AUTOMORPHISM GROUPS OF DANIELEWSKI SURFACES

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Abstract. In this note we study the automorphism group of a smooth Danielewski surface $D_p = \{(x, y, z) \in \mathbb{A}^3 \mid xy = p(z)\} \subset \mathbb{A}^3$, where $p \in \mathbb{C}[z]$ is a polynomial without multiple roots and $\deg p \geq 3$. It is known that two such generic surfaces $D_p$ and $D_q$ have isomorphic automorphism groups. Moreover, $\text{Aut}(D_p)$ is generated by algebraic subgroups and there is a natural isomorphism $\phi : \text{Aut}(D_p) \sim \rightarrow \text{Aut}(D_q)$ which restricts to an isomorphism of algebraic groups $G \sim \rightarrow \phi(G)$ for any algebraic subgroup $G \subset \text{Aut}(D_p)$. In contrast, we prove that $\text{Aut}(D_p)$ and $\text{Aut}(D_q)$ are isomorphic as ind-groups if and only if $D_p \cong D_q$ as a variety. Moreover, we show that any automorphism of the ind-group $\text{Aut}(D_p)$ is inner.

1. Introduction and Main Results

Our base field is the field of complex numbers $\mathbb{C}$. For an affine algebraic variety $X$ the group of regular automorphisms $\text{Aut}(X)$ has a natural structure of an ind-group (see section 2 for the definition). In this paper we study the following question.

Question. Let $X$ and $Y$ be affine irreducible varieties. Assume that $\text{Aut}(X)$ is generated by algebraic subgroups and there is an abstract isomorphism of groups $\phi : \text{Aut}(X) \sim \rightarrow \text{Aut}(Y)$ such that $\phi$ preserves algebraic groups, i.e. for any algebraic subgroup $G \subset \text{Aut}(X)$, the image $\phi(G)$ is again an algebraic subgroup of $\text{Aut}(Y)$ and $\phi$ restricts to an isomorphism of algebraic groups $G \rightarrow \phi(G)$. Is it true that $\text{Aut}(X)$ and $\text{Aut}(Y)$ are isomorphic as ind-groups?

It is known that for two general ind-groups, the answer is negative. For instance, let $W_1$ be the first Weyl algebra i.e. the quotient of the free associative algebra $\mathbb{C}(x, y)$ by the relation $xy - yx = 1$ and $P_1$ be the corresponding Poisson algebra i.e. the polynomial algebra $\mathbb{C}[x, y]$ endowed with the Poisson bracket $\{f, g\} := f_xg_y - f_yg_x$ for $f, g \in \mathbb{C}[x, y]$. Notice that the automorphism group of any finitely generated algebra (not necessarily commutative) has a natural structure of an ind-group (see [FK17]). In particular, $\text{Aut}(W_1)$ and $\text{Aut}(P_1)$ have natural structures of ind-groups. Moreover, there is a natural isomorphism $\phi : \text{Aut}(W_1) \sim \rightarrow \text{Aut}(P_1)$ of abstract groups (see [BK05, Section 1.1]). The group $\text{Aut}(W_1)$ is isomorphic to the subgroup $\text{SAut}(\mathbb{A}^2) := \{(F_1, F_2) \in \text{Aut}(\mathbb{A}^2) \mid \det[\frac{\partial F_i}{\partial x_j}]_{i,j = 1} = 1\} \subset \text{Aut}(\mathbb{A}^2)$ (see [Di68], [ML84, Theorem 2]). It follows from [Jun42] and [Kul53] (see also [Kam75, Theorem 2]) that $\text{SAut}(\mathbb{A}^2)$ is an amalgamated product of the group $G_1 = \text{SL}(2, \mathbb{C}) \ltimes (\mathbb{C}^*)^2$ of special affine transformations of $\mathbb{A}^2$, and the solvable group $G_2$ of polynomial transformations of the form

$$(x, y) \mapsto (\lambda x + F(y), \lambda^{-1} y), \lambda \in \mathbb{C}^*, F \in \mathbb{C}[y].$$

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In fact, it is not difficult to see that the amalgamated product structure of $\text{Aut}(P_2)$ and $\text{Aut}(W_1)$ implies that the natural isomorphism $\phi$ preserves algebraic groups. On the other hand, in [BK05] BELOV-KANEL and KONTSEVICH noticed that $\text{Aut}(W_1)$ and $\text{Aut}(P_2)$ are not isomorphic as ind-groups. Another interesting example comes from a natural isomorphism $\psi : \text{Aut}(\mathbb{C}[x,y]) \sim \rightarrow \text{Aut}(\mathbb{C}[x,y])$ of abstract groups (see [ML70]). Using the amalgamated product structure of the group $\text{Aut}(\mathbb{C}[x,y])$ as above it is not difficult to see that the $\psi$ preserves algebraic groups. However, FURTER and KRAFT \footnote{In oral communication} showed (see [FK17]) that these automorphism groups are not isomorphic as ind-groups.

In both examples at least one group is not the automorphism group of a commutative algebra. The main result of this paper is a counterexample to the question posted above (see Theorem 2 and Remark 2).

Following [AFK13], for any affine variety $X$ we define $U(X) \subset \text{Aut}(X)$ as the subgroup which is generated by all $\mathbb{C}^\times$-actions on $X$. Denote by $\mathbb{A}^2/\mu_2$ the quotient of $\mathbb{A}^2$ by the cyclic group $\mu_2 = \{ \xi \in \mathbb{C} \mid \xi^2 = 1 \}$, where $\mu_2$ acts on $\mathbb{A}^2$ by scalar multiplication, and by $T \subset \text{SL}_2 := \text{SL}_2(\mathbb{C})$ the standard subtorus of $\text{SL}_2$. The subgroups $U(\mathbb{A}^2/\mu_2) \subset \text{Aut}(\mathbb{A}^2/\mu_2)$ and $U(\text{SL}_2/T) \subset \text{Aut}(\text{SL}_2/T)$ are closed (see Section 3) and hence, they have the structure of ind-subgroups. In [Reg17, Proposition 10] it is shown that there is an abstract isomorphism $\phi : U(\text{SL}_2/T) \sim \rightarrow U(\mathbb{A}^2/\mu_2)$ which preserves algebraic groups. In contrast, we prove the following result.

**Theorem 1.** The ind-groups $U(\text{SL}_2/T)$ and $U(\mathbb{A}^2/\mu_2)$ are not isomorphic.

We denote by $D_p = \{(x,y,z) \in \mathbb{A}^3 \mid xy = p(z)\}$ the so-called DANIELEWSKI surfaces, where $p$ is a polynomial in $\mathbb{C}[z]$ with $\deg p \geq 2$ and $p$ has no multiple roots. Note that $\text{SL}_2/T \cong D_p$, where $p = z^2 - z$. In the literature surfaces given by $\{ x^n y = p(z) \} \subset \mathbb{A}^3$ are often also called DANIELEWSKI surfaces. In this text we consider only the case where $n = 1$ and the surface is smooth.

Denote by $\text{Aut}^c(D_p)$ the connected component of the neutral element of the ind-group $\text{Aut}(D_p)$. In order to prove Theorem 1 we show that the Lie subalgebra $(\text{LND}(D_p)) \subset \text{Vec}(D_p)$ generated by all locally nilpotent vector fields on $D_p$ is simple (see Proposition 10). On the other hand the Lie subalgebra $(\text{LND}(\mathbb{A}^2/\mu_2)) \subset \text{Vec}(\mathbb{A}^2/\mu_2)$ generated by all locally nilpotent vector fields is not simple. If there were an isomorphism of ind-groups $U(\text{SL}_2/T) \sim \rightarrow U(\mathbb{A}^2/\mu_2)$, then we prove that the Lie algebras $(\text{LND}(\text{SL}_2/T))$ and $(\text{LND}(\mathbb{A}^2/\mu_2))$ would be isomorphic, which is not the case.

If $\deg p > 2$, it is proven in [ML90] that the group $\text{Aut}^c(D_p)$ equals $U(D_p) \rtimes T$, where $T = \{(tx, t^{-1}y, z) \mid t \in \mathbb{C}^\times \} \subset \text{Aut}^c(D_p)$ is a one-dimensional subtorus and $U(D_p)$ is isomorphic to the free product $\mathbb{C}[x] \ast \mathbb{C}[y]$. Hence, there is a natural isomorphism $\psi : \text{Aut}^c(D_p) \sim \rightarrow \text{Aut}^c(D_q)$ of abstract groups which preserves algebraic groups (as follows from Proposition 3). On the other hand we prove the following result.

**Theorem 2.** The ind-groups $\text{Aut}^c(D_p)$ and $\text{Aut}^c(D_q)$ are isomorphic if and only if the varieties $D_p$ and $D_q$ are isomorphic. Moreover, if $\deg p \geq 3$, then any isomorphism $\text{Aut}^c(D_p) \sim \rightarrow \text{Aut}^c(D_p)$ of ind-groups is inner.

**Remark 1.** Theorem 2 has an interesting consequence. Using this, one can characterize $D_p$ by its automorphism group viewed as an ind-group i.e. to prove that
for an affine normal irreducible variety $X$, isomorphism of ind-groups $\text{Aut}(X)$ and $\text{Aut}(D_p)$ implies that $X \cong D_p$ as a variety.

**Remark 2.** If $p$ is generic in the sense that no affine automorphism $\alpha$ of $\mathbb{C}$ permutes the roots of $p$, then $\text{Aut}(D_p)$ coincides with $\text{Aut}^\circ(D_p) \rtimes I$, where $I$ is the group of order two generated by the involution $(x, y, z) \mapsto (y, x, z)$.

In order to prove Theorem 2 we show that an isomorphism of the ind-groups $\text{Aut}^\circ(D_p)$ and $\text{Aut}^\circ(D_q)$ induces an isomorphism of the Lie algebras $\langle \text{LND}(D_p) \rangle$ and $\langle \text{LND}(D_q) \rangle$ which induces an isomorphism of the DANIELEWSKI surfaces $D_p$ and $D_q$ (see Proposition 5). To prove that any automorphism of $\text{Aut}^\circ(D_p)$ is inner one uses Theorem 3.

We have the natural identification between vector fields $\text{Vec}(D_p)$ and derivations $\text{Der}(\mathcal{O}(D_p))$ of the ring of regular functions of $D_p$. For $\psi \in \text{Aut}(D_p)$ and $\delta \in \text{Vec}(D_p)$ we define

$$\text{Ad}(\psi)\delta := (\psi^*)^{-1} \circ \delta \circ \psi^*$$

where we consider $\delta$ as a derivation $\delta : \mathcal{O}(D_p) \to \mathcal{O}(D_p)$ and $\psi : \mathcal{O}(D_p) \to \mathcal{O}(D_p)$, $f \mapsto f \circ \psi$, is the co-morphism of $\psi$. It is clear that the action of $\text{Aut}(D_p)$ on the Lie algebra $\text{Vec}(D_p)$ restricts to an action on the Lie subalgebra $\langle \text{LND}(D_p) \rangle \subset \text{Vec}(D_p)$ because $\text{Ad}(\psi)$ sends locally nilpotent vector fields to locally nilpotent vector fields. It turns out that any automorphism of the Lie algebra $\langle \text{LND}(D_p) \rangle$ comes from some automorphism of $D_p$. Moreover, from Lemma 13 and Proposition 8 it follows that the action of $\text{Aut}(D_p)$ on $\text{Vec}(D_p)$ restricts to the action of $\text{Aut}(D_p)$ on the Lie subalgebra $\text{Vec}^\circ(D_p) \subset \text{Vec}(D_p)$ of all vector fields of divergence zero (see Section 6 for definition). Vice versa, we prove the following result.

**Theorem 3.** If $\deg(p) \geq 3$, then we have:

(a) Let $F$ be an automorphism of the Lie algebra $\langle \text{LND}(D_p) \rangle$. Then $F$ coincides with $\text{Ad}(\varphi)$ for some automorphism $\varphi : D_p \to D_p$.

(b) Let $F$ be an automorphism of the Lie algebra $\text{Vec}^0(D_p)$. Then $F$ coincides with $\text{Ad}(\varphi)$ for some automorphism $\varphi : D_p \to D_p$.

This is an analogue to the following result in [KR17] (see also [Reg13]): each automorphism of the Lie subalgebra $\langle \text{LND}(\mathbb{A}^n) \rangle = \{ f_1 \frac{\partial}{\partial x_1} + \ldots + f_n \frac{\partial}{\partial x_n} \in \text{Vec}(\mathbb{A}^n) | \frac{\partial f_1}{\partial x_1} + \ldots + \frac{\partial f_n}{\partial x_n} = 0 \} \subset \text{Vec}(\mathbb{A}^n)$ generated by all locally nilpotent vector fields on $\mathbb{A}^n$ is induced from an automorphism of $\mathbb{A}^n$.

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2. **Preliminaries**

The notion of an ind-group goes back to Shafarevich who called these objects infinite dimensional groups, see [Sh66]. We refer to [Kum02] and [St13] for basic notations in this context.

**Definition 1.** By an ind-variety we mean a set $V$ together with an ascending filtration $V_0 \subset V_1 \subset V_2 \subset \ldots \subset V$ such that the following is satisfied:

1. $V = \bigcup_{k \in \mathbb{N}} V_k$;
2. each $V_k$ has the structure of an algebraic variety;
3. for all $k \in \mathbb{N}$ the subset $V_k \subset V_{k+1}$ is closed in the Zariski topology.
A morphism between ind-varieties $V = \bigcup_{k \in \mathbb{N}} V_k$ and $W = \bigcup_{l \in \mathbb{N}} W_l$ is a map $\phi : V \to W$ such that for each $k$ there is an $l \in \mathbb{N}$ such that $\phi(V_k) \subset W_l$ and that the induced map $V_k \to W_l$ is a morphism of algebraic varieties. Isomorphisms of ind-varieties are defined in the usual way.

Filtrations $V = \bigcup_{k \in \mathbb{N}} V_k$ and $V = \bigcup_{k \in \mathbb{N}} V'_k$ are called equivalent if for any $k$ there is an $l$ such that $V_k \subset V'_l$ is a closed subvariety as well as $V'_k \subset V_l$.

An ind-variety $V = \bigcup_{k \in \mathbb{N}} V_k$ has a natural topology: $S \subset V$ is closed, resp. open, if $S_k := S \cap V_k \subset V_k$ is closed, resp. open, for all $k$. Obviously, a closed subset $S \subset V$ has a natural structure of an ind-variety. It is called an ind-subvariety. An ind-variety $V$ is called affine if each algebraic variety $V_k$ is affine. Later on we consider only affine ind-varieties and for simplicity we call them just ind-varieties.

For an ind-variety $V = \bigcup_{k \in \mathbb{N}} V_k$ we can define the tangent space in $x \in V$ in the obvious way. We have $x \in V_k$ for $k \geq k_0$, and $T_x V_k \subset T_x V_{k+1}$ for $k \geq k_0$, and then we define

$$T_x V := \lim_{k \geq k_0} T_x V_k,$$

which is a vector space of countable dimension. A morphism $\phi : V \to W$ induces linear maps $d\phi_x : T_x V \to T_{\phi(x)} W$ for any $x \in X$. Clearly, for a $\mathbb{C}$-vector space $V$ of a countable dimension and for any $v \in V$ we have $T_x V = V$ in a canonical way.

The product of two ind-varieties is defined in the clear way. This allows us to give the following definition.

**Definition 2.** An ind-variety $G$ is called an ind-group if the underlying set $G$ is a group such that the map $G \times G \to G$, taking $(g, h) \mapsto gh^{-1}$, is a morphism of ind-varieties.

Note that any closed subgroup $H$ of $G$, i.e. $H$ is a subgroup of $G$ and is a closed subset, is again an ind-group under the closed ind-subvariety structure on $G$. A closed subgroup $H$ of an ind-group $G$ is an algebraic group if and only if $H$ is an algebraic subset of $G$ i.e. $H$ is a closed subset of some $G_i$, where $G_1 \subset G_2 \subset \ldots$ is a filtration of $G$.

An ind-group $G$ is called connected if for each $g \in G$ there is an irreducible curve $C$ and a morphism $C \to G$ whose image contains $e$ and $g$.

If $G$ is an ind-group, then $T_x G$ has a natural structure of a Lie algebra which will be denoted by $\text{Lie} G$. The structure is obtained by showing that each $A \in T_x G$ defines a unique left-invariant vector field $\delta_A$ on $G$, see [Kum02, Proposition 4.2.2, p. 114].

The next result can be found in [St13].

**Proposition 1.** Let $X$ be an affine variety. Then $\text{Aut}(X)$ has a natural structure of an affine ind-group.

By $\text{Vec}(X)$ we denote a Lie algebra of all vector fields on an affine variety $X$. A vector field $\nu \in \text{Vec}(X)$ is called locally nilpotent if the corresponding derivation $D \in \text{Der}\mathcal{O}(X)$ is locally nilpotent i.e. if for any $f$ there exist $n \in \mathbb{N}$ such that $D^n(f) = 0$. Later on we always identify a vector field on an affine variety $X$ with its corresponding derivation. By $\langle \text{LND}(X) \rangle$ we mean a Lie subalgebra of $\text{Vec}(X)$ generated by all locally nilpotent vector fields.

The next result can be found in [Kum02, Proposition 4.2.2].

**Proposition 2.** Let $\phi : G \to H$ be a homomorphism of ind-groups. Then $\phi$ induces a homomorphism $d\phi_e : \text{Lie} G \to \text{Lie} H$ of Lie algebras.
Since Aut(X) has a structure of an ind-group for any affine variety X, we can define a Lie algebra of Aut(X). It is known that there is a homomorphism of Lie algebras \( \psi : Lie\ Aut(X) \to Vec(X) \) which is injective on each Lie \( K \subset Aut(X) \), where \( K \subset Aut(X) \) is an algebraic subgroup. Hence, \( \psi \) is injective on the Lie subalgebra \( Lie_u(Aut(X)) := (Lie\ K)|K \subset Aut(X) \) is an algebraic subgroup isomorphic to \( C^+ \) \( \subset \) Lie Aut(X) generated by Lie algebras of one-dimensional unipotent subgroups. The map \( \psi \) is injective on \( Lie_u(Aut(X)) \) and the image of \( Lie_u(Aut(X)) \) under \( \psi \) equals \( (LND(X)) \) because any locally nilpotent vector field \( \nu \in Vec(X) \) belongs to \( \psi(\text{Lie } K) \) for some one-dimensional closed unipotent subgroup \( K \) of \( Aut(X) \). In fact, one can prove that \( \ker \psi \) is trivial.

Let \( X \) and \( Y \) be affine varieties such that there is an isomorphism \( \phi : Aut(X) \overset{\sim}{\to} Aut(Y) \) of ind-groups. Then isomorphism of Lie algebras \( d\phi_e : Lie\ Aut(X) \overset{\sim}{\to} Lie\ Aut(Y) \) induces an isomorphism of Lie subalgebras \( Lie_u(Aut(X)) \cong (LND(X)) \) and \( Lie_u(Aut(Y)) \cong (LND(Y)) \). In the future we will always identify \( Lie_u(Aut(X)) \) with \( (LND(X)) \).

**Definition 3.** By \( U(X) \) we mean the subgroup of \( Aut(X) \) generated by all closed one-dimensional unipotent subgroups.

3. **Automorphisms of Danielewski surface**

Let \( p \in C[t] \) be a polynomial of degree \( d \geq 2 \) with simple roots. Define the Danielewski-surface \( D_p \subset A^3 \) to be the zero set of the irreducible polynomial \( xy - p(z) \):

\[
D_p = \{(x, y, z) \in A^3 | xy = p(z)\} \subset A^3.
\]

The following is easy (\( \hat{\mathbb{C}} := C \setminus \{0\} \)):

(a) \( D_p \) is smooth,
(b) the two projections \( \pi_x : D_p \to C, (x, y, z) \mapsto x \) and \( \pi_y : D_p \to C, (x, y, z) \mapsto y \) are both smooth,
(c) \( (D_p)_x = \pi_x^{-1}(\hat{\mathbb{C}}) \overset{\sim}{\to} \hat{\mathbb{C}} \times \mathbb{C}, (x, y, z) \mapsto (x, z) \) and similarly for \( \pi_y \),
(d) \( \pi_x^{-1}(0) \) is the disjoint union of \( d \) copies of the affine line \( \mathbb{C} \).

For the rest of this section we assume \( \deg p > 2 \) unless stated otherwise. For every nonzero \( f \in C[t] \) there is a \( C^+ \)-action \( \alpha_f \) on \( \hat{\mathbb{C}} \times \mathbb{C} \) given by \( \alpha_f(s)(x, z) := (x, x + f(x)) \), i.e., by translation with \( f(x) \) in the fiber of \( x \in \hat{\mathbb{C}} \). It is easy to see that this action extends to an action on \( D_p \) if and only if \( f(0) = 0 \). We denote the corresponding actions on \( D_p \) by \( \alpha_{x,f} \), respectively \( \alpha_{y,f} \). The explicit form is

\[
\alpha_{x,f}(s)(x, y, z) = (x, \frac{p(z + sf(x))}{x}, z + sf(x))
\]

and similarly for \( \alpha_{y,f} \). The projection \( \pi_x : D_p \to \mathbb{C} \) is the quotient for all these actions, and the action on \( \pi^{-1}(0) \) is trivial. Note that the corresponding vector fields are given by

\[
\nu_{x,f} := p'(z)\frac{f(x)}{x}\frac{\partial}{\partial y} + f(x)\frac{\partial}{\partial z} \text{ and } \nu_{y,f} := p'(z)\frac{f(y)}{y}\frac{\partial}{\partial x} + f(y)\frac{\partial}{\partial z}.
\]

Denote by \( U_x, U_y \subset Aut(D_p) \) the image of \( \alpha_x \) and \( \alpha_y \). Note that there is also a faithful \( C^+ \)-action on \( D_p \) given by \( t(x, y, z) := (tx, t^{-1}y, z) \) which normalizes \( U_x \) and \( U_y \). Denote by \( T \subset Aut(D_p) \) the image of \( C^+ \). The following result is due to Makar-Limanov.
Proposition 3. (a) The group $\text{Aut}(D_p)$ is generated by $U_x, U_y, T$ and a finite subgroup $F$ which normalizes $(U_x, U_y, T)$.

(b) $(U_x, U_y, T) \subset \text{Aut}(D_p)$ is a closed connected subgroup of finite index, hence $\langle U_x, U_y, T \rangle = \text{Aut}^s(D_p)$.

(c) $U(D_p) = U_x * U_y$ is a free product and every one-parameter unipotent subgroup of $\text{Aut}(D_p)$ is conjugate to a subgroup of $U_x$ or $U_y$.

(d) $\langle U_x, U_y, T \rangle = U(D_p) \rtimes T$ is a semi-direct product of $U(D_p)$ and torus $T$.

Proof. The fact that $\text{Aut}(D_p)$ is generated by $U_x, U_y, T$ and a finite subgroup is proved in [ML90]. The additional claims in (a), (c) and (d) about the structure of $\text{Aut}(D_p)$ are claimed in a remark of the same article and proven in [KL13]. It is clear that the subgroup $(U_x, U_y, T) \subset \text{Aut}(D_p)$ is connected. Moreover, because it has a finite index, it is closed and $\langle U_x, U_y, T \rangle = \text{Aut}^s(D_p)$. Hence, (b) follows. □

Proposition 4. $\text{Aut}(SL_2/T) = U(SL_2/T) \rtimes \mu_2$, where $\mu_2$ denotes a cyclic group of order 2. In particular, $\text{Aut}^s(SL_2/T) = U(SL_2/T)$ is an ind-group.

Proof. It is clear that $SL_2/T \cong D_z(z-1)$. Note that for any two polynomials $p, q \in \mathbb{C}[z]$ of degree 2 without multiple roots, we have $D_p \cong D_q$. It follows from [Lam05, Theorem 6] that $\text{Aut}(D_z(z-1))$ is generated by $\mathbb{C}^+$-actions and cyclic subgroup $\mu_2$ of order 2 which permute roots $\{a, b\}$ of $p$, i.e $\text{Aut}(D_z(z-1)) = \langle U(D_z(z-1)), \mu_2 \rangle$. Because $U(D_z(z-1))$ is normal subgroup of $\text{Aut}(D_z(z-1))$, we have $\text{Aut}(D_z(z-1)) = U(D_z(z-1)) \rtimes \mu_2$.

From the decomposition $\text{Aut}(SL_2/T) = U(SL_2/T) \cup g U(SL_2/T)$, where $g \in \mu_2$ is a non-neutral element, it follows that $\text{Aut}^s(SL_2/T) \subset U(SL_2/T)$. Now, the second statement follows from the fact that $U(SL_2/T)$ is generated by $\mathbb{C}^+$-actions. □

Note that $U(\mathbb{A}^2/\mu_2) \subset \text{Aut}(\mathbb{A}^2/\mu_2)$ is a closed subgroup (see [Reg17, page 9]). Hence, $U(\mathbb{A}^2/\mu_2)$ has the natural structure of an ind-group.

4. Proof of the Main Results

We denote by $\langle \text{LND}(\mathbb{A}^2/\mu_2) \rangle$ the Lie subalgebra of $\text{Vec}(\mathbb{A}^2/\mu_2)$ generated by all locally nilpotent vector fields on $\mathbb{A}^2/\mu_2$.

Proof of Theorem 1. Assume there is an isomorphism $\phi : U(SL_2/T) \cong U(\mathbb{A}^2/\mu_2)$ of ind-groups. Then it induces an isomorphism $d\phi_e : \text{Lie}(U(SL_2/T)) \cong \text{Lie}(U(\mathbb{A}^2/\mu_2))$ of Lie algebras, and because $\phi$ maps each closed unipotent subgroup $U \cong \mathbb{C}^+$ to $U \cong \mathbb{C}^+$, $d\phi_e$ induces an isomorphism of Lie algebras $\langle \text{LND}(SL_2/T) \rangle$ and $\langle \text{LND}(\mathbb{A}^2/\mu_2) \rangle$.

By Proposition 10, $\langle \text{LND}(SL_2/T) \rangle$ is simple. On the other hand, we claim that $\langle \text{LND}(\mathbb{A}^2/\mu_2) \rangle$ is not simple. Indeed, since $\mathbb{A}^2/\mu_2$ has an isolated singular point $s$, each vector field, which comes from an algebraic group action, vanishes at this singular point. In particular, each locally nilpotent vector field vanishes at $s$. Because $\langle \text{LND}(\mathbb{A}^2/\mu_2) \rangle$ is generated by locally nilpotent vector fields, each $\nu \in \langle \text{LND}(\mathbb{A}^2/\mu_2) \rangle$ vanishes at $s$. Let $I \subset \langle \text{LND}(\mathbb{A}^2/\mu_2) \rangle$ be a Lie subalgebra generated by those vector fields which vanish at $s$ with multiplicity $k > 1$. It is clear that $I \neq \langle \text{LND}(\mathbb{A}^2/\mu_2) \rangle$ because $x_{\nu}^{(k)} \in \langle \text{LND}(\mathbb{A}^2/\mu_2) \rangle \setminus I$. Moreover, it is easy to see that $[\nu, \mu] \in I$ for any $\nu \in I$ and any $\mu \in \langle \text{LND}(\mathbb{A}^2/\mu_2) \rangle$ which shows that $I$ is an ideal. The claim follows. □
Proof of Theorem 2. Assume there is an isomorphism \( \phi : \text{Aut}^e(D_p) \xrightarrow{\sim} \text{Aut}^e(D_q) \) of ind-groups. Then it induces an isomorphism \( d\phi_e : \text{Lie}\Aut^e(D_p) \xrightarrow{\sim} \text{Lie}\Aut^e(D_q) \) of Lie algebras, and because \( \phi \) maps each closed unipotent subgroup \( H \cong \mathbb{C}^+ \) to \( \phi(H) \cong \mathbb{C}^+ \), \( d\phi_e \) induces an isomorphism of Lie algebras \( \langle \text{LND}(D_p) \rangle \) and \( \langle \text{LND}(D_q) \rangle \). Now from Proposition 5(c) it follows that \( D_p \cong D_q \).

Let now \( D_q = D_p \) and \( \phi : \text{Aut}^e(D_p) \xrightarrow{\sim} \text{Aut}^e(D_p) \) be an automorphism of an ind-group \( \text{Aut}^e(D_p) \). The map \( d\phi_e \) induces an automorphism of the Lie algebra \( \langle \text{LND}(D_p) \rangle \). By Proposition 5, \( d\phi_e \nu = \text{Ad}(F)\nu \), where \( \nu \) is a locally nilpotent vector field and \( F : D_p \rightarrow D_p \) is an automorphism of the variety. This implies that \( \phi(h) = F^{-1} \circ h \circ F \) for any \( h \in H \), where \( H \subset \text{Aut}^e(D_p) \) is a closed subgroup isomorphic to \( \mathbb{C}^+ \). Hence, the restriction of \( \phi \) to \( U(D_p) \) is an inner automorphism.

Now the claim follows from Proposition 11. \( \square \)

Let \( \theta : X \rightarrow Y \) be an isomorphism of affine varieties. For \( \delta \in \text{Vec}(X) \) we define

\[
\text{Ad}(\theta)\delta = (\theta^*)^{-1} \circ \delta \circ \theta^*,
\]

where we consider \( \delta \) as a derivation \( \delta : \mathcal{O}(X) \rightarrow \mathcal{O}(X) \) and \( \theta^* : \mathcal{O}(Y) \rightarrow \mathcal{O}(X), f \mapsto f \circ \theta \), is the co-morphism of \( \theta \). It is clear that \( \text{Ad}(\theta) : \text{Vec}(X) \xrightarrow{\sim} \text{Vec}(Y) \) is an isomorphism of Lie algebras. Moreover, because \( \text{Ad}(\theta) \) sends locally nilpotent vector fields to locally nilpotent, \( \text{Ad}(\theta) \) induces an isomorphism of Lie algebras \( \langle \text{LND}(X) \rangle \subset \text{Vec}(X) \) and \( \langle \text{LND}(Y) \rangle \subset \text{Vec}(Y) \).

The next crucial result we will prove in Section 8. By \( \text{Vec}^0 D_p \), we denote the Lie subalgebra of \( \text{Vec} D_p \) of volume preserving vector fields (see Section 7).

**Proposition 5.** (a) Let \( F : \langle \text{LND}(D_p) \rangle \xrightarrow{\sim} \langle \text{LND}(D_q) \rangle \) be an isomorphism of Lie algebras and let \( \text{deg}(q) \geq 3 \). Then \( F \) is equal to \( \text{Ad}(\varphi_F) \), where \( \varphi_F : D_p \xrightarrow{\sim} D_q \) is an isomorphism of varieties.

(b) Let \( F : \text{Vec}^0 D_p \xrightarrow{\sim} \text{Vec}^0 D_q \) be an isomorphism of Lie algebras and let \( \text{deg}(q) \geq 3 \). Then \( F \) is equal to \( \text{Ad}(\varphi_F) \), where \( \varphi_F : D_p \xrightarrow{\sim} D_q \) is an isomorphism of varieties.

(c) If Lie algebras \( \langle \text{LND}(D_p) \rangle \) and \( \langle \text{LND}(D_q) \rangle \) are isomorphic then the varieties \( D_p \) and \( D_q \) are isomorphic.

**Proof of Theorem 3.** The proof follows from Proposition 5. \( \square \)

5. Module of differentials and vector fields

Since \( D_p \) is smooth, the differentials \( \Omega(D_p) \) and the vector fields \( \text{Vec}(D_p) \) are locally free \( \mathcal{O}(D_p) \)-modules, and then, projective. More precisely, we have the following description.

**Proposition 6.** (a) The module \( \Omega(D_p) \) of differentials is projective of rank 2 and is generated by \( dx, dy, dz \), with the unique relation \( ydx + xdy - p'(z)dz = 0 \).

(b) The module \( \Omega^2(D_p) := \Lambda^2 \Omega(D_p) \) is free of rank one and is generated by

\[
\omega_p := \frac{1}{x}dx \wedge dz = -\frac{1}{y}dy \wedge dz = \frac{1}{p'(z)}dx \wedge dy.
\]

**Proof.** (a) From above it is clear that \( \Omega(D_p) \) is the projective module of rank 2 = \( \dim(D_p) \). It is easy to see that \( \Omega(D_p) = (\mathcal{O}(D_p)dx \oplus \mathcal{O}(D_p)dy \oplus \mathcal{O}(D_p)dz)/(ydx + xdy - p'(z)dz) \), where \( ydx + xdy - p'(z)dz = d(xy - p(z)) \). In fact, the surface \( D_p \) is covered by the special open sets \( D_x, D_y, D_{p'(z)} \) and \( \Omega(D_p) \) is free module of
Remark 4. \( \rho \) is surjective because \( \Omega(D_{\eta}) \) is free of rank 1 (see also [KK10, Section 3] for details).

Remark 3. In fact, for any normal hypersurface \( X \subset \mathbb{C}^n \), \( \Omega^{n-1}(X) := \bigwedge^{n-1} \Omega(X) \) is a free module of rank one.

Remark 4. Note that there is no \( \delta \in \text{Vec}(D_p) \) such that \( \delta : \mathcal{O}(D_p) \to \mathcal{O}(D_p) \) is surjective because \( \Omega(D_p) \) is not free. Note also that \( \omega_p \) is unique up to a constant because \( \mathcal{O}(D_p)^* = \mathbb{C}^* \).

It is well-known that every vector field \( \delta \) on \( D_p \subset \mathbb{C}^3 \) extends to a vector field \( \tilde{\delta} \) on \( \mathbb{C}^3 \). It follows that \( \delta \) can be written in the form
\[
\delta = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z},
\]
where \( a, b, c \in \mathcal{O}(D_p) \) such that \( ay + bx - cp'(z) = 0 \) in \( \mathcal{O}(D_p) \). In fact, considering \( \delta \) as a \( \mathcal{O}(D_p) \)-linear map \( \Omega(D_p) \to \mathcal{O}(D_p) \), we have \( a = \delta(dx) \), \( b = \delta(dy) \) and \( c = \delta(dz) \). This presentation of \( \delta \) is unique.

Remark 5. In fact, the vector fields \( \text{Vec}(D_p) \) form a module over \( \mathcal{O}(D_p) \) of rank 2, generated by
\[
\nu_z := x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \quad \nu_x := p'(z) \frac{\partial}{\partial y} + y \frac{\partial}{\partial x}, \quad \nu_y := p'(z) \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}
\]
with the unique relation \( x \nu_y - y \nu_x = p'(z) \nu_z \).

The next result is clear.

Proposition 7. The sequence
\[
0 \to \mathbb{C} \to \mathcal{O}(D_p) \xrightarrow{d} \Omega(D_p) \xrightarrow{d} \Omega^2(D_p) \to 0
\]
is exact.

6. Volume Form and Divergence.

For any \( \theta \in \text{Vec}(D_p) \) we have the contraction
\[
i_\theta : \Omega^{k+1} \to \Omega^k, \quad i_\theta(\eta)(\theta_1, ..., \theta_k) := \eta(\theta, \theta_1, ..., \theta_k).
\]
In particular, for \( \eta \in \Omega(D_p) \), we have \( i_\theta(\eta) = \eta(\theta) \in \mathcal{O}(D_p) \), and so \( i_\theta(df) = \theta f \).

The vector field \( \theta \in \text{Vec}(D_p) \) acts on the differential forms \( \Omega(D_p) \) by the \( \mathcal{O}(D_p) \) derivative \( L_\theta := d \circ i_\theta + i_\theta \circ d \), extending the action on \( \mathcal{O}(D_p) \). One finds (see for details [KK10, Section 3])

\[
L_\theta(f) = \theta f, \quad L_\theta(df) = d(\theta f) \quad \text{and} \quad L_\theta(h \cdot \mu) = \theta h \cdot \mu + h \cdot L_\theta \mu \quad \text{for} \ f, h \in \mathcal{O}(D_p), \mu \in \Omega(D_p).
\]

Using the volume form \( \omega_p \) (see Proposition 6), this allows to define the divergence \( \text{div}(\theta) \) of a vector field \( \theta \):
\[
L_\theta \omega_p = d(i_\theta \omega_p) = \text{div}(\theta) \cdot \omega_p.
\]

The following Lemma is well-known:

Lemma 1. Let \( \theta = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z} \in \text{Vec}(D_p) \). Then \( \text{div}(\delta) = ax + by + cz \).
Proof. We have \( i_\theta \omega_p = \frac{1}{x} (\theta(x) dz - \theta(z)) dx = \frac{1}{x} (adz - cdx) \), hence
\[
\operatorname{div}(\theta) \cdot \omega_p = d(i_\theta \omega_p) = d(\frac{1}{x} (adz - cdx))
\]
\[
= \frac{1}{x^2} ((xda - adx) \wedge dz - (xdc \wedge dx).
\]
Now we use the following equalities: \( da \wedge dz = a_x \cdot dx \wedge dz + a_y \cdot dy \wedge dz, dx \wedge dc = c_y \cdot dx \wedge dy + c_z \cdot dx \wedge dz, dy \wedge dz = -y \cdot \omega_p, \) and \( dx \wedge dy = p'(z) \cdot \omega_p \) (see above) to get
\[
\operatorname{div}(\theta) = \frac{a}{x} + a_x - \frac{y}{x} a_y + \frac{p'(z)}{x} c_y + c_z.
\]
Since \( ya + xb - p'(z)c = 0 \) we have \( a + ya_y + xb_y - p'(z)c_y = 0 \), hence
\[
-\frac{a}{x} - \frac{y}{x} a_y + \frac{p'(z)}{x} c_y = b_y,
\]
and the claim follows. \( \square \)

There is another important formula which relates the Lie structure of \( \text{Vec}(D_p) \) with the Lie derivative (see also [KL13, Lemma 3.2]).

**Lemma 2.** For \( \theta_1, \theta_2 \in \text{Vec}(D_p) \) and \( \mu \in \Omega(D_p) \) we have
\[
i_{[\theta_1, \theta_2]} \mu = \mathcal{L}_{\theta_1}(i_{\theta_2} \mu) - i_{\theta_2}(\mathcal{L}_{\theta_1} \mu).
\]

### 7. Duality.

The volume form \( \omega_p \in \Omega^2(D_p) \) induces the usual duality between vector fields and differential forms: the \( \mathcal{O}(D_p) \)-isomorphism \( \text{Vec}(D_p) \xrightarrow{\sim} \Omega(D_p) \) is given by \( \theta \mapsto i_\theta \omega_p \).

In particular, for every \( f \in \mathcal{O}(D_p) \) we get a vector field \( \nu_f \in \text{Vec}(D_p) \) defined by \( i_{\nu_f} \omega_p = df \).

Denote by \( \text{Vec}^0(D_p) \subset \text{Vec}(D_p) \) the subspace of volume preserving vector fields, i.e. \( \text{Vec}^0(D_p) := \{ \theta \in \text{Vec}(D_p) \mid \operatorname{div} \theta = 0 \} \).

**Proposition 8.** The map \( f \mapsto \nu_f \) induces a \( \mathbb{C} \)-linear isomorphism
\[
\mathcal{O}(D_p) / \mathbb{C} \xrightarrow{\sim} \text{Vec}^0(D_p).
\]

**Proof.** Since \( d(i_\theta \omega_p) = \operatorname{div}(\theta) \cdot \omega_p \), we have the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{C} & \xrightarrow{d} & \mathcal{O}(D_p) & \xrightarrow{d} & \Omega(D_p) & \xrightarrow{d} & \Omega^2(D_p) & \longrightarrow & 0 \\
& & \uparrow \cong && \uparrow \cong & \cong && \uparrow \cong && \\
& & \mathcal{O}(D_p) & \xrightarrow{\nu} & \text{Vec}(D_p) & \xrightarrow{\text{div}} & \mathcal{O}(D_p) \\
\end{array}
\]

Now the claim follows because the first row is exact (see Proposition 7). \( \square \)

The following result can be found in [KL13, Theorem 3.26].

**Proposition 9.** Any vector field \( \nu \in \text{Vec}^0(D_p) \) on the Danielewski surface \( D_p \) is a Lie combination of locally nilpotent vector fields if and only if its corresponding function with \( i_\nu \omega_p = df \) is of the form (modulo constant)
\[
f(x, y, z) = \sum_{i=1,j=0}^k a_{ij} x^i y^j + \sum_{i=1,j=0}^l b_{ij} y^i z^j + (pr)'(z)
\]
for a polynomial $r \in \mathbb{C}[z]$. If $\deg p \geq 3$, then any $f \in \mathcal{O}(D_p) / \mathbb{C}$ bijectively corresponds to some $\nu_f \in \text{Vec}^0(D_p) := \langle \text{LND}(D_p) \rangle \oplus \bigoplus_{i=0}^{\deg p-3} \mathbb{C} z^i(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y})$.

Some corresponding functions are given as follows (see [KL13, Lemma 3.1]): Let $h$ be a polynomial in one variable and let $\nu_x, \nu_y, \nu_z$ be the vector fields from Remark 5. Then

\begin{equation}
\begin{aligned}
f_{h(x)\nu_x} &= -h(x), & f_{h(y)\nu_y} &= h(y), & f_{h(z)\nu_z} &= h(z).
\end{aligned}
\end{equation}

We also recall the useful relation that describes the corresponding function of a Lie bracket of two vector fields $\nu, \mu \in \text{Vec}^0 D_p$ (see [KL13, Lemma 3.2]):

\begin{equation}
f_{[\nu, \mu]} = \nu(f_\mu),
\end{equation}

where $\nu(f_\mu)$ is $\nu$ applied as a derivation to the function $f_\mu$. The function $f_{[\nu, \mu]}$ may also be calculated by the following formula (see [KL13, formula after Lemma 3.2]):

\begin{equation}
f_{[\nu, \mu]} = \{f_\nu, f_\mu\} := p'(z) \left( (f_\nu)_y(f_\mu)_x - (f_\nu)_x(f_\mu)_y \right) + x((f_\nu)_z(f_\mu)_x - (f_\nu)_x(f_\mu)_z) - y((f_\nu)_z(f_\mu)_y - (f_\nu)_y(f_\mu)_z),
\end{equation}

where the subindex denotes the partial derivative to the respective variable.

Let $I \subset \langle \text{LND}(D_p) \rangle$ be a non-trivial ideal and let $\tilde{I}$ be the set of functions corresponding to this ideal by the correspondence in (1). Since $I$ is an ideal, we have, using (3), that

\begin{equation}
\{ \nu \in I \iff f_\nu \in \tilde{I} \} \quad \text{and} \quad \mu \in \langle \text{LND}(D_p) \rangle \implies \nu(f_\mu), \; \mu(f_\nu) \in \tilde{I}.
\end{equation}

Our next goal is to prove the following result.

**Proposition 10.** The Lie algebra $\langle \text{LND}(D_p) \rangle$ is simple.

We prove Proposition 10 in several steps and start with the following Lemma.

**Lemma 3.** Let $f$ be a regular function on $D_p$. Then $f$ can be written uniquely as $f(x, y, z) = \sum_{i=1}^{k} a_i(z)x^i$ for some $k, l \in \mathbb{Z}$.

**Proof.** Let us take the form of $f$ as in (1) and replace $y$ by $p(z)/x$. The claim follows. \qed

Choose $l, k \in \mathbb{Z}$ such that $a_l, a_k \neq 0$ and denote by $\deg(f) = (l, k)$ the pair of min- and max-degree in $x$.

**Lemma 4.** Let $f \in \mathcal{O}(D_p)$. Then $\nu_x(f)$ and $\nu_y(f)$ are never non-zero constants.

**Proof.** Any regular function $f$ on $D_p$ can be written in the form $\sum_{i=1}^{k} a_i(z)x^i$ by Lemma 3. Then $\nu_x(f) = \sum_{i=1}^{k} a'_i(z)x^{i+1}$, in particular, $\nu_x(f)$ is constant only if $a_{-1}$ is linear, which is not the case since $a_{-1}$ is divisible by $p$. The case of $\nu_y(f)$ is analogous. \qed

**Lemma 5.** Let $\deg f = (l, k)$ and $l, k \geq 1$. Then $\deg \nu_y(f) = (l - 1, k - 1)$.

**Proof.** Let $f = \sum_{i=1}^{k} a_i(z)x^i$. Then $\nu_y(f) = \sum_{i=1}^{k} (ip'(z)a_i(z) + p(z)a'_i(z))x^{i-1}$. If $a_i(z) \neq 0$, then $ip'(z)a_i(z) + p(z)a'_i(z) \neq 0$ and the claim follows. \qed

**Lemma 6.** Let $\deg f = (l, k)$, where $k > l \geq 0$, then $\deg (p''(z)\nu_z(f)) = (\tilde{l}, k)$, where $\tilde{l} = l$ if $l \geq 1$ and $\tilde{l} > l$ if $l = 0$. 

Proof. Let \( f = \sum_{i=1}^{k} a_i(z)x^i \). Then the claim follows from the equality \( \nu_z(f) = \sum_{i=1}^{k} ia_i(z)x^i \).

\[ \]  

**Lemma 7.** Let \( \bar{I} \neq 0 \). Then there exists some \( h \in \bar{I} \) such that \( h \in \mathbb{C}[z] \setminus \mathbb{C} \) is a non-constant polynomial in \( z \).

**Proof.** Take a non-zero \( f \in \bar{I} \). Since \( \nu_z \) is locally nilpotent, there is \( k \in \mathbb{N} \) such that \( \nu_z(\nu_z^k(f)) = 0 \). Thus we have a function \( g = \nu_z^k(f) \in \mathbb{C}[x] \setminus \mathbb{C} \), which means \( \deg g = (l, k) \) for \( k, l \geq 1 \). From (5) it follows that \( g \in \bar{I} \). By applying Lemma 5 and Lemma 6 step by step, we will get that there is a non-constant \( h \in \bar{I} \) such that \( \deg h = (0, 0) \), which proves the claim.

\[ \]  

**Lemma 8.** Let \( \bar{I} \neq 0 \). Then there is some \( m \in \mathbb{N} \) such that \( r(z)x^{n+1} \in \bar{I} \) for any \( r \in \mathbb{C}[z] \) and \( n \geq m \).

**Proof.** By Lemma 7 we have a non-constant \( h(z) \in \bar{I} \). By (5), we get that \( \nu_z(h) = h'(z)x \in \bar{I} \) and \( (p(z)s(z))^m \nu_z(h'(z)x) = (p(z)s(z))^m h'(z)x \in \bar{I} \) for any \( s \in \mathbb{C}[z] \). Let \( m = \deg p'h' \) and \( n \geq m \). Then applying (5) \( m - 1 \) times for \( \mu = \nu_z \) we get

\[
\nu_z^{m-1}(p(z)s(z))^m h'(z)x = ((ps)^m h')(m-1)(z)x^m \in \bar{I}.
\]

Now apply (5) once more for \( \mu = x^{n-m} \nu_z \) and get

\[
x^{n-m} \nu_z((ps)^m h')(m-1)(z)x^m = ((ps)^m h')(m)(z)x^{n+1} \in \bar{I},
\]

and thus varying \( s(z) \) we get \( r(z)x^{n+1} \in \bar{I} \) for any \( r \in \mathbb{C}[z] \).

\[ \]  

**Lemma 9.** Let \( \bar{I} \neq 0 \). Then \( r(z)x^n \in \bar{I} \) for all \( r \in \mathbb{C}[z] \) and \( n \geq 1 \).

**Proof.** We prove this lemma by induction. By Lemma 8 we know that \( r(z)x^n \in \bar{I} \) for any \( r \in \mathbb{C}[z] \) and \( n \geq m \). Using (5) we have the following:

\[
\nu_y(z^j x^m) = mp'(z)z^j x^{m-1} + jp(z)z^{j-1} x^{m-1} \in \bar{I}
\]

for all \( j \in \mathbb{N} \cup \{0\} \). On the other hand, since \( x^m \in \bar{I} \), from (2) it follows that \( x^{m-1} \nu_x \in \bar{I} \). Hence, by (4) and (5),

\[
x^{m-1} \nu_x(y^j) = p'(z)z^j x^{m-1} + jp(z)z^{j-1} x^{m-1} \in \bar{I}
\]

for all \( j \in \mathbb{N} \cup \{0\} \). By taking suitable linear combinations of the above expressions we see that \( x^{m-1} \nu_x \in \bar{I} \) and \( x^{m-1} \nu_x \in \bar{I} \), where \( (p'(z)) \) and \( (p(z)) \) denote the ideals in \( \mathbb{C}[z] \) generated by \( p'(z) \) and \( p(z) \) respectively. Since \( p \) has only simple roots, the ideal \( (p'(z), p(z)) \) generated by both \( p'(z) \) and \( p(z) \) is equal to \( \mathbb{C}[z] \) and thus \( x^{m-1} \nu_x \in \bar{I} \) and \( \mathbb{C}[z] \subset \bar{I} \). Therefore, \( r(z)x^n \in \bar{I} \) for any \( r \in \mathbb{C}[z] \) and \( n \geq m - 1 \). The claim follows.

\[ \]  

**Proof of Proposition 10.** Let \( I \) be a nontrivial ideal of \( \langle \text{LND}(D_p) \rangle \) and let \( \bar{I} \) be the corresponding ideal on the level of functions. By Lemma 9 we have that \( r(z)x^n \in \bar{I} \) for any \( r \in \mathbb{C}[z] \) and \( n \in \mathbb{N} \). Analogously, interchanging \( x \) and \( y \) in Lemmas 8 and 9, we get that \( r(z)y^n \in \bar{I} \) for any \( r(z) \in \mathbb{C}[z] \) and \( n \in \mathbb{N} \). Since \( r(z)x \in \bar{I} \) for any \( r(z) \), from (4) it follows that \( \nu_y(r(z)x) = (p(z)r(z))' \in \bar{I} \) for any \( r(z) \). Thus, \( \bar{I} \) contains all functions that correspond to vector fields in \( \langle \text{LND}(D_p) \rangle \) or, equivalently, \( I = \langle \text{LND}(D_p) \rangle \), which concludes the proof.
8. Proof of Proposition 5.

Let \( p, q \) be two polynomials with simple zeroes and let \( \deg p \geq 2 \) and \( \deg q \geq 3 \) unless stated otherwise. Furthermore, let \( \deg p \leq \deg q \). Let us denote by \( D_p = \{ xy = p(z) \} \) and by \( D_q = \{ uv = q(w) \} \). On \( D_q \) we introduce similar vector fields as on \( D_p \):

\[
\nu_u = q'(w) \frac{\partial}{\partial v} + u \frac{\partial}{\partial w}, \quad \nu_v = q'(w) \frac{\partial}{\partial u} + v \frac{\partial}{\partial w}, \quad \nu_w = u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v}.
\]

We start with a simple Lemma:

**Lemma 10.** Let \( F \) be homomorphism from either \( \langle \text{LND}(D_p) \rangle \) or \( \text{Vec}^0(D_p) \) to \( \text{Vec}^0(D_p) \) and let \( \nu \in \text{Vec}^0(D_p) \). The kernel of \( \nu \) is mapped by \( G \) onto the kernel of \( F(\nu) \). In particular, if \( \nu \) is locally nilpotent, then \( F(\nu) \) is locally nilpotent.

**Proof.** Let \( G \) be the induced homomorphism on the level of functions. Let \( f \in \ker \nu \) then by (3)

\[
F(\nu)(G(f)) = \{ f_{F(\nu)}, G(f) \} = \{ G(f_\nu), G(f) \} = G(\{ f_\nu, f \}) = G(\nu(f)) = G(0) = 0.
\]

If \( \nu \) is locally nilpotent then for every \( f \in \mathcal{O}(D_p) \) we have \( \nu^i(f) = 0 \) for some \( i \geq 1 \). Thus, by a similar calculation as above using (3) we get \( F(\nu)^i(G(f)) = 0 \). Therefore, for every \( f \in \mathcal{O}(D_q) \) we have \( F(\nu)^i(f) = 0 \) for some \( i \geq 0 \) meaning that \( F(\nu) \) is locally nilpotent. \( \square \)

For the rest of the paper we assume that

\[
F : \langle \text{LND}(D_p) \rangle \xrightarrow{\sim} \langle \text{LND}(D_q) \rangle
\]

is an isomorphism of Lie algebras and \( G \) is the induced isomorphism on the level of functions using the correspondence (1). The following three lemmas will be needed in the future. Recall that here \( \deg(f) = (l, k) \) denotes the pair of min- and max-degree in \( u \), where \( f \in \mathcal{O}(D_q) \). The following Lemma is clear by direct calculations.

**Lemma 11.** Let \( f, g \in \mathcal{O}(D_q) \) with \( \deg(f) = (\alpha, \alpha^+) \) and \( \deg(g) = (\beta, \beta^+) \). Then \( \deg(fg) = (\alpha + \beta, \alpha^+ + \beta^+) \). Therefore, if \( f \) is divisible by \( g \) we get \( \deg(f/g) = (\alpha - \beta, \alpha^+ - \beta^+) \). Let \( p(t) \) be a polynomial of degree \( n \). Then \( \deg(p(f)) = (na^-, na^+) \).

**Lemma 12.** Let \( n = \deg(q) \geq 3 \), and let \( \varphi = (\varphi_u, \varphi_v, \varphi_w) \in \mathcal{U}(D_q) = U_u \ast U_v \). Let us denote \( \deg(\varphi_u) = (\alpha, \alpha^+) \) and \( \deg(\varphi_v) = (\beta, \beta^+) \). If \( \alpha \geq 1 \) then \( \varphi_u = u \) and if \( \beta \geq -1 \) then \( \varphi_v = v \) or \( \varphi_v = u^{-1}q(w + va_i(u)) \) for some \( a \in \mathbb{C}[u] \).

**Proof.** Let \( \varphi = \varphi_N \circ \ldots \circ \varphi_1 \) be the reduced composition of automorphisms, where \( \varphi_i \in U_u \lor \varphi_i \in U_v \). This means that \( \varphi_i \) is given either by (where \( a_i \) is a nonzero polynomial)

\[
\varphi_i(u, v, w) = \begin{cases} u, & u^{-1}q(w + va_i(u)), \, w + va_i(u) \end{cases} \quad \text{or} \quad \varphi_i(u, v, w) = \begin{cases} v^{-1}q(w + va_i(v)), \, v, \, w + va_i(v) \end{cases}
\]

depending whether it is in \( U_u \) or in \( U_v \).

First we consider the case when \( \varphi_1 \in U_u \), meaning that \( \varphi_i \in U_u \) when \( i \) is odd and \( \varphi_i \in U_v \) when \( i \) is even. Let us take a look how the degree of the components evolves when composing with the next automorphism. Let

\[
(u_i, v_i, w_i) = \varphi_i \circ \ldots \circ \varphi_1(u, v, w)
\]
and
\[(\deg u_i, \deg v_i, \deg w_i) = ((\alpha_i^-, \alpha_i^+), (\beta_i^-, \beta_i^+), (\delta_i^-, \delta_i^+))\]

We will see that the complexity of the components (measured by the degree) is growing. Looking at the first two compositions we get:
\[
\begin{align*}
(u_0, v_0, w_0) &= (u, v, w) \\
(u_1, v_1, w_1) &= (u, u^{-1}q(w + ua_1(u)), w + ua_1(u)) \\
(u_2, v_2, w_2) &= (v_1^{-1}q(w_1 + v_1a_2(v_1)), v_1, w_1 + v_1a_2(v_1))
\end{align*}
\]

We will use Lemma 11 to calculate the degrees of the component. Lemma 11 will be used for all calculations in this proof. Let \(n_i = \deg a_i + 1\), then we get:
\[
\begin{align*}
\left((\alpha_0^-, \alpha_0^+), (\beta_0^-, \beta_0^+), (\delta_0^-, \delta_0^+)\right) &= (1, 1), (-1, -1), (0, 0) \\
\left((\alpha_1^-, \alpha_1^+), (\beta_1^-, \beta_1^+), (\delta_1^-, \delta_1^+)\right) &= (1, 1), (-1, -1, n_1 - 1), (0, n_1)
\end{align*}
\]

The computation of \(\left((\alpha_2^-, \alpha_2^+), (\beta_2^-, \beta_2^+), (\delta_2^-, \delta_2^+)\right)\) is more tedious: Since \(\deg w_1 = (0, n_1)\) and \(\deg(v_1a_2(v_1)) = (-n_2, n_2(n_1 - 1))\) we see that \(\deg(w_1 + v_1a_2(v_1)) = \deg(v_1a_2(v_1))\). Therefore, \(\deg(q(w_1 + v_1a_2(v_1))) = (-n_2, n_2(n_1 - 1))\). Together with \(\deg v_1 = (-1, n_1 - 1)\) this yields \((\alpha_2^-, \alpha_2^+) = (-n_2 + 1, (n_2 - 1)(n_1 - 1))\).

Similar considerations show
\[
\left((\alpha_2^-, \alpha_2^+), (\beta_2^-, \beta_2^+), (\delta_2^-, \delta_2^+)\right) = (-n_2 + 1, (n_2 - 1)(n_1 - 1), (-n_2, n_2(n_1 - 1)).
\]

These calculations show the claim for the case \(N = 1, 2, 3\) with \(\varphi_1 \in U_n\). For larger \(N\) it is enough to show that \(\alpha_i^- < -1\) and \(\beta_i^- < -1\) for all \(4 \leq i \leq N\). We show it by induction. We claim that \(\alpha_{2k}^- < \delta_{2k}^- \leq 0\), \(\alpha_{2k}^+ > \delta_{2k}^+ \geq 0\), \(\beta_{2k-1}^- < \delta_{2k-1}^- \leq 0\) and \(\beta_{2k-1}^+ > \delta_{2k-1}^+ \geq 0\) for \(k \geq 1\). For \(k = 1\) these inequalities follow from the above. For the step \((k - 1) \rightsquigarrow k\) we need the following computations:
\[
\begin{align*}
(u_{2k-1}, v_{2k-1}, w_{2k-1}) &= (u_{2k-2}, u_{2k-1}^{-1}q(w_{2k-2} + u_{2k-2}a_{2k-1}(w_{2k-2})), w_{2k-2} + u_{2k-2}a_{2k-1}(w_{2k-2})) \\
(u_{2k}, v_{2k}, w_{2k}) &= (v_{2k-1}^{-1}q(w_{2k-1} + v_{2k-1}a_{2k}(v_{2k-1})), v_{2k-1}, w_{2k-1} + v_{2k-1}a_{2k}(v_{2k-1})) \\
\end{align*}
\]

Let us, for example, make the calculation of \(\deg v_{2k-1} = (\beta_{2k-1}^-, \beta_{2k-1}^+)\). We have \(\deg w_{2k-2} = (\delta_{2k-2}^-, \delta_{2k-2}^+)\) and
\[
\deg(u_{2k-2}a_{2k-1}(w_{2k-2})) = (n_{2k-1}a_{2k-2}, n_{2k-1}a_{2k-2}^+). \]

Since \(\alpha_{2k-2}^- < \delta_{2k-2}^-\) and \(\alpha_{2k-2}^+ > \delta_{2k-2}^+\) by induction, we get
\[
\deg(w_{2k-2} + u_{2k-2}a_{2k-1}(w_{2k-2})) = (n_{2k-1}a_{2k-2}^-, n_{2k-1}a_{2k-2}^+). \]

Together with
\[
\deg u_{2k-2} = (\alpha_{2k-2}^-, \alpha_{2k-2}^+),\]

we conclude that
\[
(\beta_{2k-1}^-, \beta_{2k-1}^+) = (nn_{2k-1}a_{2k-2}^-, \alpha_{2k-2}^-, nn_{2k-1}a_{2k-2}^+ - \alpha_{2k-2}^+).\]
Similar calculations show:
\[
\left(\alpha_{2k-1}^{-}, \alpha_{2k-1}^{+}, \beta_{2k-1}^{-}, \beta_{2k-1}^{+}, \delta_{2k-1}^{-}, \delta_{2k-1}^{+}\right) = \\
\left(\alpha_{2k-2}^{-}, \alpha_{2k-2}^{+}, (nn_{2k-1}-1)\alpha_{2k-2}^{-}, (nn_{2k-1}-1)\alpha_{2k-2}^{+}, \right) \\
\left(n_{2k-1}\alpha_{2k-2}^{-}, n_{2k-1}\alpha_{2k-2}^{+}\right)
\]
\[
\left(\alpha_{2k}^{-}, \alpha_{2k}^{+}, \beta_{2k}^{-}, \beta_{2k}^{+}, \delta_{2k}^{-}, \delta_{2k}^{+}\right) = \\
\left(((nn_{2k}-1)\beta_{2k}^{-}, (nn_{2k}-1)\beta_{2k}^{+}, (\beta_{2k-1}^{-}, \beta_{2k-1}^{+}), \right) \\
\left(n_{2k}\beta_{2k-1}^{-}, n_{2k}\beta_{2k-1}^{+}\right)
\]

Since \(\alpha_{2k-2}^{-} < 0\) by induction, we see that
\[
\beta_{2k-1}^{-} = (nn_{2k-1}-1)\alpha_{2k-2}^{-} < n_{2k-1}\alpha_{2k-2}^{-}
\]
is negative because \(n_{2k-1}\alpha_{2k-2}^{-} = \delta_{2k-1}^{-} < 0\). The inequalities \(\alpha_{2k}^{-} < \delta_{2k}^{-} \leq 0\),
\(\alpha_{2k}^{+} > \delta_{2k}^{+} \geq 0\) and \(\beta_{2k-1}^{-} > \delta_{2k-1}^{-} \geq 0\) follow in a similar way. From the same
calculations we deduce that \(\alpha_{i}^{\pm} \leq \alpha_{i-1}^{-}\), and \(\beta_{i}^{\pm} \leq \beta_{i-1}^{-}\). Together with the fact
that \(\alpha_{i} < -1\) and \(\beta_{i} < -1\) this leads as desired to \(\alpha_{i} < -1\) and \(\beta_{i} < -1\) for all
\(4 \leq i \leq N\). Therefore, the claim of the Lemma follows in the case \(\varphi_{1} \in U_{v}\).

For the case \(\varphi_{1} \in U_{v}\), a similar calculation shows that \(\varphi_{u} = u\) whenever \(\alpha^{-} \geq -1\)
and that \(\varphi_{v} = v\) whenever \(\beta^{-} \geq -1\). \(\square\)

**Lemma 13.** Let \(\varphi : D_{p} \rightarrow D_{q}\) be an isomorphism and \(f \in O(D_{p})\). Then \(\text{Ad}(\varphi)\nu_{f} = \nu_{\tilde{f}}\), for some \(\tilde{f} \in O(D_{q})\). Moreover, \(f = \varphi^{*}\tilde{f} \) \(\text{k}(\varphi)\) where \(k(\varphi) \in \mathbb{C}\) is given by \(\varphi^{*}\omega_{q} = k(\varphi)\omega_{p}\).

**Proof.** In order to find the corresponding function \(f \in O(D_{p})\) of the vector field
\(\text{Ad}(\varphi)^{-1}\nu_{f}\) we need to find \(f\) such that \(df = i_{\text{Ad}(\varphi)^{-1}\nu_{f}}\omega_{p}\). The calculation
\[
d\left(\varphi^{*}\tilde{f} \right) = \varphi^{*} \left(\frac{df}{k(\varphi)}\right) = \varphi^{*} \left(i_{\nu_{f}}\omega_{q}/k(\varphi)\right) = \varphi^{*} \left(i_{\nu_{f}}(\omega_{q}/k(\varphi))\right)
\]
\[
= \varphi^{*}i_{\nu_{f}}((\varphi^{-1})^{*}\omega_{p}) = i_{(\text{Ad}(\varphi)^{-1}\nu_{f})}\omega_{p} = i_{\nu_{f}}\omega_{p}.
\]
shows that \(f = \varphi^{*}\tilde{f} \) \(\text{k}(\varphi)\) is the desired formula. \(\square\)

In the next two lemmas we will show that there is an automorphism \(\varphi : D_{q} \xrightarrow{\sim} D_{q}\)
such that up to composition with \(\text{Ad}(\phi)\), \(F\) has a certain form.

**Lemma 14.** Up to composition with some automorphism \(\text{Ad}(\phi)\) of \(\text{LND}(D_{q})\) we
have \(G(x) = f(u)\) for some polynomial \(f \in \mathbb{C}[u]\).

**Proof.** Since \(\nu_{x}\) is locally nilpotent, \(F(\nu_{x})\) is also locally nilpotent by Lemma 10. From Proposition 3, it follows that \(F(\nu_{x})\) is conjugate to \(r(u)\nu_{a}\) by \(\text{Ad}(\phi)\) for some automorphism \(\varphi : D_{q} \rightarrow D_{q}\), where \(r \in \mathbb{C}[u]\). Hence, by (2) we get \(G(x) = f(u)\). \(\square\)

Note that equality \(G(x) = f(u)\) implies that \(F(\nu_{x}) = f'(u)\nu_{a}\).

**Lemma 15.** Up to composition with an induced automorphism \(\text{Ad}(\phi)\) of \(\text{LND}(D_{q})\) we
have \(G(x) = u\) and \(G(y) = cv\) for some \(c \in \mathbb{C}^{*}\).
Proof. By the previous Lemma we can assume that $G(x) = f(u)$ for some polynomial $f \in \mathbb{C}[u]$. Let $m = \deg p + 1$. Thus, we have $\nu_p^m(y) = 0$, and hence $F(\nu_p)^m(G(y)) = f(u)^m \nu_p^m(G(y)) = 0$. From $\nu_p^m(G(y)) = 0$ and $\deg p \leq \deg q$ we conclude that $G(y) = \sum_{i=0}^{m-1} a_i(u)w^i + \lambda(u)v$ for some $a_i \in \mathbb{C}[u]$ and $\lambda \in \mathbb{C}[u]$. Hence, we have $\deg G(y) = (\alpha^-\alpha^+) + 1$, with $\alpha^- \geq -1$. Since $\nu_y$ is locally nilpotent, by Lemma 10 $F(\nu_y)$ is also locally nilpotent. Let us choose $\varphi \in U(D_q)$ such that $(\text{Ad}(\varphi))^{-1}F(\nu_y)$ is either in $U_u$ or in $U_v$ (see [KL13, Theorem 2.15]). This means by Lemma 13 that $(\text{Ad}(\varphi))^{-1}G(y)$ is either a polynomial in $u$ or in $v$. Therefore, $G(y) = g(\varphi_u)$ or $G(y) = g(\varphi_v)$ for $\varphi = (\varphi_u, \varphi_v, \varphi_w)$ and some polynomial $g$. Since $\alpha^- \geq -1$ and since $g$ is a polynomial either the min-degree in $u$ of $\varphi_u$ or $\varphi_v$ is greater or equal than $-1$, which is exactly the assumption of Lemma 12. Thus, either $G(y) = g(u)$, $G(y) = g(v)$ or $G(y) = g(u^{-1}q(w + ua(u)))$ for some $a \in \mathbb{C}[u]$. The latter case can be excluded since $[\nu_z, \nu_y] \neq 0$ implies that $[f'(u)\nu_u, F(\nu_y)] \neq 0$. In the latter two cases we directly see that $\deg g = 1$, because the min-degree in $u$ of both $v$ and $u^{-1}q(w + ua(u))$ is equal to $-1$. Since the correspondence (2) is only up to constants we have $g(t) = ct$ for some $c_2 \in \mathbb{C}$. In the second case we have $G(y) = c_2v$. In the third case we define the automorphism

$$\varphi : (u, v, w) \mapsto (u, u^{-1}q(w + ua(u)), w + ua(u))$$

and after composition with $(\text{Ad}(\varphi))^{-1}$ we get $G(x) = f(u)$ and $G(y) = c_2v$.

Since $\nu_y^p(x) = 0$ we have that $\nu_y^m(f(u)) = 0$. Thus, $\deg f = 1$ and we have $f(u) = c_1u$ for some $c_1 \in \mathbb{C}$. After composing $G$ with the induced map of

$$\psi(u, v, w) = (c_1u, c_1^{-1}v, w)$$

the claim follows.

Since now on we assume $F$ and respectively $G$ to be as in Lemma 15.

Lemma 16. We have $G(p'(z)) = c q'(w)$. Moreover, there is a basis $\{h_i(w) \mid i \geq 0\}$ of $\mathbb{C}[w]$ such that $G((p(z)w)^i) = c(q(w)h_i(w))^i$.

Proof. The first statement follows from a direct calculation using (2),(3) and Lemma 15:

$$G(p'(z)) = G(f_{[\nu_z, \nu_y]} = \{f_{\nu_z}, f_{\nu_y}\} = F(\nu_z)(G(y)) = \nu_u(cv) = c q'(w).$$

The second statement holds since the kernel of $p''(z)\nu_z$ is mapped by $G$ onto the kernel of $F(p''(z)\nu_z) = cp''(w)\nu_w$ by Lemma 10.

Lemma 17. We have that $G(xz^i) = u(aw + b)^i, G(yz^i) = cv(aw + b)^i$ and $G((p(z)w)^i) = c(q(w)(aw + b)^i)'$ for some $a, c \in \mathbb{C}^*, b \in \mathbb{C}$.

Proof. We have $\nu_y(xz^i) = (p(z)w)^i, and thus $cv_{\nu_y}(G(xz^i)) = c(q(w)h_i(w))^i$ by Lemma 15 and Lemma 16. This can happen only if $G(xz^i) = u h_i(w) + A(v)$ for some $A \in \mathbb{C}[v]$. Similarly, we have $\nu_y(yz^i) = (p(z)w)^i, and thus $\nu_u(G(yz^i)) = c(q(w)h_i(w))^i$, which only happens if $G(yz^i) = cv h_j(w) + B(u)$ for some $B \in \mathbb{C}[u]$.

From (4) we get

$$\{xz^i, yz^j\} = -(p(z)w)^i + j$$

Thus by the above Lemma we have

$$\{uh_i(w) + A(v), cv h_j(w) + B(u)\} = -c(q(w)h_{i, j}(w))^i.$$
On the other hand, by (4) we have
\[ \{uh_i(w) + A(v), cvh_j(w) + B(u)\} = -c(q(w)h_i(w)h_j(w))' + q(w)A'(v)B'(u) + u^2h_i'(w)B'(u) + cv^2h_j'(w)A'(v). \]
These equations can only hold if \( A(v) = B(u) = 0 \) and if \( h_i(w)h_j(w) = h_{i+j} \) for all \( i, j \geq 1 \). From the latter we can conclude that \( h_i(w) = (h_1(w))^i \) for some \( h_1(w) \). Because \( \{h_i(w)\}_{i \geq 0} \) form a basis of \( \mathbb{C}[w] \) we know that \( h_1(w) = aw + b \) is of degree 1. Thus the claim follows. \( \square \)

**Lemma 18.** We have \( G(x^i) = a^{i-1}u^i \) and \( G(y^i) = a^{i-1}(cv)^i \).

**Proof.** We know that functions in \( x \) are mapped by \( G \) onto functions in \( u \) since the \( \ker \nu_x \) is mapped onto \( \ker \nu_u \) by Lemma 10. Let \( f_i(u) = G(x^i) \). From \( ixi^{i+1} = \{x^i, x \} \) we get
\[ if_i+1(u) = \{G(x^i), G(x)\} = f'_i(u)\nu_u(G(xz)) = f'_i(u)\nu_u(u(aw + b)) = af'_i(u)u^2 \]
using (2) and Lemma 17. Similarly we get for \( g_i(v) = G(y^i) \)
\[ ig_{i+1}(v) = g'_i(v)\nu_v(G(yz)) = g'_i(v)\nu_v(cv(aw + b)) = acg'_i(v)v^2. \]
Since \( f_1(u) = u \) and \( g_1(v) = cv \) by Lemma 15 the claim follows. \( \square \)

**Lemma 19.** We have \( cav^2q(w) = p(aw + b) \).

**Proof.** We have \( xix^2q(x^2) = (p^2(x))' \) which yields \( aup\nu_u(ac^2v^2) = c(q(w)p(aw + b))' \)
by the Lemma 17 and Lemma 18 . On the other hand, we have \( aup\nu_u(ac^2v^2) = a^2c^2(q^2(w))' \). Hence, the claim follows. \( \square \)

**Lemma 20.** Let \( F_1, F_2 : \langle \text{LND}(D_p) \rangle \overset{\sim}{\rightarrow} \langle \text{LND}(D_q) \rangle \) be two isomorphisms of Lie algebras such that \( F_1(x^i\nu_x) = F_2(x^i\nu_x) \) and \( F_1(y^i\nu_y) = F_2(y^i\nu_y) \) for all \( i \geq 0 \). Then \( F_1 = F_2 \).

**Proof.** The claim follows directly from the fact that the vector fields \( x^i\nu_x \) and \( y^i\nu_y \)
generate \( \langle \text{LND}(D_p) \rangle \) (see [KL13]). \( \square \)

**Lemma 21.** Let \( F_1, F_2 : \text{Vec}^0D_p \overset{\sim}{\rightarrow} \text{Vec}^0D_q \) be two isomorphisms of Lie algebras such that restrictions of \( F_1 \) and \( F_2 \) to \( \langle \text{LND}(D_p) \rangle \) coincide. Then \( F_1 = F_2 \).

**Proof.** Let \( G_1 \) and \( G_2 \) be the isomorphism on the level of functions which correspond to \( F_1 \) and \( F_2 \) respectively. Let also \( g \in \mathbb{C}[z] \) be a polynomial. From Proposition 9 and (4) it follows that \( \{f, g(z)\} = g'(z)\{f, z\} \) has form (1) for any \( f \in \mathcal{O}(D_p) \) such that \( \nu_f \in \langle \text{LND}(D_p) \rangle \). Since restrictions of \( F_1 \) and \( F_2 \) to \( \langle \text{LND}(D_p) \rangle \) coincide, \( G_1(f) = G_2(f) \) and \( G_1(\{f, g(z)\}) = G_2(\{f, g(z)\}) \) for any \( f \) such that \( \nu_f \in \langle \text{LND}(D_p) \rangle \). Hence, \( \{G_1(f), G_1(g(z)) - G_2(g(z))\} = 0 \) for any \( h \in \mathcal{O}(D_q) \) such that \( \nu_h \in \langle \text{LND}(D_q) \rangle \). Using (4) it is easy to see that the equality \( \{u, G_1(g(z)) - G_2(g(z))\} = \nu_u(G_1(g(z)) - G_2(g(z)) = 0 \) implies that \( G_1(g(z)) - G_2(g(z)) \in \mathbb{C}[u] \). Analogously, because \( \{v, G_1(g(z)) - G_2(g(z))\} = 0 \) we have \( G_1(g(z)) - G_2(g(z)) \in \mathbb{C}[v] \). Therefore, \( G_1(g(z)) - G_2(g(z)) \in \mathbb{C} \). The claim follows from Proposition 9. \( \square \)

**Proof of Proposition 5.** (a): By Lemma 18, we know that up to composition with an automorphism \( \text{Ad}(\varphi) \) of \( \langle \text{LND}(D_q) \rangle \), where \( \varphi : D_q \rightarrow D_q \) is some automorphism, the following holds: there are \( a, c \in \mathbb{C}^* \) and \( b \in \mathbb{C} \) such that \( G(x^i) = a^{i-1}u^i \) and \( G(y^i) = a^{i-1}(cv)^i \) for all \( i \). Let us define an isomorphism \( \psi : \mathbb{A}^3 \overset{\sim}{\rightarrow} \mathbb{A}^3 \) in the
following way: $\psi : (x, y, z) \mapsto (\frac{1}{a}u, \frac{1}{c}v, \frac{aw+b}{c}).$ By Lemma 19, $ca^2q(w) = p(aw+b).$

Thus, $\psi$ restricts to an isomorphism $D_p \xrightarrow{\sim} D_q$ and then $\text{Ad}(\psi) : \langle \text{LND}(D_p) \rangle \xrightarrow{\sim} \langle \text{LND}(D_q) \rangle$ is the isomorphism of the Lie algebras. Let $G_\psi$ be the corresponding map on the level of functions. By Lemma 13, we have $G_\psi(x^i) = (a\psi)^i/k(\psi)$ and $G_\psi(y^i) = (ac\psi)^i/k(\psi).$ Since $\psi^i\omega_p = a\omega_p$ and thus $k(\psi) = a,$ we have $G(x^i) = G_\psi(x^i)$ and $G(y^i) = G_\psi(y^i)$ for all $i \geq 1.$ Therefore, the conditions of Lemma 20 are satisfied, thus $F = \text{Ad}(\psi).$

The proof of (c) follows directly from (a) when either $\deg p$ or $\deg q$ is greater or equal than 3. Now assume that $\deg p = \deg q = 2.$ Since any $D_h$ is isomorphic to $D(z-1)$ (see [DP09]), where $\deg h = 2,$ (c) follows.

(b) Let $F : \text{Vec}^0 D_p \xrightarrow{\sim} \text{Vec}^0 D_q$ be an isomorphism of Lie algebras and let $\deg q \geq 3.$ The map $F$ restricts to an isomorphism $\bar{F} : \langle \text{LND}(D_p) \rangle \xrightarrow{\sim} \langle \text{LND}(D_q) \rangle,$ because $F$ and $F^{-1}$ send locally nilpotent vector fields to locally nilpotent vector fields by Lemma 10. From (a) it follows that there is an isomorphism $\varphi : D_p \xrightarrow{\sim} D_q$ such that $\bar{F} = \text{Ad}(\varphi).$ Hence, from Lemma 21 it follows that $F = \text{Ad}(\varphi)$ on $\text{Vec}^0(D_p).$

The next statement is used in the proof of Theorem 2.

**Proposition 11.** Let $\phi : \text{Aut}^0(D_p) \xrightarrow{\sim} \text{Aut}^0(D_q)$ be an automorphism of an ind-group such that restriction of $\phi$ to $\text{U}(D_p)$ is the identity map. Then $\phi$ is the identity.

**Proof.** Since the restriction of $\phi$ to $\text{U}(D_p)$ is the identity map, $\phi(T)$ acts on $\text{U}(D_p)$ by conjugations in the same way as $T$ acts on $\text{U}(D_p).$ Hence, $\text{Ad}(t)$ and $\text{Ad}(\phi(t))$ act on $\langle \text{LND}(D_p) \rangle$ in the same way, where $t \in T.$ This is equivalent to the statement that $\text{Ad}\left((\psi^{-1} \circ \phi(t))\right)$ acts identically on $\langle \text{LND}(D_p) \rangle.$ We claim that then $t^{-1} \circ \phi(\psi)$ is a trivial automorphism of $D_p.$ Indeed, because $\text{Ad}(\psi^{-1} \circ \phi(\psi))$ acts identically on $\langle \text{LND}(D_p) \rangle$ it follows that $(t^{-1} \circ \phi(t))^* (x) = x + c$ for some $c \in \mathbb{C}.$ Hence, $(\psi^{-1} \circ \phi(\psi))^* (x^2) = (x + c)^2 = x^2 + 2cx.$ Because $\text{Ad}(t^{-1} \circ \phi(t)) (xv_x) \psi(\psi^{-1} \circ \phi(\psi)(xv_x))$ has to be equal to $xv_x$ it follows that $c = 0.$ Hence, $(t^{-1} \circ \phi(t))^* (x) = x.$ Analogously, $(t^{-1} \circ \phi(t))^* (y) = y$ and $(t^{-1} \circ \phi(t))^* (z) = z.$ Therefore, $\phi(t) = t$ for any $t \in T.$ The claim follows from Proposition 3. \hfill $\square$

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