Abstract: Non-asymptotic concentration inequalities play an essential role in the finite-sample theory of machine learning and high-dimensional statistics. In this article, we obtain a sharper and constants-specified concentration inequality for the summation of independent sub-Weibull random variables, which leads to a mixture of two tails: sub-Gaussian for small deviations and sub-Weibull for large deviations from mean. These bounds improve existing bounds with sharper constants. In the application of random matrices, we derive non-asymptotic versions of Bai-Yin’s theorem for sub-Weibull entries, and it extends the previous result for sub-Gaussian entries. In the application of negative binomial regressions, we give the $\ell_2$-error of the estimated coefficients when covariate vector $X$ is sub-Weibull distributed with sparse structures, which is a new result for negative binomial regressions.

Key words: large derivation inequalities, random matrices, sub-Weibull random variables, heavy-tailed distributions, lower bounds on the least singular value.

1 Introduction

In the last two decades, with the development of modern data collection methods in science and techniques, scientists and engineers can access and load a huge number of variables in their experiments. Probability theory lays the mathematical foundation of statistics. Arising from data-driving problems, various recent statistics research advances also contribute new and challenging problems in probability for further study. For example, in recent years, the rapid development of high-dimensional statistics and machine learnings have promoted the development of the probability theory and even pure mathematics, especially the random matrices, large deviation inequalities, and geometric functional analysis, etc., see Vershynin (2018).

Motivated from sample covariance matrices, a random matrix is a certain matrix $A_{p \times p}$ with its entries $A_{jk}$ drawn from some distributions. As $p \to \infty$, random matrix theory mainly focus on studying the properties of the $p$ eigenvalues of $A_{p \times p}$, which turn out to have some limit law. Several famous limit laws in random matrix theory are different from the CLT for the summation of independent random variables since the $p$ eigenvalues are dependent and interacting with each other. For convergence in distribution, the pioneering work is the Wigner’s semicircle law for some symmetric Gaussian matrices’ eigenvalues, the Marchenko-Pastur law for Wishart distributed random matrices (sample covariance matrices), and the Tracy-Widom laws for the limit distribution for maximum eigenvalues in Wishart matrices.

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All these three laws can be regarded as the CLT of random matrix versions. However, the limit law for the empirical spectral density is some circle distribution, which shed light on the non-communicative behaviors of the random matrix, while the classic limit law in CLT is for normal distribution or infinite divisible distribution. For strong convergence, Bai-Yin’s law complements Marchenko-Pastur law, which asserts that almost surely the smallest and largest eigenvalue of a sample covariance matrix. The monographs Bai and Silverstein (2010), Yao et al. (2015) provide through the introduction of the limit law in random matrices.

The classical statistical models are faced with fixed-dimensional variables only. However, contemporary data science motivates statisticians to pay more attention to study $p \times p \to \infty$ random Hessian matrices (or sample covariance matrices), arising from the likelihood functions of high-dimensional regressions, see Vershynin (2018). When the model dimension increases with sample size, obtaining asymptotical results for the estimator is potentially more challenging than the fixed dimensional case. In statistical machine learning, concentration inequalities (large derivation inequalities) play an essential role in deriving non-asymptotic error bounds for the proposed estimator; see Wainwright (2019). Over recent decades, researchers have developed remarkable results of matrix concentration inequalities, which focuses on non-asymptotic upper and lower bounds for the largest eigenvalue of a finite sum of random matrices. For a more fascinated introduction, please refer to the book Tropp (2015).

In this work, we aim to extend non-asymptotic results from sub-Gaussian to sub-Weibull in terms of concentration inequalities and the Bai-Yin law of extreme eigenvalues in random matrices. The contributions are: (i) We derive some new results for sub-Weibull r.v.s, including sharp concentration inequalities for weighted summations of independent sub-Weibull r.v.s and negative binomial r.v.s, which are useful in many statistical applications; (ii) Based on the generalized Bernstein-Orlicz norm, a sharper concentration for sub-Weibull summations is obtained. Here we circumvent the Stirling’s approximation and derive the inequalities in a more subtle way. Our result is sharper and more accurate than that Kuchibhotla and Chakrabortty (2018) and Hao et al. (2019) gave. (iii) By using these results, we offer two applications. First, we provide a non-asymptotic version of Bai-Yin’s theorem for sub-Weibull random matrices in terms of the extreme eigenvalues. Second, from the proposed negative binomial concentration inequalities, we obtain the $\ell_2$-error for the estimated coefficients in negative binomial regressions under the increasing-dimensional framework.

2 Sharper Concentrations for Sub-Weibull Summation

As summarized in Wainwright (2019), concentration inequalities are powerful in high-dimensional statistical inference, and it can derive various explicit non-asymptotic error bounds as a function of sample size, sparsity level, and dimension. In this section, we present the result of concentration inequalities for sub-Weibull random variables.

2.1 Sub-Weibull norm and Orlicz-type norm

In empirical process theory, sub-Weibull norm (or other Orlicz-type norms) is crucial to derive the tail probability for both single sub-Weibull random variable and summation of random variables (by using the Chernoff’s inequality). A benefit of Orlicz-type norms is that the concentration does not need the zero mean assumption.
Definition 1 (Sub-Weibull norm). For $\theta > 0$, the sub-Weibull norm of $X$ is defined as

$$\|X\|_{\psi_{\theta}} := \inf \left\{ C \in (0, \infty) : E[\exp(|X|^\theta/C^\theta)] \leq 2 \right\}.$$ 

The $\| \cdot \|_{\psi_{\theta}}$ is also called the $\psi_{\theta}$-norm. We define $X$ as a sub-Weibull random variable with sub-Weibull index $\theta$ if it has a bounded $\psi_{\theta}$-norm (denoted as $X \sim \text{subW}(\theta)$). Actually, the sub-Weibull norm is a special case of Orlicz norms below.

Definition 2 (Orlicz Norms). Let $g : [0, \infty) \to [0, \infty)$ be a non-decreasing convex function with $g(0) = 0$. The "$g$-Orlicz norm" of a real-valued r.v. $X$ is given by

$$\|X\|_g := \inf \{ \eta > 0 : E[g(|X|/\eta)] \leq 1 \}.$$ 

Example 1 ($\psi_{\theta}$-norm of bounded r.v.). For a r.v. $|X| \leq M < \infty$, set $Ee^{(|X|/t)^\theta} \leq e^{(M/t)^\theta} \leq 2$ and then $t \geq M/(\log 2)^{1/\theta}$. By the definition of $\|X\|_{\psi_{\theta}}$, we have $\|X\|_{\psi_{\theta}} \geq M/(\log 2)^{1/\theta}$. Since $Ee^{(|X|/t)^\theta}$ is continuous about $t > 0$ if $Ee^{(|X|/t)^\theta} < \infty$ for $1/t$ in a neighbourhood of zero, then $\|X\|_{\psi_{\theta}} = M/(\log 2)^{1/\theta}$.

In general, we have following corollary to determine $\|X\|_{\psi_{\theta}}$ based on moment generating functions (MGF). It would be useful for doing statistical inference of $\psi_{\theta}$-norm.

Corollary 1. If $\|X\|_{\psi_{\theta}} < \infty$, then $\|X\|_{\psi_{\theta}} = (m_{|X|^\theta}^{-1}(2))^{-1/\theta}$ for the MGF $\phi_{Z}(t) := Ee^{tZ}$.

Proof. The MGF of $|X|^\theta$ is continuous in a neighbourhood of zero, by the definition of $\psi_{\theta}$-norm, $2 \geq Ee^{(|X|/\|X\|_{\psi_{\theta}})^\theta} = m_{|X|^\theta}(|X|_{\psi_{\theta}})$. Since $|X|^\theta > 0$, the MGF $m_{|X|^\theta}(t)$ is monotonic increasing. Hence the inverse function $m_{|X|^\theta}^{-1}(t)$ exists and satisfies $\|X\|_{\psi_{\theta}} = m_{|X|^\theta}^{-1}(2)$. So $\|X\|_{\psi_{\theta}} = (m_{|X|^\theta}^{-1}(2))^{-1/\theta}$. \(\square\)

Remark 1. If we observe i.i.d. $X_1, \ldots, X_n$ from some sub-Weibull distributions, we can use the empirical moment generating function [EMGF, Gbur and Collins (1989)] to estimate the sub-Weibull norm of $X$. Then since the EMGF $\hat{m}_{|X|^\theta}(t) = \frac{1}{n} \sum_{i=1}^{n} \exp\{tX_i\}$ converge to MGF $m_{|X|^\theta}(t)$ in probability for $t$ in a neighbourhood of zero, the value of the inverse function of EMGF at 2, $(\hat{m}_{|X|^\theta})^{-1}(2)$, is a consistent estimate for $\|X\|_{\psi_{\theta}}$.

In particular, if we take $\theta = 1$, we get the sub-exponential norm. That is the sub-exponential norm of $X$ is defined as $\|X\|_{\psi_{1}} = \inf\{t > 0 : E\exp(|X|/t) \leq 2 \}$. If $E X_i = 0$ and $\|X_i\|_{\psi_{1}} < \infty$, by Proposition 4.2 in Zhang and Chen (2021), we know $\forall t \geq 0$

$$P \left( \left| \sum_{i=1}^{n} X_i \right| \geq t \right) \leq 2 \exp \left\{ -\frac{1}{4} \left( \frac{t^2}{\sum_{i=1}^{n} 2\|X_i\|_{\psi_{1}}^2} \wedge \frac{t}{\max_{1 \leq i \leq n} \|X_i\|_{\psi_{1}}} \right) \right\}. \quad (2)$$ 

An explicitly calculation of the sub-exponential norm is given in Götze et al. (2019), they show that Poisson r.v. $X \sim \text{Poisson}(\lambda)$ has sub-exponential norm $\|X\|_{\psi_{1}} \leq [\log(\log(2)/\lambda^{-1} + 1)]^{-1}$. And Example 1 with triangle inequality implies

$$\|X - EX\|_{\psi_{1}} \leq \|X\|_{\psi_{1}} + \|EX\|_{\psi_{1}} = \|X\|_{\psi_{1}} + \frac{\lambda}{\log 2} \leq [\log(\log(2)/\lambda^{-1} + 1)]^{-1} + \frac{\lambda}{\log 2}.$$ 

We can also get some useful results for weighted sums of independent heterogeneous negative binomial variables $\{Y_i\}_{i=1}^{n}$ with probability mass functions:

$$P(Y_i = y) = \frac{\Gamma(y + k_i)}{\Gamma(k_i)y!} (1 - q_i)^{k_i} q_i^y \quad (q_i \in (0, 1), y \in \mathbb{N}), \quad (3)$$
where \( \{k_i\}_{i=1}^n \in (0, \infty) \) are variance-dependence parameters. Here, the mean and variance of \( \{Y_i\}_{i=1}^n \) are \( EY_i = \frac{k_i q_i}{1-q_i} \), \( \text{Var} Y_i = \frac{k_i q_i}{(1-q_i)^2} \) respectively. The MGF of \( \{Y_i\}_{i=1}^n \) are \( Ee^{s Y_i} = \left( \frac{1-q_i}{1-q_i e^s} \right)^{k_i} \) for \( i = 1, \ldots, n \).

**Corollary 2.** For any independent r.v.s \( Y_1, \ldots, Y_n \) satisfying \( \|X\|_{\psi_1} < \infty, t \geq 0 \), and non-random weight \( w = (w_1, \ldots, w_n)^T \), we have

\[
P(\| \sum_{i=1}^n w_i (Y_i - EY_i) \| \geq t) \leq 2e^{-\frac{1}{4} \left( \sum_{i=1}^n w_i^2 \|Y_i - EY_i\|_{\psi_1}^2 \right) \left( \max_{1 \leq i \leq n} |w_i| \|\|Y_i - EY_i\|_{\psi_1}\|_{\psi_1} \right)} + 2t \max_{1 \leq i \leq n} (|w_i|\|Y_i - EY_i\|_{\psi_1}) \leq 2e^{-t}
\]

In particular, if \( Y_i \) is independently distributed as \( \text{NB}(\mu_i, k_i) \), we have

\[
P(\| \sum_{i=1}^n w_i (Y_i - EY_i) \| \geq t) \leq 2e^{-\frac{1}{4} \left( \sum_{i=1}^n w_i^2 \|Y_i - EY_i\|_{\psi_1}^2 \right) \left( \max_{1 \leq i \leq n} t \|w_i\|_{\psi_1} \|\|Y_i - EY_i\|_{\psi_1}\|_{\psi_1} \right)} + 2t \max_{1 \leq i \leq n} (|w_i|\|Y_i - EY_i\|_{\psi_1}) \leq 2e^{-t}
\]

(4)

where \( a(\mu_i, k_i) := \left[ \log \frac{1 - (1 - q_i)/k_i}{q_i} \right]^{-1} + \frac{\mu_i}{\log 2} \) with \( q_i := \frac{\mu_i}{\mu_i + k_i} \).

**Proof.** The first inequality is the direct application of (2) by observing that for any constant \( a \in \mathbb{R} \), and r.v. \( Y \) with \( \|Y\|_{\psi_1} < \infty \), \( \|aY\|_{\psi_1} = |a|\|Y\|_{\psi_1} \), \( \|Y + a\|_{\psi_1} = \|Y\|_{\psi_1} + |a| \) and \( \|X + a\|_{\psi_1}^2 \leq (\|X\|_{\psi_1} + |a|/\log 2)^2 \). The second inequality is obtained from (2) by considering two rate in \( \frac{t^2}{\sum_{i=1}^n 2\|Y_i\|_{\psi_1}^2 \wedge \max_{1 \leq i \leq n} t \|Y_i\|_{\psi_1}} \) separately. For (4), only need to note that

\[
\|Y_i\|_{\psi_1} = \inf \{ t > 0 : E \exp(Y_i/t) \leq 2 \}
\]

\[
= \inf \left\{ t > 0 : \left( \frac{1 - q_i}{1 - q_i e^{1/t}} \right)^{k_i} \leq 2 \right\} = \left[ \log \frac{1 - (1 - q_i)/k_i}{q_i} \right]^{-1}
\]

Then the third inequality is obtained by the first inequality and the definition of \( a(\mu_i, k_i) \). \( \square \)

Similar to sub-exponential, the sub-Weibull r.v. \( X \) satisfies following properties as shown in Lemma 2.1 of Zajkowski (2019).

**Proposition 1** (Properties of sub-Weibull norm). If \( \|X\|_{\psi_\theta} < \infty \), then \( P\{|X| > t\} \leq 2e^{-\theta (t/\|X\|_{\psi_\theta})^\theta} \) for all \( t \geq 0 \); and then \( E|X|^k \leq \|X\|_{\psi_\theta}^k \Gamma\left( \frac{k}{\theta} + 1 \right) \) for all \( k \geq 1 \).

Particularly, when \( \theta = 1 \) or 2, sub-Weibull r.v.s reduce to sub-exponential or sub-Gaussian r.v.s, respectively. It is obvious that the smaller \( \theta \) is, the heavier tail the r.v. has. A r.v. is called heavy-tailed if its distribution function fails to be bounded by a decreasing exponential function, i.e. \( \int e^{\lambda x} dF(x) = \infty, \forall \lambda > 0 \) (the tail decays slower than some exponential r.v.s); see Foss et al. (2011). Hence for sub-Weibull r.v.s, we usually focus on the the sub-Weibull index \( \theta \in (0, 1) \). A simple example that the heavy-tailed distributions arises when we work more production on sub-Gaussian r.v.s. The next corollary explains the relation of sub-Weibull norm with parameter \( \theta \) and \( r\theta \), which is similar to Lemmas 2.7.6 of Vershynin (2018) for sub-exponential norm.
Corollary 3. For any $\theta, r \in (0, \infty)$, if $X \sim \text{subW}(\theta)$, then $|X|^r \sim \text{subW}(\theta/r)$. Moreover,
\[
\||X|^r\|_{\psi_{\theta/r}} = \|X\|_{\psi_{\theta}}.
\]
Conversely, if $X \sim \text{subW}(r\theta)$, then $X^r \sim \text{subW}(\theta)$ with $\|X^r\|_{\psi_{r\theta}} = \|X\|_{\psi_{\theta/r}}$.

Proof. By the definition of $\psi$-norm, $\mathbb{E}[\exp(|X|/\|X\|_{\psi_{\theta}}^\theta)] \leq 2$. Then $\mathbb{E}[\exp(|X|^r/\|X\|_{\psi_{\theta/r}}^{\theta/r})] \leq 2$. The result $|X|^r \sim \text{subW}(\theta/r)$ follows by the definition of $\psi$-norm again. Moreover,
\[
\|X\|_{\psi_{\theta}} = \inf\{|C| \in (0, \infty) : \mathbb{E}[\exp(|X|^\theta/C^\theta)] \leq 2\} = \left[\inf\{|C^r| \in (0, \infty) : \mathbb{E}[\exp(|X|^r/C^r)^{\theta/r}] \leq 2\}\right]^{1/r} = \|X\|_{\psi_{\theta/r}}^{1/r},
\]
which verifies (5). If $X \sim \text{subW}(r\theta)$, then $\mathbb{E}[\exp(|X|^r/\|X\|_{\psi_{r\theta}}^r)] = \mathbb{E}[\exp(|X|/\|X\|_{\psi_{r\theta}}^r)] \leq 2$, which means that $X^r \sim \text{subW}(\theta)$ with $\|X\|_{\psi_{r\theta}} = \inf\{|C| \in (0, \infty) : \mathbb{E}[\exp(|X|^r/C^r)^{r\theta}] \leq 2\} = \left[\inf\{|C^r| \in (0, \infty) : \mathbb{E}[\exp(|X|^r/C^r)^{\theta}] \leq 2\}\right]^{1/r} = \|X\|_{\psi_{r\theta}}^{1/r}$.

By Corollary 3, we obtain that $d$-th root of the absolute value of sub-Gaussian is subW(2d) by letting $r = 1/d$. Corollary 3 can be extended to product of r.v.s, from Proposition D.2. in Kuchibhotla and Chakrabortty (2018) with the equality replacing by inequality, we state it as the following proposition.

Proposition 2. If $\{W_i\}_{i=1}^d$ are (possibly dependent) r.v.s satisfying $\|W_i\|_{\psi_{\alpha_i}} < \infty$ for some $\alpha_i > 0$, then $\|\prod_{i=1}^d W_i\|_{\psi_\beta} \leq \prod_{i=1}^d \|W_i\|_{\psi_{\alpha_i}}$ where $\frac{1}{\beta} := \sum_{i=1}^d \frac{1}{\alpha_i}$.

For multi-armed bandit problems in reinforcement learning, Hao et al. (2019) move beyond sub-Gaussianity and consider the reward under sub-Weibull distribution which has a much weaker tail. The corresponding concentration inequality (Theorem 3.1 in Hao et al. (2019)) for the sum of independent sub-Weibull r.v.s will be used repeatedly.

Proposition 3 (Concentration Inequality for sub-Weibull distribution). Suppose $\{y_i\}_{i=1}^n$ are independent sub-Weibull r.v.s with $\|y_i\|_{\psi_\sigma} \leq \sigma$. Then there exists an absolute constant $C_0$ only depending on $\theta$ such that for any $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $0 < \alpha < 1/e^2$, $|\sum_{i=1}^n a_i y_i - \mathbb{E}(\sum_{i=1}^n a_i y_i)| \leq C_0 \sigma (\|a\|_2 (\log \alpha^{-1})^{1/2} + \|a\|_\infty (\log \alpha^{-1})^{1/\theta})$ with probability at least $1 - \alpha$.

The weakness in the Proposition 3 is that the upper bound of $S_n^a := \sum_{i=1}^n a_i y_i - \mathbb{E}(\sum_{i=1}^n a_i y_i)$ is up to a unknown constant $C_\theta$ and the $\alpha$ cannot tends to 0. In the next section, we will give the constants-specified tail probability upper bound for $|S_n^a|$ which is sharper than Theorem 3.1 in Kuchibhotla and Chakrabortty (2018).

2.2 Concentrations for sub-Weibull summation

The Chernoff’s inequality tricks in the derivation sub-exponential concentrations is not valid for sub-Weibull distributions, since the exponential moment conditions of sub-Weibull is about the absolute value $|X|$, but the random summation is not the sum of the absolute values. Thanks to the Bernstein’s moment condition which is the exponential moment of the absolute value, an alternative method is given by Kuchibhotla and Chakrabortty (2018) who defines the so-called Generalized Bernstein-Orlicz (GBO) norm. And the GBO norm can help us to derive tail behaviours for sub-Weibull r.v.s.
Definition 3 (GBO norm). Fix $\alpha > 0$ and $L \geq 0$. Define the function $\Psi^{-1}_{\theta,L}(\cdot)$ as the inverse function $\Psi^{-1}_{\theta,L}(t) := \sqrt{\log(t+1)} + L(\log(t+1))^{1/\theta}$ for all $t \geq 0$. The GBO norm of a r.v. $X$ is then given by $\|X\|_{\Psi_{\theta,L}} := \inf\{\eta > 0 : E[\Psi_{\theta,L}(|X|/\eta)] \leq 1\}$.

The monotone function $\Psi_{\theta,L}(\cdot)$ is motivated by the classical Bernstein’s inequality for sub-exponential r.v.s. Like the sub-Weibull norm properties Corollary 1, the following proposition in Kuchibhotla and Chakrabortty (2018) allows us to get the concentration inequality for r.v. with finite GBO norm.

Proposition 4. If $\|X\|_{\Psi_{\theta,L}} < \infty$, then $P(|X| \geq \|X\|_{\Psi_{\theta,L}}\{\sqrt{t} + Lt^{1/\theta}\}) \leq 2e^{-t} \forall t \geq 0$.

With an upper bound of GBO norm, we could easily derive the concentration inequality for a single sub-Weibull r.v. or even the sum of independent sub-Weibull r.v.s. The sharper upper bounds for the GBO norm is obtained for the sub-Weibull summation, which refines the constant in the sub-Weibull concentration inequality. Let $\|X\|_k := (E|X|^k)^{1/k}$ for all integer $k \geq 1$. First, by truncating more precisely, we obtain a sharper upper bound for $\|X\|_k$, comparing to Proposition C.1 in Kuchibhotla and Chakrabortty (2018).

Corollary 4. If $\|X\|_p \leq C_1 \sqrt{p} + C_2 p^{1/\theta}$ for $p \geq 2$ and constants $C_1$, $C_2$, then

$$\|X\|_{\Psi_{\theta,K}} \leq \gamma eC_1$$

where $K = \gamma^{2/\theta}C_2/(\gamma C_1)$ and $\gamma$ is the minimal solution of $\{k > 1 : e^{2k^{-2}} - 1 + e^{2(1-k^2)/k^2} \leq 1\}$.

Proof. Set $\Delta := \sup_{p \geq 2} \frac{\|X\|_p}{\sqrt{p} + Lp^{1/\theta}}$ so that $\|X\|_p \leq \Delta \sqrt{p} + L\Delta p^{1/\theta}$ holds for all $p \geq 2$. By Markov’s inequality for $t$-th moment $(t \geq 2)$, we have

$$P\left(|X| \geq e\Delta \sqrt{t} + eL\Delta t^{1/\theta}\right) \leq \left(\frac{\|X\|_t}{e\Delta[\sqrt{t} + Lt^{1/\theta}]^t}\right) e^{-t}, \text{ [By the definition of } \Delta].$$

So, for any $t \geq 2,

$$P\left(|X| \geq e\Delta \sqrt{t} + eL\Delta t^{1/\alpha}\right) \leq e^{-t}. \quad (6)$$

Note the definition of $\Delta$ shows $\|X\|_t \leq \Delta \sqrt{t} + L\Delta t^{1/\theta}$ holds for all $t \geq 2$ and assumption $\|X\|_t \leq C_1 \sqrt{t} + C_2 t^{1/\theta}$ for all $t \geq 2$. It gives $e\Delta \sqrt{t} + eL\Delta t^{1/\theta} \leq eC_1 \sqrt{t} + eC_2 t^{1/\theta}$. This inequality with (6) gives

$$P\left(|X| \geq eC_1 \sqrt{t} + eC_2 t^{1/\theta}\right) \leq 1\{0 < t < 2\} + e^{-t}\{t \geq 2\}, \forall t > 0. \quad (7)$$
Take $K = k^{2/\theta}C_2/(kC_1)$, and define $\delta_k := keC_1$ for a certain constant $k > 1$,

$$
\mathbb{E} \left[ \Psi_{\theta,K} \left( \frac{|X|}{\delta_k} \right) \right] = \int_{0}^{\infty} P \left( |X| \geq \delta_k \Psi_{\theta,K}^{\gamma}(s) \right) ds
$$

$$
= \int_{0}^{\infty} P(|X| \geq k \epsilon C_1 \sqrt{\log(1 + s)} + k \epsilon C_1 K \log(1 + s)^{1/\theta}) ds
$$

$$
= \int_{0}^{\infty} P(|X| \geq eC_1 \sqrt{\log(1 + s)^k + eC_2 \log(1 + s)^{k^21/\theta}}) ds
$$

|By (7)| \leq \int_{0}^{e^{2k^2 - 1}} ds + \int_{e^{2k^2 - 1}}^{\infty} \exp \left\{ -k^2 \log(1 + s) \right\} ds

\leq \int_{0}^{e^{2k^2 - 1}} dt + \int_{e^{2k^2 - 1}}^{\infty} \frac{dt}{(1 + t)^{k^2}}

= e^{2k^2 - 1} + \frac{(1 + t)^{1-k^2}}{1-k^2} \bigg|_{e^{2k^2 - 1}}^{\infty} = e^{2k^2 - 1} + e^{2(1-k^2)/k^2} \leq 1.

Therefore, $\|X\|_{\Psi_{\theta,K}} \leq \gamma eC_1$ with $\gamma$ defined as the smallest solution of the inequality \{ $k > 1 : e^{2k^2 - 1} + e^{2(1-k^2)/k^2} / k^2 - 1 \leq 1$ \}. An approximate solution is $\gamma \approx 1.78$. \hfill \Box

In below, we need the moment estimation for sums of independent symmetric r.v.s.

**Lemma 1** (Khinchin-Kahane Inequality, Theorem 1.3.1 of de la Pena and Gine (2012)). Let $\{a_i\}_{i=1}^n$ be a finite non-random sequence, $\{\varepsilon_i\}_{i=1}^n$ be a sequence of independent Rademacher variables and $1 < p < q < \infty$. Then $\|\sum_{i=1}^n \varepsilon_i a_i\|_q \leq \left( \frac{q-1}{p-1} \right)^{1/2} \|\sum_{i=1}^n \varepsilon_i a_i\|_p$.

**Lemma 2** (Theorem 2 of Latala (1997)). Let $X_1, \ldots, X_n$ be a sequence of independent symmetric r.v.s, and $p \geq 2$. Then $\frac{1}{2e^1} \sum_{i=1}^{n} \|X_i\|_p \leq \|X_1 + \cdots + X_n\|_p \leq e \sum_{i=1}^{n} \|X_i\|_p$, where $\|X_i\|_p := \inf\{t > 0 : \forall a \in \mathbb{R}, \sum_{i=1}^{n} \phi_p(X_i/t) \leq a \}$ with $\phi_p(X) := \mathbb{E}[\exp(tX)]$. \hfill \Box

**Lemma 3** (Example 3.2 and 3.3 of Latala (1997)). Assume $X$ be a symmetric r.v. satisfying $P(X \geq t) = e^{-N(t)}$. For any $t \geq 0$, we have

(a) If $N(t)$ is concave, then $\log \phi_p(e^{-2t}X) \leq p \phi_{p,X}(t) := (\mathbb{E}[|X|^p])^{1/p} \vee (pt^2 \mathbb{E}[X^2])$.

(b) For convex $N(t)$, denote the convex conjugate function $N^*(t) := \sup_{s > 0} \{ ts - N(s) \}$ and $M_{p,X}(t) := \left\{ \begin{array}{ll} p^{-1} N^*(p|t|), & \text{if } |t| \geq 2 \\ pt^2, & \text{if } |t| < 2. \end{array} \right.$ Then $\log \phi_p(tX/4) \leq p \phi_{p,X}(t)$.

With the help of three lemmas above, we can obtain the main results concerning the sharper and constant-specified concentration inequality for the sum of independent sub-Weibull r.v.s.

**Theorem 1** (Concentration for sub-Weibull summation). If $X_1, \ldots, X_n$ are independent centralized r.v.s such that $\|X_i\|_{\psi_\theta} < \infty$ for all $1 \leq i \leq n$ and some $\theta > 0$, then for any weight vector $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$, the following bounds holds true:

(a) The estimate for GBO norm of the summation:

$$
\|\sum_{i=1}^{n} w_i X_i\|_{\Psi_{\theta,K,(\theta,w)}} \leq \gamma eC(\theta) \|b\|_2,
$$

where $b = (w_1 \|X_1\|_{\psi_\theta}, \ldots, w_n \|X_n\|_{\psi_\theta})^T \in \mathbb{R}^n$, with.

$$
C(\theta) := \begin{cases} 2 \left[ \log^{1/\theta} 2 + e^{3} \left( \Gamma^{1/2} \left( \frac{2}{\theta} + 1 \right) + 3 \frac{2^{2-\theta}}{\theta^{\theta}} \sup_{p \geq 2} p^{-1/\theta} \Gamma^{1/\theta} \left( \frac{\theta}{2} + 1 \right) \right) \right], & \text{if } \theta \leq 1, \\
8e + 2(\log 2)^{1/\theta}, & \text{if } \theta > 1.
\end{cases}
$$
and $L_n(\theta, b) = \gamma^{2/\theta} A(\theta) \|b\|_2^{1/\theta} (\theta > 1)$ where $B(\theta) =: \frac{2e^{\theta - 1/\theta}(1 - \theta^{-1})^{1/\beta}}{4e + (\log 2)^{1/\theta}}$ and $A(\theta) =: \inf_{p \geq 2} \frac{e^{\beta} \Gamma^{1/p} (\frac{1}{p} + 1)}{2^{3/\theta} p^{-1/\theta} \Gamma^{1/p} (\frac{1}{p} + 1) + 3^{2/\theta} \sup_{p \geq 2} p^{-1/\theta} \Gamma^{1/p} (\frac{1}{p} + 1)}$.

For the case $\theta > 1$, $\beta$ is the Hölder conjugate satisfying $1/\theta + 1/\beta = 1$.

(b) Concentration for sub-Weibull summation

$$P \left( \left\| \sum_{i=1}^n w_i X_i \right\| \geq 2e^{C(\theta)} \|b\|_2 \left\{ \sqrt{t} + L_n(\theta, b) t^{1/\theta} \right\} \right) \leq 2e^{-t}. \quad (8)$$

(c) Another form of for $\theta \neq 2$:

$$P \left( \sum_{i=1}^n w_i X_i \geq s \right) \leq 2 \exp \left\{ -\left( \frac{s^\theta}{4eC(\theta) \|b\|_2 L_n(\theta, b)} \wedge \frac{s^2}{16eC^2(\theta) \|b\|_2^2} \right) \right\}$$

$$\begin{align*}
\theta < 2 & : \left\{ \begin{array}{ll}
2e^{-s^2/16eC^2(\theta) \|b\|_2^2}, & \text{if } s \leq 4eC(\theta) \|b\|_2 L_n^{\theta/(\theta - 2)}(\theta, b), \\
2e^{-s^\theta/[4eC(\theta) \|b\|_2^2] \theta}, & \text{if } s > 4eC(\theta) \|b\|_2 L_n^{\theta/(\theta - 2)}(\theta, b);
\end{array} \right.
\end{align*}$$

$$\begin{align*}
\theta > 2 & : \left\{ \begin{array}{ll}
2e^{-s^\theta/[4eC(\theta) \|b\|_2^2] \theta}, & \text{if } s < 4eC(\theta) \|b\|_2 L_n^{\theta/(\theta - 2)}(\theta, b), \\
2e^{-s^2/16eC^2(\theta) \|b\|_2^2}, & \text{if } s \geq 4eC(\theta) \|b\|_2 L_n^{\theta/(\theta - 2)}(\theta, b).
\end{array} \right.
\end{align*}$$

Proof. The main idea in the proof is by the sharper estimates of the GBO norm of the sum of symmetric r.v.s.

(a) Without loss of generality, we assume $\|X_i\|_{\psi_2} = 1$. Define $Y_i := (|X_i| - (\log 2)^{1/\theta})_+$, then it is easy to check that $P(|X_i| \geq t) \leq e^{-t^\theta}$ implies $P(Y_i \geq t) \leq e^{-t^\theta}$. For independent Rademacher r.v., $\{\varepsilon_i \}_{i=1}^n$, the symmetrization inequality gives $\sum_{i=1}^n w_i X_i \leq 2 \sum_{i=1}^n \varepsilon_i w_i X_i$. Note that $\varepsilon_i X_i$ is identically distributed as $\varepsilon_i |X_i|$, $\| \sum_{i=1}^n w_i X_i \|_p \leq 2 \| \sum_{i=1}^n \varepsilon_i w_i X_i \|_p \leq 2 \| \sum_{i=1}^n \varepsilon_i w_i (Y_i + (\log 2)^{1/\theta}) \|_p$

$$\leq 2 \| \sum_{i=1}^n \varepsilon_i w_i Y_i \|_p + 2(\log 2)^{1/\theta} \| \sum_{i=1}^n \varepsilon_i w_i \|_p$$

[Khinchin-Kahane inequality] $\leq 2 \left\| \sum_{i=1}^n \varepsilon_i w_i Y_i \right\|_p + 2(\log 2)^{1/\theta} \left( \frac{p - 1}{2 - 1} \right)^{1/2} \| \sum_{i=1}^n \varepsilon_i w_i \|_2$

$$< 2 \left\| \sum_{i=1}^n \varepsilon_i w_i Y_i \right\|_p + 2(\log 2)^{1/\theta} \sqrt{p} (E \left\| \sum_{i=1}^n \varepsilon_i w_i \right\|^2)^{1/2}$$

$$\left\{ \varepsilon_i \right\}_{i=1}^n \text{ are independent } = 2 \left\| \sum_{i=1}^n \varepsilon_i w_i Y_i \right\|_p + 2(\log 2)^{1/\theta} \sqrt{p} \| w \|_2.$$

From Lemma 2, we are going to handle the first term in (9) with the sum of symmetric r.v.s. Since $P(Y_i \geq t) \leq e^{-t^\theta}$, then

$$\| \sum_{i=1}^n \varepsilon_i w_i Y_i \|_p = \| \sum_{i=1}^p w_i Z_i \|_p, \quad Z_i := \varepsilon_i Y_i$$

for symmetric independent r.v.s $\{Z_i \}_{i=1}^n$ satisfying $|Z_i| \overset{d}{=} Y_i$ and $P(Z_i \geq t) = e^{-t^\theta}$ for all $t \geq 0$. 

Next, we proceed the proof by checking the moment conditions in Corollary 4.

Case $\theta \leq 1$: $N(t) = t^\theta$ is concave for $\theta \leq 1$. From Lemma 2 and Lemma 3 (a), for $p \geq 2$,
\[
\left\| \sum_{i=1}^{n} w_i Z_i \right\|_p \leq e \inf \left\{ t > 0 : \sum_{i=1}^{n} \log \phi_p \left( e^{-2 \left( \frac{w_i e^2}{t} \right) Z_i} \right) \leq p \right\} 
\leq e \inf \left\{ t > 0 : \sum_{i=1}^{n} p M_{p,Z_i} \left( \frac{w_i e^2}{t} \right) \leq p \right\} 
= e \inf \left\{ t > 0 : \sum_{i=1}^{n} \left\{ \left( \frac{w_i e^2}{t} \right)^2 \| Z_i \|_p^2 \right\} + \left\{ p \left( \frac{w_i e^2}{t} \right)^2 \| Z_i \|_p^2 \right\} \right\} \leq p \inf \left\{ t > 0 : p \Gamma \left( \frac{2}{\theta} + 1 \right) \| w \|_\infty \leq 1 \right\},
\]
where the last inequality follows from the fact that $p \geq 2$, $p \in \mathbb{N}$. Hence
\[
\left\| \sum_{i=1}^{n} w_i Z_i \right\|_p \leq e^3 \left[ \Gamma^{1/p} \left( \frac{p}{\theta} + 1 \right) \| w \|_p + \sqrt{p} \Gamma^{1/2} \left( \frac{2}{\theta} + 1 \right) \| w \|_2 \right] + 2 \left( \log 2 \right)^{1/\theta} \sqrt{p} \| w \|_2.
\]

Using homogeneity, we can assume that $\sqrt{p} \| w \|_2 + p^{1/\theta} \| w \|_\infty = 1$. Then $\| w \|_2 \leq p^{-1/2}$ and $\| w \|_\infty \leq p^{-1/\theta}$. Therefore, for $p \geq 2$,
\[
\| w \|_p \leq \left( \sum_{i=1}^{n} |w_i|^2 \| w \|_\infty^2 \right)^{1/p} \leq \left( p^{-1-(p-2)/\theta} \right)^{1/p} = \left( p^{-1-(p-2)/\theta} \right)^{1/p},
\]
\[
\leq 3 \frac{2-\theta}{3\theta} \left( p^{-1/(p-1)} \right) = 3 \frac{2-\theta}{3\theta} \left( \sqrt{p} \| w \|_2 + p^{1/\theta} \| w \|_\infty \right) \leq 3 \frac{2-\theta}{3\theta} \left( \sqrt{p} \| w \|_2 + p^{1/\theta} \| w \|_\infty \right),
\]
where the last inequality follows from the fact that $p^{1/p} \leq 3^{1/3}$ for any $p \geq 2$, $p \in \mathbb{N}$. Hence
\[
\left\| \sum_{i=1}^{n} w_i X_i \right\|_p \leq 2 e^{3+ \frac{2-\theta}{3\theta} \Gamma^{1/p} \left( \frac{p}{\theta} + 1 \right) \| w \|_\infty}
+ 2 \left[ \log^{1/\theta} 2 + e^3 \left( \Gamma^{1/2} \left( \frac{2}{\theta} + 1 \right) + 3 \frac{2-\theta}{3\theta} p^{-1/(p-1)} \Gamma \left( \frac{p}{\theta} + 1 \right) \right) \right] \sqrt{p} \| w \|_2.
\]

Following Corollary 4, we have
\[
\left\| \sum_{i=1}^{n} w_i X_i \right\|_{\psi_{\theta,L_n(\theta,p)}} \leq \gamma e D_1(\theta),
\]
where $L_n(\theta,p) = \frac{\gamma \theta D_2(\theta,p)}{\gamma D_1(\theta)}$, $D_1(\theta) := 2 \| \log^{1/\theta} 2 + e^3 \left( \Gamma^{1/2} \left( \frac{2}{\theta} + 1 \right) + 3 \frac{2-\theta}{3\theta} p^{-1/(p-1)} \Gamma \left( \frac{p}{\theta} + 1 \right) \right) \| w \|_2 < \infty$, and $D_2(\theta,p) := 2 e^{3+ \frac{2-\theta}{3\theta} \Gamma^{1/p} \left( \frac{p}{\theta} + 1 \right) \| w \|_\infty}$.

Finally, take $L_n(\theta) := \inf_{p \geq 1} L_n(\theta,p) > 0$. Indeed, the positive limit can be argued by (2.2) in Alzer (1997). Then by the monotonicity property of the GBO norm, it gives
\[
\left\| \sum_{i=1}^{n} w_i X_i \right\|_{\psi_{\theta,L_n(\theta)}} \leq \left\| \sum_{i=1}^{n} w_i X_i \right\|_{\psi_{\theta,L_n(\theta,p)}} \leq \gamma e D_1(\theta).
\]
Case $\theta > 1$: In this case $N(t) = t^\theta$ is convex with $N^*(t) = \theta^{-\frac{1}{\theta}} (1 - \theta^{-1}) t^{\theta - 1}$. By Lemmas 2 and 3 (b), for $p \geq 2$, we have

$$
\left\| \sum_{i=1}^{p} w_i Z_i \right\|_p \leq e \inf \left\{ t > 0 : \sum_{i=1}^{n} \log \phi_p \left( \frac{4w_i}{t} Z_i / 4 \right) \leq p \right\} + e \inf \left\{ t > 0 : \sum_{i=1}^{n} p M_{p, Z_i} \left( \frac{4w_i}{t} \right) \leq p \right\}
$$

$$
\leq e \inf \left\{ t > 0 : \sum_{i=1}^{p} N^* \left( \frac{4w_i}{t} \right) \leq 1 \right\} + e \inf \left\{ t > 0 : \sum_{i=1}^{p} \left( \frac{4w_i}{t} \right)^2 \leq 1 \right\}
$$

$$
= 4e \left[ \sqrt{p} \|w\|_2 + (p/\theta)^{1/\theta} (1 - \theta^{-1})^{1/\beta} \|w\|_{\beta} \right].
$$

with $\beta$ mentioned in the statement. Therefore, for $p \geq 2$, (9) implies

$$
\left\| \sum_{i=1}^{p} w_i X_i \right\|_p \leq [8e + 2(\log 2)^{1/\theta}] \sqrt{p} \|w\|_2 + 8e(p/\theta)^{1/\theta} (1 - \theta^{-1})^{1/\beta} \|w\|_{\beta}.
$$

Then the following result follows by Corollary 4,

$$
\left\| \sum_{i=1}^{n} w_i X_i \right\|_{\psi, L_1(\theta)} \leq \gamma e D'_1(\theta),
$$

where $L_n(\theta) = \frac{\gamma^{2/\theta} D'_2(\theta)}{\gamma D'_1(\theta)}$, $D'_2(\theta) = [8e + 2(\log 2)^{1/\theta}] \|w\|_2$, and $D'_2(\theta) = 8e\theta^{-1/\theta} (1 - \theta^{-1})^{1/\beta} \|w\|_{\beta}$.

Note that $w_i X_i = (w_i X_i \psi_p)(X_i / \|X_i \psi_p\|)$, we can conclude (a).

(b) It is followed from Proposition 4 and (a).

(c) For easy notation, put $L_n(\theta) = L_n(\theta, b)$ in the proof. When $\theta < 2$, by the inequality $a + b \leq 2(a \lor b)$ for $a, b > 0$, we have

$$
P \left( \left\| \sum_{i=1}^{n} w_i X_i \right\|_2 \geq 4eC(\theta) \left\| b \right\|_2 \sqrt{t} \right) \leq 2e^{-t}, \text{ if } \sqrt{t} \geq L_n(\theta) t^{1/\theta}.
$$

Put $s := 4eC(\theta) \left\| b \right\|_2 \sqrt{t}$, we have

$$
P \left( \left\| \sum_{i=1}^{n} w_i X_i \right\| \geq s \right) \leq 2 \exp \left\{ -\frac{s^2}{16e^2 C^2(\theta) \left\| b \right\|_2^2} \right\}, \text{ if } s \leq 4eC(\theta) \left\| b \right\|_2 L_n^{0/(\theta - 2)}(\theta).
$$

For $\sqrt{t} \geq L_n(\theta) t^{1/\theta}$, we get $P(\left\| \sum_{i=1}^{n} w_i X_i \right\| \geq 4eC(\theta) \left\| b \right\|_2 L_n(\theta) t^{1/\theta} \leq 2e^{-t}$. Let $s := 4eC(\theta) \left\| b \right\|_2 L_n(\theta) t^{1/\theta}$, it gives

$$
P \left( \left\| \sum_{i=1}^{n} w_i X_i \right\| \geq s \right) \leq 2 \exp \left\{ -\frac{s^\theta}{(4eC(\theta) \left\| b \right\|_2 L_n(\theta))^\theta} \right\}, \text{ if } s > 4eC(\theta) \left\| b \right\|_2 L_n^{\theta/(\theta - 2)}(\theta).
$$

Similarly, for $\theta > 2$, it implies

$$
P \left( \left\| \sum_{i=1}^{n} w_i X_i \right\| \geq s \right) \leq 2e^{-\frac{s^2}{16e^2 C^2(\theta) \left\| b \right\|_2^2}} \text{ if } s \geq 4eC(\theta) \left\| b \right\|_2 L_n^{\theta/(2-\theta)}(\theta),
$$

and $P \left( \left\| \sum_{i=1}^{n} w_i X_i \right\| \geq s \right) \leq 2e^{-\frac{16e^2 C^2(\theta) \left\| b \right\|_2^2}{16e^2 C^2(\theta) \left\| b \right\|_2^2}} \text{ if } s \geq 4eC(\theta) \left\| b \right\|_2 L_n^{\theta/(2-\theta)}(\theta).
$

\[\square\]

**Remark 2.** Theorem 1 (b) generalizes the sub-Gaussian concentration inequalities, sub-exponential concentration inequalities, and Bernstein’s concentration inequalities with Bernstein’s moment condition. For $\theta < 2$ in Theorem 1 (c), the tail behaviour of the sum is akin to a sub-Gaussian tail for small $t$ and the tail resembles the exponential tail for large $t$; For $\theta > 2$, the tail behaves like a Weibull r.v. with tail parameter $\theta$ and the tail of sums match that of the sub-Gaussian tail for large $t$. The intuition is that the sum will concentrates around zero by the Law of Large Number. Theorem 1 shows that the convergence rate will be faster for small deviations from mean and will be slower for large deviations from mean.
3.1 Non-asymptotic Bai-Yin’s theorem

In machine learning, non-asymptotic results for estimator is specially crucial to evaluate the finite-sample performance. The key topic for non-asymptotic theory is the concentration of measure or r.v.s, which can provide a sharp probabilistic upper bound on the desired estimators as a function of sample size $n$ and dimension $p$.

Let $A = A_{n,p}$ be an $n \times p$ random matrix whose entries are independent copies of a r.v. with zero mean, unit variance, and finite fourth moment. Suppose that the dimensions $n$ and $p$ both grow to infinity while the aspect ratio $p/n$ converges to a constant in $[0,1]$. Then Bai-Yin’s law (Bai and Yin, 1993) asserted that the standardized extreme eigenvalues satisfying

$$\frac{1}{\sqrt{n}} \lambda_{\min}(A) = 1 - \sqrt{\frac{p}{n}} + o\left( \sqrt{\frac{p}{n}} \right), \quad \frac{1}{\sqrt{n}} \lambda_{\max}(A) = 1 + \sqrt{\frac{p}{n}} + o\left( \sqrt{\frac{p}{n}} \right) \quad \text{a.s.}$$

Next we introduce a special counting measure for measuring the complexity of a certain set in some space. The $N_\varepsilon$ is called an $\varepsilon$-net of $K$ in $\mathbb{R}^n$ if $K$ can be covered by balls with centers in $K$ and radii $\varepsilon$ (under Euclidean distance). The covering number $N_\varepsilon(K, \varepsilon)$ is defined by the smallest number of closed balls with centers in $K$ and radii $\varepsilon$ whose union covers $K$.

For purposes of studying random matrices, we need to extend the definition of sub-Weibull r.v. to sub-Weibull random vectors. The $n$-dimensional unit Euclidean sphere $S^{n-1}$, is denoted by $S^{n-1} = \{ x \in \mathbb{R}^n : \| x \|_2 = 1 \}$. We say that a random vector $X$ in $\mathbb{R}^n$ is sub-Weibull if the one-dimensional marginals $\langle X, a \rangle$ are sub-Weibull r.v.s for all $a \in \mathbb{R}^n$. The sub-Weibull norm of a random vector $X$ is defined as $\| X \|_{\psi_0} := \sup_{a \in S^{n-1}} \| \langle X, a \rangle \|_{\psi_0}$. For simplicity, we assume that the rows in random matrices are isotropic random vectors. A random vector $Y$ in $\mathbb{R}^n$ is called isotropic if $\text{Var}(Y) = I_p$. Equivalently, $Y$ is isotropic if $E(Y, a)^2 = \| a \|_2^2$ for all $a \in \mathbb{R}^n$. In the non-asymptotic regime, Theorem 4.6.1 in Vershynin (2018) study the upper and lower bounds of maximum (minimum) eigenvalues of random matrices with independent sub-Gaussian entries which are sampled from high-dimensional distributions. As an extension of Theorem 4.6.1 in Vershynin (2018), the following result is a non-asymptotic versions of Bai-Yin’s law for sub-Weibull entries, which is useful to estimate covariance matrices from heavy-tailed data.

Theorem 2 (Non-asymptotic Bai-Yin’s law). Let $A$ be an $n \times p$ matrix whose rows $A_i$ are independent isotropic sub-Weibull random vectors in $\mathbb{R}^p$ with covariance matrix $I_p$ and $\max_{1 \leq i \leq n} \| A_i \|_{\psi_0} \leq K$. Then for every $s \geq 0$, we have

$$\mathbb{P}\left\{ \frac{1}{n} A^\top A - I_p \leq H(cp + s^2, n; \theta) \right\} \geq 1 - 2e^{-s^2}.$$

where $H(t, n; \theta) := 2eC(\theta/2)(K^2 + (\log 2)^{-1/\theta}) \left[ \sqrt{\frac{t}{n}} + \left\{ \begin{array}{ll} \frac{A(\theta/2)(\gamma^2 t)^{2/\theta}/n}{B(\theta/2)(\gamma^2 t)^{2/\theta}/n^{1/\theta}}, & \theta \leq 2 \\ \frac{\theta}{\theta - 2}, & \theta > 2 \end{array} \right. \right]$, with $A(\theta/2), B(\theta/2)$ and $C(\theta/2)$ defined in Theorem 1(a).
Moreover, the concentration inequality for extreme eigenvalues hold for \( c \geq n \log 9/p \)
\[
P\left\{ \sqrt{1 - H^2(cp + s^2, n; \theta)} \leq \frac{\lambda_{\min}(A)}{\sqrt{n}} \leq \frac{\lambda_{\max}(A)}{\sqrt{n}} \leq \sqrt{1 + H^2(cp + s^2, n; \theta)} \right\} \geq 1 - 2e^{-s^2}.
\] (10)

**Proof.** For convenience, the proof is divided into three steps.

**Step1.** Adopting the lemma

**Lemma 4** (Computing the spectral norm on a net, Lemma 5.4 in Vershynin (2018)). Let \( B \) be an \( p \times p \) matrix, and let \( \mathcal{N}_\varepsilon \) be an \( \varepsilon \)-net of \( S^{p-1} \) for some \( \varepsilon \in (0, 1) \). Then
\[
\|B\| := \max_{\|x\|=1} |Bx|_2 = \sup_{x \in S^{p-1}} |\langle Bx, x \rangle| \leq (1 - 2\varepsilon)^{-1} \sup_{x \in \mathcal{N}_\varepsilon} |\langle Bx, x \rangle|.
\]
Then show that \( \|\frac{1}{\sqrt{n}}A^\top A - I_p\| \leq 2 \max_{x \in \mathcal{N}_1} \|A\| \|x\| - 1 = \frac{1}{\sqrt{n}} \|Ax\|_2^2 - 1 \). Indeed, note that \( \langle \frac{1}{n}A^\top Ax - x, x \rangle = \langle \frac{1}{n}A^\top Ax, x \rangle - 1 = \frac{1}{n} \|Ax\|_2^2 - 1 \). By setting \( \varepsilon = 1/4 \) in Lemma 6, we can get:
\[
\left| \langle \frac{1}{n}A^\top A - I_p, x \rangle \right| \leq (1 - 2\varepsilon)^{-1} \sup_{x \in \mathcal{N}_\varepsilon} \left| \langle \frac{1}{n}A^\top Ax - x, x \rangle \right| = 2 \max_{x \in \mathcal{N}_1} |\frac{1}{n} \|Ax\|_2^2 - 1|.
\]

**Step2.** Let \( Z_i := \langle A_i, x \rangle \) fix any \( x \in S^{n-1} \). Observe that \( \|A\|_2 = \sum_{i=1}^n |\langle A_i, x \rangle|^2 = \sum_{i=1}^n Z_i^2 \). The fact that \( \{Z_i\}_{i=1}^n \) are subW(\( \theta \)) with \( E \sum_{i=1}^n Z_i^2 = 1 \), \( \max_{1 \leq i \leq n} \|Z_i\|_{\psi_0} = K \). Then by Corollary 3, \( Z_i^2 \) are independent subW(\( \theta/2 \)) r.v.s with max\( 1 \leq i \leq n \|Z_i^2\|_{\psi_0} = K^2 \). Note that \( \|Z_i^2\|_{\psi_0} = (\log 2)^{-1/\theta} \) by Example 1, then norm triangle inequality gives
\[
\max_{1 \leq i \leq n} \|Z_i^2 - 1\|_{\psi_0/2} \leq \max_{1 \leq i \leq n} \|Z_i^2\|_{\psi_0/2} + \|Z_i^2 - 1\|_{\psi_0/2} \leq K^2 + (\log 2)^{-1/\theta}.
\] (11)
Denote \( b := (\frac{1}{n}(\|Z_1^2 - 1\|_{\psi_0/2}, \ldots, \|Z_n^2 - 1\|_{\psi_0/2})^\top \) in Theorem 1. With (11), we have \( \|b\|_2 = n^{-1/2} \sum_{i=1}^n \|Z_i^2 - 1\|_{\psi_0/2} \leq \sqrt{n} \sum_{i=1}^n \|Z_i^2 - 1\|_{\psi_0/2} \leq K^2 + (\log 2)^{-1/\theta} \). For \( \beta := \theta_{\psi_0} > 1 \), we get \( \|b\|_\beta = n^{-1/\beta} \left( \sum_{i=1}^n \|Z_i^2 - 1\|_{\psi_0/2}^\beta \right)^{1/\beta} \leq n^{1/\beta - 1} [K^2 + (\log 2)^{-1/\theta}] = n^{1/\beta - 1} [K^2 + (\log 2)^{-1/\theta}] \). Write \( L_n(\theta/2, b) \) as the constant defined in Theorem 1(a). Then,
\[
\|b\|_{2L_n(\theta/2, b)} = 2^{\theta/\beta} \left\{ \begin{array}{ll} A(\theta/2)/\|b\|_{\infty}, & 0 \leq \theta \leq 2 \\ B(\theta/2)/\|b\|_{\beta}, & \theta > 2. \end{array} \right. \leq [K^2 + (\log 2)^{-1/\theta}] 2^{\theta/\beta} \left\{ \begin{array}{ll} A(\theta/2)/n, & 0 \leq \theta \leq 2 \\ B(\theta/2)/n^{1/\beta}, & \theta > 2. \end{array} \right.
\]
Hence
\[
2cC(\theta/2) \left\{ \|b\|_2 \sqrt{t} + \|b\|_{2L_n(\theta/2, b)} t^{2/\theta} \right\} \leq 2cC(\theta/2) [K^2 + (\log 2)^{-1/\theta}] \left\{ \begin{array}{ll} \sqrt{\frac{t}{n}} + A(\theta/2)(\gamma t)^{2/\theta}/n, & 0 \leq \theta \leq 2 \\ B(\theta/2)(\gamma t)^{2/\theta}/n^{1/\beta}, & \theta > 2 \end{array} \right. =: H(t, n; \theta).
\]
Therefore, \( P(\frac{1}{n} \sum_{i=1}^n (Z_i^2 - 1) \geq H(t, n; \theta)) \leq 2e^{-t} \). Let \( t = cp + s^2 \) for constant \( c \), then
\[
P\left\{ \frac{1}{n} \|Ax\|_2^2 - 1 \geq H(cp + s^2, n; \theta) \right\} \leq 2e^{-(cp + s^2)}.
\]

**Step3.** Consider the follow lemma for covering numbers in Vershynin (2018).

**Lemma 5** (Covering numbers of the sphere). For the unit Euclidean sphere \( S^{n-1} \), the covering number \( \mathcal{N}(S^{n-1}, \varepsilon) \) satisfies \( \mathcal{N}(S^{n-1}, \varepsilon) \leq (1 + \frac{2}{\varepsilon})^n \) for every \( \varepsilon > 0 \).
Then, we show the concentration for \( \| \frac{1}{n} A^\top A - I_p \| \), and (10) follows by the definition of largest and least eigenvalues. The conclusion is drawn by Step 1 and 2:

\[
P \left\{ \left\| \frac{1}{n} A^\top A - I_p \right\| \geq H(cp + s^2, n; \theta) \right\} \leq P \left\{ 2 \max_{x \in \mathbb{N}_{1/4}} \left| \frac{1}{n} \| A x \|_2^2 - 1 \right| \geq H(cp + s^2, n; \theta) \right\} \\
\leq N(s^{n-1}, 1/4)P \left\{ \left| \frac{1}{n} \| A x \|_2^2 - 1 \right| \geq H(cp + s^2, n; \theta)/2 \right\} \leq 2 \cdot 9^n e^{-(cp+s^2)},
\]

where the last inequality follows by Lemma 5 with \( \varepsilon = 1/4 \). When the \( c \geq n \log 9/p \), then \( 2 \cdot 9^n e^{-(cp+s^2)} \leq 2e^{-s^2} \), and the (10) is proved.

Moreover, note that

\[
\max_{|x|_2 = 1} \left| \frac{1}{\sqrt{n}} A x \right|_2^2 - 1 = \max_{|x|_2 = 1} \left| \frac{1}{\sqrt{n}} (A^\top A - I_p) x \right|_2^2 = \left| \frac{1}{\sqrt{n}} A^\top A - I_p \right|_2^2 \leq H^2(cp + s^2, n; \theta).
\]

implies that

\[
\sqrt{1 - H^2(cp + s^2, n; \theta)} \leq \frac{1}{\sqrt{n}} \lambda_{\text{max}}(A) \leq \sqrt{1 + H^2(cp + s^2, n; \theta)}.
\]

Similarly, for the minimal eigenvalue, we have

\[
\min_{|x|_2 = 1} \left| \frac{1}{\sqrt{n}} A x \right|_2^2 - 1 = \min_{|x|_2 = 1} \left| \frac{1}{\sqrt{n}} (A^\top A - I_p) x \right|_2^2 = \left| \frac{1}{\sqrt{n}} A^\top A - I_p \right|_2^2 \leq H^2(cp + s^2, n; \theta).
\]

This implies \( \sqrt{1 - H^2(cp + s^2, n; \theta)} \leq \frac{1}{\sqrt{n}} \lambda_{\text{min}}(A) \leq \sqrt{1 + H^2(cp + s^2, n; \theta)} \). So we obtain that the two events satisfy

\[
\left\{ \left| \frac{1}{n} A^\top A - I_p \right|_2^2 \leq H^2(cp + s^2, n; \theta) \right\} \\
\subseteq \left\{ \sqrt{1 - H^2(cp + s^2, n; \theta)} \leq \frac{1}{\sqrt{n}} \lambda_{\text{min}}(A) \leq \frac{1}{\sqrt{n}} \lambda_{\text{max}}(A) \leq \sqrt{1 + H^2(cp + s^2, n; \theta)} \right\}
\]

Then we get the second conclusion in this theorem.

\[\Box\]

### 3.2 Statistical applications of sub-Weibull concentrations

In statistical regression analysis, the responses \( \{Y_i\}_{i=1}^n \) in linear regressions are assume to be continuous Gaussian variables. However, the category in classification or grouping may be infinite with index by the non-negative integers. The categorical variables is treated as countable responses for distinction categories or groups; sometimes it can be infinite. In practice, random count responses include the number of patients, the bacterium in the unit region, or stars in the sky and so on. The responses \( \{Y_i\}_{i=1}^n \) with covariates \( \{X_i\}_{i=1}^n \) belongs to generalized linear regressions. We consider i.i.d. random variables \( \{(X_i, Y_i)\}_{i=1}^n \sim (X, Y) \in \mathbb{R}^p \times \mathbb{N} \). By the methods of the maximum likelihood or the M-estimation, The estimator \( \hat{\beta}_n \) is given by

\[
\hat{\beta}_n := \arg \min_{\beta \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^n \ell(X_i^\top \beta, Y_i)
\]

where the loss function \( \ell(\cdot, \cdot) \) is convex and twice differentiable in the first argument.

In high-dimensional regressions, the dimension \( \beta \) may be growing with sample size \( n \). When \( \{Y_i\}_{i=1}^n \) belongs to the exponential family, Portnoy (1988) studied the asymptotic behavior of \( \hat{\beta}_n \) in the generalized linear models (GLMs) as \( p_n := \dim(X) \) is increasing.
The target vector $\hat{\beta} := \arg \min_{\beta \in \mathbb{R}^p} \ell(X^T \beta, Y)$ is assumed to be the loss under the population expectation, comparing to (12). Let $\dot{\ell}(u, y) := \frac{\partial}{\partial t} \ell(t, y)|_{t = u}$, $\ddot{\ell}(u, y) := \frac{\partial}{\partial t} \dot{\ell}(t, y)|_{t = u}$ and $C(u, y) := \sup_{|s - t| \leq u} \frac{\ddot{\ell}(s, y)}{\dot{\ell}(t, y)}$. Finally, define the score function and Hessian matrix of the empirical loss function as $\hat{\mathcal{L}}_n(\beta) := \frac{1}{n} \sum_{i=1}^n \dot{\ell}(X_i^T \beta, Y_i)X_i$ and $\hat{\mathcal{Q}}_n(\beta) := \frac{1}{n} \sum_{i=1}^n \ddot{\ell}(X_i^T \beta, Y_i)X_iX_i^T$, respectively. The population version of Hessian matrix is $\mathcal{Q}(\beta) := \mathbb{E}[\ell(X^T \beta, Y)XX^T]$. The following so-called determining inequalities guarantee the $\ell_2$-error for the estimator obtained from the smooth M-estimator defined as (12).

**Lemma 6** (Corollary 3.1 in Kuchibhotla (2018)). Let $\delta_n(\beta) := \frac{3}{2} \| [\hat{\mathcal{Q}}_n(\beta)]^{-1} \hat{\mathcal{L}}_n(\beta) \|_2$ for $\beta \in \mathbb{R}^p$. If $\ell(\cdot, \cdot)$ is a twice differentiable function that is convex in the first argument and for some $\beta^* \in \mathbb{R}^p$: $\max_{1 \leq i \leq n} C(\|X_i\|_2 \delta_n(\beta^*), Y_i) \leq \frac{4}{3}$. Then there exists a vector $\hat{\beta} \in \mathbb{R}^p$ satisfying $\hat{\mathcal{L}}_n(\hat{\beta}_n) = 0$ as the estimating equation of (12),

$$
\frac{1}{2} \delta_n(\beta^*) \leq \| \hat{\beta}_n - \beta^* \|_2 \leq \delta_n(\beta^*).
$$

Applications of Lemma 6 in regression analysis is of special interest when $X$ is heavy tailed, i.e. the sub-Weibull index $\theta < 1$. For the negative binomial regression (NBR) with the known dispersion parameter $k > 0$, the loss function is

$$
\ell(u, y) = -yu + (y + k) \log(k + e^u).
$$

Thus we have $\dot{\ell}(u, y) = -\frac{k(y-e^u)}{k+e^u}$, $\ddot{\ell}(u, y) = \frac{k(y+k)e^u}{(k+e^u)^2}$, see Zhang and Jia (2022) for details.

Further computation gives $C(u, y) = \sup_{|s - t| \leq u} \frac{e^{(k+e)u}}{(k+e)^2}$ and it implies that $C(u, y) \leq e^{3u}$. Therefore, condition $\max_{1 \leq i \leq n} C(\|X_i\|_2 \delta_n(\beta^*), Y_i) \leq \frac{4}{3}$ in Lemma 6 leads to $\max_{1 \leq i \leq n} \|X_i\|_2 \delta_n(\beta^*) \leq \frac{\log(4/3)}{3}$. This condition need the assumption of the design space for $\max_{1 \leq i \leq n} \|X_i\|_2$.

In NBR with loss (13), one has

$$
\hat{\mathcal{Q}}(\beta^*) := \frac{1}{n} \sum_{i=1}^n \frac{(Y_i+k)e^y_{X_i}^T \beta^* X_iX_i^T}{(k+e^y_{X_i} \beta^* )^2} \quad \text{and} \quad \hat{\mathcal{L}}_n(\beta^*) := -\frac{1}{n} \sum_{i=1}^n \frac{k(Y_i-X_i \beta^*)X_i}{k+e^y_{X_i} \beta^*}.
$$

To guarantee that $\hat{\beta}_n$ approximates $\beta^*$ well, some regularity conditions are need.

- (C.1): For $M_Y > 0$ and $M_X > 0$, assume that $\max_{1 \leq i \leq n} \|Y_i\|_{\psi_1} \leq M_Y$ and the covariates $\{X_{ik}\}$ are uniformly sub-Weibull with $\max_{1 \leq i \leq n, 1 \leq k \leq p} \|X_{ik}\|_{\psi_0} \leq M_X$ for $0 < \theta < 1$.

- (C.2): The vector $X_i$ is sparse. Let $\mathcal{F}_Y := \{ \max_{1 \leq i \leq n} \|Y_i\|_{\psi_1} \leq B, \max_{1 \leq i \leq n} \|X_i\|_2 \leq I_n \}$ with a slowly increasing function $I_n$, we have $P(\mathcal{F}_Y^c) = \varepsilon_n \to 0$.

In addition, to bound $\max_{1 \leq i \leq n, 1 \leq k \leq p} |X_{ik}|$, the sub-Weibull concentration determines:

$$
P\left( \max_{1 \leq i \leq n, 1 \leq k \leq p} |X_{ik}| > t \right) \leq npP( |X_{11}| > t ) \leq 2np \frac{\gamma(t/\|X_{11}\|_{\psi_0})}{\gamma(t)} \leq \delta \Rightarrow t = M_X \log^{1/\theta} \left( \frac{2np}{\delta} \right),
$$

by using Corollary 1. Hence, we define the event for the maximum designs:

$$
\mathcal{F}_{max} = \left\{ \max_{1 \leq i \leq n, 1 \leq k \leq p} |X_{ik}| \leq M_X \log^{1/\theta} \left( \frac{2np}{\delta} \right) \right\} \cap \mathcal{F}_Y.
$$

To make sure that the optimization in (12) has a unique solution, we also require the minimal eigenvalue condition.
• (C.3): Suppose that $b^\top \mathbb{E}(\hat{Q}_n(\beta))b \geq C_{\min}$ is satisfied.

In the proof, to ensure that the random Hessian function has a non-singular eigenvalue, we define the event

$$F_1 = \left\{ \max_{k,j} \left\| \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{Y_i k e^{X_i^\top \beta^*} X_{ik}X_{ij}}{(k + e^{X_i^\top \beta^*})^2} - \mathbb{E} \left( \frac{Y_i k e^{X_i^\top \beta^*} X_{ik}X_{ij}}{(k + e^{X_i^\top \beta^*})^2} \right) \right] \right\| \leq \frac{C_{\min}}{4} \right\}$$

$$F_2 = \left\{ \max_{k,j} \left\| \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{ke^{X_i^\top \beta^*} X_{ik}X_{ij}}{(k + e^{X_i^\top \beta^*})^2} - \mathbb{E} \left( \frac{ke^{X_i^\top \beta^*} X_{ik}X_{ij}}{(k + e^{X_i^\top \beta^*})^2} \right) \right] \right\| \leq \frac{C_{\min}}{4} \right\}.$$

**Theorem 3** (Upper bound for ℓ_2-error). In the NBR with loss (13) and (C.1 – C.3), let

$$M_{BX} = M_X + \frac{B}{\log 2}, \quad R_n := \frac{6M_{BX}M_X}{C_{\min}} \left[ \sqrt{\frac{2p}{n} \log \left( \frac{2p}{\delta} \right)} + \frac{1}{n} \sqrt{p \log \left( \frac{2p}{\delta} \right)} \right] \log^{1/\theta} \left( \frac{2np}{\delta} \right),$$

and $b := (k/n)^2 X(1, \ldots, 1)^\top \in \mathbb{R}^n$. Under the event $F_1 \cap F_2 \cap F_{\max}$, for any $0 < \delta < 1$, if the sample size $n$ satisfies

$$R_n I_n \leq \frac{\log(4/3)}{3},$$

(14)

Let $c_n := e^{-\frac{1}{2} \left( \frac{n^2}{2M_X^2 \log^{2/\theta}(10cP)^2 M_{BX}^2} + \frac{nt}{2M_X^2 \log^{2/\theta}(10cP)^2 M_{BX}^2} \right)} + \exp( -\frac{t^2}{4cC(\theta/2)^2 \|b\|_2^2 \log^2 p} )$ with $t = C_{\min}/4$, then

$$P(\|\hat{\beta}_n - \beta^*\|_2 \leq R_n) \geq 1 - 2p^2 c_n - \delta - \varepsilon_n.$$

A few comments are made on this Theorem. First, in order to get $\|\hat{\beta}_n - \beta^*\|_2 \xrightarrow{p} 0$, we need $p = o(n)$ under sample size restriction (14) with $I_n = o(\log^{-1/\theta}(np) \cdot [n^{-1} \log p]^{-1/2})$. Second, note that the $\varepsilon_n$ in probability $1 - 2p^2 c_n - \delta - \varepsilon_n$ depends on the models size and the fluctuation of the design by the event $F_{\max}$.

**Proof.** Note that for $\forall b \in S^{p-1}$, it yields

$$b^\top \hat{Q}_n(\beta^*)b - b^\top \mathbb{E}(\hat{Q}_n(\beta^*))b \geq -\|b\| \max_{k,j} \|[\hat{Q}_n(\beta^*) - \mathbb{E}\hat{Q}_n(\beta^*)]_{kj}\| = -\max_{k,j} \left| \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{(Y_i + k) k e^{X_i^\top \beta^*} X_{ij}X_{ik}}{(k + e^{X_i^\top \beta^*})^2} - \mathbb{E} \left( \frac{(Y_i + k) k e^{X_i^\top \beta^*} X_{ij}X_{ik}}{(k + e^{X_i^\top \beta^*})^2} \right) \right] \right|.$$

(15)

Consider the decomposition

$$\frac{1}{n} \sum_{i=1}^{n} \left[ \frac{(Y_i + k) k e^{X_i^\top \beta^*} X_{ik}X_{ij}}{(k + e^{X_i^\top \beta^*})^2} - \mathbb{E} \left( \frac{(Y_i + k) k e^{X_i^\top \beta^*} X_{ik}X_{ij}}{(k + e^{X_i^\top \beta^*})^2} \right) \right] = \frac{k}{n} \sum_{i=1}^{n} \left[ \frac{Y_k e^{X_i^\top \beta^*} X_{ik}X_{ij}}{(k + e^{X_i^\top \beta^*})^2} - \mathbb{E} \left( \frac{Y_k e^{X_i^\top \beta^*} X_{ik}X_{ij}}{(k + e^{X_i^\top \beta^*})^2} \right) \right] + \frac{k^2}{n} \sum_{i=1}^{n} \frac{ke^{X_i^\top \beta^*} X_{ik}X_{ij}}{(k + e^{X_i^\top \beta^*})^2} - \mathbb{E} \left( \frac{ke^{X_i^\top \beta^*} X_{ik}X_{ij}}{(k + e^{X_i^\top \beta^*})^2} \right)$$

For the first term, we have under the $F_{\max}$ with $t = C_{\min}/4$

$$P \left( \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{Y_k e^{X_i^\top \beta^*} X_{ik}X_{ij}}{(k + e^{X_i^\top \beta^*})^2} - \mathbb{E} \left( \frac{Y_k e^{X_i^\top \beta^*} X_{ik}X_{ij}}{(k + e^{X_i^\top \beta^*})^2} \right) \right] \right) \geq t, F_{\max}$$

$$\leq 2 \exp \left\{ -\frac{1}{4} \left( \frac{n^2 t^2}{2 \sum_{i=1}^{n} (X_{ik}X_{ij})^2 (\|Y_k\|_{\psi_1} + \frac{\exp(X_k^\top \beta^*)}{\log^2})^2) \right) \right\}$$

$$\leq 2 \exp \left\{ -\frac{1}{4} \left( \frac{n^2 t^2}{2M_X^4 \log^{4/\theta}(10cP)^2 M_{BX}^2} + \frac{nt}{2M_X^2 \log^{2/\theta}(10cP)^2 M_{BX}^2} \right) \right\}.$$
where we use $ke^{X_i^T\beta_0^*}(k + e^{X_i^T\beta_0^*})^{-2} \leq 1$ and the second last inequality is from Corollary 2. For the second term, by Theorem 1 and $\|X_{ik}X_{ij}\|_{\psi_{\theta_j/2}} \leq \|X_{ik}\|_{\psi_0}X_{ij}\|_{\psi_0} \leq M_X^2$ we have

$$P\left(\sum_{i=1}^n \left[\frac{ke^{X_i^T\beta_0^*}X_{ik}X_{ij}}{(k + e^{X_i^T\beta_0^*})^2} - E\left(\frac{ke^{X_i^T\beta_0^*}X_{ik}X_{ij}}{(k + e^{X_i^T\beta_0^*})^2}\right)\right] \geq t, \mathcal{F}_{\text{max}}\right) \leq 2 \exp\left(-\left(\frac{\left[M_X^2 \log^2(\frac{2np}{\delta}) M_B^2X\right]^2 + \frac{M_X^2 \log^2(\frac{2np}{\delta}) M_B^2}{16e^2 C^2(\theta/2)||b||_2^2}}{t^2}\right)\right)$$

where $b = (k/n)M_X^2(1, \ldots, 1)^T \in \mathbb{R}^n$.

Assume that $b^T E(\hat{Q}_n(\beta))b \geq C_{\min}$. Under $\mathcal{F}_1$ and $\mathcal{F}_2$, it shows that by (15): $b^T E(\hat{Q}_n(\beta))b \geq C_{\min} - C_{\min}^2 = C_{\min}^2$. Then

$$P\{\lambda_{\min}(\hat{Q}_n(\beta)) \leq C_{\min}^2\} = P\{b^T E(\hat{Q}_n(\beta))b \leq C_{\min}^2, \forall b \in S^{p-1}\} \leq P\left\{b^T E(\hat{Q}_n(\beta))b \leq C_{\min}^2, \forall b \in S^{p-1}, \mathcal{F}_{\text{max}}\right\} + P(\mathcal{F}_{\text{max}}^c) \leq P(\mathcal{F}_1, \mathcal{F}_{\text{max}}) + P(\mathcal{F}_2, \mathcal{F}_{\text{max}}) + P(\mathcal{F}_{\text{R}}^c(n)) \leq 2p^2 \exp\left(-\frac{\left[\frac{1}{4} M_X^2 \log^2(\frac{2np}{\delta}) M_B^2X\right]^2 + \frac{M_X^2 \log^2(\frac{2np}{\delta}) M_B^2}{16e^2 C^2(\theta/2)||b||_2^2}}{t^2}\right) + P(\mathcal{F}_{\text{max}}^c).$$

Then we have by conditioning on $\mathcal{F}_1 \cap \mathcal{F}_2$

$$\delta_n(\beta) := \frac{3}{2} \|[\hat{Q}_n(\beta)]^{-1}\hat{Z}_n(\beta)\|_2 \leq \frac{3}{C_{\min}} \|[\hat{Z}_n(\beta)]\|_2.$$

By $k/(k + e^{X_i^T\beta_0^*}) \leq 1$, Corollary 2 implies for any $1 \leq k \leq p$,

$$P\left(\sqrt{n} \sum_{i=1}^n \frac{k(Y_i - e^{X_i^T\beta_0^*})X_{ik}}{k + e^{X_i^T\beta_0^*}} \geq 2\left(\frac{2tp}{n} \sum_{i=1}^n X_{ik}^2 \|Y_i - EY_i\|_{\psi_1}^2\right)^{1/2} + 2t\sqrt{\frac{p}{n}} \max_{1 \leq i \leq n, 1 \leq k \leq p} |X_{ik}| \|Y_i - EY_i\|_{\psi_1}\right) \leq 2e^{-t}. \quad (18)$$

Let

$$\lambda_{1n}(t, X) := 2\left(\frac{2tp}{n} \max_{1 \leq k \leq p} \sum_{i=1}^n X_{ik}^2 \|Y_i - EY_i\|_{\psi_1}^2\right)^{1/2} + 2t\sqrt{\frac{p}{n}} \max_{1 \leq i \leq n, 1 \leq k \leq p} |X_{ik}| \|Y_i - EY_i\|_{\psi_1}.$$

We bound $\max_{1 \leq i \leq n, 1 \leq k \leq p} |X_{ik}| \leq M_X \log^1(\frac{2np}{\delta})$ and $\max_{1 \leq k \leq n} \sum_{i=1}^n X_{ik}^2 \leq M_X^2 \log^2(\frac{2np}{\delta})$ under the event $\mathcal{F}_{\text{max}}$. Note that $M_B X = M_X + \frac{B}{\log^2}$, then (C.1) and (C.2) gives

$$\lambda_{1n}(t, X) \leq 2\left(2tpM_B^2X \max_{1 \leq k \leq p} \frac{1}{n} \sum_{i=1}^n X_{ik}^2\right)^{1/2} + 2t\sqrt{\frac{p}{n}} \max_{1 \leq i \leq n, 1 \leq k \leq p} |X_{ik}| M_B X \leq 2M_B X M_X (\sqrt{2tp + t/p} / n) \log^1(2np/\delta) =: \lambda_n(t).$$

So, $P\left(\sqrt{n} \sum_{i=1}^n \frac{k(Y_i - e^{X_i^T\beta_0^*})X_{ik}}{k + e^{X_i^T\beta_0^*}} > \lambda_n(t)\right) \leq e^{-t}$, $k = 1, 2, \cdots, p$. Thus (18) shows

$$P\left(\sqrt{n} \|[\hat{Z}_n(\beta)^*]\|_2 > \lambda_{1n}(t)\right) \leq P\left(\sqrt{n} \|[\hat{Z}_n(\beta)^*]\|_2 > \lambda_{1n}(t), \mathcal{F}_{\text{max}}\right) + P(\mathcal{F}_{\text{max}}^c) \leq P(\bigcup_{k=1}^p \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{k(Y_i - e^{X_i^T\beta_0^*})X_{ik}}{k + e^{X_i^T\beta_0^*}} > \frac{\lambda_{1n}(t)}{\sqrt{p}} \right\}) + P(\mathcal{F}_{\text{max}}^c) \leq 2pe^{-t} + P(\mathcal{F}_{\text{max}}^c) = \delta + \varepsilon_n,$$
where \( t := \log(\frac{2p}{\delta}) \). Then \( \|\hat{\beta}_n - \beta^*\|_2 \leq \delta_n(\beta^*) \leq \frac{3}{C_{\min}} \|\hat{Z}_n(\beta^*)\|_2 \leq \frac{3\Lambda_n(t)}{C_{\min} \sqrt{n}} \) via Lemma 6. Under \( \mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_{\max} \), we get

\[
\|\hat{\beta}_n - \beta^*\|_2 \leq \frac{6M_{BX}M_X}{C_{\min}} \left[ \sqrt{\frac{2p}{n} \log \left(\frac{2p}{\delta}\right)} + \frac{1}{n} \sqrt{p \log \left(\frac{2p}{\delta}\right)} \right] \log^{1/\theta} \left(\frac{2np}{\delta}\right).
\]

Besides, under \( \mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_{\max} \), it gives the condition of \( n \): (14).

4 Conclusions

Concentration inequalities are far-reaching useful in high-dimensional statistical inferences and machine learnings. They can facilitate various explicit non-asymptotic confidence intervals as a function of the sample size and model dimension. Future research suggests that an MGF-based estimation procedure for the unknown GBO norm is crucial to construct non-asymptotic and data-driven confidence intervals for the sample mean. Although we have obtained sharper upper bounds for sub-Weibull concentrations, the lower bounds on tail probabilities are also important in some statistical applications (Zhang and Zhou, 2020). Developing non-asymptotic and sharp lower tail bounds of Weibull r.v.s is left for further study.

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