Note on the existence theory for evolution equations with pseudo-monotone operators

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Abstract
In this note we present a framework which allows to prove an abstract existence result for evolution equations with pseudo-monotone operators. The assumptions on the spaces and the operators can be easily verified in concrete examples.

Keywords: Evolution equation, pseudo-monotone operator, existence result.

1. Introduction
The theory of pseudo-monotone operators is very useful in proving the existence of solutions of non-linear problems. The main theorem on pseudo-monotone operators, due to Brezis [2], shows the surjectivity of a pseudo-monotone, bounded, coercive operator. This result extends the fundamental contribution of Browder [3] and Minty [11] on monotone operators to pseudo-monotone operators. The prototype of such an operator is a sum of a monotone operator and a compact operator. A huge class of elliptic partial differential equations can be treated in this framework, since many "lower order terms" define a compact operator due to compact embedding theorems.

The theory of monotone operators can easily be generalized to the treatment of non-linear evolution equations (cf. [8], [18], [16]). In the fundamental contribution [8] Lions combines, among others, monotonicity methods with compactness methods. Even though, there exists a general existence result for evolutionary pseudo-monotone, coercive, bounded operators (cf. [16], [13], [12]), its applicability to concrete problems is limited. This is due to the fact that the treatment of "lower order terms" as a compact operator needs usually additional information on the time derivative. The incorporation of the time derivative into the function space however contradicts the required coercivity of the operator. The way out of this problem for evolution problems is to repeat and adapt the arguments given in [8] to the concrete application to be treated. This is a non-satisfactory situation. There are many contributions to develop a general existence theory for evolution equations with pseudo-monotone operators (cf. [6], [7], [15], [16], [12], [9]).
The purpose of this note is to provide an existence theory for evolution equations with pseudo-monotone operators which is easily applicable. To this end we use and extend ideas from [15] and [12]. Essentially, one has to check whether the operator is on almost all time slices pseudo-monotone, coercive and satisfies certain natural growth conditions. This enables us to prove a generalization of Hirano’s lemma, which in turn allows the limiting process in the Galerkin approximation of the evolution problem. To verify the assumptions of Hirano’s lemma we need a technical assumption on the function spaces, which can be traced back to [8]. Note that such an assumption is not present in [15]. However, we are not able to follow one argument in the proof of Lemma 3 there, which would circumvent this technical assumption. In concrete application the technical assumption is easily verified.

The paper is organized as follows: In Section 2 we introduce the notation and collect some basic results for pseudo-monotone operators and evolution problems. In Section 3 we give the assumption on the function spaces and the operator and formulate the main theorem. Section 4 is devoted to the proof of the generalization of Hirano’s lemma. The main theorem is proved in Section 5. Finally we apply the main result in Section 6 to treat the evolution $p$-Laplace equation with a lower order term and the evolution equation for generalized Newtonian fluids.

2. Preliminaries

2.1. Notation and conventions

For a Banach space $X$ with norm $\| \cdot \|_X$ we denote by $X^*$ its dual space equipped with the norm $\| \cdot \|_{X^*}$. The duality pairing is denoted by $\langle \cdot, \cdot \rangle_X$. All occurring Banach spaces are assumed to be real. By an embedding we always understand a continuous embedding. Time integrals will usually be written as $\int_0^T f(t) dt$ instead of $\int_0^T f(t) dt$. By $c$ we denote a generic constant which may change from line to line. Finally we use the standard notation for Bochner spaces (cf. [5]).

2.2. Auxiliary results

From now on $V$ always denotes a separable, reflexive Banach space and $H$ a Hilbert space. If the embedding $V \hookrightarrow H$ is dense, we call $(V, H, V^*)$ a Gelfand-Triple. Using the Riesz representation theorem we obtain $V \hookrightarrow H \cong H^* \hookrightarrow V^*$ where both embeddings are dense. An operator $A : V \rightarrow V^*$ is said to be monotone if $\langle Ax - Ay, x - y \rangle_V \geq 0$ for all $x, y \in V$. The operator $A : V \rightarrow V^*$ is said to be pseudo-monotone if $x_n \rightharpoonup x$ in $V$ and $\limsup_{n \to \infty} \langle Ax_n, x_n - x \rangle_V \leq 0$ implies

$$\langle Ax, x - y \rangle_V \leq \liminf_{n \to \infty} \langle Ax_n, x_n - y \rangle_V$$

for all $y$ in $V$.

The following proposition introduces some typical types of pseudo-monotone operators.

Proposition 2.1. Let $A, B : V \rightarrow V^*$ be operators. Then there holds:

(a) If $A$ is monotone and hemicontinuous, then $A$ is pseudo-monotone.
(b) If $A$ is strongly continuous, then $A$ is pseudo-monotone.

c) If $A$ and $B$ are pseudo-monotone, then $A + B$ is pseudo-monotone.

Proof. See [18, Proposition 27.6]

Next we will state some well known results concerning Bochner and Bochner–Sobolev spaces, which will be used in the following.

**Theorem 2.2** (Pettis). A function $u : (0,T) \to V$ is Bochner measurable if and only if $u$ is weakly measurable in the sense: $\langle v^*, u(\cdot) \rangle_V$ is Lebesgue measurable for any $v^* \in V^*$.

Proof. See [12, Theorem 1.34].

Let $W$ be a Banach space such that the embedding $V \hookrightarrow W$ is dense and let $1 < p, q < \infty$. We say that a function $u \in L^p(0,T;V)$ has a generalized derivative in $L^q(0,T;W)$ if there exists a function $g \in L^q(0,T;W)$ such that

$$\int_0^T \varphi'(t) u(t) = - \int_0^T \varphi(t) g(t), \quad \text{for all } \varphi \in C_0^\infty((0,T)).$$

If such a function $g$ exists, it is unique and we set $\frac{du}{dt} := g$. With this definition of generalized derivatives we are able to introduce Bochner–Sobolev spaces. For $1 < p, q < \infty$ we define

$$W^{1,p,q}(0,T;V,W) := \{ u \in L^p(0,T;V) \mid \frac{du}{dt} \in L^q(0,T;W) \}.$$

With the norm

$$\|u\|_{W^{1,p,q}(0,T;V,W)} := \|u\|_{L^p(0,T,V)} + \left\| \frac{du}{dt} \right\|_{L^q(0,T;W)},$$

this space is a reflexive Banach space.

**Proposition 2.3.** Any element $u \in W^{1,p,q}(0,T;V,W)$ (defined almost everywhere) possesses a continuous representation on $[0,T]$ with values in $W$, and $W^{1,p,q}(0,T;V,W)$ embeds into $C(0,T;W)$.

Proof. See [1, Proposition II.5.11].

Let $(V,H,V^*)$ be a Gelfand-Triple and $1 < p < \infty$. Then we define the Bochner–Sobolev space

$$W^1_p(0,T;V,H) := \left\{ u \in L^p(0,T;V) \mid \frac{du}{dt} \in L^p(0,T;V^*) \right\},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. With the norm

$$\|u\|_{W^1_p(0,T;V,H)} := \|u\|_{L^p(0,T;V)} + \left\| \frac{du}{dt} \right\|_{L^p(0,T;V^*)},$$

this space is a reflexive Banach space. Since the embedding $V \hookrightarrow V^*$ is dense, Proposition 2.3 is valid. In this setting we can sharpen Proposition 2.3 as follows:
Proposition 2.4. Let \((V, H, V^*)\) be a Gelfand-Triple. Then \(W^1_p(0, T; V, H)\) embeds into \(C(0, T; H)\). Moreover, the integration by parts formula

\[
(u(t), v(t))_H - (u(s), v(s))_H = \int_s^t \left\langle \frac{du}{dt}(\tau), v(\tau) \right\rangle_V + \left\langle \frac{dv}{dt}(\tau), u(\tau) \right\rangle_V
\]

holds for any \(u, v \in W^1_p(0, T; V, H)\) and arbitrary \(0 \leq s, t \leq T\).

Proof. See [17, Proposition 23.23].

Proposition 2.5. Let \((V, H, V^*)\) be a Gelfand-Triple and let \(1 < p, q < \infty\). The function \(u \in L^p(0, T; V)\) possesses a generalized derivative \(\frac{du}{dt} \in L^q(0, T; V^*)\) if there is a function \(w \in L^q(0, T; V^*)\) such that

\[
\int_0^T (u(t), v)_H \varphi'(t) = -\int_0^T \langle w(t), v \rangle_V \varphi(t)
\]

for all \(v \in V\) and all \(\varphi \in C_0^\infty((0, T))\). Then \(\frac{du}{dt} = w\).

Proof. See [17, Proposition 23.20].

3. Statement of the main theorem

Now we formulate the assumptions on the spaces and the operator which allow us to proof an abstract existence result for an evolution equation with pseudo-monotone operators.

Assumption 3.1 (Spaces). Let \((V, H, V^*)\) be a Gelfand-Triple. Assume that there exists a reflexive, separable Banach space \(Z\) such that the embedding \(Z \rightarrow V\) is dense. Moreover, assume that there exists an increasing sequence of finite dimensional subspaces \(V_n \subseteq Z\), such that \(\bigcup_{n \in \mathbb{N}} V_n\) is dense in \(V\). Additionally, assume that there exists self-adjoint projections \(P_n : H \rightarrow H\), such that \(P_n(V) = V_n\) and \(\|P_n z\|_{L(Z, Z)} \leq c\) with a constant \(c\) independent of \(n \in \mathbb{N}\).

Assumption 3.2 (Operators). Let \(\{A(t) \mid 0 \leq t \leq T\}\) be a family of operators from \(V\) to \(V^*\) with the following properties:

(A1) \(A(t) : V \rightarrow V^*\) is pseudo-monotone for almost every \(t \in [0, T]\).

(A2) For every \(u \in L^p(0, T; V) \cap L^\infty(0, T; H)\) the mapping \(t \mapsto A(t)u(t)\) from \([0, T]\) to \(V^*\) is Bochner-measurable.

(A3) There exists a positive constant \(c_1\) and a non-negative function \(C_2 \in L^1((0, T))\), such that

\[
\langle A(t)x, x \rangle_V \geq c_1 \|x\|^p_V - C_2(t)
\]

for almost every \(t \in [0, T]\) and all \(x \in V\).

(A4) There exists \(0 \leq q < \infty\), as well as constants \(c_3 > 0, c_4 \geq 0\) and a non-negative function \(C_5 \in L^p'((0, T))\), such that

\[
\|A(t)x\|_{V^*} \leq c_3 \|x\|^p_V + c_4 \|x\|^q_H \|x\|^p_{V^*} + C_5(t)
\]

for almost every \(t \in [0, T]\) and all \(x \in V\).
Thm 3.3 (Main Theorem). If the Assumptions 3.1 and 3.2 are satisfied, then for every \( u_0 \in H, f \in L^p(0,T;V^*) \) there exists a function \( u \in W^1_p(0,T;V,H) \) such that

\[
\frac{du}{dt}(t) + A(t)u(t) = f(t) \quad \text{in } V^* \text{ for a.e. } t \in [0,T]
\]

\[ u(0) = u_0 \quad \text{in } H. \]

4. Hirano’s Lemma

In this section we prove a generalized version of Hirano’s Lemma (cf. [6], [7], [15]). First we show that the induced operator is bounded between the correct function spaces.

Lm 4.1. Assume that \( \{ A(t) \mid 0 \leq t \leq T \} \) satisfy Assumption 3.2. Then the induced operator \( (Au)(t) := A(t)u(t), A: L^p(0,T;V) \cap L^\infty(0,T;H) \to L^{p'}(0,T;V^*) \) is bounded.

Proof. The Bochner-measurability holds due to (A2). With the growth condition (A4) we conclude

\[
\|Au\|_{L^{p'}(0,T;V^*)} \leq \sup_{\|\varphi\| \leq 1} \int_0^T \|A(t)u(t)\|_V \cdot \|\varphi(t)\|_V \leq \sup_{\|\varphi\| \leq 1} \int_0^T (c_3 \|u(t)\|_{V^*}^{p-1} + c_4 \|u(t)\|_H^q \|u(t)\|_{V^*}^{p-1} + c_5(t)) \|\varphi(t)\|_V \\
\leq c_3 \|u\|_{L^p(0,T;V)} \cdot \|\varphi\|_{L^{p'}(0,T;V^*)} \leq c_4 \|u\|_{L^p(0,T;V)} \cdot \|\varphi\|_{L^{p'}(0,T;V^*)} + \|C_5\|_{L^{p'}(0,T)}.
\]

From now on we denote \( V := L^p(0,T;V), X := L^p(0,T;V) \cap L^\infty(0,T;H) \) and \( W := W^{1,p'}(0,T;V,Z^*) \).

Lm 4.2. Let the Assumptions 3.1 and 3.2 be satisfied. Further assume that \( v_n \subseteq W \cap L^\infty(0,T;H) \) satisfies \( v_n \to v_0 \) in \( W \), \( \|v_n\|_{L^\infty(0,T;H)} \leq K \) and that

\[
\limsup_{n \to \infty} (Av_n, v_n - v_0)_V \leq 0. \tag{4.1}
\]

Then for any \( z \in V \) there holds

\[
(Av_0, v_0 - z)_V \leq \liminf_{n \to \infty} (Av_n, v_n - z)_V. \tag{4.2}
\]

Moreover, \( Av_n \rightharpoonup Av_0 \) in \( V^* = L^{p'}(0,T;V^*) \).

Proof. Fix \( z \in V \). First we choose a subsequence \( (v_k)_{k \in A_1}, A_1 \subseteq \mathbb{N}, \) such that

\[
\lim_{k \to \infty} (Av_k, v_k - z)_V = \liminf_{n \to \infty} (Av_n, v_n - z)_V. \tag{4.3}
\]
From (A3) and (A4) in Assumption 3.2 as well as Young’s inequality we conclude that for a.e. \( t \in [0, T] \), all \( x \in \mathcal{X} \) with \( \|x\|_{L^\infty(0,T,H)} \leq K \) and all \( y \in \mathcal{V} \) there holds
\[
\langle A(t)x(t), x(t) - y(t) \rangle_\mathcal{V} \geq k_1 \|x(t)\|_\mathcal{V}^p - k_2 \|y(t)\|_\mathcal{V}^p - K_3(t) \tag{4.4}
\]
with positive constants \( k_1, k_2 \) (depending on \( K \)) and a non-negative function \( K_3 \in L^1((0, T)) \). Especially (4.4) holds for \( x = v_n \) and \( y = v_0 \). Since \( \mathcal{W} \mapsto C(0, T; \mathcal{Z}^*) \) holds by Proposition 2.3 we get \( v_n \rightarrow v_0 \) in \( C(0, T; \mathcal{Z}^*) \). This implies
\[
v_n(t) \rightarrow v_0(t) \quad \text{in} \quad \mathcal{Z}^* \quad \text{for all} \quad t \in [0, T]. \tag{4.5}
\]
Next we show that
\[
\liminf_{k \in \Lambda_1 \atop k \rightarrow \infty} \langle A(t)v_k(t), v_k(t) - v_0(t) \rangle_\mathcal{V} \geq 0 \tag{4.6}
\]
holds for a.e. \( t \in [0, T] \). To prove this, we define
\[
B_1 := \{ t \in [0, T] \mid \langle A(t)v_k(t), v_k(t) - v_0(t) \rangle_\mathcal{V} \geq k_1 \|v_k(t)\|_\mathcal{V}^p - k_2 \|v_0(t)\|_\mathcal{V}^p - K_3(t) \quad \text{holds for any} \quad k \in \Lambda_1 \},
\]
\[
B_2 := \{ t \in [0, T] \mid \|v_0(t)\|_\mathcal{V} < K_3(t) \quad \text{is finite} \},
\]
\[
B_3 := \{ t \in [0, T] \mid A(t) \text{is pseudo-monotone} \},
\]
\[
B := B_1 \cap B_2 \cap B_3.
\]
Due to \( v_0 \in \mathcal{V}, K_3 \in L^1((0, T)) \), (4.4) and (A1) we conclude that \( B^c \) is a set of measure zero. So it is enough to prove (4.6) for all \( t \in B \). For a fixed \( t \in B \) we define \( \Lambda_2 \subseteq \Lambda_1 \) by
\[
\ell \in \Lambda_2 \iff \langle A(t)v_\ell(t), v_\ell(t) - v_0(t) \rangle_\mathcal{V} < 0. \tag{4.7}
\]
If \( \Lambda_2 \) is finite, then (4.6) holds for this fixed \( t \in B \). Thus, assume that \( \Lambda_2 \) is not finite. Then by the definition of \( B_1 \) and (4.7) we get for all \( \ell \in \Lambda_2 \)
\[
K_1 \|v_\ell(t)\|_\mathcal{V}^p \leq \langle A(t)v_\ell(t), v_\ell(t) - v_0(t) \rangle_\mathcal{V} + k_2 \|v_0(t)\|_\mathcal{V}^p + K_3(t)
\]
\[
\leq k_2 \|v_0(t)\|_\mathcal{V}^p + K_3(t)
\]
and the right-hand side is finite due to \( t \in B_2 \). Therefore \( (v_\ell(t))_{\ell \in \Lambda_2} \) is bounded in \( \mathcal{V} \) and there exists subsequence \( (v_{\ell_j}(t))_{j \in \mathbb{N}} \subseteq (v_\ell(t))_{\ell \in \Lambda_2} \) which converges weakly to \( a \in \mathcal{V} \). Since \( \mathcal{V} \mapsto \mathcal{Z}^* \) we get that \( (v_{\ell_j}(t))_{j \in \mathbb{N}} \) also converges weakly to \( a \in \mathcal{Z}^* \). This together with (4.5) implies \( v_\ell(t) \rightarrow a \) in \( \mathcal{Z}^* \). Since the embedding \( \mathcal{V} \mapsto \mathcal{Z}^* \) is injective, we obtain \( v_\ell(t) \rightarrow a \) in \( \mathcal{V} \). This argument is valid for every weakly convergent subsequence of \( (v_\ell(t))_{\ell \in \Lambda_2} \) and therefore
\[
v_\ell(t) \rightarrow v_0(t) \quad \text{in} \quad \mathcal{V} \quad (\ell \rightarrow \infty, \ell \in \Lambda_2). \tag{4.8}
\]
From (4.7) and (4.8) together with the pseudo-monotonicity of \( A(t) \) we conclude that
\[
0 \leq \liminf_{\ell \in \Lambda_2 \atop \ell \rightarrow \infty} \langle A(t)v_\ell(t), v_\ell(t) - v_0(t) \rangle_\mathcal{V}.
\]
This proves (4.6) for every \( t \in B \), since for every \( k \in \Lambda_1 \setminus \Lambda_2 \) we also have
\[
\langle A(t)v_k(t), v_k(t) - v_0(t) \rangle_\mathcal{V} \geq 0.
\]
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Using (4.6) and Fatou’s Lemma we get
\[
0 \leq \int_0^T \liminf_{k \to \infty} \langle A(t)v_k(t), v_k(t) - v_0(t) \rangle_V \leq \liminf_{k \to \infty} \int_0^T \langle A(t)v_k(t), v_k(t) - v_0(t) \rangle_V
\]
\[
\leq \limsup_{k \to \infty} \langle Av_k, v_k - v_0 \rangle_V \tag{4.1}
\]
which implies
\[
\lim_{k \to \infty} \langle Av_k, v_k - v_0 \rangle_V = 0. \tag{4.9}
\]
We set \(h_k(t) := \langle A(t)v_k(t), v_k(t) - v_0(t) \rangle_V\) for \(k \in \Lambda_1\). Thus (4.6) and (4.9) read:
\[\begin{align*}
\text{a)} & \quad \liminf_{k \to \infty} h_k(t) \geq 0 \text{ for a.e. } t \in I, \\
\text{b)} & \quad \lim_{k \to \infty} \int_0^T h_k = 0.
\end{align*}\]
By (4.4) and Lebesgue’s dominated convergence theorem, we get \(\lim_{k \to \infty} \int_0^T h_k^- = 0\), with \(h_k^-(t) := -\min\{h_k(t), 0\}\). Since \(|h_k| = h_k + 2h_k^-\), this implies \(\lim_{k \to \infty} \int_0^T |h_k| = 0\).
Thus we can choose a subsequence \(\Lambda_3 \subseteq \Lambda_1\) such that
\[
\lim_{j \to \infty} \langle A(t)v_j(t), v_j(t) - v_0(t) \rangle_V = 0 \tag{4.10}
\]
holds for a.e. \(t \in [0, T]\). From (4.4) and (4.10) follows that for a.e. \(t \in [0, T]\) the sequence \((v_j(t))_{j \in \Lambda_3}\) is bounded in \(V\) and with the same argumentation as for (4.8) we get
\[
v_j(t) \to v_0(t) \text{ in } V, \quad (j \to \infty), \in \Lambda_3 \tag{4.11}
\]
for a.e. \(t \in [0, T]\). Now (4.10) and (4.11) together with the pseudo-monotonicity of \(A(t)\) imply, that for our fixed \(z \in V\), there holds
\[
\langle A(t)v_0(t), v_0(t) - z(t) \rangle_V \leq \liminf_{j \to \infty} \langle A(t)v_j(t), v_j(t) - z(t) \rangle_V. \tag{4.12}
\]
Thus from (4.3), (4.12) and Fatou’s lemma we get
\[
\langle Av_0, v_0 - z \rangle_V \leq \int_0^T \liminf_{j \to \infty} \langle A(t)v_j(t), v_j(t) - z(t) \rangle_V
\]
\[
\leq \liminf_{j \to \infty} \langle Av_j, v_j - z \rangle_V = \lim_{k \to \infty} \langle Av_k, v_k - z \rangle_V
\]
\[
= \liminf_{n \to \infty} \langle Av_n, v_n - z \rangle_V.
\]
Therefore we proved (4.2) for any $z \in \mathcal{V}$. This together with (4.1) implies

\[
\langle A v_0, v_0 - z \rangle_\mathcal{V} \leq \liminf_{n \to \infty} \langle A v_n, v_n - z \rangle_\mathcal{V}
\leq \limsup_{n \to \infty} \langle A v_n, v_n - v_0 \rangle_\mathcal{V} + \liminf_{n \to \infty} \langle A v_n, v_0 - z \rangle_\mathcal{V}
\leq \liminf_{n \to \infty} \langle A v_n, v_0 - z \rangle_\mathcal{V}
\]

for any $z \in \mathcal{V}$. Using that we can replace $z$ by $2v_0 - z$ in (4.13), we obtain in a standard manner

\[
\langle A v_0, v_0 - z \rangle_\mathcal{V} \leq \liminf_{n \to \infty} \langle A v_n, v_n - z \rangle_\mathcal{V} \leq \limsup_{n \to \infty} \langle A v_n, v_0 - z \rangle_\mathcal{V} \leq \langle A v_0, v_0 - z \rangle_\mathcal{V}.
\]

for any $z \in \mathcal{V}$, i.e. $A v_n \rightharpoonup A v_0$ in $\mathcal{V}^*$.

\[\square\]

5. Proof of the main theorem

Let $\{v_n^k\}_{k=1}^{d_n}$ be a basis of $V_n$, $d_n = \dim V_n$. We are seeking approximative solutions

\[u_n(t) = \sum_{k=1}^{d_n} c_n^k(t) v_n^k,\]

which solve the Galerkin system

\[
\left\langle \frac{d u_n(t)}{dt}, v_n^k \right\rangle_\mathcal{V} + \langle A(t) u_n(t), v_n^k \rangle_\mathcal{V} = \langle f(t), v_n^k \rangle_\mathcal{V} \quad k = 1, \ldots, d_n
\]

\[u_n(0) = u_{0,n},\]

where $u_{0,n} \in V_n$ are chosen such that $u_{0,n} \xrightarrow{n \to \infty} u_0$ in $H$.

Since the matrix $D^n = \left((v_n^i, v_n^j)\right)_{i,j=1,\ldots,d_n}$ is positive definite, the Galerkin system (5.1) can be re-written as a system of ordinary differential equations. From (A1) and (A4) we get that $A(t)$ is a pseudo-monotone and bounded operator, which yields that $A(t)$ is demi-continuous from $V \to V^*$ [12, Lemma 2.4]. This implies that the system of ordinary differential equations fulfills the Carathéodory conditions and is therefore solvable locally in time, i.e. on a small time interval $[0, t_0]$. We multiply the Galerkin system (5.1) with $c_n^k(t)$, sum from 1 to $d_n$, integrate over $[0, t_0]$ and use (A3) as well as Young’s inequality to obtain

\[
\frac{1}{2} \|u_n(t)\|^2_H + (c_1 - \varepsilon) \int_0^{t_0} \|u_n(t)\|^2_\mathcal{V} \leq \frac{1}{2} \|u_n(0)\|^2_H + c(\varepsilon) \int_0^{t_0} \|f(t)\|^2_{\mathcal{V}^*} + \int_0^{t_0} C_2(t) \leq K(u_0, f, C_2).
\]

This proves the boundedness of $|c_n^k(t)|$ on $[0, t_0]$ for any $1 \leq k \leq d_n$ and we can iterate Carathéodory’s theorem to get the existence of an absolutely continuous solution $u_n$ on $[0, T]$, which satisfies the a priori estimates

\[
\|u_n\|_{L^\infty(0,T;H)} + \|u_n\|_{L^p(0,T;\mathcal{V})} \leq K,
\]

\[
\|A u_n\|_{L^{p'}(0,T;\mathcal{V}^*)} \leq K.
\]

(5.3)
where we used Lemma 4.1 for the second estimate.

To get an uniform estimate for the time derivative we proceed as follows. For any \( v \in L^p(0, T; Z) \) we have \( P_n v \in L^p(0, T; V_n) \), which therefore is an admissible test-function in (5.1). Since \( P_n \) is self-adjoint and \( P_n |_{V_n} = \text{Id} \), we conclude

\[
\langle du_n \frac{dt}{dt}, v \rangle_{L^p(0, T; Z)} = \langle P_n \frac{du_n}{dt}, v \rangle_{L^p(0, T; Z)} = \langle du_n, P_n v \rangle_{L^p(0, T; Z)} 
\leq c \left( \|A u_n\|_{L^p(0, T; V)} + \|f\|_{L^{p'}(0, T; V^*)} \right) \|P_n v\|_{L^p(0, T; Z)}.
\]

This yields

\[
\left\| \frac{du_n}{dt} \right\|_{L^{p'}(0, T; Z^*)} \leq K. \tag{5.4}
\]

The estimates (5.3) and (5.4) yield

\[
\begin{align*}
  u_n &\to u \quad \text{in } L^p(0, T; V), \\
  u_n &\ast u \quad \text{in } L^\infty(0, T; H), \\
  Au_n &\to \zeta \quad \text{in } L^p(0, T; V^*), \\
  \frac{du_n}{dt} &\to \frac{du}{dt} \quad \text{in } L^{p'}(0, T; Z^*).
\end{align*}
\tag{5.5}
\]

From this, the embedding \( W^{1,p,p'}(0, T; V, Z^*) \hookrightarrow C(0, T; Z^*) \) (cf. Proposition 2.3), the injectivity of the embedding \( H \hookrightarrow Z^* \), the convergence \( u_{0,n} \to u_0 \) in \( H \) and (5.3) we obtain

\[
\begin{align*}
  u_n(0) &\to u(0) = u_0 \quad \text{in } H, \\
  u_n(T) &\to u(T) \quad \text{in } H.
\end{align*}
\tag{5.6}
\]

Let \( v \in V_k \), \( k \in \mathbb{N} \), and \( \varphi \in C^\infty_0((0, T)) \). If we use integration by parts for real-valued functions, (5.1) reads as

\[
- \int_0^T (u_n(t), v)_H \varphi'(t) = \int_0^T \langle f(t) - A(t) u_n(t), v \rangle_V \varphi(t)
\]

and (5.5) allows us to pass to the limit in every term, yielding

\[
- \int_0^T (u(t), v)_H \varphi'(t) = \int_0^T \langle f(t) - \zeta(t), v \rangle_V \varphi(t). \tag{5.7}
\]

This and the density of \( \cup_{k \in \mathbb{N}} V_k \) in \( V \) yields that (5.7) is valid for every \( v \in V \) and \( C^\infty_0((0, T)) \). Now Proposition 2.5 implies

\[
\frac{du}{dt} + \zeta = f \quad \text{in } L^p(0, T; V^*) \tag{5.8}
\]

and therefore \( u \in W^{1,p}_p(0, T; V, H) \hookrightarrow C(0, T; H) \).
We want to use Lemma 4.2 to prove that $A u = \zeta$. From (5.5) we know that $(u_n) \subseteq W \cap L^\infty(0,T;H)$ satisfies $u_n \rightharpoonup u_0$ in $W$, $\|u_n\|_{L^\infty(0,T;H)} \leq K$ so we only need to check that (4.1) holds. To this end we test (5.1) with $u_n$ and use integration by parts, (5.5), (5.6) as well as the weak lower semicontinuity of the norm to conclude

$$\limsup_{n \to \infty} \langle A u_n, u_n \rangle_V \leq \langle f, u \rangle_V - \frac{1}{2} \|u(T)\|_H^2 + \frac{1}{2} \|u_0\|_H^2.$$ 

If we test (5.8) with $u$, use the integration by parts formula from Proposition 2.4 and (5.6), we obtain

$$\langle \zeta, u \rangle_V = \langle f, u \rangle_V - \frac{1}{2} \|u(T)\|_H^2 + \frac{1}{2} \|u_0\|_H^2.$$ 

The last two equalities imply (4.1) and Lemma 4.2 yields $A u = \zeta$, so that

$$\frac{du}{dt} + A u = f \quad \text{in } L^\prime(0,T;V^\prime). 
(5.9)$$ 

6. Examples

In this section we illustrate the above theory by two applications. Let us start with some notation. Let $\Omega_1 \subseteq \mathbb{R}^3$ and $\Omega_2 \subseteq \mathbb{R}^d$ be bounded domains with Lipschitz-boundary. Let $V_1 := W^1_0,\text{div}(\Omega_1)^3 = L_0^2(\Omega_1)^3$ be the completion of $\{\varphi \in C^\infty_0(\Omega_1)^3 \mid \text{div} \varphi = 0\}$ in $W^1_0$ and $L^2$ respectively. Additionally we define $V_2 := W^1_0(\Omega_2)$ and $H_2 := L^2(\Omega_2)$. We assume further that a function $g : [0,T] \times \Omega_2 \times \mathbb{R} \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfies:

(g1) $g$ is measurable in its first two variables and continuous in its third variable.

(g2) For some constant $c_0 > 0$, some non-negative function $C_7 \in L^p((0,T))$ and $1 \leq r \leq p \frac{d+2}{d-2}$ holds

$$|g(t,x,s)| \leq c_0 (1 + |s|^{r-1}) + C_7(t).$$

(g3) For some non-negative $C_8 \in L^1(0,T)$ there holds

$$g(t,x,s) \cdot s \geq -C_8(t).$$

Then we have the following two existence theorems.

**Theorem 6.1.** For $0 < T < \infty$ we set $\Omega_1^T := [0,T] \times \Omega_1$. If $\frac{2d}{d+2} \leq p$, then for any $f \in L^p(0,T;V_1^*)$ and $u_0 \in H_1$ there exists a function $u \in W^1_p(0,T;V_1,H_1)$ with $u(0) = u_0$ in $H_1$, which satisfies the equation

$$\int_0^T \left\langle \frac{du(t)}{dt}, \varphi(t) \right\rangle_{V_1} + \int_{\Omega_1^T} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi - u \otimes u : \nabla \varphi = \int_0^T \langle f(t), \varphi(t) \rangle_{V_1},$$

for any $\varphi \in L^p(0,T;V_1)$.

**Theorem 6.2.** For $0 < T < \infty$ we set $\Omega_2^T := [0,T] \times \Omega_2$. Assume that the function $g$ satisfies the properties (g1), (g2) and (g3). If $\frac{2d}{d+2} < p$, then for every $u_0 \in H_2$ and $f \in L^p(0,T;(W^1_0)^*)$ there exists a function $u \in W^1_0(0,T;V_2,H_2)$ with $u(0) = u_0$, such that for any $\varphi \in L^p(0,T;W^1_0)$

$$\int_0^T \left\langle \frac{du(t)}{dt}, \varphi(t) \right\rangle_{V_2} + \int_{\Omega_2^T} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi + g(\cdot, \cdot, u) \cdot \varphi = \int_0^T \langle f(t), \varphi(t) \rangle_{V_2}.$$
Lemma 6.3. $(V_1, H_1, (V_1)^*)$ and $(V_2, H_2, (V_2)^*)$, resp., are Gelfand-triples, which satisfy Assumption 3.1 with $Z_1 := \{ \varphi \in C_0^\infty(\Omega_1)^3 | \text{div} \varphi = 0 \}^{W_1,2}$ and $Z_2 := W_0^{s_2,2}(\Omega_2)$, respectively.

Proof. From Sobolev’s embedding theorem we conclude that $(V_1, H_1, (V_1)^*)$ and $(V_2, H_2, (V_2)^*)$ are Gelfand-triples. It is also clear that $Z_1$ and $Z_2$, resp., are densely embedded into $V_1$ and $V_2$, resp., if $s_1$ and $s_2$ are chosen appropriately. A proof for the existence of projections $P_n$ can be found e.g. in the appendix of [10].

Next we want to introduce some operators for the proofs of our theorems. We define $B_1, B_2, A_1 : V_1 \to (V_1)^*$ and $B_3, B_4(t), A_2(t) : V_2 \to (V_2)^*$ by

\[
\begin{align*}
\langle B_1 u, v \rangle_{V_1} &:= \int_\Omega |\nabla u|^{p-2} \nabla u : \nabla v \\
\langle B_2 u, v \rangle_{V_1} &:= -\int_\Omega u \otimes u : \nabla v \\
\langle B_3 u, v \rangle_{V_2} &:= \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \\
\langle B_4(t) u, v \rangle_{V_2} &:= \int_\Omega g(t, \cdot, u) \cdot v
\end{align*}
\]

The operators $B_1$ and $B_3$ are the well-known $p$-Laplace operators. The basic properties of these operators can be found e.g. [14, Theorem 17.11].

Theorem 6.4. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain, $p \in (1, \infty)$. Then the $p$-Laplace operator maps $W_0^{1,p}(\Omega)$ into $W_0^{1,p}(\Omega)^*$ and is monotone and continuous.

Note that one can replace the $p$-Laplace operators by operators $B$ having a so-called $(p, \delta)$-structure, with $\delta \in [0, \infty)$, $p \in (1, \infty)$, as e.g.

\[
\langle Bu, v \rangle_{W_0^{1,p}} := \int_\Omega (\delta + |\nabla u|)^{p-2} \nabla u : \nabla v.
\]

Proof (of Theorem 6.1). From Lemma 6.3 we know that Assumption 3.1 is satisfied. So we only have to check that $A_1 := B_1 + B_2$ satisfies Assumption 3.2. We restrict ourselves to the more complicated case $p < 3$. For $p \geq 3$ one has to use different embedding theorems and some calculation simplify.

(A1) Due to Proposition 2.1 and Theorem 6.4 we only have to prove that $B_2 : V_1 \to (V_1)^*$ is strongly continuous. Let $u_n \to u$ in $V_1$. By the Sobolev embedding theorem we get $u_n \to u$ in $L^s(\Omega_1)$ for any $1 \leq s < \frac{3p}{3-p}$. Then Hölder’s inequality yields

\[
\begin{align*}
\sup_{\|v\|_{V_1} \leq 1} |\langle A u_n - A u, v \rangle_{V_1^*}| &\leq \sup_{\|v\|_{V_1} \leq 1} \left( \|u_n - u\|_{L^s} \|v\|_{L^p} + \|u_n\|_{L^s} \|u_n - u\|_{L^s} \|\nabla v\|_{L^p} \right) \\
&\to 0.
\end{align*}
\]

(A2) Fix an arbitrary $u \in L^p(0, T; V_1) \cap L^\infty(0, T; H_1)$. Due to Pettis’ Theorem we only have to prove that $t \mapsto \langle A_1 u(t), v \rangle_{V_1}$ is Lebesgue-measurable for any

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v ∈ V_1. The functions |∇u(t, x)|^{p-1}\nabla v(x) and u(t, x) \odot u(t, x) : \nabla v(x)
are Lebesgue-measurable on [0, T] \times \Omega_1 and we easily estimate
\[
\int_0^T \int_{\Omega_1} |\nabla u(t, x)|^{p-1} |\nabla v(x)| \leq \int_0^T \|\nabla u(t)\|_{L^{p-1}_{L^p}}^{p-1} \|\nabla v\|_{L^p} \leq c \|u\|_{L^{p-1}_{L^p}(0, T; V_1)} \|v\|_{V_1},
\]
\[
\int_0^T \int_{\Omega_1} |u(t, x) \odot u(t, x) : \nabla v(x)| \leq c \|u\|^2_{L^{5/3}\pi((0, T] \times \Omega)} \|v\|_{V_1}.
\]
Since L^p(I; V_1) \cap L^\infty(I; H_1) \hookrightarrow L_4^p(I \times \Omega_1), see [4, Proposition 3.1], we get
Fubini’s Theorem that
t \mapsto \langle A_1 u(t), v \rangle_{V_1} = \int_{\Omega_1} |\nabla u(t, x)|^{p-2} \nabla v(t, x) : \nabla v(x) - \int_{\Omega_1} u(t, x) \odot u(t, x) : \nabla v(x)
is Lebesgue-measurable.

(A3) For any u ∈ V_1 there holds
\[
\langle A_1 u, u \rangle_{V_1} = \int_{\Omega_1} |\nabla u|^p - \int_{\Omega_1} u \odot u : \nabla u = \|u\|^p_{V_1},
\]
since the second term vanishes due to \text{div} u = 0.

(A4) From Hölder’s inequality we estimate
\[
\|B_1 u\|_{L^p(V_1)} \leq \|\nabla u\|_{L^{p-1}}^{p-1} = \|u\|_{V_1}^{p-1},
\]
\[
\|B_2 u\|_{L_{\pi}} \leq \|u\|_{L^{2\pi}}^{2\pi} \|u\|_{L^{2\pi}}^{p-1}.
\]
For r = \frac{12p}{5p^2 + 12p - 6} we use the Hölder interpolation
\[
\|u\|_{L^r} \leq \|u\|_{L^{2\pi}}^{\frac{2\pi}{r}} \|u\|_{L^{2\pi}}^{\frac{p-1}{r}}.
\]
Since for any p ∈ [\frac{11}{6}, 3) there holds that 2p' ≤ r we conclude that
\[
\|B_2 u\|_{L^p(V_1)} \leq C \|u\|_{L^{2\pi}}^{3-p} \|u\|_{L^{2\pi}}^{p-1} \leq C \|u\|_{H_1}^{3-p} \|u\|_{V_1}^{p-1}.
\]
So all the assumptions are satisfied and Theorem 3.3 proves Theorem 6.1.

Proof (of Theorem 6.2). This works basically the same way as the proof of Theorem 6.1. The assumption 3.1 is fulfilled by Lemma 6.3, so that we only have to check Assumption 3.2 for A_2(t) := B_3 + B_4(t) to apply Theorem 3.3. Again, we restrict ourselves to the more complicated case p < d. For p ≥ d one has to use different embedding theorems and some calculation simplify.

(A1) Again we only have to prove the pseudo-monotonicity of B_4(t) for a.e. t ∈ [0, T]. Using (g2) and Hölder’s inequality we easily estimate
\[
|\langle B_4(t) u, \varphi \rangle_{V_2} | \leq c_0 (1 + C_7(t) + \|u\|_{L^{(r-1)\sigma_\varphi}}^{r-1}) \|\varphi\|_{L^\sigma}
\]
\[
\leq c_0 (1 + C_7(t) + \|u\|_{V_2}^{r-1}) \|\varphi\|_{V_2},
\]
where σ := \frac{dp}{2d} is the Sobolev exponent and \frac{1}{\sigma} + \frac{1}{\sigma_\varphi} = 1. The last inequality holds if (r - 1)σ' ≤ σ but this is equivalent to r ≤ σ. For any p ∈ (\frac{2d}{d + 2}, d)
there holds \( \sigma = \frac{d \nu}{d\nu_p} > p \frac{d_\nu}{d\nu_p} \geq r \), due to (g2). Thus the Sobolev embedding theorem implies \( u_n \rightharpoonup u \) in \( L^r(\Omega_2) \) for \( u_n \rightharpoonup u \) in \( V_2 \). Now define \( F_\ell : L^r(\Omega_2) \to L^r(\Omega_2) \) by \( F_\ell(u)(x) := g(t, x, u(x)) \). Condition (g2) and the theory of Nemyckii-Operators, cf. [12, Theorem 1.43], implies the continuity of \( F_\ell \) and

\[
\sup_{\varphi \in V_2, \|\varphi\|_{V_2} \leq 1} |\langle A_2(t)u_n - A_2(t)u, \varphi \rangle_{V_2}| \leq \sup_{\varphi \in V_2, \|\varphi\|_{V_2} \leq 1} \|F_\ell(u_n) - F_\ell(u)\|_{L^{r'}} \|\varphi\|_{V_2} \to 0.
\]

So \( B_4(t) \) is strongly continuous and therefore pseudo-monotone.

(A2) Using Pettis’ Theorem we only have to check that

\[
t \mapsto \langle A_2(t)u(t), \varphi \rangle_{V_2} = \int_{\Omega_2} |\nabla u(t,x)|^{p-2} \nabla u(t,x) \cdot \nabla \varphi(x) + g(t, x, u(t,x)) \cdot \varphi(x)
\]

is Lebesgue-measurable for arbitrary \( \varphi \) in \( V_2 \). The functions \( g(t, x, u(t,x)) \cdot \varphi(x) \) and \( |\nabla u(t,x)|^{p-2} \nabla u(t,x) \cdot \nabla \varphi(x) \) are Lebesgue-measurable on \([0,T] \times \Omega_2 \). Using the growth condition (g2) we estimate

\[
\int_0^T \int_{\Omega_2} |\nabla u(t,x)|^{p-1} |\nabla \varphi(x)| \leq \int_0^T \|\nabla u(t)\|_{L^{p'}}^{p-1} \|\nabla \varphi\|_{L^p} \leq T^{\frac{1}{p}} \|u\|_{L^{p}(0,T;V_2)} \|\varphi\|_{V_2},
\]

\[
\int_0^T \int_{\Omega_2} g(t, x, u(t,x)) \cdot \varphi(x) \leq c_6 (1 + \|C_7\|_{L^{r'}} + \|u\|_{L^{p}(0,T;W_0^{1,p})}^{\frac{1}{p-1}}) \|u\|_{W_0^{1,p}},
\]

so that Fubini’s Theorem again proves the assertion.

(A3) Using (g3) we have

\[
\langle A_2(t)u, u \rangle_{W_0^{1,p}} = \int_{\Omega} |\nabla u(x)|^p + g(t, x, u(x)) \cdot u(x) \geq \|u\|_{V_2}^p - C_8(t).
\]

(A4) Using Hölder’s inequality we easily get

\[
\|B_3 u\|_{(V_2)'} \leq \|u\|_{V_2}^{p-1}.
\]

From (g2) we estimate for \( u \) in \( V_2 \)

\[
\|B_4(t)u\|_{(V_2)'} \leq c_6 (1 + C_7(t)) + c \|u\|_{L^{p}(0,\infty;L^{r'})}^{\frac{1}{p-1}} \leq c_6 (1 + C_7(t)) + c \|u\|_{L^{p}(0,\infty;L^{r'})}^{\frac{1}{p-1}}
\]

where \( r_0 := p \frac{d_\nu}{d\nu_p} \) and \( \sigma = \frac{d \nu}{d\nu_p} \). Using the interpolation

\[
\|u\|_{L^{r_0}(\infty;L^p)} \leq \|u\|_{L^2}^{1-\lambda} \|u\|_{L^{r_0}(\infty;L^p)}^\lambda \frac{d \nu}{d\nu_p}
\]

we see that \( \lambda(r_0 - 1) \leq p - 1 \). This and the embedding \( V_2 \hookrightarrow L^\sigma \) yields

\[
\|A_2(t)u\|_{(V_2)'} \leq c_6 (1 + C_7(t)) + c \|u\|_{L^2}^{(r_0-1)(1-\lambda)} \|u\|_{L^{r_0}(\infty;L^p)}^{\frac{1}{p-1}} \leq c_6 (1 + C_7(t)) + c \|u\|_{H_2}^{(r_0-1)(1-\lambda)} \|u\|_{V_2}^{p-1}.
\]

So Assumption 3.2 is satisfied.
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