\ell$ coefficient étale cohomology for $\ell$ a prime number, non-zero in the field $k$.

One can also define Lawson homology with $\mathbb{Z}_\ell$ coefficients in this context [Fri91]. But the construction of the analogue of $\mathbb{Z}_\ell(X)$ is more complicated. Also Lawson homology in this context is much less studied. Many of the known results for complex varieties have not been explicitly stated to hold, even though one can imagine that they are still true. For example, the author do not know a reference for the construction of the s-map, Neither could the author find the analogue of Friedlander-Mazur’s result (Theorem 4.13) explicitly stated. If one had developed all the necessary results in this general context, presumably the argument in last section works the same way.

So we decided to use another approach for the general case.

Definition 5.1. A finite correspondence from $Y$ to $X$ is a family of relative cycles of dimension 0 with proper support over $Y$.

Definition 5.2. Let $\text{Sch}_k$ be the category of finite type separated $k$-schemes. Let $\text{Cor}_k$ be category whose objects are separated finite type $k$-schemes and morphisms finite correspondences. Let $\text{SmCor}_k$ be the full subcategory whose objects are smooth $k$-varieties. In this subcategory a finite correspondence between from $X$ to $Y$ is a linear combination of closed subvarieties of $X \to Y$ that are finite surjective onto one of the irreducible components of $X$.

Recall that the h-topology is generated by covers that are universal topological epimorphisms. Since we will only deal with noetherian schemes, this is the same as the topology generated by Zariski covers and covers that are proper surjective morphisms.

The qfh-topology is generated by covers in the h-topology that are also quasi-finite.

The cdh topology is generated by two types of covers

1. Nisnevich covers: i.e., étale covers that has the lifting property for every point.
2. Covers of the form $X' \amalg Z \to X$ where $i : Z \to X$ is a closed embedding and $p : X' \to X$ is proper and $p^{-1}(X - Z) \cong X' - p^{-1}(Z)$. 

Later we will only use the fact that a surjective proper morphism is an h-cover. And a blow-up along a closed subscheme $Z \subset X$ produces a cdh cover $Bl_Z X \coprod Z \to X$.

**Definition 5.3.** We define the presheaf $Z^h(X, r)$ on the category $\text{Sch}_k$ whose value on a scheme $S$ is a formal linear combination of integral subschemes $Z \subset S \times X$ that is flat, of equidimension $r$ over $S$.

We also define $Z_{eq}(X, r)$ on the category $\text{Sch}_k$ whose value on a scheme $S$ is the group of families of cycles in $X$ of equidimension $r$ over $S$.

We also define $Z(X, r)$ on the category $\text{Sch}_k$ whose value on a scheme $S$ is the group of families of cycles of dimension $r$ in $X$ over $S$.

We define $Z^{eff}(X, r)$ (resp. $Z_{eq}^{eff}(X, r), Z^{eff}(X, r)$) on the category $\text{Sch}_k$ whose value on a scheme $S$ is the monoid of families of effective cycles in $X$ of equidimension $r$ over $S$.

Similarly, we define $C^h(X, r), C_{eq}(X, r), C(X, r), C^{eff}(X, r), C_{eq}^{eff}(X, r), C^{eff}(X, r)$ as the counterpart of the above presheaves for families of cycles with proper support over $S$.

The sheaf $C^h(X, r)$ is denoted by $\mathbb{Z}^\text{PropHilb}$ in [SV00].

Since later we will consider cycles on proper schemes, with the purpose of keeping the names consistent with previous section, we will use $Z^h(X, r)$ etc. notations.

Note that we do not require the subschemes to be equidimensional over $S$ in the definition of $Z(X, r)$ and $C(X, r)$. It is possible to have higher dimensional fibers ([SV00] Example 3.1.9). However, $Z^{eff}(X, r)$ is the same as $Z_{eq}^{eff}(X, r)$. Similarly for the properly supported version.

We have the following.

**Proposition 5.4.** [SV00] Proposition 4.2.7, 4.2.6, Lemma 4.2.13] The presheaf $Z_{eq}(X, r) \otimes \mathbb{Z}\frac{1}{p^i}$ is a qfh-sheaf and the presheaf $Z(X, r) \otimes \mathbb{Z}\frac{1}{p^i}$ is an h-sheaf. Moreover, the sheafification in the h topology of $Z_{eq}(X, r)$ is the same as that of $Z(X, r)$.

In the following, we write $Z^h_{eq}(X)$ as the h-sheaf associated to $Z^h(X, r)$ (which is the same as that of $Z_{eq}(X, r)$ or $Z(X, r)$).

In the following, given a presheaf $\mathcal{F}$, we use $C^*(\mathcal{F})$ to denote the Suslin complex of presheaves (with non-positive degrees). That is, $C^{-i}(\mathcal{F})(S) = \mathcal{F}(S \times \Delta^i)$, where $\Delta^i = \text{Spec } k[t_0, \ldots, t_i]/(\sum t_j = 1)$ is the algebraic $i$-simplex.

If $\mathcal{F}$ is a torsion, homotopy invariant étale sheaf with transfers, or a qfh, or h sheaf on the category of schemes over $X$, with torsion order prime to $p$, the Suslin rigidity theorem [SV96] Theorem 4.5] shows that $C^*(\mathcal{F})$ is isomorphic to the pullback of a complex of locally constant sheaves. Since we work over an algebraically closed field, locally constant is the same as constant. Moreover if $\mathcal{F}$ is a constant étale sheaf, we have isomorphisms ([SV96] Theorem 10.2, 10.7]

$$H^i_{\text{ét}}(\text{Spec } k, \mathcal{F}) \cong H^i_{\text{qfh}}(\text{Spec } k, \mathcal{F}^{\text{qfh}}) \cong H^i_h(\text{Spec } k, \mathcal{F}^h).$$

Since we assume that $k$ is algebraically closed, Spec $k$ has no higher cohomology for any sheaf in any of these three topologies. Therefore for any complex of constant sheaves, we also have the isomorphism of cohomologies over Spec $k$. In particular, above discussions apply to the complex $C^*(Z_{eq}(X, r)) \otimes \mathbb{Z}/NZ$. We may identify the cohomology of this complex.
**Theorem 5.5.** Let $X$ be a quasi-projective variety defined over an algebraically closed field $k$. Let $N$ be an integer, non-zero in the field. We have the following isomorphisms.

$$H^i_k(\text{Spec } k, C^*(Z_{eq}(X, r) \otimes \mathbb{Z}/N\mathbb{Z})) \cong H^i_{qfh}(\text{Spec } k, C^*(Z_{eq}^h(X, r) \otimes \mathbb{Z}/N\mathbb{Z}))$$

$$\cong H^i_{\text{cl}}(\text{Spec } k, C^*(Z_{eq}(X, r) \otimes \mathbb{Z}/N\mathbb{Z})) \cong H^i_{\text{Ab}}(C^*(Z_{eq}(X, r) \otimes \mathbb{Z}/N\mathbb{Z})(\text{Spec } k))$$

$$\cong CH_i(X, -i, \mathbb{Z}/N\mathbb{Z}).$$

**Proof.** The first three cohomology groups are isomorphic as discussed above. They are all equal to the cohomology of the complex $C^*(Z_{eq}(X, r) \otimes \mathbb{Z}/N\mathbb{Z})(\text{Spec } k)$, since this complex computes the cohomology group in the qfh and étale topology.

Finally, under the hypothesis that resolution of singularities holds, Suslin [Sus00, Theorem 3.2] proves that for any quasi-projective variety $X$, we have an isomorphism

$$H^i_{\text{Ab}}(C^*(Z_{eq}(X, r) \otimes \mathbb{Z}/N\mathbb{Z})(\text{Spec } k)) \cong CH_i(X, -i, \mathbb{Z}/N\mathbb{Z}).$$

Using Gabber’s refinement of de Jong’s alteration theorem, Kelly [Kel17, Theorem 5.6.4] removed the resolution of singularities hypothesis. □

**Corollary 5.6.** Let $X$ be a quasi-projective variety defined over an algebraically closed field $k$. Let $N$ be an integer, non-zero in the field. We have the following isomorphisms.

$$H^i_k(\text{Spec } k, C^*(Z_{eq}(X, 0) \otimes \mathbb{Z}/N\mathbb{Z})) \cong H^i_{qfh}(\text{Spec } k, C^*(Z_{eq}(X, 0) \otimes \mathbb{Z}/N\mathbb{Z}))$$

$$\cong H^i_{\text{cl}}(\text{Spec } k, C^*(Z_{eq}(X, 0) \otimes \mathbb{Z}/N\mathbb{Z})) \cong H^i_{\text{Ab}}(C^*(Z_{eq}(X, 0) \otimes \mathbb{Z}/N\mathbb{Z})(\text{Spec } k))$$

$$\cong CH_0(X, -i, \mathbb{Z}/N\mathbb{Z}) \cong H^{2d-i}_{\text{cl}}(X, \mathbb{Z}/N\mathbb{Z}),$$

where $d$ is the dimension of $X$. In particular, all the groups are finite.

**Proof.** The last equality follows from [SV96, Corollary 7.8]. Clearly the étale cohomology group is finite. □

We use $A_1(X)$ to denote the group of one cycles in $X$ modulo algebraic equivalence. For any abelian group $A$ and any integer $m$, we use $A[m]$ to denote the group of $m$-torsions in $A$, and $A/m$ to denote the quotient $A/mA$.

For any integer $N$ invertible in the field $k$, we have a homomorphism

$$CH_1(X, 1, \mathbb{Z}/N\mathbb{Z}) \to CH_1(X, 0, \mathbb{Z})[N]$$

that comes from the long exact sequence of higher Chow groups with $\mathbb{Z}$ and $\mathbb{Z}/N$ coefficients. Composing with the surjective map

$$CH_1(X, 0, \mathbb{Z})[N] \to A_1(X)[N],$$

we have a homomorphism

$$CH_1(X, 1, \mathbb{Z}/N\mathbb{Z}) \to A_1(X)[N].$$

Now we can state the counterpart of Theorem 4.8

**Theorem 5.7.** Let $X$ be a smooth projective variety defined over an algebraically closed field $k$ of characteristic $p$. Assume that $X$ is either a separably rationally connected variety or a separably rationally connected fibration over a curve. Fix an integer $N$ that is invertible in $k$. For any class $[L]$ in the kernal of the map

$$H^1_h(\text{Spec } k, C^*(Z^1(X)) \otimes \mathbb{Z}/N\mathbb{Z}) \cong CH_1(X, 1, \mathbb{Z}/N\mathbb{Z}) \to A_1(X)[N],$$
there is a smooth projective variety $Z$ and a family of one-dimensional cycles over $Z$ such that the class $[L]$ is in the image of

$$H^{-1}_k(Spec \ k, C^*(Z^0_0(Z)) \otimes \mathbb{Z}/N\mathbb{Z}) \to H^{-1}_k(Spec \ k, C^*(Z^1_1(X)) \otimes \mathbb{Z}/N\mathbb{Z})$$

induced by this family of cycles.

Remark 5.8. This result is a priori weaker than Theorem 4.8 over complex numbers. We have a short exact sequence

$$0 \to L_1H_3(X)/N \to L_1H_3(X, \mathbb{Z}/N) \to L_1H_2(X)[N] \to 0.$$  \(\text{\textsuperscript{5.7}}\)

Since $L_1H_3(X, \mathbb{Z}/N) \cong CH_1(X, 1, \mathbb{Z}/N)$ and $L_1H_2(X)[N] \cong A_1(X)[N]$, Theorem \(\text{\textsuperscript{5.7}}\) only says that classes in $L_1H_3(X)/N$ comes from a smooth projective variety.

But if we know that $L_1H_3(X)$ is finitely generated, then we can find the lift. Conjecturally, this group is isomorphism to $H_3(X, \mathbb{Z})$, thus finitely generated.

The proof of Theorem 5.7 is analogous to that of Theorem 4.8. We first prove the analogue of Lemma 4.9.

Lemma 5.9. Let $X$ be a smooth projective variety defined over an algebraically closed field $k$ of characteristic $p$. Assume that $X$ is either a separably rationally connected variety or a separably rationally connected fibration over a curve. Fix an integer $N$ that is invertible in $k$. For any class $[L]$ in

$$H^{-1}_k(Spec \ k, C^*(Z^0_0(X)) \otimes \mathbb{Z}/N\mathbb{Z}) \cong CH_1(X, 1, \mathbb{Z}/N\mathbb{Z}),$$

there is a normal projective variety $U$, a family of equidimensional one cycles $\gamma_U$ over $U$ and a morphism $f : \Delta^1 \to U$ such that $[L]$ is represented by $f^* \gamma_U$ over $\Delta^1$.

Proof. We could translate the proof of Theorem 4.8 in the context of h-sheaves. But here is an easier argument using Hilbert schemes.

The class $[L]$ is represented by a family of cycles $\sum_i m_i \Gamma_i, m_i \in \mathbb{Z}/N$ over $\Delta^1$, where $\Gamma_i \subset \Delta^1 \times X$ is an integral subvariety. Since $\Delta^1$ is one dimensional, the projection $\Gamma_i \to \Delta^1$ is flat. Thus we get a morphism $f_i$ from $\Delta^1$ to the Hilbert scheme. The universal subscheme over the Hilbert scheme gives a family of cycles over the Hilbert scheme. Therefore, we may take $U$ to be the normalization of of products of irreducible components of the Hilbert scheme and $\gamma_U$ to be the family of cycles (with appropriate multiplicity) coming from universal subschemes.

We will need the following observation later in the proof.

Lemma 5.10. Let $T$ be a connected projective algebraic set over an algebraically closed field $k$, and let $x, y$ be two points in $T$. Let $\mathcal{F}$ be a sheaf of abelian groups in the qfh or h topology, or an étale sheaf with transfers. Fix an integer $N$ invertible in $k$. Write $F_x$ (resp. $F_y$) the restriction of $F \in \mathcal{F} \otimes \mathbb{Z}/N\mathbb{Z}(T)$ to $x$ (resp. $y$). Then $F_x = F_y$ in $H^0(Spec \ k, C^*(\mathcal{F}) \otimes \mathbb{Z}/N\mathbb{Z})$, where the cohomology in taken in the étale topology, qfh topology or h topology.

Proof. Elements in $\mathcal{F} \otimes \mathbb{Z}/N\mathbb{Z}(T)$ induces a unique morphism

$$Z_0(T) \otimes \mathbb{Z}/N \to \mathcal{F} \otimes \mathbb{Z}/N\mathbb{Z}.$$  \(\text{\textsuperscript{5.11}}\)

If $\mathcal{F}$ is a sheaf with transfers, this is the Yoneda Lemma. If $\mathcal{F}$ is a qfh sheaf or h sheaf, this follows from the fact that the qfh sheafification of $Z_0(T)[\frac{1}{p}]$ is the free
sheaf $\mathbb{Z}^{[0]}_T[T]$ generated by the presheaf of sets $\text{Hom}(\cdot, T)$ (SV96 Theorem 6.7)). Thus the class $[F]_x$ (resp. $[F]_y$) is the image of $[x]$ (resp. $[y]$) under the map

$$H^0(\text{Spec } k, C^*(Z_0(T)) \otimes \mathbb{Z}/N\mathbb{Z}) \to H^0(\text{Spec } k, C^*(F) \otimes \mathbb{Z}/N\mathbb{Z}).$$

So it suffices to show that $[x] = [y]$ in $H^0(\text{Spec } k, Z_0(T) \otimes \mathbb{Z}/N\mathbb{Z})$. But the latter cohomology group is $CH_0_0(\mathbb{Z}) \otimes \mathbb{Z}/N\mathbb{Z} \cong \mathbb{Z}/N\mathbb{Z}$ by Lemma 5.6 and the isomorphism is given by the degree map. Any two points $x, y$ give the same class in $H^0(\text{Spec } k, Z_0(T) \otimes \mathbb{Z}/N\mathbb{Z})$.

Now we begin the proof Theorem 5.7.

**Proof.** Given a normal projective variety $S$ parameterizing a family of 1 dimensional cycles on $X$, there is an induced morphism of h-sheaves:

$$Z^0(S) \to Z^0(X).$$

We denote by $I(S)$ (resp. $K(S)$) the image h-sheaf (resp. kernel h-sheaf) of this map.

By Lemma 5.9 there is a normal projective variety $U$, a family of equidimensional one cycles $\gamma_U$ over $U$ and a morphism $f : \Delta^1 \to U$ such that $[L]$ is represented by $f^* \gamma_U$ over $\Delta^1$.

Denote by $\gamma_0, \gamma_1$ the restriction of the family of cycles $\gamma_U$ over $U$ to $0, 1 \in \Delta^1$. Then $\gamma_0 - \gamma_1 = N(\gamma_{0,1})$ for some cycle $\gamma_{0,1}$. The image of $[L]$ in $CH_1(X, 0, \mathbb{Z})[N]$ and $A_1(X)[N]$ is the class of $\gamma_{0,1}$.

If $\gamma_{0,1}$ is zero in $A_1(X)[N]$, that is, if $\gamma_{0,1}$ is algebraically equivalent to 0, then by Proposition 5.10 there is a smooth projective curve $D$ with a family of cycles $\gamma_D$ and two points $d, d'$ such that $\Gamma_d$ is 0 and $\Gamma_{d'}$ is $\gamma_{0,1}$.

Consider the product $U \times D$. We have a family of cycles $\gamma = \pi_U^* \gamma_U + N \pi_D^* \gamma_D$.

There are three points in $S = U \times D$:

$$x = (f(0), d), y = (f(1), d), z = (f(1), d')$$

such that

1. $\gamma_x = \gamma_z$.
2. There is a curve $C$ containing $y, z$ such that for every point $c \in C$, the cycle $\gamma_c$ is zero in $Z_1(X) \otimes \mathbb{Z}/N(\text{Spec } k)$.

As in the proof of Theorem 4.8 we apply the second part of Theorem 5.5 to find a normal projective variety $T$ with a surjective projective morphism $T \to S$ and an embedding $T \to Y$, and liftings $x_T, y_T, z_T$ of the points $x, y, z$ such that

1. The two points $x_T$ and $z_T$ are connected by a chain of curves in $Y$ parameterizing constant cycles.
2. The two points $y_T$ and $z_T$ are connected by a curve $D_T$ in $T$ parameterizing cycles divisible by $N$.

Denote by $K$ the kernel of the morphism between h sheaves

$$Z_0(T) \to Z_0(S).$$

Here $T \to S$ is proper and surjective. So the above morphism of h sheaves is surjective. Then we also have a surjection of h sheaves

$$I(T) \to I(S) \to 0.$$

It follows that we have a short exact sequence of h sheaves:

$$0 \to K \to K(T) \to K(S) \to 0.$$
We have commutative diagrams:
\[
\begin{array}{cccc}
H_h^{-1}(C^*(Z_0(Y))/N) & \longrightarrow & H_h^{-1}(C^*(I(Y))/N) & \longrightarrow & H_h^0(C^*(K(Y))/N) \\
\uparrow & & \uparrow & & \uparrow \\
H_h^{-1}(C^*(Z_0(T))/N) & \longrightarrow & H_h^{-1}(C^*(I(T))/N) & \longrightarrow & H_h^0(C^*(K(T))/N) \\
\downarrow & & \downarrow & & \downarrow \\
H_h^{-1}(C^*(Z_0(S))/N) & \longrightarrow & H_h^{-1}(C^*(I(S))/N) & \longrightarrow & H_h^0(C^*(K(S))/N)
\end{array}
\]

The obstruction of lifting the class $[L]$ in $H_h^{-1}(\text{Spec } k, C^*(I(T)) \otimes \mathbb{Z}/N\mathbb{Z})$ is in $H_h^0(\text{Spec } k, C^*(K(T)))$ and maps to $[x-y]$ in $H_h^0(\text{Spec } k, C^*(K(S)) \otimes \mathbb{Z}/N\mathbb{Z})$. The class $[x_T-y_T]$ differs from the obstruction class by an element in $H_h^0(\text{Spec } k, C^*(K) \otimes \mathbb{Z}/N\mathbb{Z})$.

Given the morphism $T \to S$, we have a long exact sequence
\[
\ldots \to H_h^{-1}(\text{Spec } k, C^*(Z_0(T)) \otimes \mathbb{Z}/N\mathbb{Z}) \to H_h^{-1}(\text{Spec } k, C^*(Z_0(S)) \otimes \mathbb{Z}/N\mathbb{Z}) \to H_h^0(\text{Spec } k, C^*(K(T)) \otimes \mathbb{Z}/N\mathbb{Z}) \to \ldots
\]
Therefore $H_h^0(\text{Spec } k, C^*(K) \otimes \mathbb{Z}/N\mathbb{Z})$ is finitely generated by Corollary 5.1. By Lemma 5.11 any class in $H_h^0(\text{Spec } k, C^*(K) \otimes \mathbb{Z}/N\mathbb{Z})$ is equivalent to the class of the form $[a-b]$, where $a, b$ are points in the fiber over a general point in $S$.

The class $[L]$ maps to $H_h^{-1}(\text{Spec } k, C^*(I(Y)) \otimes \mathbb{Z}/N\mathbb{Z})$, with obstruction class the push-forward of $[x_T - y_T]$ modulo classes in $H_h^0(\text{Spec } k, C^*(K) \otimes \mathbb{Z}/N\mathbb{Z})$.

By the existence of the families of constant cycles in the second part of Theorem 3.5 and by Lemma 5.10 we have

1. The composition

\[
H_h^0(\text{Spec } k, C^*(K)/N) \to H_h^0(\text{Spec } k, C^*(K(T))/N) \to H_h^0(\text{Spec } k, C^*(K(Y))/N)
\]

is the zero map.

2. The push-forward of the class $[x_T - y_T]$ vanishes in $H_h^0(\text{Spec } k, C^*(K(Y)) \otimes \mathbb{Z}/N\mathbb{Z})$ (by applying Lemma 5.10 to $[x_T - y_T]$ and $[x_T - x_T] = 0$).

Thus the class $[L]$ in $H_h^{-1}(\text{Spec } k, C^*(I(Y))/N)$ comes from $H_h^{-1}(\text{Spec } k, C^*(Z_0(Y))/N)$.

Finally, we use Gabber’s refinement of de Jong’s alteration to find a smooth projective variety $Z$ and a projective alteration $Z \to Y$ whose degree is relatively prime to $N$. Then
\[
CH_0(Z, 1, \mathbb{Z}/N\mathbb{Z}) \to CH_0(Y, 1, \mathbb{Z}/N\mathbb{Z})
\]
is surjective by Lemma 5.12. Pulling back the families of cycles over $Y$ gives a family of cycles over $Z$. Then the theorem follows from the following commutative diagram
\[
\begin{array}{ccc}
CH_0(Z, 1, \mathbb{Z}/N) & \longrightarrow & CH_0(Y, 1, \mathbb{Z}/N) \\
\cong \downarrow & & \cong \downarrow \\
H_1(Z, \mathbb{Z}/N) & \longrightarrow & H_1(Y, \mathbb{Z}/N)
\end{array}
\begin{array}{ccc}
\longrightarrow & CH_1(X, 1, \mathbb{Z}/N) & \longrightarrow & CH_1(X, 1, \mathbb{Z}/N) \\
\cong \downarrow & & \cong \downarrow \\
\longrightarrow & H_3(X, \mathbb{Z}/N) & \longrightarrow & H_3(X, \mathbb{Z}/N)
\end{array}
\]

The two lemmas used in the proof are the following.
Lemma 5.11. Let $p : X \to Y$ be a generically finite surjective morphism between normal projective varieties over an algebraically closed field $k$. Let $N$ be an integer invertible over $k$. Denote by $K$ the kernel sheaf of $Z_0(X) \to Z_0(Y)$. Then $H^0_h(\text{Spec } k, C^*(K) \otimes \mathbb{Z}/N\mathbb{Z})$ is generated by class of the form $[x_1 - x_2]$, for $x_1, x_2$ in the fiber of any chosen general point in $Y$.

Proof. There is a birational projective variety morphism $Y' \xrightarrow{q} Y$ such that the strict transform $X'$ of $X$ is flat over $Y'$. That is, we have a commutative diagram:

$$
\begin{array}{ccc}
X' & \xrightarrow{q'} & X \\
\downarrow p' & & \downarrow p \\
Y' & \xrightarrow{q} & Y
\end{array}
$$

We denote by $K(p)$ etc. to denote the kernel sheaf of $Z_0(X) \to Z_0(Y)$ etc. There is a commutative diagram of short exact sequences of $h$ sheaves

$$
\begin{array}{cccc}
0 & \to & K(q') & \to & Z_0(X') & \xrightarrow{q'} & Z_0(X) & \to & 0 \\
\downarrow & & \downarrow p' & & \downarrow & & \downarrow & & \\
0 & \to & K(q) & \to & Z_0(Y') & \xrightarrow{q'} & Z_0(Y) & \to & 0
\end{array}
$$

which also gives commutative diagrams after tensoring with $\mathbb{Z}/N\mathbb{Z}$. Then we have long exact sequences:

$$
CH_1(X, 1, \mathbb{Z}/N\mathbb{Z}) \to CH_1(Y, 1, \mathbb{Z}/N\mathbb{Z}) \to H^0_h(\text{Spec } k, C^*(K(p)) \otimes \mathbb{Z}/N\mathbb{Z}) \ldots ,
$$

$$
CH_1(X', 1, \mathbb{Z}/N\mathbb{Z}) \to CH_1(Y', 1, \mathbb{Z}/N\mathbb{Z}) \to H^0_h(\text{Spec } k, C^*(K(p')) \otimes \mathbb{Z}/N\mathbb{Z}) \ldots .
$$

The cohomology group $H^0_h(\text{Spec } k, C^*(K(p)) \otimes \mathbb{Z}/N\mathbb{Z})$ (resp. $H^0_h(\text{Spec } k, C^*(K(p')) \otimes \mathbb{Z}/N\mathbb{Z})$) is generated by cycles of the form $y_1 - y_2$ for $y_1, y_2$ in the same fiber. So it suffices to show that such cycles are zero.

We first show that

$$CH_0(Y', 1, \mathbb{Z}/N\mathbb{Z}) \to CH_0(Y, 1, \mathbb{Z}/N\mathbb{Z})$$

is surjective. This is because $Y' \to Y$ has connected fibers. So for any two points in the same fiber, by Lemma [5, 10] the class of the difference is zero. Since

$$CH_0(Y', 0, \mathbb{Z}/N\mathbb{Z}) \to CH_0(Y, 0, \mathbb{Z}/N\mathbb{Z})$$

is an isomorphism, we know that $H^0_h(\text{Spec } k, C^*(K(q)) \otimes \mathbb{Z}/N\mathbb{Z})$ vanishes.

By the same argument, $H^0_h(\text{Spec } k, C^*(K(q')) \otimes \mathbb{Z}/N\mathbb{Z})$ vanishes. Then a simple diagram chasing shows that

$$H^0_h(\text{Spec } k, C^*(K(p')) \otimes \mathbb{Z}/N\mathbb{Z}) \to H^0_h(\text{Spec } k, C^*(K(p)) \otimes \mathbb{Z}/N\mathbb{Z})$$

is surjective.

Thus it suffices to prove the lemma under the assumption that $X \to Y$ is flat and finite, and $Y$ is normal (but $X$ is not necessarily normal).

Let $x_1, x_2$ be two points in the fiber over $y \in Y$. We first show that for any general point $t \in Y$, $[x_1 - x_2]$ is equivalent to a class $[t_1 - t_2]$ for some points $t_1, t_2$ in the fiber over $t$. Consider the correspondence $X \times_Y X \subset X \times X$. Since $X \to Y$ is assumed to be flat, $X \times_Y X \to X$ is flat. We take an irreducible component $D$ containing $(x_1, x_2)$, which dominates (and thus surjects onto) $X$. There are two points $x_D, t_D$ in $D$ such that the following conditions are satisfied.
(1) There is a surjective morphism \( f : D \to Y \) that maps \( x_D \) (resp. \( t_D \)) to \( y \) (resp. \( t \)).
(2) There are two morphisms \( f_1, f_2 : D \to X \) such that \( f_1(x_D) = x_1, f_2(x_D) = x_2 \).
(3) The composition of \( f_1, f_2 \) with the morphism \( q : X \to Y \) gives the morphism \( f : D \to Y \).

Then by Lemma 5.10 again, the cycle \([x_1 - x_2]\) has the same class as \([f_1(t_D) - f_2(t_D)]\).

**Lemma 5.12.** Let \( p : X \to Y \) be a generically finite morphism between normal projective varieties over an algebraically closed field \( k \). Let \( N \) be an integer invertible over \( k \). Assume that \( \text{deg } p \) is relatively prime to \( N \). Then we have a surjection

\[
CH_0(X, 1, \Bbb{Z}/N\Bbb{Z}) \to CH_0(Y, 1, \Bbb{Z}/N\Bbb{Z}).
\]

**Proof.** By Lemma 5.11 and the long exact sequence

\[
CH_0(X, 1, \Bbb{Z}/N\Bbb{Z}) \to CH_0(Y, 1, \Bbb{Z}/N\Bbb{Z}) \to H^0_h(\text{Spec } k, C^*(K)/N)
\]

\[
\to CH_0(X, 0, \Bbb{Z}/N) \xrightarrow{\sim} CH_0(Y, 0, \Bbb{Z}/N),
\]

it suffices to show that for a general point \( y \in Y \) and any two points \( x_1, x_2 \) in the fiber of \( y \), the class \([x_1 - x_2]\) is zero in \( H^0_h(\text{Spec } k, C^*(K)/N) \).

By the Bertini theorem for étale fundamental groups, there is a general complete intersection curve \( H \) such that the inverse image \( H' \) in \( Y \) is irreducible. For \( H \) general, the morphism \( H' \to H \) is flat and finite of degree prime to \( N \). Thus

\[
CH_0(H', 1, \Bbb{Z}/N\Bbb{Z}) \to CH_0(H, 1, \Bbb{Z}/N\Bbb{Z}) \to H^0_h(\text{Spec } k, C^*(K_H) \otimes \Bbb{Z}/N\Bbb{Z})
\]

\[
\to CH_0(H', \Bbb{Z}/N\Bbb{Z}) \to CH_0(H, \Bbb{Z}/N\Bbb{Z}),
\]

where \( K_H \) is the kernal sheaf of \( Z_0(H') \to Z_0(H) \). The map

\[
CH_0(H', \Bbb{Z}/N\Bbb{Z}) \to CH_0(H, \Bbb{Z}/N\Bbb{Z})
\]

is an isomorphism. On the other hand, since \( p : H' \to H \) is flat and finite, we have pull-back and push-forward on all the higher Chow groups. The composition of pull-back and push-forward

\[
CH_0(H, 1, \Bbb{Z}/N\Bbb{Z}) \xrightarrow{\ell} CH_0(H', 1, \Bbb{Z}/N\Bbb{Z}) \xrightarrow{p_*} CH_0(H, 1, \Bbb{Z}/N\Bbb{Z})
\]

is multiplication by \( \text{deg } p \). Since the degree of the map is relatively prime to \( N \),

\[
CH_0(H', 1, \Bbb{Z}/N\Bbb{Z}) \xrightarrow{\ell} CH_0(H, 1, \Bbb{Z}/N\Bbb{Z})
\]

is surjective. Thus for any two points \( t_1, t_2 \) over a general point \( t \in Y \), the class \([t_1 - t_2]\) vanishes in \( H^0_h(\text{Spec } k, C^*(K_H) \otimes \Bbb{Z}/N\Bbb{Z}) \). So does its push-forward in \( H^0_h(\text{Spec } k, C^*(K) \otimes \Bbb{Z}/N\Bbb{Z}) \). \( \square \)

Fix a prime number \( \ell \) different from the characteristic of \( k \). In the following theorem, we omit all the Tate twists for simplicity of notations.

**Theorem 5.13.** Let \( X \) be a smooth projective separably rationally connected variety or a separably rationally connected fibration over a curve defined over an algebraically closed field. There is a smooth projective curve \( C \) with a family of 1-dimensional cycles \( \Gamma \subset C \times X \) such that

\[
\Gamma_* : H^{BM}_1(C, \Bbb{Z}_\ell) \to H^{BM}_3(X, \Bbb{Z}_\ell)
\]

surjects onto \( N_1H_3(X, \Bbb{Z}_\ell) \).
Proof. In the following, we use Borel-Moore homology. For simplicity of notations, we only write them as $H_1$, $H_3$. Let $NH_3(X, \mathbb{Z}/\ell^n)$ be the coniveau filtration on the homology $N_1H_3(X, \mathbb{Z}/\ell^n)$. Denote by $\tilde{N}H_3(X, \mathbb{Z}/\ell^n)$ the strong coniveau filtration $\tilde{N}_1H_3(X, \mathbb{Z}/\ell^n)$.

For a projective variety $Y$, we have $H_1(Y, \mathbb{Z}/\ell^n) \cong H_1(Y, \mathbb{Z}/\ell^n)$, since $H_0(Y, \mathbb{Z}_\ell)$ is torsion free. Therefore, $\tilde{N}H_3(X, \mathbb{Z}/\ell^n) \to \tilde{N}H_3(X, \mathbb{Z}/\ell^n)$ is surjective.

We have a commutative diagram

$$
\oplus_{(S, \Gamma)} CH_0(S, 1, \mathbb{Z}/\ell^n) \xrightarrow{\oplus \Gamma_*} CH_1(S, 1, \mathbb{Z}/\ell^n) \xrightarrow{\Delta^\vee} A_1(S)[1/\ell^n],
$$

where the direct sum is taken over all equi-dimensional family of cycles over smooth projective varieties $S$.

By Theorem 5.7, the upper row is exact. The lower row is also exact, since it comes from

$$0 \to H_3(X, \mathbb{Z}_\ell)/\ell^n \to H_3(X, \mathbb{Z}/\ell^n) \to H_2(X, \mathbb{Z}_\ell)[1/\ell^n] \to 0.$$

The vertical maps are cycle class maps.

The middle vertical map

$$CH_1(X, 1, \mathbb{Z}/\ell^n) \to NH_3(X, \mathbb{Z}/\ell^n)$$

is surjective, since for any surface $\Sigma$, not necessarily smooth, we have a surjection

$$CH_1(\Sigma, 1, \mathbb{Z}/\ell^n) \to H_3(\Sigma, \mathbb{Z}/\ell^n)$$

by the Bloch-Kato conjecture and the localization sequence for higher Chow groups and Borel-Moore homology.

The left vertical arrow is the direct sum of the composition

$$CH_0(S, 1, \mathbb{Z}/\ell^n) \to H_1(S, \mathbb{Z}/\ell^n) \cong H_1(S, \mathbb{Z}_\ell)/\ell^n \xrightarrow{\Delta^\vee} H_3(X, \mathbb{Z}_\ell)/\ell^n,$$

Since the cycle class map induces an isomorphism $CH_0(S, 1, \mathbb{Z}/\ell^n) \cong H_1(S, \mathbb{Z}/\ell^n) \cong H_1(S, \mathbb{Z}_\ell)/\ell^n$, the left vertical arrow has the same cokernel as

$$\tilde{N}H_3(X, \mathbb{Z}_\ell)/\ell^n \to H_3(X, \mathbb{Z}_\ell)/\ell^n \cap NH_3(X, \mathbb{Z}/\ell^n),$$

which, by the snake lemma, is isomorphic to the cokernel $C_n$ of

$$\text{Ker}(CH_1(X, 1, \mathbb{Z}/\ell^n) \to H_3(X, \mathbb{Z}/\ell^n)) \to \text{Ker}(A_1[1/\ell^n] \to H_2(X, \mathbb{Z}_\ell)[1/\ell^n]).$$

By the decomposition of the diagonal argument à la Bloch-Srinivas, we know that the kernel of

$$A_1[1/\ell^n] \to H_2(X, \mathbb{Z}_\ell)[1/\ell^n]$$

is $\ell^M$ torsion for a fixed integer $M$. Therefore the inverse limit $\varprojlim C_n$ is zero since the connecting maps $C_{n+1} \to C_n$ are multiplication by $\ell^n$. So we have a surjection

$$\tilde{N}H_3(X, \mathbb{Z}_\ell) \cong \varprojlim \tilde{N}H_3(X, \mathbb{Z}_\ell)/\ell^n \xrightarrow{\varprojlim} \varprojlim H_3(X, \mathbb{Z}_\ell)/\ell^n \cap NH_3(X, \mathbb{Z}/\ell^n) \cong NH_3(X, \mathbb{Z}_\ell).$$

By Theorem 5.7 there is a smooth projective variety $Y$ and a family of cycles $\Gamma_Y$ such that the induced map

$$\Gamma_Y : H_1(Y, \mathbb{Z}_\ell) \to \tilde{N}^3H_3(X, \mathbb{Z}_\ell)$$
is surjective, by taking hyperplane sections in $Y$, we may find a smooth projective curve $C$ with a family of cycles $\Gamma$ such that
$$\Gamma_* : H_1(C, \mathbb{Z}_\ell) \to \tilde{N}^1 H_3(X, \mathbb{Z}_\ell)$$
is surjective.

**Theorem 5.14.** Let $X$ be a smooth projective separably rationally connected variety or a separably rationally connected fibration over a curve defined over an algebraically closed field. Assume $X$ is a 3-fold. Then all the filtrations on $H^3(X, \mathbb{Z}_\ell)$ introduced in Definition 1.3 equal the whole cohomology group:
$$\tilde{N}_{1,cyl,eq} H^3(X, \mathbb{Z}_\ell) = \tilde{N}_1 H^3(X, \mathbb{Z}_\ell) = \tilde{N}^1 H^3(X, \mathbb{Z}_\ell) = N^1 H^3(X, \mathbb{Z}_\ell) = H^3(X, \mathbb{Z}_\ell).$$

**Corollary 5.15.** Let $X$ be a smooth projective variety of dimension $d$ defined over a finite field $\mathbb{F}_q$, that is either separably rationally connected or a separably rationally connected fibration over a curve. Assume one of the followings
1. $N_1 H^{2d-3}_\text{ét}(\tilde{X}, \mathbb{Z}_\ell(d - 1)) = H^{2d-3}_\text{ét}(\tilde{X}, \mathbb{Z}_\ell(d - 1)).$
2. The cycle class map
$$cl : \varprojlim_n CH_1(\tilde{X}, 1, \mathbb{Z}/\ell^n) \to H^{2d-3}_\text{ét}(\tilde{X}, \mathbb{Z}_\ell(d - 1))$$
is surjective.

Then every class in $H^1(\mathbb{F}_q, H^3(\tilde{X}, \mathbb{Z}_\ell(d - 1)))$ is the class of an algebraic cycle defined over $\mathbb{F}_q$. In particular, this holds if $X$ has dimension 3.

**Proof.** We first show that the surjectivity of the cycle class map
$$cl : \varprojlim_n CH_1(\tilde{X}, 1, \mathbb{Z}/\ell^n) \to H^{2d-3}_\text{ét}(\tilde{X}, \mathbb{Z}_\ell(d - 1))$$
implies that $N_1 H^{2d-3}_\text{ét}(\tilde{X}, \mathbb{Z}_\ell(d - 1)) = H^{2d-3}_\text{ét}(\tilde{X}, \mathbb{Z}_\ell(d - 1))$. In fact, we have
$$\varprojlim_n CH_1(\tilde{X}, 1, \mathbb{Z}/\ell^n) \to \varprojlim_n N_1 H^{2d-3}_\text{ét}(\tilde{X}, \mathbb{Z}/\ell^n(d - 1)) \to \varprojlim_n H^{2d-3}_\text{ét}(\tilde{X}, \mathbb{Z}/\ell^n(d - 1)) = H^{2d-3}_\text{ét}(\tilde{X}, \mathbb{Z}_\ell(d - 1)).$$

Therefore
$$\varprojlim_n N_1 H^{2d-3}_\text{ét}(\tilde{X}, \mathbb{Z}/\ell^n(d - 1)) \to \varprojlim_n H^{2d-3}_\text{ét}(\tilde{X}, \mathbb{Z}/\ell^n(d - 1)) = H^{2d-3}_\text{ét}(\tilde{X}, \mathbb{Z}_\ell(d - 1))$$
is surjective. On the other hand, since $N_1 H_3(X, \mathbb{Z}/\ell^n)$ is a subgroup of $H_3(X, \mathbb{Z}/\ell^n)$, the inverse limit is injective, hence an isomorphism. But we have an exact sequence
$$0 \to \varprojlim_n H^{2d-3}_\text{ét}(\tilde{X}, \mathbb{Z}_\ell(d - 1)/\ell^n \cap N_1 H^{2d-3}_\text{ét}(\tilde{X}, \mathbb{Z}/\ell^n(d - 1)) \to \varprojlim_n H^{2d-3}_\text{ét}(\tilde{X}, \mathbb{Z}_\ell(d - 1)) \to \varprojlim_n H^{2d-2}_\text{ét}(\tilde{X}, \mathbb{Z}_\ell(d - 1)) \mathbb{Z}/\ell^n\text{n},$$
where the first inverse limit is $N_1 H^{2d-3}_\text{ét}(\tilde{X}, \mathbb{Z}_\ell(d - 1))$, and the last inverse limit is torsion free. Since the quotient of $H^{2d-3}_\text{ét}(\tilde{X}, \mathbb{Z}_\ell(d - 1))/N_1 H^{2d-3}_\text{ét}(\tilde{X}, \mathbb{Z}_\ell(d - 1))$ is torsion for separably rationally connected varieties or separably rationally connected fibrations over a curve, we know it is surjective.
Therefore, by Theorem 5.13, there is a smooth projective curve $C$ defined over $\bar{F}_q$ with a family of 1-dimensional cycles $\Gamma \subset C \times \bar{X}$ such that

$$\Gamma_* : H^1_{\text{ét}}(C, \mathbb{Z}_\ell(1)) \to H^{2d-3}_{\text{ét}}(\bar{X}, \mathbb{Z}_\ell(d-1))$$

is surjective. Then this corollary follows from [SS22, Proposition 7.6] \hfill \Box

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