Presenting with Quantitative Inequational Theories

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Abstract

It came to the attention of myself and the coauthors of [1] that a number of process calculi can be obtained by algebraically presenting the branching structure of the transition systems they specify. For example, labelled transition systems branch into finite sets of transitions representable by terms in the free semilattice generated by the transitions and probabilistic labelled transition systems branch into finitely supported probability distributions representable by terms in the free convex algebra generated by the transitions [1]–[3].

Since the mentioned theories are equational, their presentations are given in terms of monads on the category of sets, but restricting branching structures to set-valued monads has a number of undesirable limitations. In [4], I show how to extend the framework of [1] to ordered branching structures given by monads on the category of partially ordered sets. Unfortunately, ordered analogues of a few important examples of monad presentations seem to be missing from the literature, despite their relevance to process theory beyond labelled transition systems. My goal in this article is to initiate a line of research that addresses this gap in a way that lends itself well to studying behavioural inequalities of weighted automata and related models.

In this article, I discuss examples of monads on the category of partially ordered sets coming from quantitative theories, free modules over ordered semirings, and give sufficient conditions for one of these to lift a monad on the category of sets. I also discuss monad presentations in general. In the last section, I give descriptions of ordered semirings that are useful for specifying unguarded recursive calls. Applications include ordered probability theory and ordered semilattices.

1 Monad presentations

Fix an endofunctor $S : C \to C$ on a category $C$. An ($S$-)algebra is a pair $(X, \alpha)$ consisting of an object $X$ of $C$ and an arrow $\alpha : SX \to X$. A homomorphism $h : (X, \alpha) \to (Y, \beta)$ is an arrow $h : X \to Y$ in $C$ such that $S(h) \circ \alpha = \beta \circ h$. Alg$_C(S)$ denotes the category of $S$-algebras and homomorphisms.

Intuitively, a monad is a free-algebra construction that takes some desirable algebraic properties as input and outputs the initial algebra satisfying those properties. In the most general picture, a property is simply a full subcategory of Alg$(S)$. The following notion of monad presentation is inspired by [5] (see also [6, IV.§8]).

Definition 1. Let $T$ be a full subcategory of Alg$_C(S)$. A T-presented monad is a triple $(M, \eta, \rho)$ consisting of the following ingredients: (i) An endofunctor $M : C \to C$, (ii) a natural transformation $\eta : \text{Id} \Rightarrow M$, and (iii) a natural transformation $\rho : SM \Rightarrow M$ such that

(a) $(MX, \rho_X) \in T$ for any object $X$ of $C$, and

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(b) for any \(S\)-algebra \((Y, \beta)\) and any arrow \(f : X \to Y\) in \(C\), there is a unique \(S\)-algebra homomorphism \(f^\beta : (MX, \rho_X) \to (Y, \beta)\) such that \(f^\beta \circ \eta = f\).

This notion of monad is not entirely standard. To see its connection to the usual notion, we turn to the following two results.

**Theorem 2.** Every \(T\)-presented monad \((M, \eta, \rho)\) corresponds to a unique monad \((M, \eta, \mu)\) on \(C\).

**Proof.** Consider the identity arrow \(\text{id}_{MX} : MX \to MX\). We define the natural transformation \(\mu : MM \Rightarrow M\) to have the components \(\mu_X = \text{id}_{\rho_{MX}}\), where \(\text{id}_{\rho_{MX}}\) is the unique \(S\)-algebra homomorphism \((MMX, \rho_{MX}) \to (MX, \rho_X)\) such that \(\text{id}_{\rho_{MX}} \circ \eta = \text{id}_{MX}\). It is now routine to check that \((M, \eta, \mu)\) is a monad.

Given an arbitrary monad \((M, \eta, \mu)\), one simply needs to verify that \(\mu_X : MMX \to MX\) is an \(S\)-algebra homomorphism \((MMX, \rho_{MX}) \to (MX, \rho_X)\).

**Theorem 3.** The functor \(\Phi : (X, \alpha) \mapsto (X, \text{id}_\alpha X)\) is a functor between \(T\) and the category of Eilenberg-Moore algebras \(EM(M, \eta, \mu)\), where \(\mu_X = \text{id}_{\rho_{MX}}\). If \(T\) is closed under split epis (i.e. if \((X, \alpha) \in T\) and \((\exists m, i) m : (X, \alpha) \to (Y, \beta)\) and \(m \circ i = \text{id}_Y\), then \((Y, \beta) \in T\)), then \(\Phi\) is an isomorphism of categories.

**Proof.** By assumption, \((X, \beta) \in T\), so the universal property of \(\rho\) applies to it. We start by verifying that \((X, \text{id}_\beta X)\) is indeed an Eilenberg-Moore algebra. By naturality of \(\eta\) and \(\rho\), the following diagram commutes.

\[
\begin{array}{ccc}
SMMX & \longrightarrow & SX \\
\downarrow^{\rho_M} & & \downarrow^{\beta} \\
MMX & \longrightarrow & MX \\
\downarrow^\eta & & \downarrow^\eta \\
MX & \longrightarrow & X
\end{array}
\]

By the universal property of \(\rho\), \(\text{id}_\beta X \circ M(\text{id}_\beta X)\) is the unique \(S\)-algebra homomorphism satisfying \((\text{id}_\beta X \circ M(\text{id}_\beta X)) \circ \eta = \text{id}_X\).

On the other hand, the following commutes because \(\mu \circ \eta_M = \text{id}_M\).

\[
\begin{array}{ccc}
SMMX & \longrightarrow & SX \\
\downarrow^{\rho_M} & & \downarrow^{\beta} \\
MMX & \longrightarrow & MX \\
\downarrow^\eta & & \downarrow^{\text{id}_M} \\
MX & \longrightarrow & X
\end{array}
\]

This means that \(\text{id}_\beta X \circ \mu_M\) is also an \(S\)-algebra homomorphism satisfying \((\text{id}_\beta X \circ \mu_M) \circ \eta = \text{id}_X\). It follows that \(\text{id}_\beta X \circ \mu_M = \text{id}_\beta X \circ M(\text{id}_\beta X)\). Since \(\text{id}_\beta X \circ \eta = \text{id}_X\) by definition, \((X, \text{id}_\beta X)\) is an EM-algebra for \((M, \eta, \mu)\). This establishes that \(\Phi\) is indeed a functor into \(EM(M, \eta, \mu)\).

Now we construct the inverse of \(\Phi\). Let \(\Theta(X, \gamma) = (X, \gamma \circ \rho \circ S(\eta))\). We will show that \((X, \gamma \circ \rho \circ S(\eta)) \in T\) by exhibiting a split epi \(MX \to X\). In fact, the split epi is \(\gamma\), since \(\gamma \circ \eta = \text{id}_X\), so we just need to show that \(\gamma\) is an \(S\)-algebra homomorphism. This can be seen from the following diagram.

\[
\begin{array}{ccc}
SMX & \xrightarrow{S(\eta)} & SMX \\
\downarrow^{S(\gamma)} & & \downarrow^{SM(\gamma)} \\
SX & \xrightarrow{\rho} & MX \\
& & \downarrow^\gamma \\
& & X
\end{array}
\]
This diagram commutes by naturality of $\mu$, $\rho$, $S(\eta)$, and the fact that $(X, \gamma)$ is am EM-algebra. We furthermore know that

$$
\mu \circ \rho_M \circ S(\eta) = \mu \circ \eta_M \circ \rho_M \\
= \text{id}_M \circ \rho_M \\
= \rho
$$

(naturality) (def. of monad)

Hence, $\Theta : \text{EM}(M, \eta, \mu) \to T$. Furthermore, $\gamma$ is the unique $S$-algebra homomorphism $(MX, \rho) \to \Theta(X, \gamma)$ such that $\gamma \circ \eta = \text{id}_X$, so in fact $\Phi\Theta(X, \gamma) = (X, \gamma)$. It therefore suffices to see that $\Theta\Phi(X, \beta) = (X, \beta)$. This is rather easy, fortunately:

$$
id^X_\beta \circ \rho \circ S(\eta) = \beta \circ S(id^X_\gamma) \circ S(\eta) = \beta \circ S(id^X_\gamma \circ \eta) = \beta \circ S(id_X) = \beta
$$

Remark 4. The assumption that $T$ is closed under split epis is not very strict. For example, every variety is closed under split epis, because every split epi is automatically regular [7], [8].

## 2 Quantitative theories

Discrete processes that branch with observable quantities are popular in the automata theory and process algebra literature [2], [9]–[11]. In the cited works, formal calculi are used for specifying and reasoning about behaviours of such processes. Algebraically reasoning about behaviour inevitably relies on an algebraic characterisation of the branching structures of processes.

Quantitative branching is often depicted by decorating transitions with numbers, vectors, or other quantities. Generally, quantities appear as elements of a *semiring* $P = (P, 0, 1, +, \cdot)$, a set $P$ equipped with constants 0, 1 and binary operations $+, \cdot$ such that $(P, 0, +)$ is a commutative monoid, $(P, 1, \cdot)$ is a monoid, $0r = 0$, $r(s + t) = rs + rt$, and $(s + t)r = sr + tr$ for all $r, s, t \in P$ [12].

**Definition 5.** A $P$-module is a monoid $(X, 0, +)$ equipped with an action $P \times X \to X$, written $(r, x) \mapsto rx$, such that $0x = 0$, $1x = x$, $(rs)x = r(sx)$, and $(r + s)x = rx + sx$ for $x \in X$ and $r, s \in P$.

Abstractly, $P$-modules are algebras for $S = P \times \text{Id} + \{+\} \times \text{Id}^2$ that satisfy a handful of equations, where $\text{Id}$ is the identity on $\text{Sets}$. They form a full subcategory $P\text{Mod}$ of $\text{Alg}_{\text{Sets}}(S)$. The free $P$-module on $X$ consists of finitely supported $\theta : X \to P$, where the support of $\theta$ is $\text{supp}(\theta) = \{x \in X \mid \theta(x) \neq 0\}$.

**Definition 6.** The free $P$-module functor $O_P : \text{Sets} \to \text{Sets}$ is given by $O_PX = \{\theta : X \to P \mid \text{supp}(\theta) \text{ is finite}\}$ $O_P(h)(\theta)(y) = h^*\theta(y) := \sum_{h(x) = y} \theta(x)$

for any set $X$ and any $h : X \to Y$. The $P\text{Mod}$-presented monad is the triple $(O_P, \delta, \rho)$, defined

$$
\delta_X(x) = \delta_x := \lambda y. \begin{cases} 1 & x = y \\
0 & x \neq y \end{cases} \rho(0) = 0 \rho(r, \theta) = r\theta \rho(+, \theta_1, \theta_2) = \theta_1 + \theta_2
$$

where $0(x) = 0$, and $r\theta$ and $\theta_1 + \theta_2$ are evaluated pointwise.

**Example 7.** Consider $2 = (\{0, 1\}, 0, 1, \text{max}, \text{min})$. $2\text{Mod}$ is equivalent to the category of join semilattices with bottom. In particular, $O_2$ is naturally isomorphic to the finitary powerset functor $P_\omega$.

**Example 8.** Consider $R^+ = (\mathbb{R}_{\geq 0}, 0, 1, +, \times)$. $R^+\text{Mod}$ is equivalent to the category of positive cones of finite dimensional real-valued metric spaces and linear maps with nonnegative entries.
3 Ordered quantitative theories

A notable example of the power of algebraic presentation for the purposes of reasoning about behaviour is Stark and Smolka’s algebra of probabilistic actions [9], which is used to study processes that branch probabilistically, with weights from the semiring $\mathbb{R}^+$. An important feature of their calculus is its interpretation of recursive specifications as least fixed-points.

Recently [4], I made the observation that the existence of these least fixed-points is a property of the inequational theory obtained from the theory of $\mathbb{R}^+$-modules. Where $\text{Pos}$ is the category of partially ordered sets (posets) and monotone maps, an inequational theory is a set of inequations that describes a category of $S$-algebras for a functor $S : \text{Pos} \to \text{Pos}$, or ordered algebras.

Definition 9. An ordered semiring $P = (P, \le, 0, 1, +, \cdot)$ is a semiring $(P, 0, 1, +, \cdot)$ where $(P, \le)$ is a poset with $0$ as its bottom element, and $+$ and $\cdot$ are monotone. An ordered $P$-module is a structure $(X, \le, 0, +)$ consisting of a $P$-module $(X, 0, +)$ such that $r \le s$ and $x \le y$ implies $rx + z \le sy + z$ for any $r, s \in P$ and $x, y, z \in X$.

Abstractly, an ordered $P$-module is an algebra for $S = (P, \le) \times \text{Id} + \{\{\cdot\}, =\} \times \text{Id}$ satisfying a set of inequations, where $\text{Id}$ is the identity functor on $\text{Pos}$ now. We also write $\text{PMod}$ for the full subcategory of $\text{Alg}_{\text{Pos}}(S)$ consisting of ordered $P$-modules.

If the reader is anything like myself, they might expect me to say that $\text{Op}$ constructs the free ordered $P$-module on a poset $(X, \le)$. This is not exactly the case. For example, turn the semiring 2 from Example 7 into an ordered semiring by taking $0 < 1$. Then an ordered 2-module consists of the same data as a semilattice with bottom, in the sense of order theory [13]. The free ordered semilattice on a poset $(X, \le)$ is carried by the finitely generated downwards-closed subsets of $(X, \le)$, which is not even the same size as $\mathcal{P}_X$ in general!

In many cases, quotienting the set $\text{Op}X$ by a certain preorder suffices. Call a subset $U \subseteq X$ upwards closed if $x \in U$ and $x \le y$ implies $y \in U$.

Definition 10. The heavier-higher order is the preorder $\sqsubseteq$ on $\text{Op}X$ defined so that $\theta_1 \sqsubseteq \theta_2$ if and only if for any upwards-closed $U \subseteq X$, $\sum_{x \in U} \theta_1(x) \le \sum_{x \in U} \theta_2(x)$ (see for e.g. [14]).

We write $\theta_1 \equiv \theta_2$ if $\theta_1 \sqsubseteq \theta_2 \sqsubseteq \theta_1$, $[\theta_1] = \{\theta_2 \mid \theta_1 \equiv \theta_2\}$, and $\tilde{\text{Op}}(X, \le) = (\text{Op}X/\equiv, \sqsubseteq)$. The next lemma tells us that $\tilde{\text{Op}}$ is an endofunctor on $\text{Pos}$.

Lemma 11. For any monotone map $f : (X, \le) \to (Y, \le)$, the map $f^\ast : \tilde{\text{Op}}X \to \tilde{\text{Op}}Y$ is monotone w.r.t. the heavier-higher order.

Proof. Let $U \subseteq Y$ be an upwards closed subset of $(Y, \le)$. Then $f^{-1}(U)$ is an upwards closed subset of $(X, \le)$. We have

$$f^\ast \theta_1(U) = \sum_{u \in U} f^\ast \theta_1(u) = \sum_{u \in U} \sum_{f(x) = u} \theta_1(x) = \sum_{f(x) \in U} \theta_1(x) \le \sum_{f(x) \in U} \theta_2(x) = f^\ast \theta_2(U)$$

Next we indicate the cases where free ordered modules are constructed by the functor $\tilde{\text{Op}}$. Say that a semiring is cancellative if $(\forall r, s, t) r + s \le r + t$ implies $s \le t$, difference ordered if $r \le s$ implies $(\exists t) t + t = s$, and idempotent if $r + r = r$.

\(^1\)These are sometimes called positive ordered semirings in the literature [12].
Theorem 12. If $P$ is either cancellative difference-ordered or idempotent, then $(O_p, [\delta], [\rho])$ is the PMod-presented monad.

This follows from the next three facts.

**Fact (1)** The transformation $\delta$ is monotone with respect to $\sqsubseteq$.

**Fact (2)** $(O_p(X, \leq), \rho)$ is an ordered $P$-module. This means that for $\theta_1, \theta_2, \varphi \in O_\omega X$ and $p, q \in P$,

1. if $\theta_1 \sqsubseteq \theta_2$, then $\theta_1 + \varphi \sqsubseteq \theta_2 + \varphi$, and
2. if $p \leq q$ (in $P$), then $p \varphi \sqsubseteq q \varphi$.

**Fact (3)** For any ordered $P$-module $(Y, \leq, \beta)$ and any monotone $f : (X, \leq) \to (Y, \leq)$, the homomorphism $f^\beta : (O_\omega X, \rho) \to (Y, \beta)$ defined

\[
\begin{align*}
f^\beta(0) &= 0 \\
f^\beta(\delta_x) &= f(x) \\
f^\beta(r \theta) &= rf^\beta(\theta) \\
f^\beta(\theta_1 + \theta_2) &= f^\beta(\theta_1) + f^\beta(\theta_2)
\end{align*}
\]

is monotone with respect to the heavier-higher order.

We are only going to argue for Fact (3), as it is the only one with a nontrivial proof (additionally, Facts (1) and (2) do not require the cancellative, difference ordered, or idempotent assumptions). The argument uses the following lemma, whose proof we omit.

**Lemma 13.** Let $P$ be an ordered semiring.

1. If $P$ is cancellative and difference ordered, then $r \leq s$ implies there exists a unique $t \in P$ such that $r + t = s$. (We write the witness as $t = s - r$.)

2. If $P$ is idempotent, then in every ordered $P$-module $(X, \leq, +)$, $x \leq y$ if and only if $x + y = y$.

**Proof of Fact (3).** Let $f : (X, \leq) \to (Y, \leq)$ be a monotone map into an ordered $P$-module $(Y, \leq, 0, +)$, and let $\theta_1 \sqsubseteq \theta_2$. We proceed by induction on the support of $\theta_1$.

Assume $P$ is cancellative difference-ordered. Let $z$ be any maximal element of $\text{supp}(\theta_1)$, so that $\theta_1(z) = \theta_1(\uparrow z)$. Define

\[
\theta'_1(x) = \begin{cases} 
\theta_1(z) - \theta_1(z) & x = z \\
\theta_1(x) & \text{otherwise}
\end{cases}
\]

For any upwards closed subset $U \ni z$, $\theta_1(U) = \theta_1(z) + \theta'_1(U)$, and

\[
\theta_1(z) + \theta'_1(U) = \theta_1(U) \leq \theta_2(U) = \theta_1(z) + \theta'_2(U)
\]

By cancellativity, $\theta'_1(U) \leq \theta'_2(U)$. For any upwards closed subset $U$ such that $z \notin U$, $\theta_1(U) = \theta'_1(U)$, and again $\theta'_1(U) \leq \theta'_2(U)$. It follows that $\theta'_1 \sqsubseteq \theta'_2$, so from the induction hypothesis we have $f^\beta(\theta'_1) \leq f^\beta(\theta'_2)$. Since $f^\beta$ is a homomorphism,

\[
f^\beta(\theta_1) = f^\beta(\theta_1(z)\delta_z + \theta'_1) \\
= f^\beta(\theta_1(z)\delta_z) + f^\beta(\theta'_1) \\
\leq f^\beta(\theta_1(z)\delta_z) + f^\beta(\theta'_2) \\
= f^\beta(\theta_1(z)\delta_z + \theta'_2) \\
= f^\beta(\theta_2)
\]
Therefore $f^β$ is monotone when $P$ is cancellative and difference-ordered.

Now assume $P$ is idempotent. Let $z$ be any maximal element of $\text{supp}(θ_1)$ and define

$$θ'_1(x) = \begin{cases} 0 & x = z \\ θ_1(x) & x ≠ z \end{cases}$$

Then $θ'_1 ⊆ θ_2$, since $θ_1(U) = θ'_1(U) + θ_1(z)$ if $z ∈ U$ and $θ_1(U) = θ'_1(U)$ if $z ∉ U$. By the induction hypothesis, $f^β(θ'_1) ≤ f^β(θ_2)$.

On the other hand, $θ_1(z)δ_z ⊆ θ_2$ by assumption. This means there is some $y ∈ \text{supp}(θ_2)$ such that $z ≤ y$ and $θ_1(z) ≤ θ_2(y)$. We therefore have

$$f^β(θ_1) = f^β(θ_1(z)δ_z + θ'_1)$$
$$= θ_1(z)f(z) + f^β(θ'_1)$$
$$≤ θ_1(z)f(z) + f^β(θ_2)$$
$$≤ θ_2(y)f(y) + f^β(θ_2)$$
$$≤ f^β(θ_2) + f^β(θ_2)$$
$$= f^β(θ_2)$$

this shows that $f^β$ is monotone when $P$ is idempotent.

In fact, if $P$ is cancellative, like in Example 8, then $⊆$ is a partial order. Consequently, by Theorem 12, if $P$ is cancellative and difference ordered, $(\hat{O}_P, [δ], [ρ])$ lifts the monad $(\mathcal{O}_P, δ, ρ)$ on $\text{Sets}$.

**Theorem 14.** If $P$ is cancellative, then $(\hat{O}, [δ], [ρ])$ lifts $(\mathcal{O}, δ, ρ)$.

**Proof.** We show that $⊆$ is antisymmetric. Let $θ_1 ⊆ θ_2 ⊆ θ_1$. Define $θ'_i(x) = θ_i(x)$ when $x ≠ x_0$ and $θ'_i(x_0) = θ_i(x_0) - θ_1(x_0)$, $i = 1, 2$. Since $θ_1 = θ_1(x_0) ⊗ δ_{x_0} + θ'_1$, by cancellativity of $P$

$$θ_1(x_0) ⊗ δ_{x_0} + θ'_1 ⊆ θ_1(x_0) ⊗ δ_{x_0} + θ_2 ⊆ θ_1(x_0) ⊗ δ_{x_0} + θ'_1$$

implies $θ'_1 ⊆ θ'_2 ⊆ θ'_1$. By induction, $θ'_1 = θ'_2$, so $θ_1 = θ_2$ as well.

**Remark 15.** The assumption that $P$ is cancellative is necessary here. Consider the example of $P = 2 = (\{0, 1\}, ≤, 0, 1, \text{max}, \text{min})$. The heavier-higher order is not a partial order on $\mathcal{O}_ω\{0, 1\}$, where $0 < 1$, because it cannot distinguish between $δ_1$ and $δ_0 + δ_1$.

Theorem 12 also extends to products of cancellative difference-ordered and idempotent semirings.

**Corollary 16.** Suppose $P = \prod_{i ∈ I} P_i$, where $P_i$ is either cancellative and difference ordered, or idempotent, for each $i ∈ I$. Then $(\hat{O}_P, [δ], [ρ])$ is the $P\text{Mod}$-presented monad.

Despite the apparent necessity of the extra assumptions in Theorem 12, the construction $(\hat{O}_P, [ρ])$ always produces a $P$-module, and the transformation $[δ] : \text{Id} ⇒ \hat{O}_P$ is always monotone with respect to the heavier-higher order. One might wonder if this construction is therefore more general.

**Question 1.** Are the restrictions on $P$ in Theorem 12 necessary? In other words, is there a semiring $P$ such that $\hat{O}_P$ does not construct the free ordered $P$-module?

A negative answer to this question would consist of an example of an ordered semiring $P$ (that cannot be decomposed into cancellative difference ordered and idempotent semirings) and a monotone map $f : (X, ≤) → (Y, ≤)$ into a $P$-module $(Y, ≤, β)$ such that the inductively defined homomorphism $f^β : \mathcal{O}_P X → Y$ is not monotone with respect to the heavier-higher order.
Example 17. Let \( A \) be the ordered semiring \((A, \leq, \vec{0}, \vec{1}, +, \cdot)\), where
\[
A = \{ f : \mathbb{N} \to \mathbb{N} \mid \text{either } f \text{ is constant or } (\forall n) f^{-1}(n) \text{ is finite} \}
\]
and \( \leq \) is the point-wise order. Then \( A \) is cancellative, so \( \subset \) is a partial order. On the other hand, \( A \) is not difference-ordered: consider \( f_1(n) = n \) and \( f_2(n) = n + (n \mod 2) \). Then \( f_1 \leq f_2 \), but \((f_2 - f_1)(n) = n \mod 2\), which is 0 on an infinite proper subset of \( \mathbb{N} \). This means that \( A \) cannot be of the form described in Theorem 12, as a product of difference-ordered semirings is always difference-ordered. Is this a counterexample?

4 Distribution theories

Finally, I would like to consider an application of our study of ordered semirings to subprobabilistic state-based systems. For a given ordered semiring \( P \), define
\[
S_{P}X = \left\{ \theta \in O_{P}X \mid \sum_{x \in X} \theta(x) \leq 1 \right\} \quad \bar{S}_{P}(X, \leq) = (S_{P}X/\equiv, \subseteq)
\]
With the added \( \bar{S}_{P}(h) = h^{*} \), \( \bar{S}_{P} \) is a subfunctor of \( O_{P} \). Standard subprobability distributions are obtained by setting \( P = \mathbb{R}_{+} \), and we will simply write \( S \) instead of \( S_{\mathbb{R}_{+}} \).

I define a (subprobabilistic) state-based decision process (or SDP) to be a function of the form \( \gamma : X \to SY \), where \( X \) is the state space and \( Y \) is a set, disjoint from \( X \), that represents extra data like state transitions or outputs of the machine. In [1], [4] for example, \( Y = \text{Var} + \text{Act} \times X \) for fixed sets \( \text{Var} \) and \( \text{Act} \).

For a concrete example of a subprobabilistic SDP, consider rolling a six-sided die. There is one state in this system, with six outputs.

Unfortunately, implementing this decision process in everyday life requires you to have a six-sided die on hand. As Knuth and Yao observe in [15], however, a series of coin flips can do the same job. The Knuth-Yao algorithm can be visualised as a sort of state-based system, except that it involves state-to-state transitions that carry probabilities.
Recognizing the six-sided die hidden in the SDP above amounts to determining the probability of eventually outputting one of $\ominus, \ldots, \oplus$ if the initial state is $x_1$. This produces the subprobabilistic state-based system $\gamma^1 : X \rightarrow SY$ below,

| $\gamma^1(x_i)(-)$ | $\ominus$ | $\ominus$ | $\ominus$ | $\ominus$ | $\ominus$ | $\ominus$ |
|-------------------|---------|---------|---------|---------|---------|---------|
| $x_1$             | $1/6$   | $1/6$   | $1/6$   | $1/6$   | $1/6$   | $1/6$   |
| $x_2$             | $1/4$   | $1/4$   | $1/4$   | $1/4$   | $0$     | $0$     |
| $x_3$             | $1/12$  | $1/12$  | $1/12$  | $1/12$  | $1/3$   | $1/3$   |
| $x_4$             | $1/2$   | $1/2$   | $0$     | $0$     | $0$     | $0$     |
| $x_5$             | $0$     | $0$     | $1/2$   | $1/2$   | $0$     | $0$     |
| $x_6$             | $0$     | $0$     | $0$     | $0$     | $1/2$   | $1/2$   |

Notice that $x_1$ represents the six-sided die system from before. Algebraically, the defining feature of $\gamma^1$ is that it is the unique solution to the recursive specification

$$
x_1 = (1/2)x_2 + (1/2)x_3
$$

$$
x_2 = (1/2)x_4 + (1/2)x_5
$$

$$
x_3 = (1/2)x_1 + (1/2)x_6
$$

$$
x_4 = (1/2)\ominus + (1/2)\ominus
$$

$$
x_5 = (1/2)\ominus + (1/2)\ominus
$$

$$
x_6 = (1/2)\ominus + (1/2)\ominus
$$

in the sense that if you replaced each $x_i$ with $\gamma^1(x_i)$, you would see six true identities.

Let us call an SDP $X \rightarrow SpY$ unguarded when $X \cap Y \neq \emptyset$, and note that every unguarded system can be written in the form $\gamma : X \rightarrow Sp(X + Y)$ for some disjoint $X$ and $Y$. In the case of sets, it is clear which unguarded SDPs have unique solutions and which do not: $x_1 = x_1$ has many solutions, for example, but $x_1 = (1/3)x_1 + (2/3)y$ has only one. Comparing subdistributions point-wise, one can show that every recursive specification has a least solution. What I would like to investigate next is for which ordered semirings $P$ we can guarantee that every finite\(^2\) unguarded SDP with ordered outputs $\gamma : (X, =) \rightarrow Sp((X, =) + (Y, \leq))$ has a least solution $\gamma^1 : (X, =) \rightarrow Sp(Y, \leq)$.

One way to ensure that every unguarded SDP has a least solution is to mimic the conditions of $R^+$. To this end, call an ordered semiring $P$ division if $(\forall r > 0)(\exists s) rs = sr = 1$. Note that $s$ is necessarily unique, so we write $s = r^{-1}$.

Given a poset $(Y, \leq)$, the set $Sp(Y, \leq)$ can always be equipped with the point-wise order, which simply has $\theta_1 \leq \theta_2$ if and only if $(\forall x) \theta_1(x) \leq \theta_2(x)$. Of course, if $\theta_1 \leq \theta_2$, then $\theta_1 \sqsubseteq \theta_2$.

**Theorem 18.** If $P$ is cancellative, difference ordered, and division, then every finite subprobabilistic unguarded system $\gamma : (X, =) \rightarrow Sp((X, =) + (Y, \leq))$ has a least solution $\gamma^1 : (X, =) \rightarrow Sp(Y, \leq)$ with respect to the point-wise order. Furthermore, if $\gamma(x)$ is a probability\(^3\) distribution for each $x \in X$, then $\gamma^1(x)$ is as well.

**Proof of Theorem 18.** Let $\gamma : (X, =) \rightarrow Sp((X, =) + (Y, \leq))$, $X = \{x_1, \ldots, x_n\}$, and for any $i, j \leq n$.

\(^2\)In the sense that $X$ is finite.

\(^3\)Meaning $\gamma(x)(X) = 1$. 

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write \( a_{ij} = \gamma(x_i)(x_j) \) and \( \theta_i = \gamma(x_i)\delta_Y \). We need to find a least solution to the system of equations

\[
x_1 = a_{11}x_1 + \cdots + a_{1n}x_n + \theta_1 \\
\vdots \\
x_n = a_{n1}x_1 + \cdots + a_{nn}x_n + \theta_n
\]

(1)

We proceed by induction on \( n \). If \( n = 1 \), then the system consists of the single equation \( x_1 = a_{11}x_1 + \theta_1 \). If \( a_{11} = 0 \), then \( \gamma^\dagger(x_1) = \theta_1 \) is the least solution. If \( a_{11} = 1 \), then again, \( \gamma^\dagger(x_1) = 0 \) is the least solution, as \( \theta_1 = 0 \). Otherwise, \( 0 < a_{11} < 1 \) and \( 1 - a_{11} > 0 \). Let \( b \in \mathbb{P} \) such that \( (1 - a_{11})b = 1 \), and observe that \( \gamma^\dagger(x_1) = b\theta_1 \) is a solution:

\[
a_{11}b\theta_1 + \theta_1 = (a_{11}b + 1)\theta_1 = (a_{11}b + (1 - a_{11})b)\theta_1 = (a_{11} + (1 - a_{11}))b\theta_1 = b\theta_1
\]

It is also easy to see that it is the unique solution: if \( \psi \) is another, then \( \psi(x_i) = a_{11}\psi(x_1) + \theta_1 \) and therefore \( (1 - a_{11})\psi(x_1) = \psi(x_1) - a_{11}\psi(x_1) = \theta_1 \). Multiplying both sides by \( b \),

\[
\psi(x_1) = b(1 - a_{11})\psi(x_1) = b\theta_1 = \gamma^\dagger(x_1)
\]

For the inductive step, consider the \( n \)th equation in (1). Without loss of generalisation, assume that \( a_{nn} > 0 \). If \( a_{nn} = 1 \), then \( a_{ni} = 0 \) for \( i \neq n \) and \( \theta_n = 0 \), and we let \( \gamma^\dagger(x_n) = 0 \). Substituting \( x_n \) for \( \gamma^\dagger(x_n) \), we see that (1) is equivalent to

\[
x_1 = a_{11}x_1 + \cdots + a_{1(n-1)}x_{n-1} + \theta_1 \\
\vdots \\
x_{n-1} = a_{(n-1)1}x_1 + \cdots + a_{(n-1)(n-1)}x_{n-2} + \theta_{n-1}
\]

which, by the induction hypothesis has a least solution \( \psi \). Setting \( \gamma^\dagger(x_i) = \psi(x_i) \) for \( i < n \) and \( \gamma^\dagger(x_n) = 0 \), then \( \gamma^\dagger \) is a solution to (1). If \( \phi \) is another solution to (1), then \( \gamma^\dagger(x_i) \leq \phi(x_i) \) for \( i < n \) by the induction hypothesis, and \( \gamma^\dagger(x_n) \leq \phi(x_n) \) trivially.

If \( 0 < a_{nn} < 1 \), let \( b = (1 - a_{nn})^{-1} \). Substituting \( x_n \) for \( ba_{nn}x_1 + \cdots + ba_{n(n-1)}x_{n-1} + b\theta_n \) in (1), expanding, and collecting like terms, we obtain

\[
x_1 = c_{11}x_1 + \cdots + c_{1(n-1)}x_{n-1} + (\theta_1 + b\theta_n) \\
\vdots \\
x_{n-1} = c_{(n-1)1}x_1 + \cdots + c_{(n-1)(n-1)}x_{n-2} + (\theta_{n-1} + a_{(n-1)n}b\theta_n)
\]

(2)

where \( c_{ij} = a_{ij} + a_{in}ba_{nj} \). By the induction hypothesis, (2) has a least solution \( \psi \). To obtain a solution to (1), set \( \gamma^\dagger(x_i) = \psi(x_i) \) for \( i < n \) and

\[
\gamma^\dagger(x_n) = ba_{nn}\psi(x_1) + \cdots + ba_{n(n-1)}\psi_{n-1} + b\theta_n
\]

To see that \( \gamma^\dagger \) is indeed a solution, observe that for \( i < n \),

\[
\gamma^\dagger(x_i) = c_{i1}\gamma^\dagger(x_1) + \cdots + c_{i(n-1)}\gamma^\dagger(x_{n-1}) + (\theta_i + b\theta_n) \\
= (a_{i1} + a_{in}ba_{nn})\gamma^\dagger(x_1) + \cdots + (a_{i(n-1)} + a_{in}ba_{n(n-1)})\gamma^\dagger(x_{n-1}) + (\theta_i + b\theta_n) \\
= a_{i1}\gamma^\dagger(x_1) + \cdots + a_{i(n-1)}\gamma^\dagger(x_{n-1}) + a_n\gamma^\dagger(x_n) + \theta_i
\]
as well as
\[(1 - a_{nn})\gamma \dagger (x_n) = (1 - a_{nn})ba_{n1}\gamma \dagger (x_1) + \cdots + (1 - a_{nn})ba_{n(n-1)}\gamma \dagger (x_{n-1}) + (1 - a_{nn})b\theta_n\]
\[= a_{n1}\gamma \dagger (x_1) + \cdots + a_{n(n-1)}\gamma \dagger (x_{n-1}) + \theta_n\]
\[\gamma \dagger (x_n) = a_{n1}\gamma \dagger (x_1) + \cdots + a_{n(n-1)}\gamma \dagger (x_{n-1}) + a_{nn}\gamma \dagger (x_n) + \theta_n\]

To see that it is the least solution, let \(\phi\) be any solution to (1), and observe that this implies
\[\phi(x_n) = ba_{n1}\phi(x_1) + \cdots + ba_{n(n-1)}\phi(x_{n-1}) + b\theta_1\]
and for \(i < n\),
\[\phi(x_i) = c_{i1}\phi(x_1) + \cdots + c_{i(n-1)}\phi(x_{n-1}) + (\theta_i + b\theta_n)\]
This means \(\phi\) is a solution to (2), so \(\gamma \dagger (x_i) = \psi(x_i) \leq \phi(x_i)\) for each \(i < n\) by the induction hypothesis. Finally,
\[\gamma \dagger (x_n) = ba_{n1}\gamma \dagger (x_1) + \cdots + ba_{n(n-1)}\gamma \dagger (x_{n-1}) + b\theta_n\]
\[= ba_{n1}\psi(x_1) + \cdots + ba_{n(n-1)}\psi(x_{n-1}) + b\theta_n\]
\[\leq ba_{n1}\phi(x_1) + \cdots + ba_{n(n-1)}\phi(x_{n-1}) + b\theta_n\]
\[= \phi(x_n)\]

This shows that \(\gamma \dagger\) is the least solution to (1) in the point-wise order. \(\square\)

**Example 19.** Consider the ordered semiring \(R^\omega = (R, <, \bar{0}, \bar{1}, +, \cdot)\), where
\[R = \{(r_1, \ldots, r_n) \in (\mathbb{R}^+)^n \mid r_i = 0 \implies (\forall j) r_j = 0\}\]
where \(\bar{r} <^* \bar{s}\) iff either \(\bar{r} = \bar{s}\) or \(r_i < s_i\) for all \(i \leq n\). Then \(R^\omega\) is division, and furthermore satisfies the conditions of Theorem 18. An unguarded system \((X, =) \rightarrow S_{R^\omega}((X, =), (Y, \leq))\) can be thought of as \(n\) simultaneous unguarded systems \((X, =) \rightarrow S_{R^\omega}((X, =) + (Y, \leq))\).

Buried in the proof of Theorem 18 is a proof of the following additional fact: if \(0 < \gamma(x_i)(x_j) < 1\) for all \(x_i, x_j \in X\), then \(\gamma\) has a unique solution. This somewhat trivially implies that if \(P\) is cancellative, difference ordered, and division, then every finite subprobabilistic unguarded SDP has a least solution with respect to the heavier-higher order as well.

A different approach shows that divisibility is not necessary if we assume that \(P\) is sufficiently complete. An ordered algebra is called \(\omega\)-continuous if every chain \(x_1 \leq x_2 \leq \ldots\) has a least upper bound and its algebraic operations preserve least upper bounds of chains.

**Theorem 20.** Let \(P\) be either cancellative and difference-ordered or idempotent. If \(P\) is \(\omega\)-continuous, then every unguarded system has a least solution.

**Proof.** This is nearly an application of the Kleene fixed-point theorem [16]. If we let
\[L : \mathcal{S}_P((X, =) + (Y, \leq))^n \rightarrow \mathcal{S}_P((X, =) + (Y, \leq))^n\]
\[L \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n + \theta_1 \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n + \theta_n \end{bmatrix}\]
as in (1), and \(P\) be \(\omega\)-continuous, then by monotonicity and point-wise \(\omega\)-continuity of \(L\), we just need to take the least upper bound of \(L^n(\bar{0})\) to find the least solution of (1). There is a small snag in this
approach: even if $P$ is continuous, $\bar{S}P(X, \leq)$ is usually not. Consider, for example, $X = \mathbb{N}$, and the chain defined by $\psi_n = \delta_n + \sum_{m<n} \delta_m$. Any upper bound to $\{\psi_n \mid n \in \mathbb{N}\}$ must have infinite support, and so will lie outside of $\bar{S}P(\mathbb{N}, \leq)$.

On the other hand, the support of the $i$th component of $L^n(\vec{0})$ is contained in the union $U \subseteq Y$ of the supports of the $\theta_j$. Since there are finitely many $\theta_j$ to consider, $U$ is finite. Now, $L$ is monotone, so for each $y \in U_i$, let $\psi_i(y) = \sup(L^n(\vec{0}))_i(y)$ and $\psi_i(x) = 0$ for each $x \notin U_i$. Then $\psi_i$ is a subprobability distribution because $L^n(\vec{0})_i$ is for each $n \in \mathbb{N}$. Also, a routine check reveals that $\vec{\psi}$ is the least upper bound of $L^n(\vec{0})$ for all $n > N$.

For each $y \in U_i$, let $\psi_i(y) = \sup(L^n(\vec{0}))_i(y)$ and $\psi_i(x) = 0$ for each $x \notin U_i$. Then $\vec{\psi}$ is a subprobability distribution because $L^n(\vec{0})_i$ is for each $n \in \mathbb{N}$. Also, a routine check reveals that $\vec{\psi}$ is the least upper bound of $L^n(\vec{0})$ for all $n > N$. Therefore $\vec{\psi} \subseteq \vec{\gamma}$, so indeed $\vec{\psi}$ is a least fixed-point of $L$.

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