NONCOMMUTATIVE RIGIDITY

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Abstract. In this article we prove that the numerical Grothendieck group of every smooth proper dg category is invariant under primary field extensions, and also that the mod-$n$ algebraic $K$-theory of every dg category is invariant under extensions of separably closed fields. As a byproduct, we obtain an extension of Suslin’s rigidity theorem, as well as of Yagunov-Østvær’s equivariant rigidity theorem, to singular varieties. Among other applications, we show that base-change along primary field extensions yields a faithfully flat morphism between noncommutative motivic Galois groups. Finally, along the way, we introduce the category of $n$-adic noncommutative mixed motives.

1. Introduction

Let $l/k$ be a field extension and $X$ an algebraic $k$-variety. On the one hand, it is (well-)known that when the field extension $l/k$ is primary and the algebraic $k$-variety $X$ is smooth and proper, base-change induces an isomorphism between the $\mathbb{Q}$-vector spaces of algebraic cycles up to numerical equivalence:

$$(1.1) \quad (-)_!: \mathbb{Z}^*(X)_{\mathbb{Q}/\text{num}} \xrightarrow{\cong} \mathbb{Z}^*(X_l)_{\mathbb{Q}/\text{num}}.$$  

On the other hand, when $l/k$ is an extension of algebraically closed fields, a remarkable result of Suslin [12] asserts that, for every integer $n \geq 2$ coprime to $\text{char}(k)$, base-change induces an isomorphism in mod-$n$ $G$-theory:

$$(1.2) \quad (-)_!: G_*(X; \mathbb{Z}/n) \xrightarrow{\cong} G_*(X_l; \mathbb{Z}/n).$$  

Among other applications, the isomorphism (1.2) (with $X = \text{Spec}(k)$) enabled Suslin to describe the torsion of the algebraic $K$-theory of every algebraically closed field of positive characteristic, thus solving a longstanding conjecture of Quillen-Lichtenbaum; consult Suslin’s ICM address [13] for further applications.

The main goal of this article is to establish far-reaching noncommutative generalizations of the above rigidity isomorphisms (1.1)-(1.2); consult §2 for applications.

Statement of results. A differential graded (=dg) category $\mathcal{A}$, over a base field $k$, is a category enriched over complexes of $k$-vector spaces; see §3.1. Every (dg) $k$-algebra $A$ gives naturally rise to a dg category with a single object. Another source of examples is proved by algebraic varieties (or more generally by algebraic stacks) since the category of perfect complexes $\text{perf}(X)$, resp. the bounded derived category of coherent $\mathcal{O}_X$-modules $\mathcal{D}^b(\text{coh}(X))$, of every algebraic $k$-variety $X$ admits a canonical dg enhancement $\text{perf}_{\text{dg}}(X)$, resp. $\mathcal{D}^b_{\text{dg}}(\text{coh}(X))$; see [5, §4.6]. Following

\[\text{Date}: \text{May 9, 2017.}\]
\[2010 \text{ Mathematics Subject Classification.} \quad 14A22, 14C25, 19E08, 19E15.\]
\[\text{Key words and phrases.} \quad \text{Algebraic cycles, } K\text{-theory, noncommutative algebraic geometry.}\]
\[\text{The author was partially supported by a NSF CAREER Award.}\]
\[\text{\footnote[1]{Recall that a field extension } l/k \text{ is called primary if the algebraic closure of } k \text{ in } l \text{ is purely inseparable over } k. \text{ Whenever } k \text{ is algebraically closed, every field extension } l/k \text{ is primary.}}\]
Kontsevich [6, 7, 8], a dg category $\mathcal{A}$ is called smooth if it is compact as a bimodule over itself and proper if $\sum_i \dim_k H^i(\mathcal{A}; \mathcal{x, y}) < \infty$ for every ordered pair of objects $(x, y)$. Examples include finite dimensional $k$-algebras $A$ of finite global dimension (when $k$ is perfect) as well as the dg categories of perfect complexes $\text{perf}_{dg}(X)$ associated to smooth proper algebraic $k$-varieties.

Given a smooth proper dg category $\mathcal{A}$, recall from §3.2 the definition of its numerical Grothendieck group $K_0(\mathcal{A})_{/-\text{num}}$. In the same vein, given a dg category $\mathcal{A}$ and an integer $n \geq 2$, recall from §3.3 the definition of the mod-$n$ algebraic $K$-theory groups $K_*(\mathcal{A}; \mathbb{Z}/n)$ as well as of its variants $K_*(\mathcal{A})^\wedge_n$, $K_*(\mathcal{A}; \mathbb{Z}/n)$, $K_*(\mathcal{A})/n$, and $nK_*(\mathcal{A})$. Under these notations, our main result is the following:

**Theorem 1.3** (Noncommutative rigidity). Given a field extension $l/k$, a dg $k$-linear category $\mathcal{A}$, and an integer $n \geq 2$, the following holds:

(i) When the field extension $l/k$ is primary and the dg category $\mathcal{A}$ is smooth and proper, base-change induces an isomorphism:

$$\otimes_{l/k} : K_0(\mathcal{A})_{\mathbb{Q}/-\text{num}} \xrightarrow{\sim} K_0(\mathcal{A} \otimes_{l/k} l)_{\mathbb{Q}/-\text{num}}.$$  

The same holds integrally whenever $k$ is algebraically closed.

(ii) When $l/k$ is an extension of separably closed fields and $n$ is coprime to $\text{char}(k)$, base-change induces an isomorphism:

$$\otimes_{l/k} : K_*(\mathcal{A}; \mathbb{Z}/n) \xrightarrow{\sim} K_*(\mathcal{A} \otimes_{l/k} l; \mathbb{Z}/n).$$

Similarly for the variants $K_*(-)^\wedge_n$, $K_*^\text{et}(-; \mathbb{Z}/n)$, $K_*(-)/n$, and $nK_*(\mathcal{A})$.

**Remark 1.6** (Euler pairing). As explained in §3.2, the numerical Grothendieck group $K_0(\mathcal{A})_{/-\text{num}}$ comes equipped with a non-degenerate Euler bilinear pairing $\chi$. Since base-change preserves the Euler bilinear pairing, the above rigidity isomorphism (1.4) holds moreover for all those invariants which can be extracted from the pair $(K_0(\mathcal{A})_{/-\text{num}}, \chi)$. Among others, these invariants include the Neron-Severi group $\text{NS}(\mathcal{A})$ of surface-like dg categories $\mathcal{A}$ in the sense of Kuznetsov [9, §3].

To the best of the author’s knowledge, Theorem 1.3 is new in the literature. In what follows, we illustrate its strength via several examples:

**Example 1.7** (Algebraic cycles up to numerical equivalence). Let $X$ be a smooth proper algebraic $k$-variety. Thanks to the Hirzebruch-Riemann-Roch theorem, by applying Theorem 1.3(i) to the dg category $\mathcal{A} = \text{perf}_{dg}(X)$, we recover the original rigidity isomorphism (1.1).

**Example 1.8** ($G$-theory). Let $X$ be an algebraic $k$-variety and $l/k$ an extension of algebraically closed fields. By applying Theorem 1.3(ii) to the dg category $\mathcal{A} = \mathcal{D}_{dg}(\text{coh}(X))$, we recover Suslin’s original rigidity isomorphism (1.2).

**Example 1.9** (Algebraic $K$-theory). Let $X$ be an algebraic $k$-variety. By applying Theorem 1.3(ii) to the dg category $\mathcal{A} = \text{perf}_{dg}(X)$, we obtain the isomorphism:

$$(\text{-}!) : K_*(X; \mathbb{Z}/n) \xrightarrow{\sim} K_*(X_l; \mathbb{Z}/n).$$

It is well-known that $G$-theory agrees with algebraic $K$-theory for nonsingular algebraic varieties. Therefore, in the particular case of an extension $l/k$ of algebraically closed fields, the isomorphism (1.10) may be understood as the extension of Suslin’s rigidity isomorphism (1.2) to the case of singular algebraic varieties.
Example 1.11 (Algebraic stacks). Let \( \mathcal{X} \) be an algebraic \( k \)-stack. By applying Theorem 1.3(ii) to the dg category \( \mathcal{A} = \text{perf}_{dg}(\mathcal{X}) \), we obtain the isomorphism:

\[
(\cdot)_! : K^*_{et}(\mathcal{X}; \mathbb{Z}/n) \xrightarrow{\cong} K^*_{et}(X; \mathbb{Z}/n).
\]

Similarly for all the other variants and also for mod-\( n \) \( G \)-theory.

Example 1.13 (Equivariant algebraic \( K \)-theory). Let \( G \) be an algebraic group acting on an algebraic \( k \)-variety \( X \). In the particular case where \( \mathcal{X} \) is the global orbifold \( [X/G] \), (1.12) reduces to the isomorphism in equivariant algebraic \( K \)-theory:

\[
(\cdot)_! : K^*_{et}(X; \mathbb{Z}/n) \xrightarrow{\cong} K^*_{et}(X_G; \mathbb{Z}/n).
\]

When \( X \) is smooth and \( l/k \) is an extension of algebraically closed fields, this isomorphism was originally established by Yagunov-Østvær in [18]. The rigidity isomorphism (1.14) holds similarly for all the other variants and also for mod-\( n \) \( G \)-theory.

Example 1.15 (Twisted algebraic \( K \)-theory). Let \( X \) be an algebraic \( k \)-variety and \( [\alpha] \in H^2_{et}(X; \mathbb{G}_m) \) a (not necessarily torsion) étale cohomology class. By applying Theorem 1.3(ii) to the dg category of \( \alpha \)-twisted perfect complexes \( \mathcal{A} = \text{perf}_{dg}(X; \alpha) \), we obtain the isomorphism in twisted algebraic \( K \)-theory

\[
(\cdot)_! : K^*_{et}(X; \mathbb{Z}/n) \xrightarrow{\cong} K^*_{et}(X^{\alpha}; \mathbb{Z}/n),
\]

where \( \text{res}: H^2_{et}(X; \mathbb{G}_m) \to H^2_{et}(X_G; \mathbb{G}_m) \) stands for the restriction homomorphism. Similarly for all the other variants and also for mod-\( n \) \( G \)-theory.

2. Applications

Noncommutative motivic rigidity. Recall from [16, §4.6] the construction of the category of noncommutative numerical motives \( \text{NNum}(k) \), and from [15, §4] the construction of the triangulated category of noncommutative mixed motives \( \text{NMix}(k; \mathbb{Z}/n) \) (with \( \mathbb{Z}/n \)-coefficients). In the same vein, recall from §5 the construction of the category of \( n \)-adic noncommutative mixed motives \( \text{NMix}(k)_{\mathbb{Q}}^{\wedge} \).

Theorem 2.1. (i) Given a primary field extension \( l/k \), the base-change functor

\[
- \otimes_k l: \text{NNum}(k)_{\mathbb{Q}} \to \text{NNum}(l)_{\mathbb{Q}}
\]

is fully-faithful. The same holds integrally whenever \( k \) is algebraically closed.

(ii) Given an extension \( l/k \) of separably closed fields and an integer \( n \geq 2 \) coprime to \( \text{char}(k) \), the following base-change functors are fully-faithful:

\[
- \otimes_k l: \text{NMix}(k)_{\mathbb{Q}}^{\wedge} \to \text{NMix}(l)_{\mathbb{Q}}^{\wedge} - \otimes_k l: \text{NMix}(k; \mathbb{Z}/n) \to \text{NMix}(l; \mathbb{Z}/n).
\]

On the one hand, the commutative counterpart of item (i) was established by Kahn in [4, Prop. 5.5]; consult Remark 5.6 below for an alternative proof. On the other hand, in the particular case where \( l/k \) is an extension of algebraically closed fields, the commutative counterpart of item (ii) was established by Röndigs-Østvær in [11, Thm. 1.1] (\( n \)-adic case) and by Haesemeyer-Hornbostel in [3, Thm. 30] (\( \mathbb{Z}/n \)-coefficients case).

Noncommutative motivic Galois groups. Recall from [16, §6] the definition of the (conditional) noncommutative motivic Galois group \( \text{Gal}(\text{NNum}^1(k)_{\mathbb{Q}}) \). By combining Theorem 2.1(i) with the Tannakian formalism, we obtain the result:

Theorem 2.2. Given a primary field extension \( l/k \), the induced base-change functor

\[
- \otimes_k l: \text{NNum}^1(k)_{\mathbb{Q}} \to \text{NNum}^1(l)_{\mathbb{Q}}
\]

gives rise to a faithfully flat morphism of affine group schemes \( \text{Gal}(\text{NNum}^1(l)_{\mathbb{Q}}) \to \text{Gal}(\text{NNum}^1(k)_{\mathbb{Q}}) \).
Intuitively speaking, Theorem 2.2 shows that every “\(\otimes\)-symmetry” of the category of noncommutative numerical \(k\)-linear motives can be extended to a “\(\otimes\)-symmetry” of the category of noncommutative \(l\)-linear motives. In the particular case of an extension of algebraically closed fields \(l/k\), the commutative counterpart of Theorem 2.2 was established by Deligne-Milne in [2, Prop. 6.22(b)].

**Extra functoriality.** Our last application shows that the theory of noncommutative numerical motives is equipped with an extra functoriality:

**Theorem 2.3.** Given a primary field extension \(l/k\), with \(\text{char}(k) = 0\), the base-change functor \(- \otimes_{k} l\) : \(\text{NNum}(k)_Q \to \text{NNum}(l)_Q\) admits a left=right adjoint.

**Proof.** As proved in [16, Thm. 4.27], since \(\text{char}(k) = 0\), the categories \(\text{NNum}(k)_Q\) and \(\text{NNum}(l)_Q\) are abelian semi-simple. Therefore, the proof follows from the combination of Theorem 2.1(i) with the general [4, Prop. 5.3].

Roughly speaking, the left=right adjoint functor extracts from each noncommutative numerical \(l\)-linear motive its largest \(k\)-linear sub motive. The commutative counterpart of Theorem 2.3 was established by Kahn in [4, Thm. 5.6].

**Remark 2.4.** Theorem 2.3 is false without the assumption that the field extension \(l/k\) is primary. For example, as we explain below, whenever \(l = k_{\text{sep}}\) and the field extension \(l/k\) is infinite, the functor \(- \otimes_{k} l\) : \(\text{NNum}(k)_Q \to \text{NNum}(l)_Q\) does not admit a left=right adjoint. By construction, the category of noncommutative numerical motives is equipped with a functor \(U(-)_Q : \text{dgcat}_{\text{sp}}(k) \to \text{NNum}(k)_Q\) defined on smooth proper dg categories. Let \(\text{AM}(k)_Q\) be the idempotent completion of the full subcategory of \(\text{NNum}(k)_Q\) consisting of the objects \(U(A)_Q\) with \(A\) a commutative separable \(k\)-algebra. As proved in [17, Prop. 2.3], \(\text{AM}(k)_Q\) is equivalent to the classical category of Artin motives. In particular, this latter category is abelian semi-simple. Making use once again of the general [4, Prop. 5.3], we hence conclude that the inclusion of categories \(\text{AM}(k)_Q \subset \text{NNum}(k)_Q\) admits a left=right adjoint.

Now, let us assume by absurd that the above base-change functor \(- \otimes_{k} l\) admits a left adjoint. This would imply the existence of an Artin motive \(\mathcal{N}M\) such that
\[
\dim \text{Hom}_{\text{AM}(k)_Q}(\mathcal{N}M, U(l')_Q) = \dim \text{Hom}_{\text{NNum}(l)_Q}(U(l)_Q, U(l')_Q \otimes_{k} l) = [l' : k]
\]
for every finite separable field extension \(l' / k\). By definition, \(\mathcal{N}M\) is a direct summand of \(U(l_1 \times \cdots \times l_m)_Q\), where \(l_1, \ldots, l_m\) are finite separable field extensions over \(k\). Therefore, whenever \(l'\) contains \(l_1, \ldots, l_m\), we would conclude that the left-hand side of (2.5) is \(\leq [l_1 : k] + \cdots + [l_m : k]\). This is a contradiction because, since the extension \(l/k\) is infinite, the degree \([l' : k]\) can be arbitrarily high. A similar argument shows that the base-change functor \(- \otimes_{k} l\) does not admit a right adjoint.

3. Preliminaries

**3.1. Dg categories.** Let \(k\) be a commutative ring and \(\mathcal{C}(k)\) the category of complexes of \(k\)-modules. A differential graded (=dg) category \(\mathcal{A}\) is a category enriched over \(\mathcal{C}(k)\) and a dg functor \(F : \mathcal{A} \to \mathcal{B}\) is a functor enriched over \(\mathcal{C}(k)\); consult Keller’s ICM survey [5]. We write \(\text{dgcat}(k)\) for the category of dg categories.

Let \(\mathcal{A}\) be a dg category. The opposite dg category \(\mathcal{A}^{\text{op}}\) has the same objects and \(\mathcal{A}^{\text{op}}(x, y) := \mathcal{A}(y, x)\). A right dg \(\mathcal{A}\)-module is a dg functor \(M : \mathcal{A}^{\text{op}} \to \mathcal{C}_{\text{dg}}(k)\) with values in the dg category \(\mathcal{C}_{\text{dg}}(k)\) of complexes of \(k\)-modules. Let us write \(\mathcal{C}(\mathcal{A})\) for
the category of right dg $\mathcal{A}$-modules. Following [5, §3.2], the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ is defined as the localization of $\mathcal{C}(\mathcal{A})$ with respect to the objectwise quasi-isomorphisms. Let $\mathcal{D}_c(\mathcal{A})$ be the triangulated subcategory of compact objects.

A dg functor $F: \mathcal{A} \to \mathcal{B}$ is called a Morita equivalence if it induces an equivalence on derived categories $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\mathcal{B})$; see [5, §4.6]. As explained in [16, §1.6], the category $\text{dgcat}(k)$ admits a Quillen model structure whose weak equivalences are the Morita equivalences. Let us denote by $\text{Hmo}(k)$ the associated homotopy category.

The tensor product $\mathcal{A} \otimes \mathcal{B}$ of dg categories is defined as follows: the set of objects is the cartesian product and $\mathcal{A} \otimes \mathcal{B}((x, w), (y, z)) := \mathcal{A}(x, y) \otimes \mathcal{B}(w, z)$. As explained in [5, §2.3], this construction gives rise to a symmetric monoidal structure $- \otimes -$ on $\text{dgcat}(k)$, which descends $- \otimes^L -$ to the homotopy category $\text{Hmo}(k)$.

3.2. Numerical Grothendieck group. Let $k$ be a field. Given a proper dg $k$-linear category $\mathcal{A}$, its Grothendieck group $K_0(\mathcal{A}) := K_0(\mathcal{D}_c(\mathcal{A}))$ comes equipped with the Euler bilinear pairing $\chi: K_0(\mathcal{A}) \times K_0(\mathcal{A}) \to \mathbb{Z}$ defined as follows:

$$(3.1) \quad ([M], [N]) \mapsto \sum_j (-1)^j \dim_k \text{Hom}_{\mathcal{D}_c(\mathcal{A})}(M, N[j]).$$

This bilinear pairing is, in general, not symmetric neither skew-symmetric. Nevertheless, when $\mathcal{A}$ is moreover smooth, the associated left and right kernels of $\chi$ agree; see [16, Prop. 4.24]. Consequently, under these assumptions on $\mathcal{A}$, we have a well-defined numerical Grothendieck group $K_0(\mathcal{A})_{/-\text{num}} := K_0(\mathcal{A})/\text{Ker}(\chi)$; similarly, consider the $\mathbb{Q}$-vector space $K_0(\mathcal{A})_{/\text{num}}$. Note that $K_0(\mathcal{A})_{/-\text{num}}$ is torsion-free and that $\chi$ induces a non-degenerate bilinear pairing on this latter group. When char$(k) = 0$, this latter group is known to be finitely generated; see [17, Thm. 1.2].

 Remark 3.2 (Generalization). Let $k$ be a connected\footnote{Recall that a commutative ring $k$ is called connected if Spec$(k)$ is a connected topological space or, equivalently, if $k$ does not contains non-trivial idempotent elements.} commutative ring (e.g. a field) and $\mathcal{A}$ a dg $k$-linear category such that $\sum_i \text{rank}_k H^i(\mathcal{A})(x, y) < \infty$ for every ordered pair of objects $(x, y)$. By replacing $\dim_k \text{Hom}_{\mathcal{D}_c(\mathcal{A})}(M, N[j])$ with $\text{rank}_k \text{Hom}_{\mathcal{D}_c(\mathcal{A})}(M, N[j])$ in the above definition (3.1), we obtain a natural generalization of the Euler bilinear pairing $\chi: K_0(\mathcal{A}) \times K_0(\mathcal{A}) \to \mathbb{Z}$.

3.3. Mod-$n$ algebraic $K$-theory. Recall from [16, §2.2.4] the construction of nonconnective algebraic $K$-theory $K: \text{dgcat}(k) \to \text{Ho}(\text{Spt})$, with values in the homotopy category of (symmetric) spectra. Given an integer $n \geq 2$, consider the triangle $\mathbb{S}^n \to \mathbb{S} \to \mathbb{S}/n \to \Sigma \mathbb{S}$, where $\mathbb{S}$ stands for the sphere spectrum. Following Browder [1] (and Karoubi), mod-$n$ algebraic $K$-theory is defined as follows:

$$(3.3) \quad K(-; \mathbb{Z}/n): \text{dgcat}(k) \to \text{Ho}(\text{Spt}) \quad \mathcal{A} \mapsto K(\mathcal{A}) \wedge \mathbb{S}/n.$$

In addition to mod-$n$ algebraic $K$-theory, we can also consider $n$-adic algebraic $K$-theory $K(A)_n^\wedge := \text{holim}_n K(\mathcal{A}; \mathbb{Z}/n^r)$, mod-$n$ étale $K$-theory $K^\text{et}(\mathcal{A}; \mathbb{Z}/n) := L_{K(1)} K(\mathcal{A}; \mathbb{Z}/n)$ (see [16, §2.2.6]), and the groups $K_*(\mathcal{A})/n$ and $nK_*(\mathcal{A}).$

4. Proof of Theorem 1.3

Proof of item (i). We start by describing the behavior of the numerical Grothendieck group with respect to some field extensions:
Lemma 4.1. Given a field extension \( l/k \) and a smooth proper dg \( k \)-linear category \( \mathcal{A} \), we have the following commutative diagram:

\[
\begin{array}{ccc}
K_0(\mathcal{A} \otimes_k l)_\mathbb{Q} \times K_0(\mathcal{A} \otimes_k l)_\mathbb{Q} & \xrightarrow{\chi} & \mathbb{Q} \\
\bigoplus & & \\
K_0(\mathcal{A})_\mathbb{Q} \times K_0(\mathcal{A})_\mathbb{Q} & \xrightarrow{\chi} & \mathbb{Q}.
\end{array}
\]

Proof. Given right dg \( \mathcal{A} \)-modules \( M, N \in \mathcal{D}_c(\mathcal{A}) \), we have the following natural isomorphisms of \( l \)-vector spaces

\[
\text{Hom}_{\mathcal{D}_c(\mathcal{A})}(M, N[j]) \otimes_k l \simeq \text{Hom}_{\mathcal{D}_c(\mathcal{A} \otimes_k l)}(M \otimes_k l, (N \otimes_k l)[j]) \quad j \in \mathbb{Z}.
\]

This implies that \( \chi([M \otimes_k l], [N \otimes_k l]) = \chi([M], [N]) \). Consequently, the proof follows from the fact that the \( \mathbb{Q} \)-vector space \( K_0(\mathcal{A})_\mathbb{Q} \) is generated by the elements

\[
a_1[M_1] + \cdots + a_n[M_n] \quad \text{with} \quad a_1, \ldots, a_n \in \mathbb{Q} \quad \text{and} \quad M_1, \ldots, M_n \in \mathcal{D}_c(\mathcal{A}).
\]

\( \square \)

Remark 4.2 (Generalization). Let \( k \rightarrow l \) be an homomorphism between connected commutative rings. Given a dg \( k \)-linear category \( \mathcal{A} \) such that \( \sum_i \text{rank}_k H^i(\mathcal{A}, x, y) < \infty \) for every ordered pair \( (x, y) \) (see Remark 3.2), a proof similar to the one of Lemma 4.1 yields the following commutative diagram:

\[
\begin{array}{ccc}
K_0(\mathcal{A} \otimes_k l)_\mathbb{Q} \times K_0(\mathcal{A} \otimes_k l)_\mathbb{Q} & \xrightarrow{\chi} & \mathbb{Q} \\
\bigoplus & & \\
K_0(\mathcal{A})_\mathbb{Q} \times K_0(\mathcal{A})_\mathbb{Q} & \xrightarrow{\chi} & \mathbb{Q}.
\end{array}
\]

Lemma 4.4. Let \( l/k \) be a field extension and \( \mathcal{A} \) a smooth proper dg \( k \)-linear category. Whenever the field extension \( l/k \) is algebraic or the field \( k \) is algebraically closed, base-change induces an injective homomorphism:

\[
- \otimes_k l: K_0(\mathcal{A})_\mathbb{Q}/\text{num} \longrightarrow K_0(\mathcal{A} \otimes_k l)_\mathbb{Q}/\text{num}.
\]

Proof. The \( \mathbb{Q} \)-vector space \( K_0(\mathcal{A})_\mathbb{Q} \), resp. \( K_0(\mathcal{A} \otimes_k l)_\mathbb{Q} \), is generated by the elements \( a_1[M_1] + \cdots + a_n[M_n] \), resp. \( b_1[N_1] + \cdots + b_m[N_m] \), with \( a_1, \ldots, a_n \in \mathbb{Q} \) and \( M_1, \ldots, M_n \in \mathcal{D}_c(\mathcal{A}) \), resp. \( b_1, \ldots, b_m \in \mathbb{Q} \) and \( N_1, \ldots, N_m \in \mathcal{D}_c(\mathcal{A} \otimes_k l) \). Therefore, given a right dg \( \mathcal{A} \)-module \( M \in \mathcal{D}_c(\mathcal{A}) \), such that \( [M] \in \text{Ker}(\chi) \), and a right dg \( (\mathcal{A} \otimes_k l) \)-module \( N \in \mathcal{D}_c(\mathcal{A} \otimes_k l) \), it suffices to show that \( \chi([M \otimes_k l], [N]) = 0 \); the injectivity of (4.5) follows automatically from the above Lemma 4.1.

Let us assume first that \( l/k \) is a finite (algebraic) field extension of degree \( d \). In this case, we have the following adjunction of categories:

\[
\begin{array}{ccc}
\mathcal{D}_c(\mathcal{A} \otimes_k l) & \xrightarrow{\text{res}} & \mathcal{D}(\mathcal{A})_c \\
\bigoplus & & \\
\mathcal{D}(\mathcal{A})_c & & \\
\bigoplus & & \\
\mathcal{D}(\mathcal{A}).
\end{array}
\]

This yields adjunction isomorphisms

\[
\text{Hom}_{\mathcal{D}_c(\mathcal{A} \otimes_k l)}(M \otimes_k l, N[j]) \simeq \text{Hom}_{\mathcal{D}_c(\mathcal{A})}(M, \text{res}(N)[j]) \quad j \in \mathbb{Z}
\]

and hence the following equality:

\[
\dim_k \text{Hom}_{\mathcal{D}_c(\mathcal{A} \otimes_k l)}(M \otimes_k l, N[j]) = d \cdot \dim_k \text{Hom}_{\mathcal{D}_c(\mathcal{A})}(M, \text{res}(N)[j]).
\]
Consequently, we conclude that $\chi([M \otimes_k l], [N]) = d \cdot \chi([M], [\text{res}(N)]) = 0$.

Let us now assume that $l/k$ is an infinite algebraic field extension. In this case, $l$ identifies with the colimit of the filtrant diagram $\{l_i\}_{i \in I}$ of all those intermediate field extensions $l_i/k$ which are finite over $k$. This leads to an equivalence $\colim_{i \in I} \mathcal{D}_c(A \otimes_k l_i) \simeq \mathcal{D}_c(A \otimes_k l)$. Consequently, there exists an index $i_0 \in I$ and a right dg $(A \otimes_k l_{i_0})$-module $N_{i_0} \in \mathcal{D}_c(A \otimes_k l_{i_0})$ such that $N_{i_0} \otimes_{l_{i_0}} l \simeq N$. Making use of Lemma 4.1, we hence obtain the equality $\chi([M \otimes_k l], [N]) = \chi([M \otimes_k l_{i_0}], [N_{i_0}])$. The proof follows now from the equality $\chi([M \otimes_k l_{i_0}], [N_{i_0}]) = d_{i_0} \cdot \chi([M], [\text{res}(N_{i_0})]) = 0$, where $d_{i_0}$ stands for the degree of the finite field extension $l_{i_0}/k$.

Finally, let us assume that $k$ is algebraically closed. Note that $l$ identifies with the colimit of the filtrant diagram $\{k_i\}_{i \in I}$ of all those finitely generated $k$-algebras $k_i$ which are contained in $l$; note that all these $k$-algebras $k_i$ are connected because they are contained in the field $l$. This leads to an equivalence of categories $\colim_{i \in I} \mathcal{D}_c(A \otimes_k k_i) \simeq \mathcal{D}_c(A \otimes_k l)$. Consequently, there exists an index $i_0 \in I$ and a right dg $(A \otimes_k k_{i_0})$-module $N_{i_0} \in \mathcal{D}_c(A \otimes_k k_{i_0})$ such that $N_{i_0} \otimes_{k_{i_0}} l \simeq N$. Making use of Remark 4.2, we hence obtain the equality $\chi([M \otimes_k l], [N]) = \chi([M \otimes_k k_{i_0}], [N_{i_0}])$. Since $k$ is algebraically closed and the $k$-algebra $k_{i_0}$ is finitely generated, the Hilbert’s nilstellensatz theorem implies that the $k$-scheme $\text{Spec}(k_{i_0})$ admits a rational point $p : \text{Spec}(k) \to \text{Spec}(k_{i_0})$. Hence, we can consider the Grothendieck class $[N_{i_0} \otimes_{k_{i_0}} k] = K_0(A)_{\mathbb{Q}}$. Using the fact that the composition $k \to k_{i_0} \overset{p}{\to} k$ is equal to the identity, we then conclude from Remark 4.2 that $\chi([M \otimes_k k_{i_0}], [N_{i_0}]) = \chi([M], [N_{i_0} \otimes_{k_{i_0}} k]) = 0$. This finishes the proof. □

We now describe the behavior of the numerical Grothendieck group with respect to Galois field extensions and purely inseparable field extensions.

**Notation 4.7.** Given a dg $k$-linear category $A$, let us denote by

$$- \circ - : K_0(A)_{\mathbb{Q}} \times K_0(k)_{\mathbb{Q}} \to K_0(A)_{\mathbb{Q}}$$

the bilinear pairing associated to the canonical (right) action of $\mathcal{D}_c(k)$ on $\mathcal{D}_c(A)$.

**Proposition 4.8** (Galois). Given a Galois field extension $l/k$ and a smooth proper dg $k$-linear category $A$, we have an induced isomorphism:

$$- \otimes_k l : K_0(A)_{\mathbb{Q}}/_{\sim_{\text{num}}} \xrightarrow{\simeq} (K_0(A \otimes_k l)_{\mathbb{Q}}/_{\sim_{\text{num}}})^{\text{Gal}(l/k)}.$$  \hspace{1cm} (4.9)

**Proof.** Let us assume first that the field extension $l/k$ is finite of degree $d$. In this case we have the following adjunctions of categories:

$$\xymatrix{ \mathcal{D}_c(A \otimes_k l) \ar[r]^{\text{res}} \ar[d]_{\otimes_k l} & \mathcal{D}_c(l) \ar[d]^{\text{res}} \\
\mathcal{D}_c(A) \ar[r]^{\text{res}} & \mathcal{D}_c(k).}$$

Note that the equality (4.6) in the proof of Lemma 4.4 implies not only that the homomorphism $- \otimes_k l : K_0(A)_{\mathbb{Q}} \to K_0(A \otimes_k l)_{\mathbb{Q}}$ preserves the subspaces $\text{Ker}(\chi)$, but also that the homomorphism $\text{res} : K_0(A \otimes_k l)_{\mathbb{Q}} \to K_0(A)_{\mathbb{Q}}$ preserves the subspaces $\text{Ker}(\chi)$. Hence, we can consider the following two homomorphisms:

$$K_0(A)_{\mathbb{Q}}/_{\sim_{\text{num}}} \xrightarrow{- \otimes_k l} K_0(A \otimes_k l)_{\mathbb{Q}}/_{\sim_{\text{num}}} \quad K_0(A \otimes_k l)_{\mathbb{Q}}/_{\sim_{\text{num}}} \xrightarrow{\text{res}} K_0(A)_{\mathbb{Q}}/_{\sim_{\text{num}}}.$$
Clearly, the composition \( \text{res} \circ (- \otimes_k l) \) is equal to \( d \cdot \text{id} \). Given an element \( \sigma \in G := \text{Gal}(l/k) \), let us write \( \sigma(-) \) for the associated automorphism of \( K_0(A \otimes_k l)_{\mathbb{Q}/\text{num}} \).

Under these notations, the following equalities

\[
\text{res}([N]) \otimes_k l = [N] \cdot (\text{res}([l]) \otimes_k l) \overset{(a)}{=} [N] \cdot \left( \sum_{\sigma \in G} \sigma([l]) \right) = \sum_{\sigma \in G} \sigma([N])
\]

hold for every \( N \in D_c(A \otimes_k l) \); the equality (a) follows from the fact that \( l/k \) is Galois. Consequently, since the \( \mathbb{Q} \)-vector space \( K_0(A \otimes_k l)_{\mathbb{Q}} \) is generated by the elements \( b_1[N_1] + \cdots + b_m[N_m] \), with \( b_1, \ldots, b_m \in \mathbb{Q} \) and \( N_1, \ldots, N_m \in D_c(A \otimes_k l) \), we conclude that the composition \( \text{res}(-) \otimes_k l \) is equal to \( \sum_{\sigma \in G} \sigma(-) \). The proof follows now from the fact that the \( \mathbb{Q} \)-vector space \( (K_0(A \otimes_k l)_{\mathbb{Q}/\text{num}})^G \) agrees with the image of the idempotent endomorphism \( \sum_{\sigma \in G} \sigma(-) \) of \( K_0(A \otimes_k l)_{\mathbb{Q}/\text{num}} \).

Let us now assume that the field extension \( l/k \) is infinite. In this case, \( l \) identifies with the colimit of the filtrant diagram \( \{l_i\}_{i \in I} \) of all those intermediate field extensions \( l_i/k \) which are finite and Galois over \( k \). This leads to an equivalence of categories \( \text{colim}_{i \in I} D_c(A \otimes_k l_i) \simeq D_c(A \otimes_k l) \) and hence to an isomorphism \( \text{colim}_{i \in I} K_0(A \otimes_k l_i)_{\mathbb{Q}/\text{num}} \simeq K_0(A \otimes_k l)_{\mathbb{Q}/\text{num}} \). Thanks to Lemma 4.4, note that we have a similar isomorphism \( \text{colim}_{i \in I} K_0(A \otimes_k l_i)_{\mathbb{Q}/\text{num}} \simeq K_0(A \otimes_k l)_{\mathbb{Q}/\text{num}} \). Consequently, the proof follows from the following natural isomorphisms

\[
(K_0(A \otimes_k l)_{\mathbb{Q}/\text{num}})^{\text{Gal}(l/k)} \simeq (\text{colim}_{i \in I} K_0(A \otimes_k l_i)_{\mathbb{Q}/\text{num}})^{\text{Gal}(l/k)} \\
\simeq (\text{colim}_{i \in I} K_0(A \otimes_k l_i)_{\mathbb{Q}/\text{num}})^{\text{Gal}(l/k)} \\
\simeq (\text{colim}_{i \in I} K_0(A)_{\mathbb{Q}/\text{num}})^{\text{Gal}(l/k)} \\
\simeq K_0(A)_{\mathbb{Q}/\text{num}} ,
\]

where in (4.10) we are (implicitly) using the surjection \( \text{Gal}(l/k) \to \text{Gal}(l_i/k) \).

\[\square\]

**Proposition 4.11** (Purely inseparable). Given a purely inseparable field extension \( l/k \) and a smooth proper dg \( k \)-linear category \( A \), we have an induced isomorphism:

\[
- \otimes_k l: K_0(A)_{\mathbb{Q}/\text{num}} \xrightarrow{\simeq} K_0(A \otimes_k l)_{\mathbb{Q}/\text{num}} .
\]

**Proof.** Let us assume first that the field extension \( l/k \) is finite of degree \( d \). Similarly to the proof of Proposition 4.8, the composition \( \text{res} \circ (- \otimes_k l) \) is equal to \( d \cdot \text{id} \). On the other hand, the following equalities

\[
\text{res}([N]) \otimes_k l = [N] \cdot (\text{res}([l]) \otimes_k l) \overset{(a)}{=} [N] \cdot (d \cdot [l]) = d \cdot [N]
\]

hold for every \( N \in D_c(A \otimes_k l) \); the equality (a) follows from [10, §7 Prop. 4.8]. Consequently, since the \( \mathbb{Q} \)-vector space \( K_0(A \otimes_k l)_{\mathbb{Q}} \) is generated by the \( \mathbb{Q} \)-vector space \( K_0(A \otimes_k l)_{\mathbb{Q}} \) is generated by the elements \( b_1[N_1] + \cdots + b_m[N_m] \), with \( b_1, \ldots, b_m \in \mathbb{Q} \) and \( N_1, \ldots, N_m \in D_c(A \otimes_k l) \), we conclude that the composition \( \text{res}(-) \otimes_k l \) is equal to \( d \cdot \text{id} \). This implies that the above induced homomorphism (4.12) is invertible.

Let us now assume that the field extension \( l/k \) is infinite. In this case, \( l \) identifies with the colimit of the filtrant diagram \( \{l_i\}_{i \in I} \) of all those intermediate field extensions \( l_i/k \) which are finite and purely inseparable over \( k \). This leads to an equivalence of categories \( \text{colim}_{i \in I} D_c(A \otimes_k l_i) \simeq D_c(A \otimes_k l) \) and hence to an isomorphism \( \text{colim}_{i \in I} K_0(A \otimes_k l_i)_{\mathbb{Q}/\text{num}} \simeq K_0(A \otimes_k l)_{\mathbb{Q}/\text{num}} \). Consequently, the proof follows from the preceding finite-dimensional case. \[\square\]
We now have all the ingredients necessary for the proof of item (i). Let us assume first that \( l/k \) is a field extension with \( k \) algebraically closed. The injectivity of (1.4) follows automatically from Lemma 4.4. In order to prove the surjectivity of (1.4), note first that \( l \) identifies with the colimit of the filtrant diagram \( \{ k_i \}_{i \in I} \) of all those finitely generated \( k \)-algebras \( k_i \) which are contained in \( l \); all these \( k \)-algebras \( k_i \) are connected because they are contained in the field \( l \). This leads to an equivalence of categories \( \text{colim}_{i \in I} \mathcal{D}_c(\mathcal{A} \otimes_k k_i) \simeq \mathcal{D}_c(\mathcal{A} \otimes_k l) \) and hence to an isomorphism \( \text{colim}_{i \in I} K_0(\mathcal{A} \otimes_k k_i) \cong K_0(\mathcal{A} \otimes_k l) \). Therefore, given an element \( \alpha \in K_0(\mathcal{A} \otimes_k l)_q \), there exists an index \( i_0 \in I \) and an element \( \alpha_{i_0} \in K_0(\mathcal{A} \otimes_k k_{i_0})_q \) such that \( \alpha_{i_0} \otimes_{k_{i_0}} l = \alpha \). Since \( k \) is algebraically closed and the \( k \)-algebra \( k_{i_0} \) is finitely generated, the Hilbert’s nullstellensatz theorem implies that the \( k \)-scheme \( \text{Spec}(k_{i_0}) \) admits a rational point \( p: \text{Spec}(k) \to \text{Spec}(k_{i_0}) \). Hence, we can consider the element \( \alpha_{i_0} \otimes_{k_{i_0}} l \in K_0(\mathcal{A})_q \). We now claim the following:

\[
\chi(\alpha, \beta) = \chi((\alpha_{i_0} \otimes_{k_{i_0}} l) \otimes_k l, \beta) \quad \forall \beta \in K_0(\mathcal{A} \otimes_k l)_q.
\]

Note that since \( \alpha \) is arbitrary, this claim would imply the surjectivity of (1.4). As above, given an element \( \beta \in K_0(\mathcal{A} \otimes_k l)_q \), there exists an index \( i'_0 \in I \) and an element \( \beta_{i'_0} \in K_0(\mathcal{A} \otimes_k k_{i'_0})_q \) such that \( \beta_{i'_0} \otimes_{k_{i'_0}} l = \beta \). Since \( I \) is a filtered diagram, we can (and will) assume without loss of generality that there exists a morphism \( i_0 \to i'_0 \) in \( I \). In particular, this yields the base-change homomorphism

\[
- \otimes_{k_{i_0}} l: K_0(\mathcal{A} \otimes_k k_{i_0})_q \to K_0(\mathcal{A} \otimes_k k_{i'_0})_q.
\]

Therefore, thanks to the general Remark 4.2, in order to prove the above claim (4.13), it suffices to show that

\[
\chi(\alpha_{i_0} \otimes_{k_{i_0}} l_{i'_0} \otimes_{k_{i'_0}} b), (\beta_{i_0} \otimes_{k_{i_0}} l_{i'_0} \otimes_{k_{i'_0}} b) = \chi((\alpha_{i_0} \otimes_{k_{i_0}} l) \otimes_k k_{i'_0}, \beta_{i'_0} \otimes_{k_{i'_0}} b).
\]

Since \( k \) is algebraically closed and the \( k \)-algebra \( k_{i'_0} \) is finitely generated, the \( k \)-scheme \( \text{Spec}(k_{i'_0}) \) admits a rational point \( q: \text{Spec}(k) \to \text{Spec}(k_{i'_0}) \). Hence, we can consider the base-change homomorphism

\[
- \otimes_{k_{i'_0}} l: K_0(\mathcal{A} \otimes_k k_{i'_0})_q \to K_0(\mathcal{A})_q.
\]

Making use once again of the general Remark 4.2, we observe that (4.14) holds if and only if the following equality holds:

\[
\chi((\alpha_{i_0} \otimes_{k_{i_0}} l_{i'_0} \otimes_{k_{i'_0}} l) \otimes_{k_{i'_0}} k_{i'_0}, b \otimes_{k_{i'_0}} l_{i'_0} \otimes_{k_{i'_0}} l) = \chi((\alpha_{i_0} \otimes_{k_{i_0}} l) \otimes_k k_{i'_0}) \otimes_{k_{i'_0}} l, b \otimes_{k_{i'_0}} l_{i'_0} \otimes_{k_{i'_0}} l).
\]

Thanks to the general Remark 4.2, the left-hand side is equal to

\[
\chi(\alpha_{i_0} \otimes_{k_{i_0}} l_{i'_0} \otimes_{k_{i'_0}} l, b \otimes_{k_{i'_0}} l_{i'_0} \otimes_{k_{i'_0}} l) = \chi(\alpha_{i_0}, (\beta_{i'_0} \otimes_{k_{i'_0}} l) \otimes_{k_{i'_0}} k_{i'_0})
\]

and

\[
\chi(\alpha_{i_0} \otimes_{k_{i_0}} l, b \otimes_{k_{i'_0}} l_{i'_0} \otimes_{k_{i'_0}} l) = \chi(\alpha_{i_0} \otimes_{k_{i_0}} l, b \otimes_{k_{i'_0}} l_{i'_0} \otimes_{k_{i'_0}} l),
\]

where in (4.15), resp. (4.16), we are (implicitly) using the fact that the composition \( k \to k_{i'_0} \otimes_{k_{i'_0}} l_{i'_0} \otimes_{k_{i'_0}} l \) is equal to the identity. Similarly, thanks to the general Remark 4.2, the right-hand side is equal to

\[
\chi((\alpha_{i_0} \otimes_{k_{i_0}} l) \otimes_k k_{i'_0}, (\beta_{i'_0} \otimes_{k_{i'_0}} l) \otimes_k k_{i'_0}) = \chi(\alpha_{i_0} \otimes_{k_{i_0}} l, b \otimes_{k_{i'_0}} l_{i'_0} \otimes_{k_{i'_0}} l).
\]

where in (4.17) we are (implicitly) using the fact that the composition \( k \to k_{i'_0} \otimes_{k_{i'_0}} l_{i'_0} \otimes_{k_{i'_0}} l \) is equal to the identity. This proves the above claim (4.13) and hence shows that
the base-change homomorphism (1.4) is invertible. Finally, note that the above proof also holds integrally. Consequently, base-change induces an isomorphism:

\[- \otimes_k l: K_0(A)_{/\text{num}} \xrightarrow{\sim} K_0(A \otimes_k l)_{/\text{num}}.\]

Let us now assume that \( l/k \) is a field extension of separably closed fields. Consider the associated field extension \( l_{\text{alg}}/k_{\text{alg}} \), where \( l_{\text{alg}} \) stands for “the” algebraic closure of \( l \) and \( k_{\text{alg}} \) for the algebraic closure of \( k \) inside \( l_{\text{alg}} \). Since by assumption \( k \) and \( l \) are separably closed, both field extensions \( l_{\text{alg}}/l \) and \( k_{\text{alg}}/k \) are purely inseparable. Under these notations, we have the following diagram:

\[
\begin{array}{ccc}
K_0(A \otimes_k l)_Q & \Rightarrow & K_0(A \otimes_k l_{\text{alg}})_Q \\
\downarrow & & \downarrow \\
K_0(A)_{Q} & \Rightarrow & K_0(A)_{/\text{num}} \\
\downarrow & & \downarrow \\
K_0(A)_{Q} & \Rightarrow & K_0(A)_{Q/k_{\text{alg}}} \\
\end{array}
\]

Thanks to Proposition 4.11 and to the above considerations, the three “solid” base-change homomorphisms in the central square are invertible. This implies that there exists a unique “dashed” isomorphism making the central square commute. The above diagram implies moreover that the latter “dashed” isomorphism is induced by base-change \(- \otimes_k l\). Hence, the proof is finished.

Finally, let \( l/k \) be a primary field extension. Thanks to Proposition 4.11, we can assume without loss of generality that \( l/k \) is regular. Consider the associated field extension \( l_{\text{sep}}/k_{\text{sep}} \), where \( l_{\text{sep}} \) stands for “the” separable closure of \( l \) and \( k_{\text{sep}} \) for the separable closure of \( k \) inside \( l_{\text{sep}} \). The field extensions \( l_{\text{alg}}/l \) and \( k_{\text{alg}}/k \) are Galois. Moreover, since \( l/k \) is regular, the homomorphism \( \text{Gal}(l_{\text{sep}}/l) \to \text{Gal}(k_{\text{sep}}/k) \) is surjective. Under these notations, we have the following diagram:

\[
\begin{array}{ccc}
K_0(A \otimes_k l)_Q & \Rightarrow & K_0(A \otimes_k l_{\text{alg}})_{\text{Gal}(l_{\text{sep}}/l)} \\
\downarrow & & \downarrow \\
K_0(A)_{Q} & \Rightarrow & K_0(A)_{Q/k_{\text{sep}}} \\
\downarrow & & \downarrow \\
K_0(A)_{Q} & \Rightarrow & K_0(A)_{Q/k_{\text{sep}}} \text{Gal}(k_{\text{sep}}/k). \\
\end{array}
\]

Thanks to Proposition 4.8 and to the above considerations, the three “solid” base-change homomorphisms in the central square are invertible. This implies that there
exists a unique “dashed” isomorphism making the central square commute. The above diagram implies moreover that the latter “dashed” isomorphism is induced by base-change $- \otimes_k l$. Hence, the proof is finished.

**Proof of item (ii).** We start by describing the behavior of mod-$n$ algebraic $K$-theory with respect to purely inseparable field extensions.

**Proposition 4.18 (Purely inseparable).** Given a purely inseparable field extension $l/k$, a dg category $A$, and an integer $n$ coprime to $\text{char}(k)$, we have an isomorphism:

$$- \otimes_k l: K_*(A; \mathbb{Z}/n) \xrightarrow{\sim} K_*(A \otimes_k l; \mathbb{Z}/n).$$

**Proof.** Let us assume first that the field extension $l/k$ is finite of degree $d$. As in the proof of Proposition 4.8, we have the following two homomorphisms:

$$K_*(A; \mathbb{Z}/n) \xrightarrow{- \otimes_k l} K_*(A \otimes_k l; \mathbb{Z}/n) \quad K_*(A \otimes_k l; \mathbb{Z}/n) \xrightarrow{\text{res}} K_*(A; \mathbb{Z}/n).$$

Clearly, the composition $\text{res} \circ (- \otimes_k l)$ is equal to $d \cdot \text{id}$. Similarly to the proof of Proposition 4.11, the converse composition $\text{res}(-) \otimes_k l$ is also equal to $d \cdot \text{id}$. Since the field extension $l/k$ is purely inseparable, the degree $d$ is a power of $\text{char}(k)$. Therefore, making use of the fact that the (graded) abelian groups $K_* (A; \mathbb{Z}/n)$ and $K_*(A \otimes_k l; \mathbb{Z}/n)$ are $\mathbb{Z}/n^2$-modules and that $n$ is coprime to $\text{char}(k)$, we conclude that the above induced homomorphism (4.19) is invertible.

Let us now assume that the field extension $l/k$ is infinite. In this case, $l$ identifies with the colimit of the filtrant diagram $\{ l_i \}_{i \in I}$ of all those intermediate field extensions $l/l_i/k$ which are finite and purely inseparable over $k$. Since $\text{colim}_{i \in I} l_i \simeq l$, we have $\text{colim}_{i \in I} A \otimes_k l_i \simeq A \otimes_k l$. Using the fact that the functor (3.3) preserves filtered colimits, we hence conclude that $\text{colim}_{i \in I} K_*(A \otimes_k l_i; \mathbb{Z}/n) \simeq K_*(A \otimes_k l; \mathbb{Z}/n)$. Consequently, the proof follows from the preceding finite dimensional case. \[ \square \]

Consider the field extension $l_{\text{alg}}/k_{\text{alg}}$, where $l_{\text{alg}}$ stands for “the” algebraic closure of $l$ and $k_{\text{alg}}$ for the algebraic closure of $k$ inside $l_{\text{alg}}$. Since $k$ and $l$ are separably closed, the extensions $l_{\text{alg}}/l$ and $k_{\text{alg}}/k$ are purely inseparable. Therefore, thanks to Proposition 4.18, in order to prove that (1.5) is invertible it suffices to address the particular case where $l/k$ is a field extension of algebraically closed fields.

We start by proving that (1.5) is injective. Note that $l$ identifies with the colimit of the filtrant diagram $\{ k_i \}_{i \in I}$ of all those finitely generated $k$-algebras $k_i$ which are contained in $l$. Without loss of generality, we may assume that $k_i$ is integrally closed in its field of fractions. Hence, each such $k$-algebra $k_i$ corresponds to an irreducible smooth affine $k$-curve $\text{Spec}(k_i)$. Since $\text{colim}_{i \in I} k_i \simeq l$, we have $\text{colim}_{i \in I} A \otimes_k k_i \simeq A \otimes_k l$. Using the fact that the functor (3.3) preserves filtered (homotopy) colimits, we hence obtain the following isomorphism:

$$\text{colim}_{i \in I} K_*(A \otimes_k k_i; \mathbb{Z}/n) \simeq K_*(A \otimes_k l; \mathbb{Z}/n).$$

By assumption, the field $k$ is algebraically closed. Therefore, the Hilbert’s nullstellensatz theorem implies that all such smooth affine curves $\text{Spec}(k_i)$ admit a rational point $p_i : \text{Spec}(k) \rightarrow \text{Spec}(k_i)$. Consequently, the following compositions

$$K_*(A; \mathbb{Z}/n) \xrightarrow{- \otimes_k k_i} K_*(A \otimes_k k_i; \mathbb{Z}/n) \xrightarrow{- \otimes_k k_i} K_*(A; \mathbb{Z}/n) \quad i \in I$$

are equal to the identity. This shows, in particular, that the homomorphisms $- \otimes_k k_i$ are injective. Making use of the above isomorphism (4.20), we hence conclude that the homomorphism (1.5) is also injective.
We now prove that (1.5) is surjective. Let \( \alpha \) be an element of \( K_*(A \otimes_k l; \mathbb{Z}/n) \). Thanks to the above isomorphism (4.20), there exists an index \( i_\alpha \in I \) and an element \( \alpha_{i_\alpha} \) of \( K_*(A \otimes_{k_\alpha} l; \mathbb{Z}/n) \) such that \( \alpha_{i_\alpha} \otimes_{k_{i_\alpha}} l = \alpha \). Choose a rational point \( p: \text{Spec}(k) \to \text{Spec}(k_{i_\alpha}) \) of the smooth affine \( k \)-curve \( \text{Spec}(k_{i_\alpha}) \) and consider the following commutative diagram:

\[
\begin{array}{ccc}
l & \xrightarrow{p} & k_{i_\alpha} \otimes_k l \\
k & \downarrow & \downarrow \\
l & = & l
\end{array}
\]

The two upper horizontal maps in (4.21) may be understood as two rational points of the smooth affine \( l \)-curve \( \text{Spec}(k_{i_\alpha} \otimes_k l) \). Hence, thanks to Proposition 4.22 below, the associated homomorphisms from \( K_*(A \otimes_k (k_{i_\alpha} \otimes_k l); \mathbb{Z}/n) \) to \( K_*(A \otimes_k l; \mathbb{Z}/n) \) agree. This implies that \( (\alpha_{i_\alpha} \otimes_{k_{i_\alpha}} l) \otimes_k l = \alpha_{i_\alpha} \otimes_{k_{i_\alpha}} l = \alpha \). Since \( \alpha_{i_\alpha} \otimes_{k_{i_\alpha}} l \) belongs to \( K_*(A; \mathbb{Z}/n) \), we then conclude that the homomorphism (1.5) is surjective.

**Proposition 4.22.** Let \( C = \text{Spec}(R) \) be a smooth affine \( l \)-curve. Given any two rational points \( p, q: \text{Spec}(l) \to C \), the associated homomorphisms \( \text{id} \otimes p^* = - \otimes_R l \) and \( \text{id} \otimes q^* = - \otimes_R l \) from \( K_*(A \otimes_k l; \mathbb{Z}/n) \) to \( K_*(A \otimes_k l; \mathbb{Z}/n) \) agree.

**Proof.** Consider the following homomorphism

\[
\text{Div}(C) \xrightarrow{\theta} \text{Hom}(K_*(A \otimes_k R; \mathbb{Z}/n), K_*(A \otimes_k l; \mathbb{Z}/n)) \quad p \mapsto (\text{id} \otimes p^*),
\]

where \( \text{Div}(C) \) stands for the abelian group of divisors on \( C \). Choose a smooth compactification\(^3\) \( \overline{C} \) of \( C \), with closed complement \( C_\infty \) consisting of a finite set of points, and consider the relative Picard group \( \operatorname{Pic}(\overline{C}, C_\infty) \). Recall that \( \operatorname{Pic}(\overline{C}, C_\infty) \) is defined as the quotient of \( \text{Div}(C) \) by the following equivalence relation: \( D \sim D' \) if there exists a rational function \( f: \overline{C} \to \mathbb{P}^1 \) such that \( f|_{C_\infty} = 1 \), \( f^{-1}(0) = D \) and \( f^{-1}(\infty) = D' \). Thanks to Lemma 4.23 below, \( \theta \) factors through \( \operatorname{Pic}(\overline{C}, C_\infty) \). Since the (graded) abelian group \( \text{Hom}(K_*(A \otimes_k R; \mathbb{Z}/n), K_*(A \otimes_k l; \mathbb{Z}/n)) \) is \( n^2 \)-torsion, the homomorphism \( \theta \) factors moreover through the quotient \( \operatorname{Pic}(\overline{C}, C_\infty)/n^2 \). Now, using the fact that the kernel \( \operatorname{Pic}^0(\overline{C}, C_\infty) \) of the degree map \( \operatorname{deg}: \operatorname{Pic}(\overline{C}, C_\infty) \to \mathbb{Z} \) is a \( n \)-divisible group (since it agrees with the group of \( k \)-points of the Rosenlicht Jacobian of \( \overline{C} \)) and that the difference \( p - q \) belongs to \( \operatorname{Pic}^0(\overline{C}, C_\infty) \), we hence conclude that \( (\text{id} \otimes p^*) = (\text{id} \otimes q^*) \).

**Lemma 4.23.** The above homomorphism \( \theta \) factors through \( \operatorname{Pic}(\overline{C}, C_\infty) \).

**Proof.** Since the field \( l \) is algebraically closed, the relative Picard group \( \operatorname{Pic}(\overline{C}, C_\infty) \) is generated by the unramified divisors on \( C \). Hence, it suffices to show that the homomorphism \( \theta \) vanishes on the principal divisors \( \text{div}(f) := f^{-1}(0) - f^{-1}(\infty) \) associated to those rational functions \( f: \overline{C} \to \mathbb{P}^1 \), with \( f|_{C_\infty} = 1 \), which are unramified over 0 and \( \infty \). Let \( D_0 := f^{-1}(0), D_\infty := f^{-1}(\infty) \), and \( \mathcal{U} := f^{-1}(\mathbb{P}^1 \setminus \{1\}) \). By definition, \( \mathcal{U} := \text{Spec}(S) \) is an affine open subscheme of \( C \) which contains \( D_0 \) and \( D_\infty \). Moreover, \( f \) restricts to a finite flat map \( f: \mathcal{U} \to \mathbb{P}^1 \setminus \{1\} = \mathbb{A}^1 \). Hence, without loss of generality, we can replace \( K_*(A \otimes_k R; \mathbb{Z}/n) \) by \( K_*(A \otimes_k S; \mathbb{Z}/n) \).

\(^3\)Recall that the smooth compactification \( \overline{C} \) is unique up to isomorphism.
Consider the following commutative diagrams:

\[
\begin{array}{ccc}
D_0 = f^{-1}(0) & \xrightarrow{j_0} & \mathcal{U} \\
\downarrow{f_0} & & \downarrow{f} \\
\text{Spec}(l) & \xrightarrow{i_0} & \mathbb{A}^1
\end{array} \quad \quad \begin{array}{ccc}
D_\infty = f^{-1}(\infty) & \xrightarrow{j_\infty} & \mathcal{U} \\
\downarrow{f_\infty} & & \downarrow{f} \\
\text{Spec}(l) & \xrightarrow{i_\infty} & \mathbb{A}^1
\end{array}
\]

Thanks to flat proper base-change, they yield commutative diagrams

\[
\begin{array}{ccc}
\text{perf}_{dg}(D_0) & \xrightarrow{i_0^*} & \text{perf}_{dg}(\mathcal{U}) \\
\downarrow{(f_0)_*} & & \downarrow{f_*} \\
\text{perf}_{dg}(\text{Spec}(l)) & \xrightarrow{i_0^*} & \text{perf}_{dg}(\mathbb{A}^1)
\end{array} \quad \quad \begin{array}{ccc}
\text{perf}_{dg}(D_\infty) & \xrightarrow{i_\infty^*} & \text{perf}_{dg}(\mathcal{U}) \\
\downarrow{(f_\infty)_*} & & \downarrow{f_*} \\
\text{perf}_{dg}(\text{Spec}(l)) & \xrightarrow{i_\infty^*} & \text{perf}_{dg}(\mathbb{A}^1)
\end{array}
\]

in the homotopy category \( \text{Hmo}(k) \). Since \( f \) is unramified over 0 and \( \infty \) and \( D_0 = \Pi_{f^{-1}(0)} \text{Spec}(l) \) and \( D_\infty = \Pi_{f^{-1}(\infty)} \text{Spec}(l) \), the push-forward dg functors \( (f_0)_* \) and \( (f_\infty)_* \) correspond to the fold identity dg functors from \( \Pi_{f^{-1}(0)} \text{perf}_{dg}(\text{Spec}(l)) \) and \( \Pi_{f^{-1}(\infty)} \text{perf}_{dg}(\text{Spec}(l)) \), respectively. Consequently, making use of the Morita equivalences \( \text{perf}_{dg}(\text{Spec}(l)) \simeq l \) and \( \text{perf}_{dg}(\mathcal{U}) \simeq S \), by applying the functor \( K_*(A \otimes_k -; \mathbb{Z}/n) \) to the preceding commutative diagrams, we conclude that

\[
\theta(D_0) = (\text{id} \otimes (f_0)_*) \circ (\text{id} \otimes i_0^*) = (\text{id} \otimes f_*) \circ (\text{id} \otimes i_0^*)
\]

\[
\theta(D_\infty) = (\text{id} \otimes (f_\infty)_*) \circ (\text{id} \otimes i_\infty^*) = (\text{id} \otimes f_*) \circ (\text{id} \otimes i_\infty^*)
\]

As proved in [14, Thm. 1.2(i)], since \( n \) is coprime to \( \text{char}(k) \), the functor (3.3) is \( \mathbb{A}^1 \)-homotopy invariant. Hence, the equality \( \text{id} \otimes i_0^* = \text{id} \otimes i_\infty^* \) holds. This allows us to conclude that \( \theta(\text{div}(f)) = \theta(D_0 - D_\infty) = 0 \), and so the proof is finished. \( \square \)

Finally, we prove that the isomorphism (1.5) also holds for the variants \( K_*(\mathcal{A}/n) \), \( K^n_*(\mathcal{A}/\mathbb{Z}/n) \), \( K_*/n \) and \( nK_*(-) \). The cases of \( n \)-adic algebraic \( K \)-theory and \( \text{mod-}n \text{étale } K \)-theory follow automatically from their definition. In what concerns the other two variants, note that the above proof of the injectivity of (1.5) holds \textit{mutatis mutandis} with \( \text{mod-}n \) algebraic \( K \)-theory replaced by nonconnective algebraic \( K \)-theory. This implies, in particular, that the base-change homomorphisms

\[
(4.24) - \otimes_k l: K_*(\mathcal{A}/n) \rightarrow K_*(\mathcal{A} \otimes_k l)/n \quad - \otimes_k l: nK_*(\mathcal{A}) \rightarrow nK_*(\mathcal{A} \otimes_k l)
\]

are injective. Making use of the following commutative diagrams

\[
\begin{array}{ccc}
0 & \rightarrow & K_*(\mathcal{A} \otimes_k l)/n \\
\downarrow{\otimes_k l} & & \downarrow{\otimes_k l} \\
K_*(\mathcal{A}/n) & \rightarrow & K_*(\mathcal{A}/\mathbb{Z}/n)
\end{array} \rightarrow \begin{array}{ccc}
0 & \rightarrow & nK_{* - 1}(\mathcal{A} \otimes_k l) \\
\downarrow{\otimes_k l} & & \downarrow{\otimes_k l} \\
nK_{* - 1}(\mathcal{A}) & \rightarrow & 0
\end{array}
\]

and of the snake lemma, we hence conclude that the base-change homomorphisms (4.24) are moreover surjective. This concludes the proof of Theorem 1.3.

5. \( n \)-adic Noncommutative Mixed Motives

Recall from [16, §8.3] the construction of the closed symmetric monoidal Quillen model category \( \text{NMot}(k) := L_{dg} \text{Fun}(\text{dgc}(k)^{op}, \text{Spt}) \) and of the associated symmetric monoidal functor \( U: \text{dgc}(k) \rightarrow \text{NMot}(k) \). The triangulated category of
noncommutative mixed motives $\text{NMix}(k)$ is defined as the smallest thick triangulated subcategory of $\text{Ho}(\text{NMot}(k))$ containing the objects $U(\mathcal{A})$ with $\mathcal{A}$ a smooth proper dg category; see [16, §9.1]. By construction, the category $\text{Ho}(\text{NMot}(k))$, and hence $\text{NMix}(k)$, is enriched over $\text{Ho}(\text{Spt})$.

Let $L_{S/n}(\text{Spt})$ be the closed symmetric monoidal Quillen model category of $S/n$-local symmetric spectra, $\text{NMot}(k)_{\Lambda}^n$ the closed symmetric monoidal Quillen model category $L_{\text{loc}} \text{Fun}(\text{dgcat}_l(k)^{op}, L_{S/n}(\text{Spt}))$, and $U(-)^\Lambda_n: \text{dgcat}(k) \rightarrow \text{NMot}(k)_{\Lambda}^n$ the associated symmetric monoidal functor. The triangulated category of $n$-adic noncommutative mixed motives $\text{NMix}(k)^\Lambda_n$ is defined as the smallest thick triangulated subcategory of $\text{Ho}(\text{NMot}(k)_{\Lambda}^n)$ containing the objects $U(\mathcal{A})^\Lambda_n$ with $\mathcal{A}$ a smooth proper dg category. As proved in [16, Thm. 1.43], the smooth proper dg categories can be characterized as the strongly dualizable objects of the symmetric monoidal category $H_{\text{mo}}(k)$; the dual of a smooth proper dg category $\mathcal{A}$ is given by the opposite dg category $\mathcal{A}^{op}$. Since the functor $U(-)^\Lambda_n$ is symmetric monoidal, we hence conclude that the objects $U(\mathcal{A})^\Lambda_n$, with $\mathcal{A}$ smooth proper, are strongly dualizable and consequently that the symmetric monoidal category $\text{NMix}(k)^\Lambda_n$ is rigid.

**Proposition 5.1.** Given dg categories $\mathcal{A}$ and $\mathcal{B}$, with $\mathcal{A}$ smooth and proper, we have a natural isomorphism of (symmetric) spectra:

$$
\text{RHom}(U(\mathcal{A})^\Lambda_n, U(\mathcal{B})^\Lambda_n) \simeq K(\mathcal{A}^{op} \otimes \mathcal{B})^\Lambda_n.
$$

**Proof.** Since $\mathcal{A}$ is smooth and proper, the object $U(\mathcal{A})^\Lambda_n$ is strongly dualizable with dual $U(\mathcal{A}^{op})^\Lambda_n$. Hence, we have the following isomorphisms

$$
\text{RHom}(U(\mathcal{A})^\Lambda_n, U(\mathcal{B})^\Lambda_n) \simeq \text{RHom}(U(k)^\Lambda_n, U(\mathcal{A}^{op})^\Lambda_n \otimes U(\mathcal{B})^\Lambda_n)
\simeq \text{RHom}(U(k)^\Lambda_n, U(\mathcal{A}^{op} \otimes B)^\Lambda_n)
\simeq \text{RHom}(U(k), \text{holim}_n U(\mathcal{A}^{op} \otimes \mathcal{B})/n^n)
\simeq \text{holim}_n \text{RHom}(U(k), U(\mathcal{A}^{op} \otimes \mathcal{B})/n^n) =: K(\mathcal{A}^{op} \otimes \mathcal{B})^\Lambda_n,
$$

where (5.3) follows from the induced adjunction between $\text{NMot}(k)$ and $\text{NMix}(k)^\Lambda_n$ and (5.4) from [16, Thm. 8.28].

**Proof of Theorem 2.1.** By construction, every object of $\text{NNum}(k)_Q$ is, up to a direct summand, of the form $U(\mathcal{A})_Q$; see Remark 2.4. Moreover, as explained in [16, §4.7], we have natural isomorphisms of $Q$-vector spaces:

$$
\text{Hom}(U(\mathcal{A})_Q, U(\mathcal{B})_Q) \simeq K_0(\mathcal{A}^{op} \otimes \mathcal{B})_Q/\_\text{num}.
$$

Therefore, the proof of item (i) follows from Theorem 1.3(i).

By construction, the triangulated category $\text{NMix}(k; \mathbb{Z}/n)$ comes equipped with a functor $U(-; \mathbb{Z}/n): \text{dgcat}_{sp}(k) \rightarrow \text{NNum}(k; \mathbb{Z}/n)$. Moreover, it is generated by the objects of the form $U(\mathcal{A}; \mathbb{Z}/n)$. Furthermore, as explained in [15, Prop. 4.5], we have natural isomorphisms of (symmetric) spectra

$$
\text{RHom}(U(\mathcal{A}; \mathbb{Z}/n), U(\mathcal{B}; \mathbb{Z}/n)) \simeq K(\mathcal{A}^{op} \otimes \mathcal{B}) \land H(\mathbb{Z}/n),
$$

where $H(\mathbb{Z}/n)$ stands for the Eilenberg-MacLane spectrum of $\mathbb{Z}/n$. Therefore, the fully-faithfulness of the base-change functor $- \otimes_k l: \text{NMix}(k; \mathbb{Z}/n) \rightarrow \text{NMix}(l; \mathbb{Z}/n)$ follows from Theorem 1.3(ii). The proof of the fully-faithfulness of the base-change functor $- \otimes_k l: \text{NMix}(k)^\Lambda_n \rightarrow \text{NMix}(l)^\Lambda_n$ is similar; simply replace (5.5) by (5.2).
Remark 5.6 (Alternative proof). The commutative counterpart of Theorem 2.1(i) (where the category $\text{NNum}(k)_Q$ is replaced by the classical category of numerical motives $\text{Num}(k)_Q$) was established by Kahn in [4, Prop. 5.5]. Here is an alternative proof: as explained in [16, §4.6-4.7 and §4.10], we have the commutative diagram

\[
\begin{array}{ccc}
\text{Num}(k)_Q & \xrightarrow{- \otimes k l} & \text{Num}(l)_Q \\
\downarrow & & \downarrow \\
\text{Num}(k)_Q/_{- \otimes Q(1)} & \xrightarrow{- \otimes k l} & \text{Num}(l)_Q/_{- \otimes Q(1)} \\
\Phi & & \Phi \\
\downarrow & & \downarrow \\
\text{NNum}(k)_Q & \xrightarrow{- \otimes k l} & \text{NNum}(l)_Q,
\end{array}
\]

where $\text{Num}(k)_Q/_{- \otimes Q(1)}$ stands for the orbit category with respect to the Tate motive $Q(1)$. Since the functor $\Phi$ is fully-faithful, it follows then from the combination of Theorem 2.1(i) with the definition of the orbit category that the upper base-change functor $- \otimes k l$ in (5.7) is also fully-faithful.

Acknowledgments: The author is grateful to Joseph Ayoub and Ivan Panin for useful discussions, to Oliver Röndigs and Paul Arne Østvær for references, and to Charles Vial for comments on a previous version. The author is also thankful to the Mittag-Leffler Institute for its hospitality.

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