Rotations in classical mechanics using geometric algebra

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Abstract

In geometric algebra, the rotation of a vector is described using rotors. Rotors are phasors where the imaginary number has been replaced by a oriented plane element of unit area called a unit bivector. The algebra in three dimensional space relating vectors and bivectors is the Pauli algebra. Multivectors consisting of linear combinations of scalars and bivectors are isomorphic to quaternions. The rotational dynamics can be expressed entirely in the plane of rotation using bivectors. In particular, the Poisson formula providing the time derivative of the unit vectors of a moving frame are recast in terms of the angular velocity bivector and applied to cylindrical and spherical frames. The rotational dynamics of a point particle and a rigid body are fully determined by the time evolution of rotors. The mapping of the angular velocity bivector onto the angular momentum bivector is the inertia map. In the principal axis frame of the rigid body, the inertia map is characterised by symmetric coefficients representing the moments of inertia. The Huygens-Steiner theorem, the kinetic energy of a rigid body and the Euler equations are expressed in terms of bivector components. This formalism is applied to study the rotational dynamics of a gyroscope.

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1. Introduction

Rotational motion is a key component of the classical mechanics of rigid bodies. Textbooks describing rotations in linear algebra [1, 2, 3, 4], usually write
vectors in components with respect to a given frame and represent rotations as matrices. The approach followed in geometric algebra is simpler, geometrically more meaningful and insightful, as we shall show in the article.

To understand the main idea of geometric algebra, we consider the geometric interpretation of complex numbers. The complex plane \( \mathbb{C} \) is isomorphic to \( \mathbb{R}^2 \). The rotation of any complex number \( z \in \mathbb{C} \) by an angle \( \theta \) in this plane is performed by multiplying this number by the complex phasor \( e^{i\theta} = \cos \theta + i \sin \theta \). The imaginary number \( i \) is not merely a useful algebraic concept. It has also a geometric meaning as an oriented plane element of unit norm or modulus, called a unit bivector in geometric algebra (GA) \( \mathbb{G}_3 \). To establish this geometric interpretation, we write the geometric product of two vectors \( a \) and \( b \) as the sum of the symmetric inner product \( a \cdot b \), that is identical to the scalar product, and the antisymmetric outer product \( a \wedge b \), that represents an oriented plane element and is the dual of the cross product \( a \times b \),

\[
abla = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}
\]  

(1)

In a plane spanned by the orthonormal units vectors \( \hat{e}_1 \) and \( \hat{e}_2 \), using the orthogonality of the vector and the antisymmetry of their outer product and the symmetry of their inner product,

\[
\begin{align*}
\hat{e}_1 \cdot \hat{e}_2 &= 0 \\
\hat{e}_1 \hat{e}_2 &= \hat{e}_1 \wedge \hat{e}_2 = -\hat{e}_2 \wedge \hat{e}_1 = -\hat{e}_2 \hat{e}_1 \\
\hat{e}_1 \hat{e}_1 &= \hat{e}_1 \cdot \hat{e}_1 = 1 \\
\hat{e}_2 \hat{e}_2 &= \hat{e}_2 \cdot \hat{e}_2 = 1
\end{align*}
\]  

(2)

the square of the their geometric product reads,

\[
(\hat{e}_1 \hat{e}_2)^2 = \hat{e}_1 \hat{e}_2 \hat{e}_1 \hat{e}_2 = -\hat{e}_1 \hat{e}_1 \hat{e}_2 \hat{e}_2 = -1
\]  

(3)

which demonstrates that the bivector \( \hat{e}_1 \hat{e}_2 \) behaves geometrically as an imaginary number. Similarly, in the planes spanned by the units vectors \( \hat{e}_2 \) and \( \hat{e}_3 \) and by the unit vectors \( \hat{e}_3 \) and \( \hat{e}_1 \), the bivectors \( \hat{e}_1 \hat{e}_2 \) and \( \hat{e}_3 \hat{e}_1 \) behave like two other imaginary numbers. Thus, the subalgebra of the bivectors of \( \mathbb{G}_3 \) is isomorphic to the quaternions \( \mathbb{H} \). As we shall show in this article, the rotation of any vector can be described using compositions of the three phasors \( e^{\hat{e}_1 \hat{e}_2} \), \( e^{\hat{e}_2 \hat{e}_3} \), and \( e^{\hat{e}_3 \hat{e}_1} \) called rotors in geometric algebra.
This geometric structure can be established by showing that every rotation is the result of the composition of two reflections. Each reflection of a vector over a plane can be written in a beautiful yet simple form using the geometric product of this vector with the vector orthogonal to the plane. By iterating the process twice, rotors appear.

Rotations in classical mechanics are usually described in terms of the angular velocity $\omega$ and the angular momentum $\ell$. These quantities are pseudovectors that were introduced in order to describe classical mechanics in a vector space. The right hand rule used to define the angular momentum pseudovector $\ell_O$ evaluated at the origin $O$ as a cross product of the position vector $r$ and the momentum vector $p$, i.e. $\ell_O = r \times p$, is a pure mathematical convention. In order not to break the rotational symmetry, the angular pseudovector $\ell_O$ has to be chosen along the axis orthogonal to the plane of motion. Then, all that is left is the chirality: history chose the right hand! However, when an object rotates in a plane, why should a vector orthogonal to the plane of rotation be introduced in the first place? Would it not be more natural to introduce an oriented plane in the rotation plane? It turns out that this oriented plane element is the angular momentum bivector $L_O$ evaluated at the origin $O$. It is the outer product of the position vector $r$ and the momentum vector $p$, i.e. $L_O = r \wedge p$. The pseudovector $\ell_O$ is the dual of the bivector $L_O$, i.e $\ell_O = L_O^\ast$. Similarly, the angular velocity pseudovector $\omega$ is the dual of the angular velocity bivector $\Omega$, i.e $\omega = \Omega^\ast$. The rotation of rigid bodies can thus be written in terms of these bivectors, as we shall see in this publication.

This article is structured as follows. In the first part, we begin with a geometric description of projections, rejection, reflections and rotation in geometric algebra. Then, in Sec. 4 rotors are written in terms of quaternions and their algebra is shown to be isomorphic to the Pauli algebra. In the second part, we describe the rotation of point particles. In Sec. 5 the time evolution of the vector frame is expressed in terms of the angular velocity bivector. The rotation of cylindrical and spherical frames is described in Sec. 6 and 7. The rotational dynamics of a point particles is detailed in Sec. 8. In the third part, we describe
the intrinsic rotational dynamics of rigid bodies. The rotation of rigid bodies is
discussed in Sec. 9. In Sec. 10 inertia is expressed as a mapping of unit bivectors
of the principal body frame. The Huygens-Steiner theorem is established in
Sec. 11. The angular momentum and the kinetic energy are discussed in Sec. 12
and 13. The Euler equations are established in Sec. 14 and the rotational dy-
namics of a spinning disk is described in Sec. 15. The foundations of geometric
algebra are presented in Appendix A. The duality operation in geometric alge-
bra (GA) is defined in Appendix B. Important geometric identities in geometric
algebra are established in Appendix C.

2. Projection, rejection and reflection

From a geometric perspective, a rotation can be described as the composition of
two reflections over two different planes. In order to show this in the framework
of geometric algebra, we begin by studying reflections. We consider an arbitrary
vector \( \mathbf{v} \) and an arbitrary unit vector \( \hat{n}_1 \). The geometric product (A.7) of the
vectors \( \hat{n}_1 \) and \( \mathbf{v} \) reads,

\[
\hat{n}_1 \mathbf{v} = \hat{n}_1 \cdot \mathbf{v} + \hat{n}_1 \wedge \mathbf{v}
\]

Since \( \hat{n}_1 \) is a unit vector, the vector \( \mathbf{v} \) can be resolved into a parallel and a
perpendicular part to \( \hat{n}_1 \) using the geometric product (4),

\[
\mathbf{v} = \hat{n}_1^2 \mathbf{v} = \hat{n}_1 (\hat{n}_1 \mathbf{v}) = \hat{n}_1 (\hat{n}_1 \cdot \mathbf{v}) + \hat{n}_1 (\hat{n}_1 \wedge \mathbf{v})
\]

Using the geometric product (A.10) of the vector \( \hat{n}_1 \) with the bivector \( \hat{n}_1 \wedge \mathbf{v} \),
the second part of vector \( \mathbf{v} \) can be recast as,

\[
\hat{n}_1 (\hat{n}_1 \wedge \mathbf{v}) = \hat{n}_1 \wedge \hat{n}_1 \wedge \mathbf{v} + \hat{n}_1 \cdot (\hat{n}_1 \wedge \mathbf{v}) = \hat{n}_1 \cdot (\hat{n}_1 \wedge \mathbf{v})
\]

In view of relations (5) and (6), the vector \( \mathbf{v} \) is resolved into parallel and per-
pendicular parts (Fig. 1),

\[
\mathbf{v} = \mathbf{v}_\parallel \hat{n}_1 + \mathbf{v}_\perp \hat{n}_1
\]
according to,

\[ \mathbf{v}_\parallel \hat{n}_1 = (\hat{n}_1 \cdot \mathbf{v}) \hat{n}_1 \]
\[ \mathbf{v}_\perp \hat{n}_1 = \hat{n}_1 \cdot (\hat{n}_1 \wedge \mathbf{v}) \]  

(8)

The parallel part of a vector \( \mathbf{v} \) is the result of the projection onto the unit vector \( \hat{n}_1 \). In view of relation (8), taking into account that a unit vector is its own inverse, i.e. \( \hat{n} = \hat{n}^{-1} \), the projection of the vector \( \mathbf{v} \) onto a unit vector \( \hat{n} \) is an automorphism \( P_\hat{n} : \mathbb{R}^3 \to \mathbb{R}^3 \) defined as (Fig. 1),

\[ P_\hat{n} (\mathbf{v}) = (\hat{n} \cdot \mathbf{v}) \hat{n} = \frac{\hat{n} \cdot \mathbf{v}}{\hat{n}} \]  

(10)

The perpendicular part of a vector \( \mathbf{v} \) is the result of the rejection onto the unit vector \( \hat{n}_1 \), which is a projection on the orthogonal complement of the subspace \( P_\hat{n}_1 \) in the plane spanned by \( \hat{n}_1 \) and \( \mathbf{v} \). In view of relations (6) and (8), taking into account that a unit vector is its own inverse, i.e. \( \hat{n} = \hat{n}^{-1} \), the rejection of
the vector $v$ onto the unit vector $\hat{n}$ is an automorphism $\bar{P}_{\hat{n}} : \mathbb{R}^3 \to \mathbb{R}^3$ defined as (Fig. 1),
\[ \bar{P}_{\hat{n}} (v) = \hat{n} \cdot (\hat{n} \wedge v) = \hat{n} (\hat{n} \wedge v) = \frac{\hat{n} \wedge v}{\hat{n}} \]  
(11)

\[ \begin{array}{c}
\text{\textbf{v}} \\
\bar{P}_{\hat{n}} (v)
\end{array} \]

\[ \begin{array}{c}
\hat{n} \\
\hat{n}
\end{array} \]

Figure 2: Projection $P_{\hat{n}} (v)$ and rejection $\bar{P}_{\hat{n}} (v)$ of a vector $v$ on the unit vector $\hat{n}_1$.

In view of relations (7), (10) and (11), the complementarity between the subspaces $P_{\hat{n}_1}$ and $\bar{P}_{\hat{n}_1}$ is written as,
\[ v = P_{\hat{n}_1} (v) + \bar{P}_{\hat{n}_1} (v) \]  
(12)

The reflection of a vector $v$ over the plan orthogonal to the unit vector $\hat{n}_1$ yields the vector $v'$ according to (Fig. 1),
\[ v' = - P_{\hat{n}_1} (v) + \bar{P}_{\hat{n}_1} (v) = - v_{\parallel \hat{n}_1} + v_{\perp \hat{n}_1} \]  
(13)

where the projection $P_{\hat{n}_1} (v)$ changes sign and the rejection $\bar{P}_{\hat{n}_1} (v)$ remains unchanged. Using the parallel and perpendicular parts (8) of the vector $v$ and the antisymmetry (A.19) of the inner product of the vector $\hat{n}_1$ and the bivector $\hat{n}_1 \wedge v$, the reflected vector (13) is recast as,
\[ v' = - (\hat{n}_1 \cdot v) \hat{n}_1 + \hat{n}_1 \cdot (\hat{n}_1 \wedge v) = - (\hat{n}_1 \cdot v) \hat{n}_1 - (\hat{n}_1 \wedge v) \cdot \hat{n}_1 \]  
(14)

Using the geometric product (A.11) of the bivector $\hat{n}_1 \wedge v$ with the vector $\hat{n}_1$, the second part of vector $v'$ can be recast as,
\[ (\hat{n}_1 \wedge v) \cdot \hat{n}_1 = (\hat{n}_1 \wedge v) \hat{n}_1 - \hat{n}_1 \wedge v \wedge \hat{n}_1 = (\hat{n}_1 \wedge v) \hat{n}_1 \]  
(15)
Taking into account the geometric product (4) and the identity (15), the reflected vector (14) is recast as,

\[ v' = -(\hat{n}_1 \cdot v + \hat{n}_1 \wedge v)\hat{n}_1 = -\hat{n}_1 v \hat{n}_1 \] (16)

Thus, we conclude that the reflection of a vector \( v \) over a plane orthogonal to the unit vector \( \hat{n}_1 \) is an automorphism \( F_{\hat{n}_1} : \mathbb{R}^3 \to \mathbb{R}^3 \) defined as,

\[ F_{\hat{n}_1}(v) = -\hat{n}_1 v \hat{n}_1 \] (17)

3. Rotation

The rotation of a vector \( v \) in a plane spanned by the unit vectors \( \hat{n}_1 \) and \( \hat{n}_2 \) is obtained by performing a reflection over the plane orthogonal to the unit vector \( \hat{n}_1 \) followed by a reflection over the plane orthogonal to the unit vector \( \hat{n}_2 \). According to relations (14) and (17), the composition of the reflection \( F_{\hat{n}_1}(v) \) of a vector \( v \) over a plane orthogonal to the unit vector \( \hat{n}_1 \) that yields the vector \( v' \) and a reflection \( F_{\hat{n}_1}(v) \) over a plane orthogonal to the unit vector \( \hat{n}_1 \) yields the vector \( v'' \) (Fig: 3),

\[ v'' = F_{\hat{n}_2} \circ F_{\hat{n}_1}(v) = F_{\hat{n}_2}\left(F_{\hat{n}_1}(v)\right) = F_{\hat{n}_2}(v') \]

\[ = -F_{\hat{n}_2}(\hat{n}_1 v \hat{n}_1) = \hat{n}_2 \hat{n}_1 v \hat{n}_1 \hat{n}_2 \] (18)

The key geometric object to describe rotations is the multivector rotor \( R \) and its reverse \( R^\dagger \) that are given by,

\[ R = \hat{n}_2 \hat{n}_1 \quad \text{and} \quad R^\dagger = (\hat{n}_2 \hat{n}_1)^\dagger = \hat{n}_1^\dagger \hat{n}_2^\dagger = \hat{n}_1 \hat{n}_2 \] (19)

Thus, the rotation (18) of the vector \( v \) that yields the vector \( v'' \) is written in terms of the rotor \( R \) and its reverse \( R^\dagger \) as,

\[ v'' = R v R^\dagger \] (20)

The rotor and its reverse \( R \) and its reverse \( R^\dagger \) satisfy the orthogonality relation,

\[ RR^\dagger = \hat{n}_2 \hat{n}_1 \hat{n}_1 \hat{n}_2 = 1 \] (21)

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Vector $v'$ is the reflection of vector $v$ over the plane orthogonal to vector $\hat{n}_1$ and vector $v''$ is the reflection of vector $v'$ over the plane orthogonal to vector $\hat{n}_2$. Vector $v''$ is the rotation of vector $v$ along the bivector $\hat{n}_1 \wedge \hat{n}_2$ or the unit bivector $\hat{B}$.

Thus, the rotor is a multivector of unit modulus,

$$|R|^2 = RR^\dagger = 1 \quad (22)$$

The inner and outer products of the vectors $\hat{n}_2$ and $\hat{n}_1$ can be written in terms of the angle $\phi$ between the vectors,

$$\hat{n}_2 \cdot \hat{n}_1 = \hat{n}_1 \cdot \hat{n}_2 = \cos \phi \quad \text{and} \quad \hat{n}_2 \wedge \hat{n}_1 = - \hat{n}_1 \wedge \hat{n}_2 = - \sin \phi \hat{B} \quad (23)$$

where $\hat{B}$ is the unit bivector in the rotation plane oriented in the rotation direction from vector $\hat{n}_1$ to vector $\hat{n}_2$ (Fig. 3). According to these products (23), the rotor and its reverse (19) are recast as,

$$R = \hat{n}_2 \cdot \hat{n}_1 + \hat{n}_2 \wedge \hat{n}_1 = \cos \phi - \sin \phi \hat{B}$$

$$R^\dagger = \hat{n}_1 \cdot \hat{n}_2 + \hat{n}_1 \wedge \hat{n}_2 = \cos \phi + \sin \phi \hat{B} \quad (24)$$

which shows that the rotor and its reverse are even multivectors of unit modulus which are linear combination of a scalar and a bivector. Moreover, the unit
bivector \( \hat{B} \) satisfies the relation,

\[
\hat{B}^2 = \hat{B} \hat{B} = -\hat{B} \hat{B} = -|\hat{B}|^2 = -1
\]  

(25)

which means that the unit bivector \( \hat{B} \) in the rotation plane is isomorphic to the imaginary number \( i \) in the complex plane \( \mathbb{C} \). This isomorphism yields the Euler formula in the plane of rotation where \( i \) is replaced by \( \hat{B} \),

\[
e^{\hat{B} \phi} = \cos \phi + \sin \phi \hat{B} \quad \text{and} \quad e^{-\hat{B} \phi} = \cos \phi - \sin \phi \hat{B}
\]  

(26)

In view of the Euler formula (26), the rotor and its reverse (24) are recast as,

\[
R = e^{-\hat{B} \phi} \quad \text{and} \quad R^\dagger = e^{\hat{B} \phi}
\]  

(27)

and the rotation (20) is recast as,

\[
v'' = e^{-\hat{B} \phi} v e^{\hat{B} \phi}
\]  

(28)

The rotation (28) of the vector \( v \) is expressed in terms of the angle \( \phi \) between the vectors \( \hat{n}_1 \) and \( \hat{n}_2 \). It has to be expressed now in terms of the angle \( \theta \) between the projections of the vectors \( v \) and \( v'' \) in the rotation plane. The projection of the vector \( v \) on the rotation plane is a projection on the unit bivector \( \hat{B} \). Using the geometric product (A.11) of the bivector \( \hat{B} \) with the vector \( v \), the vector \( v \) is resolved into parallel and perpendicular parts as,

\[
v = \hat{B}^{-1} \hat{B} v = \hat{B}^{-1} \left( \hat{B} \cdot v + \hat{B} \wedge v \right) = \hat{B} \cdot v \frac{\hat{B}}{|\hat{B}|} + \hat{B} \wedge v \frac{\hat{B}}{|\hat{B}|}
\]  

(29)

where \( \hat{B}^{-1} = \hat{B}^\dagger \). The projection of the vector \( v \) onto the unit bivector \( \hat{B} \) is an automorphism \( P_{\hat{B}} : \mathbb{R}^3 \to \mathbb{R}^3 \) defined as (Fig. 4),

\[
P_{\hat{B}}(v) = \hat{B}^\dagger \cdot (\hat{B} \cdot v) = \hat{B} \cdot v
\]  

(30)

The rejection of the vector \( v \) onto the unit bivector \( \hat{B} \) is an automorphism \( \bar{P}_{\hat{B}} : \mathbb{R}^3 \to \mathbb{R}^3 \) defined as (Fig. 4),

\[
\bar{P}_{\hat{B}}(v) = \hat{B}^\dagger \cdot (\hat{B} \wedge v) = \hat{B} \wedge v
\]  

(31)

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Figure 4: Projection $P_{\hat{B}}(v)$ and rejection $\bar{P}_{\hat{B}}(v)$ of a vector $v$ on the plane defined by the bivector vector $\hat{B}$.

In view of relations (29), (30) and (31), the complementarity between the subspaces $P_{\hat{B}}$ and $\bar{P}_{\hat{B}}$ is written as,

$$v = P_{\hat{B}}(v) + \bar{P}_{\hat{B}}(v)$$ (32)

In view of relations (24), the projection of the rotor $R$ and its reverse $R^\dagger$ onto the unit bivector $\hat{B}$ is the identity,

$$P_{\hat{B}}(R) = \hat{B}^\dagger \cdot (\hat{B} (\cos \phi - \sin \phi \hat{B})) = \cos \phi - \sin \phi \hat{B} = R$$
$$P_{\hat{B}}(R^\dagger) = \hat{B}^\dagger \cdot (\hat{B} (\cos \phi + \sin \phi \hat{B})) = \cos \phi + \sin \phi \hat{B} = R^\dagger$$ (33)

Taking into account relations (27) and (33), we obtain the following projection identities,

$$P_{\hat{B}}(e^{-\hat{B}\phi}) = e^{-\hat{B}\phi} \quad \text{and} \quad P_{\hat{B}}(e^{\hat{B}\phi}) = e^{\hat{B}\phi}$$ (34)

Since a projection is a linear map, the projection of the rotation (28) onto the rotation plane is written as,

$$P_{\hat{B}}(v''') = P_{\hat{B}}(e^{-\hat{B}\phi}v e^{\hat{B}\phi}) = P_{\hat{B}}(e^{-\hat{B}\phi}) P_{\hat{B}}(v) P_{\hat{B}}(e^{\hat{B}\phi})$$ (35)
In view of the identities (34), the projection (35) becomes,

\[ P_{\hat{B}}(v''') = e^{-\hat{B}\phi} P_{\hat{B}}(v) e^{\hat{B}\phi} \]  

(36)

Using the projection (30) and the Euler formula (26), the projection (36) becomes,

\[ \hat{B}^\dagger \cdot (\hat{B} \cdot v'') = (\cos \phi - \sin \phi \hat{B}) \hat{B}^\dagger \cdot (\hat{B} \cdot v) (\cos \phi + \sin \phi \hat{B}) \]  

(37)

Using the fact that the unit bivector commutes with its reverse, i.e. \( \hat{B} \hat{B}^\dagger = \hat{B}^\dagger \hat{B} \), and the antisymmetry of the inner product \( A.19 \) of the bivector \( \hat{B} \) with the vector \( \hat{B} \cdot v \), we obtain the following identity,

\[ \hat{B} (\hat{B} \cdot v) = \hat{B} \cdot (\hat{B} \cdot v) = - (\hat{B} \cdot v) \cdot \hat{B} = - (\hat{B} \cdot v) \hat{B} \]  

(38)

Using the identity (38), the projection (37) is recast as (Fig. 5),

\[ \hat{B}^\dagger \cdot (\hat{B} \cdot v') = \hat{B}^\dagger \cdot (\hat{B} \cdot v) (\cos \phi + \sin \phi \hat{B})^2 \]  

(39)

Using the projection (30) and the Euler formula (26), the projection (39) becomes,

\[ P_{\hat{B}}(v''') = P_{\hat{B}}(v) e^{2\hat{B}\phi} \]  

(40)

The projection of the vectors \( v \) and \( v''' \) on the rotation plane are defined as,

\[ v_{\parallel \hat{B}} = P_{\hat{B}}(v) \quad \text{and} \quad v''_{\parallel \hat{B}} = P_{\hat{B}}(v''') \]  

(41)

Thus, the projection (40) is recast as,

\[ v'''_{\parallel \hat{B}} = v_{\parallel \hat{B}} e^{2\hat{B}\phi} \]  

(42)

The geometric product of the vectors \( v_{\parallel \hat{B}} \) and \( v''_{\parallel \hat{B}} \) is written as (Fig. 5),

\[ v_{\parallel \hat{B}} v''_{\parallel \hat{B}} = v_{\parallel \hat{B}} \cdot v''_{\parallel \hat{B}} + v_{\parallel \hat{B}} \wedge v''_{\parallel \hat{B}} \]  

(43)

The inner and outer products of the vectors \( v_{\parallel \hat{B}} \) and \( v''_{\parallel \hat{B}} \) can be written in terms of the angle \( \theta \) between the vectors in the rotation plane,

\[ v_{\parallel \hat{B}} \cdot v''_{\parallel \hat{B}} = |v_{\parallel \hat{B}}| |v''_{\parallel \hat{B}}| \cos \theta \]

\[ v_{\parallel \hat{B}} \wedge v''_{\parallel \hat{B}} = |v_{\parallel \hat{B}}| |v''_{\parallel \hat{B}}| \sin \theta \hat{B} \]  

(44)
where the same unit bivector $\hat{B}$ rotates $v_\| \hat{B} \cdot v_\| \hat{B}$ onto $v_\| \hat{B} \cdot v''_\| \hat{B}$ and $\hat{n}_1$ onto $v$. In view of the products (44) and the Euler formula (26) for the angle $\theta$, the geometric product (43) is recast as,

$$v_\| \hat{B} v''_\| \hat{B} = |v_\| \hat{B}| |v''_\| \hat{B}| \left(\cos \theta + \sin \theta \hat{B}\right) = |v_\| \hat{B}| |v''_\| \hat{B}| e^{\hat{B}\theta} \quad (45)$$

Since the vector $v''_\| \hat{B}$ is obtained by rotating the vector $v_\| \hat{B}$ in the rotation plane, their moduli are equal,

$$|v''_\| \hat{B}| = |v_\| \hat{B}| \quad (46)$$

Multiplying relation (45) on the left by the vector $v_\| \hat{B}$ and taking into account the identity (46) yields,

$$v^2_\| \hat{B} v''_\| \hat{B} = |v_\| \hat{B}|^2 v''_\| \hat{B} = |v_\| \hat{B}|^2 v_\| \hat{B} e^{\hat{B}\theta} \quad (47)$$

which implies that,

$$v''_\| \hat{B} = v_\| \hat{B} e^{\hat{B}\theta} \quad (48)$$

Comparing the rotations (42) and (48), we conclude that the angle $\theta$ between the vectors $v_\| \hat{B}$ and $v''_\| \hat{B}$ is the double of the angle $\phi$ between the units vectors $\hat{n}_1$ and $\hat{n}_2$,

$$\theta = 2 \phi \quad (49)$$
Thus, the rotation $v''$ of the vector $v$ is recast in terms of the angle $\theta$ as,

$$v'' = e^{-\hat{B}\theta/2} v e^{\hat{B}\theta/2} \quad (50)$$

The rotor and its reverse $(27)$ can be recast in terms of the angle $\theta$,

$$R = e^{-\hat{B}\theta/2} \quad \text{and} \quad R^\dagger = e^{\hat{B}\theta/2} \quad (51)$$

In view of relation $(50)$, we conclude that the rotation of a vector $v$ by an angle $\theta$ in a plane along the unit bivector $\hat{B}$ is an automorphism $R_{\hat{B}\theta} : \mathbb{R}^3 \to \mathbb{R}^3$ defined as,

$$R_{\hat{B}\theta} (v) = e^{-\hat{B}\theta/2} v e^{\hat{B}\theta/2} \quad (52)$$

According to the orthogonality condition $(22)$, the transformation law for the geometric product of two vectors $uv$ under rotation by an angle $\theta$ in a plane along the unit bivector $\hat{B}$ is an automorphism,

$$R_{\hat{B}\theta} (uv) = R u \cdot v R^\dagger = R u R^\dagger R v R^\dagger = R_{\hat{B}\theta} (u) R_{\hat{B}\theta} (v) \quad (53)$$

According to the orthogonality condition $(22)$, the transformation law for the inner product of two vectors $u \cdot v$ under rotation by an angle $\theta$ in a plane along the unit bivector $\hat{B}$ is an innermorphism,

$$R_{\hat{B}\theta} (u \cdot v) = R u \cdot v R^\dagger = (R u R^\dagger) \cdot (R v R^\dagger) = R_{\hat{B}\theta} (u) \cdot R_{\hat{B}\theta} (v) \quad (54)$$

In view of the transformation laws $(53)$ and $(54)$, the transformation law for the outer product of two vectors $u \wedge v$ under rotation by an angle $\theta$ in a plane along the unit bivector $\hat{B}$ is an outermorphism,

$$R_{\hat{B}\theta} (u \wedge v) = R u \wedge v R^\dagger = (R u R^\dagger) \wedge (R v R^\dagger) = R_{\hat{B}\theta} (u) \wedge R_{\hat{B}\theta} (v) \quad (55)$$

In view of the rotor $(51)$ and the transformation law $(55)$, the transformation law for a bivector $A = u \wedge v$ under rotation by an angle $\theta$ in a plane along the unit bivector $\hat{B}$ is an automorphism,

$$R_{\hat{B}\theta} (A) = R A R^\dagger = e^{-\hat{B}\theta/2} A e^{\hat{B}\theta/2} \quad (56)$$
Using the transformation laws for vectors (52) and bivectors (56), the transformation law for a multivector $M = s + v + A + s' I$ (A.5) is an automorphism $R_{B \theta} : \mathbb{C}^3 \to \mathbb{C}^3$ defined as,

$$R_{B \theta} (M) = R M R^\dagger = e^{-\hat{B} \theta/2} M e^{\hat{B} \theta/2} \quad (57)$$

The composition of two rotations of a vector $v$ can be written as,

$$R_{B_2 \theta_2} \circ R_{B_1 \theta_1} (v) = R_{B_2 \theta_2} \left( R_{B_1 \theta_1} (v) \right) = R_{B_2 \theta_2} \left( e^{-\hat{B}_1 \theta_1/2} v e^{\hat{B}_1 \theta_1/2} \right) = e^{-\hat{B}_2 \theta_2/2} e^{-\hat{B}_1 \theta_1/2} v e^{\hat{B}_1 \theta_1/2} e^{\hat{B}_2 \theta_2/2} \quad (58)$$

In view of the rotors (51), the composition of rotations (58) is a rotation,

$$R_{B \theta} (v) = R_{B_2 \theta_2} \circ R_{B_1 \theta_1} (v) = R_2 R_1 v R_1^\dagger R_2^\dagger = R v R^\dagger \quad (59)$$

where the rotor characterising the composition of the two rotations is,

$$R = R_2 R_1 = e^{-\hat{B}_1 \theta_1 + \hat{B}_2 \theta_2}/2 = e^{-\hat{B} \theta/2} \quad (60)$$

and the bivector characterising the composition of the two rotations is,

$$\frac{1}{2} \left( \hat{B} \theta \right) = \frac{1}{2} \left( \hat{B}_1 \theta_1 + \hat{B}_2 \theta_2 \right) \quad (61)$$

The composition of two rotations reveals that a rotation $R_{B \theta}$ is an element of a Lie group called the rotation group. It also shows that a rotor $R$ is an element of another Lie group called the spin group, and the bivector $\hat{B} \theta/2$ is an element of the spin algebra. There is a fundamental difference between a vector and a rotor under rotation. Unlike the rotation of a vector that is subject to a double-sided transformation law (52), rotor are subject to a single-sided transformation law (60). In order to see how the consequence of this difference, we consider a rotation by an angle $\theta = 2\pi$. In view of relation (52), a 360° rotation of the vector $v$ leaves it unchanged,

$$R_{B 2\pi} (v) = e^{-\hat{B} \pi} v e^{\hat{B} \pi} = (-1) v (-1) = v \quad (62)$$

as expected. In view of relation (60), a 360° rotation of the rotor $R_1$ characterised by the rotor $R_2 = e^{-\hat{B} \pi}$ changes the sign of the rotor,

$$R = e^{-\hat{B} \pi} R_1 = (-1) R_1 = -R_1 \quad (63)$$
which implies that a 720° rotation of the rotor $R_1$ leaves it unchanged. Rotors transform under rotation as spinors that describe the state of fermions in quantum mechanics. The rotation group is isomorphic to the special orthogonal group SO (3) and the spin group Spin (3) is isomorphic to the special unitary group SU (2), i.e Spin (3) $\cong$ SU (2). The difference between a vector and a rotor under rotation is due to the fact that the spin group is the double cover of the rotation group exactly like the special unitary group that is the double cover of the special orthogonal group, i.e. SU (2) / $\mathbb{Z}_2$ $\cong$ SO (3).

4. Rotors, quaternions and the Pauli algebra

A unit bivector $\hat{B}$ in a rotation plane can be written in an orthonormal frame $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ as,

$$\hat{B} = B_{12} \hat{e}_1 \wedge \hat{e}_2 + B_{23} \hat{e}_2 \wedge \hat{e}_3 + B_{31} \hat{e}_3 \wedge \hat{e}_1 = B_{12} \hat{e}_1 \hat{e}_2 + B_{23} \hat{e}_2 \hat{e}_3 + B_{31} \hat{e}_3 \hat{e}_1$$

where the normalisation condition is,

$$B_{12}^2 + B_{23}^2 + B_{31}^2 = 1$$

The basis unit bivectors satisfy the following conditions,

$$\begin{align*}
(\hat{e}_1 \hat{e}_2)^2 &= \hat{e}_1 \hat{e}_2 \hat{e}_1 \hat{e}_2 = -\hat{e}_1 \hat{e}_2 \hat{e}_2 \hat{e}_1 = -1 \\
(\hat{e}_2 \hat{e}_3)^2 &= \hat{e}_2 \hat{e}_3 \hat{e}_2 \hat{e}_3 = -\hat{e}_2 \hat{e}_3 \hat{e}_3 \hat{e}_2 = -1 \\
(\hat{e}_3 \hat{e}_1)^2 &= \hat{e}_3 \hat{e}_1 \hat{e}_3 \hat{e}_1 = -\hat{e}_3 \hat{e}_1 \hat{e}_1 \hat{e}_3 = -1 \\
(\hat{e}_1 \hat{e}_2) (\hat{e}_2 \hat{e}_3) (\hat{e}_3 \hat{e}_1) &= 1
\end{align*}$$

In view of relations (24) and (64), a rotor $R$ generating a rotation of angle $\theta$ in a rotation plane along the unit bivector $\hat{B}$ can be expressed in the orthonormal bivector basis as,

$$R = \cos \left( \frac{\theta}{2} \right) - B_{12} \sin \left( \frac{\theta}{2} \right) \hat{e}_1 \hat{e}_2 - B_{23} \sin \left( \frac{\theta}{2} \right) \hat{e}_2 \hat{e}_3 - B_{31} \sin \left( \frac{\theta}{2} \right) \hat{e}_3 \hat{e}_1$$
The bivector identities (66) are isomorphic to the formula for the quaternions with the identifications,

\[ i \leftrightarrow -\hat{e}_1 \hat{e}_2 \quad \text{and} \quad j \leftrightarrow -\hat{e}_2 \hat{e}_3 \quad \text{and} \quad k \leftrightarrow -\hat{e}_3 \hat{e}_1 \quad (68) \]

Indeed, in view of the identifications (68), bivector identities (66) yield formula for the quaternions,

\[ i^2 = j^2 = k^2 = ijk = -1 \quad (69) \]

With the identification (68), the rotor (67) can be written as a quaternion,

\[ R = a + bi + cj + dk \quad (70) \]

where the real coefficients are,

\[ a = \cos \left( \frac{\theta}{2} \right) \quad \text{and} \quad b = -B_{12} \sin \left( \frac{\theta}{2} \right) \]
\[ c = -B_{23} \sin \left( \frac{\theta}{2} \right) \quad \text{and} \quad d = -B_{31} \sin \left( \frac{\theta}{2} \right) \quad (71) \]

and satisfy the normalisation condition,

\[ a^2 + b^2 + c^2 + d^2 = 1 \quad (72) \]

This identification shows that the spin group Spin (3) is isomorphic to the hyperunitary group Sp (1) consisting of the unit quaternions \( \mathbb{H} \), i.e. \( \text{Spin} (3) \cong \text{Sp} (1) \).

After showing that rotors are isomorphic to unit quaternions, we now show how rotors or unit bivectors are related to the Pauli algebra. The totally antisymmetric Levi-Civita symbols are defined as,

\[ \varepsilon_{\sigma(123)} = (-1)^{\text{sgn}(\sigma)} \varepsilon_{123} \quad (73) \]

where \( \sigma \) (123) is a permutation of the indices 123 and \( \text{sgn}(\sigma) \) is its signature.

In view of the pseudoscalar \( I = \hat{e}_1 \hat{e}_2 \hat{e}_3 \) and the Levi-Civita symbol (73), the unit bivectors can be recast as,

\[ \hat{e}_1 \hat{e}_2 = \hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_3 = I \hat{e}_3 = I \varepsilon_{123} \hat{e}_3 \]
\[ \hat{e}_2 \hat{e}_3 = \hat{e}_2 \hat{e}_3 \hat{e}_1 \hat{e}_1 = I \hat{e}_1 = I \varepsilon_{231} \hat{e}_1 \]
\[ \hat{e}_3 \hat{e}_1 = \hat{e}_3 \hat{e}_1 \hat{e}_2 \hat{e}_2 = I \hat{e}_2 = I \varepsilon_{312} \hat{e}_2 \quad (74) \]
where \( \varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1 \). Thus, the outer product of the unit vectors \( \hat{e}_i \) and \( \hat{e}_j \) with \( i, j, k = 1, 2, 3 \) is written as,

\[
\hat{e}_i \wedge \hat{e}_j = I \varepsilon_{ijk} \hat{e}_k
\tag{75}
\]

The geometric product of the unit vectors \( \hat{e}_i \) and \( \hat{e}_j \) reads,

\[
\hat{e}_i \hat{e}_j = \hat{e}_i \cdot \hat{e}_j + \hat{e}_i \wedge \hat{e}_j
\tag{76}
\]

where the orthonormality condition is given by,

\[
\hat{e}_i \cdot \hat{e}_j = \delta_{ij}
\tag{77}
\]

In view of the outer product (75) and the inner product (77) of unit vectors, the geometric product (76) yields the Pauli algebra,

\[
\hat{e}_i \hat{e}_j = \delta_{ij} + I \varepsilon_{ijk} \hat{e}_k
\tag{78}
\]

Therefore, we conclude that the Pauli algebra, that is usually introduced to characterise the spin in quantum mechanics, has a purely geometric interpretation as the underlying algebra of physical space.

### 5. Rotating frame and angular velocity

We consider an orthonormal frame \( \{ \hat{f}_1, \hat{f}_2, \hat{f}_3 \} \) moving with respect to a fixed orthonormal frame \( \{ \hat{e}_1, \hat{e}_2, \hat{e}_3 \} \). A translation does not affect a vector. Only a rotation of the moving frame changes the basis vectors of the frame. Thus, we consider a rotation by an angle \( \theta \) in a plane along the constant unit bivector \( \hat{B} \) (Fig. 6).

According to relation (51) and (52), the unit vector \( \hat{f}_i \) is expressed in terms of the unit vector \( \hat{e}_i \) for \( i = 1, 2, 3 \) as follows,

\[
\hat{f}_i = R \hat{e}_i R^\dagger
\tag{79}
\]

where the time dependent rotor and its reverse are given by,

\[
R = e^{-\hat{B} \theta / 2} \quad \text{and} \quad R^\dagger = e^{-\hat{B} \theta / 2}
\tag{80}
\]
The time derivative of the rotating basis vector (79) is written as,

$$ \dot{\hat{f}}_i = \dot{R} \hat{e}_i \hat{R}^\dagger + \hat{R} \hat{e}_i \dot{\hat{R}} $$  \hspace{1cm} (81)$$

In view of relation (79), it can be recast as,

$$ \dot{\hat{f}}_i = \dot{R} R^\dagger \hat{f}_i + \hat{f}_i R \dot{R}^\dagger $$  \hspace{1cm} (82)$$

In view of relation (22), the normalisation condition for a time dependent rotor reads,

$$ R R^\dagger = 1 $$  \hspace{1cm} (83)$$

The time derivative of the condition (83) is written as,

$$ \dot{R} R^\dagger + R \dot{R}^\dagger = 0 $$  \hspace{1cm} (84)$$

Using relation (84), the time derivative of the rotating basis vector (81) becomes,

$$ \dot{\hat{f}}_i = \dot{R} R^\dagger \hat{f}_i - \hat{f}_i \dot{R} R^\dagger $$  \hspace{1cm} (85)$$

In view of relation (80), we obtain the identity,

$$ \dot{R} R^\dagger = - \frac{\dot{\theta}}{2} \hat{B} e^{-\hat{B} \theta/2} e^{\hat{B} \theta/2} = - \frac{\dot{\theta}}{2} \hat{B} $$  \hspace{1cm} (86)$$

The angular velocity is a bivector is defined as,

$$ \Omega = \dot{\theta} \hat{B} $$  \hspace{1cm} (87)$$
which implies that,
\[ \dot{R} R^I = -\frac{1}{2} \Omega \]  
(88)
and can be recast as,
\[ \dot{R} = -\frac{1}{2} \Omega R \]  
(89)
In view of the reverse \((B.3)\) of the bivector \(\Omega\), the reverse of relation \((89)\) is written as,
\[ \dot{R}^I = \frac{1}{2} R^I \Omega \]  
(90)
Using relation \((88)\), the time derivative of the rotating basis vector \((85)\) is expressed in terms of the angular velocity \(\Omega\) as,
\[ \dot{\hat{f}}_i = \frac{1}{2} (\hat{f}_i \cdot \Omega - \Omega \cdot \hat{f}_i) \]  
(91)
In view of the inner product \((A.19)\) between the vector \(\hat{f}_i\) and the bivector \(\Omega\), the time derivative of the rotating basis vector \((85)\) reduces to,
\[ \dot{\hat{f}}_i = \hat{f}_i \cdot \Omega \]  
(92)
which is the Poisson formula in geometric algebra. To recover the Poisson formula in vector algebra, we define the angular velocity pseudovector \(\omega\) as the dual of the angular velocity bivector \(\Omega\) using the relations \((B.15)\) and \((B.16)\),
\[ \omega = \Omega^* \equiv \Omega I^{-1} \quad \text{and} \quad \Omega \equiv -\omega^* = -\omega I^{-1} \]  
(93)
where the unit pseudoscalar \(I^{-1} = -I = -\hat{e}_1\hat{e}_2\hat{e}_3\) and the dual of the dual is the opposite of the identity, i.e. \((\Omega^*)^* = -\Omega\), since \(I^2 = -1\). Note that the duality preserves the modulus of a multivector. Thus, the plan area covered by the angular velocity bivector \(\Omega\) is equal to the length of the angular velocity pseudovector \(\omega\),
\[ |\Omega| = |\omega| \]  
(94)
The geometric interpretation of this duality is the following: if the palm of the right hand is oriented along the angular velocity bivector \(\Omega\) in the plane of rotation, then the thumb is oriented along the angular velocity pseudovector \(\omega\) (Fig. 7).
Using identities (B.32), (B.35) and (93), the right-hand side of the Poisson formula (92) is recast as,

$$\hat{f}_i \cdot \Omega = -\hat{f}_i \cdot \omega^* = -\left(\hat{f}_i \wedge \omega\right)^* = -\hat{f}_i \times \omega$$

In view of the identity (95), since the vector cross product is antisymmetric,

$$-\hat{f}_i \times \omega = \omega \times \hat{f}_i$$

the Poisson formula (92) is recast in vector algebra as,

$$\dot{\hat{f}}_i = \omega \times \hat{f}_i$$

as expected. It is worth emphasising that the Poisson formula in geometric algebra (92) is in a sense much more natural than the Poisson formula in vector algebra (97). In geometric algebra, the geometric entities are contained entirely in the rotation plane whereas in vector algebra the angular velocity pseudovector is orthogonal to the plane. In geometric algebra, the angular velocity bivector $\Omega$ rotates the basis vector $\hat{f}_i$ by a $90^\circ$ angle in the rotation direction defined by the unit bivector $\hat{B}$, this is the geometric interpretation of the inner product $\hat{f}_i \cdot \Omega$. 

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Figure 7: Duality between the angular velocity bivector $\Omega$ and the angular velocity pseudovector $\omega$ illustrated by the right hand rule.
6. Rotating cylindrical frame

We consider an cylindrical frame \( \{ \hat{\rho}, \hat{\phi}, \hat{z} \} \) rotating around a fixed Cartesian frame \( \{ \hat{x}, \hat{y}, \hat{z} \} \). This rotation is characterised by an azimuthal angle \( \phi \) in the horizontal plane (Fig. 8).

![Figure 8: Cylindrical frame \( \{ \hat{\rho}, \hat{\phi}, \hat{z} \} \) rotating with an angular velocity \( \Omega \) around a fixed Cartesian frame \( \{ \hat{x}, \hat{y}, \hat{z} \} \) in the \( \hat{x} \wedge \hat{y} \) plane.](image)

The unit bivector oriented along the rotation in the horizontal plane \( \hat{B}_\phi \) is written in the Cartesian frame as,

\[
\hat{B}_\phi = \hat{x} \wedge \hat{y} = \hat{x} \hat{y}
\]  

(98)

where the index \( \phi \) refers to the azimuthal angle in the plane of the bivector.

In view of the unit bivector \( \hat{B}_\phi \), the rotation that maps the Cartesian frame \( \{ \hat{x}, \hat{y}, \hat{z} \} \) onto the cylindrical frame \( \{ \hat{\rho}, \hat{\theta}, \hat{\phi} \} \) is described by the rotor \( R_\phi \) in the horizontal rotation plane,

\[
R_\phi = e^{-\hat{B}_\phi \phi/2} = e^{-\hat{x} \wedge \hat{y} \phi/2} = e^{-\hat{x} \hat{y} \phi/2}
\]  

(99)

The time derivative of the rotor \( R_\phi \) is written as,

\[
\dot{R}_\phi = -\frac{\dot{\phi}}{2} \hat{x} \hat{y} R_\phi
\]  

(100)
In view of relations (83), (88), (98), (99) and (100), the angular velocity is given by,
\[ \Omega = -2 \ddot{R}_\phi R^\dagger_\phi = \dot{\phi} \hat{x} \hat{y} R^\dagger_\phi = \dot{\phi} \hat{y} = \dot{\phi} \hat{B}_\phi \] (101)
The unit vectors \( \hat{x} \) and \( \hat{y} \) are written in the cylindrical frame as (Fig. 8),
\[ \hat{x} = \cos \theta \hat{\rho} - \sin \theta \hat{\phi} \]
\[ \hat{y} = \sin \theta \hat{\rho} + \cos \theta \hat{\phi} \] (102)
In view of the change of basis (129), the unit bivector \( \hat{B}_\phi \) is recast in the cylindrical frame as,
\[ \hat{B}_\phi = \hat{x} \hat{y} = \left( \cos \theta \hat{\rho} - \sin \theta \hat{\phi} \right) \left( \sin \theta \hat{\rho} + \cos \theta \hat{\phi} \right) = \hat{\rho} \hat{\phi} \] (103)
The angular velocity bivector (101) is recast in the cylindrical frame as,
\[ \Omega = \dot{\phi} \hat{B}_\phi = \dot{\phi} \hat{\rho} \hat{\phi} \] (104)
The Poisson formula (92) for the radial unit vector \( \hat{\rho} \) and the azimuthal unit vector \( \hat{\phi} \) are written as,
\[ \dot{\hat{\rho}} = \hat{\rho} \cdot \Omega = \hat{\rho} \Omega = \hat{\rho} \left( \dot{\phi} \hat{\rho} \hat{\phi} \right) = \dot{\phi} \hat{\rho} \hat{\phi} \]
\[ \dot{\hat{\phi}} = \hat{\phi} \cdot \Omega = \hat{\phi} \Omega = \hat{\phi} \left( \dot{\phi} \hat{\rho} \hat{\phi} \right) = \dot{\phi} \hat{\rho} \hat{\phi} = -\dot{\phi} \hat{\rho} \] (105)
where \( \dot{\hat{\rho}} \hat{\phi} \hat{\phi} = \hat{\rho}^2 \hat{\phi} = \hat{\phi} \) and \( \dot{\hat{\phi}} \hat{\rho} \hat{\phi} = -\hat{\phi}^2 \hat{\rho} = -\hat{\rho} \). In view of the time derivatives of the unit vectors (105), the time derivatives of the unit bivectors in the cylindrical frame are given by,
\[ \left( \dot{\hat{\rho}} \hat{\phi} \right)^* = \dot{\hat{\rho}} \hat{\phi} + \hat{\rho} \dot{\hat{\phi}} = \dot{\phi} \hat{\rho} \hat{\phi} - \dot{\phi} \hat{\rho} \hat{\rho} = 0 \]
\[ \left( \dot{\hat{\phi}} \hat{z} \right)^* = \dot{\hat{\phi}} \hat{z} = -\dot{\phi} \hat{\phi} \hat{z} = \hat{\phi} \hat{z} \hat{\rho} \]
\[ \left( \hat{z} \hat{\rho} \right)^* = \hat{z} \hat{\rho} = \dot{\phi} \hat{\phi} \hat{ \rho} \hat{ \phi} = -\dot{\phi} \hat{\rho} \hat{\phi} \] (106)
In view of the duality (B.16) of a bivector and the pseudoscalar \( I = \hat{\rho} \hat{\phi} \hat{z} \), the dual of the unit bivectors in the cylindrical frame are given by,
\[ \left( \dot{\hat{\rho}} \hat{\phi} \right)^* = -\dot{\hat{\rho}} \hat{\phi} I = -\dot{\hat{\rho}} \hat{\phi} \hat{\rho} \hat{\phi} \hat{z} = \hat{\rho} \hat{\rho} \hat{\phi} \hat{\phi} \hat{z} = \hat{z} \]
\[ \left( \dot{\hat{\phi}} \hat{z} \right)^* = -\dot{\hat{\phi}} \hat{z} I = -\dot{\hat{\phi}} \hat{z} \hat{\rho} \hat{\phi} \hat{z} = \hat{\phi} \hat{z} \hat{\phi} \hat{z} \hat{\rho} = \hat{\rho} \]
\[ \left( \hat{z} \hat{\rho} \right)^* = -\hat{z} \hat{\rho} I = -\hat{z} \hat{\rho} \hat{\rho} \hat{\phi} \hat{z} = \hat{z} \hat{\rho} \hat{z} \hat{\rho} \hat{\phi} = \hat{\phi} \] (107)
Since the pseudoscalar \( I \) is a unit frame invariant trivector,
\[
I = \hat{x} \hat{y} \hat{z} = \hat{\rho} \hat{\phi} \hat{z}
\]  
(108)
it is constant, i.e \( \dot{I} = 0 \). Thus, for any bivector \( B \), the time derivation commutes with the duality,
\[
((B)^*)^* = (-B I)^* = -(B)^* I = ((B)^*)^*
\]  
(109)
In view of the duality (107) and the identity (109), the time derivatives of the unit vectors (105) are the dual of the time derivatives of the unit bivectors (106),
\[
\dot{\hat{\rho}} = \left( (\hat{\phi} \hat{z})^* \right)^* = \phi (\dot{\hat{\rho}})^* \quad \text{as expected.}
\]  
(110)

7. Rotating spherical frame

We consider an spherical frame \( \{ \hat{r}, \hat{\theta}, \hat{\phi} \} \) rotating around a fixed Cartesian frame \( \{ \hat{x}, \hat{y}, \hat{z} \} \). This rotation is characterised by an azimuthal angle \( \phi \) in the horizontal rotation plane and a polar angle \( \theta \) in a vertical radial rotation plane (Fig. 9). The unit bivector oriented along the rotation in the horizontal plane \( \hat{B}_\phi \) is written as,
\[
\hat{B}_\phi = \hat{x} \wedge \hat{y} = \hat{x} \hat{y}
\]  
(111)
The unit bivector oriented along the rotation in the vertical radial plane \( \hat{B}_\theta \) is obtained by rotating the reference vertical unit bivector \( \hat{z} \wedge \hat{x} = \hat{z} \hat{x} \) by an angle \( \phi \) in the horizontal plane,
\[
\hat{B}_\theta = R_\phi (\hat{z} \wedge \hat{x}) R_\phi^\dagger = R_\phi \hat{z} \hat{x} R_\phi^\dagger
\]  
(112)
The rotation that maps the Cartesian frame \( \{ \hat{x}, \hat{y}, \hat{z} \} \) onto the spherical frame \( \{ \hat{r}, \hat{\theta}, \hat{\phi} \} \) described by a composition of rotors in the horizontal and radial vertical rotation planes,
\[
R = R_\theta R_\phi
\]  
(113)
24
Figure 9: Spherical frame $\{\hat{r}, \hat{\theta}, \hat{\phi}\}$ rotating around a fixed Cartesian frame $\{\hat{x}, \hat{y}, \hat{z}\}$ with an auxiliary horizontal radial vector $\hat{\rho}$.

In view of the unit bivector (111), the azimuthal rotor $R_\phi$ is written as,

$$R_\phi = e^{-\hat{B}_\phi \phi/2} = e^{-\hat{x} \hat{y} \phi/2}$$  \hspace{1cm} (114)

In view of the unit bivector (112) and the Euler formula (26), the vertical rotor $R_\theta$ is written as,

$$R_\theta = e^{-\hat{B}_\theta \theta/2} = e^{-\hat{x} \hat{z} \hat{R}_\phi \theta/2} = R_\phi e^{-\hat{z} \hat{x} \theta/2} R_\phi^\dagger$$  \hspace{1cm} (115)

Using the rotor (114), the rotor (115) is recast as,

$$R_\theta = R_\phi e^{-\hat{z} \hat{x} \theta/2} R_\phi^\dagger = e^{-\hat{x} \hat{y} \phi/2} e^{-\hat{z} \hat{x} \theta/2} R_\phi^\dagger$$  \hspace{1cm} (116)

In view of the rotors (114) (116), the rotor (113) reduces to,

$$R = R_\phi R_\theta = e^{-\hat{x} \hat{y} \phi/2} e^{-\hat{z} \hat{x} \theta/2}$$  \hspace{1cm} (117)

The time derivative of the rotor (117) is written as,

$$\dot{R} = \left(-\frac{\dot{\phi}}{2} \hat{\phi} \hat{y}\right) R + R \left(-\frac{\dot{\theta}}{2} \hat{z} \hat{x}\right)$$  \hspace{1cm} (118)
In view of relations (83), (88), (111), (112) and (118), the angular velocity is given by,
\[
\mathbf{\Omega} = -2 \dot{R} R^\dagger = \dot{\phi} \mathbf{\hat{x}} \mathbf{\hat{y}} + \dot{\theta} \mathbf{\hat{z}} \mathbf{\hat{x}} R^\dagger \tag{119}
\]

In view of the rotors (114), (115) and (117), we obtain the following identity,
\[
R \mathbf{\hat{z}} \mathbf{\hat{x}} R^\dagger = R_\phi R_\theta \mathbf{\hat{z}} \mathbf{\hat{x}} R_\theta R_\phi^\dagger = R_\phi e^{-\mathbf{\hat{z}} \mathbf{\hat{x}} \theta/2} \mathbf{\hat{z}} \mathbf{\hat{x}} e^\mathbf{\hat{z}} \mathbf{\hat{x}} \theta/2 R_\phi^\dagger \tag{120}
\]

In view of the Euler formula (26),
\[
e^{-\mathbf{\hat{z}} \mathbf{\hat{x}} \theta/2} \mathbf{\hat{z}} \mathbf{\hat{x}} e^\mathbf{\hat{z}} \mathbf{\hat{x}} \theta/2 = \mathbf{\hat{z}} \mathbf{\hat{x}} e^{-\mathbf{\hat{z}} \mathbf{\hat{x}} \theta/2} e^{\mathbf{\hat{z}} \mathbf{\hat{x}} \theta/2} = \mathbf{\hat{z}} \mathbf{\hat{x}} \tag{121}
\]

which means that the bivector \( \mathbf{\hat{z}} \mathbf{\hat{x}} \) is invariant under a rotation in the \( \mathbf{\hat{z}} \mathbf{\hat{x}} \) plane, as expected. In view of the orthonormality condition (83) and the transformation law (121), the bivector (120) becomes,
\[
R \mathbf{\hat{z}} \mathbf{\hat{x}} R^\dagger = R_\phi \mathbf{\hat{z}} \mathbf{\hat{x}} R_\phi^\dagger = \left( R_\phi \mathbf{\hat{z}} R_\phi^\dagger \right) \left( R_\phi \mathbf{\hat{x}} R_\phi^\dagger \right) \tag{122}
\]

Since the vector \( \mathbf{\hat{z}} \) is orthogonal to the plan of rotation, this vector is invariant under a rotation (83)
\[
R_\phi \mathbf{\hat{z}} R_\phi^\dagger = \mathbf{\hat{z}} R_\phi R_\phi^\dagger = \mathbf{\hat{z}} \tag{123}
\]

In view of the Euler formula (26), since the vector \( \mathbf{\hat{x}} \) is in the plan of rotation, the rotation of this vector is a right-handed transformation (42),
\[
R_\phi \mathbf{\hat{x}} R_\phi^\dagger = \mathbf{\hat{x}} R_\phi^{1/2} \tag{124}
\]

In view of the Euler formula (26) and the rotor (99), the rotor reversed squared characterising the rotation by an angle \( \phi \) of a vector in the rotation plane is given by,
\[
R_\phi^{1/2} = \left( e^{-B_\phi \phi/2} \right)^{1/2} = e^{B_\phi \phi} = e^{B_\phi \mathbf{\hat{x}} \mathbf{\hat{y}}} = e^{\phi \mathbf{\hat{x}} \mathbf{\hat{y}}} = \cos \phi + \sin \phi \mathbf{\hat{x}} \mathbf{\hat{y}} \tag{125}
\]

In view of relations (124) and (125), we obtain the following relation,
\[
R_\phi \mathbf{\hat{x}} R_\phi^\dagger = \mathbf{\hat{x}} R_\phi^{1/2} = \cos \phi \mathbf{\hat{x}} + \sin \phi \mathbf{\hat{x}} \mathbf{\hat{y}} = \cos \phi \mathbf{\hat{x}} + \sin \phi \mathbf{\hat{y}} = \mathbf{\hat{p}} \tag{126}
\]
where \( \hat{x} \hat{y} = \hat{x}^2 \hat{y} = \hat{y} \). In view of the transformation laws (123) and (124), the bivector (122) becomes,
\[
R \hat{z} \hat{x} R_\phi^\dagger = R_\phi \hat{z} \hat{x} R_\phi^\dagger = \left( R_\phi \hat{z} \hat{x} R_\phi^\dagger \right) = \hat{z} \hat{p}
\]
(127)

In view of the bivectors (103), (112) and (127), the angular velocity (119) is recast,
\[
\Omega = \dot{\phi} \hat{B}_\phi + \dot{\theta} \hat{B}_\theta = \dot{\phi} \hat{\rho} \hat{\phi} + \dot{\theta} \hat{z} \hat{p}
\]
(128)

The auxiliary vector \( \hat{\rho} \) and the vertical vector \( \hat{z} \) are written in the spherical frame as (Fig. 9),
\[
\hat{\rho} = \sin \theta \hat{r} + \cos \theta \hat{\theta} \\
\hat{z} = \cos \theta \hat{r} - \sin \theta \hat{\theta}
\]
(129)

In view of the change of basis (129), the unit bivector \( \hat{B}_\phi \) is recast in the spherical frame as,
\[
\hat{B}_\phi = \hat{\rho} \hat{\phi} = \left( \sin \theta \hat{r} + \cos \theta \hat{\theta} \right) \hat{\phi} = \cos \theta \hat{\theta} \hat{\phi} - \sin \theta \hat{\phi} \hat{r}
\]
(130)

In view of the change of basis (129), the unit bivector \( \hat{B}_\theta \) is recast in the spherical frame as,
\[
\hat{B}_\theta = \hat{z} \hat{\rho} = \left( \cos \theta \hat{r} - \sin \theta \hat{\theta} \right) \left( \sin \theta \hat{r} + \cos \theta \hat{\theta} \right) = \hat{r} \hat{\theta}
\]
(131)

where we used the trigonometric identity \( \sin^2 \theta + \cos^2 \theta = 1 \). Using the unit bivectors (130) and (131), the angular velocity bivector (128) is recast in the spherical frame as,
\[
\Omega = \dot{\phi} \hat{B}_\phi + \dot{\theta} \hat{B}_\theta = \dot{\phi} \cos \theta \hat{\theta} \hat{\phi} - \dot{\phi} \sin \theta \hat{\phi} \hat{r} + \dot{\theta} \hat{r} \hat{\theta}
\]
(132)

The Poisson formula (92) for the radial unit vector \( \hat{r} \) reads,
\[
\dot{r} = \hat{r} \cdot \Omega = \hat{r} \cdot \left( \dot{\phi} \cos \theta \hat{\theta} \hat{\phi} - \dot{\phi} \sin \theta \hat{\phi} \hat{r} + \dot{\theta} \hat{r} \hat{\theta} \right)
\]
\[
= -\dot{\phi} \sin \theta \hat{r} \hat{\phi} \hat{r} + \dot{\theta} \hat{r} \hat{\theta} \hat{\phi} + \dot{\phi} \sin \theta \hat{\phi}
\]
(133)

where the vector \( \hat{r} \) is orthogonal to the bivector \( \hat{\theta} \hat{\phi} \) and \( \hat{r} \hat{\phi} \hat{r} = -\hat{r}^2 \hat{\phi} = -\hat{\phi} \)
and also \( \hat{r} \hat{r} \hat{\theta} = \hat{r}^2 \hat{\theta} = \hat{\theta} \). The Poisson formula (92) for the polar unit vector
\( \hat{\theta} \) reads,
\[
\dot{\hat{\theta}} = \hat{\theta} \cdot \Omega = \hat{\theta} \cdot \left( \dot{\phi} \cos \theta \hat{\phi} - \dot{\phi} \sin \theta \hat{\phi} \hat{r} + \hat{\theta} \hat{\theta} \right)
\]
\[= \dot{\phi} \cos \theta \hat{\theta} \hat{\phi} + \hat{\theta} \hat{\theta} \hat{\phi} = - \hat{\theta} \hat{r} = \dot{\phi} \cos \theta \hat{\phi}
\] (134)

where the vector \( \hat{\theta} \) is orthogonal to the bivector \( \hat{\phi} \hat{r} \) and \( \hat{\theta} \hat{\phi} = \hat{\theta}^2 \hat{\phi} = \hat{\phi} \) and also \( \hat{\theta} \hat{r} \hat{\theta} = - \hat{\theta}^2 \hat{r} = - \hat{r} \). The Poisson formula (92) for the azimuthal unit vector \( \hat{\phi} \) reads,
\[
\dot{\hat{\phi}} = \hat{\phi} \cdot \Omega = \hat{\phi} \cdot \left( \dot{\phi} \cos \theta \hat{\phi} - \dot{\phi} \sin \theta \hat{\phi} \hat{r} + \hat{\theta} \hat{\phi} \hat{\theta} \right)
\]
\[= \dot{\phi} \cos \theta \hat{\phi} \hat{\phi} - \dot{\phi} \sin \theta \hat{\phi} \hat{r} = - \dot{\phi} \left( \sin \theta \hat{r} + \cos \theta \hat{\phi} \right)
\] (135)

where the vector \( \hat{\phi} \) is orthogonal to the bivector \( \hat{r} \hat{\theta} \) and \( \hat{\phi} \hat{\phi} = - \hat{\phi}^2 \hat{\phi} = - \hat{\phi} \) and also \( \hat{\phi} \hat{r} \hat{\phi} = \hat{\phi}^2 \hat{r} = \hat{r} \). In view of the time derivatives of the unit vectors (133), (134) and (135), the time derivatives of the unit bivectors in the spherical frame are given by,
\[
\begin{align*}
\left( \hat{r} \hat{\theta} \right)^{\cdot} &= \hat{r} \hat{\theta} + \hat{\theta} \hat{r} = - \dot{\phi} \left( \sin \theta \hat{r} + \cos \theta \hat{\phi} \right) \\
\left( \hat{\theta} \hat{\phi} \right)^{\cdot} &= \hat{\theta} \hat{\phi} + \hat{\phi} \hat{\theta} = \hat{\theta} \hat{\phi} + \dot{\phi} \sin \theta \hat{r} \hat{\phi} \\
\left( \hat{\phi} \hat{r} \right)^{\cdot} &= \hat{\phi} \hat{r} + \dot{\phi} \hat{r} = - \hat{\theta} \hat{\phi} + \dot{\phi} \cos \theta \hat{r} \hat{\phi}
\end{align*}
\] (136)

In view of the duality (B.16) of a bivector and the pseudoscalar \( I = \hat{r} \hat{\theta} \hat{\phi} \), the dual of the unit bivectors in the spherical frame are given by,
\[
\begin{align*}
\left( \hat{r} \hat{\theta} \right)^{\ast} &= - \hat{r} \hat{\theta} \hat{r} = - \hat{r} \hat{r} \hat{\phi} = \hat{r} \hat{r} \hat{\theta} \hat{\phi} = \hat{\phi} \\
\left( \hat{\theta} \hat{\phi} \right)^{\ast} &= - \hat{\theta} \hat{\phi} I = - \hat{\theta} \hat{\phi} \hat{r} \hat{\theta} \hat{\phi} = \hat{\theta} \hat{\phi} \hat{r} \hat{\phi} \hat{r} = \hat{r} \\
\left( \hat{\phi} \hat{r} \right)^{\ast} &= - \hat{\phi} \hat{r} \hat{r} = - \hat{\phi} \hat{r} \hat{r} \hat{\phi} = \hat{\phi} \hat{r} \hat{r} \hat{\phi} = \hat{\phi} \hat{\phi} \hat{r} \hat{r} = \hat{\phi}
\end{align*}
\] (137)

Since the pseudoscalar \( I \) is a unit frame invariant trivector,
\[
I = \hat{x} \hat{y} \hat{z} = \hat{r} \hat{\theta} \hat{\phi}
\] (138)

it is constant, i.e \( \dot{I} = 0 \). In view of the duality (137) and the identity (109), the time derivative of the unit radial vector (133), (134) and (135) is the dual of the
time derivative of the corresponding unit bivector \((136)\),

\[
\dot{\hat{r}} = \left( (\hat{\theta} \hat{\phi})^* \right)^* = \left( (\hat{\theta} \hat{\phi})^* \right)^* = (\hat{\theta} \hat{\phi} \hat{r} + \dot{\phi} \sin \theta \hat{\phi} \hat{\theta})^* \tag{139}
\]

\[
= \hat{\theta} (\hat{\phi} \hat{r})^* + \dot{\phi} \sin \theta (\hat{\phi} \hat{\theta})^* = \hat{\theta} \hat{\theta} + \dot{\phi} \sin \hat{\phi} \hat{\phi}
\]

The time derivative of the unit polar vector \((133), (134)\) and \((135)\) is the dual of the time derivative of the corresponding unit bivector \((136)\),

\[
\dot{\hat{\theta}} = \left( (\hat{\phi} \hat{r})^* \right)^* = \left( (\hat{\phi} \hat{r})^* \right)^* = \left( -\hat{\theta} \hat{\phi} + \dot{\phi} \cos \theta \hat{\phi} \hat{\theta} \right)^* \tag{140}
\]

\[
= -\hat{\theta} (\hat{\phi} \hat{r})^* + \dot{\phi} \cos \theta (\hat{\phi} \hat{\theta})^* = -\hat{\theta} \hat{\phi} + \dot{\phi} \cos \theta \hat{\phi}
\]

The time derivative of the unit polar vector \((133), (134)\) and \((135)\) is the dual of the time derivative of the corresponding unit bivector \((136)\),

\[
\dot{\hat{\phi}} = \left( (\hat{r} \hat{\theta})^* \right)^* = \left( (\hat{r} \hat{\theta})^* \right)^* = \left( -\hat{\phi} \sin \theta \hat{\phi} + \cos \theta \hat{\phi} \hat{r} \hat{\theta} \right)^* \tag{141}
\]

\[
= -\hat{\phi} (\sin \theta \hat{\phi} \hat{r})^* + \cos \theta (\hat{\phi} \hat{r})^* = -\hat{\phi} \sin \theta \hat{\phi} + \cos \theta \hat{\phi}
\]

as expected.

8. Point particle motion

We consider the motion of a point particle \(P\) of mass \(m\) with respect to an inertial frame of reference. We describe this motion with respect to a fixed orthonormal frame \(\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}\). The motion of the point particle is given by Newton’s second law where the external forces \(f^{\text{ext}}\) are the cause of the time variation of the momentum \(p\) according to,

\[
\sum f^{\text{ext}} = \dot{p} \tag{142}
\]

which has the same structure in geometric algebra as in vector algebra since the geometric entities involved are only vectors. The rotational motion of the point particle \(P\) is described in the plane spanned by the position \(r\) and the momentum \(p\). In order to describe such a motion, we define the angular momentum bivector in the plane of motion as,

\[
L_O = r \wedge p \tag{143}
\]
It is antisymmetric under the permutation of the position $r$ and the momentum $p$

$$L_O = r \wedge p = - p \wedge r = - L_O \quad \text{(144)}$$

The geometric meaning is that if the order of two vectors in an outer product is changed the resulting bivectors turns the other way around (Fig. 10) In the fixed orthonormal frame, the position $r$ and the momentum $p = m \ u$ are written in components as,

$$r = x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3$$
$$p = m \dot{x}_1 \hat{e}_1 + m \dot{x}_2 \hat{e}_2 + m \dot{x}_3 \hat{e}_3 \quad \text{(145)}$$

and the angular momentum reads,

$$L_O = m (x_1 \dot{x}_2 - x_2 \dot{x}_1) \hat{e}_1 \hat{e}_2 + m (x_2 \dot{x}_3 - x_3 \dot{x}_2) \hat{e}_2 \hat{e}_3 + m (x_3 \dot{x}_1 - x_1 \dot{x}_3) \hat{e}_3 \hat{e}_1 \quad \text{(146)}$$

where the unit vectors $\hat{e}_1 \hat{e}_2 = \hat{e}_1 \wedge \hat{e}_2$, $\hat{e}_2 \hat{e}_3 = \hat{e}_2 \wedge \hat{e}_3$ and $\hat{e}_3 \hat{e}_1 = \hat{e}_3 \wedge \hat{e}_1$.

According to identity $[B.32]$, the dual of the angular momentum bivector is the angular momentum pseudovector $\ell_O$,

$$L_O^* = (r \wedge p)^* = r \times p = \ell_O \quad \text{(147)}$$

which is the definition of the angular momentum pseudovector in vector algebra.

In view of the duality $[B.16]$ of a bivector,

$$(\hat{e}_1 \hat{e}_2)^* = \hat{e}_3 \quad \text{and} \quad (\hat{e}_2 \hat{e}_3)^* = \hat{e}_1 \quad \text{and} \quad (\hat{e}_3 \hat{e}_1)^* = \hat{e}_2 \quad \text{(148)}$$
which implies that the angular momentum pseudovector \( \ell_O \) is written in vector algebra as,

\[
\ell_O = m (x_2 \dot{x}_3 - x_3 \dot{x}_2) \hat{e}_1 + m (x_1 \dot{x}_3 - x_3 \dot{x}_1) \hat{e}_2 + m (x_1 \dot{x}_2 - x_2 \dot{x}_1) \hat{e}_3 \quad (149)
\]
as expected. The plan area covered by the angular momentum bivector \( L_O \) is equal to the length of the angular momentum pseudovector \( \ell_O \),

\[
|L_O| = |\ell_O| \quad (150)
\]
The geometric interpretation of this duality is the following: if the palm of the right hand is oriented along the angular momentum bivector \( L_O \) in the plane of rotation, then the thumb is oriented along the angular momentum pseudovector \( \ell_O \) (Fig. 11).

\[
|L_O| = |\ell_O| \quad (150)
\]

Figure 11: Duality between the angular momentum bivector \( L_O \) and the angular momentum pseudovector \( \ell_O \) illustrated by the right hand rule.

The time derivative of the angular momentum bivector \( L_O \) is given by,

\[
\dot{L}_O = \dot{r} \wedge p + r \wedge \dot{p} \quad (151)
\]

Since the momentum \( p \) is collinear to the velocity \( \dot{r} \), the bivector spanned by these vectors vanishes,

\[
\dot{r} \wedge p = v \wedge (m v) = m v \wedge v = 0 \quad (152)
\]
In view of the identity (152) and Newton’s second law (142), the time derivative of the angular momentum bivector becomes,

\[ \dot{L}_O = r \wedge \dot{p} = r \wedge \sum f^{\text{ext}} = \sum r \wedge f^{\text{ext}} \]  

(153)

The external torque bivector \( T^\text{ext}_O \) due to the external force \( f^{\text{ext}} \) acting on the point particle \( P \) is defined as the oriented area spanned by the position \( r = OP \) and the external force \( f^{\text{ext}} \),

\[ T^\text{ext}_O = r \wedge f^{\text{ext}} \]  

(154)

In the fixed orthonormal frame, the external force \( f^{\text{ext}} \) is written in components as,

\[ f^{\text{ext}} = f^{\text{ext}}_1 \hat{e}_1 + f^{\text{ext}}_2 \hat{e}_2 + f^{\text{ext}}_3 \hat{e}_3 \]  

(155)

and in view of the position (145), the external torque (154) reads,

\[ T^\text{ext}_O = (x_1 f^{\text{ext}}_2 - x_2 f^{\text{ext}}_1) \hat{e}_1 \hat{e}_2 + (x_2 f^{\text{ext}}_3 - x_3 f^{\text{ext}}_2) \hat{e}_2 \hat{e}_3 + (x_1 f^{\text{ext}}_3 - x_3 f^{\text{ext}}_1) \hat{e}_3 \hat{e}_1 \]  

(156)

According to identity (B.32), the dual of the external torque bivector is the external torque pseudovector \( \tau^\text{ext}_O \),

\[ T^\text{ext}_O^* = (r \wedge f^{\text{ext}})^* = r \times f^{\text{ext}} = \tau^\text{ext}_O \]  

(157)

which is the definition of the external torque pseudovector in vector algebra. In view of the duals of the unit bivectors (148), the external torque pseudovector (268) is written in vector algebra as,

\[ \tau^\text{ext}_O = (x_2 f^{\text{ext}}_3 - x_3 f^{\text{ext}}_2) \hat{e}_1 + (x_1 f^{\text{ext}}_3 - x_3 f^{\text{ext}}_1) \hat{e}_2 + (x_1 f^{\text{ext}}_2 - x_2 f^{\text{ext}}_1) \hat{e}_3 \]  

(158)

as expected. The plan area covered by the external torque bivector (156) is equal to the length of the external torque pseudovector (158),

\[ |T^\text{ext}_O| = |	au^\text{ext}_O| \]  

(159)
The geometric interpretation of this duality is the following: if the palm of the right hand is oriented along the external torque bivector $T^\text{ext}_O$, then the thumb is oriented along the external torque pseudovector $\tau^\text{ext}_O$ (Fig. 12).

In view of the time derivative of the angular momentum bivector $\dot{L}_O$ and the external torque bivector $T^\text{ext}_O$, we obtain the angular momentum theorem,

$$\sum T^\text{ext}_O = \dot{L}_O$$  \hspace{1cm} (160)

Using the identity (109) and the duality (147), the dual of the time derivative of the angular momentum bivector is the time derivative of the angular momentum pseudovector,

$$\left(\dot{L}_O\right)^* = (L^*_O)^* = \dot{\ell}_O$$  \hspace{1cm} (161)

In view of the dualities (268) and (161), the dual of the angular momentum theorem (160) in geometric algebra is the angular momentum theorem in vector algebra,

$$\sum \tau^\text{ext}_O = \dot{\ell}_O$$  \hspace{1cm} (162)

as expected.
9. Rigid body motion

A rigid body is a set of point particles $P_\alpha$ such that the relative distance between every couple of points remains constant over time. The position $r_\alpha$ of every point particle $P_\alpha$ is the sum of the position of the centre of mass $r_G$ and the relative position of $r'_\alpha$,

$$r_\alpha = r_G + r'_\alpha$$  \hfill (163)

The intrinsic rotation of the rigid body is characterised by a rotor $R$. Thus, the relative position $r'_\alpha$ of point $P_\alpha$ is related to the initial relative position $r'_{\alpha,0}$ through the rotation (Fig. 13),

$$r'_\alpha = R r'_{\alpha,0} R^\dagger$$  \hfill (164)

The relative velocity $v_\alpha$ of point $P_\alpha$ is obtained by taking the time derivative of the relative position (164),

$$v'_\alpha = \dot{R} r'_{\alpha,0} R^\dagger + R r'_{\alpha,0} \dot{R}^\dagger$$  \hfill (165)

In view of the rotor equations (89) and (90), the relative velocity (165) is recast as,

$$v'_\alpha = -\frac{1}{2} \Omega R r'_{\alpha,0} R^\dagger + \frac{1}{2} R r'_{\alpha,0} R^\dagger \Omega$$  \hfill (166)

Using the relative position (164), the relative velocity (166) becomes,

$$v'_\alpha = \frac{1}{2} (r'_\alpha \Omega - \Omega r'_\alpha)$$  \hfill (167)
Using the inner product (A.20) between the relative position vector $r'_\alpha$ and the angular velocity bivector $\Omega$, the relative velocity (167) reduces to,

$$v'_\alpha = r'_\alpha \cdot \Omega$$  \hspace{1cm} (168)$$

The angular momentum of the rigid body evaluated at the centre of mass $G$ is the sum of the relative angular momenta of all the point particles of mass $m_\alpha$, relative position $r'_\alpha$ and relative momentum $p'_\alpha = m_\alpha v'_\alpha$ that belong to the rigid body. In view of the relative velocity (168) and the angular momentum (143), the angular momentum bivector of the rigid body evaluated at the centre of mass $G$ reads,

$$L_G = \sum_\alpha r'_\alpha \wedge p'_\alpha = \sum_\alpha m_\alpha r'_\alpha \wedge v'_\alpha = \sum_\alpha m_\alpha r'_\alpha \wedge (r'_\alpha \cdot \Omega)$$  \hspace{1cm} (169)$$

The linear mapping of the angular velocity bivector $\Omega$ to the angular momentum bivector $L_G$ (169) is a rotation of the inertia tensor of the rigid body. This mapping is time dependent since the relative position vector $r'_\alpha$ and the angular velocity bivector $\Omega$ are time dependent. The angular velocity bivector $\Omega$ is related to the initial angular velocity bivector $\Omega_0$ through the rotation,

$$\Omega = R \Omega_0 R^\dagger$$  \hspace{1cm} (170)$$

In view of the orthonormality condition (83), the rotations (164) and (170), the relative velocity (167) is recast as,

$$v'_\alpha = \frac{1}{2} \left( R r'_{\alpha,0} \Omega_0 R^\dagger - R \Omega_0 r'_{\alpha,0} R^\dagger \right) = R \left( r'_{\alpha,0} \cdot \Omega_0 \right) R^\dagger$$  \hspace{1cm} (171)$$

as expected in view of relation (168). Using the rotations (164) and (170), the angular momentum (169) becomes,

$$L_G = \sum_\alpha m_\alpha R r'_{\alpha,0} R^\dagger \wedge R \left( r'_{\alpha,0} \cdot \Omega_0 \right) R^\dagger$$  \hspace{1cm} (172)$$

Using the orthonormality condition (83), the angular momentum (241) is recast as,

$$L_G = \frac{1}{2} \sum_\alpha m_\alpha R \left( r'_{\alpha,0} \left( r'_{\alpha,0} \cdot \Omega_0 \right) - \left( r'_{\alpha,0} \cdot \Omega_0 \right) r'_{\alpha,0} \right) R^\dagger$$  \hspace{1cm} (173)$$
and reduces to,

$$L_G = R \left( \sum_\alpha m_\alpha r'_{\alpha,0} \wedge (r'_{\alpha,0} \cdot \Omega_0) \right) R^\dagger$$  \hspace{1cm} (174)$$

The linear mapping of the initial angular velocity bivector $\Omega_0$ to the initial angular momentum bivector is the inertia map,

$$I_G (\Omega_0) = \sum_\alpha m_\alpha r'_{\alpha,0} \wedge (r'_{\alpha,0} \cdot \Omega_0)$$  \hspace{1cm} (175)$$

Indeed, in view of the inertia map (175), the angular momentum (174) is recast as,

$$L_G = R \ L_{G,0} R^\dagger = R \ I_G (\Omega_0) R^\dagger$$  \hspace{1cm} (176)$$

where the initial angular momentum $L_{G,0}$ coincides with the inertia map $I_G (\Omega_0)$ of the initial angular velocity $\Omega_0$. Afterwards, the angular momentum $L_G$ is obtained by performing a rotation of the inertia map $I_G (\Omega_0)$ using the time dependent rotor $R$. For a rigid body, the initial inertia map (175) is a constant that depends on the mass distribution. In the continuum limit, the point particles are replaced by an infinitesimal volume $dV$ of mass density $\rho (r'_0)$. In this limit, the initial inertia map (175) is recast as an integral over the volume $V$ of the rigid body,

$$I_G (\Omega_0) = \int_V dV \rho (r'_0) \ r'_0 \wedge (r'_0 \cdot \Omega_0)$$  \hspace{1cm} (177)$$

The orientation of the inertia map $I_G (\Omega_0)$ is locally determined the bivector $r'_0 \wedge (r'_0 \cdot \Omega_0)$ around each point $r'_0$ (Fig. 14). The linearity of the inertia map (177) as a function of the initial angular velocity bivector $\Omega_0$ is straightforward to show,

$$I_G (\lambda_1 \Omega_1 + \lambda_2 \Omega_2) = \int_V dV \rho (r'_0) \ r'_0 \wedge \left( r'_0 \cdot (\lambda_1 \Omega_1 + \lambda_2 \Omega_2) \right)$$

$$= \lambda_1 \int_V dV \rho (r'_0) \ r'_0 \wedge (r'_0 \cdot \Omega_1) + \lambda_2 \int_V dV \rho (r'_0) \ r'_0 \wedge (r'_0 \cdot \Omega_2)$$  \hspace{1cm} (178)$$

$$= \lambda_1 I_G (\Omega_1) + \lambda_2 I_G (\Omega_2)$$
Figure 14: The inertia map \( I_G(\Omega_0) \) is obtained by integrating the local bivectors \( r'_0 \wedge (r'_0 \cdot \Omega_0) \) over the volume of the rigid body. These bivectors are spanned by the vectors \( r'_0 \) and \( r'_0 \cdot \Omega_0 \).

Using the identity, we write,

\[
    r'_0 \wedge (r'_0 \cdot \Omega_0) = \frac{1}{2} \left( r'_0 (r'_0 \cdot \Omega_0) - (r'_0 \cdot \Omega_0) r'_0 \right) = \frac{1}{4} \left( r'_0 r'_0 \Omega_0 - 2 r'_0 \Omega_0 r'_0 + \Omega_0 r'_0 r'_0 \right)
\]

which is recast as,

\[
    r'_0 \wedge (r'_0 \cdot \Omega_0) = \frac{1}{2} \left( r'^2_0 \Omega_0 - r'_0 \Omega_0 r'_0 \right)
\]

In view of the identity (180), the initial inertia map is recast as,

\[
    I_G(\Omega_0) = \frac{1}{2} \int_V dV \rho (r'_0) \left( r'^2_0 \Omega_0 - r'_0 \Omega_0 r'_0 \right)
\]

10. Inertia and principal body frame

In an orthonormal frame \( \{\hat{e}_1, \hat{e}_2, \hat{e}_3\} \) initially in the frame of reference of the rigid body and attached to its centre of mass \( G \), the angular velocity bivector \( \Omega_0 \) is a linear combination of the inertia bivectors,

\[
    \Omega_0 = \frac{1}{2} \sum_{i,j=1}^{3} \Omega_{ij} \hat{e}_i \wedge \hat{e}_j
\]

where \( \Omega_{ij} = -\Omega_{ji} \) is the initial scalar angular velocity in the oriented plane defined by the unit bivector \( \hat{e}_i \wedge \hat{e}_j \) defined as,

\[
    \Omega_{ij} = \hat{e}_i \cdot \Omega_0 \cdot \hat{e}_j = (\hat{e}_j \wedge \hat{e}_i) \cdot \Omega_0 = \Omega_0 \cdot (\hat{e}_j \wedge \hat{e}_i)
\]
Using the initial scalar angular velocity \( \Omega_0 \), the initial inertia map can be written as a linear combination of inertia bivectors \( I \mathbf{e}_i \wedge \hat{e}_j \),

\[
I_G (\Omega_0) = \frac{1}{2} \sum_{i,j=1}^{3} \Omega_{ij} I_G (\mathbf{e}_i \wedge \hat{e}_j) \tag{184}
\]

In view the inertia maps (181) and (184), the inertia bivectors are written as,

\[
I_G (\hat{e}_i \wedge \mathbf{e}_j) = \frac{1}{2} \int_V dV \rho (r'_0) \left( r'^2_0 (\hat{e}_i \wedge \mathbf{e}_j) - r'_0 (\mathbf{e}_i \wedge e_j) r'_0 \right) \tag{185}
\]

which shows that the inertia bivector is antisymmetric,

\[
I_G (\hat{e}_i \wedge \hat{e}_j) = -I_G (\hat{e}_j \wedge \hat{e}_i) \tag{186}
\]

as expected. In order to account for the respective orientation of the vector \( r'_0 \) with respect to the oriented plane spanned by the unit bivector \( \hat{e}_i \wedge \mathbf{e}_j \), we decompose the relative position vector \( r'_0 \) into a parallel part \( r'_{0\parallel} \) and an orthogonal part \( r'_{0\perp} \) using the projection (30) and the rejection (31),

\[
r'_0 = r'_{0\parallel} + r'_{0\perp} \tag{187}
\]

where,

\[
r'_{0\parallel} = P_{\hat{e}_i \wedge \mathbf{e}_j} (r'_0) = (\hat{e}_j \wedge \hat{e}_i) \cdot \left( (\hat{e}_i \wedge \mathbf{e}_j) \cdot r_0 \right)
\]

\[
r'_{0\perp} = \bar{P}_{\hat{e}_i \wedge \mathbf{e}_j} (r'_0) = (\hat{e}_j \wedge \hat{e}_i) \cdot \left( (\hat{e}_i \wedge \mathbf{e}_j) \wedge r_0 \right) \tag{188}
\]

and thus, \( r'_{0\parallel} \cdot r'_{0\perp} = 0 \). In view of the decomposition (187), we have,

\[
r'^2_0 = r'^2_{0\parallel} + r'^2_{0\perp} \tag{189}
\]

and,

\[
r'_0 (\hat{e}_i \wedge \hat{e}_j) r'_0 = r'_{0\parallel} (\hat{e}_i \wedge \hat{e}_j) r'_0 + r'_{0\perp} (\hat{e}_i \wedge \hat{e}_j) r'_{0\perp} + r'_{0\parallel} (\hat{e}_i \wedge \hat{e}_j) r'_{0\perp}
\]

\[
+ r'_{0\parallel} (\hat{e}_i \wedge \hat{e}_j) r'_{0\parallel} + r'_{0\perp} (\hat{e}_i \wedge \hat{e}_j) r'_{0\perp} \tag{190}
\]

The vectors \( r'_{0\parallel} \) and \( r'_{0\perp} \) are written in components in the orthonormal frame as,

\[
r'_{0\parallel} = r'_{0\parallel i} \hat{e}_i + r'_{0\parallel j} \hat{e}_j \quad \text{and} \quad r'_{0\perp} = r'_{0\perp k} \hat{e}_k \tag{191}
\]
where the unit vector $\hat{e}_k$ is the dual of the unit bivector $\hat{e}_i \cdot \hat{e}_j$ according to,

$$\hat{e}_k = (\hat{e}_i \wedge \hat{e}_j)^* \quad \text{and} \quad \hat{e}_i \wedge \hat{e}_j = -\hat{e}_k^* \quad \text{thus} \quad \hat{e}_i \cdot \hat{e}_k = \hat{e}_j \cdot \hat{e}_k = 0 \quad (192)$$

Since the vector $r'_{0\perp} = r'_{\perp, k} \hat{e}_k$ is orthogonal to the oriented plane described by the unit bivector $\hat{e}_i \wedge \hat{e}_j$, the vector $r'_{0\perp}$ commutes with this bivector,

$$r'_{0\perp} (\hat{e}_i \wedge \hat{e}_j) = r'_{\perp, k} \hat{e}_k \hat{e}_i \hat{e}_j = r'_{\perp, k} \hat{e}_i \hat{e}_j \hat{e}_k = (\hat{e}_i \wedge \hat{e}_j) r'_{0\perp} \quad (193)$$

Since the vector $r'_{0||} = r'_{||, i} \hat{e}_i + r'_{||, j} \hat{e}_j$ is coplanar to the oriented plane described by the unit bivector $\hat{e}_i \wedge \hat{e}_j$, the vector $r'_{0||}$ anticommutes with this bivector,

$$r'_{0||} (\hat{e}_i \wedge \hat{e}_j) = r'_{||, i} \hat{e}_i \hat{e}_j + r'_{||, j} \hat{e}_j \hat{e}_i = -r'_{||, i} \hat{e}_i \hat{e}_j - r'_{||, j} \hat{e}_j \hat{e}_i = -(\hat{e}_i \wedge \hat{e}_j) r'_{0||} \quad (194)$$

In view of the identities (193) and (194), the identity (190) becomes,

$$r'_0 (\hat{e}_i \wedge \hat{e}_j) r'_0 = \left(-r'_{0||} r'_0 + r'_{0\perp} - r'_{0\perp} r'_0 + r'_{0||} r'_0\right) (\hat{e}_i \wedge \hat{e}_j) = -\left(r'_{0||}^2 - r'_{0\perp}^2\right) (\hat{e}_i \wedge \hat{e}_j) + 2 r'_{0||} \cdot \left(r'_{0\perp} \wedge (\hat{e}_i \wedge \hat{e}_j)\right) \quad (195)$$

since the vector $r'_{0\perp}$ is orthogonal to the bivector $\hat{e}_i \wedge \hat{e}_j$ and the vector $r'_{0||}$ belongs to the oriented volume element defined by the trivector $r'_{0\perp} \wedge (\hat{e}_i \wedge \hat{e}_j)$. Using identities (189) and (195), the inertia bivector (185) becomes,

$$I_G (\hat{e}_i \wedge \hat{e}_j) = \int_V dV \rho (r'_0) r'_{0||}^2 \hat{e}_i \wedge \hat{e}_j - \int_V dV \rho (r'_0) r'_{0||} \cdot \left(r'_{0\perp} \wedge (\hat{e}_i \wedge \hat{e}_j)\right) \quad (196)$$

where the second integral is a bivector orthogonal to the vector $r'_0$. It is useful to introduce the inertia dual vector $i_G (\hat{e}_k)$ defined as the dual of the inertia bivector $I_G (\hat{e}_i \wedge \hat{e}_j)$,

$$i_G (\hat{e}_k) = I_G (\hat{e}_i \wedge \hat{e}_j) \quad \text{and} \quad I_G (\hat{e}_i \wedge \hat{e}_j) = -i_G^* (\hat{e}_k) \quad (197)$$

In view of relations (196) and (197), the inertia dual vector is written as,

$$i_G (\hat{e}_k) = \int_V dV \rho (r'_0) r'_{0||} (\hat{e}_i \wedge \hat{e}_j)^* - \int_V dV \rho (r'_0) \left(r'_{0||} \cdot \left(r'_{0\perp} \wedge (\hat{e}_i \wedge \hat{e}_j)\right)\right)^* \quad (198)$$
In view of the dualities (B.58) and (B.45), we obtain the identity,

\[ \left( r'_{0\parallel} \cdot ( r'_{0\perp} \land (\hat{e}_i \land \hat{e}_j)) \right) = r'_{0\parallel} \left( r'_{0\perp} \land (\hat{e}_i \land \hat{e}_j) \right) \]

(199)

In view of the dualities (192) and (199), the inertia dual vector (198) reduces to,

\[ i_G (\hat{e}_k) = \int_V dV \rho (r'_0) \left( r'_{0\parallel}^2 \hat{e}_k - \int_V dV \rho (r'_0) \right) (r'_0 \cdot \hat{e}_k) \]

(200)

For a unit vector \( \hat{e}_\ell = \cos \phi \hat{e}_i + \sin \phi \hat{e}_j \), where the angle \( \phi \in [0, 2\pi) \), in the oriented plane spanned by the bivector \( \hat{e}_i \land \hat{e}_j \) orthogonal to the unit vector \( \hat{e}_k \), the projection of the inertia dual vector (200) along the axis \( G \hat{e}_\ell \) yields the matrix element,

\[ i_G (\hat{e}_k) = \int_V dV \rho (r'_0) \left( r'_{0\parallel}^2 \hat{e}_k - \int_V dV \rho (r'_0) \right) (r'_0 \cdot \hat{e}_k) \]

(201)

since \( \hat{e}_\ell \cdot \hat{e}_k = 0 \) and thus \( r'_0 \cdot \hat{e}_k = r'_0 \cdot \hat{e}_\ell \). In view of relation (189), for any unit vector \( \hat{e}_\ell \) belonging to the orthonormal basis \( \{\hat{e}_i, \hat{e}_j, \hat{e}_k\} \), the matrix element (201) is the \( k\ell \) component of the representation of the inertia tensor of vector algebra in the orthonormal frame,

\[ i_{G, k\ell} = i_{G, \ell k} = \int_V dV \rho (r'_0) \left( r'_{0\parallel}^2 \hat{e}_k - \int_V dV \rho (r'_0) \right) (r'_0 \cdot \hat{e}_\ell) \]

(202)

According to the spectral theorem, there is always an orthonormal frame, called a principal body frame, where the inertia tensor is diagonal. Such a frame is not unique but its existence is guaranteed mathematically. From now on, we consider that the unit vectors \( \hat{e}_k \) with \( k = 1, 2, 3 \) belong initially to a principal body frame of the rigid body and thus are the unit vectors of the initial principal axis \( G \hat{e}_k \) of the rigid body. This means that the unit vector \( \hat{e}_k \) is an eigenvector of the inertia dual vector \( i_G (\hat{e}_k) \). Thus, in a principal body frame, the inertia dual vector (200) reduces to,

\[ i_G (\hat{e}_k) = i_{G, k} \hat{e}_k = \int_V dV \rho (r'_0) \left( r'_{0\parallel}^2 \hat{e}_k \right) \]

(203)

where

\[ i_{G, k} = \int_V dV \rho (r'_0) \left( r'_{0\parallel}^2 \right) \]

(204)
is the initial principal moment of inertia of the rigid body around the principal axis $G \hat{e}_k$ and the vector $r'_0\parallel$ is orthogonal of the initial principal axis and parallel to the oriented plane defined by the bivector $\hat{e}_i \wedge \hat{e}_j$. In view of relations (200) and (203), the condition for a frame to be a principal body frame is that every unit vector $\hat{e}_k$ of the frame satisfies the condition,

$$\int_V dV \rho (r'_0) r'_0\parallel (r'_0 \cdot \hat{e}_k) = 0$$  \hspace{1cm} \text{(205)}$$

If the axis $G \hat{e}_k$ is an axis of symmetry of the rigid body, the condition (205) is clearly satisfied since for every point of the rigid body described by a relative parallel position vector $r'_0\parallel$, there is a symmetric point described by a relative parallel position vector $-r'_0\parallel$ such that the integral (205) vanishes (Fig. 15). Even if the rigid body has no axis of symmetry, there is always a principal body frame or a class of principal body frames that satisfy the condition (205).

![Diagram](image)

Figure 15: For a rigid body with an axis of symmetry $G \hat{e}_k$, for every point described by a relative parallel position vector $r'_0\parallel$ there is a symmetric point described by a relative parallel position vector $-r'_0\parallel$.

The rigid body can be sliced into a continuous set of slices of infinitesimal thickness orthogonal to the unit vector $\hat{e}_k$. The material points belonging to each slice are characterised by a constant relative position vector $r'_0\perp$ orthogonal
to the slice. Geometrically, each slice has to satisfy the condition \( (205) \), which therefore reduces to,

\[
\int_V dV \rho (r_0') r_{0\parallel}' = 0 \tag{206}
\]

In the principal body frame, in view of the inertia duality \((197)\) and the inertia pseudovector \((203)\), the inertia bivector reduces \((196)\) to,

\[
I_G (\hat{e}_i \wedge \hat{e}_j) = \int_V dV \rho (r_0') r_{0\parallel}'^2 \hat{e}_i \wedge \hat{e}_j \tag{207}
\]

which means that the unit bivector \(\hat{e}_i \wedge \hat{e}_j\) is an eigenbivector of the inertia bivector \(I_G (\hat{e}_i \wedge \hat{e}_j)\),

\[
I_G (\hat{e}_i \wedge \hat{e}_j) = I_{G,ij} \hat{e}_i \wedge \hat{e}_j \tag{208}
\]

The antisymmetry of the inertia bivector \((186)\) is written in components as,

\[
I_G (\hat{e}_i \wedge \hat{e}_j) = I_{G,ij} \hat{e}_i \wedge \hat{e}_j = - I_{G,ji} \hat{e}_j \wedge \hat{e}_i = - I_G (\hat{e}_j \wedge \hat{e}_i) \tag{209}
\]

In view of the antisymmetry of the unit bivector, i.e. \(\hat{e}_i \wedge \hat{e}_j = - \hat{e}_j \wedge \hat{e}_i\), and of relations \((73),(204),(207)\) and \((209)\), the principal moment of inertia of the rigid body in the oriented plane defined by the bivector \(\hat{e}_i \wedge \hat{e}_j\) is written as,

\[
I_{G,ij} = I_{G,ji} = (\varepsilon_{ijk})^2 i_{G,k} = \int_V dV \rho (r_0') r_{0\parallel}'^2 \tag{210}
\]

where the indices \(i, k\) and \(k\) are different and \(\hat{e}_k = (\hat{e}_i \wedge \hat{e}_j)^*\).

11. Huygens-Steiner theorem

In this section, we determine the moments of inertia \(I_{A,ij}\) of the rigid body with respect to a point \(A\), that may or may not belong to the rigid body, in the rotation plane spanned by the unit bivector \(\hat{e}_i \wedge \hat{e}_j\). In order to do so, we adopt the notation \(r_{GP\parallel} = r_{0\parallel}'\) for the projection in the plane of rotation of the relative position of a point \(P\) belonging to the rigid body. Thus, the moment of inertia \((210)\) in the oriented plane spanned by the unit bivector \(\hat{e}_i \wedge \hat{e}_j\) is written as,

\[
I_{G,ij} = \int_V dV \rho (r_0') r_{GP\parallel}^2 \tag{211}
\]
and the condition \([206]\) characterising a principal body frame is given by,

\[
\int_V dV \rho (r'_0) r_{GP} = 0
\]  

(212)

In view of the geometric identity,

\[
r_{GP}^2 = (r_{AP} - r_{AG})^2 = r_{AP}^2 - 2 r_{AG} \cdot r_{AP} + r_{AG}^2
\]  

(213)

where \(r_{AG} = r_{AG}\parallel\) since points \(A\) and \(G\) belong to the rotation plane. The moment of inertia \([211]\) is recast as,

\[
I_{G,ij} = \int_V dV \rho (r'_0) r_{AP}^2 - \int_V dV \rho (r'_0) r_{AG}^2 - 2 r_{AG} \cdot \left( \int_V dV \rho (r'_0) r_{GP} \parallel \right)
\]  

(214)

In view of the moment of inertia with respect at point \(G\) \([211]\), the moment of inertia with respect to point \(A\) in the oriented plane spanned by the unit bivector \(\hat{e}_i \wedge \hat{e}_j\) is given by,

\[
I_{A,ij} = \int_V dV \rho (r'_0) r_{AP}^2
\]  

(215)

The mass \(m\) of the rigid body is obtained by integrating the mass density over its volume \(V\),

\[
m = \int_V dV \rho (r'_0)
\]  

(216)

In view of the condition \([212]\), the moment of inertia \([215]\) and the mass \([216]\), the identity \([214]\) yields the Huygens-Steiner theorem,

\[
I_{A,ij} = I_{G,ij} + m r_{AG}^2
\]  

(217)

12. Angular momentum of a rigid body

In this section, we derive the angular momentum bivector \(L_G\) and its dual the angular momentum pseudovector \(\ell_G\) evaluated at the centre of mass of a rigid body. Using the eigenbivector decomposition \([209]\), the initial inertia map \([184]\),
which is the initial angular momentum bivector evaluated at the centre of mass \(G\), is written in the principal body frame as,

\[
L_{G,0} = I_G (\Omega_0) = \frac{1}{2} \sum_{i,j=1}^{3} I_{G,ij} \Omega_{ij} \hat{e}_i \wedge \hat{e}_j
\]  

(218)

In view of the initial inertia map (218), the angular momentum bivector of the rigid body (176) evaluated at the centre of mass \(G\) is recast in the principal body frame as,

\[
L_G = \frac{1}{2} \sum_{i,j=1}^{3} I_{G,ij} \Omega_{ij} R (\hat{e}_i \wedge \hat{e}_j) R^\dagger
\]  

(219)

Now, we determine the dual of the angular momentum bivector \(L_G\). In view of the identity (A.34), the rotor \(R\) and its reverse \(R^\dagger\) which are even multivectors commute with the pseudoscalar \(I\),

\[
RI = IR \quad \text{and} \quad R^\dagger I = IR^\dagger
\]  

(220)

Using the duality (147) and taking into account the commutation (220), the dual of the angular momentum bivector (176) is the angular momentum pseudovector,

\[
\ell_G = L_G^* = (R I_G (\Omega_0) R^\dagger)^* = - R I_G (\Omega_0) R^\dagger I
\]

\[
= R (- I_G (\Omega_0) I) R^\dagger = R I_G^* (\Omega_0) R^\dagger
\]  

(221)

Using the duality (93), the dual of the initial angular velocity bivector (182) is the initial angular velocity pseudovector,

\[
\omega_0 = \Omega_0^* = \frac{1}{2} \sum_{i,j=1}^{3} \Omega_{ij} (\hat{e}_i \wedge \hat{e}_j)^*
\]  

(222)

where,

\[
\Omega_{ij} = - \Omega_{ji} = \varepsilon_{ijk} \omega_k
\]  

(223)

In view of the dual of the unit bivector (192) and the components (223), the initial angular velocity pseudovector (222) becomes,

\[
\omega_0 = \frac{1}{2} \sum_{i,j=1}^{3} \varepsilon_{ijk} \omega_k \hat{e}_k = \sum_{k=1}^{3} \omega_k \hat{e}_k
\]  

(224)
In view of the duality of the bivector (197) and the components (223) and the eigenvector decomposition (203), the dual of the unit bivector (192) and the components (223), the dual of the initial inertia map bivector (184) is the initial inertia map pseudovector, which is the initial angular momentum pseudovector evaluated at the centre of mass $G$,

$$\ell_{G,0} = \mathbf{i}_G (\omega_0) = \mathbf{I}_G^* (\Omega_0)$$

$$= \frac{1}{2} \sum_{i,j=1}^{3} \varepsilon_{ijk} \omega_k \mathbf{I}_G^* (\hat{e}_i \wedge \hat{e}_j) = \sum_{k=1}^{3} \omega_k \mathbf{i}_G (\hat{e}_k) = \sum_{k=1}^{3} i_{G,k} \omega_k \hat{e}_k \quad (225)$$

In view of the duality between the initial inertia maps (218), the angular momentum pseudovector of the rigid body (221) evaluated at the centre of mass $G$ is recast in the principal body frame as,

$$\ell_G = \sum_{k=1}^{3} i_{G,k} \omega_k R \hat{e}_k R^\dagger \quad (226)$$

To get a better picture of the angular momentum, we now introduce the orthonormal frame $\{\hat{f}_1, \hat{f}_2, \hat{f}_3\}$, which is defined as the principal body frame and related to the initial principal body frame $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ through the rotation of the rigid body described by the rotor $R$. The unit vectors of the orthonormal frames are related by,

$$\hat{f}_i = R \hat{e}_i R^\dagger \quad (227)$$

In view of the orthonormality condition (83) and the rotation (227), the unit bivectors of the orthonormal frames are related by,

$$\hat{f}_i \wedge \hat{f}_j = \hat{f}_i \hat{f}_j = (R \hat{e}_i R^\dagger) (R \hat{e}_j R^\dagger) = R (\hat{e}_i \hat{e}_j) R^\dagger = R (\hat{e}_i \wedge \hat{e}_j) R^\dagger \quad (228)$$

Using the rotation (227), the angular momentum bivector (219) becomes,

$$L_G = \frac{1}{2} \sum_{i,j=1}^{3} I_{G,ij} \Omega_{ij} \hat{f}_i \wedge \hat{f}_j \quad (229)$$

Similarly, using the rotation (228), the angular momentum pseudovector (226) becomes,

$$\ell_G = \sum_{k=1}^{3} i_{G,k} \omega_k \hat{f}_k \quad (230)$$
13. Kinetic energy of a rigid body

The kinetic energy of a rigid body reads,

\[ T = \frac{1}{2} \int_V dV \rho (r'_0) \mathbf{v}^2 \]  

(231)

where the mass density \( \rho (r'_0) \) is a function of the initial relative position \( r'_0 \) and the velocity \( \mathbf{v} \) is the sum of the velocity of the centre of mass \( \mathbf{v}_G \) and the relative velocity \( \mathbf{v}' \)

\[ \mathbf{v} = \mathbf{v}_G + \mathbf{v}' \]  

(232)

According to relation (168), the relative velocity is the continuum limit is given by,

\[ \mathbf{v}' = \mathbf{r}' \cdot \mathbf{\Omega} \]  

(233)

In view of the velocity decomposition (232) and the relative velocity (233), the kinetic energy (231) is recast as,

\[ T = \frac{1}{2} \int_V dV \rho (r'_0) \left( \mathbf{v}_G^2 + 2 \mathbf{v}_G \cdot \mathbf{v}' + (r' \cdot \mathbf{\Omega})^2 \right) \]  

(234)

Using the mass (216), the kinetic energy (234) becomes,

\[ T = \frac{1}{2} m \mathbf{v}_G^2 + \mathbf{v}_G \cdot \int_V dV \rho (r'_0) \mathbf{v}' + \frac{1}{2} \int_V dV \rho (r'_0) (r' \cdot \mathbf{\Omega})^2 \]  

(235)

By definition of the centre of mass, the relative momentum of the rigid body vanishes,

\[ \mathbf{p}' = \int_V dV \rho (r'_0) \mathbf{v}' = 0 \]  

(236)

which implies that the kinetic energy (235) reduces to,

\[ T = \frac{1}{2} m \mathbf{v}_G^2 + \frac{1}{2} \int_V dV \rho (r'_0) (r' \cdot \mathbf{\Omega})^2 \]  

(237)

In view of the dualities (B.41), (B.49) and (B.53), we obtain the identity,

\[
\begin{align*}
\left( (r' \cdot \mathbf{\Omega})^2 \right)^* &= \left( (r' \cdot \mathbf{\Omega}) \cdot (r' \cdot \mathbf{\Omega}) \right)^* = (r' \cdot \mathbf{\Omega}) \wedge (r' \cdot \mathbf{\Omega})^* \\
&= (r' \cdot \mathbf{\Omega}) \wedge r' \wedge \mathbf{\Omega}^* = -(r' \wedge (r' \cdot \mathbf{\Omega}) \wedge \mathbf{\Omega})^* = -\left( (r' \wedge (r' \cdot \mathbf{\Omega})) \cdot \mathbf{\Omega} \right)^*
\end{align*}
\]  

(238)
The dual of the identity (238) reads,

\[(r' \cdot \Omega)^2 = - \left( r' \wedge (r' \cdot \Omega) \right) \cdot \Omega \]  \hspace{1cm} (239)

Using the identity (239), the kinetic energy (237) becomes,

\[ T = \frac{1}{2} m v_G^2 - \frac{1}{2} \int_V dV \rho (r'_0) \left( r' \wedge (r' \cdot \Omega) \right) \cdot \Omega \] \hspace{1cm} (240)

In the continuum limit, the angular momentum bivector (169) evaluated at the centre of mass \( G \) is recast as,

\[ L_G = \int_V dV \rho (r'_0) \left( r' \wedge (r' \cdot \Omega) \right) \] \hspace{1cm} (241)

In view of the total momentum of the rigid body,

\[ p = m v_G \] \hspace{1cm} (242)

and the angular momentum (241), the kinetic energy (240) reduces to,

\[ T = \frac{1}{2} p \cdot v_G - \frac{1}{2} L_G \cdot \Omega \] \hspace{1cm} (243)

In view of the rotation (170) of the initial angular velocity (182) and the basis vectors (227) of the principal body frame, the angular velocity is written in this frame as,

\[ \Omega = R \Omega_0 R^\dagger = \frac{1}{2} \sum_{i,j=1}^{3} \Omega_{ij} R (\hat{e}_i \wedge \hat{e}_j) R^\dagger = \frac{1}{2} \sum_{i,j=1}^{3} \Omega_{ij} \hat{f}_i \wedge \hat{f}_j \] \hspace{1cm} (244)

The scalar product of the angular momentum and the angular velocity (244) is written in the principal body frame as,

\[ L_G \cdot \Omega = \frac{1}{4} \sum_{i,j,k,\ell=1}^{3} I_{G,ij} \Omega_{ij} \Omega_{k\ell} (\hat{f}_i \wedge \hat{f}_j) \cdot (\hat{f}_k \wedge \hat{f}_\ell) \] \hspace{1cm} (245)

where the unit vectors satisfy the scalar identity,

\[ (\hat{f}_i \wedge \hat{f}_j) \cdot (\hat{f}_k \wedge \hat{f}_\ell) = \frac{1}{4} \left( \hat{f}_i \hat{f}_j - \hat{f}_j \hat{f}_i \right) \cdot \left( \hat{f}_k \hat{f}_\ell - \hat{f}_\ell \hat{f}_k \right) \]

\[ = \delta_{jk} \delta_{i\ell} - \delta_{ik} \delta_{j\ell} \] \hspace{1cm} (246)
Using the scalar identity (246), the scalar product (247) reduces to,
\[
L_G \cdot \Omega = -\frac{1}{2} \sum_{i,j=1}^{3} I_{G,ij} \Omega_{ij}^2
\] (247)

In view of the identities (242) and (247), the kinetic energy is recast as,
\[
T = \frac{1}{2} m v_G^2 + \frac{1}{4} \sum_{i,j=1}^{3} I_{G,ij} \Omega_{ij}^2
\] (248)

In view of the dualities (210) and (223) and the property \(\varepsilon_{ijk}^2 = 1\) of the Levi-Civita symbol (73), the kinetic energy (248) is recast as,
\[
T = \frac{1}{2} m v_G^2 + \frac{1}{4} \sum_{i,j,k=1}^{3} \varepsilon_{ijk} i_{G,k} \omega_k^2
\] (249)

The antisymmetry of the Levi-Civita symbol (73) implies that the sum over the different indices \(i\) and \(j\) yields a factor 2. Thus, the kinetic energy (249) reduces to,
\[
T = \frac{1}{2} m v_G^2 + \frac{1}{2} \sum_{k=1}^{3} i_{G,k} \omega_k^2
\] (250)
as expected in vector algebra.

14. Euler equations

The rotational dynamics of a rigid body is described by the Euler equations. In order to establish this equation in geometric algebra, we begin by considering the angular momentum theorem (160) evaluated at the centre of mass \(G\),
\[
\sum T_G^{ext} = \dot{L}_G
\] (251)

Since the moments of inertia \(I_{G,ij}\) are constant for a rigid body, the time derivative of the angular momentum (219) is given by,
\[
\dot{L}_G = \frac{1}{2} \sum_{i,j=1}^{3} I_{G,ij} \dot{\Omega}_{ij} R (\dot{\hat{e}}_i \wedge \dot{\hat{e}}_j) R^i
\]
\[
+ \frac{1}{2} \sum_{i,j=1}^{3} I_{G,ij} \Omega_{ij} \left( \dot{R} (\hat{e}_i \wedge \hat{e}_j) R^i + R (\hat{e}_i \wedge \hat{e}_j) \dot{R}^i \right)
\] (252)
In view of the dynamical equations for the rotor and its reverse, the time derivative of the angular momentum becomes,

\[
\dot{L}_G = \frac{1}{2} \sum_{i,j=1}^{3} I_{G,ij} \dot{\Omega}_{ij} R (\hat{e}_i \wedge \hat{e}_j) R^\dagger + \frac{1}{4} \sum_{i,j=1}^{3} I_{G,ij} \Omega_{ij} \left( R (\hat{e}_i \wedge \hat{e}_j) R^\dagger \Omega - \Omega R (\hat{e}_i \wedge \hat{e}_j) R^\dagger \right) \tag{253}
\]

In view of the rotation and the angular momentum, the time derivative of the angular momentum is recast as,

\[
\dot{L}_G = \frac{1}{2} \sum_{i,j=1}^{3} I_{G,ij} \dot{\Omega}_{ij} \hat{f}_i \wedge \hat{f}_j + \frac{1}{2} (L_G \Omega - \Omega L_G) \tag{254}
\]

In view of the commutator of the bivectors and , the time derivative of the angular momentum is recast as,

\[
\dot{L}_G = \frac{1}{2} \sum_{i,j=1}^{3} I_{G,ij} \dot{\Omega}_{ij} \hat{f}_i \wedge \hat{f}_j + L_G \times \Omega \tag{255}
\]

In view of the initial angular velocity and the rotation, the angular velocity pseudovector is written as,

\[
\Omega = R \Omega_0 R^\dagger = \frac{1}{2} \sum_{k,\ell=1}^{3} \Omega_{k,\ell} R (\hat{e}_i \wedge \hat{e}_j) R^\dagger = \frac{1}{2} \sum_{k,\ell=1}^{3} \Omega_{k,\ell} \hat{f}_k \wedge \hat{f}_\ell \tag{256}
\]

The commutator of the angular momentum bivector and the angular velocity bivector is written in components as,

\[
L_G \times \Omega = \frac{1}{2} (L_G \Omega - \Omega L_G)
\]

\[
= \frac{1}{8} \sum_{i,j,k,\ell=1}^{3} (I_{G,ij} - I_{G,\ell k}) \Omega_{ij} \Omega_{k\ell} \left( \hat{f}_i \wedge \hat{f}_j \right) \left( \hat{f}_k \wedge \hat{f}_\ell \right) \tag{257}
\]

where the unit vectors satisfy the bivectorial identity,

\[
\left( \hat{f}_i \wedge \hat{f}_j \right) \left( \hat{f}_k \wedge \hat{f}_\ell \right) = \frac{1}{4} \left( \hat{f}_i \hat{f}_j - \hat{f}_j \hat{f}_i \right) \left( \hat{f}_k \hat{f}_\ell - \hat{f}_\ell \hat{f}_k \right) = \delta_{jk} \hat{f}_i \wedge \hat{f}_\ell - \delta_{ik} \hat{f}_j \wedge \hat{f}_\ell + \delta_{\ell k} \hat{f}_j \wedge \hat{f}_i - \delta_{\ell i} \hat{f}_j \wedge \hat{f}_k \tag{258}
\]
Thus, the commutator \((257)\) becomes,
\[
L_G \times \Omega = \frac{1}{8} \sum_{i,j,k=1}^{3} (I_{G,ij} - I_{G,jk}) \Omega_{ij} \Omega_{jk} \hat{f}_i \wedge \hat{f}_k \\
+ \frac{1}{8} \sum_{i,j,k=1}^{3} (I_{G,ij} - I_{G,ik}) \Omega_{ij} \Omega_{ik} \hat{f}_k \wedge \hat{f}_j \\
- \frac{1}{8} \sum_{i,j,k=1}^{3} (I_{G,ij} - I_{G,kj}) \Omega_{ij} \Omega_{kj} \hat{f}_i \wedge \hat{f}_k \\
- \frac{1}{8} \sum_{i,j,k=1}^{3} (I_{G,ij} - I_{G,ki}) \Omega_{ij} \Omega_{ki} \hat{f}_k \wedge \hat{f}_j
\]
\[(259)\]
where the indices \(\ell\) and \(k\) were exchanged in the first two terms on the right hand side. In view of the antisymmetric components of the angular velocity bivector, i.e. \(\Omega_{jk} = -\Omega_{kj}\) and \(\Omega_{ik} = -\Omega_{ki}\), the commutator \((259)\) reduces to,
\[
L_G \times \Omega = \frac{1}{4} \sum_{i,j,k=1}^{3} (I_{G,ij} - I_{G,jk}) \Omega_{ij} \Omega_{jk} \hat{f}_i \wedge \hat{f}_k \\
+ \frac{1}{4} \sum_{i,j,k=1}^{3} (I_{G,ij} - I_{G,ik}) \Omega_{ij} \Omega_{ik} \hat{f}_k \wedge \hat{f}_j
\]
\[(260)\]
and is recast as,
\[
L_G \times \Omega = \frac{1}{4} \sum_{i,j,k=1}^{3} (I_{G,ik} - I_{G,kj}) \Omega_{ik} \Omega_{kj} \hat{f}_i \wedge \hat{f}_j \\
+ \frac{1}{4} \sum_{i,j,k=1}^{3} (I_{G,kj} - I_{G,ki}) \Omega_{kj} \Omega_{ki} \hat{f}_i \wedge \hat{f}_j
\]
\[(261)\]
where the indices \(j\) and \(k\) were exchanged in the first term and the indices \(i\) and \(k\) were exchanged in the first term on the right hand side. In view of the antisymmetric components of the angular velocity bivector \((223)\), and the antisymmetric moments of inertia \((210)\), the commutator \((259)\) reduces to,
\[
L_G \times \Omega = \frac{1}{2} \sum_{i,j=1}^{3} \left( \sum_{k=1}^{3} (I_{G,ik} - I_{G,kj}) \Omega_{ik} \Omega_{kj} \right) \hat{f}_i \wedge \hat{f}_j
\]
\[(262)\]
50
In view of the commutator (262), the time derivative of the angular momentum (255) is recast in components as,

$$\dot{\mathbf{L}}_G = \frac{1}{2} \sum_{i,j=1}^{3} \left( I_{G,ij} \dot{\Omega}_{ij} + \sum_{k=1}^{3} (I_{G,ik} - I_{G,kj}) \Omega_{ik} \Omega_{kj} \right) \hat{f}_i \wedge \hat{f}_j$$  (263)

The net external torque bivector $\mathbf{T}^\text{ext}_G$ is written in components in the principal body frame as,

$$\sum \mathbf{T}^\text{ext}_G = \frac{1}{2} \sum_{i,j=1}^{3} T^\text{ext}_{G,ij} \hat{f}_i \wedge \hat{f}_j$$  (264)

According to the angular momentum theorem (251), the components of the bivectors (263) and (264) are equal,

$$\sum T^\text{ext}_{G,ij} = I_{G,ij} \dot{\Omega}_{ij} + \sum_{k=1}^{3} (I_{G,ik} - I_{G,kj}) \Omega_{ik} \Omega_{kj}$$  (265)

For a given unit bivector $\hat{f}_i \wedge \hat{f}_j$, where $i \neq j$, the index $k$ is the remaining index and no summation over $k$ is needed in equation (265). This equation for the bivector $\hat{f}_i \wedge \hat{f}_j$ describes the rotation of the rigid body in the oriented plane spanned by this bivector,

$$\sum T^\text{ext}_{G,ij} = I_{G,ij} \dot{\Omega}_{ij} + \sum_{k=1}^{3} (I_{G,ik} - I_{G,kj}) \Omega_{ik} \Omega_{kj}$$  (266)

Using the antisymmetric components of the angular velocity bivector (223), and the antisymmetric moments of inertia (210), equation (266) evaluated for the couples of indices $ij \in \{12, 23, 31\}$ yields the Euler equations in geometric algebra,

$$\sum T^\text{ext}_{G,12} = I_{G,12} \dot{\Omega}_{12} + (I_{G,23} - I_{G,31}) \Omega_{23} \Omega_{31}$$
$$\sum T^\text{ext}_{G,23} = I_{G,23} \dot{\Omega}_{23} + (I_{G,31} - I_{G,12}) \Omega_{31} \Omega_{12}$$
$$\sum T^\text{ext}_{G,31} = I_{G,31} \dot{\Omega}_{31} + (I_{G,12} - I_{G,23}) \Omega_{12} \Omega_{23}$$  (267)

In order to deduce from the Euler equations in geometric algebra the Euler equations in vector algebra, we use the duality of the components of the external torques,

$$T_{G,ij} = -T_{G,ji} = \varepsilon_{ijk} \tau_{G,k}$$  (268)
the duality of the components of the angular acceleration,

\[ \dot{\Omega}_{ij} = - \dot{\Omega}_{ji} = \varepsilon_{ijk} \dot{\omega}_k \quad (269) \]

and the duality of the components of the moments of inertia \([210]\). Thus, the Euler equations \([267]\) are recast as,

\[
\sum \varepsilon_{123} \tau_{G,3}^{ext} = \varepsilon_{123}^2 i_{G,3} \dot{\omega}_3 + \varepsilon_{231}^2 \varepsilon_{312} (i_{G,1} - i_{G,2}) \omega_1 \omega_2 \\
\sum \varepsilon_{231} \tau_{G,1}^{ext} = \varepsilon_{231}^2 i_{G,1} \dot{\omega}_1 + \varepsilon_{231}^2 \varepsilon_{312} (i_{G,2} - i_{G,3}) \omega_2 \omega_3 \\
\sum \varepsilon_{312} \tau_{G,2}^{ext} = \varepsilon_{312}^2 i_{G,2} \dot{\omega}_2 + \varepsilon_{312}^2 \varepsilon_{123} (i_{G,3} - i_{G,1}) \omega_3 \omega_1 \\
(270)
\]

Taking into account the invariance of the Levi-Civita symbols \([73]\) under cyclic permutation,

\[ \varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} \quad (271) \]

the Euler equations in vector algebra \([270]\) reduce to,

\[
\sum \tau_{G,1}^{ext} = i_{G,1} \dot{\omega}_1 + (i_{G,2} - i_{G,3}) \omega_2 \omega_3 \\
\sum \tau_{G,2}^{ext} = i_{G,2} \dot{\omega}_2 + (i_{G,3} - i_{G,1}) \omega_3 \omega_1 \\
\sum \tau_{G,3}^{ext} = i_{G,3} \dot{\omega}_3 + (i_{G,1} - i_{G,2}) \omega_1 \omega_2 \\
(272)
\]

15. Symmetric spinning disk

We now consider the torque free motion of a symmetric spinning disk (Fig. [16]). The disk is a gyroscope that can freely rotate around its center of mass \(G\). The principal body frame \(\{\hat{f}_1, \hat{f}_2, \hat{f}_3\}\) is oriented such that the unit vector \(\hat{f}_3\) is along the symmetry axis of the disk. It coincides initially with the fixed orthonormal frame \(\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}\). The plane spanned by the unit bivector \(\hat{f}_1 \wedge \hat{f}_2\) that is orthogonal to the unit vector \(\hat{f}_3\) is the plane of symmetry of the disk. By symmetry, the moments of inertia in the orthogonal planes spanned by the unit bivectors \(\hat{f}_2 \wedge \hat{f}_3\) and \(\hat{f}_3 \wedge \hat{f}_1\) containing the unit vector \(\hat{f}_3\) are equal,

\[ I_{G,||} \equiv I_{G,12} \quad \text{and} \quad I_{G,\perp} \equiv I_{G,23} = I_{G,31} \quad (273) \]

where the indices \(\parallel\) and \(\perp\) indicate the moment of inertia is defined with respect to a principal axis plane that is either parallel or perpendicular to the plane of
symmetry. There is no net external torque bivector evaluated at the center of mass of the freely rotating disk,

\[ \sum T_\text{ext}^G = 0 \]  \hfill (274)

We now identify the constant bivectors in order to find the rotor that describes entirely the rotator of the symmetric disk. Firstly, in view of the angular momentum theorem \hbox{(251)}, the angular momentum bivector is constant,

\[ L^G = \text{const} \]  \hfill (275)

Secondly, in the absence of an external torque \hbox{(274)}, using the symmetry \hbox{(273)}, the projection of the Euler equations \hbox{(267)} in the plane of symmetry reads,

\[ I_{G,\parallel} \dot{\Omega}_{12} = 0 \]  \hfill (276)

which means that the velocity bivector \( \Omega_{0,\parallel} \) that is initially in the plane of symmetry is constant,

\[ \Omega_{0,\parallel} = \Omega_{12} \hat{e}_1 \wedge \hat{e}_2 = \text{const} \]  \hfill (277)

Figure 16: Symmetric disk spinning around its centre of mass \( G \). The principal body frame \( \{\hat{f}_1, \hat{f}_2, \hat{f}_3\} \) is oriented such that the unit vector \( \hat{f}_3 \) is along the symmetry axis of the disk. The orthonormal frame \( \{\hat{e}_1, \hat{e}_2, \hat{e}_3\} \) is fixed.
In view of relations (229) and (273), the angular momentum bivector is given by,

\[ L_G = I_G, \parallel \Omega_{12} \hat{f}_1 \wedge \hat{f}_2 + I_G, \perp \Omega_{23} \hat{f}_2 \wedge \hat{f}_3 + I_G, \perp \Omega_{31} \hat{f}_3 \wedge \hat{f}_1 \]  

and recast as,

\[ L_G = I_G, \perp (\Omega_{12} \hat{f}_1 \wedge \hat{f}_2 + \Omega_{23} \hat{f}_2 \wedge \hat{f}_3 + \Omega_{31} \hat{f}_3 \wedge \hat{f}_1) \]  

In view of the angular velocity (244), the angular momentum bivector (279) reduces to,

\[ L_G = I_G, \perp \Omega + R \left( (I_G, \parallel - I_G, \perp) \Omega_{12} \hat{e}_1 \wedge \hat{e}_2 \right) R^\dagger \]  

Taking into account the bivector (277), the angular momentum bivector (280) becomes,

\[ L_G = I_G, \perp \Omega + R \left( (I_G, \parallel - I_G, \perp) \Omega_{0, \parallel} \right) R^\dagger \]  

Thus, the angular velocity bivector \( \Omega \) is expressed in terms of the constant bivectors \( L_G \) and \( \Omega_{0, \parallel} \) as,

\[ \Omega = \frac{1}{I_{G, \perp}} \left( L_G + R \left( (I_G, \parallel - I_G, \perp) \Omega_{0, \parallel} \right) R^\dagger \right) \]  

Using the angular momentum bivector (282) and the normalisation condition (83), the time evolution equation for the rotor (89) is recast as,

\[ \dot{R} = - \frac{1}{2} \Omega R = - \frac{1}{2 I_{G, \perp}} \left( L_G R + R \left( (I_G, \parallel - I_G, \perp) \Omega_{0, \parallel} \right) \right) \]  

Using the two constant bivectors defined as,

\[ \Omega_L \equiv - \frac{1}{2 I_{G, \perp}} L_G \quad \text{and} \quad \Omega_R \equiv - \frac{I_G, \parallel - I_G, \perp}{2 I_{G, \perp}} \Omega_{0, \parallel} \]  

the time evolution equation of the rotor (283) is recast as,

\[ \dot{R} = - \frac{1}{2} \Omega_L R - \frac{1}{2} R \Omega_R \]  

The rotor satisfies the trivial initial condition,

\[ R(0) = 1 \]
The solution of the time evolution equation (285) that satisfies the initial condition (286) is the rotor,

$$R(t) = \exp\left(-\frac{1}{2} \Omega_L t\right) \exp\left(-\frac{1}{2} \Omega_R t\right)$$

(287)
as we now show. The time derivative of the rotor (287) is,

$$\dot{R}(t) = \left(-\frac{1}{2} \Omega_L\right) \exp\left(-\frac{1}{2} \Omega_L t\right) \exp\left(-\frac{1}{2} \Omega_R t\right) + \exp\left(-\frac{1}{2} \Omega_L t\right) \exp\left(-\frac{1}{2} \Omega_R t\right) \left(-\frac{1}{2} \Omega_R\right)$$

(288)

which reduces to the dynamical equation (285) in view of the rotor (287), as it should.

16. Conclusion

Rotations can always be expressed as the composition of two reflections of intersecting planes. These planes containing the origin $O$ are determined by their respective orthogonal unit vectors $\hat{n}_1$ and $\hat{n}_2$. The geometric product of these vectors is the rotor $R_\theta = \hat{n}_2 \hat{n}_1 = e^{-\hat{B} \theta / 2} = e^{-\hat{B} \theta}$ where $\hat{B}$ is the unit bivector and $\phi = \theta / 2$ is the angle between the unit vectors $\hat{n}_1$ and $\hat{n}_2$ in the rotation plane. The rotation of a vector $v$ by an angle $\theta = 2 \phi$ in the rotation plane spanned by the unit bivector $\hat{B}$ is given by the transformation law,

$$R_{\hat{B} \theta} (v) = R v R^\dagger = e^{-\hat{B} \theta / 2} v e^{\hat{B} \theta / 2}$$

(289)

where $R$ is the rotor of angle $\theta$ in the rotation plane spanned by the unit bivector $\hat{B}$. Similarly, the rotation of a multivector $M = s + v + A + s^t I$ is given by the transformation law,

$$R_{\hat{B} \theta} (M) = R M R^\dagger = e^{-\hat{B} \theta / 2} M e^{\hat{B} \theta / 2}$$

(290)

The time evolution of the rotor $R$ and its reverse $R^\dagger$ are expressed in terms of the angular velocity bivector $\Omega = \dot{\theta} \hat{B}$ in the rotation plane as,

$$\dot{R} = -\frac{1}{2} \Omega R \quad \text{and} \quad \dot{R}^\dagger = \frac{1}{2} R^\dagger \Omega$$

(291)
where the angular bivector $\Omega$ is the dual of the angular pseudovector $\omega$, i.e. $\omega = \Omega^\ast$. In geometric algebra, the time evolution of the rotor yields the Poisson formula describing the time evolution of the rotating basis vectors $\hat{f}_i$,

$$\dot{\hat{f}}_i = \hat{f}_i \cdot \Omega$$  \hspace{1cm} (292)

The geometric meaning of the inner product $\hat{f}_i \cdot \Omega$ is that the angular velocity bivector $\Omega$ rotates the basis vector $\hat{f}_i$ by a $90^\circ$ angle in the rotation direction defined by the unit bivector $\hat{B}$.

Rotational dynamics is described by the angular momentum theorem. For a rigid body, this theorem is expressed in the rotation plane in geometric algebra in terms of bivectors evaluated with respect to the centre of mass $G$ as,

$$\sum T_{G}^{\text{ext}} = \dot{L}_G$$  \hspace{1cm} (293)

where the angular momentum bivector $L_G$ is the dual of the angular momentum pseudovector $\ell_G$, i.e. $\ell_G = L_G^\ast$, and the external torque bivector $T_{G}^{\text{ext}}$ which is the dual of the external torque pseudovector $\tau_{G}^{\text{ext}}$. According to the transformation law (290) for a multivector under rotation, the time evolution of an angular momentum bivector $L_G$ describing the intrinsic rotation of a rigid body around its centre of mass is written as,

$$L_G = R L_{G,0} R^\dagger$$  \hspace{1cm} (294)

where the initial angular momentum bivector $L_{G,0}$ is a linear map $I_G(\Omega_0)$ of the initial angular velocity bivector,

$$\Omega_0 = \frac{1}{2} \sum_{i,j=1}^{3} \Omega_{ij} \hat{e}_i \wedge \hat{e}_j \quad \text{with} \quad \Omega_{ji} = -\Omega_{ij}$$  \hspace{1cm} (295)

that can be expressed as,

$$L_{G,0} = I_G(\Omega_0) = \frac{1}{2} \sum_{i,j=1}^{3} I_{G,ij} \Omega_{ij} \hat{e}_i \wedge \hat{e}_j \quad \text{with} \quad I_{G,ji} = I_{G,ij}$$  \hspace{1cm} (296)

where $\hat{e}_i \wedge \hat{e}_j$ are the unit bivectors in the initial principal axis frame of the rigid body, $I_{G,ij} = (\varepsilon_{ijk})^2 i_{G,k}$ is the dual of the moment of inertia $i_{G,k}$ and $\Omega_{ij}$ is
the antisymmetric component of the initial angular moment bivector $\Omega_0$ in the rotation plane spanned by the bivectors $\hat{e}_i \wedge \hat{e}_j$. The vectors $\hat{e}_i$ and bivectors $\hat{e}_i \wedge \hat{e}_j$ of the principal axis frame of the rigid body are related to the vectors $\hat{f}_i$ and bivectors $\hat{f}_i \wedge \hat{f}_j$ of the initial principal axis frame of the rigid body through the rotational transformation laws (290),

$$\hat{f}_i = R \hat{e}_i R^\dagger \quad \text{and} \quad \hat{f}_i \wedge \hat{f}_j = R \hat{e}_i \wedge \hat{e}_j R^\dagger$$

(297)

In view the linear inertia map (296) and the rotational transformation laws (297), the angular momentum bivector is recast in the principal axis frame of the rigid body in terms of the unit bivectors $\hat{f}_i \wedge \hat{f}_j$ as,

$$L_G = \frac{1}{2} \sum_{i,j=1}^{3} I_{G,ij} \Omega_{ij} \hat{f}_i \wedge \hat{f}_j$$

(298)

In geometric algebra, the Huygens-Steiner theorem for a rigid body of mass $m$ in the plane spanned by the unit bivector frame $\hat{f}_i \wedge \hat{f}_j$ reads,

$$I_{A,ij} = I_{G,ij} + m r_{AG}^2$$

(299)

and the kinetic energy is given by,

$$T = \frac{1}{2} m v_G^2 + \frac{1}{4} \sum_{i,j=1}^{3} I_{G,ij} \Omega_{ij}^2$$

(300)

The Euler equations for the rigid body are expressed in terms of the components of the dynamical bivectors in three orthogonal planes spanned by the unit bivectors $\hat{f}_1 \wedge \hat{f}_2$, $\hat{f}_2 \wedge \hat{f}_3$ and $\hat{f}_3 \wedge \hat{f}_1$ as,

$$\sum T_{G,12}^{\text{ext}} = I_{G,12} \dot{\Omega}_{12} + (I_{G,23} - I_{G,31}) \Omega_{23} \Omega_{31}$$

$$\sum T_{G,23}^{\text{ext}} = I_{G,23} \dot{\Omega}_{23} + (I_{G,31} - I_{G,12}) \Omega_{31} \Omega_{12}$$

$$\sum T_{G,31}^{\text{ext}} = I_{G,31} \dot{\Omega}_{31} + (I_{G,12} - I_{G,23}) \Omega_{12} \Omega_{23}$$

(301)

The dynamics of rigid bodies in geometric algebra can be applied to the torque free motion of a gyroscope consisting of a symmetric spinning disk around its centre of mass $G$. It is described by the rotor,

$$R(t) = \exp \left(-\frac{1}{2} \Omega_L t \right) \exp \left(-\frac{1}{2} \Omega_R t \right)$$

(302)
where the angular velocity bivectors $\Omega_L$ and $\Omega_R$ are given by,

$$\Omega_L \equiv -\frac{1}{2I_G,\perp}L_G \quad \text{and} \quad \Omega_R \equiv -\frac{I_G,\parallel - I_G,\perp}{2I_G,\perp} \Omega_{0,\parallel} \quad (303)$$

$I_{G,\parallel}$ is the moment of inertia and $\Omega_{0,\parallel}$ is the angular velocity in the plane of the disk, and $I_{G,\perp}$ are the moments of inertia in the planes orthogonal to the plane of the disk.

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A. Geometric algebra (GA)

We consider an orthonormal vector spatial frame \( \{ \hat{e}_1, \hat{e}_2, \hat{e}_3 \} \). The geometric product of two basis vectors reads, \[ \hat{e}_i \hat{e}_j = \hat{e}_i \cdot \hat{e}_j + \hat{e}_i \wedge \hat{e}_j \quad \text{with} \quad i, j = 1, 2, 3 \tag{A.1} \]

where,

\[ \hat{e}_1^2 = \hat{e}_2^2 = \hat{e}_3^2 = 1 \quad \text{and} \quad \hat{e}_i \wedge \hat{e}_i = \mathbf{0} \tag{A.2} \]

The inner product of two basis vectors is symmetric and defined as,

\[ \hat{e}_i \cdot \hat{e}_j = \hat{e}_j \cdot \hat{e}_i = \frac{1}{2} (\hat{e}_i \hat{e}_j + \hat{e}_j \hat{e}_i) \tag{A.3} \]

and the outer product is antisymmetric,

\[ \hat{e}_i \wedge \hat{e}_j = - \hat{e}_j \wedge \hat{e}_i = \frac{1}{2} (\hat{e}_i \hat{e}_j - \hat{e}_j \hat{e}_i) \tag{A.4} \]

The 8 basis elements of the geometric algebra (GA) \( \mathbb{G}^3 \) called the geometric algebra are:

- one scalar : 1
- three vectors : \( \hat{e}_1, \hat{e}_2, \hat{e}_3 \)
- three bivectors : \( \hat{e}_1 \wedge \hat{e}_2, \hat{e}_2 \wedge \hat{e}_3, \hat{e}_3 \wedge \hat{e}_1 \)
- one trivector or pseudoscalar : \( \hat{e}_1 \wedge \hat{e}_2 \wedge \hat{e}_3 \)

where the number of elements of each type 1, 3, 3, 1 are the binomial coefficients in three dimensions. In \( \mathbb{G}^3 \), there are \( 2^3 \) basis elements. The geometric interpretation of the basis elements is clear (Fig. A.17). A scalar \( s \) is an oriented point, a vector is an oriented line element, a bivector is an oriented plane element and a pseudoscalar is an oriented volume element. The notation used is the following : scalars \( s \) are written in lower case letters, vectors \( \mathbf{v} \) are written in lower case bold letters, bivectors \( \mathbf{B} \) are written in upper case bold letters and pseudoscalars are written as \( s'I \). A multivector \( M \), which is linear combination of elements of \( \mathbb{G}^3 \), is written in upper case letters,

\[ M = s + \mathbf{v} + \mathbf{B} + s'I \tag{A.5} \]
Taking into account the definition (A.1), the bivectors and the pseudoscalar are also written as geometric products of the basis unit vectors $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$:

- bivectors: $\hat{e}_1 \hat{e}_2$, $\hat{e}_2 \hat{e}_3$, $\hat{e}_3 \hat{e}_1$
- pseudoscalar: $\hat{e}_1 \hat{e}_2 \hat{e}_3$

The pseudoscalar $I = \hat{e}_1 \hat{e}_2 \hat{e}_3$ behaves as an imaginary number since,

$$I^2 = (\hat{e}_1 \hat{e}_2 \hat{e}_3)(\hat{e}_1 \hat{e}_2 \hat{e}_3) = (\hat{e}_1 \hat{e}_2 \hat{e}_3)(\hat{e}_1 \hat{e}_2 \hat{e}_3) = -1$$

(A.6)

The geometric product of two basis vectors $u$ and $v$ reads,

$$u \cdot v = u \cdot v + u \wedge v$$

(A.7)

The inner product between the two basis vectors is symmetric,

$$u \cdot v = v \cdot u = \frac{1}{2} (u v + v u)$$

(A.8)
and the outer product is antisymmetric,

\[ \mathbf{u} \wedge \mathbf{v} = - \mathbf{v} \wedge \mathbf{u} = \frac{1}{2} (\mathbf{u} \cdot \mathbf{v}) \]  
\[ \quad \text{(A.9)} \]

The geometric product of a vector \( \mathbf{v} \) and a bivector \( \mathbf{B} \) is the sum of the inner product of the outer product,

\[ \mathbf{v} \mathbf{B} = \mathbf{v} \cdot \mathbf{B} + \mathbf{v} \wedge \mathbf{B} \]  
\[ \quad \text{(A.10)} \]

and the geometric product of a bivector \( \mathbf{B} \) and a vector \( \mathbf{v} \) is the sum of the inner product of the outer product,

\[ \mathbf{B} \mathbf{v} = \mathbf{B} \cdot \mathbf{v} + \mathbf{B} \wedge \mathbf{v} \]  
\[ \quad \text{(A.11)} \]

To determine the symmetry of these products, the spatial frame is oriented such that the bivector reads \( \mathbf{B} = B_{12} \hat{e}_1 \wedge \hat{e}_2 = B_{12} \hat{e}_1 \hat{e}_2 \) and the orientation of the vector \( \mathbf{v} = v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3 \) is arbitrary. The geometric product of the vector \( \mathbf{v} \) and the bivector \( \mathbf{B} \) reads,

\[ \mathbf{v} \mathbf{B} = (v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3) (B_{12} \hat{e}_1 \hat{e}_2) \\
= v_1 B_{12} \hat{e}_1 \hat{e}_2 + v_2 B_{12} \hat{e}_2 \hat{e}_1 + v_3 B_{12} \hat{e}_3 \hat{e}_1 \hat{e}_2 \]  
\[ \quad \text{(A.12)} \]

The geometric product of the bivector \( \mathbf{B} \) and the vector \( \mathbf{v} \) reads,

\[ \mathbf{B} \mathbf{v} = (B_{12} \hat{e}_1 \hat{e}_2) (v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3) \\
= B_{12} v_1 \hat{e}_1 \hat{e}_2 + B_{12} v_2 \hat{e}_2 \hat{e}_1 + B_{12} v_3 \hat{e}_3 \hat{e}_1 \hat{e}_2 \]  
\[ \quad \text{(A.13)} \]

The outer product of the vector \( \mathbf{v} \) and the bivector \( \mathbf{B} \) reads,

\[ \mathbf{v} \wedge \mathbf{B} = (v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3) \wedge (B_{12} \hat{e}_1 \hat{e}_2) \\
= v_1 B_{12} \hat{e}_1 \wedge \hat{e}_2 + v_2 B_{12} \hat{e}_2 \wedge \hat{e}_1 + v_3 B_{12} \hat{e}_3 \wedge \hat{e}_1 \wedge \hat{e}_2 \]  
\[ \quad \text{(A.14)} \]

The outer product of the bivector \( \mathbf{B} \) and the vector \( \mathbf{v} \) reads,

\[ \mathbf{B} \wedge \mathbf{v} = (B_{12} \hat{e}_1 \hat{e}_2) \wedge (v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3) \]
From relations (A.14) and (A.15) we conclude that the outer product of a vector \( v \) and a bivector \( B \) is symmetric,

\[
v \wedge B = B \wedge v
\]  
(A.16)

In view of relations (A.10), (A.12) and (A.14),

\[
v \cdot B = vB - v \wedge B = -B_{12} v_2 \hat{e}_1 + B_{12} v_1 \hat{e}_2
\]  
(A.17)

In view of relations (A.11), (A.13) and (A.15),

\[
B \cdot v = Bv - B \wedge v = B_{12} v_2 \hat{e}_1 - B_{12} v_1 \hat{e}_2
\]  
(A.18)

From relations (A.17) and (A.18) we conclude that the inner product of a vector \( v \) and a bivector \( B \) is antisymmetric,

\[
v \cdot B = -B \cdot v
\]  
(A.19)

In view of relations (A.12) and (A.13), (A.16) and (A.19),

\[
v \cdot B = \frac{1}{2} (vB - Bv)
\]  
(A.20)

\[
v \wedge B = \frac{1}{2} (vB + Bv)
\]  
(A.21)

To determine the geometric product of two bivectors \( A \) and \( B \), we write them as \( A = A_{12} \hat{e}_1 \hat{e}_2 + A_{23} \hat{e}_2 \hat{e}_3 + A_{31} \hat{e}_3 \hat{e}_1 \) and \( B = B_{12} \hat{e}_1 \hat{e}_2 + B_{23} \hat{e}_2 \hat{e}_3 + B_{31} \hat{e}_3 \hat{e}_1 \).

The geometric product of the bivectors \( A \) and \( B \) reads,

\[
AB = (A_{12} \hat{e}_1 \hat{e}_2 + A_{23} \hat{e}_2 \hat{e}_3 + A_{31} \hat{e}_3 \hat{e}_1) (B_{12} \hat{e}_1 \hat{e}_2 + B_{23} \hat{e}_2 \hat{e}_3 + B_{31} \hat{e}_3 \hat{e}_1)
\]  
(A.22)

\[
= A_{12} B_{12} \hat{e}_1 \hat{e}_2 \hat{e}_1 \hat{e}_2 + A_{12} B_{23} \hat{e}_1 \hat{e}_2 \hat{e}_2 \hat{e}_3 + A_{12} B_{31} \hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_1
\]

\[
+ A_{23} B_{12} \hat{e}_2 \hat{e}_3 \hat{e}_1 \hat{e}_2 + A_{23} B_{23} \hat{e}_2 \hat{e}_3 \hat{e}_3 \hat{e}_3 + A_{23} B_{31} \hat{e}_2 \hat{e}_3 \hat{e}_3 \hat{e}_1
\]

\[
+ A_{31} B_{12} \hat{e}_3 \hat{e}_1 \hat{e}_1 \hat{e}_2 + A_{31} B_{23} \hat{e}_3 \hat{e}_1 \hat{e}_2 \hat{e}_3 + A_{31} B_{31} \hat{e}_3 \hat{e}_1 \hat{e}_3 \hat{e}_1
\]
which reduces to,

\[ AB = - (A_{12} B_{12} + A_{23} B_{23} + A_{31} B_{31}) + (A_{31} B_{23} - A_{23} B_{31}) \hat{e}_1 \hat{e}_2 \]
\[ + (A_{12} B_{31} - A_{31} B_{12}) \hat{e}_2 \hat{e}_3 + (A_{23} B_{12} - A_{12} B_{23}) \hat{e}_3 \hat{e}_1 \]  \hfill (A.23)

The geometric product of the bivectors \( B \) and \( A \) reads,

\[ BA = (B_{12} \hat{e}_1 \hat{e}_2 + B_{23} \hat{e}_2 \hat{e}_3 + B_{31} \hat{e}_3 \hat{e}_1) (A_{12} \hat{e}_1 \hat{e}_2 + A_{23} \hat{e}_2 \hat{e}_3 + A_{31} \hat{e}_3 \hat{e}_1) \]
\[ = A_{12} B_{12} \hat{e}_1 \hat{e}_2 \hat{e}_1 \hat{e}_2 + A_{23} B_{23} \hat{e}_2 \hat{e}_3 \hat{e}_2 \hat{e}_3 + A_{31} B_{31} \hat{e}_3 \hat{e}_1 \hat{e}_3 \hat{e}_1 \]
\[ + A_{12} B_{23} \hat{e}_1 \hat{e}_3 \hat{e}_2 \hat{e}_1 + A_{23} B_{31} \hat{e}_2 \hat{e}_1 \hat{e}_3 \hat{e}_2 + A_{31} B_{12} \hat{e}_3 \hat{e}_2 \hat{e}_1 \hat{e}_3 \]  \hfill (A.24)

which reduces to,

\[ BA = - (A_{12} B_{12} + A_{23} B_{23} + A_{31} B_{31}) + (A_{23} B_{31} - A_{31} B_{23}) \hat{e}_1 \hat{e}_2 \]
\[ + (A_{31} B_{12} - A_{12} B_{31}) \hat{e}_2 \hat{e}_3 + (A_{12} B_{23} - A_{23} B_{12}) \hat{e}_3 \hat{e}_1 \]  \hfill (A.25)

The geometric product (A.23) of two bivectors \( A \) and \( B \) is the sum symmetric product and an antisymmetric product,

\[ AB = A \cdot B + A \times B \]  \hfill (A.26)

According to relations (A.23) and (A.25), the inner product, or anticommutator, of two bivectors \( A \) and \( B \) yields a scalar,

\[ A \cdot B = B \cdot A = \frac{1}{2} (AB + BA) = - (A_{12} B_{12} + A_{23} B_{23} + A_{31} B_{31}) \]  \hfill (A.27)

According to relations (A.23) and (A.25), the cross product, or commutator, of two bivectors \( A \) and \( B \) yields a bivector,

\[ A \times B = - B \times A = \frac{1}{2} (AB - BA) \]
\[ = (A_{31} B_{23} - A_{23} B_{31}) \hat{e}_1 \hat{e}_2 + (A_{12} B_{31} - A_{31} B_{12}) \hat{e}_2 \hat{e}_3 \]
\[ + (A_{23} B_{12} - A_{12} B_{23}) \hat{e}_3 \hat{e}_1 \]  \hfill (A.28)

The fact that the anticommutator of two bivectors yields a scalar and the commutator of two bivectors yields a bivector is specific to the geometric algebra.
$\mathbb{G}^3$. To determine the geometric product of a vector $v$ and the pseudoscalar $I$, we write them as $B = B_{12} \hat{e}_1 \hat{e}_2 + B_{23} \hat{e}_2 \hat{e}_3 + B_{31} \hat{e}_3 \hat{e}_1$ and $I = \hat{e}_1 \hat{e}_2 \hat{e}_3$. The geometric product of the vector $v$ and the pseudoscalar $I$ reads,

$$v I = (v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3) (\hat{e}_1 \hat{e}_2 \hat{e}_3) = v_1 \hat{e}_2 \hat{e}_3 + v_2 \hat{e}_3 \hat{e}_1 + v_3 \hat{e}_1 \hat{e}_2 \quad (A.29)$$

$$I v = (\hat{e}_1 \hat{e}_2 \hat{e}_3) (v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3) = v_1 \hat{e}_2 \hat{e}_3 + v_2 \hat{e}_3 \hat{e}_1 + v_3 \hat{e}_1 \hat{e}_2 \quad (A.30)$$

According to relation (A.29) and (A.30), the vector $v$ commutes with the pseudoscalar $I$,

$$v I = I v \quad (A.31)$$

To determine the geometric product of a vector $v$ and the pseudoscalar $I$, we write them as $v = v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3$ and $I = \hat{e}_1 \hat{e}_2 \hat{e}_3$. The geometric product of the bivector $B$ and the pseudoscalar $I$ reads,

$$B I = (B_{12} \hat{e}_1 \hat{e}_2 + B_{23} \hat{e}_2 \hat{e}_3 + B_{31} \hat{e}_3 \hat{e}_1) (\hat{e}_1 \hat{e}_2 \hat{e}_3)$$

$$= - (B_{23} \hat{e}_1 + B_{31} \hat{e}_2 + B_{12} \hat{e}_3) \quad (A.32)$$

$$I B = (\hat{e}_1 \hat{e}_2 \hat{e}_3) (B_{12} \hat{e}_1 \hat{e}_2 + B_{23} \hat{e}_2 \hat{e}_3 + B_{31} \hat{e}_3 \hat{e}_1) = - B_{12} \hat{e}_3$$

$$= - (B_{23} \hat{e}_1 + B_{31} \hat{e}_2 + B_{12} \hat{e}_3) \quad (A.33)$$

According to relation (A.32) and (A.33), the bivector $B$ commutes with the pseudoscalar $I$,

$$B I = I B \quad (A.34)$$

**B. Duality in geometric algebra**

The reverse of the scalar $s$, the vector $v = v_3 \hat{e}_3$, of the bivector $B = B_{12} \hat{e}_1 \hat{e}_2$ and of the pseudoscalar $I$ is obtained by reversing the order of the basis vectors,

$$s^\dagger = s \quad (B.1)$$

$$v^\dagger = v_3 \hat{e}_3 = v \quad (B.2)$$

$$B^\dagger = B_{12} \hat{e}_2 \hat{e}_1 = - B_{12} \hat{e}_1 \hat{e}_2 = - B \quad (B.3)$$
\[ I^\dagger = \hat{e}_3 \hat{e}_2 \hat{e}_1 = - \hat{e}_1 \hat{e}_2 \hat{e}_3 = -I \]  
(B.4)

Thus, the reverse of the multivector \((A.5)\) is given by,

\[ M^\dagger = s^\dagger + v^\dagger + B^\dagger + s' I^\dagger = s + v - B - s' I \]  
(B.5)

The square of the modulus of a vector \(v\), a bivector \(B\) and a pseudovector \(I\) are defined as,

\[ |v|^2 = v^\dagger \cdot v = v \cdot v = v^2 \]  
(B.6)
\[ |B|^2 = B^\dagger \cdot B = - B \cdot B = -B^2 \]  
(B.7)
\[ |I|^2 = I^\dagger \cdot I = - I \cdot I = -I^2 = 1 \]  
(B.8)

The geometric interpretation of the modulus is clear: the modulus of a vector \(|v|\) is the length of a line element, the modulus of a bivector \(|B|\) is the surface of a plane element and the modulus of a pseudovector \(|s'I|\) is the volume of a space element. Thus, the modulus of the bivector obtained by taking the outer product of a vector \(u\) and a vector \(v\) is the surface of the parallelogram spanned by these vectors,

\[ |u \wedge v| = |u| \cdot |v| \sin \theta \]  
(B.9)

where \(\theta\) is the acute angle between these vectors. This modulus is the same as the modulus of the cross product of these vectors,

\[ |u \times v| = |u| \cdot |v| \sin \theta \]  
(B.10)

Thus,

\[ |u \wedge v| = |u \times v| \]  
(B.11)

The inverse of a vector \(v\), a bivector \(B\) and a pseudovector \(I\) are defined as,

\[ v^{-1} = \frac{v}{v^2} = \frac{v^\dagger}{|v|^2} \]  
(B.12)
\[ B^{-1} = \frac{B}{B^2} = \frac{B^\dagger}{|B|^2} \]  
(B.13)
\[ I^{-1} = \frac{I}{I^2} = \frac{I^\dagger}{|I|^2} = -I \]  
(B.14)
The dual of a vector $v$, a bivector $B$ and a pseudovector $I$ are defined as,

$$v^* = \frac{v}{I} = v I^{-1} = -v I$$  \hspace{1cm} (B.15)

$$B^* = \frac{B}{I} = B I^{-1} = -B I$$  \hspace{1cm} (B.16)

$$I^* = \frac{I}{I} = I I^{-1} = 1$$  \hspace{1cm} (B.17)

This duality is the transformation as the Hodge duality in differential forms.

The dual of the dual of a vector $v$, a bivector $B$ and a pseudovector $I$ are their opposite,

$$\begin{align*}
(v^*)^* &= -v^* I = v I^2 = -v \\
(B^*)^* &= -B^* I = B I^2 = -B \\
(I^*)^* &= -I^* I = I I^2 = -I
\end{align*}$$  \hspace{1cm} \text{B.18, B.19, B.20}

The dual of the vector $v = v_3 \hat{e}_3$ is,

$$v^* = -v I = -(v_3 \hat{e}_3) (\hat{e}_1 \hat{e}_2 \hat{e}_3) = -v_3 \hat{e}_1 \hat{e}_2$$  \hspace{1cm} (B.21)

Defining the bivector $V$ as,

$$V = v_{12} \hat{e}_1 \hat{e}_2 \quad \text{where} \quad |V| = |v| \quad \text{and thus} \quad v_{12} = v_3$$  \hspace{1cm} (B.22)

it satisfies the duality,

$$v^* = -V = (V^*)^* \quad \text{and thus} \quad V^* = v$$  \hspace{1cm} (B.23)

The dual of the bivector $B = B_{12} \hat{e}_1 \hat{e}_2$ is,

$$B^* = -B I = - (B_{12} \hat{e}_1 \hat{e}_2) (\hat{e}_1 \hat{e}_2 \hat{e}_3) = B_{12} \hat{e}_3$$  \hspace{1cm} (B.24)

Defining the vector $b$ as,

$$b = b_3 \hat{e}_3 \quad \text{where} \quad |b| = |B| \quad \text{and thus} \quad b_3 = B_{12}$$  \hspace{1cm} (B.25)

it satisfies the duality,

$$B^* = b = -(b^*)^* \quad \text{and thus} \quad b^* = -B$$  \hspace{1cm} (B.26)
There is a duality between a vector and a bivector of same modulus and there is a duality between a scalar and a pseudoscalar of same modulus. In view of relations \([B.15]\) and \([B.26]\),

\[ B = -b^* = blI \] (B.27)

which implies that by duality the multivector \(A.5\) is recast as, \(^7\)

\[ M = (s + s'I) + (v + bI) \] (B.28)

and the reverse of the multivector \(B.5\) is recast as,

\[ M^\dagger = (s - s'I) + (v - bI) \] (B.29)

which means that the reverse of a multivector is like a complex conjugate where the pseudoscalar \(I\) is like the imaginary number \(i\). The bivector \(W\) and the vector \(w\) are defined as the wedge product and the cross product of two vectors \(u\) and \(v\) respectively,

\[ W = u \wedge v \quad \text{and} \quad w = u \times v \] (B.30)

According to relation \([B.11]\) the modulus of the bivector \(W\) and vector \(w\) are equal, which means that the vector \(w\) is the dual of the bivector \(W\),

\[ |W| = |w| \quad \text{and thus} \quad W^* = w \quad \text{and} \quad w^* = -W \] (B.31)

Thus, the cross product of the vectors \(u\) and \(v\) is the dual of the wedge product of these vectors,

\[ (u \wedge v)^* = u \times v \quad \text{and} \quad (u \times v)^* = -u \wedge v \] (B.32)

To establish the duality between the inner and outer product of two vectors, we choose a vector \(u = u_1 \hat{e}_1 + u_2 \hat{e}_2\) and a vector \(v = v_1 \hat{e}_1 + v_2 \hat{e}_2\) in the same plane. The dual of the outer product of the vectors \(u\) and \(v\) yields,

\[
(u \wedge v)^* = - \left( (u_1 \hat{e}_1 + u_2 \hat{e}_2) \wedge (v_1 \hat{e}_1 + v_2 \hat{e}_2) \right) (\hat{e}_1 \hat{e}_2 \hat{e}_3) \\
= - (u_1 v_2 \hat{e}_1 \hat{e}_2 + u_2 v_1 \hat{e}_1 \hat{e}_3) (\hat{e}_1 \hat{e}_2 \hat{e}_3) \\
= (u_1 v_2 - u_2 v_1) \hat{e}_3
\] (B.33)
The inner product of the vector $\mathbf{u}$ and $\mathbf{v}^*$ yields,

$$\mathbf{u} \cdot \mathbf{v}^* = - (u_1 \hat{e}_1 + u_2 \hat{e}_2) \cdot \left((v_1 \hat{e}_1 + v_2 \hat{e}_2)(\hat{e}_1 \hat{e}_2 \hat{e}_3)\right)$$

$$= - (u_1 \hat{e}_1 + u_2 \hat{e}_2) \cdot (v_1 \hat{e}_2 \hat{e}_3 - v_2 \hat{e}_1 \hat{e}_3)$$

$$= (u_1 v_2 - u_2 v_1) \hat{e}_3$$

(B.34)

The identification of relations (B.33) and (B.34) yields the vectorial duality,

$$(\mathbf{u} \wedge \mathbf{v})^* = \mathbf{u} \cdot \mathbf{v}^*$$

(B.35)

which is recast in terms of the dual bivector $\mathbf{V} = - \mathbf{v}^*$ as,

$$(\mathbf{u} \wedge \mathbf{v})^* = - \mathbf{u} \cdot \mathbf{V} = \mathbf{V} \cdot \mathbf{u}$$

(B.36)

In view of relations (B.32) and (B.35), we obtain,

$$\mathbf{u} \times \mathbf{v} = \mathbf{u} \cdot \mathbf{v}^*$$

(B.37)

which is recast in terms of the dual bivector $\mathbf{V} = - \mathbf{v}^*$ as,

$$\mathbf{u} \times \mathbf{v} = - \mathbf{u} \cdot \mathbf{V} = \mathbf{V} \cdot \mathbf{u}$$

(B.38)

The dual of the inner product of the vectors $\mathbf{u}$ and $\mathbf{v}$ yields,

$$(\mathbf{u} \cdot \mathbf{v})^* = - \left((u_1 \hat{e}_1 + u_2 \hat{e}_2) \cdot (v_1 \hat{e}_1 + v_2 \hat{e}_2)\right)(\hat{e}_1 \hat{e}_2 \hat{e}_3)$$

$$= - (u_1 v_1 + u_2 v_2) \hat{e}_1 \hat{e}_2 \hat{e}_3$$

(B.39)

The outer product of the vectors $\mathbf{u}$ and $\mathbf{v}^*$ yields,

$$\mathbf{u} \wedge \mathbf{v}^* = - (u_1 \hat{e}_1 + u_2 \hat{e}_2) \wedge \left((v_1 \hat{e}_1 + v_2 \hat{e}_2)(\hat{e}_1 \hat{e}_2 \hat{e}_3)\right)$$

$$= - (u_1 \hat{e}_1 + u_2 \hat{e}_2) \wedge (v_1 \hat{e}_2 \hat{e}_3 - v_2 \hat{e}_1 \hat{e}_3)$$

$$= (u_1 v_1 + u_2 v_2) \hat{e}_1 \hat{e}_2 \hat{e}_3$$

(B.40)

The identification of relations (B.39) and (B.40) yields the pseudoscalar duality,

$$(\mathbf{u} \cdot \mathbf{v})^* = \mathbf{u} \wedge \mathbf{v}^*$$

(B.41)

which is recast in terms of the dual bivector $\mathbf{V} = - \mathbf{v}^*$ as,

$$(\mathbf{u} \cdot \mathbf{v})^* = - \mathbf{u} \wedge \mathbf{V}$$

(B.42)
To establish the duality between the inner and outer product of a bivector and a vector, the spatial frame is oriented such that the bivector is given by
\[ B = B_{12} \hat{e}_1 \hat{e}_2 \] and the vector is written as \[ v = v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3. \] The dual of the outer product of the vector \( u \) and the bivector \( B \) yields,
\[
(u \wedge B)^* = - \left( (u_1 \hat{e}_1 + u_2 \hat{e}_2 + u_3 \hat{e}_3) \wedge (B_{12} \hat{e}_1 \hat{e}_2) \right) (\hat{e}_1 \hat{e}_2 \hat{e}_3)
\]
\[
= - (u_3 B_{12} \hat{e}_1 \hat{e}_2 \hat{e}_3) (\hat{e}_1 \hat{e}_2 \hat{e}_3)
\]
\[
= u_3 B_{12}
\] (B.43)

The inner product of the vector \( u \) and the bivector \( B^* \) yields,
\[
u \cdot B^* = - (u_1 \hat{e}_1 + u_2 \hat{e}_2 + u_3 \hat{e}_3) \cdot \left( (B_{12} \hat{e}_1 \hat{e}_2) \right) (\hat{e}_1 \hat{e}_2 \hat{e}_3)
\]
\[
= (u_1 \hat{e}_1 + u_2 \hat{e}_2 + u_3 \hat{e}_3) \cdot (B_{12} \hat{e}_3)
\]
\[
= u_3 B_{12}
\] (B.44)

The identification of relations (B.43) and (B.44) yields the scalar duality,
\[
(u \wedge B)^* = u \cdot B^*
\] (B.45)

which is recast in terms of the dual vector \( b = B^* \) as,
\[
(u \wedge B)^* = u \cdot b
\] (B.46)

and is the dual of the pseudoscalar duality (B.42) for \( b = v \) and \( B = V \). The dual of the inner product of the vector \( u \) and the bivector \( B \) yields,
\[
u \cdot B^* = - \left( (u_1 \hat{e}_1 + u_2 \hat{e}_2 + u_3 \hat{e}_3) \cdot (B_{12} \hat{e}_1 \hat{e}_2) \right) (\hat{e}_1 \hat{e}_2 \hat{e}_3)
\]
\[
= - (u_1 B_{12} \hat{e}_2 - u_2 B_{12} \hat{e}_1) (\hat{e}_1 \hat{e}_2 \hat{e}_3)
\]
\[
= u_1 B_{12} \hat{e}_1 \hat{e}_3 + u_2 B_{12} \hat{e}_2 \hat{e}_3
\] (B.47)

The outer product of the vector \( u \) and the dual of the bivector \( B^* \) yields,
\[
u \wedge B^* = - (u_1 \hat{e}_1 + u_2 \hat{e}_2 + u_3 \hat{e}_3) \wedge \left( (B_{12} \hat{e}_1 \hat{e}_2) (\hat{e}_1 \hat{e}_2 \hat{e}_3) \right)
\]
\[
= (u_1 \hat{e}_1 + u_2 \hat{e}_2 + u_3 \hat{e}_3) \wedge (B_{12} \hat{e}_3)
\]
\[
= u_1 B_{12} \hat{e}_1 \hat{e}_3 + u_2 B_{12} \hat{e}_2 \hat{e}_3
\] (B.48)
The identification of relations \( \text{(B.47)} \) and \( \text{(B.48)} \) yields the bivectorial duality,

\[
(u \cdot B)^* = u \land B^* \tag{B.49}
\]

which is recast in terms of the dual vector \( b = B^* \) as,

\[
(u \cdot B)^* = u \land b \tag{B.50}
\]

and is the dual of the vectorial duality \( \text{(B.36)} \) for \( b = v \) and \( B = V \). To establish the duality between the inner and outer product of bivectors in the same plane, the spatial frame is oriented such that the bivectors \( A \) and \( B \) are written as \( A = A_{12} \hat{e}_1 \hat{e}_2 \) and \( B = B_{12} \hat{e}_2 \hat{e}_3 \). The dual of the inner product of the bivectors \( A \) and \( B \) yields,

\[
(A \cdot B)^* = -(A_{12} \hat{e}_1 \hat{e}_2) \cdot (B_{12} \hat{e}_1 \hat{e}_2) \hat{e}_3 = A_{12} B_{12} \hat{e}_1 \hat{e}_2 \hat{e}_3 \tag{B.51}
\]

The outer product of the bivectors \( A \) and \( B^* \) yields,

\[
A \land B^* = -(A_{12} \hat{e}_1 \hat{e}_2) \land (B_{12} \hat{e}_1 \hat{e}_2) \hat{e}_3 = A_{12} B_{12} \hat{e}_1 \hat{e}_2 \hat{e}_3 \tag{B.52}
\]

The identification of relations \( \text{(B.51)} \) and \( \text{(B.52)} \) yields the pseudoscalar duality,

\[
(A \cdot B)^* = A \land B^* \tag{B.53}
\]

The pseudoscalar duality \( \text{(B.53)} \) is expressed in terms of the dual vector \( b = B^* \),

\[
(A \cdot B)^* = A \land b \tag{B.54}
\]

In view of the identities \( \text{(A.10)}, \text{(B.49)} \), the inner product of two bivectors \( A \) and \( B \) is expressed in terms of the dual vectors \( a = A^* \) and \( b = B^* \) as,

\[
(A \cdot B)^* = A \land b = b \land A = -b \land a^* = -(b \cdot a)^* = -(a \cdot b)^* \tag{B.55}
\]
The dual of identity (B.55) yields,

\[ A \cdot B = -a \cdot b \]  

(B.56)

To establish the duality of the inner product of a vector \( v \) and a trivector \( T = tI \), we write that trivector as \( T = tI \), where \( t \) is a scalar. Thus, in view of identity (B.17),

\[ T^* = (tI)^* = - (tI) I = t \quad \text{and} \quad t^* = -T \]  

(B.57)

Use the dualities (B.16) and (B.57), and the commutation (A.31) between the vector \( v \) and the pseudoscalar \( I \) or the trivector \( T = tI \), the dual of bivector \( v \cdot T \) is written as,

\[ (v \cdot T)^* = (vT)^* = - (vtI) I = vt = vT^* \]  

(B.58)

C. Algebraic identities in geometric algebra

The double cross product of the three vectors \( u, v \) and \( w \) is written as,

\[ u \times (v \times w) = (u \cdot w) v - (u \cdot v) w \]  

(C.1)

In view of the duality (B.32) between the cross and wedge products, the vectorial duality (B.37), the double duality (B.19) and the antisymmetry of the outer product (A.9), the left-hand side of relation (C.1) is recast as,

\[ u \times (v \times w) = u \times (v \wedge w)^* = u \cdot \left( (v \wedge w)^* \right)^* = - u \cdot (v \wedge w) = u \cdot (w \wedge v) \]  

(C.2)

According to the identity (C.2), the double cross product (C.1) yields the triple product,

\[ u \cdot (v \wedge w) = (u \cdot v) w - (u \cdot w) v \]  

(C.3)

This triple product (C.3) represents the projection of the vector \( u \) on the bivector \( v \wedge w \). The vectors \( u, v \) and \( w \) satisfy the identity,

\[ u \cdot (v \times w) = v \cdot (w \times u) = w \cdot (u \times v) \]  

(C.4)
In view of the duality (B.32) between cross product and wedge product and the scalar duality (B.45), the three terms of identity (C.4) are recast as,

\[ u \cdot (v \times w) = u \cdot (v \wedge w)^* = (u \wedge v \wedge w)^* \]

\[ v \cdot (w \times u) = v \cdot (w \wedge u)^* = (v \wedge w \wedge u)^* \]

\[ w \cdot (u \times v) = w \cdot (u \wedge v)^* = (w \wedge u \wedge v)^* \]

(C.5)

According to the relations (C.5), the scalar identity (C.4) becomes,

\[ (u \wedge v \wedge w)^* = (v \wedge w \wedge u)^* = (w \wedge u \wedge v)^* \]

(C.6)

The dual of the scalar identity (C.6) yields the pseudoscalar identity,

\[ u \wedge v \wedge w = v \wedge w \wedge u = w \wedge u \wedge v \]

(C.7)

This identity represents the fact the oriented volume of the parallelepiped spanned by the vectors \( u, v, w \) is the same as that of the parallelepiped spanned by the vectors \( v, w, u \) and also the same as that of the parallelepiped spanned by the vectors \( w, u, v \) because these three parallelepipeds have the same orientation. According to relations (A.20) and (A.21) for \( B = v \wedge w \), the inner and outer products of the vector \( u \) and the bivector \( v \wedge w \) are given by,

\[ u \cdot (v \wedge w) = \frac{1}{2} \left( u (v \wedge w) - (v \wedge w) u \right) \]

(C.8)

\[ u \wedge (v \wedge w) = \frac{1}{2} \left( u (v \wedge w) + (v \wedge w) u \right) \]

(C.9)

To determine the mixed product of two vectors \( u, v \) and a bivector \( B \), we use the dual vector defined as \( b = -BI \). Using the identity (A.19) and the triple product of vectors (C.4), we obtain,

\[ (u \wedge v) \cdot B = (u \wedge v) \cdot b I = -b \cdot (u \wedge v) I = -I (b \cdot u) v + I (b \cdot v) u \]

(C.10)

which reduces to,

\[ (u \wedge v) \cdot B = (B \cdot u) \cdot v - (B \cdot v) \cdot u = u \cdot (v \cdot B) - v \cdot (u \cdot B) \]

(C.11)

To determine the first kind of mixed product of two bivectors \( A, B \) and a vector \( v \), we use the dual vectors defined as \( a = -AI \) and \( b = -BI \). Using
the identities (A.19) and $I^2 = -1$, we obtain,

$$(A \wedge v) \cdot B = (a I \wedge v) \cdot b I = - (a \wedge v) \cdot b = b \cdot (a \wedge v)$$  \hspace{1cm} \text{(C.12)}$$

Using the triple product of vectors (C.4), the mixed product is recast as,

$$(A \wedge v) \cdot B = b \cdot (a \wedge v) = (a \cdot b) v - (b \cdot v) a$$  \hspace{1cm} \text{(C.13)}$$

$$= - (A \cdot B) v + (B \cdot v) A$$

To determine the first kind of mixed product of two bivectors $A$, $B$ and a vector $v$, we use the dual vector defined as $a = A^* = - A I$ and the duality $(v \wedge a)^* = -(v \wedge a) I$. Using the identities (A.19), (B.35) and $I^2 = -1$, we obtain,

$$(A \cdot v) \wedge B = - (v \cdot A) \wedge B = (v \cdot a^*) \wedge B = (v \wedge a)^* \wedge B$$  \hspace{1cm} \text{(C.14)}$$

$$= - (v \wedge a) I \wedge B = v \wedge (A \wedge B)$$

References

[1] H. Goldstein, Classical Mechanics, Pearson, 2001.

[2] A. J. Pettofrezzo, Matrices and Transformations, Dover Publications, 1978.

[3] D. Atanasiu, P. Mikusinski, A Bridge to linear algebra, World Scientific, 2020.

[4] G. Strang, Introduction to linear algebra, Wellesley-Cambridge Press, 2009.

[5] C. Doran, A. Lasenby, Geometric Algebra for Physicists, Cambridge University Press, 2003. \texttt{doi:10.1017/CBO9780511807497}

[6] A. Macdonald, Linear and Geometric Algebra, CreateSpace Independent Publishing Platform, 2011.

[7] D. Hestenes, Space-Time Algebra, Birkhäuser, 2015. \texttt{doi:10.1007/978-3-319-18413-5}.