On the flatness of models of certain Shimura varieties of PEL-type

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Chapter 1

Introduction

In the arithmetic theory of Shimura varieties, it is of interest to have a model of the Shimura variety over the ring of integers $O_E$, where $E$ is the completion of the reflex field at some place lying over a prime $p$. For a Shimura variety of PEL-type, which is a moduli space of abelian varieties with certain additional structure, it is a natural idea to define such a model by posing the moduli problem over $O_E$. In the case of a hyperspecial level structure, one gets a smooth model as was shown by Kottwitz. We are interested in the case of parahoric level structures, where one cannot expect to get a smooth model.

The problem reduces in some sense to the level of $p$-divisible groups. In their book \cite{RZ}, Rapoport and Zink investigate (formal) moduli schemes of $p$-divisible groups and define such models. Unfortunately, very little is known in general about the structure of these models. In most cases one does not even know if they are flat over $O_E$ — which is certainly a condition a reasonable model should satisfy.

A different approach to these problems is to look for another model which is semi-stable or at least has toroidal singularities. There is an interesting proposal of Genestier of a semi-stable model in the case of the symplectic group (cf. \cite{G}), which works in low dimensions. Recently Faltings announced some results about a model with toroidal singularities, which again works in low dimensions (cf. \cite{F2}). On the other hand, a drawback of this approach is that the new model does not have an easy description as a moduli space of $p$-divisible groups.

To examine local properties such as flatness, it is useful to work with the so-called local model, which locally for the étale topology around each point of the special fibre coincides with the corresponding moduli scheme of $p$-divisible groups, but which can be defined in terms of linear algebra and is thus much easier to handle (cf. \cite{RZ}).

In this article, we will deal with the flatness conjecture in a special case which will be explained now. We use the same notation as in \cite{RZ}.

Let $F/\mathbb{Q}_p$ be a finite unramified extension, let $B = F$, $V = F^n$. The algebraic group associated to these data is $G = \text{Res}_{F/\mathbb{Q}_p} \text{GL}_F(V)$. 
Let $L$ be an algebraically closed field of characteristic $p$, and denote by $K_0$ the quotient field of the Witt ring $W(L)$. As $F$ is unramified over $\mathbb{Q}_p$, with the notation of [RZ] we have $K_0 = K$.

Furthermore let $\mu : G_{m,K} \to G_K$ be a 1-parameter subgroup, such that the weight decomposition of $V \otimes_{\mathbb{Q}_p} K$ contains only the weights 0 and 1:

$$V \otimes_{\mathbb{Q}_p} K = V_0 \oplus V_1.$$ 

Denote by $E$ the field of definition of the conjugacy class of $\mu$.

Finally, let $\mathcal{L}$ be a periodic lattice chain in $V$.

We will call data of this type unramified data of (EL) type. Note that this is a more general notion than is used in [RZ], 3.82. There, the lattice chain consists only of multiples of one lattice, and the resulting local model is smooth. Here, ‘unramified’ relates only to the field extension.

Given these data, Rapoport and Zink define a local model (see [RZ] 3.27 or section [LT]), which is a projective $O_E$-scheme. We will give the definition in the case $F = \mathbb{Q}_p$ below.

These local models are related to Shimura varieties of PEL-type which belong to unitary groups that split over an unramified extension of $\mathbb{Q}_p$.

Our main result is the following theorem which confirms the conjecture of Rapoport and Zink in this case (see theorem [4.6.1]).

**Main Theorem.** The local model associated to an unramified (EL)-datum is flat over $O_E$, and its special fibre is reduced. The irreducible components of the special fibre are normal with rational singularities, so in particular are Cohen-Macaulay.

It is essential that we consider only unramified extensions $F/\mathbb{Q}_p$. In fact, in the ramified case the flatness conjecture has to be refined as is shown by the results of Pappas [P].

It is an interesting question if the special fibre as a whole has Cohen-Macaulay singularities. In view of the flatness this is equivalent to the local model being Cohen-Macaulay. The second remark after proposition [4.4.8] shows that this would follow if one could prove that the affine scheme (over some field) defined by the equations

$$B_{m-1}B_{m-2} \cdots B_0 = B_{m-2} \cdots B_0B_{m-1} = \cdots = B_0B_{m-1} \cdots B_1 = 0,$$

where the $B_i$ are generic $k \times k$-matrices, is Cohen-Macaulay and has the ‘right’ dimension.

Of course, the flatness question presents itself also for other groups, in particular for the symplectic group $Sp_{2n}$. In this situation flatness has been verified in special cases by Deligne and Pappas [DP], de Jong [dJ], and Chai and Norman [CN]. Their proofs rely heavily on very explicit calculations with the equations. One of the theories involved is the theory of algebras with straightening law.
It allows one to show in some cases that the special fibre of the local model is reduced, or even that it is Cohen-Macaulay. The disadvantage of this method is that only cases where the lattice chain is small, i.e. does not consist of too many lattices, can be handled.

To give an idea of the proof of the main theorem, let us first give the definition of the standard local model, where in terms of the (EL) datum we have \( F = \mathbb{Q}_p \) (compare section 4.1). In fact, it is easy to see that the local model associated to an unramified (EL)-datum is isomorphic after unramified base change to a product of standard local models.

Let \( O \) be a complete discrete valuation ring with perfect residue class field. Let \( \pi \) be a uniformizer of \( O \) and denote the field of fractions of \( O \) by \( K \). Let \( k \) be an algebraic closure of the residue class field of \( O \). Fix integers \( 0 < r < n \). Let \( e_1, \ldots, e_n \) be the canonical basis of \( K^n \). Let \( \Lambda_i, 0 \leq i \leq n - 1 \), be the free \( O \)-module of rank \( n \) with basis \( e^i_1 := \pi^{-1} e_1, \ldots, e^i_i := \pi^{-1} e_i, e^i_{i+1} := e_{i+1}, \ldots, e^i_n := e_n \). This yields a complete lattice chain

\[
\cdots \to \Lambda_0 \to \Lambda_1 \to \cdots \to \Lambda_{n-1} \to \pi^{-1} \Lambda_0 \to \cdots
\]

Choose \( I = \{ i_0 < i_1 < \cdots < i_{m-1} \} \subseteq \{ 0, \ldots, n - 1 \} \).

Then the standard local model \( M^\text{loc}_I \) is the \( O \)-scheme that represents the following functor. For an \( O \)-scheme \( S \), the \( S \)-valued points of \( M^\text{loc}_I \) are the isomorphism classes of commutative diagrams

\[
\Lambda_{i_0,S} \to \Lambda_{i_1,S} \to \cdots \to \Lambda_{i_{m-1},S} \to \pi \Lambda_{i_0,S}
\]

where \( \Lambda_{i,S} := \Lambda_i \otimes_O O_S \), and where the \( F_k \) are locally free \( O_S \)-submodules of rank \( r \) which Zariski-locally on \( S \) are direct summands of \( \Lambda_{i,S} \). We write \( M^\text{loc} := M^\text{loc}_{\{0, \ldots, n-1\}} \).

We see that this functor is representable by a closed subscheme of a product of Grassmannians. The generic fibre is a Grassmannian itself since all the maps \( \Lambda_i \to \Lambda_j \) are isomorphisms after tensoring with \( K \).

The case where \( r = n - 1 \) (the so-called Drinfeld case) is particularly simple: then \( M^\text{loc} \) has semi-stable reduction, and \( M^\text{loc} \) is obviously flat. On the other hand, if \( I \) is small, the local model is less complicated than for large \( I \). For example if \( I \) has only one element, then \( M^\text{loc}_I \) is simply a Grassmannian over \( O \), so it is even smooth. In this work we will show that \( M^\text{loc}_I \) is flat over \( O \) for general \( n, r \) and \( I \).

The most difficult part is to show that the special fibre of the local model is reduced. In positive characteristic, this question can be reduced to a question on local models for small \( m \) by embedding the special fibre of the local model
into the affine flag variety and using the technique of Frobenius splitting. Let us make this a little more precise.

**Theorem 4.5.1** Let \( \text{char } k = p > 0 \). Then the special fibre of \( M^{\text{loc}} \) is reduced.

We give an outline of the proof:

We embed the special fibre \( \widetilde{M}^{\text{loc}} \) of the local model into the affine flag variety \( F = SL_n(k((t))/B \). Set-theoretically \( \widetilde{M}^{\text{loc}} \) is a union of Schubert varieties.

Further consider the special fibres \( \widetilde{M}^{\text{loc}}_0 \) resp. \( \widetilde{M}^{\text{loc}}_{0,\kappa} \) of the local models of type \( \{0\} \) resp. \( \{0,\kappa\} \). We can embed them in \( SL_n(k((t))/P^0 \) resp. \( SL_n(k((t))/P^{0,\kappa} \), where \( P^I, I \subseteq \{0,\ldots,n-1\} \), is the stabilizer of the lattice chain corresponding to \( I \). Denote the inverse images under the canonical projections in \( F \) by \( \widetilde{M}^{\text{loc}}_0 \) resp. \( \widetilde{M}^{\text{loc}}_{0,\kappa} \).

Obviously,

\[
\widetilde{M}^{\text{loc}} = \bigcap_{\kappa=1}^{n-1} \widetilde{M}^{\text{loc}}_{0,\kappa}.
\]

Now, \( \widetilde{M}^{\text{loc}}_0 \) is invariant under the action of the Iwahori subgroup, and thus is set theoretically a union of Schubert varieties. But we also know that it is a smooth, connected scheme, so in particular, is reduced and irreducible. Thus it is a Schubert variety in \( F \).

Furthermore, it can be shown that the \( \widetilde{M}^{\text{loc}}_{0,\kappa} \) essentially are so-called varieties of circular complexes (see [MT] resp. section 4.4.5): locally, they have the form

\[
\{(X,Y) \in \text{Mat}_{N'}(k) \times \text{Mat}_{N'}(k); \ XY = YX = 0\} \times \mathbb{A}^N.
\]

In fact, locally all the \( \widetilde{M}^{\text{loc}}_I \) can be interpreted as spaces of certain homomorphisms (up to a product with affine space), see theorem 4.4.7. Now by the results of Strickland [S] respectively of Mehta and Trivedi [MT], the \( \widetilde{M}^{\text{loc}}_{0,\kappa} \) are reduced. Hence the \( \widetilde{M}^{\text{loc}}_{0,\kappa} \) are reduced as well and thus they are unions of Schubert varieties even scheme-theoretically.

To prove the theorem, we apply the technique of Frobenius splittings. As intersections and unions of compatibly split subvarieties are split again, and split schemes are reduced, the theorem follows from (cf. corollary 3.4.4):

**Theorem.** The Schubert variety \( \widetilde{M}^{\text{loc}}_0 \) is Frobenius split, and all Schubert subvarieties of \( \widetilde{M}^{\text{loc}}_0 \) are simultaneously compatibly split.

The corresponding theorem is well known for the finite dimensional flag variety (we will recall this briefly in section 2.3). Mathieu proved a similar theorem in the context of Kac-Moody algebras (cf. [M]).

Once one knows that the special fibre of the local model is reduced, it is not very difficult to show that the local model itself is flat over \( O \). Namely, an explicit calculation yields that the generic points of the irreducible components of the special fibre can be lifted to the generic fibre (see proposition 4.4.9).
The consideration of the special fibre of the standard local model leads to the following question on incidence varieties of flag varieties. Fix $n > 0$ and a partition $\underline{r}$ of $n$. Let $V_0, \ldots, V_\ell$ be vector spaces of dimension $n$ over some field $k$. Choose $(\varphi_{ij})_{ij} \in \prod_{i,j} \text{Hom}(V_i, V_j)$ and define

$$X := \{(F_i)_{i} \in \prod_{i=0}^{\ell} \text{Flag}_{\underline{r}}(V_i); \ \varphi_{ij}(F_i) \subseteq F_j \text{ for all } i, j\},$$

where $\text{Flag}_{\underline{r}}(V_i)$ is the flag variety of flags of type $\underline{r}$ in $V_i$. What are the singularities of the scheme $X$?

The most interesting question that remains open at the moment is what can be said about other groups, especially for the symplectic group. Of course the approach of embedding the special fibre of the local model in an affine flag variety should work as well. For proving the reducedness, the main problem is to establish the analogue of the results of Strickland resp. Mehta and Trivedi that we cited in the case of $GL_n$. Some results in this direction are already available (cf. [DP], [CN]), but this does not seem sufficient to get the proof started. Again, it is not difficult to show that the local model is flat, once one knows that the special fibre is reduced.

Finally, it is a pleasure to acknowledge the help I received from several people with this work. First of all, I am very grateful to M. Rapoport who initiated this work and introduced me into this area of mathematics. His mathematical advice as well as his encouragement and steady interest in my work were extremely helpful to me. Furthermore, I would like to thank O. Bültel, T. Haines, S. Orlik and T. Wedhorn for many useful discussions, and T. Wedhorn again for making a lot of valuable remarks on this text.
Chapter 2

Frobenius Splittings

In this section we give the relevant definitions and collect some basic facts about Frobenius splittings. We mostly follow the article [MR] of Mehta and Ramanathan; confer also Ramanathan’s article [Ram].

2.1 Definition

Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Let \( X \) be a \( k \)-scheme of finite type. Denote by \( X' \) the base change of \( X \) with respect to the Frobenius morphism \( \text{Spec} \ k \to \text{Spec} \ k \). The relative Frobenius morphism \( F : X \to X' \) gives us a homomorphism \( O_{X'} \to F_*O_X \) of \( O_{X'} \)-modules.

Definition 2.1.1  

i) The scheme \( X \) is called Frobenius split (or \( F \)-split), if the homomorphism \( O_{X'} \to F_*O_X \) admits a section. Such a section is called a splitting.

ii) Let \( \sigma : F_*O_X \to O_{X'} \) be a splitting. A closed subscheme \( Y \subseteq X \) with sheaf of ideals \( I \) is called compatibly \( \sigma \)-split (or simply compatibly split) if \( \sigma(F_*I) \subseteq I_{Y'} \).

If \( Y \subseteq X \) is compatibly \( \sigma \)-split, then \( \sigma \) induces a splitting of \( Y \).

Lemma 2.1.2  

Let \( \sigma : F_*O_X \to O_{X'} \) be a splitting.

i) If \( Y_1, Y_2 \subseteq X \) are compatibly \( \sigma \)-split, then \( Y_1 \cap Y_2 \) and \( Y_1 \cup Y_2 \) are compatibly \( \sigma \)-split.

ii) If \( Y = Y_1 \cup \ldots \cup Y_n \subseteq X \) is the decomposition into irreducible components and \( Y \) is compatibly \( \sigma \)-split, then \( Y_1, \ldots, Y_n \) are compatibly \( \sigma \)-split.

The following proposition is a trivial consequence of the definition, but it will be very important for us.

Proposition 2.1.3  

If \( X \) is \( F \)-split, then it is reduced.
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Proof. If \( O_{X'} \rightarrow F_*O_X \) has a section, then it must be injective. \( \square \)

**Proposition 2.1.4** Let \( f : Z \rightarrow X \) be a proper morphism of algebraic varieties over \( k \). Assume \( f_*O_Z = O_X \).

i) If \( Z \) is \( F \)-split, then \( X \) is also \( F \)-split.

ii) If \( Y \subseteq Z \) is a closed subvariety which is compatibly split, then its image \( f(Y) \) is compatibly split in \( X \). \( \square \)

Proof. Let \( \sigma : F_*O_Z \rightarrow O_{Z'} \) be a splitting. Since the Frobenius morphism commutes with any morphism, we have \( f_*F_*O_Z = F_*f_*O_Z = F_*O_X \), hence \( f_*\sigma \) is a splitting of \( X \).

Now let \( Y \subseteq Z \) be compatibly \( \sigma \)-split. Let \( I \subseteq O_Z \) (resp. \( J \subseteq O_X \)) be the ideal sheaf of \( Y \) (resp. \( f(Y) \)). Then \( f_*I = J \) (cf. [MR], Lemma 2), and it follows that \( f_*\sigma (F_*J) = J \). \( \square \)

2.2 A Criterion for Splitting

Now let \( X \) be a smooth projective variety of dimension \( n \) over \( k \). To find a splitting of \( X \), it is enough to find a homomorphism \( F_*O_X \rightarrow O_{X'} \), such that the composite \( O_{X'} \rightarrow F_*O_X \rightarrow O_{X'} \) is non-zero on the fibre at a single point (since any homomorphism \( O_{X'} \rightarrow O_{X'} \) is a constant in \( k \)). Such a homomorphism, which is a splitting up to a constant, will also be called a splitting.

So we want to understand the global sections of \( \text{Hom}_{O_{X'}}(F_*O_X, O_{X'}) = (F_*O_X)^* \).

Denote by \( \omega_X \) the canonical bundle of \( X \). Serre duality gives a correspondence between global sections of \( \omega^{-1}_X \) and global sections of \( (F_*O_X)^* \). In fact, using the Cartier operator, one can give a natural isomorphism \( F_*\omega^{-1}_X \rightarrow (F_*O_X)^* \), which can be written down explicitly in terms of local coordinates on \( X \). Analyzing this isomorphism, one arrives at the following criterion for compatible splittings:

**Proposition 2.2.1** ([MR] Prop. 8) Let \( X \) be a smooth projective variety of dimension \( n \) over \( k \). Let \( Z_1, \ldots, Z_n \subseteq X \) be irreducible closed subvarieties of codimension 1 such that for any subset \( I \subseteq \{1, \ldots, n\} \) the scheme-theoretic intersection \( Z_I = \bigcap_{i \in I} Z_i \) is reduced and irreducible of codimension \( \# I \). Let \( P = \bigcap_{i=1}^n Z_i \).

Further, suppose that there exists a global section \( s \) of \( \omega^{-1}_X \) such that \( \text{div} \ s = Z_1 + \cdots + Z_n + D \), where \( D \) is an effective divisor with \( P \notin \text{supp} \ D \).

Then the section \( s^{p-1} \in H^0(X, \omega^{-p}_X) \) gives a splitting of \( X \) which compatibly splits all the \( Z_i \). \( \square \)
2.3 Frobenius Splitting of Classical Schubert Varieties

In this section we want to collect some facts about $F$-splittings for classical Schubert varieties. We will not need them later, but it will become clear that everything we want to do for the affine flag variety has an analogue in the classical case.

Let $G$ be a semisimple algebraic group over $k$, and choose a maximal torus $T$ and a Borel subgroup $B$ of $G$ which contains $T$. We have the following theorem ([Ram]):

**Theorem 2.3.1** (Ramanathan) All Schubert varieties in $G/B$ are simultaneously compatibly $F$-split.

By lemma 2.1.2 and proposition 2.1.3, this immediately gives the following corollary.

**Corollary 2.3.2** Arbitrary intersections of unions of Schubert varieties are reduced. □

We give a sketch of the proof of the theorem.

Take a reduced expression $w_0 = s_{\alpha_1} \cdots s_{\alpha_r}$ of the longest element of the Weyl group. Let $w_i = s_{\alpha_1} \cdots s_{\alpha_i}$, and denote by $X_i$ the corresponding Schubert variety. Then $X_r = G/B$.

We have the Demazure varieties $Z_i$ which are smooth varieties of dimension $i$ and maps $\psi_i : Z_i \to X_i$. Inside $Z_i$ we have $i$ divisors $Z_{i1}, \ldots, Z_{ii}$ (cf. [Ram]). Denote the sum of these divisors by $\partial Z_i$.

Now, we want to show that $Z_r$ is Frobenius split, and that the $Z_{rij}$ are simultaneously compatibly split, by applying the criterion 2.2.1 above. We need the following lemma.

**Lemma 2.3.3** ([Ram, Prop. 2]) The canonical bundle of $Z_i$ is $\mathcal{O}(-\partial Z_i) \otimes \psi_i^*\mathcal{L}_\rho^{-1}$, where $\mathcal{L}_\rho$ is the equivariant line bundle associated to the character $\rho (= \text{half the sum of the positive roots})$. □

As $\mathcal{L}_\rho$ is a very ample line bundle on $X_r = G/B$, $\psi_r^*\mathcal{L}_\rho$ has no base point. (See corollary 3.3.7 ii.) We can then apply the criterion cited above. (Compare the proof of proposition 3.4.1.)

But as $\psi_*\mathcal{O}_{Z_r} = \mathcal{O}_{G/B}$, it follows from proposition 2.1.4 that $G/B$ is Frobenius split, and that the images of the $Z_{rij}$ are compatibly split. In particular, all Schubert subvarieties of codimension 1 are compatibly split, and using lemma 2.1.2 one can see that all Schubert varieties are compatibly split. (Compare the proof of corollary 3.4.4.) □
Chapter 3

The Affine Flag Variety

Denote by $k$ an algebraically closed field.

3.1 Definition and Basic Properties

In this section, we follow the article [BL] of Beauville and Laszlo quite closely. But we do not restrict ourselves to the case of a ground field of characteristic 0. Indeed, we are especially interested in the case $\text{char } k = p > 0$. We consider $GL_n(k((t)))$ as an ind-scheme over $k$ in the following way: Define

$$G^{(N)}(R) = \{ g(z) \in GL_n(R((t))); \ g(z) \text{ and } g(z)^{-1} \text{ have poles of order } \leq N \}.$$ 

This is an (infinite dimensional) $k$-scheme, and we have $GL_n = \lim \rightarrow G^{(N)}$ (for further details see [BL], for example). Furthermore let $B$ denote the standard Iwahori subgroup. It is an (infinite dimensional) scheme over $k$. The fppf quotient $GL_n(k((t)))/B$ is a $k$-ind-scheme.

Similarly, we have the ind-scheme $\mathcal{F} := SL_n(k((t)))/B$. (By abuse of notation, we denote the Iwahori subgroup of $SL_n$ by $B$ as well). $\mathcal{F}$ is called the affine flag variety. We will describe the ind-structure of $\mathcal{F}$ more explicitly later.

We want to identify $\mathcal{F}$ with a space of lattice chains. First, we recall some definitions.

**Definition 3.1.1** Let $R$ be a $k$-algebra. A lattice in $R((t))^n$ is a sub-$R[[t]]$-module $\mathcal{L}$ of $R((t))^n$ which is projective of rank $n$, and such that $\mathcal{L} \otimes_{R[[t]]} R((t)) = R((t))^n$. Equivalently, we can say that a lattice is a sub-$R[[t]]$-module $\mathcal{L}$ of $R((t))^n$, such that $t^N R[[t]]^n \subseteq \mathcal{L} \subseteq t^{-N} R[[t]]^n$ for some $N$, and such that the $R$-module $t^{-N} R[[t]]^n / \mathcal{L}$ is projective.

**Definition 3.1.2** Let $R$ be a $k$-algebra. A sequence $\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \cdots \mathcal{L}_{n-1} \subseteq t^{-1}\mathcal{L}_0$ of lattices in $R((t))^n$ is called a complete lattice chain, if $\mathcal{L}_{i+1}/\mathcal{L}_i$ is a locally free $R$-module of rank 1 for all $i$. 

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Proposition 3.1.3 We have a functorial isomorphism \( (GL_n(k((t))))/B)(R) \cong \{\text{complete lattice chains in } R((t))^n\} \).

Proof. Of course, the morphism is given by \( \overline{g} \mapsto g \cdot (R[[t]]^n, t^{-1}R[[t]] \oplus R[[t]]^{n-1}, \ldots t^{-1}R[[t]]^{n-1} \oplus R[[t]]) \). To prove the proposition, one shows that Zariski-locally on \( \text{Spec } R \), every lattice chain is of the form \( g \cdot (R[[t]]^n, t^{-1}R[[t]] \oplus R[[t]]^{n-1}, \ldots t^{-1}R[[t]]^{n-1} \oplus R[[t]]) \). This is done in [RZ App. to chapter 3]. \( \square \)

Definition 3.1.4 Let \( r \in \mathbb{Z} \). A lattice \( L \subseteq R((t))^n \) is called \( r \)-special, if \( \bigwedge^n L = t^r R[[t]] \) (as submodule of \( \bigwedge^n R((t))^n = R((t)) \)). A (complete) lattice chain \( (L_i) \) is called \( r \)-special, if \( L_0 \) is \( r \)-special.

Proposition 3.1.5 Fix \( r \in \mathbb{Z} \). Then we have a functorial isomorphism

\[ \mathcal{F}(R) \cong \{ r \text{-special complete lattice chains in } R((t))^n \}. \]

Proof. The morphism is given by

\[ \overline{g} \mapsto g \cdot (\lambda_i), \]

where

\[ \lambda_0 = R[[t]]^{n-r} \oplus tR[[t]]^r, \lambda_1 = R[[t]]^{n-r+1} \oplus tR[[t]]^{r-1}, \]
\[ \ldots, \lambda_{n-1} = t^{-1}R[[t]]^{n-r-1} \oplus R[[t]]^{r+1}. \]

We have to show that, if \( (L_i) \) is a \( r \)-special lattice chain, then there exists, (fppf-)locally on \( \text{Spec } R \), an element \( g \in SL_n(R[[t]]) \), such that \( (L_i) = g \cdot (\lambda_i) \).

Now by the proposition above there exists, locally on \( \text{Spec } R \) (even Zariski-locally), \( g' \in GL_n(R[[t]]) \), such that \( (L_i) = g' \cdot (\lambda_i) \). As \( L_0 \) is \( r \)-special, we have

\[ \bigwedge^n g' : t^r R[[t]] = \bigwedge^n \mathcal{L}_0 \cong \bigwedge^n \lambda_0 = t^r R[[t]], \]

so \( \det(g') \in R[[t]]^\times \). But then we can clearly find an element \( g \in SL_n(R[[t]]) \), such that \( (L_i) = g \cdot (\lambda_i) \). \( \square \)

Corollary 3.1.6 If \( (L_i) \) is \( r \)-special, then \( L_i \) is \((r-i)\)-special, for all \( i \). \( \square \)

Remark. It is known that \( \mathcal{F} \) is reduced if \( \text{char } k = 0 \) (cf. [BL]). It is easy to see that \( GL_n(k((t))))/B \) is not reduced.

Let us now describe the ind-structure of \( \mathcal{F} \) in terms of lattice chains. Identify \( \mathcal{F} \) with the space of 0-special lattice chains as above. For \( g \in SL_n(R((t))) \), we have
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$t^N R[[t]] \subset gR[[t]] \subset t^{-N} R[[t]]$ if and only if $g$ and $g^{-1}$ have poles (with respect to $t$) of order $\leq N$. Thus we define

$$\mathcal{F}^{(N)} = \{(L_i)_i \in \mathcal{F} ; \ t^N R[[t]] \subseteq L_0 \subseteq t^{-N} R[[t]]\}.$$

This is a closed subscheme of a finite flag variety (consisting of certain flags in $t^{-N-1} R[[t]]/t^N R[[t]]$), and $\mathcal{F} = \lim_{\to} \mathcal{F}^{(N)}$. Even in characteristic 0, it is not clear whether the schemes $\mathcal{F}^{(N)}$ are reduced.

If $P \supseteq B$ is a parahoric subgroup of $SL_n(k((t)))$, then of course we have the quotient $SL_n(k((t)))/P$ and can interpret it as a space of (partial) lattice chains again. Namely $P$ is the stabilizer of a partial lattice chain $(\lambda_i)_{i \in I}$, for some $I \subseteq \{0, \ldots, n-1\}$, and the ind-scheme $SL_n(k((t)))/P$ parametrizes partial special lattice chains $(L_i)_{i \in I}$.

**Proposition 3.1.7** Let $P \supseteq B$ be a parahoric subgroup of $SL_n(k((t)))$. Then the canonical projection $\mathcal{F} \to SL_n(k((t)))/P$ is a smooth morphism, the fibres of which are (finite dimensional) flag varieties.

**Proof.** It follows from the infinitesimal lifting criterion that the morphism is smooth. It is clear that the fibres are flag varieties. □

Finally, we introduce some more notations: Denote by $S = \{s_0, \ldots, s_{n-1}\}$ the set of simple reflections. For $I \subseteq S$ denote by $P_I \supseteq B$ the corresponding parahoric subgroup. (If $P^I$ denotes the stabilizer of the lattice chain $(\Lambda_i)_{i \in I}$, then we have $P^I = P\{0, \ldots, n-1\} \setminus I$.) We also write $P_i$ instead of $P\{s_i\}$. So $P_i$ is the subgroup of $SL_n(k((t)))$ that stabilizes all the $\Lambda_j$ except for $\Lambda_i$. Denote by $W_a$ the affine Weyl group, and by $W$ the finite Weyl group of $SL_n$.

3.2 Schubert Varieties

For $w \in W_a$, we have the Schubert cell $BwB/B \subseteq \mathcal{F}$. It is contained in some finite dimensional part of $\mathcal{F}$. Its Zariski closure (with the reduced scheme structure) is called the Schubert variety associated to $w$ and denoted by $X_w$.

Set theoretically, $X_w$ is the disjoint union

$$X_w = \bigcup_{v \leq w} BvB/B,$$

where $\leq$ is the Bruhat order.

3.3 Demazure Varieties

Many of the definitions and results of this section can be found in a similar form in [M]. In fact, Mathieu shows that Schubert varieties in the affine flag variety are Frobenius split, just as we want to do. But the difference is that he
gets the scheme structure in a different way, using the theory of Kac-Moody algebras. To get a relation to the local model, we need the — a priori different — scheme structure from the 'functorial approach' chosen above. It is possible that these two scheme structures coincide (in characteristic zero, this has been shown by Faltings, cf. [BL]), but I do not know how to prove this.

3.3.1 Definition

Let \( w \in W_a \) and let \( \tilde{w} = s_{i_1} \cdots s_{i_\ell} \) be a reduced expression for \( w \). (We write \( \tilde{w} \) instead of \( w \) to indicate that the following definitions really depend on the choice of a reduced decomposition.)

We define

\[
E(\tilde{w}) := P_{i_1} \times B \times \cdots \times B P_{i_\ell},
\]

\[
D(\tilde{w}) := P_{i_1} \times B \times \cdots \times B P_{i_\ell} / B.
\]

The variety \( D(\tilde{w}) \) is called the Demazure variety corresponding to \( \tilde{w} \). If \( \hat{\tilde{u}} = s_{i_1} \cdots s_{i_k} \cdots s_{i_\ell} \) is reduced, we have a closed immersion

\[
E(\hat{\tilde{u}}) \rightarrow E(\tilde{w}),
\]

\[
(g_{i_1}, \ldots, g_{i_{k-1}}, g_{i_{k+1}}, \ldots, g_{i_\ell}) \mapsto (g_{i_1}, \ldots, g_{i_{k-1}}, 1, g_{i_{k+1}}, \ldots, g_{i_\ell}),
\]

which induces a closed immersion

\[
D(\hat{\tilde{u}}) \rightarrow D(\tilde{w}). \tag{3.1}
\]

If \( \tilde{w} = \tilde{u} \bar{v} \), i.e. \( \tilde{u} = s_{i_1} \cdots s_{i_k} \), \( \bar{v} = s_{i_{k+1}} \cdots s_{i_\ell} \), for some \( k \), we get a canonical projection morphism \( D(\tilde{w}) \rightarrow D(\tilde{u}) \), which is a locally trivial fibre bundle with fibre \( D(\bar{v}) \). In particular, let \( \tilde{u} = s_{i_1} \cdots s_{i_{k-1}}, \bar{v} = s_{i_k} \). Then we get a \( \mathbb{P}^1 \)-fibration

\[
D(\tilde{u}) \rightarrow D(\tilde{w}). \tag{3.2}
\]

The closed immersion \( D(\hat{\tilde{u}}) \rightarrow D(\tilde{w}) \) defined above is a section of this fibration.

**Corollary 3.3.1** The Demazure variety \( D(\tilde{w}) \) is smooth and proper over \( k \), and has dimension \( l(w) \). \( \square \)

Multiplication gives us a morphism \( \Psi_{\tilde{w}} : D(\tilde{w}) \rightarrow X_w \).

**Proposition 3.3.2** The morphism \( \Psi_{\tilde{w}} : D(\tilde{w}) \rightarrow X_w \) is proper and birational. If \( \hat{\tilde{u}} = s_{i_1} \cdots s_{i_k} \cdots s_{i_\ell} \) is reduced, these morphisms together with the closed immersion \( (3.1) \) yield a commutative diagram

\[
\begin{array}{ccc}
D(\hat{\tilde{u}}) & \rightarrow & X_u \\
\downarrow & & \downarrow \\
D(\tilde{w}) & \rightarrow & X_w
\end{array}
\]
3.3. DEMAZURE VARIETIES

Proof. This is clear, except maybe for the birationality. But if \( \tilde{w} = s_{i_1} \cdots s_{i_\ell} \), then \( B_{s_{i_1}}B \times B \cdots \times B B_{s_{i_\ell}}B/B \) is an open part of \( D(\tilde{w}) \), and multiplication is an isomorphism

\[
B_{s_{i_1}}B \times B \cdots \times B B_{s_{i_\ell}}B/B \cong BwB/B,
\]
as is easily seen. \( \square \)

In the Demazure variety \( D(\tilde{w}) \), we have \( l(w) \) divisors \( Z_{\tilde{w}_1}, \ldots, Z_{\tilde{w}_{l(w)}} \). These are defined inductively on the length of \( w \), as follows.

Write \( \tilde{w} = \tilde{u}s_{i_\ell} \). We have a map \( \pi : D(\tilde{w}) \rightarrow D(\tilde{u}) \), which is a \( \mathbb{P}^1 \)-fibration, and we also have a section \( \sigma : D(\tilde{u}) \rightarrow D(\tilde{w}) \) of \( \pi \).

We define

\[
Z_{\tilde{w}_i} := \pi^{-1}(Z_{\tilde{u}_i}), \quad i = 1, \ldots, l(w) - 1
\]

\[
Z_{\tilde{w}_{l(w)}} := \sigma(D(\tilde{u})). \tag{3.3}
\]

We denote by \( Z_{\tilde{w}} \) the sum of the divisors \( Z_{\tilde{w}_i} \), and by \( P_{\tilde{w}} \) the intersection \( \bigcap_i Z_{\tilde{w}_i} \).

Lemma 3.3.3 Set-theoretically, \( P_{\tilde{w}} \) consists only of one point.

Proof. As \( \sigma \) is a section of \( \pi \), we have (set theoretically):

\[
Z_{\tilde{w}_i} \cap D(\tilde{u}) = Z_{\tilde{u}_i}, \quad i = 1, \ldots, l(w) - 1.
\]

So if \( \tilde{u} \neq 1 \), we have \( P_{\tilde{w}} = P_{\tilde{u}} \). But \( P_{\tilde{w}} \) is a point if \( \tilde{w} \) has length 1, so it is always only a point. It corresponds to the natural map \( D(1) \rightarrow D(\tilde{w}) \). \( \square \)

Lemma 3.3.4 The subvarieties \( Z_{\tilde{w}_i} \) are smooth of codimension 1 in \( D(\tilde{w}) \). In the tangent space \( T_{D(\tilde{w}), P_{\tilde{w}}} \) we have

\[
T_{Z_{\tilde{w}_i}, P_{\tilde{w}}} \cap \cdots \cap T_{Z_{\tilde{w}_{l(w)}}, P_{\tilde{w}}} = \{0\}.
\]

In particular, \( P_{\tilde{w}} \) is only a point even scheme-theoretically.

Proof. We prove the lemma by induction on the length of \( w \). Write \( \tilde{w} = \tilde{u}s_{i_\ell} \) as above. As the fibration \( D(\tilde{w}) \rightarrow D(\tilde{u}) \) is locally trivial, there exists a neighbourhood \( V \) of \( P_{\tilde{w}} \) in \( D(\tilde{u}) \) with a trivialization \( U := \pi^{-1}(V) = \mathbb{P}^1 \times V \).

Let \( \Delta := T_{\mathbb{P}^1, P} \subseteq T_{D(\tilde{w}), P} \).

Since \( \pi \) is locally trivial with smooth fibres, the smoothness of \( Z_{\tilde{w}_i} \) implies that \( Z_{\tilde{w}_i} \) is smooth as well \((i = 1, \ldots, l(w) - 1)\). Finally, also \( Z_{\tilde{w}_{l(w)}} = D(\tilde{u}) \) is smooth.

We have

\[
T_{Z_{\tilde{w}_i}, P} = T_{Z_{\tilde{u}_i}, P} \oplus \Delta, \quad i = 1, \ldots, l(w) - 1
\]

\[
T_{Z_{\tilde{w}_{l(w)}}, P} = T_{D(\tilde{u}), P}.
\]
Thus, for $i = 1, \ldots, l(w) - 1$ we have

$$T_{Z_d^i,p} \cap T_{Z_{l(w)}^i,p} = T_{Z_d^i,p},$$

and this immediately implies the lemma. □

**Lemma 3.3.5** If $\tilde{v} < \tilde{w}, l(v) = l(w) - 1$, then $D(\tilde{v})$ (considered as a closed subscheme of $D(\tilde{w})$ by the embedding defined above) is one of the $Z_i^\tilde{w}$. □

**Proof.** More precisely, it is easy to see by induction on $l(w)$ that $Z_j^\tilde{w}$ is the variety $P_i \times B \cdots \times B P_i$, where $P_j$ is left out. Thus all the $D(\tilde{v})$ appear as a $Z_i^\tilde{w}$. (Since (in general) not all the $s_i \cdots \check{s}_j \cdots s_i$ are reduced, not all of the $Z_i^\tilde{w}$ are of the form $D(\tilde{v})$.) □

### 3.3.2 The Canonical Bundle

As above, let $w \in W_a$ be arbitrary and choose a reduced decomposition $\tilde{w}$. We want to describe the canonical bundle of the Demazure variety $D(\tilde{w})$. To do this, we first define a certain line bundle on the Schubert variety $X_w$.

As above, we identify the affine flag variety with the space of $r$-special complete lattice chains. Again we denote by $(\lambda_i)$ the standard $r$-special lattice chain.

The Schubert variety $X_w$ consists of certain lattice chains $(L_i)$. We can find $N > 0$, such that all lattices occurring here lie between $t^{-N} k[[t]]^n$ and $t^N k[[t]]^n$.

Let $n_i = \dim_k \lambda_i / t^N k[[t]]^n$. We get maps

$$\varphi_i : X_w \longrightarrow \text{Grass}(t^{-N} k[[t]]^n / t^{N} k[[t]]^n, n_i), \ (\mathcal{L}_i)_i \mapsto \mathcal{L}_i / t^N k[[t]]^n.$$ 

These maps yield a closed embedding

$$\varphi : X_w \hookrightarrow \prod_{i=0}^{n-1} \text{Grass}(t^{-N} k[[t]]^n / t^{N} k[[t]]^n, n_i). \quad (3.4)$$

Now let $L_i$ be the very ample generator of the Picard group of $\text{Grass}(t^{-N} k[[t]]^n / t^N k[[t]]^n, n_i)$, and define

$$L_w := \varphi^* \bigotimes_{i=0}^{n-1} L_i.$$ 

This line bundle does not depend on $N$.

**Proposition 3.3.6** The line bundle $L_w$ on $X_w$ has the following properties:

i) If $u \in W_a$, such that $u \leq w$, i.e. $X_u \subseteq X_w$, then $L_u$ is the pull back of $L_w$.

ii) $L_w$ is very ample. □
Denote by $L_{\tilde{w}}$ the pull back of $L_w$ along the morphism $\Psi_{\tilde{w}} : D(\tilde{w}) \to X_w$. 

**Corollary 3.3.7** i) Let $\tilde{u} < \tilde{w}$, $l(\tilde{u}) = \tilde{w} - 1$. The pull back of $L_{\tilde{w}}$ along the embedding $\sigma : D(\tilde{u}) \to D(\tilde{w})$ is $L_{\tilde{u}}$.

ii) The line bundle $L_{\tilde{w}}$ does not have a base point.

**Proof.** i) Apply part i) of the previous proposition and proposition 3.3.2.

ii) It is easy to see that the pull back of any very ample line bundle under a morphism $Z \to Z'$, $Z' \neq \{pt\}$, is base point free. □

To establish a relation between $L_{\tilde{w}}$ and the canonical bundle of $D(\tilde{w})$, we need the following proposition.

**Proposition 3.3.8** Write $\tilde{w} = \tilde{u}s_i$. The degree of $L_{\tilde{w}}$ along the fibres of $\pi : D(\tilde{w}) \to D(\tilde{u})$ is 1.

**Proof.** Take a point $(g_1, \ldots, g_{i-1}) \in D(\tilde{u})$ and let $g = g_1 \cdots g_{i-1}$. The fibre over this point is isomorphic to $P_{\tilde{u}}/B$ and maps to $gP_{\tilde{u}}/B$ in $X_w$. Thus the degree along the fibre over $(g_1, \ldots, g_{i-1})$ equals the degree of the pull back of $L_w$ to $gP_{\tilde{u}}/B$.

We have a closed immersion $gP_{\tilde{u}}/B \to Grass(t^{-N}k[t]^{n}/t^{N}k[t]^{n}, n_{i})$, the image of which is the projective line $\mathbb{P}_g$ consisting of all subspaces of $t^{-N}k[t]/t^{N}k[t]$ (of dimension $n_i$) lying between $g\lambda_{i+1}/t^{N}k[t]^{n}$ and $g\lambda_{i-1}/t^{N}k[t]^{n}$ (note that $g\lambda_{i+1}$ and $g\lambda_{i-1}$ really lie between $t^{-N}k[t]^{n}$ and $t^{N}k[t]^{n}$, since $g(\lambda_i) \in X_w$). But in view of the lemma below, the Picard groups of the Grassmannian and of $\mathbb{P}_g$ are isomorphic (via restriction of line bundles), and this shows that $L_{\tilde{u}}|_{\mathbb{P}_g}$ has degree 1.

On the other hand, for $i \neq i_\ell$, the image of $gP_{\tilde{u}}/B$ under $\varphi_i : X_w \to Grass(t^{-N}k[t]^{n}/t^{N}k[t]^{n}, n_{i})$ is just a point, so $(\varphi_i^*L_{\tilde{u}})|_{\mathbb{P}_g}$ has degree 0. □

**Lemma 3.3.9** Let $V$ be a $k$-vector space, $n < \dim V$, $U \subseteq W \subseteq V$ subspaces, such that $\dim W = n + 1$, $\dim U = n - 1$. Denote by $P$ the projective line inside $Grass(V,n)$ consisting of all subspaces of $V$ lying between $U$ and $W$. Then restriction of line bundles gives an isomorphism of the Picard groups of $Grass(V,n)$ and $P$. □

We cite the following lemma from [Ram].

**Lemma 3.3.10** (Ramanathan) Let $\pi : X \to Y$ be a $\mathbb{P}^1$-bundle with $X$ and $Y$ smooth varieties. Let $\sigma : Y \to X$ be a section, $D$ the divisor $\sigma(Y)$ in $X$ and $L_D$ the line bundle $\mathcal{O}_X(D)$ corresponding to the divisor $D$.

i) The relative canonical bundle $\omega_{X/Y} = \omega_X \otimes \pi^*\omega_Y^{-1}$ is isomorphic to $L_D^{-2} \otimes \pi^*\sigma^*L_D$.

ii) If $L$ is any line bundle on $X$ whose degree along the fibres of $\pi$ is 1, then $\omega_{X/Y} \cong L_D^{-1} \otimes (L^{-1} \otimes \pi^*\sigma^*L)$. □
Now, the following description of the canonical bundle of $D(\tilde{w})$ is a purely formal consequence of our definitions, the corollary (part i)) and the proposition above.

**Proposition 3.3.11** The canonical bundle of $D(\tilde{w})$ is

$$\omega_{D(\tilde{w})} = \mathcal{O}(-Z^{\tilde{w}}) \otimes L_{\tilde{w}}^{-1}. $$

**Proof.** We do induction on the length of $w$. If $l(w) = 1$, then $D(\tilde{w}) = P_1/B \cong \mathbb{P}^1$. As $\mathcal{O}(-Z^{\tilde{w}}) \otimes L_{\tilde{w}}^{-1}$ has degree $-2$, it is the canonical bundle.

Now let $l(w) > 1$. Again, we write $\tilde{w} = \tilde{us}_i\pi, \pi: D(\tilde{w}) \to D(\tilde{u}), \sigma: D(\tilde{u}) \to D(\tilde{w})$ (see (3.2), (3.3)). First, note that

$$\omega_{D(\tilde{w})} = \pi^* \omega_{D(\tilde{u})} \otimes \omega_{D(\tilde{w})/D(\tilde{u})}. $$

Let $D$ be the divisor $D(\tilde{u})$ in $D(\tilde{w})$, and denote by $L_D$ the associated line bundle. As $L_{\tilde{w}}$ has degree 1 along the fibres of $\pi$, the preceding lemma gives us

$$\omega_{D(\tilde{w})/D(\tilde{u})} \cong L_D^{-1} \otimes (L_{\tilde{w}})^{-1} \otimes \pi^* \sigma^* L_{\tilde{w}}. $$

By induction hypothesis,

$$\omega_{D(\tilde{u})} = \mathcal{O}(-Z^{\tilde{u}}) \otimes (L_{\tilde{u}})^{-1}. $$

But by construction (see (3.3)),

$$\pi^* Z^{\tilde{u}} = Z^{\tilde{w}} - D. $$

Finally, we have

$$\sigma^* L_{\tilde{w}} = L_{\tilde{u}}. $$

Thus we get

$$\omega_{D(\tilde{w})} = \pi^* \omega_{D(\tilde{u})} \otimes \omega_{D(\tilde{w})/D(\tilde{u})}$$

$$\omega_{D(\tilde{w})} = \pi^* (\mathcal{O}(-Z^{\tilde{u}}) \otimes (L_{\tilde{u}})^{-1}) \otimes \omega_{D(\tilde{w})/D(\tilde{u})}$$

$$\omega_{D(\tilde{w})} = \mathcal{O}(-Z^{\tilde{u}}) \otimes L_D \otimes \pi^* (L_{\tilde{u}})^{-1} \otimes L_D^{-1} \otimes (L_{\tilde{w}})^{-1} \otimes \pi^* \sigma^* L_{\tilde{w}}$$

$$\omega_{D(\tilde{w})} = \mathcal{O}(-Z^{\tilde{u}}) \otimes \pi^* (L_{\tilde{u}})^{-1} \otimes (L_{\tilde{w}})^{-1} \otimes \pi^* L_{\tilde{u}}$$

$$\omega_{D(\tilde{w})} = \mathcal{O}(-Z^{\tilde{u}}) \otimes (L_{\tilde{w}})^{-1}. $$

This is precisely what we wanted. □

### 3.4 Normal Schubert Varieties are $F$-split

Now assume that our algebraically closed field $k$ has characteristic $p > 0$.

**Proposition 3.4.1** The Demazure variety $D(\tilde{w})$ admits a Frobenius splitting which compatibly splits all the divisors $Z_i^{\tilde{w}}$. 

3.4. NORMAL SCHUBERT VARIETIES ARE F-SPILT

Proof. We want to apply the criterion of Mehta and Ramanathan for Frobenius splitting (proposition 2.2.1).

By lemma 3.3.4, the divisors $Z_i^{\tilde{w}}$ satisfy the necessary conditions. By proposition 3.3.11 the canonical bundle on $D(\tilde{w})$ is $O(-Z) \otimes L_{\tilde{w}}^{-1}$. Since the point $P (= \bigcap_i Z_i^{\tilde{w}})$ is not a base point of $L_{\tilde{w}}$ by corollary 3.3.7 ii), we can find a global section $t$ of $L_{\tilde{w}}$, such that $P$ is not contained in the support of the (effective) divisor $\text{div}(t)$. But then we obviously get a global section $s$ of $\omega_{D(\tilde{w})}^{-1}$, such that

$$\text{div } s = Z_1^{\tilde{w}} + \cdots + Z_n^{\tilde{w}} + \text{div}(t),$$

and this shows that $D(\tilde{w})$ is Frobenius split, and that all the $Z_i^{\tilde{w}}$ are compatibly split. □

Corollary 3.4.2 Assume that $X_w \subseteq F$ is a normal Schubert variety. Then $X_w$ is F-split, and all Schubert varieties of codimension 1 in $X_w$ are simultaneously compatibly split.

Proof. Use proposition 2.1.4 and lemma 3.3.3. □

Lemma 3.4.3 Let $v, w \in W_a$, $v < w$, $l(v) = l(w) - 2$. Then there exist (precisely) two elements $v' \in W_a$ with $v < v' < w$.

Proof. This is proved for a finite Weyl group in [D1], lemma 7.7.6. The same proof applies for the affine Weyl group. □

Corollary 3.4.4 Assume that $X_w \subseteq F$ is a normal Schubert variety. Then $X_w$ is F-split, and all Schubert subvarieties in $X_w$ are simultaneously compatibly split.

Proof. This follows from the previous corollary by induction on the codimension of the Schubert subvariety. The corollary says that the codimension 1 Schubert subvarieties are compatibly split. Now assume we knew the codimension $i$ Schubert subvarieties to be compatibly split and take one of codimension $i + 1$, say $Y$. Of course, there is a Schubert variety $X' \subseteq X$ of codimension $i - 1$ which contains $Y$. By the lemma above, we find Schubert varieties $X'_1$ and $X'_2$ in $X'$, such that $X'_1 \neq X'_2$ and $Y \subseteq X'_1 \cap X'_2$. Because of dimension reasons, $Y$ must be an irreducible component of $X'_1 \cap X'_2$ and thus is compatibly split as well by lemma 2.1.2. □

Remark. It should be expected that all Schubert varieties are normal. This is true in the context of Kac-Moody algebras, cf. [M].

In any case, it is clear that the theorem holds for all Schubert varieties that are embedded in a normal Schubert variety. In fact, it follows from the above that all those Schubert varieties are themselves normal (see the next section).
3.5 Consequences

The fact that normal Schubert varieties are Frobenius split allows one to draw conclusions about their singularities. First of all, we have

**Proposition 3.5.1** Let \( X_w \subseteq F \) be a normal Schubert variety. Then all Schubert subvarieties of \( X_w \) are normal.

*Proof.* Let \( X_u \) be a Schubert subvariety of \( X_w \). Then \( X_u \) is Frobenius split, hence to prove that \( X_u \) is normal it is enough to find a normal variety \( D \) and a surjection \( D \to X_u \) with connected fibres (see [MS]). Clearly we can take the Demazure variety \( D(\tilde{u}) \) for some reduced expression \( \tilde{u} \) of \( u \). \( \square \)

Furthermore, we show that normal Schubert varieties have rational singularities. More precisely:

**Theorem 3.5.2** Let \( X_w \subseteq F \) be a normal Schubert variety, and choose a reduced expression \( \tilde{w} \). Then the morphism \( \Psi_{\tilde{w}} : D(\tilde{w}) \to X_w \) is a rational resolution. In particular, \( X_w \) is Cohen-Macaulay.

*Proof.* The proof of theorem 4 in [Ram] in principle works in our situation as well. The main steps of the proof are the following:

We have to show that

1. \( \Psi_* O_{D(\tilde{w})} = O_{X_w} \),
2. \( R^q \Psi_* O_{D(\tilde{w})} = 0 \) for \( q > 0 \),
3. \( R^q \Psi_* \omega_{D(\tilde{w})} = 0 \) for \( q > 0 \).

A morphism that satisfies the first two conditions is called trivial. The first point is clearly fulfilled since \( X_w \) is normal and \( \Psi_{\tilde{w}} \) is proper and birational.

The third point follows from the Grauert-Riemenschneider theorem for Frobenius split varieties (see [MvK]).

The second point will be proved by induction on the length of \( \tilde{w} \). We will follow the proof of theorem 4 in [Ram].

Write \( \tilde{w} = \tilde{u}s_\ell \). The following diagram is cartesian:

\[
\begin{array}{ccc}
D(\tilde{w}) & \xrightarrow{\Psi_{\tilde{w}}} & X_w \\
\downarrow \text{pr} & & \downarrow \text{pr} \\
D(\tilde{u}) & \xrightarrow{\Psi_{\tilde{u}}} & X_u & \xrightarrow{\text{pr}} & \text{pr}(X_w)
\end{array}
\]
Here \( pr \) denotes the projection \( \mathcal{F} \to SL_n(k((t)))/P_i \).

Since the composition of trivial morphisms is trivial and triviality is stable under flat base change, it is enough to show that \( pr : X_u \to pr(X_w)(= pr(X_u)) \) is trivial. To show this, we apply the following criterion of Kempf (compare [Ram], prop. 3).

**Lemma 3.5.3** Let \( f : X \to Y \) be a proper morphism of algebraic varieties and let \( L \) be an ample line bundle on \( Y \) such that \( f_* \mathcal{O}_X = \mathcal{O}_Y \) and \( H^q(X, f^* L^n) = 0 \) for \( q > 0 \) and \( n \) large. Then \( f \) is trivial. \( \square \)

Consider the embedding (3.4):

\[
\varphi : X_u \hookrightarrow \prod_{i=0}^{n-1} Grass(t^{-N}k[[t]]^n/t^Nk[[t]]^n, n_i).
\]

We denote the line bundle \( \varphi_i^* L_i \) on \( X_u \) by \( L_{u,i} \). Note that the morphism \( \varphi_i : X_u \to Grass(t^{-N}k[[t]]^n/t^Nk[[t]]^n, n_i) \) factors through \( pr(X_u) \). The pull-back \( L_{i}' \) of the line bundle \( L_i \) to \( pr(X_u) \) is very ample.

We want to apply the lemma above to the morphism \( pr : X_u \to pr(X_w) \) and the line bundle \( L_{i}' \). It can be shown as in the proof of theorem 2 in [Ram] that \( H^q(X_u, L_{u,i}) = 0 \) for \( q > 0 \), and thus the hypothesis of the lemma is fulfilled. \( \square \)

The theorem holds also in characteristic 0. Probably this can be derived from the results in positive characteristic by some kind of continuity argument, but I have not thoroughly checked this. In any case one can use the results of Mathieu in [M] since it is known that in characteristic 0 the affine flag manifolds coincide.
Chapter 4

The Local Model

4.1 Definition of the Standard Local Model

Let $O$ be a complete discrete valuation ring with perfect residue class field. Let $\pi$ be a uniformizer of $O$ and let $k$ be an algebraic closure of the residue class field of $O$.

Denote the quotient field of $O$ by $K$. Let $e_1, \ldots, e_n$ be the canonical basis of $K^n$.

Let $\Lambda_i$, $0 \leq i \leq n - 1$, be the free $O$-module of rank $n$ with basis $e_i^1 := \pi^{-1} e_1, \ldots, e_i^i := \pi^{-1} e_i, e_{i+1}^i := e_{i+1}, \ldots, e_n^i := e_n$. This yields a complete lattice chain

\[ \cdots \rightarrow \Lambda_0 \rightarrow \Lambda_1 \rightarrow \cdots \rightarrow \Lambda_{n-1} \rightarrow \pi^{-1} \Lambda_0 \rightarrow \cdots \]

Fix a dominant minuscule cocharacter $\mu = (1^r, 0^{n-r})$ of $GL_n$ (with respect to the torus of diagonal matrices and the Borel subgroup of upper triangular matrices).

Furthermore choose $I = \{i_0 < \cdots < i_{m-1}\} \subseteq \{0, \ldots, n-1\}$.

The standard local model $M_I^{\text{loc}}$ is the $O$-scheme representing the following functor (cf. [RZ], definition 3.27):

For every $O$-scheme $S$, $M_I^{\text{loc}}(S)$ is the set of isomorphism classes of commutative diagrams

\[
\begin{array}{c}
\Lambda_{i_0,S} \rightarrow \Lambda_{i_1,S} \rightarrow \cdots \rightarrow \Lambda_{i_{m-1},S} \rightarrow \Lambda_{i_0,S} \\
\downarrow \quad \downarrow \quad \cdots \quad \downarrow \quad \downarrow \quad \downarrow \\
F_0 \rightarrow F_1 \rightarrow \cdots \rightarrow F_{m-1} \rightarrow F_0
\end{array}
\]

where $\Lambda_{i,S}$ is $\Lambda_i \otimes_O O_S$, and where the $F_\kappa$ are locally free $O_S$-submodules of rank $r$ which Zariski-locally on $S$ are direct summands of $\Lambda_{i,\kappa,S}$. 
It is clear that this functor is indeed representable. In fact, $M_{I}^{\text{loc}}$ is a closed subscheme of a product of Grassmannians.

We write $M_{I}^{\text{loc}} := M_{\{0, \ldots, n-1\}}^{\text{loc}}$. Furthermore, we will often write $M_{\{i_0, \ldots, i_m-1\}}^{\text{loc}}$ instead of $M_{\{0, \ldots, n-1\}}^{\text{loc}}$.

As the stabilizer of the complete lattice chain is the Iwahori subgroup, and the stabilizer of a partial lattice chain is a parahoric subgroup, we speak also of the local model in the Iwahori case resp. in the parahoric case.

Note the following obvious fact:

**Lemma 4.1.1** Let $M_{I}^{\text{loc}}$ be the local model over $O$ as above, and let $M_{I}^{\text{loc}}'$ be the local model over $k[[t]]$. Then the special fibres of $M_{I}^{\text{loc}}$ and $M_{I}^{\text{loc}}'$ are the same. □

### 4.2 The Standard Local Model and the Affine Flag Variety

We can embed the special fibre $M_{I}^{\text{loc}}$ of $M_{I}^{\text{loc}}$ in the affine flag variety (over $k$) as follows:

We identify $F$ with the space of $(n-r)$-special lattice chains. Let $R$ be a $k$-algebra and write

$$
\lambda_0 = R[[t]]^n, \lambda_1 = t^{-1}R[[t]]^1 \oplus R[[t]]^{n-1},
\lambda_2 = t^{-1}R[[t]]^{n-1} \oplus R[[t]]^1.
$$

Let $(\mathcal{F}_i)_i$ be an $R$-valued point of $M_{I}^{\text{loc}}$. Then $\mathcal{F}_i$ is a subspace in $\Lambda_i = R^n \cong \lambda_i/t\lambda_i$. Let $L_i$ be the inverse image of $\mathcal{F}_i$ under the canonical projection $\lambda_i \rightarrow \lambda_i/t\lambda_i$. This gives us a complete lattice chain $(L_i)_i$.

**Lemma 4.2.1** The complete lattice chain defined above is $(n-r)$-special. □

Thus we have defined a point of $\mathcal{F}$ and we get a closed immersion

$$i : M_{I}^{\text{loc}} \rightarrow \mathcal{F}. \quad (4.1)$$

Via $i$, $M_{I}^{\text{loc}}$ is identified with the closed subscheme of $\mathcal{F}$ consisting of those lattice chains $(L_i)_i$ with $\lambda_i \supseteq L_i \supseteq t\lambda_i$ for all $i$.

In the same way we get closed immersions $M_{I}^{\text{loc}} \rightarrow SL_n(k((t)))/P^I$ for each subset $I \subseteq \{0, \ldots, n-1\}$, where $P^I$ denotes the (elementwise) stabilizer of the lattice chain $(\lambda_i)_{i \in I}$. It is a parahoric subgroup of $SL_n(k((t)))$. Of course, $M_{I}^{\text{loc}}$ denotes the special fibre of $M_{I}^{\text{loc}}$. 
4.3 The Stratification of the Special Fibre

Consider $\overline{M}_{\text{loc}}$ as a closed subscheme of $F$. It is invariant under the action of the Iwahori subgroup $B$ and thus set theoretically is a union of Schubert varieties. Thus the decomposition into Schubert cells gives us a stratification of the special fibre of the local model.

Consider the Bruhat-Tits building of $SL_n$ over $k((t))$. Identify the vertices of the standard apartment with $\mathbb{Z}^n/\mathbb{Z}$, such that the lattice generated by $t^{-r_1}e_1, \ldots, t^{-r_n}e_n$ corresponds to $(r_1, \ldots, r_n)$. Let $\omega = (\omega_1, \ldots, \omega_n)$ be the base alcove, i.e. $\omega_i = (1^i, 0^{n-i})$. Denote by $\tau$ the alcove $((1^r, 0^{n-r}), (1^{r+1}, 0^{n-r-1}), \ldots, (2^r, 1^{n-r}))$.

Furthermore, recall the following definitions from the article [KR] by Kottwitz and Rapoport.

**Definition 4.3.1** Let $x = (x_1, \ldots, x_n)$ be an alcove.

i) The number $\sum x_i - \sum \omega_i = \sum_j x_i(j) - \sum_j \omega_i(j)$ is independent of $i$ and is called the size of $x$.

ii) We say that $x$ is minuscule if $0 \leq x_i(m) - \omega_i(m) \leq 1$ for all $i \in \{0, \ldots, n-1\}, \ m \in \{1, \ldots, n\}$.

iii) We say that $x$ is $\mu$-admissible, if $x \leq w(\mu)$ for some $w \in W$, where $W$ denotes the finite Weyl group.

In fact, an alcove is $\mu$-admissible if and only if it is minuscule of size $r$ (see [KR], theorem 3.5).

Now let $(F_i)_i \in M_{\text{loc}}(k)$. Let $(L_i)_i$ be the associated lattice chain in $F$, as above. Then there is a uniquely determined element $w$ in the affine Weyl group $W_a$ (referred to as the relative position of $(L_i)_i$ and $\tau$), and an element $b$ in the Iwahori subgroup $B$, such that $(L_i)_i = bw\tau$. We say that $w\tau$ is the alcove associated to the point $(F_i)_i$.

The alcoves which occur here are just the minuscule ones (of size $r$), in other words the $\mu$-admissible alcoves.

We get

$$\overline{M}_{\text{loc}} = \bigcup_{x \text{ $\mu$-adm.}} S_x,$$

where $S_x = BwB/B$ is the Schubert cell associated to $x = w\tau$.

If $x = w\tau$ is a $\mu$-admissible alcove, then we denote by $l(x)$ the length of $w \in W_a$.

**Lemma 4.3.2** The stratification has the following properties:

i) The strata are just the orbits of the action of $B$ on $M_{\text{loc}}$.

ii) We have $S_x \subseteq \overline{S}_y$ if and only if $x \leq y$ with respect to the Bruhat order.

iii) The dimension of $S_x$ is $\dim S_x = l(x)$.  \(\square\)
The stratum corresponding to the alcove \( \tau \) only consists of one point. This is the worst singularity of \( \overline{M}^{\text{loc}} \). (While it is difficult to give a precise meaning to the term ‘worst singularity’, it should be intuitively clear what is meant: As the singularities cannot become better under specialization, the worst singularity has to appear in the one-point stratum \( S_\tau \), which lies in the closure of every other stratum. In other words, if \( \overline{M}^{\text{loc}} \) has a certain nice property in the point \( S_\tau \) (for example reducedness), then this should hold everywhere.)

We want to associate to each \( \mu \)-admissible alcove \( x \) an open subset \( U_x \) of \( M^{\text{loc}} \) which contains the corresponding stratum.

Let \( x = w \tau = (x_1, \ldots, x_n) \) be a \( \mu \)-admissible alcove. If \( (F_i)_i \in S_x \), then we have

\[
\mathcal{L}_i = b \cdot \begin{pmatrix}
  t^{-x_i(1)+1} & & & \\
  & t^{-x_i(2)+1} & & \\
  & & \ddots & \\
  & & & t^{-x_i(n)+1}
\end{pmatrix},
\]

for some \( b \) in the Iwahori subgroup. Here we think of the matrix on the right hand side as the submodule of \( k[[t]]^n \) generated by the column vectors (with respect to the canonical basis \( e_1, \ldots, e_n \)).

Again let \( \lambda_i = t^{-1} k[[t]]^i \oplus k[[t]]^{n-i} \). Denote by \( e^i_1 = t^{-1} e_1, \ldots, e^i_n = e_n \) the canonical basis of \( \lambda_i \).

Then \( \omega \) corresponds to the lattice chain \( (\lambda_i)_i \), i.e. \( (1^i, 0^{n-i}) \) corresponds to \( \lambda_i \). As we want to consider \( \mathcal{L}_i \) as a submodule of \( \lambda_i \) and as \( \omega_i \leq x_i \leq \omega_i + (1, \ldots, 1) \), we indeed have to take \(-x_i(\cdot) + 1\) instead of \(-x_i(\cdot)\) as the exponent in the matrix above.

The above description of \( \mathcal{L}_i \) shows that the quotient \( \lambda_i / \mathcal{L}_i \) is generated by those \( e^i_j \) with \( \omega_i(j) - (x_i(j) - 1) = 1 \), i.e. with \( \omega_i(j) = x_i(j) \).

Thus the open subset of \( \overline{M}^{\text{loc}} \), where for all \( i \) the quotient \( \lambda_i / \mathcal{L}_i \) is generated by those \( e^i_j \) with \( \omega_i(j) = x_i(j) \), contains the stratum \( S_x \).

We want to define an open subset of \( M^{\text{loc}} \) which contains the stratum \( S_x \). Thus we consider more generally the quotients \( \Lambda_i / F_i \) and define (compare lemma 4.1.1):

**Definition 4.3.3** Let \( x = (x_1, \ldots, x_n) \) be a minuscule alcove of size \( r \). Then let \( U_x \) be the open subset of \( M^{\text{loc}} \) which consists of all points \( (F_i)_i \), such that for all \( i \) the quotient \( \Lambda_i / F_i \) is generated by those \( e^i_j \) with \( \omega_i(j) = x_i(j) \).

We have

**Proposition 4.3.4** i) The stratum \( S_x \) is contained in \( U_x \).

ii) The open subset \( U_\tau \) intersects every stratum.

iii) The irreducible components of the special fibre \( \overline{M}^{\text{loc}} \) are the closures of the \( U_x \cap \overline{M}^{\text{loc}} \), where \( x \) is an extreme alcove, i.e. \( x = w(\mu) \) for some \( w \in W \).
Proof. Part i) follows from the discussion above, and ii) and iii) follow from the lemma. □

Of course, we can just as well characterize the irreducible components of \( M_{loc} \) as the closures of the strata \( S_x, x \in W(\mu) \). Thus \( M_{loc} \) has \( \# W/W_\mu = \binom{n}{r} \) irreducible components, where \( W_\mu \subseteq W \) denotes the stabilizer of \( \mu \). A similar description can be given in the parahoric case.

Finally, we note the following lemma:

**Lemma 4.3.5** Let \( x = (x_1, \ldots, x_n) \) be a minuscule alcove of size \( r \). Let \( t_i = x_i - \omega_i, i = 0, \ldots, n - 1 \). Then

i) For all \( j \), \( (t_1(j), \ldots, t_n(j)) \) is a cyclic permutation of \( (1^\kappa, 0^{n-\kappa}) \) for some \( \kappa \).

ii) If \( t_i(j) = 1 \) and \( t_{i+1}(j) = 0 \), then \( \phi_i(e_j^i) = \pi e_j^{i+1} \), where \( \phi_i \) is the map \( \Lambda_i \rightarrow \Lambda_{i+1} \). □

Of course, we can do similar things in the parahoric case. We then have to work with 'partial alcoves' \( x = (x_i)_{i \in I} \) (cf. also [KR], §9). The lemma above holds then in an analogous form.

### 4.4 The Equations of the Standard Local Model \( M^{loc} \)

We want to compute the equations describing the standard local model.

#### 4.4.1 More General Schemes of Compatible Subspaces

To establish the relation between the local model and certain spaces of homomorphism, as will be done in the next section, it is useful to introduce more general 'schemes of compatible subspaces'.

Let \( m, n > 0, 0 < r < m \), and consider free \( O \)-modules \( \Lambda_i, i = 0, \ldots, m - 1 \), of rank \( n \) with bases \( \mathcal{E}^i = (e_1^i, \ldots, e_n^i) \)

Take \( O \)-linear maps \( \phi_i : \Lambda_i \rightarrow \Lambda_{i+1}, i = 0, \ldots, m - 1 \) (\( \Lambda_m := \Lambda_0 \)).

Then we denote by \( M(m, n, r, (\phi_i)_i) \) the functor which associates to an \( O \)-scheme \( S \) the set of isomorphism classes of commutative diagrams

\[
\Lambda_{0,S} \xrightarrow{\phi_0} \Lambda_{1,S} \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{m-1}} \Lambda_{m-1,S} \xrightarrow{\phi_{m-1}} \Lambda_{0,S}
\]

\[
\bigcup_{i} F_i \quad \bigcup_{i} F_i \quad \bigcup_{i} F_i \quad \bigcup_{i} F_i \quad \bigcup_{i} F_i
\]

where as before \( \Lambda_{i,S} \) is \( \Lambda_i \otimes_O O_S \), and where the \( F_i \) are locally free \( O_S \)-submodules of rank \( r \) which Zariski-locally on \( S \) are direct summands of \( \Lambda_{i,S} \).
Again, this functor is representable by a closed subscheme of a product of Grassmannians. Obviously the local models occur as a special case of this definition. It seems to be very difficult to describe these schemes in general. The only case we will need below is the following:

**Definition 4.4.1** The scheme $M(m, n, r, (\phi_i)_i)$ is called a generalized local model, if the $\phi_i$ are diagonal matrices with only 1's and $\pi$'s on the diagonal (with respect to the fixed bases $e_i$) and their composition is $\pi$.

The only new possibility here is that $\phi_i = \text{id}$ is allowed. Strictly speaking, this does not happen for the local models. On the other hand, as the steps where $\phi_i$ is an isomorphism can be neglected, these schemes still are isomorphic to certain (parahoric) local models. Thus, in the cases we will consider, the new definition really is only another notation.

Now we want to define certain open subsets in the generalized local model $M(m, n, r, (\phi_i)_i)$. For $i = 0, \ldots, m - 1$, choose $t_i \in \{0, 1\}^n$ such that $\#\{j; t_i(j) = 1\} = r$, and such that the following two conditions are satisfied:

**Condition 4.4.2**

i) For all $j$, $(t_1(j), \ldots, t_n(j))$ is a cyclic permutation of $(1^\kappa, 0^{n-\kappa})$ for some $\kappa$.

ii) If $t_i(j) = 1$ and $t_{i+1}(j) = 0$, then $\phi_i(e_j^i) = \pi e_{j+1}^i$.

Let $U = U((t_i)_i)$ be the open subset of $M(m, n, r, (\phi_i)_i)$, where $\Lambda_{i,S}/F_i$ is generated by the $e_j^i$ with $t_i(j) = 0$.

The open subsets $U_x$ associated to an admissible alcove defined above are a special case of this definition (cf. lemma 4.3.5).

Finally, we state the following lemma.

**Lemma 4.4.3** We have

$$M(m, n, r, (\phi_i)_i) \cong M(m, n, n - r, (\phi_i)_i).$$

*Proof.* Replace $\mathcal{F}_i$ with the dual of $\Lambda_i/F_i$. \qed

### 4.4.2 Interpretation in Terms of Homomorphisms

We will see that the open subsets $U((t_i)_i)$ of a generalized local model can be related to certain spaces of homomorphisms between free modules. This will enable us to read off the equations of the local models almost immediately.

As the combinatorics involved here is quite complicated, it is probably more enlightening to look at the statement of theorem 4.4.7 and then to try to figure out the equations of the parahoric local model $M_{\mu, \kappa}^{\text{loc}}$ directly rather than to go
through the proof of the theorem. The result for this special case is stated in section 4.4.3.

Consider a generalized local model \( M = M(m, n, r, (\phi_i)_i) \).

We will represent all the subspaces \( F_i \) by giving \( r \) generating vectors (with respect to the basis \( e^i_1, \ldots, e^i_n \) of \( \Lambda_i \)) which we will arrange as column vectors in a matrix.

We want to study an open subset of the form \( U := U((t_i)_i) \), where \( (t_i)_i \) satisfies the condition 4.4.2. Choose permutations \( \sigma_i \in S_n \) such that \( t_i = \sigma_i(1^r, 0^{n-r}) \).

The conditions defining \( U \) can be stated in the following way: \( U \) is the open subset of \( M \) consisting of those \( (F_i)_i \), such that \( F_i \) can be described by a matrix

\[
M_i := (b^i_{jk})_{j=1,\ldots,n, \ k=1,\ldots,r} := \sigma_i \begin{pmatrix}
1 & 1 & \cdots & 1 \\
a^i_{11} & a^i_{12} & \cdots & a^i_{1r} \\
\vdots & \vdots & \ddots & \vdots \\
a^i_{n-r,1} & a^i_{n-r,2} & \cdots & a^i_{n-r,r}
\end{pmatrix},
\]

i. e. we have a unit matrix in the rows \( \sigma_i(1), \ldots, \sigma_i(r) \). Note that the \( a^i_{jk} \) are uniquely determined by \( F_i \).

The condition that \( F_i \) is mapped under \( \phi_i \) into \( F_{i+1} \) can be expressed in terms of matrices as follows: If \( F_i \) (resp. \( F_{i+1} \)) is described by \( M_i \) (resp. \( M_{i+1} \)), then we must have

\[
\phi_i M_i = M_{i+1} N_i,
\]

where \( N_i \) is a certain \( r \times r \)-matrix. In fact, \( N_i \) is uniquely determined by \( M_i \) and \( M_{i+1} \) since certain rows of \( M_{i+1} \) form a unit matrix.

**The first lemma**

Let

\[
S := \{ i \in \{1, \ldots, n\}; \ t_i(i) = 0 \text{ for all } i \}, \ s := \#S.
\]

We will show that \( U \) is isomorphic to the product of an open subset of some generalized local model \( M(m, n-s, r, (\psi_i)_i) \) with \( \mathbb{A}^s \).

For all \( i \) let \( \Lambda_i := \Lambda_i / \langle e^i_j; \ j \in S \rangle. \) Denote the map \( \Lambda_i \rightarrow \Lambda_{i+1} \) induced by \( \phi_i \) by \( \psi_i \).

Let \( u_i := 't_i \) with the entries \( t_i(j), j \in S, \) left out’, such that \( u_i \) is a permutation of \((1^r, 0^{n-r-s})\). The conditions 4.4.2 are satisfied again (with respect to the \( \psi_i \)).

**Lemma 4.4.4** We have

\[
U \cong V \times \mathbb{A}^s,
\]

where

\[
V := U((u_i)_i) \subseteq M(m, n-s, r, (\psi_i)_i).
\]
Proof. We construct a surjection \( U \rightarrow V \). To define the map \( U \rightarrow V \), take \((\mathcal{F}_i)_i\) in \( U(R) \) for some \( O \)-algebra \( R \), and consider the associated matrices \( M_i \).

We then define a point \((\mathcal{F}_i)_i \in V\) in the following way: The matrix associated to \( \mathcal{F}_i \) (with respect to the basis \( e_j^i, j \not\in S \)) is the matrix \( M_i' \) obtained from \( M_i \) by deleting the rows with index in \( S \). Then \( M_i' \) describes a subspace of \( \overline{\mathcal{X}}_{i,R} \). Note that \( \mathcal{F}_i \) again has rank \( r \). In this way we get a morphism \( U \rightarrow V \).

What are the fibres of this map? We have to check how to construct matrices \( M_i \), given matrices \( M_i' \). To do this, we have to fill in the entries \( b^i_{jk} \) in the \( j \)-th row of \( M_i \) for \( j \in S \). So we fix \( j \in S \) and consider the equations which arise for these \( b^i_{jk} \)'s.

We distinguish the following two cases that yield two basically different types of equations:

First case: \( \phi_i(e_j^i) = \pi e_j^{i+1} \).
Second case: \( \phi_i(e_j^i) = e_j^{i+1} \).

The first case occurs precisely once, say for \( i = i_0 \), and the condition that \( \mathcal{F}_{i_0} \) is mapped into \( \mathcal{F}_{i_0+1} \) gives equations of the form \((\kappa = 1, \ldots, r)\)

\[
\pi b^{i_0}_{j\kappa} = \text{something depending on } b^{i_0+1}_{j_1}, \ldots, b^{i_0+1}_{j_r} \text{ and } b^{i_0}_{\ell\kappa} \text{ with } \ell \not\in S.
\]

The second case yields equations of the form \((\kappa = 1, \ldots, r)\)

\[
b^{i}_{j\kappa} = \text{something depending on } b^{i+1}_{j_1}, \ldots, b^{i+1}_{j_r} \text{ and } b^{i}_{\ell\kappa} \text{ with } \ell \not\in S.
\]

Thus, if we choose \( b^{i_0}_{j_1}, \ldots, b^{i_0}_{j_r} \in R \), all \( b^i_{jk} \) are uniquely determined by the equations derived from the second case. The only question is, if the equation given in the first case is satisfied. But it follows from the fact that all the compositions of the \( \phi_i \) are \( \pi \) that this equation is automatically satisfied. Hence we can indeed choose the \( b^{i_0}_{j_1}, \ldots, b^{i_0}_{j_r} \) arbitrarily. So 'leaving out' the \( j \)-th row gives us a fibration with fibres isomorphic to \( \mathbb{A}^r \), and as \( S \) has \( s \) elements, we see that all fibres of the map \( U \rightarrow V \) (over \( R \)-valued points) are isomorphic to \( \mathbb{A}^{rs} \). Hence we get \( U \cong V \times \mathbb{A}^{rs} \). \( \Box \)

The second lemma
Let \( U = U((t_i)_i) \subseteq M(m, n, r, (\phi_i)_i) \) be as above.

Let

\[
T := \{ t \in \{1, \ldots, n\}; \ t_i(t) = 1 \text{ for all } i \}, \ t := \#T.
\]

In this second step we show that \( U = U((t_i)_i) \) is isomorphic to a product of an open subset of some \( M(m, n - t, r - t, (\xi_i)_i) \) with \( \mathbb{A}^{l(n-r)} \).

Let \( \overline{\Lambda}_i := \Lambda_i/(e^i_j; \ j \in T) \). Denote the map induced by \( \phi_i \) by \( \xi_i \). Let \( v_i := 't_i \) with the entries \( t_i(j), j \in T \), left out’. Thus \( v_i \) is a permutation of \((1^{r-t}, 0^{n-r})\), and the conditions \footnote{4.4.2} are satisfied again (with respect to the \( \xi_i \)).
Lemma 4.4.5 We have
\[ U \cong W \times \mathbb{A}^{t(n-r)}, \]
where
\[ W := U((v_i)_i) \subseteq M(m, n-t, r-t, (\xi_i)_i). \]

Proof. We follow the same strategy as before: first, we define a map \( U \to W \), then we examine its fibres.

The map \( U \to W \) is defined as follows. Take an \( R \)-valued point \( (\mathcal{F}_i)_i \) in \( U \). The \( \mathcal{F}_i \) correspond to matrices \( M_i \). Consider \( j \in T \). In each \( M_i \), the \( j \)-th row consists of \( r-1 \) 0’s and one 1, say in column \( k^i_j \). Construct matrices \( M'_i \) as follows: For all \( j \in T \), delete the \( j \)-th row from \( M_i \), and also delete the \( k^i_j \)-th column. We get a \((n-t) \times (r-t)\)-matrix, and if we denote by \( \mathcal{F}'_i \) the subspace of \( \mathcal{N}_i \) defined by \( M'_i \), we have \( (\mathcal{F}'_i)_i \in W \).

Now suppose we are given matrices \( M'_i \) associated to \( (\mathcal{F}'_i)_i \in W(R) \) and we want to define matrices \( M_i \) that give an element in \( U(R) \). Then for each \( j \in T \) we have to choose entries \( a_{i,k^i_j}^j, \ldots, a_{n-r,k^i_j}^i \) such that the corresponding subspaces are mapped into one another. Note that \( \{a_{1,k^i_j}^1, \ldots, a_{n-r,k^i_j}^i\} = \{b^i_{1,k^i_j}; t_i(\tau) = 0\} \).

We fix \( j \in T \). Again, two different types of equations appear:

First case: \( \phi_i(e^i_j) = \pi e^i_{j+1} \).

Second case: \( \phi_i(e^i_j) = e^i_{j+1} \).

As before, the first case occurs precisely once, say for \( i = i_0 \). We then get equations of the form
\[ b_{i,k^i_j}^{i_0} = \pi b_{i,k^i_j}^{i_0+1} + \text{something depending on } b_{i,k^i_j}^{i_0} \text{ and } b_{i_0,k^i_j}^{i_0+1}, \quad \tau = 1, \ldots, n. \]

In the second case, the equations have the form
\[ b_{i,k^i_j}^i = b_{i,k^i_j}^{i+1} + \text{something depending on } b_{i,k^i_j}^i \text{ and } b_{i,k^i_j}^{i+1}, \quad \tau = 1, \ldots, n. \]

One sees easily that therefore for \( i \neq i_0 \), \( b_{i,k^i_j}^{i+1} \) is determined by \( b_{i,k^i_j}^i \) and some other \( b \)'s which are 'not related to \( T' \). This yields an isomorphism between the fibre and \( \mathbb{A}^{t(n-r)} \): choose the \( b_{i,k^i_j}^{i_0+1} \) with \( \tau \) such that \( t_{i_0+1}(\tau) = 0 \) arbitrarily, then determine the other \( b_{i,k^i_j}^i \) (\( \tau \) such that \( t_{i_0}(\tau) = 0 \)) by the equations from the second case. The equations of the second case where \( t_{i_0}(\tau) = 1 \) and the equations of the first case are automatically satisfied.

This proves the lemma.

The third lemma

Again, let \( U = U((t_i)_i) \subseteq M(m, n, r, (\phi_i)_i) \) be as above. Furthermore we now assume that
\[ \{\tau \in \{1, \ldots, n\}; t_i(\tau) = 1 \text{ for all } i\} = \emptyset. \]
4.4. The Equations of the Standard Local Model $\mathbf{M}^{\text{LOC}}$

By the second lemma we can write $U$ as a product of an affine space and a new open subset of a generalized local model which has this property.

We will show that in this case $U$ can be described as a space of $m$-tuples of homomorphisms the compositions of which are $\pi$.

If $(\mathcal{F}_i)_i \in U$, then by the definition of $U$, for all $i$ the vectors $e^i_j$ with $t_i(j) = 0$ give a basis of $\Lambda_i/\mathcal{F}_i$. Here by an abuse of notation we write $\Lambda_i/\mathcal{F}_i$ instead of $\Lambda_i\mathcal{S}/\mathcal{F}_i$ for variable $S$. With respect to these bases the map $\Lambda_i/\mathcal{F}_i \to \Lambda_{i+1}/\mathcal{F}_{i+1}$ induced by $\phi_i$ is described by a $(n-r) \times (n-r)$-matrix $X_i$. The columns of the matrix correspond to the $e^i_j$ with $t_i(j) = 0$. We denote these $j'$s by $j_1', \ldots, j_{n-r}'$.

Thus we get a map

$$\Phi : U \to \prod_{i=0}^{m-1} \text{Mat}_{n-r}(O).$$

We want to determine the image of this map.

It is clear that each $X_i$ underlies the following restriction: If $t_i(j) = 0$ and $t_{i+1}(j) = 0$, then $e^i_j$ and $e^{i+1}_j$ are part of the chosen bases of $\Lambda_i/\mathcal{F}_i$ and $\Lambda_{i+1}/\mathcal{F}_{i+1}$. Now $e^i_j$ is mapped to $e^{i+1}_j$ (or $\pi e^{i+1}_j$) under the map $\Lambda_i \to \Lambda_{i+1}$, and the same must be true for $X_i : \Lambda_i/\mathcal{F}_i \to \Lambda_{i+1}/\mathcal{F}_{i+1}$. This observation leads to the following definition:

Let $\varepsilon^i_j = 0$, if $\phi_i(e^i_j) = e^{i+1}_j$ and $\varepsilon^i_j = 1$, if $\phi_i(e^i_j) = \pi e^{i+1}_j$.

Then

$$X_i := \{X = (x_{\mu,\nu})_{\mu,\nu} \in \text{Mat}_{n-r}(O), \text{ such that for all } i = 1, \ldots, n-r \text{ with } t_{i+1}(j'_i) = 0, \text{ so } j'_i = j_{i'}^{i+1} \text{ for some } i', \text{ we have (write } j := j^i_j) : \}
\begin{align*}
x_{\mu,\nu} &= \pi^{i+j} \delta_{\mu,\nu'}, \quad \mu = 1, \ldots, n-r. \}
\end{align*}$$

So $X_i \subseteq \text{Mat}_{n-r}$ is an affine subspace and the image of $\Phi$ lies in $\prod X_i$.

The image of $\Phi$ is determined in the next lemma.

**Lemma 4.4.6** The map defined above gives an isomorphism

$$U \cong \{(X_i)_i \in \prod X_i; \ X_{m-1}X_{m-2} \cdots X_0 = X_{m-2} \cdots X_0X_{m-1} = \cdots = \pi \}. \$$

**Proof.** We define the inverse map. Given a tuple of matrices $(X_i)_i$ (over some $O$-algebra $R$) in the right hand side set, we want to define an $R$-valued point $(\mathcal{F}_i)_i$ in $U$.

First we define maps $\alpha_i : \Lambda_i R \to R^{n-r}$. Afterwards we want to define $\mathcal{F}_i := \ker \alpha_i$.

Denote the canonical basis of $R^{n-r}$ by $f_1, \ldots, f_{n-r}$.

**Definition of** $\alpha_i$. If $t_i(j) = 0$, say $j = j^i_j$, we must have $\alpha_i(e^i_j) = f_i$. 


If \( t_i(j) = 1 \), then choose \( i' \) such that \( t_{i'}(j) = 0 \). Then \( j = j''_{i'} \) for some \( \iota \).
Furthermore, \( \phi_{i-1} \circ \cdots \circ \phi_{i+1} \circ \phi_i(e_j') = e_j' \) by condition 4.4.2. Now define
\[
\alpha_i(e_j') := X_{i-1} \cdots X_{i'} f_i.
\]
This is independent of the choice of \( i' \) (use condition 4.4.2 i)). It is clear that the \( \alpha_i \) are surjective.

The condition that the products of the \( X_i \) are \( \pi \) together with condition 4.4.2 ii) ensures that in this way we get a commutative diagram

\[
\begin{array}{cccccc}
\Lambda_{0,R} & \rightarrow & \Lambda_{1,R} & \rightarrow & \cdots & \rightarrow & \Lambda_{m-1,R} & \rightarrow & \Lambda_{0,R} \\
\alpha_0 & \downarrow & \alpha_1 & & & & \alpha_{m-1} & \downarrow & \alpha_0 \\
R^{n-r} & \rightarrow & X_0 & \rightarrow & X_1 & \rightarrow & \cdots & \rightarrow & X_{m-2} & \rightarrow & X_{m-1} & \rightarrow & R^{n-r} \\
\end{array}
\]

Thus defining \( F_i := \ker \alpha_i \) we indeed get a point of \( U \). It is clear that the two maps are inverse to one another. \( \square \)

The three lemmas together give the following theorem:

**Theorem 4.4.7** Let \( U = U((t_i)_i) \subseteq M(m,n,r,(\phi_i)_i) \), and define \( s \) and \( t \) as above. Then
\[
U \cong W \times \mathbb{A}_O^{rs+t(n-r-s)},
\]
where
\[
W = \{ (X_i)_i \in \prod_{i=0}^{m-1} \mathcal{X}_i; X_{m-1}X_{m-2} \cdots X_0 = X_{m-2} \cdots X_0X_{m-1} = \cdots = \pi \},
\]
and where the \( \mathcal{X}_i \) are certain affine subspaces of the space of \( (n-r-s) \times (n-r-s) \)-matrices \( \text{Mat}_{n-r-s}(O) \). \( \square \)

### 4.4.3 The Equations of \( U_\tau \)

Now we want to apply the previous theorem to find the equations for the open subset \( U_\tau \subseteq M^{\text{loc}} \), which is an open neighborhood of the worst singularity of the local model. So we take \( \underline{t} = ((t_i)_i) = ((1^r,0^{n-r}),(1^{r+1},0^{n-r-1}),\ldots,(2^{r-1},1^{n-r+1})). \)

Denote by \( A_i \) the following \( (n-r) \times (n-r) \)-matrix:
\[
\begin{pmatrix}
a_1^i & 1 \\
a_2^i & 0 & \ddots \\
& \ddots & \ddots & 1 \\
a_n^{i-r} & & & 0 \\
\end{pmatrix},
\]
where the \( a_j^i \) are indeterminates.
4.4. THE EQUATIONS OF THE STANDARD LOCAL MODEL $M^{\text{loc}}$

Proposition 4.4.8 The open subset $U_\tau$ of $M^{\text{loc}}$ is isomorphic to

$$\text{Spec} \mathcal{O}[a^i_\kappa; i = 0, \ldots, n-1, \kappa = 1, \ldots, n-r]/I,$$

where $I$ is the ideal generated by the entries of the matrices

$$A_{n-1}A_{n-2} \cdots A_0 - \pi, \ A_{n-2} \cdots A_0 A_{n-1} - \pi, \ldots, \ A_0 A_{n-1} \cdots A_1 - \pi.$$

Proof. Use theorem 4.4.7. We have $s = t = 0$, thus only lemma 3 is needed. □

The proposition gives a handy form to write down the equations $s$, but in fact these are much more equations than needed. For example in the case $n = 4$, $\mu = (1, 1, 0, 0)$, one can describe $U_\tau$ by 6 equations, whereas the description we have given here consists of 16 equations.

Remark. In particular, we re-discover here the well-known equations of the local model in the Drinfeld case (i.e. $r = n-1$). Then the $A_i$ are $1 \times 1$-matrices, so they are just indeterminates and we get only one equation:

$$A_{n-1}A_{n-2} \cdots A_0 = \pi.$$

Remark. If one wants to analyse the properties of $M^{\text{loc}}$ starting with the equations, it might be disturbing that the $A_i$ are not generic $(n-r) \times (n-r)$-matrices, but that some entries are 0 resp. 1. But in fact, there is a relation to the scheme defined in an analogous way considering generic matrices. To make this precise, let $B_i = (b^i_{jk})_{jk}$, $i = 0, \ldots, n-1$ be $(n-r) \times (n-r)$-matrices of indeterminates and consider the scheme

$$M' = \text{Spec} \mathcal{O}[b^i_{jk}]/B_{n-1} \cdots B_0 = B_{n-2} \cdots B_0 B_{n-1} = \cdots = \pi.$$

If we knew that the special fibre $\overline{M'}$ of $M'$ has the same dimension as the generic fibre, namely $(n-r)^2(n-1)$, and that $\overline{M'}$ is Cohen-Macaulay, we could conclude that $U_\tau \cap \overline{M^{\text{loc}}}$ (and thus $\overline{M^{\text{loc}}}$) is Cohen-Macaulay as well.

The reason is that $U_\tau \cap \overline{M^{\text{loc}}}$ is a closed subscheme of $\overline{M'}$ defined by $(n-r-1)(n-r)n$ equations, and that the number of equations is just the difference between the dimensions of $\overline{M'}$ and $U_\tau \cap \overline{M^{\text{loc}}}$.

See also the remarks at the end of Faltings’ article [F1].

4.4.4 The Open Subsets $U_x$ for Extreme Alcoves $x$

Proposition 4.4.9 Let $x$ be an extreme alcove, i.e. $x \in W(\mu)$. Then $U_x \cong \mathbb{A}^{r(n-r)}_O$.

In particular, $U_x$ is a smooth open subset of $M^{\text{loc}}$. It seems reasonable to expect that the special fibre of $U_x$ coincides with $S_x$ for extreme alcoves $x$.

Proof. For an extreme alcove $x$, we have $U_x = U((t_i)_i)$ with $t_1 = \cdots = t_n$. Thus, with notation as above, $s = n-r$, $t = r$ and the proposition follows immediately from theorem 4.4.7. □
4.4.5 The Equations of $M_{\mu,\kappa}^{\text{loc}}$

In this section we want to determine the equations of an open neighbourhood of the 'worst singularity' of $M_{\mu,\kappa}^{\text{loc}}$, $\mu, \kappa \in \{0, \ldots, n - 1\}$, $\mu \neq \kappa$. Recall that this is the parahoric local model, where not the complete lattice chain, but only the lattices $\Lambda_\mu, \Lambda_\kappa$ are involved. As $M_{\kappa}^{\text{loc}}$ is simply a Grassmannian, this is the first non-trivial case.

Obviously, we may assume that $\mu = 0$ and $\kappa, r \leq \frac{n}{2}$.

Let

$$U = U((1^r, 0^{n-r}), (0^\kappa, 1^r, 0^{n-r-\kappa})) = \{(F_0, F_\kappa) \in M_{0,\kappa}^{\text{loc}};$$

$$F_0 = \begin{pmatrix}
1 & 1 & & & \\
& a_{11}^0 & a_{12}^0 & \cdots & a_{1r}^0 \\
& \vdots & \vdots & \ddots & \vdots \\
& a_{n-r,1}^0 & a_{n-r,2}^0 & \cdots & a_{n-r,r}^0
\end{pmatrix},$$

$$F_\kappa = \begin{pmatrix}
a_{n-r-\kappa+1,1}^\kappa & a_{n-r-\kappa+1,2}^\kappa & \cdots & a_{n-r-\kappa+1,r}^\kappa \\
& \vdots & \vdots & \ddots & \vdots \\
& a_{n-r,1}^\kappa & a_{n-r,2}^\kappa & \cdots & a_{n-r,r}^\kappa \\
1 & & & & 1
\end{pmatrix}. $$

This is an open subset of $M_{0,\kappa}^{\text{loc}}$, which contains the 'worst singularity'. Theorem 4.4.7 gives the following description of $U$:

First case: $\kappa \leq r$

Let $A = (a_{i,j}^0)_{i,j=1,\ldots,\kappa}$, $B = (a_{i,j}^\kappa)_{i=1,\ldots,\kappa, j=r-\kappa+1,\ldots,r}$ be $\kappa \times \kappa$-matrices of indeterminates.

Then

$$U \cong \text{Spec} O[A, B]/(AB = BA = \pi) \times V,$$

where

$$V = \text{Spec} O[a_{i,j}^0; i = 1, \ldots, r, j = \kappa + 1, \ldots, n - r] \times \text{Spec} O[a_{i,j}^\kappa; i = 1, \ldots, r - \kappa, j = 1, \ldots, \kappa]$$

$$\cong \mathbb{A}^{(n-r)(r-\kappa^2)}.$$
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Second case: \( \kappa > r \)

Let \( A = (a_{i,j}^\kappa)_{i=\kappa-r+1,...,\kappa,j=1,...,r} \), \( B = (a_{i,j}^\kappa)_{i=n-r-\kappa+1,...,n-\kappa,j=1,...,r} \) be \( r \times r \)-matrices of indeterminates.

Then

\[
U \cong \text{Spec } O[A, B]/(AB = BA = \pi) \times V,
\]

where

\[
V = \text{Spec } O[a_{i,j}^0; i = 1,\ldots,r, j = \kappa+1,\ldots,n-r] \times \\
\text{Spec } O[a_{i,j}^\kappa; i = 1,\ldots,r-\kappa, j = 1,\ldots,\kappa] \\
\cong \mathbb{A}^{(n-r)r-r^2}.
\]

So we see that up to a product with an affine space the special fibre of \( U \) is a ‘variety of circular complexes’ (cf. [MT]). These varieties have been first studied by Strickland [S] using the technique of algebras with straightening law. She gives an explicit \( k \)-basis in terms of Young tableaux, and shows that these rings are algebras with straightening law. In particular we get

**Theorem 4.4.10** The special fibre of \( M_{\mu,\kappa}^{\text{loc}} \) is reduced.

This is the only result we will need, but in fact Strickland proves much more, for example that the irreducible components of these varieties are normal and have Cohen-Macaulay singularities.

Recently, Mehta and Trivedi have proved similar results (for \( \text{char } k = p > 0 \)) using the technique of Frobenius splittings (cf. [MT]). In fact, their results yield that \( M_{\mu,\kappa}^{\text{loc}} \) is Frobenius split. But they do not consider \( M_{\mu,\kappa}^{\text{loc}} \) as a subvariety of the affine flag variety (which is important for us since we want to look at all the \( M_{\mu,\kappa}^{\text{loc}} \) at the same time and at their intersections).

4.5 Flatness of the Standard Local Model

**Theorem 4.5.1** Let \( \text{char } k = p > 0 \). Then the special fibre of \( M_{\mu}^{\text{loc}} \) is reduced.

**Proof.** For \( I \subseteq \{0,\ldots,n-1\} \), we have the parahoric local model \( M_I^{\text{loc}} \). Its special fibre \( \tilde{M}_I^{\text{loc}} \) can be embedded in \( SL_n(k((t)))/P_I \), where \( P_I \) is a certain parahoric subgroup of \( SL_n(k((t))) \) (cf. section 4.2). Denote by \( \tilde{M}_I^{\text{loc}} \) the inverse image of \( \tilde{M}_I^{\text{loc}} \) under the canonical projection \( F \rightarrow SL_n(k((t)))/P_I \). Set-theoretically, it is a union of Schubert varieties.

We can describe \( \tilde{M}_I^{\text{loc}} \) in terms of lattice chains in the following way: As in section 4.2, write

\[
\lambda_0 = R[t]^n, \lambda_1 = t^{-1}R[t] \oplus R[t]^{n-1}, \\
\ldots, \lambda_{n-1} = t^{-1}R[t]^{n-1} \oplus R[t],
\]
where $R$ is a $k$-algebra. Then $\tilde{M}^{\text{loc}}_{\kappa}$ is identified via the embedding $i$ with the set of those complete lattice chains $(\mathcal{L}_i)_i \in \mathcal{F}(R)$ with $\lambda_i \supseteq \mathcal{L}_i \supseteq t\lambda_i$ for all $i$. The closed subscheme $\tilde{M}^{\text{loc}}_{I} \subseteq \mathcal{F}$ consists of the complete lattice chains $(\mathcal{L}_i)_i$ such that $\lambda_i \supseteq \mathcal{L}_i \supseteq t\lambda_i$ for all $i \in I$.

Of course, $\tilde{M}^{\text{loc}}_{\kappa}$, $\kappa \in \{0, \ldots, n-1\}$, is just a Grassmannian, so in particular a smooth variety. Hence $\tilde{M}^{\text{loc}}_{\kappa}$ is smooth as well by proposition 3.1.7. But it is also connected, hence irreducible, and hence it is just a single, smooth Schubert variety.

Furthermore, the $\tilde{M}^{\text{loc}}_{\mu,\kappa}$ are reduced by theorem 4.4.10 and proposition 3.1.7. Thus they are unions of Schubert varieties also scheme-theoretically.

Obviously, we have (scheme-theoretic intersection inside $\tilde{M}^{\text{loc}}_{0}$):

$$\tilde{M}^{\text{loc}} = \bigcap_{\kappa=1}^{n-1} \tilde{M}^{\text{loc}}_{0,\kappa}. $$

Now we apply corollary 3.4.4 to $\tilde{M}^{\text{loc}}_{0}$. We get that $\tilde{M}^{\text{loc}}_{0}$ is Frobenius split and that all the $\tilde{M}^{\text{loc}}_{0,\kappa}$ are simultaneously compatibly split. Thus their intersection is reduced by lemma 2.1.2 and proposition 2.1.3. □

**Remark.** The results of Strickland that we used to prove the theorem are not really in the spirit of this paper. One could hope that the theorem can also be proved in the following, more elegant way.

What we have used is that $\tilde{M}^{\text{loc}}$ is the intersection of the $\tilde{M}^{\text{loc}}_{0,\kappa}$ inside $\tilde{M}^{\text{loc}}_{0}$. But of course, $\tilde{M}^{\text{loc}}$ is also the intersection of the $\tilde{M}^{\text{loc}}_{\kappa}$ (inside $\mathcal{F}$, say).

Now if we could find a Schubert variety $X$ in $\mathcal{F}$ which contains all the $\tilde{M}^{\text{loc}}_{\kappa}$, and which we knew to be normal, we could apply the same reasoning as above. Not only would this be independent of Strickland’s results, it would even imply the results of Strickland (at least the part we make use of).

One way to get such an $X$ would be to find a ‘sufficiently big’ normal Schubert variety in the affine Grassmannian $SL_n/P^0$, and to take the inverse image in $\mathcal{F}$. But even in the Grassmannian, it seems difficult to see which Schubert varieties are normal.

By a constructibility argument, we can show that the previous theorem holds also in characteristic 0.

**Theorem 4.5.2** Let $k$ be of characteristic 0. Then the special fibre of $\tilde{M}^{\text{loc}}$ is reduced.

**Proof.** We first construct a $\mathbb{Z}$-scheme the fibre of which over a prime $p$ is the special fibre of the local model in characteristic $p$.

Let $\Lambda_i = \mathbb{Z}^n$, $i = 0, \ldots, n-1$ with basis $e^i_1, \ldots, e^i_n$ and define maps

$$\phi_i : \Lambda_i \longrightarrow \Lambda_{i+1}, \quad e^i_j \mapsto \begin{cases} e^{i+1}_j & j \neq i + 1 \\
0 & j = i + 1 \end{cases}$$


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(i = 0, . . . , n − 1, Λ_n := Λ_0). Denote by M the corresponding scheme of compatibly chosen subspaces of rank r.

Then the geometric fibre of M over a prime p is just the special fibre of the local model in characteristic p, and the geometric generic fibre of M is the special fibre of the local model in characteristic 0. Denote by f the morphism M → Spec Z.

We know already that all the geometric fibres over primes p are reduced. As M is a Jacobson scheme, the union of all these fibres is very dense in M. Thus the only constructible subset of M that contains all these fibres is M itself. Now the proposition follows from [EGA IV] 9.9.2, which asserts that in our situation the set of x ∈ M such that f^{-1}(f(x)) is geometrically reduced in x is constructible.

□

As a consequence of the previous theorems we get for k of arbitrary characteristic:

Theorem 4.5.3 The standard local model M^loc is flat over O.

Proof. The irreducible components of the special fibre correspond to the extreme alcoves x ∈ W(µ) (see prop. 4.3.4). A non-empty open subset of the irreducible component associated to x is contained in the open subset U_x ⊆ M^loc, which is isomorphic to affine space as we have seen in proposition 4.4.9. Thus its generic point can be lifted to the generic fibre. Now the flatness is a consequence of the reducedness of the special fibre.

□

More generally,

Corollary 4.5.4 All the parahoric local models M^loc_I, I ⊆ {1, . . . , n}, are flat over O.

Proof. The scheme M^loc_I is smooth over M^loc, as it is just the inverse image of M^loc under the morphism F → SL_n/P^I which is smooth.

As M^loc_I is reduced by what we have seen above, so is M^loc_I. Now it follows that M^loc_I is flat over O in the same way as above.

□

4.5.1 The Singularities of the Local Model

From the corresponding results about affine Schubert varieties (see section 3.3) we obtain

Proposition 4.5.5 The irreducible components of M^loc are normal. They have rational singularities, hence in particular are Cohen-Macaulay.
It would be very interesting to know if the whole special fibre of the local model is still Cohen-Macaulay.

**Remark.** As computations with the help of computer algebra programs show, in general the irreducible components of the special fibre are not locally complete intersections.

### 4.6 The General Case

In this section we want to show that our results carry over without any difficulties to the case where an unramified field extension is involved in the (EL) datum. We use the same notation as in [RZ]. First we define the notion of unramified data of (EL) type; compare the introduction.

So consider a finite *unramified* extension $F/\mathbb{Q}_p$, and let $B = F$, $V = F^n$. The algebraic group associated to these data is $G = \text{Res}_{F/\mathbb{Q}_p} GL_F(V)$.

Furthermore, choose an algebraically closed field $L$ of characteristic $p$, and let $K_0$ be the quotient field of the Witt ring $W(L)$. Since the extension $F/\mathbb{Q}_p$ is unramified, with the notation of [RZ], we have $K_0 = K$.

Now let $\mu : \mathbb{G}_{m,K} \to G_K$ be a 1-parameter subgroup, such that the weight decomposition of $V \otimes_{\mathbb{Q}_p} K$ contains only the weights 0 and 1:

$$V \otimes_{\mathbb{Q}_p} K = V_0 \oplus V_1.$$

We denote by $E$ the field of definition of the conjugacy class of $\mu$.

Finally, let $\mathcal{L}$ be a periodic lattice chain of lattices in $V$.

We will call data of this type unramified data of (EL) type. This is a more general notion than is used in [RZ], 3.82. There, the lattice chain consists only of multiples of one lattice and the resulting local model is smooth, whereas here ‘unramified’ relates only to the field extension.

The local model $M^{\text{loc}}$ associated to these data is the $O_E$-scheme defined as follows. For an $O_E$-scheme $S$, the $S$-valued points are given by

1. a functor $\Lambda \mapsto t_\Lambda$ from $\mathcal{L}$ to the category of $O_F \otimes_{\mathbb{Z}_p} O_S$-modules on $S$, and

2. a morphism of functors $\varphi_\Lambda : \Lambda \otimes_{\mathbb{Z}_p} O_S \to t_\Lambda$,

which are subject to the following conditions

a) $t_\Lambda$ is locally on $S$ a free $O_S$-module of finite rank, and we have an identity of polynomial functions on $O_F$:

$$\text{det}_{O_S}(a; t_\Lambda) = \det_K(a; V_0). \quad (4.2)$$

b) $\varphi_\Lambda$ is surjective for all $\Lambda$. 

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We have a decomposition

\[ F \otimes_{\mathbb{Q}_p} K = \bigoplus_{\varphi : F \to K} K, \]

and correspondingly we get

\[ V \otimes_{\mathbb{Q}_p} K = \bigoplus_{\varphi} V_{\varphi}, \quad V_0 = \bigoplus_{\varphi} V_{0,\varphi}, \quad V_1 = \bigoplus_{\varphi} V_{1,\varphi}. \]

All the \( V_{\varphi} \) have dimension \( n = \dim_F V \) over \( K \). We write \( r_{\varphi} = \dim V_{0,\varphi} \). Of course, the number of summands is \( d = [F : \mathbb{Q}_p] \).

As \( F/\mathbb{Q}_p \) is unramified, we even have a ring isomorphism

\[ O_F \otimes_{\mathbb{Z}_p} O_K = \bigoplus_{\varphi} O_K. \]

So a \( O_F \otimes_{\mathbb{Z}_p} O_K \)-module \( M \) is just a family \( (M_{\varphi})_{\varphi} \) of \( O_K \)-modules, and homomorphisms \( M \to N \) are families \( (M_{\varphi} \to N_{\varphi})_{\varphi} \) of homomorphisms.

To investigate properties like flatness or reducedness of the special fibre, we can just as well look at \( M^{\text{loc}} \otimes_{O_E} O_K \).

We get the following description: For an \( O_K \)-scheme \( S \), the \( S \)-valued points of the local model are given by

1. a functor which associates to each \( \Lambda \) in \( \mathcal{L} \) a family \( (t_{\Lambda,\varphi}) \) of \( O_S \)-modules, and
2. a morphism of functors

\[ \varphi_{\Lambda} : \Lambda \otimes_{\mathbb{Z}_p} O_S = \bigoplus_{\varphi} \Lambda_{\varphi} \otimes_{O_K} O_S \to \bigoplus_{\varphi} t_{\Lambda,\varphi}, \]

which are subject to the following conditions

a) \( t_{\Lambda,\varphi} \) is locally on \( S \) a free \( O_S \)-module of finite rank \( r_{\varphi} \), and

b) all the morphisms \( \Lambda_{\varphi} \otimes_{O_K} O_S \to t_{\Lambda,\varphi} \) are surjective.

Thus the local model in this case is just a product of standard local models. In particular, we get from theorem 4.5.3 and proposition 4.5.5 our main theorem.

**Theorem 4.6.1** The local model \( M \) associated to an unramified (EL)-datum is flat over \( O_E \), and its special fibre is reduced. Furthermore, the irreducible components of the special fibre of \( M \) are normal with rational singularities, so in particular are Cohen-Macaulay. \( \square \)
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