On the solution uniqueness in portfolio optimization and risk analysis

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November 2, 2018

Abstract

We consider the issue of solution uniqueness for portfolio optimization problem and its inverse for asset returns with a finite number of possible scenarios. The risk is assessed by deviation measures introduced by [Rockafellar et al., Mathematical Programming, Ser. B, 108 (2006), pp. 515–540] instead of variance as in the Markowitz optimization problem. We prove that in general one can expect uniqueness neither in forward nor in inverse problems. We discuss consequences of that non-uniqueness for several problems in risk analysis and portfolio optimization, including capital allocation, risk sharing, cooperative investment, and the Black-Litterman methodology. In all cases, the issue with non-uniqueness is closely related to the fact that subgradient of a convex function is non-unique at the points of non-differentiability. We suggest methodology to resolve this issue by identifying a unique “special” subgradient satisfying some natural axioms. This “special” subgradient happens to be the Steiner point of the subdifferential set.

Key words: Capital allocation, Risk sharing, Portfolio optimization, Cooperative investment, Black-Litterman model, Convex differentiation, Steiner point

1 Introduction

In various problems in economics and finance, including capital allocation [Kalkbrener, 2005], risk sharing [Filipović and Kupper, 2008], cooperative investment [Grechuk and Zabarankin, 2017], inverse portfolio problem [Bertsimas et al., 2012], and generalized Black-Litterman model [Palczewski and Palczewski, 2018], it is important to be able to identify a unique solution. We show that a solution to many of such problems can be expressed in an explicit way using a sub-gradient of some convex function $f : \mathbb{R}^N \to \mathbb{R}$ at some point $X \in \mathbb{R}^N$, and the solution is unique if and only if $f$ is differentiable at $X$. Because every convex function $f$ is differentiable almost everywhere [Rockafellar, 1970, Theorem 25.5], one may expect that such a solution should be unique in all “practical” cases. While this is indeed true in the context of risk sharing, we demonstrate that this intuition fails badly in other contexts. To resolve this issue, we suggest an axiomatic framework for selecting a unique special sub-gradient, which we call an extended gradient, from the subdifferential set $\partial f(X)$ of every convex function $f$ at any point $X$. In fact, our extended gradient coincides with the Steiner point [Schneider, 1971] of $\partial f(X)$. This allows us to resolve the issue of solution non-uniqueness in various applications.

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Capital allocation problem is one of the basic problems in risk management, which has been studied in a number of papers, see e.g. [Denault (2001), Kalkbrener (2005), and references therein. The problem is to distribute the risk capital among \( n \) subsidiaries or business units. Equivalently (see Cherny and Orlov (2011)), the problem is to decide how much the risk coming from each subsidiary contributes to the total (cumulative) risk. Kalkbrener (2005) established necessary and sufficient conditions on the risk measure for the existence of capital allocation with two highly desirable properties: linearity and diversification. Unfortunately, linear diversifying capital allocation may be non-unique, and, in this case, it is unclear which one to select. Cherny and Orlov (2011) suggested an additional “law-invariance” axiom, under which the capital allocation becomes unique for some specific family of risk measures, but not in general. Grechuk (2015) introduced so-called “centroid capital allocation”, which is unique but lacks axiomatic foundation. The present work suggests a capital allocation approach based on the Steiner point of a sub-differential set, which is always unique and follows from some natural axioms.

Optimal risk sharing is a classical problem which was originated by Borch (1962), Arrow (1963), and others, which asks for the optimal redistribution of risk among \( n \) agents. Such redistribution is called Pareto optimal if no agent can decrease their risk without increasing the risk for some other agents. If agents are allowed to trade, they will eventually arrive at some special Pareto optimal allocation, which is called equilibrium allocation (Filipović and Kupper, 2008). However, if equilibrium allocation is not unique, which one to choose? Our Steiner point approach can be applied to this problem as well.

In the problem of cooperative investment, \( m \) agents decide that instead of investing individually, they can form a coalition, buy a joint portfolio, and then distribute the profit of this joint portfolio in the same way as in the optimal risk sharing problem, see e.g. [Xia (2004) and Grechuk et al. (2013). The utility of investor \( i \) is \( U_i(Z_i) \), where \( U_i \) is some utility function and \( Z_i \) the random wealth of agent \( i \) at the investment horizon. Grechuk and Zabarankin (2017) show that, under some mild conditions on \( U_i \), cooperative investment is strictly preferable for all agents compared to their optimal individual investment strategies. In the cooperative investment problem, the coalition’s preferences can be represented by a cooperative utility function \( U^* \). The coalition solves an optimization problem with the utility \( U^* \) to find an optimal portfolio with the terminal wealth \( X^* \). This terminal wealth must consequently be distributed among investors: one has to find a Pareto optimal allocation \((Z_1, \ldots, Z_m)\) such that \( X^* = \sum_{i=1}^{m} Z_i \). There are usually infinitely many Pareto-optimal wealth allocations, but Grechuk and Zabarankin (2017) defined an allocation which can be considered as “fair”. The issue is that this “fair” allocation is, in general, non-unique, as we demonstrate in this paper. Because this non-uniqueness is the consequence of possible non-differentiability of \( U^* \), this issue is resolved by our Steiner point approach provided that \( U^* \) is a concave function.

In the realm of portfolio analysis we consider a market with a riskless asset and \( n \) risky assets. Portfolios are represented as combinations \( x_1 R^{(1)} + \cdots + x_n R^{(n)} \), where the vector random variable \( R = (R^{(1)}, \ldots, R^{(n)})^T \) denotes excess returns of risky assets. The objective is to find a portfolio allocation (fractions of wealth invested in the risky assets) \( x = (x_1, \ldots, x_n)^T \) that solves the following optimization problem:

\[
\min_x \rho(R^T x) \quad \text{subject to } \mu^T x \geq \Delta,
\]

where \( \rho \) measures portfolio risk, \( \Delta \) is the target excess return and

\[
\mu = (\mu_1, \ldots, \mu_n)^T = (E[R^{(1)}], \ldots, E[R^{(n)}])^T.
\]

In this paper we study the uniqueness of solutions to problem (1) and solutions to the following inverse problem: given a vector \( x^* \), the information on the distribution of \( R \) sufficient to compute
\( \rho(R^T x) \) for any \( x \), and \( \Delta > 0 \) find a vector of mean returns \( \mu \) such that \( x^* \) is a solution to problem (I) for that \( \mu \). Notice that the inverse problem we are interested in is meaningful only when the risk measure \( \rho \) is indifferent to the location parameter of distribution \( R \), e.g., standard deviation, variance of portfolio returns, or a deviation measure of Rockafellar et al. (2006a).

The problem of portfolio inverse optimization under different formulations has been investigated by several authors. Bertsimas et al. (2012) consider an inverse optimization in a robust optimization framework with the portfolio mean as the objective function and risk accounted for in constraints. The problem of uniqueness is not addressed in that paper, particularly because under their assumption of normality of asset returns the forward problem always has a unique solution. Due to the number of degrees of freedom (in the mean-variance case it is both the mean, variance and the target return \( \Delta \) that are to be inferred from the optimal portfolio), the inverse problem inherently has many solutions. Grechuk and Zabarankin (2014, 2016) attempt to infer risk preferences of an investor: assuming a complete knowledge of the distribution of \( R \) and portfolio \( x^* \), they look for a risk measure \( \rho \) for which \( x^* \) is an optimal solution to (I). They solved this inverse problem for two classes of risk measures \( \rho \): deviation measures and coherent risk measures.

A motivation for analyzing the uniqueness of forward and inverse optimization problems stated above comes from the Black-Litterman asset allocation model, cf. Black and Litterman (1992) where the model is formulated and Litterman et al. (2004) for a more detailed presentation. In the classical Black-Litterman model, the risk is modeled by the variance. The inverse optimization, used to establish the equilibrium distribution, has a unique solution. The variance, however, is a poor measure of risk for non-Gaussian distributions. Rockafellar et al. (2006a) promote deviations measures which are rooted in coherent risk measures but are indifferent to the location parameter of the distribution (as the variance). The optimization problem (I) retains its convexity in \( x \), but the uniqueness of solutions to the forward and inverse problems has not been studied. A general theory of convex optimization implies that they depend on the interplay between the distribution of \( R \) and the risk measure \( \rho \). Our Steiner point approach can be used to identify a unique “special” solution to this problem as well.

In the context of asset management, many papers assume a finite (but possibly very large) number of scenarios for the future excess return \( R \), (which for example may correspond to historical time series of returns of the corresponding portfolio at the specified times in the past, see Grechuk and Zabarankin (2018)) and this is the case that we research in this paper. Although the questions of existence of optimal solutions has been solved, the problem of uniqueness for finite number of scenarios has not been analyzed carefully enough. We perform detailed analysis of that problem for arbitrary discrete scenarios and a class of deviation measures which we call “finitely generated risk measures”. It includes Conditional Value-at-Risk (CVaR), mixed CVaR and mean absolute deviation. In our approach we use the characterization of deviation measures by their risk envelopes introduced in Rockafellar et al. (2006b).

Our contributions are based on a new link between the uniqueness of an optimal portfolio \( x^* \) in (I) and the number of risk identifiers for the deviation measure \( \rho(R^T x^*) \). This has three consequences. Firstly, the portfolio optimization problem has a unique solution for any \( \mu \in \mathbb{R}^n \) that does not belong to a finite number of hyperplanes; therefore, for practical applications the uniqueness can be safely assumed. Secondly, a unique optimal portfolio corresponds to many risk identifiers and, consequently, there are many Pareto-optimal sharing arrangements in cooperative investment, which is obviously highly inconvenient in practice. It is also suprising as this possibility was only inferred from the general convexity theory and treated as an unlikely and inconvenient case that is not of prime importance, see Grechuk and Zabarankin (2017). The third consequence is related to the extension of the Black-Litterman model to arbitrary distributions and deviation measures (Palczewski and Palczewski, 2018). Analogously as in the classical model the first step of the extended model is to solve an inverse
portfolio problem in which for a market (or benchmark) portfolio $x^*$ one establishes an equilibrium mean return $\mu_{eq}$ that yields $x^*$ as an optimal solution, cf. Palczewski and Palczewski (2018, Section 4). We demonstrate that if $x^*$ is a unique optimal solution for a particular $\mu^*$ then the inverse problem has multiple solutions. Hence, the final investment recommendation coming out of Black-Litterman methodology is not unique. Our Steiner point approach is then used to select a unique recommendation.

The rest of the paper is organized as follows. Section 2 suggests a Steiner point approach for assigning a unique “extended gradient” of every convex function on $\mathbb{R}^N$ at every point. Section 3 applies this methodology for selecting the unique solution in the capital allocation and risk sharing problems. Section 4 formulates the portfolio optimization problem in the framework of deviation measures, defines portfolio risk generators and discusses the portfolio uniqueness problem in terms of portfolio risk generators. Section 5 formulates the cooperative investment problem and resolves the issue of non-uniqueness of its solution. Section 6 discusses the dichotomy between uniqueness of solutions of the forward and inverse optimization problems. Section 7 considers consequences of non-uniqueness for the Black-Litterman model for non-Gaussian distributions. Section 8 concludes the work.

2 Extended gradient of a convex function

Let $f : \mathbb{R}^n \to \mathbb{R}$ be an arbitrary (finite valued) convex function. It is known Rockafellar (1970, Theorem 23.1) that the one-sided limit

$$\phi_{f,Y}(X) = \lim_{\epsilon \to 0^+} \frac{f(Y + \epsilon X) - f(Y)}{\epsilon}$$

exists for every $X, Y \in \mathbb{R}^n$. Limit $\phi_{f,Y}(X)$ is called the directional derivative of $f$ at $Y$ with respect to $X$. We say that $f$ is (Gâteaux) differentiable at $Y \in \mathbb{R}^n$ if the (two-sided) limit $\lim_{\epsilon \to 0} \frac{f(Y + \epsilon X) - f(Y)}{\epsilon}$ exists for every $X \in \mathbb{R}^n$. In this case, $\phi_{f,Y}(X)$ is a linear functional in $X$, and can be represented as $\phi_{f,Y}(X) = Q^T X$ for some $Q \in \mathbb{R}^n$, which is usually denoted as $Q = \nabla f(Y)$ and called the gradient of $f$ at $Y$. It is known Rockafellar (1970, Theorem 25.5) that any finite-valued convex function $f$ on $\mathbb{R}^n$ is differentiable almost everywhere.

This section develops an axiomatic framework for “extending” the notion of a gradient in such a way that the “extended gradient” is defined for every convex function $f : \mathbb{R}^n \to \mathbb{R}$ at every point $Y \in \mathbb{R}^n$. In the later sections we will demonstrate that this “extended gradient” is useful in numerous applications, including capital allocation, risk sharing, cooperative investment, inverse portfolio problem, and the analysis of Black-Litterman model.

Let $\mathcal{F}$ be a set of all convex functions $f : \mathbb{R}^n \to \mathbb{R}$. Formally, we define extended gradient as a map $G : \mathcal{F} \times \mathbb{R}^n \to \mathbb{R}^n$, which assigns to every $f \in \mathcal{F}$ and $Y \in \mathbb{R}^n$ a vector $G_Y(f) \in \mathbb{R}^n$, such that the following properties hold:

\begin{enumerate}
  \item [(G1)] Additivity: $G_Y(f + g) = G_Y(f) + G_Y(g)$ for all $f, g \in \mathcal{F}$ and all $Y \in \mathbb{R}^n$;
  \item [(G2)] Rotation invariance: Let $f \in \mathcal{F}$ and $g(Y) = f(A Y)$, $Y \in \mathbb{R}^n$, where $A$ is an $n \times n$ rotation matrix, that is, matrix such that $A^T = A^{-1}$ and $\det(A) = 1$. Then
    $$G_Y(g) = A^{-1} G_{A Y}(f), \quad \forall Y \in \mathbb{R}^n.$$ 
\end{enumerate}
(G3) Continuity: Let $Y \in \mathbb{R}^n$, $f \in \mathcal{F}$, and $f_1, f_2, \ldots$ be a sequence of functions in $\mathcal{F}$ such that
\[
\lim_{m \to \infty} \phi_{f_m,Y}(X) = \phi_{f,Y}(X) \quad \text{for all } X \in \mathbb{R}^n.
\]
Then
\[
\lim_{m \to \infty} G_Y(f_m) = G_Y(f).
\]

(G4) Linear differentiation: Let $Q \in \mathbb{R}^n$, and let $f(Y) = Q^T Y$, $\forall Y \in \mathbb{R}^n$ be a linear function. Then
\[
G_Y(f) = Q, \quad \forall Y \in \mathbb{R}^n.
\]

Properties (G1)-(G4) are desirable properties for any extension of the concepts of “derivative” or “gradient”. (G1) states that the derivative/gradient is a local property of a function at a point, and two functions which “look locally almost the same” in every direction should have “almost identical” gradients. Finally, (G4) states that the derivative/gradient of a linear function is a constant. Theorem 2.2 below states that, somewhat surprisingly, these natural properties are sufficient for the unique characterization of $G$.

Directional derivative $\phi_{f,Y}(X)$ can be represented in the form
\[
\phi_{f,Y}(X) = \sup_{Q \in \partial f(Y)} Q^T X
\]
see [Rockafellar (1970)], where $\partial f(Y)$ is called subdifferential of $f$ at $Y$, and is defined as a set of all $Q \in \mathbb{R}^n$ such that $f(X) \geq f(Y) + Q^T (X - Y)$, $\forall X \in \mathbb{R}^n$. Set $\partial f(Y)$ is always non-empty, convex, and compact, see [Rockafellar (1970), Theorem 23.4]. Let $\mathcal{K}$ be the family of all non-empty convex compact subsets of $\mathbb{R}^n$.

With $f_1 = f_2 = \cdots = f_m = \cdots = g$, property (G3) implies that $G_Y(f) = G_Y(g)$ whenever $\phi_{f,Y}(X) = \phi_{g,Y}(X)$ for all $X \in \mathbb{R}^n$. Equivalently, $G_Y(f) = G_Y(g)$ whenever $\partial f(Y) = \partial g(Y)$. Hence $G_Y(f)$ can be represented as
\[
G_Y(f) = S(\partial f(Y)),
\]
where $S$ is a map assigning to every set $K \in \mathcal{K}$ a vector $S(K) \in \mathbb{R}^n$.

Properties (G1)-(G4) of $G_Y(f)$ can be equivalently written as properties of the map $S$. For any $K_1, K_2 \subset \mathbb{R}^n$, the set $K_1 + K_2 = \{Q_1 + Q_2 | Q_1 \in K_1, Q_2 \in K_2\}$ is called (Minkowski) sum of $K_1$ and $K_2$. Theorem 23.8 in [Rockafellar (1970)] implies that $\partial ((f + g)(Y)) = \partial f(Y) + \partial g(Y)$ for all $f, g \in \mathcal{F}$ and all $Y \in \mathbb{R}^n$. Hence, property (G1) is equivalent to
\[
(S1) \quad S(K_1 + K_2) = S(K_1) + S(K_2) \quad \text{for all } K_1 \in \mathcal{K}, K_2 \in \mathcal{K}.
\]

Property (G4) is equivalent to $S(\{Q\}) = Q$. Substituting this into (S1), we get $S(K + Q) = S(K) + Q$. In other words, if the set $K$ is translated by a vector $Q \in \mathbb{R}^n$, $S(K)$ is translated by the same vector.

Let $A, f$, and $g$ be as defined in (G2). Theorem 23.9 in [Rockafellar (1970)] implies that $\partial g(Y) = A^{-1} \partial f(AY)$. Hence, property (G2) is equivalent to $S(AK) = AS(K), \forall K \in \mathcal{K}$. This implies that $S(AK + Q) = AS(K) + Q$ for all $Q \in \mathbb{R}^n$, or, equivalently,
\[
(S2) \quad S(TK) = TS(K) \quad \text{for all } K \in \mathcal{K} \text{ and all transformations } T : \mathbb{R}^n \to \mathbb{R}^n \text{ in the form } T(X) = AX + Q, \text{ where } A \text{ is a rotation matrix, and } Q \in \mathbb{R}^n. \text{ Such transformations } T \text{ are called proper motions.}
For every non-empty closed convex set $K$ in $\mathbb{R}^n$, its support function is given by $f_K(X) = \sup\{Q^T X | Q \in K\}$. In particular, (3) implies that directional derivative $\phi_{f,Y}(X)$ is a support function of the subdifferential $\partial f(Y)$. For sets $K, K_1, K_2, \ldots$ in $\mathcal{K}$, a combination of Corollary P4.1 and Corollary 3A in Salinetti and Wets (1979) implies that point-wise convergence of the support functions of $K_m$ to the support functions of $K$ is equivalent to $\lim_{m \to \infty} h(K_m, K) = 0$, where $h$ denotes the Hausdorff distance between sets. This implies the following reformulation of property (G3).

**Lemma 2.1.** Let $S : \mathcal{K} \to \mathbb{R}^n$, and let $G_Y$ be given by (4). Then $G_Y$ satisfies (G3) if and only if $S$ satisfies

(S3) Map $S$ is continuous with respect to the Hausdorff metric. That is,

$$\lim_{m \to \infty} S(K_m) = S(K)$$

whenever sets $K, K_1, K_2, \ldots \in \mathcal{K}$ are such that $\lim_{m \to \infty} h(K_m, K) = 0$.

**Proof.** First, assume that (S3) holds, $Y \in \mathbb{R}^n$, $f \in \mathcal{F}$, and $f_m$ is a sequence of functions as in (G3). Let $K = \partial f(Y)$, $K_m = \partial f_m(Y)$, and $\phi_{f,Y}(X)$ and $\phi_{f_m,Y}(X)$ are the support functions of $K$ and $K_m$, respectively, and condition $\lim_{m \to \infty} \phi_{f_m,Y}(X) = \phi_{f,Y}(X)$ implies that $\lim_{m \to \infty} h(K_m, K) = 0$. Then, by (S3), $\lim_{m \to \infty} S(K_m) = S(K)$, which, by (4), is translated to $\lim_{m \to \infty} G_Y(f_m) = G_Y(f)$ and proves (G3).

Conversely, assume that (G3) holds and let $K, K_1, K_2, \ldots \in \mathcal{K}$ be such that $\lim_{m \to \infty} h(K_m, K) = 0$. Let $f_0, f_1, f_2, \ldots$ be the support functions of these sets. Then $\lim_{m \to \infty} f_m = f_0$ point-wise. Because each $f_m$ is positively homogeneous, its directional derivative at $Y = 0$ is $\phi_m(X) = \lim_{\epsilon \to 0^+} \frac{f_m(\epsilon X) - f_m(0)}{\epsilon} = f_m(X)$. Hence, $\lim_{m \to \infty} \phi_m = \phi_0$ point-wise, and (G3) with $Y = 0$ implies that $\lim_{m \to \infty} G_0(f_m) = G_0(f)$. With (4), this translates to $\lim_{m \to \infty} S(K_m) = S(K)$ and proves (S3).

In summary, $G_Y(f)$ satisfies properties (G1)-(G4) if and only if it is representable in the form (4), with a map $S : \mathcal{K} \to \mathbb{R}^n$ satisfying (S1)-(S3). However, Theorem 1 in Schneider (1971) states that, in any dimension $n \geq 2$, there is a unique map $S$ satisfying (S1)-(S3), and it is given by

$$S(K) = \frac{n}{|S^{n-1}|} \int_{S^{n-1}} X f_K(X) dX,$$

where $S^{n-1} = \{X \in \mathbb{R}^n | \|X\| = 1\}$ denotes the unit sphere in $\mathbb{R}^n$, $|S^{n-1}|$ is its surface area, and $f_K(X)$ is the support function of $K$. $S(K)$ is known as the Steiner point of the set $K$. Equivalently (see e.g. Dentcheva (1998)),

$$S(K) = \frac{1}{|B_1|} \int_{B_1} \nabla f_K(X) dX,$$

where $B_1 = \{X \in \mathbb{R}^n | \|X\| \leq 1\}$ denotes the unit ball, $\nabla$ is the gradient, and the integral is well-defined because the support function of any $K \in \mathcal{K}$ is differentiable almost everywhere. If $K = \partial f(Y)$, its support function is $\phi_{f,Y}(X)$, and we obtain

$$G_Y(f) = S(\partial f(Y)) = \frac{1}{|B_1|} \int_{B_1} \nabla \phi_{f,Y}(X) dX.$$

We will summarise the above discussion in the following theorem.

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1The Hausdorff distance $h(K, L)$ between any subsets $K$ and $L$ of $\mathbb{R}^n$ is defined as $h(K, L) = \max\{\sup_{X \in K} \inf_{Y \in L} d(X, Y), \sup_{Y \in L} \inf_{X \in K} d(X, Y)\}$, where $d(\ldots)$ is the usual Euclidean distance in $\mathbb{R}^n$. 

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Theorem 2.2. In any dimension \( n \geq 2 \), the extended gradient \( G_Y(f) \) is uniquely characterized by properties (G1)-(G4), and it is given by (7), where \( \phi_{f,Y}(X) \) is defined in (2).

The next theorem provides an alternative formula for extended gradient \( G_Y(f) \).

Theorem 2.3. For every convex function \( f : \mathbb{R}^n \to \mathbb{R} \), and every \( Y \in \mathbb{R}^n \), the extended gradient \( G_Y(f) \) is given by

\[
G_Y(f) = \lim_{\epsilon \to 0^+} \frac{1}{|B_\epsilon(Y)|} \int_{B_\epsilon(Y)} \nabla f(X) dX,
\]

where \( B_\epsilon(Y) = \{ X \in \mathbb{R}^n \mid ||X - Y|| \leq \epsilon \} \) is the ball centred at \( Y \) with radius \( \epsilon \), \( \nabla \) is the (almost everywhere defined) gradient of \( f \), and the limit is guaranteed to exist.

Proof. Rockafellar (1970, Theorem 24.6) states that, for a sequence \( X_1, X_2, \ldots \in \mathbb{R}^n \) converging to \( Y \) but distinct from \( Y \) such that \( \lim_{i \to \infty} \frac{X_i - Y}{||X_i - Y||} = Z \in S^{n-1} \), and any \( \delta > 0 \), there exist an index \( i_0 \) such that

\[
\partial f(X_i) \subset \partial f(Y)_Z + B_\delta(0), \quad \forall i \geq i_0,
\]

where \( \partial f(Y)_Z \) is the set of points of \( \partial f(Y) \) in which \( Z \) is normal to \( \partial f(Y) \). In other words,

\[
\partial f(Y)_Z = \arg \max_{Q \in \partial f(Y)} Q^T Z.
\]

Let \( Z \in S^{n-1} \) be such that \( \partial f(Y)_Z \) is a singleton (which we denote \( g_{f,Y}(Z) \)), and \( \partial f(Y + \epsilon Z) = \{ \nabla f(Y + \epsilon Z) \} \) is a singleton for almost all sufficiently small \( \epsilon > 0 \). By Rockafellar (1970, Theorem 25.5), these conditions hold for almost every \( Z \in S^{n-1} \) (as \( g(Z) := \max_{Q \in \partial f(Y)} Q^T Z \) is convex and \( \partial f(Y)_Z = \partial g(Z) \)). In this case, (9) with \( X_i = Y + \epsilon_i Z, \epsilon_i \to 0^+ \), implies that the limit \( \lim_{\epsilon \to 0^+} \nabla f(Y + \epsilon Z) \) exists and equals to \( g_{f,Y}(Z) \).

Because the definition of \( g_{f,Y}(Z) \) depends only on \( Z \) and \( \partial f(Y) \), vector \( g_{f,Y}(Z) \) remains unchanged if we replace \( f \) by any other convex function \( h \) with \( \partial h(Y) = \partial f(Y) \). In particular, this is true for \( h(X) = \phi_{f,Y}(X - Y) \). Hence, by the same argument, \( \lim_{\epsilon \to 0^+} \nabla h(Y + \epsilon Z) = \lim_{\epsilon \to 0^+} \nabla \phi_{f,Y}(\epsilon Z) = g_{f,Y}(Z) \). However, (2) implies that \( \phi_{f,Y}(\epsilon Z) \) is a positive homogeneous function, that is, \( \phi_{f,Y}(\epsilon Z) = \epsilon \phi_{f,Y}(Z) \) for all \( \epsilon > 0 \). Hence, gradient \( \nabla \phi_{f,Y}(\epsilon Z) \) does not depend on \( \epsilon \), and in fact \( \nabla \phi_{f,Y}(\epsilon Z) = g_{f,Y}(Z) \) for all \( \epsilon > 0 \). In particular, \( \nabla \phi_{f,Y}(Z) = g_{f,Y}(Z) \).

Because \( \lim_{\epsilon \to 0^+} \nabla f(Y + \epsilon Z) = g_{f,Y}(Y) = \nabla \phi_{f,Y}(Y) \), for every \( \delta > 0 \), there exists an \( \epsilon(Z) > 0 \) such that \( ||\nabla f(Y + \epsilon Z) - \nabla \phi_{f,Y}(Z)|| < \delta \) for every \( \epsilon \leq \epsilon(Z) \). In fact, we can select \( \epsilon(Z) > 0 \), such that for \( \epsilon \leq \epsilon(Z) \), the inequality \( ||\nabla f(Y + \epsilon Z) - \nabla \phi_{f,Y}(Z)|| < \delta \) holds true for all \( Z \) from the unit sphere except of a set of measure at most \( \delta \). With \( X = Y + \epsilon Z, \nabla f(Y)(Z) = \nabla \phi_{f,Y}(\epsilon Z) = \nabla \phi_{f,Y}(X - Y), \) and the last inequality is equivalent to \( ||\nabla f(X) - \nabla \phi_{f,Y}(X - Y)|| < \delta \). Because \( ||\nabla f(X)|| \leq C, \forall X \in B_1(Y) \) for some constant \( C > 0 \) Rockafellar (1970, Theorem 24.7), this proves that

\[
\lim_{\epsilon \to 0^+} \frac{1}{|B_\epsilon(Y)|} \int_{B_\epsilon(Y)} (\nabla f(X) - \nabla \phi_{f,Y}(X - Y)) dX = 0.
\]

Since the gradient \( \nabla \phi_{f,Y}(\epsilon Z) \) does not depend on \( \epsilon \), we have

\[
\frac{1}{|B_\epsilon(Y)|} \int_{B_\epsilon(Y)} \nabla \phi_{f,Y}(X - Y) dX = \frac{1}{|B_1(Y)|} \int_{B_1(Y)} \nabla \phi_{f,Y}(X - Y) dX = \frac{1}{|B_1(0)|} \int_{B_1(0)} \nabla \phi_{f,Y}(X) dX = G_Y(f),
\]

which together with (10) completes the proof. \( \square \)
Theorem 2.3 gives a nice intuitive interpretation of $G_Y(f)$: it is an average gradient of $f$ in a small ball centred in $Y$ when the radius of the ball goes to 0. Because Steiner point of any set $K \in \mathcal{K}$ belongs to $K$, Theorem 2.3 implies that $G_Y(f) \in \partial f(Y)$, or, in words, extended gradient always belongs to the subdifferential set. In particular, $G_Y(f) = \nabla f(Y)$, whenever the latter exists. Theorem 2.2 implies that $G_Y(f)$ is the only way to define the gradient of every convex function $f$ at every point $Y$ such that the natural properties (G1)-(G4) are satisfied.

Remark 2.4. While Theorem 2.2 is applicable in dimension $n \geq 2$, extended gradient (7)-(8) can be studied in dimension $n = 1$ as well. For a convex function $f : \mathbb{R} \to \mathbb{R}$, and every $y \in \mathbb{R}$, $G_y(f)$ in (7)-(8) is given by

$$G_y(f) = \frac{f'_+(y) + f'_-(y)}{2},$$

where $f'_+(y)$ and $f'_-(y)$ are right and left derivatives of $f$ at $y$, respectively.

Alternative characterization of $G_Y(f)$ follows directly from Theorem 2.2.

**Definition 2.5.** We say that a map $h : \mathbb{R}^n \to \mathbb{R}^m$ is robust if, for all $Y \in \mathbb{R}^n$, the limit

$$\lim_{\epsilon \to 0^+} \frac{1}{|B_\epsilon(Y)|} \int_{B_\epsilon(Y)} h(X)dX$$

exists and is equal to $h(Y)$.

The term “robust” originates in the fact that if we “measure” $Y$ with an error $E$, uniformly distributed in a small ball, then $h(Y)$ is the expected value of $h$ evaluated at the point $Y + E$. Obviously, any continuous map is robust but the converse is not true. For example, take $n = m = 1$ and $f(x) = \text{sign}(x)$ (that is, $f(x) = 1$, $f(x) = 0$ and $f(x) = -1$ for $x > 0$, $x = 0$, and $x < 0$, respectively). Then $f(x)$ is discontinuous at 0 but it is robust. Theorem 2.2 together with almost everywhere differentiability of any convex function, implies that $G_Y(f)$, treated as an function of $Y$ for a fixed $f$, is the only map from $\mathbb{R}^n$ to $\mathbb{R}^n$, which is (i) robust and (ii) $G_Y(f) = \nabla f(Y)$ whenever the latter exists.

Multiplying both sides of (8) by $Z^T$ for any $Z \in \mathbb{R}^n$, we get

$$Z^TG_Y(f) = \lim_{\epsilon \to 0^+} \frac{1}{|B_\epsilon(Y)|} \int_{B_\epsilon(Y)} Z^T \nabla f(X)dX = \lim_{\epsilon \to 0^+} \frac{1}{|B_\epsilon(Y)|} \int_{B_\epsilon(Y)} \phi_{f,X}(Z)dX,$$

that is, $Z^TG_Y(f)$ is the average value of the directional derivative of $f$ in direction $Z$ in a small ball around $Y$. This fact provides another characterization of $G_Y(f)$.

### 3. An application to capital allocation and risk sharing

Assume that the probability space $\Omega$ is finite with $N = |\Omega|$ and $\mathbb{P}(\omega) > 0$ for any $\omega \in \Omega$. A random variable (r.v.) $X$ on $\Omega$ can be identified as a vector $X = (X_1, \ldots, X_N)$ in $\mathbb{R}^N$. We will, therefore, treat the space $V = L^2(\Omega)$ of all random variables as the space $\mathbb{R}^N$ with the Euclidean norm.

---

2In fact, the error may equivalently be normally distributed. This follows from the rotation invariance property of the multivariate standard normal distribution.
### 3.1 Capital allocation

Let r.v. $Y \in V$ represents a (random) profit of some portfolio, consisting of $m$ sub-portfolios, that is, $Y = \sum_{i=1}^{m} X_i$. Let $\rho : V \to \mathbb{R}$ be a function such that $\rho(X)$ represents the risk associated with any $X \in V$. The capital allocation problem is the problem of distributing risk capital $\rho(Y)$ among sub-portfolios, that is, assigning to sub-portfolio $i$ its risk contribution $k_i$ such that $\sum_{i=1}^{m} k_i = \rho(Y)$. The problem is to develop a system of “natural” axioms defining a unique capital allocation scheme.

By (12), the problem is to develop a system of “natural” axioms defining a unique capital allocation scheme.

Moreover, Theorem 4.3 in Kalkbrener (2005) guarantees that, for any positively homogeneous and sub-portfolios, that is, assigning to sub-portfolio $i$ its risk contribution $k_i$ such that $\sum_{i=1}^{m} k_i = \rho(Y)$. The problem is to develop a system of “natural” axioms defining a unique capital allocation scheme. Assume that there are $n$ agents, indexed by $i = \{1, 2, \ldots, n\}$. Each agent $i \in I$ has an initial endowment $Y_i \in V$, and an associated risk measure $\rho_i : V \to \mathbb{R}$. The agents aim to redistribute the total endowment $Y = \sum_{i=1}^{m} Y_i$ among themselves to reduce their risk. Agent $i \in I$ receives the part $X_i \in V$ of the total endowment such that $\sum_{i=1}^{m} X_i = Y$; the vector $\bar{X} = (X_1, X_2, \ldots, X_n)$ is called the risk allocation. A risk allocation $\bar{X}$ is called Pareto optimal if there is no risk allocation $\bar{Z} = (Z_1, Z_2, \ldots, Z_n)$ with $\rho_i(Z_i) \leq \rho_i(X_i)$, $i \in I$, with at least

\[
\Lambda_{\rho}(X, Y) = \rho(X) \leq \rho(Y) \quad \forall X, Y \in V.
\]

Condition (i) guarantees that $\sum_{i=1}^{m} k_i = \rho(Y)$, where $k_i = \Lambda_{\rho}(X_i, Y)$, and condition (ii) is called diversification, see Kalkbrener (2005) for further discussion and justification. Theorem 4.2 in Kalkbrener (2005) states that a linear diversifying capital allocation $\Lambda_{\rho}$ exists if and only if $\rho$ is

(i) positively homogeneous, that is, $\rho(\lambda X) = \lambda \rho(X)$ for all $\lambda \geq 0$ and $X \in V$, and

(ii) sub-additive, that is, $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for all $X, Y \in V$.

Moreover, Theorem 4.3 in Kalkbrener (2005) guarantees that, for any positively homogeneous and sub-additive $\rho$, a linear diversifying $\Lambda_{\rho}$ is unique if and only if $\rho$ is differentiable at $Y$; in this case $\Lambda_{\rho}(X, Y) = X^T \nabla \rho(Y)$.

Because any finite-valued convex function $\rho$ on $\mathbb{R}^N$ is differentiable almost everywhere by Rockafellar (1970, Theorem 25.5), this implies that linear diversifying capital allocation is unique for almost all $Y$.

However, Example 3 in Grechuk (2015) demonstrates that if the initial portfolio $Y$ is not “arbitrary” but is a result of natural risk-minimization policy, then it may happen that $Y$ is “forced” to belong to exactly the (measure zero) set in which $\rho$ is not differentiable. Hence, the problem of identifying a unique capital allocation scheme reduces to the problem of “extending” the notion of a gradient of a convex function $\rho : \mathbb{R}^n \to \mathbb{R}$, such that the “extended gradient” is defined at every point $Y \in \mathbb{R}^n$, that is, exactly to the problem we have addressed in Section 2. The resulting capital allocation is given by $\Lambda_{\rho}^*(X, Y) = X^T G_Y(\rho)$, where $G_Y(\rho)$ is the extended gradient of $\rho$ at $Y$, defined in Section 2.

In the terminology of Definition 2.5, the last property states that $\Lambda_{\rho}^*(X, Y)$ is robust in a second argument.

### 3.2 Risk sharing

Assume that there are $n$ agents, indexed by $I = \{1, 2, \ldots, n\}$. Each agent $i \in I$ has an initial endowment $Y_i \in V$, and an associated risk measure $\rho_i : V \to \mathbb{R}$.

The agents aim to redistribute the total endowment $Y = \sum_{i=1}^{m} Y_i$ among themselves to reduce their risk. Agent $i \in I$ receives the part $X_i \in V$ of the total endowment such that $\sum_{i=1}^{m} X_i = Y$; the vector $\bar{X} = (X_1, X_2, \ldots, X_n)$ is called the risk allocation. A risk allocation $\bar{X}$ is called Pareto optimal if there is no risk allocation $\bar{Z} = (Z_1, Z_2, \ldots, Z_n)$ with $\rho_i(Z_i) \leq \rho_i(X_i)$, $i \in I$, with at least
one inequality being strict. If the vector $\bar{Y}_i = (Y_1, Y_2, \ldots, Y_n)$ of all initial endowments is not Pareto optimal, it is beneficial for all agents to switch to a Pareto optimal one.

However, there are typically many Pareto optimal allocations, and the problem is how to choose a “fair” one among them? A natural approach is to allow agents to trade and select an allocation to which they arrive at an equilibrium. Consider a linear continuous functional $P : V \rightarrow \mathbb{R}$, such that $P(1) = 1$ and $P(X) \geq 0$ whenever $X \geq 0$, which we will call a price functional. For a fixed $P$, agent $i \in I$ solves the optimization problem

$$\min_{X_i \in V} \rho_i(X_i), \quad \text{s.t.} \quad P(X_i) \leq P(Y_i),$$

that is, tries to find an $X_i$ with the minimal risk she can buy in exchange of the initial endowment $Y_i$. If $X_i^\ast$, $i \in I$, are optimal solutions to (13), and $\sum_{i=1}^n X_i^\ast = Y$, then $X^\ast = (X_1^\ast, X_2^\ast, \ldots, X_n^\ast)$ is a risk allocation assigning to each agent her “optimal share”. Such risk allocation is called an equilibrium allocation, and the corresponding $P$ is called an equilibrium price. The First Welfare Theorem (Mas-Colell et al., 1995) states that, under some general conditions on risk measures, every equilibrium allocation $X^\ast$ is Pareto optimal, and such an allocation is considered to be a “natural” and “fair” choice from a Pareto optimal set. If an equilibrium allocation $X^\ast$ is unique, this solves the problem completely. However, if it is not unique, which one to choose?

If risk measures $\rho_i$ are cash-invariant, that is, $\rho_i(X + C) = \rho_i(X) - c$ for every $X \in V$ and a constant $C = (c, c, \ldots, c)$, then risk allocation $\bar{X} = (X_1, X_2, \ldots, X_n)$ is Pareto optimal if and only if it minimizes the total risk $\sum_{i=1}^n \rho_i(X_i)$ over all possible risk allocations. If, moreover, all $\rho_i$ are also positively homogeneous and sub-additive, then so is the functional

$$\rho^\ast(Y) = \inf_{\bar{X} : \sum_{i=1}^n X_i = Y} \sum_{i=1}^n \rho_i(X_i),$$

mapping the total endowment $Y$ to the corresponding total risk. In this case, it is known (see Filipović and Kupper (2008)) that equilibrium prices $P$ correspond to the elements of the sub-gradient set $\partial \rho^\ast(Y)$ of the convex function $\rho^\ast$ at $Y$. Hence, the choice of unique equilibrium allocation reduces to the problem we have solved in Section 2. Specifically, we suggest that the equilibrium price corresponding to the Steiner point (5)–(6) of $\partial \rho^\ast(Y)$ should be selected.

In the context of risk sharing, however, our contribution has only theoretical importance, because in practice we expect $\partial \rho^\ast(Y)$ to be a singleton due to differentiability of convex function $\rho^\ast$ almost everywhere.

4 Mean-deviation portfolio optimization

4.1 Finitely generated deviation measures

As in the previous section, assume that the probability space $\Omega$ is finite with $N = |\Omega|$ and $\mathbb{P}(\omega) > 0$ for any $\omega \in \Omega$. A finite probability space $\Omega$ will be called uniform, if $\mathbb{P}[\omega_1] = \cdots = \mathbb{P}[\omega_N] = \frac{1}{N}$.

Let $R^{(i)}$, $i = 1, \ldots, n$, be random variables denoting the rates or return of financial instruments. We assume that there exists also a risk-free instrument with a constant rate of return $r_0$. Following Rockafellar et al. (2006b), we also assume that

(M) any portfolio $X = \sum_{i=1}^n x_i R^{(i)}$ is a non-constant random variable for any non-zero $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. 

10
Rockafellar et al. (2006b) formulated portfolio optimization problem as follows

$$\min_{(x_0, x_1, \ldots, x_n)} D \left( \sum_{i=0}^{n} x_i R(i) \right), \quad \text{s.t.} \quad \sum_{i=0}^{n} x_i = 1, \quad \sum_{i=0}^{n} x_i E[R(i)] \geq r_0 + \Delta,$$  \hspace{1cm} (14)

where $\Delta > 0$ and $D$ is a general deviation measure, that is, a functional $D : \mathcal{L}^2(\Omega) \to [0; \infty]$ satisfying:

(D1) $D(X) = 0$ for constant $X$, but $D(X) > 0$ otherwise (non-negativity),

(D2) $D(\lambda X) = \lambda D(X)$ for all $X$ and all $\lambda > 0$ (positive homogeneity),

(D3) $D(X + Y) \leq D(X) + D(Y)$ for all $X$ and $Y$ (subadditivity),

(D4) set $\{X \in \mathcal{L}^2(\Omega) \mid D(X) \leq C\}$ is closed for all $C < \infty$ (lower semicontinuity).

With centered rates of return $\hat{R}(i) = R(i) - E[R(i)]$, $i = 1, \ldots, n$, and $\mu_i = E[R(i)] - r_0, i = 1, \ldots, n$, problem (14) can be reformulated as

$$\min_{x \in \mathbb{R}^n} D(\hat{R}^T x), \quad \text{s.t.} \quad \mu^T x \geq \Delta,$$  \hspace{1cm} (15)

where $\hat{R} = (\hat{R}^{(1)}, \ldots, \hat{R}^{(n)})^T, x = (x_1, \ldots, x_n)^T$, and $\mu = (\mu_1, \ldots, \mu_n)^T$. We chose to use a distinct notation $D$ to indicate this particular choice of the risk measure $\rho$ in (1) as this will be the default setting for the rest of the paper.

By Rockafellar et al. (2006a, Theorem 1), every deviation measure $D$ can be represented in the form

$$D(X) = EX + \sup_{Q \in \mathcal{Q}} E[-XQ],$$  \hspace{1cm} (16)

where $\mathcal{Q} \subset \mathcal{L}^2(\Omega)$ is called risk envelope and can be recovered from $D$ by

$$\mathcal{Q} = \{Q \in \mathcal{L}^2(\Omega) \mid E[X(1 - Q)] \leq D(X) \ \forall X \in \mathcal{L}^2(\Omega)\}.$$  \hspace{1cm} (17)

Moreover, the set $\mathcal{Q}$ is closed and convex in $\mathcal{L}^2(\Omega)$. Elements $Q \in \mathcal{Q}$ for which supremum in (16) is attained are called risk identifiers of $X$. The set of all risk identifiers of $X$ is denoted $\mathcal{Q}(X)$.

A deviation measure $D$ is finite, that is, $D(X) < \infty, \forall X$ if and only if the corresponding $\mathcal{Q}$ is bounded. In this case, $\mathcal{Q}(X)$ is non-empty for every $X \in \mathcal{L}^2(\Omega)$, and, due to closeness and convexity of $\mathcal{Q}$ and linearity of $Q \mapsto E[-XQ]$, every set $\mathcal{Q}(X)$ must contain at least one extreme point of $\mathcal{Q}$. Therefore, $\sup_{Q \in \mathcal{Q}} E[-XQ] = \max_{Q \in \mathcal{Q}^e} E[-XQ]$, where $\mathcal{Q}^e$ is the set of all extreme points of $\mathcal{Q}$. In fact, a bounded closed convex $\mathcal{Q}$ is the closed convex hull of $\mathcal{Q}^e$, see Theorem 2 in Phelps (1974).

Of particular importance to this paper will be the set of such risk measures for which the set $\mathcal{Q}^e$ is finite:

**Definition 4.1.** A finite deviation measure $D$ is called finitely generated if the set $\mathcal{Q}^e$ of all extreme points of $\mathcal{Q}$ is finite. We will call elements of this set extreme risk generators.

In other words, $D$ is finitely generated if and only if $\mathcal{Q}$ is a convex hull of a finite number of points.

\[^3\] Because $\mathcal{L}^2(\Omega)$ is a reflexive Banach space, it has the Radon-Nikodym property, and Theorem 2 in Phelps (1974) applies.
Example 1. For standard deviation, $\sigma(X) = ||X - E[X]||_2$, the risk envelope is given by \cite{Rockafellar06b} (Example 1)

$$Q = \{ Q \mid E[Q] = 1, \ \sigma(Q) \leq 1 \},$$

and, for $N > 2$, has infinitely many extreme points, hence $\sigma$ is not finitely generated.

Example 2. For mean absolute deviation, $\text{MAD}(X) = E[|X - E[X]|]$, the risk envelope is given by \cite{Rockafellar06b} (Example 2)

$$Q = \{ Q \mid E[Q] = 1, \ \sup Q - \inf Q \leq 2 \},$$

which is a convex polytope in $\mathbb{R}^N$ with a finite number of vertices. Hence $\text{MAD}$ is finitely generated.

Example 3. For CVaR-deviation

$$\text{CVaR}_\alpha(X) \equiv E[X] - \frac{1}{\alpha} \int_0^\alpha q_{X}(\beta) \, d\beta,$$

(18)

the risk envelope is given by \cite{Rockafellar06b} (Example 4)

$$Q_\alpha = \{ Q \mid E[Q] = 1, \ 0 \leq Q \leq \alpha^{-1} \}.$$

The linearity of constraints imply that $\text{CVaR}_\alpha$ is finitely generated. In particular, if the probability is uniform over $\Omega$ and $\alpha = k/N$ for some integer $1 \leq k < N$, extreme points $Q^e$ are

$$Q^e = \{ x \in \mathbb{R}^N \mid \exists S \subset \{1,2,\ldots,N\} : |S| = k, \ x_i = \frac{N}{k}, \ i \in S; \ x_i = 0, \ i \notin S \},$$

where the subset $S$ is taken non-empty and proper. Hence, $|Q^e| = 2^N - 2$.

Lemma 4.2. Let $D_1, D_2, \ldots, D_m$ be finitely generated deviation measures. Then functionals

(a) $D(X) = \sum_{i=1}^m \lambda_i D_i(X)$, with $\lambda_i > 0$, $i = 1, \ldots, m$;

(b) $D(X) = \max\{D_1(X), \ldots, D_m(X)\}$

are also finitely generated deviation measures.

Proof. Proof follows from \cite{Rockafellar06a} (Proposition 4), and from the fact that if sets $Q_1, Q_2, \ldots, Q_m$ are all convex hulls of a finite number of points, then so are the sets: $\lambda_1 Q_1 + \cdots + \lambda_m Q_m$, the convex hull of $Q_1 \cup \cdots \cup Q_m$; and $\{ Q \mid Q = (1 - \lambda)Q_1 + \lambda Q_i \text{ for some } Q_i \in Q_i \}, \lambda > 0, i = 1, \ldots, m$.\hfill$\square$
Example 4. Mixed CVaR-deviation

\[
\text{CVaR}_\lambda^\Delta(X) = \int_0^1 \text{CVaR}^{\Delta}_{\lambda_i}(X) \lambda(d\alpha),
\]  

(19)

where \( \lambda \) is a probability measure on \((0, 1)\), is also finitely generated. Indeed, because the probability space is finite, mixed CVaR-deviation (19) can be written as a finite mixture of CVaR-deviations

\[
\text{CVaR}_\lambda^\Delta(X) = \sum_{i=1}^m \lambda_i \text{CVaR}^{\Delta}_{\alpha_i}(X),
\]

where \( \alpha_i \in (0, 1) \), \( \lambda_i > 0 \), \( i = 1, \ldots, m \), and \( \sum_{i=1}^m \lambda_i = 1 \), which is a finitely generated deviation measure due to Example 3 and Lemma 4.2(a). \[ \square \]

4.2 Optimal portfolios and active portfolio risk generators

We make the following standing assumptions:

(A) The deviation measure \( D \) is finitely generated.

(B) \( \Delta > 0 \) and \( \mu \neq 0 \).

The latter assumption implies the following properties of the optimal solution to (15).

Lemma 4.3. The optimal objective value in (15) is positive and in optimum the constraint is binding: \( \mu^T x = \Delta \).

Proof. By Theorem 1 in Rockafellar et al. (2006b), there is an optimal solution \( x^* \). By assumption (B), \( x = 0 \) does not satisfy the constraint on the expected return, and so \( x^* \neq 0 \). Due to assumption (M), we conclude that \( \hat{R}^T x^* \) is random and hence \( D(\hat{R}^T x^*) > 0 \). For the second part of the statement, assume that \( \mu^T x^* > \Delta \). Therefore, there is \( \eta < 1 \) such that \( \mu^T (\eta x^*) \geq \Delta \) and we have \( D(\hat{R}^T (\eta x^*)) = \eta D(\hat{R}^T x^*) < D(\hat{R}^T x^*) \), a contradiction. \[ \square \]

Since \( D \) is finitely generated, the deviation measure of a centered return of portfolio \( x \in \mathbb{R}^n \) can be expressed as a maximum of a finite number of terms:

\[
D(\hat{R}^T x) = \max_{Q \in \mathcal{Q}^n} E[-\hat{R}^T x Q].
\]

(20)

As the number of extreme risk generators for \( D \) is finite, they can be enumerated: \( \mathcal{Q}^n = \{Q_1, \ldots, Q_{M'}\} \). Define \( \tilde{D}_i = E[-\hat{R}Q_i] \), \( i = 1, \ldots, M' \). It follows from (20) that the set of \( \tilde{D}_i \)'s is sufficient to evaluate \( D(\hat{R}^T x) \) for a portfolio \( x \):

\[
D(\hat{R}^T x) = \max_{i=1,\ldots,M'} \tilde{D}_i^T x.
\]

(21)

It may happen that \( \tilde{D}_i = \tilde{D}_j \) for some \( i \neq j \); for example, \( \hat{R} \) may be constant on a number of elementary events in \( \Omega \). It may also happen that \( \tilde{D}_i \) is not an extreme point of \( \text{conv}\{\tilde{D}_1, \ldots, \tilde{D}_{M'}\} \), but a subset of \( \tilde{D}_i \)'s forms all extreme points of this set (Rockafellar, 1970, Theorem IV.19.3). For the convenience of future arguments, we choose only those vectors \( \tilde{D}_i \) that are extreme points.

Definition 4.4. Extreme points of \( \text{conv}\{\tilde{D}_1, \ldots, \tilde{D}_{M'}\} \) are denoted by \( \tilde{D}_i, i = 1, \ldots, M \), and called portfolio risk generators.
Remark 4.5. Portfolio risk generators are generators (in the sense of Rockafellar [1970, Section 19]) of the polyhedral set \( \{ E[-RQ] \mid Q \in \mathcal{Q} \} \), see the proof of Theorem 19.3 in Rockafellar [1970].

By Definition 4.4 and (21), it is easy to see that for any portfolio \( x \):

\[
\mathcal{D}(\hat{R}^T x) = \max_{i=1, \ldots, M} D_i^T x.
\] (22)

Definition 4.6. Those \( D_i \) that realize the maximum in (22) are called active portfolio risk generators for the portfolio \( x \).

The following lemma shows that the set of portfolio risk generators is sufficiently rich to span the whole space \( \mathbb{R}^n \).

Lemma 4.7. \( \text{lin}(D_1, \ldots, D_M) = \mathbb{R}^n \).

Proof. Assume the opposite and take any non-zero vector \( x \) in the orthogonal complement of \( \text{lin}(D_1, \ldots, D_M) \). Then \( \mathcal{D}(\hat{R}^T x) = 0 \). However, \( \hat{R}^T x \) is non-constant by assumption (M), so its deviation measure should be strictly positive by (D1). A contradiction. \(\square\)

The representation (22) of the deviation measure of a portfolio \( x \) enables an equivalent formulation of optimization problem (15) as a linear program:

\[
\begin{align*}
\text{minimize } & A, \\
\text{subject to: } & A \geq D_i^T x, \quad i = 1, \ldots, M, \\
& \mu^T x \geq \Delta, \\
& (A, x) \in \mathbb{R} \times \mathbb{R}^n.
\end{align*}
\] (23)

The solution \((A^*, x^*)\) is related to (15) as follows: \( x^* \) is the optimal portfolio while \( A^* = \mathcal{D}(\hat{R}^T x^*) \).

Theorem 4.8. The linear program (23) as well as the optimization problem (15) have the following properties:

1. The set of optimal portfolios \( X^* \) is a bounded polyhedral subset of \( \mathbb{R}^n \). The set of solutions to (23) is of the form \( \{ A^* \} \times X^* \) for some \( A^* > 0 \).
2. If the solution is not unique then \( \mu \) is a linear combination of at most \( n-1 \) portfolio risk generators.
3. If the solution is unique, then the set of active portfolio risk generators spans the whole space \( \mathbb{R}^n \), i.e., there are \( n \) linearly independent active portfolio risk generators.

Proof. (23) is a linear program, so the set of solutions is polyhedral. The mapping \( x \mapsto \mathcal{D}(\hat{R}^T x) \) is convex, hence also continuous. Denote by \( d \) its minimum on the sphere \( \{ x \in \mathbb{R}^n \mid \|x\| = 1 \} \). This minimum is strictly positive due to assumptions (M) and (D1). Employing further assumption (D2) gives that \( \{ x \in \mathbb{R}^n \mid \mathcal{D}(\hat{R}^T x) \leq A \} \) is bounded for any \( A > 0 \); indeed, it is contained in the ball with radius \( A/d \). Hence, the set of solutions \( X' \) to (23) is a bounded polyhedral set. It is expressed by convex combinations of its extreme points at which the objective function is optimal. In each such extreme point the coordinate \( A \) is identical, so \( X' = \{ A^* \} \times X^* \) for some \( A^* > 0 \); the positivity of \( A^* \) follows from Lemma 4.3.

If \( X' \) is a single point, then it is an extreme point. Since the constraint \( \mu^T x \geq \Delta \) is active (see Lemma 4.3), Bertsimas and Tsitsiklis [1997, Theorem 2.2] implies that there are \( n \) indeces \( i_1, \ldots, i_n \).
such that $A = D^T_{ij}x$, $j = 1, \ldots, M$, and vectors $(D_{ij})_{j=1}^n$ are linearly independent, hence generate $\mathbb{R}^n$.

The proof of assertion 2 uses the dual of problem (23):

$$\begin{align*}
\text{maximize} & \quad \Delta q, \\
\text{subject to:} & \quad \sum_{i=1}^M p_i D_i - q\mu = 0, \\
& \quad \sum_{i=1}^M p_i = 1, \\
& \quad q \geq 0, \quad p_i \geq 0, \quad i = 1, \ldots, M.
\end{align*}$$

(24)

By the strong duality, $\Delta q = A^*$ and we know $A^* > 0$, hence $q > 0$. Theorem 4.5) implies that the dual variables corresponding to inactive constraints are zero. Denote by $i_1, \ldots, i_k$ the active constraints involving portfolio risk generators. Then the first constraint in the above dual problem (24) reads:

$$\mu = \frac{1}{q} \sum_{j=1}^k p_{i_j} D_{i_j}.$$  

(25)

Assume now that the solution is not unique, i.e., $\mathcal{X}'$ contains at least two extreme points and therefore a line connecting them. Fix an internal point of that line $(A^*, x^*)$. Since $(A^*, x^*)$ is not an extreme point of $\mathcal{X}'$, the linear space spanned by active portfolio risk generators $D_{i_j}, j = 1, \ldots, k$, has dimension not larger than $n - 1$ (there is at least one portfolio risk generator which is active at extreme point of $\mathcal{X}'$ and does not belong to $\text{lin}\{D_{i_1}, \ldots, D_{i_k}\}$). This proves assertion 2 of the theorem.

**Corollary 4.9.** There is a finite number of hyperplanes (of dimensions from 1 to $n - 1$) such that: $\mu$ belongs to one of them if and only if a solution to (15) is not unique. Therefore, the set of $\mu$ for which the portfolio optimization problem has a unique solution has a full Lebesgue measure.

**Proof.** By Theorem 4.8 non-uniqueness of solutions coincides with $\mu$ being a linear combination of at most $n - 1$ portfolio risk generators, i.e., belongs to a linear space spanned by at most $n - 1$ vectors in $\mathbb{R}^n$. This is a hyperplane of dimension at most $n - 1$, so it has a Lebesgue measure 0. There is a finite number of ways to choose up to $n - 1$ vectors from the set of $M$ vectors, so the number of such hyperplanes is finite. A finite sum of sets of Lebesgue measure zero has the measure zero. Its complement has therefore a full measure.

A practical consequence of the above theorem and corollary is that there is a unique optimal portfolio in (15) unless $\mu$ is specially chosen to match the distribution of returns $\hat{R}$ and the risk measure. In the following section we will show that the uniqueness of a solution, which implies multiple active portfolio risk generators, leads to problems with optimal cooperative investment. We will also show that there are natural settings when $\mu$ happens to be on one of the hyperplanes mentioned in the corollary.

5 Cooperative investment

5.1 Theoretical framework

The general problem of cooperative investment can be formulated as follows, see Grechuk and Zabarankin (2017). Let $\mathcal{F} \subset L^2(\Omega)$ be a feasible set, representing rates of return from feasible investment oppor-
tunities on the market without a riskless asset:

\[ \mathcal{F} = \left\{ X \mid X = \sum_{i=1}^{n} R^{(i)} x_i, \sum_{i=1}^{n} x_i = 1 \right\}. \]

An individual portfolio optimization problem for agent \( i, i = 1, \ldots, m \), is

\[
\max_{X \in \mathcal{F}} U_i(X),
\]

where \( U_i : \mathcal{L}^2(\Omega) \to [-\infty, \infty) \) is the utility function of agent \( i \). Instead of investing individually, \( m \) agents can buy a joint portfolio \( X \in \mathcal{F} \) and distribute it so that agent \( i \) receives a share \( Y_i \) with \( \sum Y_i = X \). An allocation \( Y = (Y_1, \ldots, Y_m) \) is called feasible if \( \sum Y_i \in \mathcal{F} \), and Pareto optimal if there is no feasible allocation \( Z = (Z_1, \ldots, Z_m) \) such that \( U_i(Y_i) \leq U_i(Z_i) \) with at least one inequality being strict.

A utility function \( U \) is called cash-invariant if \( U(X + C) = U(X) + C \) for all \( X \in \mathcal{L}^2(\Omega) \) and \( C \in \mathbb{R} \). Proposition 2 in Grechuk and Zabarankin (2017) implies that if all \( U_i, i = 1, \ldots, m \), are cash-invariant, and \( Y = (Y_1, \ldots, Y_m) \) is Pareto optimal, then \( X^* = \sum Y_i \) must solve the optimization problem

\[
\sup_{X \in \mathcal{F}} U^*(X),
\]

where

\[
U^*(X) \equiv \sup_{Z \in A(X)} \sum_{i=1}^{m} U_i(Z_i)
\]

with \( A(X) = \{ Z = (Z_1, \ldots, Z_m) : \sum_{i=1}^{m} Z_i = X, Z_i \in \mathcal{L}^2(\Omega) \} \). Furthermore, if \( Y = (Y_1, \ldots, Y_m) \) is any Pareto optimal allocation, then all Pareto optimal allocations are given by

\[
(Y_1 + C_1, \ldots, Y_m + C_m),
\]

where \( C_1, \ldots, C_m \) are constants with \( \sum_{i=1}^{m} C_i = 0 \). Hence, the coalition should (i) solve the portfolio optimization problem \((27)\) to find an optimal portfolio \( X^* \) for the whole group; (ii) find any Pareto optimal way \( Y = (Y_1, \ldots, Y_m) \) to distribute \( X \) among group members, and finally (iii) agree on constants \( C_1, \ldots, C_m \) in \((29)\) to select a specific Pareto-optimal allocation among the ones available.

Following risk sharing ideas (c.f. Section 3.2), Grechuk and Zabarankin (2017) suggested an “equilibrium” approach to (iii), resulting in a “fair” allocation \( C_i = P(-Y_i) \), where \( P : \mathcal{L}^2(\Omega) \to \mathbb{R} \) is a linear functional such that (a) \( P \in \partial U^*(X^*) \) and (b) \( P(X) \leq P(X^*) \) for all \( X \in \mathcal{F} \). Proposition 7 in Grechuk and Zabarankin (2017) guarantees that such \( P \) exists provided that \( \mathcal{F} \) is a convex set and \( U^* \) is a concave function. Because a concave function is differentiable almost everywhere, one may expect that \( \partial U^*(X^*) \) is “typically” a singleton, in which case this approach leads to the unique selection of a “fair” Pareto optimal allocation in \((29)\). Below we show, however, that this intuition may be wrong.

Mean-deviation portfolio optimization problem \((15)\) cannot be formulated as \((26)\) with a cash invariant utility function. However, instead of minimizing a deviation measure subject to a constraint on the expected return, investor \( i \) may choose to maximize \( U_i(X) = EX - D_i(X) \) for some deviation measure \( D_i, i = 1, \ldots, m \). In this formulation, \( U_i \) are cash-invariant, and the above theory applies. \( U^* \) in \((28)\) is given by \( U^*(X) = EX - D^*(X) \), where

\[
D^*(X) \equiv \inf_{Z \in A(X)} \sum_{i=1}^{m} D_i(Z_i).
\]
Lemma 5.1. Let $\mathcal{D}_i$ be deviation measures with risk envelopes $Q_i$, $i = 1, \ldots, m$. Then $\mathcal{D}^*$ is a deviation measure with risk envelope $Q^* = Q_1 \cap \cdots \cap Q_m$. In particular, if all $\mathcal{D}_i$ are finitely generated, then so is $\mathcal{D}^*$.

Proof. Proposition 3 in Rockafellar et al. (2006a) implies that $Q_1, \ldots, Q_m$ are closed, convex subsets of the closed hyperplane $H = \{Q | EQ = 1\}$ in $L^2(\Omega)$ such that constant 1 is in their quasi-interior relative to $H$. Because $Q_1, \ldots, Q_m$ have a common point in their relative interiors, Rockafellar (1970, Corollary 16.4.1) implies that $\mathcal{D}^*$ can be represented in the form (16) with $Q^* = Q_1 \cap \cdots \cap Q_m$.

Because $Q^*$ is also closed, convex subset of $H$ with constant 1 in quasi-interior relative to $H$, this implies that $\mathcal{D}^*$ is a deviation measure. Because intersection of polygons is a polygon, $\mathcal{D}^*$ is finitely generated provided that all $\mathcal{D}_i$ are.

Theorem 5.2. Assume that investors’ utility functions are of the form $U_i(X) = EX - D_i(X)$ with deviation measures $\mathcal{D}_i$ finitely generated and none of the portfolio risk generators for $\mathcal{D}^*$ is parallel to $1 := (1, \ldots, 1)^T$ or equal to $\mu = E[R]$. Then a solution $X^* = R^T x^*$ to (27) has at least two extreme risk identifiers.

Proof. We follow ideas from the proof of Theorem 4.8. Let $\hat{D}_i^* = D_i^* - E[R]$. Then (27) is equivalent to the following linear problem

$$
\begin{align*}
\text{minimize} & \quad A, \\
\text{subject to:} & \quad x^T \hat{D}_i^* - A \leq 0, \quad i = 1, \ldots, M, \\
& \quad x^T 1 = 1, \\
& \quad (A, x) \in \mathbb{R} \times \mathbb{R}^n. 
\end{align*}
$$

By assumption this program has a solution, hence its dual also has a solution:

$$
\begin{align*}
\text{maximize} & \quad q, \\
\text{subject to:} & \quad \sum_{k=1}^M p_k \hat{D}_k^* - q 1 = 0, \\
& \quad \sum_{k=1}^M p_k = 1, \\
& \quad p_k \geq 0, \quad k = 1, \ldots, M, \\
& \quad q \in \mathbb{R}.
\end{align*}
$$

If the optimal solution $q \neq 0$, then the middle equation together with the assumption that none of $\hat{D}_k^*$’s is parallel to $1$ implies that there must be at least two $p_k$’s strictly positive. Bertsimas and Tsitsiklis (1997, Theorem 4.5) states that the corresponding constraints in the primal problem are active, i.e., their respective portfolio risk generators are active for $X^*$. When $q = 0$, the assumption that none of $\hat{D}_k^*$’s is zero imply again that at least two $p_k$’s must be non-zero.

Theorem 5.2 implies that there are at least two linearly independent active portfolio risk generators. Therefore, $\partial U^*(X^*)$ is very far from being a singleton and there are multiple fair Pareto-optimal solutions to the cooperative investment problem.

Remark 5.3. The problem of non-uniqueness of a risk identifier is particular for the cooperative investment. However, in the classical risk sharing problem for agents with utility functions based on finitely generated deviation measures there is a unique fair Pareto optimal allocation for initial endowments from a set of full Lebesgue measure, c.f. Section 3.2. This is unlike the cooperative investment problem when the endowment to be distributed between agents comes from the set of measure zero.

Section 2 suggests a method for selecting a unique point in $\partial U^*(X^*)$ (a unique risk identifier), and hence a unique “fair” Pareto optimal allocation in (29). Specifically, this unique point is the Steiner point (5)-(6) of set $\partial U^*(X^*)$. 

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5.2 Explicit example

Cash-or-nothing binary option \( O \) returns some fixed amount of cash \( C(O) \) if it expires in-the-money but nothing otherwise. Assume that there are two such options \( A \) and \( B \) which expire in-the-money if \( P > C_1 \) and \( P > C_2 \), respectively, where \( P \) is the (random) price of (the same) underlying asset, and \( C_1 < C_2 \) are constants. Assume that options are offered for the same price \( p \) with \( C(A) = 2p \) and \( C(B) = 8p \). Each agent can invest a unit of capital into \( A \) and \( B \), precisely \( 1 - t \) into \( A \) and \( t \) into \( B \), to get profit \(- (1 - t) - t = -1; (1 - t) - t = 1 - 2t; \) or \((1 - t) + 7t = 1 + 6t \) depending on the relation of the price \( P \) with respect to \( C_1 \) and \( C_2 \). We assume that two agents think that these three opportunities are equally probable.

For agent 1 with \( U_1(X) = E[X] - CV aR_2^\frac{1}{2}(X) = -CV aR_2^\frac{1}{2}(X) \), an optimal individual investment can be found from the linear program

\[
\max_{a_1,t} a_1, \quad \text{s.t.} \quad X = (-1, 1 - 2t, 1 + 6t), \quad E[Q X] \geq a_1, \forall Q \in Q^1,
\]

where \( Q^1 = \{(\frac{2}{3}, 0, \frac{2}{3}), (0, \frac{2}{3}, 0) \} \), \{Perm \( (\frac{2}{3}, 0, \frac{2}{3}) \) \}, resulting in the optimum \( t = 0, X = (-1, 1, 1) \), and the optimal value \( u_1^* = 0 \).

Similarly, for agent 2 with \( U_2(X) = E[X] - \frac{1}{2}MAD(X) \), the linear program

\[
\max_{a_2,t} a_2, \quad \text{s.t.} \quad X = (-1, 1 - 2t, 1 + 6t), \quad E[Q X] \geq a_2, \forall Q \in Q^2,
\]

where \( Q^2 = \{\text{Perm} \( (\frac{2}{3}, 0, \frac{2}{3}) \), \text{Perm} \( (\frac{1}{3}, \frac{2}{3}, \frac{1}{3}) \) \}, \text{returns} \ t = \frac{1}{5}, \text{with the optimal value} \ u_2^* = \frac{1}{15} \). The cooperative investment corresponds to the linear program

\[
\max_{a_1,a_2,Y_1,Y_2,t} a_1 + a_2, \quad \text{s.t.} \quad Y_1 + Y_2 = 2(-1, 1 - 2t, 1 + 6t), \quad E[XY_j] \geq a_j, \forall Q \in Q^j, \ j = 1, 2,
\]

that is, we are simultaneously looking for optimal portfolio \( t \), and an optimal way to share it \( (Y_1, Y_2) \) to maximize the sum of agents utilities. The optimal \( t \) is \( t = \frac{1}{5}, \) with \( Y_1 + Y_2 = (-2, \frac{6}{5}, \frac{4}{5}), \) and optimal value is \( u^* = \frac{2}{15} > u_1^* + u_2^* \). The simplex method returns a solution \( Y_1 = (\frac{2}{15}, \frac{2}{15}, \frac{2}{15}), \ Y_2 = (-\frac{32}{15}, \frac{16}{15}, \frac{64}{15}) \), with \( u_1(Y_1) = \frac{2}{15} \) and \( u_2(Y_2) = 0 \), which is obviously unfair. Because the utilities are cash invariant, any solution in the form \( Y_1' = Y_1 + C, Y_2' = Y_2 - C \) is Pareto-optimal, and the question is how to select a "fair" \( C \).

A standard approach for determining an exact value of \( C \) is the following one (see Grechuk and Zabarankin (2017)). Utility of a coalition \( U^*(X) \) can be written as

\[
U^*(X) = \min_{Q \in Q^*} E[Q X], \tag{33}
\]

where \( Q^* \) can be found as (the vertices of) intersection of convex hulls of \( Q^1 \) and \( Q^2 \). In our case, \( Q^* = \{\text{Perm} \( (\frac{1}{3}, 1, \frac{1}{3}) \), \text{Perm} \( (\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) \) \}. \) The optimal portfolio \( X^* = Y_1 + Y_2 = (-2, \frac{6}{5}, \frac{22}{5}) \) can be found from optimization problem

\[
\max_{X} U^*(X), \quad \text{s.t.} \quad X = 2(-1, 1 - 2t, 1 + 6t). \tag{34}
\]

Now, let \( Q^* \) be the minimizer in (33) for \( X^* \). Then \( C \) should be selected such that

\[
E[Q^*(Y_1 + C)] = E[Q^*(Y_2 - C)]. \tag{35}
\]

An intuition is that the investors should get the same profit under the critical scenario \( Q^* \). The problem is that, for \( X^* = (-2, \frac{6}{5}, \frac{22}{5}) \), the minimizer \( Q^* \) in (33) in not unique! Indeed, \( E[Q X^*] = \frac{2}{15} \) for
$Q = \left( \frac{3}{2}, 1, \frac{1}{2} \right)$, and also for $Q = \left( \frac{1}{3}, \frac{4}{3}, \frac{1}{3} \right)$. This is not a coincidence as we have shown in Theorem 5.2. While set of random variables $X$ with non-unique risk identifier has measure 0, the optimal portfolio in (34) is guaranteed to belong to this set. Consequently, the cooperative investment does not have a unique solution in the case of finitely generated deviation measures.

In our example, the set of minimizers in (33) is the line interval with endpoints $\left( \frac{3}{2}, 1, \frac{1}{2} \right)$ and $\left( \frac{1}{3}, \frac{4}{3}, \frac{1}{3} \right)$. The Steiner point of this set is its midpoint $Q^*$, which is then $\left( \frac{17}{18}, \frac{7}{18}, \frac{5}{18} \right)$. From (35), this implies that $C = \frac{1}{2}E[Q^*(Y_2 - Y_1)] = -\frac{1}{18}$. The equilibrium endowments are then $Y_1 + C = \left( \frac{1}{18}, \frac{1}{18}, \frac{1}{18} \right)$ and $Y_2 - C = \left( -\frac{31}{18}, \frac{17}{18}, \frac{13}{18} \right)$.

6 Inverse portfolio problem

Following Palczewski and Palczewski (2018), let us formulate a problem inverse to (15) as follows. Assume that we know a solution $x^M = (x_1^M, \ldots, x_n^M) \neq 0$ to (15), together with centered rates of return $\hat{R}$, deviation measure $\mathcal{D}$, and $\Delta^M > 0$ the expected excess return of the portfolio $x^M$. Can we then “recover” $\mu$, the expected excess returns of individual instruments? Are they determined uniquely? We will give a positive answer to the first question and discuss a dichotomy faced by the second: if the solution of the inverse problem is unique then the forward problem with the computed $\mu$ has multiple solutions, while if the forward problem has a unique solution then there are many $\mu$’s solving the inverse problem.

6.1 An explicit formula using risk generators

Assume that $\Delta^M > 0$. Necessarily, $x^M \neq 0$. Theorem 4 in Rockafellar et al. (2006b) states that, the portfolio $x^M$ is a solution to (15) if and only if there is a risk identifier $Q^*$ for the random variable $\hat{R}^T x^M$ such that

$$\mu = \frac{\Delta^M}{\mathcal{D}(\hat{R}^T x^M)} E[-\hat{R}Q^*] = \frac{\Delta^M}{(x^M)^T E[-\hat{R}Q^*]} E[-\hat{R}Q^*].$$

(36)

This follows since every finite deviation measure on a discrete probability space is continuous, c.f. Rockafellar et al. (2006a, page 518).

Let $D_{ij}, j = 1, \ldots, k$, be the set of active portfolio risk generators for $x^M$. Then (36) amounts to the existence of weights $\beta_1, \ldots, \beta_k \geq 0$, such that $\sum_{j=1}^{k} \beta_j = 1$ and

$$\mu = \frac{\Delta^M}{\sum_{j=1}^{k} \beta_j D_{ij}^T x^M} \sum_{j=1}^{k} \beta_j D_{ij}. \quad (37)$$

From the above formula we immediately get the following characterization of vectors $\mu$ for which $x^M$ is a solution to (15).

Lemma 6.1. The set of solutions $\mathcal{M}$ to an inverse optimization problem is convex and spanned by points $\delta D_{ij}$, where $D_{ij}, j = 1, \ldots, k$, are active portfolio risk generators for $x^M$ and $\delta = \Delta / \mathcal{D}(\hat{R}^T x^M)$:

$$\mathcal{M} = \left\{ \delta \sum_{j=1}^{k} \beta_j D_{ij} \Bigg| \beta \in [0, \infty)^k \text{ and } \sum_{j=1}^{k} \beta_j = 1 \right\}.$$

Equipped with this characterization of the set $\mathcal{M}$ we demonstrate the link between the set of solutions of the inverse and forward optimization problems.
Theorem 6.2.

1. If \( x^M \) is a unique solution to (15) for some \( \mu \), then the set of all solutions \( M \) to the inverse optimization problem has at least \( n + 1 \) extreme points. Moreover, all extreme points are of the form \( \delta D_{ij} \), where \( \delta > 0 \) and \( D_{ij} \) is an active portfolio risk generator for \( x^M \).

2. If there is a unique active portfolio risk generator for \( x^M \), then the inverse optimization problem has a unique solution \( \mu^* \) (the set \( M \) consists of one point). However, the optimization problem (15) with \( \Delta = \Delta_M \) and \( \mu = \mu^* \) has multiple solutions: the set of solutions \( \mathcal{X}_\mu^* \) is a polyhedron of dimension \( n - 1 \) and has at least \( n \) extreme points.

The proof of the above theorem requires the following simple technical result.

Lemma 6.3. Given \( v_i \in \mathbb{R}^n, i = 1, \ldots, k \), let \( \hat{n} = \text{rank}(v_1, \ldots, v_k) = \text{dim}(\text{lin}(v_1, \ldots, v_k)) \).

Then \( \mathcal{N} = \text{conv}(v_1, \ldots, v_k) \) has at least \( \hat{n} + 1 \) extreme points and all extreme points are from the set \( \{v_1, \ldots, v_k\} \).

Proof. It follows from Rockafellar [1970, Corollary 18.3.1] that all extreme points of \( \mathcal{N} \) are in \( \{v_1, \ldots, v_k\} \). It remains to prove that there are at least \( \hat{n} + 1 \) extreme points. Assume the opposite: there are only \( n' < \hat{n} + 1 \) extreme points \( v_1, \ldots, v_{n'} \) of \( \mathcal{N} \). Then \( \mathcal{N} \subseteq A := \text{lin}(v_1, \ldots, v_{n'}) \) and \( \text{dim}(A) \leq n' + 1 \). However, \( A \) is a linear space containing all points \( v_1, \ldots, v_k \) so it also contains \( \text{lin}(v_1, \ldots, v_k) \). The latter space has dimension \( \hat{n} + 1 \) by assumption, hence a contradiction.

Proof of Theorem 6.2. From Theorem 4.8, the uniqueness of solutions to (15) implies that the set of active portfolio risk generators \( D_1, \ldots, D_k \) spans the whole space \( \mathbb{R}^n \), i.e., the dimension of a linear space generated by those vectors is \( n \). The conclusions follow from Lemma 6.3.

Assume now that there is a unique active portfolio risk generator. The uniqueness of solution to the inverse optimization problem is clear from formula (37). Consider the equivalent form (23) for the forward optimization problem. Recall that the set of all solutions to such a linear problem is a convex bounded polyhedral set, a face of a polyhedral set generated by the constraints. The portfolio \( x_M \) is a solution for which there are exactly two active constraints: one with the unique portfolio risk generator and one encoding the minimum expected return. This implies that the set of solutions is a polyhedron of dimension \( n - 1 \). By Lemma 6.3 it must have at least \( n \) extreme points.

Corollary 6.4. In the case 1 of Theorem 6.2 if \( \mu \in \text{ri} \mathcal{M} \) (\( \mu \) is in the relative interior of \( \mathcal{M} \)), then the forward optimization problem (15) has a unique solution for \( \Delta = \Delta_M \).

Proof. The implication is equivalent to: solution to (15) is not unique \( \iff \mu \notin \text{ri} \mathcal{M} \). This follows immediately from assertion 2 of Theorem 4.8 and Rockafellar [1970, Theorem 6.4].

6.2 Choice of a single solution to the inverse optimization problem

By Corollary 4.9 the solution to the portfolio optimization problem (15) is unique unless \( \mu \) belongs to a set of Lebesgue measure zero (a union of a finite number of hyperplanes). It is therefore common that the inverse optimization problem has multiple solutions (Theorem 6.2).

How to choose a unique point from the set of solutions to the inverse optimization problem? In view of (35), this is equivalent to the choice of a unique risk identifier \( Q^* \) or rather a map \( f_D : \mathcal{L}^2(\Omega) \rightarrow \mathcal{L}^2(\Omega) \) that, for a deviation measure \( D \), assigns to a random variable \( X \in \mathcal{L}^2(\Omega) \) one of its risk identifiers. We will call such a map \( f_D \) a selector corresponding to the deviation measure \( D \). We say that \( f_D \) is a robust selector if it is (i) a selector, and (ii) a robust map in sense of Definition 2.5.
Lemma 6.5. For any finite deviation measure \( D \) there exists a unique robust selector \( f_D \).

Proof. By Rockafellar et al. (2006b, Proposition 1) we have \( \partial D(X) = 1 - Q(X) \). Hence the existence and uniqueness follow from Theorem 2.3.

Example 5. For mean absolute deviation \( \text{MAD}(X) = E[|X - EX|] \), the unique robust selector is given by \( f_D(X) = 1 + EZ - Z \), where \( Z(\omega) = 1 \), \( Z(\omega) = 0 \), and \( Z(\omega) = -1 \), for \( X(\omega) > E[X] \), \( X(\omega) = E[X] \), and \( X(\omega) < E[X] \), respectively.

6.3 Explicit examples

Let \( \Omega = \{\omega_1, \ldots, \omega_N\} \) with \( P(\omega_j) = w_j \), \( j = 1, \ldots, N \), and \( \hat{R}_j = \hat{R} (\omega_j) \). Consider a given portfolio \( x^M \) and denote \( X^* = \hat{R}^M x^M \) and \( x_j^* = X^*(\omega_j) \). Without loss of generality, we assume that \( \{\omega_1, \ldots, \omega_N\} \) are ordered in such a way that \( x_1^* \leq x_2^* \leq \cdots \leq x_N^* \). Since \( E[X^*] = 0 \) and either \( x_1 = \cdots = x_N = 0 \) or \( x_1 < 0 < x_N \). The former case is impossible for a non-zero portfolio \( x^M \) under the assumption \( [M] \), therefore, we will concentrate on the non-trivial latter case of non-zero return \( X^* \). We will examine the inverse portfolio problem for risk measured by MAD and by deviation CVaR.

6.3.1 Mean absolute deviation

Let \( k \) be the maximal index such that \( x_k^* < 0 \) and \( m \) be the maximal index such that \( x_m^* \leq 0 \). It follows from the discussion above that \( 1 \leq k \leq m \). The inverse portfolio problem has solutions of the form

\[
\mu = \frac{\Delta_M}{\text{MAD}(X^*)} E[-Q^* \hat{R}],
\]

where \( Q^* \) is a risk identifier for \( X^* \). Recalling the form \( Q = 1 + E[Z] - Z \) of risk generators for MAD, see Example 5 and \( E[\hat{R}] = 0 \) we get \( E[-Q^* \hat{R}] = E[Z \hat{R}] \). If \( k = m \), there is a unique risk identifier given by \( Z(\omega_j) = 1_{(j > k)} - 1_{(j \leq k)} \), \( j = 1, \ldots, N \). Otherwise, there are \( 2(m - k) \) extreme risk identifiers corresponding to \( Z \)'s of the form \( Z(\omega_j) = 1_{(j > m)} - 1_{(j \leq k)} + z 1_{(j = j^*)} \), \( j = 1, \ldots, N \), for some \( k < j^* \leq m \) and \( z \in \{-1, 1\} \). Therefore, the set of solutions of the inverse problem is given by

\[
\left\{ \sum_{j=m+1}^N w_j \hat{R}_j - \sum_{j=1}^k w_j \hat{R}_j + \sum_{j=k+1}^m \lambda_j w_j \hat{R}_j \mid \lambda_{k+1}, \ldots, \lambda_m \in [-1, 1] \right\}.
\]

A robust selector corresponds to taking \( \lambda = 0 \) (see Example 5).

Example 6. Let \( N = 3 \) and \( P(\omega_j) = \frac{1}{3}, j = 1, 2, 3 \). There are two risky assets with centered returns \( \hat{R}_1 = (-1, -2)^T \), \( \hat{R}_2 = (-1, 1)^T \) and \( \hat{R}_3 = (2, 1)^T \). The solution to the forward portfolio optimization problem with \( \mu = (0.4, 0.6) \) and \( \Delta_M = 0.5 \) is \( x_M = (0.5, 0.5) \). Then \( X^* = (-1.5, 0, 1.5) \) and \( \text{MAD}(X^*) = 1 \). The set of risk identifiers of \( X^* \) is given by \( Z = (-1, z, 1) \) with an arbitrary number \( z \in [-1, 1] \), i.e., \( Q^* = \left( 2 + \frac{z}{3}, 1 - \frac{2}{3}z, \frac{z}{3} \right) \). The corresponding set of solutions \( \mu \) to the inverse problem takes the form:

\[
\mu = \frac{0.5}{1} \left( \frac{1}{3} (-1) \hat{R}_1 + \frac{1}{3} z \hat{R}_2 + \frac{1}{3} \hat{R}_3 \right) = \left( \frac{0.5 - z/6}{0.5 + z/6} \right), \quad z \in [-1, 1].
\]

The unique robust selector suggested in Example 5 corresponds to \( z = 0 \), resulting in \( \mu = (0.5, 0.5)^T \).
6.3.2 Conditional Value at Risk

For deviation CVaR let \( k \) be the maximal index such that \( x_k^* < -\text{VaR}_\alpha(X^*) \) (set \( k = 0 \) is no such index exists) and \( m \) be the maximal index such that \( x_m^* \leq -\text{VaR}_\alpha(X^*) \). Then any risk identifier \( Q^* = (q_1, \ldots, q_N) \) of \( X^* \) satisfies, c.f. Rockafellar et al. (2006b),

\[
\begin{cases}
0 \leq q_j \leq 1/\alpha, & \sum_{j=1}^N w_j q_j = 1, \\
q_1 = q_2 = \cdots = q_k = 1/\alpha, \\
q_{m+1} = \cdots = q_N = 0.
\end{cases}
\] (38)

Hence,

\[
\mu = \frac{\Delta_M}{\text{CVaR}_\alpha^2(X^*)} \left( \frac{1}{\alpha} \sum_{j=1}^k w_j (-\hat{R}_j) + \sum_{j=k+1}^m w_j q_j (-\hat{R}_j) \right),
\] (39)

where \( q_{k+1}, \ldots, q_m \) are arbitrary numbers satisfying linear constraints

\[
\sum_{j=k+1}^m w_j q_j = 1 - \frac{1}{\alpha} \sum_{j=1}^k w_j, \quad \text{and} \quad 0 \leq q_j \leq 1/\alpha, \quad j = k + 1, \ldots, m.
\]

If \( m = k + 1 \), the risk identifier in (38) and \( \mu \) in (39) are uniquely defined. For \( m > k + 1 \), i.e., \( x_{k+1}^* = \cdots = x_m^* = -\text{VaR}_\alpha(X^*) \), the inverse problem has infinitely many solutions. The robust selector corresponds to \( q_{k+1} = \cdots = q_m = \frac{1}{\alpha} \sum_{j=1}^{k+1} w_j (\hat{R}_j) \), that is,

\[
\mu = \frac{\Delta_M}{\text{CVaR}_\alpha^2(X^*)} \left( \frac{1}{\alpha} \sum_{j=1}^k w_j (-\hat{R}_j) + q \sum_{j=k+1}^m w_j (-\hat{R}_j) \right),
\]

where \( q = (1 - \frac{1}{\alpha} \sum_{j=1}^k w_j) / \left( \sum_{j=k+1}^m w_j \right) \).

Example 7. Let \( \Omega = \{\omega_1, \omega_2, \omega_3\} \) with uniform probability \( \mathbb{P}(\omega_j) = 1/3 \). There are two risky assets with centered returns \( \hat{R}_1 = (-1, 0)^T, \hat{R}_2 = (0, -1, 1, 1) \). Fix \( \alpha = 0.05 \). The solution to the forward portfolio optimization problem with \( \mu = (1/3, 2/3) \) and \( \Delta_M = 0.5 \) is \( x_M = (0.5, 0.5) \). Then \( X^* = (-0.5, -0.5, 1), -\text{VaR}_\alpha(X^*) = -0.5, \) and \( k = 0, m = 2 \). The set of risk identifiers of \( X^* \) comprises \( Q = (q_1, q_2, 0) \), where \( 0 \leq q_1, q_2 \leq 20 \) and \( q_1 + q_2 = 3 \). Parameterizing \( q_1 = q \) and \( q_2 = 3 - q \) for \( q \in [0, 3] \), we obtain

\[
\mu = \frac{0.5}{1} \left( \frac{1}{3} q (-\hat{R}_1) + \frac{1}{3} (3 - q) (-\hat{R}_2) \right) = \left( \frac{q/3}{(3-q)/3} \right), \quad q \in [0, 3].
\]

The robust selector is given by \( q = 1.5 \), resulting in \( \mu_1 = \mu_2 = 0.5 \).

7 An application to Black-Litterman portfolio framework

It is due to comment on the findings of Section 6.1 from the perspective of an investor applying the Black-Litterman theory to portfolio optimization. We start with a short presentation of the extension of market-based Black-Litterman model of Meucci (2005) to general discrete distributions and deviation measures. The reader is referred to Palczewski and Palczewski (2018) for a detailed discussion of a parallel extension for continuous distributions.
7.1 Theoretical view

The underlying assumption of the original Black-Litterman model (Black and Litterman, 1992) is that the market is in equilibrium in which the mutual fund theorem holds, i.e., all investors hold risky asset in the same proportions. In the general setting of deviation measures, Rockafellar et al. (2007) develop an analogous theory and call the common portfolio of risky assets a master fund. It can be recovered by solving \( \Delta = \Delta^M \). We assume, as in the original framework, that the real market is in equilibrium, so the master fund corresponds to \( x^M \). Further, acting in the spirit of Black and Litterman (1992) we assume that the centered equilibrium distribution is known, for example, it is equal to the centered empirical distribution of asset returns. The only parameter of the distribution which is unknown is its location. To recover the latter, we solve an inverse optimization problem: knowing the solution \( x^M \) to problem \( (15) \) we find the mean excess return vector \( \mu_{eq} \) for a given expected market return \( \Delta = \Delta^M \). The distribution \( \mu_{eq} + \hat{R} \) is then called equilibrium distribution or prior distribution.

Investor’s views are represented by a \( m \times n \) ‘pick matrix’ \( P \) and a vector \( v \in \mathbb{R}^m \). Each row of \( P \) specifies combinations of assets and the corresponding entry in \( v \) provides a forecasted excess return. The uncertainty (the lack of confidence) in the forecasts is represented by a zero-mean random variable \( \varepsilon \) with a continuous distribution with full support on \( \mathbb{R}^m \), for example, a normal distribution \( N(0, Q) \). The resulting Bayesian model is

\[
\text{prior: } \ R \sim \mu_{eq} + \hat{R}, \\
\text{observation: } \ V|\{R = r\} \sim Pr + \varepsilon.
\]

The posterior distribution of future returns \( R \) given \( V = v \) is concentrated on the same points as the prior distribution but with different probabilities. It can be described by a new probability measure \( Q \) on \( \Omega \), i.e., the posterior distribution of asset excess returns is that of \( \mu_{eq} + \hat{R} \) under \( Q \). Following Bayes formula, we set the unnormalized “density” of the posterior distribution:

\[
X(\omega) = f_\varepsilon(v - P\mu_{eq})P\hat{R}(\omega),
\]

where \( f_\varepsilon \) is the density of \( \varepsilon \). Then \( Q(\omega)/P(\omega) = X(\omega)/E_P[X] \). The posterior distribution of asset returns in then fed into the optimization problem \( (15) \).

Assume now that the deviation measure \( D \) is finitely generated. By Corollary 4.9 it should be expected that the market portfolio is a unique solution to \( (15) \). Consequently, the inverse optimization problem that determines the equilibrium distribution has many solutions (Theorem 6.2) resulting in multitude of posterior distributions and, in effect, multitude of optimal portfolios. This is obviously unacceptable in a financial context. Methods described in Section 6.2 can be used to select one solution of the inverse optimization problem, therefore, bringing back the uniqueness of solution of the complete portfolio optimization exercise.

7.2 Practical view

In practice, an investor models scenarios for centered returns \( \hat{R} \) first and infers the market portfolio \( x^M \), for example, from the market capitalization of assets or his current portfolio. It is therefore unlikely that there is more than one active portfolio risk generator for the market portfolio, so the inverse optimization problem has a unique solution. Indeed, portfolios with at least two active portfolio risk

\(^5\)The location vector of the posterior distribution is rarely equal to \( \mu_{eq} \) due to the reweighting of probabilities in \( Q \) relative to \( P \).
generators lie on a finite number of hyperplanes in $\mathbb{R}^n$, hence, their Lebesgue measure is zero. It is unlike the theoretical presentation above in which active portfolio risk generators for the market portfolio span the whole space $\mathbb{R}^n$. In theory, the market portfolio is a unique solution to the optimization problem (15) for some target expected excess return $\Delta^M$. In practice, it solves an unlikely portfolio optimization problem for which the space of solutions has dimension $n$, see Theorem 6.2. If the investor has no opinions about the future returns, there will be many solutions to the forward optimization problem (15). This is an essential difference with the original Black-Litterman model (Litterman et al., 2004) and its extension to general continuous distributions (Palczewski and Palczewski, 2018), where such uniqueness problems do not exist.

Concluding, a practical application of the extension of Black-Litterman approach to discrete distributions and general deviation measures does not pose any difficulties but it contradicts assumptions of the theoretical model. This is unlike the original Black-Litterman model (Litterman et al., 2004) and its extension to general continuous distributions (Palczewski and Palczewski, 2018), where such uniqueness problems do not exist.

7.3 Example

As in Example 7, let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with uniform probability $P(\omega_j) = 1/3$. There are two risky assets with centered returns $R_1 = (-1,0)^T$, $R_2 = (0, -1)$, $R_3 = (1, 1)$. Extreme risk identifiers for CVaR$_{0.5}$ are $Q^e = \{\text{Perm}(3, 0, 0)\}$. The set of portfolio risk generators consists of 3 vectors:

$$D_1 = (1,0)^T, \quad D_2 = (0,1)^T, \quad D_3 = (-1,-1)^T.$$

Consider a market portfolio $x^M = (0.2, 0.8)^T$ and its return $\Delta^M = 0.4$. The only active portfolio risk generator for $x^M$ is $D_2$. From Lemma 6.1 the inverse optimization problem has a unique solution $\mu^* = (0,0.5)$. Consider now the forward optimization problem with expected excess return $\Delta^M$ and mean excess return $\mu^*$:

$$\min_{x_1, x_2} \max \{x_1; x_2; -x_1 - x_2\}, \quad \text{subject to:} \quad x_2 \geq 0.8.$$ 

The set of solutions is $X^* = \{(x_1, 0.8) : x_1 \in [-1.6, 0.8]\}$. Each solution in $X^*$ has CVaR$_{0.5}^e$ equal to 0.8 and the expected excess return of $\Delta^M$.

Application of any investor’s view perturbs the probabilities of events in $\Omega$, therefore, changing not only the risk profile of assets but usually their expected returns too. Indeed, imagine that investor’s views shifted the probabilities to $(\frac{1}{3}, \frac{1}{4}, \frac{1}{4})$. The new mean excess return becomes $\mu = \mu^* + (\frac{1}{3}, \frac{1}{4})^T$ and the centered returns are changed to $R_1 = (-\frac{5}{4}, -\frac{1}{4})^T$, $R_2 = (-\frac{1}{2}, -\frac{5}{4})$, $R_3 = (\frac{2}{3}, \frac{2}{3})$. Consequently, the new portfolio risk generators are $D_i = -R_i$, $i = 1, \ldots, 3$. Solving the forward portfolio problem with the expected excess return $\Delta^M$ yields a unique portfolio $x^* = (\frac{2}{3}, \frac{2}{3})$ for which portfolio risk generators $D_1$ and $D_2$ are active.

8 Conclusions

We have analyzed in depth forward and inverse portfolio optimization problems when asset returns follow a finite number of scenarios and deviation measure is finitely generated (covering popular deviation measures: CVaR, mixed CVaR and MAD). We discovered a dichotomy in the uniqueness
of solutions for both problems: the forward and inverse problems cannot be simultaneously uniquely solved (for the same data). Nevertheless, the set of parameters for which the non-uniqueness holds is of measure zero. Although it may seem that the uniqueness problem is practically negligible, we have demonstrated that this is not true in many applications, like capital allocation, cooperative investment, and the generalized Black-Litterman model. In cooperative investment, the non-uniqueness affects a “fair” way of distributing profit of joint investment between participating investors: for investors with preferences described by utility functions derived from finitely generated deviation measures, when the coalition’s forward optimization problem has a unique solution (which happens on the set of model parameters of full measure), there are many risk identifiers for the optimal wealth which prevents a unique “fair” allocation of wealth between investors. For the extended Black-Litterman model, the market portfolio (up to a set of measure zero) corresponds to the model data for which there is non-uniqueness of solutions to the forward problem. Recalling that this set has measure zero, it clearly points to an internal inconsistency of the finite scenario setup for the Black-Litterman model. This result is in contrast with the classical Black-Litterman model where the uniqueness holds for both forward and inverse problems.

The above problem of non-uniqueness has been shown to be connected to the fact that a convex function (here a risk or deviation measure) may not be everywhere differentiable, and, at points of non-differentiability, has a non-unique sub-gradient. This issue has been addressed by introducing the set of axioms, such that, for any convex function and at every point, there is a unique sub-gradient satisfying these axioms. This sub-gradient happens to coincide with the Steiner point of the sub-differential set.

Acknowledgements

The first author (BG) thanks the University of Leicester for granting him the academic study leave to do this research. The research of the second (AP) and third (JP) author was supported by the National Science Centre, Poland, under Grant 2014/13/B/HS4/00176.

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### Appendix A  Law invariant selectors

Section 6.2 introduces a method for selecting a unique solution to an inverse portfolio optimization problem. The approach is based on the principle of robustness. Its advantage is that robust selector is always uniquely determined.

Here we discuss an alternative approach which is based on the principle of law-invariance. The concept of a law-invariant selector may not be unique for some deviation measures in which case the law-invariance fails to resolve the non-uniqueness of the inverse optimization problem. However, it is financially and probabilistically natural and works in some important special cases.

**Definition A.1.** A selector $f_D : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$ is called law-invariant if $E[Y_1 f_D(X)]=E[Y_2 f_D(X)]$ whenever pairs of r.v.s $(Y_1, X), (Y_2, X) \in \mathcal{L}^2(\Omega) \times \mathcal{L}^2(\Omega)$ have the same joint laws.

A deviation measure $D$ is called law-invariant if $D(X) = D(Y)$ whenever r.v.s $X$ and $Y$ have the same distribution. For example, CVaR$_\alpha$ (CVaR-deviation) is law invariant for every $\alpha \in (0,1)$. Notice that not every deviation measure is law-invariant: a simple example of a non-law-invariant deviation measure can be constructed on $\Omega = \{\omega_1, \omega_2\}$, with $P[\omega_1] = P[\omega_2] = 0.5$, and

$$D(X) := \max \{X(\omega_1) - X(\omega_2), 2(X(\omega_2) - X(\omega_1))\}. \quad (40)$$

In the framework of uniform probability spaces, we prove below the existence, but not uniqueness, of a law-invariant selector.

**Theorem A.2.** If $\Omega$ is uniform, then there exists a law-invariant selector $f_D$ for every law-invariant deviation measure $D$.

**Proof.** It follows easily from Lemmas A.3, A.4 and A.5 below. $\square$

For non-uniform finite probability spaces, the notion of law-invariance as defined above is of little use for defining a unique selector, because, for example, on $\Omega = \{\omega_1, \omega_2\}$ with $P[\omega_1] \neq 0.5$, r.v.s $X$ and $Y$ have the same distribution if and only if $X = Y$, and, by definition, every deviation measure, including (40), is law-invariant. For similar reasons, every selector $f_D$ on such probability space is law-invariant. An appropriate extension of the notion of law-invariance to non-uniform probability spaces follows from results below.
An r.v. $X$ dominates r.v. $Y$ in second order stochastic dominance, denoted $X \succeq_2 Y$, if
\[
\int_{-\infty}^{t} F_X(x)dx \leq \int_{-\infty}^{t} F_Y(x)dx, \quad \forall t \in \mathbb{R}.
\]
An r.v. $X$ dominates r.v. $Y$ in concave order, denoted $X \succeq_c Y$, if $E[X] = E[Y]$ and $X \succeq_2 Y$. A deviation measure $\mathcal{D}$ is called consistent with concave order if $\mathcal{D}(X) \leq \mathcal{D}(Y)$ whenever $X \succeq_c Y$.

Lemma A.3. If a deviation measure $\mathcal{D}$ is consistent with the concave order, it is law-invariant. If $\Omega$ is uniform, the converse statement also holds.

Proof. The first statement is trivial, and the second one is well-known, but the proof is usually presented for atomless probability space, see [Dan and 2005, Theorem 4.1]. For a discrete uniform $\Omega$, let r.v.s $X$ and $Y$ take values $x_1 \leq \cdots \leq x_N$ and $y_1 \leq \cdots \leq y_N$, respectively. Then $X \succeq_c Y$ is equivalent to
\[
\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i, \quad k = 1, \ldots, N,
\]
with equality for $k = N$. Let us prove that in this case $Y$ can be obtained from $X$ by a finite sequence of operations
\[
(z_1, z_2, \ldots, z_N) \to (z_1, \ldots, z_i - d, z_{i+1}, \ldots, z_{j-1}, z_j + d, z_{j+1}, \ldots, z_N),
\]
d > 0, \quad 1 \leq i < j \leq N. \tag{42}

The statement is trivial for $N = 2$, and the case $N > 2$ can be proved by induction. If $\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i$ for some $k < N$, we can apply induction hypothesis to pair of r.v.s $X_1 = (x_1, \ldots, x_k)$ and $Y_1 = (y_1, \ldots, y_k)$, and separately to pair $X_2 = (x_{k+1}, \ldots, x_N)$ and $Y_2 = (y_{k+1}, \ldots, y_N)$, to conclude that there exists a sequence of operations (42) transforming $X_1$ to $Y_1$ and $X_2$ to $Y_2$, and hence $X$ to $Y$. Otherwise, apply operation (42) to $X$ with $i = 1, j = N$, and $d = \min_{k} \sum_{i=1}^{k} (x_i - y_i) > 0$, to get $X = (x_1, x_2, \ldots, x_N) \to (x_1 - d, x_2, \ldots, x_N + d) = (z_1, \ldots, z_N) = Z$. Then condition (41) holds for $z_1, z_2, \ldots, z_N$ in place of $x_1, x_2, \ldots, x_N$, with equality for some $k < N$, hence $Z$ can be transformed to $Y$ by the argument above.

Because operation (42) can only increase a law-invariant deviation measure $\mathcal{D}$, $\mathcal{D}(X) \leq \mathcal{D}(Y)$ follows.

Lemma A.4. If for any r.v. $X \in L^2(\Omega)$ the selector $f_\mathcal{D}$ satisfies the condition
\[
Q(\omega_i) = Q(\omega_j) \quad \text{whenever} \quad X(\omega_i) = X(\omega_j), \tag{43}
\]
where $Q = f_\mathcal{D}(X)$, then it is law-invariant. If $\Omega$ is uniform, the converse statement also holds.

Proof. Condition (43) implies that $Q = g(X)$ for some function $g : \mathbb{R} \to \mathbb{R}$. Then $E[Y_1Q] = E[Y_1g(X)] = E[Y_2g(X)] = E[Y_2Q]$ whenever pairs of r.v.s $(Y_1, X)$ and $(Y_2, X)$ have the same joint law.

Conversely, let $\Omega$ be uniform and $X(\omega_i) = X(\omega_j)$. Then pairs of r.v.s $(I_i, X)$ and $(I_j, X)$ have the same joint law, where $I_i$ and $I_j$ are indicator functions for $\omega_i$ and $\omega_j$, respectively. If $f_\mathcal{D}$ is law-invariant, this implies $Q(\omega_i) = N \cdot E[I_iQ] = N \cdot E[I_jQ] = Q(\omega_j)$, where $N = |\Omega|$, and (43) follows.
Proof. Let robust selector \( f \) be a map interchanging indices \( i \) and \( \lambda \). The set of risk identifiers satisfying (43) corresponds to all such points in the above set for which solutions of the inverse problem is given by Lemma A.6. Let \( f \) satisfies (43), and it is unique solution. However, as demonstrated in the paper, the robust selector approach of Section 6.2 ensures uniqueness regardless of the deviation measure used. 

Example 8. For CVaR-deviation \( \mathcal{D} = \text{CVaR}_\alpha \), there exists a unique selector satisfying (43), and it is given by (see Cherny (2006))

\[
 f_D(X) = Q_\alpha = \begin{cases} 
 0, & X > -\text{VaR}_\alpha(X), \\
 c_X, & X = -\text{VaR}_\alpha(X), \\
 1/\alpha, & X < -\text{VaR}_\alpha(X), 
\end{cases} 
\]

where constant \( c_X \in [0, 1/\alpha] \) is such that \( E[Q] = 1 \). 

Example 9. For mixed CVaR-deviation \( (19) \), there exists a unique selector satisfying (43), and it is of the form \( f_D(X) = Q_\mu = \int_0^1 Q_\alpha \mu(da) \), where \( Q_\alpha \) is given by (44) (see Cherny (2006)). 

Example 10. For mean absolute deviation \( \text{MAD}(X) = \|X - EX\|_1 \) (c.f. Example 2), if \( P(X = EX) > 0 \), there are infinitely many selectors satisfying (43). Indeed, in Subsection 6.3.1, the set of solutions of the inverse problem is given by

\[
 \left\{ \sum_{j=m+1}^N w_j \hat{R}_j - \sum_{j=1}^k w_j \hat{R}_j + \sum_{j=k+1}^m \lambda_j w_j \hat{R}_j \mid \lambda_{k+1}, \ldots, \lambda_m \in [-1, 1] \right\} .
\]

The set of risk identifiers satisfying (43) corresponds to all such points in the above set for which \( \lambda_{k+1} = \cdots = \lambda_m = \lambda \in [-1, 1] \).

Example 11 demonstrates that imposing condition (43) may not be sufficient for specifying a unique solution. However, as demonstrated in the paper, the robust selector approach of Section 6.2 ensures uniqueness regardless of the deviation measure used.

The following lemma demonstrates the consistency of two suggested approaches.

Lemma A.6. Let \( \Omega \) be uniform. Then, for every law-invariant deviation measure \( \mathcal{D} \), the corresponding robust selector \( f_D \) is law-invariant.

Proof. Let \( X \in \mathcal{L}^2(\Omega) \) and \( 1 \leq i < j \leq N \) be such that \( X(\omega_i) = X(\omega_j) \). Let \( T : \mathcal{L}^2(\Omega) \rightarrow \mathcal{L}^2(\Omega) \) be a map interchanging indices \( i \) and \( j \), that is, for \( Y = (y_1, \ldots, y_N) \),

\[
 T(Y) = (y_1, \ldots, y_{i-1}, y_j, y_{i+1}, \ldots, y_{j-1}, y_i, y_{j+1}, \ldots, y_N).
\]
Then $T(X) = X$. Let $A_D \subset \mathbb{R}^N$ be the set on which $D : \mathbb{R}^N \to \mathbb{R}$ is differentiable. This set has a full Lebesgue measure due to the convexity of $D$. By law-invariance, $D(T(Y)) = D(Y)$, $\forall Y$. Hence, for every $Y \in A_D$ such that $T(Y) \in A_D$, we have $T(f_D(Y)) = f_D(T(Y))$. Eqn. (11) implies for any $Y \in \mathcal{L}^2(\Omega) \equiv \mathbb{R}^N$,

$$T(f_D(Y)) = T\left( \lim_{\epsilon \to 0} E[f_D(Y + e_\epsilon)] \right) = \lim_{\epsilon \to 0} E[T(f_D(Y + e_\epsilon))]$$

$$= \lim_{\epsilon \to 0} E[\{f_D(T(Y + e_\epsilon))\}] = f_D(T(Y)),$$

where $e_\epsilon$ is uniformly distributed on the ball $B_\epsilon(0) \subset \mathbb{R}^N$. Because $T(X) = X$, this implies $T(f_D(X)) = f_D(X)$. Hence, (43) holds, and $f_D$ is law-invariant by Lemma [A,4].

In conclusion, this paper suggests two principles for choosing a unique selector, and hence a unique solution to the inverse optimization problem if the set $M$ has more than one point. One principle states that if $D$ is law-invariant, we should have $\mu_i = \mu_j$ in (56), whenever pairs $(\hat{e}^{(i)}, \hat{R}^T x^M)$ and $(\hat{e}^{(j)}, \hat{R}^T x^M)$ have the same joint law. This principle is already sufficient to resolve the problem for CVaR-deviation, and, more generally, for mixed CVaR-deviation, but, in general, may not return a unique solution. Another principle postulates that selector should be “robust” as defined in Section 2 and has an advantage that it always returns a unique solution. However, its economic interpretation/justification is not as clear as for the law-invariance principle.