The local large deviations principle for random walk with catastrophes

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Abstract

The continuous time Markov process considered in the paper belongs to a class of population models with linear growth and uniform catastrophes, where an eliminating portion of the population is chosen uniformly. We proved low of large numbers and local large deviation principle for the process.

Keywords. Low of Large Numbers, Large Deviation Principle, Local Large Deviation Principle, rate function, Markov processes, population models with catastrophes.

Subject classification. 60F10, 60F15, 60J27

1. Introduction

Let us start by definition of the process. Throughout the paper we assume that all considered random elements are given on a probability space \((\Omega, \mathcal{F}, P)\). We construct the process in two steps. First, we define the discrete time Markov chain \(\eta(k)\) with state space \(\mathbb{Z}^+ = \mathbb{N} \cup \{0\}\) and transition probabilities

\[
P(\eta(k+1) = j|\eta(k) = i) = \begin{cases} 
\frac{\lambda}{\lambda + \mu}, & \text{if } j = i + 1, \\
\frac{\mu}{i(i+\mu)}, & \text{if } 0 \leq j < i, i \neq 0, \\
1, & \text{if } j = 1, i = 0,
\end{cases}
\]  

(1.1)

where \(\lambda\) and \(\mu\) are positive constants. Let \(\eta(0) = 0\). Second, let \(\nu(t), t \in \mathbb{R}^+\) be Poisson point process with parameter \(E\nu(t) = \alpha t\), which does not depend on the chain \(\eta(\cdot)\). The continuous time Markov process we deal with is

\[\xi(t) := \eta(\nu(t)), \ t \in \mathbb{R}^+.\]  

(1.2)

The process belongs to a well known class of Markov processes that models a population dynamics with catastrophes (random large eliminating portion of the population). The
historical comments and references can be found for example in [1, 2]. According [1] we call catastrophes of \[ (1.2) \] as \textit{uniform catastrophes}, where the eliminating portion has uniform distribution. The name for the process we borrowed from [2]. Typically researchers are interested in extinction probability, the mean time to extinction, invariant measures, convergence to invariant measures. To the best of our knowledge there are no results on the large deviations for such processes.

Our original interest came from the modeling of a dynamics of spread – the difference between the best ask and the best bid prices of some asset (share, commodity, currency, futures, option). Usually large spread and price changes are attributed to changes in some characteristics of the market, for example changes in liquidity [4]. Our interest is to understand how large fluctuations in spread occur.

We are interested in the Law of Large Numbers (LLN) and the Local Large Deviation Principle (LLDP) for the family of processes

\[ \xi_T(t) := \frac{\xi(T_t)}{T}, \quad t \in [0, 1], \]

where \( T \to \infty \) is an increasing parameter.

Trajectories of the process \( \xi_T(\cdot) \) almost surely (a.s.) belong to the space \( \mathbb{D}[0, 1] \) of càdlàg functions. Let

\[ \rho(f, g) = \sup_{t \in [0,1]} |f(t) - g(t)| \]

for \( f, g \in \mathbb{D}[0,1] \).

The work consists of four sections. The main results are formulated in Section 2; in Section 3 we prove LLN; LLDP is proved in Section 4; auxiliary statements are in Section 5.

2. Definitions and main results

Recall the definition of LLDP.

\textbf{Definition 2.1.} \textit{A family of random processes} \( \xi_T(\cdot) \) \textit{satisfies LLDP on the set} \( G \subset \mathbb{D}[0, 1] \) \textit{with a rate function} \( I = I(f) : \mathbb{D}[0,1] \to [0, \infty] \) \textit{and the normalizing function} \( \psi(T) \) \textit{such that} \( \lim_{T \to \infty} \psi(T) = \infty \), \textit{if for any function} \( f \in G \) \textit{the following equalities hold}

\[ \lim_{T \to \infty} \lim_{\varepsilon \to 0} \sup_{T \to \infty} \frac{1}{\psi(T)} \ln \mathbb{P}(\xi_T(\cdot) \in U_\varepsilon(f)) = \lim_{\varepsilon \to 0} \lim_{T \to \infty} \inf \frac{1}{\psi(T)} \ln \mathbb{P}(\xi_T(\cdot) \in U_\varepsilon(f)) = -I(f), \]

\textit{where} \( U_\varepsilon(f) \) \textit{stands for} \( \varepsilon \)-neighborhood of \( f \),

\[ U_\varepsilon(f) := \{ g \in \mathbb{D}[0,1] : \rho(f, g) < \varepsilon \}. \]
For more details about LLDP see [5], [6].

Recall the definition of Large Deviation Principle (LDP) [3]. Denote the closure and interior of a set $B$ by $[B]$ and $(B)$, respectively.

**Definition 2.2.** A family of random processes $\xi_T(\cdot)$ satisfies LDP on metric space $(\mathbb{D}[0,1], \rho)$ with a rate function $I = I(f) : \mathbb{D}[0,1] \to [0,\infty]$ and the normalizing function $\psi(T)$ such that $\lim_{T \to \infty} \psi(T) = \infty$, if, for any $c \geq 0$ the set $\{ f \in \mathbb{D}[0,1] : I(f) \leq c \}$ is a compact set on $(\mathbb{D}[0,1], \rho)$ and, for any set $B \in \mathcal{B}_{(\mathbb{D}[0,1], \rho)}$ the following inequalities hold:

$$\limsup_{T \to \infty} \frac{1}{\psi(T)} \ln \mathbb{P}(\xi_T(\cdot) \in B) \leq -I([B]),$$

$$\liminf_{T \to \infty} \frac{1}{\psi(T)} \ln \mathbb{P}(\xi_T(\cdot) \in B) \geq -I((B)),$$

where $\mathcal{B}_{(\mathbb{D}[0,1], \rho)}$ is the Borel $\sigma$-algebra constructed by open cylindrical subsets of the space $\mathbb{D}[0,1]$, $I(B) = \inf_{y \in B} I(y)$ for $B \in \mathcal{B}_{(\mathbb{D}[0,1], \rho)}$, $I(\emptyset) = \infty$.

We use the following notations: $\mathbb{AC}^M_0[0,1]$ is the set of monotonically nondecreasing, absolutely continuous on the interval $[0,1]$ functions that start from zero; $\mathbb{AC}^+_0[0,1]$ is the set of absolutely continuous on the interval $[0,1]$ functions starting from zero and taking positive values for $t \in (0,1]$; $\text{Var}_f[0,a]$ is a total variation of the function $f$ on the interval $[0,a]$; $\overline{B}$ is a complement of the set $B$; $\mathbf{1}(B)$ is the indicator function of the set $B$; $[a]$ is the integer part of the number $a$.

**Theorem 2.3.** (LLN) For every $\varepsilon > 0$ the equality

$$\mathbb{P}\left( \lim_{T \to \infty} \sup_{t \in [0,1]} \xi_T(t) > \varepsilon \right) = 0$$

holds true.

Every function $f \in \mathbb{AC}^+_0[0,1]$ can be uniquely represented as a difference of functions $f^+ \in \mathbb{AC}^M_0[0,1], f^- \in \mathbb{AC}^M_0[0,1]$ such that

$$\text{Var}_f[0,1] = \text{Var}_{f^+}[0,1] + \text{Var}_{f^-}[0,1].$$

Nondecreasing functions $f^+$ and $f^-$ are called respectively positive and negative variations of the function $f$, see [7, Ch. 1, §4].

We denote by $B_f$ the set of monotonically nondecreasing functions $g$, such that for almost all $t \in [0,1]$ the inequality $\dot{g}(t) \geq f^+(t)$ holds.

**Theorem 2.4.** (LLDP) A family of random processes $\xi_T(\cdot)$ satisfies the LLDP on the set $\mathbb{AC}^+_0[0,1]$ with the normalizing function $\psi(T) = T$ and the rate function

$$I(f) = \frac{\alpha\mu}{\lambda + \mu} + \inf_{g \in B_f} \int_0^1 \left( \dot{g}(t) \ln \left( \frac{\dot{g}(t)(\lambda + \mu)}{\alpha \lambda} \right) - \dot{g}(t) + \frac{\alpha \lambda}{\lambda + \mu} \right) dt. \quad (2.2)$$
Remark 2.5. The rate function (2.2) can be rewritten without the infimum:

\[ I(f) = \frac{\alpha \mu}{\lambda + \mu} + \int_0^1 \left( \dot{f}^+(t) \ln \left( \frac{\dot{f}^+(t)(\lambda + \mu)}{\alpha \lambda} \right) - \dot{f}^+(t) + \frac{\alpha \lambda}{\lambda + \mu} \right) \mathbf{1}(\dot{f}^+(t) \geq \frac{\alpha \lambda}{\lambda + \mu}) \, dt. \]

Remark 2.6. Note that it is impossible to obtain LDP for the family \( \xi_T(\cdot) \) in metric space \( (D[0,1], \rho_S) \), where \( \rho_S \) is the Skorokhod’s metric, because the corresponding family of measures is not exponentially tight.

3. Proof of Theorem 2.3

First let \( T \) takes only integer values.

\[
P \left( \sup_{t \in [0,1]} \xi_T(t) > \varepsilon \right) \leq P \left( \sup_{t \in [0,1]} \xi_T(t) > \varepsilon, \nu(T) \leq 3\alpha T \right)
+ P \left( \nu(T) > 3\alpha T \right) := P_1(T) + P_2(T).
\]

Let us estimate from above \( P_1(T) \). By virtue of the independence of the Poisson process \( \nu(t) \) and the Markov chain \( \eta(\cdot) \) we have

\[
P_1(T) = \sum_{n=0}^{[3\alpha T]} P \left( \sup_{0 \leq k \leq n} \eta(k) > T\varepsilon \right) P(\nu(T) = n)
\leq P \left( \sup_{0 \leq k \leq [3\alpha T]} \eta(k) > T\varepsilon \right) \leq \sum_{k=0}^{[3\alpha T]} P(\eta(k) > T\varepsilon).
\]

(3.1)

It is obvious that the equality

\[
P(\eta(k) > T\varepsilon) = P \left( \eta(k) I(\eta(k) > C_1) > T\varepsilon \right)
\]

holds true, for any constant \( C_1 \) and sufficiently large \( T \). Choose \( C_1 = C_1(\lambda, \mu) \), where \( C_1(\lambda, \mu) \) is the constant from Lemma 5.1. Using Lemma 5.1 and the Chebyshev’s inequality, we obtain

\[
P \left( \eta(k) I(\eta(k) > C_1) > T\varepsilon \right) \leq \frac{E\eta^3(k) I(\eta(k) > C_1)}{(T\varepsilon)^3} \leq \frac{C_2}{(T\varepsilon)^3}.
\]

(3.3)

From (3.1), (3.2), (3.3) and for sufficiently large \( T \) we have

\[
P_1(T) \leq \frac{3\alpha C_2}{T^2\varepsilon^3}.
\]

(3.4)

Let us now estimate from above \( P_2(T) \). Applying the Stirling formula, for sufficiently large \( T \), it follows that

\[
P_2(T) = e^{-\alpha T} \sum_{n=[3\alpha T]+1}^{\infty} \frac{(\alpha T)^n}{n!} \leq e^{-\alpha T} \sum_{n=[3\alpha T]+1}^{\infty} \frac{e^n(\alpha T)^n}{(3\alpha T)^n}
\leq e^{-\alpha T} \sum_{n=0}^{\infty} \left( \frac{e}{3} \right)^k = e^{-\alpha T} \frac{3}{3 - e}.
\]

(3.5)
Thus, the convergence of the series
\[ \sum_{T=0}^{\infty} P \left( \sup_{t \in [0,1]} \xi_T(t) > \epsilon \right) \]
follows from (3.4) and (3.5). Therefore, by virtue of the Borel-Cantelli lemma, Theorem 2.3 is proved for the case when the parameter \( T \) takes integer values.

We will show now that Theorem 2.3 holds not only for integers \( T \). For all \( T > 0 \), inequalities
\[ 0 \leq \sup_{t \in [0,1]} \xi_T(t) \leq \sup_{t \in [0,1]} \xi_T \left( \left\lfloor T \right\rfloor + 1 \right) = \frac{T+1}{T} \sup_{t \in [0,1]} \xi_{\left\lfloor T \right\rfloor + 1}(t) \]
are fulfilled. Hence, thanks to the fact that \( \sup_{t \in [0,1]} \xi_{\left\lfloor T \right\rfloor + 1}(t) \) converges a.s. to zero Theorem 2.3 is proved. \( \square \)

4. Proof of Theorem 2.4

We represent the random process \( \xi(t) \) in the following form
\[ \xi(t) = \nu_1(t) - \sum_{k=0}^{\nu_2(t)} \zeta_k(\xi(\tau_k-)) \]
where \( \nu_1(t), \nu_2(t) \) are independent Poisson processes with parameters
\[ E\nu_1(t) = \frac{\alpha \lambda}{\lambda + \mu} t, \quad E\nu_2(t) = \frac{\alpha \mu}{\lambda + \mu} t; \]
and \( 0 = \tau_0 < \tau_1 < \ldots < \tau_k \) are jump moments of the process \( \nu_2(t) \); random variables \( \zeta_k(m), k, m \in \mathbb{Z}^+ \) are mutually independent and do not depend on \( \nu_1(t) \) and \( \nu_2(t) \); \( \zeta_0(m) = 0 \), for all \( m \in \mathbb{Z}^+ \); for a fixed \( k, m \in \mathbb{N} \) the distribution of \( \zeta_k(m) \) is given by
\[ P(\zeta_k(m) = r) = \frac{1}{m}, \quad 1 \leq r \leq m, \]
and \( \zeta_k(0) = -1 \), for all \( k \in \mathbb{N} \).

Using the representation (4.1), we obtain
\[ \xi_T(t) = \frac{\nu_1(Tt)}{T} - \frac{1}{T} \sum_{k=0}^{\nu_2(Tt)} \zeta_k(\xi(\tau_k-)) := \xi_T^+(t) - \xi_T^-(t). \]

Estimate from above \( P(\xi_T(\cdot) \in U_\epsilon(f)) \). For all \( c > 0 \) and \( \delta > 0 \) we have
\[ P(\xi_T(\cdot) \in U_\epsilon(f)) \leq P \left( \sup_{t \in [\delta,1]} |\xi_T(t) - f(t)| < \epsilon, A_c \right) + P \left( \sup_{t \in [\delta,1]} |\xi_T(t) - f(t)| < \epsilon, A_c^c \right) := P_1 + P_2, \]

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where
\[ A_c := \{ \omega : \nu_2(T) - \nu_2(\delta T) \leq cT \} \].

Estimate from above \( P_1 \). Using the formula (4.2), we obtain
\[
P_1 = P \left( \sup_{t \in [\delta, 1]} |\xi_T^+(t) - \xi_T^-(t) - f(t)| < \varepsilon, A_c \right).
\]

Denote \( m_\delta := \min_{t \in [\delta, 1]} f(t) \). Note that for sufficiently small \( \varepsilon \) the inequality \( m_\delta > \varepsilon \) is fulfilled. Therefore, if \( \sup_{t \in [\delta, 1]} |\xi_T(t) - f(t)| < \varepsilon \), than \( \xi_T(t) > 0 \) for \( t \in [\delta, 1] \), and therefore the random process \( \xi_T(t) \) does not monotonically decrease on this interval. Hence, it follows from Lemma 5.2 that
\[
P_1 = P \left( \sup_{t \in [\delta, 1]} |\xi_T^+(t) - \xi_T^-(t) - f(t)| < \varepsilon, A_c \right) \leq P \left( \xi_T^+ \in B_{1, \delta}^{\delta, \varepsilon}, A_c \right),
\]

where
\[
B_{1, \delta}^{\delta, \varepsilon} := \left\{ v \in [0, 1] : \inf_{g \in B_{1, \delta}^{\delta, \varepsilon}} \sup_{t \in [\delta, 1]} |g(t) - v(t)| \leq \varepsilon \right\}.
\]

Since the random processes \( \nu_1(\cdot) \) and \( \nu_2(\cdot) \) are independent, then
\[
P_1 \leq P(\xi_T^+ \in B_{1, \delta}^{\delta, \varepsilon}, A_c) = P(\xi_T^+ \in B_{1, \delta}^{\delta, \varepsilon})P(A_c). \tag{4.3}
\]

Estimate from above \( P_2 \). Let \( \tau_{k_1}, \ldots, \tau_{k_l} \) be the first \([cT]\) jumps of the process \( \nu_2(T) \) belonging to the interval \([\delta, 1] \). For \( 1 \leq l \leq [cT] \) denote
\[
G_{k_l} := \{ \omega : \xi(\tau_{k_l}) - \in [T(f(\tau_{k_l}) - \varepsilon); T(f(\tau_{k_l}) + \varepsilon)] \},
H_{k_l} := \{ \omega : \zeta(\tau_{k_l}) < 2T \varepsilon \}.
\]

If the trajectory of the process \( \xi_T(t) \) does not leave the set \( U_\varepsilon(f) \), then \( \zeta_{k_l}(\xi(\tau_{k_l})) < 2T \varepsilon \) for \( \tau_{k_l} \in [\delta, 1] \), \( 1 \leq l \leq [cT] \). Therefore, the inequality
\[
P_2 = P \left( \sup_{t \in [\delta, 1]} |\xi_T(t) - f(t)| < \varepsilon, A_c \right)
\]
\[
\leq \sum_{r = [cT]}^{\infty} P \left( \bigcap_{l=1}^{[cT]} H_{k_l}, \bigcap_{l=1}^{[cT]} G_{k_l} \left| \nu_2(T) - \nu_2(\delta T) = r \right. \right) P(\nu_2(T) - \nu_2(\delta T) = r)
\]
is fulfilled.

Using Lemma 5.6 we obtain
\[
P_2 \leq \sum_{r = [cT]}^{\infty} \left( \frac{2T \varepsilon}{[T(m_\delta - \varepsilon)]} \right)^{[cT]} P(\nu_2(T) - \nu_2(\delta T) = r) \leq \left( \frac{2T \varepsilon}{[T(m_\delta - \varepsilon)]} \right)^{[cT]}.
\]

The inequality \( m_\delta > \sqrt{\varepsilon} \) holds for sufficiently small \( \varepsilon \), hence for sufficiently large \( T \)
\[
P_2 \leq \left( \frac{2T \varepsilon}{[T(m_\delta - \varepsilon)]} \right)^{[cT]} \leq \left( \frac{4 \sqrt{\varepsilon}}{1 - \sqrt{\varepsilon}} \right)^{[cT]} \cdot \tag{4.4}
\]
For all $c > 0$ the following holds from the inequality (4.4)

$$\lim_{\epsilon \to 0} \lim_{T \to \infty} \frac{1}{T} \ln P_2 \leq c \lim_{\epsilon \to 0} \left( \frac{4\sqrt{\epsilon}}{1 - \sqrt{\epsilon}} \right) = -\infty. \quad (4.5)$$

Therefore, using the inequalities (4.3), (4.5), Lemmas 5.3 and 5.4 and the fact that the set $B_f^{\delta, \epsilon}$ is closed, for all $c \in (0, 1), \delta > 0$ we have

$$\lim_{\epsilon \to 0} \lim_{T \to \infty} \frac{1}{T} \ln P(\xi_T(\cdot) \in U_\epsilon(f)) \leq \lim_{\epsilon \to 0} \lim_{T \to \infty} \frac{1}{T} \ln (P_1 + P_2)$$

$$\leq \lim_{\epsilon \to 0} \lim_{T \to \infty} \frac{1}{T} \ln \left(2 \max\{P_1, P_2\}\right)$$

$$\leq \lim_{\epsilon \to 0} \left(-I_1(B_f^{\delta, \epsilon}) - \frac{\alpha \mu (1 - \delta)}{\lambda + \mu} + \frac{\alpha \mu (1 - \delta) c}{\lambda + \mu} - c \ln c\right)$$

$$= -I_1(B_f^\delta) - \frac{\alpha \mu (1 - \delta)}{\lambda + \mu} + \frac{\alpha \mu (1 - \delta) c}{\lambda + \mu} - c \ln c,$$

where

$$B_f^\delta := \left\{ v \in \mathbb{D}[0, 1] : \inf_{g \in B_f} \sup_{t \in [0, 1]} |g(t) - v(t)| = 0 \right\}.$$

Taking the limits $\delta \to 0$ and $c \to 0$, we obtain

$$\lim_{\epsilon \to 0} \lim_{T \to \infty} \frac{1}{T} \ln P(\xi_T(\cdot) \in U_\epsilon(f)) \leq -I_1(B_f) - \frac{\alpha \mu}{\lambda + \mu}.$$

Estimate from below $P(\xi_T(\cdot) \in U_\epsilon(f))$. Note that, if $I_1(f^+) = \infty$, then the estimate from below naturally holds. Suppose that $I_1(f^+) < \infty$.

$$P_3 := P \left( \sup_{t \in [0, 1]} |\xi_T^+(t) - \xi_T^-(t) - f(t)| < \epsilon \right) \geq P \left( \xi_T^+(\cdot) \in U_{\frac{\epsilon}{2}}(g^*), \xi_T^-(\cdot) \in U_{\frac{\epsilon}{2}}(g^* - f) \right),$$

where $g^*$ is such function that $\inf_{g \in B_f} I_1(g) = I_1(g^*)$. Note that by Definition 2.2 this function exists because

$$\inf_{g \in B_f} I_1(g) = \inf_{g \in B_f \cap \{g : I_1(g) \leq I_1(f^+)\}} I_1(g),$$

and the set $B_f \cap \{g : I_1(g) \leq I_1(f^+)\}$ is a compact.

Since the function $g^* \in B_f$, then the function $g^* - f \in AC_0^M[0, 1]$. If $g^* - f \equiv 0$, then

$$P \left( \xi_T^+(\cdot) \in U_{\frac{\epsilon}{2}}(g^*), \xi_T^-(\cdot) \in U_{\frac{\epsilon}{2}}(g^* - f) \right) \geq P \left( \xi_T^+(\cdot) \in U_{\frac{\epsilon}{2}}(g^*), \nu_2(T) = 0 \right).$$

Therefore, since the random processes $\nu_1(\cdot)$ and $\nu_2(\cdot)$ are independent we have

$$P_3 \geq P \left( \xi_T^+(\cdot) \in U_{\frac{\epsilon}{2}}(g^*) \right) e^{-\frac{\alpha \mu}{\lambda + \mu} T}. \quad (4.6)$$

Let $g^* - f \neq 0$. Define

$$n(\epsilon) := \min \left\{ n \in \mathbb{N} : \frac{M}{n} \leq \frac{\epsilon}{8} \right\},$$

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where \( M := \max_{t \in [0,1]} (g^*(t) - f(t)) = g^*(1) - f(1) \).

Since the function \( g^* - f \) is continuous and monotonically nondecreasing, then there exists a finite set of points \( 0 = t_0 < t_1 < \cdots < t_{n(\epsilon)} = 1 \), such that the equalities
\[
g^*(t_1) - f(t_1) = \frac{M}{n(\epsilon)}, \quad g^*(t_i) - f(t_i) = \frac{2M}{n(\epsilon)}, \ldots, \quad g^*(t_{n(\epsilon)}) - f(t_{n(\epsilon)}) = M
\]
are fulfilled. Therefore, if the random process \( \nu_2(Tt) \) does not have jumps in the interval \([0, t_1]\), we will have one jump on each of the segments \([t_{k-1}, t_k]\), \( 2 \leq k \leq n(\epsilon) \), and the random variables \( \zeta_k(\xi(\tau_k-)) \) will take values from the intervals
\[
\left( \frac{TM}{n(\epsilon)} - 2T\epsilon^3; \frac{TM}{n(\epsilon)} - T\epsilon^3 \right),
\]
then for sufficiently small \( \epsilon \) the inequality
\[
\sup_{t \in [0,1]} \left| \frac{1}{T} \sum_{k=0}^{n(\epsilon)-1} \zeta_k(\xi(\tau_k-)) - (g^*(t) - f(t)) \right| < \frac{\epsilon}{2}
\]
holds true. Hence, for sufficiently small \( \epsilon \) the following inequality
\[
\mathbf{P}_3 \geq \mathbf{P} \left( \xi_T^+ (\cdot) \in U_{\epsilon^3}(g^*) \mid \bigcap_{k=1}^{n(\epsilon)-1} A_k, \bigcap_{k=1}^{n(\epsilon)} B_k \right) \geq \mathbf{P} \left( \xi_T^+ (\cdot) \in U_{\epsilon^3}(g^*), \bigcap_{k=1}^{n(\epsilon)} A_k \right) \geq \mathbf{P} \left( \xi_T^+ (\cdot) \in U_{\epsilon^3}(g^*), \bigcap_{k=1}^{n(\epsilon)-1} A_k, \bigcap_{k=1}^{n(\epsilon)} B_k \right)
\]
holds, where
\[
A_1 := \{ \omega : \nu_2(Tt_1) = 0 \}, \quad A_k := \{ \omega : \nu_2(Tt_k) - \nu_2(Tt_{k-1}) = 1 \}, \quad 2 \leq k \leq n(\epsilon),
\]
\[
B_k := \left\{ \omega : \zeta_k(\xi(\tau_k-)) \in \left( \frac{TM}{n(\epsilon)} - 2T\epsilon^3; \frac{TM}{n(\epsilon)} - T\epsilon^3 \right) \right\}, \quad 1 \leq k \leq n(\epsilon) - 1.
\]
From the inequality \( (4.7) \) it follows that
\[
\mathbf{P}_3 \geq \mathbf{P} \left( \bigcap_{k=1}^{n(\epsilon)-1} B_k \bigg| \xi_T^+ (\cdot) \in U_{\epsilon^3}(g^*), \bigcap_{k=1}^{n(\epsilon)} A_k \right) \mathbf{P} \left( \xi_T^+ (\cdot) \in U_{\epsilon^3}(g^*), \bigcap_{k=1}^{n(\epsilon)} A_k \right).
\]
And from Lemma \( 5.5 \) it follows that
\[
\mathbf{P}_3 \geq \left( \frac{\epsilon^3}{2(g^*(1) + \epsilon^3)} \right)^{n(\epsilon)-1} \mathbf{P} \left( \xi_T^+ (\cdot) \in U_{\epsilon^3}(g^*), \bigcap_{k=1}^{n(\epsilon)} A_k \right).
\]
Since the random processes \( \xi_T^+ (t) \) and \( \nu_2(Tt) \) are independent, then
\[
\mathbf{P}_3 \geq \left( \frac{\epsilon^3}{2(g^*(1) + \epsilon^3)} \right)^{n(\epsilon)-1} \mathbf{P} \left( \xi_T^+ (\cdot) \in U_{\epsilon^3}(g^*) \right) \mathbf{P} \left( \bigcap_{k=1}^{n(\epsilon)} A_k \right)
\]
\[
= \left( \frac{\epsilon^3}{2(g^*(1) + \epsilon^3)} \right)^{n(\epsilon)-1} \mathbf{P} \left( \xi_T^+ (\cdot) \in U_{\epsilon^3}(g^*) \right) \left( \frac{\alpha \mu}{\lambda + \mu} \right)^{n(\epsilon)-1} e^{-\frac{\alpha \mu T}{\lambda + \mu}} \prod_{k=2}^{n(\epsilon)} (t_k - t_{k-1}).
\]
Using the inequalities (4.6) and (4.8), we obtain
\[
\liminf_{T \to \infty} \frac{1}{T} \ln P(\xi_T(\cdot) \in U_\varepsilon(f)) \geq -\frac{\alpha \mu}{\lambda + \mu} - I_1(U_\varepsilon(g^*)).
\]

From Lemma 5.3 it follows that
\[
\lim_{\varepsilon \to 0} \liminf_{T \to \infty} \frac{1}{T} \ln P(\xi_T(\cdot) \in U_\varepsilon(f)) \geq \lim_{\varepsilon \to 0} \left(-\frac{\alpha \mu}{\lambda + \mu} - I_1(U_\varepsilon(g^*))\right) = -\frac{\alpha \mu}{\lambda + \mu} - I_1(g^*).
\]
\[\square\]

5. Auxiliary results

Here we will prove several auxiliary lemmas.

Denote
\[
C_1 := C_1(\lambda, \mu) = \left(\frac{4\lambda + 4\mu}{4\lambda + 2\mu}\right)^{1/3} - 1 + 1.
\]

Lemma 5.1. For all \( k \in \mathbb{Z}^+ \) the inequality
\[
\mathbb{E}\eta^3(k)I(\eta(k) > C_1) \leq C_2
\]
holds true, where
\[
C_2 := C_2(\lambda, \mu) = \frac{4\lambda + \mu}{\mu}(C_1)^3.
\]

Proof will be carried out by the method of mathematical induction. If \( k = 0 \), then it is obvious that the inequality (5.1) holds. Suppose that the inequality holds for \( k = m - 1 \) and show that, it holds true for \( k = m \). Note that if \( m - 1 < C_1 \), then the inequality is fulfilled for \( k = m \). Let now suppose that \( m - 1 \geq C_1 \), then
\[
\mathbb{E}\eta^3(m)I(\eta(m) > C_1) = \mathbb{E}\mathbb{E}(\eta^3(m)I(\eta(r) > C_1) \mid \eta(m - 1))
\]
\[
= \mathbb{E} \sum_{r=0}^{m-1} \mathbb{I}(\eta(m - 1) = r) \mathbb{E}(\eta^3(m)I(\eta(m) > C_1) \mid \eta(m - 1) = r)
\]
\[
= \mathbb{E} \sum_{r=C_1}^{m-1} \mathbb{I}(\eta(m - 1) = r) \mathbb{E}(\eta^3(m)I(\eta(m) > C_1) \mid \eta(m - 1) = r). \tag{5.2}
\]

We show that
\[
\mathbb{E}(\eta^3(m)I(\eta(m) > C_1) \mid \eta(m - 1) = r) \leq r^3 \frac{4\lambda + \mu}{4\lambda + 2\mu}. \tag{5.3}
\]
From the definition of the Markov chain \( \eta(\cdot) \) and the fact that \( r \geq C_1 \) we obtain
\[
E(\eta^3(m) | \eta(m) > C_1) \leq E(\eta^3(m) | \eta(m-1) = r) \\
= \frac{\lambda}{\lambda + \mu} (r + 1)^3 + \frac{\mu}{r(\lambda + \mu)} \sum_{d=0}^{r-1} d^3 = \frac{\lambda}{\lambda + \mu} (r + 1)^3 + \frac{\mu r(r - 1)^2}{4(\lambda + \mu)} \\
< 4\lambda + \mu \frac{(r + 1)^3}{4(\lambda + \mu)} \leq r^3 \frac{4\lambda + \mu}{4\lambda + 2\mu}.
\]

From the inequalities (5.2), (5.3) and the inductive assumption it follows that
\[
E\eta^3(m) | \eta(m) > C_1 \leq \frac{4\lambda + \mu}{4\lambda + 2\mu} E \sum_{r=C_1}^{m-1} r^3 I(\eta(m-1) = r) \\
\leq \frac{4\lambda + \mu}{4\lambda + 2\mu} (C_1)^3 + \frac{4\lambda + \mu}{4\lambda + 2\mu} E\eta^3(m-1) | \eta(m-1) > C_1 \leq \frac{4\lambda + \mu}{\mu} (C_1)^3.
\]

\( \square \)

**Lemma 5.2.** Let the function \( f \in AC_0^+ [0, 1] \) be represented in the form
\[
f(t) = g_1(t) - g_2(t),
\]
where \( g_1 \in AC_0^M [0, 1] \) and \( g_2 \in AC_0^M [0, 1] \). Then the inequality \( \dot{g}_1(t) \geq \dot{f}^+(t) \) holds for almost all \( t \in [0, 1] \).

*Proof.* Assume the opposite, then there exist \( t_1 < t_2 \) on the interval \([0, 1]\) such that \( g_1(t_2) - g_1(t_1) < f^+(t_2) - f^+(t_1) \). We note that in this case the inequality \( g_2(t_2) - g_2(t_1) < f^-(t_2) - f^-(t_1) \) also holds.

Since the variation of the sum of two functions does not exceed the sum of their variations, then
\[
\text{Var}(g_1(t_1, t_2)) + \text{Var}(g_2(t_1, t_2)) = g_1(t_2) - g_1(t_1) + g_2(t_2) - g_2(t_1) \geq \text{Var}(f(t_1, t_2)).
\]

On the other hand
\[
g_1(t_2) - g_1(t_1) + g_2(t_2) - g_2(t_1) < f^+(t_2) - f^+(t_1) + f^-(t_2) - f^-(t_1) = \text{Var}(f(t_1, t_2)).
\]

The obtained contradiction completes the proof. \( \square \)

**Lemma 5.3.** The family of processes \( \frac{\nu(T)}{T} \) satisfies LDP on the metric space \((\mathbb{D}[0, 1], \rho)\) with the normalizing function \( \psi(T) = T \) and the rate function
\[
I_1(f) = \begin{cases} 
\int_0^1 \left( \dot{f}(t) \ln \left( \frac{\dot{f}(t)(\lambda + \mu)}{\alpha \lambda} \right) - \dot{f}(t) + \frac{\alpha \lambda}{\lambda + \mu} \right) dt, & \text{if } f \in AC_0^M [0, 1], \\
\infty, & \text{otherwise}.
\end{cases}
\]
Proof. According to well-known results, see, for example, [8, 9, 10, pp. 13–14] in our case it is sufficient to show that the Legendre transformation of the exponential moment of the random variable \( \nu_1(1) \) has the following form

\[
\Lambda(x) = \sup_{y \in \mathbb{R}} (xy - \ln E e^{y\nu_1(1)}) = x \ln \left( \frac{x(\lambda + \mu)}{\alpha \lambda} \right) - x + \frac{\alpha \lambda}{\lambda + \mu}, \quad x \geq 0.
\]

Since

\[
E e^{y\nu_1(1)} = \exp \left\{ \frac{\alpha \mu}{\lambda + \mu} e^y - \frac{\alpha \lambda}{\lambda + \mu} \right\},
\]
then the application of methods of the differential calculus completes the proof. \( \square \)

Lemma 5.4. The inequality

\[
P(\nu_2(T) - \nu_2(\delta T) \leq cT) \leq \exp \left\{ -\frac{\alpha \mu (1 - \delta)}{\lambda + \mu} T + \frac{\alpha \mu (1 - \delta)}{\lambda + \mu} T - Tc \ln c \right\} \tag{5.4}
\]
holds for all \( c \in [0, 1), \delta \in [0, 1] \).

Proof. Using the Chebyshev inequality, for all \( r > 0 \) we obtain

\[
P(\nu_2(T) - \nu_2(\delta T) \leq cT) = \frac{\exp \left\{ -r(\nu_2(T) - \nu_2(\delta T)) \right\}}{\exp \left\{ -rcT \right\}} \geq \exp \left\{ -\frac{\alpha \mu (1 - \delta)}{\lambda + \mu} T + \frac{\alpha \mu (1 - \delta)}{\lambda + \mu} T - Tc \ln c \right\}.
\]
Choosing \( r = -\ln c \), we obtain the inequality (5.4). \( \square \)

Lemma 5.5. The inequality

\[
P \left( \bigcap_{k=1}^{n(\varepsilon)-1} B_k \bigg| \xi_T^+(\cdot) \in U_{\varepsilon^3}(g^*), \bigcap_{k=1}^{n(\varepsilon)} A_k \right) \geq \left( \frac{\varepsilon^3}{2(g^*(1) + \varepsilon^3)} \right)^{n(\varepsilon)-1},
\]
holds with \( g^*, A_k, B_k, n(\varepsilon) \) defined in Section 4.

Proof. We show that, for \( 1 \leq k \leq n(\varepsilon) - 1 \) the inequality

\[
P_k := P \left( B_k \bigg| \xi_T^+(\cdot) \in U_{\varepsilon^3}(g^*), \bigcap_{i=1}^{n(\varepsilon)} A_i, \bigcap_{j=1}^{k-1} B_j \right) \geq \frac{\varepsilon^3}{2(g^*(1) + \varepsilon^3)} \tag{5.5}
\]
holds.
If events \( \{\omega : \xi_T^+(\cdot) \in U_{\varepsilon^3}(g^*)\}, \bigcap_{i=1}^{n(\varepsilon)} A_i, \bigcap_{j=1}^{k-1} B_j \) have occurred, then
\[ T(g^*(1) + \epsilon^3) > T(g^*(\tau_k) + \epsilon^3) > \xi(\tau_k -) \geq T \left( \xi^+_T(\tau_k -) - (k - 1) \left( \frac{M}{n(\epsilon)} - \epsilon^3 \right) \right) \]
\[
> T \left( g^*(\tau_k) - \epsilon^3 - (k - 1) \left( \frac{M}{n(\epsilon)} - \epsilon^3 \right) \right) > T \left( g^*(t_k) - \epsilon^3 - (k - 1) \left( \frac{M}{n(\epsilon)} - \epsilon^3 \right) \right) (5.6) \\
> T \left( g^*(t_k) - f(t_k) - \epsilon^3 - (k - 1) \left( \frac{M}{n(\epsilon)} - \epsilon^3 \right) \right) > \frac{TM}{n(\epsilon)} - T\epsilon^3.
\]

Note that, by definition, the family of random variables \( \zeta_k(m_k), m_k \in \mathbb{N} \) does not depend on \( \nu_1(t) \) and \( \nu_2(t) \), \( \zeta_{k-1}(m_{k-1}), m_{k-1} \in \mathbb{N}, \ldots, \zeta_1(m_1), m_1 \in \mathbb{N} \), and hence on \( \xi(\tau_k -), \ldots, \xi(\tau_1 -) \).

Therefore, for sufficiently large \( T \), using the inequality [5.6], we obtain

\[
P_k = P \left( B_k \mid \xi^+_T(\cdot) \in U_{\epsilon^3}(g^*), \bigcap_{i=1}^{n(\epsilon)} A_i, \bigcap_{j=1}^{n(\epsilon)} B_j \right)
\]
\[
= P \left( \zeta_k(\xi(\tau_k -)) \in \left( \frac{TM}{n(\epsilon)} - 2T\epsilon^3, \frac{TM}{n(\epsilon)} - T\epsilon^3 \right) \mid \xi^+_T(\cdot) \in U_{\epsilon^3}(g^*), \bigcap_{i=1}^{n(\epsilon)} A_i, \bigcap_{j=1}^{n(\epsilon)} B_j \right)
\]
\[
= \sum_{r = \frac{TM}{n(\epsilon)} - T\epsilon^3}^{\frac{TM}{n(\epsilon)} + 1} P \left( \zeta_k(r) \in \left( \frac{TM}{n(\epsilon)} - 2T\epsilon^3, \frac{TM}{n(\epsilon)} - T\epsilon^3 \right) \mid \xi^+_T(\cdot) \in U_{\epsilon^3}(g^*), \bigcap_{i=1}^{n(\epsilon)} A_i, \bigcap_{j=1}^{n(\epsilon)} B_j \right)
\]
\[
\times P \left( \xi(\tau_k -) = r \mid \xi^+_T(\cdot) \in U_{\epsilon^3}(g^*), \bigcap_{i=1}^{n(\epsilon)} A_i, \bigcap_{j=1}^{n(\epsilon)} B_j \right)
\]
\[
= \sum_{r = \frac{TM}{n(\epsilon)} - T\epsilon^3}^{\frac{TM}{n(\epsilon)} + 1} P \left( \zeta_k(\xi([T(g^*(1) + \epsilon^3)] + 1) \in \left( \frac{TM}{n(\epsilon)} - 2T\epsilon^3, \frac{TM}{n(\epsilon)} - T\epsilon^3 \right) \right)
\]
\[
\times P \left( \xi(\tau_k -) = r \mid \xi^+_T(\cdot) \in U_{\epsilon^3}(g^*), \bigcap_{i=1}^{n(\epsilon)} A_i, \bigcap_{j=1}^{n(\epsilon)} B_j \right)
\]
\[
= P \left( \zeta_k([T(g^*(1) + \epsilon^3)] + 1) \in \left( \frac{TM}{n(\epsilon)} - 2T\epsilon^3, \frac{TM}{n(\epsilon)} - T\epsilon^3 \right) \right) = \frac{[T\epsilon^3]}{[T(g^*(1) + \epsilon^3)] + 1} \geq \frac{\epsilon^3}{2(g^*(1) + \epsilon^3)}.
\]

It proves the inequality (5.5). Using the inequality (5.5), we get

\[
P \left( \bigcap_{k=1}^{n(\epsilon) - 1} B_k \mid \xi^+_T(\cdot) \in U_{\epsilon^3}(g^*), \bigcap_{i=1}^{n(\epsilon)} A_i \right) = \prod_{k=1}^{n(\epsilon) - 1} P_k \geq \left( \frac{\epsilon^3}{2(g^*(1) + \epsilon^3)} \right)^{n(\epsilon) - 1}.
\]

\[ \square \]

**Lemma 5.6.** The inequality

\[
P \left( \bigcap_{i=1}^{[cT]} H_{k_i} \cap \bigcap_{l=1}^{[cT]} G_{k_l} \mid \nu_2(T) - \nu_2(\delta T) = r \right) \leq \left( \frac{[2T\epsilon]}{[T(m_\delta - \epsilon)]} \right)^{[cT]}
\]

holds with \( H_{k_i}, G_{k_l}, r, m_\delta \) defined in Section 4.
Proof. Let $G_{k_0} := \Omega, H_{k_0} := \Omega$.

We show that the inequality

$$P_l := P \left( H_{k_l}, G_{k_l} \mid \nu_2(T) - \nu_2(\delta T) = r, \bigcap_{d=0}^{l-1} G_{k_d}, \bigcap_{d=0}^{l-1} H_{k_d} \right) \leq \frac{[2T \varepsilon]}{[T(m_\delta - \varepsilon)]} \quad (5.7)$$

holds for $1 \leq l \leq \lfloor cT \rfloor$. We estimate from above $P_l$

$$P_l \leq \sum_{v=T(m_\delta - \varepsilon)}^{T(M_1 + \varepsilon)} P \left( \zeta_k(\xi(t_k)) < 2T \varepsilon \mid \nu_2(T) - \nu_2(\delta T) = r, \bigcap_{d=0}^{l-1} G_{k_d}, \bigcap_{d=0}^{l-1} H_{k_d} \right) \times P \left( \xi(t_k) = v \mid \nu_2(T) - \nu_2(\delta T) = r, \bigcap_{d=0}^{l-1} G_{k_d}, \bigcap_{d=0}^{l-1} H_{k_d} \right)$$

$$= \sum_{v=T(m_\delta - \varepsilon)}^{T(M_1 + \varepsilon)} P \left( \zeta_k(v) < 2T \varepsilon \right) P \left( \xi(t_k) = v \mid \nu_2(T) - \nu_2(\delta T) = r, \bigcap_{d=0}^{l-1} G_{k_d}, \bigcap_{d=0}^{l-1} H_{k_d} \right)$$

$$\leq \sum_{v=T(m_\delta - \varepsilon)}^{T(M_1 + \varepsilon)} P \left( \zeta_k(\lfloor T(m_\delta - \varepsilon) \rfloor) < 2T \varepsilon \right) P \left( \xi(t_k) = v \mid \nu_2(T) - \nu_2(\delta T) = r, \bigcap_{d=0}^{l-1} G_{k_d}, \bigcap_{d=0}^{l-1} H_{k_d} \right)$$

$$= P \left( \zeta_k(\lfloor T(m_\delta - \varepsilon) \rfloor) < 2T \varepsilon \right) \leq \frac{[2T \varepsilon]}{[T(m_\delta - \varepsilon)]}$$

holds for sufficiently small $\varepsilon$. Thus, the inequality $\text{(5.7)}$ is proved. Using the inequality $\text{(5.7)}$, we obtain

$$P \left( \bigcap_{l=1}^{\lfloor cT \rfloor} H_{k_l}, \bigcap_{l=1}^{\lfloor cT \rfloor} G_{k_l} \mid \nu_2(T) - \nu_2(\delta T) = r \right) = \prod_{l=1}^{\lfloor cT \rfloor} P_l \leq \left( \frac{[2T \varepsilon]}{[T(m_\delta - \varepsilon)]} \right)^{\lfloor cT \rfloor}. $$

$\square$
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