An asymptotic combinatorial construction of 2D-sphere

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1 Introduction

A sphere is a perfectly round surface. In Analysis, an Euclidean 2D-sphere is defined as a subset \( S^2 \) of three-dimensional Euclidean space \( \mathbb{R}^3 \). If a radius (say \( r \)) and a center (say \((0,0,0)\)) is fixed, then \( S^2 \) is the set of all points \((x,y,z)\) such that \( x^2 + y^2 + z^2 = r^2 \). Without \( \mathbb{R}^3 \), it is not easy to define 2D-sphere as a stand-alone geometrical space. It may be represented as a Riemannian manifold. However, necessary and sufficient conditions, for the manifold to be isometric to a sphere, are quite complex, see [1].

Perhaps a solution is to construct a sequence of finite approximations of 2D-sphere (with fixed radius) that asymptotically (in the limit) give a geometrical space isomorphic to \( S^2 \). Triangulations seem to be natural candidates for such approximations.

A triangulation on a sphere is a simple graph embedded on the sphere so that each face is triangular. The graph may be represented as a geodesic polyhedron. Triangles can then be further subdivided for new geodesic polyhedra. So that, finer and finer triangulations (graphs) may lead to solve the problem.

Similar idea was explored by Boal, Domínguez, and Sayas (2008) [2]. Also the papers by Popko (2012) [3], and Thurston (1998) [4] are of interest here. Despite these efforts, the problem is still open.

Figure 1: Class I operator (a), the kiss operator (class II) (b), a kiss–like operator (c)

The triangular Goldberg–Coxeter operators (see [5]) may be applied for constructing finer triangulations. There are three classes of such operators. Class I operators give a simple division with original edges being divided into sub-edges, see Fig. 1 (a). For Class II operators, triangles are divided with a center point; as an example, see the kiss operator in Fig. 1 (b).

Class III may be viewed as a askew combined version of the I and II classes. The class I operators correspond to the flat surfaces. It seems that the kiss operator may be appropriate to construct finer sphere triangulations.

The five Platonic solids (Tetrahedron, Cube, Octahedron, Dodecahedron and Icosahedron) are (by projection) regular tessellations of the sphere. They may be used as a basis for finer triangulations. Only Cube and Dodecahedron are not triangulations. By applying the kiss–like operator (see Fig. 1 (c)) to each of their faces, we get triangulations. Octahedron (see Fig. 2 (a)) has a nice property; its finer triangulation (by applying the kiss operator) results in equilateral spherical triangles when projected into sphere, see Fig. 2 (b). The triangulation forms the disdyakis dodecahedron that is a Catalan solid with 48 faces, and its dual is the Archimedean truncated cuboctahedron, see Fig. 2 (c). Duality means that triangle faces correspond to vertices in the dual, and adjacency of the faces corresponds to edges.

Dodecahedron has 12 pentagon faces. Icosahedron has 12 vertices of degree 5. Finer triangulations (first, using the kiss–like operator, see Fig. 1 (c), and then the kiss operator) have always exactly 12 distinguished vertices with degree of the form \( 5 \ast 2^k \).

A sphere is a perfectly round surface. Any two points of the sphere are locally isomorphic (indistinguishable), i.e. there are open neighborhoods of the points that are isomorphic. For this very reason, in the sequence of finer and finer triangulations, all vertices can not be distinguished from each other. Hence, Dodecahedron and Icosahedron cannot be considered as the basis for finer triangulations.

Figure 2: Octahedron (a), disdyakis dodecahedron projected into sphere (b), Archimedean truncated cuboctahedron (c). Source CC BY-SA 4.0 [6]

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  \item Class III may be viewed as a askew combined version of the I and II classes.
  \item The class I operators correspond to the flat surfaces.
  \item The five Platonic solids (Tetrahedron, Cube, Octahedron, Dodecahedron and Icosahedron) are (by projection) regular tessellations of the sphere. They may be used as a basis for finer triangulations. Only Cube and Dodecahedron are not triangulations. By applying the kiss–like operator (see Fig. 1 (c)) to each of their faces, we get triangulations. Octahedron (see Fig. 2 (a)) has a nice property; its finer triangulation (by applying the kiss operator) results in equilateral spherical triangles when projected into sphere, see Fig. 2 (b). The triangulation forms the disdyakis dodecahedron that is a Catalan solid with 48 faces, and its dual is the Archimedean truncated cuboctahedron, see Fig. 2 (c). Duality means that triangle faces correspond to vertices in the dual, and adjacency of the faces corresponds to edges.
  \item Dodecahedron has 12 pentagon faces. Icosahedron has 12 vertices of degree 5. Finer triangulations (first, using the kiss–like operator, see Fig. 1 (c), and then the kiss operator), have always exactly 12 distinguished vertices with degree of the form \( 5 \ast 2^k \).
\end{itemize}
For our purpose (to construct the 2D sphere), only uniform (fractal like) finer and finer triangle tessellations resulting from consecutive divisions are of interest. That is, all vertex neighborhoods (in the limit) are the same, i.e., have (as subgraphs) the same structure.

Euclidean equilateral triangles are not appropriate for asymptotic sphere triangulations. To overcome this, weights may be attached to edges. This is a complex and sophisticated technique used in numerical integration on the sphere, see [7].

The division of a spherical triangle using the kiss operator, where the point of division is the center of mass of the triangle, may be interesting for our purpose. Although the finer triangles are not equilateral, they are smaller and smaller in the consecutive divisions, and asymptotically (in the limit) converge uniformly to points on the sphere.

Our approach is based on graphs that are dual to the triangulation graphs. A vertex of dual graph (corresponding to a triangle face) is interpreted as the center of mass of a spherical triangle corresponding to that face. Octahedron (see Fig. 2(a)) is the first basic triangulation on the sphere. The next finer triangulations (by applying the kiss operator) have their counterparts as the dual graphs. Each vertex of the dual graphs is of degree 3, whereas the faces of the counterparts as the dual graphs. Each vertex of the polyhedron (by applying the kiss operator) have their counterparts as the dual graphs. Each vertex of the dual graphs is of degree 3.

Adjacency relation between triangles may be represented as the dual graph to the triangulation graph. The division of a spherical triangle using the kiss operator (see Fig. 1(b)) to each triangle face (Fig. 2) is isomorphic to an Euclidean sphere $S^2$ with radius $r$.

2 The construction

Any triangulation graph of the sphere corresponds to a geodesic polyhedron where vertices and edges correspond to the vertices and edges of the polyhedron. Adjacency relation between triangles may be represented as the dual graph to the triangulation graph.

Let $T_1$ be the triangulation graph of the sphere consisting of $8$ equilateral spherical triangles constituting Octahedron. The triangle faces are denoted by numbers $1, 2, 3, 4, 5, 6, 7, 8$. Let $G_1$ be the dual to $T_1$. Then, vertices of $G_1$ are numbers from 1 to 8, whereas the edges correspond to the adjacency between the faces. It is the graph of Cube.

The second finer triangulation graph $T_2$ is disdyakis dodecahedron, see Fig. 2(b). Here, the spherical triangles are also equilateral. Let $G_2$ be dual to this graph; then $G_2$ corresponds to the Archimedean truncated cuboctahedron, see Fig. 2(c).

$G_1$ has 6 square faces. $G_2$ is composed of 6 octagon faces, 8 hexagon faces, and 12 square faces. $G_3$ consists of six 16-gon faces, eight 12-gon faces, 12 octagon faces, 48 hexagon faces, and 72 square faces.

The inductive construction of the sequence of graphs is as follows. Suppose that $G_n$ and corresponding dual triangulation graph $T_n$ are already constructed. The graph $T_{n+1}$ is constructed by applying the kiss operator (see Fig. 1(b)) to each triangle face in $T_n$. Let $G_{n+1}$ be the dual to the $T_{n+1}$.

Let us define explicitly the graphs $G_k$ for $k = 1, 2, \ldots$. It is convenient to use the dot notation (ASN.1) to denote the sub-triangles resulting from consecutive divisions. The triangles (vertices) of $G_1$ are denoted by the numbers $1, 2, 3, 4, 5, 6, 7, 8$. The triangles (vertices) of $G_2$ are denoted by $i.a, i.b, i.c, i.d, i.e, i.f$; where $1 \leq i \leq 8$.

The next finer triangulation results in triangles (vertices in $G_3$) denoted by labels of the form $t_1, t_2, t_3$, where $1 \leq t_1 \leq 8$, and $t_2$ and $t_3$ belong to the set \{a; b; c; d; e; f\}. Fig. 3 shows the first and the second triangulation, and the labeling. In the right part of Fig. 3 for any two adjacent triangles $i$ and $j$, the labels for their sub-triangles are distributed in the symmetric way relatively to their common edge. Although, the triangles 3, 4, 7, and 8 are not visible, their division and labeling are the same.

The conjecture is that the space (with a natural metric parametrized by $r$) is isomorphic to an Euclidean sphere $S^2$. From this, weights may be attached to edges. This is a complex and sophisticated technique used in numerical integration on the sphere, see [7].

In general case, the vertices in the graph $G_k$ are denoted by finite sequences of the form $x = t_1, t_2, \ldots, t_{k-1}, t_k$. Let $C_k$ denote the set of all such $x$ of length $k$. $C_k$ is defined as the set of vertices in the graph $G_k$.

The vertices and edges of $G_k$ are defined in the following way. Suppose that the label distribution...
was already defined for \((k+1)\)-th triangulation. For a single triangle its sub-triangles are labeled in the clockwise cyclic order as shown in the left part of Fig. 4, where the black circle denotes the mass center of the triangle. The very black circle is the orientation point of any of the sub-triangle for labeling in clockwise order. In the right part of Fig. 4, the triangle \(x.d\) was chosen (as an example) for the label distribution in the next finer triangulation.

The label distribution determines the adjacency relations between triangles, and equivalently also the edges in graph \(G_k\) for any \(k\).

For our purpose it is convenient to consider the adjacency relation (denoted by \(\text{Adj}_{C_k}\)) instead of the set of edges of the graph \(G_k\).

For \(i\) less or equal to the length of \(x\), let \(x(i)\) denote the prefix (initial segment) of \(x\) of length \(i\). Note that for any \(x\) of length \(k\), the sequence \((x(1), x(2), ..., x(k-1), x(k))\) may be interpreted as a sequence of nested spherical triangles converging to a point on the sphere if \(k\) goes to infinity.

Note that \(\text{Adj}_{C_k}\) is symmetric, i.e. for any \(x\) and 
\(y\) in \(C_k\), if \(\text{Adj}_{C_k}(x; y)\), then \(\text{Adj}_{C_k}(y; x)\). It is also convenient to assume that \(\text{Adj}_{C_k}\) is reflexive, i.e. any \(x\) is adjacent to itself, formally \(\text{Adj}_{C_k}(x; x)\).

Let \(C\) denote the union of the sets \(C_k\) for \(k = 1, 2, ..., \) to the adjacency relation \(\text{Adj}_{C,}\) in the following way. For any \(x\) and \(y\) of different length (say \(n\) and \(k\) respectively, and \(n < k\)), relations \(\text{Adj}_{C}(x; y)\) and \(\text{Adj}_{C}(y; x)\) hold if there is \(\hat{x}\) of length \(k\) such that \(x\) is a prefix (initial segment) of \(\hat{x}\), and \(\text{Adj}_{C_k}(\hat{x}; y)\) holds, i.e. \(\hat{x}\) is of the same length as \(y\) and is adjacent to \(y\).

### 3 Topology and geometry

Consider the infinite sequences of the form \((t_1, t_2, t_{k-1}, t_k, ...)\), and denote the set of such sequences by \(C^\infty\). By the construction of \((G_k, k = 2, 3, ...)\) any such infinite sequence corresponds to a sequence of nested triangles (on the 2D-sphere) converging to a point. However, there may be many (also infinite many) such sequences that converge to the same point of the sphere.

For an infinite sequence, denoted by \(u\), let \(u(k)\) denote its initial finite sequences of length \(k\).

The set \(C^\infty\) may be considered as a topological space (Cantor space) with topology determined by the family of clopen sets \(U_x\) such that for any \(x \in C\) of length \(n\): \(U_x = \{u : u(n) = x\}\). Note that the adjacency relation \(\text{Adj}_{C}\) is not used in the definition.

We are going to introduce another topological and geometrical structure on the set \(C^\infty\) determined by \(\text{Adj}_{C}\).

Two infinite sequences \(u\) and \(v\) are defined as adjacent if for any \(k\), the prefixes \(u(k)\) and \(v(k)\) are adjacent, i.e. \(\text{Adj}_{C}(u(k); v(k))\) holds. Let this adjacency relation, defined on the infinite sequences, be denoted by \(\text{Adj}_{C}^\infty\). Note that any infinite sequence is adjacent to itself. Two different adjacent infinite sequences may have a common prefix.

The transitive closure of \(\text{Adj}_{C}^\infty\) is an equivalence relation denoted by ~. Let the quotient set be denoted by \(C^\infty_{/\sim}\), whereas its elements, i.e. the equivalence classes be denoted by \([v]\) and \([u]\). Actually, \(C^\infty_{/\sim}\) is the inverse limit of the graph sequence, see Smyth (1994). [11]

There are rational and irrational points (equivalence classes) in \(C^\infty_{/\sim}\). Each rational equivalence class has infinite (countable) number of elements, whereas the irrational classes are singletons. Each rational class corresponds to a vertex of degree 4 or 6 in a triangulation graph \(T_k\) (for some \(k\)), and equivalently to a square face or hexagon face in \(G_k\).

By the construction of \(G_k\), any point of the initial 2D-sphere corresponds exactly to one equivalence class (a point in \(C^\infty_{/\sim}\)) and vice versa. The set of rational classes is dense in \(C^\infty_{/\sim}\).

The relation \(\text{Adj}_{C}\) determines natural topology on the quotient set \(C^\infty_{/\sim}\).

Usually, a topology on a set is defined by a family of open subsets that is closed under finite intersections and arbitrary unions. The set and the empty set belong to that family. Equivalently (in the Kuratowski style), the topology is determined by a family of closed sets; where finite unions and arbitrary intersections belong to this family.

Let us define the base for the closed sets as the collection of the following neighborhoods of the points of \(C^\infty_{/\sim}\). For any equivalence class \([v]\), i.e. a point in \(C^\infty_{/\sim}\), the neighborhoods (indexed by \(k\)) are defined as the sets \(U^v_k\) of equivalence classes \([u]\) such that there are \(v' \in [v]\) and \(u' \in [u]\) such that \(u'(k)\) is adjacent to \(v'(k)\) i.e. \(\text{Adj}_{C}(u'(k); v'(k))\) holds. Note that \(u'(k)\) may be equal to \(v'(k)\).

A sequence \((e_1, e_2, ..., e_n, ...)\) of equivalence classes of ~ (elements of the set \(C^\infty_{/\sim}\)) converges to \(e\), if for any \(i\) there is \(j\) such that for all \(k > j\), \(e_k \in U^e_i\).

The topological space \(C^\infty_{/\sim}\) is a Hausdorff compact space. The sequence of graphs \((G_k; k = 1, 2, ...)\) may be seen as finite approximations of the space \(C^\infty_{/\sim}\), more exact if \(k\) is greater. By the construction, the space is homomorphic to 2D-sphere. Actually, the graph sequence contains a geometric structure that is much more rich than the topology.

We are going to define geodesics in graph \(G_k\) for any \(k\). Let each face of \(G_k\) be assigned one and the same unit length defined as

\[
\text{unit}^k = \frac{\pi \ast r}{\text{dia}_{k}}
\]

where \(\text{dia}_{k}\) is the diameter of the triangulation graph \(T_k\) dual to \(G_k\), and \(r\) is a positive real number corresponding to the radius of the sphere to be constructed.
Consider an edge (say \( e \)) of \( G_k \). There are exactly two faces (say \( f_1 \) and \( f_2 \)) of \( G_k \) that have \( e \) in common. Let \( 2 \ast n_1 \) and \( 2 \ast n_2 \) denote the number of edges of \( f_1 \) and \( f_2 \) respectively. The length of edge \( e \) (denoted by \( \text{length}(e) \)) is defined as the arithmetic mean of \( \frac{\text{unit}_{\ast 1}}{n_1} \) and \( \frac{\text{unit}_{\ast 2}}{n_2} \).

The geodesics in \( G_k \) are defined as the shortest paths relatively to \( \text{length}(e) \).

The length between two vertices \( x \) and \( y \) of \( G_k \) is denoted by \( \mu_k(x, y) \), and defined as the length (relatively to \( \text{length}(e) \)) of a geodesic between \( x \) and \( y \).

Metrics \( \mu_r \) on \( C_{\infty r} \) is defined as

\[
\mu_r([u], [v]) = \lim_{k \to \infty} \mu_k(u(k), v(k)).
\]

The conjecture. The metric space \( (C_{\infty r}, \mu_r) \) is isomorphic to an Euclidean sphere \( S^2 \) with radius \( r \).

4 Conclusions

The essence of the asymptotic combinatorial construction of the 2D-sphere presented above, is the sequence of graphs \( (G_k; k = 1, 2, \ldots) \). Actually the topology and the geometry were defined on the basis of the sequence alone without reference to an Euclidean sphere \( S^2 \) that serves only as a helpful intuition. The same method can be applied to any graph sequence \( (G_k' ; k = 1, 2, \ldots) \) constructed according to a regular and geodesic pattern, see [3] for details. Then, the graph \( G_k' \) may be considered as an approximation of a geometry, more exact if \( k \) is approaching infinity. Similar approach was proposed by Ruiz & Morón [9].

In the limit, we got a geometry that may be a higher dimensional compact space like 3D-sphere, or be surprising like a fractal, or be a weird geometry like the following flat tori and flat Klein bottles.

Torus can be tessellated by convex quadrilaterals, however, not by equal squares because of the curvature. It is possible to tessellate torus with 16 quadrilaterals. Each of the quadrilaterals may be divided into 4 smaller quadrilaterals, and so on. The dual of the first graph of the tessellations graphs is shown in Fig. 5 where vertices (of degree 4) represent the quadrilaterals of the tessellation. The edges correspond to the adjacency of the quadrilaterals. We may construct the infinite sequence of finer and finer tessellations and corresponding dual graphs. For any of the graphs the geodesics are defined as the minimal paths. Let the length of the geodesics be normalized by the graph diameter. Then, the inverse limit of the graphs determines a flat metric space (known as flat torus) that locally looks like 2D-Euclidean space.

Also Klein bottle can be tessellated by convex quadrilaterals. The simple argument is that the bottle may be “constructed” by joining the edges of two M"obius strips together. So that the infinite sequence, of graphs (dual to finer and finer tessellations) can be constructed. The sequence determines a geometry known as flat Klein bottle. Although it cannot be embedded in \( \mathbb{R}^3 \) (as a topological space), it is a concrete geometrical space that can be approximated by finite graphs.

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