Bound states in the continuum in a heavy fluxonium qutrit

María Hita-Pérez\textsuperscript{†}, Pedro Orellana\textsuperscript{‡}, Juan José García-Ripoll\textsuperscript{†}, Manuel Pino\textsuperscript{††}
\textsuperscript{† Institute of Fundamental Physics IFF-CSIC, Calle Serrano 113b, Madrid 28006, Spain}
\textsuperscript{‡ Departamento de Física Universidad Técnica Federico Santa María,Casilla 110-V, Valparaíso, Chile.}
\textsuperscript{* Nanotechnology Group, USAL-Nanolab, Universidad de Salamanca, E-37008 Salamanca, Spain.}

The heavy fluxonium at zero external flux has a long-lived state when coupled capacitively to any other system. We analyze it by projecting all the fluxonium relevant operators into the qutrit subspace, as this long-lived configuration corresponds to the second excited fluxonium level. This state becomes a bound-state in the continuum (BIC) when the coupling occurs to an extended state supporting a continuum of modes. In the case without noise, we find BIC decay times that can be much larger than seconds $T_1 \gg s$ when the fluxonium is coupled to superconducting waveguide, while typical device frequencies are in the order of GHz. We have also analyzed the noise in a realistic situation, arguing that the most dangerous noise source is the well-known 1/f flux noise. Even in its presence, we show that decay times could reach the range of $T_1 \sim 10^{-6}$ms.

Confined quantum excitations generally decay when coupled to a band of states with a continuous spectrum \cite{1}. There are some exceptions to those decay processes where a confined state lying at the continuum part of the spectrum lives forever. Those bound states in the continuum (BICs) were predicted long ago by von Neumann and Wigner \cite{2}. The BICs have appeared on several platforms, some of them following the laws of quantum mechanics as solid-state devices \cite{3,4}, some others—under the wave-particle duality—obeying classical wave mechanics \cite{5,6}. For instance, there have been many studies of BIC’s in photonic devices \cite{9–13} since their first experimental observation around ten years ago \cite{14}.

There are also proposals to study BICs in high-coherence quantum optical devices. Inspired by the physics of classical BICs with confined electromagnetic fields \cite{16,21}, recent works proposed using two-level systems or qubits \cite{22,23} to create an extremely long-lived and confined single-photon excitation. Another approach based on two-level emitters is to use their collective photon-mediated interactions to create extended BIC states—authors refer to as a multi-dark state—that lives on two or more separated qubits \cite{24}.

In this work, we show how to engineer a scalable, compact BIC, using a superconducting circuit, a fluxonium qutrit \cite{25}, capacitively embedded in the continuum of microwave excitations from a co-planar waveguide [cf. Fig. 1(a)]. Similar to the classical setup in Ref. \cite{20}, where the BIC lives in a photonic resonator connected to an open waveguide, our BIC is the confined plasmonic excitation that lives in the fluxonium loop and is prevented from decaying into the microwave. More precisely, the BIC state is the symmetric excited state $\left| + \right\rangle$ of the fluxonium potential at zero external flux $\Phi_{\text{ext}} = 0$ [cf. Fig.1(b)]. This state is a BIC state because the $\left| + \right\rangle \rightarrow \left| 0 \right\rangle$ transitions are suppressed due to the fact that charge operator is antisymmetric and cannot connect both states, as experimentally observed in Ref. \cite{25}. Moreover, the $\left| + \right\rangle \rightarrow \left| - \right\rangle$ transition can also be suppressed by a suitable choice of the capacitive to inductive energy ratios $E_1/E_C$, as in capacitively shunted flux qubits \cite{21} and capacitively shunted heavy fluxonium qubits at half frustra-
of this state is shown to be extremely long in the presence of realistic $1/f$ flux noise and losses into the dielectric. Finally, we also discuss the implications of these compact BIC states for applications in quantum information and quantum sensing, including open questions such as the robust preparation of the $|+\rangle$ in the open system.

I. FLUXONIUM QUTRIT

Let us formalize the intuitive picture of a BIC using a fluxonium circuit. We need to show a field coupled capacitively to the fluxonium cannot excite transitions in and out of the second excited. This information is obtained from the expansion of the charge operator and the Hamiltonian in a relevant low-energy subspace. This subspace has a qutrit structure, where the BIC mode corresponds to the second excited state, $|+\rangle$. Incidentally, in this qutrit representation of the heavy fluxonium states, flux and charge operators adopt the simple representation of $S_z$ and $S_y$ spin-1 operators, respectively.

A fluxonium [25] consists of a single Josephson junction with Josephson energy $E_J$ shunted by a capacitance $C$ and a large inductance $L$, as shown in Fig. 1(a). The Hamiltonian for such a system is given by:

$$H_f(q, \phi) = \frac{1}{2C} q^2 + V(\phi), \quad \text{with}$$

$$V(\phi) = \frac{1}{2L} \phi^2 - E_J \cos \left( \frac{\phi + \Phi_{\text{ext}}}{\Phi_0} \right).$$

(1)

Here, $q$ is the charge difference on the capacitance, $\phi$ is the conjugate flux operator, and $\Phi_{\text{ext}}$ is the external flux passing through the superconducting loop. We typically work at $\Phi_{\text{ext}} = 0$, so that the potential has the shape depicted in (1b).

The characteristic energies of the fluxonium are the junction’s Josephson energy $E_J$, the charging energy introduced by the capacitance $E_C = e^2/2C$, and the inductive energy introduced by the inductance $E_L = (h/2e)^2 / L = \Phi_0^2 / 2$. The main difference between this and other inductively shunted Josephson junction devices lies precisely in the relation between these parameters, which satisfy $E_L \ll E_J$ and $1 \lesssim E_J / E_C$ [30]. The heavy fluxonium is realized approximately for $E_J / E_C > 5$ [29].

Let us describe an analytical derivation of the qutrit Hamiltonian and relevant operators. At the symmetry point $\Phi_{\text{ext}} = 0$, the potential energy of the fluxonium has two local minima at both sides of the global one. The lowest energy eigenstates around the three minima are denoted $|L\rangle$, $|0\rangle$, $|R\rangle$, as depicted in Fig. 1(b). Intuitively, one would want to use $|L\rangle$, $|0\rangle$, $|R\rangle$ as a qutrit basis. However, the use of the $|L\rangle$ and $|R\rangle$ vectors is problematic in common situations where they have a strong overlap and are very close in energy to nearby excitations.

One solution is to replace the $|L\rangle$ and $|R\rangle$ states with slightly modified vectors that have been orthogonalized with respect to other low-energy excitations. Fig. 2(b) compares the exact eigenstates (solid lines) of a fluxonium with intermediate values of $E_J/E_C$ together with the approximated eigenstates (dashed lines) computed with a Gram-Schmidt orthogonalization up to the 4th excited state. The agreement is good, and helps in capturing the part of the excited wavefunction that tunnels to intermediate region $\phi \approx 0$, a feature that is not present in the original intuitive expansion. Moreover, these features introduce relevant qutrit interaction terms that are mediated by higher energy excitations.

To keep computations tractable, we can orthogonalize the $|L\rangle$ and $|R\rangle$ states with respect to just the third excited state $|−1\rangle = \frac{|L\rangle - a|3\rangle}{\sqrt{1 - a^2}}$ and $|1\rangle = \frac{|R\rangle + a|3\rangle}{\sqrt{1 + a^2}}$, parameterizing the overlap with a new parameter $a = \langle 3|L \rangle = −\langle 3|R \rangle$. We will verify that this order of perturbation is sufficient to recover the qualitative form of the qutrit operators. Let us denote by $±\phi$, the position of the local minima in units of flux, and assume that the capacitance

![Wave function and energy levels of the fluxonium](image-url)
Our derivation expresses all operators in terms of the harmonic states of the right, left, and central wells. Intuitively, all terms containing a function of $\phi$, $\sin(\phi)$, and $\cos(\phi)$ are mediated by the third excited state. In the limit of large charging capacitance or heavy fluxonium, the $|{-1}\rangle, |1\rangle, |0\rangle$ states become strongly localized in left, center, and right well, making the factors $a$ and $\Delta$ exponentially small. In this case, the charge and flux operators converge to $S_x$ and $S_y$, respectively and, at zero bias $\Phi_{\text{ext}} = 0$ the Hamiltonian is diagonalized by the states $|0\rangle$ and $|\pm\rangle = (|L\rangle \pm |R\rangle)/\sqrt{2}$. As explained in the introduction, in this limit the charge operator $q \approx S_y$ cannot mediate the decay of the $|+\rangle$ state to the ground state $|0\rangle$, and the second excited state can be used to construct a BIC.

We have compared the analytical estimates with numerical diagonalizations, obtaining an excellent qualitative agreement. In our method we compute the lowest energy eigenstates, project the relevant operators onto the qutrit basis and express them as combination of 1-spin operators. For greater accuracy, we use the $\sin(\phi)$ operator instead of $\phi$, and recover this and the $q$ operator as derivatives of the fluxonium's Hamiltonian with respect to flux and voltage perturbations, respectively.

Fig. 3 illustrates the excellent agreement between our predictions Eqs. (3)-(4), and the expansions of the Hamiltonian as a function of external magnetic flux $\Phi_{\text{ext}} = 0$ (a-b), and of the flux (c) and charge (d) of a fluxonium with $E_J/E_C = 21.74$ as a function of $E_J/E_C$. Our derivation expresses all operators in terms of the harmonic states of the right, left, and central wells. Intuitively, all terms containing a function of $\phi$, $\sin(\phi)$, and $\cos(\phi)$ are mediated by the third excited state. In the limit of large charging capacitance or heavy fluxonium, the $|{-1}\rangle, |1\rangle, |0\rangle$ states become strongly localized in left, center, and right well, making the factors $a$ and $\Delta$ exponentially small. In this case, the charge and flux operators converge to $S_x$ and $S_y$, respectively and, at zero bias $\Phi_{\text{ext}} = 0$ the Hamiltonian is diagonalized by the states $|0\rangle$ and $|\pm\rangle = (|L\rangle \pm |R\rangle)/\sqrt{2}$. As explained in the introduction, in this limit the charge operator $q \approx S_y$ cannot mediate the decay of the $|+\rangle$ state to the ground state $|0\rangle$, and the second excited state can be used to construct a BIC.

We have compared the analytical estimates with numerical diagonalizations, obtaining an excellent qualitative agreement. In our method we compute the lowest energy eigenstates, project the relevant operators onto the qutrit basis and express them as combination of 1-spin operators. For greater accuracy, we use the $\sin(\phi)$ operator instead of $\phi$, and recover this and the $q$ operator as derivatives of the fluxonium’s Hamiltonian with respect to flux and voltage perturbations, respectively.

Here, $E_3$ is the eigenenergy of the third excited state, $\tilde{\phi}_* = \frac{1-2a^2}{1-4a^2} \phi_*$, $b = \frac{\sqrt{2a}}{\sqrt{1-a^2}}$ with $\epsilon = U(\phi_*)$. Last but not least, we can compute the fluxonium charge operator acting on the qutrit subspace as in Ref. [33], via the Heisenberg equation $q = \frac{iC_l}{\hbar}[H, \phi]$:  
$$ q = \frac{iC_l}{\hbar} \left[ \epsilon b (\epsilon - \Delta) S_y + \Delta \tilde{\phi}_* \left( S_+^2 - S_-^2 \right) \right]. \quad(4) $$

Our derivation expresses all operators in terms of the overlap $a$ and the energy $E_3$, quantities that may be estimated using the harmonic states of the right, left, and central wells. Intuitively, all terms containing $a$ in Eqs. (3)-(4) are mediated by the third excited state. In the limit of large charging capacitance or heavy fluxonium, the $|{-1}\rangle, |1\rangle, |0\rangle$ states become strongly localized in left, center, and right well, making the factors $a$ and $\Delta$ exponentially small. In this case, the charge and flux operators converge to $S_x$ and $S_y$, respectively and, at zero bias $\Phi_{\text{ext}} = 0$ the Hamiltonian is diagonalized by the states $|0\rangle$ and $|\pm\rangle = (|L\rangle \pm |R\rangle)/\sqrt{2}$. As explained in the introduction, in this limit the charge operator $q \approx S_y$ cannot mediate the decay of the $|+\rangle$ state to the ground state $|0\rangle$, and the second excited state can be used to construct a BIC.

We have compared the analytical estimates with numerical diagonalizations, obtaining an excellent qualitative agreement. In our method we compute the lowest energy eigenstates, project the relevant operators onto the qutrit basis and express them as combination of 1-spin operators. For greater accuracy, we use the $\sin(\phi)$ operator instead of $\phi$, and recover this and the $q$ operator as derivatives of the fluxonium’s Hamiltonian with respect to flux and voltage perturbations, respectively.

Fig. 3 illustrates the excellent agreement between our predictions Eqs. (3)-(4), and the expansions of the Hamiltonian as a function of external magnetic flux $\Phi_{\text{ext}} = 0$ (a-b), and of the flux (c) and charge (d) of a fluxonium with $E_J/E_C = 21.74$ as a function of $E_J/E_C$. Following our previous discussion, we see that charge and flux effectively become $S_x, S_y$ with other terms exponentially vanishing with increasing fluxonium capacitance. Knowing that at zero flux the second excited state is $|+\rangle$, we have obtained rigorously that the matrix element of this state with the other qutrit state is suppressed exponentially fast for the heavy fluxonium. This state becomes a BIC when the fluxonium is coupled via its charge operator to an extended object with a continuous spectrum, as we show next.

II. BIC IN A FLUXONIUM QUTRIT COUPLED TO A WAVEGUIDE

Let us now discuss the dynamics of a fluxonium with a capacitive coupling $C_c$ to the continuum of propagating modes in a coplanar microwave guide, as shown in Fig. 4(a). From quantum optical considerations, the waveguide is a gapless medium supporting frequencies $\lambda / 2$-waveguide
of length $L$ [53, 54]:

$$H = \frac{1}{2C_\Sigma} q^2 + V(\phi) + \sum_{n=0}^{N-1} \hbar \omega_n \left( b_n^\dagger b_n + \frac{1}{2} \right) + \Delta H$$

$$\Delta H = C_\Sigma^2 q \sum_{n=0}^{N-1} (-1)^n \sqrt{\frac{\hbar \omega_n}{2c_0 L}} (b_n - b_n^\dagger). \quad (5)$$

We have expanded the waveguide Hamiltonian using the normal modes $[b_n, b_m] = i\hbar \delta_{nm}$ with $n, m = 0, \ldots, N - 1$, introducing waveguide’s capacitance per unit length $c_0 = C/L$ and the renormalized fluxonium’s capacitance $C_\Sigma = C_f + C_C$.

Fermi’s Golden Rule [55] is a good estimate for the transition rates $\Gamma_{ij}$ between fluxonium states $i$ and $j$ assisted by the modes of the waveguide. The formula requires the interaction Hamiltonian $H$ and the density of states $\rho(\omega)$ which, for a waveguide with a linear dispersion relation $\omega = \nu k$, is uniform $\rho(\omega) = L/(2\pi\nu)$. In this case Fermi’s Golden Rule (see [56] for an inductive coupling) predicts:

$$\Gamma_{ij} = 2\pi \left( \frac{C_\Sigma}{C_f} \right)^2 G_0 Z |\langle i | N_f | j \rangle|^2 \omega_{ij}, \quad (6)$$

as a function of the transition frequency $\omega_{ij}$, the number of Cooper pairs in the fluxonium $N_f = q/2e$ and the line’s impedance $Z = c_0 \nu$. As discussed, direct transitions from $|+\rangle \to |0\rangle$ are forbidden by symmetry $\Gamma_{+0} = 0$, and the $|+\rangle$ state can decay only via the $|-\rangle$ state.

![FIG. 4. Transition rate from the BIC state $|+\rangle$ to $|-\rangle$ due to radiative losses to the waveguide as a function of the inverse of the renormalized fluxonium charging energy, $\tilde{E}_C = \frac{e^2}{C_\Sigma}$. The Josephson energy of the fluxonium is $E_J = 10\, \text{GHz}$ and the linear inductions are chosen so that they are experimentally realizable $E_J/E_L = 17.31$ [25], $E_J/E_L = 21.74$ [31] and $E_J/E_L = 33.79$ [28]. The waveguide has impedance $Z = 50\, \Omega$ and the coupling capacitance is taken $(E_C)_c = 0.25\, \text{GHz}$. The vertical dashed lines indicate the value of $\tilde{E}_C$ for which the energy distance between levels $|+\rangle$ and $|-\rangle$ exceeds that between the high energy subspace and the qutrit subspace.](image)

The Josephson energy of the fluxonium is $E_J = 10\, \text{GHz}$ and the coupling capacitance is taken

$$\Gamma_{-+} = \frac{1}{\hbar} |\langle i | N_f | j \rangle|^2 \omega_{ij} \frac{Z}{E_C}.$$
To understand this, we rely on state-of-the-art models for how low-frequency flux noise penetrates Josephson devices in quantum information applications [39][41]. This noise has a power spectrum that can be approximated as $S(\omega) \sim 2\pi A^2/\omega$ with $A \approx (10^{-5} - 10^{-6})\Phi_0$ [12][44], which implies quasi-static fluctuations with an amplitude $\sigma \sim (10^{-5} - 10^{-6})\Phi_0$ (see Supplemental Material).

Fig. 5 displays the transition rate from the BIC to the ground state as a function of the flux deviation $\Phi_{\text{ext}} \neq 0$. Using perturbation theory, we obtain $\Gamma_+ \sim (E_J/E_C)^2\Phi_{\text{ext}}$. This dependence is reflected in Fig. 5 where in log-log scale all curves share the slope of this quadratic dependency. This figure also confirms that the BIC becomes more sensitive to external flux perturbations as the ratio $E_J/E_C$ is increased. From state-of-the-art experiments we expect low-frequency flux fluctuations with amplitude $10^{-5} - 10^{-6}\Phi_0$, denoted by a colored region in Fig. 5. In this region the noise-induced decay $\Gamma_+\Phi$ can grow substantially, unless we make a "not too heavy" fluxonium.

As example of a realistic compromise that works well for a pessimistic estimate of fluctuations $\sim 10^{-5}\Phi_0$, the purple line from Fig. 5 $E_J/E_C \approx 5$, $E_J/E_L \approx 30$, produces a BIC with a decay time $T_1 \sim 10^{-4}$ ms, where the contribution of $\Gamma_+ \approx 2 \times 10^{-2}\pi$ is small [cf. horizontal dashed line in Fig. 3]. For a moderately optimistic noise amplitude $\sim 10^{-4}\Phi_0$, a BIC with the same parameters would produce an expected decay time of $T_1 \sim 10$ms, as now the radiative losses due to the $|+\rangle \rightarrow |0\rangle$ and $|+\rangle \rightarrow |-\rangle$ are of the same order.

When the influence of $1/f$ noise is small, there are other noise sources that need to be considered. In this limit, losses into the dielectric [45] may become the most important decay channel. This has been demonstrated in Ref. [44] for some fluxonium qubits, where a detailed characterization of dielectric losses showed that they were the dominant decay channel, giving relaxation times $\sim 10^{-4}$ms. In our setup, the $|+\rangle \rightarrow |-\rangle$ decay channel could be enabled by matrix elements in the flux operator that couple to high-frequency currents in the dielectric [46]. While we cannot provide ab-initio estimates of this relaxation channel, we can extrapolate the scaling of the decay time, which—as shown in Supplemental Material and Ref. [44]—scales quadratically $\Gamma_{\text{dielectric}} \sim \omega_{+-}^\beta$ ($\beta \approx 2$) as a function of the separation between excited states $\omega_{+-}$. By working closer to the heavy fluxonium limit, both $\omega_{+-}$ and the decay rate can be significantly reduced. This way, we expect that we can improve the lifetime of the BIC, up from $T_1 \approx 100\mu$s for the fluxonium in Ref. [44], up to a lifetime that is closer to the $1/f$-limited lifetimes discussed above, $T_1 \sim 10$ms (see Supplemental Material).

III. CONCLUSIONS

In summary, in this work, we have shown how to construct a compact BIC living in a superconducting fluxonium qutrit that is capacitively connected to an open microwave guide. Furthermore, we have found that this device can be brought to a regime where the second excited state is a quasi-BIC state. That is a state with a highly long decay time that, under realistic parameters devices, can reach up to tens of milliseconds.

The possibility of creating long-lived BIC states in small fluxonium devices is exciting as a scalable platform for storing protected quantum information. However, as discussed above, there is a compromise between the lifetime of the BIC state and the possibility of accessing those states using external fields. It, therefore, remains an open question of how to scalable and coherently prepare these states in realistic setups. Some avenues that can be explored with the tools developed in this work include the engineering of $|+\rangle$ states by dynamically tuning the external magnetic fields and controlling the injection of photons or using multi-photon induced transitions via excited states.

In this study, we have also shown that the fluxonium qutrit and its BIC states are very sensitive to external magnetic fields. In particular, we believe that it is possible to build a very sensitive magnetic field sensor by monitoring the $|+\rangle \leftrightarrow |0\rangle$ resonance, as both the intensity and linewidth of that resonance depend on small deviations of the magnetic flux experienced by the fluxonium [cf. Fig. 5]. This mechanism would enable the detection of small changes $10^{-4}\Phi_0$ even in the presence of realistic experimental flux noise $10^{-6} - 10^{-5}\Phi_0$.

ACKNOWLEDGMENTS

This work is supported by the European Commission FET-Open project AVaQus GA 899561, by Proyecto Sinérgico CAM 2020 Y2020/TCS-6545 (NanoQuCo-CM) and CSIC Quantum Technologies Platform PTI-001. We acknowledge Centro de Supercomputación de Galicia (CESGA) who provided access to the supercomputer FinisTerrae for performing simulations.

Appendix A: Quasi-Static noise

We extract the typical fluctuations in external flux for a fluxonium from the power spectrum. We are interested in the low-frequency noise, which is the one that can produce a decay of the BIC. Thus, $1/f$ noise is likely to be important to understand the decay times of fluxonium BICS in realistic situations. We assume a power spectrum as explained in the Appendix of Ref. [44], based on many previous experimental results:

$$S(\omega) = 2\pi A^2/\omega$$

with $A = (10^{-5} - 10^{-6})\Phi_0$. We also need to set a low and high frequency cut-off for the $1/f$ noise, which we take $\gamma_- = 10^{-2}\text{Hz}$ and $\gamma_+ = 10^3\text{Hz}$, which is consistent
with the experimental results in Ref. [42] (red points in Fig. 3 of that reference). In any case, we will see that the values of those cut-offs do not affect that much the final results.

Once the noise model is set, we can extract the fluctuations at the high frequency cut-off, the important one for the BIC, as the real part of the Fourier transform of the power spectrum:

\[
\sigma^2 \approx \langle \Phi_{\text{ext}}(t)\Phi_{\text{ext}}(t+\tau) \rangle = A^2 \int_{\gamma_-}^{\gamma_+} \frac{d\omega}{\omega} \cos(\omega \tau) \ . \quad (A1)
\]

The time \( \tau \approx 1/\gamma_- \) should be of the order of the inverse of the low-frequency cut-off so to get the amplitude of the quasi-static fluctuations. Setting this value in the previous formula, we obtain:

\[
\sigma^2 \approx A^2 \int_{1}^{\gamma_+} \frac{dx}{x} \cos(x) \ . \quad (A2)
\]

Expressing the previous results in terms of the integral cosine integral function

\[
\text{Ci}(x) = -\int_{x}^{\infty} \frac{dx}{x} \cos(x) \ , \quad (A3)
\]

and taking \( \frac{\gamma_+}{\gamma_-} \approx 10^3 \) we get the desired amplitude as given by:

\[
\sigma^2 = A^2 \left[ -\text{Ci}(10^3) + \text{Ci}(1) \right] \sim O(1) A^2 \ . \quad (A4)
\]

As we previously state, this results do not depend strongly on the low and high-energy cut-offs. As we are interested in only the order of magnitude of these fluctuations, we can take them as \( \sigma \approx A \), up to a factor of order one, which is irrelevant because the uncertainty in \( A \) for the devices is, at least, of the same order. For this reason, we have taken the fluctuations in Fig. (4) of the main text as given by \( A = (10^{-5} - 10^{-6}) \Phi_0 \), which is the value provided in Ref. [44] based on many previous experimental works.

### Appendix B: Noise due to losses in the dielectric

It is argued in Ref. [44], that the main source of relaxation in the fluxonium qubit is tangential losses into the dielectric. Those losses have been shown to be an important source of dissipation for several devices, such as phase qubits or transmons [45,47]. For the fluxonium, the main source of dissipation seems to be fluctuations in the polarization currents of the dielectric that couples with the fluxonium phase degree of freedom. We can extract the decay time using the phenomenological model found in Ref. [44], but now describing the transition from the BIC to the \(-\) state:

\[
\Gamma = |\langle -|\varphi>||^2 \frac{\hbar \omega_{\perp}^2}{8E_C} \left( \frac{\omega_{\perp}}{12\pi\Gamma} \right)^{0.15} \left( \coth \left( \frac{\hbar \omega_{\perp}}{2k_B T} \right) + 1 \right)
\]

The experimental results validated this law, giving decay times around \( T_1 \approx 100\mu s \) at the sweet spot for direct transitions from the \( 1 \rightarrow 0 \), where \( E_{01} \approx 0.8\Gamma \). We notice that for the parameters we have seen that the fluxonium BIC has a large decay time \( \frac{\hbar \omega_{\perp}}{2k_B T} \approx 5 \), \( \frac{\hbar \omega_{\perp}}{2k_B T} \approx 30 \), the energy of the relevant transition has an small frequency \( \omega_{\perp} \sim 0.01\Gamma \), much smaller that the one for the qubit transition. Taking into account that the transition rate depends almost quadratically in previous equation, we expect a rough increase of the decay time by, at least, three orders of magnitude for the decay of the BIC due to dielectric losses \( BIC T_1 \approx 10^{-5}s \). We conclude that this source of noise does not posses a significant dangerous for the BIC stability.

---

[1] P. A. M. Dirac, The quantum theory of the emission and absorption of radiation, Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character 114, 243 (1927).

[2] J. von Neumann and E. P. Wigner, Über das verhalten von eigenwerten bei adiabatischen prozessen, in The Collected Works of Eugene Paul Wigner (Springer, 1993) pp. 294–297.

[3] F. Capasso, C. Sirtori, J. Faist, D. L. Sivco, S.-N. G. Chu, and A. Y. Cho, Observation of an electronic bound state above a potential well, Nature 358, 565 (1992).

[4] J. P. Ramos-Andrade, D. Zambrano, and P. A. Orellana, Fano-majorana effect and bound states in the continuum on a crossbar-shaped quantum dot hybrid structure, Annalen der Physik 531, 1800498 (2019).

[5] M. L. De Guevara and P. Orellana, Electronic transport through a parallel-coupled triple quantum dot molecule: Fano resonances and bound states in the continuum, Physical Review B 73, 205303 (2006).

[6] J. González, M. Pacheco, L. Rosales, and P. Orellana, Bound states in the continuum in graphene quantum dot structures, EPL (Europhysics Letters) 91, 66001 (2010).

[7] R. Parker, Resonance effects in wake shedding from parallel plates: some experimental observations, Journal of Sound and Vibration 4, 62 (1966).

[8] R. Parker, Resonance effects in wake shedding from parallel plates: calculation of resonant frequencies, Journal of Sound and Vibration 5, 330 (1967).

[9] S. Weimann, Y. Xu, R. Keil, A. E. Miroshnichenko, A. Tümmersmann, S. Nolte, A. A. Sukhorukov, A. Szameit, and Y. S. Kivshar, Compact surface fano states embedded in the continuum of waveguide arrays, Physical review letters 111, 240403 (2013).

[10] G. Corrielli, G. Della Valle, A. Crespi, R. Ostellame, and S. Longhi, Observation of surface states with algebraic localization, Physical review letters 111, 220403 (2013).

[11] Y. Yang, C. Peng, Y. Liang, Z. Li, and S. Noda, Analytical perspective for bound states in the continuum in pho-
tonic crystal slabs, Physical Review letters 113, 037401 (2014).
[12] A. I. Ovcharenko, C. Blanchard, J.-P. Hugonin, and C. Sauvan, Bound states in the continuum in symmetric and asymmetric photonic crystal slabs, Physical Review B 101, 155303 (2020).
[13] A. Kodigala, T. Lepetit, Q. Gu, B. Bahari, Y. Fainman, and B. Kanté, Lasing action from photonic bound states in continuum, Nature 541, 196 (2017).
[14] Z. Yu and X. Sun, Acousto-optic modulation of photonic bound state in the continuum, Light: Science & Applications 9, 1 (2020).
[15] Y. Plotnik, O. Peleg, F. Dreisow, M. Heinrich, S. Nolte, A. Szameit, and M. Segev, Experimental observation of optical bound states in the continuum, Physical review letters 107, 183901 (2011).
[16] H. Dong, Z. R. Gong, H. Ian, L. Zhou, and C. P. Sun, Intrinsic cavity qed and emergent quasinormal modes for a single photon, Phys. Rev. A 79, 063847 (2009).
[17] T. Tufarelli, F. Ciccarello, and M. Kim, Dynamics of spontaneous emission in a single-end photonic waveguide, Physical Review A 87, 013820 (2013).
[18] T. Tufarelli, M. Kim, and F. Ciccarello, Non-markovianity of a quantum emitter in front of a mirror, Physical Review A 90, 012113 (2014).
[19] I.-C. Hoi, A. F. Kockum, L. Tornberg, A. Pourkabirian, G. Johansson, P. Delsing, and C. Wilson, Probing the quantum vacuum with an artificial atom in front of a mirror, Nature Physics 11, 1045 (2015).
[20] M. Cotofiguero and A. Alì, Excitation of single-photon embedded eigenstates in coupled cavity–atom systems, Optica 6, 799 (2019).
[21] L. Zhou, Z. R. Gong, Y.-x. Liu, C. P. Sun, and F. Nori, Controllable scattering of a single photon inside a one-dimensional resonator waveguide, Phys. Rev. Lett. 101, 100501 (2008).
[22] G. Calajo, Y.-L. F. Fang, H. U. Baranger, F. Ciccarello, et al., Exciting a bound state in the continuum through multiphoton scattering plus delayed quantum feedback, Physical review letters 122, 073601 (2019).
[23] A. Feiguin, J. J. García-Ripoll, and A. González-Tudela, Qubit-photon corner states in all dimensions, Physical Review Research 2, 023082 (2020).
[24] M. Zanner, T. Orell, C. M. Schneider, R. Albert, S. Oleshko, M. L. Juan, M. Silveri, and G. Kirchmair, Coherent control of a multi-qubit dark state in waveguide quantum electrodynamics, Nature Physics, 1 (2022).
[25] V. E. Manucharyan, J. Koch, L. I. Glazman, and M. H. Devoret, Fluxonium: Single Cooper-pair circuit free of charge offsets, Science 326, 113 (2009).
[26] M. Ahumada, P. Orellana, and J. Retamal, Bound states in the continuum in whispering gallery resonators, Physical Review A 98, 023827 (2018).
[27] M. Hita-Pérez, G. Jaumà, M. Pino, and J. J. García-Ripoll, Ultrastrong capacitive coupling of flux qubits, arXiv preprint arXiv:2108.02549 (2021).
[28] N. Earnest, S. Chakram, Y. Lu, N. Irons, R. K. Naik, N. Leung, L. Ocola, D. A. Czaplewski, B. Baker, J. Lawrence, et al., Realization of a λ system with metastable states of a capacitively shunted fluxonium, Physical review letters 120, 150504 (2018).
[29] H. Zhang, S. Chakram, T. Roy, N. Earnest, Y. Lu, Z. Huang, D. K. Weiss, J. Koch, and D. I. Schuster, Universal fast-flux control of a coherent, low-frequency qubit, Phys. Rev. X 11, 011010 (2021).
[30] M. Peruzzo, F. Hassani, G. Szepl, A. Trioni, E. Redchenko, M. Zenlicka, and J. Fink, Geometric superinductance qubits: Controlling phase delocalization across a single josephson junction, arXiv preprint arXiv:2106.05882 (2021).
[31] U. Vool, I. M. Pop, K. Sliwa, B. Abdó, C. Wang, T. Brecht, Y. Y. Gao, S. Shankar, M. Hatridge, G. Catelani, et al., Non-poissonian quantum jumps of a fluxonium qubit due to quasiparticle excitations, Physical review letters 113, 247001 (2014).
[32] U. Vool, A. Kou, W. Smith, N. Frattini, K. Serniak, P. Reinhold, I. Pop, S. Shankar, L. Frunzio, S. Girvin, et al., Driving forbidden transitions in the fluxonium artificial atom, Physical Review Applied 9, 054046 (2018).
[33] M. Hita-Pérez, G. Jaumà, M. Pino, and J. J. García-Ripoll, 3-josephson junctions flux qubit couplings, in preparation.
[34] We use the term quasi-BIC to indicate that the BIC state has always a finite lifetime.
[35] J. J. García-Ripoll, B. Peropadre, and S. De Liberato, Light-matter decoupling and a2 term detection in superconducting circuits, Scientific reports 5, 1 (2015).
[36] A. Parra-Rodriguez, E. Rico, E. Solano, and I. Egusquiza, Quantum networks in divergence-free circuit qed, Quantum Science and Technology 3, 024012 (2018).
[37] D. J. Griffiths, Introduction to quantum mechanics (Pearson International Edition (Pearson Prentice Hall, Upper Saddle River, 2005), 1962).
[38] I. Moskalenko, I. Besedin, I. Simakov, and A. Ustinov, Tunable coupling scheme for implementing two-qubit gates on fluxonium qubits, Applied Physics Letters 119, 194001 (2021).
[39] R. H. Koch, J. Clarke, W. Gobian, J. M. Martinis, C. Pe grim, and D. J. Van Harlingen, Flicker (1/f) noise in tunnel junction de squid, Journal of low temperature physics 51, 207 (1983).
[40] R. C. Bialczak, R. McDermott, M. Ansmann, M. Hofheinz, N. Katz, E. Lucero, M. Neeley, A. D. O’Connell, H. Wang, A. N. Cleland, and J. M. Martinis, 1/f flux noise in josephson phase qubits, Phys. Rev. Lett. 99, 187006 (2007).
[41] E. Paladino, Y. M. Galperin, G. Falci, and B. L. Altshuler, 1/f noise: Implications for solid-state quantum information, Rev. Mod. Phys. 86, 361 (2014).
[42] F. Yan, S. Gustavsson, A. Kamal, J. Birenbaum, A. P. Sears, D. Hover, T. J. Gudnason, D. Rosenberg, G. Samach, S. Weber, et al., The flux qubit revisited to enhance coherence and reproducibility, Nature communications 7, 1 (2016).
[43] P. Kumar, S. Sendelbach, M. A. Beck, J. W. Freeland, Z. Wang, H. Wang, C. C. Yu, R. Q. Wu, D. P. Pappas, and R. McDermott, Origin and reduction of 1/f magnetic flux noise in superconducting devices, Phys. Rev. Applied 6, 041001 (2016).
[44] L. B. Nguyen, Y.-H. Lin, A. Somoroff, R. Mencia, N. Grabon, and V. E. Manucharyan, High-coherence fluxonium qubit, Physical Review X 9, 041041 (2019).
[45] J. M. Martinis, Surface loss calculations and design of a superconducting transmon qubit with tapered wiring, np Quantum Information 8, 1 (2022).
[46] This noise cannot produce decay from $|+\rangle$ to $|0\rangle$ as they have equal parity.
[47] J. M. Martinis, K. B. Cooper, R. McDermott, M. Steffen, M. Ansmann, K. Osborn, K. Cicak, S. Oh, D. P. Pappas, R. W. Simmonds, et al., Decoherence in josephson qubits from dielectric loss, Physical review letters 95, 210503 (2005).