Controlled Fractional Order Processes with Distributed Parameters

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Abstract. This work is devoted to the study of the problem of pursuit in systems controlled with distributed parameters of fractional order. Derivatives with respect to spatial variables are ordinary, integer, and even of arbitrary order. Designed and studied sampling schemes. A numerical method is constructed for finding strategies in suitable classes and for constructing the corresponding control laws. Theorems are proved for the possibility of completing the pursuit by the finite difference method, and they can be used to solve finite-difference optimal control problems or numerical solutions of differential games of fractional order. Problems of the type under study are encountered in modeling the processes of economic growth and in problems of stabilizing dynamic systems.

1. Introduction

There are many works devoted to differential equations of fractional order, among them we mention research [1,2]. They mainly study fractional-order differential equations with lumped parameters. Boundary-value problems are considered in detail in [3,4], and non-self-adjoint integral operators accompanying these boundary-value problems are investigated by methods of perturbation theory. Many, especially interdisciplinary problems, can be modeled using fractional derivatives. For example, a nonlinear oscillation of an earthquake can be modeled with a fractional derivative [3], and a hydrodynamic model of motion with fractional derivatives [4] can eliminate the disadvantage arising from a continuous traffic flow.

Numerous studies are devoted to pursuit and evasion problems for various classes of differential and discrete games [5-11]. In [6], linear differential games are considered, the main model for which is the process of pursuit of one controlled object by another controlled object, and fundamental results of the theory of differential games described by systems with lumped parameters are obtained. In [7, 9], the pursuit problem in controlled systems with high-order distributed parameters is investigated. Sufficient conditions are obtained for the possibility of completing the pursuit by the finite difference method. And in the article [10] the problem of pursuit of differential games is discussed. In [18], the method of processing digital images as a game problem with distributed parameters was first investigated. This article is related to works [6-10].

2. Methods

The general research methodology is based on the fundamental results of the mathematical theory and methods of sequential systems, the theory of linear multistep games, Bellman's methods of discrete control problems, the theory of dynamic and integer programming. The work also partially uses some numerical methods for solving fractional differential equations. Such methods have been investigated.
by many authors. For example, the variation iterative method, the method of homotopy perturbations, spectral methods and other methods [1-5].

In this paper, we use Pontryagin methods developed by specialists for discrete game pursuit problems. The well-known methods of discrete games are inapplicable to the corresponding discrete games, which are obtained after applying the method of finite differences to an equation with distributed parameters of fractional order. Basically, discrete games were studied as an independent problem or the corresponding differential games were of a whole order. Consequently, it was necessary to develop new sufficient conditions that could be applied to solve differential games described by an equation with distributed parameters of fractional order. By modifying the known methods, we managed to obtain such a sufficient condition. Thus, the mathematical apparatus used in this work is new.

Other methods are known as well. Currently, only algorithms for solving matrix games have been developed. Let us point out the methods of Brown, Neumann and their numerous modifications. Some approaches to solving continuous games are given in [11], but the algorithms proposed in them turn out to be very cumbersome in the case, especially when variable multidimensional vectors.

The methods for solving discrete games proposed in this paper are new; it allows solving difference game problems with initial and boundary conditions for differential games of fractional order.

3. Results

We consider the following game problem described by the fractional-order equation

$$D_0^{\alpha}z(x,t) - m^{-1} \frac{\partial^m}{\partial x^m} z(x,t) + u(x,t) = v(x,t), \quad m = 1, 2, \ldots,$$

$$z(x,0) = f(x), \quad 0 < x < 1;$$

$$\frac{\partial^2}{\partial x^2} z(0,t) = 0, \quad p = 0, 1, \ldots, m-1; \quad 0 \leq t \leq T,$$

$$z \in \mathbb{R}^1, \quad u, v \quad \text{control parameters,} \quad u \quad \text{pursuit department} \quad v \quad \text{escape control} \quad \Omega \subset \mathbb{R}^1, \quad x \in \Omega = \{x: 0 \leq x \leq 1\}, \quad (x,t) \in Q_T = \{(x,t): x \in \Omega, \quad 0 < t \leq T\}, \quad S_T = \{(x,t) | x \in \partial \Omega, \quad t \in [0,T]\}. \quad \partial \Omega \quad \text{area boundary} \quad 0 < \alpha \leq 1.$$

The fractional derivative will be understood in the usual sense. $D_0^{\alpha}z(x,t) = \frac{\partial}{\partial t} I_{0t}^{1-\alpha} z(x,t)$, where $I_{0t}^{1-\alpha}$ partial fractional integrals [1].

In $\mathbb{R}^1$ selected non-empty terminal set $\bar{M}_1 \subset \mathbb{R}^1$.

**Definition 1.** In problem (1) - (3), it is possible $\mathcal{E} \quad \text{completion} \quad \varepsilon > 0$ pursuit from starting position $f(\cdot)$, if there are number $T = T(f(\cdot))$ and function $u(\nu, x, t) \in \overline{P}, \quad v \in \overline{Q}, \quad x \in \Omega, \quad t \in [0,T]$, such that for an arbitrary function $t_0(x, t) \in \overline{Q}, \quad x \in \Omega, \quad t \in [0,T]$ solution $\mathcal{A}(x, t)$ of problem (1) - (3), where $u = u(\nu, x, t), \quad v = v_0(x, t)$, falls into the set $\mathcal{E} I + \bar{M}_1$, at some $(\tilde{x}, \tilde{t})$, $\tilde{x} \in \Omega, \quad \tilde{t} \in [0,T]$:

$$z_0(\tilde{x}, \tilde{t}) \in \mathcal{E} I + \bar{M}_1,$$

On the segment $[t_k, t_{k+1}]$ get

$$D_0^{\alpha}z(x,t) \bigg|_{t_k}^{t_{k+1}} = \frac{1}{\Gamma(1-\alpha)} \left( \frac{z(x,t_k)}{(t_{k+1}-t_k)^{1-\alpha}} + \int_{t_k}^{t_{k+1}} \frac{z'(x,s)ds}{(t_{k+1}-t_k)^{\alpha}} \right)$$

Hence we have
Problems (1) - (3), are replaced by the following equations
\[
D_0^\alpha z(x,t)\bigg|_{t_i} \approx \frac{1}{\Gamma(1-\alpha)} \left( \frac{z(x,t_k)}{(t_{k+1}-t_k)\alpha} + \frac{z(x,t_{k+i}) - z(x,t_k)}{l} \int_{t_k}^{t_{k+i}} \frac{ds}{(t_{k+i}-s)^\alpha} \right) = \frac{z(x,t_{k+1}) - \alpha z(x,t_k)}{l^\alpha (1-\alpha) \Gamma(1-\alpha)} = \frac{z(x,t_{k+1}) - \alpha z(x,t_k)}{l^\alpha (2-\alpha)}.
\]

It is easy to verify [5] that the solution satisfies the following convergence rate estimate
\[
\left| \left( z\right)_{hl} - z_{i,k} \right|_{\Phi_{hl}} \leq \frac{K_1l + K_2h^2}{a \neq \frac{1}{2}}, \quad \left| \left( z\right)_{hl} - z_{i,k} \right|_{\Phi_{hl}} \leq \frac{K_1l^2 + K_2h^2}{a = \frac{1}{2}},
\]

\( \Phi_{hl} \) - space of grid functions, \( \| \cdot \|_{\Phi_{hl}} \) - the norm of this space, \( K_1, K_1, K_2, K_2 \) - constants.

After some transformations and notation, we obtain the following equation
\[
Az_{k+1} - Bz_k = -(2-\alpha) \quad l^\alpha u_k + (2-\alpha) \quad l^\alpha v_k, \quad k = 0, 1, 2, \ldots, \theta - 1, \quad z_0 = 0,
\]

where \( z_k, u_k, v_k - H \) - matrices, \( H = r - 1 \), here
\[
z_k = (z_{i,k}, z_{i+1,k}, \ldots, z_{i+r-1,k})^T,
\]
\[
A = E + a (1 - \alpha) \frac{m \Gamma(2-\alpha)}{h^2m} C + B = a E + (1 - \alpha) \frac{m \Gamma(2-\alpha)}{h^2m} C + \frac{m \Gamma(2-\alpha)}{h^2m} C + \frac{m \Gamma(2-\alpha)}{h^2m} C,
\]

Where \( E \) - identity matrix, \( C = (c_{ij}) \) - a square matrix in which \( c_{ii} = -2 \) and \( c_{i-1,j} = c_{i+1,j} = 1 \). It is clear that there is \( A^{-1} \neq 0 \). We multiply both sides of equation (6) by \( A^{-1} \), and denoting
\[
A^{-1} B = G, \quad A^{-1} = F \text{ get}
\]
\[
z_{k+1} = Gz_k - F\Gamma(2-\alpha) l^\alpha u_k + F\Gamma(2-\alpha) l^\alpha v_k, \quad k = 0, 1, 2, \ldots, \theta - 1, \quad z_0 = 0.
\]

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Where \( E \) - identity matrix, \( C = (c_{ij}) \) - a square matrix in which \( c_{ii} = -2 \) and \( c_{i-1,j} = c_{i+1,j} = 1 \). It is clear that there is \( A^{-1} \neq 0 \). We multiply both sides of equation (6) by \( A^{-1} \), and denoting
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z_{k+1} = Gz_k - F\Gamma(2-\alpha) l^\alpha u_k + F\Gamma(2-\alpha) l^\alpha v_k, \quad k = 0, 1, 2, \ldots, \theta - 1, \quad z_0 = 0.
\]
orthogonal projection operator from $R^H$ to $L$. $A + B$ and $A^\perp B$ algebraic sum and geometric difference of sets $A, B$ respectively. Let $M_1 = \prod_{\gamma=1}^{n} M_{1\gamma}$, $1 \leq \gamma \leq H$,

$$\beta_m() = \left\{ \beta_0, \beta_1, \ldots, \beta_{m-1}; \beta_k \geq 0, \sum_{k=0}^{m-1} \beta_k = 1 \right\},$$

and

$$W(\beta_m()) = \sum_{k=0}^{m-1} \left[ (\beta_k M_1 + \Pi G^k FT(2-\alpha) \l^{\alpha} P) \ast \Pi G^k FT(2-\alpha) \l^{\alpha} Q \right], \quad n = 0, N.$$ We put

$$W_1(0) = M_1, \quad W_1(m) = \bigcup_{\beta_m()} W(\beta_m()), \quad u_n, v_n.$$ (8)

**Theorem 1.** If the inclusion takes place $\Pi G^m z_0 \in W_1(m)$, then in the game (7) the pursuit can be completed from the given initial position.

$$W(0) = W[0], \quad W(m) = \sum_{k=0}^{m-1} \left[ (\beta_k M_1 + \Pi G^k FT(2-\alpha) \l^{\alpha} P) \ast \Pi G^k FT(2-\alpha) \l^{\alpha} Q \right], \quad W_1(m) = M_1 + W(m), \quad m = 1, 2, \ldots, \theta$$ (9)

**Theorem 2.** Let’s pretend that $\Pi G^m z_0 \in W_2(m)$. Then in game (7) from $z_0 = \vec{f}$ you can complete the pursuit.

Let now $W_3(0) = M_1, \quad W_3(1) = [W_3(0) + \Pi F T(2-\alpha) \l^{\alpha} P] \ast \Pi F T(2-\alpha) \l^{\alpha} Q, \ldots, \quad W_3(m) = \left[ W_3(m-1) + \Pi G^{m-1} F T(2-\alpha) \l^{\alpha} P \right] \ast \Pi G^{m-1} F T(2-\alpha) \l^{\alpha} Q.$

**Theorem 3.** If $\Pi G^m z_0 \in W_3(m)$, then in game (7) from $z_0 = \vec{f}$ you can complete the pursuit for $m$ steps.

**Theorem 4.** Let in inequality (5) $K_1 + K_2 h^2 < \varepsilon$, at $\alpha \neq \frac{1}{2}$, and $K_1 l^2 + K_2 h^2 < \varepsilon$, at $\alpha = \frac{1}{2}$, and in game (7) from point $z_0 = \vec{f}$ the pursuit can be completed in the sense of Definition 2. Then the game can be completed in the sense of Definition 1.

4. Discussion

**Proof of Theorem 1.** Instead of including the condition of the theorem, consider the equivalent inclusion

$$\Pi G^m z_0 = \sum_{k=0}^{m-2} \left[ (\beta_k M_1 + \Pi G^k FT(2-\alpha) \l^{\alpha} P) \ast \Pi G^k FT(2-\alpha) \l^{\alpha} Q \right] + \bar{\beta}_{m-1} M_1 + \Pi G^{m-1} F T(2-\alpha) \l^{\alpha} P \ast \Pi G^{m-1} F T(2-\alpha) \l^{\alpha} Q$$ (10)

Where (10) for $\bar{U}_0$ get

$$\Pi G^m z_0 + \Pi G^{m-1} F T(2-\alpha) \l^{\alpha} v_0 = \sum_{k=0}^{m-2} \left[ (\beta_k M_1 + \Pi G^k FT(2-\alpha) \l^{\alpha} P) \ast \Pi G^k FT(2-\alpha) \l^{\alpha} Q \right] + \bar{\beta}_{m-1} M_1 + \Pi G^{m-1} F T(2-\alpha) \l^{\alpha} P$$ (11)

Control $\bar{U}_0 \in P$ construct as a solution to the following equation

$$\Pi G^{m-1} F T(2-\alpha) \l^{\alpha} v_0 - \Pi G^{m-1} F T(2-\alpha) \l^{\alpha} \bar{u}_0 = \bar{\beta}_{m-1} a_1, \quad a_1 \in M_1.$$ Further, by virtue of (11), we have

$$\Pi G^{m-1}(G z_0 - F T(2-\alpha) \l^{\alpha} u_0 + F T(2-\alpha) \l^{\alpha} v_0) \in$$
Then (7) we find
\[ M = \sum_{k=0}^{m-2} \left[ (\beta_k M_1 + \Pi G^k FT(2-\alpha) i^\alpha P_1^* \Pi G^k FT(2-\alpha) i^\alpha Q_1^* \right] + \bar{\beta}_{m-1} a_1. \]

In the same way, we obtain the inclusion
\[ \Pi G^{m-2} z_2 = \sum_{k=0}^{m-3} \left[ (\beta_k M_1 + \Pi G^k FT(2-\alpha) i^\alpha P_1^* \Pi G^k FT(2-\alpha) i^\alpha Q_1^* \right] + \bar{\beta}_{m-1} a_1 + \bar{\beta}_{m-2} a_2, \]

etc. Thus, we have
\[ a_2 \in M, \]

and we have
\[ a_2 \in M_1, \]

hence we have
\[ z_m \in M. \]

Theorem 2-4 is proved similarly.

5. Conclusion

In conclusion, it can be argued that the application of the finite difference method to solving fractional order differential pursuit games is a powerful and effective tool. The problem is investigated when the movements of the players are described by fractional order in time and high order in spatial variables. When solving the problem, using the finite difference method, we go over the game, which is described by the grid equation, after some transformations we went over to the discrete game. With the help of auxiliary theorems, sufficient conditions are obtained for solving the pursuit problem. The properties of the dynamics of change in the set of meeting points of players are studied, with the help of which the game is resolved for the case under consideration. In many works, the stability of the grid equation has been studied, which consists in finding sufficient conditions under which the error tends to zero, or at least remains bounded.

If the rounding errors generated during the counting process tend to decrease, or at least not increase, as they arise, then Eq. (4) is called stable. For the case under consideration, it is naturally assumed that the stability conditions are satisfied. The results obtained can be used to solve problems in the theory of dynamic games, in particular, in the development of the method of resolving functions.

References

[1] Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J. 2006. Theory and Applications of Fractional Differential Equations. (Amsterdam: North-Holland/American Elsevier) p 500.

[2] Mainardi F 1997 Fractional calculus: some basic problems in continuum and statistical mechanics, Fractals and fractional calculus in continuum mechanics (Springer-Verlag, New York) pp 291-348.

[3] J. He, 1998 Nonlinear oscillation with fractional derivative and its applications, International Conference on Vibrating Engineering, Dalian, China, pp 288-291.

[4] J. He, 1999 Some applications of nonlinear fractional differential equations and their approximations, Bull. Sci. Technol. Vol 15 pp 86-90.

[5] Pontreagin L. 1980 Linear differential games of pursuit (Mat. Sb. Vol I12) pp 307-330.

[6] Alimov, K, Mamatov, M 2014 Solving a pursuit problem in high-order controlled distributed systems (Siberian Advances in Mathematics, Vol 24) pp 229–239.

[7] Mamatov M, Tashmanov E, Alimov H 2015 Zwquasi Linear Discrete Games of Pursuit Described by High Order Equation Systems (Automatic Control and Computer Sciences, Vol 49) pp 148-152.

[8] Mamatov M, Sobirov K 2020 Journal of Mathematical Sciences (US), 245(3), pp 332–340.

[9] Mamatov M, Tashmanov E, Alimov K 2015 Automatic Control and Computer Sciences, 49(3), pp 148-152.
[10] Tukhtasinov M, Mamatov M 2009 Differential Equations, 45(3), pp 439-444.
[11] Yevtushenko Y, Smolyakov E 1999 Equilibria in Differential Games and Problems of Proposal Acceptance (Math and Phys, Vol 39), pp 897-905.