The Rohlin Property for $\mathbb{Z}^2$-Actions on UHF Algebras

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Abstract

We define a Rohlin property for $\mathbb{Z}^2$-actions on UHF algebras and show a non-commutative Rohlin type theorem. Among those actions with the Rohlin property, we classify product type actions up to outer conjugacy. We consider two classes of UHF algebras. For UHF algebras in one class including the CAR algebra, there is one and only one outer conjugacy class of product type actions and for UHF algebras in the other class, contrary to the case of $\mathbb{Z}$-actions, there are infinitely many outer conjugacy classes of product type actions.

1 Introduction

A non-commutative Rohlin property was introduced by A. Connes for classification of (single) automorphisms of von Neumann algebras ([3, 4]), and this property was generalized for example by A. Ocneanu ([20, 21]) to systems of commuting automorphisms and further to actions of discrete amenable groups. On the other hand, this notion has also proved useful in the framework of $C^*$-algebras ([1, 6, 7, 12, 13, 15, 16]), and in [15, 16] Kishimoto established a non-commutative Rohlin type theorem for automorphisms of UHF algebras (and some AF algebras) and classified automorphisms (i.e., $\mathbb{Z}$-actions) with the Rohlin property up to outer conjugacy.

The purpose of the paper is to extend Kishimoto’s work to $\mathbb{Z}^2$-actions. Motivated by [15, 16], in Section 2 we introduce notions of Rohlin property and uniform outerness to $\mathbb{Z}^N$-actions on unital $C^*$-algebras. In the UHF algebra case and $N = 1$, the uniform outerness was shown to be the same as the ordinary outerness of the relevant automorphism on the GNS von Neumann algebra obtained via the trace ([13]). Our main theorem here says that for $\mathbb{Z}^2$-actions on UHF algebras the Rohlin property characterizes the uniform outerness. The
main idea of the proof is similar to the one in [16], but to avoid additional technical problems we make use of the stability i.e., the vanishing of 1-cohomology obtained in [12].

In Section 3 we introduce three notions of conjugacy to $\mathbb{Z}^2$-actions, i.e., approximate conjugacy, cocycle conjugacy, and outer conjugacy. Using the generalized determinant introduced by P. de la Harpe and G. Skandalis ([14]), we show that approximate conjugacy implies cocycle conjugacy when a (unital) $C^*$-algebra is simple and possesses a unique trace.

In Section 4 we consider product type $\mathbb{Z}^2$-actions on UHF algebras, i.e., pairs $(\alpha, \beta)$ of commuting automorphisms

$$\alpha \cong \bigotimes_{k=1}^{\infty} A_{u_k}, \quad \beta \cong \bigotimes_{k=1}^{\infty} A_{v_k}.$$ Considering the case when the $n_k \times n_k$ matrices $u_k, v_k$ commute at first, we show that the Rohlin property in this case is characterized by the property of uniform distribution of the joint spectral set $\text{Sp}(\bigotimes_{k=m}^{n} u_k, \bigotimes_{k=m}^{n} v_k)$ ($m \leq n$). From this we show that any such pairs are approximately conjugate.

We then investigate two special classes of UHF algebras. The first one is of the form $\bigotimes_{k=1}^{\infty} M_{p_k}^{i_k}$, where $p_k (k \in \mathbb{N})$ are primes and non-negative (finite) integers $i_k (k \in \mathbb{N})$ satisfy $\sum_{k=1}^{\infty} i_k = \infty$. The second one is of the form $\bigotimes_{k \in K} M_{q_k}^{\infty}$ with primes $q_k (k \in K)$ (where $\#K \leq \infty$) and $M_{q_k}^{\infty}$ of course means the infinite tensor product of $M_{q_k}$. For algebras in the first class we construct infinitely many non-cocycle conjugate product type $\mathbb{Z}^2$-actions with the Rohlin property. We would like to emphasize that for $\mathbb{Z}$-actions this phenomenon does not occur. On the other hand, for algebras in the second class we show that all the product type $\mathbb{Z}^2$-actions with the Rohlin property are mutually approximately conjugate. Combining these results we get the classification of product type $\mathbb{Z}^2$-actions with the Rohlin property on UHF algebras up to outer conjugacy.

2 Rohlin type theorem

Let $N$ be a positive integer. We first define the Rohlin property for $\mathbb{Z}^N$-actions on unital $C^*$-algebras. As mentioned above this is a simple generalization of that in the case of $N = 1$ [13]. Let $\xi_1, \ldots, \xi_N$ be the canonical basis of $\mathbb{Z}^N$ i.e.,

$$\xi_i = (0, \ldots, 0, 1, 0, \ldots, 0),$$

where 1 is in the $i$-th component, and let $I = (1, \ldots, 1)$ throughout this section.

For $m = (m_1, \ldots, m_N)$ and $n = (n_1, \ldots, n_N) \in \mathbb{Z}^N$, $m \leq n$ means $m_i \leq n_i$ for each $i = 1, \ldots, N$. We define

$$m\mathbb{Z}^N = \{(m_1n_1, \ldots, m_Nn_N)||n_1, \ldots, n_N) \in \mathbb{Z}^N\}$$
for \(m = (m_1, \ldots, m_N) \in \mathbb{Z}^N\) and let \(\mathbb{Z}^N\) act on \(\mathbb{Z}^N/m\mathbb{Z}^N\) by addition modulo \(m\mathbb{Z}^N\).

**Definition 1** Let \(\alpha\) be a \(\mathbb{Z}^N\)-action on a unital \(C^*\)-algebra \(A\), i.e., \(\alpha\) is a group homomorphism from \(\mathbb{Z}^N\) into the automorphisms \(\text{Aut}(A)\) of \(A\). Then \(\alpha\) is said to have the Rohlin property if for any \(m \in \mathbb{N}^N\) there exist \(R \in \mathbb{N}\) and \(m^{(1)}, \ldots, m^{(R)} \in \mathbb{N}^N\) with \(m^{(1)}, \ldots, m^{(R)} \geq m\) and which satisfy the following condition: For any \(\varepsilon > 0\) and finite subset \(F\) of \(A\), there exist projections

\[
e^{(r)}_g \quad (r = 1, \ldots, R, \ g \in \mathbb{Z}^N/m^{(r)}\mathbb{Z}^N)
\]

in \(A\) satisfying

\[
\sum_{r=1}^R \sum_{g \in \mathbb{Z}^N/m^{(r)}\mathbb{Z}^N} e^{(r)}_g = 1 ,
\]

\[
\|[x, e^{(r)}_g]\| < \varepsilon ,
\]

\[
\|\alpha \xi_i(e^{(r)}_g) - e^{(r)}_{\xi_i + g}\| < \varepsilon
\]

for any \(x \in F, r = 1, \ldots, R, i = 1, \ldots, N\), and \(g \in \mathbb{Z}^N/m^{(r)}\mathbb{Z}^N\).

**Remark 2** When \(A\) is a UHF algebra, using Christensen’s perturbation argument ([5, Theorem 5.3.]), we can restate the definition of the Rohlin property as follows. For any \(n, m \in \mathbb{N}\) with \(1 \leq n \leq N\) there exist \(R \in \mathbb{N}\) and positive integers \(m^{(1)}, \ldots, m^{(R)} \geq m\) which satisfy the following condition: For any \(\varepsilon > 0\) and finite subset \(F\) of \(A\) there exist projections

\[
e^{(r)}_0, \ldots, e^{(r)}_{m^{(r)}-1} \quad (r = 1, \ldots, R)
\]

in \(A\) satisfying

\[
\sum_{r=1}^R \sum_{j=0}^{m^{(r)}-1} e^{(r)}_j = 1 ,
\]

\[
\|[x, e^{(r)}_j]\| < \varepsilon
\]

for each \(r = 1, \ldots, R, j = 0, \ldots, m^{(r)} - 1\) and \(x \in F\), and

\[
\|\alpha \xi_n(e^{(r)}_j) - e^{(r)}_{j+1}\| < \varepsilon ,
\]

\[
\|\alpha \xi_n(e^{(r)}_j) - e^{(r)}_j\| < \varepsilon
\]

for each \(n' = 1, \ldots, N\) with \(n' \neq n\), \(r = 1, \ldots, R\) and \(j = 0, \ldots, m^{(r)} - 1\), where \(e^{(r)}_{m^{(r)}} = e^{(r)}_0\).
For automorphisms of $C^*$-algebras a notion of uniform outerness was introduced in [15]. That is, an automorphism $\alpha$ of a unital $C^*$-algebra $A$ is said to be uniformly outer if for any $a \in A$, any nonzero projection $p \in A$ and any $\varepsilon > 0$, there exist projections $p_1, \ldots, p_n$ in $A$ such that

$$p = \sum_{i=1}^{n} p_i,$$

$$\|p_i \alpha(p_i)\| < \varepsilon \quad (i = 1, \ldots, n).$$

It was shown that this notion for automorphisms of UHF algebras is equivalent to the usual outerness for the automorphisms of the von Neumann algebras obtained through the GNS representations associated with the traces ([15, Theorem 4.5]). Based on this fact and the Rohlin type theorem for automorphisms of von Neumann algebras due to A. Connes ([4, Theorem 1.2.5]), a $C^*$-algebraic version of the theorem (for the UHF algebras) was shown by A. Kishimoto ([16, Theorem 1.3]). We extend Kishimoto’s work to $\mathbb{Z}^2$-actions.

**Theorem 3** Let $\alpha$ be a $\mathbb{Z}^2$-action on a UHF algebra $A$. Then the following conditions are equivalent:

1. $\alpha$ has the Rohlin property.
2. $\alpha_g$ is uniformly outer for each $g \in \mathbb{Z}^2 \setminus \{0\}$.

Once we establish this theorem, we have immediately

**Corollary 4** Let $\alpha$ be a $\mathbb{Z}^2$-action on a UHF algebra $A$. Then the following conditions are equivalent:

1. $\alpha$ has the Rohlin property as a $\mathbb{Z}^2$-action on $A$.
2. $\alpha_g$ has the Rohlin property as an automorphism of $A$ for each $g \in \mathbb{Z}^2 \setminus \{0\}$.

In Theorem 3 it is obvious that (1) implies (2). We devote the rest of this section to prove the converse in several steps.

**Lemma 5** Let $\alpha$ be a $\mathbb{Z}^2$-action on a UHF algebra $A$. If the condition (2) in Theorem 3 holds then for any $m = (m_1, m_2) \in \mathbb{N}^2$, $\varepsilon > 0$ and any unital full matrix subalgebra $B$ of $A$, there exists an orthogonal family $\{ e_g \mid g \in \mathbb{Z}^2, 0 \leq g \leq m - I \}$ of projections in $A \cap B'$ such that

$$\|\alpha_{\xi_i}(e_g) - e_{\xi_i + g}\| < \varepsilon$$

for any $i = 1, 2$ and $g \in \mathbb{Z}^2$ with $0 \leq g, \xi_i + g \leq m - I$, and furthermore,

$$1 < (|m| + 1)\tau(e_0),$$

where $\tau$ is the unique tracial state of $A$ and $|m| \equiv m_1 \cdot m_2$.
Proof. Let \((\pi, H)\) be the GNS representation associated with \(\tau\). By the uniqueness of a trace we can extend each \(\alpha_g (g \in Z^2)\) to an automorphism of the AFD II factor \(\pi(A)' = B(H)\) and we use the same symbol \(\alpha_g\) for this extension. Since \(\alpha_g\) is outer on \(\pi(A)'\) for \(g \in Z^2\) \(\setminus \{0\}\) by \cite[Theorem 4.5]{15}, it follows from \cite[Theorem 2]{20} that for any \(m \in N^2\) there exists a strongly central sequence

\[
\left( \left( E_{g}^{(j)} \mid g \in Z^2, 0 \leq g \leq m - I \right) \mid j \in N \right)
\]

of orthogonal families of projections in \(\pi(A)'\) such that

\[
\sum_{g \in Z^2, 0 \leq g \leq m - I} E_{g}^{(j)} = 1
\]

for each \(j \in N\) and

\[
\alpha_{\xi} (E_{g}^{(j)}) - E_{g+\xi}^{(j)} \to 0
\]

strongly as \(j \to \infty\) for each \(i = 1, 2\) and \(g \in Z^2\) with \(0 \leq g \leq m - I\), where \((m_1, k) \equiv (0, k), (k, m_2) \equiv (k, 0)\).

From this central sequence we shall construct a uniformly central sequence

\[
\left( \left( f_{g}^{(j)} \mid g \in Z^2, 0 \leq g \leq m - I \right) \mid j \in N \right)
\]

of orthogonal families of projections in \(A\) such that

\[
\pi_{\tau} \left( \sum_{g \in Z^2, 0 \leq g \leq m - I} f_{g}^{(j)} \right) \to 1
\]

strongly as \(j \to \infty\) and

\[
\|\alpha_{\xi} (f_{g}^{(j)}) - f_{g+\xi}^{(j)}\| \to 0
\]

as \(j \to \infty\) for each \(i = 1, 2\) and \(g \in Z^2\) with \(0 \leq g, g + \xi \leq m - I\). To do this, let \((A_j \mid j \in N)\) be an increasing sequence of unital full matrix subalgebras of \(A\) such that \(\cup_j A_j\) is dense in \(A\). From \cite[Lemma 4.7]{3} we find a uniformly central sequence \((e_j \mid j \in N)\) of projections in \(A\) such that

\[
\pi_{\tau} (e_j) - E_{0}^{(j)} \to 0
\]

strongly as \(j \to \infty\). Changing \(e_j\) slightly and taking a subsequence, we may assume that \(e_j \in (\cup_k A_k) \cap A_j'\) for each \(j\). Let \(\varepsilon > 0\). From \cite[Corollary 6.8]{3}, by taking inner perturbations, there are \(\alpha_1, \alpha_2 \in Aut(A)\) such that

\[
\|\alpha_i - \alpha_{\xi}\| < \varepsilon
\]
\[ \alpha_i \left( \bigcup_{j \in \mathbb{N}} A_j \right) \subseteq \bigcup_{j \in \mathbb{N}} A_j \]

for \( i = 1, 2 \). Set

\[ h_j = e_j \left( \sum_{g = (g_1, g_2) \in \mathbb{Z}^2 \setminus \{0\}} (\alpha_1)^{g_1} (\alpha_2)^{g_2} (e_j) \right) e_j , \]

\[ k_j = e_j \left( \sum_{g \in \mathbb{Z}^2 \setminus \{0\}} \alpha_g (e_j) \right) e_j , \]

\( \eta_j = \tau(h_j) \) and \( \kappa_j = \tau(k_j) \). Then it follows that

\[ \| h_j - k_j \| < \epsilon(m_1 + m_2 - 2) \{(2m_1 - 1)(2m_2 - 1) - 1\} . \]

Furthermore, \( \lim_j \kappa_j = 0 \) since

\[
\begin{align*}
\lim_{j \to \infty} \kappa_j &= \lim_{j \to \infty} \tau \left( e_j \sum_{g \in \mathbb{Z}^2 \setminus \{0\}} \sum_{-(m_1-1) \leq g \leq m-1} \alpha_g (e_j) \right) \\
&\leq \lim_{j \to \infty} \sum_{g, h \in \mathbb{Z}^2, g \neq h} \tau (e_j \alpha_g \cdot h (e_j)) \\
&\leq \lim_{j \to \infty} \sum_{0 \leq g, h \leq m-1} \tau (\alpha_g (e_j) \alpha_h (e_j)) \\
&= \lim_{j \to \infty} \sum_{g, h \in \mathbb{Z}^2, g \neq h} \tau (E_h^{(j)} E_g^{(j)}) = 0 .
\end{align*}
\]

Let \( p_j \) be the spectral projection of \( h_j \) corresponding to \( (0, \eta_j^{\frac{1}{2}}) \). Then \( p_j \in A \) since \( \text{Sp}(h_j) \) is finite. \( p_j \leq e_j \) and \( \eta_j^{\frac{1}{2}} (e_j - p_j) \leq h_j \) because \( \eta_j^{\frac{1}{2}} \chi_{[\eta_j^{\frac{1}{2}}, \infty]} (t) \leq t \) \( (t \in [0, \infty)) \), hence

\[ \tau(e_j) - \eta_j^{\frac{1}{2}} \leq \tau(p_j) \leq \tau(e_j) . \]
In addition
\[
\|p_j\left(\sum_{g=(g_1,g_2)\in\mathbb{Z}^2\setminus\{0\}}^{-(m-I)\leq g\leq m-I} (\alpha_1)^{g_1}(\alpha_2)^{g_2}(p_j)\right)\| \leq \|p_jh_jp_j\| \leq \|p_j\| \leq \|p_jh_jp_j\| \leq \eta_j^{1/2}.
\]

So for any \(g, h \in \mathbb{Z}^2\) with \(0 \leq g, h \leq m-I\) and \(g \not= h\), we have
\[
\|\alpha_g(p_j)\alpha_h(p_j)\|^2 = \|\alpha_g(p_j\alpha_{h-g}(p_j))\|^2 = \|p_j\alpha_{h-g}(p_j)\|^2
\]
\[
\leq \|p_j\left(\sum_{g=(g_1,g_2)\in\mathbb{Z}^2\setminus\{0\}}^{-(m-I)\leq g\leq m-I} \alpha_g(p_j)\right)\| \leq \varepsilon' + \eta_j^{1/2},
\]
where \(\varepsilon' = \varepsilon(m_1 + m_2 - 2)\{(2m_1 - 1)(2m_2 - 1) - 1\}\). Here \(\lim \eta_j = 0\) since \(\lim \alpha_j = 0\) and \(\|h_j - k_j\| = 0\). Therefore taking a sufficiently large \(j\) for each \(\varepsilon > 0\), we obtain the required \(f_{g(j)}\) near \(\alpha_g(p_j)\) by slight modification.

Noting that \(\sum_{0 \leq g \leq m-I} \tau(f_{g(j)}) \to 1\) and \(\tau(f_{g(j)}) = \tau(f_{h(j)})\), we have
\[
\tau(f_{g(j)}) \to \frac{1}{|m|}.
\]
Furthermore for any unital full matrix subalgebra \(B\) of \(A\), taking a sufficiently large \(j\), we may assume that \(f_{g(j)} \in A \cap B^\prime\) for any \(g\). This concludes the proof.

In Ocneanu’s result [20, Theorem 2], applied in the above proof, we have the cyclicity condition (under the action) of the projections \((E_{i,j}^g)\) \(g \in \mathbb{Z}^2, 0 \leq g \leq m-I\) in the von Neumann algebra \(\pi(A)^c\). However when approximating these projections by the projections \((f_{g(j)}^i)\) \(g \in \mathbb{Z}^2, 0 \leq g \leq m-I\) in the C*-algebra \(A\), we lose the cyclicity condition. It is our next problem to restore this cyclicity condition. To do this we need a technical lemma from [19]. Let \(K(l^2(\mathbb{Z}))\) be the compact operators on \(l^2(\mathbb{Z})\) and let \((E_{i,j})\) \(i, j \in \mathbb{Z}\) be the canonical matrix units for \(K(l^2(\mathbb{Z}))\). On \(K(l^2(\mathbb{Z}))\) we define an automorphism \(\sigma\) by \(\sigma(E_{i,j}) = E_{i+1,j+1}\) \((i, j \in \mathbb{Z})\). For any \(n, k, l \in \mathbb{N}\) with \(1 < k < l\), define
\[
N = n(2k + l - 1),
\]
\[
f = \sum_{i=1}^{k-1} \left( \frac{i}{k} E_{ni,ni} + \frac{k-i}{k} E_{n(k+l+i),n(k+l+i)} + \frac{\sqrt{i(k-i)}}{k} E_{ni,n(k+l+i)} \right).
\]
\[ e_i = \sigma^{i-n} f \quad (i = 0, \ldots, n - 1). \]

Then \((e_i \mid i = 0, \ldots, n - 1)\) is an orthogonal family of projections in \(K(l^2(\mathbb{Z}))\).

Hence for any \(\varepsilon > 0\), there exist \(k, l\) with \(1 \ll k \ll l\) such that

\[ \sum_{i=0}^{n-1} e_i \leq P_N, \quad P_N = \sum_{i=0}^{N-1} E_i, \]
\[ \| \sigma(e_i) - e_{i+1} \| < \varepsilon \quad (i = 0, \ldots, n - 1, \ e_n \equiv e_0), \]
\[ \frac{n \dim e_0}{N} > 1 - \varepsilon \]

(see \([13]\, \text{Lemma 2.1}\) for the detail). Using these estimates we have the next lemma.

**Lemma 6** Let \(\alpha\) be a \(\mathbb{Z}^2\)-action on a UHF algebra \(A\). If \(\alpha_g\) is uniformly outer for any \(g \in \mathbb{Z}^2 \setminus \{0\}\), then for any \(m \in \mathbb{N}, \varepsilon > 0\) and any unital full matrix subalgebra \(B\) of \(A\) there exists an orthogonal family \((e_i \mid i = 0, \ldots, m - 1)\) of projections in \(\mathbb{A} \cap B'\) such that

\[ \| \alpha_{\xi_1}(e_i) - e_{i+1} \| < \varepsilon, \]
\[ \| \alpha_{\xi_2}(e_i) - e_i \| < \varepsilon, \]
\[ \tau(1 - \sum_{i=0}^{m-1} e_i) \leq \varepsilon \tau(e_0) \]

for \(i = 0, \ldots, m - 1\), where \(e_m \equiv e_0\).

**Proof.** Let \(m \in \mathbb{N}, \varepsilon_1 > 0\) and let \(B_1\) be a unital full matrix subalgebra of \(A\). By the above statement there exist \(k_1, l_1 \in \mathbb{N}\) with \(1 \ll k_1 \ll l_1\) and an orthogonal family \((e_i \mid i = 0, \ldots, m - 1)\) of projections in \(K(l^2(\mathbb{Z}))\) such that

\[ \sum_{i=0}^{m-1} e_i \leq P_{N_1}, \]
\[ \| \sigma(e_i) - e_{i+1} \| < \varepsilon_1 \quad (i = 0, \ldots, m - 1), \]
\[ \frac{m \dim e_0}{N_1} > 1 - \varepsilon_1, \]

(3)

where \(N_1 \equiv m(2k_1 + l_1 - 1)\) and \(e_m \equiv e_0\). Similarly by the above statement (for \(n = 1, \varepsilon_1\) and \(B_1\)), there exist \(k_2, l_2 \in \mathbb{N}\) with \(1 \ll k_2 \ll l_2\) and a projection \(e\) in \(K(l^2(\mathbb{Z}))\) such that

\[ e \leq P_{N_2}, \]
decomposition we have

\[
\| \sigma(e) - e \| < \varepsilon_1, \quad \dim e \quad \frac{N_2}{N_2 - 1} > 1 - \varepsilon_1, \quad (4)
\]

where \( N_2 = 2k_2 + l_2 - 1 \).

Next by applying Lemma \( k \) to \((N_1, N_2) \in \mathbb{N}^2\), any \( \varepsilon_2 > 0 \) and \( B_1 \), there exists an orthogonal family \( \{p_g \mid g \in \mathbb{Z}^2, 0 \leq g \leq (N_1 - 1, N_2 - 1)\} \) of projections in \( A \cap B_1' \) such that

\[
\| \alpha_{\xi_i}(p_g) - p_{\xi_i}g \| < \varepsilon_2, \quad 1 \leq (N_1N_2 + 1)\tau(p_0)
\]

for any \( i = 1, 2 \) and \( g \in \mathbb{Z}^2 \) with \( 0 \leq g, \xi_i + g \leq (N_1 - 1, N_2 - 1) \).

If we put

\[
x_1 = \frac{1}{N_2} \sum_{j=0}^{N_2-1} \left\{ \sum_{i=0}^{N_1-2} p_{(i+1,j)}^{(i,j)}(\alpha_{\xi_1}(p_{(i,j)}))\right\},
\]

then we have

\[
x_1\alpha_{\xi_1}(p_{(i,j)}) = p_{(i+1,j)}x_1
\]

for any \( i = 0, \ldots, N_1 - 2, j = 0, \ldots, N_2 - 1 \) and

\[
x_1 - 1 = \frac{1}{N_2} \sum_{j=0}^{N_2-1} \left\{ \sum_{i=0}^{N_1-2} p_{(i+1,j)}^{(i,j)}(\alpha_{\xi_1}(p_{(i,j)}) - p_{(i+1,j)})\right\}
\]

\[
+ (1 - \sum_{i=1}^{N_1-1} \sum_{j=1}^{N_2-1} \alpha_{\xi_1}(p_{(i,j)})) \left\{ (\sum_{i=1}^{N_1-2} \alpha_{\xi_1}(p_{(i,j)}) \right\}.
\]

Noting that \( \| \alpha_{\xi_1}(p_{(i,j)}) - p_{(i+1,j)} \| < \varepsilon_2 \), we have \( \| x_1 - 1 \| < 2(N_1 - 1)\varepsilon_2 \). So taking the polar decomposition \( u_1 \mid x_1 \rangle \) of \( x_1 \) for a sufficiently small \( \varepsilon_2 > 0 \), we obtain a unitary \( u_1 \) with \( \| u_1 - 1 \| < 4(N_2 - 1)\varepsilon_2 \). By the uniqueness of the polar decomposition we have

\[
Ad u_1 \circ \alpha_{\xi_1}(p_{(i,j)}) = p_{(i+1,j)}
\]

for \( i = 0, \ldots, N_1 - 2, j = 0, \ldots, N_2 - 1 \). Similarly for \( \alpha_{\xi_2} \), we obtain a unitary \( u_2 \) in \( A \) such that

\[
\| u_2 - 1 \| < 4(N_2 - 1)\varepsilon_2,
\]

\[
Ad u_2 \circ \alpha_{\xi_2}(p_{(i,j)}) = p_{(i,j+1)}
\]

for \( i = 0, \ldots, N_1 - 1, j = 0, \ldots, N_2 - 2 \). Let \( \alpha_1 = Ad u_1 \circ \alpha_{\xi_1} \) and let \( \alpha_2 = Ad u_2 \circ \alpha_{\xi_2} \). Since \( [p_{(0,0)}] = [p_{(1,0)}] \) it follows that there exists a partial isometry
$v_1$ of $A \cap B'_1$ such that $v_1^*v_1 = p_{(0,0)}$ and $v_1v_1^* = p_{(1,0)}$. Similarly there exists a partial isometry $v_2$ of $A \cap B'_1$ such that $v_2^*v_2 = p_{(0,0)}$ and $v_2v_2^* = p_{(0,1)}$. Then $Ad v_2^* \circ \alpha_2(p_{(0,0)}) = p_{(0,0)}$, so $Ad v_2^* \circ \alpha_2 \in Aut(p_{(0,0)} Ap_{(0,0)})$. On the other hand $\alpha_3 \in Aut(A)$ has the Rohlin property as a single automorphism, and hence so does $Ad v_2^* \circ \alpha_2$. Therefore $Ad v_2^* \circ \alpha_2$ is stable by [12, 13]. More precisely for any $\varepsilon_3 > 0$, any unital full matrix subalgebra $B_2$ of $A$ and the unitary $v_2^* \alpha_2(v_1)^* \alpha_1(v_2)v_1 \in p_{(0,0)} Ap_{(0,0)}$, if $B_1$ is taken sufficiently large in advance, we have a unitary $w$ in $A \cap B'_2$ such that

$$\|v_2^* \alpha_2(v_1)^* \alpha_1(v_2)v_1 - (Ad v_2^* \circ \alpha_2(w)) \cdot w^*\| < \varepsilon_3.$$  

Let $w_1 = w_1 w$ and let $w_2 = v_2$. Then $w_1$ and $w_2$ are partial isometries in $A \cap B'_2$ such that $w_1^*w_1 = w_2^* w_2 = p_{(0,0)}$, $w_1w_1^* = p_{(1,0)}$, $w_2w_2^* = p_{(0,1)}$ and

$$\|\alpha_1(w_2)w_1 - \alpha_2(w_1)w_2\| = \|v_2^* \alpha_2(v_1)^* (\alpha_1(v_2)v_1 w - \alpha_2(v_1 w)v_2)w^*\| < \varepsilon_3  \quad (6)$$

Define

$$E_{i,j}^{(k)} = \begin{cases} 
\alpha_{i,j}^{-1}(\alpha_{i,j}^k(w_2))\alpha_{i,j}^{k-2}(\alpha_{i,j}^k(w_2))\cdots \alpha_{i,j}^{2}(\alpha_{i,j}^k(w_2)) & (i > j) \\
\alpha_{i,j}^{2}(\alpha_{i,j}^k(w_2))^*\alpha_{i,j}^{2+1}(\alpha_{i,j}^k(w_2))^*\cdots \alpha_{i,j}^{2+1}(\alpha_{i,j}^{k-1}(w_2))^* & (i < j)
\end{cases}$$

for $k = 0, 1, i, j = 0, \ldots, N_2 - 1$. Then we can easily see that $\{ E_{i,j}^{(k)} | i, j = 0, \ldots, N_2 - 1 \}$ is a system of matrix units. For any unital full matrix subalgebra $B_3$ of $A$, by taking a sufficiently large $B_2$ including $B_3$, we may assume that $\{ E_{i,j}^{(k)} | k = 0, 1; i, j = 0, \ldots, N_2 - 1 \} \subseteq A \cap B'_3$. Let $C^{(k)}$ be the $C^*$-subalgebra of $A$ generated by $\{ E_{i,j}^{(k)} | i, j = 0, \ldots, N_2 - 1 \}$ and let $\Phi_k$ be the canonical isomorphism from $C^{(k)}$ onto $P_{N_2} K(l^2(\mathbb{Z})) P_{N_2}$. Define

$$e^{(k)} = \Phi_k^{-1}(e) .$$

Since $\sigma \circ \Phi_k = \Phi_k \circ \alpha_2 \circ C^{(k)}$, we have from [4] that

$$e^{(k)} \leq \sum_{i=0}^{N_2-1} p_{(k,i)} ,$$

$$\|\alpha_2(e^{(k)}) - e^{(k)}\| < \varepsilon_1 ,$$

$$\tau(e^{(k)}) > (1 - \varepsilon_1)N_2 \tau(p_{(0,0)}) . \quad (7)$$

Furthermore define

$$W_1 = \left( \sum_{i=0}^{N_2-1} \alpha_{i,j}^{2}(w_1) \right) e^{(0)} ,$$

$$e^{(1)} = W_1 W_1^* \leq \sum_{i=0}^{N_2-1} p_{(1,i)} .$$
Again for any unital full matrix subalgebra $B_4$ of $A$, by taking a sufficiently large $B_3$ including $B_4$, we may assume that $W_1 \in A \cap B'_4$. Then recalling the formula (2), we have
\[
e^{(1)'} = \left( \sum_{i=0}^{N_2-1} \alpha_2^i(w_1) \right)^* X \left( \sum_{i'=0}^{N_2-1} \alpha_2^{i'}(w_1)^* \right)^*,
\]
where $X$ denotes
\[
\sum_{j=1}^{k_2-1} \left\{ \frac{j}{k_2} E_{j,j}^{(0)} + \frac{k_2 - j}{k_2} E_{k_2 + l_2 + j, k_2 + l_2 + j}^{(0)} + \frac{\sqrt{j(k_2 - j)}}{k_2} E_{j,k_2 + l_2 + j}^{(0)} \right\}
+ \sum_{j=k_2}^{k_2 + l_2} E_{j,j}^{(0)}.
\]
On the right hand side of (8), the nonzero terms are calculated as follows:
\[
\alpha_2^j(w_1) E_{j,j}^{(0)} \alpha_2^j(w_1)^* = \alpha_2^j(w_1) p_{0,j} \alpha_2^j(w_1)^* = \alpha_2^j(w_1) E_{j,j}^{(1)},
\]
\[
\alpha_2^j(w_1) E_{j,k_2 + l_2 + j}^{(0)} \alpha_2^{k_2 + l_2 + j}(w_1)^* \simeq \alpha_2^j(w_1) \alpha_2^j(w_2)^* \cdots \alpha_2^{k_2 + l_2 + j-1}(w_1)^* + \alpha_2^{k_2 + l_2 + j-1}(w_1)^* + \cdots + \alpha_2^{k_2 + l_2 + j-1}(w_1)^* ,
\]
where $x \simeq y$ means $\|x - y\| < \varepsilon$. Applying (8) repeatedly we have
\[
\alpha_2^j(w_1) E_{j,k_2 + l_2 + j}^{(0)} \alpha_2^{k_2 + l_2 + j}(w_1)^* \simeq \alpha_2^j(w_1) E_{j,k_2 + l_2 + j}^{(1)},
\]
We estimate the other nonzero terms similarly and obtain
\[
\|e^{(1)'} - e^{(1)}\| \leq \sum_{j=1}^{k_2-1} \sqrt{j(k_2 - j)} (k_2 + l_2) \varepsilon_3.
\]
Let $c_1(k_2, l_2, \varepsilon_3)$ be the right hand side of the above inequality. Then we have
\[
\|\alpha_2(e^{(1)'}) - e^{(1)}\| \leq 2\|e^{(1)' - e^{(1)}}\| + \|\alpha_2(e^{(1)}) - e^{(1)}\|
\leq 2c_1(k_2, l_2, \varepsilon_3) + \varepsilon_1.
\]
For any $\varepsilon_4 > 0$ and any unital full matrix subalgebra $B_5$ of $A$, applying [13, Lemma 3.5] and taking a sufficiently large $B_4$ including $B_5$, we have a partial isometry $W'_1$ of $A \cap B'_5$ such that $(W'_1)^*W'_1 = \varepsilon^{(0)}$, $W'_1(W'_1)^* = \varepsilon^{(1)}$ and
\[
\|\alpha_2(W'_1) - W'_1\| \leq \|\alpha_2(\varepsilon^{(0)}) - \varepsilon^{(0)}\| + \|\alpha_2(\varepsilon^{(1)}) - \varepsilon^{(1)}\| + \varepsilon_4
\leq \varepsilon_1 + 2\varepsilon_1(k_2, l_2, \varepsilon_3) + \varepsilon_1 + \varepsilon_4.
\] (9)

Of course we can make the last quantity very small. By using this $W'_1$, let $D$ be the $C^*$-subalgebra of $A$ generated by
\[
\{\alpha_1^{-1}(W_1)\alpha_1^{-2}(W_1)\cdots\alpha_1^j(W_1) \mid N_1 - 1 \geq i > j \geq 0\}.
\]
Here again for any unital full matrix subalgebra $B_6$, by taking a sufficiently large $B_5$ including $B_6$, we may assume that $D \subseteq A \cap B'_6$. As $D$ is isomorphic to $P_N, K(l^2(\mathbf{Z})), P_1$, let $\Psi$ be the canonical isomorphism from $D$ onto $P_N, K(l^2(\mathbf{Z})), P_1$, and let
\[
f_i = \Psi^{-1}(\varepsilon_i)
\]
for $i = 0, \ldots, m - 1$. Then $(f_i \mid i = 0, \ldots, m - 1)$ is an orthogonal family of projections in $A \cap B'_6$ such that
\[
\|\alpha_1(f_i) - f_{i+1}\| < \varepsilon_1,
\]
m\tau(f_i) > (1 - \varepsilon)N_1\tau(\varepsilon^{(0)})
\] (10)
for $i = 0, \ldots, m - 1$, where $f_m \equiv f_0$. Thus we have for $i = 0, \ldots, m - 1$,
\[
\|\alpha_{\xi i}(f_i) - f_{i+1}\| \leq \|\alpha_{\xi i}(f_i) - \alpha_1(f_i)\| + \|\alpha_1(f_i) - f_{i+1}\|
\leq 2\|u_1 - 1\| + \varepsilon_1
\leq 2 \cdot 4(N_1 - 1)\varepsilon_2 + \varepsilon_1.
\]
Using the formula (3), the formula (5) and
\[
\|\alpha_1\alpha_2 - \alpha_2\alpha_1\| \leq 2(\|\alpha_1 - \alpha_{\xi i}\| + \|\alpha_2 - \alpha_{\xi i}\|)
\leq 4(\|u_1 - 1\| + \|u_2 - 1\|),
\]
we can also make $\|\alpha_{\xi i}(f_i) - f_i\|$ very small. Finally we want to estimate $\tau(f_0)$. We have already three inequalities from (6), (7) and (11)
\[
1 \leq (N_1N_2 + 1)\tau(p_{(0,0)}),
\]
\[
\tau(\varepsilon^{(0)}) > (1 - \varepsilon_1)N_2\tau(p_{(0,0)}),
\]
m\tau(f_0) > (1 - \varepsilon_1)N_1\tau(\varepsilon^{(0)}).
\]
From these we obtain
\[
m\tau(f_0) > (1 - \varepsilon_1)^2N_1N_2\frac{1}{N_1N_2 + 1}\left(m\tau(f_0) + \tau(1 - \sum_{i=0}^{m-1}f_i)\right).
\]
Since $1 \ll k_i \ll l_i$, $(f_i \mid i = 0, \ldots, m - 1)$ satisfies the desired conditions.

Proof of Theorem 3.

Let $\alpha$ be a $\mathbb{Z}^2$-action on a UHF algebra $A$ which satisfies the condition (2). For any $m \in \mathbb{N}$ we take $m_0, m_1 \in \mathbb{N}$ such that $m \ll m_1 \ll m_0$ and $m_0$ is divided by $m_1$. Furthermore for any $\varepsilon_1 > 0$ and finite subset $F$ of $A$, we take a unital full matrix subalgebra $B_1$ of $A$ such that for any $x \in F$ there exists $y \in B_1$ with $\|x - y\| < \varepsilon_1$. If we apply Lemma 6 to any $n \in \mathbb{N}$ and $\varepsilon_2 > 0$ then we have an orthogonal family $(e_i \mid i = 0, \ldots, m - 1)$ of projections in $A \cap B_1'$ satisfying

$$\|\alpha_{\xi_1}(e_i) - e_{i+1}\| < \varepsilon_2,$$

$$\|\alpha_{\xi_2}(e_i) - e_i\| < \varepsilon_2,$$

$$\tau(e_0) \geq n\tau(1 - \sum_{i=0}^{m_0-1} e_i)$$

for $i = 0, \ldots, m_0 - 1$, where $e_{m_0} \equiv e_0$. This is not sufficient because the sum $\sum_{i=0}^{m_0-1} e_i$ of the projections $(e_i)$ may not be 1. We will cope with this problem now. Put

$$x_1 = \sum_{i=0}^{m_0-1} e_{i+1}\alpha_{\xi_1}(e_i) + \left(1 - \sum_{i=0}^{m_0-1} e_i\right) \left(1 - \sum_{i=0}^{m_0-1} \alpha_{\xi_1}(e_i)\right),$$

$$x_2 = \sum_{i=0}^{m_0-1} e_i\alpha_{\xi_2}(e_i) + \left(1 - \sum_{i=0}^{m_0-1} e_i\right) \left(1 - \sum_{i=0}^{m_0-1} \alpha_{\xi_2}(e_i)\right)$$

and let $u_1|x_1|$ and $u_2|x_2|$ be the polar decompositions of $x_1$ and $x_2$ respectively. As in the proof of Lemma 6 we can show that $u_1$ and $u_2$ are unitaries in $A$ satisfying

$$\|u_1 - 1\| < 4m_0\varepsilon_2,$$

$$Ad u_1 \circ \alpha_{\xi_1}(e_i) = e_{i+1}$$

for $i = 0, \ldots, m_0 - 1$, where $e_{m_0} \equiv e_0$, and

$$\|u_2 - 1\| < 4m_0\varepsilon_2,$$

$$Ad u_2 \circ \alpha_{\xi_2}(e_i) = e_i$$

for $i = 0, \ldots, m_0 - 1$. Let $\alpha_1 = Ad u_1 \circ \alpha_{\xi_1}$ and let $\alpha_2 = Ad u_2 \circ \alpha_{\xi_2}$. Then $\alpha_1^{m_0}$ and $\alpha_2$ are automorphisms of $e_0 Ae_0$. By Lemma 6 there are an orthogonal family $(p_j \mid j = 0, \ldots, n - 1)$ of projections in $A \cap B_1'$ and a positive number $c_1(m_0, \varepsilon_2)$ which decreases to zero as $\varepsilon_2 \to 0$ such that

$$\sum_{i=0}^{n-1} p_i \leq e_0.$$
\begin{equation*}
\|\alpha_1^{m_0}(p_i) - p_{i+1}\| < c_1(m_0, \varepsilon_2),
\|\alpha_2(p_i) - p_i\| < c_1(m_0, \varepsilon_2)
\end{equation*}
for \(i = 0, \ldots, n-1\), where \(p_n \equiv p_0\), and
\(\tau(p_i) = \tau(1 - e)\),
where \(e \equiv \sum_{i=0}^{m_0-1} e_i\). We have used the fact \(\tau(e_0) \geq n\tau(1 - e)\) here. For any \(\varepsilon_3 > 0\) and any unital full matrix subalgebra \(B_2\) of \(A\), by taking a sufficiently large \(B_1\) and by applying [13, Lemma 3.5], there exists a partial isometry \(v \in A \cap B_2\) such that
\begin{equation*}
v^*v = 1 - e, \quad vv^* = p_0,
\end{equation*}
\begin{equation*}
\|\alpha_2(v) - v\| \leq \|\alpha_2(p_0) - p_0\| + \|\alpha_2(1 - e) - (1 - e)\| + \varepsilon_3 
\leq c_1(m_0, \varepsilon_2) + m_0\varepsilon_2 + \varepsilon_3.
\end{equation*}
As before there also exists a unitary \(u_1' \in A\) satisfying that \(\|u_1' - 1\| < 4c_1(m_0, \varepsilon_2)\) and
\begin{equation*}
Ad u_1' \circ \alpha_1^{m_0}(p_i) = p_{i+1}
\end{equation*}
for \(i = 0, \ldots, n-1\). Let \(\beta = Ad u_1' \circ \alpha_1^{m_0}\) and let \(w = n^{-\frac{1}{2}} \sum_{i=0}^{n-1} \beta^i(v)\). Then we have
\begin{equation*}
w^*w = 1 - e, \quad ww^* \leq \varepsilon_0,
\|\beta(w) - w\| < n^{-\frac{1}{2}} \cdot 2, \quad (11)
\|\alpha_2(w) - w\| < c_2(m_0, n, \varepsilon_2, \varepsilon_3)
\end{equation*}
for some positive number \(c_2(m_0, n, \varepsilon_2, \varepsilon_3)\) which we can make very small. Furthermore by taking \(B_2\) very large we may assume that \(w \in A \cap B_3\) for any unital full matrix subalgebra \(B_3\) of \(A\). Let
\begin{equation*}
E_{i,j} = \begin{cases} 
\alpha_1^{i-1}(w)\alpha_2^{j-2}(w)\cdots\alpha_2^j(w) & \text{(if } i > j) \\
\alpha_1^{i-1}(ww^*) & \text{(if } i = j) \\
\alpha_1^i(w^*\alpha_1^{i+1}(w)^*\cdots\alpha_2^{j-1}(w)^*) & \text{(if } i < j)
\end{cases}
\end{equation*}
for \(0 \leq i, j \leq m_0\) and let \(C\) be the \(C^*\)-subalgebra of \(A\) generated by \(\{E_{i,j} \mid 0 \leq i, j \leq m_0 - 1\}\). Then \(C\) is isomorphic to \(M_{m_0+1}\) and we may assume that \(C\) is a subalgebra of \(A \cap B_4\) for any unital full matrix subalgebra \(B_4\) of \(A\) if \(B_3\) is very large. Let
\begin{equation*}
U = \begin{bmatrix} 
1 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix} \in M_{m_0+1}.
\end{equation*}
By simple calculation \( \alpha_1[C = Ad U] \) and

\[
\text{Sp}(U) = \{ 1 \} \cup \{ e^{\frac{2\pi i k}{m_0}} \mid k = 0, \ldots, m_0 - 1 \}.
\]

Define orthogonal families \((e^{(1)}_i \mid i = 0, \ldots, m_1 - 1), (e^{(2)}_j \mid j = 0, \ldots, m_1)\) of projections in \(A \cap B'_4\) as follows:

\[
e^{(1)}_i \equiv \sum_{k=0}^{m_0 - 1 - m_1} E_{m_1 + i((\frac{m_0}{m_1}) - 1) + k, m_1 + i((\frac{m_0}{m_1}) - 1) + k},
\]

\[
e^{(2)}_j \equiv E_{j,j}.
\]

Since \((E_{i,j} \mid 0 \leq i, j \leq m_0 - 1)\) is a system of matrix units in \(A \cap B'_4\), there are canonical systems \((e^{(1)}_{k,l} \mid 0 \leq k, l \leq m_1 - 1)\) and \((e^{(2)}_{k,l} \mid 0 \leq k, l \leq m_1)\) respectively. Define a partial isometry \(V\) in \(A \cap B'_4\) by

\[
V = \sum_{i=0}^{m_1 - 1} e^{(1)}_{i+1,i} + \sum_{j=0}^{m_1} e^{(2)}_{j+1,j},
\]

where \(e^{(1)}_{m_1,m_1-1} \equiv e^{(1)}_{0,m_1-1}\) and \(e^{(2)}_{m_1+1,m_1} \equiv e^{(2)}_{0,m_1}\). Then

\[
\sum_{i=0}^{m_1 - 1} e^{(1)}_i + \sum_{j=0}^{m_1} e^{(2)}_j = 1C \left( = \sum_{i=0}^{m_0} E_{i,i} \right),
\]

\[
Ad V(e^{(1)}_i) = e^{(1)}_{i+1},
\]

\[
Ad V(e^{(2)}_j) = e^{(2)}_{j+1}
\]

for \(i = 0, \ldots, m_1 - 1\) and \(j = 0, \ldots, m_1, m_1\) where \(e^{(1)}_{m_1} \equiv e^{(1)}_{0}\) and \(e^{(2)}_{m_1+1} \equiv e^{(2)}_{0}\). By a simple calculation

\[
\text{Sp}(V) = \{ e^{\frac{2\pi i k}{m_0 - m_1}} \mid k = 0, \ldots, m_0 - m_1 - 1 \} \cup \{ e^{\frac{2\pi i l}{m_1 + 1}} \mid l = 0, \ldots, m_1 \}.
\]

Therefore \(\text{Sp}(V)\) is very close to \(\text{Sp}(U)\) if \(m_0\) and \(m_1\) are very large, i.e., \(V\) is almost unitarily equivalent to \(U\). Consequently we can find orthogonal families \((f^{(1)}_i \mid i = 0, \ldots, m_1 - 1), (f^{(2)}_j \mid j = 0, \ldots, m_1)\) of projections in \(A \cap B'_4\) and a small enough positive number \(c_3(m_0, m_1, \varepsilon_2, \varepsilon_3)\) in such a way that

\[
\sum_{i=0}^{m_1 - 1} f^{(1)}_i + \sum_{j=0}^{m_1} f^{(2)}_j = 1C,
\]

\[
||\alpha_1(f^{(1)}_i) - f^{(1)}_{i+1}|| < c_3(m_0, m_1, \varepsilon_2, \varepsilon_3),
\]

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\[ \| \alpha_1(f^{(1)}_i) - f_i^{(1)} \| < c_3(m_0, m_1, \varepsilon_2, \varepsilon_3), \]
\[ \| \alpha_1(f^{(2)}_j) - f_j^{(2)} \| < c_3(m_0, m_1, \varepsilon_2, \varepsilon_3), \]
\[ \| \alpha_2(f^{(2)}_j) - f_j^{(2)} \| < c_3(m_0, m_1, \varepsilon_2, \varepsilon_3) \]
for \( i = 0, \ldots, m_1 - 1 \) and \( j = 0, \ldots, m_1 \), where \( f^{(1)}_{m_1} \equiv f^{(1)}_0 \) and \( f^{(2)}_{m_1+1} \equiv f^{(2)}_0 \).

Then by considering (11), if \( n \) is very large,
\[ (f_i^{(1)} | i = 0, \ldots, m_1 - 1), \]
\[ (f_j^{(2)} | j = 0, \ldots, m_1), \]
\[ (e_i - \alpha_i^1(ww^*) | i = 0, \ldots, m_0 - 1) \]
satisfy the conditions appearing in Remark 3 (for \( n = 1 \)). Hence \( \alpha \) has the Rohlin property.

3 Conjugacy

In this section we introduce three notions of conjugacy for \( \mathbb{Z}^N \)-actions on \( C^* \)-algebras and discuss their relationship. First we prepare some notations. For \( \mathbb{Z}^N \)-actions \( \alpha, \beta \) on a unital \( C^* \)-algebra \( A \), we write \( \alpha \gamma, \varepsilon \approx \beta \) when
\[ \| \alpha\xi - \gamma \circ \beta \xi \circ \gamma^{-1} \| \leq \varepsilon \]
for \( \varepsilon \geq 0 \) and \( \gamma \in \text{Aut}(A) \). For simplicity \( \gamma^0 \) will be denoted by \( \gamma \) or \( \gamma \). Recall that a 1-cocycle for \( \alpha \) means a mapping \( u \) from \( \mathbb{Z}^N \) into the unitaries \( U(A) \) of \( A \) satisfying \( u_{g+h} = u_g \alpha^g(u_h) \) for each \( g, h \in \mathbb{Z}^N \).

**Definition 7** Let \( \alpha \) and \( \beta \) be \( \mathbb{Z}^N \)-actions on a unital \( C^* \)-algebra \( A \).
(1) \( \alpha \) and \( \beta \) are **approximately conjugate** if for any \( \varepsilon > 0 \) there exists an automorphism \( \gamma \) of \( A \) such that \( \alpha \gamma, \varepsilon \approx \beta \).
(2) \( \alpha \) and \( \beta \) are **cocycle conjugate** if there exist an automorphism \( \gamma \) of \( A \) and a 1-cocycle \( u \) for \( \alpha \) such that
\[ \text{Ad } u_g \circ \alpha^g = \gamma \circ \beta^g \circ \gamma^{-1} \]
for each \( g \in \mathbb{Z}^N \).
(3) \( \alpha \) and \( \beta \) are **outer conjugate** if there exist an automorphism \( \gamma \) of \( A \) and unitaries \( u_1, \ldots, u_N \) in \( A \) such that
\[ \text{Ad } u_i \circ \alpha^i = \gamma \circ \beta^i \circ \gamma^{-1} \]
for \( i = 1, \ldots, N \).
Cocycle conjugacy of course implies outer conjugacy, and we have

**Proposition 8** Assume that $A$ is a simple separable unital $C^*$-algebra with a unique tracial state. Then approximately conjugate $\mathbb{Z}^N$-actions on $A$ are cocycle conjugate.

Our proof is based on the generalization of the determinant introduced by P. de la Harpe and G. Skandalis (see [14] for details) and the famous $2 \times 2$ matrix trick due to A. Connes (see [2]). We quickly review basic facts on the former. For a unital $C^*$-algebra $A$, we let $GL_n(A)$ the group of the invertible elements in the $n \times n$ matrices $M_n(A)$ over $A$ (equipped with the $C^*$-norm). The inductive limit of topological groups $(GL_n(A) | n \in \mathbb{N})$ with the usual embeddings $GL_n(A) \hookrightarrow GL_{n+1}(A)$ is denoted by $GL_\infty(A)$ and the connected component of the identity by $GL_0^\infty(A)$. Suppose that $\tau$ is a tracial state on $A$. If $\xi$ is a piecewise continuously differentiable mapping from $[0,1]$ into $GL_0^\infty(A)$, we define

$$\Delta_\tau(\xi) = \frac{1}{2\pi i} \int_0^1 \tau \left( \xi(t) \xi(t)^{-1} \right) dt$$

(note that the range of $\xi$ is contained in $GL_n(A)$ for some $n$ since $[0,1]$ is compact and that $\tau$ actually means $\tau \otimes \text{tr}$ on $A \otimes M_n = M_n(A)$). The determinant $\Delta_\tau$ associated with a tracial state $\tau$ is the mapping from $GL_\infty(A)$ into $\mathbb{C}/\tau^*(K_0(A))$ defined by

$$\Delta_\tau(x) = p(\Delta_\tau(\xi)).$$

Here $p$ is the quotient mapping from $\mathbb{C}$ onto $\mathbb{C}/\tau^*(K_0(A))$, and $\xi$ is a piecewise continuously differentiable mapping from $[0,1]$ into $GL_\infty(A)$ with $\xi(0) = 1$ and $\xi(1) = x$.

A crucial fact here is that $\Delta_\tau$ is a group homomorphism. For a unitary $x \in A$ with $\|x - 1\| < 1$, the logarithm $h = i^{-1} \log(x)$ (with the principal branch) makes sense and we can consider the path $\xi(t) = \exp(iht)$ from 1 to $x$. Since $\xi(t)\xi(t)^{-1} = ih$, we have

$$\Delta_\tau(x) = \frac{1}{2\pi i} p \left( \int_0^1 \tau(ih) dt \right) = \frac{1}{2\pi i} p(\tau(\log(x))).$$

Proof of Proposition [2]

Let $\alpha$ and $\beta$ be approximately conjugate $\mathbb{Z}^N$-actions on $A$. In general, if an automorphism of a simple unital $C^*$-algebra is close to the identity in norm then it is inner, and furthermore it is implemented by a unitary which is also close to the unit of the algebra. Hence for a sufficiently small $\varepsilon > 0$ there exist unitaries $u_1, \ldots, u_N$ in $A$ and an automorphism $\gamma$ of $A$ such that

$$\text{Ad} u_i \circ \alpha_{\xi_i} = \gamma \circ \beta_{\xi_i} \circ \gamma^{-1},$$

$$\|u_i - 1\| < \varepsilon$$
for \( i = 1, \ldots, N \). We want to show
\[
 u_k \alpha_{\xi_i}(u_l) = u_l \alpha_{\xi_i}(u_k) \tag{12}
\]
for any \( k, l = 1, \ldots, N \). From the commutativity of \( \alpha_{\xi_i} \), and the simplicity of \( A \), there exists \( \lambda \in T \) such that
\[
(u_l \alpha_{\xi_i}(u_k))^* u_k \alpha_{\xi_i}(u_l) = \lambda 1 .
\]
Since \( u_k, u_l \) are close to 1, we can set
\[
h_k = \frac{1}{2\pi i} \log(u_k), \quad h_l = \frac{1}{2\pi i} \log(u_l)
\]
and
\[
H(s) = \frac{1}{2\pi i} \log \{ (u_l^* \alpha_{\xi_i}(u_k))^* u_k^* \alpha_{\xi_i}(u_l) \}
\]
for \( s \in [0, 1] \). Applying \( \Delta_\tau \) to the both sides of the equality
\[
e^{-2\pi i \alpha_{\xi_i}(sh_k)} e^{-2\pi ish_l} e^{2\pi ish_k} e^{2\pi i \alpha_{\xi_i}(sh_l)} = e^{2\pi i H(s)},
\]
we have
\[
-p(\tau(\alpha_{\xi_i}(sh_k))) - p(\tau(sh_l)) + p(\tau(\alpha_{\xi_i}(sh_l))) = p(\tau(H(s))) .
\]
The uniqueness of a trace shows that the left hand side of the above equality is zero, i.e., \( \tau(H(s)) \in \tau_*(K_0(A)) \) for any \( s \in [0, 1] \). Since \( \tau_*(K_0(A)) \) is discrete in \( C \) and \( \tau(H(0)) = 0 \), we obtain \( H(1) = \tau(H(1)) = 0 \) i.e., \( \lambda = 1 \). Using these unitaries \( u_1, \ldots, u_N \), we construct a desired 1-cocycle by the method in [2]. We consider the \( \mathbb{Z}^N \)-action \( \sigma \) on \( M_2(A) \) defined by
\[
\sigma_{\xi_i} = Ad \begin{bmatrix} 1 & 0 \\ 0 & u_i \end{bmatrix} \circ \alpha_{\xi_i} .
\]
Since \( \sigma_{\xi_1}, \ldots, \sigma_{\xi_N} \) commute with each others from (12), \( \sigma \) is indeed well-defined. Note that
\[
\sigma_g \left( \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right) = \begin{bmatrix} \alpha_g(x) & 0 \\ 0 & \gamma \circ \beta_g \circ \gamma^{-1}(y) \end{bmatrix}
\]
for any \( g \in \mathbb{Z}^N \) and \( x, y \in A \). The identity
\[
\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]
shows
\[
\gamma \circ \beta_g \circ \gamma^{-1}(x) = Ad u_g \circ \alpha_g(x) ,
\]
where \( u_g \) is the desired 1-cocycle defined by
\[
\begin{bmatrix} 0 & 0 \\ u_g & 0 \end{bmatrix} = \sigma_g \left( \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right)
\]
(see [22 Lemma 8.11.2]) .
Remark 9 If $\alpha$ and $\beta$ are outer conjugate automorphisms of a UHF algebra with the Rohlin property, then they are approximately conjugate by the stability property. Hence the three notions of conjugacy defined above are the same for those automorphisms. However outer conjugacy does not imply approximate conjugacy for $\mathbb{Z}^N$-actions. See Remark 18 for a counter-example.

4 Product type actions

In this section we discuss product type $\mathbb{Z}^2$-actions on UHF algebras. As in the case of single automorphisms, the Rohlin property for these actions is closely related to a notion of uniform distribution of points in $\mathbb{T}^2$. First we say the $N$-dimensional version of [1, Lemma 4.1]. This can be shown as in the one-dimensional case, so we omit the proof.

Proposition 10 Let $(S_k \mid k \in \mathbb{N})$ be a sequence of finite sequences in $\mathbb{T}^N$ i.e.,

$$S_k = (s_{k,p} \mid p = 1, \ldots, n_k),$$

$s_{k,p} \in \mathbb{T}^N$ for each $k \in \mathbb{N}$ and $p = 1, \ldots, n_k$. Then the following conditions on $(S_k \mid k \in \mathbb{N})$ are equivalent.

1. $$\lim_{k\to\infty} \frac{1}{n_k} \sum_{p=1}^{n_k} f(s_{k,p}) = \int_{\mathbb{T}^n} f(s) ds$$

for any $f \in C(\mathbb{T}^N)$, where $ds$ denotes the normalized Haar measure on $\mathbb{T}^N$.

2. $$\lim_{k\to\infty} \frac{1}{n_k} \sum_{p=1}^{n_k} s_{k,l,p} = 0$$

for any $l = (l_1, \ldots, l_N) \in \mathbb{Z}^N \setminus \{0\}$, where $s^l = s_{1}^{l_1} \cdots s_{N}^{l_N}$ for each $s = (s_1, \ldots, s_N) \in \mathbb{T}^N$.

3. $$\lim_{k\to\infty} \frac{1}{n_k} \nu_k \left( \prod_{i=1}^{N} (\theta_1^{(i)}, \theta_2^{(i)}) \right) = (2\pi)^{-N} \prod_{i=1}^{N} (\theta_2^{(i)} - \theta_1^{(i)})$$

for any $0 \leq \theta_1^{(i)} \leq \theta_2^{(i)} < 2\pi$, where $\nu_k$ is defined by

$$\nu_k(S) = \sharp \{ p \mid 1 \leq p \leq n_k \text{ and } \text{arg}(s_{k,p}) \in S \}$$

for each subset $S$ of $\prod_{i=1}^{N} [0, 2\pi)$ and $\sharp F$ denotes the cardinality of the set $F$. 
Since there exist eigenvalues \( \lambda \) belonging to \( u \) such that for any positive integer \( k \) there exists positive integer \( n \) such that \( n \geq n_0 \) for any \( k \). Furthermore, for any \( n \in \mathbb{N} \), there exists an bijection \( \varphi \) from \( \{1, \ldots, n_k\} \) onto \( \{1, \ldots, n_k^{(1)}\} \times \cdots \times \{1, \ldots, n_k^{(N)}\} \) such that

\[
\left| s_{k,p} - \left( \exp \left( 2\pi i \cdot \frac{(\varphi(p))_1}{n_k^{(1)}} \right), \ldots, \exp \left( 2\pi i \cdot \frac{(\varphi(p))_N}{n_k^{(N)}} \right) \right) \right| < \varepsilon \tag{13}
\]

for any \( k \) and \( p \), where \( |s| = \max\{|s_p| : 1 \leq p \leq N\} \) for each \( s \in \mathbb{T}^N \) and \( (\varphi(p))_i \) denotes the \( i \)-th component of \( \varphi(p) \).

If \( S_k \) satisfies the estimate \((3)\) for some \( \varphi \) as above, then \( S_k \) is said to be \( (n_k^{(1)}, \ldots, n_k^{(N)}; \varepsilon) \)-distributed. If one of the conditions of the above proposition holds then \( (S_k | k \in \mathbb{N}) \) is said to be uniformly distributed.

**Definition 11** Let \( \alpha \) be a \( \mathbb{Z}^N \)-action on a UHF algebra \( A \). Then \( \alpha \) is said to be a **product type action** if there exists a sequence \( (m_k | k \in \mathbb{N}) \) of positive integers such that \( A \cong \otimes_{k=1}^{\infty} M_{m_k} \) and

\[
\alpha_g(A_k) = A_k
\]

for any \( g \in \mathbb{Z}^N \) and \( k \in \mathbb{N} \), where \( A_k \) denotes the \( C^* \)-subalgebra of \( A \) corresponding to \( M_{m_k} \otimes (\otimes_{l \neq k} \mathbb{C}1_{m_l}) \).

**Remark 12** In the situation above, if \( N = 2 \), then one finds unitaries \( u_k^{(1)}, u_k^{(2)} \) in \( A_k \) and \( \lambda_k \in \mathbb{T} \) such that

\[
\alpha_{(p,q)}[A_k] = Ad u_k^{(1)p} u_k^{(2)q}, \quad u_k^{(1)} u_k^{(2)} = \lambda_k u_k^{(2)} u_k^{(1)}
\]

for any \( p, q \in \mathbb{Z} \). Since \( u_k^{(1)}, u_k^{(2)} \) are unique up to a constant multiple, \( \lambda_k \) is unique. In addition \( \lambda_k^{m_k} = 1 \). For if \( \mu_1 \) is an eigenvalue of \( u_k^{(2)} \) with multiplicity \( r_1 \) then \( \mu_1 \lambda_k^{-p} \) is also an eigenvalue of \( u_k^{(2)} \) with multiplicity \( r_1 \) for each \( p \in \mathbb{N} \). Since \( M_{m_k} \) is finite-dimensional, there exists \( p_0 \in \mathbb{N} \) such that \( \lambda_k^{p_0} = 1 \) and \( \lambda_k^{p} \neq 1 \) for any \( p = 1, \ldots, p_0 - 1 \). If \( \{\mu_1 \lambda_k^{-p} | p = 0, \ldots, p_0 - 1\} \) does not exhaust all the eigenvalues of \( u_k^{(2)} \) then we take an eigenvalue \( \mu_2 \) of \( u_k^{(2)} \) not belonging to \( \{\mu_1 \lambda_k^{-p} | p = 0, \ldots, p_0 - 1\} \) and repeat the same process. Thus there exist eigenvalues \( \mu_1, \ldots, \mu_s \) of \( u_k^{(2)} \) with multiplicity \( r_1, \ldots, r_s \) respectively. Since \( m_k = (r_1 + \cdots + r_s)p_0 \), it follows that \( \lambda_k^{m_k} = 1 \).
For \( n \times n \) unitary matrices \( U \) and \( V \) with \( U V = V U \), we define \( \text{Sp}(U) \) to be a sequence consisting of the eigenvalues of \( U \), each repeated as often as multiplicity dictates and \( \text{Sp}(U, V) \) is a sequence consisting of the pairs of eigenvalues of \( U \) and \( V \) with a common eigenvector, each repeated as often as multiplicity dictates. Then the Rohlin property for the product type \( \mathbb{Z}^2 \)-actions on \( A \) with \( \lambda_k = 1 \) can be characterized as follows.

**Proposition 13** Let \( \alpha \) be a product type \( \mathbb{Z}^2 \)-action on a UHF algebra \( A \) with \( (m_k \mid k \in \mathbb{N}), (u_k^{(1)} \mid k \in \mathbb{N}), (u_k^{(2)} \mid k \in \mathbb{N}), (\lambda_k \mid k \in \mathbb{N}) \) as above. If \( \lambda_k = 1 \) for each \( k \in \mathbb{N} \) then the following conditions are equivalent:

1. \( \alpha \) has the Rohlin property.
2. \( (\text{Sp}(\bigotimes_{k=m}^n u_k^{(1)}), \bigotimes_{k=m}^n u_k^{(2)}) \mid n = m, m+1, \ldots) \) is uniformly distributed for any \( m \in \mathbb{N} \).

**Proof.** By Corollary \[1\] (1) is equivalent to the condition: \( \alpha_p^p \alpha_q^q \) has the Rohlin property as a single automorphism of \( A \) for each \( (p, q) \in \mathbb{Z}^2 \setminus \{0\} \). By \[16, Lemma 5.2\] this condition is equivalent to the condition:

\[
(\text{Sp}\left(\bigotimes_{k=m}^n u_k^{(1)}u_k^{(2)}\bigg| n = m, m+1, \ldots\right))
\]

is uniformly distributed in \( T \). By Proposition \[10\] for \( N = 1 \), the last condition is equivalent to the condition: for any \( m \in \mathbb{N} \)

\[
\lim_{n \to \infty} \frac{1}{N(m, n)} \sum_{(\lambda_1, \lambda_2) \in \text{Sp}(\bigotimes_{k=m}^n u_k^{(1)}; \bigotimes_{k=m}^n u_k^{(2)})} \lambda_1^p \lambda_2^q = 0,
\]

where \( N(m, n) \equiv \prod_{k=m}^n m_k \). Finally by Proposition \[10\] for \( N = 2 \), the last condition is equivalent to (2).

In \[16\] A.Kishimoto showed the following for a UHF algebra \( A \).

1. Any two product type \( \mathbb{Z} \)-actions on \( A \) with the Rohlin property are approximately conjugate.
2. For any \( \mathbb{Z} \)-action \( \alpha \) on \( A \) with the Rohlin property and \( \varepsilon > 0 \), there exist a product type \( \mathbb{Z} \)-action \( \beta \) on \( A \) with the Rohlin property and an automorphism \( \gamma \) of \( A \) such that \( \alpha \overset{\gamma}{\approx} \beta \).

In particular there is one and only one approximate conjugacy class of \( \mathbb{Z} \)-actions on \( A \) with the Rohlin property. In the case of \( N = 2 \) we do not know whether (2) is valid or not. In the rest of this section we state several versions of (1) for \( \mathbb{Z}^2 \).
Theorem 14 Let $\alpha$ and $\beta$ be product type $\mathbb{Z}^2$-actions on a UHF algebra $A$ with the Rohlin property. Let $\alpha$ be determined by $(m_k \mid k \in \mathbb{N})$, $(\lambda_k \mid k \in \mathbb{N})$ as in Definition 7 and Remark 12, and $\beta$ by $(n_l \mid l \in \mathbb{N})$, $(\mu_l \mid l \in \mathbb{N})$. If $\lambda_k = \mu_l = 1$ for each $k, l \in \mathbb{N}$, then $\alpha$ and $\beta$ are approximately conjugate.

Proof. By patching several parts of the $M_{m_k}$'s and the $M_{n_l}$'s respectively there exist a sequence $(N_k \mid k \in \mathbb{N})$ of integers satisfying $N_0 = 0$ and $N_k > 0$ ($k \geq 1$) and sequences $(U_k^{(1)} \mid k \in \mathbb{N})$, $(U_k^{(2)} \mid k \in \mathbb{N})$ of unitary matrices such that

$$U_k^{(i)} \in U(M_{N_k} \otimes M_{N_{k+1}}),$$

$$U_k^{(1)}U_k^{(2)} = U_k^{(2)}U_k^{(1)} \ (i = 1, 2, k \in \mathbb{N}),$$

$$(A, \alpha_{\xi_1}, \alpha_{\xi_2}) \cong \left( \bigotimes_{k=0}^{\infty} M_{N_k}, \bigotimes_{k=0}^{\infty} Ad U_{2k}^{(1)}, \bigotimes_{k=0}^{\infty} Ad U_{2k}^{(2)} \right),$$

$$(A, \beta_{\xi_1}, \beta_{\xi_2}) \cong \left( \bigotimes_{k=0}^{\infty} M_{N_k}, \bigotimes_{k=0}^{\infty} Ad U_{2k+1}^{(1)}, \bigotimes_{k=0}^{\infty} Ad U_{2k+1}^{(2)} \right).$$

Let $\varepsilon > 0$. Since $(\text{Sp} \left( \bigotimes_{k=0}^{n} Ad U_{2k+1}^{(1)}, \bigotimes_{k=0}^{n} Ad U_{2k+1}^{(2)} \right) \mid n \in \mathbb{N})$ is uniformly distributed by Proposition 13 and $U_0^{(1)}U_0^{(2)} = U_0^{(2)}U_0^{(1)}$, for a sufficiently large $n$ there exist a unitary $W_1$ in $\bigotimes_{k=0}^{2n+2} M_{N_k}$ and unitaries $V_1^{(1)}, V_1^{(2)}$ in $\bigotimes_{k=2}^{2n+2} M_{N_k}$ such that

$$\|W_1 \left( \bigotimes_{k=0}^{n} U_{2k+1}^{(1)} \right) W_1^* - U_0^{(1)} \otimes V_1^{(1)} \| < 2^{-2\varepsilon},$$

$$\|W_1 \left( \bigotimes_{k=0}^{n} U_{2k+1}^{(2)} \right) W_1^* - U_0^{(2)} \otimes V_1^{(2)} \| < 2^{-2\varepsilon}.$$
Theorem 15

then the following are equivalent:

there is an action \( \alpha \)

\( \lambda \) a relation in \( \prod (2) \)

the Rohlin property.

We let \([\alpha, \lambda] \) be the trivial sequence \( (0) \) of \( p \) increasing order. For a sequence \((\gamma_k)\) where \( \gamma \) classes of UHF algebras. Let \((\alpha_k, \beta_k, \gamma_k)\) be actions in the above class and satisfy the Rohlin property, if \( \alpha \) and \( \gamma \) in such a way that

\[
\| \text{Ad} W_k(U_k^{(1)}) - V_k^{(1)} \otimes V_k^{(1)} \| < 2^{-(k+1)} \varepsilon ,
\]

Thus

\[
(A, \alpha, \lambda) \equiv \left( \bigotimes_{k=0}^{\infty} M_{N_k}, \bigotimes_{k=0}^{\infty} \text{Ad} U_{2k}^{(1)}, \bigotimes_{k=0}^{\infty} \text{Ad} U_{2k}^{(2)} \right)
\]

\[
\gamma \equiv \bigotimes_{k=1}^{\infty} \text{Ad} W_{2k}
\]

where \( \gamma \equiv \bigotimes_{k=1}^{\infty} \text{Ad} W_{2k} \) and

\[
(A, \beta, \beta') \equiv \left( \bigotimes_{k=0}^{\infty} M_{N_k}, \bigotimes_{k=0}^{\infty} \text{Ad} U_{2k}^{(1)}, \bigotimes_{k=0}^{\infty} \text{Ad} U_{2k}^{(2)} \right)
\]

\[
\gamma' \equiv \bigotimes_{k=0}^{\infty} \text{Ad} W_{2k+1}
\]

where \( \gamma \equiv \bigotimes_{k=0}^{\infty} \text{Ad} W_{2k+1} \). This completes the proof.

As mentioned in the introduction, we discuss product type actions for two classes of UHF algebras. Let \((p_k | k \in \mathbb{N})\) be the prime numbers in the increasing order. For a sequence \((i_k | k \in \mathbb{N})\) of nonnegative integers with \( \sum_{k=1}^{\infty} i_k = \infty \), put \( q_k = p_k^{i_k} \) and let \( A = \bigotimes_{k=1}^{\infty} M_{q_k} \). We regard \( M_{q_k} \) as a \( C^* \)-subalgebra of \( A \). We consider the class of product type \( \mathbb{Z}^2 \)-actions \( \alpha \) on this \( A \). Assume that \( \alpha \) looks like

\[
\alpha_{(p,q)}[M_{q_k}] = \text{Ad} u_k^{(1)} u_k^{(2)}
\]

on \( M_{q_k} \) with unitaries \( u_k^{(1)}, u_k^{(2)} \) in \( M_{q_k} \) and \( \lambda_k \in T \) satisfying \( u_k^{(1)} u_k^{(2)} = \lambda_k u_k^{(2)} u_k^{(1)} \). Since \( \lambda_k = 1 \), we may regard \( \lambda_k \) as an element of \( G_k \equiv \mathbb{Z}/q_k \mathbb{Z} \).

We let \([\alpha]\) be the sequence \( (\lambda_k | k \in \mathbb{N}) \) in \( \prod_{k=1}^{\infty} G_k \). We define an equivalence relation in \( \prod_{k=1}^{\infty} G_k \) by: \( g \sim h \) if there is an \( n \) such that \( g_k = h_k \) for all \( k \geq n \). Let \( 0 \) be the trivial sequence \( (0,0,\ldots) \). We note that for every \( g \in \prod_{k=1}^{\infty} G_k \) there is an action \( \alpha \) in the above class with \([\alpha] = g\).

**Theorem 15** (1) If \( \alpha \) is an action in the above class and \([\alpha] \neq 0\), then \( \alpha \) has the Rohlin property.

(2) If \( \alpha \) and \( \beta \) are actions in the above class and satisfy the Rohlin property, then the following are equivalent:

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(2.1) $[\alpha] \sim [\beta].$

(2.2) $\alpha$ and $\beta$ are outer conjugate.

Before proving Theorem 15 we introduce some notations and prepare a lemma. For a positive integer $n$ and $\lambda \in T$ with $\lambda^n = 1$, we define the $n \times n$ unitary matrices $S(n)$ and $\Omega(n, \lambda)$ by

$$S(n) = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & 0 \end{bmatrix}, \quad \Omega(n, \lambda) = \begin{bmatrix} \lambda \\ \vdots \\ \lambda^{n-1} \end{bmatrix}.$$

**Lemma 16** Suppose that $U$ and $V$ are $n \times n$ unitary matrices such that

$$UV = \exp(2\pi ik/n) VU$$

for some $k \in \mathbb{N}$. Let $q/p$ be the irreducible form of $k/n$. Then there exist $\omega_i, \mu_i \in T$ ($i = 1, \ldots, n/p$) such that $(U, V)$ is conjugate to $(U_1 \otimes U_2, V_1 \otimes V_2)$, where

$$\lambda = \exp(2\pi ik/n),$$

$$U_1 = S(p), \quad V_1 = \Omega(p, \lambda^{-1}),$$

$$U_2 = \bigoplus_{i=1}^{n/p} \omega_i, \quad V_2 = \bigoplus_{i=1}^{n/p} \mu_i.$$

Moreover each $\omega_i, \mu_i$ are unique up to multiples of powers of $\lambda$.

**Proof.** Since $U^p V = V U^p$ we have a complete orthonormal system of $\mathbb{C}^n$ consisting of the common eigenvectors of $U^p$ and $V$. We take such a system $(\xi_{(\kappa, \mu)})|_{(\kappa, \mu)}$ i.e.,

$$U^p \xi_{(\kappa, \mu)} = \kappa \xi_{(\kappa, \mu)},$$

$$V^p \xi_{(\kappa, \mu)} = \mu \xi_{(\kappa, \mu)}.$$  

Then if $\omega^p = \kappa$, the space spanned by

$$\xi_{(\kappa, \mu)}, \quad \omega U \xi_{(\kappa, \mu)}, \quad \ldots, \quad \omega^{p-1} U^{p-1} \xi_{(\kappa, \mu)}$$

is invariant under $U, V$, and the matrix representation of $(U, V)$ with respect to the above basis is $(\omega U_1, \mu V_1)$. Thus $(U, V)$ is conjugate to the direct sum of $(\omega U_1, \mu V_1)$ for some sequences $\omega_i, \mu_i$ in $T$. Since $(\omega U_1, \mu V_1)$ is conjugate to $(\omega \lambda^k U_1, \mu \lambda^j V_1)$ for all $k, j$, the last statement is obvious.

Proof of Theorem 15 (1)
Let $\alpha$ be given as in the theorem. Take unitaries $u_k^{(1)}, u_k^{(2)}$ in $M_{q_k}$ and $\lambda_k \in T$ as in (14). By Corollary [4] it suffices to prove that $\alpha_{(p,q)}$ has the Rohlin property as a single automorphism for each $(p, q) \in \mathbb{Z}^2 \setminus \{0\}$. From the assumption there is a subsequence $(p_k | n \in \mathbb{N})$ of $(p_k | k \in \mathbb{N})$ such that $\lambda_k \neq 1$ for any $n$. Applying Lemma 16 to $u_k^{(1)}, u_k^{(2)}$ we have a decomposition 

$$(u_k^{(1)}, u_k^{(2)}_{k,1} \otimes u_k^{(2)}_{k,2})$$

of $(u_k^{(1)}, u_k^{(2)})$ up to conjugacy, where

$$u_k^{(1)}_{k,1} \equiv S(p_k^n), \quad u_k^{(2)}_{k,1} \equiv \Omega(p_k^n, \lambda_k^{-1})$$

for some $1 \leq j \leq i_k$. Then it is easy to see that

$$\left(\text{Sp}\left( u_{k,1}^{(1)} u_{k,2}^{(2)} \right) | \ n \in \mathbb{N} \right)$$

is uniformly distributed in $T$. This ensures that

$$\left(\text{Sp}\left( \bigotimes_{k=0}^l u_k^{(1)} u_k^{(2)} \right) \bigg| l \geq k_0 \right)$$

is also uniformly distributed for all $k_0$. Hence $\alpha_{(p,q)} = \otimes_{k=1}^{\infty} \text{Ad} u_k^{(1)} u_k^{(2)}$ has the Rohlin property by [16, Lemma 5.2].

To prove Theorem 15 (2) we introduce an invariant as a slight generalization of that in [4].

**Definition 17** For $n \times n$ unitary matrices $U, V$ and $\lambda \in T$ with $\lambda^n = 1$, if $\|\lambda UV - VU\| < 2$ then the closed complex path $\gamma(t) = \text{det}((1 - t)\lambda UV + tVU)$ ($t \in [0, 1]$) does not go through zero. We define $\omega_\lambda(U, V)$ as the winding number of the path $\gamma$ around zero.

From $\|\lambda UV - VU\| < 2$ we can define $\log(\lambda^{-1}VUV^*U^*)$, with log the principal branch of the logarithm. As is shown in [10, Lemma 3.1]

$$\omega_\lambda(U, V) = \frac{1}{2\pi i} \text{Tr}(\log(\lambda^{-1}VUV^*U^*))$$

with the nonnormalized trace $\text{Tr}$ on $M_n$.

**Proof of Theorem 15 (2).**

Take $u_k^{(1)}, u_k^{(2)} \in U(M_{q_k})$, $\lambda_k \in T$ as in (14) for $\alpha$, and $v_k^{(1)}, v_k^{(2)} \in U(M_{q_k})$, $\mu_k \in T$ for $\beta$ similarly, i.e.,

$$\beta_{(p,q)}|M_{q_k} = \text{Ad} v_k^{(1)} v_k^{(2)},$$

$$v_k^{(1)} v_k^{(2)} = \mu_k v_k^{(2)} v_k^{(1)}.$$
First we show that (2.2) implies (2.1). To get a contradiction we assume that outer conjugate $\alpha, \beta$ satisfy $[\alpha] \not\sim [\beta]$. Outer conjugacy means

$$Ad W_i \circ \alpha_{\xi_i} = \gamma^{-1} \circ \beta_{\xi_i} \circ \gamma \quad (i = 1, 2)$$

for some unitaries $W_1, W_2 \in A$ and an automorphism $\gamma$ of $A$. For any $\varepsilon > 0$ we have a positive integer $M$ and unitaries $W'_1, W'_2$ in $\bigotimes_{k=1}^M M_{q_k}$ with $\|W_i - W'_i\| < \varepsilon$ ($i = 1, 2$) so that

$$\|Ad W'_i \circ \alpha_{\xi_i} - \gamma^{-1} \circ \beta_{\xi_i} \circ \gamma\| < 2\varepsilon.$$ 

By the assumption there exists a positive integer $K > M$. Further take a sufficiently large $N > K$ such that

$$\gamma(M_{q_K}) \subseteq_\varepsilon M_{q_1} \otimes \cdots \otimes M_{q_N},$$

where $X \subseteq_\varepsilon Y$ means that for any $x \in X$ there is $y \in Y$ satisfying $\|x - y\| \leq \varepsilon \|x\|$. Here we use the perturbation theorem [3, Corollary 6.8], that is, for a sufficiently small $\varepsilon > 0$ we have a unitary $w_1$ in $A$ such that

$$Ad w_1 \circ \gamma(M_{q_K}) \subseteq M_{q_1} \otimes \cdots \otimes M_{q_N}$$

and $\|w_1 - 1\| < 28\varepsilon^{\frac{1}{2}}$. Set $\gamma_1 = Ad w_1 \circ \gamma$, $B_1 = \gamma_1(M_{q_K})$ and $B_2 = M_{q_1} \otimes \cdots \otimes M_{q_N} \cap B_1'$. Then

$$M_{q_1} \otimes \cdots \otimes M_{q_N} = B_1 \otimes B_2,$$

$$\gamma_1 \circ Ad W'_1 \circ \alpha_{\xi_i} \circ \gamma_1^{-1}(B_1) = B_1,$$

$$\gamma_1 \circ Ad W'_1 \circ \alpha_{\xi_i} \circ \gamma_1^{-1}(B_1) = Ad \gamma_1(u_{q_k}^{(i)})$$

for $i = 1, 2$. Since $\|Ad W'_1 \circ \alpha_{\xi_i} \circ \gamma_1^{-1} \circ \beta_{\xi_i} \circ \gamma_1\| \leq C_1 \varepsilon^{\frac{1}{2}}$ for some positive constant $C_1$ independent of $\varepsilon$ we have

$$\beta_{\xi_i}(B_1) \subseteq_{C_1 \varepsilon^{\frac{1}{2}}} B_1.$$ 

Noting that $B_1, \beta_{\xi_i}(B_1) \subseteq M_{q_1} \otimes \cdots \otimes M_{q_N}$, we can use [3, Corollary 6.8] again. So we have a unitary $w_2$ in $M_{q_1} \otimes \cdots \otimes M_{q_N}$ such that

$$Ad w_2 \circ \beta_{\xi_i}(B_1) \subseteq B_1,$$

$$\|w_2 - 1\| < C_2 \varepsilon^{\frac{1}{2}}$$

for some positive constant $C_2$. As $M_{q_1} \otimes \cdots \otimes M_{q_N}$ is finite-dimensional and

$$Ad w_2 \circ \beta_{\xi_i}(B_1) = B_1,$$

$$Ad w_2 \circ \beta_{\xi_i}(M_{q_1} \otimes \cdots \otimes M_{q_N}) = M_{q_1} \otimes \cdots \otimes M_{q_N},$$

we have unitaries $U_1$ in $B_1$ and $U_2$ in $B_2$ such that

$$Ad w_2 \circ \beta_{\xi_i}[B_1 = Ad U_1.$$
\[ Ad w_2 \circ \beta_{\xi_1} [B_2] = Ad U_2. \]

For these unitaries we have the following estimates:

\[
\| (Ad \gamma_1(u_K^{(1)}) - Ad U_1) \|_{B_1} = \| (\gamma_1 \circ Ad W'_1 \circ \alpha_{\xi_1} \circ \gamma_1^{-1} - Ad w_2 \circ \beta_{\xi_1}) \|_{B_1} \\
\leq \| \gamma_1 \circ Ad W'_1 \circ \alpha_{\xi_1} \circ \gamma_1^{-1} - Ad w_2 \circ \beta_{\xi_1} \| \\
\leq C_4 \varepsilon^\frac{1}{4},
\]

\[
\| Ad(U_1 \otimes U_2) - \bigotimes_{k=1}^N Ad v_k^{(1)} \| \leq \| Ad w_2 \circ \beta_{\xi_1} - \beta_{\xi_1} \| \\
\leq C_4 \varepsilon^\frac{1}{4}
\]

for some positive constant \( C_4 \). Thus we obtain scalars \( \eta_1, \eta_2 \) of \( T \) such that

\[
\| \gamma_1(u_K^{(1)}) - \eta_1 U_1 \| < 4C_4 \varepsilon^\frac{1}{4},
\]

\[
\| U_1 \otimes U_2 - \eta_2 \bigotimes_{k=1}^N v_k^{(1)} \| \leq 4C_4 \varepsilon^\frac{1}{4}.
\]

Consequently we have

\[
\| \gamma_1(u_K^{(1)}) \otimes U_2 - \eta \bigotimes_{k=1}^N v_k^{(1)} \| < C_5 \varepsilon^\frac{1}{4}
\]

for some \( \eta \in T \) and positive constant \( C_5 \). Similarly for the direction of \( \xi_2 \) we obtain a unitary \( V_2 \) in \( B_2 \) such that

\[
\| \gamma_1(u_K^{(2)}) \otimes V_2 - \zeta \bigotimes_{k=1}^N v_k^{(2)} \| < C_6 \varepsilon^\frac{1}{4}
\]

for some \( \zeta \in T \) and positive constant \( C_6 \). To use an invariant in Definition 17, we set \( \mu = \prod_{k=1}^N \mu_k \). Then there is a \( \lambda \in T \) such that \( \lambda^{q_1 \cdots q_N} = 1 \) and

\[
|\lambda \mu - 1| + 2(C_5 + C_6)\varepsilon^\frac{1}{4} < 2.
\]

Note that \( \omega_\lambda \) is invariant under homotopy of unitaries for which \( \omega_\lambda \) is defined. From the above estimates we have

\[
\omega_\lambda(\gamma_1(u_K^{(1)}) \otimes U_2, \gamma_1(u_K^{(2)}) \otimes V_2) = \omega_\lambda(\eta \bigotimes_{k=1}^N v_k^{(1)}, \zeta \bigotimes_{k=1}^N v_k^{(2)})
\]

for a sufficiently small \( \varepsilon > 0 \). We now evaluate the both sides to get a contradiction. Let

\[
\lambda_k = \exp(2\pi i \cdot \frac{sk}{q_k}), \ \mu_k = \exp(2\pi i \cdot \frac{tk}{q_k}),
\]

for a sufficiently small \( \varepsilon > 0 \). We now evaluate the both sides to get a contradiction. Let

\[
\lambda_k = \exp(2\pi i \cdot \frac{sk}{q_k}), \ \mu_k = \exp(2\pi i \cdot \frac{tk}{q_k}),
\]
\[\lambda = \exp(2\pi i \cdot \frac{s}{q_1 \cdots q_N})\]

for some \(s_k, t_k \in \{0, \ldots, q_k - 1\}\) and \(s \in \{0, \ldots, (q_1 \cdots q_{N-1}) - 1\}\). Then

\[
\lambda^{-1}(\gamma_1(u_K^{(2)}) \otimes V_2)(\gamma_1(u_K^{(1)}) \otimes U_2)(\gamma_1(u_K^{(2)}) \otimes V_2)^*(\gamma_1(u_K^{(1)}) \otimes U_2)^* = 1_{q_k} \otimes \exp\left\{2\pi i \left(\frac{-s}{q_1 \cdots q_N} + \frac{-s_k}{q_k}\right)\right\} V_2 U_2^* U_2^* = \exp\left(2\pi i \left(\bigoplus_{j=1}^{q_k} \left(\frac{-s}{q_1 \cdots q_N} + \frac{-s_k}{q_k}\right) 1_{q_1 \cdots q_{K-1} q_{K+1} \cdots q_N} + H_2\right)\right)
\]

for some \(H_2 \in M_{q_1 \cdots q_{K-1} q_{K+1} \cdots q_N}\) with \(\text{Tr}(H_2) \in \mathbb{Z}\). Thus

\[
\omega_\lambda(\gamma_1(u_K^{(1)}) \otimes U_2, \gamma_1(u_K^{(2)}) \otimes V_2) = \text{Tr}\left(\bigoplus_{j=1}^{q_k} \left(\frac{-s}{q_1 \cdots q_N} + \frac{-s_k}{q_k}\right) 1_{q_1 \cdots q_{K-1} q_{K+1} \cdots q_N} + H_2\right) = -s - s_k q_1 \cdots q_{K-1} q_{K+1} \cdots q_N + q_k \text{Tr}(H_2).
\]

On the other hand

\[
\omega_\lambda\left(\prod_{k=1}^{N} j_k^{(1)}(n_1) \otimes \prod_{k=1}^{N} j_k^{(2)}(n_2)\right) = \left(\frac{-s}{q_1 \cdots q_N} + \sum_{k=1}^{N} \frac{-t_k}{q_k} + n\right) \left(\prod_{l=1}^{N} q_l\right) = -s - \sum_{k=1}^{N} t_k q_1 \cdots q_k q_{k+1} \cdots q_N + n q_1 \cdots q_N
\]

for some \(n \in \mathbb{Z}\). Therefore

\[
(s_K - t_K) q_1 \cdots q_{K-1} q_{K+1} \cdots q_N = q_k \text{Tr}(H_2) + \sum_{k=1 \atop k \neq K}^{N} \frac{-t_k}{q_k} + n q_1 \cdots q_N.
\]

Noting that \(s_K - t_K \neq 0\) and \(s_K - t_K\) is not divided by \(q_K\), we have a contradiction.

Next we show that (2.1) implies (2.2). We assume \([\alpha] \sim [\beta]\). Then we may also assume that \(\lambda_k = \mu_k\) for any \(k\) since inner perturbation does not change outer conjugacy classes. If there is a \(k_0 \in \mathbb{N}\) such that \(\lambda_k = 1\) for any \(k \geq k_0\), then we have the result from Theorem [4]. If there is no such \(k_0\), we pick up all the \(k\)’s with \(\lambda_k \neq 1\) and make the subsequence \((p_n \mid n \in \mathbb{N})\) of \((p_k \mid k \in \mathbb{N})\). Let \(\lambda_k = \exp(2\pi i s_k/q_k)\) as before. Then for any \(N\)

\[
\prod_{k=1}^{k_N} \lambda_k = \exp\left(2\pi i \cdot \sum_{n=1}^{N} \frac{s_{k_n}}{q_{k_n}}\right).
\]

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By noting that \( s_{k_n} \neq 0 \) and \( q_{k_n} \)'s are relatively prime to each other, \( \sum_{n=1}^{N} s_{k_n}/q_{k_n} \) equals to
\[
\frac{S_N}{p_{k_1}^{j_1} \cdots p_{k_N}^{j_N}}
\]
in the irreducible form for some positive integers \( j_1, \ldots, j_N \) and \( S_N \). Here we apply Lemma 16 to two pairs \( (\otimes_{k=1}^{N} u_k^{(1)}, \otimes_{k=1}^{N} u_k^{(2)}), (\otimes_{k=1}^{N} v_k^{(1)}, \otimes_{k=1}^{N} v_k^{(2)}) \) of unitaries. Then for any \( \varepsilon > 0 \), if we take a sufficiently large \( N \), these pairs are almost conjugate i.e., there is a unitary \( w_1 \) in \( \otimes_{k=1}^{N} M_{q_k} \) such that
\[
\| \text{Ad } w_1 \left( \otimes_{k=1}^{N} u_k^{(i)} \right) - \otimes_{k=1}^{N} v_k^{(i)} \| < 2^{-1}\varepsilon
\]
for \( i = 1, 2 \). We adopt the same method for \( \otimes_{k=1}^{N+1} M_{q_k} \) and \( 2^{-1}\varepsilon \) in place of \( \varepsilon \). Repeating this procedure as in the proof of Theorem 14, we have the result.

Remark 18 Let \( A = M_3 \otimes M_2^{\infty} \) and let \( \omega_n = \exp(2\pi i/n) \) for each \( n \in \mathbb{N} \). We define \( \mathbb{Z}^2 \)-actions \( \alpha, \beta \) on \( A \) by
\[
\alpha_{\xi_1} = \text{Ad } \Omega(3, \omega_3) \otimes \left( \bigotimes_{k=1}^{\infty} \text{Ad } \Omega(2^k, \omega_{2^k}) \right), \quad \alpha_{\xi_2} = \text{Ad } S(3) \otimes \left( \bigotimes_{k=1}^{\infty} \text{Ad } S(2^k) \right),
\]
\[
\beta_{\xi_1} = \text{id}_{M_3} \otimes \left( \bigotimes_{k=1}^{\infty} \text{Ad } \Omega(2^k, \omega_{2^k}) \right), \quad \beta_{\xi_1} = \text{id}_{M_3} \otimes \left( \bigotimes_{k=1}^{\infty} \text{Ad } S(2^k) \right).
\]
Then the same arguments as in the first part of the above proof show that \( \alpha, \beta \) have the Rohlin property and they are not approximately conjugate. However they are clearly outer conjugate.

Let \( \{ q_k \mid k \in K \} \) be a finite or infinite set of prime numbers. We next consider product type \( \mathbb{Z}^2 \)-actions on the UHF algebra
\[
\bigotimes_{k \in K} M_{q_k}^{\infty},
\]
where \( M_{q_k}^{\infty} \) is understood as \( \otimes_{n=1}^{\infty} M_{q_k} \).

Theorem 19 For the above UHF algebra, any two product type \( \mathbb{Z}^2 \)-actions with the Rohlin property are approximately conjugate.

Proof. From Theorem 14 it is enough to prove that for any product type \( \mathbb{Z}^2 \)-action \( \alpha \) on \( A \) with the Rohlin property and \( \varepsilon > 0 \), there exist an automorphism \( \gamma \) of \( A \) and a product type action \( \beta \) with the Rohlin property such that \( \alpha \approx \beta \).

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and $\beta$ has the same form as in Theorem 14, i.e., there exist a sequence $(n_l | l \in \mathbb{N})$ of positive integers and sequences $(v_l^{(1)} | l \in \mathbb{N}), (v_l^{(2)} | l \in \mathbb{N})$ of unitary matrices such that

$$v_l^{(1)}, v_l^{(2)} \in M_{n_l},$$

$$v_l^{(1)} v_l^{(2)} = v_l^{(1)} v_l^{(2)},$$

$$(A, \beta_{\xi_1}, \beta_{\xi_2}) \equiv \left( \bigotimes_{l=1}^{\infty} M_{n_l}, \bigotimes_{l=1}^{\infty} A v_l^{(1)}, \bigotimes_{l=1}^{\infty} A v_l^{(2)} \right).$$

Let $(p_k | k \in \mathbb{N})$ be the prime numbers in the increasing order. By definition we find a sequence $(N_k | k \in \mathbb{N})$ of positive integers in such a way that each $k \in \mathbb{N}$ there are nonnegative integers $m_k^{(1)}, \ldots, m_k^{(N_k)}$, unitaries $u_k, v_k$ and $\lambda_k \in T$ satisfying

$$(A, \alpha_{\xi_1}, \alpha_{\xi_2}) \equiv \left( \bigotimes_{k=1}^{\infty} M_{m_k^{(1)}}, \ldots, \bigotimes_{k=1}^{\infty} M_{m_k^{(N_k)}}, \bigotimes_{k=1}^{\infty} A u_k, \bigotimes_{k=1}^{\infty} A v_k \right),$$

$$u_k, v_k \in U(M_{m_k^{(1)}}, \ldots, M_{m_k^{(N_k)}}, \bigotimes_{k=1}^{\infty} A u_k, \bigotimes_{k=1}^{\infty} A v_k),$$

$$m_k^{(1)} + \ldots + m_k^{(N_k)} \neq 0.$$

Set $Q_k = p_1^{m_k^{(1)}} \cdots p_N^{m_k^{(N_k)}}$. By Remark 12 $\lambda_k Q_k = 1$ for each $k$. If $\lambda_k = 1$ for any $k$, we are done, so we assume that $\lambda_k \neq 1$ for some $k \in \mathbb{N}$. Take $k_1 \equiv \min \{ k \in \mathbb{N} \mid \lambda_k \neq 1 \}$. For $n \geq k_1 + 1$ we define

$$N(n) = \prod_{k=k_1+1}^{n} Q_k, \quad \lambda(n) = \prod_{k=k_1+1}^{n} \lambda_k$$

and relatively prime integers $M(n)$ and $K(n)$ by

$$\lambda(n) = \exp \left( \frac{K(n)}{M(n)} \right),$$

$$M(n) \leq N(n), \quad K(n) \in \{ 0, \ldots, M(n) - 1 \}.$$

If we set $m^{(i)}(n) = \max \{ m_k^{(i)} | k_1 + 1 \leq k \leq n \} (i = 1, \ldots, N_k)$ then for each $i$ the exponent of the factor $p_i$ in $M(n)$ is less than or equal to $m^{(i)}(n)$. Hence taking a sufficiently large $n$, we can make the exponent of the factor $p_i$ in $N(n) M(n)^{-1}$ as large as we like if $m^{(i)}(n) \neq 0$. In particular $N(n) M(n)^{-1}$ is divided by $Q_k^2$ and $N(n)$ is much larger than $M(n)$.

We want to show that for any $\delta > 0$, which is much smaller than $\varepsilon$, there exist positive integers $n, m_1, m_2$ and unitary matrices $U_i, V_i (i = 1, 2, 3)$ and $W$ such that

$$n \geq k_1 + 1, \quad m_1, m_2 \geq \delta^{-1},$$

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\[ m_1 m_2 = N(n)M(n)^{-1}Q_{k_1}^{-2}, \]

\[ U_1, V_1 \in M_{Q_{k_1}}, \quad U_2, V_2 \in M_{M(n)Q_{k_1}}, \]

\[ U_3, V_3 \in M_{N(n)M(n)^{-1}Q_{k_1}^{-2}}, \quad W \in M_{N(n)}, \]

\[ U_1 V_1 = \lambda_{k_1}^{-1} V_1 U_1, \quad U_3 V_3 = V_3 U_3, \]

\[ \text{Sp}(U_3, V_3) \text{ is } (m_1, m_2; \delta) - \text{distributed} \quad (15) \]

and

\[
\begin{pmatrix}
\bigotimes_{k=k_1+1}^{n} M_{Q_k}, \quad \bigotimes_{k=k_1+1}^{n} \text{Ad} u_k, \quad \bigotimes_{k=k_1+1}^{n} \text{Ad} v_k
\end{pmatrix} \text{Ad} W 2^{-1} \varepsilon
\]

\[
( M_{Q_{k_1}} \otimes M_{M(n)Q_{k_1}} \otimes M_{N(n)M(n)^{-1}Q_{k_1}^{-2}}, \quad 
\text{Ad}(U_1 \otimes U_2 \otimes U_3), \quad \text{Ad}(V_1 \otimes V_2 \otimes V_3)) .
\]

Suppose that we have shown this statement, then we can construct the required \(\gamma\) and \(\beta\) as follows. Set

\[ n_1 = (Q_1 Q_2 \cdots Q_{k_1}) \cdot Q_{k_1} \cdot N(n)M(n)^{-1}Q_{k_1}^{-2}, \]

\[ v_1^{(1)} = \left( \bigotimes_{k=1}^{k_1} u_k \right) \otimes U_1 \otimes U_3 , \]

\[ v_1^{(2)} = \left( \bigotimes_{k=1}^{k_1} v_k \right) \otimes V_1 \otimes V_3 , \]

\[ W_1 = W . \]

Then \(v_1^{(1)}, v_1^{(2)} \in U(M_{n_1})\). Applying the same method to

\[
\begin{pmatrix}
M_{M(n)Q_{k_1}} \otimes \left( \bigotimes_{k=n+1}^{\infty} M_{Q_k} \right), \quad \text{Ad} U_2 \otimes \left( \bigotimes_{k=n+1}^{\infty} \text{Ad} u_k \right), \\
\text{Ad} V_2 \otimes \left( \bigotimes_{k=n+1}^{\infty} \text{Ad} v_k \right)
\end{pmatrix}
\]

and \(2^{-1} \varepsilon\) in place of

\[
\begin{pmatrix}
\bigotimes_{k=1}^{\infty} M_{Q_k}, \quad \bigotimes_{k=1}^{\infty} \text{Ad} u_k, \quad \bigotimes_{k=1}^{\infty} \text{Ad} v_k
\end{pmatrix}
\]

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and $\epsilon$, we get $n_2, v_2^{(1)}, v_2^{(2)}, W_2$. Repeating this procedure as in the proof of Theorem 14, we obtain an automorphism $\gamma$ as the infinite product of $Ad W_i$ ($i = 1, 2, \ldots$) and an action $\beta$ as

$$\beta_{\xi_i} = \bigotimes_{i=1}^{\infty} Ad v_i^{(i)} \quad (i = 1, 2).$$

This $\beta$ also has the Rohlin property due to (15). Hence we have the required $\gamma$ and $\beta$.

Now we show the remaining part of the proof, that is, the existence of $n, m_1, m_2 \in \mathbb{N}$, unitary matrices $U_i, V_i$ ($i = 1, 2, 3$) and $W$ satisfying the prescribed conditions. By Lemma 16 we can decompose (up to conjugacy)

$$\left( \bigotimes_{k=k_1+1}^{n} M_{Q_k}, \bigotimes_{k=k_1+1}^{n} u_k, \bigotimes_{k=k_1+1}^{n} v_k \right)$$

into

$$\left( M_{M(n)} \otimes M_{N(n)M(n)^{-1}}, U_1^{(n)} \otimes U_2^{(n)}, V_1^{(n)} \otimes V_2^{(n)} \right),$$

where $U_1^{(n)} = S(M(n)), \quad V_1^{(n)} = \Omega(M(n), \lambda(n)^{-1})$ and $U_2^{(n)}, V_2^{(n)}$ are some commuting unitary matrices. Furthermore we see

$$\left( \text{Sp}(U_2^{(n)}, V_2^{(n)}) | n \geq k_1 + 1 \right)$$

is uniformly distributed. Actually note that $\text{Sp}(U_2^{(n)}, V_2^{(n)})$ is unique up to piecewise multiples of $(\lambda(n)^k, \lambda(n)^l)$ for any $k, l \in \mathbb{N}$. So if $\sup \left\{ M(n) | n \geq k_1 + 1 \right\} = \infty$ then it is clearly uniformly distributed. If $\sup \left\{ M(n) | n \geq k_1 + 1 \right\} < \infty$ then there is a positive integer $M$ such that $M$ is divided by $M(n)$ for any $n \geq k_1 + 1$. Noting that

$$\left( \bigotimes_{k=k_1+1}^{n} u_k, \bigotimes_{k=k_1+1}^{n} v_k \right) = \exp \left( 2\pi i \cdot \frac{K(n)}{M(n)} \right) \left( \bigotimes_{k=k_1+1}^{n} u_k, \bigotimes_{k=k_1+1}^{n} v_k \right)$$

we have

$$\left( \bigotimes_{k=k_1+1}^{n} u_k^M, \bigotimes_{k=k_1+1}^{n} v_k^M \right) = \left( \bigotimes_{k=k_1+1}^{n} v_k^M, \bigotimes_{k=k_1+1}^{n} u_k^M \right)$$

for any $n \geq k_1 + 1$. This implies $u_k^M v_k^M = v_k^M u_k^M$ for any $k \geq k_1 + 1$. Hence by Proposition 14

$$\left( \text{Sp} \left( \bigotimes_{k=k_1+1}^{n} u_k^M, \bigotimes_{k=k_1+1}^{n} v_k^M \right) | n \geq k_1 + 1 \right)$$

is uniformly distributed.
is uniformly distributed. Since
\[
\operatorname{Sp}\left(\bigotimes_{k=1}^{n} u_k^M, \bigotimes_{k=1}^{n} v_k^M\right) = \operatorname{Sp}\left(1 \otimes U_2^{(n)M}, 1 \otimes V_2^{(n)M}\right),
\]
it follows that
\[
\left(\operatorname{Sp}\left(U_2^{(n)}, V_2^{(n)}\right) \mid n \geq k_1 + 1\right)
\]
should be uniformly distributed. Using this distribution of the joint spectrum, we can make for a sufficiently large \(n\)
\[
\left(M_{N(n)M(n)-1}, U_2^{(n)}, V_2^{(n)}\right)
\]
close to (up to conjugacy)
\[
\left(M_{Q_{k_1}} \otimes M_{Q_{k_1}} \otimes M_{N(n)M(n)-1}Q_{k_1}^{-2}, U_4' \otimes U_4' \otimes U_5^{(n)}, V_4' \otimes V_4' \otimes V_5^{(n)}\right)
\]
in norm, where
\[
U_4' = V_4' = S(Q_{k_1}), \quad U_4' = V_3'^* = \Omega(Q_{k_1}, \lambda_{k_1})
\]
and \(\operatorname{Sp}(U_5^{(n)}, V_5^{(n)})\) is \((m_1, m_2; \delta)-\text{distributed}\) for some \(m_1, m_2 \in \mathbb{N}\) satisfying
\[
m_1, m_2 \geq \delta^{-1}, \quad m_1m_2 = N(n)M(n)^{-1}Q_{k_1}^{-2}.
\]
Since \(U_4'V_4' = \lambda_{k_1}^{-1}V_4'U_4'\), we obtain desired unitaries in such a way that
\[
U_1 = U_4', \quad V_1 = V_4',
\]
\[
U_2 = U_4^{(n)} \otimes U_4', \quad V_2 = V_3^{(n)} \otimes V_3',
\]
\[
U_3 = U_5^{(n)}, \quad V_3 = V_5^{(n)}.
\]
We complete the proof. \(\square\)

Now we sum up the results we have shown so far. Let \((p_k \mid k \in \mathbb{N})\) be the prime numbers in the increasing order. By Glimm’s theorem (\[11\]), for any UHF algebra \(A\) there exists one and only one sequence \((i_k \mid k \in \mathbb{N})\) of nonnegative integers or \(\infty\) such that \(A \cong \otimes_{k=1}^{\infty} M_{p_k^{i_k}}\), where \(M_{p_k^{\infty}}\) is understood as \(\otimes_{n=1}^{\infty} M_{p_k}\).

Then our classification of \(\mathbb{Z}^2\)-actions on \(A\) is as follows:

**Theorem 20** Let \(A\) be a UHF algebra with the invariant \((i_k \mid k \in \mathbb{N})\) as above.
(1) If \(\sharp\{ k \in \mathbb{N} \mid 1 \leq i_k < \infty \} = \infty\) then there are infinitely many outer conjugacy classes of product type \(\mathbb{Z}^2\)-actions on \(A\) with the Rohlin property.
(2) If \(\sharp\{ k \in \mathbb{N} \mid 1 \leq i_k < \infty \} < \infty\) and \(A\) is infinite-dimensional then there is one and only one outer conjugacy class of product type \(\mathbb{Z}^2\)-actions on \(A\) with the Rohlin property.
(3) If \(A\) is finite-dimensional then there is no \(\mathbb{Z}^2\)-action on \(A\) with the Rohlin property.

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