Coherent states, constraint classes and area operators in the new spin-foam models

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Abstract
Recently, two new spin-foam models have appeared in the literature, both motivated by a desire to modify the Barrett–Crane model in such a way that the imposition of certain second class constraints, called cross-simplicity constraints, are weakened. We refer to these two models as the FKLS (Freidel–Krasnov–Livine–Speziale) model and the flipped model. Both of these models are based on a reformulation of the cross-simplicity constraints. This paper has two main parts. First, we clarify the structure of the reformulated cross-simplicity constraints and the nature of their quantum imposition in the new models. In particular, we show that in the FKLS model quantum cross-simplicity implies no restriction on states. The deeper reason for this seems to be that, with the symplectic structure relevant for FKLS, the reformulated simplicity constraints, among themselves, now form a first class system, and this seems to cause the coherent state method of imposing the constraints, key in the FKLS model, to fail to give any restriction on states. Nevertheless, the cross-simplicity can still be seen as implemented via suppression of intertwiner degrees of freedom in the dynamical propagation. In the second part of the paper, we investigate area spectra in the models. The results of these two investigations will highlight how, in the flipped model, the Hilbert space of states, as well as the spectra of area operators, exactly match those of loop quantum gravity, whereas in the FKLS (and Barrett–Crane) models the boundary Hilbert spaces and area spectra are different.

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1. Introduction

Loop quantum gravity (LQG) is a modern, background-independent approach to the canonical quantization of general relativity. For reviews, see [1–3]. The kinematics of LQG are well-understood, whereas the dynamics is less well-understood. One approach to the dynamics is...
the canonical approach [4]. A second approach, which seeks to preserve manifest spacetime covariance, is a sum-over-histories approach, leading to the spin-foam formalism [5]. In the search for a spin-foam model of quantum gravity, recent progress has been made in better understanding how to handle certain second class constraints—called the *simplicity* constraints.

In the most prominent spin-foam model, the Barrett–Crane model [6], the simplicity constraints are imposed strongly as operator equations. Because of this, all intertwiner degrees of freedom in the spin-foam model are frozen out; this has caused problems with the semiclassical limit of the theory, as investigated in [7]. In response to this problem, it was realized that the simplicity constraints should be handled more carefully, as they are second class, with the hope that the necessary intertwiner degrees of freedom would be liberated. From this motivation, two alternatives to the Barrett–Crane model have recently been proposed [8–12]. These two alternative models can be viewed as corresponding to the case of the small Barbero–Immirzi parameter $\gamma$, and the case of the infinite Barbero–Immirzi parameter $\gamma^\infty$. The value of $\gamma$ affects only the symplectic structure of the canonical theory. For small $\gamma$ and infinite $\gamma^\infty$, using the terminology of [8, 9], this symplectic structure is, respectively, the ‘flipped’ or ‘unflipped’ symplectic structure. We will therefore refer to the small $\gamma$ model [8, 9] as the ‘flipped vertex’. The infinite $\gamma^\infty$ model, as quantized in [11], will be referred to as the FKLS (Freidel–Krasnov–Livine–Speziale) model.

Both of these models can be obtained by using the coherent state approach to solving the cross-simplicity constraints developed in [10–12]. In addition, the flipped model [8, 9] can be obtained using an approach to the constraints involving Casimir operators; as we will see in this paper, this can be viewed as a sort of master constraint [14] approach to solving the simplicity constraints. This is the original way in which the flipped model was derived. The fact that the model was later derived using coherent states was a complete surprise at the time and greatly increased the confidence in the model, as well as opening a possible avenue for investigating its semiclassical limit.

We will present a clarification of the constraint analysis involved in these two models. As a consequence, we will raise some questions regarding the manner of imposing the constraints in the FKLS model. More specifically, we will note that if one tries to interpret the imposition of constraints in FKLS as an imposition of constraints on states, then FKLS does not impose cross-simplicity at all—the $SO(4)$ intertwiner spaces remain completely unconstrained. The source of this will be found to be a curious property of the reformulation of the constraints at the heart of the two new spin-foam models: in the unflipped case the cross-simplicity constraints form a closed, first class algebra, whereas in the flipped case they do not. This is relevant for the following reason. The procedure of imposing the constraints in the new models [8, 9, 11, 12] consists of two steps: first impose the simplicity constraints and then average over $SO(4)$ gauge transformations. It is in the first step that one uses the coherent state approach. First class constraints should be imposed strongly, which means, for the coherent state approach, that states should have zero spread with respect to these constraints, as well as the quantizations of the area operators. The case of the arbitrary $\gamma$ is handled in the forthcoming paper [13].

In this paper, we do not construct the quantum theories for an arbitrary $\gamma$ and then restrict to $\gamma \ll 1$ and $\gamma = \infty$. Rather, we let the assumption of $\gamma \ll 1$ and $\gamma = \infty$ influence the quantizations of the constraints, as well as the quantizations of the area operators. The case of the arbitrary $\gamma$ is handled in the forthcoming paper [13].

By ‘constraints on states’ we mean first and foremost constraints as imposed in the boundary Hilbert space of the model. In [11, 12], the boundary Hilbert space of FKLS is not discussed; in sections 2.2 and 3, we derive it in what seems to us the most straightforward way.
one can see in detail how this happens. One should note that, for the symplectic structure used in FKLS, the strong imposition of the cross-simplicity constraints leads naturally to the Barrett–Crane model.

The ‘master constraint’ approach to cross-simplicity allows one to be less concerned with the class of the constraints involved, as the master constraint program applies to any type of constraints, whether first or second class [14]. Using the master constraint approach (which, as we will see, is the same as the ‘Casimir operator’ approach of [8, 9]), for the unflipped symplectic structure, we obtain Barrett–Crane, whereas for the flipped symplectic structure, we obtain the flipped model of [8, 9].

One result of this investigation is that, in the FKLS model, as the \(\text{SO}(4)\) intertwiner spaces are completely unconstrained, the boundary space is clearly not isomorphic to the Hilbert space of LQG, in contrast with the flipped model [8, 9]. In addition, we will close with a discussion of \(\text{SO}(3)\)-gauge-fixed area operators. We will note that in the flipped model, the spectrum of this operator exactly matches that of the area operator in LQG, including numerical factors, whereas the spectra in the FKLS and BC models are different.

A final note should be said regarding the FKLS model. Although the FKLS model does not impose cross-simplicity as a constraint on boundary states, nevertheless it imposes cross-simplicity in a different, albeit less standard sense: the dynamics appear to suppress intertwiners that are far from the Barrett–Crane intertwiner. This viewpoint will be touched upon in the discussion section. Furthermore, this means that the fully physical Hilbert space of states—that is, the image of the generalized projector implicitly defined by the spin-foam sum—may very well be affected by the way cross-simplicity is incorporated into FKLS. However, we have not investigated this possibility in this paper.

The paper is organized as follows. First, we will briefly review the structure of the classical discrete theory from [8, 9]. Then we will discuss the space of states satisfying cross-simplicity in the BC, flipped and FKLS models. This will in part motivate a subsequent section discussing the relation between the coherent state method of imposing constraints, and the class of the constraints involved. Finally, the area spectra are analyzed in the BC, flipped and FKLS models and are compared with the spectra in LQG. Some final reflections on the significance of these results are then given.

2. Model and constraints

2.1. Discrete classical theory

Following [9], we introduce a Regge geometry. That is, first we introduce a simplicial decomposition \(\Lambda\) of spacetime, consisting of 4-simplices, tetrahedra and triangles. These are dual, respectively, to vertices, edges and faces in the dual 2-complex, and we shall denote them by \(v, t\) and \(f\). The geometry is flat on each 4-simplex. Curvature is concentrated on the ‘bones’ \(f\), and is coded in the holonomy around the ‘link’ of each \(f\).

The basic discrete variables for the theory can be motivated as follows. First, for each 4-simplex \(v\), introduce a tetrad field \(e_{\mu}^I(v)\), defined within \(v\), and for each tetrahedron \(t\), introduce a tetrad field \(e_{\mu}^I(t)\), defined within the two 4-simplices adjoining \(t\). We require that all of these tetrads determine the same, locally flat geometry where they overlap, and we require that they all be covariantly constant with respect to the derivative operator determined by this geometry. For each \(t\) and triangle \(f\) therein, we then define \(B_f(t) \in \mathfrak{so}(4)\) by

\[
B_f(t)^{kl} := \ast \int_f e^k(t) \wedge e^l(t) = \frac{1}{2} \epsilon^{klm} \int_f e^m(t) \wedge e^k(t),
\]

(1)
and for each 4-simplex \( v \) and tetrahedron \( t \) therein, we define \((V^{-1})_{tt'} = V_{tt'} \in \text{SO}(4)\) by
\[
e^{-I}_{tt'}(t) = (V_{tt'})^{J}J^I_{tt'}(v).
\]
Equations (1) and (2) are the basic discrete spacetime variables. For each triangle \( f \) and each pair of tetrahedra \( t, t' \) in the link of \( f \), define
\[
U_{f}(t, t') := V_{tt'}V_{v_1t}V_{v_2t} \cdots V_{v_{nt}}',
\]
where the product is around the link in the clockwise direction from \( t' \) to \( t \). The constraints on the discrete variables may then be stated as follows.

1. \( U_{f}(t, t')B_{f}(t') = B_{f}(t)U_{f}(t, t') \ \forall \ f \) and \( t, t' \in \text{Link}(f) \).
2. (closure) \( \sum_{f \in t} B_{f}(t) = 0 \ \forall \ t \).
3. Discrete simplicity constraints:
   i. \( C_{ff} := \ast B_{f}(t) \cdot B_{f}(t) \approx 0 \ \forall \ f \)
   ii. \( C_{ff'} := \ast B_{f}(t) \cdot B_{f'}(t) \approx 0 \ \forall \ f, f' \in t \)
   iii. \( \ast B_{f}(v) \cdot B_{f}(v) \approx \pm 12V(v) \ \forall \ f, f' \in v \) not in the same \( t \),

where \( \ast \) stands for the Hodge dual in the \( \text{SO}(4) \) algebra indices and the dot stands for the scalar product in this algebra. This is the traditional formulation of the constraints. The two constraints (3.i) and (3.ii) in fact have two sectors of solutions, one in which \( B = \ast e \wedge e \), and the other in which \( B = e \wedge e \). For finite, non-trivial Barbero–Immirzi parameter, both sectors in fact yield GR, but the value of the Newton constant and the Barbero–Immirzi parameter are different in each sector\(^4\). Therefore, in order to talk about one sector at a time in a coherent way, it is desirable to reformulate the simplicity constraints such that these two sectors are distinguished. In fact, this can be done: condition (3.ii) can be replaced with the condition that
\[
(3.\text{ii}') \text{ for each tetrahedron } t, \text{ there exists an internal vector } n' \text{ such that }
(\ast B_{f}(t))_{D}n' = 0 \text{ for all } f \in t.
\]

This reformulation of the constraint (3.ii) (the ‘off-diagonal’ or ‘cross-simplicity’ constraint) is central to the new models \([8, 9, 11, 12]\). When constructing the quantum theory, the above constraints are incorporated as follows: (1.) will be imposed prior to varying the action, (2.), (3.i) and (3.ii') will be imposed in quantum theory. As noted in \([9]\), (3.iii) is automatically satisfied when the rest of the constraints are satisfied, due to the choice of variables.

The classical discrete action is \([9]\)
\[
S_{\text{disc}} = -\frac{1}{2\kappa} \sum_{f \in \text{int} \Delta} \text{tr} \left[ B_{f}(t)U_{f}(t) + \frac{1}{\gamma} \ast B_{f}(t)U_{f}(t) \right]
- \frac{1}{2\kappa} \sum_{f \in \partial \Delta} \text{tr} \left[ B_{f}(t)U_{f}(t, t') + \frac{1}{\gamma} \ast B_{f}(t)U_{f}(t, t') \right],
\]
where \( U_{f}(t) := U_{f}(t, t) \) is the holonomy around the full link, starting at \( t \), and where we have set \( \kappa = 8\pi G \). From this we can read off the boundary variables as \( B_{f}(t) \in \text{so}(4), U_{f}(t, t') \in \text{SO}(4) \). One can also see that the variable conjugate to \( U_{f}(t, t') \) is proportional to \( B_{f}(t) + \frac{1}{\gamma} \ast B_{f}(t) \). The constant of proportionality is fixed in appendix C:
\[
J_{f}(t) = \frac{1}{\kappa} \left( B_{f}(t) + \frac{1}{\gamma} \ast B_{f}(t) \right).
\]

\(^4\) In the \( B = \ast e \wedge e \) sector, with coefficients in the action as in (4), one obtains the Holst \([15]\) formulation of general relativity with the Newton constant \( G \) and the Barbero–Immirzi parameter \( \gamma \). In the sector \( B = e \wedge e \), one also obtains the Holst formulation of general relativity, but this time with \( G\gamma \) acting as the Newton constant, and \( \frac{1}{\gamma} \) acting as the Barbero–Immirzi parameter.
More precisely, each matrix component $J_f(t)^{IJ}$ has as its Hamiltonian vector field the left-invariant vector field on the group $U_f(t, t')$ corresponding to the Lie algebra element $J^I$ defined in appendix A. Inverting the above equation gives

$$B_f(t) := \kappa \left( \frac{\gamma^2}{\gamma^2 - 1} \right) \left( J_f(t) - \frac{1}{\gamma} J_f(t) \right).$$

(6)

For the cases $\gamma \ll 1$ and $\gamma = \infty$, this reduces to

$$\gamma \ll 1 \rightarrow B = \kappa \gamma^2 J, \quad \gamma = \infty \rightarrow B = \kappa J,$n

corresponding, respectively, to the flipped and non-flipped Poisson structures of $SO(4)$. In terms of the new variables $J_f$, the constraints (3.i) and (3.ii') read\(^5\) (we consider $\gamma \neq 0$, 1 so that overall factors can be discarded)

$$C_{ff} := J_f \cdot J_f \left( 1 + \frac{1}{\gamma^2} \right) - \frac{2}{\gamma} J_f \cdot J_f \approx 0,$n$$

(7)

$$C_{f}^I := n_I \left( \frac{(J_f)^I}{\gamma} - \frac{1}{\gamma} J_f^I \right) \approx 0.$n$$

(8)

The closure for the $B_f$ is equivalent to the closure for the $J_f$ which, as noted in [9], will be imposed automatically by the dynamics. The strategy for imposing (8) in quantum theory will be to first impose it for a fixed $n_I$, and then to average over $SO(4)$. This is the strategy used in [8, 9, 11, 12]. For this purpose, let us fix $n_I := \delta_{I0}$ in what follows. Equation (12) then becomes

$$C_f^i = \frac{1}{2} \epsilon^{ijl} J^l_f - \frac{1}{\gamma} J^0_f = L^i_f - \frac{1}{\gamma} K^i_f,$n$$

(9)

where $\epsilon^{ijkl} := \epsilon^{ij0kl}$, $L^i_f := \frac{1}{2} \epsilon^{ijkl} J^k_f$, $K^i_f := J^0_f$. We further make the self-dual/anti-self-dual decomposition of $J_f^I$:

$$J_f^{(\pm)i} := \frac{1}{2} (\pm K^i_f + L^i_f)$$

(10)

so that $L^i_f = J_f^{(+)i} + J_f^{(-)i}$ and $K^i_f = J_f^{(+)i} - J_f^{(-)i}$. In terms of this decomposition we rewrite the constraints as follows:

$$C_{ff} = (J_f^{(+)})^2 \left( 1 - \frac{1}{\gamma} \right)^2 \left( J_f^{(-)} \right)^2 \left( 1 + \frac{1}{\gamma} \right)^2 \approx 0,$n$$

(11)

$$C_f^i = J_f^{(+)i} \left( 1 - \frac{1}{\gamma} \right) + J_f^{(-)i} \left( 1 + \frac{1}{\gamma} \right) \approx 0.$n$$

(12)

We take (11), (12) as our basic set of constraints. They will be imposed in quantum theory. Because (12) is a gauge-fixed version of the constraint (3.ii'), one will have to average over $SO(4)$ gauge transformations after its imposition\(^6\).

Let us note something which will be important later. For $\gamma = \infty$ (the unflipped symplectic structure), the gauge-fixed cross-simplicity constraints (9) become

$$C_f^i = L^i_f = J_f^{(+)i} + J_f^{(-)i} = L^i_f.$$n

(13)

\(^5\) A continuum version of the first of these equations appeared in [16].

\(^6\) One can be more precise about the gauge used in (12): it is just the usual time gauge, used, for example, in LQG. More precisely, it is implied by the usual time gauge. For, suppose one is given a spatial hypersurface $\Sigma$ and a 4-vector field $v$ thereon, transverse to $\Sigma$, and consider the fixed internal vector $n^I = \delta^I_0$. The corresponding time gauge implies that $n_I B_f^I := 0$, where the underarrow denotes pull-back to $\Sigma$. This in turn implies that for any triangle $f$ in $\Sigma$, $n_I B_f^I := n_I \int_f v^I = 0$. 

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and for $\gamma \ll 1$ (the flipped symplectic structure) they become
\[ C_f^j \propto K_f^j = -J_f^* + J_f^{-1}. \]  
(14)

In the unflipped case, constraints (13) are close and so are ‘first class’ in this sense. In fact, their algebra is just that of $so(3)$. In the flipped case, constraints (14) do not close and so are ‘second class’ in this sense.

2.2. Quantum kinematics

From the discrete boundary variables and their symplectic structure, one can write down the Hilbert space associated with a boundary or 3-slice. To do this, it is simpler to switch to the dual, 2-complex picture, $\Delta^*$. For each 3-surface $\Sigma$ intersecting no vertices of $\Delta^*$, let $\gamma_\Sigma := \Sigma \cap \Delta^*$. The Hilbert space associated with $\Sigma$ is then
\[ \mathcal{H}_\Sigma = L^2(SO(4)|L(\gamma_\Sigma)), \]  
(15)

where $|L(\gamma_\Sigma)|$ denotes the number of links in $\gamma_\Sigma$. Let $\hat{J}_f(t)^H$ denote $(-i)$ times the left-invariant vector fields, determined by the basis $J^H$ of $so(4)$, on the copy of $SO(4)$ associated with the link $l = f \cap \Sigma$ determined by $f$, with orientation such that the node $n = t \cap \Sigma$ is the source of $l$. The $B_f(t)$’s are then quantized as
\[ \hat{B}_f(t) := \kappa \left( \frac{\gamma^2}{\gamma^2 - 1} \right) \left( \hat{J}_f(t) - \frac{1}{\gamma} \hat{J}_f(t) \right). \]  
(16)

Next, we promote (11) and (12) to quantum operators. We note that the first constraint commutes with the others and can be carried directly to quantum theory. In terms of the usual (generalized) spin-network basis $\Psi_{(j^+, j^-), T}$ labeled by spins $(j_f^+, j_f^-)$ on links and tensors $T$, at nodes, the first constraint implies
\[ j_f^+(j_f^- + 1) = \left( \frac{1 + \gamma}{1 - \gamma} \right)^2 j_f^-(j_f^- + 1). \]  
(17)

For either $\gamma \ll 1$ or $\gamma = \infty$, this condition is satisfied by the simple representations of $SO(4)$, i.e., $j^+ = j^-$. In the following, we will always specialize to either $\gamma \ll 1$ or $\gamma = \infty$ and we will restrict consideration to states in the kernel of the diagonal simplicity constraints, that is, satisfying $j^+ = j^-$. 

3. Cross-simplicity in the various models, and the solution spaces

Up until now, the quantum theory is standard, and the same as in the BC model. The difference among the BC model, the flipped model and the FKLS model comes in the implementation of the cross-simplicity constraints. Whereas the diagonal simplicity constraints constrain the spins on the $SO(4)$ spin networks, the cross-simplicity constraints constrain the intertwiners. For simplicity of presentation, in this section, we will consider a fixed node $n$ and the intertwiner space at that node. We number the links at this node $1, 2, 3, 4$. For external spins $(j_1, j_1), \ldots, (j_4, j_4)$, the space of possible tensors at the node $n$ will be denoted
\[ T^{(j_1, j_1)} := \mathcal{H}_{(j_1, j_1)} \otimes \cdots \otimes \mathcal{H}_{(j_4, j_4)} = (\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_1}) \otimes \cdots \otimes (\mathcal{H}_{j_4} \otimes \mathcal{H}_{j_4}), \]  
(18)

where $\mathcal{H}_j$ denotes the carrying space for the spin $j$ irreducible representation (irrep) of $SU(2)$ and $\mathcal{H}_{(j^+, j^-)}$ denotes the carrying space for the irrep of $SO(4)$ labeled by the spins $(j^+, j^-)$. The associated intertwiner subspace at $n$ will be denoted $T^{(j_1, j_1)} \subset T^{(j_1, j_1)}$. 

In the following, we will discuss the solutions to cross-simplicity in the various models prior to averaging over $SO(4)$ gauge transformations, for simplicity of comparison. That is, we
will discuss imposing the ‘gauge-fixed’ cross-simplicity constraint (9) in the various models. In each model, this will give us a ‘gauge-fixed’ solution subspace of $T^{(j,i)}$, which, when averaged over $SO(4)$ gauge transformations will yield the final physical space of intertwiners to be summed over in the spin-foam sum, and to be used in building the physical boundary Hilbert space.

3.1. BC model

In the BC model, we take the unflipped symplectic structure—that is, $\gamma = \infty$. As noted above, the new, reformulated cross-simplicity constraints (9) in this case close, forming a first class system. Thus, constraints (12) can all be imposed simultaneously as operator equations. We have $C_f \sim L_f \approx 0$, which restricts the intertwiner space to be one dimensional, being spanned by the one unique Barrett–Crane intertwiner [17, 18].

3.2. The new spin-foam models

Both of the new spin-foam models can be viewed as arising from the use of coherent states to impose the cross-simplicity constraints. The idea of using coherent states to impose second class constraints can be found, for example, in [19] (or in more implicit form in [20]). The specific coherent states relevant for the new spin-foam models were introduced by Livine and Speziale in [10]. We review these coherent states here (see [10] and appendix B for further details).

Consider the $SU(2)$ coherent states $|j, \hat{n}\rangle$, where $j \in \mathbb{N}/2$ and $\hat{n}$ is a unit vector in $\mathbb{R}^3$. $|j, \hat{n}\rangle$ may be defined as the $m = j$ eigenstate of $\hat{n} \cdot J$, where $\hat{J}^i := \frac{1}{2} \sigma^i$ is a basis of the Lie algebra satisfying the usual angular momentum commutation relations. Starting with these, one can construct $SO(4)$ coherent states by tensoring them together giving states of the form $|j^+, \hat{n}^+\rangle \otimes |j^-, \hat{n}^-\rangle$. On such states, the expectation value of constraint (12) is given by

$$\langle \hat{C}_f \rangle = j^+ \left(1 - \frac{1}{\gamma}\right) \hat{n}^+ + j^- \left(1 + \frac{1}{\gamma}\right) \hat{n}^- \approx 0. \quad (19)$$

One can see that $\hat{n}^+ = \hat{n}^-$ and $\hat{n}^+ = -\hat{n}^-$ are solutions for $\gamma \ll 1$ and $\gamma = \infty$, respectively. The first set of states leads to the flipped model [8, 9] as shown in [12]. The second set of states leads to the FKLS model [11, 12]. In the first case, constraints (9) are actually satisfied as matrix elements: that is, given any two $SO(4)$ coherent states $|\psi\rangle$ and $|\chi\rangle$ constrained by $\hat{n}^+ = \hat{n}^-$, we have $\langle \chi | C_f | \psi \rangle \approx 0$. This fact is also true for the original form of the constraints $C_{ij}^f$, as was noted in [8, 9].

Up to now we have been considering only coherent states in the carrying space of a representation associated with a single face, but we could just as well consider intertwiner spaces between the four faces meeting in a single tetrahedron. All we have to do to get a state in this space is to tensor four coherent states and project the resulting state into the gauge-invariant subspace. Let us state this in equations. It will be convenient to first define the gauge-fixed tensors in the two models:

$$I^{+GF}_{n_{a}}(\hat{n}_{a}) = \otimes_{n_{a}=1}^{4} (|j_{\hat{n}_{a}}\rangle \otimes |j_{\hat{n}_{a}}\rangle), \quad (20)$$

$$I^{-GF}_{n_{a}}(\hat{n}_{a}) = \otimes_{n_{a}=1}^{4} (|j_{\hat{n}_{a}}\rangle \otimes |j_{\hat{n}_{a}}\rangle), \quad (21)$$

The original derivation in [8, 9] did not use coherent states, but rather a ‘master constraint’-like approach, which is reviewed in subsection 3.3. We here use coherent states for unity of initial presentation of the two new models.
so that the $+$ label corresponds to the flipped model and the $-$ label corresponds to FKLS (the notation being motivated by the sign of the second $\hat{n}$ on the right-hand side). Projection of these to the $SO(4)$ gauge-invariant subspace then yields

$$I^+_n(\hat{n}_a) = \int_{SO(4)} dGG \cdot I^{+\, GF}_n(\hat{n}_a),$$

$$I^-_n(\hat{n}_a) = \int_{SO(4)} dGG \cdot I^{-\, GF}_n(\hat{n}_a).$$

Transforming to the basis $|j_a; i^+, i^-\rangle$, one can check that these intertwiners can be written as

$$I^+_n(\hat{n}_a) = \sum_{i^+, i^-} c_{i^+}(\hat{n}_a) c_{i^-}(\hat{n}_a) |j_a; i^+, i^-\rangle,$$

$$I^-_n(\hat{n}_a) = \sum_{i^+, i^-} c_{i^+}(\hat{n}_a) \bar{c}_{i^-}(\hat{n}_a) |j_a; i^+, i^-\rangle,$$

where the coefficients $c_i(\hat{n}_a)$ are given in appendix B. The intertwiner states in the flipped model are symmetric under parity operation (that is, under the interchange of the self-dual and anti-self-dual components) and satisfy the $C_{ff'}$ as matrix elements. Intertwiners in the FKLS model are complex conjugated under the interchange of the self-dual and anti-self-dual parts, and satisfy the constraints only as expectation values.

Equations (22) and (23) specify a restriction to certain $SO(4)$ coherent states in each model. What is the corresponding space of solutions to the cross-simplicity constraints? If the coherent states are to be considered solutions, and the solution space is to be a vector space, one has to consider the span of all the constrained coherent states as the solution space. We will look at these spans in the case of the flipped model and the FKLS model and see what they are; in the FKLS case we will see that the span is in fact the entire space of $SO(4)$ intertwiners at the node. These constrained spaces of intertwiners for the models will then be the intertwiners one sums over in the spin-foam sum, as well as the spaces of intertwiners used in describing the boundary state space for each model.

For the purposes of describing these solution spaces, for each of the four links $a = 1, 2, 3, 4$ at the node of interest, let $\hat{L}_a$ denote the rotation generators introduced earlier, generating the $SO(3)$ subgroup preserving the fixed vector $n_I$. For each $a$, $\hat{L}_a \hat{L}_{ai}$ is then the Casimir operator for the representation of this subgroup on each of the four links. For a given set of fixed external spins $(j_1, j_1), \ldots, (j_4, j_4)$, the spectrum of each of these Casimir operators is $\{k_a(k_a + 1)\}$ with $k_a \in \{0, 1, \ldots, 2j_a\}$. One then has a decomposition of the tensor space at the node into the simultaneous eigenspaces of these $SO(3)$ Casimirs:

$$T^{(\vec{j}, \vec{j})} = \bigoplus_{a=1}^4 \bigoplus_{k_a=0}^{2j_a} \mathcal{H}_{k_a}. \quad (26)$$

That is, this is the decomposition of the tensor space into irreducible representations of the $SO(3)$ subgroup preserving the chosen gauge-fixed $n_I$.

### 3.3. The solution space for the flipped model

At the gauge-fixed level, the solution space for the flipped model is easy to state. In terms of the decomposition (26), it is the $k_a = 2j_a$ subspace:

$$\mathcal{H}^{\text{Flipped, GF}}_n = \bigoplus_{a=1}^4 \mathcal{H}_{2j_a} \subset T^{(\vec{j}, \vec{j})}. \quad (27)$$
See [11, 12] for proof that this is the space spanned by (20) (so that its projection to the gauge-invariant subspace is spanned by (22)). As noted above, the constraint operators corresponding to (14) (as well as the operators corresponding to the original formulation of cross-simplicity in (3.ii)) have zero matrix elements on this space. Furthermore, this space can also be obtained by imposing a sort of ‘master constraint’ [14] constructed from the gauge-fixed constraints (12). As above, label the links at the node \( n \) by \( a \in \{1, 2, 3, 4\} \), and fix the spins \( j_a \) on each of these links. For each link, labeled by \( a \in \{1, 2, 3, 4\} \), one can then define a ‘master constraint’
\[
\hat{M}_a = \hat{C}_a^i \hat{C}_a^i
\]
acting on the space \( T^{(\vec{j}, \vec{j})} \). Each operator \( \hat{M}_a \) has as minimal eigenvalue \( 2 j_a \bar{\hbar}^2 \) (a value zero in the semiclassical limit\(^8\)). Just to better follow the prescription of [14], let us sum these four master constraints into a single constraint for the node:
\[
\hat{M}_n := \sum_{a=1}^{4} \hat{M}_a.
\]
As all the constraints \( \hat{M}_a \) commute with each other, the spectrum of \( \hat{M}_n \) will just be the point-wise sum of the spectra of the operators \( \hat{M}_a \), so that the minimal eigenvalue will be \( \sum_{a=1}^{4} 2 j_a \bar{\hbar}^2 \). Thus, following the prescription of [14], we take the solution space to be the eigenspace of \( \hat{M}_n \) with minimal eigenvalue \( \sum_{a=1}^{4} 2 j_a \bar{\hbar}^2 \). One can check that this space is precisely \( H_{\text{Flipped}}^{\text{GF}}_n \). (This presentation is a different way of looking at the original derivation of \( H_{\text{Flipped}}^{\text{GF}}_n \) in [8, 9].)

### 3.4. The solution space for the FKLS model

In this subsection, we wish to understand more explicitly the intertwiner solution space for the FKLS model:
\[
\mathcal{H}^{\text{FKLS,GF}}_n := \text{span}\{ I_{n}^{-,GF} (\hat{\eta}_o) \}_{\hat{\eta}_o}.
\]
For this purpose, let us first define, for each \( j \in \mathbb{N}/2 \),
\[
G_j := d_j^2 \int_{g \in SU(2)} \text{d}g \langle j, g \cdot \hat{\eta}_o | \otimes | j, -g \cdot \hat{\eta}_o \rangle \otimes \langle j, g \cdot \hat{\eta}_o | \otimes \langle j, -g \cdot \hat{\eta}_o |,
\]
so that \( G_j \) maps \( \mathcal{H}(j,j) \) to \( \mathcal{H}(j,j) \). Here, \( d_j := 2j + 1 \) is a dimension factor, and \( \hat{\eta}_o \) is an arbitrary unit reference vector in \( \mathbb{R}^3 \). Because of the integration over the action of \( g \), the above expression is independent of the choice of \( \hat{\eta}_o \). \( G_j \) so defined is in fact the same as the \( G_j \) defined in equation (76) in [11]. Next, for a fixed node \( n \) with external spins, \( (j_1, j_1), \ldots, (j_4, j_4) \), consider the map
\[
G_j := G_{j_1} \otimes \cdots \otimes G_{j_4} : T^{(\vec{j}, \vec{j})} \rightarrow T^{(\vec{j}, \vec{j})},
\]
where \( T^{(\vec{j}, \vec{j})} \) is the tensor space introduced in (18). One can see immediately from its definition that
\[
\text{Im} G_j \subseteq \mathcal{H}_n^{\text{FKLS,GF}}.
\]
But from equation (81) in [11],
\[
G_j = d_j^2 \sum_{k=0}^{2j} C_k^j P_k,
\]
\(^8\) The semiclassical limit is the limit \( \bar{\hbar} \to 0 \) while holding the analogues of classical quantities fixed. With diagonal simplicity satisfied, as here, the eigenvalues \( A_{2j}^2 \) of the full SO(4) area squared are proportional to \( \hbar^2 (j+1) \). Thus, taking \( \hbar \to 0 \) holding \( A_4 \) fixed, \( \hbar^2 j^2 \) approaches a constant, so that \( \hbar^2 j \) goes to zero.
where $P_k : \mathcal{H}(j,j) \to \mathcal{H}(j,j)$ is the projector onto the spin $k$ representation in the decomposition of $\mathcal{H}(j,j)$ into irreducibles of the fixed $SO(3)$ subgroup ($\mathcal{H}(j,j) = \oplus_{k=0}^{2j} \mathcal{H}_k$), and where $C_k^j$ is given by

$$C_k^j = \frac{(2j)!}{(2j-k)! (2j+k+1)!} .$$

(35)

In (34), because all of the coefficients of the projection operators are nonzero, $G_j$ is manifestly invertible and hence its image is the entirety of $\mathcal{H}(j,j)$. It follows that the image of $G_j$ is the entirety of $T^{(j,j)}$. Thus, from (33), the gauge-fixed solution space for FKLS at the node $n$ is the entirety of $T^{(j,j)}$. From this, in turn, it follows that, after projecting onto the $SO(4)$ gauge-invariant subspace, we will obtain

$$\mathcal{H}^{FKLS}_n = T^{(j,j)},$$

(36)

i.e., the final solution space for the intertwiners in the FKLS model is in fact all of the $SO(4)$ intertwiners.

Thus, if one is to understand the implementation of cross-simplicity in this model as a restriction on states (in particular, boundary states), then in fact we see that no constraint is imposed. The first significance of this is that it shows clearly that the boundary Hilbert space of the FKLS model is not isomorphic to the Hilbert space of LQG, whereas there are strong indications that the boundary Hilbert space of the flipped model [8, 9] is so isomorphic. Second, it raises questions as to the method used to incorporate cross-simplicity in FKLS. It is true there is still a sense in which cross-simplicity is imposed: cross-simplicity affects the final vertex amplitude in the FKLS model. However, it is unusual for a constraint to only manifest itself in this way; it thus needs to be better understood.

One more remark is in order. As noted in subsection 3.2, the expectation values of the constraint operators $\hat{C}_f^j$ with respect to the FKLS coherent states (21) are in fact zero. However, the matrix elements of these constraint operators with respect to these states are not zero. This highlights the importance of satisfying the constraints by matrix elements, and not just by expectation values: for the former is closed under linear combinations, whereas the latter is not. The mere fact that the constraints $\hat{C}_f^j$ have zero expectation value on the ‘solution’ states (21) tells us nothing about their span, and in fact their span is the whole intertwiner space.

4. The coherent state approach and first class constraints

We suggest that the reason the coherent state approach fails to constrain the states in FKLS is due to the first class nature of the cross-simplicity subsystem of constraints in this case. The class of coherent states used have nonzero spread in these constraints. More generally, for a system with both first and second class constraints mixed, one should be careful to choose coherent states sharply peaked with respect to the first class part of the constraints. We give a simple example where this kind of problem appears, and compare with the more standard Dirac and master constraint methods.

4.1. A simple example

A simple example suffices to show why this should be the case. In order to keep things conceptually clear, we first consider a system with a single degree of freedom. The phase space is $\mathbb{R}^2 = \{(q, p)\}$ with standard Poisson brackets $\{q, p\} = 1$. The kinematical quantum state space is then the standard $\mathcal{H}_{kin} = L^2(\mathbb{R})$, with $\hat{q}$ and $\hat{p}$ acting in the usual way by
multiplication and derivation. Consider the standard (unnormalized) family of coherent states for the simple harmonic oscillator:

\[ \psi_{\text{coh}}^{\text{coh}}(q, p)(q) = e^{-ipq} e^{-(q - q_0)^2/2}. \] 

(37)

Consider the pair of second class constraints

\[ q = 0 \quad \text{and} \quad p = 0. \] 

(38)

The solution space for this pair of constraints, using the coherent states (37) is

\[ \mathcal{H}_{\text{phys}}^{\text{2nd class}} = \text{span}\{\psi_{(0,0)}^{\text{coh}}\}, \] 

(39)

i.e., the one-dimensional space spanned by the vacuum. Thus, one obtains an actual constraint on the state space, and it is in fact what one expects: classically the constraints (38) completely constrain the phase space to a point. Likewise, in the final quantum theory, there is only a single state, modulo rescaling. Consider now the first class system instead

\[ q = 0. \] 

(40)

Classically, this constraint generates the gauge \( p \mapsto p + \lambda \). Thus, after imposing the constraint and dividing by the gauge\(^9\), one expects again the degree of freedom to be completely eliminated. The quantum solution space of this constraint, using the coherent states (37), is, on the other hand

\[ \mathcal{H}_{\text{phys}}^{\text{1st class}} = \text{span}\{\psi_{(0,p)}^{\text{coh}}\}_{p \in \mathbb{R}} = \text{span}\{e^{-ipq} e^{-q^2/2}\}_{p \in \mathbb{R}}, \] 

(41)

where the overbar denotes Cauchy completion. It is not difficult to see that in fact \( \mathcal{H}_{\text{phys}}^{\text{1st class}} = \mathcal{H}_{\text{kin}} \).

**Proof 1.** We show that the orthogonal complement of \( \mathcal{H}_{\text{phys}}^{\text{1st class}} \) in \( \mathcal{H}_{\text{kin}} \) is trivial. Suppose \( \Psi \) is orthogonal to \( \mathcal{H}_{\text{phys}}^{\text{1st class}} \). Then, \( \int dq \, e^{ipq} e^{-q^2/2}\Psi(q) = 0 \) for all \( p \). Then the Fourier transform of \( e^{-q^2/2}\Psi(q) \) is zero, so that \( \Psi(q) \) is zero.\( \square \)

Thus no constraint has been imposed in \( \mathcal{H}_{\text{phys}}^{\text{1st class}} \).

Now, one may think at first glance that this is a rather trivial example. However, one can easily extend it to an arbitrary number \( n \) of degrees of freedom. In this case, the kinematical Hilbert space \( \mathcal{H}_{\text{kin}} \) decomposes into a tensor product of kinematical Hilbert spaces, one for each degree of freedom:

\[ \mathcal{H}_{\text{kin}} = \bigotimes_{i=1}^{n} \mathcal{H}_{\text{kin}}^{i} \] 

(42)

and the coherent states \( \psi_{\text{coh}}^{\text{coh}}(q_i, p_i) \) correspondingly decompose into a tensor product of coherent states

\[ \psi_{\text{coh}}^{\text{coh}}(q_i, p_i) = \bigotimes_{i=1}^{n} \psi_{(q_i, p_i)}^{\text{coh}}. \] 

(43)

Finally, for some \( m < n \), one can consider either the second class set of constraints

\[ q_i = 0 \quad \text{and} \quad p_i = 0 \quad \text{for} \quad i = 1, \ldots, m \] 

(44)

or the set of first class constraints

\[ q_i = 0 \quad \text{for} \quad i = 1, \ldots, m. \] 

(45)

Imposing these two sets of constraints using the coherent states (43) then yields conclusions similar to those for one degree of freedom. For the case of the second class constraints (44),\(^9\) The two operations simultaneously accomplished in quantum theory when imposing first class constraints a la Dirac.

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\(^9\) The two operations simultaneously accomplished in quantum theory when imposing first class constraints a la Dirac.
the state space is reduced to that of \( n - m \) degrees of freedom, as one would expect. However, for the case of the first class constraints (45), no constraint is imposed on the state space.

Let us compare this with what happens in the standard Dirac approach, and then with what happens in the master constraint approach. We will first discuss the second class constraints (44) and then the first class constraints (45). In the standard Dirac approach, in the case of the second class constraints, one modifies the Poisson brackets such that all of the constraints commute with each other. Constraints (44) are then imposed strongly as operator equations. The end result is equivalent to simply ignoring the degrees of freedom being constrained in (44), so that one has again only \( n - m \) degrees of freedom left [21]. In the master constraint approach, one selects the eigenstates of the operator

\[
\hat{M} := \sum_{i=1}^{m} (\hat{q}_i^2 + \hat{p}_i^2),
\]

with least eigenvalue. The least eigenvalue of \( \hat{M} \) in this case is \( \frac{m}{2} \), and the corresponding eigenstates are all states that are in the simple harmonic oscillator vacuum for the first \( m \) degrees of freedom, whence the solution space is identical to that obtained via the coherent state method,

\[
\mathcal{H}_{\text{phys}}^{\text{2nd class master}} = \mathcal{H}_{\text{phys}}^{\text{2nd class coh method}} = \text{span}\{\Psi_{\text{coh}}^\text{phys}(\{0,0,\ldots,0,q_{m+1},\ldots,q_n\},\{0,0,\ldots,0,p_{m+1},\ldots,p_n\})\}.
\]

This is in turn equivalent to the result with the standard Dirac approach because the first \( m \) degrees of freedom have been frozen in the vacuum state, so that one can ignore them. More precisely, of the original basic observables \( \hat{q}_1, \ldots, \hat{q}_m, \hat{p}_1, \ldots, \hat{p}_m \), only \( \hat{q}_{m+1}, \ldots, \hat{q}_n, \hat{p}_{m+1}, \ldots, \hat{p}_n \) preserve the physical Hilbert space \( \mathcal{H}_{\text{phys}}^{\text{2nd class master}}(=\mathcal{H}_{\text{phys}}^{\text{2nd class coh method}}) \); these are also the only operators that preserve the physical Hilbert space in the Dirac case. Thus, the physical observable algebra is naturally taken to be the one generated by \( \hat{q}_{m+1}, \ldots, \hat{q}_n, \hat{p}_{m+1}, \ldots, \hat{p}_n \), and the representations of this algebra on the physical Hilbert space in the Dirac approach and the physical Hilbert space in the coherent state/master constraint approaches are equivalent\(^{10}\).

Finally, in the case of the first class constraints (45), first, we note that the corresponding classical gauge is given by \( p_i \mapsto p_i + \lambda_i, i = 1, \ldots, m \), so that once we divide by the gauge, the first \( m \) degrees of freedom are eliminated. Coming to quantum mechanics, using the Dirac method—i.e., in this case, strong imposition of the constraints as operators equations with no modification of Poisson brackets—we obtain the solution space

\[
\mathcal{H}_{\text{phys}}^{\text{1st class Dirac}} = L^2(q_{m+1}, \ldots, q_n) \otimes \prod_{i=1}^{m} \delta(q_i).
\]

This is manifestly isomorphic to the Hilbert space of states corresponding to only the last \( n - m \) degrees of freedom. What happens with the master constraint in this case? The minimal eigenvalue of \( \hat{M} = \sum_{i=1}^{m} \hat{q}_i^2 \) is zero in this case, and the zero eigenspace is precisely (48) once again. However, with the coherent state method, as we saw, imposition of (45) yields no constraint on states.

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\(^{10}\) This is also related to the following potential objection: in the Dirac approach, the wavefunctions solving constraints (44) have no spread in \( q_1, \ldots, q_m \) and \( p_1, \ldots, p_m \), while in the coherent state and master constraint approaches, the physical states do have spread in these variables. Thus, one may argue, it seems that the coherent state/master constraint approach is here weaker than the standard Dirac approach. But in fact it is not: the operators \( \hat{q}_1, \ldots, \hat{q}_m \) and \( \hat{p}_1, \ldots, \hat{p}_m \) do not preserve the physical Hilbert space in either approach, and are not physical observables. Thus, probability distributions in these variables are not measurable by any physical process.
4.2. The classes of constraints in the flipped case

One can also repeat the multiple-degrees-of-freedom example of the last subsection, but mix the first and second class constraints. If one then imposes the constraints using the coherent states, one finds that the state space is reduced, but not as much as it should be. This then raises a question. In the case of the flipped model, it is true that the cross-simplicity constraints are not first class. However, as shown in appendix A, neither are they purely second class. There is a first class component given by

$$\tilde{C}_f \propto L_f^I C_{fI} = L_f^I K_{fI} = C_{ff}.$$  (49)

But as one can see, this first class component is just the diagonal simplicity constraint for the link. The coherent states (20) have zero spread in this constraint. As a consequence, the coherent state method, in this case, actually imposes (49) strongly, as it should12.

The master constraint method is ‘smart enough’ to impose first class constraints strongly and second class constraints weakly, so that the existence of a first class component is of no concern for the master constraint method. The fact that the coherent state method and master constraint method match for the flipped model can thus be taken as a further check that the chosen coherent states (20) handle correctly the issue of the class of the constraints in this case.

5. The spectra of the area operators

In this section, we will analyze the spectra of area operators in the two models, and compare these spectra with those found in loop quantum gravity. We will find that the area spectrum of the model for small $\gamma$ [8, 9, 12] matches that of loop quantum gravity. The area spectrum of the model for infinite $\gamma$ [11, 12] (the FKLS model) will be found to be quite different.

The section is organized as follows. First, a derivation of the spectra of the area operators for general $\gamma$ will be given. For $\gamma \ll 1$ and $\gamma = \infty$, we then show that the LQG spectrum, or, respectively, that of [22] is obtained.

5.1. Areas: classical analysis

Classically, the area of a triangle $f$ is given by13

$$A_3(f)^2 = \frac{1}{2} (\ast B_f)^I J_{IJ} (\ast B_f)_J = \frac{1}{2} B_f^I B_f^J$$

$$= \frac{2\kappa^2}{(\gamma^2 - 1)^2} \left[ (J_f^{(+)i})^2 \left(1 - \frac{1}{\gamma}\right)^2 + (J_f^{(-)i})^2 \left(1 + \frac{1}{\gamma}\right)^2 \right].$$  (50)

When the gauge-fixed cross-simplicity constraints (9) hold, the $B_f^I$ vanish and the above quantity is equal to

$$A_3(f)^2 := B_f^{0I} B_{f0} = \kappa^2 \frac{\gamma^4}{(\gamma^2 - 1)^2} \left[ J_f^{(+i)} \left(1 - \frac{1}{\gamma}\right) - J_f^{(-)i} \left(1 + \frac{1}{\gamma}\right) \right]^2.$$  (51)

11 The coherent states (20) (and (21)) are only partially coherent states: they are coherent states with respect to the operators $B_f(t)^I$, but not with respect to the connection operators $U_f$. This is why it is possible for the coherent states to be sharply peaked with respect to $\hat{B} \cdot \hat{B} = \hat{C}_f^4$ and $\ast \hat{B} \cdot \hat{B} = \hat{\tilde{C}}_f^4$.

12 Showing how extraordinarily well-adapted the coherent states (22) are for imposing these constraints.

13 To see this, choose the Cartesian coordinates $(x^1, x^2)$ in the triangle $f$, then

$$(*B_f)^I (*B_f)_J = \frac{1}{4} (e_1 \cdot e_2^1 \cdot e_2 - (e_1 \cdot e_2)^2) = 2 \text{Area}(f).$$
which we refer to as the gauge-fixed area of $f$. Rewriting (51) in preparation for quantum theory, using $L^I = J^{|\nu I|} + J^{|-\nu I|}$, we have

$$A_3(f)^2 = 2\kappa^2 \left( \frac{\gamma^4}{(\gamma^2 - 1)^2} \right) \left[ (J^{|\nu I|})^2 \left( 1 - \frac{1}{\gamma} \right) + (J^{|-\nu I|})^2 \left( 1 + \frac{1}{\gamma} \right) \right] - k^2 \frac{\gamma^2}{\gamma^2 - 1} (L')^2. \tag{52}$$

### 5.2. Areas: quantum analysis

Let us take a look at the quantum area operators. Working in the $SO(4)$ (generalized) spin-network basis, $\hat{A}_4(f)$ is diagonal with eigenvalues

$$\text{spec}(\hat{A}_4(f)^2) = \left\{ 2\kappa^2 \left( \frac{\gamma^2}{(\gamma^2 - 1)^2} \right) j^+ (j^+ + 1) + \frac{\gamma^2}{(\gamma - 1)^2} j^- (j^- + 1) \right\} \left| j^\pm \in \mathbb{N}/2 \right\}. \tag{53}$$

The spectrum of $\hat{A}_3(f)^2$ can be easily calculated and is given by

$$\text{spec}(\hat{A}_3(f)^2) = \left\{ \frac{2\kappa^2 \gamma^4}{(\gamma - 1)^2} \left[ j^+ (j^+ + 1) \left( 1 - \frac{1}{\gamma} \right) + j^- (j^- + 1) \left( 1 + \frac{1}{\gamma} \right) \right] \right. - \left. \frac{k^2 \gamma^2}{\gamma^2 - 1} k (k + 1) \right\} \left| j^\pm \in \mathbb{N}/2, k \in \mathbb{N} \right\}. \tag{54}$$

For small $\gamma$ and infinite $\gamma$, diagonal simplicity dictates $j_+ = j_- \equiv j (\in \mathbb{N}/2)$. In these cases, the spectrum of $\hat{A}_3(f)^2$ reduces to $4\kappa^2 \gamma^2 j (j + 1)$ for $\gamma \ll 1$, and to $4\kappa^2 j (j + 1)$ for $\gamma = \infty$.

The spectrum of $\hat{A}_3(f)^2$ reduces to

$$\gamma \ll 1 \rightarrow \text{spec}(\hat{A}_3(f)^2) = \{ \kappa^2 \gamma^2 k (k + 1) \mid k \in \mathbb{N} \}$$

$$\gamma = \infty \rightarrow \text{spec}(\hat{A}_3(f)^2) = \{ 4\kappa^2 j (j + 1) - \kappa^2 k (k + 1) \mid j = j^+ = j^- \in \mathbb{N}/2, k \in \mathbb{N} \}. \tag{55}$$

It is interesting to note that in the small $\gamma$ case, the dependence of the spectrum on $\gamma$ is exactly the same as in LQG, whereas in the large $\gamma$ limit, all the dependence of the spectrum on $\gamma$ vanishes. Furthermore, note the latter spectrum is exactly the Euclidean analogue of the Lorentzian area spectrum presented in equation (24) of [22]. This is in part not surprising, as, in [22], the $B$ variables are quantized as the generators of $SO(4)$, which, as can be seen from (16), is the symplectic structure corresponding to the $\gamma = \infty$ case.

Incorporation of the cross-simplicity constraints in the BC model leads to the $\gamma = \infty$ area spectrum (55) with $k$ set to zero. Incorporation of the cross-simplicity constraints for the flipped and FKLS models $\gamma = \infty$, however, does not change the spectra (55). In the flipped case $\gamma \ll 1$, cross-simplicity tells us that $k = 2j$, whereas in the FKLS model $\gamma = \infty$, as discussed in the last two sections, there are no restrictions on the space of states from cross-simplicity.

Let us next recall that in the classical theory, the gauge-fixed area $A_3(f)$ equals the non-gauge-fixed area $A_4(f)$. What happens to this equality in quantum theory? The fate of this equality is different in the FKLS, flipped and BC models; we first state what happens in each of the models and then shed light on why. First, in the FKLS model, the spectra are completely different. In the flipped model, the spectra differ only by a term of order of $j$, a term which is zero in the semiclassical limit. Finally, in the BC model, the spectra are exactly equal even in the quantum theory. To see why these observations are true, we first note that the difference between the gauge-fixed and non-gauge-fixed areas is proportional to the sum of the squares of the gauge-fixed cross-simplicity constraints (9)—the master constraint $M_t$ for cross-simplicity at a given $t$ and $f$. It is then easy to see why in the FKLS case the spectra are

14 Furthermore, in (24) of [22], one has not yet fully imposed cross-simplicity, as in equation (55) here one has not imposed cross-simplicity.
completely different: in FKLS, cross-simplicity is not imposed at all as a restriction on states, so that there is no restriction on the eigenvalues of $\hat{M}_{ij}$. In the flipped model, one does impose cross-simplicity on states, and as noted earlier, the method of imposing cross-simplicity can even be viewed as choosing the minimal eigenvalue of the master constraint. The minimal eigenvalue is a quantity zero in the semiclassical limit, and this is why, in the flipped model, the spectra of the gauge-fixed and non-gauge-fixed areas differ by a quantity that is zero in the semiclassical limit. Finally, in the BC model, the cross-simplicity constraints (13) are imposed strongly, directly as operator equations, so that the difference between the operators $\hat{A}_4(f)$ and $\hat{A}_3(f)$ is exactly zero after imposing of cross-simplicity.

It remains to address a natural question. $\hat{A}_3(f)$ is not an $SO(4)$-Gauss gauge-invariant quantity. One may then ask: why are we then interested in the spectra (55)? As noted in footnote 6, the gauge-fixing involved in defining $A_3(f)$ is a part of the time gauge which is used in LQG. Thus it is natural to look at the area $A_3(f)$ when comparing spectra with those in LQG. Furthermore, we are considering these spectra in the spirit of [23]. That is, $\hat{A}_3(f)$ is viewed as a partial observable: it is to be made into a complete observable by coupling it with an appropriate choice of clock [23, 24] (and not by group averaging it). In the present case, the relevant gauge freedom is the internal tetrad degrees of freedom; therefore, an appropriate clock would have to be constructed from a field that is sensitive to this internal freedom, such as a spinorial matter field. The spectrum of the resulting complete observable will probably depend on which clock is used. The kinematical spectrum is then seen as a 'clock-independent spectrum'; its physical meaning is discussed, e.g., in [23].

6. Discussion

In this paper, we have discussed properties of the two recently proposed spin-foam models [8, 9] and [11, 12], which we refer to as the flipped model and the FKLS model, respectively. In particular, we have recalled that the boundary Hilbert space of the flipped model matches the Hilbert space of LQG, whereas we have shown that the boundary space of the FKLS model is completely different. The area spectra of the two theories were also compared, and it was found that in the flipped case, the $(SO(3))$ area spectrum exactly matches that of LQG, whereas in the FKLS model, the spectra are completely different. In the case of the BC model, the boundary Hilbert space differs from that of LQG in that the BC model has no intertwiner degrees of freedom; and the BC area spectrum differs from that of LQG—it does not depend at all on the Barbero–Immirzi parameter.

Furthermore, an unusual aspect of the FKLS model was pointed out: in the FKLS model, cross-simplicity is in fact not imposed on states. This was seen to be due to the fact that the gauge-fixed cross-simplicity constraints in fact become a first class system, and the coherent states used in FKLS have nonzero spread in these constraints. Nevertheless, it is possible that this is not a fatal problem. For the strategy used in [11, 12] to construct the FKLS model is such that cross-simplicity has an effect on the final vertex of the model. More specifically, the vertex seems to suppress intertwiners that are far from the BC intertwiner. This heuristic statement comes from the fact that the coefficients $C_\ell^j$ in (35) appear in the FKLS amplitude sum:

$$\sum_{jj} \prod_{f} d_{jj}^2 \sum_{l_l} \prod_{f} d_{l_l} \sum_{k_{ff}} \prod_{f} d_{k_{ff}}^2 C_{k_{ff}}^j A_{Grav}^j(j_f, k_{ff}, l_f),$$

(57)

where $A_{Grav}^j(j_f, k_{ff}, l_f)$ is as in equation (84) of [11]. The coefficients $C_\ell^j$ are peaked for $k = 0$ and vanish for large $k$. $k = 0$ corresponds to the Barrett–Crane intertwiner, whence it appears that the FKLS model favors the propagation of the Barrett–Crane intertwiner. This seems to be
the sense in which FKLS imposes cross-simplicity. Furthermore, up to this point, the effect of the full spin-foam sum on the fully physical Hilbert space has not been considered. (Here, the ‘fully physical Hilbert space’ is understood as the image of the generalized projector implicitly defined by the spin-foam sum.) As cross-simplicity in FKLS does affect the spin-foam sum, it may be that this fully physical Hilbert space is also affected.

In summary, the flipped model exactly matches LQG at the level of Hilbert space structure and area operators, whereas the FKLS model and the BC model do not. Furthermore, in the FKLS model, it was found that no cross-simplicity is imposed on states. Forthcoming papers are in preparation on the Lorentzian case and the case of an arbitrary \( \gamma \) [13, 25]; these will further clarify these issues.

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Appendix A. Poisson algebra of the off-diagonal simplicity constraints

In this appendix, we first recall some facts about the structure of the algebra \( \mathfrak{so}(4) \) and then analyze the class structure of constraints (9).

Let \( J^{ij} \) be the generators of \( \mathfrak{so}(4) \). Concretely, one can take \( \mathfrak{so}(4) \) to be the matrix algebra of skew \( 4 \times 4 \) matrices; in terms of this viewpoint, the generators \( J^{ij} \) are defined to be the basis \( (J^{ij})_{MN} \):

\[
J^{ij} = \frac{1}{2}(\epsilon^{ijk} J^{jk} + \epsilon^{jki} J^{kj}),
\]

The \( \mathfrak{su}(2) \) structure is then easy to see, since \( [J^i, J^j] = \epsilon^{ijk} J^k \). In particular, we can choose \( n = (0, 0, 0, 1) \), and \( v_i = \delta_i^0 \), and we have

\[
J^{ij} = \frac{1}{2} \epsilon^{ijk} J^k \pm \frac{i}{2} J^{0i}.
\]

The Casimirs \( (J^{(+)})^2 \), \( (J^{(-)})^2 \) have spectra \( \{ j^+ (j^+ + 1) \mid j^+ \in \mathbb{N}/2 \} \) and \( \{ j^- (j^- + 1) \mid j^- \in \mathbb{N}/2 \} \), respectively, and irreducible representations of SO(4) are labeled by pairs \( (j^+, j^-) \).

Finally, the generators of rotations and boosts in the chosen frame \( n^I = \delta^I_0 \), \( v^I_i = \delta^I_i \) are

\[
L^i = \frac{1}{2} \epsilon^{ijk} J^k = J^{(+)i} + J^{(-)i},
\]

\[
K^i = J^{0i} = J^{(+)i} - J^{(-)i} \quad \text{and they satisfy the algebra} \]

\[
[L^i, L^j] = \epsilon^{ijk} L^k,
\]
\[ [K^i, K^j] = \epsilon^{ij} k L^k, \quad (A.6) \]
\[ [L^i, K^j] = \epsilon^{ij} k K^k. \quad (A.7) \]

With these formulæ, it is easy to analyze the algebra of the gauge-fixed off-diagonal simplicity constraints (9), \( C_f = L_f^j - \frac{1}{\gamma} K_f^j \):
\[
\{ C_f^i, C_f^j \} = \epsilon^{ij} k \left\{ \left( 1 + \frac{1}{\gamma^2} \right) L_f^k - \frac{2}{\gamma} K_f^k \right\}.
\]

The first thing we note from this expression is that, for finite nonzero \( \gamma \), the algebra of constraints \( \{ C_f \} \) does not close and so is not first class. For \( \gamma \ll 1 \), the constraints \( C_f \) are proportional to the boost generators \( K_f^j \) and so the constraint algebra again fails to close. For \( \gamma = \infty \), on the other hand, \( C_f = L_f^j \), so that the algebra closes and forms a first class system.

Let us investigate further the cases of finite \( \gamma \) and \( \gamma \ll 1 \). In these cases the constraints are second class, but we know that they cannot be purely second class, as the constraints \( C_f \) are three in number, and second class constraints always come in pairs. Thus, there must exist a first class component in the constraints. In fact, one can determine the first class component from (A.8). The (phase-space-dependent) null vector of the Poisson bracket (A.8) matrix can be read off as
\[
X^i_f = \left( 1 + \frac{1}{\gamma^2} \right) L_f^k - \frac{2}{\gamma} K_f^k
\]
so that the first class component of the cross-simplicity constraints is
\[
\tilde{C}_f = X^i_f C_{fi} = \left( 1 - \frac{1}{\gamma^2} \right) L_f^j + \frac{2}{\gamma} \left( K_f^2 + L_f^2 \right) - \frac{1}{\gamma} \left( 3 + \frac{1}{\gamma^2} \right) K_f \cdot L_f.
\]

For small \( \gamma \), this reduces to
\[
\tilde{C}_f \propto L_j f C_{fi} \propto L_f \cdot K_f,
\]
which is just the diagonal simplicity constraint for small \( \gamma \).

**Appendix B. Coherent states**

In this appendix, we review the construction of coherent states for \( SU(2) \) and \( SO(4) \) and derive the coefficients \( c_\gamma(n_0) \) referred to in the main text. For details, see [10, 26]. Coherent states for \( SU(2) \) are defined as follows. Consider the carrying space \( \mathcal{H}_j \) of the spin \( j \) representation of \( SU(2) \). This is spanned by the states \( m \in \{-j, -j+1, \ldots, j\} \). Consider now the states invariant under the \( U(1) \) subgroup that generates rotations around the \( z \)-axis \( \hat{\varepsilon}_z \). Any element of the above basis satisfies this requirement. Next, we select from among these states those that minimize the uncertainty \( \Delta^2 := \langle \hat{J}^2 \rangle - \langle \hat{J} \rangle \cdot \langle \hat{J} \rangle \). There are two such states: \( |j, \pm j \rangle \). Coherent states are defined by acting with \( SU(2)/U(1) \sim S^2 \) on each of these states. Of the two states \( |j, \pm j \rangle \), it is sufficient to restrict consideration to \( |j, j \rangle \), as the other can be obtained by a rotation of angle \( \pi \) around any vector in the \( xy \) plane. For each unit vector \( \hat{n} \in S^2 \), we then define the coherent state
\[
|j, \hat{n} \rangle := g(\hat{n})|j, j \rangle,
\]
where \( g(\hat{n}) \in SU(2) \) denotes the group element that rotates \( \hat{\varepsilon}_z \) into the direction \( \hat{n} \). Explicitly, if, in coordinates, \( \hat{n} \) is given by \( \hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \), then \( g(\hat{n}) \) may be taken to be \( e^{i\theta \hat{m} \cdot \hat{J}} \), where \( \hat{m} := (\sin \phi, -\cos \phi, 0) \) is a unit vector orthogonal to both \( \hat{\varepsilon}_z \) and \( \hat{n} \), and \( \hat{J} \) are the generators of the algebra. These coherent states so defined form an overcomplete basis of the space \( \mathcal{H}_j \).
The decomposition of the coherent state $|j, \hat{n}\rangle$ in the usual basis $|j, m\rangle$ is given by

$$|j, \hat{n}\rangle = g^*(\hat{n}) |j, m\rangle = \sum_m D_{m,j}^j(g(\hat{n})) |j, m\rangle,$$

where $D^j(g)$ denotes the representation matrix of the group element $g$ acting on the carrying space $H_j$. Furthermore, the following identity will be useful in what follows:

$$|j, -\hat{n}\rangle = \sum_m D_{m,j}^j(g(\hat{n})) (-1)^{jm} |j, -m\rangle,$$

where we have used the fact that $\bar{D}_{mn}^j(g) = (-1)^{m+n} D_{m-n}^j(g)$. We also give some useful formulae for calculating expectation values:

$$\langle j, \hat{n}|J^i|j, \hat{n}\rangle = j n^i$$

and

$$\langle j, \hat{n}|(J^i)^2|j, \hat{n}\rangle = \frac{j^2}{2} + j \left( j - \frac{1}{2} \right) (n^i)^2, \quad \text{for } i = x, y, z.$$  \hfill (B.4)

In particular, this allows us to compute

$$\Delta^2 = \langle \hat{J}^2 \rangle - \langle \hat{J} \rangle \cdot \langle \hat{J} \rangle = j.$$  \hfill (B.6)

Now, using the local isomorphism $SO(4) \simeq SU(2) \times SU(2)$, one can use $SU(2)$ coherent states to define $SO(4)$ coherent states. We then define $SO(4)$ coherent states as given by the action of $SU(2) \times SU(2) / U(1) \times U(1) \sim S^2 \times S^2$ on the states $|j^+, j^-\rangle \otimes |j^-, j^-\rangle$: $|j^+, \hat{n}^+\rangle \otimes |j^-, \hat{n}^-\rangle := (g(\hat{n}^+), g(\hat{n}^-))|j^+, j^-\rangle \otimes |j^-, j^-\rangle$. These states form an overcomplete basis of the carrying space $H_{j^+} \otimes H_{j^-}$. Among these there are two classes of states that will be of more interest, given by $j^+ = j^-$ and $\hat{n}^+ = \pm \hat{n}^-$, as they solve the simplicity constraints for the case of flipped and unflipped symplectic structure, as noted in the main text.

We are now ready to calculate the coefficients $c_i(\hat{n}_a)$ given in the main text. Remember the form of the intertwiner for both the flipped $(I^+)$ and the FKLS $(I^-)$ models:

$$I^\pm = \int_{SO(4)} dG \cdot \sum_a (|\hat{n}_a\rangle \otimes |j_a, \pm\hat{n}_a\rangle).$$  \hfill (B.7)

Let us first write these states in the $|j, m\rangle$ basis as above:

$$I^\pm = \int_{SU(2) \times SU(2)} d^4 g^+ d^4 g^- \sum_a (D_{m_a, j_a}^{j_a} (g^+) D_{m_a, j_a}^{j_a} (g_a) D_{m_a, j_a}^{j_a} (g^-) \times D_{m_a, j_a}^{j_a} (g_a) |j_a, m_a\rangle) \otimes |j_a, m_a\rangle,$$

where we have written $G = (g^+, g^-)$, $g_a := g(\hat{n}_a)$ and summation over repeated indices is understood. Integration over $g^\pm$ gives 4-valent intertwiners of $SU(2)$, $C_{m_a}^{j_a}$. Defining $|i^{\pm}; j_a\rangle := C_{[m_a]}^{j_a} \otimes |j_a, m_a\rangle$, one can rewrite the last equation as

$$I^\pm = \sum_{i, i'} C_{[m_a]}^{j_a} \otimes D_{j_a, m_a}^{j_a} (g(\hat{n}_a)) \left( C_{[n_a]}^{i_a} \otimes D_{n_a, i_a}^{i_a} (g(\hat{n}_a)) \right) |i^+; j_a\rangle \otimes |i^-; j_a\rangle.$$  \hfill (B.9)

One can then readily read the form of the coefficients $c_i^\pm(\hat{n}_a)$:

$$c_i^\pm(\hat{n}_a) = C_{[m_a]}^{j_a} \otimes D_{j_a, m_a}^{j_a} (g_a).$$  \hfill (B.10)

The fact that for $I^-$ the coefficients for the anti-self-dual part are conjugated in equation (25) comes from equation (B.3) and some elementary properties of the Clebsch–Gordan coefficients.
Appendix C. Fixing the scaling of the symplectic structure

In this appendix, we fix the scaling of the symplectic structure in the classical discrete theory and so determine the correct coefficient in (5).

There are two possible ways to fix the coefficient in the symplectic structure: (i) by first fixing the coefficient in the discrete action and then deriving the symplectic structure from the discrete action or (ii) by setting the Poisson brackets of the discrete variables equal to their Poisson brackets in the corresponding continuum theory. The former, although seemingly more systematic, is problematic, first because the manner of fixing the coefficient in the discrete action is not fully understood\(^\text{15}\), and second because the manner of deriving the symplectic structure from the action in [9] is not completely standard.

Thus, we carry out the second option. The symplectic structure of the discrete theory is implicitly specified by equation (5) in the main text, \(J_f(t)\) being defined to be the phase space function whose Hamiltonian vector field is the left-invariant vector field, on the group \(U_f(t', t)\), corresponding to the Lie algebra element \(J^{IJ}\) defined in appendix A. We begin by replacing equation (5) with

\[
J_f(t) = \lambda \left( B_f(t) + \frac{1}{\gamma} B_f(t) \right),
\]

where now \(\lambda \in \mathbb{R}\) is a coefficient to be fixed by comparison with the continuum theory (on shell with respect to the Gauss constraint/closure constraint). We will show that this manner of fixing the coefficient results in precisely the coefficient appearing in (5).

In the continuum BF theory, we start from the action

\[
S = \frac{1}{2\kappa} \int_M \left[ B_{IJ} \wedge F_{IJ} + \frac{1}{\gamma} (\ast B)_{IJ} \wedge F_{IJ} \right],
\]

where \(B_{IJ} = B_{IJ}^{ab}\) is a two-form and \(F_{IJ}^{ab}\) is the curvature of an \(SO(4)\) connection \(\omega_{IJ}^a\). Upon substituting in \(B_{IJ} = \varepsilon^{I} \wedge e^{J}\), this becomes the Holst action with all of the correct numerical factors [2]. To simplify the following derivations from the action, define

\[
P_{IJ} := \frac{1}{2\kappa} \left( B_{IJ} + \frac{1}{\gamma} (\ast B)_{IJ} \right).
\]

Following the prescription for deriving symplectic structure used in [2, 28], we vary action (C.2) to obtain the symplectic one-form

\[
\Theta(\delta) = \int_\Sigma P_{IJ} \wedge \delta \omega^{IJ}.
\]

The symplectic structure of the theory is then

\[
\Omega(\delta_1, \delta_2) = \int_\Sigma [\delta_1 P_{IJ} \wedge \delta_2 \omega^{IJ} - (1 \leftrightarrow 2)],
\]

so that the basic non-trivial Poisson brackets in the continuum theory are

\[
\{A^{IJ}(\vec{x}), P_{bcKL}(\vec{y})\} = \epsilon_{abc} \delta^{JL} \delta^{IK} (\vec{x}, \vec{y}),
\]

\(^{15}\text{Naively one would think one could fix the scaling in the action by requiring the discrete action to be approximately equal to the corresponding continuum action. However, there is a difficulty with this. In the derivation in [9], if one is more careful with numerical factors, one finds that in relating the bulk discrete action to the bulk continuum action, there is a missing factor of }12\text{, because the shape of the elementary volume associated with each face in the sum was not taken into account. However, in comparing the discrete boundary action with the continuum boundary action, there was a missing factor of }3\text{ (again because of the shape of the elementary volumes in the sum). But if one corrects the discrete bulk and boundary actions with these factors, so that they equal their continuum counterparts in the continuum limit, then the action will no longer be approximately additive, a requirement noted in [9, 27].}
where $\epsilon_{abc}$ is the Levi-Civita symbol (Levi-Civita tensor density of weight $-1$). Given an edge $\ell$, with source point $n$, let $U_\ell(n)$ denote the parallel transport defined by $\omega_\ell^I$ along $\ell$, starting at $n$. Given an oriented 2-surface $S$, define the ‘flux’ $P(S)^{IJ}$ by

$$P(S)^{IJ} = \oint_S p^{IJ}. \quad (C.7)$$

If $S$ is a 2-surface and $\ell$ is an edge ‘above’ $S$, with source node $n$ in $S$, being careful with numerical factors, one finds the Poisson bracket between $P(S)^{IJ}$ and $U_\ell(n)$ to be exactly

$$\{P(S)^{IJ}, U_\ell(n)\} = \frac{1}{2} U_\ell(n) J^{IJ}, \quad (C.8)$$

where $(J^{IJ})^{MN} := 2\delta^{M[I} \delta^{N]}_I$ is the basis (modulo the skew symmetry in the $IJ$ label) of $so(4)$ introduced in appendix A.

Now, consider a graph $\gamma$ with only 4-valent nodes. Consider the abstract triangulation $\Delta_3 = \gamma^*$ dual, within the 3-slice $\Sigma$, to the graph $\gamma$. In the following, we will motivate, by algebraic considerations, an identification of the variables of the canonical discrete theory based on $\Delta_3$, with certain variables of the present continuum theory. Let $\Psi(\vec{U})$ be a function of the parallel transports along the links in $\gamma$. Consider a node $n$ and adjacent link $\ell$ in $\gamma$. Call the other three links at $n\ell_1, \ell_2, \ell_3$. Finally, let $t, f, f_1, f_2, f_3$ denote the tetrahedron and abstract triangles in $\Delta_3$ dual to $n, \ell, \ell_1, \ell_2, \ell_3$, respectively. Construct a 2-surface $S_{n,\ell}$ which intersects $\gamma$ only at the node $n$, with $\ell$ ‘above’ and $\ell_1, \ell_2, \ell_3$ ‘below’ $S_{n,f}$. The Poisson bracket of $P(S_{n,\ell})^{IJ}$ with $\Psi(\vec{U})$ is given by

$$\{P(S_{n,\ell})^{IJ}, \Psi(\vec{U})\} = \frac{1}{4} \left( \vec{L}^{IJ}_{(n,\ell)} - \sum_{i=1}^{3} \vec{L}^{IJ}_{(n,\ell_i)} \right) \Psi(\vec{U}), \quad (C.9)$$

where $\vec{L}^{IJ}_{(n,\ell)}$ denotes the left invariant vector field, on the parallel transport $U_\ell(n)$, associated with the Lie algebra element $J^{IJ}$. A comparison of (C.9) with the discrete theory leads us to identify the following quantities in the continuum and discrete theories

$$P(S_{n,\ell})^{IJ} = \frac{1}{4} \left( J_f(t)^{IJ} - \sum_{i=1}^{3} J_{f_i}(t)^{IJ} \right), \quad (C.10)$$

where, as defined in section 2.1, $J_f(t)^{IJ}$ denotes the phase space function in the discrete theory whose Hamiltonian vector field is the left-invariant vector field on $U_\ell(n)$ corresponding to the Lie algebra element $J^{IJ}$. Imposing the closure constraint in the discrete theory reduces this to

$$P(S_{n,\ell})^{IJ} = \frac{1}{4} J_f(t)^{IJ}. \quad (C.11)$$

Now if we define $B(S)^{IJ} := \int_S B^{IJ}$, we have

$$B(S_{n,\ell})^{IJ} = 2\kappa \left( \frac{\gamma^2}{\gamma^2 - 1} \right) \left( P(S_{n,\ell})^{IJ} - \frac{1}{\gamma} \gamma^* P(S_{n,\ell})^{IJ} \right)$$

$$= \kappa \left( \frac{\gamma^2}{\gamma^2 - 1} \right) \left( J_f(t)^{IJ} - \frac{1}{\gamma} J_f(t)^{IJ} \right)$$

$$= \kappa \lambda B_f(t)^{IJ}, \quad (C.12)$$

where in the last line we have inserted (C.1). Recall the physical meaning of $B_f(t)^{IJ}$: it is the integral of the two-form $B = *e \wedge e$ over the triangle $f$ dual to $\ell$. As the triangulation $\Delta_3$ is abstract, a priori this triangle $f$ is some abstract surface dual to $\ell$. Equation (C.12), forced on us from algebraic considerations, motivates us to identify this dual surface with $S_{n,\ell}$, so that $B_f(t)^{IJ}$ is physically the same quantity as $B(S_{n,\ell})^{IJ}$. We thus conclude $\lambda = 1/\kappa$, giving us the final relation between $J_f(t)^{IJ}$ and $B_f(t)^{IJ}$:

$$J_f(t) = \frac{1}{\kappa} \left( B_f(t) + \frac{1}{\gamma} B_f(t) \right), \quad (C.13)$$

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