Nonlinear ion-acoustic (IA) waves driven in a cylindrically symmetric flow

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Abstract By employing a self-similar, two-fluid MHD model in a cylindrical geometry, we study the features of nonlinear ion-acoustic (IA) waves which propagate in the direction of external magnetic field lines in space plasmas. Numerical calculations not only expose the well-known three shapes of nonlinear structures (sinusoidal, sawtooth, and spiky or bipolar) which are observed by numerous satellites and simulated by models in a Cartesian geometry, but also illustrate new results, such as, two reversely propagating nonlinear waves, density dips and humps, diverging and converging electric shocks, etc. A case study on Cluster satellite data is also introduced.

Keywords nonlinear waves; ion-acoustic (IA); MHD; satellite

1 Introduction

Nonlinear plasma theory and approaches have been developed for more than half a century (Sagdeev & Galeev 1963; Davidson 1972). Important problems, such as the excitation, propagation, and effects of nonlinear waves, have been extensively studied since 1970s, see, e.g., Infeld & Rowlands (2000). One branch lies in cylindrically symmetric plasma systems which are ubiquitously observed in geo-space by a multitude of measurements from, e.g., ground-based imagers (Pimenta et al. 2001), rockets (Earle et al. 1989, Moore et al. 1996), and satellites (Pickett et al. 2004, Vaivads et al. 2004, De Keyser et al. 2005).

In the cylindrical frame where \( B \) is along axial \( z \)-direction, small-amplitude ion acoustic (IA) waves were firstly studied (Maxon \& Viecel 1974, Maxon 1976). Then, nonlinear surface waves were found to shift the radius of a plasma cylinder (Gradov \& Stenflo 1983, Gradov et al. 1984). After that, the deformed amplitudes and propagation velocities were confirmed to be different from those in the semi-infinite limit of plasmas (Gradov et al. 1985, 1986). In addition, the equations governing evolution of surface and body waves were obtained (Molotovshchikov \& M. S. Ruderman 1987). Furthermore, in unmagnetized systems, a set of nonlinear equations was solved to describe the temporal change of the electron density in strongly nonlinear surface waves (Stenflo 1990, Yu \& Stenflo 1991). The work was followed by a generalized study including the rotation effects of a time-dependent rigid plasma body (Stenflo \& Yu 1992, 1995). By contrast, in magnetized systems, nonlinear IA parallel-propagating electric field structures were recently calculated (Shi et al. 2001, 2005), and bipolar electric field structures were verified to be able to originate from either IA or ion cyclotron (IC) waves (Shi et al. 2008).

By examining observations by high-resolution satellites (e.g., Wind, FAST, Polar, Cluster), we found that, in addition to the bipolar shapes of nonlinear structures, there exist two other well-known nonlinear electric field envelops: sinusoidal and sawtooth. These three shapes were first recorded by S3-3 in the 1970s (Temerin et al. 1979). Since then, they have been detected by numerous satellites such as S3-3 (Temerin et al. 1982), Viking (Bostron et al. 1988), Geotail (Matsumoto et al. 1994), Wind (Bale et al. 1998), FAST (Ergun et al. 1998, McFadden et al. 1999, 2003), Polar (Mozer et al. 1997, Franz et al. 1998), bounds et al. 1999, Cattell et al. 1999, Franz et al. 2000), and Cluster (Pickett et al. 2004, 2005).

The formation of these three nonlinear structures have been studied extensively in Cartesian coordinates. For example, Temerin et al. (1979) firstly reproduced these shapes by solving a set of fluid equations in the
ion-cyclotron (IC) / ion-acoustic (IA) regime. In a unified work, Lee & Kan (1981) obtained these two important nonlinear electrostatic waves (it is worth to mention here that the authors also mentioned a third type of the nonlinear waves: “ion-acoustic solitons”. By redoing the calculations, we can easily see that it is not a new type but the simple waves.

Particularly, Reddy et al. (2002); Bharuthram et al. (1997); Das et al. (2000); Jovanovic & Shukla (2000); Mamun & Shukla (2002). Chatterjee & Roychoudhury (1997); Das et al. (2000); Nakamura & Sugai (1996); see that it is not a new type but the simple waves.

The model takes into account isothermal electron and ion fluids, with \( v_{Te} \gg v_{Ti} \), where \( v_{Te} = \sqrt{2k_B T_e/m_e} \) and \( v_{Ti} = \sqrt{2k_B T_i/m_i} \) are the electron and ion thermal speeds, respectively, in which \( k_B \) is Boltzmann’s constant, \( T_e \) and \( T_i \) are their temperatures (assumed constant), respectively, and \( m_e \) and \( m_i \) are their masses, respectively. In this study, we neglect the electron inertia because of \( m_e \ll m_i \). Both the momentum equation and the isothermal equation of state of the electron fluid provide \( N_e = e \phi \), where \( N_e = n_e/n_0 \) is the dimensionless electron density in which \( n_e \) and \( n_0 \) are the dimensional electron density and the uniform background plasma density, respectively, and \( \Phi = e \varphi / (k_B T_e) \) where \( \varphi \) is the perturbed electrostatic potential.

The ion fluid is described by a set of MHD equations expressing ion continuity, momentum (containing the equation of state), and the quasi-neutrality:

\[
\begin{align*}
\frac{\partial N_i}{\partial t} + \frac{\partial (N_i V_i)}{\partial R} + \frac{\partial (N_i V_i)}{\partial Z} &= - \frac{N_i V_i}{R} \\
\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial R} + V_z \frac{\partial V_r}{\partial Z} &= \frac{V_r^2}{R} + V_r - \zeta \frac{\partial N_i}{\partial R} \\
\frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial R} + V_z \frac{\partial V_z}{\partial Z} &= - \zeta \frac{\partial N_i}{\partial Z} \\
N_i &\approx N_e = e \phi
\end{align*}
\]

in which \( N_i = n_i/n_0 \) (where \( n_i \) is the ion density), \( \tau = \Omega_i t \) (where \( \Omega_i = eB/m_i \) is the ion gyro-frequency, and \( t \) is time), \( V_r = u_r/c_s, V_z = u_z/c_s \) (where \( \{u_r, u_\phi, u_z\} \) is the ion velocity components, and \( c_s = \sqrt{k_B T_e/m_i} \) is the ion acoustic speed), \( R = r/\rho_i, Z = z/\rho_i \) (where \( \rho_i = c_s/\Omega_i \) is the ion gyro-radius), and \( \zeta = 1 + v_{Ti}^2/(2c_s^2) \). Note that (1) due to the symmetric nature of the cylindrical system, all derivative terms along the \( \phi \)-direction do not occur; and (2) the parameter \( R \) represents the radius of curvature of the flow streamline intersecting the magnetic field lines on which the equation is going to be solved.

In this system, linear IA and IC waves can be excited. By linearizing Eq. (1), we obtain \( \omega_i^2 = \zeta k^2 c_s^2 \) (where \( k \) is the amplitude of the wave vector \( k \) in the parallel direction and \( \omega_i^2 = \Omega_i^2 \) in the transverse plane. Superimposing upon these background oscillations, there exist nonlinear waves the features of which can be obtained by solving a set of self-similar equations of Eq. (1) via introducing a self-similar parameter \( X \) (Lee & Kan, 1981; Shi et al., 2001):

\[
X = \frac{\alpha_1}{M} R + \frac{\alpha_2}{M} Z - \tau
\]

where \( M \) is the Mach number, \( \alpha_1 = \sin \theta \), and \( \alpha_2 = \cos \theta \) in which \( \theta \) is the inclination angle between the propagating wavevectors and the magnetic field. This approach provides a simple way to study the propagation of nonlinear waves in cylindrically symmetric plasma.
Fig. 1  (Color online) Ion density $N_i$ and wave-field strength $E_i$ of sinusoidal nonlinear waves excited under $V_{φ0}=0.05$ at $M=2$ and $R=1$. Upper panel: Fast nonlinear wave; Lower panel: Slow nonlinear wave. Inserted in each panel is the speed of the simple wave propagating along $B$.

ion direction and the magnetic field. Using self-similar transformations, i.e.,
\[
\frac{\partial}{\partial \tau} = -\frac{d}{dX}, \quad \frac{\partial}{\partial R} = \frac{\alpha_1}{M} \frac{d}{dX}, \quad \frac{\partial}{\partial Z} = \frac{\alpha_2}{M} \frac{d}{dX}
\]
We concentrates on parallel-propagating nonlinear waves, i.e., $\theta=0$. This means $\alpha_1=0$, and $\alpha_2=1$. Therefore, we have
\[
\frac{\partial}{\partial R} = 0, \quad \frac{\partial}{\partial Z} = \frac{1}{M} \frac{d}{dX}
\]
the first expression of which indicates that $R$ is independent of $X$. We thus obtain a set of four self-similar equations of nonlinear waves propagating along $B$:
\[
\begin{align*}
\frac{dV_r}{dX} - \frac{V_r}{M} \frac{dV_r}{dX} &= -\frac{V_r^2}{M} - V_φ \\
\frac{dV_φ}{dX} - \frac{V_r}{M} \frac{dV_φ}{dX} &= \frac{V_r V_φ}{M} + V_r \\
\left\{ \begin{array}{l}
\frac{dV_r}{dX} - \frac{V_r}{M} \frac{dV_r}{dX} = \frac{\zeta}{M} \frac{dV_r}{dX} = \frac{V_r}{M} \\
\left( 1 - \frac{V_r}{M} \right) \frac{dV_φ}{dX} - \frac{V_r}{M} \frac{dV_φ}{dX} = \frac{\zeta}{M} \frac{dV_φ}{dX} = \frac{V_r}{M}
\end{array} \right.
\end{align*}
\]
in which $R$ behaves only as an input parameter to represent the existence of geometrical effects (namely, the centrifugal and Coriolis forces) in curvilinear flows. Note that these effects are absent in rectilinear systems.

Using boundary conditions $\Phi|_{X=0} = 0$, the third equation of Eq.(3) can be integrated to give
\[
V_z = M \left( 1 \pm \sqrt{1 - \Psi} \right)
\]
where $\Psi=2\zeta\Phi/M^2$ and $V_z|_{X=0}$ have two initial values: 0 and $2M$. This leads to a new set of equations as follows:
\[
\begin{align*}
\pm \sqrt{1 - \Psi} \frac{dV_r}{dX} &= \left( \frac{V_r}{R} + 1 \right) V_φ \\
\mp \sqrt{1 - \Psi} \frac{dV_φ}{dX} &= \left( \frac{V_r}{R} + 1 \right) V_r \\
\pm \left( -\frac{M^2}{\zeta} \right) \sqrt{1 - \Psi} + \frac{V_r}{\sqrt{1 - \Psi}} \frac{dV_r}{dX} &= 2 \frac{V_φ}{R} \\
E_i &= -\frac{1}{M} \frac{dV_r}{dX} = -\frac{M}{\zeta} \frac{dV_φ}{dX}
\end{align*}
\]
where $E_i$ is the dimensionless electric field amplitude of the simple wave with a unit of $E_0=c_sB$.

3 Parameterized analysis

3.1 Prerequisite of the excitation of nonlinear waves

Cylindrically-symmetric flows are azimuthal with $V_{φ0} \neq 0$ at $X=0$. Other parameters satisfy equilibrium conditions, i.e., $V_r(0)=\Phi(0)=0$. A nonzero $V_{φ0}$ is important.
This can be seen from the first two equations of Eq.(7). They provide

\[ V_r dV_r + V_\phi dV_\phi = 0 \]  

or,

\[ V_r^2 + V_\phi^2 = V_r^2 |_{x=0} + V_\phi^2 |_{x=0} = V_{\phi 0}^2 \]  

Clearly, if \( V_{\phi 0} = 0 \), both \( V_r \) and \( V_\phi \) are zeros at any \( X \), and thus, the last equation of Eq.(7) gives \( \Psi = 0 \) at any \( X \). In this case, no nonlinear waves can develop.

As a result, only under nonzero \( V_{\phi 0} \) conditions is it possible to trigger nonlinear processes. Luckily, this condition is always met naturally in toroidal flows in geospace.

### 3.2 Evolution of nonlinear waves

At three different levels of \( V_{\phi 0} = 0.05 \) (weak), 0.5 (medium), 0.95 (strong), respectively, Figs.1,2,3 illustrate, respectively, the three well-known structures of the nonlinear wave-field \( E_i \), namely, sinusoidal, sawtooth and spiky or bipolar, as well as the accompanied nonlinear-wave density \( N_i \), under \( M = 2 \) and \( R = 1 \). Every figure contains two panels, each of which is inserted into a \( V_z \)-curve to express the propagating speed of a nonlinear wave along the axial direction. Note that the fast wave is always propagating in the direction of \( B \) (\( V_z \geq 0 \)), while the other in the opposite direction (\( V_z \leq 0 \)).

The three figures illustrate that the weak flow drives sinusoidal nonlinear structures with small amplitudes (\( |E_i|_{\text{max}} = 0.03 \)), the medium flow drives sawtooth structures with medium amplitudes (\( |E_i|_{\text{max}} = 0.3 \)), and the strong flow drives spiky (bipolar) nonlinear waves with high amplitudes (\( |E_i|_{\text{max}} = 0.5 \)). This result reproduces the results obtained by Reddy et al. (2002); Bharuthram et al. (2002) in Cartesian coordinates. This indicates that the features of nonlinear waves are strengthened increasingly with stronger drifts of azimuthal flows. Especially, in the last case, the two panels in Fig.3 reveal diverging shocks (a negative electric field followed by a positive one) of the fast nonlinear-wave packet in the upper panel, and converging shocks (a positive electric field followed by a negative one) of the slow packet in the lower panel. In addition, more calculations with a changing \( M \) expose that under \( M > 1 \), the nonlinear-wave density \( N_i \) is never larger than 1, but goes to a minimum which is smaller at a faster drift, meaning density holes are formed and their boundaries result in nonlinear waves. For example, the hole is 0.876 in amplitude with a
Fig. 3 (Color online) The same as Fig.1 but spiky (or bipolar) nonlinear waves excited under \(V_{\phi 0}=0.95\). Nonlinear waves have density holes.

3.3 Influence of \(R\).

In a cylindrical system, the frame effects produced by the centrifugal and Coriolis terms in the momentum equations decrease with radius. They should have a direct influence on the structure of nonlinear waves excited in the azimuthal flows. Fig.4 expose the role played by \(R\). At a larger radius \(R=5\) compared to Fig.3, the nonlinear waves change their appearances from a spiky (bipolar) shape to a sawtooth one, with a lower amplitude \(|E_i|_{max}=0.12\) but a higher frequency. At another radius, \(R=10\), a calculation shows that sinusoidal shape emerges, with a very small amplitude of \(|E_i|_{max}=0.06\). This indicates that the features of nonlinear waves are weakened when going farther from the center of the cylindrically symmetric flow. It is thus reasonable to propose that it is easier to detect nonlinear waves closer to the symmetric axis where the influence of the centrifugal and Coriolis forces are more dominant.

3.4 Criterion for density holes and humps.

The Mach number cannot always be larger than 1,\(^{37}\) referring to that \(M<1\) is also possible. Fig.5 illustrates a case under conditions of Fig.3 but at \(M=0.5\). Interestingly, different from the density holes, there are now density humps coming into being: the nonlinear-wave density \(N_i\) is never smaller than 1, and goes to a maxmum 1.133. In addition, the bipolar nonlinear wave becomes denser with a much higher frequency of 3, than that of 0.04 in Fig.3. However, the amplitude of \(E_i\) decreases from 0.5 to 0.35, meaning the nonlinear feature is weakened. Other calculations confirm that density holes occur at \(M>1\), while density humps appear at \(M<1\).

4 A case study

We have chosen a cylindrical geometry to study the excitation of nonlinear structures modulated by different values of the initial azimuthal flow speeds, the distance to the symmetric axis, and the Mach number. We not only reproduced the three familiar shapes (i.e., sinusoidal, sawtooth, and spiky or bipolar) of nonlinear
Fig. 4 (Color online) The same as Fig. 3 but sawtooth nonlinear waves excited at a distant radial position $R=5$. 

waves observed by satellites, but also found that nonlinear waves supported by cylindrically symmetric plasma flows have both fast and slow branches, both converging and diverging electric shocks, and both density humps and dips. This is different from the rectangular flows. By solving Eq. (1) in a Cartesian frame, we obtain that the wave potential $\Phi$ satisfies [c.f., e.g., Ma & Hirose (2009)]

$$\frac{d^2 \Phi}{dx^2} = e^\Phi - \frac{\sqrt{2}}{\sqrt{(1 + \frac{3\xi}{M^2}) + \sqrt{[1 + \frac{3\xi}{M^2} - \frac{2\Phi}{M^2}]^2 - 12 \frac{\Phi}{M^2}}} - 13 \frac{\xi}{M^2}}$$

where $\xi$ is the ratio between ion and electron initial temperatures. Taking $\xi = 0.1$ and $M=1.141.161.28$, respectively, produces a result shown in Fig. 6. Though the figure keeps exhibiting the three familiar nonlinear structures, only diverging shocks are seen to be driven.

The cylindrical model and calculations presented in last Sections are useful in gaining physical insights into the excitation and propagation of observed nonlinear waves. Fig. 7 is an example, exposing snapshots of nonlinear waves within the plasma sheet on auroral field lines, taken by the four Cluster satellites when they encountered the high altitude auroral zone at a radial distance of $\sim 4.2 R_E$ at 21:30 MLT and $\sim 62^\circ$ ILAT (south) on 31 March 2001 at $\sim 06:44$ UT (Cattell et al. 2003). The panels show both converging shocks (SC1 panel) and diverging shocks (SC2-4 panels), sawtooth structures (SC2 panel), and more complicated shapes of background hiss, at the four locations separated by $\sim 1000$ km and an $\sim 0.03$ s interval from each satellite. The detected amplitudes of waves can reach over 750 mV/m, the largest values ever reported in the outer magnetosphere.

At that time, the mission was experiencing a strong magnetic storm. The magnetosphere was highly compressed due to an intense injection of electrons $\sim 14$ min earlier (Baker et al. 2002), which could enhance the local magnetic field from a few mG of the background strength to tens even hundreds of mG (Alfaro et al. 2004; Tjulin et al. 2004; Hall et al. 2009). Although the Cluster has no single probe measurements to give nonlinear wave speeds, all four panels reveal that they excellently recorded all the bursts by double probes of high resolutions (Gustafsson et al. 1988; Cattell et al. 2003). The pulses are $\sim 2.5$ ms or its multiples in time, in good agreement with our simulation value of $X=10-30$ by using the local orbital data of the Cluster. More important, SC1 recorded an oppositely propagating nonlinear wave of converging shocks, different
Fig. 5 (Color online) The same as Fig.3 but higher-frequency spiky (or bipolar) nonlinear waves excited by a low Mach-number $M=0.5$. Nonlinear waves have density humps.

from other satellites which displayed diverging shocks, but at the same time. This is in good agreement with the illustration of two oppositely propagating nonlinear waves existing in a cylindrical system simultaneously, as shown by Fig.3. This fact notwithstanding, we are unable to find the wave speed data to double-check $V_z$ of the wave detected by SC1, which should have a smaller amplitude than those measured by other satellites.

5 Conclusion and discussion

By using a two-fluid self-similar MHD model, we studied features of parallel-propagating nonlinear waves driven in a cylindrically symmetric system. We not only reproduced the three salient shapes (i.e., sinusoidal, sawtooth, and spiky or bipolar) of parallel-propagating, IA/IC nonlinear waves, but also found following new results: (1) the prerequisite to trigger the nonlinear waves is the nonzero azimuthal speed; (2) there are always two types of nonlinear waves: a fast one propagating along $B$ (diverging shocks) and a slow one against $B$ (converging shocks); (3) the distance from the symmetric axis influences the nonlinear features: the closer to the axis, the more pronounced the characteristics, and vice versa; (4) there exist density holes for $M>1$ and humps for $M<1$, the boundaries of which constitute nonlinear structures.

This study is the first step to investigate the mechanism of nonlinear wave-particle interactions and their effects on some unusual observed phenomena, e.g., transverse ion heating, broadband noise emission, and magnetic holes (or bubbles, decreases Tsurutani et al. 2005). It offers an alternative to explain inverted-V structures in beam-precipitating regions (Pottelette & Berthomier 2009), as to be contributed in a companion paper. This is based on the considerations as follows.

When a field-aligned current is enclosed in the ionosphere by the Pedersen current, plasma turbulence enhances abnormal resistance locally which may bring about an electric field perpendicular to the magnetic field lines. This field drives naturally an azimuthal flow, and thus provides, on one hand, a boundary condition to Eq. (1). On the other hand, there also exists another magnetospheric boundary condition. Therefore, unlike the treatment employed in the present paper, the solution that yields a non-vanishing parallel potential drop has to be sought in a system where Eq. (1) should be solved as a two-point boundary condition problem, rather than an initial condition one as was the case here. From the result of this paper, we know that the
Parallel-propagating nonlinear waves developed from a rectangular flow. Although there still exist three types of shapes (sinusoidal, sawtooth, and spiky or bipolar), only diverging shocks can be evolved from such a system modulated by the Mach number $M$.

Parallel electric field decreases away from the center of the flow (see the amplitude decrease of $E_i$ from Fig.3 to Fig.4). In this way, we anticipate that an alternative solution would yield the classical inverted-V structures, with scale length comparable to the ion gyroradius.

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