Feedback Capacity Formulas of AGN Channels Driven by Nonstationary Autoregressive Moving Average Noise

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Abstract—In this paper we derive closed-form formulas of feedback capacity and nonfeedback achievable rates, for Additive Gaussian Noise (AGN) channels driven by nonstationary autoregressive moving average (ARMA) noise (with unstable one poles and zeros), based on time-invariant feedback codes and channel input distributions. From the analysis and simulations follows the surprising observations, (i) the use of time-invariant channel input distributions gives rise to multiple regimes of capacity that depend on the parameters of the ARMA noise, which may or may not use feedback, (ii) the more unstable the pole (resp. zero) of the ARMA noise the higher (resp. lower) the feedback capacity, (iii) certain conditions, known as detectability and stabilizability are necessary and sufficient to ensure the feedback capacity formulas and nonfeedback achievable rates are independent of the initial state of the ARMA noise. Another surprising observation is that Kim’s characterization of feedback capacity which is developed for Gaussian Noise random variables (RVs), respectively.

I. INTRODUCTION

The AGN channel is defined by

\[ Y_t = X_t + V_t, \quad t = 1, \ldots, n, \quad \frac{1}{n} \mathbb{E}\left\{ \sum_{t=1}^{n} (X_t)^2 \right\} \leq \kappa \]

where \( \kappa \) is the total power of the transmitter, \( X^n = \{X_1, X_2, \ldots, X_n\}, Y^n = \{Y_1, Y_2, \ldots, Y_n\} \) and \( V^n = \{V_1, \ldots, V_n\} \), are the sequences of channel input, channel output, and Gaussian noise random variables (RVs), respectively.

The feedback and nonfeedback capacity of the AGN channel, when the noise \( V^n \) is stable, stationary, or asymptotically stationary, can be considered to have been explained sufficiently in information theory \[3\], \[4\]. The most general is the Cover and Pombra formulation and coding theorems \[4\] Theorem 1], for the set of uniformly distributed messages \( W : \Omega \rightarrow \mathcal{M}^{(n)} \triangleq \{1, 2, \ldots, |\mathcal{M}_{R_n}|\} \), codewords of block length \( n, X_1 = e_1(W), \ldots, X_n = e_n(W, X^{n-1}, Y^{n-1}) \), decoder functions, \( y^n \mapsto d_n(y^n) \in \mathcal{M}^{(n)} \), with average error probability

\[ P^{(n)}_{error} = \frac{1}{|\mathcal{M}_{R_n}|} \sum_{w=1}^{|\mathcal{M}_{R_n}|} P\left(d_n(y^n) \neq w\right) | W = w \]

The objective of this paper is twofold.

1) To show that feedback and nonfeedback capacity formulas and achievable rates, may behave very different, depending on the definitions of achievable rates, in particular, whether conditions are imposed to ensure these rates are insensitive to initial states or distributions of the channel, i.e., of \( V^n \).

2) To show, the surprising result that, the consideration of an unstable noise \( V^n \) alters the mathematical formulas of feedback and nonfeedback capacity formulas and achievable rates, and that noises with unstable poles give significant gains of achievable rates, at no extra expense of power.

To keep the analysis simple, we consider the unstable and stable, autoregressive moving average, unit memory noise, denoted by ARMA(a,c), \( a \in (-\infty, \infty), c \in (-\infty, \infty), c \neq a \), as defined below. Versions of stable or marginally stable, ARMA(a,c), \( a \in [-1,1], c \in [-1,1] \) noise are considered since the early 1970’s, in \[1\], \[4\]–\[9\], where the reader may find bounds on achievable rates of feedback and nonfeedback codes, under various assumptions and formulations.

ARMA(a,c): \( V_t = cV_{t-1} + W_t - aW_{t-1}, \quad t = 1, \ldots, n \), (1.3)

\( W_t \in \mathcal{N}(0, K_W), K_W > 0, t = 1, \ldots, n, \) mutually indep, (1.4)

\( \{W_1, \ldots, W_n\} \) indep. of initial state \( S \triangleq \{V_0, W_0\} \), (1.5)

\( V_0 \in \mathcal{N}(0, K_{V_0}), W_0 \in \mathcal{N}(0, K_{W_0}), K_{V_0} \geq 0, \quad K_{W_0} \geq 0 \), (1.6)

\( a \in (-\infty, \infty), \quad c \in (-\infty, \infty), \quad c \neq a \), (1.7)

where the notation, \( Z \in \mathcal{N}(0, K_Z) \), means \( Z \) is a Gaussian RV, with zero mean, and variance \( K_Z \). The ARMA(a,c) noise is equivalently expressed in state form, with state \( S_t \) as,

\[ S_t \triangleq cV_{t-1} - aW_{t-1} - c - a, \quad t = 1, \ldots, n \], (1.8)

\( S_{t+1} = aS_t + W_t, \quad S_1 = s, \quad t = 1, \ldots, n \), (1.9)

\( V_t = (c-a)S_t + W_t, \quad t = 1, \ldots, n \), (1.10)

From the ARMA(a,c) noise, follow the two special cases,

Autoregressive: \( AR(c) | a=0 \): \( V_t = cV_{t-1} + W_t \) (1.11)

Moving Average: \( MA(a) | c=0 \): \( V_t = W_t - aW_{t-1} \) (1.12)

A. Literature Review

Due to the relevance to our investigation, of prior formulas found in \[1\], \[3\]–\[9\], we briefly discuss some of these below, with emphasis on the formulations and assumptions.

3) Formulation and Bounds with Initial State \[5\]–\[7\]. Wol- fowizit \[6\] and Butman \[7\], derived a lower bound on feedback capacity of the AGN channel, driven by the noise,
exists), where is known to the encoder and decoder. The lower bound is 

\[ \Theta (c) \]

\[ C^{\text{LB}} = \frac{1}{2} \log \chi^2, \quad \chi = \text{the positive root of} \]

\[ \chi^4 - \chi^2 - \frac{\kappa}{K_{\text{W}}} (\chi + |c|)^2 = 0, \quad |c| \leq 1, \quad K_{\text{W}} > 0. \]  

Butman conjectured that \( C^{\text{LB}} \) is the feedback capacity. 

4.1) Cover and Pombra [3] characterized the feedback (and nonfeedback) capacity of the AGN channel, driven by a nonstationary Gaussian noise \( V^n \). [3] Theorem 1], for codes that do not assume knowledge of the initial state of the noise. The feedback capacity is 

\[ C \triangleq \lim_{n \to \infty} \frac{1}{n} C_n \]  

(provided the limit exists), where \( C_n \) is the \( n \)-finite block length or transmission feedback information \( (n-\text{FTFI}) \) capacity, 

\[ C_n \triangleq \sup \left\{ \sum_{n=1}^{\infty} H(Y^n) - H(X^n) \left| \frac{1}{n} \mathbb{E} \left\{ \sum_{i=1}^{n-1} (X_i) \right\} \right| \leq \kappa \right\} \]

\[ X_n = \sum_{j=1}^{n-1} \Gamma_{n,j} V + Z_n, \quad n = 1, 2, \ldots, \quad X_1 = Z_1, \]  

and the supremum is over nonrandom \( \Gamma_{n,j}, j = 1, \ldots, n-1 \) and jointly correlated, Gaussian RVs \( \{Z_1, \ldots, Z_n\} \), independent of \( V^n \); \( H(X) \) is the differential entropy of RV \( X \).

4.2) Yang, Kavcic and Tatikonda [9] analyzed the feedback capacity of the AGN channel driven by a noise \( V^n \), with state \( S^n \), under the following assumption.

Assumption (YKT [9 page 933, I-III]): given the initial state of the noise \( S_1 = s, \) which is known to the encoder and the decoder; the channel input \( X^n \triangleq \{X_1, \ldots, X_n\} \) uniquely defines the state variables \( S^n \) and vice-versa.

[9] Theorem 7], computed the feedback rate of the AGN channel driven by ARMA\((a, c), a \in (-1, 1), c \in (-1, 1)\) noise, using the definition,

\[ C^{\text{YKT}} \triangleq \sup_{(\Lambda, K_{\text{Z}})} \left\{ \frac{1}{n} \mathbb{E} \left\{ \sum_{i=1}^{n-1} (X_i) \right\} | s_i = s \right\} \leq \kappa \]

\[ X_n = \Lambda \left( S_n - \mathbb{E} \left( S_n | V^n, S_1 = s \right) \right) + Z_n, \quad n = 2, \ldots, \]

\[ X_1 = Z_1, \quad Z_n \in N(0, K_{\text{Z}}), \quad K_{\text{Z}} \geq 0, \quad n = 1, 2, \ldots \]  

where the RVs \( \{Z_1, Z_2, \ldots, Z_n\} \) are mutually independent. The limiting problem in the steady state, is solved, and a formula of \( C^{\text{YKT}} \) is obtained. For the AR\((c), c \in (-1, 1)\) noise, \( C^{\text{YKT}} = \frac{1}{2} \log |A^*|^2 \), where \( A^* \) satisfies Butman’s equation \( (1.14) \), i.e., \( C^{\text{YKT}} = C^{\text{LB}} \) of Butman (see [9 Corollary 7.1]).

Remark 1.1. In both [9 Theorem 7, Corollary 7.1], \( C^{\text{YKT}} \), are achieved by \( \Lambda = A^* \) and \( K_{\text{Z}} = K_{\text{Z}}^* = 0 \).

4.3) Kim [17] re-visited the AGN channel driven by a stable noise, and derived characterizations of feedback capacity, in frequency domain [1, Theorem 4.1], with zero power spectral density of the innovations part of the channel input \( X_n, n = 1, 2, \ldots \), and in time-domain [1, Theorem 6.1], with zero variance of the innovations part, \( Z_n \) of the input \( X_n, n = 1, 2, \ldots \).

\[ X_n = \Lambda \left( S_n - \mathbb{E} \left( S_n | V^{n-1}, S_1 = s \right) \right) + Z_n, \quad n = 2, \ldots, \]

\[ X_1 = Z_1, \quad Z_n \in N(0, K_{\text{Z}}), \quad K_{\text{Z}} \geq 0, \quad n = 1, 2, \ldots \]  

For the AR\((c), c \in (-1, 1)\) noise, the optimal \( \Lambda = A^* \) satisfies Butman’s equation \( (1.14) \), and \( C_{\text{FB}} = \frac{1}{2} \log |A^*|^2 \), i.e., identical to Butman’s lower bound.

B. Main Problem of the Paper: Brief Discussion of Results and Comparisons

As we briefly demonstrate via simulations of our feedback capacity expressions, which we derived in Section II that even for stable ARMA\((a, c), a \in (-1, 1), c \in (-1, 1)\) noise, we arrive at completely different formulas, compared to those in [11, 9]. We show these differences are attributed to our feedback capacity problem definition, stated as Problem I.1. In particular, the limiting expression of feedback rate, \( (1.22) \), is independent of the initial state of the noise, i.e., \( C^n(\kappa, s) = C^n(\kappa), \forall s \), when compared to \( (1.17) \).

Problem I.1. Consider the AGN channel driven by the ARMA\((a, c)\) noise, with \( c \in (-\infty, \infty), \quad a \in (-\infty, \infty), \) and with
initial state $S_1 = s$, known to the encoder and the decoder. Define the feedback rate

$$C^w(\kappa, s) \triangleq \sup_{\lim_{n \to \infty} - \frac{1}{n} \sum_{t=1}^{n} H(Y_t^t|s) \leq \kappa} - H(V^n|s)$$

(I.22)

where the supremum is taken over all jointly Gaussian channel input process $X_n, n = 1, 2, \ldots$, generated by time-invariant feedback strategies, and induce distributions $P_{X_t^n|X_0, Y_0^n, S_1} = P_{X_t^n|X_0, Y_0^n, S_1} = P_{X_t^n|X_0, Y_0^n, S_1}$, such that the joint process $(X_n, Y_n), n = 1, 2, \ldots$, is jointly Gaussian, for $S_1 = s$.

We address the following questions.

(a) What are necessary and/or sufficient conditions for

(i) asymptotic stationarity of the process $(X^n, Y^n), n = 1, 2, \ldots$, or the marginal process $X^n$, that achieve $C^w(\kappa, s)$, and

(ii) $C^w(\kappa, s) = C^w(\kappa) \forall s$, i.e., independent of initial data?

(b) What are the closed form formulas of feedback capacity $C^w(\kappa, s) = C^w(\kappa) \forall s$?

Problem (11)(a),(i),(ii), captures the requirement that $C^w(\kappa, s)$ is well-defined, for unstable ARMA$(a,c), c \in (-\infty, \infty), a \in (-\infty, \infty)$ noise (as well as stable, $c \in (-1, 1), a \in (-1, 1)$), and $C^w(\kappa, s) = C^w(\kappa) \forall s$.

In the rest of the paper, we show there are multiple regimes of $C^w(\kappa)$, which depend on the parameters $(a,c, \kappa)$. At some regimes, feedback does not increase the capacity. This is attributed to the use of time-invariant channel input strategies. For these regimes we derive achievable nonfeedback rates $C^w_{LB, FB}(\kappa)$, based on a simple IID channel input $X_n = Z_n, Z_n \in \{0,1\}, n = 1, 2, \ldots$.

Fig. 11 compares our feedback capacity and achievable nonfeedback rate to Butman’s lower bound and Kim’s feedback capacity [1] Theorem 6.1, $C_{FB}$, for the AR(c), stable and unstable noise. For unstable noise, even an IID channel input outperforms the feedback rate $C_{FB} = C_{VKT} < C_{LB}$.

The verification of our objectives described in Section 1 under 1) and 2), are demonstrated in Figures 12, 13. Figure 12 shows that feedback capacity $C^w(\kappa)$ over the appropriate region, is an increasing function of the parameter $c$, i.e., the higher unstable pole, the higher the value of $C^w(\kappa)$. On the other hand, the higher the value of $a$, i.e., of the zero, the lower the value $C^w(\kappa)$.

Figure 13 shows that the lower bound on nofeedback capacity $C^w_{LB}(\kappa)$ is achievable for all $\kappa \in (0, \infty)$, for stable and unstable ARMA$(a,c)$ noise, and it is very close to the feedback capacity $C^w(\kappa)$.

II. Characterization of Feedback Capacity

We consider a the set of uniformly distributed messages $W : \Omega \to \mathcal{M}[n] \triangleq \{1, 2, \ldots, |\mathcal{M}|\}$, codewords $X_1 = e_1(W, S_1, \ldots, X_n = e_n(W, S_1, X_0^n, Y_0^n)$, decoder functions $(s, y^n) \to d_{s}(s, y^n) \in \mathcal{M}[n]$, and average error probability $\epsilon_{\text{w}}$, which is also conditioned on $S_1 = s$.

A. $n$–FTFI Capacity of Time-Invariant Channel Input Strategies

Since our code depends on the initial state of ARMA$(a,c)$ noise, $S_1 = s$, then the entropies in (I.15), (I.16), are conditional on $S_1 = s$. Theorem II.1 is easily derived from the Cover and Pombra characterization (see [10] or [9]).

Theorem II.1. Characterization of $n$–FTFI Capacity

Consider the AGN Channel driven by ARMA$(a,c)$ noise, and a feedback code with knowledge of the noise initial state $S_1 = s$, and let $S_t \triangleq \mathbb{E}\{S_t|Y^{t-1}, S_1 = s\}, K_t \triangleq \mathbb{E}\{(S_t - S_t)^2|S_1 = s\}, K_t = 0, t = 2, \ldots, n$. The analog of (1.15) and (1.16) are given as follows.

$$X_t = A_t (S_t - \hat{S}_t) + Z_t, \quad t = 1, \ldots, n, \quad X_1 = Z_1$$

$$Z_t \in N(0, K_z), \text{ a Gaussian sequence, } t = 1, \ldots, n$$

$$V_t \text{ independent of } \{Y^{t-1}, X^{t-1}, Y^{t-1}, S_1\}$$

$$Y_t = X_t + V_t = A_t (S_t - \hat{S}_t) + Z_t + V_t$$

$$= A_t (S_t - \hat{S}_t) + (c - a) S_t + W_t + Z_t$$

$$Y_t = Z_t + (c - a) S_t + W_t, \quad S_1 = s$$

$$S_{t+1} = cS_t + W_t, \quad S_1 = s$$

$$t = 0, 1, \ldots$$
Further, \( H(Y^n[s] - H(V^n[s], \vec{S}_t, K_t), t = 1, \ldots, n \) determined by the generalized time-varying Kalman-filter.

Kalman-filter recursion:
\[
\hat{S}_{t+1} = c\hat{S}_t + M_t(K_t, \Lambda_t, \Sigma_t)I_t, \quad \hat{S}_1 = s, \quad I_t = Z_t + W_t, \quad t = 1, \ldots, n,
\]
\[
E_t = (\Lambda_t + c - a)\left(\hat{S}_t - \hat{S}^\tau\right) + Z_t + W_t, \quad \hat{S}_t = E_t, \quad t = 2, \ldots, n.
\]

Error recursion, \( E_{t+1} = F_t(K_t, \Lambda_t, \Sigma_t)E_t - M_t(K_t, \Lambda_t, \Sigma_t)(Z_t + W_t), \) \( E_1 = \hat{S}_1 - \hat{S}_1 = 0, \) \( t = 2, \ldots, n. \) \hspace{1cm} (II.37)

Entropy of channel output Process:
\[
H(Y^n[s]) = \sum_{i=1}^{n} H(I_t), \quad I_t \text{ independ. innovation process}. \hspace{1cm} (II.39)
\]

Generalized time-varying difference Riccati equation (DRE):
\[
K_{t+1} = c^2K_t + K_W - \frac{(K_W + cK_t(\Lambda_t + c - a))^2}{(K_Z + K_W + (\Lambda_t + c - a)^2K_t)}, \quad K_t \geq 0, \quad K_0 = 0, \quad t = 2, \ldots, n. \hspace{1cm} (II.40)
\]

The characterization of the n−FTFI capacity \( C_n(\kappa, s) \)
\[
C_n(\kappa, s) = \sup_{(\Lambda_t, \Sigma_t), t = 1, \ldots, n} \left\{ \frac{1}{2} \sum_{i=1}^{n} \left\{ (\Lambda_t)^2K_t + K_Z + K_W \right\} \right\}. \hspace{1cm} (II.41)
\]

Proof. See [10].

Unlike the Cover and Pombra problem (1.15) and (1.16), problem (II.41) is sequential, hence easier to address.

B. Feedback Capacity of Time-Invariant Channel Input Strategies

To address Problem [1], we restrict the channel input strategies to time-invariant, \( \Lambda_t = \Lambda^\tau, \Sigma_t = K_Z^\tau, \forall t \), with corresponding \( X_t = X_t^\tau, Y_t = Y_t^\tau, I_t = I_t^\tau, E_t = E_t^\tau, K_t = K_t^\tau = K_t^\tau(\Lambda^\tau, K_Z^\tau) \). Then we have the following.

Generalized time-Invariant DRE:
\[
K_{t+1}^\tau = c^2K_t^\tau + K_W - \frac{(K_W + cK_t^\tau(\Lambda^\tau + c - a))^2}{(K_Z^\tau + K_W + (\Lambda^\tau + c - a)^2K_t^\tau)}, \quad K_t^\tau \geq 0, \quad K_0^\tau = 0, \quad t = 2, \ldots, n. \hspace{1cm} (II.42)
\]

We define feedback capacity \( C^\tau(\kappa, s) \) as in (11), (9).
\[
C^\tau(\kappa, s) = \sup_{\Lambda^\tau \in (-\infty, \infty)^{\times 2}, K_Z^\tau \in [0, \infty)^{\times 2}} \lim_{n \to \infty} \frac{1}{2n} \sum_{i=1}^{n} \log \left\{ \frac{(\Lambda^\tau + c - a)^2K_t^\tau + K_Z^\tau + K_W}{K_W} \right\}. \hspace{1cm} (II.43)
\]

To address Problem [1](a), we require conditions for convergence of DRE (II.42) to algebraic Riccati equation (ARE),
\[
K^\tau = c^2K^\tau + K_W - \frac{(K_W + cK^\tau(\Lambda^\tau + c - a))^2}{(K_Z^\tau + K_W + (\Lambda^\tau + c - a)^2K^\tau)}, \quad K^\tau \geq 0. \hspace{1cm} (II.44)
\]

Such conditions require concepts of detectability and stabilizability of DREs (11), as done in [10], using the definitions.

Asymptotic stability.
A solution \( K^\tau \geq 0 \) to the generalized ARE (II.44), assuming it exists, is called stabilizing if \( |F(K^\tau, \Lambda^\tau, K_Z^\tau)| < 1 \). In this case, we say \( F(K^\tau, \Lambda^\tau, K_Z^\tau) \) is asymptotically stable.

Definition II.1. (11), (12) Asymptotic stability.

A solution \( K^\tau \geq 0 \) to the generalized ARE (II.44), assuming it exists, is called stabilizing if \[ F(K^\tau, \Lambda^\tau, K_Z^\tau) \] is a limit process.

Definition II.2. (11), (12)
(a) The pair \( \{A, C\} \) is called detectable if there exists a \( G \in \mathbb{R}^n \) such that \( |A - GC| < 1 \) (stable).
(b) The pair \( \{A^*, B^*, \gamma\} \) is called unit circle controllable if there exists an \( \mathbb{R}^n \) such that \( |A^* - B^* \gamma| \neq 1 \).
(c) The pair \( \{A^*, B^*, \gamma\} \) is called stabilizable if there exists an \( \mathbb{R}^n \) such that \( |A^* - B^* \gamma| < 1 \).

In the next theorem we collect known results on the convergence of DREs to AREs.

Theorem II.2. (11) and (13)

Let \( K^\tau_t, t = 1, 2, \ldots, n \) denote a sequence that satisfies the DRE (II.42) with an arbitrary initial condition.

Then the following hold.
(1) Consider the DRE (II.42) with zero initial condition, i.e., \( K^\tau = 0 \), and assume the pair \( \{A, C\} \) is detectable, and the pair \( \{A^*, B^*, \gamma\} \) is unit circle controllable.

Then the sequence \( \{K^\tau_t : t = 1, 2, \ldots, n\} \) that satisfies (II.42), with \( K^\tau_0 = 0 \), converges, \( \lim_{n \to \infty} K^\tau_n = K^\tau \), where \( K^\tau \) satisfies the ARE (II.44), if only if the pair \( \{A^*, B^*, \gamma\} \) is stabilizable.

(2) Assume, the pairs, \( \{A, C\} \) is detectable, and \( \{A^*, B^*, \gamma\} \) is unit circle controllable. Then there exists a unique stabilizing solution \( K^\tau \geq 0 \) to ARE (II.42), i.e., such that, \( |F(K^\tau, \Lambda, K_Z)| < 1 \), if and only if \( \{A^*, B^*, \gamma\} \) is stabilizable.

(3) If \( \{A, C\} \) is detectable and \( \{A^*, B^*, \gamma\} \) is stabilizable, then any solution \( K^\tau_t, t = 1, 2, \ldots, n \) to the DRE (II.42) with arbitrary initial condition, \( K^\tau_0 \) is such that \( \lim_{n \to \infty} K^\tau_n = K^\tau \), where \( K^\tau \geq 0 \) is the unique solution of the generalized ARE (II.44) with \( |F(K^\tau, \Lambda, K_Z)| < 1 \), i.e., it is stabilizing.
Remark II.1. At this point we should emphasize that to address Problem II.1 we need to impose detectability and stabilizability. This is fundamentally different from [9, Theorem 7, Corollary 7.1] and [1, Theorem 6.1 and Lemma 6.1] (as explain under literature review), where the stabilizability condition is not part of the optimization problems of [1], [9], i.e., (I.7)- (I.19). Because of this, our answers are different from [1], [9].

C. Closed-Form Formulas of Feedback Capacity of AGN Channels Driven by Nonstationary Noise

Using Theorem II.2 we address Problem II.1 as stated in Theorem below.

Theorem II.3. Consider the Problem II.1. Define the set

\[ \mathcal{A}^∞ = \left\{ (\Lambda^∞, K_2^∞) \in (-\infty, \infty) \times (0, \infty) : 
\right. \\
(i) \text{the pair } \{A, C\} \text{ is detectable,} \\
(ii) \text{the pair } \{A^+, B^+\} \text{ is stabilizable} \right\}. \quad (II.46) \]

Then,

\[ C^∞(\kappa) = \frac{1}{2} \log \left( \frac{(\Lambda^∞ + c - a)^2 K^∞ + K_2^∞ + K_W}{K_W} \right) \quad (II.47) \]

\[ \mathcal{A}^∞(\kappa) = \left\{ (\Lambda^∞, K_2^∞) \in \mathcal{A}^∞ : K_2^∞ \geq 0, (\Lambda^∞)^2 K^∞ + K_2^∞ \leq \kappa \right\} \]

provided there exists \( \kappa > 0 \) such that \( \mathcal{A}^∞(\kappa) \) is non-empty. The maximum element \( (\Lambda^∞_0, K_2^∞_0) \in \mathcal{A}^∞(\kappa) \), is such that,

(i) if \( |c| < 1 \), then \( (S_t, V_t, Y_t), t = 1, \ldots \) are asymptotic stationary, and
(ii) \( (X_t, I_t), t = 1, \ldots \) are asymptotic stationary,
\[ \forall c \in (-\infty, \infty), a \in (-\infty, \infty). \]

Proof. By Theorem II.2 the limits in (II.43) converge to a unique number and \( C^∞(\kappa, s) \) is independent of \( s \). 

Theorem II.4 is obtained by solving the optimization problem of Theorem II.3.

Theorem II.4. Consider the optimization problem of Theorem II.3. Feedback increases capacity for the following regions.

A) \( c \in (1, \sqrt{2}) \cup (\sqrt{2}, \infty), a \in \left[ -\frac{\sqrt{2} - 1}{2}, \frac{\sqrt{2} - 1}{2} \right] \)

B) \( c \in (-\infty, -\sqrt{2}) \cup (-\sqrt{2}, -1), a \in (\frac{\sqrt{2} - 1}{2}, \frac{\sqrt{2} + 1}{2}) \)

provided the power \( \kappa \) satisfies

\[ \kappa > \kappa_{\min} = \frac{K_W (1 - a c)(2ac - a^2 - 2a + \sqrt{c(2a^3 - 6a^2 - 4a + 4c^3 - 3c)})}{2a^2(c - 2)^2} \]

and the value of feedback capacity \( C^∞(\kappa) \) is

\[ C^∞(\kappa) = \frac{1}{2} \log \left( \frac{(\Lambda^∞_0 + c - a)^2 K^∞_0 + K_2^∞_0 + K_W}{K_W} \right) \quad (II.48) \]

\[ = \frac{1}{2} \log \left( \frac{cK_W (c - 2a + a^2) + c^2 \kappa(c^2 - 1)}{K_W (c - 2)^2} \right), \quad (II.49) \]

\[ K^∞_0 = \frac{g}{c(c - 2)^2(a - c)^2}, \quad (II.50) \]

\[ \Lambda^∞_0 = \frac{K_W (a - c)^2(1 - ac)}{g}, \quad (II.51) \]

\[ K^∞_2 = \frac{c \kappa(c^2 - 1) g - K_W^2 (a - c)^2(1 - ac)^2}{c(c - 2)^2 g}, \quad (II.52) \]

\[ g = K_W (2a - c + a^2 c^2 - 2a^2 c) + c \kappa(c^2 - 1)^2 \quad (II.53) \]

Proof. The solution is obtained by a method similar to [13].

The discussion, conclusions and Figures of Section II.B related to feedback capacity, are based on Theorem II.4.

For the complements of Regimes A and B of Theorem II.4 or \( \kappa \leq \kappa_{\min} \), there does not exist feedback strategy. However, we can show that we can always pick \( \Lambda^∞ = 0 \) and ensure a nonfeedback achievable rate.

D. Nonfeedback Achievable Rates of IID Channel Input Processes

Letting \( \Lambda^∞ = 0 \), in (II.43), the channel input reduces to an independent innovation process \( X^0_t = Z^0_t, t = 1, \ldots, n \), and hence the code does not use feedback. For such an input the detectability and stabilizability conditions are always satisfied, and we obtain a nonfeedback achievable rate, as stated in the next theorem.

Theorem II.5. Consider (II.43), with \( \Lambda^∞ = 0 \), and an achievable lower bound on nonfeedback capacity is,

\[ C^{LB}_{\kappa}(\kappa) = \frac{1}{2} \log \left( \frac{(c - a)^2 K^∞ + \kappa + K_W}{K_W} \right), \quad \forall \kappa \in (0, \infty) \quad (II.54) \]

where \( K^∞ \) is the unique and stabilizable solution of (II.44), with \( \kappa^∞_2 = \kappa, \Lambda^∞ = 0 \), given by

\[ K^∞ = -h + \sqrt{h^2 + 4(c - a)^2 K_W \kappa} \geq 0 \]

\[ h = \kappa(1 - c^2) + K_W(1 - a^2). \]

Proof. This is straightforward to show.

III. ACKNOWLEDGEMENTS

This work was supported in parts by the European Regional Development Fund and the Republic of Cyprus through the Research Promotion Foundation Projects EXCELLENCE/1216/0365 and EXCELLENCE/1216/0296.

IV. CONCLUSION

In this paper, we characterized and derived closed form expressions of feedback capacity and achievable lower bounds on nonfeedback rates, for AGN channels driven by ARMA\((a, c), a \in (-\infty, \infty), c \in (-\infty, \infty), c \neq a \) noise, when channel input strategies or distributions are time-invariant. Simulations showed that the more unstable the noise the higher the feedback capacity, and the achievable lower bounds on nonfeedback rates.
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