A new class of austere submanifolds

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Abstract

Austere submanifolds of Euclidean space were introduced in 1982 by Harvey and Lawson in their foundational work on calibrated geometries. In general, the austerity condition is much stronger than minimality since it express that the nonzero eigenvalues of the shape operator of the submanifold appear in opposite pairs for any normal vector at any point. Thereafter, the challenging task of finding non-trivial explicit examples, other than minimal immersions of Kaehler manifolds, only turned out submanifolds of rank two, and these are of limited interest in the sense that in this special situation austerity is equivalent to minimality. In this paper, we present the first explicitly given family of austere non-Kaehler submanifolds of higher rank, and these are produced from holomorphic data by means of a Weierstrass type parametrization.

After the celebrated paper by Harvey and Lawson [8] on calibrated geometries the classification of austere Euclidean submanifolds became a rather challenging task in submanifold theory. An isometric immersion $f: M^n \to \mathbb{R}^N$ of a Riemannian manifold $M^n, n \geq 2$, into Euclidean space is called austere if the nonzero eigenvalues of the shape operator for any normal vector at any point appear in opposite pairs, or equivalently, if all odd degree symmetric polynomials on these eigenvalues vanish.

The notion of austerity was introduced by Harvey and Lawson [8] in connection with the class of special Lagrangian submanifolds in complex Euclidean space $\mathbb{C}^N$ that are not only minimal but absolutely area minimizing. Given an isometric immersion $f: M^n \to \mathbb{R}^N$, the embedding of its normal bundle $\psi: N_f M \to \mathbb{R}^N \oplus \mathbb{R}^N$ defined by

$$\psi(\xi(x)) = (f(x), \xi(x))$$

is a Lagrangian submanifold of $\mathbb{C}^N \equiv \mathbb{R}^N \oplus \mathbb{R}^N$ with respect to the complex structure $J(X,Y) = (-Y, X)$. Then $\psi$ is special Lagrangian if and only if $f$ is austere.

In the special case of a submanifold $M^n$ in $\mathbb{R}^N$ of rank $\rho = 2$, that is, when the kernel of the second fundamental form (called the relative nullity subspace) of the submanifold has constant dimension (called the index of relative nullity) $n - 2$, we have that austerity and minimality are equivalent. Notice that $\rho$ is the rank of the Gauss map with values in the Grassmannian $G_{n,N}$ of oriented subspaces. But for submanifolds for higher rank, the
austerity condition is much more demanding than minimality. This makes it rather hard to find examples of austere submanifolds other than the obvious examples of holomorphic isometric immersions of Kaehler manifolds into $\mathbb{C}^N$. In fact, we know from [6] that for an isometric immersion of a Kaehler manifold into $\mathbb{R}^N$ to be austere it suffices to be minimal, but these immersions are always the “real part” of a holomorphic one in $\mathbb{C}^N$.

The quest to construct new examples of austere submanifolds was initiated by Bryant [1] who classified the rank two submanifolds of dimension three as well as a quite simple family of examples of higher dimension called generalized helicoids. Bryant showed that the interesting examples of dimension three are “twisted cones” over minimal surfaces in spheres. As for dimension four, he provided a careful full pointwise description of the structures of all possible second fundamental forms. In a somehow dual parametric form, Bryant’s construction in the three dimensional case was extended by Dajczer and Florit [3] to submanifolds of rank two of any dimension. Roughly speaking, they showed that these submanifolds are subbundles of the normal bundles of a class of Euclidean or spherical surfaces called elliptic that, in addition, satisfy that the ellipses of curvature of a certain order are circles. But outside special cases, it is not known how to generate these surfaces. Finally, the four dimensional case was intensively studied by Ionel and Ivey [9], [10] building on Bryant’s algebraic results. In particular, they obtained a non-parametric classification in the special case of the submanifolds ruled by planes.

In this paper, we take advantage of our results in [7] in order to characterize in an explicit parametric form a class of austere submanifolds $M^n$ in $\mathbb{R}^{n+2}$ of dimension $n \geq 4$ and rank $\rho = 4$. Besides being the first non-trivial known examples, other than minimal Kaehler submanifolds, having any possible dimension and rank $\rho > 2$, what makes this new class of particular interest is that they are given in terms of a Weierstrass type parametrization depending on $n$ holomorphic functions on a domain. Consequently, the same is true for the special Lagrangian submanifolds that can be constructed from them as shown above.

Before stating our results, we first briefly recall some facts that can be seen exposed with many details in [7]. In fact, in the sequel we will make systematic use of results in that paper, sometimes without further reference.

A substantial minimal surface $g: L^2 \to \mathbb{R}^N$ is called $m$-isotropic, $m \geq 1$, if at any point of $L^2$ all ellipses of curvature (defined below) until order $m$ are circles. Being substantial means that the surface is not contained in any proper affine subspace of $\mathbb{R}^N$, in fact, not even locally since $g$ is real analytic. It is well-known that $g: L^2 \to \mathbb{C}^{N/2} \cong \mathbb{R}^N$ for $N$ even is a holomorphic curve if and only if the ellipses of curvature of any order at any point are circles; for instance see [2].

Any simply connected $m$-isotropic surface admits a Weierstrass type representation given in [5] based on results in [2]. In particular, any simply connected 2-isotropic surface is obtained as follows: Start with a nonzero holomorphic map $\alpha_0: U \to \mathbb{C}^{N-4}$
on a domain $U \subset \mathbb{C}$ and define $\alpha_1: U \to \mathbb{C}^{N-2}$ by

$$\alpha_1 = \beta_1 \left(1 - \phi_0^2, i(1 + \phi_0^2), 2 \phi_0\right)$$

where $\phi_0 = \int_U^z \alpha_0 dz$ and $\beta_1 \neq 0$ is any holomorphic function. Define $\alpha: U \to \mathbb{C}^N$ by

$$\alpha = \beta_2 \left(1 - \phi_1^2, i(1 + \phi_1^2), 2 \phi_1\right)$$

where $\phi_1 = \int_U^z \alpha_1 dz$ and $\beta_2 \neq 0$ is any holomorphic function. If $\phi = \int_U^z \alpha dz$ then $g = \text{Re} \phi$ is a 2-isotropic surface in $\mathbb{R}^N$.

It is easy to see that the above procedure yields examples of 2-isotropic surfaces with complete metrics. For instance, see the construction at the final part of [5].

Let $g: L^2 \to \mathbb{R}^{n+2}$ be a 1-isotropic oriented surface. Then let $\Lambda_g \subset N_g L$ be the vector subbundle of the normal bundle of $g$ with $(n-2)$-dimensional fibers

$$\Lambda_g(u, v) = (\text{span}\{g_u, g_v, g_{uu}, g_{uv}\})^\perp$$

where $g = g(u, v)$ is parametrized in local isothermal coordinates. If $g = \text{Re} \phi$ is as above, then

$$\Lambda_g = \alpha \wedge \alpha_z.$$

It was shown in [7] that the dimension of $\Lambda_g(u, v)$ may fail to be $n-2$ only at isolated points and that the vector bundle extends smoothly to these points. Hence, from now on $\pi: \Lambda_g \to L^2$ denotes the extended vector bundle.

Let $F_g: \Lambda_g \to \mathbb{R}^{n+2}$ be the immersion associated to $g$ defined on $\pi: \Lambda_g \to L^2$ by

$$F_g(p, \xi) = g(p) + \xi, \quad p = \pi(\xi). \quad (1)$$

In the sequel, we denote by $M^n$ the manifold $\Lambda_g$ when endowed with the metric induced by $F_g$ and by $j: L^2 \to M^n$ the immersion in $M^n$ of the zero-section of $\Lambda_g$. We have by construction that $F_g$ is a $(n-2)$-ruled submanifold, and it is easily seen that $j(L)$ is a totally geodesic cross section that is orthogonal to the rulings.

**Theorem 1.** Let $g: L^2 \to \mathbb{R}^{n+2}, \ n \geq 4$, be a 2-isotropic substantial surface. Then the associated immersion $F_g: M^n \to \mathbb{R}^{n+2}$ is an austere $(n-2)$-ruled submanifold with complete rulings that has rank $\rho = 4$ on an open dense subset of $M^n$. Moreover, the surface $j: L^2 \to M^n$ is the unique totally geodesic cross section that is orthogonal to the rulings. Furthermore, the metric of $M^n$ is complete if and only if $L^2$ is complete.

Conversely, let $F: M^n \to \mathbb{R}^{n+2}, \ n \geq 4$, be an austere $(n-2)$-ruled isometric immersion that has rank $\rho = 4$ on an open dense subset of $M^n$. If there exists a totally geodesic global cross section $j: L^2 \to M^n$ orthogonal to the rulings, then the surface $g = F \circ j: L^2 \to \mathbb{R}^{n+2}$ is 2-isotropic and $F$ can be parametrized by $F_g$. 

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Assume that $L^2$ is simply connected. By Theorem 4 in [7] there is a one-parameter family of minimal isometric immersions $F_\theta$ for $\theta \in [0, \pi]$ with $F_0 = F_g$ such that each $F_\theta$ is also austere carrying the same rulings and relative nullity subspaces as $F_g$. Consequently, we have the isometric immersions in higher codimension

$$G = (\cos \varphi F_0, \sin \varphi F_{\pi/2}): M^n \to \mathbb{R}^{n+2} \oplus \mathbb{R}^{n+2} \equiv \mathbb{R}^{2n+4}, \ \varphi \in [0, \pi],$$

that are also austere with the same rulings and relative nullity subspaces.

The following result analyzes when the submanifold $M^n$ above is Kaehler, which turns out to be always the case for $n = 4$. On the other hand, we see that the property of being Kaehler is exceptional for higher even dimensions.

**Theorem 2.** Let $F_g: M^n \to \mathbb{R}^{n+2}$, $n \geq 4$, be the austere $(n - 2)$-ruled submanifold associated to a 2-isotropic substantial surface $g: L^2 \to \mathbb{R}^{n+2}$. Then $M^n$ is Kaehler if and only if $g$ is holomorphic. In addition $F_g$ in the Kaehler case is never holomorphic.

If $M^n$ above is Kaehler and simply-connected, being $F_g$ not holomorphic it follows from a result in [4] that $F_g$ admits an non-trivial associated one-parameter family of isometric minimal immersions. It can be shown that this family coincides with the one discussed after Theorem 1.

1 The proofs

Let $g: L^2 \to \mathbb{R}^{n+2}$, $n \geq 4$, be a substantial 1-isotropic isometric immersion. Hence the surface is minimal and the first ellipse of curvature is a circle at all points. The minimality condition yields that the normal bundle of $g$ splits along an open dense subset of $L^2$ as the orthogonal sum

$$N_g L = N_1^g \oplus N_2^g \oplus \cdots \oplus N_m^g, \ m = [(n + 1)/2],$$

of the higher normal bundles and these have rank two except possible the last one that has rank one if $n$ is odd. Given an orthonormal tangent frame $\{e_1, e_2\}$ we have

$$N_k^g(p) = \text{span} \\{\alpha_{g}^{k+1}(e_1, \ldots, e_1, e_1)(p), \alpha_{g}^{k+1}(e_1, \ldots, e_1, e_2)(p)\}$$

at $p \in L^2$. Here $\alpha_g^2 = \alpha_g: TL \times TL \to N_g L$ is the second fundamental form of $g$ and $\alpha_g^s: TL \times \cdots \times TL \to N_g L$, $s \geq 3$, is the $s^{th}$-fundamental form defined inductively by

$$\alpha_g^s(X_1, \ldots, X_s) = (\nabla_{X_s} \ldots \nabla_{X_3} \alpha_g(X_2, X_1))^\perp$$

where $(\ )^\perp$ denotes the projection onto the normal complement of $N_1^g \oplus \cdots \oplus N_{s-2}^g$. 4
The $k^{th}$-order ellipse of curvature $E_k^g(p) \subset N_k^g(p)$ at $p \in L^2$ is

$$E_k^g(p) = \{ \alpha_{g}^{k+1}(e(\theta), \ldots, e(\theta))(p) : e(\theta) = \cos \theta e_1 + \sin \theta e_2 \text{ and } \theta \in [0, 2\pi) \}.$$ 

Then $E_k^g(p)$ is indeed an ellipse and is a circle if and only if the vectors

$$\alpha_{g}^{k+1}(e_1, \ldots, e_1)(p), \ \alpha_{g}^{k+1}(e_1, \ldots, e_2)(p)$$

are orthogonal with equal norm.

**Proof of Theorem** The minimal submanifold $F_g : M^n \rightarrow \mathbb{R}^{n+1}$ parametrized by (1) is $(n - 2)$-ruled of rank four. Its tangent bundle splits orthogonally as

$$TM = \mathcal{H} \oplus \mathcal{V}$$

where $\mathcal{H}$ is the tangent distribution orthogonal to the rulings and $\mathcal{V} = \ker \pi_*$ is the vertical bundle of the submersion $\pi$. Then $j_\ast T_p L = \mathcal{H}(j(p))$ at every point $p \in L^2$ and the fibers of $\mathcal{V}$ form the distribution tangent to the rulings. We also have the orthogonal splitting

$$\mathcal{V} = \mathcal{V}^1 \oplus \mathcal{V}^0$$

where $\mathcal{V}^1$ is identified with the fibers of $N_2^g$ and $\mathcal{V}^0$ is identified with $N_3^g \oplus \cdots \oplus N_m^g$ and are the relative nullity subspaces of $F_g$.

Let $e_1, \ldots, e_{n+2}$ be an orthonormal frame such that $N_r^g = \text{span}\{e_{2r+1}, e_{2r+2}\}$. Denote

$$\omega_{ij}^k = \langle \nabla_{e_i} e_j, e_k \rangle, \quad 1 \leq k \leq 2, \quad 1 \leq i, j \leq n + 2$$

where $\nabla$ stands for the connection in $\mathbb{R}^{n+2}$. By assumption

$$\alpha_{11}^2 = \alpha_{g}^2(e_1, e_1) = \kappa_1 e_3 \text{ and } \alpha_{12}^2 = \alpha_{g}^2(e_1, e_2) = \kappa_1 e_4.$$ 

Thus

$$\alpha_{111}^3 = (\nabla_{e_1}^1 \alpha_{11}^2)_{N_2^g} = \kappa_1 (\nabla_{e_1}^1 e_3)_{N_2^g} = \kappa_1 (\omega_{35}^1 e_5 + \omega_{36}^1 e_6) = \kappa_1 (a_1 e_5 + b_1 e_6)$$

$$\alpha_{112}^3 = (\nabla_{e_1}^1 \alpha_{12}^2)_{N_2^g} = \kappa_1 (\nabla_{e_1}^1 e_4)_{N_2^g} = \kappa_1 (\omega_{45}^1 e_5 + \omega_{46}^1 e_6) = \kappa_1 (a_2 e_5 + b_2 e_6). \quad (3)$$

As shown in [7] there is an orthonormal tangent frame $E_i, \ 1 \leq i \leq n$, such that

$$\mathcal{H} = \text{span}\{E_1, E_2\}, \ \mathcal{V}^1 = \text{span}\{E_3, E_4\} \text{ and } \mathcal{V}^0 = \text{span}\{E_5, \ldots, E_n\} \quad (4)$$

where $E_3, E_4$ are taken constant in each ruling and $F_* E_j = e_{j+2}, \ 3 \leq j \leq n$. Then the submanifold can be parametrized as

$$F_g = g + \sum_{j=1}^{n-2} t_j E_{j+2}.$$
where \(t_1, \ldots, t_{n-2} \in \mathbb{R}\). Moreover, there is an orthogonal normal frame \(\xi, \eta\) satisfying \(\|\xi\| = \Omega = \|\eta\|\) with \(\Omega \in C^\infty(M)\) such that the shape operators of \(F_g\) vanish on \(\mathcal{V}_0\) and restricted to \(\mathcal{H} \oplus \mathcal{V}_1\) have the form

\[
A_\xi = \begin{bmatrix}
k_1 + h_1 & h_2 & r_1 & s_1 \\
h_2 & -k_1 - h_1 & r_2 & s_2 \\
r_1 & r_2 & 0 & 0 \\
s_1 & s_2 & 0 & 0
\end{bmatrix}, \quad A_\eta = \begin{bmatrix}
h_2 & k_1 - h_1 & r_2 & s_2 \\
k_1 - h_1 & -h_2 & -r_1 & -s_1 \\
r_2 & -r_1 & 0 & 0 \\
s_2 & -s_1 & 0 & 0
\end{bmatrix}.
\] (5)

Moreover, \(r_j = -a_j/\Omega, s_j = -b_j/\Omega, j = 1, 2\), with \(k_1, a_1, a_2, b_1, b_1 \in C^\infty(L)\) whereas

\[
h_j = \frac{1}{\Omega^2} (t_1D^j_1 + \cdots + t_4D^j_4), \quad j = 1, 2,
\]

where \(D^j_i \in C^\infty(L)\), \(1 \leq i \leq 4\), and \(t_1, \ldots, t_4 \in \mathbb{R}\) are independent parameters.

We obtain from (5) that if \(F_g\) is austere then the coefficients of the terms of third order of the characteristic polynomials of both shape operators have to vanish. From this it turns out that austerity implies that

\[
2(r_1r_2 + s_1s_2)h_2 + (r_1^2 + s_1^2 - r_2^2 - s_2^2)(k_1 + h_1) = 0
\]

and

\[
2(r_1r_2 + s_1s_2)(h_1 - k_1) - (r_1^2 + s_1^2 - r_2^2 - s_2^2)h_2 = 0.
\]

It follows that austerity yields

\[
r_1r_2 + s_1s_2 = 0 \quad \text{and} \quad r_1^2 + s_1^2 = r_2^2 + s_2^2
\] (6)

that is equivalent to

\[
a_1a_2 + b_1b_2 = 0 \quad \text{and} \quad a_1^2 + b_1^2 = a_2^2 + b_2^2.
\] (7)

Using (3) it follows that

\[
\langle \alpha_{111}^3, \alpha_{112}^3 \rangle = k_1^2(a_1a_2 + b_1b_2) = 0
\] (8)

and

\[
\|\alpha_{111}^3\|^2 = k_1^2(a_1^2 + b_1^2) = k_1^2(a_2^2 + b_2^2) = \|\alpha_{112}^3\|^2,
\] (9)

hence \(g\) is 2-isotropic.

To prove the converse, we have to verify that if (6) holds then the coefficient of the term of third order of the characteristic polynomial of \(A_{\cos \varphi \xi + \sin \varphi \eta}\) vanishes for any \(\varphi \in [0, 2\pi]\). In this case, since (5) and (9) hold we can choose \(e_5\) and \(e_6\) collinear with \(\alpha_{111}^3\) and \(\alpha_{112}^3\), respectively, and the remaining of the proof is just a long but straightforward computation. \(\blacksquare\)
In the sequel, we will be dealing with the case when $M^n$ is a Kaehler manifold.

Let $F = F_g: M^n \to \mathbb{R}^{n+2}$, $n = 2m \geq 4$, be a minimal $(n-2)$-ruled submanifold associated to a 1-isotropic oriented surface $g: L^2 \to \mathbb{R}^{n+2}$. The orientation of $L^2$ induces an orientation on each plane vector bundle $N^g_k$ in (2) given by the ordered pair

$$\alpha_g^{k+1}(e_1, \ldots, e_1, e_1), \alpha_g^{k+1}(e_1, \ldots, e_1, e_2)$$

where $\{e_1, e_2\}$ is a positively oriented tangent frame. Then let the orthonormal frame $e_1, \ldots, e_{n+2}$ be such that the pairs $e_{2r+1}, e_{2r+2}$ spanning $N^g_k$ are positively oriented. Now define $T: TM \to TM$ with respect to the orthonormal frame $E_1, \ldots, E_n$ as in (4) by

$$T|_{H \oplus V} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and $T|_{V^0} = I$. Thus $T$ leaves invariant the distributions tangent to the rulings.

**Lemma 3.** The following facts are equivalent:

(i) $\alpha_T(TX, Y) = \alpha_T(X, TY)$ for all $X, Y \in TM$.

(ii) $g$ is 2-isotropic.

*Proof:* We have that (i) is equivalent to

$$A_\xi \circ T = -T \circ A_\xi \text{ and } A_\eta \circ T = -T \circ A_\eta.$$ 

It is straightforward to verify that the above is equivalent to

$$a_1 = b_2, \ a_2 = -b_1. \quad (10)$$

Thus (7) holds and $g$ is 2-isotropic. Conversely, if $g$ is 2-isotropic then the pair of orthogonal vectors with the same norm $\alpha_{111}^3, \alpha_{112}^3$ is positively oriented. Hence, we can take $\alpha_{111}^3 = \kappa e_5$ and $\alpha_{112}^3 = \kappa e_6$ and (10) holds.

*Proof of Theorem 2.* Assume that $g$ is holomorphic. Then

$$\alpha_g^{s+1}(e_1, \ldots, e_1) = \kappa_s e_{2s+1} \quad \text{and} \quad \alpha_g^{s+1}(e_1, \ldots, e_1, e_2) = \kappa_s e_{2s+2}, \ 1 \leq s \leq n/2.$$ 

Moreover, from (7) the connection forms $\omega_{\alpha, \beta} = \langle \nabla e_\alpha, e_\beta \rangle$ satisfy

$$\omega_{2s-1, 2s+1} = \omega_{2s, 2s+2} = \tau_s \omega_1,$$ 

and

$$\omega_{2s-1, 2s+2} = -\omega_{2s, 2s+1} = \tau_s \omega_2 \quad (12)$$

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where $\omega_1, \omega_2$ are dual to $e_1, e_2$, respectively, and $\tau_s = \kappa_s / \kappa_{s-1}$ with $\kappa_0 = 1, 1 \leq s \leq n/2$.

Let $E_1, \ldots, E_n$ be an orthonormal frame as in (4). We have to show that the almost complex structure $J$ defined as $J|_{H \oplus V^1} = T|_{H \oplus V^1}$ and $J E_{2i+1} = E_{2i+2}$, $J E_{2i+2} = -E_{2i+1}$, $i \geq 2$, is parallel. That is,

$$\langle \nabla_{E_k} E_i, E_j \rangle = \langle \nabla_{E_k} J E_i, J E_j \rangle,$$

or equivalently,

$$\langle \nabla_{E_k} F_s E_i, F_s E_j \rangle = \langle \nabla_{E_k} F_s J E_i, F_s J E_j \rangle, \quad k = 1, 2 \text{ and } 1 \leq i, j \leq n.$$

Since $g$ holomorphic, we have from [7] that

$$F_s E_1 = \frac{1}{\Omega}(g_* e_1 - \tau_2 (t_1 e_3 + t_2 e_4)), \quad F_s E_2 = \frac{1}{\Omega}(g_* e_2 - \tau_2 (t_2 e_3 - t_1 e_4)).$$

We only argue for nontrivial cases:

Let $i = 1$ and $j = 2s + 1$, $s \geq 2$. Then

$$\langle \nabla_{E_k} F_s E_1, F_s E_{2s+1} \rangle = -\langle \nabla_{E_k} F_s E_2, F_s E_{2s+2} \rangle \iff \langle g_* e_1 - \tau_2 (t_1 e_3 + t_2 e_4), \nabla_{E_k} e_{2s+3} \rangle = -\langle g_* e_2 - \tau_2 (t_2 e_3 - t_1 e_4), \nabla_{E_k} e_{2s+4} \rangle.$$

$$\iff t_1 \omega_{3,2s+3} + t_2 \omega_{4,2s+3} = -t_2 \omega_{3,2s+4} + t_1 \omega_{4,2s+4}.$$  

The last equality holds trivially for $s \geq 2$ and by (11) and (12) for $s = 1$. The proof for the cases $i = 1, 2$ and $j = 2s + 1, 2s + 2$, $s \geq 1$, is similar.

Let $i = 2s + 1$ and $j = 2r + 1$ with $r \neq s$. Then

$$\langle \nabla_{E_k} F_s E_{2s+1}, F_s E_{2r+1} \rangle = \langle \nabla_{E_k} F_s E_{2s+2}, F_s E_{2r+2} \rangle \iff \omega_{2s+3,2r+3} = \omega_{2s+4,2r+4}$$

where the last equality either holds trivially or follows from (11). The proof for the remaining cases is similar.

Now let us assume that $M^n$ is Kaehler. Being $F_g$ is austere we have that $g$ is 2-isotropic. Being $F_g$ minimal we have

$$\alpha_F(JX,Y) = \alpha_F(X,JY)$$

for any $X,Y \in TM$. It follows easily that the three subspaces in the decomposition $TM = \mathcal{H} \oplus \mathcal{V}^1 \oplus \mathcal{V}^0$ are $J$-invariant. Hence

$$J|_{\mathcal{H} \oplus \mathcal{V}^1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon & 0 \\ 0 & 0 & \epsilon & 0 \end{bmatrix}$$
where $\epsilon = \pm 1$.

Since $g$ is 2-isotropic we have from [7] that

\[
A_{\xi} = \begin{bmatrix}
\kappa_1 + h_1 & h_2 & r & 0 \\
h_2 & -\kappa_1 - h_1 & 0 & r \\
r & 0 & 0 & 0 \\
0 & r & 0 & 0
\end{bmatrix}, \quad A_{\eta} = \begin{bmatrix}
h_2 & \kappa_1 - h_1 & 0 & r \\
\kappa_1 - h_1 & -h_2 & -r & 0 \\
0 & -r & 0 & 0 \\
r & 0 & 0 & 0
\end{bmatrix}
\]
on $H \oplus V_1$. That $A_{\xi}J + JA_{\xi} = 0$ and $A_{\eta}J + JA_{\eta} = 0$ hold is equivalent to $\epsilon = 1$.

We define an isometry $J^\perp : N_g L \to N_g L$ by

\[
J^\perp e_3 = -e_4, \quad J^\perp e_4 = e_3 \quad \text{and} \quad J^\perp e_{j+2} = F_*JE_j, \quad j \geq 3.
\]

Then $J^\perp$ is an almost complex structure since

\[
J^\perp e_5 = F_*JE_3 = F_*E_4 = e_6, \quad J^\perp e_6 = F_*JE_4 = -F_*E_3 = -e_5
\]

and

\[
(J^\perp)^2 e_{j+2} = F_*J^2E_j = -F_*E_j = -e_{j+2}.
\]

We claim that $J^\perp$ is parallel with respect to the normal connection of $g$. Since $J$ is parallel, we have

\[
\langle \nabla_X e_i, JE_j \rangle = -\langle \nabla_X JE_i, E_j \rangle
\]

which is equivalent to

\[
\langle \bar{\nabla}_X F_* E_i, F_* JE_j \rangle = -\langle \bar{\nabla}_X F_* JE_i, F_* E_j \rangle
\]

for any $X \in TM$.

If $i, j \geq 3$, we have

\[
\langle \nabla^\perp_X e_{i+2}, J^\perp e_{j+2} \rangle = -\langle \nabla^\perp_X J^\perp e_{i+2}, e_{j+2} \rangle
\]

which gives

\[
((\nabla^\perp_X J^\perp)e_{i+2})_{N_1^g} = 0, \quad i \geq 3. \quad (13)
\]

If $i \geq 3$ we have

\[
\langle (\nabla^\perp_X J^\perp)e_{i+2}, e_3 \rangle = -\langle J^\perp e_{i+2}, \nabla^\perp_X e_3 \rangle + \langle e_{i+2}, \nabla^\perp_X e_4 \rangle.
\]

Since $\nabla^\perp_X e_3, \nabla^\perp_X e_4 \in N_1^g \oplus N_2^g$, we obtain

\[
\langle (\nabla^\perp_X J^\perp)e_{i+2}, e_3 \rangle = 0, \quad i \geq 5, \quad (14)
\]

and

\[
\langle (\nabla^\perp_X J^\perp)e_5, e_3 \rangle = \omega_{36}(X) + \omega_{45}(X) = 0. \quad (15)
\]
Similarly, we obtain
\[
\langle (\nabla^\perp_X J^\perp)e_6, e_3 \rangle = 0. \tag{16}
\]

If follows from (14), (15) and (16) that
\[
\langle (\nabla^\perp_X J^\perp)e_{i+2}, e_3 \rangle = 0, \ i \geq 3. \tag{17}
\]
and in the same way that
\[
\langle (\nabla^\perp_X J^\perp)e_{i+2}, e_4 \rangle = 0, \ i \geq 3. \tag{18}
\]

It follows from (13), (17) and (18) that
\[
(\nabla^\perp_X J^\perp)e_{i+2} = 0, \ i \geq 3.
\]
The same type of arguments yield
\[
(\nabla^\perp_X J^\perp)e_i = 0, \ i = 3, 4,
\]
and this proves the claim.

We have
\[
J^\perp \alpha(e_1, e_1) = -\kappa_1 e_4 = -\alpha_1 = -\alpha(Je_1, e_1)
\]
and
\[
J^\perp \alpha(e_1, e_2) = \kappa_1 e_3 = \alpha_1 = -\alpha(Je_2, e_1).
\]
Hence
\[
J^\perp \alpha(X, Y) = -\alpha(JX, Y).
\]
Let \(\tilde{J}: g^*T\mathbb{R}^{n+2} \to g^*T\mathbb{R}^{n+2}\) be defined as
\[
\tilde{J}|_{g^*TL} = g_\ast \circ J \text{ and } \tilde{J}|_{N_gL} = -J^\perp.
\]

It is now straightforward to verify that \(\tilde{\nabla}\tilde{J} = 0\), that is, that \(\tilde{J}\) is a complex structure in \(\mathbb{R}^{n+2}\) that satisfies \(\tilde{J} \circ g_* = g_\ast \circ J\), hence \(g\) is holomorphic.

For the last statement, observe that if we had that \(F_g\) is holomorphic then we would have in (5) that \(A_\eta = \pm J \circ A_\xi\), and it is easy to verify that this cannot be the case.

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