Locality and universality of quantum memory effects

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The modeling and analysis of the dynamics of complex systems often requires to employ non-Markovian stochastic processes. While there is a clear and well-established mathematical definition for non-Markovianity in the case of classical systems, the extension to the quantum regime recently caused a vivid debate, leading to many different proposals for the characterization and quantification of memory effects in the dynamics of open quantum systems. Here, we derive a mathematical representation for the non-Markovianity of quantum processes based on the exchange of information between the open system and its environment, which reveals the locality and universality of non-Markovianity in the quantum state space and substantially simplifies its numerical and experimental determination. We further illustrate the application of this representation by means of an all-optical experiment which allows the measurement of the degree of memory effects in a photonic quantum process with high accuracy.

In recent years the problem of characterizing non-Markovian dynamics in the quantum regime has initiated an intense debate. A series of diverse definitions along with measures of quantum memory effects have been proposed, invoking many different mathematical and physical concepts and techniques. Examples are characterizations of non-Markovianity in terms of deviations from a Lindblad semigroup, of the divisibility of the dynamical map, of the dynamics of entanglement and correlations with an ancilla system, and of the Fisher information.

In this work we focus on the measure of non-Markovianity introduced in Refs. [5, 6] which characterizes non-Markovianity in terms of the information exchanged between an open system and its environment. To quantify this information one uses the trace distance $D(\rho_1, \rho_2) = \frac{1}{2} \text{Tr} |\rho_1 - \rho_2|$ between two quantum states represented by density matrices $\rho_1$ and $\rho_2$, which provides a measure for the distinguishability of these states. The dynamics of an open quantum system can be described formally by means of a dynamical map $\Phi$, which maps any initial state $\rho(0)$ to its time-evolved state $\rho(t) = \Phi(t)\rho(0)$ at time $t$. The time evolution over some (finite or infinite) time interval $[0, T]$ is then described by the one-parameter family $\Phi = \{\Phi_t | 0 \leq t \leq T, \Phi_0 = I\}$ of dynamical maps. A quantum process given by such a family of dynamical maps is said to be Markovian if for all pairs of initial states the trace distance $D(\rho_1(t), \rho_2(t))$ is a monotonically decreasing function of time. In physical terms this means that the distinguishability between the states decreases monotonically which can be interpreted as a continuous flow of information from the open system to the environment. Conversely, a quantum process is non-Markovian if there is an initial pair of states for which the trace distance increases over certain time intervals. During these time intervals the distinguishability of the states thus increases, which corresponds to a flow of information from the environment back into the open system signifying quantum memory effects.

The above characterization of quantum non-Markovianity leads to a measure for the degree of non-Markovianity which is defined by

$$\mathcal{N}(\Phi) = \max_{\rho_1 \neq \rho_2} \int_{t > 0} dt \sigma(t, \rho_1, \rho_2),$$

where

$$\sigma(t, \rho_1, \rho_2) \equiv \frac{d}{dt} D(\Phi_t(\rho_1), \Phi_t(\rho_2))$$

and

$$\mathcal{N}(\Phi) = \frac{1}{2} \text{Tr} |\rho(0) - \rho(T)|$$

for $\rho(0)$ and $\rho(T)$ being the initial and final states, respectively. This measure captures the degree to which the information exchange between the open system and its environment is non-Markovian.
denotes the time derivative of the trace distance between the pair of states at time \( t \). In Eq. (1) the time integral is extended over all time intervals in which this derivative is positive, and the maximum is taken over all pairs of orthogonal initial states \( \rho_1 \perp \rho_2 \). Thus, the measure accumulates the total increase of the distinguishability occurring in the time evolution of two initially orthogonal quantum states. We recall that two quantum states \( \rho_1 \) and \( \rho_2 \) are said to be orthogonal if their supports, i.e. the subspaces spanned by their nonzero eigenvalues, are orthogonal which is equivalent to \( \mathcal{D}(\rho_1, \rho_2) = 1 \). This implies that optimal state pairs exhibiting a maximal backflow of information during their time evolution are initially distinguishable with certainty, and thus represent a maximal initial information content.

This measure for non-Markovianity was originally defined in Ref. [5] in terms of a maximization over all pairs of quantum states. However, as demonstrated in Ref. [11] the maximization can be restricted to pairs of orthogonal initial states. Although this result greatly simplifies the mathematical representation of the non-Markovianity measure, its determination still requires the maximization over pairs of quantum states. Here, we derive a much simpler Markovianity measure, its determination still requires the maximization over pairs of orthogonal initial states. We recall that two quantum states occurring in the time evolution of two initially orthogonal quantum states at time \( t \) denotes the time derivative of the trace distance between the pair of states. Since any state from the interior of the state space has an enclosing surface. For example, since any fixed point \( \rho_0 \in \mathcal{S}(\mathcal{H}) \) which encloses \( \rho_0 \) in all directions of state space, i.e. the set of all nonzero, Hermitian and traceless operators on \( \mathcal{H} \). Considering any fixed reference state \( \rho_0 \in \mathcal{S}(\mathcal{H}) \) we can now introduce a particular class of subsets of the state space: A set \( H(\rho_0) \subset \mathcal{S}(\mathcal{H}) \) containing \( \rho_0 \) is called an enclosing surface of \( \rho_0 \) if and only if for any operator \( A \in \mathcal{E}(\mathcal{H}) \) there exists a real number \( \lambda > 0 \) such that

\[
\rho_0 + \lambda A \in \partial U(\rho_0).
\]

Note that by definition \( \rho_0 \) itself is not contained in \( \partial U(\rho_0) \) and that the full set \( \partial U(\rho_0) \) is part of the state space. It can be easily seen that any state from the interior of the state space has an enclosing surface. For example, since \( \rho_0 \) is an interior point of the state space there is an \( \epsilon > 0 \) such that the set of states \( \rho \) defined by \( \mathcal{D}(\rho, \rho_0) = \epsilon \) represents a spherical enclosing surface with center \( \rho_0 \). However, an enclosing surface \( \partial U(\rho_0) \) can have an arbitrary geometrical shape, the only requirement being that it encloses the reference state in all directions of state space. An example is shown in Fig. 1(a). Using these definitions, we can now state our central result.

**Theorem.** Let \( \rho_0 \in \mathcal{S}(\mathcal{H}) \) be any fixed state of the interior of the state space and \( \partial U(\rho_0) \) an arbitrary enclosing surface of \( \rho_0 \). For any dynamical process \( \Phi \), the measure for quantum non-Markovianity defined by Eq. (1) is then given by

\[
\mathcal{N}(\Phi) = \max_{\rho \in \partial U(\rho_0)} \int_{t > 0} d\tau \sigma(t, \rho, \rho_0),
\]

where

\[
\sigma(t, \rho, \rho_0) = \frac{\mathcal{D}(\Phi(t)\rho, \Phi(t)\rho_0)}{\mathcal{D}(\rho, \rho_0)}
\]

is the derivative of the trace distance at time \( t \) divided by the initial trace distance.

**Proof.** Let \( \rho \in \partial U(\rho_0) \). Applying the Jordan-Hahn decomposition of the operator \( \rho - \rho_0 \) one concludes that there exists an orthogonal pair of states \( \rho_1 \) and \( \rho_2 \) such that

\[
\rho_1 - \rho_2 = \frac{\rho - \rho_0}{\mathcal{D}(\rho, \rho_0)}.
\]

and, hence, we have

\[
\mathcal{D}(\Phi(t)\rho_1, \Phi(t)\rho_2) = \frac{\mathcal{D}(\Phi(t)\rho, \Phi(t)\rho_0)}{\mathcal{D}(\rho, \rho_0)},
\]

by the linearity of the dynamical maps and the homogeneity of the trace distance. This shows that \( \sigma(t, \rho_1, \rho_2) = \sigma(t, \rho, \rho_0) \). It follows that the right-hand side of Eq. (5) is smaller than or equal to \( \mathcal{N}(\Phi) \) as defined by Eq. (1). Conversely, suppose \( \rho_1, \rho_2 \) are two orthogonal states. Since \( \rho_1 - \rho_2 \in \mathcal{E}(\mathcal{H}) \), there exists \( \lambda > 0 \) such that \( \rho = \rho_0 + \lambda (\rho_1 - \rho_2) \in U(\rho_0) \) by definition of an enclosing surface. Thus, one obtains \( \rho_1 - \rho_2 = (\rho - \rho_0)/\lambda \). Since \( \rho_1 \perp \rho_2 \) we find \( \mathcal{D}(\rho_1, \rho_2) = \mathcal{D}(\rho, \rho_0) \) and, hence, \( \lambda = \mathcal{D}(\rho_1, \rho_2) \). Thus, we are again led to Eq. (7) and to \( \sigma(t, \rho_1, \rho_2) = \sigma(t, \rho, \rho_0) \). This shows that the measure \( \mathcal{N}(\Phi) \) as defined by Eq. (1) is smaller than or equal to the right-hand side of Eq. (5) which thus concludes the proof.

The statement of the theorem can be easily understood for the case of a qubit, representing the corresponding state space by means of the Bloch ball. The representation (1) requires to perform a maximization over all pairs of orthogonal quantum states, i.e. over all pairs of antipodal points on the surface of the Bloch ball. The theorem states that this maximization can also be carried out in the following equivalent way. Choose a fixed point \( \rho_0 \) in the interior of the Bloch ball and a small surface \( \partial U(\rho_0) \) which encloses \( \rho_0 \) and take the maximum over all points \( \rho \in \partial U(\rho_0) \). To see the equivalence, consider the straight line joining \( \rho_0 \) with \( \rho \). This line can be moved by parallel translation (without changing the trace distance) in such a way that it is centered at the origin of the Bloch ball (maximally mixed state). Stretching the line by an appropriate scaling factor (which is equal to the inverse of the trace distance between \( \rho_0 \) and \( \rho \)) its endpoints become antipodal points of the Bloch ball. Conversely, any pair of antipodal points on the surface of the Bloch ball corresponds in this
way to a point \( \rho \) belonging to \( \partial U(\rho_0) \), which leads to the theorem for the particular case of a qubit.

**Experimental realization.** We have used Eq. (5) to develop an experimental scheme for the determination of the degree of non-Markovianity in a photonic quantum process. In the experiment the open quantum system is given by the polarization degree of freedom of a single photon coupled to the frequency degree of freedom representing the environment. While in many system-environment models the system and its environment are represented by different physical entities, the present situation in which system and environment are formed by different degrees of freedom of the same particle is not uncommon in the theory of open systems. A typical example in this context are experiments on trapped ions where also the internal electronic degree of freedom couples to the motional degree of freedom of the ion\(^{12}\). In our experiment the decoherence of the polarization degree of freedom is due to birefringent quartz plates in the optical path of the photon which induce a coupling between the polarization and frequency degrees of freedom and lead to dephasing of superpositions of vertical and horizontal polarization states. This dephasing strongly depends on the structure of the frequency spectrum which can be efficiently controlled by the tilt angle of a Fabry-Pérot cavity, producing a bimodal spectrum. Hence, the non-

![Figure 2](image) | Experimental setup. Key to the components: HWP – half-wave plate, QWP – quarter-wave plate, FP – Fabry-Pérot cavity, IF – interference filter, QP – quartz plate, (P)BS – (polarizing) beamsplitter, SPD – single photon detector.

![Figure 3](image) | Experimental results for the increase of the trace distance between 175\( \lambda \) and 318\( \lambda \) for \( A_\rho = 0.64 \) for states on the enclosing surface of reference state \( \rho_0^1 \) (a), \( \rho_0^2 \) (b) and pairs of orthogonal states (c). The corresponding \( \phi_{loc} \)-averaged increase with respect to local spherical coordinates is shown in (d), (e) and (f). Error bars show the standard deviations.

![Figure 4](image) | The same as Fig. 3 for \( A_\rho = 0.22 \).

Markovianity in our experiment is due to the presence of a structured environment which is a typical cause for memory effects.

The experimental setup is depicted in Fig. 2. With the help of a frequency doubler a mode-locked Ti:sapphire laser (central wavelength 780 nm) is used to pump two 1 mm thick BBO crystals to generate the maximally entangled two-qubit state \( \frac{1}{\sqrt{2}} (|H, V\rangle - |V, H\rangle) \). The cavity and a partial reflecting coating, with approximately 80% reflectivity at 780 nm, serves as a Fabry-Pérot cavity (FP) which can be tilted to generate different dynamical behavior\(^{14}\). A fused silica plate (0.1 mm thick and coated with a partial reflecting coating, with approximately 80% reflectivity at 780 nm) is used to form a Fabry-Pérot cavity (FP) which in addition can be tilted to generate different dynamical behaviour\(^{14}\). A fused silica plate (0.1 mm thick and coated with a partial reflecting coating, with approximately 80% reflectivity at 780 nm) serves as a Fabry-Pérot cavity (FP) which in addition can be tilted to generate different dynamical behavior\(^{14}\). The cavity and a consecutively placed interference filter (IF) (FWHM about 3 nm) single out two peaks near 780 nm of width \( \sigma \approx 7.7 \times 10^{11} \) Hz each which are separated by \( \Delta \theta \approx 7.2 \times 10^{12} \) Hz. The relative amplitude \( A_\rho \) of the two peaks depends strongly on the tilt angle \( \alpha \) whereas the other quantities are almost constant. A polarizing beamsplitter (PBS) together with a half-wave plate (HWP) and a quarter-wave plate (QWP) are used as a photon state analyzer\(^{15}\).

Photon 1 is directly detected in a single photon detector at the end of arm 1 as a trigger for photon 2. The optical setup in part \( a, b \) and \( c \) (see Fig. 2) is used to prepare arbitrary quantum states of photon 2 needed for the sampling process\(^{16}\). This set-up conveniently allows to prepare any single pure photon polarization state (in arm 2) and reference states (\( 2a \) along with \( 2b \) together with arbitrary etching surfaces which can be controlled by changing the relative amplitudes of the attenuators built in in each arm. The path difference between each arm is about 25 mm to ensure that the mixture of the three parts is classical.

After the preparation photon 2 passes through birefringent quartz plates of variable thickness which couple the polarization and frequency degree of freedom and lead to the decoherence of superpositions of polarization states. The birefringence is given by \( \Delta n = (8.9 \times 10^{-7}) \) at 780 nm. The thickness of the quartz plates simulating different evolution times ranges from 75\( \lambda \) to 318\( \lambda \) in units of the central wavelength of the FP cavity.

Employing the Bloch vector representation, the set of polarization states can be conveniently parametrized by means of spherical coordinates.

![Figure 5](image) | The same as Fig. 3 for \( A_\rho = 0.01 \).
The quantum non-Markovianity measure for the three dynamics obtained from the experimental data in comparison to the theoretical value

| $A_e$ | $N_{\text{theo}}$ | $N_{(\alpha)}$ | $N_{(\beta)}$ | $N_{(\gamma)}$ |
|-------|-----------------|----------------|---------------|---------------|
| 0.64  | 0.59            | $0.59 \pm 0.01$| $0.59 \pm 0.02$| $0.59 \pm 0.02$|
| 0.22  | 0.21            | $0.21 \pm 0.01$| $0.21 \pm 0.02$| $0.21 \pm 0.02$|
| 0.01  | 0               | $0.001 \pm 0.013$| $-0.005 \pm 0.008$| $-0.0002 \pm 0.0015$|


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Author contributions
S.W., A.K., I.P. and H.-P.B. developed the theory. B.-H.L., X.-M.H., C.Z., Y.-F.H., C.-F.L. and G.-C.G. devised and performed the experiment. B.-H.L. and S.W. analyzed the data. All authors contributed to the discussion and interpretation of the results and to the preparation of the manuscript.

Additional information
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