Non-Markovian effects in stochastic resonance in double quantum dot system

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(Dated: July 19, 2022)

Due to the combination of non-Markovian dissipation and external driving force, it is difficult to evaluate the correlation function and hence the power spectrum in quantum stochastic resonance. Fortunately, a recently developed algorithm, which is called time-evolving matrix product operators (TEMPO), and its extensions provide an efficient and numerically exact approach for this task. In this article, using TEMPO we investigate non-Markovian effects in stochastic resonance in a double quantum dot system.

I. INTRODUCTION

Stochastic resonance (SR) is phenomenon for which the response of the system to external driving is enhanced by noise, which was first proposed by Benzi et al.\(^1\). Since then SR has continuously attracted considerable attention over several decades. The first experimental verification of such phenomenon was obtained by Fauve and Heslot\(^2\), who studied the noise-induced transition process in a bistable system. Another key experiment in this field is the observation of SR in an optical device, the bidirectional ring laser, by McNamara et al.\(^3\). The concept of SR was extended to the quantum regime by L"{o}fstedt and Coppersmith\(^4\) by measuring conductance fluctuations in mesoscopic metals. Recently, a tunnelling-controlled quantum SR is demonstrated in the a.c.-driven charging and discharging of single electron on a quantum dot\(^5\).

The theoretical investigation for SR in terms of periodically driven dissipative system starts several decades ago, and some comprehensive reviews can be found in Refs.\(^6–9\). When the quantum coherence is suppressed, both quantum and classical SR can be well described by the classical rate equation approach\(^6\). In deep quantum regime, i.e., quantum stochastic resonance (QSR), qualitative new features arise\(^10, 11\). This regime has been investigated semiclassically by Grifoni et al.\(^10\) and numerically by Makarov and Makri\(^12, 13\). Gam-maitoni et al. proposed a control scheme for SR\(^14\). The QSR in a system driven by weak signal and white noise was studied by Joshi\(^15\).

The non-Markovian transient property of QSR is difficult to evaluate due to the combination of dissipative and external driving. To investigate QSR, Grifoni et al. focus on asymptotic property or employ semiclassical analysis\(^11\), Joshi focuses on white noise\(^15\), and Makarov and Makri employ a numerically exact method known as quasi-adiabatic propagator path integral (QUAPI) method\(^12, 13\). The computational cost of QUAPI grows exponentially with size of the system Hilbert space and memory length, therefore although numerically exact, QUAPI can become highly inefficient or even infeasible under certain circumstances. Due to this limit, the correlation function and the corresponding power spectrum, which is the quantity of fundamental interest in SR study, are not suitable to be evaluated by bare QUAPI algorithm.

Recently, Strathearn et al.\(^16\) show that QUAPI method can be represented in the framework of matrix product states (MPS)\(^17, 18\). The standard MPS compression algorithms is applicable in this framework, and thus they obtain an efficient and numerically exact method which is called time-evolving matrix product operators (TEMPO). Later Jørgensen and Pollock\(^19\) relate TEMPO to process tensor to motivate an efficient and numerically exact algorithm for simulation of correlation functions in undriven open systems. The TEMPO method is also modified for repeated computation of various sets of parameters by Fux et al.\(^20\).

In this article, we employ TEMPO to study the non-Markovian effects in QSR in a double quantum dot system, which is a periodically driven open system. The asymptotic time-averaged correlation function is evaluated to obtain the power spectrum, from which the signal-to-noise ratio is calculated. It is found that the non-Markovian effects indeed plays an important role in quantum stochastic resonance.

This article is organized as follows. The introduction of the model and method are given in Sec. II and III, respectively. The non-Markovian effects on observable and correlation function are discussed in Sec. IV and V, respectively. Section VI gives the result of signal-to-noise ratio. Finally a conclusion is given in Sec. VII.

II. MODEL

Bistable system is the simplest and also the most widely used model to study stochastic resonance. Such a bistable system can be effectively described by a double quantum dot system, when external driving is present the Hamiltonian can be written as

\[
H_S(t) = \Delta \sigma_z + \frac{E}{2} \sigma_z \cos \Omega t, \quad (1)
\]

where \(\sigma_z\) and \(\sigma_x\) are Pauli matrices, and the eigenstates of \(\sigma_z\) are the basis states in a localized representation. Here \(\Delta\) gives the tunneling amplitude between two dots, \(E\) is the strength of external driving and \(\Omega\) is driving frequency.

As pointed out by Caldeira and Leggett\(^21, 22\), a bath consisting of harmonic oscillators is rather general and provides a suitable description of a realistic quantum environment. Therefore the bath Hamiltonian and the system-bath coupling are

\[
H_B = \sum_k \omega_k b_k^\dagger b_k, \quad H_{SB} = \sigma_z \sum_k V_k (b_k^\dagger + b), \quad (2)
\]
where the operator $b_k^\dagger (b_k)$ creates (annihilates) a boson in state $k$ with frequency $\omega_k$. The total Hamiltonian is

$$H = H_S(t) + H_B + H_{SB}.$$  \hspace{1cm} (3)

This model is usually referred as spin-boson model \cite{23, 24}. The bath is characterized by a spectral function $J(\omega)$ and in this article we choose Ohmic dissipation for which

$$J(\omega) = \frac{\lambda \omega e^{-\omega/\omega_c}}{2\pi},$$  \hspace{1cm} (4)

where $\lambda$ is the coupling strength parameter and $\omega_c$ is the cutoff frequency. If the bath is in thermal equilibrium state then the bath autocorrelation function is written in terms of $J(\omega)$ as

$$\alpha(t) = \int_0^\infty J(\omega) \left[ \coth \left( \frac{\omega}{2T} \right) \cos \omega t - i \sin \omega t \right] d\omega,$$  \hspace{1cm} (5)

where $T$ is the temperature. Throughout this article we set $\hbar = k_B = 1$ and use dimensionless quantities.

The relevant theoretical quantity describing the dissipative dynamics is the expectation value of observable $\langle \sigma_z(t) \rangle$. Under periodic driving, this quantity shows coherent oscillations with periodicity $\pi/\omega_c$. Under this definition, the correlation function is symmetrized

$$C(t_1, t_2) = \frac{1}{2} \langle \sigma_z(t_1) \sigma_z(t_2) + \sigma_z(t_2) \sigma_z(t_1) \rangle.$$  \hspace{1cm} (6)

Under this definition, the correlation function is symmetrized in the sense that $C(t_1, t_2) = C(t_2, t_1)$, and we can write

$$C(t_1, t_2) = \begin{cases} \text{Re} \langle \sigma_z(t_1) \sigma_z(t_2) \rangle, & t_2 \geq t_1; \\
\text{Re} \langle \sigma_z(t_2) \sigma_z(t_1) \rangle, & t_1 \geq t_2. \end{cases}$$  \hspace{1cm} (7)

It means the symmetrized correlation function $C(t_1, t_2)$ is a real quantity and the knowledge of the correlation function $\langle \sigma_z(t_1) \sigma_z(t_2) \rangle$ for $t_2 \geq t_1$ is enough to obtain whole $C(t_1, t_2)$. Therefore we focus on $C(t_0, t_0 + t)$ for $t \geq 0$ situation. The time-averaged, asymptotic (i.e., $t_0 \to \infty$) symmetrized correlation function is defined as

$$\bar{C}(t) = \lim_{t_0 \to \infty} \frac{\omega}{2\pi} \int_0^{2\pi} C(t_0, t_0 + t) dt_0.$$  \hspace{1cm} (8)

The quantity of experimental interest for QSR is the power spectrum \cite{6, 8}, which is defined one-sided, i.e., for positive $\omega$ only, as

$$S(\omega) = \int_{-\infty}^{\infty} \bar{C}(t)(e^{i\omega t} + e^{-i\omega t}) dt = \int_{-\infty}^{\infty} \bar{C}(t) \cos \omega t dt.$$  \hspace{1cm} (9)

### III. METHOD

#### A. Quasi-Adiabatic Propagator Path Integral

Here we give a brief review of the QUAPI and TEMPO method. Let $\rho(t)$ denote the total density matrix, then the time evolution of $\rho$ is given by

$$\rho(t) = U(t, 0)\rho(0)U^\dagger(t, 0),$$  \hspace{1cm} (10)

where

$$U(t, t_0) = T \exp \left[ -i \int_{t_0}^{t} H(\tau) d\tau \right].$$  \hspace{1cm} (11)

Here $T$ denotes the chronological ordering operator. The reduced density matrix is defined as $\rho_B(t) = \text{Tr}_B[\rho(t)]$, where $\text{Tr}_B$ denote the trace over the bath degrees of freedom. Let $s$ denote the eigenvalue of $\sigma_z$, then the element of the reduced density matrix can be written as

$$\rho_B(s''; s'; t) = \text{Tr}_B \langle s''| U(t, 0)\rho(0)U^\dagger(t, 0)|s' \rangle.$$  \hspace{1cm} (12)

If at initial time $t = 0$ the total density matrix is in product state for which

$$\rho(0) = \rho_S(0) \otimes \rho_B,$$  \hspace{1cm} (13)

then the above expression can be written in path integral form by splitting the evolution into $N$ pieces for for which $\delta t = t/N$ with $N \to \infty$. Relabeling $s'', s'$ as $s_N^-, s_N^+$ yields

$$\rho_B(s_N^+; s_N^-; t) = \sum_{s_0^- \ldots s_{N-1}^-} F(s_N^+, \ldots, s_N^+) I(s_0^-, \ldots, s_{N-1}^-).$$  \hspace{1cm} (14)

where $\rho_B(s_N^+) = \langle s_N^-| \rho_B(t)|s_N^- \rangle$ and

$$F(s_N^+, \ldots, s_N^+) = K(s_N^+, \ldots, s_N^+) I(s_0^-, \ldots, s_{N-1}^-).$$  \hspace{1cm} (15)

Here $K(s_0^-, \ldots, s_N^-)$ is the bare system propagating tensor for which

$$K(s_N^+, \ldots, s_N^+) = \rho_S(s_0^-) \tilde{K}(s_N^+, s_N^+) \cdots \tilde{K}(s_{N-1}^+, s_N^+),$$  \hspace{1cm} (16)

where ($t_k = k\delta t$)

$$\tilde{K}(s_{k-1}^+, s_k^+) = \langle s_k^+| U(t_k, t_{k-1})|s_{k-1}^+ \rangle \times \langle s_{k-1}^-| U^\dagger(t_k, t_{k-1})|s_{k-1}^- \rangle.$$  \hspace{1cm} (17)

In the continuum limit $\delta t \to 0$, the collections of $(s_0^+, \ldots, s_N^+)$ and $(s_0^-, \ldots, s_N^-)$ can be regarded as a forward path $s^+(t')$ and a backward path $s^-(t')$ from 0 to $t$. If at initial time the bath is in thermal equilibrium state that $\rho_B = e^{-H_0/2}$, then the influence functional $I$ can be written as \cite{21, 25, 26}
\[ I[s^\pm(t)] = \exp \left[ -\int_0^t dt' \int_0^{t'} dt'' [s^+ (t') - s^- (t')] [\alpha (t' - t'') s^+ (t'') - \alpha^* (t' - t'') s^- (t'')] \right], \] (18)

where \( \alpha(t) \) is the autocorrelation function given by (5). When employing finite \( \delta t \) approximation, this influence functional can be discretized as

\[ I(s_0^+, \ldots, s_N^+) = \exp \left[ -\sum_{j=0}^N \sum_{k=0}^j \phi_{jk} \right], \] (19)

where

\[ \phi_{jk} = (s_j^+ - s_j^-)((\eta_{jk} s_k^+ - \eta_{jk}^* s_k^-)). \] (20)

Here the form of \( \eta_{jk} \) depends on choice of discretization scheme [27].

The discretized influence functional (19) is a tensor which can be decomposed as

\[
I(s_0^+, \ldots, s_N^+) = \prod_{k=0}^N I_0(s_k^+, s_{k+1}^+) \prod_{k=0}^{N-1} I_1(s_k^+, s_{k+1}^+) \cdots \\
\times \prod_{k=0}^{N-\Delta k} I_{k\Delta k}(s_k^+, s_{k+\Delta k}^+) \cdots \\
\times I_N(s_N^+, s_{N-1}^+),
\] (21)

where

\[ I_{k\Delta k}(s_k^+, s_{k+\Delta k}^+) = e^{-\phi_{k\Delta k}}. \] (22)

The key idea of QUAPI method is that non-locality of \( \eta_{jk} \) drops off as \( \Delta k \) increases, then \( \eta_{k+\Delta k} \) can be neglected when \( \Delta k \) is greater than a certain positive integer \( N_s \). Therefore the influence functional (21) can be truncated as

\[
I(s_0^+, \ldots, s_N^+) = \prod_{k=0}^N I_0(s_k^+, s_{k+1}^+) \prod_{k=0}^{N-1} I_1(s_k^+, s_{k+1}^+) \cdots \\
\times \prod_{k=0}^{N-\Delta k} I_{k\Delta k}(s_k^+, s_{k+\Delta k}^+) \cdots \\
\times I_{N-N_s}(s_N^+, s_{N+N_s}^+),
\] (23)

Now define a tensor \( A(s_0^+, \ldots, s_{N-1}^+) \) as

\[ \tilde{K}(s_k^+, s_{k+1}^+) \prod_{k=0}^{N-\Delta k} I_{k\Delta k}(s_{k+\Delta k}^+, s_{k+\Delta k}^+) \] (24)

then there is a recursive relation for which

\[ F(s_0^+, \ldots, s_{k+1}^+) = F(s_0^+, \ldots, s_k^+) A(s_k^+, \ldots, s_{k+1}^+). \] (25)

Employing the above recursive relation iteratively we can get the final \( F(s_0^+, \ldots, s_N^+) \) from initial condition

\[ F(s_0^+) = \rho_S(s_0^+) I_0(s_0^+). \] (26)

### B. Correlation Functions

The formalism described above only deals with the dynamics of reduced density matrix. It is, however, easy to be generalized for the correlation function calculation.

The expectation value \( \langle \sigma_z(t') \rangle \) can be obtained via

\[ \langle \sigma_z(t') \rangle = \text{Tr} [U(t,0)\rho(0)U^\dagger(t',0)\sigma_z U(t,t')] \] (27)

where \( \text{Tr} = \text{Tr}_S \text{Tr}_B \) is the trace over all degrees of freedom with \( \text{Tr}_S \) the trace over degrees of freedom of the system. Suppose \( t' = k\delta t \), we can define a one-time “correlated” reduced density matrix as

\[ \tilde{\rho}_S(s_N^+, \bar{s}_k^-) = \sum_{s_0^+, \ldots, s_{N-1}^+} F(s_0^+, \ldots, s_{N-1}^+) s_k^-, \] (28)

then the expectation value can be written as

\[ \langle \sigma_z(t') \rangle = \text{Tr}_S [\tilde{\rho}_S(s_N^+, \bar{s}_k^-)] \] (29)

As mentioned in Sec. II, we need to only calculate the correlation function \( \langle \sigma_z(t_1)\sigma_z(t_2) \rangle \) for \( t_2 \geq t_1 \). In this case, \( \langle \sigma_z(t_1)\sigma_z(t_2) \rangle \) can be written as

\[ \text{Tr} [U(t,0)\rho(0)U^\dagger(t_1,0)\sigma_z U^\dagger(t_2,t_1)\sigma_z U(t_2,t_2)U^\dagger(t,t_2)]. \] (30)

Suppose \( t_1 = k_1\delta t \) and \( t_2 = k_2\delta t \), where \( k_2 \geq k_1 \), we can define a two-time “correlated” reduced density matrix as

\[ \tilde{\rho}_S(s_N^+, s_k^-, s_{k_2}^-) = \sum_{s_0^+, \ldots, s_{N-1}^+} F(s_0^+, \ldots, s_{N-1}^+) s_{k_1}^-, s_{k_2}^- \] (31)

and the correlation function is obtained via

\[ \langle \sigma_z(t_1)\sigma_z(t_2) \rangle = \text{Tr}_S [\tilde{\rho}_S(s_N^+, s_{k_1}^-, s_{k_2}^-)] \] (32)

### C. Time-Evolving Matrix Product Operators

So far we have discussed the basic framework of calculating the correlation function in a driven spin-boson model. However, \( F(s_0^+, \ldots, s_{N-1}^+) \) is a tensor of rank \( 2(N+1) \) for which a space with size proportional to \( 2^{2(N+1)} \) is needed to store it. In practical calculation it is very difficult to handle such tensor directly unless \( N_s \) is fairly small.

In original QUAPI [28–30], an iterative tensor multiplication algorithm is employed and a tensor of rank \( 2(N_s+1) \), rather than \( 2(N+1) \), is kept in track during the time evolution process. This greatly reduce the space needed, but the computational cost still scales exponentially with \( N_s \). The value \( N_s \delta t \) is supposed to cover the range of non-locality of \( \eta_{jk} \), and to ensure a small \( N_s \) usually a relatively large \( \delta t \) is adopted, which may introduce relatively large Trotter errors.
Recently, Strathearn et al. [16] showed that the tensor $F(s_0^+, \ldots, s_N^+)$ can be naturally represented by matrix product states (MPS) [17, 18] and developed the TEMPO algorithm. The main idea is that $F(s_0^+, \ldots, s_N^+)$ can be efficiently constructed via iterative application of matrix product operator (MPO). Such iterative process is amenable to standard MPS compression algorithm, and thus computational cost scales only polynomially with $N_s$. This allows us to perform simulations to large values of $N_s$, for instance, $N_s$ in Ref. [16] is up to 200 which is impossible to simulate without tensor compression algorithm. There is also another approach for tensor network representation of discretized path integral by Oshiyama et al. [31, 32].

Later Jørgensen and Pollock [19] related the influence functional to process tensor via the influence functional $F(s_0^+, \ldots, s_N^+)$ by MPS, they use this connection to motivate a tensor network algorithm for simulation of multiple time correlation functions. Fux et al. [20] modified TEMPO method for repeated computation of various sets of parameters.

For clarity and simplicity, we abbreviate the index pair \{s_k^+, s_k^-\} as $s_k$. In this way the $2 \times 2$ reduced density matrix is represented as a vector of 4 elements. We also write $F$ as a superscripted tensor for which

$$F^{s_0, \ldots, s_k} = F(s_0, \ldots, s_k).$$

(33)

Define a $B$ tensor as

$$B^{s_0, \ldots, s_{k+1}}_{r_0, \ldots, r_k} = \left(\prod_{i=0}^k \delta_{s_i r_i}\right) A(s_0, \ldots, s_{k+1}),$$

(34)

where $A(s_0, \ldots, s_{k+1})$ is the tensor defined in (24), then the recursive relation (25) can be written in terms of Einstein summation convention way as

$$F^{s_0, \ldots, s_{k+1}} = B^{s_0, \ldots, s_{k+1}} F^{r_0, \ldots, r_k}. $$

(35)

If $F^{s_0, \ldots, s_k}$ is represented as a MPS, then $F^{s_0, \ldots, s_{k+1}}$ can keep the MPS structure if the $B$ tensor is represented as a MPO. Then during the iterative process the standard MPS compression algorithm can be applied such that the required computational resource scales polynomially. The form meets the requirement is (here Einstein summation convention still applies)

$$B^{s_0, \ldots, s_{k+1}}_{r_0, \ldots, r_k} = \begin{cases} [b_{k+1}]^{s_0,00}_{r_0} [b_k]^{r_1,01} \cdots [b_{k+1-m}]^{s_m,00}_{s_m-\alpha_m-1,00} & \text{if } m < k \text{ the rank-3 tensor in the front is defined as} \\
 [b_{k+1-m}]^{s_m,00}_{s_m-\alpha_m-1,00} \cdots [b_1]^{r_1,01} \cdots [b_0]^{s_0+1,00} & \text{otherwise} \end{cases},$$

(36)

where the rank-3 tensor in the front is defined as

$$[b_{k+1}]^{s_0,00}_{r_0} = I_{k+1}(s_0, \alpha_0) \delta_{r_0}^{r_0},$$

(37)

When $m < k$ the rank-4 tensors in the middle are defined as

$$[b_{k+1-m}]^{s_m,00}_{s_m-\alpha_m-1,00} = I_{k+1-m}(s_m, \alpha_m) \delta_{r_m}^{r_m} \delta_{s_m-1}^{s_m-1},$$

(38)

and when $m = k$ we have

$$[b_1]^{s_k,00}_{r_k,\alpha_k-1} = \tilde{K}(s_k, \alpha_k) I_1(s_k, \alpha_k) \delta_{r_k}^{r_k} \delta_{\alpha_k}^{\alpha_k}.$$
IV. NON-MARKOVIAN EFFECTS ON $\langle \sigma_z(t) \rangle$

In this section we study the non-Markovian dynamics of the expectation value $\sigma_z(t)$. Starting from an arbitrary initial state (here we start from $\langle \sigma_z(0) \rangle = 1$), the reduced density matrix would eventually reach a steady state where $\langle \sigma_z(t) \rangle$ oscillates with frequency $\Omega$.

In QUAPI and TEMPO algorithms, the non-Markovianity are controlled by the truncation $N_s$. If $N_s$ is large enough to cover the non-locality of $\alpha(t)$ (or $\eta_{jk}$) then it is supposed to capture all non-Markovian effects. On the other hand, if $N_s = 1$ then the dynamics is only relevant to one last time step and thus it gives the Markovian result. In this article, we shall call the $N_s = 1$ case Markovian, as in Ref. [19].

A set of parameters which induces large amplitude oscillation can be found in Refs. [12, 13]. If we set $\Delta = 1$, then in our model they correspond to $\Omega = 1$, $E = \frac{1}{2}$, $\omega_c = 3.75$, $T = 0.139$ and $\lambda = 0.08$. From now on, we shall fix our parameters listed here except the coupling strength $\lambda$. The autocorrelation function $\alpha(t)$ with these parameters are shown in Fig. 3. At $t = \pm 4$, the autocorrelation function already becomes very small.

![Autocorrelation Function](image)

FIG. 3. The autocorrelation function $\alpha(t)$ with $\lambda = 0.08$, $T = 0.139$ and $\omega_c = 3.75$.

Now we want to cover the non-locality of $\alpha(t)$ shown in Fig. 3, i.e., $N_s \delta t \geq 4$. Typical simulations of QUAPI are restricted to $N_s < 20$ [33, 34], and in fact when $N_s$ is greater than 10 it already become time consuming. Therefore the time interval $\delta t$ is usually not less than 0.25. By employing TEMPO algorithm we go to $N_s = 80$ in this article, and then $\delta t$ can reach a fairly small value 0.05.

Some typical results of $\langle \sigma_z(t) \rangle$ with different $N_s$ are shown in Fig. 4(a). It can be seen that the behavior of Markovian $\langle \sigma_z(t) \rangle$ is qualitatively different from the non-Markovian ones. When $N_s = 1$ (the Markovian case), $\langle \sigma_z(t) \rangle$ shows coherent decaying oscillation which decays to a very small (almost zero) oscillation eventually (see the inset of the figure). For non-Markovian cases, even with a not so large $N_s = 10$, $\langle \sigma_z(t) \rangle$ reaches steady states fast and then oscillates coherently. When $N_s$ increases, the larger amplitude coherent oscillation is induced as the more non-Markovian effects are included. The amplitudes of coherent oscillation with respect to different $N_s$ are shown in Fig. 4(b). It can be seen that the amplitude increases as $N_s$ increases and becomes stable when $N_s \geq 80$. This value corresponds $N_s \delta t \approx 4$, at this value most non-Markovian effects are captured.

Figure 5 shows the amplitudes of steady-state oscillation with respect to coupling strength $\lambda$ with different $N_s$. For non-Markovian case $N_s = 80$, a pronounced maximum is demonstrated. This is the sign of QSR phenomenon where the response of the a non-linear system to the external periodic driving is enhanced by noise. Note that the maximum of amplitudes is at $\lambda \approx 0.06$ rather than $\lambda = 0.08$.

V. NON-MARKOVIAN EFFECTS ON CORRELATION FUNCTION

In this section we study the non-Markovian effects on symmetrized correlation function $C(t_0, t_0 + t)$. As mentioned in Sec. II, it is enough to calculate the $t \geq 0$ case.

The correlation function should be evaluated at steady state
where \( \langle \sigma_z(t) \rangle \) is doing coherent oscillation, i.e., the time \( t_0 \) should be large enough. It can be seen from Fig. 2(a) that for \( \lambda = 0.08 \), the steady states are reached before \( t = 40 \). Therefore in this case \( t_0 \) should be greater than 40 when evaluating asymptotic \( C(t_0, t_0 + t) \). The correlation functions \( C(t_0, t_0 + t) \) with different \( N_s \) and some typical \( t_0 \) are shown in Fig. 6. Here we set \( t_0 \geq 200 \) which is much larger than 40 to ensure that the correlation function is evaluated in steady state.

In Markovian \((N_s = 1)\) case, the reduced dynamics only involves one last time step. Therefore the correlation function obtained in this case just corresponds the quantum regression theorem results [35–37]. It can be seen from Fig. 6 that the quantum regression theorem misses important non-Markovian effects and gives invalid results, as already mentioned in Refs. [19, 38].

In Markovian \((N_s = 1)\) case [Fig. 6(a)], correlation function \( C(t_0, t_0 + t) \) tends to almost zero no matter what value of \( t_0 \). With increasing non-Markovianity, i.e., increasing \( N_s \), asymptotic \( C(t_0, t_0 + t) \) does coherent oscillation with increasing amplitude. This is not surprising since such phenomenon is similar to asymptotic \( \langle \sigma_z(t) \rangle \) shown in Fig. 4.

The behavior of \( C(t_0, t_0 + t) \) depends on \( t_0 \), this can be seen clearly from Fig. 6(d), where the non-Markovian results are shown. Rather than always oscillating with large amplitude, the amplitude varies with different \( t_0 \). When \( t_0 = 201.6 \), the amplitude can even become very small. This shows that the correlation function \( C(t_0, t_0 + t) \) contains much more information than an observable \( \langle \sigma_z(t) \rangle \), and the property of correlation function can not be simply deduced from the behavior of observable.

It can be also seen that with different \( t_0 \), the coherent oscillations have phase difference. This means that the value of \( t_0 \) not only affects the oscillation amplitude but also the oscillation phase. Due to this phase shift, \( t_0 = 201.6 \) does not cause a minimum oscillation amplitude for \( N_s = 20 \) case. In this case, a \( t_0 \) close to 201.6 does it (not shown in the figure).

![FIG. 5. Amplitudes of steady-state oscillation with respect to coupling strength \( \lambda \) with different \( N_s \).](image)

![FIG. 6. Some typical symmetrized correlation function \( C(t_0, t_0 + t) \) with different \( N_s \) and \( t_0 \). Here \( \lambda = 0.08 \).](image)

### VI. SIGNAL-TO-NOISE RATIO

A typical way to quantify the response to the driving is the signal-to-noise ratio (SNR) [4, 6, 9, 39]. QSR occurs when SNR passes through a maximum as the noise level increases. The first papers on SR in fact focused on the behavior of the signal-to-noise ratio (SNR) [4, 6, 9, 39]. QSR occurs when SNR passes through a maximum as the noise level increases.

The time-averaged asymptotic symmetrized correlation function \( C(t) \) coherently oscillates with the driving frequency \( \Omega \) when \( t \) is large, therefore the power spectrum \( \tilde{S}(\omega) \) contains a noise background and \( \delta \)-function peaks at \( \Omega \) and its harmonics. The ratio of the coefficient of the fundamental peak and the value of noise at \( \Omega \) is the SNR.

If only considering the fundamental peak, the time-averaged power spectrum can be described as the superposition of a background noise power \( N(\omega) \) and a \( \delta \) signal term for which

\[
\tilde{S}(\omega) = G\delta(\omega - \Omega) + N(\omega),
\]

where \( G \) is the strength of the signal. The ratio \( G/N(\Omega) \) gives the SNR. For sufficiently small driving, \( N(\omega) \) does not deviate much from the power spectrum of the undriven system, while for large driving the effect of the signal on the noise need to be taken into consideration.
The correlation function $C(t_0, t_0 + t)$ can be split into two parts for which
\[ C(t_0, t_0 + t) = C_0(t_0, t_0 + t) + C_1(t_0, t_0 + t), \]  
(42)
where $C_0$ is the transient part and $C_1$ is the asymptotically coherent oscillation part. Accordingly the time-averaged asymptotic correlation function (8) can be also split into two parts as
\[ \bar{C}(t) = \bar{C}_0(t) + \bar{C}_1(t). \]  
(43)

The Fourier transform of coherent oscillation part $\bar{C}_1(t)$ just yields delta peak at $\Omega$ and its harmonics, from which the strength of signal $G$ is obtained. Here we simply set $G$ as the amplitude of coherent oscillation $C_1(t)$, and background noise power $N(\omega)$ is the Fourier transform of $C_0(t)$. Let $N_0(\omega)$ be the background noise power without periodic driving. The background noise powers $N_0(\omega)$ and $N(\omega)$ with different $N_s$ are shown in Fig. 7.

It should be noted that the shape of $N(\omega)$ presented here is different to those with Gaussian white noise [6, 9, 15] where $N(\omega)$ is roughly a Lorentzian centered at $\omega = 0$. Our simulations are in deep quantum regime, the shape of $N(\omega)$ is roughly an asymmetrical “Lorentzian” centered at nonzero $\omega$. In this sense, we are dealing with the color noise.

From Fig. 7, it is clear that the driving force alter the background noise power. In Markovian ($N_s = 1$) case, the positions of peak of $N_0(\omega)$ and $N(\omega)$ are almost the same. But when $N_s$ increases, the position of peak of $N(\omega)$ starts deviate from that of $N_0(\omega)$.

SNR with respect to $\lambda$ for different $N_s$ are shown in Fig. 8. It can be seen that, unlike amplitude of $\langle \sigma_z(t) \rangle$ shown in Fig. 5, for weak dissipation ($\lambda = 0.01$), i.e., weak noise level, SNR is almost zero no matter what value of $N_s$ is. SNR reaches its maximum when $\lambda$ is around 0.08, which is different from the position of maximum amplitude of $\langle \sigma_z(t) \rangle$.

The QSR has attracted much attention over several decades. The physical quantity of most interest in QSR is the power spectrum and accordingly the SNR. To obtain the power spectrum, the knowledge of two time correlation function is necessary. However, due to the combination of non-Markovian dissipation and external driving force, the correlation function is difficult to evaluate. Fortunately, a recently developed TEMPO algorithm and its extension provide an efficient and numerically exact approach for this task.

In this article, we employ TEMPO algorithm to investigate non-Markovian effects in QSR. The non-Markovianity is controlled by the truncation parameter $N_s$. For observable $\langle \sigma_z(t) \rangle$, the amplitude of Markovian coherent oscillation is close to zero, while more non-Markovianity (larger $N_s$) induces larger amplitude coherent oscillation. The amplitude with respect to coupling strength $\lambda$ shows a nonmonotonic behavior for which it reaches maximum at $\lambda \approx 0.06$, which is a sign of QSR.

The time-averaged correlation function $\bar{C}(t)$ can be split into a transient part $\bar{C}_0(t)$ and a coherent oscillation part $\bar{C}_1(t)$. The Fourier transform of $\bar{C}_0(t)$ gives the background noise power $N(\omega)$. The $N(\omega)$ obtained in our model differs from that from white noise approximation, which indicates that effects of environment in deep quantum regime can be viewed as color noise. When comparing $N(\omega)$ to the background noise power without driving $N_0(\omega)$, it is found that their peaks are at almost the same position when $N_s$ is small. When $N_s = 80$, i.e., in the non-Markovian case, the position of their peaks deviate from each other. This makes SNR increases faster than $\langle \sigma_z(t) \rangle$ when $N_s$ increases, and the maximum of SNR and $\langle \sigma_z(t) \rangle$ appear at different $\lambda$.

Fig. 8. The signal-to-noise ratio with respect to $\lambda$ for different $N_s$.
Acknowledgments. This work is supported by the NSFC Grant No. 12104328.

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