A ternary diophantine inequality by primes near to squares

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Abstract

Let $c$ be fixed with $1 < c < 35/34$. In this paper we prove that for every sufficiently large real number $N$ and a small constant $\varepsilon > 0$, the diophantine inequality

$$|p_1^c + p_2^c + p_3^c - N| < \varepsilon$$

is solvable in primes $p_1, p_2, p_3$ near to squares.

Keywords: Diophantine inequality; exponential sum; prime.

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1 Introduction and statement of the result

In 1952 I. I. Piatetski-Shapiro \[12\] investigated the inequality

$$|p_1^c + p_2^c + \cdots + p_r^c - N| < \varepsilon$$

where $c > 1$ is not an integer, $\varepsilon$ is a fixed small positive number, and $p_1, \ldots, p_r$ are primes. He proved the existence of an $H(c)$, depending only on $c$, such that for all sufficiently large real $N$, (1) has a solution for $H(c) \leq r$. He established that

$$\limsup_{c \to \infty} \frac{H(c)}{c \log c} \leq 4$$

and also that $H(c) \leq 5$ if $1 < c < 3/2$.

In 1992 Tolev \[14\] showed that (1) has a solution for $r = 3$ and $1 < c < 15/14$. The interval $1 < c < 15/14$ was subsequently improved by several authors \[2, 3, 4, 5, 6, 9, 10\]. The best result up to now belongs to Cai \[5\] with $1 < c < 43/36$. 
On the other hand in 1991 Tolev [13] solved the diophantine inequality
\[ |\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < \varepsilon \]
in primes \( p_1, p_2, p_3 \) near to squares. Here \( \eta \) is real, the constants \( \lambda_1, \lambda_2, \lambda_3 \) satisfy some necessary conditions and \( \varepsilon > 0 \) is a small constant.

More precisely Tolev proved the following theorem

**Theorem 1.** Suppose that \( \lambda_1, \lambda_2, \lambda_3 \) are non-zero real numbers, not all of the same sign, that \( \eta \) is real, \( \lambda_1/\lambda_2 \) is irrational and \( 0 < \tau < 1/8 \). Then there exist infinitely many triples of primes \( p_1, p_2, p_3 \) such that
\[ |\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{-\tau} \]
and
\[ \|\sqrt{p_1}\|, \|\sqrt{p_2}\|, \|\sqrt{p_3}\| < (\max p_j)^{-\frac{(1-8\tau)}{26}} \log^5(\max p_j) \]
(as usual, \( \|\alpha\| \) denotes the distance from \( \alpha \) to the nearest integer).

**Proof.** See [13]. \( \square \)

Motivated by these results and following the method of Tolev [13] we shall prove the following theorem

**Theorem 2.** Let \( c \) and \( \tau \) be fixed with \( 1 < c < \tau < 35/34 \) and \( \delta > 0 \) be a fixed sufficiently small number. Then for every sufficiently large real number \( N \), the diophantine inequality
\[ |p_1^c + p_2^c + p_3^c - N| < N^{-\frac{1}{2}(\tau-c)} \log N \]
is solvable in primes \( p_1, p_2, p_3 \) such that
\[ \|\sqrt{p_1}\|, \|\sqrt{p_2}\|, \|\sqrt{p_3}\| < N^{-\frac{1}{12}(\frac{35}{34} - \tau)} + \delta. \]

2 **Notations and lemmas**

Let \( N \) be a sufficiently large positive number. By \( \eta \) we denote an arbitrary small positive number, not the same in all appearances. For positive \( A \) and \( B \) we write \( A \asymp B \) instead of \( A \ll B \ll A \). As usual \( \mu(n) \) is Möbius’ function and \( \tau(n) \) denotes the number of positive divisors of \( n \). The letter \( p \) with or without subscript will always denote prime number. We denote by \( \Lambda(n) \) von Mangoldt’s function. Moreover \( e(y) = e^{2\pi i y} \). As usual, \([y]\) denotes the integer part of \( y \). Let \( c \) and \( \tau \) be fixed with \( 1 < c < \tau < 35/34 \). By \( \delta \) we
denote an fixed sufficiently small positive number.

Denote

\[ X = (N/2)^{1/c}; \]  
\[ \varepsilon = X^{c-\tau}; \]  
\[ r = |\log X|; \]  
\[ Y = X^{\Delta^{-1}\left(\frac{\tau}{\pi} - \tau\right)}; \]  
\[ \Delta = Y/5; \]  
\[ M = \Delta^{-1}r; \]  
\[ S(\alpha) = \sum_{X/2 < p \leq X} e(\alpha p^c) \log p; \]  
\[ U(\alpha, m) = \sum_{X/2 < p \leq X} e(\alpha p^c + m\sqrt{p}) \log p. \]

**Lemma 1.** Let \( r \in \mathbb{N} \). There exists a function \( \chi(t) \) which is \( r \)-times continuously differentiable and 1-periodic with a Fourier series of the form

\[
\chi(t) = \frac{9}{5} Y + \sum_{m = -\infty}^{\infty} g(m)e(mt),
\]

where

\[
|g(m)| \leq \min\left(\frac{1}{\pi|m|}, \frac{1}{\pi|m|}\left(\frac{r}{|m|\Delta}\right)^r\right)
\]

and

\[
\chi(t) = \begin{cases} 
1 & \text{if } ||t|| \leq Y - \Delta, \\
0 & \text{if } ||t|| \geq Y, \\
\text{between } 0 \text{ and } 1 \text{ for the other } t. 
\end{cases}
\]

**Proof.** See ([8], p. 14).

We also denote

\[
H(\alpha) = \sum_{X/2 < p \leq X} \chi(\sqrt{p})e(\alpha p^c) \log p; 
\]

\[
V(\alpha) = \sum_{m = -\infty}^{\infty} g(m)U(\alpha, m). 
\]
Further we need the function $A(x)$ used by Baker and Harman \([1]\). It is continuous and integrable on the real line such that

$$A(x) \leq \chi_{[-1,1]}(x). \quad (15)$$

Further, if we write

$$\hat{A}(\alpha) = \int_{-\infty}^{\infty} A(x)e(-\alpha x)dx,$$

then

$$\hat{A}(\alpha) = 0 \quad \text{for} \quad |\alpha| \geq \mu,$$

where $\mu$ is a constant. Therefore if

$$P = \frac{\mu}{\varepsilon}, \quad (16)$$

then

$$\hat{A}(\varepsilon \alpha) = 0 \quad \text{for} \quad |\alpha| \geq P. \quad (17)$$

**Lemma 2.** Let $1 < c < 15/14$. Then

$$\int_{-\infty}^{\infty} S^3(\alpha)e(-N\alpha)\hat{A}(\varepsilon \alpha) d\alpha \gg X^{3-c}. \quad (18)$$

**Proof.** Arguing as in \([1]\) and \([14]\) we obtain the lower bound \((18)\). \(\square\)

**Lemma 3.** (Van der Corput) Let $k \geq 2$, $K = 2^{k-1}$ and $f(x)$ be a real-valued function with $k$ continuous derivatives in $[a,b]$ such that

$$|f^{(k)}(x)| \asymp \lambda, \quad \text{uniformly in} \quad x \in [a,b].$$

Then

$$\left| \sum_{a < n \leq b} e(f(n)) \right| \ll (b-a)^{\lambda^{1/2}} \left(1 - \frac{Q}{P} \right)^{-\lambda^{1/2}} + (b-a)^{1/2} \lambda^{-1/2}. \quad (19)$$

**Proof.** See \((8), \text{Ch. 1, Th. 5}\). \(\square\)

**Lemma 4.** For any complex numbers $a(n)$ we have

$$\left| \sum_{a < n \leq b} a(n) \right|^2 \leq \left(1 + \frac{b-a}{Q} \right) \sum_{|q| \leq Q} \left(1 - \frac{|q|}{Q} \right) \sum_{a < n, n+q \leq b} a(n+q)a(n),$$

where $Q$ is any positive integer.
Proof. See (7, Lemma 8.17).

Lemma 5. For the sum denoted by (8) we have

\[ \int_{-P}^{P} |S(\alpha)|^2 d\alpha \ll PX \log^3 X. \]

Proof. See (14, Lemma 7).

Lemma 6. For the sum denoted by (14) we have

\[ \int_{-P}^{P} |V(\alpha)|^2 d\alpha \ll PX \log^5 X. \]

Proof. On the one hand

\[ \int_{-P}^{P} |V(\alpha)|^2 d\alpha \ll P \int_{0}^{1} |V(\alpha)|^2 d\alpha. \]

On the other hand arguing as in (13, Lemma 5), (14, Lemma 7) and using (4), (6), (7), (11) we obtain

\[ \int_{0}^{1} |V(\alpha)|^2 d\alpha = \sum_{|m_1|, |m_2| > 0} g(m_1)g(m_2) \]
\[ \times \sum_{X/2 < p_1, p_2 \leq X} e(m_1 \sqrt{p_1} - m_2 \sqrt{p_2}) \log p_1 \log p_2 \int_{0}^{1} \alpha(p_1^c - p_2^c) d\alpha \]
\[ \ll \sum_{|m_1|, |m_2| > 0} |g(m_1)| |g(m_2)| \sum_{X/2 < p_1, p_2 \leq X} \log p_1 \log p_2 \int_{0}^{1} \alpha(p_1^c - p_2^c) d\alpha \]
\[ \ll X \log^3 X \sum_{|m_1|, |m_2| > 0} |g(m_1)| |g(m_2)| \]
\[ = X \log^3 X \left( \sum_{|m| > 0} |g(m)|^2 + \sum_{|m_1|, |m_2| < M} |g(m_1)| |g(m_2)| \right. \]
\[ + \sum_{0 < m_1 \leq M, |m_2| > M} |g(m_1)| |g(m_2)| + \sum_{|m_1|, |m_2| > M} |g(m_1)| |g(m_2)| \)

\[ = X \log^3 X \left( \sum_{|m| > 0} |g(m)|^2 + \sum_{|m_1|, |m_2| < M} |g(m_1)| |g(m_2)| \right. \]
\[ + \sum_{0 < m_1 \leq M, |m_2| > M} |g(m_1)| |g(m_2)| + \sum_{|m_1|, |m_2| > M} |g(m_1)| |g(m_2)| \)

\[ + \sum_{0 < m_1 \leq M, |m_2| > M} |g(m_1)| |g(m_2)| + \sum_{|m_1|, |m_2| > M} |g(m_1)| |g(m_2)| \)
\[ \ll X \log^3 X \left( \sum_{|m| > 0} \frac{1}{m^2} + \sum_{0 < |m_1|, |m_2| < M} \frac{1}{|m_1| \cdot |m_2|} \right) \\
\quad + \sum_{0 < |m_1| \leq M, |m_2| > M} \frac{1}{|m_1|} |g(m_2)| + \sum_{|m_1|, |m_2| > M} |g(m_1)| \cdot |g(m_2)| \right) \ll X \log^3 X \left( \log^2 X + \left( \frac{r}{\pi M \Delta} \right)^r \log X + \left( \frac{r}{\pi M \Delta} \right)^{2r} \right) \\
\ll X \log^3 X \left( \log^2 X + \frac{\log X}{X} + \frac{1}{X^2} \right) \ll X \log^5 X. \tag{20} \]

From (19) and (20) it follows the assertion in the lemma. \[ \square \]

**Lemma 7.** For the sum denoted by (14) the upper bound

\[ \max_{|\alpha| \leq P} |V(\alpha)| \ll \left( M^{1/2} X^{7/12} + M^{1/6} X^{3/4} + X^{11/12} + P^{1/16} X^{2c+29} \right. \]
\[ \quad + P^{-3/16} M^{1/4} X^{\frac{33c-66}{32}} + P^{-1/16} M^{1/12} X^{\frac{31c-26}{32}} \left. \right) X^\eta \] \tag{21}

holds.

**Proof.** Bearing in mind (4), (6), (7), (9), (11) and (14) we write

\[ |V(\alpha)| \ll \sum_{0 < |m| \leq M} \frac{1}{|m|} |U(\alpha, m)| + X \sum_{|m| > M} |g(m)| \]
\[ \ll \sum_{0 < |m| \leq M} \frac{1}{|m|} |U(\alpha, m)| + \left( \frac{r}{\pi M \Delta} \right)^r X \]
\[ \ll \sum_{0 < |m| \leq M} \frac{1}{|m|} |U(\alpha, m)| + 1. \tag{22} \]

In order to prove the lemma we have to find the upper bound of the sum \( U(\alpha, m) \) denoted by (9). Our argument is a modification of Petrov’s and Tolev’s \[ \text{[11]} \] argument.

Assume that \( m > 0 \). For \( m < 0 \) the proof is analogous.

We denote

\[ \psi(t) = \alpha t^c + m \sqrt{t}. \tag{23} \]
\[ f(d, l) = \psi(dl) = \alpha (dl)^c + m \sqrt{dl}. \tag{24} \]
It is clear that
\[ U(\alpha, m) = \sum_{X/2 < n \leq X} \Lambda(n)e(\alpha n^c + m\sqrt{n}) + O(X^{1/2}). \]

Using Vaughan’s identity (see [15]) we get
\[ U(\alpha, m) = U_1 - U_2 - U_3 - U_4 + O(X^{1/2}), \tag{25} \]
where
\[ U_1 = \sum_{d \leq X^{1/3}} \mu(d) \sum_{X/2d < l \leq X/d} (\log l)e(f(d, l)), \tag{26} \]
\[ U_2 = \sum_{d \leq X^{1/3}} c(d) \sum_{X/2d < l \leq X/d} e(f(d, l)), \tag{27} \]
\[ U_3 = \sum_{X^{1/3} < d \leq X^{2/3}} c(d) \sum_{X/2d < l \leq X/d} e(f(d, l)), \tag{28} \]
\[ U_4 = \sum_{X/2 < dl \leq X} \sum_{d > X^{1/3}, l > X^{1/3}} a(d)\Lambda(l)e(f(d, l)), \tag{29} \]
and where
\[ |c(d)| \leq \log d, \quad |a(d)| \leq \tau(d). \tag{30} \]

**Estimation of \( U_1 \) and \( U_2 \)**

Consider first \( U_2 \) defined by (27). Bearing in mind (24) we find
\[ f''_{ll}(d, l) = \gamma_1 - \gamma_2, \tag{31} \]
where
\[ \gamma_1 = d^2\alpha c(c - 1)(dl)^{c-2}, \quad \gamma_2 = \frac{1}{4}md^2(dl)^{-3/2}. \tag{32} \]
From (32) and the restriction
\[ X/2 < dl \leq X \tag{33} \]
we obtain
\[ |\gamma_1| \asymp |\alpha|d^2X^{c-2}, \quad |\gamma_2| \asymp md^2X^{-3/2}. \tag{34} \]

On the one hand from (31) and (34) we conclude that there exists sufficiently small constant \( h_0 > 0 \) such that if \( |\alpha| \leq h_0mX^{1/2-c}, \) then \( |f''_{ll}(d, l)| \asymp md^2X^{-3/2}. \)

On the other hand from (31) and (34) it follows that there exists sufficiently large constant \( H_0 > 0 \) such that if \( |\alpha| \geq H_0mX^{1/2-c}, \) then \( |f''_{ll}(d, l)| \asymp |\alpha|d^2X^{c-2}. \)
Consider several cases.

**Case 1a.**

\[ H_0mX^{1/2-c} \leq |\alpha| \leq P. \tag{35} \]

We remind that in this case \( |f''_l(d,l)| \asymp |\alpha|d^2X^{c-2} \) and using Lemma 3 for \( k = 2 \) we get

\[
\sum_{X/2d < l \leq X/d} e(f(d,l)) \ll \frac{X}{d} \left( |\alpha|d^2X^{c-2}\right)^{1/2} \left( |\alpha|d^2X^{c-2}\right)^{-1/2} \\
= |\alpha|^{1/2}X^{c/2} + |\alpha|^{-1/2}d^{-1}X^{1-c/2}. \tag{36}
\]

From (27), (30), (35) and (36) it follows

\[ U_2 \ll \left( P^{1/2}X^{3c+2/6} + m^{-1/2}X^{3/4}\right) \log^2 X. \tag{37} \]

**Case 2a.**

\[ h_0mX^{1/2-c} < \alpha < H_0mX^{1/2-c}. \tag{38} \]

By (24) we find

\[ f'''_{ll}(d,l) = d^3\alpha c(c-1)(c-2)(dl)^c - 3/8d^3m(dl)^{-5/2}. \tag{39} \]

The formulas (31), (32) and (39) give us

\[(c-2)f''_l(d,l) - lf'''_{ll}(d,l) = \frac{1-2c}{8}d^2(dl)^{-3/2}m. \tag{40} \]

From (33) and (40) we obtain

\[ |(c-2)f''_l(d,l) - lf'''_{ll}(d,l)| \asymp md^2X^{-3/2}. \]

The above implies that there exists \( \alpha_0 > 0 \), such that for every \( l \in (X/2d, X/d) \) at least one of the following inequalities is fulfilled:

\[ |f''_l(d,l)| \geq \alpha_0md^2X^{-3/2}. \tag{41} \]

\[ |f'''_{ll}(d,l)| \geq \alpha_0md^3X^{-5/2}. \tag{42} \]

Let us consider the equation

\[ f'''_{ll}(d,l) = 0. \tag{43} \]

From (39) it is tantamount to

\[ 3m(dl)^{1/2-c} - 8\alpha c(c-1)(c-2) = 0. \tag{44} \]
It is easy to see that the equation (44) has at most 1 solution $Z \in (X^{1/2-c}, (X/2)^{1/2-c}]$. Consequently the equation (43) has at most 1 solution in real numbers $l \in (X/2d, X/d]$. According to Rolle’s Theorem if $C$ does not depend on $l$ then the equation $f''_u(d, l) = C$ has at most 2 solution in real numbers $l \in (X/2d, X/d]$. Therefore the equation $|f''_u(d, l)| = \alpha_0md^2X^{-3/2}$ has at most 4 solution in real numbers $l \in (X/2d, X/d]$. From these consideration it follows that the interval $(X/2d, X/d]$ can be divided into at most 5 intervals such that if $J$ is one of them, then at least one of the following assertions holds:

The inequality (44) is fulfilled for all $l \in J$. \hspace{1cm} (45)

The inequality (42) is fulfilled for all $l \in J$. \hspace{1cm} (46)

On the other hand from (31), (33), (34), (38) and (39) we get

\[ |f''_u(d, l)| \ll md^2X^{-3/2}, \quad |f'''_u(d, l)| \ll md^3X^{-5/2}. \] \hspace{1cm} (47)

Bearing in mind (45) – (47) we conclude that the interval $(X/2d, X/d]$ can be divided into at most 5 intervals such that if $J$ is one of them, then at least one of the following statements is fulfilled:

\[ |f''_u(d, l)| \asymp md^2X^{-3/2} \quad \text{uniformly for} \quad l \in J. \] \hspace{1cm} (48)

\[ |f'''_u(d, l)| \asymp md^3X^{-5/2} \quad \text{uniformly for} \quad l \in J. \] \hspace{1cm} (49)

If (48) holds, then we use Lemma 3 for $k = 2$ and obtain

\[ \sum_{l \in J} e(f(d, l)) \ll \frac{X}{d} \left(\frac{md^2X^{-3/2}}{X^{1/4}} + \left(\frac{md^2X^{-3/2}}{X^{1/4}}\right)^{-1/2}\right)^{1/2} + \left(\frac{md^2X^{-3/2}}{X^{1/4}}\right)^{-1/2} \ll m^{1/2}d^{-1/2}X^{7/12} + m^{-1/2}d^{-1}X^{11/12}. \] \hspace{1cm} (50)

If (49) is fulfilled, then we use Lemma 3 for $k = 3$ and find

\[ \sum_{l \in J} e(f(d, l)) \ll \frac{X}{d} \left(m^3d^3X^{-5/2} \right)^{1/6} + \left(\frac{X}{d}\right)^{1/2} \left(m^3d^3X^{-5/2}\right)^{-1/6} \ll m^{1/6}d^{1/2}X^{7/12} + m^{-1/6}d^{-1}X^{11/12}. \] \hspace{1cm} (51)

From (50) and (51) it follows

\[ \sum_{X/2d < l \leq X/d} e(f(d, l)) \ll m^{1/2}d^{-1/2}X^{7/12} + m^{-1/2}d^{-1}X^{11/12} \] \hspace{1cm} (52)
Bearing in mind (27) and (52) we get

\[ U_2 \ll \left( m^{1/2} X^{7/12} + m^{1/6} X^{3/4} + m^{-1/6} X^{11/12} \right) \log^2 X. \] (53)

**Case 3a.**

\[ |\alpha| \leq h_0 m X^{1/2-c}. \] (54)

We recall that in this case \( |f''_l(d, l)| \approx m d^2 X^{-3/2} \) and using Lemma 3 for \( k = 2 \) we obtain

\[ \sum_{X/2d < l \leq X/d} e(f(d, l)) \ll m^{1/2} X^{1/4} + m^{-1/2} d^{-1} X^{3/4}. \] (55)

Using (27) and (55) we find

\[ U_2 \ll \left( m^{1/2} X^{7/12} + m^{-1/2} X^{3/4} \right) \log^2 X. \] (56)

**Case 4a.**

\[ -H_0 m X^{1/2-c} < \alpha < -h_0 m X^{1/2-c}. \] (57)

In this case again \( |f''_l(d, l)| \approx m d^2 X^{-3/2} \). Consequently

\[ U_2 \ll \left( m^{1/2} X^{7/12} + m^{-1/2} X^{3/4} \right) \log^2 X. \] (58)

From (37), (53), (56) and (58) it follows

\[ U_2 \ll \left( m^{1/2} X^{7/12} + m^{1/6} X^{3/4} + m^{-1/6} X^{11/12} + P^{1/2} X^{3k+2} \right) \log^2 X. \] (59)

In order to estimate \( U_1 \) defined by (26) we apply Abel’s transformation. Then arguing as in the estimation of \( U_2 \) we get

\[ U_1 \ll \left( m^{1/2} X^{7/12} + m^{1/6} X^{3/4} + m^{-1/6} X^{11/12} + P^{1/2} X^{3k+2} \right) \log^2 X. \] (60)

**Estimation of \( U_3 \) and \( U_4 \)**

Consider first \( U_4 \) defined by (29). We have

\[ U_4 \ll |U_5| \log X, \] (61)

where

\[ U_5 = \sum_{L<d\leq2L} b(l) \sum_{D<d\leq2D} a(d) e(f(d, l)) \] (62)

and where

\[ a(d) \ll X^n, \quad b(l) \ll X^n, \quad X^{1/3} \ll D \ll X^{1/2} \ll L \ll X^{2/3}, \quad DL \asymp X. \] (63)
Using (62), (63) and Cauchy’s inequality we obtain

\[ |U_5|^2 \ll X^\eta L \sum_{L < d \leq 2L} \left| \sum_{D_1 < d \leq D_2} a(d) e(f(d, l)) \right|^2, \quad (64) \]

where

\[ D_1 = \max \left\{ D, X \frac{L}{2l} \right\}, \quad D_2 = \min \left\{ \frac{X}{l}, 2D \right\}. \quad (65) \]

Now from (63) – (65) and Lemma 4 with \( Q \) such that \( Q \leq D \) we find

\[ |U_5|^2 \ll X^\eta \sum_{L < d \leq 2L} \frac{D}{Q} \sum_{|q| \leq Q} \left( 1 - \frac{|q|}{Q} \right) \left| \sum_{D_1 < d \leq D_2} \sum_{D_1 < d + q \leq D_2} a(d + q) \overline{a(d)} e(f(d + q, l) - f(d, l)) \right| \]

\[ \ll \left( \frac{(LD)^2}{Q} + \frac{LD}{Q} \sum_{0 < |q| \leq Q} \sum_{D_1 < d \leq D_2} \sum_{D_1 < d + q \leq D_2} e(g_{d,q}(l)) \right) X^\eta, \quad (67) \]

where

\[ L_1 = \max \left\{ L, \frac{X}{2d}, \frac{X}{2d + q} \right\}, \quad L_2 = \min \left\{ 2L, \frac{X}{d}, \frac{X}{d + q} \right\} \quad (68) \]

and

\[ g(l) = g_{d,q}(l) = f(d + q, l) - f(d, l). \quad (69) \]

It is not hard to see that the sum over negative \( q \) in formula (67) is equal to the sum over positive \( q \). Thus

\[ |U_5|^2 \ll \left( \frac{(LD)^2}{Q} + \frac{LD}{Q} \sum_{1 \leq q \leq Q} \sum_{D_1 < d \leq D - q} \sum_{L_1 < l \leq L_2} e(g_{d,q}(l)) \right) X^\eta. \quad (70) \]

Consider the function \( g(l) \). From (23), (24) and (69) it follows

\[ g(l) = \int_d^{d+q} f'(l, t) \, dt = \int_d^{d+q} l \psi'(tl) \, dt. \]

Hence

\[ g''(l) = \int_d^{d+q} 2t \psi''(tl) + lt^2 \psi'''(tl) \, dt. \quad (71) \]
Bearing in mind (23) and (71) we obtain

\[ g''(l) = \int_{d}^{d+q} \left( \Psi_1(t,l) - \Psi_2(t,l) \right) dt, \]  

(72)

where

\[ \Psi_1(t,l) = \alpha c^2 (c-1) t^{c-1} l^{c-2}, \quad \Psi_2(t,l) = \frac{m}{8} t^{-1/2} l^{-3/2}. \]  

(73)

If \( t \in [d, d+q] \), then

\[ tl \approx X. \]  

(74)

From (73) and (74) we get

\[ |\Psi_1(t,l)| \approx |\alpha| d^2 X^{c-2}, \quad |\Psi_2(t,l)| \approx m d X^{-3/2}. \]  

(75)

On the one hand from (72) and (75) we conclude that there exists sufficiently small constant \( h_1 > 0 \) such that if \( |\alpha| \leq h_1 m X^{1/2-c} \), then \( |g''(l)| \approx q |\alpha| d X^{c-2} \).

On the other hand from (72) and (75) it follows that there exists sufficiently large constant \( H_1 > 0 \) such that if \( |\alpha| \geq H_1 m X^{1/2-c} \), then \( |g''(l)| \approx q |\alpha| d X^{c-2} \).

Consider several cases.

**Case 1b.**

\[ H_1 m X^{1/2-c} \leq |\alpha| \leq P. \]  

(76)

We recall that the constant \( H_1 \) is chosen in such a way, that if \( |\alpha| \geq H_1 m X^{1/2-c} \), then uniformly for \( l \in (L_1, L_2) \) we have \( |g''(l)| \approx q |\alpha| d X^{c-2} \). Using (63), (68) and applying Lemma 3 for \( k = 2 \) we find

\[ \sum_{L_1 < l \leq L_2} e(g(l)) \ll L (q |\alpha| d X^{c-2})^{1/2} + (q |\alpha| d X^{c-2})^{-1/2} \]

\[ = L q^{1/2} |\alpha|^{1/2} d^{1/2} X^{c/2-1} + q^{-1/2} |\alpha|^{-1/2} d^{-1/2} X^{1-c/2}. \]  

(77)

From (63), (70), (76) and (77) it follows

\[ U_5 \ll (X Q^{-1/2} + P^{1/4} X^{2k+5} Q^{1/4} + m^{-1/4} X Q^{-1/4}) X^\eta. \]  

(78)

**Case 2b.**

\[ h_1 m X^{1/2-c} < |\alpha| < H_1 m X^{1/2-c}. \]  

(79)

The formulas (72) and (73) give us

\[ g''(l) = \int_{d}^{d+q} \left( \Phi_1(t,l) + \Phi_2(t,l) \right) dt, \]  

(80)
where
\[ \Phi_1(t, l) = \alpha c^2(c - 1)(c - 2)t^{c-1}l^{-3}, \quad \Phi_2(t, l) = \frac{3m}{16} t^{-1/2}l^{-5/2}. \] (81)

From (72), (73), (80) and (81) it follows
\[ (c - 2)g''(l) - lg'''(l) = \frac{7 - 2c}{16} m \int l(t) t^{3/2} dt. \] (82)

Using (74) and (82) we obtain
\[ |(c - 2)g''(l) - lg'''(l)| \approx qmdX^{-3/2}. \]

Consequently there exists \( \alpha_1 > 0 \), such that for every \( l \in (L_1, L_2] \) at least one of the following inequalities holds:
\[ |g''(l)| \geq \alpha_1 qmdX^{-3/2}. \] (83)
\[ |g'''(l)| \geq \alpha_1 qmd^2X^{-5/2}. \] (84)

Consider the equation
\[ g'''(l) = 0. \] (85)

From (80) and (81) we get
\[ \alpha c(c - 1)(c - 2)[(d + q)c - d^c]t^{c-3} - \frac{3m}{8}[(d + q)^{1/2} - d^{1/2}]l^{-5/2} = 0 \] (86)
which is equivalent to
\[ tc^{1/2} = \frac{3m[(d + q)^{1/2} - d^{1/2}]}{8\alpha c(c - 1)(c - 2)[(d + q)c - d^c]}. \] (87)

It is not hard to see that the equation (87) has at most 1 solution \( Z \in (L_1^{c-1/2}, L_2^{c-1/2}] \).

Therefore the equation (85) has at most 1 solution in real numbers \( l \in (L_1, L_2] \). According to Rolle’s Theorem if \( C \) does not depend on \( l \) then the equation \( g''(l) = C \) has at most 2 solution in real numbers \( l \in (L_1, L_2] \). Therefore the equation \( |g''(l)| = \alpha_1 qmd^2X^{-3/2} \) has at most 4 solution in real numbers \( l \in (L_1, L_2] \). From these consideration it follows that the interval \( (L_1, L_2] \) can be divided into at most 5 intervals such that if \( J \) is one of them, then at least one of the following statements holds:

The inequality (83) is fulfilled for all \( l \in J \). (88)

The inequality (84) is fulfilled for all \( l \in J \). (89)
Using (72), (74), (75), (79), (80) and (81) we find
\[ |g''(l)| \ll qmdX^{-3/2}, \quad |g''(l)| \ll qmd^2X^{-5/2}. \tag{90} \]

From (88) – (90) it follows that the interval \((L_1, L_2]\) can be divided into at most 5 intervals such that if \(J\) is one of them, then at least one of the following assertions is fulfilled:
\[ |g''(l)| \asymp qmdX^{-3/2} \quad \text{uniformly for} \quad l \in J. \tag{91} \]
\[ |g''(l)| \asymp qmd^2X^{-5/2} \quad \text{uniformly for} \quad l \in J. \tag{92} \]

If (91) is fulfilled, then we use Lemma 3 for \(k = 2\) and get
\[
\sum_{l \in J} e(g(l)) \ll L(qmdX^{-3/2})^{1/2} + (qmdX^{-3/2})^{-1/2} \\
= Lq^{1/2}m^{1/2}d^{1/2}X^{-3/4} + q^{-1/2}m^{-1/2}d^{-1/2}X^{3/4}. \tag{93} \]

If (92) holds, then we use Lemma 3 for \(k = 3\) and obtain
\[
\sum_{l \in J} e(g(l)) \ll L(qmd^2X^{-5/2})^{1/6} + L^{1/2}(qmd^2X^{-5/2})^{-1/6} \\
= Lq^{1/6}m^{1/6}d^{1/3}X^{-5/12} + L^{1/2}q^{-1/6}m^{-1/6}d^{-1/3}X^{5/12}. \tag{94} \]

From (93) and (94) it follows
\[
\sum_{L_1 < l \leq L_2} e(g(l)) \ll Lq^{1/2}m^{1/2}d^{1/2}X^{-3/4} + q^{-1/2}m^{-1/2}d^{-1/2}X^{3/4} \\
+ Lq^{1/6}m^{1/6}d^{1/3}X^{-5/12} + L^{1/2}q^{-1/6}m^{-1/6}d^{-1/3}X^{5/12}. \tag{95} \]

Taking into account (63), (70) and (95) we find
\[
U_5 \ll (XQ^{-1/2} + m^{1/4}X^{3/4}Q^{1/4} + m^{-1/4}XQ^{-1/4} \\
+ m^{1/12}X^{7/8}Q^{1/12} + m^{-1/12}XQ^{-1/12})X^\eta. \tag{96} \]

**Case 3b.**
\[ |\alpha| \leq h_1mX^{1/2-c}. \tag{97} \]

We have chosen the constant \(h_1\) in such a way, that from (72), (74), (75) and (97) it follows that \(|g''(l)| \asymp qmdX^{-3/2}\) uniformly for \(l \in (L_1, L_2].\) Applying Lemma 3 for \(k = 2\) we get
\[
\sum_{L_1 < l \leq L_2} e(g(l)) \ll Lq^{1/2}m^{1/2}d^{1/2}X^{-3/4} + q^{-1/2}m^{-1/2}d^{-1/2}X^{3/4}. \tag{98} \]
From (70) and (98) we obtain
\[ U_5 \ll (XQ^{-1/2} + m^{1/4}X^{3/4}Q^{1/4} + m^{-1/4}XQ^{-1/4})X^\eta. \]  
(99)

**Case 4b.**
\[-H_1mX^{1/2-c} < \alpha < -h_1mX^{1/2-c}.\]  
(100)

In this case \(|g''(l)| \asymp qmdX^{-3/2}|. Arguing in a similar way we find
\[ U_5 \ll (XQ^{-1/2} + m^{1/4}X^{3/4}Q^{1/4} + m^{-1/4}XQ^{-1/4})X^\eta. \]  
(101)

From (61), (78), (96), (99) and (101) we get
\[ U_4 \ll \left( XQ^{-1/2} + P^{1/4}X^{2c+5/8}Q^{1/4} + m^{1/4}X^{3/4}Q^{1/4} + m^{-1/4}XQ^{-1/4} 
+ m^{1/12}X^{7/8}Q^{1/12} + m^{-1/12}XQ^{-1/12} \right)X^\eta. \]  
(102)

Arguing as in the estimation of \(U_4\) we obtain
\[ U_3 \ll \left( XQ^{-1/2} + P^{1/4}X^{2c+5/8}Q^{1/4} + m^{1/4}X^{3/4}Q^{1/4} + m^{-1/4}XQ^{-1/4} 
+ m^{1/12}X^{7/8}Q^{1/12} + m^{-1/12}XQ^{-1/12} \right)X^\eta. \]  
(103)

Summarizing (25), (59), (60), (102) and (103) we conclude that for \(|\alpha| \leq P\) and any integer \(m \neq 0\) the estimation
\[ |U(\alpha, m)| \ll \left( m^{1/2}X^{7/12} + m^{1/6}X^{3/4} + m^{-1/6}X^{11/12} + XQ^{-1/2} 
+ P^{1/4}X^{2c+5/8}Q^{1/4} + m^{1/4}X^{3/4}Q^{1/4} + m^{-1/4}XQ^{-1/4} 
+ m^{1/12}X^{7/8}Q^{1/12} + m^{-1/12}XQ^{-1/12} \right)X^\eta \]  
(104)

holds.

We substitute the expression (104) for \(U(\alpha, m)\) in (22) and find
\[ \max_{|\alpha| \leq P} |V(\alpha)| \ll \left( M^{1/2}X^{7/12} + M^{1/6}X^{3/4} + X^{11/12} + XQ^{-1/2} 
+ P^{1/4}X^{2c+5/8}Q^{1/4} + M^{1/4}X^{3/4}Q^{1/4} + XQ^{-1/4} 
+ M^{1/12}X^{7/8}Q^{1/12} + XQ^{-1/12} \right)X^\eta. \]  
(105)

We choose
\[ Q = \left[ P^{3/4}X^{\frac{9-6c}{8}} \right]. \]  
(106)

The direct verification assures us that the condition (66) is fulfilled.

Bearing in mind (105) and (106) we obtain the estimation (21).
3 Proof of the Theorem

Consider the sum
\[
\Gamma(X) = \sum_{X/2 < p_1, p_2, p_3 \leq X \atop |p_1^c + p_2^c + p_3^c - N| < \varepsilon} \prod_{j=1}^{3} \chi(\sqrt{p_j}) \log p_j \int_{-\infty}^{\infty} e((p_1^c + p_2^c + p_3^c - N)\alpha) \hat{A}(\varepsilon \alpha) \, d\alpha.
\] (107)

The theorem will be proved if we show that \( \Gamma(X) \to \infty \) as \( X \to \infty \).

Consider the integrals
\[
I_1 = \int_{-\infty}^{\infty} H^3(\alpha) e(-N\alpha) \hat{A}(\varepsilon \alpha) \, d\alpha
\] (108)
\[
I = \int_{-\infty}^{\infty} S^3(\alpha) e(-N\alpha) \hat{A}(\varepsilon \alpha) \, d\alpha.
\] (109)

On the one hand from (12), (13), (15), (107) and (108) it follows
\[
I_1 = \sum_{X/2 < p_1, p_2, p_3 \leq X} \prod_{j=1}^{3} \chi(\sqrt{p_j}) \log p_j \int_{-\infty}^{\infty} e((p_1^c + p_2^c + p_3^c - N)\alpha) \hat{A}(\varepsilon \alpha) \, d\alpha
\]
\[
= \sum_{X/2 < p_1, p_2, p_3 \leq X} \prod_{j=1}^{3} \chi(\sqrt{p_j}) (\log p_j) \varepsilon^{-1} A((p_1^c + p_2^c + p_3^c - N)\varepsilon^{-1}) \leq \varepsilon^{-1} \Gamma(X).
\] (110)

On the other hand (8), (10), (13), (14), (17), (108) and (109) give us
\[
I_1 = \int_{-\infty}^{\infty} \left( \frac{9}{5} Y S(\alpha) + V(\alpha) \right)^3 e(-N\alpha) \hat{A}(\varepsilon \alpha) \, d\alpha
\]
\[
= \left( \frac{9}{5} Y \right)^3 I + \mathcal{O} \left( Y^2 \int_{-P}^{P} |S^2(\alpha)V(\alpha)| \, d\alpha \right)
\]
\[
+ \mathcal{O} \left( Y \int_{-P}^{P} |S(\alpha)V^2(\alpha)| \, d\alpha \right) + \mathcal{O} \left( \int_{-P}^{P} |V^3(\alpha)| \, d\alpha \right).
\] (111)

We write
\[
\int_{-P}^{P} |S^2(\alpha)V(\alpha)| \, d\alpha \ll \max_{|\alpha| \leq P} |V(\alpha)| \int_{-P}^{P} |S(\alpha)|^2 \, d\alpha.
\] (112)
Applying Cauchy’s inequality we get

\[
\int_{-P}^{P} |S(\alpha)V^2(\alpha)| \, d\alpha \ll \max_{|\alpha| \leq P} |V(\alpha)| \left( \int_{-P}^{P} |S(\alpha)|^2 \, d\alpha \right)^{1/2} \left( \int_{-P}^{P} |V(\alpha)|^2 \, d\alpha \right)^{1/2}.
\]  

(113)

Similarly

\[
\int_{-P}^{P} |V(\alpha)|^3 \, d\alpha \ll \max_{|\alpha| \leq P} |V(\alpha)| \int_{-P}^{P} |V(\alpha)|^2 \, d\alpha.
\]  

(114)

Using Lemmas 5 6 7 and (111) – (114) we obtain

\[
I_1 = \left( \frac{9}{5} Y \right)^3 I + O \left( \left( PM^{1/2} X^{19/12} + PM^{1/6} X^{7/4} + PX^{23/12} + P^{17/16} X^{2c+61/32} \right) + P^{13/16} M^{1/4} X^{65-6\epsilon} + P^{15/16} M^{1/12} X^{63-2\epsilon} \right) X^{\eta}.
\]  

(115)

From (3), (5), (6), (7), (16), (109), (115), Lemma 2 and choosing \( \eta < \delta \) we find

\[
I_1 \gg Y^3 X^{3-c}.
\]  

(116)

Finally (110) and (116) give us

\[
\Gamma(X) \gg \varepsilon Y^3 X^{3-c}.
\]  

(117)

Bearing in mind (3), (5) and (117) we establish that \( \Gamma(X) \to \infty \) as \( X \to \infty \).

The proof of the Theorem 2 is complete.

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