Concept Paper

Nearest Neighbor search in Complex Network for Community Detection

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Received: xx / Accepted: xx / Published: xx

Abstract: Nearest neighbor search is a basic computational tool used extensively in almost research domains of computer science specially when dealing with large amount of data. However, the use of nearest neighbor search is restricted for the purpose of algorithmic development by the existence of the notion of nearness among the data points. The recent trend of research is on large, complex networks and their structural analysis, where nodes represent entities and edges represent any kind of relation between entities. Community detection in complex network is an important problem of much interest. In general, a community detection algorithm represents an objective function and captures the communities by optimizing it to extract the interesting communities for the user. In this article, we have studied the nearest neighbor search problem in complex network via the development of a suitable notion of nearness. Initially, we have studied and analyzed the exact nearest neighbor search using metric tree on proposed metric space constructed from complex network. After, the approximate nearest neighbor search problem is studied using locality sensitive hashing. For evaluation of the proposed nearest neighbor search on complex network we applied it in community detection problem. The results obtained using our methods are very competitive with most of the well known algorithms exists in the literature and this is verified on collection of real networks. On the other-hand, it can be observed that time taken by our algorithm is quite less compared to popular methods.

Keywords: Complex Network; Nearest Neighbor; Metric tree; Locality Sensitive Hashing; Community Detection
1. Introduction  

Nearest neighbor (NN) search is an important computational primitive for structural analysis of data and other query retrieval purposes. NN search is very useful for dealing with massive datasets, but it suffers with "curse of dimensionality"[1,2]. However, some recent surge of results show that it is also very efficient for high dimensional data provided a suitable space partitioning data structure is used, like, kd-tree, quad-tree, R-tree, metric-tree and locality sensitive hashing[3–6]. Some of these data structures also support approximate nearest neighbor search which hardly made any degradation of results whereas saves lot of computational times. In the NN-search problem, the goal is to pre-process a set of data points, so that later, given a query point, one can find efficiently the data point nearest to the query point on some metric space of consideration. NN search has many applications in data processing and analysis. For instance, information retrieval, searching image databases, finding duplicate pages, compression, and many others. To represent the objects and the similarity measures, one often uses geometric notions of nearness [7,8].

One important research direction of recent interest is to extract network communities in large real graphs such as social networks, web, collaboration networks and bio-networks [9–12]. The availability of large, detailed datasets representing such networks has stimulated extensive study of their basic properties, and the identification of hierarchical structural features[11,13]. Other than graphs, the complex networks are characterized by small average path length and high clustering coefficient. A network community (also known as a module or cluster) is typically a group of nodes with more interconnections among its members than the remaining part of the network [13–15]. To extract such group of nodes from a network one generally selects an objective function that captures the possible communities as a set of nodes with better internal connectivity than external [16,17]. However, very less research is done for network community detection which tries to develop nearness among the nodes of a complex network and use nearest neighbor search for partitioning the network[18–24]. Complex networks are characterized by small average path length and high clustering coefficient the way the metric is defined should be able to capture the crucial properties of complex networks. Therefore, we need to create the metric very carefully so that it can explore the underlying community structure of the real life networks[25].

In this work, we have developed the notion of nearness among the nodes of the network using some new matrices derived from modified adjacency matrix of the graph which is flexible over the networks and can be tuned to enhance the structural properties of the network required for community detection. The main contributions of this work include:

- Define a metric on complex network suitable for community detection
- Choice of data structure and approximations for NN computation (M-tree, LSH)
- Community detection algorithm developed using NN search
- Experiments on real networks

The rest of this paper is organized as follows: In Section 2 several definitions and challenges relevant to nearest neighbor search in complex network are discussed. Section 3 describes the
notion of nearness in complex network and developed a method to represent a complex network as points of a metric space. Section 4 describes the problem of nearest neighbor search over complex network and the use of metric tree data structure in this regard. In Section 5, the problem approximate nearest neighbor search on complex network is discussed with a newly developed locality sensitive hashing method. Network community detection using exact and approximate nearest neighbor search is formulated and several possible solutions are presented in Section 6, also, the initialization procedures, termination criteria, convergence are discussed in detail. The results of comparison between community detection algorithms are illustrated in Section 7. The computational aspects of the proposed framework are also discussed in this section.

2. NNSCN: Nearest neighbor search on complex network

The nearest-neighbor searching problem is to find the nearest points in a $D$ dimensional dataset $X \subset \mathbb{R}^D$ containing $n$ points to a query point $q \in \mathbb{R}^D$, usually in a metric space. It has applications in a wide range of real-world settings, in particular pattern recognition, machine learning and database querying to name a few. Several effective methods exist for this problem when the dimension $D$ is small, such as Voronoi diagrams, however, kd-trees and metric trees are common when the dimension is high. Many methods with different approach are developed for searching data and finding the nearest point. Searching the nearest neighbor in different studies are presented by different names such as post office problem, proximity search, closest point search, best match file searching problem, index for similarity search, vector quantization encoder, the light-bulb problem and etc.. The solutions for the Nearest Neighbor Search (NNS) problem usually have two parts: nearness determination in the data and and algorithmic developments. In most the NNS algorithms, the main framework is based on four fundamental algorithmic ideas: Branch-and-bound, Walks, Mapping-based techniques and Epsilon nets. There are thousands of possible framework variations and any practical application can lead to its unique problem formalization such as pattern recognition, searching in multimedia data, data compression, computational statistics, information retrieval, databases and data mining, machine learning, algorithmic theory, computational geometry, recommendation systems and etc.

2.1. NN problem definition on complex network

A NNS problem defined in a metric space is defined below.

**Definition 1** (Metric space). *Given a set $S$ of points and $d$ as a function to compute the distance between two points. Pair $(S, d)$ distinguished metric space if $d$ satisfies reflexivity, non-negativity, symmetry and triangle inequality.*

Non-metric space data are indexed by special data structures in non-metric spaces and then searching is done on these indexes. A few efficient methods exist for searching in non-metric space that in most of them, non-metric space is converted to metric space. The focus of this paper is
on the problems defined on a metric space. In a more detailed classification, NNS problems can be defined in Euclidean space as follow:

**Definition 2** (Nearest neighbor search). Given a set $S$ of points in a $D$ dimensional space, construct a data structure which given any query point finds the point in $S$ with the smallest distance with $q$.

This definition for a small dataset with low dimension has sub linear (or even logarithmic) query time, but for massive dataset with high dimension is exponential. Fortunately, some little approximation can decrease the exponential complexity into polynomial time.

Approximate NNS is defined as:

**Definition 3** (Approximate nearest neighbor). Given a set $S$ of Points in a $D$-dimensional space, construct a data structure which given any query point, reports any point within distance at most $c$ times the nearest distance from $q$.

The first requirement in order to search in a metric space is the existence of a formula to calculate the distance between each pair of objects in $S$. Different metric distance functions can be defined depending on the search space of consideration. A NN query on a complex network $G$, consists of a source node $s$ and a metric function $d(x,y)$. This computations depends on the dimension of the instance and face "curse of dimensionality' problem. The computation can be reduced drastically if instead of computing the exact nearest neighbor we compute the approximate nearest neighbor.

3. Notion of nearness in complex network

The notion of nearness among the nodes of a graph are used in several purposes in the history of literature of graph theory. Most of the time the shortest path and edge connectivity are popular choice to describe nearness of nodes. However, that edges do not give the true measure of network connectivity (proof by kleinbarg). The notion of network connectivity some times generalized to be the number of paths, of any length, that exist between two nodes. This measure, called influence by sociologists, because it measures the ability of one node to affect another, gives a better measure of connectivity between nodes of real life graphs / complex networks. Beside discovering natural groups within a network, the influence metric can also help identify the weak ties who bridge different communities. Research in this direction gained special attention in the domain of complex network analysis, some of them along with the one proposed in this article are discussed in the following subsections.

3.1. Nearness literature

Methods based on node neighborhoods. For a node $x$, let $N(x)$ denote the set of neighbors of $x$ in a graph $G(V,E)$ . A number of approaches are based on the idea that two nodes $x$ and $y$ are more likely to be affected by one another if their sets of neighbors $N(x)$ and $N(y)$ have large overlap.
Common neighbors: The most direct implementation of this idea for nearness computation is to define \( d(x, y) := |N(x) \cap N(y)| \), the number of neighbors that \( x \) and \( y \) have in common.

Jaccard coefficient: The Jaccard coefficient, a commonly used similarity metric, measures the probability that both \( x \) and \( y \) have a feature \( f \), for a randomly selected feature \( f \) that either \( x \) or \( y \) has. If we take features here to be neighbors in \( G(V, E) \), this leads to the measure \( d(x, y) := |N(x) \cap N(y)|/|N(x) \cup N(y)| \).

Preferential attachment: The probability that a new edge involves node \( x \) is proportional to \( |N(x)| \), the current number of neighbors of \( x \). The probability of co-authorship of \( x \) and \( y \) is correlated with the product of the number of collaborators of \( x \) and \( y \). This corresponds to the measure \( d(x, y) := |N(x)| \times |N(y)| \).

Katz measure: This measure directly sums over the collection of paths, exponentially damped by length to count short paths more heavily. This leads to the measure \( d(x, y) := \beta \times |\text{paths}(x, y)| \) where \( \text{paths}(x, y) \) is the set of all length paths from \( x \) to \( y \). (\( \beta \) determines the path size, since paths of length three or more contribute very little to the summation.)

Hitting time and PageRank: A random walk on \( G \) starts at a node \( x \), and iteratively moves to a neighbor of \( x \) chosen uniformly at random. The hitting time \( H(x, y) \) from \( x \) to \( y \) is the expected number of steps required for a random walk starting at \( x \) to reach \( y \). Since the hitting time is not in general symmetric, it is also natural to consider the commute time \( C(x, y) := H(x, y) + H(y, x) \). Both of these measures serve as natural proximity measures, and hence (negated) can be used as \( d(x, y) \). Random resets form the basis of the PageRank measure for Web pages, and we can adapt it for link prediction as follows: Define \( d(x, y) \) to be the stationary probability of \( y \) in a random walk that returns to \( x \) with probability \( \alpha \) each step, moving to a random neighbor with probability \( 1 - \alpha \).

Most of the methods are developed for different types of problems like information retrieval, ranking, prediction e.t.c. and developed for general graphs. In this article we studied a measure specially designed for complex network and discussed in the next subsection

3.2. Proposed metric on complex network: In this section we have demonstrated the procedure to transform a graph into points of a metric space and developed the methods of community detection with the help of metric defined for pair of points. We have also studied and analyzed the community structure of the network therein.

The nodes of the graph do not lie on a metric space. The standard Euclidean distance and spherical distance define over the adjacency or Laplacian matrices above failed to capture similarity information among the nodes of a complex network. On the other-hand, the algorithms developed based on shortest path or Jaccard similarity are computationally inefficient and have less success in terms of standard evaluation criteria (like, conductance and modularity).

In this work, we have tried to develop the notion of similarity among the nodes using some new matrices derived from adjacency matrix and degree matrix of the graph. Let \( A \) be the adjacency matrix and \( D \) the degree matrix of the graph \( G = (V, E) \). The Laplacian \( L = D - A \). We have defined two diagonal matrix of same size \( D(\lambda) \) and \( D(\lambda_x) \) where \( \lambda \) is a parameter determine
from the given graph and can be optimized from the optimization criteria of the problem under consideration. In $D(\lambda)$ a fixed optimally determine value is used in the diagonal entries of the matrix $D$ and in $D(\lambda_x)$ a variable value also optimally determine is used in the diagonal entries of the matrix $D$. The similarities are defined on matrices $L_1$ and $L_2$, where $L_1 = D(\lambda) + A$ and $L_2 = D(\lambda_x) + A$ respectively as spherical similarity among the rows and determine by applying a concave function $\phi$ over the standard notions of similarities like, Pearson coefficient($\sigma_{PC}$), Spacerman coefficient($\sigma_{SC}$) or Cosine similarity($\sigma_{CS}$). $\phi(\sigma)$ must be chosen using the chord condition to obtain a metric.

In this subsection we have demonstrated the algorithm to convert the nodes of the graph to the points of a metric space preserving the community structure of the graph. The algorithm depends on the sub modules 1) construction of $L_x$ ($L_1$ or $L_2$) and 2) obtaining a structure preserving distance function. The algorithm works as picking pair of nodes from $L_x$ and and computing distance defined in the second module.

3.2.1. $L_x$ construction

The $L_1$ is defined as $L_1 = D(\lambda) + A$, where $A$ is the adjacency matrix of the given network and $D(\lambda)$ is a diagonal matrix of same size with diagonal values equal to a non negative constant $\lambda$.

The $L_2$ is defined as $L_2 = D(\lambda_x) + A$, where $A$ is the adjacency matrix of the given network and $D(\lambda_x)$ is a diagonal matrix of same size with diagonal values determine by a non negative function $\lambda_x$ of the node $x$.

The choice of $\lambda$ and $\lambda_x$ plays a crucial role in combination with the function chosen in the second module for determination of a suitable metric and is discussed later part of this subsection.

3.2.2. Function selection

The function selection module determine the metric for pair of nodes. The function selector $\phi$ converts a similarity function (Pearson coefficient($\sigma_{PC}$), Spacerman coefficient($\sigma_{SC}$) or Cosine similarity($\sigma_{CS}$)) into a distance matrix. In general the similarity function satisfies the positivity and similarity condition of metric but not triangle inequality. $\phi$ is a metric preserving ($\phi(d(x_i, x_j) = d_\phi(x_i, x_j)$), concave and monotonically increasing function. The three conditions above refer to as chord condition. The $\phi$ function is chosen to have minimum internal area with chord.

3.2.3. Choice of $\lambda$ and $\phi(\sigma)()$

The choices in the above sub modules play a crucial role in the graph to metric transformation algorithm to be used for community detection. The complex network is characterized by small average diameter and high clustering coefficient. Several studies on network structure analysis reveal that there are hub nodes and local nodes characterizing the interesting structure of the complex network. Suppose we have taken $\phi = arccos$, $\sigma_{CS}$ and constant $\lambda \geq 0$. $\lambda = 0$ penalize the effect of direct edge in the metric and is suitable to extract communities from highly dense graph. $\lambda = 1$ place the similar weight of direct edge and common neighbor reduce the effect of
direct edge in the metric and is suitable to extract communities from moderately dense graph. \( \lambda = 2 \) set more importance to direct edge than common neighbor (this is the common case of available real networks). \( \lambda \geq 2 \) penalize the effect of common neighbor in the metric and is suitable to extract communities from very sparse graph. The choice of \( \lambda \) depends on the input graph, i.e. whether it is sparse or dense and its cluster structure. A more detailed explanation on the metric described above can be obtained in [25].

4. Nearest neighbor search on complex network using metric tree

There are a large number of methods developed to compute nearest neighbor search. However, finding nearest neighbor search on some data where dimension is high suffer from curse of dimensionality. Some recent research on this direction revealed that dimension constrained can be tackled by using efficient data structures like metric tree and locality sensitive hashing. In this section we have explored metric tree to perform nearest neighbor search on complex network with the help of metric mapping of complex network described in the previous section.

4.1. Metric-tree

A metric tree is a data structure specially designed to perform nearest neighbor query for the points residing on a metric space and perform well on high dimension particularly when some approximation is permitted. A metric tree organizes a set of points in a spatial hierarchical manner. It is a binary tree whose nodes represent a set of points. The root node represents all points, and the points represented by an internal node \( v \) is partitioned into two subsets, represented by its two children. Formally, if we use \( N(v) \) to denote the set of points represented by node \( v \), and use \( v.lc \) and \( v.rc \) to denote the left child and the right child of node \( v \), then we have \( N(v) = N(v.lc) \cup N(v.rc) \phi = N(v.lc) \cap N(v.rc) \) for all the non-leaf nodes. At the lowest level, each leaf node contains very few points.

An M-Tree [26] has these components and sub-components:

- Non-leaf nodes: A set of routing objects \( N_{RO} \), Pointer to Node’s parent object \( v_p \).
- Leaf nodes: A set of objects \( N_v \), Pointer to Node’s parent object \( v_p \).
- Routing Object: (Feature value of) routing object \( v_r \), Covering radius \( r(v_r) \), Pointer to covering tree \( T(v_r) \), Distance of \( v_r \) from its parent object \( d(v_r, P(v_r)) \)
- Object: (Feature value of the) object \( v_j \), Object identifier \( oid(v_j) \), Distance of \( v_j \) from its parent object \( d(v_j, P(v_j)) \)

Partitioning: The key to building a metric-tree is how to partition a node \( v \). A typical way is as follows: We first choose two pivot points from \( N(v) \), denoted as \( v.lp v \) and \( v.rpv \). Ideally, \( v.lp v \) and \( v.rpv \) are chosen so that the distance between them is the largest of all distances within \( N(v) \). More specifically, \( \|v.lp v - v.rpv\| = max_{p_1, p_2 \in N(v)} \|p_1 - p_2\| \). However, it takes \( O(n^2) \) time to find the optimal \( v.lp v \) and \( v.rpv \). In practice a linear-time heuristic is used to find reasonable pivot
points. \(v.lpv\) and \(v.rpv\) are then used to partition node \(v\). We first project all the points down to the vector \(u = v.rpv - v.lpv\), and then find the median point \(A\) along \(u\). Next, we assign all the points projected to the left of \(A\) to \(v.lc\), and all the points projected to the right of \(A\) to \(v.rc\). We use \(L\) to denote the \(d-1\) dimensional plane orthogonal to \(u\) and goes through \(A\). It is known as the decision boundary since all points to the left of \(L\) belong to \(v.lc\) and all points to the right of \(L\) belong to \(v.rc\). By using a median point to split the data points, we can ensure that the depth of a metric-tree is \(\log n\). Each node \(v\) also has a hypersphere \(B\), such that all points represented by \(v\) fall in the ball centered at \(v.center\) with radius \(v.r\), i.e. \(N(v) \in B(v.center, v.r)\).

Searching: A search on a metric-tree is performed using a stack. The current radius \(r\) is used to decide which child node to search first. If the query \(q\) is on the left of current point, then \(v.lc\) is searched first, otherwise, \(v.rc\) is searched first. At all times, the algorithm maintains a candidate NN and there distance determine the current radius, which is the nearest neighbor it finds so far while traversing the tree. We call this point \(x\), and denote the distance between \(q\) and \(x\) by \(r\). If algorithm is about to exploit a node \(v\), but discovers that no member of \(v\) can be within distance \(r\) of \(q\), then it skip the subtree from \(v\). This happens whenever \(v.center - |q - v.r| \geq r\).

In practice, metric tree search typically finds a very good NN candidate quickly, and then spends lots of the time verifying that it is in fact the true NN. However, in case of approximate NN we can save majority of time with moderate approximation guarantee. The algorithm for NN search using metric tree is given below 1.

4.2. Nearest Neighbor search algorithm using M-Tree algorithm

**Algorithm 1** NN search in M-Tree

- **Require:** \(M = (V, d) & q\)
- **Ensure:** \(d(q, v_q)\)

1: Insert root object \(v_r\) in stack
2: Set current radius as \(d(v_r, q)\)
3: Successively traverse the tree in search of \(q\)
4: PUSH all the objects of traversal path into stack
5: Update the current radius
6: If leaf object reached
7: POP objects from stack
8: For all points lying inside the ball of current radius centering \(q\), verify for possible nearest neighbor and update the current radius.
9: **return** \(d(q, v_q)\)

**Theorem 4.** Let \(M = (V, d)\), be a bounded metric space. Then for any fixed data \(V \in R^n\) of size \(n\), and for constant \(c \geq 1\), \(\exists \epsilon\) such that we may compute \(d(q, V)\) with at most \(c \cdot [\log(n) + 1]\) expected metric evaluations[27]

5. Nearest neighbor search on complex network using locality sensitive hashing
Metric trees, so far represent the practical state of the art for achieving efficiency in the largest dimensionality possible. However, many real-world problems are posed with very large dimensionality which are beyond the capability of such search structures to achieve sub-linear efficiency. Thus, the high-dimensional case is the long-standing frontier of the nearest-neighbor problem.

The approximate nearest neighbor can be computed very efficiently using Locality sensitive hashing.

5.1. Approximate nearest neighbor

Given a metric space \((S, d)\) and some finite subset \(S_D\) of data points \(S_D \subset S\) on which nearest neighbor queries are to be made, our aim to organize \(S_D\) s.t. NN queries can be answered more efficiently. For any \(q \in S\), NN problem consists of finding single minimal located point \(p \in S_D\) s.t. \(d(p, q)\) is minimum over all \(p \in S_D\). We denote this by \(p = NN(q, S_D)\).

An \(\epsilon\) approximate NN of \(q \in S\) is to find a point \(p \in S_D\) s.t. \(d(p, q) \leq (1 + \epsilon)d(x, d) \forall x \in S_D\).

5.2. Locality Sensitive Hashing (LSH)

Several methods to compute first nearest neighbor query exists in the literatures and locality-sensitive hashing (LSH) is most popular because of its dimension independent runtime \([28,29]\). In a locality sensitive hashing, the hash function has the property that close points are hash into same bucket with high probability and distance points are hash into same bucket with low probability. Mathematically, a family \(H = \{h : S \rightarrow U\}\) is called \((r_1, r_2, p_1, p_2)\)-sensitive if for any \(p, q \in S\)

- if \(p \in B(q, r_1)\) then \(Pr_H[|h(q) = h(p)] \geq p_1\)
- if \(p \notin B(q, r_2)\) then \(Pr_H[|h(q) = h(p)] \leq p_2\)

where \(B(q, r)\) denotes a hypersphere of radius \(r\) centered at \(q\). In order for a locality-sensitive family to be useful, it has to satisfy inequalities \(p_1 > p_2\) and \(r_1 < r_2\) when \(D\) is a dissimilarity measure, or \(p_1 > p_2\) and \(r_1 > r_2\) when \(D\) is a similarity measure\([30,31]\). The value of \(\delta = \log(1/P_1)/\log(1/P_2)\) determines search performance of LSH. Defining a LSH as \(a(r, r(1+\epsilon), p1, p2)\), the \((1 + \epsilon)\) NN problem can be solved via series of hashing and searching within the buckets \([5,32,33]\).

5.3. Locality sensitive hash function for complex network

In this sub-section, we discuss the existence of locality sensitive hash function families for the proposed metric on complex network. The LSH data structure stores all nodes in hash tables and searches for nearest neighbor via retrieval. The hash table is contain many buckets and identified by bucket id. Unlike conventional hashing the LSH approach try to maximize the probability of collision of near items and put them into same bucket. For any given the query \(q\) the bucket \(h(q)\) considered to search nearest node. In general \(k\) hash functions are chosen independently and uniformly at random from hash family \(H\). The output of nearest neighbor query is provided from the union ok \(k\) buckets. The consensus
of $k$ functions reduces the error of approximation. For metric defined in the previous section 3 we considered $k$ random points from the metric space. Each random point $r_i$ define a hash function $h_i(x) = \text{sign}(d(x, r_i))$, where $d$ is the metric and $i \in [1, k]$. These randomized hash functions are locality sensitive [34,35].

Algorithm 2 NN search in LSH

Require: $M = (V, d) \& q$

Ensure: $d(q, V)$

1: Identify buckets of query point $q$ corresponding to different hash functions.
2: Compute nearest neighbor of $q$ only for the points inside the selected buckets.
3: return $d(q, V)$

Theorem 5. Let $M = (V, d)$, be a bounded metric space. Then for any fixed data $V \in \mathbb{R}^n$ of size $n$, and for constant $c \geq 1$, $\exists \varepsilon$ such that we may compute $d(q, V) \in \varepsilon$ with at most $mnO(1/\varepsilon)$ expected metric evaluations, where $m$ is the number of dimension of the metric space. In case of complex network $m = n$ so expected time is $nO(2/\varepsilon)$ [27,36].

6. Network community detection using nearest neighbor search

Community detection in real networks aims to capture the structural organization of the network using the connectivity information as input[15,16]. Early work on this domain was attempted by Weiss and Jacobson while searching for a work group within a government agency[14].

Most of the methods developed for network community detection are based on a two-step approach. The first step is specifying a quality measure (evaluation measure, objective function) that quantifies the desired properties of communities and the second step is applying an algorithmic techniques to assign the nodes of graph into communities by optimizing the objective function.

Several measures for quantifying the quality of communities have been proposed, they mostly consider that communities are set of nodes with many edges between them and few connections with nodes of different communities(e.g. modularity, conductance, expansion, internal density, average degree, triangle precipitation ratio,...,e.t.c.).

6.1. Popular algorithms In this subsection we have given a brief list of the algorithms developed for network community detection purposes. The broad categorization of the algorithms are based on graph traversal, semidefinite programming and spectral. The basic approach and the complexity of very popular algorithms are listed in the table 1. There are more algorithms developed to solve network community detection problem a complete list can be obtained in several survey articles [13,37,38].

A partial list of algorithms developed for network community detection purpose is tabulated in 1. The algorithms are categorized into three main group as spectral (SP), graph traversal based (GT) and semi-definite programming based (SDP). The categories and complexities are also given in the table 1.
Table 1. Algorithms for network community detection and their complexities

| Author | Ref. | Cat. | Order |
|--------|------|------|-------|
| Van Dongen | (Graph clustering, 2000[39]) | GT | $O(n k^2)$, $k \leq n$ parameter |
| Eckmann & Moses | (Curvature, 2002[40]) | GT | $O(mk^2)$ |
| Girvan & Newman | (Modularity, 2002[41]) | SDP | $O(n^2m)$ |
| Zhou & Lippoksky | (Vertex Proximity, 2004[42]) | GT | $O(mn)$ |
| Reichardt & Bornholdt | (spinglass, 2004[43]) | SDP | parameter dependent |
| Clauset et al. | (fast greedy, 2004[44]) | SDP | $O(m log n)$ |
| Newman & Girvan | (eigenvector, 2004[16]) | SP | $O(nm^2)$ |
| Wu & Huangman | (linear time, 2004[45]) | SP | $O(m log n)$ |
| Fortunato et al. | (infocentrality, 2004[46]) | SP | $O(m^3/n)$ |
| Radicchi et al. | (Radicchi et al., 2004[12]) | SP | $O(m^3/n^2)$ |
| Donetti & Munoz | (Donetti and Munoz, 2004[47]) | SDP | $O(n^4)$ |
| Guimera et al. | (Simulated Annealing, 2004[48]) | SDP | parameter dependent |
| Clauset et al. | (Capecci et al., 2004[49]) | SP | $O(mn^2)$ |
| Latapy & Pons | (walktrap, 2004[50]) | SP | $O(mn)$ |
| Duch & Arenas | (Extremal Optimization, 2005[51]) | SP | $O(n^2 log n)$ |
| Bagnes & Bolli | (Local method, 2005[52]) | SP | $O(n^2)$ |
| Palla et al. | (overlapping community, 2005[53]) | GT | $O(cap(m))$ |
| Raghavan et al. | (label propagation, 2007[54]) | GT | $O(n + m)$ |
| Rosvall & Bergstrom | (Infomap, 2008[55]) | SP | $O(m)$ |
| Roncides & Nenov | (Multiresolution community, 2009[56]) | SP | $O(m log n)$, $\beta \leq 1.3$ |

6.2. k-central algorithm for network community detection using nearest neighbor search In this section we have described k-central algorithm for the purpose of network community detection by using the nearest neighbor search inside complex network. We have also studied and analyzed the advantages of the k-central method over the standard algorithm for network community detection.

6.2.1. k-central algorithm

The community detection methods based on partitioning of graph is possible using nearest neighbor search, because the nodes of the graph are converted into the points of a metric space. This algorithm for network community detection converges automatically and does not compute the value of objective function in iteration therefore reduce the computation compared to standard methods. The results of this algorithm are competitive on a large set networks shown in section 7. The k-central algorithm for community detection and its details analysis is given below.

6.2.2. k selection

Determining the optimal number of k is an important problem for community detection researchers. An extensive analysis can be found in the work of Leskovec et al. [56]. The standard practice is to solve an optimization equation with respect to k for which the optimal value of the objective function is achieved. Another method based on farthest first traversal is also very useful in terms of computational efficiency [57]. For small networks the global optimization works better and for very large network the second choice give the faster approximate solution.

6.2.3. Initialization

The set of initial nodes are also very important problem for k-central algorithm

- Input: graph $G = (V, E)$, with the node similarity $sim(x_a, x_b)$ defined on it
- Output: A partition of the nodes into $k$ communities $C_1, C_2, ..., C_k$
- Objective function: Maximize the minimum intra community similarity

\[ \min_{j \in \{1, 2, \ldots, k\}} \max_{x_a, x_b \in C_j} \text{sim}(x_a, x_b) \]

**Algorithm 3** k-central algorithm

**Require:** \( M = (V, d) \)

**Ensure:** \( T = \{C_1, C_2, \ldots, C_k\} \) with minimum \( \text{cost}(T) \)

1. Initialize centers \( z_1, \ldots, z_k \in \mathbb{R}^n \) and clusters \( T = \{C_1, C_2, \ldots, C_k\} \)
2. repeat
   3. for \( i = 1 \) to \( k \) do
      4. for \( j = 1 \) to \( k \) do
         5. \( C_i \leftarrow \{x \in V \text{ s.t. } |z_i - x| \leq |z_j - x|\} \)
      6. end for
   7. end for
   8. for \( j = 1 \) to \( k \) do
      9. \( z_i \leftarrow \text{Central}(C_i) \); where \( \text{Central}(C_i) \) returns the node with minimum total distance in the class of consideration.
   10. end for
11. until \( |\text{cost}(T_t) - \text{cost}(T_{t+1})| = 0 \)
12. return \( T = \{C_1, C_2, \ldots, C_k\} \)

6.2.4. Convergence

Convergence of the network community detection algorithms are the least studied research areas of network science. However, the rate of convergence is one of the important issues and low rate of convergence is the major pitfall of the most of the existing algorithms. Due to the transformation into the metric space, our algorithm equipped with the quick convergence facility of the k-partitioning on metric space by providing a good set of initial points. Another crucial pitfall suffer by majority of the existing algorithms is the validation of the objective function in each iteration during convergence. Our algorithm converges automatically to the optimal partition thus reduces the cost of validation during convergence.

**Theorem 6.** During the course of the k center partitioning algorithm, the cost monotonically decreases.

**Proof.** Let \( Z^t = \{z_1^t, \ldots, z_k^t\} \), \( T^t = \{C_1^t, \ldots, C_k^t\} \) denote the centers and clusters at the start of the \( t^{th} \) iteration of k partitioning algorithm. The first step of the iteration assigns each data point to its closest center; therefore \( \text{cost}(T^{t+1}, Z^t) \leq \text{cost}(T^t, Z^t) \)

On the second step, each cluster is re-centered at its mean; therefore \( \text{cost}(T^{t+1}, Z^{t+1}) \leq \text{cost}(T^{t+1}, Z^t) \)

\[ \Box \]

7. Experiments and results We performed many of experiments to test the performance of
nearest neighbor search based community detection method for complex network over several real networks. Objective of the experiment is to verify behavior of the algorithm and the time required to compute the algorithm. One of the major goals of the experiment is to verify the behavior of the algorithm with respect to the performance of other popular methods exists in the literature with respect to the standard measures like conductance and modularity. Experiments are conducted to compare the results (tables 3, 4 and 5) of our algorithm with the state of the art algorithms (table 1) available in the literature in terms of common measures mostly used by the researchers of the domain of network community detection. The details of the several experiments and the analysis of the results are given in the following subsections.

7.1. Experimental designs

Experiment for comparison: In this experiment we have compared several algorithms for network community detection with our proposed algorithm developed using nearest neighbor search in complex network. Experiment is performed on a large list of network data sets. Two version of the experiment is developed for comparison purpose based on two different quality measure conductance and modularity. The results are shown in the tables 3 and 4 respectively.

Experiment on performance and time: In this experiment we have evaluated our algorithm for performance on the network collection. We have evaluated the time taken by our algorithm on different size of networks and is shown in the table 5.

7.2. Performance indicator

Modularity: The notion of modularity is the most popular for the network community detection purpose. The modularity index assigns high scores to communities whose internal edges are more than that expected in a random-network model which preserves the degree distribution of the given network.

Conductance: Conductance is widely used for graph partitioning literature. The conductance of a set $S$ with complement $S^C$ is the ratio of the number of edges connecting nodes in $S$ to nodes in $S^C$ by the total number of edges incident to $S$ or to $S^C$ (whichever number is smaller).

7.3. Datasets

A list of real networks taken from several real life interactions is considered for our experiments and they are tabulate below. We have also listed the number of nodes, number of edges, average diameter, data complexity for community detection (DCC) and the k value used (6.2.2). The values of the last column can be used to assess the quality of detected communities.

7.4. Computational results

In this subsection we have compared two groups of algorithms for network community detection with our proposed algorithm using nearest neighbor search. Experiment is performed on a large list of network data sets. Two version of the experiment is developed for comparison purpose based on two different quality measure conductance and modularity. The results based on conductance is shown in the table 3 and the results based
Table 2. Complex network datasets and values of their parameters

| Name           | Type | # Nodes | # Edges | Diameter | k   |
|----------------|------|---------|---------|----------|-----|
| DBLP           | U    | 317,080 | 1,049,866 | 8        | 268 |
| Arxiv-AstroPh  | U    | 18,772  | 396,160  | 5        | 23  |
| web-Stanford   | D    | 284,369 | 2,322,492 | 9.1      | 187 |
| Facebook       | U    | 4,389   | 88,234   | 4.7      | 164 |
| Gephi          | D    | 107,814 | 13,671,453| 3        | 357 |
| Twitter        | D    | 81,396  | 1,768,149 | 4.5      | 256 |
| Epinions1      | D    | 75,875  | 508,837  | 5        | 128 |
| LiveJournal1   | D    | 4,877,771| 68,993,773| 6.6      | 111 |
| DBLP           | U    | 1,134,250| 2,587,224 | 6.6      | 811 |
| Pokoc          | D    | 1,632,803| 30,622,564| 5.2      | 246 |
| Slashdot0811   | D    | 17,360  | 505,468  | 4       | 81  |
| Slashdot0922   | D    | 82,168  | 948,464  | 4.7      | 87  |
| Friendster     | U    | 65,608,366| 18,066,153| 5.8      | 833 |
| Amazon0601     | D    | 401,394 | 3,387,458 | 7.5      | 92  |
| P2P-Gnutella31 | D    | 62,586  | 147,892  | 6.6      | 35  |
| RoadNet-CA     | D    | 1,365,206| 5,533,214 | 590      | 322 |
| Wiki-Vote      | D    | 7,113   | 109,569  | 3.8      | 21  |

on modularity is shown in the table 4, respectively. Regarding the two groups of algorithms; first group contain algorithms based on semi-definite programming and the second group contain algorithms based on graph traversal approaches. For each group, we have taken the best value of conductance in table 3 and best value of modularity in table 4 among all the algorithms in the groups. The results obtained with our approach are very competitive with most of the well known algorithms in the literature and this is justified over the large collection of datasets. On the other hand, it can be observed that time taken (table 5) by our algorithm is quite less compared to other methods and justify the theoretical findings.

Table 3. Comparison of our approaches with other best methods in terms of conductance

| Name           | Spectra | M-tree | LSH | Index | M-Tree | LSH |
|----------------|---------|--------|-----|-------|--------|-----|
| Facebook       | 0.0077  | 0.1974 | 0.1044 | 0.1062 | 0.0827 | 0.0340 |
| Gephi          | 0.0119  | 0.1593 | 0.1644 | 0.1602 | 0.1207 | 0.0500 |
| Twitter        | 0.0084  | 0.0480 | 0.0465 | 0.0483 | 0.0363 | 0.0150 |
| Epinions1      | 0.0087  | 0.1247 | 0.1208 | 0.1204 | 0.0741 | 0.0309 |
| LiveJournal1   | 0.0069  | 0.0626 | 0.0649 | 0.0696 | 0.0523 | 0.0218 |
| Pokoc          | 0.0069  | 0.0174 | 0.0168 | 0.0175 | 0.0129 | 0.0054 |
| Slashdot0811   | 0.0069  | 0.0247 | 0.0201 | 0.0254 | 0.0150 | 0.0064 |
| Slashdot0922   | 0.0070  | 0.0188 | 0.0183 | 0.0188 | 0.0102 | 0.0043 |
| Friendster     | 0.0042  | 0.0273 | 0.0264 | 0.0273 | 0.0208 | 0.0084 |
| Epinions1      | 0.0016  | 0.0411 | 0.0390 | 0.0412 | 0.0330 | 0.0129 |
| Youtube        | 0.0031  | 0.0899 | 0.0838 | 0.0871 | 0.0663 | 0.0262 |
| DBLP           | 0.0005  | 0.0210 | 0.0203 | 0.0211 | 0.0152 | 0.0064 |
| Arxiv-AstroPh  | 0.0024  | 0.0299 | 0.0289 | 0.0291 | 0.0169 | 0.0083 |
| web-Stanford   | 0.0007  | 0.0320 | 0.0308 | 0.0320 | 0.0229 | 0.0099 |
| Amazon0601     | 0.0048  | 0.0899 | 0.0865 | 0.0999 | 0.0643 | 0.0247 |
| P2P-Gnutella31 | 0.0009  | 0.0522 | 0.0503 | 0.0523 | 0.0373 | 0.0158 |
| RoadNet-CA     | 0.0024  | 0.1502 | 0.1445 | 0.1504 | 0.1070 | 0.0455 |
| Wiki-Vote      | 0.0026  | 0.1853 | 0.1701 | 0.1856 | 0.1318 | 0.0561 |

7.5. Results analysis and achievements In this subsection, we have described the analysis of the results obtained in our experiments shown above and also highlighted the achievements from the results. It is clearly evident from the results shown in the tables 3, 4 and 5 that, proposed nearest neighbor based method for network community detection using metric tree and locality sensitive hashing provide very good competitive performance with respect to conductance and modularity and also in terms of time. It is also evident from the results that our methods provide case base solution of network community detection depending on the requirements of time or better conductance/modularity.
8. Conclusions In this paper, we have studied the interesting problem of nearest neighbor queries in complex networks. Processing nearest neighbor search in complex networks cannot be achieved by straightforward applications of previous approaches for the Euclidean space due to the complexity of graph traversal based computations of node nearness as opposed to geometric distances. We presented the transformation of graph to metric space and efficient computation of nearest neighbor therein using metric tree and locality sensitive hashing. Our techniques can be applied for various structural analysis of complex network using geometric approaches. To validate the performance of proposed nearest neighbor search designed for complex networks we applied our approaches on community detection problem. The results obtained on several network data sets prove the usefulness of the proposed method and provide motivation for further application of other structural analysis of complex network using nearest neighbor search.

Acknowledgments

This work is supported by the Jaypee University of Information Technology.

Author Contributions
Suman Saha proposed the algorithm and prepared the manuscript. S.P. Ghrera was in charge of the overall research and critical revision of the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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