The topological entanglement entropy of black holes in loop quantum gravity

Jingbo Wang

Department of Physics and Electronic Engineering,
Hebei Normal University for Nationalities, Chengde, 067000, China

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Topological order (long-range entanglement) play important role in modern condensed matter physics. In this paper, we will show that the four dimensional black hole can also have topological order, by showing that the topological entanglement entropy is non-zero. The properties of the topological order show that the large diffeomorphisms will act as symmetry (not gauge) transformation on the physical states. More importantly, the long-range entanglement will make the Hawking radiation pure.
I. INTRODUCTION

Topological order is a new type of order that beyond Landau’s symmetry breaking order \[1, 2\]. It origins from the long-range entanglement (LRE) and has some properties that are all robust against any small perturbations. Topological order can produce quasi-particles with Fermi statistics and emergent gauge excitations, so may be used for unification of force and matter \[3\].

In the previous paper \[4\], it was shown that quantum gravity in \(AdS_3\) spacetime have topological orders. Actually it was shown that the quantum gravity in \(AdS_3\) spacetime satisfy all the three properties that define the topological order \[5\]:

1. the ground state degeneracy on torus (or other space with non-trivial topology);

2. the non-Abelian geometrical phase of those degenerate ground states (which form a representation of the mapping class group \(SL(2, Z)\) for torus);

3. the gapless edge modes.

For higher dimensional black holes, such as Kerr black holes in four dimension, the boundary modes can be described by massless scalar field, which is also the same as higher dimensional topological insulators \[6\]. So it was conjectured that black holes in higher dimensions can also have topological orders. In this paper, we will show that the four-dimensional black holes indeed have topological order. It is shown that the black hole has a non-zero topological entanglement entropy, which indicate the existence of the topological order.

II. THE TOPOLOGICAL ENTANGLEMENT ENTROPY FOR 4D BLACK HOLES

In loop quantum gravity (LQG), the degrees of freedom on the horizon of black holes can be described by Chern-Simons (CS) theory with \(U(1)\) \[7, 8\] or \(SU(2)\) \[9–12\] gauge group. Also one can use the \(SO(1, 1)\) BF theory to describe those degrees of freedom \[13–15\]. In this paper, we just consider the CS theory. By counting the admirable boundary states one can get the Bekenstein-Hawking area entropy for black holes \[16\], for a detail counting for different theories, see Ref. \[17\]. The logarithmic correction is different for those two groups, which is \(-\frac{1}{2} \log S_{BH}\) for the \(U(1)\) group and \(-\frac{3}{2} \log S_{BH}\) for the \(SU(2)\) group. This difference can be due to the gauge-fixing from the \(SU(2)\) to \(U(1)\) group \[18\]. Those two terms were obtained through complicated mathematical tools, and in this paper, we will show that the logarithmic corrections have simple explanation: they are just the topological entanglement entropy for the Chern-Simons theory with different gauge group.

A universal quantum number to characterize the topological order in a system can be provided by the topological entanglement entropy (TEE) \[19, 20\]. For a topological ordered system, the entanglement area law satisfy

\[ S \sim \alpha A - \gamma, \] (1)

where \(A\) is the area of the surface, and \(\alpha\) is a parameter that depend on the system, \(\gamma > 0\) is a UV-insensitive term. The first term show the short-range entanglement near the surface, and the second term indicates the existence of long-range entanglement that origin the topological order. This term is called the topological entanglement entropy.
For example, the TEE was calculated for Euclidean $AdS_3$ using surgery [21]. It is related to the so-called "total quantum dimension" $D$ with $\gamma = \ln D$. The total quantum dimension measure the number of the quasi-particles and is defined by

$$D = \sqrt{\sum_a d_a^2},$$

(2)

where $d_a$ is the quantum dimension for the quasi-particle type $a$.

For $U(1)$ Chern-Simons theory with level $k_1 = \frac{aH}{4\pi l^2\beta}$, which is abelian group, the quantum dimension for each quasi-particle is $d_a = 1$ for $a = 1, 2, \cdots, k$, so the TEE for this theory is

$$\gamma_1 = \ln D = \ln \sqrt{\sum_a d_a^2} = \ln \sqrt{k} = \frac{1}{2} \ln S_{BH} - \frac{1}{2} \ln(\pi \beta),$$

(3)

which gives the right logarithmic correction.

For $SU(2)$ Chern-Simons theory with level $k_2 = \frac{aH}{4\pi l^2\beta^2}$, there are $k + 1$ types of quasi-particles labeled by $j = 0, 1, \cdots, k$ (for a more detail, see for example Ref.[22]). The quantum dimension for each type is given by

$$d_j = \frac{\sin \left(\frac{(j+1)\pi}{k+2}\right)}{\sin \frac{\pi j}{k+2}},$$

so the TEE for this theory is

$$\gamma_2 = \ln D = \ln \sqrt{\sum_j d_j^2} = \ln \sqrt{\frac{k^2 + 2}{2 \sin^2 \frac{\pi}{k+2}}} \approx \frac{1}{2} \ln \frac{(k_2 + 2)^3}{2\pi^2} \approx \frac{3}{2} \ln S_{BH} - \frac{1}{2} \ln(2\pi^5\beta^3(1 - \beta^2)^3),$$

(4)

for $k$ is very large. It also matches the result for the states-counting method.

In LQG, the black hole entropy is essentially the entanglement entropy of gravitational field through the horizon [23–25]. In this paper, we show that the logarithmic correction terms can be identified with the topological entanglement entropy, which indicate that the black hole have the topological order. The $SU_k(2)$ Chern-Simons theory with large can be used for universal quantum computing, so it suggest that the black hole can be considered not just as a quantum computer [26, 27] but also as a topological quantum computer [28, 29]. In Ref.30, the non-abelian anyon on the horizon of black hole in LQG are investigated, and suggest a connection to the topological quantum computation. The relation between large diffeomorphisms and the log-correction [31] can be also confirmed. The log-correction is identified with the topological entanglement entropy, which indicate the topological order. According to the property of topological order, the degenerate ground states form the representation of the mapping class group, which is formed by the large diffeomorphisms. So the large diffeomorphisms act as symmetry transformation on the physical states.

III. TOPOLOGICAL ORDER MAKE THE HAWKING RADIATION PURE

The topological order indicate that the black hole has long-range entanglement, and this LRE can have deep influences on the Hawking radiation. Actually it will make the Hawking radiation pure [32, 33]. Due to the property of topological order, the black hole has boundary modes. The Hawking radiation can be considered as superposition of thermal radiation of right/left sector on the horizon at different temperatures $T^r_{R/L}$. The entropy and energy of black holes have the same form as phonon gas in one dimension circle, both for the BTZ black hole and the Kerr black hole. Based on those results, we proposed a simple solution to the information loss paradox for Schwarzschild black holes. The entanglement between the left-moving edge state (with angular momentum $k < 0$) and right-moving
edge state (with angular momentum $k > 0$) is the key to keep the finial state a pure state. The entanglement between the exterior and the interior of the black hole leads to the entanglement between the left-moving edge state and right-moving edge state. So similar to the fractional quantum Hall state case, the state for whole Hawking radiation is a pure state

$$|\Omega> = \frac{1}{\sqrt{Z}} \sum_{k>0} e^{-\beta_H(H_R+H_L)/2} |k, k >_R \otimes |k, -k >_L, \quad (5)$$
even through it appears as thermal. Actually this state is an example of maximally entangled state, and for any right-moving state with angular momentum $k$ there exists a left-moving edge state with opposite angular momentum $-k$. It is this entanglement that make the state pure. And this entanglement can be detected in principle. Since the Hawking radiation is pure, there is no information loss.

IV. CONCLUSION

In this paper, we show that the log-correction terms for the black hole entropy can be identified with the topological entanglement entropy of the system, so indicate the existence of the topological order for the black hole. Once the black hole has the topological order, we can apply the general properties to the black hole, such as the ground states must be degenerate for non-trivial topology, and the large diffeomorphisms should act as symmetry transformation on the physical states. More importantly, the black hole has boundary modes, and the long-range entanglement will make the Hawking radiation pure.

The boundary degrees of freedom on the horizon can be described by the $SU_k(2)$ Chern-Simons theory, which make the black hole as a universal topological quantum computer for large $k$.

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