On Poisson-Lie structure on the external algebra of the classical Lie groups

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April 1994

Abstract

The general expression for the bicovariant bracket for odd generators of the external algebra on a Poisson-Lie group is given. It is shown that the graded Poisson-Lie structures derived before for $GL(N)$ and $SL(N)$ are the special cases of this bracket. The formula is the universal one and can be applied to the case of any matrix Lie group.

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‡Supported in part by RFFR under grant N93-011-147
1 Introduction

The quantum bicovariant differential calculi first introduced by Woronowicz [1] is an object of primary importance in the noncommutative differential geometry. Despite the existence of numerous papers on this subject, up to now only the Quantum General Linear group was equipped with the proper differential structure. As it can be seen from the recent works [2, 3, 4] the noncommutative differential geometry reveals intriguing and unusual properties even for the case of $SL_q(N)$.

In this note we continue to discuss the issue of Poisson-Lie structure on external algebra of a Lie group. As it was realized in our previous papers [5, 6] the explicit description of the graded Poisson-Lie brackets appears to be a powerful tool in solving the problem of quantum bicovariant differential calculi [1], [7]-[9]. In this approach, in accordance with the general concept of Faddeev [10], objects in the theory of quantum groups appear as a result of quantization of their classical counterparts.

In [5, 6] our strategy was the following. We consider the algebra $Fun(G)$ of ordinary functions on a Lie group $G$ as a subalgebra of a graded coalgebra $M$ of external forms on $G$. The brackets on $M$ are defined as soon as one defines only three level of brackets: of the order zero – between a function and a function, of the first order – between a function and a one-form and of the second order – between a one-form and a one-form. All the other brackets can be found by using the graded Leibniz rule and linearity. The bracket of the order zero on $Fun(G)$ is obviously fixed to be the Sklyanin bracket [11] and the others we define as a solution of a bicovariance condition and the graded Jacobi identity. This solution for $G = GL(N), SL(2)$ was obtained by straightforward and tedious calculations.

In this note we are going to elaborate an effective tool for solving this problem, which can be applied to any classical matrix Lie group. Especially, we confine ourselves by considering the graded brackets of the second order. As we shall prove the bicovariance condition is strong enough to fix the general form of these brackets:

$$\{\theta_1, \theta_2\}_{\text{c.b.}} = W^0_{12} + \{\theta_1[\theta_2, r]\} + \text{Tr}_{34}(W^2_{1234}\theta_3\theta_4) + \text{Tr}_{3456}(W^4_{123456}\theta_3\theta_4\theta_5\theta_6)\ldots,$$

(1.1)

where $\theta$-s stand for the odd generators of $M$. The first term in (1.1) is universal for any group and it depends only on a canonical $r$-matrix for the group in question. All the other terms coming in eq.(1.1) have the transparent geometric meaning as the constant Ad-invariant tensors with special symmetry properties. Thus, for a given Lie group the question about the description of the brackets of the second order reduces to the issue of the classification of special tensors on this group.

We apply this technique to the case of $GL(N)$ and $SL(N)$ confining ourselves by considering the terms in eq.(1.1) of zero and second orders in $\theta$-s. Under these assumptions it is not a problem to impose the Jacobi identity that fixes in eq.(1.1) some of free parameters. The surprising fact is that the resulting Poisson brackets coincide (up to $\theta$-independent terms) with the Poisson-Lie brackets that we have obtained before. In other words, the brackets of the second order are determined without any reference to the brackets of the first order. The similar phenomena was observed directly in the quantum case in [2].
2 Graded Poisson-Lie structure

The most transparent way to define a Poisson-Lie structure is to use the notion of a coalgebra. We shall not list here all the basic definitions, they can be found in the classical works [13, 14, 15], or in our previous papers [6, 7]. We shall pick up only some necessary points below.

In the case of a matrix Lie group $G$ the appropriate coordinate system in $Fun(G)$ is given by the matrix elements of a matrix $T = [\theta_i^j]$ in the fundamental representation of $G$. In other words, $Fun(G)$ is defined to be an algebra generated by the variables $\theta_i^j$. Roughly speaking, functions on $G$ are identified with formal power series in $\theta_i^j$. The Poisson-Lie brackets in terms of $\theta_i^j$-s read [13]:

$$\{\theta_i^j, \theta_k^l\} = r_{ik}^m \theta_m^j \theta_l^i - \theta_i^m \theta_k^l r_{mn}^{ji},$$

(2.1)

where $r_{ik}^m$ is the $r$-matrix which will be specified later. In the following it will be important that the bracket (2.1) is degenerate and the function $\det T$ lies in its center:

$$\{\theta_i^j, \det T\} = 0.$$  

(2.2)

Fixing the value of $\det T$ equal to unity we obtain the Poisson-Lie structure on $SL(N)$.

One fact from the ordinary differential geometry will be necessary. Let $G^*$ be the dual space of a Lie algebra $G$. The cotangent bundle $T^*G$ on $G$ is trivialized by means of right (left) action of $G$ on itself: $T^*G \approx G \times G^*$. Let us define the Maurer-Cartan right-invariant form $\theta_g$ on $G$ which takes value in $G$:

$$\theta_g(X_g) = X_e,$$

where $X_g$ is a right-invariant vector field corresponding to the element $X_e \in G$. With respect to the left action $g \rightarrow g_1 g$ the form $\theta_g$ transforms as follows:

$$\theta_g \rightarrow \theta_{g_1 g} = g_1 \theta_g g_1^{-1} + d g_1 g_1^{-1}.$$  

(2.3)

(This equation is well-known in physical literature as the gauge transformation law.) One can treat the right hand side of eq.(2.3) as the differential form on $G \times G$. With the identification $T^*(G \times G) \approx T^*G \otimes T^*G$ eq.(2.3) can be written as

$$\theta_g \rightarrow \theta_{g_1 g} = (g_1 \otimes I)(I \otimes \theta_g)(g_1^{-1} \otimes I) + \theta_{g_1} \otimes I.$$  

(2.4)
Now we can introduce our basic object $\mathcal{M}$. To describe the external algebra of right-invariant forms we add to the system of coordinates $t_i^j$ new anticommuting variables $\theta_i^j$. To specialize the group we shall put the proper constraints on $t$-s and $\theta$-s and now we shall treat them as independent variables as it is for the case of $GL(N)$. Hence, by definition $\mathcal{M}$ is a free associative algebra generated by $t_i^j, \theta_i^j, t$ modulo the relations:

$$
t_i^j t_k^l = t_k^l t_i^j, \quad tt_i^j = t_i^j t, \quad t \det T = I,
$$

$$
t_i^j \theta_k^l = \theta_k^l t_i^j, \quad t \theta_i^j = \theta_i^j t, \quad \theta_i^j \theta_k^l = -\theta_k^l \theta_i^j.
$$

The algebra $\mathcal{M}$ has a natural grading with $\deg(t_i^j) = 0$ and $\deg(\theta_i^j) = 1$.

Let us define the multiplication law in $\mathcal{M} \otimes \mathcal{M}$ as follows:

$$(a \otimes b) (c \otimes d) = (-1)^{\deg b \deg c} (ac \otimes bd) \quad (2.5)$$

for any $a, b, c, d \in \mathcal{M}$. It is well-known that $Fun(G)$ has the Hopf algebra structure with the coproduct $\Delta$, the counit $\varepsilon$ and the antipode $S$:

$$
\Delta(t_i^j) = t_k^l \otimes t_i^j, \quad \delta_i^j, \quad S(t_i^j) = t_i^j,
$$

$$
\Delta(t) = t \otimes t, \quad \varepsilon(t) = 1, \quad S(t) = \det T, \quad (2.6)
$$

where in the usual notation one has $\|t_i^j\|^{-1} = t_i^j$. One can endow $\mathcal{M}$ with two graded coalgebra structures defining the action of $\Delta_{L,R}$ on the generators $t_i^j, \theta_i^j, t$ as follows:

$$
\Delta_R \theta_i^j = \theta_i^j \otimes I \quad (2.8)
$$

$$
\Delta_L \theta_i^j = t_k^l S(t_j^p) \otimes \theta_k^p, \quad (2.9)
$$

$$
\Delta_R t_i^j = \Delta_L t_i^j = \Delta t_i^j, \quad (2.10)
$$

i.e. the action of $\Delta_{R,L}$ coincides with the action of $\Delta$. To an arbitrary element of $\mathcal{M}$ the actions of $\Delta_{L,R}$ are extended as to be homomorphisms.

Our main goal is to equip $\mathcal{M}$ with a Poisson structure consistent with the co-product on $\mathcal{M}$. Precisely, it means the following. We introduce a bilinear operation $\{ , \}: \mathcal{M} \otimes \mathcal{M} \rightarrow \mathcal{M}$ called brackets. The algebra $\mathcal{M}$ has odd and even generators, so it is natural to require for this bracket the fulfillment of the super Jacobi identity:

$$
(-1)^{\deg a \deg c} \{ \{ a, b \}, c \} + (-1)^{\deg b \deg c} \{ \{ c, a \}, b \} + (-1)^{\deg a \deg b} \{ \{ b, c \}, a \} = 0, \quad (2.11)
$$

the graded Leibniz rule

$$
\{ a \cdot b, c \} = a \{ b, c \} + (-1)^{\deg b \deg c} \{ a, c \} b, \quad (2.12)
$$

and the graded symmetry property:

$$
\{ a, b \} = (-1)^{\deg a \deg b + 1} \{ b, a \}, \quad \deg \{ a, b \} = (\deg a + \deg b) \mod 2. \quad (2.13)
$$
Let us require now that our algebra $\mathcal{M}$ supplied with the Poisson structure would be a Poisson-Lie algebra, i.e. the Poisson brackets would satisfy
\[
\Delta_{R,L}\{a,b\}_\mathcal{M} = \{\Delta_{R,L}(a), \Delta_{R,L}(b)\}_{\mathcal{M} \otimes \mathcal{M}},
\]
(2.14)
where the bracket on $\mathcal{M} \otimes \mathcal{M}$ is defined as
\[
\{a \otimes b, c \otimes d\}_{\mathcal{M} \otimes \mathcal{M}} = (-1)^{\deg a \deg c} \{a, c\}_{\mathcal{M}} \otimes bd + (-1)^{\deg b \deg c} ac \otimes \{b, d\}
\]
(2.15)
for any elements $a, b, c, d \in \mathcal{M}$. In what follows we will call the algebra $\mathcal{M}$ equipped with the brackets defined above as the Poisson-Lie superalgebra.

A comment is in order. It would be more natural to define the coproduct $\Delta$ on right-invariant forms $\theta$ as $\Delta = \Delta_L + \Delta_R$, in that case the coproduct law $\Delta$ would mimic the transformation law for the right invariant forms on a Lie group (eqs. (2.3), (2.4)). However, as we have found for the $SL(2)$ there is no solutions of eq. (2.14) with the total $\Delta$, hence we relax (2.14) by replacing the total $\Delta$ by two separate coproducts $\Delta_R$ and $\Delta_L$. This choice of coalgebra structure strictly corresponds to the notion of bicovariant differential calculi of Woronowicz [1].

To define the brackets on $\mathcal{M}$ it is enough to define them on the set of generators $t_i^j, \theta_i^j$ and then to extend by the Leibniz rule to the whole algebra.

The linear space $\mathcal{N}$ spanned by the generators $t_i^j, \theta_i^j$ and $t$ forms the Hopf subalgebra $Fun(G)$. Hence, we equip $\mathcal{N}$ with the Sclyanin bracket (2.1) described above:
\[
\{T_1, T_2\} = [r, T_1 T_2].
\]
(2.16)
Here we use the standard tensor notation: $T_1 = T \otimes I$, $T_2 = I \otimes T$.

3 Bicovariant bracket of the second order

In this section we will derive the general expression for the bicovariant bracket between two odd generators. We will consider the situation when this bracket is required to be $Z_2$ graded one rather then quadratic.

Note once more that we are going to solve the bicovariance condition eq. (2.14) only. We shall denote this bicovariant bracket as $\{\}_\text{c.b.}$. To obtain the actual Poisson bracket one needs also to impose the Jacobi identity. We shall come back to this issue when considering the concrete Poisson-Lie groups. Up to now we do not know the general procedure to solve the Jacobi identity for the graded bracket without specifying a group.

We start with imposing the $\Delta_R$ covariance. The eq. (2.14) reads:
\[
\Delta_R\{\theta_1, \theta_2\}_\text{c.b.} = \{\Delta_R(\theta_1), \Delta_R(\theta_2)\}_{\text{c.b.}, \mathcal{M} \otimes \mathcal{M}}
\]
(3.1)
Here we use for $\theta$-s the same tensor notation as for $T$-s. By using the explicit form for $\Delta_R$ and the definition of the bracket on $\mathcal{M} \otimes \mathcal{M}$ we get
\[
\Delta_R\{\theta_1, \theta_2\}_\text{c.b.} = \{\theta_1, \theta_2\}_{\text{c.b.}} \otimes I
\]
(3.2)
that means that the \( \{ \theta_1, \theta_2 \}_{c.b.} \) is nothing but the right-invariant form as the \( \theta \)-s are. We will use this key feature below.

Now let us turn to the \( \Delta_L \)-covariance:

\[
\Delta_L \{ \theta_1, \theta_2 \}_{c.b.} = \{ \Delta_L(\theta_1), \Delta_L(\theta_2) \}_{c.b., \mathcal{M} \otimes \mathcal{M}}. \tag{3.3}
\]

In the following it will be useful to adopt the convention: we shall omit the sign of a tensor product in eq. (2.9) writing it simply as

\[
\Delta_L \theta = T \theta S(T). \tag{3.4}
\]

Clearly one should have in mind that \( \theta \) and \( T, S(T) \) belong to different factors of the tensor product in \( \mathcal{M} \otimes \mathcal{M} \). Let us find the r.h.s. of eq. (3.3):

\[
\{ \Delta_L \theta_1, \Delta_L \theta_2 \}_{c.b.} = \{ T_1 \theta_1 S(T_1), T_2 \theta_2 S(T_2) \}_{c.b.}. \tag{3.5}
\]

As it is seen from this expression to get the answer we need to calculate the ordinary brackets of the type \( \{ TS(T), TS(T) \} \). By using the Leibniz rule and making the proper arrangement of the matrix multipliers we have

\[
\{ \Delta_L(\theta_1), \Delta_L(\theta_2) \}_{c.b., \mathcal{M} \otimes \mathcal{M}} = \{ T_1, T_2 \} \theta_1 \theta_2 S(T_1) S(T_2) + T_1 \theta_1 \{ S(T_1), T_2 \} \theta_2 S(T_2) - T_2 \theta_2 \{ T_1, S(T_2) \} T_1 S(T_1) - T_1 T_2 \theta_1 \theta_2 \{ S(T_1), S(T_2) \} + T_1 T_2 \{ \theta_1, \theta_2 \}_{c.b.} S(T_1) S(T_2).
\]

All the even brackets here are defined by the Sklyanin bracket and look like

\[
\{ T_1, T_2 \} = [r, T_1 T_2], \quad \{ T_1, S(T_2) \} = -S(T_2) [r, T_1 T_2] S(T_2), \quad \{ S(T_1), S(T_2) \} = S(T_1) S(T_2) [r, T_1 T_2] S(T_1) S(T_2). \tag{3.6}
\]

Inserting eq. (3.6) into the previous formula one gets

\[
\{ \Delta_L \theta_1, \Delta_L \theta_2 \}_{c.b.} = \quad \tag{3.7}
\]

\[
T_1 T_2 \left( \{ \theta_1, \theta_2 \}_{c.b.} - \{ \theta_1[\theta_2, r] \} \right) S(T_1) S(T_2) + \{ T_1 \theta_1 S(T_1), [T_2 \theta_2 S(T_2), r] \},
\]

where \([,] \) stands for the commutator in a Lie algebra and \(\{,\} \) – for the anticommutator.

Now let us note that by using our convention eq. (3.4) one can write the last term in eq. (3.7) as

\[
\{ \Delta_L \theta_1, [\Delta_L \theta_2, r] \} = \Delta_L \{ \theta_1[\theta_2, r] \}
\]

since \( \Delta_L \) is an algebra homomorphism.

Thus, the left-covariance of the bracket \( \{ \Delta_L \theta_1, \Delta_L \theta_2 \}_{c.b.} = \Delta_L \{ \theta_1, \theta_2 \}_{c.b.} \) implies the fulfillment of the following relation

\[
T_1 T_2 \left( \{ \theta_1, \theta_2 \}_{c.b.} - \{ \theta_1[\theta_2, r] \} \right) S(T_1) S(T_2) = \Delta_L \left( \{ \theta_1, \theta_2 \}_{c.b.} - \{ \theta_1[\theta_2, r] \} \right). \tag{3.8}
\]
In terms of the new tensor $\Theta$:

$$\Theta(\theta)_{12} = \{\theta_1, \theta_2\}_{c,b} - \{\theta_1[\theta_2, r]\}$$

the last equation takes the form

$$T_1 T_2 \Theta_{12} S(T_1) S(T_2) = \Delta_L \Theta_{12} \quad (3.9)$$

or

$$\Theta_{12} = S(T_1) S(T_2) \Delta_L \Theta_{12} T_1 T_2. \quad (3.10)$$

Any tensor $\Theta$ can be written as the polynomial in $\theta$-s with coefficients in $Fun(G)$. The grading requirement means that $\Theta_{12}$ is an even element, i.e. it has the form

$$\Theta_{ij} = W^0_{ik} + W^2_{ikj} j_1 j_2 \theta^{i_1}_j \theta^{i_2}_j + W^4_{ikj_1j_2j_3j_4} j_1 j_2 \theta^{i_1}_j \theta^{i_2}_j \theta^{i_3}_j \theta^{i_4}_j + \ldots. \quad (3.11)$$

Moreover, the right-covariance of the bracket forces all the tensors $W^{2k}$ to be constant ($T$-independent) tensors. Hence, when acting on $\Theta$ the coproduct $\Delta_L$ actually acts on $\theta$-s only. Writing down this action explicitly we see that the equation (3.10) is nothing but the condition of the Ad-invariance of any tensor $W^{2k}$:

$$W^{2k} = \underbrace{(\text{Ad}_g \otimes \ldots \otimes \text{Ad}_g)}_{2k+2} W^{2k}. \quad (3.12)$$

Now let us note that the Sklyanin bracket depends only on the antisymmetric part of the $r$-matrix. We take $r$-matrix $r \in \mathcal{G} \wedge \mathcal{G}$ satisfying the MCYBE. Then the bracket $\{\theta_1[\theta_2, r]\}$ is symmetric. It means that $W^{2k} \in S^2 \mathcal{G} \otimes \wedge^k \mathcal{G}$, where $S^2 \mathcal{G}$ stands for the symmetric part of the tensor product $\mathcal{G} \otimes \mathcal{G}$ and $\wedge^k \mathcal{G}$ for the antisymmetric part respectively. Thus the general form of the bracket between two odd generators is

$$\begin{align*}
\{\theta_1, \theta_2\}_{c,b} &= W^0_{12} + \{\theta_1[\theta_2, r]\} + \text{Tr}_{34}(W^2_{1234} \theta_3 \theta_4) + \text{Tr}_{3456}(W^4_{123456} \theta_3 \theta_4 \theta_5 \theta_6) + \ldots \quad (3.13)
\end{align*}$$

### 4 Poisson bracket of the second order for $GL(N)$ and $SL(N)$

In this section we are going to show that when specifying a Poisson-Lie group in the known cases the formula (3.13) reproduces the correct answers for the Poisson brackets on the external algebra of the corresponding group. In particular, we will be interested here in the Poisson-Lie groups $GL(N)$ and $SL(N)$. We also confine ourselves by considering the quadratic brackets. It means that we choose all the tensors $W^{2k}$ equal to zero when $k > 1$. Under this assumption the bracket (3.13) takes the form

$$\begin{align*}
\{\theta_1, \theta_2\}_{c,b} &= W^0_{12} + \{\theta_1[\theta_2, r]\} + \text{Tr}_{34}(W^2_{1234} \theta_3 \theta_4) \quad (4.1)
\end{align*}$$
4.1 \( GL(N) \) case

Let us consider the Lie group \( GL(N) \) and its Lie algebra \( \mathcal{G} = gl(N) \). Let \( \{ e_{\mu} \} \) be an orthonormal basis in \( \mathcal{G} \). The Poisson-Lie structure on the function algebra of \( GL(N) \) is defined by means of the Sklyanin bracket (2.14) with the \( r \)-matrix being the trivial lift of the canonical \( sl(N) \) \( r \)-matrix. As mentioned above we choose \( r \in \mathcal{G} \wedge \mathcal{G} \) being the solution of the Modified Classical Yang-Baxter Equation (MCYBE):

\[
[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = \alpha (t_{13}, t_{23}). \tag{4.2}
\]

Here \( [t_{13}, t_{23}] \) is the Ad-invariant tensor in \( \mathcal{G} \wedge \mathcal{G} \wedge \mathcal{G} \) defined via the canonical element \( t = e_{\mu} \otimes e_{\mu} \in \mathcal{G} \otimes \mathcal{G} \). The constant \( \alpha \) in the r.h.s. of eq.(4.2) defines the normalization of the chosen \( r \)-matrix.

For the bracket (1.1) to be treated as the \( GL(N) \)-bicovariant bracket one needs to specify the \( Ad_{GL(N)} \)-invariant tensors \( W^0 \in S^2 \mathcal{G} \) and \( W^2 \in S^2 \mathcal{G} \otimes \wedge^2 \mathcal{G} \). For this purpose let us choose a basis in \( GL(N) \) consisting of the matrix unities \( e_{i}^{j} \):

\[
(e_{i}^{j})^{l}_{k} = \delta_{i}^{l} \delta_{k}^{j}, \quad i, j = 1, \ldots, N.
\]

The entries of the basis elements coincide with the matrix elements of the permutation operator \( P \), i.e. \( P^{jl}_{ik} = \delta_{i}^{l} \delta_{k}^{j} \). It is easy to see that \( P \) is Ad-invariant element in \( S^2 \mathcal{G} \). Except the permutation operator there is only one element in \( S^2 \mathcal{G} \), namely the unity \( I \), which is Ad-invariant. Thus, in our basis one can write

\[
W^{0}_{12}(\beta_{1}, \beta_{2}) = \beta_{1}I + \beta_{2}P_{12}, \tag{4.3}
\]

where \( \beta_{1}, \beta_{2} \) are arbitrary numbers.

To proceed with the construction of \( W^{2} \) let us note that two Ad-invariant elements \( I \) and \( P \) in \( \mathcal{G} \otimes \mathcal{G} \) can be written in our basis via the matrix trace as

\[
I^{jl}_{ik} = \delta_{i}^{j} \delta_{k}^{l} = \text{Tr}(e_{i}^{j}) \text{Tr}(e_{k}^{l})
\]

and

\[
P^{jl}_{ik} = \delta_{i}^{l} \delta_{k}^{j} = \text{Tr}(e_{i}^{l} e_{k}^{j}).
\]

In this form their Ad-invariance is obvious. Moreover, these formulae imply the straightforward generalization to the higher invariants of tensor products of \( \mathcal{G} \). Clearly, all invariants in \( \mathcal{G}^{\otimes 4} \) should be of the types

\[
L_{1} = \text{Tr} \text{Tr}(e) \text{Tr}(e) \text{Tr}(e),
\]

\[
L_{2} = \text{Tr}(ee) \text{Tr}(e) \text{Tr}(e),
\]

\[
L_{3} = \text{Tr}(e) \text{Tr}(eee),
\]

\[
L_{4} = \text{Tr}(eee).
\]

(4.4)

We omitted here the indexes labelling the elements of the basis. In such a way we have constructed all invariant tensors in \( \mathcal{G}^{\otimes 4} \). Now the question is how to find the invariants in the space \( S^2 \mathcal{G} \otimes \wedge^2 \mathcal{G} \). Obviously, it can be done by symmetrization of the invariant tensors (4.4) with respect to the first two spaces in \( \mathcal{G}^{\otimes 4} \) and antisymmetrization with respect to the two others. Applying successively this procedure one can easily see that the first two invariants in (4.4) give zero since there is no invariant tensors in \( \wedge^2 \mathcal{G} \). As to the \( L_{3} \) and \( L_{4} \) the explicit computation gives

\[
(L_{3})^{jlp}_{ikmn} = \delta_{i}^{j} \delta_{k}^{l} \delta_{m}^{p} + \delta_{i}^{n} \delta_{k}^{j} \delta_{m}^{l} \delta_{n}^{p} - \delta_{i}^{j} \delta_{k}^{p} \delta_{m}^{l} \delta_{n}^{p} - \delta_{i}^{p} \delta_{k}^{l} \delta_{m}^{j} \delta_{n}^{p}, \tag{4.5}
\]
and

\[(L_4)^{j_{\text{top}}} = \delta_i^j \delta^i_j \delta^m_n \delta^p_m + \delta_i^j \delta^i_j \delta^p_m - \delta_i^j \delta^i_j \delta^p_m - \delta_i^j \delta^i_j \delta^p_m. \tag{4.6}\]

Combining $L_3$ and $L_4$ with arbitrary numerical coefficients $\beta_3$ and $\beta_4$ we obtain the general expression for the tensor $W^2$:

\[W^2 = \beta_3 L_3 + \beta_4 L_4.\]

Now having at hand the explicit form of $W^0$ and $W^2$ we substitute them in (4.1) and obtain the four-parameter family of quadratic covariant brackets on the external algebra of $GL(N)$:

\[\{\theta_1, \theta_2\}_{\text{c.b.}} = \beta_1 I + \beta_2 P_{12} + \{\theta_1[\theta_2, r]\} + \beta_3 (\theta_1 \theta_1 + \theta_2 \theta_2) + \beta_4 (\theta_1 P \theta_2 + \theta_2 P \theta_1). \tag{4.7}\]

Now what we are going to do is to check the Jacobi identity:

\[J = \{\{\theta_1, \theta_2\}_{\text{c.b.}}, \theta_3\}_{\text{c.b.}} + c.p. = 0, \tag{4.8}\]

where c.p. stands for the cyclic permutations of the indexes 1, 2, 3. The direct calculation leads to the following answer for $J$

\[J = [\theta_1, \{\theta_2, [\theta_3, C(r)]\}] - \beta_3^2[\theta_1, \{\theta_2, [\theta_3, \Omega]\}] = 0, \tag{4.9}\]

where $C(r)$ is the l.h.s. of the MCYBE (4.2) and $\Omega = [P_{13}, P_{23}]$ is the Ad-invariant tensor standing in the r.h.s. of MCYBE. Thus, substituting $C(r)$ from eq.(4.2) into eq.(4.9) we arrive to

\[J = (\alpha - \beta_4^2)[\theta_1, \{\theta_2, [\theta_3, \Omega]\}] \tag{4.10}\]

Since the tensor in the l.h.s. of the last equation is not equal to zero the Jacobi identity is satisfied if and only if $\beta_4 = \pm \sqrt{\alpha}$ whereas $\beta_1, \beta_2, \beta_3$ remain to be arbitrary. Therefore, the graded bicovariant brackets given by

\[\{\theta_1, \theta_2\}_{\text{c.b.}} = (\beta_1 I + \beta_2 P_{12} + \{\theta_1[\theta_2, r]\} + \beta_3 (\theta_1 \theta_1 + \theta_2 \theta_2) \pm \sqrt{\alpha}(\theta_1 P \theta_2 + \theta_2 P \theta_1) \tag{4.11}\]

defines the genuine graded Poisson-Lie structure in quadratic sector of $\theta$ generators.

In [3] using the full coproduct law $\Delta = \Delta_R + \Delta_L$ the following one-parameter family of the graded Poisson-Lie brackets between two odd generators was derived:

\[\{\theta_1, \theta_2\}_{\text{c.b.}} = \beta (\theta_1 \theta_1 + \theta_2 \theta_2) + r_+^{12} \theta_1 \theta_2 + \theta_1 \theta_2 r_+^{12} + \theta_2 r_+^{12} \theta_1 - \theta_1 r_+^{12} \theta_2. \tag{4.12}\]

Now comparing (4.11) with (4.12) one can see that the requirement for the bracket $\{\theta, \theta\}$ to be $\Delta_{R,L}$ covariant instead of being $\Delta$-covariant leads to the appearance in (4.11) the additional terms of another degree. However, in the pure quadratic sector (when we put in (4.11) $\beta_1 = \beta_2 = 0$) by using the definition of $r_\pm = r \pm \sqrt{\alpha}P$ we conclude that equations (4.11) and (4.12) coincide.

\[
\text{In this case } M \text{ can be equipped with the graded Hopf algebra structure.}
\]
4.2 $SL(N)$ case

Here we will study the possibility of reducing the general expression (4.1) to the bicovariant bracket on the external algebra of $SL(N)$. In principle there is no difference in the procedure of description of invariant elements for $GL(N)$ and $SL(N)$ cases. To begin with let us choose the following set of matrices in the fundamental representation of $G = sl(N)$: $(E_i^j)_k^l = \delta_i^l \delta_j^k - \frac{1}{N} \delta_i^j \delta_k^l, i, j = 1, \ldots, N$. Note that the matrix elements of the matrices from this set can be written as follows

$$(E_i^j)_k^l = P_{ik}^{jl} - \frac{1}{N} I_{ik}^{jl}.$$ 

Clearly, the matrices $E_i^j$ do not constitute the basis in $G$ because they are linear dependent $E_i^i = 0$. Nevertheless, we can use them to construct invariant tensors of $SL(N)$ just in the same way as the basis of matrix unities was used for the $GL(N)$ case. There is only one invariant tensor in $S^G$:

$$\text{Tr}(E_i^j E_k^l) - \frac{1}{N} \text{Tr}(E_i^j) \text{Tr}(E_k^l) = P_{ik}^{jl} - \frac{1}{N} I_{ik}^{jl}.$$ 

So for the invariant $W^0$ one can write

$$W^0 = \beta_1 \left( P - \frac{1}{N} I \right).$$ 

Before considering the invariant tensors in higher dimensions let us note that the trace of any matrix $E_i^j \in G$ is equal to zero and therefore there is no $\text{Ad}$-invariant tensors in $G$ ($G$ is simple). It means that in $S^2 G \otimes \wedge^2 G$ there is only one invariant tensor $L$ which is obtained from $\text{Tr}(E_i^j E_k^l E_m E_n)$ by symmetrization with respect to the first pair of indexes and antysimmetrization with respect to the second one:

$$L_{ik \; mn}^{jl sp} = \text{Tr}(E_i^j E_k^l E_m E_n) + \text{Tr}(E_k^l E_i^j E_m E_n) - \text{Tr}(E_i^j E_k^l E_n E_m) - \text{Tr}(E_k^l E_i^j E_n E_m) =$$

$$= \text{Tr}(E_k^l E_i^j E_n E_m) - \text{Tr}(E_k^l E_i^j E_m E_n).$$ 

Using the definition of the basis $\{E_i^j\}$ and performing the explicit calculations we find

$$L_{ik \; mn}^{jl sp} = \delta_k^{j s} \delta^l_m \delta^p_n + \delta_i^l \delta_k^j \delta^s_m \delta^p_n - \delta_i^l \delta_k^j \delta^s_m \delta^p_n - \delta_i^l \delta_k^j \delta^s_m \delta^p_n -$$

$$- \frac{2}{N} \delta_k^{j s} \delta^l_m \delta^p_n - \frac{2}{N} \delta_i^l \delta_k^j \delta^s_m \delta^p_n + \frac{2}{N} \delta_i^l \delta_k^j \delta^s_m \delta^p_n + \frac{2}{N} \delta_i^l \delta_k^j \delta^s_m \delta^p_n.$$

Thus, the tensor $W^2 \in S^2 G \otimes \wedge^2 G$ coincides with $L$ up to the numerical constant which we call $\beta_2$: $W^2 = \beta_2 L$.

Now substituting $W^0$ and $W^2$ in (4.1) we obtain the expression for the $SL(N)$ bicovariant bracket:

$$\{\theta_1, \theta_2\}_{c.b.} = \beta_1 \left( P - \frac{1}{N} I \right) + \{\theta_1 [\theta_2, r]\} + \beta_2 \left( \theta_1 P \theta_2 + \theta_2 P \theta_1 - \frac{2}{N} \theta_1 \theta_2 - \frac{2}{N} \theta_2 \theta_1 \right).$$

Comparing these brackets with the $GL(N)$-covariant brackets (4.7) we see that they are obtained from (4.7) under special values of parameters $\beta_1, \beta_2$. Thus, the Jacobi identity for the quadratic part of eq.(4.13) is satisfied as soon as $\beta_2$ is fixed to be $\pm \sqrt{n}$. One can easily realize that the quadratic part of the brackets (4.13) reproduce the covariant bracket for two odd generators on $SL(N)$ derived in [3].
5 Concluding remarks

In this note we have derived the general expression for the bicovariant brackets for odd generators of the external algebra on a Lie group. We applied this formula to $GL(N)$ and $SL(N)$ and reproduced the Poisson-Lie brackets that was found before. It is interesting to note that previously these quadratic brackets were obtained as the solution of the total Jacobi identity involving the complete set of generators of $\mathcal{M}$. Now we see that the quadratic part of these brackets is completely fixed without any reference to the brackets of the first order.

The proposed general expression opens the way to resolve the question if there exist bicovariant Poisson brackets for other classical Lie groups. Namely, one has to classify the Ad-invariant constant tensors on a given group. As soon as these tensors are described the only task is to impose the Jacobi identity that as we suggest is also a problem of the transparent geometric nature.

The quantization of these brackets would lead to the graded noncommutative algebra that can be treated as the algebra of quantum right-invariant forms on the corresponding quantum group. The interesting question if it is possible to supply this algebra by the quantum exterior derivative $d$. As it was shown in [3] even for $SL_q(N)$ the quantum operator $d$ may be defined but it looses some usual properties. Since a bicovariant bracket on any Lie group contains the term $\{\theta_1[\theta_2, r]\}$ the quantization of this part leads to the appearance of the quantum $R$-matrix in the defining relations of the quantum algebra. As for the other terms in the bracket defined via the Ad-invariant tensors their quantization means that we construct the tensors that would be the invariants of the adjoint action of the corresponding quantum group.

ACKNOWLEDGMENT

The authors are grateful to I.Ya.Aref’eva, I.V.Volovich, A.P.Isaev and P.N.Pyatov for interesting discussions.

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