Kazhdan Constants for $\text{SL}_n(\mathbb{Z})$

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November 2, 2021

Abstract

In this article we improve the known Kazhdan constant for $\text{SL}_n(\mathbb{Z})$ with respect to the generating set of the elementary matrices. We prove that the Kazhdan constant is bounded from below by $[42\sqrt{n} + 860]^{-1}$, which gives the exact asymptotic behavior of the Kazhdan constant, as $n$ goes to infinity, since $\sqrt{2}/n$ is an upper bound.

We can use this bound to improve the bounds for the spectral gap of the Cayley graph of $\text{SL}_n(\mathbb{F}_p)$ and for the working time of the product replacement algorithm for abelian groups.

1 Introduction

Kazhdan property $T$ plays important role in the representation theory of arithmetic groups. Since the work of Kazhdan (see [5]) it is known that any higher rank arithmetic group has property $T$.

In recent years there have been several connections between property $T$ and the working time of several algorithms in componential group theory. In [6] the authors use the Kazhdan property $T$ of the group $\text{SL}_n(\mathbb{Z})$ to prove that the product replacement algorithm on abelian groups has logarithmic working time. In order to make these results quantitative one needs exact values of the Kazhdan constants for certain groups, most notably for the group $\text{SL}_n(\mathbb{Z})$, with respect to the standard generators.

Almost all methods for proving that an arithmetic group $\Gamma$ has property $T$ use Kazhdan’s result and transfer the question to Lie groups. These methods can not be made quantitative and they do not lead to any explicit Kazhdan constants. The first author to obtain partial results in this direction was M. Burger — in [1], he found a lower bound for the some constant, closely related to the Kazhdan constant for the group $\text{SL}_3(\mathbb{Z})$. Several years later, Y. Shalom (see [8]) used bounded generation to construct an explicit lower bound for Kazhdan constant for the group $\text{SL}_n(\mathbb{Z})$. This result can be combined with the result of O. Tavgen (see [9]) to obtain similar bounds for other higher rank arithmetic groups.

*The author is supported by several NSERC grants
The main result of this paper gives the exact asymptotical behavior of the Kazhdan constant of $\text{SL}_n$ with respect to the generating set $E_n$ consisting of elementary matrices with $\pm 1$ off the diagonal. The methods used in the proof are based on the ideas in [8].

Structure of the paper: in the following section we describe the main result and an outline of the basic idea of the proof; section 3 we give several applications of the main theorem; sections 4 and 5 are dedicated to the derivations of explicit Kazhdan constants for the relative property $T$ of the groups $\text{SL}_p \rtimes \mathbb{Z}^p$ and $(\text{SL}_p \times \text{SL}_q) \rtimes \mathbb{Z}^{pq}$; section 6 describes vector systems in $\mathbb{Z}^k$ and generalized elementary operations, which are used in the proof of the stronger version bound generation property of the group $\text{SL}_n(\mathbb{Z})$ with respect to the set $E_n$; section 7 concludes the proof of Theorem A. The last section is dedicated to some possible extensions of the main theorem.

Acknowledgements: The author wishes to thanks Roman Muchnik, Tal Poznansky and Misha Ershov for the useful discussions and helpful suggestions during the preparation of the manuscript. I thank Igor Pak for suggesting several applications of the main result. I wish to express my gratitude to Alex Lubotzky, Yehuda Shalom and my adviser Efim Zelmanov for introducing me to this subject. I also wish to thank to Clay Mathematics Institute for the financial support during the preparation of this paper.

2 Main Result

Let us recall the definition of Kazhdan property $T$:

**Definition 2.1.** A topological group $G$, generated by a compact set $Q$, is said to have Kazhdan property $T$, if there exists a constant $\epsilon$, such that any (continuous) unitary representation $(\pi, H)$ of the group $G$, which contains a unit vector $v$ such that $||\pi(g)v - v|| \leq \epsilon$ for any $g \in Q$, contains a $G$ invariant vector. The maximal $\epsilon$ with this property is called the Kazhdan constant of $G$ with respect to $Q$ and is denoted by $K(G, Q)$.

In [5], Kazhdan proved that any higher rank Lie group $G$ and any lattice $\Gamma$ in such a group has property $T$, without giving any values for the Kazhdan constants.\(^1\) In particular from his work follows that $\text{SL}_n(\mathbb{Z})$ has property $T$ if $n \geq 3$.

Let $E_n$ be the set of all elementary matrices with $\pm 1$ off the diagonal. It is known that the set $E_n$ generates the group $\text{SL}_n(\mathbb{Z})$ and it is natural to look for the value of the Kazhdan constant $K(\text{SL}_n, E_n)$. The main result in this paper is the following lower bound for the Kazhdan constant of the group $\text{SL}_n(\mathbb{Z})$ with respect to the set $E_n$.

**Theorem A.** The Kazhdan constant for $\text{SL}_n(\mathbb{Z})$ with respect to the set $E_n$ of elementary matrices with $\pm 1$ off the diagonal is

$$K(\text{SL}_n(\mathbb{Z}), E_n) \geq (64\sqrt{n} + 2850)^{-1}.$$\(^1\)

\(^1\)This result does not hold for some rank 1 groups. For example, the groups $\text{SL}_2(\mathbb{R})$ and $\text{SL}_2(\mathbb{Z})$ does not have Kazhdan property $T$. 

2
Using the same methods but carefully tracking all constants, allows us to obtain a slightly better result:

**Theorem A'.** The Kazhdan constant for \( \text{SL}_n(\mathbb{Z}) \) with respect to the set \( E_n \) of elementary matrices with ±1 off the diagonal is

\[
\mathcal{K}(\text{SL}_n(\mathbb{Z}), E_n) \geq (42\sqrt{n} + 860)^{-1}.
\]

If we consider the group \( \text{SL}_n(\mathbb{F}_p) \) we can improve the bound even further:

**Theorem A''.** The Kazhdan constant for \( \text{SL}_n(\mathbb{F}_p) \) with respect to the set \( E_n \) of elementary matrices with ±1 off the diagonal is

\[
\mathcal{K}(\text{SL}_n(\mathbb{F}_p), E_n) \geq (31\sqrt{n} + 700)^{-1}.
\]

Theorem A can be generalized to linear group over number fields:

**Theorem B.** Let \( \mathcal{O} \) be the ring of integers in a number field \( \mathbb{K} \), which is generated as a ring by 1 and the elements \( \alpha_i \) for \( i = 1, \ldots, s \). The Kazhdan constant for \( \text{SL}_n(\mathcal{O}) \) with respect to the set \( E_n(\mathcal{O}) \) of elementary matrices with ±1 and ±\( \alpha_i \) off the diagonal is

\[
\mathcal{K}(\text{SL}_n(\mathcal{O}), E_n(\mathcal{O})) \geq [50\sqrt{n} + (B + C\Delta)6^m]^{-1}.
\]

where \( B \) and \( C \) are universal constants and \( \Delta \) is the number of different prime divisors of the discriminant of \( \mathbb{K} : \mathbb{Q} \).

The lower bounds of the Kazhdan constant obtained in Theorems A and B are asymptotically exact. Let us consider the natural representation of \( \text{SL}_n(\mathbb{Z}) \) into \( L^2(\mathbb{Z}^n) \). There exists a unit vector \( v \in L^2(\mathbb{Z}^n) \), which is moved by any element in \( E_n \) by a distance of \( \sqrt{2/n} \). This shows that

\[
\mathcal{K}(\text{SL}_n(\mathbb{Z}), E_n) \leq \sqrt{2/n}.
\]

This upper bound for the Kazhdan constant for \( \text{SL}_n(\mathbb{Z}) \) was found by A. Zuk (unpublished) and can be found in \[\text{[8]}\].

The basic idea of the proof is similar to the one in \[\text{[5]}\]. There, Y. Shalom used the relative property \( T \) and bounded generation to prove that \((33n^2 - 11n + 1152)^{-1}\) is a lower bound for the Kazhdan constant. The main steps in his proof are the following:

The group \( \text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \) has relative property \( T \), with respect to the normal subgroup \( \mathbb{Z}^2 \) with Kazhdan constant at least 1/10 (using the elementary matrices in \( \text{SL}_2 \) and the basis vectors of \( \mathbb{Z}^2 \) as generating set). This bound gives that for any unitary representation \((\pi, \mathcal{H})\) if any elementary matrix (with ±1 off the

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\[\text{[3]}\]From the Generalized Reimann Hypothesis it follows that the bound for the Kazhdan constant does not depend on the discriminant of \( \mathbb{K} : \mathbb{Q} \), see \[\text{[8]}\] for details.

\[\text{[3]}\]This representation contains one dimensional space of invariant vectors, however the vector \( v \) lies in the orthogonal compliment of this invariant subspaces and we can restrict everything to that subspace.
diagonal) moves a fixed unit vector \( v \in \mathcal{H} \) by less than \( \epsilon \), then any elementary matrix (with any integer off the diagonal) moves the same vector \( v \) by at most \( 20\epsilon \).

Bounded generation of the group \( SL_n(\mathbb{Z}) \) with respect to the elementary matrices, proved by Carter and Keller \[3\], gives that any element \( g \in SL_n(\mathbb{Z}) \) can be written as product of at most (approximately) \( 3n^2/2 \) elementary matrices. This, together with the previous step, shows that any element \( g \in SL_n(\mathbb{Z}) \) moves the vector \( v \) by at most \( 30n^2\epsilon \).

Finally we use the observation that if a unit vector is moved by any element of a group by a distance less than 1 then the representation has an invariant vector. This leads to a lower bound for the Kazhdan constant. We use the group \( N \), because a generic element in \( SL_n(\mathbb{Z}) \) is at least \( \frac{1}{h} \) of elementary matrices. In our proof, instead of working with the group \( \mathbb{Z}^2 \), we work with larger abelian subgroups \( H_i \) of \( SL_n \). Instead of \( SL_2 \ltimes \mathbb{Z}^2 \), we use the group \( N_i \ltimes H_i \), where \( N_i \) is semi-simple and \( N_i \ltimes H_i \) is maximal parabolic. This group has a relative property \( T \) with respect to the subgroup \( H_i \). Using this constant it can be shown that if any elementary matrix moves \( v \) by at least \( \epsilon \), then any element \( g \) lying in some \( H_i \), moves \( v \) by at most \( 2k(n)\epsilon \). We can obtain this result using the relative property \( T \) of \( SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \) which can be embedded in many different ways in \( N_i \ltimes H_i \) and obtain an upper bound for \( k(n) \) of type \( O(n^2) \). However, using the whole groups \( N_i \ltimes H_i \) allows us to improve the bound for \( k(n) \) to \( O(\sqrt{n}) \), which allows us to obtain a better lower bound for the Kazhdan constant of \( SL_n(\mathbb{Z}) \).

The main result in section 4 shows that using 5 multiplications by elements lying in conjugates of some \( H_i \), every element in \( SL_n(\mathbb{Z}) \) can be transformed to an element in \( SL_{\lambda n}(\mathbb{Z}) \) (embedded in the upper left corner of \( SL_n(\mathbb{Z}) \)), where \( \lambda = 2/3 \). This, together with the result by Carter and Keller \[2,3\], saying that any matrix in \( SL_2(\mathbb{Z}) \) can be written as a product of at most 60 elementary matrices, gives that any matrix in \( SL_n(\mathbb{Z}) \) can be written as a product of at most \( s(n) = 60 + 13 \ln n \) matrices lying in some \( H_i \). This implies that any element in \( SL_n \) moves the vector \( v \) by at most \( h(n)\epsilon \), where \( h(n) = 2s(n)k(n) \). From this result, we obtain that the Kazhdan constant for \( SL_n(\mathbb{Z}) \) with respect to the set of elementary matrices is at least \( 1/h(n) \). This argument gives a bound of type \( (\sqrt{n} \ln n)^{-1} \) for the Kazhdan constant, but a more detailed consideration allow us to improve this bound to \( n^{-1/2} \).

\[4\]In almost all cases, the group \( N_i \ltimes H_i \) actually has property \( T \). Using the same methods a bound the Kazhdan constant for this group can be computed, and it is of the similar to the relative Kazhdan constant.

Using a result of L. Vaserstain (see \[10\]) it can be shown that any element in \( SL_n(\mathbb{Z}) \) can be written as a product of a fixed number of matrices in some \( H_i \). The best bound for the number of matrices need to write any element in \( SL_n \) as such product is around 100 and leads to slightly worse bound for the Kazhdan constant of \( SL_n(\mathbb{Z}) \) than the one obtained in Theorem 5.
3 Applications

The value of the Kazhdan constant is related to several other constants, like the spectral gap of the Laplacian and the mixing time of random walks on finite Cayley graphs.

3.1 Spectral gap of Cayley graph of $\text{SL}_n(\mathbb{Z})$

Applying the result by I. Pak and A. Zuk from [7], we have that the spectral gap is

$$\beta(\text{SL}_n(\mathbb{Z})) \geq \mathcal{K}(\text{SL}_n(\mathbb{Z}))^2 / 4 \geq \frac{1}{4(42\sqrt{n} + 860)^2} = O(1/n).$$

This bound is better than the previously known one which was $O(n^{-4})$. The argument from section 2 give that

$$\beta(\text{SL}_n(\mathbb{Z})) \leq 1/n.$$

3.2 Spectral gap of Cayley graph of $\text{SL}_n(\mathbb{F}_p)$

Since $\text{SL}_n(\mathbb{F}_p)$ is a factor group of $\text{SL}_n(\mathbb{Z})$, the spectral gap of $\text{SL}_n(\mathbb{F}_p)$ is bigger than the one for $\text{SL}_n(\mathbb{Z})$. We can obtain a slightly better estimate, using the fact that $\mathbb{F}_p$ is a field. The better bound for the Kazhdan constant $\mathcal{K}(\text{SL}_n(\mathbb{F}_p)) \geq (31\sqrt{n} + 700)^{-1}$, from Theorem A', yields the bound for the spectral gap

$$\frac{1}{n} \geq \beta(\text{SL}_n(\mathbb{F}_p)) \geq \frac{1}{4(31\sqrt{n} + 700)^2} = O(1/n).$$

3.3 Mixing time of random walks on $\text{SL}_n(\mathbb{F}_p)$

There is a connection between the spectral gap of the finite Cayley graph and the mixing time of the random walk on the same graph. Applying that to the group $G = \text{SL}_n(\mathbb{F}_p)$ gives

$$\text{mix} \lesssim \beta \log |G| = O(n^3 \log p),$$

which is better than the previous known bounds of $n^6 \log p$ (see [2]) and $n^4 \log^3 p$ (see [4]).
3.4 Mixing time of the product replacement algorithm for abelian groups

In [6], A. Lubotzky and I. Pak showed a connection between the working time of the Product Replacement Algorithm on \( n \)-generated abelian groups and the Kazhdan constant for \( SL_n(\mathbb{Z}) \), in particular they proved the following upper bound for the mixing time:

\[
\text{mix} \lesssim nK(SL_n(\mathbb{Z}), E_n)^{-2} \log |\Gamma|.
\]

Using the bound for the Kazhdan constant from Theorem [A] gives a bound of \( n^2 \log |\Gamma| \).

3.5 Relaxation time for particle systems

Particle systems were studied by P. Diaconis and L. Saloff-Coste (the original problem was proposed by D. Aldous). In [4] they proved that the relaxation time of a particle system is bounded by \( n^2 \log n \) and made a conjecture that it is of type \( n \log n \). Via the particle system as a random walk on \( \mathbb{Z}_p^n \) we can use the bound for the Kazhdan constant of \( SL_n(\mathbb{Z}) \), which allows us to confirm this conjecture.

4 Relative Kazhdan constant for \( SL_p(\mathbb{Z}) \rtimes \mathbb{Z}^p \)

In this section, we estimate the relative Kazhdan constant for the group \( SL_p(\mathbb{Z}) \rtimes \mathbb{Z}^p \) with respect to the group \( \mathbb{Z}^p \), considering the set of elementary matrices in \( SL_p \) together with the basis vectors of \( \mathbb{Z}^p \) as a generating set. The idea of the proof of Theorem [4.1] is based on the one used by Burger in [1] for estimating the relative Kazhdan constant for \( SL_2(\mathbb{Z}) \rtimes \mathbb{Z}_2 \). A very detailed explanation of this proof can be found in [8].

**Theorem 4.1.** Let \( F \) denote the set of elementary matrices in \( SL_p(\mathbb{Z}) \), and \( G \) denote the set of the \( p \) standard basis elements of \( \mathbb{Z}^p \). Let \((\pi, \mathcal{H})\) be a unitary representation of \( SL_p(\mathbb{Z}) \rtimes \mathbb{Z}^p \), containing a vector \( v \) which is \((F \cup G, 1/l(p))\) invariant, where

\[
l(p) = \sqrt{p + 25} + 3.
\]

Then \( \mathcal{H} \) contains \( \mathbb{Z}^p \) invariant vector, provided that \( p \geq 2 \).

**Remark 4.2.** Let us consider the standard the unitary representation of \( SL_{p+1} \) on \( L^2(\mathbb{Z}_{p+1}) \). The group \( SL_p \rtimes \mathbb{Z}^p \) is isomorphic to a maximal parabolic in \( SL_{p+1} \), thus we have a representation of that group in \( L^2(\mathbb{Z}_{p+1}) \). The representation decomposes as a sum of two representations, one is trivial and isomorphic to \( L^2(\mathbb{Z}) \), the other is a representation on \( L^2(\mathbb{Z}_{p+1} \setminus \mathbb{Z}) \), without \( \mathbb{Z}^p \) invariant vectors. Using that representation and a suitable vector \( v \), it can be shown that the Kazhdan constant is at most \( \sqrt{2/p} \). This shows that \( 1/l(n) \) is not a relative Kazhdan constant, where \( l(p) = \sqrt{p/2} \).

\[^6\]In his paper [4], Burger attributes the idea of this proof to Furstenberg.
Proof. Assume that $v \in \mathcal{H}$ is $(F \cup G, \epsilon)$ invariant unit vector, and that the Hilbert space $\mathcal{H}$ does not contain $Z^p$ invariant vector. Let $P$ be the projection valued measure on $\mathbb{T}^p = \mathbb{T}^p$, coming from the restriction of the representation $\pi$ to $\mathbb{T}^p$, and let $\mu_v$ be the measure on $\mathbb{T}^p$, defined by $\mu_v(B) = \langle P(B)v, v \rangle$. The probability measure $\mu_v$ is supported on $\mathbb{T}^p \setminus \{0\}$, because by assumption $\mathcal{H}$ does not contain an $Z^p$ invariant vector and by construction $P(\{0\})$ is the projection onto the space of $Z^p$ invariant vectors.

For an element $x \in \mathbb{T}^p$ we will write $x = (x_1, \ldots, x_p)$, where all $x_i$ are in $\mathbb{R}/\mathbb{Z}$, which we identify with the interval $(-1/2, 1/2]$.

**Lemma 4.3.** Let $K_i = \{x \mid 1/4 > |x_i|\}$, then $\mu_v(K_i) \geq 1 - \epsilon^2/2$.

**Proof.** By the definition of the measure $\mu_v$, we have

$$||\pi(g_i) v - v||^2 = \int_{\mathbb{T}^p} |e^{2\pi i x_j} - 1|^2 d\mu_v \leq \epsilon^2,$$

where $g_i$ form the standard basis of $Z^p$. Now using the fact that $|e^{2\pi i x_j} - 1|^2 \geq 2$ for $1/2 \geq |x_i| \geq 1/4$, the above inequality implies that $\mu_v(|x_i| \geq 1/4) \leq \epsilon^2/2$. □

**Lemma 4.4.** For every Borel set $B \subset \mathbb{T}^p$ and every elementary matrix $g \in F$, we have that

$$|\mu_v(gB) - \mu_v(B)| \leq 2\epsilon \sqrt{\mu_v(B)} + \epsilon^2.$$

The action of $\text{SL}_p(\mathbb{Z})$ on $\mathbb{T}^p$ is the standard one coming from the action on $\mathbb{R}^p$, via the isomorphism $\mathbb{T}^p = \mathbb{R}^p/Z^p$.

**Remark 4.5.** Similar lemma was used in [1] and in [8], but the upper bound for $|\mu_v(gB) - \mu_v(B)|$ was $2\epsilon$. If we use that lemma we could only obtain $1/p$ as a bound for the relative Kazhdan constant for $\text{SL}_p \times \mathbb{Z}^p$, which will give a Kazhdan constant for $\text{SL}_n(\mathbb{Z})$ of the form $O(1/n)$.

**Proof.** Using the properties of the projection valued measure $P$, we have

$$|\mu_v(gB) - \mu_v(B)| = |\langle \pi(g^{-1})P(B)v, v \rangle - \langle P(B)v, v \rangle| \leq$$

$$\leq |\langle \pi(g^{-1})P(B)(\pi(g)v - v), v \rangle + |\langle P(B)v, (\pi(g)v - v) \rangle| =$$

$$= 2|\langle \pi(g)v - v, P(B)v \rangle + (P(B)(\pi(g)v - v), \pi(g)v - v) \rangle| \leq$$

$$\leq 2\epsilon \sqrt{\mu_v(B)} + \epsilon^2,$$

where the final inequality follows from the facts that $v$ is $(F, \epsilon)$ invariant vector and $||P(B)v||^2 = \mu_v(B)$. □

**Lemma 4.6.** Let $\mu$ be a finitely additive measure on $\mathbb{T}^2$ such that:

- $\mu(|x| \geq 1/4) \leq \epsilon^2/2$ and $\mu(|y| \geq 1/4) \leq \epsilon^2/2$,

- $|\mu(gB) - \mu(B)| \leq 2\epsilon \sqrt{\mu(B)} + \epsilon^2$ for any Borel set $B$ and any elementary matrix $g \in \text{SL}_2(\mathbb{Z})$. 

7
Then we have

\[ \mu(T^2 \setminus \{(0,0)\}) \leq (2 + \sqrt{10})\varepsilon^2 \quad \text{and} \quad \mu(x \neq 0, y = 0) \leq (1 + \sqrt{3})\varepsilon^2. \]

**Proof.** Let us define the Borel subsets \( A_i \) and \( A'_i \) of \( T^2 \) using the picture:

\[
\begin{array}{c|c|c|c|}
\hline
-1/2 & 1/2 & -1/2 & 1/2 \\
\hline
A_3 & A'_2 & & \\
\hline
A'_4 & A_4 & A_1 & A'_1 \\
\hline
A_3 & A'_2 & A_1 & A'_1 \\
\hline
A'_4 & A_4 & & \\
\hline
\end{array}
\]

Each set \( A_i \) or \( A'_i \) consists of the interiors of two triangles and part of their boundary (not including the vertices). The sets \( A_i \) do not contain the side which is part of the small square, they also do not contain their clockwise boundary but contain the counter-clockwise one. Each set \( A'_i \) includes only the part of its boundary which lies on the small square.

From the picture it can be seen that the elementary matrices \( g_{ij}^+ = I \pm e_{ij} \in F \), act on the sets \( A_i \) as follows:

\[
\begin{align*}
g_{12}^+(A_3 \cup A'_1) &= A_3 \cup A_4 \\
g_{12}^+(A'_1 \cup A_2) &= A_1 \cup A_2 \\
g_{21}^+(A'_3 \cup A_4) &= A_3 \cup A_4 \\
g_{21}^+(A_1 \cup A'_2) &= A_1 \cup A_2.
\end{align*}
\]

Using the properties of the measure \( \mu \) the above equalities imply the inequalities:

\[
\begin{align*}
\mu(A_1) + \mu(A_2) &\leq \mu(A'_1) + \mu(A_2) + \varepsilon^2 + 2\varepsilon \sqrt{\mu(A'_1) + \mu(A_2)} \\
\mu(A_1) + \mu(A_2) &\leq \mu(A_1) + \mu(A'_2) + \varepsilon^2 + 2\varepsilon \sqrt{\mu(A_1) + \mu(A'_2)} \\
\mu(A_3) + \mu(A_4) &\leq \mu(A'_3) + \mu(A_4) + \varepsilon^2 + 2\varepsilon \sqrt{\mu(A'_3) + \mu(A_4)} \\
\mu(A_3) + \mu(A_4) &\leq \mu(A_3) + \mu(A'_4) + \varepsilon^2 + 2\varepsilon \sqrt{\mu(A_3) + \mu(A'_4)}.
\end{align*}
\]

Adding these inequalities and noticing that

\[
\begin{align*}
\mu(A'_1) + \mu(A'_2) &\leq \mu(\{|x| \geq 1/4\}) \leq \varepsilon^2/2 \\
\mu(A'_2) + \mu(A'_3) &\leq \mu(\{|y| \geq 1/4\}) \leq \varepsilon^2/2
\end{align*}
\]

we obtain

\[
\sum_i \mu(A_i) \leq 4\varepsilon^2 + \sum_i \mu(A'_i) + 2\varepsilon \sqrt{4 \left( \sum_i \mu(A_i) + \sum_i \mu(A'_i) \right)} \leq
\]

8
\[ \leq 5\epsilon^2 + 4\epsilon \sqrt{\sum \mu(A_i) + \epsilon^2}. \]

Here we have used that any positive numbers \( a_i \), satisfy the inequality
\[ \sum_{i=1}^{k} \sqrt{a_i} \leq \sqrt{k \sum_{i=1}^{k} a_i}. \]

After substituting \( c = \sqrt{\sum \mu(A_i) + \epsilon^2} \) and solving the resulting quadratic inequality we obtain \( \sum \mu(A_i) \leq (13 + 4\sqrt{10})\epsilon^2 \).

Also from the system of inequalities, taking the inequality for \( \mu(A_i) \), where the index \( i \) is such that \( \mu(A_i) \) is maximal, we have:
\[ \max_i \mu(A_i) \leq \epsilon^2 + \max_i \mu(A_i') + 2\epsilon \sqrt{\max_i \mu(A_i) + \epsilon^2} \leq 3\epsilon^2/2 + 2\epsilon \sqrt{\max_i \mu(A_i) + \epsilon^2/2}, \]
which yields \( \max_i \mu(A_i) \leq (7/2 + 2\sqrt{3})\epsilon^2 \). Finally we can use that
\[ \mu(T^n \setminus \{(0,0)\}) \leq \sum \mu(A_i) + \mu(|x| \geq 1/4) + \mu(|y| \geq 1/4) \leq (14 + 4\sqrt{10})\epsilon^2 = (2 + \sqrt{10})^2\epsilon^2, \]
and
\[ \mu(x \neq 0, y = 0) \leq \max_i \mu(A_i) + \mu(|x| \geq 1/4) \leq (4 + 2\sqrt{3})\epsilon^2 = (1 + \sqrt{3})^2\epsilon^2, \]
which completes the proof of the lemma.

**Lemma 4.7.** Let \( \mu \) be a finitely additive measure on \( T^n \), which satisfies conditions from the previous lemma (with \( SL_p \) replacing \( SL_2 \))
\[ \mu(T^n \setminus \{(0,...,0)\}) \leq (\sqrt{p + 25} + 3)^2\epsilon^2. \]

**Proof.** For a point \( y \in T^n \) we write \( y = (y_1,...,y_p) \), where \( y_i \in (1/2,1/2] \). Let us define the Borel sets
\[ B_i = \{ y \mid y_k = 0 \text{ for } k \leq i \}, \text{ and } \]
\[ C_i = \{ y \mid y_1 = y_i \neq 0, y_k = 0 \text{ for } 1 < k < i \}. \]

The elementary matrix \( g_{1i} \in SL_p \) sends \( B_{i-1} \setminus B_i \) into \( C_i \) for any \( i \geq 3 \). Therefore, we have
\[ \mu(B_{i-1} \setminus B_i) \leq \mu(C_i) + \epsilon^2 + 2\epsilon \sqrt{\mu(C_i)}. \]
Let us notice that the sets \( C_i \), for \( i = 2,...,p \), are disjoint and their union lies in the set \( C = \{ y|y_1 \neq 0, y_2 = 0 \} \). Therefore by adding these inequalities we
have
\[
\mu(B_2 \setminus B_p) = \sum_{i=3}^{p} \mu(B_i \setminus B_i) \leq \\
\leq \sum_{i=3}^{p} [\mu(C_i) + \epsilon^2 + 2\epsilon \sqrt{\mu(C_i)}] \leq \\
\leq \mu(\cup_i C_i) + (p-2)\epsilon^2 + 2\epsilon \sqrt{(p-2)\mu(\cup_i C_i)} \leq \\
\leq \mu(C) + (p-2)\epsilon^2 + 2\epsilon \sqrt{(p-2)\mu(C)}.
\]

Using the projection \(\mathbb{T}^p \to \mathbb{T}^2\) given by taking at the first two coordinates, we can project the measure \(\mu\) to a measure \(\tilde{\mu}\) on \(\mathbb{T}^2\). Applying the previous lemma to the measure \(\tilde{\mu}\) we have:
\[
\mu(\mathbb{T}^p \setminus B_2) \leq (2 + \sqrt{10})^2 \epsilon^2 \quad \text{and} \quad \mu(C) \leq (1 + \sqrt{3})^2 \epsilon^2,
\]
therefore
\[
\mu(\mathbb{T}^p \setminus \{(0, \ldots, 0)\}) = \mu(\mathbb{T}^p \setminus B_2) + \mu(B_2 \setminus B_p) \leq \\
\leq (2 + \sqrt{10})^2 \epsilon^2 + (1 + \sqrt{3})^2 \epsilon^2 + (p-2)\epsilon^2 + 2(1 + \sqrt{3})\sqrt{p-2} \epsilon^2 = \\
= (p + 16 + 4\sqrt{10} + 2\sqrt{3} + 2(1 + \sqrt{3})\sqrt{p-2}) \epsilon^2 \leq \\
\leq (p + 6\sqrt{p} + 33) \epsilon^2 \leq (\sqrt{p} + 25 + 3) \epsilon^2
\]
which completes the proof of the lemma.

We finish the proof of the Theorem 4.1 by noticing that the measure \(\mu_v\) satisfies all the conditions in Lemma 4.7, and also that \(\mu_v\) is supported on \(\mathbb{T}^p \setminus \{0\}\). This implies that
\[
(\sqrt{p} + 25 + 3)^2 \epsilon^2 \geq 1,
\]
which is equivalent to
\[
\epsilon \geq \frac{1}{\sqrt{p} + 25 + 3}.
\]
Therefore the first inequality is not satisfied if \(\epsilon \leq 1/l(p)\). This proves that, if the representation \((\pi, \mathcal{H})\), does not have \(\mathbb{Z}^p\) invariant vectors, then for any \(v\), there exists \(g \in F \cup G\) such that \(\|\pi(g)v - v\| \geq \|v\|_{l(p)}\).

**Corollary 4.8.** Let \((\pi, \mathcal{H})\) be a unitary representation of the group
\[
G = SL_p(\mathbb{Z}) \ltimes \mathbb{Z}^p.
\]
Let \(v \in \mathcal{H}\) be a \((F \cup G, \epsilon)\) invariant vector. Then for every \(g\) in \(\mathbb{Z}^p\) we have
\[
\|\pi(g)v - v\| \leq 2l(p)\epsilon.
\]

**Proof.** Let us split the Hilbert space \(\mathcal{H}\) as a direct sum of the closed subspaces \(\mathcal{H}_0\) and \(\mathcal{H}_1\), where \(\mathcal{H}_0\) contains all \(\mathbb{Z}^p\) invariant vectors and \(\mathcal{H}_1\) is the orthogonal compliment of \(\mathcal{H}_0\). We have that both \(\mathcal{H}_0\) and \(\mathcal{H}_1\) are closed under the action of the group \(G\), because \(\mathbb{Z}^p\) is a normal subgroup of \(G\). Let us write \(v = v_0 + v_1\),
where \( v_i \in \mathcal{H}_i \). Since there are no \( \mathbb{Z}^p \) invariant vectors in \( \mathcal{H}_1 \), there exists \( h \in F \cup G \) such that 
\[
||\pi(h)v_1 - v_1|| \geq ||v_1||/l(p).
\]
But we have that
\[
||\pi(h)v - v||^2 = ||\pi(h)v_0 - v_0||^2 + ||\pi(h)v_1 - v_1||^2 \leq \epsilon^2,
\]
therefore \( ||v_1|| \leq l(p)\epsilon \). For any \( g \in \mathbb{Z}^p \), we have
\[
||\pi(g)v - v||^2 = ||\pi(g)v_0 - v_0||^2 + ||\pi(g)v_1 - v_1||^2 \leq 0 + 4||v_1||^2 \leq 4(l(p)\epsilon)^2,
\]
therefore \( ||\pi(g)v - v|| \leq 2l(p)\epsilon \).

\[\square\]

5 Relative Kazhdan constant for \( (\text{SL}_p \times \text{SL}_q) \ltimes \mathbb{Z}^{pq} \)

In this section we estimate the relative Kazhdan constant for the maximal parabolic subgroup \( (\text{SL}_p(\mathbb{Z}) \times \text{SL}_q(\mathbb{Z})) \ltimes \mathbb{Z}^{pq} \) of \( \text{SL}_{p+q} \), with respect to the group \( \mathbb{Z}^{pq} \) considering the set of elementary matrices in \( \text{SL}_p \) and \( \text{SL}_q \) together with the basis vectors of \( \mathbb{Z}^{pq} \) as a generating set. The proof of Theorem 5.1 is based on Theorem 4.1.

**Theorem 5.1.** Let \( F_1 \) and \( F_2 \) denote the sets of elementary matrices in \( \text{SL}_p(\mathbb{Z}) \) and \( \text{SL}_q(\mathbb{Z}) \) respectively, and \( G \) denotes the set of the pq standard basis elements of \( \mathbb{Z}^{pq} \). Let \( (\pi, \mathcal{H}) \) be a unitary representation of \( (\text{SL}_p(\mathbb{Z}) \times \text{SL}_q(\mathbb{Z})) \ltimes \mathbb{Z}^{pq} \), containing a vector \( v \) which is \((F_1 \cup F_2 \cup G, 1/k(p+q)) \) invariant, where
\[
k(n) = \sqrt{5n/2 + 60 + 6}.
\]
Then \( \mathcal{H} \) contains \( \mathbb{Z}^{pq} \) invariant vector, provided that \( p, q \geq 2 \).

**Remark 5.2.** Let us consider the standard representation of \( \text{SL}_{p+q} \) on \( \mathcal{L}^2(\mathbb{Z}^{p+q}) \). The group \( (\text{SL}_p \times \text{SL}_q) \ltimes \mathbb{Z}^{pq} \) is isomorphic to a maximal parabolic in \( \text{SL}_{p+q} \), thus we have a representation of that group in \( \mathcal{L}^2(\mathbb{Z}^{p+q}) \). The representation decomposes as a sum of two representations, one is isomorphic to \( \mathcal{L}^2(\mathbb{Z}^p) \), where \( \text{SL}_q \) and \( \mathbb{Z}^{pq} \) act trivially, the other is a representation on \( \mathcal{L}^2(\mathbb{Z}^{p+q} \setminus \mathbb{Z}^p) \), without \( \mathbb{Z}^{pq} \) invariant vectors. Using that representation and a suitable vector \( v \), it can be shown that the Kazhdan constant is at most \( \sqrt{2}/q \). This shows that \( 1/k(n) \) is not a relative Kazhdan constant, where \( k(n) = \sqrt{n/2} \).

**Proof.** Assume that \( v \in \mathcal{H} \) is \((F_1 \cup F_2 \cup G, \epsilon) \) invariant vector, and that \( \mathcal{H} \) does not contain \( \mathbb{Z}^{pq} \) invariant vector. Let \( P \) be the projection valued measure on \( \mathcal{L}^2(\mathbb{Z}^{p+q}) = \mathcal{L}^2(\mathbb{Z}^p) \), coming form the restriction of the representation \( \pi \) to \( \mathbb{Z}^{pq} \), and let \( \mu_v \) be the measure on \( \mathbb{L}^{pq} \), defined by \( \mu_v(B) = \langle P(B)v, v \rangle \). The measure \( \mu_v \) is supported on \( \mathbb{L}^{pq} \setminus \{0\} \), because by assumption \( \mathcal{H} \) does not contain an \( \mathbb{Z}^{pq} \) invariant vectors.

We can identify the torus \( \mathbb{L}^{pq} \) with the product of \( q \) tori of dimension \( p \). For an element \( x \in \mathbb{L}^{pq} \) we will write \( x = (x_1, \ldots, x_q) \), where each \( x_i \) is in \( \mathbb{L}^p \), we will also write \( x_i = (x_{i1}, \ldots, x_{ip}) \), where all \( x_{ij} \) are in \( \mathbb{R}/\mathbb{Z} \), which we identify with the interval \((-1/2, 1/2)]\).

The proofs of the next two lemmas are similar to the ones in section 4.1 and we will omit their proofs.
Lemma 5.3. Let $K_{ij} = \{x \mid 1/4 > |x_{ij}|\}$, then $\mu_v(K_{ij}) \geq 1 - \epsilon^2/2$.

Lemma 5.4. For every Borel set $B \subset \mathbb{T}^{pq}$ and every elementary matrix $g \in F_1 \cup F_2$, we have that

$$|\mu_v(gB) - \mu_v(B)| \leq 2\epsilon \sqrt{\mu_v(B)} + \epsilon^2.$$  

The action of $\text{SL}_p(\mathbb{Z}) \times \text{SL}_q(\mathbb{Z})$ on $\mathbb{T}^{pq}$, comes from the standard action on $\mathbb{R}^{pq}$, by the isomorphism $\mathbb{T}^{pq} = \mathbb{R}^{pq}/\mathbb{Z}^{pq}$.

We need a result similar to Lemma 4.7 considering the action of $\text{SL}_p \times \text{SL}_q$ on $\mathbb{T}^{pq}$.

Lemma 5.5. Let $\mu$ be a finitely additive probability measure on $\mathbb{T}^{pq}$ such that

- $\mu(|x_{ij}| \geq 1/4) \leq \epsilon^2/2$ for any $i$ and $j$,
- $|\mu(gB) - \mu(B)| \leq 2\epsilon \sqrt{\mu(B)} + \epsilon^2$ for any Borel set $B$ and any elementary matrix $g$ in $\text{SL}_p(\mathbb{Z})$ or $\text{SL}_q(\mathbb{Z})$.

Then the measure of the origin is at least

$$\mu(\{x \mid x = 0\}) \geq 1 - (\sqrt{3p + 2q + 60} + 6)^2 \epsilon^2.$$  

Proof. For a point $x \in \mathbb{T}^{pq}$ we write $x = (x_1, \ldots, x_q)$, where $x_i \in \mathbb{T}$ and $x_i = (x_{i1}, \ldots, x_{ip})$ and $x_{ij} \in (1/2, 1/2]$. Let us define the Borel sets

$$B_i = \{x \mid x_k = 0 \text{ for } k \leq i\}, \quad C_i = \{x \mid x_i = x_i \neq 0, x_k = 0 \text{ for } 1 < k < i\}.$$  

The elementary matrix $g_{1i} \in \text{SL}_q$ sends $B_{i-1} \setminus B_i$ into $C_i$ for any $i \geq 2$. Therefore, we have

$$\mu(B_{i-1} \setminus B_i) \leq \mu(C_i) + 2\epsilon \sqrt{\mu(C_i)}.$$  

Let us notice that the sets $C_i$ are disjoint and lies in the compliment of $B_1$. Therefore

$$\mu(B_1 \setminus B_q) \leq \mu(\mathbb{T}^{pq} \setminus B_1) + (q - 1)^2 \epsilon^2 + 2\epsilon \sqrt{(q - 1)\mu(\mathbb{T}^{pq} \setminus B_1)}.$$  

Using lemma 4.7 (by considering the measure $\tilde{\mu}$ on $\mathbb{T}^p$, defined as follows: $\tilde{\mu}(K) = \mu(\{x \mid x_1 \in K\})$) we have

$$\mu(\mathbb{T}^{pq} \setminus B_1) \leq (\sqrt{p + 25} + 3)^2 \epsilon^2$$  

Finally we have

$$\mu(\mathbb{T}^{pq} \setminus B_q) \leq 2\mu(\mathbb{T}^{pq} \setminus B_1) + 2\epsilon \sqrt{(q - 1)\mu(\mathbb{T}^{pq} \setminus B_1)} + (q - 1)^2 \epsilon^2 \leq$$

$$\leq (2p + 6q + 12\sqrt{p})\epsilon^2 + 2\epsilon^2 \sqrt{(q - 1)(p + 33 + 6\sqrt{p})} + (q - 1)^2 \epsilon^2 \leq$$

$$\leq (3p + 2q + 97 + 18\sqrt{p})\epsilon^2 \leq (\sqrt{3p + 2q + 60} + 6)^2 \epsilon^2.$$  

That completes the proof of the lemma since the set $B_q$ contains only the origin.
We finish the proof of the theorem considering without loss of generality that $q \geq p \geq 2$. The measure $\mu_v$ satisfies all the conditions in the lemma and also $\mu_v$ is supported on $T^{pq} \setminus \{0\}$, because $\mathcal{H}$ does not have invariant vectors. This implies that

$$(\sqrt{3p + 2q + 60 + 6})^2 \epsilon^2 \geq 1,$$

which is equivalent to

$$\epsilon \geq \frac{1}{\sqrt{3p + 2q + 60 + 6}}.$$ 

This inequality is not satisfied if $\epsilon \leq 1/(p + q)$.

This proves that if the representation $(\pi, \mathcal{H})$, does not have $\mathbb{Z}^{pq}$ invariant vectors, then for any $v$, there exists $g \in F_1 \cup F_2 \cup G$ such that $||\pi(g)v - v|| \geq k(p + q)$.

**Corollary 5.6.** Let $(\pi, \mathcal{H})$ be a unitary representation of the group

$$G = (\text{SL}_p(\mathbb{Z}) \times \text{SL}_q(\mathbb{Z})) \ltimes \mathbb{Z}^{pq}.$$ 

Let $v \in \mathcal{H}$ be a $(F_1 \cup F_2 \cup G, \epsilon)$ invariant vector. Then for every $g$ in $\mathbb{Z}^{pq}$ we have $||\pi(g)v - v|| \leq 2k(p + q)\epsilon$.

**Proof.** The proof of this corollary is similar to the one of corollary 4.8.

## 6 Vectors systems in $\mathbb{Z}^k$

Let $v_1, \ldots, v_n$ be vectors in $\mathbb{Z}^k$, which generate the whole group $\mathbb{Z}^k$, we will call $V = \{v_1, \ldots, v_n\}$ a complete system of vectors in $\mathbb{Z}^k$. We can also consider $V$ as a left invertible $k \times n$ matrix with integer coefficients by letting $V = (v_1, \ldots, v_n)$. We can define an elementary transformation $E_{i,j,a}$ on a complete vector system $V$, which preserves all vectors except $v_j$ and sends $v_j$ to $v'_j = v_j + av_i$. It is clear that we obtain a new complete vector system after this operation.

It is well known (see [2, 3, 10]) that if $n \geq k + 2$, then using approximately $2kn$ elementary operations we can transform any vector system to the canonical vector system $U$, which contains only standard basis vectors at the first $k$ places and the zero vectors in the other places.

In this section we will show that using a few ‘generalized elementary transformations’ we can also transform any complete vector system $V$ to the canonical one $U$.

Let us partition the set of indices $\{1, \ldots, n\}$ into two disjoint parts $I$ and $J$. For any $|I| \times |J|$ matrix $\alpha$ we define a generalized elementary transforation $E_{i,j,\alpha}$ as follows: For any vector system $V = \{v_i\}$, we define a new vector system $V' = \{v'_i\}$ as follows:

$v'_i = v_i$ for all $i \in I$.

The condition $n \geq k + 2$, comes from the fact that the ring $\mathbb{Z}$ has stable range equal to 2, see [10] for details. Using the fact that $\text{SL}_3(\mathbb{Z})$ is boundedly generated by the elementary matrices it is possible to extend this result to all $n \geq k$ except $n = 2$ and $k = 1$ or $k = 2$. 

13
\[ v'_j = v_j + \sum_{i \in I} \alpha_{ij} v_i \text{ for all } j \in J. \]

If we consider \( V \) as a \( k \times n \) matrix with integer coefficients, the generalized elementary operation \( E_{l,j,\alpha} \) corresponds to left multiplication with the matrix \( A, \) obtained from \( \begin{pmatrix} I & \alpha \\ 0 & I \end{pmatrix} \), by rearranging the rows and the columns.

**Theorem 6.1.** If \( n \geq 3k \), then any complete system \( V \) of \( n \) vectors in \( \mathbb{Z}^k \) can be transformed by using at most 4 generalized elementary operations, to the ‘standard’ system of vectors \( U \), where the first \( k \) vectors in \( U \) are the standard basis vectors of \( \mathbb{Z}^k \) (in the same order) and all other vectors are zero.

**Remark 6.2.** This result for \( k = 1 \) is well known and it is used in the induction step of the proof that \( \text{SL}_n(\mathbb{Z}) \) is bounded generated by elementary matrices. In fact for \( k = 1 \), three operations are enough. For \( k > 1 \) using 3 operations we know how to transform the system \( V \), into a system \( U' \), which contains \( k \) vectors from the standard basis of \( \mathbb{Z}^k \) and \( n - k \) zero vectors, but we do not know how to control the positions of the nonzero vectors.

**Proof.** Let us first recall how to transform such system in the case \( k = 1 \). By one elementary operation we can make one of the vectors a sufficiently big prime number, after another operation we can put 1 at the first place, and using the final operation we can make all other vectors equal to 0.

In order to generalize this construction for \( k \geq 2 \) we need to define the analog of the prime number.

**Definition 6.3.** We call a finite index subgroup \( B \) of \( \mathbb{Z}^k \) a ‘prime’ subgroup if the quotient \( \mathbb{Z}^k/B \) is isomorphic to

\[ \mathbb{Z}/\pi_1 \mathbb{Z} \times \cdots \times \mathbb{Z}/\pi_k \mathbb{Z}, \]

where \( \pi_i \) are pairwise different prime numbers. Any \( k \) vectors which generate a ‘prime’ subgroup are called a ‘prime’ system.

**Remark 6.4.** The vectors \( w_1 = (\pi_1, *, *, \ldots, *) \), \( w_2 = (0, \pi_2, *, \ldots, *) \), \ldots, \( w_k = (0, 0, 0, \ldots, \pi_k) \), where \( \pi_i \) are distinct primes generate a prime subgroup of \( \mathbb{Z}^k \).

**Lemma 6.5.** Let \( V \) be a system of vectors in \( \mathbb{Z}^k \), using one generalized elementary operation we can transform \( V \) into a system \( V' \), where some \( k \) vectors form a ‘prime’ system, i.e., they generate a ‘prime’ subgroup of \( \mathbb{Z}^k \).

**Proof.** Assume that the last \( k \) vectors are linearly independent.

**Remark 6.6.** Using several elementary transformations (which modify only the first \( k \) vectors), we can transform any complete system \( V \) into system \( V' \) such that \( v'_1 = (\pi_1, *, *, \ldots, *) \), \( v'_2 = (0, \pi_2, *, \ldots, *) \), \ldots, \( v'_k = (0, 0, 0, \ldots, \pi_k) \), where \( \pi_i \) are sufficiently large distinct prime numbers — larger then the determinant of the matrix formed by the coefficients of the last \( k \) vectors.
Proof. The proof is by induction on $k$ — in the base case $k = 0$, there is nothing to prove. Suppose that the vectors $v_1, \ldots, v_{k-1}$ have the desired form. Let us consider the set of vectors

$$P = \{ v_k + \sum_{i \neq k} \alpha_i v_i | \alpha_i \in \mathbb{Z} \}.$$ 

Since the vectors $\{v_i\}$ form a complete vector system and the last $k$ vectors are linearly independent, the set $P$ contains all vectors of the form $(0, \ldots, 0, a + \lambda d)$, for all $\lambda \in \mathbb{Z}$, for some relatively prime integers $a$ and $d$. Here we use that $\pi_i$, for $i < k$, are sufficiently big prime numbers therefore the standard basis vectors $e_i$ for $i < k$ lie in the subgroup generated by the vectors $v_1, \ldots, v_{k-1}, v_{n-k+1}, \ldots, v_n$.

Using Dirichlet’s theorem about primes in the arithmetic progressions, it follows that $P$ contains the vector of the form $(0, 0, 0, \ldots, \pi_k)$, which completes the induction step. \hfill \Box

By the above remark using elementary transformations which modify only the first $k$ vectors we can make these vectors a ‘prime’ system. Doing all these elementary transformations corresponds to multiplying from the left (the matrix of the vector system $V$) with matrix $A \in \text{SL}_n(\mathbb{Z})$ of the form

$$\begin{pmatrix} * & * \\ 0 & I \end{pmatrix},$$

where the blocks are of sizes $k$ and $n-k$. Any such matrix can be written uniquely as $A = BC$, where $B, C \in \text{SL}_n(\mathbb{Z})$ and

$$B = \begin{pmatrix} * & 0 \\ 0 & I \end{pmatrix}, \quad C = \begin{pmatrix} I & * \\ 0 & I \end{pmatrix}.$$ 

If we apply the generalized elementary transformation corresponding to the multiplication by matrix $C$, we obtain vector system such that the subgroup generated by the first $k$ vectors, coincides with the subgroup generated by $v'_i$-es. Because the upper left corner of $B$ is in $\text{SL}_k(\mathbb{Z})$, and the multiplication by $B$ does not change the subgroup generated by the first $k$ vectors, this subgroup is ‘prime’, which finishes the proof of the lemma. \hfill \Box

Notice that if $B$ is a ‘prime’ subgroup in $\mathbb{Z}^k$, then any strictly increasing sequence of subgroups between $B$ and $\mathbb{Z}^k$ has at most $k$ terms. This implies that if a complete vector system contains a ‘prime’ subsystem, then there exist at most $2k$ vectors which generate the whole group $\mathbb{Z}^k$.

For such system by applying one generalized elementary operation we can generate $k$ vectors that form a standard basis of $\mathbb{Z}^k$. This is true, because we have $2k$ vectors, which generate the whole $\mathbb{Z}^k$ and putting them in the set $J$, we can transform the other vectors to any vectors in $\mathbb{Z}^k$. Since $n \geq 3k$, we have at least $k$ vectors to modify and we can make these vectors equal to the standard $15$
basis vectors $e_i$. Moreover, if any of these vectors is among the first $k$ we can make it equal to the corresponding vector in the standard basis.

Finally, we need one more transformation in order to make the first $k$ vectors equal to ‘standard’ basis vectors of $\mathbb{Z}^k$ and with one final generalized elementary operation we can make all the other vectors 0-es.

\[ \square \]

Remark 6.7. The condition $n \geq 3k$ is too strong and can be replaced by $n \geq 2k + 1$, but in that case we need 5 operations. This is true because for any ‘prime’ subgroup $B$ the quotient $\mathbb{Z}^k/B$ is cyclic and can be generated by 1 element. So using an additional generalized linear transformation (after the first in the proof), we can modify 1 vector so that some $k + 1$ vectors generate the whole group $\mathbb{Z}^k$. Also if we replace $\mathbb{Z}$ with some field then 3 generalized elementary transformations are enough, provided that $n \geq 2k$.

It is interesting whether this condition can be replaced by $n \geq k + C$, for some fixed constant $C$. Such a result will improve the Kazhdan constant for $\text{SL}_n(\mathbb{Z})$ by approximately a factor of 3 (if the number of transformations stays the same).

Corollary 6.8. If $n \geq 3k$, then any matrix $g \in \text{SL}_n(\mathbb{Z})$ can be written as a product of 6 matrices:

$$ g = g_1 g_2 g_3 g_4 g^* g_5, $$

where $g^*$ lies in the copy of $\text{SL}_{n-k}(\mathbb{Z})$ embedded in the lower right corner. Also any matrix $g_i$ can be obtained from a matrix of the type $\begin{pmatrix} I & * \\ 0 & I \end{pmatrix}$, by rearranging the rows and columns (the position of the blocks depend on the matrix $g_i$).

\[ \textbf{Proof.} \] Let us consider the first $k$ entries of each row of $g$. They form a complete system of $n$ vectors in $k$ dimensional space, because $g$ is an invertible matrix.

Every generalized elementary transformation on these vectors corresponds to multiplying the $k \times n$ matrix of their coordinates from the left by a matrix similar to $\begin{pmatrix} I & * \\ 0 & I \end{pmatrix}$. By Theorem 6.1 after 4 such multiplications we can transform this matrix to $\begin{pmatrix} I \\ 0 \end{pmatrix}$.

Therefore, by multiplying $g$ from the left with these matrices we can reduce it to a matrix of type $\begin{pmatrix} I & * \\ 0 & * \end{pmatrix}$. Finally by one multiplication from the right we can transform this matrix to $g^* = \begin{pmatrix} I & 0 \\ 0 & * \end{pmatrix}$, which lies in $\text{SL}_{n-k}$. If we ‘reverse’ this process we obtained the desired decomposition of the matrix $g$. \[ \square \]

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8This observation was made by Tal Poznansky, it is used for obtaining the bound in Theorem xx.
Corollary 6.9. If \( n \geq 3 \), then any matrix \( g \in \text{SL}_n(\mathbb{Z}) \) can be written as a product of at most \( 60 + 13 \ln n \) matrices, each of which can be obtained from a matrix of the type \( \begin{pmatrix} I & * \\ 0 & I \end{pmatrix} \), by rearranging the rows and columns.

7 Kazhdan constants for \( \text{SL}_n(\mathbb{Z}) \)

Using the fact that \( \text{SL}_n(\mathbb{Z}) \) (for \( n \geq 3 \)) is bounded generated by the elementary matrices, and using an analog of Corollary 5.6 for \( \text{SL}_2(\mathbb{Z}) \), it can be shown (see [8]), that if \((\pi, \mathcal{H})\) is a unitary representation of \( \text{SL}_n(\mathbb{Z}) \), and \( v \) is an \( \epsilon \)-invariant vector with respect to all elementary matrices, then for any \( g \in \text{SL}_n(\mathbb{Z}) \) we have that \( \|\pi(g)v - v\| \leq 22 f(n)\epsilon \), where \( f(n) = 3(n^2 - n)/2 + 51 \), which is the number of elementary matrices (with any integer off the diagonal) needed to express any element in \( \text{SL}_n(\mathbb{Z}) \), see [3]. From here it easily follows that the Kazhdan constant for \( \text{SL}_n(\mathbb{Z}) \) with respect to the elementary matrices is at least \( 1/22 f(n) \). Our goal is to improve the upper bound \( 22 f(n) \) and obtain a better Kazhdan constant.

Definition 7.1. Let \( h(n) \) be the smallest number such that for any unitary representation and any positive number \( \epsilon \), the condition \( \|\pi(g)v - v\| < \epsilon \) for any elementary matrix \( g \) in \( E_n \), implies that \( \|\pi(g)v - v\| \leq h(n)\epsilon \) for any \( g \in \text{SL}_n(\mathbb{Z}) \).

Lemma 7.2. If \( n \geq 3i \) and \( i \geq 2 \), then the function \( h(n) \) satisfies the inequality

\[
h(n) \leq h(n - i) + 10k(n) \leq h(n - i) + \sqrt{250n + 6000} + 60.
\]

Here \( k(n) \) is the function defined in Theorem 6.4.

Proof. Let \((\pi, \mathcal{H})\) be a unitary representation of \( \text{SL}_n(\mathbb{Z}) \) and \( v \in \mathcal{H} \) be a unit vector such that \( \|\pi(g)v - v\| \leq \epsilon \) for any elementary matrix \( g \).

The set of all matrices of the form \( \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \) is a subgroup of \( \text{SL}_n(\mathbb{Z}) \) isomorphic to \( (\text{SL}_p(\mathbb{Z}) \times \text{SL}_q(\mathbb{Z})) \times \mathbb{Z}^{pq} \). If we restrict the representation \( \pi \) to this subgroup, we can apply corollary 5.6 and obtain that \( \|4v - v\| \leq 2k(p+q)\epsilon \leq 2k(n) \) for any matrix \( A \) of the form \( \begin{pmatrix} I & * \\ 0 & I \end{pmatrix} \).

Let \( g \) be a matrix in \( \text{SL}_n(\mathbb{Z}) \) by lemma 6.8 we can write \( g \) as a product of 6 matrices. By the above argument, five of these matrices move the element \( v \) by less than \( 2k(p+q)\epsilon \leq 2k(n)\epsilon \). The sixth matrix lies in a copy of the group \( \text{SL}_{n-k}(\mathbb{Z}) \) and if we restrict the representation \( \pi \) to that subgroup we can see that it moves the vector \( v \) by less than \( h(n - i)\epsilon \). This implies that \( g \) moves \( v \) by less than \( (h(n - i) + 10k(n))\epsilon \), which proves the lemma.

Before completing the proof of Theorem A, we need a lemma about functions which satisfy an inequality like the one in lemma 7.2.
Lemma 7.3. Let $a, b, c$ be positive real numbers, $\lambda < 1$ and let $f : \mathbb{N} \to \mathbb{R}$ be a function. If the function $f$ satisfies the inequality

$$f(n) \leq f(i) + \sqrt{an} + b + c,$$

for any $i \geq \lambda^2 n$ and any $n \geq n_0 > 1/(1 - \lambda^2)$ then

$$f(n) \leq A(\sqrt{n} - \lambda \sqrt{n_0}) - c \left( \log_{\lambda^2} \frac{n}{n_0} + 1 \right) + \frac{B}{\sqrt{n_0}} + f(n_0)$$

where $A, B$ and $\tilde{n}_0$ are given by:

$$A = \frac{\sqrt{a}}{1 - \lambda}, \quad B = \frac{b + a/(1 - \lambda^2)}{(1 - \lambda)\sqrt{a}}, \quad \tilde{n}_0 = n_0 - \frac{1}{1 - \lambda^2}.$$

Proof. Let us define recursively the sequences $x_k$ and $y_k$ as follows: $x_0 = y_0 = n$ and $x_{i+1} = \lceil \lambda^2 x_i \rceil$, $y_{i+1} = \lambda^2 y_i$. Here $\lceil x \rceil$ denotes the smallest integer greater than $x$. By induction it follows that

$$y_i < x_i \leq y_i + \frac{1 - \lambda^{2i}}{1 - \lambda^2} < y_i + \frac{1}{1 - \lambda^2}.$$

Therefore, for $s = \lceil -\log_{\lambda^2}(n/\tilde{n}_0) \rceil$ we have $x_s \leq y_s + 1/(1 - \lambda^2) \leq \tilde{n}_0 + 1/(1 - \lambda^2) = n_0$. Using the functional inequality we have

$$f(x_{i+1}) \leq f(x_i) + \sqrt{ax_{i+1} + b} + c.$$

Adding all these inequalities for different $i$’s we obtain

$$f(n) \leq f(x_s) + \sum_{i=0}^{s-1} (\sqrt{ax_i + b} + c) \leq f(n_0) + \sum_{i=0}^{s-1} (\sqrt{ax_i + b} + c)$$

Using the inequality between $x_i$ and $y_i$ we have

$$f(n) \leq f(n_0) + cs + \sum_{i=0}^{s} \sqrt{an\lambda^{2i} + a/(1 - \lambda^2) + b} =$$

$$= f(n_0) + cs + \sqrt{an} \sum_{i=0}^{s} \lambda^i \sqrt{1 + \lambda^{-2i} a/(1 - \lambda^2) + b} \leq$$

$$\leq f(n_0) + cs + \sqrt{an} \sum_{i=0}^{s} \lambda^i + \sum \frac{a/(1 - \lambda^2) + b \lambda^{-i}}{2 \sqrt{an}} \leq$$

because $\sqrt{1 + x} \leq 1 + x/2$ for every $x$

$$\leq f(n_0) + cs + \sqrt{an/(1 - \lambda^s)} + \frac{a/(1 - \lambda^2) + b \lambda^{-s}}{1 - \lambda} \leq$$

$$\leq f(n_0) + cs + A \sqrt{n/(1 - \lambda^s)} + B \lambda^{-s}/\sqrt{n} \leq$$

$$\leq A(\sqrt{n} - \lambda \sqrt{n_0}) + c \left( \log_{\lambda^2} \frac{n}{n_0} + 1 \right) + \frac{B}{\sqrt{n_0}} + f(n_0),$$

For the last inequality we used $s \leq \log_{\lambda^2} \frac{n}{n_0} + 1$ and $\lambda^{-2s} \leq n/\tilde{n}_0$.  \qed
Applying the previous lemma to the function $h(n)$ we obtain

**Theorem 7.4.** The function $h(n)$ satisfies the inequality

$$h(n) < 90\sqrt{n} + 4000.$$ 

*Proof.* By lemma 7.2 we have that the function $h(n)$ satisfies the inequality with $a = 250$, $b = 6000$, $c = 60$ and $\lambda = \sqrt{2}/3$. Putting these constants and $n_0 = 7$ in the lemma 7.3 gives

$$A = \frac{\sqrt{250}}{1 - \sqrt{2}/3} = 15\sqrt{10} + 10\sqrt{15}, \quad B = \frac{6750}{\sqrt{150}(1 - \sqrt{2}/3)} = 675(2 + \sqrt{6}),$$

and $\tilde{n}_0 \geq 1$, which implies the inequality

$$h(n) \leq (15\sqrt{10} + 10\sqrt{15})(\sqrt{n} - \sqrt{14}/3) + 60(\log_{3/2} n + 1) + 675(2 + \sqrt{6})/\sqrt{7} + h(7).$$

By the Shalom result we have $h(n) \leq 33n^2 - 11n + 1152$, i.e., $h(7) \leq 2692$. Finally we have

$$h(n) < (15\sqrt{10} + 10\sqrt{15})\sqrt{n} + 60 \log_{3/2} n + 3900 < 90\sqrt{n} + 4000.$$ 

\[\square\]

Now we prove Theorem A.

**Theorem A.** The Kazhdan constant for $\text{SL}_n(\mathbb{Z})$ and $\text{SL}_n(\mathbb{F}_p)$ with respect to the elementary matrices is

$$\mathcal{K}(\text{SL}_n(\mathbb{Z}), E_n) \geq (64\sqrt{n} + 2850)^{-1}.$$ 

*Proof.* It is well known fact that if a representation $(\pi, \mathcal{H})$ of a group $G$ contains a unit vector $v \in \mathcal{H}$ such that $||\pi(g)v - v|| < \sqrt{2}$ for any $g \in G$ then $\mathcal{H}$ contains a $G$-invariant vector. Applying this observation gives that

$$\mathcal{K}(\text{SL}_n(\mathbb{Z}), E_n) \geq \sqrt{2}/h(n) \geq (50\sqrt{n} + 2850)^{-1}.$$ 

\[\square\]

**Remark 7.5.** More detailed consideration, using the exact size of the blocks of matrices $g_i$ in Corollary 6.8 and using the stronger version of Theorem 6.1 for $n \geq 2k + 1$, gives that

$$h(n) \leq \sqrt{2}(5\sqrt{5} + 1)(\sqrt{2} + 1)\sqrt{n} + 22 \log_2 n + 350 < \sqrt{2}(42\sqrt{n} + 860),$$

which implies the bound of the Kazhdan constant in Theorem A

$$\mathcal{K}(\text{SL}_n(\mathbb{Z}), E_n) \geq (33\sqrt{n} + 317)^{-1}.$$ 

Similarly using the version of Theorem 6.1 for vector systems over a field $\mathbb{F}_p$, we have

$$h(n) \leq 8\sqrt{3}(\sqrt{2} + 1)\sqrt{n} + 8 \log_2 n/3 + 100 < \sqrt{2}(24\sqrt{n} + 100),$$

which proves Theorem A.
8 Generalizations to other groups

In this section we show how Theorem A can be generalized to the groups $\text{SL}_n(R)$ for several classes of rings $R$. We will only sketch the proofs of the necessary lemmas. In order to do so we need to generalize Theorem 5.1 and Lemma 6.8.

The first step is the proof analogous to the proof of the lemma 4.6 for the ring $\mathbb{Z}[t_1, \ldots, t_s]$. Let $F$ denote the set of elementary matrices in $\text{SL}_2$, with $\pm 1$ and $\pm t_i$ off the diagonal.

**Lemma 8.1.** Let $\mu$ be a finitely additive measure on the dual of $\mathbb{Z}[t_1, \ldots, t_s]^2$, i.e. $\hat{\mathbb{Z}}[t_1, \ldots, t_s]^2 = (\mathbb{R}/\mathbb{Z}[[t_1^{-1}, \ldots, t_s^{-1}]])^2$ such that

- $\mu(|x_0| \geq 1/4) \leq 2\epsilon^2/2$ and $\mu(|y_0| \geq 1/4) \leq \epsilon^2/2$. Here $x_0$ denotes the constant term of the series $x$;
- $|\mu(gB) - \mu(B)| \leq 2\epsilon \sqrt{\mu(B)} + \epsilon^2$ for any Borel set $B$ and any elementary matrix $g \in F \subset \text{SL}_2(\mathbb{Z}[t_1, \ldots, t_s])$.

Then if $\epsilon < 1/12$ we have

$$\mu(\hat{\mathbb{Z}}[t_1, \ldots, t_s]^2 \setminus \{(0, 0)\}) \leq 10.6k\epsilon^2$$

**Proof.** The proof is by induction using lemma 4.6 as the base case. The proof of the induction step uses the description of $\hat{\mathbb{R}}[t]$ in term of $\hat{\mathbb{R}}$ and is based on the proof of lemma 3.3 from [8].

Using this lemma we can generalize Theorems 4.1 and 5.1 for the ring $\mathbb{Z}[t_1, \ldots, t_s]$, and therefore for any finitely generated ring, by replacing the functions $l(n)$ and $k(n)$ with

$$l_s(n) = \min\{\sqrt{3n + 21.6s}, 12\} \quad k_s(n) = \min\{\sqrt{6n + 48.6s}, 12\}$$

This can be further generalized to rings which contain a finitely generated dense sub rings – like $\mathbb{C}$ or $\mathbb{Z}[[t_1, \ldots, t_s]]$.

Theorem 6.1 can be generalized to many different classes of rings replacing $\mathbb{Z}$. In the proof of Theorem 6.1 we used the fact that for any ideal $I \triangleleft \mathbb{Z}$ and any $x$ in the ring $\mathbb{Z}$, such that $x\mathbb{Z} + I = \mathbb{Z}$, there are infinitely many elements $y \in x + I$, such that the ring $\mathbb{Z}/y\mathbb{Z}$ has a unique maximal ideal. Therefore, for any commutative ring $R$ which has the above property, Theorem 6.1 holds, and any vector system in $R^k$ consisting of more than $3n$ vectors can be transformed to the standard one using at most 4 generalized elementary transformations. An example of a ring satisfying this condition is $\mathbb{Z}[[t]]$.

Suppose that the ring $R$ satisfies the following condition: there are invertible elements in the coset $x + I$ for any element $x \in R$ and ideal $I \triangleleft R$, such that $xR + I = R$, in particular if $R$ is a local ring or a filed. Then we can transform any vector system in $R^k$ to the standard one using 3 generalized elementary
transformations of a fixed type, provided that \( n \geq 2k \). A nontrivial example of a ring satisfying the above condition is \( \text{Hom}(S^1, \mathbb{C}) \) with pointwise operations\(^9\).

Using this remark we can show that if the ring \( R \) satisfies one of the above conditions and contains a dense sub-ring \( S \) generated by \( \alpha_i \) for \( i = 1, \ldots, s \). Then the group \( \text{SL}_n(R) \) has property \( T \) and the Kazhdan constant is

\[
K(\text{SL}_n(R), E_n(R)) \geq (50\sqrt{n} + (10N + \ldots)6^s + 300)^{-1},
\]

provided that \( \text{SL}_3(R) \) is boundedly generated by the elementary matrices, and every element \( g \in \text{SL}_3(R) \) can be written as a product of \( N \) elementary matrices. In particular we have the following corollaries:

**Corollary 8.2.** For any be compact ring \( R \) such that there exist \( d \) elements which generate a dense sub-ring, the groups \( \text{SL}_n(R) \), and \( \text{SL}_n(R[t_1, \ldots, t_s]) \) have property \( T \) and the Kazhdan constant with respect to the set of elementary matrices is \( O(n^{1/2}) \).

**Theorem 8.3.** The loop group \( \mathcal{L}(\text{SL}_n(\mathbb{C})) = \text{SL}_n(\mathcal{L}(\mathbb{C})) \) of \( \text{SL}_n(\mathbb{C}) \), has property \( T \) for \( n \geq 3 \) and the Kazhdan constant with respect to the set of trivial loops \( E_n \) of elementary matrices with \( \pm 1 \) off the diagonal, is at least

\[
K(\mathcal{L}(\text{SL}_n(\mathbb{C})), E_n) \geq [50\sqrt{n} + B]^{-1},
\]

where \( B \) is a constant. Note that this is not a locally compact Lie group and the set \( E_n \) generates a finite dimensional subgroup.

**Proof.** The proof is based on the fact that the ring \( \text{Hom}(S^1, \mathbb{C}) \) contains a dense sub-ring generated by 4 elements – 1 and \( \alpha_i \). Moreover the elements \( \alpha_i \) can be chosen in any neighborhood of 0, which allows us not to include in the generating set of the group the elementary matrices with \( \pm \alpha_i \) off the diagonal. \( \square \)

**Theorem C.** Let \( \mathcal{O} \) be the ring of integers in a number field \( \mathcal{K} \), with discriminant \( \Delta \), which is generated as a ring by 1 and the elements \( \alpha_i \) for \( i = 1, \ldots, s \). The Kazhdan constant for \( \text{SL}_n(\mathcal{O}) \) with respect to the set \( E_n(\mathcal{O}) \) of elementary matrices with \( \pm 1 \) and \( \pm \alpha_i \) is

\[
K(\text{SL}_n(\mathcal{O}), E_n(\mathcal{O})) \geq [50\sqrt{n} + (B + C\Delta)6^s]^{-1},
\]

where \( B \) and \( C \) are universal constants and \( \Delta \) is the number of different prime divisors of the discriminant of \( \mathcal{K} : \mathbb{Q} \).

**Proof.** Here we used the result by Carter and Keller \( \square \) that every element in \( \text{SL}_3(\mathcal{O}) \) can be written as a product of \( 60 + \Delta \) elementary matrices. \( \square \)

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