ON A FUNCTIONAL EQUATION RELATED TO A GENERALIZATION OF FLETT’S MEAN VALUE THEOREM

T. RIEDEL and MACIEJ SABLIK

(Received 24 November 1997 and in revised form 10 June 1998)

Abstract. In this paper, we characterize all the functions that attain their Flett mean value at a particular point between the endpoints of the interval under consideration. These functions turn out to be cubic polynomials and thus, we also characterize these.

Keywords and phrases. Flett’s mean value theorem, functional equations, generalized polynomials.

2000 Mathematics Subject Classification. Primary 39B22.

1. Introduction. In [5], Sahoo and Riedel gave a generalization of Flett’s mean value theorem [2] as follows.

Theorem 1.1. Let \( f \) be a real valued function which is differentiable in \( [a, b] \), then there is a point \( c \in (a, b) \) such that

\[
f(c) - f(a) = (c - a)f'(c) - \frac{1}{2} \frac{f'(b) - f'(a)}{b - a}(c - a)^2.
\]

It is easy to see that if \( f'(b) = f'(a) \), then this reduces to Flett’s mean value theorem.

Aczél [1] and Haruki [3] used the Lagrange mean value theorem to ask the question of which functions attained their mean value at a prescribed point \( c \in (a, b) \), in particular, at the midpoint \( c = (a + b)/2 \). The answer is that only quadratic polynomials have the property that the mean value on any interval is attained at the midpoint of that interval. A natural question to ask is this same question for the above mean value theorem. It turns out that quadratic polynomials satisfy (1.1) for any \( c \), but, more interestingly, cubic polynomials satisfy it for \( c = (a + 3b)/4 \). Thus, the main question becomes whether cubic polynomials are the only functions having this property.

Following the approach in [1], we generalize (1.1) to obtain

\[
f(c) - f(a) = (c - a)h(c) - \frac{1}{2} \frac{h(b) - h(a)}{b - a}(c - a)^2,
\]

and now setting \( c = (a + 3b)/4 \) yields

\[
f\left(\frac{a + 3b}{4}\right) - f(a) = \frac{3}{4}(b - a)h\left(\frac{a + 3b}{4}\right) - \frac{9}{32}(b - a)(h(b) - h(a))
\]

or

\[
f\left(\frac{a + 3b}{4}\right) - f(a) = \frac{3}{4}(b - a)\left[h\left(\frac{a + 3b}{4}\right) - \frac{3}{8}(h(b) - h(a))\right].
\]
More generally, setting \( c = sa + tb \) with \( s + t = 1 \) and \( 0 < s, t < 1 \), we obtain

\[
f(sa + tb) - f(a) = (sa + tb - a)h(sa + tb) - \frac{1}{2} \frac{h(b) - h(a)}{b-a} (sa + tb - a)^2. \tag{1.5}
\]

The question we answer, in this paper, is: What are the functions \( f, h \) that satisfy the functional equations (1.4) and (1.5) for all \( a, b \in \mathbb{R} \)? In solving this functional equation, we do not assume any regularity conditions on \( f \) or \( h \).

2. Solution of the functional equation. The main work in solving this functional equation is to reduce (1.4) and (1.5) to a form where we can apply the following result by Székelyhidi [6, Thm. 9.5] and Wilson [7].

**Theorem 2.1.** Let \( G, S \) be commutative groups, \( n \) a nonnegative integer, \( \varphi_i, \psi_i \) additive functions from \( G \) into \( G \) and let \( \text{Ran}(\varphi_i) \subseteq \text{Ran}(\psi_i) \) \( (i = 1, \ldots, n + 1) \). Then if

\[
h(x) + \sum_{i=1}^{n+1} h_i(\varphi_i(x) + \psi_i(t)) = 0, \tag{2.1}
\]

then \( h \) is a generalized polynomial of degree at most \( n \).

Thus, we are able to prove our main result (Theorem 2.2).

**Theorem 2.2.** The real valued functions \( f \) and \( h \) are solutions of the functional equation (1.5) if and only if

\[
f(x) = \begin{cases} 
Ax^3 + Bx^2 + Cx + D \quad &\text{if } s = \frac{1}{4}, t = \frac{3}{4}, \\
Bx^2 + Cx + D \quad &\text{if } s \neq \frac{1}{4}, t = \frac{3}{4}, 
\end{cases} \tag{2.2}
\]

and

\[
h(x) = \begin{cases} 
3Ax^2 + 2Bx + C \quad &\text{if } s = \frac{1}{4}, t = \frac{3}{4}, \\
2Bx + C \quad &\text{if } s \neq \frac{1}{4}, t = \frac{3}{4}. 
\end{cases} \tag{2.3}
\]

**Proof.** It is easy to check that the functions \( f \) and \( h \), given above, do satisfy the functional equation (1.5).

To show that these are the only solutions, we start by rewriting (1.5) using \( s + t = 1 \) as follows:

\[
f(a + t(b-a)) - f(a) = t(b-a) \left[ h(a + t(b-a)) - \frac{t}{2} [h(b) - h(a)] \right]. \tag{2.4}
\]

Now, letting \( u = (b-a)/3 \), we obtain

\[
f(a + 3tu) - f(a) = 3tu \left[ h(a + 3tu) - \frac{t}{2} [h(3u+a) - h(a)] \right]. \tag{2.5}
\]

Now, we replace \( a \) by \( a - tu \) in (2.5) and get

\[
f(a + 2tu) - f(a - tu) = 3tu \left[ h(a + 2tu) - \frac{t}{2} [h((3-t)u+a) - h(a - tu)] \right]. \tag{2.6}
\]
Similarly, using \( a = a - 2tu \) in (2.5), we get
\[
f(a + tu) - f(a - 2tu) = 3tu \left[ h(a + tu) - \frac{t}{2} \left[ h((3 - 2t)u + a) - h(a - 2tu) \right] \right]. \tag{2.7}
\]
Interchanging \( u \) with \(-u\) in (2.7) gives
\[
f(a - tu) - f(a + 2tu) = -3tu \left[ h(a - tu) - \frac{t}{2} \left[ h((-3 + 2t)u + a) - h(a + 2tu) \right] \right]. \tag{2.8}
\]
Comparing (2.8) and (2.6) gives, for \( a, u \in \mathbb{R} \),
\[
\left[ h(a - tu) - \frac{t}{2} \left[ h((-3 + 2t)u + a) - h(a + 2tu) \right] \right] = \left[ h(a + 2tu) - \frac{t}{2} \left[ h((3 - t)u + a) - h(a - tu) \right] \right], \tag{2.9}
\]
which simplifies to
\[
t \left[ h((3 - t)u + a) - h((-3 + 2t)u + a) - (h(a - tu) - h(a + 2tu)) \right] = -2 \left[ h(a - tu) - h(a + 2tu) \right]. \tag{2.10}
\]
Collecting the terms of \( h \) that have the same argument, we obtain
\[
(2 - t)h(a + 2tu) - (2 - t)h(a - tu) - th((3 - t)u + a) + th((-3 + 2t)u + a) = 0. \tag{2.11}
\]
Writing \( x = a + 2tu \) and dividing (2.11) by \((2 - t)\) yields
\[
\frac{t}{2 - t} h(x) - h(x - 3tu) - \frac{t}{2 - t} h(x + 3(1 - t)u) + \frac{t}{2 - t} h(x - 3u) = 0. \tag{2.12}
\]
Thus, since \( t \neq 0 \) is fixed, (2.12) is of the form of equation (2.1) and hence, \( h(x) \) is a generalized polynomial of degree at most 2,
\[
h(x) = \beta(x, x) + \alpha(x) + C, \tag{2.13}
\]
where \( \beta \) is a symmetric, biadditive function and \( \alpha \) is an additive function and \( C \) is an arbitrary real constant.

Setting \( a = 0 \) in (2.5), we get
\[
f(x) = x \left[ h(x) - \frac{t}{2} \left[ h \left( \frac{x}{t} \right) - h(0) \right] \right] + D, \tag{2.14}
\]
and substituting from (2.13), we obtain
\[
f(x) = x \beta(x, x) + x \alpha(x) + Cx - x \frac{t}{2} \beta \left( \frac{x}{t}, \frac{x}{t} \right) - x \frac{t}{2} \alpha \left( \frac{x}{t} \right) + D. \tag{2.15}
\]
To prove the continuity of \( f \) and \( h \), let us substitute the solutions given in (2.15) into (2.5). We see that both the left- and the right-hand side of (2.5) are polynomial functions in \( a \) and \( u \). The equality of the two sides implies, therefore, the equality
of terms which are of the same degree with respect to $a$ and $u$. First, comparing the
terms of degree 1 with respect to each variable, we get
\[
3a \left[ \alpha(3tu) - \frac{t}{2} \alpha(3u) \right] + 3tu \left[ \alpha(a) - \frac{t}{2} \alpha \left( \frac{a}{t} \right) \right] = 3tu \alpha(a),
\]
whence, substituting $ta$ instead of $a$ and dividing by $t/2$, we get
\[
tu \alpha(a) = 2a \alpha(tu) - ta \alpha(u).
\] (2.17)
Dividing both sides by $tua$, we obtain
\[
\frac{\alpha(a)}{a} = 2 \frac{\alpha(tu)}{tu} - \frac{\alpha(u)}{u} \quad \forall a \neq 0 
eq u.
\] (2.18)
In particular, $\alpha(a)/a$ does not depend on $a$ and, therefore, $\alpha(a) = 2Ba$ for some
constant $B$.

Now, let us compare the terms of degree 2 with respect to $a$ and those of degree 1
with respect to $u$. We get
\[
6a \left[ \beta(a,tu) - \frac{t}{2} \beta \left( \frac{a}{t},u \right) \right] + 3tu \left[ \beta(a,a) - \frac{t}{2} \beta \left( \frac{a}{t}, \frac{a}{t} \right) \right] = 3tu \beta(a,a).
\] (2.19)
Rearranging and simplifying, we get
\[
6a \left[ \beta(a,tu) - \frac{t}{2} \beta \left( \frac{a}{t},u \right) \right] = \frac{3t^2}{2} \beta \left( \frac{a}{t}, \frac{a}{t} \right),
\]
or, after substituting $ta$ instead of $a$ and dividing by $3t/2$,
\[
4a \left[ \beta(ta,tu) - \frac{t}{2} \beta(a,u) \right] = tu \beta(a,a).
\] (2.21)
Dividing (2.21) by $a^2u$, we obtain
\[
4 \left[ \frac{\beta(ta,tu)}{au} - \frac{t}{2} \frac{\beta(a,u)}{au} \right] = \frac{t \beta(a,a)}{a^2} \quad \text{for } a \neq 0 \neq u.
\] (2.22)
Using the symmetry of $\beta$, we infer that
\[
\frac{\beta(a,a)}{a^2} = \frac{\beta(u,u)}{u^2} \quad \forall u \neq 0 \neq a,
\] (2.23)
whence, it follows that $\beta(a,a) = 3Aa^2$ for some constant $A$. Comparing this with
formulae for $f$ and $h$, we see that
\[
f(x) = 3A \left( 1 - \frac{1}{2t} \right)x^3 + Bx^2 + Cx + D,
\]
\[
h(x) = 3Ax^2 + 2Bx + C.
\] (2.24)
Inserting (2.24) into (1.5), we get, after simplifying,
\[
27t \left( 1 - \frac{1}{2t} \right) Aa^2 u + 81t^2 \left( 1 - \frac{1}{2t} \right) Aa^2 u^2 = 9ta^2 u + 27t^2 Aa^2 u^2 \quad \forall a,u \in \mathbb{R},
\]
whence, it follows that $A = 0$ provided $t \neq 3/4$. Note that, for $t = 3/4$, we have $3A(1 - (1/2t)) = A$ and the assertion follows from (2.24). \qed
REFERENCES

[1] J. Aczél, A mean value property of the derivative of quadratic polynomials—without mean values and derivatives, Math. Mag. 58 (1985), no. 1, 42–45. MR 86c:39012. Zbl 571.39005.

[2] T. M. Flett, A mean value theorem, Math. Gaz. 42 (1958), 38–39.

[3] S. Haruki, A property of quadratic polynomials, Amer. Math. Monthly 86 (1979), no. 7, 577–579. MR 80g:26010. Zbl 413.39003.

[4] M. Sablik, A remark on a mean value property, C. R. Math. Rep. Acad. Sci. Canada 14 (1992), no. 5, 207–212. MR 94b:39039. Zbl 796.39014.

[5] P. K. Sahoo and T. Riedel, Mean Value Theorems and Functional Equations, World Scientific Publishing Co., NJ, 1998. CMP 1 692 936.

[6] L. Székelyhidi, Convolution Type Functional Equations on Topological Abelian Groups, World Scientific Publishing Co., Inc., Teaneck, NJ, 1991. MR 92f:39017. Zbl 748.39003.

[7] W. H. Wilson, On a certain general class of functional equations, Amer. J. Math. 40 (1918), 263–282.

Riedel: Department of Mathematics, University of Louisville, Louisville, KY 40292, USA
E-mail address: Thomas.Riedel@louisville.edu

Sablik: Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, PL-40-007 Katowice, Poland
E-mail address: mssablik@us.edu.pl
Special Issue on
Intelligent Computational Methods for
Financial Engineering

Call for Papers
As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems).

This special issue will include (but not be limited to) the following topics:

- **Computational methods**: artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning
- **Application fields**: asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects**: decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

Authors should follow the Journal of Applied Mathematics and Decision Sciences manuscript format described at the journal site http://www.hindawi.com/journals/jamds/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

| Event                        | Date       |
|------------------------------|------------|
| Manuscript Due               | December 1, 2008 |
| First Round of Reviews       | March 1, 2009  |
| Publication Date             | June 1, 2009   |

**Guest Editors**

**Lean Yu**, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; yulean@amss.ac.cn

**Shouyang Wang**, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; sywang@amss.ac.cn

**K. K. Lai**, Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; mskklai@cityu.edu.hk