Obtaining a representation of the solution to the Cauchy problem for equation high-order fractional derivative, by the method of finding self-similar solutions

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Abstract. In this paper, with the help of previously constructed self-similar solutions, a solution of the Cauchy problem for an equation of even order with a fractional Riemann-Liouville derivative of order $1 < \alpha < 2$ is obtained.

Keywords. Higher order equation. Fractional Riemann-Liouville derivative, self-similar solutions, fundamental solution, Cauchy problem.

1. Construction of self-similar solutions

Consider the equation

$$L[u] \equiv D_{0y}^\alpha u(x, y) - d D_{0x}^\beta u(x, y) = 0, \quad (1)$$

here

$$q, p \in \mathbb{N}, \quad q - 1 < \alpha \leq q, \quad p - 1 < \beta \leq p, \quad q < p, \quad d = \pm 1,$$

and $D_{0y}^\alpha, D_{0x}^\beta$ – Riemann-Liouville fractional differentiation operators, respectively, of orders $\alpha, \beta$:

$$D_{0y}^\alpha u(x, y) = \frac{1}{\Gamma(q - \alpha)} \frac{d^q}{dy^q} \int_0^y \frac{u(x, \tau) d\tau}{(y - \tau)^{\alpha-q+1}}, \quad y > 0,$$

$$D_{0x}^\beta u(x, y) = \frac{1}{\Gamma(p - \beta)} \frac{d^p}{dx^p} \int_0^x \frac{u(\tau, y) d\tau}{(x - \tau)^{\beta-p+1}}, \quad x > 0.$$

Note that self-similar solutions of an equation with the usual derivative of the form

$$\frac{\partial^p u(x, y)}{\partial x^p} - \frac{\partial^q u(x, y)}{\partial y^q} = 0, \quad p < q,$$

were constructed in [1].

In [2], using the methods of special operators, self-similar solutions of equation (1) were found in the cases $\frac{\beta}{2} \leq \alpha < \beta \leq 2, \quad n - 1 < \beta \leq n \in \mathbb{N}$.

In this paper, we construct self-similar solutions by a method that does not require knowledge of special operators.

We seek self-similar solutions to equation (1) in the form of the series

$$u(x, y) = y^b \sum_{n=0}^{\infty} c_n (x^a y^c)^{n+\gamma} = \sum_{n=0}^{\infty} c_n x^{an+a\gamma} y^{cn+c\gamma+b}, \quad (2)$$

here the parameters $a, b, c, \gamma$ are to be defined.

We first introduce the notation

$$(a)_s = a (a - 1) ... (a - (s - 1)), \quad (a)_0 = 1, \quad (a)_1 = a.$$
Substitute (2) into (1). Then we will have

\[ D_{0y}^\alpha u(x, y) = \frac{1}{\Gamma(q - \alpha)} \sum_{n=0}^{\infty} c_n x^{an+a\gamma} \frac{d^n y^q}{dy^n} \int_{0}^{y} \frac{\tau^{cn+c\gamma+b} d\tau}{(y - \tau)^{\alpha-q+1}} = \]

\[ = \frac{1}{\Gamma(q - \alpha)} \sum_{n=0}^{\infty} c_n x^{an+a\gamma} \frac{d^n y^q}{dy^n} \left[ \frac{1}{y^{\alpha-q+1}} \right]_{0}^{y} = \]

\[ = \sum_{n=0}^{\infty} (cn + c\gamma + b - \alpha + q) q c_n (x^a y^c)^n y^{\gamma+b-\alpha} \frac{\Gamma(cn+c\gamma+b+1)}{\Gamma(cn+c\gamma+b-\alpha+q+1)}, \quad (3) \]

similarly

\[ D_{0x}^\beta u(x, y) = \frac{1}{\Gamma(p - \beta)} \sum_{n=0}^{\infty} c_n y^{cn+c\gamma+b} \frac{d^n x^p}{dx^n} \int_{0}^{x} \frac{\tau^{an+a\gamma} d\tau}{(x - \tau)^{\beta-p+1}} = \]

\[ = \frac{1}{\Gamma(p - \beta)} \sum_{n=0}^{\infty} c_n y^{cn+c\gamma+b} \frac{d^n x^p}{dx^n} \left[ \frac{x^{an+a\gamma+b+1}}{x^{\beta-p+1}(1-z)^{\beta-p+1}} \right]_{0}^{x} = \]

\[ = \sum_{n=0}^{\infty} (an + a\gamma + p - \beta) p c_n (x^a y^c)^n x^{\beta} y^{\gamma+b} \frac{\Gamma(an+a\gamma+1)}{\Gamma(an+a\gamma+1+p-\beta)}. \quad (4) \]

Substituting (3), (4) into (1), we obtain

\[ \sum_{n=0}^{\infty} (cn + c\gamma + b - \alpha + q) q c_n (x^a y^c)^n x^{\beta} y^{\gamma+b-\alpha} \frac{\Gamma(cn+c\gamma+b+1)}{\Gamma(cn+c\gamma+b-\alpha+q+1)} = \]

\[ = d \sum_{n=0}^{\infty} (an + a\gamma + p - \beta) p c_n (x^a y^c)^n \frac{\Gamma(an+a\gamma+1)}{\Gamma(an+a\gamma+1+p-\beta)} \Rightarrow \]

\[ a = \beta, \ c = -\alpha \Rightarrow \]

\[ \sum_{n=0}^{\infty} c_n (x^\beta y^{-\alpha})^{n+1} \frac{\Gamma(-\alpha(n+\gamma)+b+1)}{\Gamma(-\alpha(n+\gamma+1)+b+1)} = \]

\[ = d \sum_{n=0}^{\infty} (\beta n + \beta\gamma + p - \beta) p c_n (x^a y^c)^n \frac{\Gamma(an+a\gamma+1)}{\Gamma(\beta n + \beta\gamma+1+p-\beta)} \Rightarrow \]

got conditions for the parameter \( \gamma \):

\[ \gamma_j = 1 - \frac{j}{\beta}, \ j = 1,2,...,p. \]

Substituting the found parameters into representation (2), we find the formula for the coefficients \( c_n \). We have

\[ c_n = \frac{\Gamma(\beta n + \beta\gamma + 1)}{\Gamma(\beta n + \beta\gamma + 1 - \beta)} \frac{\Gamma(-\alpha(n+\gamma-1)+b+1)}{\Gamma(-\alpha(n+\gamma)+b+1)} \Rightarrow \]

\[ c_n = d c_{n-1} \frac{\Gamma(-\alpha(n+\gamma)+b+1)}{\Gamma(-\alpha(n+\gamma)+b+1)} \frac{\Gamma(\beta n + \beta\gamma + 1 - \beta)}{\Gamma(\beta n + \beta\gamma + 1)} \]

\[ = d^n c_0 \frac{\Gamma(-\alpha(n+\gamma)+b+1) \Gamma(\beta\gamma+1)}{\Gamma(-\alpha(n+\gamma)+b+1) \Gamma(\beta(n+\gamma)+1)}. \]
if now 
\[ c_0 = \frac{1}{\Gamma (-\alpha \gamma + b + 1) \Gamma (\beta \gamma + 1)} , \]
then 
\[ c_n = \frac{d^n}{\Gamma (-\alpha (n + 1 - \frac{j}{\beta}) + b + 1) \Gamma (\beta (n + 1 - \frac{j}{\beta}) + 1)} = \frac{d^n}{\Gamma (-\alpha n - \alpha + \frac{\alpha j}{\beta} + b + 1) \Gamma (\beta n + \beta - j + 1)} . \]

So we got the following self-similar solutions to equation (1):
\[ u_j (x, y) = y^b \left( x^{\beta} y^{-\alpha} \right)^{1-\frac{j}{\beta}} \sum_{n=0}^{\infty} \frac{(dx^{\beta} y^{-\alpha})^n}{\Gamma (-\alpha n - \alpha + \frac{\alpha j}{\beta} + b + 1) \Gamma (\beta n + \beta - j + 1)} = \]
\[ = y^b t^{\frac{\alpha}{\beta}} W_{(-\alpha, -\alpha + \frac{\alpha j}{\beta} + b + 1), (\beta, \beta - j + 1)} (dt), \ j = 1, 2, ..., p, \ t = x^{\beta} y^{-\alpha}, \] (5)

here 
\[ W_{(\mu, a), (\nu, b)} (z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma (\mu n + a) \Gamma (\nu n + b)}, \ \mu, \nu \in \mathbb{R}, \ a, b \in \mathbb{C}, \ \mu + \nu > 0 \] - generalized Wright function [2]. Series (5) converges uniformly and it can be differentiated term by term, since \( \beta - \alpha > 0 \). Note that there is no condition on the \( b \) parameter. Note that solutions of the form (5) coincide exactly with the solutions obtained in [2].

Now let \( \beta = p \in \mathbb{N} \), then the solutions of equation (1) are expressions of the form:
\[ u_s (x, y) = y^b \sum_{n=0}^{\infty} \frac{d^n \left( x y^{\frac{s}{p}} \right)^{pn+s}}{\Gamma (-\alpha n - \alpha + \frac{\alpha s}{p} + b + 1) (pn + s)!}, \ s = 0, 1, ..., p - 1. \]

Then their linear combination is also a solution to equation (1)
\[ u (x, y) = y^b \left( c_0 \sum_{n=0}^{\infty} \frac{d^n \phi_{p}^n (pn+b+1)}{(pn)! \Gamma (-\alpha n + b + 1)} + c_1 \sum_{n=0}^{\infty} \frac{d^n \phi_{p}^n (pn+1)}{(pn+1)! \Gamma (-\alpha n + \frac{1}{p} + b + 1)} + \right. \]
\[ + \left. c_{p-1} \sum_{n=0}^{\infty} \frac{d^n \phi_{p}^n (pn+p)}{(pn+p+1)! \Gamma (-\alpha n + \frac{p-1}{p} + b + 1)} \right) , \] (6)

where \( c_i \) - arbitrary real numbers, \( i = 0, 1, ..., p - 1 \).

Now let \( c_i \), be such that \( c_i \neq c_j \), for \( i \neq j \) and \( c_{p}^j = d \). Then from (6) we obtain
\[ u (x, y) = y^b \phi \left( -\frac{\alpha}{p}, b + 1, c x y^{\frac{s}{p}} \right) , \ c^p = 1, \] (7)

here
\[ \phi (-\delta, \varepsilon, z) = \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma (-\delta k + \varepsilon)} \]
- Wright function.
Consider the function
\[ U (x - \xi, y - \eta) = (y - \eta)^b \phi \left( -\frac{\alpha}{p}, b + 1, c (x - \xi) (y - \eta)^{\frac{\alpha}{p}} \right), \ y > \eta. \] (8)
Let us note some properties of function (8) that are obtained by direct computation.

**Lemma 1.**
1. \( D_y^\gamma \left( y^b \phi \left( -\frac{a}{p}, b + 1, cxy^{\frac{b}{p}} \right) \right) = y^{b-\gamma} \phi \left( -\frac{a}{p}, b + 1 - \gamma, cxy^{\frac{b}{p}} \right), \ \gamma \in R, \)
2. \( D_y^\gamma \left( (y-\eta)^b \phi \left( -\frac{a}{p}, b + 1, c(x-\xi) (y-\eta)^{-\frac{b}{p}} \right) \right) = \)
   \( = (y-\eta)^{b-\gamma} \phi \left( -\frac{a}{p}, b + 1 - \gamma, c(x-\xi) (y-\eta)^{-\frac{b}{p}} \right), \ \gamma \in R, \)
3. \( \frac{\partial^s}{\partial x^s} \left( y^b \phi \left( -\frac{a}{p}, b + 1, c(x-\xi) y^{-\frac{b}{p}} \right) \right) = \)
   \( = c^s y^{b-\frac{s}{p}} \phi \left( -\frac{a}{p}, b + 1 - \frac{\alpha}{p}, c(x-\xi) y^{-\frac{b}{p}} \right), \ \gamma \in \{0\} \cup N, \)
4. \( (D_y^\alpha - a \frac{\partial^p}{\partial x^p}) D_y^\alpha U(x-\xi, y-\eta) = 0, \)
5. \( (D_y^\alpha - (1)^p d \frac{\partial^p}{\partial x^p}) D_y^\alpha U(x-\xi, y-\eta) = 0, \)
6. \( (-1)^q D_y^\alpha \frac{\partial^p}{\partial x^p} D_y^\alpha U(x-\xi, y-\eta) = 0, \)

**2. Cauchy problem**

In this section, using the found particular solutions (8), we obtain an explicit form of the solution to the Cauchy problem for an equation of the form:

\[
D^{n-1}_y u(x, y) - (-1)^{n-1} \frac{\partial^{2n}_x u(x, y)}{\partial x^{2n}} = f(x, y), \quad 1 < \alpha < 2. \tag{9}
\]

Note that the Cauchy problem for higher order equations with a fractional derivative of order \( 0 < \alpha < 1 \) was considered in [4], [5]. We will use ideas from these papers.

Following (7), we have

\[ c_{1k}^{2n} = (-1)^{n-1} c_{1k} = e^{\frac{n-1-2k}{2n} \pi}, k = 0, (n-1), \text{ Re } c_{1k} > 0, \]

Consider the function

\[
\Gamma_b (x-\xi, y-\eta) = \left\{ \begin{array}{ll}
\Gamma^1_b (x-\xi, y-\eta), & x > \xi, \\
\Gamma^2_b (x-\xi, y-\eta), & x < \xi,
\end{array} \right.
\tag{10}
\]

where

\[
\Gamma^1_b (x-\xi, y-\eta) = \frac{(y-\eta)^b}{2n} \sum_{k=0}^{n-1} \left( -e^{\frac{n-1-2k}{2n} \pi} \right) \phi \left( -\frac{\alpha}{2n}, b + 1, -e^{\frac{n-1-2k}{2n} \pi} t \right),
\]

\[
\Gamma^2_b (x-\xi, y-\eta) = -\frac{(y-\eta)^b}{2n} \sum_{k=0}^{n-1} e^{\frac{n-1-2k}{2n} \pi} \phi \left( -\frac{\alpha}{2n}, b + 1, -e^{\frac{n-1-2k}{2n} \pi} (-t) \right),
\]

\[ t = \frac{(x-\xi)}{(y-\eta)^{2n}}. \]

We have

\[
\frac{\partial^s \Gamma^1_b (x-\xi, y-\eta)}{\partial x^s} = \frac{1}{2n} \sum_{k=0}^{n-1} \left( -e^{\frac{n-1-2k}{2n} \pi} \right) \frac{\partial^s \left( (y-\eta)^b \phi \left( -\frac{\alpha}{2n}, b + 1, -e^{\frac{n-1-2k}{2n} \pi} t \right) \right)}{\partial x^s} = \]

\[
= \frac{1}{2n} \sum_{k=0}^{n-1} \left( -e^{\frac{n-1-2k}{2n} \pi} \right)^{s+1} \phi \left( -\frac{\alpha}{2n}, b + 1 - \frac{\alpha}{2n} s, -e^{\frac{n-1-2k}{2n} \pi} t \right) \Rightarrow \]

\[
\frac{\partial^s \Gamma^1_b (x-\xi, y-\eta)}{\partial x^s} \bigg|_{x=\xi} = \frac{(y-\eta)^b - \frac{\alpha}{2n} s}{2n \Gamma \left( b + 1 - \frac{\alpha}{2n} s \right)} \sum_{k=0}^{n-1} \left( -e^{\frac{n-1-2k}{2n} \pi} \right)^{s+1}.
\]
Similarly
\[
\frac{\partial^s \Gamma_b^2(x - \xi, y - \eta)}{\partial x^s} \bigg|_{x = \xi} = -\frac{(y - \eta)^{\frac{b - \alpha}{2n}}}{2n \Gamma(b + 1 - \frac{\alpha}{2n})} \sum_{k=0}^{n-1} \left( e^{\frac{n-1-2k}{2n} i \pi} \right)^{s+1}.
\]

Hence we have if
\[
s = (2n - 1) \pmod{2n},
\]
then
\[
\frac{\partial^s \Gamma_b^1(x - \xi, y - \eta)}{\partial x^s} \bigg|_{x = \xi} - \frac{\partial^s \Gamma_b^2(x - \xi, y - \eta)}{\partial x^s} \bigg|_{x = \xi} = (-1)^{n-1} \frac{(y - \eta)^{\frac{b - \alpha}{2n}}}{\Gamma(b + 1 - \frac{\alpha}{2n})},
\]
if
\[
s \neq (2n - 1) \pmod{2n},
\]
then
\[
\frac{\partial^s \Gamma_b^1(x - \xi, y - \eta)}{\partial x^s} \bigg|_{x = \xi} - \frac{\partial^s \Gamma_b^2(x - \xi, y - \eta)}{\partial x^s} \bigg|_{x = \xi} = \frac{(y - \eta)^{\frac{b - \alpha}{2n}}}{2n \Gamma(b + 1 - \frac{\alpha}{2n})} \left( -\sum_{k=0}^{n-1} \left( e^{\frac{n-1-2k}{2n} i \pi} \right)^{s+1} + \sum_{k=0}^{n-1} \left( e^{\frac{n-1-2k}{2n} i \pi} \right)^{s+1} \right) = \frac{(y - \eta)^{\frac{b - \alpha}{2n}}}{2n \Gamma(b + 1 - \frac{\alpha}{2n})} e^{\frac{n-1}{2n} (s+1) i \pi} \sum_{k=0}^{n-1} e^{-k(s+1) \frac{i \pi}{n}} \left( (-1)^{s+1} + 1 \right) = 0.
\]

So we have proved the following lemma.

**Lemma 2.** For \( s \in \mathbb{N} \), the following relation holds:
\[
\frac{\partial^s \Gamma_b^1(x - \xi, y - \eta)}{\partial x^s} \bigg|_{x = \xi} - \frac{\partial^s \Gamma_b^2(x - \xi, y - \eta)}{\partial x^s} \bigg|_{x = \xi} = (-1)^{n-1} \frac{(y - \eta)^{\frac{b - \alpha}{2n}}}{\Gamma(b + 1 - \frac{\alpha}{2n})} \begin{cases} 1, & s = (2n - 1) \pmod{2n}, \\ 0, & s \neq (2n - 1) \pmod{2n}. \end{cases}
\]

The results of Lemma 2 coincide with the results of [5]. In what follows, we need the asymptotics of the Wright function for large values of the variable. The main results on asymptotics were obtained by Wright [see 3]. In particular, the following theorem is proved.

**Theorem 1** [see 3]. If \( |\arg y| \leq \min \left\{ \frac{\pi}{2} (1 - \sigma), \pi \right\} - \varepsilon, \ 0 < \sigma < 1 \), then
\[
\phi(-\sigma, \beta, t) = Y^{\frac{1}{2} - \beta} e^{-Y} \left\{ \sum_{m=0}^{M-1} A_m Y^{-m} + O \left( Y^{-M} \right) \right\}, \ |t| \to \infty, \quad (11)
\]

here \( Y = (1 - \sigma) (\sigma y)^{\frac{1}{1-\sigma}}, \ y = -t \), the coefficients of \( A_m \) depend on \( \sigma, \beta \).

If \( n = 3, 4, \ldots, \) to \( 0 < \frac{\alpha}{2n} < \frac{1}{n} \leq \frac{1}{3} \), and the relation
\[
|\arg y| = |\arg (-t)| = \left| \frac{n - 1 - 2k}{2n} \pi \right| \leq \pi - \varepsilon, \ k = 0, 1, \ldots, n - 1. \quad (12)
\]

If \( n = 2 \), then for \( 0 < \frac{\alpha}{4} \leq \frac{1}{3} \), we have
\[
|\arg y| = |\arg (-t)| = \left| \frac{\pi}{4} \right| < \pi, \quad (13)
\]
and at $\frac{1}{2} < \alpha < \frac{3}{2}$, get

$$|\arg y| = |\arg (-t)| = \left| \pm \frac{\pi}{4} \right| < \frac{3}{2} \pi \left( 1 - \frac{\alpha}{4} \right).$$

(14)

Taking into account (12) - (14), we conclude that for $n = 2, 3, \ldots$ we always have relation (11). Now we write (10) in the form

$$\Gamma_b (x - \xi, y - \eta) = \frac{(y - \eta)^b}{2n} \sum_{k=0}^{n-1} \left( -e^{-\frac{n-1-2k}{2n} \pi |t|} \right) \phi \left( -\frac{\alpha}{2n} b + 1, -e^{-\frac{n-1-2k}{2n} \pi |t|} |t| \right),$$

(15)

where

$$|t| = \frac{|x - \xi|}{\pi (y - \eta)^{\frac{2n}{2n - \alpha}}},$$

then, taking into account (11), (12) - (14), for large values of $|t|$, the estimate obtained in [5], [6] is valid for (15):

$$\left| \phi \left( -\frac{\alpha}{2n} b + 1, -e^{-\frac{n-1-2k}{2n} \pi |t|} |t| \right) \right| \leq C |t|^\frac{2n-\alpha}{2n - \alpha} (b + 1) \exp \left( -\sigma |t|^\frac{2n-\alpha}{2n - \alpha} \right),$$

(16)

here

$$\sigma = \left( 1 - \frac{\alpha}{2n} \right) \left( \frac{\alpha}{2n} \right)^\frac{\alpha}{2n - \alpha} \cos \frac{n-1}{2n - \alpha}, 0 < C - const,$$

or applying Lemma 1 (properties 1, 3), for $x \neq 0$, $\gamma \in R$, $s \in N \cup \{0\}$, we obtain

$$\left| \frac{\partial^s}{\partial x^s} \left( D_{0y}^s \Gamma_b (x, y) \right) \right| \leq M y^{h - \gamma - \frac{\alpha}{2n} (s-\theta)} |x|^{-\theta} \exp \left( -\sigma |x|^{\frac{2n-\alpha}{2n - \alpha}} \right),$$

(17)

here

$$\left| x \right| y^{-\frac{2n}{2n - \alpha}} \rightarrow +\infty,$$

$$\theta = \frac{2n}{2n - \alpha} \left( \frac{1}{2} + b - \gamma - \frac{\alpha}{2n} s \right).$$

Note that estimate (17) coincides with the estimate obtained in [5], for $b = \alpha - 1 - \frac{\alpha}{2n}$.

**Lemma 3.** For $j, s \in N$ and any function $h (x) \in C (R)$ such that

$$|h (x)| \leq M \exp \left( c|x|^{\frac{2n-\alpha}{2n}} \right), c < \sigma, |x| \rightarrow \infty, 0 < M - const,$$

(18)

equality holds

$$\lim_{y \rightarrow 0} \int_{-\infty}^{+\infty} h (\xi) D_{0y}^{\alpha-s} \Gamma_{\alpha-\frac{\alpha}{2n}} (x - \xi, y) d\xi = -h (x) \cdot \left\{ \begin{array}{ll}
0, & j \neq s, \\
1, & j = s.
\end{array} \right.$$  

**Proof.** First, we calculate the integral

$$\int_{-\infty}^{+\infty} D_{0y}^{\alpha-s} \Gamma_{\alpha-\frac{\alpha}{2n}} (x - \xi, y) d\xi = \int_{-\infty}^{x} D_{0y}^{\alpha-s} \Gamma_{\alpha-\frac{\alpha}{2n}} (x - \xi, y) d\xi +$$

$$+ \int_{x}^{+\infty} D_{0y}^{\alpha-s} \Gamma_{\alpha-\frac{\alpha}{2n}} (x - \xi, y) d\xi =$$

$$= - \sum_{k=0}^{n-1} \int_{-\frac{\alpha}{2n}}^{\frac{\alpha}{2n}} x D_{0y}^{\alpha-s} \left( y^{\alpha-\frac{\alpha}{2n}-j} \phi \left( -\frac{\alpha}{2n} - \alpha - \frac{\alpha}{2n} (x - \xi) y^{-\frac{\alpha}{2n}}, \alpha - \frac{\alpha}{2n}, -j + 1, -e^{\frac{n-1-2k}{2n} \pi i (x - \xi) y^{-\frac{\alpha}{2n}}} \right) \right) d\xi -$$

$$- \sum_{k=0}^{n-1} \int_{-\frac{\alpha}{2n}}^{\frac{\alpha}{2n}} x D_{0y}^{\alpha-s} \left( y^{\alpha-\frac{\alpha}{2n}-j} \phi \left( -\frac{\alpha}{2n} - \alpha - \frac{\alpha}{2n}, -j + 1, -e^{\frac{n-1-2k}{2n} \pi i (x - \xi) y^{-\frac{\alpha}{2n}}} \right) \right) d\xi =$$
Now let’s calculate each integral separately

\[-e^{\frac{n-1-2k\pi i}{2n}} \int_{-\infty}^{x} y^{s-\frac{n}{2n}} \phi \left( -\frac{\alpha}{2n}, s - \frac{\alpha}{2n} - j + 1, -e^{\frac{n-1-2k\pi i}{2n}} (x - \xi) y^{-\frac{n}{2n}} \right) d\xi -
\]

\[-\sum_{k=0}^{n-1} e^{\frac{n-1-2k\pi i}{2n}} \int_{x}^{+\infty} y^{s-\frac{n}{2n}} \phi \left( -\frac{\alpha}{2n}, s - \frac{\alpha}{2n} - j + 1, -e^{\frac{n-1-2k\pi i}{2n}} (x - \xi) y^{-\frac{n}{2n}} \right) d\xi.
\]

From here we get

\[\int_{-\infty}^{+\infty} D_{0y}^\alpha G_{\alpha - \frac{\alpha}{2n} - j} (x - \xi, y) d\xi = -\frac{y^{s-j}}{\Gamma (s - j + 1)}. (19)\]

Note that for \( s < j \) we have

\[\int_{-\infty}^{+\infty} D_{0y}^\alpha G_{\alpha - \frac{\alpha}{2n} - j} (x - \xi, y) d\xi = 0, \ s < j. \quad (20)\]

Now

\[h(\xi) D_{0y}^\alpha G_{\alpha - \frac{\alpha}{2n} - j} (x - \xi, y) d\xi = \int_{-\infty}^{x} h(\xi) D_{0y}^\alpha G_{\alpha - \frac{\alpha}{2n} - j} (x - \xi, y) d\xi +
\]

\[\int_{x}^{+\infty} h(\xi) D_{0y}^\alpha G_{\alpha - \frac{\alpha}{2n} - j} (x - \xi, y) d\xi = I_1 + I_2,
\]
We calculate each term separately

\[ I_1 = \int_{-\infty}^{x} h(\xi) D_{0y}^{x, y} \Gamma_{\alpha-y/2n}^{-j} \xi (x, \xi, y) d\xi = \]

\[ = -\sum_{k=0}^{n-1} e^{-\frac{n-1}{2n}k\pi i} y^{s-j} \frac{\alpha}{2n} \right) d\xi = \]

\[ = -\sum_{k=0}^{n-1} e^{-\frac{n-1}{2n}k\pi i} y^{s-j} \int_{0}^{+\infty} \left( h(x - ty) \phi \left(-\frac{\alpha}{2n}, s - \frac{\alpha}{2n} - j + 1, -e^{-\frac{n-1}{2n}k\pi i} t \right) d\xi = \]

\[ = -\sum_{k=0}^{n-1} e^{-\frac{n-1}{2n}k\pi i} y^{s-j} \int_{0}^{+\infty} \left( h(x - ty) - h(x) \right) \phi \left(-\frac{\alpha}{2n}, s - \frac{\alpha}{2n} - j + 1, -e^{-\frac{n-1}{2n}k\pi i} t \right) d\xi + \]

\[ + h(x) \left( -\sum_{k=0}^{n-1} e^{-\frac{n-1}{2n}k\pi i} y^{s-j} \int_{0}^{+\infty} \phi \left(-\frac{\alpha}{2n}, s - \frac{\alpha}{2n} - j + 1, -e^{-\frac{n-1}{2n}k\pi i} t \right) d\xi \right) = \]

\[ = J_1 + J_2, \]

Hence we have

\[ \lim_{y \to +0} I_1 = \lim_{y \to +0} J_1 + \lim_{y \to +0} J_2. \]

Let’s find each limit separately. Using (19), (20), we have

\[ \lim_{y \to +0} J_2 = -h(x) \lim_{y \to +0} \frac{y^{s-j}}{2\Gamma(s-j+1)} = \begin{cases} -\frac{1}{2}h(x), & s = j, \\ 0, & s \neq j. \end{cases} \]

Further, if \( s \neq j \), then \( \lim_{y \to +0} J_1 = 0 \). Let \( s = j \), then

\[ \lim_{y \to +0} \int_{0}^{+\infty} \left( h(x - ty) - h(x) \right) \phi \left(-\frac{\alpha}{2n}, s - \frac{\alpha}{2n} - j + 1, -e^{-\frac{n-1}{2n}k\pi i} t \right) d\xi = \]

\[ = \lim_{y \to +0} \int_{0}^{N} \left( h(x - ty) - h(x) \right) \phi \left(-\frac{\alpha}{2n}, s - \frac{\alpha}{2n} - j + 1, -e^{-\frac{n-1}{2n}k\pi i} t \right) d\xi + \]

\[ + \lim_{y \to +0} \int_{N}^{+\infty} \left( h(x - ty) - h(x) \right) \phi \left(-\frac{\alpha}{2n}, s - \frac{\alpha}{2n} - j + 1, -e^{-\frac{n-1}{2n}k\pi i} t \right) d\xi = 0, \]

here \( 0 < N \). The last equality holds due to the fact that for an arbitrary small number \( \epsilon > 0 \), you can choose \( N \) so that the sublimit expressions will be less than \( 1/2\epsilon \) (the first expression, due to the continuity the function \( h(x) \), and the second is due to estimates (16) and (18)).

Means

\[ \lim_{y \to +0} I_1 \begin{cases} -\frac{1}{2}h(x), & s = j, \\ 0, & s \neq j. \end{cases} \]

It is shown similarly that

\[ \lim_{y \to +0} I_2 \begin{cases} -\frac{1}{2}h(x), & s = j, \\ 0, & s \neq j. \end{cases} \]
Lemma 3 is proved.

**Cauchy problem.** Find the solution $u(x, y)$ to equation (9) in the region $D = \{(x, y) : -\infty < x < +\infty, 0 < y\}$, satisfying the following conditions:

1) $D_{0y}^\alpha u(x, y), \frac{\partial u}{\partial x}(x, y) \in C(D), D_{0y}^{-1}u(x, y), D_{0y}^{-2}u(x, y) \in C(\overline{D})$;

2) $\lim_{y\to+0} D_{0y}^{-1}u(x, y) = \varphi(x), \lim_{y\to+0} D_{0y}^{-2}u(x, y) = \psi(x)$.

The specified functions satisfy the constraints:

$$f(x, y) \in C(\overline{D}), \varphi(x), \psi(x) \in C(R),$$

$$|\varphi(x)| \leq M \exp\left(k|x|^{\frac{2a}{2\alpha-1}}\right), |x| \to +\infty,$$

$$|\psi(x)| \leq M \exp\left(k|x|^{\frac{2a}{2\alpha-1}}\right), |x| \to +\infty,$$

$$|f(x, y)| \leq M \exp\left(k|x|^{\frac{2a}{2\alpha-1}}\right), |x| \to +\infty,$$

$k < \sigma, 0 < M - \text{constant}$.

**Theorem 2.** The solution to the Cauchy problem has the form

$$u(x, y) = - \int_{-\infty}^{+\infty} \varphi(\xi) \Gamma_{\alpha-\frac{a}{2\alpha}}(x - \xi, y) d\xi - \int_{-\infty}^{+\infty} \psi(\xi) \Gamma_{\alpha-\frac{a}{2\alpha}-2}(x - \xi, y) d\xi -$$

$$- \int_{-\infty}^{+\infty} \int_{0}^{+\infty} f(\xi, \eta) \Gamma_{\alpha-\frac{a}{2\alpha}}(x - \xi, \eta - \eta) d\xi d\eta.$$

**Proof.** It is easy to verify that the first two terms satisfy homogeneous equation (9). Consider the third term

$$D_{0y}^\alpha \left( \int_{0}^{+\infty} f(\xi, \eta) \Gamma_{\alpha-\frac{a}{2\alpha}}(x - \xi, y - \eta) d\xi d\eta \right) =$$

$$= \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial y^2} \left( \int_{-\infty}^{+\infty} d\xi D_{0y}^{-2} \left( \int_{0}^{+\infty} f(\xi, \eta) \Gamma_{\alpha-\frac{a}{2\alpha}}(x - \xi, y - \eta) d\eta \right) \right) \Rightarrow$$

$$D_{0y}^{-2} \left( \int_{0}^{+\infty} f(\xi, \eta) \Gamma_{\alpha-\frac{a}{2\alpha}}(x - \xi, y - \eta) d\eta \right) =$$

$$= \int_{0}^{+\infty} \frac{d\tau}{(y - \tau)^{\alpha-1}} \left( \int_{0}^{\tau} f(\xi, \eta) \Gamma_{\alpha-\frac{a}{2\alpha}}(x - \xi, \tau - \eta) d\eta \right) =$$

$$= \int_{0}^{+\infty} f(\xi, \eta) d\eta \left( \int_{0}^{+\infty} \frac{\Gamma_{\alpha-\frac{a}{2\alpha}}(x - \xi, \tau - \eta)}{(y - \tau)^{\alpha-1}} d\tau \right) =$$

So finally we have

$$\lim_{y \to 0} \int_{-\infty}^{+\infty} h(\xi) D_{0y}^{\alpha-s} \Gamma_{\alpha-\frac{a}{2\alpha}-j}(x - \xi, y) d\xi = -h(x) \cdot \begin{cases} 0, & j \neq s, \\ 1, & j = s. \end{cases}$$
\[
\int_0^y f(\xi, \eta)\, d\eta \left( \int_0^1 \frac{(y - \eta)^{2-\alpha} \Gamma_{\alpha - \frac{\alpha}{2n} - 1}(x - \xi, (y - \eta) t)}{(1 - t)^{\alpha-1}}\, dt \right) = \\
= \Gamma (2 - \alpha) \int_0^y f(\xi, \eta) \, d\eta.
\]

Hence we have
\[
D_0^\alpha y \int_{-\infty}^{+\infty} d\xi \left( \int_0^y f(\xi, \eta) \, d\eta \right) = \\
= \int_{-\infty}^{+\infty} d\xi \left( \int_0^y f(\xi, \eta) \, d\eta \right).
\]

Note that taking into account (21), we have
\[
\lim_{y \to +0} D_0^{\alpha - 2} y \int_{-\infty}^{+\infty} d\xi \left( \int_0^y f(\xi, \eta) \, d\eta \right) = \\
= \lim_{y \to +0} \int_{-\infty}^{+\infty} d\xi \left( \int_0^y f(\xi, \eta) \, d\eta \right) = 0,
\]

\[
\lim_{y \to +0} D_0^{\alpha - 1} y \int_{-\infty}^{+\infty} d\xi \left( \int_0^y f(\xi, \eta) \, d\eta \right) = \\
= \lim_{y \to +0} \int_{-\infty}^{+\infty} d\xi \left( \int_0^y f(\xi, \eta) \, d\eta \right) = 0.
\]

From (23), (24) and Lemma 3 we have
\[
\lim_{y \to +0} D_0^{\alpha - 2} u(x, y) = \varphi(x), \\
\lim_{y \to +0} D_0^{\alpha - 1} u(x, y) = \psi(x).
\]

Now we find the partial derivatives with respect to the variable \(x\), the third term
\[
\frac{\partial}{\partial x} \left( \int_{-\infty}^{x} d\xi \left( \int_0^y f(\xi, \eta) \, d\eta \right) \right) + \\
+ \frac{\partial}{\partial x} \left( \int_{-\infty}^{+\infty} d\xi \left( \int_0^y f(\xi, \eta) \, d\eta \right) \right) = \\
= \lim_{\xi \to x} \int_0^y f(\xi, \eta) \left( \Gamma_{\alpha - \frac{\alpha}{2n} - 1}(x - \xi, y - \eta) - \Gamma_{\alpha - \frac{\alpha}{2n} - 1}(x - \xi, y - \eta) \right) d\eta + \\
+ \int_{-\infty}^{+\infty} d\xi \left( \int_0^y f(\xi, \eta) \, d\eta \right) = 
\]
continuing this process we get

\[
\frac{\partial^{2n}}{\partial x^{2n}} \left( \int_{-\infty}^{+\infty} d\xi \left( \int_{0}^{y} f(\xi, \eta) \frac{\partial}{\partial x} \Gamma_{\alpha - \frac{2n}{2n} - 1} (x - \xi, y - \eta) d\eta \right) \right) =
\]

\[
\lim_{\varepsilon \to +0} \left( \frac{\partial}{\partial x} \int_{-\infty}^{x-\varepsilon} d\xi \left( \int_{0}^{y} f(\xi, \eta) \frac{\partial^{2n-1}}{\partial x^{2n-1}} \Gamma_{\alpha - \frac{2n}{2n} - 1} (x - \xi, y - \eta) d\eta \right) \right) +
\]

\[
\lim_{\varepsilon \to +0} \left( \frac{\partial}{\partial x} \int_{x+\varepsilon}^{+\infty} d\xi \left( \int_{0}^{y} f(\xi, \eta) \frac{\partial^{2n-1}}{\partial x^{2n-1}} \Gamma_{\alpha - \frac{2n}{2n} - 1} (x - \xi, y - \eta) d\eta \right) \right) +
\]

\[
\lim_{\varepsilon \to +0} \left( \frac{\partial}{\partial x} \int_{y+\varepsilon}^{+\infty} d\xi \left( \int_{0}^{y} f(\xi, \eta) \frac{\partial^{2n-1}}{\partial x^{2n-1}} \Gamma_{\alpha - \frac{2n}{2n} - 1} (x - \xi, y - \eta) d\eta \right) \right) =
\]

\[= I_1 + I_2 + I_3, \]

We calculate each term separately

\[
I_1 = \lim_{\varepsilon \to +0} \left( \int_{0}^{y} f(x - \varepsilon, \eta) \frac{\partial^{2n-1}}{\partial x^{2n-1}} \Gamma_{\alpha - \frac{2n}{2n} - 1} (\varepsilon, y - \eta) d\eta \right) +
\]

\[
+ \int_{-\infty}^{x} d\xi \left( \int_{0}^{y} f(\xi, \eta) \frac{\partial^{2n}}{\partial x^{2n}} \Gamma_{\alpha - \frac{2n}{2n} - 1} (x - \xi, y - \eta) d\eta \right),
\]

from here

\[
\lim_{\varepsilon \to +0} \int_{0}^{y} f(x - \varepsilon, \eta) \frac{\partial^{2n-1}}{\partial x^{2n-1}} \Gamma_{\alpha - \frac{2n}{2n} - 1} (\varepsilon, y - \eta) d\eta =
\]

\[
= \frac{(-1)^{n-1}}{2n} \sum_{k=0}^{n-1} \frac{1}{2n} \sum_{k=0}^{n-1} \int_{0}^{y} f(x, \eta) (y - \eta)^{m-1} \sum_{m=1}^{\infty} \left( -\frac{y - \eta - \varepsilon}{2n} \right)^{-\frac{2n}{2n} k} \left( \frac{k! \Gamma \left( -\frac{\alpha}{2n} k \right)}{m! \Gamma \left( -\frac{\alpha}{2n} m \right)} \right) d\eta =
\]

\[
\left( z = \varepsilon (y - \eta)^{-\frac{\alpha}{2n}} \Rightarrow \eta = y - \varepsilon \frac{2n}{\alpha} z - \frac{2n}{\alpha}, \ d\eta = \frac{2n}{\alpha} \varepsilon\frac{2n}{\alpha} z - \frac{2n}{\alpha} dz \right)
\]

\[
= \frac{(-1)^{n-1}}{\alpha} f(x, y) \sum_{m=0}^{\infty} \left( \frac{(-e^{\alpha^{-2n} \pi i} z)^{m}}{m! \Gamma \left( -\frac{\alpha}{2n} m \right)} \right) \int_{0}^{+\infty} z^{m-1} dz =
\]

\[
= \frac{(-1)^{n-1}}{2n} f(x, y) \sum_{k=0}^{n-1} \left( \sum_{m=0}^{\infty} \left( \frac{(-e^{\alpha^{-2n} \pi i} z)^{m}}{m! \Gamma \left( -\frac{\alpha}{2n} m \right)} \right) \right|_{z=0}^{z=+\infty} = \frac{(-1)^{n-1}}{2} f(x, y).
\]

Means

\[
I_1 = \frac{(-1)^{n-1}}{2} f(x, y) + \int_{-\infty}^{x} d\xi \left( \int_{0}^{y} f(\xi, \eta) \frac{\partial^{2n}}{\partial x^{2n}} \Gamma_{\alpha - \frac{2n}{2n} - 1} (x - \xi, y - \eta) d\eta \right).
\]
Similarly
\[ I_3 = \frac{(-1)^{n-1}}{2} f(x, y) + \int_x^\infty d\xi \left( \int_0^y f(\xi, \eta) \frac{\partial^{2n}}{\partial x^{2n}} \Gamma_{\alpha - \frac{n}{2n}}(x - \xi, y - \eta) d\eta \right), \]

it is easy to show that
\[ I_2 = 0. \]

so
\[
\frac{\partial^{2n}}{\partial x^{2n}} \left( \int_{-\infty}^{+\infty} d\xi \left( \int_0^y f(\xi, \eta) \Gamma_{\alpha - \frac{n}{2n}}(x - \xi, y - \eta) d\eta \right) \right) = \\
= (-1)^{n-1} f(x, y) + \int_{-\infty}^{+\infty} d\xi \left( \int_0^y f(\xi, \eta) \frac{\partial^{2n}}{\partial x^{2n}} \Gamma_{\alpha - \frac{n}{2n}}(x - \xi, y - \eta) d\eta \right) = \\
= (-1)^{n-1} f(x, y) + (-1)^{n-1} \int_{-\infty}^{+\infty} d\xi \left( \int_0^y f(\xi, \eta) \Gamma_{\alpha - \frac{n}{2n}}(x - \xi, y - \eta) d\eta \right). \tag{25}
\]

From (22) and (25) we obtain that the expression
\[-\int_{0}^{+\infty} \int_{-\infty}^{+\infty} f(\xi, \eta) \Gamma_{\alpha - \frac{n}{2n}}(x - \xi, y - \eta) d\xi d\eta,\]
satisfies equation (9). The theorem is proved.

The question of the uniqueness of the Cauchy problem is still open. Function
\[-\Gamma_{\alpha - \frac{n}{2n}}(x - \xi, y - \eta) \tag{26}\]
will be called the fundamental solution of equation (9). Representation (26) coincides with the form of the fundamental solution obtained for the case \(0 < \alpha < 1\) in [5].

3. Application

In [7], for the equation of beam vibration
\[ u_{tt}(t, x) + a^2 u_{xxxx}(t, x) = 0, \]
a representation of the solution of the Cauchy problem was obtained using the previously constructed fundamental solution, which for \(a = 1\) has the form
\[ G_2(x, t, \xi) = \sqrt{\frac{t}{\pi}} \sin \left[ \frac{(\xi-x)^2}{4t} + \frac{\pi}{4} \right] + \\
+ \frac{\xi - x}{2} \left\{ S \left[ \frac{(\xi-x)^2}{4t} \right] - C \left[ \frac{(\xi-x)^2}{4t} \right] \right\}, \]
where
\[ S(z) = \frac{1}{2\sqrt{\pi}} \int_{0}^{z} \sin \frac{t}{\sqrt{t}} dt, \quad C(z) = \frac{1}{2\sqrt{\pi}} \int_{0}^{z} \cos \frac{t}{\sqrt{t}} dt - \]
Fresnel integrals.

Our fundamental solution (27), for \(\alpha = 2, n = 2\), has the form (\(y\) was replaced by \(t\), \(\eta = 0\))
\[-\Gamma_{\frac{1}{2}}(x - \xi, t) = -\frac{\sqrt{t}}{4} \left( -e^{-\frac{\pi}{4} t} \sum_{n=0}^{\infty} \frac{(-e^{-\frac{\pi}{4} t})^n}{n!} \Gamma(-\frac{1}{2} n + \frac{3}{2}) - e^{-\frac{\pi}{4} t} \sum_{n=0}^{\infty} \frac{(-e^{-\frac{\pi}{4} t})^n}{n!} \Gamma(-\frac{1}{2} n + \frac{3}{2}) \right), \]
where
\[ \tau = \frac{|x - \xi|}{\sqrt{t}}, \]
show that
\[ G_2(x, t, \xi) = -\Gamma_{1/4}(x - \xi, t). \]

Indeed, we have
\[
\sum_{n=0}^{\infty} \frac{(-e^{\frac{\pi i}{4}})^n \tau^n}{n! \Gamma(-\frac{n}{2} + \frac{3}{2})} = \sum_{n=0}^{\infty} \frac{(-e^{\frac{\pi i}{4}})^n \tau^{4n}}{(4n)! \Gamma(-2n + \frac{3}{2})} + \sum_{n=0}^{\infty} \frac{(-e^{\frac{\pi i}{4}})^{4n+1} \tau^{4n+1}}{(4n+1)! \Gamma(-2n + 1)} + \\
+ \sum_{n=0}^{\infty} \frac{(-e^{\frac{\pi i}{4}})^{4n+2} \tau^{4n+2}}{(4n+2)! \Gamma(-2n + \frac{3}{2})} + \sum_{n=0}^{\infty} \frac{(-e^{\frac{\pi i}{4}})^{4n+3} \tau^{4n+3}}{(4n+3)! \Gamma(-2n)} = \\
= \sum_{n=0}^{\infty} \frac{(-1)^n \tau^{4n}}{(4n)! \Gamma(-2n + \frac{3}{2})} + i \sum_{n=0}^{\infty} \frac{(-1)^n \tau^{4n+2}}{(4n+2)! \Gamma(-2n + \frac{1}{2})} + (-e^{\frac{\pi i}{4}}) \tau = I_1.
\]
Similarly
\[
\sum_{n=0}^{\infty} \frac{(\tau e^{\frac{\pi i}{4}})^n}{n! \Gamma(-\frac{n}{2} + \frac{3}{2})} = \sum_{n=0}^{\infty} \frac{(-1)^n \tau^{4n}}{(4n)! \Gamma(-2n + \frac{3}{2})} - i \sum_{n=0}^{\infty} \frac{(-1)^n \tau^{4n+2}}{(4n+2)! \Gamma(-2n + \frac{1}{2})} - e^{\frac{\pi i}{4}} = I_2,
\]
from here
\[ -\Gamma_{1/2}(x - \xi, t) = -\sqrt{\frac{t}{4}} \left(-e^{\frac{\pi i}{4}}I_1 - e^{\frac{\pi i}{4}}I_2\right) = \]
\[ = \frac{\sqrt{2t}}{8} ((I_1 + I_2) + i (I_1 - I_2)), \]
further
\[ I_1 + I_2 = 2 \sum_{n=0}^{\infty} \frac{(-1)^n \tau^{4n}}{(4n)! \Gamma(-2n + \frac{3}{2})} - \sqrt{2\tau}, \]
\[ (I_1 - I_2) i = -2 \sum_{n=0}^{\infty} \frac{(-1)^n \tau^{4n+2}}{(4n+2)! \Gamma(-2n + \frac{1}{2})} + \sqrt{2\tau} \Rightarrow \]
\[ -\Gamma_{1/2}(x - \xi, t) = \frac{\sqrt{2t}}{4} \left( \sum_{n=0}^{\infty} \frac{(-1)^n \tau^{4n}}{(4n)! \Gamma(-2n + \frac{3}{2})} - \sum_{n=0}^{\infty} \frac{(-1)^n \tau^{4n+2}}{(4n+2)! \Gamma(-2n + \frac{1}{2})} \right), \]
further, calculations show that
\[
\sum_{n=0}^{\infty} \frac{(-1)^n \tau^{4n}}{(4n)! \Gamma(-2n + \frac{3}{2})} = \frac{\Gamma\left(-\frac{1}{4}\right)}{\pi} _1 F_2 \left[ \begin{array}{c} \frac{1}{4}, \frac{3}{8} \\ \frac{3}{4}, \frac{5}{4} \end{array} \right],
\]
\[
\sum_{n=0}^{\infty} \frac{(-1)^n \tau^{4n+2}}{(4n+2)! \Gamma(-2n + \frac{1}{2})} = \frac{\Gamma\left(-\frac{1}{4}\right) \tau^2}{4\pi} _1 F_2 \left[ \begin{array}{c} \frac{1}{4}, \frac{3}{8} \\ \frac{3}{4}, \frac{5}{4} \end{array} \right],
\]
where
\[ _1 F_2 \left[ \begin{array}{c} a, z \\ b, c \end{array} \right] = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n(c)_n} \frac{z^n}{n!} \]
-hypergeometric function,
\[ (d)_n = d (d + 1) \ldots (d + n - 1), \]
\[ -\Gamma_{\frac{1}{2}}(x - \xi, t) = \frac{\sqrt{2t} \Gamma\left(-\frac{3}{8}\right)}{16\pi} \left( 4_1 F_2 \left[ -\frac{1}{4}, -\left(\frac{t^2}{8}\right)^2 \right] - \tau^2 F_2 \left[ \frac{1}{4}, -\left(\frac{t^2}{8}\right)^2 \right] \right). \]

Now, taking into account the relations (see [8])

\[ 4_1 F_2 \left[ -\frac{1}{4}, -\left(\frac{t^2}{8}\right)^2 \right] = 4\cos \left(\frac{\tau^2}{4}\right) + 2\sqrt{2\pi} |\tau| S \left(\frac{\tau^2}{4}\right), \]
\[ \tau^2 F_2 \left[ \frac{1}{4}, -\left(\frac{t^2}{8}\right)^2 \right] = 2 |\tau| \sqrt{2\pi} C \left(\frac{\tau^2}{4}\right) - 4 \sin \frac{\tau^2}{4}, \]

finally get

\[ -\Gamma_{\frac{1}{2}}(x - \xi, t) = \sqrt{\frac{t}{\pi}} \sin \left(\frac{\tau^2}{4} + \frac{\pi}{4}\right) + \frac{|\xi - x|}{2} \left( S \left(\frac{\tau^2}{4}\right) - C \left(\frac{\tau^2}{4}\right) \right), \]

i.e.

\[ -\Gamma_{\frac{1}{2}}(x - \xi, t) = G_2(x, t, \xi). \]

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