Effects of Lovelock terms on the final fate of gravitational collapse: analysis in dimensionally continued gravity

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Abstract

We find an exact solution in dimensionally continued gravity in arbitrary dimensions which describes the gravitational collapse of a null dust fluid. Considering the situation where a null dust fluid injects into the initially anti-de Sitter spacetime, we show that a naked singularity can be formed. In even dimensions, a massless ingoing null naked singularity emerges. In odd dimensions, meanwhile, a massive timelike naked singularity forms. These naked singularities can be globally naked if the ingoing null dust fluid is switched off at a finite time; the resulting spacetime is static and asymptotically anti-de Sitter spacetime. The curvature strength of the massive timelike naked singularity in odd dimensions is independent of the spacetime dimensions or the power of the mass function. This is a characteristic feature in Lovelock gravity.

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1. Introduction

Black holes are one of the most fascinating objects in general relativity. It is considered that they are formed by the gravitational collapse of very massive stars or density fluctuations in the very early universe. It has been shown that spacetimes necessarily have a singularity under physically reasonable conditions [1]. Gravitational collapse is one of the presumable scenarios that singularities are formed.

Since there is no way to predict information from singularities, we cannot say anything about the causal future of singularities. In order for spacetimes not to be pathological, Penrose made a celebrated proposal, the cosmic censorship hypothesis (CCH) [2], that is, singularities
which are formed in a physically reasonable gravitational collapse should not be seen by distant observers, i.e., spacetimes are asymptotically predictable (see [1]). This is the weak version of the CCH. The strong version of the CCH states that no observer can see the singularities formed by gravitational collapse [3]. Although there is a long history of research on the final fate of the gravitational collapse [4–12], we are far from having achieved consensus on the validity of the CCH, which is one of the most important open problems in general relativity (see [13] for a recent review).

Lately it has been of great importance to consider higher-dimensional spacetimes. Although there is no direct observational evidence of extra dimensions, fundamental theories such as string/M-theory predict the existence of extra dimensions. Over the past few years, the braneworld model has also reinforced the study of higher-dimensional spacetimes [14].

There exists a natural extension of general relativity in higher dimensions, Lovelock gravity [15]. The Lovelock Lagrangian is composed of dimensionally extended Euler densities, which include the higher-order curvature invariants with a special combination so that the field equations are of second order [16]. In four dimensions, such higher curvature terms do not contribute to the Einstein field equations since they culminate in total derivatives. The quadratic curvature terms are known as Gauss–Bonnet terms, which appear in the low-energy limit of heterotic string theory [17]. In Gauss–Bonnet gravity, static black hole solutions have been found and investigated in detail [18, 19].

Higher-order curvature terms come into effect where gravity becomes very strong. One of the present authors has discussed the gravitational collapse of a null dust fluid in Gauss–Bonnet gravity [20]. It was found that the final fate of the gravitational collapse in five dimensions is significantly different from that in general relativity: a massive timelike naked singularity can be formed. Such a class of singularities has never appeared in the general relativistic case [9]. The purpose of the present paper is to analyse how higher-order Lovelock terms modify the final fate of gravitational collapse in comparison to the Gauss–Bonnet or general relativistic cases.

A static vacuum solution in Lovelock gravity has been found [21, 22] and studied [23]. In this case, the metric function is obtained from an algebraic equation, which cannot be solved explicitly in general. This is because, in D-dimensional spacetimes, the Lovelock Lagrangian contains \((D + 1)/2\) independent free parameters, where the symbol \([x]\) is understood to take the integer part of \(x\). This complexity makes the analysis in Lovelock gravity quite hard to tackle, and hence it is not obvious how to extract physical information from the solution. Consequently, Bañados, Teitelboim and Zanelli have proposed a method that reduces the \((D + 1)/2\) free parameters to only 2 by taking a special relation between the coefficients of the Lovelock Lagrangian [24]. An exact solution in the theory of gravity, called dimensionally continued gravity (DC gravity), representing a static vacuum solution, has been found [24, 22]. DC gravity is very effective in estimating the effects of higher-order Lovelock terms although it is a restricted class of Lovelock gravity.

In this paper, we extend our investigations on the gravitational collapse of a null dust fluid in Gauss–Bonnet gravity [20] into DC gravity in order to see the effects of higher-order Lovelock terms on the final fate of gravitational collapse. An exact solution describing gravitational collapse of a null dust fluid is given. We discuss the final fate of gravitational collapse of a null dust fluid in comparison with the results in Gauss–Bonnet gravity [20].

The outline of this paper is as follows; in section 2, DC gravity is summarized. An exact solution for a null dust fluid is presented in section 3. Section 4 is devoted to the discussion of naked singularity formations in the gravitational collapse of a null dust fluid. The strength of naked singularities is investigated in section 5. Our conclusions and discussion follow in section 6. The discussions on the geodesics from singularities are given in appendix A. In
appendix B, the scalar curvature quantities are calculated for the estimation of singular nature. We discuss the general relativistic case in appendix C so as to compare it with the DC gravity results. We use the unit conventions $c = 1$ throughout the paper.

2. Dimensionally continued gravity

We first summarize DC gravity as a restricted class of Lovelock gravity. Then, we give the field equations in DC gravity for odd and even dimensions.

Lovelock gravity is a natural extension of general relativity in $D \geq 3$-dimensional spacetimes, whose action is given by

$$ I = \int L_D + I_m $$

$$ L_D \equiv \kappa \sum_{p=0}^{[D/2]} \alpha_p I_p, $$

$$ I_p \equiv \int_M \epsilon_{a_1 \ldots a_D} R^{a_2 b_2} \wedge \cdots \wedge R^{a_{2p-1} b_{2p-1}} \wedge e^{a_{2p}} \wedge \cdots \wedge e^{b_D}, $$

where $R^{ab} = \omega^{ab} + \omega^a \wedge \omega^b$ is the curvature 2-form, $e^a$ is the orthonormal frame, $\omega^{ab}$ is the spin connection which satisfies the torsion-free Cartan structure equations and $I_m$ is the action of the matter field. In our notation, $a_p, b_p, \ldots$, runs over all spacetime dimensions $0, \ldots, D - 1$. $\epsilon_{a_1 \ldots a_D}$ is the $D$-dimensional Levi-Civita tensor in Minkowski space ($\epsilon_{01 \ldots (D-1)} = 1$). The $[(D + 1)/2]$ real constants $\alpha_p$ in the action (2.2) have dimensions $[\text{length}]^{-(D-2p)}$; otherwise they are arbitrary. A real constant $\kappa$ has units of the action and can be set to be positive without loss of generality. $(\kappa \alpha_1)^{-1}$ is proportional to the $D$-dimensional gravitational constant with a positive coefficient, so that it is real and positive.

The Lovelock action (2.1) is the special combination of curvature invariants in that the resulting field equations do not include more than the second derivatives of the metric. They are composed of dimensionally extended Euler densities [16]. In $D$ dimensions, the first $[(D + 1)/2]$ terms contribute to the field equations; higher-order curvature terms will be the total derivatives. In even dimensions, the last term in the summation (2.1), that is $p = D/2$, is the Euler density and consequently does not contribute to the field equations. The action (2.1) reduces to the usual Einstein–Hilbert action in four dimensions.

Varying the action (2.1) with respect to the vielbein form $e^a$, one obtains the field equations

$$ -\kappa \sum_{p=0}^{[D/2]} \alpha_p (D - 2p) \epsilon_{a_1 \ldots a_D} R^{a_2 b_2} \wedge \cdots \wedge R^{a_{2p-1} b_{2p-1}} \wedge e^{a_{2p}} \wedge \cdots \wedge e^{b_D} = Q_{a_D}, $$

where $Q_{a_D}$ is the $(D - 1)$-form representing the energy–momentum tensor $T_{\mu\nu}$ of the matter,

$$ Q_{a_D} \equiv \frac{1}{(D - 1)!} T_{a_D}^{b_1 \ldots b_D} \epsilon_{b_1 \ldots b_D} e^{b_1} \wedge \cdots \wedge e^{b_D}. $$

The spherically symmetric vacuum solution in Lovelock gravity was first found by Whitt [21]. This solution has been extended to spacetime as a product manifold $\mathcal{M} \approx \mathcal{M}^2 \times \mathcal{K}^{D-2}$ by Cai [22], where $\mathcal{K}^{D-2}$ is the maximally symmetric spaces with constant curvature $(D - 2)(D - 3)k$. Without loss of generality, $k$ is normalized as $+1$ (positive curvature), $0$ (zero curvature) and $-1$ (negative curvature).
The vacuum static solution in Lovelock gravity includes \([(D + 1)/2]\) arbitrary constants [21, 22], so that the solution is given by the roots of polynomial of \([(D + 1)/2]\) degrees, which cannot be written in explicit form. In DC gravity, the arbitrary constants are reduced to 2 by embedding the Lorentz group \(SO(D - 1, 1)\) into the larger AdS group \(SO(D - 1, 2)\) [24]. The remaining two arbitrary constants are a gravitational constant and a cosmological constant. Accordingly, Lovelock gravity is separated into two distinct types of branches for odd and even dimensions. The special combinations of Lovelock coefficients are given by

\[
\alpha_p = \begin{cases} 
\frac{1}{D - 2p} \binom{n - 1}{p} l^{-D+2p}, & \text{for } D = 2n - 1, \\
\binom{n}{p} l^{-D+2p}, & \text{for } D = 2n, 
\end{cases}
\] (2.6)

where \(n \equiv [(D + 1)/2]\) and \(1/l^2\) is proportional to the cosmological constant with a negative coefficient. From equation (2.6), we obtain

\[
(\kappa \alpha_1)^{-1} = \begin{cases} 
(D - 2)/[(n - 1)kl^{-D+2}], & \text{for } D = 2n - 1, \\
1/(nk l^{-D+2}), & \text{for } D = 2n,
\end{cases}
\] (2.7)

so that \(l\) must be real and positive both for \(D = 2n - 1\) and \(D = 2n\) in order for the \(D\)-dimensional gravitational constant to be real and positive.

In odd dimensions \(D = 2n - 1\), Lagrangian \(\mathcal{L}_{2n-1}\) is given by [24]

\[
\mathcal{L}_{2n-1} = \kappa \sum_{p=0}^{n-1} \alpha_p e_a \cdots e_{2p-1} \hat{R}^{a_1b_2} \wedge \cdots \wedge \hat{R}^{a_{2p-1}b_p} \wedge e^{b_{p+1}} \wedge \cdots \wedge e^{b_{2n-1}}
\] (2.8)

with (2.6). We adopt units such that

\[
(D - 2)!\Omega_{D-2} |l|^{-1} = 1 \quad \text{for } D = 2n - 1,
\] (2.9)

where \(\Omega_{D-2}\) is the area of \((D - 2)\)-dimensional unit sphere

\[
\Omega_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)}.
\] (2.10)

Equation (2.8) is a \(D\)-dimensional Chern–Simons Lagrangian for the AdS group, whose exterior derivative is proportional to the Euler density in \((D + 1)\)-dimensions. Defining 2-forms

\[
\hat{R}^{ab} = R^{ab} + l^{-2} e^a \wedge e^b,
\] (2.11)

one can find the equations of motion for odd dimensions \(D = 2n - 1\),

\[
- l^{-1} \epsilon_{a_1 \cdots a_{2n-1}} \hat{R}^{a_1b_2} \wedge \cdots \wedge \hat{R}^{a_{2n-1}b_{2n-2}} = Q_{2n-1}.
\] (2.12)

In even dimensions \(D = 2n\), the Lagrangian is given by

\[
\mathcal{L}_{2n} = \kappa \epsilon_{a_1 \cdots a_{2n}} \hat{R}^{a_1b_2} \wedge \cdots \wedge \hat{R}^{a_{2n-1}b_{2n}},
\] (2.13)

where we adopt the units

\[
2D(D - 2)!\Omega_{D-2} l^{-2} = 1 \quad \text{for } D = 2n.
\] (2.14)

Due to the particular choice of the Lovelock coefficients (2.6), the Lagrangian (2.13) is brought into the Born–Infeld type

\[
\mathcal{L}_{2n} = \kappa \sqrt{|\det \hat{R}^{a_{2n}}|}.
\] (2.15)

The field equations for even dimensions \(D = 2n\) are then given by

\[
-2n l^{-2} \kappa \epsilon_{a_1 \cdots a_{2n}} \hat{R}^{a_1b_2} \wedge \cdots \wedge \hat{R}^{a_{2n-1}b_{2n-2}} \wedge e^{b_{2n-1}} = Q_{2n}.
\] (2.16)
3. Null dust solution

Bañados, Teitelboim and Zanelli derived the static and spherically symmetric vacuum solution of DC gravity in $D \geq 3$-dimensions [24]. The extension to spacetime $\mathcal{M} \approx \mathcal{M}^2 \times \mathcal{K}^{D-2}$ has also been considered [22]. The general metric on the spacetime $\mathcal{M}$ can be written as

$$g_{\mu \nu} = \text{diag}(g_{AB}, r^2 \gamma_{ij}),$$

(3.1)

where $g_{AB}$ is an arbitrary Lorentz metric on $\mathcal{M}^2$, $r$ is a scalar function on $\mathcal{M}^2$ with $r = 0$ defining the boundary of $\mathcal{M}^2$ and $\gamma_{ij}$ is the unit curvature metric on $\mathcal{K}^{D-2}$. When $r$ is not constant, it is shown in full Lovelock gravity that the vacuum solutions are described by the static solutions obtained by Cai [22]; i.e., the generalization of Birkhoff’s theorem into Lovelock gravity holds [26].

From this fact, when $r$ is not constant, the general vacuum solution in DC gravity is obtained as [24, 22]

$$ds^2 = -f(r) \, dr^2 + f^{-1}(r) \, dr^2 + r^2 \, d\Sigma^2_{D-2}$$

(3.2)

with

$$f(r) = \begin{cases} k - (2M/r)^{1/(n-1)} + l^{-2} r^2, & \text{for } D = 2n, \\ k - M^{1/(n-1)} + l^{-2} r^2, & \text{for } D = 2n - 1, \end{cases}$$

(3.3)

where we denote the line element of $\mathcal{K}^{D-2}$ as $d\Sigma^2_{D-2}$.

It is noted that the conventions of $M$ are slightly different from those in [24, 22]. The constant $M$ has dimensions of length in even dimensions, while it is dimensionless in odd dimensions. This is owing to the definition of the present unit conventions (see equations (2.9) and (2.14)). The solution (3.2) is asymptotically $D$-dimensional AdS spacetime because $l^2 > 0$.

The causal structures in the spherically symmetric case ($k = 1$) are shown in [24]. The solution represents a black hole for $M > 0$ and represents the $D$-dimensional AdS spacetime for $M = 0$ in even dimensions. In odd dimensions, meanwhile, the solution represents a black hole for $M^{1/(n-1)} > k$, the $D$-dimensional AdS spacetime for $M = 0$ and the naked singularity for $M^{1/(n-1)} < k$. In three dimensions, this is the so-called BTZ solution [27, 28], in which there are no curvature singularities. In four dimensions with $k = 1$, equation (3.2) reduces to the Schwarzschild–AdS solution.

We will see in five and six dimensions, the solution (3.3) belongs to the special family of solutions in Gauss–Bonnet gravity. In five and six dimensions, up to quadratic curvature terms have nontrivial effects on the field equations. The Lagrangian in Gauss–Bonnet gravity is

$$L = -2\Lambda + R + \alpha (R^2 - 4R_{\mu \nu}R^{\mu \nu} + R_{\mu \nu \rho \sigma}R^{\mu \nu \rho \sigma}).$$

(3.4)

The vacuum static solution in Gauss–Bonnet gravity [18] is given by the metric (3.2) with

$$f = k + \frac{\rho^2}{2\hat{a}} \left[ 1 - \sqrt{1 + 4\hat{a} \left( \frac{m}{r_{D-1}} + \tilde{\Lambda} \right)} \right],$$

(3.5)

where $\hat{a} \equiv (D-3)(D-4)\alpha$, $\tilde{\Lambda} \equiv 2\Lambda/[(D-1)(D-2)]$ and $m$ is the integration constant. We can easily find that the solution (3.5) reduces to the solution (3.3) with $1 + 4\hat{a} \tilde{\Lambda} = 0$ and $2\hat{a} = l^2$ in five and six dimensions.

In four-dimensional spacetimes, we can obtain the Vaidya solution by replacing the mass parameter $M$ in the Schwarzschild solution as an arbitrary mass function $M(v)$, which specifies the flux of the ingoing radiation with the advanced time coordinate $v$. In the course of this heuristic procedure, we find an exact solution of DC gravity in $D$-dimensional spacetimes by
taking $M \rightarrow M(v)$ in the solution (3.2), representing a radially ingoing null dust fluid. The energy–momentum tensor of a null dust fluid is written as

$$T_{\mu \nu} = \rho(v, r) l_\mu l_\nu,$$

where $\rho(v, r)$ represents the energy density of the null dust fluid and $l^\mu$ is the null vector normalized as $l^\mu \partial_\mu = -\partial_r$. Then, we can find the metric of the null dust solution as

$$ds^2 = -f(v, r) dv^2 + 2 dv dr + r^2 d\Sigma_{k,D-2}^2$$

with

$$f(v, r) = \begin{cases} k - (2M(v)/r)^{1/(n-1)} + l^{-2}r^2, & \text{for } D = 2n, \\ k - M(v)^{1/(n-1)} + l^{-2}r^2, & \text{for } D = 2n - 1. \end{cases}$$

We can confirm the solution (3.7) to satisfy the field equations with the help of

$$\hat{R}^{vr} = \frac{n}{2(n-1)r^2} \left( \frac{2M(v)}{r} \right)^{1/(n-1)} e^v \wedge e^i,$$

$$\hat{R}^{ri} = -\frac{1}{2(n-1)r^2} \left( \frac{2M(v)}{r} \right)^{1/(n-1)} e^v \wedge e^i + \frac{\dot{M}(v)}{2(n-1)rM(v)} \left( \frac{2M(v)}{r} \right)^{1/(n-1)} e^v \wedge e^i,$$

$$\hat{R}^{vi} = -\frac{1}{2(n-1)r^2} \left( \frac{2M(v)}{r} \right)^{1/(n-1)} e^v \wedge e^i,$$

$$\hat{R}^{ij} = \frac{1}{2} \left( \frac{2M(v)}{r} \right)^{1/(n-1)} e^i \wedge e^j,$$

for even dimensions and

$$\hat{R}^{ri} = \frac{\dot{M}(v)M(v)^{-1+1/(n-1)}}{2(n-1)r} e^v \wedge e^i,$$

$$\hat{R}^{ij} = \frac{M(v)^{1/(n-1)}}{r^2} e^i \wedge e^j,$$

for odd dimensions, where the overdot denotes differentiation with respect to the coordinate $v$ and $i, j$ represent the indices on $\mathcal{K}^{D-2}$. The matter terms are given by

$$Q_v = \frac{\rho(v, r)}{(D-2)!} \epsilon_{\alpha_1 \cdots \alpha_D} e^{\alpha_1} \wedge e^{\alpha_2} \cdots \wedge e^{\alpha_D}.$$

From the field equations, the energy density of the null dust fluid is given by

$$\rho(v, r) = \frac{1}{\Omega_{D-2} r^{D-2}} \dot{M}$$

both in odd and even dimensions. $\dot{M} \geq 0$ is required in order for the energy density of the null dust fluid to be non-negative. The solution (3.7) is a generalization of the Vaidya solution. Equation (3.7) reduces to equation (3.2) with $dv = dt + dr/f(r)$ when the mass function $M$ is constant.
4. Naked singularity formations

In this and the following sections, we study the final fate of the gravitational collapse of a null dust fluid in DC gravity by use of the solution obtained in the previous section. In what follows, we restrict attention to the spherically symmetric case, i.e., \( k = 1 \), for comparison with the result in Gauss–Bonnet gravity [20]. Hereafter, we shall call the solution (3.7) with \( k = 1 \) the DC-Vaidya solution. For the case of constant \( M \), we shall call the solution the DC-BTZ solution. In the Vaidya solution, the mass function \( M(v) \) is the Misner–Sharp mass. In the generalized Vaidya solution in Gauss–Bonnet gravity, \( M(v) \) is not the Misner–Sharp mass but more preferable quasi-local mass [20]. In this paper, we also adopt \( M(v) \) as the quasi-local mass in the DC-Vaidya solution. The odd- and even-dimensional cases are separately investigated below.

Discussions with the fixed-point method are also presented in appendix A.1.

4.1. Even dimensions

We consider the situation in which a null dust fluid radially falls into the initial AdS spacetime \((M(v) = 0)\) at \( v = 0 \). We set the mass function as a power-law form
\[
M(v) = M_0 v^q
\]
for simplicity, where \( M_0 (>0) \) and \( q (\geq 1) \) are constant. We can easily see from equation (3.16) that a central singularity appears at \( r = 0 \) for \( v > 0 \). Let us discuss more specifically the singular nature of \( v = r = 0 \).

The future-directed outgoing radial null geodesics obey
\[
\frac{d r}{d v} = \frac{1}{\sqrt{f}},
\]
so a trapped region exists in \( f \leq 0 \), which is given by
\[
2M(v) = 2M_0 v^q \geq r (1 + l^{-2} r^2)^{n-1},
\]
where the equality sign holds at the marginally trapped surface \( f = 0 \). Thus, the singularity at \( r = 0, v > 0 \) is in the trapped region and only the point \( v = r = 0 \) has the possibility of being naked. Along the trapping horizon, that is, the trajectory of the marginally trapped surface, we have
\[
d s^2 = \frac{4qM_0 v^{q-1}}{(1 + l^{-2} r^2)^{n-2}(1 + (2n - 1) l^{-2} r^2)} \, dv^2,
\]
so that it is spacelike for \( v > 0, r = 0 \).

We try to find the radial null geodesics emanating from the singularity in order to determine whether or not the singularity is naked.

It is shown that a future-directed radial null geodesic does not emanate from the singularity, nor does a future-directed causal (excluding radial null) geodesic (see appendix A.2 for the proof). So we consider here only the future-directed outgoing radial null geodesics.

Suppose the asymptotic form of the geodesics as \( v \simeq K_1 r^p \) around \( v = r = 0 \) [31], where \( p \) and \( K_1 \) are positive constants. Then the lowest order of equation (4.2) around \( v = r = 0 \) yields \( p = 1 \), so that
\[
v \simeq K_1 r.
\]

\( K_1 = 2 \) is obtained for \( q > 1 \), while \( K_1 \) satisfies the relation
\[
\left(1 - \frac{2}{K_1}\right)^{n-1} = 2M_0 K_1
\]
for \( q = 1 \). The condition for the existence of a positive \( K_1 \) satisfying equation (4.6) is
\[
0 < M_0 \leq \frac{1}{4n} \left( \frac{n-1}{n} \right)^{n-1},
\]
and then \( K_1 > 2 \) holds.

Along the radial null geodesic arising from \( v = r = 0 \) with asymptotic form (4.5), the energy density for the null dust fluid (3.16) and the Kretschmann scalar \( I_1 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \) diverge for \( r \to 0 \) as (see appendix B)
\[
\rho = O(1/r^{D-q-1})
\]
and
\[
I_1 = O(1/r^{4(1-(q-1)/(D-2))}),
\]
respectively, so they are singular null geodesics for \( 1 \leq q < D - 1 \). Thus, the spacetime represents the formation of a naked singularity for \( 1 < q < D - 1 \) with any \( M_0 (> 0) \) and for \( q = 1 \) with \( M_0 \) satisfying condition (4.7). In the limit of \( n \to \infty \), the right-hand side of equation (4.7) goes to zero; i.e., the formation of naked singularity is less plausible as the spacetime dimensions are higher. Hereafter we consider the case where \( 1 \leq q < D - 1 \), in which a naked singularity appears at \( v = r = 0 \).

The central singularity \( v = r = 0 \) is then at least locally naked. Now we consider the structure of the naked singularity. Expanding equation (4.3) around \( r = 0 \), we have
\[
v \simeq (2M_0)^{-1/q} \eta_+^{1/q}.
\]
For \( q > 1 \), there exists a spacetime region \( \mathcal{U} \) which is both the past of the trapping horizon and the future of the future-directed outgoing radial null geodesic \( \gamma \) which behaves as equation (4.5) near \( v = r = 0 \). Such a region also exists for \( q = 1 \) because \( 1 > 2M_0K_1 \) holds from equation (4.6) and \( K_1 > 2 \). Because the trapping horizon is spacelike for \( v > 0 \) and \( r > 0 \), the past-directed ingoing radial null geodesic \( \zeta \) emanating from an event in \( \mathcal{U} \) never crosses the trapping horizon. Also \( \zeta \) never crosses \( \gamma \) except for \( v = r = 0 \) because they are both future-directed outgoing radial null geodesics. Consequently, \( \zeta \) inevitably reaches the singularity at \( v = r = 0 \). Since \( \mathcal{U} \) is an open set, we then conclude that there exist an infinite number of future-directed outgoing radial null geodesics emanating from the singularity at \( v = r = 0 \). Such geodesics should correspond to the solution of equation (A.2) with \( \eta = 0 \) or \( \eta = \infty \) in appendix A.1. On the other hand, the future-directed ingoing radial null geodesic reaching the singularity \( v = r = 0 \) is only \( v = 0 \). Therefore, it is concluded that the singularity \( v = r = 0 \) has an ingoing-null structure.

When \( M_0 \) fails to satisfy condition (4.7) for \( q = 1 \) in \( D \geq 4 \) (\( n \geq 2 \)) dimensions, more detailed analyses are needed in order to determine whether the final fate of gravitational collapse is a black hole or a naked singularity since there might exist null geodesics which do not obey the power-law form asymptotically.

In order to see whether the singularity is globally naked, we consider a very simple situation in which the null dust fluid is switched off at \( v = v_f > 0 \). The solution is described by the DC-BTZ solution for \( v > v_f \), which is static and asymptotically AdS spacetime. The DC-BTZ solution is joined with the \( D \)-dimensional AdS spacetime for \( v < 0 \) by way of the DC-Vaidya solution for \( 0 \leq v \leq v_f \). The situation is depicted in figure 1.

If a null geodesic emanating from the singularity \( v = r = 0 \) reaches the surface \( v = v_f \) in the untrapped region, it can escape to infinity; then, the singularity is globally naked. If we take \( v_f \) to be sufficiently small, the singularity can be globally naked. The singularity in the outer DC-BTZ spacetime is spacelike in even dimensions [24]. Thus, the Penrose diagram of the gravitational collapse is drawn in figure 2 for the globally naked singularity formation. Of course, the locally naked singularity formation is also obtainable.
4.2. Odd dimensions

As in the even-dimensional case, we consider the situation in which a null dust fluid radially falls into the initial AdS spacetime and we choose the mass function as the power-law form $M(v) = M_0 v^q$. Then we can see from equation (3.16) that a singularity develops at $r = 0$ for $v > 0$. As shown below, the point $v = r = 0$ may also be singular. It is noted that the
singularity with $r = 0$ and $v > 0$ is also interpreted as a conical singularity; that is, a deficit angle exists at $r = 0$ (see [28] for the conical singularity in the BTZ solution).

A trapped region $f \leq 0$ is given by

$$M(v) = M_0 v^q \geq (1 + l^{-2} r^2)^{q-1},$$

(4.11)

where the equality holds at the marginally trapped surface $f = 0$. Along the trapping horizon, we have

$$ds^2 = \frac{M_0 q^2 v^{q-1}}{r(n-1)(1 + l^{-2} r^2)^{q-2}} dv^2,$$

(4.12)

so that it is spacelike for $r > 0$, $v > 0$. From equation (4.11), the singularity $r = 0$ might be naked for $0 \leq v \leq v_{AH}$, where $v_{AH} \equiv M_0^q$. Following the argument in appendix A.2, we only consider the future-directed outgoing radial null geodesics from the singularity. Suppose the asymptotic form

$$v \simeq v_0 + K_2 r^p$$

(4.13)

near $r = 0$ with $0 \leq v_0 \leq v_{AH}$, where $p$ and $K_2$ are positive constants. From the lowest order of equation (4.2), we obtain

$$v \simeq v_0 + \frac{2}{1 - (M_0 v_0^q)^{1/(q-1)}}$$

(4.14)

for $0 \leq v_0 < v_{AH}$, while there are no null geodesics with the asymptotic form, equation (4.13), for $v = v_{AH}$.

Along the radial null geodesics from the singularity $v = r = 0$ asymptotically to equation (4.14), the Kretschmann scalar diverges as

$$I_1 = O(1/r^{4(1-q/(D-1))})$$

(4.15)

for $D > 5$. In the meanwhile, it is finite for $D = 3$. In fact, the scalar quantities for the DC-Vaidya solution with $D = 3$ are the same as those for the three-dimensional AdS spacetime, which are constant everywhere; $I_1 = I_2 = 12 l^{-4}$, $R = -6 l^{-2}$. (The curvature invariants $I_1$, $I_2$ and $R$ for the metric (3.7) are calculated in appendix B.) However, the energy density of the null dust fluid (3.16) diverges along the null geodesics (4.14) as

$$\rho = O(1/r^{D-q-1})$$

(4.16)

for $D > 3$. As a result, they are singular null geodesics for $1 \leq q < D - 1$.

On the other hand, the radial null geodesics from the singularity $r = 0$, $0 < v < v_{AH}$ asymptoting to equation (4.14) are singular null geodesics for any $q \geq 1$. Along them, the Kretschmann scalar and the energy density of the null dust fluid diverge for $D \geq 5$ and $D \geq 3$, respectively.

We can consider the situation as in the even-dimensional case; that is, the null dust fluid is turned off at some finite time $v = v_f > 0$ in order to see whether the singularity is globally naked. Figure 3 illustrates the situation in which the DC-BTZ solution for $v > v_f$ is connected with the $D$-dimensional AdS spacetime for $v < 0$ via the DC-Vaidya solution for $0 \leq v < v_f$.

The structures of the singularity in the outer DC-BTZ spacetime in odd dimensions are spacelike, null and timelike for $v_f < v_{AH}$, $v_f = v_{AH}$ and $v_f > v_{AH}$, respectively [24]. Now we consider the structure of the naked singularity in the null dust region. The naked singularity with $0 \leq v < v_{AH}$ is both an endpoint of the future-directed ingoing radial null geodesic and the initial point of the future-directed outgoing radial null geodesic, therefore we conclude that it is timelike.

For $0 < v_f \leq v_{AH}$, the singularity is always globally naked. In the case of $v_f > v_{AH}$, if we take $v_f$ as sufficiently close to $v_{AH}$, a null ray emanating from the singularity reaches the surface $v = v_f$ in the untrapped region, so that it can escape to infinity; then, the singularity is
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Figure 3. Gravitational collapse in odd dimensions for $v_f > v_{AH}$ (left) and $0 < v_f \leq v_{AH}$ (right). The DC-Vaidya solution for $0 \leq v \leq v_f$ is jointed with the DC-BTZ solution for $v > v_f$ and the AdS spacetime for $v < 0$.

Figure 4. Penrose diagram in odd dimensions for $0 < v_f < v_{AH}$. The massive timelike naked singularity appears. The resultant spacetime is always globally naked.

globally naked. As in the even-dimensional case, globally naked singularities can be formed in odd dimensions.

In the case of $0 < v_f < v_{AH}$, the Penrose diagram of the gravitational collapse is shown in figure 4. The possible Penrose diagrams are depicted in figure 5 for $v_f = v_{AH}$ and in figure 6 for $v_f > v_{AH}$. From the present analysis, however, it is not clear whether there exists a null portion of the naked singularity at $r = 0$ for $v = v_{AH}$.

5. Strength of naked singularities

As have seen in the preceding section, naked singularities can be formed in DC gravity. In this section, we investigate the strength of naked singularities along the radial null geodesics
with the asymptotic form (4.5) in even dimensions and (4.14) in odd dimensions. We define

\[ \psi \equiv R_{\mu \nu} k^\mu k^\nu, \tag{5.1} \]

where \( k^\mu = d\chi^\mu / d\lambda \) is the affinely parametrized tangent vector of the future-directed outgoing radial null geodesic with an affine parameter \( \lambda \). We evaluate the strength of naked singularities by the divergent behaviour of \( \psi \) in the neighbourhood of the singularities.

We consider the radial null geodesic with the tangent vector \( k^\mu \), which emanates from a singularity at \( \lambda = 0 \). In four dimensions, the strong curvature condition and the limiting focusing condition are satisfied along an affinely parametrized geodesic if \( \lim_{\lambda \to 0} \lambda^2 \psi > 0 \)

Figure 5. Possible Penrose diagrams in odd dimensions for \( v_f = v_{\text{AH}} \), which are globally naked.

Figure 6. Possible Penrose diagrams in odd dimensions with \( v_f > v_{\text{AH}} \) for the globally naked singularity formation.
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and \( \lim_{\lambda \to 0} \lambda \psi > 0 \), respectively \([32, 33]\). However, these results have not been extended to the higher-dimensional case so far.

Along the radial null geodesics emanating from the singularity, we have

\[
k' = \frac{1}{2} f k^v. \tag{5.2}
\]

Then a straightforward calculation yields

\[
\psi = -\frac{2(D - 2)f}{rf^2}(k')^2, \tag{5.3}
\]

where \( f \) is given by equation (3.8). The geodesic equation for the radial null geodesic is written as

\[
d\frac{dk'}{d\lambda} - \frac{2f}{f^2}(k')^2 = 0. \tag{5.4}
\]

As we will see below, the asymptotic form of the differential equation (5.4) for the DC-Vaidya solution reduces to the following form:

\[
d\frac{dk'}{d\lambda} + sr^\alpha (k')^2 \simeq 0, \tag{5.5}
\]

where \( s > 0 \) and \( \alpha \) are constants. Then, \( \psi \) is expressed as

\[
\psi \simeq s(D - 2)r^{\alpha - 1}(k')^2. \tag{5.6}
\]

We suppose the asymptotic solution of equation (5.5) around \( \lambda = \lambda_0 \) as

\[
r(\lambda) \simeq D_1(\lambda - \lambda_0)^\beta, \tag{5.7}
\]

where \( D_1 \) and \( \beta \) are positive constants. The divergent behaviour of \( \psi \) is completely specified by the following two cases.

Case 1 (\( \beta \neq 1 \)). Direct substitution of equation (5.7) into equation (5.5) yields \( \alpha = -1 \) and \( \beta = 1/(1 + s) \). Redefinition of the affine parameter enables us to set \( D_1 = 1 \) and \( \lambda_0 = 0 \) without loss of generality in such a way that \( \lambda = 0 \) corresponds to the central singularity. Then, we have

\[
r(\lambda) \simeq \lambda^{1/(1+s)}, \tag{5.8}
\]

Therefore, we obtain

\[
\psi \simeq \frac{s(D - 2)}{(1 + s)^2} \lambda^{-2}, \tag{5.9}
\]

and then it is concluded

\[
\lim_{\lambda \to 0} \lambda^2 \psi = \frac{s(D - 2)}{(1 + s)^2} > 0. \tag{5.10}
\]

Note that the right-hand side of equation (5.10) varies delicately with the scaling of the affine parameter.

Case 2 (\( \beta = 1 \)). We have to take into account the next leading term such as

\[
r(\lambda) \simeq D_2(\lambda - \lambda_0) + D_3(\lambda - \lambda_0)^{\gamma'}, \tag{5.11}
\]

where \( \gamma \) is a positive constant. The most dominant order of equation (5.5) gives \( \gamma = \alpha + 1 \) and \( D_2 = -s D_2^{\alpha + 2}/[(\alpha + 1)(\alpha + 2)] \). We set \( \lambda_0 = 0 \) and \( D_2 = 1 \) by the redefinition of the affine parameter. Therefore, the asymptotic solution is given by

\[
r(\lambda) \simeq \lambda. \tag{5.12}
\]

Hence, we have

\[
\psi \simeq s(D - 2) \lambda^{\alpha - 1}. \tag{5.13}
\]
Table 1. The divergent behaviour of $\psi$ around $v = r = 0$ along the radial null geodesics with the asymptotic form (4.5) in even dimensions.

| $q$ | $q = 1$ | $1 < q < D - 1$ | $q = D - 1$ | $q > D - 1$ |
|-----|---------|-----------------|-------------|-------------|
| $\psi$ | $\lambda^{-2}$ | $\lambda^{-2(1-(q-1)/(D-2))}$ | Const | 0 |

It is then concluded that

$$\lim_{\lambda \to 0} \lambda^{1-\alpha} \psi = s(D - 2) > 0.$$  \hspace{1cm} (5.14)

Discussions for the odd- and even-dimensional cases are made separately in the following subsection by the aid of the criterion obtained above.

### 5.1. Even dimensions

For $q = 1$, we find that the asymptotic form of equation (5.4) near $v = r = 0$ becomes

$$\frac{dk'}{d\lambda} + \frac{s_1}{r} (k')^2 \simeq 0$$  \hspace{1cm} (5.15)

with

$$s_1 \equiv \frac{K_1 - 2}{D - 2} > 0,$$  \hspace{1cm} (5.16)

where the constant $K_1$ satisfies equation (4.6). This belongs to case 1. We then obtain

$$\lim_{\lambda \to 0} \lambda^2 \psi = \frac{s_1(D - 2)}{(1 + s_1)^2} > 0.$$  \hspace{1cm} (5.17)

We next examine the case where $q > 1$. In this case, the asymptotic form of the geodesic equation becomes

$$\frac{dk'}{d\lambda} + s_2 r^{(q-\sigma)/(n-1)} (k')^2 \simeq 0$$  \hspace{1cm} (5.18)

with

$$s_2 = \frac{2q(M_0 v_0)^{1/(n-1)}}{n - 1} > 0.$$  \hspace{1cm} (5.19)

This belongs to case 2 with $\alpha = -1 + (q - 1)/(n - 1)$. Thus, we obtain

$$\lim_{\lambda \to 0} \lambda^{2(1-(q-1)/(D-2))} \psi = s_2(D - 2) > 0.$$  \hspace{1cm} (5.20)

The power of $\lambda$ depends on the power of the mass function and the number of the spacetime dimensions. It takes values from 0 for $q = D - 1$ to 2 for $q = 1$. The divergent behaviour of $\psi$ is summarized in table 1.

### 5.2. Odd dimensions

The steps in the odd dimensions are quite similar to those in even dimensions. We will first consider the radial null geodesics from the singularity $r = 0$ and $0 < v < v_{AH}$ with the asymptotic form (4.14). The asymptotic form of the geodesic equation (5.4) is given by

$$\frac{dk'}{d\lambda} + s_3 (k')^2 \simeq 0,$$  \hspace{1cm} (5.21)

where the constant $s_3$ is defined by

$$s_3 = \frac{2q(M_0 v_0)^{1/(n-1)}}{(n - 1)v_0(1 - (M_0 v_0)^{1/(n-1)})^2} > 0.$$  \hspace{1cm} (5.22)
Table 2. The divergent behaviour of $\psi$ at $v = r = 0$ along the radial null geodesics with the asymptotic form (4.14) in odd dimensions. It is noted that, for the naked singularity with $r = 0$ and $0 < v < v_{\text{AH}}$, $\psi \propto \lambda^{-\frac{1}{2}}$ is satisfied independent of $D$ and $q$.

| $q$ | $q = 1$ | $1 < q < D - 1$ | $q = D - 1$ | $q > D - 1$ |
|-----|---------|-----------------|-------------|------------|
| $\psi$ | $\lambda^{-2(1-\frac{1}{D-1})}$ | $\lambda^{-2(1-\frac{q}{D-1})}$ | Const | 0 |

This corresponds to case 2 with $\alpha = 0$. Consequently, we find

$$\lim_{\lambda \to 0} \lambda \psi = s_{\lambda}(D - 2)$$

(5.23)

for the singularity $r = 0$ and $0 < v < v_{\text{AH}}$. It is emphasized that the strength is independent of both the spacetime dimensions and the power of the mass function.

We finally consider the geodesics from the singularity $v = r = 0$. The asymptotic form of the radial null geodesics (5.4) becomes

$$\frac{dk'}{d\lambda} + s_{\lambda}r^{-1+q/(n-1)}(k')^2 \simeq 0,$$

(5.24)

with

$$s_{\lambda} = \frac{q(2^{q}M_{0})^{\frac{1}{n-1}}}{n-1} > 0.$$  

(5.25)

This corresponds to case 2 with $\alpha = -1 + q/(n - 1)$. Thus, we obtain

$$\lim_{\lambda \to 0} \lambda^{2(1-q/(D-1))}\psi = s_{\lambda}(D - 2)$$

(5.26)

for $v = r = 0$. The power of $\lambda$ takes the value from 0 for $q = D - 1$ to $2(1 - 1/(D - 1))$ for $q = 1$. In particular, the strength of the singularity for $r = 0$ and $0 < v < v_{\text{AH}}$ is weaker than that for $v = r = 0$ in the case of $(D - 1)/2 < q < D - 1$, while in the case of $q = (D - 1)/2$, their strengths are the same. The divergent behaviour of $\psi$ around $v = r = 0$ is summarized in table 2.

6. Conclusions and discussions

In this paper, we analysed the $D$ ($\geq 3$)-dimensional gravitational collapse of a null dust fluid in DC gravity. We found an exact solution with the topology of $\mathcal{M} \approx M^{2} \times K^{D-2}$, which describes the gravitational collapse. Applying this solution with spherical symmetry to the situation in which a null dust fluid radially injects into initially $D$-dimensional AdS spacetime, we investigated the effects of the Lovelock terms on the final fate of gravitational collapse. Our model is an example of dynamical and inhomogeneous collapse in Lovelock gravity and should provide better insight into the CCH in the context of higher-curvature gravity theories. We supposed that (i) the power-law mass function $M(v) = M_{0}v^{q}$, where $M_{0} > 0$ and $q \geq 1$ and (ii) the null geodesics obey a power law near the singularity.

We found that globally naked singularities can be formed in DC gravity. Furthermore, the final states of the gravitational collapse differ substantially depending on whether the spacetime dimensions are odd or even. This property is seen in the homogeneous collapse of a dust fluid in DC gravity [34]. It is also noted that odd-dimensional DC gravity is an exceptional case in the thin-shell collapse investigated in [35], only in which a naked singularity can emerge.

3 The singularities shown in the Penrose diagrams in [34] are spacelike both in odd and even dimensions. However, the singularity in the dust region is actually null in odd dimensions because the scale factor depends on the cosmic time $t$ linearly in the vicinity of $t = 0$, while it is spacelike because the scale factor in even dimensions behaves as $a(t) \propto t^{1-1/(D-1)}$ near $t = 0$. 


A massless ingoing null naked singularity can appear in the even-dimensional case \( (D = 2n) \) for \( 1 \leq q < D - 1 \). In the odd-dimensional case \( (D = 2n - 1) \), on the other hand, a massive timelike naked singularity is formed for any \( q \geq 1 \). These naked singularities can be globally naked. In three dimensions, the singularity is not a curvature singularity, where the scalar curvatures are the same as those for the three-dimensional AdS spacetime although the energy density of a null dust fluid diverges. As a result, the formation of a naked singularity cannot be avoided in DC gravity, nor in general relativity.

In DC gravity, massive timelike singularities appear in odd dimensions. When \( M(v) \) is intact in equation (4.11), it is easily shown that this property is independent of the form of \( M(v) \) as long as \( \dot{M} > 0 \) and \( M(0) = 0 \). The formation of a massive timelike singularity in odd dimensions is considered to be a characteristic feature in Lovelock gravity. Our results are consistent with those in general relativity and in Gauss–Bonnet gravity. In general relativity, a massive timelike singularity appears in three dimensions, where the Einstein–Hilbert term becomes first nontrivial (see appendix C). While in Gauss–Bonnet gravity, it appears in five dimensions, where the Gauss–Bonnet term becomes first nontrivial [20].

In section 5, we investigated the strength of naked singularities. We compare the strength by the divergent behaviour of \( \psi \equiv R_{\mu\nu k^\mu k^\nu} \). The strength of the naked singularity at \( v = r = 0 \) depends on \( D \) and \( q \) both in odd and even dimensions (see tables 1 and 2). In contrast, around the massive timelike singularity at \( r = 0 \) and \( 0 < v < v_{AH} \) in odd dimensions, \( \psi \) diverges as \( \lambda^{-1} \), independent of \( D \) or \( q \). Such divergent behaviour for the massive timelike singularity can also be seen both in general relativity in three dimensions (see appendix C) and in Gauss–Bonnet gravity in five dimensions [20]. Massive timelike singularities diverging as \( \lambda^{-1} \) might be salient features in Lovelock gravity.

We have shown in this paper that naked singularity formation is generic for the power-law mass function in DC gravity. We can speculate from the results that naked singularity formation will also be inevitable in full Lovelock gravity. It is also extrapolated that if massive timelike naked singularities appear in full Lovelock gravity, they would be developed only in odd dimensions and have a portion in which \( \psi \) would diverge as \( \lambda^{-1} \) regardless of the spacetime dimensions or the form of the mass function. Additional investigations are required in order to clarify these points.

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Appendix A. Geodesics from singularities

A.1. Existence and uniqueness

We apply the fixed-point method for proving the existence and uniqueness of future-directed outgoing radial null geodesics from singularities with asymptotic form (4.5) and (4.14).

We define new coordinates \( \theta \) and \( \chi \) as

\[
\theta \equiv \frac{r}{v - v_0}, \quad \chi \equiv (v - v_0),
\]

where \( s \) is a positive constant and we only consider the region with \( v \geq v_0 \). Then the geodesic equation (4.2) becomes
\[ \frac{d\theta}{d\chi} + \frac{1}{s\chi} (\theta - \eta) = \eta \Psi(\theta, \chi) \]

(A.2)

with

\[ \Psi(\theta, \chi) = \frac{1}{s\chi} \left[ \left( \frac{1}{2\eta} - 1 \right) + \frac{\theta^2 \chi^{2/s}}{2\eta^{2/s}} - \frac{\Phi(\theta, \chi)}{2\eta} \right]. \]

(A.3)

where a new parameter \( \eta (0 < \eta < \infty) \) has been introduced. The form of \( \Phi \) is given below for each case.

If we choose the parameters \( \eta \) and \( s \) to be \( \eta_0 \) and \( s_0 \) respectively, with which \( \Psi \) is at least \( C^1 \) in \( \chi \geq 0, \theta > 0 \), then we can apply the contraction mapping principle to equation (A.2) to find that there exists the solution satisfying \( \theta(0) = \eta_0 \) and moreover that it is the unique solution of equation (A.2) which is continuous at \( \chi = 0 \) (see [5, 30] for the proof).

### A.1.1. Even-dimensional DC-Vaidya solution.

For the even-dimensional DC-Vaidya solution, we have \( v_0 = 0 \) and

\[ \Phi = \left( \frac{2M_0 v^{q-1}}{\theta} \right)^{1/(n-1)} = \left( \frac{2M_0 \chi^{(q-1)/s}}{\theta} \right)^{1/(n-1)}. \]

(A.4)

For \( q > 1 \), if we choose \( \eta = \eta_0 = 1/2 \) and \( s = s_0 \) satisfying

\[ 0 < s_0 \leq 1, \quad 0 < s_0 \leq \frac{(q-1)}{2(n-1)}, \]

\( \Psi \) becomes at least \( C^1 \) in \( \chi \geq 0, \theta > 0 \), and consequently the existence of the solution with the asymptotic form (4.5) is shown. Unfortunately, this method cannot be applied to the case with \( q = 1 \).

### A.1.2. Odd-dimensional DC-Vaidya solution.

For the odd-dimensional DC-Vaidya solution, we have

\[ \Phi = (M_0 v^q)^{1/(n-1)} = [M_0 (v_0 + \chi^{1/q})^{1/(n-1)}]. \]

(A.6)

For \( 0 < v_0 < v_{AH} \), if we choose \( \eta = \eta_0 = (1 - (M_0 v_0^{1/(n-1)})^2) / 2 \) and \( s = s_0 \) satisfying

\[ 0 < s_0 \leq 1, \quad 0 < s_0 \leq \frac{q}{2(n-1)}, \]

\( \Psi \) becomes at least \( C^1 \) in \( \chi \geq 0, \theta > 0 \). Consequently, the existence of the solution with the asymptotic form (4.14) is shown.

### A.1.3. Vaidya-AdS solution.

We also study the general relativistic case. For the \( D \)-dimensional Vaidya-AdS solution in general relativity (see appendix C), we have

\[ \Phi = \frac{M_0 v^{q-D+3}}{\theta^{D-3}} = \frac{M_0 (v_0 + \chi^{1/q})^{q-D+3}}{\theta^{D-3}}. \]

(A.8)

First we consider the case in \( D \geq 4 \) dimensions, in which we have \( v_0 = 0 \). For \( q > D-3 \), if we choose \( \eta = \eta_0 = 1/2 \) and \( s = s_0 \) satisfying

\[ 0 < s_0 \leq 1, \quad 0 < s_0 \leq \frac{q-(D-3)}{2}, \]

(A.9)
\( \Psi \) becomes at least \( C^1 \) in \( \chi \geq 0, \theta > 0 \), and consequently the existence of the solution with the asymptotic form \((C.6)\) is shown. Unfortunately, this method cannot be applied to the case with \( q = D - 3 \).

Next we consider the three-dimensional case. For \( 0 < v_0 < v_{\text{AH}} \), if we choose \( \eta = \eta_0 = (1 - M_0 v_0^q) / 2 \) and \( s = s_0 \) satisfying \( 0 < s_0 \leq 1 \), \( \Psi \) becomes at least \( C^1 \) in \( \chi \geq 0, \theta > 0 \), and consequently the existence of the solution with the asymptotic form \((C.6)\) is shown. Unfortunately, this method cannot be applied to the case with \( q = D - 3 \).

For \( v_0 = 0 \), on the other hand, if we choose \( \eta = \eta_0 = 1 / 2 \) and \( s = s_0 \) satisfying \( 0 < s_0 \leq q / 2 \), \( \Psi \) becomes at least \( C^1 \) in \( \chi \geq 0, \theta > 0 \).

A.2. On the null and causal geodesics

In section 4, we have analysed only future-directed outgoing radial null geodesics. The contraposition of the following theorem implies that it is sufficient to consider only the future-directed outgoing radial null geodesics in order to determine whether or not the singularity is naked.

We will prove the following theorem. If a future-directed causal (excluding radial null) geodesic emanates from the central singularity, then a future-directed radial null geodesic emanates from the central singularity. The proof is similar to the four-dimensional case in [36].

We consider the DC-Vaidya solution, the metric of which is given by \((3.7)\) with \( k = 1 \). Without loss of generality, we set the geodesics on the several equatorial plane thanks to the spherical symmetry. In the DC-Vaidya spacetime, the tangent to a causal geodesic satisfies

\[
-f \left( \frac{dv}{d\lambda} \right)^2 + 2 \frac{dv}{d\lambda} \frac{dr}{d\lambda} + \frac{L^2}{r^2} = \epsilon,
\]

where \( \lambda \) is an affine parameter, \( L^2 \) is the sum of the square of conserved angular momenta and \( \epsilon = 0, -1 \) for null and timelike geodesics, respectively. Then, at any point on such a geodesic,

\[
f \left( \frac{dv}{d\lambda} \right)^2 \geq 2 \frac{dv}{d\lambda} \frac{dr}{d\lambda}
\]

with equality holding only for radial null geodesics. For the future-directed outgoing geodesics, this gives

\[
\frac{dv}{dr} \geq \frac{2}{f} > 0
\]

and

\[
\frac{dv}{dr} \leq \frac{2}{f} < 0
\]

in the untrapped and trapped regions, respectively. Therefore,

\[
\frac{dv_{\text{CG}}}{dr} > \frac{dv_{\text{KNG}}}{dr} > 0
\]

and

\[
\frac{dv_{\text{CG}}}{dr} < \frac{dv_{\text{KNG}}}{dr} < 0
\]

are satisfied in the untrapped and trapped regions, respectively, where the subscripts represent causal (excluding radial null) geodesics and outgoing radial null geodesics, respectively.

Let us consider the \((r, v)\)-plane. The singularity is located at \( r = 0 \) for \( v \geq 0 \). First, we consider the singularity with \( f < 0 \). By equation \((A.16)\), the past-directed ingoing geodesics emanating from an event with \( r > 0 \) in the trapped region cannot reach the singularity at \( r = 0 \).
Therefore, there is no future-directed outgoing geodesic emanating from the singularity in the trapped region, i.e., the singularity with $f < 0$ is censored.

Next, we focus on the singularity with $0 \leq v \leq v_{AH}$, where $f \geq 0$ holds. Now suppose that $v = v_{CG}(r)$ extends back to a central singularity located at $(r, v) = (0, v_0)$, where $v_0$ satisfies $0 \leq v_0 \leq v_{AH}$. There exists a portion of $v = v_{CG}(r)$ which is in the untrapped region. Let $p$ be any point on such a portion of $v = v_{CG}(r)$ and to the future of the singularity. Applying inequality (A.15) at $p$, we see that $v = v_{RG}(r)$ through $p$ crosses $v = v_{CG}(r)$ from above and hence the points on $v = v_{RG}(r)$ prior to $p$ must lie to the future of points on $v = v_{CG}(r)$ prior to $p$, in the sense $v_{RG}(r) > v_{CG}(r)$ for $r \in (0, r_0)$, where $r_0$ corresponds to $p$. Thus, the radial null geodesics must extend back to $r = 0$ with $v = v_0$ satisfying $v_0 \leq v_0 \leq v_{AH}$ and so must emerge from the singularity.

It is noted that $v_{AH} = v_0 = 0$ holds in the even-dimensional case.

**Appendix B. Curvature tensors**

We present the scalar curvatures in order to analyse the singularities. The curvature invariants for the metric (3.7) are

\[
I_1 = R_{abcd} R^{abcd} = (f')^2 + \frac{2(D-2)}{r^2} (f')^2 + \frac{2(D-2)(D-3)}{r^4} (k-f)^2, \tag{B.1}
\]

\[
I_2 = R_{abcd} R^{abcd} = \frac{1}{2} (f'')^2 + \frac{D-2}{r} f' (f')' + \frac{D(D-2)}{2r^2} (f')^2 - \frac{2(D-2)(D-3)}{r^3} (k-f)f',
\]

\[+ \frac{(D-2)(D-3)^2}{r^4} (k-f)^2. \tag{B.2}
\]

The Ricci scalar becomes

\[
R = -f'' - \frac{2(D-2)}{r} f' + \frac{(D-2)(D-3)}{r^2} (k-f). \tag{B.3}
\]

These formulae are applicable not only for the DC-Vaidya solution but also for the DC-BTZ solution (3.2) or the Vaidya-AdS solution (C.1). These quantities remain finite, as expected, for the three-dimensional BTZ solution [27].

**Appendix C. General relativistic collapse**

In general relativity, the function $f(v, r)$ in equation (3.7) and the energy density $\rho$ for the $D$-dimensional Vaidya-AdS solution for $k = 1$ are

\[
f(v, r) = 1 - \frac{M(v)}{r^{D-3}} + r^2 \rho^2, \tag{C.1}
\]

and

\[
\rho = \frac{D-2}{2k_D^2 r^{D-2}} M, \tag{C.2}
\]

respectively, where $M(v)$ is an arbitrary function of $v$. $k_D^2$ is related to the $D$-dimensional gravitational constant $G_D$ as $k_D^2 = 8\pi G_D$. We assume the form of $M(v)$ as a power law

\[
M(v) = M_0 v^q, \tag{C.3}
\]

as that in the case of DC gravity, where $M_0(> 0)$ and $q(\geq 1)$ are constants. We can see that the central singularity appears at $r = 0$ for $v > 0$. The singular nature at $v = r = 0$ will be clarified below. The trapped region is

\[
M(v) = M_0 v^q \geq r^{D-3} + l^{-2} r^{D-1}. \tag{C.4}
\]

We will consider the $D = 3$ and $D \geq 4$ cases separately in the following.
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C.1. \( D \geq 4 \) case

From equation (C.4), we find that only the point \( v = r = 0 \) has the possibility of being naked. Thus, a massive timelike naked singularity is absent in this case for any mass function with \( M(0) = 0 \) and \( M \geq 0 \).

We assume that the asymptotic form of the future-directed outgoing null geodesics near \( v = r = 0 \) as

\[
v \simeq K_3 r^p,
\]

where \( K_3 \) and \( p \) are positive constants. After some straightforward calculations, we find the asymptotic solution

\[
v \simeq 2r
\]

for \( q > D - 3 \). For \( q = D - 3 \), which is valid for \( D > 3 \), we obtain

\[
v \simeq K_3 r,
\]

where \( K_3 \) satisfies the relation

\[
M_0 K_3^{D-2} - K_3 + 2 = 0.
\]

The condition for the existence for positive \( K_3 \) satisfying equation (C.8) is given by

\[
0 < M_0 \leq \frac{1}{D-2} \left( \frac{D-3}{2(D-2)} \right)^{D-3}.
\]

When \( M_0 \) satisfies equation (C.9) for \( q = D - 3 \), the spacetime represents the formation of the naked singularities; otherwise we need a more careful investigation to determine the final state of the gravitational collapse. The right-hand side of equation (C.9) goes to zero as \( D \to +\infty \); here the naked singularity formation becomes less feasible as the spacetime dimensions are higher, similar to the case of the even-dimensional DC-Vaidya solution. While there exist no null geodesics with the asymptotic form (C.5) for \( 1 \leq q < D - 3 \), the geodesics (C.6) and (C.7) are singular null geodesics for \( D - 3 \leq q < D - 1 \). Thus the solution represents the formation of a massless ingoing null naked singularity for \( D - 3 < q < D - 1 \) for any \( M_0 \) (>0) and \( q = D - 3 \) with \( M_0 \) satisfying equation (C.9).

The asymptotic form of the null geodesic equation near \( v = r = 0 \) is given by

\[
\frac{d}{d\lambda} k' + \frac{s_5}{r} (k')^2 \simeq 0
\]

with

\[
s_5 \equiv \frac{K_3^{D-2} M_0 (D-3)}{2}
\]

for \( q = D - 3 \), where \( K_3 \) satisfies the relation (C.8). This belongs to case 1 in section 5. For \( q > D - 3 \), the asymptotic form of the null geodesic equation near \( v = r = 0 \) is given by

\[
\frac{d}{d\lambda} k' + s_6 r^{q-D+2} (k')^2 \simeq 0
\]

with

\[
s_6 \equiv 2^\alpha M_0 q.
\]

This belongs to case 2 with \( \alpha = q - D + 2 \).

It is then concluded

\[
\lim_{\lambda \to 0} \lambda^2 \psi = \frac{s_5 (D-2)}{(1 + s_5)^2}
\]

for \( q = D - 3 \), and

\[
\lim_{\lambda \to 0} \lambda^{D-(q+1)} \psi = s_6 (D-2)
\]

for \( D - 3 < q < D - 1 \). The divergent behaviour of \( \psi \) is summarized in table 3.
Table 3. The divergent behaviour of $\psi$ around $v = r = 0$ along the radial null geodesics with the asymptotic forms (C.6) for $D > q - 3$ and (C.7) for $q = D - 3$ in $D \geq 4$ dimensions.

| $q$     | $D - 3$ | $D - 3 < q < D - 1$ | $q = D - 1$ | $q > D - 1$ |
|---------|---------|---------------------|-------------|-----------|
| $\psi$  | $\lambda^{-2}$ | $\lambda^{-Dq+1}$ | Const      | 0         |

C.2. $D = 3$ case

From equation (C.4), the point $r = 0$, $0 \leq v \leq v_{AH} \equiv M_0^{-q}$ can be naked. In this case, the discussion in section 4.2 can be applied. There exist null geodesics which behave as equation (4.14) with $n = 2$ for $0 \leq v < v_{AH}$. The scalar curvature quantities are finite everywhere along these null geodesics although the energy density of the null dust fluid diverges. From section 5.2, along the null geodesics from $v = r = 0$, $\psi \propto R_{\mu\nu}k^\mu k^\nu$ diverges as $\lambda^{-2q}$. In contrast, along the null geodesics from $r = 0$, $0 < v < v_{AH}$, $\psi$ diverges as $\lambda^{-1}$ independent of the power of the mass function.

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