LIMIT THEOREMS FOR A STABLE SAUSAGE

WOJCIECH CYGAN, NIKOLA SANDRIĆ, AND STJEPAN ŠEBEK

Abstract. In this article, we study fluctuations of the volume of a stable sausage defined via a $d$-dimensional rotationally invariant $\alpha$-stable process with $d > 3\alpha/2$, and a closed unit ball. As the main results, we establish a functional central limit theorem with a standard one-dimensional Brownian motion in the limit, and an almost sure invariance principle for the process of the volume of a stable sausage. As a consequence, we obtain Khintchine’s and Chung’s laws of the iterated logarithm for this process.

1. Introduction

Let $X = \{X_t\}_{t \geq 0}$ be a Lévy process in $\mathbb{R}^d$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A Lévy sausage associated with the process $X$ and a given compact set $K \subset \mathbb{R}^d$, on the time interval $[s, t]$, $0 \leq s \leq t$, is the random set defined as

$$S^K[s, t] = \bigcup_{s \leq u \leq t} \{X_u + K\}.$$ 

If $s = 0$ we use notation $S^K_t = S^K[0, t]$. Let $\lambda(dx)$ be the Lebesgue measure on $\mathbb{R}^d$ and let us denote by

$$\mathcal{V}^K[s, t] = \lambda(S^K[s, t])$$

the volume of the Lévy sausage $S^K[s, t]$ (we write $\mathcal{V}^K_t = \lambda(S^K_t)$). Already Spitzer [27] linked $\mathcal{V}^K$ with the first hitting time $\tau_K = \inf\{s \geq 0 : X_s \in K\}$ via the identity

$$\mathbb{E}[\mathcal{V}^K_t] = \int_{\mathbb{R}^d} \mathbb{P}_x(\tau_K \leq t) \, dx, \quad t \geq 0,$$

where $\mathbb{P}_x$ is the probability measure related to the process $X$ started at $x \in \mathbb{R}^d$. Port and Stone [21] proved that if $X$ is transient then

$$\lim_{t \to \infty} \frac{\mathbb{E}[\mathcal{V}^K_t]}{t} = \text{Cap}(K),$$

where $\text{Cap}(K)$ is the capacity of $K$ associated with the process $X$. Hawkes [11] observed that in view of the subadditivity of the process $\{\mathcal{V}^K_t\}_{t \geq 0}$, that is,

$$\mathcal{V}^K_{s+t} \leq \mathcal{V}^K_s + \mathcal{V}^K[s, s+t], \quad s, t \geq 0,$$

eq (1.2) combined with Kingman’s ergodic theorem (cf. [15, Theorem 1.5.6]) and [14, Proposition 3.12] imply the following strong law of large numbers

$$\lim_{t \to \infty} \frac{\mathcal{V}^K_t}{t} = \text{Cap}(K) \quad \mathbb{P}\text{-a.s.}$$

More involved limit theorems for the volume of a Lévy sausage are known if $X$ is a standard Brownian motion. In this case $S^K_t$ is called a Wiener sausage, and its asymptotic behavior was studied in a lot of mathematical papers by many authors. The pioneering
work [6] was due to Donsker and Varadhan were they established a large deviation principle for the volume of a Wiener sausage. Their result was extended by Eisele and Lang [8] to the case when the driving process is a standard Brownian motion with drift, and to a class of elliptic diffusions by Sznitman [28], while Ŗkura investigated similar questions for a certain class of symmetric Lévy processes. Le Gall [16] obtained a central limit theorem for the volume of a Wiener sausage in dimensions $d \geq 2$, with different normalizing sequences and distributions in the limit for $d = 2$, $d = 3$ and $d \geq 4$, respectively. More recently, van den Berg, Bolthausen and den Hollander [33] studied the problem of intersections of two Wiener sausages, see also [30], [31] and [32]. For further limit theorems for the volume of a Wiener sausage see [2], [12] and [34]. We remark that first studies on a Wiener sausage were motivated by its applications in physics [13]. We refer the reader to the book by Simon [25] for a comprehensive discussion in this direction.

In the present article we focus on the limit behavior of the volume of a stable sausage, that is, a Lévy sausage corresponding to a stable Lévy process. Asymptotic behavior of stable sausages has not been extensively studied yet. In the seminal paper [6] Donsker and Varadhan obtained a large deviation principle for the volume of a stable sausage. Some other works were concerned with the expansion of the expected volume of a stable sausage. More precisely, Getoor [9] proved eq. (1.2) for rotationally invariant $\alpha$-stable processes with $d > \alpha$ and for any compact set $K$, and he investigated the first order asymptotics of the difference $E[\gamma^K] - \tau \text{Cap}(K)$, whose shape depends on the value of the ratio $d/\alpha$, see [9, Theorem 2]. The second order terms in this expansion were found by Port [20] for all strictly stable processes satisfying some extra assumptions. In this article, we obtain a central limit theorem for the volume of a stable sausage. We then apply this result to study convergence of the volume process in the path (Skorohod) space, and establish the corresponding functional central limit theorem. At the end, we study the almost sure growth of the paths of the volume process at infinity and derive an almost sure invariance principle, and Khintchine’s and Chung’s laws of the iterated logarithm for this process.

Before we formulate our results, we briefly recall some basic notation from the potential theory. Let $X$ be a rotationally invariant stable Lévy process of index $\alpha \in (0, 2]$, that is, a Lévy process whose bounded continuous transition density $p(t, x)$ is uniquely determined by the Fourier transform

$$e^{-t|x|^{\alpha}} = \int_{\mathbb{R}^d} e^{i(x, \xi)} p(t, x) \, dx,$$

where $(x, \xi)$ stands for the inner product in $\mathbb{R}^d$, $|x| = (x, x)^{1/2}$ is the Euclidean norm, and $dx = \lambda(dx)$. We assume that $X$ is transient, which holds if $d > \alpha$. Then its Green function is given by $G(x) = \int_0^{\infty} p(t, x) \, dt$. Let $\mathcal{B}(\mathbb{R}^d)$ denote the family of all Borel subsets of $\mathbb{R}^d$. For each $B \in \mathcal{B}(\mathbb{R}^d)$ there exists a unique Borel measure $\mu_B(dx)$ supported on $B \in \mathcal{B}(\mathbb{R}^d)$ such that

$$\mathbb{P}_x(\tau_B < \infty) = \int_{\mathbb{R}^d} G(x - y) \mu_B(dy).$$

The measure $\mu_B(dx)$ is called the equilibrium measure of $B$, and its capacity $\text{Cap}(B)$ is defined as the total mass of $\mu_B(dx)$, that is, $\text{Cap}(B) = \mu_B(B)$. We denote by $\mathcal{B}(x, r)$ the closed Euclidean ball centred at $x \in \mathbb{R}^d$ of radius $r > 0$. In the case when $r = 1$ and $x = 0$, we write $\mathcal{B} = \mathcal{B}(0, 1)$. If $B = \mathcal{B}(0, r)$ then the measure $\mu_B(dy)$ has a density which is proportional to $(r - |y|^2)^{-\alpha/2}$. In particular, we have (see for instance [29])

$$\text{Cap}(\mathcal{B}) = \frac{\Gamma(d/2)}{\Gamma(\alpha/2)\Gamma(1 + (d - \alpha)/2)}.$$
In the case when $K = \mathcal{B}$, we simply write $\mathcal{V}_t$ instead of $\mathcal{V}_t^\mathcal{B}$ (and similarly $\mathcal{S}_t$ for $\mathcal{S}_t^\mathcal{B}$). Let $\mathcal{N}(0, 1)$ be the standard normal distribution. Our central limit theorem (see Theorem 2.1) for the volume of a stable sausage asserts that if $d > 3\alpha/2$ then there exists a constant $\sigma > 0$ such that

\[
\frac{\mathcal{V}_t - t \operatorname{Cap}(\mathcal{B})}{\sigma \sqrt{t}} \xrightarrow{\text{(d)}} \mathcal{N}(0, 1),
\]

where convergence holds in distribution. The cornerstone of the proof of eq. (1.4) is to represent $\mathcal{V}_t$ as a sum of independent random variables plus an error term. For this we use inclusion-exclusion formula together with the Markov property and rotational invariance of the process $X$. More precisely, for $t, s \geq 0$, we have

\[
\mathcal{V}_{t+s} = \lambda((S_t \cup S[t, t+s]) = \lambda((S_t - X_t) \cup (S[t, t+s] - X_t))
\]

\[
= \mathcal{V}_t^{(1)} + \mathcal{V}_s^{(2)} - \lambda(S_t^{(1)} \cap S_s^{(2)}),
\]

where $\mathcal{V}_t^{(1)}$ and $\mathcal{V}_s^{(2)}$ ($S_t^{(1)}$ and $S_s^{(2)}$) are independent and have the same law as $\mathcal{V}_t$ and $\mathcal{V}_s$ ($S_t$ and $S_s$), respectively. This decomposition allows us to apply the Feller-Lindeberg central limit theorem in the present context. The first key step is to find estimates for the error term $\lambda(S_t^{(1)} \cap S_s^{(2)})$, which we give in Section 2.1. The second step is to control the variance of the volume of a stable sausage which is achieved in Section 2.2.

Let us emphasize that this article is mainly motivated by Le Gall’s work [16] where he studied fluctuations of the volume of a Wiener sausage (the case $\alpha = 2$). Among other results, he established the central limit theorem eq. (1.4) for dimensions $d \geq 4$. Our another motivation was the article [17] by Le Gall and Rosen where they proved a corresponding central limit theorem for the range of stable random walks and mentioned that it is plausible that similar result holds for stable sausages, see [17, Page 654]. Both of these articles were concerned also with the lower-dimensional case $d < 4$ and $d/\alpha \leq 3/2$, respectively. In the present article we are only interested in the case when $d/\alpha > 2/3$, and we postpone the study of the remaining values of the ratio $d/\alpha$ to follow-up articles.

As an application of eq. (1.4) we obtain a functional central limit theorem (see Theorem 3.1) which states that under the same assumptions, and with the same constant $\sigma > 0$,

\[
\left\{ \frac{\mathcal{V}_{nt} - nt \operatorname{Cap}(\mathcal{B})}{\sigma \sqrt{n}} \right\}_{t \geq 0} \xrightarrow{(\mathcal{J}_1)}_{n \rightarrow \infty} \{W_t\}_{t \geq 0}.
\]

Here, convergence holds in the Skorohod space $\mathcal{D}([0, \infty), \mathbb{R})$ endowed with the $\mathcal{J}_1$ topology, and $\{W_t\}_{t \geq 0}$ is a standard Brownian motion in $\mathbb{R}$. The proof of eq. (1.6) is performed according to a general two-step scheme: (i) convergence of finite-dimensional distributions which follows from eq. (1.4); (ii) tightness which we investigate by employing the well-known Aldous criterion, see Section 3 for details.

We finally use eq. (1.4) to study the a.s. growth of the paths of the volume of a stable sausage $\mathcal{S}_t$. We prove an a.s. invariance principle which provides that the process $\{\mathcal{V}_t - t \operatorname{Cap}(\mathcal{B})\}_{t \geq 0}$ can be almost surely approximated by a Brownian path, that is, under the assumptions stated above it holds that for every $\varepsilon > 0$,

\[
\sigma^{-1}(\mathcal{V}_t - t \operatorname{Cap}(\mathcal{B})) - W_t = \begin{cases} O(t^{\alpha/2} - d/(4\alpha) + \varepsilon), & 3\alpha/2 < d < 2\alpha, \\ O(t^{1/4} + \varepsilon), & d \geq 2\alpha, \end{cases} \quad \mathbb{P}\text{-a.s.}
\]
see Theorem 4.1. As a direct consequence of Khintchine’s law of the iterated logarithm for \( \{W_t\}_{t \geq 0} \), we obtain that \( \mathbb{P} \)-a.s.

\[
\liminf_{t \to \infty} \frac{\mathcal{V}_t - t \text{Cap}(B)}{\sqrt{2\sigma^2 t \log \log t}} = -1 \quad \text{and} \quad \limsup_{t \to \infty} \frac{\mathcal{V}_t - t \text{Cap}(B)}{\sqrt{2\sigma^2 t \log \log t}} = 1.
\]

Similarly, by Chung’s law of the iterated logarithm for \( \{W_t\}_{t \geq 0} \), we conclude that \( \mathbb{P} \)-a.s.

\[
\liminf_{t \to \infty} \sup_{0 \leq s \leq t} |\mathcal{V}_s - s \text{Cap}(B)| \sqrt{\frac{\sigma^2}{t \log \log t}} = \frac{\pi}{\sqrt{8}}.
\]

Analogous results to eqs. (1.8) and (1.9) were found by a different approach in \([34]\) for the Wiener sausage in dimensions \( d \geq 4 \). To show eq. (1.7) we utilize a refined version of the decomposition eq. (1.5) to represent \( \mathcal{V}_t \) as a sum of i.i.d. random variables which we then approximate with a Brownian motion according to the Skorohod embedding theorem.

It is remarkable that results (1.4), (1.6) and (1.7)–(1.9) correspond to analogous results for the range (and its capacity) of stable random walks on the integer lattice \( \mathbb{Z}^d \) which we discussed in \([3, 4]\) and \([5]\), respectively.

The rest of the paper is organized as follows. In Section 2 we prove the central limit theorem (1.4). For this we first deal with the error terms derived from eq. (1.5), and in the second part we show that the variance of the volume of a stable sausage behaves linearly at infinity. Section 3 is devoted to the proof of eq. (1.6), and in Section 4 we prove eq. (1.7). In Section 5 we present the proofs of some technical results which we need in the course of the study.

2. Central limit theorem

The goal of this section is to prove the following central limit theorem. We assume that \( X \) is a rotationally invariant stable Lévy process in \( \mathbb{R}^d \) of index \( \alpha \in (0, 2] \) satisfying \( d > 3\alpha/2 \).

**Theorem 2.1.** Under the above assumptions, there exists a constant \( \sigma = \sigma(d, \alpha) > 0 \) such that

\[
\frac{\mathcal{V}_t - t \text{Cap}(B)}{\sigma \sqrt{t}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1).
\]

We remark that Theorem 2.1 holds for any closed ball \( B(x, r) \), with a possibly different constant \( \sigma > 0 \). Moreover, as indicated by eq. (1.2), the statement of the theorem remains valid if we replace the term \( t \text{Cap}(B) \) with \( \mathbb{E}[\mathcal{V}_t] \).

Before we embark on the proof of the theorem, which is given at the end of the section, we first need to find satisfactory estimates for the error term in decomposition eq. (1.5). Next step is to investigate the variance \( \text{Var}(\mathcal{V}_t) \) of the volume of a stable sausage and to show that it behaves as \( \sigma t \) at infinity.

2.1. Error term estimates. We assume that \( X \) is defined on the canonical space \( \Omega = \mathcal{D}([0, \infty), \mathbb{R}^d) \) of all càdlàg functions \( \omega : [0, \infty) \to \mathbb{R}^d \). It is endowed with the Borel \( \sigma \)-algebra \( \mathcal{F} \) generated by the Skorokhod \( J_1 \) topology. Then \( X \) is understood as the coordinate process, that is, \( X_t(\omega) = \omega(t) \), and the shift operator \( \theta_t \) acting on \( \Omega \) is defined by

\[
\theta_t \omega(s) = \omega(t + s), \quad t, s \geq 0.
\]

In what follows we use notation

\[
\mathcal{S}^K[t, \infty) = \bigcup_{s \geq t} \{X_s + K\}, \quad t \geq 0.
\]
We also write $S^K_t = S^K[0, \infty)$, $V^K_t = \lambda(S^K_t)$ and $V^K[t, \infty) = \lambda(S^K[t, \infty))$, $t \geq 0$. We start with a lemma which enables us to represent the expected volume of the intersection of two sausages in terms of the difference $E[V^K_t] - t \operatorname{Cap}(K)$.

**Lemma 2.2.** For any compact set $K$ and all $t \geq 0$ it holds
\[
E[V^K_t] - t \operatorname{Cap}(K) = E[\lambda(S^K_t \cap S^K[t, \infty))].
\]

**Proof.** We clearly have
\[
V^K_t = V^K_0 + V^K[t, \infty) - \lambda(S^K \cap S^K[t, \infty))
\]
which implies
\[
\lambda(S^K_t \cap S^K[t, \infty)) = V^K_t - \lambda(S^K_t \setminus S^K[t, \infty)).
\]
Hence, it suffices to show that
\[
(2.1) \quad E[\lambda(S^K_t \setminus S^K[t, \infty)) = t \operatorname{Cap}(K), \quad t \geq 0.
\]

By rotational invariance of the process $X$ we have
\[
E[\lambda(S_t^K \setminus S^K[t, \infty]) = \int_{R^d} \mathbb{P}(x \in S^K_t \setminus S^K[t, \infty)) \, dx
= \int_{R^d} \mathbb{P}_x \left( \bigcup_{0 \leq s \leq t} \{X_s \in K\}, \bigcap_{s \geq t} \{X_s \notin K\} \right) \, dx
= \int_{R^d} \mathbb{P}_x (0 < \eta_K \leq t) \, dx,
\]
where $\eta_K$ is the last exit time of the process $X$ from the set $K$, that is,
\[
\eta_K = \begin{cases} 
\sup\{t > 0 : X_t \in K\}, & \tau_K < \infty, \\
0, & \tau_K = \infty.
\end{cases}
\]
We observe that $\{\eta_K > t\} = \{\tau_K \circ \theta_t < \infty\}$ which together with eq. (1.3) yields
\[
\mathbb{P}_x (\eta_K > t) = \int_{R^d} p(t, x, y) \mathbb{P}_y (\tau_K < \infty) \, dy = \int_{R^d} \int_t^\infty p(s, x, z) \, ds \, \mu_K(dz),
\]
where we used notation $p(t, x, y) = p(t, y - x)$. We obtain
\[
\mathbb{P}_x (0 < \eta_K \leq t) = \int_{R^d} \int_0^t p(s, x, y) \, ds \, \mu_K(dy).
\]
This and eq. (2.2) imply
\[
E[\lambda(S^K_t \setminus S^K[t, \infty]) = \int_{R^d} \int_0^t \int_{R^d} p(s, y, x) \, dx \, ds \, \mu_K(dy) = t \operatorname{Cap}(K),
\]
and the proof is finished.

In the following lemma we show how one can easily estimate the higher moments of the expected volume of the intersection of two sausages through the first moment estimate.

**Lemma 2.3.** Let $X'$ be an independent copy of the process $X$ such that $X_0 = X'_0$, and let $S^t_t, t \geq 0$, denote the sausage associated with $X'$. Then for all $k \in \mathbb{N}$ and $t \geq 0$ it holds
\[
E[\lambda(S_t \cap S'_\infty)^k] \leq 2^{k-1}(k!)^2 \left( \mathbb{E}[\lambda(S_t \cap S'_\infty)] \right)^k.
\]
Proof. We observe that
\begin{equation}
\mathbb{E}[\lambda(S_t \cap S'_\infty)] = \int_{\mathbb{R}^d} \mathbb{P}(x \in S_t) \mathbb{P}(x \in S'_\infty) \, dx = \int_{\mathbb{R}^d} \mathbb{P}_x(\tau_B \leq t) \mathbb{P}_x(\tau_B < \infty) \, dx,
\end{equation}
where we used rotational invariance of X. Similarly, for \( k \geq 1 \) we have
\begin{equation}
\mathbb{E}[\lambda(S_t \cap S'_\infty)^k] = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbb{P}(x_1, \ldots, x_k \in S_t) \mathbb{P}(x_1, \ldots, x_k \in S'_\infty) \, dx_1 \cdots \, dx_k.
\end{equation}
By the strong Markov property, we obtain
\begin{equation}
\mathbb{P}(x_1, \ldots, x_k \in S_t) = \mathbb{P}(\tau_{B-x} \leq t, \ldots, \tau_{B-x_k} \leq t)
\leq k! \mathbb{P}(\tau_{B-x} \leq \tau_{B-x_k} \leq \cdots \leq \tau_{B-x_k})
\leq k! \mathbb{P}(\tau_{B-x} \leq \cdots \leq \tau_{B-x_k}) \sup_{z \in B-x_k} \mathbb{P}_z(\tau_{B-x_k} \leq t).
\end{equation}
For any \( w \in B \) we have \( B - w \subseteq B(0, 2) \) and whence
\[ \mathbb{P}_{w-x_k}(\tau_{B-x} \leq t) = \mathbb{P}_{x_k-x_k}(\tau_{B-w} \leq t) \leq \mathbb{P}_{x_k-x_k}(\tau_{B(0, 2)} \leq t). \]
For \( x \in \mathbb{R}^d \) and \( B \in \mathfrak{B}(\mathbb{R}^d) \) we set \( \tau^x_B = \inf \{ t \geq 0 : x + X_t \in B \} \) and show that \( \tau^x_B(0, 2) = 2^a \tau^{x/2}_B \), that is, the random variables \( \tau^x_B(0, 2) \) and \( 2^a \tau^{x/2}_B \) are equal in distribution. Indeed, the easy calculation yields
\[ \tau^x_B(0, 2) = \inf \{ t \geq 0 : |x + X_t| \leq 2 \} = \inf \left\{ t \geq 0 : \left| \frac{x}{2} + \frac{X_t}{2} \right| \leq 1 \right\} = 2^a \inf \left\{ s \geq 0 : \left| \frac{x}{2} + X_s \right| \leq 1 \right\} = 2^a \tau^{x/2}_B. \]
This implies that for arbitrary \( w \in B \),
\[ \mathbb{P}_{w-x_k}(\tau_{B-x} \leq t) = \mathbb{P}_{x_k-x_k}(\tau_{B-w} \leq t) \leq \mathbb{P}_{x_k-x_k}(\tau_{B(0, 2)} \leq t) \leq \mathbb{P}_{x_k-x_k}(\tau_{B} \leq t). \]
In particular,
\[ \sup_{z \in B-x_k} \mathbb{P}_z(\tau_{B-x} \leq t) \leq \mathbb{P}_{x_k-x_k}(\tau_{B} \leq t). \]
By combining this with eq. (2.5) and iterating the whole procedure, we obtain
\[ \mathbb{P}(x_1, \ldots, x_k \in S_t) \leq k! \mathbb{P}_{x_1}(\tau_B \leq t) \mathbb{P}_{x_2-x_1}(\tau_B \leq t) \cdots \mathbb{P}_{x_k-x_{k-1}}(\tau_B \leq t). \]
Similarly, it follows that
\[ \mathbb{P}(x_1, \ldots, x_k \in S_\infty \cap \cdots \cap S'_\infty) \leq k! \mathbb{P}_{x_1}(\tau_B < \infty) \mathbb{P}_{x_2-x_1}(\tau_B < \infty) \cdots \mathbb{P}_{x_k-x_{k-1}}(\tau_B < \infty). \]
Applying the last two inequalities to eq. (2.4) and using eq. (2.3) finishes the proof. \( \square \)

**Corollary 2.4.** In the notation of Lemma 2.3, for all \( k \in \mathbb{N} \) and \( t > 0 \) large enough there is a constant \( c = c(d, \alpha) > 0 \) such that
\begin{equation}
\mathbb{E}[\lambda(S_t \cap S'_\infty)^k] \leq 2^{k-1} (k!)^2 c^k h(t)^k,
\end{equation}
where the function \( h : (0, \infty) \to (0, \infty) \) is defined as
\begin{equation}
h(t) = \begin{cases} 
1, & d > 2\alpha, \\
\log(t + e), & d = 2\alpha, \\
\frac{t^{2-d/\alpha}}{d}, & d \in (\alpha, 2\alpha).
\end{cases}
\end{equation}
Proof. It follows from [9, Theorem 2] that there is a constant \( c(d, \alpha) > 0 \) such that for \( t > 0 \) large enough
\[
\int_{\mathbb{R}^d} \mathbb{P}_x(\tau_B \leq t) \, dx - t \text{Cap}(B) \leq c(d, \alpha) h(t),
\]
where the function \( h(t) \) is given in eq. (2.7). We observe that by the Markov property and translation invariance of \( \lambda(dx) \) we have
\[
\mathbb{E}[\lambda(S_t \cap S'_\infty)] = \mathbb{E}[\lambda((S_t - X_t) \cap (S[t, \infty) - X_t))] = \mathbb{E}[\lambda(S_t \cap [t, \infty))].
\]
Thus, eq. (1.1) combined with Lemma 2.2 and eq. (2.8) implies the assertion for \( k = 1 \). For \( k > 1 \) the result follows by Lemma 2.3. \( \square \)

2.2. Variance of the volume of a stable sausage. Our aim in this section is to determine the constant \( \sigma \) in Theorem 2.1. We can easily adapt the approach of [3, Lemma 4.3] to the present setting and combine it with [10, Theorem 2] to conclude that the limit below exists
\[
\lim_{t \to \infty} \frac{\text{Var}(V_t)}{t} = \sigma^2.
\]
The main difficulty is to show that \( \sigma \) is strictly positive, and this is obtained in the following crucial lemma. We adapt the proof of [16, Lemma 4.2] but let us emphasize that it is a laborious task to adjust it to the case of stable processes.

**Lemma 2.5.** The constant \( \sigma \) in eq. (2.9) is strictly positive.

**Proof.** We split the proof into several steps and we notice that it clearly suffices to restrict our attention to integer values of the parameter \( t \).

**Step 1.** We start by finding a handy decomposition of the variance \( \text{Var}(V_n) \) expressed as a sum of specific random variables, see eq. (2.18). We assume that \( X_0 = 0 \), and we set
\[
\hat{S}[s, t] = S[s, t] \setminus S_s, \quad 0 \leq s \leq t < \infty.
\]
For \( n, N \in \mathbb{N} \) such that \( 1 \leq n \leq N \) we have
\[
V_n + \lambda([\hat{S}[n, N]]) = V_N.
\]
Let \( F_t = \sigma(X_s : s \leq t) \). Since \( V_n \) is \( F_n \)-measurable, we obtain
\[
V_n + \mathbb{E}[\lambda([\hat{S}[n, N]]) \mid F_n] = \mathbb{E}[V_N \mid F_n]
\]
and by taking expectations and subtracting\(^1\)
\[
\langle V_n \rangle + \langle \mathbb{E}[\lambda([\hat{S}[n, N]]) \mid F_n] \rangle = \mathbb{E}[V_N \mid F_n] - \mathbb{E}[V_N].
\]
Hence
\[
\langle V_n \rangle + \langle \mathbb{E}[\lambda([\hat{S}[n, N]]) \mid F_n] \rangle = \sum_{k=1}^{n} U_k^N,
\]
where
\[
U_k^N = \mathbb{E}[V_N \mid F_k] - \mathbb{E}[V_N \mid F_{k-1}].
\]
We first discuss the second term on the left-hand side of eq. (2.11). We claim that
\[
\langle \mathbb{E}[\lambda([\hat{S}[n, N]]) \mid F_n] \rangle = -\langle \mathbb{E}[\lambda([S[n, N] \cap S_n) \mid F_n] \rangle.
\]
Indeed, by eq. (2.10) we have
\[
\lambda([\hat{S}[n, N]]) = \lambda([S[n, N]) - \lambda([S[n, N] \cap S_n)]
\]
\(^1\)For any random variable \( Y \in L^1 \) we write \( \langle Y \rangle = Y - \mathbb{E}[Y] \).
and the independence of the increments of the process $X$ implies that
\[
\mathbb{E}[\lambda(\hat{S}[n, N]) | \mathcal{F}_n] = \mathbb{E}[\lambda(\mathcal{S}[n, N]) | \mathcal{F}_n] - \mathbb{E}[\lambda(\mathcal{S}[n, N] \cap \mathcal{S}_n) | \mathcal{F}_n]
\]
\[
= \mathbb{E}[\lambda(\mathcal{S}[n, N] - X_n) | \mathcal{F}_n] - \mathbb{E}[\lambda(\mathcal{S}[n, N] \cap \mathcal{S}_n) | \mathcal{F}_n]
\]
\[
= \mathbb{E}[\lambda(\mathcal{S}[n, N] - X_n)] - \mathbb{E}[\lambda(\mathcal{S}[n, N] \cap \mathcal{S}_n) | \mathcal{F}_n]
\]
\[
= \mathbb{E}[\lambda(\mathcal{S}[n, N])] - \mathbb{E}[\lambda(\mathcal{S}[n, N] \cap \mathcal{S}_n) | \mathcal{F}_n].
\]
Taking expectation and then subtracting the two relations yields eq. (2.12).

Next we deal with the random variables $U^N_k$ for $k = 1, \ldots, N$. By the independence of the increments of the process $X$, we obtain
\[
U^N_k = \mathbb{E}[\lambda(\mathcal{S}_k) + \lambda(\mathcal{S}[k, N]) - \lambda(\mathcal{S}_k \cap \mathcal{S}[k, N]) | \mathcal{F}_k]
\]
\[
- \mathbb{E}[\lambda(\mathcal{S}_{k-1}) + \lambda(\mathcal{S}[k-1, N-1]) - \lambda(\mathcal{S}_{k-1} \cap \mathcal{S}[k-1, N-1]) | \mathcal{F}_{k-1}]
\]
\[
- \mathbb{E}[\lambda(\hat{S}[N - 1, N]) | \mathcal{F}_{k-1}]
\]
\[
= \lambda(\mathcal{S}_k) - \lambda(\mathcal{S}_{k-1}) + \mathbb{E}[\lambda(\mathcal{S}[k, N]) | F_{k-1}] - \mathbb{E}[\lambda(\mathcal{S}[k-1, N - 1]) | F_{k-1}]
\]
\[
- \mathbb{E}[\lambda(\hat{S}[N - 1, N]) | F_{k-1}] + \mathbb{E}[\lambda(\mathcal{S}_k \cap \mathcal{S}[k, N]) | F_k]
\]
\[
= \lambda(\hat{S}[k - 1, k]) - \mathbb{E}[\lambda(\hat{S}[N - 1, N]) | F_{k-1}] + \mathbb{E}[\lambda(\mathcal{S}_k \cap \mathcal{S}[k-1, N - 1]) | F_{k-1}]
\]
\[
- \mathbb{E}[\lambda(\mathcal{S}_k \cap \mathcal{S}[k, N]) | F_k].
\]

Let $\mathcal{F}_{k,t}$ denote the $\sigma$-algebra generated by the increments of $X$ on $[s, t]$, $0 \leq s \leq t$. Then, by a reversibility argument,
\[
\mathbb{E}[\lambda(\hat{S}[N - 1, N]) | \mathcal{F}_{k-1}] \overset{(d)}{=} \mathbb{E}[\lambda(\mathcal{S}_1 \setminus \mathcal{S}[1, N]) | \mathcal{F}_{N-k+1, N}].
\]
Moreover, the following convergence in $L^1$ holds
\[
(2.13) \quad \mathbb{E}[\lambda(\mathcal{S}_1 \setminus \mathcal{S}[1, N]) | \mathcal{F}_{N-k+1, N}] \xrightarrow{L^1} \mathbb{E}[\lambda(\mathcal{S}_1 \setminus \mathcal{S}[1, \infty])].
\]
The proof of eq. (2.13) is postponed to Section 5, Lemma 5.1. In view of eq. (2.1) it follows that
\[
\mathbb{E}[\lambda(\hat{S}[N - 1, N]) | \mathcal{F}_{k-1}] \xrightarrow{L^1} \text{Cap}(B).
\]

We thus obtain
\[
U^N_k \xrightarrow{L^1} \lambda(\hat{S}[k - 1, k]) - \text{Cap}(B) + \mathbb{E}[\lambda(\mathcal{S}_{k-1} \cap \mathcal{S}[k-1, \infty]) | \mathcal{F}_{k-1}]
\]
\[
- \mathbb{E}[\lambda(\mathcal{S}_k \cap \mathcal{S}[k, \infty]) | \mathcal{F}_k].
\]

Further, we observe that
\[
\lambda(\mathcal{S}_k \cap \mathcal{S}[k, \infty)) = \lambda(\mathcal{S}_{k-1} \cap \mathcal{S}[k-1, \infty]) - \lambda(\mathcal{S}_{k-1} \cap \mathcal{S}[k-1, k])
\]
\[
+ \lambda(\mathcal{S}_k \cap \mathcal{S}[k, \infty) \cap \mathcal{S}_{k-1} \cap \mathcal{S}[k-1, k]) + \lambda(\mathcal{S}[k-1, k] \cap \mathcal{S}[k, \infty))
\]
\[
- \lambda(\mathcal{S}_{k-1} \cap \mathcal{S}[k-1, \infty) \cap \mathcal{S}[k-1, k]) \cap \mathcal{S}[k, \infty))
\]
\[
= \lambda(\mathcal{S}_{k-1} \cap \mathcal{S}[k-1, \infty)) - \lambda(\mathcal{S}_{k-1} \cap \mathcal{S}[k-1, k])
\]
\[
+ \lambda(\mathcal{S}[k-1, k] \cap \mathcal{S}[k, \infty)).
\]

This and eq. (2.14) imply
\[
U^N_k \xrightarrow{L^1} \lambda(\mathcal{S}[k - 1, k] \cap \mathcal{S}_{k-1}^c - \text{Cap}(B) + \mathbb{E}[\lambda(\mathcal{S}_{k-1} \cap \mathcal{S}[k-1, \infty]) | \mathcal{F}_{k-1}]
\]
\[
- \mathbb{E}[\lambda(\mathcal{S}_{k-1} \cap \mathcal{S}[k-1, \infty]) | \mathcal{F}_k] + \lambda(\mathcal{S}[k-1, k] \cap \mathcal{S}_{k-1})
\]
\[ -\mathbb{E}[\lambda(S[k - 1, k] \cap S[k, \infty)) | \mathcal{F}_k] = \mathbb{E}[\lambda(S[k - 1, k] \cap S[k, \infty)) | \mathcal{F}_k] - \text{Cap}(\mathcal{B}) \]

\[ + \mathbb{E}[\lambda(S_{k-1} \cap S[k - 1, \infty)) | \mathcal{F}_{k-1}] - \mathbb{E}[\lambda(S_{k-1} \cap S[k - 1, \infty)) | \mathcal{F}_k] \]

\[ + \langle \mathbb{E}[\lambda(S[k - 1, k \setminus S[k, \infty)) | \mathcal{F}_k] \rangle + \mathbb{E}[\lambda(S[k - 1, k \setminus S[k, \infty)) | \mathcal{F}_k] \rangle - \text{Cap}(\mathcal{B}) \]

\[ \mathbb{E}[\lambda(S[k - 1, k) \setminus S[k, \infty)) | \mathcal{F}_k] \rangle + \mathbb{E}[\lambda(S_{k-1} \cap S[k - 1, \infty)) | \mathcal{F}_{k-1}] - \mathbb{E}[\lambda(S_{k-1} \cap S[k - 1, \infty)) | \mathcal{F}_k] \]

where in the last line we used eq. (2.1). Hence, by eqs. (2.11) and (2.12),

\[ \langle \mathcal{V}_n \rangle = \langle \mathbb{E}[\lambda(S_n \cap S[n, \infty)) | \mathcal{F}_n] \rangle + \sum_{k=1}^{n} Y_k, \]

where

\[ Y_k = \mathbb{E}[\lambda(S_{k-1} \cap S[k - 1, \infty)) | \mathcal{F}_{k-1}] - \mathbb{E}[\lambda(S_{k-1} \cap S[k - 1, \infty)) | \mathcal{F}_k] \]

\[ + \langle \mathbb{E}[\lambda(S[k - 1, k) \setminus S[k, \infty)) | \mathcal{F}_k] \rangle. \]

From eq. (2.15) it follows that the variance of \( \mathcal{V}_n \) is equal to

\[ \text{Var}(\mathcal{V}_n) = \text{Var}(\mathbb{E}[\lambda(S_n \cap S[n, \infty)) | \mathcal{F}_n]) + \mathbb{E}\left( \left( \sum_{k=1}^{n} Y_k \right)^2 \right) \]

\[ + 2 \mathbb{E}\left( \langle \mathbb{E}[\lambda(S_n \cap S[n, \infty)) | \mathcal{F}_n] \rangle \right) \sum_{k=1}^{n} Y_k. \]

Clearly, \( Y_k \) is \( \mathcal{F}_k \)-measurable and \( \mathbb{E}[Y_l | \mathcal{F}_k] = 0 \) for \( k < l \). It follows that

\[ \mathbb{E}\left( \left( \sum_{k=1}^{n} Y_k \right)^2 \right) = \sum_{k=1}^{n} \mathbb{E}[Y_k^2]. \]

Jensen’s inequality and eq. (2.6) with \( d > 3 \alpha/2 \) yield

\[ \lim_{n \to \infty} \frac{1}{n} \text{Var}(\mathbb{E}[\lambda(S_n \cap S[n, \infty)) | \mathcal{F}_n]) \leq \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\lambda(S_n \cap S[n, \infty))^2] = 0. \]

The sequence \( \left\{ \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}[Y_k^2] \right\} \) is bounded (the proof is given in Section 5, Lemma 5.2), and thus by the Cauchy-Schwarz inequality we conclude that

\[ \lim_{n \to \infty} \frac{1}{n} \mathbb{E}\left( \langle \mathbb{E}[\lambda(S_n \cap S[n, \infty)) | \mathcal{F}_n] \rangle \sum_{k=1}^{n} Y_k \right) = 0. \]

We have shown that

\[ \lim_{n \to \infty} \frac{\text{Var}(\mathcal{V}_n)}{n} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}[Y_k^2]. \]

**Step 2.** In this step we prove that the limit on the right-hand side of eq. (2.18) is strictly positive. Let \( X' \) be an independent copy of the original process \( X \) such that \( X_0 = 0 \) and it has càdlàg paths. We consider a process \( \bar{X} = \{ \bar{X}_t \}_{t \in \mathbb{R}} \) by setting \( \bar{X}_t = X'_{t -} \) for \( t \leq 0 \), and \( \bar{X}_t = X_t \) for \( t \geq 0 \). Clearly, the process \( \bar{X} \) has càdlàg paths, and stationary and independent increments. The sausages \( S'[s, t], S'(-\infty, s) \) and \( S'[s, \infty) \) corresponding to \( \bar{X} \) are defined for all \( s, t \in \mathbb{R}, s \leq t \). Recall that \( S_\infty = S[0, \infty) \). We assume that the process \( \bar{X} \) is defined on the canonical space \( \Omega = \mathcal{D}(\mathbb{R}, \mathbb{R}^d) \) of all càdlàg functions \( \omega: \mathbb{R} \to \mathbb{R}^d \) as the coordinate process \( \bar{X}_t(\omega) = \omega(t) \). We define a shift operator \( \vartheta \) acting on \( \Omega \) by

\[ \vartheta \omega(t) = \omega(1 + t) - \omega(1), \quad t \in \mathbb{R}, \]
and we notice that it is a \( \mathbb{P} \)-preserving mapping. For \( t \in \mathbb{R} \), we define

\[
\mathcal{G}_t = \sigma(\overline{X}_s : -\infty < s \leq t),
\]

and for \( k \in \mathbb{N} \),

\[
(2.20) \quad Z_k = \mathbb{E}[\lambda(\mathcal{S}[-k,0] \cap \mathcal{S}_\infty) \mid \mathcal{G}_0] - \mathbb{E}[\lambda(\mathcal{S}[-k,0] \cap \mathcal{S}_\infty) \mid \mathcal{G}_1] + \langle \mathbb{E}[\lambda(\mathcal{S}_1 \setminus \mathcal{S}[1,\infty)) \mid \mathcal{G}_1] \rangle.
\]

By eqs. \((2.16)\) and \((2.19)\) it follows that

\[
(2.21) \quad Y_k = Z_{k-1} \circ \vartheta^{k-1}.
\]

In the sequel, we prove that there exists a random variable \( Z \) such that \( \mathbb{E}[|Z|] > 0 \) and

\[
(2.22) \quad Z_k \xrightarrow{k \to \infty} Z.
\]

**Step 2a.** We start by proving the existence of \( Z \). This is evident if \( d > 2\alpha \) as in this case

\[
h(t) = 1 \quad \text{and whence using eq.} \; (2.6) \; \text{we obtain}
\]

\[
\mathbb{E}[\lambda(\mathcal{S}(-\infty,0] \cap \mathcal{S}_\infty)] < \infty.
\]

This implies \((2.22)\) with

\[
(2.23) \quad Z = \mathbb{E}[\lambda(\mathcal{S}(-\infty,0] \cap \mathcal{S}_\infty) \mid \mathcal{G}_0] - \mathbb{E}[\lambda(\mathcal{S}(-\infty,0] \cap \mathcal{S}_\infty) \mid \mathcal{G}_1]
+ \langle \mathbb{E}[\lambda(\mathcal{S}_1 \setminus \mathcal{S}[1,\infty)) \mid \mathcal{G}_1] \rangle.
\]

We next consider the case \( 3\alpha/2 < d \leq 2\alpha \). Then \( \mathbb{E}[\lambda(\mathcal{S}(-\infty,0] \cap \mathcal{S}_\infty)] \) is not finite and we cannot define \( Z \) as in the previous case. We notice that

\[
\lambda(\mathcal{S}[-k,0] \cap \mathcal{S}_\infty) = \lambda(\mathcal{S}[-k,0] \cap \mathcal{S}[1,\infty)) + \lambda(\mathcal{S}[-k,0] \cap (\mathcal{S}_1 \setminus \mathcal{S}[1,\infty))
\]

and thus we can rewrite \( Z_k \) as follows

\[
Z_k = \langle \mathbb{E}[\lambda(\mathcal{S}_1 \setminus \mathcal{S}[1,\infty)) \mid \mathcal{G}_1] \rangle - \mathbb{E}[\lambda((\mathcal{S}_1 \setminus \mathcal{S}[1,\infty)) \cap \mathcal{S}[-k,0)] \mid \mathcal{G}_1]
+ \mathbb{E}[\lambda(\mathcal{S}[-k,0] \cap \mathcal{S}_\infty) \mid \mathcal{G}_0] - \mathbb{E}[\lambda(\mathcal{S}[-k,0] \cap \mathcal{S}[1,\infty)) \mid \mathcal{G}_1].
\]

Before we let \( k \) tend to infinity in the above expression, we rewrite the expression from the second line. We observe that

\[
\lambda(\mathcal{S}[-k,0] \cap \mathcal{S}_\infty) = \int_{\mathbb{R}^d} 1_{\mathcal{S}[-k,0]}(y) \mathbb{1}_{\mathcal{S}_\infty}(y) \, dy.
\]

By taking conditional expectation with respect to \( \mathcal{G}_0 \), we obtain

\[
\mathbb{E}[\lambda(\mathcal{S}[-k,0] \cap \mathcal{S}_\infty) \mid \mathcal{G}_0] = \int_{\mathbb{R}^d} 1_{\mathcal{S}[-k,0]}(y) \mathbb{E}[\mathbb{1}_{\mathcal{S}_\infty}(y)] \, dy = \int_{\mathbb{R}^d} 1_{\mathcal{S}[-k,0]}(y) \phi(y) \, dy,
\]

where we set

\[
\phi(y) = \mathbb{P}(y \in \mathcal{S}_\infty).
\]

Similarly, we write

\[
\lambda(\mathcal{S}[-k,0] \cap \mathcal{S}[1,\infty)) = \int_{\mathbb{R}^d} 1_{\mathcal{S}[-k,0]-X_1}(y) \mathbb{1}_{\mathcal{S}[1,\infty)-X_1}(y) \, dy
\]

and we take conditional expectation with respect to \( \mathcal{G}_1 \) which yields

\[
\mathbb{E}[\lambda(\mathcal{S}[-k,0] \cap \mathcal{S}[1,\infty)) \mid \mathcal{G}_1] = \int_{\mathbb{R}^d} 1_{\mathcal{S}[-k,0]-X_1}(y) \mathbb{E}[\mathbb{1}_{\mathcal{S}[1,\infty)-X_1}(y)] \, dy
\]

\[
= \int_{\mathbb{R}^d} 1_{\mathcal{S}[-k,0]}(y) \phi(y - X_1) \, dy.
\]
It follows that
\[
E[\lambda(S[-k,0] \cap S_\infty) \mid G_0] - E[\lambda(S[-k,0] \cap S[1,\infty)) \mid G_1] \\
= \int_{\mathbb{R}^d} 1_{S[-k,0]}(y) (\phi(y) - \phi(y - X_1)) \, dy.
\] (2.25)

We prove in Lemma 5.4 that the right-hand side integral in eq. (2.25) is a well-defined random variable in $L^1$. Thus, the dominated convergence theorem implies eq. (2.22) with
\[
Z = (E[\lambda(S_1 \setminus S[1,\infty)) \mid G_1]) - E[\lambda((S_1 \setminus S[1,\infty]) \cap S(-\infty,0) \mid G_1]
+ \int_{\mathbb{R}_d} 1_{S(-\infty,0)}(y) (\phi(y) - \phi(y - X_1)) \, dy.
\]

**Step 2b.** We next show that $E[|Z|] > 0$ and we remark that the following arguments apply to all $d > 3\alpha/2$. From eqs. (2.15) and (2.21) we have
\[
\langle \mathcal{V}_n \rangle = \sum_{k=0}^{n-1} Z \circ \vartheta^k + H_n,
\]

where
\[
H_n = \langle E[\lambda(S_n \cap S[n,\infty)) \mid G_n]\rangle + \sum_{k=0}^{n-1} (Z_k - Z) \circ \vartheta^k.
\] (2.26)

Equation (2.1) yields
\[
\langle \mathcal{V}_n \rangle = \mathcal{V}_n - E[\mathcal{V}_n] \leq \mathcal{V}_n - E[\lambda(S_n \setminus S[n,\infty))] = \mathcal{V}_n - n \text{Cap}(\mathcal{B}).
\]

This implies
\[
\mathcal{V}_n \geq n \text{Cap}(\mathcal{B}) + \sum_{k=0}^{n-1} Z \circ \vartheta^k + H_n.
\]

We aim to prove that there is $\overline{c} > 0$ (which does not depend on $n$) such that for all $n \in \mathbb{N}$
\[
P(\{\mathcal{V}_n \leq \overline{c}\} \cap \{|H_n| \leq \overline{c}\}) > 0.
\] (2.27)

Notice that if $E[|Z|] = 0$ then the inequality $\mathcal{V}_n \geq n \text{Cap}(\mathcal{B}) + H_n$ and eq. (2.27) would imply that $\text{Cap}(\mathcal{B}) = 0$ which is a contradiction. Therefore, it is enough to show eq. (2.27).

From eqs. (2.20) and (2.23) we obtain
\[
\sum_{k=0}^{n-1} (Z - Z_k) \circ \vartheta^k = \int_{\mathbb{R}^d} 1_{S(-\infty,0]}(y) E[1_{S_\infty}(y) \mid G_0] \, dy
- \sum_{k=0}^{n-2} \int_{\mathbb{R}^d} \left( 1_{S(-\infty,0] \setminus S_k}(y) E[1_{S[k,\infty]}(y) \mid G_{k+1}] \\
- 1_{S(-\infty,0] \setminus S_{k+1}}(y) E[1_{S[k+1,\infty]}(y) \mid G_{k+1}] \right) \, dy
- \int_{\mathbb{R}^d} 1_{S(-\infty,0] \setminus S_{n-1}}(y) E[1_{S[n-1,\infty]}(y) \mid G_n] \, dy.
\] (2.28)
We next observe that
\[
\int_{\mathbb{R}^d} \left( 1_{S(-\infty,0) \setminus S_n}(y) \mathbb{E}[1_{S[k,\infty)}(y) \mid \mathcal{G}_{k+1}] 
- 1_{S(-\infty,0) \setminus S_{k+1}}(y) \mathbb{E}[1_{S[k+1,1) \setminus (S[k,k+1] \setminus S_k]}(y) \mid \mathcal{G}_{k+1}] \right) \, dy
= \int_{\mathbb{R}^d} \left( 1_{S(-\infty,0)}(y) 1_{S_{k+1}}(y) \mathbb{E}[1_{S[k,k+1]}(y) + 1_{S[k+1,1)}(y) 1_{S[k,k+1]}(y) \mid \mathcal{G}_{k+1}] 
- 1_{S(-\infty,0)}(y) 1_{S_{k+1}}(y) \mathbb{E}[1_{S[k+1,1)}(y) \mid \mathcal{G}_{k+1}] \right) \, dy
\]
(2.29)
\[
= \lambda(S(-\infty,0] \cap (S[k,k+1] \setminus S_k)).
\]
We clearly have
\[
1_{S[k-1]}(y) = 1_{S[n,\infty)}(y) + 1_{S[n-1, \infty)](y). \]
This identity and a similar argument as in eq. (2.24) yield
\[
\int_{\mathbb{R}^d} 1_{S(-\infty,0) \setminus S_{n-1}}(y) \mathbb{E}[1_{S[n-1,\infty)}(y) \mid \mathcal{G}_n] \, dy
= \int_{\mathbb{R}^d} 1_{S(-\infty,0) \setminus S_{n-1}}(y) \mathbb{E}[1_{S[n,\infty)}(y) \mid \mathcal{G}_n] \, dy
+ \int_{\mathbb{R}^d} 1_{S(-\infty,0) \setminus S_{n-1}}(y) \mathbb{E}[1_{S[n-1,\infty)}(y) \mid \mathcal{G}_n] \, dy
= \int_{\mathbb{R}^d} 1_{S(-\infty,0)}(y) \phi(y-X_n) \, dy
- \int_{\mathbb{R}^d} 1_{S(-\infty,0)] \cap S_{n-1}}(y) \phi(y-X_n) \, dy
+ \int_{\mathbb{R}^d} 1_{S(-\infty,0) \setminus S_{n-1}}(y) \mathbb{E}[1_{S[n-1,\infty)}(y) \mid \mathcal{G}_n] \, dy.
\]
(2.30)
By combining eqs. (2.28) to (2.30), we arrive at
\[
\sum_{k=0}^{n-1} (Z - Z_k) \circ \vartheta^k
\]
(2.31)
\[
= \int_{\mathbb{R}^d} 1_{S(-\infty,0)}(y) \left( \phi(y) - \phi(y-X_n) \right) \, dy
+ \int_{\mathbb{R}^d} 1_{S(-\infty,0) \cap S_{n-1}}(y) \phi(y-X_n) \, dy
- \int_{\mathbb{R}^d} 1_{S(-\infty,0) \setminus S_{n-1}}(y) \mathbb{E}[1_{S[n-1,\infty)}(y) \mid \mathcal{G}_n] \, dy
- \lambda(S(-\infty,0] \cap S_{n-1}).
\]
We claim that there is a constant \( \tilde{c} > 0 \) such that for all \( n \in \mathbb{N} \)
\[
P \left( \left\{ \sup_{0 \leq s \leq n} |X_s| \leq 1 \right\} \cap \left\{ \int_{\mathbb{R}^d} 1_{S(-\infty,0)}(y) \left( \phi(y) - \phi(y-X_n) \right) \, dy \leq \tilde{c} + 1 \right\} \right) > 0.
\]
(2.32)
If \( \sup_{0 \leq s \leq n} |X_s| \leq 1 \), then clearly \( \lambda(S_n) \leq \lambda(B(0,2)) \) and this allows us to estimate the first term on the right-hand side of eq. (2.26) and, similarly, the three last terms on the right-hand side of eq. (2.31) by a constant. We infer that there exists a constant \( \tilde{c} > 0 \) such that
\[
\left\{ \sup_{0 \leq s \leq n} |X_s| \leq 1 \right\} \cap \left\{ \int_{\mathbb{R}^d} 1_{S(-\infty,0)}(y) \left( \phi(y) - \phi(y-X_n) \right) \, dy \leq \tilde{c} + 1 \right\}
\subseteq \{ V_n \leq \tilde{c} \} \cap \{ H_n \leq \tilde{c} \}.
\]
To finish the proof of eq. (2.27), we are only left to show eq. (2.32). In view of the Markov inequality it is enough to prove that under \( \sup_{0 \leq s \leq n} |X_n| \leq 1 \) we have
\[
\mathbb{E} \left[ \int_{\mathbb{R}^d} 1_{S(-\infty,0)}(y) \left( \phi(y) - \phi(y - X_n) \right) \, dy \right] < \infty.
\]
This holds as, under \( \sup_{0 \leq s \leq n} |X_n| \leq 1 \),
\[
\mathbb{E} \left[ \int_{\mathbb{R}^d} 1_{S(-\infty,0)}(y) \left( \phi(y) - \phi(y - X_n) \right) \, dy \right] \leq \sup_{x \in \mathbb{R}} \int_{\mathbb{R}^d} \phi(y) \, dy \phi(y - x) \, dy < \infty,
\]
where convergence of the last integral is established in Lemma 5.5.

\textbf{Step 2c.} We finally show that the limit in eq. (2.18) is positive. Equation (2.22) implies
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}[|Z_k|] = \mathbb{E}[|Z|].
\]
Hence, by Jensen’s inequality,
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}[Z_k^2] \geq \lim_{n \to \infty} \mathbb{E} \left[ \left( \frac{1}{n} \sum_{k=0}^{n-1} |Z_k| \right)^2 \right] \geq \lim_{n \to \infty} \left( \mathbb{E} \left[ \frac{1}{n} \sum_{k=0}^{n-1} |Z_k| \right] \right)^2
\]
\[
= \lim_{n \to \infty} \left( \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}[|Z_k|] \right)^2 = (\mathbb{E}[|Z|])^2.
\]
By eq. (2.21), we have
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}[Y_k^2] = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}[Z_k^2] \geq (\mathbb{E}[|Z|])^2 > 0,
\]
and this finishes the proof of the lemma. \(\square\)

2.3. \textbf{Proof of the central limit theorem.} In this paragraph, we prove Theorem 2.1. In the proof we apply the Lindeberg-Feller central limit theorem which we include for completeness.

\textbf{Lemma 2.6} ([7, Theorem 3.4.5]). \textit{For each} \( n \in \mathbb{N} \) \textit{let} \( \{X_{n,i}\}_{1 \leq i \leq n} \) \textit{be a sequence of independent random variables with zero mean. If the following conditions are satisfied}
\begin{itemize}
  \item[(i)] \( \lim_{n \to \infty} \sum_{i=1}^{n} \mathbb{E}[X_{n,i}^2] = \sigma^2 > 0 \), \textit{and}
  \item[(ii)] for every \( \varepsilon > 0 \), \( \lim_{n \to \infty} \sum_{i=1}^{n} \mathbb{E} \left[ X_{n,i} 1_{\{|X_{n,i}| > \varepsilon\}} \right] = 0 \),
\end{itemize}
\textit{then} \( X_{n,1} + \cdots + X_{n,n} \xrightarrow{d} n \to \infty \sigma \mathcal{N}(0,1). \)

\textit{Proof of Theorem 2.1.} \textit{For} \( t > 0 \) \textit{large enough we choose} \( n = n(t) = \lfloor \log(t) \rfloor \). \textit{We have}
\[
\mathcal{V}_t = \lambda(S_t) = \lambda(S_{t/n} \cup S_{t/n}] = \lambda((S_{t/n} - X_{t/n}) \cup (S_{t/n}] - X_{t/n})).
\]
By the Markov property,
\[
S_{t/n}^{(1)} = S_{t/n} - X_{t/n} \quad \text{and} \quad S_{(n-1)t/n}^{(2)} = S_{t/n} - X_{t/n}
\]
are independent, and \( S_{(n-1)t/n}^{(2)} \) has the same law as \( S_{(n-1)t/n} \). Rotational invariance of \( X \) implies that \( S_{t/n}^{(1)} \) is equal in law to \( S_{t/n} \). Hence,
\[
\mathcal{V}_t = \lambda(S_{t/n}^{(1)}) + \lambda(S_{(n-1)t/n}^{(2)}) - \lambda(S_{t/n}^{(1)} \cap S_{(n-1)t/n}^{(2)}).
\]
By iterating this procedure, we obtain

\[ V_t = \sum_{i=1}^{n} \lambda(S_{t/n}^{(i)}) - \sum_{i=1}^{n-1} \lambda(S_{t/n}^{(i)} \cap S_{(n-i)t/n}^{(i+1)}), \]

(2.33)

We denote

\[ V_{t/n}^{(i)} = \lambda(S_{t/n}^{(i)}), \quad \text{and} \quad R(t) = \sum_{i=1}^{n-1} \lambda(S_{t/n}^{(i)} \cap S_{(n-i)t/n}^{(i+1)}), \]

and we notice that \( \{V_{t/n}^{(i)}\}_{1 \leq i \leq n} \) are i.i.d. random variables. By taking expectation in eq. (2.33) and then subtracting, we obtain

\[ \langle V_t \rangle = \sum_{i=1}^{n} \langle V_{t/n}^{(i)} \rangle - \langle R(t) \rangle. \]

(2.34)

We first show that

\[ \langle R(t) \rangle \sim \frac{1}{t} \rightarrow 0. \]

(2.35)

Since \( R(t) \geq 0 \), we clearly have \( \mathbb{E}[\langle R(t) \rangle] \leq 2 \mathbb{E}[R(t)] \). By Corollary 2.4,

\[ \mathbb{E}[R(t)] \leq \sum_{i=1}^{n-1} \mathbb{E} \left[ \lambda(S_{t/n}^{(i)} \cap S_{(n-i)t/n}^{(i+1)}) \right] \leq c n h(t/n), \]

for all \( t > 0 \) large enough. Hence, eq. (2.35) follows by eq. (2.7), and the fact that \( n = \lfloor \log(t) \rfloor \) and \( d > 3\alpha/2 \). Next we prove that

\[ \frac{1}{\sqrt{t}} \sum_{i=1}^{n} \langle V_{t/n}^{(i)} \rangle \quad \overset{(d)}{\rightarrow} \quad \sigma \mathcal{N}(0, 1). \]

(2.36)

For this we introduce the random variables

\[ X_{n,i} = \frac{\langle V_{t/n}^{(i)} \rangle}{\sqrt{t}}, \quad i = 1, \ldots, n, \]

and we check the validity of conditions (i) and (ii) from Lemma 2.6. Condition (i) follows by Lemma 2.5,

\[ \lim_{n \rightarrow \infty} \sum_{i=1}^{n} \mathbb{E}[X_{n,i}^2] = \lim_{n \rightarrow \infty} n \text{Var}(V_{t/n}) = \sigma^2. \]

(2.37)

To establish condition (ii) we apply the Cauchy-Schwartz inequality and obtain that for every \( \varepsilon > 0 \),

\[ \mathbb{E} \left[ X_{n,i}^2 \mathbb{1}_{|X_{n,i}| > \varepsilon} \right] \leq \frac{1}{t} \left( \mathbb{E} \left[ \langle V_{t/n} \rangle^4 \right] \right) \mathbb{P} \left( |\langle V_{t/n} \rangle| > \varepsilon \sqrt{t} \right)^{1/2}. \]

By Chebyshev's inequality combined with Lemma 2.5 and the fact that \( n = \lfloor \log(t) \rfloor \), there is a constant \( c_1 > 0 \) such that

\[ \mathbb{P} \left( |\langle V_{t/n} \rangle| > \varepsilon \sqrt{t} \right) \leq \frac{\text{Var}(V_{t/n})}{\varepsilon^2 t} \leq \frac{c_1 t/n}{\varepsilon^2 t} = \frac{c_1}{\varepsilon^2 n}. \]

This together with Lemma 5.6 imply that there are constants \( c_2, c_3 > 0 \) such that

\[ \lim_{n \rightarrow \infty} \sum_{i=1}^{n} \mathbb{E} \left[ X_{n,i}^2 \mathbb{1}_{|X_{n,i}| > \varepsilon} \right] \leq \lim_{n \rightarrow \infty} \sum_{i=1}^{n} \frac{1}{t} \left( c_2 \left( \frac{t}{n} \right)^2 c_1 \right)^{1/2} \leq \lim_{n \rightarrow \infty} c_3 \sqrt{n} = 0. \]

(2.38)
Thus, eq. (2.36) follows and we conclude that

\[
\frac{\langle V_t \rangle}{\sqrt{t}} \overset{(d)}{\underset{n \to \infty}{\to}} \sigma \mathcal{N}(0, 1).
\]

We finally observe that

\[
\frac{\mathcal{V}_t - t \text{Cap}(B)}{\sigma \sqrt{t}} = \frac{\langle \mathcal{V}_t \rangle}{\sigma \sqrt{t}} + \frac{\mathbb{E}[\mathcal{V}_t] - t \text{Cap}(B)}{\sigma \sqrt{t}},
\]

which allows us to finish the proof in view of Lemma 2.2 and Corollary 2.4.

\[\square\]

3. Functional central limit theorem

The goal of this section is to prove the functional central limit theorem in eq. (1.6). To prove this statement we adapt the proof of [4, Theorem 1.1], which is concerned with the functional central limit theorem for the capacity of the range of a stable random walk.

We again assume that \( X \) is a stable rotationally invariant Lévy process in \( \mathbb{R}^d \) of index \( \alpha \in (0, 2] \) satisfying \( d > 3\alpha/2 \). We follow the classical two-step scheme (see [14, Theorem 16.10 and Theorem 16.11]). Let \( \{Y^n\}_{n \geq 1} \) be a sequence of random elements in the Skorohod space \( \mathcal{D}([0, \infty), \mathbb{R}) \) endowed with the Skorohod \( J_1 \) topology. The sequence \( \{Y^n\}_{n \geq 1} \) converges weakly to a random element \( Y \) in \( \mathcal{D}([0, \infty), \mathbb{R}) \) if the following two conditions are satisfied:

(i) The finite dimensional distributions of \( \{Y^n\}_{n \geq 1} \) converge weakly to the finite dimensional distributions of \( Y \).

(ii) For any bounded sequence \( \{T_n\}_{n \geq 1} \) of \( \{Y^n\}_{n \geq 1} \)-stopping times and any non-negative sequence \( \{b_n\}_{n \geq 1} \) converging to zero,

\[
\lim_{n \to \infty} \mathbb{P}(\left| Y^n_{T_n + b_n} - Y^n_{T_n} \right| \geq \varepsilon) = 0, \quad \varepsilon > 0.
\]

**Theorem 3.1.** Under the above assumptions, the following convergence holds

\[
\left\{ \frac{\mathcal{V}_{nt} - nt \text{Cap}(B)}{\sigma \sqrt{n}} \right\}_{t \geq 0} \overset{(d)}{\underset{n \to \infty}{\to}} \{W_t\}_{t \geq 0},
\]

where \( \sigma \) is the constant from Theorem 2.1.

**Proof.** We consider the following sequence of random elements which are defined in the space \( \mathcal{D}([0, \infty), \mathbb{R}) \),

(3.1)

\[
Y^n_t = \frac{\mathcal{V}_{nt} - nt \text{Cap}(B)}{\sigma \sqrt{n}}, \quad n \in \mathbb{N},
\]

where \( \sigma \) is the constant from Theorem 2.1. Let us start by showing condition (i).

**Condition (i).** By Theorem 2.1, we have

\[
Y^n_t = \frac{\mathcal{V}_{nt} - nt \text{Cap}(B)}{\sigma \sqrt{n}} = \sqrt{t} \frac{\mathcal{V}_{nt} - nt \text{Cap}(B)}{\sqrt{nt}} \overset{(d)}{\underset{n \to \infty}{\to}} \mathcal{N}(0, t).
\]

Let \( k \in \mathbb{N} \) be arbitrary and choose \( 0 = t_0 < t_1 < t_2 < \cdots < t_k \). We need to prove that

\[
(Y^n_{t_1}, Y^n_{t_2}, \ldots, Y^n_{t_k}) \overset{(d)}{\underset{n \to \infty}{\to}} (W_{t_1}, W_{t_2}, \ldots, W_{t_k}).
\]

In view of the Cramér-Wold theorem [14, Corollary 5.5] it suffices to show that

\[
\sum_{j=1}^{k} \xi_j Y^n_{t_j} \overset{(d)}{\underset{n \to \infty}{\to}} \sum_{j=1}^{k} \xi_j W_{t_j}, \quad (\xi_1, \xi_2, \ldots, \xi_k) \in \mathbb{R}^k.
\]
Using a similar reasoning as in the beginning of the proof of Theorem 2.1, we obtain for \( j \in \{1, 2, \ldots, k\} \),
\[
\mathcal{V}_{nt_j} = \sum_{i=1}^{j} \mathcal{V}_{n(t_i-t_{i-1})}^{(i)} - \sum_{i=1}^{j-1} \mathcal{R}_{nt_j}^{(i)},
\]
where
\[
\mathcal{V}_{n(t_i-t_{i-1})}^{(i)} = \lambda(S_{n(t_i-t_{i-1})}^{(i)}) \quad \text{and} \quad \mathcal{R}_{nt_j}^{(i)} = \lambda(S_{n(t_i-t_{i-1})}^{(i)} \cap S_{n(t_j-t_i)}^{(i+1)}).
\]

The random variables \( \mathcal{V}_{n(t_i-t_{i-1})}^{(i)} \) for \( i \in \{1, 2, \ldots, k\} \), are independent, \( S_{n(t_i-t_{i-1})}^{(i)} \) has the same law as \( S_{n(t_i-t_{i-1})} \), and \( \mathcal{R}_{nt_j}^{(i)} \) has the same law as \( \lambda(S_{n(t_i-t_{i-1})} \cap S_{n(t_j-t_i)}^{(i+1)}) \), with \( S_{n(t_j-t_i)}^{(i+1)} \) being an independent copy of \( S_{n(t_j-t_i)} \). For arbitrary \( (\xi_1, \xi_2, \ldots, \xi_k) \in \mathbb{R}^k \) we have
\[
\sum_{j=1}^{k} \xi_j Y_{t_j}^n = \sum_{j=1}^{k} \xi_j \left( \frac{\mathcal{V}_{nt_j} - nt_j \text{Cap}(\mathcal{B})}{\sigma \sqrt{n}} \right)
\]
\[
= \frac{1}{\sigma \sqrt{n}} \sum_{j=1}^{k} \xi_j \left( \sum_{i=1}^{j} \mathcal{V}_{n(t_i-t_{i-1})}^{(i)} - \sum_{i=1}^{j-1} \mathcal{R}_{nt_j}^{(i)} - nt_i - t_{i-1} \text{Cap}(\mathcal{B}) \right)
\]
\[
= \frac{k}{\sigma \sqrt{n}} \sum_{j=1}^{k} \xi_j \mathcal{V}_{n(t_i-t_{i-1})}^{(i)} - \frac{1}{\sigma \sqrt{n}} \sum_{j=1}^{k} \xi_j \sum_{i=1}^{j-1} \mathcal{R}_{nt_j}^{(i)}
\]
\[
= \frac{k}{\sigma \sqrt{n}} \sum_{j=1}^{k} \xi_j \mathcal{V}_{n(t_i-t_{i-1})}^{(i)} - \frac{1}{\sigma \sqrt{n}} \sum_{j=1}^{k} \xi_j \sum_{i=1}^{j-1} \mathcal{R}_{nt_j}^{(i)}.
\]

Theorem 2.1 provides that
\[
\mathcal{V}_{n(t_i-t_{i-1})}^{(i)} - nt_i - t_{i-1} \text{Cap}(\mathcal{B}) \xrightarrow{\text{d}} \mathcal{N}(0, nt_i - t_{i-1}).
\]

Markov’s inequality combined with Corollary 2.4 implies that for every \( \varepsilon > 0 \),
\[
\mathbb{P} \left( \frac{\mathcal{R}_{nt_j}^{(i)}}{\sqrt{n}} > \varepsilon \right) \leq \frac{\mathbb{E}[\lambda(S_{n(t_i-t_{i-1})} \cap S_{n(t_j-t_i)}^{(i+1)})]}{\varepsilon} \leq \frac{\mathbb{E}[\lambda(S_{nt_j} \cap S_{nt_j}^{(i)})]}{\varepsilon} \leq \frac{c \lambda(nt_j)}{\varepsilon},
\]
which converges to zero, as \( n \) tends to infinity. Since for \( i \in \{1, 2, \ldots, k\} \) the random variables \( \mathcal{V}_{n(t_i-t_{i-1})}^{(i)} \) are independent, we obtain
\[
\sum_{j=1}^{k} \xi_j Y_{t_j}^n \xrightarrow{n \to \infty} \mathcal{N} \left( 0, \sum_{j=1}^{k} \left( \sum_{i=1}^{j} \xi_j \right)^2 (t_i - t_{i-1}) \right).
\]

It follows that the finite dimensional distributions of \( \{Y^n\}_{n \geq 1} \) converge weakly to the finite dimensional distributions of a one-dimensional standard Brownian motion.

Condition (ii). Let \( \{T_n\}_{n \geq 1} \) be a bounded sequence of \( \{Y^n\}_{n \geq 1} \)-stopping times, and let \( \{b_n\}_{n \geq 1} \subset [0, \infty) \) be an arbitrary sequence which converges to zero. We aim to prove that
\[
Y_{T_n + b_n}^n - Y_{T_n}^n \xrightarrow{p \ n \to \infty} 0,
\]
where the convergence holds in probability. By eq. (3.1), we have

$$Y_{nT_n+b_n}^n - Y_{T_n}^n = \frac{V_{n(T_n+b_n)} - n(T_n + b_n) \text{Cap}(B)}{\sqrt{n}} - \frac{V_{nT_n} - nT_n \text{Cap}(B)}{\sqrt{n}}. \quad (3.2)$$

The Markov property and rotational invariance of $X$ yield

$$V_{n(T_n+b_n)} - V_{nT_n} = \lambda((S_{nT_n} \cup [nT_n, n(T_n + b_n)]) - X_{nT_n}) - \lambda(S_{nT_n} - X_{nT_n})$$

$$= \lambda(S_{nT_n}^{(1)}) + \lambda(S_{nb_n}^{(2)}) - \lambda(S_{nT_n}^{(1)} \cap S_{nb_n}^{(2)}) - \lambda(S_{nT_n}^{(1)})$$

$$= \lambda(S_{nb_n}^{(2)}) - \lambda(S_{nT_n}^{(1)} \cap S_{nb_n}^{(2)}),$$

where $S_{nT_n}^{(1)}$ and $S_{nb_n}^{(2)}$ are independent and have the same distribution as $S_{nT_n}$ and $S_{nb_n}$, respectively. Equation (3.2) implies

$$Y_{nT_n+b_n}^n - Y_{T_n}^n = \frac{\lambda(S_{nb_n}^{(2)}) - \lambda(S_{nT_n}^{(1)} \cap S_{nb_n}^{(2)}) - nb_n \text{Cap}(B)}{\sqrt{n}}.$$

With a slight abuse of notation we write $V_{nb_n} = \lambda(S_{nb_n}^{(2)})$. By Lemma 2.2, we obtain

$$Y_{nT_n+b_n}^n - Y_{T_n}^n = \frac{V_{nb_n} - \mathbb{E}[V_{nb_n}]}{\sqrt{n}} + \frac{\mathbb{E}[\lambda(S_{nb_n} \cap [nb_n, \infty))]}{\sqrt{n}} - \frac{\lambda(S_{nT_n}^{(1)} \cap S_{nb_n}^{(2)})}{\sqrt{n}}. \quad (3.3)$$

We prove that the three terms on the right-hand side of eq. (3.3) converge to zero in probability. For the first term, Chebyshev’s inequality yields that for every $\varepsilon > 0$,

$$\mathbb{P} \left( \left| \frac{V_{nb_n} - \mathbb{E}[V_{nb_n}]}{\sqrt{n}} > \varepsilon \right\rangle \right) \leq \frac{\text{Var}(V_{nb_n})}{\varepsilon^2 n},$$

and we are left to show that

$$\lim_{n \to \infty} \frac{\text{Var}(V_{nb_n})}{n} = 0. \quad (3.4)$$

This follows by Lemma 2.5. Indeed, there exist $t_1, c_1 > 0$, such that for every $t \geq t_1$, we have $\text{Var}(V_t) \leq c_1 t$, and whence for $nb_n \geq t_1$, $\text{Var}(V_{nb_n}) \leq c_1 nb_n$. For $nb_n < t_1$ we observe that $\text{Var}(V_{nb_n}) \leq \mathbb{E}[V_{nb_n}^2] \leq \mathbb{E}[V_{t_1}^2]$. This trivially implies eq. (3.4).

By Corollary 2.4, similarly as above, we show that there is $t_2 > 0$ such that for all $n \in \mathbb{N}$

$$\mathbb{E}[\lambda(S_{nb_n} \cap [nb_n, \infty))] \leq c h(nb_n) + \mathbb{E}[V_{t_2}].$$

We then easily conclude that the second term on the right-hand side of eq. (3.3) converges to zero in probability.

There exists $c_2 > 0$ such that $\sup_{n \geq 1} T_n \leq c_2$. By the Markov inequality and Corollary 2.4, we obtain that for every $\varepsilon > 0$

$$\mathbb{P} \left( \left| \frac{\lambda(S_{nT_n}^{(1)} \cap S_{nb_n}^{(2)})}{\sqrt{n}} > \varepsilon \right\rangle \right) \leq \frac{\mathbb{E}[\lambda(S_{nT_n}^{(1)} \cap S_{nb_n}^{(2)})]}{\varepsilon \sqrt{n}} \leq \frac{\mathbb{E}[\lambda(S_{c_2n} \cap S_{\infty})]}{\varepsilon \sqrt{n}} \leq \frac{c h(c_2n)}{\varepsilon \sqrt{n}},$$

which converges to zero, as $n$ tends to infinity. This shows that the last term on the right-hand side of eq. (3.3) goes to zero in probability and the proof is finished. \qed

4. Almost sure invariance principle

Our goal in this section is to prove the following almost sure invariance principle for the process $\{V_t - t \text{Cap}(B)\}_{t \geq 0}$. 
**Theorem 4.1.** Assume that $X$ is a rotationally invariant $\alpha$-stable process in $\mathbb{R}^d$ and $d > 3\alpha/2$. Then there exists a standard Brownian motion $\{W_t\}_{t \geq 0}$ defined on the same probability space (possibly enlarged) as $X$, such that for every $\varepsilon > 0$ we have $\mathbb{P}$-a.s.

\begin{equation}
\sigma^{-1}(\mathcal{V}_t - t \text{ Cap}(B)) - W_t = \begin{cases} 
O(t^{7/8-d/(4\alpha)+\varepsilon}), & 3\alpha/2 < d < 2\alpha, \\
O(t^{1/4+\varepsilon}), & d \geq 2\alpha.
\end{cases}
\end{equation}

As we already mentioned in the introduction, Theorem 4.1 combined with Khintchine’s and Chung’s laws of the iterated logarithm for $\{W_t\}_{t \geq 0}$ (see [24, Chapter 11]) imply eqs. (1.8) and (1.9), respectively.

The proof is the adaptation of the proof of [5, Theorem 1.4] where a similar result for the cardinality of the range of a stable random walk was given. For $n \in \mathbb{N}$, we have

\begin{align*}
\mathcal{V}_n &= \lambda(S_{n/2} \cup S[n/2, n]) = \lambda((S_{n/2} - X_{n/2}) \cup (S[n/2, n] - X_{n/2})) \\
&= \lambda(S_{n/2}^{(1)} \cup S_{n/2}^{(2)}) = \lambda(S_{n/2}^{(1)}) + \lambda(S_{n/2}^{(2)}) - \lambda(S_{n/2}^{(1)} \cap S_{n/2}^{(2)}),
\end{align*}

where $S_{n/2}^{(1)}$ and $S_{n/2}^{(2)}$ are independent, and have the same law as $S_{n/2}$. By iterating the above procedure $L$ times, $2^L \leq n$, we arrive at

\begin{equation}
\mathcal{V}_n = \sum_{i=1}^{2^L} \lambda(S_{n/2^L}^{(i)}) - \sum_{l=1}^{L} \sum_{i=1}^{2^{l-1}} \mathcal{E}_l^{(i)}.
\end{equation}

For $i = 1, \ldots, 2^L$ the random variables $S_{n/2^L}^{(i)}$ are independent copies of $S_{n/2^L}$ and $\mathcal{E}_l^{(i)}$ have the same law as $\lambda(S_{n/2^l} \cap S_{n/2^l}')$ with $S_{n/2^l}'$ being an independent copy of $S_{n/2^l}$. We denote $\mathcal{V}_{n/2^L} = \lambda(S_{n/2^L})$. By eq. (4.2) we obtain

\begin{equation}
\langle \mathcal{V}_n \rangle = \sum_{i=1}^{2^L} \langle \mathcal{V}_{n/2^L} \rangle - \mathcal{E}(n),
\end{equation}

where

\begin{equation}
\mathcal{E}(n) = \sum_{l=1}^{L} \sum_{i=1}^{2^{l-1}} \langle \mathcal{E}_l^{(i)} \rangle.
\end{equation}

We first show that the error term $\mathcal{E}(n)$ is negligible almost surely. For convenience we set $\Delta = d/\alpha - 3/2$.

**Lemma 4.2.** In the above notation, it holds $\mathbb{P}$-a.s.

\begin{equation}
\mathcal{E}(n) = \begin{cases} 
O(n^{1/2-\varepsilon}), & \Delta \in (0, 1/2) \text{ and } \varepsilon \in (0, \Delta), \\
O(n^\varepsilon), & \Delta \geq 1/2 \text{ and } \varepsilon > 0.
\end{cases}
\end{equation}

**Proof.** For any $p \in \mathbb{N}$,

\begin{equation}
\mathbb{E}[|\mathcal{E}(n)|^p] \leq 2p^{-1} \left( \sum_{l=1}^{L} \sum_{i=1}^{2^{l-1}} \mathbb{E}[\mathcal{E}_l^{(i)}] \right)^p + 2p^{-1} \mathbb{E} \left[ \left( \sum_{l=1}^{L} \sum_{i=1}^{2^{l-1}} \mathcal{E}_l^{(i)} \right)^p \right].
\end{equation}

For any $t \geq 0$ we write $\mathcal{I}_t = \lambda(S_t \cap S_t')$, where $S_t'$ is an independent copy of $S_t$. Clearly, $\mathcal{E}_l^{(i)}$ has the same distribution as $\mathcal{I}_{n/2^l}$. By Corollary 2.4, for all $n \in \mathbb{N}$ large enough,

\begin{equation}
\left( \sum_{l=1}^{L} \sum_{i=1}^{2^{l-1}} \mathbb{E}[\mathcal{E}_l^{(i)}] \right)^p \leq 2^{pL}c^p h(n)^p.
\end{equation}
and
\[
\mathbb{E}
\left[
\left(
\sum_{t=1}^{L} \sum_{i=1}^{2^{t-1}} \mathcal{E}_l^{(i)}
\right)^p
\right]
\leq \mathbb{E}
\left[
\sum_{t=1}^{L} \sum_{i=1}^{2^{t-1}} \mathcal{E}_l^{(ii)} \cdots \sum_{l_p=1}^{I_p} \sum_{i_p=1}^{2^{p-1}} \mathcal{E}_l^{(ip)}
\right]
\leq \sum_{t=1}^{L} \sum_{i=1}^{2^{t-1}} \cdots \sum_{l_p=1}^{I_p} \sum_{i_p=1}^{2^{p-1}} \|\mathcal{E}_l^{(ii)}\|_p \cdots \|\mathcal{E}_l^{(ip)}\|_p
\leq 2^{p(L+1)-1}(p^1)^2 \mathcal{E} h(n)^p.
\]

Thus, for all \(n \in \mathbb{N}\) large enough,
\[
\mathbb{E}[|\mathcal{E}(n)|^p] \leq (1 + (pl^1)^2 2^{p-1}) 2^{p(L+1)-1} c^n \mathcal{E} h(n)^p.
\]

We choose \(L = [\log_2(n^\beta)]\) with \(\beta \in (0, 1)\). Then

\[
\mathbb{E}[|\mathcal{E}(n)|^p] \leq (1 + (pl^1)^2 2^{p-1}) 2^{p-1} c^n n^\beta h(n)^p.
\]

We assume that \(\Delta \in (0, 1/2)\) and we fix \(\epsilon \in (0, \Delta)\). We have
\[
\mathbb{P}(|\mathcal{E}(n)| > n^{1/2-\epsilon}) \leq \frac{\mathbb{E}[|\mathcal{E}(n)|^p]}{n^{p/2-\epsilon c}} \leq (1 + (pl^1)^2 2^{p-1}) 2^{p-1} c^n n^{p(\beta - \Delta + \epsilon)}.
\]

Since \(\Delta - \epsilon > 0\), we can choose \(\beta \in (0, 1)\) such that \(\beta - \Delta + \epsilon < 0\), and \(p \in \mathbb{N}\) such that \(p(\beta - \Delta + \epsilon) < -1\). Hence,
\[
\sum_{n=1}^{\infty} \mathbb{P}(|\mathcal{E}(n)| > n^{1/2-\epsilon}) < \infty.
\]

The Borel-Cantelli lemma implies that \(|\mathcal{E}(n)| > n^{1/2-\epsilon}\) only finitely many times \(\mathbb{P}\)-a.s. which gives the result in the first case.

Next we assume that \(\Delta \geq 1/2\) and we fix \(\epsilon > 0\). We have
\[
\mathbb{P}(|\mathcal{E}(n)| > n^\epsilon) \leq \frac{\mathbb{E}[|\mathcal{E}(n)|^p]}{n^{p/2-\epsilon c}} \leq (1 + (pl^1)^2 2^{p-1}) 2^{p-1} c^n n^{p(\beta + \gamma - \epsilon)}.
\]

Since in this case \(h(t)\) is slowly varying, for any \(\gamma > 0\) there is \(c_\gamma > 0\) such that \(h(n) \leq c_\gamma n^\gamma, n \in \mathbb{N}\). We thus obtain
\[
\mathbb{P}(|\mathcal{E}(n)| > n^\gamma) \leq (1 + (pl^1)^2 2^{p-1}) 2^{p-1} c^n n^{p(\beta + \gamma - \epsilon)}.
\]

By choosing \(\beta \in (0, 1), \gamma > 0\) and \(p \in \mathbb{N}\), such that \(p(\beta + \gamma - \epsilon) < -1\), the assertion again follows from the Borel-Cantelli lemma. \(\square\)

In the next step we study the asymptotic behavior of \(\sum_{i=1}^{2^L} \langle V_{n/2L}^{(i)} \rangle\). We apply the Skorohod embedding theorem (see [26]) which asserts that there exist a standard Brownian motion \(\{W_t\}_{t \geq 0}\) and non-negative independent stopping times \(T_1, \ldots, T_{2L}\), such that
\[
\{W_{T_0 + \cdots + T_i} - W_{T_0 + \cdots + T_{i-1}} \}_{i=1, \ldots, 2L} \overset{d}{=} \{\sigma^{-1} \langle V_{n/2L}^{(i)} \rangle \}_{i=1, \ldots, 2L},
\]
where \(T_0 = 0\). It follows that \(\sigma^{-1} \sum_{i=1}^{2^L} \langle V_{n/2L}^{(i)} \rangle\) has the same law as \(W_{T_0 + \cdots + T_{2L}}\). If necessary, we enlarge the probability space for \(\{W_t\}_{t \geq 0}\) and \(X\) (see [19]). Moreover, the following moment estimates hold
\[
\mathbb{E}[T_i] = \sigma^{-2} \text{Var}(\langle V_{n/2L}^{(i)} \rangle) \quad \text{and} \quad \mathbb{E}[T_i^2] \leq \tilde{c} \sigma^{-4} \mathbb{E}[\langle V_{n/2L}^{(i)} \rangle^4],
\]
for a constant \(\tilde{c} > 0\) which does not depend on \(i = 1, \ldots, 2^L\).
Lemma 4.3. For \( L = \lfloor \log_2(n^\beta) \rfloor \) with \( \beta \in (0, 1) \) we have
\[
\sum_{i=1}^{2^L} T_i = n + \mathcal{O}(n^{(1+\beta)/2} h(n^{1-\beta})) \quad \mathbb{P}\text{-a.s.}
\]

Proof. In Lemma 5.7 we obtain the estimate of the error in the asymptotics eq. (2.9). This and eq. (4.3) imply
\[
\mathbb{E}[T_i] = \sigma^{-2} \text{Var}(\mathcal{V}^{(i)}_{n/2^L}) = \frac{n}{2^L} + \mathcal{O}((n/2^L)^{1/2} h(n/2^L)).
\]
Since \( L = \lfloor \log_2(n^\beta) \rfloor \) we have \( n^{\beta}/2 \leq 2^L \leq n^{\beta} \) which yields
\[
\sum_{i=1}^{2^L} \mathbb{E}[T_i] = n + \mathcal{O}(n^{(1+\beta)/2} h(n^{1-\beta})).
\]
By Lemma 5.6 there exist constants \( c_1, t_1 > 1 \) such that
\[
\mathbb{E}[\langle \mathcal{V}_i \rangle^4] \leq c_1t^2, \quad t \geq t_1.
\]
The elementary inequality \((a - b)^4 \leq 8(a^4 + b^4)\), which holds for any \( a, b \geq 0 \), combined with Jensen’s inequality, implies
\[
\mathbb{E}[\langle \mathcal{V}_i \rangle^4] = \mathbb{E}[\langle \mathcal{V}_i - \mathbb{E}[\mathcal{V}_i] \rangle^4] \leq \mathbb{E}[8(\langle \mathcal{V}_i^4 \rangle + (\mathbb{E}[\mathcal{V}_i])^4)] = 16 \mathbb{E}[\mathcal{V}_i^4], \quad t > 0.
\]
For \( t \in [1, t_1] \) we have
\[
\mathbb{E}[\langle \mathcal{V}_i \rangle^4] \leq 16 \mathbb{E}[\mathcal{V}_i^4] \leq 16 \mathbb{E}[\mathcal{V}_i^4] \leq 16 \mathbb{E}[\mathcal{V}_i^4]t^2.
\]
Thus
\[
\mathbb{E}[\langle \mathcal{V}_i \rangle^4] \leq (c_1 + 16 \mathbb{E}[\mathcal{V}_i^4])t^2 = c_2t^2, \quad t \geq 1,
\]
with \( c_2 = c_1 + 16 \mathbb{E}[\mathcal{V}_i^4] \). Together with eq. (4.3) this gives
\[
\mathbb{E}[T_i^2] \leq \tilde{c}\sigma^{-4}\mathbb{E}[\langle \mathcal{V}^{(i)}_{n/2^L} \rangle^4] \leq c_3 \left( \frac{n}{2^L} \right)^2 \leq 4c_3n^{2(1-\beta)},
\]
where \( c_3 = \tilde{c}\sigma^{-4}c_2 \). We obtain
\[
\sum_{i=1}^{\infty} \text{Var} \left( \frac{T_i - \mathbb{E}[T_i]}{\sqrt{i \log(i + 1)}} \right) \leq 4c_3n^{2(1-\beta)} \sum_{i=1}^{\infty} \frac{1}{i \log^2(i + 1)} < \infty,
\]
and according to [7, Theorem 2.5.3] it follows that
\[
\sum_{i=1}^{\infty} \frac{T_i - \mathbb{E}[T_i]}{\sqrt{i \log(i + 1)}} < \infty \quad \mathbb{P}\text{-a.s.}
\]
Next we apply Kronecker’s lemma (see [7, Theorem 2.5.5]) to the two sequences \( \{(T_i - \mathbb{E}[T_i])/(\sqrt{i \log(1+i)})\}_{i \geq 1} \) and \( \{\sqrt{i \log(1+i)}\}_{i \geq 1} \). We conclude that
\[
\sum_{i=1}^{2^L} (T_i - \mathbb{E}[T_i]) = \mathcal{O}(n^{\beta/2} \log n) \quad \mathbb{P}\text{-a.s.}
\]
Finally, we have
\[
\sum_{i=1}^{2^L} T_i = \sum_{i=1}^{2^L} (T_i - \mathbb{E}[T_i]) + \sum_{i=1}^{2^L} \mathbb{E}[T_i]
\]
\[
= \mathcal{O}(n^{\beta/2} \log n) + n + \mathcal{O}(n^{(1+\beta)/2} h(n^{1-\beta})) \quad \mathbb{P}\text{-a.s.}
\]
and the proof is finished. \qed
Lemma 4.4. Choose $L = \lfloor \log_2(n^\beta) \rfloor$ with $\beta \in (0, 1)$. Then for any $\varepsilon > 0$ we have

$$W_{T_0 + \ldots + T_2} - W_{n} = \begin{cases} \mathcal{O}(n^{(1-\Delta+\beta\Delta)/2+\varepsilon}), & \Delta \in (0, 1/2), \\ \mathcal{O}(n^{(1+\beta)/4+\varepsilon}), & \Delta \geq 1/2, \end{cases} \mathbb{P}\text{-a.s.}$$

Proof. The proof follows along the same steps as in [5, Lemma 2.4].

Proof of Theorem 4.1. A similar combination of Lemmas 4.2 and 4.4 as that in [5, Theorem 1.4], implies that for every $\varepsilon > 0$,

$$(4.4) \quad \sigma^{-1}(\mathcal{V}_n - \mathbb{E}[\mathcal{V}_n]) - W_{n} = \begin{cases} \mathcal{O}(n^{7/8-\delta/(4\alpha)+\varepsilon}), & 3\alpha/2 < d < 2\alpha, \\ \mathcal{O}(n^{1/4+\varepsilon}), & d \geq 2\alpha, \end{cases} \mathbb{P}\text{-a.s.}$$

Next, for any $t \geq 1$ we set $n = [t]$. By Lemma 2.2 we obtain

$$|\sigma^{-1}(\mathcal{V}_t - t \text{ Cap}(\mathcal{B})) - W_t| \leq |\sigma^{-1}(\mathcal{V}_n - \mathbb{E}[\mathcal{V}_n]) - W_n| + |\sigma^{-1}\mathcal{V}_t - \mathcal{V}_n|$$

where $\mathcal{V}_n = \lambda(\mathcal{S}_t \setminus \mathcal{S}_n) \leq \lambda(\mathcal{S}[n, n+1]) = \lambda(\mathcal{S}[n, n+1] - X_n) = \lambda(S_1^n)$, where $S_1^n$ is an independent copy of $S_1$. This yields

$$(4.5) \quad \mathbb{E}||\mathcal{V}_t - \mathcal{V}_n| \leq \mathbb{E}[\lambda(S_1^n)].$$

We can apply [22, Ch. II, Excercise 1.23] to arrive at

$$\mathbb{P}(|W_t - W_n| > n^{1/4}) \leq \mathbb{P}\left(\sup_{0 \leq t \leq 1} |W_s| > n^{1/4}\right) \leq 2 e^{-n^{1/2}}.$$ 

Hence, by the Borel-Cantelli lemma, $|W_t - W_n| = \mathcal{O}(n^{1/4})$ and we easily conclude the result.

5. Technical results

Lemma 5.1. In the notation of the proof of Lemma 2.5, for any $k \in \mathbb{N}$ it holds that

$$\mathbb{E}[\lambda(\mathcal{S}_1 \setminus \mathcal{S}[1, N]) \mid \mathcal{F}_{N-k+1, N}] \xrightarrow{\mathcal{L}^1} \mathbb{E}[\lambda(\mathcal{S}_1 \setminus \mathcal{S}[1, \infty])].$$

Proof. Set $m_N = \lambda(\mathcal{S}_1 \setminus \mathcal{S}[1, N])$ and $m_\infty = \lambda(\mathcal{S}_1 \setminus \mathcal{S}[1, \infty])$. We have

$$\lim_{N \to \infty} \mathbb{E}[||m_N \mid \mathcal{F}_{N-k+1, N}] - \mathbb{E}[m_\infty||]$$

$$\leq \lim_{N \to \infty} \mathbb{E}[||m_N - m_\infty \mid \mathcal{F}_{N-k+1, N}|| + ||m_\infty \mid \mathcal{F}_{N-k+1, N} - \mathbb{E}[m_\infty]|]$$

$$\leq \lim_{N \to \infty} \left(\mathbb{E}[||m_N - m_\infty \mid \mathcal{F}_{N-k+1, N}|| + \mathbb{E}[||m_\infty \mid \mathcal{F}_{N-k+1, N} - \mathbb{E}[m_\infty]|]|\right)$$

$$= \lim_{N \to \infty} \mathbb{E}[||m_N - m_\infty||] + \lim_{N \to \infty} \mathbb{E}[||m_\infty \mid \mathcal{F}_{N-k+1, N} - \mathbb{E}[m_\infty]|].$$

Since $m_N$ clearly converges to $m_\infty$ in $L^1$, we are left to prove that the second limit in the expression above is zero. For a fixed $k \in \mathbb{N}$ we define

$$\mathcal{H}_N = \sigma(X_t : t \geq N - k + 1), \quad N \geq k.$$ 

We observe that $\mathcal{F}_{N-k+1, N} \subset \mathcal{H}_N$ and $\mathcal{H}_N$ is a decreasing family of $\sigma$-algebras. Moreover, according to Kolmogorov’s 0 – 1 law, for every $H \in \mathcal{H}_\infty = \bigcap_{N \geq 1} \mathcal{H}_N$, we have $\mathbb{P}(H) \in \{0, 1\}$. From Levi’s theorem (see [22, Ch. II, Corollary 2.4]) we infer that $\mathbb{P}$-a.s.

$$(5.1) \quad \lim_{N \to \infty} \mathbb{E}[m_\infty \mid \mathcal{H}_N] = \mathbb{E}[m_\infty \mid \mathcal{H}_\infty] = \mathbb{E}[m_\infty].$$
Notice that by eq. (2.1), \( E[m_{\infty}] = \text{Cap}(B) \). Since the family \( \{E[m_{\infty} | \mathcal{H}_N]\}_{N \geq 1} \) is uniformly integrable, we infer that the convergence in eq. (5.1) holds also in \( L^1 \), see [7, Theorem 5.5.1]. We finally obtain

\[
\lim_{N \to \infty} \mathbb{E}\left[ E\left[ m_{\infty} \mid \mathcal{F}_{N-k+1,N} \right] - E[m_{\infty}] \right] = 0,
\]

and the proof is finished. \( \square \)

**Lemma 5.2.** In the notation of the proof of Lemma 2.5, the sequence \( \left\{ \frac{1}{n} \sum_{k=1}^{n} E[Y_k^2] \right\}_{n \geq 1} \) is bounded.

**Proof.** We set \( \Delta = d/\alpha - 3/2 \) and recall that it is a positive number. We present the proof in the case \( \Delta \in (0, 1/2) \) as the proof for \( \Delta \geq 1/2 \) is similar. For \( \Delta \in (0, 1/2) \), the function \( h(t) \) defined in eq. (2.7) is given by \( h(t) = t^{1/2 - \Delta} \). By the Cauchy-Schwarz inequality,

\[
\left| E\left[ \langle \lambda(S_n \cap S[n, \infty]) \mid F_n \rangle \right] \sum_{k=1}^{n} Y_k \right| \leq \left( \text{Var}(E[\lambda(S_n \cap S[n, \infty]) \mid F_n]) \right)^{1/2} \left( \sum_{k=1}^{n} E[Y_k^2] \right)^{1/2}
\]

This combined with eq. (2.17) yields

\[
\frac{\text{Var}(Y_n)}{n} \geq \frac{1}{n} \text{Var}(E[\lambda(S_n \cap S[n, \infty]) \mid F_n]) + \frac{1}{n} \sum_{k=1}^{n} E[Y_k^2] - \frac{4\sqrt{2}c}{\Delta} \left( \frac{1}{n} \sum_{k=1}^{n} E[Y_k^2] \right)^{1/2}.
\]

We suppose, for the sake of contradiction, that there exists a subsequence \( \{n_m\}_{m \geq 1} \subseteq \mathbb{N} \) such that

\[
\lim_{m \to \infty} \frac{1}{n_m} \sum_{k=1}^{n_m} E[Y_k^2] = \infty.
\]

Since

\[
\lim_{n \to \infty} \frac{\text{Var}(Y_n)}{n} = \sigma^2 \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} \text{Var}(E[\lambda(S_n \cap S[n, \infty]) \mid F_n]) = 0,
\]

it follows that

\[
\lim_{m \to \infty} \frac{1}{n_m^{\Delta}} \left( \frac{1}{n_m} \sum_{k=1}^{n_m} E[Y_k^2] \right)^{1/2} = \infty.
\]

We deduce that \( \left\{ \frac{1}{n_m} \sum_{k=1}^{n_m} E[Y_k^2] \right\}_{m \geq 1} \) diverges faster to infinity than \( \left\{ n_m^{2\Delta} \right\}_{m \geq 1} \). Since

\[
\lim_{n \to \infty} \frac{\text{Var}(Y_n)}{n^{1+2\Delta}} = 0,
\]
we can again use eq. (2.17) to obtain
\[
\frac{\text{Var}(Y_{nm})}{n_m^{1+2\Delta}} \geq \frac{1}{n_m^{1+2\Delta}} \text{Var}(\mathbb{E}[\lambda(S_{nm} \cap S[n_m, \infty)) | \mathcal{F}_{nm}]) + \frac{1}{n_m^{1+2\Delta}} \sum_{k=1}^{n_m} \mathbb{E}[Y_k^2]
\]
- \frac{4\sqrt{2}c}{n_m^{1+2\Delta}} \left( \frac{1}{n_m^{1+2\Delta}} \sum_{k=1}^{n_m} \mathbb{E}[Y_k^2] \right)^{1/2}.
\]

We infer that \( \left\{ \frac{1}{n_m^{1+2\Delta}} \sum_{k=1}^{n_m} \mathbb{E}[Y_k^2] \right\}_{m \geq 1} \) grows faster to infinity than \( \left\{ n_m^{6\Delta} \right\}_{m \geq 1} \). By iterating this procedure, we conclude that
\[
(5.2) \quad \lim_{m \to \infty} \frac{1}{n_m^{1+2\Delta}} \sum_{k=1}^{n_m} \mathbb{E}[Y_k^2] = \infty.
\]

On the other hand, from eq. (2.16) we have
\[
|Y_k| \leq \mathbb{E}[\lambda(S_{k-1} \cap S[k-1, \infty))) | \mathcal{F}_{k-1}] + \mathbb{E}[\lambda(S_{k-1} \cap S[k-1, \infty))) | \mathcal{F}_k] + \mathbb{E}[\lambda(S[k-1, k] \cap S[k, \infty))) | \mathcal{F}_k] + \lambda(S[k-1, k]) + \text{Cap}(\mathcal{B}),
\]
where in the last line we used monotonicity and eq. (2.1). By Jensen’s inequality,
\[
\mathbb{E}[|Y_k|^2] \leq 4(2|\mathbb{E}[\lambda(S_{k-1} \cap S[k-1, \infty)))^2] + \mathbb{E}[\lambda(S[k-1, k])^2]) + \text{Cap}(\mathcal{B}))^2 \leq 64c^2h(k-1)^4 + 4\mathbb{E}[|Y_k|^2] + 4(\text{Cap}(\mathcal{B}))^2 \leq c_1k,
\]
for a constant \( c_1 > 0 \). This yields \( \sum_{k=1}^{n_m} \mathbb{E}[Y_k^2] \leq c_1n_m^2 \), which gives a contradiction. \( \square \)

**Lemma 5.3.** In the notation of the proof of Lemma 2.5, for any \( \beta \in (0, 1) \) there exists a constant \( c(d, \alpha, \beta) > 0 \) such that
\[
|\phi(y) - \phi(y - x)| \leq c(d, \alpha, \beta)(\phi(y) + \phi(y - x)) \left( \frac{1 + |y|^\beta}{|y|^\beta} \wedge 1 \right), \quad x, y \in \mathbb{R}^d.
\]

**Proof.** Recall that \( \phi(y) = \mathbb{P}(y \in S_\infty) \). This yields
\[
|\phi(y) - \phi(y - x)| \leq \phi(y) + \phi(y - x), \quad x, y \in \mathbb{R}^d.
\]

To establish the second non-trivial part of the claimed inequality, that is, for \( |y|^\beta > 1 + |x|^\beta \), we first observe that by rotational invariance of \( X \) it holds \( \phi(y) = \mathbb{P}_y(\tau_{\mathcal{B}} < \infty) \). Moreover, by eq. (1.3),
\[
\phi(y) = a_{d, \alpha} \int_\mathbb{B} |y - w|^{\alpha - d}(1 - |w|^2)^{-\alpha/2} dw, \quad y \in \mathbb{R}^d,
\]
where
\[
a_{d, \alpha} = \frac{\sin \frac{\pi \alpha}{2} \Gamma((d - \alpha)/2) \Gamma(d/2)}{2^{\alpha - d + 1} \Gamma(\alpha/2)},
\]
see e.g. [29]. We fix \( \beta \in (0, 1) \) and \( x \in \mathbb{R}^d \). For any \( y \in \mathcal{B}^c(0, 1 + |x|) \) we have
\[
|y - w| \geq |y| - |w| \geq |y| - 1 > |x|, \quad w \in \mathcal{B}.
\]

There exists \( x_0 \in \mathcal{B}(0, |y - w|) \) lying on the line going through the origin, determined by the vector \( y - w \), and such that
\[
\frac{|y - w|^{\alpha - d} - |y - w - x|^{\alpha - d}}{|y - w|^{\alpha - d} + |y - w - x|^{\alpha - d}} \frac{|y - w|^\beta}{1 + |x|^\beta} = \frac{|y - w|^{d - \alpha} - |y - w - x|^{d - \alpha}}{|y - w|^{d - \alpha} + |y - w - x|^{d - \alpha}} \frac{|y - w|^\beta}{1 + |x|^\beta}.
\]
and we obtain
\[
\frac{d}{|y-w|^{d-\alpha} - |y-w - x|^{d-\alpha}} |y-w|^\beta \leq \frac{|y-w|^{d-\alpha} - (|y-w| - \varrho)^{d-\alpha}}{|y-w|^{d-\alpha} + (|y-w| - \varrho)^{d-\alpha}} |y-w|^\beta.
\]

Since \( x_0 \) is necessarily of the form \( x_0 = \frac{y-w}{|y-w|} \varrho \), for some \( \varrho \in [-|y-w|, |y-w|] \), we have
\[
\frac{|y-w|^{\alpha-d} - |y-w - x|^{\alpha-d}}{|y-w|^{\alpha-d} + |y-w - x|^{\alpha-d}} \leq \frac{|y-w|^{\alpha-d} - (|y-w| - \varrho)^{\alpha-d}}{|y-w|^{\alpha-d} + (|y-w| - \varrho)^{\alpha-d}}.
\]

We investigate the two following cases.

Case 1. We first assume that \( d - \alpha \leq 1 \). If \( \varrho \in [0, |y-w|/2] \) then, by the concavity of the function \( r \mapsto r^{d-\alpha} \), we obtain
\[
\frac{|y-w|^{\alpha-d} - |y-w - x|^{\alpha-d}}{|y-w|^{\alpha-d} + |y-w - x|^{\alpha-d}} \leq \frac{(d - \alpha) \varrho (|y-w| - \varrho)^{d-\alpha-1}}{|y-w|^{d-\alpha} + (|y-w| - \varrho)^{d-\alpha}} |y-w|^\beta.
\]
\[
\leq (d - \alpha) \frac{\varrho}{|y-w|^{d-\alpha}} |y-w|^\beta.
\]
\[
\leq 2(d - \alpha) \frac{\varrho}{|y-w|^{d-\alpha}} |y-w|^\beta.
\]
\[
\leq 2(d - \alpha) \frac{\varrho^3}{|y-w|^{3(d-\alpha)}} |y-w|^\beta.
\]
\[
\leq 2(d - \alpha).
\]

If \( \varrho \in [|y-w|/2, |y-w|] \), then
\[
\frac{|y-w|^{\alpha-d} - |y-w - x|^{\alpha-d}}{|y-w|^{\alpha-d} + |y-w - x|^{\alpha-d}} \leq \frac{|y-w|^\beta}{1 + 2^\beta |y-w|^\beta} \leq 2^\beta.
\]

If \( \varrho \in [-|y-w|, 0] \) then we again use the concavity argument which yields
\[
\frac{|y-w|^{\alpha-d} - |y-w - x|^{\alpha-d}}{|y-w|^{\alpha-d} + |y-w - x|^{\alpha-d}} \leq \frac{(d - \alpha) |\varrho|}{|y-w|^{d-\alpha}} |y-w|^{\beta}.
\]
\[
\leq (d - \alpha) \frac{|\varrho|}{|y-w|^\beta} |y-w|^\beta.
\]
\[
\leq (d - \alpha) \frac{|\varrho|^3}{|y-w|^3} |y-w|^\beta.
\]
\[
\leq d - \alpha.
\]

Case 2. Assume that \( d - \alpha > 1 \). If \( \varrho \in [0, |y-w|] \) then the function \( r \mapsto r^{d-\alpha} \) is convex and we obtain
\[
\frac{|y-w|^{\alpha-d} - |y-w - x|^{\alpha-d}}{|y-w|^{\alpha-d} + |y-w - x|^{\alpha-d}} \leq \frac{(d - \alpha) \varrho |y-w|^{d-\alpha-1}}{|y-w|^{d-\alpha} + (|y-w| - \varrho)^{d-\alpha}} |y-w|^\beta.
\]
\[
\leq (d - \alpha) \frac{\varrho}{|y-w|^{d-\alpha}} |y-w|^\beta.
\]
\[
\leq (d - \alpha) \frac{\varrho^3}{|y-w|^3} |y-w|^\beta.
\]
\[
\leq d - \alpha.
\]
If \( q \in [-|y-w|,0] \) then again in view of the convexity we have

\[
\frac{|y-w|^{a-d} - |y-w-x|^{a-d}}{|y-w|^{a-d} + |y-w-x|^{a-d}} \frac{1}{1 + |x|^\beta} \leq \frac{(d-\alpha) |q| (|y-w| + |q|)^{d-\alpha} - 1}{|y-w|^{d-\alpha} + (|y-w| + |q|)^{d-\alpha} 1 + |q|^\beta} \frac{(d-\alpha) |q|}{|y-w|^{\beta}} \leq \frac{(d-\alpha) |q|^\beta}{|y-w|^{\beta} 1 + |q|^\beta} \leq d - \alpha.
\]

Finally, for \( y \in B^c(0,1 + |x|) \cap B^c(0,2) \) we obtain

\[
\frac{|y-w|^{a-d} - |y-w-x|^{a-d}}{|y-w|^{a-d} + |y-w-x|^{a-d}} \frac{1}{1 + |x|^\beta} \leq \frac{|y-w|^{\beta}}{|y-w|^{\beta} 1 + |q|^\beta} \leq 2^{1+\beta}(d-\alpha) \lor 2^{2^\beta}.
\]

On the other hand, if \( y \in B^c(0,1 + |x|) \cap B(0,2) \) then \( x \in \mathcal{B} \) and

\[
\frac{1}{|y|^\beta} \geq 2^{-\beta}.
\]

Equations (5.3) to (5.5) imply the result. \(\square\)

**Lemma 5.4.** In the notation of the proof of Lemma 2.5, it holds that

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p(1, x) \phi(y) \left| \phi(y) - \phi(y - x) \right| \, dy \, dx < \infty.
\]

**Proof.** We split the integral into three parts

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p(1, x) \phi(y) \left| \phi(y) - \phi(y - x) \right| \, dy \, dx = \int_{\mathbb{R}^d} \int_{B^c(0,1+|x|)} p(1, x) \phi(y) \left| \phi(y) - \phi(y - x) \right| \, dy \, dx \\
+ \int_{\mathbb{R}^d} \int_{B(0,1+|x|) \cap B} p(1, x) \phi(y) \left| \phi(y) - \phi(y - x) \right| \, dy \, dx \\
+ \int_{\mathbb{R}^d} \int_{B(0,1+|x|) \cap B^c} p(1, x) \phi(y) \left| \phi(y) - \phi(y - x) \right| \, dy \, dx.
\]

According to Lemma 5.3, by setting \( \beta = \alpha/2 \) we obtain

\[
\left| \phi(y) - \phi(y - x) \right| \leq c_1 \left( \phi(y) + \phi(y - x) \right) \frac{1 + |x|^\alpha/2}{|y|^\alpha/2}, \quad y \in B^c(0,1 + |x|),
\]

where \( c_1 = c(d, \alpha, \beta) \). By [18, Lemma 2.5], there exists a constant \( c_2 = c_2(d, \alpha) > 0 \) such that \( \phi(w) \leq c_2 |w|^{\alpha-d} \), for any \( w \in B^c \). Thus,

\[
\int_{\mathbb{R}^d} \int_{B^c(0,1+|x|)} p(1, x) \phi(y) \left| \phi(y) - \phi(y - x) \right| \, dy \, dx
\]
\[ \begin{align*}
&\leq c_1 c_2^2 \int_{\mathbb{R}^d} \int_{B^c(0,1+|x|)} p(1,x) \frac{1}{|y|^{\alpha/2}} \left( \frac{1}{|y|^{d-\alpha}} + \frac{1}{|y-x|^{d-\alpha}} \right) \frac{1 + |x|^{\alpha/2}}{|y|^{\alpha/2}} \ dy \ dx \\
&\leq 2^{d-\alpha/2}(1 + 2^{d-\alpha}) c_1 c_2^2 \int_{\mathbb{R}^d} \int_{B^c(0,1+|x|)} p(1,x)(1 + |x|^{\alpha/2}) \frac{1}{|y-x|^{2d-3\alpha/2}} \ dy \ dx \\
&\leq 2^{d-\alpha/2}(1 + 2^{d-\alpha}) c_1 c_2^2 \int_{\mathbb{R}^d} p(1,x)(1 + |x|^{\alpha/2}) \ dx \int_{B^c} \frac{1}{|z|^{2d-3\alpha/2}} \ dz \\
&= 2^{d-\alpha/2}(1 + 2^{d-\alpha}) c_1 c_2^2 \lambda(\mathcal{B}) \int_{\mathbb{R}^d} p(1,x)(1 + |x|^{\alpha/2}) \ dx \int_{1}^{\infty} \frac{1}{r^{d-3\alpha/2+1}} \ dr,
\end{align*} \]

where in the second step we used the fact that \( |y-x| \leq |y| + |x| \leq |y| + |y| - 1 \leq 2|y| \). The last integral is finite as \( d > 3\alpha/2 \) and \( X \) has finite \( \beta \)-moment for any \( \beta < \alpha \) (see [23, Example 25.10]).

For the second integral on the right-hand side of eq. (5.6) we observe that

\[ \int_{\mathbb{R}^d} \int_{B(0,1+|x|) \cap \mathbb{B}} p(1,x) \phi(y) \left| \phi(y) - \phi(y-x) \right| \ dy \ dx \leq 2 \lambda(\mathcal{B}). \]

The third integral on the right-hand side of eq. (5.6) is most demanding. We start by splitting this integral into two parts

\[ \begin{align*}
(5.7) \quad &\int_{\mathbb{R}^d} \int_{B(0,1+|x|) \cap \mathbb{B}^c} p(1,x) \phi(y) \left| \phi(y) - \phi(y-x) \right| \ dy \ dx \\
&= \int_{B(0,\Lambda)} \int_{B(0,1+|x|) \cap \mathbb{B}^c} p(1,x) \phi(y) \left| \phi(y) - \phi(y-x) \right| \ dy \ dx \\
&\quad + \int_{\mathbb{B}^c(0,\Lambda)} \int_{B(0,1+|x|) \cap \mathbb{B}^c} p(1,x) \phi(y) \left| \phi(y) - \phi(y-x) \right| \ dy \ dx.
\end{align*} \]

For the first integral in this decomposition we have

\[ \int_{B(0,\Lambda)} \int_{B(0,1+|x|) \cap \mathbb{B}^c} p(1,x) \phi(y) \left| \phi(y) - \phi(y-x) \right| \ dy \ dx \]

\[ \leq 2 c_2 \int_{B(0,\Lambda)} \int_{B(0,1+|x|) \cap \mathbb{B}^c} p(1,x) \frac{1}{|y|^{d-\alpha}} \ dy \ dx \]

\[ \leq 2 c_2 \int_{B(0,1+\Lambda) \cap \mathbb{B}^c} \frac{1}{|y|^{d-\alpha}} \ dy \]

\[ \leq 2 c_2 \lambda(\mathcal{B}) (0, 1 + \Lambda). \]

The second integral on the right-hand side of eq. (5.7) we decompose further as follows

\[ \begin{align*}
&\int_{\mathbb{B}^c(0,\Lambda)} \int_{B(0,1+|x|) \cap \mathbb{B}^c} p(1,x) \phi(y) \left| \phi(y) - \phi(y-x) \right| \ dy \ dx \\
&= \int_{\mathbb{B}^c(0,\Lambda)} \int_{B(0,1+|x|) \cap \mathbb{B}^c \cap \{z: |z-x| > |z|\}} p(1,x) \phi(y) \left| \phi(y) - \phi(y-x) \right| \ dy \ dx \\
&\quad + \int_{\mathbb{B}^c(0,\Lambda)} \int_{B(0,1+|x|) \cap \mathbb{B}^c \cap \{z: |z-x| \leq |z|\}} p(1,x) \phi(y) \left| \phi(y) - \phi(y-x) \right| \ dy \ dx \\
&\quad + \int_{\mathbb{B}^c(0,\Lambda)} \int_{B(0,1+|x|) \cap \mathbb{B}^c \cap \{z: |z-x| < |z|\}} p(1,x) \phi(y) \left| \phi(y) - \phi(y-x) \right| \ dy \ dx.
\end{align*} \]

(5.8)

It is well-known that for any \( \Lambda > 1 \) large enough there is \( c_3 = c_3(d, \alpha, \Lambda) > 0 \) such that

\[ p(1,x) \leq \frac{c_3}{|x|^{d+\alpha}}, \quad x \in \mathbb{B}^c(0, \Lambda). \]
We set $A_1 = \mathcal{B}(0, 1 + |x|) \cap \mathcal{B}^c \cap \{z : |z - x| > |z|\}$ and estimate the first integral on the right-hand side of eq. (5.8) as follows
\[
\int_{\mathcal{B}(0, \Delta)} \int_{A_1} p(1, x) \phi(y) \left| \phi(y) - \phi(y - x) \right| dy \, dx \\
\leq c_2^2 c_3 \int_{\mathcal{B}(0, \Delta)} \int_{A_1} \frac{1}{|x|^{d+\alpha}} \frac{1}{|y|^{d-\alpha}} \left( \frac{1}{|y|^{d-\alpha}} + \frac{1}{|y - x|^{d-\alpha}} \right) dy \, dx \\
\leq 2 c_2^2 c_3 \int_{\mathcal{B}(0, \Delta)} \int_{A_1} \frac{1}{|x|^{d+\alpha}} \frac{1}{|y|^{2d-2\alpha}} dy \, dx \\
\leq 2 c_2^2 c_3 \int_{\mathcal{B}(0, \Delta)} \int_{\mathcal{B}(0, 1+|x|) \cap \mathcal{B}^c} \frac{1}{|x|^{d+\alpha}} \frac{1}{|y|^{2d-2\alpha}} dy \, dx \\
\leq 2 c_2^2 c_3 \int_{\mathcal{B}(0, \Delta)} \frac{1}{|x|^{d+\alpha}} \int_1^{1+|x|} \frac{1}{r^{d-2\alpha+1}} dr \, dx.
\]
The last integral is finite as $d > 3\alpha/2$.

We set $A_2 = \mathcal{B}(0, 1 + |x|) \cap \mathcal{B}^c \cap \{z : 1 \leq |z - x| \leq |z|\}$ and for the second integral on the right-hand side of eq. (5.8) we have
\[
\int_{\mathcal{B}(0, \Delta)} \int_{A_2} p(1, x) \phi(y) \left| \phi(y) - \phi(y - x) \right| dy \, dx \\
\leq c_2^2 c_3 \int_{\mathcal{B}(0, \Delta)} \int_{A_2} \frac{1}{|x|^{d+\alpha}} \frac{1}{|y|^{d-\alpha}} \left( \frac{1}{|y|^{d-\alpha}} + \frac{1}{|y - x|^{d-\alpha}} \right) dy \, dx \\
\leq 2^{d+\alpha+1} c_2^2 c_3 \int_{\mathcal{B}(0, \Delta)} \int_{A_2} \frac{1}{|y|^{2d}} \frac{1}{|y - x|^{d-\alpha}} dy \, dx \\
\leq 2^{d+\alpha+1} c_2^2 c_3 \int_{\mathcal{B}(0, |y|) \cap \mathcal{B}^c} \frac{1}{|y|^{2d}} \frac{1}{|z|^{d-\alpha}} dz \, dy \\
\leq 2^{d+\alpha+1} c_2^2 c_3 \int_{\mathcal{B}(0, \Delta)} \frac{1}{|y|^{2d-\alpha}} dy \, dx \\
= 2^{d+\alpha+1} c_2^2 c_3 \int_1^{\infty} \frac{1}{r^{d-\alpha+1}} dr,
\]
where in the second step we used the fact that $|y| \leq 2|x|$.

Finally, we set $A_3 = \mathcal{B}(0, 1 + |x|) \cap \mathcal{B}^c \cap \{z : |z - x| < 1\}$ and for the third integral on the right-hand side of eq. (5.8) we proceed as follows
\[
\int_{\mathcal{B}(0, \Delta)} \int_{A_3} p(1, x) \phi(y) \left| \phi(y) - \phi(y - x) \right| dy \, dx \\
\leq 2 c_2 c_3 \int_{\mathcal{B}(0, \Delta)} \int_{A_3} \frac{1}{|x|^{d+\alpha}} \frac{1}{|y|^{d-\alpha}} dy \, dx \\
\leq 2^{1+d-\alpha} c_2 c_3 \int_{\mathcal{B}(0, \Delta)} \int_{\{z : |z - x| < 1\}} \frac{1}{|x|^{2d}} dy \, dx \\
\leq 2^{1+d-\alpha} c_2 c_3 \int_1^{\infty} \frac{1}{r^{d+1}} dr,
\]
where in the second step we used the fact that $|x| \leq |y - x| + |y| \leq 1 + |y| \leq 2|y|$.
Lemma 5.5. In the notation of the proof of Lemma 2.5, it holds that
\[ \sup_{x \in B} \int_{\mathbb{R}^d} \phi(y) |\phi(y) - \phi(y - x)| \, dy < \infty. \]

Proof. We split the integral as follows
\[ \int_{\mathbb{R}^d} \phi(y) |\phi(y) - \phi(y - x)| \, dy = \int_{B(0,1+|x|)} \phi(y) |\phi(y) - \phi(y - x)| \, dy \]
\[ + \int_{B^c(0,1+|x|)} \phi(y) |\phi(y) - \phi(y - x)| \, dy. \]

For any \( x \in B \) one has \( B(0,1+|x|) \subseteq B(0,2) \), and whence
\[ \sup_{x \in B} \int_{B(0,1+|x|)} \phi(y) |\phi(y) - \phi(y - x)| \, dy \leq 2 \lambda(B(0,2)). \]

By Lemma 5.3 with \( \beta = \alpha/2 \), for a constant \( c_1 = c(d, \alpha, \beta) \),
\[ |\phi(y) - \phi(y - x)| \leq c_1 (\phi(y) + \phi(y - x)) \frac{1 + |x|^{\alpha/2}}{|y|^{\alpha/2}}, \quad y \in B^c(0,1+|x|). \]

By [18, Lemma 2.5], there exists a constant \( c_2 = c_2(d, \alpha) > 0 \) such that \( \phi(w) \leq c_2 |w|^{-d} \),
for any \( w \in B^c \). Thus, for \( x \in B \) we have
\[ \int_{B^c(0,1+|x|)} \phi(y) |\phi(y) - \phi(y - x)| \, dy \]
\[ \leq 2^{d-\alpha/2+1} (1 + 2^{d-\alpha}) c_1 c_2^2 \int_{B^c(0,1+|x|)} \frac{1}{|y-x|^{2d-3\alpha/2}} \, dy \]
\[ \leq 2^{d-\alpha/2+1} (1 + 2^{d-\alpha}) c_1 c_2^2 d \lambda(B) \int_1^\infty \frac{1}{r^{d-3\alpha/2+1}} \, dr, \]
where we used the fact that \( |y-x| \leq 2|y| \). The assertion follows as \( d > 3\alpha/2 \). \( \square \)

Lemma 5.6. There exists a constant \( \tilde{c} > 0 \) such that for all \( t > 0 \) large enough,
\[ \mathbb{E}[(\mathcal{V}_t)^4] \leq \tilde{c} t^2. \]

Proof. By setting \( n = 2 \) in eq. (2.33), we have
\[ \mathcal{V}_t = \lambda(S_{t/2}^{(1)}) + \lambda(S_{t/2}^{(2)}) - \lambda(S_{t/2}^{(1)} \cap S_{t/2}^{(2)}), \]
where \( S_{t/2}^{(1)} \) and \( S_{t/2}^{(2)} \) are independent, and have the same law as \( S_{t/2} \). Let \( \mathcal{V}_{t/2} = \lambda(S_{t/2}^{(i)}) \), for \( i = 1, 2 \). Taking expectation in the last equation and then subtracting the two relations yields
\[ \langle \mathcal{V}_t \rangle = \langle \mathcal{V}_{t/2}^{(1)} \rangle + \langle \mathcal{V}_{t/2}^{(2)} \rangle - \langle \lambda(S_{t/2}^{(1)} \cap S_{t/2}^{(2)}) \rangle. \]

By the triangle inequality,\(^2\)
\[ \|\langle \mathcal{V}_t \rangle\|_4 \leq \|\langle \mathcal{V}_{t/2}^{(1)} \rangle + \langle \mathcal{V}_{t/2}^{(2)} \rangle\|_4 + \|\langle \lambda(S_{t/2}^{(1)} \cap S_{t/2}^{(2)}) \rangle\|_4, \]
Jensen’s inequality and Lemma 2.3 imply that there is a constant \( c_1 > 0 \) such that
\[ \|\langle \lambda(S_{t/2}^{(1)} \cap S_{t/2}^{(2)}) \rangle\|_4 \leq \|\lambda(S_{t/2}^{(1)} \cap S_{t/2}^{(2)})\|_4 \leq 2\|\lambda(S_{t/2}^{(1)} \cap S_{t/2}^{(2)})\|_4 \]
\[ \leq 2\|\lambda(S_{t/2}^{(1)} \cap S_{t/2}^{(2)})\|_4 \leq c_1 h(t/2) \leq c_1 \sqrt{t}. \]
\[ ^2\text{For a random variable } Y \text{ we write } \|Y\|_p = (\mathbb{E}[|Y|^p])^{1/p}, \text{ for any } p \geq 1. \]
By the independence of the variables $\langle V_{t/2}^{(1)} \rangle$ and $\langle V_{t/2}^{(2)} \rangle$, we have
\[
E \left[ \left( \langle V_{t/2}^{(1)} \rangle + \langle V_{t/2}^{(2)} \rangle \right)^4 \right] = E \left[ \langle V_{t/2}^{(1)} \rangle^4 \right] + E \left[ \langle V_{t/2}^{(2)} \rangle^4 \right] + 6 E \left[ \langle V_{t/2}^{(1)} \rangle^2 \right] E \left[ \langle V_{t/2}^{(2)} \rangle^2 \right].
\]
By Lemma 2.5, there exists $N \in \mathbb{N}$ large enough such that $\text{Var}(V_t) \leq c_2 t$ for all $t \geq 2^N$ and some $c_2 > 0$. Hence, for $t \geq 2^{N+1}$,
\[
E \left[ \left( \langle V_{t/2}^{(1)} \rangle + \langle V_{t/2}^{(2)} \rangle \right)^4 \right] \leq E \left[ \langle V_{t/2}^{(1)} \rangle^4 \right] + E \left[ \langle V_{t/2}^{(2)} \rangle^4 \right] + 6 \left( \frac{c_2 t}{2} \right)^2.
\]
By combining this with the elementary inequality $(a + b)^{1/4} \leq a^{1/4} + b^{1/4}$, we arrive at
\[
\| \langle V_{t/2}^{(1)} \rangle + \langle V_{t/2}^{(2)} \rangle \|_4 \leq \left( E \left[ \langle V_{t/2}^{(1)} \rangle^4 \right] + E \left[ \langle V_{t/2}^{(2)} \rangle^4 \right] \right)^{1/4} + c_3 \sqrt{t}
\]
with $c_3 = (3c_2^2/2)^{1/4}$. From eqs. (5.9) to (5.11) it follows that there is $c_4 > 0$ such that
\[
\| \langle V_t \rangle \|_4 \leq \left( E \left[ \langle V_{t/2}^{(1)} \rangle^4 \right] + E \left[ \langle V_{t/2}^{(2)} \rangle^4 \right] \right)^{1/4} + c_4 \sqrt{t}
\]
For $k \geq N$ we set
\[
\gamma_k = \sup \{ \| \langle V_t \rangle \|_4 : 2^k \leq t < 2^{k+1} \}.
\]
Thus, for $k \geq N + 1$ and for every $2^k \leq t < 2^{k+1}$ we have
\[
\| \langle V_t \rangle \|_4 \leq (\gamma_{k-1} + \gamma_{k-1})^{1/4} + c_5 2^{k/2}
\]
with $c_5 = \sqrt{2} c_4$. Taking supremum over $2^k \leq t < 2^{k+1}$ yields
\[
\gamma_k \leq 2^{1/4} \gamma_{k-1} + c_5 2^{k/2}.
\]
We set $\delta_k = \gamma_k/2^{k/2}$ and we divide the last inequality by $2^{k/2}$. We thus have
\[
\delta_k \leq \frac{2^{1/4} \gamma_{k-1}}{2^{1/2} 2^{(k-1)/2}} + c_5 = 2^{-1/4} \delta_{k-1} + c_5.
\]
By iterating this inequality we finally conclude the result. □

**Lemma 5.7.** The following expansion is valid
\[
\text{Var}(V_t) = \sigma^2 t + O(t^{1/2} h(t)), \quad t \geq 1,
\]
where the function $h(t)$ is defined in eq. (2.7).

**Proof.** For every $s, t \geq 0$ we have
\[
\langle V_{s+t} \rangle = \lambda(\mathcal{S}_s \cup \mathcal{S}[s, s + t]) = \lambda(\mathcal{S}_s^{(1)}) + \lambda(\mathcal{S}_t^{(2)}) - \lambda(\mathcal{S}_s^{(1)} \cap \mathcal{S}_t^{(2)}).
\]
This implies
\[
\langle V_s^{(1)} \rangle + \langle V_t^{(2)} \rangle - \lambda(\mathcal{S}_s^{(1)} \cap \mathcal{S}_t^{(2)}) \leq \langle V_{s+t} \rangle \leq \langle V_s^{(1)} \rangle + \langle V_t^{(2)} \rangle,
\]
and whence
\[
\langle V_s^{(1)} \rangle + \langle V_t^{(2)} \rangle - \lambda(\mathcal{S}_s^{(1)} \cap \mathcal{S}_t^{(2)}) \leq \langle V_{s+t} \rangle \leq \langle V_s^{(1)} \rangle + \langle V_t^{(2)} \rangle + \mathbb{L} \left[ \lambda(\mathcal{S}_s^{(1)} \cap \mathcal{S}_t^{(2)}) \right].
\]
Here $\mathcal{S}_s^{(1)}$ and $\mathcal{S}_t^{(2)}$ are independent and have the same law as $\mathcal{S}_s$ and $\mathcal{S}_t$, respectively. We set $\mathcal{I}_t = \lambda(\mathcal{S}_t \cap \mathcal{S}_t')$. From the previous relation we obtain
\[
\| \langle V_{s+t} \rangle - (\langle V_s^{(1)} \rangle + \langle V_t^{(2)} \rangle) \|_2 \leq \mathbb{E} \left[ \mathcal{I}_{s+t} \right] + \mathbb{E} \left[ \mathcal{I}_{s+t} \right] \leq \mathcal{I}_{s+t} + \mathbb{E} \| \mathcal{I}_{s+t} \|_2.
\]
Hence
\[
(5.12) \quad \| \langle V_{s+t} \rangle - (\langle V_s^{(1)} \rangle + \langle V_t^{(2)} \rangle) \|_2 \leq 2 \| \mathcal{I}_{s+t} \|_2,
\]
and
\[
(5.13) \quad \| \langle V_{s+t} \rangle \|_2^2 \leq \| \langle V_s \rangle \|_2^2 + \| \langle V_t \rangle \|_2^2 + 4 \left[ (\| \langle V_s \rangle \|_2^2 + \| \langle V_t \rangle \|_2^2) \right]^{1/2} \| \mathcal{I}_{s+t} \|_2 + 4 \| \mathcal{I}_{s+t} \|_2^2.
\]
By eq. (2.6), there are $c_1 > 0$ and $t_1 > 1$ such that $\| I_t \|_2 \leq c_1 h(t)$ for $t \geq t_1$. For $t \in [1, t_1]$ we clearly have $I_t \leq I_{t_1}$. Thus, there is a constant $c_2 > 0$ such that,

$$\| I_t \|_2 \leq c_2 h(t), \quad t \geq 1.$$ 

Moreover, from eq. (2.9) we have that there exist $c_3 > 0$ and $t_2 > 1$ such that $\operatorname{Var}(V_t) \leq c_3 t$ for $t \geq t_2$, and for $t \in [1, t_2]$ we have $\operatorname{Var}(V_t) \leq \mathbb{E}[V_t^2] \leq \mathbb{E}[V_{t_2}^2] t$. Hence

$$\operatorname{Var}(V_t) \leq (c_3 + \mathbb{E}[V_{t_2}^2]) t, \quad t \geq 1.$$ 

We conclude that there is $c_4 > 0$ such that

$$\| \langle V_t \rangle \|_2 \leq c_4 \sqrt{t} \quad \text{and} \quad \| I_t \|_2 \leq c_4 h(t), \quad t \geq 1.$$ 

By eq. (5.13), we obtain

$$\| \langle V_{s+t} \rangle \|_2^2 \leq \| \langle V_s \rangle \|_2^2 + \| \langle V_t \rangle \|_2^2 + 4 c_5 \sqrt{s + t} h(s + t) + 4 c_5^2 (h(s + t))^2$$

$$\leq \| \langle V_s \rangle \|_2^2 + \| \langle V_t \rangle \|_2^2 + c_5 \sqrt{s + t} h(s + t),$$

for some constant $c_5 > 0$. Similarly as above, in view of eq. (5.12) we have

$$\| \langle V_s^{(1)} \rangle + \langle V_t^{(2)} \rangle \|_2 \leq \| \langle V_{s+t} \rangle \|_2 + \| \langle V_{s+t} \rangle - (\langle V_s^{(1)} \rangle + \langle V_t^{(2)} \rangle) \|_2$$

$$\leq \| \langle V_{s+t} \rangle \|_2 + 2 \| I_{s+t} \|_2,$$

which implies

$$\| \langle V_s \rangle \|_2^2 + \| \langle V_t \rangle \|_2^2 \leq \| \langle V_{s+t} \rangle \|_2^2 + 4 \| \langle V_{s+t} \rangle \|_2 \| I_{s+t} \|_2 + 4 \| I_{s+t} \|_2^2.$$ 

By eq. (5.14),

$$\| \langle V_s \rangle \|_2^2 + \| \langle V_t \rangle \|_2^2 \leq \| \langle V_{s+t} \rangle \|_2^2 + 4 c_5^2 \sqrt{s + t} h(s + t) + 4 c_5^2 (h(s + t))^2$$

$$\leq \| \langle V_{s+t} \rangle \|_2^2 + c_5 \sqrt{s + t} h(s + t).$$

We set

$$x_t = \operatorname{Var}(V_t) = \| \langle V_t \rangle \|_2^2 \quad \text{and} \quad b_t = c_5 \sqrt{t} h(t), \quad t > 0,$$

and we have shown that

$$x_s + x_t - b_{s+t} \leq x_{s+t} \leq x_s + x_t + b_{s+t}, \quad s, t \geq 1.$$ 

By Lemma 2.5 we know that

$$\lim_{t \to \infty} \frac{x_t}{t} = \sigma^2 > 0.$$ 

Take $s = t = 2^{k-1} r$ for $k \in \mathbb{N}$ and $r \in \mathbb{R}, r \geq 1$. We easily verify that

$$\left| \frac{x_{2^k r}}{2^k r} - \frac{x_{2^{k-1} r}}{2^{k-1} r} \right| \leq \frac{b_{2^k r}}{2^k r}, \quad k \in \mathbb{N}, \ r \geq 1.$$ 

Next, we observe that

$$\sum_{k=1}^{\infty} \left( \frac{x_{2^k r}}{2^k r} - \frac{x_{2^{k-1} r}}{2^{k-1} r} \right) = \lim_{N \to \infty} \sum_{k=1}^{N} \left( \frac{x_{2^k r}}{2^k r} - \frac{x_{2^{k-1} r}}{2^{k-1} r} \right) = \sigma^2 - \frac{x_r}{r}, \quad r \geq 1,$$

and whence

$$\left| \frac{x_t}{t} - \sigma^2 \right| = \left| \sum_{k=1}^{\infty} \left( \frac{x_{2^k t}}{2^k t} - \frac{x_{2^{k-1} t}}{2^{k-1} t} \right) \right| \leq \sum_{k=1}^{\infty} \frac{b_{2^k t}}{2^k t}, \quad t \geq 1.$$ 

This yields

$$\left| \frac{x_t}{t} - \sigma^2 \right| \leq \sum_{k=1}^{\infty} \frac{c_5 \sqrt{2^k t} h(2^k t)}{2^k t} \leq \frac{c_5}{\sqrt{t}} \sum_{k=1}^{\infty} \frac{h(2^k t)}{2^{k/2}}, \quad t \geq 1.$$
Case (i). For $\Delta \in (0, 1/2)$ we have
\[
\left| \frac{x_t}{t} - \sigma^2 \right| \leq \frac{c_5}{\sqrt{t}} \sum_{k=1}^{\infty} \frac{(2^k t)^{1/2 - \Delta}}{2^{k/2}} = \frac{c_5}{t^{1/2}} \sum_{k=1}^{\infty} (2^{-\Delta})^k = c_6 t^{-\Delta}, \quad t \geq 1,
\]
where $c_6 = c_5 \sum_{k=1}^{\infty} 2^{-\Delta k}$. It follows that
\[
|x_t - \sigma^2 t| \leq c_6 t^{1-\Delta} = c_6 t^{1/2} h(t), \quad t \geq 1.
\]

Case (ii). If $\Delta \geq 1/2$, then $h(t)$ is slowly varying. According to [1, Theorem 1.5.6] there is a constant $c_7 > 0$ such that $h(2^k t) \leq c_7 2^{k/4} h(t)$ for all $k \in \mathbb{N}$ and $t \geq 1$. We obtain
\[
\left| \frac{x_t}{t} - \sigma^2 \right| \leq \frac{c_5}{\sqrt{t}} \sum_{k=1}^{\infty} \frac{c_7 2^{k/4} h(t)}{2^{k/2}} = \frac{c_5 c_7 h(t)}{\sqrt{t}} \sum_{k=1}^{\infty} 2^{-k/4} = c_8 t^{-1/2} h(t), \quad t \geq 1,
\]
with $c_8 = c_5 c_7 \sum_{k=1}^{\infty} 2^{-k/4}$, and the proof is finished. \(\square\)

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(Wojciech Cygan) Institut für Mathematische Stochastik, Technische Universität Dresden, Dresden, Germany & Instytut Matematyczny, Uniwersytet Wrocławski, Wrocław, Poland
E-mail address: wojciech.cygan@uwr.edu.pl

(Nikola Sandrić) Department of Mathematics, University of Zagreb, Zagreb, Croatia
E-mail address: nsandric@math.hr

(Stjepan Šebek) Institute of Discrete Mathematics, Graz University of Technology, Graz, Austria & Department of Applied Mathematics, Faculty of Electrical Engineering and Computing, University of Zagreb, Zagreb, Croatia
E-mail address: stjepan.sebek@fer.hr