Multiparticle $\mathcal{N}=8$ mechanics with $F(4)$ superconformal symmetry

Sergey Fedoruk, Evgeny Ivanov

Bogoliubov Laboratory of Theoretical Physics, JINR, Joliot-Curie 6, 141980 Dubna, Moscow region, Russia

fedoruk,eivanov@theor.jinr.ru

Abstract

We present a new multiparticle model of $\mathcal{N}=8$ mechanics with superconformal $F(4)$ symmetry. The system is constructed in terms of two matrix $\mathcal{N}=4$ multiplets. One of them is a dynamical matrix $(1, 4, 3)$ multiplet and another is a semi-dynamical (spin) $(0, 4, 4)$ one. Off-diagonal bosonic components of the $(1, 4, 3)$ multiplet are chosen to take values in the flag manifold $U(n)/[U(1)]^n$ and they carry additional gauge symmetries. The explicit form of the $F(4)$ supersymmetry generators is found. We demonstrate that the $F(4)$ superalgebra constructed contains as subalgebras two different $D(2, 1; \alpha=-1/3)$ superalgebras intersecting over the common $sl(2, \mathbb{R}) \oplus su(2)$ subalgebra.

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1 Introduction

The models of superconformal mechanics occupy a notable place in the study of the AdS/CFT correspondence in supersymmetric gauge theories. This is basically due to the fact that the one-dimensional conformal SL(2,\(\mathbb{R}\)) symmetry naturally emerges as a symmetry of the near horizon geometries of the black-hole solutions of the appropriate supergravities.

The superconformal mechanics systems, pioneered in eighties in the papers \cite{1,2,3}, have been so far worked out mainly up to the case of \(\mathcal{N}=4\), \(d=1\) extended supersymmetry (see, e.g., refs. \cite{4,5,6,7,8,9} and the review \cite{10}). The models with \(\mathcal{N}=8\) superconformal symmetry were studied to much less extent \cite{11,12,13,14,15} But it is just \(\mathcal{N}=8\) superconformal mechanics which is most important from the standpoint of the AdS/CFT correspondence (see a recent review \cite{16} and references therein). Moreover, the important role in this context is played by the exceptional \(\mathcal{N}=8\) superconformal symmetry \(F(4)\) (see, for example, recent papers \cite{17,18}).

The first example of \(\mathcal{N}=8\) superconformal mechanics with \(F(4)\) supersymmetry was presented in \cite{12}. The one-particle system considered there was underlain by an interaction of two \(\mathcal{N}=8\) multiplets: the dynamical \((1,4,3)\) and the semi-dynamical \((0,4,4)\) ones. In the present paper we consider a matrix generalization of this system and, as an outcome, obtain a new model of the multiparticle \(\mathcal{N}=8\) superconformal mechanics.

The matrix models are an efficient tool of constructing conformally invariant systems \cite{22,23,24}. In particular, it was found in \cite{25,26,27} that the matrix one-dimensional superfield models yield Calogero-like systems with \(\mathcal{N}=4\) supersymmetries after exploiting the appropriate gauging procedure \cite{28}. The physical bosonic degrees of freedom were described by the diagonal elements of dynamical bosonic matrix of the \((1,4,3)\) multiplets. The off-diagonal entries of this matrix proved to represent the purely gauge degrees of freedom.

As opposed to the gauging approach of refs. \cite{25,26,27}, in this paper all bosonic variables including the off-diagonal components of the \((1,4,3)\) matrix multiplet are treated as dynamical. These off-diagonal fields parametrize the target space of flags \(U(n_1) \otimes \ldots \otimes U(n)\). So, they can be interpreted as a kind of the target harmonics, while the corresponding part of the worldline action as that of supersymmetric \(d=1\) sigma model on such a manifold.

The plan of the paper is as follows. In Section 2 we present \(\mathcal{N}=4\) harmonic superfield description of the matrix multiplets \((1,4,3)\) and \((0,4,4)\). Also we introduce \(\mathcal{N}=8\) superconformally invariant interaction of these multiplet and find, by Noether procedure, the supercharges generating the relevant \(\mathcal{N}=8\) conformal superalgebra. In Section 3 we split the matrix \(n^2\) bosonic fields into the sets of \(n\) diagonal (radial) and \(n^2-n\) non-diagonal (angular) ones. The latter fields are identified with the target \(U(n_1) \otimes \ldots \otimes U(n)\) harmonics on which some additional gauge symmetries are realized. In Section 4 we eliminate auxiliary fields and pass to the physical variables. Then we fulfill the Hamiltonian analysis of the system. We find the corresponding Hamiltonian and the relevant set of constraints. With respect to

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1 In \cite{15,16} the \(\mathcal{N}=8\) superconformal algebras were derived in terms of the so called \(D\)-module representations.

2 One of the first applications of the exceptional superalgebra \(F(4)\) comes back to refs. \cite{20,21}.

3 The two-dimensional bosonic flag-manifold sigma models were studied, e.g., in \cite{29,30}. The \(SU(3)/(U(1) \otimes U(1))\) harmonics play the crucial role in the off-shell formulation of \(\mathcal{N}=3\), \(d=4\) super Yang-Mills theory \cite{31}.
the second-class constraints, we introduce Dirac brackets. In Section 5 we show that $\mathcal{N}=8$ conformal superalgebra of our system is just $F(4)$ with $so(7)$ as R-symmetry algebra and write down the explicit form of $\mathcal{N}=8$ supercharges. We demonstrate that the underlying $F(4)$ superalgebra contains two $D(2,1;\alpha=-1/3)$ superalgebras with the common $sl(2,\mathbb{R})$ and $su(2)$ subalgebras.\footnote{An analogous closure property for some other $d=1$ superconformal algebras was observed in [32, 33, 34].} Some concluding remarks are collected in the last Section 6.

2 Superconformal coupling of the matrix multiplets $(1, 4, 3)$ and $(0, 4, 4)$

The powerful approach to constructing $\mathcal{N}=4$, $d=1$ supersymmetric models and finding interrelations between them is $\mathcal{N}=4$, $d=1$ harmonic formalism which was proposed in [9]. As compared to the description in the usual superspace with the coordinates $z=(t, \theta_i, \bar{\theta}_i)$, $(\theta_i)^* = \bar{\theta}^i$ and covariant derivatives

$$D^i = \frac{\partial}{\partial \theta^i} - i\bar{\theta}^i \partial t, \quad \bar{D}_i = \frac{\partial}{\partial \bar{\theta}^i} - i\theta^i \partial t, \quad (D^i)^* = -\bar{D}_i, \quad \{D^i, \bar{D}_k\} = -2i \delta^i_k \partial t, \quad (2.1)$$

the harmonic description involves additional commuting harmonic variables

$$u^\pm_i, \quad (u^+_i)^* = u^-_i, \quad u^+_i u^-_i = 1. \quad (2.2)$$

In the harmonic analytic basis

$$z_A = (t_A, \theta^\pm, \bar{\theta}^\pm, u^\pm_i), \quad t_A = t + i(\theta^+ \bar{\theta}^- + \theta^- \bar{\theta}^+), \quad \theta^\pm = \theta^i u^\pm_i, \quad \bar{\theta}^\pm = \bar{\theta}^i u^\pm_i \quad (2.3)$$

half of the $\mathcal{N}=4$ covariant spinor derivatives $D^\pm = u^\pm_i D^i$, $\bar{D}^\pm = u^\pm_i \bar{D}^i$ becomes short:

$$D^+ = \frac{\partial}{\partial \theta^-}, \quad \bar{D}^+ = -\frac{\partial}{\partial \bar{\theta}^-}. \quad (2.4)$$

This implies the existence of the harmonic analytic superfields defined on the analytic subspace of the full harmonic superspace:

$$(\zeta, u) = (t_A, \theta^+, \bar{\theta}^+, u^+_i), \quad u^+_i u^-_i = 1. \quad (2.5)$$

It is closed under both $\mathcal{N}=4$ supersymmetry and $\mathcal{N}=4$ superconformal symmetry. The integration measure in the harmonic analytic subspace is defined as $du d\bar{u} d\zeta^{(-2)} = du_A d\theta^+ d\bar{\theta}^+$. An important tool of the formalism is the harmonic derivatives:

$$D^{\pm \pm} = \partial^{\pm \pm} + 2i \theta^\pm \bar{\theta}^\pm \partial_{t_A} + \theta^\pm \frac{\partial}{\partial \theta^\pm} + \bar{\theta}^\pm \frac{\partial}{\partial \bar{\theta}^\pm}, \quad \partial^{\pm \pm} = u^\pm_i \frac{\partial}{\partial u^\pm_i}. \quad (2.6)$$

The harmonic derivative $D^{++}$ is distinguished in that it commutes with the spinor derivatives \cite{2.4} and so preserves the analyticity.

Here we presented only the definitions of the basic notions which will be used below. The full description of the harmonic superspace approach to $d=1$ models is given in ref. [6] (details of the harmonic formulation of the multiplets which will be considered in this paper can be found in [10, 35]).
2.1 Fermionic matrix multiplet (0, 4, 4)

The multiplet (0, 4, 4) is a fermionic analog of the multiplet (4, 4, 0). We consider $n^2$ multiplets (0, 4, 4) described off shell by the fermionic $n \times n$ matrix analytic superfield $\Psi^+ := \|\Psi^+ \|_a$, $(\Psi^+)^A = \Psi^+_A$, $A = 1, 2$, $a = 1, \ldots, n$, which satisfies the constraint [6]:

$$D^{++}\Psi^+ = 0 \quad \Rightarrow \quad \Psi^+ = \phi^+ u^+_i + \theta^+ F^A + \bar{\theta}^+ F^A - 2i\theta^+ \bar{\theta}^+ \phi^+ u^-_i , \quad (2.7)$$

where $(\phi^A)^\dagger = -\phi_{iA}$, $(F^A)^\dagger = F_A$ or, in terms of the matrix entries,

$$(\phi_{iA}^A)^\dagger = -\phi_{iAb}^a, \quad (F^A_a)^\dagger = F_{Ab}^a .$$

On the doublet index $A = 1, 2$ the appropriate $SU(2)_{PG}$ group acts. It commutes with the $\mathcal{N} = 4$ superconformal $D(2, 1; \alpha)$ transformations which are realized on the component fields as (see, e.g., [35]):

$$\begin{align*}
\delta \phi^A &= - \left( \omega^i F^A + \bar{\omega}^i \bar{F}^A \right), \\
\delta F^A &= 2i \bar{\omega}^i \phi^A_k + 2i\alpha \bar{\eta}^k \phi^A_k , \quad \delta \bar{F}_A = 2i \omega_k \phi^A_k + 2i\alpha \eta_k \phi^A_A .
\end{align*} \quad (2.8)$$

In the central basis, the constraint (2.7) and the analyticity conditions $D^+\Psi^+ = D^+\Psi^+ = 0$ imply

$$\Psi^+ A(z, u) = \Psi^+ A(z) u_i^+ , \quad D^+ i \Psi^+ A(z) = D^+ i \Psi^+ A(z) = 0 . \quad (2.9)$$

The free action of the matrix superfield $\Psi^+$, $S_{\text{free}}^{(\Psi)} \sim \text{Tr} \int dud\bar{u} \left( -2 \right) \Psi^+ \Psi^+_A$, is not invariant under $\mathcal{N} = 4$, $d = 1$ superconformal group $D(2, 1; \alpha)$, except for the special case of $\alpha = 0$, which will be of no interest for us here. As we will see, the superconformal versions of the free $\Psi^+$ action, which are valid for any $\alpha$, can be constructed by means of coupling this multiplet to those considered in the next sections.

2.2 Bosonic matrix multiplet (1, 4, 3)

The off-shell $n^2$ multiplets (1, 4, 3) are described by an Hermitian $\mathcal{N} = 4$ matrix superfield $\mathcal{M}(z) := \|\mathcal{M}_a^b(z)\|$, $\bar{\mathcal{M}} = \mathcal{M}$, obeying the constraints [36]

$$D^i \mathcal{M} = \bar{D}_i \mathcal{M} = 0 , \quad [D^i , \bar{D}_j] \mathcal{M} = 0 . \quad (2.10)$$

These constraints are solved by

$$\mathcal{M}(t, \theta, \bar{\theta}) = M + \theta_i \varphi^i + \bar{\varphi}_i \bar{\theta}^i + i \theta^i \bar{\theta}^k A_{ik} - \frac{i}{2} (\theta)^2 \varphi_i \bar{\theta}^i - \frac{i}{2} (\bar{\theta})^2 \theta_i \varphi^i + \frac{1}{4} (\theta)^2 (\bar{\theta})^2 \bar{\mathcal{M}} , \quad (2.11)$$

where $(\theta)^2 = \theta_i \theta^i$, $(\bar{\theta})^2 = \bar{\theta}^k \bar{\theta}_k$ and $M^\dagger = M$, $(\varphi^i)^\dagger = \bar{\varphi}_i$, $(A_{ik})^\dagger = A_{ik} = A_{(ik)}$ or, in the more detailed notation,

$$(M_a^b)^* = M_b^a , \quad (\varphi_{iA}^b)^* = \bar{\varphi}_{ib}^a , \quad (A_{ik}^b)^* = A_{ikb}^a .$$
The same constraints (2.10), being rewritten in the harmonic superspace, read
\[ D^{++} \mathcal{M} = 0, \quad D^+ D^- \mathcal{M} = \bar{D}^+ \bar{D}^- \mathcal{M} = 0, \quad (D^+ \bar{D}^- + \bar{D}^+ D^-) \mathcal{M} = 0. \tag{2.12} \]
The extra harmonic constraint guarantees the harmonic independence of \( \mathcal{M} \) in the central basis.

As was shown in [28], the \((1, 4, 3)\) multiplet can be also described in terms of the real analytic gauge superfield prepotential. In the matrix case, we can introduce the matrix analytic superfield \( \mathcal{V}(\zeta, u) \) defined up to the abelian gauge transformations
\[ \mathcal{V} \Rightarrow \mathcal{V}' = \mathcal{V} + D^{++} \Lambda^{-}, \quad \Lambda^{-} = \Lambda^{-}(\zeta, u). \tag{2.13} \]
In the Wess-Zumino gauge, just the irreducible \((1, 4, 3)\) content is recovered
\[ \mathcal{V}_{WZ}(\zeta, u) = M(t_A) - 2 \theta^+ \varphi^i(t_A) u_i^- - 2 \bar{\theta}^+ \bar{\varphi}^i(t_A) u_i^- + 3 i \theta^+ \bar{\theta}^+ A^{(ik)}(t_A) u_i^- u_k^- . \tag{2.14} \]
The original matrix superfield \( \mathcal{M}(z) \) is related to \( \mathcal{V}(\zeta, u) \) by the transform
\[ \mathcal{M}(t, \theta^i, \bar{\theta}_k) = \int du \mathcal{V} \left( t + 2 i \theta^i \bar{\theta}^k u^{+}_{i} u^{-}_{k} \right) . \tag{2.15} \]
The constraints (2.10) emerge as a consequence of the harmonic analyticity of \( \mathcal{V} \),
\[ D^+ \mathcal{V} = \bar{D}^+ \mathcal{V} = 0. \tag{2.16} \]

The transformation properties of the component fields in the expansion (2.11) under \( \mathcal{N}=4 \) conformal supersymmetry \( D(2,1;\alpha) \) are given by
\[ \delta M = -\omega_i \varphi^i + \bar{\omega}^i \bar{\varphi}_i , \]
\[ \delta \varphi^i = i \bar{\omega}^i \bar{M} - i \omega_k A^{ki} - 2i \alpha \bar{\eta}^i M , \quad \delta \bar{\varphi}_i = -i \omega_i \bar{M} - i \omega_k A_{ki} + 2i \alpha \eta_i M , \tag{2.17} \]
\[ \delta A_{ik} = -2 \left( \omega_i \bar{\varphi}_k + \omega_k \bar{\varphi}_i \right) + 2(1 + 2\alpha) \left( \eta_i \varphi_k + \bar{\eta}_i \bar{\varphi}_k \right). \]
The general \( \varepsilon_i, \bar{\varepsilon}_k \)-invariant superfield action of the multiplet \((1, 4, 3)\) can be written as
\[ S_{\text{gen}}^{(\mathcal{M})} = \int dt d^4 \theta \mathcal{L}_{\text{gen}}(\mathcal{M}) . \tag{2.18} \]
The \( D(2,1;\alpha) \) invariant action (excepting the values of \( \alpha = 0, -1 \)) is as follows [9]
\[ S_{\text{sc}}^{(\mathcal{M})} \sim \int dt d^4 \theta \text{ Tr} \left( \mathcal{M}^{1/\alpha} \right). \tag{2.19} \]
In this paper we will deal with the choice \( \alpha = -1/3 \). The \( D(2,1;\alpha = -1/3) \) superconformally invariant action for the \((1, 4, 3)\) multiplet is so given by
\[ S_{\text{sc}}^{(\mathcal{M})(\alpha = -1/3)} = -\frac{1}{6} \int dt d^4 \theta \text{ Tr} \left( \mathcal{M}^3 \right). \tag{2.20} \]
In the component notation, the action (2.20) takes the form
\[ S_{\text{sc}}^{(X)} = \int dt \text{ Tr} \left( M \dot{M} \dot{M} - \frac{i}{2} \{ M, \varphi^k \} \dot{\varphi}_k - \frac{i}{2} \{ M, \varphi_k \} \dot{\varphi}^k + \frac{1}{2} M A_{ik} A_{ik} + \frac{i}{2} A^{ik} [\varphi_i, \varphi_k] \right) . \tag{2.21} \]
Using the transformations (2.17) with \( \alpha = -1/3 \), it is easy to directly check \( D(2,1;\alpha = -1/3) \) invariance of the action (2.21).
2.3 Superconformal coupling

Proceeding from the description of the multiplet \((1, 4, 3)\) through the analytic prepotential \(\mathcal{V}\), it is easy to construct its superconformal coupling to \(\Psi^+\) \cite{28}

\[
S^{(M, \Psi)}_{sc} = \frac{1}{2} \int dudc^{(-2)} \, \text{Tr} \left( \mathcal{V} \Psi^+ T^+ \right).
\] (2.22)

This action is superconformal at any \(\alpha \neq 0\) and it also respects the gauge invariance \cite{21, 23} as a consequence of the constraint \((2.7)\). An analysis based on dimensionality and the Grassmann character of the superfields \(\psi^A, \psi^- = D^- \Psi^+\) shows that the coupling \((2.22)\) is the only possible coupling of this fermionic multiplet to the multiplet \((1, 4, 3)\), such that it preserves the canonical number of time derivatives in the component action (no more than two for bosons and no more than one for fermions).

It is easy to find the component-field representation of \((2.22)\)

\[
S^{(M, \Psi)}_{sc} = \frac{1}{2} \int dt \, \text{Tr} \left( M \{ F^A, \dot{F}_A \} - i \{ M, \phi^{iA} \} \dot{\phi}_{iA} \right.
\]
\[
+ iA^{ik} \phi^k \phi_{iA} + \bar{\varphi}^k \{ \phi_{kA}, F^A \} - \phi_{k} \{ \phi^{kA}, F_A \} \right).
\] (2.23)

The total action is the sum of the superconformal \((1, 4, 3)\) action \((2.20)\) and the action \((2.22)\) describing the superconformal coupling \((1, 4, 3) + (0, 4, 4)\) (or the sum of the component actions \((2.21)\) and \((2.23)\)):

\[
S_{(M+\Psi)} = S^{(M)}_{sc} + S^{(M, \Psi)}_{sc}.
\] (2.24)

The variation of the total component action \((2.24)\) with respect to the transformations \((2.8), (2.17)\) can be represented as the integral

\[
\delta_\omega S_{(M+\Psi)} = \int dt \, \Lambda_\omega,
\] (2.25)

where

\[
\Lambda_\omega = -\frac{1}{2} \omega_i \text{Tr} \left( \{ M, \dot{M} \} \varphi^i + \{ M, \varphi_k \} A^i - i \varphi^i \varphi^k \varphi_k + i \{ M, \phi^{iA} \} F_A + 2i \varphi_k \phi^{iA} \phi_A^k \right)
\]
\[
+ \frac{1}{2} \bar{\omega}^i \text{Tr} \left( \{ M, \dot{M} \} \bar{\varphi}_i - \{ M, \bar{\varphi}_k \} A^i_k + i \bar{\varphi}_i \bar{\varphi}^k \varphi_k - i \{ M, \phi_i^{A} \} F^A - 2i \varphi^k \phi^{A}_i \phi_A^{k} \right)
\]
\[
+ \frac{2}{3} \eta_i \text{Tr} \left( M^2 \varphi^i \right) - \frac{2}{3} \bar{\eta}^i \text{Tr} \left( M^2 \bar{\varphi}_i \right).
\] (2.26)

Using this property, the Noether \(\mathcal{N}=4\) supercharges generating the \(\varepsilon\)-transformations are computed to be

\[
Q^i = \text{Tr} \left( P \varphi^i - i \frac{1}{3} \varphi_k \varphi^{(i,k)} - i \varphi_k \phi^{(iA,\phi_A)} \right),
\]
\[
\bar{Q}_i = \text{Tr} \left( P \bar{\varphi}_i + i \frac{1}{3} \bar{\varphi}_k \varphi^{(i,k)} + i \bar{\varphi}_k \phi^{(iA,\phi_A)} \right),
\] (2.27)

where \(P := \{ M, \dot{M} \}\) stands for the matrix momenta. Note that \(\text{Tr} \left( \varphi_k \varphi^{(i,k)} \right) = -\frac{3}{2} \text{Tr} \left( \varphi^i \varphi^k \varphi_k \right),\)
\(\text{Tr} \left( \varphi^k \bar{\varphi}_k \bar{\varphi} \right) = -\frac{3}{2} \text{Tr} \left( \bar{\varphi}_i \bar{\varphi}^k \varphi_k \right).\) The Noether charges associated with the odd \(\eta\)-transformations are

\[
S^i = \frac{4}{3} \text{Tr} \left( M^2 \varphi^i \right) - tQ^i, \quad \bar{S}_i = \frac{4}{3} \text{Tr} \left( M^2 \bar{\varphi}_i \right) - t \bar{Q}_i.
\] (2.28)
2.4 Implicit $\mathcal{N}=4$, $d=1$ supersymmetry

As was shown in [12], in the one-particle case ($n=1$) the total action (2.24) is invariant with respect to extra implicit $\mathcal{N}=4$ supersymmetry transformations. The multiparticle (matrix) generalization of these transformations [37] reads

$$\delta \xi M = -\xi_{iA} \phi^{iA}, \quad \delta \xi \phi^{iA} = \xi_{iA} F_A, \quad \delta \xi \bar{\varphi}_i = -\xi_{iA} \bar{F}_A, \quad \delta \xi A_{ik} = 2 \xi_{iA} \dot{\phi}^A_k,$$

(2.29)

where $\xi_{iA}$ are fermionic parameters. The superfield transformations (2.29) amount to the following ones for the component fields

$$\delta \xi M = -\xi_{iA} \phi^{iA}, \quad \delta \xi \phi^{iA} = \xi_{iA} F_A, \quad \delta \xi \bar{\varphi}_i = -\xi_{iA} \bar{F}_A, \quad \delta \xi A_{ik} = 2 \xi_{iA} \dot{\phi}^A_k,$$

(2.30)

Thus the matrix $\mathcal{N}=4$ multiplets $(1,4,3)$ and $(0,4,4)$ in the model under consideration constitute together $n^2$ matrix $\mathcal{N}=8$ multiplets $(1,8,7)$.

The variation of the total action (2.24) written in terms of components (that is, the sum of the component actions (2.21) and (2.23)) with respect to the transformations (2.30) is the integral

$$\delta \xi S(\mathcal{M}+\Psi) = \int dt \lambda \xi,$$

(2.31)

where

$$\lambda \xi = \frac{1}{2} \xi_{iA} \text{Tr}\left( -M \{ \dot{M}, \phi^{iA} \} + M \{ A_{ik}, \phi^{A}_{k} \} + iM \{ F_{A}, \varphi^{i} \} - iM \{ F^{A}, \varphi^{i} \} 
+ \frac{2i}{3} \phi^{A}_{k} \phi^{(iB)k} + 2i \phi^{A}_{k} \phi^{(i)k} \right).$$

(2.32)

The Noether charges of this hidden supersymmetry are then easily computed to be

$$Q^{iA} = \text{Tr}\left( P \phi^{iA} + \frac{i}{3} \phi^{A}_{k} \phi^{(iB)k} + i \phi^A_k [\varphi^{(i)}, \varphi^{(k)}] \right).$$

(2.33)

In the next sections we will prove that the closure of the supersymmetry transformations (2.28), (2.30) generated by the supercharges (2.27), (2.28), (2.33) is just $\mathcal{N}=8$ conformal superalgebra $F(4)$ [38, 39].

3 Harmonic variables in the Hermitian matrix model

The kinetic terms of bosonic and fermionic fields in the component actions (2.20) and (2.22) are not fully flat. In this section we extract, from the complete set of bosonic fields, $n$ fields having flat kinetic terms. The residual $n(n-1)$ bosonic variables are described by a non-trivial $d=1$ non-linear sigma models and admit a suggestive interpretation as the target
harmonics. After the appropriate redefinitions, the kinetic terms of all fermionic fields will acquire the flat form.

The basic step in proving these assertions will be the spectral decomposition of the matrix $M$. The Hermitian matrix $M = M^\dagger$ is unitarily diagonalizable and its eigenvalue-decomposition takes the form (see, for example, [40])

$$M = U Y U^\dagger,$$

where $n \times n$ matrix $U$ is unitary,

$$U U^\dagger = I.$$

In terms of the entries of $U$,

$$U = \|u_{\alpha}^\beta\|, \quad U^\dagger = \|\bar{u}_{\alpha}^b\|, \quad a = 1, \ldots, n, \quad \alpha = 1, \ldots, n,$$

$$u_{\alpha}^\beta \equiv u_{\alpha}^\beta, \quad \bar{u}_{\alpha}^b \equiv \bar{u}_{\alpha}^b, \quad \bar{u}_{\alpha}^b = (u_{b}^\alpha)^*,$$

the unitarity condition amounts to the relations

$$u_{\alpha}^\gamma \bar{u}_{\gamma}^b = \delta_{\alpha}^b, \quad u_{\alpha}^c \bar{u}_{\beta}^c = \delta_{\alpha}^\beta.$$

The matrix $Y$ in the decomposition (3.1) is a diagonal matrix,

$$Y = \|y_{\alpha}\delta_{\alpha}^\beta\|,$$

with the real eigenvalues $y_{\alpha} = (y_{\alpha})^*$. Thus, the components of the matrix $M$ defined in (3.1) are expressed as

$$M_{a}^b = \sum_{\gamma=1}^{N} y_{\gamma} u_{\gamma}^\alpha \bar{u}_{\gamma}^b.$$

In this paper we will consider the option with unequal eigenvalues of the matrix $M$ (3.1), i.e. with $y_{\alpha} \neq y_{\beta}$ for all $\alpha, \beta$.

Taking into account the diagonal form of the matrix $Y$ and the decomposition (3.1), we observe that the components of the unitary matrix $U$ are defined up to local $[U(1)]^n$ transformations

$$u_{\alpha}^\beta \rightarrow e^{i\vartheta_{\alpha}^\beta} u_{\alpha}^\beta, \quad \bar{u}_{\beta}^b \rightarrow e^{-i\vartheta_{\beta}^b} \bar{u}_{\beta}^b,$$

where $\vartheta_{\alpha}(t) (\alpha = 1, \ldots, n)$ are local real parameters. Thus, the matrix $U$ is defined up to the right local transformations

$$U \rightarrow U h,$$

where the matrix $h$ is diagonal with the components $e^{i\vartheta_{\alpha}}, \alpha = 1, \ldots, n$. In a fixed gauge with respect to these right local shifts, the matrices $U$ involve $n^2 - n$ essential parameters and so parametrize the cosets $U(n)/H$ with the abelian stability subgroups $H = U_1(1) \otimes \ldots \otimes U_n(1)$.

Then the variables $u_{\alpha}^\beta$ and $\bar{u}_{\beta}^b$ can be interpreted as the $U(n)/U(1)$ target harmonics. Similar harmonics were considered in [31] for the case $n = 3$ and in [41, 42] for arbitrary $n$.

In the $n = 2$ case we face just the target space analogs of the standard $SU(2)$ harmonics defined in (2.2) [43, 44].

In accord with what has been said above, the $U(n)$ transformations act on the indices $a, b$ whereas the indices $\alpha, \beta$ are subject to the $U_1(1) \otimes \ldots \otimes U_n(1)$ transformations. The
harmonics $u^a_\alpha$, $\tilde{u}^\beta_\alpha$ play the role of the bridges connecting the quantities with different types of symmetry, $U(n)$ and $[U(1)]^n$.

Let us rewrite the total action (2.24) (the sum of the component actions (2.21) and (2.23)) in terms of the variables $y_\alpha$, $u^\beta_\alpha$, $\tilde{u}^\beta_\alpha$. The crucial role will be played by the relation [24]

$$\dot{M} = U \left( \dot{Y} + [K,Y] \right) U^\dagger, \quad (3.9)$$

where

$$K := \frac{1}{2} \left( U^\dagger \dot{U} - \dot{U}^\dagger U \right), \quad K_{\alpha\beta} = \frac{1}{2} \left( \tilde{u}^\beta_\alpha \dot{u}^\beta_\alpha - \dot{u}^\alpha_\beta u^\alpha_\beta \right). \quad (3.10)$$

The first term in the Lagrangian of the action (2.21) takes the form

$$\text{Tr} \left( M \dot{M} \dot{M} \right) = \text{Tr} \left( Y \ddot{Y} \right) + \text{Tr} \left( Y [K,Y]^2 \right) = \sum_\alpha y_\alpha \dot{y}_\alpha y_\alpha - \frac{1}{2} \sum_{\alpha \neq \beta} (y_\alpha + y_\beta)(y_\alpha - y_\beta)^2 K_{\alpha\beta} K_{\beta\alpha}. \quad (3.11)$$

The remaining terms in the actions (2.21) and (2.23) can be simplified after introducing new fermionic matrix variables,

$$\tilde{\phi}^A_k := U^\dagger \dot{\phi}^A_k U, \quad \tilde{\varphi}^k := U^\dagger \varphi^k U, \quad \tilde{\varphi}_k := U^\dagger \varphi_k U, \quad (3.12)$$

and new bosonic matrix variables,

$$A^{ik} := U^\dagger A^{ik} U, \quad F^A := U^\dagger F^A U, \quad \tilde{F}_A := U^\dagger \tilde{F}_A U. \quad (3.13)$$

New matrix quantities (3.12), (3.13) carry the $U_1(1) \otimes \ldots \otimes U_n(1)$ indices: $(A^{ik})_{\alpha\beta}$, $(\tilde{\varphi}^A)_{\alpha\beta}$, $(\varphi^k)_{\alpha\beta}$, etc.

The definitions (3.12), (3.13) imply the relations

$$U^\dagger \dot{\phi}^A_k U = \dot{\tilde{\phi}}^A_k + [K, \tilde{\phi}^A_k], \quad U^\dagger \varphi^k U = \tilde{\varphi}^k + [K, \tilde{\varphi}^k], \quad U^\dagger \varphi_k U = \tilde{\varphi}_k + [K, \tilde{\varphi}_k]. \quad (3.14)$$

Using them, the total action (2.24) can be rewritten in terms of the new variables (3.5), (3.3), (3.12), (3.13) as

$$S_{(M+\Psi)} = \int dt \text{Tr} \left( Y \ddot{Y} + Y [K,Y]^2 + \frac{1}{2} Y \dot{A}^{ik} \dot{A}_{ik} + \frac{1}{2} Y \{ F^A, \tilde{F}_A \} \right)$$

$$- \frac{i}{2} \int dt \text{Tr} \left( Y, \tilde{\varphi}^k \right) \dot{\tilde{\varphi}}_k + \left\{ Y, \tilde{\varphi}^k \right\} \tilde{\varphi}_k + \left\{ Y, \phi^A \right\} \dot{\phi}_k$$

$$- \frac{i}{2} \int dt \text{Tr} \left( Y, \tilde{\varphi}^k \right) [K, \tilde{\varphi}_k] + \left\{ Y, \varphi^A \right\} \{ [K, \tilde{\varphi}_k] + \left\{ Y, \phi^A \right\} \left\{ K, \phi_k \right\} \right)$$

$$+ \frac{1}{2} \int dt \text{Tr} \left( \left\{ \phi^A, \tilde{\varphi}_k \right\} + \left[ \phi^A, \phi_k \right], \tilde{F}_A \right) A^{ik} + \left[ \tilde{\varphi}^k, \phi_k \right] F^A - \left[ \varphi^k, \phi^A \right] \tilde{F}_A \right) \ . \quad (3.15)$$

The fields $\tilde{A}^{ik}, \tilde{F}^A, \tilde{F}_A$ are auxiliary. In the next section, before performing the Hamiltonian analysis, we will eliminate them and diagonalize the kinetic terms for the bosonic $y$-variables and for the fermionic ones.
4 The physical-variable form of the system

4.1 On-shell action

Elimination of the auxiliary fields $A_{ik}$ and $F^A_i$ in the component action \( S(\mathbb{M}+\Psi) \) by their equations of motion,

\[
\begin{align*}
A_{ik\alpha}^\beta &= -i(y_\alpha + y_\beta)^{-1}\left( [\dot{\varphi}_{i\beta}, \varphi^k_{\cdot\beta}] + \dot{\varphi}^A_{i\cdot\beta} \phi_k^A \right)_{\alpha}^\beta, \\
F^A_{i\alpha} &= (y_\alpha + y_\beta)^{-1}[\varphi^k_{\cdot\alpha}, \dot{\varphi}^{kA}_{\cdot\alpha}], \\
F^A_{\dot{\alpha}A} &= -(y_\alpha + y_\beta)^{-1}[\varphi^k_{\cdot\cdot\alpha}, \dot{\varphi}^{kA}_{\cdot\alpha}],
\end{align*}
\]

(4.1)

produces the 4-fermionic terms. As the result, we obtain the total on-shell superconformal action in the form

\[
S_{(\mathbb{M}+\Psi)} = \int dt \left( \sum_\alpha y_\alpha \dot{y}_\alpha - \frac{1}{2} \sum_{\alpha,\beta} (y_\alpha + y_\beta)(y_\alpha - y_\beta)^2 \mathcal{K}_\alpha^\beta K_\alpha^\beta \right) - \frac{i}{2} \sum_{\alpha,\beta} (y_\alpha + y_\beta) \left( \bar{\varphi}_k^\alpha \bar{\varphi}_k^\alpha + \varphi_k^\alpha \varphi_k^\alpha + \dot{\varphi}^{A\alpha}_{\cdot} \dot{\phi}_{A\alpha}^k \right) \mathcal{K}_\alpha^\beta
\]

\[
+ \frac{i}{2} \sum_{\alpha,\beta,\gamma} (y_\alpha + y_\beta) \left( \bar{\varphi}_k^\alpha \bar{\varphi}_{k\gamma}^\alpha + \varphi_k^\alpha \varphi_{k\gamma}^\alpha + \dot{\varphi}^{A\alpha}_{\cdot} \dot{\phi}_{A\alpha}^k \right) \mathcal{K}_\alpha^\beta \mathcal{K}_\gamma^\beta
\]

\[
+ \frac{i}{2} \sum_{\alpha,\beta} (y_\alpha + y_\beta)^{-1}[\varphi^k_{\cdot\alpha}, \dot{\varphi}^{kA}_{\cdot\alpha}] \left( [\varphi_{i\beta}, \dot{\varphi}^k_{\cdot\beta}] + \phi^B_{i\cdot\beta} \phi_{kB} \right) \beta^\alpha. \quad (4.2)
\]

Redefining field variables as\(^5\)

\[
x_\alpha = \frac{2\sqrt{2}}{3} (y_\alpha)^{3/2}, \quad \psi_{k\alpha}^\beta = (y_\alpha + y_\beta)^{1/2} \varphi_k^\alpha, \quad \chi_{iA}^\alpha = (y_\alpha + y_\beta)^{1/2} \tilde{\phi}_{i\cdot\alpha}^{kA}, \quad \chi_{iA}^{\dot{\alpha}} = (y_\alpha + y_\beta)^{1/2} \bar{\phi}_{i\cdot\alpha}^{kA}, \quad (4.3)
\]

we cast the action \((4.2)\) in the more convenient form

\[
S_{(\mathbb{M}+\Psi)} = \int dt \, L_{(\mathbb{M}+\Psi)}, \quad (4.4)
\]

\[
L_{(\mathbb{M}+\Psi)} = \frac{1}{2} \sum_\alpha \dot{x}_\alpha \dot{x}_\alpha - \frac{i}{2} \sum_{\alpha,\beta} \left( \dot{\psi}_k^\alpha \psi_{k\beta}^\alpha + \psi_k^\alpha \dot{\psi}_{k\beta}^\alpha + \chi_{iA}^\alpha \chi_{iA\beta}^\alpha \right)
\]

\[
- \frac{9}{16} \sum_{\alpha,\beta} \Delta_{\alpha\beta}^+ (\Delta_{\alpha\beta}^-)^2 \mathcal{K}_\alpha^\beta K_\alpha^\beta + \frac{i}{2} \sum_{\alpha,\beta} \Omega_{\alpha\beta}^\beta K_\beta^\alpha + L_{(\text{d.f})}. \quad (4.5)
\]

Here

\[
\Delta_{\alpha\beta}^\pm := x_\alpha^{2/3} \pm x_\beta^{2/3}, \quad (4.6)
\]

\(^5\) Similar fractional-degree redefinitions of the target coordinates appeared in \([35, 19]\).
\[ \Omega^\beta_{\alpha} := \sum_{\gamma} \frac{\Delta^{+}_{\alpha\gamma} + \Delta^{+}_{\beta\gamma}}{(\Delta^{+}_{\alpha\gamma} \Delta^{+}_{\beta\gamma})^{1/2}} \left( \psi_{k\alpha} \gamma^k \gamma^\beta + \bar{\psi}_{k}^\beta \gamma^k \gamma^\alpha + \chi^{iA}_{\alpha} \gamma^{iA}_{\beta} \right) \] (4.7)

and the 4-fermionic term \( L_{(4-f)} \) reads

\[ L_{(4-f)} = \frac{2}{9} \sum_{\alpha,\beta} \frac{1}{(\Delta^{+}_{\alpha\beta})^{1/2}(\Delta^{+}_{\beta\alpha})^{1/2}(\Delta^{+}_{\beta\alpha})^{1/2}} \left\{ \chi^{iA}_{\alpha} \gamma^k \gamma^k \chi^{iA}_{\beta} \chi^{iA}_{\delta} \frac{\delta^\beta}{\delta^\alpha} + \left( \psi_{\alpha}^k \gamma^k \gamma^\beta - \bar{\psi}_{k}^\beta \gamma^k \gamma^\alpha \right) \left( \psi_{\beta}^k \gamma^k \gamma^\alpha - \bar{\psi}_{k}^\beta \gamma^k \gamma^\beta \right) \right\}. \] (4.8)

The Lagrangian \( (4.5) \) contains flat kinetic terms for \( x \)-variables and fermions. Harmonics are dynamical in this model: their second order kinetic term is proportional to \( K^2 \) which is just the relevant target space sigma-model metric.

In the one-particle case \( (n=1) \), the action \( (4.4) \) is reduced to the on-shell action from ref. \[12\] and, at \( \alpha = -1/3 \), to the action from ref. \[35\].

### 4.2 Hamiltonian and harmonic constraints

The Lagrangian \( (4.5) \) yields the following explicit expressions for the momenta:

\[ p_\alpha = \frac{\partial L_{(M+\Psi)}}{\partial \dot{x}_\alpha} = \dot{x}_\alpha, \] (4.9)

\[ \Pi_{\alpha} = \frac{\partial L_{(M+\Psi)}}{\partial \dot{\psi}_{k\alpha}} = \dot{\psi}_{k\alpha}, \quad \bar{\Pi}_{\alpha} = \frac{\partial L_{(M+\Psi)}}{\partial \dot{\bar{\psi}}_{k\beta}} = \dot{\bar{\psi}}_{k\beta}, \] (4.10)

\[ \Pi_{k\alpha} = \frac{\partial L_{(M+\Psi)}}{\partial \dot{\chi}_{k\alpha}} = -\frac{i}{2} \chi_{k\alpha}, \quad \Pi_{k\beta} = \frac{\partial L_{(M+\Psi)}}{\partial \dot{\chi}_{k\beta}} = \frac{i}{2} \chi_{k\beta}, \] (4.10)

\[ \Pi_{k\alpha} = \frac{\partial L_{(M+\Psi)}}{\partial \dot{\chi}_{k\alpha}} = -\frac{i}{2} \chi_{k\alpha}, \quad \Pi_{k\beta} = \frac{\partial L_{(M+\Psi)}}{\partial \dot{\chi}_{k\beta}} = \frac{i}{2} \chi_{k\beta}, \] (4.10)

\[ \pi^b_{\alpha} = \frac{\partial L_{(M+\Psi)}}{\partial \dot{\bar{\psi}}_{k\alpha}} = -\frac{1}{16} \sum_{\beta} \left( 9 \Delta^{+}_{\alpha\beta} \Delta^{-}_{\alpha\beta} \chi_{k\beta}^2 \chi_{k\beta}^2 - 4i \chi_{k\beta}^2 \chi_{k\beta}^2 \right) \bar{u}_{b\alpha}, \] (4.11)

\[ \bar{\pi}^b_{\alpha} = \frac{\partial L_{(M+\Psi)}}{\partial \dot{\bar{\psi}}_{k\alpha}} = \frac{1}{16} \sum_{\beta} \left( 9 \Delta^{+}_{\alpha\beta} \Delta^{-}_{\alpha\beta} \chi_{k\beta}^2 \chi_{k\beta}^2 - 4i \chi_{k\beta}^2 \chi_{k\beta}^2 \right) \bar{u}_{b\alpha}. \] (4.11)

The canonical Hamiltonian

\[ H = p_\alpha \dot{x}_\alpha + \Pi_{\alpha} \dot{\psi}_{k\alpha} + \bar{\Pi}_{\alpha} \dot{\bar{\psi}}_{k\alpha} + \Pi_{k\alpha} \dot{\chi}_{k\alpha} + \bar{\Pi}_{k\beta} \dot{\bar{\chi}}_{k\beta} + \pi^b_{\alpha} \dot{\bar{u}}_{b\alpha} + \bar{\pi}^b_{\alpha} \dot{u}_{b\alpha} - L_{(M+\Psi)} \] (4.12)

takes the form

\[ H = \frac{1}{2} \sum_{\alpha} p_\alpha \dot{p}_\alpha - \frac{4}{9} \sum_{\alpha \neq \beta} \frac{D^\alpha_{\beta} D^\beta_{\alpha}}{\Delta^{+}_{\alpha\beta} (\Delta^{-}_{\alpha\beta})^2} - L_{(4-f)}, \] (4.13)

where

\[ D^\alpha_{\beta} := D^\alpha_{\beta} - \frac{i}{2} \Omega^\alpha_{\beta}, \quad D^\beta_{\alpha} := u^\alpha_{\beta} \bar{\pi}^\alpha_{\beta} - \bar{u}^\alpha_{\beta} \pi^\alpha_{\beta} \] (4.14)
and \( \Omega_\beta^\alpha \) was defined in (4.7). We point out that only the components \( \mathcal{K}^\alpha_\beta \) at \( \alpha \neq \beta \) were used in the calculation of the Hamiltonian (4.13). These quantities are expressed as

\[
\mathcal{K}^\alpha_\beta = -\frac{8 D^\alpha_\beta}{9 \Delta^\alpha_\beta (\Delta^\beta_\alpha)^2}.
\]  

(4.15)

Remind that we consider the case with \( y_\alpha \neq y_\beta \) for any \( \alpha \neq \beta \) and so all \( \Delta^\alpha_\beta \) in (4.15) are non-vanishing.

The expressions (4.10) produce the standard second class constraints for odd variables. After introducing Dirac brackets for them, odd momenta \( \Pi_{k\alpha}^\beta \) are removed from the phase space. The non-vanishing canonical Dirac brackets for the residual variables (at equal times) are

\[
\{x_\alpha, p_\beta\}^* = \delta^\alpha_\beta, \quad \{u_\alpha^\alpha, \pi^\beta_\beta\}^* = \delta^\alpha_\beta , \quad \{\tilde{u}_\alpha^\alpha, \tilde{\pi}^\beta_\beta\}^* = \delta^\alpha_\beta, 
\]

(4.16)

\[
\{\bar{\psi}_\alpha^i, \bar{k}^i_\gamma\}^* = -i \delta^i_\gamma \delta^\alpha_\beta \delta^\beta_\gamma, \quad \{\chi^i\alpha^A_\beta, \chi^j\beta^B_\gamma\}^* = i \epsilon^{ij} \epsilon^{AB} \delta^{\alpha}_\delta \delta^\beta_\gamma ,
\]

(4.17)

where \( \epsilon_{12} = \epsilon^{21} = 1 \).

Taking into account (3.4), we note that in the expressions for the harmonic momenta (4.11) the following conditions are implicitly used

\[
u^\alpha_\alpha \pi^\alpha_\beta = -\bar{u}^\alpha_\beta \bar{\pi}^\alpha_\beta \quad (\alpha, \beta \text{ are arbitrary; sum over } c),
\]

(4.18)

\[
u^\alpha_\alpha \pi^\beta_\beta - \bar{u}^\beta_\alpha \bar{\pi}^\alpha_\beta = i \sum_\gamma \left( \bar{\psi}_\alpha^\gamma \bar{k}^i_\gamma \bar{\pi}^\alpha_\gamma + \bar{\psi}_\alpha^\gamma \bar{\psi}_\gamma^\alpha \pi^\alpha_\gamma + \chi^i_\alpha^A \gamma \chi_i^\alpha \gamma^A \right) \quad (\text{no sum over } \alpha; \text{ sum over } c).
\]

(4.19)

As a consequence, the considered system possesses \( n \) harmonic constraints

\[
\mathcal{D}^\alpha_\alpha := \nu^\alpha_\alpha \pi^\alpha_\alpha - \bar{u}^\alpha_\beta \bar{\pi}^\alpha_\beta - i \sum_\gamma \left( \psi_{\alpha}^\gamma \bar{\psi}_\gamma^\alpha \pi^\alpha_\gamma + \bar{\psi}_\gamma^\alpha \bar{\psi}_\gamma^\alpha \pi^\alpha_\gamma + \chi^i_\alpha^A \gamma \chi_i^\alpha \gamma^A \right) \approx 0
\]

(4.20)

(no sum over \( \alpha \); sum over \( c \)), \( n^2 \) harmonic constraints

\[
G^\alpha_\beta := \nu^\alpha_\alpha \pi^\alpha_\beta + \bar{u}^\beta_\alpha \bar{\pi}^\alpha_\beta \approx 0,
\]

(4.21)

(\( \alpha \) and \( \beta \) are arbitrary, summation over \( c \)) and \( n^2 \) kinematic constraints (3.4)

\[
g^\alpha_\beta := \nu^\alpha_\alpha \bar{u}^\beta_\beta - \delta^\alpha_\beta \approx 0.
\]

(4.22)

It should be pointed out that the quantities \( \mathcal{D}^\alpha_\alpha \) appearing in (4.20) coincide with \( \mathcal{D}^\alpha_\alpha \) from (4.14) at \( \alpha = \beta \): \( \mathcal{D}^\alpha_\alpha = \mathcal{D}^\alpha_\alpha \).

The quantities \( D^\alpha_\beta \) defined in (4.14) form \( u(n) \) algebra with respect to Dirac brackets (4.17),

\[
\{D^\alpha_\beta, D^\gamma_\delta\}^* = \delta^\alpha_\delta D^\gamma_\beta - \delta^\gamma_\beta D^\alpha_\delta,
\]

(4.23)

and commute, in a weak sense, with the quantities \( G^\alpha_\beta \), defined in (4.21),

\[
\{G^\alpha_\beta, D^\gamma_\delta\}^* = \delta^\alpha_\delta G^\gamma_\beta - \delta^\gamma_\beta G^\alpha_\delta.
\]

(4.24)
The non-vanishing Dirac brackets of the constraints (4.21) and (4.22) are

\[
\{G_\alpha^\beta, G_\delta^\gamma\}^* = \delta_\alpha^\gamma D_\delta^\beta - \delta_\delta^\gamma D_\alpha^\beta, \quad (4.25)
\]

\[
\{g_\alpha^\beta, G_\delta^\gamma\}^* = 2\delta_\alpha^\gamma \delta_\beta^\delta + \delta_\delta^\gamma g_\beta^\alpha + \delta_\beta^\gamma g_\delta^\alpha. \quad (4.26)
\]

Therefore, \( n \) constraints \( D_\alpha \) (4.20) are first class, whereas the constraints \( G_\alpha^\beta \) (4.21) and \( g_\beta^\alpha \) (4.22) form \( n^2 \) pairs of second class constraints.

We take account of the second class constraints (4.21), (4.22) by introducing Dirac brackets for them:

\[
\{A, B\}^{**} = \{A, B\}^* + \frac{1}{2} \{A, g_\beta^\alpha\}^*\{G_\alpha^\beta, B\}^* - \frac{1}{2} \{A, G_\alpha^\beta\}^*\{g_\beta^\alpha, B\}^*
\]

\[
- \frac{1}{4} \{A, g_\beta^\alpha\}^* D_\alpha^\gamma\{g_\beta^\gamma, B\} - \frac{1}{4} \{A, g_\beta^\alpha\}^* D_\gamma^\beta\{g_\alpha^\gamma, B\}^*. \quad (4.27)
\]

The bracket (4.27) gives the necessary conditions \( \{A, g_\alpha^\beta\}^{**} = \{A, G_\alpha^\beta\}^{**} = 0 \) for an arbitrary phase variable \( A \). For the non-harmonic variables new Dirac brackets coincide with the brackets (4.16), (4.17):

\[
\{x_\alpha, p_\beta\}^{**} = \delta_\alpha^\beta, \quad \{\psi^i_\alpha^\beta, \bar{\psi}_\gamma^\delta\}^{**} = -i \delta_\delta^\gamma \delta_\beta^\alpha, \quad \{\chi^{iA}_\alpha^\beta, \chi^{jB}_\gamma^\delta\}^{**} = i \epsilon^{ij} \epsilon^{AB} \delta_\delta^\alpha \delta_\gamma^\beta. \quad (4.28)
\]

The new Dirac brackets for the \( u(n) \) generators (4.14) also retain the old form:

\[
\{D_\alpha^\beta, D_\delta^\gamma\}^{**} = \delta_\alpha^\gamma D_\delta^\beta - \delta_\delta^\gamma D_\alpha^\beta. \quad (4.29)
\]

In what follows, this property will be crucial for analyzing the superconformal symmetry.

5 \( \mathcal{N}=8 \) supersymmetry generators

5.1 Odd generators of the superalgebra \( D(2, 1; \alpha = -1/3) \)

Using the relations (3.1), (3.9), (3.12) and (4.3) we find that the generators of the \( \mathcal{N}=4 \) supersymmetry (2.27) take the following form in the new variables

\[
Q_i = \sum_\alpha p_\alpha \psi^i_\alpha + \frac{2\sqrt{2}}{3} \sum_{\alpha \neq \beta} \frac{D_\alpha^\beta \psi^i_\beta}{(\Delta^+_{\alpha\beta})^{1/2} \Delta^-_{\alpha\beta}} - \frac{i2\sqrt{2}}{9} \sum_{\alpha, \beta, \gamma} \psi_{k\alpha}^\beta \left( \bar{\psi}^\beta(\bar{\psi}^\gamma k)^\epsilon_\gamma - \bar{\psi}(\bar{\psi}^\gamma k)^\epsilon_\gamma + 3 \chi^{iA}(\bar{\psi}_A)^\epsilon_\gamma \right),
\]

\[
\bar{Q}_i = \sum_\alpha p_\alpha \bar{\psi}_i^\alpha + \frac{2\sqrt{2}}{3} \sum_{\alpha \neq \beta} \frac{D_\alpha^\beta \bar{\psi}_i^\beta}{(\Delta^+_{\alpha\beta})^{1/2} \Delta^-_{\alpha\beta}} + \frac{i2\sqrt{2}}{9} \sum_{\alpha, \beta, \gamma} \bar{\psi}_{k\alpha}^\beta \left( \bar{\psi}(\bar{\psi}^\gamma k)^\epsilon_\gamma - \bar{\psi}^\gamma k)^\epsilon_\gamma + 3 \chi^{iA}(\bar{\psi}_A)^\epsilon_\gamma \right), \quad (5.1)
\]
where \( p_\alpha = \dot{x}_\alpha \) as in (4.9) and \( \Delta_{\alpha\beta} \) were defined in (4.6). The generators of the superconformal boosts (2.28) are

\[
S^i = \sum_\alpha x_\alpha \psi^i_\alpha - t Q^i, \quad \bar{S}_i = \sum_\alpha x_\alpha \bar{\psi}_i^\alpha - t \bar{Q}_i.
\]

With taking into account the relations (3.1), (3.9), (3.12) and (4.3), the generators of the \( \mathcal{N}=4 \) supersymmetry (2.33) acquire the following form in the new variables

\[
Q_{iA} = \sum_\alpha p_\alpha \chi^{iA}_\alpha + \frac{2\sqrt{2}}{3} \sum_{\alpha \neq \beta} \frac{D_\alpha \chi^{iA}_\beta}{(\Delta_{\alpha\beta})^{1/2} \Delta_{\alpha\beta}}
\]

\[
+ \frac{i2\sqrt{2}}{9} \sum_{\alpha,\beta,\gamma} \chi^{iA}_\alpha \beta \left( i\chi^{iB}_\beta \gamma \chi^{iA}_\gamma \right) + 3 \psi^{(i\beta \gamma} \bar{\psi}^{k)} \gamma - 3 \bar{\psi}^{(i\beta \gamma} \psi^{k)} \gamma \right).
\]

5.2 \( \mathcal{N}=8 \) superalgebras in the \( n=1 \) case

In the one-particle case the indices \( \alpha, \beta \) take only one value and the harmonic variables are absent. The Hamiltonian (4.13) and the supercharges (5.1), (5.2), (5.3) in this case read

\[
H = \frac{1}{2} p^2 + \frac{3}{36} \left( 2 \psi^i \bar{\psi}_k \bar{\psi}^k - 12 \psi^i \bar{\psi}_k \chi^{iA} \chi^{A} \chi^{iB} \chi^{B} \right),
\]

\[
Q^i = p \psi^i - \frac{i \psi^i \chi^{iA}}{9},
\]

\[
\bar{Q}_i = p \bar{\psi}_i + \frac{i \bar{\psi}_i \chi^{iA}}{9},
\]

\[
S^i = x \psi^i - t Q^i, \quad \bar{S}_i = x \bar{\psi}_i - t \bar{Q}_i,
\]

\[
Q^{iA} = p \chi^{iA} + \frac{i \chi^{iA} \psi^i \bar{\psi}_k \bar{\psi}^k}{9}.
\]

Using the \( n=1 \) case form of (1.28),

\[
\{x, p\}^* = 1, \quad \{\psi^i, \bar{\psi}_k\}^* = -i \delta^i_k, \quad \{\chi^{iA}, \chi^{jB}\}^* = i \epsilon^{ij} \epsilon^{AB},
\]

we arrive at the following Dirac brackets for the fermionic generators (5.5), (5.6), (5.7)

\[
\{Q^i, Q^k\}^* = -2i \delta^i_k H, \quad \{Q^i, Q^k\}^* = 0, \quad \{Q^i, \bar{Q}_k\}^* = 0,
\]

\[
\{S^i, \bar{S}_k\}^* = -2i \delta^i_k K, \quad \{S^i, S^k\}^* = 0, \quad \{\bar{S}_i, \bar{S}_k\}^* = 0,
\]

\[
\{Q^i, S^k\}^* = \frac{4i}{3} \epsilon^{ik} I(\psi), \quad \{Q^i, \bar{S}_k\}^* = -\frac{4i}{3} \epsilon^{ik} I(\bar{\psi}),
\]

\[
\{Q^i, \bar{S}_k\}^* = 2i \delta^i_k D - \frac{2i}{3} J^i_k - \frac{4i}{3} \delta^i_k I(\psi),
\]

\[
\{Q^i, S^k\}^* = 2i \delta^k_i J^i_k + \frac{4i}{3} \delta^k_i J^i_k.
\]
Here, the bosonic generators $H$ (defined in (4.13)) and the generators
\[
K = \frac{1}{2} \dot{x}^2 - t \dot{x}p + t^2 H , \quad D = -\frac{1}{2} x^2 + t H ,
\]
\[
I^{(\psi)} = \frac{i}{2} \psi_k \psi^k , \quad \bar{I}^{(\psi)} = \frac{i}{2} \bar{\psi}_k \bar{\psi}^k , \quad I_3^{(\psi)} = \frac{i}{2} \psi_k \bar{\psi}^k
\]
\[
J_{ik} = -i \left[ \psi_i \bar{\psi}^k - \frac{1}{2} \chi_i^A \chi_k A \right] ,
\]
form the following algebra
\[
\{H, K\} = 2D , \quad \{H, D\} = H , \quad \{K, D\} = -K ,
\]
\[
\{I^{(\psi)}, \bar{I}^{(\psi)}\} = 2I_3^{(\psi)} , \quad \{I^{(\psi)}, I_3^{(\psi)}\} = \bar{I}^{(\psi)} , \quad \{\bar{I}^{(\psi)}, I_3^{(\psi)}\} = -I^{(\psi)} ,
\]
\[
\{J_{ij}, J_{kl}\} = \epsilon_{ik} J_{jl} + \epsilon_{jl} J_{ik} .
\]
Finally, defining the quantities $Q_{\mu i}, T_{\mu \nu}, I^{(\psi)}_{\nu \nu \nu}$ as
\[
Q_{1i'} = Q_i , \quad Q_{i2'} = \bar{Q}_i , \quad Q_{2i'} = -S_i , \quad Q_{2i'} = -\bar{S}_i ,
\]
\[
T_{11} = H , \quad T_{22} = K , \quad T_{12} = D ,
\]
\[
I^{(\psi)}_{1i'1'} = I^{(\psi)} , \quad I^{(\psi)}_{2i'2'} = \bar{I}^{(\psi)} , \quad I^{(\psi)}_{1i'2'} = I_3^{(\psi)} ,
\]
we find the closed superalgebra of the full set of generators:
\[
\{Q_{\mu i'}, Q_{\nu kk'}\} = -2i \left( \epsilon_{ik} \epsilon_{i'k'} T_{\mu \nu} - \frac{1}{3} \epsilon_{\mu \nu} \epsilon_{i'k'} J_{ik} - \frac{2}{3} \epsilon_{\mu \nu} \epsilon_{ik} I^{(\psi)}_{i'k'} \right) ,
\]
\[
\{T_{\mu \nu}, T_{\lambda \rho}\} = \epsilon_{\mu \lambda} T_{\nu \rho} + \epsilon_{\nu \rho} T_{\mu \lambda} ,
\]
\[
\{I^{(\psi)}_{i'i'}, I^{(\psi)}_{k'k''}\} = \epsilon_{i'k'} I^{(\psi)}_{j'j''} + \epsilon_{j'j''} I^{(\psi)}_{i'k''} ,
\]
\[
\{T_{\mu \nu}, Q_{\lambda i'i'}\} = -\epsilon_{\lambda (\mu} Q_{\nu i'i'} ,
\]
\[
\{J_{ij}, Q_{\mu k'k}\} = -\epsilon_{k'(i} Q_{\mu k')j} ,
\]
\[
\{I^{(\psi)}_{i'i'}, Q_{\mu k'k}\} = -\epsilon_{k'(i'} Q_{\mu k')i} .
\]
This is none other than the standard form of the superalgebra $D(2,1;\alpha = -1/3)$.

In terms of the quantities
\[
J^{(\psi)}_{ik} = -i \psi_i \bar{\psi}^k , \quad J^{(\chi)}_{ik} = \frac{i}{2} \chi_i^A \chi_k A ,
\]
the “currents” (5.13) are written as
\[
J^{(\psi)}_{ik} = J^{(\psi)}_{ik} + J^{(\chi)}_{ik} .
\]
Using the definition (5.24), we can represent the Hamiltonian (5.4) and the supercharges (5.5), (5.6) in the form
\[
H = \frac{1}{2} \dot{p}^2 + \frac{J^{(\psi)}_{ik} J^{(\psi)}_{ik} + J^{(\chi)}_{ik} J^{(\chi)}_{ik}}{9 x^2} - 6 J^{(\psi)}_{ik} J^{(\chi)}_{ik} ,
\]
\[
Q^i = p \psi^i + \frac{2}{9 x} \psi_k \left( J^{(\psi)}_{ik} - 3 J^{(\chi)}_{ik} \right) ,
\]
\[
\bar{Q}_i = p \bar{\psi}_i - \frac{2}{9 x} \bar{\psi}^k \left( J^{(\psi)}_{ik} - 3 J^{(\chi)}_{ik} \right) .
\]
After passing to the notation

\[ \chi_k = i \chi_{k,A=1}, \quad \bar{\chi}_i = i \chi_{k,A=2}, \quad (\chi^i)^* = \bar{\chi}_i, \] (5.29)

the “current” \( J^{(\chi)}_{ik} \) can be cast in the form similar to \( J^{(\psi)}_{ik} \),

\[ J^{(\chi)}_{ik} = -i \chi_{(i} \bar{\chi}_{k)} , \] (5.30)

and the supercharges (5.8) become

\[ Q^i = p \chi^i + \frac{2}{9x} \chi_k \left( J^{ik}_{(\chi)} - 3 J^{ik}_{(\psi)} \right) , \] (5.31)

\[ \bar{Q}_i = p \bar{\chi}_i - \frac{2}{9x} \bar{\chi}_k \left( J_{ik}^{(\chi)} - 3 J_{ik}^{(\psi)} \right) , \] (5.32)

where \( Q_k = i Q_{k,A=1}, \quad \bar{Q}_k = i Q_{k,A=2} \).

The supercharges (5.31), (5.32) are obtained from the supercharges (5.27), (5.28) via the substitutions \( \chi^i \leftrightarrow \psi^i, \quad \bar{\chi}_i \leftrightarrow \bar{\psi}_i \). Dirac brackets of the supercharges (5.31), (5.32) with \( T_{\mu\nu} \) yield the additional fermionic charges

\[ S^i = x \chi^i - t Q^i , \quad \bar{S}_i = x \bar{\chi}_i - t \bar{Q}_i . \] (5.33)

Defining, similarly to (5.17), the quantities \( Q_{\mu A} \) by

\[ Q_{1i,A=1} = Q_i , \quad Q_{1i,A=2} = \bar{Q}_i , \quad Q_{2i,A=1} = -S_i , \quad Q_{2i,A=2} = -\bar{S}_i , \] (5.34)

we obtain Dirac brackets which are analogous to (5.20)

\[ \{ Q_{\mu A}, Q_{\nu B} \}^{**} = -2i \left( \epsilon_{ik} \epsilon_{AB} T_{\mu\nu} - \frac{1}{3} \epsilon_{\mu\nu} \epsilon_{AB} J_{ik}^{(\chi)} - \frac{2}{3} \epsilon_{\mu\nu} \epsilon_{ik} I^{(\chi)}_{AB} \right) . \] (5.35)

Therefore, the supercharges (5.31), (5.32) and (5.33) also constitute the superalgebra \( D(2, 1; \alpha = -1/3) \), with the bosonic generators \( T_{\mu\nu}, \quad J_{ij} \) defined by the same expressions (5.27), (5.28) as in the previously discussed \( D(2, 1; \alpha = -1/3) \) superalgebra, and with the generators

\[ I^{(\chi)} = \frac{i}{2} \chi_k \bar{\chi}^k , \quad \bar{I}^{(\chi)} = \frac{i}{2} \bar{\chi}_k \chi^k , \quad \bar{I}_3^{(\chi)} = \frac{i}{2} \bar{\chi}_k \chi^k \] (5.36)

instead of \( I^{(\psi)}, \quad \bar{I}^{(\psi)}, \quad I_3^{(\psi)} \). The quantities defined similarly to (5.19),

\[ I^{(\chi)}_{11} = I^{(\chi)} , \quad I^{(\chi)}_{22} = \bar{I}^{(\chi)} , \quad I^{(\chi)}_{12} = I_3^{(\chi)} , \] (5.37)

are combined into the generators

\[ I^{(\chi)}_{AB} = \frac{i}{2} \chi_A \bar{\chi}_B , \] (5.38)

which satisfy the relations

\[ \{ I^{(\chi)}_{AB}, I^{(\chi)}_{CD} \}^{**} = \epsilon_{AC} I^{(\chi)}_{BD} + \epsilon_{BD} I^{(\chi)}_{AC} . \] (5.39)

The crossing Dirac brackets among the supercharges of the two \( \mathcal{N}=4 \) supersymmetries are vanishing:

\[ \{ Q_i, Q_j \}^{**} = \{ \bar{Q}_i, \bar{Q}_j \}^{**} = \{ Q_i, \bar{Q}_j \}^{**} = \{ \bar{Q}_i, Q_j \}^{**} = 0 . \] (5.40)
Also,  
\[ \{S_i, S_j\}^{**} = \{S_i, \bar{S}_j\}^{**} = \{\bar{S}_i, S_j\}^{**} = \{\bar{S}_i, \bar{S}_j\}^{**} = 0. \]  
(5.41)
The only non-vanishing Dirac brackets are  
\[ \{Q_i, S_j\}^{**} = -\{Q_i, S_j\}^{**} = -\frac{4}{3} N_{ij}, \quad \{\bar{Q}_i, \bar{S}_j\}^{**} = -\frac{4}{3} \bar{N}_{ij}, \]  
(5.42)
\[ \{Q_i, \bar{S}_j\}^{**} = -\{\bar{Q}_i, S_j\}^{**} = -\frac{4}{3} K_{ij}, \quad \{\bar{Q}_i, \bar{S}_j\}^{**} = -\bar{N}_{ij}, \]  
(5.43)
where  
\[ N_{ij} := \psi(i\chi_j), \quad \bar{N}_{ij} := \bar{\psi}(i\bar{\chi}_j), \quad K_{ij} := \psi(i\bar{\chi}_j), \quad \bar{K}_{ij} := \bar{\psi}(i\chi_j). \]  
(5.44)
Upon using the three-spinor supercharges (5.17), (5.34), the anticommutators (5.40), (5.41), (5.42), (5.43) can be succinctly written as  
\[ \{Q_{\mu i'}, Q_{\nu kB}\}^{**} = \frac{4}{3} N_{(ik)i'A}, \]  
(5.45)
where  
\[ N_{(ik)1'1} = N_{ik}, \quad N_{(ik)2'2} = \bar{N}_{ik}, \quad N_{(ik)1'2} = K_{ik}, \quad N_{(ik)2'1} = \bar{K}_{ik}. \]  
(5.46)
Using (5.29) and introducing the second two-rank spinor \( \psi_{kk'} \) as  
\[ \psi_k = i\psi_{k,k'=1'}, \quad \bar{\psi}_i = i\psi_{k,k'=2'}, \quad (\psi_{kk'}^{ij})^* = -\psi_{i'i'}, \]  
(5.47)
we rewrite the generators (5.44), (5.46) in the form  
\[ N_{(ij)k'A} = -\psi_{(ik')\chi_jA}. \]  
(5.48)
Also, in terms of the quantities (5.47) the \( su(2) \) generators (5.12), (5.19) become  
\[ I_{i'j'}^{(\psi)} = \frac{i}{2} \psi_k^{i'}\psi_{kj'}. \]  
(5.49)
The bosonic generators \( T_{\mu\nu} \) form \( sl(2, \mathbb{R}) \) algebra. The generators \( N_{(ij)k'A} \) refer to the \( SU(2) \otimes SU(2) \otimes SU(2) \) coset and, together with the \( su(2) \oplus su(2) \oplus su(2) \) generators \( J_{ij}, I_{i'j'}^{(\psi)}, I_{AB}^{(\chi)} \), form just \( so(7) \) R-symmetry algebra.

\[ so(7) : \quad J_{ij} \quad I_{i'j'}^{(\psi)} \quad I_{AB}^{(\chi)} \quad N_{(ij)k'A}. \]  
(5.50)
These bosonic generators and \( T_{\mu\nu} \), together with the odd generators \( Q_{\mu i'}, Q_{\mu i'A} \), constitute \( F(4) \) superalgebra. We observe that the \( F(4) \) superalgebra obtained in this way has the following notable structure

\[ F(4) : \quad I_{i'j'}^{(\psi)} \quad Q_{\mu i'} \quad T_{\mu\nu} \quad J_{ij} \quad Q_{\mu i'A} \quad I_{AB}^{(\chi)} \quad N_{(ij)k'A}. \]  
(5.51)
It includes two \( D(2,1;\alpha=-1/3) \) superalgebras with the common \( sl(2, \mathbb{R}) \) and \( su(2) \) generators \( T_{\mu\nu} \) and \( J_{ij} \) and so can be treated as a closure of these two superalgebras.

\[ \text{Such a decomposition of } so(7) \text{ algebra was used in \cite{45,46}.} \]
5.3 $\mathcal{N}=8$ superalgebras for $n > 1$

Let us introduce, similarly to (5.29), the quantities

$$\chi_{k\alpha}^\beta = i\chi_{k,A=1,\alpha}^\beta, \quad \bar{\chi}_{k\alpha}^\beta = i\chi_{k,A=2,\alpha}^\beta,$$  

(5.52)

which satisfy the relations

$$(\chi_i^\alpha)^* = \bar{\chi}_{i\alpha}, \quad \{\chi_i^\alpha, \bar{\chi}_{k\gamma}\}^{**} = -i\delta_i^\alpha\delta_k^\gamma\delta_\gamma.$$

(5.53)

In terms of the newly defined objects the supercharges (5.1) take the form

$$Q^i = \sum_\alpha p_\alpha\psi_i^\alpha + \frac{2\sqrt{2}}{3} \sum_{\alpha\neq\beta} \frac{D_\beta^\alpha\tilde{\psi}_i^\alpha}{(\Delta_{\alpha\beta}^+)^{1/2}\Delta_{\alpha\beta}}$$

$$- \frac{i2\sqrt{2}}{9} \sum_{\alpha,\beta,\gamma} \tilde{\psi}_k^\beta \left(\tilde{\psi}_i^\gamma\bar{\psi}_{k}\gamma - \tilde{\psi}_i^\gamma\bar{\psi}_{k}\gamma - 3\chi(i_{\beta}^\gamma\bar{\chi}_{k}\gamma)^\alpha + 3\bar{\chi}(i_{\beta}^\gamma\chi_{k}\gamma)^\alpha\right)$$

$$\left(\Delta_{\alpha\beta}^+)^{1/2}(\Delta_{\alpha\beta}^-)^{1/2}(\Delta_{\alpha\beta}^+)^{1/2}\right)$$

(5.54)

$$\tilde{Q}_i = \sum_\alpha p_\alpha\tilde{\psi}_i^\alpha + \frac{2\sqrt{2}}{3} \sum_{\alpha\neq\beta} \frac{D_\alpha^\beta\tilde{\psi}_i^\alpha}{(\Delta_{\alpha\beta}^+)^{1/2}\Delta_{\alpha\beta}}$$

$$+ \frac{i2\sqrt{2}}{9} \sum_{\alpha,\beta,\gamma} \tilde{\psi}_k^\beta \left(\tilde{\psi}_i^\gamma\bar{\psi}_{k}\gamma - \tilde{\psi}_i^\gamma\bar{\psi}_{k}\gamma - 3\chi(i_{\beta}^\gamma\bar{\chi}_{k}\gamma)^\alpha + 3\bar{\chi}(i_{\beta}^\gamma\chi_{k}\gamma)^\alpha\right)$$

$$\left(\Delta_{\alpha\beta}^+)^{1/2}(\Delta_{\alpha\beta}^+)^{1/2}(\Delta_{\alpha\beta}^-)^{1/2}\right).$$

(5.55)

The supercharges (5.54) and (5.55) go into each other through the replacements

$$\left(\psi_{k\alpha}, \bar{\psi}_{k\alpha}^\beta \right) \leftrightarrow \left(\chi_{k\alpha}^\beta, \bar{\chi}_{k\alpha}^\beta \right).$$

(5.56)

The supercharges (5.54), (5.55), together with the generators (5.2), produce the conformal $F(4)$ superalgebra considered in the previous subsection. The Hamiltonian in the multiparticle case is given in (4.13). The remaining even generators have the same form as in (5.19), (5.39), (5.25), (5.24), (5.44), with the matrix variables and the traces of their products. For example, one of the involved quantities is

$$J_{ik} = -i\text{Tr} \left(\psi(i\bar{\psi}_k) + \chi(i\bar{\chi}_k)\right).$$

(5.57)
The basic difference of the multiparticle \((n>1)\) case from its one-particle \((n=1)\) prototype lies in the property that the corresponding fermionic charges form a closed algebra in a weak sense, that is, on the shell of the first class constraints \((4.20)\); for example,

\[
\{Q^i, Q^j\} = -\frac{8}{9} \sum_{\alpha \neq \beta} \frac{\psi^i_\alpha \psi^j_\beta}{\Delta_{\alpha \beta}} \left( D_\alpha - D_\beta \right).
\] (5.58)

Such terms arise from the Dirac brackets \(\{D^\beta_\alpha, D^\delta_\gamma\}\) of the quantities \(D^\beta_\alpha\) which are present in the supercharges \((5.54), (5.55)\). A similar situation occurred in the matrix model of refs. \([26, 27]\).

6 Concluding remarks

In this paper we have presented multiparticle \(\mathcal{N}=8\) superconformal mechanics with the underlying \(F(4)\) supersymmetry. The initial system in our construction is the \(\mathcal{N}=4\) superfield system with the matrix dynamical \((1, 4, 3)\) and the matrix semi-dynamical \((0, 4, 4)\) multiplets. This system exhibits an implicit \(\mathcal{N}=4\) supersymmetry that extends \(\mathcal{N}=4\) superconformal symmetry \(D(2, 1; \alpha = -1/3)\) to \(\mathcal{N}=8\) superconformal symmetry \(F(4)\) with respect to which the \(\mathcal{N}=4\) multiplets are combined into irreducible \((1, 8, 7)\) supermultiplets.

The off-diagonal bosonic components of matrix \((1, 4, 3)\) multiplet take values in the flag coset manifold \(U(n)/[U(1)]^n\) and can be identified with a kind of the target space harmonics. We presented the full set of the first class constraints generating gauge transformations of these variables.

In our model the harmonic variables are dynamical – the Lagrangian is of the second order in the harmonic velocities. Moreover, the covariant harmonic momenta are present in the Calogero-like terms in the supercharges (the second terms in \((5.54), (5.55)\)) and in the Hamiltonian (the second term in \((4.13)\)). It would be interesting to inquire whether some \(\mathcal{N}=8\) supersymmetric reduction in the harmonic sector is possible, such that the numerator(s) in the second terms of the Hamiltonian \((4.13)\) become constants and the latter acquires a more Calogero-like form.

One more interesting problem for the future study is to find a quantum realization of the \(\mathcal{N}=8\) superalgebras we have discussed.

Let us emphasize once more that in this article we dealt with the case when all eigenvalues of the matrix \(M (3.1)\) are unequal, \(y_\alpha \neq y_\beta\) for any \(\alpha \neq \beta\). This case is most general, in the sense that all options with coincident eigenvalues can be recovered as some special limits of it. On the other hand, the cases with equal eigenvalues could be equally considered on their own right along the same lines as here. The basic new point will be the appearance of the equalities \(\Delta_{\alpha \beta} = 0\) when \(y_\alpha = y_\beta\). This will amount to additional first-class constraints apart from the \(U(1)\)-constraints \((4.20)\). If \(k_1\) eigenvalues coincide with each other, there will arise the constraints that generate local symmetry \(U(k_1)\). In general, in the system with \(y_{\alpha_1} = y_{\alpha_2} = \ldots = y_{\alpha_{k_1}}, y_{\beta_1} = y_{\beta_2} = \ldots = y_{\beta_{k_2}}, etc.\), the relevant harmonics parametrize the coset \(U(n)/U(k_1) \otimes U(k_2) \otimes \ldots \otimes U(k_m)\), \(k_1 + k_2 + \ldots + k_m = n\). We plan to investigate multiparticle models with such harmonic sets elsewhere.

\(^7\) Such a coset is in fact a most general flag manifold of \(SU(n)\), while the coset which we dealt with in this paper is the maximal-dimension flag manifold containing all others as submanifolds.
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Appendix A: Calogero-like form of the supercharges in non-flat target space

In terms of the odd variables $\zeta^i \zeta^j$, $\tilde{\zeta}^i \zeta^j$, $\tilde{\phi}^i \zeta^j$, $\bar{\phi}^i \zeta^j$ defined in (3.12) [similarly to (5.52)], $\tilde{\phi}^i \zeta^j$, $\bar{\phi}^i \bar{\phi}^j$ are the components of $\tilde{\phi}^{iA} \zeta^B$, the supercharges (5.54) and (5.55) take the form

$$Q^i = \sum_{\alpha} p_{\alpha}^{(y)} \zeta^i \zeta^\alpha + \sum_{\alpha \neq \beta} (y_\alpha - y_\beta)^{-1} \bar{D}_\alpha^{\beta} \zeta^i \zeta^\alpha - \frac{i}{3} \sum_{\alpha, \beta} \bar{\varphi}^k \zeta^\alpha \left( \bar{\zeta}^{(\alpha)}, \bar{\zeta}^{(k)} \right) \bar{\zeta}^\beta, \quad (A.1)$$

$$Q^j = \sum_{\alpha} p_{\alpha}^{(y)} \tilde{\zeta}^i \zeta^j + \sum_{\alpha \neq \beta} (y_\alpha - y_\beta)^{-1} \bar{D}_\alpha^{\beta} \tilde{\zeta}^i \zeta^j - \frac{i}{3} \sum_{\alpha, \beta} \bar{\varphi}^k \tilde{\zeta}^\alpha \left( \bar{\tilde{\zeta}}^{(\alpha)}, \bar{\tilde{\zeta}}^{(k)} \right) \bar{\zeta}^\beta, \quad (A.2)$$

(and c.c.). The supercharges (A.1) (A.2) act in the phase space parametrized by the bosonic variables $y_\alpha$ defined in (3.5), their momenta $p_{\alpha}^{(y)}$ and harmonic variables.

The supercharges (A.1), (A.2) have the Calogero-like form, as is seen from their second terms involving the typical denominators $(y_\alpha - y_\beta)$. But, as opposed to the “flat” Dirac brackets (4.28), Dirac brackets for the “untilded” variables used here have extra non-vanishing terms: the full set of the non-vanishing Dirac brackets is encompassed by the relations

$$\{y_\alpha, p_{\beta}^{(y)}\}^{**} = \delta_{\alpha \beta}, \quad (A.3)$$

$$\{\zeta^i \zeta^j, p_{\gamma}^{(y)}\}^{**} = -\frac{\delta_\alpha \gamma + \delta_\beta \gamma}{2(y_\alpha + y_\beta)} \zeta^i \zeta^\gamma, \quad \{\bar{\zeta}^{iA} \bar{\zeta}^{jB}, p_{\gamma}^{(y)}\}^{**} = -\frac{\delta_\alpha \gamma + \delta_\beta \gamma}{2(y_\alpha + y_\beta)} \bar{\zeta}^{iA} \bar{\zeta}^{jB}, \quad (A.4)$$

$$\{\phi^i \zeta^j, p_{\gamma}^{(y)}\}^{**} = -\frac{\delta_\alpha \gamma + \delta_\beta \gamma}{2(y_\alpha + y_\beta)} \phi^i \zeta^\gamma, \quad \{\bar{\phi}^{iA} \bar{\phi}^{jB}, p_{\gamma}^{(y)}\}^{**} = -\frac{\delta_\alpha \gamma + \delta_\beta \gamma}{2(y_\alpha + y_\beta)} \bar{\phi}^{iA} \bar{\phi}^{jB}, \quad (A.5)$$

$$\{\zeta^i \zeta^j, \zeta^k \zeta^\gamma\}^{**} = -i \frac{\delta^i_k \delta^j_\beta \delta^\gamma_\alpha}{y_\alpha + y_\beta}, \quad \{\bar{\zeta}^{iA} \bar{\zeta}^{jB}, \bar{\zeta}^{kC} \bar{\zeta}^{\gammaD}\}^{**} = -i \frac{\delta^{iA}_{kC} \delta^{jB}_{\betaD} \delta^{\gammaD}_\alpha}{y_\alpha + y_\beta}. \quad (A.6)$$

Despite this complication, the calculation of the supercharge algebra in the “untilded” variables (A.3)-(A.6) is simpler than in the original “untilded” ones (4.28).

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