Ultraviolet cut off, black hole-radiation equilibrium and big bang.

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In the presence of a minimal uncertainty in length, there exists a critical temperature above which
the thermodynamics of a gas of radiation changes drastically.

We find that the equilibrium temperature of a system composed of a Schwarzschild black hole
surrounded by radiation is unaffected by these modifications. This is in agreement with works
related to the robustness of the Hawking evaporation. The only change the deformation introduces
concerns the critical volume at which the system ceases to be stable.

On the contrary, the evolution of the very early universe is sensitive to the new behavior. We
readdress the shortcomings of the standard big bang model (flatness, entropy and horizon problems)
in this context, assuming a minimal coupling to general relativity. Although they are not solved,
some qualitative differences set in.

I. INTRODUCTION

The ultimate structure of space time has been at the core of many works. Some begin with a fundamental con-
struction like string theory and find that when particular fields are turned on, the effective theory can be described
as built on a space time which has modified commutation relations [1–3] or dispersion relations [4]. The same occurs
with loop quantum gravity [5]. Another approach consists in using toy models with ad hoc modifications in order to
study, in a simplified way, the influence of possible departures from usual symmetries at high scale. This last approach
has been adopted in the study of the trans-Planckian problem of black hole evaporation [6–12]. Our work fits in the
second approach although the commutators on which we work can be seen as coming from string theory [2].

If one modifies the commutators, one changes the Heisenberg uncertainty relations. The measure on the phase space
is no more the same; this results in new partition functions and consequently different thermodynamical behaviors.
From the quantum point of view, the energy spectra of systems are modified by the change in the commutation
relations.

The influence of this “new” thermodynamics in the early universe has been analyzed in some models with modified
dispersion relations [13,14]. However, the equations of states used came only from considerations on bosons. In the
unmodified theory this is justified by the fact that the difference between bosons and fermions reduces to a factor 7/8
in the energy densities, pressures, etc. As pointed out in [15], the difference between bosons and fermions in theories
with ultraviolet cut offs is much more pronounced. This naturally raises the question of the way the picture is modified
when one consider them altogether. In this work, we provide analytical approximations for the equation of state, the
entropy of such a mixture and we quantify the flatness, entropy and horizon problem in this framework. Many studies
have been devoted to cosmological perturbations in transplanckian physics [16–27]; our treatment tackles some of the
reasons which led to the inflationary paradigm.

Concerning black holes, It was first realized that, on purely classical grounds, an entropy and a temperature could
be associated to these objects [28]. This was confirmed using Q.F.T on a curved background; the exact factor for the
temperature was also found [29]. It was then realized that in this derivation, photons were emitted with transplanckian
frequencies, raising doubts about their treatment as non interacting particles. It was pointed out afterward that taking
somehow into account this effect trough dispersion relations which depart from the usual theory at transplanckian
scale, the Hawking radiation was not intrinsically changed while its derivation became more reliable [6–10].

When the dispersion relation of the photons is the usual one, an equilibrium can be achieved for a system in which
a neutral, non rotating and non charged black hole is in a fixed box filled with radiation [30,31]. Moreover, this
temperature is the one obtained by Hawking. We analyze how this is affected when non trivial dispersion relations
are considered, in the spirit of [6–10].

The article is organized as follows. In the second section we briefly present a model exhibiting a minimal uncertainty
in length and derive its black-body radiation. We find sensible differences between fermions and bosons at very high
temperatures [15]. The third section treats a system consisting of a black hole in equilibrium with radiation in the new

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framework. The last part investigates, quantitatively, how the problems of the standard big bang (flatness, horizon, entropy) are affected.

II. BLACK BODY RADIATION

A very simple modification of the position-momentum commutation relation leads to a theory possessing a minimal uncertainty in length [32]. Some high dimensional extensions of this algebra preserve rotational and translational invariance. The model we shall study is given by the following non vanishing commutators:

\[ [\hat{x}_j, \hat{p}_k] = i\hbar \left( f(p^2)\delta_{jk} + g(p^2)\hat{p}_j\hat{p}_k \right), \quad g(p^2) = \beta, \quad f(p^2) = \frac{\beta p^2}{1 + \sqrt{1 + 2\beta p^2}}. \]  

(1)

This theory has no position representation; the best way to recover information on positions is through the so called quasi-position representation in which the momentum operators read:

\[ p_k = -i\hbar \sum_{r=0}^{\infty} \left( \frac{\hbar^2 \beta}{2} \Delta \right)^r \frac{\partial}{\partial \xi_k}, \quad \text{where} \quad \Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial \xi_i^2}. \]  

(2)

Introducing the momentum scale \( \beta \), it is straightforward that with the Boltzmann constant \( k \), the light velocity \( c \), one can construct on purely dimensional grounds the characteristic temperature

\[ T_c = \frac{c}{k\sqrt{\beta}}. \]  

(3)

Let us now analyze how radiation gets affected by the new scale We will use the conventions of [37,38]. Thanks to the deformation of the Klein-Gordon equation [33], the dispersion relation in our case reads:

\[ E = \frac{chk}{(1 + \frac{1}{2}h^2 \beta k^2)}. \]  

(4)

The action of the momentum operators (Eq.(2)) on plane waves of wave vectors \( \vec{k} \) is finite only if the condition

\[ h^2 k^2 \leq \frac{2}{\beta} \]  

is satisfied; this is our cut off. The important quantities are

\[ q_{bo} = \sum_{\vec{l}} \log \left( 1 - \exp \left( -\frac{\epsilon_{\vec{l}}}{kT} \right) \right) = -\log Z_{bo}, \quad q_{fe} = -\sum_{\vec{l}} \log \left( 1 + \exp \left( -\frac{\epsilon_{\vec{l}}}{kT} \right) \right) = -\log Z_{fe}, \]  

(6)

where \( Z \) is the grand partition function. The entropy is given by

\[ S = -\frac{\partial \Phi}{\partial T}, \quad \text{with} \quad \Phi = kT \log Z, \]  

(7)

while the energy and the particle number read

\[ U = \sum_{\vec{l}} \frac{\epsilon_{\vec{l}}}{\exp \left( \frac{\epsilon_{\vec{l}}}{kT} \right) + 1}, \quad N = \sum_{\vec{l}} \frac{1}{\exp \left( \frac{\epsilon_{\vec{l}}}{kT} \right) + 1}. \]  

(8)

For bosons, the quantity \( q \) linked to the partition function by Eq.(6) can be written as

\[ q = 4\pi V \left( \frac{kT}{hc} \right)^3 \int_0^{\sqrt{2\pi}} dx^2 x^2 \log \left( 1 - \exp \left( -\frac{x}{1 + \frac{1}{2}x^2} \right) \right), \]  

(9)

while the energy assumes the following form
\[ U = 4\pi V \left( \frac{kT}{\hbar c} \right)^4 \int_0^{\sqrt{T_0^2/T_c}} \frac{x^3}{1 + \frac{1}{2} \frac{x^2}{T_c^2}} \left[ \exp \left( \frac{x}{1 + \frac{1}{2} \frac{x^2}{T_c^2}} \right) - 1 \right]^{-1} . \] (10)

The particle number admits a similar integral expression.

For temperatures greater than or comparable to \( T_c \), the interval of integration is small and a Taylor expansion can be used to obtain an approximation. Working to fourth order, we are led to the following expressions:

\[ p_{bo} = \sigma T_c \left[ 2 - \frac{2}{15} \sqrt{2} T_c \frac{T_c}{T} + \frac{2}{15} \sqrt{2} \frac{T_c}{T} \left( \frac{8}{3} \log \frac{T}{T_c} + \frac{112}{45} - \frac{4}{3} \log 2 \right) \right] , \]
\[ \rho_{bo} = \sigma T_c \left[ -2 + \frac{8}{3} \sqrt{2} \frac{T_c}{T} + \frac{4}{15} \sqrt{2} T_c \frac{T_c}{T} \right] , \quad s_{bo} = \sigma \left[ \frac{2}{15} \sqrt{2} \left( \frac{T_c}{T} \right)^2 + \frac{2}{45} \sqrt{2} \left( 60 \log \frac{T}{T_c} + 116 - 30 \log 2 \right) \right] , \]
\[ N_{bo} = \frac{\sigma}{k} V \left[ \frac{1}{3} \frac{T_c}{T} - \frac{4}{3} \sqrt{2} + \frac{6}{T_c} \right] , \quad \text{where} \quad \sigma = \pi k^4 T_c^3 \frac{3}{\hbar c} \ . \] (11)

The corresponding quantities for fermions (except the pressure) are dominated by constants:

\[ p_{fe} = \sigma T_c \left[ \frac{2}{5} \sqrt{2} T_c \frac{T_c}{T} - 2 + \frac{8}{3} \log 2 \sqrt{2} \frac{T_c}{T} \right] , \quad \rho_{fe} = \sigma T_c \left[ 2 - \frac{4}{5} \sqrt{2} \frac{T_c}{T} \right] , \]
\[ s_{fe} = \sigma \left[ \frac{8}{3} \sqrt{2} \log 2 - \frac{2}{5} \sqrt{2} \left( \frac{T_c}{T} \right)^2 \right] , \quad N_{fe} = \frac{\sigma}{k} V \left( \frac{4}{3} \sqrt{2} - \frac{T_c}{T} \right) . \] (12)

The behavior of the energy is depicted in Fig2.

![FIG. 1. The energy densities for fermions and bosons are plotted in terms of \( \frac{T}{T_c} \). The unit for energies is \( 16\pi \left( \frac{kT}{\hbar c} \right)^4 \).](image)

At temperatures below \( T_c \), the energy density is polynomial (\( \sim T^4 \)). Above \( T_c \), it becomes linear as obtained in Eq.(11) for bosons while it goes to a constant for fermions, as shown in Eq.(12). The difference between bosons and fermions in the unmodified theory is encoded in the factor 7/8, for the energy contributions for example. One sees this is drastically changed here.

### III. BLACK HOLE - RADIATION EQUILIBRIUM

We have seen in the past section how thermodynamics is influenced by the existence of a minimum uncertainty in length. We now wish to apply our results to the only systems in which extremely high temperatures can be obtained: black holes and the very early universe.

Using purely classical considerations, it was argued by Beckenstein that the area of a black hole can be interpreted as an entropy while its mass is identified with the energy [28]. The point of view considered in this paper is that the entropy of a black hole comes from a classical reasoning and so is essentially the same as in the unmodified theory. This is linked to the fact that in most phenomenological approaches to trans-Planckian physics, one suppose the particles evolve on a classical background but are subject to non trivial dispersion relations for example [6–12].

Let us consider a Schwarzschild black hole surrounded by radiation. The entropy and the energy of such a system read:
Computing its derivative one finds

\[ S_{tot} = \frac{4\pi}{l_{pl}^2} M^2 + S_{rad} \quad , \quad E_{tot} = M c^2 + E_{rad} \quad , \]

where \( l_{pl} \) is the Planck length and we use units in which \( k = 1 \) [35]. If the system is isolated, the total energy is conserved. According to the second law of thermodynamics, equilibrium configurations correspond to maxima of the entropy. Therefore, if the volume of the system is fixed, the derivatives of \( S_{tot} \) and \( E_{tot} \) vanish at equilibrium. This can be used to obtain a relation between the mass of the black hole and the equilibrium temperature:

\[ M = \frac{l_{pl}^2 c^2}{8\pi} \left[ \frac{dS_{rad}}{dT} \left( \frac{dE_{rad}}{dT} \right)^{-1} \right]_{eq} \quad . \]  

In the usual theory, one has

\[ \frac{dS_{rad}}{dT} \left( \frac{dE_{rad}}{dT} \right)^{-1} = \frac{d}{dT} \left( \frac{4\pi^2}{45} \frac{T^3}{c^3 h^3} \left( n_{bo} + \frac{7}{8} n_{fe} \right) V \right) \left[ \frac{d}{dT} \left( \frac{\pi^2}{15} \frac{T^4}{c^3 h^3} \left( n_{bo} + \frac{7}{8} n_{fe} \right) V \right) \right]^{-1} = \frac{1}{T} \quad . \]

\[ \Rightarrow T_{eq} = \frac{l_{pl}^2 c^2}{8\pi M} \quad , \]  

so that the mass of a black hole is inversely proportional to its temperature; this is the Hawking temperature. Including fermions introduces \( 7/8 \) factors but doesn’t change the final result.

This temperature is not affected by the presence of a minimal length uncertainty. To show this, let us first consider the regime in which a black hole is in equilibrium with a radiation consisting uniquely of bosons in the extremely high temperatures. Using the formulas displayed in Eq.(11) and retaining only the dominant contributions one has

\[ \frac{dS_{rad}}{dT} \left( \frac{dE_{rad}}{dT} \right)^{-1} = \frac{d}{dT} \left( \frac{8}{3} \sqrt{2} \log \frac{T}{T_c} \right) \quad . \]  

Similarly, one has for a gas containing only fermions

\[ \frac{dS_{rad}}{dT} \left( \frac{dE_{rad}}{dT} \right)^{-1} = \frac{d}{dT} \left( -\frac{2}{5} \sqrt{2} \frac{T_c^2}{T^2} \right) \quad . \]

so that for a mixture the temperature of equilibrium is unchanged.

This relation does not get corrections as one goes beyond the dominant contributions; it is true at all temperatures. This can be seen explicitly for bosons by writing the integral related to the partition function in the following way:

\[ q = 4\pi V \left( \frac{T_c}{h c} \right)^3 \int_0^1 dy \sqrt{2} y^2 \log (1 - \exp (g(y))) \quad , \quad \text{where} \quad g(y) = -\sqrt{2} \frac{T_c}{T} \frac{y}{1 + y^2} \quad . \]

Contrary to Eq.(9), the temperature does not appear in the upper bound of the integral but in the integrand only. One can carry the derivative with respect to temperature trough the integral to obtain the entropy. The energy can in the same way be rewritten as

\[ E_{rad} = 16\pi V \frac{T_c^4}{(h c)^3} \int_0^1 dy \frac{y^3}{1 + y^3} \frac{1}{\exp (-g(y)) - 1} \quad . \]

Computing its derivative one finds

\[ \frac{dE_{rad}}{dT} = 4\pi \sqrt{2} V \frac{T_c^5}{c^3 h^3 T^2} \int_0^1 \frac{y^4}{\left(1 + y^2\right)^2} \cosh^2 \left( \frac{1}{\sqrt{2}} \frac{T_c}{T} \frac{y}{1 + y^2} \right) \quad . \]

The derivative of the entropy

\[ \frac{dS_{rad}}{dT} = - \left( \frac{\partial q}{\partial T} + T \frac{\partial^2 q}{\partial T^2} \right) \quad . \]

is found to have the same expression, with an extra factor \( T \), so that going back to Eq.(14) the term under square brackets is exactly \( 1/T \). A similar situation occurs for fermions.
This is in agreement with the idea that the Hawking black hole temperature measured by an observer at infinity is not affected by a modification of the dispersion relation at transplanckian energies [6–10]. In fact, the result we just showed is generic and does not rely very much on the modified dispersion relation. The only think one needs is a fixed volume for the black hole- radiation system and a vanishing chemical potential for the particles composing the radiation. In fact, using Eq.(6,7) without specifying any dispersion relation, one obtains

$$\frac{dS_{rad}}{dT} = \frac{1}{kT^3} \sum_n \epsilon_n^2 \frac{\exp \left( \frac{-\epsilon_n}{kT} \right)}{1 - \exp \left( \frac{-\epsilon_n}{kT} \right)} . \tag{22}$$

Computing the derivative of the energy given in Eq.(8) leads to a cancellation in Eq.(14) which preserves exactly the last part of Eq.(15). This reasoning is valid for any kind of "reasonable" dispersion relation i.e such that $q$ and its derivatives are finite and the derivation can be carried into the infinite sum.

There is a more direct way of obtaining the same result. The first law of thermodynamics relates in the following way the changes in the energy $E_{rad}$, the entropy $S_{rad}$ and the number of particles $N$:

$$dE_{rad} = T \, dS_{rad} - p \, dV + \mu \, dN \tag{23},$$

$T$ being the temperature, $p$ the pressure and $\mu$ the chemical potential. The gas of radiation we consider is contained in a fixed volume($dV = 0$) and has zero chemical potential($\mu = 0$); this gives

$$\frac{dS_{rad}}{dE_{rad}} = \frac{1}{T} \), \tag{24}$$

so that the term under square brackets in Eq.(14) takes the value $1/T$.

The equilibrium is stable if the second derivative of the entropy is positive. This gives a particular volume such that the system tends to a black hole surrounded by radiation below it and to pure radiation above it. In our case, this critical volume is

$$V_o = \frac{15c^7h^3l_{pl}^2M^2}{\sqrt{2}T_c^3 \left( \left(5c^4l_{pl}^4 - 32\pi^2T_c^2M^2 \right)n_{bo} + 96\pi^2T_c^2M^2n_{fe} \right)} . \tag{25}$$

where again we have put $k = 1$. Contrary to the equilibrium temperature which does not feel the modified dispersion relation, the volume which fixes the final fate of the system depends on it. In particular, one sees that it goes like the second power of the black hole mass, contrary to the usual case where it behaves like its fifth power. Small black holes are the ones which display higher temperatures. The preceding formula is applicable only to such black holes. The coefficients of $n_{fe}$ and $n_{bo}$ are thus positive and $V_o$ never blows up.

**IV. THE VERY EARLY UNIVERSE**

The influence of transplanckian physics on the CMB predictions has been the subject of many studies [16–25]. It was also pointed out that the existence of a new physical scale changes the equation of state for radiation and thus the evolution of the cosmic scale factor [13,14]. To our knowledge, all these studies restricted themselves to the bosonic case. We showed in section 2 that fermions, compared to bosons have a drastically different behavior above $T_c$. We will include them to the picture and quantify the horizon, entropy and flatness problem in this context.

We consider a Robertson-Walker space time

$$ds^2 = -c^2dt^2 + a(t)^2 \left( \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right) \tag{26};$$

the constant $K$ can take the values 1, 0, −1, corresponding respectively to the closed, flat, open cases.

To use general relativity one needs to restore covariance somehow. One of the ways to tackle this issue is by introducing a unit vector field $u_\mu$ which is timelike and fixes the reference frame in which the modified dispersion relation hold. This field is made dynamical by the incorporation of a Maxwell type Lagrangian for example. A multiplier is also used to impose the norm of this vector to be equal to one. In a cosmological solution, this vector depends only on time. For a theory exhibiting a modified dispersion relation of the form $E^2 = p^2 + ap^4$, a coupling to general relativity has been constructed in which the vector field does not contribute to the energy-momentum tensor while the multiplier vanishes. The details of this procedure can be found in [13]. Such a coupling will be said to be
minimal. For simplicity, we shall assume such a coupling can be constructed in our model. The possibility of more subtle kinetic terms for the vector field is also possible. If one chooses another coupling to gravity our analysis can be seen as giving a heuristic view of the fact that the bosonic and fermionic content of the universe has, in the presence of an ultraviolet cut off, an influence which is different from the one to which we are accustomed. The same thing holds if ultimately one is not able to build the minimal coupling to gravity we suggested above.

We shall then use the following equations for the evolution of the universe:

\[ \dot{a} + 3 \frac{\dot{a}}{a} (\rho + p) = 0 \quad , \quad \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3c^2} \rho + \frac{K}{a^2} \; ; \]  

(27)

the dot means a derivative with respect to time. An important characteristic displayed by the thermodynamics of the system endowed with a minimal uncertainty in length is that the equations of states (see Eq.(11,12)) are not of the type \( p = \gamma \rho \), with \( \gamma \) a constant. This class of equations of states is common in cosmology and leads to the scale factor dependence \( a \sim t^{4/5} \). Therefore, in our case, the derivation of the relation between the scale factor, time and temperature requires a slightly different treatment. Introducing a prime to denote the derivative with respect to temperature, from the second part of Eq.(27), one reads similarly the link between time and temperature:

\[ \frac{a'}{a} = -\frac{\rho'(T)}{p(T) + \rho(T)} \]  

(28)

which admits a solution by quadratures:

\[ a(T) = a_* \exp \left[ -\frac{1}{3} \int_{T_*}^{T} \frac{d\xi}{\rho(\xi)(\rho(\xi) + \rho(\xi))} \right] . \]  

(29)

As one knows the expressions for density and pressure in terms of the temperature, one can infer the scale factor dependence on temperature. From the second part of Eq.(27), one reads similarly the link between time and temperature:

\[ t(T) - t_* \sim -\frac{1}{3} \sqrt{\frac{3c^2}{8\pi G}} \int_{T_*}^{T} \frac{d\xi}{\sqrt{\rho(\xi)(\rho(\xi) + \rho(\xi))}} . \]  

(30)

In the usual case, one has \( p = cT^4 \) and \( \rho = 3p \). The last two equations then give

\[ a_2(T) = \tilde{a}_2 \frac{T_{pl}^2}{T} \quad \text{and} \quad t_2(T) = \tilde{t}_2 \left( \frac{T_{pl}}{T} \right)^2 + \tilde{t}_3 , \quad \tilde{t}_2 = \frac{3\sqrt{5}}{4\pi \sqrt{11}} \frac{1}{\sqrt{h_{bo} + \frac{2}{3}h_{fc}}} t_{pl} , \]  

where

\[ T_{pl} = \frac{1}{k} \sqrt{\frac{c^4 h}{G}} \quad \text{and} \quad t_{pl} = \sqrt{\frac{Gh}{c^5}} . \]  

(31)

are the Planck temperature and time. From this one finds the usual relation \( t \sim 1/T^2 \).

The reason of the subscript for the scale factor is the following. The big bang in this model has a radiation period composed of two stages: the first one corresponds to extremely high temperatures and feels the presence of the minimal uncertainty in length. The evolution of the scale factor during that epoch will be denoted \( a_1(t) \) and given in Eqs.(33). The second period is the one in which the usual theory becomes valid; its evolution is given by Eqs.(31) and its scale factor is denoted \( a_2(t) \).

At very high temperatures, bosons dominate. The evolution of the scale factor in terms of the temperature is then essentially given by the following parametric relations

\[ t \sim \left( \frac{T}{T_c} \right)^{-1/2} \left( \log \frac{T}{T_c} \right)^{-1} , \quad a \sim \left( \log \frac{T}{T_c} \right)^{-1/3} . \]  

(32)

Compared to the unmodified theory, we find that the scale factor’s evolution in terms of temperature is much slower. To put it differently, at the same temperature, the usual big bang would predict a much bigger scale factor, for temperatures well above \( T_c \). The time spent to attain this temperature is also more important in our model.

For future use, we will need the behavior at high temperatures but at an epoch when fermions begin to play a role. Eqs.(11,12,29,30) lead to the following formulas:
\[
a_1(T) = \tilde{a}_1 \left[ \bar{a} \left( \frac{T}{T_c} \right)^{-2} + 2 \left( \bar{d} + \bar{c} \log \frac{T}{T_c} \right) \right]^{-1/3},
\]
\[
t_1(T) = \tilde{t}_1 \int_{T/T_c}^{\infty} \frac{dx \sqrt{x \bar{c} x^2 + bx + \tilde{a} \left[ \frac{1}{2} + (\bar{d} + \bar{c} \log x) x^2 \right]}}{\sqrt{\bar{c} x^2 + bx + \tilde{a} \left[ \frac{1}{2} + (\bar{d} + \bar{c} \log x) x^2 \right]}}, \quad \text{with } \tilde{t}_1 = \frac{1}{\sqrt{3\pi}} \left( \frac{T_{pl}}{T_c} \right)^2 t_{pl}. \tag{33}
\]

The constants $\bar{a}, \ldots \bar{d}$, embody the dependence of the system on the bosonic-fermionic content at very high temperatures:
\[
\bar{a} = \frac{4}{15} \sqrt{2} (n_{bo} - 3 n_{fe}), \quad \bar{b} = 2 (-n_{bo} + n_{fe}), \quad \bar{c} = \frac{8}{3} \sqrt{2} n_{bo}, \quad \bar{d} = \frac{2}{45} \sqrt{2} \left( 116 n_{bo} - 30 (n_{bo} - 2 n_{fe}) \log 2 \right). \tag{34}
\]

We have used a superscript $(n_{Ibo})$ to emphasize that the degrees of freedom appearing here are the ones present at $T > T_c$.

The two periods must join smoothly at some point. For illustrative purposes, we shall make the approximation that this occurs at the critical temperature: we shall equalize the scale factors ($\tilde{a}_1 = \tilde{a}_2$) and the times ($\tilde{t}_1 = \tilde{t}_2$) at this value. The first equation leads to a relation between the constants fixing the scales of the geometries in the two regions:
\[
\tilde{a}_2 \tilde{a}_1 = (\bar{a} + 2\bar{d}) \frac{T_c}{T_{pl}}, \tag{35}
\]

while the second can be used to extract the value of $\tilde{t}_3$.

### A. The flatness/entropy problem

The critical density $\rho_{cr}$ is a fictitious value which gives the same evolution of the scale factor but with $K = 0$ in formula (27). Introducing $\Omega = \rho/\rho_{cr}$, one has
\[
\Omega - 1 = \frac{K}{a^2 H^2}. \tag{36}
\]

In the usual theory, one can use Eq.(31) to show that
\[
|\Omega - 1|_T = 4 \left( \frac{\tilde{t}_2}{\tilde{a}_2} \right)^2 \left( \frac{T_{pl}}{T} \right)^2, \tag{37}
\]

from which one deduces
\[
\frac{|\Omega - 1|_T}{|\Omega - 1|_{T_{pl}}} = \left( \frac{T_{pl}}{T_c} \right)^2 = 10^{-64} \quad \text{for } T = T_{pl}. \tag{38}
\]

In this formula $T_{pl}$ is the present day temperature of the CMB radiation and $T_{pl}$ is the Planck temperature. As the denominator of the left hand side of the preceding formula is known to be close to unity today, its numerator must have been incredibly close to one at that Planck scale: this is the flatness problem; it can be solved by inflation.

Retaining the dominant contributions in Eqs.(33), one obtains for the first part of the radiation period
\[
|\Omega - 1|_T = 9 \cdot 2^{2/3} (\bar{c})^{-1/3} \left( \frac{\tilde{t}_1}{\tilde{a}_1} \right)^2 \left( \frac{T}{T_c} \right)^{-1} \left( \log \frac{T}{T_c} \right)^{2/3}, \tag{39}
\]

while Eq.(37) is valid for the second period. As we don’t have explicitly the scalar factor time dependence, we have obviously used the equation
\[
H = \frac{\dot{a}}{a} = \frac{1}{a} \frac{a'}{v}. \tag{40}
\]

Using the matching condition displayed in Eq.(35), one finds the ratio
\[
\frac{|\Omega - 1|_T}{|\Omega - 1|_{T_0}} = Z \left( \frac{T_o}{T_e} \right)^2 \left( \frac{T}{T_e} \right)^{-1} \left( \log \frac{T}{T_e} \right)^{2/3}, \quad \text{where} \quad Z = \frac{4}{15} 2^{2/3}(\bar{c})^{-1/3} (\bar{a} + 2\bar{d})^2 \left( n_{b0}^{I} + \frac{7}{8} n_{bc}^{I} \right). \quad (41)
\]

We have used a superscript \((n_{bc}^{I})\) to emphasize that the particles appearing in this formula are the degrees of freedom present at \(T_0\); their numbers are of course smaller than the ones at the high temperature \(T\) which enter into the constants \(\bar{a}, \ldots\) as displayed in Eq.(34).

From the last equation one draws two conclusions. First, as the temperature goes to infinity, the ratio vanishes so that the flatness problem is not solved in this context. This was remarked in a different model [13], relying on numerical computations and in [14] in the presence of a minimal uncertainty in length. The analytical approach followed in our work enables us to say two more things. First, although the flatness problem is not solved, Eq.(41) shows that it is less severe in the presence of a minimal uncertainty in length. Second, the content of the theory in terms of bosons and fermions now plays a role, contrary to the usual theory in which Eq.(38) applies. One also sees that assuming the critical and the Planck temperatures to be different or equal doesn’t matter; above the maximum of the two, our formulas show that the flatness problem remains unsolved.

At very high temperatures, one can rewrite the departure from flatness in terms of the entropy in the following way:

\[
|\Omega - 1|_T \sim \frac{1}{s^{2/3} \exp \left( \frac{3s}{\sqrt{2} \pi} \right)} \quad ;
\]

this differs from the undeformed theory for which one has

\[
|\Omega - 1|_T \sim \frac{1}{s^{2/3}} \quad .
\]

The same conclusion than in the unmodified theory holds: to be almost flat today, the universe had to have a huge entropy density at the initial times.

In our model, one can verify that the total entropy is conserved, order by order. For example, the dominant contribution to the entropy contained in a comoving volume is

\[
-a^3 s \sim c^{st} \left[ \left( \log \frac{T}{T_e} \right)^{-1/3} \right]^3 \left( \log \frac{T}{T_e} \right) \quad (44)
\]

which is independent of the temperature.

**B. The horizon problem**

The distance a photon has traveled since the big bang is given by

\[
R_H = c a(t_o) \int_0^{t_o} \frac{dt}{a(t)} = -c a(t_o) \left[ \int_{T_o}^{T_e} t_2'(T) \frac{dT}{a_2(T)} + \int_{T_o}^{\infty} t_1'(T) \frac{dT}{a_1(T)} \right] \quad .
\]

The dominant contributions can be written as

\[
R_H(T_o) = R_H^{\star}(T_o) \left( 1 + \frac{T_o}{T_e} \right) \quad ,
\]

where \(R_H^{\star}(T_o)\) is the horizon in the unmodified theory while \(\gamma\) contains the information about the fermionic/bosonic content:

\[
\gamma = 2^{-2/3} \frac{4\pi}{3\sqrt{15}} \sqrt{n_{f_\infty}^{II} + \frac{7}{8} n_{f_\infty}^{I}} (\bar{a} + 2\bar{d}) \int_0^\infty dx \sqrt{x^7/\sqrt{\bar{e}x^2 + \bar{b}x + \bar{a}}} \left[ \frac{3}{2} + (\bar{d} + \bar{c}\log x)x^2 \right]^{2/3} \quad .
\]

The smallness of today’s temperature \(T_o\) compared to the critical temperature \(T_e\) is such that the correction to the horizon will not be significant. Note, however that once again the spin content of the universe enters into play in a non trivial way.

The various quantities which appear in our formulas have the correct behaviours. Considering for example Eq.(47), the term under square root in the integral must obviously be positive. One can rewrite it as a linear combination of \(n_{f_\infty}^{I}\) and \(n_{b_\infty}^{I}\); the first coefficient vanishes at \(x = 0.56\) while the second has no zero. This means that the matching temperature can not be taken to be lower than \(0.56 T_e\). A precise numerical evaluation of this quantity is possible but one has then to work in a specific model, with \(n_{f_\infty}^{I}, n_{b_\infty}^{I}\) known. One can nevertheless expect that the departure from usual physics will take place around \(T_e\).
V. CONCLUSIONS

We have studied the thermodynamics induced by a non local theory which exhibits a minimal uncertainty in length. We have obtained that a new behavior sets in at very high temperatures. The difference between fermions and bosons is more important than in the usual case. When studying the equilibrium of a gas of radiation surrounding a black hole, we have suggested a generic argument which assures the universality of the temperature of equilibrium, for reasonable deformations. Our work is in agreement with previous studies which, using the machinery of quantum field theory in curved backgrounds, showed that the Hawking radiation as perceived by a remote observer is not affected when the dispersion relation gets modified at Planckian energies. The only difference we found so far is in the volume which is at the frontier separating the cases in which the final stage contains only radiation from the ones in which the two are present. On the contrary, we have shown that the flatness problem is less sever in this context, contrary to the horizon problem which remains roughly speaking untouched. The scale factor and time growths in function of the temperature are slower than in the usual big bang. We also saw that the entropy per comoving volume was conserved.

One of the important ingredients in our cosmological analysis is the way the theory is coupled to gravity. We assumed, inspired by [13], that there is a "minimal" coupling i.e one for which, in the cosmological solution, the unit time-like vector field does not contribute to the energy density. One of the challenges is now to construct explicitly such a model. If one considers other possibilities, quantitative differences are likely to appear. We nevertheless suspect that, qualitatively, the fact that bosons and fermions behave differently at very high temperatures is captured, at least partially, by the treatment we have presented. One of the key issues in a more complete treatment will be the adiabatic expansion of the universe.

The commutation relations studied here can be interpreted as phenomenological consequences of string or M theory [2]. Our work, after others, suggests that string cosmology may not be uniquely characterized by the evolution of the fields which appear at the lowest order(like the dilaton) but also by some non trivial statistical effects. Finally, the theory exhibiting a minimal length uncertainty may forbid the singularity present in the standard big bang scenario. A similar reasoning was advocated to argue that the Hawking evaporation of a black hole may halt without using the complementarity hypothesis [39].

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