HÖRMANDER’S THEOREM FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

N.V. KRYLOV

Abstract. We prove Hörmander’s type hypoellipticity theorem for stochastic partial differential equations when the coefficients are only measurable with respect to the time variable. The need for such kind of results comes from filtering theory of partially observable diffusion processes, when even if the initial system is autonomous, the observation process enters the coefficients of the filtering equation and makes them time-dependent with no good control on the smoothness of the coefficients with respect to the time variable.

1. Introduction

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space with an increasing filtration \(\{\mathcal{F}_t, t \geq 0\}\) of complete with respect to \((\mathcal{F}, P)\) \(\sigma\)-fields \(\mathcal{F}_t \subset \mathcal{F}\). Let \(d_1 \geq 1\) be an integer and let \(w^k_t, k = 1, 2, ..., d_1\), be independent one-dimensional Wiener processes with respect to \(\{\mathcal{F}_t\}\).

Fix an integer \(d \geq 1\) and introduce \(\mathbb{R}^d\) as a Euclidean space of column-vectors (written in a common abuse of notation as) \(x = (x^1, ..., x^d)\). Denote

\[
D_i = \partial / \partial x^i, \quad D_{ij} = D_i D_j
\]

and for an \(\mathbb{R}^d\)-valued function \(\sigma_t(x) = \sigma_t(\omega, x)\) on \(\Omega \times [0, \infty) \times \mathbb{R}^d\) and functions \(u_t(x) = u_t(\omega, x)\) on \(\Omega \times [0, \infty) \times \mathbb{R}^d\) set

\[
L_{\sigma_t} u_t(x) = [D_i u_t(x)] \sigma^i_t(x).
\]

Next take an integer \(d_2 \geq 1\), assume that we are given \(\mathbb{R}^d\)-valued functions \(\sigma^k_t = (\sigma^k_t(x)), k = 0, ..., d_1 + d_2\), on \(\Omega \times [0, \infty) \times \mathbb{R}^d\), which are infinitely differentiable with respect to \(x\) for any \((\omega, t)\), and define the operator

\[
L_t = (1/2) \sum_{k=1}^{d_2 + d_1} L^2_{\sigma^k_t} + L_{\sigma^0_t}.
\] (1.1)

Assume that on \(\Omega \times [0, \infty) \times \mathbb{R}^d\) we are also given certain real-valued functions \(c_t(x)\) and \(\nu^k_t(x), k = 1, ..., d_1\), which are infinitely differentiable with respect to \(x\), and that on \(\Omega \times [0, \infty) \times \mathbb{R}^d\) we are given real-valued functions \(f_t\) and

2010 Mathematics Subject Classification. 60H15, 35R60.

Key words and phrases. Hypoellipticity, SPDEs, Hörmander’s theorem.

The author was partially supported by NSF Grant DMS-1160569.
Then under natural additional assumptions which will be specified later the SPDE

$$du_t = (L_t u_t + c_t u_t + f_t) \, dt + (L_{\sigma^k_t} u_t + \nu^k_t u_t + g^k_t) \, dw^k_t$$  \hspace{1cm} (1.2)

makes sense (where and below the summation convention over repeated indices is enforced regardless of whether they stand at the same level or at different ones).

The main goal of this paper is to show, somewhat loosely speaking, that, if $\Omega_0 \in \mathcal{F}$, $(s_1, s_2) \in (0, \infty)$ and for any $\omega \in \Omega_0$ and $t \in (s_1, s_2)$ the Lie algebra generated by the vector-fields $\sigma^{d_1+k}_t$, $k = 1, ..., d_2$, has dimension $d$ everywhere in a ball $B$ in $\mathbb{R}^d$ and $f_t$ and $g^k_t$ are infinitely differentiable in $B$ for any $\omega \in \Omega_0$ and $t \in (s_1, s_2)$, then any function $u_t$ satisfying (1.2) in $\Omega_0 \times (s_1, s_2) \times B$, for almost any $\omega \in \Omega_0$, coincides on $(s_1, s_2) \times B$ with a function which is infinitely differentiable with respect to $x$. Thus, under a local Hörmander’s type condition we claim the local hypoellipticity of the equation.

It is worth mentioning article [5] where the authors prove the hypoellipticity for SPDEs whose coefficients do not explicitly depend on time and $\omega$ under Hörmander’s type condition which is global but otherwise much weaker than ours. The dependence on the time variable $t$ and $\omega$ of the coefficients in [5] is allowed only through an argument in which a Wiener process is substituted. However, it seems to the author of the present article that there is a gap in the arguments in [5] when the authors claim that one can estimate derivatives of order $s + \varepsilon$ ($\varepsilon > 0$) of solutions through derivatives of order $s$ for any $s \in (-\infty, \infty)$ and not only for $s = 0$. The claim is only proved for $s = 0$ in [5] and even if there are no stochastic terms the proof of the claim is not completely trivial (see the comment below formula (5.2) in [8]). It is worth noting that our methods are absolutely different from the methods in [5]. Our main method of proving Theorems 2.3 and 2.4 is based on an observation by A. Wentzell [15] who discovered the Itô-Wentzell formula and used it to make a random change of coordinates in such a way that the stochastic terms in the transformed equation disappear so that we can use the results from [8]. We apply this method locally.

Kunita in [11] also uses Wentzell’s reduction of SPDEs with even time-inhomogeneous coefficients to deterministic equations with random and time-dependent coefficients satisfying a global Hörmander’s type condition. He writes that the probabilistic approach to proving Hörmander’s theorem developed by Malliavin [14], Ikeda and Watanabe [6], Stroock [16], and Bismut [1] can be applied to the case of operators continuously depending on the time parameter $t$. In [12] he replaces this list of references with [14], [6], [17], and [2]. However, to the best of the author’s knowledge until now the best result in proving Hörmander’s theorem by using the Malliavin calculus for parabolic equations with the coefficients only continuous with respect to $t$ are obtained in [4] where equations with coefficients that are Hölder continuous in $t$ are considered. In our case the coefficients are only assumed
to be predictable, so that if they are not random, then their measurability with respect to $t$ suffices. Another objection against the arguments in [11] and [12] is that the reduction of SPDEs is done globally and yields deterministic parabolic equations with random coefficients without any control on their behavior as $|x| \to \infty$, which is needed for any existing theory of unique solvability of such equations.

Wentzell’s method allows us to derive from a local version of Hörmander’s type condition infinite differentiability of solutions at the same locality, whereas in [5], [11], and [12] a global condition is imposed and the way $\omega$ and $t$ enter the coefficients is quite restrictive. Another difference between our results and those in [5] is that we prove infinite differentiability of any generalized solution and not only of measure-valued ones.

Speaking about generalized solution, our functions $u_t, f_t, g^k_t$ are, actually, assumed to be given on a subset of $\Omega \times (0, \infty)$ and take values in $\mathcal{D}$, which is the space of generalized functions on $\mathbb{R}^d$.

One more issue worth noting is that we derive a priori estimates which will allow us in a subsequent article not only show that the filtering density for $t > 0$ is in $C^\infty$ if the unobservable process starts at any fixed point $x$ but also prove that it is infinitely differentiable with respect to $x$. As far as the author is aware such kind of results was never proved for degenerate SPDEs.

We finish the introduction with a few more notation and a description of the structure of the article. For (generalized) functions $u$ on $\mathbb{R}^d$ by $Du$ we mean the row-vector $(D_1 u, ..., D_d u)$ and when we write $Du \phi$ we always mean $(Du) \phi$. In this notation

$$L_{\sigma_t} u_t = [D_i u_t] \sigma^i_t = Du_t \sigma_t.$$

One knows that the product of any generalized function and an infinitely differentiable one is again a generalized function and that any generalized function is infinitely differentiable in the generalized sense, so that what is said above has perfect sense.

For $R, t \in (0, \infty)$ set

$$B_R = \{x \in \mathbb{R}^d : |x| < R\}, \quad C_{t, R} = (0, t) \times B_R,$$

and denote by $\mathcal{D}_R$ the set of generalized functions on $B_R$. In the whole article $T, R_0$ are fixed numbers from $(0, \infty)$.

The rest of the article is organized as follows. In Section 2 we state our main results, Theorems 2.3 and 2.4. Section 3 contains a computation of the determinant of a matrix-valued process satisfying a linear stochastic equation. A very short Section 4 reminds the reader one of properties of stochastic integrals of Hilbert-space valued processes. In Section 5 we discuss some facts related to stochastic flows of diffeomorphisms and change of variables. The reader can find in [13] much more information about stochastic flows of diffeomorphisms in a much more general setting. Our discussion is more elementary than in [13] albeit it is only valid in a particular case we
need. In Section 6 we prove a version of the Itô-Wentzell formula we need. Finally, in Sections 7 and 8 we prove Theorems 2.3 and 2.4, respectively.

2. Main results

Denote by \( \mathcal{P} \) the predictable \( \sigma \)-field in \( \Omega \times (0, \infty) \) associated with \( \{ \mathcal{F}_t \} \).

**Definition 2.1.** Denote by \( \mathcal{D}(C_{T,R_0}) \) the set of all \( \mathcal{D}_{R_0} \)-valued functions \( u \) (written as \( u_t(x) \) in a common abuse of notation) on \( \Omega \times [0,T] \) such that, for any \( \phi \in C^\infty_0(B_{R_0}) \), the restriction of the function \( (u_t, \phi) \) on \( \Omega \times (0,T] \) is \( \mathcal{P} \)-measurable and \( (u_0, \phi) \) is \( \mathcal{F}_0 \)-measurable. For \( p = 1,2 \) denote by \( \mathcal{D}^{-\infty}_p(C_{T,R_0}) \) the subset of \( \mathcal{D}(C_{T,R_0}) \) consisting of \( u \) such that for any \( \zeta \in C^\infty_0(B_{R_0}) \) there exists an \( m \in \mathbb{R} \) such that for any \( \omega \in \Omega \), for almost all \( t \in [0,T] \), we have \( \zeta u_t \in L^p_{T,R} \) \((1 - \Delta)^{-m/2} L^2, \ L_2 = L^2(\mathbb{R}^d) \) and

\[
\int_0^T \| u_t \zeta \|^p_{L^p_{T,R}} \, dt < \infty. \tag{2.1}
\]

**Definition 2.2.** Assume that we are given some \( u, f, g^k \in \mathcal{D}(C_{T,R_0}) \), \( k = 1, \ldots, d_1 \) (not necessarily those from Section 1). We say that the equality

\[
d u_t(x) = f_t(x) \, dt + g_t^k(x) \, dw_t^k, \quad (t, x) \in C_{T,R_0},
\]

holds *in the sense of distributions* if \( f \in \mathcal{D}_1^{-\infty}(C_{T,R_0}) \), \( g^k \in \mathcal{D}_2^{-\infty}(C_{T,R_0}) \), \( k = 1, \ldots, d_1 \), and for any \( \phi \in C^\infty_0(B_{R_0}) \), with probability one we have

\[
(u_t, \phi) = (u_0, \phi) + \int_0^t (f_s, \phi) \, ds + \sum_{k=1}^{d_1} \int_0^t (g_s^k, \phi) \, dw_s^k \tag{2.3}
\]

for all \( t \in [0,T] \), where, as usual, \((\cdot, \cdot)\) stands for pairing of generalized and test functions.

**Remark 2.1.** Observe that if \( g^k \in \mathcal{D}_2^{-\infty}(C_{T,R_0}) \), \( \phi, \zeta \in C^\infty_0(B_{R_0}) \), and \( \zeta = 1 \) on the support of \( \phi \), then

\[
|\langle g_s^k, \phi \rangle|^2 = |\langle \zeta g_s^k, \phi \rangle|^2 \leq \| \zeta g_s^k \|_{L^p_{T,R}}^2 \| \phi \|_{L^p_{T,R}}^2
\]

and the right-hand side has finite integral over \([0, T] \) (a.s.) if \( m \) is chosen appropriately. This and a similar estimate concerning \((f_s, \phi)\) shows that the right-hand side in (2.3) makes sense.

In the following assumption we are talking about the objects from Section 1.

**Assumption 2.1.** (i) The functions \( \sigma_t^k(x) \), \( k = 0, \ldots, d_1 + d_2, c_t, \nu_t^k \), \( k = 1, \ldots, d_1 \), are infinitely differentiable with respect to \( x \) and each of their derivatives of any order is bounded on \( \Omega \times [0,T] \times B_{R_0} \). These functions are predictable with respect to \((\omega, t)\) for any \( x \in B_{R_0} \);

(ii) We have \( u, f, g^k \in \mathcal{D}_2^{-\infty}(C_{T,R_0}) \), \( k = 1, \ldots, d_1 \);

(iii) Equation (1.2) holds on \( C_{T,R_0} \) in the sense of Definition 2.2;

(iv) for any \( \zeta \in C^\infty_0(B_{R_0}) \) there exists an \( m \in \mathbb{R} \) such that for any \( \omega \in \Omega \), we have \( u_0 \zeta \in H^m_2 \).
Remark 2.2. The argument in Remark 2.1 shows that (1.2) has perfect sense owing to Assumptions 2.1 (i), (ii), and we need $u \in \mathcal{D}^{-\infty}_{\omega}(C_{T,R_0})$ in contrast with Definition 2.2 because $Du$ and $u$ enter the stochastic part in (1.2).

Furthermore, under Assumption 2.1 for any $\zeta \in C_0^\infty(B_{R_0})$ there is an $m$ such that $u_0\zeta \in H_m^2$. It follows by a classical continuity result that (a.s.) $u_t\zeta$ is a continuous $H_{m-1/2}$-valued function on $[0,T]$. If we drop Assumption 2.4 (iv), then the same will be true with $(0,T]$ in place of $[0,T]$ since $u_t\zeta \in H_m^2$ for almost all $t \in (0,T)$.

Next, as usual, for two smooth $\mathbb{R}^d$-valued functions $\sigma, \gamma$ on $\mathbb{R}^d$ we set

$$[\sigma, \gamma] = D\gamma \sigma - D\sigma \gamma,$$

where for instance $D\gamma$ is the matrix with entries $(D\gamma)^{ij} = \frac{\partial}{\partial x_j} \gamma^i$, so that

$$[\sigma, \gamma]^i = \sigma^j D\gamma^i - \gamma^j D\sigma^i.$$

Then introduce collections of $\mathbb{R}^d$-valued functions defined on $\Omega \times [0,T] \times B_{R_0}$ inductively as

$$L_0 = \{\sigma^{d_1+1}, \ldots, \sigma^{d_1+d_2}\},$$

$$L_{n+1} = L_n \cup \{[\sigma^{d_1+k}, M] : k = 1, \ldots, d_2, M \in L_n\}, \quad n \geq 0.$$

For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \{0, 1, \ldots\}$, introduce as usual

$$D^\alpha = D_1^{\alpha_1} \cdot \ldots \cdot D_d^{\alpha_d}, \quad |\alpha| = \alpha_1 + \ldots + \alpha_d.$$

Also define $BC^\infty_b$ as the set of real-valued measurable functions $a$ on $\Omega \times [0,T] \times \mathbb{R}^d$ such that, for each $t \in [0,T]$ and $\omega \in \Omega$, $a_t(x)$ is infinitely differentiable with respect to $x$, and for any $\omega \in \Omega$ and multi-index $\alpha$ we have

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |D^\alpha a_t(x)| < \infty.$$

Finally we denote by $\text{Lie}_n$ the set of (finite) linear combinations of elements of $L_n$ with coefficients which are of class $BC^\infty_b$. Observe that the vector-field $\sigma^0$ is not explicitly included into $\text{Lie}_n$. Finally, fix $\Omega_0 \in \mathcal{F}$, $S \in [0,T)$, and introduce

$$G = (S,T) \times B_{R_0}.$$

Assumption 2.2. For every $\omega \in \Omega_0$, $\eta \in C_0^\infty(S,T)$, and $\zeta \in C_0^\infty(B_{R_0})$ there exists an $n \in \{0, 1, \ldots\}$ such that we have $\xi \eta \zeta \in \text{Lie}_n$ for any $\xi \in \mathbb{R}^d$.

Here is our first main result which is proved in Section 7. We remind the reader that the common way of saying that a generalized function in a domain is smooth means that there is a smooth function which, as a generalized function, coincides with the given generalized one in the domain under consideration.
**Theorem 2.3.** Assume that for any \( \omega \in \Omega_0, n = 1, 2, \ldots \), and \( \zeta \in C_0^\infty(G) \), for almost any \( t \in [S, T] \) we have \( f_t \zeta \in H_2^r \) and

\[
\int_S^T \|f_t \zeta\|^2_{H_2^r} \, dt < \infty
\]

and for any \( \omega \in \Omega, n = 1, 2, \ldots \), and \( \zeta \in C_0^\infty(G) \), for almost any \( t \in [S, T] \) we have \( g_t^k \zeta \in H_2^r, \ k = 1, \ldots, d_1 \), and

\[
\sum_{k=1}^{d_1} \int_S^T \|g_t^k \zeta\|^2_{H_2^r} \, dt < \infty.
\]

Then, for almost all \( \omega \in \Omega_0, u_t(x) \) is infinitely differentiable with respect to \( x \) for \((t, x) \in G\) and each derivative is a continuous function in \( G \).

Furthermore, let \([s_0, t_0] \subseteq (S, T), r \in (0, R_0),\) take a \( \zeta \in C_0^\infty(G) \) such that \( \zeta = 1 \) on a neighborhood of \([s_0, t_0] \times B_r \), and take an \( m \) (which exists by definition) such that (2.1) holds with \( p = 2 \). Then, for any multi-index \( \alpha \) and \( l \) such that

\[
2(l - |\alpha| - 2) > d + 1
\]

there exists a (random, finite) constant \( N \), independent of \( u, f, \) and \( g^k \), such that, for almost any \( \omega \in \Omega_0, \)

\[
\sup_{(t, x) \in [s_0, t_0] \times B_r} |D^\alpha u_t(x)|^2 \leq N \int_S^T \left[ \|f_t \zeta\|^2_{H_2^r} + \|u_t \zeta\|^2_{H_2^m} \right] \, dt,
\]

provided that \( g_t^k \zeta I_{\Omega_0} \equiv 0, k = 1, \ldots, d_1. \)

Here is a result which is “global” in \( t \). We derive it from Theorem 2.3 in Section 8.

**Theorem 2.4.** Suppose that a stronger assumption than Assumption 2.2 is satisfied: For every \( \omega \in \Omega_0 \) and \( \zeta \in C_0^\infty(B_{R_0}) \) there exists an \( n \in \{0, 1, \ldots\} \) such that we have \( \xi I_{[S, T]} \zeta \in \text{Lie}_n \) for any \( \xi \in \mathbb{R}^d \). Also suppose that the assumption stated in Theorem 2.3 is satisfied for with \( \zeta \in C_0^\infty(B_{R_0}) \) rather than \( \zeta \in C_0^\infty(G) \).

Then the first assertion of Theorem 2.3 holds true with \( (S, T) \times B_{R_0} \) in place of \( G \), and the second assertion holds with \( s_0 \in (S, T), t_0 = T, \) and \( \zeta \in C_0^\infty(B_{R_0}), \) which equals one in a neighborhood of \( B_r. \)

If we additionally assume that \( u_S \) is infinitely differentiable in \( B_{R_0} \) for every \( \omega \in \Omega_0, \) then the first assertion of Theorem 2.3 holds true with \( [S, T] \times B_{R_0} \) in place of \( G, \) and the second assertion holds with \( s_0 = S, t_0 = T, \) and \( \zeta \in C_0^\infty(B_{R_0}), \) which equals one in a neighborhood of \( B_r, \) if we add to the right-hand side of (2.5) a constant (independent of \( u \)) times \( \|u_S\|^2_{H_2^{l+1}}. \)

**Remark 2.5.** The reader will see that Assumption 2.1 (iv) will be used only in the proof of the second assertion of Theorem 2.4 for \( S = 0. \)
3. On linear stochastic equations

Let \( z_t \) be a \( d \times d \) matrix-valued continuous \( \mathcal{F}_t \)-adapted process satisfying

\[
z_t = I + \int_0^t \alpha_s z_s \, dw_s^k + \int_0^t \beta_s z_s \, ds, \quad s \geq 0,
\]

where \( \alpha_s^k, \, k = 1, \ldots, d_1 \), and \( \beta_s \) are bounded predictable \( d \times d \) matrix-valued processes and \( I \) is the identity \( d \times d \) matrix. The goal of this section is to prove the following result, which is probably known, but the author could not find an appropriate reference. In any case the proof is short.

**Lemma 3.1.** For \( s \geq 0 \)

\[
det z_t = \exp \left( \int_0^t \tr \alpha_s^k \, dw_s^k + \int_0^t \left[ \tr \beta_s - \frac{1}{2} \sum_{k=1}^{d_1} \tr ((\alpha_s^k)^2) \right] \, ds \right). \tag{3.1}
\]

**Proof.** Take a \( d \times d \) nondegenerate matrix \( A = (A^{ij}) \) and consider it as a function of its entries \( A^{ij}, \, i, j = 1, \ldots, d \). Then \( \det A \) is also a function of \( A^{ij} \). One knows that (we write \( f_x \) to denote the derivative of \( f \) with respect to \( x \))

\[
(\det A)^{A_{ij}} = B^{ji} \det A,
\]

where \( B = A^{-1} \). Also as with derivatives with respect to any parameter

\[
B^{A_{ij}} = -B A_{ij} B.
\]

Observe that \( A^{nm}_{A_{ij}} = \delta^n \delta^m \). It follows that

\[
B^{ji}_{A_{ij}} = -B^{jn} \delta^n \delta^m B^{mi} = -B^{ji} B^{pi},
\]

\[
(\det A)^{A_{ij} A_{ij}} = -B^{ji} B^{pi} \det A + B^{ji} B^{pi} \det A.
\]

Now we can use Itô’s formula. Denote \( x_t = z_t^{-1} \). Then

\[
d \det z_t = x_t^{ji} \alpha_t^{ink} z_t^{nj} \det z_t \, dw_t^k + x_t^{ji} \beta_t^{nk} z_t^{nj} \det z_t \, dt
\]

\[
+ (1/2) \left[ x_t^{ji} x_t^{pp} - x_t^{ji} x_t^{pr} \right] \alpha_t^{ink} z_t^{nj} \alpha_t^{mk} z_t^{mp} \det z_t \, dt.
\]

We note that

\[
x_t^{ji} z_t^{nj} = \delta^{in}, \quad x_t^{pp} z_t^{mp} = \delta^{rm}, \quad x_t^{jr} z_t^{nj} = \delta^{rn}, \quad x_t^{pi} z_t^{mp} = \delta^{im}
\]

and conclude that

\[
d \det z_t = \det z_t \left[ \tr \alpha_t^k \, dw_t^k + \tr \beta_t \, dt + (1/2) \left( \sum_{k=1}^{d_1} (\tr \alpha_t^k)^2 - \tr ((\alpha_t^k)^2) \right) \right].
\]

We see that \( \det z_t \) satisfies a linear equation as long as it stays strictly positive. A unique solution of this equation which equals one at \( t = 0 \) is given by the right-hand side of (3.1), which does not vanish for \( t \geq 0 \). This shows that (3.1) holds for all \( t \geq 0 \) and the lemma is proved.
4. On stochastic integrals of Hilbert-space valued processes

Let $H$ be a separable Hilbert space (in our applications $H$ is one of $H_2^{-n}$ with large $n > 0$). Take a predictable $H$-valued process $h_t, t \in [0,T]$, such that (a.s.)

$$\int_0^T \|h_t\|^2_H dt < \infty$$

for any $\omega$ and set $w_t = w^1_t$.

**Lemma 4.1.** The stochastic integral

$$\int_0^t h_s dw_s$$

has a (continuous) modification such that, if there is a $\phi \in H$, $(s_0,t_0) \subset (0,T)$, and $\omega \in \Omega$ for which $(\phi,h_r(\omega))_H = 0$ for $r \in (s_0,t_0)$, then

$$(\phi, \int_0^t h_s dw_s)_H$$

is constant on that $\omega$ for $r \in [s_0,t_0]$.

The proof of this lemma is achieved immediately after one recalls that there exists a sequence $n_k \to \infty$ and a $c \in (0,1)$ such that (a.s.) uniformly on $[0,T]$

$$\int_0^t h_{\kappa(n_k,s+c)-c} dw_s := \sum_{m=1}^{\infty} I_{s \leq t h_{t_{mk}-c} I_{t_{mk} \leq s+c < t_{m+1,k}}} (w_{t_{m+1,k}-c} - w_{t_{mk}-c}) \to \int_0^t h_s dw_s,$$

in $H$, where $t_{mk} = m2^{-ns}$, $\kappa(n,s) = 2^{-n|2^ns|}$, and $h_t$ is extended as zero outside $[0,T]$.

5. On some random mappings

Here we suppose that Assumption 2.1 (i) is satisfied with $R_0 = \infty$ and, moreover, there is an $R \in (0,\infty)$ such that, for any $k = 0,1,\ldots,d_1$ and $\omega,t$, we have $\sigma^k_t(x) = 0$ if $|x| \geq R$.

Consider the equation

$$x_t = x - \int_0^t \sigma^k_s(x_s) dw^k_s - \int_0^t b_t(x_s) ds, \quad (5.1)$$

where

$$b_t(x) = \sigma^0_t(x) - (1/2) \sum_{k=1}^{d_1} D \sigma^k_t(x) \sigma^k_t(x).$$

As follows from [3] (see [13] for more advanced treatment of the subject), there exists a function $X_t(x)$ on $\Omega \times [0,T] \times \mathbb{R}^d$ such that

(i) it is continuous in $(t,x)$ for any $\omega$ along with each derivative of $X_t(x)$ of any order with respect to $x$. 

(ii) it is $\mathcal{F}_t$-adapted for any $(t,x)$,
(iii) for each $x$ with probability one it satisfies (5.1) for all $t \in [0,T]$,
(iv) the matrix $D X_t(x)$ for any $x$ with probability one satisfies

$$D X_t(x) = I - \int_0^t D\sigma^k_s(X_s(x)) DX_s(x) \, dw^k_s - \int_0^t D b_s(X_s(x)) DX_s(x) \, ds$$

for all $t \in [0,T]$.

By Lemma 3.1 we obtain that for any $x$ with probability one

$$\det DX_t(x) = \exp \left( - \int_0^t \tr D\sigma^k_s(X_s(x)) \, dw^k_s ight. \\
- \left. \int_0^t \left[ \tr D b_s - (1/2) \sum_{k=1}^{d_1} \tr ((D\sigma^k_s)^2)(X_s(x)) \right] \, ds \right)$$

for all $t \in [0,T]$. By formally considering the system consisting of equation (5.1) and the “equation”

$$y_t = y + \int_0^t \tr D\sigma^k_s(x_s) \, dw^k_s$$

and applying what is said above, we see that there exists a function $I_t(x) = I_t(\omega,x)$ which is continuous with respect to $(t,x) \in [0,T] \times \mathbb{R}^d$ for each $\omega$ and such that for each $x$

$$I_t(x) = \int_0^t \tr D\sigma^k_s(x_s) \, dw^k_s$$

with probability one for all $t \in [0,T]$. Then for each $(t,x)$ with probability one

$$\det DX_t(x) = \exp \left( - I_t(x) \\
- \int_0^t \left[ \tr D b_s - (1/2) \sum_{k=1}^{d_1} \tr ((D\sigma^k_s)^2)(X_s(x)) \right] \, ds \right)$$

and since both parts of these equality are continuous with respect to $(t,x)$ the equality holds for all $(t,x)$ at once with probability one.

It follows that, perhaps after modifying $X_t(x)$ on a set of probability zero, we may assume that $\det DX_t(x) > 0$ for all $(\omega,t,x)$. Also observe that obviously $X_t(x) = x$ for $|x| \geq R$ and $|X_t(x)| \leq R$ for $|x| \leq R$. Hence, there is a random variable $\varepsilon = \varepsilon(\omega) > 0$ such that $\det DX_t \geq \varepsilon$ and

$$\det ([DX_t]^{*} DX_t) \geq \varepsilon$$

for all $(\omega,t,x)$. Since $DX_t(x)$ is a bounded function of $(t,x)$ for each $\omega$, it follows that the smallest eigenvalue of the symmetric matrix $[DX_t]^{*} DX_t$ is bounded below by a $\delta = \delta(\omega) > 0$, that is

$$|DX_t \xi|^2 \geq \delta |\xi|^2$$

for all $(\omega,t,x)$ and $\xi \in \mathbb{R}^d$. 

Now we need the following consequence of (5.2), which is proved in a much more general case of quasi-isometric mappings of Banach spaces in Corollary of Theorem II of [9] (see also [10]).

**Lemma 5.1.** For all \((\omega, t)\), the mapping \(X_t(x)\) of \(\mathbb{R}^d\) is one-to-one and onto \(\mathbb{R}^d\).

Kunita [12] gives a different proof of Lemma 5.1 in a much more general case based on the fact that the mapping \(X_t(x)\) is obviously homotopic to the identity mapping (but still in his case an additional effort is applied because \(\mathbb{R}^d\) is not compact). Yet another proof provides the following result, in which the nondegeneracy of the Jacobian is not required and which may have an independent interest.

**Lemma 5.2.** Let \(D\) be a connected bounded domain in \(\mathbb{R}^d\) and \(X : \bar{D} \to \bar{D}\) be a continuous mapping which has bounded and continuous first-order derivatives in \(D\). Assume that \(X(x) = x\) if \(x \in \partial D\), \(\det DX(x)\) does not change sign in \(D\), and for any \(x_0 \in D\) the mapping \(X(x)\) is a homeomorphism if restricted to a neighborhood of \(x_0\) (for instance, \(\det DX(x) > 0\) on \(D\)). Then the mapping \(X\) is one-to-one and onto \(\bar{D}\).

Proof. The fact that the mapping is onto is an easy consequence of the fact that \(\partial X(D) = X(\partial D) = \partial D\).

To prove that \(X\) is one-to-one, for \(n = 1, 2, \ldots, i = (i_1, \ldots, i_d), i_k = 0, \pm 1, \ldots\), introduce

\[
C_{i,n} = (i_1/2^n, (i_1 + 1)/2^n) \times \cdots \times (i_d/2^n, (i_d + 1)/2^n).
\]

Take a domain \(D' \subset \bar{D'} \subset D\) and observe that, because of our assumption that \(X\) is a local homeomorphism, there exists an \(n\) such that \(X\) restricted to \(C_{i,n} \cap D'\) is one-to-one whenever this intersection is nonempty. In that case also

\[
\text{Vol} X(C_{i,n} \cap D') = \int_{C_{i,n} \cap D'} |\det DX(x)| \, dx.
\]

Summing up these relations and then letting \(D' \uparrow D\) we obtain

\[
\text{Vol} D = \text{Vol} X(D) \leq \int_D |\det DX(x)| \, dx. \tag{5.3}
\]

Note that (5.3) holds without the assumption that \(X\) does not move the points on the boundary of \(D\). Also note for the future that \(X\) is Lipschitz continuous in \(\bar{D}\). Indeed, if \(x_1, x_2 \in \bar{D}\) and the open straight segment connecting \(x_1\) and \(x_2\) belongs to \(D\), then \(|X(x_1) - X(x_2)| \leq N_0|x_1 - x_2|\), where \(N_0\) is the supremum of \(\|DX\|\) over \(D\). If not the whole of the segment is in \(D\), then denote by \(y_1 \in \partial D\) and \(y_2 \in \partial D\) the closest points to \(x_1\) and \(x_2\), respectively, on the closure of this segment. Then

\[
|X(x_1) - X(x_2)| \leq N_0|x_1 - y_1| + |y_1 - y_2| + N_0|y_2 - x_2| \leq (N_0 + 1)|x_1 - x_2|.
\]
Next, concentrate on the case that \( \det DX(x) \geq 0 \). The other case is treated similarly. It turns out that for \( t \in [0,1] \) sufficiently close to 1

\[
\int_D \det (I + (1-t)DX(x)) \, dx = \text{Vol } D. \tag{5.4}
\]

To prove this observe that for such \( t \) the Jacobian of the mapping \( X_t(x) := tx + (1-t)X(x) \) is positive on \( D \) and, therefore, the image \( D_t \) of \( D \) under \( X_t \) is a domain. For \( t \) close to one also \( D_t \cap D \neq \emptyset \) and the mapping \( X_t \) is invertible (since \( X(x) \) is Lipschitz continuous in \( D \)). Take such a \( t \).

Notice that if \( y_0 \in \partial D_t \), then there exist \( y_n \to y_0 \), \( y_n \in D_t \). Then there exist \( x_n \in D \) such that \( y_n = X_t(x_n) \) and for any convergent subsequence of \( x_n \) its limit, say \( x_0 \) is not in \( D \), because \( y_0 = X_t(x_0) \not\in D_t \). Hence \( x_0 \in \partial D \), \( y_0 = x_0 \) and \( \partial D_t \subset \partial D \).

Similarly, if \( x_0 \in \partial D \), then there exist \( x_n \to x_0 \), \( x_n \in D \). Then \( y_n := X_t(y_n) \in D_t \) and \( y_n \to y_0 = X_t(x_0) = x_0 \). If \( y_0 \in D_t \) then there is a \( z \in D \) such that \( y_0 = X_t(z) = X_t(x_0) \), which is impossible since \( \partial D \ni x_0 \neq z \) and \( X_t \) is a one-to-one mapping in \( D \). Hence, \( x_0 = y_0 \in \partial D_t \), \( \partial D \subset \partial D_t \), and \( \partial D_t = \partial D \).

This fact combined with the fact that \( D \) is connected and \( D_t \cap D \neq \emptyset \) easily implies that \( D_t = D \) for \( t \) close to one. Now (5.4) follows. Once (5.4) is true for \( t \) close to 1 it is true for all \( t \in \mathbb{R} \) since the left-hand side is a polynomial with respect to \( t \). By plugging in \( t = 1 \) we get that

\[
\text{Vol } D = \int_D \det DX(x) \, dx. \tag{5.5}
\]

Now assume that there are points \( x_0, y_0 \in D \) such that \( x_0 \neq y_0 \) and \( z_0 := X(x_0) = X(y_0) \). Then there exists a (small) ball \( B \) centered at \( x_0 \) which is mapped to an open set containing \( z_0 \), such that this set is also covered by an image of a neighborhood of \( y_0 \). It follows that the image of \( D \setminus B \) under the mapping \( X \) is still \( D \). Then (5.3) applied to \( D \setminus B \) in place of \( D \) shows that \( \text{Vol } D \) is less than or equal to the integral of \( \det DX \) over \( D \setminus B \) which is strictly less than the right hand side of (5.5) since \( \det DX \neq 0 \) in \( B \), because the said neighborhood of \( z_0 \) has nonzero volume. This is a desired contradiction and the lemma is proved.

We now know that, for each \( (\omega, t) \in \Omega \times [0,T] \), the mapping \( x \to X_t(x) \) is one-to-one and onto and there exists the inverse mapping \( X_t^{-1}(x) \), which is infinitely differentiable in \( x \) by the implicit function theorem. In addition, from formulas for derivatives of \( X_t^{-1}(x) \) we conclude that these derivatives are continuous and bounded as functions of \( (t, x) \) for each \( \omega \).

Next, define the operations “hat” and “check” which transform any function \( \phi_t(x) \) into

\[
\hat{\phi}_t(x) := \phi_t(X_t(x)), \quad \check{\phi} = \phi_t(X_t^{-1}(x)). \tag{5.6}
\]

Also define \( \rho_t(x) \) from the equation

\[
\rho_t(X_t(y)) \det DX_t(y) = 1
\]
and observe that by the change of variables formula
\[ \int_{\mathbb{R}^d} F(X_t(y))\phi(y)\,dy = \int_{\mathbb{R}^d} F(x)\hat{\phi}_t(x)\rho_t(x)\,dx, \] (5.7)
whenever at least one side of the equation makes sense.

We are going to make change of variables \( x \to X_t(x) \) in (1.2) and therefore we need to understand how the equation transforms under this change. Define the mapping “bar” which transforms any \( \mathbb{R}^d \)-valued function \( \sigma_t(x) \) into
\[ \bar{\sigma}_t(x) = Y_t(x)\hat{\sigma}_t(x) = Y_t(x)\sigma_t(X_t(x)), \] (5.8)
where
\[ Y = (DX)^{-1}. \]

Observe that for real-valued functions
\[ D_j\hat{\phi}_t(x) = D_j[\phi_t(X_t(x))] = \hat{D}_j\phi_t(x)D_jX_t^i(x), \]
that is
\[ \hat{D}\hat{\phi} = \hat{D}\phi DX, \quad \hat{D}\phi = D\hat{\phi}Y. \]

It follows that for \( k = 0, 1, \ldots, d_1 + d_2 \)
\[ \hat{L}_{ak}u = D\hat{\sigma}^k = L_{ak}\hat{u}. \] (5.9)

One more standard fact is the following.

**Lemma 5.3.** For any smooth \( \mathbb{R}^d \)-valued functions \( \alpha \) and \( \beta \) on \( \mathbb{R}^d \), for all values of arguments we have
\[ \overline{[\alpha, \beta]} = [\hat{\alpha}, \hat{\beta}]. \] (5.10)

Proof. Dropping the obvious values of arguments we have that by definition the right-hand side of (5.10) equals
\[ D\hat{\beta}\hat{\alpha} - D\hat{\alpha}\hat{\beta} = Y[D\hat{\beta}DX\hat{\alpha} - D\hat{\alpha}DX\hat{\beta}] + D_iY\hat{\alpha}^i\hat{\beta} - D_jY\hat{\beta}^j\hat{\alpha}. \]
Furthermore, since \( YDX = I, \)
\[ D_iY\hat{\alpha}^iDX + YDD_iX\hat{\alpha}^i = 0, \quad D_iY\hat{\alpha}^i = -YDD_iX\hat{\alpha}^iY; \]
\[ D_iY\hat{\alpha}^i\hat{\beta} = -YDD_iX\hat{\alpha}^i\hat{\beta} = -YD_jX\hat{\alpha}^i\hat{\beta}^j = D_jY\hat{\beta}^j\hat{\alpha}. \]
This and the facts that \( DX\hat{\alpha} = \hat{\alpha} \) and \( DX\hat{\beta} = \hat{\beta} \) prove the lemma.

**6. Itô-Wentzell formula**

Here we suppose that Assumption 2.1 (i) is satisfied with \( R_0 = \infty \) and define \( C_T = C_{T,\infty} \). In this section we show what happens with the stochastic differential of a \( D \)-valued process under a random change of variables.

We make the following assumption which is justified in the situation of Section 5 but certainly not justified in a much more general setting in [13].
Assumption 6.1. There exists a function $X_t(x)$ on $\Omega \times [0, T] \times \mathbb{R}^d$ which has properties (i)-(iv) listed in Section 5 and such that, for any $(\omega, t)$, $\det DX_t(x) > 0$ for any $x$, the mapping $x \to X_t(x)$ is one-to-one and onto, so that there exists an inverse mapping $X_t^{-1}(x)$, and for any $R \in (0, \infty)$

$$\sup_{\omega} \sup_{t \in [0, T]} \sup_{|x| \leq R} |X_t(x)| < \infty.$$ 

We start by discussing Definition 2.1 (recall that $R_0 = \infty$).

**Remark 6.1.** Since $\| \cdot \|_{H^2_\omega} \leq \| \cdot \|_{H^2_{\omega_0}}$ for $n \leq m$ one can always assume that (2.1) holds with any $n \leq m$. Also note that, as is well known and easily derived by using the Fourier transform, for any $r \in \{0, 1, \ldots\}$ there is a constant $N$ depending only on $r$ and $d$ such that for any $\phi \in H^2_\omega$

$$\|\phi\|_{H^2_\omega} := \| (1-\Delta)^r \phi \|_{L^2_\omega} \leq N \sum_{|\alpha| \leq 2r} \|D^\alpha \phi\|_{L^2_\omega}, \quad \sum_{|\alpha| \leq 2r} \|D^\alpha \phi\|_{L^2_\omega} \leq N\|\phi\|_{H^2_\omega}.$$ 

**Remark 6.2.** Let $u \in \mathcal{D}_p^{-\infty}(C_T)$ and let $\mathcal{M}$ be a set of $\mathcal{F} \otimes \mathcal{B}(0, T) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functions $\phi_t = \phi_t(x) = \phi_t(\omega, x)$ on $\Omega \times (0, T) \times \mathbb{R}^d$ such that

(i) For any $\phi \in \mathcal{M}$ and $\omega$ and $t$, $\phi_t \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, and there exists an $R_1 \in (0, \infty)$ such that, for any $t \in (0, T)$, $\phi_t \in \mathcal{M}$, and $\omega$, we have $\phi_t(x) = 0$ if $|x| \geq R_1$;

(ii) There is an $r \in \{0, 1, \ldots\}$ such that, for $\phi \in \mathcal{M}$ and $\omega \in \Omega$, the $L^2_\omega$-norm of any derivative of $\phi_t(x)$ with respect to $x$ up to order $2r$ is bounded on $(0, T)$ uniformly with respect to $\phi \in \mathcal{M}$.

It turns out that for any $\omega$ and any $\zeta \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ which equals one for $|x| \leq R_1$ there is a constant $N$ such that for all $t \in (0, T)$

$$\sup_{\phi \in \mathcal{M}} |(u_t, \phi_t)| \leq N\|\zeta u_t\|_{H^{-2r}}.$$ 

In particular, if $m$ is such that (2.1) holds and $-r \leq m/2$, then

$$\int_0^T \sup_{\phi \in \mathcal{M}} |(u_t, \phi_t)|^p dt < \infty.$$ 

Indeed, in light of Remark 6.1

$$\sup_{\phi \in \mathcal{M}} |(u_t, \phi_t)| = \sup_{\phi \in \mathcal{M}} |(\zeta u_t, \phi_t)| \leq N\|\zeta u_t\|_{H^{-2r}} \sup_{\phi \in \mathcal{M}} \|\phi_t\|_{H^2_\omega}.$$ 

We use the notation from Section 5 and observe that $\rho_t(x)$ is infinitely differentiable with respect to $x$ and for any $\omega$ any its derivatives are bounded on $[0, T] \times B_R$ for any $R \in (0, \infty)$. Hence, the following definition makes sense: for $u_t \in \mathcal{D}$, $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, and $t \in [0, T]$ let

$$(\hat{u}_t, \phi) := (u_t, \hat{\phi}_t \rho_t).$$

(6.1)

Observe that, if $u_t$ is a locally summable function, this definition coincides with the one given in (5.6) due to (5.7).

**Lemma 6.3.** If $u \in \mathcal{D}$ and $t \in [0, T]$, then $(\hat{u}, \hat{\phi}_t)$ is a generalized function for each $\omega$. Furthermore, if $u \in \mathcal{D}_p^{-\infty}(C_T)$, then $\hat{u} \in \mathcal{D}_p^{-\infty}(C_T)$. 


Proof. To prove the first assertion observe that if \( \phi^n \) converge to \( \phi \) as test functions, then their supports are in the same compact set and \( \phi^n \to \phi \) uniformly on \( \mathbb{R}^d \) along with each derivative in \( x \). From calculus we conclude that the same is true about \( \phi^n_t \rho_t(x) \) for each \( t \) and \( \omega \) and then by definition 

\[
(u, \tilde{\phi}_t \rho_t) = (u, \phi_t \rho_t).
\]

To prove the second assertion, first of all take a \( \zeta \in C_0^\infty(\mathbb{R}^d) \) with unit integral, define \( \zeta_n(x) = n^d \zeta(nx) \) and let \( u^n_t = u_t * \zeta_n \). One knows that \( u^n_t(x) \) is an infinitely differentiable function of \( x \) for each \( n, t, \) and \( \omega \) and \( u^n_t \to u_t \) as \( n \to \infty \) in the sense of generalized functions for each \( t \) and \( \omega \). In particular,

\[
(u_t, \tilde{\phi}_t \rho_t) = \lim_{n \to \infty} (u^n_t, \tilde{\phi}_t \rho_t) = \lim_{n \to \infty} \int_{\mathbb{R}^d} u^n_t(x) \tilde{\phi}_t(x) \rho_t(x) \, dx.
\]

This formula and the fact, that for each \( x \) the function \( u^n_t(x) \), continuous in \( x \), is predictable by definition, show that \( \hat{u}_t \) possesses the measurability properties required in Definition 2.1.

Next, take an open ball \( B \subset \mathbb{R}^d \) and take \( \phi \in C_0^\infty(B) \). Observe that by assumption there is an \( R \in (0, \infty) \) such that \( X_t(x) \in B_R \) for all \( t \in (0, T] \), \( x \in B \), and \( \omega \). Take an \( r \in \{0, 1, \ldots\} \) such that \( -r \leq m/2 \), where \( m \) is taken from Definition 2.1 corresponding to the ball \( B_{2R} \), and let \( \mathcal{M} = \{ \psi \in C_0^\infty(\mathbb{R}^d) : \| \psi \|_{H_2^2} = 1 \} \).

Since the inequality \( \tilde{\phi}_t(x) \neq 0 \) implies that \( X_t^{-1}(x) \in B \), that is \( x \in X_t(B) \) and \( x \in B_R \), the supports of \( \tilde{\phi}_t \hat{\psi}_t \) lie in \( B_R \) for all \( t \in (0, T) \) and \( \psi \in \mathcal{M} \). It follows by Remark 6.2 that

\[
\| \hat{u}_t \phi \|_{H_2^{-2r}} = \sup_{\psi \in \mathcal{M}} |(\hat{u}_t, \phi \psi)| = \sup_{\psi \in \mathcal{M}} |(u_t, \tilde{\phi}_t \hat{\psi}_t)| \leq N \| \zeta u_t \|_{H_2^{-2r}},
\]

where \( N \) is independent of \( t \), and \( \zeta \) is any function of class \( C_0^\infty(B_{2R}) \) which equals one on \( B_R \). This obviously shows that \( \hat{u}_t \) satisfies the condition related to (2.1) if \( u_t \) does, and the lemma is proved.

Here is the version of Itô-Wentzell formula we need.

**Theorem 6.4.** \( f \in \mathcal{D}_{-1}^\infty(C_T), u, g^k \in \mathcal{D}_{-2}^\infty(C_T), k = 1, \ldots, d_1, \) and assume that (2.2) holds (in the sense of distributions). Then

\[
d\hat{u}_t = [\hat{f}_t + \hat{a}^{ij}_t \hat{D}_i \hat{u}_t - \hat{b}^{ij}_t \hat{D}_i u_t - \hat{D}_i \hat{g}^k_t \hat{\sigma}^{ik}_t] \, dt
\]

\[
+ [\hat{g}^k_t - \hat{D}_i \hat{u}_t \hat{\sigma}^{ik}_t] \, dw^k_t, \quad t \leq T
\]

(6.2)

(in the sense of distributions), where

\[
a^{ij}_t = (1/2) \sum_{k=1}^{d_1} \sigma^{ik}_t \sigma^{jk}_t.
\]

Proof. Take an \( \eta \in C_0^\infty(\mathbb{R}^d) \) and fix a \( y \in \mathbb{R}^d \). Then by Theorem 1.1 of [7] the equation

\[
d(u_t, \eta(\cdot + X_t(y))) = ([g^k_t - D_i u_t \sigma^{ik}_t(X_t(y))], \eta(\cdot + X_t(y))] \, dw^k_t
\]
very little to do with Fubini's theorem) that where, for each \( \omega \) as

\[
\int_0^t [f_t + a^{ij}_t(X_t(y))D_{ij}u_t - b^i_t(X_t(y))D_iu_t - D_i g^{ik}_t \sigma^k_t(x)] \eta(\cdot + X_t(y)) \, dt
\]

holds, after being integrated from 0 to \( t \), with probability one for all \( t \in [0, T] \).

Then we take a \( \phi \in C_0^\infty(\mathbb{R}^d) \), multiply both parts of (6.3) by \( \phi(y) \), and apply usual and stochastic Fubini's theorems (see, for instance, [7]). Owing to the fact that for each \( \omega \) and \( R > 0 \) the set \( \{ X_t(y) : t \in [0, T], |y| \leq R \} \) is bounded, in order to be able to apply Fubini's theorems it suffices to show that for any \( R > 0 \) (a.s.)

\[
\int_0^T \sup_{|x| \leq R} (|G_t(x)| + \sum_k |H^k_t(x)|^2) \, dt < \infty,
\]

where

\[
G_t(x) = ([f_t + a^{ij}_t(X_t(y))D_{ij}u_t - b^i_t(X_t(y))D_iu_t - D_i g^{ik}_t \sigma^k_t(x)], \eta(\cdot + x)),
\]

\[
H^k_t(x) = ([g^k_t - D_i u_t \sigma^k_t(x)], \eta(\cdot + x)).
\]

The fact that all terms entering \( G \) and \( H \) apart from one admit needed estimates easily follows from Remark 6.2. The remaining one is

\[
\sum_k \int_0^T \sup_{|x| \leq R} (D_i u_t \sigma^k_t(x), \eta(\cdot + x))^2 \, dt \leq N \sup_{t \leq T, |x| \leq R} (D u_t, \eta(\cdot + x))^2,
\]

where \( N < \infty \) and the last supremum is finite (a.s.) by Lemma 4.1 of [7].

Thus, we are in the position to apply Fubini's theorems. We also use (5.7). Then we obtain

\[
d \int_{\mathbb{R}^d} (u_t, \eta(\cdot + x)) \hat{\phi}_t(x) \rho_t(x) \, dx
\]

\[
= \int_{\mathbb{R}^d} ([g^k_t - D_i u_t \sigma^k_t(x), \eta(\cdot + x)] \hat{\phi}_t(x) \rho_t(x) \, dx \, dw^k_t
\]

\[
+ \int_{\mathbb{R}^d} ([f_t + a^{ij}_t(x)D_{ij}u_t - b^i_t(x)D_iu_t - D_i g^{ik}_t \sigma^k_t(x)], \eta(\cdot + x)) \hat{\phi}_t(x) \rho_t(x) \, dx \, dt
\]

(6.4)

We substitute here \( \eta^n \) in place of \( \eta \), where \( \eta^n \) tend to the delta-function as \( n \to \infty \) in the sense of distributions. Then we use the simple fact (having very little to do with Fubini's theorem) that

\[
\int_{\mathbb{R}^d} (u_t, \eta^n(\cdot + x)) \hat{\phi}_t(x) \rho_t(x) \, dx = (u_t, \int_{\mathbb{R}^d} \eta^n(\cdot + x)) \hat{\phi}_t(x) \rho_t(x) \, dx,
\]

where, for each \( \omega \), the test functions

\[
\int_{\mathbb{R}^d} \eta_h(y + x)) \hat{\phi}_t(x) \rho_t(x) \, dx
\]
as functions of \( y \) vanish outside the same ball and converge to \( \tilde{\phi}_t(y)\rho_t(y) \) uniformly on \( \mathbb{R}^d \) along with each derivative. Similar statements are true about other terms entering (6.4), for instance,

\[
\int_{\mathbb{R}^d} \left( D_i u_t, \sigma_t^{ik} (x), \eta^n (\cdot + x) \right) \tilde{\phi}_t(x) \rho_t(x) \, dx = \left( D_i u_t, \int_{\mathbb{R}^d} \eta^n (\cdot + x) \sigma_t^{ik} (x) \tilde{\phi}_t(x) \rho_t(x) \, dx \right).
\]

We want to use the dominated convergence theorem to pass to the limit in (6.4) with \( \eta^n \) in place of \( \eta \). Notice that the supports of \( \tilde{\phi}_t(y) \) and

\[
\int_{\mathbb{R}^d} \eta^n(y + x) \sigma_t^{ik}(x) \tilde{\phi}_t(x) \rho_t(x) \, dx
\]

lie in the same ball for all \( \omega, t, n \). By Remark 6.2 for any \( \omega \)

\[
|(u_t, D_i \int_{\mathbb{R}^d} \eta^n (\cdot + x) \sigma_t^{ik}(x) \tilde{\phi}_t(x) \rho_t(x) \, dx)|^2 \leq N \| \zeta u_t \|^2_{H^2_2} \tag{6.6}
\]

with \( N \) independent of \( t \) and \( n \) if \( \zeta \in C_0^\infty(\mathbb{R}^d) \) equals one on the supports of (6.5).

Since \( u \in \mathcal{D}_2^{-\infty}(C_T) \), the right-hand side of (6.6) has finite integral over \( [0,T] \) if \( r \) is chosen appropriately, and this allows us to pass to the limit in the stochastic integral containing \( D_i u_t \). Similarly one deals with other integrals and concludes that

\[
d(u_t, \tilde{\phi}_t \rho_t) = ([g_t^k - D_i u_t \sigma_t^{ik}], \tilde{\phi}_t \rho_t) \, dw_t^k
\]

\[
+ ([f_t + a_t^{ij} D_{ij} u_t - b_t^i D_i u_t - D_t g_t^k \sigma_t^{ik}], \tilde{\phi}_t \rho_t) \, dt. \tag{6.7}
\]

This yields (6.2) by definition and the theorem is proved.

**Corollary 6.5.** Assume that \( u \) satisfies (1.2) in \( C_T \). Then

\[
d \hat{u}_t = \sum_{k=1}^{d} L^2_{\sigma_t^{d_1+k}} \hat{u}_t + \hat{c}_t \hat{u}_t + \hat{f}_t - \frac{1}{2} \sum_{k=1}^{d_1} L^2_{\sigma_t^k} u_t - L_{\sigma_t^d} u_t,
\]

Indeed, as is easy to see

\[
a_t^{ij} D_{ij} u_t - b_t^i D_i u_t = (1/2) \sum_{k=1}^{d_1} L^2_{\sigma_t^k} u_t - L_{\sigma_t^d} u_t,
\]

\[
-L_{\sigma_t^d} u_t \sigma_t^{ik} = - \sum_{k=1}^{d_1} L^2_{\sigma_t^k} u_t, \quad L_{\sigma_t^d} u_t - D_i u_t \sigma_t^{ik} = 0,
\]

and thanks to (5.9)

\[
L^2_{\sigma_t^{d_1+k}} u_t = L^2_{\sigma_t^{d_1+k}} \hat{u}_t.
\]
Remark 7.1. While proving Theorem 2.3 we may assume that \( u_S = 0 \). Indeed, take an \( s_0 \in (S, T) \) and take an infinitely differentiable function \( \chi_t \), \( t \geq 0 \), such that \( \chi_t = 0 \) on \([0, S]\) and \( \chi_t = 1 \) for \( t \geq s_0 \). Then the function \( \chi_t u_t \) satisfies an easily derived equation and equals zero at \( t = S \). Furthermore, \( f_t \) and \( g^k_t \) remain unchanged for \( t \geq s_0 \) under this change of \( u \) and hence if the theorem is true when \( u_S = 0 \), then in the general case its assertions are true if we replace in them \( S \) with \( s_0 \). Due to the arbitrariness of \( s_0 \), then the theorem is true as it is stated.

Our next observation is that, while proving Theorem 2.3, we may assume that \( S = 0 \). Indeed, if is not, we can always make an appropriate shift of the origin of the time axis.

In light of Remark 7.1 everywhere below we assume that \( S = 0 \) and \( u_0 = 0 \). The rest of the proof we split into a few steps.

**Step 1.** First suppose that, for \( k = 1, \ldots, d_1 \), \( g^k_t(x) = 0 \) if \( |x| < R_0 \) and \( \nu^k_t \equiv 0 \). Also suppose that, for any \( k = 0, 1, \ldots, d_1 \), we have \( \sigma^k_t(x) = 0 \) if \( |x| \geq 2R_0 \) and \( f_t(x) = u_t(x) = 0 \), if \( |x| > R_0 - \varepsilon \), where the constant \( \varepsilon > 0 \). Then equation (1.2) holds on \( C_T \) in the sense of Definition 2.2 with \( \nu^k_t \equiv g^k_t \equiv 0 \) for \( k = 1, \ldots, d_1 \).

By Corollary 6.5

\[
(\hat{u}_t, \phi) = \int_0^t \left( \sum_{k=1}^{d_2} L^2_{\sigma^k_{t}, d_1+k} \hat{u}_s + \hat{c}_s \hat{u}_s + \hat{f}_s, \phi \right) ds
\]  

and this holds for any \( \phi \in C^\infty_0(\mathbb{R}^d) \) with probability one for all \( t \in [0, T] \). Let \( \Phi \) be a countable subset of \( C^\infty_0(\mathbb{R}^d) \) which is everywhere dense in \( H^2_2 \) for any \( n \in \mathbb{R}^d \). Then there exists a set \( \Omega' \) of full probability such that for any \( \omega \in \Omega' \) and any \( \phi \in \Phi \) equation (7.1) holds for all \( t \in [0, T] \). By setting \( u \) and \( f \) to be zero if necessary for \( \omega \notin \Omega' \) we may assume that equation (7.1) holds for any \( \phi \in \Phi \), \( t \in [0, T] \), and \( \omega \). Furthermore, observe that by assumption \( u_t(x) = 0 \) if \( |x| \geq R_0 - \varepsilon \). Hence, (2.1) holds with \( \phi \equiv 1 \) with probability one for an appropriate \( m \) (depending on \( \omega \)). By redefining, if necessary, \( u \) one more time we may assume that for any \( \omega \) there exists an integer \( r \) such that

\[
\int_0^T \|u_t\|_{H^2_2}^2 \, dt < \infty, \quad \int_0^T \|\hat{u}_t\|_{H^2_2}^2 \, dt < \infty.
\]  

Having this and similar relations for \( f \) and remembering that \( \Phi \) is dense in \( H^2_2 \) we easily conclude that (7.1) holds for any \( \phi \in C^\infty_0(\mathbb{R}^d) \), \( t \in [0, T] \), and \( \omega \).

Next argument is conducted for a fixed \( \omega \in \Omega_0 \). Introduce

\[
\hat{G} = \{(t, x) : t \in (0, T), x \in X_t^{-1}(B_{R_0})\}.
\]

Since \( X_t(x) \) is a diffeomorphism continuous with respect to \( t \), \( \hat{G} \) is a domain. Furthermore, it follows from the assumptions of the theorem that for any
\( \zeta \in C_0^\infty(\hat{G}) \) and any \( n = 1, 2, \ldots \), we have
\[
\int_0^T \| \hat{\zeta} \|_{H_2^2}^2 \, dt < \infty.
\]

Next let \( \bar{\mathbb{L}}_0 = \{ \sigma^{d_1+1}, \ldots, \sigma^{d_1+d_2} \} \),
\[
\bar{\mathbb{L}}_{n+1} = \bar{\mathbb{L}}_n \cup \{ [\sigma^{d_1+k}, M] : k = 1, \ldots, d_2, M \in \bar{\mathbb{L}}_n \}, \quad n \geq 0.
\]

Note that by Lemma 5.3, if \( \sigma \in \mathbb{L}_n \), then \( \hat{\sigma} \in \bar{\mathbb{L}}_n \).

Now, take \( \zeta \in C_0^\infty(\hat{G}) \) and \( \zeta_1 \in C_0^\infty(\hat{G}) \) so that
\[
\zeta_1 = 1 \quad \text{on} \quad \text{supp} \hat{\zeta}.
\]

By Assumption 2.2 there exists an \( n \in \{ 0, 1, \ldots \} \) such that for any \( i = 1, 2, \ldots, d \) there exist \( r \in \{ 0, 1, \ldots \} \) and \( \sigma^{(i)}, \ldots, \sigma^{(r)} \in \mathbb{L}_n \) and real-valued functions \( \gamma^{(i)}, \ldots, \gamma^{(r)} \) of class \( BC_b^\infty \) such that
\[
\zeta_1 e_i = \gamma^{(i)}(\sigma^{(i)}) + \ldots + \gamma^{(r)}(\sigma^{(r)}).
\]

Obviously one may assume that \( r \) is common for all \( i = 1, 2, \ldots, d \). It follows that
\[
\hat{\zeta_1} Y e_i = \hat{\gamma}^{(i)}(\hat{\sigma}^{(i)}) + \ldots + \hat{\gamma}^{(r)}(\hat{\sigma}^{(r)}),
\]
which after being multiplied by \( \zeta \) yields
\[
\zeta Y e_i = \zeta \hat{\gamma}^{(i)}(\hat{\sigma}^{(i)}) + \ldots + \zeta \hat{\gamma}^{(r)}(\hat{\sigma}^{(r)}),
\]

Observe that for \( \xi \in \mathbb{R}^d \) and \( \lambda = DX \xi \) we have \( Y e_i \lambda^i = \xi \), so that
\[
\zeta \xi = \lambda^i \hat{\gamma}^{(i)}(\hat{\sigma}^{(i)}) + \ldots + \lambda^i \hat{\gamma}^{(r)}(\hat{\sigma}^{(r)}).
\]

Hence, for any \( \xi \in \mathbb{R}^d \) and \( \zeta \in C_0^\infty(\hat{G}) \), \( \zeta \xi \) is represented as a linear combination of elements of \( \bar{\mathbb{L}}_n \) with coefficients of class \( BC_b^\infty \).

We checked the assumptions of Theorem 2.7 of [8] and by that theorem conclude that \( u_t(x) \) is infinitely differentiable with respect to \( x \) for \( (t, x) \in \hat{G} \), each of its derivative is a continuous function in \( \hat{G} \), and an estimate similar to (2.5) holds. Changing back the coordinates we get the first assertion of our theorem and, in addition, the fact that in (2.4) the right-hand side can be taken to be \( d \) in place of \( d + 1 \).

Step 2. We keep the assumption of Step 1 that, for \( k = 1, \ldots, d_1 \), \( g_k^T(x) = 0 \) if \( |x| < R_0 \) and \( \nu_k^T \equiv 0 \). We will cut-off \( u_t \) for \( x \) near the boundary of \( B_{R_0} \), so that the new function will satisfy an equation in \( C_T \), to which we can then apply the Itô-Wentzell formula. The only difficulty which appears after that is that we will get a new \( g_k^T \) which is not vanishing in \( C_{T,R_0} \). Partial help comes from the fact that if we cut-off close to the boundary, then the new \( g_k^T \) will be not vanishing only near the boundary. Due to this fact the transformations made in Step 1 will not lead exactly to a deterministic equation like (7.1) with random coefficients but to an equation containing the stochastic integral of \( g_k^T \) \( du_k^T \). This integral can be, so to speak, locally in time neglected near the lateral boundary of a domain like \( \hat{G} \). This yields a deterministic situation where we apply Theorem 2.7 of [8].
Take a sequence $\zeta^n \in C_0^\infty(B_{R_0})$ such that $\zeta^n = 1$ on $B_{R_0-1/n}$ and $\zeta^n = 0$ on $B_{R_0-1/(n+1)}$ and define $u^n_t = u_t \zeta^n$. Then as is easy to see

$$du^n_t = (L_t u^n_t + c_t u^n_t + f^n_t) dt + (L_{\sigma^k_t} u^n_t + \sigma^{nk}_t) dw^k_t$$

in $C_T$, where

$$f^n_t = f_t \zeta^n - u_t L_t \zeta^n - (L_{\sigma^k_t} u^n_t) \zeta^n, \quad g^{nk}_t = -u_t L_{\sigma^k_t} \zeta^n.$$

Also take a $\zeta \in C_0^\infty(\mathbb{R}^d)$ such that $\zeta = 1$ on $B_{R_0}$ and $\zeta = 0$ outside $B_{2R_0}$. Obviously, in (7.3) one can replace the operator $L_t$ with the one denoted by $\tilde{L}_t$ and constructed on the basis of $\tilde{\sigma}^k_t := \sigma^k_t$. Thus,

$$du^n_t = (\tilde{L}_t u^n_t + c_t u^n_t + f^n_t) dt + (L_{\tilde{\sigma}^k_t} u^n_t + \tilde{\sigma}^{nk}_t) dw^k_t, \quad (7.4)$$

Next we change the coordinates by defining $X_t(x)$ as a unique solution of

$$x_t = -\int_0^t \tilde{s}^k_s(x_s) dw^k_s - \int_0^t \tilde{b}_t(x_s) ds, \quad (7.5)$$

where

$$\tilde{b}_t(x) = \tilde{\sigma}^0_t(x) - (1/2) \sum_{k=1}^{d_1} D\tilde{\sigma}^k_t(x) \tilde{\sigma}^k_t(x).$$

We also recall that $u \in \mathcal{D}_{-\infty}^\infty(C_{T,R_0})$ so that the stochastic integral

$$m_t^n := \int_0^t \hat{u}_t L_{\tilde{\sigma}^k_t} \zeta^n dw^k_s$$

is well-defined as a stochastic integral of a Hilbert-space valued function and is continuous with respect to $t$ for all $\omega$. Then similarly to (7.1) we come to the conclusion that for any $\phi \in C_0^\infty(\mathbb{R}^d)$ with probability one

$$(\hat{u}_t^n, \phi) = (\hat{u}_0^n, \phi) + \int_0^t \left( \sum_{k=1}^{d_2} L_{\tilde{\sigma}^k_{t+k}} \hat{u}_s^n + \tilde{c}_s \hat{u}_s^n + \tilde{\sigma}^{nk}_s \zeta_s^n, \phi \right) ds - (\phi, m_t^n) \quad (7.6)$$

for all $t \in [0, T]$, where $\tilde{\sigma}^k_s$ are constructed from $\tilde{s}^k_s$ as in (5.8) starting with equation (7.5) instead of (5.1).

After that by doing the same manipulations as below (7.1) we convince ourselves that without losing generality we may assume that (7.6) holds for all $\phi \in C_0^\infty(\mathbb{R}^d)$, $t \in [0, T]$, and $\omega$. This and our result about $\hat{L}_m$ are the only facts which we need from the arguments in Step 1.

Then we again argue with $\omega \in \Omega_0$ fixed. Take $t_0 \in (0, T)$ and $y_0 \in B_{R_0-2/n}$. Then there is an $\varepsilon > 0$ such that, for $x_0 = X^{t_0}_t(y_0)$ we have

$$X_t(B_{\varepsilon}(x_0)) \subset B_{R_0-1/n}$$

for any $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$. For $\phi \in C_0^\infty(B_{\varepsilon}(x_0))$ and $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ we have $(\phi, m^n_t) = (\phi, m^n_{t_0-\varepsilon})$ by Lemma 4.1 since, for those $t$, $L_{\sigma^k_t} \zeta^n = 0$ in $B_{R_0-1/n}$, $\hat{\phi} = 0$ outside $B_{R_0-1/n}$, $\hat{\phi} L_{\sigma^k_t} \zeta^n \equiv 0$, and

$$(\phi, \hat{u}_t L_{\sigma^k_t} \zeta^n) = (u_t, \rho \zeta^n L_{\sigma^k_t} \zeta^n) = 0.$$
It follows that for $\phi \in C_0^\infty(B_{\varepsilon}(x_0^n))$ and $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$

$$(\hat{u}_t^n, \phi) = (\hat{u}_{t_0 - \varepsilon}, \phi) + \int_{t_0 - \varepsilon}^{t} \left( \sum_{k=1}^{d_2} L_{d_1+k}^{2} \hat{u}_s^n + \hat{\sigma}_s \hat{\nu}_s^n + \hat{\sigma}_s \hat{\nu}_s^n, \phi \right) ds.$$  

As in Step 1 we conclude by Theorem 2.7 of [8] that $\hat{u}_t^n(x)$ is infinitely differentiable with respect to $x$ for $(t, x) \in G_\varepsilon := (t_0 - \varepsilon, t_0 + \varepsilon) \times B_\varepsilon(x_0)$ and each derivative is a continuous function in $G_\varepsilon$. Furthermore, an estimate similar to (2.5) is available for any closed cylinder inside $G_\varepsilon$. Actually, Theorem 2.7 of [8] is formally applicable only if $t_0 - \varepsilon = 0$ and $\hat{u}_{t_0 - \varepsilon} = 0$. Our explanations given in Remark 7.1 take care of the general case.

Changing back the coordinates we get that $u_t^n(y)$ is infinitely differentiable with respect to $y$ for $y$ in a neighborhood of $y_0$ and $t$ in a neighborhood of $t_0$ and each derivative is a continuous function of $(t, y)$ for those $(t, y)$. Estimate (2.5) is also valid in any closed cylinder lying in that neighborhood. Since $y_0 \in B_{R_0-2/n}$, the said neighborhood of $y_0$ can be taken to belong to $B_{R_0-1/n}$, where $u_t^n = u_t$ and $f_t^n = f_t$. Now the assertion of the theorem follows owing to the arbitrariness of $y_0$, which is provided by the possibility to take $n$ as large as we wish. Again as in Step 1 it suffices that condition (2.4) be satisfied with $d$ in place of $d + 1$.

**Step 3.** Now we abandon the assumption of Step 2 that, for $k = 1, ..., d_1$, $\nu_t^k \equiv 0$, but still assume that $\hat{g}_t^k(x) = 0$ if $|x| < R_0$ for $k = 1, ..., d_1$. Introduce the function $v_t(x, y) = yu_t(x)$ and the $d + 1$-dimensional vectors

$$\hat{\sigma}_t^k(x, y) = \begin{pmatrix} \sigma_t^k(x) \\ y \nu_t^k(x) \end{pmatrix}, \quad k \leq d_1, \quad \hat{\sigma}_t^k(x, y) = \begin{pmatrix} \sigma_t^k(x) \\ 0 \end{pmatrix}, \quad k \leq d_1 + d_2,$$

$$\hat{\sigma}_t^{d_1+d_2+1}(x, y) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

Obviously, Assumption 2.2 is satisfied if we replace $G$, $d$, and $d_2$ with $G \times (0, 1)$, $d + 1$, and $d_2 + 1$, respectively. Also, routine computations yield that $v_t$ satisfies

$$dv_t = \left( \frac{1}{2} \sum_{k=1}^{d_1} L_{d_1}^{2} \nu_t^k v_t - \nu_t^k v_t - L_{d_1}^{2} (\nu_t^k v_t) - \nu_t^k v_t \nu_t^k v_t \right) + \left( \frac{1}{2} \sum_{k=1}^{d_2+1} L_{d_1+1}^{2} \nu_t^k v_t + L_{d_1+1}^{2} \nu_t^k v_t + c_t v_t + y f_t \right) dt + L_{d_1}^{2} v_t dw_t^i.$$  

The result of Step 2 shows that $v_t$ is infinitely differentiable with respect to $(x, y)$ in $B_{R_0} \times (0, 1)$ for any $t \in (0, T)$ and the derivatives are continuous with respect to $(t, x, y)$. Also the corresponding counterpart of (2.5) holds for $v_t$ under condition (2.4). This obviously proves the theorem in this particular case.

**Step 4.** Now we consider the general case. Take an $R_0^* \in (0, R_0)$ and $\zeta \in C_0^\infty(B_{R_0^*})$ such that $\zeta = 1$ on $B_{R_0}$, Then according to classical results,
for sufficiently large constant $K > 0$, there exists a function $v \in \mathcal{D}_{2}^{\infty}(C_T)$ such that $v_0 = 0$,

$$\int_0^T \|v_t\|_{H_n^2}^2 dt < \infty$$

(a.s.) for any $n$, and

$$dv_t = K \Delta v_t \, dt + (L_{\zeta \sigma^k_t} v_t + \zeta \nu t^k v_t + \zeta g^k_t) \, dw^k_t.$$

Then the function $w_t = u_t - v_t$ satisfies an equation which falls into the scheme of Step 3 with $R'_0$ in place of $R_0$ and a different $f$ but still satisfying the assumption of Theorem 2.3.

The assertion of the theorem now follows and the theorem is proved.

8. Proof of Theorem 2.4

The idea of the proof is to find a neighborhood of $[S,T] \times B_r$ to which Theorem 2.3 is applicable. First, we extend $u_t$ beyond $T$. To do that we take $R_1 \in (0, R_0)$, $\zeta \in C_0^\infty(B_{R_0})$, which equals one in $B_{R_1}$ and consider the function $v_t = \zeta u_t$ for $t \in [0,T]$. By Remark 2.2 there is an $m \in \mathbb{R}$ such that, with probability one, $v_t$ is a continuous $H_m^2$-valued process.

It follows that $v_T \in H_m^2$ (a.s.), so that solving the heat equation

$$dv_t = \Delta v_t \, dt, \quad t > T, x \in \mathbb{R}^d,$$

with initial data $v_T$, which is possible by classical results, allows us to extend $v_t$ beyond $T$ as an $H_m^2$-valued continuous functions of $t$. If we now accordingly define $c, f, \nu, g, \sigma$ for $t \geq T$, then we will see that the assumptions of Theorem 2.3 are satisfied with $(S,T + 1) \times B_{R_1}$ in place of $G$. This proves the first assertion of Theorem 2.4.

Passing to the second one we assume that $u_S$ is infinitely differentiable in $B_{R_0}$ for every $\omega \in \Omega_0$. Then we want to reduce the general case to the one in which $u_S = 0$ in $B_{R_0}$ for $\omega \in \Omega_0$. To achieve that take $R_1 \in (r, R_0)$ and $\zeta \in C_0^\infty(B_{R_0})$ as in the beginning of the proof and solve the equation

$$dv_t = [\Delta v_t + (1/2) \sum_{k=1}^{d_1} L_{\zeta \sigma^k_t} v_t] \, dt + L_{\zeta \sigma^k_t} v_t \, dw^k_t, \quad t \in (S,T), x \in \mathbb{R}^d \quad (8.1)$$

with initial data $v_S = \zeta u_S$. After making an appropriate random change of coordinates according to Corollary 6.5 we reduce this SPDE to a usual parabolic equation with random coefficients which is uniformly nondegenerate for any $\omega \in \Omega$ (we said more about this in the beginning of Section 7). By classical results there is a solution $v_t$ of this new equation with initial data $\zeta u_S$, which, for any $\omega \in \Omega_0$, is continuous in $[S,T] \times \mathbb{R}^d$ along with each its derivative of any order with respect to $x$. This is true because $\zeta u_S \in C_0^\infty(\mathbb{R}^d)$ for $\omega \in \Omega_0$. The same holds for equation (8.1). Furthermore,

$$\sup_{(t,x) \in [S,T] \times \mathbb{R}^d} |D^a v_t(x)|^2 + \int_S^T \|v_t\|_{H^{l+2}}^2 \, dt \leq N \|\zeta u_S\|_{H_x^{l+1}}^2, \quad (8.2)$$
provided that $2(l + 1 - |\alpha|) > d$ and $\omega \in \Omega_0$.

We set $v_t = \zeta u_t$ for $t \in [0, S]$ and then we see that in $[0, T] \times B_{R_1}$ the function $u_t - v_t$ satisfies the same equation as $u_t$ with $g^k_{1t}I_{(S,T)}$ in place of $g^k_t$ and with a new $f_t$, whose norms for $\omega \in \Omega_0$ admit and obvious estimates through the norms of the old one and the right-hand side of (8.2). Hence, the assumptions of the present theorem are satisfied with $B_{R_1}$ in place of $B_{R_0}$.

By replacing $u_t$ and $R_0$ with $u_t - v_t$ and $R_1$, we see that without loosing generality we may assume that $u_t = f_t = g^k_t = 0$ for $t \in [0, S]$ on $B_{R_0}$. In that case, we define $u_t = f_t = g^k_t = 0$, $\sigma^k_t = 0$, $k = 0, 1, ..., d_1 + d_2$, for $t \in [-1, S)$. We also introduce new $\sigma^k_t$ for $k = d_1 + d_2 + i$, $i = 1, ..., d$, by setting $\sigma^k_t = c_i I_{[-1,S]}(t)$, where the $c_i$'s form the standard orthonormal basis in $\mathbb{R}^d$. After that we define $I_t$ for $t \in [-1, S)$ according to (1.1), where we replace $d_1 + d_2$ with $d_1 + d_2 + d$ and observe that the new $u_t$ now satisfies (1.2) in $(-1, T) \times B_{R_0}$. The reader may object that $w^k_t$ are not defined for negative $t$, but since $du_t^k$ for negative $t$ are multiplied by zeros, one can just take independent Wiener processes and glue them to $w^k_t$ from $-1$ to $0$. As is easy to see, the first assumption of the present theorem is satisfied with $I_{-1,T}$ in place of $I_{S,T}$, and this proves the present theorem.

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E-mail address: krylov@math.umn.edu

127 Vincent Hall, University of Minnesota, Minneapolis, MN, 55455