Model Misspecification in ABC: Consequences and Diagnostics.

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Abstract

We analyze the behavior of approximate Bayesian computation (ABC) when the model generating the simulated data differs from the actual data generating process; i.e., when the data simulator in ABC is misspecified. We demonstrate both theoretically and in simple, but practically relevant, examples that when the model is misspecified different versions of ABC can lead to substantially different results. Our theoretical results demonstrate that under regularity conditions a version of the accept/reject ABC approach concentrates posterior mass on an appropriately defined pseudo-true parameter value. However, under model misspecification the ABC posterior does not yield credible sets with valid frequentist coverage and has non-standard asymptotic behavior. We also examine the theoretical behavior of the popular linear regression adjustment to ABC under model misspecification and demonstrate that this approach concentrates posterior mass on a completely different pseudo-true value than that obtained by the accept/reject approach to ABC. Using our theoretical results, we suggest two approaches to diagnose model misspecification in ABC. All theoretical results and diagnostics are illustrated in a simple running example.

1 Introduction

It is now routine in the astronomic, ecological and genetic sciences, as well as in economics and finance, that the models used to describe observed data are so complex that the likelihoods associated with these model are computationally intractable. In a Bayesian inference paradigm, these settings have led to the rise of approximate Bayesian computation (ABC) methods that eschew calculation of the likelihood in favor of simulation; for reviews on ABC methods see, e.g., Marin et al. (2012) and Robert (2016).

ABC is predicated on the belief that the observed data $y := (y_1, y_2, ..., y_n)'$ is drawn from the class of models $\{\theta \in \Theta : P_\theta\}$, where $\theta \in \Theta \subset \mathbb{R}^{k_\theta}$ is an unknown vector of parameters and where
The goal of ABC is to conduct inference on the unknown \( \theta \) by simulating pseudo-data \( \mathbf{z}, \mathbf{z} := (z_1, ..., z_n)^T \sim P_\theta \), and then “comparing” \( \mathbf{y} \) and \( \mathbf{z} \). In most cases, this comparison is carried out using a vector of summary statistics \( \eta(\cdot) \) and a metric \( d\{\cdot, \cdot\} \). Simulated values \( \theta \sim \pi(\theta) \) are then accepted, and used to build an approximation to the exact posterior if they satisfy an acceptance rule that depends on a tolerance parameter \( \epsilon \).

Algorithm 1 ABC Algorithm

1. Simulate \( \theta^i, i = 1, 2, ..., N \), from \( \pi(\theta) \),
2. Simulate \( \mathbf{z}^i = (z_1^i, z_2^i, ..., z_n^i)^T, i = 1, 2, ..., N \), from \( P_\theta \);
3. For each \( i = 1, ..., N \), accept \( \theta^i \) with probability one if \( d\{\eta(\mathbf{z}^i), \eta(\mathbf{y})\} \leq \epsilon \), where \( \epsilon \) denotes an user chosen tolerance parameter \( \epsilon \).

Algorithm 1 details the common accept/reject implementation of ABC, which can be augmented with additional steps to increase sampling efficiency; see, e.g., the MCMC-ABC approach of Marjoram et al. (2003), or the SMC-ABC approach of Sisson et al. (2007). Post-processing of the simulated pairs \( \{\theta^i, \eta(\mathbf{z}^i)\} \) has also been proposed as a means of obtaining more accurate posterior approximations (for reviews of ABC post-processing methods see Marin et al., 2012 and Blum et al., 2013).

Regardless of the ABC algorithm chosen, the very nature of ABC is such that the researcher must believe there are values of \( \theta \) in the prior support that can yield simulated summaries \( \eta(\mathbf{z}) \) ‘close to’ the observed summaries \( \eta(\mathbf{y}) \). Therefore, in order for ABC to yield meaningful inference about \( \theta \) there must exist values of \( \theta \in \Theta \) such that \( \eta(\mathbf{z}) \) and \( \eta(\mathbf{y}) \) are similar.

While complex models allow us to explain many features of the observed data, it is unlikely that any \( P_\theta \) will be able to produce simulated data that perfectly reproduces all features of \( \mathbf{y} \). In other words, by the very nature of the complex models to which ABC is applied, the class of models \( \{\theta \in \Theta : P_\theta\} \) used to simulate pseudo-data \( \mathbf{z} \) is likely misspecified. Even when accounting for the use of summary statistics that are not sufficient, and which might be compatible with several models, the value these summaries take for the observed data may well be incompatible with the realised values of these statistics for the model of interest.

Consequently, understanding the behavior of popular ABC approaches under model misspecification is of paramount importance for practitioners. Indeed, as the following example illustrates, when the model is misspecified the behavior of popular ABC approaches can vary drastically.

Example 1: To demonstrate the impact of model misspecification in ABC, we consider an artificially simple example where the assumed data generating process (DGP) is \( \mathbf{z} \sim \mathcal{N}(\theta, 1) \) but the actual DGP is \( \mathbf{y} \sim \mathcal{N}(\theta, \sigma^2) \). That is, for \( \sigma^2 \neq 1 \), the DGP for \( \mathbf{z} \) maintains an incorrect assumption about the variance of \( \mathbf{y} \) and thus differs from the actual DGP for \( \mathbf{y} \). We consider as the basis of our ABC analysis the following summary statistics:

- the sample mean \( \eta_1(\mathbf{y}) = \frac{1}{n} \sum_{i=1}^n y_i \)
- the centered summary \( \eta_2(\mathbf{y}) = \frac{1}{n-1} \sum_{i=1}^n (y_i - \eta_1(\mathbf{y}))^2 - 1 \)

1In particular, ABC post-processing approaches follow steps (1)-(2) in Algorithm 1 but replace step (3) with a step that accepts all values generated in step (2) but re-weights them according to some criteria. For example, in the case of linear regression adjustment, \( \theta \sim \pi(\theta) \) is replaced by \( \hat{\theta} = \theta - \hat{\beta}^T \{\eta(\mathbf{z}) - \eta(\mathbf{y})\} \), with \( \hat{\beta} \) obtained by a (weighted) least squares regression of \( \theta \) on \( \eta(\mathbf{z}) - \eta(\mathbf{y}) \).
For this experiment we consider two separate versions of ABC: the accept/reject approach, where we take \( d(x, y) = \|x - y\| \) to be the Euclidean norm; and a post-processing ABC approach that uses a weighted linear regression adjustment step in place of the selection step in Algorithm 1. We refer to these approaches as ABC-AR, and ABC-Reg, respectively.

To demonstrate how these ABC approaches react to model misspecification, we fix \( \theta = 1 \) and simulate “observed data sets” \( y \) according to different values of \( \tilde{\sigma}^2 \). We consider one hundred simulated data sets for \( y \) such that each corresponds to a different value of \( \tilde{\sigma}^2 \), with \( \tilde{\sigma}^2 \) taking values from \( \tilde{\sigma}^2 = .5 \) to \( \tilde{\sigma}^2 = 5 \) with evenly spaced increments. Across all the data sets we fix the random numbers used to generate the simulated data and only change the value of \( \tilde{\sigma}^2 \) to isolate the impact of model misspecification; i.e., we generate one common set of random numbers \( \nu_i \sim N(0,1), i = 1, ..., n \), for all data sets, then for a value of \( \tilde{\sigma}^2 \) we generate observed data from \( y_i = 1 + \nu_i \cdot \tilde{\sigma} \). The sample size across the experiments is taken to be \( n = 50 \).

Figure 1 compares the posterior mean, \( E_{\Pi}[\theta|\eta(y)] \), of ABC-AR, and ABC-Reg across different values for \( \tilde{\sigma}^2 \). The results demonstrate that misspecification in ABC can have drastic consequences, even at a relatively small sample sizes. Two useful conclusions can be drawn from Figure 1: one, the performance of the ABC-AR procedure remains stable regardless of the level of misspecification; two, the behavior of the linear regression adjustment approach to ABC, ABC-Reg, becomes volatile even at relatively small levels of misspecification. We formally explore these issues in Sections two and three but note here that when \( \tilde{\sigma}^2 \approx 1 \) (i.e., correct model specification) the two ABC approaches give similar results.

\[
\begin{align*}
\text{Figure 1: Comparison of posterior means for ABC-AR, and ABC-Reg across varying levels of model misspecification. Note that ABC-Reg ends after a certain point and does not continue; after this value all posterior means continued on the same trajectory and hence are not reported. Both ABC approaches used } & N = 50,000 \text{ simulated data sets generated according to } z_i^j \sim N(\theta^j, 1), \\
& \text{with } \theta^j \sim N(0,25). \text{ ABC-AR retained draws that yielded } \|\eta(y) - \eta(z^j)\| \text{ in the } \alpha_n = n^{-5/9} \text{ quantile. The bandwidth for ABC-Reg was taken as } n^{-5/9}. \\
\end{align*}
\]

\[2\] Even though the DGP for \( z \) is misspecified, because of the nature of the model misspecification and the limiting behavior of \( \eta(y) \), if one were to only use the first summary statistic (the sample mean) model misspecification would have little impact in this example. However, in general both the nature of the model misspecification and the precise limiting form of \( \eta(y) \) are unknown. Therefore, choosing a set of summaries that can mitigate the impact of model misspecification will be difficult, if not impossible, in practical applications of ABC.
It is interesting to note that the linear regression post processing approach behaves poorly under misspecification. This is particularly interesting since the post-processing linear regression adjustment approach has theoretical advantages over the standard ABC-AR approach, i.e., Algorithm 1, when the model is correctly specified; see Li and Fearnhead (2018a) for details.

In the remainder, we elaborate on the above issues and rigorously characterizes the asymptotic behavior of ABC when the model generating the pseudo-data is misspecified. In Section two, we consider model misspecification in the ABC context and demonstrate that under model misspecification, for a certain choice of the tolerance, the posterior associated with Algorithm 1 concentrates all mass on an appropriately defined pseudo-true value. In addition, we find that the asymptotic shape of the ABC posterior is non-standard under model misspecification, and will lead to credible sets with arbitrary levels of coverage. Section three demonstrates that under model misspecification, the regression adjustment ABC approach yields a posterior that concentrates posterior mass on a completely different region of the parameter space than ABC based on Algorithm 1. We then use these theoretical results to devise an alternative regression adjustment approach that performs well regardless of model specification. Motivated by our asymptotic results, in Section four we develop two model misspecification detection procedures: a graphical detection approach based on comparing acceptance probabilities from Algorithm 1 and an approach based on comparing the output from Algorithm 1 and its linear regression adjustment counterpart. Proofs of all theoretical results are contained in the appendix.

2 Model Misspecification in ABC

2.1 On the Notion of Model Misspecification in ABC

Let \( y \) denote the observed data and define \( P_0 \) to be the true distribution generating \( y \). Let \( \mathcal{P} := \{ \theta \in \Theta \subset \mathbb{R}^{k_\theta} : P_\theta \} \) be the class of model implied distributions used in ABC to simulate pseudo-data so that \( z \sim P_\theta \); \( \mathcal{Z} \) represents the space of simulated data; \( \eta(y) = (\eta_1(y), \ldots, \eta_{k_\eta}(y))' \) is a \( k_\eta \)-dimension vector of summary statistics; \( \mathcal{B} := \{ \eta(z) : z \in \mathcal{Z} \} \subset \mathbb{R}^{k_\eta} \) is the range of the simulated summaries; \( d_1(\cdot, \cdot) \) is a metric on \( \Theta \); \( d_2(\cdot, \cdot) \) is a metric on \( \mathcal{B} \). When no confusion will result we simply denote a generic metric by \( d(\cdot, \cdot) \). \( \Pi(\theta) \) denotes a prior measure and \( \pi(\theta) \) its corresponding density.

In likelihood-based inference, model misspecification means that \( P_0 \notin \mathcal{P} \). The result of which is that the Kullback-Leibler divergence satisfies:

\[
\inf_{\theta \in \Theta} \mathcal{D}(P_0||P_\theta) = \inf_{\theta \in \Theta} - \int \log \left\{ \frac{dP_0(y)}{dP_\theta(y)} \right\} dP_0(y) > 0,
\]

and

\[
\theta^* = \arg \inf_{\theta \in \Theta} \mathcal{D}(P_0||P_\theta)
\]

is defined to be the pseudo-true value. Under regularity conditions Bayesian procedures predicated on the likelihood of \( P_\theta \) yield posteriors that concentrate on \( \theta^* \); see, e.g., Kleijn and van der Vaart (2012) and Muller (2013).

In this paper, we assume the researcher conducts posterior inference on \( \theta \) via ABC when the
observed sample is generated according to \( y \sim P_0 \). However, in contrast to previous research on ABC, we are explicitly interested in the case where \( P_0 \notin \mathcal{P} \). Because ABC algorithms are not based on the full data \( y \) but on two types of approximations, the summary statistics \( \eta(y) \) and the threshold \( \epsilon \), even if \( P_0 \notin \mathcal{P} \) the model class \( \mathcal{P} \) may be capable of generating a simulated summary that is compatible with the observed summary \( \eta(y) \), or is within an \( \epsilon \) neighbourhood of \( \eta(y) \). Therefore, the approximate nature of ABC means that \( \mathcal{D}(P_0||P_\theta) \) does not yield a meaningful notion of model misspecification associated with the output of an ABC algorithm, or ABC posterior distributions.

Recalling that the ABC posterior measure is given by, for \( A \subset \Theta \),

\[
\Pi_\epsilon[A|\eta(y)] = \frac{\int_A P_\theta [d\{\eta(y), \eta(z)\} \leq \epsilon] d\Pi(\theta)}{\int_\Theta P_\theta [d\{\eta(y), \eta(z)\} \leq \epsilon] d\Pi(\theta)},
\]

we see that misspecification in ABC will be driven by the behavior of \( \eta(y), \eta(z) \) and the set \( \{\theta \in \Theta : d\{\eta(y), \eta(z)\} \leq \epsilon\} \). To rigorously formulate the notion of model misspecification associated with the output of a given ABC algorithm, we must study the limiting behaviour of the ABC likelihood \( P_\theta [d\{\eta(y), \eta(z)\} \leq \epsilon] \) as the amount of information in the data accumulates.

To this end, we follow the framework of Marin et al. (2014), Frazier et al. (2018) and Li and Fearnhead (2018b), where it is assumed that the summary statistics concentrate around some fixed value, namely, \( b_0 \) under \( P_0 \) and \( b(\theta) \) under \( P_\theta \). In Marin et al. (2014), the authors study the case where \( \epsilon = 0 \), while Frazier et al. (2018) and Li and Fearnhead (2018b) study \( \epsilon > 0 \) but allow \( \epsilon \) to vary with \( n \) and set \( \epsilon = \epsilon_n \). In the latter two papers the authors demonstrate that the amount of information ABC obtains about a given \( \theta \) depends on: (1) the rate at which the observed (resp. simulated) summaries converge to a well-defined limit counterpart \( b_0 \) (resp., \( b(\theta) \)); (2) the rate at which the tolerance \( \epsilon_n \) (or bandwidth) goes to zero; (3) the link between \( b_0 \) and \( b(\theta) \). When \( P_0 \in \mathcal{P} \), there exists some \( \theta_0 \) such that \( b(\theta_0) = b_0 \) and the results of Frazier et al. (2018) completely characterize the asymptotic behaviour of the ABC posterior distribution. Furthermore, this analysis remains correct even if \( P_0 \notin \mathcal{P} \), so long as there exists some \( \theta_0 \in \Theta \) such that \( b_0 = b(\theta_0) \).

Therefore, the meaningful concept of model misspecification in ABC is when there does not exist any \( \theta_0 \in \Theta \) satisfying \( b_0 = b(\theta_0) \), which is precisely the notion of model incompatibility defined in Marin et al. (2014). Throughout the remainder, we say that the model is (ABC) misspecified if

\[
\epsilon^* = \inf_{\theta \in \Theta} d\{b_0, b(\theta)\} > 0 \tag{1}
\]

and note here that this condition is more likely to occur when \( k_0 < k_\eta \).

Heuristically the implication of misspecification in ABC is that, under regularity and since

\[
d\{\eta(y), \eta(z)\} \geq d\{b_0, b(\theta)\} - o_{P_0}(1) - o_{P_\theta}(1) \geq \epsilon^* - o(1),
\]

the event \( \{\theta \in \Theta : d\{\eta(y), \eta(z)\} \leq \epsilon_n\} \) becomes extremely rare, and corresponds to \( d\{\eta(z), b(\theta)\} > \epsilon^* - o(1) \). Therefore, for a sequence of tolerances \( \epsilon_n = o(1) \), or even if \( \epsilon_n < \epsilon^* \) no draws of \( \theta \) will be selected regardless of how many simulated samples from \( \pi(\theta) \) we generate, and \( \Pi_\epsilon[A|\eta(y)] \) will be ill-behaved.

While tolerance sequences \( \epsilon_n = o(1) \) will eventually cause \( \Pi_\epsilon[A|\eta(y)] \) to be ill-behaved, it is
possible that other choices for $\epsilon_n$ will produce a well-behaved posterior. In the following section we show that (certain) tolerance sequences satisfying $\epsilon_n \to \epsilon^*$, as $n \to +\infty$, yield well-behaved posteriors that concentrate posterior mass on the pseudo-true value $\theta^*$.

2.2 ABC Posterior Concentration Under Misspecification

Building on the intuition in the previous section, in this and the following section we rigorously characterize the asymptotic behaviour of

$$
\Pi_\epsilon[A|\eta(y)] = \int_A P_\theta [d\{\eta(y), \eta(z)\} \leq \epsilon_n] d\Pi(\theta) / \int_\Theta P_\theta [d\{\eta(y), \eta(z)\} \leq \epsilon_n] d\Pi(\theta)
$$

when $P_\theta \notin \mathcal{P}$ and $\epsilon^* > 0$, with $\epsilon^*$ defined by (1). To do so, we first define the following: for sequences $\{a_n\}$ and $\{b_n\}$, real valued, $a_n \lesssim b_n$ denotes $a_n \leq C b_n$ for some $C > 0$, $a_n \asymp b_n$ denotes equivalent order of magnitude, $a_n \gg b_n$ indicates a larger order of magnitude and the symbols $o_P(a_n), O_P(b_n)$ have their usual meaning.

We consider the following assumptions.

[A0] $d\{\eta(y), b_0\} = o_{P_\theta}(1)$ and there exists a positive sequence $v_{0,n} \to +\infty$ such that

$$
\liminf_{n \to +\infty} P_\theta [d\{\eta(y), b_0\} \geq v_{0,n}^{-1}] = 1.
$$

[A1] There exist a continuous, injective map $b : \Theta \to B \subset \mathbb{R}^{k_n}$ and a function $\rho_n(\cdot)$ satisfying: $\rho_n(u) \to 0$ as $n \to +\infty$ for all $u > 0$, and $\rho_n(u)$ monotone non-increasing in $u$ (for any given $n$), such that, for all $\theta \in \Theta$,

$$
P_\theta [d\{\eta(z), b(\theta)\} > u] \leq c(\theta)\rho_n(u), \quad \int_\Theta c(\theta)d\Pi(\theta) < +\infty
$$

where $z \sim P_\theta$, and we assume either of the following:

(i) **Polynomial deviations:** There exist a positive sequence $v_n \to +\infty$ and $u_0, \kappa > 0$ such that $\rho_n(u) = v_n^{-\kappa}u^{-\kappa}$, for $u \leq u_0$.

(ii) **Exponential deviations:** There exists $h_\theta(\cdot) > 0$ such that $P_\theta[d\{\eta(z), b(\theta)\} > u] \leq c(\theta)e^{-h_\theta(uv_n)}$ and there exists $c, C > 0$ such that

$$
\int_\Theta c(\theta)e^{-h_\theta(uv_n)} d\Pi(\theta) \leq Ce^{-c(uv_n)^r}, \text{ for } u \leq u_0.
$$

[A2] There exists some $D > 0$ and $M_0, \delta_0 > 0$ such that, for all $\delta_0 \geq \delta > 0$ and $M \geq M_0$, there exists $S_\delta \subset \{\theta \in \Theta : d\{b(\theta), b_0\} - \epsilon^* \leq \delta\}$ for which

(i) In case (i) of [A1], $D < \kappa$ and

$$
\int_{S_\delta} \left(1 - \frac{c(\theta)}{M}\right) d\Pi(\theta) \gtrsim \delta^D.
$$
(ii) In case (ii) of [A1],
\[ \int_{S_s} (1 - c(\theta) e^{-h_0(M)}) \, d\Pi(\theta) \gtrsim \delta^D. \]

We then have the following result.

**Theorem 1.** The data generating process for \( y \) satisfies [A0] and assume that (1) holds. Assume also that conditions [A1] and [A2] are satisfied and \( \epsilon_n \downarrow \epsilon^* \) with
\[ \epsilon_n \geq \epsilon^* + M \nu_n^{-1} + \nu_0^{-1} \]
and \( M \) is large enough. Let \( M_n \) be any positive sequence going to infinity and \( \delta_n \geq M_n \max\{\epsilon_n - \epsilon^*, \nu_0^{-1}, \nu_n^{-1}\} \), then
\[ \Pi_\epsilon \left[ d\{b(\theta), b_0\} \geq \epsilon^* + \delta_n |\eta(y)| \right] = o_{P_0}(1), \]
as soon as
\[ \delta_n \geq M_n \nu_n^{-1} \ln(u_n)^{1/\tau} = o(1) \quad \text{in case (ii) of assumption [A1]}, \]
with
\[ u_n = \epsilon_n - (\epsilon^* + M \nu_n^{-1} + \nu_0^{-1}). \]

**Remark 1.** Theorem 1 gives conditions so that the ABC posterior concentrates on
\[ \arg \min_{\theta \in \Theta} d\{b(\theta), b_0\}, \]
at least under the assumption that \( \epsilon_n \) is slightly larger than \( \epsilon^* \). Under the more precise framework of Theorem 2, where the asymptotic shape of the posterior distribution is studied, this condition can be refined to allow \( \epsilon_n \) to be slightly smaller than \( \epsilon^* \). However, if \( \epsilon^* - \epsilon_n \) is bounded below by a positive constant, then the posterior distribution does not necessarily concentrate.

**Corollary 1.** Assume the hypotheses of Theorem 1 are satisfied and \( \theta^* \in \Theta \) uniquely satisfies
\[ \theta^* = \arg \inf_{\theta \in \Theta} d\{b_0, b(\theta)\}. \]
For any \( \delta > 0 \), \( \Pi_\epsilon[d_1\{\theta, \theta^*\} > \delta |\eta(y)|] = o_{P_0}(1). \)

**Remark 2.** Theorem 1 and Corollary 1 demonstrate that \( \Pi_\epsilon[\cdot |\eta(y)] \) concentrates on \( \theta^* \) if the model is misspecified. Therefore, Theorem 1 is an extension of Theorem 1 in Frazier et al. (2018) to the case of misspecified models. In addition, we note that Theorem 1 above is similar to Theorem 4.3 in Bernton et al. (2017) for ABC inference based on the Wasserstein distance.

**Remark 3.** It is crucial to note that the pseudo-true value \( \theta^* \) directly depends on the choice of \( d_2\{\cdot, \cdot\} \). Indeed, ABC based on two different metrics \( d_2\{\cdot, \cdot\} \) and \( \tilde{d}_2\{\cdot, \cdot\} \) will produce different pseudo-true values, unless if by happenstance \( \inf\{\theta \in \Theta : d_2\{b(\theta), b_0\}\} \) and \( \inf\{\theta \in \Theta : \tilde{d}_2\{b(\theta), b_0\}\} \) coincide. This lies in stark contrast to the posterior concentration result in Frazier et al. (2018), which demonstrated that, under correct model specification, \( \Pi_\epsilon[\cdot |\eta(y)] \) concentrates on the same true value regardless of the choice of \( d_2\{\cdot, \cdot\} \).
## 2.3 Shape of the Asymptotic Posterior Distribution

In this section, we analyse the asymptotic shape of the ABC posterior under model misspecification. For simplicity, we take the rate at which the simulated and observed summaries converge to their limit counterparts to be the same, i.e., we take $v_{0,n} = v_n$ and we consider as the distance $d\{\eta(z), \eta(y)\} = \|\eta(z) - \eta(y)\|$ where $\| \cdot \|$ is the norm associated to a given scalar product $< \cdot, \cdot >$. Denote by $I_k$ the $(k \times k)$ dimensional identity matrix and let

$$\Phi_k(B) = \Pr[N(0, I_k) \in B]$$

for any measurable subset $B$ of $\mathbb{R}^k$.

The following conditions are needed to establish the results of this section.

[A0] Assumption [A0] is satisfied, $\epsilon^* = \inf_{\theta \in \Theta} d\{b(\theta), b_0\} > 0$ and $\theta^* = \arg\inf_{\theta \in \Theta} \|b(\theta) - b_0\|$ exists and is unique.

[A1] Assumption [A1] holds and for some positive-definite matrix $\Sigma_n(\theta^*)$, $c_0 > 0$, $\kappa > 1$ and $\delta > 0$, for all $\|\theta - \theta^*\| \leq \delta$, $\Pr[\|\Sigma_n(\theta^*)\{\eta(z) - b(\theta)\}\| > u] \leq c_0 u^{-\kappa}$ for all $0 < u \leq \delta v_n$.

[A2] The map $\theta \mapsto b(\theta)$ is twice continuously differentiable at $\theta^*$ and the Jacobian $\nabla_\theta b(\theta^*)$ has full column rank $k_\theta$. The Hessian of $\|b(\theta) - b_0\|^2$ evaluated at $\theta^*$, and denoted by $H^*$, is a positive-definite matrix.

[A3] There exists a sequence of $(k_n \times n)$ positive-definite matrices $\Sigma_n(\theta)$ such that for all $M > 0$ there exists $u_0 > 0$ for which

$$\sup_{|x| \leq M} \sup_{\|\theta - \theta^*\| \leq u_0} |P_\theta(\langle Z_n, e \rangle \leq x) - \Phi(x)| = o(1),$$

where $Z_n = \Sigma_n(\theta)(\eta(z) - b(\theta))$ and $e = (b(\theta^*) - b_0) / \|b(\theta^*) - b_0\|$.  

[A5] There exists $v_n$ going to infinity and $u_0 > 0$ such that for all $\|\theta - \theta^*\| \leq u_0$, the sequence of functions $\theta \mapsto \Sigma_n(\theta)v_n^{-1}$ converges to some positive-definite matrix $A(\theta)$ and is equicontinuous at $\theta^*$.

[A6] $\pi(\theta)$, the density of the prior measure $\Pi(\theta)$, is continuous and positive at $\theta^*$.

[A7] For $Z_n^0 = \Sigma_n(\theta^*)\{\eta(y) - b_0\}$ and all $M_n$ going to infinity

$$P_0(\|Z_n^0\| > M_n) = o(1).$$

**Theorem 2.** Assumptions [A0]', [A1]', (with $\kappa \geq k_\theta$), [A2] and [A3]-[A7] are satisfied. We then have the following results.

(i) If $\lim_n v_n(\epsilon_n - \epsilon^*) = 2c$, with $c \in \mathbb{R}$, then for $\| \cdot \|_{TV}$ the total-variation norm

$$\|\Pi_{v_n^{1/2}, \epsilon} - Q_\epsilon\|_{TV} = o_{R_n}(1)$$

where $\Pi_{z_n, \epsilon}$ is the ABC posterior distribution of $z_n(\theta - \theta^*)$ for any sequence $z_n > 0$ and $Q_c$
has density $q_\varepsilon$ with respect to Lebesgue measure on $\mathbb{R}^{k_\theta}$ proportional to
\[
q_\varepsilon(x) \propto \Phi \left( \frac{c - < Z^0_n, A(\theta^*) e >}{\| A(\theta^*) e \|} - \frac{x^\top H^* x}{4 \| A(\theta^*) e \|} \right)
\]

(ii) If $\lim_n v_n(\varepsilon_n - \varepsilon^*) = +\infty$ with $u_n = \varepsilon_n - \varepsilon^* = o(1)$, for $U_{\{\|x\| \leq M\}}$ the uniform measure over the set $\{\|x\| \leq M\}$,
\[
\|\Pi_{u_n^{-1,\varepsilon}} - U_{\{x^\top H^* x \leq 2\}}\|_{TV} = o_{P_h}(1).
\]

Remark 4. As is true in the case where the model is correctly specified, if $\varepsilon_n$ is too large, which here means that $(\varepsilon_n - \varepsilon^*) \gg 1/v_n$, then the asymptotic distribution of the ABC posterior is uniform with a radius that is of the order $\varepsilon_n - \varepsilon^*$. In contrast to the case of correct model specification, if $\varepsilon^* > 0$ and if $v_n \{\varepsilon_n - \varepsilon^*\} \to 2c \in \mathbb{R}$, then the limiting distribution is no longer Gaussian. Moreover, this result maintains even if $c = 0$.

Remark 5. In likelihood-based Bayesian inference, credible sets are not generally valid confidence sets if the model is misspecified (see, e.g., Kleijn and van der Vaart, 2012 and Muller, 2013). However, it remains true in likelihood-based settings that the resulting posterior is still asymptotically normal. In the case of ABC, not only will credible sets not be valid confidence sets, but the asymptotic shape of the ABC posterior is not even Gaussian.

Remark 6. In practice $\varepsilon^*$ is unknown, and it is therefore not possible to choose $\varepsilon_n$ directly. However, we note that the application of ABC is most often implemented by accept draws of $\theta$ within some pre-specified (and asymptotically shrinking) quantile threshold; i.e., one accepts a simulated draw $\theta^*$ if $d\{\eta(z^j), \eta(y)\}$ is smaller than the $\alpha$-th empirical quantile of the simulated values $d\{\eta(z^j), \eta(y)\}$, $j \leq N$. However, as discussed in Section 6 of Frazier et al., 2018, the two representations of the ABC approach are dual in the sense that choosing a value of $\alpha$ on the order of $\delta v_n^{-k_\theta}$, with $\delta$ small, corresponds to choosing $|\varepsilon_n - \varepsilon^*| \lesssim \delta^{1/k_\theta} v_n$ and choosing $\alpha_n \gtrsim M v_n^{k_\theta}$ corresponds to choosing $\varepsilon_n - \varepsilon^* \gtrsim M v_n$. We further elaborate on the equivalence between the two approaches in Section 4.1.

Interestingly, the proof of Theorem 2 demonstrates that if $v_n(\varepsilon_n - \varepsilon^*) \to -\infty$, in particular when $\varepsilon_n = o(1)$ and $\varepsilon^* > 0$, posterior concentration of $\Pi_{\{\|\eta(y)\|\}}$ need not occur. We present an illustration of this phenomena in the following simple example.

Example 2: Consider the case where $k_\theta = 1$ and $k_\eta = 2$. Let $\tilde{Z}_y = \sqrt{n}(\eta(y) - b_0)$ and $\tilde{Z}_n = \sqrt{n}(\eta(z) - b(\theta))$, where $\tilde{Z}_n \sim \mathcal{N}(0, v_\theta^2 I_2)$, for $v_\theta$ some known function of $\theta$, and $b(\theta) = (\theta, \theta)^\top$. In addition, assume that $b_0 = (\tilde{b}_0, -\tilde{b}_0)$, with $\tilde{b}_0 \neq 0$. Under this setting, and when $\| \cdot \|$ is the Euclidean norm, it follows that the unique pseudo-true value is $\theta^* = 0$. However, depending on $v_\theta$, the approximate posterior need not concentrate on $\theta^* = 0$. This is summarized in the following Proposition.

Proposition 1. In the setup described above, if $v_\theta/v_\sigma = \sigma(\theta)$, for $v_\theta$ some known function, such that $\sigma$ is continuous and $\sigma(\tilde{b}_0/2)^2 \geq 3$ and if the prior has positive and continuous density on $[-\tilde{b}_0, \tilde{b}_0]$, then
\[
\Pi_{\{\|\theta - \theta^*\| \leq \delta |\eta(y)|\}} = o \left( \Pi_{\{\|\theta - \tilde{b}_0/2\| \leq \delta |\eta(y)|\}} \right) = o(1).
\]


3 Regression Adjustment under Misspecification

3.1 Posterior Concentration

Linear regression post-processing methods are a common means of adjusting the ABC output. First proposed by Beaumont et al. (2002), this method has found broad applicability with ABC practitioners.

However, as demonstrated in the introductory example, we caution against the blind application of these post-processing methods when one is willing to entertain the idea of model misspecification. In particular, the use of post-processing steps in ABC can lead to point estimators that have very different behavior than those obtained from Algorithm 1, even in small samples.

In this section, we rigorously characterize posterior concentration of the linear regression adjustment ABC approach (hereafter, ABC-Reg) under model misspecification. For simplicity, we only consider the case of scalar \( \theta \), however, we allow \( \eta(y) \) to be multi-dimensional.

We consider an ABC-Reg approach that first runs Algorithm 1, with tolerance \( \epsilon_n \), to obtain a set of selected draws and summaries \( \{\theta^i, \eta(z^i)\} \) and then uses a linear regression model to predict the accepted values of \( \theta \). The accepted value \( \theta^i \) is then artificially related to \( \eta(y) \) and \( \eta(z) \) through the linear regression model

\[
\theta^i = \mu + \beta^\top \{\eta(y) - \eta(z^i)\} + \nu_i,
\]

where \( \nu_i \) is the model residual. Define \( \bar{\theta} = \frac{1}{N} \sum_{i=1}^{N} \theta^i/N \) and \( \bar{\eta} = \frac{1}{N} \sum_{i=1}^{N} \eta(z^i)/N \). ABC-Reg defines the adjusted parameter draw according to

\[
\tilde{\theta}^i = \bar{\theta} + \hat{\beta}^\top \{\eta(y) - \eta(z^i)\},
\]

where \( \hat{\beta} \) is estimated as

\[
\hat{\beta} = \left[ \frac{1}{N} \sum_{i=1}^{N} (\eta(z^i) - \bar{\eta}) (\eta(z^i) - \bar{\eta})^\top \right]^{-1} \left[ \frac{1}{N} \sum_{i=1}^{N} (\eta(z^i) - \bar{\eta}) (\theta^i - \bar{\theta}) \right] = \frac{\text{Var}^{-1}(\eta(z^i))}{\text{Cov}(\eta(z^i), \theta^i)}.
\]

Therefore, for \( \tilde{\theta}^i \sim \Pi[\theta|\eta(y)] \), the posterior measure for \( \tilde{\theta}^i \) is nothing but a scaled and shifted version of \( \Pi[\cdot|\eta(y)] \). Consequently, the asymptotic behavior of the ABC-Reg posterior, denoted by \( \widetilde{\Pi}[\cdot|\eta(y)] \), is determined by the behavior of \( \Pi[\cdot|\eta(y)] \), \( \hat{\beta} \), and \( \{\eta(y) - \eta(z^i)\} \).

Corollary 2. Assumptions \([A0']\), \([A1]\) and \([A2]\) are satisfied and \( \epsilon_n \downarrow \epsilon^* \) with

\[
\epsilon_n \geq \epsilon^* + Mv_n^{-1} + v_0^{-1},
\]

and \( M \) large enough. Furthermore, for some \( \beta_0 \) with \( \|\beta_0\| > 0 \), \( \|\hat{\beta} - \beta_0\| = o_{P_\theta}(1) \). Define \( \hat{\theta}^* = \hat{\theta} + \beta_0^\top (b(\theta^*) - b_0) \). For any \( \delta > 0 \),

\[
\widetilde{\Pi}(|\theta - \hat{\theta}^*| > \delta|\eta(y)|) = o_{P_\theta}(1),
\]

\[\text{3This result can be extended at the cost of more complicated arguments but we refrain from this setting to simplify the interpretation of our results.}\]
as soon as
\[
\rho_n(\epsilon_n - \epsilon^*) \geq (\epsilon_n - \epsilon^*)^{-D/\kappa} \quad \text{in case (i)}
\]
\[
\rho_n(\epsilon_n - \epsilon^*) \geq |\log(\epsilon_n - \epsilon^*)|^{1/\tau} \quad \text{in case (ii)}.
\]

**Remark 7.** An immediate consequence of Theorems 1 and Corollary 2 is that \( \Pi_{\epsilon}[\cdot | \eta(y)] \) concentrates posterior mass on
\[
\theta^* = \arg \min_{\theta \in \Theta} d\{b(\theta), b_0\},
\]
while \( \tilde{\Pi}_\epsilon[\cdot | \eta(y)] \) concentrates posterior mass on
\[
\tilde{\theta}^* = \theta^* + \beta_0^T (b(\theta^*) - b_0).
\]

It is also important to realize that, for \( \|\beta_0\| \) large, the pseudo-true value \( \tilde{\theta}^* \) can lie outside \( \Theta \). Therefore, if the model is misspecified, the ABC-Reg procedure can return parameter values that do not have a sensible interpretation in terms of the assumed model.

**Remark 8.** An additional consequence of Theorem 1 and Corollary 2 is that \( \Pi_{\epsilon}[\cdot | \eta(y)] \) and \( \tilde{\Pi}_\epsilon[\cdot | \eta(y)] \) yield different posterior expectations. We use this point in the next section to derive a procedure for detecting model misspecification.

### 3.2 Adjusting Regression Adjustment

The difference between accept/reject ABC and ABC-Reg under model misspecification is related to the regression adjustments re-centering of the accepted draws \( \theta^i \) by \( \hat{\beta}^T \{\eta(y) - \eta(z)\} \). Whilst useful under correct model specification, when the model is misspecified the adjustment can force \( \theta^i \) away from \( \theta^* \) and towards \( \tilde{\theta}^* \), which need not lie in \( \Theta \).

The cause of this behavior is the inability of \( \eta(z) \) to replicate the asymptotic behavior of \( \eta(y) \), which in the terminology of Marin et al. (2014) means that the model is incompatible with the observed summaries. This incompatibility of the summary statistics ensures that the influence of the centering term \( \hat{\beta}^T \{\eta(y) - \eta(z)\} \) can easily dominate that of the accepted draws \( \theta^i \), with the introductory example being just one example of this behavior (an additional example is given in Section 4.3).

In an attempt to maintain the broad applicability of linear regression adjustment in ABC, and still ensure it gives sensible results under model misspecification, we propose a useful modification of the regression adjustment approach. To motivate this modification recall that, under correct model specification and regularity conditions, at first-order the regression adjustment approach ensures (see Theorem 4 in Frazier et al., 2018):

\[
\tilde{\theta}^i = \theta^i + \hat{\beta}^T \{\eta(y) - \eta(z)\}
= \theta^i + \hat{\beta}^T \{b_0 - b(\theta^i)\} + O_p(1/v_n)
= \theta^i - [\nabla_\theta b(\theta^*)^T V_0^{-1} \nabla_\theta b(\theta^*)]^{-1} \nabla_\theta b(\theta^*)^T V_0^{-1} \nabla_\theta b(\theta)(\theta^i - \theta^*) + O_p(1/v_n),
\]

where \( b_0 = b(\theta^*) \) by correct model specification, \( \tilde{\theta} \) is an intermediate value satisfying \( |\tilde{\theta} - \theta^*| \leq |\theta^i - \theta^*| \), \( V_0 = \text{plim} \text{Var} \{\sqrt{n} \{\eta(y) - b_0\} \} \), and the third line follows from the definition of \( \beta_0 \) and a
mean-value expansion. Therefore, it follows from (3) that, even if \( k_\eta > k_\theta \), the dimension of \( \eta(y) \) will not affect the asymptotic variance of the ABC-Reg posterior mean. This result, at least in part, helps explain (from a technical standpoint) the popularity of the ABC-Reg approach as a dimension reduction method. However, under model misspecification, \( b_0 \neq b(\hat{\theta}) \) for any \( \theta \in \Theta \), and hence the intermediate value \( \theta \) will be such that

\[
b_0 - b(\theta^*) \neq \nabla_\theta b(\hat{\theta})(\theta^* - \theta^i).
\]

As a consequence, equation (3) can not be valid (in general) if the model is misspecified.

The behavior of ABC-Reg under correct and incorrect model specification suggests that the methods poor behavior under the latter can be mitigated by replacing \( \eta(y) \) with an appropriate term. To this end, define \( \hat{\theta} = \mathbb{E}_\eta[\theta|\eta(y)] \) to be the posterior mean of standard ABC; let \( \hat{z}^m, m = 1, \ldots, M \), be a pseudo-data set of length \( n \) simulated under the assumed DGP and at the value \( \hat{\theta} \); and define

\[
\hat{\eta} = \frac{1}{M} \sum_{m=1}^M \eta(\hat{z}^m)/M.
\]

Using \( \hat{\eta} \), we can then implement the regression adjustment approach

\[
\hat{\theta}^i = \theta^i + \hat{\beta}^T \{\hat{\eta} - \eta(z^i)\}.
\]

The key to this approach is that under correct specification \( \hat{\eta} \) behaves like \( \eta(y) \), while under incorrect specification \( \hat{\eta} \) behaves like \( \eta(z) \). A direct consequence of this construction is that this approach avoids the incompatibility issue that arises from model misspecification.

To demonstrate the robustness of this new regression adjustment approach to ABC, we return to the simple normal example.

**Example 1 (Continued):** The assumed DGP is \( z \sim \mathcal{N}(\theta, 1) \) but the actual DGP is \( y \sim \mathcal{N}(1, \tilde{\sigma}^2) \). ABC is conducted using the following summary statistics:

- the sample mean \( \eta_1(y) = \frac{1}{n} \sum_{i=1}^n y_i \);
- the centered summary \( \eta_2(y) = \frac{1}{n-1} \sum_{i=1}^n (y_i - \eta_1(y))^2 - 1 \).

We consider two different DGPs corresponding to \( \tilde{\sigma}^2 = 1 \) and \( \tilde{\sigma}^2 = 2 \), which respectively correspond to correct and incorrect specification for the DGP of \( z \). For each of these two cases we generate 100 artificial samples for \( y \) of length \( n = 100 \) and apply three different ABC approaches: ABC-AR, ABC-Reg and our new ABC-Reg procedure (referred to as ABC-Reg-New). Each procedure relies on \( N = 50,000 \) pseudo-data sets generated according to \( z \sim \mathcal{N}(\theta, 1) \); for ABC-AR we take \( d\{\cdot, \cdot\} \) to be the Euclidean norm \( \| \cdot \| \); and we retain draws that yield \( \| \eta(y) - \eta(z^i) \| \) in the \( \alpha_n = n^{-5/9} \) quantile.

Figure 2 plots the posterior mean of each approach across the Monte Carlo replications and across both designs. The results demonstrates that the new regression adjustment maintains stable performance across both correct and incorrect model specification. Table 1 reports the corresponding coverage and average credible set length across the two Monte Carlo designs. Jointly, these results demonstrate that under correct specification the ABC-Reg-New approach performs
Figure 2: Posterior mean comparison of ABC-AR (AR), standard regression adjustment (Reg) and the new regression adjustment approach (Reg-New) across the two Monte Carlo designs. $\tilde{\sigma}^2 = 1$ (resp., $\tilde{\sigma}^2 = 2$) corresponds to correct (resp., incorrect) model specification.

just as well as the other ABC approaches in terms of accuracy and precision, while under model misspecification the procedure behaves similar to ABC-AR.

Before concluding, we compare the Monte Carlo coverage of these methods across the two Monte Carlo designs. The results in Table 1 demonstrate that ABC-Reg and ABC-Reg-New give much shorter credible sets than ABC-AR on average. However, when the model is misspecified, this behavior gives researchers a false sense of the procedures precision, which is reflected by the poor coverage rates (for the pseudo-true value) of both regression adjustment procedures. Therefore, even though this new regression adjustment procedure gives stable performance under correct and incorrect model specification, it still suffers from the coverage issues alluded to in the remarks proceeding Theorem 2.

Table 1: Monte Carlo coverage (Cov.) and average credible set length (Length) for the simple normal example under correct (resp., incorrect) model specification. COV is the percentage of times that the 95% credible set contained $\theta^* = 1$. Length is the average length of the credible set, across the Monte Carlo trials.

|       | $\tilde{\sigma}^2 = 1$ |       | $\tilde{\sigma}^2 = 2$ |
|-------|-------------------------|-------|-------------------------|
| Cov.  | 98%                     | 100%  |
| Length| 0.97                    | 0.86  |
4 Detecting Misspecification

In this section we propose two methods to detect model misspecification in ABC. The first approach is based on the behavior of the acceptance probability under correct and incorrect model specification. The second approach is based on comparing posterior expectations calculated under \( \Pi_0[\eta(y)] \) (obtained from Algorithm 1) and \( \tilde{\Pi}_0[\eta(y)] \) (obtained using the linear regression adjustment approach, i.e., ABC-Reg).

4.1 A Simple Graphical Approach to Detecting Misspecification

From the results of Frazier et al. (2018), under regularity and correct model specification, the acceptance probability

\[
\alpha_n = \Pr \left[ d(\eta(y), \eta(z)) \leq \epsilon_n \right]
\]

satisfies, for \( n \) large and \( \epsilon_n \gg v_n^{-1} \),

\[
\alpha_n = \Pr \left[ d(\eta(y), \eta(z)) \leq \epsilon_n \right] \approx \epsilon_n^{k_\theta}.
\]

In this way, as \( \epsilon_n \to 0 \) the acceptance probability \( \alpha_n \to 0 \) in a manner that is approximately linear in \( \epsilon_n^{k_\theta} \).

However, this relationship between \( \alpha_n \) and \( \epsilon_n \) does not extend to the case where \( \lim_n \epsilon_n > 0 \). In particular, if \( \epsilon^* > 0 \), once \( \epsilon_n < \epsilon^* \) we will often obtain an acceptance probability \( \alpha_n \) that is small or zero, even for a large number of simulations \( N \).

The behavior of \( \alpha_n \) under correct and incorrect model specification means that one can potentially diagnose model misspecification graphically by comparing the behavior of \( \alpha_n \) over a decreasing sequence of tolerance values. In particular, by taking a decreasing sequence of tolerances \( \epsilon_1,n \leq \epsilon_2,n \leq \cdots \leq \epsilon_J,n \) we can construct and plot the resulting sequence \( \{ \alpha_{j,n} \}_j \) to determine if \( \{ \alpha_{j,n} \}_j \) decays in an (approximately) linear fashion.

While \( \alpha_n \) is infeasible to obtain in practice, the same procedure can be applied with \( \alpha_n \) replaced by the estimator

\[
\hat{\alpha}_{j,n} = \frac{1}{N} \sum_{i=1}^N \mathbb{I}[d(\eta(y), \eta(z)) \leq \epsilon_{j,n}] / N.
\]

Once \( \hat{\alpha}_{j,n} \) has been obtained, it can be plotted against \( \epsilon_{j,n} \) (in some fashion) and the relationship can be analyzed to determine if deviations from linearity are in evidence.

To understand exactly how such a procedure can be implemented, we return to the simple normal example.

Example 1 (Continued): The assumed DGP is \( z \sim \mathcal{N}(\theta,1) \) but the actual DGP is \( y \sim \mathcal{N}(1,\tilde{\sigma}^2) \). We again consider ABC analysis using the following summary statistics:

- the sample mean \( \eta_1(y) = \frac{1}{n} \sum_{i=1}^n y_i \);
- the centered summary \( \eta_2(y) = \frac{1}{n-1} \sum_{i=1}^n (y_i - \eta_1(y))^2 - 1 \).

Taking \( \tilde{\sigma}^2 \in \{1,1+2/9,1+3/9,...,2\} \), we generate observed samples of size \( n = 100 \) according to \( y \sim \mathcal{N}(1,\tilde{\sigma}^2) \), where, for each of the nine different simulated data sets, we keep the random
numbers fixed and only change $\bar{\sigma}^2$. $N = 50,000$ simulated data sets are again generated according to $z_i \sim \mathcal{N}(\theta^i, 1)$, with $\theta^i \sim \mathcal{N}(0, 25)$, and for $d\{\cdot, \cdot\}$ we consider the Euclidean norm $\| \cdot \|$. The results are presented in Figure 3.

![Graphical comparison of estimated acceptance probabilities $\hat{\alpha}_{j,n}$ against decreasing tolerance values $\epsilon_{j,n}$.](image)

Figure 3: Graphical comparison of estimated acceptance probabilities $\hat{\alpha}_{j,n}$ against decreasing tolerance values $\epsilon_{j,n}$.

The figure demonstrates that for $n = 100$ this procedure has difficulty detecting model misspecification if $|\bar{\sigma}^2 - 1| \leq 1/3$. However, for $|\bar{\sigma}^2 - 1| \geq 4/9$ the procedure can detect model misspecification, which shows up as an exponential decay in $\hat{\alpha}_{n,j}$.

Clearly, obtaining broad conclusions about model misspecification from this graphical approach depends on many features of the underlying model, the dimension of $\theta_i$, and the exact nature of misspecification. While potentially useful, this approach should only be used as a tool to help diagnose model misspecification. □

4.2 Detecting Model Misspecification Using Regression Adjustment

Theorem 1 and Corollary 2 demonstrate that basic ABC, as described in Algorithm 1, and ABC-Reg place posterior mass in different regions of the parameter space. Therefore, the posterior

4In this simple example correct specification would warrant a linear relationship between $\alpha_n$ and $\epsilon_n$, since we are only conducting inference on one parameter. More generally, under correct model specification, we would expect a linear relationship between $\alpha_n$ and $\epsilon_n^k$. 

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expectations

\[ \hat{h} = \int h(\theta) d\Pi_\epsilon[\theta|\eta(y)], \quad \tilde{h} = \int h(\theta) d\tilde{\Pi}_\epsilon[\theta|\eta(y)] \]

converge, as \( n \to +\infty \) and \( \epsilon_n \downarrow \epsilon^* \), to distinct values. However, if the model is correctly specified, \( \hat{h} \) and \( \tilde{h} \) will not differ, up to first order, so long as \( \epsilon_n = o(1/\sqrt{n}) \). Therefore, a useful approach for detecting model misspecification is to compare various posterior expectations, such as moments or quantiles, calculated from the two posteriors.

More specifically, under regularity conditions given in Li and Fearnhead (2018b) and Li and Fearnhead (2018a), if the model is correctly specified and if we use Algorithm 1 based on quantile thresholding with \( \alpha_n = \delta n^{-k_\theta/2} \) with \( \delta > 0 \) small then

\[ \sqrt{n}||\hat{h} - \tilde{h}|| = o_{P_0}(1). \]

However, if \( \epsilon^* = \inf_{\theta \in \Theta} d\{b_0, b(\theta)\} > 0 \), under regularity conditions, we can deduce that

\[ \liminf_n ||\hat{h} - \tilde{h}|| > 0 \]

Therefore a not small \( ||\hat{h} - \tilde{h}|| \) is meaningful evidence that the model may be misspecified.

To demonstrate this approach to diagnosing model misspecification, we return to our simple running example.

**Example 1 (Continued):** The assumed DGP is \( z \sim N(\theta, 1) \), but the actual DGP is \( y \sim N(\theta, \tilde{\sigma}^2) \). We again consider the following summary statistics:

- the sample mean \( \eta_1(y) = \frac{1}{n} \sum_{i=1}^n y_i; \)
- the centered summary \( \eta_2(y) = \frac{1}{n-1} \sum_{i=1}^n (y_i - \eta_1(y))^2 - 1. \)

We simulate \( n = 100 \) observed data points from a normal random variable with mean \( \theta = 1 \) and variance \( \tilde{\sigma}^2 = 2 \), so as to capture a mild level of model misspecification, and generate one-thousand independent Monte Carlo replications. We again take \( N = 50,000 \) simulated data sets generated according to \( z_i^j \sim N(\theta^j, 1) \), with \( \theta^j \sim N(0, 25) \). For \( d\{\cdot, \cdot\} \) we take the Euclidean norm \( \|\cdot\| \) and we accept values of \( \theta \) that lead to distances lower than the corresponding \( \alpha_n = n^{-5/9} \) quantile.

Across the Monte Carlo replications, we compare the non-centered second and third posterior moments calculated under ABC-AR and ABC-Reg:

\[
\hat{h} = \left( \int \theta^2 d\Pi_\epsilon[\theta|\eta(y)], \int \theta^3 d\Pi_\epsilon[\theta|\eta(y)] \right)^\top, \quad \tilde{h} = \left( \int \theta^2 d\tilde{\Pi}_\epsilon[\theta|\eta(y)], \int \theta^3 d\tilde{\Pi}_\epsilon[\theta|\eta(y)] \right)^\top
\]

The sampling distribution of \( \sqrt{n}||\hat{h} - \tilde{h}|| \), across the Monte Carlo replications, is presented in Figure 4.
Figure 4: Monte Carlo sampling distribution of $\sqrt{n}∥\hat{h} - \tilde{h}∥$ in the normal example with $\tilde{\sigma}^2 = 2$.  

Recall that under correct specification $\sqrt{n}∥\hat{h} - \tilde{h}∥ = o_P(1)$. Thus, if the model was correctly specified, we would expect a majority of the realizations for $\sqrt{n}∥\hat{h} - \tilde{h}∥$ to be relatively small. It is then clear from Figure 4 that there is a substantial difference between $\hat{h}$ and $\tilde{h}$ even under moderate model misspecification.

As further evidence on the difference between the behavior of $\sqrt{n}∥\hat{h} - \tilde{h}∥$ under correct and incorrect model specification, Figure 5 plots, across the Monte Carlo replications, the sampling distributions of $\sqrt{n}∥\hat{h} - \tilde{h}∥$ when $\tilde{\sigma}^2 = 1$ (correct specification) and when $\tilde{\sigma}^2 = 2$ (incorrect specification). From Figure 5 it is clear that the distribution of $\sqrt{n}∥\hat{h} - \tilde{h}∥$ is drastically different under correct and incorrect specification, even at this relatively minor level of misspecification.

Figure 5: Monte Carlo sampling distributions for $\sqrt{n}∥\hat{h} - \tilde{h}∥$ under correct specification ($\tilde{\sigma}^2 = 1$) and a mild level of model misspecification ($\tilde{\sigma}^2 = 2$). $\hat{h}$ (resp., $\tilde{h}$) is the vector of second and third posterior moments calculated from ABC-AR (resp., ABC-Reg). $∥ \cdot ∥$ is the Euclidean norm.
Before concluding this section, we note that, at least for smooth functions $h(\theta)$, a formal testing strategy based on comparing $\hat{h}$ and $\tilde{h}$ can be constructed. For example, a test could be constructed to determine whether or not the (population) means of $h(\theta)$ under the two different sample measures, $\Pi_{\epsilon}$ and $\tilde{\Pi}_{\epsilon}$ are the same, against the alternative hypothesis that these (population) means differ. A natural test statistic for such a hypothesis test is an appropriately scaled version of $\hat{h} - \tilde{h}$. While potentially useful, we do not explore this topic in any formal way within this paper but leave its analysis for future research.

4.3 Additional Monte Carlo Evidence

In this section we demonstrate the consequences of model misspecification in ABC using the MA(2) model.

4.3.1 Moving Average Model

When the behavior of the observed data $y$ displays short memory properties, a moving average model is capable of capturing these features in a parsimonious fashion. If the researcher believes $y$ is generated according to an MA($q$) model, then ABC requires the generation of pseudo-data according to

$$z_t = e_t + \sum_{i=1}^{q} \theta_i e_{t-i},$$

where, say, $e_t \sim \mathcal{N}(0, 1)$ i.i.d and $\theta_1, ..., \theta_q$ are such that the roots of the polynomial

$$p(x) = 1 - \sum_{i=1}^{q} \theta_i x^i$$

all lie outside the unit circle.

Specializing this model to the case where $q = 2$ we have that

$$z_t = e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2}, \quad (4)$$

and the unknown parameters $\theta = (\theta_1, \theta_2)^T$ are assumed to obey

$$-2 < \theta_1 < 2, \quad \theta_1 + \theta_2 > -1, \quad \theta_1 - \theta_2 < 1. \quad (5)$$

Our prior information on $\theta = (\theta_1, \theta_2)^T$ is uniform over the invertibility region in (5). A useful choice of summary statistics for the MA(2) model are the sample autocovariances $\gamma_j(z) = \frac{1}{T} \sum_{t=1+j}^{T} z_t z_{t-j}$, for $j = 0, 1, 2$. Throughout the remainder of this subsection we let $\eta(z)$ denote the summaries $\eta(z) = (\gamma_0(z), \gamma_1(z), \gamma_2(z))^T$. It is simple to show that, under the DGP in equations (4)-(5), the limit map $\theta \mapsto b(\theta)$ is

$$b(\theta) = (1 + \theta_1^2 + \theta_2^2, \quad \theta_1(1 + \theta_2), \quad \theta_2)^T$$

and $\eta(z)$ satisfies the sufficient conditions for posterior concentration previously outlined, assuming $y$ is sufficiently regular.
While short memory properties can exist in the levels of many economic and financial time series, the observed data $y$ can often display conditional heteroskedasticity. In such cases, the dynamics of the level series $\{y_t\}_{t \geq 1}$ displays short memory properties but the autocorrelations of the squared or absolute series, $\{y^2_t\}_{t \geq 1}$ or $\{|y_t|\}_{t \geq 1}$, display persistence that can not be captured by the MA(2) model. Therefore, if one disregards these conditional dynamics, the moving average model will be misspecified.

More concretely, consider the artificially simple situation where the researcher believes the data is generated according to an MA(2) model, equation (4), but the actual DGP for $y$ evolves according to the stochastic volatility model

$$y_t = \exp(h_t/2)u_t$$
$$h_t = \omega + \rho h_{t-1} + v_t \sigma_v$$

$|\rho| < 1$, $0 < \sigma_v < 1$, $u_t$ and $v_t$ and both iid standard Gaussian. In this case, if one takes $\eta(y) = (\gamma_0(y), \gamma_1(y), \gamma_2(y))^\top$ it follows that, under the DGP in (6),

$$\eta(y) \rightarrow_P b_0 = \left(\frac{\sigma^2_v}{1-\rho^2}, 0, 0\right)^\top.$$

For $d\{\cdot, \cdot\}$ the Euclidean norm we then have

$$\theta^* = \arg \inf_{\theta \in \Theta} d\{b_0, b(\theta)\} = (0, 0)^\top \text{ and } \epsilon^* = \sqrt{\left(\frac{\sigma^2_v}{1-\rho^2} - 1\right)^2}.$$  

### 4.3.2 Monte Carlo

We are interested in comparing the behavior of ABC-AR, ABC-Reg and ABC-Reg-New when the true model generating $y$ is actually a stochastic volatility model, as above, but the model used for simulating pseudo-data in ABC is an MA(2) model. We carry out this comparison across two simulation designs: one, $(\omega, \rho, \sigma_v)^\top = (-.736, .90, \sqrt{.363})^\top$ and two, $(\omega, \rho, \sigma_v)^\top = (-.147, .98, \sqrt{.0614})^\top$.

These particular values are related to the unconditional coefficient of variation $\kappa$ for the unobserved level of volatility $h_t$ in the observed data, with

$$\kappa^2 = \frac{\Var(h_t)}{(E[h_t])^2} = \exp\left(\frac{\sigma^2_v}{1-\rho^2}\right) - 1.$$  

In the first design, i.e., $(\omega, \rho, \sigma_v)^\top = (-.736, .90, \sqrt{.363})^\top$, we have $\kappa^2 = 1$, which roughly represents the behavior exhibited by lower-frequency financial returns (say, weekly or monthly returns); for the second design, i.e., $(\omega, \rho, \sigma_v)^\top = (-.147, .98, \sqrt{.0614})^\top$, we have $\kappa^2 = .1$, which roughly corresponds to higher-frequency financial returns (say, daily returns).

Across the two different designs, we generate $n = 1000$ observations for $y$ and consider one-hundred Monte Carlo replications. Across the replications we apply ABC-AR, with $d\{\cdot, \cdot\} = \| \cdot \|$, ABC-Reg and ABC-Reg-New to estimate the parameters of the MA(2) model. Given the theoretical results deduced in Sections two and three, it should be the case that the ABC-AR and ABC-Reg-New approaches gives estimators close to the pseudo-true value $(\theta_1^*, \theta_2^*)^\top = (0, 0)^\top$,
while ABC-Reg is likely to deliver point estimates with different behavior.

Figure 6 plots the resulting posterior means across the two designs, and across the ABC approaches. From Figure 6 we note that the behavior of ABC-Reg represents a substantial departure from the stable performance of ABC-AR and ABC-Reg-New. This is further evidence that standard post-processing ABC methods are more susceptible to misspecification than more basic ABC approaches.

![Design=1,theta1](image1)

![Design=1,theta2](image2)

Figure 6: Posterior mean plots for ABC-AR (AR), ABC-Reg (Reg) and ABC-Reg-New (Reg-New) across the Monte Carlo trials for designs one and two. For Design-1, observed data was simulated according to \((\omega, \rho, \sigma_v)^\top = (-.736, .90, \sqrt{.36})^\top\) and for Design-2 \((\omega, \rho, \sigma_v)^\top = (-.146, .98, \sqrt{.0614})^\top\).

In addition, Table 2 presents the Monte Carlo coverage and average credible set length across the three procedures, and for both Monte Carlo designs. Similar to the simple normal example, we see that the regression adjustment procedures give much smaller confidence intervals than those obtained from ABC-AR. Again, however, we emphasize that this result is not desirable since the smaller confidence sets leave researchers with a false sense of precision regarding the uncertainty of point estimators in misspecified models.
Table 2: Monte Carlo coverage (Cov.) and average credible set length (Length) for the MA(2) model. Design-1 corresponds to data simulated under $(\omega, \rho, \sigma_v) = (-.736, .9, \sqrt{.36})^\top$, while Design-2 takes $(\omega, \rho, \sigma_v) = (-.146, .98, \sqrt{.0614})^\top$. Calculations are based on the marginal ABC posteriors.

|           | Design-1 | Design-2 |
|-----------|----------|----------|
|           | AR Reg-New | Reg | AR Reg-New | Reg |
| $\theta_1$ Cov. | 98% | 95% | 95% | 100% | 100% | 49.6% |
| Length    | 0.97 | 0.38 | 0.38 | 0.38 | 0.12 | 0.13 |
| $\theta_2$ Cov. | 98% | 95% | 95% | 100% | 100% | 49.4% |
| Length    | 0.97 | 0.38 | 0.37 | 0.41 | 0.12 | 0.13 |

Figure 7 depicts the results of the proposed graphical check for detecting model misspecification in ABC associated with an arbitrarily chosen Monte Carlo trial. It is clear from this figure that the distances calculated in ABC display the distinct exponential decay that is expected when the model generating the pseudo-data in ABC is not correctly specified.

![Figure 7](image)

Figure 7: Graphical comparison of estimated acceptance probabilities against decreasing tolerance values for the MA(2) model. In this example $k_{\theta} = 2$ and so $\alpha_n$ should decay linearly in $\epsilon_n^2$ under correct specification.

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A Proofs

Theorem 1

Proof. This theorem is an adaptation of Frazier et al. (2018).

Let \( \delta_n \geq M_n(\epsilon_n - \epsilon^*) \geq 3M_nv_{0,n}^{-1} \); then \( P_0(\Omega_d) = 1 + o(1) \) for \( \Omega_d := \{ y : d\{ \eta(y), b_0 \} \leq \delta_n/2 \} \). Assume that \( y \in \Omega_d \). Consider the event

\[
A_d(\delta_n) := \{(z, \theta) : \{ d\{ \eta(z), \eta(y) \} \leq \epsilon_n \} \cap \{ d\{ b(\theta), b_0 \} \geq \epsilon^* + \delta_n \}.
\]
Note that, by definition $d\{b(\theta), b_0\} \geq \epsilon^*$, with $\epsilon^* > 0$. For all $(z, \theta) \in A_d(\delta_n)$ and if $y \in \Omega_d$,

$$
\delta_n < d\{b(\theta), b_0\} - \epsilon^* \leq d\{b(\theta), \eta(z)\} + d\{\eta(z), \eta(y)\} + d\{\eta(y), b_0\} - \epsilon^* \\
\leq d\{b(\theta), \eta(z)\} + \epsilon_n - \epsilon^* + \delta_n/2
$$

so that

$$
\delta_n \leq 4d\{b(\theta), \eta(z)\}.
$$

This implies in particular that

$$
\Pr(A_d(\delta_n)) = \int_{\{d\{b(\theta), b_0\} \geq \epsilon^* + \delta_n\}} P_\theta [d\{\eta(z), \eta(y)\} \leq \epsilon_n] \, d\Pi(\theta) \\
\leq \int P_\theta (d\{b(\theta), \eta(z)\} \geq \delta_n/4) \, d\Pi(\theta).
$$

(7)

In case (i) of polynomial tails,

$$
\Pr(A_d(\delta_n)) \leq (\nu_n \delta_n)^{-\kappa} \int_{\Theta} c(\theta) \, d\Pi(\theta) = o(1)
$$

(8)

as soon as $\nu_n \delta_n \to +\infty$, or in case (ii) of exponential tails

$$
\Pr(A_d(\delta_n)) \leq C e^{-c(\delta_n \nu_n)^r}.
$$

(9)

Moreover, we can bound from below

$$
\alpha_n = \int_{\Theta} P_\theta [d\{\eta(z), \eta(y)\} \leq \epsilon_n] \, d\Pi(\theta)
$$

Note that on $\{d\{\eta(z), b(\theta)\} \leq M \nu_n^{-1}/2\} \cap \Omega_d$

$$
d\{\eta(y), \eta(z)\} \leq d\{\eta(z), b(\theta)\} + d\{\eta(y), b_0\} + d\{b(\theta), b_0\} \leq \nu_0^{-1} + M \nu_n^{-1}/2 + d\{b(\theta), b_0\} \leq \epsilon_n
$$

as soon as $\epsilon^* \leq d\{b(\theta), b_0\} \leq \epsilon_n - \nu_0^{-1} + M \nu_n^{-1}/2$. Since $\epsilon_n - \epsilon^* \geq \nu_0^{-1} + M \nu_n^{-1}$, on $\Omega_d$,

$$
\int_{\Theta} P_\theta (d\{\eta(z), \eta(y)\} \leq \epsilon_n) \, d\Pi(\theta) \geq \int_{d\{b(\theta), b_0\} \leq (\epsilon_n - \epsilon^*)/4 \vee \nu_n^{-1}/2 \vee M/2} (1 - P_\theta (d\{\eta(z), b(\theta)\} \geq M \nu_n^{-1}/2)) \, d\Pi(\theta)
\geq \int_{d\{b(\theta), b_0\} \leq (\epsilon_n - \epsilon^*)/4 \vee \nu_n^{-1}/2 \vee M/2} \left(1 - \frac{c(\theta)2^{\kappa}}{M^{\kappa}}\right) \, d\Pi(\theta)
\gtrsim (\epsilon_n - \epsilon^*) D \vee \nu_n^{-D} \gtrsim (\epsilon_n - \epsilon^*) D
$$

in case (i) of [A1], under [A2]. If case (ii) of [A1] holds, under [A2], we have

$$
\int_{\Theta} P_\theta (d\{\eta(z), \eta(y)\} \leq \epsilon_n) \, d\Pi(\theta) \geq \int_{d\{b(\theta), b_0\} \leq (\epsilon_n - \epsilon^*)/4 \vee \nu_n^{-1}/2 \vee M/2} (1 - c(\theta) e^{-h_\theta(M/2)}) \, d\Pi(\theta)
\gtrsim (\epsilon_n - \epsilon^*) D
$$
Combining these two inequality with the upper bounds (8) or (9) leads to
\[
\Pi_\epsilon [d\{b(\theta), b_0\} \geq \epsilon^* + \delta_n |\eta(y)|] \lesssim (\epsilon_n - \epsilon^*)^{-D}(v_n \delta_n)^{-\kappa},
\]
in case (i) and
\[
\Pi_\epsilon [d\{b(\theta), b_0\} \geq \epsilon^* + \delta_n |\eta(y)|] \lesssim (\epsilon_n - \epsilon^*)^{-D} e^{-c(\delta_n v_n)^\tau},
\]
in case (ii). These are of order \(o(1)\) if
\[
\begin{align*}
\delta_n &\geq M_n v_n^{-1}(\epsilon_n - \epsilon^*)^{-D/\kappa} \quad \text{in case (i)} \\
\delta_n &\geq M_n v_n^{-1}|\log(\epsilon_n - \epsilon^*)|^{1/\tau} \quad \text{in case (ii)}
\end{align*}
\]
\(\square\)

**Corollary 1**

**Proof.** Define \(Q(\theta) = |d\{b(\theta), b_0\} - d\{b(\theta^*), b_0\}|\). From the continuity of \(\theta \mapsto b(\theta)\) and the definition of \(\theta^*\), for any \(\delta > 0\) there exists a \(\gamma(\delta) > 0\) such that
\[
\inf_{\theta, d(\theta, \theta^*) > \delta} Q(\theta) \geq \gamma(\delta) > 0.
\]
Then,
\[
\Pi_\epsilon[d\{\theta, \theta^*\} > \delta |\eta(y)|] \leq \Pi_\epsilon[|Q(\theta) - Q(\theta^*)| > \gamma(\delta) |\eta(y)|] = \Pi_\epsilon[|d\{b(\theta), b_0\} - d\{b(\theta^*), b_0\}| > \gamma(\delta) |\eta(y)|] = \Pi_\epsilon[d\{b(\theta), b_0\} > \epsilon^* + \gamma(\delta) |\eta(y)|].
\]
The result follows if \(\Pi_\epsilon[d_2\{b(\theta), b_0\} > \epsilon^* + \gamma(\delta) |\eta(y)|] = o_P(1)\). For \(\delta_n > 0\) and \(\delta_n = o(1)\) as defined in Theorem 1, by the conclusion of Theorem 1, the result follows once \(\gamma(\delta) \geq \delta_n\). \(\square\)

**Proof of Theorem 2**

**Proof.** For the sake of simplicity and without loss of generality we write \(v_n = \sqrt{n}, \hat{Z}_n = \sqrt{n}(\eta(z) - b(\theta))\) and \(\tilde{Z}_y = \sqrt{n}(\eta(y) - b_0)\). Denote by \(B_n(K) = \{||\theta - \theta^*|| \leq K\}\). Throughout the proof \(C\) denotes a generic constant which may vary from line to line. We have for all \(\theta\)
\[
P_\theta \left(\|\eta(z) - \eta(y)\|^2 \leq \epsilon_n^2\right) = P_\theta \left(\|\hat{Z}_n - \tilde{Z}_y + \sqrt{n}(b(\theta) - b_0)\|^2 \leq n\epsilon_n^2\right)
\]
\[
= P_\theta \left(\|\hat{Z}_n\|^2 + 2 < Z_n, \sqrt{n}(b(\theta) - b_0) - \tilde{Z}_y > \leq n[\epsilon_n^2 - \|b(\theta) - b_0 - \tilde{Z}_y/\sqrt{n}\|^2]\right)
\]
\[
= P_\theta \left(< \hat{Z}_n, b(\theta) - b_0 > \leq \frac{\sqrt{n}\epsilon_n^2 - \|b(\theta) - b_0 - \tilde{Z}_y/\sqrt{n}\|^2}{2} - \frac{||\hat{Z}_n\|^2 - 2 < \hat{Z}_n, \tilde{Z}_y >}{2\sqrt{n}}\right)
\]
Now on $\Omega_n = \{|\tilde{Z}_y| \leq M_n/2\}$ with $M_n$ a sequence going to infinity arbitrarily slowly and such that $M_n = o(n^{1/4})$,
\[
\sqrt{n}||b(\theta) - b_0 - \tilde{Z}_y/\sqrt{n}||^2 = \sqrt{n}||b(\theta^*) - b_0||^2 + \sqrt{n}(\theta - \theta^*)^T H^*(\theta - \theta^*)^T \geq 2 < b(\theta^*) - b_0, \tilde{Z}_y > + O(M_n^2/\sqrt{n}) + O(\sqrt{n}||\theta - \theta^*||^3 + ||\theta - \theta^*||M_n)
\]
where $H^*$ is the second derivative of $\theta \mapsto ||b(\theta) - b_0||^2$ at $\theta^*$, noting that the first derivative is equal to 0 at $\theta^*$. Let $\epsilon^* = ||b(\theta^*) - b_0||$, $\epsilon' = (b(\theta^*) - b_0)$, and $\epsilon > 0$. If $||\theta - \theta^*|| \leq \epsilon$ and on the event $\Omega_n = \{|Z_y| \leq M_n\}$ where $M_n^2 = o(\sqrt{n})$,
\[
P_\theta(\|\eta(z) - \eta(y)\|^2 \leq \epsilon_n^2) 
\leq P_\theta(< \tilde{Z}_n, e > \leq \sqrt{n}(\epsilon_n^2 - \epsilon^2)^2 - (1 + C\epsilon)(\theta - \theta^*)^T H^*(\theta - \theta^*/2) + < \tilde{Z}_y, e' > + C\epsilon + ||\theta - \theta^*||M_n) + P_\theta(\|Z_n\|^2 > \epsilon_n^2 / 2 \sqrt{n} / 4)
\]
\[
\geq P_\theta(< \tilde{Z}_n, e > \leq \sqrt{n}(\epsilon_n^2 - \epsilon^2)^2 - (1 - C\epsilon)(\theta - \theta^*)^T H^*(\theta - \theta^*/2) + < \tilde{Z}_y, e > - C\epsilon - ||\theta - \theta^*||M_n) - P_\theta(\|\tilde{Z}_n\|^2 > \epsilon_n^2 / 2 \sqrt{n} / 4)
\]
(10)

Consider the case where $\sqrt{n}(\epsilon_n^2 - \epsilon^2) \rightarrow 2c \in \mathbb{R}$. We split $\Theta$ into $\{|||\theta - \theta^*||^2 \sqrt{n} \leq M\}$, $\{\epsilon > ||\theta - \theta^*|| > M/n^{1/4}\}$ and $\{\epsilon \leq ||\theta - \theta^*||\}$, where $\epsilon$ is arbitrarily small.

First if $||\theta - \theta^*||^2 \sqrt{n} \leq M$,
\[
\sqrt{n}(\epsilon_n^2 - \epsilon^2)^2 - (1 - C\epsilon)(\theta - \theta^*)^T H^*(\theta - \theta^*/2) + < \tilde{Z}_y, e' > + M_n^2 / \sqrt{n} + ||\theta - \theta^*||M_n 
\leq c+ < \tilde{Z}_y, e' > - (1 - C\epsilon)\sqrt{n}(\theta - \theta^*)^T H^*(\theta - \theta^*) / 4 + C\epsilon
\]
and
\[
\sqrt{n}(\epsilon_n^2 - \epsilon^2)^2 - (1 + C\epsilon)(\theta - \theta^*)^T H^*(\theta - \theta^*/2) + M^2 / \sqrt{n} + ||\theta - \theta^*||M_n 
\geq c+ < \tilde{Z}_y, e > - (1 + C\epsilon)\sqrt{n}(\theta - \theta^*)^T H^*(\theta - \theta^*) / 4 - C\epsilon
\]
Moreover, using assumption [A5],
\[
< \tilde{Z}_n, e' > = \sqrt{n} < \Sigma_n(\theta)^{-1}Z_n, e' > = < Z_n, \sqrt{n}\Sigma_n(\theta)^{-1}e' > = < Z_n, A(\theta)e' > + o(||Z_n||).
\]
We then have with $e' = c+ < \tilde{Z}_y, e' >$, $x = n^{1/4}(1 - \epsilon)^{1/2}(\theta - \theta^*)$ and $||\theta - \theta^*|| \leq M/n^{1/4} \leq u_0$ if
\( n \) is large enough

\[
P_\theta \left( \| \eta(z) - \eta(y) \|^2 \leq \epsilon_n^2 \right) \leq P_\theta \left( < Z_n, A(\theta^*)e' > \leq c' - \frac{x^T H^* x}{4} + M \epsilon \right) + \frac{c_0 \epsilon^{-\kappa}}{n^{\kappa/4}}
\]

\[
\leq \Phi \left( \frac{c' + M \epsilon}{\| A(\theta^*)e' \|} - \frac{x^T H^* x}{4\| A(\theta^*)e' \|} \right) + \frac{c_0 \epsilon^{-\kappa}}{n^{\kappa/4}}
\]

\[
+ \sup_{\| \theta' - \theta^* \| \leq u} \left| P_{\theta'} \left( < Z_n, A(\theta^*)e' > \leq c' - \frac{x^T H^* x}{4} + M \epsilon \right) - \Phi \left( \frac{c' + M \epsilon}{\| A(\theta^*)e' \|} - \frac{x^T H^* x}{4\| A(\theta^*)e' \|} \right) \right|
\]

\[
\leq \Phi \left( \frac{c' + M \epsilon}{\| A(\theta^*)e' \|} - \frac{x^T H^* x}{4\| A(\theta^*)e' \|} \right) + o(1) + \frac{c_0 \epsilon^{-\kappa}}{n^{\kappa/4}}.
\]

Similarly with \( y = n^{1/4}(1 + \epsilon)^{1/2}(\theta - \theta^*) \)

\[
P_\theta \left( \| \eta(z) - \eta(y) \|^2 \leq \epsilon_n^2 \right) \]

\[
\geq \Phi \left( \frac{c' - M \epsilon}{\| A(\theta^*)e' \|} - \frac{y^T H^* y}{4\| A(\theta^*)e' \|} \right) - \frac{c_0 \epsilon^{-\kappa}}{n^{\kappa/4}}
\]

\[- \sup_{\| \theta' - \theta^* \| \leq u} \left| P_{\theta'} \left( < Z_n, A(\theta^*)e' > \leq c' - \frac{y^T H^* y}{4} - M \epsilon \right) - \Phi \left( \frac{c' - M \epsilon}{\| A(\theta^*)e' \|} - \frac{y^T H^* y}{4\| A(\theta^*)e' \|} \right) \right|
\]

\[
\geq \Phi \left( \frac{c' - M \epsilon}{\| A(\theta^*)e' \|} - \frac{y^T H^* y}{4\| A(\theta^*)e' \|} \right) + o(1) - \frac{c_0 \epsilon^{-\kappa}}{n^{\kappa/4}}
\]

Moreover for all \( t \in \mathbb{R} \), writing \( x(\theta) \) to emphasize its dependence in \( \theta \),

\[
\Delta_1 = \int_{B_u(M/n^{1/4})} \sup_{\| \theta' - \theta^* \| \leq u_0} \left| P_{\theta'} \left( < Z_n, A(\theta^*)e' > \leq t - \frac{x(\theta)^T H^* x(\theta)}{4} \right) - \Phi \left( \frac{t}{\| A(\theta^*)e' \|} - \frac{x(\theta)^T H^* x(\theta)}{4\| A(\theta^*)e' \|} \right) \right| d\theta
\]

\[
= n^{-k_\theta/4} \int_{\| x \| \leq M} \sup_{\| \theta' - \theta^* \| \leq u_0} \left| P_{\theta'} \left( < Z_n, A(\theta^*)e' > \leq t - \frac{x^T H^* x}{4} \right) - \Phi \left( \frac{t}{\| A(\theta^*)e' \|} - \frac{x^T H^* x}{4\| A(\theta^*)e' \|} \right) \right| dx
\]

\[
= n^{-k_\theta/4} M o(1)
\]

where the last inequality follows from the dominated convergence theorem. We then have

\[
\int_{B_u(M/n^{1/4})} P_\theta \left( \| \eta(z) - \eta(y) \|^2 \leq \epsilon_n^2 \right) \pi(\theta) d\theta
\]

\[
\leq \pi(\theta^*)(1 + o(1)) n^{-k_\theta/4} \int_{\| x \| \leq M} \Phi \left( \frac{c' + M \epsilon}{\| A(\theta^*)e' \|} - \frac{x^T H^* x}{4\| A(\theta^*)e' \|} \right) dx + o(n^{-k_\theta/4} M)
\]

(11)
Similarly
\[
\int \Theta P_\theta \left( \|\eta(z) - \eta(y)\|^2 \leq \epsilon_n^2 \right) \pi(\theta) d\theta \geq \int_{\|\theta - \theta^*\| \leq n^{-1/4}M} \Phi \left( \frac{c - M\epsilon}{\|A(\theta^*)e'\|} - \frac{x^TH^*x}{4\|A(\theta^*)e'\|} \right) dx (1 + o_{P_0}(1)).
\]

Also if \( \epsilon > \|\theta - \theta^*\| > M/n^{1/4} \), since there exists \( a > 0 \) such that \( z^THz \geq a\|z\|^2 \) for all \( z \), if \( M \) is large enough,
\[
P_\theta \left( \|\eta(z) - \eta(y)\|^2 \leq \epsilon_n^2 \right) \leq P_\theta \left( \langle Z_n, A(\theta^*)e' \rangle \leq -\frac{aM^2}{8} \right) \leq P_\theta \left( \|Z_n\| > \frac{aM^2}{8\|A(\theta^*)e'\|} \right) \lesssim M^{-2\kappa}
\]

Now let \( j \geq 0 \) and set \( M_j = 2^j M \). On \( M_jn^{-1/4} \leq \|\theta - \theta^*\| \leq M_{j+1}n^{-1/4} \)
\[
P_\theta \left( \|\eta(z) - \eta(y)\|^2 \leq \epsilon_n^2 \right) \leq P_\theta \left( \langle Z_n, A(\theta^*)e' \rangle \leq -\frac{aM_j^2}{8} \right) \leq P_\theta \left( \|Z_n\| > \frac{aM_j^2}{8\|A(\theta^*)e'\|} \right) \lesssim M^{-2\kappa}_j
\]
so that
\[
\int_{M_jn^{-1/4} \leq \|\theta - \theta^*\| \leq \epsilon} P_\theta \left( \|\eta(z) - \eta(y)\|^2 \leq \epsilon_n^2 \right) \pi(\theta) d\theta \lesssim n^{-k_0/4} \sum_{j=0}^{J_n} M_j^{-2\kappa} (M_{j+1} - M_j) \lesssim n^{-k_0/4} \sum_{j=0}^{J_n} M_j^{-2\kappa+1} \lesssim n^{-k_0/4} M^{-2\kappa+1}.
\]

Finally if \( \|\theta - \theta^*\| > \epsilon, \|b(\theta) - b_0 - Z_n/\sqrt{n}\|^2 - (\epsilon^*)^2 \geq C\epsilon \) on \( \Omega_n \) and when \( n \) is large enough \( \sqrt{n}(\epsilon_n^2 - (\epsilon^*)^2) \leq c + C\epsilon \) so that
\[
P_\theta \left( \|\eta(z) - \eta(y)\|^2 \leq \epsilon_n^2 \right) \leq P_\theta \left( \langle \tilde{Z}_n, b(\theta) - b_0 \rangle \leq -\sqrt{n}C\epsilon + c' + C\epsilon \right) + O(n^{-\kappa/4}) \leq P_\theta(\|Z_n\| > Cn^{1/2}\|b(\theta) - b_0\|^{-1}/2) + O(n^{-\kappa/4}) \leq c(\theta)C^{-\kappa}n^{-\kappa/2}\|b(\theta) - b_0\|^\kappa + O(n^{-\kappa/4}).
\]
Therefore
\[
\int_{\|\theta - \theta^*\| \geq \epsilon} P_\theta (\|\eta(z) - \eta(y)\|^2 \leq \epsilon_n^2) \pi(\theta)d\theta \leq C^{-\kappa} n^{-\kappa/2} \int_{\|\theta - \theta^*\| \geq \epsilon} c(\theta)\|b(\theta) - b_0\|^\kappa \pi(\theta)d\theta + C n^{-\kappa/4} \int_{\|\theta - \theta^*\| \geq \epsilon} c(\theta)\pi(\theta)d\theta
\]
(14)

Finally combining (11), (13), (14) and (12) we obtain that if \( \kappa > k_\theta \)
\[
\int_{\theta} P_\theta (\|\eta(z) - \eta(y)\|^2 \leq \epsilon_n^2) \pi(\theta)d\theta = n^{-k_\theta/4} \int_{\mathbb{R}} \Phi \left( \frac{c'}{\|A(\theta)^*c\|} - \frac{x^\top H^*x}{4\|A(\theta)^*c\|} \right) dx + o(n^{-k_\theta/4})
\]
and for all \( x = n^{1/4}(\theta - \theta^*) \in \mathbb{R}^{k_\theta} \) fixed, writing \( \pi_{n,\epsilon}(\cdot) \) the density of \( \Pi_{z_n, \epsilon} \), the ABC posterior distribution of \( z_n(\theta - \theta^*) \),
\[
\pi_{n,\epsilon}(x) = \frac{\Phi \left( \frac{c'}{\|A(\theta)^*c\|} - \frac{x^\top H^*x}{4\|A(\theta)^*c\|} \right)}{\int_{\mathbb{R}} \Phi \left( \frac{c'}{\|A(\theta)^*c\|} - \frac{x^\top H^*x}{4\|A(\theta)^*c\|} \right) dx} + o(1) \quad \text{with} \quad o(1) := q_c(x) + o(1)
\]
so that
\[
\|\pi_{n,\epsilon} - q_c\|_1 = o(1).
\]

We now study the case where \( \sqrt{\frac{\epsilon^2}{n}} : = \sqrt{n}u^2_n \to +\infty \) with \( u_n = o(1) \) and we show that the limiting distribution is uniform. Using (10), we have that if \( B_{0,n} = \{ (\theta - \theta^*)^\top H^*(\theta - \theta^*) \leq 2u^2_n - 4M_n/\sqrt{n} \} \), with \( M_n < u^2_n/\sqrt{n} \) going to infinity
\[
P_\theta (\|\eta(z) - \eta(y)\| \leq \epsilon_n) \leq 1
\]
\[
\geq P_\theta \left( < \tilde{Z}_n, c' > \leq 2M_n + < \tilde{Z}_y, c' > - \epsilon - \|\theta - \theta^*\|M_n \right)
\]
\[
\geq P_\theta \left( < \tilde{Z}_n, c' > \leq M_n/2 \right) \geq 1 - c_1 \frac{1}{M_n^{-\kappa}}
\]
for some \( c_1 > 0 \) on the event \( \{ | < \tilde{Z}_y, c' > | \leq M_n/2 \} \), which has probability going to 1.

This implies in particular that
\[
\int_{B_{0,n}} P_\theta (\|\eta(z) - \eta(y)\| \leq \epsilon_n)\pi(\theta)d\theta \leq \pi(\theta^*)(1 + o(1))\text{Vol}(B_{0,n})
\]
(15)
\[
\geq \pi(\theta^*)(1 + o(1))\text{Vol}(B_{0,n})(1 - \frac{c_1}{M_n^{-\kappa}})
\]

Also
\[
\text{Vol}(B_{0,n}) \asymp u^{k_\theta}_n
\]
(16)

Let \( K_n \geq 4 \), if \( K_n u^2_n \geq (\theta - \theta^*)^\top H^*(\theta - \theta^*) > 2u^2_n(1 - C\epsilon)^{-1} + 4M_n/\sqrt{n} \), then there exists
\( C' > 0 \) such that

\[
\frac{\sqrt{n}[u_n^2 - (1 - C\epsilon)(\theta - \theta^*)^2H^*(\theta - \theta^*)/2]}{2} + <\tilde{Z}_y, e'> > -\epsilon - \|\theta - \theta^*\|M_n
\]

\[
\leq -2M + <\tilde{Z}_y, e'> > -\epsilon - C'\epsilon^2M_n \leq -M_n
\]

on the event \(| <\tilde{Z}_y, e'> | \leq M_n/2\). Therefore writing \(B_{1,n} = \{K_nu_n^2 \geq (\theta - \theta^*)^2H^*(\theta - \theta^*) > 2u_n^2(1 - C\epsilon)^{-1} + 4M_n/\sqrt{n} \}\)

\[
\int_{B_{1,n}} P_\theta(\|\eta(z) - \eta(y)\| \leq \epsilon_n)\pi(\theta)d\theta \lesssim M_n^{-\kappa}\text{Vol}(B_{1,n}) \lesssim M_n^{-\kappa}K_{n}^{k_{\theta}/2}\text{Vol}(B_{0,n})
\]

(17)

Moreover

\[
\text{Vol}(\{(\theta - \theta^*)^2H^*(\theta - \theta^*) \leq 2u_n^2(1 - C\epsilon)^{-1} + 4M_n/\sqrt{n}\}) - \text{Vol}(B_{0,n}) \lesssim c\text{Vol}(B_{0,n}).
\]

(18)

If \(K_nu_n^2 \leq (\theta - \theta^*)^2H^*(\theta - \theta^*) \leq \epsilon^2\), then

\[
\frac{\sqrt{n}[u_n^2 - (1 - C\epsilon)(\theta - \theta^*)^2H^*(\theta - \theta^*)/2]}{2} + <\tilde{Z}_y, e'> > -\epsilon - \|\theta - \theta^*\|M_n
\]

\[
\leq \frac{-\sqrt{n}(\theta - \theta^*)^2H^*(\theta - \theta^*)}{8}
\]

when \(n\) is large enough and there exists \(b > 0\) such that

\[
\int_{B_{1,n}} 1_{(\theta - \theta^*)^2H^*(\theta - \theta^*) \leq \epsilon^2}P_\theta(\|\eta(z) - \eta(y)\| \leq \epsilon_n)\pi(\theta)d\theta \lesssim n^{-\kappa/2}\int_{B_n(A_{\epsilon^2})} 1_{\|\theta - \theta^*\| \geq b\sqrt{\kappa_nu_n}} \|\theta - \theta^*\|^{-2\kappa}d\theta
\]

\[
\lesssim n^{-\kappa/2}\int_{b\sqrt{\kappa_nu_n}} r^{k_{\theta} - 2\kappa - 1}dr
\]

(19)

Since \(k_{\theta} < 2\kappa\) then the above term is of order

\[
K_n^{(k_{\theta} - 2\kappa)/2}(\sqrt{n}u_n^2)^{-\kappa/2}u_n^{k_{\theta}} \approx K_n^{(k_{\theta} - 2\kappa)/2}(\sqrt{n}u_n^2)^{-\kappa/2}\text{Vol}(B_{0,n}) = o(\text{Vol}(B_{0,n}))
\]

Finally if \(\|\theta - \theta^*\| \geq \epsilon\), similarly to the case where \(\sqrt{n}u_n^2 \to c \in \mathbb{R}\), we obtain (14) and this term is \(o(\text{Vol}(B_{0,n}))\) as soon as \(n^{-\kappa/4} = o(u_n^{k_{\theta}})\). Since \(n^{-1/4} = o(u_n)\) the latter is true as soon as \(\kappa \geq k_{\theta}\). Combining (15), (17), (19) , (18) and (14), we obtain that

\[
\left\|\int_{\Theta} P_\theta(\|\eta(Z) - \eta(y)\| \leq \epsilon_n)\pi(\theta)d\theta - \pi(\theta^*)\text{Vol}(\tilde{B}_{0,n})\right\| = o_p(1)
\]

(20)

where \(\tilde{B}_{0,n} = \{(\theta - \theta^*)^2H^*(\theta - \theta^*) \leq 2u_n^2\}\). Let \(x = u_n^{-1}(\theta - \theta^*)\) be fixed and \(x^\top H^*x < 2\), then for \(n\) large enough \(x^\top H^*x \leq 2 - 4M_n^2/(\sqrt{n}u_n^2)\) and using

\[
\pi_{u_n^{-1},x}(x) = \pi(x + u_nx|y)u_n^{k_{\theta}}
\]

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\[ \pi_{u_n^{-1}, \epsilon}(x) = 1 + o_p(1). \]

If \( x^\top H^* x > 2 \), then if \( \epsilon > 0 \) is small enough and \( n \) is large enough \( x^\top H^* x \geq 2(1 - C\epsilon)^{-1} + 4M_n^2/(\sqrt{n}u_n^2) \) and

\[ \pi_{u_n^{-1}, \epsilon}(x) = o_p(1). \]

This implies that the ABC posterior distribution of \( u_n^{-1}(\theta - \theta^*) \) converges to the Uniform distribution over the ellipsoid \( \{x^\top H^* x \leq 2\} \) in total variation. \( \square \)

**Proposition 1**

*Proof.* To prove Proposition 1, we prove that the approximate likelihood

\[ P_{\theta}\left( \left\| Z - Z_y + \sqrt{n}(b(\theta) - b_0)\right\|^2 \leq \epsilon_n^2 \right) \]

is highly peaked around \( \theta \neq 0 \), and, as such, concentration around \( \theta^* = 0 \) cannot result.

As in the proof of Theorem 2, writing \( Z_n = Z = (Z_1, Z_2)^\top \), we can define \( W = Z/\|Z\| \) and \( R = \|Z\|/v_\theta \) and we have that \( W \) and \( R \) are independent and that their distribution does not depend on \( \theta \). In particular \( R^2 \sim \chi^2(2) \).

Now, set \( h = b(\theta) - b_0 - Z_y/\sqrt{n} \), so that

\[ \left\| Z - Z_y + \sqrt{n}(b(\theta) - b_0)\right\|^2 - n\epsilon_n^2 = v_\theta^2 R^2 + 2\sqrt{n}Rv_\theta < W, h > + n(\|h\|^2 - \epsilon_n^2) \leq 0 \]

if and only if

\[ \Delta(W) = v_\theta^2 n(< W, h >^2 - \|h\|^2 + \epsilon_n^2) = v_\theta^2 n\tilde{\Delta}(W) \geq 0, \quad R \in (r_1(W), r_2(W)) \cap \mathbb{R}_+, \]

where

\[ r_1(W) = \sqrt{\frac{n}{v_\theta}} \left[ -< W, h > - \sqrt{\tilde{\Delta}(W)} \right], \quad r_2(W) = \sqrt{\frac{n}{v_\theta}} \left[ -< W, h > + \sqrt{\tilde{\Delta}(W)} \right]. \]

Note that \( \tilde{\Delta}(W) \leq \epsilon_n^2 \) so that if \( \tilde{\Delta}(W) \geq 0 \) then \( |< W, h >| = \|h\|(1 + O(\epsilon_n)) \), given that \( \|h\| \asymp 1 \) on the event \( \|Z_y\| \leq M \) for some arbitrarily large \( M \). Therefore if \( < -W, h > \leq 0 \), then \( < -W, h > \geq -\|h\| \) and there is no solution for \( R \) in (21). Hence (21) holds if and only if \( < -W, h > \geq 0 \), \( \tilde{\Delta}(W) \geq 0 \) and \( R \in (r_1(W), r_2(W)) \).

By symmetry we can set \( W = -W \) and, on the set \( < W, h > \geq 0 \), using the fact that \( R^2 \sim \chi^2(2) \),

\[ P_{\theta}\left( \left\| Z - Z_y + \sqrt{n}(b(\theta) - b_0)\right\|^2 \leq n\epsilon_n^2|W'\right) = e^{-r_1(W)^2/2}(1 - \frac{n\tilde{\Delta}(W)}{v_\theta}) \]

\[ r_1(W) \in \left\{ \sqrt{n}\|h\|/v_\theta \{1 - 2\epsilon_n\}, \sqrt{n}\|h\|/v_\theta \} \right\}. \]

To derive an approximation of \( P_{\theta}\left( \left\| Z - Z_y + \sqrt{n}(b(\theta) - b_0)\right\|^2 \leq n\epsilon_n^2 \right) \) we study more precisely \( r_1(W) \). For the sake of simplicity we assume that \( \sqrt{n}\epsilon_n = o(1) \), since the case where \( \sqrt{n}\epsilon_n = O(1) \)

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can be treated similarly. Then
\[
P_\theta \left( \| Z - Z_y + \sqrt{n}(b(\theta) - b_0) \|^2 \leq n\epsilon_n^2 \right) \leq e^{-\frac{n\|h\|^2}{n^2\theta}(1-2\epsilon_n)^2} P_\theta \left( \tilde{\Delta}(W) \geq 0 \right) \\
\geq e^{-\frac{n\|h\|^2}{n^2\theta}} P_\theta \left( \tilde{\Delta}(W) \geq 0 \right)
\]

Consider \( \theta = r\tilde{b}_0 \) with \( r \in [-1, 1] \) so that \( r = 0 \) corresponds to \( \theta = \theta^* \equiv 0 \), then \( h = \tilde{b}_0(r - 1, r + 1)^T + O_{p_\theta}(1/\sqrt{n}) \) and
\[
P_\theta \left( \tilde{\Delta}(W) \geq 0 \right) \leq 2P_\theta \left( 0 \leq W - h/\|h\|, h > \epsilon_n^2/2(\|h\|^2)(1 + \epsilon_n^2/\|h\|^2) \right) \\
\geq 2P_\theta \left( 0 \leq W - h/\|h\|, h > \epsilon_n^2/2(\|h\|^2) \right) \\
= \frac{\epsilon_n^2}{(1 + r^2)b_0^2} g \left\{ \frac{(1 - r)}{\sqrt{2(1 + r^2)}} \right\} \left\{ 1 + O(\epsilon_n^2 \vee 1/\sqrt{n}) \right\}
\]
where \( g(\cdot) \) is the density of \( W_1 \), with \( W = (W_1, W_2) \). We thus obtain that for \( n \) large enough
\[
P_\theta \left( \| Z - Z_y + \sqrt{n}(b(\theta) - b_0) \|^2 \leq n\epsilon_n^2 \right) \leq \frac{\epsilon_n^2}{(1 + r^2)b_0^2} e^{-\frac{n\|h\|^2}{n^2\theta}(1-3\epsilon_n)^2} g \left\{ \frac{\tilde{b}_0(1 - r)}{\sqrt{2(1 + r^2)}} \right\} \left\{ 1 + O(\epsilon_n^2 \vee 1/\sqrt{n}) \right\} \\
\geq \frac{\epsilon_n^2}{(1 + r^2)b_0^2} e^{-\frac{n\|h\|^2}{n^2\theta}(1+\epsilon_n)^2} g \left\{ \frac{\tilde{b}_0(1 - r)}{\sqrt{2(1 + r^2)}} \right\} \left\{ 1 + O(\epsilon_n^2 \vee 1/\sqrt{n}) \right\}
\]
Take \( v_{\tilde{b}_0} = \tilde{b}_0v(r) \) such that \( (1 + 1/4)/v(1/2)^2 \leq 1/(2v(0)^2) \), then for \( \delta > 0 \) small enough,
\[
\Pi_c \left\{ |\theta - \theta^*| \leq \delta |\eta(y)\right\} = o \left( \Pi_c \left\{ |\theta - b_0/2| \leq \delta |\eta(y)\right\} \right)
\]
since there exists \( c > 0 \) such that
\[
\int_{|\theta| \leq \delta} e^{-\frac{n\|h\|^2}{n^2\theta}(1+\epsilon_n)^2} \pi(\theta)d\theta \leq c \int_{|\theta - b_0/2| \leq \delta} e^{-\frac{n\|h\|^2}{n^2\theta}(1+\epsilon_n)^2} \pi(\theta)d\theta.
\]

\[ \square \]

**Corollary 2**

*Proof.* The proof is a consequence of Theorem 1 and the structure of \( \tilde{\theta} = \theta + \beta^T \{ \eta(z) - \eta(y) \} \), and \( \tilde{\theta}^* = \theta^* + \beta^T \{ b(\theta^*) - b_0 \} \). Therefore, we only sketch the idea here.

Take \( \delta_n \geq M_n(\epsilon_n - \epsilon^*) \geq M_n\epsilon_n^{-1} \). By assumption \( \epsilon^* > 0 \) and \( \|\beta_0\| > 0 \). Define \( \Omega_d = \{ y : \)
\[ \| \eta(y) - b_0 \| \leq \delta_n / u_0 \] for some \( u_0 \geq 2(1 + \| \beta_0 \|) \). By the result of Theorem 1 we have that
\[
\bar{\Pi}_e \left[ |\bar{\theta} - \hat{\theta}^*| > \delta_n \| \right] = \Pi_e \left[ \left\{ \theta : |\bar{\theta} - \hat{\theta}^*| > \delta_n \right\} \cap \left\{ \theta : |\theta - \theta^*| \leq \delta_n / u_0 \right\} \right] + o_{P_b}(1)
\]
\[
= \int_{|\theta - \theta^*| \leq \delta_n / u_0} \Pi \left( |\bar{\theta} - \hat{\theta}^*| \geq \delta_n \right) P_\theta (\| \eta(z) - \eta(y) \| \leq \epsilon_n) d\Pi(\theta) + o_{P_b}(1),
\]
where both equalities follow by posterior concentration of \( |\theta - \theta^*| \) at rate \( \delta_n \gg v_0^{-1} \).

Similar steps to that of Theorem 1 yield
\[
D_n = \int_{|\theta - \theta^*| \leq \delta_n / u_0} P_\theta (|\eta(z) - \eta(y)\| \leq \epsilon_n) d\Pi(\theta) \gtrsim \delta_n^D,
\]
under case (i) or case (ii) of [A1].

Define the event
\[
S(\delta_n) = \left\{ (z, \theta) : \{ \theta : |\bar{\theta} - \hat{\theta}^*| > \delta_n \} \cap \{ \theta : |\theta - \theta^*| \leq \delta_n / u_0 \} \cap \{ z : \| \eta(z) - \eta(y) \| \leq \epsilon_n \} \right\}
\]

Note that
\[
|\bar{\theta} - \hat{\theta}^*| = |\theta - \theta^*| + |\hat{\beta} - \beta^0|\|b(\theta) - b_0\| + |\hat{\beta} - \beta^0|\|b_0 - \eta(y)\| + |\hat{\beta} - \beta^0|\|\eta(z) - b(\theta)\|
\]
\[
+ |\beta_0|\|b_0 - \eta(y)\| + |\beta_0|\|\eta(z) - b(\theta)\| + |\beta_0|\|\delta_n / u_0 + o(\delta_n) + o(\delta_n) + (O(\delta_n) + \| \beta_0 \|)\|\eta(z) - b(\theta)\|
\]
where the last inequality follows from \( \| \hat{\beta} - \beta_0 \| = o_{P_b}(1) \) and concentration of \( |\theta - \theta^*| \) at rate \( \delta_n \gg v_0^{-1} \). Therefore, take \( u_0 \geq 2(1 + \| \beta_0 \|) \) and rearrange the above to obtain
\[
0 < \frac{\delta_n}{2( O(\delta_n) + \| \beta_0 \| )} < \| \eta(z) - b(\theta) \| + o(\delta_n).
\]

This then implies that
\[
\Pr \left[ S(\delta_n) \right] = \int_{\{ \theta : |\theta - \theta^*| \leq \delta_n / u_0 \}} \Pi \left[ |\bar{\theta} - \hat{\theta}^*| > \delta_n \right] P_\theta (|\eta(z) - \eta(y)\| \leq \epsilon_n) d\Pi(\theta)
\]
\[
= N_n \leq \int P_\theta (|\eta(z) - b(\theta)\| > c \cdot \delta_n) d\Pi(\theta)
\]
\[
\leq (v_n \delta_n)^{-\kappa} \text{ under case (i) of [A1]}
\]
\[
\leq \exp(-c v_n^\kappa \delta_n^\kappa) \text{ under case (ii) of [A1]}
\]

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The remainder of the proof now follows along the lines of Theorem 1.