SYMMETRY OF SOLUTIONS TO A CLASS OF MONGE-AMPÈRE EQUATIONS

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Abstract. We study the symmetry of solutions to a class of Monge-Ampère type equations from a few geometric problems. We use a new transform to analyze the asymptotic behavior of the solutions near the infinity. By this and a moving plane method, we prove the radially symmetry of the solutions.

1. Introduction. The radial symmetry for the positive solutions of the equations \( \Delta u + f(u) = 0 \) for \( x \in \mathbb{R}^n \) was studied in [14] under the ground state conditions: \( u(x) \to 0 \) as \( |x| \to \infty \). Afterwards, a lot of works have been published along this direction, dealing with various symmetry problems for semi-linear elliptic equations. See, for example [1, 2, 4, 6, 11, 25] and the references therein. The radial symmetry for quasi-linear elliptic equations and fully nonlinear elliptic equations were also studied in [13, 12, 29] and in [2, 7, 24, 26] respectively. However, all those symmetric results, up to authors’ knowledge, need the a priori ground state condition.

In this paper, we study the radial symmetry for the solutions to the following equations of Monge-Ampère equation type:
\[
\det D^2 u(x) = F(u(x), x \cdot Du(x) - u(x), |Du(x)|^2), \quad x \in \mathbb{R}^n, \quad (1)
\]
where \( n \geq 2 \) is a positive integer.

Recall that a \( C^2 \)-function \( h(x) \) is called strictly convex in \( \Omega \) if its hessian matrix \( [D^2h(x)] \) is positive definite in \( \Omega \).

Since the solutions we considered for equation (1) are strictly convex, we may assume that the origin is the minimum point of \( u \) so that
\[
u(0) = 0, \quad Du(0) = 0 \quad (2)
\]
and
\[
D^2u(0) = I \quad (the \ unit \ matrix) \quad (3)
\]
by translating, rotating and scaling the coordinates. Note that equation (1) is invariant under these coordinates transforms.

The solutions of (1) do not satisfy the ground state condition. In fact, because of the convexity it tends to infinity as \( x \to \infty \). Usually, it satisfies
\[
\lim _{x \to \infty} Du(x) = \infty, \quad (4)
\]

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which will bring us many technique difficulties. Our new idea to overcome the difficulties is to use the transform

\[ y = \frac{x}{u(x)}, \quad v(y) = \frac{1}{u(x)} \quad (5) \]

introduced in [22], which is similar to the Kelvin transform in conformal geometry. This transform reduces equation (1) to an equation of the same type but with an isolated singular point in \( R^n \). However, by the advantages of this transform we will obtain the behavior of \( v \) near \( \infty \) from that of \( u \) near 0. The latter can be easily understood because of (2), (3) and the convexity.

To state our main results, we need the following structure assumptions on \( F \):

**A1.** \( F(t) = F(t_1, t_2, t_3) \) is well-defined in \( R^3 \), \( F(t) > 0 \) for all \( t \in R^3_+ \equiv \{ t = (t_1, t_2, t_3) \in R^3 : t_3 > 0, t_3 \geq 0 \} \) and \( F \in C^1(R^3_+) \);

**A2.** For any \( b_1 > 0 \) and \( b_3 \geq 0 \), the function \( f(s) \equiv s^{n+2}F(b_1, \frac{1}{s}, \frac{b_3}{s^2}) \) is strictly decreasing for \( s \in (0, \infty) \).

**Theorem 1.1.** Suppose that \( F \) satisfies **A1** and **A2**. If \( u \in C^4(R^n) \) is strictly convex and satisfies (1)-(4) and

\[ u_{x_i, x_j, x_k}(0) = 0, \forall i, j, k = 1, 2, \cdots, n, \quad (6) \]

then \( u \) is radially symmetric about the origin.

For the existence, uniqueness and the regularity of radially symmetric solution for (1), see [5] and the references therein.

Observe that the convexity and (2) implies that

\[ u(x) \geq 0, \quad x \cdot Du(x) - u(x) \geq 0, \quad \forall x \in R^n, \quad (7) \]

which will be used in Section 2 and Section 3 for the proof of Theorem 1.1.

Our study is motivated by a few geometric equations. The first one is the equation of the form

\[ \det D^2 v(x) = \frac{K_1(x)}{v^p}. \quad (8) \]

An important special case is \( p = n + 2 \), which describes the Minkowski problem in Centroaffine Geometry, and \( K_1(x) \) is the Centroaffine Gauss curvature function of the convex hypersurface as the graph of \( u \). See, for example [9, 17, 20, 21, 23]. In particular, when \( K_1 \) is a positive constant, (8) is exactly the equation for affine hyperbolic affine spheres [8]. For general \( p \), Equation (8) is reduced from the study of Minkowski problem for complete noncompact convex hypersurfaces [10] and \( L_p \)-Minkowski problem [18, 19, 21, 27].

Using Legendre transform

\[ u(x) = y \cdot Du(y) - v(y), \quad x = Du(y), \quad (9) \]

we see that (8) is reduced to

\[ \det D^2 u(x) = \frac{[x \cdot Du(x) - u(x)]^p}{K_1(Du(x))}, \quad (10) \]

which is the special case of (1) if \( K_1(x) = K_1(|x|) \).

**Example 1.2.** Equation (10) satisfies A1 – A2 if \( K_1(x) = K_1(|x|) \) satisfies one of the followings:

(i) \( K_1(|x|) = |x|^\alpha \) for \( p \geq 0 \) and \( p - 2 - n < \alpha \leq -2 \);

(ii) \( K_1(|x|) = 1 + |x|^\alpha \) for \( 0 \leq p \leq n + 2 \) and \( \alpha \geq 2 \);

(iii) \( K_1(|x|) = (1 + |x|^\alpha)^{-1} \) for \( 0 \leq p \leq n + 2 - \alpha \) and \( \alpha \geq 2 \).
The second model motivating our study is the equation in the form of
\[ \det D^2 v(x) = (1 + |Dv(x)|^2)^p K_2(x, v(x), Du(x)) \] in \( \mathbb{R}^n \), \( n \geq 2 \), (11)
where \( p = \frac{n+2}{2} - \frac{1}{2\alpha} \) for some number \( \alpha \). Equation (11) is the equation of translating solutions to Gauss curvature flow \( v = [ K_3(x, v(x), Du(x)) ]^{\alpha} \) where \( \nu \) is the velocity along the normal direction of the moving hyper-surface and \( K \) is its Gauss curvature. See \([22, 30]\) for the details. When \( K \) is a positive constant and \( \alpha = 1 + 2 n \), the classical results of Jorgens \((n = 2, [16])\), Calabi \((n \leq 5, [3])\) and Pogorelov \((n \geq 2, [28])\) assert that any convex solution to (11) must a quadratic polynomial. Obviously, it is radially symmetric under (2) and (3). When \( K \) is a positive constant and \( \alpha \in (0, \frac{1}{2}) \), there exists a radially symmetric solution to (11) as well as infinitely many smooth, non-rotationally symmetric convex solutions to (11). See Theorem 6 in \([30]\) and Theorem 1.2 in \([22]\) respectively.

**Example 1.3.** Equation (11) satisfies \( A_1 - A_2 \) if \( K_3(x, v(x), Du(x)) = [xDv(x) - v(x)]^{\beta} \hat{K}(v(x)) \) with \( \hat{K} \in C^1(0, \infty), \hat{K} > 0 \) and \( \beta \leq n + 1 - 2p \).

The existence for (1), especially for (8) (or (10)) and (11), are better understood. See the references mentioned above. But little is known for the uniqueness or the behavior of the solutions near \( \infty \). Theorem 1.1 can be viewed as the first step in understanding this respect. In fact it can be viewed as a generalization of the classical results of Jorgens, Calabi and Pogorelov as mentioned above.

### 2. A transform and asymptotic behavior

In this section, we use the transform introduced in \([21]\) to reduce the behavior of \( u \) near the origin to that of the new function near \( \infty \). The advantage of this transform is to preserve the form of equation (1). We will study the properties of the new functions.

Consider the transform
\[ y = \frac{x}{u(x)} \quad \text{and} \quad v(y) = \frac{1}{u(x)}, \] (12)
where we assume \( u \in C^2 \) and \( u > 0 \). Then
\[ x = \frac{y}{v(y)} \quad \text{and} \quad u(x) = \frac{1}{v(y)}. \]

Denoting \( u_i = \frac{\partial u}{\partial x_i} \) and \( v_i = \frac{\partial v}{\partial y_i} \), we have
\[ u_i = -\frac{1}{v^2} \sum_{j=1}^{n} \frac{\partial v}{\partial y_j} \frac{\partial}{\partial x_i} \left[ \frac{x_j}{u(x)} \right] = -\frac{v_i}{v} + \frac{y \cdot Du u_i}{v}. \]
From this we solve \( u_i \) as
\[ u_i = \frac{v_i}{y \cdot Du - v}, \] (13)
which implies
\[ x \cdot Du - u = \frac{1}{y \cdot Du - v} \] (14)
and
\[ u_{ij} := u_{x_i x_j} = \frac{v}{y \cdot Du-v} (\delta_{ik} - v_i b_k) (v_{kl} - b_l v_j), \] (15)
where
\[ v_{ij} := v_{y_i y_j} \quad \text{and} \quad b_k = \frac{y_k}{y \cdot Du - v}. \]
Since
\[ \det(\delta_{jk} - v_jb_k) = 1 - \sum_{i=1}^{n} v_ib_i, \]
we have
\[ \det D^2u = \left[ \frac{v}{y \cdot Dv - v^2} \right]^{n+2} \det(D^2v). \quad (16) \]

**Lemma 2.1.** Suppose that a strictly convex function \( u \in C^2(\mathbb{R}^n) \) satisfies (1), (2) and (4). Let \( y \) and \( v \) be given by (12). Then
\[ \{ y = \frac{x}{u(x)} : x \in \mathbb{R}^n \setminus \{0\} \} = \mathbb{R}^n \setminus \{0\} \]
and \( v \in C^2(\mathbb{R}^n \setminus \{0\}) \) is a strictly convex and positive function, satisfying equation
\[ \det D^2v = \left[ \frac{y \cdot Dv - v^2}{v^2} \right]^{n+2} \frac{1}{y \cdot Dv - v^2} \left( \frac{|Dv|^2}{(y \cdot Dv - v^2)^2} \right), \quad \forall y \in \mathbb{R}^n \setminus \{0\}. \quad (17) \]

*Proof.* Under transform (12) we have \( \{ y = \frac{x}{u(x)} : \frac{u(x)}{|x|} = \frac{x}{|x|} : x \in \mathbb{R}^n \setminus \{0\} \} = S^{n-1}. \) It follows from (2) and (3) that
\[ \lim_{x \to 0} \frac{x}{u(x)} = \lim_{x \to 0} \frac{2x}{\mathbb{R}^2 u(\xi) x^\top} = \infty, \]
where \( \xi \) is a point in the segment connecting 0 and \( x \). On the other hand, we have, by (4) and the strictly convexity, that \( \lim_{x \to \infty} \frac{u(x)}{|x|} = 0 \). Hence,
\[ \{ y = \frac{x}{u(x)} : x \in \mathbb{R}^n \setminus \{0\} \} = \mathbb{R}^n \setminus \{0\}. \]
Since
\[ u(0) = u(x) - x \cdot Du(x) + \frac{1}{2} xD^2u(\xi)x^\top, \]
x \( \cdot \) Du(x) - u(x) > 0 for all \( x \in \mathbb{R}^n \setminus \{0\} \) by the convexity of \( u \) and (2). Hence
\[ v > 0 \text{ and } y \cdot Dv(y) - v(y) > 0, \forall y \in \mathbb{R}^n \setminus \{0\} \]
by (14). Therefore, (15) implies that \( [v_{ij}]_{n \times n} \) is positive definite and so \( v \) is strictly convex for \( y \in \mathbb{R}^n \setminus \{0\} \). Finally, (17) follows directly from (1), and (12)-(16). \( \square \)

**Lemma 2.2.** Let \( u \in C^3(B_r(0)) \) for some \( \gamma > 0 \). If \( u \) satisfies (2) and (3), there is a function \( h \in C^1(B_r(0)) \) such that \( h(0) = 0 \) and
\[ |Dh(x)| \leq M, \quad u(x) = \frac{|x|^2}{2} + |x|^2 h(x) \]
for any \( x \) near 0, where
\[ M := 1 + 4n \sum_{i,j,k=1}^{n} |u_{x_i x_j x_k}(0)|. \quad (19) \]

*Proof.* By (2), (3) and Taylor expansion,
\[ u(x) = \frac{|x|^2}{2} + \theta(x), \quad \text{as} \quad x \to 0 \]
where \( \theta(x) = \sum_{i,j,k=1}^{n} u_{x_i x_j x_k}(\xi)x_i x_j x_k \) and \( \xi \to 0 \) as \( x \to 0 \). Since
\[ u \in C^3(B_r(0)), \quad D\theta(x) = Du(x) - x, \]
we have, by Taylor expansion again, that
\[
\theta_\xi(x) = 3 \sum_{j,k=1}^n u_{x_jx_k}(\xi)x_kx_j
\]
for some $\xi$ depending on $x$. Hence,
\[
\frac{|D\theta|}{|x|^2} \leq 3n \sum_{i,j,k=1}^n |u_{x_ix_j}(\xi)| \text{ when } x \text{ near } 0.
\]

Letting
\[
h(x) = \begin{cases} \frac{\partial}{\partial x'}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}
\]
we obtain the desired result. \hfill \square

**Lemma 2.3.** Let $u \in C^4(B_0(0))$ for some $\gamma > 0$, satisfying (2) and (3). If $y$ and $v(y)$ are given by (12), then there exists a $r_0 > 0$ such that for all $\lambda \geq 3M$,
\[
v(2\lambda - y, y') - v(y) > 0, \forall y = (y_1, y') \in \mathbb{R}^n \setminus B_{r_0}(0) \text{ with } y_1 < \lambda,
\]
where $M$ is the same constant as in Lemma 2.2.

**Proof.** Let $y = (y_1, y_2, \ldots, y_n) := (y_1, y')$ and
\[
y^\lambda = (2\lambda - y_1, y'), \quad v_\lambda(y) = v(y^\lambda).
\]

It is obvious that
\[
x \to 0 \quad \text{and} \quad u(x) \to 0 \iff y \to \infty \quad \text{and} \quad v(y) \to \infty.
\]

Since Lemma 2.2 implies
\[
v(y) = \frac{|y|^2}{2} + |y|^2h\left(\frac{y}{v(y)}\right), \quad \text{as} \quad y \to \infty,
\]
we have, for any $\lambda > 0$, any $|y| > r$ (large enough) such that $y_1 < \lambda$, that
\[
v_\lambda(y) - v(y) = \frac{1}{2}(|y^\lambda|^2 - |y|^2) + |y|^2h\left(\frac{y^\lambda}{v_\lambda(y)}\right) - |y|^2h\left(\frac{y}{v(y)}\right)
\]
\[
= 2\lambda(\lambda - y_1) + 4\lambda(\lambda - y_1)h\left(\frac{y^\lambda}{v_\lambda(y)}\right) + |y|^2Dh(\xi)\left(\frac{y^\lambda}{v_\lambda(y)} - \frac{y}{v(y)}\right)
\]
where $\xi$ is a point in the segment connecting $\frac{y}{v(y)}$ and $\frac{y^\lambda}{v_\lambda(y)}$.

Observing that
\[
\lim_{y \to \infty} \frac{y}{v(y)} = 0, \quad \lim_{x \to 0} h(x) = 0, \quad \lim_{y \to \infty} \frac{v(y)}{|y|^2} = \frac{1}{2}
\]
and this, together with the fact $|y| < |y^\lambda|$ for all $\lambda > 0$ with $y_1 < \lambda$, implies
\[
\lim_{y \to \infty} \frac{y^\lambda}{v_\lambda(y)} = 0 \quad \text{uniformly for all } \lambda > 0,
\]
we can choose large $r$ such that for all $\lambda > 0$ and all $|y| > r$ with $y_1 < \lambda$
\[
v_\lambda(y) - v(y) > \frac{9}{5}\lambda(\lambda - y_1) + |y|^2Dh(\xi)\left(\frac{y^\lambda}{v_\lambda(y)} - \frac{y}{v(y)}\right), \forall|y| > r \text{ with } y_1 < \lambda
\]
and
\[
\frac{3}{2} \leq \frac{|y|^2}{v(y)} \leq \frac{5}{2}, \quad \forall|y| > r.
\]
Hence, we have, by Lemma 2.2, that for all $|y| > r$ with $y_1 < \lambda$,
\[
|y|^2 Dh(\xi)(\frac{y^\lambda}{v_\lambda(y)} - \frac{y}{v(y)}) = \frac{|y|^2}{v(y)} Dh(\xi)[2(\lambda - y_1, 0) - \frac{y^\lambda(v_\lambda(y) - v(y))}{v_\lambda(y)}] \\
\geq -5M(\lambda - y_1) - \frac{|y|^2}{v(y)} Dh(\xi)\frac{y^\lambda}{v_\lambda(y)}(v(y) - v_\lambda(y)).
\]
Substituting this inequality into (22) and using (20) and (21), we prove the Lemma.

**Lemma 2.4.** Let $u \in C^4(B_r(0))$ for some $\gamma > 0$, and satisfying (2) and (3). If $y$ and $v(y)$ are given by (12), then there is a function $H(y)$, differentiable for large $y$, such that as $y \to \infty$, 
\[
v(y) = \frac{|y|^2}{2} + \frac{A(y)}{3|y|^2} + H(y)
\]
and
\[
H(y) = O(1), \quad DH(y) = O\left(\frac{1}{|y|}\right),
\]
where $A(y) := \sum_{i,j,k=1}^n a_{ijk}y_iy_jy_k$ and $a_{ijk} = u_{x_i,x_j,x_k}(0)$.

**Proof.** It follows from (2) and (3) that as $x \to 0$,
\[
u(x) = \frac{|x|^2}{2} + \frac{1}{6} \sum_{i,j,k=1}^n a_{ijk}x_i x_j x_k + O(|x|^4)
\]
(24)
and
\[
u_{x_i}(x) = x_i + \frac{1}{2} \sum_{j,k=1}^n a_{ijk}x_j x_k + O(|x|^3).
\]
The second equality, together with (12), implies that as $y \to \infty$,
\[
v_{y_i} = -\frac{1}{u^2(x)} \sum_{j=1}^n u_{x_j} \left[\frac{y_j}{v(y)}\right] y_i \\
= -[x_j + \frac{1}{2} \sum_{l,k=1}^n a_{ilk}x_l x_k + O(|x|^3)] [\delta_{ij} v(y) - y_j v_{y_i}] \\
= -y_i - \frac{1}{2v(y)} \sum_{j,k=1}^n a_{ilk}y_k y_k - v(y)O\left(\frac{y}{v(y)} |^3\right) \\
+ v_{y_i} \left[\frac{|y|^2}{v(y)} + \frac{A(y)}{2v^2(y)} + O\left(\frac{y}{v(y)} |^3\right) \sum_{j=1}^n y_j\right].
\]
(25)
It is obvious from (23) that as $y \to \infty$,
\[
\frac{A(y)}{2v^2(y)} = O\left(\frac{1}{|y|}\right), \quad v(y)O\left(\frac{y}{v(y)} |^3\right) = O\left(\frac{1}{|y|}\right), \quad O\left(\frac{y}{v(y)} |^3\right) \sum_{j=1}^n y_j = O\left(\frac{1}{\sqrt{|y|}}\right).
\]
This and (25) yields
\[
v_{y_i} = -y_i - \frac{1}{2v(y)} \sum_{l,k=1}^n a_{ilk}y_k y_k + O\left(\frac{1}{|y|}\right).
\]
(26)
Lemma 3.1. Let
\[ \frac{1}{v(y)} = \frac{|y|^2}{2v^2(y)} + \frac{A(y)}{6v^3(y)} + O\left(\frac{1}{|y|^4}\right) = \frac{|y|^2}{2} \left(\frac{1}{v(y)}\right)^2 + O\left(\frac{1}{|y|^3}\right). \] (27)

Solving this for \( \frac{1}{v(y)} \) yields
\[ \frac{1}{v(y)} = \frac{2}{|y|^2} + O\left(\frac{1}{|y|^3}\right) : = \frac{2}{|y|^2} + h_1(y) \]
and so
\[ \frac{1}{v^2(y)} = \frac{8}{|y|^6} + O\left(\frac{1}{|y|^7}\right). \]
Substituting the last two equalities in (27), we have
\[ \frac{|y|^2}{2} h_1^2(y) + h_1(y) + \frac{4A(y)}{3|y|^6} + O\left(\frac{1}{|y|^4}\right) = 0, \]
which yields
\[ h_1(y) = -\frac{4A}{3|y|^6} + O\left(\frac{1}{|y|^4}\right). \]
Hence, we obtain, as \( y \to \infty \), that
\[ \frac{1}{v(y)} = \frac{1}{|y|^2} \left[ 2 - \frac{4A(y)}{3|y|^4} + O\left(\frac{1}{|y|^2}\right) \right], \quad v(y) = \frac{|y|^2}{2} + \frac{A(y)}{3|y|^2} + O(1) \]
and
\[ v_{y_i} = y_i + \left(\frac{A(y)}{3|y|^2}\right) x_i + O\left(\frac{1}{|y|}\right) \] (28)
by (26).

Write \( v(y) = \frac{|y|^2}{2} + \frac{A(y)}{3|y|^2} + H(y) \). Then
\[ H(y) = O(1), \quad DH(y) = O\left(\frac{1}{|y|}\right) \]
by (28). This proves the Lemma. \( \Box \)

3. Proof of Theorem 1.1. In this section, we state two preliminary results for the sake of convenience.

**Lemma 3.1.** Let \( w \in C^2(\Omega) \) be a nonnegative solution to
\[ \sum_{i,j=1}^{n} a^{ij}(x) \partial_{ij} w + \sum_{i=1}^{n} b^i(x) \partial_i w + C(x)w \leq 0, \forall x \in \Omega, \] (29)
where \( \Omega \) is an open set in \( \mathbb{R}^n \), \( a^{ij}, b^i, C(x) \in L^\infty(\Omega') \) and the matrix \( [a^{ij}(x)] \) is positive in \( \Omega' \) for any compact set \( \Omega' \subset \Omega \). Then either \( w \equiv 0 \) in \( \Omega \) or \( w(x) > 0 \) for all \( x \in \Omega \). Moreover, if \( w(x_0) > 0 \) for some \( x_0 \in \Omega \) and \( w(\bar{x}) = 0 \) for some \( \bar{x} \in \partial \Omega \) which is smooth near \( \bar{x} \), then \( \frac{\partial w}{\partial \nu}(\bar{x}) < 0 \) where \( \nu \) is the unit outer normal of \( \partial \Omega \).

The proof can be found in [15].

**Lemma 3.2.** Suppose that \( \text{diam}(\Omega) \leq d \), \( a^{ij}, b^i, C(x) \in L^\infty(\Omega) \) and the matrix \( [a^{ij}(x)] \) is positive in \( \Omega \). Let \( w \in C^2(\Omega) \) satisfies (29) and \( \lim_{x \to \partial \Omega} w(x) \geq 0 \). There exists a \( \delta > 0 \) depending only on \( n, d \) and the bound of the coefficients such that \( w(x) \geq 0 \) in \( \Omega \) provided that the measure \( |\Omega| < \delta \).
This is exactly Proposition 1.1 in [2].

**Proof of Theorem 1.1.** Assume \( u \in C^4(\mathbb{R}^n) \) is strictly convex and satisfies (1)-(4) and (6). Let \( y \) and \( v(y) \) be given by (12). It is enough to prove that \( v \) is radially symmetric about the origin in \( \mathbb{R}^n \setminus \{0\} \).

It follows Lemma 2.1 that \( v \in C^4(\mathbb{R}^n \setminus \{0\}) \) is strictly convex, satisfying

\[
\det D^2 v(x) = \left[ \frac{I(v)(x)}{v(x)} \right]^{n+2} F\left( \frac{1}{v(x)}, \frac{1}{I(v)(x)}, \frac{|Dv(x)|^2}{(I(v)(x))^2} \right) \tag{30}
\]

for all \( x \in \mathbb{R}^n \setminus \{0\} \), where

\[
I(v)(x) := x \cdot Dv(x) - v(x) \tag{31}
\]

and

\[
v > 0 \text{ in } \mathbb{R}^n \setminus \{0\}, \lim_{x \to 0} v(x) = 0, \text{ and } \lim_{x \to \infty} v(x) = \infty. \tag{32}
\]

We will use the moving planes method to prove that \( v \) is radially symmetric with respect to the origin. This needs to show that \( v \) is symmetric in any direction with respect to the origin. Since equation (30) is invariant under orthogonal transforms, it is sufficient to do this in one direction. Without loss of generality, we will do it in \( e_1 \)-direction. In a word, to show Theorem 1.1, it is enough to prove that

\[
v(x_1, x_2, \cdots, x_n) = v(x_1, x_2, \cdots, x_n), \quad \forall x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \setminus \{0\}. \tag{33}
\]

Use \((x_1, x') := (x_1, x_2, \cdots, x_n)\) to denote a point \( x \) in \( \mathbb{R}^n \). For any \( \lambda \in \mathbb{R} \), define

\[
\sum(\lambda) = \{ x = (x_1, x') \in \mathbb{R}^n : x_1 > \lambda \}, \quad T(\lambda) = \{ x \in \mathbb{R}^n : x_1 = \lambda \},
\]

and

\[
x^\lambda = (2\lambda - x_1, x'), \quad v^\lambda(x) = v(x^\lambda), \quad w^\lambda(x) = v(x) - v^\lambda(x).
\]

Denote the term on the right hand side of equation (30) by \( G(v)(x) \), i.e.,

\[
G(v)(x) := \left[ \frac{I(v)(x)}{v(x)} \right]^{n+2} F\left( \frac{1}{v(x)}, \frac{1}{I(v)(x)}, \frac{|Dv(x)|^2}{(I(v)(x))^2} \right)
\]

where \( I(v) \) is defined as (31). Let

\[
G_\lambda(v)(x) := \left[ \frac{I_\lambda(v)(x)}{v_\lambda(x)} \right]^{n+2} F\left( \frac{1}{v_\lambda(x)}, \frac{1}{I_\lambda(v)(x)}, \frac{|Dv_\lambda(x)|^2}{(I_\lambda(v)(x))^2} \right)
\]

where

\[
I_\lambda(v)(x) := \begin{cases} x_1 v_{x_1}(x) + \sum_{i=2}^n x_i v_{x_i}(x_1) - v_{x_1}(x), & \text{if } v_{x_1}(x^\lambda) \geq 0, \\ I(v_\lambda)(x) = \sum_{i=1}^n x_i v_\lambda(x_1) - v_\lambda(x), & \text{if } v_{x_1}(x^\lambda) < 0. \end{cases}
\]

One key to use the moving planes method is to introduce the following differential operator:

\[
L_\lambda(v)(x) = a_\lambda^i(x) \partial_{x^i} w_\lambda(x) + b_\lambda^i(x) \partial_{x^i} w_\lambda(x) + C_\lambda(x) w_\lambda(x), \forall x \in \Omega \bigcap \sum(\lambda)
\]

By assumption A1 and a mean value theorem, we have the following obvious result.

**Lemma 3.3.** Let \( r \) be any positive constant. If \( v \in C^2(\overline{B_r}(0) \setminus \{0\}) \) is bounded, positive and strictly convex, then for any compact \( \Omega \) in \( \overline{B_r}(0) \setminus \{0^4\} \), there exist a constant \( C_1 > 0 \) independent of \( \lambda \in (0, \infty) \) (but depending on \( \Omega \)) and piecewise continuous functions \( \{a_\lambda^i(x)\}, \{b_\lambda^i(x)\}, C_\lambda(x) \), (all depending on the \( v \) and its derivatives up to second order in \( \Omega \)), such that

\[
L_\lambda(v)(x) = a_\lambda^i(x) \partial_{x^i} w_\lambda(x) + b_\lambda^i(x) \partial_{x^i} w_\lambda(x) + C_\lambda(x) w_\lambda(x), \forall x \in \Omega \bigcap \sum(\lambda)
\]
and
\[ C_1^{-1}I \leq (\sigma^j_i(x)) \leq C_1I, \quad |\theta^j_k(x)| + |\theta^j_k(x)| \leq C_1, \forall x \in \Omega \cap \sum(\lambda). \]

We will complete the proof of Theorem 1.1 by four Steps.

**Step 1. (i)** There exists a \( \lambda > 4M \) (depending only on the function \( v \)) such that
\[ w_\lambda > 0 \text{ in } \sum(\lambda) \setminus \{0^\lambda\}, \forall \lambda \in [\bar{\lambda}, \infty). \]

To show (i), we use the fact \( \lim_{y \to \infty} v(y) = +\infty \) in (32) to find a \( \bar{\lambda} > 3M \) such that for all \( \lambda \in [\bar{\lambda}, \infty) \)
\[ v(x) > v_\lambda(x) = v(x^\lambda), \quad \forall x \in \sum(\lambda) \setminus \{0^\lambda\} \ such \ that \ x^\lambda \in B_{r_0}(0), \]

where \( M, r_0 \) are the same as in Lemmas 2.2 and 2.3. This, together with Lemma 2.3 again, implies
\[ w_\lambda > 0 \text{ in } \sum(\lambda) \setminus \{0^\lambda\}, \forall \lambda \in [\bar{\lambda}, \infty). \quad (34) \]

This proves (i).

For the proof of (ii), we notice, for \( \lambda > 0 \) and \( \lambda < x_1 < 2\lambda \), that
\[ (2\lambda - x_1)v_{x_1}(x^\lambda) = \begin{cases} x_1v_{x_1}(x) & \text{if } v_{x_1}(x^\lambda) \geq 0, \\ -x_1v_{x_1}(x^\lambda) = x_1(v_\lambda)_{x_1}(x) & \text{if } v_{x_1}(x^\lambda) < 0, \end{cases} \]

where we have used the fact \( v_{x_1}(x^\lambda) < v_{x_1}(x) \) by the convexity and (32). It follows that
\[
I(v)(x^\lambda) = x^\lambda \cdot Dv(x^\lambda) - v(x^\lambda)
\]
\[ = (2\lambda - x_1)v_{x_1}(x^\lambda) + \sum_{i=2}^{n} x_i v_i(x^\lambda) - v(x^\lambda) \]
\[ = (2\lambda - x_1)v_{x_1}(x^\lambda) + \sum_{i=2}^{n} x_i(v_\lambda)_{x_i}(x) - v_\lambda(x) \]
\[ < I_\lambda(v)(x). \quad (35) \]

This, together with assumptions A2, implies
\[ G(v)(x^\lambda) > G_\lambda(v)(x), \quad \forall \lambda > 0, \quad \forall x \in \sum(\lambda) \setminus (\sum(2\lambda) \cup \{0^\lambda\}). \quad (36) \]

Using (35) and (36), we have, by equation (30), that
\[ 0 = \det D^2 v(x) - G(v)(x) - \det D^2 v(x^\lambda) + G(v)(x^\lambda) \]
\[ > L_\lambda(v)(x), \quad \forall \lambda > 0, \quad \forall x \in \sum(\lambda) \setminus (\sum(2\lambda) \cup \{0^\lambda\}). \quad (37) \]

Now for each \( \lambda \in (0, \infty) \), noting \( w_\lambda = 0 \) on \( T(\lambda) \), using the assumption of (ii) and applying Lemmas 3.3 and 3.1 to (37) in the bounded domains chosen suitably, we obtain \( \frac{\partial w}{\partial x_1} > 0 \) on \( T(\lambda) \), which yields \( \frac{\partial v}{\partial x_1} > 0 \) on \( T(\lambda) \). This proves (ii).
Step 2. If $0 < a < b$, then there exists a $R = R(a,b) > 0$ such that for any $\lambda \in (a,b)$, 

$$w_{\lambda}(x) > 0, \forall x \in \sum(\lambda) \setminus (B_{R}(0) \bigcup \{0^\lambda\}).$$

To show this, we use (6) and Lemma 2.4 to find that for any $\lambda > 0$ and any $x \in \mathbb{R}^n$ such that $x_1 > \lambda$ and $|x|$ is large enough,

$$w_{\lambda}(x) = v(x) - v_{\lambda}(x) = \frac{1}{2}(|x|^2 - |x|^2) + H(x) - H(x^\lambda) = 2\lambda(x_1 - \lambda) + H(x) - H(x^\lambda).$$

Since $H(x) = O(1)$ as $x \to \infty$ and $\lim_{x \to \infty} \frac{|x|}{|x^\lambda|} = 1$ uniformly for $\lambda \in (a,b)$, there is a $R_1 > 2b$ such that for all $\lambda \in (a,b)$

$$w_{\lambda}(x) > \frac{\lambda}{2}(x_1 - \lambda), \forall x \in \sum(\lambda) \setminus \{0^\lambda\} \text{ such that } x_1 > R_1. \quad (38)$$

On the other hand, it follows from Lemma 2.4 again that

$$H(x) - H(x^\lambda) = 2(x_1 - \lambda)H_{x_1}(\xi) = O\left(\frac{1}{|\xi|}\right)(x_1 - \lambda),$$

where $\xi$ is a point in the segment connecting $x$ and $x^\lambda$. Observing that $\xi \to \infty$ as $x \to \infty$, we find a $R = R(a,b) > R_1$ such that $O\left(\frac{1}{|\xi|}\right) > -\frac{\lambda}{2}$ and so

$$H(x) - H(x^\lambda) > -\frac{\lambda}{2}(x_1 - \lambda), \forall x \in \sum(\lambda) \setminus (B_{R}(0) \bigcup \{0^\lambda\}) \text{ such that } \lambda \leq x_1 \leq R_1.$$ 

Consequently, (38) is also true for all $x \in \sum(\lambda) \setminus (B_{R}(0) \bigcup \{0^\lambda\})$. This proves Step 2.

Step 3. Let $Q := \{\lambda \in (0, \infty) : w_{\lambda} > 0 \text{ in } \sum(\lambda) \setminus \{0^\lambda\}\}$. Then Step 1 means $[\lambda, \infty) \subset Q$. Moreover, the set $Q$ is open.

Suppose this is false, i.e., there exist a $\lambda' \in Q$ and a number sequence $\lambda_k$ and a sequence of points $\{x^k\} \subset \sum(\lambda_k) \setminus \{0^\lambda_k\}$ such that

$$\lim_{k \to \infty} \lambda_k = \lambda' > 0, \text{ and } w_{\lambda_k}(x^k) \leq 0, \quad k = 1, 2, \cdots. \quad (39)$$

Then $\{x^k\}$ must be bounded due to Step 2.

By choosing a subsequence, we may assume

$$x^k \to x^0 \text{ as } k \to \infty. \quad (40)$$

Then

$$w_{\lambda_k}(x^0) \leq 0 \text{ and } x^0 \in T(\lambda').$$

Since $w_{\lambda'}(x) > 0$ for $x \in \sum(\lambda) \setminus \{0^\lambda'\}$. Here we have used (32) to exclude $x^0 \neq 0^\lambda'$. Because $\lambda' \in Q$, we have, by Step 1 (ii), that

$$\frac{\partial v}{\partial x_1}(x^0) > 0. \quad (41)$$

On the other hand, (39) means that $v(x^k) \leq v((x^k)^{\lambda_k})$, which implies $\frac{\partial v}{\partial x_1}(\xi^k) \leq 0$ for some $\xi^k$ in the segment connecting $x^k$ and $(x^k)^{\lambda_k}$ for $k = 1, 2, \cdots$. Moreover, $\xi^k \to x^0$ by (40) and the fact $x^0 \in T(\lambda')$. Then $\frac{\partial v}{\partial x_1}(x^0) \leq 0$, contradicting (41). This finishes the proof of Step 3.

Step 4. Let $(\lambda_0, \infty)$ be the connected component of $Q$ in $(0, \infty)$ containing $[\lambda, \infty)$. Then $\lambda_0 = 0$. 

By the definition of $\lambda_0$, we have
\[
w_{\lambda_0} \geq 0 \text{ in } \sum(\lambda_0)
\] (42)
and
\[
w_\lambda > 0 \text{ in } \sum(\lambda), \quad \forall \lambda \in (\lambda_0, \infty).
\]
Hence, \(\frac{\partial w}{\partial x_1} > 0\) on each \(T(\lambda)\) for all \(\lambda \in (\lambda_0, \infty)\) due to Step 1 (ii), i.e.,
\[
\frac{\partial v}{\partial x_1} > 0 \text{ in } \sum(\lambda_0).
\] (43)
Suppose the contrary \(\lambda_0 > 0\). We claim that
\[
w_{\lambda_0} > 0 \text{ in } \sum(\lambda_0).
\] (44)
Weren’t (44) false, \(w_{\lambda_0}(\bar{x}) = 0\) for some \(\bar{x} \in \sum(\lambda_0)\), which is a minimum point of \(w_{\lambda_0}\) in \(\sum(\lambda_0)\) by (42). Then \(\frac{\partial w_{\lambda_0}}{\partial x_1}(\bar{x}) = 0\), which, together with (43), implies
\[
\frac{\partial v}{\partial x_1}(x^{\lambda_0}) = -\frac{\partial v}{\partial x_1}(\bar{x}) < 0.
\]
Set
\[
\Omega_{0}^0 = \{x: x \in \mathbb{R}^n \setminus \{0\} : x_1 < \lambda_0 \text{ and } \frac{\partial v}{\partial x_1}(x) < 0\}.
\]
Let \(\Omega_0^0\) be the symmetric set of \(\Omega_0^0\) with respect to the plane \(x_1 = \lambda_0\). Then \(\Omega_0^0\) is an open set and \(\bar{x}\) is its interior point, and
\[
\frac{\partial v}{\partial x_1}(x^{\lambda_0}) < 0 \quad \forall x \in \Omega_0^0.
\]
Recalling (35) and the definition of \(L_{\lambda_0}(v)(x)\), we have, by the assumption \(\lambda_0 > 0\), that
\[
I(v)(x^{\lambda_0}) = x^{\lambda_0} \cdot Dv(x^{\lambda_0}) - v(x^{\lambda_0})
\]
\[
\quad < -x_1v_{x_1}(x^{\lambda_0}) + \sum_{i=2}^{n} x_i v_{x_i}(x^{\lambda_0}) - v(x^{\lambda_0})
\]
\[
\quad = x_1(v_{\lambda_0})_{x_1}(x) + \sum_{i=2}^{n} x_i(v_{\lambda_0})_{x_i}(x) - v_{\lambda_0}(x)
\]
\[
\quad = I_{\lambda_0}(v)(x)
\]
for all \(x \in \Omega_0^0\).
Hence, (35)-(37) hold for \(\lambda = \lambda_0\) and all \(x \in \Omega_0^0\), i.e.,
\[
L_{\lambda_0}(v)(x) < 0, \quad \forall x \in \Omega_0.
\] (45)
Using Lemmas 3.3 and 3.1 we obtain that \(w_{\lambda_0} \equiv 0\) in a ball \(B_\delta(\bar{x})\) contained in \(\Omega_0^0\). Therefore \(v \equiv v_{\lambda_0}\) and so \(L_{\lambda_0}(v) \equiv 0\) in \(B_\delta(\bar{x})\), contradicting (45). This proves (44).
But (44) means \(\lambda_0 \in Q\), contradicting the definition of \(\lambda_0\). In this way, we have completed the proof of Step 4.
Now we finish the proof of Theorem 1.1. Since \(\lambda_0 = 0\), by Step 4 and (42), we have
\[
v(x_1, x') \geq v(-x_1, x'), \forall x = (x_1, x') \in \mathbb{R}^n, \quad x_1 > 0.
\]
The opposite inequality is also true, because \(V(x) := v(-x_1, x')\) is a solution to (30) in \(\mathbb{R}^n \setminus \{0\}\) and the same conditions as \(v\) holds for \(V\). This proves (33) and thus Theorem 1.1.
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