On Calabi-Yau supermanifolds II

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**ABSTRACT**

We study when Calabi-Yau supermanifolds $\mathbb{M}^{1|2}$ with one complex bosonic coordinate and two complex fermionic coordinates are super Ricci-flat, and find that if the bosonic manifold is compact, it must have constant scalar curvature.
In [1], we found that super Ricci-flat Kähler manifolds with one fermionic dimension and an arbitrary number of bosonic dimensions exist above a bosonic manifold with a vanishing Ricci scalar. This paper explores super Calabi-Yau manifolds with one bosonic dimension and two fermionic dimensions. We find that the supermetric is super Ricci-flat implies several interesting constraints that are familiar from other contexts, including the field equation of the WZW-model on \( AdS_3 \). Locally, these constraints imply that the super Kähler potential has the form

\[
K(z, \bar{z}, \theta, \bar{\theta}) = K_0(z, \bar{z}) + \sqrt{K_0(z, \bar{z})} (\theta^i \bar{\theta}^i)^2 + \frac{1}{4} \left( \ln [K_0(z, \bar{z})] \right)_{z \bar{z}} (\theta^i \bar{\theta}^i)^2 ,
\]

where \( K_0(z, \bar{z}) \) is the Kähler potential of the bosonic manifold. We find the further constraint that the scalar curvature of the bosonic manifold is harmonic; on a complete compact space, this implies that the scalar curvature is constant.

Consider the super Kähler potential \( K \) of the manifold \( M^{1|2} \) with 1 bosonic coordinate and 2 fermionic coordinates:

\[
K = f_0 + if_1 \theta^2 + i \bar{f}_1 \bar{\theta}^2 + f_{ij} \theta^i \bar{\theta}^j + f_2 \theta^2 \bar{\theta}^2
\]

We use the notation \( \theta^2 = \frac{1}{2} \epsilon_{ij} \theta^i \theta^j \), where \( \epsilon_{ij} = -\epsilon_{ji} \). Since \( \bar{\theta}^i \theta^i = \bar{\theta}^i \bar{\theta}^i \), the factor of \( i \) is needed to make \( K \) real. The supermetric of this manifold is

\[
\begin{pmatrix}
  f_{0, z\bar{z}} + if_{1, z\bar{z}} \theta^2 + i \bar{f}_{1, z\bar{z}} \bar{\theta}^2 + if_{ij, z\bar{z}} \theta^i \theta^j + f_{2, z\bar{z}} \theta^2 \bar{\theta}^2 & if_{1, z\bar{z}} \epsilon_{ij} \bar{\theta}^i + f_{ij, z\bar{z}} \theta^j \bar{\theta}^i + f_{2, z\bar{z}} \theta^2 \epsilon_{ij} \bar{\theta}^i \\
if_{1, z\bar{z}} \epsilon_{ij} \theta^i + f_{ij, z\bar{z}} \bar{\theta}^j + \frac{1}{2} f_{2, z\bar{z}} \epsilon_{ij} \theta^i \bar{\theta}^j & f_{ij} + f_{2} \epsilon_{id} \epsilon_{kj} \bar{\theta}^k
\end{pmatrix}
\]

The superdeterminant can be set to 1 by a holomorphic coordinate transformation as described in [1]. Looking at only the bosonic part of the superdeterminant yields

\[
f_{0, z\bar{z}} = \det(f_{ij})
\]

Equating the coefficients of the purely holomorphic and purely anti-holomorphic fermions gives the equation

\[
\frac{f_{1, z\bar{z}}}{f_{1, z}} = \frac{\det(f_{ij})}{f_{0, z\bar{z}}}
\]

as well as its conjugate. This is equivalent to \( \ln(f_{1, z})_z = \ln(\det(f_{ij}))_z \) or

\[
f_{1, z} = \lambda(\bar{z}) \det(f_{ij})
\]

where \( \lambda(\bar{z}) \) is an arbitrary function of \( \bar{z} \). This implies that locally \( f_1 \) can always be removed by a holomorphic coordinate transformation: If \( \lambda = 0 \), \( f_1 = f_1(z) \) is holomorphic and
contributes a holomorphic term to the super Kähler potential that does not change the supermetric. Otherwise, the coordinate transformation \( \frac{1}{\lambda(z)} \frac{\partial}{\partial \bar{z}} \rightarrow \frac{\partial}{\partial \bar{z}'} \), results in the equation

\[
f_{1,z} = \det(f_{i\bar{j}}) = f_{0,z\bar{z}} ,
\]

where we have used (4) in the last step. Then the coordinate transformation, \( z + i \theta^2 \rightarrow z \) and \( \bar{z} + i \bar{\theta}^2 \rightarrow \bar{z} \) eliminates \( f_1 \) (up to purely holomorphic terms), as can be verified by a Taylor expansion. This coordinate transformation is only valid locally, as \( z \) and \( \bar{z} \) are not globally defined functions on all such manifolds.

Thus the Kähler potential can be assumed to have the form:

\[
K = f_0 + f_{ij} \theta^i \bar{\theta}^j + f_2 \theta^2 \bar{\theta}^2
\]

and the simplified supermetric is

\[
\begin{pmatrix}
  f_{0,z\bar{z}} + f_{ij,z\bar{z}} \theta^i \bar{\theta}^j + f_{2,z\bar{z}} \theta^2 \bar{\theta}^2 & f_{ij,z} \theta^i + \frac{1}{2} f_{2,z} \theta^2 \epsilon_{ij} \bar{\theta}^2 \\
  f_{ij,z} \bar{\theta}^j + \frac{1}{2} f_{2,z} \epsilon_{ij} \theta^i \bar{\theta}^2 & f_{ij} + f_{2} \epsilon_{ij} \epsilon_{k\bar{j}k}
\end{pmatrix}
\]

By looking at the coefficient of the \( \theta \bar{\theta} \) terms, one can find that

\[
\frac{f_{bc} f_{i\bar{a},z} f_{d\bar{j},z} \epsilon^{db} \epsilon^{ca}}{\det(f_{i\bar{j}})} + f_{ij,z\bar{z}} = - f_{2} f_{ij} ,
\]

where \( \epsilon_{abc} \delta^c_a = \delta^a_b \). Multiplying by the inverse \( f_{i\bar{j}} \) and contracting yields the equation

\[
f_2 = - \left( \ln \sqrt{\det(f_{i\bar{j}})} \right)_{z\bar{z}} .
\]

Substituting this into equation (10) gives

\[
(f_{ij,z} f^{i\bar{k}})_{\bar{z}} = \left( \ln \sqrt{\det(f_{i\bar{j}})} \right)_{z\bar{z}} \delta^{i}_{\bar{k}} ,
\]

where \( f^{i\bar{j}} \) is the inverse of \( f_{ij} \). Now let \( M := \frac{f_{i\bar{j}}}{\sqrt{\det(f_{i\bar{j}})}} \); then \( M \) is a hermitian matrix whose determinant is 1. It can be written as

\[
M = \begin{pmatrix}
x + y & u + iv \\
u - iv & x - y
\end{pmatrix} .
\]

The condition \( \det(M) = 1 \) implies \( x^2 - y^2 - u^2 - v^2 = 1 \), which is a hyperboloid, and is well known in the physics literature as \( AdS_3 \) (see, e.g., [2], where this space is used to study Black holes).

Equation (12) implies

\[
(M^{-1} M_{\bar{z}})_z = 0 \leftrightarrow (M_{\bar{z}} M^{-1})_{\bar{z}} = 0 .
\]
This matrix differential equation is well studied and appears in many physical systems– it is
the classical equation of motion of the WZW-model on $AdS_3$ \[^3\]. If we use the parameteriza-
tion

$$M = \begin{pmatrix} e^{-\phi} + \gamma \bar{\gamma} e^{\phi} & \gamma e^{\phi} \\ \bar{\gamma} e^{\phi} & e^{\phi} \end{pmatrix},$$

then the four resulting equations give the functional gradient of equation (2.9) in \[^3\].

Equation (14) implies that

$$M = \mathcal{M}(z) Y \bar{\mathcal{M}}(\bar{z}) ,$$

where $\mathcal{M}(z)$ is a holomorphic matrix and $\bar{\mathcal{M}}$ is its adjoint. In this context, it is possible to

go further and (locally) eliminate $M$ by applying the coordinate transformation $\theta M Y \bar{\theta} \rightarrow \theta$
and $Y^{1/2} \bar{\mathcal{M}} \bar{\theta} \rightarrow \bar{\theta}$. This simplifies the super Kähler potential further to the form \[^1\].

The remaining terms in the superdeterminant, once one assumes $M_{ij} = \delta_{ij}$, imply

$$R_{zz\bar{z}} = 0 ,$$

where $R$ is the Ricci scalar of the bosonic part. On a complete compact manifold, the only
bosonic functions that obey (17) are constant functions. This proves that the Ricci scalar is
a constant on this type of manifold.

On a noncompact manifold, there are other solutions which may prove to be interesting;
the results of \[^4\] relate closely to equation (17), and potentially could be useful in pursuing
this further.

An example of a space obeying all the conditions that we have found is $S\mathbb{P}^{1|2}$. This
space is compact and satisfies all of the constraints that we have found, guaranteeing that
its supermetric is super Ricci-flat. Its Kähler potential is

$$\ln(1 + z\bar{z}) + \frac{\theta^1 \bar{\theta}^1 + \theta^2 \bar{\theta}^2}{1 + z\bar{z}} + \frac{\theta^1 \theta^2 \bar{\theta}^1 \bar{\theta}^2}{(1 + z\bar{z})^2} ,$$

The Ricci scalar of the bosonic part is 2 and $\ln(1 + z\bar{z})_{z\bar{z}} = 1/(1 + z\bar{z})^2$. Another example
is given by a Riemann surface $\Sigma$ with a constant negative Ricci scalar.

In this paper, many coordinate transformations are used that could be obstructed glob-
ally, e.g., it may not be possible to remove the $\theta^2$ and $\bar{\theta}^2$ terms as well to transform $M$
to the identity matrix; it would be interesting to see how such terms modify (17) and if other
interesting solutions arise.

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Note: While writing up our results after completing our calculations, we became aware of [5], which has considerable overlap with our work. It studies the case with an arbitrary number of bosonic dimensions, but with a super Kähler potential assumed to have the simple form (8). Because of the greater complexity of bosonic manifolds in higher dimensions, [5] does not find as complete results as those presented here.

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