Counterexample of the Riemann Hypothesis

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Abstract

Under the assumption that the Riemann hypothesis is true, von Koch deduced the improved asymptotic formula \( \theta(x) = x + O(\sqrt{x} \log^2 x) \), where \( \theta(x) \) is the Chebyshev function. We obtain a result which contradicts this asymptotic formula. By contraposition, we deduce that the Riemann hypothesis is false.

Keywords: Riemann hypothesis, Nicolas inequality, Chebyshev function, prime numbers

2000 MSC: 11M26, 11A41, 11A25

1. Introduction

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part \( \frac{1}{2} \) [1]. On the other hand, \( h(x) + O(f(x)) \) denotes the collection of functions having the growth of \( h(x) \) plus a part whose growth is limited to that of \( f(x) \). Thus,

\[
g(x) = h(x) + O(f(x))
\]

expresses the same as

\[
g(x) - h(x) = O(f(x)).
\]

In mathematics, the Chebyshev function \( \theta(x) \) is given by

\[
\theta(x) = \sum_{p \leq x} \log p
\]

where \( p \leq x \) means all the prime numbers \( p \) that are less than or equal to \( x \). Say \( \text{Nicolas}(p_n) \) holds provided

\[
\prod_{q \leq p_n} \frac{q}{q - 1} > e^\gamma \times \log \theta(p_n).
\]

The constant \( \gamma \approx 0.57721 \) is the Euler-Mascheroni constant, \( \log \) is the natural logarithm, and \( p_n \) is the \( n^{th} \) prime number. The importance of this property is:

**Theorem 1.1.** [2], [3]. \( \text{Nicolas}(p_n) \) holds for all prime numbers \( p_n > 2 \) if and only if the Riemann hypothesis is true.

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We know the following properties for the Chebyshev function:

**Theorem 1.2.** [4]. If the Riemann hypothesis holds, then

\[ \theta(x) = x + O(\sqrt{x} \log^2 x) \]

for all \( x \geq 10^8 \).

**Theorem 1.3.** [5]. For \( 2 \leq x \leq 10^8 \)

\[ \theta(x) < x. \]

We also know that

**Theorem 1.4.** [6]. If the Riemann hypothesis holds, then

\[
\frac{e^{-\gamma}}{\log x} \prod_{q \leq x} \frac{1}{q - 1} \left( 1 + \frac{\log \frac{q}{q-1} - \frac{1}{q}}{\log \frac{x}{x-1}} \right) < \frac{3 \log x + 5}{8\pi \sqrt{x}}
\]

for all numbers \( x \geq 13.1 \).

Let’s define \( H = \gamma - B \) such that \( B \approx 0.2614972128 \) is the Meissel-Mertens constant [7]. We know from the constant \( H \), the following formula:

**Theorem 1.5.** [8].

\[
\sum_{q} \left( \log \frac{q}{q-1} - \frac{1}{q} \right) = \gamma - B = H.
\]

For \( x \geq 2 \), the function \( u(x) \) is defined as follows

\[
u(x) = \sum_{q \leq x} \left( \log \frac{q}{q-1} - \frac{1}{q} \right).
\]

We use the following theorems:

**Theorem 1.6.** [9]. For \( x > -1 \):

\[
\frac{x}{x + 1} \leq \log(1 + x).
\]

**Theorem 1.7.** [10]. For \( x \geq 1 \):

\[
\log(1 + \frac{1}{x}) < \frac{1}{x + 0.4}.
\]

Let’s define:

\[
\delta(x) = \left( \sum_{q \leq x} \frac{1}{q} - \log \log x - B \right).
\]

**Definition 1.8.** We define another function:

\[
\sigma(x) = \left( \sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B \right).
\]

Putting all together yields the proof that the inequality \( \sigma(x) > u(x) \) is satisfied for a number \( x \geq 3 \) if and only if Nicolas\((p)\) holds, where \( p \) is the greatest prime number such that \( p \leq x \). In this way, we introduce another criterion for the Riemann hypothesis based on the Nicolas criterion and deduce some of its consequences.
2. Results

**Theorem 2.1.** The Riemann hypothesis is true if and only if the inequality \( \varpi(x) > u(x) \) is satisfied for all numbers \( x \geq 3 \).

*Proof.* In the paper [3] is defined the function:

\[
    f(x) = e^\gamma \times (\log \theta(x)) \times \prod_{q \leq x} \frac{q-1}{q}.
\]

We know that \( f(x) \) is lesser than 1 when Nicolas\((p)\) holds, where \( p \) is the greatest prime number such that \( 2 < p \leq x \). In the same paper, we found that

\[
    \log f(x) = U(x) + u(x)
\]

where \( U(x) = -\varpi(x) \) [3]. When \( f(x) \) is lesser than 1, then \( \log f(x) < 0 \). Consequently, we obtain that

\[
    -\varpi(x) + u(x) < 0
\]

which is the same as \( \varpi(x) > u(x) \). Therefore, this is a consequence of the theorem 1.1. \( \square \)

**Theorem 2.2.** If the Riemann hypothesis holds, then

\[
    \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log \theta(x)} > 1
\]

for all numbers \( x \geq 13.1 \).

*Proof.* Under the assumption that the Riemann hypothesis is true, then we would have

\[
    \prod_{q \leq x} \frac{q}{q-1} < e^\gamma \times \log x \times \left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}} \right)
\]

after of distributing the terms based on the theorem 1.4 for all numbers \( x \geq 13.1 \). If we apply the logarithm to the both sides of the previous inequality, then we obtain that

\[
    \sum_{q \leq x} \log \left( \frac{q}{q-1} \right) < \gamma + \log \log x + \log \left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}} \right).
\]

That would be equivalent to

\[
    \sum_{q \leq x} \frac{1}{q} + \sum_{q \leq x} \left( \log \left( \frac{q}{q-1} \right) - \frac{1}{q} \right) < \gamma + \log \log x + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}
\]

where we know that

\[
    \log \left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}} \right) < \frac{1}{\frac{8 \times \pi \times \sqrt{x}}{3 \times \log x + 5} + 0.4} = \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 0.4 \times (3 \times \log x + 5)} = \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} \times 3
\]
according to theorem 1.7 since $\frac{8x\log x}{x \log x} \geq 1$ for all numbers $x \geq 13.1$. We use the theorem 1.5 to show that

$$\sum_{q \leq x} \left( \log \left( \frac{q}{q-1} \right) - \frac{1}{q} \right) = H - u(x)$$

and $\gamma = H + B$. So,

$$H - u(x) < H + B + \log \log x - \sum_{q \leq x} \frac{1}{q} + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

which is the same as

$$H - u(x) < H - \delta(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}.$$  

We eliminate the value of $H$ and thus,

$$-u(x) < -\delta(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

which is equal to

$$u(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \delta(x).$$

Under the assumption that the Riemann hypothesis is true, we know from the theorem 2.1 that $\varpi(x) > u(x)$ for all numbers $x \geq 13.1$ and therefore,

$$\varpi(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \delta(x).$$

Hence,

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \log \log \theta(x) - \log \log x.$$  

Suppose that $\theta(x) = \epsilon \times x$ for some constant $\epsilon > 1$. Then,

$$\log \log \theta(x) - \log \log x = \log (\log (\epsilon \times x)) - \log \log x$$

$$= \log (\log x + \log \epsilon) - \log \log x$$

$$= \log \left( \log x \times (1 + \frac{\log \epsilon}{\log x}) \right) - \log \log x$$

$$= \log \log x + \log (1 + \frac{\log \epsilon}{\log x}) - \log \log x$$

$$= \log (1 + \frac{\log \epsilon}{\log x}).$$

In addition, we know that

$$\log (1 + \frac{\log \epsilon}{\log x}) \geq \frac{\log \epsilon}{\log \theta(x)}$$

using the theorem 1.6 since $\frac{\log \epsilon}{\log x} > -1$ when $\epsilon > 1$. Certainly, we will have that

$$\log (1 + \frac{\log \epsilon}{\log x}) \geq \frac{\log \epsilon}{\log \epsilon + \log x} \geq \frac{\log \epsilon}{\log \epsilon + \log x} = \frac{\log \epsilon}{\log \theta(x)}.$$
Thus,
\[
\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \frac{\log \epsilon}{\log \theta(x)}.
\]
If we add the following value of \( \frac{\log x}{\log \theta(x)} \) to the both sides of the inequality, then
\[
\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log \theta(x)} > \frac{\log \epsilon}{\log \theta(x)} + \frac{\log x}{\log \theta(x)} = \frac{\log \epsilon + \log x}{\log \theta(x)} = \frac{\log \epsilon}{\log \theta(x)} + \frac{\log x}{\log \theta(x)} = 1.
\]
We know this inequality is satisfied when \( 0 < \epsilon \leq 1 \) since we would obtain that \( \frac{\log x}{\log \theta(x)} \geq 1 \). Therefore, the proof is done.

**Theorem 2.3.** The Riemann hypothesis is false.

**Proof.** If the Riemann hypothesis holds, then
\[
\theta(x) = x + O(\sqrt{x} \times \log^2 x)
\]
for all \( x \geq 10^8 \) due to the theorem 1.2. Now, suppose there is a real number \( x \geq 10^8 \) such that \( \theta(x) > x + \sqrt{x} \times \log^{1.9} x \). That would be equivalent to
\[
\log \theta(x) > \log(x + \sqrt{x} \times \log^{1.9} x)
\]
and so,
\[
\frac{1}{\log \theta(x)} < \frac{1}{\log(x + \sqrt{x} \times \log^{1.9} x)}
\]
for all numbers \( x \geq 10^8 \). Hence,
\[
\frac{\log x}{\log \theta(x)} < \frac{\log x}{\log(x + \sqrt{x} \times \log^{1.9} x)}.
\]
If the Riemann hypothesis holds, then
\[
\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + \sqrt{x} \times \log^{1.9} x)} > 1
\]
for those values of \( x \) that complies with
\[
\theta(x) > x + \sqrt{x} \times \log^{1.9} x
\]
due to the theorem 2.2. By contraposition, if there exists some number \( y \geq 10^8 \) such that for all \( x \geq y \) the inequality
\[
\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + \sqrt{x} \times \log^{1.9} x)} \leq 1
\]
is satisfied, then the Riemann hypothesis should be false. Let’s define the function

\[ \nu(x) = \frac{3 \log x + 5}{8 \pi \sqrt{x} + 1.2 \log x + 2} + \frac{\log x}{\log(x + \sqrt{x \log^1.9 x})} - 1. \]

The Riemann hypothesis would be false when there exists some number \( y \geq 10^8 \) such that for all \( x \geq y \) the inequality \( \nu(x) \leq 0 \) is always satisfied. We ignore when \( 2 \leq x \leq 10^8 \) since \( \theta(x) < x \) according to the theorem 1.3. We know that the function \( \nu(x) \) is monotonically decreasing for every number \( x \geq 10^8 \). The derivative of \( \nu(x) \) is negative for all \( x \geq 10^8 \). The function \( \nu(x) \) of a real variable \( x \) is monotonically decreasing in some interval if the derivative of \( \nu(x) \) is lesser than zero and the function \( \nu(x) \) is continuous over that interval [11]. It is enough to find a value of \( y \geq 10^8 \) such that \( \nu(y) \leq 0 \) for all \( x \geq y \) we would have that \( \nu(x) \leq \nu(y) \leq 0 \), because of \( \nu(x) \) is monotonically decreasing. We found the value \( y = 10^8 \) complies with \( \nu(y) \leq 0 \). In this way, we obtain that \( \nu(x) \leq 0 \) for every number \( x \geq 10^8 \). Consequently, under the assumption that the Riemann hypothesis is true, then

\[ \theta(x) < x + \sqrt{x \log^1.9 x} \]

for all \( x \geq 10^8 \). Hence, this implies that the Riemann hypothesis is false using the theorem 1.2. \( \square \)

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