Finite Difference Method For Laplace Equation In Irregular Domain

E. Juarlin
Department of Physics, Faculty Mathematics and Natural Science, Hasanuddin University. Makassar. South Sulawesi. Indonesia.
E-mail: ekojuarlin@unhas.ac.id

Abstract. Laplace equation on irregular domain has been solved by finite difference method. The program for calculating distribution of electric potential inside has been developed. Symmetrical Dirichlet boundary condition and Cartesius Coordinates are applied. A trapezoid and a quarter of a circle are chosen as irregular domain example. The contour of electric potential shows symmetrical result to its domain. The convergence and stability output program shows a good result.

1. Introduction
Electric potential is a scalar variable, appearing from one or many electric sources that put in typical media. Electric potential is special property of electric field to reduce a vector problem down to a scalar problem. Electric potential is an useful physical quantity in many fields concerned with electromagnetism.

Laplace equation is a second order partial differential equation (PDE) that appears in many areas of science. In physics it uses to find some physical quantities, like temperature, electrics potential, amplitude of wave and so on. Solution of this equation, in a domain, requires the specification of certain conditions that the unknown function must satisfy at the boundary of the domain. The function itself is specified on a part of the boundary called Dirichlet boundary but the normal derivative of the function is specified on a part of the boundary, called Neumann boundary condition.

The easiest method of numerical techniques to get solution of Laplace equation is Finite Difference Method (FDM). There are commonly two reasons why FDM is used. First, the shape is irregular. Second, the solution can not be calculated analytically. FDM proceeds by replacing the derivatives in the differential equations with finite difference approximations according to Taylor expansions [1]. A mathematical relation between neighbourhood points inside is calculated. The solution can be calculated simultaneously or iterative. Widely, FDM has been implemented in many field, like vibration [2], thermal spreading of geothermal systems [3], electronic field [4].

2. Laplace Equation in FDM
The Laplace equation for electric potential is

\[
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x^2} = 0 \tag{1}
\]
The second derivatives for two dimensions can be approximated as

\[
\frac{\partial^2 V}{\partial x^2} \approx \frac{V(i-1,j) - 2V(i,j) + V(i+1,j)}{\Delta x^2}
\]  

(2)

So, two dimension FDM approximation for Laplace equation is

\[
\frac{V(i-1,j) - 2V(i,j) + V(i+1,j)}{\Delta x^2} + \frac{V(i,j-1) - 2V(i,j) + V(i,j+1)}{\Delta y^2} = 0
\]  

(3)

From equation 3, after all points are made into, finally a matrix containing relation electric potential between points and a column matrix containing boundary value are constructed. The solution for a set of linear algebraic equations is calculated by Gauss-Seidel iteration until electric potential value is convergent.

3. The Algorithm for Numbering Points
Numbering points inside shape needs a special technique because shape is irregular. The numbering technique for both domains is almost same.

A bigger rectangle than domain is constructed outside trapezoid. Then, trapezoid is constructed by four lines mathematically determined by four equations of all circumference line. One circumference line is negative gradient, one circumference line is positive gradient and two others are zero gradient. Points are discretized horizontally and vertically along to \( \Delta x \) and \( \Delta y \). There are points outside trapezoid. Perhaps, there is one or more points nearest circumference and there is one or more points lying exactly on circumference. The domain is cut so its shape is formed from a collection of horizontal and vertical grid lines having same interval distance. On every horizontal grid line, there are two end points inside trapezoid such as one point is near to negative gradient line and one point is near to positive gradient line. On every vertical grid line, there are two end points inside trapezoid such as one point lies on x-coordinate and one point lies on top of trapezoid. The rest points are calculated points. All points have been classified.

In quarter circle, the technique for numbering is similar to trapezoid numbering. Quarter circle lies on first quadrant. There are three circumference lines that one vertical line lying on y coordinate, one horizontal line lying x coordinate and one as an arc. Vertical and horizontal grid lines are made up into domain such as every intersection has a discrete point. After putting discrete points inside, there must be an end point on top at a certain x and an end point on right at a certain y. On all end points, Dirichlet boundary condition is applied. The rest points are calculated points.

4. Simulation Results
Domain is figured in figure 1.

![Figure 1. Domain of a square of circle (left) and a trapezoid (right)](image)

The center of coordinate is as point of center of a circle in this paper. Electric potential source 10 V is placed on circumference of quarter circle line. Two functions are used such as
\( f(x, y) = 10 \) on first quadrant points which obeys equation \( x^2 + y^2 = r^2 \) and \( f(x, y) = 0 \) for \( x = 0 \) and \( y = 0 \) which are applied as boundary conditions. Radius of a square of a circle is 10, and value of X interval and Y interval is 1. The contour of electric potential for a quarter circle is shown on figure 2.

![Electric Potential Distribution](image1.png)

**Figure 2.** Electric Potential Distribution in Square of Circle with Radius 10

Figure 2 displays the electric potential distribution is symmetry to line \( y = x \). Electric potential becomes greater as it becomes near to arc. The domain shape is not a perfect quarter circle because its edge cutting effect. The electric scalar potential in all points reaches convergence in about 100 iterations with 56 calculated points and simulation results a good stability as in figure 3.

![Convergence and Stability of Electric Potential](image2.png)

**Figure 3.** Convergence and Stability of Square of Circle with Radius 10

In a trapezoid, interval one segment for both axes is equal which is 0.5, 1 and 2. Length on bottom is 10 and length on its top is 6 and length on its height is 6. Base of trapezoid lie on x axis. The isosceles trapezoid is taken as example. The value of electricity potential is 10 at bottom and base but 0 at hypotenuse. The electric potential distribution at interval 0.25 is illustrated in figure 4.
Figure 4. Electric Potential Distribution in Trapezoid with Interval 0.25

As a symmetrical boundary condition to y axis, the simulation shows symmetrical result. From center of trapezoid, electricity potential becomes greater to the base and top but becomes lower to the hypotenuse. This case has 39 calculated points and reaches stability after about 250 iteration. The convergence and stability show good result as showed in figure 5.

Figure 5. Convergence and Stability of Trapezoid with interval 0.25
5. Conclusion
We have constructed a simulation of electricity potential on irregular domain with symmetrical Dirichlet boundary condition and symmetrical domain. The simulation result is agreed with theorem. Convergence and stability as analyzed variable to determine quality of simulation shows good result.

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