FUNCTIONS WITH LARGE ADDITIVE ENERGY SUPPORTED ON A HAMMING SPHERE

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Abstract. In this note, we prove that, among functions $f$ supported on a Hamming Sphere in $\mathbb{F}_2^n$ with fixed $\ell^2$ norm, the additive energy is maximised when $f$ is constant. This answers a question asked by Kirshner and Samorodnitsky.

1. Introduction

For a function $f : \mathbb{F}_2^n \to \mathbb{R}$, we define its Gowers $u_2$ norm to be

$$||f||_{u_2} = \left( \mathbb{E}_{a_1, a_2, a_3, a_4 \in \mathbb{F}_2^n} f(a_1)f(a_2)f(a_3)f(a_4) \right)^{1/4}. \tag{1.1}$$

This is also called the additive energy of $f$. This agrees with the usual notion of energy for sets (up to a scaling), in the sense that $||1_A||_{u_2}^4 = \frac{E(A)}{N^3}$, where $N = 2^n$ is the size of $\mathbb{F}_2^n$. Similarly, we will define the $\ell^2$ norm to be

$$||f||_2 = \left( \mathbb{E}_{a \in \mathbb{F}_2^n} f(a)^2 \right)^{1/2}.$$

For a set $A \subseteq \mathbb{F}_2^n$, define $\mu(A)$ by

$$\mu(A) = \max_{f : \mathbb{F}_2^n \to \mathbb{R}} \frac{||f||_{u_2}^4}{||f||_2}, \tag{1.2}$$

where $\text{supp}(f)$ denotes the support of $f$. Let the Hamming Sphere $S(n, k) \subseteq \mathbb{F}_2^n$ consist of those vectors of weight $k$; in other words, $S(n, k)$ consists of those vectors with exactly $k$ ones.

In [2], Kirshner and Samorodnitsky made the following conjecture:

Conjecture (Conjecture 1.9 from [2]). Let $A = S(n, k)$. Then, $\mu(A) = \frac{1}{N} \frac{E(A)}{|A|^2}$.

In other words, the ratio $\frac{||f||_{u_2}^4}{||f||_2}$ achieves its maximum when $f$ is constant.

The purpose of this note is to establish this conjecture.

Theorem 1.1. Let $A = S(n, k)$. Then, $\mu(A) = \frac{1}{N} \frac{E(A)}{|A|^2}$.

Remark. Kirshner and Samorodnitsky define $\mu(A)$ as the maximal value of $\frac{||f||_{u_2}^4}{||f||_2}$ among functions whose Fourier transform is supported on $A$. However, it can easily be seen that these two formulations are equivalent (up to normalisation) by taking a Fourier transform, and using Parseval’s identity and the relation that $||f||_{u_2}^4 = ||\hat{f}||_{u_2}^4$. 

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2. Proof of Theorem 1.1

Throughout the proof, let $e_1, \ldots, e_n$ denote the standard basis for $\mathbb{F}_2^n$, so that any element of $\mathbb{F}_2^n$ may be written $\sum_i \varepsilon_i e_i$, where $\varepsilon_i \in \{0, 1\}$. If $v, w \in \mathbb{F}_2^n$, let $\langle v, w \rangle$ denote the standard inner product of $v$ and $w$. In other words,

$$\left\langle \sum_i \varepsilon_i^{(1)} e_i, \sum_i \varepsilon_i^{(2)} e_i \right\rangle = \sum_i \varepsilon_i^{(1)} \varepsilon_i^{(2)}.$$

Our approach for proving Theorem 1.1 relies on the following lemma about compressions.

Definition 2.1. For a function $f : \mathbb{F}_2^n \to \mathbb{R}$ and $i < j \leq n$, define the $i, j$ compression $f^{(ij)}$ as follows:

$$f^{(ij)}(x) = \begin{cases} f(x) & \langle x, e_i + e_j \rangle = 0 \\ \sqrt{\frac{f(x)^2 + f(x + e_i + e_j)^2}{2}} & \text{otherwise}, \end{cases}$$

In other words, let $\pi_{ij} : \mathbb{F}_2^n \to \mathbb{F}_2^{(n-2)}$ denote the projection given by ignoring the coefficients of $e_i$ and $e_j$. Then, $f^{(ij)}(x)$ is the $\ell^2$-average of $f$ over elements of the coset of $\ker \pi_{ij}$ containing $x$, which have the same Hamming weight as $x$.

The proof of Theorem 1.1 relies on the following lemma about compressions.

Lemma 2.2. Let $A = S(n, k)$, and suppose that $f$ is supported on $A$.

- (1) $f^{(ij)}$ is also supported on $A$.
- (2) $\|f^{(ij)}\|_2 = \|f\|_2$.
- (3) $\|f^{(ij)}\|_{w_2} \geq \|f\|_{w_2}$.
- (4) $\|f^{(ij)}\|_{w_2} > \|f\|_{w_2}$ unless $f = f^{(ij)}$.

Proof. The proofs of (1) and (2) follow immediately from Definition 2.1.

For (3), observe that we may rewrite (1.1) as follows.

$$\|f\|_{w_2}^4 = \frac{1}{4^3} \sum_{b_1, b_2, b_3, b_4 \in \pi_{ij}(\mathbb{F}_2^n)} \left( \sum_{a_1 \in \pi_{ij}^{-1}(b_1) \cap A} f(a_1) f(a_2) f(a_3) f(a_4) \right), \quad (2.1)$$

where the outer expectation is over cosets of $\ker \pi_{ij}$, and the factor of $\frac{1}{4^3}$ comes from the fact that we have renormalised the inner expectation to be a summation. Our strategy will be to prove that each bracketed term on the right hand side of (2.1) does not decrease when we pass from $f$ to $f^{(ij)}$.

Observe that, if $f$ is supported on $A$, then the outer expectation of (2.1) may be restricted to terms such that each $b_i$ has Hamming weight either $k, k-1$ or $k-2$, and the size of $\pi_{ij}^{-1}(b_i) \cap A$ depends on whether $b_i$ has weight $k-1$ or not. Thus, we split naturally into three cases.

Case 1: None of $b_1, b_2, b_3$ or $b_4$ has Hamming weight $k-1$. In this case, the bracketed term is a sum over exactly one term, and is unchanged as we pass from $f$ to $f^{(ij)}$.

Case 2: Exactly two of $b_1, b_2, b_3$ and $b_4$ have Hamming weight $k-1$. Without loss of generality, it is $b_1$ and $b_2$ which have Hamming weight $k-1$. Then, there are two
possibilities for the bracketed term, depending on how many of $b_3$ and $b_4$ have weight $k - 2$. If neither or both of them do, then we may write the bracketed term as
\[f(b_1 + e_i)f(b_2 + e_i)f(a_3)f(a_4) + f(b_1 + e_j)f(b_2 + e_j)f(a_3)f(a_4),\]
where $a_3$ denotes the unique element of $\pi^{-1}_{ij}(b_3) \cap A$ (and likewise for $a_4$). The conclusion then follows from the assertion that
\[f(b_1 + e_i)f(b_2 + e_i) + f(b_1 + e_j)f(b_2 + e_j) \leq 2f^{(ij)}(b_1 + e_i)f^{(ij)}(b_2 + e_i),\]
which is a consequence of the Cauchy Schwarz inequality. A similar argument applies if exactly one of $b_3$ and $b_4$ have weight $k - 2$.

**Case 3:** All four of $b_1, b_2, b_3$ and $b_4$ have Hamming weight $k - 1$. In this case, the bracketed term is now a sum of eight terms. One of the terms is
\[f(a_1)f(a_2)f(a_3)f(a_4) = f(b_1 + e_i)f(b_2 + e_i)f(b_3 + e_i)f(b_4 + e_i),\]
and the others can be obtained by replacing two or four of the $e_i$ with $e_j$.

Group the terms into four pairs, according to the values of $a_3$ and $a_4$. If $a_3 = b_3 + e_1$ and $a_4 = b_4 + e_1$, for example, then we have
\[f(b_1 + e_i)f(b_2 + e_i)f(a_3)f(a_4) + f(b_1 + e_j)f(b_2 + e_j)f(a_3)f(a_4) \leq f^{(ij)}(a_1)f^{(ij)}(a_2)f(a_3)f(a_4),\]
as in case 2. The conclusion then follows from the fact that
\[f(b_3 + e_i) + f(b_3 + e_j) \leq 2f^{(ij)}(a_3),\]
which follows from Cauchy-Schwarz.

Finally, it remains to prove (1). But this is easy to do. Suppose that $f \neq f^{(ij)}$; in other words, there is some vector $v$ of weight $k - 1$, such that $f(v + e_i) \neq f(v + e_j)$. Then, consider the term of (2.1) coming from $b_1 = b_2 = b_3 = b_4 = v$. It is easy to see that equality will not hold in the relation
\[\sum f(a_1)f(a_2)f(a_3)f(a_4) \leq 8f^{(ij)}(a_1)f^{(ij)}(a_2)f^{(ij)}(a_3)f^{(ij)}(a_4).\]

We can now complete the proof of Theorem 1.1. By compactness, there must exist some function $f$ achieving the maximal value of $\|f\|_{u_2}$, for fixed $\|f\|_2$. Suppose that this maximal value is achieved for a function $f$ which is not constant.

Consider the Hamming Sphere as a graph, where we join two elements $v$ and $w$ with an edge if and only if $w = v + e_i + e_j$ for some $i$ and $j$. Then, the Hamming Sphere is connected. Thus, there must be two adjacent elements $v$ and $w$ for which $f(v) \neq f(w)$.

Thus, if $w = v + e_i + e_j$, then $f \neq f^{(ij)}$, and so Lemma 2.2 (1) tells us that $\|f\|_{u_2} < \|f^{(ij)}\|_{u_2}$, contradicting the maximality of $\|f\|_{u_2}$.

Therefore, $f$ must be constant, yielding Theorem 1.1.

References

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