DENSITY FUNCTIONS IN HIGH-DIMENSIONAL BASKET OPTIONS

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Abstract. We consider an important class of derivative contracts written on multiple assets (so-called spread options) which are traded on a wide range of financial markets. The present paper introduces a new approximation method of density functions arising in high-dimensional basket options which is based on applications of generalised Nyquist-Whitakker-Kotel’nikov-Shannon theorem we established. It is shown that the method of approximation we propose has an exponential rate of convergence in various situations.

1. INTRODUCTION

Consider a frictionless market with no arbitrage opportunities with a constant riskless interest rate \( r > 0 \). Let \( S_{1,t}, t \geq 0 \) and \( S_{2,t}, t \geq 0 \), be two asset price processes. Consider a European call option on the price spread \( S_{1,T} - S_{2,T} \). The common spread option with maturity \( T > 0 \) and strike \( K \geq 0 \) is the contract that pays \((S_{1,T} - S_{2,T} - K)_+\) at time \( T \), where \((a)_+ := \max\{a, 0\}\). There is a wide range of such options traded across different sectors of financial markets. For instance, the crack spread and crush spread options in the commodity markets \([31], [39]\), credit spread options in the fixed income markets, index spread options in the equity markets \([11]\) and the spark (fuel/electricity) spread options in the energy markets \([9], [34]\).

Assuming the existence of a risk-neutral equivalent martingale measure we get the following pricing formula for the call value at time 0.

\[
V = V(S_{1,0}, S_{2,0}, T) = e^{-rT} \mathbb{E}\left[(S_{1,T} - S_{2,T} - K)_+\right],
\]

where the expectation is taken with respect to the equivalent martingale measure. There is an extensive literature on spread options and their applications. In particular, if \( K = 0 \) a spread option is the same as an option to exchange one asset for another. An explicit solution in this case has been obtained by Margrabe \([27]\). Margrabe’s model assumes that \( S_{1,t} \) and \( S_{2,t} \) follow a geometric Brownian motion whose volatilities \( \sigma_1 \) and \( \sigma_2 \) do not need to be constant, but the volatility \( \sigma \) of \( S_{1,t}/S_{2,t} \) is a constant, \( \sigma = (\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho) \), where \( \rho \) is the correlation coefficient of the Brownian motions \( S_{1,t} \) and \( S_{2,t} \). Margrabe’s formula states that

\[
V = e^{-q_1 T} S_{1,0} N(d_1) - e^{-q_2 T} S_{2,0} N(d_2),
\]
where \( N \) denotes the cumulative distribution for a standard Normal distribution,

\[
d_1 = \frac{1}{\sigma T^{1/2}} \left( \ln \left( \frac{S_{1,0}}{S_{2,0}} \right) + \left( q_1 - q_2 + \frac{\sigma}{2} \right) T \right)
\]

and \( d_2 = d_1 - \sigma T^{1/2} \).

Unfortunately, in the case where \( K > 0 \) and \( S_{1,t}, S_{2,t} \) are geometric Brownian motions, no explicit pricing formula is known. In this case various approximation methods have been developed. There are three main approaches: Monte Carlo techniques which are most convenient for high-dimensional situations because the convergence is independent of the dimension, fast Fourier transform methods studied in [3] and PDEs. Observe that PDE based methods are suitable if the dimension of the PDE is low (see, e.g., [33], [12], [40] and [42] for more information). The usual PDE’s approach is based on numerical approximation resulting in a large system of ordinary differential equations which can then be solved numerically.

Approximation formulas usually allow quick calculations. In particular, a popular among practitioners Kirk formula [19] gives a good approximation to the spread call (see also Carmona-Durrleman procedure [7], [23]). Various applications of fast Fourier transform have been considered in [8] and [24].

It is well-known that Merton-Black-Scholes theory becomes much more efficient if additional stochastic factors are introduced. Consequently, it is important to consider a wider family of Lévy processes. Stable Lévy processes have been used first in this context by Mandelbrot [26] and Fama [17].

From the 90th Lévy processes became very popular (see, e.g., [28], [29], [4], [5] and references therein). Usually the reward function has a simple structure, hence the main problem in computation of integral (1.1) is to approximate well the respective density function. In the present article we develop a general method of approximation of density functions. This method is saturation free and can be applied in high-dimensional situations.

2. THEORETICAL BACKGROUND

Let \( C(\mathbb{R}^n) \) be the space of continuous functions on \( \mathbb{R}^n \) and \( L_p(\mathbb{R}^n) \) be the usual space of \( p \)-integrable functions equipped with the norm

\[
\|f\|_p = \|f\|_{L_p(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}, \quad 1 \leq p < \infty,
\]

\[
\|f\|_\infty = \sup_{x \in \mathbb{R}^n} |f(x)|, \quad p = \infty.
\]

Let \( x, y \in \mathbb{R}^n, x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n), \) and \( \langle x, y \rangle \) be the usual scalar product in \( \mathbb{R}^n \), i.e.,

\[
\langle x, y \rangle = \sum_{k=1}^n x_k y_k \in \mathbb{R}.
\]

For an integrable on \( \mathbb{R}^n \) function, i.e., \( f(x) \in L_1(\mathbb{R}^n) \) define its Fourier transform

\[
\mathbf{F}f(y) = \int_{\mathbb{R}^n} \exp(-i \langle x, y \rangle) f(x)dx.
\]

and its formal inverse as

\[
(\mathbf{F}^{-1} f)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp(i \langle x, y \rangle) f(y)dy.
\]
Remark that in the periodic case the most natural (and in many important cases optimal in the sense of the respective $n$-widths) method to approximate sets of smooth functions is to use trigonometric approximations. In the case of approximation on the whole real line $\mathbb{R}$ the role of subspaces of trigonometric polynomials play functions from the Wiener spaces $W_\sigma(\mathbb{R})$, i.e., entire functions from $L_2(\mathbb{R})$ whose Fourier transform has support $[-\sigma, \sigma]$. Such functions have an exponential type $\sigma > 0$. Remind that an entire function $f(z)$ defined on the complex plane $\mathbb{C}$ can be represented as

$$f(z) = \sum_{k=0}^{\infty} c_k z^k$$

for any $z \in \mathbb{C}$. Assume that $f(z)$ has such coefficients $c_k$ that

$$\lim_{k \to \infty} (k! |c_k|)^{1/k} = \sigma < \infty.$$ 

Then for some constant $M > 0$ we have

$$|f(z)| \leq \sum_{k=0}^{\infty} |c_k| |z|^k = \sum_{k=0}^{\infty} \frac{1}{k!} \left( |z| (k! |c_k|)^{1/k} \right)^k \leq M \sum_{k=0}^{\infty} \frac{1}{k!} (|\sigma z|)^k = Me^{\sigma |z|}.$$

We say that a function $f(z)$ defined on the complex plane $\mathbb{C}$ is of exponential type $\sigma > 0$ if there exists a constant $M$ such that for any $\theta \in [0, 2\pi)$,

$$|f(z)| \leq Me^{\sigma r}, \quad z = re^{i\theta}, \quad r \to \infty.$$ 

(2.1) In the limit of $r \to \infty$. The key role here plays the classical Paley-Wiener theorem which relates decay properties of a function at infinity with analyticity of its Fourier transform. It makes use of the holomorphic Fourier transform defined on the space of square-integrable functions on $\mathbb{R}$.

**Theorem 1.** (Paley-Wiener) Suppose that $F$ is supported in $[-\sigma, \sigma]$, so that $F \in L_2[-\sigma, \sigma]$. Then the holomorphic Fourier transform

$$f(z) = \int_{[-\sigma, \sigma]} F(\xi)e^{-iz\xi}d\xi$$

is an entire function of exponential type $\sigma$ as defined in (2.1).

Remark that entire functions of exponential type $\sigma$ as an apparatus of approximation was first considered by Bernstein $\cite{1}$.

In the $n$-dimensional settings we use entire functions $f(z) : \mathbb{C}^n \to \mathbb{C}$ of $n$ variables $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ which satisfy the condition

$$|f(z)| \leq M \exp \left( \sum_{k=1}^{n} \sigma_k |z_k| \right), \forall z \in \mathbb{C}^n,$$

where $M$ is a fixed constant. Here $\sigma : = (\sigma_1, \ldots, \sigma_n)$ is the exponential type of $f(z)$. To justify an inversion formula we will need Planchrel’s theorem (see, e.g., $\cite{10}$).
Theorem 2. (Plancherel) The Fourier transform is a linear continuous operator from $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$. The inverse Fourier transform, $F^{-1}$, can be obtained by letting

$$(F^{-1}g)(x) = \frac{1}{(2\pi)^n} (Fg)(-x)$$

for any $g \in L^2(\mathbb{R}^n)$.

3. \(\lambda\)-deformation of entire basis functions of exponential type

In this section we discuss a multidimensional generalisation of a well-known Nyquist-Whitekker-Kotel’nikov-Shannon theorem which explains why Wiener spaces $W_\sigma(\mathbb{R})$ are so important. Observe that Nyquist-Whitekker-Kotel’nikov-Shannon theorem has its roots in the Information Theory first developed by Shannon [36], [37], [38]. There are various extensions of the mention above theorem to a more general sets of lattice points in $\mathbb{R}^n$ [30]. However, these results are aside of our main line of research. In particular, almost uniformly distributed lattice points in $\mathbb{R}^n$ can be used to reproduce trigonometric polynomials just with the spectrum inside a properly scaled symmetric hyperbolic cross [21], [20], [22]. Unfortunately, characteristic exponents of density functions which correspond to a jump-diffusion process, which can be used in pricing formulas, should admit an analytic extension into a proper domain to guarantee the existence of the pricing integral. The shape of such characteristic exponents is quite far from the hyperbolic cross. Hence, number theoretic lattice points can not be effective in such situations. The problem of constructing of lattice points which will reproduce trigonometric polynomials inside the respective domain which corresponds to the shape of a given characteristic exponent is a deep problem of Geometry of Numbers which remains unsolved.

Consider several examples of characteristic functions which illustrate this situation.

Example 1. Let $W_t^1$ and $W_t^2$ are risk-neutral Brownian motions with correlation $\rho$ and $\sigma_1, \sigma_2 > 0$. Consider the vector $S_t = (S_{1,t}, S_{2,t})$ with components

$$S_{k,t} = S_{k,0} \exp \left( (r - \sigma_k^2/2) t + \sigma_k W_t^k \right), \quad k = 1, 2.$$ 

The joint characteristic function of $X_T = (\ln S_{1,T}, \ln S_{2,T})$ has the form

$$\Phi_1(u, T) = \exp \left( \frac{1}{2} \left( u \cdot \sigma T \right)^2 - \frac{T}{2} \left( u, \Sigma u^T \right) \right),$$

where $u = (u_1, u_2), e = (1, 1), \sigma^2 = (\sigma_1^2, \sigma_2^2), \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_1^2 \sigma_2 \rho \\ \sigma_2^2 \sigma_2 \rho & \sigma_2^2 \end{pmatrix}$ and $u^T$ means $u$ transposed. Direct calculation shows

$$\Phi_1(u, T) = \Phi_1(u_1, u_2)$$

$$= \exp \left( -\frac{1}{2} T \left( i r \sigma_1^2 u_1 + 2 \rho \sigma_1 \sigma_2 u_1 u_2 + \sigma_1 u_1^2 + i r \sigma_2^2 u_2 + 2 \sigma_2 u_2^2 - 2 i r u_1 - 2 i r u_2 \right) \right)$$

The parameters are [8], [18]: $r = 0.1, T = 1, \rho = 0.5, \sigma_1 = 0.2, \sigma_2 = 0.1$. For such set of parameters the function $\Phi_1$ simplifies as

$$\Phi_1(u_1, u_2)$$
Let us fix parameters as in \( [8] \) where
\[
  dW = \exp \left( \frac{\theta}{\sigma} \right) dt + \sigma v^{1/2} dW^v,
\]
where \( dW^1, dW^2 \) and \( dW^v \) have correlations
\[
  E [dW^1, dW^2] = \rho dt,
\]
\[
  E [dW^1, dW^v] = \rho_1 dt,
\]
\[
  E [dW^2, dW^v] = \rho_2 dt,
\]
\( X_t = (\log S_{1,t}, \log S_{2,t}) \) and \( v_t \) is the squared volatility. The characteristic function has the form
\[
  \Phi_2(u) = \Phi_2(u_1, u_2) = \left( ix \ln S_{1,0} + iy \ln S_{2,0} + \frac{2\omega (1 - e^{-\theta T})}{2\theta - (\theta - \gamma) (1 - e^{-\theta T})} v_0 \right.
  + i \langle u, (re - \delta) \rangle T
  - \frac{\kappa \mu}{\sigma^2} \left( 2 \log \left( \frac{2\theta - (\theta - \gamma) (1 - e^{-\theta T})}{2\theta} + (\theta - \gamma) T \right) \right),
\]
where
\[
  \omega := -\frac{1}{2} \left( \left( \sigma_1^2 u_1^2 + \sigma_2^2 u_2^2 + 2\rho \sigma_1 \sigma_2 u_1 u_2 \right) + i \left( \sigma_1^2 u_1 + \sigma_2^2 u_2 \right) \right),
\]
\[
  \gamma := \kappa - i \left( \rho_1 \sigma_1 u_1 + \rho_2 \sigma_2 u_2 \right) \sigma_v,
\]
\[
  \theta := \left( \gamma^2 - 2\sigma_\omega^2 \omega \right).
\]
Let us fix parameters as in \([8], p.16\): \( r = 0.1, T = 1, \rho = 0.5, \rho_1 = 0.25, \rho_2 = -0.5, \delta_1 = 0.05, \delta_2 = 0.05, \sigma_1 = 0.5, \sigma_2 = 1.0, v_0 = 0.04, \kappa = 1, \mu = 0.04, \sigma_v = 0.05, S_{1,0} = 96, S_{2,0} = 100.\)

**Example 3.** Following \([25]\) consider the VG process. The Lévy measure in this case is
\[
  \Pi(x) = \frac{\lambda \left( e^{-a+x} \chi_{[0,\infty)}(x) + e^{a-x} \chi_{(-\infty,0)}(x) \right)}{x}, \lambda > 0, a_+ > 0, a_- > 0,
\]
where
\[
  \chi_A(x) := \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}
\]

\( A \in R \) and the characteristic function is
\[
  \Phi_Y_1(u) = \left( 1 + i \left( \frac{1}{a_-} - \frac{1}{a_+} \right) u + \frac{u^2}{a_- a_+} \right)^{-\lambda t}.
\]

Let \( Y_{k,t}, k = 1, 2, 3 \) be three independent VG processes with common parameters \( a_+, a_-, \lambda_1 = \lambda_2 = (1 - \alpha) \lambda, \lambda_3 = \alpha \lambda, \alpha \in [0, 1] \). The log return \( X_{k,t} = \log S_{k,t}, k = 1, 2 \) is given by
\[
  X_{k,t} = X_{k,0} + Y_{k,t} + Y_{3,t}, k = 1, 2.
\]
The characteristic function has the form
\[ \Phi_3 (u, T) = \Phi_3 (u_1, u_2, T) \]
\[ = \left( 1 + i \frac{1 - a_-}{a_+} \right) \left( u_1 + u_2 + \frac{(u_1 + u_2)^2}{T} \right)^{-\alpha \lambda T} \]
\[ \times \left( 1 + i \frac{1 - a_-}{a_+} \right) u_1 + \frac{u_1^2}{a_+} \]
\[ \times \left( 1 + i \frac{1 - a_-}{a_+} \right) u_2 + \frac{u_2^2}{a_+} \]
\[ = \left( 1 + i \frac{1 - a_-}{a_+} \right) \left( u_1 + u_2 + \frac{(u_1 + u_2)^2}{T} \right)^{-\alpha \lambda T} \]

Let us put \( T = 1, a_+ = 2, a_- = 3, \lambda = 1, \alpha = 0.5 \).

Observe that all three examples show high concentration of characteristic functions around the origin.

Consider a general case now. Let \( a \) be a fixed positive vector in \( \mathbb{R}^n \), i.e. \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n, a_k > 0, 1 \leq k \leq n \) and \( A = \text{diag}(a_1^{-1}, \ldots, a_n^{-1}) \) be a diagonal matrix generated by \( a \). Consider the set of points in \( \mathbb{R}^n \).

\[ \Omega_a = \{ z_m = Am^T \mid m \in \mathbb{Z}^n \}, \quad m \in \mathbb{Z}^n, \]

where \( m^T \) means transpose of \( m \). Observe that
\[ z_m = \left( \frac{m_1}{a_1}, \ldots, \frac{m_n}{a_n} \right) \in \mathbb{R}^n. \]

for any fixed \( m \in \mathbb{Z}^n \). Let
\[ Q_a := \{ x \mid x = (x_1, \ldots, x_n) \in \mathbb{R}^n, |x_k| \leq a_k, 1 \leq k \leq n \}. \]

Denote by \( W_a(\mathbb{R}^n) \) the space of functions \( f \in L_2(\mathbb{R}^n) \) such that \( \text{supp} F f \subset Q_a \).

Let \( C(\mathbb{C}^n) \) and \( C(\mathbb{R}^n) \) be the spaces of continuous functions on \( \mathbb{C}^n \) and \( \mathbb{R}^n \) respectively. We construct a family of linear operators \( \mathbf{P}_a \),
\[ \mathbf{P}_a : C(\mathbb{C}^n) \rightarrow W_{2a}(\mathbb{R}^n) + iW_{2a}(\mathbb{R}^n) \subset C(\mathbb{R}^n) + iC(\mathbb{R}^n) \]
\[ f(z) \rightarrow (\mathbf{P}_a f)(z) \]
such that
\[ \| \mathbf{P}_a f \|_{C(\mathbb{C}^n)} \rightarrow C(\mathbb{R}^n) + iC(\mathbb{R}^n) \| < \infty \]
and \((\mathbf{P}_a f)(z) = f(z)\) for any \( f(z) \in W_a(\mathbb{R}^n) \). The sign ”+” means the Minkowski sum of two vector spaces \( C(\mathbb{R}^n) \) and \( iC(\mathbb{R}^n) \) endowed with the induced topology of \( C(\mathbb{C}^n) \supset C(\mathbb{R}^n) + iC(\mathbb{R}^n) \).

**Theorem 3.** Let \( f(z) \in W_a(\mathbb{R}^n) \) and \( \lambda_a : \mathbb{R}^n \rightarrow \mathbb{R} \) be any continuous function such that \( \lambda_a(y) = 1 \) if \( y \in Q_a \) and \( \lambda_a(y) = 0 \) if \( y \in \mathbb{R}^n \setminus Q_{2a} \), then
\[ f(x) = \sum_{m \in \mathbb{Z}^n} f(\pi Am^T) J_{m,\lambda_a}(x) \]
\[ = \sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_n \in \mathbb{Z}} f \left( \frac{m_1}{a_1}, \ldots, \frac{m_n}{a_n} \right) J_{m_1,\ldots,m_n,\lambda_a,\ldots,\lambda_a}(x_1, \ldots, x_n), \]
where
\[ J_{m,\lambda_a}(x) = \pi^n \text{det} A \left( F^{-1} \lambda_a \right) \left( x - i\pi \frac{m}{a} \right) \]
\[ = 2^{-n} \text{det} A \left( F \lambda_a \right) \left( -x + \pi \frac{m}{a} \right) \]
Proof. For any \( f \in \mathcal{W}_a(\mathbb{R}^n) \) we have
\[
    f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (F f)(y)e^{iy \cdot x} \, dy
\]
\[
    = \frac{1}{(2\pi)^n} \int_{Q_{2a}} \lambda_a(y)(F f)(y)e^{iy \cdot x} \, dy
\]
because \( \text{supp} F f \subset Q_a \) and \( \lambda_a(y) = 1 \) if \( y \in Q_a \) and \( \lambda_a(y) = 0 \) if \( y \in \mathbb{R}^n \setminus Q_{2a} \).
Since the set
\[
    \varrho_m(y, a) := 2^{-n/2}(\det A)^{1/2} \exp \left( i\pi \langle Am^T, y \rangle \right)
\]
\[
    = \left( \prod_{k=1}^{n} \left( \frac{1}{(2a_k)^{1/2}} \right) \right) \prod_{k=1}^{n} \exp \left( \frac{i\pi}{a_k} m_k y_k \right), \; m \in \mathbb{Z}^n
\]
is an orthonormal basis in \( L_2(Q_a) \) then \( (F f)(y) \) can be represented as
\[
    (F f)(y) = \sum_{m \in \mathbb{Z}^n} \alpha_m \varrho_m(y, a).
\]
Remind that \( f \in \mathcal{W}_a(\mathbb{R}^n) \subset L_2(\mathbb{R}^n) \). We understand the convergence in \( L_2(Q_a) \) in the sense that
\[
    \lim_{N \to \infty} \left\| (F f)(y) - \sum_{m \in NQ_a} \alpha_m \varrho_m(y, a) \right\|_{L_2(Q_a)} = 0.
\]
Observe that instead of \( Q_a \) we can take any neighborhood of \( 0 \in \mathbb{R}^n \). Using Plancherel’s theorem we find that
\[
    \alpha_m = \int_{Q_a} (F f)(y) \varrho_m(-y, a) \, dy
\]
\[
    = \int_{Q_a} (F f)(y) \overline{\varrho_m(y, a)} \, dy
\]
\[
    = \int_{\mathbb{R}^n} (F f)(y) \varrho_m(-y, a) \, dy
\]
\[
    = (2\pi)^n 2^{-n/2} \left( \det A \right)^{1/2} (F^{-1} \circ F f)(-\pi Am^T)
\]
\[
    = (2\pi)^n 2^{-n/2} \left( \det A \right)^{-1/2} f(-\pi Am^T).
\]
Applying Plancherel’s theorem again we get
\[
    f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} (2\pi)^n 2^{-n/2} \left( \det A \right)^{1/2} f(-\pi Am^T)
\]
\[
    \times \lambda_a(y) \varrho_m(y, a) e^{i(x \cdot y)} \, dy
\]
\[
    = \int_{Q_{2a}} \sum_{m \in \mathbb{Z}^n} 2^{-n/2} \left( \det A \right)^{1/2} f(-\pi Am^T)
\]
\[
    \times 2^{-n/2} \left( \det A \right)^{1/2} \lambda_a(y) \exp \left( i(\langle x, y \rangle + i\pi \langle Am^T, y \rangle) \right) \, dy
\]
\[
    = 2^{-n} \det A \sum_{m \in \mathbb{Z}^n} f(-\pi Am^T) \int_{Q_{2a}} \lambda_a(y) \exp \left( i(\langle x, y \rangle + i\pi \langle Am^T, y \rangle) \right) \, dy
\]
Changing the index of summation and simplifying we get
\[
    f(x) = \sum_{m \in \mathbb{Z}^n} f(\pi Am^T) J_m \lambda_a(x),
\]
where
\[ J_{m, \lambda}(x) = 2^{-n} \det A \int_{Q_{2n}} \lambda_n(y) \exp \left( i \langle x, y \rangle - i\pi \langle Am^T, y \rangle \right) dy \]
\[ = 2^{-n} \det A \int_{Q_{2n}} \lambda_n(y) \exp \left( i \langle x - \pi Am^T, y \rangle \right) dy \]
\[ = 2^{-n} \det A (2\pi)^n (F^{-1}\lambda_n)(x - \pi Am^T). \]

Consider a particular form of \(\lambda\)-deformation. Let
\begin{equation}
\lambda_n(x) = \lambda_{a_1, \ldots, a_n}(x_1, \ldots, x_n) := \prod_{k=1}^{n} \lambda_{a_k}(x_k),
\end{equation}
where
\[ \lambda_{a_k}(x_k) := \begin{cases} 
0, & x \leq -a, \\
2a^{-1}x + 2, & -a \leq x \leq -a/2, \\
1, & -a/2 \leq x < a/2, \\
-2a^{-1}x + 2, & a/2 \leq x < a, \\
0, & x \geq a,
\end{cases} \]

Direct calculation shows that
\[ (F\lambda_{a_k})(y_k) = \left( \int_{-a_k}^{-a_k/2} + \int_{a_k/2}^{a_k} \right) e^{-ixy} \lambda_{a_k}(x) dx \]
\[ = \int_{-a_k}^{-a_k/2} e^{-ixy} (2x + 2) dx + \int_{-a_k/2}^{a_k/2} e^{-ixy} dx + \int_{a_k/2}^{a_k} e^{-ixy} (-2x + 2) dx \]
\[ = \frac{2}{a_k y^2} e^{-i(-\frac{a_k}{2})y} \left( i \left( -\frac{a_k}{2} \right) y + 1 \right) - \frac{2}{a_k y^2} e^{-i(-a_k)y} \left( i \left( -a_k \right) y + 1 \right) \]
\[ + 2 \left( \frac{i}{y} e^{-i(-\frac{a_k}{2})y} - \frac{i}{y} e^{-i(-a_k)y} \right) \]
\[ + \frac{i}{y} e^{-i(\frac{a_k}{2})y} - \frac{i}{y} e^{-i(\frac{a_k}{2})y} \]
\[ + \frac{2}{a_k y^2} e^{-i\alpha_k y} \left( i\alpha_k y + 1 \right) - \frac{2}{a_k y^2} e^{-i(\alpha_k/2)y} \left( i \left( \frac{\alpha_k}{2} \right) y + 1 \right) \]
\[ + 2 \left( \frac{i}{y} e^{-i\alpha_k y} - \frac{i}{y} e^{-i(\alpha_k/2)y} \right) \]
\[ = \frac{i}{y} e^{i\alpha_k y} - \frac{2}{a_k y^2} e^{i\alpha_k y} + \frac{2}{a_k y^2} e^{i\alpha_k y} \]
\[ + \frac{i}{y} e^{-i(\frac{\alpha_k}{2})y} - \frac{i}{y} e^{-i(\frac{\alpha_k}{2})y} \]
\[ = 2 \frac{1}{a_k y^2} e^{\frac{1}{2}i\alpha_k y} - \frac{2}{a_k y^2} e^{-i\alpha_k y} - \frac{i}{y} e^{\frac{1}{2}i\alpha_k y} \]
\[ = 4 \frac{1}{a_k y^2} \left( \cos \frac{1}{2} \alpha_k y - \cos \alpha_k y \right). \]
Lemma 1. \[ \| P_{\lambda_a} \|_{C(\mathbb{R})} \rightarrow C(\mathbb{R}) \| < 2.834. \]

Proof. \[ \| P_{\lambda_a} \|_{C(\mathbb{R})} \rightarrow C(\mathbb{R}) \| := \sup \left\{ P_{\lambda_a} f \mid \| f \|_{C(\mathbb{R})} \leq 1 \right\} \]

\[ = \sup \left\{ \sup_{x \in \mathbb{R}} \sum_{m \in \mathbb{Z}} f \left( \frac{\pi m}{a} \right) |J_{m,\lambda_a}(x)| \mid \| f \|_{C(\mathbb{R})} \leq 1 \right\} \]

\[ \leq \sup_{x \in \mathbb{R}} \sum_{m \in \mathbb{Z}} |J_{m,\lambda_a}(x)| \]

\[ = \sum_{x \in \mathbb{R}} \sum_{m \in \mathbb{Z}} |2^{-1} (a^{-1})^{-1} (F_{\lambda_a}) \left( -x + \frac{\pi m}{a} \right)| \]

\[ = 2^{-1} a \sup_{x \in \mathbb{R}} \sum_{m \in \mathbb{Z}} \left| \frac{4}{a (ax - \pi m)^2} \left( \cos \left( \frac{ax - \pi m}{2} \right) - \cos \left( ax - \pi m \right) \right) \right| \]

\[ = 2 \sup_{x \in \mathbb{R}} \sum_{m \in \mathbb{Z}} \left| \frac{1}{(ax - \pi m)^2} \left( \cos \left( \frac{ax - \pi m}{2} \right) - \cos \left( ax - \pi m \right) \right) \right|. \]

Observe that the function \[ \varphi(x) := 2 \sum_{m \in \mathbb{Z}} \left| \frac{1}{(ax - \pi m)^2} \left( \cos \left( \frac{ax - \pi m}{2} \right) - \cos \left( ax - \pi m \right) \right) \right| \]

is \( \pi/a \)-periodic. Consequently \[ \sup \{ \varphi(x) \mid x \in \mathbb{R} \} = \sup \{ \varphi(x) \mid x \in [0, \pi/a) \} \]

\[ \leq 2 \left( \sup_{x \in [0, \pi/a)} \sum_{|m| \geq 2} + \sup_{x \in [0, \pi/a)} \sum_{m \in \{-1, 0, 1\}} \right) \]

\[ \left| \frac{1}{(ax - \pi m)^2} \left( \cos \left( \frac{ax - \pi m}{2} \right) - \cos \left( ax - \pi m \right) \right) \right|. \]

Clearly, \[ 2 \sup_{x \in [0, \pi/a)} \sum_{m \leq -2} \left| \frac{1}{(ax - \pi m)^2} \left( \cos \left( \frac{ax - \pi m}{2} \right) - \cos \left( ax - \pi m \right) \right) \right| \]

\[ \leq 4 \sum_{m \geq 2} \frac{1}{\pi^2 m^2} = \frac{4}{\pi^2} \left( \frac{\pi^2}{6} - 1 \right). \]

Similarly \[ 2 \sup_{x \in [0, \pi/a)} \sum_{m \geq 2} \left| \frac{1}{(ax - \pi m)^2} \left( \cos \left( \frac{ax - \pi m}{2} \right) - \cos \left( ax - \pi m \right) \right) \right| \]

\[ \leq 4 \sum_{m \geq 1} \frac{1}{\pi^2 m^2} = \frac{4}{\pi^2} \frac{\pi^2}{6} = \frac{2}{3}. \]

and
\[
2 \sup_{x \in [0, \pi/a]} \sum_{m=\{-1,0,1\}} \left| \frac{1}{(ax - \pi m)^2} \left( \cos \left( \frac{ax - \pi m}{2} \right) - \cos (ax - \pi m) \right) \right| \\
\leq 2 \left( 2 \cdot 0.375 + 2 \cdot \frac{1}{\pi^2} \right) \approx 1.905 \, 284 \, 735.
\]

Hence

\[
\|P_{\lambda a} | C (\mathbb{R}) \to C (\mathbb{R}) \| \leq \frac{4}{\pi^2} \left( \frac{\pi^2}{6} - 1 \right) + \frac{2}{3} + 1.905 \, 284 \, 735 \approx 2.834. \]

\[\square\]

Corollary 1.

\[
\|P_{\lambda a} | C (\mathbb{R}^n) \to C (\mathbb{R}^n) \| \leq \prod_{k=1}^n \|P_{\lambda_{a_k}} | C (\mathbb{R}) \to C (\mathbb{R}) \| \leq 2.834^n
\]

and

\[
\|P_{\lambda a} | C (\mathbb{C}^n) \to C (\mathbb{C}^n) \| \leq 2 \times 2.834^n.
\]

Another example of \( \lambda \)-deformation is given by the function

\[
\phi (x) = \frac{1}{\omega} \int_{-1}^x \left( g \left( 4t + \frac{3}{4} \right) - g \left( 4t - \frac{3}{4} \right) \right) dt,
\]

where

\[
g (t) = \begin{cases} 
\exp \left( - (1 - t^2)^{-1} \right), & t \in [-1, 1], \\
0, & t \in \mathbb{R} \setminus [-1, 1],
\end{cases}
\]

and

\[
\omega = \frac{1}{4} \int_{-1}^1 g (t) dt.
\]

Clearly, Fourier transform of \( \phi (x) \) is an entire function of type 1 since \( \phi (x) \equiv 0, x \in \mathbb{R} \setminus [-1, 1] \). Applying the method of saddle point it is possible to show that

\[
\mathbf{F}g (y) \approx 2 \Re \left( \left( \frac{-i\pi}{(2i)^{1/2}} \right)^{1/2} y^{-3/2} \exp \left( iy - 4^{-1} - (2iy)^{1/2} \right) \right) \approx y^{-3/2} \exp \left( -y^{1/2} \right).
\]

Hence

\[
\mathbf{F} \phi (y) \approx |y|^{-5/2} \exp \left( -2 |y|^{1/2} \right).
\]

Let

\[
\lambda_a (x) = \prod_{k=1}^n \phi \left( \frac{y_k}{a_k} \right), a = (a_1, \ldots, a_n)
\]

then

\[
J_{m, \lambda_a} (x) = \left( \prod_{k=1}^n \frac{1}{2a_k} \right) \mathbf{F}^{-1} \lambda_a (x - \pi Am^T)
\]

and

\[
= 2^{-n} \mathbf{F}^{-1} \lambda_1 (A^{-1} x - \pi m^T) = 2^{-n} \mathbf{F}^{-1} \lambda_1 (A^{-1} (x - \pi Am^T)), 1 = (1, \ldots, 1).
\]
and such that

\[ \Omega(\delta) = \{ z \in \mathbb{C}^n \mid \Im z_k \leq \delta_k, 1 \leq k \leq n \} \]

and the norm of the interpolation operator can be bounded as

\[ \| P_{\lambda} | C(\mathbb{C}^n) \rightarrow C(\mathbb{C}^n) \| \leq 2 \sup_{x \in \mathbb{R}} \sum_{m \in \mathbb{Z}^n} |J_{m, \lambda}(x)| \ll 1. \]

4. APPROXIMATION OF DENSITY FUNCTIONS

Consider the set \( A_{\infty, \delta} \mathcal{U} M \) of analytic functions \( f(z) = u(x, y) + iv(x, y), z = x + iy \) in the strip \( |\Im z| \leq \delta \) such that \( |u(x, y)| \leq M \) for any \( z \in \{ z \mid |\Im z| \leq \delta \} \) and \( f(x) \in \mathbb{R} \) for any \( x \in \mathbb{R} \). It is known (see, \[41\], p. 150) that for almost all \( x \) there are limits

\[ \lim_{y \to \delta} u(x, y) = \lim_{y \to -\delta} u(x, y) = g(x) \]

and such functions can be represented in the form

\[ f(z) = \frac{1}{2\delta} \int_{\mathbb{R}} \left( \cosh \left( \frac{\pi (z - x)}{2\delta} \right) \right)^{-1} g(x) \, dx, \]

where \( |g(x)| \leq M \). Conversely, since the function \( (\cosh (\pi (x - iy) / 2\delta))^{-1} \) is analytic in the strip \( |y| < \delta, x \in \mathbb{R} \) and

\[ \Re \left( \left( \cosh \left( \frac{\pi (z - x)}{2\delta} \right) \right)^{-1} \right) > 0, \]

\[ \frac{1}{2\delta} \int_{\mathbb{R}} \left( \cosh \left( \frac{\pi (z - x)}{2\delta} \right) \right)^{-1} dx = 1 \]

then the function

\[ f(x + iy) = \frac{1}{2\delta} \int_{\mathbb{R}} \left( \cosh \left( \frac{\pi (x + iy - s)}{2\delta} \right) \right)^{-1} g(s) \, ds \]

is analytic in the strip \( |y| < \delta, x \in \mathbb{R} \) and \( |\Re f(x + iy)| \leq M \) if \( \text{ess sup} |g(x)| \leq M \). Observe that the function

\[ f(z) = \frac{1}{2i} \ln \left( \frac{\exp \pi z/(2\delta) + i}{1 + i \exp \pi z/(2\delta)} \right) \in A_{\infty, \delta} \mathcal{U} M \]

is not bounded in the strip \( \{ z \mid |z - x + iy| < \delta, x \in \mathbb{R} \} \). It means that the set of functions \( A_{\infty, \delta} \mathcal{U} M \) which are bounded and analytically extendable into the same strip is a proper subset of \( A_{\infty, \delta} \mathcal{U} M \).

Similarly, in the multidimensional settings, the set \( A_{\infty, \delta} \mathcal{U} M, \delta = (\delta_1, \cdots, \delta_n) \) of functions \( f(z), z = (z_1, \cdots, z_n) \in \mathbb{C}^n \) which are analytically extendable into the tube \( \Omega(\delta) := \{ z \mid z = (z_1, \cdots, z_n) \in \mathbb{C}^n, |\Im z_k| \leq \delta_k, 1 \leq k \leq n \} \)

\[ \Omega(\delta) = \Omega(\delta_1, \cdots, \delta_n) := \{ z \mid z = (z_1, \cdots, z_n) \in \mathbb{C}^n, |\Im z_k| \leq \delta_k, 1 \leq k \leq n \} \]

and such that \( |\Re f(z)| \leq M \), admits representation

\[ f(z) = (K * g)(z) := \int_{\mathbb{R}^n} K(z - x) g(x) \, dx, \]
where
\[ K(z) := \prod_{k=1}^{n} K_{(k)}(z_k) \]
and
\[ K_{(k)}(x_k) := \frac{1}{2\delta_k} \left( \cosh \left( \frac{\pi x_k}{2\delta_k} \right) \right)^{-1}. \]

Let
\[ E(f, W_a(\mathbb{R}^n), L_p(\mathbb{R}^n)) := \inf \{ \| f - g \|_{L_p(\mathbb{R}^n)} \mid g \in W_a(\mathbb{R}^n) \} \]
be the best approximation of \( f(z) \) by the space \( W_a(\mathbb{R}^n) \) in \( L_p(\mathbb{R}^n) \) and
\[ E(K, W_a(\mathbb{R}^n), L_p(\mathbb{R}^n)) := \sup \{ E(f, W_a(\mathbb{R}^n), L_p(\mathbb{R}^n)) \mid f \in K \}. \]
be the best approximation of the function class \( K \subseteq L_p(\mathbb{R}^n) \) by \( W_a(\mathbb{R}^n) \) in \( L_p(\mathbb{R}^n) \).

**Lemma 2.** Let
\[ \vartheta_a(x) := \prod_{k=1}^{n} \vartheta_{a_k}(x_k), \]
where \( \vartheta_{a_k}(x_k) \) is an integrable entire function of exponential type \( a_k > 0 \). Let \( g(y) \) be a bounded function, i.e. \( |g(y)| \leq M, \forall y \in \mathbb{R}^n \). Then the function
\[ \vartheta_a(x) := \int_{\mathbb{R}^n} \vartheta_a(x-y) g(y) \, dy \]
is of exponential type \( \mathfrak{a} = (a_1, \cdots, a_n) \).

**Proof.** It follows from the definition that it is sufficient to show that
\[ |\vartheta_a(x)| = |\vartheta(x_1, \cdots, x_1, \cdots, x_n)| \leq M e^{a_l|x_l|}, 1 \leq l \leq n \]
for some absolute constant \( M_l \) and fixed \( x_k, k \neq l \). Expanding \( \vartheta_{a_l}(x_l-y_l), 1 \leq l \leq n \) into the power series with respect to \( x_k \) we get
\[ \vartheta_{a_l}(x_l-y_l) = \sum_{s=0}^{\infty} \frac{(\vartheta_{a_l}(-y_l))^{(s)}}{s!} x_l^s. \]

Hence
\[ |\vartheta_a(x)| = \left| \int_{\mathbb{R}^n} \prod_{k=1}^{n} \vartheta_{a_k}(x_k-y_k) g(y_1, \cdots, y_n) \prod_{k=1}^{n} dy_k \right| \]
\[ = \int_{\mathbb{R}^n} \left( \sum_{s=0}^{\infty} \frac{(\vartheta_{a_l}(-y_l))^{(s)}}{s!} x_l^s \right) \left( \prod_{k=1, k \neq l}^{n} \vartheta_{a_k}(x_k-y_k) \right) g(y_1, \cdots, y_n) \prod_{k=1}^{n} dy_k \]
\[ \leq M \int_{\mathbb{R}^n} \left( \sum_{s=0}^{\infty} \frac{|(\vartheta_{a_l}(-y_l))^{(s)}|}{s!} x_l^s \right) \left( \prod_{k=1, k \neq l}^{n} |\vartheta_{a_k}(x_k-y_k)| \right) \prod_{k=1}^{n} dy_k \]
\[ \leq M K_l \int_{\mathbb{R}^n} \left( \sum_{s=0}^{\infty} \frac{|(\vartheta_{a_l}(y_l))^{(s)}|}{s!} x_l^s \right) \prod_{k=1, k \neq l}^{n} dy_k, \]
where
\[ K_l := \int_{\mathbb{R}^{n-1}} \prod_{k=1, k \neq l}^{n} |\vartheta_{a_k}(x_k-y_k)| \prod_{k=1, k \neq l}^{n} dy_k. \]
Lemma 3.

where

It means that \( q_a(x) \) is of exponential type \( a \). \( \square \)

**Lemma 3.**

\[ E \left( (K \ast g) (z) , W_a (R^n) , L_\infty (R^n) \right) \leq M \| K - \vartheta_a \|_1 \]

\[ \leq 2Mn \prod_{k=1}^{n} \left( \frac{S_{ak}}{2\delta_k} \right) \exp \left( - \min \{a_k\delta_k \mid 1 \leq k \leq n \} \right) , \]

and

\[ E \left( (K \ast g) (z) , W_a (R^n) , L_1 (R^n) \right) \leq L \| K - \vartheta_a \|_1 \]

\[ \leq 2Ln \prod_{k=1}^{n} \left( \frac{S_{ak}}{2\delta_k} \right) \exp \left( - \min \{a_k\delta_k \mid 1 \leq k \leq n \} \right) , \]

where \( K \) is defined by (4.1), \( |g| \leq M , g \in L_1(R^n) \) and

\[ \vartheta_a (x) = \prod_{k=1}^{n} \vartheta_{ak} (x_k) , \]

where

\[ \vartheta_{ak} (x_k) := 2\delta_k \int_{0}^{ak} \left( \mu_k (\xi) - C_k (ak - \xi) - C_k (ak + \xi) \right) \cos \xi x_k d\xi , \]

\[ \mu_k (\xi) := (\pi \cosh (\delta_k \xi))^{-1} , \]

\[ C_k (\xi) := \sum_{s=0}^{\infty} (-1)^s \mu_k ((2s + 1)ak + \xi) , 1 \leq k \leq n . \]

and

\[ S_{ak} := 2\delta_k ak \left( \frac{1}{2\delta_k} + \mu_k (2ak) + \mu_k (ak) \right) . \]

**Proof.** We prove Lemma just in the case \( p = \infty \). The case \( p = 1 \) follows in the similar way. It is easy to check that \( \vartheta_{ak} (x_k) \) is a function of exponential type \( a_k \) and \( \vartheta_{ak} (x_k) \in L_2 (R) \). Hence \( \vartheta_a (x) \in L_2 (R^n) \).

Consider the set of functions \( K \ast g \), where \( |g| \leq M \) and \( \|g\|_1 \leq L \). Assume that \( \vartheta_a \in L_2 (R^n) \). In this case,

\[ \left\| \int_{R^n} \vartheta_a \ast g \right\|_2 \leq \| \vartheta_a \|_2 \|g\|_1 \leq L \| \vartheta_a \|_2 . \]

Hence, by Lemma 2, \( (\vartheta_a \ast g) \in W_a (R^n) \) and

\[ E \left( (K \ast g) , W_a (R^n) , L_\infty (R^n) \right) \leq \sup \{ \|K \ast g - \vartheta_a \ast g\|_\infty \mid |g| \leq M , \|g\|_1 \leq L \} \]

Applying Bernstein’s inequality \( \text{[41]} \) p.230

\[ \int_{R} \left| (\vartheta_a (y))^{(s)} \right| dy \leq a^s \int_{R} |\vartheta_a (y)| dy, s \in \mathbb{N} , \]

which is valid for any entire function \( \vartheta_a (y) \) of exponential type \( a \), we get

\[ |q_a (x)| \leq MK_1 P_1 \sum_{s=0}^{\infty} \frac{|s|^s}{s!} |x|^s \]

\[ = MK_1 P_1 e^{a|x|} , \]

where

\[ P_1 := \int_{R} |\vartheta_a (y)| dy . \]
Observe that for any complex numbers \( \rho_{1,m} \) and \( \rho_{2,m}, 1 \leq m \leq n \) we have
\[
\prod_{m=1}^{n} \rho_{1,m} - \prod_{m=1}^{n} \rho_{2,m} = \sum_{m=1}^{n} (\rho_{1,m} - \rho_{2,m}) \prod_{r=1}^{m-1} \rho_{2,r} \prod_{r=m+1}^{n} \rho_{1,r}.
\]
It easy to check that
\[
\sup_{z_k \in \mathbb{R}} \frac{1}{2\delta_k} \left( \cosh \left( \frac{\pi z_k}{2\delta_k} \right) \right)^{-1} \leq \frac{1}{2\delta_k},
\]
\[0 \leq \mu_k (\xi) \leq \frac{1}{\pi}, \xi \in \mathbb{R},\]
\[|C_k (a_k + \xi)| \leq \mu_k (2a_k), 0 \leq \xi \leq a_k\]
and
\[|C_k (a_k - \xi)| \leq \mu_k (a_k), 0 \leq \xi \leq a_k.
\]
Hence
\[
|\vartheta_{a_k} (x_k)| \leq S_{a_k}.
\]
Comparing (4.2) - (4.5) we get
\[
E ((K * g), W_a (\mathbb{R}^n), L_\infty (\mathbb{R}^n)) \leq 2Mn \prod_{k=1}^{n} \left( \frac{S_{a_k}}{2\delta_k} \right) \exp (- \min \{a_k\delta_k | 1 \leq k \leq n \}).
\]

**Lemma 4.** Let \( B (\mathbb{R}^n) \) be the set of bounded measurable functions on \( \mathbb{R}^n \) then for any density function \( p_t \in B (\mathbb{R}^n) \) we have
\[
p_t = F^{-1} \left( e^{-t\psi (\cdot)} \right) \in \bigcap_{q=1}^{\infty} L_q (\mathbb{R}^n).
\]
Lemma 5. In our notations

\[ p_\sim (x) = \int_0^\infty \chi_{\{y: f(y) > t\}}(x) \, dt \]

Since

\[ \int_{\mathbb{R}^n} p_t(x) \, dx = 1, \quad p_t(x) \geq 0, \, \forall x \in \mathbb{R}^n \]

then \( p_t \in \mathcal{B} \cap L_1(\mathbb{R}^n) \) and \( \lim_{(x, R) \to \infty} p_t(x) = 0 \). It means that there is a set \( B \subset \mathbb{R}^n \) such that \( \text{Vol}_n B < \infty \) and \( p_t(x) \leq 1, \forall x \in \mathbb{R}^n \setminus B \). Hence for any \( 1 \leq q < \infty \) we get

\[ \int_{\mathbb{R}^n} p_t^q(x) \, dx = \int_{\mathbb{R}^n} (p_t(x))^q \, dx = \left( \int_B + \int_{\mathbb{R}^n \setminus B} \right) (p_t(x))^q \, dx \]

\[ \leq C^q \text{Vol}_n B + \int_{\mathbb{R}^n \setminus B} p_t(x) \, dx \leq C^q \text{Vol}_n B + 1. \]

Therefore, \( p_t(x) \in \mathcal{B} \cap L_1(\mathbb{R}^n) \cap L_q(\mathbb{R}^n) \) and applying Plancherel’s theorem we get \( p_t(x) = F^{-1} (e^{-t \psi(\cdot)})(x) \).

\[ \square \]

Lemma 5. In our notations

\[ \left\| e^{-t \psi(\cdot)} - \sum_{m \in \mathbb{Z}^n} \exp \left( -t \psi \left( \pi A m^T \right) \right) J_{m, \lambda}(\cdot) \right\|_\infty \]

\[ \leq 2 (1 + 2 \times 2.834^n) M n \prod_{k=1}^n \left( \frac{S_{a_k}}{2\delta_k} \right) \exp \left( -\min \{ a_k \delta_k | 1 \leq k \leq n \} \right). \]

Proof. Since for any \( \rho_{a} \in W_n(\mathbb{R}^n) \) we have

\[ \rho_{a}(\cdot) = \sum_{m \in \mathbb{Z}^n} \rho_{a}(\pi A m^T) J_{m, \lambda}(\cdot) \]

then applying Lemma 3 and Corollary 1 we get

\[ \left\| e^{-t \psi(\cdot)} - \sum_{m \in \mathbb{Z}^n} \exp \left( -t \psi \left( \pi A m^T \right) \right) J_{m, \lambda}(\cdot) \right\|_\infty \]

\[ = \left\| e^{-t \psi(\cdot)} - \rho_{a}(\cdot) + \rho_{a}(\cdot) - \sum_{m \in \mathbb{Z}^n} \exp \left( -t \psi \left( \pi A m^T \right) \right) J_{m, \lambda}(\cdot) \right\|_\infty \]

\[ \leq E \left( (K * g), W_n(\mathbb{R}^n), L_\infty(\mathbb{R}^n) \right) \]

\[ + \left\| \rho_{a}(\cdot) - \sum_{m \in \mathbb{Z}^n} \exp \left( -t \psi \left( \pi A m^T \right) \right) J_{m, \lambda}(\cdot) \right\|_\infty \]

\[ \leq E \left( (K * g), W_n(\mathbb{R}^n), L_\infty(\mathbb{R}^n) \right) (1 + \| P_{\lambda} \| C(\mathbb{R}) \to C(\mathbb{R})) \]

\[ \leq 2 (1 + 2 \times 2.834^n) M n \prod_{k=1}^n \left( \frac{S_{a_k}}{2\delta_k} \right) \exp \left( -\min \{ a_k \delta_k | 1 \leq k \leq n \} \right). \]

\[ \square \]
Similarly, where

\[
\phi_\infty (\cdot) = \begin{cases} 
\frac{1}{2} (2\pi)^{-n} (\cosh(\langle \alpha, \cdot \rangle))^{-1} & \int_{\mathbb{R}^n} \exp(i \langle x, \cdot \rangle - t\psi(x)) \, dx \\
(2\pi)^{-n} & \int_{\mathbb{R}^n+i\alpha} \exp(i \langle x, \cdot \rangle - t\psi(x)) \, dx \\
(2\pi)^{-n} & \int_{\mathbb{R}^n} \exp(i \langle z+i\alpha, \cdot \rangle - t\psi(z+i\alpha)) \, dz \\
(2\pi)^{-n} \exp(-\langle \alpha, \cdot \rangle) & \int_{\mathbb{R}^n} \exp(i \langle z, \cdot \rangle - t\psi(z+i\alpha)) \, dz, 
\end{cases}
\]

and applying Caychy’s theorem we get

\[
p_t (\cdot) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(i \langle x, \cdot \rangle - t\psi(x)) \, dx
\]

Prove. Since the characteristic function \( \Phi(x,t) := \exp(-t\psi(x)) \) admits an analytic extension into the tube \( \Omega(\delta) \) and \( \Phi(x) \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n) \) then

\[
\lim_{(x,x) \to \infty} \Phi(x) = 0
\]

and applying Caychy’s theorem we get

\[
p_t (\cdot) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(i \langle x, \cdot \rangle - t\psi(x)) \, dx
\]

Similarly,

\[
p_t (\cdot) \exp(\langle \alpha, \cdot \rangle) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(i \langle z, \cdot \rangle - t\psi(z+i\alpha)) \, dz.
\]

It means that

\[
p_t (\cdot) = \frac{1}{2} (2\pi)^{-n} (\cosh(\langle \alpha, \cdot \rangle))^{-1} \int_{\mathbb{R}^n} \exp(i \langle z, \cdot \rangle - t\psi(z+i\alpha)) \, dz.
\]

\[\Box\]

Lemma 7. Assume that the characteristic function \( \Phi(x,t) := \exp(-Tt\psi(x)) \) admits an analytic extension into the tube \( \Omega(\delta) \) and \( \Phi(x) \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n) \) then for any \( \alpha \in \mathbb{R}^n, ||\alpha||_\infty < ||\delta||_\infty \) we have

\[
p_{T,a} (\cdot) := (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(i \langle z, \cdot \rangle) g_a(z,T) \, dz,
\]

where \( g_a(z,T) \in W_a(\mathbb{R}^n) \) interpolates

\[
\Xi(\cdot,T) := \exp(-t(\psi(z+i\alpha) + \psi(z-i\alpha)))
\]

at the points \( z_m = Am^T, m \in \mathbb{Z}^n \). Then

\[
\left\| p_{T} (\cdot) - p_{T,a} (\cdot) \right\|_1 \leq (2\pi)^{-n} \prod_{k=1}^{n} \alpha_k^{-1} \text{Vol}(A) \|\Xi(z,T) - g_a(z,T)\|_\infty + \varepsilon,
\]

where \( A \subset \mathbb{R}^n \) is such that

\[
(2\pi)^{-n} \prod_{k=1}^{n} \alpha_k^{-1} \left( \int_{\mathbb{R}^n \setminus A} |\Xi(z,T)| \, dz + \int_{\mathbb{R}^n \setminus A} \sum_{m \in \mathbb{Z}^n} |\Xi(\pi Am,T)| + \int_{\mathbb{R}^n \setminus A} |Jm,\lambda_n(x)| \, dx \right) \leq \varepsilon
\]
and, in our notations,
\[ \| \Xi(z,T) - g_n(z,T) \|_{\infty} \]
\[ \leq 2(1 + 2 \times 2.834^n) \left( \prod_{k=1}^{n} \left( \frac{S_k}{2(\delta_k - \alpha_k)} \exp \left( - \min \{ a_k (\delta_k - \alpha_k) | 1 \leq k \leq n \} \right) \right) \]

Proof. Applying Lemma 6 we get
\[ \left\| p_T(\cdot) - p_{T,a}(\cdot) \right\|_1 \]
\[ = \frac{1}{2} (2\pi)^{-n} \left\| \frac{1}{\cosh(\langle \alpha, \cdot \rangle)} \int_{\mathbb{R}^n} \exp(i \langle z, \cdot \rangle) (\Xi(z,T) - g_n(z,T, A)) dz \right\|_1 \]
\[ \leq \frac{1}{2} (2\pi)^{-n} \left\| \frac{1}{\cosh(\langle \alpha, \cdot \rangle)} \int_{\mathbb{R}^n} \Xi(z,T) - g_n(z,T, A) dz \right\|_1 \]
\[ = \frac{1}{2} (2\pi)^{-n} \prod_{k=1}^{n} \alpha_k^{-1} \int_{\mathbb{R}^n} \Xi(z,T) - g_n(z,T) dz \]
\[ \leq \frac{1}{2} (2\pi)^{-n} \prod_{k=1}^{n} \alpha_k^{-1} \text{Vol}(A) \| \Xi(z,T) - g_n(z,T) \|_{\infty} \]
\[ + \frac{1}{2} (2\pi)^{-n} \prod_{k=1}^{n} \alpha_k^{-1} \left( \int_{\mathbb{R}^n \setminus A} | \Xi(z,T) | dz + \int_{\mathbb{R}^n \setminus A} | g_n(z,T) | dz \right) , \]
where C is some absolute constant. Since \( \Phi(x,T) \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n) \) then \( \Xi(x,T) \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n) \). Hence for any \( \varepsilon > 0 \) there is such \( A := A_\varepsilon \) that
\[ \int_{\mathbb{R}^n \setminus A_\varepsilon} | \Xi(z,T) | dz \leq \varepsilon \]
and
\[ \int_{\mathbb{R}^n \setminus A_\varepsilon} | g_n(z,T) | dz \leq \sum_{m \in \mathbb{Z}^n} | \Xi(\pi A_m, T) | \int_{\mathbb{R}^n \setminus A_\varepsilon} | J_{m, \lambda_n}(x) | dx. \]
Finally, we apply Lemma 5.

Observe that a similar estimate can be obtained in the case \( \| p_T(\cdot) - p_{T,a}(\cdot) \|_{\infty} \).

Finally, examples 1-3 show that the function \( \Xi(\pi A \cdot , T) \) is rapidly decaying. Consequently, we can effectively truncate the Fourier series in the representation of \( p_{T,a}(\cdot) \) to reduce significantly the number of point evaluation.

5. Appendix. Characteristic Exponents and Density Functions

Let \((\Omega, \mathcal{F}, P)\) be a probability space. Let \( \mathcal{B}(\mathbb{R}^n) \) be the collection of all Borel sets on \( \mathbb{R}^n \) which is the \( \sigma \)–algebra generated by all open sets in \( \mathbb{R}^n \), i.e., the smallest \( \sigma \)–algebra that contains all open sets in \( \mathbb{R}^n \). A real valued function is called measurable (Borel measurable) if it is \( \mathcal{B}(\mathbb{R}^n) \) measurable. A mapping \( X : \Omega \rightarrow \mathbb{R}^n \) is an \( \mathbb{R}^n \)–valued random variable if it is \( \mathcal{F} \)–measurable, i.e., for any \( B \in \mathcal{B}(\mathbb{R}^n) \) we have \( \{ \omega | X(\omega) \in B \} \in \mathcal{F} \). A stochastic process \( X = \{ X_t \}_{t \in \mathbb{R}_+} \) is a one-parametric family
of random variables on a common probability space \((\Omega, \mathcal{F}, P)\). The \textit{trajectory} of the process \(X\) is a map

\[
\begin{array}{ccl}
\mathbb{R}_+ & \longrightarrow & \mathbb{R}^n \\
t & \mapsto & X_t(\omega),
\end{array}
\]

where \(\omega \in \Omega\) and \(X_t = (X_{t1}, \ldots, X_{nt})\).

\(X = \{X_t\}_{t \in \mathbb{R}_+}\) is called a \textit{Lévy process} (process with stationary independent increments) if

1. The random variables \(X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_m} - X_{t_{m-1}}\), for any \(0 \leq t_0 < t_1 < \cdots < t_m\) and \(m \in \mathbb{N}\) are independent (\textit{independent increment property}).
2. \(X_0 = 0\) a.s.
3. The distribution of \(X_{t+\tau} - X_t\) is independent of \(\tau\) (\textit{temporal homogeneity} or \textit{stationary increments property}).
4. It is \textit{stochastically continuous}, i.e.

\[
\lim_{\tau \to t} P[|X_\tau - X_t| > \epsilon] = 0
\]

for any \(\epsilon > 0\) and \(t \geq 0\).
5. There is \(\Omega_0 \in \mathcal{F}\) with \(P(\Omega_0) = 1\) such that, for any \(\omega \in \Omega_0\), \(X_t(\omega)\) is right-continuous on \([0, \infty)\) and has left limits on \((0, \infty)\).

A process satisfying (1–4) is called a \textit{Lévy process in law}. An \textit{additive process} is a stochastic process which satisfies (1, 2, 4, 5) and an \textit{additive process in law} satisfies (1, 2, 4).

The \textit{convolution} \(\mu = \mu_1 * \mu_2\) of two distributions \(\mu_1\) and \(\mu_2\) on \(\mathbb{R}^n\) is defined as

\[
\mu(B) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \chi_B(x+y)\mu_1(dx)\mu_2(dy) < \infty,
\]

where

\[
\chi_B(x) := \begin{cases} 1, & x \in B, \\ 0, & x \notin B \end{cases}
\]

is the \textit{characteristic function} of a Borel (Lebesgue) measurable set \(B \subset \mathbb{R}^n\). A probability measure \(\mu\) is called \textit{infinitely divisible} if for any \(m \in \mathbb{N}\) there is a probability measure \(\mu_{(m)}\) such that

\[
\mu = \mu_{(m)} * \cdots * \mu_{(m)}.
\]

Consider the set \(\mathcal{L}\) of Lévy process \(X = \{X_t\}_{t \in \mathbb{R}_+}\) on a probability space \((\Omega, \mathcal{F}, P)\). For a finite measure \(\mu\) on \(\mathbb{R}^n\) (i.e., if \(\mu(\mathbb{R}^n) < \infty\)) we define its Fourier transform as

\[
\hat{\mu}(y) = F\mu(y) = \int_{\mathbb{R}^n} \exp(-i\langle x,y \rangle) \mu(dx)
\]

and its formal inverse

\[
\mu(dx) = F^{-1}\hat{\mu}(dx) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp(i\langle x,y \rangle) \hat{\mu}(y) dy.
\]

It is known that if \(\mu\) is infinitely divisible then there exists a unique continuous function \(\phi: \mathbb{R}^n \to \mathbb{C}\) such that \(\phi(0) = 0\) and \(e^{\phi(y)} = \hat{\mu}(y)\). Hence, the \textit{characteristic function} of the distribution of \(X_t\) of any Lévy process can be represented in the form

\[
\mathbb{E}[\exp(i\langle x, X_t \rangle)] = e^{-t\phi(x)},
\]
where \( x \in \mathbb{R}^n \), \( t \in \mathbb{R}_+ \) and the function \( \psi(x) \) is uniquely determined. This function is called the characteristic exponent. Vice versa, a Lévy process \( X = \{X_t\}_{t \in \mathbb{R}_+} \) is determined uniquely by its characteristic exponent \( \psi(x) \). In particular, density function \( p_t \) can be expressed as

\[
(5.1) \quad p_t(\cdot) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp(i \langle \cdot, x \rangle - t\psi(x)) = F^{-1}(\exp(-t\psi(x)))(\cdot).
\]

We say that a matrix \( A \) is nonnegative-definite (or positive-semidefinite) if \( \langle x^*, Ax \rangle \geq 0 \) for all \( x \in \mathbb{C}^n \) (or for all \( x \in \mathbb{R}^n \) for the real matrix), where \( x^* \) is the conjugate transpose.

The key role in our analysis plays the following classical result known as the Lévy-Khintchine formula which gives a representation of characteristic functions of all infinitely divisible distributions.

**Theorem 4.** Let \( X = \{X_t\}_{t \in \mathbb{R}_+} \) be a Lévy process on \( \mathbb{R}^n \). Then its characteristic exponent admits the representation

\[
\psi(y) = -\frac{1}{2} \langle Ay, y \rangle - i \langle b, y \rangle - \int_{\mathbb{R}^n} \left( 1 - e^{i\langle y, x \rangle} + i\langle y, x \rangle \chi_D(x) \right) \Pi(dx),
\]

where \( \chi_D(x) \) is the characteristic function of \( D := \{x \in \mathbb{R}^n, |x| \leq 1\} \), \( A \) is a symmetric nonnegative-definite \( n \times n \) matrix, \( b \in \mathbb{R}^n \) and \( \Pi(dx) \) is a measure on \( \mathbb{R}^n \) such that

\[
(5.2) \quad \int_{\mathbb{R}^n} \min\{1, \langle x, x \rangle\} \Pi(x) < \infty, \quad \Pi(\{0\}) = 0.
\]

Hence \( \hat{\mu}(y) = e^{\psi(y)} \).

The density of \( \Pi \) is known as the Lévy density and \( A \) is the covariance matrix. In particular, if \( A = 0 \) (i.e. \( A = (a_{j,k})_{1 \leq j, k \leq n}, a_{j,k} = 0 \)) then the Lévy process is a pure non-Gaussian process and if \( \Pi = 0 \) the process is Gaussian.

We say that the Lévy process has bounded variation if its sample paths have bounded variation on every compact time interval. A Lévy process has bounded variation iff \( A = 0 \) and

\[
\int_{\mathbb{R}^n} \min\{\langle x, x \rangle, 1\} \Pi(dx) < \infty, \quad \Pi(\{0\}) = 0,
\]

(see, e.g., [3], p.15).

The systematic exposition of the theory of Lévy processes can be found in [13], [14], [15], [35], [2], [32].

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