Numerical study of linear plasma dynamics in a spherical tokamak

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Abstract. The magnetohydrodynamic equilibrium is the starting point to study macro-instabilities in a confined plasma; for the particular case, where the system is axially symmetric, the static and stationary equilibrium is due by the Grad-Shafranov equation. We present the equilibrium state for a toroidal plasma confined by a spherical Tokamak with aspect ratio $A \sim 1.6$, total plasma current 1.3 MA and beta parameter $\beta \sim 0.35$. The Grad-Shafranov equation is solved numerically in a rectangular region of the poloidal plane, using the finite differences method under a successive over-relaxation scheme. Profiles of poloidal magnetic flux, pressure, safety factor and magnetic field are presented. Subsequently, by using the resistive magnetohydrodynamic model, said equilibrium is subjected to perturbations in the velocity to study the dynamics of the plasma in the linear regime. The plasma dynamics simulation is carried out under a fourth order finite difference scheme for the spatial derivatives and implementing the Runge-Kutta algorithm as a temporal integrator. The results show that the perturbations are located in the plasma outer edge; however, some poloidal modes move toward the central zone around the magnetic axis.

1. Introduction

Plasma physics in confinement devices like the tokamak is not completely understood; for example, instabilities on the outer edge of a toroidal device that generates plasma eruptions are no exception. These eruptions occur when, on the poloidal section, the plasma presents a filamented structure in the high confinement regime. This phenomenon known as edge localized modes has been observed in devices such as the mega ampere spherical torus (MAST) [1, 2]. Theoretical works based on the nonlinear theory of ballooning modes predict this phenomenon [3]. Many works are centered to describe the instabilities in these devices and find ways to have control over them. In a macroscopic point of view, the plasma can be described by a set of equations well know as magnetohydrodynamics (MHD) model [4]. To solve said equations set are necessary to employ numerical techniques. Typically, the first step to understand the dynamics of a physical system is studying the linear regime with small perturbations and define regions of stability. There are two ways to do this, the first one is through normal modes analysis, usually for ideal MHD, and the second one as an initial value problem [4,5]. For the last option, simulations based on finite volume methods and finite differences schemes usually are employed. In this paper, we present the numerical simulation of the dynamics for a confined plasma with an axially symmetry toroidal device. We show the equilibrium state in this configuration for a spherical tokamak and the time evolution in the linear regime.
2. Physical model and geometry
The plasma dynamics from a microscopy point of view is typically described through a set of equations well known as MHD model, where the plasma like a collection of electrons and ions with collective behavior is approximate as a charged fluid due to the high collisionality. In the ideal case, the plasma has no dissipation and the plasma has a behavior like a perfect conductor without resistive loses and the model is typically named as ideal MHD. In a static ($\partial t = 0$) and stationary $\vec{v} = 0$ equilibrium state and for an axially symmetric system, the MHD model leads to the Equation (1) [6], known as the Grad-Shafranov equation.

$$\Delta^* \psi = -r^2 \frac{dP(\psi)}{d\psi} - \frac{1}{r} \frac{d}{dr} \left( \frac{d}{dr} \right) g(\psi),$$

(1)

where $\psi$ is the poloidal magnetic flux, $P(\psi)$ and $g(\psi)$ are the equilibrium pressure and the poloidal current function, and $\Delta^* \equiv \frac{\partial^2}{\partial z^2} - \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} \right)$ the elliptical toroidal operator.

The geometry of the toroidal systems is simplified when considering symmetry on the toroidal axis, thus, the simulations are made on the poloidal plane of the reactor, which for simplicity is limited by a rectangular region that includes both, the open and closed magnetic field lines of the system. With these ideas in mind it is appropriate to work on the cylindrical coordinate system $(r, \phi, z)$, where the radial coordinate coincides with the axis of the major radius of the toroid, the angular coordinate $\phi$ with the toroidal angle and the axis $z$ represents the vertical distance of the system from the radial axis as shown in the Figure 1, where $r_m, z_m, r_M$ and $z_M$ represent the minimum and maximum value along $r$ and $z$ axis respectively, that delimits the rectangular simulation region. Due to the aforementioned symmetry, any function that describes the state of the plasma is entirely described by the variables that represent the poloidal plane $(r, z)$.

![Figure 1. Geometric representation of the system to simulate: Toroidal system with rectangular cross-section.](image)

The study of the processes that occur in thermonuclear fusion devices is usually approached by different plasma models at different levels of approximation, depending essentially on the phenomenon to be analyzed [7,8]. We mentioned the magnetohydrodynamic model is the most appropriate when it is desired to study macroinstabilities. In order to simulate the early phase of high confinement of a tokamak reactor, it was decided to model the plasma under the MHD resistive model in the linear regimen, defined by Equation (2) to Equation (7).
\[ \frac{\partial \rho_1}{\partial t} = -\nabla \cdot (\rho_o \vec{v}) \]  

(2)

\[ \rho_o \frac{\partial \vec{v}}{\partial t} = -\nabla P_o + \vec{J}_o \times \vec{B}_1 + \nu \left[ \nabla^2 (\vec{v}) + \frac{1}{3} \nabla (\nabla \cdot \vec{v}) \right] \]  

(3)

\[ \frac{\partial P_1}{\partial t} = -\vec{v} \cdot \nabla P_o - \gamma P_o \nabla \cdot (\vec{v}) - (\gamma - 1) \eta (\vec{J}_o^2 + 2 \vec{J}_o \cdot \vec{J}_1) \]  

(4)

\[ \frac{\partial \vec{B}_1}{\partial t} = -\nabla \times \vec{E} \]  

(5)

\[ \vec{E} = -\vec{v} \times \vec{B}_o + \eta (\vec{J}_o + \vec{J}_1) \]  

(6)

\[ \nabla \times \vec{B}_1 = \vec{J}_1 \]  

(7)

The parameters \( \eta \) and \( \nu \) represent resistivity and viscosity respectively, which are considered uniform and constant throughout the plasma region. The adiabatic compressibility constant \( \gamma \) takes the value of \( 5/3 \).

3. Numerical scheme

The MHD linearized model guarantees the continuity of all the variables of the system, this is an advantage in the implementation of any numerical scheme, since no special treatments are required to model the abrupt changes. Based on this idea, the poloidal plane is discretized with an equidistant meshgrid on the radial axis and the azimuthal axis. Spatial derivatives are expanded in a fourth-order finite differences scheme, Equation (8), where \( i \) represent a meshgrid index and \( \Delta x \) is the spatial step. Over time, the differential equations are integrated with the fourth-order Runge-Kutta scheme, Equation (9), to maintain consistency in precision with the spatial derivatives.

\[ \left( \frac{\partial f}{\partial x} \right)_i \approx \frac{1}{12 \Delta x} [f_{i-2} - 8f_{i-1} + 8f_{i+1} - f_{i+2}] \]  

(8)

\[ Y^{n+1} \approx Y^n + \frac{\Delta t}{6} (k_1 + 2k_2 + 2k_3 + k_4); \text{ with:} \]

\[ \begin{align*}
    k_1 &= R_h (Y^n) \\
    k_2 &= R_h (Y^n + 0.5 \Delta x k_1) \\
    k_3 &= R_h (Y^n + 0.5 \Delta x k_2) \\
    k_4 &= R_h (Y^n + \Delta x k_3)
\end{align*} \]  

(9)

where \( Y \) denotes the set of the independent variables (\( \rho_1, \vec{v}, P_1, \vec{B}_1 \)) and \( R_h \) have the information of the right side in Equations (2)-(7); \( n \) and \( \Delta x \) denotes the temporal index and temporal step respectively.

3.1. Numerical boundary

The rectangular section in the poloidal plane is sufficiently representative since it includes open and closed field lines, which indicates that the plasma-vacuum interaction is considered in the simulation. The poloidal section is bounded by the innermost border of the system: \( r = r_m \), the outer one is located in \( r = r_M \) and the upper and lower borders in \( z = z_M \) and \( z = z_m \) respectively, satisfying \( z_m = -z_M \). In the experience, the devices do not have this type of border, it is usually more complex and plays a very important role, however, they are made of metal. If complex effects such as recombination, secondary emission of electrons and so on are ignored, it is most appropriate to subject the borders as perfect conductive walls. In this way, the tangential component of the electric field must be annulled in the walls as well as the normal component of \( \vec{B}_1 \) and \( \vec{v} \). In addition to these conditions that arise when considering the conductive walls, one more is added, and consists of cancelling the tangential component of the speed at the border, this condition is known as non-slip walls.
4. Results and discussion

4.1. Equilibrium state

To solve this problem, we consider the Equation (1) as a conventional Poisson problem and is solved through an iterative algorithm, which is developed in order to keep fixed the total current and the pressure on the magnetic axis in the plasma, similar like the algorithm presented in [9]; besides setting the criteria to differentiate the region in which the confined plasma is located and the vacuum, employing the ideas presented by Jardin in his book [10].

Applying the appropriate boundary conditions to the rectangular region and incorporating the ideas presented by S Jardin and Young Mu [10,11], we obtain the poloidal flow profile of the system, Figure 2. In this simulation, the total plasma flow and pressure in the magnetic axis are chosen very close to the typical MAST reactor values: 1.3 MA and 35 kPa respectively [12].

The magnetic field and current density in the equilibrium state are recovered through the following expressions, presented in Equation (10).

\[ B_r = -\frac{1}{r} \frac{\partial \psi}{\partial z} \quad B_z = \frac{1}{r} \frac{\partial \psi}{\partial r} \quad B_\phi = \frac{g(\psi)}{r} \quad J_r = -\frac{1}{r} \frac{\partial g}{\partial z} \quad J_z = \frac{1}{r} \frac{\partial g}{\partial r} \quad J_\phi = -\frac{1}{\mu_0 r} \Delta^* \psi \]  \hspace{1cm} (10)

Profiles of the magnetic field components, pressure, safety factor and toroidal current density are presented in Figure 2. It is possible to note that the typical behaviour of the toroidal magnetic field for tokamak devices is obtained, where its typical value in the inner plasma is near to 2T and decays to values about of 0.2T in the outer edge. We can note that the pressure profile is flat around the magnetic axis, this because we impose that \( dP(\psi)/d\psi_n = 0 \) on the magnetic axis. Equation (11) show the free functions selected, similar to Riaz Khan work [13]:

\[ P(\psi_n) = P_o \left(1 - (1 - \psi_n)^2\right)^2 \quad \text{and} \quad g^2(\psi_n) = \frac{3}{2} \frac{2(\psi_n^2 - \frac{2}{10} \psi_n^4)}{\psi_n^4} \]  \hspace{1cm} (11)
where $\psi_n$ is the normalized poloidal flux function and is equal to unity at the magnetic axis and zero at the plasma vacuum boundary.

4.2. Linear dynamics

In these simulations, $\eta = 10^{-6}\Omega \cdot m$ and $\nu = 1.5 \times 10^{-5} Pa \cdot s$ the numerical values for the resistivity and viscosity respectively. The initial conditions of the system are defined according to the previous results, obtained from the numerical solution of the Grad-Shafranov equation; however, it is necessary to specify some details. The results show a zero pressure in the empty region of the plasma, but for numerical convenience, the external region is modelled as a plasma at uniform pressure, this is achieved by adding a constant value to the solution obtained from the equilibrium, as show the Equation (12).

$$P_o = P_{GS} + P_a,$$

(12)

where $P_{GS}$ is the pressure already calculated and $P_a$ the additional pressure on the entire simulation region. The addition of this constant does not alter the equilibrium obtained since the dynamics of the fluid element depends on the pressure gradient, so this can be understood as an extra treatment to prevent the pressure from taking negative values throughout the simulation. The value of $P_a$ considered in this paper is 30% of the pressure at magnetic axis in the solution of the Grad-Shafranov equation: ($P_a = 0.3P_{GS}^{max}$). The mass density at equilibrium is considered uniform throughout all the simulation region, equal to $\rho_o(r,z) = 6.7 \times 10^{-8} kg/m^3$ characteristic of the MAST reactor.

To study the dynamics of the system, the plasma breaks slightly from its initial state of equilibrium through a slight perturbation in all components of the velocity for every fluid element; in this way, the system will evolve automatically. The perturbation employed has the Gaussian form and the maximum is found in the magnetic axis. The explicit form of the perturbation is given by Equation (13).

$$v_i = A_p e^{-\kappa[(r-r_o)^2+(z-z_o)^2]},$$

(13)

where the subscript i represents the components of the velocity ($v_r, v_\phi, v_z$) and $A_p$ is the amplitude of the perturbation, whose value is approximately 1% of the Alfven velocity at magnetic axis. The parameter $\kappa$ is chosen appropriately so that the Gaussian does not have a very sharp peak and the finite difference scheme is entirely applicable. $\kappa = 15$.

The study of plasma dynamics is developed in the framework of the linearized MHD equations, where the evolution of the plasma is governed by the evolution of the perturbations according to the model described in section 2; however, it is necessary to impose the magnetic field divergence equation as a constraint in order to avoid ambiguities in the simulation. We implemented the restricted flow transport method, proposed by Evans & Hawley [14], applicable to rectangular meshes, performed to cylindrical coordinates. The evolution of kinetic pressure is presented in Figure 3 over the poloidal plane.

We can note that the perturbation the kinetic pressure in the first 30 Alfven periods tends to increase, the perturbation is dissipated in the 30 subsequent periods. These results in turn show regions in which the perturbations agglomerates more in certain regions than in others. This is a resonant effect in the plasma, and like the movement of a string or the dynamics of the membrane in a drum, in the plasma, some modes are excited on the poloidal plane. The evolution of the perturbations in the first instants seems to be located in the outer edge region of the plasma (separatrix), but gradually, this is located in the central area, around the magnetic axis.

In order to study the structure of the poloidal modes of magnetic pressure, it expands into Fourier series as Equation (14) and Equation (15).
\[ P_m(\psi, \theta) = \sum_{m=-\infty}^{m=\infty} [a_m(\psi)\sin(m\theta) + b_m(\psi)\cos(m\theta)] \quad (14) \]

\[ a_m(\psi) = \frac{1}{\pi} \int_{0}^{2\pi} P_m(\psi, \theta)\sin(m\theta)\,d\theta; \quad b_m(\psi) = \frac{1}{\pi} \int_{0}^{2\pi} P_m(\psi, \theta)\cos(m\theta)\,d\theta, \quad (15) \]

where \( \theta \) represents the poloidal angle. The coefficients \( a_m(\psi) \) and \( b_m(\psi) \) of the expansion are obtained through the law of orthogonality: Equation (15).

Figure 3. Time evolution for kinetic pressure over all computational domain at \( t=10\tau_a, 20\tau_a, 30\tau_a, 40\tau_a, 50\tau_a \) y \( 60\tau_a \).

To quantify the damping effect, the evolution of the magnetic pressure for the first modes was plotted (\( m = 3-10 \)) (see Figure 4). The curves show two important things: 1) the mode that is most excited is the third, and 2) the exponential decay in the amplitude of each mode shows that some modes have the similar ratio of decrease, however, there are modes that dampen faster than others (see Table 1).

Figure 4. (a) Temporal evolution of the poloidal modes, exalting the damping effect. (b) Curves fitted to the exponential decay for each mode.

Table 1. Damping factor in exponential decay for each pole mode: \( m=3, 4, ...10 \).

| \( m \) | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|---|---|---|---|---|----|
| \( \gamma_m \times 10^{-2} \) | 8.4 | 9.4 | 9.1 | 6.8 | 11.9 | 7.9 | 8.5 | 10.9 |
Finally, we present the evolution of the maximum value of the magnetic field divergence, (see Figure (5)), under the implementation of the restricted flow, guaranteeing that the perturbation of the magnetic fields is consistent.

Figure 5. Evolution of the maximum value for the magnetic field divergence, implementing the restricted flow scheme.

5. Conclusions
In this work, the magnetohydrodynamic equilibrium was first studied in axially symmetric toroidal geometry plasmas, which is characteristic of fusion devices such as tokamak. The obtained results manage to predict in a great approximation the stationary equilibrium of the MAST reactor, suitable for analyzing the high confinement phase. The typical current value in the plasma $\sim 1.3$ MA, the value of the safety factor in the plasma edge $\sim 8$ and pressure in the magnetic axis $\beta_o \approx 39\%$, strongly coincide with the experimental observations. Details in the geometry of the plasma column differ slightly to the characteristic D-shape in this device.

The dynamic evolution of the perturbations presents a damping effect, managing to establish that the perturbations at about 60 Alfvén periods are undetectable. In these simulations, it is not possible to determine the cause that generates this damping, but based on the results of previous investigations and reports of simulations, it seems to indicate that the dissipation of energy is generated by inappropriate viscosity values.

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