Extensions of Barrier Sets to Nonzero Roots of the Matching Polynomials

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Abstract

In matching theory, barrier sets (also known as Tutte sets) have been studied extensively due to its connection to maximum matchings in a graph. In this paper, we first define \( \theta \)-barrier sets. Our definition of a \( \theta \)-barrier set is slightly different from that of a barrier set. However we show that \( \theta \)-barrier sets and barrier sets have similar properties. In particular, we prove a generalized Berge’s Formula and give a characterization for the set of all \( \theta \)-special vertices in a graph.

KEYWORDS: matching polynomial, Gallai-Edmonds Decomposition, barrier sets, extreme sets

1 Introduction

All the graphs in this paper are simple and finite.

Definition 1.1. An \( r \)-matching in a graph \( G \) is a set of \( r \) edges, no two of which have a vertex in common. The number of \( r \)-matchings in \( G \) will be denoted by \( p(G, r) \). We set \( p(G, 0) = 1 \) and define the matching polynomial of \( G \) by

\[
\mu(G, x) = \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r p(G, r)x^{n-2r}.
\]

We shall denote the multiplicity of \( \theta \) as a root of \( \mu(G, x) \) by \( \text{mult}(\theta, G) \). Let \( u \in V(G) \), the graph obtained from \( G \) by deleting the vertex \( u \) and all edges that contain \( u \) will be denoted by \( G \setminus u \). Inductively if \( u_1, \ldots, u_k \in V(G) \), \( G \setminus u_1 \ldots u_k = (G \setminus u_1 \ldots u_{k-1}) \setminus u_k \). Note that the order of which vertex is being deleted first is not important, that is, if \( i_1, \ldots, i_k \) is a permutation of \( 1, \ldots, k \), we have \( G \setminus u_1 \ldots u_k = G \setminus u_{i_1} \ldots u_{i_k} \). Furthermore if \( X = \{u_1, \ldots, u_k\} \), \( G \setminus X = G \setminus u_1 \ldots u_k \).

The followings are properties of \( \mu(G, x) \).

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Theorem 1.2. (Theorem 1.1 on p. 2 of [1])

(a) \( \mu(G \cup H, x) = \mu(G, x)\mu(H, x) \) where \( G \) and \( H \) are disjoint graphs,

(b) \( \mu(G, x) = \mu(G \setminus e, x) - \mu(G \setminus uv, x) \) if \( e = \{u, v\} \) is an edge of \( G \),

(c) \( \mu(G, x) = xu(G \setminus u, x) - \sum_{i \sim u} \mu(G \setminus ui, x) \) where \( i \sim u \) means \( i \) is adjacent to \( u \),

(d) \( \frac{d}{dx} \mu(G, x) = \sum_{i \in V(G)} \mu(G \setminus i, x) \) where \( V(G) \) is the vertex set of \( G \).

It is well known that all roots of \( \mu(G, x) \) are real. Throughout, let \( \theta \) be a real number and \( \text{mult}(\theta, G) \) denote the multiplicity of \( \theta \) as a root of \( \mu(G, x) \). In particular, \( \text{mult}(\theta, G) = 0 \) if and only if \( \theta \) is not a root of \( \mu(G, x) \). By Theorem 5.3 on p. 29 and Theorem 1.1 on p. 96 of [1], one can easily deduce the following lemma.

Lemma 1.3. Let \( G \) be a graph and \( u \in V(G) \). Then

\[
\text{mult}(\theta, G) - 1 \leq \text{mult}(\theta, G \setminus u) \leq \text{mult}(\theta, G) + 1.
\]

As a consequence of Lemma 1.3, we can classify the vertices in a graph with respect to \( \theta \) as follows:

Definition 1.4. (see [2, Section 3]) For any \( u \in V(G) \),

(a) \( u \) is \( \theta \)-essential if \( \text{mult}(\theta, G \setminus u) = \text{mult}(\theta, G) - 1 \),

(b) \( u \) is \( \theta \)-neutral if \( \text{mult}(\theta, G \setminus u) = \text{mult}(\theta, G) \),

(c) \( u \) is \( \theta \)-positive if \( \text{mult}(\theta, G \setminus u) = \text{mult}(\theta, G) + 1 \).

Furthermore if \( u \) is not \( \theta \)-essential but it is adjacent to some \( \theta \)-essential vertex, we say \( u \) is \( \theta \)-special.

It turns out that \( \theta \)-special vertices play an important role in the Gallai-Edmonds Decomposition of a graph (see [3]). One of our main result is a characterization of the set of these vertices in terms of \( \theta \)-barriers.

Note that if \( \text{mult}(\theta, G) = 0 \) then for any \( u \in V(G) \), \( u \) is either \( \theta \)-neutral or \( \theta \)-positive and no vertices in \( G \) can be \( \theta \)-special. By Corollary 4.3 of [2], a \( \theta \)-special vertex is \( \theta \)-positive. Therefore

\[
V(G) = D_\theta(G) \cup A_\theta(G) \cup P_\theta(G) \cup N_\theta(G),
\]

where

\( D_\theta(G) \) is the set of all \( \theta \)-essential vertices in \( G \),

\( A_\theta(G) \) is the set of all \( \theta \)-special vertices in \( G \),

\( N_\theta(G) \) is the set of all \( \theta \)-neutral vertices in \( G \),

\( P_\theta(G) = Q_\theta(G) \setminus A_\theta(G) \), where \( Q_\theta(G) \) is the set of all \( \theta \)-positive vertices in \( G \),
is a partition of $V(G)$. Note that there is no 0-neutral vertices. So $N_0(G) = \emptyset$ and $V(G) = D_0(G) \cup A_0(G) \cup P_0(G).

**Definition 1.5.** (see [2] Section 3) A graph $G$ is said to be $\theta$-critical if all vertices in $G$ are $\theta$-essential and $\text{mult}(\theta, G) = 1$.

The Gallai-Edmonds Structure Theorem describes a certain canonical decomposition of $V(G)$ with respect to the zero root of $\mu(G, x)$. In [3], Chen and Ku proved the Gallai-Edmonds Structure Theorem for graph with any root $\theta$.

**Theorem 1.6.** (Theorem 1.5 of [3]) Let $G$ be a graph with $\theta$ a root of $\mu(G, x)$. If $u \in A_\theta(G)$ then

(i) $D_\theta(G \setminus u) = D_\theta(G),$
(ii) $P_\theta(G \setminus u) = P_\theta(G),$
(iii) $N_\theta(G \setminus u) = N_\theta(G),$
(iv) $A_\theta(G \setminus u) = A_\theta(G) \setminus \{u\}.$

**Theorem 1.7.** (Theorem 1.7 of [3]) If $G$ is connected and every vertex of $G$ is $\theta$-essential then $\text{mult}(\theta, G) = 1$.

By Theorem 1.6 and Theorem 1.7, it is not hard to deduce the following whose proof is omitted. For convenience, a connected component will be called a component.

**Corollary 1.8.**

(i) $A_\theta(G \setminus A_\theta(G)) = \emptyset, D_\theta(G \setminus A_\theta(G)) = D_\theta(G), P_\theta(G \setminus A_\theta(G)) = P_\theta(G), \text{ and } N_\theta(G \setminus A_\theta(G)) = N_\theta(G).

(ii) $G \setminus A_\theta(G)$ has exactly $|A_\theta(G)| + \text{mult}(\theta, G)$ $\theta$-critical components.

(iii) If $H$ is a component of $G \setminus A_\theta(G)$ then either $H$ is $\theta$-critical or $\text{mult}(\theta, H) = 0$.

(iv) The subgraph induced by $D_\theta(G)$ consists of all the $\theta$-critical components in $G \setminus A_\theta(G)$.

Let $G$ be a graph. The number of odd components in $G$ is denoted by $c_{\text{odd}}(G)$. Recall the following famous Berge’s Formula.

**Theorem 1.9.** $\text{mult}(0, G) = \max_{X \subseteq V(G)} c_{\text{odd}}(G \setminus X) - |X|.$

**Definition 1.10.** Motivated by the Berge’s Formula, a barrier set is defined to be a set $X \subseteq V(G)$ for which $\text{mult}(0, G) = c_{\text{odd}}(G \setminus X) - |X|$. An extreme set is defined to be the set for which $\text{mult}(0, G \setminus X) = \text{mult}(0, G) + |X|$.

Properties of extreme and barrier sets can be found in [3] Section 3.3. In fact a barrier set is an extreme set. An extreme set is not necessary a barrier set, but it can be shown that an extreme set is contained in some barrier set. In general the union or intersection of two barrier sets is not a barrier set. However it can be shown that the intersection of two (inclusionwise) maximal barriers set is a barrier set. $A_0(G)$ is a barrier and extreme set. It can be shown that $A_0(G)$ is in fact the intersection of all the maximal barrier sets in $G$. Here we extend this fact to $A_\theta(G)$:

**Theorem 1.11.** Suppose $N_\theta(G) = \emptyset$. Then $A_\theta(G)$ is the intersection of all maximal $\theta$-barrier sets in $G$. 

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2 Properties of $\theta$-barrier sets

The number of $\theta$-critical components in $G$ is denoted by $c_\theta(G)$. An immediate consequence of part (a) of Theorem 1.2 and Theorem 1.7 is the following inequality which is used frequently.

$$\text{mult}(\theta, G \setminus X) \geq c_\theta(G \setminus X) \quad \text{for any } X \subseteq V(G).$$

We prove the following analogue of Berge’s Formula.

**Theorem 2.1.** [Generalized Berge’s Formula]

$$\text{mult}(\theta, G) = \max_{X \subseteq V(G)} c_\theta(G \setminus X) - |X|.$$

**Proof.** We claim that, $c_\theta(G \setminus X) \leq |X| + \text{mult}(\theta, G)$ for all $X \subseteq V(G)$. Suppose the contrary. Then $c_\theta(G \setminus X) > |X| + \text{mult}(\theta, G)$ for some $X \subseteq V(G)$. Recall that $\text{mult}(\theta, G \setminus X) \geq c_\theta(G \setminus X)$. Together with Lemma 1.3, we have $\text{mult}(\theta, G) \geq \text{mult}(\theta, G \setminus X) - |X| > \text{mult}(\theta, G)$, a contradiction. Hence $c_\theta(G \setminus X) \leq |X| + \text{mult}(\theta, G)$ for all $X \subseteq V(G)$.

Now it is sufficient to show that there is a set $X \subseteq V(G)$ for which $\text{mult}(\theta, G) = c_\theta(G \setminus X) - |X|$. By (ii) of Corollary 1.8 and taking $X = A_\theta(G)$, we are done.

**Definition 2.2.** Motivated by the Generalized Berge’s Formula, we define a $\theta$-barrier set to be a set $X \subseteq V(G)$ for which $\text{mult}(\theta, G) = c_\theta(G \setminus X) - |X|$.

We define a $\theta$-extreme set to be a set $X \subseteq V(G)$ for which $\text{mult}(\theta, G \setminus X) = \text{mult}(\theta, G) + |X|$.

Note that the definitions of 0-extreme set and extreme set coincide. But the definitions of 0-barrier set and barrier set are different. Our next proposition shows that a 0-barrier set is a barrier set.

**Proposition 2.3.** A 0-barrier set is a barrier set.

**Proof.** Let $X$ be a 0-barrier set. Then $c_0(G \setminus X) = \text{mult}(0, G) + |X|$. Note that $c_0(G \setminus X) \leq c_{\text{odd}}(G \setminus X)$. Using Theorem 1.9, we conclude that $c_{\text{odd}}(G \setminus X) = \text{mult}(0, G) + |X|$. Hence $X$ is a barrier set.

The converse of Proposition 2.3 is not true. In Figure 1, $X = \{u, v\}$ is a barrier set in $G$ but it is not a 0-barrier set.

![Figure 1](image.png)
However we have a weak converse of Proposition 2.3.

**Proposition 2.4.** A (inclusionwise) maximal barrier set is a maximal 0-barrier set.

**Proof.** Let $X$ be a maximal barrier set. Note that $|X| + \text{mult}(0, G) \geq \text{mult}(0, G \setminus X) \geq c_{\text{odd}}(G \setminus X) = |X| + \text{mult}(0, G)$, where the first inequality follows from Lemma 1.3 and the last inequality follows from the fact that $X$ is a barrier set. Therefore, equality holds throughout whence $\text{mult}(0, G \setminus X) = c_{\text{odd}}(G \setminus X)$ and 0 is a root of multiplicity 1 in each of the odd components in $G \setminus X$.

We claim that an odd component in $G \setminus X$ is 0-critical. Suppose the contrary. Let $H$ be an odd component in $G \setminus X$ and $H$ is not 0-critical. Then $A_0(H) \neq \emptyset$. Now $\text{mult}(0, H) = 1$. By (ii) of Corollary 1.8 $c_0(H \setminus A_0(H)) = |A_0(H)| + 1$. Since $c_0(H \setminus A_0(H)) \leq c_{\text{odd}}(H \setminus A_0(H))$, by Theorem 1.9 we conclude that $c_{\text{odd}}(H \setminus A_0(H)) = |A_0(H)| + 1$. Therefore $c_{\text{odd}}(G \setminus (X \cup A_0(H))) = c_{\text{odd}}(G \setminus X) - 1 + c_{\text{odd}}(H \setminus A_0(H)) = |X| + \text{mult}(0, G) - 1 + |A_0(H)| + 1 = |X \cup A_0(H)| + \text{mult}(0, G)$. But then $X \cup A_0(H)$ is a barrier set in $G$, a contrary to the maximality of $X$. Hence an odd component in $G \setminus X$ must be 0-critical. This means that $c_{\text{odd}}(G \setminus X) = c_0(G \setminus X)$ and $X$ is a 0-barrier set. By Proposition 2.3 we conclude that $X$ must be a maximal 0-barrier set. \qed

Now we shall study the properties of $\theta$-barrier and $\theta$-extreme sets.

**Lemma 2.5.** A subset of a $\theta$-extreme set is a $\theta$-extreme set.

**Proof.** Let $X$ be an $\theta$-extreme set and $Y \subseteq X$. Now $\text{mult}(\theta, G \setminus X) = \text{mult}(\theta, G) + |X|$. By Lemma 1.3 $\text{mult}(\theta, G \setminus Y) \leq \text{mult}(\theta, G) + |Y|$. If $Y$ is not $\theta$-extreme then $\text{mult}(\theta, G \setminus Y) < \text{mult}(\theta, G) + |Y|$, and by Lemma 1.3 again, $\text{mult}(\theta, G \setminus X) \leq \text{mult}(\theta, G \setminus Y) + |X \setminus Y| < \text{mult}(\theta, G) + |X|$, a contradiction. Hence a subset of an $\theta$-extreme set is $\theta$-extreme. \qed

**Lemma 2.6.** If $X$ is a $\theta$-barrier [\theta-extreme] set and $Y \subseteq X$ then $X \setminus Y$ is a $\theta$-barrier [\theta-extreme] set in $G \setminus Y$.

**Proof.** Note that $c_0(G \setminus X) = |X| + \text{mult}(\theta, G)$. By Theorem 2.1 and Lemma 1.3 $c_0(G \setminus X) \leq |X \setminus Y| + \text{mult}(\theta, G \setminus Y) \leq |X \setminus Y| + \text{mult}(\theta, G) + |Y| = |X| + \text{mult}(\theta, G)$. Hence $c_0(G \setminus X) = |X \setminus Y| + \text{mult}(\theta, G \setminus Y)$ and $X \setminus Y$ is a $\theta$-barrier set in $G \setminus Y$. \qed

**Lemma 2.7.** Every $\theta$-extreme set of $G$ lies in a $\theta$-barrier set.

**Proof.** Let $X$ be a $\theta$-extreme set and $T = A_0(G \setminus X) \cup X$. Then

$$c_0(G \setminus T) = c_0(G \setminus (A_0(G \setminus X) \cup X))$$
$$= c_0((G \setminus X) \setminus A_0(G \setminus X))$$
$$= |A_0(G \setminus X)| + \text{mult}(\theta, G \setminus X) \quad \text{(by (ii) of Corollary 1.8)}$$
$$= |A_0(G \setminus X)| + \text{mult}(\theta, G) + |X| \quad \text{(\(X\) is \(\theta\)-extreme)}$$
$$= |T| + \text{mult}(\theta, G),$$

and hence $T$ is a $\theta$-barrier set. \qed

**Lemma 2.8.** Let $X$ be a $\theta$-barrier set. Then $X$ is a $\theta$-extreme set.
Proof. Recall from (1) that $\text{mult}(\theta, G \setminus X) \geq c_\theta(G \setminus X)$. Since $c_\theta(G \setminus X) = |X| + \text{mult}(\theta, G)$, by Lemma 2.8 we have

$$\text{mult}(\theta, G) \geq \text{mult}(\theta, G \setminus X) - |X| \geq c_\theta(G \setminus X) - |X| = \text{mult}(\theta, G).$$

Hence $\text{mult}(\theta, G \setminus X) = \text{mult}(\theta, G) + |X|$ and so $X$ is a $\theta$-extreme set. \hfill \Box

Note that in general a $\theta$-extreme set is not a $\theta$-barrier set. In Figure 1, $X_1 = \{u\}$ is a 0-extreme set but it is not a 0-barrier set.

**Lemma 2.9.** Let $X$ be a $\theta$-barrier set and $H$ be a component of $G \setminus X$. Then either $H$ is $\theta$-critical or $\text{mult}(\theta, H) = 0$.

**Proof.** Note that $c_\theta(G \setminus X) = |X| + \text{mult}(\theta, G)$. By Lemma 2.8 $X$ is a $\theta$-extreme set. Therefore $\text{mult}(\theta, G \setminus X) = \text{mult}(\theta, G) + |X| = c_\theta(G \setminus X)$. Now if $H$ is not $\theta$-critical and $\text{mult}(\theta, H) > 0$, then by (1), $\text{mult}(\theta, G \setminus X) > c_\theta(G \setminus X)$, a contradiction. Hence either $H$ is $\theta$-critical or $\text{mult}(\theta, H) = 0$. \hfill \Box

**Lemma 2.10.** Let $X$ be a maximal $\theta$-barrier set. Let $H$ be a component of $G \setminus X$ and $\text{mult}(\theta, H) = 0$. Then for all $u \in V(H)$, $u$ is $\theta$-neutral in $H$. Furthermore for all $Y \subseteq V(H)$ and $Y \neq \emptyset$, $c_\theta(H \setminus Y) \leq |Y| - 1$.

**Proof.** Suppose $H$ has a $\theta$-positive vertex, say $u$. Then $\text{mult}(\theta, H \setminus u) = 1$. By (ii) of Corollary 1.8 $c_\theta((H \setminus u) \setminus A_\theta(H \setminus u)) = |A_\theta(H \setminus u)| + \text{mult}(\theta, H \setminus u) = |A_\theta(H \setminus u)| + 1$. But then

$$c_\theta(G \setminus (X \cup \{u\} \cup A_\theta(H \setminus u))) = c_\theta(G \setminus X) + c_\theta((H \setminus u) \setminus A_\theta(H \setminus u)) = |X| + \text{mult}(\theta, G) + |A_\theta(H \setminus u)| + 1 = |X \cup \{u\} \cup A_\theta(H \setminus u)| + \text{mult}(\theta, G),$$

and so $X \cup \{u\} \cup A_\theta(H \setminus u)$ is a $\theta$-barrier in $G$, a contrary to the maximality of $X$. Hence for all $u \in V(H)$, $u$ is $\theta$-neutral in $H$.

Since $Y \neq \emptyset$, there is a $y \in Y$. Let $Y' = Y \setminus y$ and $H' = H \setminus y$. Note that $\text{mult}(\theta, H \setminus y) = 0$ since $y$ is $\theta$-neutral in $H$. By Theorem 2.1 $c_\theta(H' \setminus Y') \leq |Y'|$. Since $H \setminus Y = H' \setminus Y'$, we have $c_\theta(H \setminus Y) \leq |Y| - 1$. \hfill \Box

**Lemma 2.11.** Let $G$ be $\theta$-critical. Then for all $Y \subseteq V(G)$ and $Y \neq \emptyset$, $c_\theta(G \setminus Y) \leq |Y| - 1$.

**Proof.** Since $Y \neq \emptyset$, there is a $y \in Y$. Let $Y' = Y \setminus y$ and $G' = G \setminus y$. Note that $\text{mult}(\theta, G \setminus y) = 0$ since $y$ is $\theta$-essential in $G$. By Theorem 2.1 $c_\theta(G' \setminus Y') \leq |Y'|$. Since $G \setminus Y = G' \setminus Y'$, we have $c_\theta(G \setminus Y) \leq |Y| - 1$. \hfill \Box

In general the union or intersection of two $\theta$-barrier sets is not necessary a $\theta$-barrier set. In Figure 1, $X_2 = \{u, v, w\}$ and $X_3 = \{v, w, z\}$ are two 0-barrier sets. But $X_2 \cap X_3$ and $X_2 \cup X_3$ are not a 0-barrier set. However the intersection of two maximal $\theta$-barrier sets is a $\theta$-barrier set.

**Theorem 2.12.** The intersection of two maximal $\theta$-barrier sets is a $\theta$-barrier set.
Proof. Let $X$ and $Y$ be two maximal $\theta$-barrier sets. Let $G_1, G_2, \ldots, G_k$ be all the $\theta$-critical components of $G \setminus X$ and $H_1, H_2, \ldots, H_m$ be all the components of $G \setminus Y$. Note that $k = |X| + \mult(\theta, G)$. Let $X_i = X \cap V(H_i)$, $Y_i = Y \cap V(G_i)$ and $Z = X \cap Y$. By relabelling if necessary we may assume that $X_1, \ldots, X_{m_1} \neq \emptyset$, $Y_1, \ldots, Y_{k_1} \neq \emptyset$, but $X_{m_1+1} = \cdots = X_m = Y_{k_1+1} = \cdots = Y_k = \emptyset$, and also that $k_1 \leq m_1$. Note that $G_{k_1+1}, \ldots, G_k$ are $\theta$-critical components in $(G \setminus X) \setminus Y$. So each of them is contained in a component of $G \setminus Y$. Now let us count the number of $G_i$’s where $k_1 + 1 \leq i \leq k$ that are contained in some $H_j$.

Suppose $m_1 + 1 \leq j \leq m$. Then $H_j$ is a component in $(G \setminus X) \setminus Y$. So if $G_i \subseteq H_j$, we must have $G_i = H_j$. Furthermore $G_i$ is a component of $G \setminus Z$. By Theorem 2.1 the number of such $G_i$’s is at most $c_\theta(G \setminus Z) \leq |Z| + \mult(\theta, G)$.

Suppose $1 \leq j \leq m_1$. Let $G_{i_1}, \ldots, G_{i_t}$ be all the $G_i$’s that are contained in $H_j$. Then $G_{i_1}, \ldots, G_{i_t}$ are $\theta$-critical components in $H_j \setminus X_j$. By Lemma 2.9 $H_j$ is either $\theta$-critical or $\mult(\theta, H) = 0$. If $\mult(\theta, H) = 0$, we have, by Lemma 2.10 $c_\theta(H_j \setminus X_j) \leq |X_j| - 1$. If $H_i$ is $\theta$-critical, we have, by Lemma 2.11 $c_\theta(H_j \setminus X_j) \leq |X_j| - 1$. Therefore in either cases, we have $t \leq |X_j| - 1$.

The number of $G_i$’s where $k_1 + 1 \leq i \leq k$ that are disjoint from $Y$ is at most

\[
c_\theta(G \setminus Z) + \sum_{j=1}^{m_1} (|X_j| - 1) \leq |Z| + \mult(\theta, G) + |X \setminus Z| - m_1
\]

\[
= |X| + \mult(\theta, G) - m_1
\]

\[
= k - m_1
\]

\[
\leq k - k_1.
\]

Since this number is exactly $k - k_1$, we infer that equality must hold throughout. Hence $c_\theta(G \setminus Z) = |Z| + \mult(\theta, G)$ and $Z$ is a $\theta$-barrier set.

$\Box$

3 Characterizations of $A_\theta(G)$

A characterization of $A_\theta(G)$ is that it is the minimal (inclusionwise) $\theta$-barrier set (see Theorem 3.5). Furthermore if $N_\theta(G) = \emptyset$, we have another characterization of $A_\theta(G)$, that is, it is the intersection of all maximal $\theta$-barrier sets in $G$ (see Theorem 3.6).

Lemma 3.1. If $X$ is a $\theta$-barrier or a $\theta$-extreme set then $X \subseteq A_\theta(G) \cup P_\theta(G)$.

Proof. By Lemma 2.8 we may assume $X$ is a $\theta$-extreme. Let $x \in X$. By Lemma 2.5 $\{x\}$ is a $\theta$-extreme set. Therefore $\mult(\theta, G \setminus x) = \mult(\theta, G) + 1$ and $x$ is $\theta$-positive. So $x \in A_\theta(G) \cup P_\theta(G)$ and $X \subseteq A_\theta(G) \cup P_\theta(G)$.

$\Box$

Lemma 3.2. Let $X$ be a $\theta$-barrier set. If $X \subseteq A_\theta(G)$ then $X = A_\theta(G)$.

Proof. Note that $c_\theta(G \setminus X) = \mult(\theta, G) + |X|$. By Lemma 2.7 we conclude that $A_\theta(G \setminus X) = \emptyset$. By Theorem 1.6 $A_\theta(G \setminus X) = A_\theta(G) \setminus X$. Hence $X = A_\theta(G)$.

We shall require the following result of Godsil [2].
**Theorem 3.3.** (Theorem 4.2 of [2]) Let $\theta$ be a root of $\mu(G, x)$ with non-zero multiplicity $k$ and let $u$ be a $\theta$-positive vertex in $G$. Then

(a) if $v$ is $\theta$-essential in $G$ then it is $\theta$-essential in $G \setminus u$;

(b) if $v$ is $\theta$-positive in $G$ then it is $\theta$-essential or $\theta$-positive in $G \setminus u$;

(c) if $u$ is $\theta$-neutral in $G$ then it is $\theta$-essential or $\theta$-neutral in $G \setminus u$.

**Lemma 3.4.** Let $u \in P_\theta(G)$. Then $A_\theta(G) \subseteq A_\theta(G \setminus u)$.

**Proof.** If $A_\theta(G) = \emptyset$, then we are done. Suppose $A_\theta(G) \neq \emptyset$. Let $v \in A_\theta(G)$. Then $v$ is adjacent to a $\theta$-essential vertex $w$. By Theorem $3.3$, $w$ is $\theta$-essential in $G \setminus u$ and $v$ is either $\theta$-positive or $\theta$-essential in $G \setminus u$. Suppose $v$ is $\theta$-essential in $G \setminus u$. Then $\text{mult}(\theta, G \setminus uv) = \text{mult}(\theta, G)$. By Theorem $1.6$, $u \in P_\theta(G) = P_\theta(G \setminus v)$. Since $v$ is $\theta$-special in $G$, $v$ is $\theta$-positive in $G$ (see Corollary 4.3 of [2]). So $\text{mult}(\theta, G \setminus uv) = \text{mult}(\theta, G) + 2$, a contradiction. Therefore $v$ is $\theta$-positive in $G \setminus u$. Since $v$ is adjacent to $w$, we must have $v \in A_\theta(G \setminus u)$. Hence $A_\theta(G) \subseteq A_\theta(G \setminus u)$. \hfill \Box

**Theorem 3.5.** Let $X$ be a $\theta$-barrier set in $G$. Then $A_\theta(G) \subseteq X$. In particular, $A_\theta(G)$ is the minimal $\theta$-barrier set.

**Proof.** By Lemma $3.1$, $X \subseteq A_\theta(G) \cup P_\theta(G)$. We shall prove the result by induction on $|X \cap P_\theta(G)|$. Suppose $|X \cap P_\theta(G)| = 0$. Then $X \subseteq A_\theta(G)$ and by Lemma $3.2$, $X = A_\theta(G)$. Suppose $|X \cap P_\theta(G)| \geq 1$. We may assume that if $X'$ is a $\theta$-barrier set in $G'$ with $|X' \cap P_\theta(G')| < |X \cap P_\theta(G)|$, then $A_\theta(G') \subseteq X'$.

Let $x \in X \cap P_\theta(G)$. By Lemma $2.4$, $X' = X \setminus x$ is a $\theta$-barrier set in $G' = G \setminus x$. By Lemma $3.1$ and Lemma $3.4$, we have $X' \subseteq A_\theta(G') \cup P_\theta(G')$ and $A_\theta(G) \subseteq A_\theta(G')$. Therefore $|X' \cap P_\theta(G')| < |X \cap P_\theta(G)|$. By induction $A_\theta(G') \subseteq X'$. Hence $A_\theta(G) \subseteq X$. \hfill \Box

In general, $A_\theta(G)$ is not the intersection of all maximal $\theta$-barrier sets in $G$. For instance, in Figure 2, $\text{mult}(\sqrt{3}, G) = 0$ and $A_{\sqrt{3}}(G) = \emptyset$. Now $\{u\}$ is the only maximal $\sqrt{3}$-barrier set. But $A_{\sqrt{3}}(G) \neq \{u\}$. However we can show that $A_\theta(G)$ is the intersection of all maximal $\theta$-barrier sets in $G$ if $N_\theta(G) = \emptyset$.

![Figure 2](image-url)
Theorem 3.6. Suppose $N_\theta(G) = \emptyset$. Then $A_\theta(G)$ is the intersection of all maximal $\theta$-barrier sets in $G$.

Proof. By Theorem 3.5, $A_\theta(G)$ is contained in the intersection of all maximal $\theta$-barriers in $G$. It is sufficient to show that for each $x \in V(G) \setminus A_\theta(G)$ there is a maximal barrier that does not contain $x$. If $x \in D_\theta(G)$, by Lemma 3.1, $x$ is not contained in any $\theta$-barriers and thus any maximal $\theta$-barriers. Suppose $x \in P_\theta(G)$. Then $x$ is contained in a component $H$ in $G \setminus A_\theta(G)$ with $\text{mult}(\theta, H) = 0$. Note that $|V(H)| \geq 2$, for $x \in P_\theta(G) = P(G \setminus A_\theta(G))$ and $\text{mult}(\theta, H \setminus x) = 1$ (see Theorem 1.2). By (c) of Theorem 1.2 and the fact that $\text{mult}(\theta, H) = 0$, we deduce that there is a vertex $y \in V(H \setminus x)$ for which $\text{mult}(\theta, H \setminus xy) = 0$. Now $y \in P_\theta(G)$ for $N_\theta(G) = \emptyset$. Furthermore $x$ is $\theta$-essential in $H \setminus y$. Therefore $x \notin A_\theta(H \setminus y)$ and by (ii) of Corollary 1.8, $c_\theta((H \setminus y) \setminus A_\theta(H \setminus y)) = |A_\theta(H \setminus y)| + 1$. Hence

$$c_\theta(G \setminus (A_\theta(G) \cup \{y\} \cup A_\theta(H \setminus y))) = c_\theta(G \setminus A_\theta(G)) + c_\theta((H \setminus y) \setminus A_\theta(H \setminus y))$$

$$= |A_\theta(G)| + \text{mult}(\theta, G) + |A_\theta(H \setminus y)| + 1$$

$$= |A_\theta(G) \cup \{y\} \cup A_\theta(H \setminus y)| + \text{mult}(\theta, G),$$

and so $A_\theta(G) \cup \{y\} \cup A_\theta(H \setminus y)$ is a $\theta$-barrier set not containing $x$. Let $Z$ be a maximal $\theta$-barrier set containing $Y = A_\theta(G) \cup \{y\} \cup A_\theta(H \setminus y)$. By Lemma 2.6, $Z \setminus Y$ is a $\theta$-barrier set in $G \setminus Y$. Using Theorem 1.6 and the fact that $x$ is $\theta$-essential in $H \setminus y$, we can deduce that $x \notin D_\theta(G \setminus Y)$. By Lemma 3.1, we conclude that $x \notin Z \setminus Y$ and hence $x \notin Z$. The proof of the theorem is completed.

Since $N_0(G) = \emptyset$, by Theorem 3.6 and Proposition 2.4, we deduce the following classical result.

Corollary 3.7. (Theorem 3.3.15 of [4]) $A_0(G)$ is the intersection of all maximal barrier sets in $G$.

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