ON THE SHARP CONSTANT IN THE BIANCHI–EGNELL
STABILITY INEQUALITY

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ABSTRACT. This note is concerned with the Bianchi-Egnell inequality, which quantifies the stability of the Sobolev inequality, and its generalization to fractional exponents $s \in (0, \frac{d}{2})$. We prove that in dimension $d \geq 2$ the best constant $c_{BE}(s) = \inf_{f \in H^s(\mathbb{R}^d) \setminus M} \frac{\|(-\Delta)^{s/2} f\|_{L^2(\mathbb{R}^d)}^2 - S_{d,s} \|f\|_{L^{2^*}(\mathbb{R}^d)}^2}{\text{dist}_{H^s(\mathbb{R}^d)}(f,M)^2}$ is strictly smaller than the spectral gap constant $\frac{4s}{4s + 2} = \frac{4s}{4s + 2}$ associated to sequences which converge to the manifold $M$ of Sobolev optimizers. In particular, $c_{BE}(s)$ cannot be asymptotically attained by such sequences. Our proof relies on a precise expansion of the Bianchi-Egnell quotient along a well-chosen sequence of test functions converging to $M$.

1. Introduction and main result

For $0 < s < \frac{d}{2}$, the (fractional) Sobolev inequality states that

$$\|(-\Delta)^{s/2} f\|_{L^2(\mathbb{R}^d)}^2 \geq S_{d,s} \|f\|_{L^{2^*}(\mathbb{R}^d)}^2,$$

for every $f$ in the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$. Here

$$2^* = \frac{2d}{d - 2s}$$

is the critical Sobolev exponent and the best constant $S_{d,s}$ is given by

$$S_{d,s} = 2^{2s} \pi^s \frac{\Gamma\left(\frac{d+2s}{2}\right)}{\Gamma\left(\frac{d-2s}{2}\right)} \left(\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma(d)}\right)^{\frac{2s}{d}}$$

(1.2)

It was shown by Lieb [10] in an equivalent dual formulation (and before by Aubin [2] and Talenti [12] for $s = 1$) that $S_{d,s}$ is optimal and that it is achieved precisely by the so-called Talenti bubbles, i.e. functions belonging to the $(d + 2)$-dimensional manifold

$$\mathcal{M} := \left\{ x \mapsto c(a + |x - b|^2)^{-\frac{d-2s}{2}} : a > 0, b \in \mathbb{R}^d, c \in \mathbb{R} \setminus \{0\} \right\}.$$
In recent years, the study of the stability properties of the Sobolev and other geometric inequalities has attracted much interest and remarkable advances. The natural stability inequality associated to (1.1) has been established by Bianchi and Egnell [4] for \( s = 1 \) and \( d \geq 3 \) and was later extended in [6] to all \( s \in (0, \frac{d}{2}) \). It states that there is a constant \( c_{BE}(s) > 0 \) such that
\[
\mathcal{E}(f) \geq c_{BE}(s)
\]
for all \( f \in \dot{H}^s(\mathbb{R}^d) \setminus \mathcal{M} \). (1.4)

Here and in the rest of this paper, dist denotes the distance in \( \dot{H}^s(\mathbb{R}^d) \), i.e.
\[
\text{dist}(f, \mathcal{M}) := \inf_{g \in \mathcal{M}} \|(-\Delta)^{s/2}(f-g)\|_2.
\]
Moreover, we abbreviate \( \| \cdot \|_p = \| \cdot \|_{L^p(\mathbb{R}^d)} \) for ease of notation.

Since the proof of (1.4) in [4, 6] proceeds by compactness, it yields no explicit lower bound on the constant \( c_{BE}(s) \). For \( s = 1 \), the first constructive lower bound on \( c_{BE}(1) \) is proved in the recent preprint [9].

On the other hand, it is standard to derive an upper bound on \( c_{BE}(s) \) by using an explicit sequence of test functions of the form
\[
f_\varepsilon(x) = (1 + |x|^2)^{-\frac{d+2s}{2}} + \varepsilon \rho
\]
converging to \( \mathcal{M} \) as \( \varepsilon \to 0 \). As observed in [6], a suitable choice of \( \rho \) then yields the bound
\[
c_{BE}(s) \leq \frac{4s}{d+2s+2}.
\](1.5)

The purpose of this note is to prove that the inequality (1.5) must in fact be strict.

**Theorem 1.** Let \( s \in (0, \frac{d}{2}) \) with \( d \geq 2 \) and let \( c_{BE}(s) \) be the optimal constant in (1.4). Then
\[
c_{BE}(s) < \frac{4s}{d+2s+2}.
\]

Let us emphasize that Theorem 1 covers in particular the classical case \( s = 1, d \geq 3 \), and that its conclusion is new also for this case.

Theorem 1 shows that the study of the sharp constant \( c_{BE}(s) \) cannot be reduced to a local analysis near the manifold \( \mathcal{M} \). This phenomenon is analogous to the situation for the planar isoperimetric inequality and its associated stability inequality; we refer to the introduction of [9] for more details and references about this case.

We finally mention that the recent preprint [7] gives an abstract condition under which the global best constant of a general stability inequality is equal to the local best constant corresponding to the right side of (1.5). However, consistently with Theorem 1 this condition is not fulfilled for the Sobolev stability inequality (1.4).
2. Proof of Theorem 1

We prove Theorem 1 by using the same idea as for inequality (1.5), but we manage to be more precise in the asymptotic expansion near $\mathcal{M}$ and in the choice of the perturbation $\rho$.

To be more precise, let us introduce some more notation. We fix the standard Talenti bubble

$$U(y) := (1 + |y|^2)^{-\frac{d-2s}{2}}.$$ 

and denote its dilated and translated version by

$$U_{x,\lambda}(y) := \lambda^{\frac{d-2s}{2}} U(\lambda(y-x)).$$

Then the tangent space $T$ of $\mathcal{M}$ at $U$ is spanned by $U$ and the $d+1$ functions

$$V_0 := \partial_{\lambda}|_{\lambda=1} U_{0,\lambda}, \quad V_i := \partial_{x_i}|_{x=0} U_{x,1}, \quad i = 1, ..., d$$

We denote by $T^\perp$ the orthogonal of $T$ with respect to the $\dot{H}^s$ scalar product $\langle u, v \rangle_{\dot{H}^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (-\Delta)^{s/2} u(-\Delta)^{s/2} v \, dx$.

We also introduce the (inverse) stereographic projection $S : \mathbb{R}^d \to S^d$ given, in Cartesian coordinates of $S^d \subset \mathbb{R}^{d+1}$, by

$$(S(x))_i = \frac{2x_i}{1+|x|^2} \quad (i = 1, ..., d), \quad (S(x))_{d+1} = \frac{1-|x|^2}{1+|x|^2}.$$ (2.1)

It is convenient to denote by $J_S(x) = |\det DS(x)| = \left(\frac{2}{1+|x|^2}\right)^d = 2^d U(x)^2$ its Jacobian. The relevant transformation properties of the stereographic projection and discussed to some greater extent, e.g., in [6, Section 2].

The following spectral gap inequality plays a crucial role in [4] and [6] and seems to appear for the first time in [11], for $s = 1$. For $s \in (0, \frac{d}{2})$, a detailed statement and proof can be found in [8].

**Proposition 2.** Let $\rho \in T^\perp$. Then

$$\|(-\Delta)^{s/2}\rho\|^2_2 - S_{d,s}(2^s - 1)\|U\|^2_2 2^s \int_{\mathbb{R}^d} U^{2^s-2} \rho^2 \, dx \geq \frac{4s}{d+2s+2} \|(-\Delta)^{s/2}\rho\|^2_2.$$ 

Moreover, equality holds if and only if

$$\rho(x) = J_S(x)^{\frac{d}{2s}} v_2(S(x)),$$ (2.2)

with $v_2$ a spherical harmonic of degree $\ell = 2$ (i.e. the restriction to $S^d$ of a homogeneous harmonic polynomial on $\mathbb{R}^{d+1}$ of degree 2).

We can now prove our main result.
Proof of Theorem 1. We build a sequence of test functions of the form

\[ f_\varepsilon = U + \varepsilon \rho, \]

where \( \varepsilon \to 0 \) and \( \rho \in T^1 \) is as in (2.2), with a certain spherical harmonic \( v_2 \) to be determined.

The orthogonality relations, and the fact that \( (-\Delta)^{s/2} U = c_{d,s} U^{2^*-1} \), for \( c_{d,s} = \frac{S_{d,s}}{\|U\|_{2^*}^{2^*-2}} \), easily imply

\[ \|f_\varepsilon\|_2^2 = \|(-\Delta)^{s/2} U\|_2^2 + \varepsilon^2 \rho^2 \]

and

\[ \int_{\mathbb{R}^d} U^{2^*-1} \rho \, dx = \frac{1}{c_{d,s}} \int_{\mathbb{R}^d} (-\Delta)^{s/2} U(-\Delta)^{s/2} \rho \, dx = 0. \]

On the other hand, a Taylor expansion yields

\[ (U + \varepsilon \rho)^{2^*} = U^{2^*} + \varepsilon 2^* U^{2^*-1} \rho + \varepsilon^2 \frac{2^*(2^*-1)}{2} U^{2^*-2} \rho^2 + \varepsilon^3 \frac{2^*(2^*-1)(2^*-2)}{6} U^{2^*-3} \rho^3 + o(\varepsilon^3), \]

Notice that the Taylor expansion up to third order is justified no matter the value of \( 2^* \), because \( \varepsilon |\rho(x)| << U(x) \) in every point \( x \in \mathbb{R}^d \), as \( \varepsilon \to 0 \).

Hence, using \( \int_{\mathbb{R}^d} U^{2^*-1} \rho \, dx = 0 \),

\[ \|U + \varepsilon \rho\|_2^{2^*} = \left( \int_{\mathbb{R}^d} \left( U^{2^*} + \varepsilon^2 \frac{2^*(2^*-1)}{2} U^{2^*-2} \rho^2 + \varepsilon^3 \frac{2^*(2^*-1)(2^*-2)}{6} U^{2^*-3} \rho^3 \right) \, dx \right)^{\frac{1}{2^*}} + o(\varepsilon^3) \]

\[ = \|U\|_2^{2^*} + \varepsilon^2 (2^*-1) \|U\|_2^{2^*-2} \int_{\mathbb{R}^d} U^{2^*-2} \rho^2 \, dx \]

\[ + \varepsilon^3 \frac{(2^*-1)(2^*-2)}{3} \|U\|_2^{2^*-3} \int_{\mathbb{R}^d} U^{2^*-3} \rho^3 \, dx + o(\varepsilon^3). \]

Now recall that \( \|(-\Delta)^{s/2} U\|_2^2 = S_{d,s} \|U\|_2^{2^*} \), and that we have chosen \( \rho \) to achieve equality in Proposition 2. We thus find that the numerator of the Bianchi–Egnell quotient in (1.4) equals

\[ \|(-\Delta)^{s/2} f_\varepsilon\|_2^2 = S_{d,s} \|f_\varepsilon\|_2^2, \]

\[ = \left( \|(-\Delta)^{s/2} U\|_2^2 - S_{d,s} \|U\|_2^{2^*} \right) + \varepsilon^2 \left( \|(-\Delta)^{s/2} \rho\|_2^2 - S_{d,s}(2^*-1) \|U\|_2^{2^*-2} \int_{\mathbb{R}^d} U^{2^*-2} \rho^2 \, dx \right) \]

\[ - \varepsilon^3 S_{d,s} \frac{(2^*-1)(2^*-2)}{3} \|U\|_2^{2^*-3} \int_{\mathbb{R}^d} U^{2^*-3} \rho^3 \, dx + o(\varepsilon^3) \]

\[ = \frac{4}{d+2s+2} \varepsilon^2 \|(-\Delta)^{s/2} \rho\|_2^2 - \varepsilon^3 S_{d,s} \frac{(2^*-1)(2^*-2)}{3} \|U\|_2^{2^*-2} \int_{\mathbb{R}^d} U^{2^*-3} \rho^3 \, dx + o(\varepsilon^3). \]

It follows by the implicit function theorem (see [3] for a similar argument for \( s = 1 \), which remains valid for all \( s \in (0, \frac{d}{2}) \) as observed in [1]) that for \( \varepsilon > 0 \) small enough
the minimum of the distance $\text{dist}(U + \varepsilon \rho, \mathcal{M})$ is in fact achieved in $U$. Hence
\[ \text{dist}(U + \varepsilon \rho, \mathcal{M})^2 = \varepsilon^2 \| (-\Delta)^{s/2} \rho \|^2_2. \]

Thus the above expansions yield, for $\varepsilon > 0$ small enough, the desired strict inequality
\[ \mathcal{E}(U + \varepsilon \rho) < \frac{4s}{d + 2s + 2}, \]
provided we can choose $v_2$ in a way such that (with the relation (2.2) between $v_2$ and $\rho$)
\[ \int_{\mathbb{R}^d} U^{2s-3} \rho^3 \, dx > 0. \quad (2.3) \]

To achieve this, we make the choice
\[ v_2(\omega) = \omega_1 \omega_2 + \omega_2 \omega_3 + \omega_3 \omega_1, \]
which is clearly a spherical harmonic of degree $\ell = 2$. (This choice of $v_2$ is what necessitates the additional hypothesis $d \geq 2$.) We have
\[ v_2(\omega)^3 = I_1(\omega) + I_2(\omega) + I_3(\omega), \]
where
\[ I_1(\omega) = 6 \omega_1^2 \omega_2^2 \omega_3^2, \]
\[ I_2(\omega) = 3(\omega_1 \omega_2^3 \omega_3^2 + \omega_1 \omega_3^2 \omega_2^3 + \omega_2 \omega_1^2 \omega_3^3 + \omega_3 \omega_1^2 \omega_2^3 + \omega_3 \omega_2^3 \omega_1^2 + \omega_2 \omega_3^2 \omega_1^3 + \omega_3 \omega_2^3 \omega_1^3), \]
\[ I_3(\omega) = \omega_1^2 \omega_2^3 + \omega_2^3 \omega_3^2 + \omega_3^2 \omega_2^3. \]

Writing (2.2) as $\rho(x) = 2^{\frac{d-2s}{2}} U(x) v_2(S(x))$, we get
\[
\int_{\mathbb{R}^d} U^{2s-3} \rho^3 \, dx = 2^{\frac{3(d-2s)}{2} - d} \int_{\mathbb{S}^d} v_2(\omega)^3 \, d\omega = 2^{\frac{3(d-2s)}{2} - d} \int_{\mathbb{S}^d} (I_1(\omega) + I_2(\omega) + I_3(\omega)) \, d\omega
\]
\[ = 6 \times 2^{\frac{3(d-2s)}{2} - d} \int_{\mathbb{S}^d} \omega_1^2 \omega_2^3 \omega_3^2 \, d\omega > 0, \]
because all the monomials in $I_2$ and $I_3$ are odd functions of at least one coordinate $\omega_i$ and thus cancel in the integral. This completes the proof of Theorem 1. \(\square\)

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