Resummation in Fractional APT: How many loops do we need to take into account? *

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We give a short introduction to the Analytic Perturbation Theory (APT) [1] and its generalization to Fractional powers — FAPT [2, 3]. We describe how to treat heavy-quark thresholds in FAPT and then show how to resum perturbative series in both the one-loop APT and FAPT. As an application we consider FAPT description of the Higgs boson decay $H^0 \rightarrow b\bar{b}$.

1. APT and FAPT in QCD

In the standard QCD Perturbation Theory (PT) we know the Renormalization Group (RG) equation $da_s[L]/dL = -\alpha_s^2 - \ldots$ for the effective coupling $\alpha_s(Q^2) = a_s[L]/\beta_f$ with $L = \ln(Q^2/\Lambda^2)$, $\beta_f = b_0(N_f)/(4\pi) = (11 - 2N_f/3)/(4\pi)$. Then the one-loop solution generates Landau pole singularity, $a_s[L] = 1/L$.

In the Analytic Perturbation Theory (APT) we have different effective couplings in Minkowskian (Radyushkin [4], and Krasnikov and Pivovarov [5]) and Euclidean (Shirkov and Solovtsov [1]) regions. In Euclidean domain, $-q^2 = Q^2$, $L = \ln Q^2/\Lambda^2$, APT generates the following set of images for the effective coupling and its $n$-th powers, $\{A_n[L]\}_{n\in\mathbb{N}}$, whereas in Minkowskian domain, $q^2 = s$, $L_s = \ln s/\Lambda^2$, it generates another set, $\{Q_n[L_s]\}_{n\in\mathbb{N}}$. APT is based on the RG and causality that guarantees standard perturbative UV asymptotics and spectral properties. Power series $\sum_m d_m a_s^n[L]$ transforms into non-power series $\sum_m d_m A_m[L]$ in APT.

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1 We use notations $f(Q^2)$ and $f[L]$ in order to specify the arguments we mean — squared momentum $Q^2$ or its logarithm $L = \ln(Q^2/\Lambda^2)$, that is $f[L] = f(\Lambda^2 \cdot e^L)$ and $\Lambda^2$ is usually referred to $N_f = 3$ region.
By the analytization in APT for an observable \( f(Q^2) \) we mean the “Källen–Lehman” representation

\[
[f(Q^2)]_{an} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon} \, d\sigma \quad \text{with} \quad \rho_f(\sigma) = \frac{1}{\pi} \text{Im} [f(-\sigma)].
\]  

(1)

Then in the one-loop approximation for the running coupling its spectral
density is \( \rho_1(\sigma) = \frac{1}{\sqrt{\sigma + Q^2}} \) and

\[
A_1[L] = \int_0^\infty \frac{\rho_1(\sigma)}{\sigma + Q^2} \, d\sigma = \frac{1}{L} - \frac{1}{e^L - 1},
\]

(2a)

\[
A_1[L_s] = \int_0^\infty \frac{\rho_1(\sigma)}{\sigma} \, d\sigma = \frac{1}{\pi} \arccos \frac{L_s}{\sqrt{\pi^2 + L_s^2}},
\]

(2b)

whereas analytic images of the higher powers \((n \geq 2, n \in \mathbb{N})\) are:

\[
\left( \frac{A_n[L]}{A_n[L_s]} \right) = \frac{1}{(n-1)!} \left( -\frac{d}{dL} \right)^{n-1} \left( A_1[L] \right),
\]

(3)

In the standard QCD PT we have also:

(i) the factorization procedure in QCD that gives rise to the appearance of logarithmic factors of the type: \( a_\nu[L] L^\nu \).

(ii) the RG evolution that generates evolution factors of the type: \( B(Q^2) = \left[ Z(Q^2)/Z(\mu^2) \right] B(\mu^2) \), which reduce in the one-loop approximation to \( Z(Q^2) \sim a_\nu[L] \) with \( \nu = \gamma_0/(2b_0) \) being a fractional number.

All these means we need to construct analytic images of new functions: \( a_\nu, a_\nu^m, \ldots \).

In the one-loop approximation using recursive relation (3) we can obtain explicit expressions for \( A_\nu[L] \) and \( A_\nu[L_s] \):

\[
A_\nu[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1 - \nu)}{\Gamma(\nu)};
\]

\[
A_\nu[L_s] = \sin \left[ (\nu - 1) \arccos \left( \frac{L}{\sqrt{\pi^2 + L^2}} \right) \right] / (\pi(\nu - 1)(\pi^2 + L^2)^{(\nu-1)/2}).
\]

(4)

Here \( F(z, \nu) \) is the reduced Lerch transcendental function, which is an analytic function in \( \nu \). They have very interesting properties, which we discussed extensively in our previous papers [2, 7].

Construction of FAPT with fixed number of quark flavors, \( N_f \), is a two-step procedure: we start with the perturbative result \( [a_s(Q^2)]^\nu \), generate the spectral density \( \rho_\nu(\sigma) \) using Eq. (1), and then obtain analytic couplings \( A_\nu[L] \) and \( A_\nu[L_s] \) via Eqs. (2). Here \( N_f \) is fixed and factorized out. We can

\footnote{First indication that a special “analytization” procedure is needed to handle these logarithmic terms appeared in [6].}
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proceed in the same manner for $N_f$-dependent quantities: $[\alpha_s(Q^2; N_f)]'' \Rightarrow \bar{\rho}_\nu(\sigma; N_f) = \bar{\rho}_\nu[L_\sigma; N_f] \equiv \rho_\nu(\sigma)/\beta_f' \Rightarrow \tilde{\mathfrak{A}}_\nu[L; N_f]$ and $\tilde{\mathfrak{A}}_\nu[L; N_f]$ — here $N_f$ is fixed, but not factorized out.

Global version of FAPT, which takes into account heavy-quark thresholds, is constructed along the same lines but starting from global perturbative coupling $[\alpha_s^{\text{glob}}(Q^2)]''$, being a continuous function of $Q^2$ due to choosing different values of QCD scales $\Lambda_f$, corresponding to different values of $N_f$. We illustrate here the case of only one heavy-quark threshold at $s = m^2_f$, corresponding to the transition $N_f = 3 \to N_f = 4$. Then we obtain the discontinuous spectral density

$$\rho_n^{\text{glob}}(\sigma) = \theta(L_{\sigma} < L_4) \bar{\rho}_n[L_{\sigma}; 3] + \theta(L_4 \leq L_{\sigma}) \bar{\rho}_n[L_{\sigma} + \lambda_4; 4] \quad (5)$$

with $L_{\sigma} \equiv \ln(\sigma/\Lambda^2_3)$, $L_f \equiv \ln(m^2_f/\Lambda^2_3)$ and $\lambda_f \equiv \ln(\Lambda^2_3/\Lambda^2_f)$ for $f = 4$, which is expressed in terms of fixed-flavor spectral densities with 3 and 4 flavors, $\bar{\rho}_n[L; 3]$ and $\bar{\rho}_n[L + \lambda_4; 4]$. However it generates the continuous Minkowskian coupling

$$\mathfrak{A}_\nu^{\text{glob}}[L_s] = \theta(L_s < L_4) \left( \tilde{\mathfrak{A}}_\nu[L_s; 3] - \tilde{\mathfrak{A}}_\nu[L_4; 3] + \tilde{\mathfrak{A}}_\nu[L_4 + \lambda_4; 4] \right) + \theta(L_4 \leq L_s) \tilde{\mathfrak{A}}_\nu[L_s + \lambda_4; 4] \quad (6)$$

and the analytic Euclidean coupling $\mathcal{A}_\nu^{\text{glob}}[L]$ (for more detail see in [7]).

2. Resummation in the one-loop APT and FAPT

We consider now the perturbative expansion of a typical physical quantity, like the Adler function and the ratio $R$, in the one-loop APT. Due to the limited space of our presentation we provide all formulas only for quantities in Minkowski region:

$$R[L] = d_0 + \sum_{n=1}^{\infty} d_n \mathfrak{A}_n[L]. \quad (7)$$

We suggest that there exist the generating function $P(t)$ for coefficients $d_n = d_n/d_1$:

$$\tilde{d}_n = \int_0^\infty P(t) t^{n-1} dt \quad \text{with} \quad \int_0^\infty P(t) dt = 1. \quad (8)$$

3 Remind here that $\beta_f = b_0(N_f)/(4\pi)$. 
To shorten our formulae, we use for the integral $\int_0^\infty f(t)P(t)dt$ the following notation: $\langle \langle f(t) \rangle \rangle_{P(t)}$. Then coefficients $d_n = d_1 \langle \langle t^{n-1} \rangle \rangle_{P(t)}$ and as has been shown in [8] we have the exact result for the sum in (7)

$$R[L] = d_0 + d_1 \langle \langle \mathfrak{A}_1[L - t] \rangle \rangle_{P(t)}. \quad (9)$$

The integral in variable $t$ here has a rigorous meaning, ensured by the finiteness of the coupling $A_1[t] \leq 1$ and fast fall-off of the generating function $P(t)$.

In our previous publications [7, 9] we have constructed generalizations of these results, first, to the case of the global APT, when heavy-quark thresholds are taken into account. Then one starts with the series of the type (7), where $A_n[L]$ are substituted by their global analogs $A_{\text{glob}}^n[L]$ (note that due to different normalizations of global couplings, $A_{\text{glob}}^n[L] \approx A_n[L]/\beta_f$, the coefficients $d_n$ should be also changed). Then

$$R^\text{glob}[L] = d_0 + d_1 \langle \langle \theta(L < L_4) \Delta_4 \mathfrak{A}_1[L] \rangle \rangle_{P(t)} + d_1 \langle \langle \theta(L \geq L_4) \bar{\mathfrak{A}}_1[L + \lambda_4 - t/\beta_4; 4] \rangle \rangle_{P(t)}; \quad (10)$$

where $\Delta_4 \mathfrak{A}_\nu[t] \equiv \bar{\mathfrak{A}}_\nu[L_4 + \lambda_4 - t/\beta_4; 4] - \bar{\mathfrak{A}}_\nu[L_3 - t/\beta_3; 3]$.

The second generalization has been obtained for the case of the global FAPT. Then the starting point is the series of the type $\sum_{n=0}^\infty d_n A_{\text{global}}^n[L]$ and the result of summation is a complete analog of Eq. (10) with substitutions

$$P(t) \Rightarrow P_\nu(t) = \int_0^1 P \left( \frac{t}{1 - x} \right) \frac{\nu x^{\nu - 1}dx}{1 - x}, \quad (11)$$

$$d_0 \Rightarrow d_0 \bar{\mathfrak{A}}_\nu[L], \quad \bar{\mathfrak{A}}_1[L - t] \Rightarrow \bar{\mathfrak{A}}_{1+\nu}[L - t], \quad \text{and} \quad \Delta_4 \bar{\mathfrak{A}}_1[t] \Rightarrow \Delta_4 \bar{\mathfrak{A}}_{1+\nu}[t].$$

All needed formulas have been also obtained in parallel for the Euclidean case.

3. Applications to Higgs boson decay

Here we analyze the Higgs boson decay to a $\bar{b}b$ pair. For its width we have

$$\Gamma(H \rightarrow \bar{b}b) = \frac{G_F}{4\sqrt{2}\pi} M_H \tilde{R}_s(M_H^2) \text{ with } \tilde{R}_s(M_H^2) = m_0^2(M_H^2) R_s(M_H^2) \quad (12)$$
and $R_s(s)$ is the $R$-ratio for the scalar correlator, see for details in \[2, 10\]. In the one-loop FAPT this generates the following non-power expansion:

$$
\tilde{R}_s[L] = 3 \tilde{m}^2_{(1)} \left\{ \mathcal{A}^{\text{glob}}_{0} [L] + d^S_1 \sum_{n \geq 1} \frac{d^S_n}{\pi^n} \mathcal{A}^{\text{glob}}_{n+1} [L] \right\},
$$

where $\tilde{m}^2_{(1)} = 8.45 \text{ GeV}^2$ is the RG-invariant of the one-loop $m_0^2(\mu^2)$ evolution $m_0^2(Q^2) = \tilde{m}^2_{(1)} \alpha_s^{\text{LO}}(Q^2)$ with $\nu_0 = 2\gamma_0/b_0(5) = 1.04$ and $\gamma_0$ is the quark-mass anomalous dimension (for a discussion — see in \[11\]).

![Relative errors and width](image)

Fig. 1. Left: The relative errors $\Delta^S_N[L]$, $N = 2, 3$ and 4, of the truncated FAPT in comparison with the exact summation result, Eq. (15). Right: The width $\Gamma^\infty_{H \rightarrow b\bar{b}}$ as a function of the Higgs boson mass $M_H$ in the resummed FAPT (solid line).

We take for the generating function $P(t)$ the Lipatov-like model of \[9\] with $\{c = 2.4, \beta = -0.52\}$

$$
\tilde{d}^S_n = c^{n-1} \frac{\Gamma(n+1) + \beta \Gamma(n)}{1 + \beta}; \quad P_s(t) = \frac{(t/c) + \beta}{c(1 + \beta)} e^{-t/c}. \quad (14)
$$

It gives a very good prediction for $\tilde{d}^S_n$ with $n = 2, 3, 4$, calculated in the QCD PT \[10\]: 7.50, 61.1, and 625 in comparison with 7.42, 62.3, and 620. Then we apply FAPT resummation technique to estimate how good is FAPT in approximating the whole sum $\tilde{R}_s[L]$ in the range $L \in [11.5, 13.7]$ which corresponds to the range $M_H \in [60, 180] \text{ GeV}$ with $\Lambda_{\text{QCD}} = 189 \text{ MeV}$ and $\mathcal{A}^{\text{glob}}_{0} (m_Z^2) = 0.122$. In this range we have ($L_6 = \ln(m^2_{Z}/\Lambda^2)$)

$$
\frac{\tilde{R}_s[L]}{3 \tilde{m}^2_{(1)}} = \mathcal{A}^{\text{glob}}_{0} [L] + \frac{d^S_1}{\pi} \left( \langle \bar{A}_{1+1+0} \left[ L + \lambda_5 - \frac{t}{\pi \beta_5} ; 5 \right] + \Delta_0 \bar{A}_{1+1+0} \left[ \frac{t}{\pi} \right] \rangle \right) P^S_{0} \quad (15)
$$

\footnote{Appearance of denominators $\pi^n$ in association with the coefficients $d_n$ is due to $d_n$ normalization.}
with $P_{t_0}^S(t)$ defined via Eqs. (14) and (11). Now we analyze the accuracy of the truncated FAPT expressions

$$\tilde{R}_S[L; N] = 3 \hat{m}^2(1) \left[ \mathfrak{A}_{t_0}^{\text{glob}}[L] + d_1^S \sum_{n=1}^{N} \frac{d_n^S}{\tau^n} \mathfrak{A}_{n+t_0}^{\text{glob}}[L] \right]$$

(16)

and compare them with the total sum $R_S[L]$ in Eq. (15) using relative errors

$$\Delta_S[N] = 1 - \frac{R_S[L; N]}{R_S[L]}.$$ 

In the left panel of Fig. 1 we show these errors for $N = 2$, $N = 3$, and $N = 4$ in the analyzed range of $L \in [11, 13.8]$. We see that already $R_S[L; 2]$ gives accuracy of the order of 2.5%, whereas $R_S[L; 3]$ of the order of 1%. That means that there is no need to calculate further corrections: at the level of accuracy of 1% it is quite enough to take into account only coefficients up to $d_3$. This conclusion is stable with respect to the variation of parameters of the model $P_S(t)$ and is in a complete agreement with Kataev–Kim conclusion [11].

4. Conclusions

In this report we described the resummation approach in the global versions of the one-loop APT and FAPT and argued that it produces finite answers, provided the generating function $P(t)$ of perturbative coefficients $d_n$ is known. The main conclusion is: To achieve an accuracy of the order of 1% we do not need to calculate more than four loops and $d_4$ coefficients are needed only to estimate corresponding generating functions $P(t)$.

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