PACKINGS OF REGULAR PENTAGONS IN THE PLANE

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ABSTRACT. We show that every packing of congruent regular pentagons in the Euclidean plane has density at most \((5 - \sqrt{5})/3 \approx 0.92\). More specifically, this article proves the pentagonal ice-ray conjecture of Henley (1986), and Kuperberg and Kuperberg (1990), which asserts that an optimal packing of congruent regular pentagons in the plane is a double lattice, formed by aligned vertical columns of upward pointing pentagons alternating with aligned vertical columns of downward pointing pentagons. The strategy is based on estimates of the areas of Delaunay triangles. Our strategy reduces the pentagonal ice-ray conjecture to area minimization problems that involve at most four Delaunay triangles. These minimization problems are solved by computer. The computer-assisted portions of the proof use techniques such as interval arithmetic, automatic differentiation, and a meet-in-the-middle algorithm.

We dedicate this article to W. Kuperberg.

1. Introduction

A fundamental problem in discrete geometry is to determine the highest density of a packing in Euclidean space by congruent copies of a convex body \(C\). For example, when \(C\) is a ball of given radius, this problem reduces to the sphere-packing problem in Euclidean space. Besides a sphere, the simplest shape to consider is a regular polygon \(C\) in the plane. If the regular polygon is an equilateral triangle, square, or hexagon, then copies of \(C\) tile the plane. In these cases, the packing problem is trivial. The first nontrivial case is the packing problem for congruent regular pentagons. This article solves that problem.

Henley and Kuperberg and Kuperberg have conjectured that the densest packing of congruent regular pentagons in the plane is achieved by a double-lattice arrangement: vertical columns of aligned pentagons pointing upward, alternating with vertical columns of aligned pentagons pointing downward (Figure 1) [7] [6, p.801]. This packing of pentagons is called the pentagonal ice-ray in Dye’s book on Chinese lattice designs [4]. Two plates (Y3b and Y3c) in Dye’s book depict the pentagonal ice-ray, originating from Chengdu, China around 1900 CE.

We call this the pentagonal ice-ray conjecture. This packing has density

\[
\frac{5 - \sqrt{5}}{3} \approx 0.921311.
\]

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Before our work, the best known bound on the density of packings of regular pentagons was $0.98103$, obtained through the representation theory of the group of isometries of the plane [2]. Our research is a continuation of W. Kusner’s thesis [8], which proves the local optimality of the pentagonal ice-ray.

As far as we know, our methods are adequate for the solution of other related problems in geometric optimization. The limiting factor seems to be the availability of sufficient computer resources.

![Figure 1. The pentagonal ice-ray. All figures show pentagons in red and Delaunay triangles in blue.](image)

This article gives a computer-assisted proof of the pentagonal ice-ray conjecture. The proof appears at the end of Section 10.

**Theorem 1.** No packing of congruent regular pentagons in the Euclidean plane has density greater than that of the pentagonal ice-ray. The pentagonal ice-ray is the unique periodic packing of congruent regular pentagons that attains optimal density.

In this article, we consider packings of congruent regular pentagons in the Euclidean plane. Density is scale-invariant. Without loss of generality, we may assume that all pentagons are regular pentagons of fixed circumradius 1. The inradius of each pentagon is $\kappa := \cos(\pi/5) = (1 + \sqrt{5})/4 \approx 0.809$. We set $\sigma := \sin(\pi/5) \approx 0.5878$. The length of each pentagon edge is $2\sigma$. (All of the numerical calculations in this article have been checked in a file calcs.ml, which is available for download from the project code repository [10].)

All pentagon packings will be assumed to be saturated; that is, no further regular pentagons can be added to the packing without overlap. The assumption of saturation can be made without loss of generality, because our ultimate aim is to give upper bounds on the density of pentagon packings, and the saturation of a packing cannot decrease its density.

We form the Delaunay triangulations of the pentagon packings. The vertices of the triangles are always taken to be the centers of the pentagons. The saturation hypothesis implies that no Delaunay triangle has circumradius greater than 2. This property of Delaunay triangles is crucial. Every edge of a Delaunay triangle has length at most 4.

A Delaunay triangle in a pentagon packing has edge lengths at least $2\kappa$. This minimum Delaunay edge length is attained exactly when the two pentagons have a full edge in common.
For most of the article, we consider a fixed saturated packing and its Delaunay triangulation. Generally, unless otherwise stated, every triangle is a Delaunay triangle. Statements of lemmas and theorems implicitly assume this fixed context.

Initially, pentagons in a packing play two roles: they constrain the shapes of the Delaunay triangles and they carry mass for the density. We prefer to change our model slightly so pentagons are only used to constrain the shapes of triangles. We replace the mass of each pentagon by a small, massive, circular disk (each of the same small radius) centered at the center of the pentagon, and of uniform density and the same total mass as the pentagon. In this model, each Delaunay triangle contains exactly one-half the mass of a pentagon. By distributing the pentagon mass uniformly among the Delaunay triangles, we may replace density maximization with Delaunay triangle area minimization.

We write

\[ a_{\text{crit}} := \frac{3}{2} \sigma \kappa (1 + \kappa) \approx 1.29036 \]

for the common (critical) area of every Delaunay triangle in the pentagonal ice-ray.

2. Clusters of Delaunay Triangles

The area of a Delaunay triangle in a saturated pentagon packing can be smaller than \( a_{\text{crit}} \). Our strategy for proving the pentagonal ice-ray conjecture is to collect triangles into finite clusters such that the average area over each cluster is at least \( a_{\text{crit}} \).

We say that a Delaunay triangle (in any pentagon packing) is subcritical if its area is at most \( a_{\text{crit}} \). We will obtain a lower bound \( a_{\text{min}} := 1.237 \) on the area of a nonobtuse Delaunay triangle (Lemma 35). This is a very good bound. It is very close to the numerically smallest achievable area, which is approximately 1.23719.\(^1\) We write \( \epsilon_N := a_{\text{crit}} - a_{\text{min}} \approx 0.05336 \), for the difference between the desired bound \( a_{\text{crit}} \) on averages of triangles and the bound \( a_{\text{min}} \) for a single nonobtuse triangle. Let \( \epsilon_M = 0.008 \).

2.1. examples. While reading this article, it is useful to carry along a series of examples, illustrated in Figures 2 – 7. Otherwise, later definitions such as the modified area function \( b(T) \) and the construction of clusters might appear to be unmotivated. Some of these examples serve as counterexamples to naive approaches to this problem. These different examples can interact with one another in potentially complicated ways. The proof of the main theorem must sort through these interactions.

(Another essential ingredient in the understanding of the proof is a rather large body of computer code that is used to carry out the computer-assisted portions of the proof. We will have more to say about this later.)

\(^1\)In the notation of the appendix, the numerical minimum is achieved by a pinwheel with parameters \( \alpha = \beta = 0 \) and \( x_\gamma \approx 0.16246 \).
Figure 2. Ice-ray triangles and ice-ray dimers. All Delaunay triangles in the pentagonal ice-ray are congruent and have area $a_{\text{crit}}$. We call them ice-ray triangles. The pentagonal ice-ray can be partitioned into pairs of Delaunay triangles (called ice-ray dimers) in which the two triangles in each dimer share their common longest edge.

$\text{area} \approx 1.23719$

Figure 3. Subcritical triangles. Experimentally, the first triangle has the smallest area among all acute Delaunay triangles. The second triangle is a cloverleaf that has two edges of minimal length $2\kappa$. Experimentally, the third triangle minimizes the longest edge among subcritical acute Delaunay triangles. It is equilateral with edge length about 1.72256.

$\text{area} = a_{\text{crit}}$

$\text{area} \approx 1.248$

$\text{area} \approx 1.23719$

Figure 4. Ice-ray triangle deformation. The ice-ray triangle is not a local minimum of the area function. The ice-ray triangle (left) can be continuously deformed along an area decreasing curve to the subcritical acute triangle of numerically minimum area (right).

2.2. attachment, modified area, and clusters. As mentioned above, our strategy for proving the pentagonal ice-ray conjecture is to collect triangles into finite clusters such that the average area over each cluster is at least $a_{\text{crit}}$. The clusters are defined by an equivalence relation. The equivalence relation, in turn, is defined as the reflexive, symmetric, transitive closure of a further relation on the set of Delaunay triangles in a pentagon packing.

A second strategy is to replace the area function $\text{area}(T)$ on Delaunay triangles with a modified area function $b(T)$. The modification steals area from nearby triangles that have area to spare and gives to triangles in need. It will be sufficient to prove that the average of $b(T)$ over each cluster is at least $a_{\text{crit}}$. 



area \approx 2a_{\text{crit}} + 0.03 \\
area = 2a_{\text{crit}} \\
area \approx 2a_{\text{crit}} + 0.03

Figure 5. Ice-ray dimer deformation. The ice-ray dimer (center) admits a shear motion that preserves all edges of contact. This deformation increases the area of the dimer. It is obvious by the symmetry of the figures on the left and right that the area function along this deformation has a critical point at the ice-ray dimer. It is known that the ice-ray dimer is a local minimum of the area function [8].

Figure 6. Pseudo-dimer. There exist pairs of acute triangles \((T_1, T_0)\) such that the sum of their two areas is at most \(2a_{\text{crit}}\) and such that (1) \(T_1\) is subcritical and is adjacent to \(T_0\) along the longest edge of \(T_1\), but such that (2) the longest edge of \(T_0\) is not the edge shared with \(T_1\). We call such pairs pseudo-dimers. The illustrated pseudo-dimer has area about \(2a_{\text{crit}} - 10^{-5}\), and the triangle \(T_0\) has longest edge length about 1.84. The correction term \(\epsilon_M = 0.008\) is based on this and closely related examples of pseudo-dimers. Pseudo-dimers add significant complications to the proof.

In more detail, below, we define a particular relation \((\Rightarrow_b)\), viewing a relation in the usual way as a set of ordered pairs. For this relation \((\Rightarrow_b)\), we write \((\equiv_b)\) for the equivalence relation obtained as the reflexive, symmetric, transitive closure of \((\Rightarrow_b)\). We call a corresponding equivalence classes \(C\) a cluster. In other words, the relation \((\Rightarrow_b)\) defines a directed graph whose nodes are the Delaunay triangles of a pentagon packing, with directed edges given as arrows \(T_1 \Rightarrow_b T_0\). A cluster is the set of nodes in a connected component of the underlying undirected graph.

We say that Delaunay triangle \(T_1\) attaches to Delaunay triangle \(T_0\) when the following condition holds: \(T_0\) is the adjacent triangle to \(T_1\) along the longest edge of \(T_1\). (If the triangle \(T_1\) has more than one equally longest edge, fix once and for all a choice among
Figure 7. Obtuse Delaunay triangles can have small area. A triangle with circumradius about 2 and area about 0.98 is shown. The adjacent Delaunay triangle cannot have a vertex inside the circumcircle of the first. The neighbor of a subcritical obtuse Delaunay triangle tends to have large area.

them, and use this choice to determine the triangle that $T_1$ attaches to. Thus, $T_1$ always attaches to exactly one triangle $T_0$. We can assume that the tie-breaking choices are made according to a translation invariant rule.) We write $T_1 \Rightarrow T_0$ for the attachment relation $T_1$ attaches to $T_0$. Although it is not always possible to adhere to the convention, note our general convention to use descending subscripts $i > j$ for attachment $T_i \Rightarrow T_j$ of triangles. We also follow a general convention of letting $T_0$ denote a triangle that is the target of other triangles in the same cluster.

If $T$ is a finite set of Delaunay triangles, we set $\text{area}(T) := \sum_{T \in T} \text{area}(T)$. The following are key definitions of this article: dimer pair, pseudo-dimer, $\mathcal{N}$, $\mathcal{M}$, $n_\pm$, $m_\pm$, $b(T)$, $\Rightarrow$, and cluster.

**Definition 2 (dimer pair).** We define a dimer pair to be an ordered pair $(T_1, T_0)$ of Delaunay triangles such that

1. $T_0$ and $T_1$ are both nonobtuse.
2. $T_1 \Rightarrow T_0$, and $T_1$ is subcritical.
3. $T_0 \Rightarrow T_1$.
4. $\text{area}(T_1, T_0) \leq 2a_{\text{crit}}$.

We write $DP$ for the set of dimer pairs.

**Definition 3 (pseudo-dimer).** We define a pseudo-dimer to be an ordered pair $(T_1, T_0)$ of Delaunay triangles such that

1. $T_0$ and $T_1$ are both nonobtuse;
2. $T_1 \Rightarrow T_0$, and $T_1$ is subcritical.
3. $T_0 \Rightarrow T_1$;
4. $\text{area}(T_1, T_0) \leq 2a_{\text{crit}}$.

We write $\Psi D$ for the set of pseudo-dimers.

We observe that dimer pairs differ from pseudo-dimers in the third defining condition, which is a condition on the location of the longest edge of $T_0$. Every pseudo-dimer determines a third triangle $T_-$ by the condition $T_0 \Rightarrow T_-$. The shared edge of $T_0$ and $T_-$, leading
out of the pseudo-dimer is called the *egressive* edge of the pseudo-dimer or of the triangle $T_0$.

For any set $S$ of ordered pairs, and any Delaunay triangles $T_+$ and $T_-$, let

$$n_+(T_+, S) = \text{card}\{T_+ : (T_+, T_-) \in S\} \quad \text{and} \quad n_-(T_-, S) = \text{card}\{T_- : (T_+, T_-) \in S\}.$$  

We define $N$ as the disjoint union of two sets of ordered pairs: $N = N_{\text{obtuse}} \sqcup N_{\text{nonobtuse}}$. The set $N_{\text{obtuse}}$ consists of those pairs with obtuse target and $N_{\text{nonobtuse}}$ are those pairs with nonobtuse target $T_-$, as follows.

Define $N_{\text{obtuse}}$ to be the set of pairs $(T_+, T_-)$ of Delaunay triangles such that

1. $T_-$ is obtuse;
2. $T_+ \Rightarrow T_-$;
3. $T_+$ is nonobtuse and the longest edge of $T_+$ has length at least 1.72;

Define $N_{\text{nonobtuse}}$ to be the set of pairs $(T_+, T_-)$ of Delaunay triangles such that

1. $T_-$ is nonobtuse;
2. $T_+ \Rightarrow T_-$;
3. $T_+$ is nonobtuse and the longest edge of $T_+$ has length at least 1.72;
4. there exists an obtuse triangle $T$ such that $T \Rightarrow T_-$ and

$$\text{area}(T) - n_-(T, N_{\text{obtuse}}) \epsilon N \leq a_{\text{Crit}}.$$

Let $M$ be the set of pairs $(T_+, T_-)$ of Delaunay triangles such that

1. $(T_+, T_-) \not\in N$;
2. $T_+ \Rightarrow T_-$;
3. The longest edge of $T_+$ has length at least 1.72.
4. There exists a unique $T_1$ such that $(T_1, T_+) \in \Psi D$.

We abbreviate $m_+(T) = n_+(T, M)$, $m_-(T) = n_-(T, M)$, $n_+(T) = n_+(T, N)$, and $n_-(T) = n_-(T, N)$.

**Remark 4.** Note that $n_+(T) \leq 1$, because each triangle attaches to exactly one other triangle. Also, $n_-(T) \leq 3$, because each attachment forms along an edge of the triangle $T$. Similarly, $m_+(T) \leq 1$ and $m_-(T) \leq 3$.

We define the modified area function

$$(5) \quad b(T) := \text{area}(T) + \epsilon_M(n_+(T) - n_-(T)) + \epsilon_M(m_+(T) - m_-(T)).$$

We say that $T$ is $b$-subcritical if $b(T) \leq a_{\text{Crit}}$. We write $T_1 \Rightarrow_b T_2$, if $T_1 \Rightarrow T_2$ and $T_1$ is $b$-subcritical. An equivalence classes of triangles under the corresponding equivalence relation ($\equiv_b$) is called a *cluster*.
We note that \((\Rightarrow b)\) is given by a translation-invariant rule. The function \(b(T)\) is also translation invariant and depends only on local information in the pentagon packing near the triangle \(T\).

The intuitive basis of using \(b(T) \leq a_{\text{crit}}\) as the condition for cluster formation with \((\Rightarrow b)\) is the following. Eventually, we wish to show that the average of the modified areas \(b(T)\) over each cluster is greater than \(a_{\text{crit}}\). (See Lemma 11.) If \(b(T) \leq a_{\text{crit}}\), then this means that its modified area \(b(T)\) is less than our desired goal for the average over the cluster, so that \(T\) needs to be part of a larger cluster. This suggests we should define \((\Rightarrow b)\) in such a way that a further triangle is added whenever \(b(T) \leq a_{\text{crit}}\). That is what our definition of \((\Rightarrow b)\) does.

We remark that the modification \(b(T)\) of the area function has two correction terms. The first term \(\epsilon_N(n_+(T) - n_-(T))\) takes away from obtuse triangles (and their neighbors) and gives to nonobtuse triangles. The intuition behind this correction term is that the triangle adjacent to an obtuse triangle along its long edge has a very large surplus area that can be beneficially redistributed (Figure 7). It allows us to make a clean separation of the proof of the main inequality into two cases: clusters that contain an obtuse triangle and clusters that do not.

The second term \(\epsilon_M(m_+(T) - m_-(T))\) augments the area of a pseudo-dimer, by taking from a neighboring triangle. The intuition behind this correction term is that a pseudo-dimer can have area strictly less than the ice-ray dimer, and we need to boost its area with a correction term to make it satisfy the main inequality (Figure 6). A calculation given below shows that the neighbor of the pseudo-dimer has area to spare (Corollary 41).

The correction terms allow us to keep the size of each cluster small. Eventually, we show that each cluster contains at most four triangles (Lemma 84). This small size will be helpful when we turn to the computer calculations. If \((T_+, T_-)\) is a member of \(\mathcal{N}\) or \(\mathcal{M}\), the rough expectation is that there should not be an arrow \(T_+ \Rightarrow b T_-\) and that \(T_+\) and \(T_-\) should belong to different clusters. That is, \(\mathcal{N}\) and \(\mathcal{M}\) are designed to mark cluster boundaries. Some lemmas in this article make this expectation more precise (for example, Lemma 76).

We give some simple consequences of our definitions.

**Lemma 6.** If \((T_+, T_-) \in \mathcal{M}\), then both \(T_+\) and \(T_-\) are nonobtuse.

**Proof.** By the definition of pseudo-dimer, \(T_+\) is nonobtuse, because \((T_1, T_+) \in \Psi D\) for some \(T_1\). If \(T_-\) were obtuse, then we would satisfy all the membership conditions for \((T_+, T_-) \in \mathcal{N}\) (obtuse target), which is impossible because \(\mathcal{N} \cap \mathcal{M} = \emptyset\). \(\square\)

**Lemma 7 (obtuse b).** If \(T\) is an obtuse Delaunay triangle, then \(m_+(T) = m_-(T) = n_+(T) = 0\). Thus, \(b(T) = \text{area}(T) - \epsilon_N n_-(T)\).

**Proof.** The previous lemma gives \(m_+(T) = m_-(T) = 0\). The first component of \(\mathcal{N}\) is nonobtuse by definition, so \(n_+(T) = 0\). \(\square\)

**Corollary 8.** If \((T_+, T_-) \in \mathcal{N}\) with nonobtuse target \(T_-\), then there exists an obtuse triangle \(T\) such that \(T \Rightarrow b T_-\).
Proof. Let $T$ be the obtuse triangle such that $T \Rightarrow T_-$ given by $N$ (nonobtuse target), condition 4. Then $n_-.T, N_{\text{obtuse}} = n_-(T)$, and by Lemma 7, we have $b(T) = \text{area}(T) - \epsilon_n n_-(T) \leq \alpha_{\text{crit}}$. The result follows from the definition of $(\Rightarrow b)$. □

Lemma 9. Suppose that $n_+(T) > 0$. Then $m_+(T) = 0$.

Proof. This follows directly from the disjointness of $M$ and $N$. □

Lemma 10. Suppose $n_-(T_-) > 0$. Then $m_-(T_-) = 0$.

Proof. If $T_-$ is obtuse, then $m_-(T_-) = 0$ by Lemma 7. We may assume that $T_-$ is nonobtuse. By the definition of $N$ (nonobtuse target), the inequality $n_-(T_-) > 0$ implies the existence of an obtuse $T$ with $T \Rightarrow T_-$ (by $N$ condition 4). If (for a contradiction) $m_-(T_-) > 0$, then there exists $(T_+, T_-) \in M$. We complete the proof by checking that $(T_+, T_-)$ satisfies each membership condition of $N$ (nonobtuse target), so that $(T_+, T_-) \in N$. This contradicts disjointness: $(T_+, T_-) \in N \cap M = \emptyset$. □

2.3. the main inequality.

Lemma 11. If every cluster $C$ in every saturated packing of regular pentagons is finite, and if for some $a$ every cluster average satisfies

\begin{equation}
\frac{\sum_{T \in C} b(T)}{\text{card}(C)} \geq a,
\end{equation}

then the density of a packing of regular pentagons never exceeds

$$\frac{\text{area}_P}{2a},$$

where $\text{area}_P = 5\kappa \sigma$ is the area of a regular pentagon of circumradius 1. In particular, if the inequality holds for $a = \alpha_{\text{crit}}$, then the density never exceeds

$$\frac{\text{area}_P}{2\alpha_{\text{crit}}} = \frac{5 - \sqrt{5}}{3},$$

the density of the pentagonal ice-ray.

For any finite set $C$ of triangles, we will call the inequality (12) with the constant $a = \alpha_{\text{crit}}$ the main inequality (for $C$). We call the strict inequality,

\begin{equation}
\frac{\sum_{T \in C} b(T)}{\text{card}(C)} > \alpha_{\text{crit}},
\end{equation}

the strict main inequality (for $C$).

Proof. The maximum density can be obtained as the limit of the densities of a sequence of saturated periodic packings. Thus, it is enough to consider the case when the packing is periodic. A periodic packing descends to a packing on a flat torus $\mathbb{R}^2/\Lambda$, for some lattice $\Lambda$. The rule defining $N$ is translation invariant, and $N$ descends to the torus. On the torus, the set of pentagons, the set of triangles, and the set $N$ are finite. The equivalence relation $(\equiv_p)$ defining clusters is translation invariant, and each cluster is finite, so that no cluster contains both a triangle and a translate of the triangle under a nonzero element of $\Lambda$. Thus, each cluster $C$ in $\mathbb{R}^2$ maps bijectively to a cluster $C$ in the flat torus. The functions $b, n_+$, 

\[ \text{area}_P = 5\kappa \sigma, \]

\[ \alpha_{\text{crit}} = \frac{5 - \sqrt{5}}{3}, \]

the density of the pentagonal ice-ray.
$m_\pm$ are the same whether computed on $\mathbb{R}^2$ or $\mathbb{R}^2/\Lambda$. Let $p$ be the number of pentagons in the torus. By the Euler formula for a torus triangulation, the number of Delaunay triangles is $2p$. We have

$$\sum_T n_+(T) = \sum_T n_-(T) = \text{card}(\mathcal{N}); \quad \sum_T m_+(T) = \sum_T m_-(T) = \text{card}(\mathcal{M}).$$

Thus, the terms in $b(T)$ involving $n_+(T), n_-(T), m_+(T),$ and $m_-(T)$ cancel:

$$\text{area}(\mathbb{R}^2/\Lambda) = \sum_T \text{area}(T) = \sum_T b(T).$$

Let $\text{area}_P$ be the area of a regular pentagon. Making use of the hypothesis of the lemma, we see that the density is

$$\frac{p \text{ area}_P}{\sum_T \text{area}(T)} = \frac{p \text{ area}_P}{2 \text{ area}_P} \leq \frac{2 \text{ area}_P}{2a}.$$

When $a = a_{\text{crit}}$, the term on the right is the density of the pentagonal ice-ray, as desired. □

This article gives a proof of the following theorem. In view of Lemma 11, it implies the main result, Theorem 1. The proof of this result appears at the end of Section 10.

**Theorem 14.** Let $C$ be a cluster of Delaunay triangles in a saturated packing of regular pentagons. Then $C$ is finite and the average of $b(T)$ over the cluster is at least $a_{\text{crit}}$. That is, $C$ satisfies the main inequality. Equality holds exactly when $C$ consists of two adjacent Delaunay triangles from the pentagonal ice-ray, attached along their common longest edge, forming an ice-ray dimer pair.

**Remark 15.** Analyzing the proof of Lemma 11, we see that for a periodic packing, the maximum density is achieved exactly when each cluster in the packing gives exact equality in the main inequality. Thus, the theorem implies that the pentagonal ice-ray is the unique periodic packing that achieves maximal density.

3. **Pentagons in Contact**

3.1. **notation.** By way of general notation, we use uppercase $A, B, C, \ldots$ for pentagons; $v_A, v_B, \ldots$ for the vertices of pentagons; $c_A, c_B, \ldots$ for centers of pentagons; $p, q, \ldots$ for general points in the plane; $||p||$ for the Euclidean norm; $d_{AB} = ||c_A - c_B||$ for center-to-center distances; $\alpha, \beta, \gamma, \phi, \psi, \ldots$ for angles; and $T, T', T_+, T_-, T_0, T_1, T_2, \ldots$ for Delaunay triangles.

We let $\eta(T) = \eta(d_1, d_2, d_3)$ be the circumradius of a triangle $T$ with edge lengths $d_1, d_2,$ and $d_3$.

Let $\angle(p, q, r)$ be the angle at $p$ of the triangle with vertices $p, q,$ and $r$. Let $\text{arc}(d_1, d_2, d_3)$ be the angle of a triangle (when it exists) with edge lengths $d_1, d_2,$ and $d_3$, where $d_1$ is the edge length of the edge opposite the calculated angle. We write $\text{area}(T) = \text{area}(d_1, d_2, d_3)$ for the area of triangle $T$ with edge lengths $d_1, d_2,$ and $d_3$. 

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3.2. **triple contact.** In this subsection, we describe possible contacts between pentagons.

We consider a single Delaunay triangle and the three nonoverlapping pentagons centered at the triangle’s vertices (Figure 9). We call such a configuration a *P-triangle*. A *P*-triangle is determined up to congruence by six parameters: the lengths of the edges of the Delaunay triangle and the rotation angles of the regular pentagons. When we refer to the area or edges of a *P*-triangle, we mean the area or edges of the underlying Delaunay triangle. More generally, we allow *P*-triangles to inherit properties from Delaunay triangles, such as obtuseness or nonobtuseness, the relation (⇒), clusters, and so forth. In clusters of *P*-triangles it is to be understood that the pentagons agree at coincident vertices of the triangles. When there is a fixed backdrop of a Delaunay triangulation of a pentagon packing, it is not necessary to make a careful distinction between a *P*-triangle and its underlying Delaunay triangle.

When two pentagons touch each other, some vertex of one meets an edge of the other. We call the pentagon with the vertex contact the *pointer* pentagon, and the pentagon with the edge contact the *receptor* pentagon (Figure 8). We also call the vertex in contact the *pointer vertex* of the pointer pentagon. There are degenerate cases, when the contact set between two pentagons contains of a vertex of both pentagons. In these degenerate cases, the designation of one pentagon as a pointer and the other as a receptor is ambiguous.

![Figure 8. Pointer and receptor pairs of pentagons. In each pair, the pentagon on the left can be considered a pointer pentagon, with receptor on the right. The first pair is nondegenerate, and the other two pairs are degenerate.](image)

We say that a *P*-triangle is 3*C* (triple contact), if each of the three pentagons contacts the other two.

We may direct the edges of a 3*C* triangle by drawing an arrow from the pointer pentagon to the receptor pentagon. We may classify 3*C* triangles according to the types of triangles with directed edges. There are two possibilities for the directed graph.

1. Some vertex of the triangle is a source of two directed edges and another vertex is the target of two directed edges (*LJ*-junction, *TJ*-junction or *Δ*-junction).
2. Every vertex of the triangle is both a source and a target (pinwheel, pin-*T*).

As indicated in parentheses, we have named each of the various contact types. An example of each of the contact types is shown in Figure 9. An exact description of these contact types appears later. The name *LJ*-junction is suggested by the *L*-shaped region bounded by the three pentagons. Similarly, the name *TJ*-junction is suggested by the *T*-shaped region bounded by the three pentagons. Similarly, for *Δ*-junctions. This section shows that the types in the figure exhaust the geometric types of 3*C* contact.
A cloverleaf arrangement is a 3C triangle that has a point at which vertices from all three pentagons meet (Figure 10). This is degenerate because this shared vertex can be considered as a pointer or receptor.

In general, in this article, a non-anomaly lemma refers to a geometrical lemma that shows that certain geometric configurations are impossible. Generally, it is obvious from the informal pictures that various configurations cannot exist. The non-anomaly lemmas then translate the intuitive impossibilities into mathematically precise statements. We give a few non-anomaly lemmas as follows. They are expressed as separation results, asserting that two pentagons $A$ and $C$ do not touch.

**Lemma 16.** Let $T$ be a 3C-triangle with pentagons $A$, $B$, and $C$ such that $B$ is a pointer to both of the other pentagons $A$ and $C$. Assume that $T$ is not a cloverleaf. Then the two pointer vertices $v_B$ and $v'_B$ are adjacent vertices of $B$.

**Lemma 17.** Let $T$ be a 3C-triangle with pentagons $A$, $B$, and $C$. Suppose that pentagon $A$ is a pointer to $B$ at $v_A$ and that $B$ is a pointer to $C$ at $v_B$. Then on $B$, the vertex $v_B$ is not opposite to the edge of $B$ containing $v_A$.

**Lemma 18.** Let $T$ be a 3C-triangle with pentagons $A$, $B$, and $C$ such that $B$ is a receptor of both of the other pentagons. Then the two pointer vertices $v_A$ and $v_C$ lie on the same edge or adjacent pentagon edges of $B$. 

Figure 9. Types of 3C-contact from left-to-right: a pinwheel, a pin-$T$ junction, a $\Delta$-junction, an $LJ$-junction, and a $TJ$-junction.

Figure 10. A cloverleaf (degenerate pinwheel). The Delaunay triangle in this particular example is not subcritical.

Figure 11. A line through the center of the middle pentagon $B$ through one of its vertices separates the two extremal pentagons $A$ and $C$. 
Proof. The Lemmas 16, 17, and 18 can be proved in the same way. In each case, we prove the contrapositive, assuming the negation of the geometric conclusion, and proving that the configuration is not $3C$. We show that the configuration is not $3C$ by constructing a separating hyperplane between the pentagons $A$ and $C$. In each case, the separating hyperplane is a line through the center of the middle pentagon and passing through a vertex $v$ of that pentagon. See Figure 11. In the case of Lemma 16, there is a degenerate case of a cloverleaf, where all three pentagons meet at the vertex $v$ on the separating line. □

Definition 19 ($\Delta$). We say that a $3C$-triangle has type $\Delta$ if we are in the first case of Lemma 18 (the two pointer vertices $v_A$ and $v_C$ of $A$ and $C$ lie on the same edge of $B$) provided the line $\lambda$ through that edge of $B$ separates $B$ from $A$ and $C$. (See Figure 12.)

![Figure 12](image_url)

Figure 12. In type $\Delta$, a line separates pentagon $B$ from the other two pentagons. The second figure (which is degenerate of type $LJ$) does not have type $\Delta$.

In type $\Delta$, say $A$ is a pointer into $C$ at $v$. Then $v_A$ and $v$ are the two endpoints of some edge of $A$. Also, $v_C$ and $v$ lie on the same edge of $C$. If the line $\lambda$ does not separate $B$ from $A$ and $C$, then $v_C$ is a shared vertex of $B$ and $C$, and we have a degeneracy that can also be viewed as $v_A$ and $v_C$ on adjacent pentagon edges of $B$. This case will be classified as a degenerate $LJ$-junction below.

Definition 20. Let $T$ be a $P$-triangle with pentagons $A$, $B$, and $C$. Assume that $A$ points to $B$ at $v_{AB}$, and $B$ points to $C$ at $v_{BC}$. An inner vertex $v$ of $B$ is a vertex $v \neq v_{BC}$ of $B$ such that $v$ lies between $v_{AB}$ and $v_{BC}$ (along the short run of the perimeter of $B$ from $v_{AB}$ to $v_{BC}$). We allow the degeneracy $v = v_{AB}$. See Figure 13.

Definition 21. Let $T$ be a $3C$-triangle. Fix the pointer directions on $T$, if ambiguous. We say $T$ has type $pin-k$, for $k \in \{0, 1, 2, 3\}$ if $A$ points into $B$, $B$ points into $C$, $C$ points into $A$, and if there are exactly $k$ pentagons among $A$, $B$, $C$ that have an inner vertex. We use $pinwheel$ as a synonym for $pin-0$ and $pin-T$ as a synonym for $pin-2$.

Lemma 22. There does not exist a $3C$-triangle $T$ of type $pin-1$ with pentagons $A$, $B$, and $C$.

Proof. For a contradiction, we draw a (distorted) picture of a $pin-1$ configuration (Figure 13). We let $v$ be the inner vertex of $B$; that is, the vertex that is interior to the triangle $(v_{CA}, v_{AB}, v_{BC})$ with vertices at the pointers $X$ into $Y$. It is an endpoint of the edge of the pentagon $B$ containing $v_{AB}$. We have

$$\angle(v, v_{AB}, v_{CA}) \leq \pi, \quad \angle(v, v_{BC}, v_{AB}) = \frac{3\pi}{5}, \quad \angle(v_{CA}, v_{BC}, v_{AB}) \leq \frac{2\pi}{5}.$$ 

(The last inequality uses the fact that $T$ is not a degenerate $pin-2$, so that $v_{CA}$ is not a vertex of $A$.) We also have

$$\angle(v, v_{BC}, v_{CA}) \geq \frac{2\pi}{5} \geq \angle(v_{CA}, v_{BC}, v_{AB}) \geq \angle(v_{CA}, v_{BC}, v).$$
The law of sines applied to the triangle $(v, v_{BC}, v_{CA})$ then gives

$$2\sigma = |v_{BC} - v| \leq |v_{BC} - v_{CA}| \leq 2\sigma.$$ 

Thus, we have equality everywhere. In particular, $\angle(v, v_{AB}, v_{CA}) = \pi$, and $v_{BC}$ is a vertex of $C$. Hence $v_{BC}$ is a degenerate inner vertex of $C$, and $T$ has type pin-$k$, for some $k \geq 2$. □

**Lemma 23.** The type pin-3 does not exist.

**Proof.** Suppose for a contradiction that a $P$-triangle $T$ of type pin-3 exists. The region $X$ bounded by the three pentagons is a nonconvex star-shaped hexagon, with interior angles $\alpha', \pi/5, \beta', \pi/5, \gamma'$, and $\pi/5$. The vertices of $X$ with angles $7\pi/5$ are the inner vertices of the three pentagons of $T$. The sum of the interior angles in a hexagon is $4\pi$:

$$4\pi = \alpha' + \beta' + \gamma' + 3(\pi/5),$$

which implies that $\alpha' + \beta' + \gamma' = -\pi/5$, which is impossible. □

**Definition 24 (TJ and LJ-junction).** We say that a 3C-triangle is a type TJ- or LJ-junction if it is not type $\Delta$ and if we are in the second case of Lemma 18 (both $A$ and $C$ point into $B$, and the two pointer vertices $v_A$ and $v_C$ lie on adjacent edges of $B$). Say $A$ is a pointer into $C$ at $v$. We say that it has type LJ-junction if $v_C$ and $v$ lie on the same pentagon edge of $C$, and otherwise we say it has type TJ-junction.

We can be more precise about the structure of a TJ-junction. In the context of the definition, Lemma 17 implies that $v$ and $v_C$ lie on adjacent pentagon edges of $C$.

This completes the classification of 3C-triangles: $\Delta$, pinwheel, pin-$T$, LJ, and TJ. Useful coordinate systems for the various types can be found in the appendix (Section 11).

### 4. Delaunay Triangle Areas

As an application of the classification from the previous section, this section makes a computer calculation of a lower bound on the longest edge length of a subcritical triangle. We also obtain a lower bound on the area of a nonobtuse Delaunay triangle.

**Lemma 25.** A nonobtuse subcritical Delaunay triangle has edge lengths at most 2.1.
Proof. By the monotonicity of area as a function of edge length for nonobtuse triangles, a triangle with an edge length at least 2.1 has area at least
\[ \text{area}(2.1, 2\kappa, 2\kappa) > a_{\text{crit}}, \]
which is not subcritical. \qed

Lemma 26. A nonobtuse subcritical triangle has edge lengths less than \( \kappa \sqrt{8} \). In particular, a right-angled Delaunay triangle is not subcritical.

Proof. This is a corollary of the previous lemma, because \( 2.1 < \kappa \sqrt{8} \approx 2.288 \). \qed

Remark 27. A motion of a pentagon in the plane can be described by an element of the isometry group of the plane, which is a semidirect product of a translation group and an orthogonal group. Because of the dihedral symmetries of the regular pentagon, each motion can be realized as a translation followed by a rotation by angle between 0 and \( 2\pi/5 \). In particular, a translation of a pentagon is a motion of a pentagon such that the rotational part is the identity.

Definition 28. In a \( P \)-triangle, we say that a pentagon \( A \) has primary contact if one or more of the following three conditions hold:

1. (slider contact) The pentagon \( A \) and one \( B \) of the other two share a positive length edge segment;
2. (midpointer contact) A vertex of one of the other two pentagons is the midpoint of one of the edges of the pentagon \( A \); or
3. (double contact) The pentagon \( A \) is in contact with both of the other pentagons.

The next lemma is used to give area estimates when an edge has length at most \( \kappa \sqrt{8} \). We give two forms of the lemma. We prove them together.

Lemma 29. Let \( A \) be a pentagon in a nonobtuse \( P \)-triangle. Assume that the triangle edge opposite \( c_A \) has length at most \( \kappa \sqrt{8} \). Then the \( P \)-triangle can be continuously deformed until \( A \) is in primary contact, while preserving the following constraints: the deformation (1) maintains nonobtuseness, (2) is non-increasing in the edge lengths, and (3) keeps fixed the other two pentagons \( B \) and \( C \).

Lemma 30. Let \( A, B, C \) be pentagons in a nonobtuse \( P \)-triangle. Assume that the triangle edges opposite \( c_A \) and \( c_C \) have length at most \( \kappa \sqrt{8} \). Then the \( P \)-triangle can be continuously deformed until \( A \) is in primary contact, while preserving the following constraints: the deformation (1) maintains nonobtuseness, (2) fixes the edge length \( d_{AB} \) and does not increasing the area of the triangle, and (3) keeps fixed the other two pentagons \( B \) and \( C \).

Proof. Fixing \( B \) and \( C \), we translate \( A \) to contract the two edges of the triangle at \( c_A \), where we keep \( d_{AB} \) fixed in Lemma 30. For a contradiction, assume that none of the primary contact conditions occur throughout the deformation. Continue the contractions, until \( A \) contacts another pentagon, then continue by rotating \( A \) about its center \( c_A \) to break the contact and continue. Eventually, the assumption of nonobtuseness must be violated. However, this triangle cannot be obtuse at \( c_A \), because the triangle edge lengths are at least \( 2\kappa, 2\kappa \) with opposite edge at most \( \kappa \sqrt{8} \). This is a contradiction.
In the second lemma, after \( A \) is rotated to break the contact, we translate \( A \) along the circle such that \( c_A \) stays at fixed distance from \( c_B \).

**Lemma 31.** A subcritical 3C-triangle does not have type \( \Delta \). In fact, such a \( P \)-triangle \( T \) has area greater than 1.5.

**Proof.** The proof is computer-assisted. The 3C-triangles of type \( \Delta \) form a three-dimensional configuration space. The appendix (Section 11) introduces good coordinate systems for each of the various 3C-triangle types. We make a computer calculation of the area of the Delaunay triangle as a function of these coordinates. We use interval arithmetic to control the computer error. The lemma follows from these computer calculations.

**Lemma 32.** If a pentagon \( A \) has midpointer contact with a pentagon \( B \), then \( d_{AB} > 1.72 \).

**Proof.** Suppose a pointer vertex \( v_A \) of \( A \) is the midpoint of an edge of pentagon \( B \). Rotating \( A \) about the vertex \( v_A \), keeping \( B \) fixed, we may decrease \( d_{AB} \) until \( A \) and \( B \) have slider contact. This determines the configuration of \( A \) and \( B \) up to rigid motion. By the Pythagorean theorem, the distance between pentagon centers is

\[
d_{AB} = \sqrt{(2\kappa)^2 + \sigma^2} \approx 1.72149 > 1.72.
\]

**Lemma 33.** If every edge of a \( P \)-triangle \( T \) is at most 1.72, then the triangle is not subcritical.

**Proof.** This is a computer-assisted proof.\(^2\) Such a triangle is nonobtuse. By Lemma 29, we may deform \( T \), decreasing its edge lengths and area, until each pentagon is in primary contact with the other two. By the previous lemma, we may assume that the contact is not midpointer contact. Thus, each pentagon has double contact or slider contact with the other pentagons.

If the \( P \)-triangle does not have 3C contact, then obvious geometry forces one pentagon to have double contact and the other two pentagons to have slider contact (Figure 14). The nonoverlapping of the pentagons forces one of slider contacts to be such that a sliding motion along the edges of contact decreases area and edge lengths of \( T \). Thus, the \( P \)-contact can be deformed until 3C contact results.

\(^2\)The constant 1.72 is nearly optimal. For example, in the notation of the appendix, the pinwheel with parameters \( \alpha = \beta = \pi/15 \), \( x_Y = 0.18 \) is subcritical equilateral with edge lengths approximately 1.72256.
Now assume that the \(P\)-triangle has 3C contact. We have classified all 3C triangles. We obtain the proof by expressing each type of triangle in terms of explicit coordinates from the appendix (Section 11) and computing bounds on the areas and edge lengths of the triangles using interval arithmetic. The result follows.

**Lemma 34.** Let \(T\) be a subcritical nonobtuse \(P\)-triangle with pentagons \(A\), \(B\), and \(C\). Then fixing \(B\) and \(C\), we may deform \(T\) by moving the third pentagon \(A\), without increasing the area of \(T\), until \(A\) has double contact (with \(B\) and \(C\)).

**Proof.** By Lemma 26, the edge lengths of \(T\) are at most \(\kappa \sqrt{8}\). By Lemma 29, we may assume that the pentagon \(A\) has primary contact. If the primary contact of a pentagon \(A\) is slider contact, we may slide \(A\) along the edge segment of contact in the direction to decrease the area of \(T\) until it has double contact. If the contact of \(A\) is midpointer contact, then we may rotate \(A\) about the point of contact with a second pentagon \(B\), in the direction to decrease the area of \(T\) until it has double contact. These area-decreasing deformations never transform the nonobtuse subcritical triangle into a right triangle (Lemma 26).

**Lemma 35.** A nonobtuse \(P\)-triangle \(T\) has area greater than \(a_{\text{min}}\).

**Proof.** This is a computer-assisted proof. We may assume for a contradiction that \(T\) has area less than \(a_{\text{min}}\). In particular, it is subcritical. By Lemma 34, we may assume that each pentagon has double contact with the other two, and that \(T\) is 3C. We have classified all 3C triangles. We obtain the proof by expressing each type of triangle in terms of explicit coordinates from the appendix (Section 11) and computing bounds on the areas and edge lengths of the triangles using interval arithmetic. The result follows.

### 5. Computer Calculations

The proofs of the theorems in this article rely heavily on computer calculations. These computer calculations are discussed further at the end of the article in Section 12. In this section, we make use of the following lemmas, which are proved by computer. (Although further discussion appears at the end of the article, there is no circular reasoning involved in using those calculations here.)

**Lemma 36** (computer-assisted). Let \((T_1, T_0) \in \Psi D\). The edge shared between \(T_0\) and \(T_1\) has length less than 1.8.

**Lemma 37** (computer-assisted). Let \((T_1, T_0) \in \Psi D\). The longest edge of \(T_0\) (that is, its egressive edge) has length greater than 1.8.

**Lemma 38.** Let \((T_1, T_0) \in \Psi D\). The two edges of \(T_0\) other than the longest edge have lengths less than 1.8.

**Proof.** The shared edge between \(T_0\) and \(T_1\) has length less than 1.8 by Lemma 36. It has length at least 1.72 by Lemma 33. If (for a contradiction) the third edge of \(T_0\) has length at least 1.8, then by the previous lemmas, its three edges have lengths at least 1.72, 1.8, 1.8. Then

\[
\text{area}(T_1, T_0) > a_{\text{min}} + \text{area}(1.72, 1.8, 1.8) > a_{\text{min}} + (a_{\text{crit}} + \epsilon_N) = 2a_{\text{crit}}.
\]
This area inequality contradicts a defining property of pseudo-dimers.

Lemma 39 (computer-assisted). Let \((T_1, T_0) \in \Psi D\). Then \(\text{area}(T_0, T_1) \geq 2a_{\text{crit}} - \epsilon_M\).

Lemma 40 (computer-assisted). Let \((T_1, T_0) \in \Psi D\). Assume \(T_0 \Rightarrow T_-\). Then \(\text{area}(T_0, T_1, T_-) > 3a_{\text{crit}} + \epsilon_M\).

Corollary 41. Let \((T_1, T_0) \in \Psi D\). Assume that \(T_0 \Rightarrow T_-\). Then \(\text{area}(T_-) > a_{\text{crit}} + \epsilon_M\).

Proof. By Lemma 40 and the definition of pseudo-dimer,
\[
\text{area}(T_-) = \text{area}(T_0, T_1, T_-) - \text{area}(T_0, T_1) > (3a_{\text{crit}} + \epsilon_M) - 2a_{\text{crit}} = a_{\text{crit}} + \epsilon_M.
\]

Corollary 42. Suppose that \(m_-(T_-) > 0\). Then \(\text{area}(T_-) > a_{\text{crit}} + \epsilon_M\).

Proof. If \(m_-(T_-) > 0\), there exists \((T_1, T_0) \in \Psi D\) such that \(T_0 \Rightarrow T_-\). The result follows from the previous corollary.

Definition 43 (long isosceles). We say that a triangle is long isosceles if the two longest edges of the triangle have equal length. We include equilateral triangles as a special case of long isosceles.

Definition 44 (O2C). We say that a triangle \(T = T_0\) or \(T = T_1\) in a dimer pair or a pseudo-dimer pair has outside double contact (O2C) if the pentagon \(A\) at the vertex of \(T\) that is not shared with the other triangle in the pair has double contact.

Lemma 45 (computer-assisted). Let \((T_1, T_0) \in DP\). Then \(T_1\) is not both O2C and long isosceles.

Definition 46 (large angle). Let \(T\) be a \(P\)-triangle. Let \(e\) be an edge of the triangle with pentagons \(A\) and \(B\) at its endpoints. Let \(\alpha = \alpha(T, e)\) be the angle between the edges of the two pentagons \(A\) and \(B\). (See Figure 15.) Modulo \(2\pi/5\), we can assume that \(\alpha \in [0, 2\pi/5]\). We say that the angle is large along \((T, e)\) if \(\pi/5 < \alpha < 2\pi/5\).

Figure 15. A Delaunay triangle \(T\) with a large angle \(\alpha\) along \((T, e)\).

Let \(T\) be a \(P\)-triangle and let \(T'\) be the adjacent \(P\)-triangle along edge \(e\). We have the invariant \(\alpha(T, e) + \alpha(T', e) = 2\pi/5\). Hence if the angle is large along \((T, e)\) then it is not large along \((T', e)\).

Lemma 47 (computer-assisted). Let \(\{T_0, T_1\}\) be given \(P\)-triangles (not necessarily a pseudo-dimer) such that \(\text{area}(T_1) \leq a_{\text{crit}}\) and \(T_1 \Rightarrow T_0\). Assume that there is a nonshared edge \(e\) of \(T_0\) of length greater than 1.8 and such that the angle is not large along \((T_0, e)\). Then \(\text{area}(T_0, T_1) > 2a_{\text{crit}} + \epsilon_M\).
Corollary 48. If $(T_1, T_0) \in \Psi D$ and $e$ is the longest (that is, egressive) edge of $T_0$. Then the angle is large along $(T_0, e)$.

Proof. If the angle is not large, then we can apply the lemma to find that the area of the pseudo-dimer is greater than $2a_{\text{crit}} + \epsilon_M$, which contradicts one of the defining properties of a pseudo-dimer. □

Lemma 49 (computer-assisted). Let $T_1' \Rightarrow T_0$ and area$(T_1') \leq a_{\text{crit}}$ for distinct $P$-triangles $T_0^0$ and $T_1^1$. Then area$(T_0, T_0^0, T_1^1) > 3a_{\text{crit}} + \epsilon_M$.

Lemma 50 (computer-assisted). Let $T_1' \Rightarrow T_0$ and area$(T_1') \leq a_{\text{crit}}$ for distinct $P$-triangles $T_0^0$, $T_1^1$, and $T_2^1$. Then area$(T_0, T_0^0, T_1^1, T_2^1) > 4a_{\text{crit}}$.

6. Dimer Pairs

The purpose of this section is to give a proof of the following theorem. This theorem is the principal optimization problem of this article in the sense that all other optimizations deal with configurations that are far from optimal.

Theorem 51. Let $(T_1, T_0)$ be a dimer pair. (In particular, we assume that $T_1$ is subcritical, that area$(T_0, T_1) \leq 2a_{\text{crit}}$ and that $T_0$ and $T_1$ share a common longest edge.) Then $(T_1, T_0)$ is the ice-ray dimer of area exactly $2a_{\text{crit}}$.

The proof will fill the entire section. The strategy of the proof is to give a sequence of area decreasing deformations to $(T_1, T_0)$, until the ice-ray dimer is reached.

We fix notation that will be used throughout this section. Let $(T_1, T_0)$ be a dimer pair. The $P$-triangle $T_1$ has a pentagon centered at each vertex. We label the pentagons of $T_1$ as $A$, $B$, $C$, with $A$ and $C$ shared with $T_0$. We call $B$ the outer pentagon of $T_1$. Similarly, we label the pentagons of $T_0$ as $A$, $C$, $D$, with outer pentagon $D$ of $T_0$.

Lemma 52. Let $(T_1, T_0)$ be a dimer pair, then every edge of $T_1$ and $T_0$ has length less than $\kappa \sqrt{8}$. In particular $T_1$ and $T_0$ are both acute (and not just merely nonobtuse).

Proof. The shared edge between $T_0$ and $T_1$ is the common longest edge of the two triangles. It is enough to show that this edge has length less than $\kappa \sqrt{8}$. This is an edge of a subcritical triangle $T_1$. The result follows from Lemma 26. □

In particular, area non-increasing deformations of a dimer pair, never transform an acute triangle into a right or obtuse triangle. In other words, the nonobtuseness constraint in the definition of a dimer pair is never a binding constraint in a deformation.

The deformation of a general dimer pair to the ice-ray dimer takes place in several stages. We give a summary of the stages here, before going into details. Here is the proof sketch:

1. We deform the dimer pair so that each of $T_0$ and $T_1$ is $O2C$ or long isosceles.
(2) We deform so that each of $T_0$ and $T_1$ is $O2C$.
(3) We show that both triangles are triple contact.
(4) Working with triple contact triangles, we compute that the condition area{$T_0, T_1$} ≤ $2\alpha_{\text{crit}}$ implies that $(T_1, T_0)$ lies in a small explicit neighborhood of the ice-ray dimer.
(5) We construct a curve $\Gamma$ in the configuration space of dimer pairs, with parameter $t$ such that $t = 0$ defines the ice-ray dimer.
(6) Working with triple contact triangles in a small explicit neighborhood of the ice-ray dimer, and for some small explicit constant $M$, each dimer pair can be connected by a path (in the dimer configuration space) to a dimer on the curve $\Gamma$ with parameter $|t| < M$. A computation shows that the area of the dimer decreases along the path to $\Gamma$. Thus, every area-minimizing dimer pair lies on the curve $\Gamma$.
(7) The unique global minimum of the area function along $\Gamma$ occurs at $t = 0$; that is, the ice-ray dimer is the unique global minimizer along $\Gamma$ for $|t| < M$.

6.1. reduction to $O2C$ or long isosceles. In this section, we show that each of $T = T_0$ and $T = T_1$ can be deformed in an area decreasing way until $T$ is either $O2C$ (that is, the outer pentagon has double contact) or long isosceles (that is, the two longest edges of the triangle have the same length).

To show this, we assume that $T$ and its deformations are not long isosceles; that is, it and its deformations have a unique longest edge that is shared with the other triangle. Fixing the two pentagons of $T$ (A and C) along the shared edge, we show we can deform the outer pentagon until it has 2C contact.

This is easy to carry out. By Lemmas 29 and 30, we can move B in an area decreasing way until B has primary contact. We can continue to move the outer pentagon by translation (meaning no rotation) that preserves contact with either A or C and that is non-increasing in triangle area until double contact is achieved. This is $O2C$. We do this for both $T = T_0$ and $T = T_1$.

6.2. reduction to $O2C$. In this subsection we show that each of $T = T_0$ and $T = T_1$ can be deformed in an area decreasing way so that it is $O2C$.

We begin with the case $T = T_0$. If the deformations in the previous section made $T_0$ into a long isosceles triangle, then $(T_1, T_0)$ is a boundary case that can also be classified as a pseudo-dimer. By earlier calculations, the longest edge of a pseudo-dimer is greater than 1.8 and has strictly greater length than the shared edge between $T_0$ and $T_1$. Thus, it is not long isosceles.

We now consider the case $T = T_1$. In view of the reductions of the previous subsection, we may assume that $T_1$ has primary contact and that $T_1$ is long isosceles. We may further assume that translation of the outer pentagon B while maintaining contact (with A or C) in an area decreasing direction would violate the constraint that the longest edge of $T_1$ is the shared edge. (In other words, the translation that decreases area would increase the edge length of the long nonshared edge.) For a contradiction, we may assume that the primary contact is not $O2C$. 

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We claim that these conditions force $T_1$ not to be subcritical. This is contrary to the defining conditions of a dimer pair. Thus, we complete this stage of the proof by proving non-subcriticality. For the rest of the proof, we disregard $T_0$.

The proof is computer assisted. We deform $T_1$ in an area decreasing way into a configuration that can be easily computed. Without loss of generality, we assume that the outer pentagon $B$ is in contact with pentagon $A$. Because we are now disregarding $T_0$, we may deform the triangle $T_1$ by moving $C$, preserving the long isosceles constraint and decreasing area, until $C$ has primary contact. In particular, $C$ is in contact with $A$ or $B$.

We consider two cases, depending on whether the primary contact of $B$ has slider contact or midpointer contact with $A$. We need two non-anomaly lemmas, one for slider contact and one for midpointer contact. In both cases, the lemmas imply that the edge of contact between $A$ and $B$ (whether slider or midpointer) is one of the long edges of the isosceles triangle.

**Lemma 53 (slider-non-anomaly).** Let $T$ be a subcritical nonobtuse $P$-triangle with pentagons $A$, $B$, and $C$. Assume that $B$ has slider contact with $A$. Then the translation of $B$ along the slider contact in the direction to decrease the area also decreases the length $d_{BC}$.

**Proof.** Assume to the contrary that the translation is increasing in $d_{BC}$. Choose coordinates so that the $x$-axis passes through $c_A$ and $c_C$, with the origin at $c_A$, with $c_C$ in the right half-plane, and with $c_B$ in the positive half-plane (Figure 16). Our contrary assumption means the the line $\lambda$ through the edge of contact between $A$ and $B$ has positive slope. Slider contact implies that the center $c_B$ lies on the line $\lambda'$ parallel to $\lambda$ at distance $2\kappa$ from the origin. Every point on $\lambda'$ either has $y$-coordinate at least $2\kappa$ or negative $x$-coordinate. If the $y$-coordinate is at least $2\kappa$ the triangle is not subcritical. (The area is at least $2\kappa^2 > a_{\text{crit}}$.) If the $x$-coordinate of $c_B$ is negative, then the triangle $T$ is obtuse. $\square$

![Figure 16](image_url)

**Figure 16.** The center $c_B$ of the nonobtuse triangle lands in the first quadrant and gives a triangle $(c_A, c_B, c_C)$ of height and base both at least $2\kappa$.

**Lemma 54 (midpointer-non-anomaly).** Let $T$ be a subcritical nonobtuse $P$-triangle with pentagons $A$, $B$, and $C$. Assume that $A$ has midpointer contact with $B$, with $A$ pointing to $B$ at $v_{AB}$. Then the rotation of $B$ about the point of contact in the direction to decrease the area also decreases the distance $d_{BC}$.
Proof. Suppose to the contrary that the rotation is increasing in $d_{BC}$. Choose coordinates so that the $x$-axis passes through $c_A$ and $c_C$, with origin at $c_A$, with $c_C$ in the right half-plane, and with $c_B$ in the first quadrant. We claim that the distance from $c_B$ to the $x$-axis is at least $2\kappa$ so that the area of the triangle is at least $2\kappa^2 > a_{\text{crit}}$, contrary to the assumption that the triangle $T$ is subcritical. To prove the claim, we disregard the pentagon $C$. We translate $B$ directly downward, rotating $A$ as needed about $c_A$ so that it maintains pointer contact with $B$. Eventually, slider contact is established between $A$ and $B$, and the configuration falls into the setting of the previous lemma. □

As a corollary of these two lemmas, if $d_{AB}$ is not a longest edge, then there exists a deformation decreasing area and $d_{BC}$. This allows us to reduce the long isosceles triangle to a triangle with double contact and such that a long edge runs from $c_A$ to $c_B$. The contact type between $A$ and $B$ is either slider contact or midpointer contact. We choose coordinates and compute with interval arithmetic to show that no such triangle is subcritical. This completes this reduction.

6.3. reduction to triple contact. In this stage, we initially assume that both triangles $T_0$ and $T_1$ are $O2C$. We deform so that both triangles are triple contact.

We briefly describe the argument. Our deformations will preserve the $O2C$ contacts. By an argument made in the second paragraph of Section 6.2, we may assume without loss of generality that the triangle $T_0$ is not long isosceles. By Lemma 45, we have that $T_1$ is not long isosceles. Assuming that neither triangle is long isosceles, we show that both triangles can be brought into triple contact. Because both triangles are already $O2C$, this amounts to decreasing the edge between $c_A$ and $c_C$ until the pentagons $A$ and $C$ come into contact. This deformation consists of a translation of all four pentagons in a motion we call squeezing. It suffices to describe the deformation separately on each triangle $T_0$ and $T_1$ and to prove that this deformation decreases area.

Let $T$ be a triangle with double contact at $B$. We assume that $c_A$ and $c_C$ lie on the $x$-axis with $c_A$ to the left of $c_C$, and with $c_B$ in the upper half-plane. If $c_A$ is free to translate to the right or if $c_C$ is free to translate to the left without overlapping pentagons, then we do so. We assume that we are not in this trivial case.

The squeezing deformation is defined as a motion that translates $B$ directly upward away from the $x$-axis, while translating $A$ to the right and $C$ to the left to maintain double contact at $B$. Note that $A$ and $C$ move by translation along the $x$-axis. It is clear that by adjusting the rates at which $B$ (on $T_1$) moves upward and $D$ (on $T_0$) moves downward, the motions of $T_0$ and $T_1$ are concordant and give a motion of a dimer pair.

Lemma 55. Let $T$ be a nonobtuse $P$-triangle with pentagons $A$, $B$, and $C$, where $B$ has double contact. Assume that the area of $T$ is at most $a_{\text{crit}} + \epsilon_N$. Assume that $T$ is not in the trivial situation of free translation mentioned above. Then the squeezing deformation decreases area.

---

3 calculations $iso_{2C}$ and $iso_{2C'}$
Proof. We analyze the effect on the area by moving \( B \) directly upward by \( \Delta y > 0 \), \( A \) to the right by \( \Delta x_A > 0 \) and \( C \) to the left by \( \Delta x_C > 0 \). Recall that \( d_{xy} = |e_x - e_y| \), and let \( \arcc \) be the angle of the triangle at \( e_x \). The area of \( T \) is \( \text{area}(T) = d_{AC}d_{AB} \sin(\arcc) / 2 \). The transformed area is \((d_{AC} - \Delta x_A - \Delta x_C)(d_{AB} \arcc_A + \Delta y) / 2 \). Let \( \sigma_A = \Delta y / \Delta x_A \) and \( \sigma_C = \Delta y / \Delta x_C \). Passing to the limit as \( \Delta y \to 0 \), we find that squeezing decreases the area exactly when

\[
d_{AC} < \left( \frac{1}{\sigma_A} + \frac{1}{\sigma_C} \right) d_{AB} \sin(\arcc_A).
\]

It is enough to prove this inequality. We do this with a computer calculation\(^4\) using interval arithmetic.

We defined \( \sigma_A \) and \( \sigma_C \) as derivatives, but in fact no differentiation is required. For example, consider \( \sigma_A \). The pentagon \( B \) has contact with \( A \). Thus, \( B \) points into \( A \) or \( A \) points into \( B \). If \( A \) points into \( B \), then the point of contact lies along an edge \( e \) of \( B \). The squeeze transformation translates \( A \) and \( B \) maintaining the contact. Viewed from a coordinate system that fixes \( B \), the squeezing lemma translates \( A \) parallel to the line through \( e \). That is, \( \sigma_A \) is simply the absolute value of the slope of the line through \( e \). Elementary coordinate calculations described in the coordinate section of this article give an explicit formula for the slope of this line. There are two cases, depending on which pentagon points to the other. There are no difficulties in carrying out the computer calculations with these explicit formulas.

In a degenerate situations, the pentagons \( A \) and \( B \) might have vertex to vertex contact. But even in this degenerate case, the squeezing deformation determines an edge \( e \) of \( A \) or \( B \) that determines the slope \( \sigma_A \). \( \square \)

The triangle \( T_1 \) is subcritical. The area of \( T_0 \) is given by

\[
\text{area}(T_0) = \text{area}(T_0, T_1) - \text{area}(T_1) < 2a_{\text{crit}} - a_{\text{min}} = a_{\text{crit}} + \epsilon_N.
\]

Thus, the assumption of the lemma holds. We continue the squeezing deformation until \( A \) comes into contact with \( C \). This completes the reduction to \( 3C \) contact.

6.4. reduction to a small neighborhood of the ice-ray dimer. From this stage forward, \( T_0 \) and \( T_1 \) are both triangles with triple contact. We show that the condition \( \text{area}(T_0, T_1) \leq 2a_{\text{crit}} \) implies that \( (T_1, T_0) \) lies in a small explicit neighborhood of the ice-ray dimer.

The shared pentagons \( A \) and \( C \) are in contact. By symmetry, we may assume that \( A \) points into \( C \). We have classified all \( P \)-triangles with triple contact in Section 3. At this stage, we rely heavily on this classification. The triple contact type \( \Delta \) has area at least \( 1.5 > a_{\text{crit}} + \epsilon_N \) by Lemma 31. This is too large an area to be part of a dimer pair. There are eight combinatorial ways that the pair \((A, C)\) with \( A \) pointing to \( C \) can be extended to a triple contact triangle: a pinwheel, a pin-\( T \) junction, an \( LJ \)-junction (3 ways), and a \( TJ \)-junction (3 ways). There are three ways of extending the edge along \((A, C)\) to an \( LJ \)junction depending on which of the three triangle edges of the \( LJ \)-junction is placed along \((A, C)\). A similar remark applies to \( TJ \)-junctions. A pinwheel has cyclic symmetry, so it gives rise to a single case.

\(^{4}\)calculation squeeze calc
We need an argument to show that only a single combinatorial type of pin-$T$ triangle needs to be considered. In this paragraph only, we shift notation and refer to labels on pentagons in the pin-$T$ configuration shown in Figure 28. We claim that if the area the triangle has area less than $a_{\text{crit}} + \epsilon_N$, then the edge length $d_{BC}$ in that figure is the unique longest edge of the triangle. This claim is established by computer calculation. (See Section 11.12.1.) This means that the shared edge of the pin-$T$ triangle is determined.

Because $T_0$ and $T_1$ both have triple contact, the dimer pair $(T_1, T_0)$ lies in a four-dimensional configuration space. The strategy of the proof is to check by computer that there does not exist a dimer pair outside a small explicit neighborhood of the ice-ray dimer. In other words, the area constrains area($T_1$) $\leq a_{\text{crit}}$ and area($T_0$, $T_1$) $\leq 2a_{\text{crit}}$ are impossible to satisfy when the longest edge on both triangles is the shared edge. We run over $64 = 8 \times 8$ cases depending on the combinatorial types of $T_0$ and $T_1$. In each case, $T_1$ runs over a three-dimensional configuration space. Most of the 64 cases do not contain the ice-ray dimer. In these cases, it is not necessary to specify a small explicit neighborhood. In the cases that do contain the ice-ray dimer, the neighborhood is described in local coordinates.

We say a word about the coordinate system used to carry out these calculations. The triangles $T_0$ and $T_1$ separately have good coordinate systems described in Section 11. Three variables each running over a bounded closed interval parameterize the configuration space for each configuration type. These coordinates are always numerically stable. The quantities $x_{AC}$ and $\alpha_{AC}$ (that parameterize two pentagons in contact) can be computed from $T_0$ or $T_1$ alone. A natural way to try to parameterize the dimer pairs of a given combinatorial type is to use the three coordinates $x_1, x_2, x_3$ from Section 11 for $T_1$, then to choose an appropriate quantity $x_4$ on $T_0$ such that $T_0$ is determined by $x_{AC}, \alpha_{AC}$ (viewed as functions of $x_1, x_2, x_3$) and $x_4$. Then $x_1, \ldots, x_4$ give coordinates for the dimer pair that can be used to do the computer calculations.

Usually, this strategy works, but in a few situations, there is no obvious way to pick the fourth coordinate $x_4$ in a numerically stable way. Fortunately, we can give a characterization of all situations where it is difficult to pick a numerically stable coordinate $x_4$. These are expressed as conditions on the combinatorial type of $T_0$ and as bounds on $x_{AC}$ and $\alpha_{AC}$. In each situation, we use good coordinates provided by Section 11 to show that these conditions of numerical instability force $T_0$ to have area greater than $a_{\text{crit}} + \epsilon_N$, which is incompatible with the conditions defining a dimer pair. These too are computer calculations. (See Sections 11.10.3 and 11.11.1.) Thus, we are justified in excluding a few situations, where coordinates become unstable. (The underlying source of numerical instability is our use of the law of sines

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$$

to compute the length of one triangle edge $a$ in terms of another $b$, which encounters instability for $\beta$ near 0.)

In terms of the local coordinates of Section 11, if $x_1 = x_2 = x_3 = x_4 = 0$ defines the ice-ray dimer, then the explicit small neighborhood we exclude is given by $|x_i| \leq 0.01$, for $i = 1, 2, 3, 4$. For example, in the pinwheel type on $T_1$, we have $(x_1, x_2, x_3) = (\alpha, \beta, x_4)$ as given in Section 11.12, and the fourth variable $x_4$ is determined by the type of $T_0$. These computer calculations are used to complete this stage of the optimization.
6.5. defining a curve. At this stage, we define a curve $\Gamma$ in the configuration space of triple contact dimer pairs such that the ice-ray dimer is defined by parameter value $t = 0$. We represent the ice-ray dimer by a pair of triple contact triangles with shared pentagons $A$ and $C$, where $A$ points to $C$. The curve is described by the shear motion that slides along all four edges of contact, illustrated in Figures 5 and 17. The parameter $t$ is the signed distance between the pointer vertex $A \rightarrow C$ and the midpoint of the receptor edge on $C \leftarrow A$. See Figure 17.

$$t = 0.25 \hspace{2cm} t = 0 \hspace{2cm} t = -0.25$$

Figure 17. Three configurations along the curve $\Gamma$.

In terms of the coordinates $(x_\alpha, \alpha)$ of Section 11 for two pentagons in contact, the relative position of $A$ and $C$ is described by the curve $\alpha = \pi/5$ and $x_\alpha = \sigma + t$.

We call a $\Gamma$-dimer to be a dimer on the curve $\Gamma$. We define a $\Gamma$-triangle to be a $P$-triangle that occurs as one of the two triangles in a $\Gamma$-dimer.

6.6. reduction to the curve. At this stage, we show that we can reduce to points on the curve in the following sense. Working with triple contact triangles in a small explicit neighborhood $U$ of the ice-ray dimer ($|x_i| < M$, where $M = 0.01$, for $i = 1, 2, 3, 4$ in appropriate coordinates), each dimer pair $(T_1, T_0)$ can be connected by a path (in the dimer configuration space) to a $\Gamma$-dimer with parameter $|t| < 0.01$. The area function is decreasing along this path. We have two tasks. First, we construct the path $P$, and then we show that the area function decreases along the path. We will use $s$ as a local parameter on the path ($s \mapsto P(s)$), which we need to keep separate from the parameter $t$ for $\Gamma$. We construct a path such that $s = 0$ determines the initial dimer pair $(T_1, T_0)$ and such that the path $P$ is defined for $s \in [0, s_0]$, for some $s_0 > 0$.

Section 11.7 accomplishes these tasks. The path $P$ is constructed and it terminates on $\Gamma$. If the initial point of the path lies in the neighborhood $U$, then the path stays in $U$ and terminates at a point on $\Gamma$ with parameter $|t| < M$.

Finally, we need to check that the area function is decreasing. This, we prove by a computer calculation of the derivative of the area function along the path at $s = 0$. For this, we use automatic differentiation algorithms, as described in Section 12.2. This completes the reduction to points on the curve $\Gamma(t)$.

6.7. global minimization along the curve. The final stage of the proof of Theorem 51 is the minimization of the area of a $\Gamma$-dimer, for parameters $|t| < 0.01$. We have now reduced to an optimization problem in a single variable that is relatively easy to solve. The area
function is obviously analytic. By automatic differentiation, we take the second derivative of the area function as a function of $t$ and calculate that it is always positive. Thus, the area function has a unique global minimum on $|t| \leq 0.01$. By symmetry in the underlying geometry, it is clear that the area function has derivative zero along $\Gamma$ at $t = 0$. (See Figure 5.) The global minimum is therefore given at $t = 0$, which is the ice-ray dimer.

7. PSEUDO-DIMERS

In this section, we determine the structure of pseudo-dimers and specialize the function $b$ to this context.

For a nonobtuse triangle, the area is a monotonic function of its edge lengths. This makes it easy to give lower bounds on triangle areas. Here are some simple area calculations that will be used.

\[
\begin{align*}
\text{area}(1.8, 1.8, 1.8) &> a_{\text{cut}} + 0.112 \\
\text{area}(1.8, 1.8, 1.72) &> a_{\text{cut}} + 0.069 \\
\text{area}(1.8, 1.72, 1.72) &> a_{\text{cut}} + 3\epsilon_M \\
\text{area}(1.8, 1.8, 2\kappa) &> a_{\text{cut}} + \epsilon_M \\
\text{area}(\kappa \sqrt{3}, 2\kappa, 1.72) &> a_{\text{cut}} + \epsilon_N.
\end{align*}
\]

**Lemma 56.** If $(T_1, T_0) \in \Psi D$, then $m_-(T_0) = 0$.

**Proof.** Assume for a contradiction that $m_-(T_0) > 0$. Then there exists $(T'_1, T'_0) \in \Psi D$ and $(T'_0, T_0) \in M$. The shared edge between $T_0$ and $T'_0$ has length greater than 1.8. This is the egressive edge $e$ of $T_0$. This is impossible by Lemma 47, which states that the condition of having a large angle is not symmetrical for two pseudo-dimers $(T'_1, T'_0)$ and $(T_1, T_0)$. □

**Corollary 57.** For every triangle $T$, either $m_+(T) = 0$ or $m_-(T) = 0$.

**Proof.** If $m_+(T) > 0$, then there exists some $(T_1, T) \in \Psi D$. The lemma gives $m_+(T) = 0$. □

**Lemma 58** (pseudo-dimer disjointness). Assume $(T_1, T_0) \in \Psi D$ and $(T'_1, T'_0) \in \Psi D$ and $\{T_0, T_1\} \cap \{T'_0, T'_1\} \neq \emptyset$. Then $T_0 = T'_0$.

**Proof.** If we dismiss the other three cases $T_1 = T'_1$ and $T_1 = T'_0$ and $T_0 = T'_1$ of a nonempty intersection, then the conclusion $T_0 = T'_0$ will stand.

Assume first that $T_1 = T'_1$. Then $T_1 \Rightarrow T_0$ and $T_1 \Rightarrow T'_0$, which gives $T_0 = T'_0$.

Next assume that $T_1 = T'_0$. We have $T'_1 \Rightarrow T'_0 = T_1 \Rightarrow T_0$. By the calculations above (Lemma 36 and Lemma 37), this puts incompatible constraints on the length of the shared edge between $T_0$ and $T_1$. It must have length less than 1.8 and greater than 1.8.

The case $T_0 = T'_1$ follows from the previous case by symmetry. □
Lemma 59 (pseudo-dimer-obtuse). Assume \((T_1, T_0) \in \Psi D\). Then each edge of \(T_0\) and \(T_1\) has length less than \(\kappa \sqrt{8}\). In particular, if \(T\) is obtuse then \(T \Rightarrow T_0\) and \(T \Rightarrow T_1\).

Proof. The shared edge between \(T_0\) and \(T_1\) has length less than \(1.8 < \kappa \sqrt{8}\) (Lemma 36). The shared edge is the longest edge of \(T_1\), so that each edge of \(T_1\) has length less than 1.8. The only possibility is the egressive edge of \(T_0\). But if the egressive edge of \(T_0\) has length at least \(\kappa \sqrt{8}\), then

\[
\text{area}(T_1, T_0) > a_{\text{min}} + \text{area}(\kappa \sqrt{8}, 2\kappa, 1.72) > a_{\text{min}} + (a_{\text{crit}} + \epsilon_N) = 2a_{\text{crit}}.
\]

This contradicts the area condition in the definition of pseudo-dimer. This completes the first claim of the lemma.

If \(T\) is obtuse, then its longest edge has length at least \(\kappa \sqrt{8}\). If \(T \Rightarrow T_0\), then the shared edge has length at least \(\kappa \sqrt{8}\), contrary to the first claim of the lemma. \(\square\)

Corollary 60. If \((T_1, T_0) \in \Psi D\), then \(n_-(T_0) = 0\).

Proof. Suppose for a contradiction that \(n_-(T_0) > 0\). Then \((T_+, T_0) \in \mathcal{N}\) for some \(T_+\). Because \(T_0\) is nonobtuse, Lemma 8 implies that there exists some obtuse triangle \(T\) such that \(T \Rightarrow T_0\), which is contrary to the previous lemma. \(\square\)

Lemma 61. Let \((T_1, T_0) \in \Psi D\). Then \(n_+(T_1) = n_-(T_1) = m_+(T_1) = m_-(T_1) = 0\). In particular, \(b(T_1) = \text{area}(T_1)\).

Proof. We claim that \(m_+(T_1) = m_-(T_1) = 0\). Otherwise \(T_1\) shares an egressive edge of length greater than 1.8 with some pseudo-dimer. However, the longest edge of \(T_1\) is its shared edge with \(T_0\), which has length less than 1.8. This gives the claim.

Next we claim that \(n_+(T_1) = 0\). Otherwise, \((T_1, T_0) \in \mathcal{N}\). Because \(T_0\) is nonobtuse, Lemma 8 implies that there exists an obtuse triangle \(T\) such that \(T \Rightarrow T_0\), which contradicts Lemma 59.

Finally, we claim that \(n_-(T_1) = 0\). Otherwise, \((T_+, T_1) \in \mathcal{N}\) for some \(T_+\). Because \(T_1\) is nonobtuse, this implies that there exists an obtuse triangle \(T\) such that \(T \Rightarrow T_1\), which contradicts Lemma 59. \(\square\)

Lemma 62. Let \((T_1, T_0) \in \Psi D\). Then \(b(T_0) > a_{\text{crit}}\). That is, \(T_0\) is not \(b\)-subcritical.

Proof. We recall from Corollary 60 that \(n_-(T_0) = 0\). From Lemma 56 we have \(m_-(T_0) = 0\). Thus, \(b(T_0) = \text{area}(T_0) + \epsilon_N n_+(T_0) + \epsilon_M m_+(T_0)\). We have (by Lemma 39)

\[
\text{area}(T_0) = \text{area}(T_0, T_1) - \text{area}(T_1) > (2a_{\text{crit}} - \epsilon_M) - a_{\text{crit}} = a_{\text{crit}} - \epsilon_M.
\]

We first treat the case that \(n_+(T_0) > 0\) or \(m_+(T_0) > 0\). Then we have \(n_+(T_0) + m_+(T_0) \geq 1\). Thus,

\[
b(T_0) \geq \text{area}(T_0) + \epsilon_M (n_+(T_0) + m_+(T_0)) > (a_{\text{crit}} - \epsilon_M) + \epsilon_N > a_{\text{crit}}.
\]

This completes this case.
For the remainder of the proof, we assume that \( n_+(T_0) = m_+(T_0) = 0 \). In particular, we have \( b(T_0) = \text{area}(T_0) \).

Let \( T_0 \Rightarrow T_- \). We have \((T_0, T_-) \in \mathcal{M}\) by the definition of \( \mathcal{M} \), unless the uniqueness property of \( \mathcal{M} \) condition 4 fails. That is, there exists \( T'_1 \neq T_1 \) such that \((T'_1, T_0) \in \Psi D\). The edges of \( T_0 \) shared with \( T_1 \) and \( T'_1 \) have length at least 1.72 and less than 1.8. The egressive edge of \( T_0 \) has length at least 1.8. Thus,
\[
b(T_0) = \text{area}(T_0) \geq \text{area}(1.72, 1.72, 1.8) > a_{\text{crit}}.
\]
This completes the proof. □

8. Main inequality for dimers

Recall that the previous section shows that there exists a unique dimer pair (up to congruence): the ice-ray dimer. In particular, if \((T_1, T_0)\) is a dimer, then \( \text{area}(T_0) = \text{area}(T_1) = a_{\text{crit}}\), and \( T_0 \Rightarrow T_1 \), and \( T_1 \Rightarrow T_0 \), and \((T_1, T_0)\) is a dimer too. Thus, the relationship between \( T_0 \) and \( T_1 \) in a dimer pair is symmetrical. This section proves the following theorem.

**Theorem 63.** Let \((T_1, T_0)\) be a dimer pair. Then for \( T \in \{T_0, T_1\} \), we have \( m_+(T) = m_-(T) = n_+(T) = n_-(T) = 0 \); and \( b(T) = \text{area}(T) \). Moreover, \((T_0, T_1)\) is a cluster, and the main inequality holds for \((T_0, T_1)\).

The proof will occupy the entire section. Before treating dimers, we treat the easy case of singletons.

**Lemma 64.** Assume that all the edges of a \( P \)-triangle \( T \) have length less than 1.72. Then the cluster of \( T \) is the singleton \( \{T\}\) and \( b(T) = \text{area}(T) \). Generally, if \( \{T\}\) is a singleton cluster, then \( b(T) > a_{\text{crit}} \) and the main inequality holds.

**Proof.** Assume that all the edges of \( T \) have length less than 1.72. The constant 1.72 is built into the definition of the constants \( n_+, m_+ \). This gives \( n_+(T) = n_-(T) = m_+(T) = m_-(T) = 0 \). Thus, \( b(T) = \text{area}(T) \). We have \( \text{area}(T) > a_{\text{crit}} \) by Lemma 33. Because \( T \) is not \( b \)-subcritical, there are no arrows \( T \Rightarrow_{b} \). —

We claim that there cannot exist an arrow \( T_+ \Rightarrow_{b} T \). Otherwise, the longest edge of \( T_+ \) has length less than 1.72 and again \( T_+ \) is not \( b \)-subcritical. The claim follows and the cluster is a singleton.

In general, if \( \{T\}\) is any singleton cluster, there is no arrow \( T \Rightarrow_{b} \). This implies that \( T \) is not \( b \)-subcritical, so that \( b(T) > a_{\text{crit}} \). The result follows. □

8.1. Interaction of dimers and pseudo-dimers. Next, we show the disjointness of dimers from pseudo-dimers.

**Lemma 65.** Let \((T_1, T_0)\) \( \in \mathcal{D} \mathcal{P} \) and let \((T'_1, T'_0)\) \( \in \Psi \mathcal{D} \). Then for \( T \in \{T_0, T_1\} \), we have \( T'_0 \Rightarrow T \).
Lemma 68. If \( T \in \{ \text{Lemma 65.} \} \), then \( \text{area}(T) = a_{\text{crit}} \). This gives the result.

Proof. Assume \( T_0' \Rightarrow T \). By Lemma 41, we have \( \text{area}(T) > a_{\text{crit}} \). But if \( T \in \{ T_0, T_1 \} \), we have \( \text{area}(T) = a_{\text{crit}} \). This gives the result.

Corollary 66. Let \((T_1, T_0) \in DP\) and let \((T_1', T_0') \in \Psi D\). Then \( m_-(T_0) = m_-(T_1) = 0 \).

Proof. If \( m_-(T_0) > 0 \), then by the definition of the set \( M \), there exists \((T_1, T_0) \in \Psi D\), such that \( T_0 \Rightarrow T_- \). However, \( T_0' \Rightarrow T_- \), for \( T_- \in \{ T_0, T_1 \} \) by Lemma 65. The result follows.

Lemma 67. Let \((T_1, T_0) \in DP\) and let \((T_1', T_0') \in \Psi D\). Then \((T_0, T_1) \cap (T_0', T_1') = \emptyset \). Moreover, \( m_+(T_0) = m_+(T_1) = 0 \).

Proof. The relationship between \( T_0 \) and \( T_1 \) in a dimer pair is symmetrical. It is enough to show \( T_1 \notin \{ T_0, T_1' \} \). We claim that \( T_1 \neq T_1' \). Otherwise, \( T_0' \Rightarrow T_0 \), which is contrary to Lemma 65.

We claim that \( T_1' \neq T_1 \). Otherwise, if \( T_1 = T_1' \), then \( T_1' = T_1 \Rightarrow T_0 \) and \( T_1' = T_0 \), so that \( T_0 = T_0' \), which have shown impossible. This proves the disjointness result.

If \( m_+(T) > 0 \), then there exists \( T_1'' \) such that \((T_1'', T) \in \Psi D\). This is impossible for \( T \in \{ T_0, T_1 \} \) by the disjointness result.

8.2. interaction with obtuse triangles.

Lemma 68. If \( T_1 \) is obtuse, \( T_0 \) is nonobtuse, and if \( T_1 \Rightarrow T_0 \), then \( T_0 \) is not \( b \)-subcritical.

Proof. If \( T_1 \) is obtuse, then its longest edge, which is shared with \( T_0 \), has length at least \( \kappa \sqrt{8} \).

Assume for a contradiction that \( T_0 \) is nonobtuse and \( b \)-subcritical. By Lemma 26, \( \text{area}(T_0) > a_{\text{crit}} \). Thus, we must have \( n_+(T_0) > 0 \) or \( m_-(T_0) > 0 \). We consider two cases, according to which of these inequalities occurs.

Assume in the first case that \( n_+(T_0) > 0 \); that is, \((T', T_0) \in N \). Recall that this implies \( m_-(T_0) = 0 \) by Lemma 10. We have \( T_1 \Rightarrow T_0 \) and \((T_1, T_0) \notin N \), because \( T_1 \) is obtuse. It follows that \( n_+(T_0) \leq 2 \). In fact, \( n_+(T_0) \) is equal to the number of nonobtuse \( T \) with longest edge of length at least 1.72 such that \( T \Rightarrow T_0 \). If \( n_+(T_0) = 1 \), we have

\[
b(T_0) \geq \text{area}(T_0) - \epsilon_N > \text{area}(\kappa \sqrt{8}, 1.72, 2\kappa) - \epsilon_N > a_{\text{crit}}.
\]

If \( n_+(T_0) = 2 \), we have

\[
b(T_0) \geq \text{area}(T_0) - 2\epsilon_N > \text{area}(\kappa \sqrt{8}, 1.72, 1.72) - 2\epsilon_N > a_{\text{crit}}.
\]

This completes the first case.

Finally, we consider the case that \( m_-(T_0) > 0 \) (and \( n_+(T_0) = 0 \)). We have \( m_-(T_0) \leq 2 \), because \( T_1 \Rightarrow T_0 \), and \( T_1 \) is obtuse, and cannot be part of a pseudo-dimer. The following area estimate gives the result.

\[
b(T_0) \geq \text{area}(T_0) - 2\epsilon_M > \text{area}(\kappa \sqrt{8}, 1.8, 2\kappa) - 2\epsilon_M > a_{\text{crit}}.
\]
The constant 1.8 comes from the egressive edge of a pseudo-dimer in the definition of \(M\) and \(m\).

**Corollary 69.** If \(n_-(T_\bullet) > 0\) with \(T_\bullet\) nonobtuse, then \(b(T_\bullet) > a_{\text{crit}}\).

**Proof.** By the definition of \(N\) (nonobtuse target), there exists an obtuse triangle \(T\) such that \(T \Rightarrow T_\bullet\). The result follows from the lemma. \(\square\)

**Lemma 70.** Let \(T\) be \(b\)-subcritical and nonobtuse. Then \(n_-(T) = 0\). Moreover, assume that there exists \(T'\) that is \(b\)-subcritical such that \(T' \Rightarrow_b T\) or \(T \Rightarrow_b T'\). Then \(\text{area}(T) \leq a_{\text{crit}}\).

**Proof.** By the contrapositive of the corollary, it follows that \(n_-(T) = 0\).

Assume for a contradiction that \(\text{area}(T) > a_{\text{crit}}\). We have

\[
\text{area}(T) > a_{\text{crit}} \geq b(T) \geq \text{area}(T) - \epsilon_M m_-(T).
\]

Thus, \(m_-(T) > 0\). By the definition of \(M\), there exists \((T'_1, T'_0)\) in \(\Psi D\) with \((T'_0, T)\) in \(M\) and \(T'_0 \Rightarrow T\). By Corollary 41, we have \(\text{area}(T) > a_{\text{crit}} + \epsilon_M\). Combined with Inequality 71, this gives \(m_-(T) \geq 2\). Repeating the argument, we have \((T''_1, T''_0)\) in \(\Psi D\) with \((T''_0, T)\) in \(M\) and \(T''_0 \Rightarrow T\). The triangles \(T', T'_0\), and \(T''_0\) are distinct, because for example \(b(T''_0) > a_{\text{crit}} \geq b(T')\) (Lemma 62). Because of the arrow \(T' \Rightarrow_b T\) or \(T \Rightarrow_b T'\), the triangles \(T\) and \(T'\) belong to the same cluster. The longest edge of \(T'\) has length at least 1.72. Then

\[
b(T) \geq \text{area}(T) - m_-(T)\epsilon_M > \text{area}(1.8, 1.8, 1.72) - 3\epsilon_M > a_{\text{crit}}.
\]

This contradicts the assumption that \(T\) is \(b\)-subcritical. \(\square\)

**Lemma 72.** Assume that \(T_1\) and \(T_0\) are both nonobtuse. Then there does not exist a sequence \(T_1 \Rightarrow_b T_0 \Rightarrow_b T\), with \(T_1 \neq T\).

**Proof.** Assume for a contradiction that the sequence exists. By the Lemma 70, \(n_-(T_0) = 0\) and \(\text{area}(T_0) \leq a_{\text{crit}}\).

We claim that \(m_+(T_0) = 0\). Otherwise, there exists \((T'_1, T'_0)\) in \(\Psi D\), and by Lemma 62, \(T_0\) is not \(b\)-subcritical, contradicting the assumptions of the lemma.

We claim that \(m_-(T_0) = 0\). This follows by Lemma 42 and the claim \(\text{area}(T_0) \leq a_{\text{crit}}\).

We have that \(n_+(T_0) = 0\). Otherwise, we reach the contradiction,

\[
a_{\text{crit}} \geq b(T_0) = \text{area}(T_0) + \epsilon_N n_+(T_0) > a_{\text{min}} + \epsilon_N = a_{\text{crit}}.
\]

This shows that \(\text{area}(T_0) = b(T_0)\).

By Lemma 70, we have \(\text{area}(T_1) \leq a_{\text{crit}}\). It follows that \((T_1, T_0)\) in \(\Psi D\), and by Lemma 62, we reach a contradiction to the assumption that \(T_0\) is \(b\)-subcritical. \(\square\)

We are finally in a position to prove Theorem 63.
Proof of dimer theorem 63. Let \((T_1, T_0)\) be a dimer pair, and let \(T \in \{T_0, T_1\}\). We have proved that \(m_+(T) = m_-(T) = 0\) in Lemma 67 and Corollary 66.

We claim that \(n_-(T_0) = n_-(T_1) = 0\). Otherwise, \(T_1\) or \(T_0\) has an edge of length at least \(\kappa \sqrt{8}\), coming from an edge shared with an obtuse triangle, and the triangle \(T_0\) or \(T_1\) is not subcritical.

We claim that \(n_+(T_0) = n_+(T_1) = 0\). Otherwise, say \((T_1, T_0) \in \mathcal{N}\), and we have the contradiction \(n_-(T_0) > 0\).

This shows that \(b(T_i) = \text{area}(T_i)\), for \(i = 0, 1\). Because \(T_0\) and \(T_1\) are both subcritical, they are also \(b\)-subcritical, and fall into the same cluster. This is the full cluster, for otherwise, we would have say \(T \Rightarrow T_0 \Rightarrow T_1\), which is impossible by Lemma 72, for \(T'\) nonobtuse. And if \(T'\) is obtuse, this contradicts Lemma 68). This gives the proof. □

8.3. Cluster structure. We continue with our analysis of the clusters in a fixed saturated packing of regular pentagons.

**Lemma 73.** Let \(T'\) and \(T\) be \(P\)-triangles, such that \(T' \Rightarrow T\), where \(T'\) is nonobtuse subcritical and \(T\) is obtuse. Then the edge of attachment is not the longest edge of \(T\).

*Proof.* We have seen that each edge of a nonobtuse subcritical triangle has length less than \(\kappa \sqrt{8}\) and that the longest edge of an obtuse Delaunay triangle has length at least \(\kappa \sqrt{8}\). These are incompatible conditions on a shared edge. □

**Corollary 74.** Let \((T_+, T_-) \in \mathcal{N}\), where \(T_-\) is obtuse. Then the edge shared between the triangles is not the longest edge of \(T_-\). In particular, \(n_-(T_-) \leq 2\).

*Proof.* If \((T_+, T_-) \in \mathcal{N}\) (obtuse target), then \(T_+ \Rightarrow T_-\). Also, \(T_+\) and \(T_-\) satisfy the assumptions of Lemma 73. □

**Lemma 75.** If \(m_-(T) = 3\), then \(b(T) > a_{crit} + \epsilon_N\).

*Proof.* By Lemma 10, \(n_-(T) = 0\). There is a longest (egressive) edge of a pseudo-dimer along each edge of \(T\), each of length at least 1.8. This gives
\[
b(T) \geq \text{area}(T) - 3\epsilon_M \geq \text{area}(1.8, 1.8, 1.8) - 3\epsilon_M > a_{crit} + \epsilon_N.
\]
□

**Lemma 76.** If \((T_+, T_-) \in \mathcal{N}\), then \(b(T_+) > a_{crit}\).

*Proof.* Otherwise, \(n_+(T_+) \geq 1\), and
\[
a_{crit} \geq b(T_+) \geq \text{area}(T_+) + \epsilon_N(1 - n_-(T_+)) - \epsilon_M m_-(T_+) > a_{crit} - \epsilon_N n_-(T_+) - \epsilon_M m_-(T_+).
\]
So \(n_-(T_+) > 0\) or \(m_-(T_+) > 0\). This gives two cases.

Suppose that \(n_-(T_+) > 0\). Recall that \(T_+\) is nonobtuse. Then \((T, T_+) \in \mathcal{N}\) for some nonobtuse \(T\) whose shared edge with \(T_+\) has length at least 1.72. Thus, by the definition
of \( N \) (nonobtuse target), there exists \( T' \) obtuse such that \( T' \Rightarrow T_+ \). The shared edge has length at least \( \kappa \sqrt{8} \). Also, \( m_+(T_+) \leq 2 \) (because \( (T', T_+) \notin N \)). By Lemma 10, we have \( m_- (T_+) = 0 \). Then

\[
b(T_+) \geq \text{area}(T_+) + \epsilon_N (1 - 2) \geq \text{area}(2 \kappa, 1.72 \sqrt{8}) - \epsilon_N > a_{\text{crit}}.
\]

This completes this case.

Finally, suppose that \( m_-(T_+) > 0 \) and \( n_-(T_+) = 0 \). There exists a pseudo-dimer \((T'_1, T'_0)\) such that \( T'_0 \Rightarrow T_+ \). We have by Lemma 42,

\[
b(T_+) \geq \text{area}(T_+) + \epsilon_N - \epsilon_M m_-(T_+) > (a_{\text{crit}} + \epsilon_M) + \epsilon_N - \epsilon_M^3 \geq a_{\text{crit}}.
\]

\(\square\)

**Lemma 77.** There is no arrow \( T_1 \Rightarrow T_0 \) with \( T_1 \) nonobtuse and \( T_0 \) obtuse.

**Proof.** Assume for a contradiction that such a pair \((T_1, T_0)\) exists. By Lemma 76, \((T_1, T_0) \notin N\). By the definition of \( N \) (obtuse target), the longest edge of \( T_1 \) has length less than 1.72. By Lemma 64, \( T_1 \) forms a singleton cluster. This contradicts \( T_1 \Rightarrow T_0 \). \(\square\)

9. **Obtuse Clusters**

In this section we prove the strict main inequality for clusters that contain an obtuse triangle. This will involve several cases, but in every case the strict main inequality will be found to hold by a large margin. This allows us to use rather crude approximations of area in this section. In particular, we are able to disregard most constraints on the shapes of Delaunay triangles imposed by the pentagons. Instead, we use generic features of the triangles such as the fact that the circumradius is at most two and the edge lengths of the triangle are at least \( 2 \kappa \).

**Remark 78.** Recall that the Delaunay property implies that two adjacent Delaunay triangles \( T_1 \) and \( T_2 \) have the property that \( \alpha_1 + \alpha_2 \leq \pi \), where \( \alpha_i \) is the angle of \( T_i \) that is not at the shared edge of \( T_1 \) and \( T_2 \). In particular, two obtuse Delaunay triangles cannot be joined along an edge that is the longest on both triangles. The extreme case \( \alpha_1 + \alpha_2 = \pi \) corresponds to the degenerate situation where \( T_1 \) and \( T_2 \) form a cocircular quadrilateral. When cocircular, either diagonal of the quadrilateral gives an acceptable Delaunay triangulation.

We write \( \text{area}_a(d_1, d_2, h) \) for the area of a triangle with two edges \( d_1, d_2 \) and circumradius \( h \). In general, two noncongruent triangles have data \( d_1, d_2, h \). We choose \( \text{area}_a(d_1, d_2, h) \) to give the area of that triangle such that its third edge \( d_3 \) is as long as possible.

The following lemma shows that under quite general conditions, we are justified in our decision to choose \( d_3 \) as long as possible in the definition of the function \( \text{area}_a(d_1, d_2, h) \). It is justified in the sense that the other choice does not usually give a Delaunay triangle of a pentagon packing, according to the following simple test.

**Lemma 79.** Let \( d_1, d_2 \) and \( \eta \) be positive real numbers. Assume that \( T \) and \( T' \) are triangles with edge lengths \( d_1, d_2, d_3 \) and \( d_1, d_2, d_3' \), and with the same circumradius \( \eta \). Assume \( 2 \kappa \leq d_1 \leq d_2 \). Set \( \theta = \arccos(\eta, \eta, d_1) + \arccos(\eta, \eta, 2 \kappa) \). If \( 2 \kappa \leq d_3' < d_3 \), then \( \theta < \pi \) and \( 2 \eta \sin(\theta/2) \leq d_2 \).
As a corollary, in contraposition, if \( \theta \geq \pi \) or if \( d_2 < 2\eta \sin(\theta/2) \), then the triangle \( T' \neq T \), with \( d'_3 < d_3 \), cannot satisfy the constraint \( 2\kappa \leq d'_3 \) of a Delaunay triangle.

**Proof.** See Figure 18. Let \( p, q, \) and \( r \) (resp. \( p, q, \) and \( r' \)) be the vertices of \( T \) (resp. \( T' \)) on a common circle, with

\[
|p - q| = d_1, \quad |p - r| = |p - r'| = d_2, \quad \text{and} \quad |q - r'| = d'_3 \leq d_3 = |q - r|.
\]

The angle on the circumcircle from \( p \) to \( q \) is \( \arcsin(\eta, \eta, d_1) \), and \( \theta \) is the angle on the circumcircle from \( p \) to the first point \( s \) beyond \( q \) such that \( |s - q| = 2\kappa \). If \( \theta \geq \pi \), a point \( r' \neq r \) satisfying the constraints does not exist. Assume \( \theta < \pi \). As the figure indicates, \( |p - r'| \) is minimized (as a function of \( d_2 \)) when \( r' = s \), and \( d'_3 = 2\kappa \), the lower constraint. Then \( d_2 = |p - r'| \geq |p - s| = 2\eta \sin(\theta/2) \). \( \square \)

![Figure 18](image)

**Figure 18.** There can be two positions \( r, r' \) on the circumcircle for the third vertex of the triangle. Here, \( d_1 = |p - q| \).

**Lemma 80.** If \( T_1 \) is obtuse, and \( T_1 \Rightarrow T_0 \), then \( T_0 \) is not \( b \)-subcritical.

**Proof.** If \( T_0 \) is nonobtuse, then this is Lemma 68.

Assume that \( T_0 \) is obtuse. By basic properties of Delaunay triangles (Remark 78), Delaunay triangles never join along an edge that is the longest on both triangles. Thus, \( T_1 \) attaches to \( T_0 \) along an edge adjacent to the obtuse angle of \( T_0 \). The shared edge has length at least \( \kappa \sqrt{5} \). To bound the area of \( T_0 \), we deform \( T_0 \) decreasing its area and increasing its longest edge and its circumradius, until we obtain a triangle of circumradius \( \eta = 2 \), and shortest edges \( 2\kappa \) and \( \kappa \sqrt{5} \). Then a numerical calculation (using Corollary 74 and Lemma 7) gives

\[
b(T_0) \geq \text{area}(T_0) - \epsilon_N n(T_0) \geq \text{area}_q(2\kappa, \kappa \sqrt{5}, 2) - 2\epsilon_N > a_{crit}.
\]

The use of the function \( \text{area}_q \) is justified by Lemma 79 and the numerical estimate

\[
d_2 = \kappa \sqrt{5} < 4 \sin(\arcsin(2, 2, 2\kappa)) = 2\eta \sin(\theta/2).
\]

\( \square \)

In future uses of the function \( \text{area}_q \), we always check that the conditions of Lemma 79 justify the use of the function. We do not show these calculations.

**Lemma 81.** There does not exist a three term sequence \(- \Rightarrow b - \Rightarrow b - \) where the three triangles are distinct.
Proof. Assume for a contradiction, that such a sequence exists. By Lemma 80, there does not exist a sequence $\Rightarrow b \Rightarrow_0 \Rightarrow b \Rightarrow$, where the first triangle is obtuse. Thus, we may assume that the first triangle is nonobtuse. By Lemma 77, there does not exist $T_1 \Rightarrow b T_0$, where $T_0$ is nonobtuse and $T_0$ is obtuse. Thus, we may assume that every triangle in the sequence is nonobtuse. This is impossible by Lemma 72. □

If $C$ is a cluster that is not a singleton, then there is some arrow $T_1 \Rightarrow b T_0$. We have the following structure theorem for clusters.

**Theorem 82.** Let $C$ be a cluster, and let $T_1 \Rightarrow b T_0$ be an arrow between triangles in $C$. Then

$$C = \{T_0\} \cup \{T : T \Rightarrow b T_0\}.$$  

**Proof.** We use Lemma 81. Assume that $T \Rightarrow b T_0$. There is no arrow $T' \Rightarrow b T$, with $T' \neq T_0$, because that would also produce a sequence $T' \Rightarrow b T \Rightarrow b T_0$ of three distinct triangles. There is a unique arrow out of $T$. Thus, we have accounted for all of the arrows in and out of $T$.

There is no arrow $T_0 \Rightarrow b T''$, with $T'' \neq T_1$, because that would produce a sequence $T_1 \Rightarrow b T_0 \Rightarrow b T''$ of three distinct triangles. We have accounted for all the arrows in and out of $T_0$. Thus, the full cluster has been identified. □

**Corollary 84.** Every cluster is finite of cardinality at most 4.

**Proof.** At most three triangles attach to $T_0$. □

The following theorem is the main result of this section.

**Theorem 85.** Let $C$ be any cluster that contains an obtuse triangle. Then the strict main inequality (13) holds for $C$.

We prepare for the proof of the theorem with some lemmas.

**Lemma 86.** Let $T_1 \Rightarrow b T_0$ be an arrow between two triangles in a cluster that contains an obtuse triangle. Then $T_1$ is obtuse. Moreover, for every obtuse triangle $T$ in the cluster, $b(T) = \text{area}(T) - \varepsilon N_\infty(T)$.

**Proof.** Assume for a contradiction that $T_1$ is nonobtuse. By Lemma 77 applied to the arrow $T_1 \Rightarrow b T_0$, the triangle $T_0$ is nonobtuse. By assumption and the structure theorem for clusters, there exists $T' \Rightarrow b T_0$, where $T'$ is obtuse. The singleton lemma (Lemma 64) implies that the longest edge of $T_1$, which is shared with $T_0$ has length at least 1.72. By the definition of $N$ (nonobtuse target), we have $(T_1, T_0) \in N$. By Lemma 76, we have $b(T_1) > a_{\text{crit}}$, and $T_1$ is not $b$-subcritical. Thus, we obtain a contradiction to $T_1 \Rightarrow b T_0$.

Thus, $T_1$ is obtuse. Moreover, if $T$ is obtuse, then $b(T) = \text{area}(T) - \varepsilon N_\infty(T)$ by Lemma 7. □
Let \( C \) be a cluster. By \textit{external edge} of the cluster, we mean an edge of a triangle in the cluster that is not shared with another triangle in the cluster. Let \( \tilde{n}(C) \) be the total number of external edges of length at least 1.72 of the cluster \( C \). We have

\[
\tilde{n}(C) = \sum_{T \in C} \tilde{n}(T),
\]

where \( \tilde{n}(T) \) is the number of edges of \( T \) of length at least 1.72 that are external edges of its cluster.

Define \( b(T) := \text{area}(T) - \epsilon_N \tilde{n}(T) \). We use \( b(T) \) to give an easily computed lower bound given in the following lemma.

\textbf{Lemma 87}. Let \( C \) be a cluster of cardinality at least 2 that contains an obtuse triangle. Then

\[
\sum_{T \in C} b(T) \geq \sum_{T \in C} b(T).
\]

\textit{Proof}. Let \( N_{\text{ext}} \subseteq N \) be the subset consisting of pairs \((T_+, T_-)\) such that at least one of \( T_+ \) and \( T_- \) is not in \( C \). Define \( M_{\text{ext}} \subseteq M \) similarly. If \((T_+, T_-)\) \(\in\) \(N \setminus N_{\text{ext}}\), then \( T_+ \), \( T_- \in C \) and the pair \((T_+, T_-)\) contributes \( +\epsilon_N \) to the value of \( b(T_+) \) and \( -\epsilon_N \) to the value of \( b(T_-) \). These contributions cancel. Similar comments apply to \((T_+, T_-)\) \(\in\) \(M \setminus M_{\text{ext}}\). Thus,

\[
\sum_{T \in C} b(T) = \sum_{T \in C} b_{\text{ext}}(T),
\]

where \( b_{\text{ext}}(T) \) is defined as \( b(T) \), but using \( N_{\text{ext}} \) and \( M_{\text{ext}} \) instead of \( N \) and \( M \). It is enough to show that

\[
b_{\text{ext}}(T) \geq b(T),
\]

for all \( T \in C \).

Let \( T \in C \). To prove the inequality for \( T \), it is enough to show that \( \tilde{n}(T) \geq n_{\text{ext},-}(T) + m_{\text{ext},-} \). In fact, this inequality gives

\[
b_{\text{ext}}(T) \geq \text{area}(T) - \epsilon_N n_{\text{ext},-}(T) - \epsilon_M m_{\text{ext},-}(T) \geq \text{area}(T) - \epsilon_N (n_{\text{ext},-}(T) + m_{\text{ext},-}(T)) \geq \text{area}(T) - \epsilon_N \tilde{n}(T) = b(T).
\]

Note that \( n_{\text{ext},-}(T) + m_{\text{ext},-}(T) \) counts pairs \((T_+, T_-)\) \(\in\) \(N_{\text{ext}} \cup M_{\text{ext}}\) and every such shared edge is external and has length at least 1.72. Thus every pair counted in \( n_{\text{ext},-}(T) + m_{\text{ext},-}(T) \) is also counted in \( \tilde{n}(T) \). This gives the inequality and completes the proof of the lemma. 

\textit{proof of Theorem 85}. The proof involves several relatively simple cases. We recall that each Delaunay triangle has edge lengths at least \( 2\kappa \) and circumradius at most 2.

If the cluster is a singleton \([T]\), where \( T \) is obtuse, then the singleton lemma (Lemma 64) gives the result. We now assume that \( C \) is not a singleton. By Theorem 82, the cluster \( C \) has the form of Equation 83 for some triangle \( T_0 \). Each \( T \) such that \( T \Rightarrow_b T_0 \) is obtuse by Lemma 85.

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We break the proof into six cases depending on whether $T_0$ is nonobtuse, and depending on card($C$) $\in \{2, 3, 4\}$. In each case we prove inequality

\[(88) \quad \text{area}(C) > a_{\text{crit}} \text{card}(C) + \epsilon_n \bar{n}(C).\]

By Lemma 87, this implies that

\[
\sum_{T \in C} b(T) \geq \sum_{T \in C} b(T) \geq \text{area}(C) - \epsilon_n \bar{n}(C) > a_{\text{crit}} \text{card}(C),
\]

which is the strong main inequality.

**Case 1.** The triangle $T_0$ is a nonobtuse triangle, and $C = \{T_0, T_1\}$. The triangle $T_0$ has a vertex $v$ that is not shared with $T_1$. By the Delaunay property, $v$ lies outside the circumcircle of $T_1$. The triangles $T_0$ and $T_1$ form a quadrilateral $Q$ whose diagonal is the shared edge of $T_0$ and $T_1$. We deform the quadrilateral $Q$ to decrease its area while maintaining the following constraints:

1. The vertex $v$ lies on or outside the circumcircle of $T_1$. The circumradius of $T_1$ is at most 2.
2. The edge length of the $i$th edge of $Q$ is at least $d_i \in \{2\kappa, 1.72\}$, where the number of $d_i$ that equal 1.72 is $\bar{n}(C)$; and
3. $T_1$ is not acute; $T_0$ is not obtuse.

We drop all other constraints as we deform. (In particular, we do not enforce the nonoverlapping of pentagons in the $P$-triangles.) We continue to deform $Q$ until one of the following two subcases hold:

1. $Q$ is cocircular; or
2. For all $i = 1, 2, 3, 4$, the $i$th edge of $Q$ has reached its lower bound $d_i$.

In the first subcase (cocircularity), we drop the third constraint (acute/obtuse) and continue area decreasing deformations for $Q$ under the constraint of a fixed circumcircle. We note that the area of a cocircular quadrilateral $Q$ depends only on the lengths of the edges and not on their cyclic order on $Q$. We may thus rearrange the edge order as we deform. For a given circumcircle, the area is minimized when three of the edges attain their lower bound $d_i$. By suitable reordering of the edges, we may assume that $Q$ is an isosceles trapezoid and that the fourth (free) edge is parallel to and at least as long as its opposite edge on $Q$. For such $Q$, the area as a function of the circumradius is concave, so that the minimum occurs when the circumradius is as small (that is, all edges attain the minimum $d_i$) or as large (that is, $\eta(Q) = 2$) as possible. When $\eta(Q) = 2$, we relax the edge lengths constraints further to allow three edges to have length $2\kappa$. Explicit numerical calculations in these two extremal configurations show that the inequality (88) is satisfied for each $\bar{n} \in \{1, 2, 3, 4\}$.

In the second subcase (every edge attains its minimal length $d_i$), the four edge lengths are fixed. We drop the constraint that $T_0$ is not obtuse. The area of $Q$ is a concave function of the length of the diagonal. We thus minimize the area of $Q$ when the diagonal is as small as possible (that is, $T_1$ is a right triangle – when this satisfies the constraint that $v$ is outside the circumcircle of $T_1$) or as large as possible (that is, $Q$ is cocircular). The cocircular case
has already been considered. Explicit numerical calculations of $Q$ when $T_1$ is right gives the inequality (88) in each case.

**Case 2. The triangle $T_0$ is a nonobtuse triangle, and $C = \{T_0, T_1, T'_1\}$.** The long edges of the obtuse triangles $T_1$ and $T'_1$ have length at least $\kappa \sqrt{8}$.

We consider a subcase where $\eta(T_1) \leq 1.7$ and $\eta(T'_1) \leq 1.7$. Then calculations based on the monotonicity of the area functions give

$$\text{area}(T_0) \geq \text{area}(2\kappa, \kappa \sqrt{8}, \kappa \sqrt{8}) > \begin{cases} 1.73, & \text{if } d = 2\kappa \\ 1.73 + \epsilon_N, & \text{if } d = 1.72 \end{cases}.$$  

The areas of $T = T_1, T'_1$ are at least

$$\text{area}(T) \geq \text{area}(2\kappa, d, 1.7) = \begin{cases} 1.08, & \text{if } d = 2\kappa \\ 1.08 + 2\epsilon_N, & \text{if } d = 1.72 \end{cases}. $$  

These bounds give inequality (88):

$$\text{area}(T_0) + \text{area}(T_1, T'_1) > 1.73 + 2(1.08) + \epsilon_N \bar{n} > 3a_{\text{crit}} + \epsilon_N \bar{n}.$$  

By symmetry, we may now assume that $\eta(T_1) \geq 1.7$. The areas of $T_1$ and $T'_1$ are at least

$$\text{area}(2\kappa, 2\kappa, 2) > 0.968.$$  

By the Delaunay condition, because $T_1 \Rightarrow T_0, T_1$ is obtuse, and $T_0$ is nonobtuse, this forces $\eta(T_0) \geq 1.7$. We minimize the area of $T_0$ subject to the constraints that its circumradius is at least 1.7, that it is nonobtuse, and its edge lengths are at least $\kappa \sqrt{8}$, $\kappa \sqrt{8}$, and $2\kappa$. If two edges are $2\kappa$, $\kappa \sqrt{8}$ (or even if two edges are $\kappa \sqrt{8}, \kappa \sqrt{8}$), then $T_0$ is obtuse by the circumradius constraint. The binding constraints for the minimization become $\eta(T_0) = 1.7$, $2\kappa$ edge length, and a right triangle. Such a triangle has area at least

$$2\kappa \sqrt{\eta^2 - \kappa^2} \geq 2.41.$$  

There are five external edges, and we have $\bar{n}(C) \leq 5$. This completes this case:

$$\text{area}(T_1) + \text{area}(T'_1) + \text{area}(T_0) > 2(0.968) + 2.41 > 3a_{\text{crit}} + 5\epsilon_N \geq 3a_{\text{crit}} + \epsilon_N \bar{n}.$$  

**Case 3. The triangle $T_0$ is a nonobtuse triangle, and $C' = \{T_0, T_1, T'_1, T''_1\}$.**

This case is almost identical to case 2. We use the same bounds (Equations (89) and (90)) on area($T$) as before, for $T = T_1, T'_1, T''_1$. We can improve the bound on the area of $T_0$:

$$\text{area}(T_0) \geq \text{area}(\kappa \sqrt{8}, \kappa \sqrt{8}, \kappa \sqrt{8}) > 2.2668.$$  

Moreover, in the subcase where $\eta(T_0) \geq 1.7$, we have (even after dropping the nonobtuse-ness constraint):

$$\text{area}(T_0) \geq \text{area}(\kappa \sqrt{8}, \kappa \sqrt{8}, 1.7) > 2.6.$$  

In this case, $\bar{n} \leq 6$. Proceeding as before, we get

$$\text{area}(T_0, T_1, T'_1, T''_1) > \begin{cases} 2.2668 + 3(1.08) \\ 2.6 + 3(0.968) \end{cases} > 4a_{\text{crit}} + \epsilon_N \bar{n}.$$
This completes the proof for cases involving a nonobtuse triangle $T_0$. In the remaining cases, we assume that $T_0$ is obtuse. In the remaining cases, every triangle in $C$ is obtuse.

**Case 4.** The triangle $T_0$ is an obtuse triangle, and $C = \{T_0, T_1\}$.

In this case, $\tilde{n} \leq 4$. It will not be necessary to create subcases according to whether short edges are at least $2\kappa$ or 1.72. We will show that we can relax the lower bound on the short edges to $2\kappa$ and still obtain the bound (88).

We minimize area by flattening $T_0$ by stretching its long edge until $\eta(T_0) = 2$. We further decrease area, keeping the circumradius fixed, by contracting the shorter edge not shared with $T_1$, until that edge has length $2\kappa$.

Next continue to minimize area by contracting an edge of $T_1$, keeping its circumradius fixed, until an edge has length $2\kappa$. Then, allowing the circumradius of $T_1$ to increase, we continue until both shorter edges have length $2\kappa$ or until the circumradius reaches 2.

First assume that both shorter edges of $T_1$ have length $2\kappa$. We have reduced to a one-parameter family of quadrilaterals. We can choose the parameter to be the length $x$ of the diagonal, the common edge of $T_1$ and $T_0$. The parameter $x$ ranges between $\kappa \sqrt{8}$ and $x_{\text{max}} \approx 2.9594$, determined by the condition $\eta(2\kappa, 2\kappa, x_{\text{max}}) = 2$. We check numerically that

\[
\text{area}(T_1) + \text{area}(T_0) \geq \text{area}(2\kappa, 2\kappa, x) + \text{area}_{\eta}(2\kappa, x, 2) > 2a_{\text{crit}} + 4\varepsilon_N \geq 2a_{\text{crit}} + \varepsilon_N \tilde{n}.
\]

Next, assume the circumradius of $\eta(T_1)$ reaches 2, then we have a cocircular quadrilateral that can be treated as in Case 1. In particular, the minimizing cocircular quadrilateral has three edges of length $2\kappa$ and circumradius 2, which has area

\[
\text{area}(T_0, T_1) > 2a_{\text{crit}} + 4\varepsilon_N \geq 2a_{\text{crit}} + \varepsilon_N \tilde{n}.
\]

This completes the argument in this case.

**Case 5.** The triangle $T_0$ is an obtuse triangle, and $C = \{T_0, T_1, T_1'\}$.

By Remark 78, there is no arrow $T \Rightarrow_b T_0$ in $C$ such that the shared edge is the long edge of $T_0$. In particular, there cannot exist (Case 6) with $C = \{T_0, T_1, T_1', T_1''\}$ with every triangle obtuse. Thus, Case 5 is the last case to be considered.

We have $\tilde{n} \leq 5$. The area of $T_0$ is at least $\text{area}_{\eta}(\kappa \sqrt{8}, \kappa \sqrt{8}, 2) > 2.45$. Using our earlier estimates (90) for $\text{area}(T)$, for $T = T_1, T_1'$, we have

\[
\text{area}(T_1, T_1', T_0) > 2(0.968) + 2.45 > 3a_{\text{crit}} + \varepsilon_N \tilde{n}.
\]

(As mentioned earlier, each use of the function $\text{area}_{\eta}$ is justified by a calculation based on Lemma 79.) This completes the proof of the theorem. $\square$
10. **Nonobtuse Clusters**

In this section we prove the main inequality for clusters in which every triangle is nonobtuse. By Corollary 69 and Lemma 76, if $T$ is $b$-subcritical and nonobtuse, then $n_+(T) = n_-(T) = 0$.

**Lemma 91.** Let $T \Rightarrow_b T_0$. Assume that $T$ and $T_0$ are nonobtuse. Then $m_-(T) = 0$.

**Proof.** Assume for a contradiction that $m_-(T) > 0$. We have just observed that $n_+(T) = n_-(T) = 0$. By Lemma 57, we have $m_+(T) = 0$. By Lemma 75, we have $m_-(T) \leq 2$.

We have $m_-(T) = 2$. Otherwise, if $m_-(T) = 1$, we have a contradiction (by Corollary 42):

$$a_{\text{crit}} \geq b(T) \geq \text{area}(T) - \epsilon_M > (a_{\text{crit}} + \epsilon_M) - \epsilon_M = a_{\text{crit}}.$$  

Because $m_-(T) = 2$, there exist two pseudo-dimers $(T'_1, T'_0)$ and $(T''_1, T''_0)$ such that $T'_0 \Rightarrow T$ and $T''_0 \Rightarrow T$. The shared edges $e'$ and $e''$ have length at least 1.8. Moreover, the angles are large along $(T'_0, e')$ and $(T''_0, e'')$, but not large along $(T, e')$ and $(T, e'')$. (See Definition 46 and Lemma 48.) Lemma 92 (below) and the estimate

$$a_{\text{crit}} \geq b(T) \geq \text{area}(T) - 2\epsilon_M > (a_{\text{crit}} + 2\epsilon_M) - 2\epsilon_M = a_{\text{crit}}$$  

complete the proof. □

**Lemma 92.** Let $T$ be a nonobtuse $P$-triangle. Suppose that two of its edges $e'$ and $e''$ have length at least 1.8 and that the angles are not large along $(T, e')$ and $(T, e'')$. Then $\text{area}(T) > a_{\text{crit}} + 2\epsilon_M$.

**Proof.** If the third edge has length at least 1.63, then the result easily follows:

$$\text{area}(T) \geq \text{area}(1.8, 1.8, 1.63) > a_{\text{crit}} + 2\epsilon_M.$$  

We remark that 1.63 is close to the minimum edge length $2\kappa \approx 1.618$. This leaves hardly any flexibility in the relative position of the two pentagons along this edge.

We may assume without loss of generality that the third edge has length in the range $[2\kappa, 1.63]$. Let the pentagons at the vertices of $T$ be $A$, $B$, and $C$, with $d_{AB} \leq 1.63$. For the moment, we disregard the pentagon $C$ and parallel translate $B$, decreasing $d_{AB}$ until $A$ and $B$ come into contact. We assume without loss of generality that $B$ points into $A$ at $v_B$. Draw the configuration as in Figure 19 with a vertical receptor edge $e$ on $A$. There are two cases, depending on whether $v_B$ lies above or below the midpoint of the edge $e$. (The pointer cannot be at the midpoint of $e$ by Lemma 32, and $1.72 > 1.63 \geq d_{AB}$. ) Let $\gamma \in [0, 2\pi/5)$ be the angle formed by edges of $A$ and $B$ at the pointer $v_B$ as in Figure 19.

We have the constraint

$$\cos(\pi/5 - \gamma) = \sin(\gamma + 3\pi/10) \leq 1.63 - \kappa.$$  

This constraint expresses the fact the distance from $e_B$ to the edge $e$ of pentagon $A$ can be at most $1.63 - \kappa$. The constraint implies that

$$\gamma \geq \pi/5 + \arccos(1.63 - \kappa) > 2\pi/5 - 0.021 \quad \text{or} \quad \gamma \leq \pi/5 - \arccos(1.63 - \kappa) < 0.021,$$
according to whether \( v_B \) is below or above the midpoint of \( e \).

Now we return to the original \( P \)-triangle with pentagons \( A, B, C \) in their original position. Because \( \gamma \) was obtained after a parallel translation of \( B \), it equals the incidence angle of lines through edges of the original \( A \) and \( B \). Changing the choice of edges on \( A \) and \( B \), we find an incidence angle \( \gamma' \in [\frac{4\pi}{5} - 0.021, \frac{4\pi}{5} + 0.021] \). See Figure 19.

We form a triangle with angles \( \alpha, \beta, \) and \( \gamma' \) by extending edges of \( A, B, \) and \( C \). The assumption that angles are not large along \((T, e')\) and \((T, e'')\) gives \( \alpha \geq \frac{\pi}{5} \) and \( \beta \geq \frac{\pi}{5} \).

The angle sum of the triangle gives a contradiction
\[
\pi = \alpha + \beta + \gamma' \geq \frac{\pi}{5} + \frac{\pi}{5} + \left(\frac{4\pi}{5} - 0.021\right) > \pi.
\]
\[\square\]

Figure 19. The angle between the nearly vertical edges of \( A \) and \( B \) is \( \gamma \). On the left, \( \gamma = 0.02 \) and \( 2\kappa < d_{AB} \approx 1.6296 < 1.63 \). There is almost no play in the configuration. The figure on the right does not satisfy the constraints of the proof of Lemma 92, but illustrates the notation.

Next, we turn our attention to the target \( T_0 \) of an arrow \( T \Rightarrow b T_0 \), where both \( T \) and \( T_0 \) are nonobtuse. By Lemma 64, the shared edge has length at least 1.72.

**Lemma 93.** Let \( T_0 \) be a triangle in a cluster \( C \) containing only nonobtuse triangles. Then \( n_-(T_0) = 0 \).

**Proof.** If \( n_-(T_0) > 0 \), then by Corollary 8 there exists an obtuse triangle \( T' \) such that \( T' \Rightarrow b T_0 \) and the cluster \( C \) contains an obtuse triangle. \[\square\]

**Lemma 94.** Let \( T \Rightarrow b T_0 \). Assume that \( T \) and \( T_0 \) belong to a cluster \( C \) containing only nonobtuse triangles. Then \( m_+(T) = 0 \). Moreover, \( n_+(T) = n_-(T) = m_+(T) = m_-(T) = 0 \) and \( b(T) = \text{area}(T) \leq a_{\text{crit}} \).

**Proof.** If \( m_+(T) > 0 \), then there exists a pseudo-dimer \((T', T'_0)\) such that \( T = T'_0 \). Then Lemma 62 implies the contradiction that there is no arrow \( T \Rightarrow b T_0 \).

The final statement is a summary of the preceding series of lemmas. If \( n_+(T) > 0 \), then \((T, T_0) \in \mathcal{N} \) and \( n_-(T_0) > 0 \), which is contrary to Lemma 93. The equalities \( n_-(T) = m_-(T) = 0 \) are Lemmas 69 and 91.

**Lemma 95.** Let \( C \) be a cluster consisting of nonobtuse triangles. Assume that the cardinality of \( C \) is four. Then the strict main inequality holds for \( C \).
Proof. Let $\mathcal{C} = \{T_0\} \cup \{T'_i \mid i = 1, 2, 3\}$. We have $b(T'_i) = \text{area}(T'_i) \leq a_{\text{crit}}$ by Lemma 94. By Lemma 93, we have $n_-(T_0) = 0$.

We claim $m_-(T_0) = 0$. Otherwise, if $m_-(T_0) > 0$, then $(T, T'_i) \in \Psi D$ for some $T$ and some $i$. The arrow $T'_i \Rightarrow b T_0$ is inconsistent with Lemma 62.

We claim $n_+(T_0) = 0$. Otherwise, if $(T_0, T'_i) \in \mathcal{N}$, then we get $n_-(T'_i) > 0$, which is contrary to Lemma 94.

Hence all the negative coefficients $n_-, m_-$ are zero on the cluster: $b(T'_i) = \text{area}(T'_i)$ and $b(T_0) \geq \text{area}(T_0)$. The result now follows from Lemma 50.

Lemma 96. Let $\mathcal{C}$ be a cluster consisting of nonobtuse triangles. Assume that the cardinality of $\mathcal{C}$ is three. Then the strict main inequality holds for $\mathcal{C}$.

Proof. Let $\mathcal{C} = \{T_0\} \cup \{T'_i \mid i = 1, 2\}$. We have $b(T'_i) = \text{area}(T'_i)$ by Lemma 94. By Lemma 93, we have $n_-(T_0) = 0$.

We claim that $m_-(T_0) \leq 1$. Otherwise, by the definition of $\mathcal{M}$, there exists $(T, T'_i) \in \Psi D$ for some $T$ and some $i$. The arrow $T'_i \Rightarrow b T_0$ is inconsistent with Lemma 62.

This gives

$$b(T_0) \geq \text{area}(T_0) - \epsilon_M$$

By Lemma 49, we have

$$\sum_{T \in \mathcal{C}} b(T) \geq (\text{area}(T_0) - \epsilon_M) + \text{area}(\mathcal{C} \setminus \{T_0\}) = -\epsilon_M + \text{area}(\mathcal{C}) > 3a_{\text{crit}}.$$  

This is the strict main inequality for $\mathcal{C}$.

Lemma 97. Let $\mathcal{C}$ be a cluster consisting of nonobtuse triangles. Assume that the cardinality of $\mathcal{C}$ is two. If the cluster is not a dimer pair, then the strict main inequality holds for $\mathcal{C}$.

Proof. Let $\mathcal{C} = \{T_1, T_0\}$, with $T_1 \Rightarrow b T_0$. By Lemma 94, $b(T_1) = \text{area}(T_1) \leq a_{\text{crit}}$.

We assume that $\mathcal{C}$ is not a dimer pair $(T_1, T_0)$.

We consider the case of a pseudo-dimer. If $(T_1, T_0) \in \Psi D$, then $m_-(T_0) = n_-(T_0) = 0$ (by Lemmas 56 and 60). Thus, by Lemma 39 and Lemma 94,

$$b(T_1) + b(T_0) \geq \text{area}(T_1, T_0) + \epsilon_M(n_+(T_0) + m_+(T_0)) > 2a_{\text{crit}} - \epsilon_M + \epsilon_M(n_+(T_0) + m_+(T_0)).$$

The main inequality follows if we show that $n_+(T_0) > 0$ or $m_+(T_0) > 0$. Assume for a contradiction that $n_+(T_0) = m_+(T_0) = 0$. Pick $T$ such that $T_0 \Rightarrow T$. The condition $n_+(T_0) = 0$ implies that $(T_0, T) \notin \mathcal{N}$. The longest edge of $T_0$ is at least 1.72 by Lemma 37. According to the definition of $\mathcal{M}$, we have $m_+(T_0) > 0$ unless uniqueness fails: there exists $T'_i \neq T_1$ such that $(T'_i, T_0) \in \Psi D$. This is impossible by Lemma 61, because the cardinality of $\mathcal{C}$ is only two. This completes the case of a pseudo-dimer.
By Lemma 93, we have \( n_-(T_0) = 0 \). Thus, by Lemma 94,

\[
b(T_0) \geq \text{area}(T_0) - \epsilon_M m_-(T_0) \quad b(T_1) = \text{area}(T_1) \leq a_{\text{crit}}.
\]

We may assume that \( \text{area}(T_0 \cup T_1) > 2a_{\text{crit}} \). Otherwise, \((T_1, T_0)\) is a dimer pair or a pseudo-dimer, and these cases have already been handled. We assume for a contradiction that the strict main inequality is false:

\[
2a_{\text{crit}} \geq b(T_1) + b(T_0).
\]

Combined with the Inequalities 98, this gives

\[
2a_{\text{crit}} \geq b(T_1) + b(T_0) \\
\geq \text{area}(T_1) + \text{area}(T_0) - \epsilon_M m_-(T_0) \\
> 2a_{\text{crit}} - \epsilon_M m_-(T_0).
\]

This implies that \( m_-(T_0) > 0 \).

We consider the case \( m_-(T_0) = 3 \). By Lemma 75, we have \( b(T_1) + b(T_0) > a_{\text{min}} + (a_{\text{crit}} + \epsilon_N) = 2a_{\text{crit}} \), which completes this case.

We consider the case \( m_-(T_0) = 2 \). In this case, there are two pseudo-dimers that share a longest edge with \( T_0 \). These edges have length at least 1.8. The third edge is shared with \( T_1 \) and has length at least 1.72. Then

\[
b(T_1) + b(T_0) \geq a_{\text{min}} + (\text{area}(1.8, 1.8, 1.72) - 2\epsilon_M) > 2a_{\text{crit}}.
\]

Finally, we consider the case \( m_-(T_0) = 1 \). There exists a pseudo-dimer whose long edge \( e \) is shared with \( T_0 \). That edge has length at least 1.8, and the angle is not large along \((T_0, e)\). We are in the context covered by Lemma 47. That lemma implies

\[
b(T_1) + b(T_0) \geq \text{area}(T_1, T_0) - \epsilon_M > (2a_{\text{crit}} + \epsilon_M) - \epsilon_M = 2a_{\text{crit}}.
\]

This completes the proof. \( \square \)

We are ready to give a proof of the pentagonal ice-ray conjecture. We repeat the statement of the theorem (Theorem 1) from the introduction of the article.

**Theorem 99.** No packing of congruent regular pentagons in the Euclidean plane has density greater than that of the pentagonal ice-ray. The pentagonal ice-ray is the unique periodic packing of congruent regular pentagons that attains optimal density.

We combine the proof with a proof of Theorem 14.

**Proof.** By the main inequality in Lemma 11 applied to \( a = a_{\text{crit}} \), together with Remark 15 it is enough to give a proof of Theorem 14. Specifically, we show that every cluster in every saturated packing is finite of cardinality at most 4. This is Corollary 84. If \( C \) is a dimer pair, then \( C \) is the ice-ray dimer in the pentagonal ice-ray and the (weak) main inequality holds for \( C \), with equality exactly for the ice-ray dimer. This is Theorem 63 and Theorem 51. If \( C \) is not a dimer pair, then we show that \( C \) satisfies the strict main inequality. If \( C \) has an
obtuse triangle, then this is found in Lemma 85. If every triangle in C is nonobtuse, and if C has cardinality 4, 3, 2, or 1, then the result is found in Lemmas 95, 96, 97, 64.

This completes the proof of the main theorem. □

11. Appendix on Explicit Coordinates

11.1. Two pentagons in contact. Let A and B be pentagons in contact, with B the pointer at vertex $v_B$ to the receptor pentagon A. Label vertices $(u_A, w_A, u_B, v_B, w_B)$ of A and B as in Figure 20. Let $x = x_\alpha = |v_B - w_A|$ and $\alpha = \angle(v_B, u_B, u_A)$. We have $0 \leq x_\alpha \leq 2\sigma$ and $0 \leq \alpha \leq 2\pi/5$.

![Figure 20. coordinates for a pair of pentagons in contact](image)

Let $\ell = \ell(x_\alpha, \alpha) = |c_A - c_B|$, viewed as a function of $x_\alpha$ and $\alpha$. We omit the explicit formula for $\ell$, but it is obtained by simple trigonometry. Under the symmetry $u_A \leftrightarrow w_A$, $u_B \leftrightarrow w_B$, we have the transformation

\begin{equation}
(100) \quad x \leftrightarrow 2\sigma - x, \quad \alpha \leftrightarrow 2\pi/5 - \alpha
\end{equation}

and

$\ell(x, \alpha) = \ell(2\sigma - x, 2\pi/5 - \alpha)$.

We use $(x_\alpha, \alpha)$ as the standard coordinates on the configuration space of two pentagons in contact. These coordinates are particularly convenient, because they present the configuration space as a rectangle $[0, 2\sigma] \times [0, 2\pi/5]$.

11.2. Angles of two pentagons in contact. Let A and B be two pentagons in contact, given in coordinates by $(x_\alpha, \alpha)$ as in the previous subsection. Referring to Figure 20, $v_B$ is the pointer vertex of B into A, and we have an angle $\theta' = \angle(c_B, c_A, v_B)$ that specifies the location of $v_B$ relative to the segment $(c_B, c_A)$. The oriented angle $\angle(c_B, c_A, v) \in [0, 2\pi/5]$ to another vertex $v = u_B, w_B, \ldots$ of B is $\theta' + 2\pi k/5$ for some integer $k$. We thus consider $\theta'$ as an angle defined modulo $2\pi\mathbb{Z}/5$. By subtracting a multiple of $2\pi/5$, we choose the angle to lie in the range $[-\pi/5, \pi/5]$. With these conventions, in the figure, $\theta'$ is positive.

Referring to Figure 20, $u_A$ is a vertex of A, and we have an angle $\theta = \angle(c_A, c_B, u_A)$ that gives the location of $u_A$ relative to the segment $(c_A, c_B)$. The angle $\angle(c_A, c_B, u)$ to another vertex of A is $\theta + 2\pi k/5$ for some integer $k$. Adjusting by a multiple of $2\pi/5$, we choose $\theta$ to lie in the range $[0, 2\pi/5]$. 
We stress our convention that $\theta'$ refers to the angle on the pointer pentagon and that $\theta$ refers to the angle on the receptor pentagon. It can be easily checked that

$$\theta + \theta' \equiv \alpha \mod (2\pi \mathbb{Z}/5).$$

(We remark in passing that we can define angles $\theta$, $\theta'$, and $\alpha$ even when $A$ and $B$ are not in contact in such a way that this relation still holds.) We may consider $\theta$ and $\theta'$ functions of the standard variables $(x_\alpha, \alpha)$. With our conventions $\theta' \in [-\pi/5, \pi/5]$ and $\theta \in [0, 2\pi/5]$ on the range of these angles, $\theta'$ and $\theta$ are both determined as continuous functions of $(x_\alpha, \alpha)$. To obtain continuity, it is necessary for $\theta$ to take both values 0 and $2\pi/5$, even though these values are equal modulo $2\pi \mathbb{Z}/5$.

In general, when two pentagons $A$ and $B$ come into contact, sometimes $A$ is the pointer and sometimes $B$ is the pointer. We describe an extended coordinate system $(x_\alpha, \alpha)$, with a domain $\alpha \in [0, 4\pi/5]$ and $x_\alpha \in [0, 2\sigma]$ of twice the size that unifies both pointer directions. See Figure 21. When $\alpha \leq 2\pi/5$, the coordinates are precisely as before. Note that configurations with $\alpha = 2\pi/5$ are ambiguous, with $B$ pointing into $A$ with coordinates $(x_\alpha, \alpha)$, or with $A$ pointing into $B$ with coordinates $(x'_\alpha, \alpha') = (2\sigma - x_\alpha, 0)$. When $\alpha > 2\pi/5$, we let $(x_\alpha, \alpha)$ represent the configuration with $A$ pointing into $B$ and coordinates $(x'_\alpha, \alpha') = (2\sigma - x_\alpha, \alpha - 2\pi/5)$. The configuration of two pentagons in contact depends continuously on the coordinates $(x_\alpha, \alpha)$. The dependence is analytic except along $\alpha = 2\pi/5$.

Using the continuous dependence of the configuration on the coordinates, we may uniquely extend the functions $\theta$ and $\theta'$ to continuous functions on the extended domain. The functions still represent the inclination angle of a vertex of $A$ (resp. $B$) with respect to the edge $(e_A, e_B)$. However, when $\alpha \geq 2\pi/5$, the range of $\theta'$ is $[0, 2\pi/5]$ and the range of $\theta$ is $[\pi/5, 3\pi/5]$. Also, $\theta$ becomes the coordinate related to the pointer vertex (of $A$ into $B$).

Here is an explicit formula for the extension on the domain $\alpha \geq 2\pi/5$:

$$d_{AB}(x_\alpha, \alpha) = d_{AB}(2\sigma - x_\alpha, \alpha - 2\pi/5),$$

$$\theta'(x_\alpha, \alpha) = \theta(2\sigma - x_\alpha, \alpha - 2\pi/5),$$

$$\theta(x_\alpha, \alpha) = 2\pi/5 + \theta'(2\sigma - x_\alpha, \alpha - 2\pi/5).$$

By unifying both directions of pointing, extended coordinates lead to a significant reduction in the number of cases to be considered. In fact, a single calculation can involve multi-triangle configurations with several ($k$) edges in contact, and without extended coordinates this leads to $2^k$ times the number of cases.

11.3. **two pentagons in contact, alternative coordinates.** Inversely, $d_{AB} = \|e_A - e_B\|$, $\theta$, and $\theta'$ determine $(x_\alpha, \alpha)$. In fact, any two of $d_{AB}$, $\theta$, $\theta'$ determine $(x_\alpha, \alpha)$ up to finite ambiguity. In general, there can be two configurations of pentagons for a given $d_{AB}$ and $\theta'$. See Figure 22. They can be distinguished by a boolean variable giving the sign of $h := x_\alpha - \sigma$.

The computer code implements a function that generates a configuration of two pentagons $(A, B)$ in contact as a function of $d_{AB}$ and $\theta'$, for contact type $B \rightarrow A$ and $h \geq 0$. (This function does not use extended coordinates.) Our proof of the pentagonal ice-ray conjecture depends on having fast, numerically-stable algorithms for computing configurations in terms of these variables on intervals. It is a matter of simple trigonometry to express $\theta$ in terms of $d_{AB}$ and $\theta'$. However, it is somewhat more work to give good interval
\[ \alpha = 2\pi/5 - 0.25 \] \[ \alpha = 2\pi/5 \] \[ \alpha = 2\pi/5 + 0.25 \]

Figure 21. Extended coordinates give a continuous transition from pairs \((A, B)\) with pointer \(B \to A\) to pointer \(A \to B\). The functions \(\theta, \theta'\) are continuous in \(\alpha, x_\alpha\). The function \(\theta\) is the angle \(\angle(c_A, c_B, v)\), and \(\theta' = \angle(c_B, c_A, w)\). The vertices \(v\) and \(w\) transition into and away from the pointer vertex. In these figures, \(\alpha\) varies and \(x_\alpha = 0.8 = |v - w|\) is fixed.

Figure 22. With given fixed center \(c_A\) and fixed pointer pentagon \(B\), there might be two pentagons \(A\) in contact with \(B\). They are distinguished by a boolean variable indicating whether the pointer \(B \to A\) lies above or below the midpoint of the edge on \(A\).

bounds on \(\theta\) as a function of intervals \(d_{AB}\) and \(\theta'\). The implementation of this function is based on detailed information about the image of \((d_{AB}, \theta')\) as functions on the set of pairs \((A, B)\) with \(B \to A\) and \(h \geq 0\). The image is a convex region in the plane with a piecewise analytic boundary.

11.4. **pentagon existence test.** The values \((d_{AB}, \theta, \theta')\) can be defined for a pair of nonoverlapping pentagons, even when \(A\) and \(B\) do not touch. The computer code implements a test to determine whether a given triple \((d, \theta, \theta')\) is equal to a triple \((d_{AB}, \theta, \theta')\) associated with some pair \(A, B\) of pentagons in contact. More generally, when \((d, \theta, \theta')\) are interval-valued variables, the computer code implements a test to determine if there exist pentagons \(A\) and \(B\) that do not overlap and whose values lie in the given intervals. The interested reader can consult the computer code for the details of the test.

11.5. **coordinates for zero, single, and double contact triangles.** The configuration space of \(P\)-triangles in which no pair of pentagons is in contact is six dimensional. Let \(A, B, C\) be the pentagons centered at the vertices \(c_A, c_B, c_C\) of the triangle. For example, we can use the six coordinates \(d_{AB}, d_{BC}, d_{AC}, \theta_{ABC}, \theta_{BCA}, \theta_{CAB}\). The three edge lengths \(d_{XY}\) of the triangle determine the triangle up to congruence, and the three angles \(\theta_{XYZ}\) determine the inclination of each pentagon \(X\) relative to the edge \((c_X, c_Y)\) of the triangle. See Figure 23. Other quantities can be easily computed from these six coordinates. For example, to compute \(\theta_{ACB}\), the angle of pentagon \(A\) relative to the edge \((c_A, c_C)\), we use the
relation (102) \[ \text{arc}_A + \theta_{ABC} + \theta_{ACB} \equiv 0 \mod 2\pi/5. \]

where \( \text{arc}_A \) is the angle of the triangle at vertex \( c_A \). Similar equations hold for the angles at \( c_B \) and \( c_C \).

\[ \theta_{BAC}, \text{arc}_B, \theta_{BCA} \]

\[ \theta_{CBA}, \text{arc}_C, \theta_{CAB} \]

Figure 23. Angle conventions. The edge lengths of the Delaunay triangle are \( d_{AC}, d_{AB}, \) and \( d_{BC} \). The angles at \( c_A \) are \( \theta_{ABC}, \text{arc}_A, \) and \( \theta_{ACB} \). The angles at \( c_B \) are \( \theta_{BAC}, \text{arc}_B, \) and \( \theta_{BCA} \). The angles at \( c_C \) are \( \theta_{CBA}, \text{arc}_C, \) and \( \theta_{CAB} \).

Each \( P \)-triangle determines a six-tuple \((d_{AB}, d_{BC}, \ldots)\). We can algorithmically test whether a six-tuple of real numbers (or of intervals) comes from \( P \)-triangles by using the triangle inequality and the test of Section 11.4.

The configuration space of \( P \)-triangles in which a single pair \((A, B)\) of pentagons comes into contact is five-dimensional. Let \( A, B, C \) be the pentagons centered at the vertices \( c_A, c_B, c_C \) of the triangle. Assume that \( A \) is in contact with \( C \), with pointer \( A \to C \). For example, we can use the five coordinates \( d_{AB}, d_{BC}, d_{AC}, \theta_{ACB}, \theta_{BAC} \). The three edge lengths \( d_{XY} \) of the triangle determine the triangle up to congruence, and the two angles \( \theta_{XYZ} \) determine the inclination of each pentagon relative to the triangle. It is clear that the \( \theta_{ACB} \) fixes the inclination of \( A \) and that \( \theta_{BAC} \) fixes the inclination of \( B \). The variable \( \theta_{CAB} \) is computed from \( d_{AC} \) and \( \theta_{ACB} \) by the procedure in Section 11.3. Other quantities can be easily computed from these quantities. Again, we can algorithmically test whether a given five-tuple \((d_{AB}, d_{BC}, \ldots)\) comes from a \( P \)-triangle.

The configuration space of \( P \)-triangles in which two separate pairs \((A, B)\) and \((B, C)\) of pentagons comes into contact is four-dimensional. Let \( A, B, C \) be the pentagons centered at the vertices \( c_A, c_B, c_C \) of the triangle. We can use the extended coordinates \((x_{\alpha}, \alpha)\) of Section 11.2 to give the relative position of \( A \) and \( B \) and similar coordinates \((x_{\gamma}, \gamma)\) to give the relative position of \( B \) and \( C \). These four coordinates uniquely determine the \( P \)-triangle.
up to finite ambiguity. The ambiguity is resolved by specifying which edge of B is in contact with A and which edge is in contact with C. These four coordinates determine other quantities such as lengths $d_{AB}$ and $d_{BC}$ and angles $\theta_{ABC}$, $\theta_{BAC}$, $\theta_{BCA}$, $\theta_{CBA}$. Once again, we can algorithmically test whether a given four-tuple $(\alpha, \alpha, \gamma, \gamma)$ comes from a P-triangle.

11.6. triple contact. If we have coordinates on a 3C-triangle that determine the variables $(\alpha, \alpha)$ for each of the pairs $\{A, B\}$, $\{B, C\}$, and $\{A, C\}$ of pentagons, then we may use the function $\ell$ of Section 11.1 to calculate the edge lengths and area of the 3C-triangle.

The configuration space of 3C-triangles is three dimensional, obtained by imposing three contact constraints between pairs of pentagons on the six-dimensional configuration space of all P-triangles.

11.7. 3C triangles with a shared edge. Some calculations deal with a dimer pair of 3C triangles sharing a triangle edge and two pentagons $\bar{A}$ and $\bar{C}$, say with $\bar{A}$ pointing into $\bar{C}$. (We place bar accents on symbols in this subsection for compatibility with the sections that follow.) Let $\bar{B}$ and $\bar{D}$ be the two outer pentagons of the dimer pair. The configuration space of dimer in which both triangles have triple contact is four dimensional. In this situation, it is best to develop coordinate systems that make efficient use of the shared information. Associated with the pentagons $(\bar{A}, \bar{C})$ in contact are two variables $(\bar{\alpha}, \bar{\beta})$ (that we rename from $(\alpha, \alpha)$ in Section 11.2). It is generally advantageous to make $(\bar{\alpha}, \bar{\beta})$ two of the four coordinates on the dimer pair. We supplement this with one further angle $\bar{\alpha}$ to determine the position of $\bar{B}$ and a further angle to determine $\bar{D}$.

We focus on the P-triangle $(\bar{A}, \bar{B}, \bar{C})$, with coordinates $(\bar{\alpha}, \bar{\beta}, \bar{\alpha})$. Similar considerations apply to the other P-triangle $(\bar{A}, \bar{D}, \bar{C})$. We assume that $\bar{A}$ points into $\bar{C}$.

For each triple contact, the following sections give further details about its coordinate systems. There will be a shared edge coordinate system for each contact type and each pair of pentagons $(\bar{A}, \bar{C})$ such that $\bar{A}$ points into $\bar{C}$.

We call a $\Gamma$-triangle to be a P-triangle that appears as one of the two triangles (with given shared edge) in the curve $\Gamma$ of Section 6.5. We set up coordinates in a uniform manner so that regardless of the contact type of the 3C triangle, the curve $\Gamma$ is parameterized by variable $t$ and is given as

$$t = \bar{\alpha} - \sigma, \quad \bar{\alpha} = 0, \quad \bar{\beta} = \pi/5.$$  

(This is the curve restricted to the P-triangle $(\bar{A}, \bar{B}, \bar{C})$, with a similar description on the other triangle $(\bar{A}, \bar{D}, \bar{C})$ in the dimer.)

When the the configuration space of a give 3C-type contains $\Gamma$-triangles and the ice-ray triangle, we also describe a path $P$ with parameter $s$ from an arbitrary triangle in that configuration space to a $\Gamma$-triangle. Again, we set up coordinates uniformly so that the formulas are independent of the contact type of the P-triangle. Coordinates will be defined in such a way that for all points of the domain, the relation $\bar{\alpha} \geq 0$ holds. The path $P$ from an arbitrary point in the domain $(\bar{\alpha}_0, \bar{\beta}_0, \bar{\alpha}_0)$ to the curve $\Gamma$ is defined by functions.
This means that the paths for the two triangles are coherent along the shared pentagons initial point. The dimer pair has a second triangle (¯\(\alpha\), \(\bar{\beta}\)) to \(\bar{\gamma}\). Starting from these coordinates, we define \(\sigma\) and \(\gamma\) by \(\alpha + \gamma = \pi/5\), and angles \(\alpha', \beta', \gamma'\) of the inner triangle \(\Delta\) by

\[
\alpha + \alpha' = 2\pi/5, \\
\beta + \beta' = 2\pi/5, \\
\gamma + \gamma' = 2\pi/5.
\]

The edges \(y_\alpha, y_\beta, y_\gamma\) of the triangle \(\Delta\) opposite the angles \(\alpha', \beta', \gamma'\), respectively are easily computed by the law of sines. Define \(x_\beta\) by \(x_\alpha + y_\gamma + x_\beta = 2\sigma\), and \(x_\gamma\) by \(y_\alpha + x_\gamma = 2\sigma\). The value \(x_\beta\) is the distance between the nearly coincident vertices of pentagons \(A\) and \(C\), and \(x_\gamma\) is the distance between the nearly coincident vertices of pentagons \(A\) and \(B\). The edges of the 3C Delaunay triangle have lengths

\[
\ell(x_\alpha, \alpha'), \quad \ell(x_\beta, \beta'), \quad \ell(x_\gamma, \gamma').
\]
By Lemma 31, triangles of type $\Delta$ have area too large to be relevant for calculations of dimers. There is no need to describe the shared coordinates $(\bar{\alpha}, \bar{\beta}, \bar{x}_\gamma)$ in this case.

![Figure 24. Coordinates for $\Delta$-types](image)

11.9. **triple contact at a pinwheel type.** We describe a coordinate system on 3C-triangles of pinwheel type. We assume that $C$ points into $B$, that $B$ points into $A$, and that $A$ points into $C$. As indicated in Figure 25, we use coordinates $(\alpha, \beta, x_\gamma)$. The angles $\alpha$ and $\beta$ are angles between pentagon edges on touching pentagons. The value $x_\gamma$ is the distance between the pointer vertex of pentagon $A$ and the pointer vertex of pentagon $B$. The coordinates satisfy constraints: $0 \leq \alpha, 0 \leq \beta, \alpha + \beta \leq \pi/5$, and $0 \leq x_\gamma \leq 2\sigma$. Define $\gamma$ by $\alpha + \beta + \gamma = \pi/5$. The angles $\alpha', \beta'$, and $\gamma'$ of the inner background triangle $P$ of the pinwheel are given by Equation 103. The edge lengths $x_\alpha, x_\beta, x_\gamma$ of the inner triangle $P$ are easily computed from $(\alpha, \beta, x_\gamma)$ by the law of sines. The edges of the 3C-triangle have lengths

$$
\ell(x_\alpha, \alpha), \quad \ell(x_\beta, \beta), \quad \ell(x_\gamma, \gamma).
$$

![Figure 25. Coordinates for pinwheel type](image)

11.9.1. **pinwheel type with a shared edge.** Now we specialize this discussion of Section 11.7 to pinwheels. Pinwheels have a rotational symmetry, so we may assume without loss of generality that the shared edge is $A \rightarrow C$ (meaning, $A$ and $C$ are the shared pentagons and $A$ points to $C$). This choice of shared edge gives $(\bar{A}, \bar{C}) = (A, C)$. The shared variables are $(\bar{x}_\beta, \bar{\beta}) = (x_\beta, \beta)$. The nonshared variable is $\bar{\alpha} = \alpha$. 

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11.10. **triple contact at a LJ-junction type.** We describe a coordinate system on 3C-triangles of LJ-junction type. As indicated in Figure 26, we use coordinates \((\alpha, \beta, x_\alpha)\). The angles \(\alpha\) and \(\beta\) are each formed by edges of two pentagons in contact. Let \(x_\alpha\) be the distance between the pointer vertex of \(C\) to \(A\) and the pointer vertex of \(B\) to \(C\). The coordinates satisfy relations: \(\alpha, \beta \in [0, 2\pi/5]\), \(\pi/5 \leq \alpha + \beta \leq 3\pi/5\), and \(0 \leq x_\alpha \leq 2\sigma\). Define \(\gamma\) by \(\alpha + \beta + \gamma = 3\pi/5\). The three acute angles \(\alpha', \beta', \gamma'\) of the inner L-shaped quadrilateral are given by Equation 103. The edge lengths of the L-shaped quadrilateral are easily computed by triangulating the quadrilateral into two triangles and applying the law of sines. (Triangulate \(L\) by extending the line through the edge of \(A\) containing the pointer vertex of \(C\) into \(A\).) This gives \(x_\beta\), the distance between the pointer vertex of \(C\) to \(A\) and the inner vertex of \(A\). This gives \(x_\gamma\), the distance between the pointer vertex of \(B\) and the inner vertex of pentagon \(A\). As before, the edges of the 3C-triangle have lengths 
\[ \ell(x_\alpha, \alpha), \quad \ell(x_\beta, \beta), \quad \ell(x_\gamma, \gamma). \]

**Figure 26.** Coordinates for LJ-junction type

11.10.1. **LJ\(_1\)-junction type.** Now we specialize to LJ-junctions with a shared edge \(C \to A\). Then \((\bar{A}, \bar{B}, \bar{C}) = (C, B, A)\).

The shared variables are \((\bar{x}_\beta, \bar{x}_\beta) = (\beta, x_\beta)\). The nonshared variable is \(\bar{\alpha} = \gamma' = 2\pi/5 - \gamma \geq 0\). These coordinates are numerically stable. We compute other angles and edges by triangulating the \(L\) by extending the receptor edge of \(A\) and using the law of sines.

11.10.2. **LJ\(_2\)-junction type.** We specialize to LJ-junctions with a shared edge \(B \to C\). In this case, \((\bar{A}, \bar{B}, \bar{C}) = (B, A, C)\).

The shared variables are \((\bar{x}_\alpha, \bar{\beta}) = (x_\alpha, \alpha)\). The nonshared variable is \(\bar{\alpha} = \beta\). These coordinates are numerically stable. They are exactly the standard variables given above for a general LJ-junction.

11.10.3. **LJ\(_3\)-junction type.** We specialize to LJ-junctions with a shared edge \(B \to A\). In this case, \((\bar{A}, \bar{B}, \bar{C}) = (B, C, A)\).
The shared variables are \((\bar{x}_\beta, \bar{\beta}) = (x_\gamma, \gamma)\). The nonshared variable is \(\beta\). If \(\beta > 0.9\) a calculation\(^5\) shows that the triangle is not subcritical. We may therefore assume that \(\beta \leq 0.9\). Under this additional assumption, these coordinates are numerically stable. We compute other lengths and angles by triangulating the \(L\)-region by extending the receptor edge of \(A\).

There does not exist an ice-ray triangle with longest edge along the edge \((A, B)\).

11.11. **triple contact at a \(TJ\)-junction type.** We describe a coordinate system on 3\(C\)-triangles of \(TJ\)-junction type. As indicated in Figure 27, we use coordinates \((\alpha, \beta, x_\gamma)\). The angles \(\alpha\) and \(\beta\) are each formed by edges of two pentagons in contact. The length \(x_\gamma\) is the distance between the pointer vertex of \(B\) into \(A\) and the inner vertex of \(A\). The coordinates satisfy: \(\alpha, \beta \in [\pi/5, 2\pi/5], 3\pi/5 \leq \alpha + \beta \leq 4\pi/5, 0 \leq x_\gamma \leq 2\sigma\). Define \(\gamma\) by \(\alpha + \beta + \gamma = \pi\). Three of the angles \(\alpha', \beta',\) and \(\gamma'\) of the inner irregular \(TJ\)-shaped pentagon \(P\) are given by Equation 103. The edge lengths of the \(TJ\)-shaped pentagon are easily computed by triangulating \(P\) into three triangles and applying the law of sines. (Triangulate by extending the edge of \(P\) shared with \(A\) that ends at the pointer vertex of \(A\) into \(C\) and by extending the edge of \(P\) shared with \(C\) that contains pointer vertex of \(B\) into \(C\).) This gives \(x_\alpha\), the distance between the pointer vertex of \(A\) to \(C\) and the inner vertex of \(C\). This gives \(x_\beta\), the distance between the pointer vertex of \(B\) to \(C\) and the inner vertex of \(C\). As before, the edges of the 3\(C\)-triangle have lengths

\[
\ell(x_\alpha, \alpha), \quad \ell(x_\beta, \beta), \quad \ell(x_\gamma, \gamma).
\]

\[\text{Figure 27. Coordinates for } TJ\text{-junction types}\]

11.11.1. **\(TJ_1\)-junction type.** We specialize to \(TJ\)-junctions with a shared edge \(A \rightarrow C\). In this case, \((\bar{A}, \bar{B}, \bar{C}) = (A, B, C)\).

The shared variables are \((\bar{x}_\beta, \bar{\beta}) = (x_\beta, \beta)\). The nonshared variable is \(\bar{\alpha} = \alpha\). If \(\beta < 1.0\) a calculation\(^6\) shows that the triangle is not subcritical. We may therefore assume that \(\beta \geq 1.0\). Under this additional assumption, these coordinates are numerically stable. We compute other lengths and angles by triangulating the \(TJ\)-region by extending the receptor edge of \(A\) and extending the receptor edge of \(C\).

\(^5\)once\_jx

\(^6\)once\_jx
There does not exist an ice-ray triangle with longest edge along the edge \((A, C)\).

11.11.2. **\(TJ_2\)-junction type.** We specialize to \(TJ\)-junctions with a shared edge \(B \rightarrow A\). In this case, \((\bar{A}, \bar{B}, \bar{C}) = (B, C, A)\).

The shared variables are \((\bar{x}_0, \bar{\beta}) = (x_0, \gamma)\). The nonshared variable is \(\bar{\alpha} = \beta\). From \(\alpha + \beta + \gamma = \pi\), we obtain \((\alpha, \beta, x_\gamma)\), which are the standard coordinates described above for \(TJ\)-junction triangles. These coordinates are numerically stable.

There does not exist an ice-ray triangle with longest edge along the edge \((A, B)\).

11.11.3. **\(TJ_3\)-junction type.** We specialize to \(TJ\)-junctions with a shared edge \(B \rightarrow C\). In this case, \((\bar{A}, \bar{B}, \bar{C}) = (A, B, C)\).

The shared variables are \((\bar{x}_0, \bar{\beta}) = (x_0, \alpha)\). The nonshared variable is \(\bar{\alpha} = \beta' = 2\pi/5 - \beta\). These coordinates are numerically stable. We compute other lengths and angles by triangulating the \(T\)-region by extending the receptor edge of \(A\) (of \(B \rightarrow A\)) and extending the receptor edge of \(C\) (of \(A \rightarrow C\)).

11.12. **triple contact at a pin-\(T\) junction type.** We describe a coordinate system on 3\(C\)-triangles of pin-\(T\) junction type. As indicated in Figure 28, we use coordinates \(\alpha, \beta\), and \(x_\alpha\). The angles \(\alpha\) and \(\beta\) are each formed by edges of two pentagons in contact. The length \(x_\alpha\) is the distance between the nearly coincident vertices of \(B\) and \(C\). The coordinates satisfy \(\pi/5 \leq \alpha \leq 2\pi/5, \pi/5 \leq \beta \leq 2\pi/5\), and \(3\pi/5 \leq \alpha + \beta\). Lemma 104 shows that \(0 \leq x_\alpha \leq 0.0605\). Define \(\gamma\) by \(\alpha + \beta + \gamma = \pi\). Three of the angles \(\alpha', \beta', \) and \(\gamma'\) of the inner irregular \(T\)-shaped pentagon are given by Equation 103. The edge lengths of the \(T\)-shaped pentagon \(P\) are easily computed by triangulating \(P\) into three triangles and applying the law of sines. (Triangulate by extending the two edges of \(P\) that meet at the pointer vertex of \(C\) into \(A\).) This gives \(x_\beta\), the distance between the pointer vertex of \(C\) to \(A\) and the inner vertex of \(A\). This gives \(x_\gamma\), the distance between the pointer vertex of \(A\) to \(B\) and the pointer vertex of \(B\) into \(C\). As before, the edges of the 3\(C\)-triangle have lengths
\[
\ell(x_\alpha, \alpha), \quad \ell(x_\beta, \beta), \quad \ell(x_\gamma, \gamma).
\]

**Lemma 104.** Let \(T\) be a 3\(C\) triangle of type pin-\(T\). The coordinates \(\alpha, \beta, \) and \(x_\alpha\) satisfy the relation
\[
x_\alpha \sin(2\pi/5) \leq 2\sigma(\sin(\alpha + \pi/5) - \sin(\beta + \pi/5)).
\]
In particular, \(x_\alpha \leq 0.0605\).

**Proof.** Let \(v_{AB}\) be the pointer vertex of \(A\) to \(B\), and let \(v_{BC}\) be the pointer vertex of \(B\) to \(C\). Let \(v\) and \(v_{BC}\) be the endpoints of the edge of \(B\) containing \(v_{AB}\). We represent \(T\) as in Figure 28, with the lower edge of \(C\) along the \(x\)-axis. Because \(v_{AB}\) lies on the segment between \(v\) and \(v_{BC}\), the \(y\)-coordinate \(y(v_{AB})\) of \(v_{AB}\) is nonpositive and lies between the \(y\)-coordinates \(y(v)\) and \(y(v_{BC})\). We have
\[
y(v) = x_\alpha \sin(2\pi/5) - 2\sigma \sin(\alpha + \pi/5)
y(v_{AB}) = -x_\beta \sin(\beta') - 2\sigma \sin(\beta + \pi/5).
\]
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Figure 28. Coordinates for pin-T junction types. Although it is difficult to tell from the figure, A points into B, B into C, and C into A. The parameter $\beta' \approx 0$ measures the incidence angle between the nearly horizontal edges of A and C. The region bounded by the three pentagons is a T-shaped pentagon, with stem between the nearly parallel edges of A and C and two arms along B. The arm between B and C is almost imperceptible. The distance between the neighboring vertices of B and C is $x_\alpha$. The figure on the right distorts the pentagons to make the incidence relations more apparent.

Using $x_\beta \sin(\beta') \geq 0$ and $y(\mathbf{v}) \leq y(\mathbf{v}_{AB})$, we obtain the claimed inequality.

Recall that $\pi/5 \leq \beta \leq 2\pi/5$. In particular, we have $\sin(\alpha + \pi/5) \leq 1$ and $\sin(\beta + \pi/5) \geq \sin(2\pi/5)$. This gives

$$x_\alpha \leq 2\sigma(1/\sin(2\pi/5) - 1) < 0.0605.$$  
□

11.12.1. pin-T-junction type. A computer calculation\(^7\) shows that the longest edge in a pinto-T-junction is always $B \rightarrow C$. We specialize to TJ-junctions with a shared edge $B \rightarrow C$. We use the standard coordinates, with $(\bar{x}_\beta, \bar{\beta}) = (x_\alpha, \alpha)$ shared. An ice-ray triangle does not have type pin-T.

This completes our discussion of coordinates used for computations.

12. Appendix on Computer Calculations

The code for the computer-assisted proofs is written in Objective Caml. There are about five thousand lines of code, available for download from github [10]. The computer calculations for this article take about 60 hours in total to run on an Intel quad 2.6 GHz processor with 3.7GB memory, running the Ubuntu operating system. The hashtables occupy between 1 and 2 GB of memory.

12.1. interval arithmetic. To control rounding errors on the computer, we use an interval arithmetic package for OCaml by Alliot and Gotteland, which runs on the Linux operating system.
12.2. **automatic differentiation.** Recall that there are several ways to compute derivatives by computer, such as numerical approximation by a difference quotient \((f(y) - f(x))/(y - x)\), symbolic differentiation (as in computer algebra systems), and automatic differentiation. In this project, we differentiate functions of a single variable using automatic differentiation. The value of a function and its derivative are represented as a pair \((f, f')\) of intervals (the 1-jet of the function at an interval-valued point \(x_0\)), where \(f\) is an interval bound on the function at \(x_0\), and \(f'\) is an interval bound on the derivative of the function at \(x_0\). More complex expressions can be built from simpler expressions by extending arithmetic operations (+, -, *, /) to 1-jets. For example,

\[
(f, f') + (g, g') = (f + g, f' + g'),
\]
\[
(f, f') \times (g, g') = (fg, f'g + f'g).
\]
\[
(f, f')/ (g, g') = (f/g, (f'g - f g')/g^2).
\]

where the component-wise arithmetic operations on the right are computed by interval arithmetic. Automatic differentiation extends standard functions \(F\) to functions \(F^D\) on 1-jets. For example,

\[
sqrt^D(f, f') = (\sqrt{f}, f'/2\sqrt{f}), \quad \sin^D(f, f') = (\sin(f), \cos(f)f').
\]

Sections 6 and Section 11 describe the proof of the local minimality of the ice-ray dimer. We review that argument here with an emphasis on automatic differentiation. Automatic differentiation allows us to show that the ice-ray dimer is the unique minimizer of area in an explicit neighborhood of the ice-ray dimer. In Section 6, we give a curve \(\Gamma\) (in the configuration space of dimers) with parameter \(t \in \mathbb{R}\) that passes through the ice-ray dimer point at \(t = 0\).

For any point \(x_0\) in an explicit neighborhood of the ice-ray dimer in the dimer configuration space, that section describes a path from \(x_0\) to a point on the curve \(\Gamma\). Using automatic differentiation algorithms, we show by computer that area decreases as we move along \(P\) from \(x_0\) towards \(\Gamma\). Thus, the area minimizer, lies on \(\Gamma\) for some parameter \(|t| \leq M\).

By symmetry in the underlying geometry, it is clear that the area function has derivative zero along \(\Gamma\) at \(t = 0\). By taking a second derivative with automatic differentiation, we find that the second derivative of the area function along this curve is positive when \(|t| \leq M\). Thus, the ice-ray dimer is the unique area minimizer on this curve within this explicit neighborhood.

We remark that many of the functions that are used in the proof of the pentagonal ice-ray conjecture are not differentiable. Our use of automatic differentiation is restricted to a small neighborhood of the ice-ray dimer, where all the relevant functions are analytic.

12.3. **meet-in-the-middle.** A common algorithmic technique for reducing the time complexity of an algorithms through greater space complexity is called **meet-in-the-middle**
MITM). This is very closely related to the linear assembly algorithms used in the solution to the Kepler problem [5]. (Both methods break the problem into subproblems of smaller complexity that are later recombined. MITM stores the data for recombination in a hashtable. Linear assembly encodes the data for recombination as linear programs. We did not try to solve the pentagonal ice-ray conjecture with linear programming techniques. Such an approach might also work.)

Some introductory examples of meet-in-the-middle algorithms can be found at the blog post [9]. A simple example from there is to find if there are four numbers in a given finite set \( S \) of integers that sum to zero, where repetitions of integers are allowed. If we calculate all possible sums of four integers, testing if each is zero, then there are \( n^4 \) sums, where \( n \) is the cardinality of \( S \). The MITM solution to the problem computes and stores (in a hashset) all sums \( a + b \) of unordered pairs of elements from \( S \). We then search the hashset for a collision, meaning a sum \( a + b \) that is the negative of another sum \( c + d \): \( a + b = -(c + d) \). Any such collision gives \( a + b + c + d = 0 \). The MITM solution involves the computation of \( n^2 \) sums \( a + b \), rather than \( n^4 \), for a substantial reduction in complexity. MITM techniques have numerous applications to cryptography, and it is there that we first encountered the technique. See for example, [3] which applies MITM to a general class of dissection problems, including the Rubik’s cube.

We obtain computational bounds on the area of clusters of Delaunay triangles (or more accurately, \( P \)-triangles). Each of our clusters will be assume to consist of one triangle (called the central triangle), flanked by one, two, or three additional triangles along its edges. We call the flanking triangles peripheral triangles. The aim of each computation is to give a lower bound on the sum of the areas of the triangles, subject to a collection of constraints. Two types of constraints are allowed: (1) constraints that can be expressed in terms of a single triangle, and (2) assembly constraints. An assembly constraint states that the central \( P \)-triangle fits together with a flanking triangle. In more detail, the central triangle \( T_0 \) shares an edge and two pentagons \( A, B \) with a flanking triangle \( T_1 \). Associated with \( T_0 \) are parameters \( d_{AB}^0, \theta_{ABC}^0, \theta_{BAC}^0 \) giving the edge length of the common edge with \( T_1 \), and the inclination angles of the pentagons \( A \) and \( B \) with respect to that common edge. Similarly, associated with \( T_1 \) are parameters \( d_{AB}^1, \theta_{ABC}^1, \theta_{BAC}^1 \). The assembly constraint along the common edge is

\[
\begin{align*}
d_{AB}^0 &= d_{AB}^1, & \theta_{ABC}^0 &= -\theta_{ABC}^1, & \theta_{BAC}^0 &= -\theta_{BAC}^1.
\end{align*}
\]

The negative sign comes from the opposite orientations of the common edge with respect to \( T_0 \) and \( T_1 \).

A cluster consisting of one central \( P \)-triangle and \( k \) flanking triangles is a point in a configuration space of dimension \( 6 + 3k \). If we cover the configuration space with cubes of edge-length \( \epsilon \), then there are order \( (1/\epsilon)^{6+3k} \) cubes. This is generally beyond our computational reach when \( k > 0 \).

We can use MITM techniques to reduce to order \( (1/\epsilon)^6 \) cubes, and this puts all our computations (barely) within the reach of a laptop computer. Specifically, we fix a edge size \( \epsilon \) and cover the configuration space of peripheral triangles by cubes of size \( \epsilon \), calculating area, edge lengths, inclination angles, and other relevant quantities (using interval arithmetic) over each cube.
The idea of MITM is to place the peripheral triangle data into a hash table, keyed by the variables \( d_{AB}, \theta_{ABC}, \theta_{BAC} \) that are shared with the central triangle through Equation 105. We view Equation 105 as the analogue of the collision condition \((a + b) = -(c + d)\) in the simple example of MITM given above. Of course, the variables \( d_{AB} \) etc. are represented as intervals with floating point endpoints and cannot be used directly as keys to a hashtable. Instead we discretize the keys in such a way that a collision of real numbers implies a collision of the keys.

Once the hash is created, we divide the configuration space of central triangles into cubes and compute the relevant quantities (edge lengths, triangle area, and inclination angles) over each cube using interval arithmetic. The areas of the peripheral triangles are recovered from the hash table and combined with the area of the central triangle to get a lower bound on the sum of the areas of the triangles in the cluster.

The entire process is iterated for smaller and smaller \( \epsilon \) until the desired bound on the areas of the triangles in the cluster is obtained. Each time \( \epsilon \) is made smaller, only the peripheral cubes that were involved in a key collision with a central cube are carried into the next iteration for subdivision into smaller cubes. Only the cubes with suitably small triangle area bounds are carried into the next iteration. In practice, to achieve our bounds in Section 5, the smallest \( \epsilon \) that was required was approximately 0.00024.

12.4. preparation of the inequalities. Section 5 gives a sequence of inequalities that have been proved by MITM algorithms. We organize these calculations to permit a more or less uniform proof of all of them. The triangle \( T_0 \) is the central triangle. The other triangles \( T_1, \ldots \) in the cluster are the peripheral triangles.

We prove each inequality by contradiction. Specifically, we negate the conclusion and add it to the set of assumptions. Then we show in each case that the domain defined by the set of assumptions is empty. In the computer code, we implement out-of-domain functions that return true when the interval input lies entirely outside the given domain.

We deform the cluster of triangles to make computations easier. Note that in every case except for the triangle \( T_- \) in Lemma 40, the peripheral triangles are all subcritical. We prove the lemmas of Section 5 in the sequential order given in that section. In particular, we may assuming the previous lemmas to simplify what is to be proved in those that follow.

**Lemma 106.** We can assume without loss of generality that the triangle \( T_- \) in Lemma 40 is \( O2C \).

**Proof.** Assume for a contradiction, that we cannot deform into a \( O2C \). Let the three pentagons of \( T_- \) be \( A, B, C \), where \( A, C \) are shared with the central triangle \( T_0 \). By Lemmas 29 and 30, we can assume that \( B \) has primary contact. By translating \( B \), we can continue to deform \( T_- \), decreasing its area, until it is a right triangle (because \( O2C \) is assumed not to occur). By Lemma 37, the shared edge with \( T_0 \) has length at least 1.8, so a right angle gives the proof of Lemma 40 in this case:

\[
\text{area}(T_0, T_1, T_-) > 2a_{\min} + \text{area}(T_-) \geq 2a_{\min} + 1.8k > 3a_{\text{crit}} + \epsilon_M.
\]

\( \square \)
We note that an earlier lemma (Lemma 34) shows that subcritical peripheral triangles can be assumed to be \( O2C \). Thus, we will assume in the computations that all peripheral triangles are \( O2C \).

In the rest of this section, we describe how each calculation has been prepared, to reduce the dimension of the configuration space, prior to computation.

12.4.1. *calculation of Lemma 36*. We repeat Lemma 36. Let \((T_1, T_0) \in \Psi D\). The edge shared between \( T_0 \) and \( T_1 \) has length less than 1.8. (The reference code [NKQNXUN] links this statement to the relevant body of computer code.)

Negating the conclusion, we may assume that the shared edge between triangles has length at least 1.8. Let \( A, B, C \) be the pentagons of \( T_0 \), with \( A, C \) shared with \( T_1 \). We drop the constraint \( T_0 \Rightarrow T_1 \). (That is, we no longer assume that the longest edge of \( T_0 \) is shared with \( T_1 \). Instead, we merely assume that \( \max(d_{AB}, d_{BC}) \geq 1 \). We may assume that \( B \) has primary contact. By symmetry, we may assume contact between \( A \) and \( B \).

As long as \( B \) is not \( O2C \), we may continue to move \( B \) (decreasing area as always) until \( \max(d_{AB}, d_{BC}) = 1 \). The calculation reduces to three subcases:

1. \( T_0 \) has \( O2C \) contact.
2. \( B \) has midpointer contact along \( AB \), with \( d_{AB} = 1.8 \). (We eliminate the case \( d_{BC} = 1.8 \) because this would give using Lemma 32,

\[
\text{area}(T_0, T_1) > a_{\min} + \text{area}(T_0) \geq a_{\min} + \text{area}(1.8, 1.8, 1.72) > 2a_{\text{crit}}.
\]
3. \( B \) has slider contact with \( A \) and \( \max(d_{AB}, d_{BC}) = 1 \).

12.4.2. *calculation of Lemma 37*. We repeat the statement of Lemma 37. Let \((T_1, T_0) \in \Psi D\). The longest edge of \( T_0 \) has length greater than 1.8. (reference code [RWWHLQT]).

Negating the conclusion, we may assume that all edges of \( T_0 \) have length at most 1.8. As in the previous calculation, we let \( A, C \) be the shared pentagons, and we reduce to three cases:

1. \( T_0 \) has \( O2C \) contact.
2. \( B \) has midpointer contact along \( AB \). We continue to deform by translating \( B \) until \( T_0 \) is long isosceles.
3. \( B \) has slider contact with \( A \), and \( T_0 \) is long isosceles.

12.4.3. *calculation of Lemma 39*. We repeat the statement of Lemma 39. Let \((T_1, T_0) \in \Psi D\). Then area\([T_0, T_1] \geq 2a_{\text{crit}} - \epsilon_M\) (reference code [BXZBPJW]).

Negating the conclusion, we may assume that area\([T_0, T_1] \leq 2a_{\text{crit}} - \epsilon_M\). Let \( A, B, C \) be the pentagons of \( T_0 \), with \( A, C \) shared with \( T_1 \). We deform \( B \) until primary contact. If it is not \( O2C \), then we continue to translate \( B \). We never encounter the long isosceles constraint while translating \( B \) because if the longest edge and shared length have equal lengths, we have area\([T_0, T_1] > 2a_{\text{crit}} \) by Lemmas 36 and 37. Thus, we always reduce to \( O2C \) on \( T_0 \). We may continue with a squeeze transformation (Section 6.3) until \( T_1 \) is long isosceles or
3C. We consider two long isosceles subcases, depending on which of the two nonshared edges of $T_1$ has the same length as the shared edge.

The squeeze transformation may result in $T_0$ and $T_1$ becoming triple contact. The calculations for dimer pairs in triple contact were carried out with assumptions that were sufficiently relaxed to include this case of pseudo-dimer triple contact.

12.4.4. calculation of Lemma 40. We repeat the statement of Lemma 40. Let $(T_1, T_0) \in \Psi D$. Assume $T_0 \Rightarrow T_-$. Then $\text{area}(T_0, T_1, T_-) > 3a_{\text{crit}} + \epsilon_M$ (reference code [JQMRXTH]).

As noted above, we can assume that both peripheral triangles, $T_1$ and $T_-$ are $O2C$. Negating the conclusion, we assume that $\text{area}(T_0, T_1, T_-) \leq 3a_{\text{crit}} + \epsilon_M$. This implies that

$$\text{area}(T_-) = \text{area}(T_0, T_1, T_-) - \text{area}(T_0, T_1) \leq (3a_{\text{crit}} + \epsilon_M) - (2a_{\text{crit}} - \epsilon_M) = a_{\text{crit}} + 2\epsilon_M.$$

We let $A, B, C$ be the pentagons in the central triangle $T_0$, where $A$ is shared with $T_0, T_1, T_-$ and $B$ is shared between $T_0$ and $T_-$. While $B$ is not in contact with another pentagon, we may translate $B$ in a squeeze transformation, moving it along the segment joining $c_A$ and $c_B$. This decreases the areas of $T_0$ and $T_-$. We continue until $B$ contacts $A$ or $C$.

Renaming pentagons of the central triangle, we assume that $\bar{A}$ and $\bar{C}$ are in contact, with $A$ pointing to $\bar{C}$. We consider six cases: each permutation on three letters determines a choice for the edge of $T_0$ shared with $T_1$, and a choice of the edge of $T_0$ shared with $T_-$. We claim that we can always deform to one of these five cases. To see this, assume to the contrary that none of these cases hold. If the 1.8 constraint binds, we translate the outer
pentagon $\tilde{B}$ of $T_0$ while maintaining the 1.8 constraint until it comes into contact with one of the shared pentagons $\tilde{A}$ or $\tilde{C}$. This falls into one of the last four cases. If the 1.8 constraint does not bind, then we may translate $\tilde{B}$ until $T_0$ is $O2C$. Here $\{\tilde{A}, \tilde{B}, \tilde{C}\} = \{A, B, C\}$ is a relabeling of the pentagons.

12.4.7. calculation of Lemma 49. We repeat the statement of Lemma 49. Let $T_i \to T_0$ and $\text{area}(T_i) \leq a_{\text{crit}}$ for $i = 0, 1$ for distinct $P$-triangles $T_0^0$ and $T_1^1$. Then $\text{area}(T_0, T_1^0, T_1^1) > 3a_{\text{crit}} + \epsilon_M$ (reference code [HUQEJAT]).

We assume that $\text{area}(T_0, T_0^0, T_1^1) \leq 3a_{\text{crit}} + \epsilon_M$. We consider four cases:

1. There exists a pair $(A, C)$ of pentagons of $T_0$ in contact. We assume that $A$ points to $C$. This becomes three cases according to the edge of $T_0$ that is not shared.
2. $T_0$ has no pentagons in contact. Let $B$ be the pentagon of $T_0$ that is shared with $T_0^0$ and $T_1^1$. We squeeze $A$ along $(c_A, c_B)$ and squeeze $C$ along $(c_B, c_C)$. This allows us to assume that both $T_0^0$ and $T_1^1$ are long isosceles.

12.4.8. calculation of Lemma 50. We repeat the statement of Lemma 50. Let $T_i \to T_0$ and $\text{area}(T_i) \leq a_{\text{crit}}$ for $i = 0, 1, 2$ for distinct $P$-triangles $T_0^0$, $T_1^1$, and $T_2^2$. Then $\text{area}(T_0, T_0^0, T_1^1, T_2^2) > 4a_{\text{crit}}$ (reference code [QPJDYDB]).

Each $T_i^i$ is $O2C$. We carry out the calculation as a single case, using MITM as usual as described above. There is an $S_3$-symmetry to the situation that we can exploit to reduce the search space.

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