Computations of Greeks in stochastic volatility models via the Malliavin calculus

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Abstract
We compute Greeks for stochastic volatility models driven by Brownian informations. We use the Malliavin method introduced by [1] for deterministic volatility models.

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1 INTRODUCTION

The application of the Malliavin calculus to the computations of price sensitivities were introduced by [1] for models with deterministic volatility. In this work, we compute the Greeks, for stochastic volatility models where the underlying asset price is driven by Brownian information. We consider stochastic volatility models, since these models, unlike those with deterministic volatility, take into account the smile effect.

Let \((B_t)_{t \in [0,T]}\) and \((B'_t)_{t \in [0,T]}\) be two independent Brownian motions. We work in a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)\), where \((\mathcal{F}_t)_{t \in [0,T]}\) is the naturel filtration generated by \(B\) and \(B'\). We consider a market with two assets, a riskless one with price \((e^{-\tilde{R}_t r_s ds})_{t \in [0,T]}\), where \(r\) is deterministic and denotes the interest rate. And a risky asset \((S_t)_{t \in [0,T]}\) to which is related an option with payoff \(f(S_T)\). \((S_t)_{t \in [0,T]}\) has a stochastic volatility and is given by

\[
\frac{dS_t}{S_t} = \mu_t dt + \sigma(t, Y_t) dB_t, \quad \text{where} \quad dY_t = \mu'_t dt + \sigma'_t [\rho dB_t + dY'_t], \quad t \in [0,T],
\]

with \(S_0 = x > 0\) and \(Y_0 = y \in \mathbb{R}\). \(\mu, \mu', \sigma'\) are deterministic functions, \(\rho \in \mathbb{R}\) and \(\sigma \in C^2([0,T] \times \mathbb{R})\) such that for any \(t \in [0,T]\), \(\sigma(t, .) \neq 0\). The market considered here is incomplete. There is an infinity of E.M.M.-Equivalent Martingale Measure-% (i.e a probability equivalent to \(P\) under which the actualized price \((S_t e^{-\int_0^t r_s ds})_{t \in [0,T]}\) is a martingale). Let \(Q\) be a fixed \(P\)-E.M.M. \(Q\) is identified by its Radon-Nikodym density w.r.t \(P\), denoted \(\rho_T\) and given by

\[
\rho_T = \exp \left( \int_0^T \alpha_s dB_s + \beta_s dB'_s - \frac{1}{2} \int_0^T (\alpha_s^2 + \beta_s^2) ds \right),
\]
where \((\alpha_t)_{t\in[0,T]}\) and \((\beta_t)_{t\in[0,T]}\) are two predictable processes s.t. \(\alpha_t = -\frac{\mu_t - \sigma_t}{\sigma_t(\xi_t)}\), and \(\beta\) is arbitrary. Now let, for any \(t \in [0,T]\), \(W_t = B_t - \int_0^t \alpha_s ds\) and \(W'_t = B'_t - \int_0^t \beta_s ds\) then by the Girsanov theorem \(W\) and \(W'\) are two \(Q\)-Brownian motions. In the following we work with a fixed \(P\)-E.M.M. \(Q\) and we will use \(E[\cdot]\) (instead of \(E_Q[\cdot]\)) as the expectation under the probability \(Q\). We have, under \(Q\), for any \(t \in [0,T]\)

\[
\frac{dS_t}{S_t} = r_t dt + \sigma(t, Y_t) dW_t, \quad dY_t = \left(\mu_t^Y + \sigma_t^Y \frac{\mu_t - \mu_t^Y}{\sigma_t^Y} + \beta_t \sigma_t^Y\right) dt + \rho \sigma_t^Y dW_t + \sigma_t^Y dW'_t. \tag{1.1}
\]

### 2 MALLIAVIN DERIVATIVE ON WIENER SPACE

In this section we give an introduction to the malliavin derivative in Wiener space and to its adjoint the Skorohod integral. We refer for example to [2]. Let \((D_t^W)_{t\in[0,T]}\) be the Malliavin derivative on the direction of \(W\). We denote by \(\mathbb{P}\) the set of random variables \(F : \Omega \to \mathbb{R}\), such that \(F\) has the representation

\[
F(\omega) = f \left( \int_0^T f_1(t) dW_t, \ldots, \int_0^T f_n(t) dW_t \right),
\]

where \(f(x_1, \ldots, x_n) = \sum \alpha x^\alpha\) is a polynomial in \(n\) variables \(x_1, \ldots, x_n\) and deterministic functions \(f_i \in L^2([0,T])\). Let \(\|\cdot\|_{1,2}\) be the norm

\[
\|F\|_{1,2} := \|F\|_{L^2(\Omega)} + \|D^W F\|_{L^2([0,T] \times \Omega)}, \quad F \in L^2(\Omega).
\]

Thus the domaine of the operator \(D^W\), \(\text{Dom}(D^W)\), coincide with \(\mathbb{P}\) w.r.t the norm \(\|\cdot\|_{1,2}\). The next proposition will be useful.

**Proposition 1** Given \(F = f \left( \int_0^T f_1(t) dW_t, \ldots, \int_0^T f_n(t) dW_t \right) \in \mathbb{P}\). We have

\[
D^W F = \sum_{k=0}^{k=n} \frac{\partial f}{\partial x_k} \left( \int_0^T f_1(t) dW_t, \ldots, \int_0^T f_n(t) dW_t \right) f_k(t).
\]

To calculate the Malliavin derivative for Itô integral, we will use the following Proposition.

**Proposition 2** Let \((u_t)_{t\in[0,T]}\) be a \(\mathcal{F}_t\)-adapted process, such that \(u_t \in \text{Dom}(D^W)\), we have

\[
D^W \int_0^T u_t dW_t = \int_0^T (D^W u_s) dW_s + u_T.
\]

From now on, for any stochastic process \(u\) and for \(F \in \text{Dom}(D^W)\) such that \(u D^W F \in L^2([0,T])\) we let

\[
D^W F := \langle D^W F, u \rangle_{L^2([0,T])} := \int_0^T u_t D^W F dt.
\]

Let \(\delta^W\) be the Skorohod integral in Wiener space. We have \(\delta^W\) is the adjoint of \(D^W\) as showing in the next proposition, moreover its an extension of the Itô integral.
Proposition 3  a) Let \( u \in \text{Dom}(\delta^W) \) and \( F \in \text{Dom}(D^W) \), we have \( E[D^W_u F] \leq C(u)\|F\|_{1,2} \) and \( E[F \delta^W(u)] = E[D^W_u F] \).

b) Consider a \( L^2(\Omega \times [0,T]) \)-adapted stochastic process \( u = (u_t)_{t \in [0,T]} \). We have \( \delta^W(u) = \int_0^T u_tdW_t \).

c) Let \( F \in \text{Dom}(D^W) \) and \( u \in \text{Dom}(\delta^W) \) such that \( uF \in \text{Dom}(\delta^W) \) thus \( \delta^W(uF) = F \delta^W(u) - D^W_u F \).

3 COMPUTATIONS OF THE GREEKS

The computations of Greeks by Malliavin approach rest on a known integration by parts formula -cf. \[1\]- given in the following proposition.

Proposition 4 Let \( I \) be an open interval of \( \mathbb{R} \). Let \( (F^\zeta)_{\zeta \in I} \) and \( (H^\zeta)_{\zeta \in I} \), be two families of random functionals, continuously differentiable in \( \text{Dom}(D^W) \) in the parameter \( \zeta \in I \). Let \( (u_t)_{t \in [0,T]} \) be a process satisfying

\[
D^W_u F^\zeta \neq 0, \quad \text{a.s. on } \{ \partial_{\zeta} F^\zeta \neq 0 \}, \quad \zeta \in I,
\]

and such that \( a H^\zeta \partial_{\zeta} F^\zeta / D^W_u F^\zeta \) is continuous in \( \zeta \) in \( \text{Dom}(\delta^W) \). We have

\[
\frac{\partial}{\partial \zeta} E\left[ H^\zeta f \left( F^\zeta \right) \right] = E\left[ f \left( F^\zeta \right) \left( \frac{H^\zeta \partial_{\zeta} F^\zeta}{D^W_u F^\zeta} \delta^W(u) - D^W_u \left( \frac{H^\zeta \partial_{\zeta} F^\zeta}{D^W_u F^\zeta} \right) + \partial_{\zeta} H^\zeta \right) \right]
\]

for any function \( f \) such that \( f \left( F^\zeta \right) \in L^2(\Omega), \zeta \in I \).

Our aim is to compute the Greeks for options with payoff \( f(S_T) \), where \( (S_t)_{t \in [0,T]} \) denotes the underlying asset price given by \[11\]. We have

\[
S_T = x \exp \left( \int_0^T \sigma(s,Y_s)dW_s + \int_0^T (r_s - \frac{1}{2}\sigma^2(s,Y_s))ds \right), \quad (3.1)
\]

Let \( \zeta \) be a parameter taking the values: the initial asset price \( x = S_0 \), the volatility \( \sigma \), or the interest rate \( r \). Let \( C = E[f(S_T^\zeta)] \) be the price of the option. We will compute the following Greeks:

\[
\Delta = \frac{\partial C}{\partial x}, \quad \Gamma = \frac{\partial^2 C}{\partial x^2}, \quad \Rho = \frac{\partial C}{\partial r}, \quad \Vega = \frac{\partial C}{\partial \sigma}
\]

The next Proposition gives the first, second and third order derivatives of \( S_T \) w.r.t \( D^W \), needed for the computations of the different Greeks.

Proposition 5 For \( 0 \leq t \leq T \), we let \( G(t,T) := \sigma(t,Y_t) + \int_t^T \frac{\delta^W}{\delta_y}(v,Y_v)D^W_vY_v(dW_v - \sigma(v,Y_v)dv) \). We have \( D^W_t S_T = S_T G(t,T) \) and thus

\[
D^W_u S_T = S_T \int_0^T u_t G(t,T)dt \quad (3.2)
\]

\[
D^W_u D^W_t S_T = S_T \left( \left( \int_0^T u_t G(t,T)dt \right)^2 + \int_0^T \int_s^T u_s u_t D^W_s G(t,T)dtds \right) \quad (3.3)
\]
\[ D_w^WD_u^WD_u^WS_T = S_T \left( \left( \int_0^T u_t G(t,T)dt \right)^3 + 3 \int_0^T u_t G(t,T)dt \int_0^T \int_s^T u_s u_t D_s G(t,T)dt ds \right) \]

+ \int_0^T \int_s^T u_r u_s u_t D_r D_s G(t,T)dt ds dr \right) \quad (3.4) \]

where

\[ D_s^W G(t,T) = \frac{\partial \sigma}{\partial y}(t,Y_t) D_s^W Y_t + \int_t^T D_s^W Y_v D_t^W Y_v \frac{\partial}{\partial y} \left( \frac{\partial \sigma}{\partial y}(v,Y_v)(dW_v - \sigma(v,Y_v)dv) \right) \]

+ \int_t^T \frac{\partial \sigma}{\partial y}(v,Y_v) D_s^W D_t^W Y_v (dW_v - \sigma(v,Y_v)dv) \quad (3.5) \]

\[ D_r^W D_s^W G(t,T) = \int_t^T \left( (D_r^W D_s^W Y_v D_t^W Y_v + D_s^W Y_v D_r^W D_t^W Y_v) \frac{\partial^2 \sigma}{\partial y^2}(v,Y_v) \right. \]

+ \int_t^T \frac{\partial \sigma}{\partial y}(v,Y_v) D_s^W D_t^W Y_v \frac{\partial^2 \sigma}{\partial y^2}(v,Y_v) \frac{\partial}{\partial y} \left( \sigma(v,Y_v) \frac{\partial \sigma}{\partial y}(v,Y_v) \right) \]

\[ D_r^W D_s^W G(t,T) = \int_t^T \left( (D_r^W D_s^W Y_v D_t^W Y_v + D_s^W Y_v D_r^W D_t^W Y_v) \frac{\partial^2 \sigma}{\partial y^2}(v,Y_v) \right. \]

+ \int_t^T \frac{\partial \sigma}{\partial y}(v,Y_v) D_s^W D_t^W Y_v \frac{\partial^2 \sigma}{\partial y^2}(v,Y_v) \frac{\partial}{\partial y} \left( \sigma(v,Y_v) \frac{\partial \sigma}{\partial y}(v,Y_v) \right) \]

+ \sigma(v,Y_v) \frac{\partial \sigma}{\partial y}(v,Y_v) D_s^W D_t^W Y_v \int_t^T dv \frac{\partial^2 \sigma}{\partial y^2}(t,Y_t) D_r^W D_s^W Y_t + \frac{\partial \sigma}{\partial y}(t,Y_t) D_r^W D_s^W Y_t \quad (3.7) \]

**Proof.** By the chain rule of \( D_t^W \) and thanks to Proposition 2 we obtain

\[ D_t^W S_T = S_T \left( (r_s - \frac{\sigma^2(s,Y_s)}{2}) ds + D_t^W S_T \int_0^T \sigma(s,Y_s)dW_s \right) = S_T \left( - \int_t^T D_t^W \frac{\sigma^2(s,Y_s)}{2} ds + \int_t^T D_t^W \sigma(s,Y_s)dW_s + \sigma(t,Y_t) \right) = S_T G(t,T), \]

which gives (3.2), (3.3) and (3.4) are immediate by the chain rule of \( D^W \). Concerning (3.5) and (3.7) we have for \( 0 \leq r \leq s \leq t \leq T \)

\[ D_s^W G(t,T) = \frac{\partial \sigma}{\partial y}(t,Y_t) D_s^W Y_t + \int_t^T D_s^W Y_v \left( \frac{\partial \sigma}{\partial y}(v,Y_v) D_t^W Y_v \right) dW_v \]

\[ - \int_t^T D_s^W \left( \sigma(v,Y_v) \frac{\partial \sigma}{\partial y}(v,Y_v) D_t^W Y_v \right) dv \]

\[ = \frac{\partial \sigma}{\partial y}(t,Y_t) D_s^W Y_t + \int_t^T \left( D_s^W Y_v D_t^W Y_v \frac{\partial^2 \sigma}{\partial y^2}(v,Y_v) + \frac{\partial \sigma}{\partial y}(v,Y_v) D_s^W D_t^W Y_v \right) dW_v \]

\[ - \int_t^T \left[ D_s^W Y_v D_t^W Y_v \frac{\partial}{\partial y} \left( \sigma(v,Y_v) \frac{\partial \sigma}{\partial y}(v,Y_v) \right) + \sigma(v,Y_v) \frac{\partial \sigma}{\partial y}(v,Y_v) D_s^W D_t^W Y_v \right] dv. \]
And
\[
D^W_r D^W_s G(t, T) = \int_t^T D^W_r \left(D^W_s Y_v D^W_t Y_v \frac{\partial^2 \sigma}{\partial y^2}(v, Y_v) + \frac{\partial \sigma}{\partial y}(v, Y_v) D^W_t Y_v \right) dW_v
- \int_t^T D^W_r \left[D^W_s Y_v D^W_t Y_v \frac{\partial \sigma}{\partial y}(v, Y_v) \right] \sigma(v, Y_v) \frac{\partial \sigma}{\partial y}(v, Y_v) \right] dv
+ \sigma(v, Y_v) \frac{\partial \sigma}{\partial y}(v, Y_v) D^W_t Y_v \right] dv
+ \frac{\partial^2 \sigma}{\partial y^2}(t, Y_t) D^W_r Y_v D^W_s Y_t + \frac{\partial \sigma}{\partial y}(t, Y_t) D^W_r Y_t
\]
\[
= \int_t^T \left( D^W_r D^W_s Y_v D^W_t Y_v + D^W_r Y_v D^W_s Y_t \right) \frac{\partial^2 \sigma}{\partial y^2}(v, Y_v) \right] dW_v
+ D^W_r Y_v D^W_s Y_v \frac{\partial^3 \sigma}{\partial y^3}(v, Y_v) \right] dW_v
- \int_t^T \left[D^W_r D^W_s Y_v D^W_t Y_v + D^W_r Y_v D^W_s D^W_t Y_v \right] \frac{\partial \sigma}{\partial y}(v, Y_v) \right] dv
+ \sigma(v, Y_v) \frac{\partial \sigma}{\partial y}(v, Y_v) D^W_s Y_v D^W_t Y_v \right] dv
+ \frac{\partial^2 \sigma}{\partial y^2}(t, Y_t) D^W_r Y_v D^W_s Y_t + \frac{\partial \sigma}{\partial y}(t, Y_t) D^W_r Y_t \right] dv
+ \frac{\partial^2 \sigma}{\partial y^2}(t, Y_t) D^W_r Y_t dW_t.
\]

Concerning $D^W_t Y_v$, it can be explicitly computed when $\beta_v = \beta(v, Y_v)$. We have for $0 \leq t \leq v \leq T$
\[
D^W_t Y_v = D^W_t \left( \int_0^t \left( \mu^Y + \sigma^Y \frac{r_a - \mu_a}{\sigma(\alpha, Y_a)} + \beta(\alpha, Y_v) \sigma^Y \right) d\alpha + \int_0^t \rho \sigma^Y dW_a + \int_0^t \sigma^Y dW^\alpha \right)
= \int_t^v D^W_t \left( \mu^Y + \sigma^Y \frac{r_a - \mu_a}{\sigma(\alpha, Y_a)} + \beta(\alpha, Y_v) \sigma^Y \right) d\alpha + \int_t^v \rho \sigma^Y dW_a
= \int_t^v \sigma^Y D^W_t \left( \frac{r_a - \mu_a}{\sigma(\alpha, Y_a)} + \beta(\alpha, Y_v) \sigma^Y \right) d\alpha + \int_t^v D^W_t \rho \sigma^Y dW_a + \rho \sigma^Y
= \rho \sigma^Y - \int_t^v \sigma^Y \left( \frac{r_a - \mu_a}{\sigma(\alpha, Y_a)} - \frac{\partial \beta(\alpha, Y_a)}{\partial y} \right) D^W_t Y_v d\alpha.
\]
So for $t$ fixed in $[0, T]$, the Malliavin derivative of $Y_v$, for $v \in [t, T] : (D^W_t Y_v)_{v \in [t, T]}$, satisfies a stochastic differential equation, its solution is precisely
\[
D^W_t Y_v = \rho \sigma^Y \exp \left( - \int_t^v \left( \sigma^Y \frac{r_a - \mu_a}{\sigma^2(\alpha, Y_a)} - \frac{\partial \beta(\alpha, Y_a)}{\partial y} \right) d\alpha \right) \quad v \in [t, T].
\]
In this case for $0 \leq r \leq s \leq t \leq v \leq T$, $D^W_s D^W_t Y_v$ and $D^W_r D^W_s D^W_t Y_v$ are as follow
\[
D^W_s D^W_t Y_v = D^W_t Y_v D^W_s \left( - \int_t^v \left( \sigma^Y \frac{r_a - \mu_a}{\sigma^2(\alpha, Y_a)} - \frac{\partial \beta(\alpha, Y_a)}{\partial y} \right) d\alpha \right)
\]
Next we compute the Delta, Rho, and Vega can be computed by the same way. The Delta corresponds to $\frac{\partial C}{\partial S}$, the Rho to $\frac{\partial C}{\partial \rho}$, and the Vega to $\frac{\partial^2 C}{\partial \rho^2}$. The Gamma is computed using the second order derivative of $C = E[f(S_T^x)]$ w.r.t $\xi$. We have using Proposition 3.1

$$\frac{\partial}{\partial \xi} E[f(S_T^x)] = E\left[f(S_T^x) \left( \frac{\partial_C S_T^x}{D_u} \delta^W W + \frac{\partial^2_C S_T^x}{D_{uu}} \right)\right].$$

Next we compute the Delta, the Rho and Vega can be computed by the same way. The Delta corresponds to $\frac{\partial C}{\partial S} = \frac{\partial_x C}{S}$ and we have

$$\Delta = E\left[f(S_T) \left( \frac{\partial_C S_T}{D_u} \delta^W W - \frac{\partial^2_C S_T}{D_{uu}} \right)\right] = E\left[f(S_T) \left( \frac{S_T}{x D_u^2} \delta^W W - \frac{S_T^2}{x^2 D_u^2} \right)\right]$$

$$= \frac{1}{x} E\left[f(S_T) \left( \frac{1}{\int_0^T u(t,T)dt} \delta^W W - 2S_T + \frac{S_T^2 D_u^2 D_{uu}}{(D_u^2)^2} \right)\right]$$

$$= \frac{1}{x} E\left[f(S_T) \left( \frac{1}{\int_0^T u(t,G(t,T))dt} \delta^W W - 2S_T + \frac{S_T^2 D_u^2 D_{uu}}{(D_u^2)^2} \right)\right]$$

where $D_u^2 G(t,T)$ is given by 3.5.

### 3.2 Gamma

The Gamma is computed using the second order derivative of $C = E[f(S_T)]$ w.r.t $x$ given by

$$\frac{\partial^2}{\partial x^2} E[f(S_T)] = \Delta = \frac{1}{x} \frac{\partial}{\partial x} E[f(S_T)H]$$

$$= \frac{1}{x} E\left[f(S_T) \left( \frac{H \partial_x S_T}{D_{uu} S_T} \delta^W W - \frac{H \partial_x S_T}{D_{uu} S_T} \right)\right]$$
\[
\begin{align*}
&= \frac{1}{x} E \left[ f(S_T) \left( \frac{HS_T}{x D_u S_T} \delta^W(u) - \left( \frac{H(D_u W S_T)^2 + S_T D_u W S_T D_u W H + H S_T D_u W S_T}{x D_u S_T^2} \right) \right. \\
&\quad \left. + \partial_x H \right) \right],
\end{align*}
\]

where

\[
H = \frac{1}{\int_0^T u_s G(t, T) dt} \delta^W(u) - S_T \left( 1 - \frac{\int_0^T u_s (\int_0^T u_t D_s G(t, T) dt) ds}{(\int_0^T u_t G(t, T) dt)^2} \right)
\]

\[
D_u^W H = -\frac{\int_0^T u_s (\int_0^T u_t D_s G(t, T) dt) ds}{(\int_0^T u_t G(t, T) dt)^2} \delta^W(u)
\]

\[
- S_T \left( \int_0^T u_t G(t, T) dt - \frac{\int_0^T u_s (\int_0^T u_t D_s G(t, T) dt) ds}{\int_0^T u_t G(t, T) dt} \right)
\]

\[
\quad - \frac{\int_0^T u_r \int_0^T u_s (\int_0^T u_t D_r D_s G(t, T) dt) ds}{(\int_0^T u_t G(t, T) dt)^2} + 2 \left( \frac{\int_0^T u_s (\int_0^T u_t D_s G(t, T) dt) ds}{(\int_0^T u_t G(t, T) dt)^3} \right)^2
\]

\[
\partial_x H = -\frac{1}{x} S_T \left( 1 - \frac{\int_0^T u_s (\int_0^T u_t D_s G(t, T) dt) ds}{(\int_0^T u_t G(t, T) dt)^2} \right),
\]

\[D_s^W G(t, T)\) and \(D_r^W D_s^W G(t, T)\) are given by formulas (3.5) and (3.7).

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