Neck Pinching Dynamics Under Mean Curvature Flow

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Abstract

In this paper we study motion of surfaces of revolution under the mean curvature flow. For an open set of initial conditions close to cylindrical surfaces we show that the solution forms a “neck” which pinches in a finite time at a single point. We also obtain a detailed description of the neck pinching process.

1 Introduction

In this paper we study motion of surfaces of revolution under the mean curvature flow. The mean curvature flow of an initial hypersurface $M_0 \in \mathbb{R}^{d+1}$ parameterized by $\psi_0 : U \to M_0$ is a family of hypersurfaces $M_t \in \mathbb{R}^{d+1}$ whose local parametrizations $\psi(\cdot, t) : U \to \mathbb{R}^{d+1}$ satisfy the partial differential equation

$$\partial_t \psi(z, t) = -h(\psi(z, t))$$

where $h(y)$ is the mean curvature vector of $M_t$ at a point $y \in M_t$, with the initial condition

$$\psi(z, 0) = \psi_0(z).$$

If $d \geq 2$ and an initial surface $M_0$ is a surface of revolution around the axis $x = x_{d+1}$, given by a map $r = u_0(x)$ where $r = \left(\sum_{j=1}^{d} x_j^2\right)^{\frac{1}{2}}$, then the surface $M_t$ is also a surface of revolution and, as long as it is smooth, it is defined by the map $r = u(x, t)$ which satisfies the partial differential equation

$$\frac{\partial_t u}{u(x, 0)} = \frac{\partial^2 u}{1+(\partial_x u)^2} - \frac{d-1}{u}$$  \hspace{1cm} (1)

This equation follows from the mean curvature equation above by a standard computation.

The initial conditions for (1) can be divided into two basic groups. In the first group, $u_0(x) > 0$ for $a < x < b$ and either $u_0(a) = u_0(b) = 0$ or $\partial_x u_0(a) = \partial_x u_0(b) = 0$, for some $-\infty < a < b < \infty$. In the second

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group, \( u_0(x) > 0 \) \( \forall x \in \mathbb{R} \) and \( \liminf_{|x| \to \infty} u_0(x) > 0 \). In the first case we deal with compact or periodic initial surfaces and Eqn (1) is considered on the bounded interval \([a, b]\) with the Dirichlet or Neumann boundary conditions. In the second case, the initial surface as well as solution surfaces are noncompact and Equation (1) should be considered on \( \mathbb{R} \). In this paper we study the second, more difficult case and consequently we consider Eqn (1) on \( \mathbb{R} \). Our goal is to describe the phenomenon of collapse or neckpinching of such surfaces. We say \( u(x, t) \) collapses at time \( t^* \) if \( \| \frac{1}{u(x, t)} \|_\infty < \infty \) for \( t < t^* \) and \( \| \frac{1}{u(x, t)} \|_\infty \to \infty \) as \( t \to t^* \).

The study of the mean curvature flow goes back at least to the work of Brakke [8]. The short time existence in \( L^\infty \) was proved in [8, 20, 14, 26]. In [20, 21] Huisken has shown that compact convex surfaces shrink under the mean curvature flow into a point approaching spheres asymptotically. In some of the first works on collapse Grayson [13] and Ecker [14] have constructed rotationally symmetric barriers which can be used to determine a class of 2-dimensional hypersurfaces of barbell shape which develop a singularity under the mean curvature flow before they shrink to a point.

Huisken [22] showed that periodic rotationally symmetric, positive mean curvature surfaces of the barbell shapes always develop singularities in finite time \( t^* \), and that their blow-up at a point, where the maximal curvature blows up, converges to a cylinder of unit radius. (No information on the set of blow-up points was given.) These results were generalized to higher dimensions in [28].

Dziuk and Kawohl [12] showed that periodic surfaces of revolution of positive mean curvature which have one minimum per period and satisfy certain monotonicity conditions, including one on the derivative of curvature, pinch at exactly the point of minimum.

H.M.Soner and P.E.Souganidis [30] considered Equation (1) on a bounded, symmetric interval and showed that if \( u(x, t) \) is even and satisfies \( x \partial_x u(x, t) \geq 0 \) (i.e. \( u \) has a single minimum at \( x = 0 \)), then, along a subsequence,

\[
(t^* - t) - \frac{1}{2} u((t^* - t)^{-\frac{1}{2}} y, t) \to \sqrt{2(d - 1)},
\]

as \( t \to t^* \) (a compactness result). Smoczyk [29] showed pinching of certain periodic rotationally symmetric surfaces with the mean curvatures greater than 2, which are embedded in Euclidean space.

S.Altschuler, S.B.Angenent and Y.Giga [2] have showed that any compact, connected, rotationally symmetric hypersurface that pinches under the mean curvature flow does so at finitely many discrete points.

A collapsing solution is called of type I if the square root, \(|A|\), of the sum of squares of principal curvatures is bounded as \(|A| \leq C(t^* - t)^{-\frac{1}{2}}\). Otherwise, it is called of type II (see Huisken [24]). It was conjectured that the generic collapse is of type I. Indeed, all collapses investigated in the papers above are of type I.

S.B.Angenent and J.J.L.Velázquez [5] have constructed non-generic, type II solutions, first suggested by R.Hamilton and investigated by the level-set methods of Evans and Spruck and Chen, Giga and Goto in [14, 10, 2], neckpinching at \( x = 0 \) at a prescribed time \( t^* \). Their solutions have the asymptotics, as \( t \to t^* \),

\[
(t^* - t)^{-\frac{1}{2}} u((t^* - t)^{-\frac{1}{2}} y, t) = \sqrt{2(d - 1)} + (t^* - t)^{\frac{m}{2} - 1} \frac{K}{2\sqrt{2(d - 1)} H_m(y)} + o((t^* - t)^{\frac{m}{2} - 1})
\]

where \( K > 0 \), \( m \) is an odd integer \( \geq 3 \) and \( H_m(y) \) is a multiple of the \( m \)th Hermite polynomial.

Athanassenas [6, 7] has shown neckpinching of certain class of rotationally symmetric surfaces under the volume preserving modification of the mean curvature flow. The latter was also studied in Alikakos and Frere [1].

For other related works we refer to [15, 23, 31, 25, 7].

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Most of these works rely on parabolic maximum principle going back to Hamilton [19] and monotonicity formulae for an entropy functional (Huisken [22], Giga and Kohn [17]).

The scaling and asymptotics in (2) originate in the following key properties of (1):

1. (1) is invariant with respect to the scaling transformation,

\[ u(x, t) \rightarrow \lambda u(\lambda^{-1}x, \lambda^{-2}t) \]

for any constant \( \lambda > 0 \), i.e. if \( u(x, t) \) is a solution, then so is \( \lambda u(\lambda^{-1}x, \lambda^{-2}t) \).

2. (1) has \( x \)-independent (cylindrical) solutions:

\[ u_{cyl} = \left[ u_0^2 - 2(d - 1)t \right]^{\frac{1}{2}}. \]

These solutions collapse in finite time \( t^* = \frac{1}{2(d-1)} u_0^2 \).

In this paper we consider Equation (1) with initial conditions which are positive, have, modulo small perturbations, global minima at the origin, are slowly varying near the origin and are even. The latter condition is of a purely technical nature and will be addressed elsewhere. We show that for such initial conditions the solutions collapse in a finite time and we characterize asymptotic dynamics of the collapse.

As it turns out, the leading term is given by the expression

\[ u(x, t) = \lambda(t) \left[ \frac{2(d - 1) + b(t)\lambda^{-2}(t)x^2}{c(t)} \right]^{\frac{1}{2}} + \zeta(x, t) \]

with the parameters \( \lambda(t), b(t) \) and \( c(t) \) satisfying the estimates

\[ \lambda(t) = (t^* - t)^{\frac{1}{2}} (1 + o(1)); \]
\[ b(t) = -\frac{d-1}{m(t-1)}(1 + O(\frac{1}{m(t-1)})); \]
\[ c(t) = 1 + \frac{1}{m(t-1)}(1 + O(\frac{1}{m(t-1)})). \]

Here \( \lambda_0 = \frac{1}{\sqrt{2\varsigma_0 + \varepsilon_0}} \) with \( \varsigma_0, \varepsilon_0 > 0 \) depending on the initial datum and \( o(1) \) is in \( t^* - t \). Moreover, we estimate the remainder \( \zeta(x, t) \) as

\[ \sum_{m+n=3, n \leq 2} \| (\lambda^{-1}(t)x)^{-m} \partial_x^n \zeta(x, t) \|_\infty \leq cb^2(t) \]

for some constant \( c \).

To give more precise formulation of results we introduce some notation. Let \( L^\infty \) denote the space \( L^\infty(\mathbb{R}) \) with the standard norm \( \| u \|_\infty = \sup_x |u(x)| \). To formulate our main result we define the spaces \( L^\infty_{m,n} \) with the norm

\[ \| u \|_{m,n} = \| (x)^{-m} \partial_x^n u(x) \|_\infty \]

and define the function \( g(x, b), \ b > 0 \), as

\[ g(x, b) := \begin{cases} \frac{2}{10} \sqrt{2(d - 1)} & \text{if } bx^2 < 20(d - 1) \\ \frac{4}{4\sqrt{d - 1}} & \text{if } bx^2 \geq 20(d - 1) \end{cases}. \]

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We will also deal, without specifying it, with weak solutions of Equation (11) in some appropriate sense (see the next section for more precise formulation). These solutions can be shown to be classical for \( t > 0 \). The following is the main result of our paper.

**Theorem 1.1.** Assume the initial datum \( u_0(x) \) in (11) is even and satisfy for \((m,n) = (3,0), (4,0), (1,2)\) and \((2,1)\) the estimates

\[
\|u_0(x) - \left( \frac{2(d-1)+c_0 x^2}{2c_0} \right) \|_{m,n} \leq C \varepsilon_{0}^{\frac{m+n+1}{2}},
\]

\[
u_0(x) \geq \frac{1}{{\sqrt{2c_0 + d - 1}}}, \quad g(\sqrt{2c_0 + \frac{c_0}{d-1} x}, \frac{c_0}{2c_0}),
\]

\( \langle x \rangle^{-1} u_0 \in L^\infty, \partial_x u_0 \in L^\infty, |\partial_x u_0| \leq \kappa_0 \varepsilon_0^{\frac{1}{2}}, |\partial^n_x u_0| \leq \kappa_0 \varepsilon_0^{n/2}, n = 2,3,4, \) for some \( C, \kappa_0 \geq 2, \) and \( \frac{1}{2} \leq \varepsilon_0 \leq 2. \) There exists a constant \( \delta \) such that if \( \varepsilon_0 \leq \delta, \) then

(i) there exists a finite time \( t^* \) such that \( \frac{1}{u(t,x)} \|_{\infty} < \infty \) for \( t < t^* \) and \( \lim_{t \to t^*} \frac{1}{u(t,x)} \|_{\infty} \to \infty; \)

(ii) there exist \( C^1 \) functions \( \zeta(x,t), \lambda(t), c(t) \) and \( b(t) \) such that (15) and (7) hold;

(iii) the parameters \( \lambda(t), b(t) \) and \( c(t) \) satisfy the estimates (13);

(iv) if \( u_0 \partial_z^2 u_0 \geq -1 \) then there exists a function \( u_\ast(x) > 0 \) such that \( u(x,t) \geq u_\ast(x) \) for \( \mathbb{R} \setminus \{0\} \) and \( t \leq t^*. \)

Moreover, if the mean curvature of the initial surface is non-negative, i.e., if \( \frac{\partial^2 u_0}{\partial u_0} \leq \frac{-1}{u_0} \leq 0, \) then for any \( x, \lim_{t \to t^*} u(x,t) \) exists and is positive \( \forall x \neq 0. \)

Thus our main new results are

1) Proof of neckpinching and neckpinching asymptotics at a single given point for a new open set of initial conditions, which includes in particular surfaces whose mean curvature changes sign and which might have many necks,

2) Determination of the subleading term in the asymptotic and estimation of the remainder.

**Remarks**

1) A result similar to (iv) but for a different set of initial conditions (see above) was proven in H.M.Soner and P.E.Souganidis [30];

2) One can compute more precise asymptotics of the parameters \( \lambda(t), b(t) \) and \( c(t); \)

3) It is not hard to show using \( \langle 1 \rangle \) that the collapse in our case is of type I.

The previous result closest to our result is that by Angenent and Knopf [4, 8] on the neckpinching for the Ricci flow of \( SO(n + 1) \) invariant metrics on \( S^{n+1}. \)

Our techniques are different from those in the papers mentioned above. They rely to much lesser degree on the maximum principle and they do not use entropy monotonicity formulae. Our main point is that we do not fix the time-dependent scale in the self-similarity (collapse) variables but let its behaviour, as well as behaviour of other parameters \( (b \) and \( c), \) be determined by the original equation. Then we use a nonlinear
Lyapunov-Schmidt decomposition (the modulation method) and the method of majorants together with powerful linear estimates. We expect that our techniques can be extended to non-axisymmetric surfaces and to Ricci flows.

This paper is organized as follows. In Section 2 we prove the local well-posedness of Equation (1) in the space \( \langle x \rangle L^\infty \) which is used in this paper. In Sections 3-5 we present some preliminary derivations and some motivations for our analysis. In Section 6, we formulate a priori bounds on solutions to (1). In Section 8 we use these bounds and a lower bound proved in Section 7 to prove our main result, Theorem 1.1. A priori bounds of Section 6 are proved in Sections 10-15.

For any functions \( A \) and \( B \) we use the notation \( A \lesssim B \) to signify that there is a universal constant \( c \) such that \( A \leq cB \).

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2 Local Well-posedness of (1)

In this section we prove the local well posedness of (1) in the space adapted to our needs. The result below is standard (cf \([14, 26, 8]\)).

**Theorem 2.1.** If \( u_0(x) \in \langle x \rangle L^\infty \) and \( u_0(x) \geq C_0 \) for some \( C_0 > 0 \) and \( \partial^n_x u_0 \in L^\infty \), \( n = 1, 2, 3, 4 \) then there exists a time \( T = T(C_0, u_0) \) such that for any time \( 0 \leq t \leq T \), (1) has a unique solution \( u(\cdot, t) \in \langle x \rangle L^\infty \) with \( u(x, 0) = u_0(x) \), \( u(\cdot, t) \geq C_0 \) and \( \partial^n_x u(\cdot, t) \in L^\infty \), \( n = 1, 2, 3, 4 \). Moreover, if \( t_* \) is the supremum of such \( T(k_0, u_0) \) then either \( t_* = \infty \) or \( \| \partial^n_x u(\cdot, t) \|_\infty \to \infty \) as \( t \to t_* \).

**Proof.** First we consider the equation

\[
\begin{align*}
\partial_t u_1 &= g_1(\partial_x u_1) \partial^2_x u_1 - (d - 1)g_2(u_1)u_1 \\
u_1(x, 0) &= u_0(x)
\end{align*}
\]  

(10)

where \( g_1 \) and \( g_2 \) are strictly positive and smooth functions satisfying the conditions

\[
g_1(s) := \begin{cases} 
\frac{1}{1+s^2} & \text{if } s \leq 10\|\partial_x u_0\|_\infty, \\
1 & \text{if } s \geq 20\|\partial_x u_0\|_\infty,
\end{cases}
\]

and

\[
g_2(s) := \begin{cases} 
\frac{1}{s^2} & \text{if } s \geq \frac{C_0}{20}, \\
1 & \text{if } s \leq \frac{C_0}{20}.
\end{cases}
\]

By standard results (see [27]) there exists a time \( T > 0 \) such that (10) has a unique solution \( u_1(x, t) \) in the time interval \( t \in [0, T] \) such that \( u_1 \geq \frac{1}{2}C_0 \) and \( \partial^n_x u_1(\cdot, t) \in L^\infty \), \( n = 1, 2, 3, 4 \), and \( \| \partial^n_x u_1(\cdot, t) \|_\infty \leq 2\|\partial_x u_0\|_\infty \).
Moreover by the definition of \( g_1 \) and \( g_2 \) we have that for this solution

\[
g_1(\partial_x u_1) = \frac{1}{1 + (\partial_x u_1)^2}, \quad g_2(u_1) = \frac{1}{u_1^3},
\]

in the time interval \( t \in [0, T] \). Thus \( u(x, t) = u_1(x, t) \) is a solution to (1) and satisfies all the conditions of the theorem.

3 Collapse Variables and Almost Solutions

In this section we pass from the original variables \( x \) and \( t \) to the collapse variables \( y := \lambda^{-1}(t)(x - x_0(t)) \) and \( \tau := \int_0^t \lambda^{-2}(s) \, ds \). The point here is that we do not fix \( \lambda(t) \) and \( x_0(t) \) but consider them as free parameters to be found from the evolution of (1). Suppose \( u(x, t) \) is a solution to (1) with an initial condition \( u_0(x) \), which has a minima at \( x = 0 \) and is even with respect to \( x = 0 \). We define the new unknown function

\[
v(y, \tau) := \lambda^{-1}(t)u(x, t)
\]

with \( y := \lambda^{-1}(t)x \) and \( \tau := \int_0^t \lambda^{-2}(s) \, ds \). The function \( v \) satisfies the equation

\[
\partial_\tau v = \left( \frac{1}{1 + (\partial_y v)^2} \partial_y^2 - ay\partial_y + a \right)v - \frac{d - 1}{v},
\]

where \( a := -\lambda \partial_t \lambda \). The initial condition is \( v(y, 0) = \lambda_0^{-1}u_0(\lambda_0 y) \), where \( \lambda_0 \) is the initial condition for the scaling parameter \( \lambda \).

If \( \lambda_0 = 1 \), then the initial conditions for \( u \) given in Theorem [11] implies that there exists a constant \( \delta \) such that the initial condition \( v_0(y) \) is even and satisfy for \( (m, n) = (3, 0), (\frac{11}{10}, 0), (1, 2) \) and \( (2, 1) \) the estimates

\[
\|v_0(y) - \left( \frac{2(d-1)+\varepsilon_0 y^2}{1-4\varepsilon_0} \right)^{\frac{1}{2}} \|_{m, n} \leq C\varepsilon_0^{\frac{m+n+1}{2}},
\]

\[
v_0(y) \geq g(y, \varepsilon_0),
\]

\[
\langle y \rangle^{-1}v_0 \in L^\infty, \partial_y v_0 \in L^\infty, |\partial_y v_0 v_0^{-\frac{1}{2}}| \leq \kappa_0 \varepsilon_0^{\frac{1}{2}}, |\partial_y^n v_0| \leq \kappa_0 \varepsilon_0^{\frac{n}{2}}, \text{ for some } C, \kappa_0 \geq 2, \frac{1}{2} \leq \varepsilon_0 < 2 \text{ and } \varepsilon_0 \leq \delta.
\]

If the parameter \( a \) is a constant, then (12) has the following cylindrical, static (i.e. \( y \) and \( \tau \)-independent) solution

\[
v_a := \left( \frac{d-1}{a} \right)^{\frac{1}{2}}.
\]

In the original variables \( t \) and \( x \), this family of solutions corresponds to the homogeneous solution \( \# \) of (1) with the parabolic scaling \( \lambda^2 = 2a(T - t) \), where the collapse time, \( T := \frac{u_0^2}{2(d-1)} \), is determined by \( u_0 \), the initial value of the homogeneous solution \( u_{hom}(t) \).

If the parameter \( a \) is \( \tau \)-dependent but \( |a_\tau| \) is small, then the above solutions are good approximations to the exact solutions. A larger family of approximate solution is obtained by solving the equation \( ayv_y - \)
\[ av + \frac{d-1}{v} = 0, \] which is derived from (12) by neglecting the \( \tau \) derivative and second order derivative in \( y \) (adiabatic, slowly varying approximation). This equation has the general solution

\[ v_{bc} := \left( \frac{2(d-1) + by^2}{c} \right)^{1/2} \] (15)

for all \( b \in \mathbb{R} \) and for \( c = 2a \). In what follows we take \( b \geq 0 \) so that \( v_{bc} \) is smooth. Note that \( v_{0,2a} = v_a \). Since \( v_{bc}, c = 2a \), is not an exact solution to (12) we should leave the parameter \( c \) free, to be determined by the best overall approximation. Jumping ahead, it turns out that a convenient choice of \( c \) is

\[ c = a + \frac{1}{2}. \]

Thus we introduce

\[ V_{ab}(y) := v_{b,a+\frac{1}{2}}(y) = \left( \frac{2(d-1)+by^2}{a+\frac{1}{2}} \right)^{1/2}. \]

Note that this function is not a solution to the equation

\[ ay\partial_y v - av + \frac{d-1}{v} = 0. \]

4 “Gauge” Transform

In order to convert the non-self-adjoint linear part of Equation (12) into a more tractable self-adjoint one we perform a gauge transform. Let

\[ w(y, \tau) := \exp -\frac{a}{4} y^2 v(y, \tau). \] (16)

Then \( w \) satisfies the equation

\[ \partial_\tau w = \left( \frac{1}{1 + (\partial_y v)^2} \partial_y^2 - \omega^2 \frac{3}{4} \frac{y^2}{2} \right) w - \exp -\frac{a}{4} y^2 \frac{d-1}{w}, \] (17)

where \( \omega^2 = a^2 + a_\tau \). The approximate solution \( v_{bc} \) to (12) transforms to \( v_{abc} \) where

\[ v_{abc} := v_{bc} \exp -\frac{a}{4} y^2, \]

explicitly

\[ v_{abc} := \left( \frac{2(d-1) + by^2}{c} \right)^{1/2} \exp -\frac{a}{4} y^2. \] (18)

As was permitted above we will choose \( c = a + \frac{1}{2} \).

Note that the linear part of Equation (17) is self-adjoint in the space \( L^2(\mathbb{R}, dy) \). Hence it is natural to consider the linear part of Equation (12) in the space \( L^2(\mathbb{R}, e^{-\frac{a}{4} y^2} dy) \).

5 Reparametrization of Solutions

In this section we split solutions to Equation (12) into the leading term - the almost solution \( V_{ab} \) - and a fluctuation \( \eta \) around it. More precisely, we would like to parametrize a solution by a point on the manifold \( M_{as} := \{ V_{ab} \mid a, b \in \mathbb{R}_+, b \leq \epsilon \} \) of almost solutions and the fluctuation orthogonal to this manifold (large slow moving and small fast moving parts of the solution). We equip \( M_{as} \) with the Riemannian metric

\[ \langle \eta, \eta' \rangle := \int \eta \eta' e^{-\frac{a}{4} y^2} dy. \] (19)

For technical reasons, it is more convenient to require the fluctuation to be almost orthogonal to the manifold \( M_{as} \). More precisely, we require \( \eta \) to be orthogonal to the vectors 1 and \( (1 - ay^2) \) which are almost tangent
vectors to the above manifold, provided \( b \) is sufficiently small. Note that \( \eta \) is already orthogonal to \( \sqrt{ay} \) since our initial conditions, and therefore, the solutions are even in \( x \).

The next result will give a convenient reparametrization of the initial condition \( v_0(y) := \lambda_0^{-1}u_0(\lambda_0y) \). Recall the definition \( V_{ab} := v_{bc}|_{c=a+\frac{1}{2}} = \left(\frac{2(1+d)+by}{a+\frac{1}{2}}\right)\frac{1}{2} \). We define a neighborhood:

\[
U_{\epsilon} := \{ v \in \langle y \rangle^3 L^\infty(\mathbb{R}) \mid \| v - V_{ab} \|_{3,0} \ll b \text{ for some } a \in [1/4,1], \ b \in (0, \epsilon_0) \}.
\]

**Proposition 5.1.** There exist an \( \epsilon_0 > 0 \) and a unique \( C^1 \) functional \( g : U_{\epsilon_0} \to \mathbb{R}^+ \times \mathbb{R}^+ \), such that any function \( v \in U_{\epsilon_0} \) can be uniquely written in the form

\[
v = V_{g(v)} + \eta, \tag{20}
\]

with \( \eta \perp 1 - ay^2 \) in \( L^2(\mathbb{R}, e^{-\frac{ax^2}{2}}dy) \), \( (a, b) = g(v) \). Moreover, if \((a_0, b_0) \in [1/4, 1] \times (0, \epsilon_0) \) and \( \| v - V_{a_0,b_0} \|_{3,0} \ll b_0 \), then

\[
|g(v) - (a_0, b_0)| \lesssim \| v - V_{a_0,b_0} \|_{3,0}. \tag{21}
\]

**Proof.** Let \( X := \langle y \rangle^3 L^\infty \) with the corresponding norm. The orthogonality conditions on the fluctuation can be written as \( G(\mu, v) = 0 \), where \( \mu = (a, b) \) and \( G : \mathbb{R}^+ \times \mathbb{R}^+ \times X \to \mathbb{R}^2 \) is defined as (we use the Riemannian metric \([19]\))

\[
G(\mu, v) := \left( \begin{array}{c}
\langle V_{\mu} - v, 1 \rangle \\
\langle V_{\mu} - v, 1 - ay^2 \rangle
\end{array} \right).
\]

Using the implicit function theorem we will prove that for any \( \mu_0 := (a_0, b_0) \in [1/4, 1] \times (0, \epsilon_0) \) there exists a unique \( C^1 \) function \( g : U_{\mu_0} \to \mathbb{R}^+ \times \mathbb{R}^+ \) defined in a neighborhood \( U_{\mu_0} \subset X \) of \( V_{\mu_0} \) such that \( G(g(v), v) = 0 \) for all \( v \in U_{\mu_0} \).

Note first that the mapping \( G \) is \( C^1 \) and \( G(\mu_0, V_{\mu_0}) = 0 \) for all \( \mu_0 \). We claim that the linear map \( \partial_{\mu} G(\mu_0, V_{\mu_0}) \) is invertible. Indeed, let \( B_3(V_{\mu_0}) \) and \( B_3(\mu_0) \) be the balls in \( X \) and \( \mathbb{R}^2 \) around \( V_{\mu_0} \) and \( \mu_0 \) and of the radii \( \epsilon \) and \( \delta \), respectively. We compute

\[
\partial_{\mu} G(\mu, v) = A_1(\mu) + A_2(\mu, v) \tag{22}
\]

where

\[
A_1(\mu) := \left( \begin{array}{cc}
\langle \partial_a V_{\mu}, 1 \rangle & \langle \partial_b V_{\mu}, 1 \rangle \\
\langle \partial_a V_{\mu}, 1 - ay^2 \rangle & \langle \partial_b V_{\mu}, 1 - ay^2 \rangle
\end{array} \right)
\]

and

\[
A_2(\mu, v) := -\frac{1}{4} \left( \begin{array}{cc}
\langle V_{\mu} - v, y^2 \rangle & 0 \\
\langle V_{\mu} - v, (1 - ay^2) y^2 \rangle & 0
\end{array} \right).
\]

For \( b > 0 \) and small, we expand the matrix \( A_1 \) in \( b \) to get \( A_1 = G_1 + O(b) \), where the matrices \( G_1 \) is defined as

\[
G_1 := \left( \begin{array}{cc}
-\frac{1}{2}(a + \frac{1}{2})^{-3/2} \langle 1, 1 \rangle & \frac{1}{2(a + \frac{1}{2})} \langle y^2, 1 \rangle \\
-\frac{1}{2}(a + \frac{1}{2})^{-3/2} \langle 1, 1 - ay^2 \rangle & \frac{1}{2(a + \frac{1}{2})} \langle y^2, 1 - ay^2 \rangle
\end{array} \right).
\]

Obviously the matrices \( G_1 \) has uniformly (in \( a \in [1/4, 1] \)) bounded inverses. Furthermore, by the Schwarz inequality

\[
\| A_2(\mu, v) \| \lesssim \| v - V_{ab} \|_X.
\]
Therefore there exist $\epsilon_0$ and $\epsilon_1$ s.t. the matrix $\partial_\mu G(\mu, v)$ has a uniformly bounded inverse for any $v \in B_{\epsilon_1}(V_\mu)$ and $\mu \in \left[\frac{1}{4}, 1\right] \times (0, \epsilon_0]$. Hence by the implicit function theorem, the equation $G(\mu, v) = 0$ has a unique solution $\mu = g(v)$ on a neighborhood of every $v, \mu \in \left[\frac{1}{4}, 1\right] \times (0, \epsilon_0]$, which is $C^1$ in $v$. Our next goal is to determine these neighborhoods.

To determine a domain of the function $\mu = g(v)$, we examine closely a proof of the implicit function theorem. Proceeding in a standard way, we expand the function $G(\mu, v)$ in $\mu$ around $\mu_0$:

$$G(\mu, v) = G(\mu_0, v) + \partial_\mu G(\mu_0, v)(\mu - \mu_0) + R(\mu, v),$$

where $R(\mu, v) = O\left(|\mu - \mu_0|^2\right)$ uniformly in $v \in X$. Here $|\mu|^2 = |a|^2 + |b|^2$ for $\mu = (a, b)$. Inserting this into the equation $G(\mu, v) = 0$ and inverting the matrix $\partial_\mu G(\mu_0, v)$, we arrive at the fixed point problem

$$\alpha = \Phi_\epsilon(\alpha),$$

where $\alpha := \mu - \mu_0$ and $\Phi_\epsilon(\alpha) := -\partial_\alpha G(\mu_0, v)^{-1}[G(\mu_0, v) + R(\mu, v)]$. By the above estimates there exists an $\epsilon_1$ such that the matrix $\partial_\mu G(\mu_0, v)^{-1}$ is bounded uniformly in $v \in B_{\epsilon_1}(V_{\mu_0})$. Hence we obtain from the remainder estimate above that

$$|\Phi_\epsilon(\alpha)| \leq |G(\mu_0, v)| + |\alpha|^2. \tag{23}$$

Furthermore, using that $\partial_\mu \Phi_\epsilon(\alpha) = -\partial_\alpha G(\mu_0, v)^{-1}[G(\mu, v) - G(\mu_0, v) + R(\mu, v)]$ we obtain that there exist $\epsilon \leq \epsilon_1$ and $\delta$ such that $||\partial_\mu \Phi_\epsilon(\alpha)|| \leq \frac{1}{2}$ for all $v \in B_\delta(V_{\mu_0})$ and $\alpha \in B_\delta(0)$. Let $\mu_0 = (a_0, b_0)$. Pick $\epsilon$ and $\delta$ so that $\epsilon \leq \delta \leq \min(b_0, \epsilon_1) \leq 1$. Then, for all $v \in B_\epsilon(V_{\mu_0})$, $\phi_\epsilon$ is a contraction on the ball $B_\delta(0)$ and consequently has a unique fixed point in this ball. This gives a $C^1$ function $\mu = g(v)$ on $B_\epsilon(V_{\mu_0})$ satisfying $|\mu - \mu_0| \leq \delta$. An important point here is that since $\epsilon \leq b_0$ we have that $b > 0$ for all $V_{ab} \in B_\epsilon(V_{\mu_0})$ (we use here that $|b' - b| \leq \frac{1}{2}\|g\|^{-3}(V_{a'b'} - V_{ab})\|\infty$). Now, clearly, the balls $B_\epsilon(V_{\mu_0})$ with $\mu_0 \in \left[\frac{1}{4}, 1\right] \times (0, \epsilon_0]$ cover the neighbourhood $U_{\epsilon_0}$. Hence, the map $g$ is defined on $U_{\epsilon_0}$ and is unique, which implies the first part of the proposition.

Now we prove the second part of the proposition. The definition of the function $G(\mu, v)$ implies $G(\mu_0, v) = G(\mu_0, v - V_{\mu_0})$ and

$$|G(\mu_0, v)| \leq \|g\|^{-3}(v - V_{\mu_0})\|\infty. \tag{24}$$

This inequality together with the estimate (23) and the fixed point equation $\alpha = \Phi_\epsilon(\alpha)$, where $\alpha = \mu - \mu_0$ and $\mu = g(v)$, implies $|\alpha| \leq \|g\|^{-3}(v - V_{\mu_0})\|\infty + |\alpha|^2$ which, in turn, yields (21).

**Proposition 5.2.** In the notation of Proposition 5.1, if $\|v - V_{\mu_0}\|_{m, n} \lesssim b_0^{\frac{m+n+1}{2}}$ where $b_0 > 0$ is small and $(m, n) = (3, 0), (\frac{1}{16}, 0), (2, 1), (1, 2)$, then

$$|g(v) - \mu_0| \lesssim |v - V_{\mu_0}|_{3, 0}; \tag{25}$$

$$\|v - g(v)\|_{3, 0} \lesssim |v - V_{\mu_0}|_{3, 0}; \tag{26}$$

$$\|v - g(v)\|_{m', n'} \lesssim b_0^{\frac{m'+n'+1}{2}} \tag{27}$$

with $(m', n') = (\frac{1}{16}, 0), (1, 2), (2, 1)$.

**Proof.** Equation (21) implies (25) with $\mu_0 = (a_0, b_0)$. Moreover we observe

$$\|v - V_{g(v)}\|_{3, 0} \leq \|v - V_{\mu_0}\|_{3, 0} + \|V_{g(v)} - V_{\mu_0}\|_{3, 0} \lesssim |v - V_{\mu_0}|_{3, 0} + |\mu_0 - g(v)|$$

\[\forall v \in V_{\mu_0}\]
which is (26).

For Equation (27) we only prove the case \((m', n') = \left(\frac{11}{10}, 0\right)\), the other cases are proved similarly. We write
\[
\|v - V_{g(v)}\|_{0, 0} \leq \|v - V_{\mu_0}\|_{0, 0} + \|V_{g(v)} - V_{\mu_0}\|_{0, 0}.
\]
By the definition of \(V_{a, \delta}\) we have
\[
\|V_{g(v)} - V_{\mu_0}\|_{0, 0} \lesssim |g(v) - \mu_0| b_0^{-\frac{2}{10}}.
\]
This together with (25) implies
\[
\|V_{g(v)} - V_{\mu_0}\|_{0, 0} \lesssim b_0^{\frac{2}{10}}.
\]
Using \(\|v - V_{\mu_0}\|_{0, 0} \lesssim b_0^{-\frac{2}{10}}\) we complete the proof of (27) for \((m', n') = \left(\frac{11}{10}, 0\right)\).

Now we establish a reparametrization of solution \(u(x, t)\) on small time intervals. In Section 5 we convert this result into a global reparametrization. In the rest of the section it is convenient to work with the original time \(t\), instead of rescaled time \(\tau\). We denote \(I_{t_0, \delta} := [t_0, t_0 + \delta]\) and define for any time \(t_0\) and constant \(\delta > 0\) two sets:
\[
\mathcal{A}_{t_0, \delta} := C^1(I_{t_0, \delta}, \frac{1}{4}, 1]) \quad \text{and} \quad \mathcal{B}_{t_0, \delta, \epsilon_0} := C^1(I_{t_0, \delta}, (0, \epsilon_0])
\]
where, recall, the constant \(\epsilon_0\) is the same as in Proposition 5.1.

Denote \(u_\lambda(y, t) := \lambda^{-1}(t)u(\lambda(t)y, t)\). Suppose \(u(\cdot, t)\) is a function such that for some \(\lambda_0 > 0\)
\[
\sup_{t \in I_{t_0, \delta}} b^{-1}(t)\|u_\lambda(\cdot, t) - V_{a(t), \delta(t)}\|_{3, 0} \ll 1
\]
for some \(a \in \mathcal{A}_{t_0, \delta}, b \in \mathcal{B}_{t_0, \delta, \epsilon_0}\), and \(\lambda(t)\) satisfying \(\lambda(t_0) = \lambda_0\) and \(- \lambda(t) \partial_t \lambda(t) = a(t)\). We define the set
\[
\mathcal{U}_{t_0, \delta, \epsilon_0, \lambda_0} := \{u \in C^1(I_{t_0, \delta}, (y)^3 L^\infty) \mid (28) \text{ holds for some } a(t), b(t)\}.
\]

**Proposition 5.3.** Suppose \(u \in \mathcal{U}_{t_0, \delta, \epsilon_0, \lambda_0}\) and \(\lambda_0^2 \delta \ll 1\). Then there exists a unique \(C^1\) map \(g_\# : \mathcal{U}_{t_0, \delta, \epsilon_0, \lambda_0} \to \mathcal{A}_{t_0, \delta} \times \mathcal{B}_{t_0, \delta, \epsilon_0}\), such that for \(t \in I_{t_0, \delta}\), \(u(\cdot, t)\) can be uniquely represented in the form
\[
u_\lambda(y, t) = V_{g_\#(u)(t)}(y) + \phi(y, \tau(t)),
\]
with \(\tau(t) := \int_0^t \lambda^{-2}(t)dt, (a(t), b(t)) = g_\#(u)(t)\) and
\[
\phi(\cdot, \tau(t)) \perp 1, a(t)y^2 - 1 \text{ in } L^2(\mathbb{R}, e^{-\frac{4m^2}{\lambda_0^2}}dy), \lambda(t_0) = \lambda_0 \text{ and } - \lambda(t) \partial_t \lambda(t) = a(t).
\]

**Proof.** Recall the definition \(X := (y)^3 L^\infty\) with the corresponding norm. For any function \(a \in \mathcal{A}_{t_0, \delta}\), we define a function
\[
\lambda(a, t) := (\lambda_0^2 - 2 \int_{t_0}^t a(s)ds)^{\frac{1}{2}}.
\]
Let \(\lambda(a)(t) := \lambda(a, t)\). Define the \(C^1\) map \(G_\# : C^1(I_{t_0, \delta}, \mathbb{R}^+) \times C^1(I_{t_0, \delta}, \mathbb{R}^+) \times C^1(I_{t_0, \delta}, X) \to C^1(I_{t_0, \delta}, \mathbb{R}) \times C^1(I_{t_0, \delta}, \mathbb{R})\) as
\[
G_\#(\mu, u)(t) := G(\mu(t), u_{\lambda(a)}(\cdot, t)),
\]
where \( t \in I_{t_0,\delta}, \mu = (a, b) \) and \( G(\mu, u) \) is the same as in the proof of Proposition 5.1. The orthogonality conditions on the fluctuation can be written as \( G_\#(\mu, u) = 0 \). Using the implicit function theorem we will first prove that for any \( \mu_0 := (a_0, b_0) \in \mathcal{A}_{t_0,\delta} \times \mathcal{B}_{t_0,\delta,\epsilon_0} \) there exists a neighborhood \( \mathcal{U}_{\mu_0} \) of \( V_\mu \) and a unique \( C^1 \) map \( g_\# : \mathcal{U}_{\mu_0} \rightarrow \mathcal{A}_{t_0,\delta} \times \mathcal{B}_{t_0,\delta,\epsilon_0} \) such that \( G_\#(g_\#(v), v) = 0 \) for all \( v \in \mathcal{U}_{\mu_0} \).

We claim that \( \partial_\mu G_\#(\mu, u) \) is invertible, provided \( u_{\lambda(\alpha)} \) is close to \( V_\mu \). We compute
\[
\partial_\mu G_\#(\mu, u)(t) = \partial_\mu G(\mu(t), u_{\lambda(\alpha)}(\cdot, t)) = A(t) + B(t),
\]
where
\[
A(t) := \partial_\mu G(v, \mu)|_{v=u_{\lambda(\alpha)}}, \quad B(t) := \partial_\mu G(v, \mu)|_{v=u_{\lambda(\alpha)}} \partial_\mu u_{\lambda(\alpha)}.
\]
Note that in (32) \( \partial_\mu G(\mu, u)|_{v=u_{\lambda(\alpha)}} \) is acting on \( \partial_\mu u_{\lambda(\alpha)} \) as an integral w.r. to \( y \). We have shown in the proof of Proposition 5.1 that the first term on the r.h.s. is invertible, provided \( u_{\lambda(\alpha)} \) is close to \( V_\mu \).

Now we show that for \( \delta > 0 \) sufficiently small the second term on the r.h.s. is small. Let \( v := u_{\lambda(\alpha)} \). Assuming for the moment that \( v \) is differentiable, we compute \( \partial_\alpha v = \partial_\alpha(\lambda)\lambda^{-1}[-v + y\partial_y v] \). Furthermore, \( \partial_\alpha(\lambda)\alpha = -\lambda^{-1}(t) \int_0^t \alpha(s)ds \). Combining the last two equations together with Equation (32) we obtain
\[
[B(t)\alpha](t) = -\int B(t)(y)(-v + y\partial_y v)(y, t)dy \lambda^{-2}(t) \int_0^t \alpha(s)ds.
\]
Integrating by parts the second term in parenthesis gives
\[
[B(t)\alpha](t) = \lambda^{-2}(t) \int_0^t \alpha(s)ds \int (1 + \partial_y y) B(t)(y)v(y, t)dy.
\]
Now, using a density, or any other, argument we remove the assumption of the differentiability on \( v \) and conclude that this expression holds without this assumption. Using this expression and the inequality \( \lambda(t) \geq \sqrt{2}\lambda_0 \), provided \( \delta \leq (4\sup a)^{-1}\lambda_0^2 \leq 1/4\lambda_0^2 \), we estimate
\[
\|B(t)\alpha\|_{L^\infty([t_0, t_0+\delta])} \leq \delta \lambda_0^{-2}\|\alpha\|_{L^\infty([t_0, t_0+\delta])}.
\]
(34)

So \( B(t) \) is small, if \( \delta \lesssim (\lambda_0^{-2}\|\alpha\|_{L^\infty})^{-1} \), as claimed. This shows that \( \partial_\mu G_\#(\mu, u) \) is invertible, provided \( u_{\lambda(\alpha)} \) is close to \( V_\mu \). Proceeding as in the proof of Proposition 5.1 we conclude the proof of Proposition 5.3.

We say that \( \lambda(t) \) is admissible on \( I_{t_0,\delta} \) if \( \lambda \in C^2(I_{t_0,\delta}, \mathbb{R}^+) \) and \( -\lambda \partial_t \lambda \in [1/4, 1] \).

Lemma 5.4. Assume \( u \in C^1([0, t_*], (x)^3 L^\infty) \) and \( \inf_{x \in \mathbb{R}} u(\cdot, t) > 0 \). Furthermore, assume there is a \( t_0 \in [0, t_*] \) and \( u_{\lambda_0}(\cdot, t_0) \in U_{\epsilon_0/2} \) for some \( \lambda_0 \) and for \( \epsilon_0 \) given in Proposition 5.7. Then there are \( \delta = \delta(\lambda_0, u) > 0 \) and \( \lambda(t) \), admissible on \( I_{t_0,\delta} \), s.t. (29) and (30) hold on \( I_{t_0,\delta} \).

Proof. The conditions \( u \in C^1([0, t_*], (x)^3 L^\infty) \), \( \inf_{x \in \mathbb{R}} u(\cdot, t) > 0 \) and \( u_{\lambda_0}(t_0) \in U_{\epsilon_0/2} \) imply that there is a \( \delta = \delta(\lambda_0, u) \) s.t. \( u \in U_{t_0,\delta,\epsilon_0,\lambda_0} \). By Lemma 5.3 the latter inclusion implies that there is \( \lambda(t) \), admissible on \( I_{t_0,\delta} \), \( \lambda(t_0) = \lambda_0 \), s.t. (29) and (30) hold on \( I_{t_0,\delta} \).
6 A priori Estimates

In this section we assume that \( u(x, t) \) is a solution to (1) satisfying the following conditions

(A) For \( 0 \leq t \leq t_\# \) there exist \( C^1 \) functions \( a(t) \) and \( b(t) \) such that \( u(x, t) \) can be represented as

\[
u(x, t) = \lambda(t) \left[ \left( \frac{2(d - 1) + b(t)y^2}{a(t) + \frac{y}{2}} \right)^{\frac{1}{2}} + \phi(y, \tau) \right]
\]

where \( \phi(\cdot, \tau) \perp e^{-\frac{a(t)}{d}y^2}, \ (1 - a(t)y^2)e^{-\frac{a(t)}{d}y^2} \) (see (20)), \( y = \lambda^{-1}(t)x \) and \( \tau(t) := \int_0^t \lambda^{-2}(s)ds, \ -\lambda(t)\partial_t\lambda(t) = a(t) \).

In the following we define estimating functions to control the functions \( \phi(y, \tau), a(t(\tau)) \) and \( b(t(\tau)) \).

\[
M_{m,n}(T) := \max_{\tau \leq T} \beta^{-\frac{m-n+1}{2}}(\tau)\|\phi(\cdot, \tau)\|_{m,n},
\]

\[
A(T) := \max_{\tau \leq T} \beta^{-2}(\tau)|a(t(\tau)) - \frac{1}{2} + \frac{1}{d-1}b(t(\tau))|,
\]

\[
B(T) := \max_{\tau \leq T} \beta^{-7/4}(\tau)|b(t(\tau)) - \beta(\tau)|.
\]

with \( (m, n) = (3, 0), (\frac{11}{10}, 0), (2, 1), (1, 2) \) and with the function \( \beta(\tau) \) defined as

\[
\beta(\tau) := \frac{1}{\sqrt{\tau}} + \frac{\tau}{d-1}.
\]

Furthermore we define a vector \( M \) as

\[
M := (M_{i,j}), \ (i, j) = (3, 0), (\frac{11}{10}, 0), (1, 2), (2, 1)
\]

and its sum \( |M| := \sum_{i,j} M_{i,j} \).

We say that a polynomial \( P(M, A) \) is monotonically nondecreasing if \( P(M_1, A_1) \geq P(M_2, A_2) \) whenever \( A_1 \geq A_2 \) and \( M_{i,j}(1) \geq M_{i,j}(2) \) for all \( i, j \), with \( M_1 := (M_{i,j}(1)), M_2 := (M_{i,j}(2)). \) In what follows the symbols \( P(M, A) \) and \( P(M) \) to stand for different monotonically nondecreasing polynomials of the vector \( M \) and the variable \( A \).

In this section we present a priori bounds on the fluctuation \( \phi \) proved in later sections.

**Proposition 6.1.** Suppose that \( u(x, t) \) is a solution to (1) satisfying Condition (A) and its datum \( u_0(x) \) satisfies all the conditions in Theorem 7 except the ones in Statement (4). Let the parameters \( a(t), b(t) \) and the function \( \phi(y, \tau) \) be the same as in (20). Then there exists a nondecreasing polynomial \( P(M, Z) \) of the 4-vector \( M \) and variable \( A \) such that the functions \( a, b \) and \( \phi \) satisfy the estimates

\[
B(\tau) \lesssim 1 + P(M(\tau), A(\tau)),
\]

\[
A(\tau) \lesssim A(0) + 1 + \beta(0)P(M(\tau), A(\tau)),
\]

\[
M_{3,0}(\tau) \lesssim M_{3,0}(0) + \beta^{\frac{1}{2}}(0)P(M(\tau), A(\tau)),
\]

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Corollary 6.2. Let \( \phi \) be defined in (39) and assume \(|M(0)|, A(0), B(0) \lesssim 1\). Assume there exists an interval \([0,T]\) such that for \( \tau \in [0, T] \),

\[
|M(\tau)|, A(\tau), B(\tau) \lesssim \beta^{-\frac{1}{4}}(\tau).
\]

Then on the same time interval the parameters \( a, b \) and the function \( \phi \) satisfy the following estimates

\[
|M(\tau)|, A(\tau), B(\tau) \lesssim 1.
\]

Proof. By replacing \( M_{3,0}(\tau), M_{2,1}(\tau) \) on the right hand sides of (12)-(14) by the estimates (11), (13) we rewrite Equations (12)-(14) as

\[
A(\tau) + |M(\tau)| \lesssim A(0) + 1 + |M(0)| + \beta^{3/2}(0)P(|M(\tau)|, A(\tau))
\]

with \( P \) being some polynomial. This implies (15) by the assumptions on \(|M(0)|, A(0)\) and \( B(0) \).

\[
\square
\]

7 Lower and Upper Bounds of \( v \)

In this section we prove lower and upper bounds for \( v \) defined in (11). The main tool we use is a generalized form of maximum principle from [27].

Lemma 7.1. Suppose \( u(y, \tau) \) is a smooth function satisfying the estimates

\[
u_\tau - a_0(y, \tau)u_{yy} - [a_1(y, \tau) + m(\tau)]u_y - a_2(y, \tau)u \leq 0; \quad (y)^{-1}u(y, \tau) \in L^\infty; \quad u(y, 0) \leq 0 \text{ if } |y| \geq c(0) \text{ and } u(y, \tau) \leq 0 \text{ if } \tau \leq T \text{ and } |y| = c(\tau)
\]

for some smooth, bounded functions \( a_0, a_1, a_2, m, c \), such that \( a_0(y, \tau) \geq 0 \) and \( c(\tau) \geq 0 \). Then for any \( \tau \leq T \)

\[
u(y, \tau) \leq 0 \text{ if } |y| \geq c(\tau).
\]

Moreover, if we replace the condition (46) by the condition that \( u(y, 0) \leq 0 \) for any \( y \), then instead of (47) we have \( u(y, \tau) \leq 0 \).
Proof. In what follows we only prove the estimate for the region $|y| \geq c(\tau)$, the estimate for $y \in \mathbb{R}$ is almost the same. We start with transforming the function $u$ so that the standard maximum principle can be used. Define a new function $w$ by

$$e^{\kappa \tau}(z)w(z, \tau) := u(y, \tau)$$

(48)

with $z := ye^{\int_0^\tau m(s)ds}$ and the scalar $\kappa$ to be chosen later. Then $w$ is a smooth, bounded function satisfying the inequality

$$w_t - a_3(z, \tau)w_{zz} - a_4(z, \tau)w_z - a_5(z, \tau)w \leq 0$$

for some bounded, smooth functions $a_3$, $a_4$, $a_5$ and especially $a_3$, $a_5 \geq 0$ by choosing appropriate $\kappa$. Moreover

$$w(z, 0) \leq 0 \text{ for } |z| \geq c(0), \text{ and } w(z, \tau) \leq 0 \text{ for } \tau \leq T, \text{ } |z| = c(\tau)e^{\int_0^\tau m(s)ds}.$$

By the standard maximum principle we have

$$w(z, \tau) \leq 0 \text{ if } \tau \leq T \text{ and } |z| \geq c(\tau)e^{\int_0^\tau m(s)ds}.$$

This estimate and the relation between $w$ and $u$ in (48) imply the desired result in the case $|y| \geq c(\tau)$. □

Recall the definition of function $g(y, \beta)$ from (45). The following proposition plays an important role in our analysis.

**Proposition 7.2.** Assume $v$ satisfies Condition (A) in Section 6, $v(y, 0) \geq g(y, b_0)$, $v(y, 0) \in \langle y \rangle L^\infty$, $|\partial_y v(y, 0)v^{-1/2}(y, 0)| \leq \kappa_0\beta^{1/2}(0)$ and $|\partial_y^n v(y, 0)| \leq \kappa_0\beta^{n+1}(0)$, $n = 2, 3, 4$, and assume there exists a time $\tau(t_0) \geq \tau_1 > 0$ such that for any $\tau \leq \tau_1$, $\langle M(\tau), A(\tau), B(\tau) \leq \beta^{-1/2}(\tau)$ and

$$v(\cdot, \tau) \in \langle y \rangle L^\infty, \partial_y v(\cdot, \tau), \partial_y^n v(\cdot, \tau) \in L^\infty, \text{ and } v(y, \tau) \geq c(\tau)$$

(49)

for some $c(\tau) > 0$. Then we have

$$v(y, \tau) \geq g(y, \beta(\tau)), \ |v^{-1/2}(y, \tau)\partial_y v(y, \tau)| \leq \beta^{1/2}(\tau), \ |\partial_y^n v(y, \tau)| \leq \beta^{n+1}(\tau), \ n = 2, 3, 4 \ (50)$$

on the same interval.

**Proof.** We start with proving the first estimate in (50) by verifying that the equation for $v$ satisfies all the conditions in Lemma 7.1. Since $M_{3,0}(\tau)$, $A(\tau)$, $B(\tau) \leq \beta^{-1/2}(\tau)$ and $v(y, \tau) = \left(\frac{2(\beta+O(\beta^2))}{a(\tau)+\beta/2}\right)^{1/2} + \phi(y, \tau)$ we have

$$v(y_1, \tau) \geq g(y_1, \tau)$$

(51)

for $y_1$ satisfying $|\beta y_1^2| \leq 20(d - 1)$ and $\tau \in [0, \tau_1]$.

On the other hand we observe that $a = \frac{1}{2} - \beta + O(\beta^2)$ by the assumption on $A(\tau)$. By a direct computation we have that on the domain $[0, \tau_1] \times \{ |\beta y^2| \geq 20(d - 1) \}$

$$H (g) \leq 0 \text{ and } H (v) = 0 \ (52)$$

where the map $H (g)$ is defined as

$$H (g) = g_\tau - \frac{g_{yy}}{1 + g_y^2} + \frac{d - 1}{g} + ayyg_y - ag.$$

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In order to use Lemma \ref{lem:1} derive an equation for \( g - v \). By the forms of \( H(g) \) and \( H(v) \) there exist functions \( b_n, n = 1, 2, 3 \), such that

\[
\partial_x (g - v) - b_1 \partial_x^2 (g - v) + ay \partial_y (g - v) - b_2 \partial_y (g - v) - b_3 (g - v) = H(g) - H(v)
\]  

(53) where \( b_1 > 0 \) and \( b_n, n = 1, 2, 3 \), are bounded functions.

Equations (53), the condition (49) and the assumption \( v(y, 0) \geq g(y, b_0) \) enable us to use Lemma \ref{lem:1} on the equation for \( g - v \). This leads to the inequality

\[
v(y, \tau) \geq g(y, \beta(\tau)) \text{ if } \beta y^2 \geq 20(d - 1).
\]  

(54)

For the region \( \beta y^2 \leq 20(d - 1) \) we use the estimate (51), which together with (54) yields the first estimate in (50).

Now we prove the estimate on \( \partial_y v \). Differentiating Equation (12) we obtain an equation for \( \partial_y v \). However this equation is not accessible directly to a maximum principle. To overcome this problem we use the fact that \( v \) is large for large \( |y| \) which is proved above and use instead the equation for \( v^{-\frac{1}{2}} \partial_y v \). We define a new function \( h(y, \tau) := v^\tau(y, \tau) \). Then \( \partial_y h = \frac{1}{2} v^{-\frac{1}{2}} \partial_y v \) satisfies the equation

\[
\mathcal{K}(\partial_y h) = 0,
\]

where the map \( \mathcal{K}(\chi) \) is defined as

\[
\mathcal{K}(\chi) := \frac{d}{d\tau} \chi - \frac{1}{1+(\partial_y v)^2} \partial_y^2 \chi + \frac{1}{1+(\partial_y v)^2} \frac{1}{1+2\partial_y v} \partial_y \chi + \frac{1}{1+(\partial_y v)^2} \partial_y \chi + \frac{2}{3} \chi
\]

\[-\frac{3(d-1)}{2v^2} \chi + \frac{1}{1+(\partial_y v)^2} \frac{8}{1+2\partial_y v} \partial_y \chi + \frac{1}{1+(\partial_y v)^2} \chi^3.
\]

On the other hand since \( \frac{8}{3} \geq \frac{1}{2} \), implied by the assumption on \( A \), and \( h^2 = v \geq 4\sqrt{d-1} \) on the region \( \beta y^2 \geq 20(d - 1) \), we have that

\[
\mathcal{K}(\kappa_0 \beta^\frac{1}{2}) > 0 \text{ and } \mathcal{K}(-\kappa_0 \beta^\frac{1}{2}) < 0
\]

provided that \( b(0) > 0 \), (and therefore \( \beta(\tau) \leq b(0) \)), is sufficiently small. Recall that the constant \( \kappa_0 > 2 \) defined in Theorem \ref{thm:1}. Moreover, by the assumption \( M_{2,1} \leq \beta^\frac{-1}{4} \) we have

\[-\kappa_0 \beta^\frac{1}{2} (\tau) < \partial_y h(y, \tau)|_{\beta(\tau)y^2 = 20(d - 1)} < \kappa_0 \beta^\frac{1}{2} (\tau).
\]

By the condition on \( v(y, 0) \) we have that

\[-\kappa_0 \beta^\frac{1}{2} (0) < \partial_y h(y, 0)|_{\beta(0)y^2 \geq 20(d - 1)} < \kappa_0 \beta^\frac{1}{2} (0).
\]

Lastly we derive equations for \( \partial_y h \pm \kappa_0 \beta^{3/2} \) from \( \mathcal{K}(\partial_y h) - \mathcal{K}(\pm \kappa_0 \beta^{3/2}) \) whose proof is almost identical to (53), thus omitted.

Collecting the facts above and using that \( h = \sqrt{v} \), we have by Lemma \ref{lem:1} the second part of (50).

By almost the same reasoning on the equation for \( \partial_y^3 v \) we prove that \( |\partial_y^3 v| \leq \beta \).

Next, we present the proof of the estimate on \( \partial_y^3 v \). The estimates for \( \partial_y^2 v \) is easier, thus omitted. We compute to get

\[
W(\partial_y^3 v) = g_1
\]
where the map $W(h)$ is defined as

$$W(h) := \partial_r h - \frac{1}{1 + v_y^2} \partial_y^2 h - g_4 \partial_y h + g_1 \partial_y h + 2a_0 h - \frac{d - 1}{v^2} h + \frac{6 \partial_y v}{1 + (v_y)^2} h^2 - g_3 h,$$

where $g_4$ is a function of $\partial_y^a v$, $n = 1, 2, 3$, and $g_1, g_3$ are functions of $v^{-\frac{1}{2}} \partial_y v$ and $\partial_y^2 v$. Moreover

$$g_4 \in L^\infty, |g_3| \leq \beta^3,$$ $g_1 = g_1(\frac{\partial_y v}{\sqrt{v}}, \partial_y^2 v)$ satisfies $\|g_1\|_{\infty} \leq \kappa \beta^{2/3}$

for some constant $\kappa > 0$ by the facts $\partial_y^a v \in L^\infty, n = 1, 2, 3$, and their various estimates above. Recall that $\|\partial_y^3 v(-, 0)\|_{\infty} \leq \kappa_0 \beta^{3/2}(0)$ for some $\kappa_0 > 0$. We define a new constant $\kappa_1 := \max\{\kappa_0, \kappa\}$. By the assumption on $A$ we have $-2a - \frac{d - 1}{v^2} + g_3 \leq -\frac{1}{2}$ hence

$$W(2\kappa_1 \beta^{3/2}) > 0 \quad \text{and} \quad W(-2\kappa_1 \beta^{3/2}) < 0.$$ 

As in (53), we derive equations for $\partial_y^a v = 2\kappa_1 \beta^{3/2}$ from $W(\partial_y^3 v) - W(\pm 2\kappa_1 \beta^{3/2})$, on which we use the maximum principle to have the estimate for $\partial_y^3 v$.

The proof is complete. \(\square\)

The following proposition is used in the proof of the statement (4) of Theorem 1.1. We define a function $\varrho$ as

$$\varrho(y, \tau) := \frac{v \partial_y^2 v}{1 + (\partial_y v)^2} = \frac{u(x, t) \partial_x^2 u(x, t)}{1 + (\partial_x u)^2},$$ (55)

where the last equality follows from the definition of $v$.

**Proposition 7.3.** Suppose that $v$ satisfies all the conditions in Proposition 7.2. We have

$$|\varrho(y, \tau)| \leq 4\beta(\tau) \quad \text{for} \quad y \in [-\frac{1}{10}, \frac{1}{10}],$$ (56)

and if $v_0 \partial_y^2 v_0 \geq -1$ then

$$\varrho(y, \tau) \geq -1 \quad \text{for} \quad \beta y^2 \geq 2(d - 1).$$ (57)

If $\rho(\cdot, 0) \leq d - 1$ then

$$\rho(\cdot, \tau) \leq d - 1.$$ (58)

**Proof.** Recall that $v(y, \tau) = V_{a, b} + \phi(y, \tau)$ with $|\phi(y, \tau)| \leq \beta^2(y)^3$ by the assumption on $M_{3, 0} \leq \beta^{-\frac{3}{2}}$. This implies (56) and that $\varrho(y, \tau) \geq -1$ when $\beta y^2 = 2(d - 1)$.

We use the maximum principle to prove (57). First we derive an inequality for the function $\varpi := \frac{\partial_x^2 u}{1 + (\partial_x u)^2} + 1$. We show below that

$$\mathcal{Y}_u(\varpi) = 0,$$ (59)

where the linear mapping $\mathcal{Y}_u$ is defined as

$$\mathcal{Y}_u(\psi) = \partial_x \psi - \frac{1}{1 + (\partial_x u)^2} \partial_x^2 \psi + \frac{2 \partial_x u}{u[1 + (\partial_x u)^2]} \left[ \frac{\partial_x^2 u}{1 + (\partial_x u)^2} + 1 \right] \partial_x \psi - \frac{2(\partial_x u)^2}{u^2[1 + (\partial_x u)^2]} \chi \psi,$$

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where the function $\chi$ is defined as $\chi(x,t) := \frac{\partial^2 u(x,t)}{\partial x \partial t} - (d-1)$. Recall that $\varpi(x,t) = g(y,\tau) + 1 \geq 0$ if $\beta(\tau(t))y^2 = 2(d-1)$ or $t = 0$. Using the maximum principle Lemma 2.1 on (59) we have that $\varpi \geq 0$ for $\beta y^2 \geq 2(d-1)$, which is (57).

Now Equation (60) follows from considering $K_u(\chi(\cdot,t)) - K_u(-d)$, as in (58), and the observations that

$$K_u(\chi(\cdot,t)) = 0 \quad \text{and} \quad K_u(-d) = 0$$

where the map $K_u$ is defined as

$$K_u(h) := \partial_t h - \frac{1}{1+(\partial_x u)^2} \partial_x^2 h + \frac{2\partial_x u}{u(1+(\partial_x u)^2)} (h + d) \partial_x h - \frac{2(\partial_x u)^2}{u^2(1+(\partial_x u)^2)} (h + d) h.$$

The proof of (58) is similar by using the observations

$$K_u(\chi(\cdot,t)) = 0 \quad \text{and} \quad K_u(0) = 0$$

and the initial condition $\chi(\cdot,0) = \rho(\cdot,0) \leq 0$.

This completes the proof of Proposition 5.3.

8 Proof of Main Theorem 1.1

Choose $b_0$ so that $C\ell_0^2 \leq \frac{1}{2} \epsilon_0$ with $C$ the same as in (69) and with $\epsilon_0$ given in Proposition 5.1. Let $v_0(y) := \lambda_0^{-1} u_0(\lambda_0 y)$. Then $v_0 \in U_{2\epsilon_0}$, by the condition (69) on the initial conditions with $(m,n) = (3,0)$. Hence Proposition 5.1 holds for $v_0$ and we have the splitting $v_0 = V_{g(\eta)} + \eta_0$. Denote $g(\eta) =: (a(0),b(0))$.

Furthermore, by Lemma 5.4 there are $\delta_1 > 0$ and $\lambda_1(t)$, admissible on $[0,\delta_1]$, s.t. $\lambda_1(0) = \lambda_0$ and Equations (29) and (30) hold on the interval $[0,\delta_1]$. Hence, in particular, the estimating functions $M(\tau) = (M_{m,n}(\tau)), (m,n) = (3,0), (1,0), (2,1), (1,2), A(\tau)$ and $B(\tau)$ of Section 5 are defined on the interval $[0,\delta_1]$. We will write these functions in the original time $t$, i.e. we will write $M(t)$ for $M(\tau(t))$ where $\tau(t) = \int_0^t \lambda_1^{-2}(s) ds$.

Recall the definitions of $\beta(\tau)$ and $\kappa$ are given in (37). By the relation $\beta(0) = b(0)$, Equation (3) and Proposition 5.2, $A(0), |M(0)| \leq 1$, while $B(0) = 0$ and $v_0(y) \geq g(y,b_0)$, by the definition. We have, by the continuity, that for a sufficiently small time interval, which we can take to be $[0,\delta_1]$, Condition (A) in Section 6 holds and

$$|M(t)|, A(t), B(t) \leq \beta^{-\frac{1}{2}}(\tau(t)), \quad u_{\lambda_1}(\cdot,t) \geq \frac{1}{4} \sqrt{2(d-1)}, \quad \text{(60)}$$

the last fact together with Theorem 2.1 and the initial conditions implies that

$$v(\cdot,\tau) \in (y)L^{\infty}, \partial_y v(\cdot,\tau), \partial_y^2 v(\cdot,\tau) \in L^{\infty}.$$

Then by Proposition 6.1 Corollary 6.2 we have that for the same time interval

$$|M(t)|, A(t), B(t) \leq 1, \quad \text{and} \quad u_{\lambda_1}(\cdot,t) \geq g(y,\beta(\tau)). \quad \text{(61)}$$

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Equation \((61)\) implies that \(u_{\lambda_1}(\cdot, \delta_1) \in U_{c_0/2} \) and \(u_{\lambda_1}(\cdot, \delta_1) \geq g(y, \beta(\tau(t)))\) (indeed, by the definition of \(M_{3,0}(t)\) we have \(||u_{\lambda_1}(\cdot, t) - V_{a(t), b(t)}||_{3,0} \leq M_{3,0}(t)b^2(t)\)). Now we can apply Lemma 8.49 again and find \(\delta_2 > 0\) and \(\lambda_2(t)\), admissible on \([0, \delta_1 + \delta_2]\), s.t. \(\lambda_2(t) = \lambda_1(t)\) for \(t \in [0, \delta_1]\) and Equations \((29)\) and \((30)\) hold on the interval \([0, \delta_1 + \delta_2]\).

We iterate the procedure above to show that there is a maximal time \(t^* \leq t_*\) (the maximal existence time), and a function \(\lambda(t)\), admissible on \([0, t^*]\), s.t. \((29)\) and \((30)\) hold on \([0, t^*]\). We claim that \(t^* = t_*\) and \(t^* < \infty\) and, consequently, \(\lambda(t^*) = 0\). Indeed, if \(t^* < t_*\) and \(\lambda(t^*) > 0\), then by the a priori estimate \(u_{\lambda}(t) \in U_{c_0/2}\) and \(u(x, t) \geq \frac{1}{2} \lambda(t) \sqrt{2(d - 1)}\) for any \(t \leq t^*\). By Lemma 5.4 this implies that there is \(\delta > 0\) and \(\lambda_{\#}(t)\), admissible on \([0, t^* + \delta]\), s.t. \((29)\) and \((30)\) hold on \([0, t^* + \delta]\) and \(\lambda_{\#}(t) = \lambda(t)\) on \([0, t^*]\), which would contradict the assumption that the time \(t^*\) is maximal. Hence

\[
either t^* = t_* or t^* < t_* and \lambda(t^*) = 0. \tag{62}
\]

The second case is ruled out as follows. Using the relation between the functions \(u(x, t)\) and \(v(y, \tau)\) and Equation \((61)\) we obtain the following a priori estimate on the (non-rescaled) solution \(u(x, t)\) of equation \((1)\):

\[
u(x, t) \geq \lambda(t)g(y, \beta(\tau(t))) \geq \lambda(t) \frac{1}{4} \sqrt{2(d - 1)}. \tag{63}
\]

Moreover by \((38)\) and the fact \(\|\phi(y)\|_{\infty} \leq b^2(t)\) implied by \(M_1 \leq 1\),

\[
u(0, t) \leq \lambda(t) \left( \frac{2(d - 1)}{c(t)} \right)^{\frac{1}{2}} + Cb(t)^2 \to 0, \tag{64}
\]
as \(t \uparrow t^*\), which implies that \(t^* \geq t_*\) and therefore \(t_* = t^*\) is the collapsing time as claimed.

Now we consider the first case in \((62)\). In this case we must have either \(t^* = t_* = \infty\) or \(t^* = t_* < \infty\) and \(\lambda(t^*) = 0\), since otherwise we would have existence of the solution on an interval greater than \([0, t_*]\). Finally, the case \(t^* = t_* = \infty\) is ruled out in the next paragraph. This proves the claim which can reformulated as: there is a function \(\lambda(t)\), admissible on \([0, t_*]\), \(t_* < \infty\) s.t. \((29)\) and \((30)\) hold on \([0, t_*]\) and \(\lambda(t) \to 0\) as \(t \to t_*\).

By the definitions of \(A(t)\) and \(B(t)\) in \((36)\) and the facts that \(A(t), B(t) \leq 1\) proved above, we have that

\[
a(t) - \frac{1}{2} = -\frac{1}{d - 1} b(t) + O(\beta^{2}(\tau(t))), \quad b(t) = \beta(\tau(t))(1 + O(\beta^{2}(\tau(t)))), \tag{65}
\]

where, recall, \(\tau = \tau(t) = \int_{0}^{t} \lambda^{-2}(s)ds\). Hence \(a(t) - \frac{1}{2} = O(\beta(\tau))\). Recall that \(a = -\lambda \partial_t \lambda\), which can be rewritten as \(\lambda^2(t) = \lambda_0^2 - 2 \int_{0}^{t} a(s)ds\) or \(\lambda(t) = |\lambda_0^2 - 2 \int_{0}^{t} a(s)ds|^{1/2}\). Since \(|a(t) - \frac{1}{2}| = O(b(t))\), there exists a time \(t^* \leq t_{**} < \infty\) such that \(\lambda(t) = 2 \int_{0}^{t^*} a(s)ds\), i.e. \(\lambda(t) \to 0\) as \(t \to t_{**}\). Furthermore, by the definition of \(\tau\) and the estimate \(|a(t) - \frac{1}{2}| = O(b(t))\) we have that \(\tau(t) \to \infty\) as \(t \to t_{**}\) (precise expressions are given in the next paragraph). Since \(\lambda(t^*) = 0\) we must have \(t^* = t_{**}\). Thus we have shown that \(t^* < \infty\).

This completes the proof of Statement \((1)\) and \((2)\) of Theorem 1.11

Now we prove Statement \((3)\) of Theorem 1.11 which establishes the asymptotics of the parameter functions. Equation \((65)\) implies \(b(t) \to 0\) and \(a(t) \to \frac{1}{2}\) as \(t \to t^*\). By the analysis above and the definitions of \(a, \tau\) and \(\beta\) (see \((37)\)) we have

\[
\lambda(t) = (t^* - t)^{1/2}(1 + o(1)), \quad \tau(t) = -\ln |t^* - t|(1 + o(1)),
\]

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and
\[ \beta(t) = -\frac{1}{(d-1)n[t^*-t]}(1 + o(1)). \]

This gives the first equation in (64). By (64) and the relation \( c = a + \frac{1}{2} \) we have the last two equations in (63). This proves Statement (3) of Theorem 1.1.

Now we prove the fourth statement of Theorem 1.1. First we show for \( x \neq 0 \lim_{t \to t_*} u(x, t) \geq 0 \). We transform (11) as
\[ \partial_t u = \left[ \frac{u \partial^2_x u}{1 + (\partial_x u)^2} - (d-1) \right] \frac{1}{u}. \]  

By the estimate (56) and the definition \( g(y, \tau) := \frac{u \partial^2_x u}{1 + (\partial_x u)^2} \) we have that
\[ \partial_t u(x, t) \leq [4\beta(0) - (d-1)] \frac{1}{u(x, t)} \]
or
\[ u^2(0, t) \leq u^2(0, t_1) - 2[(d-1) - 4\beta(0)](t - t_1). \]

This together with the fact \( u(0, t_*) = 0 \) yields
\[ t_* - t_1 \leq \frac{u^2(0, t_1)}{2[(d-1) - 4\beta(0)]}. \]  

On the other hand if a fixed \( x_1 \neq 0 \) satisfies \( \lambda^{-2}(t)\beta(\tau(t))x_1^2 = \beta(\tau)y_1^2 \geq 20(d-1) \) for \( t = t_1 \) and therefore for all \( t \geq t_1 \) then by (55) (67) and (66) we have
\[ \partial_t u(x_1, t) \geq -\frac{d}{u(x_1, t)}, \]
or
\[ u^2(x_1, t) \geq u^2(x_1, t_1) - 2d(t - t_1). \]  

Now we compare \( u(x_1, t_1) \) and \( u(0, t_1) \) to see that \( u(0, t) \) goes to zero first. Recall that \( u(x, t) = \lambda(t)v(y, \tau) \) and \( v(y, \tau) \) has the lower bound \( g(y, b(\tau)) \) defined in (58), moreover the estimate \( M_{3,0} \leq 1 \) implies \( u(0, t) = \lambda(t)v(0, \tau(t)) \leq \lambda(t)2\sqrt{d} - 1 \), thus we have
\[ u(x_1, t_1) \geq 2u(0, t_1). \]  

Equations (67) -(69) and the fact \( d \geq 2 \) yield for any \( t < t_* \)
\[ u^2(x_1, t) \geq \frac{3d - 4 - 16\beta(0)}{d-1 - 4\beta(0)} u^2(0, t_1) \geq u^2(0, t_1) > 0 \]
i.e. there exists a constant \( u_*(x_1) > 0 \) such that \( u(x_1, t) \geq u_*(x_1) \) before the collapsing time, i.e. \( t < t^* \).

Moreover, by the fast decay of \( \lambda(t) \) and slow decay of \( \beta(\tau(t)) \) we have that for any \( x_1 \neq 0 \), there exists a time \( t_1 \) such that \( \lambda^{-2}(t)\beta(\tau(t))x_1^2 = \beta(\tau)y_1^2 \geq 20(d-1) \) for \( t \geq t_1 \). This implies that there exists a \( u_*(x_1) \) such that
\[ u(x_1, t) \geq u_*(x_1) > 0 \]
Lemma 5.4 and Equation (16) imply that there is a time $0 \leq t_* < \infty$ such that the solution $w(x, t) = u(x, t)$ exists and $> 0$ for any $x \neq 0$. This proves Statement (4) and with it completes the proof of Theorem 1.1.

9 Lyapunov-Schmidt Splitting (Effective Equations)

Lemma 5.4 and Equation (16) imply that there is a time $0 < t_\# \leq \infty$ such that the solution $w(y, \tau) = v(y, \tau)e^{-\frac{4}{2}\phi^2}$ of (17) can be decomposed as:

$$w = w_{ab} + \xi, \quad \xi \perp \phi_{0,a}, \phi_{2a},$$

with the functions $\phi_{0,a} := \left(\frac{a}{2}\right)^\frac{1}{2}e^{-\frac{a^2}{4}}, \phi_{2a} := \left(\frac{a}{2}\right)^\frac{1}{2}(1-ay^2)e^{-\frac{a^2}{4}},$ the parameters $a, b$ being $C^1$ functions of $t$, $w_{ab} := V_{ab}e^{-\frac{4}{2}\phi^2}$, the fluctuation $\xi := e^{-\frac{4}{2}\phi^2}e$ and the orthogonality understood in the $L^2$ norm.

According to their definition in Section 5 the parameters $a, b$ and $\xi$ depend on the rescaled time $\tau$ through the original time $t$: $a(t(\tau)), b(t(\tau))$ and $c(t(\tau))$. To simplify the notation we will write $a(\tau), b(\tau)$ and $c(\tau)$ for $a(t(\tau)), b(t(\tau))$ and $c(t(\tau))$. This will not cause confusion as the original parameter functions $a(t), b(t)$ and $c(t)$ are not used in what follows. In this section we derive equations for the parameters functions $a(\tau), b(\tau)$ and $c(\tau)$ and the fluctuation $\xi(\tau, \tau)$.

Substitute (71) into (17) to obtain the following equation for $\xi$

$$\partial_\tau \xi(y, \tau) = -L(a, b)\xi + F(a, b) + N_1(a, b, \xi) + N_2(a, b, \xi)$$

where $L(a, b)$ is the linear operator given by

$$L(a, b) := -\partial_y^2 + \frac{a^2 + \partial_y^2}{4} - \frac{3a}{2} - \frac{(d-1)(b^2 + a)}{2(d-1) + by^2}$$

and the functions $F(a, b), N_1(a, b, \xi)$ and $N_2(a, b, \xi)$ are defined as

$$F := \frac{1}{2} \exp\left\{-\frac{ay^2}{4} \left(\frac{2(d-1) + by^2}{a + \frac{b^2}{2}}\right) \left[\Gamma_1 + \Gamma_2 \frac{g^2}{2(d-1) + by^2} + F_1\right]\right\}$$

with

$$\Gamma_1 := \frac{b}{\partial_y + \frac{b}{2d}} + \frac{a}{\partial_y} + \frac{b}{d} - \frac{b^2}{d-1};$$

$$\Gamma_2 := -\partial_y b - b(a - \frac{1}{2} + \frac{b}{d-1}) - \frac{b^2}{d-1};$$

$$F_1 := -\frac{1}{d-1} \frac{b^2 y^2}{(2(d-1) + by^2)^2};$$

$$N_1(a, b, \xi) := -\frac{d-1}{a} \frac{a + \frac{b^2}{2}}{2(d-1) + by^2} \exp\frac{ay^2}{4}e^2;$$

$$N_2(a, b, \xi) := -\exp\frac{ay^2}{4} \frac{(\partial_y^2 + \partial_y^2)^2\phi^2}{2+\phi^2}.$$
Here $v$ is the same as in (11) and is related to $\xi$ be (16) and (71), and we ordered the terms in $F$ according to the leading power in $y^2$.

In the next three lemmas we prove estimates on the terms $N_1$, $N_2$, $\Gamma_1$, $\Gamma_2$ and $F$. These estimates will be used in later sections.

**Lemma 9.1.** Assume $v$ satisfies Condition (A) in Section 6, $v(y,0) \geq g(y,b_0)$, $v(y,0) \in \langle y \rangle L^\infty$, $|\partial_y v(y,0)v^{-1/2}(y,0)| \leq \kappa_0 \beta^{3/2}(0)$ and $|\partial_y^m v(y,0)| \leq \kappa_0 \beta^m(0)$, $n = 2, 3, 4$; and assume there exists a time $T \leq \tau(t_y)$ such that for any $\tau \leq T$, $|M(\tau)|$, $A(\tau), B(\tau) \leq \beta^{-\tau/4}(\tau)$ and

$$v(\cdot, \tau) \in \langle y \rangle L^\infty, \partial_y v(\cdot, \tau), \partial_y^2 v(\cdot, \tau) \in L^\infty$$

and $v(y, \tau) \geq \frac{1}{4}\sqrt{2(d - 1)}$. Then we have

$$|N_1(a, b, \xi)| \lesssim \frac{1}{1 + \beta y^2} \exp \frac{ay^2}{4}|\xi|^2,$$

(75)

$$\|\exp \frac{ay^2}{4} N_1(a, b, \xi)\|_{m,n} \lesssim \beta^{m+n+2}(\tau) P(M(\tau))$$

(76)

for $(m, n) = (3, 0), (\frac{11}{10}, 0), (2, 1)$;

$$\|\exp \frac{ay^2}{4} N_1(a, b, \xi)\|_{1,2} \lesssim \beta^2[M_{2,1} + M_{3,0}] + \beta^{5/2} M_{1,0}^2;$$

(77)

$$\|\langle y \rangle^{-5} \exp \frac{ay^2}{4} N_2(a, b, \xi)\|_{\infty} \lesssim \beta^3[1 + M_{1,2} + M_{2,1}^2 + M_{1,2} M_{2,1}];$$

(78)

and, for $(m, n) = (3, 0), (\frac{11}{10}, 0), (2, 1), (1, 2)$

$$\|\exp \frac{ay^2}{4} N_2(a, b, \xi)\|_{m,n} \lesssim \beta^{m+n+2}(\tau) P(M(\tau)).$$

(79)

**Proof.** By the explicit form of $N_1$ we have

$$|N_1(a, b, \xi)| \lesssim \frac{1}{|v| 1 + b(\tau) y^2} \exp \frac{ay^2}{4}|\xi|^2.$$

The assumptions $A(\tau), B(\tau) \leq \beta^{-\tau/4}(\tau)$ imply that $b = \beta + o(\beta)$ and $a = \frac{1}{2} + O(\beta)$ which together with the assumption on $v$ yield (75).

Now we prove (76), $(m, n) = (3, 0), (2, 1)$. The estimate of $(m, n) = (\frac{11}{10}, 0)$ is similar to that of $(3, 0)$, and is omitted.

We start with $\|\exp \frac{ay^2}{4} N_1(a, b, \xi)\|_{3,0}$. Recall the definitions $V_{ab} = (\frac{2(d-1) + b^2}{a + y^2})^{1/2}$ and $\xi(y, \tau) = e^{-\frac{a y^2}{2}} \phi(y, \tau)$, the definition of $N_1$ in (74) yields

$$e^{ay^2} N_1 = \frac{d - 1}{v} V_{ab}^{-2} \phi^2.$$
By direct computations we obtain 

$$\| \exp \frac{a y^2}{4} N_1(a, b, \xi) \|_{3,0} \leq \| \phi \|_{3,0} \| V_{ab} \|^{-\frac{M}{2}} \| \phi \|_{\infty}.$$ 

By the definition of $V_{ab}$ we have 

$$\| V_{ab} \|^{-\frac{M}{2}} \lesssim (y)^{-\frac{M}{2}} \beta^{-\frac{M}{2}},$$ 

thus, recalling the definition (7.4) of estimating functions, 

$$\| \exp \frac{a y^2}{4} N_1(a, b, \xi) \|_{3,0} \lesssim \beta^{-\frac{M}{2}} \| \phi \|_{3,0} \| \phi \|_{\frac{M}{2},0} \leq \beta^{5/2} M_{3,0} M_{\frac{M}{2},0}.$$

Before estimating for the pairs $(1, 2)$, $(2, 1)$, we recall the decomposition of $v$ as $v(y, \tau) = V_{a(b)} + \phi(y, \tau)$ with the function $V_{ab}$ admitting the estimates 

$$V_{ab}^{-k} \lesssim \beta^{-\frac{k}{2}} (y)^{-k}$$

for any $k \geq 0$, $|\partial_y^n V_{ab}^{-1}| \lesssim \beta^{\frac{n}{2}} V_{ab}^{-1}$, $|\partial_y^n V_{ab}| \lesssim \beta^{\frac{n}{2}}$, $n = 1, 2, 3$, (80) by the assumptions on the estimating functions $A$ and $B$. Also recall the inequalities $v \geq \frac{1}{4} \sqrt{2(d-1)}$ and $|\partial_y^n (v(y, \tau)| \leq 2 \beta^{\frac{n}{2}} (\tau), |\partial_y^n v(y, \tau)| \lesssim \beta^{\frac{n}{2}} (\tau), n = 2, 3, 4$ proved in Proposition 7.2. By direct computation and the recalled facts above we have 

$$|\partial_y \exp \frac{a y^2}{4} N_1(a, b, \xi)| \lesssim |2V^{-3} \partial_y V| |\phi^2| + |2V^{-2} \partial_y \phi| |\phi| + |V^{-2} \phi^2| |\partial_y v|$$

consequently 

$$\| \exp \frac{a y^2}{4} N_1(a, b, \xi) \|_{2,1} \lesssim \beta^{-\frac{M}{2}} \| \phi \|_{2,1} \| \phi \|_{\frac{M}{2},0} + \beta^{2/5} \| \phi \|_{\frac{M}{2},0}$$

$$\lesssim \beta^{5/2} [M_{2,1} M_{\frac{M}{2},0} + M_{\frac{M}{2},0}].$$

The proof of (7.7) is more involved. By direct computation and the recalled facts in and after (7.0) we have 

$$|\partial_y^2 \exp \frac{a y^2}{4} N_1(a, b, \xi)| \lesssim V_{ab}^{-2} |[\partial_y^2 \phi] + |\beta^{3/2} |\phi \partial_y \phi| v^{-1} + |(\partial_y \phi)^2 v^{-1}| + \beta \phi^2|.$$ 

Again by the facts in and after (7.0) $|\partial_y \phi| v^{-\frac{3}{2}} \leq |\partial_y \phi| v^{-\frac{1}{2}} + |\partial_y V_{ab}| v^{-\frac{1}{2}} \lesssim \beta^{\frac{1}{2}}$ and $|\partial^2 \phi| \leq |\partial^2 v| + |\partial^2 V_{ab}| \lesssim \beta$, which implies the estimate for the first two terms 

$$\| (y)^{-1} \partial^2_y \phi \phi V_{ab}^{-2} \|_{\infty} + \| (y)^{-1} \beta^{\frac{3}{2}} \phi \partial_y \phi \phi V_{ab}^{-2} \|_{\infty} \lesssim \| \phi \|_{3,0} \leq \beta^2 M_{3,0}.$$

Similarly for the third term 

$$\| (y)^{-1} (\partial_y \phi)^2 v^{-1} V_{ab}^{-2} \|_{\infty} \lesssim \| (y)^{-1} V_{ab}^{-1} \partial_y \phi \|_{\infty} \| v^{-1} \partial_y \phi \|_{\infty} \lesssim \| \phi \|_{2,1} \leq \beta^2 M_{2,1}.$$

For the last term we have 

$$\beta \| (y)^{-1} \phi^2 V_{ab}^{-2} \|_{\infty} \lesssim \beta^{2/5} \| \phi \|_{\frac{M}{2},0} \leq \beta^{5/2} M_{\frac{M}{2},0}.$$

Collecting the estimates above we have (7.7).
Now we turn to (78) and (79). By the definition of $N_2$ we have

$$\left| \exp \frac{a_n^2}{4} N_2(a, b, \xi) \right| \lesssim |\partial_y v|^2 |\partial_y^2 v|.$$  

Recall that $v = V_{a,b} + \phi$ with $V_{a,b} := \left( \frac{2(d-1) + a^2}{a + \frac{2}{3}} \right)^{\frac{1}{2}}$ and $\phi := e^{-\frac{2}{a^2}} \xi$. The bounds $|\partial_y^2 V_{a,b}|$, $\langle y \rangle^{-1} |\partial_y V_{a,b}| \lesssim \beta$ yield

$$|\langle y \rangle^{-2} \partial_y v| \leq \langle y \rangle^{-2} |\partial_y V_{a,b}| + \langle y \rangle^{-2} |\partial_y \phi| \leq \beta + \beta^2 M_{2,1}$$

and

$$|\langle y \rangle^{-1} \partial_y^2 v| \leq \langle y \rangle^{-1} |\partial_y^2 V_{a,b}| + \langle y \rangle^{-1} |\partial_y \phi| \leq \beta + \beta^2 M_{1,2}$$

which yield the estimate on $\langle y \rangle^{-5} e^{\frac{a_n^2}{4}} N_2(a, b, \xi)$ or Equation (78).

In what follows we only prove the cases of $(m, n) = (3, 0), (1, 2)$ of (79). The other cases are similar.

By the definition of $N_2$ we have

$$\| e^{\frac{a_n^2}{4}} N_2(a, b, \xi) \|_{3, 0} = \| \langle y \rangle^{-3} \partial_y^2 v(\partial_y v)^2 \| \lesssim \| \langle x \rangle^{-2} \partial_y v \|^{3/2} \| \partial_y^2 v \| \lesssim \beta^{5/2} (1 + M_{2,1}^2)$$

where recall the facts $|\partial_y^n v(y, \tau)| \lesssim \beta^{\frac{3}{2}}(\tau)$, $n = 2, 3, 4$ proved in Proposition 7.2 and recall the definition of $v$ as $v(y, \tau) = V_{a,b} + \phi(y, \tau)$ with $V_{a,b}$ admitting the estimates

$$|\partial_y^2 V_{a,b}| \lesssim \beta^2, \quad n = 1, 2, 3, \quad \| \langle y \rangle^{-1} \partial_y V_{a,b} \| \lesssim \beta.$$  

By direct computation we have

$$|\partial_y^3 N_2(a, b, \xi)| \lesssim |\partial_y^4 v| |\frac{\partial_y v}{1 + (\partial_y v)^2}| + |\partial_y^2 v| |\partial_y^2 v| + |(\partial_y^2 v)^3|$$

Using the estimates on $\partial_y^n v$ listed above and the estimate $\| \langle y \rangle^{-2} \partial_y v \| \lesssim \beta + \beta^2 M_{2,1}$ we have

$$\| N_2(a, b, \xi) \|_{1, 2} \lesssim \| v \|_{0,4} \| v \|_{2,1}^{2} + \| v \|_{0,3} \| v \|_{0,2} + \| v \|_{0,2}^3 \lesssim \beta^{5/2} (1 + M_{2,1}).$$

Thus the proof is complete. □

Now we establish some estimates for $\Gamma_1$ and $\Gamma_2$ defined after Equation (79).

**Lemma 9.2.** If $|M(\tau)|, A(\tau), B(\tau) \leq \beta^{-\frac{3}{2}}(\tau)$ and $v(y, \tau) \geq \frac{1}{2} \sqrt{2(d-1)}$ then we have

$$|\Gamma_1|, |\Gamma_2| \lesssim \beta^3 P(M, A)$$

(81)

where $P(M, A)$ is a nondecreasing polynomial of the vector $M$ and $A$.  

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Proof. Taking the inner products of (72) with the functions \( \phi_{0,a} := \exp -\frac{a y^2}{4} \) and \( \phi_{2,a} := (ay^2 - 1) \exp -\frac{a y^2}{4} \) and using the orthogonality conditions in (85) we have

\[
|\langle F(a,b), \phi_{0,a} \rangle| \leq G_1 \\
|\langle F(a,b), \phi_{2,a} \rangle| \leq G_2
\]  

(82)

where the functions \( G_1, G_2 \) are defined as

\[
G_1 := \langle |by^2 \xi| + |\partial_x ay^2 \xi| + |N_1(a,b,\xi)| + |N_2(a,b,\xi)|, \exp -\frac{a y^2}{4} \rangle
\]

and

\[
G_2 := \langle |by^2 \xi| + |\partial_x ay^2 \xi| + |N_1(a,b,\xi)| + |N_2(a,b,\xi)|, ((ay^2 - 1) \exp -\frac{a y^2}{4}) \rangle.
\]

By Equations (75), (78) and the assumptions on \( A(\tau) \) and \( B(\tau) \) we have that

\[
G_1, G_2 \lesssim |\partial_\tau a| \beta^2 M_{3,0} \beta^3 [1 + M_{1,2} + M_{2,1}^2 + M_{1,2} M_{2,1}^2 + M_{2,1}^2].
\]

We rewrite the function \( F(a,b) \) in (72) as

\[
F(a,b) = \chi(a,b)[\Gamma_1 + \Gamma_2 - \frac{1}{a[2(d-1)+by^2]} + \Gamma_2 - \frac{ay^2 - 1}{a[2(d-1)+by^2]} + F_1].
\]

where, recall, \( F_1 \) is defined in (74), and \( \chi(a,b) \) is defined as

\[
\chi(a,b) := \left(\frac{(2(d-1)+by^2)}{a + \frac{by^2}{2}}\right)^\frac{1}{2} \exp -\frac{ay^2}{4}.
\]

By using the fact that \( \phi_{0,a} \perp \phi_{2,a} \) we have

\[
|\langle F(a,b), \phi_{0,a} \rangle| \gtrsim |\Gamma_1 + \frac{1}{2a(d-1)} \Gamma_2| - |b||\Gamma_1| + |\Gamma_2| - |b|^3
\]

and

\[
|\langle F(a,b), \phi_{2,a} \rangle| \gtrsim |\Gamma_2| - |b||\Gamma_1| + |\Gamma_2| - |b|^3,
\]

which together with (82) implies that

\[
|\Gamma_1|, |\Gamma_2| \lesssim \beta^3 [1 + M_{1,2} + M_{2,1}^2 + M_{1,2} M_{2,1}^2 + M_{2,1}^2] + |\partial_\tau a| \beta^2 M_{3,0}.
\]

Moreover by the definition of \( \Gamma_1 \) \( \partial_\tau a \) has the bound

\[
|\partial_\tau a| \lesssim |\Gamma_1| + \beta^2 A.
\]

Consequently

\[
|\Gamma_1|, |\Gamma_2| \lesssim \beta^3 [1 + M_{1,2} + M_{2,1}^2 + M_{1,2} M_{2,1}^2 + M_{2,1}^2] + |\Gamma_1| \beta^2 M_{3,0} + \beta^4 AM_{3,0}.
\]

This together with the assumption that \( |M(\tau)| \leq \beta^{-\frac{3}{4}}(\tau) \) implies (SI). \( \square \)
To facilitate the later estimates we list the estimates of $F$ in the following lemma.

**Lemma 9.3.** If $A(\tau), B(\tau) \leq \beta^{-\frac{3}{4}}(\tau)$ and $(m, n) = (3, 0), (\frac{11}{10}, 0), (1, 2), (2, 1)$, then

$$\| \exp \frac{ay^2}{4} F(a,b) \|_{m,n} \lesssim \beta^{\frac{m+n+2}{2}} P(M, A).$$

(83)

**Proof.** In the following we only prove the cases $(m, n) = (\frac{11}{10}, 0), (3, 0)$. The proof of the remaining cases are similar.

We start with $(m, n) = (\frac{11}{10}, 0)$. Recall the definition of $F$ in Equation (18). We observe that

$$\| \langle y \rangle^{-\frac{11}{10}}\frac{y^2}{(2(d-1)+by^2)^{\frac{1}{2}}} \|_{\infty} \leq b^{-\frac{11}{20}} \lesssim \beta^{-\frac{5}{20}}.$$ 

Thus the estimates on $\Gamma_1$ and $\Gamma_2$ in (81) imply that

$$\| \langle y \rangle^{-\frac{11}{10}}(\frac{2(d-1)+by^2}{a+\frac{1}{2}})^{\frac{1}{2}}[\Gamma_1 + \Gamma_2 y^{2(d-1)+by^2}] \|_{\infty} \lesssim (|\Gamma_1| + |\Gamma_2|) \beta^{-\frac{5}{20}} \lesssim \beta^{\frac{25}{20}} P(M, A).$$

For the term $F_1$ by similar reasoning we have

$$\| \langle y \rangle^{-\frac{11}{10}} F_1 (\frac{2(d-1)+by^2}{a+\frac{1}{2}})^{\frac{1}{2}} \|_{\infty} \lesssim \beta^{\frac{25}{20}}.$$ 

Combining the estimates above we complete the estimate for $(m, n) = (\frac{11}{10}, 0)$.

The estimate for $(m, n) = (3, 0)$ is easier by using the observation

$$\| \langle y \rangle^{-3}(\frac{2(d-1)+by^2}{a+\frac{1}{2}})^{\frac{1}{2}}[\Gamma_1 + \Gamma_2 y^{2(d-1)+by^2}] \|_{\infty} \lesssim |\Gamma_1| + |\Gamma_2| \lesssim \beta^3 P(M, A).$$

and

$$\| \langle y \rangle^{-3} F_1 (\frac{2(d-1)+by^2}{a+\frac{1}{2}})^{\frac{1}{2}} \|_{\infty} \lesssim \beta^{5/2}.$$ 

Collecting the estimates above we complete the proof. \qed

## 10 Proof of Estimates (39)-(40)

The following lemmas show that $b$ and $\beta$ are closely related.

**Lemma 10.1.** If $B(\tau) \leq \beta^{-\frac{3}{4}}(\tau)$ for $\tau \in [0, T]$, then (39) holds.

**Proof.** We begin with rewriting equation (81) as

$$|\partial_\tau b + \frac{1}{d-1}b^2| \lesssim b^2 \frac{1}{2} - a + \frac{1}{d-1}b + \beta^3 P(M, A).$$
The first term on the right hand side is bounded by \( b^3A \lesssim \beta^3A \) by the definition of \( A \). Hence we have

\[
\left| \partial_\tau b + \frac{1}{d-1} b^2 \right| \lesssim \beta^3 P(M, A), \tag{84}
\]

To prove (89) we begin by dividing (84) by \( b^2 \) and using the inequality \( \beta \lesssim b \) to obtain the estimate

\[
\left| \partial_\tau b + \frac{1}{d-1} b \right| \lesssim \beta P(M, A). \tag{85}
\]

Since \( \beta \) satisfies \( \nabla \beta - \frac{1}{d-1} + \frac{1}{\beta} = 0 \), Equation (85) implies that

\[
|\partial_\tau \frac{1}{\beta} - \frac{1}{\beta}| \lesssim \beta P(M, A).
\]

Integrating this equation over \([0, \tau]\), multiplying the result by \( \beta^{-\frac{7}{4}} \) and using that \( b \lesssim \beta \) and \( \beta(0) = b(0) \) give the estimate

\[
\beta^{-\frac{7}{4}} |\beta - b| \lesssim \beta^\frac{7}{4} \int_0^\tau \beta P(M, A) ds.
\]

which together with definitions of \( \beta \) and \( B \) implies (89).

**Lemma 10.2.** If \( A(\tau), B(\tau) \leq \beta^{-\frac{4}{3}}(\tau) \) for \( \tau \in [0, T] \), then (40) holds.

**Proof.** Define the quantity \( \Gamma := a - \frac{1}{2} + \frac{1}{d-1} b \). We prove the proposition by integrating a differential inequality for \( \Gamma \). Differentiating \( \Gamma \) with respect to \( \tau \) and substituting for \( \partial_\tau b \) and \( \partial_\tau a \) the expression in terms of \( \Gamma_1 \) and \( \Gamma_2 \) (see 81) and using Equation (81), we obtain

\[
\partial_\tau \Gamma + (a + \frac{1}{2} + \frac{1}{d-1} b) \Gamma = -\frac{1}{(d-1)^2} b^2 + \mathcal{R}_b.
\]

where \( \mathcal{R}_b \) has the bound

\[
|\mathcal{R}_b| \leq \beta^3 P(M, A).
\]

Let \( \mu = \exp \int_0^\tau a(s) + \frac{1}{2} + \frac{1}{d-1} b(s) ds \). Then the above equation implies that

\[
\mu \Gamma = \Gamma_0 - \int_0^\tau \frac{\mu b^2}{(d-1)^2} ds + \int_0^\tau \mu \mathcal{R}_b ds.
\]

We now use the inequality \( b \lesssim \beta \) and the estimate of \( \mathcal{R}_b \) to estimate over \([0, \tau] \leq [0, T] \):

\[
|\Gamma| \lesssim \mu^{-1} \Gamma_0 + \mu^{-1} \int_0^\tau \mu b^2 ds + \mu^{-1} \int_0^\tau \mu \beta^3 ds P(M, A).
\]

For our purpose, it is sufficient to use the less sharp inequality

\[
|\Gamma| \lesssim \mu^{-1} \Gamma(0) + \mu^{-1} \int_0^\tau \mu b^2 ds[1 + \beta(0) P(M, A)].
\]

The assumption that \( A(\tau), B(\tau) \leq \beta^{-\frac{4}{3}}(\tau) \) implies that \( a + \frac{1}{2} + \frac{1}{d-1} b \geq \frac{1}{\beta} \). Thus, it is not difficult to show that \( \beta^{-2} \mu^{-1} \Gamma(0) \leq A(0) \) and \( \beta^{-2} \mu^{-1} \int_0^\tau \mu b^2 ds \) are bounded and hence

\[
A \lesssim A(0) + 1 + \beta(0) P(M, A)
\]

which is (40).
11 Rescaling of Fluctuations on a Fixed Time Interval

We return to our key equation (72). In this section we re-parameterize the unknown function $\xi(y, \tau)$ in such a way that the $y^2$-term in the linear part of the new equation has a time-independent coefficient (cf [11]).

Let $t(\tau)$ be the inverse function to $\tau(t)$, where $\tau(t) = \int_0^t \lambda^{-2}(s) \, ds$ for any $\tau \geq 0$. Pick $T > 0$ and approximate $\lambda(t(\tau))$ on the interval $0 \leq \tau \leq T$ by the new trajectory, $\lambda_1(t(\tau))$, tangent to $\lambda(t(\tau))$ at the point $\tau = T$: $\lambda_1(t(T)) = \lambda(t(T))$, and $\alpha := -\lambda_1(t(\tau)) \partial_t \lambda_1(t(\tau)) = a(T)$ where, recall $a(\tau) := -\lambda(t(\tau)) \partial_t \lambda(t(\tau))$. Now we introduce the new independent variables $z$ and $\sigma$ as $z(x,t) := \lambda_1^{-1}(t)x$ and $\sigma(t) := \int_0^t \lambda_1^{-2}(s) \, ds$ and the new unknown function $\eta(z, \sigma)$ as

$$
\lambda_1(t) \exp \frac{\alpha}{4} z^2 \eta(z, \sigma) := \lambda(t) \exp \frac{a(\tau)}{4} y^2 \xi(y, \tau).
$$

(86)

In this relation one has to think of the variables $z$ and $y$, $\sigma$, $\tau$ and $t$ as related by $z = \frac{\lambda(t)}{\lambda_1(t)} y$, $\sigma(t) := \int_0^t \lambda_1^{-2}(s) \, ds$ and $\tau = \int_0^t \lambda^{-2}(s) \, ds$, and moreover $a(\tau) = -\lambda(t(\tau)) \partial_t \lambda(t(\tau))$ and $\alpha = a(T)$.

For any $\tau = \int_0^{t(\tau)} \lambda^{-2}(s) \, ds$ with $t(\tau) \leq t(T)$ (or equivalently $\tau \leq T$) we define a new function $\sigma(\tau) := \int_0^{t(\tau)} \lambda_1^{-2}(s) \, ds$. Observe the function $\sigma$ is invertible, we denote by $\tau(\sigma)$ as its inverse. We define

$$
S := \int_0^{t(T)} \lambda_1^{-2}(s) \, ds.
$$

(87)

The new function $\eta$ satisfies the equation

$$
\frac{d}{d\sigma} \eta(\sigma) = -\mathcal{L}_\alpha \eta(\sigma) + \mathcal{W}(a,b) \eta(\sigma) + \mathcal{F}(a,b)(\sigma) + N_1(a,b,\eta) + N_2(a,b,\eta)
$$

(88)

with the operators

$$
\mathcal{L}_\alpha := L_\alpha + V
$$

where

$$
L_\alpha := -\partial_z^2 + \frac{a^2}{4} z^2 - \frac{3\alpha}{2},
$$

$$
V := -\frac{2(d-1)\alpha}{2(d-1)+\beta(\sigma)} z^2;
$$

and

$$
\mathcal{W}(a,b) := -\frac{\lambda^2}{\lambda_1^2} \frac{(d-1)(a+\frac{1}{2})}{2(d-1)+b(\sigma)y^2} + \frac{2(d-1)\alpha}{2(d-1)+\beta(\sigma)} z^2;
$$

with the function

$$
\mathcal{F}(a,b) := \exp \frac{\alpha}{4} z^2 \exp \frac{a}{4} \frac{\lambda_1}{\lambda} F,
$$

and with the nonlinear terms

$$
N_1(a,b,\eta) := \frac{\lambda_1}{\lambda} \exp \frac{-a}{4} z^2 e^{\frac{a}{2} y^2} N_1(a,b,\xi),
$$

$$
N_2(a,b,\eta) := \frac{\lambda_1}{\lambda} \exp \frac{-a}{4} z^2 e^{\frac{a}{2} y^2} N_2(a,b,\xi)
$$

(89)

where, recall $F$, $N_1$ and $N_2$ are defined after (73) and where $\tau$ and $y$ are expressed in terms of $\sigma$ and $z$. In the next proposition we prove that the new trajectory is a good approximation of the old one.
Proposition 11.1. For any \( \tau \leq T \) we have that if
\[
A(\tau) \leq \beta^{-\frac{1}{4}}(\tau)
\]
then
\[
|\frac{\lambda}{\lambda_1}(t(\tau)) - 1| \lesssim \beta(\tau) \tag{90}
\]
for some constant \( c \) independent of \( \tau \).

Proof. By the properties of \( \lambda \) and \( \lambda_1 \) we have
\[
\partial_\tau \left[ \frac{\lambda}{\lambda_1}(t(\tau)) - 1 \right] = 2a(\tau)(\frac{\lambda}{\lambda_1}(t(\tau)) - 1) + G(\tau) \tag{91}
\]
with
\[
G := \alpha - a + (\alpha - a)(\frac{\lambda}{\lambda_1} - 1)[(\frac{\lambda}{\lambda_1})^2 + \frac{\lambda}{\lambda_1} + 1] + \alpha(\frac{\lambda}{\lambda_1} - 1)^2[\frac{\lambda}{\lambda_1} + 2].
\]
By the definition of \( A(\tau) \), if \( A(\tau) \leq \beta^{-\frac{1}{4}}(\tau) \) then
\[
|a(\tau) - \alpha|, \quad |a(\tau) - \frac{1}{2}| \lesssim \beta(\tau) \tag{92}
\]
in the time interval \( \tau \in [0, T] \). Thus
\[
|G| \lesssim \beta + (\frac{\lambda}{\lambda_1} - 1)^2 + (\frac{\lambda}{\lambda_1} - 1)^3 + \beta|\frac{\lambda}{\lambda_1} - 1|. \tag{93}
\]
Observe that \( \frac{\lambda}{\lambda_1}(t(\tau)) - 1 = 0 \) when \( \tau = T \). Thus Equations (91) can be rewritten as
\[
\frac{\lambda}{\lambda_1}(t(\tau)) - 1 = -\int_\tau^T e^{-\int_s^T 2a(t)dt}G(s)ds. \tag{94}
\]
We claim that Equations (92) and (93) imply (90). Indeed, define an estimating function \( \Lambda(\tau) \) as
\[
\Lambda(\tau) := \sup_{\tau \leq s \leq T} \beta^{-1}(s)|\frac{\lambda}{\lambda_1}(t(s)) - 1|.
\]
Then (94) and the assumption \( A(\tau), B(\tau) \leq \beta^{-\frac{1}{4}}(\tau) \) imply \( 2a \geq \frac{1}{4} \) and
\[
|\frac{\lambda}{\lambda_1}(t(\tau)) - 1| \lesssim \int_\tau^T e^{-\frac{1}{4}(T-s)}[\beta(s) + \beta^2(s)\Lambda^2(\tau) + \beta^3(s)\Lambda(\tau)]ds \lesssim \beta(\tau) + \beta^2(\tau)\Lambda(\tau) + \beta^3(\tau)\Lambda^2(\tau) + \beta^2(\tau)\Lambda(\tau),
\]
or equivalently
\[
\beta^{-1}(\tau)|\frac{\lambda}{\lambda_1}(t(\tau)) - 1| \lesssim 1 + \beta(\tau)\Lambda^2(\tau) + \beta^2(\tau)\Lambda^3(\tau) + \beta(\tau)\Lambda(\tau).
\]
Consequently by the fact that \( \beta(\tau) \) and \( \Lambda(\tau) \) are decreasing functions we have
\[
\Lambda(\tau) \lesssim 1 + \beta(\tau)\Lambda^2(\tau) + \beta^2(\tau)\Lambda^3(\tau) + \beta(\tau)\Lambda(\tau)
\]
which together with \( \Lambda(T) = 0 \) implies \( \Lambda(\tau) \lesssim 1 \) for any time \( \tau \in [0, T] \). This estimate and the definition of \( \Lambda(\tau) \) imply (90).
Now we prove lemmas which will be used in the proofs of (111) and (112). Recall the definition of the function $v(y, \tau)$ in (111).

**Lemma 11.2.** Assume all the conditions in Lemma (9.7). Then for $\tau \leq T$ we have

\[
\| \exp \frac{\alpha}{4} z^2 \eta(\cdot, \sigma) \|_{m,n} \lesssim \beta^{\frac{m+n+1}{2}} \tau M_{m,n}(\sigma); \tag{95}
\]

\[
\| \exp \frac{\alpha}{4} z^2 N_2(a, b, \eta) \|_{m,n} \lesssim \beta^{\frac{m+n+2}{2}} \tau P(M(T)); \tag{96}
\]

\[
\| \exp \frac{\alpha}{4} z^2 F(a, b)(\sigma) \|_{m,n} \lesssim \beta^{\frac{m+n+2}{2}} \tau P(M(T), A(T)); \tag{97}
\]

\[
\| \exp \frac{\alpha}{4} z^2 \mathcal{W}(\eta) \|_{m,n} \lesssim \beta^{\frac{m+n+2}{2}} \tau P(M(T)) \tag{98}
\]

for $(m, n) = (3, 0)$, $(1, 0)$, $(1, 2)$, and $(2, 1)$;

\[
\| \exp \frac{\alpha}{4} z^2 N_1(a, b, \eta) \|_{m,n} \lesssim \beta^{\frac{m+n+2}{2}} P(M(T)); \tag{99}
\]

for $(m, n) = (3, 0)$, $(1, 0)$, $(2, 1)$;

\[
\| \exp \frac{\alpha}{4} z^2 N_1(a, b, \eta) \|_{1,2} \lesssim \beta^2 [M_{3,0}(T) + M_{2,1}(T)] + \beta^{5/2} M_{3,0}^2. \tag{100}
\]

**Proof.** In what follows we use implicitly that

\[
\frac{\lambda_1}{\lambda} \left( t(\tau) \right) - 1 = O(\beta(\tau)) \text{ and therefore } \frac{\lambda_1}{\lambda}(t(\tau)), \frac{\lambda}{\lambda_1}(t(\tau)) \leq 2 \tag{101}
\]

implied by (90), and from which $\langle z \rangle^{-n} \lesssim \langle y \rangle^{-n}$, $n = 1, 2, 3$.

Recall that $e^{\frac{z^2}{2}} \xi = \phi$ after (71). By the definitions of $\eta$ and $M_{m,n}$ in (80) and (83) we have

\[
\| \exp \frac{\alpha}{4} z^2 \eta(\sigma) \|_{m,n} \lesssim \| \phi(\tau(\sigma)) \|_{m,n} \lesssim \beta^{\frac{m+n+1}{2}} \tau M_{m,n}(\tau(\sigma))
\]

which is (95).

The relation between $N_2$ and $N_2$ in (83) and the estimates of $N_2$ in (79) imply

\[
\| \exp \frac{\alpha}{4} z^2 N_2(a, b, \eta) \|_{m,n} \lesssim \| \exp \frac{\alpha y^2}{4} N_2 \|_{m,n} \lesssim \beta^{\frac{m+n+2}{2}} \tau P(M(T))
\]

which is (96). Similarly we prove (99) and (100).

By the definition of $F$ and the estimate of $F$ in (83) we have

\[
\| \exp \frac{\alpha}{4} z^2 F(a, b)(\sigma) \|_{m,n} \lesssim \| e^{\frac{z^2}{2}} F(a, b)(\tau(\sigma)) \|_{m,n} \lesssim \beta^{\frac{m+n+2}{2}} \tau P(M(T), A(T))
\]

which is (97).

Now we prove (98). Equation (98) and the fact $y = \frac{\lambda(t)}{\lambda_1(t)} z$ after (80) yield

\[
\| \exp \frac{\alpha}{4} z^2 \mathcal{W}(\eta) \|_{m,n} \lesssim [\Omega_1 + \Omega_2]\| \exp \frac{\alpha}{4} z^2 \eta(\sigma) \|_{m,n} \lesssim \beta^{\frac{m+n+1}{2}} \tau M_{m,n}(T)[\Omega_1 + \Omega_2]
\]

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with
\[ \Omega_1 := |\frac{\beta}{\lambda_1} - 1| + |a(\tau(\sigma)) - \alpha|, \quad \Omega_2 := |\frac{b(\tau(\sigma)) - \beta(\tau(\sigma))}{\beta(\tau(\sigma))}|. \]

Equations (101) and (92) imply that \( \Omega_1 \leq \beta \); the assumption on \( B \) and its definition imply \( \Omega_2 \leq \beta^{\frac{4}{4}}(\tau(\sigma)) \).

Consequently
\[ \Omega_1 + \Omega_2 \leq \beta^{\frac{4}{4}}(\tau(\sigma)). \]

Collecting the estimates above we have (98).

\[ \square \]

**Lemma 11.3.** For any \( c_1, c_2 > 0 \) there exists a constant \( c(c_1, c_2) \) such that
\[ \int_0^S \exp(-c_1(S-s)\beta^2(\tau(s)))ds \leq c(c_1, c_2)\beta^2(T). \]  
(102)

**Proof.** By the definition of \( \tau(\sigma) \) we have that \( \sigma = \int_0^{\lambda(\tau)} \lambda^{-2}(s)ds \) and \( \tau = \tau(\sigma) = \int_0^{\lambda(\tau)} \lambda^{-2}(k)dk \). By (101) we have
\[ 4\sigma \geq \tau(\sigma) \geq \frac{1}{4}\sigma \]  
(103)

which implies
\[ \frac{1}{b_0 + \frac{\alpha S}{2} + \frac{1}{\alpha}} \leq 4 \frac{1}{b_0 + \frac{\alpha S}{2}}, \]
which in turn gives
\[ \int_0^S \exp(-c_1(S-s)\beta^2(\tau(s)))ds \leq c(c_1, c_2)\frac{1}{(b_0 + \frac{\alpha S}{2})^2}. \]
(104)

Using (103) again we obtain \( 4S \geq \tau(S) = T \geq \frac{1}{4}S \) which together with (104) implies that
\[ \int_0^S \exp(-c_1(S-s)\beta^2(\tau(s)))ds \leq c(c_1, c_2)\frac{1}{(b_0 + \frac{\alpha S}{2})^2} \leq c(c_1, c_2)\beta^2(T). \]

Hence
\[ \int_0^S \exp(-c_1(S-s)\beta^2(\tau(s)))ds \leq c(c_1, c_2)\beta^2(T) \]
which is (102).  
\[ \square \]

We consider the spectrum of the operator \( L_\alpha \). Due to the quadratic term \( \frac{1}{4}\alpha z^2 \), the operator \( L_\alpha \) has a discrete spectrum. For \( \beta z^2 \ll 1 \) it is closed to the harmonic oscillator Hamiltonian
\[ L_\alpha - \alpha := -\partial_z^2 + \frac{1}{4}\alpha^2 z^2 - \frac{5\alpha}{2}. \]  
(105)

The spectrum of the operator \( L_\alpha - \alpha \) is
\[ \sigma(L_\alpha - \alpha) = \{n\alpha| \ n = -2, -1, 0, 1, \ldots \}. \]  
(106)

Thus it is essential that we solve the evolution equation (28) on the subspace orthogonal to the first three eigenvectors of \( L_\alpha \). These eigenvectors, normalized, are
\[ \phi_{0,\alpha}(z) := \frac{\alpha}{2\pi} \frac{1}{\sqrt{\alpha}} \exp(-\frac{\alpha}{4} z^2), \quad \phi_{1,\alpha}(z) := \frac{\alpha}{2\pi} \frac{1}{\sqrt{\alpha}} \sqrt{\alpha} z \exp(-\frac{\alpha}{4} z^2), \quad \phi_{2,\alpha}(z) := \frac{\alpha}{8\pi} \frac{1}{\sqrt{\alpha}} (1 - \alpha z^2) \exp(-\frac{\alpha}{4} z^2). \]  
(107)
We define the orthogonal projection $P_n^\alpha$ onto the space spanned by the first $n$ eigenvectors of $L_\alpha$,

$$P_n^\alpha = \sum_{m=0}^{n-1} |\phi_{m,\alpha}\rangle \langle \phi_{m,\alpha}|$$

(108)

and the orthogonal projection

$$P_n^\alpha := 1 - P_n^\alpha, n = 1, 2, 3.$$

The following lemma establishes a relation between the functions $\phi := e^{\frac{\alpha}{2}z^2} \xi$ and $\eta$ at the times $\sigma = S$, $\tau = T$.

**Lemma 11.4.** If $m + n \leq 3$ and $l \geq 0$ then

$$\langle z \rangle^{-l} \exp \frac{\alpha z^2}{4} P_m^\alpha (\partial_z + \frac{\alpha}{2} z)^n \eta(z, S) = \langle y \rangle^{-l} \partial_y^n \phi(y, T).$$

(109)

**Proof.** By the various definitions in (86) we have $\lambda_1(t(T)) = \lambda(t(T)) = \alpha$, and hence $z = y$ and, $e^{\frac{\alpha}{2}z^2} \eta(z, S) = \phi(y, T)$. Thus by the fact $e^{\frac{\alpha}{2}z^2} (\partial_z + \frac{\alpha}{2} z) e^{-\frac{\alpha}{2}z^2} = \partial_z$ we have

$$\langle z \rangle^{-l} \exp \frac{\alpha z^2}{4} P_m^\alpha (\partial_z + \frac{\alpha}{2} z)^n \eta(z, S) = \langle y \rangle^{-l} \exp \frac{a(T)y^2}{4} P_m^\alpha(y) \exp -\frac{a(T)y^2}{4} \partial_y^n \phi(y, T).$$

(110)

By a standard integrating by part technique, the condition $\xi(\cdot, \tau) \perp \phi_{k,a(\tau)}$, $k = 0, 1, 2$, and the definitions of $\phi_{k,a}$ above yield

$$\exp -\frac{a(T)y^2}{4} \partial_y^n \phi(\cdot, \tau) \perp \phi_{k,a(\tau)}, \quad 0 \leq k \leq 2 - n,$$

i.e. $P_m^\alpha \exp -\frac{a(T)y^2}{4} \partial_y^n \phi(y, T) = \exp -\frac{a(T)y^2}{4} \partial_y^n \phi(y, T)$ if $m + n \leq 3$. This together with (110) implies (109).

The following proposition provides various decay estimates on the propagators generated by $-L_\alpha$ and $-L_\alpha$.

**Proposition 11.5.** For any function $g$ and times $\tau$, $\sigma$ with $\tau \geq \sigma \geq 0$ we have

$$\| \langle z \rangle^{-n} \exp \frac{\alpha z^2}{4} \exp -L_\alpha \sigma P_n^\alpha g \|_\infty \lesssim \exp (1 - n) \alpha \sigma \| \langle z \rangle^{-n} \exp \frac{\alpha z^2}{4} g \|_\infty$$

(111)

with $2 \geq n \geq 1$; and there exist constants $c_0, \delta > 0$ such that if $\beta(0) \leq \delta$, then

$$\| \langle z \rangle^{-n} \exp \frac{\alpha z^2}{4} P_n^\alpha U_n(\tau, \sigma) P_n^\alpha g \|_\infty \lesssim \exp -(c_0 + (n - 3) \alpha) (\tau - \sigma) \| \langle z \rangle^{-n} \exp \frac{\alpha z^2}{4} g \|_\infty$$

(112)

where $U_n(\tau, \sigma)$ denotes the propagator generated by the operator $-P_n^\alpha L_\alpha P_n^\alpha$, $n = 1, 2, 3$.

**Proof.** By results of [9] [11] [16] we have

$$\| \langle z \rangle^{-n} e^{\frac{\alpha}{2}z^2} e^{-L_\alpha \sigma} P_n^\alpha g \|_\infty \lesssim e^{-\sigma(n-1)} \| \langle z \rangle^{-n} e^{\frac{\alpha}{2}z^2} g \|_\infty, \quad n = 1, 2.$$

(113)
In particular by the estimate \( n = 1 \),

\[
\| \langle z \rangle^{-1} e^{\frac{\alpha}{2} z^2} e^{-L_\alpha \sigma} P_{n_2}^\alpha g \|_\infty \lesssim \| \langle z \rangle^{-1} e^{\frac{\alpha}{2} z^2} P_{n_2}^\alpha g \|_\infty \lesssim \| \langle z \rangle^{-1} e^{\frac{\alpha}{2} z^2} g \|_\infty.
\]

using the fact \( P_0^\alpha P_0^\alpha = P_2^\alpha \) and the explicit form of \( P_2^\alpha \). For \( 2 \geq n \geq 1 \) we use the interpolation technique to get Equation (111).

The case \( n = 3 \) of (112) was proved in [11], Proposition 10 (cf [9, 10]). The proof of the other cases is similar, thus is omitted.

12 Estimate of \( M_{3,0} \)

In this section we prove Estimate (111) on the function \( M_{3,0} \). Given any time \( \tau \), choose \( T = \tau \). Then we have the estimates of Proposition 11.1 for \( \tau \leq T \). We start from estimating \( \eta \) defined in Equation (86). We observe that the function \( \eta \) is not orthogonal to the first three eigenvectors of the operator \( L_\alpha \). Therefore we derive the equation for \( P_3^\alpha \eta \):

\[
\frac{d}{d \sigma} P_3^\alpha \eta = -P_3^\alpha L_\alpha P_3^\alpha \eta + \sum_{k=1}^{5} D^{(k)}_{3,0}(\sigma)
\]  \hspace{1cm} (114)

where the functions \( D^{(k)}_{m,n} \equiv D^{(k)}_{m,n}(\sigma) \), \( k = 1, 2, 3, 4, 5 \), \( (m,n) = (3,0), (2,0), (1,2), (2,1) \), are defined as

\[
D^{(1)}_{m,n} := -P_m^\alpha V \exp -\frac{\alpha}{4} z^2 \partial_z \exp \frac{\alpha}{4} z^2 \eta + P_m^\alpha V P_m^\alpha \exp -\frac{\alpha}{4} z^2 \partial_z \exp \frac{\alpha}{4} z^2 \eta,
\]

\[
D^{(2)}_{m,n} := P_m^\alpha \exp -\frac{\alpha}{4} z^2 \partial_z \exp \frac{\alpha}{4} z^2 W \eta,
\]

\[
D^{(3)}_{m,n} := P_m^\alpha \exp -\frac{\alpha}{4} z^2 \partial_z \exp \frac{\alpha}{4} z^2 \mathcal{F}(a,b),
\]

\[
D^{(4)}_{m,n} := P_m^\alpha \exp -\frac{\alpha}{4} z^2 \partial_z \exp \frac{\alpha}{4} z^2 N_1(a,b,\alpha,\eta),
\]

\[
D^{(5)}_{m,n} := P_m^\alpha \exp -\frac{\alpha}{4} z^2 \partial_z \exp \frac{\alpha}{4} z^2 N_2.
\]

where, recall the definitions of the function \( \mathcal{F} \), the operator \( W \) after Equation (88) and the definition of \( P_m^\alpha \) before (108).

Now we start with estimating the terms \( D^{(k)}_{3,0} \), \( k = 1, 2, 3, 4, 5 \), on the right hand side of (114).

Lemma 12.1. If \( A(\tau) \), \( B(\tau) \leq b^\pm(\tau) \) and if \( \sigma \leq S \) (equivalently \( \tau \leq T \)) then we have

\[
\sum_{k=1}^{5} \| \exp \frac{\alpha}{4} z^2 P_3^\alpha D^{(k)}_{3,0}(\sigma) \|_{3,0} \lesssim b^\pm(\tau(\sigma)) P(M(T), A(T)).
\]  \hspace{1cm} (115)

Proof. We rewrite \( P_3^\alpha D^{(1)}_{3,0} \) as

\[
P_3^\alpha D^{(1)}_{3,0}(\sigma) = P_3^\alpha \frac{\alpha + \frac{1}{2}}{2(d-1) + b(\tau(\sigma))z^2} b(\tau(\sigma))z^2 (1 - P_3^\alpha) \eta(\sigma)
\]
which admits the estimate
\[ \| \exp \frac{\alpha}{4} z^2 P_3^n D^{(1)}_{3,0}(\sigma) \|_{3,0} \lesssim \| z^{-1} b(z(\sigma)) z^2 \| \| \exp \frac{\alpha}{4} z^2 (1 - P_3^n) \eta(\sigma) \|_{2,0} \lesssim b^2(\tau(\sigma)) \| \exp \frac{\alpha}{4} z^2 \eta(\sigma) \|_{3,0} \lesssim \beta \frac{m+\frac{1}{2}}{(\tau(\sigma)) M_{3,0}(T)} \]
where we use (95), the fact that \( |b(\tau)| \leq 2\beta(\tau) \) implied by \( B(\tau) \leq \beta - \frac{1}{4}(\tau) \) and the fact for any \( m \geq 1 \)
\[ \| \exp \frac{\alpha}{4} z^2 (1 - P_m^n) g \|_{m-1,0} \lesssim \| \exp \frac{\alpha}{4} z^2 g \|_{m,0} \] (116)

by (108). Thus we have the estimate for \( D^{(1)}_{3,0} \).

Now we estimate \( D^{(k)}_{3,0} \), \( k = 2, 3, 4, 5 \). First by (116) we observe
\[ \sum_{k=2}^{5} \| \exp \frac{\alpha}{4} z^2 P_3^n D^{(k)}_{3,0}(\sigma) \|_{3,0} \leq \| \exp \frac{\alpha}{4} z^2 F(a, b)(\sigma) \|_{3,0} + \| \exp \frac{\alpha}{4} z^2 W \eta(\sigma) \|_{3,0} + \| \exp \frac{\alpha}{4} z^2 \eta(\sigma) \|_{3,0} + \| \exp \frac{\alpha}{4} z^2 \eta(\sigma) \|_{3,0}. \]

The estimates of \( F, N_1, N_2 \) and \( W \eta \) in (96)-(99) imply
\[ \sum_{k=2}^{5} \| \exp \frac{\alpha}{4} z^2 P_3^n D^{(k)}_{3,0}(\sigma) \|_{3,0} \lesssim \beta^2(\tau(\sigma)) P(M(T), A(T)). \]

Collecting the estimates above we complete the proof.

Now we prove Equation (111). Let \( S \) and \( T \) be the same as in Section 11. By Duhamel principle we rewrite Equation (114) as
\[ P_3^n \eta(S) = P_3^n U_3(S, 0) P_3^n \eta(0) + \sum_{n=1}^{5} \int_0^S P_3^n U_3(S, \sigma) P_3^n D^{(n)}_{3,0}(\sigma) d\sigma, \]
(117)

where, recall, \( U_3(\tau, \sigma) \) is defined and estimated in (112), from which we obtain
\[ \beta^{-2}(T) \| \exp \frac{\alpha}{4} z^2 P_3^n \eta(S) \|_{3,0} \lesssim \exp -c_0 S \beta^{-2}(T) \| \exp \frac{\alpha}{4} z^2 \eta(0) \|_{3,0} + \beta^{-2}(T) \sum_{k=1}^{5} \int_0^S \exp -c_0 (S - \sigma) \| \exp \frac{\alpha}{4} z^2 D^{(k)}_{3,0}(\sigma) \|_{3,0} d\sigma. \]
(118)

Now we estimate each term on the right hand side. We begin with the first term. By the slow decay of \( \beta(\tau) \) and Equation (95) we have
\[ \exp -c_0 S \beta^{-2}(T) \| \exp \frac{\alpha}{4} z^2 \eta(0) \|_{3,0} \lesssim \beta^{-2}(0) \| \exp \frac{\alpha}{4} z^2 \eta(0) \|_{3,0} \lesssim M_{3,0}(0). \]
(119)

For the second term we use the integral estimate (112) and the estimate of \( D^{(k)}_{3,0} \) in Equation (115) to obtain
\[ \sum_{k=1}^{5} \int_0^S \exp -c_0 (S - \sigma) \| \exp \frac{\alpha}{4} z^2 D^{(k)}_{3,0}(\sigma) \|_{3,0} d\sigma \lesssim \beta^{5/2}(T) P(M(T), A(T)). \]
(120)
By (109) we have \( \|\exp \frac{\alpha}{4} z^2 P_2^\alpha \eta(S)\|_{3,0} = \| \phi(\cdot, T) \|_{3,0} \) which together with (119) and (120) implies
\[
\beta^{-2}(T)\|\phi(\cdot, T)\|_{3,0} \lesssim M_{3,0}(0) + \beta^\frac{7}{2}(T)P(M(T), A(T))
\]
where \( P \) is a nondecreasing polynomial. By the definition of \( M_{3,0} \) in (36) we obtain
\[
M_{3,0}(T) \lesssim M_{3,0}(0) + \beta^\frac{7}{2}(0)P(M(T), A(T)).
\]
Since \( T \) is an arbitrary Equation (41) follows.

13 Proof of Equation (42)

We derive an equation for \( P_2^\alpha \eta(\sigma) \) from Equation (88) as
\[
\frac{d}{d\sigma} P_2^\alpha \eta = -L_\alpha P_2^\alpha \eta - P_2^\alpha V\eta + P_2^\alpha \sum_{k=2}^{5} D_{2,0}^{(k)}
\]
(121)

where the functions \( D_{2,0}^{(k)} \) and the operator \( L_\alpha \) are defined after (114) and (88) respectively.

Lemma 13.1. If \( A(\tau), B(\tau) \leq \beta^{-\frac{3}{2}}(\tau) \), then
\[
\| \exp \frac{\alpha}{4} z^2 V\eta(\sigma)\|_{11,0,0} \lesssim \beta^\frac{3}{2}(\tau(\sigma))M_{3,0}(T).
\]
(122)

\[
\sum_{k=1}^{5} \| \exp \frac{\alpha}{4} z^2 P_2^\alpha D_{2,0}^{(k)}(\sigma)\|_{11,0,0} \lesssim \beta^\frac{3}{2}(\tau)P(M(T), A(T))
\]
(123)

Proof. By the assumption on \( B \) we have \( \frac{1}{b} \lesssim \frac{1}{\tau} \) hence
\[
\langle z \rangle^{-\frac{3}{40}} \frac{1}{1 + b(\tau(\sigma))z^2} \lesssim \langle z \rangle^{-\frac{3}{40}} (1 + b(\tau(\sigma))z^2)^{-\frac{3}{2}} \lesssim \beta^{-19/20}(\tau(\sigma))\langle z \rangle^{-3}
\]
which together with the definition of \( V \) after (88) and the estimate in (95) yields
\[
\| \exp \frac{\alpha}{4} z^2 V\eta(\sigma)\|_{11,0,0} \lesssim \| \frac{1}{1 + b(\tau(\sigma))z^2} \exp \frac{\alpha}{4} z^2 \eta(\sigma)\|_{11,0,0}
\]
\[
\lesssim \beta^{-\frac{3}{20}}(\tau(\sigma))\exp \frac{\alpha}{4} z^2 \eta(\sigma)\|_{3,0}
\]
\[
\lesssim \beta^\frac{3}{20}(\tau(\sigma))M_{3,0}(T).
\]

This gives (122). The proof of (123) is almost the same to that of (115) and, thus omitted.

Rewrite (121) to have
\[
P_2^\alpha \eta(S) = \exp -L_\alpha SP_2^\alpha \eta(0) + \int_{0}^{S} \exp -L_\alpha (S-\sigma)P_2^\alpha [-V \eta + \sum_{k=2}^{5} D_{2,0}^{(k)}]d\sigma,
\]
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where, recall the definition of $S$ in (87). By the propagator estimate of $\exp -L_\alpha \sigma P_2^\alpha$ in (111), we have
\[
\| \exp \frac{\alpha}{4} z^2 P_2^\alpha \eta(S) \|_{10,0} \lesssim K_0 + K_1 + K_2
\] (124)
where the functions $K_n$’s are given by
\[
K_0 := \exp -\alpha S \| \exp \frac{\alpha}{4} z^2 \gamma(0) \|_{10,0}, \quad K_1 := \int_0^S \exp -\alpha (S - \sigma) \| \exp \frac{\alpha}{4} z^2 V \eta(\sigma) \|_{10,0} d\sigma,
\]
\[
K_2 := \sum_{k=2}^5 \int_0^S \exp -\alpha (S - \sigma) \| \exp \frac{\alpha}{4} z^2 D_{2,0}^{(k)} \|_{10,0} d\sigma.
\]
Next, we estimate $K_n$’s, $n = 0, 1, 2$.

(K0) Equation (95) and the slow decay of $\beta$ yield
\[
K_0 \lesssim \beta^\frac{21}{20} (T) \beta^{-\frac{21}{20}} (0) \| \exp \frac{\alpha}{4} z^2 \eta(0) \|_{10,0} \lesssim \beta^\frac{21}{20} (T) M_{10,0}(0).
\] (125)

(K1) The estimate in (122) and the integral estimate in (102) imply
\[
K_1 \lesssim \int_0^S \exp -\alpha (S - \sigma) \beta^\frac{31}{20} (\tau(\sigma)) d\sigma M_{3,0}(T) \lesssim \beta^\frac{21}{20} (T) M_{3,0}(T).
\] (126)

(K2) The estimates of $D_{2,0}^{(k)}$, $k = 2, 3, 4, 5$, in Equation (123) yield the bound
\[
K_2 \lesssim \int_0^S \exp -\alpha (S - \sigma) \beta^\frac{31}{20} + \frac{1}{2} (\tau(\sigma)) d\sigma P(M(T), A(T)) \lesssim \beta^\frac{21}{20} (T) P(M(T), A(T)).
\] (127)

Collecting the estimates (124)-(127) we have
\[
\beta^{-\frac{31}{20}} (T) \| \phi(T) \|_{10,0} \lesssim M_{10,0}(0) + M_{3,0}(T) + \beta^\frac{1}{2} (0) P(M(T), A(T)).
\] (128)

By Equation (109) we have
\[
\beta^{-\frac{31}{20}} (T) \| \phi(T) \|_{10,0} = \beta^{-\frac{31}{20}} (T) \| \exp \frac{\alpha}{4} z^2 P_2^\alpha \eta(S) \|_{10,0}
\]
which together with (128) and the definition of $M_{10,0}$ implies
\[
M_{10,0}(T) \lesssim M_{10,0}(0) + M_{3,0}(T) + \beta^\frac{1}{2} (0) P(M(T), A(T)).
\]
Since $T$ is an arbitrary time, the proof is complete.
14 Proof of Equation (43)

By Equation (43) and the observation

\[ \exp \left( -\frac{\alpha z^2}{4} \partial_z \left[ \exp \frac{\alpha}{4} z^2 g \right] \right) = (\partial_z + \frac{\alpha}{2} z) g \] (129)

for any function \( g \), the function \( P_\alpha^\eta (\partial_z + \frac{\alpha}{2} z) \eta \) satisfies

\[ \frac{d}{d\sigma} P_\alpha^\eta (\partial_z + \frac{\alpha}{2} z) \eta = -P_\alpha^\eta (L_\alpha + \alpha) P_\alpha^\eta (\partial_z + \frac{\alpha}{2} z) \eta + \sum_{k=1}^{5} D_{2,1}^{(k)} + D_6 \] (130)

with \( D_{2,1}^{(k)} \) defined after (114) and \( D_6 := -P_\alpha^\eta \partial_z V \).

Thus applying the operator \( \partial_z + \frac{\alpha}{2} z \) leads to the equation with improved linear part.

Lemma 14.1. If \( A(\tau), B(\tau) \leq \beta - \frac{\tau}{4} \), then we have

\[ \| \exp \frac{\alpha}{4} z^2 \partial_\sigma \|_{2,0} \lesssim \beta^2 (\tau(\sigma)) M_{3,0}(T). \] (131)

\[ \| e^{\frac{\alpha z^2}{4}} \sum_{k=1}^{5} D_{2,1}^{(k)}(\sigma) \|_{2,0} \lesssim \beta^{5/2} (\tau(\sigma)) P(M(T), A(T)). \] (132)

The proofs are the same as those of (122) and (115) and, thus are omitted.

By the Duhamel principle we rewrite Equation (130) as

\[ P_\alpha^\eta (\partial_z + \frac{\alpha}{2} z) \eta(S) = P_\alpha^\eta U_2(S,0) \exp -\alpha S P_\alpha^\eta (\partial_z + \frac{\alpha}{2} z) \eta(0) + \int_0^S P_\alpha^\eta U_2(S,\sigma) \exp -\alpha (S - \sigma) P_\alpha^\eta \sum_{n=1}^{5} D_{2,1}^{(n)} + D_6 \] d\sigma,

where \( U_2 \) is defined and estimated in (112), from which we have

\[ \| \exp \frac{\alpha}{4} z^2 P_\alpha^\eta (\partial_z + \frac{\alpha}{2} z) \eta(S) \|_{2,0} \lesssim Y_1 + Y_2 + Y_3 \] (133)

with

\[ Y_1 := \exp -c_0 S \| \exp \frac{\alpha z^2}{4} \eta(0) \|_{2,0}; \]

\[ Y_2 := \int_0^S \exp -c_0 (S - \sigma) \sum_{k=1}^{5} \| \exp \frac{\alpha z^2}{4} D_{2,1}^{(k)}(\sigma) \|_{2,0} d\sigma; \]

\[ Y_3 := \int_0^S \exp -c_0 (S - \sigma) \| \exp \frac{\alpha z^2}{4} D_6(\sigma) \|_{2,0} d\sigma. \]

Next, we estimate \( Y_n, \ n = 1, 2, 3. \) By (132) and the integral estimate (102) we have

\[ Y_2 \lesssim \int_0^S \exp -c_0 (S - \sigma) \beta^{5/2} (\tau(\sigma)) d\sigma P(M(T), A(T)) \lesssim \beta^{5/2}(T) P(M(T), A(T)); \] (134)

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by (131). By similar reasoning,
\[ Y_3 \lesssim \beta^2(T)M_{3,0}(T); \]  
and by (93) and the slow decay of \( \beta \),
\[ Y_1 \lesssim \exp -c_0S\| \phi(.0)\|_{2,1} \lesssim \beta^2(T)M_{2,1}(0). \]  
Collecting the estimates (133)-(136) we obtain
\[ \beta^{-2}(T)\| \exp \frac{\alpha z^2}{4}P_2^\alpha(\partial_z + \frac{\alpha}{2}z)\eta(S)\|_{2,0} \lesssim M_{2,1}(0) + M_{3,0}(T) + \beta^2(0)P(M(T),A(T)). \]  
Moreover by Equation (109) we have
\[ \| \exp \frac{\alpha z^2}{4}P_2^\alpha(\partial_z + \frac{\alpha}{2}z)\eta(S)\|_{2,0} = \| \phi(T)\|_{2,1}. \]  
Thus by the definition of \( M_{2,1} \)
\[ M_{2,1}(T) \lesssim M_{2,1}(0) + M_{3,0}(T) + \beta^2(0)P(M(T),A(T)) \]  
which together with fact that \( T \) is arbitrary implies (43).

15 Proof of Equation (44)

By Equation (88) the function
\[ \frac{d}{d\sigma}P_1^\alpha(\partial_z + \frac{\alpha}{2}z)^2\eta \]  
satisfies the equation
\[ \frac{d}{d\sigma}P_1^\alpha(\partial_z + \frac{\alpha}{2}z)^2\eta = -P_1^\alpha(L_\alpha + 2\alpha)P_1^\alpha(\partial_z + \frac{\alpha}{2}z)^2\eta + P_1^\alpha \sum_{k=1}^{5} D_{1,2}^{(k)} + D_7 \]  
with \( D_{1,2}^{(k)} \) defined after (114) and
\[ D_7 := -P_1^\alpha \exp -\frac{\alpha z^2}{4} [\exp \frac{\alpha}{4} z^2 \eta(\partial_z^2 V + 2\partial_z[\exp \frac{\alpha}{4} z^2 \eta] \partial_z V]. \]

Lemma 15.1. If \( A(\tau), B(\tau) \leq \beta^{-4}(\tau) \), then we have
\[ \| \exp \frac{\alpha z^2}{4}2D_7(\sigma)\|_{1,0} \lesssim \beta^2(\tau(\sigma))[M_{3,0}(T) + M_{2,1}(T)], \]  
\[ \| e^{\frac{\alpha z^2}{4}} \sum_{k=1}^{5} D_{1,2}^{(k)}(\sigma)\|_{1,0} \lesssim \beta^2(\tau(\sigma))[M_{3,0}(T) + M_{2,1}(T)] + \beta^{5/2}(\tau(\sigma))P(M(T),A(T)). \]

The proofs are almost the same as those of (122) and (115), thus omitted.

By Duhamel principle we rewrite Equation (137) as
\[ P_1^\alpha(\partial_z + \frac{\alpha}{2}z)^2\eta(S) = P_1^\alpha U_1(S,0) \exp -2\alpha SP_1^\alpha(\partial_z + \frac{\alpha}{2}z)^2\eta(0) \]  
\[ + \int_0^S P_1^\alpha U_1(S,\tau) \exp -2\alpha(S-\tau)P_1^\alpha \sum_{n=1} D_{1,2}^{(n)} + D_7(\tau) d\tau \]  
37
where, recall $U_1(t, s)$ is defined and estimated in (112), from which we have

$$\| \exp \frac{\alpha z^2}{4} P_{1}^n (\partial_z + \frac{\alpha}{2} z^2) \eta(S) \|_{1,0} \lesssim Z_1 + Z_2$$

with

$$Z_1 := \exp -c_0 S \| \exp \frac{\alpha z^2}{4} \eta(0) \|_{1,2};$$

$$Z_2 := \int_0^S \exp -c_0 (S - \sigma) \sum_{k=1}^5 \| \exp \frac{\alpha z^2}{4} D_{1,2}(\sigma) \|_{1,0} d\sigma + \int_0^S \exp -c_0 (S - \sigma) \| \exp \frac{\alpha z^2}{4} D_7(\sigma) \|_{1,0} d\sigma.$$

By (138), (139) we have

$$Z_2 \lesssim \int_0^S \exp -c_0 (S - \sigma) \beta^{5/2}(\sigma) P(M(T), A(T)) + \beta^2(\sigma)[M_{2,1}(T) + M_{3,0}(T)] d\sigma \lesssim \beta^{5/2}(T) P(M(T), A(T)) + \beta^2(T)[M_{3,0}(T) + M_{2,1}(T)];$$

and the slow decay of $\beta$

$$Z_1 \lesssim \exp -c_0 S \| \phi(\cdot, 0) \|_{1,2} \lesssim \beta^2(T) M_{1,2}(0).$$

Estimates (140)-(142) yield the bound

$$\beta^{-2}(T) \| \exp \frac{\alpha z^2}{4} P_{1}^n (\partial_z + \frac{\alpha}{2} z^2) \eta(S) \|_{1,0} \lesssim M_{1,2}(0) + M_{3,0}(T) + M_{2,1}(T) + \beta^4(0) P(M(T), A(T)).$$

By Equation (109) we obtain $\| \exp \frac{\alpha z^2}{4} P_{1}^n (\partial_z + \frac{\alpha}{2} z^2) \eta(S) \|_{1,0} = \| \partial^2_y \phi(T) \|_{1,0}$ which together with the definition of $M_{1,2}$ yields

$$M_{1,2}(T) \lesssim M_{1,2}(0) + M_{3,0}(T) + M_{2,1}(T) + \beta^4(0) P(M(T), A(T)).$$

Since $T$ is arbitrary (144) follows.

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