The local description of the Ricci and Bianchi identities for an h-normal N-linear connection on the dual 1-jet space $J^1^*(\mathcal{T}, M)$

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Abstract

In this paper we describe the local Ricci and Bianchi identities for an h-normal N-linear connection $\mathcal{D}\Gamma(N)$ on the dual 1-jet space $J^1^*(\mathcal{T}, M)$. To reach this aim, we firstly give the expressions of the local distinguished (d-) adapted components of torsion and curvature tensors produced by $\mathcal{D}\Gamma(N)$, and then we analyze their attached local Ricci identities. The derived deflection d-tensor identities are also presented. Finally, we expose the local expressions of the Bianchi identities (in the particular case of an h-normal N-linear connection of Cartan type), which geometrically connect the local torsion and curvature d-tensors of the linear connection $\mathcal{D}\Gamma(N)$.

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1 Introduction

According to Olver’s opinion [21], we consider that the 1-jet spaces and their duals are natural houses for the study of classical and quantum field theories. For such a reason, the differential geometry of 1-jet spaces was intensively studied, in a contravariant approach, by a lot of authors: Saunders [22], Asanov [1], Neagu and Udriște (see [18], [19], [20]), and many others.

In the last decades, numerous physicists and geometers were preoccupied by the development of that so-called the covariant Hamiltonian geometry of physical fields, which is a multi-parameter, or multi-time, extension of the classical Hamiltonian formulation from Mechanics. In such a perspective, we point out that the covariant Hamiltonian geometry of physical fields appears in the literature of specialty in three distinct variants: (1) the multisymplectic geometry — developed by Gotay, Isenberg, Marsden, Montgomery and their co-workers (see [9], [8]) on a finite-dimensional multisymplectic phase space; (2) the polysymplectic geometry — elaborated by Giachetta, Mangiarotti and Sardanashvily (see [6], [7]), which emphasizes the relations between the equations of first order Lagrangian field theory on fiber bundles and the covariant Hamilton equations...
on a finite-dimensional polysymplectic phase space; (3) the De Donder-Weyl Hamiltonian geometry — studied by Kanatchikov (see [10], [11], [12]) as opposed to the conventional field-theoretical Hamiltonian formalism, which requires the space + time decomposition and leads to the picture of a field as a mechanical system with infinitely degrees of freedom.

From a geometrical point of view, following the ideas initially stated by Asanov [1], a multi-time Lagrange contravariant geometry on 1-jet spaces (in the sense of d-linear connections, d-torsions and d-curvatures) was recently developed by Neagu and Udrište in [18], [19], and [20]. This 1-jet geometrical theory is a natural multi-time extension of the classical Lagrangian geometry on tangent bundles, initiated and developed by Miron and Anastasiei [14].

On the other hand, suggested by the field theoretical extension of the basic structures of classical Analytical Mechanics within the framework of the De Donder-Weyl covariant Hamiltonian formulation, the geometrical studies of Miron [15], Atanasiu [3], [2] and others led to the development of the Hamilton geometry on cotangent bundles, which is synthesized in the book [15]. Note that the Miron-Atanasiu Hamiltonian geometrical ideas from cotangent bundles represent the point start for the development of the jet covariant Riemann-Hamilton geometry depending on polymomenta, which is presented in the Atanasiu-Neagu papers [4] and [5]. In this paper we are going on the jet multi-time Hamiltonian geometrical studies from [4] and [5].

2 Components of $N$-linear connections on dual 1-jet bundle $J^{1*}(\mathcal{T}, M)$

Let $\mathcal{T}$ and $M$ be a temporal (resp. spatial) real, smooth manifold of dimension $m$ (resp. $n$), whose coordinates are $(t^a)_{a=1}^{m}$, respectively $(x^i)_{i=1}^{n}$. Note that, throughout this paper, the indices $a, b, c, \ldots$ run from 1 to $m$, while the indices $i, j, k, \ldots$ run from 1 to $n$. The Einstein convention of summation is also adopted all over this work.

Let $J^{1*}(\mathcal{T}, M)$ be the dual 1-jet fibre bundle, whose coordinates $(t^a, x^i, p^a_i)$ are induced from $\mathcal{T}$ and $M$. The coordinate transformations from the product manifold $\mathcal{T} \times M$ produce on $J^{1*}(\mathcal{T}, M)$ the following coordinate transformations:

$$
\tilde{t}^a = \tilde{t}^a (t^b), \quad \tilde{x}^i = \tilde{x}^i (x^j), \quad \tilde{p}^a_i = \frac{\partial x^j}{\partial \tilde{x}^i} \frac{\partial \tilde{t}^a}{\partial t^b} p^b_j,
$$

where $\det (\partial \tilde{t}^a / \partial t^b) \neq 0$ and $\det (\partial \tilde{x}^i / \partial x^j) \neq 0$.

**Definition 2.1** A pair of local functions on $E^* = J^{1*}(\mathcal{T}, M)$, denoted by

$$
N = \left( N_{(a)_{i}}, N_{(a)_{j}} \right),
$$

whose local components obey the transformation rules

$$
\tilde{N}_{(b)_{c}} \frac{\delta \tilde{t}^c}{\delta \tilde{x}^e} = N_{(c)_{i}} \frac{\delta t^b}{\delta x^k} \frac{\partial x^k}{\partial x^j} - \frac{\partial \tilde{p}^b_i}{\partial t^c}.
$$
\[ N_{(2)}^{(b)} \frac{\partial x^k}{\partial x^i} = N_{(1)}^{(c)} \frac{\partial \tilde{b}^b}{\partial \tilde{t}^c} \frac{\partial x^k}{\partial \tilde{x}^j} \frac{\partial \tilde{p}_j^b}{\partial \tilde{p}_i^c}, \]

is called a **nonlinear connection** on \( E^* \). The components \( N_{(1)}^{(a)} \) (resp. \( N_{(2)}^{(a)} \)) are called the **temporal** (resp. **spatial** components) of \( N \).

**Example 2.2** Let \( h_{ab}(t^f) \) (resp. \( \varphi_{ij}(x^b) \)) be a semi-Riemannian metric on the temporal manifold \( T \) (resp. spatial manifold \( M \)). Taking into account the local transformation rules of the Christoffel symbols \( \chi^{a}_{bc}(t) \) (resp. \( \Gamma^{a}_{ij}(x) \)) of the metrics \( h_{ab}(t) \) (resp. \( \varphi_{ij}(x) \)), then the pair of local functions

\[ N_0 = \left( \frac{0}{N_{(1)}^{(a)}}, \frac{0}{N_{(2)}^{(a)}} \right), \]

where

\[ \frac{0}{N_{(1)}^{(a)}} = \chi^{a}_{bc} p^c, \quad \frac{0}{N_{(2)}^{(a)}} = -\Gamma^{a}_{ij} p^i, \]

represents a nonlinear connection on \( E^* \). This is called the **canonical nonlinear connection attached to the metrics** \( h_{ab}(t) \) and \( \varphi_{ij}(x) \).

In what follows, we fix a nonlinear connection on \( E^* \), and we consider the **adapted bases** of the nonlinear connection \( N \), defined by

\[ \left\{ \frac{\delta}{\delta t^a}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_j^b} \right\} \subset \mathcal{X}(E^*), \quad \left\{ dt^a, dx^i, dp_j^b \right\} \subset \mathcal{X}^*(E^*), \tag{2.1} \]

where

\[ \frac{\delta}{\delta t^a} = \frac{\partial}{\partial t^a} - N_{(1)}^{(a)} \frac{\partial}{\partial p_j^b}, \]
\[ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_{(2)}^{(a)} \frac{\partial}{\partial p_j^b} \]
\[ \delta p_j^b = dp_j^b + N_{(1)}^{(a)} dt^a + N_{(2)}^{(a)} dx^i. \]

It is important to note that the transformation rules of the elements of the adapted bases \( 2.1 \) are tensorial ones:

\[ \frac{\delta}{\delta t^a} = \frac{\partial \tilde{t}^a}{\partial t^a} \frac{\delta}{\partial \tilde{t}^a}, \quad \frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^i}{\partial x^i} \frac{\delta}{\partial \tilde{x}^i}, \quad \frac{\partial}{\partial p_j^b} = \frac{\partial \tilde{p}_j^b}{\partial p_j^b} \frac{\delta}{\partial \tilde{p}_j^b}. \]

**Remark 2.3** The simple tensorial transformation rules \( 2.2 \) of the adapted bases \( 2.1 \) determined us to describe in what follows all geometrical objects on the dual 1-jet space \( J^1(T, M) \) in adapted local components.

In order to develop the geometrical theory of \( N \)-linear connections on the dual 1-jet space \( E^* \), we need the following result:
Proposition 2.4  (i) The Lie algebra $\mathcal{X}(E^*)$ of vector fields decomposes as
\[ \mathcal{X}(E^*) = \mathcal{X}(\mathcal{H}_T) \oplus \mathcal{X}(\mathcal{H}_M) \oplus \mathcal{X}(\mathcal{V}) , \]
where
\[ \mathcal{X}(\mathcal{H}_T) = \text{Span} \left\{ \frac{\delta}{\delta t^a} \right\}, \quad \mathcal{X}(\mathcal{H}_M) = \text{Span} \left\{ \frac{\delta}{\delta x^i} \right\}, \quad \mathcal{X}(\mathcal{V}) = \text{Span} \left\{ \frac{\partial}{\partial p_i^a} \right\} . \]
(ii) The Lie algebra $\mathcal{X}^*(E^*)$ of covector fields decomposes as
\[ \mathcal{X}^*(E^*) = \mathcal{X}^*(\mathcal{H}_T) \oplus \mathcal{X}^*(\mathcal{H}_M) \oplus \mathcal{X}^*(\mathcal{V}) , \]
where
\[ \mathcal{X}^*(\mathcal{H}_T) = \text{Span} \{ dt^a \}, \quad \mathcal{X}^*(\mathcal{H}_M) = \text{Span} \{ dx^i \}, \quad \mathcal{X}^*(\mathcal{V}) = \text{Span} \{ \delta p_i^a \} . \]

Let us consider that $h_T, h_M$ (horizontal) and $v$ (vertical) are the canonical projections of the above decompositions. In this context, we introduce the following geometrical concept:

**Definition 2.5** A linear connection $D : \mathcal{X}(E^*) \times \mathcal{X}(E^*) \rightarrow \mathcal{X}(E^*)$ is called an \textbf{$N$-linear connection} on $E^*$ if and only if $D h_T = 0, D h_M = 0$ and $D v = 0$.

It is obvious that the local description of the $N$-linear connection $D$ on $E^*$ is accomplished by \textit{nine} unique adapted components
\[ D \Gamma (N) = \left( A^a_{bc}, A^i_{jc}, - A^{(a)(j)}_{(i)(b)c}, H^a_{bk}, H^i_{jk}, - H^{(a)(j)}_{(i)(b)k}, C^{(k)}_{b(c)}, C^{(k)}_{j(c)}, - C^{(a)(j)(k)}_{(i)(b)(c)} \right) , \tag{2.3} \]
which are locally defined by the relations:
\[
\begin{align*}
D \frac{\delta}{\delta t^c} &= A^a_{bc} \frac{\delta}{\delta t^a}, \quad D \frac{\delta}{\delta x^i} = A^i_{jc} \frac{\delta}{\delta x^j}, \quad D \frac{\delta}{\delta x^i} = - A^{(a)(j)}_{(i)(b)c} \frac{\delta}{\delta p_i^c}, \\
D \frac{\delta}{\delta x^k} &= H^a_{bk} \frac{\delta}{\delta t^a}, \quad D \frac{\delta}{\delta x^i} = H^i_{jk} \frac{\delta}{\delta x^j}, \quad D \frac{\delta}{\delta x^i} = - H^{(a)(j)}_{(i)(b)k} \frac{\delta}{\delta p_i^c}, \\
D \frac{\partial}{\partial p_i^k} &= C^{(k)}_{b(c)} \frac{\delta}{\delta t^a}, \quad D \frac{\partial}{\partial p_i^k} = C^{(k)}_{j(c)} \frac{\delta}{\delta x^j}, \quad D \frac{\partial}{\partial p_i^k} = - C^{(a)(j)(k)}_{(i)(b)(c)} \frac{\partial}{\partial p_i^c} .
\end{align*}
\]

**Example 2.6** Let $N_0 = \begin{pmatrix} 0 & N^a_{(i)b} \ 0 & N^a_{(i)j} \end{pmatrix}$ be the canonical nonlinear connection produced by the semi-Riemannian metrics $(h_{ab}, \varphi_{ij})$. Taking into account the transformation rules of the Christoffel symbols $\chi^a_{bc}$ and $\Gamma^i_{jk}$, by local computations, we can show that the local components
\[ B \Gamma (N_0) = \left( \chi^a_{bc}, 0, - A^{(a)(j)}_{(i)(b)c}, 0, \Gamma^i_{jk}, - H^{(a)(j)}_{(i)(b)k}, 0, 0, 0 \right) . \]
where

\[ A^{(a)(i)}_{(i)(b)c} = -\delta^{i}_c \delta^{a}_b c, \quad H^{(a)(i)}_{(i)(b)k} = \delta^{a}_i \delta^{c}_k, \]

verify the transformation rules of the components of an \( N \)-linear connection (for more details, see [5]). Consequently, \( B \Gamma (N_0) \) is an \( N_0 \)-linear connection on \( E^* \), which is called the Bernwald connection of the metric pair \((h_{ab}, \varphi_{ij})\).

Now, let \( D \Gamma (N) \) be an \( N \)-linear connection on \( E^* \), locally defined by (2.5). The linear connection \( D \Gamma (N) \) induces a linear connection on the set of \( d \)-tensors on the dual 1-jet fibre bundle \( E^* = J^1^* (\mathcal{T}, M) \), in a natural way. Thus, starting with a \( d \)-vector field \( X \) and a \( d \)-tensor field \( T \), locally expressed by

\[
X = X^a \frac{\partial}{\partial t^a} + X^i \frac{\partial}{\partial x^i} + X^{(a)} \frac{\partial}{\partial p^a},
\]

\[
T = T_{cj(b)(l)...}^{ai(k)(d)...} \frac{\partial}{\partial t^a} \otimes \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial p^c} \otimes dt \otimes dx_l \otimes \delta p_k \otimes ...,\]

we can define the covariant derivative

\[
DXT = X^a \frac{\partial}{\partial t^a} + X^i \frac{\partial}{\partial x^i} + X^{(a)} \frac{\partial}{\partial p^a},
\]

\[
T \frac{\partial}{\partial p^c} \otimes dt \otimes dx_l \otimes \delta p_k \otimes ...,\]

where

- the \( \mathcal{T} \)-horizontal covariant derivative of \( D \Gamma (N) \):

\[
(h_T) \left\{ \begin{array}{l}
T_{cj(b)(l)...}^{ai(k)(d)...} = \frac{\delta T_{cj(b)(l)...}^{ai(k)(d)...}}{\partial t^a} + T_{cj(f)(l)...}^{ai(k)(d)...} A^a_{fg} + \\
+ T_{cj(f)(l)...}^{ai(k)(d)...} A_{fg} + T_{cj(f)(l)...}^{ai(k)(d)...} A^{(r)}_{fg} + ... \\
- T_{cj(b)(l)...}^{ai(k)(d)...} A_{fg} - T_{cj(b)(l)...}^{ai(k)(d)...} A^r_{fg} - T_{cj(b)(r)...}^{ai(k)(d)...} A^{(r)}_{fg} - ..., \\
\end{array} \right.
\]

- the \( M \)-horizontal covariant derivative of \( D \Gamma (N) \):

\[
(h_M) \left\{ \begin{array}{l}
T_{cj(b)(l)...}^{ai(k)(d)...} = \frac{\delta T_{cj(b)(l)...}^{ai(k)(d)...}}{\partial x^a} + T_{cj(b)(l)...}^{ai(k)(d)...} H^{a}_{fs} + \\
+ T_{cj(f)(l)...}^{ai(k)(d)...} H^{a}_{fs} + T_{cj(f)(l)...}^{ai(k)(d)...} H^{r}_{fs} + ... \\
- T_{cj(b)(l)...}^{ai(k)(d)...} H^{a}_{fs} - T_{cj(f)(l)...}^{ai(k)(d)...} H^{a}_{fs} - T_{cj(b)(r)...}^{ai(k)(d)...} H^{r}_{fs} - ..., \\
\end{array} \right.
\]
• the vertical covariant derivative of $D\Gamma(N)$:

\[
\begin{align*}
T^{a}(k)(d) & = \frac{\partial T^{a}(k)(d)}{\partial p_{s}} + T^{a}(k)(d) F_{(s)T} + \\
+ T^{a}(k)(d) & - T^{a}(r)(d) - C^{(d)}(k)(s) + ... \\
- T^{a}(k)(d) & - T^{f}(b)(l) Y_{(i)l} + ... \\
\end{align*}
\]

Remark 2.7 If $T = Y$ is a d-vector field on $E^{*}$, locally expressed by

\[
Y = Y^{a} \frac{\delta}{\delta t^{a}} + Y^{i} \frac{\delta}{\delta x^{i}} + Y^{(a)} \frac{\partial}{\partial p_{i}^{a}},
\]

then the following expressions of the local covariant derivatives hold good:

\[
\begin{align*}
Y_{j,c}^{a} & = \frac{\delta Y^{a}}{\delta t^{c}} + Y^{b} A_{bc}^{a}, \\
Y_{j,c}^{i} & = \frac{\delta Y^{i}}{\delta t^{c}} + Y^{j} A_{jc}^{i}, \\
Y^{(a)} & = \frac{\delta Y^{(a)}}{\delta (t)_{c}} + Y^{(b)} A_{(i)j}^{(a)(b)}, \\
Y_{(i)l}^{a} & = \frac{\delta Y_{(i)l}^{a}}{\delta (t)_{c}} - Y_{(j)k}^{(a)(b)} H_{(i)l}^{(a)(j)(b)}, \\
Y_{(i)l}^{i} & = \frac{\delta Y_{(i)l}^{i}}{\delta (t)_{c}} - Y_{(j)k}^{(a)(b)} H_{(i)l}^{(a)(j)(b)}, \\
Y^{(a)(b)} & = \frac{\delta Y^{(a)(b)}}{\delta (t)_{c}} - Y_{(j)k}^{(a)(b)} H_{(i)l}^{(a)(j)(b)}, \\
\end{align*}
\]

3 Components of $h$-normal $N$-linear connections on dual 1-jet spaces

Because the number of components which characterize an $N$-linear connection on $E^{*}$ is big one (nine local components), we are constrained to study only a particular class of $N$-linear connections on $E^{*}$, which must be characterized by a reduced number of components. In this direction, let us fix on the temporal manifold $T$ a semi-Riemannian metric $h_{ab}$, together with its Christoffel symbols $\chi^{a}_{bc}$. Let $J$ be the $h$-normalization d-tensor field on $E^{*}$, locally expressed by

\[
J = J^{(i)}_{(a)b} k_{i}^{a} \otimes dt^{b} \otimes dx^{i},
\]

where $J^{(i)}_{(a)b} = h_{ab} \delta^{i}_{j}$. In this context, we introduce the following geometrical concept:
Definition 3.1  An \( N \)-linear connection \( D\Gamma(N) \) on \( E^* \), whose local components verify the relations

\[
A^a_{bc} = \chi^a_{bc}, \quad H^a_{bi} = 0, \quad C^{a(i)}_{b(c)} = 0, \quad D\mathcal{J} = 0,
\]
is called an \textbf{h-normal} \( N \)-linear connection on the dual 1-jet fibre bundle \( E^* \).

Theorem 3.2  The adapted components of an \( h \)-normal \( N \)-linear connection \( D\Gamma(N) \) verify the following identities:

\[
\begin{align*}
A^a_{bc} &= \chi^a_{bc}, \\
H^a_{bi} &= 0, \\
C^{a(i)}_{b(c)} &= 0, \\
A^a_{(j)(a)b(j)c} &= \delta^a_b A^c_{ic} - \delta^c_j \chi^a_{bc}, \\
H^a_{(j)(a)b(j)k} &= \delta^a_b H^c_{ic}, \\
C^{(a)(i)(k)}_{(j)(b)(c)} &= \delta^a_b C^{(i)(k)}_{j(c)}.
\end{align*}
\]

\( (3.1) \)

Proof.  It is obvious that the first three relations come immediately from the definition of an \( h \)-normal \( N \)-linear connection. To prove the other three relations, we emphasize that, taking into account the definition of the local \( \mathcal{T} \)-horizontal ("/g"), \( M \)-horizontal ("|s") and vertical ("|g") covariant derivatives produced by \( D\Gamma(N) \), the condition \( D\mathcal{J} = 0 \) is equivalent to

\[
J^{(i)}_{(a)b(j)/g} = 0, \quad J^{(i)}_{(a)b(j)|s} = 0, \quad J^{(i)}_{(a)b(j)|g} = 0.
\]

Consequently, the condition \( D\mathcal{J} = 0 \) provides the local identities

\[
\begin{align*}
h_{bf} A_{(j)(a)c}^{(f)(i)} &= h_{ab} A_{jc}^i - \delta^i_j \left( \frac{\partial h_{ab}}{\partial c} - h_{ag} \chi^g_{bc} \right), \\
h_{bf} H_{(j)(a)k}^{(f)(i)} &= h_{ba} H_{jk}^i, \quad h_{bf} C_{(j)(a)c}^{(f)(i)(k)} &= h_{ba} C_{j(c)}^{i(k)}.
\end{align*}
\]

Contracting now the above relations by \( h^{bc} \), we obtain the last required identities from \( (3.1) \).  \( \blacksquare \)

Remark 3.3  The above theorem says us that an \( h \)-normal \( N \)-linear connection on \( E^* \) is an \( N \)-linear connection determined by \textbf{four} effective components (instead of nine in the general case):

\[
D\Gamma(N) = \left( \chi^a_{bc}, A^i_{jc}, H^i_{jk}, C^{i(k)}_{j(c)} \right).
\]

The other five components either vanish or are provided by the relations \( (3.1) \). Consequently, we can assert that the Berwald \( N_0 \)-linear connection associated to the pair of metrics (\( h_{ab}, \varphi_{ij} \)) is an \( h \)-normal \( N_0 \)-linear connection on \( E^* \), whose four effective components are

\[
B\Gamma(N_0) = \left( \chi^a_{bc}, 0, \Gamma^{i}_{jk}, 0 \right).
\]
4 Adapted components of torsion and curvature tensors

The study of the adapted components of the torsion and curvature tensors of an arbitrary $N$-linear connection $D\Gamma(N)$ on $E^r$ was done in [5]. In that context, one proves that the torsion tensor $T$ is determined by twelve effective local adapted d-tensors, while the curvature tensor $\mathbb{R}$ is determined by eighteen local adapted d-tensors. In what follows, we study the adapted components of the torsion and curvature tensors for an $h$-normal $N$-linear connection $D\Gamma(N)$.

**Theorem 4.1** The torsion tensor $T$ of an $h$-normal $N$-linear connection $D\Gamma(N)$ is determined by nine effective local adapted d-tensors (instead of twelve in the general case):

| $h_T$ | $h_M$ | $v$ |
|------|------|-----|
| $T_{a_j}^r$ | $-A_{a_j}^r$ | 0 |
| $T_{i}^r$ | $H_{ij}^r - H_{ji}^r$ | $P_{(r)i}^{(j)}$ |
| $P_{(r)a(b)}^{(j)}$ | $C_{(r)i}^{(j)}$ | $S_{(r)(a)(b)}^{(j)(k)}$ |

where

$$
T_{a_j}^r = -A_{a_j}^r, \quad T_{i}^r = H_{ij}^r - H_{ji}^r, \quad P_{(r)i}^{(j)} = C_{(r)i}^{(j)},
$$

$$
P_{(r)a(b)}^{(j)} = \frac{\partial N_{1(r)a}}{\partial p_j^b} + \frac{\delta^f_b}{\delta t^a} A_{ra}^i - \delta^i_r A_{ba}^f, \quad P_{(r)i}^{(j)} = \frac{\partial N_{1(r)i}}{\partial p_j^b} + \frac{\delta^f_b}{\delta x^a} H_{ri}^j,
$$

$$
R_{(r)ab}^{(j)} = \frac{\delta N_{1(r)a}}{\delta t^b} - \frac{\delta N_{1(r)b}}{\delta t^a}, \quad R_{(r)aj}^{(j)} = \frac{\delta N_{(r)a}}{\delta x^j} - \frac{\delta N_{(r)j}}{\delta t^a},
$$

$$
R_{(r)ij}^{(j)} = \frac{\delta N_{(r)i}}{\delta x^j} - \frac{\delta N_{(r)j}}{\delta x^a}, \quad S_{(r)(a)(b)}^{(j)(k)} = -\left(\frac{\delta^f_b}{\delta t^a} C_{(r)b}^{(i)} - \frac{\delta^f_b}{\delta x^a} C_{(r)a}^{(i)}\right).
$$

**Proof.** Particularizing the general local expressions from [5], which generally give those twelve d-components of the torsion tensor of an $N$-linear connection, an $h$-normal $N$-linear connection $D\Gamma(N)$, we deduce that the adapted components $T_{b(c)}^a, T_{b_j}^a$ and $P_{b(c)}^{(k)}$ vanish, while the other nine are given by the formulas from theorem. ■

**Remark 4.2** All torsion d-tensors of the Berwald $h$-normal $N_0$-linear connection $B\Gamma(N_0)$ (associated to the metrics $h_{ab}$ and $\varphi_{ij}$) are zero, except

$$
R_{(r)ab}^{(j)} = \chi^f_{gab} p^p_\tau, \quad R_{(r)ij}^{(j)} = -R^s_{(r)ij} p^{(k)}_s,
$$
where $\gamma^f_{ab}(t)$ (resp. $R^e_{ij}(x)$) are the local curvature tensors of the semi-Riemannian metric $h_{ab}$ (resp. $\varphi_{ij}$).

**Theorem 4.3** The curvature tensor $\mathbb{R}$ of an h-normal $N$-linear connection $D\Gamma(N)$ is characterized by seven effective adapted local d-tensors (instead of eighteen in the general case):

$$
\begin{array}{|c|c|c|c|}
\hline
 & h_T & R_M & \varphi \\
\hline
h_T h_T & \chi_{abc}^d & R_{abc}^d & -R_{(i)(a)b}^d = \delta^d_i \chi_{abc}^d - \delta^d_a R_{ibc}^d \\
\hline
h_M h_T & 0 & R_{ibk}^d & -R_{(i)(a)b}^d = -\delta^d_a R_{ibk}^d \\
\hline
w h_T & 0 & P_{(k)}^d_{ib(c)} & -P_{(i)(a)b}^d = -\delta^d_a P_{ib(c)}^d \\
\hline
h_M h_M & 0 & R_{ijkl}^d & -R_{(i)(a)jk}^d = -\delta^d_a R_{ijkl}^d \\
\hline
w h_M & 0 & P_{(k)}^d_{ijkl} & -P_{(i)(a)jk}^d = -\delta^d_a P_{ijkl}^d \\
\hline
w w & 0 & S_{(i)(j)(k)}^d_{bc} & -S_{(i)(a)bc}^d = -\delta^d_a S_{(i)(j)(k)}^d_{bc} \\
\hline
\end{array}
$$

(4.2)

where

$$
R_{abc}^d := \chi_{abc}^d - \frac{\delta \chi_{ab}^d}{\delta t^c} - \frac{\delta \chi_{ac}^d}{\delta t^b} + \chi_{abc}^f - \chi_{abc}^f J_{ab},
$$

$$
R_{ibc}^d = \frac{\delta A_{ib}^d}{\delta t^b} + A_{ib}^d A_{rc}^d - A_{ic}^d A_{rb}^d + C_{i(f)}^d R_{(f)(r)bc}^d,
$$

$$
R_{ibk}^d = \frac{\delta H_{ib}^d}{\delta x^k} + A_{ib}^d H_{rk}^d - H_{ib}^d A_{rk}^d + C_{i(f)}^d R_{(f)(r)bk}^d,
$$

$$
P_{(k)}_{ib(c)}^d = \frac{\partial A_{ib}^d}{\partial p_k^d} - C_{i(c)/b}^d + C_{i(f)}^d p_{(f) (k)}^d,
$$

$$
R_{ijk}^d = \frac{\delta H_{ij}^d}{\delta x^j} + H_{ij}^d H_{rk}^d - H_{ik}^d H_{rj}^d + C_{i(f)}^d R_{(f)(r)jk}^d,
$$

$$
P_{(k)}_{ij(c)}^d = \frac{\partial H_{ij}^d}{\partial p_k^d} - C_{i(c)ij}^d + C_{i(r)}^d P_{(r) (k)}^d,
$$

$$
S_{(i)(j)(k)}^d_{bc} = \frac{\partial C_{i(b)}^d}{\partial p_k^d} + C_{i(b) c}^d C_{r(c)}^d - C_{i(c)}^d C_{r(b)}^d,
$$

**Proof.** The general formulas that express the local curvature d-tensors of an arbitrary $N$-linear connection (for more details, see [5]), applied to the particular case of an $h$-normal $N$-linear connection $D\Gamma(N)$, imply the above formulas and the relations from the Table (4.2).

**Remark 4.4** In the case of the Berwald $h$-normal $N_0$-linear connection $B\Gamma(N_0)$ (associated to the pair of metrics $(h_{ab}, \varphi_{ij})$), all curvature d-tensors are zero, except

$$
R_{abc}^d = \chi_{abc}^d, \quad R_{(i)(a)bc}^d = -\delta^d_i \chi_{abc}^d, \quad R_{ijkl}^d = R_{ijkl}^d, \quad R_{(i)(a)jk}^d = \delta^d_a R_{ijkl}^d.
$$
where $\chi_{gab}^f(t)$ (resp. $R_{rij}^a(x)$) are the local curvature tensors of the semi-Riemannian metric $h_{ab}$ (resp. $\varphi_{ij}$).

## 5 Local Ricci identities. Non-metrical deflection d-tensor identities

Let us consider now the following more particular geometrical concept:

**Definition 5.1** An $h$-normal $N$-linear connection, whose local components

$$CD\Gamma(N) = \left(\chi_{bc}^a, A_{j;c}^i, H_{jk}^i, C_{j(c)}^{(k)}\right),$$

verify the relations

$$H_{jk}^i = H_{kj}^i, \quad C_{j(c)}^{(k)} = C_{j(c)}^{(k)},$$

is called an $h$-normal $N$-linear connection of Cartan type or a $CD\Gamma(N)$-linear connection on $E^* = J^{1*}(T, M)$.

**Remark 5.2** The torsion tensor $T$ of an $h$-normal $N$-linear connection of Cartan type $CD\Gamma(N)$ is characterized only by eight adapted local d-tensors because the torsion components $T_{jk}^i = H_{jk}^i - H_{kj}^i$ from the Table (77) are vanishing.

**Example 5.3** Taking into account that the Christoffel symbols $\Gamma_{ijk}^a(x)$ of the spatial metric $\varphi_{ij}(x)$ are symmetric, it follows that the Berwald $h$-normal $N_0$-linear connection $B\Gamma(N_0)$ is of Cartan type.

**Theorem 5.4** The following local Ricci identities for a $CD\Gamma(N)$-linear connection are true:

- the $h_T$-Ricci identities:

$$X_{jk/c}^a - X_{j/c|b}^a = X_f^a \chi_{fbc} - X^a_{(r)} R_{(f)(r)bc}^i,$$

$$X_{jk/b}^a - X_{j|b/k}^a = -\chi_{r}^a T_{rk}^i - X_{(r)}^a R_{(f)(r)bk}^i,$$

$$X_{ij|k}^a - X_{i|jk}^a = -X_{(r)}^a R_{(f)(r)jk}^i,$$

$$X_{j|b}^{(k)} - X_{i|b}^{(k)} = -X_{(r)}^{(f)} P_{(f)}^{(k)},$$

$$X_{j|c}^{(k)} - X_{i|c}^{(k)} = -X_{(r)}^{(f)} C_{(f)}^{(r)} - X_{(f)}^{(r)} P_{(f)}^{(k)},$$

$$X_{j|c}^{(k)} - X_{i|c}^{(k)} = -X_{(r)}^{(f)} g_{(f)(j)(k)},$$

$$X_{j|c}^{(k)} - X_{i|c}^{(k)} = -X_{(r)}^{(f)} g_{(f)(j)(k)}.$$
• the $h_M$-Ricci identities:

\[
X^i_{/b/c} - X^i_{/c/b} = X^i_{/r} R^r_{/bc} - X^i_{/r} R^i_{/rf} (f)_{/bc},
\]

\[
X^i_{/b/k} - X^i_{/k/b} = X^i_{/r} R^r_{/bk} - X^i_{/r} R^i_{/rf} (f)_{/bk},
\]

\[
X^i_{/j/k} - X^i_{/k/j} = X^i_{/r} R^r_{/jk} - X^i_{/r} R^i_{/rf} (f)_{/jk},
\]

\[
X^i_{/b/c} - X^i_{/c/b} = X^i_{/r} P^r_{/bc} - X^i_{/r} P^i_{/rf} (f)_{/bc},
\]

\[
X^i_{/j/c} - X^i_{/c/j} = X^i_{/r} P^r_{/jc} - X^i_{/r} P^i_{/rf} (f)_{/jc},
\]

\[
X^i_{/b/c} - X^i_{/c/b} = X^i_{/r} S^i_{/rc} (f)_{/bc} - X^i_{/r} S^i_{/rf} (f)_{/bc}.
\]

• the $v$-Ricci identities:

\[
X^{(a)}_{/i/b/c} - X^{(a)}_{/i/c/b} = X^{(a)}_{/r} R^r_{/bc} - X^{(a)}_{/r} R^{(a)}_{/rf} (f)_{/bc},
\]

\[
X^{(a)}_{/i/b/\kappa} - X^{(a)}_{/i/\kappa/b} = X^{(a)}_{/r} R^r_{/bk} - X^{(a)}_{/r} R^{(a)}_{/rf} (f)_{/bk},
\]

\[
X^{(a)}_{/i/\kappa/\kappa} = X^{(a)}_{/r} R^r_{/jk} - X^{(a)}_{/r} R^{(a)}_{/rf} (f)_{/jk},
\]

\[
X^{(a)}_{/i/\kappa/\kappa} = X^{(a)}_{/r} P^r_{/bc} - X^{(a)}_{/r} P^{(a)}_{/rf} (f)_{/bc},
\]

\[
X^{(a)}_{/i/\kappa/\kappa} = X^{(a)}_{/r} P^r_{/jc} - X^{(a)}_{/r} P^{(a)}_{/rf} (f)_{/jc},
\]

\[
X^{(a)}_{/i/\kappa/\kappa} = X^{(a)}_{/r} S^i_{/rc} (f)_{/bc} - X^{(a)}_{/r} S^i_{/rf} (f)_{/bc},
\]

where

\[
X = X^a \frac{\delta}{\delta x^a} + X^i \frac{\delta}{\delta x^i} + X^{(a)} \frac{\partial}{\partial p_i^{(a)}},
\]

is an arbitrary $d$-vector field on the dual 1-jet space $E^* = J^1*(T, M)$.

**Proof.** Let $(Y_A)$ and $(\omega^A)$, where $A \in \{a, i, (a)\}$, be on $E^* = J^1*(T, M)$ the dual bases adapted to the nonlinear connection $N$, and let $X = X^F Y_F$ be a d-vector field on $E^*$. In this context, using the following true equalities (applied for a $CDG(N)$-linear connection $D$):

1. $D_{Y_a} Y_B = \Gamma^F_{BC} Y_F$,
2. $[Y_B, Y_C] = R^F_{BC} Y_F$,
3. $T(Y_C, Y_B) = T^F_{BC} Y_F = \{\Gamma^F_{BC} - \Gamma^F_{CB} - R^F_{CB}\} Y_F,$

4. $R(Y_C, Y_B)Y_A = R^F_{ABC} Y_F,$

5. $D_Y \omega^B = -\Gamma^B_{FC} \omega^F,$

6. $[R(Y_C, Y_B)X] \otimes \omega^B \otimes \omega^C = \{D_Y D_Y b X - D_Y D_Y b X\} \otimes \omega^B \otimes \omega^C,$

by a direct calculation, we find that

$$X^A_{B;C} - X^A_{C;B} = X^F R^A_{FBC} - X^F \pi^F_{BC}, \quad (5.1)$$

where "$;^G$" represents one from the local covariant derivatives "$;^b$", "$;^j$" or "$(j)^{(b)}$" produced by the h-normal N-linear connection of Cartan type CD $\Gamma(N).$

Taking into account in (5.1) that the indices $A, B, C, \ldots$ belong to the set $\{a, i, (a), (i)\},$

and using the particular features of an h-normal N-linear connection of Cartan type CD $\Gamma(N)$ (i.e., the torsion d-components $T^i_{jk}$ are zero; we have the curvature relations from the Table (4.2)), by complicated computations, we find what we were looking for (see also the Table (4.1)).

In order to find an interesting application of the preceding Ricci identities, let us consider the canonical Liouville-Hamilton d-vector field of polymomenta on $E^* = J^1(T, M),$ which is given by

$$\mathbb{C}^* = p^a_i \frac{\partial}{\partial p^i}.$$ 

In this context, for an h-normal N-linear connection of Cartan type CD $\Gamma(N),$ we can construct the non-metrical deflection d-tensors, setting

$$\Delta^{(a)}_{(i)b} = p^a_i \frac{\partial}{\partial p^b}, \quad \Delta^{(a)}_{(i)j} = p^a_{ij}, \quad \psi^{(a)(j)}_{(i)b} = p^a_{i|b},$$

where "$;^b$, "$;^j$" and "$(j)^{(b)}$" are the local covariant derivatives produced by CD $\Gamma(N).$

By direct local computations, we deduce that the non-metrical deflection d-tensors of CD $\Gamma(N)$ have the expressions:

$$\Delta^{(a)}_{(i)b} = -N^{(a)}_1(i)b - A^a_{jb} p^i + \chi^a_{jb} p^i, \quad \Delta^{(a)}_{(i)j} = -N^{(a)}_2(i)j - H^a_{ij} p^i,$$

$$\psi^{(a)(j)}_{(i)b} = \delta^a_i \delta^j_b - C^{(i)(j)}_{i(b)} p^a_i.$$ 

Applying now the preceding (v)-set of Ricci identities (attached to an h-normal N-linear connection of Cartan type) to the components of the canonical Liouville-Hamilton d-vector field of polymomenta, we get
Corollary 5.5 The following deflection d-tensor identities, associated to an h-normal N-linear connection of Cartan type, are true:

\[
\begin{align*}
\Delta_{(i)b/c}^{(a)} - \Delta_{(i)c/b}^{(a)} &= p_c^b R_{ibc}^{(f)} - p_b^c R_{icb}^{(f)} - \frac{1}{2} \Delta_{(i)(j)(k)}^{(f)} R_{jkb}^{(f)}, \\
\Delta_{(i)b/k}^{(a)} - \Delta_{(i)k/b}^{(a)} &= p_a^c R_{ibk}^{(f)} - \frac{1}{2} \Delta_{(i)jk}^{(a)} R_{i(k)jk}^{(f)}, \\
\Delta_{(i)j}^{(a)} - \Delta_{(i)j}^{(a)} &= p_c^b R_{ijc}^{(f)} - \frac{1}{2} \Delta_{(i)jk}^{(a)} R_{i(k)jk}^{(f)}, \\
\Delta_{(i)(k)}^{(a)} - \Delta_{(i)(k)}^{(a)} &= p_a^c R_{ikb}^{(f)} - \frac{1}{2} \Delta_{(i)j}^{(a)} R_{i(j)jk}^{(f)}, \\
\Delta_{(i)(h)c}^{(a)} - \frac{1}{2} \Delta_{(i)(c)j}^{(a)} &= p_a^c R_{ihc}^{(f)} - \frac{1}{2} \Delta_{(i)(j)c}^{(a)} R_{i(j)jc}^{(f)}, \\
\Delta_{(i)b/c}^{(a)} - \frac{1}{2} \Delta_{(i)c/b}^{(a)} &= p_a^c R_{i(b)c}^{(f)} - \frac{1}{2} \Delta_{(i)j}^{(a)} R_{i(j)c}^{(f)}, \\
\Delta_{(i)(k)}^{(a)} - \Delta_{(i)(k)}^{(a)} &= p_a^c R_{ikb}^{(f)} - \frac{1}{2} \Delta_{(i)j}^{(a)} R_{i(j)c}^{(f)}, \\
\Delta_{(i)(h)c}^{(a)} - \frac{1}{2} \Delta_{(i)(c)j}^{(a)} &= p_a^c R_{ihc}^{(f)} - \frac{1}{2} \Delta_{(i)(j)c}^{(a)} R_{i(j)c}^{(f)}, \\
\Delta_{(i)b/c}^{(a)} - \Delta_{(i)c/b}^{(a)} &= p_a^c R_{i(b)c}^{(f)} - \frac{1}{2} \Delta_{(i)j}^{(a)} R_{i(j)c}^{(f)}.
\end{align*}
\]

(5.2)

Remark 5.6 The deflection d-tensor identities (5.2) will be used in the near future for the construction of the geometrical Maxwell equations that will govern the abstract multi-time geometrical “electromagnetism” produced by a quadratic Hamiltonian depending on polymomenta (this is our work in progress).

6 The local Bianchi identities of the $CD\Gamma(N)$-connections on the dual jet bundle $J^{1*}(T, M)$

From the general theory of linear connections on a vector bundle, one knows that the torsions $T$ and curvature $R$ of a connection $D$ on the dual 1-jet space $E^* = J^{1*}(T, M)$ are not independent. In other words, they are interrelated by the following general Bianchi identities (for any $X, Y, Z, U \in \mathcal{X}(E^*)$):

\[
\sum_{\{X, Y, Z\}} \{(D_X T)(Y, Z) - R(X, Y)Z + T(T(X, Y), Z)\} = 0,
\]

\[
\sum_{\{X, Y, Z\}} (D_X R)(Y, Z, U) + R(T(X, Y), Z) U = 0,
\]

where $\Sigma_{\{X, Y, Z\}}$ means a cyclic sum. Obviously, working with a $CD\Gamma(N)$-linear connection and the local adapted basis of d-vector fields $(X_A) \subset \mathcal{X}(E^*)$ (associated to the given nonlinear connection $N$ on $E^*$), the above Bianchi identities are locally described by the equalities:

\[
\sum_{\{A, B, C\}} \{R^F_{ABC} - T^F_{AB, C} - T^G_{AB} T^F_{CG} \} = 0,
\]

\[
\sum_{\{A, B, C\}} \{R^F_{DABC} + T^G_{AB} R^F_{DCG} \} = 0,
\]

(6.1)

where $R(X_A, X_B)X_C = R^D_{CBA}X_D$, $T(X_A, X_B) = T^P_{BA}X_D$, and “$c$” represents one from the local covariant derivatives “$/a$”, “$i$” or “$|(a)$” of the $CD\Gamma(N)$-linear connection $D$ (for similar details, see the works [14], [15] and [17]). Consequently, we find:
Theorem 6.1  The following thirty effective local Bianchi identities for an \(h\)-normal \(N\)-linear connection of Cartan type \(CD\Gamma(N)\) are true on the dual 1-jet space \(E^* = J^1(T, M)\):

- the first set:

1. \(\sum_{a,b,c} \chi_{abc}^d = 0,\)

2. \(A_{\{a,b\}} \left\{ T_{ar}^d T_{bk} - T_{ak/b}^d \right\} = R_{kab}^d - C_{k(f)}^{d(r)} R_{(r)ab}^f,\)

3. \(A_{\{j,k\}} \left\{ C_{k(f)}^{d(r)} R_{(r)aj}^f + R_{jak}^f + T_{aj/k}^d \right\} = 0,\)

4. \(\sum_{i,j,k} \left\{ C_{k(f)}^{d(r)} R_{(r)ij}^f - R_{ijk}^f \right\} = 0,\)

- the second set:

5. \(\sum_{a,b,c} \left\{ R_{(l)ab/c}^{d(f)} + P_{(l)c(f)}^{d(r)} R_{(r)ab}^f \right\} = 0,\)

6. \(A_{\{a,b\}} \left\{ R_{(l)ak/b}^{d(f)} + P_{(l)b(f)}^{d(r)} R_{(r)ak}^f + R_{(l)br}^d T_{bk}^r \right\} = R_{(l)ab}^{d(f)} + P_{(l)k(f)}^{d(r)} R_{(r)ab}^f,\)

7. \(A_{\{j,k\}} \left\{ R_{(l)aj/k}^{d(f)} + P_{(l)k(f)}^{d(r)} R_{(r)aj}^f + R_{(l)kr}^d T_{jk}^r \right\} = -R_{(l)jk/a}^{d(f)} - P_{(l)a(f)}^{d(r)} R_{(r)jk}^f,\)

8. \(\sum_{i,j,k} \left\{ R_{(l)ij/k}^{d(f)} + P_{(l)j(k)}^{d(r)} R_{(r)ij}^f \right\} = 0,\)

- the third set:

9. \(T_{ak(l)}^{(p)} - C_{r(e)}^{d(p)} T_{ak}^r + P_{ka(e)}^{d(p)} + C_{k(e)/a}^{d(p)} - C_{k(f)}^{d(r)} P_{(r)a(e)}^{d(p)} + C_{k(e)}^{d(r)} T_{ar} = 0,\)

10. \(A_{\{j,k\}} \left\{ C_{j(e)}^{d(p)} P_{(r)j(e)}^{(p)} + P_{jk(e)}^{d(p)} \right\} = 0,\)

- the fourth set:

11. \(A_{\{a,b\}} \left\{ P_{(l)a(e)/b}^{d(p)} + P_{(l)b(f)}^{d(r)} P_{(r)a(e)}^{(p)} \right\} = R_{(l)ab}^{d(p)} + R_{(l)e/ab}^{d(p)} + S_{(l)e(f)}^{d(r)} R_{(r)ab}^f,\)
12. \( A\{a,k\} \left\{ P^{(d)}(p)_{(l)ak(c)} + P^{(d)}(r)_{(l)ak(r)} \right\} = R^{(d)}(p)_{(l)ak(c)} + R^{(d)}(p)_{(l)ak(r)} + S^{(d)}(p)(r)_{(l)ak(r)}R^{(f)}_{(l)ak(r)} + R^{(d)}(r)_{(l)ak(c)}C^{(p)}_{(l)ak(r)} - T^{(r)}_{ak}P^{(d)}(p)_{(l)ak(r)} \)

13. \( A\{j,k\} \left\{ P^{(d)}(p)_{(l)jk(c)} + P^{(d)}(r)_{(l)jk(r)} + R^{(d)}(p)_{(l)jk(c)}C^{(p)}_{(l)jk(r)} \right\} = R^{(d)}(p)_{(l)jk(c)} + R^{(d)}(p)_{(l)jk(r)} + S^{(d)}(p)(r)_{(l)jk(r)}R^{(f)}_{(l)jk(r)} \)

- the fifth set:

14. \( A\{(j)\{k\}\{(b)\{c\}\} \left\{ C^{(l)}_{(j)(k)} + C^{(r)}_{(j)(k)}C^{(l)}_{(j)(c)} \right\} = S^{(l)}_{(j)(k)}S^{(l)}_{(r)(b)(c)} - C^{(l)}_{(j)(k)}C^{(l)}_{(r)(b)(c)} \)

- the sixth set:

15. \( A\{(j)\{k\}\{(b)\{c\}\} \left\{ P^{(d)}(j)(k)_{(j)(b)(c)} + P^{(f)}(j)(j)_{(j)(b)(c)} + P^{(d)}(j)(k)_{(j)(l)ak(c)} \right\} = -S^{(d)}(j)(k)_{(j)(b)(c)} - S^{(f)}(j)(j)_{(j)(b)(c)}P^{(d)}(r)(r)_{(j)(a)(c)} \)

16. \( A\{(j)\{k\}\{(b)\{c\}\} \left\{ P^{(d)}(j)(k)_{(j)(b)(c)} + P^{(f)}(j)(j)_{(j)(b)(c)}S^{(d)}(j)(k)_{(j)(b)(c)} - P^{(d)}(j)(k)_{(j)(l)ak(c)} \right\} = -S^{(d)}(j)(k)_{(j)(b)(c)} - S^{(f)}(j)(j)_{(j)(b)(c)}P^{(d)}(r)(r)_{(j)(l)ak(c)} \)

- the seventh set:

17. \( \sum\{(i)\{j\}\{k\}\{(a)\{b\}\{c\}\} \left\{ S^{(d)}(i)(j)(k)_{(i)(a)(b)(c)} + S^{(f)}(i)(j)_{(i)(a)(b)(c)}S^{(d)}(j)(k)_{(j)(a)(b)(c)} + S^{(d)}(i)(j)(k)_{(i)(a)(b)(c)} \right\} = 0, \)

- the eight set:

18. \( \sum_{(a,b,c)} \chi^{d}_{cab/c} = 0, \)

19. \( \chi^{d}_{cab/k} = 0, \)

20. \( \chi^{d}_{cab/(c)} = 0, \)

21. \( \sum_{(a,b,c)} \left\{ R^{l}_{pab/c} + R^{f}_{(r)ab}P^{l}_{pc(f)} \right\} = 0, \)

22. \( A\{(a)\{b\}\left\{ R^{l}_{pak/b} + R^{f}_{(r)ak}P^{l}_{pb(f)} + T^{r}_{ak}R^{l}_{pk(f)} \right\} = R^{l}_{pab/k} + R^{f}_{(r)ab}P^{l}_{pk(f)}, \)

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23. $A_{\{j,k\}} \left\{ R^{l}_{\alpha l j k} + T^{(f)}_{(r)_{\alpha j} \, p k (f)} + T^{(r)}_{(r)_{\alpha l} \, p k r} \right\} = - R^{l}_{\beta j k} - R^{l}_{(r) j k} \cdot P^{l}_{\alpha (r)}$,

24. $\sum_{\{i,j,k\}} \left\{ R^{l}_{\alpha i j k} + R^{l}_{(r) i j} \cdot P^{l}_{k (r)} \right\} = 0$,

- the ninth set:

25. $A_{\{a,b\}} \left\{ P^{l}_{\alpha (p)} \right\} = R^{l}_{(p) \alpha (c)} + R^{l}_{(r) \alpha b} \cdot S^{l}_{(p) (r) (c) (f)}$,

26. $A_{\{a,b\}} \left\{ P^{l}_{\alpha (p)} \right\} = R^{l}_{(p) \alpha (c)} + R^{l}_{(r) \alpha b} \cdot S^{l}_{(p) (r) (c) (f)} = R^{l}_{(p) \alpha (c)} + R^{l}_{(r) \alpha b} \cdot S^{l}_{(p) (r) (c) (f)}$,

27. $A_{\{j,k\}} \left\{ P^{l}_{\alpha j i k} + R^{l}_{(r) j i} \cdot P^{l}_{k (r)} \right\} = R^{l}_{(p) \alpha (c)} + R^{l}_{(r) \alpha b} \cdot S^{l}_{(p) (r) (c) (f)}$,

- the tenth set:

28. $A_{\{i,j\}} \left\{ P^{l}_{\alpha (p)} \right\} = R^{l}_{(p) \alpha (c)} + R^{l}_{(r) \alpha b} \cdot S^{l}_{(p) (r) (c) (f)}$

29. $A_{\{i,j\}} \left\{ P^{l}_{\alpha (p)} \right\} = R^{l}_{(p) \alpha (c)} + R^{l}_{(r) \alpha b} \cdot S^{l}_{(p) (r) (c) (f)} = R^{l}_{(p) \alpha (c)} + R^{l}_{(r) \alpha b} \cdot S^{l}_{(p) (r) (c) (f)}$

- the eleventh set:

30. $\sum_{\{i,j,k\}} \left\{ S^{l}_{\alpha (p) (b) (c)} + S^{l}_{(f) \alpha (b) (c) \cdot S^{l}_{(p) (r)} (r)} \right\} = 0$,

where, if $\{A, B, C\}$ are indices of type $\{a, i, (a)\}$, then $\sum_{\{A,B,C\}}$ represents a cyclic sum, and $A_{\{A,B\}}$ represents an alternate sum.

**Proof.** Taking into account that the indices $A, B, C, D...$ are of type $\{a, i, (a)\}$, and the torsion $T^{C}_{A B}$ and curvature $R^{D}_{A B C}$ adapted components are given in the Tables $[1.1]$ and $[1.2]$, after laborious local computations, the formulas $[6.1]$ imply the required Bianchi identities. 

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Remark 6.2 We point out that, in the particular single-time case

\((T, h) = (\mathbb{R}, \delta = 1)\),

the last identity of our each set of local Bianchi identities reduces to one of the classical eleven Bianchi identities that characterize the \(N\)-linear connections in the classical Hamilton geometry on cotangent bundles (see [15]).

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