Abstract. In this note we sharpen the lower bound from [LLP10] on the spectrum of the 2D Schrödinger operator with a \( \delta \)-interaction supported on a planar angle. Using the same method we obtain the lower bound on the spectrum of the 2D Schrödinger operator with a \( \delta \)-interaction supported on crossing straight lines. The latter operators arise in the three-body quantum problem with \( \delta \)-interactions between particles.

1. Introduction

Self-adjoint Schrödinger operators with \( \delta \)-interactions supported on sufficiently regular hypersurfaces can be defined via closed, densely defined, symmetric and lower-semibounded quadratic forms using the first representation theorem, see [BEKS94] and also [BLL13].

\( \delta \)-interactions on angles. In our first model the support of the \( \delta \)-interaction is the set \( \Sigma_\varphi \subset \mathbb{R}^2 \), which consists of two rays meeting at the common origin and constituting the angle \( \varphi \in (0, \pi] \) as in Figure 1.

![Figure 1](image)

The angle \( \Sigma_\varphi \) of degree \( \varphi \in (0, \pi] \).

The quadratic form in \( L^2(\mathbb{R}^2) \)

\[
\alpha_\varphi[f] := \| \nabla f \|_{L^2(\mathbb{R}^2)}^2 - \alpha \| f|_{\Sigma_\varphi} \|_{L^2(\Sigma_\varphi)}^2, \quad \text{dom} \alpha_\varphi := H^1(\mathbb{R}^2),
\]

is closed, densely defined, symmetric and lower-semibounded, where \( f|_{\Sigma_\varphi} \) is the trace of \( f \) on \( \Sigma_\varphi \), and the constant \( \alpha > 0 \) is called the strength of interaction. The corresponding self-adjoint operator in \( L^2(\mathbb{R}^2) \) we denote by \( A_\varphi \). Known spectral properties of this operator include explicit representation...
of the essential spectrum $\sigma_{\text{ess}}(A_\varphi) = [-\alpha^2/4, +\infty)$ and some information on the discrete spectrum: $\sharp \sigma_d(A_\varphi) \geq 1$ if and only if $\varphi \neq \pi$. These two statements can be deduced from more general results by Exner and Ichinose [EI01]. They are complemented by Exner and Nemčová in [EN03] with the limiting property $\sharp \sigma_d(A_\varphi) \to +\infty$ as $\varphi \to 0+$.

In [LLP10] the author obtained jointly with Igor Lobanov and Igor Yu. Popov a general result, which implies the lower bound on the spectrum of $A_\varphi$

$$\inf \sigma(A_\varphi) \geq -\frac{\alpha^2}{4 \sin^2(\varphi/2)}.$$  

This bound is close to optimal for $\varphi$ close to $\pi$, whereas in the limit $\varphi \to 0+$ the bound tends to $-\infty$. In the present note we sharpen this bound. Namely, we obtain

$$\inf \sigma(A_\varphi) \geq -\frac{\alpha^2}{(1 + \sin(\varphi/2))^2}. \tag{1.3}$$

The new bound yields that the operators $A_\varphi$ are uniformly lower-semibounded with respect to $\varphi$ and

$$\inf \sigma(A_\varphi) \geq -\alpha^2$$

holds for all $\varphi \in (0, \pi]$. This observation agrees well with physical expectations. Note that separation of variables yields that $\inf \sigma(A_\pi) = -\alpha^2/4$ and in this case the lower bound in (1.3) coincides with the exact spectral bottom.

For sufficiently sharp angles upper bounds on $\inf \sigma(A_\varphi)$ were obtained by Brown, Eastham and Wood in [BEW08]. See also Open Problem 7.3 in [E08] related to the discrete spectrum of $A_\varphi$ for $\varphi$ close to $\pi$.

**δ-interactions on crossing straight lines.** We also consider an analogous model with the δ-interaction supported on the set $\Gamma_\varphi = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1$ and $\Gamma_2$ are two straight lines, which cross at the angle $\varphi \in (0, \pi)$ as in Figure 2.

\[ \begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure2}
\caption{The straight lines $\Gamma_1$ and $\Gamma_2$ crossing at the angle of degree $\varphi \in (0, \pi)$.}
\end{figure} \]
The corresponding self-adjoint operator $B_\varphi$ in $L^2(\mathbb{R}^2)$ can be defined via the closed, densely defined, symmetric and lower-semibounded quadratic form

$$ b_\varphi[f] := \|\nabla f\|^2_{L^2(\mathbb{R}^2; C^2)} - \alpha \|f|_{\Gamma_\varphi}\|^2_{L^2(\Gamma_\varphi)}, \quad \text{dom } b_\varphi := H^1(\mathbb{R}^2), $$

in $L^2(\mathbb{R}^2)$, where $\alpha > 0$ is the strength of interaction. According to [EN03] it is known that $\sigma_{\text{ess}}(B_\varphi) = [-\alpha^2/4, +\infty)$ and that $\#\sigma_d(B_\varphi) \geq 1$.

In this note we obtain the lower bound

$$ \inf \sigma(B_\varphi) \geq -\frac{\alpha^2}{1 + \sin \varphi}, $$

using the same method as for the operator $A_\varphi$. Separation of variables yields

$$ \inf \sigma(B_{\pi/2}) = -\frac{\alpha^2}{2}, $$

and in this case the lower bound in the estimate (1.5) coincides with the exact spectral bottom.

Upper bounds on $\inf \sigma(B_\varphi)$ were obtained in [BEW08, BEW09]. The operators of the type $B_\varphi$ arise in the one-dimensional quantum three-body problem after excluding the center of mass, see Cornean, Duclos and Ricaud [CDR06, CDR08] and the references therein.

We want to stress that our proofs are of elementary nature and we do not use any reduction to integral operators acting on interaction supports $\Sigma_\varphi$ and $\Gamma_\varphi$.

### 2. Sobolev spaces on wedges

In this section $\Omega \subset \mathbb{R}^2$ is a wedge with the angle of degree $\varphi \in (0, 2\pi)$. The Sobolev space $H^1(\Omega)$ is defined as usual, see [McL, Chapter 3]. For any $f \in H^1(\Omega)$ the trace $f|_{\partial \Omega} \in L^2(\partial \Omega)$ is well-defined as in [McL, Chapter 3] and [M87].

**Proposition 2.1.** [LP08, Lemma 2.6] Let $\Omega$ be a wedge with angle of degree $\varphi \in (0, \pi]$. Then for any $f \in H^1(\Omega)$ the estimate

$$ \|\nabla f\|^2_{L^2(\Omega; C^2)} - \gamma \|f|_{\partial \Omega}\|^2_{L^2(\partial \Omega)} \geq -\frac{\gamma^2}{\sin^2(\varphi/2)} \|f\|^2_{L^2(\Omega)} $n

holds for all $\gamma > 0$.

**Proposition 2.2.** [LP08, Lemma 2.8] Let $\Omega$ be a wedge with angle of degree $\varphi \in (\pi, 2\pi)$. Then for any $f \in H^1(\Omega)$ the estimate

$$ \|\nabla f\|^2_{L^2(\Omega; C^2)} - \gamma \|f|_{\partial \Omega}\|^2_{L^2(\partial \Omega)} \geq -\gamma^2 \|f\|^2_{L^2(\Omega)} $n

holds for all $\gamma > 0$.

Propositions 2.1 and 2.2 are variational equivalents of spectral results from [LP08].
3. A LOWER BOUND ON THE SPECTRUM OF $A_\varphi$

In the next theorem we sharpen the bound (1.2) using only properties of the Sobolev space $H^1$ on wedges and some optimization.

**Theorem 3.1.** Let the self-adjoint operator $A_\varphi$ be associated with the quadratic form given in (1.1). Then the estimate

$$\inf \sigma(A_\varphi) \geq -\frac{\alpha^2}{(1 + \sin(\varphi/2))^2}$$

holds.

**Proof.** The angle $\Sigma_\varphi$ separates the Euclidean space $\mathbb{R}^2$ into two wedges $\Omega_1$ and $\Omega_2$ with angles of degrees $\varphi$ and $2\pi - \varphi$, see Figure 3.

The underlying Hilbert space can be decomposed as

$$L^2(\mathbb{R}^2) = L^2(\Omega_1) \oplus L^2(\Omega_2).$$

Any $f \in \text{dom } a_\varphi$ can be written as the orthogonal sum $f_1 \oplus f_2$ with respect to that decomposition of $L^2(\mathbb{R}^2)$. Note that $f_1 \in H^1(\Omega_1)$ and that $f_2 \in H^1(\Omega_2)$. Clearly,

$$\|f\|^2_{L^2(\mathbb{R}^2)} = \|f_1\|^2_{L^2(\Omega_1)} + \|f_2\|^2_{L^2(\Omega_2)},$$

$$\|\nabla f\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} = \|\nabla f_1\|^2_{L^2(\Omega_1;\mathbb{C}^2)} + \|\nabla f_2\|^2_{L^2(\Omega_2;\mathbb{C}^2)}.$$  \hfill (3.1)

The coupling constant can be decomposed as $\alpha = \beta + (\alpha - \beta)$ with some optimization parameter $\beta \in [0, \alpha]$ and the relation

$$\alpha\|f|_{\Sigma_\varphi}\|^2_{L^2(\Sigma_\varphi)} = \beta\|f_1|_{\partial \Omega_1}\|^2_{L^2(\partial \Omega_1)} + (\alpha - \beta)\|f_2|_{\partial \Omega_2}\|^2_{L^2(\partial \Omega_2)},$$

holds. According to Proposition 2.1

$$\|\nabla f_1\|^2_{L^2(\Omega_1;\mathbb{C}^2)} - \beta\|f_1|_{\partial \Omega_1}\|^2_{L^2(\partial \Omega_1)} \geq -\frac{\beta^2}{\sin^2(\varphi/2)}\|f_1\|^2_{L^2(\Omega_1)},$$

and according to Proposition 2.2

$$\|\nabla f_2\|^2_{L^2(\Omega_2;\mathbb{C}^2)} - (\alpha - \beta)\|f_2|_{\partial \Omega_2}\|^2_{L^2(\partial \Omega_2)} \geq -(\alpha - \beta)^2\|f_2\|^2_{L^2(\Omega_2)},$$

(3.3) holds.
The observations (3.1), (3.2) and the estimates (3.3), (3.4) imply
\[ a_\varphi[f] \geq -\max \left\{ \frac{\beta^2}{\sin^2(\varphi/2)}, (\alpha - \beta)^2 \right\} \| f \|^2_{L^2(\mathbb{R}^2)}. \]
Making optimization with respect to \( \beta \), we observe that the maximum between the two values in the estimate above is minimal, when these two values coincide. That is
\[ \frac{\beta^2}{\sin^2(\varphi/2)} = (\alpha - \beta)^2, \]
which is equivalent to
\[ \beta = \frac{\alpha \sin(\varphi/2)}{(1+\sin(\varphi/2))}, \]
resulting in the final estimate
\[ a_\varphi[f] \geq -\frac{\alpha^2}{(1+\sin(\varphi/2))^2} \| f \|^2_{L^2(\mathbb{R}^2)}. \]
This final estimate implies the desired spectral bound. \( \square \)

Remark 3.2. Note that the previously known lower bound (1.2) comes out from the proof of the last theorem if we choose \( \beta = \alpha/2 \), which is the optimal choice in our proof only for \( \varphi = \pi \) as we see from (3.5).

4. A LOWER BOUND ON THE SPECTRUM OF \( B_\varphi \)

In the next theorem we obtain a lower bound on the spectrum of the self-adjoint operator \( B_\varphi \) using the same idea as in Theorem 3.1.

Theorem 4.1. Let the self-adjoint operator \( B_\varphi \) be associated with the quadratic form given in (1.4). Then the estimate
\[ \inf \sigma(B_\varphi) \geq -\frac{\alpha^2}{1+\sin \varphi} \]
holds.

Proof. The crossing straight lines \( \Gamma_1 \) and \( \Gamma_2 \) separate the Euclidean space \( \mathbb{R}^2 \) into four wedges \( \{\Omega_k\}_{k=1}^4 \). Namely, the wedges \( \Omega_1 \) and \( \Omega_2 \) with angles of degree \( \varphi \) and the wedges \( \Omega_3 \) and \( \Omega_4 \) with angles of degree \( \pi - \varphi \), see Figure 4.

![Figure 4](image-url)

Figure 4. The crossing straight lines \( \Gamma_1 \) and \( \Gamma_2 \) separate the Euclidean space \( \mathbb{R}^2 \) into four wedges \( \{\Omega_k\}_{k=1}^4 \).
The underlying Hilbert space can be decomposed as

\[ L^2(\mathbb{R}^2) = \bigoplus_{k=1}^{4} L^2(\Omega_k). \]

Any \( f \in \text{dom} \ b_\varphi \) can be written as the orthogonal sum \( \oplus_{k=1}^{4} f_k \) with respect to that decomposition of \( L^2(\mathbb{R}^2) \). Note that \( f_k \in H^1(\Omega_k) \) for \( k = 1, 2, 3, 4 \).

Clearly,

\[ (4.1) \quad \|f\|_{L^2(\mathbb{R}^2)}^2 = \sum_{k=1}^{4} \|f_k\|_{L^2(\Omega_k)}^2, \quad \|
abla f\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)}^2 = \sum_{k=1}^{4} \|
abla f_k\|_{L^2(\Omega_k;\mathbb{C}^2)}^2. \]

The coupling constant can be decomposed as \( \alpha = \beta + (\alpha - \beta) \) with some optimization parameter \( \beta \in [0, \alpha] \) and the relation

\[ (4.2) \quad \alpha \|f_1\|_{L^2(\Omega;\mathbb{C}^2)}^2 = \beta \|f_1\|_{L^2(\Omega)}^2 + \|f_2\|_{L^2(\partial \Omega_2)}^2 + (\alpha - \beta) \|f_3\|_{L^2(\partial \Omega_3)}^2 + (\alpha - \beta) \|f_4\|_{L^2(\partial \Omega_4)}^2 \]

holds. According to Proposition 2.1

\[ (4.3) \quad \|
abla f_1\|_{L^2(\Omega_1;\mathbb{C}^2)} - \beta \|f_1\|_{L^2(\partial \Omega_1)} \geq -\frac{\beta^2}{\sin^2(\varphi/2)} \|f_1\|_{L^2(\Omega_1)}^2, \]

\[ \|
abla f_2\|_{L^2(\Omega_2;\mathbb{C}^2)} - \beta \|f_2\|_{L^2(\partial \Omega_2)} \geq -\frac{\beta^2}{\sin^2(\varphi/2)} \|f_2\|_{L^2(\Omega_2)}^2. \]

Also according to Proposition 2.1

\[ (4.4) \quad \|
abla f_3\|_{L^2(\Omega_3;\mathbb{C}^2)} - (\alpha - \beta) \|f_3\|_{L^2(\partial \Omega_3)} \geq -\frac{(\alpha - \beta)^2}{\cos^2(\varphi/2)} \|f_3\|_{L^2(\Omega_3)}^2, \]

\[ \|
abla f_4\|_{L^2(\Omega_4;\mathbb{C}^2)} - (\alpha - \beta) \|f_4\|_{L^2(\partial \Omega_4)} \geq -\frac{(\alpha - \beta)^2}{\cos^2(\varphi/2)} \|f_4\|_{L^2(\Omega_4)}^2. \]

The observations (4.1), (4.2) and the estimates (4.3), (4.4) imply

\[ b_\varphi[f] \geq -\max \left\{ -\frac{\beta^2}{\sin^2(\varphi/2)} - \frac{(\alpha - \beta)^2}{\cos^2(\varphi/2)} \right\} \|f\|_{L^2(\mathbb{R}^2)}^2. \]

Making optimization with respect to \( \beta \), we observe that the maximum between the two values in the estimate above is minimal, when these two values coincide. That is

\[ \frac{\beta^2}{\sin^2(\varphi/2)} = \frac{(\alpha - \beta)^2}{\cos^2(\varphi/2)}, \]

which is equivalent to

\[ (4.5) \quad \beta = \frac{\alpha \tan(\varphi/2)}{(1+\tan(\varphi/2))}, \]

resulting in the final estimate

\[ b_\varphi[f] \geq -\frac{\alpha^2}{1+\tan(\varphi)} \|f\|_{L^2(\mathbb{R}^2)}^2. \]

This final estimate implies the desired spectral bound. \( \square \)

**Remark 4.2.** The result of Theorem 4.1 complements [CDR08, Theorem 4.6 (iv)], where the bound

\[ \inf \sigma(B_\varphi) \geq -\alpha^2, \]

for all \( \varphi \in (0, \pi) \) was obtained.
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