A BLOW-UP RESULT FOR THE CHEMOTAXIS SYSTEM WITH NONLINEAR SIGNAL PRODUCTION AND LOGISTIC SOURCE

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Abstract. In this paper we consider the following chemotaxis-growth system with nonlinear signal production and logistic source

\[
\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + \lambda u - \mu u^\alpha, & x &\in \Omega, t > 0, \\
    0 &= \Delta v - \mu(t) + f(u), & x &\in \Omega, t > 0,
\end{align*}
\]

with homogeneous Neumann boundary conditions in the ball \( \Omega = B_R(0) \subset \mathbb{R}^n \) for \( n \geq 1 \) and \( R > 0 \), where \( \chi, \lambda, \mu > 0, \alpha > 1 \), and \( f \) is an appropriate regular function satisfying \( f(u) \geq ku^\kappa \) for all \( u \geq 1, \kappa > 0 \) with some constants \( k > 0 \).

If the number \( \kappa \) and \( \alpha \) satisfy

\[
\alpha + 1 > \alpha \left( \frac{2}{n} + 1 \right),
\]

then there exists appropriate initial data such that the corresponding solution \((u, v)\) of the system blow up in finite time. This result extends the blow-up result of the chemotaxis model without logistic cell kinetics in [45]. Apparently, for the case \( \kappa = 1 \), this provides a rigorous detection for blow-up of solution in spaces-dimensions three and four.

1. Introduction. Chemotaxis is the oriented movement of biological cells or organisms toward concentration gradients of the chemical signal substance. The classical mathematical model was initially proposed by Keller and Segel [16] to describe the aggregation of cellular slime molds, which is widely known as the chemotaxis model

\[
\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v), & x &\in \Omega, t > 0, \\
    v_t &= \Delta v - v + u, & x &\in \Omega, t > 0, \\
    \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, & x &\in \partial \Omega, t > 0, \\
    u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), & x &\in \Omega
\end{align*}
\]

in a bounded domain \( \Omega \subset \mathbb{R}^n \), where \( u = u(x, t) \) and \( v = v(x, t) \) denote the cell density and the concentration of the chemical, respectively. After this pioneering work, many authors study the chemotaxis models from the mathematical biology viewpoint, and the main issue is to determine when blow-up occurs and whether there is a global in-time bounded solution. Such as it was shown that the solutions
of problem (1) are global and bounded if \( n = 1 \) \([31]\) or \( n = 2 \) and \( \int_{\Omega} u_0 dx < \frac{4\pi}{n} \) \([30, 7]\), whereas the solutions may blow up in finite time in the case \( n = 2 \) and \( \int_{\Omega} u_0 dx > \frac{4\pi}{n} \) \([12, 32]\) or in the higher dimensional case \( n \geq 3 \) \([40, 43]\). As the chemicals diffuse much faster than cells, a parabolic-elliptic system was derived

\[
\begin{align*}
  u_t &= \Delta u - \chi \nabla \cdot (u \nabla v), & x \in \Omega, & t > 0, \\
  0 &= \Delta v - v + u, & x \in \Omega, & t > 0.
\end{align*}
\]

It is known that the explosion phenomena may occur in the higher-dimensional case (see \([28, 29]\)). With regard to more variants of (2), we would like to mention papers \([14, 5, 47, 17, 33, 11]\) for interested readers.

In many biological processes, we consider the proliferation and death of cells, from which one derived the related chemotaxis-growth model

\[
\begin{align*}
  u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + g(u), & x \in \Omega, & t > 0, \\
  0 &= \Delta v - v + u, & x \in \Omega, & t > 0
\end{align*}
\]

for a prototypical logistic growth term \( g(s) = \lambda s - \mu s^\alpha, \ s \geq 0 \) with \( \lambda \geq 0, \ \mu > 0, \ \alpha > 1 \). The literature for this model focus on understanding how the logistic source affects the behavior of solutions. Intuitively, the superlinear degrading term \( \mu s^\alpha \) should somewhat decrease the possibility of blow-up. On the one hand, a large number of studies indicated that the boundedness of solutions may be ensured by the strong logistic damping (i.e., \( \mu \) is suitably large), especially in the higher-dimensional setting. More precisely, when \( \alpha = 2 \), it is shown that for all suitably regular initial data, problem (3) has a unique global bounded classical solution if \( \mu > \max\{0, \frac{n-2}{2} \chi\} \) \([36]\). Recently, in \([38, 51, 50]\), this global existence and boundedness results have been extended to a more generalized nonlinear diffusion case. On the other hand, such global solutions to problem (3) are also known to exist for all suitably regular initial data \( u_0 \) and suitable large \( \alpha \) \([8, 3]\). For the parabolic-parabolic case (3), the results on the global existence and boundedness of the classical solutions can be referred to \([39, 24, 48, 9]\). Furthermore, turning to the nonlinear cell diffusion, many results about the question whether the solutions are bounded or blow up can be found in \([37, 21, 2, 49, 30, 7, 1, 4, 13, 20, 34, 35, 41]\).

However, “logistic source” does not always prevent chemotactic collapse. For example, such logistic source was shown in \([46]\) to assert the chemotactic collapse of solutions when \( \alpha = 2 \) in one-dimensional case, and we can also refer to \([19, 15]\) in the higher-dimensional setting. Recently, Winkler \([44]\) obtained a condition on initial data to ensure the occurrence of finite-time blow-up in (3) for

\[
\alpha < \begin{cases} 
\frac{7}{6}, & \text{if } n \in \{3, 4\}, \\
1 + \frac{1}{(n-1)}, & \text{if } n \geq 5.
\end{cases}
\]

Replacing the second equation in (3) by \( 0 = \Delta v - \mu(t) + u \) with the spatially constant average \( \mu(t) := \frac{1}{|\Omega|} \int_{\Omega} u(\cdot, t) dx \), the detection of explosion phenomena for corresponding initial-boundary value problem could be accomplished only in high-dimensional domain. Accurately, when \( 1 < \alpha < \frac{1}{2} + \frac{1}{2(n-2)} \) with \( n \geq 5 \), the chemotactic collapse need not be ruled out by any logistic-type growth term \([42]\). Later on, the results in \([51, 25]\) extended this blow-up arguments to more general quasilinear case with \( n \geq 5 \). Recently, Fuest in \([6]\) studied that the condition \( \int_{\Omega} u_0 dx > 8\pi \) ensures finite-time blow-up for \( n = 2 \). We note that in those papers the process of signal production through cells depends on the population density in a linear manner. Recently, the following chemotaxis system with nonlinear signal
production
\[
\begin{cases}
    u_t = \Delta u - \chi \nabla \cdot (u \nabla v), & x \in \Omega, t > 0, \\
    0 = \Delta v - \mu(t) + f(u), & x \in \Omega, t > 0, \\
    \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\
    u(x, 0) = u_0(x), & x \in \Omega,
\end{cases}
\]
was studied by Winkler in [45], where \( f(u) = u^\kappa \) with \( u \geq 0, \kappa > 0 \), which asserted that the radially symmetric solution blows up in finite time if the number meets \( \kappa > \frac{2}{n} \). Moreover, the author proved that this assumption on \( \kappa \) is essentially optimal in the sense that, if \( \kappa < \frac{2}{n} \), there exists widely initial data such that the corresponding solution is global and bounded. Recently, Li [22] generalized the results of [45] to the quasilinear case. For more related works about nonlinear signal production one can see [10, 26, 27].

Based on the above results, we note that there are few results for the chemotaxis system with nonlinear signal production and logistic source at the same time. Hence, the purpose of this paper is to explore the interaction between superlinear signal production and logistic source on the solution behavior of the following chemotaxis system
\[
\begin{cases}
    u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \lambda u - \mu u^\alpha, & x \in \Omega, t > 0, \\
    0 = \Delta v - \mu(t) + f(u), & x \in \Omega, t > 0, \\
    \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\
    u(x, 0) = u_0(x), & x \in \Omega,
\end{cases}
\]
in the ball \( \Omega := B_R(0) \subset \mathbb{R}^n \) for \( n \geq 1 \) and \( R > 0 \), where the coefficients \( \chi, \lambda, \mu \) are positive and \( \alpha > 1 \), and initial data satisfies
\[
\begin{align*}
    &u_0 \in \bigcup_{\varphi \in (0, 1)} C^0(\overline{\Omega}) \text{ is nonnegative, radially symmetric} \\
    &\text{and nonincreasing with respect to } |x|
\end{align*}
\]
as well as
\[
\begin{align*}
    &f \in \bigcup_{\varphi \in (0, 1)} C^0_{\text{loc}}([0, +\infty)) \cap C^1((0, +\infty)) \text{ is nonnegative and nondecreasing} \\
    &f(s) \geq ks^\kappa \text{ for all } s \geq 1, \kappa > 0
\end{align*}
\]
with some \( k > 0 \).

In this paper, inspired by the pioneering works in [5, 42, 44, 45], we will show that there exists suitable initial data such that the corresponding solutions blow up in finite time when \( \kappa + 1 > \alpha(\frac{2}{n} + 1) \) for any \( n \geq 1 \). Specially, when \( \kappa = 1 \), the result suggests that finite-time blow-up does occur not only in higher dimensions (\( n \geq 5 \)) [51, 42, 25] but also for \( n = 3, 4 \) to the model (5). In order to prove our main result, we first make some substitutions of solution pair, which transform model (5) into the corresponding the Dirichlet problem, and derive a superlinearly forced ODI for \( \phi \) defined by (37) (see Lemma 4.11). Unlike the situation in [45], the presence of the logistic death term in this paper gives rise to considerable mathematical difficulties. The author derived the pointwise inequality of \( w \) [45, Lemma 3.5] by estimating the crucial integral \( \int_0^s f(nw_x(\sigma, t))d\sigma \) therein (e.g. via the Hölder inequality). However, we must establish a suitable relationship between the cross-diffusive and the logistic death term (see (60)) to overcome the technical analysis. Hence, we take a slightly
different approach to achieve an upper bound for the function \( w \) (Lemma 4.6 below) by some differential inequality skills.

In order to precisely formulate our main result, we first state one result concerning local existence, uniqueness and extensibility of classical solutions to problem (5), which can be proved by the fixed point frameworks, we refer the readers to [5, 28].

**Proposition 1.** Let \( \Omega = B_R(0) \subset \mathbb{R}^n \) with \( n \geq 1 \), \( R > 0 \), and \( \chi, \lambda, \mu \) are positive and \( \alpha > 1 \). Suppose that \( u_0 \) and \( f \) fulfill (6) and (7). Then there exist \( T_{\max} \in (0, +\infty) \) and for each \( T \in (0, T_{\max}) \) uniquely determined radially symmetric nonnegative functions

\[
\begin{aligned}
&u \in C^0(\bar{\Omega} \times [0, T]) \cap C^{2,1}(\bar{\Omega} \times (0, T)), \\
v \in \cap_{q>n} L^\infty((0, T); W^{1,q}(\Omega)) \cap C^2(\Omega \times (0, T)),
\end{aligned}
\]

such that \((u, v)\) forms a classical solution of (5) in \( \Omega \times (0, T_{\max}) \), and that

\[
\text{if } T_{\max} < +\infty, \text{ then } \limsup_{t \uparrow T_{\max}} ||u(\cdot, t)||_{L^\infty(\Omega)} = +\infty.\]

Our main result on blow-up reads as follows.

**Theorem 1.1.** Let \( \Omega = B_R(0) \subset \mathbb{R}^n \) with \( n \geq 1 \) and \( R > 0 \). Assume that \( u_0 \) fulfills (6) and the function \( f \) satisfies (7) and (8). Let \( \chi, \lambda, \mu > 0 \) and \( \alpha > 1 \) be such that

\[
\kappa + 1 > \alpha \left( \frac{2}{n} + 1 \right).
\]

Then for any \( m > \tilde{C} \) defined in Lemma 4.1, there exist \( \epsilon = \epsilon(k, \kappa, \alpha, m, R) \in (0, m) \) and \( r_* = r_*(k, \kappa, \alpha, m, R) \in (0, R) \) such that

\[
\int_{\Omega} u_0 dx = m \quad \text{but} \quad \int_{B_{r_*(0)}} u_0 dx \geq m - \epsilon,
\]

the corresponding solution \((u, v)\) of (5) blows up in finite time.

**Remark 1.** If \( \lambda = \mu \) and \( \alpha = 1 \) in (5), (11) becomes to \( \kappa > \frac{2}{n} \), so Theorem 1.1 is consistent with the result of [45].

**Remark 2.** For the case \( \kappa = 1 \), the blow-up results to (5) in [51, 42, 25] have been obtained under the condition \( 1 < \alpha < \frac{3}{2} + \frac{1}{2n-2} \) with \( n \geq 5 \). However, when \( \kappa = 1 \), (11) indicates that \( 1 < \alpha < \frac{2n}{n+2} \) with \( n \geq 3 \), which apparently provides a rigorous detection of blow-up result for \( n = 3, 4 \) in the Keller-Segel chemotaxis model with logistic cell kinetics and linear signal production. It should also be point out that, for the sufficient large \( n \), the upper bound of \( \alpha \) is closer to 2.

Theorem 1.1 leaves some challenging questions for the chemotaxis-growth model (5):

**(Q1)** Whether the condition (11) is optimal for the blow-up result to problem (5).

**(Q2)** What happens for small initial data \( m < \tilde{C} \), boundedness or blow-up?

This paper is organized as follows. In Section 2, we will transform (5) into a Dirichlet problem for the scalar parabolic equation. In Section 3, we prove the concavity of \( w \) defined in (14). In Section 4, according to the previous preparation, we can establish an autonomous superlinear differential inequality for \( \phi \) given in (37) and prove Theorem 1.1.
Combining (14) and (16), the first equation in (5) shows that

$$B_2.$$ Transformation to a scalar equation.

as well as

and

asserted in Proposition 1. Then system (5) becomes

$$\begin{cases}
  u_t = r^{1-n}(r^{n-1}u_r)_r - \chi r^{1-n}(r^{n-1}uv)_r + \lambda u - \mu u^n, & r \in (0, R), t > 0, \\
  0 = r^{1-n}(r^{n-1}v_r)_r - \mu(t) + f(u), & r \in (0, R), t > 0, \\
  u_r = v_r = 0, & r = R, t > 0, \\
  u(r, 0) = u_0(r), & r \in (0, R).
\end{cases}$$

(13)

Here, we set

$$w(s, t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1}u(\rho, t)d\rho, \quad s \in [0, R^n], \quad t \in [0, T_{\text{max}})$$

and

$$z(s, t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1}v(\rho, t)d\rho, \quad s \in [0, R^n], \quad t \in [0, T_{\text{max}}).$$

(15)

Then after detailed calculation, we obtain

$$w_s(s, t) = \frac{1}{n}u(s^{\frac{1}{n}}, t) \quad \text{and} \quad w_{ss}(s, t) = \frac{1}{n^2}s^{\frac{1}{n}-1}u_r(s^{\frac{1}{n}}, t), \quad s \in (0, R^n), \quad t \in (0, T_{\text{max}})$$

(16)

as well as

$$z_s(s, t) = \frac{1}{n}v(s^{\frac{1}{n}}, t) \quad \text{and} \quad z_{ss}(s, t) = \frac{1}{n^2}s^{\frac{1}{n}-1}v_r(s^{\frac{1}{n}}, t), \quad s \in (0, R^n), \quad t \in (0, T_{\text{max}}).$$

(17)

Thus $w$ fulfills

$$w_t(s, t) = \int_0^{s^{\frac{1}{n}}} \rho^{n-1}u_t(\rho, t)d\rho$$

$$= \int_0^{s^{\frac{1}{n}}} (\rho^{n-1}u_r)_r(\rho, t)d\rho - \chi \int_0^{s^{\frac{1}{n}}} (\rho^{n-1}uv_r)_r(\rho, t)d\rho$$

$$+ \lambda \int_0^{s^{\frac{1}{n}}} \rho^{n-1}u(\rho, t)d\rho - \mu \int_0^{s^{\frac{1}{n}}} \rho^{n-1}u^n(\rho, t)d\rho$$

$$= s^{1-\frac{1}{n}}u_r(s^{\frac{1}{n}}, t) - \chi u(s^{\frac{1}{n}}, t) \cdot \left( \frac{\mu(t)s}{n} - \int_0^{s^{\frac{1}{n}}} \rho^{n-1}f(u(\rho, t))d\rho \right)$$

$$+ \lambda \int_0^{s^{\frac{1}{n}}} \rho^{n-1}u(\rho, t)d\rho - \mu \int_0^{s^{\frac{1}{n}}} \rho^{n-1}u^n(\rho, t)d\rho.$$

Combining (14) and (16), the first equation in (5) shows that

$$w_t = n^2s^{2-\frac{2}{n}}w_{ss} - \chi \mu(t)s w_s + \lambda w$$

$$+ \chi w_s \int_0^{s} f(nw_s(\sigma, t))d\sigma - \mu n^{\alpha-1} \int_0^{s} w_{ss}(\sigma, t)d\sigma$$

(18)

for all $s \in (0, R^n)$ and $t \in (0, T_{\text{max}})$ with the properties that

$$0 = w(0, t) \leq w(s, t) \leq w(R^n, t) = \frac{1}{\omega_n} \int_{\Omega} u(\cdot, t)$$

(19)
as well as
\[ w_s(s, t) \geq 0 \]  
(20)
for all \( s \in (0, R^n) \) and \( t \in (0, T_{\text{max}}) \). Using the definition of \( \mu(t) \) and (16), we have
\[ \mu(t) = \frac{1}{|\Omega|} \int_{\Omega} f(u(x, t)) \, dx = \frac{1}{R^n} \int_0^{R^n} f(nw_s(s, t)) \, ds, \quad t \in (0, T_{\text{max}}). \]  
(21)
For simplicity, throughout the paper we use the abbreviation \( \omega_n := n |B_1(0)|. \)

3. **Concavity of \( w \).** In this section, we show that \( w(\cdot, t) \) actually remains concave throughout evolution as a consequence of (6). As a preparation for this, we first derive some estimates for \( v \) which solves the elliptic equation with zero-flux boundary condition. Since the proof of the following Lemma is in [45, Lemma 2.1], we only provide the statements of the Lemma.

**Lemma 3.1.** Let \( n \geq 1, R > 0, \lambda > 0, \mu > 0 \) and \( \alpha > 1 \). Suppose that (6) and (7) hold. Then
\[ v_r(r, t) = \frac{1}{n} \mu(t) r - r^{n-1} \int_0^r \rho^{n-1} f(u(\rho, t)) \, d\rho \quad \text{for all } r \in (0, R) \text{ and } t \in (0, T_{\text{max}}). \]  
(22)
In particular,
\[ v_r(r, t) \leq \frac{1}{n} \mu(t) r \quad \text{for all } r \in (0, R) \text{ and } t \in (0, T_{\text{max}}). \]  
(23)

**Lemma 3.2.** Let \( n \geq 1, R > 0, \lambda > 0, \mu > 0 \) and \( \alpha > 1 \). Assume (6) and (7). Then
\[ u_r(r, t) \leq 0 \quad \text{for all } r \in (0, R) \text{ and } t \in (0, T_{\text{max}}) \]  
(24)
as well as
\[ w_{ss}(s, t) \leq 0 \quad \text{for all } s \in (0, R^n) \text{ and } t \in (0, T_{\text{max}}) \]  
(25)
with \( w \) as in (14).

**Proof.** In view of (16), we only need to establish (24). Next, we will give the proof by dividing it into two steps.

**Step 1.** We first verify (24) under the assumption that beyond (6) and (7), in addition \( f \) and \( u_0 \) have the properties that \( f \in C^2([0, +\infty)) \) and
\[ u_0 \in C^2(\Omega) \quad \text{with } \frac{\partial u_0}{\partial \nu} = 0 \text{ on } \partial \Omega. \]  
(26)
Then by the well-known theory on higher regularity in scalar parabolic equations [23, 18], \( u_r \) belongs to \( C^0([0, R] \times [0, T_{\text{max}}]) \cap C^{2,1}((0, R) \times (0, T_{\text{max}})) \), and according to (13) we obtain that
\[ v_{rr} + \frac{n-1}{r} v_r = \mu(t) - f(u) \]  
(27)
as well as
\[ u_t = u_{rr} + \frac{n-1}{r} u_r - \chi u_r v_r - \chi u (v_{rr} + \frac{n-1}{r} v_r) + \lambda u - \mu u^\alpha, \]  
(28)
which, combined with (27), yields
\[ u_t = u_{rr} + \frac{n-1}{r} u_r - \chi u_r v_r - \chi u \mu(t) + \chi u f(u) + \lambda u - \mu u^\alpha. \]
for all $r \in (0, R)$ and $t \in (0, T_{\text{max}})$. Moreover, differentiating the above identity with respect to $r$ we see that

$$u_{rt} = u_{rrt} + a(r, t)u_{rr} + b(r, t)u_r$$

(29)

with

$$a(r, t) := \frac{n-1}{r} - \chi v_r$$

(30)

and

$$b(r, t) := \frac{n-1}{r^2} - \chi v_{rr} - \mu(t) + \chi f(u) + \chi uf'(u) + \lambda - \mu u \alpha^{-1}$$

(31)

for all $r \in (0, R)$ and $t \in (0, T_{\text{max}})$.

Now fixed $T \in (0, T_{\text{max}})$, in accordance with the continuity of $u$ in $[0, R] \times [0, T]$, we can choose suitable $\lambda_1 > 0$ such that

$$\frac{\lambda_1}{2} > \lambda + 2\|f(u)\|_{L^\infty(0, R) \times (0, T)} + \chi \|uf'(u)\|_{L^\infty(0, R) \times (0, T)}$$

(32)

and for $\eta > 0$ we let

$$z(r, t) = u_r(r, t) - \eta e^{\lambda_1 t}, \quad r \in [0, R], \quad t \in [0, T].$$

(33)

It is clear that $z \in C^0([0, R] \times [0, T])$. Then by the monotonicity of $u_0$ in (6), we obtain

$$z(r, 0) = u_0(r) - \eta < 0 \text{ for all } r \in [0, R]$$

and also

$$z(0, t) = z(R, t) = -\eta e^{\lambda_1 t} < 0 \text{ for all } t \in [0, T]$$

due to $u_r(0, t) = u_r(R, t) = 0$ for all $t \in (0, T_{\text{max}})$. Now, we claim that

$$z(r, t) < 0 \text{ for all } r \in [0, R] \text{ and } t \in [0, T].$$

(34)

If this were false, by following the standard initial step of maximum principle we could therefore find some $r_0 \in (0, R)$ and $t_0 \in (0, T]$ such that

$$z(r_0, t_0) = 0, \quad z_r(r_0, t_0) = 0, \quad z_{rr}(r_0, t_0) \leq 0 \text{ and } z_t(r_0, t_0) \geq 0. \quad (35)$$

From (29) we know that

$$z_t = z_{rr} + a(r, t)z_r + b(r, t)z + \{b(r, t) - \lambda_1\} \eta e^{\lambda_1 t}.$$ 

It follows from (35) that at $(r_0, t_0)$ we have

$$0 \leq z_t(r_0, t_0) \leq \{b(r_0, t_0) - \lambda_1\} \eta e^{\lambda_1 t_0}.$$ 

(36)

Here we recall the definition of $b(r, t)$ in (31), which together with (23), (27) and (32) infers that

$$b(r, t) = -\frac{n-1}{r^2} - \chi \left\{-\frac{n-1}{r} v_r(r, t) + \mu(t) - f(u(r, t))\right\} - \chi \mu(t)$$

$$+ \chi f(u(r, t)) + \chi u(r, t)f'(u(r, t)) + \lambda - \alpha \mu u \alpha^{-1}(r, t)$$

$$= -\frac{n-1}{r^2} + \frac{n-1}{r} v_r(r, t) - 2\chi \mu(t) + 2\chi f(u(r, t))$$

$$+ \chi u(r, t)f'(u(r, t)) + \lambda - \alpha \mu u \alpha^{-1}(r, t)$$

$$\leq -\frac{n-1}{r^2} - \frac{n+1}{n} \chi \mu(t) + 2\chi f(u(r, t))$$

$$+ \chi u(r, t)f'(u(r, t)) + \lambda - \alpha \mu u \alpha^{-1}(r, t)$$

$$\leq -\frac{n-1}{r^2} - \frac{n+1}{n} \chi \mu(t) + 2\chi f(u(r, t))$$

$$+ \chi u(r, t)f'(u(r, t)) + \lambda - \alpha \mu u \alpha^{-1}(r, t)$$
\[ \leq 2\chi f(u(r,t)) + \chi u(r,t)f'(u(r,t)) + \lambda \]
\[ \leq \frac{\lambda_1}{2} \text{ for all } r \in (0, R) \text{ and } t \in (0, T) \]

This along with (36) leads to the absurd conclusion that

\[ 0 \leq -\frac{\lambda_1}{2} \eta e^{\lambda t_0}. \]

Hence the claim (34) is true, which implies

\[ u_r(r,t) < \eta e^{\lambda t} \text{ for all } r \in [0, R] \text{ and } t \in [0, T]. \]

Firstly, let \( \eta \searrow 0 \), which yields

\[ u_r(r,t) \leq 0 \text{ for all } r \in (0, R] \text{ and } t \in [0, T]. \]

Then let \( T \nearrow T_{\max} \), which arrives at (24).

**Step 2.** We proceed to derive (24) for arbitrary \( f \) and \( u_0 \) merely satisfying (6) and (7). The proof is similar to the step 2 in [45, Lemma 2.2], so we omit it here. \( \square \)

4. **A conditional superlinear ODI for \( \phi \).** The proof of the blow-up result in Theorem 1.1 will be based on a contradictory argument. Its core is to analyze the time evolution of the generalized moment functional \( \phi \) defined by

\[ \phi(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s)w(s,t)ds, \quad t \in [0, T_{\max}), \quad (37) \]

for suitably chosen values of \( \gamma \in (-\infty, 1) \) and \( s_0 \in (0, R^n) \), and where \( w \) defined as (14). It follows from Proposition 1 that \( u \) and \( u_t \) are continuous in \( \bar{\Omega} \times [0, T_{\max}) \) and in \( \bar{\Omega} \times (0, T_{\max}) \), respectively. For any \( \gamma \in (-\infty, 1) \) and \( s_0 \in (0, R^n) \) the mapping \((0, s_0) \ni s \mapsto s^{-\gamma}(s_0 - s) \) is integrable. Then based on the dominated convergence theorem shows that the function \( \phi \) is well-defined and lies in \( C^0([0, T_{\max}) \cap C^1((0, T_{\max}) \cap C^1([0, T_{\max})). \)

As a preparation of the subsequent analysis of \( \phi \), let us define

\[ \psi(t) := \frac{\chi}{2} \int_0^{s_0} s^{1-\gamma}(s_0 - s)w(s,t)f(mw_s(s,t))ds, \quad t \in (0, T_{\max}), \quad (38) \]

with \( \gamma \in (-\infty, 1) \) and \( s_0 \in (0, R^n) \), and the sets

\[ S_{\phi} := \left\{ t \in (0, T_{\max}) \mid \phi(t) \geq \frac{1}{(1-\gamma)(2-\gamma)\omega_n} \cdot \{M - s_0\} \cdot s_0^{2-\gamma} \right\}, \quad (39) \]

where \( M \) is a positive constant given in Lemma 4.1, and

\[ S_{\psi} := \left\{ t \in (0, T_{\max}) \mid \psi(t) \geq s_0^{3-\gamma - \frac{\alpha + 1}{\alpha}} \right\}. \quad (40) \]

Our goal toward Lemma 4.11 and the proof of Theorem 1.1 consists in deriving a superlinearly forced ODI for \( \phi \), and its condition is a set of times that are limited to two conditions which are defined in (39) and (40). As preparation, let us first establish an upper bound for \( \mu(t) \) in (21).

**Lemma 4.1.** Let \( n \geq 1, R > 0, \lambda > 0, \mu > 0 \) and \( \alpha > 1 \). Then

\[ \int_{\Omega} u(\cdot,t)dx \leq M := \max\{\bar{C}, m\} \text{ for all } t \in (0, T_{\max}), \quad (41) \]

where \( \bar{C} = |\Omega| \left( \frac{\lambda}{\mu} \right)^{\frac{1}{n-1}} \).
Proof. Integrating the first equation in (5), we obtain
\[
\frac{d}{dt} \int_{\Omega} u = \lambda \int_{\Omega} u - \mu \int_{\Omega} u^\alpha \quad \text{for all } t \in (0, T_{\text{max}})
\] (42)
and hence
\[
\frac{d}{dt} \int_{\Omega} u \leq \lambda \int_{\Omega} u - \frac{\mu}{|\Omega|^{\alpha-1}} \left( \int_{\Omega} u^\alpha \right) \quad \text{for all } t \in (0, T_{\text{max}}),
\] (43)
due to \( \frac{1}{|\Omega|^{\alpha-1}} (\int_{\Omega} u^\alpha) \leq \int_{\Omega} u^\alpha \) for all \( t \in (0, T_{\text{max}}) \) by the Hölder inequality. This implies (41) with \( M := \max\{C, m\} \) where \( C = |\Omega| \left( \frac{\lambda}{\mu} \right)^{\frac{1}{\alpha-1}} \).

**Lemma 4.2.** Let \( n \geq 1, R > 0, \lambda > 0, \mu > 0 \) and \( \alpha > 1 \). Suppose that (6) and (7) hold, and \( \gamma \in (-\infty, 1) \) and \( s_0 \in (0, R^n) \). Then recalling the \( M \) defined in Lemma 4.1 we have
\[
w(s_0, t) \geq \frac{1}{\omega_n} \left( M - \frac{4s_0}{2^n(3-\gamma)} \right) \quad \text{for all } t \in S_{\phi}.
\] (44)

**Proof.** If (44) was false, then there exists \( t_0 \in S_{\phi} \) such that
\[
w(s_0, t_0) \leq \frac{1}{\omega_n} (M - \delta)
\]
where \( \delta := \frac{4s_0}{2^n(3-\gamma)} \). Furthermore, we obtain that \( w(s, t_0) < \frac{M - \delta}{\omega_n} \) for all \( s \in (0, \frac{M}{\omega_n}) \) due to the monotonicity of \( w(\cdot, t_0) \), and by Lemma 4.1, we obtain \( w(s, t_0) \leq \frac{1}{\omega_n} \int_{\Omega} u(\cdot, t_0) \leq \frac{M}{\omega_n} \) for all \( s \in (0, s_0) \), thereby
\[
\frac{(M - s_0) \cdot s_0^{2-\gamma}}{(1-\gamma)(2-\gamma)\omega_n} \leq \phi(t_0) = \frac{M - \delta}{\omega_n} \int_0^{s_0} s^{-\gamma} (s_0 - s) ds + \frac{M}{\omega_n} \int_0^{s_0} s^{-\gamma} (s_0 - s) ds
\]
\[
= M \int_0^{s_0} s^{-\gamma} (s_0 - s) ds - \frac{\delta}{\omega_n} \int_0^{s_0} s^{-\gamma} (s_0 - s) ds
\]
\[
= M \left\{ s_0 \frac{1-\gamma}{1-\gamma} - s_0 \frac{2-\gamma}{2-\gamma} \right\} - \frac{\delta}{\omega_n} \left\{ s_0 \frac{(\frac{\omega_n}{2})^{1-\gamma}}{1-\gamma} - \frac{(\frac{\omega_n}{2})^{2-\gamma}}{2-\gamma} \right\}
\]
\[
= M \frac{1}{(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma} - \frac{2^n(3-\gamma)\delta}{4(1-\gamma)(2-\gamma)\omega_n^{5-2\gamma}}.
\]
This infers that
\[
M - s_0 < M - \frac{2^n(3-\gamma)\delta}{4},
\]
which contradicted our definition of \( \delta \).

**Lemma 4.3.** Let \( n \geq 1, R > 0, \lambda > 0, \mu > 0 \) and \( \alpha > 1 \). Suppose that (6) and (7) hold, and let \( \gamma \in (-\infty, 1) \) and \( s_0 > 0 \) satisfy \( s_0 \leq \frac{R^n}{4} \). Then we have the property of \( \mu(t) \) in (21),
\[
\mu(t) \leq f_\gamma + \frac{1}{2s} \int_0^s f(nw_\sigma(s, t)) d\sigma \quad \text{for all } s \in (0, s_0) \quad \text{and } t \in S_{\phi},
\] (45)
where
\[
f_\gamma := f\left( \frac{8n}{2^n(3-\gamma)\omega_n} \right).
\] (46)

**Proof.** The proof is almost the same as [45, Lemma 3.2].
Lemma 4.4. Let $n \geq 1$, $R > 0$, $\lambda > 0$, $\mu > 0$ and $\alpha > 1$, and let $\gamma \in (-\infty, 1)$ and $s_0 > 0$ satisfy $s_0 \leq \frac{n}{4}$. Assume that (6) and (7) hold. Then

$$
\begin{align*}
\phi_t &\geq n^2 s^{2-\gamma} w_{ss}(s,t) - \chi f_s w_s(s,t) + \lambda w(s,t) + \frac{\chi}{2} w_s(s,t) \cdot \int_0^s f(n w_s(\sigma,t)) d\sigma \\
&\quad - \mu n^{\alpha-1} \int_0^s w_\sigma(\sigma,t) d\sigma \text{ for all } s \in (0,s_0) \text{ and } t \in S_\phi,
\end{align*}
$$

(47)

where $S_\phi$ and $f_\gamma$ are as given by (39) and (46), respectively.

Proof. This is an immediate consequence of Lemma 4.3, (18) and the nonnegativity of $w_s$.

Based on (47) and several integrations by parts, and the concavity of $w$, we can establish a basic differential inequality for our target functional $\phi$.

Lemma 4.5. Let $n \geq 1$, $R > 0$, $\lambda > 0$, $\mu > 0$ and $\alpha > 1$, and let $\gamma \in (-\infty, 1)$ and $s_0 > 0$ satisfy $\gamma < 2 - \frac{2}{n}$ and $s_0 \leq \frac{n}{4}$. Suppose that (6) and (7) hold. Then the function $\phi$ defined in (37) satisfies

$$
\begin{align*}
\phi'(t) &\geq \frac{\chi}{2} \int_0^{s_0} s^{1-\gamma}(s_0 - s) w_s(s,t) f(n w_s(s,t)) ds \\
&\quad - 2n^2 (2 - \frac{2}{n} - \gamma) s_0 \int_0^{s_0} s^{-\gamma - \frac{2}{n}} w(s,t) ds \\
&\quad - \chi f_\gamma \int_0^{s_0} s^{1-\gamma} w(s,t) ds - \mu n^{\alpha-1} s_0^{1-\gamma} \int_0^{s_0} (s_0 - s) w_\sigma(\sigma,t) ds
\end{align*}
$$

(48)

for all $t \in S_\phi$, where $f_\gamma$ and $S_\phi$ are as given by (46) and (39), respectively.

Proof. Using the definition of $\phi$ in (37) and (47), and removing the positive term $\lambda w$, we have

$$
\begin{align*}
\phi'(t) &\geq \int_0^{s_0} s^{-\gamma}(s_0 - s) \left\{ n^2 s^{2-\gamma} w_{ss} - \chi f_s w_s \\
&\quad + \frac{\chi}{2} w_s \int_0^s f(n w_s(\sigma,t)) d\sigma - \mu n^{\alpha-1} \int_0^s w_\sigma(\sigma,t) d\sigma \right\} ds \\
&= n^2 \int_0^{s_0} s^{2-\gamma}(s_0 - s) w_{ss} ds \\
&\quad + \frac{\chi}{2} \int_0^{s_0} s^{-\gamma}(s_0 - s) w_s \left\{ \int_0^s f(n w_s(\sigma,t)) d\sigma \right\} ds \\
&\quad - \chi f_\gamma \int_0^{s_0} s^{1-\gamma}(s_0 - s) w_s ds \\
&\quad - \mu n^{\alpha-1} \int_0^{s_0} s^{-\gamma}(s_0 - s) \left\{ \int_0^s w_\sigma(\sigma,t) d\sigma \right\} ds \\
&:= I_1 + I_2 - I_3 - I_4 \text{ for all } t \in S_\phi.
\end{align*}
$$

(49)

Since $\gamma < 2 - \frac{2}{n}$, then $s^{1-\frac{2}{n}} w \to 0$ as $s \to 0$ for each $t \in (0,T_{\max})$. An integration by parts yields

$$
I_1 = n^2 \int_0^{s_0} s^{2-\frac{2}{n}}(s_0 - s) w_{ss} ds
$$
for all $t$ of $f$ we obtain

Once more integrating by parts we find that

From Fubini's theorem, we see that for all $t$

for all $t \in S_\phi$, where we used the the fact that $\int_0^{s_0} s^{1-\frac{2}{n}-\gamma} wds \leq s_0 \int_0^{s_0} s^{1-\frac{2}{n}-\gamma} wds$.

Due to the fact that $w_{ss} \leq 0$ in $(0, R^n) \times (0, T_{max})$ from (25) and the monotonicity of $f$, we can estimate

for all $s \in (0, R^n)$ and $t \in (0, T_{max})$. In addition, in view of the nonnegativity of $w_s$, we obtain

Once more integrating by parts we find that

due to the fact that $\gamma < 1$ and $f_\gamma \geq 0$.

From Fubini’s theorem, we see that for all $t \in (0, T_{max}),$

$$I_4 = \mu n^{a-1} \int_0^{s_0} s^{-\gamma}(s_0-s) \cdot \left\{ \int_0^s w_s^\alpha(\sigma,t) d\sigma \right\} ds$$

$$= \mu n^{a-1} \int_0^{s_0} \left\{ \int_0^s s^{-\gamma}(s_0-s) ds \right\} \cdot w_s^\alpha(\sigma,t) d\sigma.$$
Using the fact that $\gamma < 1$ we obtain
\[
\int_{s}^{s_0} s^{-\gamma}(s_0 - s)ds \leq (s_0 - \sigma) \int_{0}^{s_0} s^{-\gamma}ds = \frac{s_0^{1-\gamma}}{1-\gamma}(s_0 - \sigma) \quad \text{for all } \sigma \in (0, s_0).
\] (54)

Therefore, combining (50)-(54) this implies (48). \hfill \square

In order to estimate the second and the third integrals in (48), we need to achieve the following pointwise inequality for $w$.

**Lemma 4.6.** Let $n \geq 1$, $R > 0$, $\lambda > 0$, $\mu > 0$, $\alpha > 1$. Assume that $f$ fulfill (7) and (8) with some $k > 0$ and $\kappa > 0$, and let $\gamma \in (-\infty, 1)$ satisfy $\gamma < 2 - \frac{2}{n}$ and
\[
2 - \frac{\kappa + 1}{\alpha} < \gamma.
\] (55)

Then for all $\epsilon > 0$, and for any choice of $s_0 \in (0, R^n)$, there exists $C_1(\epsilon) > 0$ such that for each $u_0$ satisfying (6), the function $\psi$ defined by (38) has the following property:
\[
w(s, t) \leq \frac{s}{n} + \epsilon(s_0 - s)^{-\frac{1}{2}} + C_1(\epsilon)C_2(s_0 - s)^{-\frac{1}{2}} + \frac{1}{n} s_0^{1-\frac{\alpha}{\alpha + 2}} s^{\frac{\kappa + 1 - \alpha}{\alpha + 2}} \left(\frac{\kappa - \alpha}{\kappa + 1 - \alpha} \right)^{\frac{\alpha + 1 + \kappa}{\alpha + 2}} \pi^\alpha(t)
\] (56)

for all $s \in (0, s_0)$ and $t \in (0, T_{\max})$ with $C_2 = \left(\frac{2}{\chi \kappa \alpha} \right)^{\frac{\alpha + 1 + \kappa}{\alpha + 2}} \left(\frac{\kappa + 1 - \alpha}{\kappa + 1 - \alpha} \right)^{\frac{\alpha + 1 + \kappa}{\alpha + 2}}$.

**Proof.** By nonnegativity of both $w_s$ and $f$, we apply (8) to see that
\[
\psi(t) \geq \frac{\chi}{2} \int_{0}^{s_0} \chi_{\{nw_\sigma(s,t) \geq 1\}}(s) \cdot s^{1-\gamma}(s_0 - s) w_\sigma(s, t) f(nw_s(s, t))ds
\]
\[
\geq \frac{\chi \kappa \alpha}{2} \int_{0}^{s_0} \chi_{\{nw_\sigma(s,t) \geq 1\}}(s) \cdot s^{1-\gamma}(s_0 - s) w_\sigma^{\kappa+1}(s, t)ds
\] (57)
for all $t \in (0, T_{\max})$, where $\chi_M$ denotes the characteristic function of the set $M \subset \mathbb{R}$. Using $w(0,t) = 0$ we obtain that for all $s \in (0, s_0)$ and $t \in (0, T_{\max})$
\[
w(s, t) = \int_{0}^{s} w_\sigma(\sigma, t)d\sigma
\]
\[
= \int_{0}^{s} \chi_{\{nw_\sigma(s,t) < 1\}}(\sigma) \cdot w_\sigma(\sigma, t)d\sigma + \int_{0}^{s} \chi_{\{nw_\sigma(s,t) \geq 1\}}(\sigma) \cdot w_\sigma(\sigma, t)d\sigma
\] (58)
\[
:= I_5 + I_6,
\]
where
\[
I_5 \leq \frac{1}{n} \int_{0}^{s} \chi_{\{nw_\sigma(s,t) < 1\}}(\sigma)d\sigma \leq \frac{1}{n} \cdot s.
\] (59)

Moreover, given $\epsilon > 0$, from an application of the Hölder inequality and Young’s inequality we obtain $C_1 = C_1(\epsilon) > 0$ such that
\[
I_6 = \int_{0}^{s} \chi_{\{nw_\sigma(s,t) \geq 1\}}(\sigma) \cdot w_\sigma(\sigma, t)d\sigma
\]
\[
= \int_{0}^{s} \{\chi_{\{nw_\sigma(s,t) \geq 1\}}(\sigma) \cdot (s_0 - \sigma) w_\sigma^\alpha(\sigma, t)\}^{\frac{1}{\alpha}} \cdot (s_0 - \sigma)^{-\frac{1}{\alpha}} d\sigma
\]
\[
\leq (s_0 - s)^{-\frac{1}{\alpha}} \cdot \int_{0}^{s} \{\chi_{\{nw_\sigma(s,t) \geq 1\}}(\sigma) \cdot (s_0 - \sigma) w_\sigma^\alpha(\sigma, t)\}^{\frac{1}{\alpha}} d\sigma
\]
\[ \leq (s_0 - s)^{-\frac{\alpha}{2}} \left[ C_1(\epsilon) \int_0^s \chi_{\{\nu_{w_s}(\cdot,t) \geq 1\}}(\sigma) \cdot (s_0 - \sigma)w_s^\alpha(\sigma,t) \, d\sigma + \epsilon \int_0^s 1^{-\frac{\alpha}{2}} \, d\sigma \right] \]

\[ = \epsilon(s_0 - s)^{-\frac{\alpha}{2}} \cdot s + C_1(\epsilon)(s_0 - s)^{-\frac{1}{2}} \int_0^s \chi_{\{\nu_{w_s}(\cdot,t) \geq 1\}}(\sigma) \cdot (s_0 - \sigma)w_s^\alpha(\sigma,t) \, d\sigma \]

\[ := I_7 + C_1(\epsilon)(s_0 - s)^{-\frac{1}{2}} I_8 \quad (60) \]

for all \( s \in (0, s_0) \) and \( t \in (0, T_{\text{max}}) \), where \( \int_0^s \chi_{\{\nu_{w_s}(\cdot,t) \geq 1\}}(\sigma) \cdot (s_0 - \sigma)w_s^\alpha(\sigma,t) \, d\sigma \) is related to the logistic death term (Lemma 4.6 below).

Since our assumption (55) implies that \( 1 - \frac{\alpha}{\kappa + 1} > 0 \) and \( \frac{\alpha(1 - \gamma)}{\kappa + 1 - \alpha} < 1 \), using the Hölder inequality once more we obtain

\[ I_8 = \int_0^s \chi_{\{\nu_{w_s}(\cdot,t) \geq 1\}}(\sigma) \cdot (s_0 - \sigma)w_s^\alpha(\sigma,t) \, d\sigma \]

\[ = \int_0^s \left\{ \chi_{\{\nu_{w_s}(\cdot,t) \geq 1\}}(\sigma) \cdot \sigma^{-1-\gamma}(s_0 - \sigma)w_s^{\kappa+1}(\sigma,t) \right\} \frac{\sigma^{-\frac{\alpha(1-\gamma)}{\kappa+1-\alpha}}}{1 - \frac{\alpha}{\kappa+1}} (s_0 - \sigma)^{1 - \frac{\alpha}{\kappa+1}} \, d\sigma \]

\[ \leq s_0^{-\frac{\alpha(1-\gamma)}{\kappa+1-\alpha}} \left\{ \int_0^s \chi_{\{\nu_{w_s}(\cdot,t) \geq 1\}}(\sigma) \cdot \sigma^{-1-\gamma}(s_0 - \sigma)w_s^{\kappa+1}(\sigma,t) \, d\sigma \right\} \]

\[ \cdot \left\{ \int_0^s \sigma^{-\frac{\alpha(1-\gamma)}{\kappa+1-\alpha}} \, d\sigma \right\}^{\frac{\kappa+1-\alpha}{\kappa+1}} , \quad (61) \]

where using that \( \frac{\alpha(1-\gamma)}{\kappa+1-\alpha} < 1 \) we may estimate

\[ \left\{ \int_0^s \sigma^{-\frac{\alpha(1-\gamma)}{\kappa+1-\alpha}} \, d\sigma \right\}^{\frac{\kappa+1-\alpha}{\kappa+1}} = \left\{ \frac{\kappa+1-\alpha}{\kappa+1-\alpha(2-\gamma)} \right\}^{\frac{\kappa+1-\alpha}{\kappa+1}} \cdot s_0^{-\frac{\alpha(1-\gamma)}{\kappa+1} \cdot s_0^{-\frac{\alpha(1-\gamma)}{\kappa+1}}} . \quad (62) \]

So applying (57) we obtain

\[ I_8 \leq \left\{ \frac{2\psi(t)}{\chi k^n} \right\} \frac{\kappa+1}{\kappa+1-\alpha} \left\{ \frac{\kappa+1-\alpha}{\kappa+1-\alpha(2-\gamma)} \right\}^{\frac{\kappa+1-\alpha}{\kappa+1}} \cdot s_0^{-\frac{\alpha(1-\gamma)}{\kappa+1}} \cdot s_0^{-\frac{\alpha(1-\gamma)}{\kappa+1}} . \quad (63) \]

for all \( s \in (0, s_0) \) and \( t \in (0, T_{\text{max}}) \). Therefore, together with (60)-(62) this implies (56).

**Corollary 1.** Suppose that (6), (7) and (8) hold with some \( k > 0 \) and \( \kappa > 0 \).

Let \( n \geq 1, R > 0, \lambda > 0, \mu > 0, \alpha > 1 \) and \( \epsilon > 0 \), and let \( \gamma \in (-\infty, 1) \) satisfy \( 2 - \frac{\alpha+1}{\alpha} < \gamma < 2 - \frac{2}{n} \) and \( s_0 \in (0, R^n) \). Then we have

\[ w(s,t) \leq L(s_0 - s)^{-\frac{1}{n}} s_0^{-\frac{\alpha(1-\gamma)}{\kappa+1}} s^{-\frac{\alpha(1-\gamma)(2-\gamma)}{\kappa+1}} \psi^{-\frac{\alpha}{\kappa+1}}(t) \quad \text{for all } s \in (0, s_0) \quad \text{and } t \in S_\psi , \quad (64) \]

where

\[ L \equiv L(k, R, \kappa, \alpha, \gamma, n) := \frac{1}{n} R^n + \epsilon + C_1(\epsilon)C_2 . \quad (65) \]

**Proof.** By Lemma 4.6 and the definition of \( S_\psi \) and the fact that \( \gamma < 2 \), the first summand on the right of (56) can be controlled by the third one according to

\[ \frac{s}{(s_0 - s)^{-\frac{\alpha}{2}} s_0^{-\frac{\alpha}{2}} s^{-\frac{\alpha(1-\gamma)(2-\gamma)}{\kappa+1}} \psi^{-\frac{\alpha}{\kappa+1}}(t)} = s^{\frac{\alpha(2-\gamma)}{\kappa+1}} (s_0 - s)^{\frac{\alpha}{\kappa+1} - 1} \psi^{\frac{\alpha}{\kappa+1} - 1}(t) \]

\[ \leq s_0^{-\frac{\alpha(2-\gamma)}{\kappa+1} + \frac{1}{\kappa+1} - 1} \psi^{\frac{\alpha}{\kappa+1} - 1}(t) \]

\[ \leq R^n \quad \text{for all } s \in (0, s_0) \quad \text{and } t \in S_\psi . \]
Likewise, for the second summand on the right of (56) we find that

\[
\frac{(s_0 - s)^{\frac{1}{2}}}{(s_0 - s)^{\frac{1}{2} s_0^{\frac{1}{n + 1}} s^{\frac{n + 1 - \alpha (2 - \gamma)}{n + 1}}}} \leq s_0^{\frac{1}{n + 1} - 1} s_0^{\frac{1}{2}} t^{\frac{1}{n + 1}} (t) \\
\leq 1 \quad \text{for all } s \in (0, s_0) \text{ and } t \in S_\psi.
\]

The proof of Corollary 1 is thus finished. \[\square\]

**Lemma 4.7.** Suppose that (6), (7) and (8) hold with some \( k > 0 \) and \( \kappa > 0 \). Let \( n \geq 1, R > 0, \lambda > 0, \mu > 0, \alpha > 1, \) and let \( \gamma \in (-\infty, 1) \) satisfy \( 2 - \frac{\kappa + 1}{\alpha} < \gamma < 2 - \frac{2}{n} \) and \( s_0 \in (0, R^n) \). Then taking \( \phi, \psi, S_\psi, L \) from (37), (38), (40) and (65), we have

\[
\psi(t) \geq C_3 s_0^{\frac{(3 - \gamma)(\kappa + 1 - \alpha)}{\kappa + 1} + (1 - \frac{1}{\alpha})} \phi^{\frac{1}{\alpha}} (t)
\]

with \( C_3 = \left\{ \frac{(2 - \gamma)(\kappa + 1 - \alpha)}{\kappa + 1} \right\} \frac{s_0^{\frac{1}{n + 1}}}{(\kappa + 1 - \alpha)} \) for all \( t \in S_\psi \).

**Proof.** From Corollary 1, we obtain that for any \( t \in S_\psi \)

\[
\phi(t) = \int_0^{s_0} s^{-\gamma} (s_0 - s) w(s, t) ds
\]

\[
\leq L s_0^{\frac{1}{n + 1}} \left\{ \int_0^{s_0} (s_0 - s)^{\frac{1}{2}} s^{-\gamma + \frac{n + 1 - \alpha (2 - \gamma)}{n + 1}} ds \right\} \psi s_0^{\frac{1}{n + 1}} (t),
\]

which, combined the fact that \( \gamma < 2 \), gives

\[
\int_0^{s_0} (s_0 - s)^{\frac{1}{2}} s^{-\gamma + \frac{n + 1 - \alpha (2 - \gamma)}{n + 1}} ds \leq s_0^{\frac{1}{n}} \int_0^{s_0} s^{-\gamma + \frac{n + 1 - \alpha (2 - \gamma)}{n + 1}} ds
\]

\[
= s_0^{\frac{1}{n}} \left( \frac{\kappa + 1}{(2 - \gamma)(\kappa + 1 - \alpha)} \right) s_0^{\frac{(3 - \gamma)(\kappa + 1 - \alpha)}{\kappa + 1} + (1 - \frac{1}{\alpha})}.
\]

Then substituting (68) into (67), we directly arrive at the result (66). \[\square\]

**Lemma 4.8.** Suppose that (6), (7) and (8) hold with some \( k > 0 \) and \( \kappa > 0 \). Let \( n \geq 1, R > 0, \lambda > 0, \mu > 0 \) and \( \alpha > 1 \), and let \( \gamma \in (-\infty, 1) \) satisfy \( 2 - \frac{\kappa + 1}{\alpha} < \gamma < 2 - \frac{2}{n} \) and

\[
\gamma < 2 - \frac{2}{n} - \frac{2\alpha}{n(\kappa + 1 - \alpha)}.
\]

Then one has

\[
s_0 \int_0^{s_0} s^{-\gamma - \frac{2}{n}} w(s, t) ds \leq C_4 s_0 \left[ 3^{-\gamma - \frac{2}{n} + \frac{n + 1 - \alpha (2 - \gamma)}{n + 1}} - \frac{1}{n} \right] \cdot \psi s_0^{\frac{1}{n + 1}} (t)
\]

with \( C_4 = L \cdot B(1 - \gamma - \frac{2}{n} + \frac{n + 1 - \alpha (2 - \gamma)}{n + 1}, 1 - \frac{1}{\alpha}) \) for all \( s_0 \in (0, R^n) \) and \( t \in S_\psi \), where \( B \) denotes Euler’s Beta function and \( \psi, S_\psi, L \) are defined as in (38), (40) and (65), respectively.

**Proof.** Once more by Corollary 1, we see that

\[
s_0 \int_0^{s_0} s^{-\gamma - \frac{2}{n}} w(s, t) ds
\]

\[
\leq L s_0^{\frac{2}{n + 1}} \left\{ \int_0^{s_0} s^{-\gamma - \frac{2}{n} + \frac{n + 1 - \alpha (2 - \gamma)}{n + 1}} (s_0 - s)^{-\frac{1}{n}} ds \right\} \cdot \psi s_0^{\frac{1}{n + 1}} (t)
\]

(71)
for all $t \in S_\psi$. By (69), we have
\[
1 - \frac{2}{n} + \frac{\kappa + 1 - \alpha(2 - \gamma)}{\kappa + 1} = \frac{(\kappa + 1)(2 - \frac{2}{n}) - 2\alpha - (\kappa + 1 - \alpha)\gamma}{\kappa + 1} > 0.
\]
Thereby we have
\[
s^0 \left\{ \int_0^s s^{-\gamma - \frac{2}{n} + \frac{\kappa + 1 - \alpha(2 - \gamma)}{\kappa + 1}} (s_0 - s)^{-\frac{1}{\gamma}} ds \right\} = B(1 - \frac{2}{n} + \frac{\kappa + 1 - \alpha(2 - \gamma)}{\kappa + 1}, 1 - \frac{1}{\alpha}s_0^{3 - \gamma - \frac{2}{n} + \frac{\kappa + 1 - \alpha(2 - \gamma)}{\kappa + 1}} - \frac{1}{\alpha}),
\]
which implies (70).

Lemma 4.9. Suppose that (6), (7) and (8) hold with some $k > 0$ and $\kappa > 0$. Let $n \geq 1$, $R > 0$, $\lambda > 0$, $\mu > 0$ and $\alpha > 1$, and let $\gamma \in (-\infty, 1)$ satisfy $2 - \frac{\kappa + 1}{\alpha} < \gamma < 2 - \frac{2}{n}$. Then for $s_0 \in (0, R^n)$, we have
\[
\int_0^{s_0} s^{1 - \gamma} w(s, t) ds \leq \frac{\alpha L}{\alpha - 1}s_0^{3 - \gamma + \frac{\kappa + 1 - \alpha(2 - \gamma)}{\kappa + 1} - \frac{1}{\alpha}} \psi^{\frac{\kappa + 1}{\alpha}}(t) \tag{72}
\]
for all $t \in S_\psi$, where $\psi, S_\psi, L$ are defined as in (38), (40) and (65), respectively.

Proof. Again by means of corollary 1, we obtain
\[
\int_0^{s_0} s^{1 - \gamma} w(s, t) ds \leq L \int_0^{s_0} s^{1 - \gamma + \frac{\kappa + 1 - \alpha(2 - \gamma)}{\kappa + 1}} (s_0 - s)^{-\frac{1}{\gamma}} ds \psi^{\frac{\kappa + 1}{\alpha}}(t)
\]
for all $t \in S_\psi$. Due to $2 - \frac{\kappa + 1}{\alpha} < \gamma < 1$, we obtain
\[
\int_0^{s_0} s^{1 - \gamma + \frac{\kappa + 1 - \alpha(2 - \gamma)}{\kappa + 1}} (s_0 - s)^{-\frac{1}{\gamma}} ds \leq s_0^{1 - \gamma + \frac{\kappa + 1 - \alpha(2 - \gamma)}{\kappa + 1}} \int_0^{s_0} (s_0 - s)^{-\frac{1}{\gamma}} ds = \frac{\alpha}{\alpha - 1}s_0^{1 - \gamma + \frac{\kappa + 1 - \alpha(2 - \gamma)}{\kappa + 1}} s_0^{1 - \frac{1}{\alpha}},
\]
which immediately entails (72).

Using the similar idea in Lemma 4.6, the damping action of the logistic death term can be estimated in terms of the positive summand on the right of (48).

Lemma 4.10. Suppose that (6), (7) and (8) hold with some $k > 0$ and $\kappa > 0$. Let $n \geq 1$, $R > 0$, $\lambda > 0$, $\mu > 0$ and $\alpha > 1$, and let $\gamma \in (-\infty, 1)$ satisfy
\[
2 - \frac{\kappa + 1}{\alpha} < \gamma < 2 - \frac{2}{n} \tag{73}
\]
Then for any $s_0 \in (0, R^n)$, we have
\[
\frac{\mu n^{\alpha - 1}}{1 - \gamma}s_0^{1 - \gamma} \int_0^{s_0} (s_0 - s) \omega_s(s, t) ds \leq C_6 s_0^{2 - \gamma + \frac{\kappa + 1 - \alpha(2 - \gamma)}{\kappa + 1}} \psi^{\frac{\kappa + 1}{\alpha}}(t) \tag{74}
\]
for all $s \in (0, s_0)$ and $t \in S_\psi$ with
\[
C_6 = \frac{\mu}{(1 - \gamma)n}R^{\frac{\kappa + 1}{\alpha + 1}} + \frac{\mu n^{\alpha - 1}}{1 - \gamma}(\frac{2}{\chi kn^\kappa})^{\frac{\kappa + 1 - \alpha}{\alpha + 1}}(\frac{\kappa + 1 - \alpha(2 - \gamma)}{\kappa + 1 - \alpha(2 - \gamma)}).
Proof. We first observe that for all $t \in (0, T_{\text{max}})$
\begin{align*}
\int_0^{s_0} (s_0 - s) u_\alpha^\alpha(s, t) ds &= \int_0^{s_0} \chi_{\{n\omega_n(s, t) < 1\}}(s) \cdot (s_0 - s) u_\alpha^\alpha(s, t) ds \\
&\quad + \int_0^{s_0} \chi_{\{n\omega_n(s, t) \geq 1\}}(s) \cdot (s_0 - s) u_\alpha^\alpha(s, t) ds \\
&= I_0 + I_{10}.
\end{align*}
where $\chi_M$ denotes the characteristic function of the set $M \subset \mathbb{R}$. It is obvious to see that
\begin{align*}
I_0 &\leq \frac{s_0}{n^\alpha} \int_0^{s_0} \chi_{\{n\omega_n(s, t) < 1\}}(s) ds \leq \frac{s_0^2}{n^\alpha}.
\end{align*}
Moreover, the left inequality of (73) ensure that $1 - \frac{\alpha}{\kappa + 1} > 0$ and $\frac{\alpha(1 - \gamma)}{\kappa + 1 - \alpha} < 1$, so we estimate from the Hölder inequality
\begin{align*}
I_{10} &= \int_0^{s_0} \chi_{\{n\omega_n(s, t) \geq 1\}}(s) (s_0 - s) u_\alpha^\alpha(s, t) ds \\
&= \int_0^{s_0} \left\{ \chi_{\{n\omega_n(s, t) \geq 1\}}(s) s^{1 - \gamma}(s_0 - s) u_\alpha^{\kappa + 1}(s, t) \right\} \frac{\kappa + 1}{\kappa + \alpha} \cdot s^{-\frac{\alpha(1 - \gamma)}{\kappa + 1 - \alpha}} (s_0 - s)^{1 - \frac{\alpha}{\kappa + 1}} ds \\
&\leq s_0^{1 - \frac{\alpha}{\kappa + 1}} \left\{ \int_0^{s_0} \chi_{\{n\omega_n(s, t) \geq 1\}}(s) s^{1 - \gamma}(s_0 - s) u_\alpha^{\kappa + 1}(s, t) ds \right\} \frac{\kappa + 1}{\kappa + \alpha} \\
&\cdot \left\{ \int_0^{s_0} s^{-\frac{\alpha(1 - \gamma)}{\kappa + 1 - \alpha}} ds \right\} \frac{\kappa + 1 - \alpha}{\kappa + \alpha}.
\end{align*}
Thus, we can estimate
\begin{align*}
\left\{ \int_0^{s_0} s^{-\frac{\alpha(1 - \gamma)}{\kappa + 1 - \alpha}} ds \right\} \frac{\kappa + 1 - \alpha}{\kappa + \alpha} = \left\{ \frac{\kappa + 1 - \alpha}{\kappa + 1 - \alpha(2 - \gamma)} \right\} \frac{\kappa + 1 - \alpha}{\kappa + \alpha} \cdot s_0^{\frac{\alpha(1 - \gamma)}{\kappa + 1 - \alpha}} (77)
\end{align*}
for all $s \in (0, s_0)$ and $t \in (0, T_{\text{max}})$. Then using (57), we obtain
\begin{align*}
\frac{\mu n^\alpha - 1}{1 - \gamma} s_0^{1 - \gamma} \int_0^{s_0} (s_0 - s) u_\alpha^\alpha(s, t) ds \\
&\leq \frac{\mu}{1 - \gamma} s_0^{3 - \gamma} + C_5 s_0^{2 - \gamma - \frac{\alpha}{\kappa + 1} + \frac{\alpha + 1 - \alpha(2 - \gamma)}{\kappa + \alpha}} \cdot \psi^\alpha(t) \\
&\leq C_6 s_0^{2 - \gamma - \frac{\alpha}{\kappa + 1} + \frac{\alpha + 1 - \alpha(2 - \gamma)}{\kappa + \alpha}} \cdot \psi^\alpha(t)
\end{align*}
with $C_5 = \frac{\mu n^\alpha - 1}{1 - \gamma} \left( \frac{2}{\lambda n^\mu} \right)^{\frac{\alpha}{\kappa + 1 - \alpha(2 - \gamma)}} \left( \frac{\kappa + 1 - \alpha}{\kappa + 1 - \alpha(2 - \gamma)} \right) \frac{\kappa + 1 - \alpha}{\kappa + \alpha}$ and $C_6 = \frac{\mu}{1 - \gamma} n R^n + C_5$ for all $s \in (0, s_0)$ and $t \in (0, T_{\text{max}})$, where we have used the fact that
\begin{align*}
\frac{s_0^{3 - \gamma}}{s_0^{2 - \gamma - \frac{\alpha}{\kappa + 1} + \frac{\alpha + 1 - \alpha(2 - \gamma)}{\kappa + \alpha} \cdot \psi^\alpha(t)}} = s_0^{\frac{\alpha(1 - \gamma)}{\kappa + 1} - \frac{\alpha}{\kappa + 1}} \psi^\alpha(t) \leq s_0 \leq R^n,
\end{align*}
and thereby complete the proof. \hfill \Box

Lemma 4.11. Let $n \geq 1$, $R > 0$, $\lambda > 0$, $\mu > 0$ and $\alpha > 1$. Suppose that (6), (7) and (8) hold with some $k > 0$, $\kappa > 0$ and
\begin{align*}
\kappa + 1 > \alpha \left( \frac{n}{2} + 1 \right).
\end{align*}

Then there exists $\gamma = \gamma(\kappa, \alpha) \in (-\infty, 1)$ such that for any choice of $s_0 > 0$ such that $s_0 \leq \frac{8}{7}$, the function $\phi$ defined as (37) fulfills

$$\phi'(t) \geq \frac{C_3}{2} s_0 \frac{\phi(s_0)}{\phi(t)} - C_{13} s_0 \phi^{\gamma - 2\phi(t)}$$

for all $t \in S_0$, where $\phi$ fulfills

$$\phi(0) = \frac{4}{\kappa(\kappa + 1 - \alpha)}$$

and also

$$2 - \frac{\kappa + 1}{\alpha} < 2 - \frac{2}{n} - \frac{2\alpha}{n(\kappa + 1 - \alpha)}$$

such that $\gamma < 2 - \frac{2}{n}$ as well as

$$2 - \frac{\kappa + 1}{\alpha} < \gamma < 2 - \frac{2}{n} - \frac{2\alpha}{n(\kappa + 1 - \alpha)}$$

Moreover, fixing large enough $C_7(\kappa, \alpha) > 0$ such that according to the Young inequality we have

$$\xi \eta \leq \frac{1}{6}\xi^{\frac{\kappa + 1}{\alpha}} + C_7 \eta^{\frac{\kappa + 1}{\alpha - 1}}$$

for all $\xi \geq 0$ and $\eta \geq 0$.

From (38) and (48), we see

$$\phi'(t) \geq \psi(t) - C_8 s_0 \int_0^{s_0} s^{-\frac{\gamma}{2}} \psi(s, t) ds - \chi f \int_0^{s_0} s^{-1-\gamma} w(s, t) ds - C_9 s_0^{1-\gamma} \int_0^{s_0} (s_0 - s) \psi(s, t) ds$$

for all $t \in S_0$, where $C_8 = 2n^2(2 - \frac{2}{n} - \gamma)$ and $C_9 = \frac{4n^{\alpha - 1}}{1 - \gamma}$. The inequality in (83) enables us to use Lemma 4.8 to know that

$$C_8 s_0 \int_0^{s_0} s^{-\frac{\gamma}{2}} \psi(s, t) ds \leq C_4 C_8 s_0^{\frac{3-\gamma}{2} + \frac{(\kappa + 1 - \alpha)(3-\gamma)}{(\kappa + 1)}} \cdot \psi(t)$$

for all $t \in S_0$. Then

$$\psi(t) \leq \frac{1}{6} \psi(t) + C_{10} s_0 \phi^\gamma(t)$$

due to (84) for all $t \in S_0$ with $C_{10} = C_7 \cdot (C_4 C_8)^{\frac{\kappa + 1}{\alpha - 1}}$. Similarly, by virtue of the inequality in (83) we may employ Lemma 4.9 to estimate
\[
\chi f_\gamma \int_0^{s_0} s^{1-\gamma} w(s, t) \, ds \leq \frac{\alpha \chi L f_\gamma}{\alpha - 1} s_0^{3-\gamma + \frac{\gamma + 1 - \alpha(3-\gamma)}{\kappa+1} - \frac{1}{\alpha}} \cdot \psi \frac{a}{\alpha} (t) \leq \frac{1}{6} \psi(t) + C_{11}s_0^{(3-\gamma)(\kappa + 1) + \alpha(3-\gamma) - \frac{1}{\alpha}}
\]

(87)

for all \( t \in S_\psi \) with \( C_{11} = C_7 \cdot \left( \frac{\alpha \chi L f_\gamma}{\alpha - 1} \right)^{\frac{\gamma + 1}{\alpha}} \). Moreover, thanks to the inequality in (83), we may apply Lemma 4.10 to conclude that

\[
C_9 s_0^{1-\gamma} \int_0^{s_0} (s_0 - s) w^\alpha(s, t) \, ds \leq C_6 s_0^{2-\gamma + \frac{\gamma + 1 - \alpha(2-\gamma)}{\kappa+1}} \cdot \psi \frac{a}{\alpha} (t) \leq \frac{1}{6} \psi(t) + C_{12}s_0^{3-\gamma}
\]

(88)

for all \( t \in S_\psi \) with \( C_{12} = C_7 \cdot (C_6)^{\frac{\gamma + 1}{\alpha}} \). Due to the inequality in (83), we may deduce from Lemma 4.7 that

\[
\psi(t) \geq C_{3} s_0^{(3-\gamma)(\kappa + 1) + (1 - \frac{1}{\alpha})\kappa} \cdot \phi \frac{a}{\alpha} (t) \text{ for all } t \in S_\psi.
\]

(89)

In a combination of (85)-(89), we see

\[
\phi'(t) \geq \frac{C_3}{2} s_0^{\frac{(3-\gamma)(\kappa + 1) + (1 - \frac{1}{\alpha})\kappa}{\alpha}} \cdot \phi \frac{a}{\alpha} (t) - C_{10}s_0^{(3-\gamma)(\kappa + 1) + \alpha(3-\gamma) - \frac{1}{\alpha}} \text{ for all } t \in S_\psi \cap S_\varphi.
\]

(90)

for all \( t \in S_\psi \cap S_\varphi \). Since

\[
C_{10}s_0^{\frac{(3-\gamma)(\kappa + 1) + \alpha(3-\gamma) - \frac{1}{\alpha}}{\alpha}} + C_{11}s_0^{(3-\gamma)(\kappa + 1) + \alpha(3-\gamma) - \frac{1}{\alpha}} + C_{12}s_0^{3-\gamma}
\]

\[
\leq \left\{ C_{10} + C_{11} R^{2(\kappa + 1)} \right\} s_0^{\frac{(3-\gamma)(\kappa + 1) + \alpha(3-\gamma) - \frac{1}{\alpha}}{\alpha}} + C_{12}s_0^{3-\gamma},
\]

(91)

where \( C_{13} = \max \left\{ C_{12}, C_{10} + C_{11} R^{2(\kappa + 1)} \right\} \), we arrive at (81).

\[
\square
\]

Based on the above preparations, next, we will prove the blow-up result in Theorem 1.1 by contradictory argument.

**Proof of Theorem 1.1.** We start by only fixing \( \Omega, f \) such that (7) and (8) hold with some \( k > 0 \) and \( \kappa + 1 > \alpha(\frac{2}{n} + 1) \). Applying Lemma 4.11 we can find \( \gamma = \gamma(\kappa, \alpha) \in (\infty, 1) \) such that for any \( s_0 \in (0, \frac{R^n}{4}] \) and arbitrary \( u_0 \) satisfying \( \int_\Omega u_0 = m \geq C \) and (6), the function \( \phi \) introduced in (37) fulfills

\[
\phi'(t) \geq \frac{C_3}{2} s_0^{\frac{(3-\gamma)(\kappa + 1) + (1 - \frac{1}{\alpha})\kappa}{\alpha}} \cdot \phi \frac{a}{\alpha} (t)
\]

(92)

for all \( t \in S_\psi \cap S_\varphi \) with \( \psi, S_\psi \) and \( S_\varphi \) given by (38), (39) and (40). Moreover, Lemma 4.7 provides that there exists \( C_3 = C_3(k, R, \kappa, \alpha, \gamma, n) > 0 \) such that

\[
\psi(t) \geq C_3 s_0^{\frac{(3-\gamma)(\kappa + 1) + (1 - \frac{1}{\alpha})\kappa}{\alpha}} \cdot \phi \frac{a}{\alpha} (t) \text{ for all } t \in S_\psi.
\]

(93)
Next, to specify our choice of $s_0$, we choose $s_0 \in (0, \frac{R^n}{4}]$ small enough such that
\begin{equation}
  s_0 \leq \frac{M}{2} \tag{94}
\end{equation}
and
\begin{align}
  s_0 &\leq \min \left\{ \left( \frac{C_3}{8C_{13}} \right)^{\frac{1}{2}} \cdot \left( \frac{M}{2(1-\gamma)(2-\gamma)\omega_n} \right)^{\frac{n+1}{2n}} \right. \\
  &\quad \left. \left( \frac{C_3}{8C_{13}} \right)^{\frac{1}{2n}(1+\varepsilon+\alpha)} \cdot \left( \frac{M}{2(1-\gamma)(2-\gamma)\omega_n} \right)^{\frac{2-\gamma}{2n-1}} \right\}, \tag{95}
\end{align}
where $A$ is positive constants given in (104), as well as
\begin{equation}
  s_0 \leq \left\{ C_3^{\frac{1}{2n}} \frac{M}{2(1-\gamma)(2-\gamma)\omega_n} \right\}^{\frac{2-\gamma}{2n-1}}. \tag{96}
\end{equation}
Then, we fix $\epsilon = \epsilon(k, \alpha, \kappa, m, R)$ to be small such that $\epsilon < s_0^2$ and see that
\begin{equation}
  \int_{s_0}^{s_0} s^{-\gamma}(s_0 - s) ds \not\rightarrow \frac{s_0^{2-\gamma}}{(1-\gamma)(2-\gamma)} \text{ as } \delta \searrow 0. \tag{97}
\end{equation}
It’s possible to find $s_* = s_*(k, \kappa, \alpha, m, R)$ such that
\begin{equation}
  \frac{m - \epsilon}{\omega_n} \cdot \int_{s_*}^{s_0} s^{-\gamma}(s_0 - s) ds > \frac{m - s_0}{(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma}. \tag{98}
\end{equation}
Setting $r_* = r_*(k, \kappa, \alpha, m, R) := s_*^\frac{1}{2} \in (0, R)$, we assume for contradiction that then we have $T_{\text{max}} = +\infty$. The expression of $\phi$ in (37) is defined throughout $[0, +\infty)$ and letting
\begin{equation}
  S := \left\{ T > 0 \mid \phi(t) > \frac{M - s_0}{(1-\gamma)(2-\gamma)\omega_n} \cdot s_0^{2-\gamma} \text{ for all } t \in [0, T] \right\}, \tag{99}
\end{equation}
we note that $S$ is not empty due to the continuity of $\phi$ at 0. In fact, by (12) we estimate
\begin{equation}
  w(s, 0) \geq w(s_*, 0) = \frac{1}{\omega_n} \int_{B_{r_*}(0)} u_0 dx \geq \frac{m - \epsilon}{\omega_n} \text{ for all } s \in (s_*, R^n). \tag{100}
\end{equation}
Recalling $M := \max\{C, m\}$ in Lemma 4.1 and the assumption $m > \tilde{C}$, we have $M = m$. This, along with (97), yields
\begin{equation}
  \phi(0) \geq \int_{s_*}^{s_0} s^{-\gamma}(s_0 - s) w(s, 0) ds \\
  \geq \frac{m - \epsilon}{\omega_n} \cdot \int_{s_*}^{s_0} s^{-\gamma}(s_0 - s) ds \\
  > \frac{m - s_0}{(1-\gamma)(2-\gamma)\omega_n} \cdot s_0^{2-\gamma} \\
  = \frac{M - s_0}{(1-\gamma)(2-\gamma)\omega_n} \cdot s_0^{2-\gamma}. \tag{101}
\end{equation}
Therefore, $T := \sup S$ is well-defined element of $(0, +\infty]$ which is such that
\begin{equation}
  \phi(t) > \frac{M - s_0}{(1-\gamma)(2-\gamma)\omega_n} \cdot s_0^{2-\gamma} \text{ for all } t \in (0, T). \tag{102}
\end{equation}
In particular, thanks to (94), we arrive at
\[
\phi(t) \geq \frac{M}{2(1 - \gamma)(2 - \gamma)\omega_n} \cdot s_0^{2 - \gamma} \text{ for all } t \in (0, T).
\] (102)

Moreover, (95) and (102) ensure that
\[
C_3 \frac{8}{s_0} \left[ \psi \left( \frac{\frac{3}{2} - \frac{1}{\alpha} + (1 - \frac{1}{\alpha})}{M} + \alpha \right) \right] \geq \left[ 3 - \gamma - \frac{2\alpha}{n} + \frac{1}{\alpha}(\kappa + 1) \right] \left[ \frac{\kappa + 1 - \alpha}{\kappa + 1} \right] \left[ \frac{2\alpha}{n} + 1 + (\kappa + 1)(1 - \frac{1}{\alpha}) \right] > 0.
\] (104)

Similarly, (95) and (102) ensure that
\[
C_3 \frac{8}{s_0} \left[ \psi \left( \frac{M}{2(1 - \gamma)(2 - \gamma)\omega_n} \right) \right] \geq \left[ 3 - \gamma - \frac{2\alpha}{n} + \frac{1}{\alpha}(\kappa + 1) \right] \left[ \frac{\kappa + 1 - \alpha}{\kappa + 1} \right] \left[ \frac{2\alpha}{n} + 1 + (\kappa + 1)(1 - \frac{1}{\alpha}) \right] > 0\text{ for all } t \in (0, T).
\] (105)

From Lemma 4.7, (96) and (102), the defined function \( \psi \) in (38) satisfies
\[
\psi(t) \geq C_3 \frac{8}{s_0} \left[ \psi \left( \frac{M}{2(1 - \gamma)(2 - \gamma)\omega_n} \right) \right] \geq \left[ 3 - \gamma - \frac{2\alpha}{n} + \frac{1}{\alpha}(\kappa + 1) \right] \geq \frac{3}{s_0} \text{ for all } t \in (0, T).
\] (106)

Particularly, (101) and (106) ensure that for the sets \( S_\phi \) and \( S_\psi \) from (39) and (40) we have
\[
(0, T) \subset S_\phi \cap S_\psi.
\]
which in conjunction with (92), (103) and (105) shows that for all $t \in (0, T)$
\[
\phi'(t) \geq \frac{C_3}{4} - \frac{(3-\gamma)(\kappa+1-\alpha)+(1-\frac{1}{\alpha})(\kappa+1)}{\alpha} \phi^{\frac{\alpha+1}{\alpha}}(t).
\] (107)

On the one hand, we can claim $T = +\infty$. Otherwise, we have $T < +\infty$ such that $\phi(T) = \frac{M - s_0}{(1-\gamma)(2-\gamma)\omega_n} \cdot s_0^{2-\gamma}$ by definition of $S$ and the continuity of $\phi$, which is incompatible with the fact that $\phi(t) \geq \phi(0) > \frac{M - s_0}{(1-\gamma)(2-\gamma)\omega_n} \cdot s_0^{2-\gamma}$ due to (107) and (100). On the other hand, integrating (107) over $(0, t)$ and let $t \to T$, we obtain
\[
T \leq \frac{4\alpha s_0^{(3-\gamma)(\kappa+1-\alpha)+(1-\frac{1}{\alpha})(\kappa+1)}}{C_3(\kappa+1-\alpha)\phi^{\frac{\alpha+1}{\alpha}}(0) < +\infty,
\]
where we have used the positivity of $\kappa + 1 - \alpha$ and $\phi(0)$. This contract with the claim $T = +\infty$. Consequently, our assumption $T_{\max} = +\infty$ is false, which implies that $T_{\max}$ must be finite. □

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