Universal localisations of hereditary rings

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Abstract
We describe all possible universal localisations of a hereditary ring in terms of suitable full subcategories of the category of finitely presented modules. For these universal localisations we then identify the category of finitely presented bound modules over the universal localisation as being equivalent to a certain full subcategory of the category of finitely presented bound modules over the original ring. We also describe the abelian monoid of finitely generated projective modules over the universal localisation.

1 Introduction
The purpose of this paper is to study the universal localisations of a hereditary ring. We shall describe all possible universal localisations and for each of these we determine the kernel of the homomorphism from $R$ to its universal localisation $R_{\Sigma}$ and the kernel of the homomorphism from an $R$ module $M$ to $M \otimes R_{\Sigma}$, the structure of a finitely presented module over $R_{\Sigma}$ as an $R$ module and the structure of the category of finitely presented modules over the universal localisation.

Over a hereditary ring, every finitely presented module is a direct sum of a finitely generated projective module and a bound module (a module $M$ such that $\text{Hom}(M, R) = 0$). The universal localisation of a hereditary ring is a hereditary ring and we show that the category of finitely presented bound modules over $R_{\Sigma}$ is equivalent to a full subcategory of $\text{fpmod}(R)$ defined in terms of the set of maps $\Sigma$. It remains to understand the finitely generated projective modules over the universal localisation and we are able to describe the abelian monoid of isomorphism classes of finitely generated projective modules over the universal localisation in terms of a suitable subcategory of the category of finitely presented modules over $R$, $\text{fpmod}(R)$.

This paper follows on from [4] which should be read before this one.

This paper should be regarded as extending P. M. Cohn’s and others’ investigation of the universal localisation of firs (see [2]). We find that essentially everything carries over to this more general situation and our results clarify the problem of determining when any universal localisation of a fir is itself a fir by looking at modules instead of matrices. In fact, the methods developed here will often allow us to determine the isomorphism classes of finitely generated
projective modules over such a universal localisation of a fir which is a task unattempted by previous theory.

In a subsequent paper, we shall see that these more general hereditary rings and their universal localisations are of great interest since we shall show that the right perpendicular category to a vector bundle over a smooth projective curve is equivalent to $\text{Mod}(R)$ for a suitable (usually noncommutative) hereditary ring. The description of these rings and their investigation heavily use the techniques of this paper.

In the next section, we characterise all possible universal localisations of a hereditary ring. The following section describes the finitely presented bound modules over the universal localisation in terms of a suitable subcategory of finitely presented modules and the last section studies the finitely generated projective modules over the universal localisation again in terms of a suitable subcategory of finitely presented modules.

Throughout this paper, a hereditary ring will mean a left and right hereditary ring.

2 Characterising the universal localisations

In the paper [4], we saw that a universal localisation $R_\Sigma$ of a ring $R$ was best studied in terms of the full subcategory of finitely presented modules of homological dimension at most 1 that are left perpendicular to the $R_\Sigma$ modules. We say that a module $M$ of homological dimension at most 1 is left perpendicular to $N$ if and only if $\text{Hom}(M, N) = 0 = \text{Ext}(M, N)$. We call this the category of $R_\Sigma$ trivial modules, $S(\Sigma)$. The best cases in that paper were shown to be those where the kernel of a map in $S(\Sigma)$ was itself a factor of another such module.

In the case of a hereditary ring, we shall show that a better result holds. This category must be an exact abelian subcategory of $\text{fpmod}(R)$. By this we mean that it is an abelian category and the inclusion is exact. Equivalently, it is an additive subcategory closed under images, kernels and cokernels. Of course, it is also closed under extensions and we call a full exact abelian subcategory of $\text{fpmod}(R)$ closed under extensions a well-placed subcategory of $\text{fpmod}(R)$. Our main result is that the set of well-placed subcategories of $\text{fpmod}(R)$ parametrise the universal localisations of a hereditary ring $R$. These categories have further structure which we end this section by describing since it is important for the subsequent discussion of the finitely presented bound modules and the finitely generated projective modules over the universal localisation. We also give a precise description of the structure of a finitely presented module over the universal localisation as a directed union of modules over $R$.

In general, any universal localisation of a ring need not be a universal localisation at an injective set of maps between finitely generated projective modules. One can reduce to this case in principle by factoring out a suitable ideal which may however be difficult to determine in practice. For a hereditary ring we can avoid this problem as our first lemma shows.
Lemma 2.1. Let \( R \) be a right hereditary ring and let \( \Sigma \) be a set of maps between finitely generated projective modules over \( R \). Then there exists a set of injective maps between finitely generated projective modules \( \Sigma' \) such that \( R\Sigma \cong R\Sigma' \).

Proof. Let \( f : P \to Q \) be a map in \( \Sigma \). Let \( I = \text{im}(f) \) and \( K = \ker(f) \). Then \( f = pf' \) where the induced map \( p : P \to I \) is split surjective with kernel \( K \). Therefore, \( p \otimes R\Sigma \) and \( f' \otimes R\Sigma \) are invertible. Conversely, if we invert \( p \) and \( f' \) then we must also invert \( f \). However, adjoining a universal inverse to the split surjective map \( p \) is equivalent to adjoining an inverse to its left inverse, a split injective map \( \iota \) from \( I \) to \( P \) with cokernel \( K \) and vice versa. Thus we may replace \( \Sigma \) by \( \Sigma_1 = \Sigma - \{f\} \cup \{f', \iota\} \). Applying this to each non-injective map in \( \Sigma \) gives us a set of injective maps \( \Sigma' \) such that \( R\Sigma \cong R\Sigma' \). \( \square \)

We have already seen in [4] that whenever \( \Sigma \) is a set of injective maps between finitely generated projective modules, it is useful to replace consideration of \( \Sigma \) by the full subcategory of \( R\Sigma \)-trivial modules. This is the full subcategory of \( \text{fpmod}(R) \) whose objects are those modules of homological dimension at most 1 such that their presentations are inverted by \( \otimes R\Sigma \) or equivalently those \( R \) modules \( M \) of homological dimension at most 1 such that \( \text{Hom}(M, \cdot) \) and \( \text{Ext}(M, \cdot) \) vanish on \( R\Sigma \) modules (see theorem 5.2 in [4]). In that paper, we showed that particularly sensible results held whenever this category has the property that kernels of maps in this category are torsion with respect to the torsion theory generated by this category. In fact, we can do better here. Over a right hereditary ring, the category of \( R\Sigma \) trivial modules is an abelian subcategory of \( \text{fpmod}(R) \) whose inclusion in \( \text{fpmod}(R) \) is exact.

Theorem 2.2. Let \( R \) be a right hereditary ring and let \( \Sigma \) be a set of maps between finitely generated projective modules over \( R \). Then the category of \( R\Sigma \) trivial modules, \( S(\Sigma) \), is an exact abelian subcategory of \( \text{fpmod}(R) \) closed under extensions.

Proof. We need to show that \( S(\Sigma) \) is closed under images. However, for any factor \( F \) of a module in \( S(\Sigma) \), \( \text{Hom}(F, \cdot) \) vanishes on \( R\Sigma \) modules and for any submodule \( S \) of a module in \( S(\Sigma) \), \( \text{Ext}(S, \cdot) \) vanishes on \( R\Sigma \) modules (because \( R \) is right hereditary). Since the image of a map in \( S(\Sigma) \) is both a submodule and a factor module, we see that \( S(\Sigma) \) is closed under images. The rest follows once we note that given a short exact sequence \( 0 \to A \to B \to C \to 0 \) where two of \( A, B, C \) lie in \( S(\Sigma) \) then so does the remaining term. \( \square \)

Thus to any universal localisation of a right hereditary ring we may associate a full exact abelian subcategory of \( \text{fpmod}(R) \) closed under extensions. We recall that we have named such a subcategory a well-placed subcategory.

Theorem 2.3. Let \( R \) be a right hereditary ring. Then the universal localisations of \( R \) are parametrised by the well-placed subcategories of \( \text{fpmod}(R) \) by associating to a universal localisation \( R\Sigma \) the category of \( R\Sigma \) trivial modules.
Proof. We have already seen one half of this bijection in the previous theorem and the following remark. Conversely, let $E$ be a well-placed subcategory of $\text{fpmod}(R)$. Then in the terminology of \cite{4} $E$ is certainly a pre-localising category such that the kernel of any map in $E$ is torsion with respect to the torsion theory generated by $E$. Therefore, we may form the universal localisation $R_E$ and by theorem 5.6 of \cite{4}, the category of $R_E$-trivial modules is just $E$. Thus we have the stated bijection.

Since all universal localisations of a right hereditary ring are parametrised by well-placed subcategories of $\text{fpmod}(R)$ we shall use the notation $R_E$ for the universal localisation corresponding to the well-placed subcategory $E$. Note that the category of $R_E$ modules considered as $R$ modules can be identified with the category $E^\perp$ which is the full subcategory of $R$ modules $M$ such that $\text{Hom}(E, M) = 0 = \text{Ext}(E, M)$ for every $E \in E$.

A well-placed subcategory $E$ of $\text{fpmod}(R)$ for a right hereditary ring is fairly easy to understand. A finitely presented module over a right hereditary ring is a direct sum of a bound module and a finitely generated projective module. Clearly, a well-placed subcategory of $\text{fpmod}(R)$ is closed under direct summands and so these summands lie in $E$. Any finitely presented bound module over a right hereditary ring satisfies the ascending chain condition on bound submodules by corollary 5.1.7 of \cite{2}. Thus in the case where $E$ contains no projective module (which is clearly equivalent to assuming that $R$ embeds in the universal localisation), we know that $E$ is a Noetherian abelian category. The best results occur when we assume that $R$ is a hereditary ring by which we mean that is both left and right hereditary.

For the rest of the paper we shall usually assume that $R$ embeds in the universal localisation. This is equivalent to assuming that $E$ contains no nonzero finitely generated projective module. We can reduce to this situation by factoring out by the trace ideal of the finitely generated projective modules in $E$. This is not ideal but the results are most simply demonstrated in this context.

**Theorem 2.4.** Let $R$ be a hereditary ring and let $E$ be a well-placed subcategory of $\text{fpmod}(R)$ of modules of homological dimension 1. Then $E$ is a finite length category.

Proof. We assumed that every module in $E$ has homological dimension 1; therefore no such module can have a nonzero projective summand since $E$ is closed under direct summands. It follows that all modules in $E$ are bound. However, by theorem 5.2.3 of \cite{2}, any bound module over a hereditary ring satisfies the descending chain condition on bound submodules as well as the ascending chain condition and so $E$ is a finite length category.

We say that a set $S$ of modules is Hom-perpendicular if and only if for $T, U \in S$, and a map $f: T \to U$ then $f$ is invertible or 0. Given a Hom-perpendicular set of bound finitely presented modules, its extension closure is a well-placed subcategory of $\text{fpmod}(R)$. Conversely, given a well-placed subcategory of $\text{fpmod}(R)$ of modules of homological dimension 1, it is a finite length category.
category and its simple objects form a Hom-perpendicular set of bound finitely presented modules. Moreover, since \( E \) is closed under extensions and every object has a composition series in \( E \), \( E \) is the closure of its set of simple objects under extensions. We summarise this discussion in the following theorem.

**Theorem 2.5.** Let \( R \) be a hereditary ring. Then its universal localisations into which it embeds are parametrised by the set of Hom-perpendicular sets of isomorphism classes of finitely presented bound modules. This bijection arises by assigning to the universal localisation \( R_\Sigma \) the set of simple objects in the category of \( R_\Sigma \) trivial modules.

We use this to obtain a better description of a module \( M \otimes R_E \) for a finitely presented module \( M \) over the hereditary ring \( R \). For any universal localisation, we showed (see theorem 4.2 of [4]) how to find the \( R \) module structure of \( M \otimes R_\Sigma \) as a direct limit of finitely presented modules over \( R \). The maps in this direct limit are not all injective but in the case where \( R \) is a hereditary ring we can represent \( M \otimes R_E \) as a direct limit of finitely presented modules where the maps in the system are injective.

**Theorem 2.6.** Let \( R \) be a hereditary ring and let \( E \) be a well-placed subcategory of \( \text{fpmod}(R) \) of modules of homological dimension 1. Let \( M \) be a finitely presented module over \( R \). Then the kernel of the map from \( M \) to \( M \otimes R_E \) is finitely generated.

If \( M \) embeds in \( M \otimes R_E \) then as \( R \) module \( M \otimes R_E \) is a directed union of modules \( M_s \geq M \) where \( M_s/M \in E \).

**Proof.** By theorem 5.5 of [4] we know that the kernel of the map from \( M \) to \( M \otimes R_E \) is the torsion submodule of \( M \) with respect to the torsion theory generated by \( E \). However, \( M \) satisfies the ascending chain condition on bound submodules and so the torsion submodule must be finitely generated.

Now suppose that \( M \) embeds in \( M \otimes R_E \) so that \( M \) is torsion-free with respect to the torsion theory generated by \( E \). By theorem 4.2 of [4], \( M \otimes R_E \) is a direct limit of modules \( M_s \) where the map from \( M \) to \( M_s \) is in the severe right Ore set generated by a set of presentations of the modules in \( E \). Each of these factors as a good pushout followed by a good surjection (see [4] for these terms). In our present case, a good pushout is injective.

Thus we have a short exact sequence \( 0 \rightarrow M \rightarrow N_s \rightarrow E_s \rightarrow 0 \) where \( E_s \in E \) and a surjection from \( N_s \) to \( M_s \) whose kernel is torsion with respect to the torsion theory generated by \( E \). The inclusion of \( M \) in \( N_s \) is inverted by \( \otimes R_E \) and so the image of \( N_s \) in \( M \otimes R_E \) is \( N_s/T_s \) where \( T_s \) is the torsion submodule of \( N_s \) with respect to the torsion theory generated by \( E \) and so \( T_s \) is finitely generated by the first paragraph of this proof. We note that \( M \otimes R_E \) is the directed union of the modules \( N_s/T_s \) and we wish to show that the cokernel of the inclusion of \( M \) in each of these modules lies in \( E \).

We choose some \( G_s \in E \) that maps onto \( T_s \) and consider the induced map from \( G_s \) to \( E_s \). Then the image \( I_s \) must be in \( E \) and so must the kernel \( K_s \). Under the map from \( G_s \) to \( N_s \) the image of \( K_s \) must lie in \( M \) and so must be
zero since $\text{Hom}(E, M) = 0$ for every $E \in \mathcal{E}$. It follows that $T_s \cong I_s$ and so we have a short exact sequence $0 \to M \to N_s/T_s \to E_s/I_s \to 0$ where $E_s/I_s \in \mathcal{E}$ as we set out to prove.

This allows us to give a simple criterion for when two finitely presented module over $R$ induce up to isomorphic modules over $R_E$ which is better than the results we prove for general universal localisations in [4].

**Theorem 2.7.** Let $R$ be a hereditary ring and let $\mathcal{E}$ be a well-placed subcategory of $\text{fpmod}(R)$ all of whose modules are bound. Let $M$ and $N$ be finitely presented module over $R$ that are torsion-free with respect to the torsion theory generated by $E$. Then $M \otimes R_E$ is isomorphic to $N \otimes R_E$ if and only if there exist a module $L$ and short exact sequences $0 \to M \to L \to E \to 0$ and $0 \to N \to L \to F \to 0$ where $E, F \in \mathcal{E}$.

**Proof.** Certainly, the existence of such short exact sequences implies that $M \otimes R_E \cong N \otimes R_E$.

Now suppose that $M \otimes R_E \cong N \otimes R_E$. Since $M \otimes R_E$ is a directed union of modules $M_t$ where $M \subset M_t$ and $M_t/M \in \mathcal{E}$ (by the last theorem), the isomorphism of $M \otimes R_E$ and $N \otimes R_E$ gives an embedding of $N$ in $M$ and hence of $N$ in some $M_t$; moreover, the inclusion of $N$ in $M_t$ becomes an isomorphism under $\otimes R_E$.

Since $\text{Tor}_1^R(M \otimes R_E, R_E) = 0$ and $M_t \subset M \otimes R_E$, it follows that $\text{Tor}_1^R(M_t, M \otimes R_E) = 0$. So applying $\otimes R_E$ to the short exact sequence $0 \to N \to M \to G \to 0$ shows that $G \otimes R_E = 0 = \text{Tor}_1^R(G, R_E)$ from which we conclude that $G$ must be an $R_E$-trivial module and hence $G \in \mathcal{E}$. Thus our proof is complete.

3 **Bound modules over a universal localisation**

In this section, $R$ is a hereditary ring and $\mathcal{E}$ is a well-placed subcategory of $\text{fpmod}(R)$. We shall be interested in describing the category of finitely presented bound modules over $R_E$. We shall assume that $\mathcal{E}$ contains no projective modules since we can replace $R$ by $R/T$ where $T$ is the trace ideal of the projective modules in $\mathcal{E}$ which is still a hereditary ring and we replace $\mathcal{E}$ by its image under $\otimes R/T$.

In order to describe our theorem we introduce some terminology. Let $S$ be a set of modules or $\mathcal{S}$ the full subcategory of modules isomorphic to a module in $S$. Then the Hom-perpendicular category to $S$ (or to $\mathcal{S}$), written as $\text{Hom}_{\mathcal{S}}(S)$ (or $\text{Hom}_{\mathcal{S}}(S)$) is the full subcategory of bound modules $X$ such that $\text{Hom}(X, M) = 0 = \text{Hom}(M, X)$ for every $M \in S$ (or $M \in \mathcal{S}$). We shall prove that the category of finitely presented bound modules over $R_E$ is equivalent to $\text{Hom}_{\mathcal{S}}(\mathcal{E})$.

Before we begin, we should not that certain finitely presented modules over $R$ cannot induce to bound modules over the universal localisation.

**Lemma 3.1.** Let $M$ be a strict submodule of $E$, a simple object in $\mathcal{E}$. Then $M \otimes R_E$ is a nonzero projective module over $R_E$.

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Proof. We note that $\text{Ext}(M, ) = 0$ on $R_E$ modules because $R$ is hereditary and $M \subset E$ for which by assumption $\text{Ext}(E, )$ vanishes on $R_E$ modules. However, $\text{Hom}(M, )$ cannot vanish on $R_E$ modules since $M$ is not in $E$ and theorem 5.6 shows that $E$ is the category of $R$-trivial modules. But $\text{Hom}_{R_E}(M \otimes R_E, ) = \text{Hom}(M, )$ on $R_E$ modules so that $M \otimes R_E$ cannot be 0.

We begin by showing that that every finitely presented bound module over the universal localisation is induced from a module in $\text{Hom}_{\text{perp}}(E)$.

**Theorem 3.2.** Let $R$ be a hereditary ring and $E$ a well-placed subcategory of $\text{fpmod}(R)$ all of whose modules are bound. Then every finitely presented bound module over $R_E$ is isomorphic to a module of the form $M \otimes R_E$ where $M \in \text{Hom}_{\text{perp}}(E)$.

**Proof.** Every finitely presented module over $R_E$ is isomorphic to a module of the form $M \otimes R_E$ where $M \in \text{fpmod}(R)$ and $\text{Hom}(E, M) = 0$ for every $E \in E$ by theorem 2.6. Clearly, $M$ is a bound module since $P \otimes R_E$ is nonzero for any projective module and we are assuming that $M \otimes R_E$ is bound.

Now suppose that $\text{Hom}(M, E) \neq 0$ for some module $E \in E$. Then since $E$ is a finite length category we may assume that $E$ is a simple object in $E$. If the homomorphism is not surjective then we obtain a short exact sequence $0 \to K \to M \to F \to 0$ where $F$ is a strict submodule of $E$ and so $F \otimes R_E$ is nonzero projective by lemma 3.4. Therefore, applying $\otimes R_E$ to our short exact sequence shows that $M \otimes R_E$ is not bound. This contradiction implies that we have a surjection from $M$ to $E$ whenever $E$ is a simple object in $E$ and there is a nonzero homomorphism from $M$ to $E$. Thus we have a short exact sequence $0 \to M_1 \to M \to E_1 \to 0$ whenever $M$ has a nonzero homomorphism to some module in $E$. From this short exact sequence, we see that $M \otimes R_E \cong M_1 \otimes R_E$. Applying this argument as many times as necessary we either obtain a submodule $M_n \subset M$ such that $M_n \otimes R_E \cong M \otimes R_E$ and $\text{Hom}(M_n, E) = 0$ for every $E \in E$ or else a strictly descending sequence of submodules of $M$, $M \supset M_1 \supset \ldots \supset M_n \ldots$ where $M/M_n \in E$ and $M_n \otimes R_E \cong M \otimes R_E$. However, any bound module over a hereditary ring satisfies the descending chain condition on submodules whose factor is bound and so the second case cannot arise. Thus we have some submodule $M' \subset M$ such that $M' \otimes R_E \cong M \otimes R_E$ and $\text{Hom}(M', E) = 0$ for every $E \in E$. But $M'$ must also satisfy $\text{Hom}(E, M) = 0$ for every $E \in E$. So $M' \in \text{Hom}_{\text{perp}}(E)$ which completes our proof.

Next we restrict the $R_E$ homomorphisms from a module $M \otimes R_E$ where $M \in \text{Hom}_{\text{perp}}(E)$.

**Lemma 3.3.** Let $R$ be a hereditary ring and $E$ a well-placed subcategory of $\text{fpmod}(R)$ all of whose modules are bound. Let $M \in \text{Hom}_{\text{perp}}(E)$ and let $N$ be some finitely presented module over $R$ torsion-free with respect to the torsion theory generated by $E$. Let $\phi : M \otimes R_E \to N \otimes R_E$ be a homomorphism of $R_E$ modules. Then $\phi = f \otimes R_E$ for some $f : M \to N$. 

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Proof. We showed in theorem 2.6 that $N \otimes R_E$ is isomorphic to a direct limit of modules $N_i \supset N$ such that $N_i/N \in E$. So the image of $\phi$ must lie in some $N_i$. Since $N_i/N \in E$ and $M \in \text{Hom}_{\text{perp}}(E)$, $\text{Hom}(M, N_i/N) = 0$ and so the image of $M$ lies in $N$. Let $f: M \to N$ be this induced map. Then $\phi - f \otimes R_E$ vanishes on $M$ and so must be the zero map as required.

We note a consequence of this.

Lemma 3.4. Let $M \in \text{Hom}_{\text{perp}}(E)$. Then $M \otimes R_E$ is bound

Proof. From the last lemma we see that $\text{Hom}_{R_E}(M \otimes R_E, R_E) = \text{Hom}_R(M, R) = 0$.

We conclude the main theorem of this section.

Theorem 3.5. Let $R$ be a hereditary ring and $E$ a well-placed subcategory of $\text{fpmod}(R)$ all of whose modules are bound. Then $\otimes R_E$ induces an equivalence of categories between $\text{Hom}_{\text{perp}}(E)$ and the full subcategory of finitely presented bound modules over $R_E$.

Proof. We have already seen in theorem 3.2 and lemma 3.3 that $\otimes R_E$ induces a functor from $\text{Hom}_{\text{perp}}(E)$ to the full subcategory of finitely presented bound modules over $R_E$ that is surjective on isomorphism classes of objects.

However, by lemma 3.3 any homomorphism from $M \otimes R_E$ to $N \otimes R_E$ for $M, N \in \text{Hom}_{\text{perp}}(E)$ must be of the form $f \otimes R_E$ for some $f: M \to N$. Thus $\otimes R_E$ is surjective on Hom-sets as a functor from $\text{Hom}_{\text{perp}}(E)$ to the full subcategory of finitely presented bound modules over $R_E$. However, it is visibly injective since $N$ is a submodule of $N \otimes R_E$. Thus $\otimes R_E$ induces the stated equivalence.

In some ways, this result is very surprising although its proof is simple enough. One consequence is that the category of finitely presented bound modules over the free associative algebra over a field is equivalent to a category of finite dimensional representations over a generalised Kronecker quiver since a 2 by 2 matrix ring over the free associative algebra is a universal localisation of the path algebra of a generalised Kronecker quiver. Perhaps more shocking is that the category of finitely presented bound modules over the universal algebra with an isomorphism from the free module of rank 1 to the free module of rank 2 is equivalent to a category of finite dimensional representations over a generalised Kronecker quiver because it too is a universal localisation of a free algebra and hence Morita equivalent to a universal localisation of the path algebra of a generalised Kronecker quiver. Thus the category of finitely presented bound modules over a hereditary ring has no knowledge of the pathology of the finitely generated projective modules.

4 Finitely generated projective modules

We wish to calculate the finitely generated projective modules over the universal localisation of a hereditary ring. We begin by pointing out that the map from $K_0(R)$ to $K_0(R_E)$ is surjective which is not guaranteed for general rings.
Lemma 4.1. The map from $K_0(R)$ to $K_0(R_E)$ is surjective.

Proof. Every finitely presented module over $R_E$ is induced. So given a finitely generated projective module over $R_E$ we can write it in the form $M \otimes R_E$ for some finitely presented module $M$ that embeds in $M \otimes R_E$. It follows that $\text{Tor}^1(M, R_E) = 0$. We consider a short exact sequence $0 \to P \to Q \to M \to 0$ where $P, Q$ are finitely generated projective modules over $R$. Tensoring gives a short exact sequence $0 \to P \otimes R_E \to Q \otimes R_E \to M \otimes R_E \to 0$ and completed the proof of the lemma.

Thus we know what the $K$-theory of the universal localisation is and so we know the finitely generated projective modules stably but we can get far more precise information; in principle, we can determine the abelian monoid of the finitely generated projective modules up to isomorphism. In order to describe our main result, we need to introduce a couple of categories and a construction.

Let $R$ be a hereditary ring and let $E$ be a well-placed subcategory of $\text{fpmod}(R)$. Our first category is the full subcategory of $\text{fpmod}(R)$ whose objects are those modules $M$ such that $\text{Ext}(M, \_)$ vanishes on $E^\perp$; the category of $R_E$ modules in $\text{Mod}(R)$; we call this $\text{relproj}(E^\perp)$, the category of finitely presented modules projective with respect to $E^\perp$. This category is clearly closed under extensions and submodules and contains the finitely generated projective modules and $E$; we shall see shortly that it is the smallest such full subcategory. Our second category is the full subcategory of $\text{fpmod}(R)$ whose objects are those modules $M$ such that $\text{Hom}(M, \_)$ vanishes on $E^\perp$; equivalently, $M \otimes R_E = 0$ from which we see that it is precisely the full subcategory of $\text{fpmod}(R)$ whose objects are the finitely presented modules that are factors of modules in $E$; equivalently, these are the finitely presented torsion modules with respect to the torsion theory generated by $E$. We shall refer to this category $\text{fac}(E)$ as the category of factors of $E$. Both of these categories are exact categories where the short exact sequences are all short exact sequences in $\text{fpmod}(R)$ whose objects lie in the relevant full subcategory.

Given an exact category $F$ we associate to this an abelian monoid $A^+(F)$ generated by the isomorphism classes of objects in $F$ and whose relations are given by the short exact sequences in $F$; that is, given a short exact sequence $0 \to A \to B \to C \to 0$ in $F$, we have a relation $[B] = [A] + [C]$. In the special case where $F$ is $\text{proj}(R)$, $A^+(\text{proj}(R))$ is also known as $P_0(R)$ and here the elements of the abelian monoid are just the isomorphism classes of objects in $\text{proj}(R)$. In general, however, the map from the set of isomorphism classes of objects in $F$ to $A^+(F)$ is only surjective.

Since $R$ is a hereditary ring we can show easily that $\text{Tor}^1(M, R_E) = 0$ for $M$ in $\text{relproj}(E^\perp)$ so that $\otimes R_E$ is an exact functor from $\text{relproj}(E^\perp)$ to $\text{fpmod}(R_E)$ and in fact as we shall show to $\text{proj}(R_E)$. Similarly, since $\otimes R_E$ vanishes on $\text{fac}(E)$, $\text{Tor}^1(\_, R_E)$ is an exact functor from $\text{fac}(E)$ to $\text{Mod}(R)$ and in fact to $\text{proj}(R_E)$. In the first case, the map on objects will be shown to be surjective and hence we have a surjective map of abelian monoids from $A^+(\text{relproj}(E^\perp))$ and a map from $A^+(\text{fac}(E))$ to $P_0(R_E)$. In both cases, we shall show that the additional relations in each case take the form $[E] = 0$ for every $E \in E$. 

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Although this answer is useful, it helps in practice to have other techniques available. First of all, one can show that every finitely generated projective module over $R_E$ is isomorphic to a direct sum of modules of the form $S \otimes R_E$ where either $S$ is a submodule of some simple object in $E$ (recall that $E$ is a finite length category) or else $S$ is itself a finitely generated projective module. Secondly we show that every module in $\text{relproj}(E^\perp)$ has a finite filtration such that the subfactors are either finitely generated projective or else have the property that any homomorphism to a module in $E$ must be injective. This last kind of module is sensible to work with since these are the modules for which $S \otimes R_E$ is finitely generated projective and it is possible that $S \otimes R_E$ is decomposable. If $S$ and $T$ are two such modules, we find a useful way to investigate whether or not $S \otimes R_E$ and $T \otimes R_E$ are isomorphic (which is a little technical to describe at this moment). This is most useful when we know that there are indecomposable finitely generated projective modules over the universal localisation.

We begin with our first description of $P_\oplus(R_E)$. We need to develop the properties of the categories $\text{relproj}(E^\perp)$ and $\text{fac}(E)$.

We first need to show that the modules in $\text{relproj}(E^\perp)$ induce up to finitely generated projective modules over $R_E$.

**Lemma 4.2.** Let $M \in \text{relproj}(E^\perp)$. Then $M \otimes R_E$ is a finitely generated projective module over $R_E$ and this module is 0 if and only if $M \in E$. More generally, the kernel of the homomorphism from $M$ to $M \otimes R_E$ is in $E$ and the image of $M$ in $M \otimes R_E$ also lies in $\text{relproj}(E^\perp)$.

**Proof.** We note that $\text{Hom}_{R_E}(M \otimes R_E, \ ) \cong \text{Hom}_R(M, \ )$ on $E^\perp$ which is exact by assumption. So $M \otimes R_E$ is a projective module. It is finitely generated because $M$ is.

Further, $M \otimes R_E = 0$ if and only if $\text{Hom}(M, \ )$ vanishes on $E^\perp$. However, $\text{Ext}(M, \ )$ vanishes by assumption. Theorem 5.6 of [4] now completes the proof that this forces $M$ to lie in $E$.

Since $R$ is coherent, a finitely generated submodule of $M$ is finitely presented and because $R$ is hereditary, $M$ satisfies the ascending chain condition on bound submodules. Therefore, the maximal torsion submodule $T$ of $M$ (with respect to the torsion theory generated by $E$) is finitely presented. Since $\text{relproj}(E^\perp)$ is closed under submodules, $T \in \text{relproj}(E^\perp)$ and since $T$ is torsion with respect to the torsion theory generated by $E$, $\text{Hom}(T, \ ) = 0$ on $E^\perp$ and the previous paragraph completes the proof that $T$ lies in $E$. Finally, we know by theorem 4.7 of [3] that $\text{Ext}_R(\ , \ ) = \text{Ext}_{R_E}(\ , \ )$ for $R_E$ modules and so, $\text{Ext}_R(M \otimes R_E, \ ) = \text{Ext}_{R_E}(M \otimes R_E, \ ) = 0$ and so any finitely presented submodule and in particular the image of $M$ in $M \otimes R_E$ must be in $\text{relproj}(E^\perp)$. □

Now we can characterise the category $\text{relproj}(E^\perp)$.

**Lemma 4.3.** The category $\text{relproj}(E^\perp)$ is the smallest full subcategory of $\text{fpmod}(R)$ containing $\text{proj}(R)$ and $E$ that is closed under extensions and submodules.
Proof. It is clear that \( \text{relproj}(E^\perp) \) is closed under extensions and submodules and contains \( \text{proj}(R) \) and \( E \). Let us for the moment name the smallest category with these properties \( B \). Let \( M \in \text{relproj}(E^\perp) \). The kernel \( K \) of the homomorphism from \( M \) to \( M \otimes R_E \) lies in \( E \) so it is enough to show that \( M/K \in B \). However, \( M/K \) is an \( R \) submodule of a finitely generated projective module over \( R_E \) and hence of some finitely generated free module over \( R_E \). However, by theorem 2.6 \( R_E^\perp \) is a direct limit of \( R \) modules \( N_i \) for which we have a short exact sequence \( 0 \to R^n \to N_i \to E_i \to 0 \) where \( E_i \in E \). So \( M/K \subset N_i \) for some \( i \) and \( N_i \in B \) which implies \( M/K \in B \).

Next we point out the connection between \( \text{fac}(E) \) and \( \text{relproj}(E^\perp) \).

Lemma 4.4. Let \( T \in \text{fac}(E) \). Then \( \text{Tor}_1^R(T, R_E) \) is a finitely generated projective module over \( R_E \).

In fact, if we have a short exact sequence \( 0 \to K \to E \to T \to 0 \) where \( E \in E \) then \( \text{Tor}_1^R(T, R_E) \cong K \otimes R_E \).

Proof. We begin with the second statement which follows at once because \( \text{Tor}_1^R(E, R_E) = 0 = E \otimes R_E \).

Now the first statement follows because we can find \( E \in E \) and a surjection from \( E \) onto \( T \).

We come to our first main theorem.

Theorem 4.5. Let \( R \) be a hereditary ring and let \( E \) be a well-placed subcategory of \( \text{fpmod}(R) \). Then every finitely generated projective module over \( R_E \) is isomorphic to a module of the form \( M \otimes R_E \) for some \( M \in \text{relproj}(E^\perp) \).

Therefore, \( P_0(R_E) \cong A^+(\text{relproj}(E^\perp))/\{[E] = 0, E \in E \} \).

Proof. Every finitely presented module over \( R_E \) is induced from some finitely presented module over \( R \) which applies to any finitely generated projective module over \( R_E \). Moreover, we can assume that it is of the form \( M \otimes R_E \) where \( M \) embeds in \( M \otimes R_E \). But as we saw in the argument of lemma 4.4 this implies that \( M \in \text{relproj}(E^\perp) \).

Now suppose that \( M \otimes R_E \cong N \otimes R_E \). Let \( K_1 \) and \( K_2 \) be the kernels of the homomorphisms from \( M \) to \( M \otimes R_E \) and from \( N \) to \( N \otimes R_E \) respectively. Then \( K_1, K_2 \in E \) and \( M \otimes R_E \cong M/K_1 \otimes R_E \) whilst \( N \otimes R_E \cong N/K_2 \otimes R_E \). So \( [M] = [M/K_1] + [K_1] \) and \( [N] = [N/K_2] + [K_2] \).

We saw in theorem 2.6 that \( M/K_1 \otimes R_E \cong N/K_2 \otimes R_E \) if and only if there exist two short exact sequences \( 0 \to M/K_1 \to L \to E \to 0 \) and \( 0 \to N/K_2 \to L \to E' \to 0 \) where \( E, E' \in E \). So \( [M/K_1] + [E] = [N/K_2] + [E'] \).

It follows that \( [M] + [E] + [K_2] = [N] + [E'] + [K_1] \) in \( A^+(\text{relproj}(E^\perp)) \) which shows that the relations are as stated.

We now link in what happens to the category \( \text{fac}(E) \).

Lemma 4.6. Let \( M \in \text{fac}(E) \). Then \( M \) has a unique submodule \( T \) such that \( M/T \in E \) and \( \text{Hom}(T, E) = 0 \) for every \( E \in E \). \( T \) is itself in \( \text{fac}(E) \). Further, \( \text{Tor}_1^R(T, R_E) \cong \text{Tor}_1^R(M, R_E) \).
Proof. Suppose that Hom(M, E) \neq 0. Then choose some \( F \in \mathcal{E} \) that maps onto M and some nonzero \( f : M \to E \) where \( E \in \mathcal{E} \). The image of \( f \) in \( E \) is also the image of the composition of the surjection from \( F \) to \( M \) with \( f \) and since \( E \) is closed under images, the image of \( f \) lies in \( E \) and we obtain a short exact sequence \( 0 \to M_1 \to M \to I_1 \to 0 \) where \( 0 \neq I_1 \in \mathcal{E} \). Applying \( \otimes R_E \) to this short exact sequence, we see that \( M_1 \otimes R_E = 0 \) and so \( M_1 \in \text{fac}(\mathcal{E}) \) and also \( \text{Tor}^R_1(M_1, R_E) \cong \text{Tor}^R_1(M, R_E) \).

Iterating this procedure either gives a descending chain of submodules of \( M, M \supset M_1 \supset \cdots \supset M_n \supset \) or else some Hom(M_i, \cdot) vanishes on \( E \). Since \( M \) satisfies the descending chain condition on bound submodules we cannot have an infinite descending chain and therefore we obtain some \( M_i \) where Hom(M_i, \cdot) vanishes on \( E \). By construction, \( M/M_i \in \mathcal{E} \). So we take \( T = M_i \). However, the uniqueness of \( T \) is clear when it exists since \( T \) must lie in the kernel of any homomorphism from \( M \) to a module in \( \mathcal{E} \).

The isomorphism of \( \text{Tor}^R_1(T, R_E) \) with \( \text{Tor}^R_1(M, R_E) \) follows from the composition of the isomorphisms of \( \text{Tor}^R_1(M_j, R_E) \) with \( \text{Tor}^R_1(M_{j+1}, R_E) \) for each \( j \).

The point of this lemma is to remove surplus copies of modules in \( \mathcal{E} \) from the top of modules in \( \text{fac}(\mathcal{E}) \) as a dual process to the removal of modules in \( \mathcal{E} \) that occur as submodules in modules in \( \text{relproj}(\mathcal{E}^+) \).

**Lemma 4.7.** Let \( M, N \) be modules in \( \text{fac}(\mathcal{E}) \) such that Hom(M, \cdot) and Hom(N, \cdot) both vanish on \( \mathcal{E} \). Then \( \text{Tor}_1^R(M, R_E) \cong \text{Tor}_1^R(N, R_E) \) if and only if there exist two short exact sequences \( 0 \to K \to E_1 \to M \to 0 \) and \( 0 \to K \to E_2 \to N \to 0 \) where \( E_1, E_2 \in \mathcal{E} \) if and only if there exist two short exact sequences \( 0 \to E \to L \to M \to 0 \) and \( 0 \to E' \to L \to N \to 0 \) where \( E, E' \in \mathcal{E} \).

**Proof.** Assume that \( \text{Tor}_1^R(M, R_E) \cong \text{Tor}_1^R(N, R_E) \). We choose short exact sequences \( 0 \to A \to F \to M \to 0 \) and \( 0 \to B \to F' \to N \to 0 \) where \( A, B \) are torsion-free with respect to the torsion theory generated by \( \mathcal{E} \). We know that \( A \otimes R_E \cong \text{Tor}_1^R(M, R_E) \cong \text{Tor}_1^R(N, R_E) \cong B \otimes R_E \) and so by [27] there exist two short exact sequences \( 0 \to A \to K \to G_1 \to 0 \) and \( 0 \to B \to K \to G_2 \to 0 \) where \( G_i \in \mathcal{E} \). We form the pushout of the short exact sequence \( 0 \to A \to F \to M \to 0 \) along the map from \( A \) to \( K \) and the pushout of the short exact sequence \( 0 \to B \to F' \to N \to 0 \) along the map from \( B \) to \( K \) to obtain two short exact sequences \( 0 \to K \to E_1 \to M \to 0 \) and \( 0 \to K \to E_2 \to N \to 0 \) where \( E_i \in \mathcal{E} \) because \( E_1 \) is an extension of \( G_1 \) on \( F \) and \( E_2 \) is an extension of \( G_2 \) on \( F' \). Thus the first condition implies the second.

Now assume that there exist two short exact sequences \( 0 \to K \to E_1 \to M \to 0 \) and \( 0 \to K \to E_2 \to N \to 0 \) where \( E_1, E_2 \in \mathcal{E} \). Forming the common pushout of these two short exact sequences gives a module \( L \) and two short exact sequences \( 0 \to E_2 \to L \to M \to 0 \) and \( 0 \to E_1 \to L \to N \to 0 \) which completes the proof that the second condition implies the third.

Finally, if we assume that there exist two short exact sequences \( 0 \to E \to L \to M \to 0 \) and \( 0 \to E' \to L \to N \to 0 \) where \( E, E' \in \mathcal{E} \) then applying \( \otimes R_E \)
shows that \( \text{Tor}_1^R(M, R_E) \cong \text{Tor}_1^R(L, R_E) \cong \text{Tor}_1^R(N, R_E) \) and demonstrates that the third condition implies the first.

We are in a position to prove the results we want about the category \( \text{fac}(E) \).

**Theorem 4.8.** Let \( R \) be a hereditary ring and let \( E \) be a well-placed subcategory of \( \text{fpmod}(R) \). Then \( \text{Tor}_1^R(\cdot, R_E) \) gives an exact functor from \( \text{fac}(E) \) to \( \text{proj}(R_E) \). Its image is the same as the image of the full subcategory of finitely presented submodules of modules in \( E \).

The relations imposed on \( A^+(\text{fac}(E)) \) by the induced map to \( P_{\oplus}(R_E) \) are precisely \( \{ [E] = 0; E \in E \} \).

**Proof.** We have already noted that \( \text{Tor}_1^R(\cdot, R_E) \) gives an exact functor from \( \text{fac}(E) \) to \( \text{proj}(R_E) \). Choosing a surjection from some module \( E \in E \) onto a given \( M \in \text{fac}(E) \) with kernel \( K \) identifies \( \text{Tor}_1^R(M, R_E) \) with \( K \otimes R_E \) and proves that the image of this functor is the image of the full subcategory of finitely presented submodules of modules in \( E \).

Finally, we see that lemmas 4.7 and 4.6 give the relations imposed on \( A^+(\text{fac}(E)) \) to be as required.

**Theorem 4.9.** Let \( R \) be a hereditary ring and let \( E \) be a well-placed subcategory of \( \text{fpmod}(R) \) all of whose modules are bound. Let \( S \) be the set of modules simple as objects of \( E \). Then every finitely generated projective module over \( R_E \) is isomorphic to a direct sum of modules \( M \otimes R_E \) where either \( M \) is a finitely generated projective module over \( R \) or else \( M \) is a submodule of a module in \( S \).

**Proof.** We may take our finitely generated projective module over \( R_E \) to be of the form \( K \otimes R_E \) where \( K \) is a finitely presented module over \( R \) torsion-free with respect to the torsion theory generated by \( E \) so that \( K \) is a submodule of \( K \otimes R_E \) which is a submodule of \( R_E^n \) for some \( n \). Since \( R_E^n \) is a directed union of modules \( F_t \) where \( F_t \supset R^n \) and \( F_t/R^n \in E \), we know that \( K \) is a submodule of some \( F_t \). But \( F_t \) has a filtration whose subfactors are either finitely generated free or else isomorphic to a module in \( S \). Intersecting this filtration with \( K \) gives a filtration of \( K \) where the subfactors are isomorphic to submodules of free modules (so projective) or else submodules of modules in \( S \). Moreover, these subfactors are finitely presented because \( R \) is hereditary and so coherent.

Because both of these kinds of modules induce to finitely generated projective module over \( R_E \), it follows that this filtration is split by \( \otimes R_E \) and so \( K \otimes R_E \) is isomorphic to the direct sum of the modules induced from the subfactors which gives what we set out to prove.

In order to get further information, we introduce a new kind of module in \( \text{relproj}(E^+) \). We say that a module in some full subcategory of \( \text{fpmod}(R) \) is late in that category if and only if any map in the full subcategory from it is injective. If the category is an exact abelian subcategory then this is equivalent to being a simple object in the category but in general it is not. Such a module may have subobjects in the subcategory if the subcategory is not closed under factors. We have no guarantee that late objects exist; however, when they do they can be useful.
Lemma 4.10. Let $M$ be torsion-free with respect to the torsion theory generated by $E$ and assume that $M$ has no factor in $E$. Then if $M \otimes_{R_E} R$ is an indecomposable finitely generated projective module over $R_E$, $M$ is a late object in $\text{relproj}(E^\perp)$.

Proof. Since $M$ is torsion-free, $M$ embeds in $M \otimes_{R_E} R$ and so $M \in \text{relproj}(E^\perp)$. Suppose that we have a map in $\text{relproj}(E^\perp)$, $f: M \to N$. Since $\text{relproj}(E^\perp)$ is closed under submodules, we have a short exact sequence $0 \to \ker(f) \to M \to \im(f) \to 0$ all of whose terms lie in $\text{relproj}(E^\perp)$. Applying $\otimes_{R_E}$ shows that $M \otimes_{R_E} \cong \im(f) \otimes_{R_E} R_E \oplus \ker(f) \otimes_{R_E} R_E$ and since $M \otimes_{R_E} R$ is indecomposable either $\im(f) \otimes_{R_E} R_E = 0$ or else $\ker(f) \otimes_{R_E} R_E = 0$. In the first case, this implies that $\im(f) \in E$ and so $\im(f) = 0$ since $M$ has no factor in $E$ and this means that $f$ is the zero map. In the second case, this implies that $\ker(f) \in E$ and so $\ker(f) = 0$ which means that $f$ is injective as stated.

Thus if we happen to know that every finitely generated projective module over $R_E$ is a direct sum of indecomposable finitely generated projective modules we will have a large supply of late objects in $\text{relproj}(E^\perp)$.

We need the dual concept for the category $\text{fac}(E)$. We say that a module in a full subcategory is an early object in the subcategory if and only if any nonzero map in the subcategory to it must be surjective as a homomorphism of modules. We simply state the next lemma since its proof is no different from the preceding lemma.

Lemma 4.11. Let $M \in \text{fac}(E)$ have no factors nor submodules in $E$. Then if $\text{Tor}_1^R(M, R_E)$ is an indecomposable finitely generated projective module over $R_E$, $M$ is an early object in $\text{fac}(E)$.

Let $S$ be the set of simple objects in $E$. We say that the modules $A$ and $B$ are directly $S$-related if and only if there exists either a short exact sequence $0 \to A \to S_i \to B \to 0$ or a short exact sequence $0 \to B \to S_i \to A \to 0$ where $S_i \in S$. We say that $A$ and $B$ are $S$-related if they are equivalent under the equivalence relation generated by this relation. Clearly, if $A, B \in \text{relproj}(E^\perp)$ and are $S$-related then $A \otimes_{R_E} \cong B \otimes_{R_E}$. We should like a converse to this observation since attempting to find extensions of $A$ and $B$ by modules in $E$ that are isomorphic is hard whereas we may well have a clear grasp of the submodule structure of the modules in $S$ and thus an ability to calculate which modules are $S$-related. The next theorem provides us with a partial converse which in practice is frequently decisive.

Theorem 4.12. Let $R$ be a hereditary ring and let $E$ be a full subcategory of $\text{fpmod}(R)$ all of whose modules are bound. Let $S$ be the set of simple objects in $E$. Let $A, B$ be late objects in $\text{relproj}(E^\perp)$ such that $A \otimes_{R_E} \cong B \otimes_{R_E}$. Then either $A$ is $S$-related to $B$ or else one of $A$ and $B$ is $S$-related to a module in $\text{relproj}(E^\perp)$ that is not late or to a module in $\text{fac}(E)$ that is not early, either of which imply that $A \otimes_{R_E}$ is a decomposable projective module over $R_E$.

Thus if in addition $A \otimes_{R_E}$ is an indecomposable finitely generated projective module over $R_E$ then $A$ and $B$ are $S$-related.
Proof. Since \( A \otimes R_E \cong B \otimes R_E \), there exist short exact sequences \( 0 \to A \to C \to E \to 0 \) and \( 0 \to B \to C \to F \to 0 \) where \( E, F \in E \). Since \( E \) is a finite length category we proceed by induction on the sum of the lengths of \( E \) and \( F \) as objects in \( E \). Neither of these lengths can be zero since this forces one of \( A \) and \( B \) to be a submodule of the other and hence makes them isomorphic.

Let \( 0 \to F' \to F \to T \to 0 \) be a short exact sequence where \( T \) is a simple object in \( E \). Consider the composition of the maps from \( A \) to \( C \) to \( F \) to \( T \). If this is 0, we take \( C' \) to be the kernel of the surjection from \( C \) to \( T \) and \( E' \) to be the kernel of the induced surjection from \( E \) to \( T \) and obtain the short exact sequences \( 0 \to A \to C' \to E' \to 0 \) and \( 0 \to B \to C' \to F' \to 0 \) where \( E' \) and \( F' \) have shorter length as objects in \( E \). Otherwise, the map from \( A \) to \( T \) is nonzero. It cannot be surjective since \( A \) is late and not in \( E \). So it must be injective because \( A \) is late. Let \( A' \) be the factor so that \( A \) is directly \( S \)-related to \( A' \); we note that \( A' = C/A + C' \), \( A \cap C' = 0 \) and so we have a short exact sequence \( 0 \to F' \to C/A + B \to A' \to 0 \) where \( F' \in E \) of length 1 less than the length of \( F \). Interchanging the role of \( A \) and \( B \), we obtain \( B' \) directly \( S \)-related to \( B \), and a short exact sequence \( 0 \to E' \to C/A + B \to B' \to 0 \) where \( E' \in E \) and has length 1 less than the length of \( E \).

We now note that we have obtained a dual situation to the one we began with for \( A \) and \( B \) whenever both \( A' \) and \( B' \) are early objects in \( \text{fac}(E) \) but since the sum of the lengths of \( E' \) and \( F' \) is 2 less than the sum of the lengths of \( E \) and \( F \), induction and the dual argument to the one above apply to complete the proof.

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