Unity Mach number axial dispersion model for heat exchanger design

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Abstract. Recently a new axial dispersion model for the thermal design of heat exchangers with deviations from plug flow has been developed by considering the limiting case of unity dispersive Mach number of the hyperbolic dispersion model. The main advantage of the new dispersion model is that known and proved design charts and formulae for plug flow can further be applied. Only the numbers of transfer units have to be corrected. The corrections are simple functions of the dispersive Peclet numbers of the hot and the cold flow stream. In this paper a method for the determination of the dispersive Peclet numbers by transient tracer experiments is presented. The experimental inlet signals and outlet responses are evaluated in the frequency domain using Laplace transforms. Together with the analytical solution of the model equations the dispersive Peclet numbers are obtained. For comparison the evaluation procedure in the frequency domain is also applied to the parabolic dispersion model and the model of stirred tanks in series.

1. Introduction

Commonly idealized plug flow is assumed for the thermal design of heat exchangers [1-3]. But the real flow fields of heat exchangers in operation will always deviate more or less from idealized plug flow. The deviations from idealized plug flow, i.e. backmixing, backflow and maldistribution, reduce the mean temperature difference of the fluids and therefore the efficiency of the heat exchanger. Objective of this work is to take into account this negative effect by a simple model for an improved thermal design of heat exchangers.

Basis of the new unity Mach number dispersion model [4] is the dispersion model in its general form [5]. Equations (1) and (2) are the energy balance equations for steady state processes in counterflow heat exchangers. Deviations from idealized plug flow are taken into account by axial dispersive energy fluxes. They appear in the second addends of both energy equations (1) and (2) as reduced axial dispersive energy fluxes \( \varphi = \dot{q}_iL/\dot{x} \) with the dimensions of temperatures.

\[
\frac{dT_1}{dx} + \frac{1}{Pe_1} \frac{d\varphi_1}{dx} + NTU_1(t_1-t_2) = 0
\] (1)
\[
\frac{d t_2}{d x} - \frac{1}{P e_2} \frac{d \varphi_2}{d x} + N T U_2 (t_1 - t_2) = 0
\]

(2)

For the axial dispersive energy fluxes an empirical approach, equations (3) and (4), has been developed on the basis of the hyperbolic heat conduction law [6, 7].

\[
\varphi_1 - \frac{M_1}{P e_1} \frac{d \varphi_1}{d x} = - \frac{d t_1}{d x} 
\]

(3)

\[
\varphi_2 - \frac{M_2}{P e_2} \frac{d \varphi_2}{d x} = - \frac{d t_2}{d x} 
\]

(4)

The general hyperbolic dispersion model contains four parameters \(M_1\), \(M_2\), \(P e_1\) and \(P e_2\), i.e., two dispersive Peclet numbers and two dispersive thermal Mach numbers [8 – 10]. The dispersive Peclet number is given by

\[
P e = \frac{w L \rho c_p}{\lambda^*}.
\]

(5)

The property \(\lambda^*\) is an apparent thermal conductivity which is composed of the molecular conductivity and the convective dispersive contribution due to backflow, backmixing and maldistribution. The molecular part is a fluid property and the convective one a flow property. Normally the molecular contribution is negligible.

The dispersive thermal Mach number is given by

\[
M = \frac{w}{C}.
\]

(6)

For pure convective phenomena of backflow, backmixing and maldistribution it can be expected that the propagation velocity of thermal disturbances \(C\) is in the order of the mean flow velocity \(w\). For pure molecular axial heat conduction according to Fourier’s law the propagation velocity of thermal disturbances is infinite and \(M = 0\). Table 1 shows the appropriate regions of application of the different dispersion models with respect to the dispersive thermal Mach number, although all models can be applied to arbitrary deviations from plug flow depending on the required accuracy [11 – 14].

| \(M\) | type | appropriate application |
|-------|------|------------------------|
| 0     | parabolic | axial heat conduction as in liquid metals |
| 0 < \(M\) < 1 | hyperbolic | pure backmixing and backflow |
| \(M\) > 1 | hyperbolic | pure maldistribution in tube bundles or plate heat exchangers |

2. Unity Mach number dispersion model

The analytical solution (\(M \neq 1\)) even for the simple case of only one dispersive stream in a counterflow heat exchanger is very complicated and not suited for practical design calculations [9, 10]. Thus a simplified less accurate model with a single fixed value of the thermal dispersive Mach number is more appropriate. Since in industrial heat exchangers backmixing and backflow (\(M < 1\)) [9] as well as maldistribution (\(M > 1\)) [9] takes place simultaneously, the value \(M = 1\) seems to be a realistic mean
value which is at least better than $M = 0$ of the parabolic model. The special case $M = 1$ of the hyperbolic axial dispersion model is considered in the following. In equations (3) and (4) it becomes $M^2 = 1$.

Figure 1. Temperature profiles in case of counterflow, $NTU_1 = 5$, $Pe_1 = 7$, $NTU_2 = 3$, $Pe_2 = 10$. $NTU_{1,d} = 2.48$ and $NTU_{2,d} = 1.49$.

To solve the governing equations (1) – (4) for the four dependent variables $t_1$, $t_2$, $\phi_1$ and $\phi_2$ only two boundary conditions are necessary. The energy balance requires temperature jumps between the non-dispersive region outside the heat exchanger and the dispersive region inside the heat exchanger (Figure 1). With given inlet temperatures of the fluids the two boundary conditions are

$$T'_1 = t_1(x = 0) + \frac{\phi_1(x = 0)}{Pe_1} \quad (7)$$

and

$$T'_2 = t_2(x = 1) - \frac{\phi_2(x = 1)}{Pe_2}. \quad (8)$$

Finally, the outlet temperatures are calculated with exit conditions (9) and (10).

$$T''_1 = t_1(x = 1) + \frac{\phi_1(x = 1)}{Pe_1} \quad (9)$$

$$T''_2 = t_2(x = 0) - \frac{\phi_2(x = 0)}{Pe_2} \quad (10)$$
3. Alternative formulation of the governing equations

The main advantage of the new model becomes obvious if the governing equations (1) – (4) are formulated in an alternative way by introducing hypothetic temperatures \( T_1(x) \) and \( T_2(x) \) according to equations (11) and (12). The results are equations (13) and (14).

\[
T_1 = t_1 + \frac{\phi_1}{\text{Pe}_1} \tag{11}
\]

\[
T_2 = t_2 - \frac{\phi_2}{\text{Pe}_2} \tag{12}
\]

\[
\frac{dT_1}{dx} + \text{NTU}_{1,d}(T_1 - T_2) = 0 \tag{13}
\]

\[
\frac{dT_2}{dx} + \text{NTU}_{2,d}(T_2 - T_1) = 0 \tag{13}
\]

\[
\text{NTU}_{i,d} = \frac{\text{NTU}_i}{1 + \frac{\text{NTU}_1}{\text{Pe}_1} + \frac{\text{NTU}_2}{\text{Pe}_2}}, \quad i = 1,2 \tag{14}
\]

Equation (13) has the mathematical form of the well-known energy equations of non-dispersive plug flow. This means that known analytical solutions and design charts [1 – 3] can be used and that the dispersion effect can be accounted for solely by a correction of the number of transfer units according to equation (14). Figure 1 also shows the hypothetic temperatures which can be interpreted as the temperature profiles in an equivalent non-dispersive plug flow heat exchanger.

Corresponding considerations [4] show that corrections (14) are also exactly valid for parallel flow and pure cross-flow. For other flow configurations the correction (14) yields sufficiently accurate approximations [4]. The unity Mach number dispersion model has two parameters \( \text{Pe}_1 \) and \( \text{Pe}_2 \). Its application can be based on known plug flow solutions, diagrams and charts.

4. Experimental determination of dispersive Peclet numbers

To apply the unity Mach number dispersion model, numerical values of dispersive Peclet numbers must be available for the different types heat exchangers. In this paper a method which has been originally applied to determine Peclet numbers of the parabolic dispersion model (\( M = 0 \)) is considered [14] and transferred to the new model. The Peclet numbers of each flow side, e.g. shell side and tube side of a shell and tube heat exchanger, can separately be determined by transient tracer experiments. Taking the analogy between heat and mass transfer into account, the transient tracer experiment corresponds to an adiabatic thermal experiment.

4.1 Concept of adiabatic experiments

The energy equation of a transient process in an adiabatic flow channel [9] with axial dispersion in case of \( M = 1 \) is

\[
\frac{\partial t}{\partial z} + \frac{\partial t}{\partial x} = -\frac{1}{\text{Pe}} \frac{\partial \phi}{\partial x} \tag{15}
\]

In principle, the energy equations (1) and (2) are on the one hand extended to allow for transient
process by the first addend of equation (15) and on the other hand they are simplified because NTU_1 = NTU_2 = 0 because the process is adiabatic. The index denoting the flow stream can be omitted since there is no interaction between the flow streams in this case.

The empirical approach for the reduced axial dispersive energy flux [9] in case of M = 1 is

\[ \phi + \frac{1}{Pe} \left( \frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial x}\right) = -\frac{\partial t}{\partial x}. \]  

As for the steady state process the energy balance requires temperature jumps between the non-dispersive region outside the heat exchanger and the dispersive region inside the heat exchanger. At the initial state the complete system has a uniform temperature which is normalized to zero and there is no axial dispersive energy flux. The complete set of boundary and the initial conditions to solve partial differential equations (15) and (16) is

\[ x = 0: \quad T'(x = 0, z) - t(x = 0, z) = \frac{1}{Pe} \phi(x = 0, z) \]
\[ z = 0: \quad t(x, z = 0) = 0 \]
\[ z = 0: \quad \phi(x, z = 0) = 0 \]

At the inlet the temperature \( T'(z) \) may change arbitrarily with time. Common inlet signals are step change and Dirac pulse [13, 14]. The signal travels through the flow channel while it is changed due to dispersion effects. The outlet signal of the process \( T'(z) \) is finally obtained from the exit condition

\[ x = 1: \quad T^*(x = 1, z) = t(x = 1, z) + \frac{1}{Pe} \phi(x = 1, z). \]

The basic idea of the experiment is to calculate the dispersive Peclet number from measured inlet and outlet signals. Because a simple analytic solution of equations (15) – (17) exists in the frequency domain the dispersive Peclet number will be determined in the frequency domain from the Laplace transforms of inlet and outlet signals.

4.2. Determination of the dispersive Peclet number in the frequency domain

In the frequency domain the analytical relationship between inlet signal \( T'(z) \) und outlet signal \( T^*(z) \) is

\[ \frac{T^*(s)}{T'(s)} = \exp\left(-s \frac{Pe + s}{Pe + 2s}\right). \]  

Equation (19) can be solved explicitly for the dispersive Peclet number:

\[ Pe = -s \frac{s + 2 \ln \frac{T^*(s)}{T'(s)}}{s + \ln \frac{T^*(s)}{T'(s)}}. \]

If the inlet signals and outlet signals are known, the dispersive Peclet number can be directly calculated. Since the dispersive Peclet number is a constant model parameter evaluation of equation (19) with arbitrary Laplace variables \( s \) must yield the same numerical of \( Pe \), if the model exactly fits.
4.3 Evaluation procedure

The inlet and outlet signals can only be measured in the time domain. For the proposed method they have to be Laplace transformed into the frequency domain according to

$$T(s) = \int_{z=0}^{\infty} T(z) \exp(-sz) dz. \quad (21)$$

![Image of inlet and outlet signals](image)

**Figure 2.** Example of inlet and outlet signals of an adiabatic transient experiments. Numerical integration according to equation (22) with denoted discrete measuring points.

The experimental data will consist of a number of \(n\) equidistant discrete measuring points (figure 2). Therefore, the integration according to equation (21) becomes a summation and the ratio between outlet and inlet signal of equation (20) is approximately

$$\frac{T'(s)}{T''(s)} \approx \frac{1}{2} T'(s) \exp(-sz_1) + \sum_{i=2}^{n-1} T'(s) \exp(-sz_i) + \frac{1}{2} T''(s) \exp(-sz_{n}). \quad (22)$$

Now equation (20) can be evaluated for different Laplace variables \(s\). Since in general the dispersion model will not exactly fit, the dispersive Peclet number will not be the same for different Laplace variables \(s\), i.e.

$$\text{Pe} = \text{Pe}(s) \neq \text{const.} \quad (23)$$

For practical applications a mean dispersive Peclet number \(\text{Pe}_{m}\) is proposed. Equation (24) requires the evaluation of equation (20) for four different values of \(s\), table 2:
\[
\frac{1}{\text{Pe}_m} = \frac{2}{3} \left[ \frac{1}{\text{Pe}_{+1/2}} + \frac{1}{\text{Pe}_{-1/2}} \right] - \frac{1}{6} \left[ \frac{1}{\text{Pe}_{+1}} + \frac{1}{\text{Pe}_{-1}} \right]
\]

(24)

Table 2. Evaluation points for chosen value of \( s_1 \)

| \( \text{Pe} \) | \( s \)  |
|----------------|--------|
| \( \text{Pe}_{+1} \) | \( s_1 \) |
| \( \text{Pe}_{-1/2} \) | \( s_1/2 \) |
| \( \text{Pe}_{-1/2} \) | \( -s_1/2 \) |
| \( \text{Pe}_{-1} \)   | \( -s_1 \) |

4.4 Example

Instead of an adiabatic thermal experiment Balzereit [14] performed tracer experiments which are analogous to adiabatic thermal experiments. The inlet signal is a Dirac pulse of concentration which has been realized with great experimental effort. The outlet signal for the shell side of a shell and tube heat exchanger with 11 flow baffles is shown in Figure 3.

The evaluation procedure after section 4.3 yields a dispersive Peclet number \( \text{Pe}_m = 38.6 \). For the parabolic dispersion model (\( M = 0 \)) as in the original work of Balzereit [14] the dispersive Peclet number is 37.6, confirming the following relationship which can be derived from a comparison of both models:

\[
\text{Pe}^{(M=1)} = \frac{\left[ \text{Pe}^{(M=0)} \right]^2}{\text{Pe}^{(M=0)} - 1 + \exp[-\text{Pe}^{(M=0)}]}
\]

(25)

Finally, the heat exchanger can also be modeled as a cascade of stirred tanks in series [4]. The transfer function relating the outlet signal to the inlet signal in the frequency domain is

\[
\frac{\mathcal{F}(s)}{\mathcal{F}(s)} = \frac{1}{\left(1 + s/N\right)^N}.
\]

(26)
In contrast to equation (19) the equation (26) cannot be solved explicitly for the model parameter of interest, i.e. the number of stirred tanks $N$. Data regression yields $N = 19$ for this example confirming the following relationship derived from steady state investigations and comparison between the unity Mach number dispersion model and the stirred tank in series model:

$$\text{Pe}^{(M=1)} \approx 2N,$$  \hspace{1cm} (27)

i.e. the Peclet number is directly proportional to the number of stirred tanks.

For clarity the outlet responses of the different models are presented in the time domain after numerical inversion of the Laplace transforms together with the experimental data in figure 3 showing good agreement between model and experiment.

5. Conclusions
- The unity Mach number dispersion model allows a simple analytical solution of the energy equations for counterflow, parallel flow and pure cross-flow with dispersion on both flow sides.
- For all flow configurations deviations from idealized plug flow can be taken into account by simple corrections of the number of transfer units.
- All known analytical solutions and charts for idealized plug flow can be further used.
- Peclet numbers can be determined for each flow side separately by tracer experiments.

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List of symbols

$c$ dimensionless concentration
\( C \) Propagation velocity of thermal disturbances (m s\(^{-1}\))
\( c_p \) Specific isobaric heat capacity (J kg\(^{-1}\) K\(^{-1}\))
\( L \) Length of heat exchanger (m)
\( l \) Space coordinate (m)
\( M \) Dispersive thermal Mach number, \( M = w/C \)
\( \text{NTU} \) Number of transfer units
\( N \) number of stirred tanks in series
\( n \) number of measuring points
\( \text{Pe} \) Dispersive Peclet number, \( \text{Pe} = w L \rho c_p / \lambda^* \)
\( \dot{q}_l \) Axial dispersive energy flux (W m\(^2\))
\( s \) Laplace variable
\( T \) Hypothetic temperature (K)
\( t \) Fluid temperature inside the exchanger (K)
\( V \) Volume of fluid side (m\(^3\))
\( \dot{V} \) Volumetric flow rate (m\(^3\) s\(^{-1}\))
\( w \) Mean flow velocity (m s\(^{-1}\))
\( x \) Dimensionless space coordinate (\( x = l/L \))
\( z \) Dimensionless time coordinate (\( z = \tau(\dot{V}/V) \))

Greek symbols

\( \lambda^* \) apparent thermal conductivity (W m\(^{-1}\) K\(^{-1}\))
\( \varphi \) Reduced axial dispersive heat flux (K)
\( \rho \) Density (kg m\(^{-3}\))
\( \tau \) Time coordinate (s)

Super- and subscripts

\( d \) dispersive
\( 1 \) Fluid 1
\( 2 \) Fluid 2
\( \text{\textquotesingle} \) Inlet
\( \text{"} \) Outlet
\( - \) Laplace transform