Propagation and interaction of shock waves of quasilinear equation

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Abstract

We propose a new regularization method for constructing a shock wave type solution with nonsmooth front (interaction of shock waves) for quasilinear equations in the one-dimensional case.

1 Introduction and Main Results.

1. Consider the quasilinear first order equation

\[ L[u] = u_t + (f(u))_x = 0, \]

where \( f(u) \) is a smooth function of at most polynomial growth, \( u = u(x,t), \ x \in \mathbb{R}, \ t \in (0, T), \) with the initial value \( u(x,0) = u^0(x) \). The Hopf equation

\[ L_H[u] = u_t + (u^2)_x = 0. \]

is a special case of this equation.

It is known that equation (1.1) can have a discontinuous shock wave type solution even in the case of the smooth initial condition \( u^0(x) \). Such piecewise smooth solutions are determined by integral identities, namely, the function \( u(x,t) \) is called a generalized solution of equation (1.1) in the domain \( \Omega \subset \mathbb{R}^2 \) if for all functions with compact supports \( \varphi(x,t) \in \mathcal{D}(\Omega) \) we have

\[ \int_{\Omega} \left[ u(x,t)\varphi_t(x,t) + f(u(x,t))\varphi_x(x,t) \right] \, dx \, dt = 0. \]

O. A. Oleinik proved the existence of a solution of the Cauchy problem for equation (1.1) in the class of piecewise smooth functions provided that certain additional stability conditions hold. A theory of such problems for quasilinear equations and systems of equations is described in [1] – [6].

However, the Cauchy problem can be posed for equation (1.1) with the singular initial value \( u^0(x) \) if its singularity is stronger than that of the Heaviside function

\[ H(\xi) = \begin{cases} 1, & \xi > 0, \\ 0, & \xi < 0, \end{cases} \]

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i.e. a jump.

Thus, in [3] – [6] the problem about the propagation of an infinitely narrow $\delta$-soliton was considered. In the simplest case nonlinear waves of this type arise if we consider the weak asymptotics, as $\varepsilon \to +0$, of the one-soliton solution

$$u(x, t, \varepsilon) = \frac{3\alpha}{2} \exp\left(\frac{\sqrt{3}}{\varepsilon}(x - vt)\right),$$

of the Korteweg-de-Vries equation (KdV)

$$L_{KdV}[u] = u_t + (u^2)_x + \varepsilon^2 u_{xxx} = 0.$$  

Up to $O_D(\varepsilon^2)$, this asymptotics gives the infinitely narrow $\delta$-soliton

$$u_\varepsilon(x, t) = A\varepsilon \delta(x - vt), \quad \varepsilon \to +0,$$

where $\delta(x)$ is the Dirac $\delta$-function, $A = \frac{3\pi}{2} \int \exp\left(-\frac{\sqrt{3}}{\varepsilon}(x - vt)\right) d\xi = 6\sqrt{v}$ and by $O_D(\varepsilon^2)$ we denote a distribution from $D'(\mathbb{R})$ such that for any test function $\varphi(x) \in D$

$$\langle O_D(\varepsilon^2), \varphi(x) \rangle = O(\varepsilon^2),$$

and $O(\varepsilon^2)$ is understood in the ordinary sense. We stress that here and below we consider all distributions as distributions depending on the argument $x$. All other arguments are considered as parameters.

Since $u(x, t, \varepsilon) = O_D(\varepsilon)$ and $\varepsilon^2 u_{xxx} = O_D(\varepsilon^3)$ as $\varepsilon \to +0$, the limit expression $u_\varepsilon(x, t)$ was interpreted by V. P. Maslov and V. A. Tsupin [7], V. P. Maslov and G. A. Omelyanov [8] as a generalized solution (asymptotic up to $O_D(\varepsilon^2)$ of the Hopf equation (1.2) which is the limit problem for the KdV equation. In [3], [8] the corresponding generalized Hugoniot conditions similar to those at a shock wave front were obtained. Here the initial value $u^0(x) = A\varepsilon \delta(x)$, $\varepsilon \to +0$ is not a distribution but an asymptotic distribution (see [2] and below). In [3] asymptotic generalized solutions of the quasilinear equations (1.4) and systems of equations were considered in the form of infinitely narrow $\delta$-solitons in the algebra of asymptotic distributions.

In [8] the problem on propagation of infinitely narrow $P$-solitons was also considered. This is a new type of nonlinear waves which arise in solving the Cauchy problem for equation (1.4) with the initial value in the form of the asymptotic distribution $u^0(x) = u_0^0(x) + g^0(x)P(x^{-1})$, $\varepsilon \to +0$, where $P(x^{-1})$ is the principal value of the function $x^{-1}$.

In [10] – [12] problems on propagation and interaction of $\delta$-waves for semilinear hyperbolic systems are studied, i.e. the Cauchy problem is solved for the initial value $u^0(x)$ whose singularity is of the Dirac $\delta$-function type.

Generally speaking, the class of problems involving determination of singular solutions of quasilinear and semilinear equations can lead to the problem to define multiplication of distributions (generalized functions) and to construct associative algebras containing distributions. In particular, if we rewrite equation (1.3) as $u_t + f(u)u_x = 0$ which is equivalent to the divergent form for the case of smooth solutions, then even for piecewise smooth solutions of this equation with jumps there arises a problem to define the product of the Dirac $\delta$-function by the Heaviside function $\delta(x)H(x)$.

These problems require a development of special analytic methods. There exist various approaches to their solution [9], [11], [13] – [19].

In [10], [13], and especially in [9] was developed a new analytical method, the weak asymptotic method, which enables investigation of the dynamics of propagation of various types of singularities of quasilinear differential equations and hyperbolic first order systems. The fundamental ideas of this method originate from the papers by Y. B. Livchak [15], Li Bang-He [10], V. K. Ivanov [17] (where the product of distributions was defined as the weak asymptotics of the product of approximations of distributions being multiplied with respect to the approximation parameter) and V. P. Maslov [3] (where the direct substitution of a singular ansatz into a quasilinear equation was used).

In the present paper the weak asymptotics method is applied to the investigation of the dynamics of both propagation and interaction of initial discontinuities, i.e. shock waves. Further we describe the essence of the weak asymptotics method and give the definitions [2], [3] of a generalized asymptotic solution and a generalized solution to equation (1.4).

In Theorem (4.4) and Corollary (4.4) Corollary 1.4 using the weak asymptotics method, we write out a system of equations to determine the dynamics of propagation of one shock wave for equation (1.4) and the Hopf equation (1.2), i.e. in the class of piecewise smooth functions we solve the Cauchy problem with the following initial value:

$$u^0(x) = \begin{cases} 
  u_0^0(x) + e^0(x), & x < x_0, \\
  u_0^0(x), & x > x_0,
\end{cases} \tag{1.4}$$
where $x_0$ is the initial position of the shock wave front, $u_0^0(x)$, $e^0(x) > 0$ are smooth functions. Analogous results for strictly hyperbolic first order systems were derived in [3]. The proof of Theorem 1.1 is given in Section 3.

In Theorem 1.3 we write a system of equations to determine the dynamics of propagation and interaction of two initial discontinuities (shock waves with constant amplitudes) for the Hopf equation (1.2) is considered since it is a typical model of a nonlinear hyperbolic equation. In Theorem 1.3 the results obtained are extended to the case of equation (1.1). Namely, we solve, in the sense of Definition 1.3, the Cauchy problem for equations (1.2) and (1.1), in the class of piecewise constant functions, with the following initial condition:

$$u^0(x) = \begin{cases} 
  u_0 + e_1 + e_2, & x < x_0^1, \\
  u_0 + e_2, & x_1^0 < x < x_2^0, \\
  u_0, & x > x_2^0, 
\end{cases} \quad (1.5)$$

where $u_0$, $e_1$, $e_2$ are constants. The solution of the Cauchy problem is given by the formula which can be used for all $t \geq 0$.

To prove Theorems 1.3, 1.5 we construct asymptotic solutions of equations (1.2) and (1.1) in the sense of Definition 1.2. These asymptotic solutions are given by Theorems 1.2, 1.4 proved in Sections 4, 5.

In Theorems 1.2, 1.4 we describe the dynamics of the shock wave merging process.

In what follows we assume that $f''(u) \geq 0$ and the solutions $u(x,t)$ are piecewise smooth functions satisfying the Oleinik stability conditions at every point $(x,t)$ of a discontinuity line $\delta$:

$$u(x - 0, t) > u(x + 0, t). \quad (1.6)$$

The one-phase and two-phase generalized solutions constructed in Theorems 1.3, 1.5 in the sense of Definition 1.3 are generalized solutions in the sense of the standard definition by the integral identity (1.4). Thus, provided that the stability conditions are fulfilled, the Oleinik uniqueness theorem holds for them.

Note that our methods are also applicable for solving a similar problem in the case of systems of quasilinear first order equations.

2. The weak asymptotics method. When solving the problem on propagation and interaction of singularities of quasilinear equations one has to extend the Schwartz distribution space. Therefore, in [3] we constructed an associative algebra of asymptotic distributions with the identity and free of zero divisors generated by the linear span of (one-dimensional) associated homogeneous distributions. The elements $f^*_\epsilon(x)$ from the algebra of asymptotic distributions are defined as weak asymptotic expansions of expansions of elements $f^*(x, \epsilon)$ from the linear span of approximations of one-dimensional associated homogeneous distributions as the approximation parameter $\epsilon$ tends to zero. Each element of this algebra has a unique representation in the form of the asymptotics (in the weak sense) whose coefficients are distributions.

The product of associated homogeneous distributions is defined as a weak asymptotic expansion of the product of approximations of multiplied distributions as $\epsilon \to +0$. This product is an element of the associative algebra of asymptotic distributions.

In particular, infinitely narrow $\delta$ and $P$-solitons are elements of the algebra of asymptotic distributions.

Now turn to the description of our technique omitting the algebraic aspects which are given in detail in [3] and [19].

To study the interaction of (two) singularities of equation (1.1), we solve the initial value problem with distribution initial data

$$u^*0(x) = u_0^0(x) + \sum_{j=1}^2 e_j^0(x)s_j(x - x_j),$$

where $u_0^0(x)$, $e_j^0(x)$ are smooth functions, $s_j(\xi)$ are distributions (generalized functions) or asymptotic distributions (see [3], [13]) and $x_j$ are constants.

The solution of this initial value problem is found in the form of the singular ansatz

$$u^*_\epsilon(x, t) = u_0(x, t) + \sum_{j=1}^2 e_j(x, t)s_j(x - \phi_j(t)) \quad (1.7)$$

where $u_0(x, t)$, $e_j(x, t)$, $\phi_j(t)$ are functions to be found and in the general case $\phi_j(t)$, $e_j(x, t)$ and $s_j(\xi)$ may depend on the small parameter $\epsilon$. The singular ansatz $u^*_\epsilon(x, t)$ belongs to the associative and commutative differential algebra of asymptotic distributions, However, actually, we do not use this fact in this paper directly,
since to define the associative and commutative product of distributions, we need to calculate all terms of the asymptotic expansion (i.e., to calculate the weak asymptotics of the product of approximations up to $O_D(\varepsilon^\infty)$). But in view of the above remarks and Definition 4.3 below, to construct a generalized asymptotic solution, it is sufficient to calculate the weak asymptotics of the above-mentioned product of approximations up to $O_D(\varepsilon^\infty)$, $N = 1$ or $N = 2$ (see below).

To study the interaction of two nonlinear waves we assume: 1) $s_j(\xi) = \delta(\xi)$, $j = 1, 2$ in the case of the $\delta$-waves; 2) $s_j(\xi) = H(\xi)$, $j = 1, 2$ in the case of the shock waves; 3) $s_j(\xi) = \varepsilon \delta(\xi)$, $j = 1, 2$ and $s_j(\xi) = \varepsilon H(\xi)$, $j = 3, 4$ in the case of infinitely narrow $\delta$-solitons; 4) $s_j(\xi) = \varepsilon P(\xi^{-1})$, $j = 1, 2$ in the case of infinitely narrow $P$-solitons.

To find the solution of the form $u_\varepsilon(x, t)$ we construct the smooth ansatz

$$u^*(x, t, \varepsilon) = u_0(x, t) + \sum_{j=1}^{2} e_j(x, t)s_j(x - \phi_j(t, \varepsilon), \varepsilon),$$

(1.8)

which is a smooth approximation of the singular ansatz (1.7). Here $s_j(\xi, \varepsilon)$ is a smooth approximation of the distribution or asymptotic distribution $s_j(\xi)$ and $\varepsilon$ is the approximation parameter.

To construct the approximations $s_j(\xi, \varepsilon)$ we use the fact that a correspondence can be set up between each distribution (generalized function) $f(x) \in D'$ and its approximation:

$$f(x, \varepsilon) = f(x) \ast K(x, \varepsilon) = \langle f(t), K(x - t, \varepsilon) \rangle, \quad \varepsilon > 0,$$

(1.9)

where $\ast$ is a convolution, the kernel $K(x, \varepsilon) = \frac{1}{\varepsilon} \omega\left(\frac{x}{\varepsilon}\right)$ is a $\delta$-type function such that $0 \leq \omega(z) \in C^\infty(\mathbb{R})$, $\int \omega(z) \, dz = 1$, $\omega(z)$ has a compact support or decreases sufficiently rapidly as $|z| \to \infty$, for example, $|\omega(z)| \leq C(1 + |z|)^{-N}$ for all positive integers $N$. [4, ch.I, §4.6.]

For all test functions $\phi(x) \in D$ we have:

$$\lim_{\varepsilon \to +0} \langle f(x, \varepsilon), \phi(x) \rangle = \langle f(x), \phi(x) \rangle.$$

For example, for an approximation of $\delta$-function we have from (1.8):

$$\delta(x, \varepsilon) = \frac{1}{\varepsilon} \omega\left(\frac{x}{\varepsilon}\right).$$

(1.10)

For an approximation of the Heaviside function $H(x)$ we have from (1.9):

$$H(x, \varepsilon) = H(x) \ast \frac{1}{\varepsilon} \omega\left(\frac{x}{\varepsilon}\right) = \int_{0}^{\infty} \omega\left(\frac{x}{\varepsilon} - t\right) \, dt.$$

Hence we find

$$H(x, \varepsilon) = \omega_0\left(\frac{x}{\varepsilon}\right) = \int_{-\infty}^{\frac{x}{\varepsilon}} \omega(\eta) \, d\eta,$$

(1.11)

where $\omega_0(z) \in C^\infty(\mathbb{R})$, $\lim_{z \to +\infty} \omega_0(z) = 1$, $\lim_{z \to -\infty} \omega_0(z) = 0$.

The smooth ansatz $u(x, t, \varepsilon)$ is substituted into the equation (1.1), and the weak asymptotics (in terms of the space of the distributions $D'$) up to $O_D(\varepsilon^N)$, where $N = 1$ for delta and shock waves, $N = 2$ for infinitely narrow solitons, is found of the left-hand side of this equation as $\varepsilon \to +0$. We define the 4 generalized solution (asymptotics solution) to equation (1.1) as the weak asymptotics $u_\varepsilon(x, t)$ of the smooth ansatz $u(x, t, \varepsilon)$, as $\varepsilon \to +0$.

The key role in solving the problem of the interaction of nonlinear waves of the equation (1.1) plays the construction of the weak asymptotics of the smooth function from the smooth ansatz (1.8), which can be represented in the form

$$f(u_0(x, t) + \sum_{j=1}^{2} e_j(x, t)s_j(x - \phi_j(t, \varepsilon), \varepsilon)$$

$$= f(u_0(x, t)) + \sum_{j=1}^{2} B_j(u_0(x, t), e_1(x, t), e_2(x, t), \rho)s_j(x - \phi_j(t)) + O_D(\varepsilon^N),$$

(1.12)
where the estimate $O_{DP}(\varepsilon^N)$ is uniform with respect to $\psi(t, \varepsilon)$, where $\psi(t, \varepsilon) = \phi_2(t, \varepsilon) - \phi_1(t, \varepsilon)$, $\rho = \frac{\psi(t, \varepsilon)}{\varepsilon}$, and the interaction switches $B_j(u_0(x, t), e_1(x, t), e_2(x, t), \rho)$, $j = 1, 2$ which are smooth functions, can be computed explicitly.

Expansion (1.12) allows one to separate the singularities of the ansatz. Therefore, substituting expansion (1.12) and the singular ansatz (1.7) into a quasilinear equation and setting equal to zero the coefficients of the different powers of the small parameter $\varepsilon$ and of the linear independent distributions we obtain a system of equations (in particular, the Rankine–Hugoniot type condition) which describes the dynamics of singularities and defines the smooth functions $u_0(x, t)$, $e_1(x, t)$, $\phi_j(t)$, $j = 1, 2$.

If the propagation of one singularity is studied we set $e_0^0(x) = 0$ in the initial value $u^*0(x)$, and we set $e_2^2(x, t) = 0$ in (1.7), (1.8).

3. The weak asymptotics method and shock waves. It follows from (1.7), (1.8) that to study the interaction switches $B_j(u_0(x, t), e_1(x, t), e_2(x, t), \rho)$, $j = 1, 2$.

It should be noted that an the algebraic approach to the construction of solutions of the shock wave type, that is, the method of direct substitution the singular ansatz (1.13) into a quasilinear equation in divergent form was first used by V. P. Maslov [3]. His approach was based on the fact that a smooth function $f$ of the sum $u_0(x) + e(x)H(x)$, where $u_0(x)$, $e(x)$ are smooth functions, can be represented in the form as its argument:

$$f(u_0(x) + e(x)H(x)) = f(u_0(x)) + \left[f(u_0(x) + e(x)) - f(u_0(x))\right]H(x).$$

After substituting this expression into a quasilinear first order equation written in the divergent form, and differentiating, one can obtain a linear combination of the unit, the Dirac $\delta$-function and the Heaviside function with smooth coefficients. Setting this expression to zero and separating the singularities, we obtain the relations which determine the solution.

In order to investigate the interaction of two shock waves, that is, to solve the Cauchy problem (1.1), (1.5), we should seek a solution in the form of the singular ansatz

$$u^*_2(x, t) = u_0(x, t) + e(x, t) H(x - \phi(t))$$

and substitute into equation (1.1) the smooth ansatz

$$u^*(x, t, \varepsilon) = u_0(x, t) + e(x, t) H(-x + \phi(t), \varepsilon),$$

where $u_0(x, t)$, $e(x, t)$, $\phi(t)$ are functions to be found, $\phi(0) = x_0$ and $H(\xi, \varepsilon)$ is the approximation (1.13) of the Heaviside function $H(\xi)$.

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$$u^*_2(x, t) = u_0(x, t) + \sum_{k=1}^{2} \left[e_k(x, t) H(-x + \phi_k(t, \varepsilon))\right],$$

and substitute the smooth ansatz

$$u^*(x, t, \varepsilon) = u_0(x, t) + \sum_{k=1}^{2} \left[e_k(x, t) H_k(-x + \phi_k(t, \varepsilon), \varepsilon)\right],$$

into equation (1.1). Here $u_0(x, t)$, $e_k(x, t)$ are smooth functions to be found, $\lim_{\varepsilon \to +0} \phi_k(0, \varepsilon) = x_0^0$, $H_k(\xi, \varepsilon)$ are approximations (1.11) of the Heaviside function $H(\xi)$, $k = 1, 2$.

**Definition 1.1** Let $f(u)$ be a smooth function, and let $u^*_2(x, t)$ is an asymptotic distribution (singular ansatz) (1.13) or (1.14), and $u^*(x, t, \varepsilon)$ is a smooth ansatz (1.14) or (1.16).

By the $O_{DP}(\varepsilon)$-substitution we call the weak asymptotics of the expression $f(u^*(x, t, \varepsilon))$ as $\varepsilon \to +0$ computed up to $O_{DP}(\varepsilon)$. We denote this substitution $f(u^*_2(x, t))$.

Now we present the main lemma which gives an asymptotic expansion of the type (1.12) for the case of shock waves. The proof of Lemma 1.1 is given in Section 2.

**Lemma 1.1** Let

$$H(-x, \varepsilon) = \int_{-\infty}^{x} \omega_1(\eta) d\eta, \quad H(-x + a, \varepsilon) = \int_{-\infty}^{x-a} \omega_2(\eta) d\eta$$
be approximations of the Heaviside functions \( H(-x) \), \( H(-x + a) \), respectively, the approximating functions \( \omega_k(z) \in C^\infty(\mathbb{R}) \) are nonnegative and either have compact supports or decrease sufficiently rapidly, as \( |z| \to \infty \) and \( \int \omega_k(z) \, dz = 1, \ k = 1, 2. \)

Let \( f(u) \) be a smooth function of at most polynomial growth and let \( u_0(x,t), \ e_1(x,t), \ e_2(x,t) \) be bounded functions. Then we have the asymptotics:

\[
f(u_0(x,t) + e_1(x,t)H_1(-x, \varepsilon) + e_2(x,t)H_2(-x + a, \varepsilon)) = f(u_0(x,t)) + \left[ f(u_0(x,t) + e_1(x,t)) - f(u_0(x,t)) \right] H(-x) + \left[ f(u_0(x,t) + e_1(x,t)) - f(u_0(x,t)) \right] H(-x + a)
\]

\[+ B_1(x,t, \frac{a}{\varepsilon})H(-x) + B_2(x,t, -\frac{a}{\varepsilon})H(-x + a) + O_D(\varepsilon), \ \varepsilon \to +0, \quad (1.17)
\]

where the estimate \( O_D(\varepsilon) \) is uniform with respect to \( a. \)

The functions \( B_k(x,t, \rho), \ k = 1, 2 \) called interaction switch functions have the following form

\[
B_1(x,t, \rho) = \int \left[ f'(u_0(x,t) + e_1(x,t) + e_2(x,t) - u_0(x,t)) + e_1(x,t)\omega_1(-\eta + \rho) \right]
\]

\[\eta f'(u_0(x,t) + e_1(x,t) + e_2(x,t) - u_0(x,t)) + e_2(x,t)\omega_2(-\eta) d\eta,
\]

\[B_2(x,t, -\rho) = \int \left[ f'(u_0(x,t) + e_1(x,t) - u_0(x,t)) + e_1(x,t)\omega_1(-\eta - \rho) \right]
\]

\[\eta f'(u_0(x,t) + e_1(x,t) - u_0(x,t)) + e_2(x,t)\omega_2(-\eta) d\eta.
\]

In addition, the interaction switch functions satisfy the following relations: for each \( \rho \in \mathbb{R} \)

\[
B_1(x,t, \rho) + B_2(x,t, -\rho) = f(u_0(x,t) + e_1(x,t) + e_2(x,t) - u_0(x,t)) - f(u_0(x,t) + e_1(x,t) + e_2(x,t) + f(u_0(x,t)),
\]

\[
\lim_{\rho \to +\infty} B_k(x,t, \rho) = f(u_0(x,t) + e_1(x,t) + e_2(x,t) - f(u_0(x,t) + e_1(x,t))
\]

\[f(u_0(x,t) + e_2(x,t)) + f(u_0(x,t)) + f(u_0(x,t)), \quad \lim_{\rho \to -\infty} B_k(x,t, \rho) = 0, \quad k = 1, 2. \quad (1.19)
\]

Remark 1.1 For simplicity, we have selected the approximating functions \( \omega_k(z) \) such that they either have compact supports or decrease sufficiently rapidly as \( |z| \to \infty \), for example, \( |\omega_k(z)| \leq C_k(1 + |z|)^{-N} \) for all positive integers \( N, \ k = 1, 2. \) Therefore, we have

\[
\omega_{01}(z) = \int_{-\infty}^{z} \omega_k(\eta) d\eta = 1 + O(z^{-N}), \quad z \to +\infty,
\]

\[
\omega_{02}(z) = O(|z|^{-N}), \quad z \to -\infty.
\]

Consequently, applying the Lagrange theorem we obtain the estimate

\[
\left[ f'(u_0 + e_1\omega_{01}(-\eta) + e_2\omega_{02}(-\eta + \rho) - f'(u_0 + e_1\omega_{01}(-\eta) - f'(u_0 + e_1\omega_{01}(-\eta)) \right] e_1\omega_1(-\eta)
\]

\[f''(u_0 + e_1\omega_{01}(-\eta)) + \Theta e_2\omega_{02}(-\eta + \rho) e_1 e_2 \omega_1(-\eta) \omega_2(-\eta + \rho), \quad 0 < \Theta < 1.
\]

It follows from this estimate and (1.18) that

\[
B_1(x,t, \rho) = f(u_0 + e_1 + e_2) - f(u_0 + e_1) - f(u_0 + e_2) + f(u_0) + O(\rho^{-N}), \quad \rho \to +\infty,
\]

\[
B_1(x,t, \rho) = O(|\rho|^{-N}), \quad \rho \to -\infty,
\]

where \( N = 1, 2, \ldots; \) if \( \rho_0 \) is a constant then

\[
B_1(x,t, \rho) = B_1(\rho_0) + O(\rho - \rho_0), \quad \rho \to \rho_0.
\]

Corollary 1.1 Let \( f(u) \) be a smooth function of at most polynomial growth and let \( u_0(x,t), \ e(x,t) \) be bounded functions and \( H(-x, \varepsilon) \) be approximations of the Heaviside functions \( H(-x) \). Then

\[
f(u_0(x,t) + e(x,t)H(-x + \phi(t), \varepsilon)) = f(u_0(x,t))
\]

\[+ \left[ f(u_0(x,t) + e(x,t)) - f(u_0(x,t)) \right] H(-x + \phi(t)) + O_D(\varepsilon), \quad \varepsilon \to +0. \quad (1.20)
\]
If we set \( e_1(x, t) = \epsilon(x, t), \ e_2(x, t) = 0 \) in (1.18) then \( B_1(x, t, \rho) = B_2(x, t, -\rho) = 0 \) and expansion (1.20) immediately follows from (1.17).

**Remark 1.2** In view of the obvious relation

\[
[H(x, \epsilon)]^r = H(x) + O_D(\epsilon), \quad \epsilon \to +0, \quad r = 2, 3, \ldots, \tag{1.21}
\]

for the case when \( f(u) \) is a polynomial, formula (1.20) was proved in [8.1].

**Corollary 1.2**

Let

\[
H(-x, \epsilon) = \int_{-\infty}^{\omega_1(\eta)} d\eta, \quad H(-x + a, \epsilon) = \int_{-\infty}^{\omega(\eta)} d\eta
\]

be the approximations of the Heaviside functions \( H(-x), \ H(-x + a), \) respectively, where the functions \( \omega_k(z) \in C^\infty(\mathbb{R}) \) either have compact supports or decrease sufficiently rapidly for \( |z| \to \infty, \) and \( \int \omega_k(z) \, dz = 1, \ k = 1, 2. \)

Then

\[
H(-x, \epsilon)H(-x + a, \epsilon) = B_1(\frac{a}{\epsilon})H(-x) + B_2(-\frac{a}{\epsilon})H(-x + a) + O_D(\epsilon), \quad \epsilon \to +0, \tag{1.22}
\]

where the estimate \( O_D(\epsilon) \) is uniform with respect to \( a. \)

The functions \( B_k(\rho) \) called interaction switch functions, have the form

\[
B_1(\rho) = \int_{-\infty}^{\rho} (\omega_1 * \omega_2)(\eta) \, d\eta, \quad B_2(-\rho) = \int_{-\infty}^{-\rho} (\omega_1 * \omega_2)(\eta) \, d\eta, \tag{1.23}
\]

where \( \hat{\omega}(\eta) = \omega(-\eta), \ * \) is the operation of convolution.

In this case,

\[
B_1(\rho) + B_2(-\rho) = 1, \quad \rho \in \mathbb{R},
\]

\[
B_k(\infty) = \lim_{\rho \to +\infty} B_k(\rho) = 1,
\]

\[
B_k(-\infty) = \lim_{\rho \to -\infty} B_k(\rho) = 0. \tag{1.24}
\]

To prove Corollary 1.2 we find the asymptotics of the expression

\[
H(-x, \epsilon)H(-x + a, \epsilon) = \frac{1}{4} \left( f(H_1(-x, \epsilon) + H_2(-x + a, \epsilon)) - f(H_1(-x, \epsilon)) - f(H_2(-x + a, \epsilon)) \right),
\]

where \( f(u) = u^2. \)

Setting in (1.17) \( u_0(x, t) = 0, \ e_1(x, t) = e_2(x, t) = 1 \) and taking into account relation (1.21), we obtain (1.22). In this case it follows from (1.18) and (1.11):

\[
B_k((-1)^{k-1}\rho) = \int \omega_3-\kappa(-\eta + (-1)^{k-1}\rho) \omega_k(-\eta) \, d\eta
\]

\[
= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\eta + (-1)^{k-1}\rho} \omega_3-\kappa(v) \, dv \right] \omega_k(-\eta) \, d\eta, \quad k = 1, 2.
\]

After the substitution \( v = z - \xi \) in the second integral, we reduce \( B_k(\rho) \) to the form

\[
B_k((-1)^{k-1}\rho) = \int_{-\infty}^{(-1)^{k-1}\rho} \left[ \int_{-\infty}^{\infty} \omega_k(-\xi) \omega_3-\kappa(z - \xi) \, d\xi \right] \, dz
\]

and thus obtain (1.23).

In this case (1.19) implies (1.24).

4. **Generalized solutions.** Let us define a generalized discontinuous solution and rules for substituting ansatzs of the type (1.13), (1.14) into equation (1.1) (see Definition 1.1).
Consider the Cauchy problem \( u(x, t) \) which coincides with the initial value (1.4). In the framework of our approach we shall seek the solution in the form of the singular ansatz (1.13), (1.15) in equation (1.1), where
\[
L[u(x, t, \varepsilon)] = O_D(\varepsilon),
\]
where the first estimate is uniform with respect to \( t \in [0, T] \).

\textbf{Definition 1.2} Let \( u^*(x, t, \varepsilon) \) be the smooth ansatz (1.14), (1.16). We call the corresponding asymptotic distribution (singular ansatz) (1.13), (1.15), \( u^*_\varepsilon(x, t) \), by a generalized asymptotic shock wave type solution of the equation \( L[u] = 0 \), for \( t \in [0, T] \), with the initial condition \( u^0_\varepsilon(x) \), if the following relations hold:
\[
\begin{align*}
L[u^*(x, t, \varepsilon)] &= O_D(\varepsilon), \\
u^*_\varepsilon(x) &= u^0_\varepsilon(x) + O_D(\varepsilon),
\end{align*}
\]

After substitution of expansion (1.20) and the singular ansatz (1.13) (or expansion (1.17) and the singular ansatz (1.13), (1.15)), \( \phi(x, t, \varepsilon) \) of the left hand side of equation (1.1) and to obtain a system of equations which describes the dynamics of singularities and determines the smooth functions \( u_0(x, t), \; e_k(x, t), \; \phi_k(t), \; k = 1, 2 \) we use the following Lemma.

\textbf{Lemma 1.2} If \( A(x), \; B_k(x), \; C_k(x) \) are smooth functions \( k = 1, 2 \) and \( a > 0 \). Then
\[
A(x) + B_1(x)\theta(x) + C_1(x)\delta(x) + B_2(x)\theta(x-a) + C_2(x)\delta(x-a) = 0
\]
if and only if
\[
\begin{align*}
A(x) &= 0, \quad \text{for} \quad x < 0, \\
A(x) + B_1(x) &= 0, \quad \text{for} \quad 0 < x < a, \\
A(x) + B_1(x) + B_2(x) &= 0, \quad \text{for} \quad x > a, \\
C_1(0) &= 0, \\
C_2(a) &= 0.
\end{align*}
\]

5. \textbf{Propagaiton of shock wave of equation (1.1)}.

Let us consider the propagation of the single shock wave of the equation \( L[u] = u + (f(u))_x = 0 \) with the initial value (1.4). In the framework of our approach we shall seek the solution in the form of the singular ansatz (1.13), (1.15) which coincides with the initial value \( u^0_\varepsilon(x) \) for \( t = 0 \). This means that one substitute the smooth ansatz (1.13) approximating (1.13) into the equation \( L[u] = 0 \) and find the weak asymptotics, up to \( O_D(\varepsilon) \), of the left hand side of the equation, as \( \varepsilon \to +0 \).

\textbf{Theorem 1.1} Consider the Cauchy problem (1.1), (1.4), where \( f(u) \) is a smooth function, \( f'''(u) \geq 0 \) and \( u^0_\varepsilon(x) \), \( e^0(x) \geq 0 \) are smooth functions. Suppose there exists \( T > 0 \), such that
\[
- \inf_{\xi > \phi(0)} |u^0(\xi)f''(u^0(\xi))| < T^{-1}, \quad - \inf_{\xi < \phi(0)} \left[ u^0(\xi) + e^0(\xi) \right] f''(u^0(\xi) + e^0(\xi)) < T^{-1}.
\]

Then in the sense of Definitions 1.2, 1.3 equation (1.1) for \( t \in [0, T] \) has a discontinuous solution of the form
\[
u^*(x, t) = u_0(x, t) + e(x, t)H(-x + \phi(t)),
\]
if and only if the unknown functions \( u_0(x, t), \; e(x, t), \; \phi(t) \in C^\infty \) satisfy the following system of equations
\[
\begin{align*}
L[u_0(x, t)] &= 0, \quad x > \phi(t), \\
L[u_0(x, t) + e(x, t)] &= 0, \quad x < \phi(t), \\
\frac{d\phi(t)}{dt} &= f(u_0(x, t) + e(x, t)) - f(u_0(x, t)) \bigg|_{x=\phi(t)},
\end{align*}
\]
where \( u^*(x, 0) = [u_0(x, t) + e(x, t)]_{t=0} = u^0_0(x) + e^0(x) \) for \( x < \phi(0) \) and \( u^*(x, 0) = u_0(x, t) \bigg|_{t=0} = u^0_0(x) \) for \( x > \phi(0), \; \phi(0) = x_0 \).
Corollary 1.3 In the case of the Hopf equation \( L_H[u] = u_t + (u^2)_x = 0 \) system (1.26) has the form

\[
\begin{align*}
L_H[u_0(x, t)] &= 0, \quad x > \phi(t), \\
L_H[u_0(x, t) + e(x, t)] &= 0, \quad x < \phi(t), \\
\frac{d\phi(t)}{dt} &= 2u_0(\phi(t), t) + e(\phi(t), t).
\end{align*}
\]

(1.27)

The last equations in systems (1.26) and (1.27) represent the Rankine-Hugoniot conditions along the discontinuity line \( x = \phi(t) \) for shock waves [13, Ch. 4, §1.2.3] :

\[
\frac{dx}{dt} = \left[ \frac{f(u)}{|u|} \right],
\]

where

\[
\begin{align*}
[u] &= u(\phi(t) + 0, t) - u(\phi(t) - 0, t), \\
[f(u)] &= f(u(\phi(t) + 0, t)) - f(u(\phi(t) - 0, t))
\end{align*}
\]

are jumps of functions \( u(x, t) \) and \( f(u) \), respectively, along the discontinuity line \( x = \phi(t) \). In our case, \( u = u_0 + \varepsilon \) to the left of the discontinuity, and \( u = u_0 \) to the right of the discontinuity.

Corollary 1.4 Solution (1.13) constructed in the sense of Definition 1.3 to the Cauchy problem for equation (1.1) with the initial value (1.4), that is, a function of the form \( u^*(x, t, \varepsilon) = u_0(x, t) + e(x, t)H(-x + \phi(t)) \), whose smooth components satisfy system of equations (1.20) from Theorem 1.1, is the unique generalized solution of the same Cauchy problem in the sense of the standard definition by integral identity (1.3).

Proof. To prove that the solution of the Cauchy problem in the sense of Definition 1.3 is the solution in the sense of the standard definition by the integral identity (1.3) we remind that in view of Theorem 1.1 the approximation \( u^*(x, t, \varepsilon) \) (1.14) of the solution (1.13) satisfies the equation \( L[u^*(x, t, \varepsilon)] = O_D(\varepsilon) \). Let us apply the left-hand and right-hand sides of this relation to an arbitrary test function \( \varphi(x, t) \in D(\Omega), \ \Omega \subset \mathbb{R}^2 \).

Since for \( \varepsilon > 0 \) the functions \( u^*(x, t, \varepsilon) \) is smooth, integrating by parts, we obtain

\[
\int_{\Omega} \left[ u^*(x, t, \varepsilon)\varphi_t(x, t) + f(u^*(x, t, \varepsilon))\varphi_x(x, t) \right] dxdt = \langle O_D(\varepsilon), \varphi(x, t) \rangle = O(\varepsilon).
\]

Since the limit \( u^*(x, t) \) of the family \( u^*(x, t, \varepsilon) \) as \( \varepsilon \to +0 \) in the \( D' \) sense is a bounded locally integrable function, we have \( \int_{\Omega} u^*(x, t)\varphi_t(x, t) + f(u^*(x, t))\varphi_x(x, t) dxdt = 0 \), as \( \varepsilon \to +0 \), which coincides with the usual integral identity (1.3) (see (1)–(3)).

Due to our assumptions, the stability condition (1.6) is satisfied and by Theorem of O. A. Oleinik this solution is unique.

6. Interaction of the shock waves of the Hopf equation (1.2).

To study the interaction of two shock waves of the Hopf equation (1.2) \( L_H[u] = u_t + (u^2)_x = 0 \) we shall seek its solution in the form of the singular ansatz (1.13) which, for \( t = 0 \) coincides with distribution (1.5).

First we construct an asymptotic solution of problem (1.2), (1.5) in the sense of Definition 1.2, which is given by Theorem 1.2 proved in Sections 4.

Theorem 1.2 For all \( t \in [0, +\infty) \) in the sense of Definition 1.2 there exists the asymptotic solution

\[
u^*(x, t, \varepsilon) = u_0 + \sum_{k=1}^{2} c_k H_k \left( -x + \phi_k(t, \varepsilon), \varepsilon \right),
\]

for the Hopf equation (1.2) such that:

1) for all \( t > 0 \): \( L_H[u^*(x, t, \varepsilon)] = O_D(\varepsilon) \),

2) \( u^{*0}(x) - u^*(x, 0, \varepsilon) = O_D(\varepsilon) \), where

\[
u^{*0}(x) = \begin{cases} 
  u_0 + e_1 + e_2, & x < x_0^1, \\
  u_0 + e_2, & x_1^0 < x < x_0^2, \\
  u_0, & x > x_0^2,
\end{cases}
\]

\( x_1^0 < x_0^2 \) and \( u_0, \ e_1 > 0, \ e_2 > 0 \) are constants.
3) for \( t \in (0, t^*) \)

\[
\lim_{\varepsilon \to +0} u^*(x, t, \varepsilon) = u_0 + \sum_{k=1}^{2} e_k H(-x + \phi_k(t)),
\]

where

\[
\phi_1(t) = \lim_{\varepsilon \to +0} \phi_1(t, \varepsilon) = \phi_{10}(0) + (2u_0 + e_1 + 2e_2)t,
\]

\[
\phi_2(t) = \lim_{\varepsilon \to +0} \phi_2(t, \varepsilon) = \phi_{20}(0) + (2u_0 + e_2)t,
\]

\[
\phi_{k0}(0) = x_k^0, \quad k = 1, 2,
\]

\[
t^* = \frac{x_k^0 - x_1^0}{e_1 + e_2}
\]

is the shock wave merging time;

4) for \( t \in (t^*, \infty) \)

\[
\lim_{\varepsilon \to +0} u^*(x, t, \varepsilon) = u_0 + (e_1 + e_2)H(-x + \phi_0(t)),
\]

where

\[
\phi_0(t) = \lim_{\varepsilon \to +0} \phi_0(t, \varepsilon) = x^* + (2u_0 + e_1 + e_2)(t - t^*),
\]

\[
x^* = \phi_{k0}(t^*) = \frac{(2u_0 + e_1 + 2e_2)x_2^0 - (2u_0 + e_2)x_1^0}{e_1 + e_2}
\]

are approximations of the Heaviside function \( H(-x) \), where \( \omega_k(z) \) are smooth functions which either have compact supports or decrease sufficiently rapidly as \( |z| \to \infty \), \( \int \omega_k(z) \, dz = 1, \ k = 1, 2 \).

The phases \( \phi_k(t, \varepsilon) \) have the form

\[
\phi_k(t, \varepsilon) = \phi_{k0}(t) + \psi_0(t)\phi_{k1}(\tau)\Big|_{\tau = \frac{\omega_k(\eta)}{\omega_k(\eta)}}
\]

where \( \psi_0(t) = \phi_{20}(t) - \phi_{10}(t) \) and the perturbations of phase functions are given by formulae (4.61):

\[
\phi_{k1}(\tau) = (-1)^{k-1} \frac{2e_3 - k}{(e_1 + e_2)\tau} \int_{0}^{\tau} \left( 1 - B_1(\rho(\tau')) \right) d\tau'.
\]

According to (1.23),

\[
B_1(\rho) = \int_{-\infty}^{\rho} (\omega_1 * \omega_2)(\eta) \, d\eta,
\]

and \( \rho(\tau) = [\phi_2(t, \varepsilon) - \phi_1(t, \varepsilon)]/\varepsilon \) is a solution of the differential equation with the boundary condition (1.52), (4.53):

\[
\frac{d\rho}{d\tau} = 2B_1(\rho) - 1,
\]

\[
\rho(\tau)\bigg|_{\tau \to +\infty} = 1.
\]

Note that, in fact, Theorem 1.2 is based on the remark that the variable \( t \) is considered as a parameter. If we considered the variable \( t \) as a variable having the same rights as the variable \( x \), we also had to calculate a weak asymptotics with respect to the variable \( t \). In this case we would obtain Heaviside functions instead of interaction switch functions and could not obtain a formula that were uniform in \( t \) for the solution \( u^*(x, t, \varepsilon) \).

Theorem 1.2 implies the following result.

**Theorem 1.3** The Cauchy problem for the Hopf equation (1.2), (1.3) in the sense of Definition 1.3 has a discontinuous solution of the form (1.13)

\[
u^*(x, t) = u_0 + \sum_{k=1}^{2} e_k H(-x + \phi_k(t)),
\]

where

\[
\phi_1(t) = \phi_{10}(0) + (2u_0 + e_1 + 2e_2)t - e_2(t - t^*)H(t - t^*),
\]

\[
\phi_2(t) = \phi_{20}(0) + (2u_0 + e_2)t + e_1(t - t^*)H(t - t^*)
\]

are phases.

This solution of the initial value problem (1.2), (1.3) is the unique generalized solution of this Cauchy problem in the sense of the integral identity (1.3).
Proof. It follows from Theorem 1.2 that distribution \((1.29)\) is a generalized solution of the initial value problem \((1.2), (1.5)\) in the sense of Definition 1.3.

In view of Theorem 1.2, the approximation \(u^\varepsilon(x, t)\) of the solution \((1.29)\) uniformly in \(t\) satisfies the equation \(L[u^\varepsilon(x, t, \varepsilon)] = O_T(\varepsilon)\). By applying the left- and right-hand sides of the last relation to an arbitrary function \(\varphi(x, t) \in D(\Omega)\), we obtain

\[
\int_\Omega \left[ u^\varepsilon(x, t, \varepsilon)\varphi_t(x, t) + f(u^\varepsilon(x, t, \varepsilon))\varphi_x(x, t) \right] \, dx \, dt = O(\varepsilon).
\]

By integrating by parts for \(\varepsilon > 0\) and then passing to the limit as \(\varepsilon \to +0\), just as in the proof of Theorem 1.1, we prove that the functions \(u^\varepsilon(x, t)\) determined by \((1.29)\) satisfies the integral identity

\[
\int_\Omega \left[ u^\varepsilon(x, t)\varphi_t(x, t) + f(u^\varepsilon(x, t))\varphi_x(x, t) \right] \, dx \, dt = 0.
\]

Since for the piecewise constant solution \(u^\varepsilon(x, t)\) the stability condition \(\varepsilon_k > 0, \ k = 1, 2\), holds, this solution is unique by the Theorem of O. A. Oleinik [1]. Theorem 1.3 is proved.

The solution described in Theorem 1.3 is of rather simple natural form. In fact, it is sewed together of the following parts: two shock waves till the time instant \(t^*\) and one shock wave for \(t > t^*\). We cannot say anything about the equation at the point \(x = x^*, \ t = t^*\), but we see that the trajectories of discontinuities at this point are continuous. Moreover, one can easily see that the function \((1.30)\) itself is continuous w.r.t. to \(t\) in the weak sense. Thus we have the following “naive” situation: everywhere outside the curves \(x = \phi_k(t), \ k = 1, 2,\) the solution \((1.30)\) can be found as a function satisfying the equation \(L[u] = 0\), while the curves \(x = \phi_k(t), \ k = 1, 2,\) and the solution \((1.30)\) are determined by the Rankine–Hugoniot conditions for \(t \neq t^*\) and the condition that they are continuous at the point \(t = t^*\).

However, as we shall see now, one can construct a great deal of such points. As an example, we consider the following Cauchy problem for the Hopf equation:

\[
u_t + (u^2)x = 0, \quad u(x, t)|_{t=0} = H(-x - 1) - H(x - 1).
\]

One can easily verify that for \(t < t^* = 1\) the solution \(u(x, t)\) consists of two shock wave approaching each other:

\[
u(x, t) = H(-x + t - 1) - H(x + t - 1).
\]

By using the techniques used in Theorem 1.2 for calculating the solution \(u^\varepsilon(x, t)\), we see that the solution satisfying the integral identity for \(t \geq t^* = 1\) (this solution is one of “naive” solutions) is the stationary solution of the Hopf equation

\[
u(x, t) = H(-x) - H(x)
\]

consisting of two shock waves sticking together.

Another different “naive” solution can be constructed as follows. Let us consider the solution of the Hopf equation with the initial condition

\[
W(x, t)|_{t=0} = \begin{cases} 
H(-x - 1), & x \leq -1, \\
ax, & -1 \leq x \leq 1, \\
-H(x - 1), & x > 1.
\end{cases}
\]

The solution \(W(x, t)\) of this problem has the form

\[
W(x, t) = H(-x - \phi(t)) - H(x - \phi(t)) + \frac{ax}{1 + 2at} [1 - H(-x - \phi(t)) - H(x - \phi(t))].
\]

Its plot consists of two horizontal lines \((y = 1\) to the left of the point \(x_- = -\phi\) and \(y = 1\) to the right of the point \(x_+ = \phi\)) and the inclined line \(y = ax/(1 + 2at)\) between the points \(x_-\) and \(x_+\).

At the points \(x_\pm\) the solution has a jump whose coordinates \(x_\pm = \pm\phi(t)\) can be calculated from the Rankine–Hugoniot conditions

\[
\phi_t = \left(\frac{(-1)^2 - (ax/(1 + 2at))^2}{-1 - ax/(1 + 2at)}\right)_{x=\phi} = \frac{a\phi}{1 + 2at} - 1, \quad \phi(0) = 1.
\]

Assume that \(a > 1\). Then, obviously, \(\phi_t > 0\) on some interval \([0, t_0], \ t_0 > 0\).
We consider the limit of the function $W(x,t)$ in $\mathcal{D}'(\mathbb{R}_x^+)$ as $t \to -1/(2a) + 0$. For any test function $\varphi(x)$ we have
\[
\lim_{t \to -1/(2a)+0} \langle W(x,t), \varphi \rangle = \lim_{t \to -1/(2a)+0} \left[ \frac{1}{1+2at} \int_{x-\phi}^{\phi} x \varphi(x) \, dx + \int_{-\infty}^{-\phi} \varphi(x) \, dx - \int_{\phi}^{\infty} \varphi(x) \, dx \right].
\]
By using the L'Hospital rule, we obtain
\[
\lim_{t \to -1/(2a)+0} \langle W, \varphi(x) \rangle = \lim_{t \to -1/(2a)+0} \left[ \frac{\phi_t(\phi \varphi(\phi)) - \phi_t(\phi \varphi(-\phi))}{2a} \right] + \int_{0}^{\infty} \varphi(x) \, dx - \int_{\infty}^{0} \varphi(x) \, dx
= \int_{\infty}^{-\infty} (H(-x) - H(x)) \varphi(x) \, dx.
\]
Consider the function
\[
U(x,t) = \begin{cases} u(x,t) & 0 \leq t \leq t^*, \\ W(x,t - 1/(2a) - t^*) & t > t^*.
\end{cases}
\]
This function is continuous in $\mathcal{D}'$ and satisfies the classical Hopf equation everywhere except for the discontinuity trajectories
\[
x_\pm = \begin{cases} \pm(t - 1) & t \leq t^* = 1, \\ \pm \phi(x,t - 1/(2a) - t^*) & t > 1,
\end{cases}
\]
which are continuous curves. Thus this is the second “naive” solution of the problem of discontinuity merging. In this solution, the discontinuities seem to interact elastically and then to run away after the interaction.

Thus Theorem 1.3 is not useless. This theorem allows one to choose a “true naive” solution among all possible solutions.

The example of solution $W(x,t)$ is of independent interest, since this is one of few solutions with nonmonotone motion of its singularity. Namely, if $a > 1$, then, as was already pointed out, the discontinuities run away from each other but, starting from the time instant at which the inclined line $y = ax/(1+2at)$ enters the interior of the strip $y = \pm 1$, the directions of their motion are changed to the opposite, and the discontinuities merge into the stationary solution
\[
u(x,t) = H(-x) - H(x).
\]
Moreover, at the time instant of merging, we again can “paste in” a $W(x,t)$ type solution at the discontinuity point, repeat this process, and obtain the picture of $t$-periodic elastic interaction of shock waves.

7. Interaction of shock waves of equation (1.1).

Study the process of interaction of two shock waves of the equation $u_t + (f(u))_x = 0$, where $f(u)$ is a convex (downwards, i.e., $f''(u) > 0$ on the range of the solution $u$) smooth function. To this end, consider the Cauchy problem with the initial value (1.3). As in the case of the Hopf equation, substitute the smooth ansatz (1.16), where $\phi_1(0) = x_1^0$, $\phi_2(0) = x_2^0$ and $e_1$, $e_2 > 0$, approximating the singular ansatz (1.13), into equation (1.1) and find the weak asymptotics of its left hand side.

Eventually, we construct the asymptotic solution of problem (1.4), (1.3) in the sense of Definition 1.2, which is given by Theorem 1.4 proved in Sections 8.

**Theorem 1.4** For all $t \in [0, +\infty)$ in the sense of the Definition 1.2 there exists the asymptotic solution
\[
u^*(x,t,\varepsilon) = u_0 + \sum_{k=1}^{2} e_k H_k(-x + \phi_k(t,\varepsilon),\varepsilon),
\]
of the equation (1.1) such that:
1) for all $t > 0$: $L[u^*(x,t,\varepsilon)] = O_\mathcal{D}(\varepsilon)$,
2) $u^{*0}(x) - u^*(x,0,\varepsilon) = O_\mathcal{D}(\varepsilon)$, where
\[
u^{*0}(x) = \begin{cases} u_0 + e_1 + e_2, & x < x_0^1, \\ u_0 + e_2, & x_1^0 < x < x_0^2, \\ u_0, & x > x_0^2,
\end{cases}
\]
$x_1^0 < x_2^0$ and $u_0$, $e_1 > 0$, $e_2 > 0$ are constants.

\[\text{1G. A. Omel'yanov helped the authors to clear up this situation.}\]
3) for \( t \in (0, t^*) \)

\[
\lim_{\varepsilon \to +0} u^*(x, t, \varepsilon) = u_0 + \sum_{k=1}^{2} e_k H(-x + \phi_k(t)),
\]

where

\[
\phi_{10}(t) = \lim_{\varepsilon \to +0} \phi_1(t, \varepsilon) = \phi_{10}(0) + \frac{f(u_0 + e_1 + e_2) - f(u_0 + e_2) t}{e_1} + f(u_0 + e_2) - f(u_0) t,
\]

\[
\phi_{20}(t) = \lim_{\varepsilon \to +0} \phi_2(t, \varepsilon) = \phi_{20}(0) + \frac{f(u_0 + e_2) - f(u_0) t}{e_2},
\]

\[
\phi_k(0) = x_k^0, \quad k = 1, 2,
\]

\[
t^* = e_1 e_2 f(u_0 + e_1 + e_2) - (e_1 + e_2) f(u_0 + e_2) + e_1 f(u_0)
\]

is the shock wave merging time; 

4) for \( t \in (t^*, \infty) \)

\[
\lim_{\varepsilon \to +0} u^*(x, t, \varepsilon) = u_0 + (e_1 + e_2) H(-x + \hat{\rho}-(t)),
\]

where

\[
\hat{\rho}-(t) = \lim_{\varepsilon \to +0} \phi_k(t, \varepsilon) = x^* + \frac{f(u_0 + e_1 + e_2) - f(u_0)}{e_1 + e_2}(t - t^*), \quad k = 1, 2,
\]

\[
x^* = \phi_{k0}(t^*) = \frac{f(u_0 + e_1 + e_2) - f(u_0 + e_2)}{e_2 f(u_0 + e_1 + e_2) - (e_1 + e_2) f(u_0 + e_2) + e_1 f(u_0)} e_2 x_2^0.
\]

Here

\[
H_k(-x, \varepsilon) = \int_{-\infty}^x \omega_k(\eta) d\eta,
\]

are approximations of the Heaviside function \( H(-x) \), where \( \omega_k(z) \) are smooth functions which either have compact supports or decrease sufficiently rapidly as \( |z| \to \infty \), \( \int \omega_k(z) dz = 1, \quad k = 1, 2 \).

The phases \( \phi_k(t, \varepsilon) \) are set in the form

\[
\phi_k(t, \varepsilon) = \phi_{k0}(t) + \psi_0(t) \phi_{k1}(\tau) - \psi_0(t),
\]

where \( \psi_0(t) = \phi_{20}(t) - \phi_{10}(t) \) and the perturbations of phase functions are given by the formulas

\[
\phi_{k1}(\tau)
\]

\[
= (-1)^{k-1} \frac{e_3 - e_k}{\tau} \int_0^\tau \frac{f(u_0 + e_1 + e_2) - f(u_0 + e_1) - f(u_0 + e_2) + f(u_0) - B_1(\rho(\tau'))}{f(u_0 + e_1 + e_2) - f(u_0 + e_2) e_2 - f(u_0 + e_2) - f(u_0) e_1} e_1 d\tau',
\]

\[
B_1(\rho) = \int \left[ f'(u_0 + e_1 \omega_0(\eta) + e_2 \omega_02(\eta + \rho)) - f'(u_0 + e_1 \omega_01(\eta)) \right] e_1 \omega_1(\eta) d\eta,
\]

where \( \rho = \rho(\tau) \) has the same meaning as in Theorem 1.2 and satisfies the differential equation with the boundary condition:

\[
\frac{d\rho}{d\tau} = \frac{f(u_0 + e_1 + e_2 - f(u_0 + e_1)) e_1 - f(u_0 + e_1 - f(u_0)) e_1 - (e_1 + e_2) B_1(\rho)}{f(u_0 + e_2 - f(u_0)) e_1 - f(u_0 + e_1 + e_2 - f(u_0 + e_2)) e_2},
\]

\[
\rho(\tau) \bigg|_{\tau \to +\infty} = 1.
\]

This implies the following result.

**Theorem 1.5** The Cauchy problem for the equation (1.1), (1.5) in the sense of Definition 1.3 has a discontinuous solution of the form

\[
u^*(x, t) = u_0 + \sum_{k=1}^{2} e_k H(-x + \phi_k(t)),
\]

(1.31)
where
\[
\phi_1(t) = \phi_{10}(0) + f(u_0 + e_1 + e_2) - f(u_0 + e_2)t \\
- \left[ \frac{f(u_0 + e_1 + e_2) - f(u_0 + e_2)}{e_1} - \frac{f(u_0 + e_2) - f(u_0)}{e_2} \right] \frac{e_2}{e_1 + e_2} (t - t^*) H(t - t^*),
\]
\[
\phi_2(t) = \phi_{20}(0) + f(u_0 + e_2) - f(u_0)t \\
+ \left[ \frac{f(u_0 + e_1 + e_2) - f(u_0 + e_2)}{e_1} - \frac{f(u_0 + e_2) - f(u_0)}{e_2} \right] \frac{e_1}{e_1 + e_2} (t - t^*) H(t - t^*)
\]
are phases.

This solution of the initial value problem (1.1), (1.3) is the unique generalized solution of the Cauchy problem in the sense of the integral identity (1.3).

This Theorem is proved in the same way as Corollary 1.4 from Theorem 1.1 and Theorem 1.3.

8. Description of the shock wave merging process.

Thus, for \( t \in (0, t^*) \) two shock waves whose fronts move from the initial positions \( x_1^0 = \phi_1(0), \quad x_2^0 = \phi_2(0) \), where \( x_1^0 < x_2^0 \) along the straight lines
\[
x_1 = \phi_{10}(t) = x_1^0 + f(u_0 + e_1 + e_2) - f(u_0 + e_2) t, \\
x_2 = \phi_{20}(t) = x_2^0 + f(u_0 + e_2) - f(u_0) t,
\]
with the velocities
\[
\frac{d \phi_{10}(t)}{dt} = \frac{f(u_0 + e_1 + e_2) - f(u_0 + e_2)}{e_1}, \\
\frac{d \phi_{20}(t)}{dt} = \frac{f(u_0 + e_2) - f(u_0)}{e_2},
\]
where \( \frac{d \phi_{10}(t)}{dt} > \frac{d \phi_{20}(t)}{dt} \).

At the instant \( t = t^* \) after the interaction shock waves merge constituting one new shock wave which for \( t \in (t^*, \infty) \) propagates from the point \( (x^*, t^*) \) along the straight line
\[
x = x^* + \frac{f(u_0 + e_1 + e_2) - f(u_0)}{e_1 + e_2} (t - t^*)
\]
at the velocity
\[
\frac{d}{dt} \tilde{\phi}^{-}(t) = \frac{f(u_0 + e_1 + e_2) - f(u_0)}{e_1 + e_2},
\]
where
\[
t^* = e_1 e_2 \left( \frac{x_1^0}{e_1} - \frac{x_2^0}{e_2} \right) f(u_0 + e_1 + e_2) - (e_1 + e_2) f(u_0 + e_2) + e_1 f(u_0), \\
x^* = \phi_{k0}(t^*) = \frac{f(u_0 + e_1 + e_2) - f(u_0 + e_2)}{e_2} x_2^0 - \left[ \frac{f(u_0 + e_2) - f(u_0)}{e_2} \right] \frac{e_1 x_1^0}{e_1 + e_2} f(u_0 + e_2) + e_1 f(u_0).
\]

Remark 1.3 The same result for the Hopf equation can be obtained using the solution of the Burgers equation
\[
L_B[u] = u_t + (u^2)_x + \varepsilon u_{xx} = 0,
\]
which is a regularization of the Hopf equation. It turns out that a simple solution of the Burgers equation can be found whose limit, as \( \varepsilon \to +0 \), describes the process of merging two shock waves (see [3, 4.7]).

2 Proof of Lemma 1.1.

First, consider the case when \( f(u) \) is an analytical function. We find the weak asymptotics of the product
\[
g(x, a, \varepsilon) = \omega_0^m\frac{-x}{\varepsilon}\omega_0^n\frac{-x + a}{\varepsilon},
\]
which, according to Section 1, approximates the product of the distributions \( H^m(-x)H^n(-x + a) \), where \( m, n = 1, 2, \ldots \).
It follows from the definition of the primitive of a distribution [20, Ch.I, §2.2.] that:

\[ J(a, \varepsilon) = \left \langle g(x, a, \varepsilon), \varphi(x) \right \rangle = -\left \langle g_x(x, a, \varepsilon), \varphi^{(-1)}(x) \right \rangle, \]

where \( \varphi^{(-1)}(x) = \int_{-\infty}^{-x} \varphi(\xi) \, d\xi \) is the primitive of the function \( \varphi(x) \), \( \varphi(x) \in \mathcal{D}' \), \( \int_{-\infty}^{\infty} \varphi(\xi) \, d\xi = 0 \) and

\[ J_k(a, \varepsilon) = -\left \langle g_{kx}(x, a, \varepsilon), \varphi^{(-1)}(x) \right \rangle, \quad k = 1, 2, \]

\[ g_{1x}(x, a, \varepsilon) = -m \omega_{01}^{m-1} \left( -\frac{x}{\varepsilon} \right) \omega_{02}^{n-1} \left( -\frac{x + a}{\varepsilon} \right) \frac{1}{\varepsilon} \omega_1 \left( -\frac{x}{\varepsilon} \right), \]

\[ g_{2x}(x, a, \varepsilon) = -n \omega_{01}^{m-1} \left( -\frac{x}{\varepsilon} \right) \frac{1}{\varepsilon} \omega_2 \left( -\frac{x + a}{\varepsilon} \right). \]

Making the change of variables \( x = \varepsilon \eta \) we have

\[ J_1(a, \varepsilon) = m \int \omega_{01}^{m-1}(-\eta) \omega_{02}^{n-1}(-\eta + \frac{a}{\varepsilon}) \omega_1(-\eta) \varphi^{(-1)}(\varepsilon \eta) \, d\eta. \]

Since the integrand decreases rapidly as \( |\eta| \to \infty \), we have

\[ J_1(a, \varepsilon) = \varphi^{(-1)}(0)m \int \omega_{01}^{m-1}(-\eta) \omega_{02}^{n-1}(-\eta + \frac{a}{\varepsilon}) \omega_1(-\eta) \, d\eta + O(\varepsilon), \quad \varepsilon \to +0. \]  \hspace{1cm} (2.34)

In an analogous way, changing variables \( x = a + \varepsilon \eta \) we derive the asymptotics

\[ J_2(a, \varepsilon) = \left \langle g_{2x}(x, a, \varepsilon), \varphi(x) \right \rangle \]

\[ = \varphi^{(-1)}(a)n \int \omega_{01}^{m-1}(-\eta - \frac{a}{\varepsilon}) \omega_{02}^{n-1}(-\eta) \omega_2(-\eta) \, d\eta + O(\varepsilon), \quad \varepsilon \to +0. \]  \hspace{1cm} (2.35)

Summing asymptotics (2.34) and (2.35) we find the asymptotics

\[ J(a, \varepsilon) = \left \langle g(x, a, \varepsilon), \varphi(x) \right \rangle \]

\[ = B_1^{m,n} \left( \frac{a}{\varepsilon} \right) \varphi^{(-1)}(0) + B_2^{m,n} \left( -\frac{a}{\varepsilon} \right) \varphi^{(-1)}(a) + O(\varepsilon), \quad \varepsilon \to +0, \]  \hspace{1cm} (2.36)

where

\[ B_1^{m,n}(\rho) = m \int \omega_{01}^{m-1}(-\eta) \omega_{02}^{n}(\eta + \rho) \omega_1(-\eta) \, d\eta, \]

\[ B_2^{m,n}(\rho) = n \int \omega_{01}^{m}(\eta - \rho) \omega_{02}^{n-1}(-\eta) \omega_2(-\eta) \, d\eta. \]  \hspace{1cm} (2.37)

Here, as in the proof of Lemma 1.1, asymptotics (2.36) can be extended to the whole space \( \mathcal{D} \).

Considering that

\[ \varphi^{(-1)}(0) = \int_{-\infty}^{0} \varphi(\xi) \, d\xi = \left \langle H(-x), \varphi(x) \right \rangle, \]

formula (2.36) can be rewritten in the weak sense as asymptotics (2.33):

\[ g(x, a, \varepsilon) = B_1^{m,n} \left( \frac{a}{\varepsilon} \right) H(-x) + B_2^{m,n} \left( -\frac{a}{\varepsilon} \right) H(-x + a) + O_D(\varepsilon), \quad \varepsilon \to +0. \]  \hspace{1cm} (2.38)

Using (2.33), (2.38), (1.20) and the binomial formula we find the weak asymptotics of the expression

\[ \left[ c_1(x, t)\omega_1 \left( -\frac{x}{\varepsilon} \right) + c_2(x, t)\omega_2 \left( -\frac{x + a}{\varepsilon} \right) \right] = e_1^n(x, t)H(-x) + e_2^n(x, t)H(-x + a) + B_1(x, t, \frac{a}{\varepsilon})H(-x) + B_2(x, t, -\frac{a}{\varepsilon})H(-x + a) + O_D(\varepsilon), \quad \varepsilon \to +0. \]  \hspace{1cm} (2.39)

Here, as it follows from (2.37),

\[ \tilde{B}_1(x, t, \rho) = \sum_{k=1}^{n-1} C_k \left[ e_1^k(x, t)e_2^{n-k}(x, t)B_1^{k,n-k}(\rho) \right]. \]
and obtain (1.17) and formulae for the interaction switch functions (1.18). The first formula from (1.19) is thus proved.

To prove the first formula from (1.19) we convert $\tilde{B}_1(x, t, -\rho) = 0$ we find that the first integral is equal to $f(u_0) - f(u_0 + e_1)$ and $f(u_0) - f(u_0 + e_2)$.

The first formula from (1.19) is thus proved. Since $\lim_{\rho \to +\infty} \omega_{02}(-\eta + \rho) = 1$, relation (1.18) implies:

$$B_1(x, t, \infty) = \lim_{\rho \to +\infty} B_1(x, t, \rho) = \int \left[ f'(u_0 + e_1 \omega_{01}(-\eta)) + e_2 \right] e_1 \omega_1(-\eta) \, d\eta.$$ 

Integrating the last expression and taking into account that $\lim_{\eta \to +\infty} \omega_{01}(\eta) = 1$, $\lim_{\eta \to -\infty} \omega_{01}(\eta) = 0$, we obtain

$$B_1(x, t, \infty) = \lim_{\rho \to +\infty} B_1(x, t, \rho) = f(u_0 + e_1 + e_2) - f(u_0 + e_1) - f(u_0 + e_2) + f(u_0),$$

which is the second relation from (1.19). The other relations (1.19) are proved in an analogous way.

The proof of Lemma is completed by using the Theorem on approximating a smooth function on a finite interval by analytic ones.

### 3 Proof of Theorem 1.1.

Substituting $u^*_t(x, t)$ into the left hand side of equation (1.1) and using (1.20) from Corollary 1.1, we have

$$u^*_t + [f(u^*_t)]_x = u_{0t} + [f(u_0)]_x + e_t H(-x + \phi) + \phi e\delta(-x + \phi)$$

$$+ \left[ f(u_0(x, t) + e(x, t)) - f(u_0(x, t)) \right] H(-x + \phi(t))$$

$$- \left[ f(u_0(x, t) + e(x, t)) - f(u_0(x, t)) \right] \delta(-x + \phi(t)) + O_D^d(\varepsilon), \quad \varepsilon \to +0.$$
Thus,
\[
L[u^*(x,t)] = L[u_0(x,t)] + \left\{L[u_0(x,t) + \epsilon(x,t)] - L[u_0(x,t)]\right\} H(-x + \phi(t))
+ \left\{\phi_t \epsilon(x,t) - \left[f(u_0(x,t) + \epsilon(x,t)) - f(u_0(x,t))\right]\right\} \delta(-x + \phi(t)) + O_\eta(\epsilon), \quad \epsilon \to +0.
\]

(3.40)

Setting the right hand side of (3.40) equal to zero, using Lemma 1.2 (to separate the singularities) and Definitions 1.2, 1.3 of generalized solution, we derive Theorem 1.1 whose analog for strictly hyperbolic systems (when \(f(u)\) is a polynomial) was obtained in [8.1].

4 Proof of Theorem 1.2.

1. In the framework of our approach, to substitute the singular ansatz (1.15) into the Hopf equation means to substitute the smooth ansatz (1.16) \(u^*(x,t,\epsilon)\) into the Hopf equation, and then to find a weak asymptotics, up to \(O_\eta(\epsilon)\), of the left hand side of the equation \(L_H[u^*(x,t,\epsilon)] = 0\).

According to (1.21),
\[
[H(-x,\epsilon)]^2 = H(-x) + O_\eta(\epsilon), \quad \epsilon \to +0.
\]

(4.41)

To substitute ansatz (1.16) into the equation we use asymptotics (1.22) from Corollary 1.2 for the pair product \(H(-x,\epsilon)H(-x + a,\epsilon)\) as \(\epsilon \to +0\).

In view of the aforesaid and (1.22) one should involve the dependence on \(\epsilon\) into shock wave phases, that is, seek them in the form \(\phi_k = \phi_k(t,\epsilon), \quad k = 1, 2\).

We consider the case when the shock waves amplitudes \(e_k(x,t)\) and the background \(u_0(x,t)\) are constant. Using (1.41) – (1.42) we find
\[
[u^*(x,t,\epsilon)]^2 = u_0^2 + \left(2u_0e_1 + e_1^2 + 2e_1e_2B_1 \left(\frac{\psi(t,\epsilon)}{\epsilon}\right)\right) H(-x + \phi_1(t,\epsilon))
+ \left(2u_0e_2 + e_2^2 + 2e_1e_2B_2 \left(-\frac{\psi(t,\epsilon)}{\epsilon}\right)\right) H(-x + \phi_2(t,\epsilon)) + O_\eta(\epsilon), \quad \epsilon \to +0,
\]

where \(\psi(t,\epsilon) = \phi_2(t,\epsilon) - \phi_1(t,\epsilon)\) is the distance between shock wave fronts (discontinuities).

Substituting the singular ansatz \(u^*_e(x,t)\) and the asymptotics \([u^*_e(x,t)]^2\) into the Hopf equation (1.2) we have
\[
L_H[u^*(x,t,\epsilon)] = \sum_{k=1}^{2} \left[e_k \frac{d}{dt} \phi_k(t,\epsilon) - 2u_0e_k - e_k^2 - 2e_1e_2B_k \left((-1)^{k-1}\rho\right)\right] \delta(-x + \phi_k(t,\epsilon))
+ O_\eta(\epsilon), \quad \epsilon \to +0,
\]

(4.42)

where \(\rho = \frac{\psi(t,\epsilon)}{\epsilon}\) is the independent variable of the interaction switch function and the estimate \(O_\eta(\epsilon)\) in this representation is uniform with respect to the distance \(\psi(t,\epsilon)\).

From (4.42), using Lemma 1.2 (to separate the singularities), we derive the necessary and sufficient conditions for the relation \(L_H[u^*(x,t,\epsilon)] = O_\eta(\epsilon)\) to be valid:
\[
\frac{d}{dt} \phi_k(t,\epsilon) = 2u_0 + e_k + 2e_{3-k}B_k \left((-1)^{k-1}\rho\right), \quad k = 1, 2.
\]

(4.43)

Before the interaction of shock waves, when \(\psi(t,\epsilon) = \phi_2(t,\epsilon) - \phi_1(t,\epsilon) > ce^{1-\alpha}\), where \(c > 0\), \(0 < \alpha \leq 1\), according to Remark 1.3 and (1.24) we have, at least up to \(O(\epsilon)\), \(B_1(\frac{\psi}{\epsilon}) = 1\) and \(B_2 \left(-\frac{\psi}{\epsilon}\right) = 0\). Therefore, in this case system (4.43) becomes the system of equations which, according to Corollary 1.2 from Theorem 1.1, describes the dynamics of two non-interacting shock waves (taking into account that both the amplitudes \(e_k\) and the background \(u_0\) are constant):
\[
\frac{d\phi_{10}(t)}{dt} = 2u_0 + e_1 + 2e_2,
\]
\[
\frac{d\phi_{20}(t)}{dt} = 2u_0 + e_2,
\]

(4.44)

where \(\phi_{10}(t), \phi_{20}(t)\) denote the phases of non-interacting shock waves.
Let $\psi_0(t) = \phi_{20}(t) - \phi_{10}(t)$ be the distance between the fronts of non-interacting shock waves. Define the interaction time $t = t^*$ as the solution of the equation $\psi_0(t^*) = 0$.

Equations (4.44) describe the propagation of two shock waves before the interaction time, for $t \in [0, t^*)$ and represent the Hugoniot conditions (4.27) along the discontinuity lines. Indeed, in our case for the lagging shock wave we have $[u] = e_1$, $[f(u)] = 2u_0e_1 + e_1^2 + 2e_1e_2$, and for the advanced one we have $[u] = e_2$, $[f(u)] = 2u_0e_2 + e_2^2$.

Thus, before the interaction, i.e. for $t \in [0, t^*)$, the shock waves move with velocities (4.44) along the straight lines
\[
\begin{align*}
    x_1 &= \phi_{10}(t) = \phi_{10}(0) + (2u_0 + e_1 + 2e_2)t, \\
    x_2 &= \phi_{20}(t) = \phi_{20}(0) + (2u_0 + e_2)t,
\end{align*}
\]
which intersect at the point with the following coordinates:
\[
\begin{align*}
    t^* &= \frac{\phi_{20}(0) - \phi_{10}(0)}{e_1 + e_2}, \\
    x^* &= \phi_{10}(0) + (2u_0 + e_1 + 2e_2)t^*,
\end{align*}
\]
where $\phi_{k0}(0)$ are the initial positions of the discontinuities.

2. To describe the interaction dynamics we shall seek the phases of shock waves as functions of the fast variable $\tau = \frac{t^*}{\epsilon} \in \mathbb{R}$ and the slow one $t \geq 0$:

\[
\begin{align*}
    \phi_k(t, \epsilon) \overset{def}{=} \phi_k(\tau, t) &= \phi_{k0}(t) + \psi_0(t)\phi_{k1}(\tau),
\end{align*}
\]
where the functions $\phi_{k0}(t)$ defined by equations (4.43), for $t \in [0, t^*)$, are extended by these equations (4.43) for all $t \in [t^*, +\infty)$. If $\tau > 0$ then $t < t^*$, i.e. the interaction has not occurred yet; if $\tau < 0$ then $t > t^*$, i.e. the interaction has occurred.

Since we consider piecewise constant solutions, we can take the velocities of all the shock waves formed before and after the interaction to be constant, and seek the perturbations for the phases $\phi_{k0}(t)$ as functions $\phi_{k1}(\tau)$ only depending on $\tau$.

We set the following boundary values:

\[
\begin{align*}
    \phi_{k1}(\tau) \bigg|_{\tau \to +\infty} &= 0, \\
    \frac{d\phi_{k1}(\tau)}{d\tau} \bigg|_{\tau \to -\infty} &= o(\tau^{-1}),
\end{align*}
\]
That is, the derivatives of the phases with respect to the fast variable $\tau$ tend to zero as $|\tau| \to \infty$, while the phases themselves tend to zero as $\tau \to \infty$.

Finding the limit values of the perturbations as $\tau \to -\infty$
\[
\phi_{k1}(\tau) \bigg|_{\tau \to -\infty} = \phi_{k1, -},
\]
we find the phase limit values
\[
\hat{\phi}_k(\tau, t) \bigg|_{\tau \to -\infty} = \hat{\phi}_{k, -}(t) = \phi_{k0}(t) + \psi_0(t)\phi_{k1, -}
\]
and thus define ”the result” of the interaction of shock waves for $t > t^*$.

Let $\psi_1(\tau) = \phi_{21}(\tau) - \phi_{11}(\tau)$, then the full phase difference is
\[
\psi(t, \epsilon) = \psi_0(t)(1 + \psi_1(\tau)),
\]
the independent variable of the interaction switch function (1.23) has the form:
\[
\rho(\tau) = \frac{\psi(t, \epsilon)}{\epsilon} = \tau(1 + \psi_1(\tau)).
\]

The phase derivatives with respect to time are represented by the following relations:
\[
\frac{d}{dt} \phi_k(t, \epsilon) \overset{def}{=} \frac{d}{dt} \hat{\phi}_k(\tau, t) = \frac{d\phi_{k0}(t)}{dt} + \frac{d\psi_0(t)}{dt} \frac{d}{d\tau}[\tau\phi_{k1}(\tau)].
\]
Taking into account the boundary conditions (4.47), we obtain the limit values of the phases and their derivatives with respect to time as \( \tau \to -\infty \):

\[
\begin{align*}
\frac{d}{dt} \hat{\phi}_k(t) - t_0(t) + \psi_0(t) \phi_k(t) &= \frac{d}{dt} \hat{\phi}_k(\tau, t),
\end{align*}
\quad (4.50)
\]

Substituting (4.40), (4.44) into (4.43), we derive, for all \( t \geq 0 \) and \( \tau \in (-\infty, \infty) \), the following system of equations with the boundary conditions (4.47):

\[
\begin{align*}
\frac{d\phi_{10}(t)}{dt} + \frac{d\psi_0(t)}{dt} \frac{d}{dt} \phi_{11}(\tau) &= 2u_0 + e_1 + 2e_2B_1(\rho), \\
\frac{d\phi_{20}(t)}{dt} + \frac{d\psi_0(t)}{dt} \frac{d}{dt} \phi_{21}(\tau) &= 2u_0 + e_2 + 2e_1B_2(-\rho),
\end{align*}
\quad (4.51)
\]

where \( \phi_{k0}(t) \) are the functions defined, for all \( t \geq 0 \) by equations (4.45).

Subtracting one of the equations (4.51) from the other we find the differential equation satisfied by the function \( \rho(\tau) \):

\[
\rho_\tau = F(\rho),
\]

\[
\frac{\rho}{\tau} \bigg|_{\tau \to -\infty} = 1,
\quad (4.52)
\]

where the boundary condition (4.52) follows from (4.47).

Taking into account the first relation (1.24) and the fact that, according to (4.44), \( \psi_{0t}(t) = -(e_1 + e_2) \), we find the right hand side of the differential equation

\[
F(\rho) = \frac{1}{\psi_{0t}(t)} \left[ e_2 \left( 1 - 2B_1(\rho) \right) - e_1 \left( 1 - 2B_2(-\rho) \right) \right] = 2B_1(\rho) - 1.
\quad (4.53)
\]

According to Corollary 1.2, the function \( B_1(\rho) \) is smooth and its values belong to the interval \((0, 1)\). Therefore, independent of the choice of the approximating functions \( \omega_k(\xi) \), \( k = 1, 2 \), the equation \( F(\rho) = 0 \) has at least one root \( \rho_0 \) which solves the equation \( B_1(\rho) = 1/2 \).

The differential equation (4.52), (4.53) is autonomous. For this equation the following statement holds.

**Proposition 4.1** For the autonomous equation

\[
\frac{d\rho}{d\tau} = F(\rho)
\quad (4.54)
\]

with the smooth right hand side \( F(\rho) \) to have a solution such that

\[
\frac{\rho(\tau)}{\tau} \bigg|_{\tau \to +\infty} = 1,
\]

\[
\frac{\rho(\tau)}{\tau} \bigg|_{\tau \to -\infty} = \rho_0,
\quad (4.55)
\]

where \( \rho_0 \) is a constant, it is necessary and sufficient that the following conditions hold:

\[
F(\rho) \bigg|_{\rho \to +\infty} = 1,
\]

\[
F(\rho_0) = 0,
\]

\[
F(\rho) > 0 \text{ for } \rho > \rho_0,
\quad (4.56)
\]

where \( \rho_0 \) is the maximal root of the equation \( F(\rho) = 0 \).

In addition, if \( \rho_0 \) is an ordinary (nonmultiple) root of the equation \( F(\rho) = 0 \) then \( \rho - \rho_0 = O(\tau^{-N}) \), \( \tau \to -\infty \) for all \( N = 1, 2, \ldots \).

**Proof.** Suppose that the limit relations (4.55) hold, and \( \tilde{\tau}, \tilde{\tau} + 1 \in (\rho_0, +\infty) \). Then, integrating (4.54), we have

\[
\rho(\tilde{\tau} + 1) - \rho(\tilde{\tau}) = \int_{\tilde{\tau}}^{\tilde{\tau} + 1} F(\rho(t + \tilde{\tau})) dt.
\]
Since according to the second relation (1.53), the left hand side of this equality tends to zero as \( \hat{\tau} \to -\infty \), the limit of its right hand side is also equal to zero:

\[
\lim_{\tau \to -\infty} \int_0^1 F(\rho(\tau + \hat{\tau})) \, d\tau = \int_0^1 F(\rho_0) \, d\tau = F(\rho_0) = 0.
\]

That is, \( \rho_0 \) is a root of the equation \( F(\rho_0) = 0 \).

It follows from the first relation (4.53) that \( F(\rho) > 0 \) for \( \rho > \rho_0 \), and consequently, \( \rho_0 \) is the maximal root.

Since for \( \tau_1, \tau \in (\rho_0, +\infty) \)

\[
\rho(\tau) = \rho(\tau_1) + \frac{1}{\tau} \int_{\tau_1}^\tau F(\rho(\tau')) \, d\tau',
\]

passing in (4.57) to the limit as \( \tau \to +\infty \), taking into account the first relation (4.55) and using the L'Hospital rule, we prove that \( \lim_{\rho \to +\infty} F(\rho) = 1 \).

Conversely, if (4.56) holds then \( \rho = \rho_0 \) is a solution of equation (4.54). Integrating (4.54), we have

\[
\tau = \tau_1 + \int_{\rho_1}^{\rho} \frac{d\rho'}{F(\rho')},
\]

Since the function \( F(\rho) \) is smooth, the function \( 1/F(\rho) \) has a non-integrable singularity at the point \( \rho = \rho_0 \), and the integral \( \int_{\rho_1}^{\rho} \frac{d\rho'}{F(\rho')} \) diverges as \( \rho \to \rho_0 \).

That is, as \( \rho \) approaches \( \rho_0 \), we have \( \tau \to -\infty \) and therefore, \( \rho = \rho_0 \) is an asymptote for all integral curves within the region \( \rho_0 < \rho < +\infty \).

The first relation (1.53) follows from (4.57). If \( \rho_0 \) is a simple root of the equation \( F(\rho) = 0 \) then \( F(\rho) = (\rho - \rho_0)g(\rho) \), where \( g(\rho) > 0 \) for \( \rho > \rho_0 \).

The second statement is proved by passing from the differential equation (1.54) to the differential inequality

\[
\frac{d\rho}{d\tau} \leq (\rho - \rho_0)M, \quad \text{where} \quad M = \max_{\rho \geq \rho_0} g(\rho).
\]

Integrating this inequality, we find that \( \rho - \rho_0 \leq Ke^7 \). Thus, \( \rho \to \rho_0 \) more rapidly than any power of \( |\tau|^{-1} \) as \( \tau \to -\infty \).

The proposition is proved.

By Proposition 4.1, as \( \tau \to -\infty \), we have \( \rho = \tau(1 + \psi_1(\tau)) \to \rho_0 \), here \( B_1(\rho) \to B_1(\rho_0) = 1/2 \) and \( B_2(-\rho) \to B_2(-\rho_0) = 1/2 \).

Thus, passing to the limit in (4.51), as \( \tau \to -\infty \), and taking into account that, according to (4.47), the phase derivatives with respect to the fast variable \( \tau \) tend to zero, we derive the following limit system of equations which describes the shock wave evolution after the interaction, i.e. for \( t > t^* \):

\[
\frac{d\phi_{10}(t)}{dt} + \frac{d\psi_0(t)}{dt} \phi_{11,-} = 2u_0 + e_1 + e_2,
\]

\[
\frac{d\phi_{20}(t)}{dt} + \frac{d\psi_0(t)}{dt} \phi_{21,-} = 2u_0 + e_2 + e_1.
\]

(4.58)

Remind that here the functions \( \phi_{k0}(t) \) are determined by equations (4.45) for all \( t \geq 0 \).

It is clear from (4.58) that phase limit values of both shock waves coincide, that is,

\[
\hat{\phi}_{2,-}(t) = \hat{\phi}_{1,-}(t) \overset{de}{=} \hat{\phi}_{-}(t).
\]

(4.59)

Thus, it follows from (4.58), (4.59) that after the interaction, i.e. for \( t > t^* \), the discontinuities merge together constituting a new shock wave whose dynamics is determined by the equation

\[
\frac{d\hat{\phi}_{-}(t)}{dt} = 2u_0 + e_1 + e_2.
\]

(4.60)

with the initial value \( \hat{\phi}_{-}(t^*) = \phi_{k0}(t^*) \).

Equation (4.60) represents the Hugoniot condition (1.27) along the discontinuity line after the interaction, since \( |u| = u_0 - (u_0 + e_1 + e_2) = -(e_1 + e_2) \) and \( |f(u)| = (u_0)^2 - (u_0 + e_1 + e_2)^2 \).

3. Now we describe the dynamics of the shock wave merging process. To this end, substituting the phases \( \phi_{k0}(t) \) from (4.44) into (4.51), and using the relations \( B_1(\rho) + B_2(-\rho) = 1 \) from (1.24) and \( \psi_{0t}(t) = -\left(e_1 + e_2\right) \), we derive the equations determining the perturbations for the phases \( \phi_{k1}(\tau) \):

\[
\frac{d}{d\tau} \left[ \tau\phi_{11}(\tau) \right] = (-1)^{k-1} \frac{2e_{3-k}}{e_1 + e_2} \left(1 - B_1(\rho)\right), \quad k = 1, 2.
\]
Integrating these equations, we have the following expression for the phase perturbations

$$
\phi_{k1}(\tau) = (-1)^{k-1} \frac{2e_{3-k}}{(e_1 + e_2)\tau} \int_0^\tau \left(1 - B_1(\rho(\tau'))\right) d\tau', \quad k = 1, 2,
$$

(4.61)

where $\rho = \rho(\tau)$ solves the differential equation with the boundary condition [4.52], [4.53]:

$$
\frac{d\rho}{d\tau} = 2B_1(\rho) - 1,

\rho(\tau) \bigg|_{\tau \to +\infty} = 1.
$$

Verify that the functions $\phi_{k1}(\tau)$ found in (4.61) satisfy the required properties.

We have chosen the approximating functions $\omega_k(z)$ so that they either have compact supports or decrease sufficiently rapidly as $|z| \to \infty$, for example, $|\omega_k(z)| \leq C_k(1 + |z|)^{-N}$, $N = 1, 2, \ldots, k = 1, 2$. Therefore, we have $(\dot{\omega}_1 \ast \omega_2)(\eta) \leq K(1 + |\eta|)^{-N}$ as $|\eta| \to \infty$, $N = 1, 2, \ldots$ and $B_1(\rho) = \int_{-\infty}^\infty (\dot{\omega}_1 \ast \omega_2)(\eta) d\eta = 1 + O(\rho^{-N})$ as $\rho \to +\infty$, $B_1(\rho) = O(\rho^{-N})$ as $\rho \to -\infty$.

If $\tau \to \infty$ then $\rho \to \infty$ and, according to (4.61) and Remark 1.1, we have

$$
\phi_{k1}(\tau) = O(\tau^{-1}),

\frac{d\phi_{k1}(\tau)}{d\tau} = O(\tau^{-1}).
$$

Note that $\rho = \rho_0$ is an ordinary (nonmultiple) root of the right-hand side of the differential equation (4.52), (4.53). Indeed, $F'(\rho_0) = 2B_1(\rho_0) - 1 = 0$ and accordingly to (1.23),

$$
F'(\rho_0) = 2(\dot{\omega}_1 \ast \omega_2)(\rho_0) = 2 \int_{-\infty}^{+\infty} \omega_k(\xi)\omega_{3-k}(\rho_0 + \xi) d\xi > 0
$$

since $\omega_k(\xi) \geq 0$, $k = 1, 2$.

Thus, if $\tau \to -\infty$ then $\rho \to \rho_0$ and according to Proposition 4.1, $\rho - \rho_0 = O(|\tau|^{-N})$ for all $N = 1, 2, \ldots$ Therefore, using Remark 1.1, we have $B_1(\rho) = 1/2 + O(|\tau|^{-N})$, $\tau \to -\infty$. According to (4.61), we obtain

$$
\phi_{k1}(\tau) = (-1)^{k-1} \frac{e_{3-k}}{e_1 + e_2} + O(\tau^{-1}),

\psi_1(\tau) = -1 + O(\tau^{-1}).
$$

Calculate the limit of the expression $\tau \frac{d\phi_{k1}(\tau)}{d\tau}$ as $\tau \to -\infty$. To this end, rewrite this expression in the form

$$
\tau \frac{d\phi_{k1}(\tau)}{d\tau} = (-1)^{k-1} \frac{e_{3-k}}{e_1 + e_2} (1 - B_1(\rho)) \left[1 - \frac{\int_0^\tau \left(1 - B_1(\rho(\tau'))\right) d\tau'}{\tau(1 - B_1(\rho))}\right].
$$

Applying the L’Hospital rule to the limit in square brackets we find

$$
\lim_{\tau \to -\infty} \tau \phi_{k1}(\tau) = 0.
$$

As one can see, the functions $\phi_{k1}(\tau)$ obtained in (4.61) have all the properties required. Thus, we have proved the Theorem 1.2.

5 Proof of Theorem 1.4.

1. We consider the case when $u_0(x,t)$ and $e_2(x,t)$ in ansatz (1.13) are constants. In this case, therefore the interaction switch functions from Lemma 1.1 $B_k(x,t,\rho) = B_k(\rho)$. As in Section 4, we shall seek shock wave phases in ansatz (1.13) as functions of $\varepsilon$, i.e. $\phi_k = \phi_k(t,\varepsilon)$, $k = 1, 2$. Denote the distance between shock wave fronts by $(\psi(t,\varepsilon) = \phi_2(t,\varepsilon) - \phi_1(t,\varepsilon)$.

Substituting the singular ansatz (1.13) and asymptotics (1.17), given by Lemma 1.1, into equation (1.1) we have

$$
L[u^*(x,t,\varepsilon)] = u^*_{xx} + [f(u^*)]_x
$$
\[ + \sum_{k=1}^{2} \left\{ e_k \frac{d\phi_k(t, \varepsilon)}{dt} - \left[ f(u_0 + e_k) - f(u_0) \right] - B_k((-1)^{k-1}\rho) \right\} \delta(-x + \phi_k(t, \varepsilon)) + O_P'(\varepsilon), \quad \varepsilon \to +0. \] (5.62)

where \( \rho = \phi(t, \varepsilon) \) and the estimate \( O_P'(\varepsilon) \) in this representation is uniform with respect to the distance \( \psi(t, \varepsilon) \).

From (5.62), using Lemma 1.2, we derive the necessary and sufficient conditions for the relation \( L[u'_*^c(x, t)] = O_P'(\varepsilon) \) to be valid:

\[
\begin{align*}
e_1 \frac{d\phi_1(t, \varepsilon)}{dt} &= f(u_0 + e_1) - f(u_0) + B_1(\rho), \\
e_2 \frac{d\phi_2(t, \varepsilon)}{dt} &= f(u_0 + e_2) - f(u_0) + B_2(-\rho).
\end{align*}
\] (5.63)

When

\[ \psi(t, \varepsilon) = \phi_2(t, \varepsilon) - \phi_1(t, \varepsilon) > c\varepsilon^{1-\alpha}, \]

where \( c > 0, \ 0 < \alpha \leq 1 \), that is, before the interaction of shock waves, according to Remark 1.1 and (1.19), we have, up to \( O(\varepsilon) \):

\[ B_1 \left( \frac{\psi}{\varepsilon} \right) = f(u_0 + e_1 + e_2) - f(u_0 + e_1) - f(u_0 + e_2) + f(u_0), \]

\[ B_2 \left( \frac{\psi}{\varepsilon} \right) = 0. \]

It follows from the above that system (5.63) turns into the system of equations which, according to Theorem 1.1, describes the dynamics of two noninteracting shock waves:

\[
\begin{align*}
e_1 \frac{d\phi_{10}(t)}{dt} &= f(u_0 + e_1 + e_2) - f(u_0 + e_2), \\
e_2 \frac{d\phi_{20}(t)}{dt} &= f(u_0 + e_2) - f(u_0).
\end{align*}
\] (5.64)

Equations (5.64) obtained above represent the Hugoniot conditions (1.27) along the lines: \([u] = e_1, \ [f(u)] = f(u_0 + e_1 + e_2) - f(u_0 + e_2)\) for the lagging shock wave, and \([u] = e_2, \ [f(u)] = f(u_0 + e_2) - f(u_0)\) for the advanced one.

Denote by \( \psi_0(t) = \phi_{20}(t) - \phi_{10}(t) \) the distance between the shock waves before the instant of the interaction \( t = t^* \) which we define as a solution of the equation \( \psi_0(t^*) = 0 \).

Thus, before the instant of the interaction, as \( t \in (0, t^*) \), the shock waves move with velocities (5.64) along the straight lines

\[ x_1 = \phi_{10}(t) = x_1^0 + \frac{f(u_0 + e_1 + e_2) - f(u_0 + e_2)}{e_1} t, \]

\[ x_2 = \phi_{20}(t) = x_2^0 + \frac{f(u_0 + e_2) - f(u_0)}{e_2} t, \] (5.65)

which intersect at a point with the following coordinates:

\[ t^* = \frac{x_1^0 - x_2^0}{e_2 f(u_0 + e_1 + e_2) - e_1 f(u_0 + e_2) + e_1 f(u_0)}, \]

\[ x^* = \frac{f(u_0 + e_1 + e_2) - f(u_0 + e_2)}{e_2 f(u_0 + e_1 + e_2) - e_1 f(u_0 + e_2) + e_1 f(u_0)} x_1^0. \] (5.66)

where \( \phi_{k0}(0) \) are the initial positions of the discontinuities.

2. As in Section 3, define shock wave phases by formula (4.46) where the functions \( \phi_{k0}(t) \) determined by equations (5.63), as \( t \in (0, t^*) \), are expanded by the same equations (5.63) for \( t \in (t^*, +\infty) \), as was done previously, and the phase derivatives with respect to time are given by formulae (4.46). The full phase difference is

\[ \psi(t, \varepsilon) = \psi_0(t) \left( 1 + \psi_1(\tau) \right) \]

where \( \psi_1(\tau) = \phi_{21}(\tau) - \phi_{11}(\tau) \). Here the independent variable of the interaction switch function is set in (4.48).

For \( \phi_{k1}(\tau) \) we set the boundary conditions (4.47). The limit values of phases and their derivatives with respect to time, as \( \tau \to -\infty \), were determined in (4.50).
Substituting (4.46), (4.47) into (5.63), we obtain, for all $t > 0$ and $\tau \in (-\infty, \infty)$, the system of equations with boundary conditions (4.47):

$$
\begin{align*}
&\frac{d\phi_{10}(t)}{dt} + \frac{d\psi_{10}(t)}{dt} \left[ \tau \phi_{11}(\tau) \right] = f(u_0 + e_1) - f(u_0) + B_1(\rho), \\
&\frac{d\phi_{20}(t)}{dt} + \frac{d\psi_{20}(t)}{dt} \left[ \tau \phi_{21}(\tau) \right] = f(u_0 + e_2) - f(u_0) + B_2(-\rho),
\end{align*}
$$

(5.67)

where $\phi_{ik}(t)$ are the extensions of the functions $\phi_{ik}(t)$ determined by equations (5.64) for all $t \geq t^*$, the extension being determined by the same equations (5.64).

Subtracting one of the equations in system (5.67) from the other we obtain the following differential equation with the boundary condition for $\rho(\tau)$:

$$
\frac{\rho}{\tau \rightarrow \infty} = F(\rho),
$$

(5.68)

where

$$
F(\rho) = \frac{1}{\rho_{\tau \rightarrow \infty}} \left[ f(u_0 + e_2) - f(u_0) + \frac{B_2(-\rho)}{e_2} - f(u_0 + e_1) - f(u_0) + \frac{B_1(\rho)}{e_1} \right],
$$

(5.69)

here the boundary condition follows from (4.47).

Formula (5.64) implies:

$$
\psi_{0t}(t) = \frac{f(u_0 + e_2) - f(u_0)}{e_2} - \frac{f(u_0 + e_1 + e_2) - f(u_0 + e_2)}{e_1}.
$$

(5.70)

Taking into account the first correlation from (1.18), convert function (5.69) to the form

$$
F(\rho) = \frac{f(u_0 + e_1 + e_2) - f(u_0 + e_1)}{e_1} - \frac{f(u_0 + e_1 + e_2) - f(u_0 + e_2)}{e_2} - (e_1 + e_2)B_1(\rho).
$$

(5.71)

Using the limit values of the switch function $B_1(\pm \infty)$ from Lemma 1.1, we find the limit values of the function $F(\rho)$ from (5.71):

$$
F(\rho) \bigg|_{\rho \rightarrow \infty} = 1,
$$

(5.72)

$$
F(\rho) \bigg|_{\rho \rightarrow -\infty} = \frac{f(u_0 + e_1 + e_2) - f(u_0 + e_1)}{e_1} - \frac{f(u_0 + e_1 + e_2) - f(u_0 + e_2)}{e_2} + \frac{f(u_0 + e_2) - f(u_0 + e_2)}{e_1}.
$$

(5.73)

In view of our assumptions, the function $f(u)$ is convex downwards, that is, $f''(u) > 0$ on the range of the solution $u$, and the amplitudes are positive, $e_1, e_2 > 0$. Under these assumptions, both the numerator and denominator of the last fraction are strictly positive, since for a function convex downwards the inequality $(x_2 - x)f(x_1) + (x_1 - x_2)f(x) + (x - x_1)f(x_2) > 0$ holds for $x_1 < x < x_2$. In our case, $x_1 = u_0, x = u_0 + e_1, x_2 = u_0 + e_1 + e_2$.

Thus, as $\rho \rightarrow \pm \infty$, the limit values of the right hand side $F(\rho)$ of the differential equation (5.68) have opposite signs.

According to (1.18), the derivative of the interaction switch function has the form

$$
\frac{d}{d\rho} B_1(\rho) = e_1 e_2 \int f''(u_0 + e_1 \omega_01(-\eta) + e_2 \omega_02(-\eta + \rho)) \omega_1(-\eta) \omega_2(-\eta + \rho) d\eta.
$$

It is positive since $f''(u) \geq 0$, $\omega_k(\eta) > 0$ and $e_1, e_2 > 0$. It follows from here and from (5.71) that the derivative

$$
\frac{d}{d\rho} F(\rho) = \frac{(e_1 + e_2)}{f(u_0 + e_1 + e_2) - f(u_0 + e_2)(e_1 + e_2) + f(u_0 + e_2) + f(u_0 + e_1)} \frac{d}{d\rho} B_1(\rho)
$$

(5.73)
is positive, and therefore $F(\rho)$ is an increasing function.

Since the function $F(\rho)$ is smooth, the equation $F(\rho) = 0$ has a root $\rho_0$. Then, according to Proposition 1.1, the solution $\rho$ of equation (5.68) tends to $\rho_0$, as $\tau \to -\infty$.

Due to the smoothness of the function $F(\rho)$ and the aforesaid, the equation $F(\rho) = 0$ has at least one root $\rho_0$. Let $\rho_0$ be the maximal root of the equation $F(\rho) = 0$. Then, according to Proposition 1.1 as $\tau \to -\infty$, the solution of equation (5.68), $\rho$ tends to $\rho_0$.

Pass to the limit in (5.67), as $\tau \to -\infty$, taking into account that by (4.47), $\tau \phi_{k1}(\tau) \to 0$. From here we derive a system of equation describing the evolution of shock waves after the interaction, for $t > t^*$:

\[
\frac{d\phi_{10}(t)}{dt} + \phi_{11, -} \frac{d\psi_0(t)}{dt} = \frac{f(u_0 + e_1) - f(u_0) + B_1(\rho_0)}{e_1},
\]
\[
\frac{d\phi_{20}(t)}{dt} + \phi_{21, -} \frac{d\psi_0(t)}{dt} = \frac{f(u_0 + e_2) - f(u_0) + B_2(-\rho_0)}{e_2}.
\]

(5.74)

As shown above, we have $\rho = \tau(1 + \psi_1(\tau)) \to \rho_0$ as $\tau \to -\infty$. It follows that $\psi_1(\tau) \to -1$, that is $\psi_1(\tau) = -1 + O(\tau^{-1})$.

Since $\psi_0(t) = \phi_{20}(t) - \phi_{10}(t)$ and $\psi_0(t)O(\tau^{-1}) = O(\varepsilon)$, we have as $\tau \to -\infty$:

\[
\hat{\phi}_2(\tau) = \phi_{20}(t) + \psi_0(t)\phi_{21}(\tau)
\]

\[
= \phi_{20}(t) + \psi_0(t)\left(\phi_{11}(\tau) - 1 + O(\tau^{-1})\right) = \hat{\phi}_1(\tau) + O(\varepsilon).
\]

That is, up to $O(\varepsilon)$ we have

\[
\hat{\phi}_{2, -}(t) = \hat{\phi}_{1, -}(t) \overset{df}{=} \hat{\phi}_{-}(t).
\]

(5.75)

It is clear from (5.74), (5.75) that after the interaction, i.e. for $t > t^*$, discontinuities merge constituting a new shock wave whose dynamics is determined by the equation

\[
\frac{d\hat{\phi}_{-}(t)}{dt} = \frac{f(u_0 + e_1) - f(u_0) + B_1(\rho_0)}{e_1}.
\]

(5.76)

Here the following relation holds

\[
\frac{f(u_0 + e_1) - f(u_0)}{e_1} + B_1(\rho_0) = \frac{f(u_0 + e_2) - f(u_0)}{e_2} + B_2(-\rho_0).
\]

(5.77)

From (5.77) using the first relation from (1.19), by Lemma 1.1, we find the value

\[
B_1(\rho_0) = \frac{\left[f(u_0 + e_1 + e_2) - f(u_0 + e_1)\right]e_1 - \left[f(u_0 + e_1) - f(u_0)\right]e_2}{e_1 + e_2}.
\]

(5.78)

Substituting value (5.78) into (5.74) we obtain the equation which describes the propagation of the shock wave resulting from merging the initial waves:

\[
\frac{d\hat{\phi}_{-}(t)}{dt} = \frac{f(u_0 + e_1 + e_2) - f(u_0)}{e_1 + e_2}
\]

(5.79)

with the initial value $\hat{\phi}_{-}(t^*) = \phi_{k0}(t^*) = x^*$.

Equation (5.79) is the Hugoniot condition (1.27) along the line of discontinuity after merging the shock waves, since $|u| = u_0 - (u_0 + e_1 + e_2) = -(e_1 + e_2)$ and $f(u) = f(u_0 + e_1 + e_2) - f(u_0)$.

3. Consider the dynamics of the process of merging two shock waves into a new one. Substitute the phases of noninteracting shock waves $\phi_{k0}(t)$ from (5.66) into (5.67). From here, using the first correlation from (1.19) for the interaction switch functions $B_k(\rho)$ and expression (5.64) for $\psi_0(t)$ we obtain equations to determine $\phi_{k1}(\tau), \ k = 1, 2$:

\[
\frac{d}{d\tau} [\tau \phi_{k1}(\tau)]
\]
\[ = (-1)^{k-1} e^{-k} \frac{f(u_0 + e_1 + e_2) - f(u_0 + e_1) - f(u_0 + e_2) + f(u_0) - B_1(\rho(\tau))}{f(u_0 + e_1 + e_2) - f(u_0 + e_2) - f(u_0 + e_1) - f(u_0)} e_2 - \frac{f(u_0 + e_2) - f(u_0)}{e_1}. \]  
\hspace{2cm} (5.80)

Integrating equations (5.80) we derive the following expression for \( \phi_{k_1}(\tau) \):

\[ \phi_{k_1}(\tau) = (-1)^{k-1} e^{-k} \int_0^\tau \frac{f(u_0 + e_1 + e_2) - f(u_0 + e_1) - f(u_0 + e_2) + f(u_0) - B_1(\rho(\tau))}{f(u_0 + e_1 + e_2) - f(u_0 + e_2) - f(u_0 + e_1) - f(u_0)} e_2 - \frac{f(u_0 + e_2) - f(u_0)}{e_1} d\tau', \]  
\hspace{2cm} (5.81)

where \( \rho = \rho(\tau) \) solves the differential equation with boundary condition (5.68), (5.69).

As in the case of the Hopf equation, it can be verified that the functions \( \phi_{k_1}(\tau) \) found in (5.81) satisfy the presupposed properties.

Let \( \tau \to \infty \), then \( \rho \to \infty \), \( \rho \sim \tau \) and, according to (5.81) and Remark 1.1,

\[ \frac{d\phi_{k_1}(\tau)}{d\tau} = O(\tau^{-1}), \]
\[ \frac{\phi_{k_1}(\tau)}{\tau} = O(\tau^{-1}), \]

According to (5.78), \( \rho = \rho_0 \) is an ordinary (nonmultiple) root of the right-hand side of the differential equation (5.68), (5.71), i.e. \( F(\rho_0) = 0 \). But in view of to (5.73), \( \frac{d}{d\rho} F(\rho) > 0 \). Consequently, \( \rho = \rho_0 \) is an ordinary (nonmultiple) root of the right-hand side of the differential equation (5.68), (5.71).

Thus, if \( \tau \to -\infty \), then \( \rho \to \rho_0 \) and \( B_1(\rho) \) tends to the limit value \( B_1(\rho_0) \) given by formula (5.78).

According to Proposition 1.1 \( \rho - \rho_0 = O(|\tau|^{-N}) \) for all \( N = 1, 2, \ldots \). Thus, using Remark 1.1, we have

\( B_1(\rho) = B_1(\rho_0) + O(|\tau|^{-N}), \quad \tau \to -\infty. \) Therefore, (5.80), (5.81) imply the same estimates as in the case of the Hopf equation:

\[ \phi_{k_1}(\tau) = (-1)^{k-1} e^{-k} \frac{e_3 - e_1 - e_2}{e_1} + O(\tau^{-1}), \]
\[ \psi_1(\tau) = -1 + O(\tau^{-1}), \]
\[ \frac{d\phi_{k_1}(\tau)}{d\tau} = O(\tau^{-1}). \]

Thus, the functions \( \phi_{k_1}(\tau) \) constructed in (5.81) have all presupposed properties.

Thus, we have proved the Theorem 1.4.

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