INDEFINITE HALMOS, EGERVARY AND SZ.-NAGY DILATIONS

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Abstract: Let \( M \) be an indefinite inner product module over a \(*\)-ring of characteristic 2. We show that every self-adjoint operator on \( M \) admits Halmos, Egervary and Sz.-Nagy dilations.

Keywords: Dilation, Indefinite inner product space, Module.

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1. Introduction

In 1950, Halmos [22] made a deep insight into structure theory of operators on Hilbert space by exhibiting any contraction as a part of a unitary. In 1953, Sz.-Nagy [39] showed that Halmos result can be extended to powers of contractions using a unitary operator. In 1963, T. Ando [5] showed that there is a version of Sz.-Nagy dilation for commuting contractions. Combined with spectral theory and theory of (several) complex variables, today, dilation theory of contractions is a rapidly evolving area of research and for a comprehensive look, we refer [1, 4–7, 9–16, 19–21, 27, 28, 31–37, 40–43]. Started in 1970’s, dilations of contractions acting on Lebesgue spaces and Banach spaces followed Hilbert space developments [2, 3, 17, 18, 24, 30, 38].

In 2021, by identifying essential mechanisms of dilation theory, Bhat, De and Rakshit [8] obtained surprising results in the set theory context and vector spaces. In 2022, further study in the context of vector spaces was carried by Krishna and Johnson [26]. We note that another vector space variant is also studied by Han, Larson, Liu and Liu [23]. Recently Krishna introduced the notion of magic contractions and derived Sz.-Nagy dilation for \( p \)-adic Hilbert spaces and modules [25].

In this paper, we derive indefinite inner product module versions of Halmos dilation (Theorem 2.2), Egervary N-dilation (Theorem 2.3), Sz.-Nagy dilation (Theorem 2.4). Our article is highly motivated from the paper of Halmos [22], Egervary [16], Schaffer [36], Sz.-Nagy [39], Bhat, De and Rakshit [8], Krishna and Johnson [26] and Krishna [25].

2. Indefinite Halmos, Egervary and Sz.-Nagy Dilations

We are going to use the following notions. A ring \( R \) with an automorphism \(*\) which is either identity or of order 2 is called as an \(*\)-ring. Throughout the paper we assume that characteristic of ring is 2.

**Definition 2.1.** [22] Let \( V \) be a module over \( R \). We say that \( V \) is an indefinite inner product module (we write IIPM) if there is a map (called as indefinite inner product) \( \langle \cdot, \cdot \rangle : V \times V \to R \) satisfying following.

(i) If \( x \in V \) is such that \( \langle x, y \rangle = 0 \) for all \( y \in V \), then \( x = 0 \).

(ii) \( \langle x, y \rangle = \langle y, x \rangle^* \) for all \( x, y \in V \).

(iii) \( \langle ax + y, z \rangle = a \langle x, z \rangle + \langle y, z \rangle \) for all \( a \in R \), for all \( x, y, z \in V \).
Let \( \mathcal{V} \) be a IIPM and \( T : \mathcal{V} \to \mathcal{V} \) be a morphism. We say that \( T \) is adjointable if there is a morphism, denoted by \( T^* : \mathcal{V} \to \mathcal{V} \) such that \( \langle Tx, y \rangle = \langle x, T^*y \rangle, \forall x, y \in \mathcal{V} \). Note that (i) in Definition 2.1 says that adjoint, if exists, is unique. An adjointable morphism \( U \) is said to be a unitary if \( UU^* = U^*U = I_{\mathcal{V}} \), the identity operator on \( \mathcal{V} \). An adjointable morphism \( P \) is said to be projection if \( P^2 = P^* = P \). An adjointable morphism \( T \) is said to be an isometry if \( T^*T = I_{\mathcal{V}} \). An adjointable morphism \( T \) is said to be self-adjoint if \( T^* = T \). We denote the identity operator on \( \mathcal{V} \) by \( I_{\mathcal{V}} \).

Our first result is the indefinite Halmos dilation.

**Theorem 2.2.** (Indefinite Halmos dilation) Let \( \mathcal{V} \) be a IIPM over a *-ring of characteristic 2 and \( T : \mathcal{V} \to \mathcal{V} \) be a self-adjoint morphism. Then the morphism

\[
U := \begin{pmatrix}
T & 0 & 0 & \cdots & 0 & 0 & I_{\mathcal{V}} + T \\
I_{\mathcal{V}} + T & 0 & 0 & \cdots & 0 & 0 & T \\
0 & I_{\mathcal{V}} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & I_{\mathcal{V}} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & I_{\mathcal{V}} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & I_{\mathcal{V}} & 0 \\
\end{pmatrix}_{(N+1) \times (N+1)}
\]

is unitary on \( \oplus_{k=1}^{N+1} \mathcal{V} \). In other words,

\[
T = P_{\mathcal{V}}U|_{\mathcal{V}}, \quad T^* = P_{\mathcal{V}}U^*|_{\mathcal{V}},
\]

where \( P_{\mathcal{V}} : \oplus_{k=1}^{N+1} \mathcal{V} \ni (x_{k})_{k=1}^{N+1} \mapsto x_1 \in \mathcal{V} \).

**Proof.** A direct calculation says that

\[
V := \begin{pmatrix}
T & 0 & 0 & \cdots & 0 & 0 & I_{\mathcal{V}} + T \\
I_{\mathcal{V}} + T & 0 & 0 & \cdots & 0 & 0 & T \\
0 & I_{\mathcal{V}} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & I_{\mathcal{V}} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & I_{\mathcal{V}} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & I_{\mathcal{V}} & 0 \\
\end{pmatrix}_{(N+1) \times (N+1)}
\]

is the inverse and adjoint of \( U \).

\[\square\]

Our second result is the indefinite Egervary N-dilation.

**Theorem 2.3.** (Indefinite Egervary N-dilation) Let \( \mathcal{V} \) be a IIPM over a *-ring of characteristic 2 and \( T : \mathcal{V} \to \mathcal{V} \) be a self-adjoint morphism. Let \( N \) be a natural number. Then the morphism

\[
U := \begin{pmatrix}
T & 0 & 0 & \cdots & 0 & 0 & I_{\mathcal{V}} + T \\
I_{\mathcal{V}} + T & 0 & 0 & \cdots & 0 & 0 & T \\
0 & I_{\mathcal{V}} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & I_{\mathcal{V}} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & I_{\mathcal{V}} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & I_{\mathcal{V}} & 0 \\
\end{pmatrix}_{(N+1) \times (N+1)}
\]

is unitary on \( \oplus_{k=1}^{N+1} \mathcal{V} \) and

\[
T^k = P_{\mathcal{V}}U^k|_{\mathcal{V}}, \quad \forall k = 1, \ldots, N, \quad (T^*)^k = P_{\mathcal{V}}(U^*)^k|_{\mathcal{V}}, \quad \forall k = 1, \ldots, N,
\]

where \( P_{\mathcal{V}} : \oplus_{k=1}^{N+1} \mathcal{V} \ni (x_{k})_{k=1}^{N+1} \mapsto x_1 \in \mathcal{V} \).
Proof. A direct calculation of power of $U$ gives Equation (1). To complete the proof, now we need show that $U$ is unitary. Define

$$V := \begin{pmatrix}
T & I_V + T & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & I_V & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & I_V & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & I_V \\
I_V + T & T & 0 & \cdots & 0 & 0 & 0 \\
\end{pmatrix}_{(N+1) \times (N+1)}.$$ 

Then $UV = VU = I_{\oplus_{k=1}^{N+1} V}$ and $U^* = V$.

Note that the Equation (1) holds only up to $N$ and not for $N + 1$ and higher natural numbers. In the following theorem, given a IIPM $V$, $\oplus_{n=-\infty}^\infty V$ is the IIPM defined by $\oplus_{n=-\infty}^\infty V := \{\{x_n\}_{n=-\infty}^\infty, x_n \in V, \forall n \in \mathbb{Z}, x_n \neq 0 \text{ only for finitely many } n's\}$ equipped with inner product

$$\langle \{x_n\}_{n=-\infty}^\infty, \{y_n\}_{n=-\infty}^\infty \rangle := \sum_{n=-\infty}^\infty \langle x_n, y_n \rangle, \ \forall \{x_n\}_{n=-\infty}^\infty, \{y_n\}_{n=-\infty}^\infty \in \oplus_{n=-\infty}^\infty V.$$

Our third result is the indefinite Sz.-Nagy dilation.

**Theorem 2.4. (Indefinite Sz.-Nagy dilation)** Let $V$ be a IIPM over a *-ring of characteristic 2 and $T : V \to V$ be a self-adjoint morphism. Let $U := (u_{n,m})_{-\infty \leq n,m \leq \infty}$ be the morphism defined on $\oplus_{n=-\infty}^\infty V$ given by the infinite matrix defined as follows:

$$u_{0,0} := T, \ u_{0,1} := I_V + T, \ u_{-1,0} := I_V + T, \ u_{-1,1} := T,$$

$$u_{n,n+1} := I_V, \ \forall n \in \mathbb{Z}, n \neq 0,1, \ u_{n,m} := 0 \text{ otherwise},$$

i.e.,

$$U = \begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & I_V & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & I_V & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & I_V + T & T & 0 & \cdots \\
\cdots & 0 & 0 & 0 & I_V + T & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & I_V & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}_{\infty \times \infty}$$

where $T$ is in the $(0,0)$ position (which is boxed), is unitary on $\oplus_{n=-\infty}^\infty V$ and

$$(T^n = P_V U^n|_V, \ \forall n \in \mathbb{N}, \ (T^*)^n = P_V (U^*)^n|_V, \ \forall n \in \mathbb{N},$$

where $P_V : \oplus_{n=-\infty}^\infty V \ni \langle x_n \rangle_{n=-\infty}^\infty \mapsto x_0 \in V.$

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Proof. We get Equation (2) by calculation of powers of $U$. The matrix $V := (v_{n,m})_{-\infty \leq n,m \leq \infty}$ defined by

\begin{equation}
\begin{aligned}
v_{0,0} &:= T, \quad v_{0,-1} := I_V + T, \quad v_{1,0} := I_V + T, \quad v_{1,-1} := T, \\
v_{n,n-1} &:= I_V, \quad \forall n \in \mathbb{Z}, n \neq 0, 1, \quad v_{n,m} := 0 \quad \text{otherwise},
\end{aligned}
\end{equation}

i.e.,

$$V = \begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & I_V & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & I_V & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & I_V + T & T & 0 & \cdots \\
\cdots & 0 & 0 & T & I_V + T & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & I_V & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}_{\infty \times \infty}
$$

where $T$ is in the $(0,0)$ position (which is boxed), satisfies $UV = VU = I_{\oplus_{n=-\infty}^{\infty} V}$ and $U^* = V$. □

We note that explicit sequential form of $U$ is

$$U(x_n)_{n=-\infty}^{\infty} = (\ldots, x_{-2}, x_{-1}, (I_V + T)x_0 + Tx_1, Tx_0 + (I_V + T)x_1, x_2, x_3, \ldots)$$

where $Tx_0 + (I_V + T)x_1$ is in the 0 position (which is boxed) and $U^*$ is

$$U^*(x_n)_{n=-\infty}^{\infty} = (\ldots, x_{-3}, x_{-2}, (I_V + T)x_{-1} + Tx_0, Tx_{-1} + (I_V + T)x_0, x_1, \ldots),$$

where $(I_V + T)x_{-1} + Tx_0$ is in the 0 position (which is boxed). We next wish to derive indefinite isometric Sz.-Nagy dilation.

**Theorem 2.5. (Indefinite isometric Sz.-Nagy dilation)** Let $\mathcal{V}$ be a II-PM over a *-ring of characteristic 2 and $T : \mathcal{V} \to \mathcal{V}$ be a self-adjoint morphism. Let $U := (u_{n,m})_{0 \leq n,m \leq \infty}$ be the morphism defined on $\oplus_{n=0}^{\infty} \mathcal{V}$ given by the infinite matrix defined as follows:

\begin{equation}
\begin{aligned}
u_{0,0} &:= T, \quad u_{2,1} := I_V + T, \quad u_{n+1,n} := I_V, \quad \forall n \geq 2, \quad u_{n,m} := 0 \quad \text{otherwise},
\end{aligned}
\end{equation}

i.e.,

$$U = \begin{pmatrix}
T & 0 & 0 & 0 & 0 & 0 & \cdots \\
I_V + T & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & I_V & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & I_V & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & I_V & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}_{\infty \times \infty}
$$

where $T$ is in the $(0,0)$ position (which is boxed), is isometry on $\oplus_{n=0}^{\infty} \mathcal{V}$ and

\begin{equation}
T^n = P_{\mathcal{V}} U^n|_{\mathcal{V}}, \quad \forall n \in \mathbb{N}, \quad (T^*)^n = P_{\mathcal{V}} (U^*)^n|_{\mathcal{V}}, \quad \forall n \in \mathbb{N},
\end{equation}

where $P_{\mathcal{V}} : \oplus_{n=0}^{\infty} \mathcal{V} \ni (x_n)_{n=0}^{\infty} \mapsto x_0 \in \mathcal{V}$.
Proof. It suffices to note the adjoint of $U$ is

$$
U^* = \begin{pmatrix}
T & I_V + T & 0 & 0 & 0 & \cdots \\
0 & 0 & I_V & 0 & 0 & \cdots \\
0 & 0 & 0 & I_V & 0 & \cdots \\
0 & 0 & 0 & 0 & I_V & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}_{\infty \times \infty}
$$

where $T$ is in the $(0,0)$ position (which is boxed). \qed

We now formulate following problems.

**Problem 2.6.**

(i) **Whether there is an indefinite Ando dilation?** If yes, whether one can dilate commuting three, four, ... commuting self-adjoint morphisms to commuting unitaries?

(ii) **Whether there is (a kind of) uniqueness of indefinite Halmos dilation?**

(iii) **Whether there is a indefinite intertwining-lifting theorem (commutant lifting theorem)?**

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