CHANNEL FLOW OF FRACTIONALIZED H2O-BASED CNTS NANOFLUIDS WITH NEWTONIAN HEATING

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ABSTRACT. The present article deals to study heat transfer analysis due to convection occurs in a fractionalized H2O-based CNTs nanofluids flowing through a vertical channel. The problem is modeled in terms of fractional partial differential equations using a modern trend of the fractional derivative of Atangana and Baleanu. The governing equation (momentum and energy equations) are subjected to physical initial and boundary conditions. The fractional Laplace transformation is used to obtain solutions in the transform domain. To obtain semi-analytical solutions for velocity and temperature distributions, the Zakian’s algorithm is utilized for the Laplace inversions. For validation, the obtained solutions are compared in tabular form using Tzou’s and Stehfest’s numerical methods for Laplace inversion. The influence of fractional parameter is studied and presented in graphs and discussed.

1. Introduction. In thermal engineering system, enhanced heat transfer is of great concern in various industrial application due to the lack of energy recourses and high cost. Hybrid electric vehicles (HEV), microprocessor heat flux and aerospace applications etc. are the system of high power in thermal management, which became a challenging issue for the engineers and need a modern technology to maintain high performance. The technologies of extended surfaces of fins and microchannel etc. are reached to their limits to handle these issues. However, the modern trends of nanofluids have the capability to enhance the cooling fluids thermal properties and became a hot area for the repeaters. Conventional /regular fluids such as water, alcohol, kerosene oil and ethylene glycol possess poor thermal conductivity. Choi [10] initiated an interesting and effective technique for improving the thermal properties of regular base fluids by dispersing nano-meter size particles (solid) of metals and non-metals, oxides, carbonates, and carbon nanotubes (CNTs) into them are referred as nanofluid. The major aim of nanofluids is to obtain highest possible
thermal conductivity at a low concentration of nanoparticles. Because of the chemical structure of the nanoparticles, when they are dropped into the base fluid, some significant results were noted such as higher heat conduction, reduced chances of erosion, microchannel cooling with no clogging, pumping power with an increase in thermal conductivity, and mixture stability. These features of nonfluids, play equally important contribution in heat transfer enhancement and energy efficiency in several fields. Some of these fields include power generation, defense, nuclear, vehicular cooling in transportation, space, microelectronics, and biomedical devices [12]. Hassan et al. [15] consider the flow of polyvinyl alcohol (PVA) based alumina nanofluid over a wedge. They concluded that during the flow of nanofluid, the resistance between the adjacent layer increases because of nanoparticles which tends to a decrease in the velocity distribution. Alrashed et al. [1] analyzed numerically the two-dimensional channel flow of water-based MWCNTs nanofluid along with heat transfer in weight percentage of nanoparticles 0.00, 0.12, 0.25 and Reynolds number 1 to 150 using finite volume method (FVM.). They investigated an interesting observation that when the Reynolds number is increased, velocity increases. Furthermore, the nanofluid velocity at channel wall reduces due to the increase in the momentum. The heat transfer of kerosene oil based MWCNTs nanofluid in channel heat sink is studied by Arabpour et al. [4]. They explored that for Reynold number 10-100, significantly affect the profile of Nusselt number.

The carbon nanotubes (CNTs) form a cylinder of carbon atoms possess significant electrical, mechanical thermal performance. It is a valuable material in nanotechnology because of its structural backdrop, baby admeasurement and mass, stronger, higher electrical and thermal conductivity and so on [16]. In CNTs the carbon atoms form a hexagonal cylindrical network 1nm in bore and 100 nm in length. There are three kinds of CNTs: Single wall carbon nanotubes (SWCNTs), double wall carbon nanotubes (DWCNTs) and multi-wall carbon nanotubes (MWCNTs). According to Murshed et al. [19], CNT’s based nanofluids, have higher thermal conductivity (six times) as compared to material nanofluids at room temperature. The main applications of CNTs include nanolithography, lithium-battery anodes, additives in polymers, supercapacitor, drug discovery, hydrogen storage, electromagnetic wave absorption and shielding gas-discharge tubes in telecom networks [14]. Aman et al. [2] used CNTs together with three different types of base fluid. They found that the heat transfer in all three types of base fluids is higher for SWCNTs and with increasing volume fraction of CNTs nanofluid velocity decreases. An interesting investigation was reported by Safaei et al. [20] for heat transfer flow of water-based fractionalized multiwall carbon nanotube (FMWCNTs) nanofluid. They concluded that by increasing the concentration of FMWCNTs the boundary layer thickness increases which cause a decrease in the fluid motion.

The above literature revealed that all the studies of nanofluids are based on the classical partial differential equations. In the open literature, the analysis of nanofluids is very rare considering the memory effect using the modern approach of fractional derivatives. Recently, several researchers claimed that the idea of fractional derivatives is far better than a classical idea and provides an efficient tool for the description of memory effect and flow parameters [21, 22, 17, 8, 11, 9, 23, 7, 5]. Hence, the present study is focused on the channel flow of water-based CNTs nanofluids using a modern technique of Atangan-Baleanu fractional derivatives. Analytical solutions with numerical inversion via Laplace transform
are obtained and then plotted for embedded flow parameters. Solutions obtained here may widen the horizon of new research.

2. **Problem’s description.** Let us assume the flow of fractionalized water based CNTs nanofluid in a vertical microchannel consists of two parallel plates apart a distance \(d\). At \(t = 0\), both the fluid and plates are stationary. After a time \(t = 0^+\), the fluid starts motion along the \(x\)-axis with no-slip boundary condition and the \(y\)-axis is taken normal to it. The flow is due to the buoyancy forces and sudden jerk (Stock’s first problem) of the plate at \(y = d\) while Newtonian heating is taken at the plate at \(y = 0\). The physical configuration of the flow is sketched in figure 1. Under the usual Boussinesq approximation, the momentum equation is given as follows [21]:

\[
\rho_{nf} \frac{\partial u(y,t)}{\partial t} = \mu_{nf} \frac{\partial^2 u(y,t)}{\partial y^2} + g (\rho \beta T)_{nf} (T(y,t) - T_0) .
\] (1)

Brinkman [9], presented the relationship between the dynamic viscosity of nanofluid and base fluid which is to be used here is:

\[
\mu_{nf} = \frac{\mu_f}{(1 - \phi)^{2.5}}.
\] (2)

The density of a nanofluid is given by (Aminossadati and Ghasemi [3], and Matin and Pop [18]):

\[
\rho_{nf} = (1 - \phi) \rho_f + \phi \rho_{CNT}.
\] (3)

The energy equation is as under:

\[
(\rho C_p)_{nf} \left[ \frac{\partial T(y,t)}{\partial t} \right] = K_{nf} \left[ \frac{\partial^2 T(y,t)}{\partial y^2} \right],
\] (4)

where \((\rho C_p)_{nf}\) and \(K_{nf}\) are the heat capacity at a constant pressure and thermal conductivity of nanofluids defined as [26]:

\[
\left[ (\rho C_p)_{nf} \right] = \left[ (1 - \phi) (\rho C_p)_f + \phi (\rho C_p)_{CNT} \right],
\] (5)

\[
\frac{K_{nf}}{K_f} = \left[ (1 - \phi) + 2\phi \frac{K_{CNT}}{K_f} \ln \frac{K_{CNT} - K_f}{2K_f} \right].
\] (6)

Imposed physical conditions (initial and boundary conditions):

\[
[u(y,0) = 0, \ T(y,0) = T_0, \ \forall \ y \geq 0, ]
\] (7)

\[
\begin{align*}
 u(0,t) &= 0, \ \frac{\partial T(y,0)}{\partial y} \bigg|_{y=0} = -\frac{h}{k}, \ t > 0, \\
 u(d,t) &= U_0 H(t) \cos \omega t, \ T(d,t) = T_0, \ t > 0.
\end{align*}
\] (8)

Dimensionless quantities:

\[
\left[ v = \frac{d}{\nu_f} \frac{d}{y}, \ u = \frac{h}{k} u, \ \tau = \nu_f \left( \frac{h}{k} \right) t, \ \Phi = \frac{T - T_0}{T_0} \right],
\] (9)

into equation (1), (4), (7) and (8) the following dimensionless system is obtained:

\[
\left( 1 - \phi + \phi \frac{\rho_{CNT}}{\rho_f} \right) \frac{\partial u(\xi,\tau)}{\partial \tau} = \left( \frac{1}{(1 - \phi)^{2.5}} \right) \frac{\partial^2 u(\xi,\tau)}{\partial \xi^2} + \left( 1 - \phi + \phi \frac{(\rho \beta T)_{CNT}}{(\rho \beta T)_f} \right) Gr \Phi(\xi,\tau),
\] (10)
The Laplace transform of equations (16) can be obtained as:

$$\left(1 - \phi + \phi \left(\frac{\rho C_p}{\rho C_p}\right)\right) \frac{\partial \Phi (\xi, \tau)}{\partial \tau} = \frac{1}{Pr} \left(\frac{K_n f}{K_f}\right) \frac{\partial^2 \Phi (\xi, \tau)}{\partial \xi^2},$$  \tag{11}

where

$$[v (\xi, 0) = 0, \quad \Phi (\xi, 0) =, \forall \xi \geq 0,]$$  \tag{12}

and the corresponding initial and boundary conditions from (12), (13) we obtain

$$\begin{bmatrix}
  v (0, t) = 0, \quad \frac{\partial \Phi (\xi, \tau)}{\partial \tau} \\
  v (1, t) = \frac{H (\tau) \cos \omega t}{\tau}, \quad \Phi (1, \tau) = 0, \quad \tau > 0,
\end{bmatrix}$$  \tag{13}

with $Gr = \frac{(k_b^2 \cdot g(r_0) \cdot (T_0 - T_0))}{\mu_f (C_p)_f}$, $Pr = \frac{\mu f (C_p)_f}{K_f}$.

The corresponding fractionalized model using Atagana-Baleanu fractional derivatives is given by the following equation:

$$a_0 D^\alpha_{\xi} u (\xi, \tau) = a_1 \frac{\partial^2 u (\xi, \tau)}{\partial \xi^2} + a_2 Gr \Phi (\xi, \tau), \quad \tag{14}$$

$$a_3 D^\alpha_{\xi} \Phi (\xi, \tau) = \frac{\lambda_n f}{Pr} \frac{\partial^2 \Phi (\xi, \tau)}{\partial \xi^2}. \quad \tag{15}$$

Here

$$a_0 = (1 - \phi + \phi \frac{\rho C_p}{\rho C_p}), \quad a_1 = \frac{1}{(1-\phi)^2}, \quad a_2 = (1 - \phi + \phi \frac{\rho C_p}{\rho C_p}),$$

$$a_3 = (1 - \phi + \phi \frac{\rho C_p}{\rho C_p}), \quad \lambda_n f = \frac{K_n f}{K_f}.$$  

$D^\alpha_{\xi} f (\xi, \tau)$ is Atagana-Baleanu fractional derivative defined by [6]:

$$D^\alpha_{\xi} f (\xi, \tau) = \frac{N (\alpha)}{1 - \alpha} \int_0^\tau E_{\alpha} \left\{-\alpha (\tau - t)^\alpha \right\} f' (\xi, \tau) \, dt,$$  \tag{16}

where $N (\alpha)$ is normalization function such that $N (1) = N (0) = 1$.

The Laplace transform of equations (16) can be obtained as:

$$L \{f (\xi, \tau) \} = \frac{q^\alpha}{(1 - \alpha) q^\alpha + \alpha} \tilde{f} (\xi, q).$$  \tag{17}

3. Semi-analytical solutions. Applying the Laplace transform on equations (14), (15) and the corresponding initial and boundary conditions from (12), (13) we obtain the following exact solutions in the transformed domain:

$$\tilde{v} (\xi, q) = \left[\frac{q^{1+\alpha}}{q^{2+\omega} + 1} \frac{b_4 q^{\sqrt{b_2} \sqrt{W_0 (q)}}}{W_0 (q)} \sinh \frac{\sqrt{b_2} \sqrt{W_0 (q)}}{q} \right] + \left[\frac{b_4 q^{\sqrt{b_2} \sqrt{W_0 (q)}}}{W_0 (q)} \sinh \frac{\sqrt{b_2} \sqrt{W_0 (q)}}{q} \right]$$

$$\tilde{v} (\xi, q) = \frac{1}{q} \sinh \frac{\sqrt{b_2} \sqrt{W_0 (q)}}{q};$$  \tag{18}

$$W_0 (q) = \frac{b_0 q^\alpha}{q^\alpha + b_1}, \quad b_0 = \frac{1}{q^\alpha + b_1}, \quad b_1 = \alpha b_0, \quad b_2 = \frac{a_3 Pr}{\lambda_n f}, \quad b_3 = \frac{a_1}{a_1}, \quad b_4 = \frac{a_2}{\alpha} Gr.$$  \tag{19}
3.1. **The inverse Laplace transforms.** Equations (18) and (19) are the solutions of velocity and temperature distributions respectively in the transformed domain. The inverse Laplace transformations of these equations are in complex form and are not easy to use in practical applications. Therefore, the inverse Laplace transform is numerically obtained via the Zakian explicit formula using the weighted function to approximate the time domain function and presented graphically. The Zakian’s algorithm is defined by [27]:

$$\tilde{f}(y,t) = \left[ \frac{t}{2} \sum_{j=1}^{N} \text{Re} \left\{ K_j \tilde{f} \left( y \frac{\alpha_j}{t} \right) \right\} \right].$$

The constants \((K_j, \alpha_j)\) used here, can be real or pairs of complex conjugate. Here \(N\) is used to show that how many terms have been used in the summation such that \(\tilde{f}(t)\) approaches to \(f(t)\) when \(N\) tends to infinity. According to Halsted et al. [13], when one takes \(N=5\), the truncated error can be ignored for most of the multiplication. For the corresponding values of \(K_j\) and \(\alpha_j\), the reader is referred to Refs. [25]. Furthermore, the results obtained by using Eq. (20) are compared with the numerical inverse Laplace transformation calculated via Tzou’s and Stehfest’s graphically [24]:

$$\tilde{f}(y,t) = e^{4.7t} \left[ \frac{1}{2} \tilde{f} \left( y, \frac{4.7}{t} \right) + \text{Re} \left\{ \sum_{l=1}^{N_1} (-1)^l \tilde{f} \left( y, \frac{4.7 + k\pi i}{t} \right) \right\} \right],$$

$$\tilde{f}(y,t) = \frac{\ln \left( \frac{2}{t} \right)}{t} \sum_{j=1}^{2m} d_j \tilde{f} \left( y, j \frac{\ln \left( \frac{2}{t} \right)}{t} \right)$$

$$d_j = (-1)^{j+m} \sum_{i=[\frac{j+1}{2}]}^{\min(j,m)} \frac{i^m (2i)!}{(m-i)! (i-1)! (j-i)! (2i-j)!}.$$

4. **Discussion and physical interpretation of the results.** This work is focused on the fractionalized model of nanofluid. H2O is chosen as a regular base fluid and CNTs (SWCNT and MWCNT) as nanoparticles have been dropped into the based fluid for the sake of heat transfer enhancement. Analytical solutions are generated using integral transform technique of Laplace and for its inversion being complicated few numerical algorithms have been used. To highlight the influences of the fractional parameter, volume fraction, SWCNT and MWCNT on velocity, temperature, and Nusselt number, the computational technique is applied, and results are displayed in tables and graphs.

Physical sketch of the problem is shown in Figure 1. Figure 2 depicts the effect of velocity profile. It is observed that \(\alpha\) significantly influences the velocity profile of the nanofluid. For the smaller value of \(\alpha\), the velocity of nanofluid is less than the regular velocity at \(\alpha = 1\) (Classical nanofluid). This effect can be physically justified as the thermal boundary layer is thicker at \(\alpha = 1\) and gradually decreases for smaller values of \(\alpha\). The influence of \(\alpha\) on temperature distribution is represented in figure 3. Temperature profile increases with an increase in \(\alpha\) and is maximum at \(\alpha = 1\). Thermal and momentum boundary layers show same behavior with variation in \(\alpha\). This behavior of temperature profile shows a strong agreement with that of the velocity profile.
Figures 4 and 5 illustrate the effect of $\phi$ on velocity and temperature distributions. The nanofluid velocity decreases with increase in $\phi$. At $\phi = 0$ (regular fluid or pure water) the velocity is maximum. This means that by adding nanoparticles in the base fluids, the resulting nanofluid becomes denser. Furthermore, the boundary layer of regular fluids is thinner than nanofluids, as a result, the velocity shows a decreasing behavior within increasing values of $\phi$. The impact of $\phi$ on the temperature distributions is small. But by zooming a small portion of temperature distributions, the effect of $\phi$ can be observed. It is noted that the temperature distributions increase with an increase in $\phi$. This is because of the thermal boundary layer thickness which increases with increasing $\phi$.

Figures 6 and 7 represent the comparison of MWCNTs and SWCNTs. The velocity of MWCNTs is less than the velocity of SWCNTs. Physically, this is true because the density of MWCNTs is greater than the density of SWCNTs which is a factor to decrease the velocity. However, the temperature distribution shows totally opposite behavior to that of the velocity profile. As MWCNTs has higher thermal conductivity than SWCNTs but less density. Consequently, MWCNTs conduct more heat than SWCNTs and are less dense, as a result, the temperature distributions of MWCNTs is higher than SWCNTs. Figures 8, 9 are plotted to show the comparisons of velocity and temperature distributions using Zakian’s, Tzou’s and Sehfest’s algorithms. From these figures, it is quite clear that all the algorithms approximately give same results.

5. Concluding remarks. In this study, a time fractional model for the flow of water-based CNTs nanofluid is developed in a vertical microchannel. For the generalization of the classical model, Atangana-Baleanu fractional approach is used. The Laplace transform and Zakian’s algorithm is utilized to obtain semi-analytical solutions. To validate the correctness of Laplace inversion, the solutions are compared graphically using Tzou’s and Sehfest’s algorithms. Finally, the solutions are plotted in the different figure to analyze the influence of pertinent parameters. It is concluded that the fractional solutions are more general than classical solutions and can be reduced to classical solutions. The velocity and temperature profiles show a decreasing trend for smaller values of $\alpha$ and are maximum for classical fluids model. The viscosity of the nanofluid increases with increase in $\phi$ which tends to a decrease in the velocity profile. The temperature distribution increases with the increase $\phi$ because of the improvement of thermal conductivity of nanofluid. The MWCNTs are more efficient in the enhancement of thermal conductivity of the nanofluids as compared to SWCNTs.

Table 1: Thermophysical properties of water and CNTs nanoparticles

| Material | Base fluid | Nanoparticles |
|----------|------------|---------------|
|          | Water      | MWCNT         | SWCNT         |
| $\rho$ (kg/m$^3$) | 997        | 1600          | 2600          |
| $C_p$ (J/kg K)    | 4179       | 796           | 425           |
| $K$ (W/m K)       | 0.613      | 3000          | 6600          |
| Pr                 | 6.2        | -             | -             |
Figure 1. Flow configuration and coordinate system.

Figure 2. Variation of the velocity profile for water-based MWCNT nanofluid due to $\alpha$ when $\phi = 0.03$ and $Gr = 5$.

Figure 3. Variation of the temperature profile for water-based MWCNT nanofluid due to $\alpha$ when $\phi = 0.03$. 
Figure 4. Variation of the velocity profile for water based MWCNT nanofluid due to $\phi$ when $\phi = 0.5$ and $Gr = 5$.

Figure 5. Variation of the temperature profile for water-based MWCNT nanofluid due to $\phi$ when $\phi = 0.5$ and $Gr = 5$.

Figure 6. Comparison of velocity profiles for water-based MWCNT and SWCNT nanofluids when $\alpha = 0.5$, $\phi = 0.3$ and $Gr = 5$. 
Figure 7. Comparison of temperature profiles for water-based MWCNT and SWCNT nanofluids when $\alpha = 0.5$, $\phi = 0.3$ and $Gr = 5$.

Figure 8. Comparison of velocity profile using different algorithms

Figure 9. Comparison of temperature distribution using different algorithms
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