QUANTITATIVE UNIQUENESS FOR ELLIPTIC EQUATIONS WITH SINGULAR LOWER ORDER TERMS

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Abstract. We use a Carleman type inequality of Koch and Tataru to obtain quantitative estimates of unique continuation for solutions of second order elliptic equations with singular lower order terms. First we prove a three sphere inequality and then describe two methods of propagation of smallness from sets of positive measure.

1. Introduction

In this work we deal with second-order uniformly elliptic equations in a bounded domain \( \Omega \subset \mathbb{R}^n \), \( n \geq 3 \). We assume that the equation is in divergence form and terms of order one and zero may have singularities. The conditions we impose on the lower order terms imply the strong unique continuation property, we refer the reader to the article of H. Koch and D. Tataru, [10], and references therein for the history of the Carleman inequalities and strong unique continuation for second-order elliptic equations.

1.1. Problem of quantitative propagation of smallness. Assume that a solution to a second-order uniformly elliptic equation is bounded on the domain and is small on a subset of positive measure, our aim is to estimate such a solution on an arbitrary compact subset of the domain. We refer to estimates of this nature as quantitative propagation of smallness. Three sphere inequalities for elliptic equations provide classical examples of quantitative propagation of smallness; various versions of the inequality can be found, for example, in [1–3, 6, 8, 11]. Our first result is a version of the three sphere inequality for equations with singular coefficients in lower order terms, it is derived from the inequality of H. Koch and D. Tataru.

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We refer the reader to articles of N.Nadirashvili and S.Vessella, [14] and [19], in which the problem of propagation of smallness for second-order elliptic equations from a set $E$ of positive measure was considered. We mention also that similar problems for the case of elliptic equations with analytic coefficient were discussed in [12, 13, 18], methods used in these works are of complex analytic nature.

In the second part of the article we give two approaches to propagation of smallness for the case of singular coefficients, both of them use the three sphere inequality obtained in the first part of the work. The first is an improvement of the one in [19]. It uses Carleman type inequality and gives estimates of $L^2$-norms, all constants can be estimated explicitly. The second approach repeats a clever argument of N.Nadirashvili, [14], we assume a slightly better integrability of the lower order terms than for the first approach, and estimate $L^\infty$-norms, the constants are not explicit here but the asymptotic of the decay of solution is better. The precise formulations of the results are given in the next section.

1.2. Formulation of the result. We consider the equation

\begin{equation}
Pu = Vu + W_1 \cdot \nabla u + \nabla \cdot (W_2 u),
\end{equation}

where $P = \text{div}(g \nabla u)$, $g(x) = \{g^{ij}(x)\}_{i,j=1}^n$ is a real-valued symmetric matrix such that it satisfies, for a given constant $\lambda \in (0, 1]$, the uniform ellipticity condition in $\Omega$,

\begin{equation}
\lambda |\zeta|^2 \leq g(x)\zeta \cdot \zeta \leq \lambda^{-1}|\zeta|^2, \quad x \in \Omega, \zeta \in \mathbb{R}^n.
\end{equation}

We also assume that, for a given constant $\Lambda_0 > 0$, the following Lipschitz condition holds

\begin{equation}
|g(x) - g(y)| \leq \Lambda_0 |x - y|, \quad x, y \in \Omega.
\end{equation}

Finally, the lower order terms are assumed to satisfy the following integrability conditions:

\begin{equation}
V \in L^{n/2}(\Omega) \quad \text{and} \quad W_1, W_2 \in L^s(\Omega) \quad \text{with} \quad s > n,
\end{equation}

here $W_1, W_2 : \Omega \to \mathbb{R}^n$ and $V : \Omega \to \mathbb{R}$.

The main aim of the work is to obtain quantitative propagation of smallness from sets of positive measure for solutions of (1.1). The problem setting is the following:

Let $E, K$ be compact subsets of $\Omega$ and let $E$ have positive measure. Find a function $\phi(\epsilon)$, $\lim_{\epsilon \to 0} \phi(\epsilon) = 0$, such that any solution $u$ of (1.1) that satisfies

\begin{equation}
\|u\|_{L^2(\Omega)} \leq 1, \quad \|u\|_{L^2(E)} \leq \epsilon
\end{equation}
is bounded in $L^2(K)$ by
\[ \|u\|_{L^2(K)} \leq \phi(\epsilon). \]

The existence of such a function $\phi$ can be proved in the following way. Assume that there is a sequence $\{u_j\}$ of solutions to (1.1) such that $\|u_j\|_{L^2(\Omega)} \leq |\Omega|^{1/2}$, $\|u_j\|_{L^2(E)} \leq \epsilon_j$, where $\epsilon_j \to 0$ and $\|u_j\|_{L^2(K)} \geq c > 0$ for each $j$. Applying the Caccioppoli inequality (see below), we obtain that $\{u_j\}$ is bounded in $W^{1,2}(\Omega')$, where $\Omega' \subset \subset \Omega$. By choosing a subsequence $\{u_{j_l}\}$, we find a solution $u$ to (1.1) in $\Omega'$ such that $\{u_{j_l}\}$ weakly converges to $u$ in $W^{1,2}(\Omega')$ and in $L^2(\Omega)$. Then $u = 0$ on $E$ while $\|u\|_{L^2(K)} \geq c > 0$. According to a result of R.Regbaoui [15], if $u$ vanishes on a set of positive measure then $u$, in particular, has a zero of infinite order at some point and by the strong unique continuation property proved by Koch and Tataru [10], $u \equiv 0$.

In this work we describe constructive schemes that provide quantitative estimates of $\phi$. We remark also that in our schemes $\phi$ does not depend on $E$ but only on $K$, the measure of $E$, and the distance from $E$ to the boundary of $\Omega$.

Let $\Omega(\rho) = \{x \in \Omega : \text{dist}\{x, \partial \Omega\} > 4\rho\}$ for each $\rho > 0$. Since in what follows we shall assume that $\Omega$ is a bounded connected open set with Lipschitz boundary, we may consider only $\rho < \rho^*$ such that $\Omega(\rho)$ is also connected. Our main results are the following:

**Theorem 1.1.** Let $\Omega$ be a bounded domain with Lipschitz boundary, $u \in W^{1,2}(\Omega)$ be a solution of (1.1), and the coefficients of the equation satisfy (1.2-1.4). Further, let $\rho < \rho^*$ and let $E$ be a measurable subset of $\Omega(\rho)$ of positive measure such that (1.5) holds. Then
\[ (1.6) \quad \|u\|_{L^2(\Omega(\rho))} \leq C \log \epsilon^{-c}, \]
where $C$ and $c$ depend on $\Omega$, $\lambda$, $A_0$, $V$, $W_1$, $W_2$, $|E|$, and $\rho$ only.

**Theorem 1.2.** Let $\Omega$ be a bounded domain with Lipschitz boundary, $u \in W^{1,2}(\Omega)$ be a solution of (1.1), where $P$ satisfies (1.2,1.3) and
\[ (1.7) \quad V \in L^{s/2}(\Omega) \quad \text{and} \quad W_1, W_2 \in L^s(\Omega) \quad \text{with} \quad s > n, \]
and let $E$ be a measurable subset of $\Omega(\rho)$, $\rho < \rho^*$, of positive measure such that
\[ (1.8) \quad \|u\|_{L^\infty(\Omega)} \leq 1, \quad \|u\|_{L^\infty(E)} \leq \epsilon \]
holds. Then
\[ (1.9) \quad \|u\|_{L^\infty(\Omega(\rho))} \leq C \exp(-c(|\log \epsilon|)^\mu), \]
where $c, C$ and $\mu$ depend on $\Omega$, $\lambda$, $A_0$, $V$, $W_1$, $W_2$, $|E|$, and $\rho$ only.
1.3. The structure of the article. Preliminary results are collected in the next section, we formulate a version of the Carleman inequality due to Koch and Tataru that implies strong unique continuation for equations we consider; we also prove the Caccioppoli inequality for solutions of this equations. In Section 3 we obtain the doubling property for solutions of our equations and prove a three sphere inequality. The proof of Theorem 1.1 appears at the end of Section 4. First in this section we show that the Caccioppoli inequality for solution $u$ and the doubling property yield Muckenhoupt condition for the weight $|u|^2$, then we apply the three sphere inequality. Finally, in Section 5 we reproduce (a slightly modified) argument of Nadirashvili that, in combination with the three sphere inequality for the class of equations we consider, gives a proof of Theorem 1.2.

2. Preliminaries

In this section we introduce the notation and formulate the results needed in the sequel.

2.1. Notation and an inequality of Koch and Tataru. We work with standard functional spaces $L^s(\Omega)$ and $W^{l,m}_1(\Omega)$ and always assume that $\Omega$ is a bounded domain; our results reflect local properties of functions inside the domain, so without loss of generality we consider only domains with Lipschitz boundary. Solutions of (1.1) are defined as weak solutions in $W^{1,2}_1(\Omega)$, standard definitions and notation that we use can be found in [7].

Our basic tool is the following version of Carleman inequality.

**Theorem I.** Assume that coefficients of (1.1) satisfy (1.2-1.4) Then there exists $H_0$, $r$, and $\tau_0$ such that for every $\tau > \tau_0$ and each function $v$ vanishing at $x_0 \in \Omega$, $\text{dist}(x_0, \bar{\Omega}) > 2r$, and with $\text{supp} v \subset B(x_0, r)$ that solves

$$Pv - Vv - W_1 \cdot \nabla v - \nabla \cdot (W_2 v) = f$$

there exists $\phi$ such that

$$\tau \leq -r \partial_r \phi \leq H_0 \tau, \quad |\partial_\theta \phi| \leq |r \partial_r \phi|$$

and the following Carleman estimate holds

$$\|e^{\phi(x)} v\|_{L^p} \leq a_n \|e^{\phi(x)} f\|_{L^q},$$

where $q = \frac{2n}{n+2}$, $p = \frac{2n}{n-2}$ and $a_n$ depends only on the dimension $n$ of the space.
The proof of this theorem repeats that of Corollary 3.3 in [10]. We use stronger assumptions on the gradient terms and the matrix $g$. First we note that, after a simple change of the coordinates, metric $g^*$ satisfies $g^*(x_0) = I_n$ and we have

$$|g^*(x) - I_n| + |x - x_0|\|\nabla g^*(x)\| \leq C(\lambda, \Lambda_0) |x - x_0|.$$  

Then we consider the construction of function $h$ in Sections 6-7 of [10]. We claim that for our assumptions on the coefficients one can choose $a_j = q_j$, where $q = q(s) < 1$. Indeed, then both inequalities (6.5) and (7.1) in [10] are satisfied (provided that $q$ is close enough to 1). Then function $h$ in Lemma 6.1 of [10] satisfies $h'(s) \in [\tau, H_0\tau]$ for some $H_0 = H_0(q)$.

2.2. Caccioppoli’s inequality. The Caccioppoli inequality holds for solutions of elliptic equations that we consider and will be used several times in our calculations. We give a proof for the convenience of the reader.

Proposition 2.1. Let $R \subset \subset \tilde{R} \subset \Omega$, assume that coefficients of equation (1.1) satisfy the conditions (1.2-1.4) and

$$\|V\|_{L^{n/2}(\tilde{R})} < \varepsilon, \quad \|W_1\|_{L^n(\tilde{R})} + \|W_2\|_{L^n(\tilde{R})} < \varepsilon.$$  

Then there exist $\varepsilon_0$ and $C_1$, depending on $R$, $\tilde{R}$, and $\lambda$ only, such that if $0 < \varepsilon < \varepsilon_0$ then the following inequality holds

$$\|\nabla u\|_{L^2(R)} \leq C_1 \|u\|_{L^2(\tilde{R})}$$

for any solution $u$ to (1.1).

Proof. Let $\eta \in C_0^\infty(\tilde{R})$, $\eta = 1$ on $R$. Then

$$\|\eta \nabla u\|_{L^2(\tilde{R})}^2 \leq \lambda^{-1} \int_{\tilde{R}} \eta^2 g \nabla u \cdot \nabla u \leq \lambda^{-1} \left( \int_{\tilde{R}} (g \nabla u) \cdot \nabla (\eta^2 u) \right) + \lambda^{-1} \left( \int_{\tilde{R}} (g \nabla u) \cdot (2\eta u \nabla \eta) \right).$$

By the Cauchy inequality, the last term admits an estimate

$$\lambda^{-1} \left( \int_{\tilde{R}} (g \nabla u) \cdot (2\eta u \nabla \eta) \right) \leq 2\lambda^{-2} \int_{\tilde{R}} |\eta \nabla u||u \nabla \eta| \leq \frac{1}{2} \|\eta \nabla u\|^2_2 + 2\lambda^{-4} \|u \nabla \eta\|^2_2.$$
Thus, using (1.1), we obtain
\[ \| \eta \nabla u \|_{L^2(\tilde{R})}^2 \leq 2\lambda^{-1} \left( \int_{\tilde{R}} (g \nabla u) \cdot \nabla (\eta^2 u) \right) + 4\lambda^{-4} \| u \nabla \eta \|_{L^2(\tilde{R})}^2 \]
\[ \leq 2\lambda^{-1} \left( \int_{\tilde{R}} (|V||\eta u|^2 + |W_1||\eta \nabla u||\eta u| + |W_2||\nabla (\eta^2 u)|||u|) \right) + 4\lambda^{-4} \| u \nabla \eta \|_{L^2(\tilde{R})}^2. \]

Next, we apply Hölder’s inequality
\[ \| \eta \nabla u \|_{L^2(\tilde{R})}^2 \leq 2\lambda^{-1} (\| V \|_{L^{n/2}(\tilde{R})} \| \eta u \|_{L^p} + \| W_1 \|_{L^n(\tilde{R})} \| \eta \nabla u \|_{L^2} \| \eta u \|_{L^p} + \| W_2 \|_{L^2(\tilde{R})} \| \eta \nabla u \|_{L^2}) + 4\lambda^{-4} \| \nabla \eta \|_{L^\infty}^2 \| u \|_{L^2(\tilde{R})}^2. \]

Finally by Sobolev’s embedding inequality, see for example [7] page 74,
\[ \| \eta \nabla u \|_{L^2(\tilde{R})}^2 \leq 2\lambda^{-1} C \| V \|_{L^{n/2}(\tilde{R})} (\| \eta \nabla u \|_{L^2}^2 + \| \nabla \eta \|_{L^\infty}^2 \| u \|_{L^2(\tilde{R})}^2) + 2\lambda^{-1} C \| W_1 \|_{L^n(\tilde{R})} (\| \eta \nabla u \|_{L^2}^2 + \| \nabla \eta \|_{L^\infty}^2 \| u(x) \|_{L^2(\tilde{R})}^2) + 4\lambda^{-4} \| \nabla \eta \|_{L^\infty}^2 \| u \|_{L^2(\tilde{R})}^2. \]

Taking \( \varepsilon \) small enough, we absorb all the terms with \( \| \eta \nabla u \|_{L^2} \) in the left hand side of the inequality and obtain (2.3).

**Remark.** It follows from the calculations above that (2.3) holds with
\[ C_1 = C_1(\lambda, \Omega, V, W_1, W_2) \| \nabla \eta \|_{L^\infty}. \]

In particular, we will apply the inequality to two concentric balls and then
\[ \| \nabla u \|_{L^2(B_{ar}(x))} \leq \tilde{C}_1 r^{-1} \| u \|_{L^2(B_r(x))}, \]
where \( a < 1, \tilde{C}_1 \) depends on \( \lambda, \Omega, V, W_1, W_2 \) and on \( a \) only.

### 3. Doubling property and three sphere inequality

Using inequalities from the previous section, we prove the doubling property and the three sphere theorem for solutions of elliptic equations with singular lower order terms.

#### 3.1. Doubling inequality.
First we obtain a doubling inequality for solutions of (1.1).
Proposition 3.1. Suppose that $P, V, W_1$, and $W_2$ satisfy the conditions $(1.2-1.4)$. Then there exists $\rho_0 > 0$ and $\kappa < \frac{1}{4}$ such that for any solution $u \in W_2^1(\Omega)$ of $(1.1)$, any $\rho < \rho_0$, any $\bar{x} \in \Omega(\rho)$, and $r < \kappa \rho$, we have

$$
(3.1) \quad \int_{B_{2r}(\bar{x})} |u|^2 \leq C(u, \Omega, \rho) \int_{B_r(\bar{x})} |u|^2,
$$

where $r < \kappa \rho$ and

$$
(3.2) \quad C(u, \Omega, \rho) = C_0 \max_{x \in \Omega(\rho/2)} \left( \frac{\|u\|_{L^2(B_{\rho}(x))}}{\|u\|_{L^2(B_{2\rho}(x))}} \right)^{H_1},
$$

$H_1 = 5H_0$ ($H_0$ is defined in Theorem I) and $C_0$ depends only on $\lambda, \Lambda_0, \Omega, V, W_1$, and $W_2$.

The inequality above shows that doubling constants for small scales are controlled by doubling constants on some fixed scale. General discussions of the doubling property for solutions of elliptic equations and the frequency of solutions were initiated by the works of N.Garofalo and F.-H.Lin, see [4, 9]. We will derive the doubling inequality from Theorem I. Similar result was proved recently by B.Su in [17], where calculations are performed for the case $P = \Delta, V = W_2 = 0$; the author also mentions that the general case of equation $(1.1)$ can be treated similarly. We consider the general case and obtain doubling for $L^2$-norms; we also write down the precise expression for $C(u, \Omega, \rho)$ in the doubling inequality since we will need it for propagation of smallness estimates.

3.2. Localization of the problem. Let $0 < \varepsilon < \varepsilon_0$, where $\varepsilon_0$ is defined in Proposition 2.1. Let $\rho > 0$ and let $\alpha_\rho$ be a mollifier such that $\alpha_\rho(x) = \alpha(\rho^{-1}x)$, where $\alpha$ is a radial function, $\alpha(y) = 1$ when $|y| < 1$, and supp $\alpha \subset B_2(0)$. We fix $\bar{x} \in \Omega(\rho)$ and define

$$
\tilde{g}(x) = g(\bar{x} + x)\alpha_\rho(x) + g(\bar{x})(1 - \alpha_\rho(x)).
$$

Then $\tilde{g} \in W^{1,\infty}(\mathbb{R}^n)$, moreover

$$
(3.3) \quad \rho \|
abla g\|_{L^\infty} + \|g(\cdot) - g(\bar{x})\|_{L^\infty(B_{2\rho}(\bar{x}))} < \varepsilon,
$$

when $\rho$ is small enough. Further we define $\tilde{Y}(x) = Y(\bar{x} + x)\alpha_\rho(x)$, where $Y = V, W_1, W_2$. Once again, when $\rho$ is sufficiently small we achieve

$$
(3.4) \quad \|
\tilde{V}\|_{L^{n/2}} \leq \varepsilon, \quad \|
\tilde{W}_1, \tilde{W}_2\|_{L^\infty} \leq \varepsilon.
$$

At this point we choose $\rho_0$ small enough and always assume that $\rho < \rho_0$, then $(3.3, 3.4)$ hold.
Now let $r < \kappa \rho$ (we will choose $\kappa < 1/4$ later) and let $\eta \in C_0^\infty(B_\rho(0))$ be such that $\eta(x) = \eta(|x|)$, $0 \leq \eta \leq 1$ and
\[
\eta = 0 \quad \text{for } |x| < \frac{r}{2} \quad \text{and } |x| > \frac{3\rho}{4},
\]
\[
\eta = 1 \quad \text{for } \frac{3r}{4} < |x| < \frac{\rho}{2},
\]
\[
|\nabla \eta| \leq \frac{A}{r}, \quad |D^2 \eta| \leq \frac{A^2}{r^2} \quad \text{in } B_{3r/4} \setminus B_{r/2},
\]
\[
|\nabla \eta| \leq \frac{A}{\rho}, \quad |D^2 \eta| \leq \frac{A^2}{\rho^2} \quad \text{in } B_{3\rho/4} \setminus B_{\rho/2}.
\]

Let $u \in W^{1,2}(\Omega)$ be a solution of (1.1) and let $v(x) = \eta(x)u(\tilde{x} + x)$. We consider
\[
L(v) = \text{div}(\tilde{g}\nabla v) - (\tilde{V}v + \tilde{W}_1 \cdot \nabla v + \nabla \cdot (\tilde{W}_2 v)).
\]
Using (1.1), we note that
\[
L(v) = 2(\tilde{g}\nabla \eta) \cdot \nabla u(\cdot + \tilde{x}) - u(\cdot + \tilde{x})\nabla \eta \cdot (\tilde{W}_1 + \tilde{W}_2) + u(\cdot + \tilde{x})\tilde{P}\eta,
\]
where $\tilde{P}f = \text{div}(\tilde{g}\nabla f)$; note that the right-hand side of the last equation is a function in $L^2(\Omega)$ with a compact support.

We apply the inequality (2.2) to the function $v$ that vanishes at the origin and infinity and the operator $L$. By Theorem I for every $\tau > 0$ there exists $\phi = \phi(\tau, v, \tilde{W}_1, \tilde{W}_2)$ such that (2.1) is satisfied and
\[
(3.6) \quad \|e^{\phi(x)} v\|_{L^p} \leq a_n \|e^{\phi(x)}(Lv)\|_{L^q}.
\]

It is a simple, but tedious, matter to check that relations (2.1) imply that for some $b_0$, depending on $n$ only, we have
\[
(3.7) \quad \min_{|x| \leq d} e^{\phi(x)} \geq \max_{|x| \geq b d} e^{\phi(x)} \quad \text{and then} \quad \min_{|x| \leq d} e^{\phi(x)} \geq e^{\beta \tau} \max_{|x| \geq b d} e^{\phi(x)}
\]
for any $b \geq b_0$ and $\beta = \ln b - \ln b_0$.

### 3.3. Doubling from Carleman’s estimate.

We rewrite (3.6) taking into account (3.5),
\[
(3.8) \quad \|e^{\phi(x)} v\|_{L^p} \leq 2a_n(\|e^{\phi(x)} \tilde{g}\nabla \eta \cdot \nabla u(x + \tilde{x})\|_{L^q} + \|e^{\phi(x)} u(x + \tilde{x})\tilde{P}\eta\|_{L^q} + \|e^{\phi(x)}(\nabla \eta(x))u(x + \tilde{x})(\tilde{W}_1 + \tilde{W}_2)\|_{L^q}) = 2a_n(S_1 + S_2 + S_3).
\]
Let us denote, up to the end of the present section, by $B_t$ the ball with center at $\tilde{x}$ and radius $t$. The first term in the right hand side of (3.8) has the following estimate

$$S_1 = \|e^{\phi(x)} \tilde{g} \nabla \eta \cdot \nabla u(x + \tilde{x})\|_{L^q} \leq \lambda^{-1} \|e^{\phi(x)} \nabla \eta\|_{L^q} \leq \lambda^{-1} \lambda A \left( \rho^{-1} \|u\|_{L^q(B_{3\rho/4})} \max_{|x| \geq \rho/2} e^{\phi(x)} + r^{-1} \|u\|_{L^q(B_{3r/4})} \max_{|x| \geq r/2} e^{\phi(x)} \right).$$

Now we apply the Hölder inequality and the Caccioppoli inequality, (2.4),

$$\|\nabla u\|_{L^q(B_{3t/4})} \leq c_n t \|\nabla u\|_{L^2(B_{3t/4})} \leq C_2 \|u\|_{L^2(B_t)},$$

where $C_2$ depends on $\lambda, V, W_1$ and $W_2$ only. Then

$$S_1 \leq C_3 \left( \rho^{-1} \|u\|_{L^2(B_{3\rho/4})} \max_{|x| \geq \rho/2} e^{\phi(x)} + r^{-1} \|u\|_{L^2(B_r)} \max_{|x| \geq r/2} e^{\phi(x)} \right),$$

where $C_3$ depends on $\lambda, V, W_1$ and $W_2$ only.

The next term is bounded by

$$S_2 = \|e^{\phi(x)} u(x + \tilde{x}) \tilde{P} \nabla \eta\|_{L^q} \leq \|e^{\phi(x)} u(x + \tilde{x})\|_{L^q} \|\nabla \eta\|_{L^q} \leq C_4 \left( \rho^{-1} \|u\|_{L^2(B_{3\rho/4})} \max_{|x| \geq \rho/2} e^{\phi(x)} + r^{-1} \|u\|_{L^2(B_r)} \max_{|x| \geq r/2} e^{\phi(x)} \right),$$

where $C_4$ depends only on $\lambda, \Lambda_0, \Omega, V, W_1$ and $W_2$. Finally, for the last term we have

$$S_3 = \|e^{\phi(x)} (\nabla \eta(x)) u(x + \tilde{x}) (\tilde{W}_1 + \tilde{W}_2)\|_{L^q} \leq A(\|\tilde{W}_1\|_{L^q} + \|\tilde{W}_2\|_{L^q}) (\rho^{-1} \|u\|_{L^2(B_{3\rho/4})} \max_{|x| \geq \rho/2} e^{\phi(x)} + r^{-1} \|u\|_{L^2(B_r)} \max_{|x| \geq r/2} e^{\phi(x)}).$$

Thus the inequality (3.8) becomes

$$\|e^{\phi(x)} \eta(x) u(x)\|_{L^p} \leq C_5 \left( \rho^{-1} \|u\|_{L^2(B_{3\rho/4})} \max_{|x| \geq \rho/2} e^{\phi(x)} + r^{-1} \|u\|_{L^2(B_r)} \max_{|x| \geq r/2} e^{\phi(x)} \right).$$

On the other hand for any $\delta \in (0, 1)$

$$\|e^{\phi(x)} \eta(x) u(x)\|_{L^p} \geq \frac{1}{2} \min_{|x| \leq \delta \rho} e^{\phi(x)} \|\eta u\|_{L^p(B_{3\rho})} + \frac{1}{2} \min_{|x| \leq 2r} e^{\phi(x)} \|u\|_{L^p(B_{2r} \setminus B_1)}.$$

Clearly, by Hölder’s inequality, we have for any function $\psi$

$$\|\psi\|_{L^p(B_{2t} \setminus B_1)} \geq \frac{C_n}{t} \|\psi\|_{L^2(B_{2t} \setminus B_1)}.$$
If we assume that $\delta \rho \in (2r, \rho/2)$ and combine the last three estimates, we get

\[
(3.10) \quad \frac{c_n}{2\delta} \min_{|x| \leq \delta \rho} e^{\phi(x)} \| u \|_{L^2(B_{2\rho})} + \frac{c_n}{2\delta \rho} \min_{|x| \leq \delta \rho} e^{\phi(x)} \| u \|_{L^2(B_{\delta \rho})} \leq C_5 \left( \rho^{-1} \| u \|_{L^2(B_{\delta \rho})} \max_{|x| \geq r/2} e^{\phi(x)} + r^{-1} \| u \|_{L^2(B_{\rho})} \max_{|x| \geq r/2} e^{\phi(x)} \right).
\]

Now we fix $\rho < \rho_0$ and choose $\delta < \min\{2e\delta, c_n(2C_5)^{-1}, 1/9\}$ (see (3.7) and (3.10)); define $\kappa = \delta/8$ and

\[
\tau = \max_{x \in \Omega(\rho/2)} \log \frac{\| u \|_{L^2(B_{\rho}(x))}}{\| u \|_{L^2(B_{2\rho}(x))}}.
\]

We assume that $\phi$ corresponds to this $\tau$ (see Theorem I) and continue the estimates. We have

\[
\| u \|_{L^2(B_{\delta \rho})} \geq \| u \|_{L^2(B_{\rho}(x))},
\]

where $|x - \bar{x}| = 6\kappa \rho$. Clearly, $x \in \Omega(\rho/2)$ and the definition of $\tau$ gives

\[
\frac{c_n}{2\delta} e^\tau \| u \|_{L^2(B_{\delta \rho})} \geq C_5 \| u \|_{L^2(B_{\rho})} \geq C_5 \| u \|_{L^2(B_{\rho/2})},
\]

since $\rho - 6\kappa \rho > 5\rho/6$. This yields

\[
\frac{c_n}{2\delta \rho} \min_{|x| \leq \delta \rho} e^{\phi(x)} \| u \|_{L^2(B_{\delta \rho})} \geq C_5 \delta^{-1} \max_{|x| \geq \delta \rho} e^{\phi(x)} \| u \|_{L^2(B_{\rho/2})}.
\]

We remark that $b_0 \delta \rho < \rho/2$ and then (3.10) implies

\[
\min_{|x| \leq 2\rho} e^{\phi(x)} \| u \|_{L^2(B_{2\rho})} \leq C_6 \max_{|x| \geq r/2} e^{\phi(x)} \| u \|_{L^2(B_{\rho})}.
\]

Thus we obtain

\[
\| u \|_{L^2(B_{2\rho})} \leq C_6 e^{5H_0 \tau} \| u \|_{L^2(B_{\rho})},
\]

for $\tau = \tau(u)$ defined by (3.11). Proposition 3.1 is proved.

3.4. Three sphere inequality. Our next result is a version of three sphere inequality for equations with singular coefficients, once again we use Theorem I.

**Theorem 3.1.** Let $u$ be a solution of (1.4) in $\Omega$, we assume that (1.2) holds. Let $2r < R < \rho < \rho_0$, $R < \kappa \rho$, and $x \in \Omega(\rho)$ ($\rho_0$ and $\kappa$ are as in Proposition 3.1). Then the following inequality holds

\[
(3.12) \quad R^{-1} \| u \|_{L^2(B_{R}(x))} \leq C_7 M^\alpha \sigma^{1-\alpha},
\]

where $C_7$ depends on the coefficients of the differential equation but does not depend on $u$,

\[
M = \rho^{-1} \| u \|_{L^2(B_{\rho}(x))}, \quad \sigma = r^{-1} \| u \|_{L^2(B_{r}(x))},
\]
and
\[ \alpha = \frac{2H_0 \log \frac{2R}{r}}{2H_0 \log \frac{2R}{r} + \log \frac{\rho}{2b_0 R}}. \]

**Proof.** The proof of inequality (3.12) follows by standard arguments. Here we give only a sketch of such a proof for the reader convenience. For each \( \tau > \tau_0 \) (where \( \tau_0 \) is defined in Theorem I) we apply the inequality of Theorem I to \( v = \eta u \) in a small enough ball and repeat the estimates above. Inequality (3.9) implies

\[ R^{-1} \| u \|_{L^2(B_{R})} \min_{|x| \leq R} e^{\phi(x)} \leq C_9 \left( e^{-\tau \log \frac{\rho}{2b_0 R}} \rho^{-1} \| u \|_{L^2(B_{\rho})} + e^{2H_0 \tau \log \frac{2R}{r} - 1} \| u \|_{L^2(B_{r})} \right). \]

We add \( R^{-1} \| u \|_{L^2(B_{r})} \) to both sides and, using the properties of \( \phi \), see (3.7), we obtain

\[ R^{-1} \| u \|_{L^2(B_{r})} \leq C_9 \left( e^{-\tau m} M + e^{\tau l} \right). \]

Now, denote \( \log \frac{\rho}{2b_0 R} = m \) and \( 2H_0 \log \frac{2R}{r} = l \), we get

\[ R^{-1} \| u \|_{L^2(B_{r})} \leq C_9 \left( e^{-\tau m} M + e^{\tau l} \right). \]

Now, let us denote
\[ \tau_1 = \frac{\log M - \log \sigma}{l + m}. \]

Notice that \( e^{-\tau_1 m} M = e^{\tau_1 l} \sigma \). If \( \tau_1 > \tau_0 \), then we choose \( \tau = \tau_1 \) in (3.13) and we get

\[ R^{-1} \| u \|_{L^2(B_{r})} \leq 2C_9 M^\alpha \sigma^{1-\alpha}. \]

Otherwise, if \( \tau_1 \leq \tau_0 \) inequality (3.12) follows trivially. \( \square \)

**Remark.** We note that if \( s > n \) and \( V \in L^{s/2}(\Omega) \), \( W_1, W_2 \in L^s(\Omega) \) then the constants in (3.2) and (3.12) may be chosen to depend only on \( \lambda, \Lambda_0, \Omega, s \), and \( \| V \|_{L^{s/2}(\Omega)}, \| W_1 \|_{L^s(\Omega)}, \| W_2 \|_{L^s(\Omega)}. \)

4. Propagation of Smallness

In this section we prove the main result formulated in the introduction. First we show how to prolongate the smallness of a solution to a certain ball and then we apply Theorem 3.1. By normalization we may always assume that \( |\Omega| \leq 1 \).
4.1. From Doubling and Caccioppoli to Reverse Hölder and Muckenhoupt. Throughout this section $u$ is a solution of (1.1) and $C(u, \Omega, \rho)$ is given by (3.2). The Caccioppoli inequality (2.4) and the doubling inequality imply
\[ \| \nabla u \|_{L^2(B_r(x))} \leq \tilde{C}_1 r^{-1} \| u \|_{L^2(B_{2r}(x))} \leq \tilde{C}_1 r^{-1} C(u, \Omega, \rho) \| u \|_{L^2(B_r(x))}, \]
for $r < \kappa \rho$ and $\text{dist}(x, \partial \Omega) > 4 \rho, \rho < \rho_0$. Then by Sobolev’s embedding theorem
\[ \| u \|_{L^p(B_r(x))} \leq \tilde{C}_1 r^{-1} C(u, \Omega, \rho) \| u \|_{L^2(B_r(x))}, \]
where $p = \frac{2n}{n-2} > 2$, so we obtain the Reverse Hölder inequality, the constant is uniform for $x \in \Omega(\rho)$ provided that $r < \kappa \rho$.

It is well known that Reverse Hölder’s inequality implies the Muckenhoupt condition for the weight $|u|^2$. Let $F$ be a measurable subset of a ball $B := B_r(x)$, where $r < \kappa \rho$ and $x \in \Omega(\rho)$. We apply the Hölder inequality and then the Reverse Hölder inequality obtained above
\[ \left( \int_F |u|^2 \right)^{1/2} \leq \left( \int_B |u|^p \right)^{1/p} |F|^{1/n} \leq C_* \left( \int_B |u|^2 \right)^{1/2} \left( \frac{|F|}{|B|} \right)^{1/n}, \]
where $C_* = C_{10} C(u, \Omega, \rho)$, once again $C_{10}$ depends only on $\lambda, \Lambda_0, \Omega, V, W_1, W_2$.

Assume now that $G \subset B$ and
\[ |G| > (1 - \alpha)|B|, \]
where
\[ \alpha = \alpha(u, \Omega, \rho) = 2^{-n/2} C_{10}^{-n} C(u, \Omega, \rho)^{-n}. \]
Then applying the last inequality to $F = B \setminus G$ we get
\[ \int_G |u|^2 = \int_B |u|^2 - \int_F |u|^2 \geq \int_B |u|^2 (1 - C_*^2 \alpha^{2/n}) \geq \frac{1}{2} \int_B |u|^2. \]

4.2. Lemma of Nadirashvili. Let a cube $Q_0$ be fixed. We consider all dyadic sub-cubes of $Q_0$. First, $Q_0$ is divided into $2^n$ sub-cubes with the length of the side one half of that of $Q_0$, we denote them $Q^{(l)}$, $1 \leq l \leq 2^n$, and call cubes of rank one. Each cube $Q^{(l_1, \ldots, l_r)}$ of rank $r$ is divided into $2^n$ sub-cubes of rank $r + 1$ that are denoted by $Q^{(l_1, \ldots, l_r, l_{r+1})}$, $1 \leq l_{r+1} \leq 2^n$. We will also say that $Q^{(l_1, \ldots, l_r)}$ is the dyadic parent of $Q^{(l_1, \ldots, l_r, l_{r+1})}$. The dyadic parent of $Q_0$ is $Q_0$ itself.

We will use the following statement that can be found in [13], the proof is included for the convenience of the reader.

**Lemma 4.1.** Let $\mathcal{F}$ be a finite family of disjoint dyadic cubes and let $\beta \in (0, 1)$. We define $\overline{\mathcal{F}}$ to be the family of maximal dyadic cubes $R$ that satisfy
\[ |R \cap (\cup_{Q \in \mathcal{F}} Q)| > \beta |R| \]
and $\mathcal{F}_1$ to be the family of dyadic parents of the cubes from $\overline{\mathcal{F}}$. Now let $E = \cup_{Q \in \mathcal{F}} Q$ and $E_1 = \cup_{Q \in \mathcal{F}_1} Q$. Then either

(i) $|E_1| \geq \beta^{-1}|E|$ or

(ii) $\beta^{-1}|E| > |Q_0|$ and $E_1 = Q_0$.

Proof. We prove this statement using induction on the rank of the smallest cube in $\mathcal{F}$. If $Q_0 \in \mathcal{F}$ then (ii) holds. Otherwise we divide $\mathcal{F}$ into $2^n$ subfamilies $\mathcal{F}^{(l)} = \{Q \in \mathcal{F} : Q \subset Q^{(l)}\}$.

We have one of the following cases:

(A) $Q_0 \in \mathcal{F}$,

(B) $Q_0 \not\in \mathcal{F}$ but $Q^{(l)} \in \mathcal{F}$ for some $l$, or

(C) $Q_0 \not\in \mathcal{F}$ and $Q^{(l)} \not\in \mathcal{F}$ for each $l$.

For (A) we have $| \cup_{Q \in \mathcal{F}} Q | > \beta |Q_0|$ and (ii) holds.

For (B) we have $Q_0 \in \mathcal{F}_1$ and $E_1 = Q_0$. At the same time $|Q_0| \geq \beta^{-1}|E|$ since $Q_0 \not\in \mathcal{F}$ and (i) holds.

For (C) we have by the induction hypothesis the statement is true for each family $\mathcal{F}^{(l)}$ and $E^{(l)} = \cup_{Q \in \mathcal{F}^{(l)}} Q$ if we replace $Q_0$ by $Q^{(l)}$. We see that (ii) does not hold for $E^{(l)}$ since $Q^{(l)} \not\in \mathcal{F}$, i.e. $\beta^{-1}|E^{(l)}| \leq |Q^{(l)}|$. Thus (i) holds for each $l$ and

$$|E_1| = \sum_l |E_1^{(l)}| \geq \beta^{-1} \sum_l |E^{(l)}| = \beta^{-1}|E|,$$

where $E^{(l)}_1 = E_1 \cup Q^{(l)}$. $\square$

Remark. Note that (i) may be true even if $E_1 = Q_0$.

4.3. From a set of positive measure to a ball. A cube $R$ is called $\delta$-good for a function $v \in L^2(R)$ if

$$\int_R |v|^2 \leq \delta |R|.$$ 

Proposition 4.1. Suppose that the coefficients of (1.1) satisfy (1.2-1.4). Let $E$ be a compact measurable subset of $\Omega(\rho_1)$, $\rho_1 < \rho_0$, and let $u$ be a solution of (1.1). Assume that $\|u\|^2_{L^2(E)} \leq \epsilon^2 |E|$. Then there exists a cube $Q_0 \subset \Omega$ with side length $r_1 = \kappa \rho_1$ and a finite set $\{Q_j\}$ of dyadic sub-cubes of $Q_0$ such that

$$|\cup Q_j \cap E| > c(\Omega, \rho_1)|E|$$

and each $Q_j$ is $D \epsilon^2$-good for $u$, where

(4.2) $D = a_n C(u, \Omega, \rho_1)^{\gamma_n}$

and $a_n, \gamma_n$ depend only on the dimension $n$. 
Proof. We consider the set

$$E_1 = \{ x \in E : |u(x)| \leq \sqrt{2} \epsilon \}.$$

Clearly, $|E_1| \leq |E|/2$. We may cover $E_1$ by finitely many cubes with side-length $r_1$ and distance to the boundary of $\Omega$ greater than $\rho_1$. We choose one of those $Q_0$ such that $|E_1 \cap Q_0| > c(\Omega, \rho_1)|E|$.

A dyadic sub-cube $K$ of $Q_0$ is called $\beta$-filled if $|K \cap E_1| > \beta|K|$. Consider the set $\{Q_j\}$ of maximal $\beta$-filled cubes (those $\beta$-filled cubes that are not contained in any bigger $\beta$-filled cube). Since almost each point of $E_1 \cap Q_0$ is its point of density, we know that $|(E_1 \cap Q_0) \cup jQ_j| = 0$. Thus we can take finitely many of cubes $\{Q_j\}$ such that

$$|\bigcup_j Q_j \cap E| > |\bigcup_j Q_j \cap E_1| > \frac{1}{2}|E \cap Q_0| > \frac{1}{2}c(\Omega, \rho_1)|E|$$

and $|Q_j \cap E_1| > \beta|Q_j|$.

Note that $\rho_1$ is fixed and let $\alpha(u) = \alpha(u, \Omega, \rho_1)$, we can choose $\beta = \beta(u)$ such that the last inequality implies

$$|B_j \cap E_1| > (1 - \alpha)|B_j|,$$

for the ball $B_j$ inscribed in $Q_j$, i.e., $B_j$ has the same center as $Q_j$ and radius $l_j/2$, where $l_j$ is the side length of $Q_j$. Indeed

$$|B_j \cap E_1| \geq |B_j| - |Q_j \setminus E_1| \geq |B_j| - (1 - \beta)|Q_j| = |B_j|(1 - (1 - \beta)c_n).$$

Thus $|B_j \cap E_1| \geq (1 - kc_nC(u, \Omega, \rho_1)^{-n})|B_j| \geq (1 - \alpha)|B_j|$ if

$$\beta = 1 - kC(u, \Omega, \rho_1)^{-n},$$

where $k < k_0$, $k_0$ does not depend on $u$ but depends on the coefficients of the differential operator and on $\Omega$. Then, using (3.1) and (4.1), we get

$$\int_{Q_j} |u|^2 \leq \int_{\sqrt{n}B_j} |u|^2 \leq C(u, \Omega, \rho_1)^{1+\log n} \int_{B_j} |u|^2 \leq 2C(u, \Omega, \rho_1)^{1+\log n} \int_{B_j \cap E_1} |u|^2 \leq 2C(u, \Omega, \rho_1)^{1+\log n} |B_j \cap E_1| 2\epsilon^2 \leq De^2|Q_j|.$$

□

Our aim is to estimate $\int_{Q_0} |u|^2$, where $Q_0$ is the cube from the last proposition, so $u$ and $Q_0$ are fixed, we consider dyadic sub-cubes of $Q_0$. Let $D$ and $\beta < 1$ be defined by (4.2) and (4.3). We note that

- if $R$ is $\delta$-good then its dyadic parent is $D\delta$ good (by the doubling inequality (3.1)),
- if $\{R_j\}$ are disjoint $\delta$-good cubes and $|R \cap (\cup R_j)| > \beta|R|$ then $R$ is $D\delta$-good; this follows from (4.1) and (3.1).
Let $Q_1$ be the family of cubes $Q_j$ obtained in Proposition 4.1. We define by induction $\overline{Q}_j$ to be the family of maximal dyadic cubes $R$ that satisfy $|R \cap (\cup_{Q \in Q_j} Q)| > \beta |R|$ and $Q_{j+1}$ to be the family of dyadic parents of the cubes from $\overline{Q}_j$. Then by induction all cubes from $Q_j$ are $D_{2j-1}^{2j}$ good.

Let $E_j = \cup_{Q \in Q_j} Q$. By Lemma 4.1 we have

either (i) $|E_j| \geq \beta^{-j+1} |E_1|$ or (ii) $E_j = Q_0$.

By Proposition 4.1, $|E_1| \geq c(\Omega, \rho_1)|E|$. Therefore, taking (4.3) into account, we see that there exists $N = N(|E|, \rho_1, P, V, W_1, W_2)C(u, \Omega, \rho_1)^n = N_0 C(u, \Omega, \rho_1)^n$ such that $E_N = Q_0$, where $N_0$ depends only on $|E|$ on $\rho_1$ and on $\Omega$. Thus

(4.4) \[ \int_{Q_0} |u|^2 \leq (a_n C(u, \Omega, \rho_1) \gamma_n)^N \epsilon^2. \]

Now we can prove the following statement

**Proposition 4.2.** Assume that the equation (1.1) is given and its coefficients satisfy (1.2-1.4) and let $\sigma, m, \rho_1$ be positive, $\sigma < 1/n$. There exists $\epsilon_0 = \epsilon_0(\sigma, m, \rho_1)$ such that the following holds:
If $E$ is a measurable subset of $\Omega(\rho_1)$, $|E| \geq m$, and $u$ is a solution of (1.1) that satisfies

$\|u\|_{L^2(E)} |E|^{-1/2} \leq \epsilon < \epsilon_0$ and $\|u\|_{L^2(\Omega)} \leq 1$

then there exists a ball $B_0$ of radius $r_1 = \kappa \rho_1$ and center in $\Omega(\rho_1)$ such that

(4.6) \[ \int_{B_0} |u|^2 \leq (|\log \epsilon|)^{-2\sigma/H_1}, \]

where $H_1$ is as in Proposition 3.1.

**Proof.** By the definition of $C(u, \Omega, \rho_1)$, there exists a ball of radius $r_1$ such that

(4.7) \[ \int_B |u|^2 \leq M^2 \left((C_0 C^{-1}(u, \Omega, \rho_1))^{1/H_1}, \right. \]

where $M = \|u\|_{L^2(\Omega)} \leq 1$.

If $C(u, \Omega, \rho_1) \geq C_0 |\log \epsilon|^\sigma$ then (4.7) implies the desired estimate. Otherwise we use the ball $B_0$ inscribed in the cube $Q_0$ in (4.5) and
from (4.4) and (3.1) we obtain
\[
\int_{B_0} |u|^2 \leq \exp \left( 2 \log \epsilon + (N + 1) \log(a_n C_0^\gamma n) + (N + 1) \gamma_n \sigma \log |\log \epsilon| \right)
\]
\[
\leq \exp \left( -2 |\log \epsilon| + 2N_0 C_0^n |\log \epsilon| \sigma ( \log(a_n C_0^\gamma n) + \gamma_n \sigma \log |\log \epsilon|) \right),
\]
then (4.6) follows for \( \epsilon \) small enough since \( n \sigma < 1 \). \( \square \)

**Remark.** The statement of Proposition implies also that there exists \( A = A(\sigma, m, \rho_1) \) such that if \( E \) is a compact measurable subset of \( \Omega(\rho_1) \), \( |E| > m \), and \( u \) is a solution of (1.1) that satisfies \( \|u\|_{L^2(E)} |E|^{-1/2} \leq \epsilon < 1 \) and \( \|u\|_{L^2(\Omega)} \leq 1 \) then there exists a ball \( B_0 \) of radius \( r_1 = \kappa \rho_1 \) and center in \( \Omega(\rho_1) \) such that
\[
\int_{B_0} |u|^2 \leq A |\log \epsilon|^{-2 \sigma / H_1}.
\]

4.4. **Proof of the Main result.** In this section we prove Theorem 1.1 formulated in the introduction. We use standard argument of smallness propagation, see [2].

**Proof.** First, we may assume that \( \rho < \rho_0 \), then we cover \( \Omega(\rho) \) by finitely many balls of radii \( r = \kappa \rho \) and with centers in \( \Omega(\rho) \). It is enough to prove a similar inequality for \( L^2 \)-norm of \( u \) over each of those balls. Now we refer to Proposition 4.2 to find one ball \( B_0 = B(x_0, r) \) with desired estimate and \( c = \sigma / H_1 \) and \( C \) that depends on \( |E| \) and \( \rho \), we note that \( r \) depends only on \( \rho \). Using Theorem 3.1 we can obtain an estimate for the norm of \( u \) in \( 2B_0 = B(x_0, 2r) \) and then in a ball of radius \( r \) with center \( x_1 \) such that \( |x_0 - x_1| = r \). By Theorem 3.1 we get
\[
\|u\|_{L^2(B_1)} \leq C |\log \epsilon|^{-c}.
\]
Where \( C \) and \( c \) do not depend on \( u \). \( \square \)

**Remark.** We choose to work with \( L^2 \)-norms since the solutions we consider include unbounded functions (see Section 2 of Introduction in [7] for corresponding examples and general discussions). If we assume that \( V \in L^t(\Omega), t > n/2 \), then \( L^2 \) inequalities and elliptic estimates yield \( L^\infty \)-results (see [7, chapter III, §13]).

5. **On a theorem of Nadirashvili**

We prove Theorem 1.2 in this section. The proof follows the argument of N. Nadirashvili [14] (in the way we understand it). Our version of the proof differs from the original in some technical details, it is adjusted to our assumptions on coefficients. For example we use elliptic estimate in the place of the growth lemma of Landis, which appeared
in the original proof, we also apply the three sphere inequality obtained in Section 3.4 of this work.

5.1. First reduction. Once again we consider elliptic equations of the form (1.1) such that the main term satisfies inequalities (1.2), (1.3), and (1.7) holds for the lower order terms. The statement of the Theorem (1.2) follows from the lemma below.

Lemma 5.1. Let $P = \text{div}(g \nabla u)$, where $g(x) = \{g_{ij}(x)\}_{i,j=1}^n$ is a real-valued symmetric matrix satisfying (1.2) and (1.3). Let $s > n$ and assume that $\rho > 0$ and $V, W_1$ and $W_2$ satisfy

$$\|V\|_{L^{s/2}(\Omega)}, \|W_1\|_{L^s(\Omega)}, \|W_2\|_{L^s(\Omega)} \leq \Lambda_1.$$  

Then there exist positive numbers $\delta$ and $c$ that depend on $\Omega, \lambda, \Lambda_0, \Lambda_1$ and $\rho$, such that if

(5.1) \hspace{1cm} Pu = Vu + W_1 \cdot \nabla u + \nabla \cdot (W_2u), \quad \text{in } \Omega,

$$\|u\|_{L^s(\Omega)} \leq 1, \text{ } E \text{ is a measurable subset of } B_{r/2}(x), \hspace{1cm} |E| > (1 - \delta)|B_{r/2}|,$$

where $B_r(x) \subset \Omega(\rho), r \leq 1$ and $\|u\|_{L^\infty(E)} \leq \epsilon, \epsilon \in (0, 1/2)$ then

$$\|u\|_{L^\infty(B_{r/2}(x))} \leq \exp(-c|\log \epsilon|^{\alpha}),$$  

where $\alpha < 1$ and depends on the dimension of the space, $\Omega, \lambda, \Lambda_0, \Lambda_1$.

We want to show that Lemma 5.1 implies Theorem 1.2. The three sphere inequality (3.12) for $L^2$-norms implies similar inequality for $L^\infty$ norms since we assume that $V \in L^s(\Omega)$ and $s > n/2$. Then we obtain the following version of Lemma 5.1 for cubes:

There exist positive numbers $\delta_1$ and $c_1$ that depend on $\Omega, \lambda, \Lambda_0, \Lambda_1$ and $\rho$ and $\tau_0$, such that if (5.1) holds, $\|u\|_{L^\infty(\Omega)} \leq 1, \text{ } E \text{ is a measurable subset of } Q_{\rho}(x), \hspace{1cm} |E| > (1 - \delta_1)|Q_{\rho}(x)|$, where $Q_{\rho}(x) \subset \Omega(\rho), r \leq \tau_0$ and $\|u\|_{L^\infty(E)} \leq \epsilon, \epsilon \in (0, 1/2)$ then

$$\|u\|_{L^\infty(Q_{\tau_0}(x))} \leq \exp(-c_1|\log \epsilon|^{\alpha}),$$

where $Q_{\tau_0}(x)$ is a cube with side length $t$ and center $x$.

Now we find a cube $Q_0 \subset \Omega(\rho)$ with side length $l = l(\Omega, \rho)$ and a finite collection of its disjoint dyadic sub-cubes $Q_j = Q_{\tau_j}(x_j)$ such that $|E \cap Q_j| > (1 - \delta)|E|$ and $|E \cap (\cup_j Q_j)| > a|E|$, where $a = a(\rho, \Omega, n)$. Using Lemma 5.1 three sphere inequality and Lemma 4.1 we conclude that there exists $c_0, C_0$ and $\mu_0$ that depend on $\Omega, \lambda, \Lambda_0, \Lambda_1, |E|$ and $\rho$, such that

$$\|u\|_{L^\infty(Q_0)} \leq C_0 \exp(-c_0|\log \epsilon|^{\mu_0}).$$  

By applying three sphere inequality once again we obtain (1.9).
5.2. **Second reduction.** We shall formulate another statement that implies Lemma 5.1. Assume that Lemma is false, then for any \( k \in \mathbb{N} \cup \{0\} \) we can find \( E_k \subset B_{r_0/2}(x_k) \) and \( u_k \) such that \( u_k \) satisfies (5.1), \( \|u_k\|_{L^\infty(\Omega)} \leq 1 \), \( |E_k| \geq (1 - 2^{-k})|B_{r_0/2}(x_k)| \), where \( B_{r_0/2}(x_k) \subset \Omega \), \( r \leq r_0 \), \( \|u_k\|_{L^\infty(E)} \leq \epsilon_k \), \( \epsilon_k \in (0, 1/2) \) and

\[
\|u_k\|_{L^\infty(B_{r_0/2}(x_k))} > \exp(-k|\log \epsilon_k|^\alpha).
\]

We consider \( v_k(x) = u_k(x_k + rx) \) then \( v_k \) satisfies an equation of the form

\[(5.2) \quad \text{div}(\tilde{g}v) = \tilde{V}v + \tilde{W}_1 \cdot \nabla v + \nabla \cdot (\tilde{W}_2v) \quad \text{in} \ B_1,
\]

where \( B_1 \) is the unit ball with center at the origin. Moreover \( \tilde{g} \) satisfies (1.2) and (1.3) in \( B_1 \) and

\[(5.3) \quad \|\tilde{V}\|_{L^{s/2}(B_1)}, \|\tilde{W}_1\|_{L^{s}(B_1)}, \|\tilde{W}_2\|_{L^{s}(B_1)} \leq \Lambda_1.
\]

Define \( F_k = \{x : x_k + rx \in E_k\} \subset B_{1/2} \), clearly \( |F_k| \geq (1 - 2^{-k})|B_{1/2}| \). Let further \( F_0 = \cap_{k \geq 2} F_k \), we have \( |F_0| \geq \frac{1}{2}|B_{1/2}| \). Assuming that Lemma 5.1 does not hold, we see that the following statement should be false:

**Lemma 5.2.** Let \( F_0 \) be a subset of \( B_{1/2} \) with \( |F_0| > \frac{1}{2}|B_{1/2}| \). There exist \( c = c(F_0, \lambda, \Lambda_0, \Lambda_1, s) \) such that if \( u \) is a solution of (5.2), for which (1.2), (1.3), and (5.3) holds, \( |u| \leq 1 \) in \( B_1 \), and \( |u| \leq \epsilon \) on \( F_0 \), \( \epsilon \in (0, 1/2) \) then

\[
\|u\|_{L^\infty(B_{1/2})} \leq \exp(-c|\log \epsilon|^\alpha).
\]

where \( \alpha < 1 \) and \( c \) depend on the dimension of the space, \( \Omega, \lambda, \Lambda_0, \) and \( \Lambda_1 \).

Thus the argument of this subsection shows that it is enough to prove Lemma 5.2 and get an estimate with \( c \) that depends on \( F_0 \), Lemma 5.1 will follow. For the rest of the proof \( F_0 \) is a fixed subset of \( B_{1/2} \).

5.3. **Elliptic estimate.** The following elliptic estimate holds

\[(5.4) \quad \max_{y \in B_r(x)} |u(y)|^2 \leq \frac{A^2}{|B_{2r}(x)|} \int_{B_{2r}(x)} u^2, \quad \text{where} \ B_{2r}(x) \subset B_1,
\]

where \( A \) depends only on \( n, \lambda, \Lambda_0, \Lambda_1, \) see for example [7, chapter III, §13]. The next result follows from the elliptic estimate and will be used repeatedly in the sequel.

**Claim.** Let \( u \) be a solution to (5.2) in \( B_1, \ F \subset B_{1/2} \) and \( |u| < \epsilon \) on \( F \). There exist \( \gamma \in (0, 1) \) and \( \beta > 0 \) \( \beta \) depends only on \( \gamma \) and on \( A \), such that:
\[ |u(y^*)| > c > 2A\epsilon, \quad |F \cap B_{r^*}(x^*)| > \gamma |B_{r^*}(x^*)|, \quad |x^*| < 1 - 4r^* \text{ and } y^* \in B_{r^*/2}(x^*) \text{ imply} \]

\[ \sup_{B_{r^*}(x^*)} |u| > (1 + \beta)c. \]

**Proof.** Denote \( \max_{y \in B_{r^*}(x^*)} |u(y)| = d \), inequality (5.4) for \( B_{r^*/2}(x^*) \) gives

\[ c^2 < \max_{B_{r^*}(r^*/2)} |u|^2 \leq \frac{A^2}{|B_{r^*}(x^*)|} \int_{B_{r^*}(x^*)} u^2 \leq A^2(\epsilon^2 + d^2(1 - \gamma)). \]

Then

\[ d^2 > c^2 - A^2\epsilon^2 \gamma \quad \frac{c^2}{(1 - \gamma)A^2} \geq \frac{c^2}{2(1 - \gamma)A^2}. \]

If \( \gamma \) is close to 1, \( \gamma = \gamma(A) \), then the claim is justified. \( \square \)

### 5.4. Points of density and Marcinkiewicz integral

Let \( \gamma \) be from the claim above, \( \gamma \in (0, 1) \). Since almost all the points of \( F_0 \) are points of density, there exist a positive number \( r_1 \), \( r_1 \) depends on \( F_0 \), and a set \( F_1 \subset F_0 \) such that \( |F_1| > |F_0|/2 \) and for each \( x \in F_1 \) we have

\[ \frac{|F_0 \cap B_r(x)|}{|B_r(x)|} > \gamma \text{ whenever } r \leq r_1. \]

Now, \( F_1 \) has positive measure and by the Marcinkiewicz theorem (see for example, [16, chapter I] ) for almost each point of \( x \) of \( F_1 \) we have

\[ \int_{|y| \leq 1} \frac{\text{dist}(x + y, F_1)}{|y|^{n+1}} < +\infty. \]

We fix a point \( x_0 \) in \( F_1 \) for which (5.3) holds.

For each \( r \in (0, r_1) \) let \( h(r) = \max_{|y| = r} \text{dist}(x_0 + y, F_1) \). Our choice of \( x_0 \) implies that

\[ \int_0^{r_1} \frac{(h(r))^n}{r^{n+1}} dr < +\infty. \]

Indeed, let us check that (5.3) implies (5.4), Let \( \overline{y} \) be such that \( |\overline{y}| = r \) and \( \text{dist}(x_0 + \overline{y}, F_1) = h(r) \). We have \( \text{dist}(x_0 + z, F_1) \geq h(r)/2 \) for all \( z \) such that \( |x_0 - z| = r \) and \( |\overline{y} - z| < h(r)/2 \). Then

\[ \int_{S^n} \text{dist}(x_0 + ry', F_1) dy' \geq C h(r) \left( \frac{h(r)}{r} \right)^{n-1}. \]

Where \( S^n = \{ y' \in \mathbb{R}^n : |y'| = 1 \} \). Finally, polar integration gives (5.6).
Let further $h_t = \max_{r \in (2^{-l-1}, 2^{-l})} h(r) = h(r_t)$. We note that $h(r_t + t) \geq h(r_t) - |t| > h_t/2$ when $|t| < h_t/2$. Then (5.6) implies that

\begin{equation}
(5.7) \quad + \infty > \sum_{l>l_0} \int_{2^{-l-1}}^{2^{-l-1}} \frac{(h(r))^{n}}{r^{n+1}} \geq \sum_{l>l_0} (h_t/2)^{n+1} \leq \frac{l h_t}{2}.
\end{equation}

The following property holds:

for any $x$ such that $|x - x_0| < r_1/2$ there is a ball $B_r(x) = (s(x))$ such that $x \in B_{r(x)}(s(x))$, $r(x) \leq 2h(|x - x_0|)$

and

\begin{equation}
(5.8) \quad \frac{|F_0 \cap B_r(x)|}{|B_r(x)|} > \gamma.
\end{equation}

Indeed, we just take $s(x) \in B_{h(|x - x_0|)}(x) \cap F_1$ and $r(x) = 2|x - s(x)|$.

5.5. Growth properties of $u$. Let $r < r_1$ and $m(r) = \max_{|x - x_0| = r} |u(x)|$. Assume that $m(r) > 2A \epsilon$ and let $x, |x - x_0| = r$, be such that $m(r) = |u(x)|$, further let $s(x)$ and $r(x)$ be as in (5.8). We apply the Claim to the ball $B_{r(x)}(s(x))$ and get

$$
\sup_{B_{r(x)}(s(x))} |u| > (1 + \beta) m(r).
$$

We have also $|s(x) - x_0| \leq |x - x_0| + |x - s(x)| \leq r + r(x)/2 \leq r + h(r)$

and

$$
\max_{t \in [-3h(r), 3h(r)]} m(r + t) > (1 + \beta) m(r).
$$

Let us define

$$
r(M) := \min \{ r : m(r) \geq M \}
$$

for $M \leq \sup_{|x - x_0| \leq r_1} |u(x)| = M_1$. For any $M > 2A \epsilon$ we have either

$$(1 + \beta) M > M_1, \quad \text{and} \quad r(M) > r_1/2
$$

(see inequality for $h(r)$ below) or

\begin{equation}
(5.9) \quad r((1 + \beta)M) \leq r(M) + 3h(r(M)).
\end{equation}

We remark that (5.7) implies that $\lim_{l \to \infty} h_t 2^l = 0$ and there exists $l_1 = l_1(F_0, x_0)$ such that $h_t 2^l < 1/12$ when $l > l_1$. Consequently, $h(r) < r/6$ for $r < 2^{-l_1}$.

We say that $l$ is good (and the corresponding interval $(2^{-l-1}, 2^{-l})$ is good) if $h_l < l^{-1/(n+1)} 2^{-l}$. Then (5.7) implies that there exists $N_0$, $N_0 = N_0(F_0, x_0)$, such that for $N \geq N_0$ at least $2^{N-1}$ of the numbers $2^N + 1, \ldots, 2^{N+1}$ are good.

Assume that $r_0 = r(2A \epsilon)$ we want to prove that

\begin{equation}
(5.10) \quad r_0 \geq \exp (-B \log \epsilon (n+1)/(n+2)),
\end{equation}

where $B$ is such that $h_1 > (1 + \beta) B \epsilon$.
where $B$ depends on $F_0, x_0, A$. We assume also that $r_0 \in (2^{-l_0 - 1}, 2^{-l_0})$, where $l_0 > l_1$ and $l_0 \in (2^{N+1}, 2^{N+2})$, $N \geq N_0$. (Otherwise (5.10) is satisfied provided that $\epsilon < 1/2$ and $B$ is large enough.)

Further let $r_j = r((1 + \beta)^j 2A\epsilon)$, when $(1 + \beta)^j 2A\epsilon < M_1$. Then (5.9) implies $r_{j+1} \leq r_j + 3h(r_j)$. The sequence $\{r_j\}$ is increasing and $r_{j+1} - r_j < 3h(r_j) < r_j/2$; moreover if $r_j \in (2^{-l-1}, 2^{-l})$, where $l$ is good then

\begin{equation}
(5.11) \quad r_{j+1} - r_j < 3h < 2^{-l+2}l^{-1/(n+1)}.
\end{equation}

Let $K$ be the number of $j$ such that $r_j < 2^{-2N}$. For each good $l$, $2^N < l \leq 2^{N+1}$, there exists $j_l = \min\{j : r_j \in (2^{-l-1}, 2^{-l})\}$. Thus $r_{j_l} - 1 < 2^{-l-1}$ and $r_{j_l} - 3/2 r_{j_l} - 1 < 3 \cdot 2^{-l-2}$. Then (5.11) implies that there are at least $\frac{1}{4} 2^{l/(n+1)}$ elements of the sequence $\{r_j\}$ in $(2^{-l-1}, 2^{-l})$. Now, since there are at least $2^{N-1}$ good numbers $l \in \{2^N + 1, \ldots, 2^{N+1}\}$, we have

$K \geq 2^N - 1 \cdot \frac{1}{4} 2^{N/(n+1)} = \frac{1}{8} 2^{N(n+2)/(n+1)}$.

From the other hand $m(r_K) \geq (1 + \beta)^K 2A\epsilon$ and $m(r_K) \leq 1$. We get the following inequality:

$2A\epsilon (1 + \beta)^K \epsilon \leq 1$.

It implies $K \leq a|\log \epsilon|$, where $a$ depends on $A$ and on $\beta$. We combine the last inequality with the estimate we have for $K$ from below and obtain

$8a|\log \epsilon| \geq 2^{N+2}/n$. 

Now,

$r_0 \geq 2^{-2N} = \exp(-2^N \log 2) \geq \exp(-B|\log \epsilon|^{(n+1)/(n+2)})$.

Inequality (5.10) is established.

We have

$\max_{B_{r_0}(x_0)} |u| \leq 2A\epsilon$.

Now we apply Theorem 3.1 and obtain

$\|u\|_{L^2(B_{r_0}(x_0))} \leq C \exp(-B_1|\log \epsilon|^{1/(n+2)})$.

Finally, using standard technique, we complete the proof of Lemma 5.2. We note that $\alpha$ depends on $n$ and $\kappa$ from Proposition 3.1.
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