HSU-ROBBINS AND SPITZER’S THEOREMS FOR THE VARIATIONS OF FRACTIONAL BROWNIAN MOTION

CIPRIAN A. TUDOR
SAMOS/MATISSE, Centre d’Economie de La Sorbonne, Université de Panthéon-Sorbonne Paris 1, 90, rue de Tolbiac, 75634 Paris Cedex 13, France
e-mail: tudor@univ-paris1.fr

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Abstract
Using recent results on the behavior of multiple Wiener-Itô integrals based on Stein’s method, we prove Hsu-Robbins and Spitzer’s theorems for sequences of correlated random variables related to the increments of the fractional Brownian motion.

1 Introduction

A famous result by Hsu and Robbins [7] says that if \( X_1, X_2, \ldots \) is a sequence of independent identically distributed random variables with zero mean and finite variance and \( S_n := X_1 + \ldots + X_n \), then

\[
\sum_{n \geq 1} P \left( |S_n| > \varepsilon n \right) < \infty
\]

for every \( \varepsilon > 0 \). Later, Erdös ([3], [4]) showed that the converse implication also holds, namely if the above series is finite for every \( \varepsilon > 0 \) and \( X_1, X_2, \ldots \) are independent and identically distributed, then \( \mathbb{E}X_1 = 0 \) and \( \mathbb{E}X_1^2 < \infty \). Since then, many authors extended this result in several directions. Spitzer’s showed in [13] that

\[
\sum_{n \geq 1} \frac{1}{n} P \left( |S_n| > \varepsilon n \right) < \infty
\]

for every \( \varepsilon > 0 \) if and only if \( \mathbb{E}X_1 = 0 \) and \( \mathbb{E}|X_1| < \infty \). Also, Spitzer’s theorem has been the object of various generalizations and variants. One of the problems related to the Hsu-Robbins’ and Spitzer’s theorems is to find the precise asymptotic as \( \varepsilon \to 0 \) of the quantities \( \sum_{n \geq 1} P \left( |S_n| > \varepsilon n \right) \) and \( \sum_{n \geq 1} \frac{1}{n} P \left( |S_n| > \varepsilon n \right) \). Heyde [5] showed that

\[
\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{n \geq 1} P \left( |S_n| > \varepsilon n \right) = \mathbb{E}X_1^2
\]
whenever \( \mathbb{E}X = 0 \) and \( \mathbb{E}X^2 < \infty \). In the case when \( X \) is attracted to a stable distribution of exponent \( \alpha > 1 \), Spataru \([12]\) proved that

\[
\lim_{\epsilon \to 0} \frac{1}{-\log \epsilon} \sum_{n \geq 1} \frac{1}{n} P \left( |S_n| > \epsilon n \right) = \frac{\alpha}{\alpha - 1}.
\]

The purpose of this paper is to prove Hsu-Robbins and Spitzer’s theorems for sequences of correlated random variables, related to the increments of fractional Brownian motion, in the spirit of [5] or [12]. Recall that the fractional Brownian motion \( (B^{H}_t)_{t \in [0,1]} \) is a centered Gaussian process with covariance function \( R^{H}(t,s) = \mathbb{E}(B^{H}_t B^{H}_s) = \frac{1}{2} (s^{2H} + t^{2H} - |t-s|^{2H}) \). It can be also defined as the unique self-similar Gaussian process with stationary increments. Concretely, in this paper we will study the behavior of the tail probabilities of the sequence

\[
V_n = \sum_{k=0}^{n-1} H_q \left( n^H \left( B_{\frac{k+1}{n}} - B_{\frac{k}{n}} \right) \right)
\]

where \( B \) is a fractional Brownian motion with Hurst parameter \( H \in (0,1) \) (in the sequel we will omit the superscript \( H \) for \( B \)) and \( H_q \) is the Hermite polynomial of degree \( q \geq 1 \) given by

\[
H_q(x) = (-1)^q x^q \frac{d^q}{dt^q} (e^{-x^2}).
\]

The sequence \( V_n \) behaves as follows (see e.g. [9], Theorem 1; the result is also recalled in Section 3 of our paper): if \( 0 < H < 1 - \frac{1}{2q} \), a central limit theorem holds for the renormalized sequence \( Z_n^{(1)} = \frac{V_n}{c_{1,q,H}\sqrt{n}} \) while if \( 1 - \frac{1}{2q} < H < 1 \), the sequence \( Z_n^{(2)} = \frac{V_n}{c_{2,q,H}\sqrt{n}} \) converges in \( L^2(\Omega) \) to a Hermite random variable of order \( q \) (see Section 2 for the definition of the Hermite random variable and Section 3 for a rigorous statement concerning the convergence of \( V_n \)). Here \( c_{1,q,H}, c_{2,q,H} \) are explicit positive constants depending on \( q \) and \( H \).

We note that the techniques generally used in the literature to prove the Hsu-Robbins and Spitzer’s results are strongly related to the independence of the random variables \( X_1, X_2, \ldots \). In our case the variables are correlated. Indeed, for any \( k, l \geq 1 \) we have

\[
\mathbb{E} \left( H_q(B_{k+1} - B_k)H_q(B_{l+1} - B_l) \right) = \frac{1}{(q!)^2} \rho_H(k-l)
\]

where the correlation function is \( \rho_H(k) = \frac{1}{2} \left( (k+1)^{2H} + (k-1)^{2H} - 2k^{2H} \right) \) which is not equal to zero unless \( H = \frac{1}{2} \) (which is the case of the standard Brownian motion). We use new techniques based on the estimates for the multiple Wiener-Itô integrals obtained in [2], [10] via Stein’s method and Malliavin calculus. Concretely, we study in this paper the behavior as \( \epsilon \to 0 \) of the quantities

\[
\sum_{n \geq 1} \frac{1}{n} P \left( V_n > \epsilon n \right) = \sum_{n \geq 1} \frac{1}{n} P \left( Z_n^{(1)} > c_{1,q,H}^{-1} \epsilon \sqrt{n} \right),
\]

and

\[
\sum_{n \geq 1} P \left( V_n > \epsilon n \right) = \sum_{n \geq 1} P \left( Z_n^{(1)} > c_{1,q,H}^{-1} \epsilon \sqrt{n} \right),
\]

if \( 0 < H < 1 - \frac{1}{2q} \) and of

\[
\sum_{n \geq 1} \frac{1}{n} P \left( V_n > \epsilon n^{2-2q(1-H)} \right) = \sum_{n \geq 1} \frac{1}{n} P \left( Z_n^{(2)} > c_{2,q,H}^{-1} \epsilon n^{1-2q(1-H)} \right)
\]
and
\[ \sum_{n \geq 1} P \left( V_n > \epsilon n^{2-2q(1-H)} \right) = \sum_{n \geq 1} P \left( Z_n^{(2)} > c_n^{-1} \epsilon n^{1-q(1-H)} \right) \]  
(7)

if \( 1 - \frac{1}{2q} < H < 1 \). The basic idea in our proofs is that, if we replace \( Z_n^{(1)} \) and \( Z_n^{(2)} \) by their limits (standard normal random variable or Hermite random variable) in the above expressions, the behavior as \( \epsilon \to 0 \) can be obtained by standard calculations. Then we need to estimate the difference between the tail probabilities of \( Z_n^{(1)}, Z_n^{(2)} \) and the tail probabilities of their limits. To this end, we will use the estimates obtained in [2], [10] via Malliavin calculus and we are able to prove that this difference converges to zero in all cases. We obtain that, as \( \epsilon \to 0 \), the quantities (4) and (6) are of order of \( \log \epsilon \) while the functions (5) and (7) are of order of \( \epsilon^2 \) and \( \epsilon^{1-q(1-H)} \) respectively.

The paper is organized as follows. Section 2 contains some preliminaries on the stochastic analysis on Wiener chaos. In Section 3 we prove the Spitzer’s theorem for the variations of the fractional Brownian motion while Section 4 is devoted to the Hsu-Robbins theorem for this sequence.

Throughout the paper we will denote by \( c \) a generic strictly positive constant which may vary from line to line (and even on the same line).

2 Preliminaries

Let \( (W_t)_{t \in [0,1]} \) be a classical Wiener process on a standard Wiener space \( (\Omega, \mathcal{F}, \mathbb{P}) \). If \( f \in L^2([0,1]^n) \) with \( n \geq 1 \) integer, we introduce the multiple Wiener-Itô integral of \( f \) with respect to \( W \). The basic reference is [11].

Let \( f \in \mathcal{S}_m \) be an elementary function with \( m \) variables that can be written as
\[ f = \sum_{i_1, \ldots, i_m} c_{i_1, \ldots, i_m} 1_{A_{i_1} \times \cdots \times A_{i_m}} \]
where the coefficients satisfy \( c_{i_1, \ldots, i_m} = 0 \) if two indices \( i_k \) and \( i_l \) are equal and the sets \( A_i \in \mathcal{B}([0,1]) \) are disjoint. For such a step function \( f \) we define
\[ I_m(f) = \sum_{i_1, \ldots, i_m} c_{i_1, \ldots, i_m} W(A_{i_1}) \cdots W(A_{i_m}) \]
where we put \( W(A) = \int_0^1 1_A(s) dW_s \) if \( A \in \mathcal{B}([0,1]) \). It can be seen that the mapping \( I_n \) constructed above from \( \mathcal{S}_m \) to \( L^2(\Omega) \) is an isometry on \( \mathcal{S}_m \), i.e.
\[ \mathbb{E} \left[ I_n(f) I_m(g) \right] = n! \langle f, g \rangle_{L^2([0,1]^n)} \] if \( m = n \)  
(8)

and
\[ \mathbb{E} \left[ I_n(f) I_m(g) \right] = 0 \] if \( m \neq n \).

Since the set \( \mathcal{S}_n \) is dense in \( L^2([0,1]^n) \) for every \( n \geq 1 \) the mapping \( I_n \) can be extended to an isometry from \( L^2([0,1]^n) \) to \( L^2(\Omega) \) and the above properties hold true for this extension.

We will need the following bound for the tail probabilities of multiple Wiener-Itô integrals (see [8], Theorem 4.1)
\[ P \left( |I_n(f)| > u \right) \leq c \exp \left( -\frac{c u}{\sigma} \right) \]  
(9)
for all $u > 0$, $n \geq 1$, with $\sigma = \|f\|_{L^2([0,1]^n)}$.

The Hermite random variable of order $q \geq 1$ that appears as limit in Theorem 1 point ii. is defined as (see [9])

$$Z = d(q,H)\mathcal{I}_q(L)$$ (10)

where the kernel $L \in L^2([0,1]^q)$ is given by

$$L(y_1,\ldots,y_q) = \int_0^1 \partial_1 K^H(u,y_1) \cdots \partial_q K^H(u,y_q) du.$$  

The constant $d(q,H)$ is a positive normalizing constant that guarantees that $E[Z^2] = 1$ and $K^H$ is the standard kernel of the fractional Brownian motion (see [11], Section 5). We will not need the explicit expression of this kernel. Note that the case $q = 1$ corresponds to the fractional Brownian motion and the case $q = 2$ corresponds to the Rosenblatt process.

### 3 Spitzer’s theorem

Let us start by recalling the following result on the convergence of the sequence $V_n$ (3) (see [9], Theorem 1).

**Theorem 1.** Let $q \geq 2$ an integer and let $(B_t)_{t \geq 0}$ a fractional Brownian motion with Hurst parameter $H \in (0,1)$. Then, with some explicit positive constants $c_{1,q,H}, c_{2,q,H}$ depending only on $q$ and $H$ we have

- **i.** If $0 < H < 1 - \frac{1}{2q}$ then
  $$\frac{V_n}{c_{1,q,H} \sqrt{n}} \overset{\text{Law}}{\to} N(0,1)$$ (11)

- **ii.** If $1 - \frac{1}{2q} < H < 1$ then
  $$\frac{V_n}{c_{2,q,H} n^{1-q(1-H)}} \overset{L^2}{\to} n \to \infty Z$$ (12)

where $Z$ is a Hermite random variable given by (10).

In the case $H = 1 - \frac{1}{2q}$ the limit is still Gaussian but the normalization is different. However we will not treat this case in the present work.

We set

$$Z_n^{(1)} = \frac{V_n}{c_{1,q,H} \sqrt{n}}, \quad Z_n^{(2)} = \frac{V_n}{c_{2,q,H} n^{1-q(1-H)}}$$ (13)

with the constants $c_{1,q,H}, c_{2,q,H}$ from Theorem 1.

Let us denote, for every $\varepsilon > 0$,

$$f_1(\varepsilon) = \sum_{n \geq 1} \frac{1}{n} P(V_n > \varepsilon n) = \sum_{n \geq 1} \frac{1}{n} P\left(Z_n^{(1)} > c_{1,q,H}^{-1} \varepsilon \sqrt{n}\right)$$ (14)

and

$$f_2(\varepsilon) = \sum_{n \geq 1} \frac{1}{n} P(V_n > \varepsilon n^{2-2q(1-H)}) = \sum_{n \geq 1} \frac{1}{n} P\left(Z_n^{(2)} > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)}\right)$$ (15)
Remark 1. It is natural to consider the tail probability of order $n^{2-2q(1-H)}$ in (15) because the $L^2$ norm of the sequence $V_n$ is in this case of order $n^{1-2q(1-H)}$.

We are interested to study the behavior of $f_i(\epsilon)$ ($i = 1, 2$) as $\epsilon \to 0$. For a given random variable $X$, we set $\Phi_X(z) = 1 - P(X < z) + P(X < -z)$.

The first lemma gives the asymptotics of the functions $f_i(\epsilon)$ as $\epsilon \to 0$ when $Z_n^{(i)}$ are replaced by their limits.

Lemma 1. Consider $c > 0$.

\begin{enumerate}
  \item Let $Z^{(1)}$ be a standard normal random variable. Then as 
  \[ \frac{1}{-\log c\epsilon} \sum_{n \geq 1} \frac{1}{n} \Phi_{Z^{(1)}}(c\epsilon \sqrt{n}) \to_{\epsilon \to 0} 2. \]
  \item Let $Z^{(2)}$ be a Hermite random variable of order $q$ given by (10). Then, for any integer $q \geq 1$ 
  \[ \frac{1}{-\log c\epsilon} \sum_{n \geq 1} \frac{1}{n} \Phi_{Z^{(2)}}(c\epsilon n^{1-q(1-H)}) \to_{\epsilon \to 0} \frac{1}{1 - q(1-H)}. \]
\end{enumerate}

Proof: The case when $Z^{(1)}$ follows the standard normal law is hidden in (12). We will give the ideas of the proof. We can write (see (12))

\[ \sum_{n \geq 1} \frac{1}{n} \Phi_{Z^{(1)}}(c\epsilon \sqrt{n}) = \int_1^{\infty} \frac{1}{x} \Phi_{Z^{(1)}}(c\epsilon \sqrt{x}) dx - \frac{1}{2} \Phi_{Z^{(1)}}(c\epsilon) - \int_1^{\infty} P_1(x) d \left( \frac{1}{x} \Phi_{Z^{(1)}}(c\epsilon \sqrt{x}) \right). \]

with $P_1(x) = [x] - x + \frac{1}{2}$. Clearly as $\epsilon \to 0$, $\frac{1}{\log c\epsilon} \Phi_{Z^{(1)}}(c\epsilon) \to 0$ because $\Phi_{Z^{(1)}}$ is a bounded function and concerning the last term it is also trivial to observe that

\[ \frac{1}{-\log c\epsilon} \int_1^{\infty} P_1(x) d \left( \frac{1}{x} \Phi_{Z^{(1)}}(c\epsilon \sqrt{x}) \right) = \frac{1}{-\log c\epsilon} \left( - \int_1^{\infty} P_1(x) \left( \frac{1}{x^2} \Phi_{Z^{(1)}}(c\epsilon \sqrt{x}) dx + c\epsilon - x^{-\frac{1}{2}} \frac{1}{x} \Phi_{Z^{(1)}}(c\epsilon \sqrt{x}) dx \right) \right) \to_{\epsilon \to 0} 0 \]

since $\Phi_{Z^{(1)}}$ and $\Phi'_{Z^{(1)}}$ are bounded. Therefore the asymptotics of the function $f_i(\epsilon)$ as $\epsilon \to 0$ will be given by $\int_1^{\infty} \frac{1}{x} \Phi_{Z^{(1)}}(c\epsilon \sqrt{x}) dx$. By making the change of variables $c\epsilon \sqrt{x} = y$, we get

\[ \lim_{\epsilon \to 0} \frac{1}{\log c\epsilon} \int_1^{\infty} \frac{1}{x} \Phi_{Z^{(1)}}(c\epsilon \sqrt{x}) dx = \lim_{\epsilon \to 0} \frac{1}{\log c\epsilon} 2 \int_{c\epsilon}^{\infty} \frac{1}{y} \Phi_{Z^{(1)}}(y) dy = \lim_{\epsilon \to 0} 2 \Phi_{Z^{(1)}}(c\epsilon) = 2. \]

Let us consider now the case of the Hermite random variable. We will have as above

\[ \lim_{\epsilon \to 0} \frac{1}{-\log c\epsilon} \sum_{n \geq 1} \frac{1}{n} \Phi_{Z^{(2)}}(c\epsilon n^{1-q(1-H)}) \]

\[ = \lim_{\epsilon \to 0} \frac{1}{-\log c\epsilon} \left( \int_1^{\infty} \frac{1}{x} \Phi_{Z^{(2)}}(c\epsilon x^{1-q(1-H)}) dx - \int_1^{\infty} P_1(x) d \left( \frac{1}{x} \Phi_{Z^{(2)}}(c\epsilon x^{1-q(1-H)}) \right) \right) \]
By making the change of variables $c\varepsilon x^{1-q(1-H)} = y$ we will obtain
\[
\lim_{\varepsilon \to 0} \frac{1}{-\log c\varepsilon} \int_1^{\infty} \frac{1}{x} \Phi_{\varepsilon^{(1)}\mathcal{E}x^{1-q(1-H)}}(c\varepsilon x^{1-q(1-H)}) \, dx
\]
\[
= \lim_{\varepsilon \to 0} \frac{1}{-\log c\varepsilon} \frac{1}{1 - q(1-H)} \int_{c\varepsilon}^{\infty} \frac{1}{y} \Phi_{\varepsilon^{(1)}\mathcal{E}}(y) \, dy = \lim_{\varepsilon \to 0} \frac{1}{1 - q(1-H)} \Phi_{\varepsilon^{(1)}\mathcal{E}}(c\varepsilon) = \frac{1}{1 - q(1-H)}
\]
where we used the fact that $\Phi_{\varepsilon^{(1)}\mathcal{E}}(y) \leq y^{-2}E[|Z^{(2)}|^2]$ and so $\lim_{y \to \infty} \log y \Phi_{\varepsilon^{(1)}\mathcal{E}}(y) = 0$.

It remains to show that $\frac{1}{-\log c\varepsilon} \int_1^{\infty} P_1(x) d\left[ \frac{1}{x} \Phi_{\varepsilon^{(1)}\mathcal{E}x^{1-q(1-H)}} \right]$ converges to zero as $\varepsilon$ tends to $0$ (note that actually it follows from a result by [11] that a Hermite random variable has a density, but we don’t need it explicitly, we only use the fact that $\Phi_{\varepsilon^{(1)}\mathcal{E}}$ is differentiable almost everywhere).

This is equal to
\[
\lim_{\varepsilon \to 0} \frac{1}{-\log c\varepsilon} \int_{c\varepsilon}^{\infty} P_1(x) c\varepsilon (1 - q(1-H)) x^{-q(1-H)-1} \Phi_{\varepsilon^{(1)}\mathcal{E}x^{1-q(1-H)}}(c\varepsilon x^{1-q(1-H)}) \, dx
\]
\[
= c \frac{\varepsilon}{-\log c\varepsilon} (c\varepsilon)^{d(1-h)} \int_{c\varepsilon}^{\infty} P_1 \left( \frac{y}{c\varepsilon} \right) \frac{1}{y^{1-q(1-h)}} \Phi_{\varepsilon^{(1)}\mathcal{E}}(y) y^{-\frac{1}{1-q(1-h)}} \, dy
\]
\[
\leq c \frac{1}{-\log \varepsilon} \int_{c\varepsilon}^{\infty} P_1 \left( \frac{y}{c\varepsilon} \right) \frac{1}{y^{1-q(1-h)}} \Phi_{\varepsilon^{(1)}\mathcal{E}}(y) \, dy
\]
which clearly goes to zero since $P_1$ is bounded and $\int_0^{\infty} \Phi_{\varepsilon^{(1)}\mathcal{E}}(y) \, dy = 1$. □

The next result estimates the limit of the difference between the functions $f(\varepsilon)$ given by (14), (15) and the sequence in Lemma 1.

**Proposition 1.** Let $q \geq 2$ and $c > 0$.

i. If $H < 1 - \frac{1}{2q}$, let $Z_n^{(1)}$ be given by (13) and let $Z^{(1)}$ be standard normal random variable. Then it holds

\[
\frac{1}{-\log c\varepsilon} \left[ \sum_{n \geq 1} \frac{1}{n} P \left( |Z_n^{(1)}| > c\varepsilon \sqrt{n} \right) - \sum_{n \geq 1} \frac{1}{n} P \left( |Z^{(1)}| > c\varepsilon \sqrt{n} \right) \right] \to_{\varepsilon \to 0} 0.
\]

ii. Let $Z^{(2)}$ be a Hermite random variable of order $q \geq 2$ and $H > 1 - \frac{1}{2q}$. Then

\[
\frac{1}{-\log c\varepsilon} \left[ \sum_{n \geq 1} \frac{1}{n} P \left( |Z_n^{(2)}| > c\varepsilon n^{-q(1-H)} \right) - \sum_{n \geq 1} \frac{1}{n} P \left( |Z^{(2)}| > c\varepsilon n^{-q(1-H)} \right) \right] \to_{\varepsilon \to 0} 0.
\]

**Proof:** Let us start with the point i. Assume $H < 1 - \frac{1}{2q}$. We can write

\[
\sum_{n \geq 1} \frac{1}{n} P \left( |Z_n^{(1)}| > c\varepsilon \sqrt{n} \right) - \sum_{n \geq 1} \frac{1}{n} P \left( |Z^{(1)}| > c\varepsilon \sqrt{n} \right)
\]
\[
= \sum_{n \geq 1} \left[ P \left( Z_n^{(1)} > c\varepsilon \sqrt{n} \right) - P \left( Z^{(1)} > c\varepsilon \sqrt{n} \right) \right] + \sum_{n \geq 1} \left[ \frac{1}{n} P \left( Z_n^{(1)} < -c\varepsilon \sqrt{n} \right) - P \left( Z^{(1)} < -c\varepsilon \sqrt{n} \right) \right]
\]
\[
\leq 2 \sum_{n \geq 1} \sup_{x \in \mathbb{R}} \left| P \left( Z_n^{(1)} > x \right) - P \left( Z^{(1)} > x \right) \right|.
\]
It follows from [10], Theorem 4.1 that

\[
\sup_{x \in \mathbb{R}} \left| P \left( Z_n^{(i)} > x \right) - P \left( Z^{(i)} > x \right) \right| \leq c \begin{cases} \frac{1}{\sqrt{n}}, & H \in (0, \frac{1}{2}] \\ n^{H-1}, & H \in \left[ \frac{1}{2}, \frac{2q-3}{2q-2} \right) \\ n^{qH-q+\frac{1}{2}}, & H \in \left[ \frac{2q-3}{2q-2}, 1 - \frac{1}{2q} \right). \end{cases}
\]  

(16)

and this implies that

\[
\sum_{n \geq 1} \frac{1}{n} \sup_{x \in \mathbb{R}} \left| P \left( Z_n^{(i)} > x \right) - P \left( Z^{(i)} > x \right) \right| \leq c \begin{cases} \sum_{n \geq 1} \frac{1}{n^{\frac{1}{2}}}, & H \in (0, \frac{1}{2}] \\ \sum_{n \geq 1} n^{H-2}, & H \in \left[ \frac{1}{2}, \frac{2q-3}{2q-2} \right) \\ \sum_{n \geq 1} n^{qH-q-\frac{1}{2}}, & H \in \left[ \frac{2q-3}{2q-2}, 1 - \frac{1}{2q} \right). \end{cases}
\]  

(17)

and the last sums are finite (for the last one we use \( H < 1 - \frac{1}{2q} \)). The conclusion follows.

Concerning the point ii. (the case \( H > 1 - \frac{1}{2q} \)), by using a result in Proposition 3.1 of [2] we have

\[
\sup_{x \in \mathbb{R}} \left| P \left( Z_n^{(i)} > x \right) - P \left( Z^{(i)} > x \right) \right| \leq c \left( \mathbb{E} \left| (Z_n^{(2)} - Z^{(2)}) \right|^2 \right)^{\frac{1}{2q}} \leq cn^{1-\frac{1}{q-H}}
\]  

(18)

and as a consequence

\[
\sum_{n \geq 1} \frac{1}{n} \left| Z_n^{(2)} - c \varepsilon T^{1-q(1-H)} \right| - \sum_{n \geq 1} \frac{1}{n} P \left( \left| Z^{(2)} \right| > c \varepsilon n^{1-q(1-H)} \right) \leq c \sum_{n \geq 1} n^{-\frac{1}{q-H}}
\]

and the above series is convergent because \( H > 1 - \frac{1}{2q} \). \( \blacksquare \)

We state now the Spitzer's theorem for the variations of the fractional Brownian motion.

**Theorem 2.** Let \( f_1, f_2 \) be given by (14), (15) and the constants \( c_{1,q,H}, c_{2,q,H} \) be those from Theorem [1]

i. If \( 0 < H < 1 - \frac{1}{2q} \) then

\[
\lim_{\varepsilon \to 0} \frac{1}{\log(c_{1,H,q})} f_1(\varepsilon) = 2.
\]

ii. If \( 1 > H > 1 - \frac{1}{2q} \) then

\[
\lim_{\varepsilon \to 0} \frac{1}{\log(c_{2,H,q})} f_2(\varepsilon) = \frac{1}{1-q(1-H)}.
\]

**Proof:** It is a consequence of Lemma [1] and Proposition [1] \( \blacksquare \)

**Remark 2.** Concerning the case \( H = 1 - \frac{1}{2q} \), note that the correct normalization of \( V_n \) (3) is \( \frac{1}{(\log n)^{1/2}} \).

Because of the appearance of the term \( \log n \) our approach is not directly applicable to this case.
4 Hsu-Robbins theorem for the variations of fractional Brownian motion

In this section we prove a version of the Hsu-Robbins theorem for the variations of the fractional Brownian motion. Concretely, we denote here by, for every \( \varepsilon > 0 \)

\[
g_1(\varepsilon) = \sum_{n \geq 1} P \left( |V_n| > \varepsilon n \right)
\]

(19)

if \( H < 1 - \frac{1}{2q} \) and by

\[
g_2(\varepsilon) = \sum_{n \geq 1} P \left( |V_n| > \varepsilon n^{2q(1-H)} \right)
\]

(20)

if \( H > 1 - \frac{1}{2q} \) and we estimate the behavior of the functions \( g_i(\varepsilon) \) as \( \varepsilon \to 0 \). Note that we can write

\[
g_1(\varepsilon) = \sum_{n \geq 1} P \left( |Z_n^{(1)}| > c_{1,q,H}^{-1} \varepsilon \sqrt{n} \right)
\]

\[
g_2(\varepsilon) = \sum_{n \geq 1} P \left( |Z_n^{(2)}| > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)} \right)
\]

with \( Z_n^{(1)}, Z_n^{(2)} \) given by (13).

We decompose it as: for \( H < 1 - \frac{1}{2q} \)

\[
g_1(\varepsilon) = \sum_{n \geq 1} P \left( |Z_n^{(1)}| > c_{1,q,H}^{-1} \varepsilon \sqrt{n} \right)
\]

\[
+ \sum_{n \geq 1} \left[ P \left( |Z_n^{(1)}| > c_{1,q,H}^{-1} \varepsilon \sqrt{n} \right) - P \left( |Z_n^{(1)}| > c_{1,q,H}^{-1} \varepsilon \sqrt{n} \right) \right]
\]

and for \( H > 1 - \frac{1}{2q} \)

\[
g_2(\varepsilon) = \sum_{n \geq 1} P \left( |Z_n^{(2)}| > \varepsilon c_{2,q,H} n^{1-q(1-H)} \right)
\]

\[
+ \sum_{n \geq 1} \left[ P \left( |Z_n^{(2)}| > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)} \right) - P \left( |Z_n^{(2)}| > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)} \right) \right]
\]

We start again by consider the situation when \( Z_n^{(1)} \) are replaced by their limits.

**Lemma 2.** i. Let \( Z^{(1)} \) be a standard normal random variable. Then

\[
\lim_{\varepsilon \to 0} (c \varepsilon)^2 \sum_{n \geq 1} P \left( |Z^{(1)}| > c \varepsilon \sqrt{n} \right) = 1.
\]

ii. Let \( Z^{(2)} \) be a Hermite random variable with \( H > 1 - \frac{1}{2q} \). Then

\[
\lim_{\varepsilon \to 0} (c \varepsilon)^{1-q(1-H)} \sum_{n \geq 1} P \left( |Z^{(2)}| > c \varepsilon n^{1-q(1-H)} \right) = E|Z^{(2)}|^{1-q(1-H)}.
\]
Concerning part ii. we can write their use are not convergent.

Note that the bounds (16), (18) does not help here because the series that appear after Remark 4.

The Hermite random variable has moments of all orders (in particular the moment of Remark 3.

Proof: The part i. is a consequence of the result of Heyde [5]. Indeed take $X_i \sim N(0, 1)$ in (1).

Concerning part ii. we can write

$$
\lim_{\epsilon \to 0} (ce)^{\frac{1}{1 - \epsilon n(1-H)}} \sum_{n \geq 1} \Phi_{Z_2}(c \epsilon n^{1-q(1-H)})
$$

$$
= \lim_{\epsilon \to 0} (ce)^{\frac{1}{1 - \epsilon n(1-H)}} \left[ \int_1^{\infty} \Phi_{Z_2}(c \epsilon x^{1-q(1-H)}) dx - \int_1^{\infty} P_1(x) d \left[ \Phi_{Z_2}(c \epsilon x^{1-q(1-H)}) \right] \right]
$$

$$
:= \lim_{\epsilon \to 0} (A(\epsilon) + B(\epsilon))
$$

with $P_i(x) = \lfloor x \rfloor - x + \frac{1}{2}$. Moreover

$$
A(\epsilon) = (ce)^{\frac{1}{1 - \epsilon n(1-H)}} \int_1^{\infty} \Phi_{Z_2}(c \epsilon x^{1-q(1-H)}) dx
$$

$$
= \frac{1}{1 - q(1-H)} \int_1^{\infty} \Phi_{Z_2}(y) y^{\frac{1}{1 - q(1-H)} - 1} dy.
$$

Since $\Phi_{Z_2}(y) \leq y^{-2}$ we have $\Phi_{Z_2}(y) y^{\frac{1}{1 - q(1-H)}} \to_{y \to \infty} 0$ and therefore

$$
A(\epsilon) = - \Phi_{Z_2}(c \epsilon)(ce)^{\frac{1}{1 - \epsilon n(1-H)}} - \int_{ce}^{\infty} \Phi_{Z_2}(y) y^{\frac{1}{1 - q(1-H)}} dy
$$

where the first terms goes to zero and the second to $E \left| Z^{(2)} \right|^{\frac{1}{1 - q(1-H)}}$. The proof that the term $B(\epsilon)$ converges to zero is similar to the proof of Lemma 2 point ii. \[\blacksquare\]

Remark 3. The Hermite random variable has moments of all orders (in particular the moment of order $\frac{1}{1 - q(1-H)}$ exists) since it is the value at time 1 of a selfsimilar process with stationary increments.

Proposition 2. i. Let $H < 1 - \frac{1}{2q}$ and let $Z^{(1)}_n$ be given by (13). Let also $Z^{(1)}$ be a standard normal random variable. Then

$$
(c \epsilon)^2 \sum_{n \geq 1} \left[ P \left( \left| Z^{(1)}_n \right| > c \epsilon \sqrt{n} \right) - P \left( \left| Z^{(1)}_n \right| > c \epsilon \sqrt{n} \right) \right] \to_{\epsilon \to 0} 0
$$

ii. Let $H > 1 - \frac{1}{2q}$ and let $Z^{(2)}_n$ be given by (13). Let $Z^{(2)}$ be a Hermite random variable. Then

$$
(c \epsilon)^{\frac{1}{1 - q(1-H)}} \sum_{n \geq 1} \left[ P \left( \left| Z^{(2)}_n \right| > c \epsilon n^{1-q(1-H)} \right) - P \left( \left| Z^{(2)}_n \right| > c \epsilon n^{1-q(1-H)} \right) \right] \to_{\epsilon \to 0} 0.
$$

Remark 4. Note that the bounds (16), (18) does not help here because the series that appear after their use are not convergent.
Proof of Proposition 2] Case $H < 1 - \frac{1}{2q}$. We have, for some $\beta > 0$ to be chosen later,
\[
e^2 \sum_{n \geq 1} \left[ P \left( |Z_n^{(1)}| > c e \sqrt{n} \right) - P \left( |Z_n^{(1)}| > c e \sqrt{n} \right) \right]
= e^2 \sum_{n \geq 1} \left[ P \left( |Z_n^{(1)}| > c e \sqrt{n} \right) - P \left( |Z_n^{(1)}| > c e \sqrt{n} \right) \right]
+ e^2 \sum_{n > \beta} \left[ P \left( |Z_n^{(1)}| > c e \sqrt{n} \right) - P \left( |Z_n^{(1)}| > c e \sqrt{n} \right) \right]
:= I_1(\varepsilon) + J_1(\varepsilon).
\]
Consider first the situation when $\beta < 0$. Let us choose a real number $\beta$ such that $2 < \beta < 4$. By using (16),
\[
I_1(\varepsilon) \leq c e^2 \sum_{n \geq 1} n^{-\frac{1}{2}} \leq c e^2 e^{-\frac{\varepsilon}{2}} \to_{\varepsilon \to 0} 0
\]
since $\beta < 4$. Next, by using the bound for the tail probabilities of multiple integrals and since $E |Z_n^{(1)}|^2$ converges to 1 as $n \to \infty$
\[
J_1(\varepsilon) = e^2 \sum_{n > \beta} P \left( |Z_n^{(1)}| > c e \sqrt{n} \right) \leq c e^{-2} \sum_{n > \beta} \exp \left( -\frac{-c e \sqrt{n}}{E |Z_n^{(1)}|^2} \right)^{\frac{2}{q}}
\leq e^2 \sum_{n > \beta} \exp \left( \left( -c n^{-\frac{1}{2}} \sqrt{n} \right)^{\frac{2}{q}} \right)
\]
and since converges to zero for $\beta > 2$. The same argument shows that $e^2 \sum_{n > \beta} P \left( Z_n^{(1)} > c e \sqrt{n} \right)$ converges to zero.

The case when $H \in \left( \frac{1}{2}, \frac{2q - 2}{2q - 2} \right)$ can be obtained by taking $2 < \beta < \frac{2}{H}$ (it is possible since $H < 1$) while in the case $H \in \left( \frac{2q - 3}{2q - 2}, 1 - \frac{1}{2q} \right)$ we have to choose $2 < \beta < \frac{2}{qH - q + 1}$ (which is possible because $H < 1 - \frac{1}{2q}$).

Case $H > 1 - \frac{1}{2q}$. We have, with some suitable $\beta > 0$
\[
e^{\frac{1}{1-q(1-H)}} \sum_{n \geq 1} \left[ P \left( |Z_n^{(2)}| > c e n^{1-q(1-H)} \right) - P \left( |Z_n^{(2)}| > c e n^{1-q(1-H)} \right) \right]
= e^{\frac{1}{1-q(1-H)}} \sum_{n \geq 1} \left[ P \left( |Z_n^{(2)}| > c e n^{1-q(1-H)} \right) - P \left( |Z_n^{(2)}| > c e n^{1-q(1-H)} \right) \right]
+ e^{\frac{1}{1-q(1-H)}} \sum_{n \geq 1} \left[ P \left( |Z_n^{(2)}| > c e n^{1-q(1-H)} \right) - P \left( |Z_n^{(2)}| > c e n^{1-q(1-H)} \right) \right]
:= I_2(\varepsilon) + J_2(\varepsilon).
\]
Choose $\frac{1}{1-q(1-H)} < \beta < \frac{1}{(1-q(1-H))(2-H - \frac{1}{2q})}$ (again, this is always possible when $H > 1 - \frac{1}{2q}$). Then
\[
I_2(\varepsilon) \leq c e^{\frac{1}{1-q(1-H)}} e^{(-\beta)(2-H - \frac{1}{2q})} \to_{\varepsilon \to 0} 0
\]
and by (9)

\[ J_2(\epsilon) \leq c \sum_{n>\lceil \epsilon^{-p} \rceil} \exp \left( \frac{-c n^{1-q(1-H)}}{E[Z_n^{(2)}]^2} \right)^{\frac{3}{2}} \leq c \sum_{n>\lceil \epsilon^{-p} \rceil} \exp \left( \frac{cn^{-\frac{1}{2}} n^{1-q(1-H)}}{3} \right)^{\frac{3}{2}} \to_{\epsilon \to 0} 0. \]

We state the main result of this section which is a consequence of Lemma 2 and Proposition 2.

**Theorem 3.** Let \( q \geq 2 \) and let \( c_{1,q,H}, c_{2,q,H} \) be the constants from Theorem 1. Let \( Z^{(1)} \) be a standard normal random variable, \( Z^{(2)} \) a Hermite random variable of order \( q \geq 2 \) and let \( g_1, g_2 \) be given by (19) and (20). Then

i. If \( 0 < H < 1 - \frac{1}{2q} \), we have \( (c_{1,q,H}^{-1} \epsilon^2) g_1(\epsilon) \to_{\epsilon \to 0} 1 = E Z^{(1)} \).

ii. If \( 1 - \frac{1}{2q} < H < 1 \) we have \( (c_{2,q,H}^{-1} \epsilon^2)^{1-\frac{2-2H}{2}} g_2(\epsilon) \to_{\epsilon \to 0} E |Z^{(2)}|^{1-\frac{2}{H}} \).

**Remark 5.** In the case \( H = \frac{1}{2} \) we retrieve the result (11) of [5]. The case \( q = 1 \) is trivial, because in this case, since \( V_n = B_n \) and \( E V_n^2 = n^{2H} \), we obtain the following (by applying Lemma 1 and 2 with \( q = 1 \))

\[ \frac{1}{\log \epsilon} \sum_{n \geq 1} \frac{1}{n} P \left( |V_n| > \epsilon n^{2H} \right) \to_{\epsilon \to 0} \frac{1}{H}. \]

and

\[ \epsilon^2 \sum_{n \geq 1} P \left( |V_n| > \epsilon n^{2H} \right) \to_{\epsilon \to 0} E \left| Z^{(1)} \right|^{\frac{1}{2}}. \]

**Remark 6.** Let \( (\epsilon_i)_{i \in \mathbb{Z}} \) be a sequence of i.i.d. centered random variable with finite variance and let \( (a_i)_{i \geq 1} \) a square summable real sequence. Define \( X_n = \sum_{i \geq 1} a_i \epsilon_{n-i} \). Then the sequence \( S_N = \sum_{n=1}^{N} [K(X_n) - E K(X_n)] \) satisfies a central limit theorem or a non-central limit theorem according to the properties of the measurable function \( K \) (see [6] or [14]). We think that our tools can be applied to investigate the tail probabilities of the sequence \( S_N \) in the spirit of [5] or [12] at least in particular cases (for example, when \( \epsilon_i \) represents the increment \( W_{i+1} - W_i \) of a Wiener process because in this case \( \epsilon_i \) can be written as a multiple integral of order one and \( X_n \) can be decomposed into a sum of multiple integrals. We thank the referee for mentioning the references [6] and [14].

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