Gilbreath’s sequences and proof of conditions for Gilbreath’s conjecture

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Abstract

The conjecture attributed to Norman L. Gilbreath, but formulated by Francois Proth in the second half of the 1800s, concerns an interesting property of the ordered sequence of prime numbers $P$. Gilbreath’s conjecture stated that, computing the absolute value of differences of consecutive primes on ordered sequence of prime numbers, and if this calculation is done for the terms in the new sequence and so on, every sequence will starts with 1. In this paper is defined the concept of Gilbreath’s sequence, Gilbreath’s triangle and Gilbreath’s equation. On the basis of the results obtained from the proof of properties, an inductive proof is produced thanks to which it is possible to establish the necessary condition to state that the Gilbreath’s conjecture is true.

1 Introduction to Gilbreath’s conjecture

Let the ordered sequence $P = \{2, 3, 5, 7, 11, 13, 17, \ldots \} = \{p_1, \ldots, p_n, \ldots \}$ formed by prime numbers, and set $p^b_a = |p^{b-1}_{a+1} - p^{b-1}_a|$ where $b \geq 1$. Gilbreath conjectured that every term $p^1_1 = 1$. In this notation, the elements of $P$ should be indicated with $\{p^0_1, \ldots, p^0_n, \ldots \}$. For brevity, the superscript with $b = 0$ is omitted. It is likely that this conjecture is satisfied by many other sequences of integers, so it is necessary to define the general properties of all sequences that satisfy this conjecture.
2 Gilbreath’s sequence

Definition 1. Let a sequence $S = \{s_1, s_2, s_3, \ldots, s_n\}$ a sequence formed by integer number and $s_a^b = |s_{a+1}^{b-1} - s_a^{b-1}|$. $S$ is defined a Gilbreath’s sequence if $s_1^b = 1 \forall b \geq 1$. If $S$ is a Gilbreath’s sequence hence $S \in \mathbb{G}_n$ where $\mathbb{G}_n$ is the set of all the Gilbreath’s sequences of length $n$.

Let, for example, $S = \{2, 3, 5, 7, 11, 13, 17\}$ a sequence of length $n = 7$, the Gilbreath’s triangle of $S$ is defined by $s_a^b = |s_{a+1}^{b-1} - s_a^{b-1}|$. Hence:

\[
\begin{array}{ccccccc}
s_1 & s_2 & s_3 & s_4 & \ldots & s_{n-3} & s_{n-2} & s_{n-1} & s_n \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
s_1^{n-2} & s_2^{n-2} & s_3^{n-2} & s_4^{n-2} & \ldots & s_{n-3}^1 & s_{n-2}^1 & s_{n-1}^1 & s_n^1 \\
\end{array}
\]

or

\[
\begin{array}{ccccccc}
1 & 2 & 2 & 4 & 2 & 4 \\
1 & 0 & 2 & 2 & 2 \\
1 & 2 & 0 & 0 \\
1 & 2 & 0 \\
1 & 2 \\
1 \\
\end{array}
\]

The first term of every sequence is equal to 1, hence $S \in \mathbb{G}_7$.

Theorem 1. Let a sequence $S = \{s_1, \ldots, s_n\} \in \mathbb{G}_n$ and a sequence $S' = \{s_1, \ldots, s_{n-1}\}$, then $S' \in \mathbb{G}_{n-1}$.

Proof. The Gilbreath’s triangle associated with $S$:

\[
\begin{array}{ccccccc}
s_1 & s_2 & s_3 & s_4 & \ldots & s_{n-3} & s_{n-2} & s_{n-1} & s_n \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
s_1^{n-2} & s_2^{n-2} & s_3^{n-2} & s_4^{n-2} & \ldots & s_{n-3}^1 & s_{n-2}^1 & s_{n-1}^1 & s_n^1 \\
\end{array}
\]

where $s_1^1 = \ldots = s_1^{n-2} = s_1^{n-1} = 1$ as a consequence of $S \in \mathbb{G}_n$. Removing the last element of each sequence gives:

\[
\begin{array}{ccccccc}
s_1 & s_2 & s_3 & s_4 & \ldots & s_{n-3} & s_{n-2} & s_{n-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
s_1^{n-2} & s_2^{n-2} & s_3^{n-2} & s_4^{n-2} & \ldots & s_{n-3}^1 & s_{n-2}^1 & s_{n-1}^1 \\
\end{array}
\]

which is the Gilbreath’s triangle of $S'$, $s_1^1 = \ldots = s_1^{n-2} = 1$ as a consequence of $S \in \mathbb{G}_n$, hence $S' \in \mathbb{G}_{n-1}$.
Theorem 2. Let a sequence $S = \{s_1, \ldots, s_n\} \in \mathbb{G}_n$ and a sequence $S' = \{s_1, \ldots, s_n, k\}$, then $S' \in \mathbb{G}_{n+1} \iff k \in \mathbb{K}_S$.

Proof. The Gilbreath’s triangle associated with $S$:

\[
\begin{array}{cccccc}
& s_1 & s_2 & s_3 & s_4 & \ldots & s_{n-3} & s_{n-2} & s_{n-1} & s_n \\
& s_1' & s_2' & s_3' & s_4' & \ldots & s_{n-3}' & s_{n-2}' & s_{n-1}' & s_n' \\
& & & & & \vdots & & & & \\
& & s_1^{n-2} & s_2^{n-2} & \ldots & s_1 \end{array}
\]

where $s_1' = \ldots = s_1^{n-2} = s_1^{n-1} = 1$ as a consequence of $S \in \mathbb{G}_n$. The Gilbreath’s triangle of $S'$:

\[
\begin{array}{cccccc}
& s_1 & s_2 & s_3 & s_4 & \ldots & s_{n-3} & s_{n-2} & s_{n-1} & s_n & k \\
& s_1^1 & s_2^1 & s_3^1 & s_4^1 & \ldots & s_{n-3}^1 & s_{n-2}^1 & s_{n-1}^1 & |s_n - k| \\
& & & & & \vdots & & & & \\
& & s_1^{n-2} & s_2^{n-2} & |s_3^{n-3} - |s_4^{n-4} - |\ldots - |s_{n-1}^{1} - |s_n - k||| \ldots||| \\
& s_1^n & |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - |\ldots - |s_{n-1}^{1} - |s_n - k||| \ldots||| \\
& |s_1^{n-1} - |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - |\ldots - |s_{n-1}^{1} - |s_n - k||| \ldots||| \
\end{array}
\]

where $s_1' = \ldots = s_1^{n-2} = s_1^{n-1} = 1$ as a consequence of $S \in \mathbb{G}_n$ and if also $s_1^n = |s_1^{n-1} - |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - |\ldots - |s_{n-1}^{1} - |s_n - k||| \ldots||| = 1$, then $S' \in \mathbb{G}_{n+1}$.

\[
|s_1^{n-1} - |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - |\ldots - |s_{n-1}^{1} - |s_n - k||| \ldots||| = 1 \tag{1}
\]

is defined as the Gilbreath’s equation of $S$ and $\mathbb{K}_S = \{k_1, \ldots, k_{2^n}\}$ is defined as the set of all solutions for $k$.

\[\square\]

Corollary 1. Let a sequence $S = \{s_1, \ldots, s_n\} \in \mathbb{G}_n$ and a sequence $S' = \{s_1, \ldots, s_n, k\}$, then there are $2^n$ values of $k$ that satisfy $S' \in \mathbb{G}_{n+1}$.

Proof. The Gilbreath’s equation is a $2^n$ degree equation, then there are $2^n$ value of $k$ that satisfy the equation (1). The solutions are:

\[
k_{1, \ldots, 2^n} = \pm s_1^{n-1} \pm s_2^{n-2} \pm s_3^{n-3} \pm s_4^{n-4} \pm \ldots \pm s_{n-1}^1 + s_n \pm 1 \tag{2}
\]

\[\square\]

With respect to $\mathbb{K}_S$, the largest value that solves the equation is $\max \mathbb{K}_S = s_1^{n-1} + s_2^{n-2} + s_3^{n-3} + s_4^{n-4} + \ldots + s_{n-1}^1 + s_n + 1$ and the smallest value that solves the equation is $\min \mathbb{K}_S = -s_1^{n-1} - s_2^{n-2} - s_3^{n-3} - s_4^{n-4} - \ldots - s_{n-1}^1 + s_n - 1 = 2s_n - \max \mathbb{K}_S$. A remarkable relation is:

\[
\max \mathbb{K}_S + \min \mathbb{K}_S = 2s_n \tag{3}
\]
Corollary 2. Let a sequence $S = \{s_1, ..., s_n\} \in \mathbb{G}_n$ and a sequence $S' = \{s_1, ..., s_n, s_n\}$, then $S' \in \mathbb{G}_{n+1}$.

Proof. The Gilbreath’s equations of $S'$ with $k = s_n$ is:

$$|s_1^{n-1} - |s_2^{n-2} - |s_3^{n-3} - |s_4^{n-4} - ... - |s_{n-1}^{1}|||| = 1$$

which is true because $S \in \mathbb{G}_n$, hence $S' \in \mathbb{G}_{n+1}$. $\square$

It is useful, for the following proofs to introduce the definition of two important Gilbreath’s sequences. Let a sequence $S = \{s_1, ..., s_n\}$, from relation (2), any value of $k$ cannot be greater than $\max K_S$, so the sequence $\{s_1, ..., s_n, \max K_S\}$ is the upper bound sequence for the sequence $\{s_1, ..., s_n\}$.

The new sequence $S' = \{s_1, ..., s_n, \max K_S, k\}$ will have the upper limit for $k = \max K_{\{s_1, ..., s_n, \max K_S\}}$. Equally, let a sequence $S = \{s_1, ..., s_n\}$, from relation (2), any value of $k$ cannot be smaller than $\min K_S$ and the new sequence $S' = \{s_1, ..., s_n, \min K_S, k\}$ will have the lower limit for $k = \min K_{\{s_1, ..., s_n, \min K_S\}}$. From this example, it is now possible to introduce the definition of upper bound sequence and lower bound sequence.

Definition 2. Let the sequence $S = \{s_1, ..., s_n\} \in \mathbb{G}_n$

$$U_S := \{s_1, ..., s_n, \max K_{\{s_1, ..., s_n\}}, \max K_{\{s_1, ..., s_n, \max K_{\{s_1, ..., s_n\}}\}}, ...\} = \{u_{S_1}, ..., u_{S_m}, ...\} \quad (4)$$

is the upper bounds sequence for any $S$.

Definition 3. Let the sequence $S = \{s_1, ..., s_n\} \in \mathbb{G}_n$

$$L_S := \{s_1, ..., s_n, \min K_{\{s_1, ..., s_n\}}, \min K_{\{s_1, ..., s_n, \min K_{\{s_1, ..., s_n\}}\}}, ...\} = \{l_{S_1}, ..., l_{S_m}, ...\} \quad (5)$$

is the lower bounds sequence for any $S$.

From definition 2, definition 3, corollary 2 and (3)

$$S = \{s_1, ..., s_n\} \in \mathbb{G}_n \implies l_{\{s_1, ..., s_m\}} \leq s_{m+1} \leq u_{\{s_1, ..., s_m\}}, \text{ where } m \leq n - 1 \quad (6)$$

Lemma 1. Let a sequence $S = \{s_1, ..., s_n\} \in \mathbb{G}_n$ where $s_1 \in 2\mathbb{Z}$, then $\{s_2, ..., s_n\} \subset (2\mathbb{Z} + 1)^{n-1}$

Proof. Let $S_1 = \{s_1\}$, where $s_1 \in 2\mathbb{Z}$. From theorem 2, corollary 1: $S_2 = \{s_1, k\} \in \mathbb{G}_2$ if $k = s_1 \pm 1 \in 2\mathbb{Z} + 1$. Now let the sequence $S_3 = \{s_1, s_1 \pm 1, k\}$, from theorem 2, corollary 1, $S_3 \in \mathbb{G}_3$ if $k = |\pm 1| + (s_1 \pm 1) \pm 1 = 1 + s_1 \pm 1 \pm 1$. 

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From the previous step, \( s_1 \pm 1 \in 2\mathbb{Z} + 1 \), hence \( 1 + s_1 \pm 1 \in 2\mathbb{Z} \) and \( 1 + s_1 \pm 1 \pm 1 \in 2\mathbb{Z} + 1 \). Iteratively, this can be proved for every element of \( S \). Hence if \( S \in \mathbb{G}_n \) and the first element of \( S \) is an even number, then all the other numbers of the sequence must be odd.

**Lemma 2.** Let a sequence \( S = \{s_1, \ldots, s_n\} \in \mathbb{G}_n \) where \( s_1 \in 2\mathbb{Z} + 1 \), then \( \{s_2, \ldots, s_n\} \subset (2\mathbb{Z})^{n-1} \)

*Proof.* Let \( S_1 = \{s_1\} \), where \( s_1 \in 2\mathbb{Z} + 1 \). From theorem 2, corollary 1: \( S_2 = \{s_1, k\} \in \mathbb{G}_2 \) if \( k = s_1 \pm 1 \in 2\mathbb{Z} \). Now let the sequence \( S_3 = \{s_1, s_1 \pm k, k\} \), from theorem 2, corollary 1, \( S_3 \in \mathbb{G}_3 \) if \( k = |\pm 1| + (s_1 \pm 1) \pm 1 = 1 + s_1 \pm 1 \pm 1 \). From the previous step, \( s_1 \pm 1 \in 2\mathbb{Z} \), hence \( 1 + s_1 \pm 1 \in 2\mathbb{Z} + 1 \) and \( 1 + s_1 \pm 1 + 1 \in 2\mathbb{Z} \). Iteratively, this can be proved for every element of \( S \). Hence if \( S \in \mathbb{G}_n \) and the first element of \( S \) is an odd number, then all the other numbers of the sequence must be even.

**Definition 4.** Let \( A_1 = 2\mathbb{Z} \) and \( A_2 = 2\mathbb{Z} + 1 \), both sets are represented as \( A_{1,2} = 2\mathbb{Z} + (\frac{1}{2} \pm \frac{1}{2}) \).

**Lemma 3.** Let \( S = \{s_1, \ldots, s_n\} \in \mathbb{G}_n \) where \( s_1 \in 2\mathbb{Z} + (\frac{1}{2} \pm \frac{1}{2}) \), then \( \{s_2, \ldots, s_n\} \subset [2\mathbb{Z} + (\frac{1}{2} \mp \frac{1}{2})]^{n-1} \)

*Proof.* Is a direct consequence of lemma 1 and lemma 2.

**Lemma 4.** Let a sequence \( S = \{s_1, \ldots, s_n, k\} \in \mathbb{G}_{n+1} \) where \( s_1 \in 2\mathbb{Z} + (\frac{1}{2} \pm \frac{1}{2}) \), then \( K_S = \{x \in \min K_S; \max K_S/\forall x \in 2\mathbb{Z} + (\frac{1}{2} \mp \frac{1}{2})\} \)

*Proof.* From theorem 2, corollary 1, there are \( 2^n \) values of \( k \) that satisfy \( S \in \mathbb{G}_{n+1} \) and from definition 2 and definition 3, \( \min K_S \leq k \leq \max K_S \). \( K_S \) is defined as the set of all solutions of \( k \), hence it contains elements between \( \min K_S \) and \( \max K_S \). From lemma 3 it has already been shown that if \( s_1 \in 2\mathbb{Z} \), than \( s_0 \in 2\mathbb{Z} + 1, \forall a > 1 \) and if \( s_1 \in 2\mathbb{Z} + 1 \), than \( s_0 \in 2\mathbb{Z}, \forall a > 1 \), the theorem is proved from theorem 2, definition 2, definition 3 and lemma 3.

From lemma 4 is proved an important result regarding (2). (2) generates \( 2^n \) solutions for a sequence \( S = \{s_1, \ldots, s_n, k\} \), but from lemma 43 it has been proved that these solutions are only even or only odd according to the nature of the sequence. Therefore, the number of distinct solutions generated by (2) is \( 2^{n-1} \) since half solutions between \( \min K_S \) and \( \max K_S \) are equally divided between even and odd: \( \dim K_S = 2^{n-1} \).
Theorem 3. Let a sequence \( S = \{s_1, ..., s_n\} \in \mathbb{G}_n \) and \( S' = \{s_1, ..., s_n, k\} \), where \( s_1 \in 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right) \), then \( k \in \min \mathbb{K}_S; \max \mathbb{K}_S[\land k \in 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right) ] \Leftrightarrow S' \in \mathbb{G}_{n+1} \).

Proof. From definition 2 and definition 3, \( k \in \min \mathbb{K}_S; \max \mathbb{K}_S[\land k \in 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right) ] \) and from lemma 4, \( k \in 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right) \). Hence it is true \( k \in \min \mathbb{K}_S; \max \mathbb{K}_S[\land k \in 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right) ] \Rightarrow S' = \{s_1, ..., s_n, k\} \in \mathbb{G}_{n+1} \).

Suppose that \( S' \in \mathbb{G}_{n+1} \) but \( k \in \min \mathbb{K}_S; \max \mathbb{K}_S[\land k \in 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right) ] \) is false, so it is true \( k \notin \min \mathbb{K}_S; \max \mathbb{K}_S[\lor k \notin 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right) ] \). From definition 2, definition 3 and lemma 4 it is not possible to have \( S' \in \mathbb{G}_{n+1} \) if \( k \geq \max \mathbb{K}_S \lor k \leq \min \mathbb{K}_S \lor k \notin 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right) \). Hence it is also true \( k \in \min \mathbb{K}_S; \max \mathbb{K}_S[\land k \in 2\mathbb{Z} + \left(\frac{1}{2} \pm \frac{1}{2}\right) ] \Leftrightarrow S' = \{s_1, ..., s_n, k\} \in \mathbb{G}_{n+1} \). \( \Box \)

2.1 Notable upper and lower bound sequence

Definition 5. Let the sequence \( S = \{s_1, ..., s_n\} \in \mathbb{G}_n \),

\[
U'_S := \{\max \mathbb{K}_{\{s_1, ..., s_n\}}, \max \mathbb{K}_{\{s_1, ..., s_n, \max \mathbb{K}_{\{s_1, ..., s_n\}}\}}, \ldots\} = \{u'_s, ..., u'_{s_n}, \ldots\}
\]

(7)

Definition 6. Let the sequence \( S = \{s_1, ..., s_n\} \in \mathbb{G}_n \),

\[
L'_S := \{\min \mathbb{K}_{\{s_1, ..., s_n\}}, \min \mathbb{K}_{\{s_1, ..., s_n, \min \mathbb{K}_{\{s_1, ..., s_n\}}\}}, \ldots\} = \{l'_s, ..., l'_{s_n}, \ldots\}
\]

(8)

From definition 5 and definition 6 \( U_S = \{S, U'_S\} \) and \( L_S = \{S, L'_S\} \). Let \( S = \{s_1\}, U'_S = \{s_1 + 1, s_1 + 3, ..., s_1 + 2^{n-1} - 1\} \) and \( L'_S = \{s_1 - 1, s_1 - 3, ..., s_1 - 2^{n-1} + 1\} \).

No remarkable expression was found to be to analytically define the trend of \( U'_S \) and \( L'_S \) for a generic sequence \( S \) but it was observed that the exponential trend is preserved. However, this trend varies with the number of terms of \( U'_S \) and \( L'_S \) so it does not seem possible to establish what will be the \( n + 1 \)-th term of \( U'_S \) and \( L'_S \) given the previous \( n \) terms through an analytical formula. However, it is always possible use the recursive expression (2).

Let the sequence \( S = \{s_1, ..., s_n\} \in \mathbb{G}_n \), using definition 2, definition 3, definition 5, definition 6, the (3) can be rewritten as:

\[
u_{S_{n+m}} + l_{S_{n+m}} = u'_{S_m} + l'_{S_m} = 2s_n
\]

(9)

equivalent to (6).
If it is true that exponential trend is preserved, elements of $U'_S$ can be written in the form $u'_S = \alpha e^{\beta n}$ or $\log u'_S = \log \alpha + \beta n$.

The best fit for a dataset $D = \{d_1, ..., d_n\}$ in a linear regression model is

$$\beta = \frac{n \sum_{i=1}^{n} i \log d_i - \sum_{i=1}^{n} i \sum_{i=1}^{n} \log d_i}{n \sum_{i=1}^{n} i^2 - (\sum_{i=1}^{n} i)^2} = \frac{12}{n(n^2 - 1)} \left( \sum_{i=1}^{n} i \log d_i - \frac{n+1}{2} \sum_{i=1}^{n} \log d_i \right)$$

(10)

$$\log \alpha = 1 - \frac{n \sum_{i=1}^{n} \log d_i - \beta(n+1)}{2}$$

(11)

hence

$$\alpha = e^{-\frac{\beta(n+1)}{2}} \left( \prod_{i=1}^{n} d_i \right)^{\frac{1}{n}}$$

(12)

and the coefficient of determination is

$$R^2 = 1 - \frac{\sum_{i=1}^{n} (d_i - \alpha e^{\beta n})^2}{\sum_{i=1}^{n} (d_i - \bar{d})^2}$$

(13)

Note that in $D$ if $|d_a| < |d_{a+1}|$ and $d_a < 0$, it is not possible to calculate $\log d_b$, where $b > a$. To avoid this problem the transformation $d_i \to d_i + \frac{d_1}{|d_a|} \left( \frac{d_1}{|d_a|} - 1 \right)$ is performed. In this way, if $d_1 > 0$, $d_i \to d_i$ and if $d_1 < 0$, $d_i \to d_i - d_1$.

After that, the fitting curve will be $d_n = \alpha e^{\beta n} - \frac{d_1}{2} \left( \frac{d_1}{|d_a|} - 1 \right)$.

**Example 1.** Let the sequence $S = \{2, 3, 5, 7, 11, 13\}$ of length 6, the first 18 terms of the upper bound sequence are $U'_S = \{21, 47, 119, 297, 705, 1595, 3475, 7365, 15309, 31399, 63823, 128961, 259577, 521203, 1044907, 2092829, 4189253, 8382751\}$ and the first 18 terms of the lower bound sequence are $L'_S = \{5, -21, -93, -271, -679, -1569, -3449, -7339, -15283, -31373, -63797, -128935, -259551, -521177, -1044881, -2092803, -4189227, -8382725\}$.

According to (6) and to (9), $21 + 5 = 47 - 21 = 119 - 93 = 297 - 271 = \ldots = 8382751 - 8382725 = 26 = 2s_6$. Let begin fitting $U'_S$ to $\alpha U'_S e^{\beta U'_S n}$. From (10), $\beta_{U'_S} = \frac{6}{2907} \left( \sum_{i=1}^{18} i \log u'_S - \frac{19}{2} \sum_{i=1}^{18} \log u'_S \right) \approx 0.75$, from (12) $\alpha_{U'_S} = e^{-\beta_{U'_S}} \left( \prod_{i=1}^{18} u'_S \right)^{\frac{1}{19}} \approx 14.42$. The model fits the trend of $u'_S$ with $R^2 \approx 0.92$ from (13). As regards $L'_S$, from (6), $l'_S = 2s_6 - \alpha U'_S e^{\beta U'_S n} \approx 26 - 14.42e^{0.75n} \approx -7.97e^{0.80n}$ with $R^2 \approx 0.99$. 

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As explained above, the addition of a term to $U'_S$ leads to new values of $\alpha$ and $\beta$, therefore this analysis can be carried out without pretending to evaluate the $n + 1$-th element of a given $U'_S$ of length $n$.

The numerical analysis of the values of the upper limit sequence was added only to show that no analytical formula has been found for the generations of the values of this sequence, with exception of (2).

3 Proof of conditions for $P = \{p_1, \ldots, p_n\}$

Let $S = \{s_1, \ldots, s_n\} = \{f(n)\}$, from theorem 1 and 2, is true the following. The relationship $s_2 = s_1 \pm 1$ must be true, otherwise it would not be true that $s_1 = 1$, hence $f(2) = f(1) \pm 1$. As a consequence of theorem 3, for all elements subsequent to $s_1$, the absolute difference of two successive elements must be an integer multiple of 2 so as to maintain the absolute difference of two successive elements as an even value. So, if the first element in the sequence is even, the subsequent elements must be odd and if the first element is odd, the subsequent elements must be even.

As a consequence of theorem 2, solution of the Gilbreath’s equation (2) and definition 2 and definition 3: each $n$-th element of a sequence $S$ must be within the range between the upper and the lower sequences calculated on all the elements prior to the $n$-th ones. Hence, from theorem 2 and according to the solution of the Gilbreath’s equation at (2), cannot exists a Gilbreath’s sequence in which the $n$-th is larger than $\max K_{\{s_1, \ldots, s_{n-1}\}}$, since $\max K_{\{s_1, \ldots, s_{n-1}\}}$ is the maximum value that the $n$-th value can take according to (2). The same goes for $\min K_{\{s_1, \ldots, s_{n-1}\}}$, since it is the smallest value that the $n$-th value can take, according to (2). Hence:

$$l_{\{s_1, \ldots, s_{n-1}\}} \leq f(n) \leq u_{\{s_1, \ldots, s_{n-1}\}}$$

(14)

Following the results obtained in the previous paragraphs about Gilbreath’s sequence and Gilbreath’s equation, let proceed discussing the Gilbreath’s conjecture. The results obtained so far will be used to establish if theorem (4) is true for an ordered sequence of prime numbers $P$.

**Theorem 4.** For every $n$-th prime number, $n > 1$ it is true that $l_{\{p_1, \ldots, p_{n-1}\}} \leq p_n \leq u_{\{p_1, \ldots, p_{n-1}\}}$. 

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Proof. By definition of $L$ and $U$, $l_{\{p_1,\ldots,p_{n-1}\}} = \min K_{\{p_1,\ldots,p_{n-1}\}}$ and $u_{\{p_1,\ldots,p_{n-1}\}} = \max K_{\{p_1,\ldots,p_{n-1}\}}$, hence the statement becomes:

$$\min K_{\{p_1,\ldots,p_{n-1}\}} \leq p_n \leq \max K_{\{p_1,\ldots,p_{n-1}\}} \quad (15)$$

Let $S = \{p_1, p_2\} = \{2, 3\} \in \mathbb{G}_2$ a Gilbreath’s sequence formed by the first two prime numbers. As $S \in \mathbb{G}_2$, from (3), it is true that $\min K_{\{p_1, p_2\}} \leq p_2 \leq \max K_{\{p_1, p_2\}}$, it is also true, for theorem 2 corollary 3, that $\{p_1, p_2, p_3\} \in \mathbb{G}_3$ and for every prime number is true that $p_n > p_{n-1}$. Since $\min K_{\{p_1, p_2\}} \leq p_2$, it is certainly true that $\min K_{\{p_1, p_2\}} \leq p_3$. The left inequality of (15) is proved for $n = 3$.

If $p_3 \leq \max K_{\{p_1, p_2\}}$, then, subtracting the quantity $2p_2$ from both sides, $p_3 - 2p_2 \leq \max K_{\{p_1, p_2\}} - 2p_2$. From Bertrand’s postulate, $p_n < 2p_{n-1}$, hence $p_3 - 2p_2 < 0$. Replacing (3) in the previous inequation: $\min K_{\{p_1, p_2\}} \leq \alpha$, where $\alpha > 0$. Hence, exist a value $\alpha > 0$ such that $\min K_{\{p_1, p_2\}} \leq \alpha$. The value of $\min K_{\{p_1, p_2\}}$ can be replaced using (2): $\min K_{\{p_1, p_2\}} = -p_1^2 + p_2 - 1$ where $-p_1^2 - 1 < 0$ and $p_2 > 0$. If $\alpha = p_2$, the relation $\min K_{\{p_1, p_2\}} \leq \alpha$, where $\alpha > 0$ is true, hence it is true that $p_3 \leq \max K_{\{p_1, p_2\}}$. The right inequality of (15) is proved for $n = 3$.

At this point the proof can process showing that (15) is true for $n = 4$. Since, $p_4 > p_3$ and $\min K_{\{p_1, p_2, p_3\}} \leq p_3 \leq \max K_{\{p_1, p_2, p_3\}}$, it is true that $\min K_{\{p_1, p_2, p_3\}} \leq p_4$. Again, the statement $p_4 \leq \max K_{\{p_1, p_2, p_3\}}$ is equivalent to the statement $\min K_{\{p_1, p_2, p_3\}} \leq \alpha$, where $\alpha > 0$. From the equation (2), $\min K_{\{p_1, p_2, p_3\}} = -p_1^2 + p_2 + p_3 - 1$ where $-p_1^2 - p_2^2 - 1 < 0$ and $p_3 > 0$. If $\alpha = p_3$, the relation $\min K_{\{p_1, p_2, p_3\}} \leq \alpha$, where $\alpha > 0$. Iteratively, this can be proved to verify (15) for every prime.

Lemma 3 is already proved since $p_1 = 2$ is even and all other elements are odd: by definition of prime number, there are no even prime numbers except for 2. From theorem 4 and lemma 3 is proved theorem 3 for $P$.

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