$T_c$ for homogeneous dilute Bose gases: a second-order result

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Abstract

The transition temperature for a dilute, homogeneous, three-dimensional Bose gas has the expansion

$$T_c = T_0 \{1 + c_1 an^{1/3} + [c_2' \ln(an^{1/3}) + c_2'']a^2 n^{2/3} + O(a^3n)\},$$

where $a$ is the scattering length, $n$ the number density, and $T_0$ the ideal gas result. The first-order coefficient $c_1$ depends on non-perturbative physics. In this paper, we show that the coefficient $c_2'$ can be computed perturbatively. We also show that the remaining second-order coefficient $c_2''$ depends on non-perturbative physics but can be related, by a perturbative calculation, to quantities that have previously been measured using lattice simulations of three-dimensional O(2) scalar field theory. Making use of those simulation results, we find

$$T_c \approx T_0 \{1 + (1.32 \pm 0.02) an^{1/3} + [19.7518 \ln(an^{1/3}) + (75.7 \pm 0.4)]a^2 n^{2/3} + O(a^3n)\}.$$

I. INTRODUCTION

Long-distance physics at a second-order phase transition is generically non-perturbative. For this reason, researchers have found it non-trivial to compute corrections to the ideal gas result for the critical temperature $T_c$ for Bose-Einstein condensation (BEC) of a dilute, homogeneous Bose gas in three dimensions. It is currently understood that the correction $\Delta T_c \equiv T_c - T_0$ to the ideal gas result $T_0$ behaves parametrically as

$$\frac{\Delta T_c}{T_0} \to c_1 an^{1/3} \quad (1.1)$$

in the dilute (or, equivalently, weak-interaction) limit, where $c_1$ is a numerical constant, $a$ is the scattering length, which parameterizes the low energy 2-particle scattering cross-section, and $n$ is the density of the homogeneous gas. We will assume that the interactions are repulsive ($a > 0$). A clean argument for (1.1) may be found in Ref. [1], which also shows how the problem of calculating the constant $c_1$ can be reduced to a problem in three-dimensional O(2) field theory. Recent numerical simulations of that theory have obtained the results $c_1 = 1.29 \pm 0.05$ [2] and $c_1 = 1.32 \pm 0.02$ [3].

In this paper, we shall extend the result for $T_c(n)$ to second order in $a$ for a homogeneous Bose gas. This is also the relationship between $T_c$ and the central number density for a
Bose gas in an arbitrarily wide trap. (In contrast, the relationship between $T_c$ and the total number $N_p$ of particles in a trap depends on somewhat different physics. A second-order result for $T_c(N_p)$ in an arbitrarily wide trap may be found in Ref. [6].)

In the homogeneous case, Holzmann, Baym, and Laloe [5] have recently argued that a logarithmic term appears at second order,

$$\frac{\Delta T_c}{T_0} \longrightarrow c_1 a n^{1/3} + c_2' a^2 n^{2/3} \ln(a n^{1/3}) + \cdots,$$

and they made a rough estimate of the coefficient $c_2'$ using large-$N$ arguments. A similar logarithm has been found for $T_c(N_p)$ in the case of trapped gases [6]. We will show that, in contrast to $c_1$, the coefficient $c_2'$ of the logarithm can be computed exactly using perturbation theory. Our result is

$$c_2' = -\frac{64\pi \zeta\left(\frac{1}{2}\right)}{3 \left[\zeta\left(\frac{3}{2}\right)\right]^{5/3}},$$

where $\zeta(s)$ is the Riemann zeta function. We will compute this result by performing a second-order perturbative calculation to match the physics of the transition onto that of three-dimensional O(2) scalar field theory. The same matching calculation will also perturbatively determine the relationship between the non-logarithmic term at second order and certain non-perturbative quantities in O(2) scalar theory which have been previously measured in lattice simulations. As a result, we will determine all the coefficients in the second-order expansion

$$\frac{\Delta T_c}{T_0} = c_1 a n^{1/3} + \left[c_2' \ln(a n^{1/3}) + c_2''\right] a^2 n^{2/3} + O(a^3 n),$$

[where the notation $O(a^3 n)$ is not intended to make any particular claim about what powers of logarithms might appear at third order].

We should emphasize that when we refer to the “first order” and “second order” terms in (1.4), we do not mean first and second order in perturbation theory. Perturbation theory breaks down for these quantities, and that breakdown manifests as the appearance of infrared (IR) infinities beyond a certain order in the perturbative expansion.

A portion of the required perturbative matching calculations has already been performed in Ref. [6], which also gives a discussion of the philosophy and methods of perturbative matching calculations between Bose gases and three-dimensional O(2) field theory at the phase transition. However, we find it convenient to use slightly different conventions than Ref. [6]. For the sake of introducing conventions and notation, and for the sake of making this article somewhat self-contained, we will use the remainder of this introduction to briefly review matching and the different distance scales associated with physics at the transition. In section II, we will fix the ultraviolet (UV) regularization and renormalization schemes we will use for our calculation. We then proceed to do the matching calculations and assemble all the matching results in section III, though we leave the details of the more intricate diagrammatic calculations for later sections. In section IV, we then put together our final results for the relationship between $T$ and $n$ at the transition. Section V gives the details of how to calculate the most complicated diagram that was needed for matching. Section VI
reproduces some previous results related to the critical value \( \mu_c(T) \) of the chemical potential \( \mu \) in a new form that is needed for our analysis. Finally, section VII explains how our result for the second-order logarithm is modified in theories with \( N \) fields, for the sake of readers who may wish to compare exact results to the approximate large-\( N \) analysis of Holzmann, Baym, and Laloë [5]. A very brief outline of how to calculate a few simple finite-temperature integrals in dimensional regularization is left for an appendix.

For the calculations in this paper, it will be convenient to follow Baym et al. [1] and calculate the critical density \( n_c(T) \) as a function of \( T \) rather than the critical temperature \( T_c(n) \) as a function of \( n \). We will then obtain the formula for \( T_c(n) \) by inverting the relationship. The ideal gas result \( n_0(T) \) for \( n_c \) is

\[
n_0(T) = \frac{\zeta(\frac{3}{2})}{\lambda^3}, \tag{1.5}
\]

where

\[
\lambda \equiv \sqrt{\frac{2\pi}{mT}} \tag{1.6}
\]

is the thermal wavelength. (In this paper, we work in units where \( \hbar = k_B = 1 \), where \( k_B \) is Boltzmann’s constant.) The diluteness condition \( an^{1/3} \ll 1 \) for the expansions discussed above can therefore alternatively be expressed as

\[
a \ll \lambda \tag{1.7}
\]

at the transition.

A. Overview of matching to 3-dimensional O(2) theory

Baym et al. [1] were the first to use an effective three-dimensional O(2) scalar field theory to study non-universal long-distance physics of the BEC transition of a dilute Bose gas. A more systematic discussion of how to match the parameters of the O(2) theory to the original problem, in order to study interaction effects beyond leading order, may be found in Ref. [6]. Here, we will briefly review these issues in preparation for doing the matching calculations that we will need to obtain \( T_c(n) \) at second order. Perturbative matching calculations, which allow effective theories to be used to calculate non-universal quantities, can be performed whenever the short-distance physics described by the effective theory is perturbative, at a scale where the effective theory is still applicable. Such calculations have a long history that includes lattice field theory [7], Bose condensation at zero temperature [8], relativistic corrections to non-relativistic QED [9], heavy quark physics [10], ultra-relativistic plasmas [11], and non-relativistic plasma physics [12]. For a general discussion, see also Ref. [13].

The starting point is the well-known description of a dilute Bose gas by a second-quantized Schrödinger equation, together with a chemical potential \( \mu \) that couples to particle number density \( \psi^* \psi \), and a \( |\psi|^4 \) contact interaction that reproduces low-energy scattering [14]. The corresponding Lagrangian is

\[
\mathcal{L} = \psi^* \left( i \partial_t + \frac{1}{2m} \nabla^2 + \mu \right) \psi - \frac{2\pi a}{m} (\psi^* \psi)^2. \tag{1.8}
\]
This effective description is valid for distance scales large compared to the scattering length \(a\). Corrections to this description, due to the energy dependence of the cross-section or 3-body interactions or so forth, do not affect \(T_c(n)\) at second order (see section [IV]). At finite temperature, it is convenient to study \((1.8)\) using the imaginary time formalism, in which \(t\) becomes \(-i\tau\) and imaginary time \(\tau\) is periodic with period \(\beta = 1/T\). The imaginary-time action is then

\[
S_I = \int_0^\beta d\tau \int d^3x \left[ \psi^* \left( \partial_\tau - \frac{1}{2m} \nabla^2 - \mu \right) \psi + \frac{2\pi a}{m} (\psi^* \psi)^2 \right].
\] (1.9)

We shall call this the 3+1 dimensional theory, referring to three spatial dimensions plus one (imaginary) time dimension. The expectation value of the number density is given by

\[
n = \langle \psi^* \psi \rangle.
\] (1.10)

As usual, the field \(\psi\) can be decomposed into imaginary-time frequency modes \(\psi_j\) with discrete Matsubara frequencies \(\omega_j = 2\pi j/\beta\), where \(j\) is an integer. If we ignore interactions for a moment, and treat \(\mu\) as small, then non-zero Matsubara frequency modes in \((1.9)\) are associated with a correlation length of order \((2m\omega_j)^{-1/2} \lesssim \lambda\), where \(\lambda\) is the thermal wavelength \((1.6)\). Near the transition, at distance scales large compared to the thermal wavelength \(\lambda\), all the modes with non-zero Matsubara frequencies decouple, leaving behind an effective theory of just the zero-frequency modes \(\psi_0(x)\). If one were to naively throw away the non-zero frequency modes from the original 3+1 dimensional action \((1.9)\), it would reduce to

\[
S_I \rightarrow \beta \int d^3x \left[ \psi_0^* \left( -\frac{1}{2m} \nabla^2 - \mu \right) \psi_0 + \frac{2\pi a}{m} (\psi_0^* \psi_0)^2 \right].
\] (1.11)

This is the rough form of the effective 3-dimensional theory of \(\psi_0\), which describes long-distance physics at the transition for time-independent quantities. However, completely ignoring the effects of non-zero frequency modes was an oversimplification. In field theories, the short-distance and/or high-frequency modes do have effects on long-distance physics, but those effects can be absorbed into (1) a modification of the strengths of relevant interactions (in the sense of the renormalization group) between the long-distance/zero-frequency fields, and (2) the appearance of additional marginal and irrelevant interactions between the long-distance/zero-frequency modes. The latter effect will not be relevant at second order for \(T_c(n)\) (see section [IV]). Because of the first effect, the correct three-dimensional effective theory is of the more general form

\[
S_3 = \beta \int d^3x \left[ \psi_0^* \left( -\frac{Z_\psi}{2m} \nabla^2 - \mu_3 \right) \psi_0 + Z_a \frac{2\pi a}{m} (\psi_0^* \psi_0)^2 + f_3 \right],
\] (1.12)

where the difference of the \((\psi_0\text{-independent})\) parameters \(Z_\psi, \mu_3, Z_a, \) and \(f_3\) from the naive values \(Z_\psi = Z_a = 1, \mu_3 = \mu, \) and \(f_3 = 0\) of \((1.11)\) incorporates the effects of short-distance physics on long-distance physics. These parameters can be computed perturbatively because short-distance physics is perturbative. The above three-dimensional theory is super-renormalizable and has UV divergences associated with the parameters \(\mu_3\) and \(f_3\). The coefficients of the other terms, however, have a simple finite relationship to the parameters of the original theory. The parameter \(f_3\) represents the contributions of the non-zero Matsubara
frequency modes to the free energy density (along with any associated UV counterterms of the three-dimensional theory).

It is often conventional to rescale the field of the effective three-dimensional theory (1.12) as

$$\psi = \left( \frac{mT}{Z_\psi} \right)^{1/2} (\phi_1 + i\phi_2)$$  \hspace{1cm} (1.13)

and write

$$S_3 = \int d^3 x \left[ \frac{1}{2} |\nabla \phi|^2 + \frac{r_{\text{bare}}}{2} \phi^2 + \frac{u}{4!} (\phi^2)^2 + F_3 \right]$$,  \hspace{1cm} (1.14)

where $\phi = (\phi_1, \phi_2)$ is a real 2-vector, $\phi^2 \equiv \phi_1^2 + \phi_2^2$, and

$$r_{\text{bare}} = -\frac{2m\mu_3}{Z_\psi}, \quad u = \frac{96\pi^2 a}{\lambda^2} \frac{Z_a}{Z_\psi^2}, \quad F_3 = \beta f_3.$$  \hspace{1cm} (1.15)

This is O(2) scalar field theory in three dimensions. This form makes it easy to understand the scale at which physics becomes non-perturbative. The only parameters of the $\phi$-dependent part of $S_3$ are $r_{\text{bare}}$ and $u$. For fixed $u$, imagine finding the $r_{\text{bare}}(u)$ that corresponds to the phase transition. Then all correlations at the phase transition can be considered as determined by $u$. By dimensional analysis, the distance scale of non-perturbative physics is therefore $1/u$. $Z_a$ and $Z_\psi$ turn out to be perturbatively close to 1, and so this scale is $1/u \sim \lambda^2/a$ by (1.15).

Perturbation theory in the three-dimensional theory (1.14) is an expansion in $u$. Consider the dimensionless cost of each order of perturbation theory. By dimensional analysis, the contribution to that cost by physics at a momentum scale of order $p$ must be order $u/p$. This means that perturbation theory breaks down for distance scales $p^{-1} \gtrsim u^{-1}$. It also means that, thanks to (1.14), perturbation theory works fine for distance scales $p^{-1} \lesssim \lambda$ at the transition, which are the distance scales of the non-zero Matsubara frequency modes. This is the reason that those modes can be treated perturbatively and a perturbative matching calculation is possible.

To compute the number density at a given temperature and chemical potential, it is convenient to rewrite $n = \langle \psi^* \psi \rangle$ as

$$n = (\beta V)^{-1} \frac{\partial}{\partial \mu} \ln Z = - (\beta V)^{-1} \left\langle \frac{\partial S_3}{\partial \mu} \right\rangle.$$  \hspace{1cm} (1.16)

In the equivalent three-dimensional description (1.12), this becomes

$$n \simeq - (\beta V)^{-1} \left\langle \frac{\partial S_3}{\partial \mu} \right\rangle = - \frac{\partial Z_\psi}{\partial \mu} \frac{1}{2m} \langle |\nabla \psi_0|^2 \rangle + \frac{\partial \mu_3}{\partial \mu} \langle \psi_0^* \psi_0 \rangle - \frac{\partial Z_a}{\partial \mu} \frac{2\pi a}{m} \langle (\psi_0^* \psi_0)^2 \rangle - \frac{\partial f_3}{\partial \mu},$$  \hspace{1cm} (1.17)

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1 The rescaling used here differs from that of Ref. [6] by a factor of $Z_\psi^{-1/2}$. This difference of convention is actually moot since $Z_\psi = 1$ at second order.
where all the expectations are taken in the purely three-dimensional theory (and UV-
regularized, as necessary). The only approximation made here is ignoring corrections that
would appear as higher-dimensional interactions in (1.12) which, as we’ve already said we
will review later, do not affect the calculation of $T_c$ at second order. Also, we shall see that
the derivative $\langle |\nabla \phi|^2 \rangle$ term and the quartic $\langle (\psi_0^* \psi_0)^2 \rangle$ term are not relevant at second order, so that one may simply take

$$n \simeq \frac{\partial \mu_3}{\partial \mu} \langle \psi_0^* \psi_0 \rangle - \frac{\partial f_3}{\partial \mu}. \quad (1.18)$$

A very similarly structured matching calculation was undertaken in Ref. [4] for matching
the continuum 3-dimensional theory to a lattice 3-dimensional theory, where the purpose
was to match the two theories order by order in the lattice spacing, in order to improve
approach to the continuum limit in numerical simulations. The structure of that calculation,
and the topology of the required perturbative diagrams, is identical to what we will need
for the present task. The only substantial difference is that we will be evaluating those
diagrams in the 3+1 dimensional theory, rather than 3-dimensional lattice theory, and that
the perturbative expansion will correspond to an expansion in the scattering length (or $an^{1/3}$)
rather than the lattice spacing. There is additionally a trivial difference in presentation: Ref.
[4] did not keep track of $\phi$-independent terms analogous to $f_3$, which we use in (1.18), but
instead discussed the matching of $\psi^* \psi$ directly.

The generic technology of matching calculations was reviewed in the context of our
current problem in section IV.A of Ref. [6]. The idea is to perturbatively calculate an
identical finite set of physical infrared quantities in the 3+1 dimensional and 3 dimensional
theories, and then equate the answers to determine the parameters of the 3 dimensional
effective theory. Each of the calculations must be IR-regulated, but the dependence on the
choice of IR regulator will disappear in the final result of the matching (provided the same
IR regulator is used for both theories). In our case, this is because matching is accounting
for the differences of the two theories at the short distance scales ($\lesssim \lambda$) associated with the
non-zero Matsubara frequency modes. For the specific purpose of the matching calculation,
$\mu$ can be formally treated as a perturbation, along with the quartic interaction proportional
to $a$. That is because the short distance scales $\lesssim \lambda$ are associated with energies per particle
$\gtrsim T$, which is large compared to the chemical potential $\mu$ at (or very near) the transition.
With $\mu$ treated perturbatively, the imaginary time Feynman rules are given in Table I. We
will use the notation $k_0, l_0, p_0, \ldots$ to designate the Matsubara (imaginary time) frequencies
associated with propagators with momenta $k, l, p, \ldots$, and have introduced the short-hand
notation

$$\omega_k \equiv \frac{k^2}{2m}. \quad (1.19)$$

In order to streamline calculations later on, it is useful to review the fact that the critical
value $\mu_c(T)$ of chemical potential has a somewhat different dependence on non-perturbative
physics than the critical value $n_c(T)$ of the density. Whereas $n_c(T)$ becomes non-perturbative
at first order in interactions, non-perturbative effects do not enter the calculation of $\mu_c(T)$
until second order. (See Ref. [6] for a discussion.) To determine $\mu_c(T)$ to first order, it is
adequate to do a purely perturbative calculation directly in the original 3+1 dimensional
TABLE I. Feynman rules, appropriate for a matching calculation in a uniform Bose gas, for the original 3+1 dimensional theory (1.9) of $\psi$ and the effective three-dimensional theory (1.12) of $\psi_0$. The variable $k_0$ represents the Matsubara frequency of the field, while $\omega_k \equiv k^2/2m$. At finite temperature, loop frequencies $l_0$ are summed over the discrete values $l_0 = 2\pi n T$ with $n$ any integer. In dimensional regularization with the MS renormalization scheme, a factor of $M^\epsilon = (e^{\pi\epsilon/2M/\sqrt{4\pi}})^\epsilon$ should also be associated with each 4-point vertex but has not been explicitly shown above.

$$\chi^{-1} = \mu c + O(a^2)$$

FIG. 1. The vanishing of the inverse susceptibility at the phase transition, expressed in terms of diagrams at first order in $a$.

theory. In particular, one can calculate the inverse susceptibility $\chi^{-1}$ of $\psi$ and set it to zero to determine the transition point, as in Fig. 1. Such a purely perturbative calculation would be inadequate at second order, for which one can instead marry perturbative matching calculations with non-perturbative results from the 3 dimensional theory [6].

II. UV REGULARIZATION

Before starting a detailed calculation, we have to choose our convention for regulating ultraviolet divergences in our 3+1 and 3 dimensional effective theories. We will use dimensional regularization, replacing the 3 spatial dimensions by $d = 3 - \epsilon$ dimensions. One convenience of this choice is that, in the 3+1 dimensional theory, loop corrections to the zero-energy $2 \rightarrow 2$ scattering amplitude vanish at zero temperature and zero chemical potential, so that there are no corrections to the identification of the $a$ in (1.9) with the scattering length $\frac{\sqrt{a}}{\bar{A}}$.

We will define UV-renormalized parameters using the modified minimum subtraction (MS) scheme, and we shall call the associated renormalization momentum scale $\bar{M}$. At second order, this will not require any UV subtractions for the 3+1 dimensional theory, which can then be taken to be...
\[
S_1 = \int_0^\beta d\tau \int d^3x \left[ \bar{\psi} \left( \partial_\tau - \frac{1}{2m} \nabla^2 - \mu \right) \psi + M^\epsilon \frac{2\pi a}{m} (\bar{\psi}\psi)^2 \right], \tag{2.1}
\]
where
\[
M \equiv \frac{e^{\gamma_E}/2}{\sqrt{4\pi}} \bar{M}. \tag{2.2}
\]

[The factor of \(e^{\gamma_E}/2/\sqrt{4\pi}\) in (2.2) is what distinguishes modified minimal subtraction (\(\overline{\text{MS}}\)) from unmodified minimal subtraction (\(\text{MS}\)); the difference between the two schemes amounts to nothing more than a multiplicative redefinition of the renormalization scale.] The three-dimensional theory (1.12) is
\[
S_3 = \beta \int d^3x \left[ \bar{\psi} \left( -\frac{Z_\psi}{2m} \nabla^2 - \mu_3 \right) \psi + M^\epsilon Z_\psi \frac{2\pi a}{m} (\bar{\psi}\psi)^2 + M^{\epsilon} f_3 \right], \tag{2.3}
\]
or equivalently
\[
S_3 = \int d^3x \left[ \frac{1}{2} |\nabla \phi|^2 + \frac{r_{\text{bare}}}{2} \phi^2 + M^\epsilon \frac{u}{4!} (\phi^2)^2 + M^{\epsilon} f_3 \right]. \tag{2.4}
\]
This theory is super-renormalizable and requires only a finite number of UV counter-terms. In particular, in the \(\overline{\text{MS}}\) renormalization scheme, only the coefficient of \(\phi^2\) is explicitly renormalized, with the exact relation
\[
r_{\text{bare}} = r_{\overline{\text{MS}}} + \frac{1}{(4\pi)^2\epsilon} \left( \frac{u}{3} \right)^2 \tag{2.5}
\]
between the bare coupling \(r_{\text{bare}}\) and the renormalized coupling \(r_{\overline{\text{MS}}} (\bar{M})\).

The reader may wonder why we bother with the continuum 3-dimensional theory, since one will use a lattice-regulated 3-dimensional theory for actual computations of non-perturbative results. One could instead skip the continuum 3-dimensional theory and directly match the 3+1 dimensional theory to the particular lattice theory used for a particular simulation. However, it is more convenient to split this matching into two steps: (1) 3+1 dimensions to continuum 3 dimensions, and (2) continuum 3 dimensions to lattice 3 dimensions. The first step has the virtue of not depending on the details of how the theory is put on the lattice.

### III. THE MATCHING CALCULATION

#### A. What we need

There are two lattice simulation results of the three-dimensional \(O(2)\) theory (1.14) that will turn out to be relevant to our evaluation of the number density \(n\) via (1.18). Quoting values from Ref. [4], they are
\[
\kappa \equiv \frac{\Delta (\phi^2)_c}{u} = -0.001198(17), \tag{3.1a}
\]
\[ \mathcal{R} \equiv \frac{r_c^\text{MS}}{u^2} (\bar{M} = u/3) = 0.001920(2), \]  

(3.1b)

where

\[ \Delta \langle \phi^2 \rangle_c \equiv [\langle \phi^2 \rangle_c]_u - [\langle \phi^2 \rangle_c]_0 \]  

(3.2)

is the difference between the effective theory value of \( \langle \phi^2 \rangle \), at the critical point, for the cases of (i) \( u \) small and (ii) the ideal gas \( u = 0 \). Unlike \( \langle \phi^2 \rangle_c \), the difference \( \Delta \langle \phi^2 \rangle_c \) is an infrared quantity, independent of how the effective theory (1.14) is regularized in the ultraviolet. (Ref. [2] gives an independent and statistically compatible value of \( \kappa \) but did not analyze \( r_c \).) The reason that \( \kappa \) and \( \mathcal{R} \) are pure numbers is dimensional analysis. If one picks the renormalization scale to be of order \( u \), then, at the transition, the only parameter of the O(2) theory is the dimensionful parameter \( u \). The dependence of \( \Delta \langle \phi^2 \rangle_c \) and \( r_c \) on \( u \) is then determined by their dimensions.

In dimensional regularization in the 3 dimensional theory, \( \Delta \langle \phi^2 \rangle_c \) is the same as \( \langle \phi^2 \rangle_c \). This is because, for the case \( u = 0 \), the transition takes place at \( \mu_3 = 0 \), and then

\[ [\langle \phi^2 \rangle_c]_0 = 2 \int \frac{d^3 - \epsilon p}{p^2} = 0 \]  

(3.3)

in dimensional regularization. (The last integral vanishes by dimensional analysis, since in dimensional regularization there is no dimensionful parameter to make up the dimensions of the integral.) Therefore, in the formula (1.18) for \( n(T) \), we can replace the three-dimensional \( \langle \psi_0^\dagger \psi_0 \rangle = Z_\psi^{-1} m T \langle \phi^2 \rangle \) at the phase transition by \( Z_\psi^{-1} m T \kappa u \), to obtain

\[ n_c(T) \simeq \left[ 192 \pi^3 \kappa \frac{a Z_\psi}{\lambda^4 Z_\psi^3} \frac{\partial \mu_3}{\partial \mu} - \frac{\partial f_3}{\partial \mu} \right]_{\mu = \mu_c(T)}, \]  

(3.4)

where we have used (1.13) for \( u \). To evaluate \( n_c(T) \) to second order in \( a \propto u \), we therefore need

\[ Z_a, \ Z_\psi, \ \text{and} \ \left( \frac{\partial \mu_3}{\partial \mu} \right)_{\mu_c}, \]

to first order in \( a \) and

\[ \left( \frac{\partial f_3}{\partial \mu} \right)_{\mu_c}, \]

to second order. In evaluating the last quantity, we will find that we will also want \( \mu_3 \) to second order in \( a \), which was computed in Ref. [3].

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2 Technically, for the non-interacting case we must take the limit as the critical point is approached from negative \( \mu \).
FIG. 2. The self energy at first order.

B. Matching $Z_\psi$

$Z_\psi$ can be matched at first order by matching the infrared momentum dependence of the inverse Green function for $\psi_0$ between the 3+1 and 3 dimensional theories. The one-loop contribution to the inverse Green function, shown in Fig. 2, is momentum independent. So

$$Z_\psi = 1 + O(a^2),$$

where $O(a^2)$ indicates corrections that are formally second order in perturbation theory. A slightly more detailed discussion is given in Ref. 6.

In this paper, we will write $O(\cdots)$ when displaying the full parameter dependence of a correction (except possibly for logarithmic factors) and write $O(\cdots)$ when just showing the dependence on a particular parameter. So $32a^2/\lambda^2 = O(a^2/\lambda^2) = O(a^2)$. In matching calculations, where we are formally doing perturbation theory with IR regularization, $O(a^n)$ will just mean $n$-th order in perturbation theory.

We now return to why we could drop the gradient term

$$- \frac{\partial Z_\psi}{\partial \mu} \frac{1}{2m} \langle |\nabla \psi_0|^2 \rangle = - \frac{\partial Z_\psi}{\partial \mu} \frac{T}{2Z_\psi} \langle |\nabla \phi|^2 \rangle$$

(3.6)

from the original matching formula (1.17) for $n$. At the transition, the dimensionally regulated $\langle |\nabla \phi|^2 \rangle$ must be $O(u^3)$ by dimensional analysis. This means that the expression (3.6) is at least third order in the interaction strength $a$ and so irrelevant to our second-order calculation of $n_c$. In fact, it is fifth order, since the dominant contribution to $\partial Z_\psi/\partial \mu$ is $O(a^2)$.

C. Matching $Z_a$

We will determine $Z_a$ by perturbatively matching the four-point Green function of $\psi_0$ at zero momentum. Specifically, to match $Z_a$ at first order, we will compute the (amputated) diagrams of Fig. 3 in both the 3+1 and 3 dimensional theories. We need to choose an IR regulator for these two computations. For this case, we find the most convenient choice to be dimensional regularization, which will now be used to regulate both IR and UV infinities.

In the 3+1 dimensional theory, the diagrams of Fig. 3 with zero external momenta give

$$- \Gamma^{(4)} = - \frac{8\pi a}{m} + \left(- \frac{8\pi a}{m} \right)^2 \left[ \frac{1}{2} \oint \frac{1}{(ip_0 + \omega_p)(-ip_0 + \omega_p)} + 2 \oint \frac{1}{ip_0 + \omega_p} \right] + O(a^3).$$

(3.7)
We introduce the short-hand notations

\[ \mathcal{F}_p \equiv T \sum_{p_0} \int_p, \quad (3.8) \]

\[ \int_p \equiv M^e \int \frac{d^d p}{(2\pi)^d}, \quad (3.9) \]

where \(d = 3 - \epsilon\) is the number of spatial dimensions. The 3-dimensional theory result is the same as (3.7) but with only zero-mode contributions included and factors of \(Z_a\) and \(Z_\psi\) inserted, so that

\[ -\Gamma^{(4)} = -Z_a \frac{8\pi a}{m} + \left( -Z_a \frac{8\pi a}{m} \right)^2 \left[ \frac{T}{2} \int_p \frac{Z_\psi^{-2}}{\omega_p^2} + 2T \int_p \frac{Z_\psi^{-2}}{\omega_p^2} \right] + \mathcal{O}(a^3). \quad (3.10) \]

Equating the 3+1 dimensional result (3.7) and 3 dimensional result (3.10), and keeping in mind that \(Z_a = 1 + \mathcal{O}(a)\) and \(Z_\psi = 1 + \mathcal{O}(a^2)\), we can now solve for \(Z_a\) through second order:

\[ Z_a = 1 - 8\pi a \left[ \frac{1}{2} \int_p \frac{1 - \delta_{p_0,0}}{(ip_0 + \omega_p)(-ip_0 + \omega_p)} \right] + 2 \int_p \frac{1 - \delta_{p_0,0}}{(ip_0 + \omega_p)^2} \right] + \mathcal{O}(a^3), \quad (3.11) \]

where \(\delta_{i,j}\) is the Kronecker delta function. This expression is IR convergent, as it should be. In Appendix A, we derive the dimensionally regulated results

\[ \int_p \frac{1 - \delta_{p_0,0}}{(ip_0 + \omega_p)(-ip_0 + \omega_p)} = M^e \beta \lambda^{-d} \zeta \left( \frac{d}{2} - 1 \right) \frac{a}{\frac{a}{\lambda^2} - 1}, \quad (3.12) \]

\[ \int_p \frac{1 - \delta_{p_0,0}}{(ip_0 + \omega_p)^2} = M^e \beta \lambda^{-d} \zeta \left( \frac{d}{2} - 1 \right), \quad (3.13) \]

in \(d = 3 - \epsilon\) spatial dimensions. The result, after taking \(d \to 3\), is

\[ Z_a = 1 - 12 \zeta \left( \frac{1}{2} \right) \frac{a}{\lambda} + \mathcal{O} \left( \frac{a^2}{\lambda^2} \right). \quad (3.14) \]
For future reference, it’s worth briefly stepping through the same calculation if we had separately evaluated the 3+1 dimensional and 3 dimensional contributions to the matching (the 1 and $\delta_{p,0}$ pieces of $1 - \delta_{p,0}$), which are individually ill-defined without specifying a consistent IR regulator. We will often find it convenient to use dimensional regularization to regulate the IR (as well as the UV). The three-dimensional integrals are very simple in dimensional regularization,

$$\int \frac{1}{p^2 \omega^2} \propto \int \frac{d^d p}{p^4} = 0,$$

which follows by dimensional analysis. The full 3+1 dimensional piece would then be the same as the $1 - \delta_{p,0}$ results above. For example,

$$\sum \int_{\hat{P}} \frac{1}{(ip_0 + \omega_p)^2} = M^* \beta \lambda^{-d} \zeta \left( \frac{d}{2} - 1 \right).$$

(3.16)

We are now in a position to explain why we could drop the quartic term

$$- \frac{\partial Z_a}{\partial \mu} \frac{2\pi a}{m} \langle (\psi^*_0 \psi_0)^2 \rangle$$

(3.17)

from the original matching formula (1.17) for $n$. From (3.14), or the diagrams of Fig. 4, we see that $Z_a$ is $\mu$-independent at first order. The leading $\mu$-dependent contribution comes from graphs such as Fig. 4, which will produce an $O(\mu a)$ contribution to $Z_a$ and so an $O(a)$ contribution to $\partial Z_a/\partial \mu$. There is an explicit $a$ in (3.17), which brings us up to $O(a^2)$. Finally, there is the factor of

$$\langle (\psi^*_0 \psi_0)^2 \rangle = Z_{\psi}^{-2} m^2 T^2 \langle \phi^4 \rangle.$$

(3.18)

At the transition, the dimensionally-regulated result for $\langle \phi^4 \rangle$ must be order $u^2$ by dimensional analysis. This means that the contribution (3.17) to $n$ is fourth-order in the interaction strength and so irrelevant to a second-order calculation of $n_c(T)$. This argument is almost identical to a similar argument given in Ref. [4] for the matching of the 3 dimensional continuum theory to a 3 dimensional lattice theory.

**D. Matching the $\mu$ dependence of $\mu_3$**

The matching of $\mu_3$ can be accomplished by computing the inverse susceptibility $\chi^{-1} = \Gamma^{(2)}$ in both theories. At second order in $a$, this corresponds to the diagrams of Fig. 5.
For the purpose of counting orders of \( a \), we treat the chemical potential \( \mu \) as \( \mathcal{O}(a) \). That’s because we are ultimately interested in using the 3 dimensional effective theory at the phase transition – that is, for \( \mu = \mu_c \). The ideal gas result is \( \mu_c = 0 \), and the effect of interactions is that \( \mu_c = \mathcal{O}(a) \).

We discussed earlier that we need \( \partial \mu_3 / \partial \mu \) to first order. For this computation, we will find we can ignore the \( \mu \)-independent diagrams of Fig. 5. Specifically, Fig. 5 gives

\[
- \Gamma^{(2)} = \mu - \frac{8\pi a}{m} \mu \sum_{\mathcal{P}} \frac{1}{(i\mathbf{p}_0 + \mathbf{\omega}_p)^2} + (\mu\text{-independent}) + \mathcal{O}(a^3) \tag{3.19}
\]

for the 3+1 dimensional theory and

\[
- \Gamma^{(2)} = \mu_3 - Z_a \frac{8\pi a}{m} \mu_3 T \int_{\mathcal{P}} \frac{Z^{-2}_\mu}{\omega_p^2} + (\mu_3\text{-independent}) + \mathcal{O}(a^3) \tag{3.20}
\]

for the 3 dimensional theory. Equating the two results, order by order in \( a \), yields

\[
\mu_3 = Z_\mu \mu + (\mu\text{-independent}) + \mathcal{O}(a^3), \tag{3.21}
\]

with

\[
Z_\mu = 1 - \frac{8\pi a}{m} \sum_{\mathcal{P}} \frac{1 - \delta_{p_0,0}}{(i\mathbf{p}_0 + \mathbf{\omega}_p)^2}. \tag{3.22}
\]

We then have

\[
\frac{\partial \mu_3}{\partial \mu} = Z_\mu + \mathcal{O}(a^3). \tag{3.23}
\]

The integral in (3.22) is the same as one of those encountered matching \( Z_a \), and the result is

\[
\frac{\partial \mu_3}{\partial \mu} = 1 - 4 \zeta(1/2) \frac{a}{\lambda} + \mathcal{O} \left( \frac{a^2}{\lambda^2} \right). \tag{3.24}
\]

A full result for \( \mu_3 \) at exactly \( \mu = \mu_c \) was derived in Ref. 6. We will rederive the crux of that result in section 6, since we will need it in a different analytic form than presented in the original derivation.
E. Matching the $\mu$ dependence of $f_3$

We now come to the more intricate part of our calculation, which is the calculation of $\partial f_3/\partial \mu$ at the transition. We can match $f_3$ by computing the free energy density $f$ in both theories. Some examples of diagrams which contribute to the $\mu$-dependence of the free energy are shown in Fig. 6. We will find it convenient to instead compute the derivative directly, a diagrammatic representation of which is given by Figs. 7–9, which contain all diagrams contributing through $\mathcal{O}(a^2)$ in perturbation theory. The filled circles are still associated with a factor of $\mu$, but the crosses, which represent factors of $\mu$ that have been hit by a derivative, are associated with a factor of 1. One can also think of these diagrams as representing mixing of the operator $\phi^2$ (represented by the crosses) with the unit operator, which were the words used to describe analogous calculations in Ref. [4].
FIG. 9. The remaining second-order contributions to the $\mu$-derivative of the free energy, which also cancel at second order at the transition.

At the transition, the diagrams of Fig. 8 cancel at second order in $a$, as do the diagrams of Fig. 9, because of the first-order relation of Fig. 1. So we may simplify our task by focusing on just the diagrams of Fig. 7.

At this many loops, organizing all the terms of the matching calculation becomes tedious unless one from the start uses dimensional regularization for the IR (as well as the UV). In the 3-dimensional theory, the loop integrals in Fig. 7 then all vanish by dimensional analysis, leaving us with the formal result

$$-\frac{\partial f}{\partial \mu} = -\frac{\partial f_3}{\partial \mu} \quad (3.25)$$

for dimensionally regulated perturbation theory. In the 3+1 dimensional theory, we have

$$-\left(\frac{\partial f}{\partial \mu}\right)_{\mu_c} = \frac{1}{\int P} \frac{1}{(i p_0 + \omega_p)} + (\mu_c - \mu_c^{[1]}) \frac{1}{\int P} \frac{1}{(i p_0 + \omega_p)^2} + \delta n_{\text{ball}} + O(a^3) \quad (3.26)$$

at the transition, where

$$\mu_c^{[1]} = \frac{8\pi a}{m} \frac{1}{\int P} \frac{1}{i p_0 + \omega_p} \quad (3.27)$$

is the first-order result for $\mu_c$, determined by Fig. 1, and

$$\delta n_{\text{ball}} \equiv \frac{1}{2} \left( -\frac{8\pi a}{m} \right)^2 \frac{1}{\int P Q K} \frac{1}{(i p_0 + \omega_p)^2 (i q_0 + \omega_q) (i k_0 + \omega_k)(i(p_0 + q_0 + k_0) + \omega_{p+q+k})} \quad (3.28)$$

is the contribution of the last diagram of Fig. 7, which we refer to as the basketball diagram.

In section V, we will show how to evaluate this integral in dimensional regularization, with the result that

$$\delta n_{\text{ball}} = \frac{32\pi a^2}{\lambda^{5}} \left[ \frac{\sqrt{\pi}}{2} K_2 + (2K_1 + \ln 2) \zeta(\frac{1}{2}) - \frac{\ln 2}{2\sqrt{\pi}} \left[ \zeta(\frac{1}{2}) \right]^2 \right] + O(\epsilon), \quad (3.29)$$

where $K_1$ and $K_2$ are numerical constants defined by.

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3 Historical note: The origin of this terminology in the literature is associated with the physical appearance of old American Basketball Association basketballs, not current National Basketball Association ones.

4 For efficient numerical evaluation of $K_1$, it’s useful to make a change of integration variables in (3.30), such as $u = \sqrt{1-t}$, which makes the integrand analytic at both endpoints.
\[ K_1 \equiv \frac{1}{4\pi} \int_0^1 \frac{dt}{t} \left[ \left( \text{Li}_{1/2}(t) \right)^2 - \frac{\pi t}{1-t} \right] \approx -0.630\ 568\ 207\ 496\ 069, \quad (3.30) \]

\[ K_2 \equiv \frac{1}{4\pi} \int_0^1 \frac{ds}{s} \frac{dt}{t} \left\{ \text{Li}_{1/2}(s) \text{Li}_{1/2}(t) \text{Li}_{-1/2}(st) \right\} \\
- \frac{st\sqrt{\pi}}{2(1-st)^{3/2}} \left[ \sqrt{\frac{\pi}{1-s}} + \zeta(\frac{1}{2}) \right] \left[ \sqrt{\frac{\pi}{1-t}} + \zeta(\frac{1}{2}) \right] \}
\approx -0.135\ 083\ 353\ 73. \quad (3.31) \]

\( \text{Li}_\nu(z) \) is the polylogarithm function defined by
\[ \text{Li}_\nu(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\nu}. \quad (3.32) \]

[The polylogarithm function is often called \( g_\nu(z) \) in statistical mechanics.]

The other integrals in the matching (3.26) are given by (3.16) and
\[ \sum \int P \frac{1}{ip_0 + \omega_p} = M^2 \lambda^{-d} \zeta \left( \frac{d}{2} \right), \quad (3.33) \]

which is discussed in Appendix [A]. Putting it all together and taking \( d \to 3 \),
\[ - \left( \frac{\partial f_3}{\partial \mu} \right)_{\mu_c} = \zeta(\frac{3}{2}) \lambda^{-3} + \zeta(\frac{1}{2}) \lambda^{-3} (\beta \mu_c - \beta \mu_c^{[1]}) + \delta n_{\text{ball}} + O(a^3). \quad (3.34) \]

In Ref. [8], a second-order matching calculation was carried out for \( \mu_3 \) that determined \( \mu_c \) in terms of the critical value of \( r \) in the 3 dimensional theory. The result was
\[ \beta \mu_c = \beta \mu_c^{[1]} + \frac{32\pi a^2}{\lambda^2} \left[ \ln(\bar{M}\lambda) + C_1 - 72\pi^2 \frac{r_c(\bar{M})}{u^2} \right], \quad (3.35) \]

with \( C_1 \) a numerical constant that was given in that reference in terms of a somewhat inelegant double integral. In section [VI], we show that \( C_1 \) can also be expressed as
\[ C_1 = \frac{1}{2} - \frac{1}{2} \ln(32\pi) - 2K_1. \quad (3.36) \]

Combining the last several formulas, and choosing \( \bar{M} = u/3 \) to make contact with the quoted lattice measurement (B.11) of \( r_c \), we have
\[ - \left( \frac{\partial f_3}{\partial \mu} \right)_{\mu_c} = \zeta(\frac{3}{2}) \lambda^{-3} + \frac{32\pi a^2}{\lambda^5} \left\{ \left[ \ln \left( \frac{a}{\lambda} \right) + \frac{1}{2} \ln(128\pi^3) + \frac{1}{2} - 72\pi^2 R \right] \zeta(\frac{1}{2}) \right\} \\
+ \frac{\sqrt{\pi}}{2} - K_2 - \frac{\ln 2}{2\sqrt{\pi}} \left[ \zeta(\frac{1}{2}) \right]^2 + O \left( \frac{a^3}{\lambda^6} \right). \quad (3.37) \]
IV. FINAL RESULTS

We now have all the elements we need for \( n_c(T) \) as determined by (3.4). The result is

\[
n_c(T) = \lambda^{-3} \left\{ b_0 + b_1 a \lambda^{-1} + \left[ b'_2 \ln(a\lambda^{-1}) + b''_2 a^2 \lambda^{-2} + O(a^3\lambda^{-3}) \right] \right\}, \tag{4.1}
\]

with

\[
b_0 = \zeta(\frac{3}{2}), \tag{4.2a}
\]
\[
b_1 = 192\pi^3 \kappa, \tag{4.2b}
\]
\[
b'_2 = 32\pi \zeta(\frac{1}{2}), \tag{4.2c}
\]
\[
b''_2 = 32\pi \left\{ \left[ \frac{1}{2} \ln(128\pi^3) + \frac{1}{2} - 72\pi^2 \mathcal{R} - 96\pi^2 \kappa \right] \zeta(\frac{1}{2}) + \frac{\sqrt{\pi}}{2} - K_2 - \frac{\ln 2}{2\sqrt{\pi}} \left[ \zeta(\frac{1}{2}) \right]^2 \right\}. \tag{4.2d}
\]

Inverting this formula,

\[
T_c(n) = T_0(n) \left\{ 1 + c_1 a n^{1/3} + \left[ c'_2 \ln(a n^{1/3}) + c''_2 a^2 n^{2/3} + O(a^3 n) \right] \right\}, \tag{4.3}
\]

with

\[
c_1 = -\frac{2}{3} b_0^{-4/3} b_1, \tag{4.4}
\]
\[
c'_2 = -\frac{2}{3} b_0^{-5/3} b'_2, \tag{4.5}
\]
\[
c''_2 = -\frac{2}{3} b_0^{-5/3} b''_2 + \frac{7}{9} b_0^{-8/3} b_1^2 - \frac{1}{3} c'_2 \ln b_0, \tag{4.6}
\]

where the \( \ln(b_0) \) term arises because we have changed the argument of the log from \( \ln(a/\lambda) \) to \( \ln(a n^{1/3}) \). Putting in the lattice results of (3.1), we get the numerical values

\[
b_1 = 7.1(1), \quad b'_2 = -146.8108 \cdots, \quad b''_2 = -587(2), \tag{4.7}
\]
\[
c_1 = 1.32(2), \quad c'_2 = 19.7518 \cdots, \quad c''_2 = 75.7(4). \tag{4.8}
\]

These are our final results.

In our discussion of effective theories, we left out a variety of corrections, such as 3-body interactions or cross-section energy dependence in the original 3+1 dimensional theory, or \( \phi^6 \) and higher-dimensional operators in the 3-dimensional effective theory. Ref. [6] contains a detailed discussion of the parametric size of the resulting corrections to \( T_c(N_p) \) for an arbitrarily wide harmonic trap. The same analysis holds for \( T_c(n) \), with the result that there are no corrections at second order. However, a third-order result would depend not only on the scattering length \( a \) but also on the effective range of the two-body scattering potential.

The relative size of the second-order result obviously depends on the diluteness of the gas and the value of the scattering length, which will vary from experiment to experiment. However, just for fun, let us produce numbers for one particular case. In 1996, Ensher et al. [16] studied the BEC transition for for dilute gases of \(^{87}\)Rb atoms in the \( F=2 \) hyperfine state, trapped in a harmonic trap. The relevant scattering length is \( a = (103 \pm 5) a_0 \).
where $a_0 = 0.0529177 \text{ nm}$ is the Bohr radius. Their transition was at $T \simeq 280 \text{ nK}$. These parameters correspond to $a/\lambda_0 \simeq 0.015$. Let’s now consider a theoretical prediction of the central density $n$ (which, sadly, is not directly accessible experimentally). Eq. (4.1) gives a first-order correction to the ideal gas result $n_0(T)$ of roughly $-4.2\%$ and a second-order correction of roughly $+0.2\%$. As discussed in Ref. [3], where a similar analysis is made of $T_c(N_p)$, this particular trap may not actually be wide enough for our second-order result to be valid.

V. EVALUATING $\delta n_{\text{bball}}$ WITH DIMENSIONAL REGULATION

We will now explain how to obtain the result (3.29) for the basketball diagram in dimensional regularization. We do not know how to carry out the integrations for the basketball (3.28) in arbitrary dimension $d$. Instead, we will find it convenient to manipulate the integrals into a form where we can dispense with dimensional regularization and do integrations in three dimensions. We are currently relying on dimensional regularization to regulate the IR and UV divergences of our calculation. Our first step on the road to dispensing with dimensional regularization of the basketball diagram will be to convert the IR role of dimensional regularization to mass regularization, which will turn out to be more convenient for this particular calculation. To make this conversion consistently, we will first relate the basketball diagram to a combination of diagrams that is IR converge nt. In particular, consider the combination of diagrams $\delta n'_{\text{bball}}$ depicted in Figs. [10] and [11] (ignoring all $Z_\psi$ and $Z_a$ factors). In equations, this is

$$\delta n'_{\text{bball}} \equiv \frac{1}{2} \left( -\frac{8\pi a}{m} \right)^2 \int d^4P d^4Q d^4K \frac{1}{(ip_0 + \omega_p)^2(iq_0 + \omega_q)(ik_0 + \omega_k)(i(p_0 + q_0 + k_0) + \omega_{p+q+k})}$$

$$- T^3 \int d^4p d^4q d^4k \omega_p^2 \omega_q \omega_k \omega_{p+q+k} \Delta \Pi_{\text{sun}}(0) T \int \frac{1}{p \omega_p^2}, \quad (5.1)$$

where

$$\Delta \Pi_{\text{sun}}(0) = \frac{1}{2} \left( -\frac{8\pi a}{m} \right)^2 \int d^4P d^4Q d^4K \frac{1}{(ip_0 + \omega_p)(ik_0 + \omega_k)(i(q_0 + k_0) + \omega_{q+k})}$$

$$- T^2 \int d^4q d^4k \omega_q \omega_k \omega_{q+k} \Delta \Pi_{\text{sun}}(0) \quad (5.2)$$

is the difference between the “sunset” diagram contributions to the self-energy at zero momentum in the 3+1 dimensional and 3 dimensional theories. The above can be combined into

$$\delta n'_{\text{bball}} = \frac{1}{2} \left( -\frac{8\pi a}{m} \right)^2 \int d^4P d^4Q d^4K \frac{1 - \delta_{p_0,0}\delta_{q_0,0}\delta_{k_0,0}}{(ip_0 + \omega_p)^2(iq_0 + \omega_q)(ik_0 + \omega_k)}$$

$$\times \left[ \frac{1}{i(p_0 + q_0 + k_0) + \omega_{p+q+k}} - \frac{\delta_{p_0,0}}{i(q_0 + k_0) + \omega_{q+k}} \right]. \quad (5.3)$$

This expression is IR convergent. (Actually, it is absolutely convergent in the IR only if one first averages the integrand over $p \to -p$. This is a technical point that won’t have any impact, and we shall implicitly assume such averaging wherever required.)
In fact, in dimensional regularization, the original basketball (3.28) is equal to the IR-convergent version (5.1). That’s because the explicit three-dimensional integrals of the terms that have been added to $\delta n_{\text{bball}}$ in (5.1) all vanish in dimensional regularization by dimensional analysis, as in (3.15).

So we can now focus on (5.3). Because it is IR convergent, it will not be changed if we add an infinitesimal negative chemical potential $\delta \mu$ to the propagators. We then re-expand the terms into the form of (5.1), where the IR divergences of the individual terms are now regulated by $\delta \mu$:

$$\delta n_{\text{bball}} = \lim_{\delta \mu \to 0^-} \left[ \text{Eq. (5.1) with } \omega \to \omega^+ \text{ and } \Pi \to \Pi^+ \right],$$

(5.4)

$$\Delta \Pi^+_\text{sun}(0) \equiv \left[ \text{Eq. (5.2) with } \omega \to \omega^+ \right],$$

(5.5)

where

$$\omega^+ \equiv \omega - \delta \mu.$$  

(5.6)

Henceforth, the limit $\delta \mu \to 0^-$ will be implicit when we discuss $\delta n_{\text{bball}}$.

A. The mass-regulated basketball

We will now focus on the first term of (5.4),

$$\delta n_1 \equiv \frac{1}{2} \left( \frac{8\pi a}{m} \right)^2 \int_{PQK} \frac{1}{(ip_0 + \omega_p^+)(iq_0 + \omega_q^+)(ik_0 + \omega_k^+)(i(p_0 + q_0 + k_0) + \omega_{p+q+k}^+)},$$

(5.7)
which is just our original basketball diagram with an infinitesimal chemical potential $\delta \mu$, and with the UV still regulated with dimensional regularization. This can be expressed as the derivative with respect to $\delta \mu$ of the corresponding contribution

$$P_1 \equiv \frac{1}{8} \left( -\frac{8\pi a}{m} \right)^2 \int_{PQK} \frac{1}{(ip_0 + \omega_+^*)(iq_0 + \omega_+^*)(ik_0 + \omega_+^*)(i(p_0 + q_0 + k_0) + \omega_+^{p+q+k})} \quad (5.8)$$

to the pressure, corresponding to the last diagram of Fig. 6. For arbitrary chemical potential (not just the infinitesimal case), this contribution was derived in Ref. [6] for UV dimensional regularization, and simply reproduces a corresponding portion of a 1957 calculation by Huang, Yang, and Luttinger [18], which used a different UV regulator. The result (after taking $d \to 3$) is

$$P_1 = \frac{8Ta^2}{\lambda^5} \sum_i \sum_j \sum_k z^{i+j+k} \quad (5.9)$$

where $z$ is the corresponding fugacity. In our case,

$$z = e^{\beta \delta \mu}, \quad (5.10)$$

which is infinitesimally less than one and is serving the role of an IR regulator. Differentiation gives

$$\delta n_1 = \frac{8a^2}{\lambda^5} S(z), \quad (5.11)$$

with

$$S(z) \equiv \sum_{ijk} \frac{(i + j + k) z^{i+j+k}}{(i + k)(j + k)(ijk)^{1/2}}. \quad (5.12)$$

Our task is to find the $\delta \mu \to 0^-(z \to 1^-)$ behavior of this sum, extracting any divergences and the finite remainder. It will be convenient to define

$$\alpha \equiv -\beta \delta \mu, \quad z = e^{-\alpha}, \quad (5.13)$$

so that the limit of interest is $\alpha \to 0^+$.

Now rewrite the sum as

$$S(z) = \sum_{ijk} \frac{(i + j + k) z^{i+j+k}}{(ijk)^{1/2}} \int_0^1 ds \int t^s \ s^{i+k+j+k} \quad (5.14)$$

If one does the sums before the integrals, the sums can then be factorized, giving

$$S(z) = \int_0^1 ds \int t^s \sum_{ijk} s^{i+k+j+k} \left[ 2 \left( \frac{i}{jk} \right)^{1/2} + \left( \frac{k}{ij} \right)^{1/2} \right]. \quad (5.15)$$
If we removed the mass regularization by simply setting $z = 1$, the above integral would have singularities associated both with (i) $s \to 1$ with $t$ fixed, and (ii) $s$ and $t$ approaching 1 simultaneously. The first type of divergence arises only from the first term in the regulated integral (5.13) and can be eliminated by integrating this term by parts. Using
\[
\frac{d}{ds} \text{Li}_\nu(sz) = \frac{1}{s} \text{Li}_{\nu-1}(sz),
\]
integration by parts gives
\[
\int_0^1 \frac{ds}{s} \text{Li}_{1/2}(sz) \text{Li}_{1/2}(stz) = \text{Li}_{1/2}(z) \text{Li}_{1/2}(tz) - \int_0^1 \frac{ds}{s} \text{Li}_{1/2}(sz) \text{Li}_{1/2}(stz).
\]
(5.17)

Application to (5.15) then yields
\[
S(z) = S_1(z) - \int_0^1 \frac{ds}{s} \frac{dt}{t} \text{Li}_{1/2}(sz) \text{Li}_{1/2}(tz) \text{Li}_{1/2}(stz),
\]
where
\[
S_1(z) = 2 \text{Li}_{1/2}(z) \int_0^1 \frac{dt}{t} \left[ \text{Li}_{1/2}(tz) \right]^2.
\]
(5.19)

To analyze the behavior of this expression as $z \to 1$, it will be useful to have the following series expansion of polylogarithms [19],
\[
\text{Li}_\nu(e^{-\alpha}) = \Gamma(1 - \nu) \alpha^{\nu - 1} + \sum_{n=0}^{\infty} (-)^n n! \zeta(\nu - n) \alpha^n.
\]
(5.20)

The relevant special cases are
\[
\text{Li}_{1/2}(z) = \sqrt{\frac{\pi}{1 - z}} + \zeta\left(\frac{1}{2}\right) + O(\sqrt{1 - z}) = \sqrt{\frac{\pi}{\alpha}} + \zeta\left(\frac{1}{2}\right) + O(\sqrt{\alpha}),
\]
(5.21)
\[
\text{Li}_{-1/2}(z) = \frac{\sqrt{\pi}}{2(1 - z)^{3/2}} + O((1 - z)^{-1/2}) = \frac{\sqrt{\pi}}{2\alpha^{3/2}} + O(\alpha^{-1/2}).
\]
(5.22)

Let’s separate out the regularization dependence of (5.13) for $S_1$ by writing
\[
\int_0^1 \frac{dt}{t} \left[ \text{Li}_{1/2}(tz) \right]^2 = \int_0^1 \frac{\pi \, dt}{1 - zt} + \int_0^1 \frac{dt}{t} \left\{ \left[ \text{Li}_{1/2}(tz) \right]^2 - \frac{\pi t}{1 - zt} \right\}.
\]
(5.23)

If we just wanted an expression through $O((1 - z)^0) = O(\alpha^0)$, we could now set $z = 1$ in the very last integral. However, the $\text{Li}_{1/2}(z)$ factor in (5.19) has an $O(\alpha^{-1/2})$ singularity, which means we need the expansion of (5.23) through $O(\alpha^{1/2})$. This term is easily obtained by differentiating the last integral in (5.23) with respect to $z$ and then analyzing the dominant piece of the result, which is a singularity (cut off by $z$) as $t \to 1$. The result is
\[
\int_0^1 \frac{dt}{t} \left[ \text{Li}_{1/2}(tz) \right]^2 = -\pi \ln \alpha + 4\pi K_1 - 4 \zeta\left(\frac{1}{2}\right) \sqrt{\pi \alpha} + O(\alpha),
\]
(5.24)
where $K_1$ is as defined in (3.30). So
\[
S_1 = \left[ \sqrt{\frac{\pi}{\alpha}} + \zeta\left(\frac{1}{2}\right) \right] (-2\pi \ln \alpha + 8\pi K_1) - 8\pi \zeta\left(\frac{1}{2}\right) + O(\sqrt{\alpha}).
\] (5.25)

Now let’s return to the expression (5.18) for the sum $S$ and work on isolating the remaining (regulated) divergences associated with $s$ and $t$ simultaneously approaching 1. We isolate the singular pieces of the integrand in (5.18) by writing
\[
S(z) = S_1(z) + S_2(z) - 4\pi K_2 + O(\sqrt{\alpha}),
\] (5.26)

with $K_2$ defined as in (3.31). [The $\zeta\left(\frac{1}{2}\right)^2$ term in $S_2$ doesn’t actually give a singular piece, but including it makes the remainder (3.31) a little more compact to write.] Explicit integration gives
\[
S_2 = -\frac{\pi^{3/2} \ln 4}{\sqrt{\alpha}} + 2\pi \zeta\left(\frac{1}{2}\right) \ln(4\alpha) - \sqrt{\pi} \left[ \zeta\left(\frac{1}{2}\right) \right]^2 \ln 4 + 2\pi^{3/2} + O(\sqrt{\alpha}),
\] (5.28)

and so
\[
S_1 + S_2 = 4\pi \left\{ \sqrt{\frac{\pi}{\alpha}} \left[ -\frac{1}{2} \ln(2\alpha) + 2K_1 \right] + \frac{\sqrt{\pi}}{2} \right\} + (2K_1 - 2 + \ln 2) \zeta\left(\frac{1}{2}\right) - \frac{\ln 2}{2\sqrt{\pi}} \left[ \zeta\left(\frac{1}{2}\right) \right]^2 \right\} + O(\sqrt{\alpha}).
\] (5.29)

The combination of (5.11), (5.26) and (5.29) then gives our result for the mass-regulated 3+1 dimensional basketball,
\[
\delta n_1 = \frac{32\pi a^2}{\lambda^5} \left\{ \sqrt{\frac{\pi}{\alpha}} \left[ -\frac{1}{2} \ln(2\alpha) + 2K_1 \right] + \frac{\sqrt{\pi}}{2} - K_2 + (2K_1 - 2 + \ln 2) \zeta\left(\frac{1}{2}\right) - \frac{\ln 2}{2\sqrt{\pi}} \left[ \zeta\left(\frac{1}{2}\right) \right]^2 \right\} + O(\sqrt{\alpha}),
\] (5.30)

where $K_2$ is defined as in (3.31).

**B. The remaining pieces**

The second diagram of Fig. 10, when mass regulated, gives a contribution to $\delta n_{\text{ball}}$ of
\[
\delta n_2 \equiv -\frac{1}{2} \left( \frac{8\pi a}{m} \right)^2 T^3 \int_{\text{pqk}} \frac{1}{(\omega_1^+)^2 \omega_1^+ \omega_1^+ \omega_1^+ \omega_1^+ \omega_1^+ \omega_1^+ \omega_1^+ \omega_1^+} \frac{1}{(p^2 + M^2)(q^2 + M^2)(k^2 + M^2)((p + q + k)^2 + M^2)},
\] (5.31)

where
\[ \mathcal{M}^2 \equiv -2m \delta \mu = \frac{4\pi \alpha}{\lambda^2} \]  

is infinitesimal. It is not strictly necessary to calculate this term because, by dimension analysis, the last integral is proportional to \(1/\mathcal{M}\) (in three dimensions) and so will only contribute to the cancellation of IR divergences in (5.1) and not the finite remainder. However, it’s reassuring to check the cancellation. From Ref. [11],

\[
\int \frac{1}{(p^2 + \mathcal{M}^2)(q^2 + \mathcal{M}^2)(k^2 + \mathcal{M}^2)(|p + q + k|^2 + \mathcal{M}^2)} = \frac{\mathcal{M}}{16\pi^3} \left[ -\frac{1}{2\epsilon} + \frac{3}{2} \ln \left( \frac{2\mathcal{M}}{\mathcal{M}} \right) + \ln 2 \right] + O(\epsilon). 
\]  

(5.33)

Now differentiate with respect to \(\mathcal{M}^2\) to obtain minus 4 times the corresponding integral in (5.31). Then

\[
\delta n_2 = \frac{32\pi a^2}{\lambda^5} \sqrt{\frac{\pi}{\alpha}} \left[ -\frac{1}{2\epsilon} + \frac{3}{4} \ln \left( \frac{16\pi \alpha}{\mathcal{M}^2 \lambda^2} \right) + \ln 2 - \frac{1}{2} \right].
\]  

(5.34)

Finally, we need the last diagram of Fig. [10] with mass regularization, corresponding to

\[
\delta n_3 = -\Delta \Pi_{\text{sun}}^+(0) T \int_\mathcal{P} \frac{1}{(\omega_p^+)^2}.
\]  

(5.35)

In dimensional regularization,

\[
\int_\mathcal{P} \frac{1}{(p^2 + \mathcal{M}^2)^2} = \frac{\Gamma \left( 2 - \frac{d}{2} \right)}{(4\pi)^{d/2}} \mathcal{M}^{d-4} = \frac{1}{8\pi \mathcal{M}} \left[ 1 - \epsilon \ln \left( \frac{2\mathcal{M}}{M} \right) \right] + O(\epsilon^2),
\]  

(5.36)

giving

\[
\delta n_3 = -\frac{\Delta \Pi_{\text{sun}}^+(0)}{\lambda^3 T} \sqrt{\frac{\pi}{\alpha}} \left[ 1 - \frac{\epsilon}{2} \ln \left( \frac{16\pi \alpha}{\mathcal{M}^2 \lambda^2} \right) + O(\epsilon^2) \right].
\]  

(5.37)

Because of the \(\alpha^{-1/2}\) in this equation, we will need the result for the mass-regulated \(\Delta \Pi_{\text{sun}}^+(0)\) through \(O(\sqrt{\alpha})\). The \(O(\alpha^0)\) piece, which is independent of the IR regulator, was calculated in Ref. [3] as part of calculating \(\mu_c\), and we will rederive it in section [VI]. The result is

\[
\Delta \Pi_{\text{sun}}(0) = -\frac{32\pi a^2 T}{\lambda^2} \left\{ \frac{1}{2\epsilon} + \ln(M \lambda) + C_1 \right\},
\]  

(5.38)

with \(C_1\) as in (5.30). To get the \(O(\sqrt{\alpha})\) piece in the mass-regulated version, we start with the integrals corresponding to evaluating Fig. [11] with an infinitesimal chemical potential, with an infinitesimal chemical potential,

\[
\Delta \Pi_{\text{sun}}^+(0) = \frac{1}{2} \left( -\frac{8\pi a}{m} \right)^2 \int_{\mathcal{Q}\mathcal{K}} \frac{1 - \delta_{q_0,0} \delta_{k_0,0}}{(i\omega_q^+)(i\omega_k^+)(i\omega_{q+k}^+)}. 
\]  

(5.39)

\(^5\) See also Ref. [20], which has a useful collection of dimensionally regulated three dimensional integrals.
Now differentiate with respect to $\alpha = -\beta \delta \mu$,

$$
\frac{\partial}{\partial \alpha} \Delta \Pi^+_{\text{sun}}(0) = -T \left( -\frac{8\pi a}{m} \right)^2 \left\{ \frac{\mathcal{F}}{\mathcal{F}_K} \left[ \frac{1}{(iq_0 + \omega_q^+)^2(ik_0 + \omega_k^+)(i(q_0 + k_0) + \omega_{q+k}^+)} \right] 
\right.
+ \left. \frac{1}{2} \frac{\mathcal{F}}{\mathcal{F}_K} \left[ \frac{1}{(iq_0 + \omega_q^+) (ik_0 + \omega_k^+)(i(q_0 + k_0) + \omega_{q+k}^+)} \right] \right\}. \tag{5.40}
$$

We want to find the divergent $O(\alpha^{-1/2})$ pieces of this expression in order to obtain the $O(\sqrt{\alpha})$ piece of (5.39). The first integral in (5.40) has an IR divergence associated with $q_0 = 0$ and $q \to 0$. In this limit, it can be simplified to

$$
T \int_q \left( \frac{1}{(\omega_q^+)^2} \right) \frac{1}{(ik_0 + \omega_k^+)^2} + O(\alpha^0). \tag{5.41}
$$

The infinitesimal chemical potential can be dropped in the $k$ integral, since $k_0 \neq 0$ cuts off the infrared, and then the integral is given by (3.13). The $q$ integral is proportional to (5.36). So

$$
T \int_q \left( \frac{1}{(\omega_q^+)^2} \right) \frac{1}{(ik_0 + \omega_k^+)^2} = \frac{m^2 \zeta(\frac{1}{2})}{4\pi \lambda^2 \sqrt{\pi \alpha}} + O(\sqrt{\alpha}) + O(\epsilon). \tag{5.42}
$$

The second integral in (5.39) can be evaluated similarly by first making the change of variables $Q \to Q - K$, to get

$$
\mathcal{F}_Q \left[ \frac{1}{(iq_0 + \omega_q^+)^2(ik_0 + \omega_k^+)(i(q_0 - k_0) + \omega_{q-k}^+)} \right] = T \int_q \left( \frac{1}{(\omega_q^+)^2} \right) \frac{1}{(ik_0 + \omega_k^+)(-ik_0 + \omega_k^+)} + O(\alpha^0)
= \frac{m^2 \zeta(\frac{1}{2})}{2\pi \lambda^2 \sqrt{\pi \alpha}} + O(\alpha^0) + O(\epsilon). \tag{5.43}
$$

Putting it all together to get $\partial \Delta \Pi^+_{\text{sun}}(0)/\partial \alpha$ and then integrating gives

$$
\Delta \Pi^+_{\text{sun}}(0) = \Delta \Pi_{\text{sun}}(0) - \frac{64\pi a^2 T}{\lambda^2} \zeta(\frac{1}{2}) \sqrt{\frac{T}{\pi}} + O(\alpha) + O(\epsilon). \tag{5.44}
$$

Combining with (5.37) and (5.38),

$$
\delta n_3 = \frac{32\pi a^2}{\lambda^5} \left\{ \sqrt{\frac{\pi}{\alpha}} \left[ \frac{1}{2\epsilon} + \ln(M\lambda) + C_1 - \frac{1}{4} \ln \left( \frac{16\pi \alpha}{M^2 \lambda^2} \right) \right] + 2 \zeta(\frac{1}{2}) \right\}. \tag{5.45}
$$

Combining (5.30), (5.34), and (5.45) for the three diagrams of Fig. 10 as

$$
\delta n_{bball} = \delta n_1 + \delta n_2 + \delta n_3, \tag{5.46}
$$

together with (3.36) for $C_1$, then yields our final result (3.29) for the dimensionally regulated basketball $\delta n_{bball}$. All the IR divergences cancel, as they should.
VI. REDERIVATION OF $\mu_c$ AND $\Delta \Pi_{\text{sun}}(0)$

Second-order matching results for $\mu_c$ and $\Delta \Pi_{\text{sun}}(0)$ are derived in Ref. [3]. However, for the purposes of this paper, we want them expressed in terms of the same sorts of polylogarithm integrals that we used in our evaluation of the basketball diagram. Here, we shall show how to obtain that form.

Start with Fig. [1] and Eq. (3.2) for $\Delta \Pi(0)$. This expression is infrared convergent and so independent of the choice of IR regulator. [We will not concern ourselves here with vanishing corrections, such as the $O(\sqrt{\alpha})$ piece that was important in section [VII].] As we have done before, it will be convenient to nonetheless introduce an IR regulator and evaluate the 3+1 dimensional and 3 dimensional pieces separately. For these particular diagrams, the convenient choice of IR regulator will be to introduce an infinitesimal chemical potential on just two of the three internal propagators:

$$
\Delta \Pi_{\text{sun}}(0) = -\Pi_{\text{sun}}^{(3+1)}(0) + \Pi_{\text{sun}}^{(3)}(0) = \frac{1}{2} \left( \frac{8\pi a}{m} \right)^2 \int_{qk} \left[ (i\omega_q)(i\omega_k)(i\omega_l) (i(q_0 + k_0) + \omega^+_{q+k}) \right] \left( 1 - \delta_{q_0,0} \delta_{k_0,0} \right). \tag{6.1}
$$

The frequency sums in the 3+1 dimensional piece can be evaluated using standard contour tricks as in Ref. [4] to yield

$$
- \Pi_{\text{sun}}^{(3+1)}(0) = \frac{1}{2} \left( \frac{8\pi a}{m} \right)^2 \int_{qkl} n(\omega_q) n(\omega^+_k) n(\omega^+_l) \frac{e^{\beta \omega^+_q} - e^{\beta (\omega^+_q + \omega^+_k)}}{\omega^+_l - \omega_q - \omega_k} (2\pi)^d \delta^{(d)}(1 - q - k).
= \frac{1}{2} \left( \frac{8\pi a}{m} \right)^2 \int_{qkl} \left[ n(\omega_q) n(\omega^+_k) - n(\omega_q) n(\omega^+_l) - n(\omega^+_k) n(\omega^+_l) - n(\omega^+_l) \right]
\times \frac{(2\pi)^d \delta^{(d)}(1 - q - k)}{\omega_l - \omega_q - \omega_k}. \tag{6.2}
$$

Note that the infinitesimal chemical potential $\delta \mu$ cancels out in the denominator $\omega^+_l - \omega_q - \omega^+_k$. This fact will simplify our analysis later on, and it is the reason that we chose to put regulator masses on only two of the internal lines rather than all three. Now apply a redundant principal part (P.P.) prescription to this denominator so that we can separately evaluate the integrals of each term. The first term in (6.2) vanishes on angular integration. The last term in (6.2), involving just one factor of $n$, is $O(\epsilon)$ in dimensional regularization for the same reasons discussed in Ref. [6], which are that

$$
\int_{qk} \text{P.P.} \frac{(2\pi)^d \delta^{(d)}(1 - q - k)}{\omega_l - \omega_q - \omega_k} = O(\epsilon) \tag{6.3}
$$

and that the remaining 1 integration is convergent and cannot generate a compensating $1/\epsilon$. Exchanging integration variables in the second term, we are then left with

$$
- \Pi_{\text{sun}}^{(3+1)}(0) = -\frac{1}{2} \left( \frac{8\pi a}{m} \right)^2 \int_{qkl} [n(\omega^+_q) + n(\omega_q)] n(\omega^+_l) \text{P.P.} \frac{(2\pi)^3 \delta^{(3)}(1 - q - k)}{\omega_l - \omega_q - \omega_k} + O(\epsilon). \tag{6.4}
$$
This integral has no UV divergences, and so we can set \( d = 3 \) in the integral. Now expand the Bose distribution functions as a series in the regulator fugacity \( z \),

\[
n(\omega_P - \mu) = \sum_{a=1}^{\infty} z^a e^{-a\beta\omega_P},
\]

(6.5)
giving

\[
- \Pi^{(3+1)}_{\text{sun}}(0) = -\frac{1}{2} \left( \frac{8\pi a}{m} \right)^2 \sum_{ab} (z^{a+b} + z^b) \int_{\mathbf{qkl}} e^{-a\beta\omega_q} e^{-b\beta\omega_l} \frac{(2\pi)^3 \delta^{(3)}(1 - \mathbf{q} - \mathbf{k})}{\omega_l - \omega_q - \omega_k} + O(\epsilon).
\]

(6.6)

Rescaling integration variables to be dimensionless,

\[
- \Pi^{(3+1)}_{\text{sun}}(0) = -\frac{8a^2T}{\lambda^2} \sum_{ab} (z^{a+b} + z^b) I_{ab} + O(\epsilon),
\]

(6.7)

where

\[
I_{ab} \equiv (2\pi)^3 \int_{\mathbf{qkl}} e^{-aq^2/2} e^{-bt^2/2} P.P. \frac{(2\pi)^3 \delta^{(3)}(1 - \mathbf{q} - \mathbf{k})}{\frac{1}{2}(l^2 - q^2 - k^2)}.
\]

(6.8)

Using the methods of Appendix A of Ref. [6], one may evaluate this integral, obtaining

\[
I_{ab} = -\frac{1}{(a + b)\sqrt{ab}}.
\]

(6.9)

We can now extract the IR divergences of the sum using the same method as in section VA:

\[
\sum_{ab} z^{a+b} I_{ab} = -\int_0^1 \frac{dt}{t} \left[ \text{Li}_{1/2}(zt) \right]^2 = \pi \ln \alpha - 4\pi K_1 + O(\sqrt{\alpha}),
\]

(6.10)

\[
\sum_{ab} z^b I_{ab} = -\int_0^1 \frac{dt}{t} \text{Li}_{1/2}(t) \text{Li}_{1/2}(zt) = \pi \ln \left( \frac{\alpha}{4} \right) - 4\pi K_1 + O(\sqrt{\alpha}),
\]

(6.11)

So

\[
- \Pi^{(3+1)}_{\text{sun}}(0) = \frac{16\pi a^2T}{\lambda^2} \left\{ \ln \left( \frac{\alpha}{2} \right) - 4K_1 \right\} + O(\sqrt{\alpha}) + O(\epsilon).
\]

(6.12)

The corresponding result in the 3-dimensional theory is

\[
- \Pi^{(3)}_{\text{sun}}(0) = -\frac{512\pi^3 a^2T}{\lambda^2} J(\mathcal{M}),
\]

(6.13)

where

\[
J(\mathcal{M}) \equiv \int_{\mathbf{qk}} \frac{1}{q^2(k^2 + \mathcal{M}^2)(|\mathbf{q} + \mathbf{k}|^2 + \mathcal{M}^2)} = \frac{1}{(4\pi)^2} \left[ \frac{1}{2\epsilon} + \ln \frac{\mathcal{M}}{2\mathcal{M}} + \frac{1}{2} \right] + O(\epsilon).
\]

(6.14)

Putting everything together, one obtains the previously quoted result (5.38) for \( \Delta \Pi_{\text{sun}}(0) \), with the constant \( C_1 \) given by (3.36). This new form of \( C_1 \), which is equal to the value derived in Ref. [1], can then also be used in the result of Ref. [1] for \( \mu_c \), which we quoted in (3.35).
VII. THE SECOND-ORDER LOGARITHM FOR ARBITRARY $N$

As mentioned earlier, Holzmann, Baym, and Laloë argued for the existence of the logarithmic term at second order. In order to make their general argument more concrete, they also presented an approximate large $N$ calculation of the coefficient $c'_2$ of that logarithm. It’s interesting to compare exact results for the coefficient $c'_2$ to their approximate large $N$ calculation. For this reason, let us consider generalizing our 3+1 dimensional theory (1.9) to a theory with $N_c$ complex fields with $U(N_c)$ symmetry:

\[ S_I = \int_0^\beta d\tau \int d^3 x \left[ \sum_i \psi_i^* \left( \frac{1}{2m} \nabla^2 - \mu \right) \psi_i + \frac{2\pi a}{m} \left( \sum_i \psi_i^* \psi_i \right)^2 \right]. \]  

(7.1)

The corresponding 3 dimensional theory can be considered a theory of $N=2N_c$ real fields and has $O(N)$ symmetry. The action is again (1.12), with $u$ as before, but now $\phi^2 = \phi_1^2 + \cdots + \phi_N^2$. For $N_c \sim 1$, we reviewed before that the dimensionless cost of each order of perturbation theory gets a contribution of order $u/p$ from physics at momentum scale $p$. For large $N_c$, the contribution is order $N_c u/p$. (See Refs. [5,22] for a discussion of large $N$.) In both cases, the momentum scale of non-perturbative physics can therefore be characterized as order $N_c u$, and the condition for the theory to be perturbative at the scale $p \sim \lambda^{-1}$ associated with non-zero Matsubara modes is $N_c u \lambda \ll 1$. This generalizes the previous condition (1.7) for useful expansions of $T_c(n)$ and the applicability of perturbative matching to

\[ N_c a \ll \lambda \]  

(7.2) at the transition. We shall assume this in what follows.

The second-order logarithm in our calculation of $n_c(T)$ arose from the second diagram of Fig. 7, which is proportional to $\mu_c$ at the transition. More specifically, it comes from the sunset diagram $\Delta \Pi_{\text{sun}}(0)$ contribution to $\mu_c$. (The sunset diagram is depicted in Fig. 11.) The value of the sunset diagram for general $N_c$ is simply the value for $N_c = 1$ multiplied by

\[ \frac{N_c + 1}{2} = \frac{N + 2}{4}. \]  

(7.3)

So, ignoring non-logarithmic second order corrections, Eq. (3.33) for $\mu_c$ is modified to

\[ \beta \mu_c = \beta \mu_c^{[1]} + \frac{(N_c + 1)}{2} \frac{32\pi a^2}{\lambda^2} \ln(\bar{M}\lambda) + \cdots. \]  

(7.4)

Now recall that for the case $N_c = 1$ analyzed in the rest of this paper, we chose the renormalization momentum scale $\bar{M}$ to be of order the momentum scale $u$ for non-perturbative physics. The reason goes back to section III A: such a choice makes the critical value of $r_c(M)$ proportional to $u^2$ by dimensional analysis. That meant that the $r_c(\bar{M})/u^2$ term in the formula (3.35) for $\mu_c$ really gives a straight $O(a^2)$ correction without any logarithmic enhancements. So, if we want to remove the possibility of implicit logarithms in the $r_c$ term hiding in the $\cdots$ in (7.4), we should again choose $\bar{M}$ of order the momentum scale of non-perturbative physics, which in the present context is
\[ M \sim N_c u. \quad (7.5) \]

So

\[ \beta \mu_c = \beta \mu_c^{[1]} + \frac{(N_c + 1)}{2} 32 \pi a^2 \frac{\ln(N_c a)}{\lambda^2} + O \left( \frac{N_c a^2}{\lambda^2} \right), \quad (7.6) \]

where, for this section only, the notation \( O(N_c a^2/\lambda^2) \) is meant to assert that there are no additional factors of logarithms.

The second diagram of Fig. 7 gives a factor of \( N_c \) as well as the factor of \( \mu_c \). The ideal gas result \( n_0(T) \), depicted by the first diagram, also has a factor of \( N_c \). The generalization of (4.1) for \( n_c(T) \) is

\[ n_c(T) = N_c \lambda^{-3} \left\{ b_0 + b_1 a \lambda^{-1} + \frac{(N_c + 1)}{2} b'_2 a^2 \lambda^{-2} \ln(N_c a \lambda^{-1}) + b''_2 + O(N_c a^2 \lambda^{-2}) \right\}, \quad (7.7) \]

where \( b_0, b_1, \) and \( b'_2 \) are as in (4.2) but with

\[ \kappa \equiv \frac{\Delta \langle \phi^2 \rangle_c}{N_c u}. \quad (7.8) \]

The solution for \( T_c(n) \) can be written in the form

\[ \Delta T_c \frac{T_0}{T_0} = -\frac{128 \pi^3 \kappa}{\zeta(\frac{3}{2})} \frac{a}{\lambda_0} + A \frac{a^2}{\lambda_0^2} \ln \left( \frac{N_c a}{\lambda_0} \right) + O \left( \frac{a^2}{\lambda_0^2} \right), \quad (7.9) \]

where

\[ A = -(N_c + 1) \frac{32 \pi \zeta(\frac{1}{2})}{3 \zeta(\frac{3}{2})} \quad (7.10) \]

and \( \lambda_0 \equiv \lambda(T_0) \). In the large \( N \) limit, \( \kappa \to -1/96 \pi^2 \) \[22\] and the coefficient \( A \) of the logarithm becomes

\[ A \to -N_c \frac{32 \pi \zeta(\frac{1}{2})}{3 \zeta(\frac{3}{2})} \approx 18.7327 N_c. \quad (7.11) \]

This is the exact large \( N \) result for this coefficient if large \( N \) is defined to mean taking a large number of fields in the original 3+1 dimensional theory, as we have above. For comparison, the calculation of Ref. [3] was an approximate large \( N \) calculation made in the O(\( N \)) three-dimensional theory using a rough physically-motivated UV momentum cut-off of \( \Lambda \approx (2\pi)^{1/2}/\lambda \). Their approximate result for \( A \), expressed in terms of \( N_c = 2N \), was

\[ A_{\text{Ref. [3]}} \to N_c \frac{256\pi}{3 \zeta(\frac{2}{2}) \lambda \lambda} \approx N_c \frac{256\pi}{3 \zeta(\frac{3}{2}) (2\pi)^{1/2}} \approx 40.9 N_c. \quad (7.12) \]

This differs by roughly a factor of two from (7.11). Amusingly, the approximation (7.12) of Ref. [3] does well if naively applied to \( N_c = 1 \), giving \( A_{\text{Ref. [3]}} \approx 40.9 \). In contrast, the exact answer (7.11) derived in this paper is \( A \approx 37.4565 \).
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APPENDIX A: SOME DIMENSIONALLY REGULATED INTEGRALS

In this appendix, we show how to do basic single-momentum integrals used in the main text, such as (3.12), (3.13), and (3.33). We will do the $p_0$ frequency sums first, using standard contour tricks. One can just as easily get the same results by doing the $p$ integrations first (though one must be careful in that case about cuts).

Start with the basic integral

$$\sum \int \frac{1}{i p_0 + \omega_p - \mu} = \int [n(\omega_p) - \mu + \frac{1}{2}].$$

(A1)

The $\frac{1}{2}$ comes from the contour integration at infinity but can be ignored, since its integral vanishes in dimensional regularization. The Bose distribution function can then be expanded as in (6.5). If each term is integrated in $d=3-\epsilon$ dimensions, using the $t=0$ case of

$$\int \frac{d^d p}{(2\pi)^d} p^t e^{-\lambda p^2} = \frac{\lambda^{-(d+t)/2} \Gamma \left( \frac{d+t}{2} \right)}{(4\pi)^{d/2} \Gamma \left( \frac{d}{2} \right)},$$

(A2)

one obtains

$$\sum \int \frac{1}{i p_0 + \omega_p - \mu} = M^t \lambda^{-d} \text{Li}_d (z),$$

(A3)

where $z = \exp(\beta \mu)$. If we set $\mu = 0$ in dimensions where the result will still converge (and then analytically continue to other dimensions), we get the integral (3.33) used in the main text. If we first differentiate with respect to $\mu$ and then set $\mu = 0$, we get (3.16) [which, as discussed in the text, is equivalent to (3.13) in dimensional regularization].

We can do the integral (3.12) of the main text as

$$\sum \int \frac{1}{(i p_0 + \omega_p)(-i p_0 + \omega_p)} = \sum \int \frac{1}{2 \omega_p} \left[ \frac{1}{i p_0 + \omega_p} + \frac{1}{-i p_0 + \omega_p} \right] = \int \frac{n(\omega_p) + \frac{1}{2}}{\omega_p} = \int \frac{n(\omega_p)}{\omega_p}.$$

(A4)

The last integral can also be done by expanding $n$ as above, using the $t=-2$ case of (A2), with the result quoted in the text.
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