ACDM as a Noether Symmetry in Cosmology

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The standard ΛCDM model of cosmology is formulated as a simple modified gravity coupled to a single scalar field (“darkon”) possessing a non-trivial hidden nonlinear Noether symmetry. The main ingredient in the construction is the use of the formalism of non-Riemannian spacetime volume-elements. The associated Noether conserved current produces stress-energy tensor consisting of two additive parts – dynamically generated dark energy and dark matter components non-interacting among themselves. Noether symmetry breaking via an additional scalar “darkon” potential introduces naturally an interaction between dark energy and dark matter. The correspondence between the ΛCDM model and the present “darkon” Noether symmetry is exhibited up to linear order w.r.t. gravity-matter perturbations. By breaking the Noether symmetry we obtain $H_0 = 69.18±4.293$ and $\sigma_8 = 0.7860 ± 0.1106$ from combined data of the direct measurements of the Hubble expansion and growth of matter results, which is closer to latest PLANCK results.

I. INTRODUCTION

The recent realization that the Universe expansion is accelerating [1, 2] has puzzled cosmologists to this day and has lead them to conjecture the existence of dark energy (in the form of a non-zero cosmological constant $\Lambda$) and cold dark matter (CDM) – called ΛCDM cosmological model. Even though the ΛCDM model presents a good fit to the present observations, it has some conceptual problems [3, 4] motivating us to explore other possibilities for the dark sector. One enticing possibility is a form of dynamical dark energy [5, 6] in which the acceleration is induced by a scalar field, usually referred to as quintessence models [7–18]. Dark matter can also be described via a scalar field as weakly-interacting massive particles (WIMPs) – still undiscovered at colliders and dark matter detection experiments. Models for dark matter can also be based on other kinds of scalar fields. This is for example the case of fuzzy dark matter [19]. Interaction between dark matter and dark energy was considered in many cases [20–25]. Interacting scenarios prove to be efficient in alleviating the known tension of modern cosmology [26–40].

In order to provide a unified description of dark energy and dark matter through a simple scalar field one can use different extensions of the canonical scalar field action [41–56]. Ref. [49] uses the formalism of non-Riemannian spacetime volume-forms (NRVF – see Section II below) in addition to the canonical Riemannian volume-element $\sqrt{-g}$ defined by the square-root of the determinant of the Riemannian metric. This NRVF construction yields a simple model of a modified gravity coupled to a single scalar field with two main features: (i) It dynamically generates non-zero cosmological constant as a free integration constant not present in the original model; (ii) It produces a non-trivial hidden nonlinear Noether symmetry of the modified scalar field action, whose associated conserved Noether current yields the CDM part of the pertinent energy density. Thereby the scalar field is called “darkon” and the associated nonlinear Noether symmetry - “darkon” symmetry.

In the present paper we investigate the cosmological solutions of the above “darkon” model. We show that up to linear order of the metric and “darkon” field perturbations the hidden nonlinear “darkon” Noether symmetry yields energy density consisting of two separate dark energy and dark matter components. Breaking of the Noether symmetry is introduced by an additional “darkon” field potential leading to an interaction between dark energy and dark matter components. The implications of the breaking of “darkon” Noether symmetry for a possible explanation of the cosmic tensions are briefly discussed.

The plan of the paper is as follows. Section II briefly introduces the main features of the NRVF formalism. In Section III the basics of the “darkon” model are presented, specifically the emergence and the role of the hidden nonlinear “darkon” Noether symmetry, including the dynamical generation of the dark matter component of the energy density as a dust fluid flowing along geodesics. Section IV describes the homogeneous cosmological solution of the unperturbed “darkon” model whereas in Section V the perturbations of the latter are derived. In Section VI a plausible form of a ΛCDM Noether symmetry-breaking “darkon” potential is proposed and the corresponding solutions compared with some observational data. Finally, Section VII summarizes the results and discusses possible solutions to the cosmic tensions using the above formalism.
II. THE ESSENCE OF THE NON-RIEMANNIAN VOLUME-FORM FORMALISM

Volume-forms define generally covariant integration measures on differentiable manifolds (not necessarily Riemannian ones, so no metric is needed) [57]. They are given by nonsingular maximal-rank differential forms $\omega$ (for definiteness we will consider the case of $D = 4$ spacetime dimensions):

$$\int_{\mathcal{M}} \omega(\ldots) = \int_{\mathcal{M}} dx^4 \Omega(\ldots) \quad (1)$$

with:

$$\omega \equiv \frac{1}{4!} \omega_{\mu
u\kappa\lambda} dx^\mu \wedge dx^\nu \wedge dx^\kappa \wedge dx^\lambda$$

$$\omega_{\mu
u\kappa\lambda} = -\varepsilon_{\mu
u\kappa\lambda} \Omega,$$

$$\Omega = \frac{1}{4!} \varepsilon_{\mu
u\kappa\lambda} \omega_{\mu
u\kappa\lambda}. \quad (2)$$

The conventions for the alternating symbols $\varepsilon_{\mu
u\kappa\lambda}$ and $\varepsilon_{\mu
u\kappa\lambda}$ are: $\varepsilon^{0123} = 1$ and $\varepsilon_{0123} = -1$. The volume-element density (integration measure density) $\Omega$ transforms as scalar density under general coordinate reparametrizations.

In standard generally-covariant theories the Riemannian spacetime volume-form is defined through the tetrad non-singular exact 4-form $\varepsilon = e^A dx^\mu$ ($A = 0, 1, 2, 3$):

$$\omega = \det \| e^A_\mu \| \, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \quad (3)$$

which yields:

$$\Omega = \det \| e^A_\mu \| = \sqrt{-\det g_{\mu\nu}}. \quad (4)$$

Instead of $\sqrt{-g} dx^4$ we can employ another alternative non-Riemannian volume-element as in (1) given by a non-singular exact 4-form $\omega = dB$ where:

$$B = \frac{1}{3!} B_{\mu
u\kappa} dx^\mu \wedge dx^\nu \wedge dx^\kappa. \quad (5)$$

Therefore, the corresponding non-Riemannian volume-element density

$$\Omega \equiv \mathcal{F}(B) = \frac{1}{3!} \varepsilon_{\mu
u\kappa\lambda} \partial_\mu B_{\nu\kappa\lambda}. \quad (6)$$

is defined in terms of the dual field-strength scalar density of an auxiliary rank 3 tensor gauge field $B_{\mu\nu\kappa}$.

The systematic application of non-Riemannian volume-elements to construct modified gravity-matter models was originally proposed in Refs.[58–62], with a subsequent concise geometric formulation in [63, 64]. Let us particularly note the following important property of Lagrangian action terms involving (one or more independent) non-Riemannian volume-elements as in (6):

$$S = \int d^4x \sum_j \mathcal{F}(B^{(j)}) \mathcal{L}^{(j)}(\text{other fields}) + \ldots \quad (7)$$

The equations of motion of (7) with respect to the auxiliary tensor gauge fields $B^{(j)}_{\mu\nu\kappa}$ according to (6) imply:

$$\partial_\mu \mathcal{L}^{(j)}(\text{other fields}) = 0 \quad \rightarrow \quad \mathcal{L}^{(j)}(\text{other fields}) = \mathcal{M}_j, \quad (8)$$

where $\mathcal{M}_j$ are free integration constants not present in the original action (7).

The appearance of the free integration constants in (8) plays instrumental role in the application of the NRVF formalism as a basis for constructing modified gravity-matter models describing unified dark energy and dark matter scenario [49, 65] (see also Section III below), quintessential cosmological models with gravity-assisted and inflaton-assisted dynamical suppression (in the “early” universe) or dynamical generation (in the post-inflationary universe) of electroweak spontaneous symmetry breaking and charge confinement [66–68], as well as a novel mechanism for the supersymmetric Brout-Englert-Higgs effect (dynamical spontaneous supersymmetry breaking) in supergravity [63]. For a systematic numerical study of some of the cosmological models proposed above on the basis of NRVF formalism, see [69, 70].

III. HIDDEN NONLINEAR NOETHER SYMMETRY

A. “Darkon” Model

Our starting point is a modified gravity-matter model where the scalar field action consists of two terms – one coupled to the standard Riemannian volume-element (4) and a second one coupled to a non-canonical non-Riemannian one (6) (using units with $16\pi G_{\text{Newton}} = 1$):

$$S = \int d^4x \left[ \sqrt{-g}(R + X - V_1(\phi)) + \mathcal{F}(B)(X - V_2(\phi)) \right], \quad (9)$$

where $R$ is the Ricci scalar, and $X$ is the kinetic term of a scalar field:

$$X \equiv -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \quad (10)$$

The model (9), first considered in Refs.[49, 65], is a simple special case of the broad class of modified gravity-matter models based on the NRVF formalism as in Eq.(7).

We can equivalently reformulate the action (9) as:

$$S = \int d^4x \sqrt{-g}(R - U(\phi)) + \int d^4x \left( \sqrt{-g} + \mathcal{F}(B) \right)(X - V(\phi)) \quad (11)$$

using the notations:

$$V \equiv V_2, \quad U \equiv V_1 - V_2. \quad (12)$$
Variation of the action (11) w.r.t. auxiliary gauge field $B_{\mu\nu}$ inside $\mathcal{F}(B)$ (6) yields (cf. the general Eq.(8)):

$$\partial_{\mu}(X - V(\phi)) = 0 \rightarrow X - V(\phi) = -2M,$$

(13)

where $M$ is free integration constant not present in the original action (11).

The variation of (11) w.r.t. scalar field $\phi$ can be written in the following suggestive form:

$$\nabla_{\mu}J^{\mu} = -\sqrt{2X}U'(\phi),$$

(14)

$$J_{\mu} \equiv - (1 + \chi)\sqrt{2X}\partial_{\mu}\phi , \ \chi \equiv F(B)/\sqrt{-g}.$$  

(15)

The dynamics of $\phi$ is entirely determined by the dynamical constraint (13), completely independent of the potential $U(\phi)$. On the other hand, the $\phi$-equation of motion written in the form (14) is in fact an equation determining the dynamics of $\chi$.

The energy-momentum tensor $T_{\mu\nu}$ in the Einstein equations following from (11) ($R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}T_{\mu\nu}$), upon taking into account (13) and (15), reads:

$$T_{\mu\nu} = g_{\mu\nu}(-2M - U(\phi)) + (1 + \chi)\partial_{\mu}\phi\partial_{\nu}\phi.$$  

(16)

Both (16) and (15) can be represented in a relativistic hydrodynamical form for an ideal fluid:

$$T_{\mu\nu} = \rho_{0}u_{\mu}u_{\nu} + g_{\mu\nu}\tilde{p}, \quad J_{\mu} = \rho_{0}u_{\mu}$$

(17)

where $u_{\mu}$ is the fluid velocity unit vector:

$$u_{\mu} \equiv - \frac{\partial_{\mu}\phi}{\sqrt{2X}} \quad \text{(note } u^{\mu}u_{\mu} = -1),$$

(18)

the energy density $\tilde{\rho}$ and pressure $\tilde{p}$ are given as:

$$\tilde{\rho} = \rho_{0} + 2M + U(\phi), \quad \tilde{p} = -2M - U(\phi)$$

(19)

with:

$$\rho_{0} \equiv (1 + \chi)2X = \tilde{\rho} + \tilde{p}.$$  

(20)

Energy-momentum conservation $\nabla^{\mu}T_{\mu\nu} = 0$ implies:

$$\nabla^{\mu}(\rho_{0}u_{\mu}) = -\sqrt{2X}U'(\phi) \quad \text{(Eq. (14))}, \quad u_{\mu}\nabla^{\mu}u_{\mu} = 0,$$

(21)

the last Eq.(21) meaning that the matter fluid flows along geodesics.

### B. Hidden Nonlinear Noether Symmetry

In Ref.[49] a crucial property of the model (11) has been uncovered for the special case with the potential $U(\phi) = 0$:

$$S^{(0)} = \int d^{4}x \left[ \sqrt{-g}R + (\sqrt{-g} + \mathcal{F}(B))(X - V(\phi)) \right]$$

(22)

The variation with respect to the scalar field yields a conserved current (cf. Eqs.(14)-(15)):

$$\nabla^{\mu}J_{\mu} = 0, \quad J_{\mu} = -(1 + \chi)\sqrt{2X}\partial_{\mu}\phi = \rho_{0}u_{\mu}.$$  

(23)

$J_{\mu}$ (23) is a genuine Noether conserved current of the action (22) corresponding to the following hidden strongly nonlinear symmetry transformations:

$$\delta_{\epsilon}\phi = \epsilon\sqrt{X}, \quad \delta_{\epsilon}g_{\mu\nu} = 0,$$

(24)

$$\delta_{\epsilon}B^{\mu} = -\frac{1}{2\sqrt{X}}\phi^{\mu}((\Phi(B) + \sqrt{-g}),$$

with $B^{\mu} \equiv \frac{1}{3!}\epsilon^{\mu\nu\kappa\lambda}B_{\nu\kappa\lambda}$. Under (24) the action (22) transforms as total derivative of:

$$\delta_{\epsilon}S^{(0)} = \int d^{4}x \partial_{\mu}(L(\varphi, X)\delta_{\epsilon}B^{\mu}).$$  

(25)

The existence of the hidden Noether symmetry (24) of the action (22) does not depend on the specific form of the potential $V(\phi)$ in the scalar field Lagrangian. The only requirement is that the kinetic term $X$ must be positive.

The hidden Noether symmetry (24) is valid also for the action (11) in the particular case $U(\phi) = \text{const}$.

The energy-momentum tensor corresponding to $S^{(0)}$ (22), i.e., Eq.(17) with (19) for $U(\phi) = 0$, simplifies to:

$$T_{\mu\nu}^{(0)} = \rho_{0}u_{\mu}u_{\nu} - 2Mg_{\mu\nu} \equiv (\rho + p) + g_{\mu\nu}p,$$

(26)

with $\rho_{0}$ as in (20). Now the fluid tension $p = -2M$ is constant and negative, whereas the (total) fluid energy density $\rho = \rho_{0} + 2M$, so that $\rho_{0}$ (19) and 2M are the rest-mass and internal fluid energy densities, respectively (for general definitions, see e.g. [71]).

The energy-momentum tensor (26) is an exact sum of two additive parts with the following interpretation of $\rho$ and $p$ in (26) according to the standard $\Lambda$CDM model [72–74]:

$$p = -2M = p_{\text{DE}} + p_{\text{DE}} , \quad \rho = \rho_{0} + 2M = \rho_{\text{DM}} + \rho_{\text{DE}}.$$  

(27)

Namely, taking into account (23) and last Eq.(21) we have:

- Dark energy part $\rho_{\text{DE}} = -\rho_{\text{DE}} = 2M$, which arises due to the dynamical constraint on the scalar field Lagrangian (13).
- Dark matter part $p_{\text{DM}} = 0$ and $\rho_{\text{DM}} = \rho_{0} \equiv (1 + \chi)2X$, i.e., dark matter appears as a dust-like fluid flowing along geodesics and with conserved particle number density.

The above interpretation justifies the alias “darkon” for the scalar field $\phi$. Let us specifically emphasize that both dark energy and dark matter components of the energy density have been dynamically generated thanks to the non-Riemannian volume-element construction – both due to the appearance of the free integration constant $M$ and of the hidden nonlinear Noether symmetry.
On the other hand, when we start with the initial action (11) with the addition of a Noether symmetry breaking potential $U(\phi) \neq 0$, Eqs.(17)-(19) tell us that $U(\phi)$ triggers an interaction (energy transfer) between the dark energy and dark matter components due to the “darkon” $\phi$-dynamics:

$$
\rho_{DE} = -\rho_{DE} = 2M + U(\phi) , \\
\rho_{DM} = \rho_0 \equiv (1 + \chi)2X , \\
p_{DM} = 0 . \quad (28)
$$

Dark matter fluid is again dust-like fluid flowing along geodesics (second Eq.(21)), however now because of the breakdown (first Eq.(21) – non-conservation of $J_\mu$ (15)) of the hidden nonlinear Noether symmetry the dark matter particle number density is not any more conserved.

**IV. HOMOGENEOUS UNPERTURBED EVOLUTION**

Let us now perform a reduction of the action (11) to the FLRW (Friedmann-LeMaitre-Robertson-Walker) metric:

$$
ds^2 = -dt^2 + a(t)^2\delta_{ij}dx^idx^j . \quad (29)
$$

Variation of (11) w.r.t. $B$ yields the FLRW-reduced form of the dynamical constraint (13):

$$
\frac{d}{dt}\left( \frac{1}{2} \phi^2 - V(\phi) \right) = 0 \quad \rightarrow \quad \frac{1}{2} \phi^2 - V(\phi) = -2M . \quad (30)
$$

Taking time-derivative of (30) implies:

$$
\ddot{\phi} = V' (\phi) , \quad (31)
$$

note the opposite sign in the “force” term on the r.h.s. of (31). According to (30) the solution for $\phi(t)$ reads:

$$
\int_{\phi(0)}^{\phi(t)} \frac{d\phi}{\sqrt{2(V(\phi) - 2M)}} = t . \quad (32)
$$

The equation of motion of (11) w.r.t. $\phi$ is equivalent to the FLRW-reduction of (14), which amounts to an equation for the dark matter energy density $\rho_0$:

$$
\left( \frac{d}{dt} + 3H \right) \rho_0 + \frac{d}{dt}U(\phi) = 0 \quad (33)
$$

$$
\rho_0(t) \equiv (1 + \chi)\phi^2 = \frac{c_0}{a^3(t)} - \frac{1}{a^3(t)} \int dt' a^3(t') U(\phi(t')) \quad (34)
$$

Here $c_0$ is an integration constant, $\dot{U} \equiv U'(\phi) \dot{\phi}$, and $\chi \equiv B/a^3$ is the FLRW-reduced form of the ratio of volume-element densities $\chi \equiv \frac{\mathcal{V}(B)}{\sqrt{\gamma}}$ (last Eq.(15)).

In the case of $U(\phi) = 0$ when the nonlinear “darkon” Noether symmetry is intact Eqs.(33)-(34) reduce to:

$$
\left( \frac{d}{dt} + 3H \right) \rho_0 = 0 \quad \rightarrow \quad \rho_0 \equiv (1 + \chi)\dot{\phi}^2 = \frac{c_0}{a^3} . \quad (35)
$$

where the Hubble parameter $H = \frac{\dot{a}}{a}$. The last Eq.(35) explicitly exhibits the dust-like nature of the “darkon” dark matter energy density $\rho_0$.

The Friedmann equations read accordingly:

$$
6H^2 = \ddot{\rho} , \quad \ddot{p} = \rho_0 + 2M + U(\phi) \quad (36)
$$

$$
\dot{H} = -\frac{1}{4}(\ddot{\rho} + \ddot{p}) \equiv -\frac{1}{4}\dot{\rho}_0 , \quad \ddot{p} = -2M - U(\phi) , \quad (37)
$$

where $\ddot{\rho}$ and $\ddot{p}$ are as in (17)-(19) and $\rho_0$ is given now by the homogeneous solution (34).

In the case of $U(\phi) = 0$ when the nonlinear “darkon” Noether symmetry is intact, taking into account (35), Eqs.(36)-(37) simplify to:

$$
6H^2 = \rho , \quad \rho = \rho_0 + 2M \equiv \frac{c_0}{a^3} + 2M \quad , \quad (38)
$$

$$
\dot{H} = -\frac{1}{4}(\rho + p) \equiv -\frac{c_0}{4a^3} , \quad p = -2M . \quad (39)
$$

For comparison with the observational data it is convenient to rewrite Eqs.(32)-(34) and (36) in terms of function w.r.t. red-shift variable $z$:

$$
1 + z = \frac{a(t)}{a_0} , \quad \frac{d}{dz} = -(1 + z)H(z) \frac{d}{dz} , \quad (40)
$$

as follows:

- Eq.(32) is equivalent to introducing the “darkon” field redefinition:

$$
\phi \rightarrow \tilde{\phi} = \tilde{\phi}(\phi) , \quad \frac{d\tilde{\phi}}{dz} = \left[ 2(V(\phi) - 2M) \right]^{-1/2} , \quad (41)
$$

so that:

$$
\left( \frac{d\tilde{\phi}}{dt} \right)^2 = 1 \quad \rightarrow \quad \frac{d\tilde{\phi}}{dz} = -\frac{1}{(1 + z)H(z)} . \quad (42)
$$

- Eq.(33) is equivalent to:

$$
\frac{d}{dz} \rho_0(z) - \frac{3}{1 + z} \rho_0(z) + \frac{d}{dz} U(\tilde{\phi}(z)) = 0 , \quad (43)
$$

with a solution corresponding to (34):

$$
\rho_0(z) = \frac{c_0}{a_0^3} (1 + z)^3 - (1 + z)^3 \int^z d\zeta (1 + \zeta)^{-3} \frac{d}{d\zeta} U(\tilde{\phi}(\zeta)) . \quad (44)
$$

- The Friedmann Eqs.(36)-(37) are equivalent to:

$$
6H^2(z) = \rho_0(z) + 2M + U(\tilde{\phi}(z)) , \quad (45)
$$

$$
\frac{d}{dz} H^2(z) = \frac{2}{1 + z} \rho_0(z) , \quad (46)
$$

with $\rho_0(z)$ as in (44).
For the sake of confronting the observational data, Eq.(45) may be rewritten in terms of the various density \( \Omega \)-parameters:

\[
H^2(z) = H_0^2 \left[ \Omega_{dm}(z) + \Omega_\Lambda^{(0)}(z) + \Omega_\Lambda^{(1)}(z) + \Omega_r^{(0)}(1+z)^4 \right],
\]

where \( \Omega_{dm}(z) \) stands for the “darkon” dark matter density parameter:

\[
\Omega_{dm}(z) \equiv \frac{\rho_0(z)}{6H_0^2}, \quad \Omega_{dm}(z) - \frac{3}{1+z} \Omega_{dm}(z) + \Omega_{dm}^{(1)}(z);
\]

for the dark energy density parameter:

\[
\Omega_\Lambda^{(0)} = \frac{2M}{6H_0^2}, \quad \Omega_\Lambda^{(1)}(z) = \frac{U(\tilde{\phi}(z))}{6H_0^2};
\]

and where also the contributions of radiation \( \Omega_r^{(0)} \) and baryon matter \( \Omega_b^{(0)} \) have been added.

Let us now consider scalar perturbations of the FLRW metric (29) (in Newtonian gauge):

\[
ds^2 = -(1 + 2\Psi)dt^2 + a(t)^2(1 - 2\Psi)\delta_g dx^i dx^j,
\]

with perturbations of the fields:

\[
\phi = \tilde{\phi}(t) + \delta\phi(t, \vec{x}), \quad \chi = \chi(t) + \delta\chi(t, \vec{x}),
\]

where \( \tilde{\phi} \) and \( \chi \) are the unperturbed ("background") solutions for \( \phi \) and \( \chi \) from Eqs.(32)-(35), as well as perturbations of the energy density and pressure:

\[
\rho = \tilde{\rho}(t) + \rho(t, \vec{x}), \quad p = \tilde{p}(t) + \delta p(t, \vec{x}),
\]

where \( \tilde{\rho} \) and \( \tilde{p} \) are the unperturbed background values of \( \rho \) and \( p \) in (36) and (37). Explicitly:

\[
\delta\rho = \rho_0 + U'\delta\phi, \quad \delta p = -U'\delta\phi,
\]

\[
\delta\rho_0 = \rho_0 \frac{\delta\chi}{1 + \chi} + 2(1 + \chi)V'\delta\phi.
\]

The perturbation of fluid velocity unit vector (18) reads:

\[
\delta u_{\mu} = (-\Psi, \delta u_i), \quad \delta u_i = -\frac{\partial \delta\phi}{\phi}
\]

V. PERTURBATIONS

The perturbation of the dynamical constraint Eq.(13) around the FLRW background::

\[
\dot{\phi}\delta\phi - \dot{\phi}^2\Psi - V'(\phi)\delta\phi = 0
\]

or, equivalently using (31):

\[
\delta\dot{\phi} = \frac{\phi}{\rho_0} \delta\phi
\]

yields solution for \( \delta\phi(t, \vec{x}) \):

\[
\delta\phi(t, \vec{x}) = \phi \int dt' \Psi(\vec{x}', t) + C_0(\vec{x})
\]

with \( C_0(\vec{x}) \) some infinitesimal function of the spacelike coordinates.

The perturbations of the stress-energy tensor components (16) read:

\[
\delta T_{0}^0 = -\delta\rho = -\rho_0 - U'\delta\phi, \quad \delta T_{i}^i = -\frac{1}{a^2}\delta\rho u_i = \frac{1}{a^2}\rho_0 \frac{\partial\delta\phi}{\phi}, \quad \delta T_{ij} = -\delta\rho\delta p = \delta\rho\delta T_{ij}
\]

which upon inserting (59)-(61) becomes:

\[
\left( \frac{d}{dt} + 3H \right) \delta\rho + 3H\delta\rho - \frac{\rho_0}{a^2} \nabla^2 \delta\phi - 3\rho_0 \Psi = 0.
\]

Introducing the dark matter energy density contrast:

\[
\delta_{DM} = \frac{\delta\rho_0}{\rho_0}
\]

and using Eq.(63) by taking into account (33) and last Eq.(53) we obtain:

\[
\frac{d}{dt} \delta_{DM} - \nabla^2 \delta\phi - 3\Psi - \frac{U}{\rho_0} \delta_{DM} + \frac{1}{\rho_0} \frac{d}{dt}(U'\delta\phi) = 0.
\]

Applying time-derivative \( \frac{d}{dt} \) on Eq.(65) and using Eq.(57) – specific perturbation equation for the present “darkon” model of dynamical dark matter, as well as using one of the perturbed Einstein equations for the metric perturbation component \( \Psi \) (see e.g. [76]):

\[
1 \frac{d^2 \Psi}{a^2} = \frac{1}{4} \left( \delta\rho_0 + U'\delta\phi - 3aH\delta\rho_0 \delta\phi \right)
\]
we obtain the second-order differential equation for the dark matter contrast:

\[
\frac{d^2}{dt^2}\delta_{DM} + 2H \frac{d}{dt}\delta_{DM} + \frac{1}{4}\rho_0 \delta_{DM} - 3 \left[ \ddot{\psi} + 2H \dot{\psi} - \frac{1}{4}aH \rho_0 \delta \phi \right] = \frac{1}{4}U' \delta \phi - \left( \frac{d}{dt} + 2H \right) \left[ \frac{1}{\rho_0} \frac{d}{dt} (U' \delta \phi) - \frac{U'}{\rho_0} \delta_{DM} \right].
\] (67)

Recall that \(\rho_0\) and \(\delta \phi\) are explicitly given by (33) and (58), respectively.

In the case \(U(\phi) = 0\) (or \(U(\phi) = \text{const}\)) when the “darkon” nonlinear Noether symmetry is intact (23), the r.h.s. of Eq.(67) vanishes and it reduces to:

\[
\frac{d^2}{dt^2}\delta_{DM} + 2H \frac{d}{dt}\delta_{DM} + \frac{1}{4}\rho_0 \delta_{DM} - 3 \left[ \ddot{\psi} + 2H \dot{\psi} - \frac{1}{4}aH \rho_0 \delta \phi \right] = 0,
\] (68)
where \( \rho_0 \) is now given by (35) and \( \delta \phi \) is expressed through the metric perturbation \( \Psi \) according to (58). Eq.(68) is the general relativistic form of the equation for the dark matter density contrast over \( \Lambda \)CDM FLRW background. In the subhorizon limit where the metric perturbation \( \Psi \) is small [76] the terms in the square brackets on the l.h.s. of (68) can be ignored, so that the latter simplifies to the familiar form of the equation for the energy density contrast of generic dark matter perturbations on \( \Lambda \)CDM background in the Newtonian limit [76] (recall, we are using units with \( 16\pi G_{\text{Newton}} = 1 \)):

\[
\frac{d^2}{dt^2} \delta_{DM} + 2H \frac{d}{dt} \delta_{DM} + \frac{1}{4} \rho_0 \delta_{DM} = 0.
\]

(69)

In terms of redshift \( z \) Eq.(67) takes the form:

\[
\delta_{DM}' + \delta_{DM}'(H'(z) - \frac{1}{1+z}) + \frac{\rho_0(z)\delta_{DM}}{4(1+z)^2H^2(z)}
\]

\[
= \frac{d}{dz} \left( \frac{U'(z)}{\rho_0(z)} \delta_{DM} \right) + \frac{U'(z)}{\rho_0(z)} \delta_{DM}'(H'(z) - \frac{1}{1+z})
\]

(70)

with primes indicating \( \frac{d}{dz} \) and where \( \rho_0(z), H^2(z), H'(z) \) are to be replaced by the expressions (44), (45) and (46), respectively. Here again, as in (69) above, the subhorizon approximation (Newtonian limit) [76] was used (i.e., the terms involving the metric perturbation \( \Psi \) are ignored).

Let us recall that the growth rate function is defined as:

\[
f \equiv \frac{d \ln \delta}{d \ln a} \quad \text{or} \quad f \equiv \frac{\delta'}{\delta},
\]

(71)

with \( \delta = \delta \rho/\rho \) denoting the pertinent matter density contrast, which depicts how quickly the perturbations evolve. Typically, observational data on the growth of structure are presented as constraints on the parameter

\[
f_{\sigma_8}(z) = -(z+1)\sigma_8(0)\frac{\delta'(z)}{\delta(0)}
\]

(72)

which can directly be extracted from redshift space distortion data. The \( \sigma_8(0) \) is the present amplitude of the matter power spectrum at the scale of \( 8h^{-1} \)Mpc [77, 78].

VI. STATISTICAL ANALYSIS

In order to assess the viability of the model, we confront with the observational data the solutions for Eq.(47) (the homogeneous one within the FLRW framework) and Eq.(67) (for the perturbations above the FLRW background).

We examine the following “darkon” Noether symmetry-breaking potential (with \( \phi - \) the redefined “darkon” field (41)):

\[
U(\phi) = 2M \exp(-\phi^2/\beta^2).
\]

(73)

For the limit \( \beta \to 0 \) the potential goes to zero, and we recover the \( \Lambda \)CDM model both in the homogeneous solution as well as on the linear perturbation level. This form of (73) also preserves the property that in the late universe, where \( \phi = t \to \infty \), the total energy density reduces to the vacuum energy density \( 2M \), cf. Eq.(36).

We test the solutions that are provided by the present “darkon” model with two data sets: the direct measurements of the Hubble expansion [79, 80] and the growth rate data set [81–86].

The direct measurements of the Hubble expansion set contains \( N = 36 \) measurements of the Hubble expansion in the redshift range \( 0.07 \leq z \leq 2.33 \). 5 measurements are based on Baryonic Acoustic Oscillations (BAOs), and the other estimated via the differential age of passive evolving galaxies. Here, the corresponding \( \chi^2 \) function reads:

\[
\chi^2_H = H \mathbf{C}^{-1}_{H,\text{cov}} H^T,
\]

(74)

where \( H = \{H_1 - H_0E(z_1, \phi^\nu), \ldots, H_N - H_0E(z_N, \phi^\nu)\} \) and \( H_i \) are the observed Hubble rates at redshift \( z_i \) \( (i = 1, \ldots, N) \). The matrix \( \mathbf{C} \) denotes the covariance matrix, and \( \phi^\nu \) denotes the other parameters on which the Hubble rate depends.

A model-independent cosmological probe, the \( f\sigma_8 \) product, is estimated from the analysis of redshift-space distortions [87]. There is a big number of data points. We choose to use a compilation of \( f\sigma_8 \) data that checked in terms of its robustness using information theoretical methods. The relevant chi-square function reads

\[
\chi^2_{f\sigma_8} = f\sigma_8 \mathbf{C}^{-1}_{f\sigma_8,\text{cov}} f\sigma_8^T,
\]

(75)

where \( f\sigma_8(a_i, \phi^{\nu+1})_{\text{theor}} = \sigma_8 \delta'(a_i, \phi^\nu)/\delta(1, \phi^\nu)a_i \) and a prime denotes derivative of the scale factor \( a \) with the corresponding correlation matrix. The quantity \( \sigma_8 \) is a free parameter. The statistical vector \( \phi^\nu \) contains the other free parameters of the statistical model. The values \( \delta'(a_i), \delta(1) \) are calculated by the numerical solution of Eq. (67) for a given set of cosmological parameters.

To obtain the joint constraints on the cosmological parameters from 2 cosmological probes, we define the total \( \chi^2 \) expression:

\[
\chi^2_{\text{tot}} = \chi^2_H + \chi^2_{f\sigma_8}.
\]

(76)

Regarding the problem of data fit, we use a nested sampler as it is implemented within the open-source Polyhord [88] with the GetDist packaged [89] to present the results. The limit we set for the samples is 10⁴.

Fig. 1 presents the corner plot of the joint statistical analyses. Table I summarizes the joint statistics. One can see that the \( \sigma_8 \) that the potential (73) predicts is closer to the value predicted by PLANCK collaboration \( \sigma_8 = 0.811 \pm 0.006 \) and \( H_0 = 67.4 \pm 0.5 \text{km/s/Mpc} \). Therefore the present model is a good candidate for description of the dynamical dark energy and dark matter and the related effect of the tension between \( \sigma_8 \) values.
However, the $\chi^2$ is larger then the $\Lambda$CDM $\chi^2$ values. Therefore additional test should be done in the future research to test the viability of the model.

VII. CONCLUSIONS

This paper connects the standard $\Lambda$CDM model of cosmology to the hidden nonlinear Noether symmetry of a simple modified gravity-matter model with a single scalar field based on the formalism of non-Riemannian spacetime volume-elements. Via the Noether symmetry of its modified action the scalar field, called “darkon”, dynamically generates both cosmological constant (not present in the original action), as well as dust-like dark matter component of the pertinent stress-energy tensor – a simplest explicit realization of the $\Lambda$CDM framework. Adding Noether symmetry-breaking “darkon” potential introduces interaction (energy transfer) between dark energy and dark matter.

We calculate up to linear order of perturbations the solution for the above theory confirming that in the absence of “darkon” Noether symmetry breaking the known equation for the dark matter density contrast for the $\Lambda$CDM scenario is recovered.

We also studied the homogeneous background and linearly perturbed solutions with a specific plausible choice of “darkon” Noether symmetry-breaking potential. Using the direct measurements of the Hubble expansion and the growth matter perturbations data we find that: $H_0 = 69.18 \pm 4.293$ and $\sigma_8 = 0.7860 \pm 0.1106$. However, to alleviate the cosmic tensions completely we should test more data sets as pantheon Type Ia supernova and measurements from the early universe as the CMB data. This is an ongoing research.

Acknowledgments

We all are grateful for support by COST Action CA-15117 (CANTATA), COST Action CA-16104 and COST Action CA-18108. D.B thanks Ben-Gurion University of the Negev and Frankfurt Institute for Advanced Studies for generous support. E.N. and S.P. are partially supported by Bulgarian National Science Fund Grant DN 18/1.

[1] S. Perlmutter et al. [Supernova Cosmology Project Collaboration], Astrophys. J. 517, 565 (1999) [astro-ph/9812133].
[2] A. G. Riess et al. [Supernova Search Team], Astron. J. 116, 1009 (1998) doi:10.1086/300499 [astro-ph/9805201].
[3] S. Weinberg, astro-ph/0005265.
[4] P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. 75, 559 (2003) [astro-ph/0207347].
[5] B. Ratra and P. J. E. Peebles, Phys. Rev. D 37, 3406 (1988).
[6] C. Wetterich, Astron. Astrophys. 301, 321 (1995) [hep-th/9408025].
[7] I. Zlatev, L. M. Wang and P. J. Steinhardt, Phys. Rev. Lett. 82, 896 (1999) [astro-ph/9807002].
[8] T. Chiba, T. Okabe and M. Yamaguchi, Phys. Rev. D 62, 023511 (2000) [astro-ph/9912463].
[9] T. Barreiro, E. J. Copeland and N. J. Nunes, Phys. Rev. D 61, 127301 (2000) [astro-ph/9910214].
[10] R. R. Caldwell, R. Dave and P. J. Steinhardt, Phys. Rev. Lett. 80, 1582 (1998) [astro-ph/9708069].
[11] R. de Putter and E. V. Linder, Astropart. Phys. 28, 263 (2007) [arXiv:0705.0490 [astro-ph]].
[12] S. Tsujikawa, Class. Quant. Grav. 30, 214003 (2013) [arXiv:1304.1961 [gr-qc]].
[13] E. Babichev, S. Ramazanov and A. Vikman, JCAP 1811, 023 (2018) [arXiv:1807.10281 [gr-qc]].
[14] J. Kehayias and R. J. Scherrer, Phys. Rev. D 100, no. 2, 023525 (2019) [arXiv:1905.05628 [gr-qc]].
[15] V. K. Oikonomou and N. Chatzarakis, arXiv:1905.01904 [gr-qc].
[16] A. Chakraborty, A. Ghosh and N. Banerjee, Phys. Rev. D 99, no. 10, 103513 (2019) [arXiv:1904.10149 [gr-qc]].
[17] S. Chervon, I. Fomin, V. Yurov and A. Yurov, [18] M. C. Bento, O. Bertolami and A. A. Sen, Phys. Rev. D 66, 043507 (2002) [gr-qc/0202064].
[19] W. Hu, R. Barkana and A. Gruzinov, Phys. Rev. Lett. 85, 1158 (2000) [astro-ph/0003365].
[20] F. Arevalo, A. Cid and J. Moya, Eur. Phys. J. C 77, no. 8, 565 (2017) [arXiv:1610.09330 [astro-ph.CO]].
[21] F. K. Anagnostopoulos and S. Basilakos, Phys. Rev. D 97, no. 6, 063503 (2018) [arXiv:1709.02356 [astro-ph.CO]].
[22] E. N. Saridakis, K. Bamba, R. Myrzakulov and F. K. Anagnostopoulos, JCAP 1812 (2018) 012 [arXiv:1806.01301 [gr-qc]].
[23] F. K. Anagnostopoulos, S. Basilakos, G. Kofinas and V. Zarikas, JCAP 1902, 053 (2019) [arXiv:1806.10580 [astro-ph.CO]].
[24] S. Vagnozzi, arXiv:1907.07569 [astro-ph.CO].
[25] D. Vasak, J. Kirsch and J. Struckmeier, arXiv:1910.01088 [gr-qc].
[26] W. Yang, S. Pan, E. Di Valentino, R. C. Nunes, S. Vagnozzi and D. F. Mota, JCAP 1809, 019 (2018) [arXiv:1805.08252 [astro-ph.CO]].
[27] W. Yang, A. Mukherjee, E. Di Valentino and S. Pan, Phys. Rev. D 98, no. 12, 123527 (2018) [arXiv:1809.06883 [astro-ph.CO]].
[28] R. Y. Guo, J. F. Zhang and X. Zhang, JCAP 1902, 054 (2019) [arXiv:1809.02340 [astro-ph.CO]].
[29] D. Benisty and E. I. Guendelman, Phys. Rev. D 98, no. 4, 043522 (2018) [arXiv:1805.09314 [gr-qc]].
[30] S. Kumar, R. C. Nunes and S. K. Yadav, Eur. Phys. J. C 79, no. 7, 576 (2019) [arXiv:1903.04865 [astro-ph.CO]].
[31] P. Agrawal, F. Y. Cyr-Racine, D. Pinner and L. Randall, arXiv:1904.01016 [astro-ph.CO].
[32] D. Benisty, E. I. Guendelman and E. N. Saridakis, arXiv:1909.01982 [gr-qc].
[33] D. Benisty, E. I. Guendelman, E. N. Saridakis,
[86] L. Kazantzidis, L. Perivolaropoulos and F. Skara, Phys. Rev. D 99, no. 6, 063537 (2019) [arXiv:1812.05356 [astro-ph.CO]].

[87] Y. S. Song and W. J. Percival, JCAP 0910, 004 (2009) [arXiv:0807.0810 [astro-ph]].

[88] W. J. Handley, M. P. Hobson and A. N. Lasenby, Mon. Not. Roy. Astron. Soc. 450, no. 1, L61 (2015) [arXiv:1502.01856 [astro-ph.CO]].

[89] A. Lewis, arXiv:1910.13970 [astro-ph.IM].