FORMALLY RENORMALIZABLE
GRAVITATIONALLY
SELF INTERACTING STRING MODELS

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Abstract

It has recently been shown how the effect of the divergent part of the
gravitational self interaction for a classical string model in 4 dimensions can
be allowed for by a renormalisation of its stress energy tensor and in the
elastic case a corresponding renormalisation of the off shell action. It is
shown here that that it is possible to construct a new category of elastic
string models for which this effect is describable as a renormalisation in the
stricter “formal” sense, meaning that it only requires a rescaling of one of
the fixed parameters characterising the model.

1 Introduction

The gravitational self interaction of for a classical string model in a four dimen-
sional background has recently been shown [1] to have a divergent part that, when
suitably regularised, can be effectively be absorbed by a renormalisation of the
surface stress energy tensor of the worldsheet. In the particularly simple case of
a Nambu-Goto model the renormalisation is trivial in the sense that (contrary to
what had been suggested by earlier work) the divergent part actually vanishes [2].
For more general elastic string models, the renormalisation of the “on-shell” stress
energy tensor has been shown [3] to correspond to a non-trivial renormalisation of
the relevant variational action, in which the appropriate adjustment has precisely
the form that is obtained as the four dimensional specialisation of a very general
formula [4] that recently been obtained from a very different approach based on
an analysis of the “off shell” action in a background of arbitrary dimension.

The physical example that provided the original motivation for this line of
investigation was the case of cosmic strings described by the “transonic” string
model [5] that provides an effective large scale description of the effect of short
wavelength “wiggles” in an underlying Nambu-Goto model. In this case the string model is qualitatively modified by the effect of the renormalisation, in the sense that the resulting “dressed” model no longer has the special transonic character (and the ensuing integrability properties) of the original “bare” model. The purpose of the present article is to reply to the question of whether there is a category of gravitationally self interacting elastic string models that is renormalisable in the stricter “formal” sense, meaning that the algebraic form of the “bare” model is qualitatively preserved in the corresponding “dressed” model in the sense that the renormalisation is describable just as a readjustment of the free parameters specifying the particular model within the category.

2 “Bare” string models.

As explained in more detail in the preceding references [3, 4], we are concerned with models governed by an action $I$ that is specified as an integral over the string worldsheet that is specifiable as the 2-dimensional two dimensional imbedding given by $x^\mu = \bar{x}^\mu\{\sigma\}$ in terms of intrinsic coordinates $\sigma^i$ ($i = 0, 1$), so that the induced surface metric will have the form $\gamma_{ij} = g_{\mu\nu}\bar{x}_{,i}^\mu\bar{x}_{,j}^\nu$, where $g_{\mu\nu}$ is the spacetime metric of the 4-dimensional background with local coordinates $x^\mu$. In terms of such a worldsheet, the action integral will have the form,

$$I = \int \sqrt{\gamma} \|1/2 \, d^2 \sigma, \quad (1)$$

where $|\gamma|$ is the determinant of the induced metric, and $\Lambda$ is the relevant Lagrangian scalar. In the absence of gravitational interaction – other than what is automatically allowed for by large scale the curvature of the background metric $g_{\mu\nu}$ – the Lagrangian scalar for a model of the simple elastic kind under consideration here would be given by a master function $\Lambda$ depending just on the gradient of a freely variable scalar stream function $\psi$ on the worldsheet. However to allow for the effect of shortwave gravitational perturbations $g_{\mu\nu} \leftrightarrow g_{\mu\nu} + \delta g_{\mu\nu}$ in terms of a linearised gravitational field tensor $h_{\mu\nu} = \delta g_{\mu\nu}$ it is evidently necessary to augment the Lagrangian by a corresponding gravitational coupling term so that it takes the form

$$\mathcal{L} = \Lambda + \frac{1}{2} T^{\mu\nu} h_{\mu\nu}, \quad (2)$$

where $T^{\mu\nu}$ is the relevant surface stress energy tensor, which is given by $T^{\mu\nu} = 2\|\gamma\|^{-1/2} \partial(\Lambda\|\gamma\|^{1/2})/\partial g_{\mu\nu}$. The corresponding dynamical equations are given by the requirement that the action integral should be invariant with respect to local variations of the worldsheet imbedding and of the internal field $\psi$. 

2
3 “Dressed” string models

If the field $h_{\mu\nu}$ were due just to passing gravitational waves from an external source, the model that has just been described would automatically be well defined and well behaved as it stands. Allowance for self gravitation gives rise to difficulties of two kinds. The hardest part is the evaluation of the finite long range contribution including backreaction from emitted radiation, which, except for very simple configurations will be tractable in practice only in a rather approximate manner. However although it may be important in the long run, this finite contribution will usually have a negligible effect on the short timescale dynamics, for which it is the divergent short range part of the self interaction that will dominate. The subject of the present discussion is this latter part, whose treatment requires the introduction of a regularisation, whereby it will be obtained \[1\] in the form

$$\hat{h}_{\mu\nu} = 2G \hat{I} (2\mathcal{T}_{\mu\nu} - \mathcal{T}_{\sigma\rho} g_{\mu\nu}),$$  \hspace{1cm} (3)$$

where, as usual for a string self interaction in 4 dimensions, the proportionality factor has the form $\hat{I} = \ln \{\Delta^2/\delta_2^*\}$ in terms of an “ultraviolet” cut off lengthscale $\delta_*$ representing the effective thickness of the string, and a much larger “infrared” cut off $\Delta$, given by a lengthscale characterising the large scale geometry of the string configuration.

The preceding work \[1\] provides a system that will be describable in terms of the finite part $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \hat{h}_{\mu\nu}$ that is left over when the divergent part is subtracted out, by a renormalised Lagrangian of the form

$$\overline{\mathcal{L}} = \tilde{\Lambda} + \frac{1}{2} \mathcal{T}_{\mu\nu} \tilde{h}_{\mu\nu},$$  \hspace{1cm} (4)$$

in which the original “bare” master function has been replaced by a renormalised “dressed” master function expressible \[3, 4\] by

$$\tilde{\Lambda} = \Lambda + \hat{\Lambda}_g,$$  \hspace{1cm} (5)$$

in which the divergent part of the self interaction has been absorbed in an adjustment term of the form

$$\hat{\Lambda}_g = \frac{1}{4} \mathcal{T}_{\mu\nu} \hat{h}_{\mu\nu}.$$  \hspace{1cm} (6)$$

For any simple elastic string model of the kind considered here, the master function $\Lambda$ will depend just on the scalar magnitude that is specifiable \[4\] as $\chi = -\gamma^{ij} \psi_i \psi_j = -p_\mu p^\mu$ where the relevant momentum vector is defined by $p^\mu = \bar{x}^{\mu,j} \psi_j$. It can be seen that the surface stress energy tensor of the “bare” model will be expressible in the form

$$\mathcal{T}_{\mu\nu} = \Lambda \gamma^{\mu\nu} + 2 \frac{d\Lambda}{d\chi} p^\mu p^\nu,$$  \hspace{1cm} (7)$$
using the notation $\gamma^\mu_\nu = \gamma^{ij} \bar{x}^i_{\mu} \bar{x}^j_\nu$ for the fundamental tensor of the worldsheet, i.e. the background spacetime projection of its internal metric. It can thus be seen from (3) that the self gravitational action contribution (6) will be expressible simply as
\[ \tilde{\Lambda}_g = 2G\tilde{l}\left(\frac{d\Lambda}{d\chi}\right)^2. \] (8)

4 Physical admissibility conditions.

The preceding application [3] of the formula (8) was to the a smoothed average description of the effect of shortwavelength wiggles on an underlying Nambu-Goto string as described by the transonic model for which the relevant Lagrangian master function will be specified by a constant mass parameter as a function of the special form $\Lambda = -m\sqrt{m^2 - \chi}$, which (unfortunately as far as its convenient integrability properties are concerned) will not be preserved by the renormalisation. The purpose of the present article is to show however that there does exist a simple but non-trivial category of models whose algebraic character is preserved by the adjustment given by (8), and that are thus “formally” renormalisable in the sense that all that is required is a rescaling of the constant parameters characterising the model.

The simplest example of a sub category that is “formally” renormalisable in this sense is of course the one constituted by the Nambu-Goto models for which the master function is just a constant, $\Lambda = -m^2$ where $m$ has the dimensions of mass. There has never been any doubt about the “formal” renormalisability of this subcategory, since it was always supposed that the appropriately renormalised model would be given by another constant, $\tilde{\Lambda} = -\tilde{m}^2$. However according to the formula that was commonly quoted in textbooks [7] for many years the “dressed” value was given in terms of the logarithmic regularisation factor $\tilde{l}$ specified above by $m^2 = m^2(1 - 4G\tilde{m}^2\tilde{l})$ whereas a more careful calculation [2] has recently shown that the correct value, as obtainable directly from (8), will simply be $\tilde{m}^2 = m^2$. In other words the renormalisation in the Nambu-Goto case is trivial in the sense that it has no effect at all.

Having made the observation that the Nambu-Goto category is – from this point of view – trivial, one is left with the question of the existence of a category that would be “formally” renormalisable in a non-trivial manner. As will be shown explicitly below, it is very easy to construct a master function that satisfies the “formal” renormalisability condition, but what is not quite so easy is to ensure that the resulting model also satisfies the further requirements needed for physical admissibility. In order for the energy density and tension to be positive it is necessary that the (on shell) value of the master function $\Lambda$ and of its dual [8] (in the Hodge sense with respect to the 2-dimensional geometry of the string world
sheet) as given \cite{1} by

\[ \Lambda = \Lambda - 2\chi \frac{d\Lambda}{d\chi}, \]  

(9)

should both be negative,

\[ \Lambda < 0 \quad \Lambda < 0, \]  

(10)

and in order to avoid microscopic instability one hand, and causality violation on the other hand, both the extrinsic ("wiggle" type) perturbation propagation speed \( c_E \) and the longitudinal (sound type) perturbation propagation speed \( c_L \) must be real and less than unity (assuming units such that the speed of light is itself unity) the range of physical validity of an elastic string model is restricted by the conditions \cite{2}

\[ 0 < c_E^2 \leq 1, \quad 0 < c_L^2 \leq 1, \]  

(11)

in which the quantities \( c_E^2 \) and \( c_L^2 \) are given by the formulae

\[ c_{E}^{\pm 2} = \frac{\Lambda}{\Lambda}, \quad c_{L}^{\pm 2} = \frac{-d*\Lambda}{d\Lambda}, \]  

(12)

where the sign \( \pm \) is defined to be positive, \( \pm = + \), wherever the current is timelike, i.e. \( \chi < 0 \), and negative, \( \pm = - \), wherever the current is spacelike. i.e. \( \chi > 0 \), while in the null limit \( \chi = 0 \), the requirement reduces to \( *\Lambda = \Lambda \) and \( d*\Lambda/d\Lambda = -1 \).

It will be convenient to simplify the foregoing formulae by a change of variable whereby \( \chi \) is replaced by a new variable \( \Xi = \ln \{\chi/\chi_0\} \) for some fixed value \( \chi_0 \) and to use a dot to denote differentiation with respect to \( \Xi \) so that in particular one has

\[ \dot{\Lambda} = d\Lambda/d\Xi = \chi d\Lambda/d\chi. \]  

(13)

This enables us to express the dual master function in the form

\[ *\Lambda = \Lambda - 2\dot{\Lambda}, \]  

(14)

so that one obtains

\[ *\Lambda/\Lambda = 1 - 2\dot{\Lambda}/\Lambda, \]  

(15)

\[ d*\Lambda/d\Lambda = 1 - 2\ddot{\Lambda}/\Lambda. \]  

(16)

The formula for the “dressed” master function will be similarly expressible in the form

\[ \ddot{\Lambda} = \Lambda + 2G \tilde{F} \dot{\Lambda}^2. \]  

(17)

This notation can also be used to express the preceeding conditions \cite{3} for good physical behaviour (causality and local stability) as

\[ \Lambda < 2\dot{\Lambda} \leq 2\ddot{\Lambda} < \dot{\Lambda} \leq 0, \]  

(18)

in the timelike current regime, \( \chi > 0, \) and as

\[ \Lambda < 0 \leq 2\dot{\Lambda} \leq 2\ddot{\Lambda}, \]  

(19)

in the spacelike current regime, \( \chi > 0. \)
5 “Formally” renormalisable models

It is evident from (17) that, as has already been remarked, the action renormalisation \( \Lambda \mapsto \tilde{\Lambda} \) will have no effect at all on a master function that is constant (i.e. of Nambu-Goto type). Clearly the simplest category of functions that will be non trivially preserved by such a transformation consists of those that are linear in \( \Xi \), i.e. those of the form

\[
\Lambda = -m^2 + A\Xi, \tag{20}
\]

where \( A \) like \( m^2 \) is a fixed parameter. For a master function of this type one obtains \( \dot{\Lambda} = A \) and \( \ddot{\Lambda} = 0 \), so the effect of the action renormalisation will be expressible as a simple parameter renormalisation \( m^2 \mapsto \tilde{m}^2 \), \( A \mapsto \tilde{A} \), of which the latter part is trivial, \( \tilde{A} = A \) while the only non trivial part will be a mass renormalisation given by the formula \( \tilde{m}^2 = m^2 - 2G\tilde{\Lambda} A^2 \). It can be seen however that although it thus satisfies the requirement of being renormalisable in the strict “formal” sense, this linear master function does not provide a physically admissible string model, since (unless \( \ddot{\Lambda} \) also vanishes) the restriction \( \ddot{\Lambda} = 0 \) is compatible neither with (18) nor (19): as can be seen directly from (16) it is simply inconsistent with the requirement that the “sound” (longitudinal perturbation) speed should be real.

Although the simplest mathematical possibility is thus excluded on physical grounds, we can obtain a “formally renormalisable” category of physically admissible models by going on from linear to quadratic order in \( \Xi \). There is no loss of generality in writing a quadratic function of \( \Xi \) in the form

\[
\Lambda = -m^2 + C\Xi^2, \tag{21}
\]

where \( C \) like \( m^2 \) is a fixed parameter. (The reason why, provided the coefficient \( C \) is non-zero, no generality can be gained by adding in an extra linear term \( A\Xi \), is that such a term could be absorbed into the homogeneously quadratic part by a rescaling of the constant parameter \( \chi_0 \) that was used to fix the calibration of \( \Xi \).)

For a master function of this quadratic type one obtains \( \dot{\Lambda} = 2C\Xi \) and \( \ddot{\Lambda} = 2C \). The effect of the action renormalisation will therefore be expressible as a simple parameter renormalisation \( m^2 \mapsto \tilde{m}^2 \), \( C \mapsto \tilde{C} \), of which in this case it is the first part that is trivial, \( \tilde{m}^2 = m^2 \), while the second part will have the non-trivial form

\[
\tilde{C} = C + 8G\tilde{\Lambda}C^2. \tag{22}
\]

In order for a master function of the form (21) to provide a string model satisfying the physical admissibility conditions recapitulated above, it can be seen to be necessary and sufficient that the parameter \( m \) should be non zero, and that the other parameter \( C \) should satisfies the condition

\[
3C > -m^2. \tag{23}
\]
When $C$ is negative, $\chi$ will also have to be negative, i.e. the current is restricted to be timelike, and the permissible range for $\Xi$ will be given by

$$-\frac{m^2}{3} < C < -\frac{m^2}{4} \quad \Rightarrow \quad 1 \leq \Xi < 2 \left(1 - \sqrt{1 - \frac{m^2}{4|C|}}\right), \quad (24)$$

(the lower limit being where $c_L \to 1$ and the upper limit where $c_E \to 0$) and

$$-\frac{m^2}{4} \leq C < 0 \quad \Rightarrow \quad 1 \leq \Xi < 2, \quad (25)$$

(the lower limit being again where $c_L \to 1$ and the upper limit where $c_E \to 0$).

When $C$ is positive, $\chi$ will also have to be positive, i.e. the current is restricted to be spacelike, and the permissible range for $\Xi$ will be given by

$$0 < C \leq m^2 \quad \Rightarrow \quad 0 \leq \Xi \leq 1, \quad (26)$$

(the lower limit being where $c_E \to 1$ and the upper limit where $c_L \to 1$) and

$$C > m^2 \quad \Rightarrow \quad 0 \leq \Xi < \frac{m}{\sqrt{C}}, \quad (27)$$

(the lower limit being again where $c_E \to 1$ and the upper limit where $c_E \to 0$).

The quadratic formula (21) (for $C > -m^2/3$) provides not only the lowest order function of $\Xi$ that gives a “formally” renormable string model that is physically admissible, but also the highest order function with this property: for example if we were to include a cubic order term in the “bare” master function, the corresponding “dressed” master function would be of not cubic but quartic order.

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