ON NON-COMMUTATIVE EULER SYSTEMS

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Abstract. Let $p$ be a prime, $T$ a $p$-adic representation over a number field $K$ and $\mathcal{K}$ an arbitrary Galois extension of $K$. Then for each non-negative integer $r$ we define a natural notion of a ‘non-commutative Euler system of rank $r$’ for $T$ relative to the extension $\mathcal{K}/K$. We prove that if $p$ is odd and $T$ and $\mathcal{K}/K$ satisfy certain mild hypotheses, then there exist non-commutative Euler systems that control the Galois structure of cohomology groups of $T$ over intermediate fields of $\mathcal{K}/K$ and have rank that depends explicitly on $T$. As a first concrete application of this approach, we (unconditionally) extend the classical Euler system of cyclotomic units to the setting of arbitrary totally real Galois extensions of $\mathbb{Q}$ and describe explicit links between this extended cyclotomic Euler system, the values at zero of derivatives of Artin $L$-series and the Galois structures of ideal class groups. As an important preliminary to the formulation and proof of these results, we introduce natural non-commutative generalizations of several standard constructions in commutative algebra including higher Fitting invariants, higher exterior powers and the Grothendieck-Knudsen-Mumford determinant functor on perfect complexes that are of independent interest.

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1. Introduction

1.1. Background and the main result. Ever since its introduction by Kolyvagin, Rubin and Thaine in the late 1980’s, the theory of Euler systems has played a vital role in the proof of results concerning the structure of Selmer groups attached to \( p \)-adic representations \( T \) that are defined over a number field \( K \) and satisfy a variety of technical hypotheses.

In an attempt to axiomatise, and extend, the use of Euler systems, Mazur and Rubin \[26\] developed an associated theory of ‘Kolyvagin systems’ and showed both that Kolyvagin systems play the key role in obtaining structural results about Selmer groups and that the link to Euler systems is pivotal for the supply of Kolyvagin systems that are related to the special values of \( L \)-series.

However, for many natural representations, Euler and Kolyvagin systems are not sufficient to control Selmer groups and in such ‘higher rank’ cases authors have considered collections of elements in suitable higher exterior powers of the relevant cohomology groups.

The theory of higher rank Euler systems was first studied systematically by Perrin-Riou \[36\] after significant contributions were made by Rubin \[40\] in an important case related to Stark’s Conjecture (and hence to the multiplicative group). An associated theory of higher rank Kolyvagin systems was subsequently developed by Mazur and Rubin \[27\] and Sakamoto and the present authors have recently \[11\] strengthened this theory of Mazur and Rubin and also described a canonical higher rank extension of the fundamental link between classical Euler and Kolyvagin systems.

All of these important theories are, however, intrinsically ‘commutative’ in nature (since systems arise as families of elements in exterior powers of cohomology modules over abelian extensions of \( K \)) and are therefore not well-suited to the study of leading term conjectures that are relevant to non-abelian Galois extensions.
We remind the reader that such conjectures include the equivariant refinement of the Tamagawa number conjecture of Bloch and Kato [2] that has been described by Flach and the first author [5], the ‘non-commutative Tamagawa number conjecture’ formulated by Fukaya and Kato [19] and the ‘main conjecture of non-commutative Iwasawa theory’ for elliptic curves without complex multiplication formulated by Coates, Fukaya, Kato, Sujatha and Venjakob [15] as well as, in a more classical setting, the ‘Ω(3)-Conjecture’ of Galois module theory formulated by Chinburg [14] and the ‘non-abelian Brumer-Stark Conjecture’ that was formulated, independently, by Nickel [33] and by the first author [4].

For this reason at least, it would seem to be of interest to extend the theories of (higher rank) Euler and Kolyvagin systems beyond the setting of abelian extensions. With this in mind, the main aim of the present article is to take the first steps in the development of a theory of (higher rank) Euler systems in the setting of arbitrary Galois extensions.

In order to do so, we find it necessary to first describe natural non-commutative generalizations of several well-known constructions in commutative algebra including higher Fitting invariants (as discussed, for example, by Northcott [35]), higher exterior powers, the notion of ‘Rubin lattice’ that was introduced in [40] (and plays a key role in the theory of [11] via the associated notion of ‘exterior power bidual’) and the Grothendieck-Knudsen-Mumford determinant functor on perfect complexes (from [22, 23]).

These constructions are of independent interest and have arithmetic uses beyond the theory of Euler systems. For example, the ‘reduced determinant functor’ described here can be used to give a more concrete formulation of the central conjectures of [5] (thereby avoiding techniques of relative $K$-theory and the use of Deligne’s category of ‘virtual objects’) and [19] (thereby avoiding the theory of ‘localized $K_1$-groups’ developed in loc. cit). For brevity, however, these applications will not be discussed here.

To discuss Euler systems we assume $T$ is endowed with the action of a commutative Gorenstein $\mathbb{Z}_p$-order $A$. We also fix an algebraic closure $\mathbb{Q}^c$ of $\mathbb{Q}$ and an arbitrary Galois extension $\mathcal{K}$ of $K$ and for each $\mathbb{Q}^c$-valued character $\chi$ of $\text{Gal}(\mathcal{K}/K)$ with open kernel, we fix an associated representation $\text{Gal}(\mathcal{K}/K) \to \text{GL}_{\chi(1)}(\mathbb{Q}^c)$.

To each such collection of representations we can then associate a compatible family of ‘reduced exterior powers’ and ‘reduced Rubin lattices’ relative to the group rings over $A$ of finite Galois extensions of $K$ in $\mathcal{K}$.

These constructions will in turn allow us to define for each non-negative integer $r$ a natural notion of ‘non-commutative Euler system of rank $r$’ for the $A$-module $T$ and extension $\mathcal{K}/K$.

In our main result (Theorems 6.11 and 6.14), we shall then show that if $p$ is odd, all archimedean places of $K$ split completely in $\mathcal{K}$ and $T$ is both projective as an $A$-module and satisfies certain mild hypotheses relating to its invariants at archimedean places and to the extension $\mathcal{K}/K$, then there exist non-commutative Euler systems of rank depending explicitly on $T$ that control the Galois structure of cohomology groups of $T$ over $p$-adic analytic extensions of $K$ in $\mathcal{K}$.

The proof of this result is modelled on an argument used in [12] (in the setting of abelian extensions) and relies on the construction of the reduced determinant functor, on an analysis of the compactly supported $p$-adic étale cohomology complexes of $T$ over finite extensions of $K$ in $\mathcal{K}$ and on the Hermite-Minkowski Theorem (that there are only finitely many number fields of any given absolute discriminant).
This result leads one naturally to the questions of whether it is reasonable to expect the development of an analogous theory of ‘(higher rank) non-commutative Kolyvagin systems’ for $p$-adic representations and whether there exist non-commutative Euler systems that are explicitly linked to the leading terms of $L$-series?

At this stage we have nothing to say about the first of these questions but can, fortunately, say something about the second question, at least in an important special case.

To state a result in this direction we assume to be given for each natural number $n$ a choice of primitive $n$-th root of unity $\zeta_n$ in $\mathbb{Q}^c$ with the property that $\zeta_m = (\zeta_n)^{n/m}$ for all divisors $m$ of $n$. We then recall that for any finite abelian totally real extension $F$ of $\mathbb{Q}$ of conductor $f$ (so that $F \subset \mathbb{Q}(\zeta_f)$) the classical cyclotomic element of $F$ is the element of $F^\times$ obtained by setting

$$\epsilon_F := \text{Norm}_{\mathbb{Q}(\zeta_f)/F}(1 - \zeta_f).$$

In the following result we use for each finite Galois extension $F$ of $\mathbb{Q}$, with $\mathcal{G}_F := \text{Gal}(F/\mathbb{Q})$, the canonical ‘ideal of denominators’ $\delta(\mathbb{Z}[\mathcal{G}_F])$ in the centre of $\mathbb{Z}[\mathcal{G}_F]$ that is defined in §3.1.2 and the construction of non-commutative exterior powers $\bigwedge^1_{\mathbb{Q}[\mathcal{G}_F]}(\cdots)$ from §4.2.4. For each finite set $\Sigma$ of places of $\mathbb{Q}$ that contains $\infty$ we write $\mathcal{O}_{F,\Sigma}$ for the subring of $F$ comprising elements that are integral at all places of $F$ that do not restrict to give a place in $\Sigma$. For each integer $a$ and each irreducible complex character $\chi$ of $\mathcal{G}_F$ we also write $L^\Sigma_\mathbb{C}(\chi,0)$ for the coefficient of $z^a$ in the Laurent expansion at $z = 0$ of the $\Sigma$-truncated Artin $L$-series $L^\Sigma_\mathbb{C}(\chi,z)$ of $\chi$. Finally, we fix an isomorphism of fields $\mathbb{C} \cong \mathbb{C}_p$ (that we do not explicitly indicate) and so regard each complex character of $\mathcal{G}_F$ as taking values in $\mathbb{C}_p$. We fix embeddings of $\mathbb{Q}^c$ into $\mathbb{C}$ and into $\mathbb{Q}_p^c$ and use the restriction of these embeddings to define an archimedean place $w_{\infty,F}$ and a $p$-adic place $w_{p,F}$ of each field $F$ as above. For each such field we also write $S(F)$ for the set of places of $\mathbb{Q}$ comprising $\infty, p$ and the primes that ramify in $F$.

This result will follow as an easy consequence of stronger results that are derived in §8.2 from Theorem 6.11 in the case that $T = \mathbb{Z}_p(1)$ and $\mathcal{A} = \mathbb{Z}_p$.

**Theorem 1.1.** There exists a rank one non-commutative Euler system

$$\varepsilon^\text{cyc} = (\varepsilon^\text{cyc}_F)_F$$

for the Galois representation $\mathbb{Z}_p(1)$ and the maximal totally real extension of $\mathbb{Q}$ in $\mathbb{Q}^c$ that has all of the following properties at each finite totally real Galois extension $F$ of $\mathbb{Q}$ in $\mathbb{Q}^c$.

(i) If $\mathcal{G}_F$ is abelian, then one has

$$\varepsilon_F^\text{cyc} = \begin{cases} 
\epsilon_F, & \text{if } p \text{ ramifies in } F, \\
(\epsilon_F)^{1 - \sigma_{p,F}}, & \text{if } p \text{ is unramified in } F
\end{cases}$$

where, in the second case, $\sigma_{p,F}$ denotes the inverse of the Frobenius automorphism of $p$ in $\mathcal{G}_F$.

(ii) For every $\varphi$ in $\text{Hom}_{\mathcal{G}_F}(\mathcal{O}_{F,S(F)}^\times,\mathbb{Z}[\mathcal{G}_F])$ and every $x$ in $\delta(\mathbb{Z}[\mathcal{G}_F])$ the element

$$x \cdot \left(\bigwedge^1_{\mathbb{Q}[\mathcal{G}_F]}\varphi(\varepsilon^\text{cyc}_F)\right)$$

belongs to $\mathbb{Z}_p[\mathcal{G}_F]$ and annihilates $\mathbb{Z}_p \otimes_\mathbb{Z} \text{Cl}(\mathcal{O}_F[1/\ell])$ for every prime $\ell$ in $S(F)$. 

(iii) For every irreducible complex character $\chi$ of $G_F$ one has

$$
\left( \bigwedge_{C_p}^{\chi(1)} \text{Reg}^\chi_F \right)(e_{\chi(e_{\chi}^{\text{cyc}})}) = u_{F,\chi} \cdot L_{S(F)}^{\chi(1)}(\bar{\chi}, 0) \cdot e_{\chi} \left( \bigwedge_{C_p[G_F]}^1 (w_{\infty, F} - w_{p, F}) \right).
$$

Here $\text{Reg}^\chi_F$ is a canonical isomorphism of $\mathbb{C}_p$-vector spaces that is induced by the Dirichlet regulator map (and defined in §8.2.2), $e_{\chi}$ is the primitive central idempotent $\chi(1)|G_F|^{-1} \cdot \sum_{g \in G_F} \chi(g)g^{-1}$ of $\mathbb{C}[G_F]$, $\bar{\chi}$ is the contragredient of $\chi$ and $u_{F,\chi}$ is an element of $\mathbb{C}_p^\times$ that satisfies

$$
\prod_{\omega \in \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} u_{F,\chi \omega} \in \mathbb{Z}_p^\times.
$$

We shall refer to an Euler system of the sort constructed in the above result as an ‘extended cyclotomic Euler system’. We note that, since the map $\text{Reg}^\chi_F$ is injective, claim (iii) implies that for any such system $e_{\chi}^{\text{cyc}}$ one has

$$
e_{\chi}(e_{\chi}^{\text{cyc}}) \neq 0 \iff L_{S(F)}^{\chi(1)}(\bar{\chi}, 0) \neq 0$$

for all $F$ and all $\chi$. In addition, if all values of $\chi$ belong to $\mathbb{Q}$ (as is automatically the case, for example, if $G_F$ is isomorphic to a quotient of a symmetric group), then it implies $u_{F,\chi}$ belongs to $\mathbb{Z}_p^\times$.

More generally, the proof of Theorem 1.1 will show that the ‘Strong-Stark Conjecture’ of Chinburg [14] implies that the equality in claim (iii) should be valid with each element $u_{F,\chi}$ a $p$-adic unit in any subfield of $\mathbb{Q}_p^\times$ over which $\chi$ can be realised (cf. Remark 8.10) and also strongly suggests the existence of a non-commutative Euler system $e_{\chi}^{\text{cyc}}$ for which the same is true with $u_{F,\chi} = 1$ for all $F$ and $\chi$.

We will return to consider these, and other related, issues in a sequel [10] to this article, jointly authored with Macias Castillo.

In particular, in [10] we will further develop the $p$-adic methods used here and apply them to the ‘Weil-étale cohomology complexes for $\mathbb{Z}$’ that are constructed by Kurihara and the present authors in [7] (in place of the $p$-adic étale cohomology complexes of $\mathbb{Z}_p(1)$ that are used here) in order to define a notion of non-commutative Euler system for $G_m$ and to prove a stronger version of Theorem 1.1 in which the factor $1 - \sigma_{p, F}$ is omitted from the equality in claim (i) and the set $S(F)$ is replaced by the subset comprising $\infty$ and all primes that ramify in $F$ in the statement of claim (iii).

We will also show that the algebraic methods developed here can be used to define (in general, conjecturally) for any number field $K$ a canonical ‘extended Rubin-Stark Euler system’ over the maximal algebraic extension of $K$ in which all archimedean places split completely and to formulate a natural generalization of the ‘refined class number formula for $G_m$’ that was independently conjectured (in the setting of abelian extensions) by Mazur and Rubin [28] and by the second author [42]. We will then show that this approach leads to a concrete strategy for proving the equivariant Tamagawa number conjecture for the untwisted Tate motive over arbitrary finite Galois extensions of totally real fields, thereby extending the main result of Kurihara and the present authors in [8].

At the same time we will also explain in [10] how the methods developed here can be used to give a unifying approach to, and a significant refinement of, the existing theory of
refined versions of Stark’s Conjecture, thereby strongly improving the main results of the first author in [4].

1.2. The main contents. For the reader’s convenience we shall now give a little more detail concerning some of the general results that are obtained in this article.

1.2.1. In the first part of the article (§2.5) we generalize several well-known constructions in commutative algebra to the categories of modules over certain non-commutative rings.

To be more precise we let $R$ be a Dedekind domain with field of fractions $F$ that is a finite extension of either $Q$ or $Q_p$ for some prime $p$. Then our generalized algebraic constructions are made in the setting of modules over $R$-orders $A$ that span finite dimensional separable $F$-algebras. However, for simplicity, we shall in this summary restrict attention to the case that $R$ is either $Z$ or $Z_p$ for some prime $p$ and that $A = R[G]$ for a finite group $G$.

In this setting we shall (in [3]) use the reduced norms of matrices with coefficients in $R[G]$ to define a canonical order $\xi(R[G])$ in $\zeta(F[G])$ that will play a key role in our theory. Here, and in the sequel, we write $\zeta(A)$ for the centre of a ring $\Lambda$.

If $G$ is abelian, then one has $\xi(R[G]) = \zeta(R[G]) = R[G]$. In general, however, $\xi(R[G])$ is not contained in $\zeta(R[G])$ but, motivated by results of Nickel in [22], we can show that $\delta(R[G]) \cdot \xi(R[G]) \subseteq \zeta(R[G])$ for a certain explicitly defined ideal $\delta(R[G])$ of finite index in $\zeta(R[G])$ (this motivates us to refer to $\delta(R[G])$ as the ‘ideal of denominators’ of $\zeta(R[G])$).

Then, as a first application of $\xi(R[G])$, we use it to construct a natural theory of ‘higher non-commutative Fitting invariants’ for finitely generated $R[G]$-modules, as summarized in the following result.

**Theorem 1.2** (See Theorem 3.17). For any finitely generated $R[G]$-module $M$ and any non-negative integer $a$, there exists a canonical ‘$a$-th non-commutative Fitting invariant’ $\text{Fit}^a_{R[G]}(M)$ of $M$ that has all of the following properties.

(i) $\text{Fit}^a_{R[G]}(M)$ is an ideal of $\xi(R[G])$.

(ii) $\text{Fit}^a_{R[G]}(M) \subseteq \text{Fit}^{a+1}_{R[G]}(M)$.

(iii) $\text{Fit}^a_{R[G]}(M) = \xi(R[G])$ for all sufficiently large $a$.

(iv) For each $x$ in $\delta(R[G])$, one has $x \cdot \text{Fit}^0_{R[G]}(M) \subseteq \text{Ann}_{R[G]}(M)$.

(v) For any surjective homomorphism of $R[G]$-modules $M \to M'$ one has $\text{Fit}^a_{R[G]}(M) \subseteq \text{Fit}^a_{R[G]}(M')$.

(vi) If $G$ is abelian, then $\text{Fit}^a_{R[G]}(M)$ is equal to the classical $a$-th Fitting ideal of the $R[G]$-module $M$.

We shall next use $\xi(R[G])$ in [4] to construct a well-behaved theory of ‘(reduced) exterior powers’ on the category of $R[G]$-modules. To do this we fix an algebraic closure $F^c$ of $F$ and a set of representations of the form

$$(1.2.1) \quad \{ \rho_\chi : G \to \text{GL}_{\chi(1)}(F^c) \}_{\chi \in \text{Ir}_F(G)},$$

where $\text{Ir}_F(G)$ is the set of irreducible $F^c$-valued characters of $G$ and $\rho_\chi$ has character $\chi$.

We show that each such choice of representations can be used to define, for each finitely generated $F[G]$-module $W$ and each non-negative integer $r$, a canonical ‘$r$-th reduced exterior power’ $\wedge^r_{F[G]}W$ of $W$ that is a finitely generated module over $\zeta(F[G])$, for each $s$ with...
0 ≤ s ≤ r a canonical pairing of \( \zeta(F[G]) \)-modules
\[
\bigwedge_{F[G]}^r W \times \bigwedge_{F[G]}^s \text{Hom}_{F[G]}(W, F[G]) \to \bigwedge_{F[G]}^{r-s} W,
\]
and for each subset \( \{w_i\}_{1 \leq i \leq r} \) of \( W \) a canonical ‘reduced exterior product’ \( \bigwedge_{j=1}^{r-s} w_j \) in \( \bigwedge_{F[G]}^r W \) of the elements \( w_j \).

A key difficulty is then to show that the choice of a full \( R[G] \)-lattice \( M \) in \( W \) gives rise to a useful integral structure on \( \bigwedge_{F[G]}^r W \). We resolve this problem by using \( \xi(R[G]) \) to construct a natural generalization of the notion of ‘Rubin lattice’ in commutative algebra, as summarized in the following result.

**Theorem 1.3** (See Theorem 4.17). For any finitely generated \( R[G] \)-module \( M \) and any non-negative integer \( r \), there exists an ‘\( r \)-th reduced Rubin lattice’ \( \bigcap_{R[G]}^r M \) of \( M \) that depends only on \( M, r \) and the representations \( \{\mathbf{1}, \mathbf{2}, \mathbf{7}\} \) and has all of the following properties.

(i) \( \bigcap_{R[G]}^r M \) is a finitely generated \( \xi(R[G]) \)-submodule of \( \bigwedge_{F[G]}^r (F \otimes_R M) \).

(ii) For any subset \( \{w_j\}_{1 \leq j \leq s} \) of \( \text{Hom}_{R[G]}(M, R[G]) \) with \( s \leq r \), the reduced exterior product \( \bigwedge_{j=1}^{s} w_j \) induces a homomorphism of \( \xi(R[G]) \)-modules \( \bigcap_{R[G]}^r M \to \bigcap_{R[G]}^{r-s} M \).

(iii) If \( M \) is locally-free of rank \( r \), then \( \bigcap_{R[G]}^r M \) is an invertible \( \xi(R[G]) \)-module.

(iv) If \( G \) is abelian and \( R = \mathbb{Z} \), then \( \bigcap_{R[G]}^r M \) is the lattice \( \bigcap_{\mathbb{P}}^r M \) introduced by Rubin in §1.2.

The result of Theorem 1.3(iii) will then play an important role in a construction made in [5] in order to establish an explicit, and computationally useful, theory of ‘non-commutative determinants’ for perfect complexes of \( R[G] \)-modules, as summarized in the following result.

**Theorem 1.4** (See Theorem 5.2). Let \( D^R_{\text{lf}}(R[G])_{\text{lf}} \) denote the subcategory of the derived category of finitely generated locally-free \( R[G] \)-modules in which morphisms are restricted to be isomorphisms, and \( \mathcal{P}(\xi(R[G])) \) the category of graded invertible \( \xi(R[G]) \)-modules.

Then for each set of representations \( \{\mathbf{1}, \mathbf{2}, \mathbf{7}\} \) there exists a canonical determinant functor
\[
d_{R[G]} : D^R_{\text{lf}}(R[G])_{\text{lf}} \to \mathcal{P}(\xi(R[G]))
\]
with the property that for each locally-free \( R[G] \)-module \( P \) of rank \( r \) one has
\[
d_{R[G]}(P[0]) = (\bigcap_{R[G]}^r P, r'),
\]
where \( r' \) is the ‘reduced rank’ of the \( F[G] \)-module \( F \otimes_R P \) (as defined in 2.2).

1.2.2. In the second part ([6],[8]) of the article we describe arithmetic applications of the algebraic theory developed in earlier sections and finally deduce a proof of Theorem 1.1.

To describe the general arithmetic setting we fix a number field \( K \) and a \( p \)-adic representation \( T \) of \( \text{Gal}(K^c/K) \) that is unramified outside a finite set \( S \) of places of \( K \) that contains all archimedean and \( p \)-adic places and (for simplicity) has coefficients in \( \mathbb{Z}_p \).

We also fix a Galois extension \( K/K \) such that all archimedean places of \( K \) split completely in \( K \), and let \( \Omega(K/K) \) be the set of all finite Galois subextensions of \( K \) inside \( K \). For each \( F \) in \( \Omega(K/K) \) we set \( S(F) := S \cup S_{\text{ram}}(F/K) \), where \( S_{\text{ram}}(F/K) \) is the set of places of \( K \) that ramify in \( F \), and write \( \mathcal{G}_F \) in place of \( \text{Gal}(F/K) \).
We first use the notion of reduced Rubin lattice from Theorem 1.3 to define for each non-negative integer \( r \) a module of ‘non-commutative Euler systems of rank \( r \) for \((T, K)\)’

\[
\text{ES}_r(T, K) \subseteq \prod_{F \in \Omega(K/K)} H^1(O_{F, S}(F), T)
\]

(see Definition 6.4). This module has a natural action of the algebra

\[
\xi(\mathbb{Z}_p[[\text{Gal}(K/K)]]):= \lim_{\leftarrow F \in \Omega(K/K)} \xi(\mathbb{Z}_p[G_F]),
\]

where the transition morphisms are induced by the natural projection maps \( G_F \to G_{F'} \) for \( F \) and \( F' \) in \( \Omega(K/K) \) with \( F' \subseteq F \).

Our main arithmetic contribution is then to describe an unconditional construction of elements of \( \text{ES}_r(T, K) \) in the case that the rank \( r \) is equal to

\[
r_T := \sum_v \text{rank}_{\mathbb{Z}_p}(H^0(K_v, T^*(1))),
\]

where in the sum \( v \) runs over all archimedean places of \( K \).

The key idea for this is to apply the reduced determinant functors from Theorem 1.4 to complexes that arise from the compactly supported \( p \)-adic étale cohomology of \( T \) over the fields \( F \) in \( \Omega(K/K) \).

In this way we shall construct a non-commutative generalization \( \text{VS}(T, K) \) of the module of ‘vertical determinantal systems’ that was introduced by the present authors in [12]. The module \( \text{VS}(T, K) \) is constructed as an explicit inverse limit and has a natural action of \( \xi(\mathbb{Z}_p[[\text{Gal}(K/K)]] \) (for details see Definition 7.2).

Some of the main aspects of the general theory that we establish in this setting are summarized in the following result (which also relies on aspects of Theorem 1.2).

**Theorem 1.5** (See Theorem 6.11, Proposition 7.4 and Theorem 7.5). We assume that for every \( F \) in \( \Omega(K/K) \) the module \( H^0(F, T) \) vanishes and the module \( H^1(O_{F, S}(F), T) \) is \( \mathbb{Z}_p \)-free. We also set \( r := r_T \).

Then the \( \xi(\mathbb{Z}_p[[\text{Gal}(K/K)]] \)-module \( \text{VS}(T, K) \) is free of rank one and there is a canonical homomorphism of \( \xi(\mathbb{Z}_p[[\text{Gal}(K/K)]] \)-modules

\[
\Theta_{T, K} : \text{VS}(T, K) \to \text{ES}_r(T, K)
\]

such that each generator \( \varepsilon = (\varepsilon_F)_F \) of \( \text{im}(\Theta_{T, K}) \) has the following properties.

(i) The annihilator of \( \varepsilon \) in \( \xi(\mathbb{Z}_p[[\text{Gal}(K/K)]] \) is equal to

\[
\xi(\mathbb{Z}_p[[\text{Gal}(K/K)]] \cap \prod_{F \in \Omega(K/K)} \xi(\mathbb{Q}_p[G_F])(1 - e_F),
\]

where \( e_F \) is equal to the sum of primitive idempotents in \( \xi(\mathbb{Q}_p[G_F]) \) that annihilate \( H^2(O_{F, S}(F), T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \).

(ii) For every \( F \) in \( \Omega(K/K) \), every subset \( \{\varphi_j\}_{1 \leq j \leq r} \) of \( \text{Hom}_{\mathbb{Z}_p[G_F]}(H^1(O_{F, S}(F), T), \mathbb{Z}_p[G_F]) \) and every \( x \) in \( \delta(\mathbb{Z}_p[G_F]) \) one has

\[
(\Lambda_{j=1}^r \varphi_j)(\varepsilon_F) \in \text{Fit}_0^{\varepsilon_F}(H^2(O_{F, S}(F), T)),
\]
and hence also
\[ x \cdot (\Lambda_{j=1}^{r} \varphi_j)(\varepsilon_F) \in \text{Ann}_{\mathbb{Z}_p[\mathcal{G}_F]}(H^2(\mathcal{O}_{F,S(F)}, T)). \]

The proof of claim (ii) of Theorem 1.5 will actually establish a more precise result in which elements of the form \((\Lambda_{j=1}^{r} \varphi_j)(\varepsilon_F)\) completely determine the \(r\)-th non-commutative Fitting invariant of a presentation of the \(\mathbb{Z}_p[\mathcal{G}_F]\)-module \(H^2(\mathcal{O}_{F,S(F)}, T) \oplus \mathbb{Z}_p[\mathcal{G}_F]^r\) (for details see Remark 6.12). In addition, we will show (in Theorem 6.14) that any system \(\varepsilon\) constructed as in Theorem 1.5 also encodes information about the concrete structure of Iwasawa modules that arise from the cohomology of \(T\) in arbitrary \(p\)-adic analytic extensions of \(K\) in \(\mathcal{K}\).

We can now finally sketch the construction of the extended cyclotomic system \(\varepsilon_{\text{cyc}}\) in Theorem 1.1. In this regard we first apply Theorem 1.5 with \(K = \mathbb{Q}\), \(T = \mathbb{Z}_p(1)\) (so that \(r_T = 1\)) and \(K\) equal to either the maximal totally real extension \(\mathbb{Q}_{c,+}\) of \(\mathbb{Q}\) in \(\mathbb{Q}_c\) or the maximal absolutely abelian subfield \(\mathbb{Q}_{ab,+}\) of \(\mathbb{Q}_{c,+}\) in order to obtain canonical homomorphisms of \(\xi(\mathbb{Z}_p[[\text{Gal}(\mathbb{Q}_{c,+}/\mathbb{Q})]])\)-modules \(\Theta_{\mathbb{Z}_p(1), \mathbb{Q}_{c,+}}\) and \(\Theta_{\mathbb{Z}_p(1), \mathbb{Q}_{ab,+}}\).

Next we note that the natural restriction map \(\text{Gal}(\mathbb{Q}_{c,+}/\mathbb{Q}) \rightarrow \text{Gal}(\mathbb{Q}_{ab,+}/\mathbb{Q})\) induces a surjective homomorphism of \(\xi(\mathbb{Z}_p[[\text{Gal}(\mathbb{Q}_{c,+}/\mathbb{Q})]])\)-modules

\[ \text{VS}(\mathbb{Z}_p(1), \mathbb{Q}_{c,+}) \rightarrow \text{VS}(\mathbb{Z}_p(1), \mathbb{Q}_{ab,+}). \]  

In addition, one can check that the known validity of the equivariant Tamagawa Number Conjecture in this setting implies the existence of a canonical \(\xi(\mathbb{Z}_p[[\text{Gal}(\mathbb{Q}_{ab,+}/\mathbb{Q})]])\)-basis \(\eta_{ab}^{\text{ab}} \in \text{VS}(\mathbb{Z}_p(1), \mathbb{Q}_{ab,+})\) of \(\text{VS}(\mathbb{Z}_p(1), \mathbb{Q}_{ab,+})\) which \(\Theta_{\mathbb{Z}_p(1), \mathbb{Q}_{ab,+}}\) sends to the classical cyclotomic unit Euler system.

We can therefore fix a lift \(\eta \in \text{VS}(\mathbb{Z}_p(1), \mathbb{Q}_{c,+})\) of \(\eta_{ab}^{\text{ab}}\) through the surjective homomorphism (1.2.2) and then set

\[ \varepsilon_{\text{cyc}} := \Theta_{\mathbb{Z}_p(1), \mathbb{Q}_{c,+}}(\eta) \in \text{ES}_1(\mathbb{Z}_p(1), \mathbb{Q}_{c,+}). \]

In this way we obtain an unconditional construction of a non-commutative Euler system that ‘lifts’ the classical cyclotomic unit Euler system.

We can then check that this system \(\varepsilon_{\text{cyc}}\) has all of the properties stated in Theorem 1.1 and thereby complete the proof of the latter result.

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PART I: NON-COMMUTATIVE ALGEBRA

2. SEMISIMPLE ALGEBRAS

For a ring $R$ we write $R^{\text{op}}$ for its opposite ring and $\zeta(R)$ for its centre (so that $\zeta(R^{\text{op}}) = \zeta(R)$).

For natural numbers $d'$ and $d$ we write $M_{d',d}(R)$ for the set of $d' \times d$ matrices over $R$. In the case that $d = d'$ we often abbreviate the ring $M_{d,d}(R)$ to $M_d(R)$ and write $\text{GL}_d(R)$ for its unit group.

2.1. Simple rings.

2.1.1. We first review relevant facts from Morita theory (and for more details see, for example, [16, §3]).

Let $E$ be a field and $V$ an $E$-vector space of dimension $d$. Then $V$ has a natural structure as a (simple) left module over the $E$-algebra $A := \text{End}_E(V)$.

The linear dual $V^* := \text{Hom}_E(V,E)$ of $V$ has a structure of right $A$-module via the rule

$$(v^* \cdot a)(v) := v^*(a \cdot v),$$

for $a \in A, v^* \in V$, and $v \in V$.

There are also natural pairings

$$( -, - )_E : V^* \times V \to E,$$

$$( -, - )_A : V \times V^* \to A,$$

given by

$$(v^*,v)_E := v^*(v) \quad \text{and} \quad (v,v^*)_A(v') := v^*(v') \cdot v,$$

for $v, v' \in V$ and $v^* \in V^*$.

The pairing $(-,-)_E$ (resp. $(-,-)_A$) induces an isomorphism of $E$-vector spaces (resp. two-sided $A$-modules):

$$V^* \otimes_A V \xrightarrow{\sim} E$$

(resp. $V \otimes_E V^* \xrightarrow{\sim} A$).

The ‘Morita functor’ $V^* \otimes_A -$ from the category of left $A$-modules to that of $E$-vector spaces gives an equivalence of categories.

2.1.2. Let now $K$ be a field of characteristic zero and $A$ a finite-dimensional simple $K$-algebra.

All simple left $A$-modules are isomorphic and for any such module $M$ the $K$-algebra

$$(2.1.1) \quad D := \text{End}_A(M)$$

is a division ring (that is unique up to isomorphism).

By using a slightly more general version of the Morita theory recalled above one finds that there exists a canonical ring isomorphism

$$A \xrightarrow{\sim} \text{End}_D(M); \quad a \mapsto (m \mapsto am).$$
The centre \( F := \zeta(D) \) of \( D \) is a field and is canonically isomorphic to \( \zeta(A) \). An extension field \( E \) of \( F \) is said to be a ‘splitting field’ for \( A \) if \( D \otimes_F E \) is isomorphic to a matrix ring \( M_m(E) \) for some \( m \).

Such a field \( E \) always exists and can be taken to be of finite degree over \( K \) (see Remark 2.1 below) and the integer \( m \) is independent of \( E \) and referred to as the ‘Schur index’ of \( A \).

In addition, there exists a composite isomorphism

\[
A \otimes_F E \cong \text{End}_D(M) \otimes_F E \cong M_n(D^{\text{op}}) \otimes_F E \cong M_n(M_m(E)) = M_{nm}(E),
\]

where \( n \) is the dimension of the (free) left \( D \)-module \( M \), via which one can regard \( A \) as a subalgebra of \( M_{nm}(E) \).

**Remark 2.1.** Any choice of an algebraic closure \( K^c \) of \( K \) is a splitting field for \( A \). In addition, there are canonical choices of finite extensions of \( K \) in \( K^c \) that are splitting fields for \( A \). For example, the composite of all extensions of \( K \) in \( K^c \) that are isomorphic (as a \( K \)-algebra) to a maximal subfield of any division ring \( D \) as in (2.1.1) is a splitting field for \( A \) that is of finite degree and Galois over \( K \). In this regard see also Remark 4.9.

### 2.1.3. The behaviour under scalar extension of a finite-dimensional simple \( K \)-algebra \( A \)

is described in the following result.

**Lemma 2.2.** Let \( K' \) be an extension of \( K \) and \( \Omega \) an algebraic closure of \( K' \). Set \( F := \zeta(A) \) and consider the (finite) set \( \Sigma(F/K,K') \) of equivalence classes of \( K \)-embeddings \( F \rightarrow \Omega \) under the relation

\[
\sigma \sim \sigma' \iff \sigma = \tau \circ \sigma' \quad \text{for some} \quad \tau \in \text{Aut}_{K'}(\Omega).
\]

Then, for each \( \sigma \) in \( \Sigma(F/K,K') \), the \( K' \)-algebra \( A \otimes_F \sigma(F)K' \) is a simple artinian ring with center the composite field \( \sigma(F)K' \) of \( \sigma(F) \) and \( K' \) (this is independent of a representative of \( \sigma \)), and there is a product decomposition of \( K' \)-algebras

\[
A \otimes_K K' \cong \prod_{\sigma \in \Sigma(F/K,K')} (A \otimes_F \sigma(F)K').
\]

**Proof.** Since \( F \) is separable over \( K \), we have an isomorphism

\[
F \otimes_K K' \cong \prod_{\sigma \in \Sigma(F/K,K')} \sigma(F)K'.
\]

Hence we have

\[
A \otimes_K K' \cong A \otimes_F (F \otimes_K K') \cong \prod_{\sigma \in \Sigma(F/K,K')} (A \otimes_F \sigma(F)K').
\]

Since \( A \) is a central simple algebra over \( F \), \( A \otimes_F \sigma(F)K' \) is also a central simple algebra over \( \sigma(F)K' \). \( \square \)

### 2.1.4. If \( M \) is a finitely generated left \( A \)-module, then there exists a canonical composite homomorphism

\[
\text{End}_A(M) \rightarrow \text{End}_{A \otimes_F E}(M \otimes_F E) \rightarrow \text{End}_E(V^* \otimes (A \otimes_F E)(M \otimes_F E)) \rightarrow E,
\]

in which the second arrow is induced by the Morita functor and the last by taking determinants.
One checks that the image of this map factors through the inclusion $F \subseteq E$ and that the induced ‘reduced norm’ map

$$\text{Nrd}_{\text{End}_A(M)} : \text{End}_A(M) \to F$$

is such that

$$\text{Nrd}_{\text{End}_A(M)}(\theta_1 \circ \theta_2) = \text{Nrd}_{\text{End}_A(M)}(\theta_1) \cdot \text{Nrd}_{\text{End}_A(M)}(\theta_2)$$

for all $\theta_1$ and $\theta_2$ in $\text{End}_A(M)$.

**Remark 2.3.** If $M = A^{\text{op}}$ and one identifies $A$ with $\text{End}_{A^{\text{op}}}(A^{\text{op}})$, then for each element $a$ of $A$ one can check that $\text{Nrd}_A(a)$ is equal to the determinant of the image of $a$ under the isomorphism (2.1.2). This is the classical definition of reduced norm.

**Remark 2.4.** For a natural number $n$ we shall often abbreviate $\text{Nrd}_{M_n(A)}(A)$ to $\text{Nrd}_A$. Since the algebras $M_n(A^{\text{op}})$ and $M_n(A)^{\text{op}}$ are isomorphic and $\text{Nrd}_A = \text{Nrd}_{A^{\text{op}}}$, we shall also sometimes write $\text{Nrd}_A$ for $\text{Nrd}_{M_n(A^{\text{op}})}$.

The ‘reduced rank’ of a finitely generated left $A$-module $M$ is the non-negative integer obtained by setting

$$rr_A(M) := \dim_E(V^* \otimes_{(A \otimes_F E)} (M \otimes_F E)).$$

**Remark 2.5.** One can check, by explicit computation, that if $M$ is a simple left $A$-module, then $rr_A(M)$ is equal to the Schur index $\sqrt{\dim_F(D)}$ of $A$. This implies, in particular, that $rr_A(A) = \dim_D(M) \cdot \sqrt{\dim_F(D)}$.

### 2.2. Semisimple rings.

In the sequel we shall use ‘module’ to mean ‘left module’.

A module $M$ over a ring $A$ is said to be semisimple if it is a direct sum of simple modules. A ring $A$ is said to be semisimple if every nonzero $A$-module is semisimple and this is true if and only if $A$ decomposes as a direct product

$$A \cong \prod_{i \in I} A_i,$$

in which the index set $I$ is finite and each ring $A_i$ is simple (and unique up to isomorphism). In particular, Lemma 2.2 shows that simple rings naturally give rise to semisimple rings under scalar extension.

The ‘Wedderburn decomposition’ (2.2.1) of $A$ induces an identification $\zeta(A) = \prod_{i \in I} \zeta(A_i)$ and can be used to define (componentwise) generalizations of the above notions of reduced norm and reduced rank. In this way one obtains a reduced norm $\text{Nrd}_A$ for the algebra $A$ that is valued in $\zeta(A)$ and defines a reduced rank $rr_A(M)$ of a finitely generated $A$-module $M$ that is an integer-valued function on $\text{Spec}(\zeta(A))$.

It is known that this reduced norm in turn induces a reduced norm (which we denote by the same symbol)

$$\text{Nrd}_A : K_1(A) \to \zeta(A)^\times$$

from the Whitehead group $K_1(A)$ of $A$ (cf. [16 §45A]).
3. Higher non-commutative Fitting invariants

In this section we introduce a non-commutative generalization of the classical notion of ‘higher Fitting invariant’, as discussed by Northcott in [35].

This construction is natural, has many of the same properties as the classical commutative construction (see Proposition 3.17) and is also, as we will see later, well-suited to arithmetic applications.

Throughout the section we fix a Dedekind domain \( R \) with field of fractions \( F \) that is a finite extension of either \( \mathbb{Q} \) or \( \mathbb{Q}_p \) for some prime \( p \). We also fix a finite-dimensional semisimple \( F \)-algebra \( A \) and an \( R \)-order \( \mathcal{A} \) in \( A \) (in the sense of [16] Def. (23.2)).

For each prime ideal \( p \) of \( R \) we respectively write \( R_p \) and \( R \) for the localization and completion of \( R \) at \( p \). For each \( \mathcal{A} \)-module \( M \) and each \( p \) we then set \( M_{(p)} := R_{(p)} \otimes_R M \) and \( M_p := R_p \otimes_R M \). We regard these modules as respectively endowed with natural actions of the algebras \( \mathcal{A}_{(p)} = R_{(p)} \otimes_R \mathcal{A} \) and \( \mathcal{A}_p = R_p \otimes_R \mathcal{A} \).

In particular, the localisation \( M_{(0)} \) of \( M \) at the zero prime ideal of \( R \) is equal to the \( A \)-module generated by \( M \) and will often be written as \( M_F \).

3.1. The Whitehead order.

3.1.1. We first introduce a canonical \( R \)-submodule of \( \zeta(A) \).

**Definition 3.1.** For each prime ideal \( p \) of \( R \) the ‘Whitehead order’ \( \zeta(A_{(p)}) \) of \( A_{(p)} \) is the \( R_{(p)} \)-submodule of \( \zeta(A) \) that is generated by the elements \( \text{Nrd}_A(M) \) as \( M \) runs over all matrices in \( \bigcup_{n \geq 1} M_n(A_{(p)}) \).

The ‘Whitehead order’ of \( A \) is then defined by the intersection

\[
\zeta(A) := \bigcap_p \zeta(A_{(p)}),
\]

where \( p \) runs over all prime ideals of \( R \).

The basic properties of this module are described in the following result.

**Lemma 3.2.** The following claims are valid.

(i) \( \zeta(A) \) is an \( R \)-order in \( \zeta(A) \).

(ii) For every prime ideal \( p \) of \( R \) one has \( \zeta(A)_{(p)} = \zeta(A_{(p)}) \).

(iii) If \( A \) is commutative, then \( \zeta(A) = \zeta(A) \).

(iv) If \( A \) is a maximal \( R \)-order, then \( \zeta(A) \) is the (unique) maximal \( R \)-order in \( \zeta(A) \).

(v) Any surjective homomorphism of \( R \)-orders \( \varphi : A \rightarrow B \) induces, upon restriction, a surjective homomorphism \( \zeta(A) \rightarrow \zeta(B) \).

**Proof.** The integral closure \( \mathfrak{M} \) of \( R \) in \( \zeta(A) \) is the maximal \( R \)-order in \( \zeta(A) \) and \( \mathfrak{M}_{(p)} \) is the maximal \( R_{(p)} \)-order in \( \zeta(A) \) for every \( p \). In addition, by [16] Th. (26.5), we can fix a choice of maximal \( R \)-order \( M \) in \( A \) that contains \( \mathcal{A} \) as a submodule of finite index. Then, for every \( p \), the localisation \( M_{(p)} \) is a maximal \( R_{(p)} \)-order in \( A \) (by [16] Th. (26.21)(ii)) and so the arithmetic of local division algebras (as in the proof of [16] Prop. (45.8)) implies that \( \zeta(M_{(p)}) = \mathfrak{M}_{(p)} \).

In particular, for every \( p \) the \( R_{(p)} \)-module \( \zeta(A_{(p)}) \) is contained in \( \mathfrak{M}_{(p)} \) and so is finitely generated. It is also straightforward to check that \( \zeta(A_{(p)}) \) is closed under multiplication,
that it contains $R_p$ and that it has finite index in $\mathfrak{M}_p$ (since it contains $\text{Nrd}_A(x \cdot \mathcal{M}_p)$ for a non-zero element of $R$). This shows that each $\xi(A_p)$ is an $R_p$-order in $\zeta(A)$ and also implies that $\xi(A)$ is an $R$-order in $\zeta(A)$ if and only if one has $\xi(A)_F = \zeta(A)$.

At this stage, it is thus clear that the assertions of both claims (i) and (ii) follow from the general result of [16] Prop. (4.21)(vii) and the fact that $A_p = \mathcal{M}_p$, and hence also $\xi(A_p) = \mathfrak{M}_p$, for almost all $p$.

Next we note that if $A$ is commutative, then for every $p$ and every matrix $M$ in $M_n(A_p)$ one has $\text{Nrd}_A(M) = \det(M) \in A_p$. In this case it is therefore clear that $\xi(A)$ is equal to $\bigcap_p A_p$ and hence (by [16] Prop. (4.21)(vi)) to $A = \zeta(A)$, as required to prove claim (iii).

Claim (iv) is true since if $A$ is a maximal $R$-order in $A$, then $A_p$ is a maximal $R_p$-order in $A$ for every $p$ and so

$$\xi(A) = \bigcap_p \xi(A_p) = \bigcap_p \mathfrak{M}_p = \mathfrak{M},$$

where in both intersections $p$ runs over all prime ideals.

We next note claim (v) makes sense since the surjectivity of $\varrho$ implies that the $F$-algebra $B = B_F$ is a quotient of $A$ so that $B$ is semisimple (and hence the order $\xi(B)$ is defined).

This also implies that $\varrho$ restricts to give a surjective homomorphism $\varrho' : \zeta(A) \rightarrow \zeta(B)$ and claim (ii) implies that the claimed equality $\varrho'(\xi(A)) = \xi(B)$ is valid if for every $p$ one has $\varrho'(\xi(A_p)) = \xi(B_p)$. This equality is in turn true since $\varrho$ induces, for each $n$, a surjective ring homomorphism $\varrho_n : M_n(A_p) \rightarrow M_n(B_p)$ with the property that $\varrho'(\text{Nrd}_A(M)) = \text{Nrd}_B(\varrho_n(M))$ for every $M$ in $M_n(A_p)$.

**Remark 3.3.** In the case that $R$ is a discrete valuation ring, Johnston and Nickel [21, §3.4] consider the $R$-order $\mathcal{I}(A)$ in $\zeta(A)$ that is generated over $\zeta(A)$ by the elements $\text{Nrd}_A(M)$ as $M$ runs over all matrices in $\bigcup_{n \geq 1} M_n(A)$. In this case one therefore has $\mathcal{I}(A) = \zeta(A) \cdot \xi(A)$ and also $\mathcal{I}(A) = \xi(A)$ if and only if $\zeta(A) \subseteq \xi(A)$. Whilst it seems likely that there exist $R$-orders $\mathcal{A}$ for which $\zeta(A) \not\subseteq \xi(A)$, we do not at this stage know a concrete example for which this is true.

**3.1.2.** The order $\xi(A)$ is not, in general, contained in $\zeta(A)$. However, to bound the ‘denominators’ of its elements one can proceed as follows.

For every natural number $m$ and every matrix $M$ in $M_m(A)$ there is a unique matrix $M^*$ in $M_m(A)$ with

$$M \cdot M^* = M^* \cdot M = \text{Nrd}_A(M) \cdot I_m$$

and such that for every primitive central idempotent $e$ of $A$ the matrix $M^*e$ is non-zero if and only if $\text{Nrd}_A(M)e$ is non-zero. The following definition is motivated by a result of Nickel (see the proof of Proposition 3.1.17(iii) below).

**Definition 3.4.** For each prime ideal $p$ of $R$ the ‘ideal of denominators’ of $A_p$ is the subset of $\zeta(A)$ obtained by setting

$$\delta(A_p) := \{ x \in \zeta(A) : \forall d \geq 1, \forall M \in M_d(A_p) \text{ one has } x \cdot M^* \in M_d(A_p) \}.$$

The ‘ideal of denominators’ of $A$ is then defined by the intersection

$$\delta(A) = \bigcap_p \delta(A_p)$$
where \( p \) runs over all primes.

The basic properties of these sets are described in the following result.

**Lemma 3.5.**

(i) \( \delta(A) \) is an ideal of finite index in \( \zeta(A) \).

(ii) For every prime ideal \( p \) of \( R \) one has \( \delta(A)_p = \delta(A_p) \).

(iii) For each prime \( p \) an element \( x \) of \( \zeta(A) \) belongs to \( \delta(A)_p \) if and only if there exists a non-negative integer \( m_x = m_{p,x} \) such that for all \( a \geq m_x \) and all \( M \in M_a(A_p) \) one has \( x \cdot M \) in \( M_a(A_p) \).

(iv) \( \delta(A) \cdot \xi(A) = \delta(A) \).

(vi) If \( A \) is commutative, then \( \delta(A) = \xi(A) = A \).

(vii) Any surjective homomorphism of \( R \)-orders \( \varphi : A \rightarrow B \) induces, upon restriction, a homomorphism \( \delta(A) \rightarrow \delta(B) \).

**Proof.** For each \( p \) the set \( \delta(A_p) \) is clearly an additive subgroup of \( \zeta(A) \) that is stable under multiplication by \( \zeta(A_p) \). One also has \( \delta(A_p) \subseteq \zeta(A_p) \) since if \( M \) is the \( 1 \times 1 \) identity matrix, then \( x = x \cdot M = x \cdot M^* \) and so \( x = x \cdot M^* \in M_1(A_p) \) implies \( x \in A_p \). This proves \( \delta(A_p) \) is an ideal of \( \zeta(A_p) \) and we next show it has finite index.

First consider the special case that \( A_p \) is maximal (as will be the case for all but finitely many \( p \)). In this case \( \zeta(A_p) = \mathfrak{M}(p) \), where \( \mathfrak{M} \) is the integral closure of \( R \) in \( \zeta(A) \), and for every \( M \) in \( M_d(A_p) \) and each primitive idempotent \( e \) of \( \zeta(A) \) for which \( \text{Nrd}(M) e \) is non-zero, the defining property \[3.11\] implies that \( eM^* \) belongs to \( M_d(eA_p) \subseteq M_d(A_p) \) (see, for example, the discussion of \[21 \] §3.6)). In this case, therefore, it follows that \( \delta(A_p) \) contains, and is therefore equal to, \( \mathfrak{M}(p) \). To deal with the general case, we fix a maximal \( R(p) \)-order \( M \) that contains \( A_p \) and write \( n \) for the (finite) index of \( A_p \) in \( M \). Then for each \( M \) in \( M_d(A_p) \), the above argument implies that \( M^* \) belongs to \( M_d(M) \) and hence that \( n \cdot M^* \) belongs to \( M_d(A_p) \). This implies \( n \cdot \zeta(A_p) \subseteq \delta(A_p) \) and hence that \( \delta(A_p) \) has finite index in \( \zeta(A_p) = \zeta(A_p) \).

At this stage we know that \( \delta(A) = \bigcap_p \delta(A_p) \) is an ideal of \( \bigcap_p \zeta(A_p) = \zeta(A) \) and that (as a consequence of \[16 \] Prop. (4.21)(vi)) its index is finite and for every \( p \) one has \( \delta(A)_p = \delta(A_p) \). This proves claims (i) and (ii).

To prove claim (iii) it obviously suffices (in view of claim (ii)) to show that the stated condition is sufficient to imply \( x \) belongs to \( \delta(A_p) \). To do this we fix a natural number \( d \) and a matrix \( M \) in \( M_d(A_p) \) and note that in \( M_{d+m_x}(A) \) one has

\[
x\left( \begin{array}{cc} M & 0 \\ 0 & I_{m_x} \end{array} \right)^* = x\left( \begin{array}{cc} M^* & 0 \\ 0 & \text{Nrd}_A(M) \cdot I_{m_x} \end{array} \right) = \left( \begin{array}{cc} x \cdot M^* & 0 \\ 0 & x \cdot \text{Nrd}_A(M) \cdot I_{m_x} \end{array} \right).
\]

In particular, since \( d + m_x > m_x \), the stated condition on \( x \) (with \( a = d + m_x \) and \( M \) replaced by \( \left( \begin{array}{cc} M & 0 \\ 0 & I_{m_x} \end{array} \right) \)) implies that \( x \cdot M^* \) belongs to \( M_d(A_p) \), as required.

In view of claim (ii) and Lemma \[32 \] (iii), it is enough to prove the equality in claim (iv) after replacing \( A \) by \( A_p \) for each \( p \). Since 1 belongs to \( \xi(A_p) \), it is then enough to show that for any \( x \) in \( \delta(A_p) \), any natural number \( n \), and any matrix \( N \) in \( M_n(A_p) \), the element...
We do this by showing that \( x' \) satisfies the condition described in claim (iii) with \( m_{x'} \) taken to be \( n \).

We thus fix an integer \( d \) with \( d \geq n \) and choose \( N' \) in \( M_d(A(p)) \) with \( \text{Nrd}_A(N') = \text{Nrd}_A(N) \). Then, for any \( M \) in \( M_d(A(p)) \) one has \( M^* \cdot (N')^* = (N' \cdot M)^* \) and hence

\[
x' \cdot M^* = x \cdot \text{Nrd}_A(N)M^* = x \cdot \text{Nrd}_A(N')M^* = x \cdot M^*((N')^*N') = (x \cdot (N'M)^*)N'.
\]

In particular, since \( x \) belongs to \( \delta(A(p)) \) one has \( x \cdot (N'M)^* \in M_d(A(p)) \), and hence \( x' \cdot M^* \in M_d(A(p)) \), as required.

In view of claim (i) and Lemma 3.2(iii), claim (v) is reduced to showing that if \( A \) is commutative, then \( \delta(A) \) contains \( A \). This follows directly from the fact that, in this case, for every prime \( p \) and every \( M \) in \( M_d(A(p)) \) the adjoint matrix \( M^* \) also belongs to \( M_d(A(p)) \).

Claim (vi) is true since (as already observed above) if \( A(p) \) is a maximal \( R(p) \)-order in \( A \), then \( \delta(A(p)) = \delta(A(p)) = \text{Nrd}_p (= \delta(A(p)) = \delta(A(p)), \) where the last two equalities follow from the proof of Lemma 3.2(iv) and from Lemma 3.2(ii) respectively.

Finally, to prove claim (vii) we write \( A \) and \( B \) for the \( F \)-algebras that are respectively spanned by \( A \) and \( B \) and we consider the ring homomorphisms \( \varrho' : \zeta(A) \rightarrow \zeta(B) \) and \( \varrho_d : M_d(A) \rightarrow M_d(B) \) for each natural number \( d \) that are induced by \( \varrho \).

It is enough to show that \( \varrho' \delta(A(p)) = \delta(B(p)) \) for all \( p \). Then, since \( \zeta(B) \) is a direct factor of \( \zeta(A) \) (see the proof of Lemma 3.2(v)), for any matrix \( M \in M_d(A(p)) \) the defining equality 3.1 implies that \( \varrho_d(M^*) = \varrho_d(M)^* \). By using this last equality, the required equality \( \varrho' \delta(A(p)) = \delta(B(p)) \) follows directly from the definition of the respective ideals \( \delta(A(p)) \) and \( \delta(B(p)) \) and the fact that \( \varrho_d(M_d(A(p))) = M_d(B(p)) \).

**Remark 3.6.** The ideal \( \delta(A) \) differs from an ideal \( \mathcal{H}(A) \) defined (in the case that \( R \) is a discrete valuation ring) by Johnston and Nickel in [21] (since our definition of \( M^* \) differs slightly from the 'generalized adjoint matrices' defined in loc. cit.) Nevertheless, the extensive computations of \( \mathcal{H}(A) \) made in loc. cit. can be used to give concrete information about \( \delta(A) \). For example, the arguments of [21] Prop. 4.3 and 4.8] combine with claim (ii) of Lemmas 3.2 and 3.5 to imply that if \( \Gamma \) is a finite group and \( p \) a prime ideal of \( R \) with residue characteristic prime to the order of the commutator subgroup of \( \Gamma \), then one has \( \xi(R[\Gamma]) = \xi(R[\Gamma]) = \xi(R[\Gamma]) = \xi(R[\Gamma]) = \delta(R[\Gamma])(\Gamma) \).

### 3.2. Locally-free modules.

**Definition 3.7.** A finitely generated module \( M \) over an \( R \)-order \( A \) will be said to be 'locally-free' if \( M(p) \) is a free \( A(p) \)-module, or equivalently (as an easy consequence of Maranda’s Theorem - see [16] Prop. (30.17))] if \( M(p) \) is a free \( A(p) \)-module, for all prime ideals \( p \) of \( R \). In the sequel we write \( \text{Mod}^f(A) \) for the category of locally-free \( A \)-modules.

For any module \( M \) in \( \text{Mod}^f(A) \) the rank of the (finitely generated) free \( A(p) \)-module \( M(p) \) is independent of \( p \) and equal to the rank of the (free) \( A \)-module \( M_F \). We refer to this common rank as the 'rank' of \( M \) and denote it \( \text{rk}_A(M) \).

A locally-free \( A \)-module of rank one will sometimes be referred to as an 'invertible' \( A \)-module.

Since localization at \( p \) is an exact functor a locally-free \( A \)-module is automatically projective. We record two important examples for which the converse is also true.
Example 3.8.

(i) If \( A = R \), then every finitely generated torsion-free \( A \)-module \( M \) is locally-free, with \( \text{rk}_A(M) \) equal to the dimension of the \( F \)-space spanned by \( M \).

(ii) If \( G \) is a finite group for which no prime divisor of \(|G|\) is invertible in \( R \) and \( A = R[G] \) then, by a fundamental result of Swan [43] (see also [16, Th. (32.11)]), a finitely generated projective \( A \)-module is locally-free. For any such module \( M \) the product \( \text{rk}_{R[G]}(M) \cdot |G| \) is equal to the dimension of the \( F \)-space spanned by \( M \).

3.3. Fitting invariants of locally-free presentations.

3.3.1. Let \( M \) be a matrix in \( M_{d',d}(A) \) with \( d' \geq d \). Then for any integer \( t \) with \( 0 \leq t \leq d \) and any \( \varphi = (\varphi_i)_{1 \leq i \leq t} \) in \( \text{Hom}_A(A^{d'},A)^t \) we write \( \text{Min}_d^t(M) \) for the set of all \( d \times d \) minors of the matrices \( M(J,\varphi) \) that are obtained from \( M \) by choosing any \( t \)-tuple of integers \( J = \{i_1,i_2,\ldots,i_t\} \) with \( 1 \leq i_1 < i_2 < \cdots < i_t \leq d \), and setting

\[
M(J,\varphi)_{ij} := \begin{cases} 
\varphi_a(b_i), & \text{if } j = i_a \text{ with } 1 \leq a \leq t \\
M_{ij}, & \text{otherwise.}
\end{cases}
\]

(3.3.1)

where, for any natural number \( n \), we write \( \{b_i\}_{1 \leq i \leq n} \) for the standard basis of the free \( A \)-module \( A^n \).

Then the set of all \( d \times d \) minors of all matrices that are obtained from \( M \) by replacing at most \( a \) of its columns by arbitrary elements of \( A \) is equal to

\[
\mathcal{S}^a(M) := \bigcup_{0 \leq t \leq a} \bigcup_{\varphi \in \text{Hom}_A(A^{d'},A)^t} \text{Min}_d^t(M).
\]

We note, in particular, that \( \mathcal{S}^0(M) \) is the set of all \( d \times d \) minors of \( M \).

Definition 3.9. For any non-negative integer \( a \) the ‘\( a \)-th (non-commutative) Fitting invariant of \( M \)’ is the ideal of \( \xi(A) \) obtained by setting

\[
\text{Fit}^a_A(M) := \xi(A) \cdot \{\text{Nrd}_A(N) : N \in \mathcal{S}^a(M)\}.
\]

3.3.2. A ‘free presentation’ \( \Pi \) of a finitely generated \( A \)-module \( X \) is an exact sequence of \( A \)-modules of the form

\[
\Pi : A^r'_{\Pi} \xrightarrow{\theta_{\Pi}} A^r_{\Pi} \xrightarrow{\rho_{\Pi}} X \rightarrow 0
\]

in which (without loss of generality) one has \( r'_{\Pi} \geq r_{\Pi} \).

We write \( M_{\Pi} \) for the matrix of the homomorphism \( \theta_{\Pi} \) with respect to the standard bases of \( A^r'_{\Pi} \) and \( A^r_{\Pi} \).

3.3.3. A ‘locally-free presentation’ \( \Pi \) of a finitely generated \( A \)-module \( X \) is a collection of data of the following form:

- an exact sequence of \( A \)-modules

\[
\Pi^{\text{seq}} : P' \xrightarrow{\theta_{\Pi}} P \xrightarrow{\rho_{\Pi}} X \rightarrow 0
\]

in which \( P' \) and \( P \) belong to \( \text{Mod}^{lf}(A) \);
- for each prime ideal \( p \) of \( R \) fixed isomorphisms of \( A_{(p)} \)-modules

\[
i'_{\Pi,p} : P'_p \cong A_{(p)}^{\text{rk}(P')} \quad \text{and} \quad i_{\Pi,p} : P_p \cong A_{(p)}^{\text{rk}(P)}.
\]

Such a presentation will be said to be ‘locally-quadratic’ if \( \text{rk}(P') = \text{rk}(P) \).

**Example 3.10.** Let \( \varrho : A \rightarrow B \) be a surjective homomorphism of \( R \)-orders. Then the induced exact sequence of \( B \)-modules

\[
B \otimes_{A_{\varrho}} \Pi_{\text{seq}} : B \otimes_{A_{\varrho}} P' \xrightarrow{id \otimes \vartheta_1} B \otimes_{A_{\varrho}} P \rightarrow B \otimes_{A_{\varrho}} X \rightarrow 0
\]

and isomorphisms \( B \otimes_{A_{\varrho}} i'_{\Pi,p} \) and \( B \otimes_{A_{\varrho}} i_{\Pi,p} \) together constitute a locally-free presentation \( B \otimes_{A_{\varrho}} \Pi \) of \( B \otimes_{A_{\varrho}} X \) that is locally-quadratic if \( \Pi \) is locally-quadratic.

**Definition 3.11.** For each non-negative integer \( a \), the \( a \)-th Fitting invariant of the locally-free presentation \( \Pi \) is the ideal of \( \xi(A) \) obtained by setting

\[
\text{Fit}^a_{\Pi}(\Pi) := \bigcap_p \text{Fit}^a_{A_{(p)}}(M_{\Pi(p)}),
\]

where the intersection is over all prime ideals \( p \) of \( R \) and \( \Pi(p) \) denotes the free resolution of the \( A_{(p)} \)-module \( X_p \) that is obtained by localising \( \Pi_{\text{seq}} \) and using the isomorphisms \( i'_{\Pi,p} \) and \( i_{\Pi,p} \).

The basic properties of these ideals are recorded in the following result.

**Lemma 3.12.** Let \( \Pi \) be a locally-free presentation of an \( A \)-module. Then the following claims are valid for every non-negative integer \( a \).

1. \( \text{Fit}^a_{\Pi}(\Pi) \) is contained in \( \text{Fit}^{a+1}_{\Pi}(\Pi) \).
2. \( \text{Fit}^a_{\Pi}(\Pi) = \xi(A) \) for all large enough \( a \).
3. For any homomorphism \( \varrho \) as in Example 3.10 one has \( \varrho(\text{Fit}^a_{\Pi}(\Pi)) \subseteq \text{Fit}^a_{\varrho}(B \otimes_{A_{\varrho}} \Pi) \).

**Proof.** For each prime \( p \) we set \( M_p := M_{\Pi(p)} \).

Then claim (i) follows directly from the fact that in the definition of the set of matrices \( \mathcal{G}^a(M_p) \) that occurs in Definition 3.9 the variable \( t \) runs over all integers in the range \( 0 \leq t \leq a \).

For the module \( P \) in \( \text{Example 3.3} \) we set \( n := \text{rk}(P) \). Then to prove claim (ii) it is enough to show that for every \( a \geq n \) and every prime \( p \) one has \( \text{Fit}^a_{A_{(p)}}(M_p) = \xi(A_{(p)}) \). This is true because for any such \( a \) the \( n \times n \) identity matrix belongs to \( \mathcal{G}^a(M_p) \).

In a similar way, claim (iii) is true since for every prime \( p \) the induced projection map \( \varrho_n : M_n(A) \rightarrow M_n(B) \) sends any matrix in \( \mathcal{G}^a(M_p) \) to a matrix in \( \mathcal{G}^a(\varrho_n(M_p)) \). \( \square \)

### 3.3.4
In the next result we explain the connection between this definition and the notion of non-commutative Fitting invariants of presentations introduced (in the case that \( R \) is a discrete valuation ring and \( a = 0 \)) by Nickel in \( [32] \) and then subsequently studied by Johnston and Nickel in \( [21] \).

**Proposition 3.13.** Assume that \( R \) is a discrete valuation ring and let \( \Pi \) be a free presentation of an \( A \)-module \( X \). Then all of the following claims are valid.

1. \( \xi(A) \cdot \text{Fit}^0_{\Pi}(\Pi) = \xi(A) \cdot \text{Fit}_{\Pi}(\Pi) \), where \( \text{Fit}_{\Pi}(\Pi) \) is the noncommutative Fitting invariant of Nickel.
(ii) If \( \Pi \) is quadratic, then \( \text{Fit}_A^0(\Pi) \) is equal to \( \xi(\mathcal{A}) \cdot \text{Nrd}_A(M_{\Pi}) \) and depends only on the isomorphism class of the \( A \)-module \( X \).

(iii) Let \( 0 \to X_1 \to X_2 \to X_3 \to 0 \) be a short exact sequence of \( A \)-modules. If \( X_1 \) and \( X_3 \) have quadratic presentations \( \Pi_1 \) and \( \Pi_3 \), then there exists a quadratic presentation \( \Pi_2 \) of \( X_2 \) and one has \( \text{Fit}_A^0(\Pi_2) = \text{Fit}_A^0(\Pi_1) \cdot \text{Fit}_A^0(\Pi_3) \).

**Proof.** We write \( \xi'(\mathcal{A}) \) for the \( R \)-order in \( \zeta(\mathcal{A}) \) that is generated over \( \zeta(\mathcal{A}) \) by the elements \( \text{Nrd}_A(M) \) as \( M \) runs over matrices in \( \bigcup_{n \geq 1} \text{GL}_n(\mathcal{A}) \).

Then, setting \( r := r_{\Pi} \), the invariant \( \text{Fit}_A(\Pi) \) is defined in [21, (3.3)] to be the \( \xi'(\mathcal{A}) \)-submodule of \( \zeta(\mathcal{A}) \) that is generated by the elements \( \text{Nrd}_A(N) \) as \( N \) runs over all \( r \times r \) minors of the matrix \( M_{\Pi} \). Thus, since \( \text{Fit}_A^0(\Pi) \) is defined to be the ideal of \( \xi(\mathcal{A}) \) that is generated over \( R \) by the same elements \( \text{Nrd}_A(N) \), the required equality \( \zeta(\mathcal{A}) \cdot \text{Fit}_A^0(\Pi) = \xi(\mathcal{A}) \cdot \text{Fit}_A(\Pi) \) of claim (i) follows directly from the obvious fact that \( \xi(\mathcal{A}) \cdot \xi(\mathcal{A}) = \xi(\mathcal{A}) \cdot \xi'(\mathcal{A}) \).

In the context of claim (ii) one has \( r_{\Pi} = r_{\Pi} \) (in the notation of [32, (3.3.2)]) and so \( \text{Fit}_A^0(\Pi) \) is, by its very definition, equal to \( \xi(\mathcal{A}) \cdot \text{Nrd}_A(M_{\Pi}) \). Claim (ii) is therefore true provided that the latter ideal depends only on the isomorphism class of the \( A \)-module given by the cokernel of \( \theta_{\Pi} \) and this follows from the argument used by Nickel to prove [32, Th. 3.2ii)].

The key idea in the proof of claim (iii) is to construct a suitable locally-quadratic presentation \( \Pi_2 \) of \( X_2 \) from given locally-quadratic presentations of \( X_1 \) and \( X_3 \) and then to compute the respective zeroth Fitting invariants via the formula in claim (ii). The precise argument mimics that of Nickel in [32, Prop. 3.5iii)] and so will be left to the reader. \( \square \)

### 3.4. Fitting invariants of modules.

#### 3.4.1. In this section we assume to be given a finitely generated \( A \)-module \( Z \).

**Definition 3.14.** For each non-negative integer \( a \), the ‘\( a \)-th Fitting invariant’ of \( Z \) is the ideal of \( \xi(\mathcal{A}) \) obtained by setting

\[
\text{Fit}_A^a(Z) := \sum_\Pi \text{Fit}_A^a(\Pi),
\]

where in the sum \( \Pi \) runs over all locally-free presentations of finitely generated \( A \)-modules \( Z' \) for which there exists a surjective homomorphism of \( A \)-modules of the form \( Z' \to Z \).

The basic properties of these ideals are described in the next result. Before stating this result we introduce the following useful definition.

**Definition 3.15.** The ‘central pre-annihilator’ of an \( A \)-module \( Z \) is the \( \xi(\mathcal{A}) \)-submodule of \( \zeta(\mathcal{A}) \) obtained by setting

\[
\text{pAnn}_A(Z) := \{ x \in \zeta(\mathcal{A}) : x \cdot \delta(\mathcal{A}) \subseteq \text{Ann}_A(Z) \},
\]

where \( \delta(\mathcal{A}) \) is the ideal of \( \zeta(\mathcal{A}) \) from Definition 3.4 and \( \text{Ann}_A(Z) \) denotes the annihilator of \( Z \) in \( A \).

**Remark 3.16.** If \( \mathcal{A} \) is commutative, then Lemma 3.5(v) implies that \( \text{pAnn}_A(Z) = \text{Ann}_A(Z) \).

**Theorem 3.17.** The following claims are valid for every non-negative integer \( a \).

(i) \( \text{Fit}_A^a(Z) \) is contained in \( \text{Fit}_A^{a+1}(Z) \).

(ii) \( \text{Fit}_A^a(Z) = \xi(\mathcal{A}) \) if \( a \) is large enough.
(iii) $\text{Fit}^0_A(Z)$ is contained in $\text{pAnn}_A(Z)$.

(iv) Let $e$ be a primitive central idempotent of $A$. Then the ideal $e \cdot \text{Fit}^0_A(Z)$ vanishes if $rr_{Ae}(e \cdot Z_F) > a \cdot rr_{Ae}(eA)$.

(v) For any surjective homomorphism of $R$-orders $\varphi : A \to B$ one has $\varphi(\text{Fit}^0_A(Z)) \subseteq \text{Fit}^0_B(B \otimes_{A,\varphi} Z)$.

(vi) For any surjective homomorphism of $A$-modules $Z \to Z'$ one has $\text{Fit}^0_A(Z) \subseteq \text{Fit}^0_A(Z')$.

(vii) If $A$ is commutative, then $\text{Fit}^0_A(Z)$ is equal to the $a$-th Fitting ideal of the $A$-module $Z$, as discussed by Northcott in [33, 3].

**Proof.** Claims (i), (ii) and (v) follow directly from the corresponding results in Lemma 3.12 and claim (vi) from the nature of the sum in Definition 3.14.

Since each of the remaining claims can be proved after localizing at each prime ideal $p$ of $R$ we shall in the sequel assume (without explicit comment) that $R$ is local. We also then fix a free presentation $\Pi$ of a finitely generated $A$-module $X$ of the form (3.3.2) and set $r := r_\Pi$ and $r' := r'_\Pi$.

Claim (iii) is quickly reduced to proving that if $\Pi$ is quadratic, so that $r' = r$, then for any given element $a$ of $\delta(A)$ the product $a \cdot \text{Nrd}_A(M_\Pi)$ belongs to $\zeta(A)$ and annihilates $X$.

We set $M := M_\Pi$. Then claims (i) and (iv) of Lemma 3.15 combine to imply $a \cdot \text{Nrd}_A(M)$ belongs to $\zeta(A)$ and so it suffices to prove this element annihilates $X$. To do this we follow an argument used by Nickel to prove [32, Th. 4.2].

Specifically, it is easy to show that for every element $y$ of $A^r$ the product

$$a \cdot \text{Nrd}_A(M) \cdot y = a \cdot M M^*(y) = M(a \cdot M^*(y))$$

belongs to $\text{im}(\theta_\Pi)$, where the matrix $M^*$ is as defined in (3.1.1). This is in turn a direct consequence of the fact that the definition of $\delta(A)$ ensures $a \cdot M^*$ belongs to $M_{ir,r}(A)$ and hence that $M(a \cdot M^*(y))$ belongs to $\text{im}(\theta_\Pi)$.

To prove claim (iv) we set $m := rr_{Ae}(eA)$. Then it is enough to prove that for any non-negative integer $a$ one has $e \cdot \text{Fit}^0_A(\Pi) = 0$ whenever $rr_{Ae}(e \cdot X_F) > a \cdot m$.

To show this we let $M$ denote any matrix obtained from $M_\Pi$ by replacing at most $a$ of its columns by arbitrary elements of $A$. We set $d := r_\Pi$, fix a $d \times d$ minor $N$ of $M$ (so that $N$ is a typical matrix of the set $G_a(M_\Pi)$) and write $N_\Pi$ for the corresponding minor of $M_\Pi$.

We fix a splitting field $E$ for $Ae$ and an isomorphism of algebras of the form $Ae \otimes_{\zeta(A)e} E \cong M_{m,m}(E)$. This isomorphism induces a map $\iota : M_{d,d}(A) \to M_{d,m,m}(E)$ and $e \cdot \text{Fit}^0_A(\Pi)$ is, by its definition, generated over $\zeta(A)$ by the determinants of all matrices of the form $\iota(N)$. In addition, it is clear that

$$\text{rank}(\iota(N)) \leq \text{rank}(\iota(N_\Pi)) + a \cdot m \leq (d \cdot m - rr_{Ae}(e \cdot X_F)) + a \cdot m.$$ 

Hence, if $rr_{Ae}(e \cdot X_F) > a \cdot m$, then $\text{rank}(\iota(N)) < d \cdot m$ and so $\text{det}(\iota(N)) = 0$, as required to prove claim (iv).

Finally, to prove claim (vii), we assume $A$ is commutative and note that $\text{Fit}^a_A(\Pi)$ is generated over $A$ by elements of the form $\text{det}(N)$ where $N$ is an $r \times r$ matrix, at least $r - a$ columns of which coincide with the columns of an $r \times r$ minor of $M_\Pi$.

The Laplace expansion of $\text{det}(N)$ therefore shows that it is contained in the ideal of $A$ generated by the set of $(r - a) \times (r - a)$ minors of $M_\Pi$. Thus, since the latter ideal is, by definition, equal to $\text{Fit}^0_A(X)$ one has $\text{Fit}^0_A(\Pi) \subseteq \text{Fit}^0_A(X)$. 


To prove the reverse inclusion it suffices to show that for each \((r - a) \times (r - a)\) minor \(N\) of \(M_{\Pi}\) the term \(\det(N)\) belongs to \(\text{Fit}^a_{\mathbb{A}}(\Pi)\).

For any natural number \(n\) and any non-negative integer \(t\) we write \([n]_t\) for the set of subsets of \(\{1, 2, \ldots, n\}\) that are of cardinality \(\min\{t, n\}\).

Then we assume that \(N\) is obtained by first deleting from \(M_{\Pi}\) the columns corresponding to a subset \(J = \{i_1, i_2, \ldots, i_a\}\) of \([r]_a\) with \(i_1 < i_2 < \cdots < i_a\), and then taking the rows corresponding to an element \(J_1\) of \([r]_{r-a}\). We choose an element \(J'_1\) of \([r]'_r\) that contains \(J_1\), label the elements of \(J'_1 \setminus J_1\) as \(k_1 < k_2 < \cdots < k_a\) and then define an element \(\varphi_i(1 \leq i \leq a)\) of \(\text{Hom}_A(A', A)^a\) by setting \(\varphi_i(b_j) = \delta_{j,k_i}\) for each \(j\) with \(1 \leq j \leq r'\).

Then an explicit computation shows that, with these choices, the determinant of the \((\varphi)\) that at every \((\Pi_{\mathbb{A}}(\mathbb{A}', \mathbb{A}))\) with the product \(\prod_{p} \mathbb{C}_p\) where \(\psi\) runs over all irreducible \(\mathbb{C}_p\)-valued characters of \(\mathbb{A}\). For each \(x\) in \(\zeta(\mathbb{C}_p[\mathbb{A}])\) we write the corresponding element of the product as \((x_\psi)_\psi\).

We then write \(x \mapsto x^\#\) for the involution of \(\zeta(\mathbb{C}_p[\mathbb{A}])\) that is specified by the condition that at every \(\psi\) one has \((x^\#)_\psi = x_\psi^\#\) where \(\psi^\#\) denotes the contragredient of \(\psi\).

3.4.2. Let \(\Gamma\) be a finite group. In this section we discuss a construction of presentations for modules over the Gorenstein order \(\mathcal{A} = R[\Gamma]\). The observations made here are subsequently useful in arithmetic applications, both in this article and elsewhere.

If \(\Pi\) is a locally-free presentation of an \(R[\Gamma]\)-module \(X\) (as described in \([3.3.3]\)), then the ‘transpose’ \(\Pi^t\) of \(\Pi\) is the locally-free presentation of the \(R[\Gamma]\)-module \(\text{cok}(\text{Hom}_R(\theta_{\Pi}, R))\) that is given by the following data:

- the exact sequence of \(R[\Gamma]\)-modules
  \[\Pi^t] : \text{Hom}_R(P, R) \xrightarrow{\text{Hom}_R(\theta_{\Pi}, R)} \text{Hom}_R(P', R) \rightarrow \text{cok}(\text{Hom}_R(\theta_{\Pi}, R)) \rightarrow 0,\]
  where the linear duals are endowed with the contragredient action of \(\Gamma\);
- for each prime \(p\) the composite isomorphisms
  \[\text{Hom}_R(P, R)(p) \cong \text{Hom}_R(R[\Gamma])^{r_k(P)}, R)(p) \cong R(p)[\Gamma]^{r_k(P)}\]
  and
  \[\text{Hom}_R(P', R)(p) \cong \text{Hom}_R(R[\Gamma])^{r_k(P')}, R)(p) \cong R(p)[\Gamma]^{r_k(P')}\]
  where the first maps are respectively induced by the \(R(p)\)-linear duals of \(\theta_{\Pi, p}\) and \(\theta_{\Pi, p}'\) and the second by the standard isomorphism \(\text{Hom}_R(R[\Gamma], R) \cong R[\Gamma]\).

**Remark 3.18.** It is clear \(\Pi^t\) is locally-quadratic if and only if \(\Pi\) is locally-quadratic. In addition, since \(\text{Hom}_R(\text{Hom}_R(\theta_{\Pi}, R), R)\) identifies naturally with \(\theta_{\Pi}\), there exists a choice of isomorphism of \(R[\Gamma]\)-modules \(\text{cok}(\theta_{\Pi}) \cong X\) that induces an identification of \((\Pi^t)^t\) with \(\Pi\).

**Definition 3.19.** For a finitely generated \(R[\Gamma]\)-module \(Z\) and non-negative integer \(a\), the ‘transpose’ of the \(a\)-th Fitting invariant of \(Z\) is the ideal

\[\text{Fit}^a_{R[\Gamma]}(Z)^t := \sum_{\Pi} \text{Fit}^a_{R[\Gamma]}(\Pi^t),\]

of \(\zeta(\mathcal{A})\), where the sum is as in Definition \([3.14]\).

We next recall that the Wedderburn decomposition of \(\mathbb{C}_p[\Gamma]\) induces an identification of \(\zeta(\mathbb{C}_p[\Gamma])\) with the product \(\prod_{\psi} \mathbb{C}_p\) where \(\psi\) runs over all irreducible \(\mathbb{C}_p\)-valued characters of \(\Gamma\). For each \(x\) in \(\zeta(\mathbb{C}_p[\Gamma])\) we write the corresponding element of the product as \((x_\psi)_\psi\).
Lemma 3.20. Assume $R$ is contained in $C_p$. Let $a$ be a non-negative integer. Then if either $a = 0$ or $\Gamma$ is abelian one has

$$\text{Fit}_{R[\Gamma]}^a(Z)^{tr} = \text{Fit}_{R[\Gamma]}^a(Z)^\#$$

for every finitely generated $R[\Gamma]$-module $Z$.

Proof. For each matrix $M$ in $M_{d,d'}(R[\Gamma])$ we write $M^{tr}$ for the transpose matrix in $M_{d,d'}(R[\Gamma])$ and $\iota_\#(M)$ for the matrix in $M_{d,d'}(R[\Gamma])$ that is obtained from $M$ by applying to each of its components the $R$-linear anti-involution of $R[\Gamma]$ that inverts elements of $\Gamma$.

It is then easily checked that for all $M$ in $M_d(R[\Gamma])$ one has

$$(3.4.1) \ Nrd_{F[\Gamma]}(\iota_\#(M^{tr})) = Nrd_{F[\Gamma]}(\iota_\#(M)) = Nrd_{F[\Gamma]}(M)^\#.$$ 

Now, after localizing at each prime ideal of $R$, it is enough to prove the claimed equality in the case that $Z$ has a free presentation.

To consider this case we fix a homomorphism of $R[\Gamma]$-modules $\theta : R[\Gamma]^r \to R[\Gamma]^r$ and write $M_\theta$ for its matrix with respect to the standard bases of $R[\Gamma]^r$ and $R[\Gamma]^r$. Then the matrix of $\text{Hom}_R(\theta,R)$ with respect to the standard (dual) bases of $\text{Hom}_R(R[\Gamma]^r,R)$ and $\text{Hom}_R(R[\Gamma]^r,R)$ is equal to $\iota_\#(M^{tr})$.

In the case $a = 0$, the claimed result therefore follows directly from the equality $(3.4.1)$. In the case that $\Gamma$ is abelian (and $a > 0$) it follows in a similar way after taking account of Theorem 3.17(vii). \qed

Remark 3.21. The second equality in $(3.4.1)$ implies directly that $\xi(R[\Gamma]) = \xi(R[\Gamma])^\#$. In a similar way, since the defining equality $(3.1.1)$ implies that $(\iota_\#(M))^* = \iota_\#(M^*)$ for every $M$ in $M_d(R[\Gamma])$ one has $\delta(R[\Gamma]) = \delta(R[\Gamma])^\#$. In particular, if $Z$ is an $R[\Gamma]$-module, then the last equality combines with the exactness of Pontryagin duality to imply that, with respect to the contragredient action of $R[\Gamma]$ on $Z^\vee = \text{Hom}_R(Z,F/R)$, one has $p\text{Ann}_{R[\Gamma]}(Z^\vee) = p\text{Ann}_{R[\Gamma]}(Z)^\#$.

4. Reduced exterior powers

In this section we discuss the basic properties of an explicit construction of ‘exterior powers’ over semisimple rings.

4.1. Exterior powers over commutative rings. We first quickly review relevant aspects of the classical theory of exterior powers over commutative rings.

Let $R$ be a commutative ring, and $M$ be an $R$-module. Then for every positive integer $r$, an element $f \in \text{Hom}_R(M,R)$ induces the homomorphism

$$\bigwedge^R_r M \to \bigwedge^{r-1}_R M$$

which is defined by

$$m_1 \wedge \cdots \wedge m_r \mapsto \sum_{i=1}^r (-1)^{i+1} f(m_i)m_1 \wedge \cdots \wedge m_{i-1} \wedge m_{i+1} \wedge \cdots \wedge m_r.$$
This homomorphism is also denoted by $f$. Using this construction, we define the following pairing:

$$\bigwedge^r_R M \times \bigwedge^s_R \text{Hom}_R(M, R) \to \bigwedge^{r-s}_R M; \quad (m, \bigwedge_{i=1}^{i=s} f_i) \mapsto (f_s \circ \cdots \circ f_1)(m),$$

where $r$ and $s$ are non-negative integers with $r \geq s$. We then set

$$(\bigwedge_{i=1}^{i=s} f_i)(m) := (f_s \circ \cdots \circ f_1)(m).$$

We shall also use the following convenient notation: for any natural numbers $r$ and $s$ with $s \leq r$ we write $[r]_s$ for the subset of $S_r$ comprising permutations $\sigma$ which satisfy both $\sigma(1) < \cdots < \sigma(s)$ and $\sigma(s+1) < \cdots < \sigma(r)$.

(This notation is motivated by the fact that the cardinality of $[r]_s$ is equal to the binomial coefficient $\binom{r}{s}$.)

We can now record two results that play an important role in the sequel.

**Lemma 4.1.** If $s \leq r$, then for all subsets $\{f_i\}_{1 \leq i \leq s}$ of $\text{Hom}_R(M, R)$ and $\{m_j\}_{1 \leq j \leq r}$ of $M$ one has

$$(\bigwedge_{i=1}^{i=s} f_i)(\bigwedge_{j=1}^{j=r} m_j) = \sum_{\sigma \in [r]_s} \text{sgn}(\sigma) \det(f_i(m_{\sigma(j)}))_{1 \leq i, j \leq r} \bigwedge m_{\sigma(s+1)} \wedge \cdots \wedge m_{\sigma(r)}.$$ 

In particular, if $r = s$, then we have

$$(\bigwedge_{i=1}^{i=r} f_i)(\bigwedge_{j=1}^{j=r} m_j) = \det(f_i(m_j))_{1 \leq i, j \leq r}.$$ 

**Proof.** This is verified by means of an easy and explicit computation. \hfill $\Box$

**Lemma 4.2** ([Lem. 4.2]). Let $E$ be a field and $W$ an $n$-dimensional $E$-vector space. Fix a non-negative integer $m$ with $m \leq n$ and a subset $\{\varphi_i\}_{1 \leq i \leq m}$ of $\text{Hom}_E(W, E)$. Then the $E$-linear map

$$\Phi = \bigoplus_{i=1}^{m} \varphi_i : W \to E^\oplus m$$

is such that

$$\text{im}(\bigwedge_{1 \leq i \leq m} \varphi_i : \bigwedge_{E}^{n} W \to \bigwedge_{E}^{n-m} W) = \begin{cases} \bigwedge_{E}^{n-m} \ker(\Phi), & \text{if } \Phi \text{ is surjective,} \\ 0, & \text{otherwise.} \end{cases}$$

**4.2. Reduced exterior powers over semisimple rings.** In this subsection, we define a notion of exterior powers for finitely generated modules over semisimple rings.

The underlying idea is as follows. If $\Lambda$ is a non-commutative ring for which there exists a functor $\Phi$ from the category of $\Lambda$-modules to the category of modules over some commutative ring $\Omega$ that gives an equivalence of the categories, then the exterior power of a $\Lambda$-module $M$ should be defined via a suitable exterior power of the $\Omega$-module $\Phi(M)$. (In our case, we shall take $\Phi$ to be the Morita functor defined at the end of §2.1.1.)
4.2.1. Let $K$ be a field of characteristic zero and $A$ a finite-dimensional simple $K$-algebra.

**Definition 4.3.** Take a splitting field $E$ of $A$, set $A_E := A \otimes_{\zeta(A)} E$ and fix a simple left $A_E$-module $V$. Then for each left $A$-module $M$ and each non-negative integer $r$, we define the `$r$-th reduced exterior power' of $M$ over $A$ to be the $E$-vector space

$$\bigwedge^r_A M := \bigwedge^d_E (V^* \otimes_{A_E} M_E),$$

where $d := \dim_E(V)$, $M_E := M \otimes_{\zeta(A)} E$, and $V^* := \text{Hom}_E(V,E)$.

**Remark 4.4.** If $A$ is commutative, and hence a field, then our convention will always be to take $E$, and hence also $V$, to be $A$ so that the above definition coincides with the standard $r$-th exterior power of $M$ as an $A$-module. In the general case, whilst our chosen notation $\bigwedge^r_A M$ suppresses the dependence of the definition on the splitting field $E$ and simple $A_E$-module $V$ we feel that this should not lead to confusion. Firstly, for a fixed $E$, all simple $A_E$-modules $V$ are isomorphic and so lead to isomorphic reduced exterior powers. In addition, each $K$-embedding of splitting fields $\sigma : E \rightarrow E'$ induces a canonical isomorphism $E' \otimes_{E,\sigma} \bigwedge^r_A M \cong \bigwedge^r_{A'} M$ where the exterior powers are respectively defined via the pairs $(E,V)$ and $(E',E' \otimes_{E,\sigma} V)$. Aside from this, one can also define $\bigwedge^r_A M$ with respect to a `canonical' choice of splitting field $E$ as in Remark 2.1 (and see also Remark 4.9 below in this regard).

**Remark 4.5.** Reduced exterior powers are functorial in the following sense. Any homomorphism of $A$-modules $\theta : M \rightarrow M'$ induces for each natural number $r$ a homomorphism of $E$-modules

$$\bigwedge^r_A M = \bigwedge^d_E (V^* \otimes_{A_E} M_E) \xrightarrow{\bigwedge^d_E (\text{id}_{V^*} \otimes \theta)} \bigwedge^d_E (V^* \otimes_{A_E} M'_E) = \bigwedge^r_A M'.$$

4.2.2. To make analogous constructions for linear duals we use the fact that $\text{Hom}_A(M,A)$ has a natural structure as left $A_E^{\text{op}}$-module. In particular, since $V^*$ is a simple left $A_E^{\text{op}}$-module and $V^{**}$ is canonically isomorphic to $V$, the $r$-th reduced exterior power (in the sense of Definition 4.3) of $\text{Hom}_A(M,A)$ is equal to

$$\bigwedge^r_{A^{\text{op}}} \text{Hom}_A(M,A) = \bigwedge^d_E (V \otimes_{A_E^{\text{op}}} \text{Hom}_{A_E}(M_E,E)).$$

The natural isomorphism

$$V \otimes_{A_E^{\text{op}}} \text{Hom}_{A_E}(M_E,E) \sim \text{Hom}_E(V^* \otimes_{A_E} M_E,E); \quad v \otimes f \mapsto (v^* \otimes m \mapsto v^*(f(m)v))$$

therefore induces an identification

$$\bigwedge^r_{A^{\text{op}}} \text{Hom}_A(M,A) = \bigwedge^d_E \text{Hom}_E(V^* \otimes_{A_E} M_E,E) = \bigwedge^d_E \text{Hom}_{\zeta(A)}(V^* \otimes_{A_E} M_E,\zeta(A)),$$

where the last identification is induced by the trace map $E \rightarrow \zeta(A)$.

By using this identification, the construction discussed in 4.1 applies to the $E$-vector space $V^* \otimes_{A_E} M_E$ to give a pairing

$$(4.2.1) \quad \bigwedge^r_A M \times \bigwedge^s_{A^{\text{op}}} \text{Hom}_A(M,A) \rightarrow \bigwedge^{r-s}_A M.$$  

We shall denote the image under this pairing of a pair $(m,\varphi)$ by $\varphi(m)$. 


4.2.3. We now fix an ordered $E$-basis $\{v_i\}_{1 \leq i \leq d}$ of $V$ and write $\{v^*_i\}_{1 \leq i \leq d}$ for the corresponding dual basis of $V^*$.

For any subsets $\{m_i\}_{1 \leq i \leq r}$ of $M$ and $\{\varphi_i\}_{1 \leq i \leq r}$ of $\text{Hom}_A(M, A)$ we then set

\begin{equation}
\wedge_{i=1}^{r} m_i := \wedge_{1 \leq i \leq r}(\wedge_{1 \leq j \leq d} v^*_j \otimes m_i) \in \bigwedge_{E}^{r}(V^* \otimes_{A_E} M) = \bigwedge_{A}^{r} M
\end{equation}

and

\begin{equation}
\wedge_{i=1}^{r} \varphi_i := \wedge_{1 \leq i \leq r}(\wedge_{1 \leq j \leq d} v^*_j \otimes \varphi_i) \in \bigwedge_{A^{\text{op}}}^{r} \text{Hom}_A(M, A),
\end{equation}

where $m_i$ and $\varphi_i$ are regarded as elements of $M_E$ and $\text{Hom}_{A_E}(M_E, A_E)$ in the obvious way.

**Lemma 4.6.** The $E$-vector spaces $\bigwedge_{A}^{r} M$ and $\bigwedge_{A^{\text{op}}}^{r} \text{Hom}_A(M, A)$ are respectively spanned by the sets $\{\wedge_{i=1}^{r} m_i : m_i \in M\}$ and $\{\wedge_{i=1}^{r} \varphi_i : \varphi_i \in \text{Hom}_A(M, A)\}$.

**Proof.** We only prove the claim for $M$ since exactly the same argument works for $\text{Hom}_A(M, A)$ (after replacing $A$ by $A^{\text{op}}$).

Then, since there exists a surjective homomorphism of $A$-modules of the form $A^t \to M$ (for any large enough $t$) it is enough to prove the claim in the case that $M$ is free of rank $t$.

In this case, if we fix an $A$-basis $\{b_j\}_{1 \leq i \leq r}$ of $M$, then $V^* \otimes_{A_E} M$ has as an $E$-basis the set $\{x_{ij} := v^*_j \otimes b_i\}_{1 \leq j \leq d, 1 \leq i \leq t}$ and so $\bigwedge_{A}^{r} M$ is generated over $E$ by the exterior powers of all $rd$-tuples of distinct elements in this set.

It is thus enough to show that for any $d$ distinct elements $X := \{x_{ikj}\}_{1 \leq k \leq d}$ of the above set, there exists an element $m_X$ of $M_E$ with $\wedge_{y=1}^{d}(v^*_y \otimes m_X) = \pm \wedge_{k=1}^{d} x_{ikj}$.

To do this we set $X_c := \{x_{ikj} : i_k = c\}$ and $n_c := |X_c|$ for each index $c$ with $1 \leq c \leq t$. We also set $Y := \{v^*_i\}_{1 \leq i \leq d}$.

We order the indices $c$ for which $n_c \neq 0$ as $c_1 < c_2 < \cdots < c_s$ (for a suitable integer $s$) and for each integer $\ell$ with $0 \leq \ell \leq s$ we set $N_\ell := \sum_{j=\ell}^{s} n_c$ (so that $N_0 = 0$ and $N_s = d$).

For each index $k$ we then choose $a_{ck}$ to be an element of $A_E$ whose image under the isomorphism (2.1.2) maps the elements $\{v^*_i : N_{k-1} < i \leq N_k\}$ to the $(N_k - N_{k-1} = n_{ck})$ distinct elements $v^*_j$ that occur as the first components of the elements in $X_{ck}$ and maps the remaining $d - n_{ck}$ elements of $Y$ to zero.

It is then straightforward to check the element $m_X := \sum_{k=1}^{s} a_{ck} \cdot b_{ck}$ of $M_E$ is such that

\[
\wedge_{y=1}^{d}(v^*_y \otimes m_X) = \wedge_{y=1}^{d}(\sum_{k=1}^{s} v^*_y a_{ck} \otimes b_{ck})
= \pm \wedge_{k=1}^{d}(\wedge_{N_{k-1} < y \leq N_k} (v^*_y a_{ck} \otimes b_{ck}))
= \pm \wedge_{k=1}^{d} x_{ikj},
\]
as required. \(\square\)

**Remark 4.7.** If $A$ is commutative, then our convention is that $E = V = A$ (see Remark 3.4). In particular, in this case $d = 1$ and we will always take the basis $\{v_i\}$ fixed above to comprise the identity element of $A$ so that the elements defined in (4.2.2) and (4.2.3) coincide with the classical definition of exterior products.
4.2.4. We assume now that $A$ is a semisimple ring, with Wedderburn decomposition \([2.2.1]\).

In this case, each finitely generated $A$-module $M$ decomposes as a direct sum $M = \bigoplus_{i \in I} M_i$ where each summand $M_i := A_i \otimes_A M$ is a finitely generated $A_i$-module. For any non-negative integer $r$, we then define the $r$-th reduced exterior power of the $A$-module $M$ by setting

$$\bigwedge^r_A M := \bigoplus_{i \in I} \bigwedge_{A_i}^r (A_i \otimes_A M),$$

where each component exterior power in the direct sum is defined with respect to a given choice of splitting field $E_i$ for $A_i$ over $\zeta(A_i)$ and a given choice of simple $E_i \otimes_{\zeta(A_i)} A_i$-module $V_i$. The associated reduced exterior power $\bigwedge^r_{A_{\text{op}}} \text{Hom}_A(M, A)$ is defined in a similar way.

Whilst these constructions depend on the choice of splitting fields, the following result shows that they behave functorially under scalar extension.

**Lemma 4.8.** For any extension $K'$ of $K$ there exists an injective homomorphism from $\bigwedge^r_A M$ to $\bigwedge^r_{A \otimes_K K'} (M \otimes_K K')$.

**Proof.** We can assume, without loss of generality, that $A$ is simple. We then set $F := \zeta(A)$ and $A' := A \otimes_K K'$ and $M' := M \otimes_K K'$. We assume $\bigwedge^r_A M$ is defined by using an algebraic extension $E$ of $F$ in $K'$ and a simple left $A_E$-module $V$, and we set $d = \text{dim}_E(V)$. We also use the notation of Lemma \([2.2]\). In particular, we write $\Omega$ for an algebraic closure of $K'$.

For each $\sigma$ in $\Sigma(F/K, K')$ we fix a $K$-embedding $\bar{\sigma}$ of $E$ into $\Omega$ that extends $\sigma$. Then the field $E_\sigma := \bar{\sigma}(E)K'$ splits the simple ring $A \otimes_F \sigma(F)K'$. In addition, $V_\sigma := V \otimes_E E_\sigma$ is a simple left module over $A_{E_\sigma} := A \otimes_F E_\sigma$ and so, by Lemma \([2.2]\) and the definition of reduced exterior powers, one has

$$\bigwedge^r_{A'} M' = \bigoplus_{\sigma \in \Sigma(F/K, K')} \bigwedge^{rd}_{E_\sigma} (V_\sigma^* \otimes_{A_{E_\sigma}} M_{E_\sigma}),$$

with $M_{E_\sigma} := M \otimes_F E_\sigma$.

For each $\sigma \in \Sigma(F/K, K')$, there is a canonical embedding

$$V^* \otimes_{A_E} M_E \rightarrow V_\sigma^* \otimes_{A_{E_\sigma}} M_{E_\sigma}.$$  

This induces an embedding

$$f_\sigma : \bigwedge^{rd}_E (V^* \otimes_{A_E} M_E) \rightarrow \bigwedge^{rd}_{E_\sigma} (V_\sigma^* \otimes_{A_{E_\sigma}} M_{E_\sigma})$$

and we define the required scalar extension

$$\bigwedge^r_A M = \bigwedge^{rd}_E (V^* \otimes_{A_E} M_E) \rightarrow \bigoplus_{\sigma \in \Sigma(F/K, K')} \bigwedge^{rd}_{E_\sigma} (V_\sigma^* \otimes_{A_{E_\sigma}} M_{E_\sigma}) = \bigwedge^r_A M'$$

to be the tuple $\bigoplus_\sigma f_\sigma$.

Upon combining the duality pairings \([4.2.1]\) for each simple component $A_i$ of $A$ one obtains a duality pairing

$$\bigwedge^r_A M \times \bigwedge^s_{A_{\text{op}}} \text{Hom}_A(M, A) \rightarrow \bigwedge^{r-s}_A M$$

that we continue to denote by $(m, \varphi) \mapsto \varphi(m)$.\[\square\]
For each subset of elements \( \{m_a\}_{1 \leq a \leq r} \) of \( M \) and \( \{\varphi_a\}_{1 \leq a \leq r} \) of \( \text{Hom}_A(M, A) \) we also set
\[
\bigwedge_{a=1}^{a=r} m_a := \bigwedge_{1 \leq a \leq r} m_{ai} \in \bigwedge_{A}^{r} M
\]
and
\[
\bigwedge_{a=1}^{a=r} \varphi_a := \bigwedge_{1 \leq a \leq r} \varphi_{ai} \in \bigwedge_{A}^{r} \text{Hom}_A(M, A),
\]
where \( m_{ai} \) and \( \varphi_{ai} \) are the projection of \( m_a \) and \( \varphi_a \) to \( M_i \) and \( \text{Hom}_{A_i}(M_i, A_i) \) and the component exterior powers are defined (via (4.2.2) and (4.2.3)) with respect to a fixed ordered \( E_i \)-basis of \( V_i \) (and its dual basis).

**Remark 4.9.** In each simple component of \( A \) that is commutative, we will always fix conventions regarding the bases used in (4.2.5) and (4.2.6) as in Remark 4.7. In addition, for the non-commutative \( A \) that arise in the arithmetic settings that are considered in later sections, the specification of a splitting field for \( A \), of a simple \( A_E \)-module \( V \) and of an ordered \( E \)-basis of \( V \) arises naturally in the following way.

Let \( G \) be a finite group of exponent \( e \) and write \( E \) for the field generated over \( \mathbb{Q}_p \) by a primitive \( e \)-th root of unity and \( \mathbb{I}_p(G) \) for the set of irreducible \( \mathbb{Q}_p^e \)-valued characters of \( G \). Then, by a classical result of Brauer [3], for each \( \chi \) in \( \mathbb{I}_p(G) \) there exists a representation
\[
\rho_\chi : G \rightarrow \text{GL}_{\chi(1)}(E)
\]
of character \( \chi \). The induced homomorphisms of \( E \)-algebras \( \rho_{\chi,*} : E[G] \rightarrow M_{\chi(1)}(E) \) combine to give an isomorphism
\[
E[G] \xrightarrow{(\rho_{\chi,*})_{\chi}} \prod_{\chi \in \mathbb{I}_p(G)} M_{\chi(1)}(E).
\]
This decomposition shows \( E \) is a splitting field for \( \mathbb{Q}_p[G] \), that the spaces \( V_{\chi} := E^{\chi(1)} \), considered as the first columns of the component \( M_{\chi(1)}(E) \), are a set of representatives of the simple \( E[G] \)-modules and that one can specify the standard basis of \( E^{\chi(1)} \) to be the ordered basis of \( V_{\chi} \). In this way, the specification of a representation \( \rho_\chi \) for each \( \chi \) in \( \mathbb{I}_p(G) \) leads to a canonical choice of the data necessary to define reduced exterior powers.

4.3. **Basic properties.** In this section we record several useful technical properties of the reduced exteriors powers defined above.

**Lemma 4.10.** Let \( A \) be a semisimple ring and \( W \) be an \( A \)-module. Then for all subsets \( \{w_i\}_{1 \leq i \leq r} \) of \( W \) and \( \{\varphi_j\}_{1 \leq j \leq s} \) of \( \text{Hom}_A(W, A) \) one has
\[
(\bigwedge_{i=1}^{i=r} \varphi_i)(\bigwedge_{j=1}^{j=s} w_j) = \text{Nrd}_{M_i(A \otimes \mathbb{A})(\varphi_i(w_j))_{1 \leq i,j \leq r}}.
\]
**Proof.** We may assume that \( A \) is simple (and use the notation of Definition 4.3) so that there is a canonical isomorphism \( A_E := A \otimes_{\mathbb{A}} E \cong \text{End}_E(V) \). In particular, after fixing an ordered \( E \)-basis \( \{v_i\}_{1 \leq i \leq d} \) of \( V \), we can identify \( A_E \) with the matrix ring \( M_d(E) \).

Then the definitions (4.2.2) and (4.2.3) combine to imply
\[
(\bigwedge_{i=1}^{i=r} \varphi_i)(\bigwedge_{j=1}^{j=r} w_j) = (\bigwedge_{1 \leq i \leq r}(\bigwedge_{1 \leq j \leq d} v_j \otimes \varphi_i))(\bigwedge_{1 \leq i \leq r}(\bigwedge_{1 \leq j \leq d} v_j^* \otimes w_i)).
\]
Next we note that the element \( (v_j' \otimes \varphi_i)(v_j^* \otimes w_j) = v_j^*(\varphi_i(w_j)v_j') \) of \( E \) is equal to the \((j', i')\)-component of the matrix \( \varphi_i(w_j) \in A \subset M_d(E) \). Hence, writing \( i' \varphi_i(w_j) \) for the
transpose of \( \varphi_i(w_j) \in M_d(E) \) and regarding \( (^t\varphi_i(w_j))_{1 \leq i,j \leq r} \) as a matrix in \( M_{rd}(E) \), Lemma 4.11 implies that
\[
(\wedge_{i=1}^r \varphi_i)(\wedge_{j=1}^r w_j) = \det((^t\varphi_i(w_j))_{1 \leq i,j \leq r}).
\]
It is therefore enough to note that the last expression is equal to \( \text{Nrd}_{M_r(A^{\text{op}})}((\varphi_i(w_j))_{1 \leq i,j \leq r}) \) by the definition of reduced norm.

\[\text{Remark 4.11.}\] Lemma 4.10 implies both that the value \( (\wedge_{i=1}^r \varphi_i)(\wedge_{j=1}^r w_j) \) belongs to \( \zeta(A) \) and also only depends on the elements \( w_1, \ldots, w_r \) and homomorphisms \( \varphi_1, \ldots, \varphi_r \).

**Lemma 4.12.** Let \( A \) be a semisimple ring and \( W \) a free \( A \)-module of rank \( r \). Then there is a canonical isomorphism of \( \zeta(A) \)-modules
\[
\iota_W : \bigwedge_{A^{\text{op}}}^r \text{Hom}_A(W,A) \cong \text{Hom}_{\zeta(A)}(\bigwedge_A^r W, \zeta(A))
\]
with the following property: for any \( A \)-basis \( \{b_i\}_{1 \leq i \leq r} \) of \( W \) one has \( \iota_W(\wedge_{i=1}^r b_i^*)_{(\wedge_{j=1}^r j b_j)} = 1 \) where for each index \( i \) we write \( b_i^* \) for the element of \( \text{Hom}_A(W,A) \) that is dual to \( b_i \).

**Proof.** In this case the pairing \( \langle 4.2.3 \rangle \) with \( s = r \) induces a homomorphism of free rank one \( \zeta(A) \)-modules \( \iota_W : \bigwedge_{A^{\text{op}}}^r \text{Hom}_A(W,A) \cong \text{Hom}_{\zeta(A)}(\bigwedge_A^r W, \zeta(A)) \).

Both the bijectivity of this homomorphism and the equality \( \iota_W(\wedge_{i=1}^r b_i^*)(\wedge_{j=1}^r j b_j) = 1 \) follow directly from Lemma 4.10.

**Lemma 4.13.** Let \( A \) be a semisimple ring and \( W \) a free \( A \)-module of rank \( r \). Fix an \( A \)-basis \( \{b_i\}_{1 \leq i \leq r} \) of \( W \). Then for each \( \varphi \) in \( \text{End}_A(W) \) one has
\[
(\wedge_{i=1}^r \varphi(b_i)) = \text{Nrd}_{\text{End}_A(W)}(\varphi) \cdot (\wedge_{i=1}^r b_i) \in \bigwedge_A^r W.
\]

**Proof.** The algebra isomorphism
\[
\text{End}_A(W) \cong M_r(A^{\text{op}}); \quad \psi \mapsto (b_i^*(\psi(b_j)))_{1 \leq i,j \leq r}
\]
implies that
\[
\text{Nrd}_{\text{End}_A(W)}(\varphi) = \text{Nrd}_{M_r(A^{\text{op}})}((b_i^*(\varphi(b_j)))_{1 \leq i,j \leq r}).
\]
By applying Lemma 4.10 one therefore has
\[
(\wedge_{i=1}^r b_i^*)_{(\wedge_{j=1}^r j \varphi(b_j))} = \text{Nrd}_{\text{End}_A(W)}(\varphi).
\]
This in turn implies the claimed equality since one also has \( (\wedge_{i=1}^r b_i^*)_{(\wedge_{j=1}^r j b_j)} = 1 \) as a consequence of Lemma 4.10.

Finally we establish a useful non-commutative generalization of Lemma 4.12.

**Lemma 4.14.** Let \( A \) be a semisimple ring and \( W \) a free \( A \)-module of rank \( r \). For a natural number \( s \) with \( s \leq r \) and a subset \( \{\varphi_i\}_{1 \leq i \leq s} \) of \( \text{Hom}_A(W,A) \) consider the map
\[
\Phi := \bigoplus_{i=1}^s \varphi_i : W \to A^{\oplus s}.
\]
Then the image of the map
\[
\bigwedge_A^r W \to \bigwedge_A^{r-s} W; \quad b \mapsto (\wedge_{1 \leq i \leq s} \varphi_i)(b)
\]
is contained in $\bigwedge_{A}^{r-s} \ker(\Phi)$.

In addition, if $A$ is simple and $\Phi$ is surjective, respectively not surjective, then the image of this map is equal to $\bigwedge_{A}^{r-s} \ker(\Phi)$, respectively vanishes.

Proof. We can assume, without loss of generality, that $A$ is simple (and then use the notation of Definition 4.3). Then, by Morita equivalence, the kernel of the induced linear map of $E$-vector spaces

$$\Phi_1 := \bigoplus_{i=s}^{i=d}(\text{id} \otimes \varphi_i) : V^* \otimes_{A_R} W_E \to (V^*)^{\otimes s}.$$  

is equal to $V^* \otimes_{A_E} \ker(\Phi)_E$ and $\Phi_1$ is surjective if and only if $\Phi$ is surjective.

We write \{v_j\}_{1 \leq j \leq d}$ for the ordered basis of $V$ with respect to which the exterior product $\wedge_{1 \leq i \leq s} \varphi_i$ is defined (in (12.3)) and consider the $E$-linear map

$$\Phi_2 := \bigoplus_{i=s}^{i=d} \bigoplus_{j=d}^{j=s} v_j \otimes \varphi_i : V^* \otimes_{A_E} W_E \to E^{\otimes sd}.$$  

Then it is clear that $\ker(\Phi_2) = \ker(\Phi_1) = V^* \otimes_{A_E} \ker(\Phi)_E$ and that $\Phi_2$ is surjective if and only if $\Phi_1$ is surjective, and hence if and only if the given map $\Phi$ is surjective.

Given this facts, and the explicit definition of the reduced exterior power $\bigwedge_{A}^{r-s} \ker(\Phi)$, the claimed results follow directly upon applying Lemma 4.2 with the data $W, n, m$ and $\Phi$ respectively replaced by $V^* \otimes_{A_E} W_E, dr, ds$ and the above map $\Phi_2$. \hfill \Box

4.4. Reduced Rubin lattices. We now define a canonical integral structure on the reduced exterior powers of $A$-modules. This structure will play a key role in the sequel.

4.4.1. For the reader's convenience we first review notation. Just as in [38] we assume to be given a Dedekind domain $R$ whose fraction field $F$ is a finite extension of either $\mathbb{Q}$ or $\mathbb{Q}_p$ for some prime $p$. We also fix an $R$-order $A$ that spans a (finite-dimensional) semisimple $F$-algebra $A$ and for each finitely generated left $A$-module $M$ we set $M_F := F \otimes_R M$.

Definition 4.15. Let $r$ be a non-negative integer. Then the $r$-th reduced Rubin lattice of the $A$-module $M$ is the $(A)$-module obtained by setting

$$\cap_{A}^{r} M := \{a \in \bigwedge_{A}^{r} M_F : (\wedge_{i=1}^{r} \varphi_i)(a) \in \xi(A) \text{ for all } \varphi_1, \ldots, \varphi_r \in \text{Hom}_A(M, A)\}.$$  

Remark 4.16. If $A$ is a split semisimple $F$-algebra, then the results of Lemmas 4.6 and 4.12 combine to imply that the map

$$\cap_{A}^{r} M \xrightarrow{\sim} \text{Hom}_{\xi(A)}(\xi(A) : \{\wedge_{i=1}^{r} \varphi_i : \varphi_i \in \text{Hom}_A(M, A)\}, \xi(A)) ; a \mapsto (\Phi \mapsto (\Phi(a))$$  

is an isomorphism of $\xi(A)$-modules. In particular, if $A$ is commutative, then $\xi(A) = A$ (by Lemma 3.2(iii)) and, with the conventions fixed in Remark 4.7, there is a canonical isomorphism of $A$-modules

$$\cap_{A}^{r} M \cong \text{Hom}_A\left(\bigwedge_{A}^{r} \text{Hom}_A(M, A), A\right).$$  

We recall that modules of this form were first considered (in the setting of group rings of abelian groups) by Rubin in [40] in order to formulate refined versions of Stark's Conjecture. These lattices are in turn a special case of the formalism of 'exterior power biduals' that
The basic properties of reduced Rubin lattices in the general case are recorded in the next result.

**Theorem 4.17.** For each finitely generated $A$-module $M$ and each non-negative integer $r$ the following claims are valid.

(i) If $r = 0$, then $\bigcap^r_A M = \xi(A)$.

(ii) The module $\bigcap^r_A M$ is both finitely generated and torsion-free over $R$. It is also independent, in a natural sense, of the choice of $E$-bases (of the simple $A_E$-modules $V$) that occur in the definition \( \{4.2.3\} \) of exterior powers.

(iii) For every prime ideal $p$ of $R$ one has $\left( \bigcap^r_A M \right)_{(p)} = \bigcap^r_{A_{(p)}} M_{(p)}$. Hence one has

$$\bigcap^r_A M := \bigcap_p \left( \bigcap^r_{A_{(p)}} M_{(p)} \right),$$

where the intersection runs over all $p$ and takes place in $\left( \bigcap^r_A M \right)_F$.

(iv) An injective homomorphism $\iota : M \to M'$ of finitely generated $A$-modules induces injective homomorphisms of $\xi(A)$-modules

$$\iota^r_{F,F} : \bigwedge^r_A M_F \to \bigwedge^r_{A_F} M'_F \quad \text{and} \quad \iota^r_{-} : \bigcap^r_A M \to \bigcap^r_{A_F} M'.$$

If $\operatorname{Ext}_A^1(cok(\iota), A)$ vanishes, then one has

$$\iota^r_{-} \left( \bigcap^r_A M \right) = \iota^r_{F,F} \left( \bigwedge^r_A M \right) \cap \bigcap^r_{A_F} M'.$$

(v) Let $s$ be a natural number with $s \leq r$ and $\{\varphi_i\}_{1 \leq i \leq s}$ a subset of $\operatorname{Hom}_A(M, A)$. Then the map

$$\bigwedge^r_A M_F \to \bigwedge^{r-s}_A M_F; \quad x \mapsto (\bigwedge_{1 \leq i \leq s} \varphi_i)(x)$$

sends $\bigcap^r_A M$ into $\bigcap^{r-s}_A M$.

(vi) If $M$ is a free $A$-module of rank $d$ with $d \geq r$, then for any choice of basis $b = \{b_j\}_{1 \leq j \leq d}$ of $M$ there is a natural split surjective homomorphism of $\xi(A)$-modules

$$\theta_b : \bigcap^r_A M \to \bigoplus_{\sigma \in \mathcal{P}} \xi(A).$$

This homomorphism is bijective if and only if either $A$ is commutative or $r = d$.

**Proof.** To prove claim (i) we fix for each index $i$ that occurs in the Wedderburn decomposition \( \{2.11\} \) of $A$ a splitting field $E_i$ for $A_i$ over $\zeta(A_i)$. Then, with respect to these choices, one has $\bigwedge^0_A M_F = \bigoplus_{i \in I} E_i \otimes_{\zeta(A_i)} A_i$ and, by convention, the exterior power of the empty subset of $\operatorname{Hom}_A(M, A)$ is the identity element of the algebra $\bigwedge^0_{A_{op}} \operatorname{Hom}_A(M_F, A) = \bigoplus_{i \in I} E_i \otimes_{\zeta(A_i)} A_{op}$. In view of these descriptions, claim (i) follows directly from the explicit definition of $\bigcap^r_A M$ in the case $r = 0$.

Given this result, we can assume in the rest of the proof that $r > 0$. It is also convenient to prove claim (iv) next. To do this we note that the existence of an injective homomorphism of $\zeta(A)$-modules $\iota^r_{F,F}$ of the stated form is a consequence of the fact that for every simple
$A_E$-module $V$ the given map $\iota$ induces an injective homomorphism of $E$-vector spaces $V^* \otimes_{A_E} M_E \to V^* \otimes_{A_E} M'_E$.

To prove the rest of claim (iv) we apply the functor $\text{Hom}_A(-, A)$ to the tautological exact sequence $0 \to M' \xrightarrow{\iota} M' \xrightarrow{\text{cok}(\iota)} 0$ to obtain an exact sequence

$$\text{Hom}_A(M', A) \xrightarrow{\iota^*} \text{Hom}_A(M, A) \to \text{Ext}^1_A(\text{cok}(\iota), A).$$

The fact that the restriction of $\iota_E^{\ast}$ to $\bigwedge^r A M$ factors through the inclusion $\bigwedge^r_A M' \subset \bigwedge^r_A M'_F$, thereby inducing a map $\iota^r$ of the required form, is then a consequence of the fact that for each $x$ in $\bigwedge^r_A M$ and each subset $\{\phi_i^r\}_{1 \leq i \leq r}$ of $\text{Hom}_A(M', A)$ one has

$$(\bigwedge_{i=1}^r \phi_i^r)(\iota_E^r(x)) = \bigwedge_{i=1}^r (\iota^r(\phi_i^r))(x).$$

This equality also implies the final assertion of claim (iii) since if $\text{Ext}_A^1(\text{cok}(\iota), A)$ vanishes, then the exact sequence displayed above implies $\iota^r$ is surjective.

Turning now to claim (ii), it is clear that $\bigwedge^r A M$ is $R$-torsion-free and to prove it is finitely generated we observe there exists a natural number $d$ and an injective homomorphism of $A$-modules of the form $M_{t_f} \to A^d$. To justify this we note that the semisimplicity of $A$ implies the existence, for any large enough $d$, of an injective homomorphism of $A$-modules $\iota^r : M_F \to A^d$. Hence, since $M_{t_f}$ is a finitely generated $R$-submodule of $M_F$, there exists a non-zero element $x$ of $R$ such that the composite homomorphism $M_{t_f} \subset M_F \xrightarrow{\iota^r} A^d \xrightarrow{x} A^d$ factors through the inclusion $A^d \subset A^d$ and so gives an injective homomorphism $\iota : M_{t_f} \to A^d$ of the required form.

Upon applying claim (iv) to $\iota$ one sees that it is enough to prove the finite generation of $\bigwedge^r_A M = \bigwedge^r_A M_{t_f}$ in the case $M = A^d$. To deal with this case we fix a splitting field $E$ for $A$ that has finite degree over $F$ and set $A_E := E \otimes_{\zeta(A)} A$. Then Lemma 4.12 implies it is enough to prove $\bigwedge^r_{A_E} \text{Hom}_A(A^d, A)$ is the $\zeta(A_E)$-linear span of elements of the form $\bigwedge_{i=1}^r \phi_i$ as $\phi_i$ ranges over $\text{Hom}_A(A, M)$. In view of Lemma 4.12 it thus suffices to show that for any subset $\{\phi_j^r\}_{1 \leq j \leq r}$ of $\text{Hom}_{A_E}(A^d, A_E)$ there exists a subset $\{\phi_i\}_{1 \leq i \leq r}$ of $\text{Hom}_A(A^d, A)$ and an element $x$ of $\zeta(A_E)$ such that $\bigwedge_{j=1}^r \phi_j^r = x \cdot \bigwedge_{i=1}^r \phi_i$.

To prove this we show, as we may, a free $A_E^{op}$-submodule $\Theta$ of $\text{Hom}_{A_E}(A^d, A_E)$ that has rank $r$ and contains $\{\phi_j^r\}_{1 \leq j \leq r}$. Then $\Theta \cap \text{Hom}_A(A^d, A)$ is a free $A_E^{op}$-module of rank $r$ and we can choose a basis $\{\phi_i\}_{1 \leq i \leq r}$ contained in $\text{Hom}_A(A^d, A)$. Then $\{\phi_i\}_{1 \leq i \leq r}$ is an $A_E^{op}$-basis of $\Theta$ and, writing $\lambda$ for the $A_E^{op}$-module endomorphism of $\Theta$ that sends each element $\phi_j$ to $\phi_j$, Lemma 4.13 implies that $\bigwedge_{j=1}^r \phi_j^r = x \cdot (\bigwedge_{i=1}^r \phi_i) = x \cdot (\zeta(A_E))$ with $x = \text{Nrd}_{A_E^{op}}(\Theta)(\lambda) \in \zeta(A_E)$, as required. This therefore completes the proof that $\bigwedge^r_A M$ is finitely generated.

To (make precise and) prove the second assertion of claim (ii) we assume to be given some other choice of ordered $E$-bases $\{\xi_V\}_V$ of the simple $A_E$-modules $V$ and write $\tau$ for the automorphism of the $\zeta(A_E)$-module $\bigwedge^r_A M$ that sends each element $\lambda_{j=1}^r m_j$ to $\tau_{j=1}^r m_j$, where each $m_j$ belongs to $M$ and $\tau$ indicates that the exterior power is defined with respect to the bases $\{\xi_V\}_V$. Then, writing $\bigwedge^r_A M$ for the reduced Rubin lattice defined using exterior powers with respect to $\{\xi_V\}_V$, Remark 4.11 implies that $\tau(\bigwedge^r_A M) = \bigwedge^r_A M$. 
We note next that $\text{Hom}_A(M_1,M_2)$ is contained in $\text{Hom}_A(M_1,M_2)$ (by Lemma 3.2(ii)) it is easily seen that $(\cap_1^r A_i)$ contains $\cap_1^r A_i(p)$.

To show the reverse inclusion we fix an element $a$ of $(\cap_1^r A_i)$. Then Lemma 4.13 implies

$$(\cap_1^r A_i) = \text{Hom}_A(M_1,M_2),$$

where the containment is valid because $a$ belongs to $(\cap_1^r A_i)$ and the last equality because $\text{Hom}_A(M_1,M_2)$ is a unit of $(\cap_1^r A_i)$. This shows $\cap_1^r A_i(p)$ contains $\cap_1^r A_i$ and hence completes the proof that $\cap_1^r A_i(p) = (\cap_1^r A_i)_p$. Given this, the displayed equality in claim (iii) then follows directly from the general result of [16] Prop. (4.21)(vi).

Next we note that claim (v) is true because the definition of the lattice $\cap_1^r A_i$ ensures that for any subset $\{b_j\}_{1 \leq j \leq r}$ of $\text{Hom}_A(M_1,M_2)$ and any $x$ in $\cap_1^r A_i$ one has

$$(\cap_1^r A_i) \rightarrow (\cap_1^r A_i)_p$$

for all $x$ in $\cap_1^r A_i$.

We also write $\text{Hom}_A(M_1,M_2)$ for the homomorphism of $\text{Hom}_A(M_1,M_2)$-modules that satisfies

$$\theta_b(x) = ((\cap_1^r A_i)_p(\cap_1^r A_i))_{\sigma \in [\tau]}$$

for all $x$ in $\cap_1^r A_i$.

Then Lemma 4.14 implies that $(\cap_1^r A_i)_p(\cap_1^r A_i)$ is a unit of $(\cap_1^r A_i)_p$ and so the composite $\theta_b \circ \theta_b'$ is the identity on $\cap_1^r A_i$. This shows that $\theta_b'$ is a section to $\theta_b$, as required.

Next we note that if $A$ is commutative, then $\cap_1^r A_i = \cap_1^r A_i(p)$ (as $M$ is free) and $\text{Hom}_A(M_1,M_2) = A$ and using these equalities it is easily seen that $\theta_b$ is an isomorphism.

To complete the proof of claim (vi) it is therefore enough to fix a primitive central idempotent $e$ of $A$ for which $eA$ is not commutative and to show that $e(F \otimes R \ker(\theta_b))$ vanishes if and only if $r = d$.

We fix a splitting field for $Ae$ and a simple $E \otimes \text{Hom}_A(M_1,M_2)$ module $V$. We set $n := \dim(V)$ so that $n > 1$. Then $V^* \otimes_A M_E$ is an $E$-vector space of dimension $nd$ and so $e(\cap_1^r A_i)$ spans an $E$-vector space of dimension $(\frac{d}{r})$.

Since $e\left(\bigoplus_{\sigma \in [\tau]} \text{Hom}_A(M_1,M_2)\right)$ spans an $E$-vector space of dimension $(\frac{d}{r})$ it is therefore enough to show that $(\frac{d}{r})$ is equal to $(\frac{d}{r})$ if and only if $r = d$ and we leave this as an (easy) exercise for the reader.
4.4.2. Lemma 4.14 gives rise to a useful construction of elements in reduced Rubin lattices. To describe this we identify each matrix \( M \) in \( M_{d',d}(A) \) with the homomorphism of \( A \)-modules

\[
\theta_M : A^{d'} \to A^d
\]

that sends each (row) vector \( x \) to \( x \cdot M \).

For each primitive central idempotent \( e \) of \( A \) we fix a non-zero simple (left) \( A \)-module \( V(e) \) upon which \( e \) acts as the identity and we write \( D(e) \) for the associated division ring \( \text{End}_A(V(e)) \). (We recall that such a module \( V(e) \) is unique up to isomorphism.)

Then Remark 2.3 implies that for any non-negative integer \( r \) and any finitely generated \( A \)-module \( W \), one has \( \text{rr}_A(eW) = \text{rr}_A((Ae)^r) \) if and only if \( V(e) \) occurs with exact multiplicity \( r \cdot \dim_{D(e)}(V(e)) \) in the Wedderburn decomposition of \( W \).

**Proposition 4.18.** Fix natural numbers \( d' \) and \( d \) with \( d' > d \) and set \( r := d' - d > 0 \). Then for each matrix \( M \) in \( M_{d',d}(A) \) for which \( \text{Ext}_A^1(\text{im}(\theta_M),A) \) vanishes, there exists a canonical element \( \epsilon_M \) of \( \bigcap_A \ker(\theta_M) \) that has both of the following properties.

(i) For each primitive central idempotent \( e \) of \( A \) one has

\[
e \cdot \epsilon_M \neq 0 \iff \text{rr}_A(e(\ker(\theta_M)_F)) = \text{rr}_A((Ae)^r).
\]

(ii) Write \( 0_{d',r} \) for the \( d' \times r \) zero matrix. Then for the block matrix \( (0_{d',r} \mid M) \) in \( M_{d',d}(A) \) one has

\[
\text{Fit}_A^0((0_{d',r} \mid M)) = \xi(A) \cdot \{(\Lambda_{i=1}^r \varphi_i)(\epsilon_M) : \varphi_i \in \text{Hom}_A(\ker(\theta_M),A)\}.
\]

In particular, for each subset \( \{\varphi_i\}_{1 \leq i \leq r} \) of \( \text{Hom}_A(\ker(\theta_M),A) \) one has

\[
(\Lambda_{i=1}^r \varphi_i)(\epsilon_M) \in \text{Fit}_A^0(\text{cok}(\theta_M)).
\]

**Proof.** For each integer \( i \) with \( 1 \leq i \leq d \) we write \( \theta_M^i \) for the element of \( \text{Hom}_A(A^{d'},A) \) that sends \( x \) to the \( i \)-th component of the element \( \theta_M(x) \) of \( A^d \).

Then Lemma 4.14 implies that the image of the homomorphism

\[
\Theta_M : \bigcap_A^d A^{d'} \rightarrow \bigcap_A^r A^{d'}, \quad b \mapsto (\Lambda_{1 \leq i \leq d} \theta_M^i)(b)
\]

coming from Theorem 4.17(iv) is contained in the submodule \( \bigcap_A^r A^{d'} \).

Thus, since \( \text{Ext}_A^1(\text{im}(\theta_M),A) \) is assumed to vanish, Theorem 4.17(iii) implies that \( \text{im}(\Theta_M) \) is contained in \( \bigcap_A^r \ker(\theta_M) \).

In particular, if we write \( \{b_i\}_{1 \leq i \leq d'} \) for the canonical basis of \( A^{d'} \), then \( \Lambda_{i=1}^{d'} b_i \) belongs to \( \bigcap_A^d A^{d'} \) (by Lemma 4.10) and so we can define

\[
\epsilon_M := \Theta_M(\Lambda_{i=1}^{d'} b_i) \in \bigcap_A^r \ker(\theta_M).
\]

We now fix a primitive central idempotent \( e \) of \( A \) and a splitting field \( E \) for the simple algebra \( Ae \). Then one has \( \text{rr}_A(e(\ker(\theta_M)_F)) \geq \text{rr}_A((Ae)^r) \) and so the \( E \)-space \( e \cdot \bigcap_A^r \ker(\theta_M)_F \) does not vanish. Thus, since \( e \cdot \Lambda_{i=1}^{d'} b_i \) is an \( E \)-basis of \( e \cdot \bigcap_A^d A^{d'} \), the final assertion of Lemma 4.14 implies that \( e \cdot \epsilon_M \neq 0 \) if and only if one has \( \text{rr}_A(e(\ker(\theta_M)_F)) = \text{rr}_A((Ae)^r) \), as required to prove claim (i).
To prove claim (ii), we note first that the assumed vanishing of $\text{Ext}^1_A(\text{im}(\theta_M), A)$ implies the restriction map $\text{Hom}_A(A^{d',} A) \to \text{Hom}_A(\ker(\theta_M), A)$ is surjective. This fact combines with the definition of $\varepsilon_M$ and the result of Lemma 4.10 to imply that

\begin{equation}
(4.4.1) \quad \xi(A) \cdot \{(\wedge_{i=1}^r \varphi_i)(\varepsilon_M) : \varphi_i \in \text{Hom}_A(\ker(\theta_M), A)\}
= \xi(A) \cdot \{\text{Nrd}_A((M' \mid M)) : M' \in M_{d',r}(A)\},
\end{equation}

and Definition 3.9 implies directly that the latter ideal is equal to $\text{Fit}^0_A((0_{d',r} \mid M))$.

For each $M'$ in $M_{d',r}(A)$ we write $\theta_{M',M}$ for the endomorphism of $A^{d'}$ represented, with respect to the standard basis, by the block matrix $(M' \mid M)$. Then $\text{Nrd}_A((M' \mid M))$ belongs to $\text{Fit}^0_A(\text{cok}(\theta_{M',M}))$ (see Definition 3.14) and so Theorem 3.17(vi) implies that the final assertion of claim (ii) will follow as a consequence of (4.4.1) if there exists a surjective homomorphism of $A$-modules from $\text{cok}(\theta_{M',M})$ to $\text{cok}(\theta_M)$. The existence of such a homomorphism is in turn a consequence of the commutative diagram of $A$-modules

\[
\begin{array}{ccc}
A^{d'} & \xrightarrow{\theta_{M',M}} & A^{d'} \\
\parallel & & \parallel \\
A^{d'} & \xrightarrow{\theta_M} & A^{d}
\end{array}
\]

in which $\varrho$ is the (surjective) map that sends $b_i$ for each $1 \leq i \leq r$, respectively $r < i \leq d'$, to zero, respectively to the $(i-r)$-th element of the standard basis of $A^{d}$. □

5. Reduced determinant functors

The theory of determinant functors for complexes of modules over commutative noetherian rings was established by Knudsen and Mumford in [22], with later clarifications provided by Knudsen in [23], in both cases following suggestions of Grothendieck.

It was subsequently shown by Deligne in [17] that the category of ‘virtual objects’ provides a universal determinant functor for any exact category (cf. Remark 5.5). For the category of projective modules over certain non-commutative rings, Deligne’s construction has played a key role in the formulation of refined ‘special value conjectures’ in arithmetic, such as the equivariant Tamagawa number conjecture from [5].

The latter conjecture takes the form of an equality in a relative algebraic $K$-group, and an alternative approach to the formulation of conjectures in such groups was later described by Fukaya and Kato in [19] via a theory of ‘localized $K_1$-groups’.

In this section we shall use the theory of reduced Rubin lattices to prove the existence of a ‘reduced determinant functor’ on the derived category of perfect complexes over a non-commutative order that is much more explicit than either of the approaches of Deligne or Fukaya and Kato (but will depend on the same sort of auxiliary data as was fixed in our construction of reduced exterior powers in [11].

This determinant functor will later play a key role in our construction of non-commutative Euler systems.

In addition, in the supplementary article [13], the reduced determinant functor constructed here will be used to define a natural non-commutative generalization of the notion of ‘zeta element’ that underlies all of the work of Kurihara and the present authors in
and thereby to shed new light on the content of the equivariant Tamagawa number conjecture relative to non-abelian Galois extensions.

Throughout this section we fix data $R, F, A$ and $A$ as in $\mathbb{3}$

5.1. **Statement of the main result.**

5.1.1. For a commutative ring $\Lambda$ we write $\mathcal{P}(\Lambda)$ for the category of graded invertible $\Lambda$-modules. This is a Picard category and, for the reader’s convenience, we first quickly review its basic properties.

An object of $\mathcal{P}(\Lambda)$ comprises a pair $(L, \alpha)$ where $L$ is an invertible $\Lambda$-module and $\alpha$ is a continuous function from $\text{Spec}(\Lambda)$ to $\mathbb{Z}$. (Here we recall that an $\Lambda$-module $L$ is said to be ‘invertible’ if for every prime ideal $\wp$ of $\Lambda$ the $\Lambda_{(\wp)}$-module $L_{(\wp)}$ is free of rank one.)

A homomorphism $\theta : (L, \alpha) \to (M, \beta)$ in $\mathcal{P}(\Lambda)$ is a homomorphism of $\Lambda$-modules such that $\theta_{(\wp)} = 0$ whenever $\alpha(\wp) \neq \beta(\wp)$. The tensor product of two objects $(L, \alpha)$ and $(M, \beta)$ in $\mathcal{P}(\Lambda)$ is given by

$$(L, \alpha) \otimes (M, \beta) = (L \otimes_\Lambda M, \alpha + \beta)$$

and for each such pair there is an isomorphism

$$\psi_{(L, \alpha), (M, \beta)} : (L, \alpha) \otimes (M, \beta) \cong (M, \beta) \otimes (L, \alpha)$$

in $\mathcal{P}(\Lambda)$ such that for every $\wp$ and every $\ell$ in $L_{(\wp)}$ and $m$ in $M_{(\wp)}$ one has

$$\psi_{(L, \alpha), (M, \beta)}(\ell \otimes m) = (-1)^{\alpha(\wp) \cdot \beta(\wp)} \cdot (m \otimes \ell).$$

The unit object $1_{\mathcal{P}(\Lambda)}$ is the pair $(\Lambda, 0)$ and the natural ‘evaluation map’ isomorphism $L \otimes_\Lambda \text{Hom}_\Lambda(L, \Lambda) \cong \Lambda$ induces an isomorphism in $\mathcal{P}(\Lambda)$

$$(L, \alpha) \otimes (\text{Hom}_\Lambda(L, \Lambda), -\alpha) \cong 1_{\mathcal{P}(\Lambda)}.$$

This isomorphism is used to regard $(\text{Hom}_\Lambda(L, \Lambda), -\alpha)$ as a right inverse to $(L, \alpha)$ and it is then also regarded as a left inverse by means of the isomorphism $\psi_{(\text{Hom}_\Lambda(L, \Lambda), -\alpha), (L, \alpha)}$.

5.1.2. In the sequel we write the Wedderburn decomposition of $A$ as $\prod_{i \in I} A_i$ so that each algebra $A_i$ is of the form $M_{n_i}(D_i)$ for a division ring $D_i$.

For each index $i$ we choose a splitting field $E_i$ for $D_i$ so that $D_i \otimes_{\zeta(D_i)} E_i = M_{n_i}(E_i)$. We then fix an indecomposable idempotent $f_i$ of $M_{n_i}(E_i)$ and an $E_i$-basis $\{w_{ij}\}_{1 \leq a \leq n_i}$ of $W_i := f_i \cdot M_{n_i}(E_i)$. (When making such a choice we always follow the convention of Remark 4.1 on each simple component $A_i$ that is commutative.)

Then the direct sum $V_i := W_i^{n_i}$ of $n_i$-copies of $W_i$ is a simple left $A_i \otimes_{\zeta(A_i)} E_i$-module and has as an $E_i$-basis the set $\overline{w}_i = \{\overline{w}_{ij}\}_{1 \leq a \leq n_i, 1 \leq j \leq m_i}$ where $\overline{w}_{ij}$ denotes the element of $V_i$ that is equal to $w_{ij}$ in its $a$-th component and is zero in all other components.

We order each set $\overline{w}_i$ lexicographically and will apply the constructions of $\mathbb{4}$ with respect to the collection $\overline{w}$ of ordered bases $\{\overline{w}_i : i \in I\}$.

The following straightforward observation will also be useful.

**Lemma 5.1.** Let $\mathcal{R}$ be an $R$-order in $\zeta(A)$. Then the reduced rank $\text{rr}_A(Z)$ of a finitely generated $A$-module $Z$ determines a locally-constant function on $\text{Spec}(\mathcal{R})$. 
Proof. The maximal $R$-order $\mathcal{M}$ in $\zeta(A)$ is $\prod_{i \in I} \mathcal{O}_i$, where $\mathcal{O}_i$ is the integral closure of $R$ in the field $\zeta(A_i)$.

Since the inclusion $\mathcal{R} \to \mathcal{M}$ is an integral ring extension, the going-up theorem implies that every prime ideal of $\mathcal{A}$ has the form $\mathfrak{p} = A \cap \mathfrak{p}'$ for a prime ideal $\mathfrak{p}'$ of $\mathcal{M}$, and then $A_{(\mathfrak{p})}$ is a finite index subgroup of $\mathcal{M}(\mathfrak{p}')$. The key point now is that there exists a unique index $i(\mathfrak{p})$ in $I$ such that the kernel of the projection $\mathcal{M} \to \mathcal{O}_{i(\mathfrak{p})}$ is contained in $\mathfrak{p}'$ and hence that $(A_{(\mathfrak{p})})_F = (\mathcal{M}(\mathfrak{p}'))_F$ identifies with $A_{i(\mathfrak{p})}$.

One then obtains a well-defined locally-constant function on $\text{Spec}(\mathcal{R})$ by sending each $\mathfrak{p}$ to the $i(\mathfrak{p})$-th component $\text{rr}_{A_{i(\mathfrak{p})}}(A_{i(\mathfrak{p})} \otimes_A Z)$ of $\text{rr}_A(Z)$. \hfill $\square$

5.1.3. We write $D(\mathcal{A})$ for the derived category of (left) $\mathcal{A}$-modules. We also write $C^{\text{lf}}(\mathcal{A})$ for the bounded complexes of objects of the category $\text{Mod}^{\text{lf}}(\mathcal{A})$ of locally-free $\mathcal{A}$-modules, as discussed in §3.2, and $D^{\text{lf}}(\mathcal{A})$ for the full triangulated subcategory of $D(\mathcal{A})$ comprising complexes that are isomorphic to an object of $C^{\text{lf}}(\mathcal{A})$. We then write $D^{\text{lf}}(\mathcal{A})_{\text{is}}$ for the subcategory of $D^{\text{lf}}(\mathcal{A})$ in which morphisms are restricted to be isomorphisms.

We next write $K^0(\mathcal{A})$ for the Grothendieck group of the category of finitely generated locally-free $\mathcal{A}$-modules. We observe that each object $C$ of $D^{\text{lf}}(\mathcal{A})$ gives rise to a canonical ‘Euler characteristic’ in $K^0(\mathcal{A})$ and we write this element as $\chi_A(C)$.

We recall that the ‘reduced locally-free classgroup’ $SK_0(\mathcal{A})$ of $\mathcal{A}$ is defined to be the kernel of the homomorphism $K^0(\mathcal{A}) \to \mathbb{Z}$ that is induced by sending each locally-free module $M$ to $\text{rk}_A(M)$.

We write $C^{\text{lf},0}(\mathcal{A})$ for the subcategory of $C^{\text{lf}}(\mathcal{A})$ comprising complexes $P^\bullet$ for which $\chi_A(P^\bullet)$ belongs to $SK_0(\mathcal{A})$ and $D^{\text{lf},0}(\mathcal{A})$ for the full triangulated subcategory of $D^{\text{lf}}(\mathcal{A})$ comprising complexes $C$ for which $\chi_A(C)$ belongs to $SK_0(\mathcal{A})$. (The latter condition is equivalent to requiring that $C$ be isomorphic in $D(\mathcal{A})$ to an object of $C^{\text{lf},0}(\mathcal{A})$.)

Finally, we note that the concept of ‘extended determinant functor’ is made precise in Definition 5.10 below.

We can now state the main result of §5.

**Theorem 5.2.** For each set of ordered bases $\varpi$ as above, there exists a canonical extended determinant functor

$$d_{A,\varpi} : D^{\text{lf}}(\mathcal{A})_{\text{is}} \to \mathcal{P}(\zeta(\mathcal{A}))$$

that has all of the following properties.

(i) For each exact triangle

$$C' \xrightarrow{u} C \xrightarrow{v} C'' \xrightarrow{w} C'[1]$$

in $D^{\text{lf}}(\mathcal{A})$ there exists a canonical isomorphism

$$d_{A,\varpi}(C') \otimes d_{A,\varpi}(C'') \xrightarrow{\sim} d_{A,\varpi}(C)$$

in $\mathcal{P}(\zeta(\mathcal{A}))$ that is functorial with respect to isomorphisms of triangles.

(ii) Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a surjective homomorphism of $R$-orders and $\varphi_* : \zeta(\mathcal{A}) \to \zeta(\mathcal{B})$ the induced homomorphism. Then for each $C$ in $D^{\text{lf}}(\mathcal{A})$ the complex $\mathcal{B} \otimes_{\mathcal{A},\varphi} C$ belongs to $D^{\text{lf}}(\mathcal{B})$ and there exists a canonical isomorphism

$$\zeta(\mathcal{B}) \otimes_{\zeta(\mathcal{A}),\varphi_*} d_{A,\varpi}(C) \cong d_{B,\varpi'}(\mathcal{B} \otimes_{\mathcal{A},\varphi} C)$$
in \( P(\xi(B)) \), where \( \varpi' \) is the ordered subset of \( \varpi \) that corresponds to all simple modules \( V_i \) that factor through the scalar extension of \( \varrho \).

(iii) If \( P \) belongs to \( \text{Mod}^{lf}(A) \), then \( \bigcap_A^{rk_A(P)} P \) is an invertible \( \xi(A) \)-module and one has

\[
d_{A,\varpi}(P[0]) = \bigcap_A^{rk_A(P)} P, \text{rr}_A(P_F).
\]

Here the reduced Rubin lattice \( \bigcap_A^{rk_A(P)} P \) is defined with respect to \( \varpi \) and \( \text{rr}_A(P_F) \) is regarded as a locally-constant function on \( \text{Spec}(\xi(A)) \) via Lemma 5.1.

(iv) The restriction of \( d_{A,\varpi} \) to \( D^{lf,0}(A)_{\text{is}} \) is independent of the choice of \( \varpi \).

Our proof of this result adapts an argument used by Flach and the first author in [5, §2] and so is closely modelled on the original constructions of Knudsen and Mumford in [22].

Remark 5.3. Let \( R \) be any \( R \)-order in \( \xi(A) \) with the property that for each prime ideal \( p \) of \( R \) the localization \( R_p \) contains the reduced norms of all matrices in \( \bigcup_{n \geq 1} \text{GL}_n(A[p]) \).

(Note that \( \xi(A) \) is, by its very definition, an example of such an order \( R \) but that there can also, in principle, exist such orders that are strictly contained in \( \xi(A) \).) Then a closer analysis of our proof of Theorem 5.2 will show that there exists a determinant functor

\[
d_{A,\varpi,R} : D^{lf}(A)_{\text{is}} \to P(R)
\]

that satisfies analogues of all of the properties of \( d_{A,\varpi} \) listed above and is, in addition, such that if \( R' \) is any order in \( \xi(A) \) that contains \( R \), then one has

\[
d_{A,\varpi,R'} = \iota_{R',R} \circ d_{A,\varpi,R}
\]

where \( \iota_{R',R} \) is the natural scalar extension functor \( P(R) \to P(R') \). However, we shall make no use of this additional generality in the sequel and so, for simplicity, only consider \( \xi(A) \).

5.2. Determinant functors.

5.2.1. Let \( E \) be an exact category in the sense of Quillen [37, p. 91] and write \( E_{\text{is}} \) for the subcategory of \( E \) in which morphisms are restricted to isomorphisms.

Then the following definition is equivalent to that given in [5, §2.3].

Definition 5.4. A determinant functor on \( E \) is a Picard category \( P \), with unit object \( 1_P \) and product \( \boxtimes \), together with the following data.

(a) A covariant functor \( d : E_{\text{is}} \to P \).

(b) For each short exact sequence \( E' \overset{\alpha}{\to} E \overset{\beta}{\to} E'' \) in \( E \) a morphism

\[
i(\alpha,\beta) : d(E) \xrightarrow{\sim} d(E') \boxtimes d(E'')
\]

in \( P \) that is functorial for isomorphisms of short exact sequences.

(c) For each zero object \( 0 \) in \( E \) an isomorphism

\[
\zeta(0) : d(0) \xrightarrow{\sim} 1_P.
\]

This data is subject to the following axioms.
(d) For each isomorphism $\phi : E \to E'$ in $\mathcal{E}$, the induced exact sequences

\[ 0 \to E \xrightarrow{\phi} E' \quad \text{and} \quad E \xrightarrow{\phi^{-1}} E' \to 0 \]

are such that $d(\phi)$ and $d(\phi^{-1})$ respectively coincide with the composite maps

\[ d(E) \xrightarrow{i(0,\phi)} d(0) \boxtimes d(E') \xrightarrow{\zeta(0) \boxtimes \text{id}} d(E') \]

and

\[ d(E') \xrightarrow{i(\phi,0)} d(E) \boxtimes d(0) \xrightarrow{\text{id} \boxtimes \zeta(0)} d(E) \]

(e) Given a commutative diagram of objects in $\mathcal{E}$

\[ E_1' \xrightarrow{\alpha'} E_2' \xrightarrow{\beta} E_3' \]
\[ \downarrow \gamma' \quad \downarrow \gamma \quad \downarrow \gamma'' \]
\[ E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \]
\[ \downarrow \delta' \quad \downarrow \delta \quad \downarrow \delta'' \]
\[ E_1'' \xrightarrow{\alpha''} E_2'' \xrightarrow{\beta''} E_3'' \]

in which each row and column is a short exact sequence, the diagram

\[ d(E_2) \xrightarrow{i(\gamma,\delta)} d(E_2') \boxtimes d(E_2') \]
\[ d(E_1) \boxtimes d(E_3) \xrightarrow{i(\gamma',\delta') \boxtimes i(\gamma'',\delta'')} d(E_1') \boxtimes d(E_1') \boxtimes d(E_3') \]

commutes.

**Remark 5.5.** The terminology of ‘determinant functor’ used above is borrowed from the key example in which $\mathcal{E}$ is the category of vector bundles on a scheme, $\mathcal{P}$ is the category of line bundles and the functor is taking the highest exterior power. However, as was shown by Deligne in [17, §4], there exists a universal determinant functor for any given exact category $\mathcal{E}$. More precisely, there exists a Picard category $V(\mathcal{E})$, called the ‘category of virtual objects’ of $\mathcal{E}$, together with data (a)-(c) which in addition to (d) and (e) also satisfies the following universal property.

(f) For any Picard category $\mathcal{P}$ the category of monoidal functors $\text{Hom}^\boxtimes(\mathcal{E}, \mathcal{P})$ is naturally equivalent to the category of determinant functors $\mathcal{E}_{\text{is}} \to \mathcal{P}$.

Although comparatively inexplicit, this construction has played a key role in the formulation of special value conjectures in the literature.

The category $\text{Mod}^{\text{lf}}(A)$ is exact (in the sense of [17]) and in the remainder of this section we shall construct a canonical determinant functor

\[ d^\circ_{A,\omega} : \text{Mod}^{\text{lf}}(A)_{\text{is}} \to \mathcal{P}(\xi(A)). \]
5.2.2. We start by establishing several useful technical properties of the reduced Rubin lattices of modules in $\text{Mod}^\lf(A)$.

In the following result we assume that all reduced exterior power constructions are made with respect to the fixed bases $\mathbb{E}$ but will not usually indicate this dependence explicitly.

**Proposition 5.6.** Fix an object $P$ of $\text{Mod}^\lf(A)$ and set $r := \text{rk}_A(P)$. Then the following claims are valid.

(i) If $P$ is a free $A$-module, with basis $\{b_j\}_{1 \leq j \leq r}$, then $\bigcap_A^r P$ is a free rank one $\xi(A)$-module with basis $\wedge_{j=1}^r b_j$.

(ii) For each prime ideal $p$ of $R$ fix an $A(p)$-basis $\{b_{pj}\}_{1 \leq j \leq r}$ of $P(p)$. Then the $\xi(A(p))$-module $(\bigcap_A^r P)|_p$ is free of rank one, with basis $\wedge_{j=1}^r b_{pj}$. Hence one has

$$\bigcap_A^r P = \bigcap_p (\xi(A(p)) \cdot \wedge_{j=1}^r b_{pj}),$$

where the intersection runs over all $p$ and takes place in $(\bigcap_A^r P)_F$.

(iii) $\bigcap_A^r P$ is an invertible $\xi(A)$-module.

(iv) Let $\varrho : A \to B$ be a surjective homomorphism of $R$-orders. Write $B$ for the $F$-algebra spanned by $B$ and $\varrho_1 : A \to B$, $\varrho_2 : \xi(A) \to \xi(B)$ and $\varrho_3 : \xi(A) \to \xi(B)$ for the surjective ring homomorphisms induced by $\varrho$. Then $B \otimes_{A,\varrho} P$ is a locally-free $B$-module and the natural isomorphism of $\xi(B)$-modules $\xi(B) \otimes_{\xi(A)} \varrho_2 \bigwedge_A^r P_F \cong \bigwedge_B^r (B \otimes_{A,\varrho} P_F)$ restricts to give an isomorphism of invertible $\xi(B)$-modules $\xi(B) \otimes_{\xi(A)} \varrho_1 \bigwedge_A^r P \cong \bigwedge_B^r (B \otimes_{A,\varrho} P)$, where the exterior powers in the latter module are defined with respect to the same ordered $E$-bases of those simple $A_E$-modules that factor through the scalar extension of $\varrho_1$.

(v) If $P_1 \xrightarrow{\theta} P_2 \xrightarrow{\phi} P_3$ is a (split) short exact sequence in $\text{Mod}^\lf(A)$, and we set $r_i := \text{rk}_A(P_i)$ for $i = 1, 2, 3$, then there exists an isomorphism of $\xi(A)$-modules

$$i_{\varpi}^\varphi(\theta, \phi) : \bigcap_A^{r_2} P_2 \cong \bigcap_A^{r_1} P_1 \otimes_{\xi(A)} \bigcap_A^{r_3} P_3$$

that has the following properties:

(a) $i_{\varpi}^\varphi(\theta, \phi)$ is functorial with respect to isomorphisms of short exact sequences;

(b) If $P_3 \xrightarrow{\varphi'} P_2 \xrightarrow{\phi'} P_1$ is any exact sequence of $A$-modules obtained by choosing a splitting of the given sequence, then the following diagram commutes

$$\begin{align*}
\bigcap_A^{r_2} P_2 & \xrightarrow{i_{\varpi}^\varphi(\theta, \phi)} \bigcap_A^{r_1} P_1 \otimes_{\xi(A)} \bigcap_A^{r_3} P_3 \\
\bigcap_A^{r_2} P_2 & \xrightarrow{i_{\varpi}^\varphi(\varphi', \theta)} \bigcap_A^{r_3} P_3 \otimes_{\xi(A)} \bigcap_A^{r_1} P_1.
\end{align*}$$

Here $\alpha$ is the element $((-1)^{\rho_{j,i}} - \rho_{j+1,i})_{i \in I}$ of $\prod_{i \in I} \xi(A_i) = \xi(A)$, where $\rho_{j,i}$ denotes the $i$-th component of the reduced rank $\text{rk}_A(P_{j,F})$.

**Proof.** Claim (i) follows directly from the proof of Theorem 4.17(vi). (We note also that Lemma 4.13 implies that the $\xi(A)$-module generated by $\wedge_{j=1}^r b_j$ is indeed independent of the choice of basis $\{b_j\}_{1 \leq j \leq r}$.)
After replacing $\mathcal{A}$ and $P$ by $\mathcal{A}(p)$ and $P(p)$ for a prime ideal $p$, the same argument implies that the $\xi(\mathcal{A}(p))$-module $\bigcap_{j=1}^{r} P_{p,j}$ is free of rank one, with basis $\bigwedge_{j=1}^{r} b_{p,j}$ where the elements $b_{p,j}$ are chosen as in claim (ii). Given this fact, claim (ii) follows directly from the general result of Theorem 4.17(iii).

To prove claim (iii) we fix a prime ideal $\varnothing$ of $\xi(\mathcal{A})$. Then $p := R \cap \varnothing$ is a prime ideal of $R$ and by Roiter’s Lemma (cf. [16, Lem. (31.6)]) there exists a free $\mathcal{A}$-submodule $P'$ of $P$ such that $P(p) = P'(p)$. This equality combines with Theorem 4.17(iii) to imply that

$$\left(\bigcap_{j=1}^{r} P\right)(p) = \left(\left(\bigcap_{j=1}^{r} P\right)(p)\right)(p) = \left(\left(\bigcap_{j=1}^{r} P\right)(p)\right)(p) = \left(\bigcap_{j=1}^{r} P\right)(p).$$

In particular, since claim (i) implies $\bigcap_{j=1}^{r} P'$ is a free $\xi(\mathcal{A})$-module of rank one, the $\xi(\mathcal{A}(p))$-module $\left(\bigcap_{j=1}^{r} P\right)(p)$ is also free of rank one, as required to prove claim (iii).

Claim (iv) is verified by a straightforward exercise and, for brevity, we leave this to the reader.

Turning to claim (v) we fix an $\mathcal{A}$-module section $\sigma$ to $\varnothing$. We note that the given exact sequence implies $r_2 = r_1 + r_3$ and also that for any given $\mathcal{A}$-basis $b_j$ := $\{b_{j,a}\}_{1 \leq a \leq r_j}$ of $P_{j,F}$ for $j = 1, 3$ we obtain an $\mathcal{A}$-basis $b_{j,F}$ := $\{b_{j,a}\}_{1 \leq a \leq r_j}$ of $P_{j,F}$ by setting $b_{j,a} := \theta(b_{j,a})$ if $1 \leq a \leq r_j$ and $b_{j,a} := \sigma_i(b_{j,a-r_j})$ if $r_j < a \leq r_j$.

Write $E$ for the algebra $\prod_{i \in I} E_i$. Then for each $j \in \{1, 2, 3\}$, the $\mathcal{A}$-module $\bigcap_{i=1}^{r_j} P_{j,F}$ is free of rank one, with basis $\bigwedge_{a=1}^{t_j} b_{j,a}$, and so there is a unique isomorphism of $\mathcal{A}$-modules

$$\Delta : \bigcap_{i=1}^{r_2} P_{2,F} = \left(\bigcap_{i=1}^{r_1} P_{1,F}\right) \otimes E \left(\bigcap_{i=1}^{r_3} P_{3,F}\right)$$

that sends $\bigwedge_{j=1}^{r_j} b_{j,F}$ to $(\bigwedge_{s=1}^{r_1} b_{1,s}) \otimes (\bigwedge_{t=1}^{r_3} b_{3,t})$. In addition, by using Lemma 4.13, one checks that this isomorphism is independent both of the choices of bases $b_1$ and $b_3$ and of the choice of section $\sigma$.

In particular, if one fixes a prime ideal $p$ of $R$ and then chooses the elements $\{b_{1,s}\}_{1 \leq s \leq t_1}$ and $\{b_{3,t}\}_{1 \leq t \leq t_3}$ to be $\mathcal{A}(p)$-bases of $P_{1,p}$ and $P_{3,p}$, then the choice of $\sigma$ implies that the set $\{b_{2,j}\}_{1 \leq j \leq r_2}$ defined above is an $\mathcal{A}(p)$-basis of $P_{2,p}$ and so the explicit descriptions in claim (ii) imply that

$$\Delta\left(\bigcap_{i=1}^{r_2} P_i\right) = \Delta\left(\bigcap_{i=1}^{r_2} P_i\right) = \Delta\left(\bigcap_{i=1}^{r_2} P_i\right) = \Delta\left(\bigcap_{i=1}^{r_2} P_i\right).$$

Since this is true for all primes $p$, one therefore has $\Delta\left(\bigcap_{i=1}^{r_2} P_i\right) = \bigcap_{i=1}^{r_2} P_i \otimes \xi(\mathcal{A}) \bigcap_{i=1}^{r_3} P_3$ and so we can define $i_\varnothing^\omega(\theta, \varnothing)$ to be the isomorphism induced by restricting $\Delta$.

It is then straightforward to see that this isomorphism is functorial with respect to isomorphisms of short exact sequences, as required by claim (v)(a).

To justify the property in claim (v)(b) we set $\rho_{2,i} := \text{rr}_{\mathcal{A}}\left(A_i \otimes_{\mathcal{A}} P_{2,F}\right)$ for each index $i$ in $I$. Then the definition (2.1) of reduced rank combines with the given exact sequence to imply $\rho_{2,i} = \rho_{1,i} + \rho_{3,i}$. In addition, it combines with the explicit definitions of reduced exterior powers to imply $\bigwedge_{i=1}^{r_2} (A_i \otimes_{\mathcal{A}} P_{2,F}) = \bigwedge_{i=1}^{r_2} W_i$, with $W_i$ the $E_i$-space $V_i^{*} \otimes_{\mathcal{A}} \xi(\mathcal{A})_i E_i P_{2,e_i}$ of dimension $\rho_{2,i}$, and that the $i$-th components $(\bigwedge_{s=1}^{r_1} b_{2,s})_i$ and $(\bigwedge_{t=r_1}^{r_3} b_{2,t})_i$ of the elements
\(\wedge_{s=1}^{r_1} b_{2,s} \) and \(\wedge_{t=r_1+1}^{r_3} b_{2,t} \) are respectively the exterior products of \(\rho_{1,i} \) and \(\rho_{3,i} \) distinct elements of \(W_i \). In the space \(\bigwedge^j_{\Lambda}(A_i \otimes_A P_{2,\varnothing}) \) one therefore has

\[
(\wedge_{s=1}^{r_1} b_{2,s})_i \wedge (\wedge_{t=r_1+1}^{r_3} b_{2,t})_i = (-1)^{\rho_{1,i} \cdot \rho_{3,i}} \cdot (\wedge_{t=r_1+1}^{r_3} b_{2,t})_i \wedge (\wedge_{s=1}^{r_1} b_{2,s})_i.
\]

Taken together, these equalities imply that the diagram in claim (v)(b) commutes, as required to complete the proof. \(\square\)

**Remark 5.7.** Let \(P \) be a free \(A\)-module of rank one. If \(A\) is commutative, then there is a natural identification \(\bigwedge^1_A P \cong P\). In general, however, \(\bigwedge^1_A P\) is a module over \(\xi(A)\) and hence different from \(P\).

### 5.2.3. The results of Lemma 5.1 with \(R = \xi(A)\) and Proposition 5.6(iii) combine to imply that for each \(P\) in \(\text{Mod}^{\ell f}(A)\) one obtains a well-defined object of \(\mathcal{P}(\xi(A))\) by setting

\[
d^\circ_{A,\varnothing}(P) := (\bigwedge^i_{\Lambda} P, \tau_{\Lambda}(P_F)).
\]

For each short exact sequence \(P_1 \xrightarrow{\theta} P_2 \xrightarrow{\phi} P_3 \) in \(\text{Mod}^{\ell f}(A)\) the construction in Proposition 5.6(v) also gives rise to a commutative diagram of isomorphisms in \(\mathcal{P}(A)\) of the form

\[
\begin{array}{ccc}
d^\circ(P_2) & \xrightarrow{i^\circ_{\varnothing}(\theta, \phi)} & d^\circ(P_1) \otimes d^\circ(P_3) \\
\parallel & & \downarrow \psi_{d^\circ(P_1), d^\circ(P_3)} \\
d^\circ(P_2) & \xrightarrow{i^\circ_{\varnothing}(\phi', \psi')} & d^\circ(P_3) \otimes d^\circ(P_1).
\end{array}
\]

in which we abbreviate \(d^\circ_{A,\varnothing}\) to \(d^\circ\).

**Proposition 5.8.** The associations \(P \mapsto d^\circ_{A,\varnothing}(P)\) and \((\theta, \phi) \mapsto i^\circ_{\varnothing}(\theta, \phi)\) give a well-defined determinant functor \(d^\circ_{A,\varnothing} : \text{Mod}^{\ell f}(A)_{\varnothing} \to \mathcal{P}(\xi(A))\).

In addition, for any homomorphism \(\varrho : A \to B\) as in Theorem 5.2(ii), and any module \(P\) in \(\text{Mod}^{\ell f}(A)\), there exists a canonical isomorphism in \(\mathcal{P}(\xi(B))\) of the form

\[
\xi(B) \otimes_{\xi(A), \varrho} d^\circ_{A,\varnothing}(P) \cong d^\circ_{B,\varnothing}(B \otimes_A \varrho P).
\]

**Proof.** The above associations combine with the result of Theorem 4.17(i) to give data as in (a), (b) and (c) of Definition 5.4.

It is clear that this data satisfies condition (d) in the latter definition and also straightforward to check that it satisfies condition (e) by using the general result of Lemma 5.9 below (with \(\Lambda = A\)) to make a compatible choice of sections when computing each of the maps \(i^\circ_{\varnothing}(\gamma', \delta'), i^\circ_{\varnothing}(\gamma'', \delta''), i^\circ_{\varnothing}(\alpha', \beta'), i^\circ_{\varnothing}(\alpha'', \beta''), i^\circ_{\varnothing}(\alpha, \beta)\) and \(i^\circ_{\varnothing}(\gamma, \delta)\).

Finally, the existence of the displayed isomorphism follows directly from the result of Proposition 5.6(iv). \(\square\)

**Lemma 5.9.** We assume to be given a ring \(\Lambda\) and a commutative diagram of short exact sequences of finitely generated projective \(\Lambda\)-modules of the form
Then there exist $\Lambda$-equivariant sections $\sigma_i : P_i \to N_i$ to $d_i$ for $i = 1, 2$ and $3$ such that there are commutative diagrams of $\Lambda$-modules

\[
\begin{array}{ccc}
N_1 & \xrightarrow{\sigma_1} & P_1 \\
\downarrow \phi_1 & & \downarrow \kappa_1 \\
N_2 & \xleftarrow{\sigma_2} & P_2 \\
\downarrow \phi_2 & & \downarrow \kappa_2 \\
N_3 & \xleftarrow{\sigma_3} & P_3,
\end{array}
\]

(5.2.1)

Proof. First choose any $\Lambda$-equivariant section $\sigma$ to $d_2$ and write $\theta$ for the composite homomorphism $\phi_2 \circ \sigma \circ \kappa_1 : P_1 \to N_3$.

The commutativity of the given diagram implies that there exists a unique homomorphism $\theta_1$ in $\text{Hom}_\Lambda(P_1, M_2)$ such that $\theta = d_3^\ast \circ \theta_1$. Since $P_1$ is a projective $\Lambda$-module we can then choose a homomorphism $\theta_2$ in $\text{Hom}_\Lambda(P_1, M_2)$ with $\theta_1 = \epsilon_2 \circ \theta_2$.

Next we note that, since $P_3$ is a projective $\Lambda$-module, the group $\text{Ext}_\Lambda^1(P_3, M_2)$ vanishes and so there exists a homomorphism $\theta_3$ in $\text{Hom}_\Lambda(P_2, M_2)$ with $\theta_2 = \theta_3 \circ \kappa_1$.

We now set $\sigma_2 := \sigma - d_2^\ast \circ \theta_3 \in \text{Hom}_\Lambda(P_2, N_2)$. Then $\sigma_2$ is a section to $d_2$ since $d_2 \circ \sigma_2 = d_2 \circ \sigma - (d_2 \circ d_2^\ast) \circ \theta_3 = d_2 \circ \sigma$. In addition, for $x$ in $P_1$ one has

\[
\phi_2(\sigma_2(\kappa_1(x))) = \phi_2(\sigma(\kappa_1(x))) - \phi_2(d_3^\ast \circ \theta_3(\kappa_1(x))) \\
= \theta(x) - d_3^\ast((\epsilon_2 \circ \kappa_1)(x)) \\
= \theta(x) - d_3^\ast((\phi_2 \circ \kappa_1)(x)) \\
= \theta(x) - d_2^\ast(\theta_3)(x) \\
= \theta(x) - (d_2^\ast \circ \theta_1)(x) \\
= \theta(x) - (d_3^\ast \circ \theta_2)(x) \\
= \theta(x) - (\epsilon_2 \circ \theta_3)(x) \\
= \theta(x) - (\epsilon_2 \circ \kappa_1)(x) \\
= \theta(x) - \theta(\kappa_1(x)) = 0.
\]

Since $P_1$ is a projective $\Lambda$-module this implies there exists a unique homomorphism $\sigma_1$ in $\text{Hom}_\Lambda(P_1, N_1)$ which makes the first diagram in (5.2.1) commute (with respect to our fixed map $\sigma_2$) and hence that $\kappa_1(d_1 \circ \sigma_1) = (d_2 \circ \sigma_2) \circ \kappa_3 = \kappa_1$ so that $\sigma_1$ is a section to $d_1$.

Finally we note that the commutativity of the first diagram in (5.2.1) implies there exists a (unique) homomorphism $\sigma_3$ in $\text{Hom}_\Lambda(P_3, N_3)$ which makes the second diagram in (5.2.1) commute and one checks easily that this homomorphism is a section to $d_3$, as required. \qed

5.3. Extended determinant functors.
5.3.1. Let \( \Lambda \) be a noetherian ring. We write \( \text{Mod}(\Lambda) \) for category of finitely generated (left) \( \Lambda \)-modules.

We assume to be given an abelian subcategory \( \text{Mod}^\dagger(\Lambda) \) of \( \text{Mod}(\Lambda) \) that is exact in the sense of Quillen \cite{quillen} and a determinant functor on \( \text{Mod}^\dagger(\Lambda) \) in the sense of Definition 5.4.

We write \( \mathcal{P} \) for the target category of this determinant functor and \( d^\circ, i^\circ \) (and \( \zeta \)) for the associated data as in Definition 5.4 (a), (b) (and (c)).

We write \( D^\dagger(\Lambda) \) for the full triangulated subcategory of \( D(\Lambda) \) comprising complexes that are isomorphic to a bounded complex of modules belonging to \( \text{Mod}^\dagger(\Lambda) \). We also write \( D^\dagger(\Lambda)_{\text{is}} \) for the subcategory of \( D^\dagger(\Lambda) \) in which morphisms are restricted to be quasi-isomorphisms of complexes.

We regard \( \text{Mod}^\dagger(\Lambda)_{\text{is}} \) as a subcategory of \( D^\dagger(\Lambda)_{\text{is}} \) by identifying each object \( M \) of \( \text{Mod}^\dagger(\Lambda) \) with the complex that comprises \( M \) in degree zero and is zero in all other degrees.

In what follows we use the term ‘true triangle’ as synonymous for ‘short exact sequence of complexes’.

**Definition 5.10.** An ‘extension’ of the determinant functor comprising \( d^\circ \) and \( i^\circ \) to the category \( D^\dagger(\Lambda) \) comprises data of the following form.

(a) A covariant functor \( d : D^\dagger(\Lambda)_{\text{is}} \to \mathcal{P} \).

(b) For each true triangle \( X \xrightarrow{u} Y \xrightarrow{v} Z \) in which \( X, Y \) and \( Z \) are objects of \( D^\dagger(\Lambda) \) an isomorphism in \( \mathcal{P} \)

\[
i(u, v) : d(Y) \sim d(X) \otimes d(Z).
\]

This data is subject to the following axioms:

(i) If

\[
\begin{array}{c}
X \\ f \downarrow \\
\end{array}
\begin{array}{c}
Y \\ g \downarrow \\
\end{array}
\begin{array}{c}
Z \\ h \downarrow \\
\end{array}
\begin{array}{c}
X' \\ u' \downarrow \\
\end{array}
\begin{array}{c}
Y' \\ g' \downarrow \\
\end{array}
\begin{array}{c}
Z' \\ h' \downarrow \\
\end{array}
\]

is a commutative diagram of true triangles and \( f, g, h \) are all quasi-isomorphisms, then \( d(f) \otimes d(h) \circ i(u, v) \circ d(g)^{-1} = i(u', v') \).

(ii) If \( u \), respectively \( v \), is a quasi-isomorphism, then \( i(u, v) = d(u)^{-1} \), respectively \( i(u, v) = d(v) \).

(iii) For any commutative diagram of complexes

\[
\begin{array}{c}
X \\ f \downarrow \\
\end{array}
\begin{array}{c}
Y \\ g \downarrow \\
\end{array}
\begin{array}{c}
Z \\ h \downarrow \\
\end{array}
\begin{array}{c}
X' \\ u' \downarrow \\
\end{array}
\begin{array}{c}
Y' \\ g' \downarrow \\
\end{array}
\begin{array}{c}
Z' \\ h' \downarrow \\
\end{array}
\begin{array}{c}
X'' \\ u'' \downarrow \\
\end{array}
\begin{array}{c}
Y'' \\ g'' \downarrow \\
\end{array}
\begin{array}{c}
Z'' \\ h'' \downarrow \\
\end{array}
\]

in which all of the rows and columns are true triangles and all terms are objects of \( D^\dagger(\Lambda) \), the following diagram in \( \mathcal{P} \) commutes.
Proposition 5.11. The determinant functor constructed in Proposition 5.8 has a canonical extension to the category $\mathcal{D}^\mathbf{I}(\Lambda)\mathbf{S}$.

We write $d_{A,\varpi}$ and $i_{A,\varpi}$ for the data associated to this extension as in Definition 5.10(a) and (b). Then the extended determinant functor has the following additional properties.

(i) Fix a homomorphism $\varrho : A \to B$ as in Theorem 5.2(ii). Then for any complex $X$ in $\mathcal{D}^\mathbf{I}(\Lambda)$ the complex $\mathcal{B} \otimes^{\mathbb{L}}_{A,\varrho} X$ belongs to $\mathcal{D}^\mathbf{I}(\mathcal{B})$ and there exists a canonical isomorphism in $\mathcal{P}(\xi(\mathcal{B}))$ of the form

$$\xi(\mathcal{B}) \otimes \xi(\mathcal{A}), \varrho \cdot d_{A,\varpi}(X) \cong d_{B,\varpi}(\mathcal{B} \otimes^{\mathbb{L}}_{A,\varrho} X).$$

(ii) The restriction of $d_{A,\varpi}$ to $\mathcal{D}^\mathbf{I,0}(\Lambda)\mathbf{S}$ is independent of the choice of bases $\varpi$.

Proof. This argument follows the approach used by Flach and the first author to prove the same result in the setting of virtual objects (see [5, Prop. 2.1]).

The essential point is therefore that, excluding the assertions (i) and (ii), the claimed result follows directly from the general result [22, Prop. 4] of Knudsen and Mumford and the formal constructions that are used to prove [22, Th. 2], via which the same statements are proved for the determinant functor over a commutative ring.

To be more precise, if one removes the condition (iv) (regarding compatibility with scalar extensions) from the definition of extended determinant functor that is given in [22, Def. 4], then the only properties of the determinant functor that are used in the constructions that underlie the proof of [22, Th. 2] are those listed in [loc. cit., Prop. 1 (excluding claim (iii))] and Proposition 5.8 implies that $d_{A,\varpi}^0$ and $i_{A,\varpi}^0$ have all of these necessary properties.

In addition, since the restriction of $d_{A,\varpi}$ to $\mathcal{D}^\mathbf{I}(\Lambda)\mathbf{S}$ is equal to $d_{A,\varpi}^0$, the property in claim (i) follows from the final assertion of Proposition 5.8.

Lastly, we note that the property in claim (ii) follows directly from the result of Lemma 5.13(ii) below. □

Remark 5.12. Proposition 5.11 implies that for any object $X$ of $\mathcal{D}^\mathbf{I}(\Lambda)$ there is a natural isomorphism $d_{A,\varpi}(X[1]) \cong d_{A,\varpi}(X)^{-1}$ in $\mathcal{P}(\xi(\mathcal{A}))$. This is because if we write $\text{Cone}_X$ for the mapping cone of the identity endomorphism of $X$, then the associated true triangle $X \rightarrow \text{Cone}_X \rightarrow X[1]$ induces a composite isomorphism

$$d_{A,\varpi}(X) \otimes d_{A,\varpi}(X[1]) \xrightarrow{i_{A,\varpi}(u,v)} d_{A,\varpi}(\text{Cone}_X) \cong 1_{\mathcal{P}(\xi(\mathcal{A}))},$$
where the second isomorphism is induced by the acyclicity of $\operatorname{Con}_X$.

5.3.2. For the reader’s convenience, we give a more explicit description of the construction in Proposition 5.11.

**Lemma 5.13.**

(i) Let $P^\bullet$ be a complex in $C^\operatorname{ff}(\mathcal{A})$. Then for each quasi-isomorphism $u : P^\bullet \to X$ of complexes of $\mathcal{A}$-modules, the map $d_{\mathcal{A},v}(u)$ induces an isomorphism

$$
\bigotimes_{i \in \mathbb{Z}} (\bigcap_{A}^{\operatorname{rk}(P^i)} P^i, r_{\mathcal{A}}(P^i_F))^{(-1)i} \sim d_{\mathcal{A},v}(X)
$$

in $\mathcal{P}(\xi(\mathcal{A}))$, where each lattice $\bigcap_{A}^{\operatorname{rk}(P^i)} P^i$ is defined with respect to the bases $v$.

(ii) If $X$ belongs to $D^\operatorname{ff,0}(\mathcal{A})$, then $d_{\mathcal{A},v}(X)$ is independent of the choice of bases $v$.

**Proof.** To prove claim (i) it is enough to show that there is a canonical isomorphism in $\mathcal{P}(\xi(\mathcal{A}))$ of the form

$$
\bigotimes_{i \in \mathbb{Z}} (d_{\mathcal{A},v}^\circ(P^i))^{(-1)i} \sim d_{\mathcal{A},v}(P^\bullet).
$$

To show we can clearly assume that $P^\bullet$ in non-zero. We then let $d$ denote the largest integer for which $P^d$ is non-zero and write $P^\bullet_{< d}$ for the complex obtained from $P^\bullet$ by replacing $P^d$ by 0 and leaving all other terms unchanged.

Then there is a natural true triangle $P^d[-d] \xrightarrow{u} P^\bullet \xrightarrow{v} P^\bullet_{< d}$ in $D^\operatorname{ff}(\mathcal{A})$ and hence an isomorphism

$$
i_{\mathcal{A}}(u, v) : d_{\mathcal{A},v}(P^d[-d]) \otimes d_{\mathcal{A},v}(P^\bullet_{< d}) \sim d_{\mathcal{A},v}(P^\bullet)
$$

in $\mathcal{P}(\xi(\mathcal{A}))$.

By an induction on the number of non-zero terms of $P^\bullet$ we are therefore reduced to proving that $d_{\mathcal{A},v}(P^d[-d])$ identifies with $(d_{\mathcal{A},v}^\circ(P^d))^{(-1)d}$. This follows directly from (repeated application of) Remark 5.12 and the fact that $d_{\mathcal{A},v}(P^d[0]) = d_{\mathcal{A},v}^\circ(P^d)$.

To prove claim (ii) it is enough to fix a quasi-isomorphism $P^\bullet \to X$ as in claim (i) and show that the graded module $\bigotimes_{i \in \mathbb{Z}} (d_{\mathcal{A},v}^\circ(P^i))^{(-1)i}$ is independent of the choice of ordered bases $v$. Since it is enough to prove this after localising at each prime ideal of $R$ we can also assume, without loss of generality, that each $\mathcal{A}$-module $P^i$ is free. In each degree $i$ we then set $r_i := \operatorname{rk}_A(P^i)$, fix an ordered $\mathcal{A}$-basis $\{b_{i,j}\}_{1 \leq j \leq r_i}$ of $P^i$ and write $\{b_{i,j}^*\}_{1 \leq j \leq r_i}$ for the $\mathcal{A}$-basis of $\operatorname{Hom}_\mathcal{A}(P^i, A)$ that is dual to $\{b_{i,j}\}_{1 \leq j \leq r_i}$.

Then Lemma 4.12 combines with Proposition 5.10(i) to imply the natural isomorphism $\operatorname{Hom}_\mathcal{A}(\bigwedge_{A}^{j=1} P^i, \xi(\mathcal{A})) \cong \bigwedge_{\mathcal{A}}^{\delta=0} \operatorname{Hom}_{\mathcal{A}}(P^i, \mathcal{A})$ identifies the module $\operatorname{Hom}_\mathcal{A}(\bigwedge_{A}^{j=1} P^i, \xi(\mathcal{A}))$ with $\xi(\mathcal{A}) \cdot \bigwedge_{j=1}^{r_i} b_{i,j}^*$. Setting $b_{i,j}^* := b_{i,j}$ and $b_{i,j}^{-1} := b_{i,j}^*$ for each $i$ and $j$, it is therefore enough to prove that if the image of $\chi_A(P^\bullet)$ in $K_0(A)$ vanishes, then the element

$$
\bigotimes_{i \in \mathbb{Z}} \bigwedge_{j=1}^{r_i} b_{i,j}^{(-1)i}
$$

is independent of the choice of bases $v$ used in the definition of exterior products.
It suffices to verify this if projecting to each simple component of $A$ and so we shall assume $A$ is simple (and use the notation of Definition 1.3. We then fix bases $\{v_j\}_{1 \leq j \leq d}$ and $\{\tilde{v}_s\}_{1 \leq s \leq t}$ of the $E$-space $V$ and write $M = (M_{i,j})$ for the matrix in $\text{GL}_d(E)$ that satisfies $\tilde{v}_s = \sum_{t=1}^{t} M_{st} v_t$ for each integer $s$. Then, writing $N$ for the matrix in $\text{GL}_d(E)$ that is equal to the inverse of the transpose of $M$, one has $\tilde{v}_s^* = \sum_{t=1}^{t} N_{st} v^*_t$ for each integer $s$ and by using these equalities one computes that in each even degree $i$ there is an equality

$$\wedge_{1 \leq j \leq r_i} (\wedge_{1 \leq s \leq d} \tilde{v}_s^* \otimes b_{i,j}) = \det(N)^{r_i} \cdot \wedge_{1 \leq j \leq r_i} (\wedge_{1 \leq s \leq d} v_s^* \otimes b_{i,j}).$$

and in each odd degree $i$ an equality

$$\wedge_{1 \leq j \leq r_i} (\wedge_{1 \leq s \leq d} \tilde{v}_s^* \otimes b_{i,j}) = \det(M)^{r_i} \cdot \wedge_{1 \leq j \leq r_i} (\wedge_{1 \leq s \leq d} v_s^* \otimes b_{i,j}).$$

Since $\det(M) = \det(N)^{-1}$ this implies that the difference between the elements (5.4.2) when the exterior products are computing using the basis $\{v_j\}_{1 \leq j \leq d}$, respectively $\{\tilde{v}_j\}_{1 \leq j \leq d}$, of $V$ is the factor $\det(M) \sum_{i \in \mathbb{Z}} (-1)^{ri_i}$ to complete the proof it is therefore enough to note that if the image of $\chi_A(P^*)$ in $K_0(A)$ vanishes, then the sum $\sum_{i \in \mathbb{Z}} (-1)^{ri_i}$ is equal to 0.

5.4. The proof of Theorem 5.2. To complete the proof of Theorem 5.2 we must show that the construction of Proposition 5.11 retains the properties (i), (ii) and (iii) in Definition 5.4.1. As a key preliminary step, we consider similar results for the semisimple algebra $A$, first that the construction of Proposition 5.11 retains the properties (i), (ii) and (iii) in Definition 5.4.1 (with $\text{Mod}^\dagger(\Lambda) = \text{Mod}^\dagger(A)$) after one replaces true triangles and quasi-isomorphisms by arbitrary exact triangles and isomorphisms in $D^\dagger(A)$.

5.4.1. As a key preliminary step, we consider similar results for the semisimple algebra $A$. To do this we recall that, for any choice of ordered bases $\varpi$ as in (5.1.2) the argument of Flach and the first author in [3, Lem. 2] constructs a determinant functor on $\text{Mod}(A)$.

To recall the explicit construction we use the notation of (5.1.2). In particular, we assume first that $A = A_1$ is simple and hence equal to $\text{M}_{m_1}(D_1)$ for a division ring $D_1$ with Schur index $m_1$ and splitting field $E_1$. We set $n := n_1$, $m := m_1$, $D := D_1$, $E := E_1$, $W := W_1$, $V := V_1 = W_1^\dagger$ and use the ordered $E$-basis $\{w_a\}_{1 \leq a \leq m}$ of $W$ and $\varpi = \varpi_1$ of $V$.

Then for each $M$ in $\text{Mod}(A)$ the (left) $D$-module $D^n \otimes_A M$ is free of rank $r := \text{rk}_D(M)$ and so, since $\text{rr}_D(D^n \otimes_A M) = r_A(M)$, one has $\text{rr}_A(M) = r \cdot m$. If we fix an ordered $D$-basis $\{b_a\}_{1 \leq a \leq r}$ of $D^n \otimes_A M$, then Lemma 4.13 implies that the dimension one $\zeta(D)$-space spanned by $\wedge_{a=1}^{a=r} (\wedge_{s=1}^{s=m} (w_s^* \otimes b_a))$ is independent of the choice of $\{b_a\}_{1 \leq a \leq r}$ and we set

$$d^r_{A,\varpi}(M) := (\zeta(D) \cdot \wedge_{a=1}^{a=r} (\wedge_{s=1}^{s=m} (w_s^* \otimes b_a)), r_A(M)).$$

We next assume to be given a short exact sequence $M_1 \to M_2 \to M_3$ in $\text{Mod}(A)$ and set $r_j := \text{rk}_D(M_j)$ for each $j = 1, 2, 3$. Then, by following the same approach as the proof of Proposition 5.6 (v), we can use a choice of splitting $\sigma$ of the induced short exact sequence of free $D$-modules $D^n \otimes_A M_1 \to D^n \otimes_A M_2 \to D^n \otimes_A M_3$ to construct a basis $\{b_{2a}\}_{1 \leq a \leq r_2}$ of $D^n \otimes_A M_2$ from given bases $\{b_{ja}\}_{1 \leq a \leq r_j}$ of $D^n \otimes_A M_j$ for $j = 1, 3$. We then define

$$i^r_{A,\varpi} : d^r_{A,\varpi}(M_2) \to d^r_{A,\varpi}(M_1) \otimes d^r_{A,\varpi}(M_3)$$

to be the unique isomorphism of graded $\zeta(D)$-spaces with

$$i^r_{A,\varpi} (\wedge_{a=1}^{a=r_2} (\wedge_{s=1}^{s=m} (w_s^* \otimes b_{2a})), r_A(M_2)) = (\wedge_{a=1}^{a=r_1} (\wedge_{s=1}^{s=m} (w_s^* \otimes b_{1a})), r_A(M_1)) \otimes (\wedge_{a=1}^{a=r_3} (\wedge_{s=1}^{s=m} (w_s^* \otimes b_{3a})), r_A(M_3)).$$
Lemma 4.13 implies that this map is independent of the splitting \( \sigma \) and bases \( \{ b_{ja} \}_{1 \leq a \leq r_j} \) for \( j = 1, 3 \) that are used.)

In the more general case that \( A \) is not simple, one uses its Wedderburn decomposition

\[ A = \prod_{i \in I} A_i \]

to define \( d^A_{A,\omega} \) and \( i^A_{A,\omega} \) componentwise. With this construction, the image under \( d^A_{A,\omega} \) of a module \( M \) in \( \text{Mod}(A) \) has grading equal to \( \text{rr}_A(M) \), regarded as a function on \( \text{Spec}(\zeta(A)) = \bigcup_{i \in I} \text{Spec}(\zeta(D_i)) \) in the obvious way.

In the following result we write \( D_{\text{perf}}(A) \) for the full triangulated subcategory of \( D(A) \) comprising complexes that are isomorphic to a bounded complex of finitely generated \( A \)-modules.

**Proposition 5.14.** There exists a canonical extension to \( D_{\text{perf}}(A) \) of the determinant functor given by \( d^A_{A,\omega} \) and \( i^A_{A,\omega} \). The associated functor

\[ d_{A,\omega} : D_{\text{perf}}(A)_{\text{is}} \to \mathcal{P}(\zeta(A)) \]

has the following properties.

(i) For any object \( X \) of \( D_{\text{perf}}(A) \) there exists a canonical isomorphism

\[ d_{A,\omega}(X) \cong \bigotimes_{i \in \mathbb{Z}} d^A_{A,\omega}(H^i(X))^{(-1)^i} \]

in \( \mathcal{P}(\zeta(A)) \) that is functorial with respect to quasi-isomorphisms.

(ii) The following diagram

\[
\begin{array}{ccc}
D_{\text{perf}}(A)_{\text{is}} & \xrightarrow{d_{A,\omega}} & \mathcal{P}(\zeta(A)) \\
\uparrow & & \uparrow \\
D^\ell(A)_{\text{is}} & \xrightarrow{d_{A,\omega}} & \mathcal{P}(\zeta(A))
\end{array}
\]

commutes, where the vertical maps are the natural scalar extension functors.

**Proof.** The first claim is proved by using exactly the same formal argument that establishes the first claim of Proposition 5.11.

Then, since \( A \) is semisimple, every cohomology module \( H^i(X) \) of an object \( X \) in \( D_{\text{perf}}(A) \) is also itself an object of \( D_{\text{perf}}(A) \) (regarded as a complex concentrated in degree zero) and so the general argument of Knudsen and Mumford in [22] Rem. b) after Th. 2] shows that the associated functor \( d_{A,\omega} \) has the property in claim (i).

Finally, the commutativity of the scalar extension diagram in claim (ii) is verified by means of a direct comparison of the explicit construction of the functors \( d^A_{A,\omega} \) and \( d^A_{A,\omega} \) (the latter from Proposition 5.8).

The key point in this comparison is that if \( A = A_1 = M_n(D) \) is simple (as in the explicit construction made above) and \( M \) is a free \( A \)-module of rank \( k \), with basis \( \{ m_j \}_{1 \leq j \leq k} \), then the \( D \)-module \( D^n \otimes_A M \) has as a basis the lexicographically-ordered set \( \{ b_{j,t} \}_{1 \leq j \leq k, 1 \leq t \leq n} \), with \( b_{j,t} := x_t \otimes m_j \) where \( \{ x_t \}_{1 \leq t \leq n} \) is the standard \( D \)-basis of \( D^n \). In particular, if one uses this basis as the set \( \{ b_a \}_{1 \leq a \leq nk} \) that occurs in the definition (5.4.1) of \( d^A_{A,\omega}(M) \), then one has

\[
l_{a=1}^{a=nk} (\bigwedge_{j=1}^{j=k} (w_s^* \otimes b_a)) = \bigwedge_{j=1}^{j=k} (\bigwedge_{t=1}^{t=n} (\bigwedge_{s=1}^{s=m} (w_s^* \otimes x_t))) \otimes m_j = \bigwedge_{j=1}^{j=k} m_j.
\]
where the exterior product on the right hand side is as defined in \([4.2.2]\) with respect to the ordered basis \(\omega = \omega_1\) specified in \([5.1.2]\).

5.4.2. Turning now to the proof of Theorem \([5.2]\) we note that Proposition \([5.11]\) directly implies all assertions except for claim (i).

In addition, the argument used by Knudsen and Mumford to prove \([22, \text{Prop. } 6]\) shows that claim (i) of Theorem \([5.2]\) will also follow formally upon combining Proposition \([5.11]\) with the following technical observation.

**Proposition 5.15.** Let \(X\) and \(Y\) be complexes in \(D^\text{lf}(A)\) and \(\alpha\) and \(\beta\) quasi-isomorphisms \(X \to Y\) of complexes of \(A\)-modules with the following property: in each degree \(i\) there are finite filtrations \(F^\bullet(H^i(X))\) and \(F^\bullet(H^i(Y))\) that are compatible with the maps \(H^i(\alpha)\) and \(H^i(\beta)\) and such that \(\text{gr}(H^i(\alpha)) = \text{gr}(H^i(\beta))\) for all \(i\).

Then the morphisms \(d_{A,\omega}(\alpha)\) and \(d_{A,\omega}(\beta)\) coincide.

**Proof.** We note first that, since \(d_{A,\omega}(\alpha)\) and \(d_{A,\omega}(\beta)\) are homomorphisms between invertible \(\xi(A)\)-modules, they coincide if and only if they are equal after applying the scalar extension functor \(\zeta(A) \otimes_{\xi(A)} -\).

Given the commutativity of the diagram in Proposition \([5.14]\)(ii) we are therefore reduced to showing that the given hypotheses imply \(d_{A,\omega}(\alpha') = d_{A,\omega}(\beta')\) with \(\alpha' := \xi(A) \otimes_{\xi(A)} \alpha\) and \(\beta' := \xi(A) \otimes_{\xi(A)} \beta\).

The result of Proposition \([5.14]\)(i) then reduces us to showing that in each degree \(i\) the maps \(d^\omega_{A,\omega}(H^i(\alpha'))\) and \(d^\omega_{A,\omega}(H^i(\beta'))\) coincide.

This in turn follows easily from the given hypotheses and the general property of the functor \(d^\omega_{A,\omega}\) that is described in Definition \([5.4]\)(b).

This completes the proof of Theorem \([5.2]\).

**PART II: NON-COMMUTATIVE EULER SYSTEMS**

6. The general theory

6.1. The definition of non-commutative Euler systems.

6.1.1. In this section we fix a number field \(K\), with algebraic closure \(K^c\), and set \(G_K := \text{Gal}(K^c/K)\).

For each non-archimedean place \(v\) of \(K\) we write \(\sigma_v\) for the inverse of a fixed choice of Frobenius automorphism of \(v\) in the Galois group of the maximal extension of \(K\) in \(K^c\) that is unramified at \(v\).

We also write \(I_p(K)\) for the set of distinct irreducible \(\mathbb{Q}_p\)-valued characters of \(G_K\) that have open kernel.

For a Galois extension \(\mathcal{K}\) of \(K\) in \(K^c\) we write \(\Omega(\mathcal{K}/K)\) for the set of finite Galois extensions of \(K\) in \(\mathcal{K}\).

For \(F\) in \(\Omega(K^c/K)\) we set \(\mathcal{G}_F := \text{Gal}(F/K)\) and write \(I_p(\mathcal{G}_F)\) for the subset of \(I_p(K)\) comprising characters that factor through the restriction map \(G_K \to \mathcal{G}_F\).

For \(\chi\) in \(I_p(K)\) we write \(K(\chi)\) for the subfield of \(K^c\) that is fixed by \(\ker(\chi)\) and \(n_\chi\) for the exponent of \(\mathcal{G}_{K(\chi)}\). We also write \(E_\chi\) for the subfield of \(\mathbb{Q}_p^c\) generated by a choice of
primitive $n_\chi$-th root of unity and, following \[3\], we fix a representation
\[
\rho_\chi : \mathcal{G}_K(\chi) \to \text{GL}_\chi(1)(E_\chi)
\]
of character $\chi$.

For $F$ in $\Omega(K/K)$ we write $E_F$ for the composite of the field $E_\chi$ as $\chi$ runs over $\text{Ir}_p(\mathcal{G}_F)$. Then, for any subfield $L$ of $\mathbb{Q}_p^c$, the discussion of Remark \[4.9\] shows that the fixed choice of representation $\rho_\chi$ for each $\chi$ in $\text{Ir}_p(\mathcal{G}_F)$ induces for each index $i$ an isomorphism of $L E_F$-algebras
\[
(LE_F)[G_F] \cong \prod_{\chi \in \text{Ir}_p(\mathcal{G}_F)} M_{\chi(1)}(LE_F),
\]
and hence determines a natural choice of the auxiliary data needed to define reduced exterior powers (as in \[4.2\]), reduced Rubin lattices (as in \[4.3\]) and reduced determinant functors (as in \[5\] over the semisimple algebra $L[G_F]$.

We assume throughout the sequel, and without further explicit comment, that for any finite set of extensions $\{L_i\}_{i \in I}$ of $K$ in $K^c$ and any $F$ in $\Omega(K^c/K)$ the constructions of \[4.2\], \[4.3\] and \[5\] for the semisimple algebra $\prod_{i \in I} L_i[G_F]$ are made relative to this choice of data.

6.1.2. We next assume to be given a finite extension $\mathcal{O}$ of $\mathbb{Q}_p$ in $\mathbb{Q}_p^c$, with valuation ring $\mathcal{O}$, and an $\mathcal{O}$-order $\mathcal{A}$. We assume that $\mathcal{A}$ spans a finite-dimensional semisimple commutative $\mathcal{Q}$-algebra $\mathcal{A}$ and also satisfies the following condition.

**Hypothesis 6.1.** $\text{Hom}_\mathcal{O}(\mathcal{A}, \mathcal{O})$ is a free module of rank one with respect to the natural action of $\mathcal{A}$.

**Remark 6.2.** If Hypothesis \[6.1\] is satisfied, then for each field $F$ in $\Omega(K^c/K)$ the $\mathcal{O}$-order $\mathcal{A}[G_F]$ is a one-dimensional Gorenstein ring. In particular, in each such case, the group $\text{Ext}^1_{\mathcal{A}[G_F]}(M, \mathcal{A}[G_F])$ vanishes for every finitely generated $\mathcal{A}[G_F]$-module $M$ that is $\mathcal{O}$-torsion-free. (For more details see either \[12\] §A.3 or \[16\] §37.)

We also assume to be given a continuous $\mathcal{O}$-representation $T$ of $G_K$ that satisfies the following condition.

**Hypothesis 6.3.** $T$ is endowed with a (commuting) action of $\mathcal{A}$ with respect to which it is projective.

We write $S_\infty(K)$ and $S_p(K)$ for the sets of archimedean and $p$-adic places of $K$ and $S_{\text{bad}}(T)$ for the (finite) set of places of $K$ at which $T$ has bad reduction. For each field $F$ in $\Omega(K^c/K)$ we write $S_{\text{ram}}(F/K)$ for the set of places of $K$ that ramify in $F$ and then consider the finite set of places of $K$ given by
\[
S(F) = S(T, F) := S_\infty(K) \cup S_p(K) \cup S_{\text{bad}}(T) \cup S_{\text{ram}}(F/K).
\]

Now for each $F$ in $\Omega(K^c/K)$ the induced representation
\[
T_F := \text{Ind}_{G_K}^{G_F}(T)
\]
is naturally a module over $\mathcal{A}[G_F]$, and identifies with the tensor product $\mathcal{A}[G_F] \otimes_\mathcal{A} T$ upon which $\mathcal{A}[G_F]$ acts via left multiplication on the first factor whilst $G_K$ acts by
\[
\sigma \cdot (a \otimes t) := a\sigma^{-1} \otimes \sigma t \ (\sigma \in G_K, a \in \mathcal{A}[G_F], \text{ and } t \in T),
\]
where $\overline{\sigma} \in G_F$ is the image of $\sigma$.

In particular, for each place $v$ of $K$ outside $S(F)$ there is a natural action of the automorphism $\sigma_v$ on $T_F$ (that commutes with the action of $A[G_F]$).

Hypothesis 6.1 implies that for each integer $a$ the representation $T(a) := T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(a)$ is a projective $A$-module and hence that the induced representation

$$T(a)_F \cong (T_F)(a) \cong A[G_F] \otimes_A T(a)$$

is a projective $A[G_F]$-module.

Setting $T^*(1) := \text{Hom}_O(T, O)$, Hypotheses 6.1 and 6.3 combine to imply that the Kummer dual representations $T^*(1)$ and $T^*(1)_F$ are respectively projective modules over $A$ and $A[G_F]$.

We shall also use the $A$-linear representations $V := \mathbb{Q} \otimes_O T$ and $V^*(1) := \mathbb{Q} \otimes_O T^*(1)$ and for each $F$ in $\Omega(K/K)$ also $V_F := \mathbb{Q} \otimes_O T_F$ and $V^*(1)_F := \mathbb{Q} \otimes_O T^*(1)_F$.

6.1.3. In the sequel we fix a Wedderburn decomposition $A = \prod_{i \in I} L_i$ where each $L_i$ is a finite extension of $\mathbb{Q}_p$ in $\mathbb{Q}_p^c$ that contains $\mathbb{Q}$.

Then for each place $v$ of $K$ outside $S(F)$ the reduced norm

$$\text{Nrd}_{A[G_F]}(1 - \sigma_v \mid V^*(1)_F)$$

over the semisimple algebra $A[G_F] = \prod_{i \in I} L_i[G_F]$ can be computed as the reduced norm of an endomorphism of the projective $A[G_F]$-module $T^*(1)_F$ and so belongs to $\xi(A[G_F])$.

We also note that for any pair of fields $F$ and $F'$ in $\Omega(K/K)$ with $F \subseteq F'$, Remark 4.3 applies to each simple component of $A[G_F]$ to imply that the natural corestriction map

$$\text{Cor}_{F'/F} : H^1(O_{F', S(F')}, T) \to H^1(O_{F, S(F')}, T)$$

induces a homomorphism of $\zeta(A)$-modules

$$\text{Cor}_{F'/F}^a : \bigwedge^a_{A[G_F]} H^1(O_{F', S(F')}, V) \to \bigwedge^a_{A[G_F]} H^1(O_{F, S(F')}, V).$$

In the sequel we write $x \mapsto x^\#$ for the involution of $\zeta(A[G_F])$ that is induced by restricting the $A$-linear anti-involution of $A[G_F]$ that inverts elements of $G_F$ (cf. Remark 3.21).

We can now introduce the definition of non-commutative ($p$-adic) Euler systems.

**Definition 6.4.** Let $a$ be a non-negative integer. Then a **non-commutative Euler system of rank $a$** for the pair $(T, \mathcal{K})$ is a collection of elements

$$\left\{ c_F \in \bigwedge^a_{A[G_F]} H^1(O_{F, S(F)}, T) \mid F \in \Omega(K/K) \right\}$$

with the property that for every $F$ and $F'$ in $\Omega(K/K)$ with $F \subseteq F'$ one has

$$(6.1.1) \quad \text{Cor}_{F'/F}^a(c_{F'}) = \left( \prod_{v \in S(F') \setminus S(F)} \text{Nrd}_{A[G_F]}(1 - \sigma_v \mid V^*(1)_F)^\# \right) (c_F)$$

in $\bigwedge^a_{A[G_F]} H^1(O_{F, S(F')}, V)$.

We write $\text{ES}_a(T, \mathcal{K})$ for the set of Euler systems of rank $a$ for $(T, \mathcal{K})$. 


Remark 6.5. If $K/K$ is abelian, then Remark 4.16 implies that the above definition of an Euler system of rank $a$ for $(T,K)$ agrees with that given in [12, Def. 2.3] (with $T$ replaced by $T^*(1)$). In particular, if $K/K$ is abelian and $a = 1$, then the above definition recovers the classical definition of Euler systems for $p$-adic representations given by Rubin in [11, Def. 2.1.1].

Remark 6.6. It is clear that, in all cases, the set $ES_a(T,K)$ is an abelian group that is endowed with a natural action of the algebra

$$\xi(\mathcal{A}[[\text{Gal}(K/K)]]) := \lim_{L \in \Omega(K/K)} \xi(\mathcal{A}[G_L])$$

where the (surjective) transition morphisms are induced by the natural projection maps $\mathcal{A}[G_L] \to \mathcal{A}[G_{L'}]$ for $L \subseteq L'$ (and Lemma 3.2(v)).

Remark 6.7. If $T = \mathbb{Z}_p(1)$, then Kummer theory identifies $H^1(O_{F,S(F)}, T)$ with the pro-$p$ completion of the group of $S(F)$-units of $F$. In this classical setting, it is possible to develop a finer version of the theory that we describe below by considering compatible families of elements that are defined just as above but with $S(F)$ replaced by the subset comprising $S_\infty(K)$ and all places that ramify in $F$. Such ‘non-commutative Euler systems for $G_m$’ will be studied in the supplementary article [10].

6.1.4. In the sequel it is often convenient to assume that $T$ and $K$ satisfy the following hypothesis.

Hypothesis 6.8. For all fields $F$ in $\Omega(K/K)$

(i) $H^0(O_{F,S(F)}, T)$ vanishes, and

(ii) $H^1(O_{F,S(F)}, T)$ is $O$-torsion-free.

Remark 6.9. This hypothesis is automatically satisfied in many cases of interest. (For example, if $p$ is odd and $K$ is the maximal totally real extension of $K = \mathbb{Q}$, then it is satisfied in the context of Remark 6.7). In general, if Hypothesis 6.8(ii) fails to be valid but there exists a finite non-empty set of places $\Sigma$ of $K$ that is unramified in $K$, then the ‘$\Sigma$-modification’ construction described in [12, §2.3] defines a canonical torsion-free submodule $H^1_{\Sigma}(O_{F,S(F)}, T)$ of $H^1(O_{F,S(F)}, T)$ for each $F$ in $\Omega(K/K)$. By systematically replacing groups of the form $H^1(O_{F,S(F)}, T)$ by $H^1_{\Sigma}(O_{F,S(F)}, T)$ in what follows, one can establish an analogue of the theory below without assuming the validity of Hypothesis 6.8(ii). However, since this extended theory is obtained in just the same way, we prefer to avoid the extra technicalities and do not discuss it further.

Lemma 6.10. Assume $T$ and $K$ satisfy Hypothesis 6.8. Then for every pair of fields $F$ and $F'$ in $\Omega(K/K)$ with $F \subset F'$ and every natural number $a$ the following claims are valid.

(i) The restriction map $H^1(O_{F,S(F')}, T) \to H^1(O_{F',S(F')}, T)$ identifies $H^1(O_{F,S(F')}, T)$ with the submodule of $\text{Gal}(F'/F)$-invariant elements of $H^1(O_{F,S(F')}, T)$.

(ii) The corestriction map $\text{Cor}^a_{F'/F}$ restricts to give a homomorphism of $\xi(\mathcal{A}[G_F])$-modules

$$\bigcap_{A[G_F]} H^1(O_{F',S(F')}, T) \to \bigcap_{A[G_F]} H^1(O_{F,S(F')}, T).$$
(iii) The inflation map \(H^1(O_{F,S(F)}, T) \to H^1(O_{F,S(F')}, T)\) induces an identification
\[
\bigcap_{\mathcal{A}[G_F]} H^1(O_{F,S(F)}, T) = \left( \bigcap_{\mathcal{A}[G_F]} H^1(O_{F,S(F')}, T) \right) \cap \left( \bigcap_{\mathcal{A}[G_F]} H^1(O_{F,S(F)}, V) \right).
\]

Proof. We set \(\Delta := \text{Gal}(F'/F)\), \(X_1 := H^1(O_{F,S(F)}, T)\) and \(X_2 := H^1(O_{F',S(F')}, T)\).

Hypothesis \(6.8\)(i) implies that the complex \(R\Gamma(O_{F',S(F')}, T)\) is acyclic in degrees less than one. Given this fact, claim (i) follows from the fact that the fixed point functor \(M \to \text{Hom}_{\mathcal{O}([\Delta])}(O, M) = H^0(\Delta, M)\) is left exact and that there is a canonical isomorphism
\[
R\text{Hom}_{\mathcal{O}([\Delta])}(O, R\Gamma(O_{F',S(F')}, T)) \cong R\Gamma(O_{F,S(F')}, T)
\]
in \(D(\mathcal{O}[G_F])\).

We next recall a general fact: for any commutative ring \(R\), finite group \(G\) and left \(R[G]\)-module \(M\), there is a natural isomorphism of \(R[G]\)-modules
\[
(6.1.2) \quad \text{Hom}_R(M, R) \cong \text{Hom}_{R[G]}(M, R[G]); \quad f \mapsto \sum_{\sigma \in G} f(\sigma(-))\sigma^{-1},
\]
where \(G\) acts on the dual modules \(\text{Hom}_R(M, R)\) and \(\text{Hom}_{R[G]}(M, R[G])\) via the rules
\[
(\sigma \cdot f)(m) := f(\sigma^{-1}m) \quad (\sigma \in G, \ f \in \text{Hom}_R(M, R), \ m \in M),
\]
and
\[
(\sigma \cdot \theta)(m) := \theta(m)\sigma^{-1} \quad (\sigma \in G, \ \theta \in \text{Hom}_{R[G]}(M, R[G]), \ m \in M).
\]

Turning to claim (ii) we note first that claim (i) combines with Hypothesis \(6.8\)(ii) to imply the cokernel of the restriction map \(\varrho_{F'/F} : X_1 \to X_2\) is \(\mathcal{O}\)-torsion-free. (This is because if \(x\) is any element of \(X_2\) such that for some \(n > 1\) one has \((\delta - 1)(n \cdot x) = 0\) for all \(\delta \in \Delta\), then \(n((\delta - 1)(x)) = 0\) and hence, since \(X_2\) is torsion-free, also \((\delta - 1)(x) = 0\) for all \(\delta \in \Delta\).)

The assumption that \(\mathcal{A}\) is Gorenstein therefore implies that \(\text{Ext}^1_{\mathcal{A}}(\text{cok}(\varrho_{F'/F}), \mathcal{A})\) vanishes (cf. Remark \(6.2\)) and hence that the composite homomorphism
\[
\varrho_{F'/F} : \text{Hom}_{\mathcal{A}[G_{F'}]}(X_2, \mathcal{A}[G_{F'}]) \cong \text{Hom}_{\mathcal{A}}(X_2, \mathcal{A}) \to \text{Hom}_{\mathcal{A}}(X_1, \mathcal{A}) \cong \text{Hom}_{\mathcal{A}[G_{F'}]}(X_1, \mathcal{A}[G_{F'}])
\]
is surjective, where the two isomorphisms are as in \((6.1.2)\) and the central map is \(\text{Hom}_{\mathcal{A}}(\varrho_{F'/F}, \mathcal{A})\).

One can also check that for each \(\theta\) in \(\text{Hom}_{\mathcal{A}[G_{F'}]}(X_2, \mathcal{A}[G_{F'}])\) the diagram
\[
(6.1.3)
\begin{align*}
\begin{array}{ccc}
X_2 & \xrightarrow{\theta} & \mathcal{A}[G_{F'}] \\
\downarrow & & \downarrow \\
\text{Cor}_{F'/F} & \xrightarrow{\varrho_{F'/F}(\theta)} & \mathcal{A}[G_{F'}]
\end{array}
\end{align*}
\]
commutes, where the unlabelled arrow is the natural projection map. This diagram in turn implies that for every subset \(\{\theta_i\}_{1 \leq i \leq a}\) of \(\text{Hom}_{\mathcal{A}[G_{F'}]}(X_2, \mathcal{A}[G_{F'}])\) and every element \(x\) of
\[
(6.1.4)
\left( \bigwedge_{i=1}^a \varrho_{F'/F}(\theta_i) \right) \left( \text{Cor}_{F'/F}^{a}(x) \right) = \pi_{F'/F} \left( \left( \bigwedge_{i=1}^a \theta_i \right)(x) \right)
\]
where \(\pi_{F'/F}\) is the natural projection map \(\zeta(\mathcal{A}[G_{F'}]) \to \zeta(\mathcal{A}[G_{F}])\).

In particular, this equality combines with the surjectivity of \(\varrho_{F'/F}\) to imply if \(x\) belongs to \(\bigcap_{\mathcal{A}[G_{F'}]} X_2\), then \(\text{Cor}_{F'/F}^{a}(x)\) belongs to \(\bigcap_{\mathcal{A}[G_{F'}]} X_1\), as required to prove claim (ii).
To prove claim (iii) we use the canonical exact sequence

\[ 0 \to H^1(\mathcal{O}_{F,S}(F), T) \xrightarrow{i_{F,S,S'}} H^1(\mathcal{O}_{F,S}(F'), T) \to \bigoplus_{w \in (S(F) \setminus S(F))_F} H^1_f(F_w, T), \]

in which \( H^1_f(F_w, T) \) denotes the cokernel of the inflation map \( H^1(F_w/F, T) \to H^1(F_w, T) \), where \( F_w \) is the maximal unramified extension of \( F_w \) in \( F_c \).

In addition, since each \( \mathcal{O} \)-module \( H^1_f(F_w, T) \) is free (by [41, Lem. 1.3.5(ii)]) the \( \mathcal{O} \)-module \( \text{cok}(i_{F,S,S'}) \) is also free and so the group \( \text{Ext}^1_A[\mathcal{G}_F](\text{cok}(i_{F,S,S'}), A[\mathcal{G}_F]) \) vanishes (by Remark 6.2).

Given this last fact, the identification in claim (iii) follows immediately from the result of Theorem 4.17(iv) with \( i = i_{F,S,S'} \). \( \square \)

6.2. Statement of the main results.

6.2.1. For each \( E \) in \( \Omega(K/K) \) we define an idempotent in \( \zeta(A[\mathcal{G}_E]) \) by setting

\[ e_E = e_{E,T} := \sum_e e \]

where in the sum \( e \) runs over all primitive central idempotents of \( A[\mathcal{G}_E] \) that annihilate the space \( H^2(\mathcal{O}_{E,S(E),V}) \), and we use the \( A[\mathcal{G}_E] \)-module

\[ Y_E(T^*(1)) := \bigoplus_{w \in S_w(E)} H^0(E_w, T^*(1)). \]

**Theorem 6.11.** Assume that \( T \) and \( K \) satisfy all of following conditions:

(a) Hypothesis 6.8 is satisfied;
(b) the \( A \)-module \( T \) is projective;
(c) the \( A \)-module \( Y_K(T^*(1)) \) is free of rank \( r = r_T \);
(d) all archimedean places of \( K \) split completely in \( K \).

Then there exists an Euler system

\[ \varepsilon = \varepsilon_{K/K}(T) \]

in \( ES_r(T,K) \) that is canonical up to multiplication by an element of \( \zeta(A[[\Gal(K/K)]]) \) and has both of the following properties.

(i) The annihilator of \( \varepsilon \) in \( \xi(A[[\Gal(K/K)]]) \) is equal to

\[ \xi(A[[\Gal(K/K)]]) \cap \prod_{L \in \Omega(K/K)} \zeta(A[\mathcal{G}_L])(1 - e_L). \]

(ii) For each \( L \) in \( \Omega(K/K) \) one has

\[ (\wedge_{i=1}^{r_T} \varphi_i)(\varepsilon_L) \in \text{Fit}_L^0(\mathcal{O}_{L,S(L),T}^0, H^2(\mathcal{O}_{L,S(L),T}^0, T)) \]

for every subset \( \{\varphi_i\}_{1 \leq i \leq r_T} \) of \( \text{Hom}_{A[\mathcal{G}_L]}(H^1(\mathcal{O}_{L,S(L),T}, T), A[\mathcal{G}_L]) \).

**Remark 6.12.** The proof of claim (ii) of Theorem 6.11 that is given in [7.3] will establish (in the equality (7.3.2)) a more precise result in which elements of the form \( (\wedge_{i=1}^{r_T} \varphi_i)(\varepsilon_L) \) determine the \( r \)-th Fitting invariant of a presentation of the module \( H^2(\mathcal{O}_{L,S(L),T}) \otimes A[\mathcal{G}_L]^r \).
6.2.2. We next describe some Iwasawa-theoretic properties of the Euler systems constructed in Theorem 6.11.

To do this we fix a $p$-adic analytic extension $\mathcal{L}$ of $K$ in $\mathcal{K}$ and use the Iwasawa algebra

$$\mathcal{A}[[\mathcal{L}/K]] := \lim_{\mathcal{L} \in \Omega(\mathcal{L}/K)} \mathcal{A}[[\mathcal{G}_L]]$$

where the transition morphisms are induced by corestriction.

We recall that this ring is both left and right noetherian (by [25, V, 2.2.4]) and has a total quotient ring that we denote by $Q(\mathcal{A}[[\mathcal{L}/K]])$. In fact, if $\text{Gal}(\mathcal{L}/K)$ is a torsion-free pro-$p$ group, then $\mathcal{O}[[\mathcal{L}/K]]$ has no proper zero-divisors (by [31]) and so $Q(\mathcal{O}[[\mathcal{L}/K]])$ is a skew field (see [20]).

In the general case, there exists a field $L_0$ in $\Omega(\mathcal{L}/K)$ such that $\text{Gal}(\mathcal{L}/L_0)$ is a torsion-free pro-$p$ group. One then has $S(L) = S(L_0)$ for all $L$ in $\Omega(\mathcal{L}/L_0)$ and in each degree $i$ we set

$$H^i(\mathcal{O}_L, T) := \lim_{\mathcal{L} \in \Omega(\mathcal{L}/L_0)} H^i(\mathcal{O}_{L,S(L)}, T)$$

where the limit is taken with respect to the natural corestriction maps.

Taking account of Lemma 6.10(ii), we also define

$$\bigcap_{\mathcal{A}[[\mathcal{L}/K]]} H^1(\mathcal{O}_L, T) := \lim_{\mathcal{L} \in \Omega(\mathcal{L}/L_0)} \bigcap_{\mathcal{A}[[\mathcal{G}_L]]} H^1(\mathcal{O}_{L,S(L)}, T)$$

where the transition morphisms are induced by corestriction.

Finally, we set

$$\text{Hom}^*_{\mathcal{A}[[\mathcal{L}/K]]}(H^1(\mathcal{O}_L, T), \mathcal{A}[[\mathcal{L}/K]]) := \lim_{\mathcal{L} \in \Omega(\mathcal{L}/L_0)} \text{Hom}_{\mathcal{A}[[\mathcal{G}_L]]}(H^1(\mathcal{O}_{L,S(L)}, T), \mathcal{A}[[\mathcal{G}_L]])$$

where the limit is taken with respect to the maps $\mathcal{G}_L/L$ that occur in diagram [6.1.3] with $F'/F$ replaced by $L'/L$ for $L_0 \subseteq L \subseteq L' \subseteq \mathcal{L}$.

Then for any subset $\{\varphi_i\}_{1 \leq i \leq r}$ of $\text{Hom}^*_{\mathcal{A}[[\mathcal{L}/K]]}(H^1(\mathcal{O}_L, T), \mathcal{A}[[\mathcal{L}/K]])$ and any element $\eta = (\eta_L)_L$ of $\bigcap_{\mathcal{A}[[\mathcal{L}/K]]} H^1(\mathcal{O}_L, T)$ the commutativity of (6.1.4) implies that we obtain a well-defined element of the limit

$$\lim_{\mathcal{L} \in \Omega(\mathcal{L}/L_0)} \zeta(\mathcal{A}[[\mathcal{G}_L]])$$

where the transition morphisms are the natural projection maps, by setting

$$(\bigwedge_{i=1}^r \varphi_i)(\eta) := ((\bigwedge_{i=1}^r \varphi_{i,L})(\eta_L)_L$$

where $\varphi_{i,L}$ is the projection of $\varphi_i$ to $\text{Hom}_{\mathcal{A}[[\mathcal{G}_L]]}(H^1(\mathcal{O}_{L,S(L)}, T), \mathcal{A}[[\mathcal{G}_L]])$.

**Remark 6.13.** It is easily seen that all of the definitions made above are independent of the choice of the field $L_0$. In addition, an explicit description of the image of the natural homomorphism

$$\text{Hom}^*_{\mathcal{A}[[\mathcal{L}/K]]}(H^1(\mathcal{O}_L, T), \mathcal{A}[[\mathcal{L}/K]]) \to \text{Hom}_{\mathcal{A}[[\mathcal{L}/K]]}(H^1(\mathcal{O}_L, T), \mathcal{A}[[\mathcal{L}/K]])$$

is given in Lemma 7.11 below.
In the following result we shall say that an \( A[[\mathcal{L}/K]] \)-module is ‘central torsion’ if it is annihilated by a non-zero divisor of \( \zeta(A[[\mathcal{L}/K]]) \).

We will also write \( \delta(A[[\mathcal{L}/K]]) \) for the ideal of \( \zeta(A[[\mathcal{L}/K]]) \) that is given by the limit \( \lim_{L \to \mathcal{L}} \delta(A[G_L]) \) as \( L \) runs over \( \Omega(\mathcal{L}/K) \) and the transition morphisms are induced by (Lemma 3.5 (vii) and) the natural projection maps \( A[G_L] \to A[G_L'] \) for \( L \subseteq L' \).

\[ \textbf{Theorem 6.14.} \quad \text{Fix a } p\text{-adic analytic extension } \mathcal{L} \text{ of } K. \]

Then the Euler system \( \varepsilon \) that is constructed (under the stated hypotheses) in Theorem 6.11 has all of the following properties.

(i) The element

\[ \varepsilon_L := (\varepsilon_L)_{L \in \Omega(\mathcal{L}/L_0)} \]

belongs to \( \bigcap_{A[[\mathcal{L}/K]]} H^1(O_L, T) \).

In the remainder of the result we assume \( A = O \) and set \( R_L := O[[\mathcal{L}/K]]. \)

(ii) Assume that \( \text{Gal}(\mathcal{L}/K) \) has rank one. Then the \( R_L \)-module \( H^2(O_L, T) \) is a torsion module if and only if there exists a subset \( \{ \varphi_i \}_{1 \leq i \leq r} \) of \( \text{Hom}_{R_L}(H^1(O_L, T), R_L) \) for which \( \left( \bigwedge_{i=1}^r \varphi_i \right) (\varepsilon_L) \) is a unit of \( Q(R_L) \).

(iii) Assume that \( \text{Gal}(\mathcal{L}/K) \) has rank at least two. Then the \( R_L \)-module \( H^2(O_L, T) \) is a central torsion module if there exists a subset \( \{ \varphi_i \}_{1 \leq i \leq r} \) of \( \text{Hom}_{R_L}(H^1(O_L, T), R_L) \) for which \( \left( \bigwedge_{i=1}^r \varphi_i \right) (\varepsilon_L) \) is a non-zero divisor in the ideal \( \delta(R_L) \) of \( \zeta(R_L) \).

(iv) If \( \mathcal{L}/K \) is abelian, then the following conditions are equivalent:

(a) \( \varepsilon_L \) has the property stated in claim (ii), respectively claim (iii).

(b) \( \varepsilon_L \) is a generator of the \( Q(R_L) \)-module generated by \( \bigwedge_{i=1}^r H^1(O_L, T) \).

(c) \( \varepsilon_L \) is not annihilated by any non-zero divisor of \( R_L \).

\[ \textbf{7. The proofs of Theorems 6.11 and 6.14} \]

Throughout this section we assume the conditions stated in Theorem 6.11.

\[ \textbf{7.1. Vertical reduced determinantal systems.} \quad \text{For each bounded below complex of } O\text{-modules } C^* := \text{RHom}_O(C, O). \]

\[ \textbf{7.1.1.} \quad \text{For each field } F \in \Omega(K/K) \text{ and a finite set } \Sigma \text{ of places of } K \text{ with } S(F) \subseteq \Sigma, \text{ we write } \text{RG}_c(O_{F, \Sigma}, T) \text{ for the compactly supported étale cohomology of } T \text{ on } \text{Spec}(O_{F, \Sigma}). \]

We then define an object of \( D(A[G_F]) \) by setting

\[ C_{F, \Sigma}(T) := \text{RG}_c(O_{F, \Sigma}, T^*(1))^{*}[-2], \]

regarded as endowed with the natural action of \( A \) and the contragredient action of \( G_F \).

We also write \( D^{\text{perf}}(A[G_F]) \) for the subcategory of \( D(A[G_F]) \) comprising perfect complexes.

\[ \textbf{Lemma 7.1.} \quad \text{For each } E \in \Omega(K/K) \text{ and each finite set of places } \Sigma \text{ of } K \text{ that contains } S(E) \text{ the complex } C_{E, \Sigma}(T) \text{ has all of the following properties.} \]

(i) \( C_{E, \Sigma}(T) \) belongs to \( D^{\text{perf}}(A[G_E]). \)

(ii) The Euler characteristic of \( C_{E, \Sigma}(T) \) in \( K_0(A[G_E]) \) vanishes.
(iii) \( C_{E,\Sigma}(T) \) is acyclic outside degrees zero and one and there exists a canonical identification

\[
H^0(C_{E,\Sigma}(T)) = H^1(\mathcal{O}_{E,\Sigma}, T)
\]

and short exact sequence of \( \mathcal{A}[\mathcal{G}_E] \)-modules

\[
0 \to H^2(\mathcal{O}_{E,\Sigma}, T) \to H^1(C_{E,\Sigma}(T)) \to Y_E(T^*(1))^* \to 0.
\]

(iv) Given a finite set \( \Sigma' \) of places of \( K \) that contains \( \Sigma \) there exists a canonical exact triangle in \( D^\text{perf}(\mathcal{A}[\mathcal{G}_E]) \) of the form

\[
\bigoplus_{v \in \Sigma' \setminus \Sigma} \Gamma(K_v^\text{ur}/K_v, T^*(1)_E)^*[-2] \to C_{E,\Sigma}(T) \to C_{E,\Sigma'}(T) \to.
\]

(v) For all fields \( E \) and \( E' \) in \( \Omega(K/K) \) with \( E \subseteq E' \) there exists a natural isomorphism

\[
\mathcal{A}[\mathcal{G}_E] \otimes_{\mathcal{A}[\mathcal{G}_{E'}]} C_{E',\mathcal{S}(E')}(T) \cong C_{E,\mathcal{S}(E')}(T) \text{ in } D^\text{perf}(\mathcal{A}[\mathcal{G}_E]).
\]

Proof. Shapiro’s Lemma identifies \( \Gamma_c(\mathcal{O}_{E,\Sigma}, T^*(1)) \) with \( C(E) := \Gamma_c(\mathcal{O}_{K,\Sigma}, T^*(1)_E) \) and the result of Flach [18] Th. 5.1 implies \( C(E) \) belongs to \( D^\text{perf}(\mathcal{A}[\mathcal{G}_E]) \). Claim (i) is therefore true since \( \mathcal{A}[\mathcal{G}_E] \) is Gorenstein and so \( D^\text{perf}(\mathcal{A}[\mathcal{G}_E]) \) is preserved by the exact functor \( C \mapsto C^*[-2] \).

For the same reason, claim (ii) will follow if one can show that the Euler characteristic of \( C(E) \) in \( K_0(\mathcal{A}[\mathcal{G}_E]) \) vanishes. Set \( \mathfrak{a} := \mathcal{A}/(p) \). Then, since \( p \) is contained in the Jacobson radical of \( \mathcal{A}[\mathcal{G}_E] \) the natural reduction map \( K_0(\mathcal{A}[\mathcal{G}_E]) \to K_0(\mathfrak{a}[\mathcal{G}_E]) \) is injective (by [1 Chap. IX, Prop. 1.3]) and so it is enough to note [18] Th. 5.1 also implies that the Euler characteristic in \( K_0(\mathfrak{a}[\mathcal{G}_E]) \) of the complex \( \mathbb{Z}/(p) \otimes_{\mathbb{Z}} C(E) \cong \Gamma_c(\mathcal{O}_{K,\Sigma}, (T^*(1)_E)/(p)) \) vanishes.

The fact that \( C_{E,\Sigma}(T) \) is acyclic outside degrees zero and one is well-known and the existence of a canonical isomorphism and short exact sequence as in claim (iii) follows directly from the Artin-Verdier Duality theorem (and is also well-known).

The exact triangle in claim (iv) is obtained by applying the exact functor \( X \mapsto X^*[-2] \) to the canonical exact triangle in \( D^\text{perf}(\mathcal{A}[\mathcal{G}_E]) \)

\[
\Gamma_c(\mathcal{O}_{E,\Sigma}, T^*(1)) \to \Gamma_c(\mathcal{O}_{E,\Sigma}, T^*(1)) \to \bigoplus_{v \in \Sigma' \setminus \Sigma} \Gamma(K_v^\text{ur}/K_v, T^*(1)_E) \to.
\]

The isomorphisms in claim (v) result from combining the canonical isomorphisms

\[
\mathcal{A}[\mathcal{G}_E] \otimes_{\mathcal{A}[\mathcal{G}_{E'}]} C(E')^*[-2] \cong \text{RHom}_{\mathcal{A}[\mathcal{G}_{E'}]}(\mathcal{A}[\mathcal{G}_E], C(E'))^*[-2]
\]

and \( \text{RHom}_{\mathcal{A}[\mathcal{G}_{E'}]}(\mathcal{A}[\mathcal{G}_E], C(E')) \cong C(E) \).

7.1.2. For each field \( F \) in \( \Omega(K^c/K) \) we abbreviate the determinant functor \( d_{\mathcal{A}[\mathcal{G}_E],\psi}(\cdot) \) constructed in [5] to \( d_{\mathcal{A}[\mathcal{G}_E]}(\cdot) \), where we assume that the set of ordered bases \( \psi \) are as specified in [6.1.1].

For each pair of fields \( F \) and \( F' \) in \( \Omega(K^c/K) \) with \( F \subseteq F' \) we then write

\[
\nu_{F'/F} : d_{\mathcal{A}[\mathcal{G}_{E'}]}(C_{F',\mathcal{S}(F')}(T)) \to d_{\mathcal{A}[\mathcal{G}_E]}(C_{F',\mathcal{S}(F)}(T))
\]
For each natural number $n$ considered in [12, Proposition 7.4]. We also recall that the terminology of 'vertical determinantal systems' inside $K$ for the composite of all finite extensions of $K$ with the property that the absolute value of the discriminant of $K/Q$ is at most $n$.

Remark 7.3. If $K/K$ is abelian, the above definition recovers the module $VS(T, K)$ discussed in [12, §2.4]. We also recall that the terminology of 'vertical determinantal systems' is introduced in [12] in order to contrast these systems with the 'horizontal determinantal systems' that play a key role in loc. cit. (but for which we currently know of no non-commutative analogue).

It is clear that $VS(T, K)$ has a natural action of the algebra $\xi(\mathcal{A}[\text{Gal}(K/K)])$ and the following result describes this structure explicitly.

Proposition 7.4. The $\xi(\mathcal{A}[\text{Gal}(K/K)])$-module $VS(T, K)$ is free of rank one.

Proof. For each natural number $n$ we write $K_{(n)}$ for the composite of all finite extensions of $K$ inside $\mathcal{A}$ with the property that the absolute value of the discriminant of $K/Q$ is at most $n$. 
Then $K_{(n)}/K$ is a Galois extension and has finite degree as a consequence of the Hermite-Minkowski Theorem (cf. [30] §III.2).

In addition, one has $K_{(n)} \subset K_{(n+1)}$ for all $n$ and the normal subgroups $\{\text{Gal}(K/K_{(n)})\}_{n \geq 1}$ form a base of neighbourhoods of the identity in $\text{Gal}(K/K)$.

Thus, if for each $n$ we set $G_n := G_{K_{(n)}}$ and $\Xi_n := d_{A[G_n]}(C_{K_{(n)}}(T))$ and write $\tau_n$ for the (surjective) transition morphism $\Xi_{n+1} \rightarrow \Xi_n$ used in the definition of $\text{VS}(T,K)$, then there is a canonical identification

$$\text{VS}(T,K) = \lim_{\rightarrow \ n \geq 1} \Xi_n,$$

where the limit is taken with respect to the morphisms $\tau_n$.

For every $n$ the $\xi(A[G_n])$-module $\Xi_n$ is, by construction, free of rank one. We may therefore assume that, for some fixed $n$, we have made a choice for each natural number $m$ with $m \leq n$ of an $\xi(A[G_m])$-basis $x_m$ of $\Xi_m$ in such a way that $\tau_m(x_{m+1}) = x_m$ for all $m < n$.

If now $x'_{n+1}$ is any choice of $\xi(A[G_{n+1}])$-basis of $\Xi_{n+1}$, then $\tau_n(x'_{n+1})$ is a $\xi(A[G_n])$-basis of $\Xi_n$ and so there exists a unit $u_n$ of $\xi(A[G_n])$ such that $x_n = u_n \cdot \tau_n(x'_{n+1})$.

But, since $\xi(A[G_{n+1}])$ is semi-local and the projection map $\xi(A[G_{n+1}]) \rightarrow \xi(A[G_n])$ is surjective (by Lemma 3.2(v)), Bass’ Theorem (cf. [24] Chap. 7, (20.9)) implies that the homomorphism $\xi(A[G_{n+1}]) \rightarrow \xi(A[G_n])$ is surjective and so we may fix a pre-image $u_{n+1}$ of $u_n$ under this map.

It is then easily checked that the element $x_{n+1} := u_{n+1} \cdot x'_{n+1}$ is a $\xi(A[G_{n+1}])$-basis of $\Xi_{n+1}$ with the property that $\tau_m \circ \tau_{m+1} \circ \cdots \circ \tau_n(x_{n+1}) = x_m$ for all $m < n + 1$.

Continuing in this way we inductively define an element $(x_n)_{n \geq 1}$ that the isomorphism (7.1.2) implies is a $\xi(A[[\text{Gal}(K/K)]])$-basis of $\text{VS}(T,K)$, as required.  

7.2. A construction of non-commutative Euler systems. In this section we describe the crucial link between reduced determinantal systems and non-commutative Euler systems.

7.2.1. At the outset we note that condition (c) of Theorem 6.11 combines with Hypothesis 6.1 to imply that the $A$-module $Y_K(T^*(1))^*$ is free and we fix an (ordered) basis $\{a_i\}_{1 \leq i \leq r}$.

In addition, for each $E$ in $\Omega(K/K)$ there is a decomposition of $A[G_E]$-modules

$$Y_E(T^*(1)) = \bigoplus_{v \in S_\infty(K)} \bigoplus_{w|v} H^0(E_w, T^*(1))$$

where $w$ runs over all places of $E$ above $v$. This decomposition implies, in particular, that if we assume condition (d) of Theorem 6.11 and then fix a set of representatives of the $G_K$-orbits of embeddings $K \rightarrow \mathbb{Q}^c$, we obtain (by restriction of the embeddings) a compatible family of isomorphisms of $A[G_E]$-modules

$$Y_E(T^*(1))^* \cong A[G_E] \otimes_A Y_K(T^*(1))^* \cong A[G_E]^*,$$

where the second map is induced by the chosen $A$-basis $\{a_i\}_{1 \leq i \leq r}$ of $Y_K(T^*(1))^*$.
Theorem 7.5. If the conditions (a), (b), (c) and (d) of Theorem 6.11 are satisfied, then for each fixed set of isomorphisms \( (7.2.1) \) as above there exists a canonical homomorphism of \( \xi(A[[Gal(K/K)]]-modules) \)

\[ \Theta_{T,K} : VS(T,K) \to ES_r(T,K). \]

This homomorphism is non-zero if and only if there exists a field \( F \) in \( \Omega(K/K) \) and a non-zero primitive idempotent \( e \) of \( \xi(A[G_F]) \) for which the space \( e \cdot H^2(O_{F,S(F)}, V) \) vanishes.

The proof of this result will occupy the rest of this section. The basic strategy will be to define for each \( L \) in \( \Omega(K/K) \) a canonical homomorphism of \( \xi(A[[Gal(K/K)]]-modules) \)

\[ \Theta_L : VS(T,K) \to Q \otimes O \bigcap^r_{A[G_L]} H^1(O_{L,S(L)}, T) \]

and then to prove that the image of the diagonal homomorphism

\[ \Theta_{T,K} : VS(T,K) \to \prod_{L \in \Omega(K/K)} Q \otimes O \bigcap^r_{A[G_L]} H^1(O_{L,S(L)}, T) \]

belongs to \( ES_r(T,K) \) and to determine when this homomorphism is zero.

7.2.2. In this section we use the idempotent \( e_L \) of \( \xi(A[G_L]) \) defined in (6.2.1). We also set

\[ C_{L,S(L)}(V) := Q \otimes O C_{L,S(L)}(T) \]

Lemma 7.6. For each \( L \) in \( \Omega(K/K) \) set \( rr_L := rr_{A[G_L]}(e_L)^r \). Then the following claims are valid.

(i) The isomorphism \( (7.2.1) \) induces an isomorphism in \( \mathcal{P}(\xi(A[G_L])e_L) \)

\[ d^\circ_{A[G_L]}(e_L \cdot H^1(C_{L,S(L)}(V))) \cong (\xi(A[G_L])e_L, rr_L). \]

(ii) \( d^\circ_{A[G_L]}(e_L \cdot H^0(C_{L,S(L)}(V))) \subseteq e_L(Q \otimes O \bigcap^r_{A[G_L]} H^1(O_{L,S(L)}, T), rr_L). \)

Proof. Set \( A_L := A[G_L], A'_L := A[G_L] \) and \( rr'_L := rr_{A_L}(A'_L) \).

To prove claim (i) we note that the definition of \( e_L \) combines with the exact sequence in Lemma 7.4(iii) to give an identification of spaces \( e_L \cdot H^1(C_{L,S(L)}(V)) = e_L(Q \otimes O Y_L(T^*(1)*)) \) and hence also an identification in \( \mathcal{P}(\xi(A_L)e_L) \)

\[ d^\circ_{A_L}(e_L \cdot H^1(C_{L,S(L)}(V))) = d^\circ_{A_L}(e_L(Q \otimes O Y_L(T^*(1)*))) \]

\[ = e_L(Q \otimes O d^\circ_{A_L}(Y_L(T^*(1)*))). \]

Given this, the isomorphism in claim (i) is induced by the isomorphism in \( \mathcal{P}(\xi(A_L)) \)

\[ (\xi(A_L), rr'_L) \cong (\bigcap^r_{A_L} Y_L(T^*(1)*), rr'_L) = d^\circ_{A_L}(Y_L(T^*(1)*)) \]

obtained by applying Proposition 5.6(i) to the module \( M = Y_L(T^*(1)*) \) with \( \{ b_j \}_{1 \leq j \leq r} \) equal to the basis that \( (7.2.1) \) sends to the standard basis of \( A'_L. \)

To prove claim (ii) we note that Lemma 7.4(ii) and (iii) combine to imply the existence of an \( A_L \)-submodule \( X \) of \( H^1(O_{L,S(L)}, T) \) that is free of rank \( r \) and such that \( e_L \cdot (Q \otimes O X) = \)
We define $\Theta'_L$ to be the composite homomorphism of $\zeta(A[G_L])$-modules
\begin{equation}
\Theta'_L := \delta_{A[e_L]}(C_{L,S(L)}(V)) \rightarrow \delta_{A[e_L]}(H^0(C_{L,S(L)}(V))) \otimes \delta_{A[e_L]}(H^1(C_{L,S(L)}(V))^{-1})
\end{equation}
where the first map is the 'passage to cohomology' map from Proposition 5.14(i), the second is induced by multiplication by $e_L$ and the final two by the results in Lemma 7.6.

We can now finally define $\Theta_L$ to be the composite homomorphism
\begin{align*}
\Theta_L &= \Theta'(e_L) \in \bigcap_{A[G_L]} H^1(O_{L,S(L)}, T),
\end{align*}
where the first arrow is the canonical projection.

Then we need to prove that this definition implies that for every $\eta$ in $\mathcal{V}(T, K)$ and every pair of fields $F$ and $F'$ in $\Omega(K/K)$ with $F \subset F'$ one has both
\begin{align*}
\Theta_{F'}(\eta) &\in \bigcap_{A[G_F]} H^1(O_{F', S(F')}, T),
\end{align*}
and
\begin{align*}
\text{Cor}_{F'/F}^r(\Theta_{F'}(\eta)) &= \left( \prod_{v \in \Sigma} P_v \right) \cdot \Theta_F(\eta),
\end{align*}
where we set $\Sigma := S(F') \setminus S(F)$ and
\begin{align*}
P_v &:= \text{Nrd}_{A[G_F]}(1 - \sigma_v | V^*(1)_F)^\#,
\end{align*}
for each $v$ in $\Sigma$. 

7.2.3. The following result plays a key role in the proof of these facts.

**Lemma 7.7.** For each \( L \) in \( \Omega(K/K) \) there exists an exact sequence of \( \mathcal{A}[G_L] \)-modules

\[
0 \to H^1(\mathcal{O}_{L,S(L)}, T) \xrightarrow{i} P_L \xrightarrow{\theta_L} P_L \xrightarrow{\pi_L} H^1(C_{L,S(L)}(T)) \to 0
\]

that satisfies all of the following properties.

(i) \( P_L \) is finitely generated and free of rank \( d_L > r \).

(ii) Consider the composite surjective homomorphism of \( \mathcal{A}[G_L] \)-modules

\[
P_L \xrightarrow{\pi_L} H^1(C_{L,S(L)}(T)) \to Y_L(T^*(1))^* \cong \mathcal{A}[G_L]^r,
\]

where the second map comes from the exact sequence in Lemma 7.1(iii) and the isomorphism is induced by (7.2.1) and Hypothesis 6.1. Then there exists an ordered basis \( \{ b_{i,L} \}_{1 \leq i \leq d_L} \) of \( P_L \) such that the above homomorphism sends \( b_{i,L} \) to the \( i \)-th element of the standard basis of \( \mathcal{A}[G_L]^r \) if \( 1 \leq i \leq r \) and to zero otherwise.

(iii) Write \( P_L^* \) for the complex \( P_L \to P_L \), where the first term is placed in degree zero, the differential is \( \theta_L \) and \( H^0(P_L^*) \) and \( H^1(P_L^*) \) are identified with \( H^0(C_{L,S(L)}(T)) = H^1(\mathcal{O}_{L,S(L)}, T) \) and \( H^1(C_{L,S(L)}(T)) \) by using the maps \( \iota_L \) and \( \pi_L \). Then there exists an isomorphism in \( D_{\text{perfect}}(\mathcal{A}[G_L]) \) from \( P_L^* \) to \( C_{L,S(L)}(T) \) that induces the identity map in all degrees of cohomology.

(iv) The matrix in \( M_{d_L}(\mathcal{A}[G_L]) \) that represents \( \theta_L \) with respect to the basis \( \{ b_{i,L} \}_{1 \leq i \leq d_L} \) is a block matrix \( (0_{d_L,r} \mid M_L) \) where \( M_L \) belongs to \( M_{d_L,d_L-r}(\mathcal{A}[G_L]) \) and is such that, in the notation of Proposition 7.18, one has \( \ker(\theta_{M_L}) = H^1(\mathcal{O}_{L,S(L)}, T) \).

**Proof.** Claims (i), (ii) and (iii) of Lemma 7.1 combine to imply that the \((-1)\)-shift of \( C_{L,S(L)}(T) \) is an ‘admissible’ complex in the sense of [9]. Given this fact, the existence of an exact sequence with all of the properties in (i), (ii) and (iii) follows from the general construction of [9 §3.1].

The property in claim (ii) implies that \( \text{im}(\theta_L) \) is contained in the submodule of \( P_L \) generated by \( \{ b_{i,L} \}_{1 \leq i \leq d_L} \) and hence that \( \theta_L \) is represented by a block matrix \( M_L^* \) of the form \( (0_{d_L,r} \mid M_L) \) stated in claim (iv). For this representation it is then also clear that \( H^1(\mathcal{O}_{L,S(L)}, T) = \ker(\theta_{M_L}) \) is equal to \( \ker(\theta_{M_L}) \), as required. \( \square \)

Since the image of the endomorphism \( \theta = \theta_{F'} \) in Lemma 7.7 (with \( L = F' \)) is O-free, and the algebra \( \mathcal{A} \) is Gorenstein, the group \( \text{Ext}^1_{A[G_{F'}]}(\text{im}(\theta), A[G_{F'}]) \) vanishes (cf. Remark 6.2). The construction of Proposition 4.18 can therefore be applied to the matrix \( M = M_{F'} \) in Lemma 7.7(iv).

In view of the latter result, the containment (7.2.3) will follow if we can show the existence of an element \( x_\eta \) of \( \xi(\mathcal{A}[G_{F'}]) \) such that

\[
\Theta_{F'}(\eta) = x_\eta \cdot \varepsilon_M,
\]

where \( \varepsilon_M \) is the element constructed in Proposition 4.18.

To prove this we note that claims (ii) and (iii) of Lemma 7.1 combine to imply the existence of an (in general, non-canonical) isomorphism of \( A[G_{F'}]e_{F'} \)-modules

\[
e_{F'} \cdot H^1(\mathcal{O}_{F',S(F')}, V) \cong e_{F'} \cdot (\mathbb{Q} \otimes \mathcal{O} Y_{F'}(T^*(1))^*) \cong (A[G_{F'}]e_{F'})^r
\]

so that \( e_{F'} \cdot H^1(\mathcal{O}_{F',S(F')}, V) \) is a free \( A[G_{F'}]e_{F'} \)-module of rank \( r \).
Thus, since $\Theta_{F'}(\eta)$ and $\varepsilon_M$ both belong to $e_{F'} \cdot \bigwedge^r_{A[G_F]} H^1(O_{F',S(F')}, V) = \bigwedge^r_{A[G_F]} e_{F'} \cdot H^1(O_{F',S(F')}, V)$, Lemma 4.10 combines with the argument of Theorem 4.17(ii) to imply that the equality (7.2.6) is valid if $\Theta_{F'}(\eta)$ and $x_{\eta} \cdot \varepsilon_M$ have the same image under the map $\Lambda_{i=1}^r \phi_i$ for every $\phi_i$ in $\text{Hom}_{A[G_F]}(H^1(O_{F',S(F')}, T), A[G_{F'}])$.

To check this we note that, setting $d := d_{F'}$, the explicit definition of $\varepsilon_M$ implies (via Lemma 4.10) that

$$\left(\Lambda_{i=1}^r \phi_i\right)(\varepsilon_M) = \text{Nrd}_{A[G_{F'}]}\left(M' \mid M\right)$$

where $M' = M'(\{\phi_i\}_{1 \leq i \leq r})$ is the matrix in $M_{d,F'}(A[G_{F'}])$ with $ji$-entry equal to $\phi_i(b_{j,F'})$.

On the other hand, if we set $C_{F'} := C_{F',S(F')}(T)$ and $\beta_{F'} := ((\Lambda_{i=1}^d b_{i,F'}) \otimes (\Lambda_{i=1}^d b_{i,F'}'), 0)$, then Lemma 7.8(iii) allows us to identify $d_{A[G_{F'}]}(C_{F'})$ with

$$d_{A[G_{F'}]}(P^*_{F'}) = d_{A[G_{F'}]}(P^*_{F'}) \otimes \text{Hom}_{\xi(A[G_{F'}])}(d_{A[G_{F'}]}^\circ(P^*_{F'}), \xi(A[G_{F'}])) = \xi(A[G_{F'}]) \cdot \beta_{F'}$$

(where the first equality follows from (5.3.1) and the second from Proposition 5.6(i)), and then the argument of [4] Lem. 7.3.1] implies that

$$\left(\Lambda_{i=1}^r \phi_i\right)(\Theta_{F'}(\beta_{F'})) = \text{Nrd}_{A[G_{F'}]}\left(M' \mid M\right).$$

This equality combines with (7.2.8) to imply that the equality (7.2.6), and hence also the containment (7.2.3), is valid since the image of any element $\eta$ of $\text{VS}(T,K)$ in $d_{A[G_{F'}]}(C_{F'}) = d_{A[G_{F'}]}(P^*_{F'})$ is of the form $x_\eta \cdot \beta_{F'}$ for a unique element $x_\eta$ of $\xi(A[G_{F'}])$.

To prove (7.2.9) it is enough to prove an equality of maps

$$\text{Cor}^r_{F'/F} \circ \Theta'_{F'} = \left(\prod_{v \in \Sigma} P_v\right) \cdot (\Theta_F \circ \nu_{F'/F}).$$

To do this we set $\Delta := \text{Gal}(F'/F)$ and write $T_\Delta$ for the (central) element $\sum_{\delta \in \Delta} \delta$ of $A[G_F]$. We then identify $A[G_F]$ with the subalgebra $T_\Delta \cdot A[G_{F'}]$ of $A[G_{F'}]$.

Then, since an idempotent $e$ of $\zeta(A[G_{F'}])$ annihilates im($\Theta_{F'}$), respectively im($\Theta'_{F'}$), if and only if $e \cdot e_{F'} = 0$, respectively $e \cdot e_{F'} = 0$, the result of Lemma 7.8 below implies it is enough to verify the above equality after multiplying by a primitive idempotent $e$ of $\zeta(A[G_{F'}])$ with the property that $e \cdot P_v \neq 0$ for all $v$ in $\Sigma$.

But, for any such $e$, the required equality is true since the result of Lemma 7.9 below combines with the explicit definitions of the maps $\Theta'_{F'}, \nu_{F'/F}$ and $\Theta'_{F'}$ to imply the commutativity of the diagram

$$e \cdot d_{A[G_{F'}]}(C_{F',S(F')}(V)) \xrightarrow{\Theta'_{F'}} e \cdot \bigwedge^r_{A[G_{F'}]} H^1(O_{F',S(F')}, V) \xrightarrow{\text{Cor}^r_{F'/F}} e \cdot \bigwedge^r_{A[G_{F}]} H^1(O_{F,S(F')}, V).$$
At this stage we have shown that the image of the diagonal map $\Theta_{T,K'} = (\Theta_L)_L$ is contained in $ES_T(T,K)$ and so to complete the proof of Theorem 7.5 it is enough to prove its final claim.

However, this claim is equivalent to asserting that $\Theta_{T,K}$ is non-zero if and only if there exists a field $F'$ in $\Omega(K/K)$ for which $e_{F'}$ is non-zero and this follows directly from the fact that the isomorphism (7.2.7) implies that the annihilator in $\zeta(A[G_{F'}])$ of the image of $\Theta_{F'}$ is equal to $\zeta(A[G_{F'}])(1 - e_{F'})$.

This completes the proof of Theorem 7.5.

**Lemma 7.8.** For each primitive idempotent $e$ of $\zeta(A[G_{F}])$ one has $e \cdot e_{F'} \neq 0$ if and only if both $e \cdot P_v \neq 0$ for all $v$ in $\Sigma$ and also $e \cdot e_{F} \neq 0$.

**Proof.** Claims (iii) and (v) of Lemma 7.1 combine to induce an identification

$$H^2(O_{F,S(F')}, V) \cong A[G_{F}] \otimes_{A[G_{F'}]} H^2(O_{F',S(F')}, V) = T_\Delta \cdot H^2(O_{F',S(F')}, V)$$

and so the definition of $e_{F'}$ implies that $e \cdot e_{F'} \neq 0$ if and only if $e \cdot H^2(O_{F',S(F')}, V) = 0$.

To study this condition we set $W_F := \text{Hom}_Q(V^*(1)_F, Q)$ and note that the exact cohomology sequence of the triangle in Lemma 7.1(iv) gives rise to an exact sequence

$$\bigoplus_{v \in \Sigma} \ker(\phi_v | W_F) \xrightarrow{\theta} H^2(O_{F,S(F)}, V) \rightarrow H^2(O_{F,S(F')}, V) \rightarrow \bigoplus_{v \in \Sigma} \cok(\phi_v | W_F) \rightarrow 0,$$

where we set $\phi_v := 1 - \sigma_v$.

This sequence implies $e \cdot H^2(O_{F,S(F')}, V) = 0$ if and only if both $e \cdot \cok(\phi_v | W_F) = 0$ for all $v \in \Sigma$ and also $e \cdot \ker(\theta) = 0$. In addition, for each such $v$ the condition $e \cdot \cok(\phi_v | W_F) = 0$ is equivalent to $e \cdot \ker(\phi_v | W_F) = 0$ and hence also to the non-vanishing of $e \cdot P_v$.

Taken together, these facts imply that $e \cdot H^2(O_{F,S(F')}, V) = 0$ if and only if one has both $e \cdot H^2(O_{F,S(F')}, V) = 0$ (or equivalently, $e \cdot e_{F'} \neq 0$) and also $e \cdot P_v \neq 0$ for each $v$ in $\Sigma$, as claimed. \hfill \square

**Lemma 7.9.** Fix a field $F$ in $\Omega(K/K)$, a place $v$ of $K$ outside $S(F)$ and a primitive idempotent $e$ of $\zeta(A[G_{F}])$ such that $e \cdot P_v \neq 0$. Write $C_v$ for the complex $R\Gamma(K_v^{ur}/K_v, V^*(1)_F)^*[−2]$ and set $W_F := \text{Hom}_Q(V^*(1)_F, Q)$.

Then $C_v$ is represented by the complex $W_F \xrightarrow{1-\sigma_v} W_F$, where the first term is placed in degree one. In addition, the complex $e \cdot C_v$ is acyclic and $(e \cdot P_v, 0)$ is equal to the image of $(e, 0)$ under the composite isomorphism

$$(\zeta(A[G_{F}])e, 0) \cong e \cdot (d^\circ_{A[G_{F}]}(W_F) \otimes d^\circ_{A[G_{F}]}(W_F)^{-1})$$

$$\cong e \cdot d^\circ_{A[G_{F}]}(C_v)^{-1}$$

$$= d^\circ_{A[G_{F}]}(e \cdot C_v)^{-1}$$

$$\cong d^\circ_{A[G_{F}]}(e \cdot 0) \otimes d^\circ_{A[G_{F}]}(0)^{-1}$$

$$= (\zeta(A[G_{F}])e, 0).$$

Here the first isomorphism is induced by the canonical identification

$$d^\circ_{A[G_{F}]}(W_F) \otimes d^\circ_{A[G_{F}]}(W_F)^{-1} \cong (\zeta(A[G_{F}]), 0),$$
the second by the identification \(5.3.1\) (with \(A\) replaced by \(A[G_F]\)) and the third by Proposition \(5.14(i)\) (with \(A\) replaced by \(A[G_F]e\)) and the acyclicity of \(e \cdot C_v\).

Proof. It is clear that \(C_v\) is represented by the given complex \(W_F \xrightarrow{1-\sigma_v} W_F\) and hence that \(e \cdot C_v\) is acyclic whenever \(e \cdot P_v \neq 0\).

The remaining assertion is then verified by a straightforward, and explicit, computation (following \[5\] Lem. 10]). The key point is that, since \(W_F = \text{Hom}_Q(V^*(1)_F, Q)\) is a finitely generated free \(Q[G_F]\)-module, the same argument as in Lemma \(3.20\) implies that

\[
\text{Nrd}_{A[G_F]}(1 - \sigma_v | W_F) = \text{Nrd}_{A[G_F]}(1 - \sigma_v | V^*(1)_F)^* = P_v.
\]

\(\Box\)

7.3. The proof of Theorem 6.11. Following Proposition 7.3 we can fix a basis element \(\eta \in VS(T,K)\)
of the \(\xi(\mathcal{A[[Gal(K/K)]]})\)-module \(VS(T,K)\) and then use the homomorphism

\[
\Theta_{T,K} : VS(T,K) \to ES_r(T,K)
\]

constructed in Theorem 7.5 to define a system

\[
\varepsilon := \Theta_{T,K}(\eta) \in ES_r(T,K).
\]

It suffices to show that this system has the properties in claims (i) and (ii) of Theorem 6.11.

In addition, the fact that \(\varepsilon\) has the property in claim (i) follows directly from the argument used to verify the final assertion of Theorem 7.5.

Turning to claim (ii) we note that Lemma 7.1(iii) combines with the isomorphism \(7.2.1\) for \(E = L\) to imply the existence of an isomorphism of \(A[G_L]\)-modules

\[
(7.3.1) \quad H^1(C_{L,S(L)}(T)) \cong H^2(\mathcal{O}_{L,S(L)}, T) \oplus A[G_L]^\tau =: X.
\]

This isomorphism implies that the block matrix \((0_{dL,r} | M_L)\) that occurs in Lemma \(7.4\) iv) constitutes a free presentation \(\Pi_{T,L}\) of \(X\) and that (in the notation of \(4.1.2\)) the \(A[G_L]\)-module \(\text{cok}(\theta_{M_L})\) is isomorphic to \(H^2(\mathcal{O}_{L,S(L)}, T)\).

Thus, since the choice of \(\eta\) implies the element \(x_\eta\) in \(7.2.6\) (with \(F'\) replaced by \(L\)) is a unit of \(\xi(\mathcal{A}[G_L])\), the result of Proposition 4.18(ii) can be applied to deduce both that

\[
(7.3.2) \quad \xi(A[G_L]) \cdot \{(\wedge_{i=1}^{r} \varphi_i)(\varepsilon_L) : \varphi_i \in \text{Hom}_{A[G_L]}(H^1(\mathcal{O}_{L,S(L)}, T), A[G_L])\} = \xi(A[G_L]) \cdot \{(\wedge_{i=1}^{r} \varphi_i)(\varepsilon_{M_L}) : \varphi_i \in \text{Hom}_{A[G_L]}(H^1(\mathcal{O}_{L,S(L)}, T), A[G_L])\}
\]

and that

\[
(\wedge_{i=1}^{r} \varphi_i)(\varepsilon_L) = x_\eta \cdot (\wedge_{i=1}^{r} \varphi_i)(\varepsilon_{M_L}) \in \text{ Fit}_{A[G_L]}^0(H^2(\mathcal{O}_{L,S(L)}, T))
\]

for every subset \(\{\varphi_i\}_{1 \leq i \leq r}\) of \(\text{Hom}_{A[G_L]}(H^1(\mathcal{O}_{L,S(L)}, T), A[G_L])\).

This completes the proof of Theorem 6.11.
7.4. **The proof of Theorem 6.14.** Claim (i) is a direct consequence of the distribution relation (6.1.1) and the fact that $S(L) = S(L_0)$ for all fields $L$ in $\Omega(\mathcal{L}/L_0)$.

However, in order to prove the remaining assertions of Theorem 6.14 we must first refine the construction of Lemma 7.7.

In the rest of this section we shall assume that $\mathcal{A} = \mathcal{O}$. We also set $R_L := \mathcal{O}[\mathcal{G}_L]$ and $R_L := \mathcal{O}[\mathcal{G}_L]$ for each $L$ in $\Omega(\mathcal{L}/K)$.

7.4.1. For each pair of fields $L$ and $L'$ in $\Omega(\mathcal{L}/L_0)$ with $L \subseteq L'$ the isomorphism

$$R_L \otimes_{\mathcal{O}_{\mathcal{L}'}} C_{L',L_0}(T) \cong C_{L,S(L_0)}(T)$$

coming from Lemma 7.1(v) induces a surjective homomorphism

$$H^1(C_{L,S(L_0)}(T)) \to H^1(C_{L,S(L_0)}(T)).$$

The isomorphisms (7.3.1) can be chosen to be compatible with these homomorphisms as $L$ varies and hence combine to induce an isomorphism of $R_L$-modules

$$\lim_{L \in \Omega(\mathcal{L}/L_0)} H^1(C_{L,S(L)}(T)) \cong H^2(\mathcal{O}_L, T) \oplus R_L'$$

where the inverse limit is taken with respect to the homomorphisms (7.3.2).

**Lemma 7.10.** There exists an exact sequence of $R_L$-modules

$$0 \to H^1(\mathcal{O}_L, T) \xrightarrow{\iota_L} P_L \xrightarrow{\theta_L} P_L \xrightarrow{\pi_L} \lim_{L \in \Omega(\mathcal{L}/L_0)} H^1(C_{L,S(L)}(T)) \to 0$$

that has all of the following properties.

(i) The $R_L$-module $P_L$ is free of (finite) rank $d$ with $d > r$.

(ii) There exists an ordered basis $\{b_i\}_{1 \leq i \leq d}$ of $P_L$ with the property that the composite of $\pi_L$ and the isomorphism (7.4.3) sends $b_i$ to the $i$-th element of the standard basis of $R_L'$ if $1 \leq i \leq r$ and to an element of $H^2(\mathcal{O}_L, T)$ otherwise.

(iii) For each field $L$ in $\Omega(\mathcal{L}/L_0)$, with $\Gamma_L := \text{Gal}(\mathcal{L}/L)$, there exists an exact sequence of the form (7.2.3) with the following properties: $P_L = H_0(\Gamma_L, P_L)$ (so $d_L = d$), $\theta_L = H_0(\Gamma_L, \theta_L)$, $\pi_L$ is the map induced by (taking $\Gamma_L$-coinvariants of) the surjective composite homomorphism

$$P_L \xrightarrow{\pi_L} \lim_{L' \in \Omega(\mathcal{L}/L_0)} H^1(C_{L',S(L')}\right) \to H^1(C_{L,S(L)}(T))$$

where the last map is induced by the surjections (7.4.2), and the image of $\{b_i\}_{1 \leq i \leq d}$ in $P_L$ is an $R_L$-basis of $P_L$ with all of the properties required by Lemma 7.7(ii).

**Proof.** At the outset we fix a surjective homomorphism of $R_L$-modules

$$\pi_L : P_L \to \lim_{L \in \Omega(\mathcal{L}/L_0)} H^1(C_{L,S(L)}(T))$$

that has the properties in claims (i) and (ii).

For each $L$ in $\Omega(\mathcal{L}/L_0)$ we then set $P_L := H_0(\mathcal{G}(\mathcal{L}/L), P_L)$ and define $\pi_L$ to be the displayed composite homomorphism in claim (iii).
We now set $S_0 := S(L_0)$ and choose an ordered set of fields $\{L_i : i \in \mathbb{N}\}$ in $\Omega(\mathcal{L}/L_0)$ that is cofinal with respect to inclusion and for each $n$ abbreviate $C_{L_n, S_0}(T), R_{L_n}, P_{L_n}$ and $\pi_{L_n}$ to $C_n(T), R_n, P_n$ and $\pi_n$. We then consider the diagrams

\[
0 \to H^1(\mathcal{O}_{L_{n+1}, S_0}, T) \xrightarrow{i_{n+1}} P_{n+1} \xrightarrow{\theta_{n+1}} P_n \xrightarrow{\pi_{n+1}} H^1(C_{n+1}(T)) \to 0
\]

(7.4.5)

Here the rows are the respective sequences constructed in Lemma 7.7 (with $\pi_{n+1}$ and $\pi_n$ taken to be the maps specified above), $\varrho$ is the natural projection map, $\lambda$ the relevant case of (7.4.4), and $\kappa$ the corestriction map. Finally, $\varrho'$ is a homomorphism of $R_{n+1}$-modules chosen so that the second square commutes and the resulting morphism of complexes represented by the pair $(\varrho', \varrho)$ induces the isomorphism (7.4.1) for the extension $L_{n+1}/L_n$.

One checks that this construction guarantees that the diagram commutes and also that $\varrho'$ induces an isomorphism of $R_n$-modules $H_0(\text{Gal}(L_{n+1}/L_n), P_{n+1}) \cong P_n$.

Hence, if by induction we fix a compatible family of such diagrams for all $n$, then we can pass to the inverse limit over $n$ of the diagrams to obtain an exact sequence of the form (7.4.6). We note that exactness is preserved by this limiting process since each module in the diagram (7.4.5) is compact, and also that the resulting morphism $\pi_{\mathcal{L}}$ coincides with the morphism fixed at the start of this argument.

For each $L$ in $\Omega(\mathcal{L}/L_0)$ we now choose $n$ so that $L \subseteq L_n$. Then the isomorphism (7.4.1) with $L' = L_n$ implies that one obtains an exact sequence of the form (7.2.3) for the field $L$ by taking $\text{Gal}(L_n/L)$-coinvariants of the complex $P_n \xrightarrow{\theta_n} P_n$ fixed above.

With this construction it is also clear that the image of $\{b_i, L\}_{1 \leq i \leq d}$ in $P_L$ is an $R_L$-basis of $P_L$ with the properties stated in Lemma 7.11.

To proceed we note that the exactness of the sequence (7.4.1) combines with Lemma 7.10 (ii) to imply that the matrix of the endomorphism $\theta_L$ with respect to the basis $\{b_i, L\}_{1 \leq i \leq d}$ of $P_L$ is a block matrix of the form $[0_{d,r} \mid M_L]$. Here $M_L$ belongs to $M_{d,d-r}(R_L)$ and is such that for each $L$ in $\Omega(\mathcal{L}/L_0)$ its projection to $M_{d,d-r}(\mathcal{O}[G_L])$ is the matrix $M_L$ that occurs in Lemma 7.7 (iv).

The sequence (7.4.1) therefore combines with the isomorphism (7.4.3) to imply the $R_{\mathcal{L}}$-module $H^2(\mathcal{O}_{L}, T)$ is torsion if and only if there exists a matrix $M'_{\mathcal{L}}$ in $M_{d,r}(R_{\mathcal{L}})$ such that

\[
(M'_{\mathcal{L}} \mid M_L) \in \text{GL}_d(Q(R_{\mathcal{L}})).
\]

Finally we note that each column of $M'_{\mathcal{L}}$ corresponds (via the fixed basis of $P_{\mathcal{L}}$) to an element of $\text{Hom}_{R_{\mathcal{L}}}(P_{\mathcal{L}}, R_{\mathcal{L}})$ and hence, by restriction through $\iota_{\mathcal{L}}$, to an element of $\text{Hom}_{R_{\mathcal{L}}}(H^1(\mathcal{O}_{\mathcal{L}}, T), R_{\mathcal{L}})$.

The following result describes the set of homomorphisms that can arise in this way. This result uses the inverse limit $\text{Hom}_{R_{\mathcal{L}}}(H^1(\mathcal{O}_{\mathcal{L}}, T), R_{\mathcal{L}})$ defined in (6.2.2).

Lemma 7.11. The image of the restriction map

\[
\text{Hom}_{R_{\mathcal{L}}}(P_{\mathcal{L}}, R_{\mathcal{L}}) \to \text{Hom}_{R_{\mathcal{L}}}(H^1(\mathcal{O}_{\mathcal{L}}, T), R_{\mathcal{L}})
\]
that is induced by \( \iota_L \) coincides with the image of the natural map
\[
\text{Hom}^*_R(\Omega^1(O_L, T), R_L) \to \text{Hom}_{R_L}(\Omega^1(O_L, T), R_L).
\]

Proof. The exact sequence (7.4.4) gives rise to an exact commutative diagram
\[
\begin{array}{ccc}
\text{Hom}_{R_L}(P_L, R_L) & \xrightarrow{\iota^*_L} & \text{Hom}_{R_L}(\Omega^1(O_L, T), R_L) \\
\alpha & \approx & \beta \\
\lim_L \text{Hom}_{R_L}(P_L, R_L) & \xrightarrow{(\iota^*_L)_L} & \text{Hom}^*_{R_L}(\Omega^1(O_L, T), R_L).
\end{array}
\]

Here the inverse limit runs over all \( L \) in \( \Omega(L/L_0) \) and is taken with respect to the projection maps that are induced by the fact each \( R_L \)-module \( P_L = H_0(\Gamma_L, P_L) \) is free of rank \( d \), \( \iota^*_L \) denotes the restriction map induced by \( \iota_L \), \( \alpha \) the natural isomorphism (which exists since \( P_L \approx \lim L \in \Omega(L/L_0) L \) is a free \( R_L \)-module), \( (\iota^*_L)_L \) the inverse limit of the homomorphisms
\[
\iota^*_L : \text{Hom}_{R_L}(P_L, R_L) \to \text{Hom}_{R_L}(\Omega^1(O_{L,S(L)}, T), R_L)
\]
that are induced by restriction through \( \iota_L \) and \( \beta \) is the natural map.

To prove the claimed result it is therefore enough to show that \( (\iota^*_L)_L \) is surjective. But, for each \( L \), the image of the endomorphism \( \theta_L \) in the sequence (7.2.5) is torsion-free and so, since \( R_L \) is Gorenstein, the group \( \text{Ext}^1_{R_L}(\text{im}(\theta_L), R_L) \) vanishes (cf. Remark 6.2). From the exactness of (7.2.5) one can therefore deduce that \( \iota^*_L \) is surjective and this surjectivity is then preserved upon passing to the inverse limit over \( L \) since each of the modules \( \ker(\iota^*_L) \) is compact.

### 7.4.2.
To prove claim (ii) of Theorem 6.14 we assume until further notice that \( \text{Gal}(L/K) \) has rank one. In this case we also assume, as we may, that \( \text{Gal}(L/L_0) \) is central in \( \text{Gal}(L/K) \).

Since \( \text{Gal}(L/K) \) has rank one the central conductor formula of Nickel [34] Th. 3.5 implies (via [34] Cor. 4.1) that the group \( \text{Ext}^1_{R_L}(\text{im}(\theta_L), R_L) \) is annihilated by a power of \( p \).

From the diagram (7.4.7) it therefore follows that the condition (7.4.6) is satisfied by a matrix \( M'_L \) in \( M_{d,r}(R_L) \) if and only if it is satisfied by such a matrix with the property that, for each \( i \) with \( 1 \leq i \leq r \), its \( i \)-th column corresponds to an element of \( \text{Hom}_{R_L}(P_L, R_L) \) whose image under \( \iota^*_L \) is equal to \( \beta(\varphi_i) \) for some \( \varphi_i = (\varphi_{i,L})_L \) in \( \text{Hom}^*_{R_L}(\Omega^1(O_L, T), R_L) \).

In addition, in this case the algebra \( Q(R_L) \) is semisimple and so (7.4.6) is satisfied by any such matrix \( M'_L \) if and only if \( \text{Nrd}_{Q(R_L)}(\left( M'_L \mid M_L \right) \) belongs to \( \mathcal{C}(Q(R_L))^\times \).

To complete the proof of Theorem 6.14(ii) it is thus enough to prove the existence of a unit \( u \) of \( \xi(R_L) \) for which one has
\[
(\wedge_{i=1}^r \varphi_i)(\varepsilon_L) = u \cdot \text{Nrd}_{Q(R_L)}(\left( M'_L \mid M_L \right) \).
\]

To show this we note that the explicit description of \( \text{Nrd}_{Q(R_L)} \) that is given by Ritter and Weiss in [38], and reviewed by Nickel in [34] §1], implies that
\[
\text{Nrd}_{Q(R_L)}(\left( M'_L \mid M_L \right) = (\text{Nrd}_{Q(L)}(\left( M'_L \mid M_L \right)))_{L \in \Omega(L/L_0)}
\]
where \( M'_L \) denotes the image of \( M'_L \) in \( M_{d,r}(R_L) \) and the right hand side is regarded as an element of \( \xi(R_L) \).
In addition, for each $L$ in $\Omega(\mathcal{L}/L_0)$ the equality (7.2.8) combines with our choice of homomorphisms $\varphi_i = (\varphi_{i,L})$ to imply that

$$\text{Nrd}_{\mathcal{Q}|\mathcal{L}}((M'_L \mid M_L)) = (\bigwedge^i_{i=1} \varphi_{i,L})(\varepsilon_{M_L}).$$

Next we note that, by an explicit comparison of the definitions of the elements $\varepsilon_L$ and $\varepsilon_{M_L}$ for each $L$ in $\Omega(\mathcal{L}/L_0)$, one deduces the existence of a (unique) unit $u$ of $\xi(R_L)$ for which one has $\varepsilon_L = u \cdot \varepsilon_{M_L}$ for all such $L$.

Hence, since

$$(\bigwedge^i_{i=1} \varphi_{i,L})(\varepsilon_L) = ((\bigwedge^i_{i=1} \varphi_{i,L})(\varepsilon_{L}))_{L \in \Omega(\mathcal{L}/L_0)},$$

the last two displayed equalities then combine to imply that this element $u$ validates the required equality (7.4.8), and hence completes the proof of Theorem 6.14(ii).

7.4.3. To prove claim (iii) of Theorem 6.14 we no longer assume that $\text{Gal}(\mathcal{L}/K)$ has rank one but we do assume to be given a subset $\{\varphi_i\}_{1 \leq i \leq r}$ of $\text{Hom}^*_\mathcal{L}(H^1(\mathcal{O}_{\mathcal{L}}, T), R_L)$ for which $$(\bigwedge^i_{i=1} \varphi_i)(\varepsilon_{\mathcal{L}})$$ is a non-zero divisor in the ideal $\delta(R_L) = \lim_{L \in \Omega(\mathcal{L}/L_0)} \delta(R_L)$ of $\zeta(R_L)$.

Then the square of $$(\bigwedge^i_{i=1} \varphi_i)(\varepsilon_{\mathcal{L}})$$ is a non-zero divisor in $\zeta(R_L)$ and so it is enough to prove that this element annihilates $H^2(\mathcal{O}_{\mathcal{L}}, T)$.

To show this we note the given assumption on the homomorphisms $\varphi_i = (\varphi_{i,L})_{L \in \Omega(\mathcal{L}/L_0)}$ implies that for all $L$ in $\Omega(\mathcal{L}/L_0)$ one has $$(\bigwedge^i_{i=1} \varphi_{i,L})(\varepsilon_{L}) \in \delta(R_L).$$ For each such $L$, Theorem 6.11(ii) therefore implies that

$$( (\bigwedge^i_{i=1} \varphi_{i,L})(\varepsilon_{L}) )^2 \in \delta(R_L) \cdot \text{Fit}^0_{R_L}(H^2(\mathcal{O}_{L,S(L)}, T)) \subseteq \text{Ann}_{R_L}(H^2(\mathcal{O}_{L,S(L)}, T)),$$

where the inclusion follows from the general result of Theorem 3.17(iii).

This containment in turn implies that the square of $$(\bigwedge^i_{i=1} \varphi_i)(\varepsilon_{L})$$ annihilates the module $H^2(\mathcal{O}_{\mathcal{L}}, T) = \lim_{L \in \Omega(\mathcal{L}/L_0)} H^2(\mathcal{O}_{L,S(L)}, T)$, as required to prove claim (iii).

7.4.4. Turning now to claim (iv) of Theorem 6.14 we assume that $\mathcal{L}/K$ is abelian so that $Q(R_L)$ is a finite product of fields.

In this case the exact sequence (7.4.4) combines with the isomorphism (7.4.3) to imply that $H^1(\mathcal{O}_{\mathcal{L}}, T)$ spans over each field component of $Q(R_L)$ a vector space of dimension at least $r$ and that this dimension is equal to $r$ in each component if and only if $H^2(\mathcal{O}_{\mathcal{L}}, T)$ is a torsion $R_L$-module.

This implies, in particular, that $H^2(\mathcal{O}_{\mathcal{L}}, T)$ is a torsion $R_L$-module if and only if the $Q(R_L)$-module spanned by $\bigwedge^r_{i=1} H^1(\mathcal{O}_{\mathcal{L}}, T)$ is free of rank one.

Thus, since the earlier argument showed that $H^2(\mathcal{O}_{\mathcal{L}}, T)$ is a torsion $R_L$-module if $\varepsilon_{\mathcal{L}}$ satisfies the condition stated in claim (ii), respectively in claim (iii), it is clear that the condition in (iv)(a) implies the condition in (iv)(b). Since it is obvious that (iv)(b) implies the condition in (iv)(c) it is therefore enough to show that the latter condition implies that $\varepsilon_{\mathcal{L}}$ satisfies the condition in either claim (ii) or claim (iii).

To do this we note that the exact sequence (7.4.4) implies the existence of a unit $u$ of $R_L$ such that

$$\varepsilon_{\mathcal{L}} = u \cdot (\bigwedge^{r \leq i \leq d} (b_i \circ \theta_{\mathcal{L}})(\bigwedge^d_{i=1} b_{i,\mathcal{L}})).$$
In particular, if this element is not annihilated by any non-zero divisor of \( R_\mathcal{L} \), then for each field component of \( Q(R_\mathcal{L}) \) there exists an \( r \)-tuple \( \{i_1, \ldots, i_r\} \) of integers with \( 1 \leq i_j \leq d \) such that the corresponding component of \( (\land_{1 \leq j \leq r}(b_{i_j}^\mathcal{L}))(\epsilon_\mathcal{L}) \) is non-zero.

By using this fact it is then easy to construct a subset \( \{\varphi_1\}_{1 \leq i \leq r} \) of \( \text{Hom}_{R_\mathcal{L}}(P_\mathcal{L}, R_\mathcal{L}) \), and hence via the diagram (7.4.7), of \( \text{Hom}_{R_\mathcal{L}}(H^1(O_\mathcal{L}, T), R_\mathcal{L}) \) such that \( (\land_{1 \leq i \leq r}(\varphi_i))(\epsilon_\mathcal{L}) \) is a non-zero divisor in \( R_\mathcal{L} = \delta(R_\mathcal{L}) \), as required to verify the condition in (ii)(a).

This completes the proof of Theorem 6.14.

8. Extended cyclotomic Euler systems

In this final section we shall use the techniques developed above to construct (unconditionally) an extension of the classical Euler system of cyclotomic units to the setting of general totally real Galois extensions of \( \mathbb{Q} \) and verify that this extended system has all of the properties that are stated in Theorem 1.1.

We fix an odd prime \( p \) and write \( A_p \) for the pro-\( p \) completion of an abelian group \( A \). We also fix an isomorphism of fields \( \mathbb{C} \cong \mathbb{C}_p \) and use it to identity (without further explicit comment) the set \( \text{Ir}(\Gamma) \) of irreducible complex characters of a finite group \( \Gamma \) with the corresponding set \( \text{Ir}_p(\Gamma) \) of irreducible \( \mathbb{C}_p \)-valued characters of \( \Gamma \).

8.1. The construction.

8.1.1. For any finite non-empty set of places \( \Sigma \) of \( \mathbb{Q} \) and any number field \( F \) we write \( \Sigma_F \) for the set of places of \( F \) lying above those in \( \Sigma \), \( Y_{F,\Sigma} \) for the free abelian group on the set \( \Sigma_F \) and \( X_{F,\Sigma} \) for the submodule of \( Y_{F,\Sigma} \) comprising elements whose coefficients sum to zero.

If \( \Sigma \) contains \( \infty \), then we write \( O_{F,\Sigma} \) for the subring of \( F \) comprising elements integral at all places outside \( \Sigma_F \). (If \( \Sigma = \{\infty\} \), then we usually abbreviate \( O_{F,\Sigma} \) to \( O_F \).)

For any such \( \Sigma \) the \( \Sigma \)-relative Selmer group \( \text{Sel}_{\Sigma}(F) \) for \( \mathbb{G}_m \) over \( F \) is defined in [7, §2.1] to be the cokernel of the homomorphism

\[
\prod_{w \not\in \Sigma_F} \mathbb{Z} \rightarrow \text{Hom}_{\mathbb{Z}}(F^\times, \mathbb{Z}),
\]

where \( w \) runs over all places of \( F \) that do not belong to \( \Sigma_F \) and the arrow sends \( (x_w)_w \) to the map \( u \mapsto \sum_w \text{ord}_w(u)x_w \) with \( \text{ord}_w \) the normalised additive valuation at \( u \). (This group is a natural analogue for \( \mathbb{G}_m \) of the integral Selmer groups of abelian varieties that are defined by Mazur and Tate in [29,].)

We recall that \( \text{Sel}_{\Sigma}(F)_p \) lies in a canonical exact sequence

\[
0 \rightarrow \text{Cl}(O_{F,\Sigma})^\vee_p \rightarrow \text{Sel}_{\Sigma}(F)_p \rightarrow \text{Hom}_{\mathbb{Z}_p}(O_{F,\Sigma, p}^\times, \mathbb{Z}_p) \rightarrow 0
\]

and has a subquotient that is canonically isomorphic to \( \text{Cl}(O_F)^\vee_p \) (cf. [7 Prop. 2.2]).

If \( F \) is Galois over \( \mathbb{Q} \) (so that \( F \in \Omega(Q^c/\mathbb{Q}) \)), then we set \( \mathcal{G}_F := \text{Gal}(F/\mathbb{Q}) \) and

\[
C_F := C_{F,\Sigma_F}(\mathbb{Z}_p(1)),
\]

where the latter complex is as defined in [7.1.1] and also \( C_F^* := \text{RHom}_{\mathbb{Z}_p}(\mathbb{C}_F, \mathbb{Z}_p) \).

In this case, for each character \( \chi \) in \( \text{Ir}(\mathcal{G}_F) \), and each integer \( a \), we also write \( L_{\mathcal{S}(F)}^a(\chi, 0) \) for the coefficient of \( z^a \) in the Laurent expansion of \( L_{\mathcal{S}(F)}^a(\chi, z) \) at \( z = 0 \). We recall that \( \check{\chi} \) denotes the contragredient of \( \chi \).
Lemma 8.1. For every finite Galois extension $F$ of $\mathbb{Q}$ the following claims are valid.

(i) $H^0(\mathcal{O}_{F,S(F)}, \mathbb{Z}_p(1))$ vanishes.

(ii) $H^1(\mathcal{O}_{F,S(F)}, \mathbb{Z}_p(1)) = H^1(C_F)$ identifies with $O^\times_{F,S(F),p}$.

(iii) There exists a canonical exact sequence
\[ 0 \to \text{Cl}(\mathcal{O}_{F,S(F)})_p \to H^1(C_F) \to X_{F,S(F),p} \to 0. \]

(iv) $H^2(C_F^*[-2]) = H^2(\mathcal{O}_{F,S(F)}, \mathbb{Z}_p)$ identifies with $\text{Sel}_{S(F)}(F)_p$.

(v) $Y_F(\mathbb{Z}_p) = Y_{F,S_{\infty}(F),p}$ and if $F$ is totally real, then the idempotent $e_F = e_{F,\mathbb{Z}_p(1)}$ defined in \eqref{6.2.1} is equal to $\sum \chi e_\chi$ where $\chi$ runs over all characters in $\text{Irr}(\mathcal{G}_F)$ for which $I_{\chi}^{(1)}(\chi, 0) \neq 0$.

Proof. Claim (i) is obvious and the existence of the identification in claim (ii) and exact sequence in claim (iii) are well-known, being respectively induced by Kummer theory and class field theory.

Since the complex $C_F^*[-2] = (\mathcal{O}_{F,S(F)}, \mathbb{Z}_p)^*[-2]$ is canonically isomorphic to $\mathcal{O}_{F,S(F)}(\mathbb{Z}_p)$ the identification in claim (iv) is proved by the argument of \cite[Prop. 2.4(iii)]{7}.

The first assertion of claim (v) is obvious. To prove the second assertion, we note that the exact sequence in claim (iii) combines with that in Lemma \ref{7.1}(iii) to imply $e_F$ is equal to $\sum \chi e_\chi$ where $\chi$ runs over all characters in $\mathcal{G}_F$ for which the natural map $e_\chi : (\mathbb{C} \otimes X_{F,S(F)}) \to e_\chi(\mathbb{C} \otimes Y_{F,S_{\infty}(F)})$ is bijective.

In addition, if $F$ is totally real, then the $\mathcal{G}_F$-module $Y_{F,S_{\infty}(F)}$ is free of rank one and so Remark \ref{2.5} implies $r_{\mathbb{C}[\mathcal{G}_F]}e_\chi(e_\chi(\mathbb{C} \otimes Y_{F,S_{\infty}(F)})) = \chi(1)$.

Given these observations, the second assertion of claim (v) follows directly from the formula
\begin{equation}
\text{ord}_{z=0} L_{\mathcal{G}_F}(\chi, z) = r_{\mathbb{C}[\mathcal{G}_F]}e_\chi(e_\chi(\mathbb{C} \otimes X_{F,S(F)}))
\end{equation}
that is proved, for example, in \cite[Chap. I, Prop. 3.4]{44}.

\begin{flushright}
\Box
\end{flushright}

Remark 8.2. If $\Sigma$ is any finite set of places of $F$ that contains all archimedean and $p$-adic places, then the same approach as above shows that $H^0(C_{F,\Sigma}(\mathbb{Z}_p(1))) = O^\times_{F,\Sigma,p}$ and that there is a canonical exact sequence
\[ 0 \to \text{Cl}(\mathcal{O}_{F,\Sigma})_p \to H^1(C_{F,\Sigma}(\mathbb{Z}_p(1))) \to X_{F,\Sigma,p} \to 0. \]

8.1.2. Since $p$ is fixed we shall in the sequel set
\[ \Lambda := \mathbb{Z}_p[[\text{Gal}(\mathbb{Q}^{c,+}/\mathbb{Q})]] \text{ and } \Lambda_{ab} := \mathbb{Z}_p[[\text{Gal}(\mathbb{Q}^{ab,+}/\mathbb{Q})]]. \]
We also abbreviate the sets $\Omega(\mathbb{Q}^{c,+}/\mathbb{Q})$ and $\Omega(\mathbb{Q}^{ab,+}/\mathbb{Q})$ to $\Omega$ and $\Omega_{ab}$ respectively.

For each field $F$ in $\Omega$ we assume that the constructions of reduced exterior powers, reduced Rubin lattices and reduced determinants over the group ring $\mathbb{Q}_p[\mathcal{G}_F]$ are normalized via the choice of data fixed in \cite[6.11]{6.11} (with $K = \mathbb{Q}$).

At the outset we check that the conditions of Theorem \ref{6.11} and Theorem \ref{7.5} are satisfied with $K = \mathbb{Q}, \mathcal{K} = \mathbb{Q}^{c,+}, T = \mathbb{Z}_p(1)$ (so that $T^*(1) = \mathbb{Z}_p$) and $A = \mathbb{Z}_p$.

Firstly, since $p$ is odd, for each $L$ in $\Omega$ the group $O^\times_{L,S(L),p}$ is torsion-free and so Lemma \ref{8.1}(ii) implies condition (a) of Theorem \ref{6.11} is satisfied. In addition, since $A = \mathbb{Z}_p$ is
local, condition (b) is clear and condition (c) is satisfied with \( r := r_{\mathbb{Z}_p(1)} \) equal to 1 since \( Y_Q(\mathbb{Z}_p) = \mathbb{Z}_p \). Finally, condition (d) is true by the very definition of \( \mathbb{K} = \mathbb{Q}^{c,+} \).

Next we fix a field embedding

\[
\sigma : \mathbb{Q}^c \to \mathbb{C}.
\]

Then for each \( L \) in \( \Omega \) the restriction of \( \sigma \) gives an embedding \( \sigma_L : L \to \mathbb{R} \). This embedding \( \sigma_L \) defines a place \( w_L \) in \( S_\infty(L) \) that constitutes, via Lemma 8.1(v), a \( \mathbb{Z}_p[\mathcal{G}_L] \)-basis of \( Y_L(\mathbb{Z}_p) \). In this way the fixed embedding \( \sigma \) gives rise to a compatible family of isomorphisms of the form \( \mathbf{7.2.1} \). By applying Theorem 7.5 with respect to this family of isomorphisms, we obtain canonical homomorphisms of \( \xi(\Lambda) \)-modules

\[
\Theta := \Theta_{\mathbb{Z}_p(1), \mathbb{Q}^{c,+}} \text{ and } \Theta^{ab} := \Theta_{\mathbb{Z}_p(1), \mathbb{Q}^{ab,+}}.
\]

For each natural number \( n \) we write \( \zeta_n \) for the unique primitive \( n \)-th root of unity in \( \mathbb{Q}^c \) that satisfies

\[
\sigma(\zeta_n) = e^{2\pi i/n}.
\]

(It is then clear that \( \zeta_n = (\zeta_n)^{n/m} \) for all divisors \( m \) of \( n \).) For each field \( L \) in \( \Omega^{ab} \), of conductor \( f(L) \), we then define an element of \( L^\times \) by setting

\[
\epsilon_L := \text{Norm}_{\mathbb{Q}(\zeta_f(L))/\mathbb{Q}}(1 - \zeta_f(L)).
\]

The following result will be deduced from the validity of the equivariant Tamagawa Number Conjecture for abelian fields, as proved by the first author and Greither in [6].

**Theorem 8.3.** There exists a basis element \( \eta^{ab} \) of the \( \Lambda^{ab} \)-module \( \text{VS}(\mathbb{Z}_p(1), \mathbb{Q}^{ab,+}) \) with the property that for every \( L \) in \( \Omega^{ab} \) one has

\[
\Theta_L(\eta^{ab}) = \begin{cases} 
\epsilon_L, & \text{if } p \text{ ramifies in } L, \\
(\epsilon_L)^{1-\sigma_{p,L}}, & \text{if } p \text{ is unramified in } L,
\end{cases}
\]

where \( \Theta_L \) denotes the \( L \)-component of \( \Theta^{ab} \) and \( \sigma_{p,L} \) the restriction of \( \sigma_p \) to \( L \).

**Proof.** For \( E \) in \( \Omega^{ab} \) we write \( \vartheta_E \) for the composite morphism in \( \mathcal{P}(\mathbb{C}_p[\mathcal{G}_E]) \)

\[
\mathbb{C}_p \otimes_{\mathbb{Z}_p} \mathbb{d}_{\mathbb{Z}_p[\mathcal{G}_E]}(C_E) \sim \mathbb{d}_{\mathbb{C}_p[\mathcal{G}_E]}(\mathbb{C}_p \otimes_{\mathbb{Z}} \mathcal{O}_{E,S(E)}^\times) \otimes \mathbb{d}_{\mathbb{C}_p[\mathcal{G}_E]}(\mathbb{C}_p \otimes_{\mathbb{Z}} X_{E,S(E)})^{-1} \sim (\mathbb{C}_p[\mathcal{G}_E], 0).
\]

Here the first morphism is induced by the descriptions in Lemma 8.1(ii) and (iii) and the natural passage-to-cohomology map (as in Proposition 5.14(i)), the final morphism is the canonical evaluation map and the second morphism is induced by the scalar extension of the Dirichlet regulator isomorphism

\[
\text{Reg}_{E,S(E)} : \mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_{E,S(E)}^\times \cong \mathbb{R} \otimes_{\mathbb{Z}} X_{E,S(E)}
\]

that sends \( u \) in \( \mathcal{O}_{E,S(E)}^\times \) to \( -\sum_w \log(|u|_w) \cdot w \), where in the sum \( w \) runs over \( S(E)_E \) and \( | \cdot |_w \) denotes the absolute value with respect to \( w \) (normalized as in [44] Chap. 0, 0.2]).
We write \( \eta_E \) for the pre-image under \( \vartheta_E \) of the element

\[
\theta^*_{E,S(E)}(0) := \left( \sum_{\chi \in \eta(G_E)} L^*_S(\chi^{-1}, 0) \epsilon_\chi, 0 \right)
\]

doing \((\mathbb{C}_p[G_E])^\times, 0)\), where \( L^*_S(\chi^{-1}, 0) \) denotes the leading term in the Taylor expansion at \( z = 0 \) of the series \( L_S(\chi^{-1}, z) \).

Then, since \( p \) is odd, Lemma 8.4 below implies that the collection

\[
\eta^{ab} := (\eta_E^2)_{E \in \Omega^{ab}}
\]

constitutes a \( \Lambda^{ab} \)-basis of \( \text{VS}(\mathbb{Z}_p(1), \mathbb{Q}^{ab,+}_E) \). The claimed result will therefore follow if we can show that for every \( L \) in \( \Omega^{ab} \) one has

\[
\Theta_L(\eta^{ab}) = (\epsilon_L)^x_L
\]

with \( x_L = 1 \) if \( p \) ramifies in \( L \) and \( x_L = 1 - \sigma_p \) otherwise.

To do this we fix \( L \), abbreviate its conductor \( f(L) \) to \( f \) and write \( \mathbb{Q}(f) \) for the maximal real subfield of \( \mathbb{Q}(\zeta_f) \). We also write \( S(f) \) for the subset of \( S(L) \) comprising \( \infty \) and all prime divisors of \( f \) (so that \( S(L) = S(f) \) if \( p \) is ramified in \( F \) and otherwise \( S(L) \setminus S(f) = \{ p \} \)).

We set \( G_f := \mathcal{G}_f \), \( x_f := x_{\mathbb{Q}(f)} \) and \( \epsilon_f := \epsilon_{\mathbb{Q}(f)} \).

Then it is enough to prove the above equality with \( L \) replaced by \( \mathbb{Q}(f) \). The key point now is to recall (from, for example [44 Chap. 3, §5]) that for each \( \chi \) in \( \mathbb{I}(G_f) \) the first derivative \( L^1_S(\chi, z) \) of \( L_S(\chi, z) \) is holomorphic at \( z = 0 \) and that the normalization \( (8.1.4) \) implies

\[
L^1_S(\chi, 0) = \frac{-1}{2} \sum_{g \in G_f} \chi(g) \log |\sigma(1 - \zeta_g^0)^{1+\tau} |
\]

where \( \tau \) denotes complex conjugation.

Now \( L^1_S(\mathbb{Q}(f))(\chi, 0) = \chi(x_f) \cdot L^1_S(\chi, 0) \) and so Lemma 8.1(v) implies that \( \epsilon_\chi \cdot \epsilon_f \neq 0 \) if and only if both \( \chi(x_f) \neq 0 \) and \( L^1_S(\chi, 0) \neq 0 \).

The above displayed equality therefore implies, firstly, that the image in \( \mathbb{Q} \cdot \mathcal{O}^\times_{\mathbb{Q}(f), S(f)} \) of \( (1 - \zeta_f)^{x_f (1+\tau)/2} \) is stable under the action of the idempotent \( \epsilon_f \) then, secondly, that its image under the isomorphism \( (8.1.5) \) is equal to \( (\epsilon_f \cdot \theta^*_{\mathbb{Q}(f), S(f)}(0)) \cdot (w_{\mathbb{Q}(f)} - w_0) \cdot 0 \) where \( w_0 \) is any choice of \( p \)-adic place of \( \mathbb{Q}(f) \). This latter fact then combines with the explicit definition (via \( (7.2.2) \)) of the map \( \Theta_{\mathbb{Q}(f)} \) to imply that

\[
\Theta_{\mathbb{Q}(f)}(\eta^{ab}) = \epsilon_f (1 - \zeta_f)^{x_f (1+\tau)} = (1 - \zeta_f)^{(1+\tau)x_f} = (\epsilon_{\mathbb{Q}(f)})^{x_f},
\]

as required. \( \square \)

**Lemma 8.4.** The \( \Lambda^{ab} \)-module \( \text{VS}(\mathbb{Z}_p(1), \mathbb{Q}^{ab,+}_E) \) is free of rank one, with basis \( (\eta_E)_{E \in \Omega^{ab}} \).

**Proof.** At the outset, we fix \( E \) in \( \Omega^{ab} \) and recall (from [7 Prop. 3.4]) that the equivariant Tamagawa Number Conjecture for the pair \((h^E(\text{Spec}(E)), \mathbb{Z}_p[G_E])\) asserts that \( d_{\mathbb{Z}_p[G_E]}(C_E) \) is a free (graded) \( \mathbb{Z}_p[G_E] \)-module with basis \( \eta_E \). We further recall that, since \( p \) is odd, this conjecture is known to be valid by the main result of [6].
Given the explicit definition (in Definition 7.2) of VS($\mathbb{Z}_p(1), \mathbb{Q}^{ab,+}$) as an inverse limit, the claimed result will therefore follow if we can show that for each pair of fields $E$ and $E'$ in $\Omega^{ab}$ with $E \subset E'$ one has $\nu_{E'/E}(\eta_{E'}) = \eta_{E}$.

To prove this we note that for each place $v$ in $S(E') \setminus S(E)$ the discussion before (7.11) identifies $\text{RG}(v, \mathbb{Z}_p[\mathcal{G}_E])^*[-1]$ with the complex $\Psi_v$ that is equal to $\mathbb{Z}_p[\mathcal{G}_E]$ in degrees zero and one (upon which each element $g$ of $\mathcal{G}_E$ acts as multiplication by $g^{-1}$) and has the differential $x \mapsto (1 - \sigma_v)x$.

We write $Y_v$ for the free abelian group on the set of places of $E$ above $v$ and, fixing a place $w_v$ of $E$ above $v$, note there are isomorphisms $\iota_v^0: H^i(\Psi_v) \cong Y_v$ for $i \in \{0, 1\}$ with $\iota_v^0(x) = |\mathcal{G}_E|^{-1}x(w_v)$ and $\iota_v^1(x) = x(w_v)$ where $\mathcal{G}_{E,v}$ denotes the decomposition subgroup of $v$ in $\mathcal{G}_E$.

The key fact now is that the $\text{Gal}(E'/E)$-invariants of $\mathcal{G}_E$ differs from the composite $\mathcal{G}_E \circ (\mathbb{C}_p \otimes \mathbb{Q}_{E'/E})$ only in that for each $v$ in $S(E') \setminus S(E)$ and $\chi$ in $\text{Hom}(\mathcal{G}_E, \mathbb{C}_p)$ these maps respectively use the upper and lower composite homomorphisms in the diagram

\[\begin{align*}
e^\chi(\mathbb{C}_p \cdot d_{\mathbb{Z}_p[\mathcal{G}_E]}(\Psi_v)) & \xrightarrow{\alpha_1} d_{\mathbb{C}_p}^\chi(e^\chi(\mathbb{C}_p \cdot Y_v)) \otimes d_{\mathbb{C}_p}^\chi(e^\chi(\mathbb{C}_p \cdot Y_v))^{-1} \xrightarrow{\alpha_2} (\mathbb{C}_p[\mathcal{G}_E]e^\chi, 0) \\
e^\chi(\mathbb{C}_p \cdot d_{\mathbb{Z}_p[\mathcal{G}_E]}(\Psi_v)) & \xrightarrow{\alpha_1} d_{\mathbb{C}_p}^\chi(\mathbb{C}_p[\mathcal{G}_E]e^\chi) \otimes d_{\mathbb{C}_p}^\chi(\mathbb{C}_p[\mathcal{G}_E]e^\chi)^{-1} \xrightarrow{\alpha_3} (\mathbb{C}_p[\mathcal{G}_E]e^\chi, 0).
\end{align*}\]

Here we abbreviate $d_{\mathbb{C}_p[\mathcal{G}_E]}^\chi(-)$ to $d_{\mathbb{C}_p}^\chi(-)$, $\alpha_1$ is the standard ‘passage to cohomology’ isomorphism induced by (Proposition 5.14(i) and) the maps $\iota_v^0$ and $\iota_v^1$, $\alpha_2$ is the morphism induced by multiplication by $\log(N(w_v))$ on $e^\chi(\mathbb{C}_p \cdot Y_v)$, $\alpha_3$ is the identification resulting from (5.3.1) and the fact that each non-zero term of $e^\chi(\mathbb{C}_p \otimes \Psi_v)$ identifies with $\mathbb{C}_p[\mathcal{G}_E]e^\chi$, $\alpha_4$ is the standard isomorphism and we have set

\[\epsilon_v^\chi := \begin{cases} 1 - \chi^{-1}(\sigma_v), & \text{if } \chi(\sigma_v) \neq 1 \\ |\mathcal{G}_E|^{-1} \cdot \log(N(w_v)) = \log(N(v)), & \text{otherwise}. \end{cases}\]

The claimed result then follows from the fact that the argument of Lemma 7.9 implies the above diagram commutes, whilst an explicit computation shows that for every $\chi$ in $\text{Hom}(\mathcal{G}_E, \mathbb{C}_p)$ one has

\[L^*_S(E')(\chi^{-1}, 0) = \prod_{v \in S(E') \setminus S(E)} \epsilon_v^\chi \cdot L^*_S(E)(\chi^{-1}, 0).\]

\[\square\]

8.1.3. Following Proposition 7.4 we fix a $\Lambda$-basis element $\eta'$ of VS($\mathbb{Z}_p(1), \mathbb{Q}^{c,+}$).

Then the image of $\eta'$ under the natural projection map

\[(8.1.6) \quad \text{VS}(\mathbb{Z}_p(1), \mathbb{Q}^{c,+}) \to \text{VS}(\mathbb{Z}_p(1), \mathbb{Q}^{ab,+})\]

is a basis of the latter module over $\xi(\Lambda^{ab}) = \Lambda^{ab}$ and hence, following Theorem 5.3, equal to $v \cdot \eta^{ab}$ for some element $v$ of $\Lambda^{ab,x}$.
Since the natural projection map $\xi(\Lambda)^{\times} \to \xi(\Lambda^{ab})^{\times} = \Lambda^{ab,\times}$ is surjective (by Lemma 8.5 below) we can then choose a pre-image $u$ of $\nu^{-1}$ under this map. The element

$$\eta := u \cdot \eta'$$

is then a pre-image of $\eta^{ab}$ under the map \[8.1.6\].

By applying Theorem \[6.11\] in this setting we therefore obtain an element

\begin{equation}
\varepsilon^{\text{cyc}} := \Theta(\eta)
\end{equation}

of $\text{ES}_1(Z_p(1), \mathbb{Q}^c)^\pm$ and, with this definition, Theorem \[8.3\] implies directly that $\varepsilon^{\text{cyc}}$ has the property stated in Theorem \[1.1\](i).

**Lemma 8.5.** The natural projection map $\Lambda \to \Lambda^{ab}$ induces a surjective homomorphism from $\xi(\Lambda)^{\times}$ to $\xi(\Lambda^{ab})^{\times} = \Lambda^{ab,\times}$.

**Proof.** We use the same notation as in the proof of Proposition \[7.3\] (with $K = \mathbb{Q}$). For each natural number $n$ we also set $\Gamma_n := \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$.

Then, since for each $n$ the $\mathbb{Z}_p$-algebra $\xi(Z_p[\Gamma_n])$ is semi-local, Bass’s Theorem implies that the natural (surjective) projection map $\pi_n : \xi(Z_p[\Gamma_n]) \to \xi(Z_p[\Gamma^{ab}_n]) = Z_p[\Gamma^{ab}_n]$ restricts to give a surjective homomorphism

$$\pi_n^\times : \xi(Z_p[\Gamma_n])^\times \to Z_p[\Gamma^{ab}_n]^\times.$$ 

It suffices for us to show that the inverse limit over $n$ of these maps $\pi_n^\times$ is itself surjective, and for this it is enough to show, by the Mittag-Leffler criterion, that the natural projection map $\varrho_n : \xi(Z_p[\Gamma_n]) \to \xi(Z_p[\Gamma_n])$ is such that $\varrho_n(\ker(\pi_n)) = \ker(\pi_{n-1})$. 

As a first step we claim that $\varrho_n(\ker(\pi_n)) = \ker(\varrho_{n-1})$. To show this we consider the exact commutative diagram

$$
\begin{array}{cccccccc}
0 & \longrightarrow & \ker(\pi_n) & \longrightarrow & \xi(Z_p[\Gamma_n]) & \longrightarrow & Z_p[\Gamma^{ab}_n] & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \varrho_n & & \varrho_n^{ab} & & \\
0 & \longrightarrow & \ker(\pi_{n-1}) & \longrightarrow & \xi(Z_p[\Gamma_{n-1}]) & \longrightarrow & Z_p[\Gamma^{ab}_{n-1}] & \longrightarrow & 0
\end{array}
$$

in which the first vertical arrow is the restriction of $\varrho_n$ and $\varrho_n^{ab}$ is the natural projection map. In particular, since $\varrho_n$ is surjective, the Snake Lemma implies it is enough to show that $\pi_n(\ker(\varrho_n)) = \ker(\varrho_n^{ab})$.

Next we note that the kernel of $\varrho_n^{ab}$ is generated over $\mathbb{Z}_p$ by elements of the form $\gamma(\delta - 1)$ where $\gamma$ belongs to $\Gamma_n^{ab}$ and $\delta$ to the kernel of the projection $\Gamma_n^{ab} \to \Gamma_{n-1}^{ab}$. 

We choose elements $\gamma'$ and $\delta'$ of $\Gamma_n$ which project to $\gamma$ and $\delta$ in $\Gamma_n^{ab}$. Then $\varrho_n(\delta')$ belongs to the commutator subgroup of $\Gamma_{n-1}$ and so we can choose a finite set of commutators $\{[\delta'_{1,1}, \delta'_{1,2}]\}_{1 \leq i \leq m}$ in $\Gamma_n$ such that the element $\delta' \prod_{1 \leq i \leq m} [\delta'_{1,1}, \delta'_{1,2}]^{-1}$ projects to the identity element of $\Gamma_{n-1}$.

It is then easy to check that, writing $M_1$ and $M_2$ for the $1 \times 1$ matrices with entries $\gamma' \delta' \prod_{1 \leq i \leq m} [\delta'_{1,1}, \delta'_{1,2}]^{-1}$ and $\gamma'$ respectively, the element $\text{Nrd}_{\mathbb{Q}[\Gamma_{n}]}(M_1) - \text{Nrd}_{\mathbb{Q}[\Gamma_{n}]}(M_2)$ of $\xi(Z_p[\Gamma_n])$ belongs to $\ker(\varrho_n)$ and is sent by $\pi_n$ to $\gamma(\delta - 1)$. It follows that $\pi_n(\ker(\varrho_n)) = \ker(\varrho_n^{ab})$, as claimed above.
To proceed we now decompose $\xi(\mathbb{Z}_p[\Gamma_n])$ as a finite product of local rings $\prod_{j \in J} R_j$.
Then for each index $j$ there are ideals $I_j$, $I'_j$ and $I''_j$ of $R_j$ such that $\ker(\pi_n) = \prod_{j \in J} I''_j$,
$\xi(\mathbb{Z}_p[\Gamma_{n-1}]) = \prod_{j \in J} R_j/I_j$ and $\ker(\pi_{n-1}) = \prod_{j \in J} I'_j/I_j$.

In addition, for any index $j$ and any proper ideal $I$ of $R_j$ one has $1 + I \subset R_j^\times$ and so

$$\ker(\pi_n^\times) = \prod_{j \in J_1} R_j^\times \times \prod_{j \in J \setminus J_1} (1 + I'_j/I_j)$$

and

$$\ker(\pi_{n-1}^\times) = \prod_{j \in J_2} (R_j/I_j)^\times \times \prod_{j \in J \setminus J_2} (1 + I'_j/I_j)$$

with $J_1 := \{ j \in J : I'_j = R_j \}$, $J_2 := \{ j \in J : I'_j = R_j \neq I_j \}$ and $J_3 := \{ j \in J : I'_j \neq I_j \}$.

Now, since $\varrho_n(\ker(\pi_n)) = \ker(\pi_{n-1})$, one has $J_2 \subseteq J_1$, $J_3 \setminus J_2 \subseteq J \setminus J_1$ and $I'_j + I_j = I'_j$ for all $j \in J_3 \setminus J_2$. These facts then combine with the above decompositions to imply that

$$\varrho_n(\ker(\pi_n^\times)) = \ker(\pi_{n-1}^\times),$$

as required to complete the proof.

\[\square\]

8.2. The proof of Theorem 1.1. It remains to check that the system $\varphi_{\text{cyc}}$ defined by (8.1.7) has the properties described in Theorem 1.1(ii) and (iii).

Regarding the field $F$ as fixed we henceforth set $G := G_F$ and $S := S(F)$. In this sequel we also write $M^*$ for the linear dual $\text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ of a $\mathbb{Z}_p[G]$-module $M$ and regard it as endowed with the contragredient action of $\mathbb{Z}_p[G]$.

8.2.1. The fact that $\varphi_{\text{cyc}}$ has the property in Theorem 1.1(ii) follows directly from the following stronger result.

In this result we use the notion of transpose Fitting invariant from 3.4.2. For each $\varphi$ in $\text{Hom}_{\mathbb{Z}_p[G]}(\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \mathcal{O}_{F,S,p}, \mathbb{Z}_p[G])$ we also set

$$\bigwedge^1_{\mathbb{Q}_p[G]} \varphi := \bigwedge_{i=1}^1 \varphi_i,$$

where $\varphi_1 := \varphi$ and the right hand side is defined as in 1.2.6.

\textbf{Proposition 8.6.} For each $\varphi$ in $\text{Hom}_{\mathbb{Z}_p[G]}(\mathcal{O}_{F,S,p}^\times, \mathbb{Z}_p[G])$ and each prime $\ell$ in $S$ one has

$$\bigwedge^1_{\mathbb{Q}_p[G]} \varphi(\varphi_{\text{cyc}}) \in \text{Fit}^1_{\mathbb{Z}_p[G]}(\text{Sel}_{S}(F)_p)^{\text{tr}} \cap \text{pAnn}_{\mathbb{Z}_p[G]}(\text{Cl}(\mathcal{O}_F[1/\ell])_p).$$

\textbf{Proof.} Set $\epsilon := \varphi_{\text{cyc}}$. Then for each $\varphi$ in $\text{Hom}_{\mathbb{Z}_p[G]}(\mathcal{O}_{F,S,p}^\times, \mathbb{Z}_p[G])$ the equality (7.3.2) that is used to prove Theorem 6.11(ii) implies the existence of an element $u$ of $\xi(\mathbb{Z}_p[G])^\times$ that is independent of $\varphi$ and such that

$$(\bigwedge^1_{\mathbb{Q}_p[G]} \varphi)(\epsilon) = u \cdot \text{Nrd}_{\mathbb{Q}_p[G]}(\langle M_\varphi \mid M_F \rangle).$$

Here $M_\varphi$ is the $d \times 1$ column vector with $i$-th entry equal to $\varphi(b_{i,F})$ and $M_F$ is the matrix in $M_{d,d-1}(\mathbb{Z}_p[G])$ such that the block matrix $(0_{d,1} \mid M_F)$ is the matrix of the differential in the complex $P_F^\bullet$ in Lemma 7.1(iii) with respect to the basis $\{ b_{i,F} \}_{1 \leq i \leq d}$ of $P_F$. 

Now the sequence \((7.2.5)\) gives a free presentation \(\Pi\) of the \(\mathbb{Z}_p[G]\)-module \(H^1(C_{F,S}(\mathbb{Z}_p(1)))\) and Lemma \(8.1.4\) implies that the transpose \(\Pi^\tau\) of \(\Pi\) (in the sense of \(3.4.2\)) is a presentation of the \(\mathbb{Z}_p[G]\)-module \(\text{Sel}_S(F)_p\). One therefore has

\[
(\bigwedge^1_{\mathbb{Q}_p[G]} \varphi)(\epsilon) \in \text{Fit}^1_{\mathbb{Z}_p[G]}(\Pi) \subseteq \text{Fit}^1_{\mathbb{Z}_p[G]}(\text{Sel}_S(F)_p)^\tau
\]

where the containment follows directly from \((7.3.2)\) and the inclusion from the definition of transpose Fitting invariant and the fact \(\Pi\) identifies with \((\Pi^\tau)^\tau\) (by Remark \(3.18\)).

To prove the second containment in the claimed result we first note that \((8.2.1)\) implies

\[
(\bigwedge^1_{\mathbb{Q}_p[G]} \varphi)(\epsilon) = u^\# \cdot \text{Nrd}_{\mathbb{Q}_p[G]}( (M_\varphi \mid M_F))^\# = u^\# \cdot \text{Nrd}_{\mathbb{Q}_p[G]}( \left( \frac{t^\#(M_\varphi^\tau)}{t^\#(M_F^\tau)} \right)),
\]

where we have used the same notation as in the proof of Lemma \(8.20\) and the second equality follows directly from \((3.4.1)\).

Now the argument used above implies that the block matrix \(\left( \frac{0_{d,d}}{\pi(M_\varphi^\tau)} \right)\) represents, with respect to the dual basis \(\{b_{i,F}\}_{1 \leq i \leq d}\) of \(P^*_F\), the endomorphism \(\theta^*_F\) in an exact sequence of the form

\[
P^*_F \xrightarrow{\theta^*_F} P^*_F \xrightarrow{\pi} \text{Sel}_S(F)_p \to 0.
\]

It is also easily checked that \(\pi\) sends the element of \(P^*_F\) whose co-ordinate vector with respect to \(\{b_{i,F}\}_{1 \leq i \leq d}\) is \(t^\#(M_\varphi)\) to an element \(\hat{\varphi}\) that the homomorphism \(\text{Sel}_S(F)_p \to (\mathcal{O}_{F,S,p}^\times)_z\) in \((8.1.1)\) sends to \(\varphi\).

The block matrix \(\left( \frac{t^\#(M_\varphi^\tau)}{t^\#(M_F^\tau)} \right)\) is therefore the matrix of a free presentation of the quotient module \(\text{Sel}_S(F)_p/(\mathbb{Z}_p[G] \cdot \hat{\varphi})\) and so \((8.2.2)\) combines with Theorem \(3.17\)(iii) to imply that

\[
(\bigwedge^1_{\mathbb{Q}_p[G]} \varphi)(\epsilon) \in \text{pAnn}_{\mathbb{Z}_p[G]}(\text{Sel}_S(F)_p/(\mathbb{Z}_p[G] \cdot \hat{\varphi})).
\]

We now write \(\Sigma\) for the subset \(\{\infty, \ell\}\) of \(S\) and recall that the results of \([4\,\text{Prop. 2.4(ii)}\) and (iii)] combine to imply the existence of a canonical surjective homomorphism of \(\mathbb{Z}_p[G]\)-modules \(\rho^*_\ell: \text{Sel}_S(F)_p \to \text{Sel}_S(F)_p\) and hence also a surjective homomorphism

\[
\text{Sel}_S(F)_p/(\mathbb{Z}_p[G] \cdot \hat{\varphi}) \to \text{Sel}_S(F)_p/(\mathbb{Z}_p[G] \cdot \rho^*_\ell(\hat{\varphi})).
\]

In addition, since the exact sequences \((8.1.1)\) imply that \(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \rho^*_\ell(\hat{\varphi})\) is torsion-free and so the natural map \(\text{Sel}_S(F)_p,\text{tor} \to \text{Sel}_S(F)_p/(\mathbb{Z}_p[G] \cdot \rho^*_\ell(\hat{\varphi}))\) is injective. Since \(\text{Sel}_S(F)_p,\text{tor}\) is isomorphic to \(\text{Cl}(\mathcal{O}_{F,1/\ell})\) (by \((8.1.1)\)), the homomorphism \((8.2.3)\) therefore implies that \(\text{Sel}_S(F)_p/(\mathbb{Z}_p[G] \cdot \hat{\varphi})\) has a subquotient isomorphic to \(\text{Cl}(\mathcal{O}_{F,1/\ell})\) and so the containment \((8.2.3)\) implies

\[
(\bigwedge^1_{\mathbb{Q}_p[G]} \varphi)(\epsilon) \# \in \text{pAnn}_{\mathbb{Z}_p[G]}(\text{Cl}(\mathcal{O}_{F,1/\ell})\#) = \text{pAnn}_{\mathbb{Z}_p[G]}(\text{Cl}(\mathcal{O}_{F,1/\ell}))^\#,
\]

where the last equality follows from Remark \(3.21\).
Upon applying the involution $x \mapsto x^\#$ to this containment we deduce the second claimed containment, as required to complete the proof. \hfill \qed

In the next two results we set $U := \mathcal{O}_{F,S,p}^\times$.

**Lemma 8.7.** It suffices to prove the second containment of Proposition 8.6 in the case that $\rho_\ell(\varphi)$ spans a free $\mathbb{Q}_p[G]$-module.

**Proof.** We set $\epsilon := \varepsilon_F^{\text{cyc}}$ and claim first that there exists a natural number $n$ with the property that if $\varphi$ and $\varphi'$ are any elements of $U^*$, then one has

\[(8.2.5) \quad \varphi - \varphi' \in n \cdot U^* \Rightarrow \left(\bigwedge_{\varphi \in \mathbb{Q}_p[G]}^1 \varphi\right)(\epsilon) - \left(\bigwedge_{\varphi' \in \mathbb{Q}_p[G]}^1 \varphi'\right)(\epsilon) \in |\text{Cl}(\mathcal{O}_F[1/\ell])| \cdot \xi(\mathbb{Z}_p[G]).\]

To see this, we note that the equality (8.2.1) implies that
\[
\left(\bigwedge_{\varphi \in \mathbb{Q}_p[G]}^1 \varphi\right)(\epsilon) - \left(\bigwedge_{\varphi' \in \mathbb{Q}_p[G]}^1 \varphi'\right)(\epsilon) = u \cdot (\text{Nrd}_{\mathbb{Q}_p[G]}((M_p \ | M_L)) - \text{Nrd}_{\mathbb{Q}_p[G]}((M_{\varphi'} \ | M_L))),
\]
where $M_{\varphi'}$ is the $d \times 1$ column vector with $i$-th entry $\varphi'(b_{i,F})$. To prove (8.2.5) it is therefore enough to prove the existence of a natural number $n$ such that for any matrices $M$ and $M'$ in $\mathcal{M}_d(\mathbb{Z}_p[G])$ one has
\[
M - M' \in n \cdot \mathcal{M}_d(\mathbb{Z}_p[G]) \Rightarrow \text{Nrd}_{\mathbb{Q}_p[G]}(M) - \text{Nrd}_{\mathbb{Q}_p[G]}(M') \in |\text{Cl}(\mathcal{O}_F[1/\ell])| \cdot \xi(\mathbb{Z}_p[G]).
\]

This is in turn an easy exercise that we leave to the reader.

With $n$ as in (8.2.5), we can then apply the result of Lemma 8.8 below to deduce the existence for any given $\varphi$ in $U^*$ of an element $\varphi'$ of $U^*$ with the property that $\rho_\ell(\varphi')$ spans a free $\mathbb{Q}_p[G]$-module and, in addition, one has
\[
\left(\bigwedge_{\varphi \in \mathbb{Q}_p[G]}^1 \varphi\right)(\epsilon) - \left(\bigwedge_{\varphi' \in \mathbb{Q}_p[G]}^1 \varphi'\right)(\epsilon) \in |\text{Cl}(\mathcal{O}_F[1/\ell])| \cdot \xi(\mathbb{Z}_p[G]).
\]

Since any element of $|\text{Cl}(\mathcal{O}_F[1/\ell])| \cdot \xi(\mathbb{Z}_p[G])$ belongs to $\text{pAnn}_{\mathbb{Z}_p[G]}(\text{Cl}(\mathcal{O}_F[1/\ell]) \cdot \mathbb{Z}_p[G])$, the claimed result is therefore clear. \hfill \qed

**Lemma 8.8.** Fix $\varphi$ in $U^*$ and a natural number $n$. Then there exists $\varphi'$ in $U^*$ such that $\varphi' - \varphi \in n \cdot U^*$ and $\rho_\ell(\varphi')$ spans a free $\mathbb{Q}_p[G]$-module.

**Proof.** Set $V := \mathcal{O}_{F,S,p}^\times$. Then, since $F$ is totally real, we may choose a free $\mathbb{Z}_p[G]$-submodule $\mathcal{F}$ of $V^*$ of rank one. We then choose a homomorphism $f$ in $U^*$ with $\mathbb{Q}_p[G] \cdot \rho_\ell(f) = \mathcal{Q}_p \cdot \mathcal{F}$.

For any integer $m$ we set $\varphi_m := \varphi + mnf$ and note it suffices to show that for any sufficiently large $m$ the element $\rho_\ell(\varphi_m)$ spans a free $\mathbb{Q}_p[G]$-module.

Consider the composite homomorphism of $\mathbb{Q}_p[G]$-modules $\mathbb{Q}_p \cdot \mathcal{F} \rightarrow \mathbb{Q}_p \cdot V^* \rightarrow \mathbb{Q}_p \cdot \mathcal{F}$ where the first arrow sends the basis element $\rho_\ell(f)$ to $\rho_\ell(\varphi_m)$ and the second is induced by a choice of $\mathbb{Q}_p[G]$-equivariant section to the projection $\mathbb{Q}_p \cdot V^* \rightarrow \mathbb{Q}_p \cdot (V^*/\mathcal{F})$.

This map sends $\rho_\ell(f)$ to $(\lambda_\varphi + mn) \cdot \rho_\ell(f)$ for an element $\lambda_\varphi$ of $\mathbb{Q}_p[G]$ that is independent of $m$. In particular, if $m$ is large enough to ensure $\lambda_\varphi - mn$ is invertible in $\mathbb{Q}_p[G]$, then the composite homomorphism is injective and so $\rho_\ell(\varphi_m)$ must span a free $\mathbb{Q}_p[G]$-module, as required. \hfill \qed
8.2.2. In order to prove claim (iii) of Theorem [1.1] we must first specify the map $\text{Reg}_F^\chi$ that occurs in that result.

The representation $\rho_\chi$ fixed in [8.1.1] gives rise to a (left) $\mathbb{C}_p[G]$-module $V_\chi$ of character $\chi$ and we write $V_\chi^*$ for $\text{Hom}_{\mathbb{C}_p}(V_\chi, \mathbb{C}_p)$, endowed with the natural right action of $\mathbb{C}_p[G]$. For each $\mathbb{Z}_p[G]$-module $M$ we write $\mathbb{C}_p \cdot M$ for the $\mathbb{C}_p[G]$-module generated by $M$ and consider the associated $\mathbb{C}_p$-vector space

$$(\mathbb{C}_p \cdot M)^\chi := V_\chi^* \otimes_{\mathbb{C}_p[G]} (\mathbb{C}_p \cdot M).$$

In particular, setting $U := \mathcal{O}_{F,S,\rho}$ and $X := X_{F,S,\rho}$ we write

$$\text{Reg}_F^\chi : (\mathbb{C}_p \cdot U)^\chi \cong (\mathbb{C}_p \cdot X)^\chi$$

for the isomorphism of $\mathbb{C}_p$-vector spaces that is induced by the Dirichlet regulator map $\text{Reg}_F = \text{Reg}_{F,S}$ recalled in [8.1.5].

Now, in view of the description of $H^0(C_F)$ given in Lemma [8.1(ii)], the construction of Lemma [7.7] (with $T = \mathbb{Z}_p(1)$ and $L = F$) gives an exact sequence of $\mathbb{Z}_p[G]$-modules

$$(8.2.6) \quad 0 \rightarrow U \xrightarrow{i} P \xrightarrow{\theta} P \xrightarrow{\xi} H^1(C_F) \rightarrow 0.$$  

Since $Y_F(\mathbb{Z}_p) = Y_{F,S_{\infty}(F),p}$ we can also ensure that the $\mathbb{Z}_p[G]$-basis $\{b_i\}_{1 \leq i \leq d}$ of $P$ fixed in Lemma [7.7(ii)] is such that the surjective composite homomorphism

$$\omega : P \xrightarrow{\xi} H^1(C_F) \rightarrow X,$$

in which the second map comes from the exact sequence in Lemma [8.1(iii)], sends $b_1$ to $w_{\infty,F} - w_{p,F}$.

Since the algebra $\mathbb{Q}_p[G]$ is semisimple we can fix $\mathbb{Q}_p[G]$-equivariant sections $\iota_1$ and $\iota_2$ to the surjections $\mathbb{Q}_p \cdot P \rightarrow \mathbb{Q}_p \cdot \text{im}(\theta)$ and $\mathbb{Q}_p \cdot P \rightarrow \mathbb{Q}_p \cdot X$ that are respectively induced by $\theta$ and $\omega$ and we can assume that $\iota_2(w_{\infty,F} - w_{p,F}) = b_1$.

These sections then give a direct sum decomposition of $\mathbb{C}_p[G]$-modules

$$\mathbb{C}_p \cdot P = (\mathbb{C}_p \cdot U) \oplus (\mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota_1)(\mathbb{C}_p \cdot \text{im}(\theta))$$

and, via this decomposition, we define

$$\langle \theta, \iota_1, \iota_2 \rangle$$

to be the automorphism of $\mathbb{C}_p \cdot P$ that is equal to $\left((\mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota_2) \circ (\mathbb{C}_p \otimes_{\mathbb{R}} \text{Reg}_F) \right) \mathbb{C}_p \cdot U$ and to $\mathbb{C}_p \otimes_{\mathbb{Z}_p} \theta$ on $\left((\mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota_1)(\mathbb{C}_p \cdot \text{im}(\theta))\right)$.

Finally, as in [3.4.2] we use the canonical Wedderburn decomposition $\zeta(\mathbb{C}_p[G]) = \prod_{\text{Ir}_p(G)} \mathbb{C}_p$ to decompose each element $x$ of $\zeta(\mathbb{C}_p[G])$ as a vector $(x_\psi)_{\psi \in \text{Ir}_p(G)}$ with each $x_\psi$ in $\mathbb{C}_p$.

The key point in the proof of Theorem [1.1(iii)] is then the following observation.

**Lemma 8.9.** One has $e_\chi(\varepsilon_F^{\text{nc}}) \neq 0$ if and only if $L_S^{(1)}(\chi, 0) \neq 0$. In addition, if $e_\chi(\varepsilon_F^{\text{nc}}) \neq 0$, then the element $u_{F,\chi}$ that occurs in the equality of Theorem [1.1(iii)] is non-zero and there exists a unit $v$ of $\zeta(\mathbb{Z}_p[G])$ that is independent of $\chi$ and such that

$$u_{F,\chi} = v_\chi \cdot (\text{Nrd}_{\mathbb{C}_p[G]}((\theta, \iota_1, \iota_2)))_\chi \cdot L_S^*(\chi, 0)^{-1},$$

where $L_S^*(\chi, 0)$ denotes the leading coefficient in the Laurent expansion of $L_S(\chi, z)$ at $z = 0$.  

Proof. Given the explicit definition of $\varepsilon^\text{cyc}_F$, the first claim follows directly from the proof of Theorem 7.5(i) and the description of the idempotent $e_F$ given in Lemma 8.17(iii).

We therefore assume $e_\chi(\varepsilon^\text{cyc}_F) \neq 0$ and hence that the element $u_{F,\chi}$ is both non-zero and uniquely specified by the equality in Theorem 1.1(iii).

To prove the existence of a unit $u$ then the equality in Theorem 1.1(iii) implies

\[
\varepsilon^\text{cyc}_F \cdot \varepsilon^\text{cyc}_F = (\varepsilon^\text{cyc}_F)^2 \cdot \varepsilon^\text{cyc}_F = \chi(\varepsilon^\text{cyc}_F) \cdot \varepsilon^\text{cyc}_F.
\]

In addition, since $L_S^{(1)}(\chi,0)$ does not vanish, the formula (8.1.2) implies both that $L_S^{(1)}(\chi,0) = L_S^*(\chi,0)$ and $\dim_{\mathbb{C}}((\mathbb{C}_p \cdot X)^\chi) = \chi(1)$ and then the isomorphism $\text{Reg}_F^\chi$ implies that $\dim_{\mathbb{C}}((\mathbb{C}_p \cdot U)^\chi) = \chi(1)$. The $\mathbb{C}_p$-spaces $\bigwedge_{\mathbb{C}_p}^{(1)}(\mathbb{C}_p \cdot X)^\chi$ and $\bigwedge_{\mathbb{C}_p}^{(1)}(\mathbb{C}_p \cdot U)^\chi$ are therefore of dimension one and have as respective bases the elements $e_\chi(\bigwedge_{\mathbb{C}_p}^{(1)}(\mathbb{C}_p \cdot Y))$ and $e_\chi(\varepsilon^\text{cyc}_F)$.

In particular, if we define $\lambda$ to be the non-zero element of $\mathbb{C}_p \cdot e_\chi$ that is specified by

\[
(\bigwedge_{\mathbb{C}_p}^{(1)} \text{Reg}_F^\chi)(e_\chi(\varepsilon^\text{cyc}_F)) = \lambda(\bigwedge_{\mathbb{C}_p[1]}^{1}(\mathbb{C}_p[1] Y)),
\]

then the equality in Theorem 1.1(iii) implies $u_{F,\chi} \cdot e_\chi = \lambda \cdot L_S^*((\chi,0))^{-1}$. It is therefore enough to prove the existence of a unit $v$ of $\xi(\mathbb{Z}_p[\mathbb{G}])$ that is independent of $\chi$ and such that

\[
\lambda = e_\chi(v \cdot \text{Nrd}_{\mathbb{C}_p[\mathbb{G}]}((\Theta, \iota_1, \iota_2))).
\]

To do this we use the composite homomorphism of $\mathbb{C}_p[\mathbb{G}]$-modules

\[
\Theta : \mathbb{C}_p \cdot P \to \mathbb{C}_p \cdot U \to \mathbb{C}_p \cdot X,
\]

where the first and second maps are respectively induced by $\iota_1$ and $\text{Reg}_F$. We also fix a section $\varrho : \mathbb{C}_p \cdot X \to \mathbb{C}_p \cdot Y$ to the inclusion $\mathbb{C}_p \cdot Y \leq \mathbb{C}_p \cdot X$ and write $\Theta_1$ for the homomorphism $\mathbb{C}_p \cdot P \to \mathbb{C}_p[\mathbb{G}]$ given by $y^* \circ \varrho \circ \Theta$.

Then Lemma 4.10 implies that

\[
\lambda = (\bigwedge_{\mathbb{C}_p[1]}^{1} y^*) (\lambda(\bigwedge_{\mathbb{C}_p[1]}^{1} y)) = (\bigwedge_{\mathbb{C}_p[1]}^{1} y^*) (\bigwedge_{\mathbb{C}_p[1]}^{1} \text{Reg}_F^\chi)(e_\chi(\varepsilon^\text{cyc}_F)) \lambda(\bigwedge_{\mathbb{C}_p[1]}^{1} y),
\]

where the third equality follows from the explicit construction of the reduced exterior products $\bigwedge_{\mathbb{C}_p[1]}^{1} (-)$, the fourth from the fact that $\varepsilon^\text{cyc}_F$ belongs to $\bigwedge_{\mathbb{C}_p[1]}^{1} (\mathbb{C}_p \cdot U)$ and the last from the fact that the surjective homomorphism $e_\chi(\mathbb{C}_p \cdot X) \to e_\chi(\mathbb{C}_p \cdot Y)$ induced by $\varrho$ is bijective since both $\dim_{\mathbb{C}_p}((\mathbb{C}_p \cdot X)^\chi)$ and $\dim_{\mathbb{C}_p}((\mathbb{C}_p \cdot Y)^\chi)$ are equal to $\chi(1)$.

On the other hand, the argument used to prove (8.2.1) implies the existence of a unit $v$ of $\xi(\mathbb{Z}_p[\mathbb{G}])$ (that is independent of $\chi$ and) such that

\[
(\bigwedge_{\mathbb{C}_p[1]}^{1} \Theta_1)(\varepsilon^\text{cyc}_F) = v \cdot \text{Nrd}_{\mathbb{C}_p[1]}((M' \mid M)).
\]
where we write \( \zeta \) and \( S \) (8.2.10) submodule of \( M \). Therefore imply that if \( e \) is valid if and only if the element \( (8.2.9) \) \( e \)-module \( e(\mathbb{C}_p \cdot P) \) that is induced by \( \langle \theta, \iota_1, \iota_2 \rangle \).

Given these facts, the required equality [8.2.7] follows directly from the computation [8.2.8].

\[ \psi \left( \sum \left( \text{Nrd}_{\mathbb{C}_p[G]}(\langle \theta, \iota_1, \iota_2 \rangle) \right) \chi \cdot L_S^*(\bar{\chi}, 0)^{-1} \right)_{\chi \in \text{Ir}_p(G)} \]

of \( \zeta(\mathbb{C}_p[G]) \) belongs to \( \text{Nrd}_{\mathbb{Q}_p[G]}(K_1(\mathbb{Z}_p[G])) \). The observations made in [8.1 Rem. 6.1.1] therefore imply that if \( M \) is any maximal \( \mathbb{Z}_p \)-order in \( \mathbb{Q}_p[G] \) with \( \mathbb{Z}_p[G] \subseteq M \), then the `Strong-Stark Conjecture' of Chinburg [14 Con. 2.2] implies \( \psi \) belongs to \( \text{Nrd}_{\mathbb{Q}_p[G]}(K_1(M)) \). In view of the formula in Lemma 8.9, it would therefore follow from the latter conjecture that if \( \zeta(\mathbb{Q}_p[G]) \neq 0 \), then the element \( u_{F, \chi} = v_{\chi} \cdot \bar{\chi} \) in Theorem 1.1(iii) is a \( p \)-adic unit in any extension of \( \mathbb{Q}_p \) over which \( \chi \) can be realised.

Remark 8.10. The general result of [4 Lem. A.1(iii)] combines with Lemma 7.7(iii) (with \( L = F \)) to imply that the equivariant Tamagawa number conjecture for \( (h^0(\text{Spec}(F)), \mathbb{Z}_p[G]) \) is valid if and only if the element \( \omega \) takes values in \( \mathbb{Q} \). Hence, by the Artin induction theorem (see [44 Chap. II, Th. 1.2]), there exists a natural number \( m \) and an integer \( n_L \) for each field \( L \) in \( \Omega(F/Q) \) for which one has an equality of characters

\[ m \cdot \sum_{\omega \in \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} \chi^\omega = \sum_{L \in \Omega(F/Q)} n_L \cdot 1^F_L, \]

where we write \( 1^F_L \) for the induction to \( G \) of the trivial character \( 1_L \) of \( G(L) : = \text{Gal}(F/L) \).

In particular, if we consider each of the maps from \( \text{Ir}_p(G) \) to \( \mathbb{C}_p^\times \) that are respectively given by sending \( \psi \) to \( u_{F, \psi} \), to \( v_{\psi} \), to \( (\text{Nrd}_{\mathbb{C}_p[G]}(\langle \theta, \iota_1, \iota_2 \rangle)) \psi \) and to \( L_S^*(\psi, 0) \) as homomorphisms from the group of virtual \( \mathbb{C}_p \)-valued characters of \( G \) to \( \mathbb{C}_p^\times \) (in the obvious way), then the result of Lemma 8.9 implies that

\[ \left( \prod_{\omega \in \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} u_{F, \chi^\omega} \right)^m = \prod_{L \in \Omega(F/Q)} \left( \frac{v_{1^F_L} \cdot (\text{Nrd}_{\mathbb{C}_p[G]}(\langle \theta, \iota_1, \iota_2 \rangle)) \psi}{L_S^*(1^F_L, 0)} \right)^{n_L}. \]

Now the behaviour of Artin \( L \)-series under induction of characters implies that

\[ L_S^*(1^F_L, 0) = \zeta_{L,S}^*(0) = -(1/2) \cdot R_{L,S} \cdot |\text{Cl}(\mathcal{O}_{L,S})|, \]

where we write \( \zeta_{L,S}(0) \) for the leading coefficient in the Laurent expansion at \( z = 0 \) of the \( S \)-truncated zeta function of \( L \) and \( R_{L,S} \) for the \( S \)-regulator of \( L \), and the second equality follows directly from the analytic class number formula for \( L \) (as recalled, for example, in [44 Chap. I, Cor. 2.2]).

Next we recall that there exists a commutative diagram
The exact sequence (8.2.6) gives rise to an exact commutative diagram of

\[ K_1(\mathbb{C}_p[G]) \longrightarrow K_1(\mathbb{C}_p[G(L)]) \]

(8.2.11)

\[
\begin{array}{ccc}
\zeta(\mathbb{C}_p[G])^\times & \longrightarrow & \zeta(\mathbb{C}_p[G(L)])^\times \\
\downarrow \text{Nrd}_{\mathbb{C}_p[G]} & & \downarrow \text{Nrd}_{\mathbb{C}_p[G(L)]} \\
\end{array}
\]

in which the upper and lower horizontal maps are respectively the natural restriction of scalars map and the composite homomorphism

\[ \zeta(\mathbb{C}_p[G])^\times = \prod_{\text{Ir}_p(G)} \mathbb{C}_p^\times \]

where for each \( x \in \zeta(\mathbb{C}_p[G])^\times \) and each \( \phi \in \text{Ir}_p(G(L)) \) one has

\[ i_k^L(x) = \prod_{\chi \in \text{Ir}_p(G)} x_\chi^{\langle \chi, \text{Ind}_{G(L)}^G(\phi) \rangle} \]

with \( \langle \cdot, \cdot \rangle_G \) the standard scalar product on characters of \( G \).

We write \( e_{G(L)} \) for the central idempotent \( |G(L)|^{-1} \sum_{g \in G(L)} g \) of \( \mathbb{Q}_p[G(L)] \). Then the commutativity of (8.2.11) implies firstly that

(8.2.12) \[ (\text{Nrd}_{\mathbb{C}_p[G]}(\langle \theta, \iota_1, \iota_2 \rangle))_L^a = (\text{Nrd}_{\mathbb{C}_p[G(L)]}(\langle \theta, \iota_1, \iota_2 \rangle))_L^a = \det_{\mathbb{C}_p}(\langle \theta, \iota_1, \iota_2 \rangle_{G(L)}) \]

where \( \langle \theta, \iota_1, \iota_2 \rangle_{G(L)} \) denotes the \( \mathbb{C}_p \)-automorphism of \( e_{G(L)}(\mathbb{C}_p \cdot P) \) induced by \( \langle \theta, \iota_1, \iota_2 \rangle \).

In addition, if \( \mathcal{M} \) is as in Remark 8.10, then Lemma 8.2(iv) implies the existence of an element \( x \in \mathcal{M}^\times \) such that \( v = \text{Nrd}_{\mathbb{C}_p[G]}(x) \) and so the commutativity of (8.2.11) implies

\[ v_L^a = (\text{Nrd}_{\mathbb{C}_p[G]}(x))_L^a = (\text{Nrd}_{\mathbb{C}_p[G(L)]}(x))_L^a = \text{Nrd}_{\mathbb{C}_p}(x_{G(L)}) \]

where \( x_{G(L)} \) denotes the automorphism of \( \mathbb{C}_p[G(L)]e_{G(L)} \) that is induced by (right) multiplication by \( x \). In particular, since \( x_{G(L)} \) restricts to give an automorphism of the free rank one \( \mathbb{Z}_p \)-module \( \mathcal{M}e_{G(L)} \), the last displayed equalities imply that \( v_L^a \) belongs to \( \mathbb{Z}_p^\times \).

Given this fact (and the fact that \( p \) is assumed to be odd), the result of Theorem 1.1(iii) is obtained by substituting into the equality (8.2.9) each of (8.2.10), (8.2.12) and the containment established in the following result.

**Lemma 8.11.** For each \( L \) in \( \Omega(F/Q) \) one has

\[ \det_{\mathbb{C}_p}(\langle \theta, \iota_1, \iota_2 \rangle_{G(L)}/(R_{L,S} \cdot |\text{Cl}(O_{L,S})|)^{-1} \in \mathbb{Z}_p^\times, \]

where \( R_{L,S} \) is the \( S \)-regulator of \( L \).

**Proof.** We set \( J := G(L) \) and for each \( G \)-module \( M \) abbreviate the modules of \( J \)-invariants \( H^0(J, M) \) and \( J \)-coinvariants \( H_0(J, M) \) to \( M^J \) and \( M_J \) respectively.

Then, since the functors \( M \rightarrow M^J \) and \( M \rightarrow M_J \) are respectively left and right exact, the exact sequence (8.2.3) gives rise to an exact commutative diagram of \( \mathbb{Z}_p[G/J] \)-modules
since this computation is entirely routine we leave details to the reader.

is induced by the right hand vertical map in the diagram above.

$$\det_C$$ of $$T$$ in Lemma 7.7(iii) (with $$\sum$$ replaced by

in which the arrow denotes the tautological map, the second isomorphism is induced by the

$$(\ast)$$ in which the right hand vertical homomorphism is induced by sending each place $$v$$ of $$S_L$$ to $$\sum_{g \in J} g(w)$$ where $$w$$ is any fixed place of $$F$$ lying above $$v$$.

In particular, the above two diagrams combine to imply that $$\langle \theta, \iota_1, \iota_2 \rangle_J$$ identifies with the automorphism of

$$\mathbb{C}_p \cdot P^J = (\mathbb{C}_p \cdot P)^J = (\mathbb{C}_p \cdot \mathcal{O}_{L,S,p}^\times) \oplus (\mathbb{C}_p \otimes \mathbb{Q}_p \iota_1)(\mathbb{C}_p \cdot \text{im}(\theta^J))$$

that is equal to the composite $$(\mathbb{C}_p \otimes \mathbb{Q}_p \iota^J_1) \circ (\mathbb{C}_p \otimes \mathbb{R} \text{Reg}_L)$$ on $$\mathbb{C}_p \cdot \mathcal{O}_{L,S,p}^\times$$, and to $$\mathbb{C}_p \otimes \mathbb{Q}_p \theta^J$$ on $$(\mathbb{C}_p \otimes \mathbb{Q}_p \iota_1)(\mathbb{C}_p \cdot \text{im}(\theta^J))$$. In addition, if we write $$P^\bullet_J$$ for the complex $$P^J$$ that is considered in Lemma 7.7(iii) (with $$T = \mathbb{Z}_p(1)$$), then Lemma 7.7(v) implies that the complex $$(P^\bullet)^J$$ identifies with $$C_{L,S}(\mathbb{Z}_p(1))$$ in such a way that $$\mathbb{C}_p \otimes \mathbb{Q}_p \iota^J_2$$ is a section to the composite homomorphism

$$\mathbb{C}_p \cdot P^J \cong \mathbb{C}_p \cdot P_J \rightarrow \mathbb{C}_p \cdot H^1((P^\bullet)^J) = \mathbb{C}_p \cdot H^1(C_{L,S}(\mathbb{Z}_p(1))) \cong \mathbb{C}_p \cdot X_{L,S,p} \cong \mathbb{C}_p \cdot X^J$$

in which the arrow denotes the tautological map, the second isomorphism is induced by the exact sequence in Remark 8.2 (with $$F$$ and $$\Sigma$$ replaced by $$L$$ and $$S$$) and the final isomorphism is induced by the right hand vertical map in the diagram above.

Given this explicit description of $$\langle \theta, \iota_1, \iota_2 \rangle_J$$, and the description of the cohomology groups of $$C_{L,S}(\mathbb{Z}_p(1))$$ in Remark 8.2, the claimed result is verified by an explicit computation of $$\det_{C_p}(\langle \theta, \iota_1, \iota_2 \rangle_J)$$ using the methods, for example, of 4 Lem. A.1 and Lem. A.3. However, since this computation is entirely routine we leave details to the reader.

This completes the proof of Theorem 1.1.

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