YANGIAN OF $AdS_3/CFT_2$ AND ITS DEFORMATION

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Abstract. We construct highest-weight modules and a Yangian extension of the centrally extended superalgebra appearing in the worldsheet scattering associated with the $AdS_3/CFT_2$ duality, and show a link to the Yangian of $AdS_5/CFT_4$. We also consider a quantum deformation of this superalgebra, its modules and a quantum affine extension of the Drinfeld-Jimbo type which describes deformed worldsheet scattering.

1. Introduction

Recent progress in exploring integrability in $AdS/CFT$ dualities has led to the discovery of many new algebraic structures. One of the most notable is a class of the so-called $u$-deformed Hopf superalgebras, which emerge in the worldsheet scattering theory in various backgrounds \cite{BSS, BSSST, HPT, PST}. These superalgebras are deformed in the direction of their central extensions and lead to $R$-matrices of a non-relativistic type closely resembling that of the one-dimensional Hubbard chain \cite{Be1}.

For example, the worldsheet superalgebra of the $AdS_3/CFT_4$ duality is the centrally extended superalgebra $\mathfrak{sl}_2 \oplus \mathfrak{Cu}^\pm$ admitting a $u$-deformed Hopf algebra structure \cite{ST} and having a non-conventional representation theory \cite{MM}. Interestingly, it admits a non-standard $u$-deformed Yangian extension, which was constructed in various realizations: the Drinfeld $J$-basis \cite{Be2}, Drinfeld New presentation \cite{ST}, and RTT-presentation \cite{LL}. Moreover, this superalgebra can be further $q$-deformed in the Cartan direction. In such a way one obtains a double-deformed Hopf superalgebra. In particular, a $q$-deformation of the $u$-deformed $\mathfrak{sl}_2 \oplus \mathfrak{Cu}^\pm$ and its affinization of the Drinfeld-Jimbo type were constructed in \cite{BK} and \cite{BGM}, respectively.

In this paper we consider the centrally extended superalgebra $(\mathfrak{sl}_{1|1})^2 \oplus \mathfrak{Cu}^\pm$, which was shown to be the symmetry of the worldsheet scattering of massive modes in the $AdS_3/CFT_2$ duality \cite{BSZ} for the $AdS_3 \times S^3 \times S^3 \times S^1$ background \cite{BSS}. This superalgebra also serves as a prototype for the symmetries of the duality on the $AdS_3 \times S^1 \times T^4$ background \cite{BSSST}, which essentially contains two copies of this superalgebra with their central elements identified. For this reason, in this paper we will focus on the algebraic constructions associated with the former case only.

This paper contains two parts. In the first part we construct the highest-weight modules and Yangian extension of the extended superalgebra $(\mathfrak{sl}_{1|1})^2 \oplus \mathfrak{Cu}^\pm$. We then demonstrate the role of this algebra in the $AdS_3/CFT_2$ duality. In particular, we construct typical and atypical Kac modules $K(\lambda,\nu)$ and $A(\lambda,\nu)$, and the one-dimensional module $1$ of $(\mathfrak{sl}_{1|1})^2 \oplus \mathfrak{Cu}^\pm$. We show that the tensor product of two atypical modules is isomorphic to the typical one and obtain the corresponding $R$-matrix $R(\lambda,\nu;\bar{\lambda},\bar{\nu})$. We then show how to obtain the left and right modules describing left and right excitations of the worldsheet scattering in the $AdS_3/CFT_2$ duality, singlet states, and the left-left and left-right $R$-matrices. We also show that the new Yangian is “closely related” to the $(\mathfrak{sl}_{1|1})^2 \oplus \mathfrak{Cu}^\pm$ subsector of the Yangian obtained in \cite{Be2}.

In the second part of the paper we consider a double-deformation of $(\mathfrak{sl}_{1|1})^2 \oplus \mathfrak{Cu}^\pm$ and obtain its affine extension of the Drinfeld-Jimbo type. The structure of the second part closely resembles that of the first part. We construct $q$-deformed analogues of our highest-weight modules constructed before and obtain the $q$-deformed $R$-matrix. We then demonstrate how to obtain $q$-deformed analogues of the left and right modules, singlet states, and the left-left and left-right $R$-matrices. It is worth noting that a somewhat similar two-parameter $q$-deformation of this duality was recently proposed in \cite{Ho}. Since our approach differs from the one presented in cit. loc., we hope the reader will find our paper interesting and instructive.

The affinization presented in this paper is inspired by a similar construction presented in \cite{BGM}, however we formulate deformations in a slightly different way. This creates an obstruction to a direct comparison between our results and the $(\mathfrak{sl}_{1|1})^2 \oplus \mathfrak{Cu}^\pm$ subsector of the quantum affine algebra obtained in loc. cit.

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2. The Superalgebra \((\mathfrak{sl}_{1|1})^2 \oplus \mathbb{C} u^\pm\) and Its Highest-Weight Modules

In this section we present the algebra \(a = (\mathfrak{sl}_{1|1})^2 \oplus \mathbb{C} u^\pm\) and the associated Hopf algebra which arises as the worldsheet symmetry in the AdS3/CFT3 duality [BSS]. We then construct highest-weight modules of this algebra that are important in the aforementioned duality.

2.1. Algebra. Let \([\cdot, \cdot]\) denote the \(\mathbb{Z}_2\)-graded commutator, i.e. \([a, b] = ab - (-1)^{\deg a \deg b} ba\) for \(a, b \in \mathfrak{g}\), where \(\mathfrak{g}\) is a Lie superalgebra and \(p = \deg g\) denotes the \(\mathbb{Z}_2\)-grading on \(\mathfrak{g}\). We will use the notation \(C^x = C \backslash \{0\}\) and also \(i\) for \(i \in \{1, 2\}\) such that \(1 = 2\) and \(2 = 1\).

We start by considering the double centrally extended superalgebra \((\mathfrak{sl}_{1|1})^2 \oplus \mathbb{C}^2\). We then obtain \((\mathfrak{sl}_{1|1})^2 \oplus \mathbb{C} u^\pm\) as the quotient of an extension of the former algebra. The motivation for this approach is explained in Remark 2.2 given below.

Definition 2.1. The double centrally extended superalgebra \((\mathfrak{sl}_{1|1})^2 \oplus \mathbb{C}^2\) is given by elements \(e_i, f_i\) and central elements \(h_i, k_i\) with \(i, j \in \{1, 2\}\) and satisfying
\[
\{e_i, f_j\} = \delta_{ij} h_i + (1 - \delta_{ij}) k_i.
\]
The remaining relations are trivial. The \(\mathbb{Z}_2\)-grading is given by \(\deg h_i = \deg k_i = 0\) and \(\deg e_i = \deg f_i = 1\).

Our focus will be on the tensor product of two atypical highest-weight modules and the \(R\)-matrix. Bearing in mind this goal we extend the algebra above by the algebra \(C u^\pm = C u^+ \oplus C u^-\) such that \(u^\pm u^\mp = 1\) and introduce the book keeping notation \(\mathfrak{a}_0 = (\mathfrak{sl}_{1|1})^2 \oplus \mathbb{C}^2 \oplus \mathbb{C} u^\pm\). Note that \(u^\pm\) are central in \(\mathfrak{a}_0\). Let \(\mathfrak{a}_0\) denote the universal enveloping algebra of \(\mathfrak{a}_0\). The next observation follows straightforwardly.

Proposition 2.1. The vector space basis of \(\mathfrak{a}_0\) is given in terms of monomials
\[
(f_2)^{r_2}(f_1)^{r_1}(h_1)^{s_1}(h_2)^{s_2}(k_1)^{t_1}(k_2)^{t_2}(u)^{e_1}(e_2)^{e_2}
\]
with \(r_i, s_i, t_i \in \{0, 1\}\), \(l_i \in \mathbb{Z}_{\geq 0}\) and \(t \in \mathbb{Z}\).

The monomials \(2.2\) give a Poincaré–Birkhoff–Witt type basis of \(\mathfrak{a}_0\). Moreover, by the standard arguments, \(\mathfrak{a}_0 \cong \mathfrak{a}_0^n \mathfrak{a}_0^n\) as vector spaces, where \(\mathfrak{a}_0^n\) and \(\mathfrak{a}_0^n\) are the nilpotent subalgebras generated by elements \(f_i\) and \(e_i\) with \(i = 1, 2\), respectively, and \(\mathfrak{a}_0^0\) is generated by the remaining elements of \(\mathfrak{a}_0\).

Remark 2.1. The algebra \(\mathfrak{a}_0\) also admits a \(\mathbb{Z}\)-grading given by
\[
\deg h_i = \deg j = \deg f_i = 0, \quad \deg e_i = \deg f_i = 1, \quad \deg k_i = \pm 2,
\]
where the upper sign is for \(i = 1\) and the lower sign is for \(i = 2\) (we will use this dotted notation throughout this paper). Note that we could equivalently define the \(\mathbb{Z}\)-grading by inverting the grading, namely \(\deg \rightarrow -\deg\).

Let \(I_0\) be the ideal of \(\mathfrak{a}_0\) generated by the relation
\[
\Delta(k_i) = k_i \otimes u^\mp + u^{\mp} \otimes k_i,
\]
where \(\alpha_i \in \mathbb{C}^\times\). We define the quotient algebra \(\mathfrak{a} = \mathfrak{a}_0/I_0\). Then one can introduce a Hopf algebra structure on \(\mathfrak{a}\) given by the coproduct
\[
\Delta(e_i) = e_i \otimes u^\mp + u^\mp \otimes e_i, \quad \Delta(k_i) = k_i \otimes u^\mp + u^{\mp} \otimes k_i, \quad \Delta(f_i) = f_i \otimes u^+ + u^+ \otimes f_i, \quad \Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i, \quad \Delta(u^\pm) = u^\pm \otimes u^\mp,
\]
the counit \(\epsilon(a) = 0\) and the antipode \(S(a) = -a\) for all \(a \in \mathfrak{a}_0\) except \(\epsilon(u^\pm) = 1\), \(S(u^\pm) = u^\mp\).

Let us explain some properties of the algebra \(\mathfrak{a}\). First, the grading \(g_{\mathfrak{a}}\) is associated with the powers of \(u^\pm\) in the coproduct \(2.4\). Second, we want to explain the role of the ideal \(I_0\). Set \(\Delta^p = \sigma \circ \Delta\), where \(\sigma : a \otimes b \mapsto (-1)^{\deg a \deg b} b \otimes a\) is the graded permutation operator. The central elements of \(\mathfrak{a}\) are co-commutative:
\[
\Delta(c) = \Delta^p(c) \quad \text{for} \quad c \in \{h_i, k_i, u^\pm\}.
\]
This is obvious for \(h_i\) and \(u^\pm\), while for \(k_i\) we have \(\Delta(k_i) - \Delta^p(k_i) = k_i \otimes (u^{\mp 2} - u^{\pm 2}) + (u^{\pm 2} - u^\mp 2) \otimes k_i\), which is equal to zero only if \(2.3\) holds. Also note that due to \(2.3\) the algebra \(\mathfrak{a}\) is actually a two-parameter family of Hopf algebras parametrized by \(\alpha_i\). Moreover, we need to set \(\deg\alpha_i = \pm 2\), for \(2.3\) to respect the \(\mathbb{Z}\)-grading. (In section 4 we will set \(\alpha_1 = -\alpha_2 = h\), where \(h \in \mathbb{C}^\times\) plays the role of the coupling constant in the underlying field theory and the minus sign ensures unitarity.)
Besides the Chevalley anti-automorphism \( e_i \mapsto -f_i, f_i \mapsto -e_i, h_i \mapsto -h_i, k_i \mapsto -k_i, u^\pm \mapsto u^{\mp} \), there are a number of involutive automorphisms of \( \mathfrak{Ua} \) given by

\[
\begin{align*}
  f_i &\mapsto e_i, \quad e_i \mapsto f_i, \quad h_i \mapsto h_i, \quad k_i \mapsto k_i, \quad u^\pm \mapsto u^{\mp}, \\
  f_i &\mapsto e_i, \quad e_i \mapsto f_i, \quad h_i \mapsto h_i, \quad k_i \mapsto k_i, \quad u^\pm \mapsto u^{\pm}, \\
  f_i &\mapsto f_i, \quad e_i \mapsto e_i, \quad h_i \mapsto h_i, \quad k_i \mapsto k_i, \quad u^\pm \mapsto u^{\mp},
\end{align*}
\]

(2.5)

which form the Klein-four outer-automorphism group \( \text{Aut}(\mathfrak{Ua}) \) of \( \mathfrak{Ua} \). Note that \( \text{Aut}(\mathfrak{Ua}) \) is also the group of the Hopf algebra outer-automorphisms of \( \mathfrak{Ua} \).

**Remark 2.2.** The relation (2.1) in the algebra \( \mathfrak{Ua} \) is equivalent to saying that

\[
[h_0, e_i] = 2e_i, \quad [h_0, f_i] = -2e_i,
\]

(2.6)

and the remaining relations are trivial. The grading is \( \text{deg}_2(h_0) = \text{deg}(h_0) = 0 \). Note that each \( \mathfrak{Ua}' \) module is also a \( \mathfrak{Ua} \) module, since \( \mathfrak{a}' = \mathbb{C}h_0 \times \mathfrak{a} \).

2.2. **Typical module.** The typical module \( K(\lambda, \nu) \) is the four-dimensional highest-weight Kac module of \( \mathfrak{Ua}' \) defined as follows: let \( v_0 \in K(\lambda, \nu) \) be the highest-weight vector such that

\[
h_i v_0 = \lambda_i v_0, \quad k_i v_0 = \mu_i v_0, \quad u^\pm v_0 = \nu^\pm v_0, \quad e_i v_0 = 0, \quad h_0 v_0 = 0
\]

(2.7)

for \( i = 1, 2 \), where \( \lambda_i, \nu \in \mathbb{C}^\times \) are generic and \( \mu_i = \alpha_i(\nu^2 - \nu^{-2}) \) due to (2.3). Set \( v_i = f_i v_0 \) and \( v_{21} = f_2 f_1 v_0 \). Then \( K(\lambda, \nu) \cong \text{span}_{\mathbb{C}} \{v_0, v_1, v_2, v_{21}\} \) as a vector space. Moreover \( h_0 v_i = -2v_i \) and \( h_0 v_{21} = -4v_{21} \). This allows us to write the following weight space decomposition:

\[
K(\lambda, \nu) = K_0(\lambda, \nu) \oplus K_{-2}(\lambda, \nu) \oplus K_{-4}(\lambda, \nu),
\]

where \( K_0(\lambda, \nu) \) and \( K_{-4}(\lambda, \nu) \) are one-dimensional weight spaces accommodating vectors \( v_0 \) and \( v_{21} \), respectively, and \( K_{-2}(\lambda, \nu) = \text{span}_{\mathbb{C}} \{v_1, v_2\} \).

Observe that \( f_2 v_1 = -f_1 v_2 = v_{21} \) and

\[
\begin{align*}
  e_1 v_{21} &= \mu_1 v_1 - \lambda_1 v_2, \quad e_1 v_1 = \lambda_1 v_0, \quad e_1 v_2 = \mu_1 v_0, \\
  e_2 v_{21} &= \lambda_2 v_1 - \mu_2 v_2, \quad e_2 v_1 = \mu_2 v_0, \quad e_2 v_2 = \lambda_2 v_0.
\end{align*}
\]

(2.8)

Set

\[
\begin{align*}
  v'_1 &= \mu_1 v_1 - \lambda_1 v_2, \quad f'_1 = \mu_1 f_1 - \lambda_1 f_2, \quad e'_1 = \mu_1 e_1 - \lambda_1 e_2, \\
  v'_2 &= \lambda_2 v_1 - \mu_2 v_2, \quad f'_2 = \lambda_2 f_1 - \mu_2 f_2, \quad e'_2 = \lambda_2 e_1 - \mu_1 e_2.
\end{align*}
\]

(2.9)

Then

\[
\begin{align*}
  f'_1 v_0 &= v'_1, \quad f_1 v'_1 &= \lambda_1 v_{21}, \quad f_1 v'_2 &= \mu_2 v_{21}, \quad e_1 v'_2 &= (\lambda_1 \lambda_2 - \mu_1 \mu_2) v_0, \\
  f'_2 v_0 &= v'_2, \quad f_2 v'_1 &= \mu_1 v_{21}, \quad f_2 v'_2 &= \lambda_2 v_{21}, \quad e_2 v'_2 &= (\mu_1 \mu_2 - \lambda_1 \lambda_2) v_0.
\end{align*}
\]

(2.10)

Note that vectors \( f'_1, e'_1 \) and \( v'_1 \) are pairwise linearly independent for generic \( \lambda_i \) and \( \nu \). We call the set \( \{v_0, v_1, v_2, v_{21}\} \) the \textit{up-down} basis and \( \{v_0, v'_1, v'_2, v_{21}\} \) the \textit{down-up} basis of \( K(\lambda, \nu) \). The module diagram for both bases is shown in Figure 1 (a) and (b).
2.3. Atypical module. The atypical module $A(\lambda, \nu)$ is the two-dimensional submodule of $K(\lambda, \nu)$ when $\lambda_1 \lambda_2 = \mu_1 \mu_2$. This relation implies that

\begin{equation}
(2.12) \quad v_1' = \gamma v_2', \quad f_1' = \gamma f_2' \quad \text{and} \quad e_1 v_2 = 0 \quad \text{where} \quad \gamma = \frac{\mu_1}{\lambda_2} = \frac{\lambda_1}{\mu_2}.
\end{equation}

Set $v_1'' = \mu_1 v_1 + \lambda_1 v_2$ and $f_1'' = \mu_1 f_1 + \lambda_1 f_2$. Then clearly both $v_1''$, $v_2''$ and $f_1''$, $f_2''$ are linearly independent and

\begin{equation}
(2.13) \quad f_1'' v_0 = v_1'', \quad f_2'' v_0 = -(\lambda_1 \lambda_2 - \mu_1 \mu_2) v_0, \quad e_1 v_0 = 2 \lambda_1 \mu_1 v_0, \quad e_2 v_0 = (\lambda_1 \lambda_2 + \mu_1 \mu_2) v_0.
\end{equation}

The module diagram of $(\lambda, \nu)$ when $\lambda_1 \lambda_2 = \mu_1 \mu_2$ is shown in Figure 1(c). Thus $A(\lambda, \nu) \cong \mathbb{C} \langle v_1', v_2' \rangle$ as a vector space. Moreover, it is easy to see that $A(\lambda, \nu) \cong K(\lambda, \nu)/A(\lambda, \nu)$.

Let us make a remark regarding the similarity of the diagrams in Figure 1 with the ones for $\mathfrak{s}l_1$. Recall that the highest-weight module $K(\lambda)$ of $\mathfrak{s}l_1 = \text{span}_C \{ f, h, e \}$ is the two-dimensional Kac module defined by $e.v_0 = 0$, $h.v_0 = \lambda v_0$, $v_1 = f.v_0$, $e.v_1 = v_0$, and the projective module $P(\lambda)$ of $\mathfrak{s}l_1$ is four-dimensional defined by $u_1 = f.u_0$, $u_2 = e.u_0$, $u_3 = f.e.u_0$ and satisfying $h.u_0 = \lambda u_0$, $e.u_1 = u_0 - u_0', \text{e} u_0' = \lambda u_0$; here $u_0$ is the zero vector. Then the diagram in Figure 1(c) exactly coincides with the one for $P(\lambda)$ upon identification $v_1' \to u_0, v_0 \to u_1, v_2' \to u_0', v_1 \to u_2, e \to f$ and $f_1, f_2', f_1' \to e$. The diagram of $(\lambda, \nu)$ is the completion of the one for $P(\lambda)$ (and clearly the diagram of $A(\lambda, \nu)$ is equivalent to the one of $K(\lambda)$).

2.4. Tensor product of atypical modules. Let us denote vectors of the atypical module $A(\lambda, \nu)$ by $u_0 = v_1'$ and $u_1 = v_2'$. Then

\begin{equation}
(2.14) \quad f_1 u_0 = \mu_2 u_1, \quad f_2 u_0 = \lambda_2 u_1, \quad e_1 u_1 = \gamma u_0, \quad e_2 u_1 = u_0
\end{equation}

and (the remaining action is trivial, i.e. $e_i u_0 = 0$ and $f_i u_1 = 0$). Let $v_i \otimes w_j \in A(\lambda, \nu) \otimes A(\tilde{\lambda}, \tilde{\nu})$ with $i, j \in \{0, 1\}$. Then a simple computation gives

\begin{equation}
\Delta(f_1). (u_0 \otimes \tilde{u}_0) = \mu_2 \bar{v} u_1 \otimes \tilde{v}_0 + (-1)^{p(u_0)} \mu_2 \nu^{-1} u_0 \otimes \tilde{v}_1,
\end{equation}

\begin{equation}
\Delta(f_2). (u_0 \otimes \tilde{u}_0) = \lambda_2 \tilde{v}^{-1} u_1 \otimes \tilde{v}_0 + (-1)^{p(u_0)} \lambda_2 \nu u_0 \otimes \tilde{v}_1,
\end{equation}

\begin{equation}
\Delta(f_2 f_1). (u_0 \otimes \tilde{u}_0) = (-1)^{p(u_0)} (\lambda_2 \tilde{\nu} \nu^{-1} \bar{v}^{-1} - \mu_2 \tilde{\nu} \nu^{-1} \bar{v}) u_1 \otimes \tilde{v}_1
\end{equation}

and

\begin{equation}
\Delta(e_1). (u_1 \otimes \tilde{v}_0 + \tilde{v}_1) = \gamma \bar{v}^{-1} u_0 \otimes \tilde{v}_0 + (-1)^{p(u_0)} \gamma \bar{v} u_0 \otimes \tilde{v}_1
\end{equation}

\begin{equation}
\Delta(e_2). (u_1 \otimes \tilde{v}_0) = \tilde{\nu} u_0 \otimes \tilde{v}_0 + (-1)^{p(u_0)} \tilde{\nu} u_0 \otimes \tilde{v}_1
\end{equation}

\begin{equation}
\Delta(e_1 e_2). (u_1 \otimes \tilde{v}_0) = (-1)^{p(u_0)} (\tilde{\nu} \nu^{-1} \gamma - \mu_2 \tilde{\nu} \nu^{-1} \gamma^{-1}) u_0 \otimes \tilde{v}_0.
\end{equation}

Set $\tilde{v}_0 = u_0 \otimes \tilde{v}_0$ and $\tilde{v}_1 = \Delta(f_1) \tilde{v}_0, \tilde{v}_{21} = \Delta(f_2 f_1) \tilde{v}_0$. Then

\begin{equation}
\Delta(h_1) \tilde{v}_0 = (\lambda_1 + \tilde{\lambda}_1) \tilde{v}_0 = \tilde{\lambda}_1 \tilde{v}_0, \quad \Delta(k_1) \tilde{v}_0 = (\mu_2 \bar{v}^2 + \tilde{\mu}_2 \nu^2) \tilde{v}_0 = \alpha_1 (\nu^2 \bar{v}^2 - \nu^{-2} \bar{v}^{-2}) \tilde{v}_0 =: \tilde{\mu}_2 \tilde{v}_0,
\end{equation}

where the last equality is due to (2.23). Thus

\begin{equation}
\Delta(e_1) \tilde{v}_{21} = (\gamma_2 \lambda_2 \bar{v}^{-2} + \tilde{\gamma}_2 \tilde{\lambda}_2 \nu^2) \tilde{v}_1 - (\mu_2 \nu^{-2} + \tilde{\mu}_2 \nu^2) \tilde{v}_2 = \tilde{\mu}_1 \tilde{v}_1 - \tilde{\lambda}_1 \tilde{v}_2,
\end{equation}

\begin{equation}
\Delta(e_2) \tilde{v}_{21} = (\lambda_2 + \tilde{\lambda}_2) \tilde{v}_1 - (\mu_2 \nu^2 + \tilde{\mu}_2 \bar{v}^{-2}) \tilde{v}_2 = \tilde{\lambda}_2 \tilde{v}_1 - \tilde{\mu}_2 \tilde{v}_2,
\end{equation}

which compared with (2.23) implies that $A(\lambda, \nu) \otimes A(\tilde{\lambda}, \tilde{\nu}) \cong K(\tilde{\lambda}, \tilde{\nu})$ with $\tilde{\lambda}_i = \lambda_i + \tilde{\lambda}_i$ and $\tilde{\nu} = \nu \bar{v}$.
2.5. One-dimensional module. Set

\[(2.19) \quad 1 = \bar{w}_1 \otimes \bar{w}_0 + (-1)^{p(w_1)} \nu \bar{w}_0 \otimes \bar{w}_1.\]

Let us show that \(a.1 = 0\) if \(\tilde{\lambda}_1 = \tilde{\mu}_1 = 0\) for all \(a \in \mathfrak{u}a, a \neq a^\pm.\) Notice, that it is enough to consider the action of \(\Delta(e_1)\) and \(\Delta(f_1)\) on \(1\) only. It follows straightforwardly that \(\Delta(e_2).1 = 0\). For \(f_2\) we have \(\Delta(f_2).1 = (-1)^{p(w_1)}\nu (\bar{\lambda}_2 + \bar{\lambda}_3) w_1 \otimes \bar{w}_1,\) which is equal to zero if \(\tilde{\lambda}_2 = 0.\) A simple computation for \(f_1\) gives

\[\Delta(f_1).1 = (-1)^{p(w_1)} \nu (\bar{\mu}_2 \nu^2 + \mu_2 \nu^2) w_1 \otimes \bar{w}_1 = (-1)^{p(w_1)} \nu \bar{\mu}_2 w_1 \otimes \bar{w}_1 = 0,
\]

if \(\tilde{\mu}_2 = 0.\) It remains to consider the action of \(e_1.\) We have \(\Delta(e_1).1 = (\gamma \nu^2 - \gamma_1^2 \nu^2) w_0 \otimes \bar{w}_0.\) Using \(\gamma = \mu_1 / \lambda_2, \tilde{\gamma} = \tilde{\mu}_1 / \tilde{\lambda}_2\) and \(\tilde{\lambda}_2 = -\lambda_2\) we find

\[\Delta(e_1).1 = \tilde{\nu}_2 \gamma \nu w_0 \otimes \bar{w}_0 = \tilde{\nu}_1 \nu w_0 \otimes \bar{w}_0 = 0,
\]

if \(\tilde{\gamma}_1 = 0.\) Finally, requiring \(\Delta(w^\pm).1 = 1\) implies \(\nu \bar{\nu} = 1,\) which agrees with \(\tilde{\nu} = 0.\)

2.6. \(R\)-matrix. Consider the \(\mathfrak{g}_2\)-graded vector space \(C_1^{11}.\) Let \(E_{ij} \in \text{End} C_1^{11}\) denote the usual supermatrix units. The algebra \(\mathfrak{gl}_{11}\) is a \(\mathfrak{g}_2\)-graded matrix algebra spanned by supermatrices \(E_{ij}\) with \(1 \leq i, j \leq 2\) so that the grading is given by \(\deg_2 E_{ij} = p(i) + p(j),\) where \(p(1) = 0, p(2) = 1,\) and satisfying

\[[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - (-1)^{(i+j)(k+l)} \delta_{il} E_{jk}.
\]

Moreover, for any \(X, X', Y, Y' \in \mathfrak{gl}_{11}\) we have \((X \otimes Y)(X' \otimes Y') = (-1)^{\deg_2 X \deg_2 Y} XX' \otimes YY'\) and \(\deg_2(X \otimes Y) = \deg_2 X + \deg_2 Y.\) In particular, \((E_{ij} \otimes E_{kl})(E_{pr} \otimes E_{st}) = (-1)^{(k+l)(p+r)} E_{ir} \otimes E_{kt}.\) The graded permutation operator \(P \in \text{End}(C_1^{11} \otimes C_1^{11})\) is given by \(P = \sum_{1 \leq i < j \leq 2} (-1)^{(i-1)j-1} E_{ij} \otimes E_{ji}.\)

Let \(\deg_2 w_0 = 0.\) Then \(\deg_2 w_1 = 0\) and \(\deg_2 w_0 = 1.\) Set \(V = \text{span}_C\{\bar{w}_1, \bar{w}_0\}\) and identify \(V\) with \(C_1^{11}\) in the natural way. Denote \(I = E_{11} + E_{22}.\) Then the two dimensional (atypical) representation \(\pi : a \to \text{End}(V), a \to \pi(a)\) is given by

\[\pi(e_1) = \gamma E_{21}, \quad \pi(f_1) = \mu_2 E_{12}, \quad \pi(f_2) = \mu_1 / \gamma E_{12}, \quad \pi(e_2) = E_{21},\]

\[\pi(h_1) = \mu_2 \gamma I, \quad \pi(h_2) = \mu_1 / \gamma I, \quad \pi(k_1) = I, \quad \pi(k_2) = I,
\]

where \(\mu_i = \alpha_i (\nu^2 - \nu^{-2}).\) We will refer to the set of parameters \(\{\gamma, \mu_i, \nu\}\) as the representation labels. Let \(V = \text{span}_C\{\bar{w}_1, \bar{w}_0\}\) and the representation \(\pi : a \to \text{End}^V\) with the labels \(\{\gamma, \mu_i, \nu\}\).

**Proposition 2.2.** The \(R\)-matrix \(R(\gamma, \nu; \gamma_1, \nu) \in \text{End}(V \otimes V)\) intertwining the tensor product of atypical representations \(\pi, \tilde{\pi}\) is given by

\[R(\gamma, \nu; \gamma_1, \nu) = \gamma(\gamma - \gamma_1^2 \nu^2) E_{11} \otimes E_{11} + (\gamma^2 - \gamma_1^2 \nu^2) E_{21} \otimes E_{21} + \gamma \nu \nu(\nu^2 - \nu^{-2}) E_{12} \otimes E_{21} - \gamma \nu \nu(\nu^2 - \nu^{-2}) E_{21} \otimes E_{12} + (\gamma^2 - \gamma_1^2 \nu^2) E_{22} \otimes E_{22} + (\gamma - \gamma_1^2 \nu^2) E_{22} \otimes E_{22}.
\]

**Proof.** Let \(R \in \text{End}(V \otimes V)\) be an arbitrary \(4 \times 4\) matrix with elements \(r_{ij}\) and \(1 \leq i, j \leq 4.\) We need to solve the intertwining equation

\[(\pi \otimes \tilde{\pi})(\Delta^{op}(a)) = R(\pi \otimes \tilde{\pi})(\Delta^{op}(a))\]

for all \(a \in \mathfrak{u}a.\) Since the tensor product of two atypical modules is isomorphic to the typical module we can restrict the matrix \(R\) to

\[(2.23) \quad R = r_{11} E_{11} \otimes E_{11} + r_{22} E_{11} \otimes E_{22} + r_{23} E_{12} \otimes E_{21} + r_{32} E_{21} \otimes E_{12} + r_{33} E_{22} \otimes E_{11} + r_{44} E_{22} \otimes E_{22}.
\]

Let \(a = e_1.\) Then \((2.22)\) is equivalent to the following set of linear equations:

\[\gamma \nu (r_{11} - \nu^2 r_{22}) + \gamma \nu r_{32} = 0, \quad \gamma \nu (r_{22} - r_{44}) - \gamma \nu r_{32} = 0,\]

\[\gamma (r_{21} - r_{33}) - \gamma \nu r_{32} = 0, \quad \gamma (r_{33} - \nu^2 r_{44}) + \gamma \nu r_{23} = 0,\]

having a solution \(r_{23} = \gamma \nu (\nu^2 - 1)(\nu^2 r_{22} - r_{11}), r_{33} = \nu (\nu^2 - 1) r_{23}, r_{44} = \nu^2 r_{22} - \gamma \nu (\nu^2 - 1) r_{32}.\) Next, let \(a = e_2.\) Now \((2.22)\) gives

\[(\gamma^2 \nu^3 - \gamma_1^2 \nu^3) r_{11} + (\gamma - \gamma_1^2 \nu^2) r_{22} = 0, \quad \nu (\gamma^2 - \gamma \nu^2) r_{32} + \gamma \nu (\nu^2 - 1) r_{22} = 0,\]

\[\nu (\gamma^2 - \gamma \nu^2) r_{32} + \gamma \nu (\nu^2 - 1) r_{11} = 0, \quad \nu \nu^2 (r_{22} - \nu^2 r_{11}) + \gamma (\nu^2 - 1) r_{32} - \nu \nu^2 r_{22} + \nu r_{11} = 0,
\]
the solution of which is \( r_{22} = (\gamma \nu^2 - \tilde{\gamma} \tilde{\nu}^2)(\gamma - \tilde{\gamma}^2 \tilde{\nu}^2)^{-1} r_{11} \), \( r_{32} = (\gamma \nu(\tilde{\nu}^2 - 1))(\gamma \nu^2 \tilde{\nu}^2 - \tilde{\gamma} \tilde{\nu})^{-1} r_{11} \). Finally, upon setting \( r_{11} = \gamma - \tilde{\gamma}^2 \tilde{\nu}^2 \), we obtain \( (2.21) \). It remains to check that \( (2.22) \) holds when \( \alpha = f_{i,r} \), which follows by a direct computation.

It is a lengthy but direct computation to check that \( R(\gamma, \nu; \tilde{\gamma}, \tilde{\nu}) \) satisfies the Yang-Baxter equation, namely, if we set \( R_{12} = R(\gamma, \nu; \tilde{\gamma}, \tilde{\nu}) \otimes I \), \( R_{23} = I \otimes R(\gamma, \nu; \tilde{\gamma}, \tilde{\nu}) \) and \( R_{13} = (I \otimes P)(R(\gamma, \nu; \tilde{\gamma}, \tilde{\nu}) \otimes I)(I \otimes P) \) for some (generic) set of parameters \( \{ \gamma, \nu; \tilde{\gamma}, \tilde{\nu} \} \), then \( R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \).

3. Yangian

In this section we construct a Yangian \( \mathcal{Y}(\mathfrak{a}) \) of Drinfeld-New type \( \mathcal{D}_2 \) having \( \mathfrak{h} \mathfrak{a} \) as its horizontal subalgebra, and obtain an evaluation homomorphism \( e_{V,P} : \mathcal{Y}(\mathfrak{a}) \to \mathfrak{h} \mathfrak{a} \).

3.1. Yangian \( \mathcal{Y}(\mathfrak{a}) \).

**Definition 3.1.** The algebra \( \mathcal{Y}(\mathfrak{a}) \) is the unital associative superalgebra generated by elements \( e_{i,r}, f_{i,r} \) and \( k_{i,r} \) with \( i, j \in \{1, 2\} \), \( r \geq 0 \), and satisfying

\[
[e_{i,r}, f_{j,s}] = \delta_{ij} h_{i,r+s} + (1 - \delta_{ij}) k_{i,r+s}.
\]

The remaining relations are trivial.

The \( \mathbb{Z}_2 \)- and \( \mathbb{Z} \)-grading on \( \mathcal{Y}(\mathfrak{a}) \) are induced by the ones on \( \mathfrak{h} \mathfrak{a} \), namely \( \deg_2 (a_{i,r}) = \deg_2 (a_i) \) and \( \deg (a_{i,r}) = \deg (a_i) \) for all \( a_{i,r} \in \mathcal{Y}(\mathfrak{a}) \) and \( a_i \in \mathfrak{h} \mathfrak{a} \). The algebra \( \mathcal{Y}(\mathfrak{a}) \) admits a unique coproduct respecting the \( \mathbb{Z} \)-grading.

**Proposition 3.1.** There is a unique homomorphism of \( \mathbb{C} \)-algebra \( \Delta : \mathcal{Y}(\mathfrak{a}) \to \mathcal{Y}(\mathfrak{a}) \otimes \mathcal{Y}(\mathfrak{a}) \) respecting the \( \mathbb{Z}_2 \)- and \( \mathbb{Z} \)-grading given by, for \( i \in \{1, 2\} \) and \( r \geq 0 \):

\[
\Delta (e_{i,r}) = e_{i,r} \otimes u^{+} + u^{+} \otimes e_{i,r} + \sum_{l=1}^{r} (u^{+} h_{i,r-l} \otimes e_{i,l-1} + u^{+} k_{i,r-l} \otimes u^{-2} e_{i,l-1}) ,
\]

\[
\Delta (f_{i,r}) = f_{i,r} \otimes u^{+} + u^{+} \otimes f_{i,r} + \sum_{l=1}^{r} (f_{i,r-l} \otimes u^{+} h_{i,l-1} + u^{+2} f_{i,r-l} \otimes u^{+} k_{i,l-1}) ,
\]

\[
\Delta (h_{i,r}) = h_{i,r} \otimes 1 + 1 \otimes h_{i,r} + \sum_{l=1}^{r} (h_{i,r-l} \otimes h_{i,l-1} + u^{-2} k_{i,r-l} \otimes u^{-2} h_{i,l-1}) ,
\]

\[
\Delta (k_{i,r}) = k_{i,r} \otimes u^{+2} + u^{+2} \otimes k_{i,r} + \sum_{l=1}^{r} (k_{i,r-l} \otimes u^{+2} h_{i,l-1} + u^{+2} k_{i,r-l} \otimes k_{i,l-1}) .
\]

**Proof.** Let \( \Delta : \mathcal{Y}(\mathfrak{a}) \to \mathcal{Y}(\mathfrak{a}) \otimes \mathcal{Y}(\mathfrak{a}) \) be homomorphism of \( \mathbb{C} \)-algebra. By requiring \( \Delta \) to respect the \( \mathbb{Z}_2 \)-grading we obtain the following ansatz

\[
\Delta (e_{i,r}) = e_{i,r} \otimes u^{+} + u^{+} \otimes e_{i,r} + \sum_{l=1}^{r} (e_{i,l} u^{+} h_{i,r-l} \otimes e_{i,l-1} + e_{i,2} u^{+} k_{i,r-l} \otimes u^{+2} e_{i,l-1}) ,
\]

\[
\Delta (f_{i,r}) = f_{i,r} \otimes u^{+} + u^{+} \otimes f_{i,r} + \sum_{l=1}^{r} (e_{i,3} f_{i,r-l} \otimes u^{+} h_{i,l-1} + e_{i,4} u^{+2} f_{i,r-l} \otimes u^{+} k_{i,l-1}) ,
\]

\[
\Delta (h_{i,r}) = h_{i,r} \otimes 1 + 1 \otimes h_{i,r} + \sum_{l=1}^{r} (e_{i,5} h_{i,r-l} \otimes h_{i,l-1} + e_{i,6} u^{+2} k_{i,r-l} \otimes u^{+2} h_{i,l-1}) ,
\]

\[
\Delta (k_{i,r}) = k_{i,r} \otimes u^{+2} + u^{+2} \otimes k_{i,r} + \sum_{l=1}^{r} (e_{i,7} k_{i,r-l} \otimes u^{+2} h_{i,l-1} + e_{i,8} u^{+2} k_{i,r-l} \otimes k_{i,l-1}) ,
\]

with certain \( e_{i,j} \in \mathbb{C} \). To find \( e_{i,j} \) we compute the graded commutators

\[
[\Delta_{e}(e_{i,r}), \Delta_{f}(f_{j,s})] = h_{i,r+s} \otimes 1 + 1 \otimes h_{i,r+s}
\]

\[
+ \sum_{l=1}^{r+s} (e_{i,1} h_{i,r-l} \otimes h_{i,s+l-1} + e_{i,2} u^{+2} k_{i,r-l} \otimes u^{+2} k_{i,s+l-1})
\]

\[
+ \sum_{l=1}^{s} (e_{i,3} h_{i,r+s-l} \otimes h_{i,l-1} + e_{i,4} u^{+2} k_{i,r+s-l} \otimes u^{+2} k_{i,l-1})
\]
and
\[
[\Delta_r(e_{i,r}), \Delta_s(f_{i,s})] = k_{i,r+s} \otimes u^2 + u^4 \otimes k_{i,r+s}
\]
\[+ \sum_{l=1}^r (\epsilon_{i,1} u^2 k_{i,r-l} \otimes k_{i,s+l-1} + \epsilon_{i,2} \alpha_i k_{i,r-l} \otimes u^2 k_{i,s+l-1})
\]
\[+ \sum_{l=1}^r (\epsilon_{i,3} k_{i,r+s-l} \otimes u^2 k_{i,l-1} + \epsilon_{i,4} \alpha_i u^2 k_{i,r+s-l} \otimes k_{i,l-1}).
\]

By requiring \(\Delta_r\) to be algebra homomorphism we find \(\epsilon_{i,1} = \epsilon_{i,3} = \epsilon_{i,5} = \epsilon_{i,4} = \epsilon_{i,8}\) and \(\epsilon_{i,2} = \epsilon_{i,4} = \epsilon_{i,6} = \epsilon_{i,7}\). Set \(\epsilon_i = \epsilon_{i,1}\) (c.f. (3.1)) and consider the algebra map given by
\[
\omega : e_{i,r} \mapsto e_{i,r}, \hspace{1em} f_{i,r} \mapsto \epsilon_i f_{i,r}, \hspace{1em} h_{i,r} \mapsto \epsilon_i h_{i,r}, \hspace{1em} k_{i,r} \mapsto \epsilon_i k_{i,r}.
\]
It is easy to see that (3.1) is invariant under this map, thus \(\omega\) is an automorphism of \(\mathcal{Y}(a)\). Moreover, \(\Delta(a) = (\omega \otimes \omega)((\Delta \circ \omega)(a))\) for all \(a \in \mathcal{Y}(a)\). Let us demonstrate this property when \(a = f_{i,r}:
\]
\[
(\omega^{-1} \otimes \omega^{-1})((\Delta \circ \omega)(f_{i,r})) = e_i (w^{-1} \otimes w^{-1})(f_{i,r} \otimes f_{i,r}) + \sum_{l=1}^r (\epsilon_{i,1} u^2 f_{i,r-l} \otimes u^2 k_{i,l-1}) + \sum_{l=1}^r (\epsilon_{i,3} k_{i,r+l-1} \otimes u^2 k_{i,l-1}) = \Delta(f_{i,r}),
\]
and similarly for the remaining elements.

Next, we want to write the defining relation (3.1) and coproduct (3.2) in terms of generating series (Drinfeld currents). Consider the following series in \(\mathcal{Y}(a)[[z^{-1}]]:\n\]
\[
e_i(z) = \sum_{r \geq 0} e_{i,r} z^{-r-1}, \hspace{1em} f_i(z) = \sum_{r \geq 0} f_{i,r} z^{-r-1}, \hspace{1em} h_i(z) = 1 + \sum_{r \geq 0} h_{i,r} z^{-r-1}, \hspace{1em} k_i(z) = \sum_{r \geq 0} k_{i,r} z^{-r-1}.
\]
Then the relations (3.1) are equivalent to
\[
(w - z) [e_i(z), f_j(w)] = \delta_{ij} (h_i(z) - h_i(w)) + (1 - \delta_{ij})(k_i(z) - k_i(w),
\]
and the coproduct (3.2) obtains the Drinfeld form:
\[
\Delta(e_i(z)) = e_i(z) \otimes u^2 + u^2 h_i(z) \otimes e_i(z) + u^2 k_i(z) \otimes u^2 e_i(z),
\]
\[
\Delta(f_i(z)) = u^2 \otimes f_i(z) + f_i(z) \otimes u^2 h_i(z) + u^2 f_i(z) \otimes u^2 k_i(z),
\]
\[
\Delta(h_i(z)) = h_i(z) \otimes h_i(z) + u^2 h_i(z) \otimes u^2 k_i(z),
\]
\[
\Delta(k_i(z)) = k_i(z) \otimes u^2 h_i(z) + u^2 k_i(z) \otimes k_i(z).
\]
Set \(H(z) = h_1(z)h_2(z) - k_1(z)k_2(z)\). The current \(H(z)\) has a well-defined inverse. Indeed,
\[
H(z)^{-1} = \frac{h_1(z)^{-1}h_2(z)^{-1}}{1 - k_1(z)k_2(z)h_1(z)^{-1}h_2(z)^{-1}} = \sum_{l=0}^{\infty} (k_1(z)k_2(z))^l (h_1(z)^{-1}h_2(z)^{-1})^{l+1}.
\]
Since \(h_i(z)\) are invertible the series on the right hand side are elements in \(\mathcal{Y}(a)[[z^{-1}]]\), and the coefficients of \(z^{-r}\) are finite sums of monomials \((k_{i,s})^l(k_{j,r})^l(h_{i,s})^l(h_{j,r})^l\) such that \(\sum_{i=1}^{l} i_j r_i < r\). We are now ready to define the Hopf algebra structure on \(\mathcal{Y}(a)\).

**Proposition 3.2.** The Hopf algebra structure on \(\mathcal{Y}(a)\) is given by the coproduct (3.10), counit
\[
(\omega^{-1} \otimes \omega^{-1})((\Delta \circ \omega)(f_{i,r})) = e_i (w^{-1} \otimes w^{-1})(f_{i,r} \otimes f_{i,r}) + \sum_{l=1}^r (\epsilon_{i,1} u^2 f_{i,r-l} \otimes u^2 k_{i,l-1}) + \sum_{l=1}^r (\epsilon_{i,3} k_{i,r+l-1} \otimes u^2 k_{i,l-1}) = \Delta(f_{i,r}),
\]
and the antipode \(S(u^\pm) = u^\mp\) and
\[
S(e_i(z)) = -e_i(z)h_i(z) - e_i(z)k_i(z), \hspace{1em} S(f_i(z)) = -f_i(z)h_i(z) - f_i(z)k_i(z),
\]
\[
S(h_i(z)) = h_i(z)H(z)^{-1}, \hspace{1em} S(k_i(z)) = -k_i(z)H(z)^{-1}.
\]

**Proof.** The coproduct is a homomorphism by Proposition 3.1. The counit follows from (3.1) and (3.8). Thus we only need to check that
\[
M \circ (S \otimes id) \circ \Delta(a_i(z)) = \iota \circ \varepsilon(a_i(z)), \hspace{1em} M \circ (id \otimes S) \circ \Delta(a_i(z)) = \iota \circ \varepsilon(a_i(z)),
\]
for all \(a(z) \in \mathcal{Y}(a)\); here \(id\) is the identity map, \(M : \mathcal{Y}(a) \otimes \mathcal{Y}(a) \to \mathcal{Y}(a)\) denotes the associative multiplication and \(\iota : \mathbb{C} \to \mathcal{Y}(a)\) is the inclusion map. Let us compute (3.1) explicitly for all \(a(z)\):
There is a homomorphism of algebras given by

\( h_i(z) \): \((h_i(z)h_i(z) - k_i(z)k_i(z))H(z)^{-1} = 1\) for both equalities,

\( k_i(z) \): \((-k_i(z)h_i(z) + h_i(z)k_i(z))u^\mp 2H(z)^{-1} = 0\) and \((k_i(z)h_i(z) - h_i(z)k_i(z))u^\mp 2H(z)^{-1} = 0\),

\( e_i(z) \): \((-e_i(z)h_i(z) - e_i(z)k_i(z)) + h_i(z)e_i(z) - k_i(z)(e_i(z)h_i(z) - e_i(z)k_i(z))u^\mp H(z)^{-1} = 0\) and

\( e_i(z)H(z) - h_i(z)(e_i(z)h_i(z) - e_i(z)k_i(z)) - k_i(z)(e_i(z)h_i(z) - e_i(z)k_i(z))u^\mp H(z)^{-1}\)

\( = (e_i(z)H(z) - e_i(z)h_i(z)h_i(z) + e_i(z)k_i(z)k_i(z))u^\mp H(z)^{-1} = 0\),

\( f_i(z) \): \((f_i(z)H(z) - (f_i(z)h_i(z) - f_i(z)k_i(z)))h_i(z) - (f_i(z)h_i(z) - f_i(z)k_i(z))k_i(z)\) \(u^\mp H(z)^{-1}\)

\( = (f_i(z)H(z) - f_i(z)h_i(z)h_i(z) - k_i(z)k_i(z))u^\mp H(z)^{-1} = 0\)

and \((- (f_i(z)h_i(z) - f_i(z)k_i(z)) + f_i(z)h_i(z) - f_i(z)k_i(z))u^\mp H(z)^{-1} = 0\). \( \square \)

In the previous section we noted that \( \mathfrak{u} \) is a two-parameter family of Hopf algebras. The same is true for \( \mathcal{Y}(a) \). Indeed, by requiring \( \Delta(k_i(z)) = \Delta^{op}(k_i(z)) \) we find

\( k_i(z) \otimes (u^\mp 2h_i(z) - u^\mp 2h_i(z)) = (u^\mp 2h_i(z) - u^\mp 2h_i(z)) \otimes k_i(z) \),

which, together with (2.3), implies that

\( k_i(z) = \alpha_i (u^2h_i(z) - u^{-2}h_i(z))z^{-\frac{1}{2}} \).

Moreover, it follows that \( \Delta(h_i(z)) = \Delta^{op}(h_i(z)) \) is also true. Finally, the coefficients of \( z^{-1} \) in (3.15) reproduce (2.3) as required.

3.2. Evaluation representation. Recall that the algebra \( \mathfrak{u} \) is isomorphic to the subalgebra of \( \mathcal{Y}(a) \) generated by the elements \( e_{i,0}, f_{i,0}, h_{i,0}, k_{i,0} \).

**Proposition 3.3.** There is a homomorphism of algebras given by

\( ev_\rho : \mathcal{Y}(a) \rightarrow \mathfrak{u} \), \( a_{\rho, r} \mapsto \rho^a_i \) where \( \rho = \frac{u^2h_1 - u^{-2}h_2}{u^2 - u^{-2}} \).

for all \( a, r \in \mathfrak{u} \) and \( a_{\rho, r} \in \mathcal{Y}(a) \) with \( i = 1, 2 \), \( r \geq 0 \), \( a \in \{ e, f, h, k \} \).

**Proof.** It follows from (3.1) and (2.1), (2.3) that \( ev_\rho(a_{\rho, r}) = \rho^a_i \) for some \( \rho \in \mathbb{C}[h_i, k_i, u^\pm] \). Taking the coefficients of \( z^{-r-2} \) at both sides of (3.16) we find

\( k_{i, r+1} = \alpha_i (u^2h_{1, r} - u^{-2}h_{2, r}) \)

for \( r \geq 0 \). Since \( k_i = \alpha_i (u^2 - u^{-2}) \), the evaluation map applied to (3.18) gives

\( \rho^{r+1} \alpha_i(u^2 - u^{-2}) = \rho^a_i \alpha_i (u^2h_1 - u^{-2}h_2) \),

which implies (3.17) as required. \( \square \)

Note that evaluation homomorphism for generating series can be written as

\( ev_\rho : a_i(z) \mapsto i_z \frac{1}{z - \rho} a_i \),

where \( i_z \) denotes the expansion in the domain \( |z| \rightarrow \infty \).

4. The role in \( AdS_3/CFT_2 \)

An alternating double-row spin chain respecting the \( \mathfrak{u} \) symmetry was constructed in [BSS]. Each spin chain site carries a tensor product of two three-dimensional vector spaces called odd and even sites, each spanned by two types of excitations called magnons and the vacuum state (see Figure 2). In momentum space, the spin chain magnons are identified with the weight vectors of the atypical module and are conveniently called massive modes. The top row of the spin chain is called the left sector and the bottom row called the right sector. These sectors are equivalent to each other and only differ by the choice of the base (we will choose \( \deg_2 v_0 = 0 \) for the left sector and \( \deg_2 v_0 = 1 \) for the right one).

The excitations living in the sites \( L_i \) (resp. \( R_i \)) and in the sites \( L_i' \) (resp. \( R_i' \)) differ only by their masses, namely odd/even magnons have masses inversely proportional to the radius of the \( S^3_{odd/\text{even}} \) sphere of the underlying \( AdS_3 \times S^3_{\text{odd}} \times S^3_{\text{even}} \times S^3 \) spacetime. Thus there are ten \( R \)-matrices (denoted by \( R_{ab} \) with \( a, b \in \{ L, L', R, R' \} \)) associated with this model; however, only two of them (i.e. \( R_{1L} \) and \( R_{L,R} \)) are not explicitly equivalent (for complete details of the spin chain we refer the reader to [BSS], Sections 2 and 3).
Appendix B: most symmetric frame) can be identified with our notation by some of the physical properties. Before we begin, let us remark that generators of the atypical module \( \pi \rightarrow - \pi \)

\[ \nu \equiv \frac{\text{magnon}}{2} \]

and similarly as before we assume that \( \bar{h} \) and thus is non-dynamical, i.e. independent of \( x \). A direct computation gives

\[ R \equiv \frac{\pi}{2} \]

\[ \nu \equiv \frac{\phi}{2} \]

\[ \phi \equiv \pi \]

\[ \psi \equiv \phi \]

\[ \bar{\psi} \equiv \psi \]

\[ \varphi \equiv \{ \phi, \psi \} \]

Section 4.1

Representations and the R-matrices. Let us construct the left AdS\(_5\)/CFT\(_2\) module. Consider the atypical module \( A(\lambda, \nu) = \text{span}_{\mathbb{C}}\{ \psi_\nu \} \) choose \( \text{deg} \left. \sum \right| v_0 = 0 \) (this corresponds to the same setup as in Section 2.6). We introduce the notation \( |\phi_\nu\rangle = d_p w_1, |\psi_\nu\rangle = w_0 \) with some \( d_p \in \mathbb{C}^+ \), where \( p \) denotes the momentum of the magnon. Thus we have \( \text{deg} |\phi_\nu\rangle = 0 \) and \( \text{deg} |\psi_\nu\rangle = 1 \). Moreover, set \( a_p = \gamma d_p, b_p = \mu_2/d_p \) and \( c_p = \lambda_2/d_p \). Then it follows from (2.14) that

\[ e \equiv |\phi_\nu\rangle \equiv a_p |\psi_\nu\rangle, \quad f \equiv |\psi_\nu\rangle \equiv b_p |\phi_\nu\rangle, \quad h \equiv |\phi_\nu\rangle \equiv c_p |\phi_\nu\rangle, \quad k \equiv |\psi_\nu\rangle \equiv d_p |\psi_\nu\rangle. \]

Recall that \( \mu_1 = \alpha_1 (\nu^2 - \nu^{-2}) \). By requiring the module to be unitary we find \( \mu_1 = \alpha_1, a_p = b_p, c_p = d_p \) and \( \nu^* = \nu^{-1} \). Thus, without the loss of generality, we can choose the parametrization \( \alpha_1 = -\alpha_2 = -h \) and \( \nu^4 = \frac{x_p^+}{x_p^-} \) such that \( h^* = h \) and \( (x_p^+)^* = x_p^- \). This gives

\[ (x_p^+)^* = x_p^- \]

\[ \mu_1 = \frac{\text{h}^2 \left( \frac{x_p^+}{x_p^-} - 1 \right)}{a_p c_p}, \quad \mu_2 = \frac{\text{h}^2 \left( \frac{x_p^+}{x_p^-} - 1 \right)}{b_p d_p}. \]

Set \( \eta_p^2 = i(x_p^+ - x_p^-) \). Then the parametrization

\[ a_p = \sqrt{\text{h}} \eta_p \nu_p, \quad b_p = \sqrt{\text{h}} \eta_p^* \nu_p, \quad c_p = -\sqrt{\text{h}} \frac{i \eta_p \nu_p}{x_p^+}, \quad d_p = \sqrt{\text{h}} \frac{i \eta_p \nu_p}{x_p^+} \]

satisfies the required constraints, since \( \eta_p^* = \eta_p \). Here we have denoted \( \nu_p = \nu \) for homogeneity of the notation. A direct computation gives

\[ \eta = (a_p/d_p) = -i \nu_p^2 x_p^+, \quad \lambda_1 = (a_p b_p) = i h (x_p^+ - x_p^-), \quad \lambda_2 = (c_p d_p) = i h \left( \frac{1}{x_p^+} - \frac{1}{x_p^-} \right). \]

The parameters \( x_p^\pm \) are related to the momentum of the spin chain excitations as \( x_p^+/x_p^- = e^{ip} \) and are called dynamical variables, while the parameter \( h \) plays the role of the coupling constant of the model and thus is non-dynamical, i.e. independent of \( x_p^\pm \). The difference \( \delta = \lambda_1 - \lambda_2 \) is also required to be non-dynamical (this means that we can assume \( \lambda_1 = \lambda_i(\delta, p) \) with \( \delta \) and \( p \) being independent), and is conveniently called the mass of the magnon. This requirement gives the dispersion relation (see [BSS], Section 4.1)

\[ x_p^+ + \frac{1}{x_p^+} - x_p^- - \frac{1}{x_p^-} = i \frac{\delta}{h}. \]

Set \( V_p = \text{span}_{\mathbb{C}}\{ |\phi_\nu\rangle, |\psi_\nu\rangle \} \) and let \( V_\nu \) be defined in the same way. Here \( r \) denotes momentum of another magnon, and similarly as before we assume that \( \lambda_1 = \bar{\lambda}_i(\delta, r) \). Then, upon identifying \( \pi_r \in \text{End} V_p \) with \( \pi \) and \( \pi_r \in \text{End} V_\nu \) with \( \pi \) in the natural way, and using (1.24), (1.43) and Proposition (2.21) i.e. substituting \( \gamma \rightarrow -i \nu_p^2 x_p^+ \), \( \gamma \rightarrow -i \nu_r^2 x_r^- \) and \( \nu \rightarrow \nu_p \), \( \nu \rightarrow \nu_r \) into (2.24), rescaling \( E_{21} \otimes E_{12} \rightarrow d_p / d_p, E_{21} \otimes E_{12} \),
and $E_{12} \otimes E_{21} \to d_p/d_r$ and $E_{12} \otimes E_{21}$ and multiplying by an overall factor $-i$, we obtain the left-left $R$-matrix for massive modes

$$R_{\text{LL}}(p,r) = \nu_r^2(x_r^- - x_r^+) E_{11} \otimes E_{11} + (x_r^+ - x_r^-) E_{11} \otimes E_{22} + i\nu_r \eta_r \nu_r^2 E_{12} \otimes E_{21}$$

(4.6)

$$- i\nu_r \eta_r \nu_r^2 E_{21} \otimes E_{12} + \nu_r^2 \nu_r^2 (x_r^- - x_r^+) E_{22} \otimes E_{11} + \nu_r^2 (x_r^- - x_r^+) E_{22} \otimes E_{22}. $$

We will now construct the right $AdS_3/CFT_2$ module. Let $\text{deg}_{2} w_1 = 1$ and let its weights be $\tilde{\lambda}_1$ and $\mu_i$ (i.e. we want left and right modules to have the same weights $\mu_i$, but we allow to have different $\lambda_i$).

Introduce the notation $|\psi_p\rangle = b_p w_1, |\phi_r\rangle = w_0$. Then, by requiring (4.12) to hold, namely $[c_1, f_2] |\phi_r\rangle = a_p c_p |\phi_r\rangle$ and $[c_2, f_1] |\phi_r\rangle = b_p d_p |\phi_r\rangle$ for $\tilde{\gamma} \in \{\phi, \psi\}$, we find

$$e_1 |\psi_p\rangle = c_p |\phi_r\rangle, \quad f_1 |\phi_r\rangle = d_p |\psi_p\rangle, \quad f_2 |\phi_r\rangle = a_p |\psi_p\rangle, \quad e_2 |\psi_p\rangle = b_p |\phi_r\rangle$$

which combined with (2.14) gives

$$\tilde{\gamma} = (c_p/b_p) = -\frac{i \nu_r^2}{x_p^-}, \quad \tilde{\lambda}_1 = (c_p d_p) = i h \left(\frac{1}{x_p^-} - \frac{1}{x_r^+}\right), \quad \tilde{\lambda}_2 = (a_p d_p) = i h(x_r^- - x_r^+).$$

This is equivalent to saying that $\tilde{\lambda}_1 = \lambda_1$ and $\tilde{\lambda}_2 = \lambda_1$ and also $\tilde{\gamma} = \lambda_1 - \lambda_3 \equiv -\delta$. The last relation should be read as “the absolute value of the mass of the right magnon with momentum $p$ is equal to the negative absolute value of the mass of the left magnon with momentum $p$", which means that right magnons are antiparticles of the left ones; this can also be observed from the antipode map (see [BSSST], Appendix B for more details).

Next, set $\tilde{V}_r = \text{span}_c \{(|\phi_r\rangle, |\psi_r\rangle\}$. We want to identify $\tilde{\pi}_r = \text{End} \tilde{V}_r$ with $\pi$. The identification requires to transpose $\pi$, since now $\text{deg}_{2} w_1 = 1$ and $\text{deg}_{2} w_0 = 0$. Thus using (4.2), (4.3) and (4.5) together with Proposition 2.25 (i.e. substituting $\gamma \to -i \nu_r^2 x_p^+, \gamma \to -i \nu_r^2/x_r^+$ and $\nu \to \nu_p$, $\nu \to \nu_r$ into (2.21) and rescaling (and transposing the second tensor space with $\tau : E_{ij} \to E_{\tau(i)\tau(j)}$ with $\tau(1) = 2$, $\tau(2) = 1$ $E_{11} \otimes E_{22} \to E_{11} \otimes E_{11}, E_{11} \otimes E_{12} \to E_{11} \otimes E_{22}$, $E_{21} \otimes E_{22} \to E_{21} \otimes E_{12}$) and multiplying by an overall factor $-i$, we obtain the left-right $R$-matrix for massive modes

$$R_{\text{LR}}(p,r) = \left(x_p^+ - \frac{1}{x_r^+}\right) E_{11} \otimes E_{11} + \nu_r^2 \left(x_p^- - \frac{1}{x_r^+}\right) E_{11} \otimes E_{22} + \frac{\eta_r \eta_r \nu_r^2}{x_r^+} E_{12} \otimes E_{12}$$

(4.9)

$$+ \frac{\eta_r \eta_r \nu_r^2}{x_r^+} E_{21} \otimes E_{21} + \nu_r^2 \left(x_p^+ - \frac{1}{x_r^+}\right) E_{22} \otimes E_{11} + \nu_r^2 \nu_r^2 \left(x_p^- - \frac{1}{x_r^+}\right) E_{22} \otimes E_{22}. $$

Finally, we remark that both $R$-matrices in this parametrization do not explicitly depend on the parameters $\delta$ and $\tilde{\delta}$ or $\tilde{\gamma}$. (This will no longer be true in the $q$-deformed model presented in Section 7).

4.2. Yang-Baxter equation and unitarity. There is a number of Yang-Baxter equations that $R$-matrices $R_{\text{LL}}(p,r)$ and $R_{\text{LR}}(p,r)$ satisfy. Namely, set $R_{(a)12} = R_a(p,r) \otimes I$, $R_{(b)23} = I \otimes R_b(q,r)$ and $R_{(b)c} = (I \otimes P)(R_b(p,r) \otimes I)(I \otimes P)$ with $a, b, c \in \{l, r\}$. Then $R_{(a)12} R_{(b)23} R_{(c)23} = R_{(c)23} R_{(b)23} R_{(a)12}$ if $a = b = c = l$ or two indices are $l$ and one is $r$. This follows from the fact that $R_{\text{LL}}(p,r)$ and $R_{\text{LR}}(p,r)$ are specializations of $R(\nu, \lambda; \nu, \lambda)$. In order to demonstrate the unitarity property, let us set

$$R_{\text{LL}}^0(p,r) = \frac{1}{(x_p^+ - x_r^+) \nu_r^2} R_{\text{LL}}(p,r), \quad R_{\text{LR}}^0(p,r) = \left((x_p^+ x_r^- - 1) \left(1 - \frac{1}{x_p^+ x_r^-}\right) R_{\text{LR}}(p,r) \right)^{-\frac{1}{2}}$$

(4.10)

Then a direct computation gives

$$R_{\text{LL}}^0(p,r) P R_{\text{LL}}^0(r,p) P = I, \quad R_{\text{LR}}^0(p,r) P R_{\text{LR}}^0(r,p) P = I.$$

The $R$-matrices obtained above are written in the so-called “most-symmetric frame of the $u$-deformation", see ([BSSST], Appendix B) for details on different choices of the frame of $u$-deformation.

4.3. Singlet and bound states. Consider the left-left sector. We introduce the notation $|\phi_p \chi_r\rangle = |\phi_p\rangle \otimes |\chi_r\rangle$ with $\phi, \chi \in \{\phi, \psi\}$. The states $|\phi_p \chi_r\rangle$ span the typical module of $\Omega \mathfrak{m}$ with $|\psi_p \psi_r\rangle$ and $|\phi_p \phi_r\rangle$ being the highest and the lowest weight vectors, respectively. Due to the choice of the basis there are two module shortening conditions, namely

$$\Delta(e_1 e_2) |\phi_p \phi_r\rangle = \left(a_p d_r \nu_r^2 - a_r d_p \nu_p \nu_r\right) |\psi_p \psi_r\rangle = i h \left(\frac{x_p^- - x_r^+}{x_p^+ x_r^-}\right) \eta_r \eta_r \nu_r^2 |\psi_p \psi_r\rangle = 0 \quad \text{if} \quad x_r^- = x_r^+.$$
Let $\text{AdS}$ the since

\begin{equation}
\Delta(f_1 f_2) |\psi_p\rangle = (b_p c_p \nu_p \nu_r - b_p c_r \nu_p \nu_r) |\phi_p \phi_r\rangle = i\hbar \frac{\delta_p + \delta_r}{x_p x_r} \eta_p \eta_r \nu_p^2 |\phi_p \phi_r\rangle = 0 \quad \text{if} \quad x_p^+ = x_r^-.
\end{equation}

Notice that $x_p^+ = x_r^-$ is the zero and $x_p^- = x_r^+$ is the pole of $R^\nu_{\text{LR}}(p, r)$. Thus the states $|\phi_s\rangle := |\phi_p \phi_r\rangle$ and $|\psi_s\rangle := \Delta(e_2) |\phi_p \phi_r\rangle$ when $x_p^- = x_r^+$ are the bound states of the spin chain. They are parametrized

\begin{equation}
\text{Similarly, the states } |\varphi_p \varphi_r\rangle \text{ span the typical module in the left-right sector with } |\phi_p \tilde{\psi}_r\rangle \text{ and } |\psi_p \tilde{\psi}_r\rangle \text{ being the highest and the lowest weight vectors, respectively. We find }
\end{equation}

\begin{equation}
\Delta(f_1 f_2) |\psi_p \tilde{\psi}_r\rangle = \left( d_p c_p \nu_p \nu_r - c_p d_r \nu_p \nu_r \right) |\phi_p \tilde{\psi}_r\rangle = h \left( 1 - \frac{1}{x_p x_r} \right) \eta_p \eta_r \nu_p^2 |\phi_p \tilde{\psi}_r\rangle = 0 \quad \text{if} \quad x_p^- = x_r^+.
\end{equation}

Since $x_p^\pm = x_r^\pm$ are zeros of $R^\nu_{\text{LR}}(p, r)$ we conclude that there are no bound states of mixed type.

In Section 2.5 we constructed a trivial module of $\mathfrak{Y}(a)$. The vector 1 defined in (2.19) plays the role of the $\text{AdS}_5/CFT_2$ spin chain singlet state. In particular, there are two singlet states, the left-right singlet and the right-left singlet:

\begin{equation}
1_{\text{LR}} = |\phi_p \tilde{\psi}_r\rangle + d_p \nu_p \nu_r |\psi_p \tilde{\psi}_r\rangle, \quad 1_{\text{RL}} = |\tilde{\phi}_p \phi_r\rangle - d_p \nu_p \nu_r |\bar{\psi}_p \phi_r\rangle,
\end{equation}

which are annihilated by all generators of $\mathfrak{Y}(a)$ (except $u^\pm$) provided $x_p^\pm = 1/x_p^\mp$. (We refer the reader to Section 4.3 in [BSS] for more details on singlet and bound states).

4.4. Yangian symmetry and a link to $\text{AdS}_5/CFT_4$. We want to obtain evaluation modules for $\mathfrak{Y}(a)$. Let $\varphi \in \{\phi, \psi\}$. Combine (3.17) with (4.1) and (4.7). Then it follows that

\begin{equation}
\begin{align*}
a_{i, s} |\varphi_p\rangle &= (\rho_p)^s a_i |\varphi_p\rangle \quad \text{with} \quad \rho_p = -i\hbar \left( \frac{x_p^+ - 1}{x_p^-} \right), \\
a_{i, s} |\tilde{\varphi}_p\rangle &= \tilde{\rho}_p = -i\hbar \left( \frac{1}{x_p^+} - \frac{1}{x_p^-} \right)
\end{align*}
\end{equation}

for all $a_{i, s} \in \mathfrak{Y}(a)$ and $a_i \in \mathfrak{Y}(a)$.

Let us now make a link to the $\text{AdS}_5/CFT_4$ Yangian presented in [Pe2]. For this purpose we need to use the coproduct $\Delta_{\epsilon}$ (see Appendix A) and choose $\epsilon = \pm 1$ (we will denote $\Delta_{\epsilon}$ by $\Delta_{\pm}$ for this choice of $\epsilon$). Observe that evaluation map (3.17) is now given by $\rho = \frac{u^2 h_1 + u^{-2} h_2}{u^2 - u^{-2}}$, which follows from (3.6). Introduce elements

\begin{equation}
\begin{align*}
J(e_i) &= e_{i, 1} = \frac{1}{2} (h_{i, 0} e_{i, 0} - k_{i, 0} e_{i, 0}), \\
J(h_i) &= h_{i, 1} = \frac{1}{2} (h_{i, 0} h_{i, 0} - k_{i, 0} k_{i, 0}), \\
J(f_i) &= f_{i, 1} = \frac{1}{2} (f_{i, 0} h_{i, 0} - f_{i, 0} k_{i, 0}), \\
J(k_i) &= k_{i, 1} = \frac{1}{2} (k_{i, 0} h_{i, 0} - h_{i, 0} k_{i, 0}).
\end{align*}
\end{equation}

Then

\begin{equation}
\begin{align*}
J(a_i) |\varphi_p\rangle &= g_p a_i |\varphi_p\rangle, \quad J(a_i) |\tilde{\varphi}_p\rangle = -g_p a_i |\tilde{\varphi}_p\rangle \quad \text{with} \quad g_p = -\frac{i\hbar}{2} \left( \frac{x_p^+ + 1}{x_p^+} + \frac{1}{x_p^-} \right),
\end{align*}
\end{equation}
and
\[ \Delta_\pm(J(e_i)) = J(e_i) \otimes u^\pm + u^\pm \otimes J(e_i) \]
\[ = \pm \frac{1}{2} (u^\pm h_{i,0} \otimes e_{i,0} - e_{i,0} \otimes u^\pm h_{i,0} - u^\mp k_{i,0} \otimes u^\pm e_{i,0} + u^\pm e_{i,0} \otimes u^\mp k_{i,0}), \]
\[ \Delta_\pm(J(f_i)) = J(f_i) \otimes u^\pm + u^\pm \otimes J(f_i) \]
(4.20)  
\[ = \pm \frac{1}{2} (f_{i,0} \otimes u^\pm h_{i,0} - u^\mp h_{i,0} \otimes f_{i,0} - u^\pm f_{i,0} \otimes u^\mp k_{i,0} + u^\pm k_{i,0} \otimes u^\mp f_{i,0}), \]
\[ \Delta_\pm(J(h_i)) = J(h_i) \otimes 1 + 1 \otimes J(h_i) + \frac{1}{2} (u^\pm k_{2,0} \otimes u^\mp k_{1,0} - u^\mp k_{1,0} \otimes u^\pm k_{2,0}), \]
\[ \Delta_\pm(J(k_i)) = J(k_i) \otimes u^\pm \mp u^\pm \otimes J(k_i) \pm \frac{1}{2} (u^\pm (h_{1,0} + b_{2,0}) \otimes k_{i,0} - k_{i,0} \otimes u^\pm (h_{1,0} + b_{2,0})). \]

Upon identification
\[ e_{1,0} \rightarrow \mathfrak{h}_1, \quad e_{2,0} \rightarrow \mathfrak{h}_2, \quad f_{1,0} \rightarrow \mathfrak{h}_3, \quad f_{2,0} \rightarrow \mathfrak{h}_4, \quad u^\pm \rightarrow U, \quad \mathfrak{h}_i \rightarrow \mathfrak{h}, \]
\[ J(e_l) \rightarrow \mathfrak{j}_l^1, \quad J(e_r) \rightarrow \mathfrak{j}_r^1, \quad J(f_l) \rightarrow \mathfrak{j}_l^1, \quad J(f_r) \rightarrow \mathfrak{j}_r^1, \]
\[ J(h_l) + J(h_r) \rightarrow \widehat{\mathfrak{h}}_l, \quad J(k_l) \rightarrow \widehat{\mathfrak{k}}_l, \quad J(k_r) \rightarrow \widehat{\mathfrak{k}}_r, \]
the coproducts obtained above coincide with the ones presented in (12) (after setting \( R_i^2 = R_i^3 = 0 \), \( 2L_1^2 = L_3^2 = 0 \) and \( 2L_2^1 = L_3^1 = 0 \), which place restriction upon the subalgebra \( \mathfrak{sl}_{2|2} \otimes \mathbb{C} u^\pm \rightarrow (\mathfrak{sl}_{1|1})^2 \otimes \mathbb{C} u^\pm \) up to the overall sign of the tail, which accounts to the opposite sign in the evaluation parameter \( \vartheta_p \) with respect to the one given in (12), (3.16 and 3.17), and a different choice of the \( u \)-deformation frame.

5. The Deformed Superalgebra \((\mathfrak{sl}_{1|1})^2 \otimes \mathbb{C} U^\pm\) and Its Highest-Weight Modules

In this section we present a quantum deformation of the superalgebra considered in Section 2. This section follows a similar strategy to the one presented in Section 2. Namely, we start by considering a \( q \)-deformation of the universal enveloping superalgebra \( U_q((\mathfrak{sl}_{1|1})^2 \otimes \mathbb{C}^2) \), and then we obtain \( U_q a = U_q((\mathfrak{sl}_{1|1})^2 \otimes \mathbb{C} U^\pm) \) as the quotient of an extension of the this superalgebra. (Here \( U \) is considered as the \( q \)-deformed analogue of \( a \) in \( \mathfrak{U} = U((\mathfrak{sl}_{1|1})^2 \otimes \mathbb{C} U^\pm) \).)

5.1. Algebra. Let \( q \in \mathbb{C}^\times \) be generic (not a root of unity). Set
\[ [x]_q = \frac{x - x^{-1}}{q - q^{-1}}. \]

**Definition 5.1.** The centrally extended superalgebra \( U_q((\mathfrak{sl}_{1|1})^2 \otimes \mathbb{C}^2) \) is the unital associative Lie superalgebra generated by elements \( E_i, F_i \) and central elements \( K^\pm, L^\pm \) with \( i, j \in \{1, 2\} \) satisfying
\[ K^+ L^+ = L^+ K^+ = 1 \quad \text{and} \quad K^- L^- = L^- K^- = 1 \quad \text{and} \]
\[ [E_i, F_j] = \delta_{ij} \frac{K_{-i}^+ K_{-j}^- - K_{-j}^+ K_{-i}^-}{q - q^{-1}} + \alpha_i (1 - \delta_{ij}) \frac{L_{-i}^+ - L_{-j}^-}{q - q^{-1}}. \]

The remaining relations are trivial. The \( \mathbb{Z}_2 \)-grading is given by \( \deg_2(E_i) = \deg_2(F_i) = 1 \quad \text{and} \quad \deg_2(K^\pm) = \deg_2(L^\pm) = 0 \).

Following similar steps as we did in Section 2, we want to enlarge the algebra by central elements \( U^\pm \) (satisfying \( U^\pm U^\mp = 1 \)). We will denote this extended algebra by \( U_q a_0 = U_q((\mathfrak{sl}_{1|1})^2 \otimes \mathbb{C}^2 \otimes \mathbb{C} U^\pm) \). The next observation follows straightforwardly.

**Proposition 5.1.** The vector space basis of \( U_q a_0 \) is given in terms of monomials
\[ (E_2)^{r_2}(E_1)^{r_1}(K_1^+)^{s_1}(K_2^+)^{s_2}(L_1^+)^{l_1}(L_2^+)^{l_2}(U^+)^{s_3}(E_1)^{s_1}(E_2)^{s_2} \]

with \( r_i, s_i \in \{0, 1\} \) and \( l_i \in \mathbb{Z} \).

The monomials \( (5.2) \) give a Poincaré–Birkhoff–Witt type basis of \( U_q a_0 \), and \( U_q a_0 \cong U_{q_0} U_{q_0}^0 \) as vector spaces, where \( U_{q_0} \) and \( U_{q_0}^0 \) are the nilpotent subalgebras generated by elements \( F_i \) and \( E_i \) with \( i = 1, 2 \), respectively, and \( U_{q_0}^0 \) is generated by the central elements. Next we give a remark which follows from analogous Remark 2.1 for \( U a_0 \).
Remark 5.1. The algebra \( U_q a_0 \) admits a \( Z \)-grading given by
\[
\deg(K_i^\pm) = \deg(U_i^\pm) = 0, \quad \deg(E_i) = \deg(F_i) = \pm 1, \quad \deg(\alpha_i) = \pm 2.
\]

Let \( I_{q0} \) be the ideal of \( U_q a_0 \) generated by the relations
\[
L_i^+ = K_i^+ K_2^+ U_i^\pm, \quad L_i^- = K_i^- K_2^- U_i^\mp.
\]
Set \( U_q a = U_q a_0 / I_{q0} \). Then one can define a Hopf algebra structure on \( U_q a \) by introducing the coproduct
\[
\Delta(E_i) = E_i \otimes U_i^\pm K_i^\pm + U_i^\pm K_i^\mp \otimes E_i, \quad \Delta(F_i) = F_i \otimes U_i^\pm K_i^\mp + U_i^\pm K_i^\mp \otimes F_i, \quad \Delta(C) = C \otimes C,
\]and the counit and antipode
\[
\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad S(E_i) = -E_i, \quad S(F_i) = -F_i, \quad \varepsilon(C) = 1, \quad S(C) = C^{-1}
\]for \( C \in \{ K_i^\pm, L_i^\pm, U_i^\pm \} \).

The coproduct (5.5) is a morphism of \( \mathbb{C} \)-algebra \( \Delta : U_q a \to U_q a \otimes U_q a \). Indeed,
\[
[\Delta(E_i), \Delta(F_i)] = \frac{\alpha_i (L_i^+ - L_i^-) \otimes U_i^\pm K_i^\pm + U_i^\pm K_i^\mp \otimes \alpha_i (L_i^+ - L_i^-)}{q - q^{-1}},
\]and
\[
\Delta([E_i, F_i]) = \frac{\alpha_i (L_i^+ \otimes L_i^+ - L_i^- \otimes L_i^-)}{q - q^{-1}},
\]which agree with each other provided (5.3) holds.

Besides the Chevalley anti-automorphism \( E_i \mapsto E_i, F_i \mapsto F_i, K_i^\pm \mapsto K_i^\mp, L_i^\pm \mapsto L_i^- \), \( U_i^\pm \to U_i^\mp \), there are a number of involutive automorphisms of \( U_q a \) given by
\[
E_i \mapsto F_i, \quad F_i \mapsto E_i, \quad K_i^\pm \mapsto K_i^\mp, \quad L_i^\pm \mapsto L_i^\mp, \quad U_i^\pm \to U_i^- \mp, \quad \alpha_i \mapsto \alpha_i,
\](5.7)
\[
E_i \mapsto E_i, \quad F_i \mapsto F_i, \quad K_i^\pm \mapsto K_i^{\pm}, \quad L_i^\pm \mapsto L_i^\pm, \quad U_i^\pm \to U_i^\pm, \quad \alpha_i \mapsto \alpha_i,
\]
which form the Klein-four outer-automorphism group \( \text{Aut}(a) \) of \( U_q a \). Note that \( \text{Aut}(a) \) is also the group of the Hopf algebra outer-automorphisms of \( U_q a \).

In a similar spirit as we did in Section 2, it will be convenient to consider the algebra \( U_q a_0 \) extended by the elements \( K_i^\pm, L_i^\pm, U_i^\pm \) satisfying \( K_i^\pm K_i^\pm = 1 \) and
\[
K_i^\pm E_i K_i^\pm = q^1 E_i, \quad K_i^\pm F_i K_i^\pm = q^{-1} F_i
\]
and the remaining relations being trivial. The grading is \( \deg_2(K_i^\pm) = \deg(K_i^\pm) = 0 \). The Hopf structure is the same as for \( K_i^\pm \).

5.2. Typical module. The typical module \( K_q(\lambda, \nu) \) is the four-dimensional highest-weight Kac module of \( U_q a \) defined as follows: let \( v_0 \in K_q(\lambda, \nu) \) be the highest-weight vector such that
\[
K_i^\pm v_0 = q^{\pm \lambda_i / 2} v_0, \quad L_i^\pm v_0 = q^{\pm \nu_i / 2} v_0, \quad U_i^\pm v_0 = q^{\pm \nu_i} v_0, \quad E_i v_0 = 0, \quad K_0^\pm v_0 = 0
\]
for \( i = 1, 2 \), where \( \lambda_i, \nu \in \mathbb{C}^{\times} \) are generic and
\[
q^{\lambda_1} = q^{(\lambda_1 + \lambda_2)/2} q^{\nu_1}, \quad q^{\nu_2} = q^{(\lambda_1 + \lambda_2)/2} q^{-\nu_2}
\]
due to (5.3). Set \( v_1 = F_1 v_0 \) and \( v_2 = F_2 F_1 v_0 \). Thus \( K_q(\lambda, \nu) \cong \text{span}_{C} \{ v_0, v_1, v_2, v_{21} \} \) as a vector space. Since \( K_0^\pm v_i = q^{-1} v_i \) and \( K_i^\pm v_{21} = q^{-2} v_{21} \) we obtain the following weight space decomposition
\[
K_q(\lambda, \nu) = K_q(0, \lambda, \nu) \oplus K_{q_{-1}}(\lambda, \nu) \oplus K_{q_{-2}}(\lambda, \nu),
\]
satisfying \( v_0 \in K_{q_{-0}}(\lambda, \nu), \quad v_i \in K_{q_{-1}}(\lambda, \nu) \) and \( v_{21} \in K_{q_{-2}}(\lambda, \nu) \). We have
\[
E_1 v_{21} = \alpha_1 [\mu_1]_{q} v_1 - [\lambda_1]_{q} v_2, \quad E_1 v_0 = [\lambda_1]_{q} v_0, \quad E_1 v_1 = \alpha_1 [\mu_1]_{q} v_0, \quad E_1 v_2 = \alpha_1 [\mu_1]_{q} v_0,
\]
\[
E_2 v_{21} = [\lambda_2]_{q} v_{21} - \alpha_2 [\mu_2]_{q} v_2, \quad E_2 v_0 = \alpha_2 [\mu_2]_{q} v_{21}, \quad E_2 v_1 = \alpha_2 [\mu_2]_{q} v_0, \quad E_2 v_2 = [\lambda_2]_{q} v_0.
\]
Set
\[
v_1' = \alpha_1 [\mu_1]_{q} v_1 - [\lambda_1]_{q} v_2, \quad v_2' = [\lambda_2]_{q} v_{21} - \alpha_2 [\mu_2]_{q} v_2, \quad v_0' = [\lambda_1]_{q} v_0.
\]
Then
\[
F_1 v_0 = v_1', \quad F_1 v_1' = [\lambda_1]_{q} v_{21}, \quad F_1 v_2 = \alpha_2 [\mu_2]_{q} v_0, \quad E_1 v_0 = 0, \quad v_0 = -v_0.
\]
where we have introduced a short-hand notation \( \vartheta_{\pm} = [\lambda_1]_q[\lambda_2]_q \pm \alpha_1 \alpha_2 [\mu_1]_q[\mu_2]_q \). Clearly, vectors \( F'_i \), \( E'_i \) and \( v'_i \) are pairwise linearly independent for generic \( \lambda_1 \) and \( \mu_1 \). In the same way as for \( \Delta a' \), we call the set \( \{ v_0, v_1, v_2, v_{21} \} \) the up-down basis and \( \{ v_0, v'_1, v'_2, v_{21} \} \) the down-up basis of \( K_q(\lambda, \nu) \). The module diagram for both bases is equivalent to the ones shown in Figure 4(c) and (b).

5.3. Atypical module. The atypical module \( A_q(\lambda, \nu) \) is the two-dimensional submodule of \( K_q(\lambda, \nu) \) when

\[
[\lambda_1]_q[\lambda_2]_q = \alpha_1 \alpha_2 [\mu_1]_q[\mu_2]_q
\]

giving \( \vartheta_{-} = 0 \). Thus

\[
v'_1 = \gamma v'_2, \quad F'_1 = \gamma F'_2 \quad \text{and} \quad E_1 E_2 v_{21} = 0 \quad \text{where} \quad \gamma = \frac{\alpha_1 [\mu_1]_q}{[\lambda_2]_q} \frac{[\lambda_1]_q}{\alpha_2 [\mu_2]_q}.
\]

Set \( v''_0 = \alpha_1 [\mu_1]_q v_1 + [\lambda_1]_q v_2 \) and \( F''_1 = \alpha_1 [\mu_1]_q F_1 + [\lambda_1]_q F_2 \). Then clearly both \( v''_0, v'_2 \) and \( F''_1, F'_2 \) are linearly independent and

\[
F''_1 v_0 = v''_0, \quad E_1 v'_2 = 2 \alpha_1 [\lambda_1]_q [\mu_1]_q v_0, \quad v''_1 = \vartheta_{+} v_{21}, \quad E_2 v''_1 = \vartheta_{+} v_0.
\]

The module diagram of \( K_q(\lambda, \nu) \) when (5.14) holds is equivalent to the one shown in Figure 4(c). Hence \( A_q(\lambda, \nu) \cong \text{span}_C \left\{ v'_2, v_{21} \right\} \) as a vector space, and \( A_q(\lambda, \nu) \cong K_q(\lambda, \nu)/A_q(\lambda, \nu) \).

5.4. Tensor product of atypical modules. We employ the same notation as in Section 2.4. Namely, we set \( w_0 = v'_2 \) and \( w_1 = v_{21} \). Then

\[
F_1 w_0 = \alpha_2 [\mu_2]_q w_1, \quad F_2 w_0 = [\lambda_2]_q w_1, \quad E_1 w_1 = \gamma w_0, \quad E_2 w_1 = w_0, \quad E_1 w_0 = 0, \quad F_1 w_1 = 0.
\]

Let \( \bar{\lambda}_1, \bar{\mu}_1 \) denote the weights of the weight-vector \( \bar{w}_i \). We also have \( U^\pm w_1 = \nu^\pm w_1, U^\pm w_i = \nu^\pm w_i \). Then a simple computation gives

\[
\Delta(F_1) . (w_0 \otimes \bar{w}_0) = \alpha_2 \nu q^{-1/2} [\mu_2]_q w_1 \otimes \bar{w}_0 + (-1)^{p(w_0)} \alpha_2 \nu q^{-1/2} [\bar{\mu}_2]_q w_0 \otimes \bar{w}_1,
\]

\[
\Delta(F_2) . (w_0 \otimes \bar{w}_0) = \nu^{-1} q^{-1/2} [\lambda_2]_q w_1 \otimes \bar{w}_0 + (-1)^{p(w_0)} \nu q^{1/2} [\bar{\lambda}_2]_q w_0 \otimes \bar{w}_1,
\]

\[
\Delta(F_2 F_1) . (w_0 \otimes \bar{w}_0) = \alpha_2 (-1)^{p(w_0)} (\nu^{-1} q^{-1/2} [\lambda_2]_q [\bar{\mu}_2]_q - \nu \nu q^{1/2} [\bar{\lambda}_2]_q [\lambda_2]_q) w_1 \otimes \bar{w}_1\]

and

\[
\Delta(E_1) . (w_1 \otimes \bar{w}_1) = \gamma \nu^{-1} q^{-1/2} w_0 \otimes \bar{w}_1 - (-1)^{p(w_0)} \gamma \nu \nu q^{1/2} w_0 \otimes \bar{w}_0,
\]

\[
\Delta(E_2) . (w_1 \otimes \bar{w}_1) = \nu q^{-1/2} w_0 \otimes \bar{w}_1 - (-1)^{p(w_0)} \nu^{-1} q^{1/2} w_1 \otimes \bar{w}_0,
\]

\[
\Delta(E_1 E_2) . (w_1 \otimes \bar{w}_1) = (-1)^{p(w_0)} (\gamma \nu \nu q^{1/2} - \gamma \nu \nu q^{-1/2}) w_0 \otimes \bar{w}_0.
\]

Set \( \bar{v}_0 = w_0 \otimes \bar{w}_0 \) and \( \bar{v}_i = \Delta(F_i) \bar{v}_0) = \Delta(F_1 F_2) \bar{v}_0 \). Then

\[
\Delta(K^{\pm}) \bar{v}_0 = q^{\pm(1/2 + \bar{\lambda}_1/2)} \bar{v} = q^{\pm(1/2 \bar{\lambda}_1/2)} \bar{v} = q^{\mp(1/2 \bar{\lambda}_1/2)} \bar{v}, \quad \Delta(L^{\pm}_1) \bar{v}_0 = q^{\mp(1/2 + \bar{\mu}_1/2)} \bar{v} = q^{\pm(1/2 \bar{\mu}_1/2)} \bar{v}.
\]

Using the relation \( q^{\mu_1} = q^{(1/2 + \bar{\lambda}_2/2)/2} \nu^{\pm 2} \) we find

\[
\Delta(E_1) \bar{v}_{21} = \alpha_1 (\nu^{-2} q^{-1/2} [\mu_1]_q + \nu^{2} q^{1/2} [\bar{\mu}_1]_q) \bar{v}_1 - \alpha_2 (\gamma \nu^{-1} q^{-1/2} [\mu_2]_q + \gamma \nu \nu q^{1/2} [\bar{\mu}_2]_q) \bar{v}_2
\]

\[
= \alpha_1 (q^{1/2} [\mu_1]_q + q^{1/2} [\bar{\mu}_1]_q) \bar{v}_1 - (q^{-1/2} [\lambda_1]_q + q^{1/2} [\bar{\lambda}_1]_q) \bar{v}_2 = \alpha_1 [\bar{\mu}_1]_q \bar{v}_1 - [\bar{\lambda}_1]_q \bar{v}_2,
\]

\[
\Delta(E_2) \bar{v}_{21} = \alpha_2 (\nu^{2} q^{-1/2} [\mu_2]_q + \nu^{-2} q^{1/2} [\bar{\mu}_2]_q) \bar{v}_2 - \alpha_1 [\bar{\lambda}_2]_q \bar{v}_2 = [\bar{\lambda}_2]_q \bar{v}_2 - [\bar{\lambda}_2]_q \nu \nu \bar{v}_2,
\]

which compared with (5.14) implies that \( A_q(\lambda, \nu) \otimes A_q(\bar{\lambda}, \bar{\nu}) \cong K_q(\bar{\lambda}, \bar{\nu}) \) with \( \bar{\lambda}_i = \lambda_i + \bar{\lambda}_i \) and \( \bar{\nu} = \nu \bar{\nu} \). Moreover, \( q^{\mu_1} = q^{(1/2 + \bar{\lambda}_2/2)/2} \nu^{\pm 2} \).
5.5. One-dimensional module. Set
\begin{equation}
1_q = w_1 \otimes w_0 + (-1)\rho(w_1) q^{-\lambda_1/2} \nu \bar{\nu} w_0 \otimes w_1.
\end{equation}
We need to show that $a.1_q = 0$ if $\tilde{\lambda}_1 = \tilde{\mu}_1 = 0$ for all $a \in \mathfrak{u}_q a'$, $a \neq U^\pm$. It follows straightforwardly that $\Delta(E_2).1_q = 0$. For $F_1$ and $F_2$ we have
\begin{align*}
\Delta(F_2).1_q &= (-1)^{\rho(w_1)} \nu q^{-\lambda_2/2} (q^{\lambda_2}[\lambda_2]_q + q^{-\lambda_2}[\lambda_2]_q) w_1 \otimes w_1 = (-1)^{\rho(w_1)} \nu q^{-\lambda_2/2} [\lambda_2]_q w_1 \otimes w_1, \\
\Delta(F_1).1_q &= \alpha_2(-1)^{\rho(w_1)} (\nu^{-1} q^{\lambda_1/2}[\mu_2]_q + \nu q^{\lambda_1/2}[\mu_2]_q) w_0 \otimes w_1 \\
&= \alpha_2(-1)^{\rho(w_1)} (\nu^{-1} q^{\lambda_1/2}[\mu_2]_q + \nu q^{\lambda_1/2}[\mu_2]_q) w_0 \otimes w_1 \\
&= \alpha_2(-1)^{\rho(w_1)} \nu q^{-\lambda_2/2} [\mu_2]_q w_1 \otimes w_1,
\end{align*}
which are zero only if $\tilde{\lambda}_2 = \tilde{\mu}_2 = 0$. Now consider the action of $E_1$: $\Delta(E_1).1_q = (\gamma \rho^{-1} q^{-\lambda_1/2} - \gamma \nu \bar{\nu} q^{\lambda_1/2}) w_0 \otimes w_0$. Using $\gamma = \alpha_1[\mu_1]_q/[\lambda_2]_q$, $\gamma = \alpha_1[\mu_1]_q/[\lambda_2]_q$ and $\lambda_2 = -\lambda_2$, $\mu_2 = -\mu_2$, we get
\begin{align*}
\Delta(E_1).1_q &= \frac{\alpha_1}{\alpha_2[\mu_2]_q} (\rho^{-2} q^{-\lambda_1/2} [\lambda_1]_q + \nu^2 q^{\lambda_1/2} [\lambda_1]_q) w_0 \otimes w_0 \\
&= \frac{\rho^{-2} q^{-\lambda_1/2} [\lambda_1]_q + \nu^2 q^{\lambda_1/2} [\lambda_1]_q}{\alpha_2[\mu_2]_q} w_0 \otimes w_0,
\end{align*}
which gives zero if $\tilde{\lambda}_1 = 0$. Now $\tilde{\lambda}_1 = \tilde{\mu}_1 = 0$ and requiring $\Delta(U^\pm).1_q = 1_q$ implies $\tilde{\nu} = 1$.

5.6. $R$-matrix. We use the same notation as in Section 2.6. The two dimensional (atypical) representation $\pi: \mathfrak{u}_q a \rightarrow \text{End}(V)$, $a \mapsto \pi(a)$ is given by (c.f. (2.20))
\begin{align}
\pi(E_1) &= \gamma E_{21}, \quad \pi(F_1) = \alpha_2[\mu_2]_q E_{12}, \quad \pi(K^\pm) = q^{\pm \lambda_1/2} I, \quad \pi(U^\pm) = \nu^\pm I, \\
\pi(E_2) &= E_{22}, \quad \pi(F_2) = \alpha_1 \gamma^{-1} E_{12}, \quad \pi(L^\pm) = q^{\pm \mu_1/2} I.
\end{align}
As previously, we will denote by $\tilde{\pi}: a \rightarrow \text{End}V$ the representation with the labels $\{\gamma, \lambda_1, \mu_1, \nu\}$.

Proposition 5.2. The $R$-matrix $R_q(\gamma, \nu; \gamma, \bar{\nu}) \in \text{End}(V \otimes V)$ intertwining the tensor product of atypical representations $\pi$ and $\tilde{\pi}$ is given by
\begin{align}
R_q(\gamma, \nu; \gamma, \bar{\nu}) &= \left( q^{(\lambda_2 - \lambda_1)/2} - q^{(\lambda_2 - \lambda_1)/2} \gamma \nu \bar{\nu} \right) E_{11} \otimes E_{11} + \left( \gamma \nu^2 q^{-(\lambda_1 + \lambda_2)/2} - q^{-(\lambda_1 + \lambda_2)/2} \gamma \nu \bar{\nu} \right) E_{11} \otimes E_{22} \\
&+ q^{-(\lambda_1 - \lambda_2)/2} \gamma \nu \bar{\nu} \left( q^{(\lambda_2 - \lambda_1)/2} q^{-(\lambda_1 - \lambda_2)/2} \gamma \nu \bar{\nu} - 1 \right) E_{12} \otimes E_{21} + q^{-(\lambda_1 - \lambda_2)/2} \gamma \nu \bar{\nu} \left( 1 - q^{\lambda_1 - \lambda_2} \nu \bar{\nu} \right) E_{21} \otimes E_{12} \\
&+ \left( q^{(\lambda_1 + \lambda_2)/2} \gamma \nu \bar{\nu} - q^{(\lambda_1 + \lambda_2)/2} \gamma \nu \bar{\nu} \right) E_{22} \otimes E_{11} + \left( q^{(\lambda_1 - \lambda_2)/2} \gamma \nu \bar{\nu} - q^{(\lambda_1 - \lambda_2)/2} \gamma \nu \bar{\nu} \right) E_{22} \otimes E_{22}.
\end{align}
Note that $\lambda_1 = \lambda_1(\gamma, \nu)$ due to (5.10) and (5.14).

Proof. The proof is similar to that of Proposition 2.2. Set $R \in \text{End}(V \otimes V)$ to be an arbitrary matrix with elements $r_{ij}$ and $1 \leq i, j \leq 4$. We need to solve the intertwining equation (2.22) for all $a \in \mathfrak{u}_q a$. We restrict the matrix $R$ to the form given in (2.22) and choose $a = E_1$. Then (2.22) gives
\begin{align*}
q^{\lambda_1/2} \gamma \bar{\nu}q^{\lambda_1} \nu \bar{\nu} r_{22} - r_{11} - q^{\lambda_1/2} \gamma \nu \bar{\nu} r_{22} = 0, \quad \gamma q^{\lambda_1 \nu \bar{\nu}^2} r_{11} - r_{33} - q^{(\lambda_1 + \lambda_2)/2} \gamma \nu \bar{\nu} r_{32} = 0, \\
q^{\lambda_1/2} \gamma \nu \bar{\nu} r_{32} + q^{\lambda_1/2} \gamma \nu \bar{\nu} r_{44} - q^{\lambda_1/2} \gamma \nu \bar{\nu} r_{22} = 0, \quad q^{(\lambda_1 + \lambda_2)/2} \gamma \nu \bar{\nu} r_{23} + \gamma r_{33} - q^{\lambda_1 \nu \bar{\nu}^2} r_{44} = 0,
\end{align*}
having a solution $r_{32} = q^{(\lambda_1 - \lambda_2)/2} \gamma \nu \bar{\nu}^{-1} (q^{\lambda_1 \nu \bar{\nu}^2} r_{11} - r_{33})$, $r_{33} = q^{\lambda_1 \nu \bar{\nu}^2} r_{11} - q^{(\lambda_1 + \lambda_2)/2} \gamma \nu \bar{\nu} r_{32}$ and $r_{44} = q^{\lambda_1 \nu \bar{\nu}^2} r_{22} - q^{(\lambda_1 - \lambda_2)/2} \gamma \nu \bar{\nu}^{-1} r_{33}$. Now let $a = E_2$. Then (2.22) for the remaining elements
gives
\[
\left( q^{(\lambda_1 - \lambda_1 - \lambda_2)/2} - q^{(\lambda_1 - \lambda_2)/2} \right) r_{11} + \left( q^{\lambda_2 - 2\gamma} - q^{(\lambda_1 + \lambda_1 - \lambda_2)/2} \right) \gamma v^2 r_{22} = 0,
\]
\[
\left( q^{\lambda_1 - \lambda_2/2} - q^{\lambda_2/2} \right) r_{11} + \left( q^{\lambda_2 - 2\gamma - 1} \gamma v^2 \right) \gamma v^2 r_{32} = 0,
\]
\[
\left( q^{\lambda_1 - \lambda_2/2} - q^{\lambda_2/2} \right) r_{22} + \left( q^{\lambda_2 - 2\gamma} - q^{\lambda_1 - \lambda_2/2} \right) \gamma v^2 r_{32} = 0,
\]
\[
g^{(\lambda_1 + \lambda_1)/2} \gamma v^2 \left( v^2 r_{11} - q^{\lambda_2} r_{22} \right) + \bar{\gamma} \left( q^{(\lambda_1 + \lambda_1 + \lambda_2)/2} \gamma v^2 r_{22} - q^{(\lambda_2 + \lambda_2)/2} \gamma v^2 r_{32} \right) = 0,
\]
the solution of which is
\[
r_{22} = \frac{q^{(\lambda_1 + \lambda_1)/2} \gamma v^2 - q^{(\lambda_1 + \lambda_2)/2} \gamma v^2}{q^{(\lambda_1 + \lambda_2)/2} - q^{(\lambda_1 + \lambda_1)/2} \gamma v^2} r_{11}, \quad r_{32} = \frac{\gamma v^2 q^{(\lambda_2 - 2\gamma)}/q^{\lambda_1}}{q^{(\lambda_2 - 2\gamma)}/q^{\lambda_1}} r_{11}.
\]
Then, upon setting \( r_{11} = q^{(\lambda_2 - \lambda_1)/2} - q^{(\lambda_1 - \lambda_2)/2} \gamma v^2 \), we obtain (5.24). It remains to check that (2.22) holds when \( a = F_i \), which follows by a lengthy but direct computation and the usage of the identities 5.10 and 5.15 for parameters \( \gamma \) and \( \bar{\gamma} \).

Checking that the deformed R-matrix \( (5.24) \) satisfies the Yang-Baxter equation uses the same method described just below Proposition 2.22.

6. Affinization

We will affinize the algebra \( \mathfrak{g} \) by doubling its nodes. Our approach is inspired by a similar affinization presented in [BGM]. In this section we will use the additional notation \( [i] = (-1)^{i-1} \), which will appear in powers of central elements only.

**Definition 6.1.** The quantum affine algebra \( \mathfrak{g} \) is the unital associative Lie superalgebra generated by elements \( E_i, F_i \) and central elements \( K_i^\pm, U^\pm, V^\pm \) with \( 1 \leq i \leq 4 \) satisfying \( K_i^\pm K_j^\pm = U^\pm U^\pm = V^\pm V^\pm = 1 \) and
\[
[E_i, F_j] = \delta_{ij} K_i^+ K_i^- q - q^{-1} + \alpha_i (1 - \delta_{ij}) \frac{L_i^+ - L_i^-}{q - q^{-1}} \quad \text{for} \quad i, j \in \{1, 2\} \text{ or } \{3, 4\},
\]
\[
[[E_i, F_2], [E_4, F_1]] = \frac{K^+ - K^-}{q - q^{-1}}, \quad [[E_i, F_{i+2}], [E_{i+2}, F_i]] = \frac{L^+_i L^+_{i+2} - L^-_i L^-_{i+2}}{q - q^{-1}} \quad \text{and}
\]
\[
[E_i, F_{i+2}] = \alpha_i \frac{U^+ V^+ K_i^{+[i]} K_i^{-[i]} - U^- V^- K_i^{[i]} K_i^{-[-i]}}{q - q^{-1}} \quad \text{for} \quad i \in \{1, 2\},
\]
where
\[
K_i^\pm = K_2^\pm K_3^\pm K_4^\pm, \quad L_i^\pm = U^{\pm[i]} K_i^\pm K_i^\pm, \quad L_j^\pm = V^{\pm[j]} K_j^\pm K_j^\pm \quad \text{for} \quad i \in \{1, 2\}, j \in \{3, 4\}.
\]
The remaining relations are trivial. The \( \mathbb{Z}_2 \)-grading is given by \( \text{deg}_2(E_i) = \text{deg}_2(F_i) = 1 \) and \( \text{deg}_2(K_i^\pm) = \text{deg}_2(U^\pm) = \text{deg}_2(V^\pm) = 0 \).

Notice that the affine extension is such that elements with \( i = 3, 4 \) together with \( V^\pm \) generate a Hopf subalgebra of \( \mathfrak{g}_a \) isomorphic to \( \mathfrak{g} \). We will refer to the relations in the second line of (6.1) as the quantum Serre relations and to the relation in the third line as the compatibility relation. The choice of these additional relations will be explained a little bit further.

**Remark 6.1.** Assuming \( i \in \{1, 2, 3, 4\} \) and \( j \in \{1, 2\} \) the \( \mathbb{Z}_2 \)-grading on \( \mathfrak{g}_a \) is
\[
\text{deg}(E_{2j-1}) = \text{deg}(F_{2j}) = 1, \quad \text{deg}(E_{2j}) = \text{deg}(F_{2j-1}) = -1, \quad \text{deg}(K_i^\pm) = \text{deg}(U^\pm) = \text{deg}(V^\pm) = 0, \quad \text{deg}(\alpha_{2j-1}) = 2, \quad \text{deg}(\alpha_{2j}) = 0.
\]

We can define a Hopf algebra structure on \( \mathfrak{g}_a \) as follows.

**Proposition 6.1.** The Hopf algebra structure on \( \mathfrak{g}_a \) is given by the coproduct \( \Delta(C) = C \otimes C \) and
\[
\Delta(E_i) = E_i \otimes U^{-[i]} K_i^- + U^{+[i]} K_i^+ \otimes E_i, \quad \Delta(F_i) = F_i \otimes U^{+[i]} K_i^- + U^{-[i]} K_i^+ \otimes F_i \quad \text{for} \quad i = 1, 2, \quad \Delta(E_i) = E_i \otimes V^{+[i]} K_i^+ + V^{-[i]} K_i^- \otimes E_i, \quad \Delta(F_i) = F_i \otimes V^{+[i]} K_i^- + V^{-[i]} K_i^+ \otimes F_i \quad \text{for} \quad i = 3, 4,
\]
\[
\text{counit and antipode}
\]
\[
\epsilon(E_i) = \epsilon(F_i) = 0, \quad S(E_i) = -E_i, \quad S(F_i) = -F_i, \quad \epsilon(C) = 1, \quad S(C) = C^{(-1)}.
\]
for $C \in \{K^\pm, U^\pm, V^\pm\}$.

The proof of the proposition above follows by a direct computation.

6.1. Evaluation homomorphism.

**Proposition 6.2.** There exists a homomorphisms of algebras $\mathfrak{U}_q \hat{\mathfrak{a}} \to \mathfrak{U}_q \mathfrak{a}$ given by

\[
ev_{\rho} : \begin{cases} E_{i+2} \mapsto -\rho^{-1}(L_i^+ - L_i^-)E_i, & K_{i+2}^\pm \mapsto K_i^\mp, & \alpha_{i+2} \mapsto \alpha_i, \\ F_{i+2} \mapsto \rho(L_i^+ - L_i^-)^{-1}F_i, & V^\pm \mapsto U^\pm, \end{cases}
\]

for $i = 1, 2$, where

\[
\rho = U^2K_1^+K_2^- - U^{-2}K_1^-K_2^+.
\]

**Proof.** It is easy to see that $\ev_{\rho}(E_{i+2}, F_{i+2}) = [\ev_{\rho}(E_{i+2}), \ev_{\rho}(F_{i+2})]$ and the same is true for the quantum Serre relations, since $\ev_{\rho}(K_i^\pm) = 1$ and $\ev_{\rho}(L_i^\pm L_{i+2}^\pm) = 1$, which can be deduced from (6.3) (and is true only if $\ev_{\rho}(V^\pm) = U^\pm$ and $\ev_{\rho}(K_3^\pm K_4^\pm) = K_3^\mp K_4^\pm$). Now consider the compatibility relation. We have

\[
\ev_{\rho}([E_i, F_{i+2}]) = \alpha_i \frac{U^2K_i^{[+]i}K_i^{-[i]} - U^{-2}K_i^{-[i]}K_i^{[+i]}}{q - q^{-1}} = \alpha_i \frac{\rho}{q - q^{-1}},
\]

and

\[
[\ev_{\rho}(E_i), \ev_{\rho}(F_{i+2})] = \frac{\rho}{L_i^+ - L_i^-} [E_i, F_i] = \alpha_i \frac{\rho}{q - q^{-1}}.
\]

Finally,

\[
\ev_{\rho}([E_{i+2}, F_{i+2}]) = \alpha_i \frac{U^{[+]2i}K_i^-K_i^- - U^{-[i]}K_i^+K_i^+}{q - q^{-1}} = \alpha_i \frac{L_i^+ - L_i^-}{q - q^{-1}}
\]

and

\[
[\ev_{\rho}(E_{i+2}), \ev_{\rho}(F_{i+2})] = \frac{L_i^+ - L_i^-}{L_i^+ - L_i^-} [E_i, F_i] = \alpha_i \frac{L_i^+ - L_i^-}{q - q^{-1}}.
\]

\[\square\]

**Remark 6.2** (1). The affinization of $\mathfrak{U}_q \mathfrak{a}$ presented above is unique up to an isomorphism. For example, one could choose the additional relations in (6.1) to be

\[
[[E_3, F_1], [E_4, F_2]] = \frac{K^+ - K^-}{q - q^{-1}}, \quad [[E_i, F_{i+2}], [E_{i+2}, F_i]] = \frac{L_i^+ L_i^{+2} - L_i^- L_i^{-2}}{q - q^{-1}}
\]

and

\[
[E_i, F_{i+2}] = \alpha_i \frac{U^{+V}K_i^{[+]i}K_i^{[+i]} - U^{-V}K_i^{-[i]}K_i^{[-i]}}{q - q^{-1}}
\]

for $i \in \{1, 2\}$, leading to

\[
ev_{\rho} : \begin{cases} E_{i+2} \mapsto -\rho^{-1}(L_i^+ - L_i^-)E_i, & K_{i+2}^\pm \mapsto K_i^\mp, & \alpha_{i+2} \mapsto \alpha_i, \\ F_{i+2} \mapsto \rho(L_i^+ - L_i^-)^{-1}F_i, & V^\pm \mapsto U^\pm. \end{cases}
\]

(2). Let $\beta \in \mathbb{C}$ be such that $\beta^2 = 1$. Then one could substitute the map $V^\pm \mapsto U^\pm$ by $V^\pm \mapsto \beta U^\pm$ in (6.3), since $U^\pm$ only appear squared in the algebra $\mathfrak{U}_q \mathfrak{a}$ (via (5.3)).

7. Deformations of $AdS_3/CFT_2$

A two-parameter deformation of the $AdS_3/CFT_2$ superstring was recently put forward in [Ho]. In this section we present a deformation of the spin chain and the $R$-matrices obtained in Section 3. Our approach differs slightly from the one presented in loc. cit. The deformed $R$-matrices obtained below can be identified with the $\mathfrak{gl}_{1|1}$ subsectors of the $R$-matrix of the deformed Hubbard chain [LMR] and are equivalent to the ones in presented in [Ho], Section 5.1 up to a $u$-gauge transformation. The “dictionary” between the algebra $\mathfrak{U}_q \mathfrak{a}$ and its analogue in loc. cit. is given by $U^\pm \sim U^\pm$ and

\[
E_1 \approx (\mathfrak{U}_q)\mathfrak{k}^{\pm} \Omega_+, \quad F_1 \approx (\mathfrak{U}_q)\mathfrak{k}^{\pm} \Omega_-, \quad K_1^\pm \approx \Omega^{\pm 1}, \quad \alpha_1(L_1^+ - L_1^-) \approx (q - q^{-1}) \Omega^{-1} (\mathfrak{U}_q)\mathfrak{k}_R \Omega^{\pm 1},
\]

\[
E_2 \approx (\mathfrak{U}_q)\mathfrak{k}^{\pm} \Omega_+, \quad F_2 \approx (\mathfrak{U}_q)\mathfrak{k}^{\pm} \Omega_-, \quad K_2^\pm \approx \Omega^{\pm 2}, \quad \alpha_2(L_2^+ - L_2^-) \approx (q - q^{-1}) \Omega^{-1} (\mathfrak{U}_q)\mathfrak{k}_R \Omega^{\pm 2}.
\]
7.1. Representations and the $R$-matrices. Consider the atypical module $A_q(\lambda, \nu)$ and choose the \( \mathbb{Z}_2 \)-grading of \( v_0 \) to be \( \deg_2 v_0 = 0 \). We introduce the notation \( |\phi_p\rangle = d_p w_1 \), \( |\bar{\psi}_p\rangle = w_0 \) with some \( d_p \in \mathbb{C}^\times \). Set \( a_p = \gamma d_p, b_p = \alpha_2 [\mu_2]_q/d_p \) and \( c_p = [\lambda_2]_q/d_p \). It follows from (7.14) that

\[
E_1 |\phi_p\rangle = a_p |\phi_p\rangle, \quad F_1 |\psi_p\rangle = b_p |\phi_p\rangle, \quad F_2 |\psi_p\rangle = c_p |\phi_p\rangle, \quad E_2 |\phi_p\rangle = d_p |\psi_p\rangle.
\]

Denote \( \sigma = q^{(\lambda_1 + \lambda_2)}/4 \) and \( \delta = \lambda_1 - \lambda_2 \), and set \( \alpha_1 = \alpha_2 = h \). We have

\[
[E_1, F_1] |\phi_p\rangle = [\lambda_1]_q |\phi_p\rangle, \quad [E_1, F_1] |\varphi_p\rangle = h [\mu_1]_q |\varphi_p\rangle \quad \text{for} \quad \varphi_p \in \{\phi_p, \psi_p\}.
\]

Thus the representation labels must satisfy the following set of identities

\[
a_p b_p (=[\lambda_1]_q) = q^{\delta/2} \frac{\sigma^2 - \varphi^2}{q - q^{-1}}, \quad a_p c_p (=[\alpha_1]_q) = h \frac{\varphi^2 - \sigma^2}{q - q^{-1}}, \quad c_p d_p (=[\lambda_2]_q) = q^{\delta/2} \frac{\sigma^2 - \varphi^2}{q - q^{-1}}, \quad b_p d_p (=[\alpha_2]_q) = h \frac{\varphi^2 - \sigma^2}{q - q^{-1}}.
\]

Moreover, the module shortening constraint given in (7.15) becomes

\[
h^2 (\varphi^2 - \sigma^2) (\sigma^2 - \varphi^2) = (\sigma^2 - \delta^2) (\sigma^2 - \delta^2).
\]

Inspired by [BGM] we choose the following \( x^\pm \)-parametrization:

\[
\begin{align*}
\varphi^2 &= q^{2} x^+ x^- + 1, \\
\sigma^2 &= q^{2} x^+ x^- + 1, \\
\delta^2 &= \frac{x^+ + x^-}{\xi - 1}, \\
h^2 &= \frac{\xi^2}{\xi - 1}.
\end{align*}
\]

where parameters \( x^\pm \) and \( \xi \) satisfy

\[
q^{-2} \xi x^+ = q^2 \xi x^-, \quad \xi (x) = \frac{x + x^{-1} + \xi - 1}{\xi - 1}, \quad h^2 = \frac{\xi^2}{\xi - 1}.
\]

It is a direct computation to verify that this parametrization satisfies (7.4). Using an analog to (1.3) we set \( \eta_p^p = i(x^-_p - x^+_p) \) and fix the expression for \( a_p \) to give

\[
a_p = \sqrt{h} \eta_p \nu_p \sigma_p, \quad b_p = i \sqrt{h} \frac{q^{\delta^2/2} \eta_p \sigma_p}{q - q^{-1} (\xi x^+_p + 1)}, \quad c_p = i \sqrt{h} \frac{\eta_p \nu_p \sigma_p (q - q^{-1})}{(q - q^{-1}) x^+_p}, \quad d_p = \sqrt{h} \frac{q^{\delta^2/2} \eta_p \sigma_p}{\nu_p (x^p + x^-_p)},
\]

where we have added extra indices to all the dynamical elements.

We define vectors spaces \( V_p \) and \( V_r \) in the same way as we did in Section 3.1. To obtain the deformed \( \text{AdS}_3/CFT_2 \) \( R \)-matrix we substitute \( \gamma \to a_p/d_p, \quad \tilde{\gamma} \to a_r/d_r \) and \( \nu \to \nu_p, \quad \tilde{\nu} \to \nu_r \) in (7.24), rescale \( E_1 \otimes E_1 \to d_p/d_r, \ E_2 \otimes E_1 \to d_r/d_p, \ E_1 \otimes E_2 \to d_r/d_p, \ E_1 \otimes E_2 \to d_r/d_p, \) multiply by an overall factor \( q^{(\delta - \delta')/4} \xi \sigma_p \sigma_r / h \) and use the identities \( \sigma_p = q^{(\lambda_1 + \lambda_2)/4}, \quad \sigma_r = q^{(\lambda_1 + \lambda_2)/4} \) together with (7.3) and assumptions \( \delta = \lambda_1 - \lambda_2 \) and \( \delta = \lambda_1 - \lambda_2 \). This gives the deformed left-left \( R \)-matrix for massive modes

\[
R_{11}^{q}(p, r) = d_p^2 q^{2} \sigma_r^2 (x^-_r - x^+_r) E_1 \otimes E_1 + q^{-2} (x^-_r - x^+_r) E_1 \otimes E_2 + q^{2} \nu_r \nu_p \sigma_r^2 \sigma_p^2 (x^-_r - x^+_r) E_2 \otimes E_2 + q^{2} \nu_r \nu_p \sigma_r^2 \sigma_p^2 (x^-_r - x^+_r) E_2 \otimes E_{12} + q^{2} \nu_r \nu_p \sigma_r^2 \sigma_p^2 (x^-_r - x^+_r) E_2 \otimes E_{21} + q^{2} \nu_r \nu_p \sigma_r^2 \sigma_p^2 (x^-_r - x^+_r) E_2 \otimes E_{22}.
\]

Next we construct the right module. Consider the module \( A_q(\tilde{\lambda}, \tilde{\nu}) \) such that \( \deg_2 v_0 = 1 \) and assume that \( \tilde{\mu}_1 = \mu_1 \). We introduce the notation \( |\tilde{\psi}_p\rangle = (q^{-1} - q) b_p w_1 \) and \( |\tilde{\phi}_p\rangle = w_0 \). Similarly as before, we must have

\[
(E_1, F_1) |\varphi_p\rangle = [\tilde{\lambda}_1]_q |\varphi_p\rangle, \quad (E_1, F_1) |\varphi_p\rangle = h [\mu_1]_q |\varphi_p\rangle \quad \text{for} \quad \varphi_p \in \{\tilde{\phi}, \tilde{\psi}\}.
\]

Requiring \( a_p c_p = h [\mu_1]_q \) and \( b_p d_p = h [\mu_2]_q \) (c.f. (7.3)) we find

\[
E_1 |\tilde{\psi}_p\rangle = (q^{-1} - q) c_p |\tilde{\phi}_p\rangle, \quad E_2 |\tilde{\psi}_p\rangle = (q^{-1} - q) b_p |\tilde{\phi}_p\rangle,
\]

\[
F_1 |\tilde{\psi}_p\rangle = \frac{d_p}{q - q^{-1}} |\tilde{\phi}_p\rangle, \quad F_2 |\tilde{\psi}_p\rangle = \frac{d_p}{q - q^{-1}} |\tilde{\phi}_p\rangle,
\]

and \( \tilde{\lambda}_1 = \lambda_1 \) and \( \tilde{\lambda}_2 = \lambda_2 \). Hence \( \sigma = \tilde{\sigma} = q^{(\tilde{\lambda}_1 + \tilde{\lambda}_2)/4} \), \( \delta = - \delta = \tilde{\lambda}_1 - \tilde{\lambda}_2 \) and \( \tilde{\gamma} = c_p/b_p \), thus the right magnon with momentum \( p \) is the antiparticle of the left magnon with momentum \( p \) (this is in agreement with the non-deformed case, c.f. with (1.3) and (3.8)).

Next, we define the vector space \( V_r \) in the same way as we did in Section 3.1 and identify \( \tilde{\pi}_r \in \text{End}V_r \) with the transpose of \( \tilde{\pi}_r \). Then we substitute \( \gamma \to a_p/d_p, \quad \tilde{\gamma} \to a_r/d_r \) and \( \nu \to \nu_p, \quad \tilde{\nu} \to \nu_r \) into (5.24).


rename (and transpose the second tensor space with \( \tau \)) \( E_{11} \otimes E_{22} \rightarrow E_{12} \otimes E_{11} \), \( E_{11} \otimes E_{11} \rightarrow E_{11} \otimes E_{22} \),
\( E_{21} \otimes E_{12} \rightarrow (q^{-1} - q) \frac{d_p}{b_r} E_{21} \otimes E_{21} \), \( E_{12} \otimes E_{21} \rightarrow (q^{-1} - q)^{-1} \frac{b_r}{d_p} E_{12} \otimes E_{21} \), multiply by an overall factor \( q^{(\delta - \bar{\delta})/4} \xi_{\sigma p, \sigma r} / \hbar \) and use (7.5) together with the identities for \( \sigma_p, \sigma_r \) and assumption on \( \delta, \bar{\delta} \) as above. This gives the deformed left-right \( R \)-matrix for massive modes
\[
R_{LR}^q(p, r) = \left( x_p^r - \frac{1}{x_r^p} \right) E_{11} \otimes E_{11} + q^{(\delta - \bar{\delta})/4} \frac{1}{x_r^p} \left( x_p^r + \frac{1}{x_r^p} \right) E_{22} \otimes E_{22}
\]
(7.11)
\[
+ q^{(\delta - \bar{\delta})/4} \frac{1}{x_r^p} \left( x_p^r - \frac{1}{x_r^p} \right) E_{11} \otimes E_{12} + q^{(\delta - \bar{\delta})/4} \frac{1}{x_r^p} \left( x_p^r + \frac{1}{x_r^p} \right) E_{12} \otimes E_{22}
\]
\[
+ q^{(\delta - \bar{\delta})/4} \frac{1}{x_r^p} \left( x_p^r - \frac{1}{x_r^p} \right) E_{22} \otimes E_{12} + q^{(\delta - \bar{\delta})/4} \frac{1}{x_r^p} \left( x_p^r + \frac{1}{x_r^p} \right) E_{22} \otimes E_{22}
\]

The deformed \( R \)-matrices depend explicitly on the masses of the magnons. Parameters \( \delta \) and \( \bar{\delta} \) in (7.8) represent masses of magnons with momenta \( p \) and \( r \), respectively; and equivalently for \( \delta \) and \( \bar{\delta} \) in (7.11).

7.2. Yang-Baxter equation and unitarity. The deformed \( R \)-matrices \( R_{LR}^q(p, r) \) and \( R_{RL}^q(p, r) \) satisfy an equivalent set of Yang-Baxter equations as their non-deformed counterparts \( R_{LR}(p, r) \) and \( R_{RL}(p, r) \), as it was discussed in Section 7.2. This follows from the properties of \( R_q(\nu, \lambda; \nu, \lambda) \).

To demonstrate the unitarity property we set
\[
R_{LR}^q(p, r) = \frac{1}{(x_r^p - x_p^r) \nu_r \sigma_r} R_{LR}^q(p, r), \quad R_{RL}^q(p, r) = \left( x_p^r - x_r^p - 1 \right) \left( 1 - \frac{1}{x_r^p x_p^r} \right)^{-\frac{1}{4}} R_{LR}^q(p, r).
\]

Then a direct computation gives
\[
R_{LR}^q(p, r) R_{RL}^q(r, p) P = I, \quad R_{RL}^q(p, r) R_{LR}^q(r, p) P = I.
\]

7.3. Singlet and bound states. Consider the left-right sector. We use the same notation as before, namely \( |\varphi_p \chi_r \rangle = |\varphi_p \rangle \otimes |\chi_r \rangle \) with \( \varphi, \chi \in \{ \phi, \psi \} \). The states \( |\varphi_p \chi_r \rangle \) span the typical module of \( \mathfrak{u}_p \mathfrak{a} \) with \( |\psi_p \chi_r \rangle \) and \( |\varphi_p \phi_r \rangle \) being the highest and the lowest weight vectors, respectively. There are two module shortening conditions:
\[
\Delta(E_1 E_2) |\varphi_p \phi_r \rangle = q^{-(\delta + \bar{\delta})/4} \frac{\sigma_p}{\sigma_r} \left( a_p b_r \nu_r - q^{(\delta + \bar{\delta})/2} a_r b_p \nu_p \nu_r \right) |\psi_p \psi_r \rangle
\]
(7.14)
\[
= q^{(\delta - \bar{\delta})/4} \frac{\xi(x_p^r - x_r^p)}{(x_r^p + \xi)(x_p^r + \xi)} \eta_p \eta_r \nu_r^2 \sigma_r^2 |\psi_p \psi_r \rangle = 0
\]
if \( x_p^r = x_r^p \) and
\[
\Delta(F_1 F_2) |\psi_p \psi_r \rangle = -q^{-(\delta + \bar{\delta})/4} \frac{\sigma_p}{\sigma_r} \left( q^{(\delta + \bar{\delta})/2} b_r \nu_r - b_p c_r \nu_p \nu_r \right) |\varphi_p \phi_r \rangle
\]
(7.15)
\[
= \frac{q^{(\delta - \bar{\delta})/4} \xi (x_p^r - x_r^p) \eta_p \eta_r \nu_r^2 \sigma_r^2}{(q - q^{-1})^2 (x_p^r + 1)(x_r^p + 1)} |\varphi_p \phi_r \rangle = 0
\]
if \( x_p^r = x_r^p \). Notice that \( x_p^r = x_r^p \) is the zero and \( x_p^r = x_r^p \) is the pole of \( R_{LR}^q(p, r) \). Thus the states \( |\varphi_p \rangle := |\varphi_p \phi_r \rangle \) and \( |\psi_r \rangle := \Delta(E_2) |\varphi_p \phi_r \rangle \) when \( x_p^r = x_r^p \) are the bound states of the deformed spin chain. They are parametrized by \( x_p^r := x_p^r \) and \( x_r^p := x_r^p \), and their mass is equal to the total mass \( \delta + \bar{\delta} \) (see discussion at the end of Section 7.1).

Similarly, the states \( |\varphi_p \chi_r \rangle \) span the typical module in the left-right sector with \( |\varphi_p \psi_r \rangle \) and \( |\varphi_p \phi_r \rangle \) being the highest and the lowest weight vectors, respectively. We find
\[
\Delta(E_1 E_2) |\varphi_p \psi_r \rangle = \frac{q^{-1} - q}{q^{(\delta + \bar{\delta})/4} \sigma_p} \left( q^{\delta/2} a_p b_r \nu_r - q^{\bar{\delta}/2} c_r d_p \nu_p \nu_r \right) |\psi_p \phi_r \rangle
\]
(7.16)
\[
= \frac{\eta_p \eta_r \nu_r^2 \sigma_r^2}{(x_p^r + \xi)(x_r^p + 1)x_p^r} |\psi_p \phi_r \rangle = 0
\]
if \( x_p^\pm x_r^\pm = 1 \), and

\[
\Delta(F_2F_1) |\psi_p\tilde{\phi}_r\rangle = -\frac{q^{-(\delta+\delta)/4} \sigma_p}{(q^{-1} - q)} \left( \frac{q^{\delta/2} c_p d_r}{\nu_p \nu_r} - q^{-\delta/2} \alpha_r b_p \nu_p \nu_r \right) |\phi_p\tilde{\psi}_r\rangle
\]

(7.17)

\[
= \frac{i q^{(\delta+\delta)/4} (x_p^+ x_r^+ - 1) \eta_p \eta_r \nu^2 \sigma_p}{(q^{-1} - q)(x^+_p + \xi)(x^+_r + \xi)} |\phi_p\tilde{\psi}_r\rangle = 0
\]

if \( x_p^\pm x_r^\pm = 1 \). Since \( x^\pm = (x^\pm)^{-1} \) are zeros of \( R^o_{LR}(p, r) \) we conclude that there are no bound states of mixed type.

Consider the one-dimensional module \( \mathfrak{u}_q \mathfrak{a} \) of \( \mathfrak{u}_q \mathfrak{a} \) (Section 6). There are two singlet states in the deformed spin chain, the left-right singlet and the right-left singlet:

\[
1^q_{LR} = |\phi_p\tilde{\phi}_p\rangle + \frac{q^{(\delta+\delta)/4} d_p v_p u_p}{(q^{-1} - q) b_p \sigma_p \sigma_p} |\psi_p\tilde{\psi}_p\rangle, \quad 1^q_{RL} = |\phi_p\tilde{\phi}_p\rangle - \frac{q^{-(\delta+\delta)/4} d_p \sigma_p \sigma_p}{(q^{-1} - q) b_p \nu_p \nu_p} |\psi_p\tilde{\psi}_p\rangle,
\]

which are annihilated by all generators of \( \mathfrak{u}_q \mathfrak{a} \) (except \( U^\pm \)) provided \( x_p^\pm = (x^\pm)^{-1} \).

7.4. Affine symmetry. We want to obtain evaluation modules for \( \mathfrak{u}_q \mathfrak{a} \) in the \( AdS_3/CFT_2 \) parameterization. Recall that \( K_{\pm}^1 K_{\pm}^2 \cdot w_i = \sigma^{\pm 2} \cdot w_i \) and \( K_{\pm}^1 K_{\pm}^2 \cdot w_i = q^{\pm 2} \cdot w_i \) and \( U^\pm \cdot w_i = \nu^\pm \cdot w_i \) for the left module and \( K_{\pm}^1 K_{\pm}^2 \cdot w_i = q^{\pm 2} \cdot w_i \) for the right module. Thus

\[
E_{i+2} |\varphi_p\rangle = -\frac{\nu_p^2 \sigma_p^2}{\nu_p^2 q^{g/2} - \nu_p^{-2} q^{g/2}} E_i |\varphi_p\rangle, \quad F_{i+2} |\varphi_p\rangle = -\frac{\nu_p^2 \sigma_p^2}{\nu_p^2 q^{g/2} - \nu_p^{-2} q^{g/2}} F_i |\varphi_p\rangle
\]

for the left module and

\[
E_{i+2} |\tilde{\varphi}_p\rangle = -\frac{\nu_p^2 \sigma_p^2}{\nu_p^2 q^{-g/2} - \nu_p^{-2} q^{-g/2}} E_i |\tilde{\varphi}_p\rangle, \quad F_{i+2} |\tilde{\varphi}_p\rangle = -\frac{\nu_p^2 \sigma_p^2}{\nu_p^2 q^{-g/2} - \nu_p^{-2} q^{-g/2}} F_i |\tilde{\varphi}_p\rangle
\]

for the right module.

8. CONCLUSIONS AND OUTLOOK

The results of this paper are twofold. First, we demonstrated novel algebraic structures that arise in the \( AdS_3/CFT_2 \) duality. The main results are the Yangian \( \mathcal{Y}(a) \) presented in Section 3, the quantum affine algebra \( \mathfrak{u}_q \mathfrak{a} \) presented in Section 6, and the \( R \)-matrices given by Propositions (2.2) and (5.2). Second, we showed how to obtain modules and \( R \)-matrices investigated in [BSS] [BSST] from the “canonical” ones presented in this paper, and demonstrated highest weight modules and some properties of physical states of the \( q \)-deformed model of the \( AdS_3/CFT_2 \) spin chain considered in [He].

There are several possible directions of further study. First, it would be interesting to generalize the constructions presented in this paper for centrally extended superalgebras \( \mathfrak{sl}(1|1)^+ \otimes \mathbb{C}^n \), with \( n > 2 \) for different “linkings” by extending (2.1), namely \( e_i f_j = \delta_{ij} h_i + a_{ij}(1 - \delta_{ij}) k_i \), where \( (a_{ij})_{1 \leq i, j \leq n} \) is the matrix of a connected graph and \( [\cdot, \cdot] \) denotes the graded commutator; the question we want to ask is what types of graphs lead to “interesting” \( \varrho \)-deformed Hopf algebras having a \( \varrho \)-deformed coproduct (2.4), and we would then like to compare their representations with the classification obtained in [Kac]. Second, it would be interesting to construct a Drinfeld New presentation of \( \mathfrak{u}_q \mathfrak{a} \) following the construction presented in [He]. Third, a similar superalgebra \( \mathfrak{u}_{1|1} \otimes \mathbb{C}^n \) emerges in the \( AdS_2 \times S^2 \) duality [HPT]. We hope the present paper will serve as a guideline for analogous constructions in this duality. Moreover, the \( \varrho \)-deformed algebras are known to have additional so-called “secret symmetries” [LMMRT] [PTW]; it would be interesting to find such symmetries for the algebras obtained in the this paper. Lastly, it is well known that both \( AdS_3/CFT_4 \) spin chain and its \( q \)-deformed model are closely related to the one-dimensional Hubbard model [Bel] [RG] and its deformation [BSS] [BGM]; we believe that the \( AdS_3 \times S^3 \) spin chain and its deformation can also be linked to the one-dimensional Hubbard model or some generalization thereof (e.g. [FTR]). For example, one could consider the quotient of (2.1) by the ideal \( e_i - a_{ij}, f_i - a^\dagger_{ij}, h_i - 1, k_i - \alpha_i \), with \( \alpha_i \in \mathbb{C} \), giving the algebra \( [a_i, a_j] = 0, [a^\dagger_i, a^\dagger_j] = 0 \), \( [a_i, a^\dagger_j] = \delta_{ij} + (1 - \delta_{ij}) \alpha_i \), which can be interpreted as an “algebra of interacting electrons”.

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\[ \text{Page 20} \]
APPENDIX A. COPRODUCT $\Delta$.

The coproduct $\Delta_i$ is given by

$$
\Delta_i(e_{i,r}) = e_{i,r} \otimes u^+ + u^\pm \otimes e_{i,r} + \sum_{l=1}^{r} (\epsilon_i u^\pm h_{i,r,l-1} \otimes e_{i,l-1} + \epsilon_i u^\mp h_{i,r,l-1} \otimes u^\mp e_{i,l-1}),
$$

$$
\Delta_i(f_{i,r}) = f_{i,r} \otimes u^+ + u^\pm \otimes f_{i,r} + \sum_{l=1}^{r} (\epsilon_i f_{i,r,l-1} \otimes u^\pm h_{i,r,l-1} + \epsilon_i u^\pm f_{i,r,l-1} \otimes u^\pm h_{i,r,l-1}),
$$

(A.1)

$$
\Delta_i(h_{i,r}) = h_{i,r} \otimes 1 + 1 \otimes h_{i,r} + \sum_{l=1}^{r} (\epsilon_i h_{i,r,l-1} \otimes h_{i,l-1} + \epsilon_i u^\mp h_{i,r,l-1} \otimes u^\mp h_{i,l-1}),
$$

$$
\Delta_i(k_{i,r}) = k_{i,r} \otimes u^\pm + u^\pm \otimes k_{i,r} + \sum_{l=1}^{r} (\epsilon_i k_{i,r,l-1} \otimes u^\pm h_{i,l-1} + \epsilon_i u^\pm k_{i,r,l-1} \otimes h_{i,l-1}).
$$

Setting $\epsilon_i = 1$ one obtains $\Delta$, while $\epsilon_i = \pm 1$ gives $\Delta_\pm$ used in [120].

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