A New Phase at Finite Quark Density from AdS/CFT

Shin Nakamura, a,b Yunseok Seo, a Sang-Jin Sin, a, and K. P. Yogendran, a,b
a Department of physics, BK21 Program Division, Hanyang University, Seoul 133-791, Korea
b Center for Quantum Spacetime, Sogang University, Seoul, 121-742, Korea

(Dated: July 19, 2018)

We explore phases of an $\mathcal{N} = 2$ super Yang-Mills theory at finite quark density by introducing quark chemical potential in a D3-D7 setup. We formulate the thermodynamics of brane embeddings and we find that the density versus chemical potential equation of state has rich structure. This yields two distinct first order phase transitions in a small window of quark density. In order words, there is a new first order phase transition in the region of deconfined quarks. In this new phase, the chemical potential is a decreasing function of the density. We suggest that this might be relevant to the difference in sQGP–wQGP phases of QCD.

PACS numbers: Valid PACS appear here

Introduction: There has been much hope that one might be able to use AdS/CFT [1] to describe the real systems after certain amount of deformations. For example, it has been suggested that the fireball in Relativistic Heavy Ion Collision (RHIC) viewed as a strongly interacting system [2, 3], has been studied using dual gravity models [4–8]. There have been many attempts to construct models phenomenologically closer to QCD [9].

More recently, there has been renewed interest in $\mathcal{N} = 2$ super Yang-Mills (SYM) systems with quenched fundamental quark flavors studied by using a holographic description with probe D7-branes in the $AdS_5$ black hole background [10–15]. The key observation is that we have confinement of quarks even in the absence of gluon confinement or area law [10]. The phases of this theory are characterized by the brane embeddings: whether the D7-brane touches the black hole horizon (black hole embedding) or not (Minkowski embedding). Different types of embedding lead to different meson spectra.

In this letter, we explore the phases of this theory at finite quark density by introducing quark chemical potential along the lines of [16, 17]. We will first establish a clear formulation of the thermodynamics of brane embeddings. We find that we need to renormalize the finite chemical potential due to the divergence of the thermodynamic potentials. We will also find that apart from the type of first order phase transition described in [10, 11] at zero chemical potential, there is another class of first order phase transition within the black hole embedding category: it is indicated in Fig. 1 as a hopping between two black hole embeddings.

Since black hole embeddings correspond to the deconfined phase, we cautiously suggest that this new type of first order phase transitions might be relevant to the difference between sQGP–wQGP in RHIC experiments. In particular, we find that the chemical potential in this new phase is a decreasing function of the density.

We emphasize that depending on whether we control the system by the chemical potential (grand canonical ensemble) or by the density (canonical ensemble), the phase diagram is different. In this letter, we present the analysis and the results for the system based on the canonical ensemble. The details including the description of the other process based on the grand canonical ensemble will be reported in detailed publication [18].

The Euclidean AdS black hole metric is given by

$$ds^2 = \frac{U^2}{R^2} \left( f(U) dt^2 + d\vec{x}^2 \right) + R^2 \left( \frac{dU^2}{f(U)U^2} + d\Omega_5^2 \right), \quad (1)$$

where $f(U) = 1 - (U_0/U)^4$. The Hawking temperature of this geometry is given by $T = U_0/\pi R^2 = U_0/\sqrt{2\lambda} \pi \alpha'$ where $\lambda = g_5^2 N_c$. We introduce a dimensionless coordinate $\xi$ defined by $d\xi^2/\xi^2 = dU^2/(U^2 f)$, so that the bulk geometry is

$$ds^2 = \frac{U^2(\xi)}{R^2} \left( f(\xi) dt^2 + d\vec{x}^2 \right) + \frac{R^2}{\xi^2} ds_6^2, \quad \text{with} \quad (2)$$

$$ds_6^2 = d\xi^2 + \xi^2 d\vec{x}^2 = d\rho^2 + \rho^2 d\Omega_3^2 + dy^2 + y^2 d\phi^2,$$

where we have defined $\xi^2 \equiv y^2 + \rho^2$ and $\rho$ is the radius of the 3-sphere. The black hole horizon is located at $\xi = 1$. The induced metric on the D7-brane is

$$ds_{D7}^2 = \frac{U^2}{R^2} (f dt^2 + d\vec{x}^2) + \frac{R^2}{\xi^2} \left( (1 + y^2) d\rho^2 + \rho^2 d\Omega_3^2 \right), \quad (3)$$

where $y' = \partial_{\rho} y(\rho)$. It is interesting to notice that the bulk metric (1) and the induced metric (3) have the same

![FIG. 1: Brane embedding and phase transitions. Solid lines: Minkowski embedding (red) to black hole embedding (blue). Dashed lines: Hopping from a black hole embedding to another one.](image-url)
Hawking temperature. This means that the bulk and the
brane are in equilibrium. The Euclidean DBI action of
the D7-brane in the presence of the gauge field strength
$F_{\mu\nu}$ is

$$S = N_f \mu T \int d^3 x d\rho d\Omega_3 \sqrt{\det(\mathcal{G} + 2\pi a' F)} = \beta V_3 \int d\rho \mathcal{L}, (4)$$

with

$$\mathcal{L} = \tau_7 \rho^{3/2} \omega_+^{3/2} \sqrt{\frac{\omega^2}{\omega^+} (1 + y^2) - (F/m_T)^2}, (5)$$

where $\tau_7 = N_f N_c T^4 g_s^2 \xi / 32$, $\omega_+(\xi) = 1 \pm \xi^{-4}$ and $m_T = \frac{1}{2} \sqrt{\lambda T}$ (the Chern-Simons term vanishes in the present case).

**Conserved charge and equations of motion:**
Since $\mathcal{L}$ does not depend on $A_0$ explicitly, its conjugate momentum is a conserved quantity:

$$\Pi_{A_0} = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = -Q, (6)$$

in terms of which we can write

$$F_{\mu\nu} = m_T \dot{Q} \omega - \sqrt{(1 + y^2) / \omega_+ (\dot{Q}^2 + \omega_+^2 \rho^2)}, (7)$$

where $\dot{Q} = \frac{m_T}{\tau_7} Q$. Since we have a constraint (6), to obtain a Lagrangian for $y$, we should not substitute $F_{\mu\nu}$ into the original Lagrangian (5). The correct procedure is to perform a Legendre transformation

$$\mathcal{H} = \mathcal{L} - \Pi_{A_0} \dot{A}_0', (8)$$

and then impose the conservation equation (6) to eliminate the electric field completely. The resulting “Hamiltonian” is given by

$$\mathcal{H} = T(y, \rho) \sqrt{1 + y^2}, \quad T(y, \rho) = \tau_7 \sqrt{\frac{\omega^2}{\omega^+} (\dot{Q}^2 + \omega_+^2 \rho^2)}. (9)$$

We can take this Hamiltonian as our effective Lagrangian for $y$ and $T$ may be regarded as the effective tension of the D7-brane. Resulting equation of motion is

$$\frac{y''}{1 + y^2} + \frac{\partial \log T}{\partial \rho} y' - \frac{\partial \log T}{\partial y} = 0. (10)$$

One can check that both the original and the effective Lagrangian give the same equation of motion for $y$.

**Chemical potential in gravity dual:** In Refs. [16, 17], the quark chemical potential was introduced as the value of $A_0$ on the D7-brane worldvolume. Here we define the chemical potential in a gauge-invariant fashion:

$$\mu = \int_{\rho_{\text{min}}}^{\infty} d\rho F_{\mu\nu} = \lim_{\rho \rightarrow \infty} A_0. (11)$$

For the last equality, we need to gauge fix $A_0 = 0$ and set $A_0|_{\rho_{\text{min}}} = 0$, which agrees with [16, 17].

Notice that $\mu = \int F_{\mu\nu}$ is the work to bring a unit charge from the UV region ($\rho = \infty$) to the IR region against electric field $F_{\mu\nu}$. This definition (11) agrees with our intuition of the chemical potential as work done to add a quark to the system.

For the Minkowski embedding, $\rho_{\text{min}} = 0$. For the black hole embedding, $\rho_{\text{min}} = \cos \theta$ where $\theta$ is the angular coordinate on the $y$-$\rho$ plane.

**Thermodynamic potentials:** A generic grand potential (density) is defined by $e^{-\beta \mathcal{H}(\mu)} = \text{Tr} e^{-\beta \mathcal{H}(\mu)}$. Here we identify the DBI action which is a functional of $A_0$ as the grand potential $S_{\text{DBI}} = \beta V_3 \Omega$. Then, integrating the Legendre transformation (8),

$$\int d\rho \mathcal{H} = \int d\rho \mathcal{L} - \int d\rho \Pi_{A_0} A_0'. (12)$$

Using the fact that $\Pi_{A_0} = -Q$ is a constant (in $\rho$ evolution), one can rewrite the above as

$$F(Q) = \Omega(\mu) + \mu Q. (13)$$

It is remarkable that the Legendre transformation in the bulk classical field theory is reinterpreted as the Legendre transformation between the canonical and the grand canonical ensembles in the boundary thermodynamics.

The chemical potential enters the Hamiltonian density of the gauge theory at the boundary as a coupling to the baryon number density:

$$\Delta \mathcal{H}_{YM} = -\mu \langle \psi^\dagger \psi \rangle. (14)$$

Therefore $Q$ which has been originally defined as a first integral of the DBI action should be identified as the number density of quarks/baryons. Notice that the effective tension of the brane increases as we increase the quark density which may have been expected. More precisely, after considering various scale factors, we have

$$Q = \langle \psi^\dagger \psi \rangle. (15)$$

**Renormalization of chemical potential:** The Helmholtz free energy $F$ and the grand potential are in fact divergent quantities since they contain a divergent $\rho$ integral. Therefore we need to regularize them. We choose to subtract $F_0(Q)$, the value of $F$ for the D7-brane configuration that touches the black hole on the equatorial plane ($y = 0$). This is like a Pauli-Villars regularization in the baryon setup. We call this brane as the reference brane. So the renormalized free energy which we will calculate is defined by

$$F_R(Q) = F(Q) - F_0(Q). (16)$$

This has a far reaching effect to the chemical potential. To see this notice

$$F_R(Q) = \Omega(\mu) - \Omega(\mu_0) + (\mu - \mu_0)Q := \Omega_R(\mu R) + \mu R Q. (17)$$
where we have used the fact that \( F_0(Q) = \Omega(\mu_0) + \mu_0 Q \). Notice that \( \mu_0 \) is the chemical potential at the reference-brane configuration. The point is that when we deal with the renormalized quantities like \( F_R \) and \( \Omega_R \), we also have to use the renormalized chemical potential, although \( \mu \) itself is finite quantity from the beginning. In all numerical analysis, we need to use \( \mu_R \) for the chemical potential.

From now on, we delete subindex \( R \) unless it is confusing.

**The phase structure of D3-D7:** In this article, we study the system based on the canonical ensemble, where the number density \( Q \) is continuous while the chemical potential is allowed to jump.

Let us expand \( y \) and \( A_0 \) in the form of \( y(\rho) = L + \frac{\bar{c}}{\rho^2} + O(\rho^{-4}), A_0(\rho) = \mu + \frac{c}{\rho^2} + O(\rho^{-4}) \). We have \( L = \frac{m_q^3}{\sqrt{2} \pi \tau T}, \bar{c} = c \frac{\sqrt{2} \pi m_q T}{\tau} \) where \( m_q \) is a quark mass and \( c = \langle \bar{\psi} \psi \rangle \) [14, 15]. One can interpret \( L \) as the quark mass at fixed temperature or as inverse temperature at fixed quark mass. We also find that \( c' = -\frac{1}{2} \frac{m_q^3}{\sqrt{2} \pi T} Q \). The standard AdS/CFT dictionary establishes that we have two pairs of conjugate variables, namely \( (\bar{c}, L) \) and \( (Q, \mu) \).

To analyse the phase transition, we plot the free energy \( F/T^4 \) as a function of \( L \) for a given \( Q \) (Fig. 2) and track the least free energy configuration. In Figure 2, we start from \( P' \) on the Minkowski branch. As \( L \) decreases, the free energy decreases until we intersect the black hole branch at \( P \). For smaller values of \( L \), the black hole configurations have lower free energy. At \( P \) therefore, the D7-brane jumps from a Minkowski embedding to touch the black hole horizon. This is the same type of first order phase transition first found in [10] for zero chemical potential. Decreasing \( L \) further, decreases the latitude (\( \theta \)) of the intersection of the brane with the horizon smoothly. For large \( Q \), the story ends here.

For very small \( Q \), however, as we decrease \( L \) further, initially the latitude of intersection of brane-black hole goes down smoothly. But, at a critical latitude \( \theta_1 \), the embedding suddenly jumps to a smaller latitude \( \theta_2 \) as indicated schematically in Fig. 1 (the kink near \( U' \) in Fig. 2). An embedding with touching latitudes between these two values has higher free energy as seen in Fig. 3, and hence is never realized. This phenomenon happens only at finite density within a small density window,

\[
\log \tilde{Q} < -6.812 < \log \tilde{Q} < -4.726 = \log \tilde{Q}_2. \tag{18}
\]

The full phase diagram therefore looks as in Fig. 4. Notice that the horizontal axis is \( \log \tilde{Q} \). The chemical potential jumps across the phase boundary.

For \( Q > Q_2 \), there is no kink and hence the second phase transition disappears as we can see in Fig. 5. Furthermore, we have a second order phase transition at \( \tilde{Q} = Q_2 \). The slopes of the phase boundary lines are discontinuous at \( \tilde{Q} = \tilde{Q}_2 \), where three lines meet.

**Equations of state:** The relations between the variables \( (\bar{c}, L) \) (equivalently \( (c, m_q) \)) and \( (Q, \mu) \) can be considered as the diagram of equation of state, which is much like the \( P-V \) diagram of Van der Waals in liquid-gas phase transition. We may determine these by using the thermodynamic relations

\[
c = -\frac{\partial F}{\partial m_q}, \quad \mu = \frac{\partial F}{\partial Q}.
\]
Numerical analysis shows that the relationship between \( \mu \) and \( Q \) given by \( Q = -\frac{\partial (\mu)}{\partial \mu} \) is not monotonic, further \( \mu \) is not a single-valued function of \( Q \).

In Fig. 6 and 7, we show the relation between \( \tilde{c} \) and \( L \) for a representative value of \( Q \) and the relation between \( \tilde{Q} \) and \( \mu \) for a representative value of \( L \). The equations of state represented in these diagrams show that we have much richer structure than Van der Waals \( P-V \) diagram.

![FIG. 6: \( \mu-Q \) relation: We plot \( \mu/T \) vs. \( Q/T^3 \).](image)

We track the phase diagram from \( M' \) in Fig. 6 at a fixed value of \( L \) which may be thought of inverse temperature (or quark mass). The first phase transition takes place at a certain critical value of \( Q \) at point \( P' \). Here, the brane embedding jumps from a Minkowski embedding to a black hole embedding which results in a jump of the chemical potential from \( P' \) to \( R' \) on the diagram. If we increase \( Q \) further, we have a second phase transition which is realized as a jump from \( S' \) to \( U' \). We have also indicated the Maxwell’s construction, which allows us to determine the location of the phase transition (i.e., equality of the area of the shaded region on either side of a particular jump). For larger values of \( L \) however, we have only a single phase transition.

The point to be noted is that in a small window between \( R' \) and \( S' \), the chemical potential is a decreasing function of the density (in the deconfined phase). One way to achieve this is to have bound states. If this is true, our new phase is likely to have many bounded pairs of quarks.

Referring to the \( \tilde{c}-L \) equation of state in Fig. 7, the two phase transitions are as labelled. In this case however, the chiral condensate decreases uniformly as a function of \( L \) (excepting of course, for the jumps). For large quark mass \( L \), it seems to vanish as expected.

**Discussion:** Although characterized by nonstandard behaviour of the chemical potential, the nature of the second phase is not very clear from the gauge theory point of view. However, since both phases belong to the black hole embedding corresponding to deconfined quarks, it might be relevant to the famous difference between sQGP and wQGP discussed within RHIC physics [2]. In fact if we take the temperature and \( Q \) of the relevant region to be 200MeV and \( 5 \times 10^{-3} \) respectively and choose \( N_f N_c \sqrt{\lambda}/16 \sim O(1) \), then the density \( Q \sim 1 \text{fm}^{-3} \), which is in the RHIC ballpark.

![FIG. 7: \( \tilde{c}-L \) relation: Here we interpret \( L \) as quark mass.](image)

An immediate question is the universality of this second phase. Since most of the properties of the branes, for small \( Q \), are determined in the neighborhood of the horizon, we might expect that this phenomenon persists in (perhaps more realistic) models so long as we have a black hole.

One can also look at a process which corresponds to a horizontal Maxwell’s construction in \( \mu-Q \) diagram. This is a process where the chemical potential is used as the control parameter. Such a process is more typical in the literature [19]. Some other related questions are about the effect of the density and temperature on the meson spectrum and on the heavy quark potentials. A limitation of this model is the absence of gluon confinement. We can also ask whether what we found in this paper is universal feature of black hole embeddings. These issues are currently under investigation [18].

**Acknowledgments:** The authors want to thank Kazuaki Ohnishi, Mannque Rho and Motoi Tachibana for useful discussions. This work was supported by KOSEF Grant R01-2004-000-10520-0 and the SRC Program of the KOSEF through the Center for Quantum Space-time of Sogang University with grant number R11-2005-021.

---

[1] J. M. Maldacena, Adv. Theor. Math. Phys. 2 231 (1998), hep-th/9711200.

[2] E. V. Shuryak, Nucl. Phys. A750 64 (2005), hep-ph/0405066.

[3] M. J. Tannenbaum, Rept. Prog. Phys. 69 2005 (2006), nucl-ex/0603003.

[4] G. Policastro, D. T. Son and A. O. Starinets, Phys. Rev. Lett. 87 081601 (2001), hep-th/0104066.

[5] S. J. Sin and I. Zahed, Phys. Lett. B608 265 (2005), hep-th/0407215; E. Shuryak, S-J. Sin and I. Zahed, hep-th/0511199.

[6] H. Nastase, hep-th/0501068.

[7] C. P. Herzog, A. Karch, P. Kovtun, C. Kozcaz and L. G. Yaffe, JHEP 0607 013 (2006), hep-th/0605158.

[8] R. A. Janik and R. Peschanski, Phys. Rev. D 73 045013 (2006), hep-th/0512162; S. Nakamura and S. J. Sin,
A short list of the references is: T. Sakai and S. Sugimoto, Prog. Theor. Phys. 113 843 (2005), hep-th/0412141; J. Erlich, E. Katz, D. T. Son and M. A. Stephanov, Phys. Rev. Lett. 95 261602 (2005), hep-ph/0501128; L. Da Rold and A. Pomarol, Nucl. Phys. B721 79 (2005), hep-ph/0501218.

D. Mateos, R. C. Myers and R. M. Thomson, Phys. Rev. Lett. 97, 091601 (2006), hep-th/0605046.

T. Albash, V. Filev, C. V. Johnson and A. Kundu, hep-th/0605088.

A. Karch and E. Katz, JHEP06 043 (2002), hep-th/0205236.

M. Kruczenski, D. Mateos, R. C. Myers, and D. J. Winters, JHEP07 049 (2003), hep-th/0304032.

J. Babington, J. Erdmenger, N. J. Evans, Z. Guralnik, and I. Kirsch, Phys. Rev. D69 066007 (2004), hep-th/0306018.

M. Kruczenski, D. Mateos, R. C. Myers, and D. J. Winters, JHEP 05 041 (2004) 041, hep-th/0311270.

K. Y. Kim, S. J. Sin and I. Zahed, hep-th/0608046.

N. Horigome and Y. Tanii, hep-th/0608198.

S. Nakamura, Y. Seo, S-J. Sin and P. Yogendran “to appear.”

See for example, K. Yagi, T. Hatsuda and Y. Miake, “Quark-Gluon Plasma: From Big Bang To Little Bang,” Cambridge Univ. Pr. (2005).