Hydrodynamics of a Bose condensate: beyond the mean field approximation (II)

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Abstract

Self-consistent hydrodynamic one-loop quantum corrections to the Gross-Pitaevskii equation due to the interaction of the condensate with collective excitations are calculated. It is done by making use a formalism of effective action and \( \varsigma \)-function regularization for a contribution from Bogoliubov particles in hydrodynamic approximation. It turns out to be possible to reduce the problem to the investigation of a determinant of Laplace operator on curved space where a metric is defined by density and velocity of the condensate. Standard methods of quantum gravity let us get the leading logarithmic contribution of the determinant and corresponding quantum corrections. They describe an additional quantum pressure in the condensate, local heating-cooling and evaporation-condensation effects of the Bose-condensed fraction. The effects of these corrections are studied for the correction from excited states in equilibrium situation. Response functions and form factors are discussed in the same approximation.

1 Introduction

The recent success of the experimental observation of Bose-Einstein condensation for systems of spin polarized magnetically trapped atomic gases at ultra-low temperatures \[1, 2, 3\] and further investigation of their collective properties \[4\] are stimulating development of the theory of a nonuniform Bose condensate and its collective excitations. Moreover, new exciting problems such as a description of the evolution of Bose condensate from relaxed trap,

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dynamics of a collapse of the condensate for Li\textsuperscript{7} atoms, heating-cooling phenomena, various coherence effects for the condensate and so on are being posed both theoretically and experimentally.

A lot of theoretical efforts were spent to solve these problems. Evolution of the condensate from relaxed trap was considered in a number of papers both for symmetric and asymmetric cases \cite{5, 6, 7}. All of these calculations were performed in Hartree approximation, i.e. based on Nonlinear Schrodinger equation or Ginzburg-Gross-Pitaevskii equation (GPE) \cite{8}. It is obvious that the GPE describes strongly interacting Bose condensed gas well since it is possible to ignore non-condensed fraction. However next three points should be considered:

1. At least for experiments with Rb\textsuperscript{87} number of atoms is not such that strong interaction approximation (or Thomas-Fermi) approximation is sufficiently accurate to produce a robust ground state profile and realistic values of experimental observables.

2. When the condensate expands after the release from a trap the density inevitably decreases and quantum corrections due to the interaction with particles in excited states should be considered even if the original density was high enough to be described by GPE.

3. Effects like heating or cooling of the system and the collapse of atoms with attraction such as Li\textsuperscript{7} \cite{10, 11, 12} cannot be described even in principle by GPE.

Therefore excited states and their influence on the condensate require consideration.

Firstly, excitations of a trapped system, using the technique of Bogoliubov transformation, have been considered by Fetter \cite{13} (see also \cite{14}). The spectrum and wave functions of excited states were derived by Stringari \cite{15} in the hydrodynamic approximation where the low energy excited states are sound waves in inhomogeneous condensate media. An analogous spectrum was obtained by Burnett and coauthors \cite{16} who investigated response of the condensate to a time-dependent perturbation. Using Stringari’s approach, Wu and Griffin \cite{17} proceeded by quantizing the low energy excitations in a trap and diagonalizing the hydrodynamic Hamiltonian in terms of normal modes associated with the amplitude and phase. The depletion of the condensate and observable quantities (such as response functions) are some series over all excited states. It means that they cannot actually be evaluated analytically since contributions of various types of excited states are the same order of magnitude (as can be seen from Fig.1 of Ref.\cite{17}; see also discussion in section 5). Hence an effective method of summation as well as the generalization of the treatment to time-dependent and untrapped cases are needed. These problems are solved in the present paper (see also \cite{18}).

In recent paper Ref.\cite{19} the evolution of a Bose condensed gas under temporal variation of the confining potential was considered. It was found that there exist exact scaling relations for a 2-dimensional gas trapped in a harmonic potential whose frequency depends on time. Approximate relations also exist in the 3-dimensional case if the interaction is very small or very large (such that the kinetic energy can be neglected). This allows the authors to discuss principal questions of the condensate evolution from a relaxing trap and the depletion of the condensate. However this elegant work is restricted by the requirement of a harmonic potential as a confining potential.
We want to note also work [20] which considered nucleation of the condensate. It is possible to say that the present work is devoted to the consideration of opposite case where the condensate may evaporate.

In the paper we derive a generalization of Gross-Pitaevskii equation for the condensate fraction taking into account one loop quantum corrections caused by the interaction of the condensate with the noncondensed fraction. However, there is other possibility to generalize GP equation. Indeed, GP equation in mean field approximation gives dynamical equation for both condensed fraction and full density of the gas since the deplition is ignored in the approximation. One loop correction for full density is obtained in Ref. [18] that together with the results of the present paper allows to calculate the deplition in one loop approximation simply as a difference of the full density and density of the condensate fraction.

The paper is organized as follows. In next section we consider the effective action and deduce the most significant terms. In Section 3 we introduce functional integration to calculate the effective hydrodynamic action for the system and define one-loop quantum corrections which are responsible for the interaction of the condensate with excited modes. These quantum corrections are evaluated using \( \zeta \)-function regularization for functional determinants and an explicit and very simple expression for the one-loop quantum corrections is presented. This permits in Section 4 the derivation of the quantum corrections to the equations of motion for the condensate in closed and explicit form readily used in numerical calculations. Here corrections to the Nonlinear Schrödinger equation (due to the interaction of condensate with non-condensed fraction) are obtained. Using interpolation between low and high density limits, we show that the contribution from excitations are of order of the main terms in a region around the classical turning points and can define the condensate profile there. Section 5 is devoted to a consideration of Green functions and response functions. It is shown that all Green functions in the 1-loop approximation can be obtained from the effective action of Section 3 by functional differentiation. We conclude the paper with a discussion of the possible applications of our results. Two appendices contain basic facts and some technical details of \( \zeta \)-function regularizations of determinants of elliptic operators and methods of covariant variational differential calculus.

Let us emphasize that our analysis does not depend on the details of the confining potential and can be used in a variety of problems. The only simplification is usual hydrodynamic approximation.

2 Exact dynamical equations for the condensate — review

In this section we discuss relevant contributions from the non-condensed fraction which have to include in the exact equations of motion for the condensate. The estimate of the contributions is more transparent if we follow the usual approach of functional integral for the condensate dynamics [21].

The Hamiltonian of the system can be written as

\[
H = \frac{\hbar^2}{2m} \int dV \left[ \nabla \psi \nabla \psi + \nu(x,t)\psi^+ \psi + 4\pi l \psi^+ \psi \right],
\]  

(1)
where $v(x, t)$ is an external potential. For the case of trapped atoms the potential is supposed to be harmonic and can be varied or even switched off at some moment, say $t = 0$. More precisely it means that for the case of magnetically trapped atoms $v = \theta(-t)\alpha^2(t)(\rho^2 + \lambda^2(t)z^2)$, $\rho^2 = x^2 + y^2$, ($a_\perp$ is oscillator lengths) has to be put in Eqn(1). Another parameter in Eqn(1) is $l$ — the $s$-wave scattering length for atoms in the system [22]. The operators $\psi^+, \psi$ are Bose creation and annihilation operators. The corresponding vacuum-vacuum transition amplitude is given by the following functional integral:

$$Z(\mu) = \int D\psi^+(x, \tau) D\psi(x, \tau) \exp[iS/\hbar] ,$$

with the action

$$S = \int_\infty^\infty d\tau \int dV \left\{ i\hbar \psi^+ \frac{\partial \psi}{\partial \tau} - \frac{\hbar^2}{2m} \left[ \nabla \psi^+ \nabla \psi + (v - \frac{2m}{\hbar^2} \mu)\psi^+ \psi + 4\pi l \psi^+ \psi \psi \right] \right\} .$$

where the parameter $\mu$ is the chemical potential, controlling the number of particles in the system.

As it is well-known the appearance of the Bose condensate is equivalent to the fact that besides “normal” quantities some “anomalous” ones have nonzero values. The principle “anomalous” quantities are one-point Green’s functions $\langle \psi(t) \rangle$ and $\langle \psi^+(t) \rangle$ which are the amplitude of the condensate and have to be taken into account accurately. The standard approach to deal with the condensate is to make a shift of the fields $\psi^+, \psi$ by some functions $\alpha^+, \alpha$ choosing from the condition of the vanishing of the “anomalous” Green’s functions (vanishing of the one-point function provides analogies vanishing of other “anomalous” Green’s functions). Formally, we consider the change of variables

$$\psi^+ = \alpha^+ + \varphi^+ , \quad \psi = \alpha + \varphi ,$$

which leads to a new form of the action (3):

$$S = S_1 + S_2 + S_3 + S_4 + S_5 + S_6$$

\begin{align*}
S_1 &= \int_\infty^\infty d\tau \int dV \left\{ \hbar \alpha^+ i\frac{\partial \alpha}{\partial \tau} - \frac{\hbar^2}{2m} \left[ \nabla \alpha^+ \cdot \nabla \alpha + (v - \frac{2m}{\hbar^2} \mu)\alpha^+ \alpha + 4\pi l (\alpha^+ \alpha)^2 \right] \right\} , \\
S_2 &= \int_\infty^\infty d\tau \int dV \left\{ \hbar \varphi^+ i\frac{\partial \varphi}{\partial \tau} - \frac{\hbar^2}{2m} \left[ \nabla \varphi^+ \cdot \nabla \varphi + (v - \frac{2m}{\hbar^2} \mu)\varphi^+ \varphi \right] \right\} , \\
S_3 &= \int_\infty^\infty d\tau \int dV \left\{ \hbar \alpha^+ i\frac{\partial \alpha}{\partial \tau} - \frac{\hbar^2}{2m} \left[ \nabla \alpha^+ \cdot \nabla \alpha + (v - \frac{2m}{\hbar^2} \mu)\varphi^+ \alpha + 8\pi l \varphi^+ \alpha \alpha \right] \right\} , \\
S_4 &= \int_\infty^\infty d\tau \int dV \left\{ \hbar \alpha^+ i\frac{\partial \varphi}{\partial \tau} - \frac{\hbar^2}{2m} \left[ \nabla \alpha^+ \cdot \nabla \varphi + (v - \frac{2m}{\hbar^2} \mu)\varphi^+ \alpha + 8\pi l \varphi^+ \alpha \alpha \right] \right\} , \\
S_5 &= \int_\infty^\infty d\tau \int dV \left\{ -4\pi l \frac{\hbar^2}{2m} \left[ 4\alpha^+ \alpha \varphi^+ \varphi + \alpha \alpha \varphi^+ \varphi + \alpha^+ \alpha \varphi^+ \varphi \right] \right\} , \\
S_6 &= \int_\infty^\infty d\tau \int dV \left\{ -4\pi l \frac{\hbar^2}{2m} \left[ (\varphi^+ \varphi)^2 + 2\alpha \varphi^+ \varphi + 2\alpha^+ \varphi^+ \varphi \right] \right\} .
\end{align*}
By the definition of the shifts $\alpha^+, \alpha$ “normal” quantities for $\varphi^+, \varphi$ — fields have to be zero, i.e.

$$\langle \varphi \rangle = \int D\varphi^+(x, \tau) D\varphi(x, \tau) \varphi \exp[iS/\hbar] = 0$$

and

$$\langle \varphi^+ \rangle = \int D\varphi^+(x, \tau) D\varphi(x, \tau) \varphi^+ \exp[iS/\hbar] = 0.$$

The last two equations are the main dynamical equations for the condensate wave functions (shift functions):

$$\gamma + \Gamma = 0, \quad \bar{\gamma} + \bar{\Gamma} = 0$$  \hspace{1cm} (8)

where

$$\gamma = -i\hbar \frac{\partial \alpha}{\partial \tau} - \frac{\hbar^2}{2m} \nabla^2 \alpha + (v - \frac{2m}{\hbar^2} \mu) \alpha + 8\pi l \alpha^+ \alpha \alpha$$  \hspace{1cm} (9)

and $\Gamma$ is the sum of all diagrams with only one external outgoing line. Equation (8), in the mean-field approximation (neglecting $\Gamma$), becomes the Gross-Pitaevskii equation since the form of $\gamma$ (9) is:

$$-i\hbar \frac{\partial \alpha}{\partial \tau} - \frac{\hbar^2}{2m} \nabla^2 \alpha + (v - \frac{2m}{\hbar^2} \mu) \alpha + 8\pi l \alpha^+ \alpha \alpha = 0.$$  \hspace{1cm} (10)

In previous work on Bose condensates in traps only equation (10) has been studied.

Let us now estimate corrections to $\Gamma$ from the interaction of the condensate with the non-condensed fraction. For the case of magnetically trapped atoms, i.e. a harmonic potential $v = a_\perp^4 (\rho^2 + \lambda^2 z^2)$ ( $\lambda$ is the anisotropy parameter) there is natural unit of length in the system, which is $a_\perp$. After rescaling the coordinates in units $a_\perp$, the interaction constant $4\pi l$ is changed to $g = 4\pi l/a_\perp$ (in experimental conditions $g \sim 0.05$). Starting with the mean field solution for the condensate fraction in the trap, we can estimate effective interactions (6) and (7). Indeed, the mean field solution for a number of particles $N \sim 5000$ at the center of the trap is of order of 10. From this it follows that at the initial stage, the main contribution to the Gross-Pitaevskii equation is the term $2g|\alpha|^2 \alpha \sim 100$. However the next contribution from the interaction with non-condensed excited states with vertex $4g|\alpha|^2 \sim 20 \gg 1$ cannot be neglected. By comparison all other terms are quite small ($g|\alpha| \sim 0.5, \ g \sim 0.05$) and correspond to the weakly interacting non-condensed Bose gas contributions (for the original experimental setting with $N \sim 2000$ the numbers are $2g|\alpha|^2 \alpha \sim 30, \ 4g|\alpha|^2 \sim 10, \ g|\alpha| \sim 0.3, \ g \sim 0.05$). Hence the description of the transition stage from the original trapped condensate to the dilute phase require taking into account interaction effects due to terms (6) while the interactions (7) can be ignored. Moreover, cooling-heating effects, dissipation from the condensate and so on come from the interaction terms (6) and cannot be neglected in the resulting description of the dynamics of the condensate.

It is easy to see that the interaction (6) together with the quadratic part (3) correspond to a Gaussian approximation in the integration over oscillations around mean field solutions, i.e. Bogoliubov’s excitations. It is quite hard to integrate over Bogoliubov’s excitations in general but this significantly simplifies in the hydrodynamic limit. As it was shown in [7] it is possible to obtain equations for hydrodynamic excitations straightforward from Bogoliubov’s equations and then second quantize the excitations. Instead we will derive the
hydrodynamic description directly from the functional integral formalism, which simplifies the description.

3 Quantum corrections to mean field equation in hydrodynamic regime

We are looking for the effective action which describes all physical quantities for the system (for easy introduction to the formalism of the effective action see, for example, Appendix A of Ref. [18]). For example, the effective action provides all Green’s functions in the same approximation used to calculate the effective action itself. We will obtain it in a hydrodynamic one-loop approximation. To clarify the description hydrodynamical variables, density and velocity, should be used. More precisely, we start with the action (3)

\[ \frac{S}{\hbar} = \frac{1}{\hbar} \int_{-\infty}^{\infty} d\tau \int dV \left\{ i\hbar \psi^+ \frac{\partial \psi}{\partial \tau} - \frac{\hbar^2}{2m} \left[ \nabla \psi^+ \nabla \psi + (v - \frac{2m}{\hbar^2} \mu) \psi^+ \psi + 4\pi \mu (\psi^+ \psi)^2 \right] \right\} , \tag{11} \]

and rescale variables as \( \tau \to \frac{2m}{\hbar} \tau, \mu \to \frac{\hbar^2}{2m} \mu, l \to \frac{l}{4\pi} \) to get the following action form:

\[ \frac{S}{\hbar} = \int_{-\infty}^{\infty} d\tau \int dV \left\{ i\psi^+ \frac{\partial \psi}{\partial \tau} - \left[ \nabla \psi^+ \nabla \psi + (v - \mu) \psi^+ \psi + l(\psi^+ \psi)^2 \right] \right\} . \tag{12} \]

(We will measure the action in terms of \( \hbar \) everywhere below). Now we change field variables to the hydrodynamic ones:

\[ \psi(x, \tau) = \sqrt{\rho(x, \tau)} e^{-i\phi(x, \tau)}, \quad \psi^+(x, \tau) = \sqrt{\rho(x, \tau)} e^{i\phi(x, \tau)}. \tag{13} \]

Then the action (12) takes the form (up to a complete derivative term):

\[ S = \int_{-\infty}^{\infty} d\tau \int dV \left\{ \frac{\partial \phi}{\partial \tau} \rho - (\nabla \sqrt{\rho})^2 - \rho (\nabla \phi)^2 - (v - \mu) \rho - l \rho^2 \right\} . \tag{14} \]

It is not difficult to see that the classical equations of motion for the action lead to the equations:

\[ \frac{\partial \phi}{\partial \tau} - (\nabla \phi)^2 - v + \mu - 2l \rho - \frac{1}{\sqrt{\rho}} \nabla^2 \sqrt{\rho} = 0, \tag{15} \]

\[ - \frac{\partial \rho}{\partial \tau} + 2 \nabla (\nabla \phi \cdot \rho) = 0, \tag{16} \]

which are equivalent to the Gross-Pitaevskii equation [15] after the introduction of the velocity variable \( c = -2 \nabla \phi \) instead of \( \phi \). In velocity-density variables these equations look as hydrodynamic equations for an irrotational compressible fluid:

\[ \frac{\partial c}{\partial \tau} + \nabla \left( \frac{c^2}{2} + 2v - 2\mu + 4l \rho - \frac{2}{\sqrt{\rho}} \nabla^2 \sqrt{\rho} \right) = 0, \tag{17} \]
Now we fulfill the following canonical transformation:

\[
\hat{\chi} = \sqrt{\rho} e^{i\varphi} \left[ \frac{\dot{\sigma}}{2\rho} + i\dot{\alpha} \right], \quad \hat{\chi}^+ = \sqrt{\rho} e^{-i\varphi} \left[ \frac{\dot{\sigma}}{2\rho} - i\dot{\alpha} \right]
\]

The hamiltonian takes the form

\[
H = \int dV \left[ \nabla \hat{\chi}^+ \nabla \hat{\chi} + \{ \varphi - (\nabla \varphi)^2 + 2l\rho \} \hat{\chi}^+ \hat{\chi} - le^{2i\varphi} \rho \hat{\chi}^+ \hat{\chi} + le^{-2i\varphi} \rho \hat{\chi} \hat{\chi} \right].
\]  

(22)

The corresponding action (in feynmann functional integral) is

\[
S_1 = \int_{-\infty}^{\infty} d\tau \int dV \left\{ \frac{\partial \alpha}{\partial \tau} \sigma - [\rho(\nabla \alpha)^2 + \rho \dot{\alpha}^2 + \{l + \frac{\dot{\sigma}}{4\rho} \} \sigma^2 + 2\sigma(\nabla \varphi \nabla \alpha)] \right\}.
\]  

(23)
Integrating in functional integral over the field $\sigma$ we get the action for $\alpha$ field only:

$$S_1 = \int_{-\infty}^{\infty} d\tau \int dV \left\{ -\rho (\nabla \alpha)^2 + f \left( \frac{\partial \alpha}{\partial \tau} - 2 \nabla \varphi \nabla \alpha \right)^2 - \rho \dot{\varphi} \alpha^2 \right\} .$$  \hspace{1cm} (24)

where

$$f = \frac{1}{4l + \dot{\varphi} / \rho} .$$

The corresponding determinant does not contribute.

Now we will use the large parameter $\rho$. Let us introduce the following large number $\rho_0 \equiv \max \{ \rho \}$ such that $\tilde{\rho} = \frac{\rho}{\rho_0} \sim 1$ and new space-time variables $t \equiv 4l \rho_0 \tau, \ y_i \equiv \sqrt{4l \rho_0} x_i$. Hence

$$S_1 = \frac{1}{\sqrt{64 \rho_0^3}} \int_{-\infty}^{\infty} dt \int d\tilde{V} \left\{ -\tilde{\rho} (\tilde{\nabla} \alpha)^2 + \tilde{f} \left( \frac{\partial \alpha}{\partial \tau} - \tilde{\nabla} \tilde{\varphi} \tilde{\nabla} \alpha \right)^2 - \tilde{\rho} \tilde{\varphi} \alpha^2 \right\}$$

$$\equiv - \frac{1}{\sqrt{64 \rho_0^3}} \int_{-\infty}^{\infty} dt \int d\tilde{V} \left[ A^{\mu \nu} \partial_\alpha \rho_{\mu} \partial_\tau \alpha + \tilde{\rho} \tilde{\varphi} \alpha^2 \right] .$$  \hspace{1cm} (25)

where

$$\tilde{v} = 2\tilde{\nabla} \varphi , \quad \tilde{f} = 4lf = \frac{1}{1 + \dot{\varphi} / \tilde{\rho}}$$

and the matrix $A$ has a form

$$A = \begin{pmatrix}
-\tilde{f} & \tilde{f} \tilde{v}_1 & \tilde{f} \tilde{v}_2 & \tilde{f} \tilde{v}_3 \\
\tilde{f} \tilde{v}_1 & \tilde{\rho} - \tilde{f} \tilde{v}_1^2 & -\tilde{f} \tilde{v}_1 \tilde{v}_2 & -\tilde{f} \tilde{v}_1 \tilde{v}_3 \\
\tilde{f} \tilde{v}_2 & -\tilde{f} \tilde{v}_1 \tilde{v}_2 & \tilde{\rho} - \tilde{f} \tilde{v}_2^2 & -\tilde{f} \tilde{v}_2 \tilde{v}_3 \\
\tilde{f} \tilde{v}_3 & -\tilde{f} \tilde{v}_1 \tilde{v}_3 & -\tilde{f} \tilde{v}_2 \tilde{v}_3 & \tilde{\rho} - \tilde{f} \tilde{v}_3^2 \\
\end{pmatrix}$$

with its determinant det $A = -\tilde{\rho}^3 \tilde{f}$.

Our next step is to cast the action in the covariant form. To do this we introduce auxiliary metric $\tilde{g}_{\mu \nu}$ such that

$$A^{\mu \nu} = \frac{\tilde{g}^{\mu \nu}}{\sqrt{-\det (||\tilde{g}^{\mu \nu}||)}} .$$

One can easy to find the covariant metric from the equation above:

$$\tilde{g}^{\mu \nu} = \frac{A^{\mu \nu}}{\sqrt{-\det (||A^{\mu \nu}||)}}$$

that leads to the form:

$$||\tilde{g}^{\mu \nu}|| = \tilde{f}^{-1/2} \tilde{\rho}^{-3/2} \begin{pmatrix}
-\tilde{f} & \tilde{f} \tilde{v}_1 & \tilde{f} \tilde{v}_2 & \tilde{f} \tilde{v}_3 \\
\tilde{f} \tilde{v}_1 & \tilde{\rho} - \tilde{f} \tilde{v}_1^2 & -\tilde{f} \tilde{v}_1 \tilde{v}_2 & -\tilde{f} \tilde{v}_1 \tilde{v}_3 \\
\tilde{f} \tilde{v}_2 & -\tilde{f} \tilde{v}_1 \tilde{v}_2 & \tilde{\rho} - \tilde{f} \tilde{v}_2^2 & -\tilde{f} \tilde{v}_2 \tilde{v}_3 \\
\tilde{f} \tilde{v}_3 & -\tilde{f} \tilde{v}_1 \tilde{v}_3 & -\tilde{f} \tilde{v}_2 \tilde{v}_3 & \tilde{\rho} - \tilde{f} \tilde{v}_3^2 \\
\end{pmatrix} .$$
Finally we get the metric $\|\tilde{g}_{\mu\nu}\|$:

$$
\|\tilde{g}_{\mu\nu}\| = \tilde{\rho}^{1/2} \tilde{f}^{1/2} \begin{pmatrix}
-\tilde{\rho}/\tilde{f} + \tilde{v}^2 & \tilde{v}_1 & \tilde{v}_2 & \tilde{v}_3 \\
\tilde{v}_1 & 1 & 0 & 0 \\
\tilde{v}_2 & 0 & 1 & 0 \\
\tilde{v}_3 & 0 & 0 & 1
\end{pmatrix}.
$$

with the determinant $\tilde{g} \equiv \text{det}(\|\tilde{g}_{\mu\nu}\|) = -\tilde{\rho}^{3/2} \tilde{f}$.

In this metric the action takes a covariant form

$$
S_1 = -\frac{1}{\sqrt{64 \rho_0 l^3}} \int dy \sqrt{-\tilde{g}} \left[ \alpha \tilde{\Box} \alpha + \tilde{E} \alpha^2 \right] = \frac{1}{\sqrt{64 \rho_0 l^3}} \int dy \sqrt{-\tilde{g}} \alpha \left[ \frac{1}{\sqrt{-\tilde{g}}} \tilde{g}^{\mu\nu} \sqrt{-\tilde{g}} \partial_\nu \tilde{E} + \tilde{E} \right] \alpha, \tag{26}
$$

where

$$
\tilde{E} = \frac{\tilde{\rho} \dot{\phi}}{\sqrt{-\tilde{g}}} = \frac{\dot{\phi}}{\sqrt{\tilde{\rho} \tilde{f}}}. 
$$

This is the central equation of the paper.

Now the effective action is

$$
\Gamma_1 = -\frac{1}{2} \text{tr} \ln[(64 \rho_0 l^3)^{-1/2}(\tilde{\Box} + \tilde{E})] 
$$

with Laplace operator

$$
\tilde{\Box} = -\frac{1}{\sqrt{-\tilde{g}}} \partial_\mu \tilde{g}^{\mu\nu} \sqrt{-\tilde{g}} \partial_\nu. 
$$

Let us again make use the existence of large parameter in the system. Indeed, the multiplier $(64 \rho_0 l^3)^{-1/2}$ gives a possibility to express main contributions to the determinant such as (see Appendix A):

$$
\text{tr} \ln[(64 \rho_0 l^3)^{-1/2}(\tilde{\Box} + \tilde{E})] = \text{tr} \ln[\tilde{\Box} + \tilde{E}] - \frac{1}{2} \text{tr} \ln(64 \rho_0 l^3) \left( \Phi_0(\tilde{\Box} + \tilde{E}) - L(\tilde{\Box} + \tilde{E}) \right) 
$$

since in our regularization $\text{tr} \ln[\tilde{\Box} + \tilde{E}]$ is order of the zeroth Seeley coefficient $\Phi_0(\tilde{\Box} + \tilde{E}) \equiv \int \text{d}y \text{d}\tau \sqrt{-\tilde{g}} \Psi_0$. We will consider only cases where the number of zero-modes does not change. So in what follows we will drop the term with $L(\tilde{\Box} + \tilde{E})$. Returning to the initial variables all curvature tensors and the metric are written in the “physical” variables (i.e. without tildes). Moreover $\ln(64 \rho_0 l^3) \gg \ln(\tilde{\rho})$ so that we can substitute $\ln(64 \rho_0 l^3)$ instead of $\ln(64 \rho_0 l^3)$. Summarizing, we obtain an expression for the first quantum correction to the effective action:

$$
\Gamma_1 = \frac{1}{4} \int \text{d}x \text{d}\tau \sqrt{-g} \ln(64 \rho_0 l^3) \Psi_0(\Box + E), \tag{27}
$$

with the metric

$$
\|g_{\mu\nu}\| = \rho^{1/2} f^{1/2} \begin{pmatrix}
-\rho/f + v^2 & v_1 & v_2 & v_3 \\
v_1 & 1 & 0 & 0 \\
v_2 & 0 & 1 & 0 \\
v_3 & 0 & 0 & 1
\end{pmatrix},
$$

and

$$
v_i \equiv 2 \partial_i \varphi, \quad f = \frac{1}{4 l + \varphi/\rho}, \quad E = \frac{\dot{\phi}}{\sqrt{\rho f}}.
$$
4 Quantum corrections to equations of motion

In this section we derive (27) required quantum corrections to equations of motion (see, for example, [18]). To make the consideration self-consistent we should not differentiate \(\ln(64\rho_l^3)\) because in our approximation it behaves like a constant. In that case, the action we have to vary is covariant again. We find:

\[
\frac{\delta \Gamma_1}{\delta \rho} = \frac{1}{4} \ln(64\rho_l^3) \left[ \frac{\partial g_{\mu\nu}}{\partial \rho} \frac{\delta \Phi_0}{\delta g_{\mu\nu}} + \frac{\partial E}{\partial \rho} \frac{\delta \Phi_0}{\delta E} \right],
\]

\[
\frac{\delta \Gamma_1}{\delta \varphi} = \frac{1}{4} \ln(64\rho_l^3) \int d^4 y \left[ \frac{\delta g_{\mu\nu}(x)}{\delta \varphi(y)} \frac{\delta \Phi_0}{\delta g_{\mu\nu}(x)} + \frac{\delta E(x)}{\delta \varphi(y)} \frac{\delta \Phi_0}{\delta E(x)} \right].
\]

Let us remember that

\[
\Phi_0(\Box + E) = \frac{1}{(4\pi)^2} \int dx \, \sqrt{-g} \left[ -\frac{1}{30} \nabla^2 R + \frac{1}{72} R^2 - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} \right. \\
+ \left. \frac{1}{180} R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} + \frac{1}{6} R E + \frac{1}{2} E^2 - \frac{1}{6} \nabla^2 E \right]
\]

We note that two of the terms above are irrelevant:

\[
\int dx \, \sqrt{-g} \nabla^2 F = \int dx \, \sqrt{-g} \frac{1}{\sqrt{-g}} \partial_{\mu} \sqrt{-g} g^{\mu\nu} \partial_{\nu} F = \int dx \, \partial_{\mu} (\sqrt{-g} \delta_{\mu} F)
\]

since this is just a complete derivative. Hence for our purposes it is sufficient to consider only

\[
\Phi_0(\Box + E) = \frac{1}{(4\pi)^2} \int dx \, \sqrt{-g} \left[ \frac{1}{72} R^2 - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} + \frac{1}{180} R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} + \frac{1}{6} R E + \frac{1}{2} E^2 \right]
\]

Long but quite straightforward calculations lead us to the following result for the functional derivative (see Appendix B for details of the calculation):

\[
\frac{\delta}{\delta g_{\mu\nu}} \Phi_0 = \frac{1}{(4\pi)^2} \sqrt{-g} \left\{ -\frac{1}{2} g_{\mu\nu} \left[ \frac{1}{72} R^2 - \frac{1}{180} R_{\sigma\rho} R^{\sigma\rho} + \frac{1}{180} R_{\sigma\rho\alpha\beta} R^{\sigma\rho\alpha\beta} + \frac{1}{6} R E + \frac{1}{2} E^2 \right] \\
+ \frac{1}{6} R_{\mu\nu} E + \frac{1}{36} R_{\mu\nu} R - \frac{1}{90} R_{\mu\nu} R_{\rho}^{\rho} + \frac{1}{90} R_{\mu\sigma\rho\alpha} R_{\nu}^{\rho\alpha} - \frac{1}{36} g_{\mu\nu} \Box R \\
+ \frac{1}{72} \{\nabla_{\mu}, \nabla_{\nu}\} R - \frac{1}{6} g_{\mu\nu} \Box E + \frac{1}{12} \{\nabla_{\mu}, \nabla_{\nu}\} E \\
+ \frac{1}{180} \left[ -2 \nabla^{\sigma} \nabla_{\nu} R_{\mu\sigma} + \Box R_{\mu\nu} + g_{\mu\nu} \nabla^{\sigma} \nabla_{\rho} R_{\sigma\rho} \right] + \frac{1}{45} \nabla^{\sigma} \nabla^{\sigma} R_{\mu\nu\rho\sigma} \right\}
\]

taking into account

\[
\frac{\delta}{\delta g_{\mu\nu}} \Phi_0(\Box + E) = -g^{\mu\sigma} g^{\nu\rho} \frac{\delta}{\delta g^{\sigma\rho}} \Phi_0(\Box + E).
\]
This means that the equations of motions with 1-loop quantum correction take the form:

\[
\frac{\partial \varphi}{\partial \tau} - (\nabla \varphi)^2 - v + \mu - 2l\rho + \frac{1}{4} \ln(64\rho l^3) \left[ \frac{\partial g_{\mu\nu}}{\partial \rho} \frac{\delta \Phi_0}{\delta g_{\mu\nu}} + \frac{\partial E}{\partial \rho} \frac{\delta \Phi_0}{\delta E} \right] = 0 ,
\]

(28)

\[
- \frac{\partial \rho}{\partial \tau} + 2\nabla (\nabla \cdot \rho) + \frac{1}{4} \ln(64\rho l^3) \left[ \int d^4y \frac{\delta g_{\mu\nu}(x)}{\delta \varphi(y)} \frac{\delta \Phi_0}{\delta g_{\mu\nu}(x)} - \frac{\partial}{\partial \tau} \left( \frac{\partial E}{\partial \varphi} \frac{\delta \Phi_0}{\delta E} \right) \right] = 0 ,
\]

(29)

These two equations (28) and (29) replace the mean field hydrodynamic equations (15,16) for the density and velocity of the condensate and are the main result of the paper. They contain contributions of the excited states to the dynamics of the condensed fraction and provide a description of effects such as depletion and heating-cooling. For example, we note that the additional quantum pressure is due to dependence of the Bogoliubov particles determinant on the density of the condensate while an evaporation of the condensate (see continuity equation (29)) comes from phase-dependence of the determinant. In particular, for the stationary situation when the equilibrium condensate velocity is equal to zero there is no evaporation of the condensate and the continuity equation holds for the fraction. This case was considered in Ref.[18]. However we think that equations (28) and (29) are of the main interest in non-equilibrium problems such as evolution of the condensate under changing of trap shapes, collapse of condensate cloud and corresponding heating and evaporation of the condensate and so on. We hope to consider these problems in forthcoming papers.

5 Green functions, response functions and the like

Equations of motion and quantum corrections for them are the central questions of previous consideration. However in many applications related to experiment it is very important to analyze various response functions and form-factors. They can be expressed in terms of some combinations of Green functions of the theory. That is why in this section we shortly consider a calculation of Green functions in the effective action formalism.

As it is shown in Ref [18] the effective action (in any self-consistent approximation) allows the evaluation of all Green functions in the same approximation by just taking of variational derivatives. Since the effective action in 1-loop approximation was obtained the problem of Green function calculation is the problem of functional differentiation only [24].

We now formalize all said above and give formulas to evaluate Green functions in effective action formalism. Let \( W(J) \) being generating functional of connected Green functions. Then quantities

\[
W_n(x_1, \ldots, x_n) = \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} W(J)
\]

(30)

are connected Green functions in the external field \( J \). The effective action is defined then by the Legendre transformation of \( W \):

\[
\Gamma(\alpha) = W(J(\alpha)) - \alpha J(\alpha) , \quad \alpha = \frac{\delta W(J)}{\delta J}
\]

(31)
where the function \( \alpha(x) = \langle \hat{\phi}(x) \rangle = W_1(x; J) \) is the first connected Green function \([30]\). It is easy to see that the functions \( \alpha \) and \( A \) are related by the second relation \([31]\). Then the quantities
\[
\Gamma_n(x_1, \ldots, x_n) = \frac{\delta}{\delta \alpha(x_1)} \cdots \frac{\delta}{\delta \alpha(x_n)} \Gamma(\alpha)
\]
(32)
define 1PI (one-particle-irreducible) Green functions for the theory with mean field \( \alpha(x) = \langle \hat{\phi}(x) \rangle \). Knowledge of 1PI Green function is equivalent to knowledge of any (whole or connected) Green functions for the corresponding system. For the first 1PI Green function we have from \([31]\)
\[
\Gamma_1(x) = -J(x) .
\]
(33)
All connected Green functions can be expressed in terms of 1PI ones. Indeed, differentiating \( J \) we obtain
\[
W_2 \Gamma_2 = -1 , \quad W_2 = -\Gamma_2^{-1}
\]
(34)
or, in expanded form,
\[
\int dz \ W_2(x, z) \Gamma_2(z, y) = -\delta(x - y) .
\]
(35)
Differentiating now the second relation in \([34]\) on \( J \) and using the following rule
\[
\frac{\delta}{\delta J} = -\Gamma_2^{-1} \frac{\delta}{\delta \alpha}
\]
we can derive expressions for all the higher connected Green functions through the 1PI ones. For example, for the third connected function we have
\[
W_3 = -\left[ \Gamma_2^{-1} \right]^3 \Gamma_3
\]
or
\[
W_3(x_1, x_2, x_3) = -\int dy_1 dy_2 dy_3 \ \Gamma_2^{-1}(x_1, y_1) \Gamma_2^{-1}(x_2, y_2) \Gamma_2^{-1}(x_3, y_3) \Gamma_3(y_1, y_2, y_3)
\]
(36)
and so on. It means that since we know 1PI functions we have to solve the differential equation \([33]\) for the two-point correlation function and then find all other connected Green functions by integration of 1PI functions with two-point correlators as in Eqn.(36).

Putting \( J = 0 \) in \([33]\) we get equations of motion for \( \psi \) and \( \psi^+ \) \([28, 29]\). To obtain, for example, the Green function \( \langle (\psi^+(x) - \langle \psi^+(x) \rangle) (\psi(y) - \langle \psi(y) \rangle) \rangle \) we twice differentiate the effective action on \( \psi, \psi^+ \), substitute solution of equations of motion and finally invert the result in the sense of the kernel of an integral operator.

As an example of usage of this technique we calculate two-point Green function in the mean field approximation (tree or 0-loop approximation). In this approximation the effective action has the form:
\[
\Gamma = \int_{-\infty}^{\infty} d\tau \int dV \left\{ -i \frac{\partial \psi^+}{\partial \tau} \psi - \nabla \psi^+ \nabla \psi - (v - \mu) \psi^+ \psi - l(\psi^+ \psi)^2 \right\} .
\]
(37)
such that, in the equilibrium hydrodynamic picture, the matrix of second variational derivatives of the effective action can be written as

\[
\delta^2 \Gamma = \begin{pmatrix}
\frac{\delta^2 \Gamma}{\delta \psi^+(x) \delta \psi^+(y)} & \frac{\delta^2 \Gamma}{\delta \psi^+(x) \delta \psi(y)} \\
\frac{\delta^2 \Gamma}{\delta \psi(x) \delta \psi^+(y)} & \frac{\delta^2 \Gamma}{\delta \psi(x) \delta \psi(y)}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-2l \psi^2 & -i \frac{\partial}{\partial \tau} - \nabla^2 - (v - \mu) - 2I(\psi^+ \psi) \\
-i \frac{\partial}{\partial \tau} - \nabla^2 - (v - \mu) - 2I(\psi^+ \psi) & -2I(\psi^+)^2
\end{pmatrix}.
\]

where \(\psi^+ \equiv \langle \hat{\psi}^+ \rangle\). The matrix of two-point connected correlation functions of fluctuations \(\hat{\psi}(x) = \hat{\psi}(x) - \langle \hat{\psi}(x) \rangle\) and \(\hat{\psi}^+(x) = \hat{\psi}^+(x) - \langle \hat{\psi}^+(x) \rangle\)

\[
G = \begin{pmatrix}
\langle \hat{\psi}^+(x) \hat{\psi}^+(y) \rangle & \langle \hat{\psi}^+(x) \hat{\psi}(y) \rangle \\
\langle \hat{\psi}(x) \hat{\psi}^+(y) \rangle & \langle \hat{\psi}(x) \hat{\psi}(y) \rangle
\end{pmatrix}
\]

is defined as a solution of the equation

\[
\delta^2 \Gamma \cdot G = -I.
\]

Solving the equation we get the following expressions for the correlators. One can check that the Green functions calculated in this manner coincide with the Green functions \(\langle \hat{\rho}(x) \hat{\rho}(y) \rangle\), \(\langle \hat{\rho}(x) \hat{\varphi}(y) \rangle\), \(\langle \hat{\varphi}(x) \hat{\varphi}(y) \rangle\) analyzed in spectral representation by Wu and Griffin [17]. To include the first quantum correction (1-loop quantum correction) term (27) should be added to action (37) [23].

In summary, various response functions, form-factors and Green functions are evaluated from effective action in any given approximation taking variational derivatives and integrating them with the two-point correlation function. This can be done in general using formulas of this section.

6 Conclusion

In the paper a self-consistent approach to the calculation of 1-loop quantum corrections to the Gross-Pitaevskii equation due to the interaction of the condensate with collective excitations is considered. To do this the hydrodynamic approximation was used. In this approximation excitations are equivalent to sound waves on a condensate background. This opens the possibility to use methods of quantum gravity and theory of quantum gauge fields where the problem of calculation of effective action for gravitational and gauge backgrounds (or more precise, quantum corrections due to other quantum fields) is common problem.

Many of the methods to approach the problem were developed during the last four decades. They are \(\zeta\)-function regularization for determinants of operators, Schwinger-De-Witt-Seeley expansion of heat kernels [27, 28, 29], covariant methods of calculation of Seeley’s
coefficients and the covariant perturbation technique for effective actions \[30, 31\]. Although the methods are well-known in field theory they are not so familiar in condensed matter physics and the theory of coherent systems. An accurate account of corrections is very complicated even for a few first orders and the solutions are obtained by using the covariant perturbation technique and curvature expansions. On the way effects nonlocal in space and time appeared. In quantum gravity framework such effects play significant role in the gravitational collapse problem, Hawking radiation and so on.

However to benefit from it sometimes very general and only basic information about principle scales in systems in question is required. Indeed, this is the only information needed to extract leading logarithmic contributions while the calculation of other corrections is much more complicated. That is why we think that the developed approach can be widely used in many problems which have not much to do with Bose-condensation of trapped atoms or liquid Helium.

At the end let us stop on other applications of the presented method. It is easy to imagine other condensed matter examples of problems treatable by the same technique. Clearest and simplest of them are antiferromagnets, Josephson arrays and superconductors in nonuniform (in space and time) external magnetic fields. However many of other physical systems in a non-equilibrium background are potential field of applications.

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Appendices

A Determinants of elliptic operators

Usually the first quantum correction to the effective action is ill defined. The point is it is divergent. This is just the well-known ultraviolet divergence of the quantum field theory. Indeed, we can rewrite the functional determinant as

$$\ln \det A = \ln \prod_n \lambda_n = \sum_n \ln \lambda_n$$

where $\lambda_n$ are the eigenvalues of the operator $A$. This series is easy to show to be divergent.

That means that one needs a regularization. This point was investigated very thoroughly by many authors and it is found that in quantum gravity and gauge theories the most appropriate regularizations are the analytical ones. The functional determinants can be well defined in terms of the so called $\zeta$-function.

At first let us use a Wick rotation to produce a Laplace operator from the wave operator. It allows to make use of methods for the evaluation of the determinants of elliptic operators\cite{28}. Now we are ready to introduce the $\zeta$-function of an elliptic operator $A$:

$$\zeta(s,A) = \sum_i \lambda_i^{-s}$$

where $\{\lambda_i\}$ are eigenvalues of the operator $A$. Then

$$\Tr \ln A = - \left. \frac{d}{ds} \zeta(s,A) \right|_{s=0}$$

To study the behavior of the $\zeta$-function it is common to use the formula

$$\lambda_i^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} \exp(-t \lambda_i) .$$

Summing over all nonzero eigenvalues of $A$ we get

$$\zeta(s,A) = \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} (\Tr \exp(-tA) - L(A))$$

where $L(A)$ is a number of zero-modes of $A$. Representation (A.2) is valid when the integral converges. For nonnegative operator it always does converge as $t \to +\infty$. Conditions of the convergency of the integral as $t \to +0$ depend on details of the operator $A$. For Laplace operator in 4D it converges as $t \to +0$ if $\mathrm{Re}\ s > 2$.

There exists the well-known Seeley expansion for the $\Tr (\exp(-tA))$:

$$\Tr \exp(-tA) = \sum_{k \geq 0} \Phi_{-k}(A) t^{-k} + \rho(t) ,$$
where \( |\rho(t)| \) is bounded by a constant times \( t \) as \( t \to +0 \). \( \Phi_{-k}(A) \) are called Seeley coefficients and play important role in the investigation of elliptic operators and their topological properties (for example in the Index theory). For example, for 4D Laplace operator in an external potential \( \Phi - k(A) = 0 \) \( k \geq 3 \), and

\[
\Phi_{-2} = (4\pi)^{-2} \, , \quad \Phi_{-1} = -(4\pi)^{-2}(E + \frac{1}{6}R) \, , \quad \text{where } R \text{ — scalar curvature} \, ,
\]

and

\[
\Phi_0 = (4\pi)^{-2} \left( -\frac{1}{30} \nabla^2 R + \frac{1}{72} R^2 - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} + \frac{1}{180} R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} + \frac{1}{6} E R + \frac{1}{2} E^2 - \frac{1}{6} \nabla^2 E \right) .
\]

Here \( R_{\mu\nu\sigma\rho} \) and \( R_{\mu\nu} \) are Riemann and Ricci tensors correspondingly. In the paper we use the following definitions for them [26]:

\[
R = R_{\mu}^{\mu}, \quad R_{\mu\nu} = R_{\mu\sigma\nu}^{\sigma} ,
\]

and

\[
R_{\mu\nu\sigma\rho} = \frac{\partial \Gamma_{\mu\nu}^{\lambda}}{\partial x^\rho} - \frac{\partial \Gamma_{\mu\rho}^{\lambda}}{\partial x^\nu} + \Gamma^{\lambda}_{\mu\nu} \Gamma_{\rho\sigma}^{\gamma} - \Gamma^{\lambda}_{\mu\rho} \Gamma_{\nu\sigma}^{\gamma} ,
\]

with the Christoffel symbols:

\[
\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left( \frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} \right) .
\]

They are defined by a metric of a curved space. The latter coefficient \( \Phi_0 \) is particular important for the calculation of determinants.

Splitting (A.2) into an integral over \([0,1]\) and one over \([1, +\infty]\) we get

\[
\zeta(s) = \frac{1}{\Gamma(s)} \left( \sum_{k>0} \frac{\Phi_{-k}(A)}{s-k} + \frac{\Phi_0(A) - L(A)}{s} \right) + \int_1^{\infty} dt \, \text{Tr} \exp(-tA)t^{s-1} + \int_0^1 dt \, \rho(t)t^{s-1} \right)
\]

The singularity at \( s = 0 \) turns out to be removable since \( \lim_{s \to 0} s \Gamma(s) = 1 \) and \( \zeta(s) \) is defined by analytical continuation from the half-plane \( \text{Re} \, s > 0 \).

From the relation \( \ln \det A = -\zeta'(0) \) we find that

\[
\ln \det A = \sum_{k>0} \frac{\Phi_{-k}(A)}{k} + \Gamma'(1) \left( \Phi_0(A) - L(A) \right) - \int_1^{\infty} \frac{dt}{t} \text{Tr} \exp(-tA) - \int_0^1 \frac{dt}{t} \left( \text{Tr} \exp(-tA) - \sum_{k<0} \Phi_{-k}(A)t^{-k} \right) .
\]

This is \( \zeta \)-regularized \( \ln \det A \) which we use in the paper.

To conclude this section we give the relation between \( \ln \det A \) and \( \ln \det \alpha A \) where \( \alpha \) is a number parameter. It is easy to see that

\[
\zeta(s, \alpha A) = \zeta(s, A) \cdot \alpha^{-s}
\]

because \( \lambda_i(\alpha A) = \alpha \lambda_i(A) \). This leads to the relation:

\[
\ln \det \alpha A = \ln \det A + \ln \alpha \cdot \zeta(0, A) = \ln \det A + \ln \alpha \cdot \left( \Phi_0(A) - L(A) \right)
\]

(A.3) which we use intensively in the paper.
B Variational derivative of the effective action

In this appendix we give details of the calculation omitted in section 3 to do not interrupt the main stream of the consideration. To make the calculation more efficient we make use covariant form of the quantum correction since there exists well-know way to simplify covariant variation calculation [27].

We are interested in the following variational derivative

$$\frac{\delta}{\delta g^\mu\nu}\Phi_0(\Box)$$

where

$$\Phi_0(\Box) = \frac{1}{(4\pi)^2} \int dx \sqrt{-g} \left( \frac{1}{72} R^2 - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} + \frac{1}{180} R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} + \frac{1}{6} R E + \frac{1}{2} E^2 \right)$$

For variation of the volume element we have

$$\delta \sqrt{-g} = -\frac{1}{2\sqrt{-g}} g_{\mu\nu} \Delta^\mu\nu = \frac{\sqrt{-g}}{2} \delta g_{\mu\nu} g^{\mu\nu} = -\frac{\sqrt{-g}}{2} g_{\mu\nu} \delta g^{\mu\nu}$$

where $\Delta^\mu\nu$ is defined by as

$$\Delta^\mu\nu = g^{\mu\nu}$$

and the following relation was used

$$\delta g_{\mu\nu} g^{\nu\lambda} = -g_{\mu\nu} \delta g^{\nu\lambda}.$$  

For variations $R^2, R_{\mu\nu} R^{\mu\nu}, R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho}$ one can obtain

$$\delta (R E) = \delta R \cdot E = \delta g^{12} R_{12} E + g^{12} \delta R_{12} \cdot E$$

$$\delta R^2 = 2 \delta R \cdot R = 2 g^{12} R_{12} R + 2 g^{12} \delta R_{12} \cdot R$$

$$\delta R_{12} R^{12} = \delta (g^{13} g^{24} R_{12} R_{34}) = 2 \delta g^{12} R_{12} R_{23}^3 + 2 \delta R_{12} R^{12}$$

$$\delta R_{1234} R^{1234} = \delta (g^{15} g^{26} g^{37} g^{48} R_{1234} R_{5678}) = 4 \delta g^{12} R_{1345} R_{2345} + 2 \delta R_{1234} R^{1234}$$

where numbers 1, 2, 3, . . . mean indices $\mu_1, \mu_2, \mu_3, \ldots$ respectively. Hence,

$$\delta \Phi_0 = \frac{1}{(4\pi)^2} \int dx \sqrt{-g} \delta g^{\mu\nu} \left[ -\frac{1}{2} g_{\mu\nu} \left\{ \frac{1}{72} R^2 - \frac{1}{180} R_{12} R^{12} + \frac{1}{180} R_{1234} R^{1234} + \frac{1}{6} R E + \frac{1}{2} E^2 \right\} \right.$$

$$+ \frac{1}{36} R_{\mu\nu} R - \frac{1}{90} R_{\mu12} R_{k\nu}^{\cdot 123} + \frac{1}{45} R_{\mu\lambda123} R_{\nu\cdot 123} + \frac{1}{6} R_{12} E \right]$$

$$+ \frac{1}{(4\pi)^2} \int dx \sqrt{-g} \left[ \frac{1}{36} g^{12} \delta R_{12} R - \frac{1}{90} \delta R_{12} R^{12} + \frac{1}{90} \delta R_{1234} R^{1234} + \frac{1}{6} g^{12} \delta R_{12} E \right]$$

So we need to calculate last three terms.
Hence

For variations of Ricci and Riemann tensors we have [25]:

\[
\delta R_{\mu\nu} = \nabla_\nu (\delta \Gamma^1_{\mu\lambda}) - \nabla_\lambda (\delta \Gamma^1_{\mu\nu}) \]
\[
= \frac{1}{2} g^{12} [\nabla_\mu \nabla_\nu \delta g_{12} - \nabla_1 \nabla_\mu \delta g_{2\nu} - \nabla_1 \nabla_\nu \delta g_{2\mu} + \nabla_1 \nabla_2 \delta g_{\mu\nu}] \quad (B.3)
\]

\[
\delta R_{\mu\nu\rho\sigma} = -g_{\mu\lambda} \delta g^{12} R_{2\nu\rho\sigma} + g_{\mu\lambda} \nabla_\rho (\delta \Gamma^1_{\nu\sigma}) - g_{\mu\lambda} \nabla_\sigma (\delta \Gamma^1_{\nu\rho}) \]
\[
= -g_{\mu\lambda} \delta g^{12} R_{2\nu\rho\sigma} + \frac{1}{2} [\nabla_\rho \nabla_\sigma \delta g_{\mu\nu} + \nabla_\rho \nabla_\nu \delta g_{\mu\sigma} - \nabla_\rho \nabla_\nu \delta g_{\mu\sigma} - \nabla_\sigma \nabla_\rho \delta g_{\mu\nu} - \nabla_\sigma \nabla_\nu \delta g_{\mu\rho} + \nabla_\sigma \nabla_\mu \delta g_{\nu\rho}] \quad (B.4)
\]

Let us consider covariant divergence of some vector \( T^k \):

\[
\nabla_\mu T^\mu = \nabla^\mu T_\mu = \partial_\mu T^\mu + \Gamma^\nu_{\mu\nu} T^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} T^\mu)
\]

Hence

\[
\sqrt{-g} \nabla_\mu T^\mu = \partial_\mu (\sqrt{-g} T^\mu)
\]

is a pure divergence and can be dropped under integration. Using this we can easily derive the following expression for variations of the integrals we are interested in

\[
\int dx \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} R = \int dx \sqrt{-g} \delta g^{\mu\nu} \left[ \frac{1}{2} \{\nabla_\mu, \nabla_\nu\} R - g_{\mu\nu} \Box R \right]
\]

\[
\int dx \sqrt{-g} \delta R_{\mu\nu} R^{\mu\nu} = \frac{1}{2} \int dx \sqrt{-g} \delta g^{\mu\nu} \left[ -\Box R_{\mu\nu} - g_{\mu\nu} \nabla_1 \nabla_2 R^{12} + \nabla_1 \nabla_\mu R^{1\nu} + \nabla_1 \nabla_\nu R^{1\mu} \right]
\]

\[
\int dx \sqrt{-g} \delta R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \int dx \sqrt{-g} \delta g^{\mu\nu} \left[ -R_{\mu123} R^{123} + 2 \nabla^2 R_{\nu12\mu} \right]
\]

\[
\int dx \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} E = \int dx \sqrt{-g} \delta g^{\mu\nu} \left[ \frac{1}{2} \{\nabla_\mu, \nabla_\nu\} E - g_{\mu\nu} \Box E \right]
\]

Summing all terms we obtain

\[
\frac{\delta}{\delta g^{\mu\nu}} \Phi_0 = \frac{1}{2(4\pi)^2} \sqrt{-g} \left\{ -\frac{1}{2} g_{\mu\nu} \left[ \frac{1}{72} R^2 - \frac{1}{180} R_{\sigma\rho} R^{\sigma\rho} + \frac{1}{180} R_{\sigma\rho\gamma\beta} R^{\sigma\rho\gamma\beta} + \frac{1}{6} R E + \frac{1}{2} E^2 \right]
+ \frac{1}{6} R_{\mu\nu} E + \frac{1}{36} R_{\mu\nu} R - \frac{1}{90} R_{\mu\nu} R^{\sigma} + \frac{1}{90} R_{\mu\sigma\rho\alpha} R^{\sigma\rho\gamma\beta} - \frac{1}{36} g_{\mu\nu} \Box R
+ \frac{1}{72} \{\nabla_\mu, \nabla_\nu\} R - \frac{1}{6} g_{\mu\nu} \Box E + \frac{1}{12} \{\nabla_\mu, \nabla_\nu\} E
+ \frac{1}{180} \left[ -2 \nabla^\sigma \nabla_\nu R_{\mu\sigma} + \Box R_{\mu\nu} + g_{\mu\nu} \nabla^\sigma \nabla^\rho R_{\sigma\rho} \right] + \frac{1}{45} \nabla^\rho \nabla^\sigma R_{\mu\sigma\rho\nu} \right\}
\]

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[23] Using field-theoretical terminology the analysis by Wu and Griffin [17] corresponds to a classical level while term (27) is a first quantum correction. Since paper [17] has a title “Quantized hydrodynamic model...” a confusion can occur. In this way, the only self-consistent possibility to introduce non-condensed fraction in the consideration is to study the action (37) with (at least 1-loop) quantum corrections.

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