NOTES ON COMPLEX HYPERBOLIC TRIANGLE GROUPS OF TYPE \((m, n, \infty)\)

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Abstract. In this paper we mainly pay attention to the complex hyperbolic triangle groups of type \((m, n, \infty)\) and discuss the discreteness. From the results more explicit conclusions about the triangle groups of type \((n, \infty, \infty)\) will also be given.

1. Introduction

A complex hyperbolic triangle is a triple \((C_1, C_2, C_3)\) of complex geodesics in \(H^2\). If the complex geodesics \(C_{k-1}\) and \(C_k\) meet at the angle \(\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}\) \((p, q, r \in \mathbb{Z})\), where the indices are taken mod 3, we call the triangle \((C_1, C_2, C_3)\) a \((p,q,r)\)-triangle. We call \(\Gamma\) a \((p,q,r)\)-triangle group, if \(\Gamma\) is generated by three complex reflections \(I_1, I_2, I_3\) in the sides \(C_1, C_2, C_3\) of a \((p,q,r)\)-triangle. Throughout this paper we will use \(C_j, I_j, \Gamma\) to denote the complex geodesic, complex reflection, and the complex hyperbolic triangle group respectively, unless otherwise stated.

The deformation theory of complex hyperbolic triangle groups was begun in [2]. Goldman and Parker investigated \(\Gamma\) of type \((\infty, \infty, \infty)\) (complex hyperbolic ideal triangle group) and gave the necessary and sufficient conditions for ideal triangle group \(\Gamma\) to be discrete embedded. Especially the necessary condition for \(\Gamma\) of type \((\infty, \infty, \infty)\) to be discretely embedded in \(PU(2,1)\) is that the product of the three generators \(I_1I_2I_3\) is not elliptic. They conjectured the necessary condition is also sufficient. Since then there have been many findings. Schwartz in [15] proved this conjecture and also verified that such a group is non-discrete if \(I_1I_2I_3\) is elliptic. Recently Parker, Wang and Xie in [12] show that the group of type \((3, 3, n)\) is discrete if and only if \(I_1I_2I_3\) is non-elliptic which is a conjecture in [16]. Parker [10] explored groups of type \((n, n, n)\) such that \(I_1I_2I_3\) is regular elliptic. In this case there are some discrete groups. In the same fashion as the proof due to Schwartz, Wyss-Gallifent proved the Schwartz’s statement for \(\Gamma\) of type \((n, n, \infty)\) in [17] Lemma 3.4.0.19. Pratoussevitch improved the result of Wyss-Gallifent in [14]. Also Kamiya, Parker and Thompson identified the non-discrete classes of \((n, n, \infty)\)-triangle groups using the result, see [6]. It is interesting to think about whether the same statement holds
for other type complex hyperbolic triangle groups, such as type \((m, n, \infty)\) \(m \neq n\).
In this paper, we give the affirmative answer by the similar argument.

This paper is arranged as follows. Section 2 mainly consists of background about complex hyperbolic space and its holomorphic isometry group. Then we give three necessary conditions for \((m, n, \infty)\) - triangle groups to be discrete in Section 3. After that we consider non-discrete cases of \((n, \infty, \infty)\) - triangle groups in Section 4.

2. Preliminaries

Let \(\mathbb{C}^{2,1}\) denote the vector space \(\mathbb{C}^3\) equipped with the Hermitian form
\[
\langle z, w \rangle = z_1 \overline{w}_1 + z_2 \overline{w}_2 - z_3 \overline{w}_3
\]
of signature \((2,1)\). We denote by \(\mathbb{CP}^2\) the complex projectivisation of \(\mathbb{C}^{2,1}\) and by \(\mathbb{P} : \mathbb{C}^{2,1} \setminus \{0\} \to \mathbb{CP}^2\) a natural projectivisation map. We call a vector \(z \in \mathbb{C}^{2,1}\) negative, null, or positive, according as \(\langle z, z \rangle\) is negative, zero, or positive respectively. The complex hyperbolic 2-space \(\mathbb{H}^2_C\) is defined as the complex projectivisation of the set of negative vectors in \(\mathbb{C}^{2,1}\). It is called the standard projective model of complex hyperbolic space. Its boundary \(\partial \mathbb{H}^2_C\) is defined as the complex projectivisation of the set of null vectors in \(\mathbb{C}^{2,1}\). This will also form the unit ball model whose boundary is the sphere \(S^3\).

The complex hyperbolic plane \(\mathbb{H}^2_C\) is a Kähler manifold of constant holomorphic sectional curvature -1. The holomorphic isometry group of \(\mathbb{H}^2_C\) is the projectivisation \(\text{PU}(2,1)\) of the group \(\text{U}(2,1)\) of complex linear transformation on \(\mathbb{C}^{2,1}\), which preserves the Hermitian form.

Let \(x, y \in \mathbb{H}^2_C\) be points corresponding to vectors \(\tilde{x}, \tilde{y} \in \mathbb{C}^{2,1}\). Then the Bergman metric \(\rho\) on \(\mathbb{H}^2_C\) is given by
\[
\cosh^2 \left( \frac{\rho(x, y)}{2} \right) = \frac{\langle \tilde{x}, \tilde{y} \rangle \langle \tilde{y}, \tilde{x} \rangle}{\langle \tilde{x}, \tilde{x} \rangle \langle \tilde{y}, \tilde{y} \rangle}.
\]

It will be convenient for us to choose a particular model of the complex hyperbolic space which is adapted for our requirements; namely, one with the distinguished point \(q_\infty\) on the boundary and a set of coordinates with respect to this point. This set-up is generalised by the the Siegel domain model \(\mathfrak{S}\) of \(\mathbb{H}^2_C\) with horospherical coordinates, see [3]. In these coordinates \(z \in \mathfrak{S}\) is given by \(z = (\xi, v, u) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+\). Similarly, points in \(\partial \mathbb{H}^2_C = \mathbb{C} \times \mathbb{R} \cup \{q_\infty\}\) are either \(z = (\xi, v, 0) \in \mathbb{C} \times \mathbb{R} \times \{0\}\) or the point at infinity \(q_\infty\). There is unique complex projective hyperplane \(H_\infty \subset \mathbb{CP}^2\) that is tangent to \(\partial \mathbb{H}^2_C\) at \(q_\infty\). Using affine coordinates on \(\mathbb{CP}^2 - H_\infty\) complex hyperbolic space is realised as a Siegel domain.

The 3-dimensional Heisenberg group \(\mathfrak{H}\) is the set \(\mathbb{C} \times \mathbb{R}\) with the group law
\[
(\xi_1, v_1) \circ (\xi_2, v_2) = (\xi_1 + \xi_2, v_1 + v_2 + 2 \text{Im}(\xi_1 \overline{\xi}_2)).
\]
The inverse of \((\xi_1, v_1)\) is
\[
(\xi_1, v_1)^{-1} = (-\xi_1, -v_1).
\]

The boundary of the half-space model of real hyperbolic geometry is identified with the one-point compactification of Euclidean space. In the same way, the boundary of the Siegel domain may be identified with the one-point compactification of the Heisenberg group. In order to see how \(\mathcal{S}\) relates to the standard projective model of \(\mathbb{H}_2^C\) we define the map \(\psi : \overline{\mathcal{S}} \rightarrow \mathbb{C}P^2\) by
\[
\psi : (\xi, v, u) \mapsto \begin{bmatrix}
\frac{1}{2}(1 - |\xi|^2 - u + iv) \\
\frac{1}{2}(1 + |\xi|^2 + u - iv)
\end{bmatrix}
\text{ for } (\xi, v, u) \in \overline{\mathcal{S}} - \{q_{\infty}\},
\]
and \(\psi(q_{\infty}) = [0, -1, 1]^t\).

The Heisenberg norm is given by
\[
|\langle \xi, v \rangle| = \left| |\xi|^2 - iv \right|^\frac{1}{2}.
\]
This gives rise to a metric, the Cygan metric \(\rho_0\) on the Heisenberg group \(\mathfrak{H}\) by
\[
\rho_0((\xi_1, v_1), (\xi_2, v_2)) = \left|\langle \xi_1, v_1 \rangle^{-1} \diamond \langle \xi_2, v_2 \rangle\right| = \left|\langle \xi_1 - \xi_2 \rangle^2 - iv_1 + iv_2 - 2i \text{ Im}(\langle \xi_1 \xi_2 \rangle)\right|^\frac{1}{2}.
\]
We can extend the Cygan metric to \(\mathbb{H}_2^C - \{q_{\infty}\}\) as follows (\(\mathcal{S}\))
\[
\rho_0((\xi_1, v_1, u_1), (\xi_2, v_2, u_2)) = \left|\langle \xi_1 - \xi_2 \rangle^2 + |u_1 - u_2| - iv_1 + iv_2 - 2i \text{ Im}(\langle \xi_1 \xi_2 \rangle)\right|^\frac{1}{2}.
\]

A complex geodesic is a complex projectivisation of a two dimensional complex subspace of \(\mathbb{C}^{2,1}\). Given two points \(x\) and \(y\) in \(\mathbb{H}_2^C \cup \partial\mathbb{H}_2^C\), lifting \(x\) and \(y\) to \(\hat{x}\) and \(\hat{y}\) in \(\mathbb{C}^{2,1}\) respectively, and then taking \(\hat{C}\) to be the complex span of \(\hat{x}\) and \(\hat{y}\). We define the complex geodesic \(C\) to be the projectivisation of \(\hat{C}\), which can be uniquely determined by a positive vector \(p \in \mathbb{C}^{2,1}\), i.e. \(C = \pi(\{z \in \mathbb{C}^{2,1} | (z, p) = 0\})\). We call \(p\) a polar vector to \(C\).

Recall that a chain is the intersection of a complex geodesic with \(\partial\mathbb{H}_2^C\). For \(z \in \mathbb{C}\), the \(z\)-chain is the chain having polar vector \((1, -z, z)^t\). The \(z\)-chain is the vertical chain in \(\mathfrak{H}\) through the point \((z, 0)\). For \(z, r \in \mathbb{R}\), the \((z, r)\)-chain is the chain having polar vector \((0, 1 + r^2 + iz, 1 - r^2 - iz)^t\). The \((z, r)\)-chain is the circle with radius \(r\) centered at the origin in \(\mathbb{C} \times \{z\} \subset \mathfrak{H}\). One can see more details in §4.3 of [1]. It is straightforward to show that the only chains through \(\infty\) are vertical. Other chains are various ellipses (perhaps circles) which project to circle via \(\mathbb{C} \times \mathbb{R} \rightarrow \mathbb{C}\). Specifically, the unit circle in \(\mathbb{C} \times \{0\}\) and vertical lines (with the infinite point) are all chains.
The involution (complex reflection of order 2) in $C$ is represented by an element $I_C \in \text{SU}(2,1)$ that is given by

$$I_C = -z + 2 \frac{\langle z, p \rangle}{\langle p, p \rangle} p,$$

where $p$ is a polar vector of $C$. There is a one-to-one correspondence between complex geodesics and chains, therefore we also say $I_C$ is the involution on $\partial C$.

An automorphism $g$ of $\mathbb{H}_C^2$ lifts to a unitary transformation $\tilde{g}$ of $\mathbb{C}^2$ and the fixed points of $g$ on $\mathbb{P}(\mathbb{C}^2)$ correspond to eigenvalues of $\tilde{g}$. An automorphism $g$ is elliptic if it fixes at least one point in $\mathbb{H}_C^2$, parabolic if it has a unique fixed point on $\partial \mathbb{H}_C^2$, and loxodromic if it fixes a unique pair of points on $\partial \mathbb{H}_C^2$. An elliptic element $g$ is called regular elliptic if its eigenvalues are pairwise distinct. Otherwise we call it boundary elliptic, in which case the element has a multiple eigenvalue with a two dimensional eigenspace.

Define the discriminant polynomial

$$(2.1) \quad f(z) = |z|^4 - 8 \text{Re}(z^3) + 18|z|^2 - 27.$$

From [1, Theorem 6.2.4], we know an element $g \in \text{SU}(2,1)$ is regular elliptic if and only if $f(\tau(g)) < 0$, where $\tau(g)$ is the trace of $g$.

See [1] for more details about complex hyperbolic space.

3. Complex hyperbolic triangle group of type $(m, n, \infty)$

By conjugation in $\text{PU}(2,1)$, we can take three involutions $I_j$ in $C_j$ such that $\partial C_1$, $\partial C_2$, $\partial C_3$ are $(0,1)$-chain, $z_1$-chain, $z_2$-chain respectively, where $z_1 = \cos(\pi/n)$, $z_2 = e^{i\theta} \cos(\pi/m)$ according to [17, Lemma 3.1.0.7]. Then the three polar vectors correspondingly are

$$p_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad p_2 = \begin{bmatrix} 1 \\ -z_1 \\ z_1 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 1 \\ -\bar{z}_2 \\ \bar{z}_2 \end{bmatrix}.$$

It is easy to obtain the three vertices

$$\tilde{u}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \tilde{u}_2 = \begin{bmatrix} z_2 \\ 0 \\ 1 \end{bmatrix}, \quad \tilde{u}_3 = \begin{bmatrix} \bar{z}_1 \\ 0 \\ 1 \end{bmatrix}.$$

The involutions on the complex chains $\partial C_1$, $\partial C_2$, $\partial C_3$ are respectively as follows

$$I_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & -2s_1 & -2s_1 \\ -2s_1 & 2s_1^2 - 1 & 2s_1^2 \\ 2s_1 & -2s_1^2 & -2s_1^2 - 1 \end{bmatrix},$$

where $s_1 = e^{i\theta} \cos(\pi/m)$. 

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\[
I_3 = \begin{bmatrix}
1 & -2s_2e^{i\theta} & -2s_2e^{i\theta} \\
-2s_2e^{-i\theta} & 2s_2^2 - 1 & 2s_2 \\
2s_2e^{i\theta} & -2s_2 & -2s_2^2 - 1
\end{bmatrix},
\]

where \(s_1 = \cos(\pi/n), s_2 = \cos(\pi/m)\). Define the parameter of the \((m, n, \infty)\)-triangle angular invariant \(\alpha\) by

\[
\alpha = \arg\left(\prod_{i=1}^{3}(p_{i-1}, p_{i+1})\right) = \arg(z_1 z_2) = \theta.
\]

**Remark 3.1.** 1. For complex hyperbolic triangle group \(\Gamma\) of type \((n, n, \infty)\), we can take the following three polar vectors

\[
p_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad p_2 = \begin{bmatrix} 1 \\ -\frac{z}{\bar{z}} \\ \bar{z} \end{bmatrix}, \quad p_3 = \begin{bmatrix} 1 \\ \bar{z} \\ -z \end{bmatrix},
\]

where \(z = \cos(\frac{\pi}{n})e^{i\theta_0}\). These three normalised polar vectors were also used in [4], [6].

2. One can compare our parameter \(\alpha\) of the space of complex hyperbolic triangles with another parameter \(t\) by Wyss-Gallifent [17] and shall obtain

\[
\cos \alpha = \frac{t^2 - 1}{t^2 + 1}.
\]

Now we give the affirmative answer about the Schwartz’s statement [15, Section 3.3] for the \((m, n, \infty)\)-triangle groups.

**Theorem 3.2.** The complex triangle group \(\Gamma\) of type \((m, n, \infty)\) is not discrete if \(I_1 I_2 I_3\) is regular elliptic.

**Proof.** If the element \(I_1 I_2 I_3\) is of infinite order, then the cyclic group generated by this element is certainly not discrete. Hence it suffices for us to prove that \(I_1 I_2 I_3\) can not be regular elliptic of finite order. We will only consider the result when \(m \neq n\) using the similar method given by Pratoussevitch (see [14]), in which the author proved the case for \(m = n\).

Assume that the element \(I_1 I_2 I_3\) is regular elliptic of finite order. Without loss of generality, we can write

\[
\tau = \text{tr}(I_1 I_2 I_3) = \omega_1^{k_1} + \omega_1^{k_2} + \omega_1^{k_3},
\]

for some integers \(k_1, k_2, k_3\), with \(k_1 + k_2 + k_3 = 0\). Here \(\omega_l = e^{2\pi i l}\) and a positive integer \(l\) is taken as small as possible.

Let \(N\) be the least common multiple of \(l, 2m, 2n\). Let \(k\) be relatively prime to \(N\), \(\sigma_k\) be the Galois automorphism of \(\mathbb{Q}[\omega_N]\) given by \(\sigma_k(\omega_N) = \omega_N^k\). Obviously \(\sigma_k(t) = t\), for \(t \in \mathbb{N}\).

**Lemma 3.3.** \(\Re(\sigma_k(\tau)) < -1\).
Proof. From the explicit form of three involutions \( I_1, I_2, I_3 \), we can rewrite the trace of \( I_1 I_2 I_3 \) as \( \tau = 8s_1 s_2 e^{i\alpha} - (4(s_1^2 + s_2^2) + 1) \), i.e.

\[
|\tau + 4(s_1^2 + s_2^2) + 1| = (8s_1 s_2)^2.
\]

By considering (3.3) the expression of \( \tau = \omega_k l + \omega k_2 l + \omega k_3 l \), we know \( \tau \in \mathbb{Q}[\omega_l] \subseteq \mathbb{Q}[\omega_N] \).

Similarly, \( 2s_2 \in \mathbb{Q}[\omega_N] \). Let \( s_j' = \sigma_k(s_j) \) for \( j = 1, 2 \) (\( s_j' \) could be equal to \( s_j \)). Then the equation (3.4) implies that

\[
(\sigma_k(\tau) + 4(s_1^2 + s_2^2) + 1)(\sigma_k(\tau) + 4(s_1^2 + s_2^2) + 1) = (8s_1 s_2)^2.
\]

Since \( \sigma_k \) commutes with complex conjugation, we know \( s_j' \in \mathbb{R} \) and

\[
|\sigma_k(\tau) + 4(s_1^2 + s_2^2) + 1| = 8s_1 s_2.
\]

It follows that \( \sigma_k(\tau) \) lies on the circle with center at \(-4(s_1^2 + s_2^2) + 1) \) and radius \( |8s_1 s_2| \). It is easy to compute

\[
-4(s_1^2 + s_2^2) + 1 + |8s_1 s_2| = -4(s_1^2 - 2|s_1 s_2'| + s_2^2) - 1
= -4(s_1^2 - s_2^2) - 1 < -1.
\]

The last strict inequality is from \( s_1' \neq s_2' \), because \( s_1 \neq s_2 \) for \( m \neq n \). Hence \( \text{Re} (\sigma_k(\tau)) < -1 \). \( \square \)

Note that The following lemma is essentially Lemma 2 of [14]. We clarified it here again by taking different values for \( k \) from [8].

**Lemma 3.4.** For \( i \in \{1, 2, 3\} \) let \( d_i \frac{1}{(k_i, d)} \), where \( (k_i, l) \) is the greatest common divisor of \( k_i \) and \( l \). Then

\[
\frac{1}{\varphi(d_1)} + \frac{1}{\varphi(d_2)} + \frac{1}{\varphi(d_3)} > 1.
\]

Here \( \varphi \) is the Euler phi function.

**Proof.** Let \( S(N) = \{ k \in \mathbb{Z} \mid 1 \leq k < N \text{ and } (k, N) = 1 \} \) (note that one can also have the similar definition for \( S(d_i) \)). It follows from Lemma [3.3] that

\[
\text{Re} \left( \sum_{k \in S(N)} \sigma_k(\tau) \right) < -\varphi(N).
\]
By assuming $N = l \cdot l'$, we obtain $d_i = \frac{N}{(k_i l, N)}$. Note that the root of unity $\omega_i^{k_i} = \omega_N^{k_i}$ is a primitive $d_i$th root of unity and

$$\sum_{k \in S(d_i)} \omega_i^k \in \{-1, 0, 1\}.$$ 

The map $(\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/d_i\mathbb{Z})^\times$ induced by $\mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}/d_i\mathbb{Z}$ is surjective and with multiplicity $\varphi(N)/\varphi(d_i)$. Therefore we obtain the inequality

$$(3.7) \quad \left| \sum_{k \in S(N)} \sigma_k(\omega_N^{l k_i}) \right| \leq \frac{\varphi(N)}{\varphi(d_i)},$$

for $i \in \{1, 2, 3\}$. Combing (3.6) and (3.7) we get

$$\varphi(N) < \left| \sum_{k \in S(N)} \sigma_k(\tau) \right| = \left| \sum_{k \in S(N)} \sigma_k(\omega_N^{l k_1} + \omega_N^{l k_2} + \omega_N^{l k_3}) \right| \leq \left( \frac{1}{\varphi(d_1)} + \frac{1}{\varphi(d_2)} + \frac{1}{\varphi(d_3)} \right) \cdot \varphi(N).$$

Then the result can be obtained immediately. $\square$

Using the previous lemma, we could totally follow the statement due to [14] to show that there do not exist appropriate values for $l, k_1, k_2$ and $k_3$ such that (3.5) holds, i.e. $I_1I_2I_3$ can not be regular elliptic of finite order. So $\Gamma$ of type $(m, n, \infty)$ is not discrete when $I_1I_2I_3$ is regular elliptic. $\square$

Applying this theorem, we can work out some conditions on $\cos \theta$ for $\Gamma$ with angular invariant $\theta$ of type $(m, n, \infty)$ to be non-discrete. A simple calculation yields that

$$\tau = -5 - 2 \cos(2\pi/m) - 2 \cos(2\pi/n) + 8e^{i\theta} \cos(\pi/m) \cos(\pi/n)$$

by seeing (3.4). Consequently we can obtain the interval of $a = \cos \theta$ ($-1 \leq a \leq 1$) corresponding to the non-discrete $\Gamma$ by using the discriminant function (2.1). In the remaining content $\theta$ is the angular invariant of complex triangle group $\Gamma$ of type $(m, n, \infty)$, unless otherwise stated.

We give an example for $m = 8$ showing that $\Gamma$ of type $(8, n, \infty)$ is non-discrete if $a \in (a_n, b_n)$. Note that there are no solutions for $a$ when $n \leq 10$. 

Table 1. Approximate values of $a_n$, $b_n$.

| $n$ | $a_n$   | $b_n$   |
|-----|---------|---------|
| 11  | 0.93067 | 0.93114 |
| 12  | 0.93226 | 0.93268 |
| 13  | 0.93318 | 0.93377 |
| 14  | 0.93386 | 0.93454 |
| 15  | 0.93437 | 0.93512 |
| 20  | 0.93575 | 0.93654 |
| 30  | 0.93662 | 0.93733 |
| 40  | 0.93690 | 0.93757 |
| 100 | 0.93719 | 0.93780 |
| 200 | 0.93723 | 0.93783 |

In the following we will use other different ways to find sufficient conditions on $a$ for $\Gamma$ to be non-discrete. Let $g \in PU(2, 1)$ be a parabolic element. Define the translation length $t_g(z)$ of $g$ at $z \in \mathcal{R}$ by $t_g(z) = \rho_0(g(z), z)$. For the following discussion, it is necessary to give the explicit form of Ford isometric spheres. Let $h = (a_{ij})_{1 \leq i,j \leq 3}$ be an element of $PU(2, 1)$ not fixing $\infty$ (let the null vector $\infty$ represent the point $q_\infty$ at $\partial H_2^\mathbb{C}$). The isometry sphere of $h$ is the sphere in the Cygan metric with center at $h^{-1}(\infty)$ and radius

$$r_h = \sqrt{\frac{2}{|a_{22} - a_{23} + a_{32} - a_{33}|}}.$$  (see [7]). Now let’s recall the complex hyperbolic versions of Jørgensen’s inequality and Shimizu’s lemma.

**Lemma 3.5.** [5] Lemma 2.2] Let $A \in SU(2, 1)$ be a regular elliptic element of order $n \geq 7$ that preserves a Lagrangian plane (i.e. $tr(A)$ is real). Suppose that $A$ fixes a point $z \in \mathbb{H}_2^\mathbb{C}$. Let $B$ be any element of $SU(2, 1)$ with $B(z) \neq z$. If

$$\cosh\left(\frac{\rho(Bz, z)}{2}\right) \sin\left(\frac{\pi}{n}\right) < \frac{1}{2},$$

then $(A, B)$ is not discrete and consequently any group containing $A$ and $B$ is not discrete.

**Lemma 3.6.** [8] Theorem 2.1] Let $G$ be a discrete subgroup of $PU(2, 1)$ that contains the Heisenberg translation $g$ by $(\xi, \nu)$. Let $h$ be any element of $G$ not fixing $\infty$ and with isometric sphere of radius $r_h$. Then

$$r_h^2 \leq t_g(h^{-1}(\infty))t_g(h(\infty)) + 4|\xi|^2.$$  

In the sequel we give two necessary conditions for $(m, n, \infty)$—triangle groups to be discrete using the previous two lemmas.
Theorem 3.7. The complex hyperbolic triangle group $\Gamma$ of type $(m, n, \infty)$ is not discrete if $m, n, \theta$ satisfy one of the two following conditions

1. $7 \leq n < \infty$ and

\[ \left| \cos^2 \left( \frac{\pi}{n} \right) + 2 \cos^2 \left( \frac{\pi}{m} \right) - 4 \cos \left( \frac{\pi}{n} \right) \cos \left( \frac{\pi}{m} \right) \cos \theta + 1 \right| < \frac{1}{2} \sin \left( \frac{\pi}{n} \right); \]

2. (Suppose that $u = \cos^2 \left( \frac{\pi}{m} \right) + \cos^2 \left( \frac{\pi}{n} \right) - 2 \cos \left( \frac{\pi}{m} \right) \cos \left( \frac{\pi}{n} \right) \cos \theta$, $v = \cos \left( \frac{\pi}{m} \right) \cos \left( \frac{\pi}{n} \right) \sin \theta$.)

\[ |u - 2iv| + 4u < \frac{1}{4}. \]

Proof. (1) Let $A = I_1 I_2$, $B = I_3$ and $z = z_0$ (fixed point of $I_{12}$), where

\[ z_0 = \begin{bmatrix} s_1 \\ 0 \\ 1 \end{bmatrix}. \]

By computation, we know

\[ B(z_0) = \begin{bmatrix} s_1 - 2s_2 e^{i\theta} \\ -2s_1 s_2 e^{-i\theta} + 2s_2^2 \\ 2s_1 s_2 e^{-i\theta} - 2s_2^2 - 1 \end{bmatrix}. \]

It is easy to see $s_2 \neq s_1 e^{-i\theta}$ which means $B$ does not fix $z_0$, otherwise $C_1, C_2$ will coincide.

Using Jørgensen’s inequality stated previously (Lemma 3.5), we deduce if

\[ \frac{|\langle B(z_0), z_0 \rangle|}{\langle z_0, z_0 \rangle} \sin \left( \frac{\pi}{n} \right) < \frac{1}{2}, \]

then $\Gamma$ is not discrete. Simplifying the inequality above, we will obtain (3.8) stated in the theorem.

(2) Let $g = I_2 I_3$ and $h = I_1$. We see that $g$ is a Heisenberg translation in the form

\[ \left( 2 - e^{i\theta} \cos \left( \frac{\pi}{m} \right) + \cos \left( \frac{\pi}{n} \right) \right), 8 \sin \theta \cos \left( \frac{\pi}{m} \right) \cos \left( \frac{\pi}{n} \right) \] fixing $\infty$. It is clear that $h$ does not fix $\infty$ and has the isometric sphere with radius 1. By computing

\[ h^{-1}(\infty) = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, \quad h(\infty) = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, \]

we know

\[ t_g(h(\infty)) t_g(h^{-1}(\infty)) = \rho_0^2(g(h(\infty)), h(\infty)) = \left| 4 \left( \cos^2 \left( \frac{\pi}{m} \right) + \cos^2 \left( \frac{\pi}{n} \right) + 2 \cos \left( \frac{\pi}{m} \right) \cos \left( \frac{\pi}{n} \right) \cos \theta \right) - 8i \sin \theta \cos \left( \frac{\pi}{m} \right) \cos \left( \frac{\pi}{n} \right) \right|. \]
Then the inequality (3.9) can be obtained by applying Shimizu’s lemma for complex hyperbolic space (Lemma 3.6).

Following the preceding example listed in Table 1, we show the different intervals of $a$ such that $\Gamma$ to be non-discrete when $m = 8$. We will see the corresponding solutions $a \in (c_n, 1)$ by condition (1) and $a \in (d_n, 1)$ by condition (2). However there are no solutions for $a$ when $n \leq 6$ or $n \geq 130$ by condition (1) and also no solutions for $a$ when $n \leq 3$ by condition (2).

| $n$ | $c_n$ | $d_n$ |
|-----|------|------|
| 4   | —    | 0.99961 |
| 5   | —    | 0.99419 |
| 6   | —    | 0.99289 |
| 7   | 0.99170 | 0.99279 |
| 8   | 0.98685 | 0.99299 |
| 9   | 0.98459 | 0.99323 |
| 10  | 0.98363 | 0.99346 |
| 20  | 0.98750 | 0.99442 |
| 30  | 0.99147 | 0.99464 |
| 100 | 0.99911 | 0.99480 |
| 200 | —    | 0.99481 |

**Remark 3.8.** Non-discrete complex hyperbolic triangle groups of type $(n, n, \infty)$ has been investigated by some authors, one can refer to [4,5,6]. Table 1 and Table 2 are extension of their results for different type of complex hyperbolic triangle groups.

4. **Complex hyperbolic triangle groups of type $(n, \infty, \infty)$**

In this section, the aim is to consider the non-discrete classes of $\Gamma$ of type $(n, \infty, \infty)$. For convenience, we rewrite the three normalised polar vectors of $\Gamma$$p_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $p_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $p_3 = \begin{bmatrix} 1 \\ -se^{-i\theta} \\ se^{-i\theta} \end{bmatrix}$, where $s = \cos(\pi/n)$. Then the matrix representation of the three corresponding complex reflections can easily be obtained. In what follows we still assume that $a = \cos \theta$. A simple computation yields $\tau = \text{tr}(I_1I_2I_3) = -7 + 8e^{i\theta} \cos(\pi/n) - 2\cos(2\pi/n)$ and
the discriminant function (2.1)

\[ f(\tau) = 2048 - 10240a^2s^2 + 21760a^2s^2 - 16384as^3 - 16384a^3s^3 + 7680s^4 \\
+ 22528a^2s^4 - 18944as^5 + 3840s^6 + 4096a^2s^6 - 2048as^7 + 256s^8. \]

For different \( n \), the interval of \( a \) and the value of the parameter angular invariant \( \theta \) such that \( \Gamma \) to be non-discrete can be solved by Theorem 3.2. We observe that for \( n \geq 8 \), there are solutions \((\alpha_n, \beta_n)\) for \( a \), which lie between 0 and 1. But otherwise there are no solutions. Later we tabulate this interval of cos \( \theta \) with other two intervals which are defined after Corollary 4.1.

**Corollary 4.1.** If \( \Gamma \) of type \( (n, \infty, \infty) \) satisfies

1. \( 7 \leq n < \infty \) and \( |\cos^2(\pi/n) - 4\cos(\pi/n)\cos \theta + 3| < \frac{1}{2} \sin(\pi/n) \), or
2. \( |u - 2iv| + 4u < \frac{1}{4} \), where \( u = \cos^2(\pi/n) - 2\cos(\pi/n)\cos \theta + 1 \), \( v = \cos(\pi/n)\sin \theta \).

Then \( \Gamma \) will be non-discrete.

The proof of this theorem is obvious by letting \( m \) to be \( \infty \) if we see Theorem 3.7.

Thence from the assumption \( a = \cos \theta \) and \( s = \cos(\pi/n) \), we know if

\[ 35 - 96as + 25s^2 + 64a^2s^2 - 32as^3 + 4s^4 < 0, \quad \text{or} \]

\[ \sqrt{1 - 4as + 6s^2 - 4as^3 + s^4} < \frac{-15 + 32as - 16s^2}{4}, \]

then there are intervals \((\gamma_n, 1), (\eta_n, 1)\) of \( a \) for \( \Gamma \) to be non-discrete. The following Table 3 shows the intervals of cos \( \theta \).

Let \( \Gamma \) be a complex hyperbolic triangle group of type \( (n, \infty, \infty; k) \), where \( k \) is the order of \( I_3I_1I_3I_2 \). By simple computation, we have

\[ \text{tr}(I_3I_1I_3I_2) = 3 + 16s^2 - 16sa. \]

Denote \( \text{tr}(I_3I_1I_3I_2) \) by \( t \), then \( f(t) = 16384(a - s)^3s^3(-1 + 4(a - s)s) \). Therefore \( I_3I_1I_3I_2 \) will be an regular elliptic when \( a \in \left(s, \frac{1 + 4s^2}{16}\right) \). Especially \( a = s \) leads \( I_3I_1I_3I_2 \) to be unipotent parabolic, while if \( a = \frac{1 + 4s^2}{16} \) then \( I_3I_1I_3I_2 \) will be a boundary elliptic. In the following we will give two related examples.

**Example 4.2.** Discreteness of \( \Gamma_n \) of type \( (n, \infty, \infty) \) whose angular invariant \( \alpha = \frac{\pi}{n} \) (i.e. \( a = s \)).

By computing \( \tau = \text{tr}(I_1I_2I_3) = -3 + 2\cos(2\pi/n) + 4i\sin(2\pi/n) \), we have

\[ f(\tau) = 128(7 - 9\cos(2\pi/n))(\sin(\pi/n))^6. \]
$I_1I_2I_3$ will be a regular elliptic element when $n \geq 10$ which leads $\Gamma_n$ to be non-discrete. Additionally the inequality stated in Corollary 4.1 (1) is equivalent to

$$\sin\left(\frac{\pi}{n}\right) < \frac{1}{6},$$

i.e. $n \geq 19$. Meanwhile the condition (2) yields

$$\sqrt{16 + 32a^2 - 48a^4} < -15 + 16a^2,$$

i.e. $n \geq 61$. Therefore $\Gamma_n$ will be non-discrete when $n \geq 10$.

Specifically, when $n = 4$, See [5, Theorem 2.1], we will see

$$I_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & 2 \\ 2 & -2 & -3 \end{bmatrix},$$

$$I_3 = \begin{bmatrix} 1 & -1 - i & -1 - i \\ -1 + i & 0 & 1 \\ 1 - i & -1 & -2 \end{bmatrix}.$$ 

Obviously all of the matrix entries are in $\mathbb{Z}[i]$ which is a discrete subring of $\mathbb{C}$. Therefore $\Gamma_4$ is discrete. Here $I_1I_2I_3$ is a loxodromic element.

| n   | $\alpha_n$ | $\beta_n$ | $\gamma_n$ | $\eta_n$ |
|-----|------------|------------|------------|---------|
| 4   | —          | —          | —          | 0.99959 |
| 5   | —          | —          | —          | 0.99857 |
| 6   | —          | —          | —          | 0.99624 |
| 7   | —          | —          | 0.99748    | 0.99524 |
| 8   | 0.93724    | 0.93784    | 0.99099    | 0.99482 |
| 9   | 0.94201    | 0.94794    | 0.98756    | 0.99463 |
| 10  | 0.94476    | 0.95631    | 0.98575    | 0.99454 |
| 15  | 0.94993    | 0.97914    | 0.98472    | 0.99451 |
| 20  | 0.95142    | 0.98799    | 0.98647    | 0.99455 |
| 40  | 0.95272    | 0.99694    | 0.99171    | 0.99461 |
| 100 | 0.95306    | 0.99951    | 0.99632    | 0.99463 |
| 200 | 0.95311    | 0.99988    | 0.99809    | 0.99464 |

**Example 4.3.** $(7, \infty, \infty; 5)$ is non-discrete.
From the assumption about the trace of $I_{3132}$, we can deduce $3 + 16s^2 - 16sa = 1 + 2\cos(2\pi/k)$, i.e.

$$\cos(\theta) = \frac{8s^2 - \cos(2\pi/k) + 1}{8s},$$

where $s = \cos(\pi/7)$. It follows from Table 3 that $\Gamma$ is non-discrete when $0.28621 \leq \cos(2\pi/k) \leq 0.32052$. Then it is easy to see that $\Gamma$ of type $(7, \infty, \infty; 5)$ is non-discrete.

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