The Geodetic Hull Number is Hard for Chordal Graphs

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Abstract
We show the hardness of the geodetic hull number for chordal graphs.

Keywords: Geodetic convexity; shortest path; hull number; chordal graphs

1 Introduction

One of the most well studied convexity notions for graphs is the shortest path convexity or geodetic convexity, where a set $X$ of vertices of a graph $G$ is considered convex if no vertex outside of $S$ lies on a shortest path between two vertices inside of $S$. Defining the convex hull of a set $S$ of vertices as the smallest convex set containing $S$, a natural parameter of $G$ is its hull number $h(G)$, which is the minimum order of a set of vertices whose convex hull is the entire vertex set of $G$. The hull number is NP-hard for bipartite graphs, partial cubes, and $P_9$-free graphs, but it can be computed in polynomial time for cographs, $(q, q-4)$-graphs, paw, $P_5$-free graphs, and distance-hereditary graphs. Bounds on the hull number are given in.

In Kanté and Nourine present a polynomial time algorithm for the computation of the hull number of chordal graphs. Unfortunately, their correctness proof contains a gap described in detail at the end of the present paper. As our main result we show that computing the hull number of a chordal graph is NP-hard, which most likely rules out the existence of a polynomial time algorithm.

Before we proceed to our results, we collect some notation and terminology. We consider finite, simple, and undirected graphs. A graph $G$ has vertex set $V(G)$ and edge set $E(G)$. A graph $G$ is chordal if it does not contain an induced cycle of order at least 4. A clique in $G$ is the vertex set of a complete subgraph of $G$. A vertex of a graph $G$ is simplicial in $G$ if its neighborhood is a clique. The distance $\text{dist}_G(u,v)$ between two vertices $u$ and $v$ in $G$ is the minimum number of edges of a path in $G$ between $u$ and $v$. The diameter $\text{diam}(G)$ of $G$ is the maximum distance between any two vertices of $G$. The eccentricity $e_G(u)$ of a vertex $u$ of $G$ is the maximum distance between $u$ and any other vertex of $G$. For a positive integer $k$, let $[k]$ be the set of the positive integers at most $k$.

Let $G$ be a graph, and let $S$ be a set of vertices of $G$. The interval $I_G(S)$ of $S$ in $G$ is the set of all vertices of $G$ that lie on shortest paths in $G$ between vertices from $S$. Note that $S \subseteq I_G(S)$, and that $S$ is convex in $G$ if $I_G(S) = S$. The set $S$ is concave in $G$ if $V(G) \setminus S$ is convex. Note that $S$ is concave
if and only if \( S \cap I_G(\{v, w\}) = \emptyset \) for every two vertices \( v \) and \( w \) in \( V(G) \setminus S \). The hull \( H_G(S) \) of \( S \) in \( G \), defined as the smallest convex set in \( G \) that contains \( S \), equals the intersection of all convex sets that contain \( S \). The set \( S \) is a hull set if \( H_G(S) = V(G) \), and the hull number \( h(G) \) of \( G \) is the smallest order of a hull set of \( G \).

2 Result

We immediately proceed to our main result.

**Theorem 2.1.** For a given chordal graph \( G \), and a given integer \( k \), it is NP-complete to decide whether the hull number \( h(G) \) of \( G \) is at most \( k \).

**Proof.** Since the hull of a set of vertices of \( G \) can be computed in polynomial time, the considered decision problem belongs to NP. In order to prove NP-completeness, we describe a polynomial reduction from a restricted version of SATISFIABILITY. Therefore, let \( C \) be an instance of SATISFIABILITY consisting of \( m \) clauses \( C_1, \ldots, C_m \) over \( n \) boolean variables \( x_1, \ldots, x_n \) such that every clause in \( C \) contains at most three literals, and, for every variable \( x_i \), there are exactly two clauses in \( C \), say \( C_{j_i}^{(1)} \) and \( C_{j_i}^{(2)} \), that contain the literal \( x_i \), and exactly one clause in \( C \), say \( C_{j_i}^{(3)} \), that contains the literal \( \overline{x_i} \), and these three clauses are distinct. Using a polynomial reduction from [LO1] [8], it has been shown in [5] that SATISFIABILITY restricted to such instances is still NP-complete.

Let the graph \( G \) be constructed as follows starting with the empty graph:

- For every \( j \in [m] \), add a vertex \( c_j \).
- For every \( i \in [n] \), add three \( y_i \), \( \overline{y_i} \), and \( z_i \).
- Add edges such that \( B \cup Z \) is a clique, where

\[
B = \{c_j : j \in [m]\} \cup \{y_i : i \in [n]\} \cup \{\overline{y_i} : i \in [n]\} \text{ and } Z = \{z_i : i \in [n]\},
\]

- For every \( i \in [n] \), add 9 vertices and 25 edges to obtain the subgraph indicated in Figure 2.

![Figure 1: The vertices and edge added for the variable \( x_i \), where \( j_i^{(1)} = j \), \( j_i^{(2)} = k \), and \( j_i^{(3)} = \ell \).](image-url)
Note that \( \text{dist}_G(x_i, \bar{x}_i) = \text{dist}_G(x_I^I, x'_I) = 3 \) for every \( i \in [n] \). Since every vertex of \( G \) has a neighbor in the clique \( B \cup Z \), the diameter of \( G \) is 3. Furthermore, since no vertex is universal, all vertices in \( B \cup Z \) have eccentricity 2.

Let \( k = 4n \).

Note that the order of \( G \) is \( 12n + m \).

It remains to show that \( G \) is chordal, and that \( C \) is satisfiable if and only if \( h(G) \leq k \).

In order to show that \( G \) is chordal, we indicate a perfect elimination ordering, which is a linear ordering \( v_1, \ldots, v_{12n+m} \) of its vertices such that \( v_i \) is simplicial in \( G - \{v_1, \ldots, v_{i-1}\} \) for every \( i \in [12n+m] \).

Such an ordering is obtained by

- starting with the vertices \( x_I^I, x'_I, x''_I \) for all \( i \in [n] \) (in any order),
- continuing with the vertices \( x_I^I, x'_I, x''_I \) for all \( i \in [n] \),
- continuing with the vertices \( x'_I \) for all \( i \in [n] \),
- continuing with the vertices \( x_i \) and \( \bar{x}_i \) for all \( i \in [n] \), and
- ending with the vertices in the clique \( B \cup Z \).

Now, let \( S \) be a satisfying truth assignment for \( C \).

Let

\[
S = \bigcup_{i \in [n]} \{ x_I^I, x'_I, x''_I \} \cup \bigcup_{i \in [n]: x_i \text{ true in } S} \{ x_i \} \cup \bigcup_{i \in [n]: x_i \text{ false in } S} \{ \bar{x}_i \}.
\]

Clearly, \( |S| = k = 4n \). For every \( i \) in \( [n] \), we have \( \{ z_i, \bar{y}_i \} \subseteq I_G(\{ x_i, x''_i \}) \), \( \{ z_i, y_i \} \subseteq I_G(\{ \bar{x}_i, x_I^I \}) \), \( y_i \in I_G(\{ \bar{y}_I, x''_I \}) \), and \( \bar{y}_i \in I_G(\{ y_i, x''_I \}) \), which implies \( \{ z_i, y_i, \bar{y}_i \} \subseteq H_G(S) \). Since \( S \) is a satisfying truth assignment, for every \( j \) in \( [m] \), there is a neighbor, say \( v \), of \( c_j \) in

\[
\bigcup_{i \in [n]: x_i \text{ true in } S} \{ x_i \} \cup \bigcup_{i \in [n]: x_i \text{ false in } S} \{ \bar{x}_i \}.
\]

If \( v \in \bigcup_{i \in [n]: x_i \text{ true in } S} \{ x_i \} \), then \( c_j \in I_G(\{ v, x''_I \}) \), otherwise \( c_j \in I_G(\{ v, x_I^I \}) \). Hence, \( B \cup Z \subseteq H_G(S) \).

Now, for some \( i \) in \( [n] \), let \( c_j, c_k \), and \( c_\ell \) be the neighbors in \( B \setminus \{ y_i, \bar{y}_i \} \) of \( x_I^I, x''_I \) and \( \bar{x}_I \), respectively, similarly as in Figure 2. We have \( x_I^I \in I_G(\{ x_I^I, c_j \}) \), \( x''_I \in I_G(\{ x''_I, c_k \}) \), \( x'_I \in I_G(\{ x_I^I, x''_I \}) \), \( x''_I \in I_G(\{ x'_I, c_\ell \}) \), \( x_i \in I_G(\{ x_I^I, z_i \}) \), and \( \bar{x}_i \in I_G(\{ \bar{x}_I, z_i \}) \).

Altogether, we obtain that \( S \) is a hull set of \( G \) of order \( 4n \).

Finally, let \( S \) be a hull set of \( G \) of order at most \( 4n \).

**Claim 1.** For every \( i \in [n] \), the set \( \{ x_i, z_i, \bar{x}_i \} \) is concave.

**Proof of Claim 1.** For a contradiction, suppose that some vertex in \( S' = \{ x_i, z_i, \bar{x}_i \} \) lies on a shortest path \( P \) in \( G \) between two vertices \( v \) and \( w \) in \( V(G) \setminus S' \). Since the diameter of \( G \) is 3, the path \( P \) contains at most 2 vertices of \( S' \). Since the neighbors outside of \( S' \) of each vertex in \( S' \) form a clique, the path \( P \) contains exactly 2 adjacent vertices of \( S' \), that is, either \( P = vx_i z_i w \) or \( P = v\bar{x}_i z_i w \). In both cases, the vertex \( w \) has eccentricity at least 3. However, every neighbor \( w \) of \( z_i \) outside \( S' \) belongs to \( B \cup Z \), and thus, has eccentricity 2, a contradiction. □
Claim 2. For every $j \in [m]$, the set

$$V_j = \{c_j\} \cup \bigcup_{i \in [n]: j = j_i^{(1)}} \{x_i, x'_i, x''_i\} \cup \bigcup_{i \in [n]: j = j_i^{(2)}} \{x_i, x'_i, x''_i\} \cup \bigcup_{i \in [n]: j = j_i^{(3)}} \{x_i, x'_i, x''_i\}$$

is concave.

**Proof of Claim 2.** First, suppose that $C_j$ contains the positive literal $x_i$. By symmetry, we may assume that $j = j_i^{(1)}$ and $j_i^{(2)} = k$ for some $k \in [m] \setminus \{j\}$.

First, suppose that some shortest path $P$ between two vertices $v$ and $w$ in $\tilde{V}_j = V(G) \setminus V_j$ contains $x_i$. Choosing $P$ of minimum length, it follows that $v$ and $w$ are the only vertices of $P$ in $\tilde{V}_j$. Since the diameter of $G$ is 3, the length of $P$ is at most 3, and we may assume that $v$ is a neighbor of $x_i$, which implies $v \in \{z_i, c_k, y_i\}$. Since $\{z_i, c_k, y_i\}$ is a clique, the vertex $w$ is not a neighbor of $x_i$, and $P$ contains exactly one vertex $u$ of $V_j$ different of $x_i$, which implies $P = vx_iuw$ and $u \in \{x'_i, c_j\}$. Suppose that $u = x'_i$. This implies $w \in \{x''_i, c_k, y_i\}$. Since $c_k, y_i \in N_G(x_i)$, we obtain $w = x''_i$ and $v = z_i$. However, $\text{dist}_G(x_i, x''_i) = 2$, which is a contradiction. Hence, $u = c_j$ and $w \in B \cup Z$. However, every vertex in $B \cup Z$ has eccentricity 2, which is a contradiction. Hence, no shortest path between two vertices in $\tilde{V}_j$ contains $x_i$.

Next, suppose that some shortest path $P$ between two vertices $v$ and $w$ in $\tilde{V}_j$ contains $x'_i$. Similarly as above, we may assume that $v$ and $w$ are the only vertices of $P$ in $\tilde{V}_j$, the length of $P$ is at most 3, and $v$ is a neighbor of $x'_i$, which implies $v \in \{x''_i, y_i\}$. Since $\{x''_i, y_i\}$ is a clique, the path $P$ contains exactly one vertex $u$ of $V_j$ different of $x'_i$, which implies $P = vx'_iuw$ and $u \in \{x'_i, c_j\}$, where we use that $P$ does not contain $x_i$. Suppose that $u = x''_i$. This implies $w \in \{x''_i, y_i\}$. Since $y_i \in N_G(x'_i)$, we obtain $w = x''_i$ and $v = x''_i$. However, $\text{dist}_G(x''_i, x''_i) = 2$, which is a contradiction. Hence, $u = c_j$ and $w \in B \cup Z$. However, every vertex in $B \cup Z$ has eccentricity 2, which is a contradiction. Hence, no shortest path between two vertices in $\tilde{V}_j$ contains $x'_i$.

Next, suppose that some shortest path $P$ between two vertices $v$ and $w$ in $\tilde{V}_j$ contains $x''_i$. Similarly as above, we may assume that $v$ and $w$ are the only vertices of $P$ in $\tilde{V}_j$, the length of $P$ is at most 3, and $v$ is a neighbor of $x''_i$, which implies $v \in \{z_i, y_i\}$. Since $\{z_i, y_i\}$ is a clique, the vertex $w$ is not a neighbor of $x''_i$, and $P$ contains exactly one vertex $u$ of $V_j$ different of $x''_i$, which implies $P = vx''_iuw$ and $u \in \{x''_i, c_j\}$. Suppose that $u = x''_i$. This implies $w \in \{x''_i, y_i\}$. Since $y_i \in N_G(x''_i)$, we obtain $v = z_i$ and $w = x''_i$. However, $\text{dist}_G(z_i, x''_i) = 2$, which is a contradiction. Hence, $u = c_j$ and $w \in B \cup Z$. However, every vertex in $B \cup Z$ has eccentricity 2, which is a contradiction. Hence, no shortest path between two vertices in $\tilde{V}_j$ contains $x''_i$.

Next, suppose that $C_j$ contains the negative literal $\bar{x}_i$, that is, $j = j_i^{(3)}$.

First, suppose that some shortest path $P$ between two vertices $v$ and $w$ in $\tilde{V}_j$ contains $\bar{x}_i$. Similarly as above, we may assume that $v$ and $w$ are the only vertices of $P$ in $\tilde{V}_j$, the length of $P$ is at most 3, and $v$ is a neighbor of $\bar{x}_i$, which implies $v \in \{z_i, \bar{y}_i\}$. Since $\{z_i, \bar{y}_i\}$ is a clique, the vertex $w$ is not a neighbor of $\bar{x}_i$, and $P$ contains exactly one vertex $u$ of $V_j$ different of $\bar{x}_i$, which implies $P = v\bar{x}_iuw$ and $u \in \{\bar{x}_i, c_j\}$. Suppose that $u = \bar{x}_i$. This implies $w \in \{\bar{x}_i, \bar{y}_i\}$. Since $\bar{y}_i \in N_G(\bar{x}_i)$, we obtain $v = z_i$ and $w = \bar{x}_i$. However, $\text{dist}_G(z_i, \bar{x}_i) = 2$, which is a contradiction. Hence, $u = c_j$ and $w \in B \cup Z$. However, every vertex in $B \cup Z$ has eccentricity 2, which is a contradiction. Hence, no shortest path between two vertices in $\tilde{V}_j$ contains $\bar{x}_i$.

Next, suppose that some shortest path $P$ between two vertices $v$ and $w$ in $\tilde{V}_j$ contains $\bar{x'}_i$. Similarly as above, we may assume that $v$ and $w$ are the only vertices of $P$ in $\tilde{V}_j$, the length of $P$ is at most 3, and $v$ is a neighbor of $\bar{x'}_i$, which implies $v \in \{\bar{x'}_i, \bar{y}_i\}$. Since $\{\bar{x'}_i, \bar{y}_i\}$ is a clique, the path $P$ contains exactly one vertex $u$ of $V_j$ different of $\bar{x'}_i$, which implies $P = v\bar{x'}_i,c_jw$ and $v \in B \cup Z$, where we use that
$P$ does not contain $\bar{x}_i$. However, every vertex in $B \cup Z$ has eccentricity 2, which is a contradiction. Hence, no shortest path between two vertices in $\bar{V}_j$ contains $\bar{x}'_i$.

Finally, since the neighbors of $c_j$ outside of $V_j$ form a clique, no shortest path between two vertices in $\bar{V}_j$ contains $c_j$, which completes the proof of the claim. □

Note that all $3n$ simplicial vertices in $\bigcup_{i \in [n]} \{x'^1_i, x'^2_i, \bar{x}'_i\}$ belong to $S$.

Since $S$ contains at most $n$ non-simplicial vertices, Claim 1 implies that, for every $i$ in $[n]$, the set $S$ contains exactly one of the three vertices in $\{x_i, z_i, \bar{x}_i\}$, and that these are the only non-simplicial vertices in $S$. Now, Claim 2 implies that, for every $j$ in $[m]$, there is some $i \in [n]$ such that

- either $C_j$ contains the literal $x_i$ and the vertex $x_i$ belongs to $S$
- or $C_j$ contains the literal $\bar{x}_i$ and the vertex $\bar{x}_i$ belongs to $S$.

Therefore, setting the variable $x_i$ to true if and only if the vertex $x_i$ belongs to $S$ yields a satisfying truth assignment $S$ for $C$, which completes the proof. □

As pointed out in the introduction, the correctness proof in [9] contains a gap. In lines 14 and 15 on page 322 of [9] it says

"At iteration $i+1$, the vertex $x_{i+1}$ is a simplicial vertex in $G_{i+1}$. We first claim that there exists no functional dependency of the form $zt \rightarrow x_{i+1}$ in $\Sigma$.”

Consider applying the algorithm from [9] to the graph in Figure 2. In iteration 1, it would decide to add $x_1$ to $K$. In iteration 2, it would decide not to add $x_2$ to $K$, because of $t \rightarrow x_2$. Furthermore, because of $t \rightarrow x_2$ and $z, x_2 \rightarrow x_3$, it would replace $z, x_2 \rightarrow x_3$ within $\Sigma$ with $z, t \rightarrow x_3$. Therefore, in iteration 3, $\Sigma$ would actually contain $z, t \rightarrow x_3$, contrary to the claim cited above.

Figure 2: A small chordal graph.

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