On weakly $b$-continuous functions in bitopological spaces

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Abstract: In this article we introduce the notion of weakly $b$-continuous functions in bitopological spaces as a generalization of $b$-continuous functions. We prove several properties of these functions. AMS Classification No: 54A10; 54C10; 54C08; 54D15.

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Introduction

The concept of bitopological spaces $(X, \tau_1, \tau_2)$ was first introduced by Kelly (1963), where $X$ is a non-empty set and $\tau_1, \tau_2$ are topologies on $X$. The notion of $b$-open sets is due to Andrijevic (1996) plays a significant role in general topology. A subset $A$ of $(X, \tau)$ is called $b$-open, if $A \subset Int(Cl(A)) \cup Cl(Int(A))$ and called $b$-closed if $X - A$ is $b$-open. Sengul (2008) and Sengul (2009) defined the notion of almost $b$-continuous functions and weakly $b$-continuous functions in topological spaces. T. Noiri and N. Rajesh have investigated some properties of the concept of $b$-open sets and $b$-continuous functions in bitopological spaces. Recently Sarsak and Rajesh (2009), Banerjee (1987), Bose and Sinha (1981) and Tripathy and Sarma (2011, 2012) have done some works on bitopological spaces. Bitopological spaces has been istudied bt Bose (1981), Bose and Sinha (1982), Jelic (1992), Kariofillies (1986), Kelly (1963), Khedr et al. (1992), Noiri and Popa (2007), Popa and Noiri (2004) and others.

In this paper, we introduce the notion of weakly $b$-continuous functions in bitopological spaces and investigate their different properties.

Preliminaries

Throughout the present paper $(X, \tau)$ denotes a topological space and $(X, \tau_1, \tau_2)$ denotes a bitopological space on which no separation axioms are assumed. Let $(X, \tau_1, \tau_2)$ be a bitopological space and $A$ be a subset of $X$. The closure(resp. interior) of $A$ with respect to the topology $\tau_i (i = 1, 2)$ will be denoted by $i Cl(A)$ (resp. $i Int(A)$).

Now we list some known definitions and results those will be used throughout this article.

Definition 1. A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be

(i) $(i, j)$-$b$-open if $A \subset i Cl(j Cl(A)) \cup j Cl(i Cl(A)).$

(ii) $(i, j)$-regular open if $A = i Cl(j Cl(A)).$

(iii) $(i, j)$-regular closed if $A = i Cl(j Int(A)).$

The complement of $(i, j)$-$b$-open set is said to be $(i, j)$-$b$-closed.

Definition 2. Let $A$ be a subset of a bitopological space $(X, \tau_1, \tau_2)$. Then:

(i) The $(i, j)$-$b$-closure of $A$ denoted by $(i, j)$-$b$ $Cl(A)$, is defined by the intersection of all $(i, j)$-$b$-closed sets containing $A$.

(ii) The $(i, j)$-$b$-interior of $A$ denoted by $(i, j)$-$b$ $Int(A)$, is defined by the union of all $(i, j)$-$b$-open sets contained in $A$.

Lemma 2.1. Let $(X, \tau_1, \tau_2)$ be a bitopological space. Then:

(i) the arbitrary union of $(i, j)$-$b$-open sets is $(i, j)$-$b$-open.

(ii) the arbitrary intersection of $(i, j)$-$b$-closed sets is $(i, j)$-$b$-closed.

Lemma 2.2. Let $(X, \tau_1, \tau_2)$ be a bitopological space and $A$ be a subset of $X$.

(i) $(i, j)$-$b$ $Int(A)$ is $(i, j)$-$b$-open.
(ii) \((i, j)\)-weakly \(b\)-continuous if for every \(x \in X\) and each \(\tau_i\)-open set \(B\) containing \(x\), there exists a \(\tau_i\)-open set \(B\) such that \(x \in B \subseteq \tau_i\)-Cl(B) \(\subseteq A\).

**Lemma 3.** A bitopological space \((X, \tau_i, \tau_j)\) is said to be \((i, j)\)-regular if for each \(x \in X\) and each \(\tau_i\)-open set \(A\) containing \(x\), there exists a \(\tau_i\)-open set \(B\) such that \(x \in B \subseteq \tau_i\)-Cl(B) \(\subseteq A\).

**Definition 4.** Let \((X, \tau_i, \tau_j)\) be a bitopological space and \(A\) be a subset of \(X\). A point \(x\) of \(X\) is said to be in the \((i, j)\)-\(\theta\)-closure of \(A\), denoted by \((i, j)\)-Cl\(_\theta\)(A), if \(A \cap j\)-Cl(B) \(\neq \emptyset\) for every \(\tau_j\)-open set \(B\) containing \(x\), where \(i, j = 1, 2\) and \(i \neq j\).

A subset \(A\) of \(X\) is said to be \((i, j)\)-\(\theta\)-closed if \(A = (i, j)\)-Cl\(_\theta\)(A). A subset \(A\) of \(X\) is said to be \((i, j)\)-\(\theta\)-open if \(X - A\) is \((i, j)\)-\(\theta\)-closed. The \((i, j)\)-\(\theta\)-interior of \(A\), denoted by \((i, j)\)-Int\(_\theta\)(A), is defined as the union of all \((i, j)\)-\(\theta\)-open sets contained in \(A\). Hence \(x \in (i, j)\)-Int\(_\theta\)(A) if and only if there exists a \(\tau_i\)-open set \(B\) containing \(x\) such that \(x \in B \subseteq (i, j)\)-Cl(B) \(\subseteq A\).

**Lemma 2.5.** For a subset \(A\) of a bitopological space \((X, \tau_i, \tau_j)\), the following properties hold:

(i) \(X - (i, j)\)-Int\(_\theta\)(A) = \((i, j)\)-Cl\(_\theta\)(X - A).
(ii) \(X - (i, j)\)-Cl\(_\theta\)(A) = \((i, j)\)-Int\(_\theta\)(X - A).

**Lemma 2.6.** Let \((X, \tau_i, \tau_j)\) be a bitopological space. If \(B\) is a \(\tau_i\)-open set of \(X\), then \((i, j)\)-Cl\(_\theta\)(B) = \(i\)-Cl(B).

**Definition 5.** A function \(f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)\) is said to be

1. \((i, j)\)-semi-continuous if \(f^{-1}(A)\) is \((i, j)\)-semi-open in \(X\) for each \(\sigma_i\)-open set \(A\) of \(Y\).
2. \((i, j)\)-\(\theta\)-continuous if for each \(x \in X\) and each \(\sigma_i\)-open set \(B\) of \(Y\) containing \(f(x)\), there exists a \(\tau_i\)-open set \(A\) containing \(x\) such that \(f(A) \subseteq j\)-Cl(B).
3. \((i, j)\)-\(\theta\)-continuous if \(f^{-1}(A)\) is \((i, j)\)-\(\theta\)-open in \(X\) for each \(\sigma_i\)-open set \(A\) of \(Y\).

Now we introduce the following definition in this article.

**Definition 6.** A function \(f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)\) is said to be \((i, j)\)-\(\theta\)-continuous if for each \(x \in X\) and each \(\sigma_i\)-open set \(V\) of \(Y\) containing \(f(x)\), there exists an \((i, j)\)-\(\theta\)-open set \(U\) containing \(x\) such that \(f(U) \subseteq j\)-Cl(V).

### Main results

**Theorem 1.** A mapping \(f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)\) is \((i, j)\)-\(\theta\)-continuous if and only if for every open set \(V\) in \(Y\), \(f^{-1}(V) \subseteq (i, j)\)-Int\(_\theta\)(j\)-Cl(V)).

**Proof:** Let \(x \in X\) and \(V\) be a \(\sigma_i\)-open set containing \(f(x)\). Then \(x \in f^{-1}(V) \subseteq (i, j)\)-Int\(_\theta\)(j\)-Cl(V)).

**Theorem 2.** For a function \(f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)\), the following statements are equivalent:

1. \(f\) is \((i, j)\)-\(\theta\)-continuous;
2. \(f^{-1}(V) \subseteq (i, j)\)-Int\(_\theta\)(j\)-Cl(V)) for every \(\sigma_i\)-open set \(V\) of \(Y\);
3. \((i, j)\)-\(\theta\)-continuous if and only if \(x \in f^{-1}(j\)-Cl(V)) \(\subseteq (i, j)\)-Int\(_\theta\)(j\)-Cl(V)) for every \(\sigma_i\)-closed set \(V\) of \(Y\), for all \(i, j = 1, 2\).

**Proof:** (1) \(\Rightarrow\) (2) The proof follows from Theorem 3.1.

(2) \(\Rightarrow\) (3) Let \(V\) be a \(\sigma_i\)-closed subset of \(Y\).

Then \(Y - V\) is an \(\sigma_i\)-open subset of \(Y\).

By hypothesis we have

\[
f^{-1}(Y - V) \subseteq (i, j)\)-Int\(_\theta\)(j\)-Cl(Y - V))
\[
= (i, j)\)-Int\(_\theta\)(f^{-1}(Y - f^{-1}(V)))
\[
= (i, j)\)-Int\(_\theta\)(X - f^{-1}(f^{-1}(V)))
\[
= X - (i, j)\)-Cl\(_\theta\)(f^{-1}(f^{-1}(V))).
\]

Thus \(f^{-1}(Y - V) \subseteq X - (i, j)\)-Cl\(_\theta\)(f^{-1}(f^{-1}(V))).

\((i, j)\)-\(\theta\)-continuous if \(f^{-1}(A) \subseteq (i, j)\)-Cl\(_\theta\)(f^{-1}(f^{-1}(V))).

(3) \(\Rightarrow\) (1) Let \(x \in X\) and \(L\) be a \(\sigma_i\)-open set of \(Y\) containing \(f(x)\). Then \(Y - V\) is a \(\sigma_i\)-closed subset of \(Y\).

By hypothesis we have

\[
f^{-1}(Y - V) \subseteq (i, j)\)-Int\(_\theta\)(j\)-Cl(Y - V))
\[
= (i, j)\)-Int\(_\theta\)(f^{-1}(Y - f^{-1}(V)))
\[
= (i, j)\)-Int\(_\theta\)(X - f^{-1}(f^{-1}(V)))
\[
= X - (i, j)\)-Cl\(_\theta\)(f^{-1}(f^{-1}(V))).
\]

Thus \(f^{-1}(Y - V) \subseteq X - (i, j)\)-Cl\(_\theta\)(f^{-1}(f^{-1}(V))).

This shows that \(f\) is \((i, j)\)-\(\theta\)-continuous.

**Theorem 3.** For a function \(f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)\), the following statements are equivalent:

1. \(f\) is \((i, j)\)-\(\theta\)-continuous;
2. \((i, j)\)-\(\theta\)-continuous for every \(\sigma_i\)-open set \(V\);
3. \((i, j)\)-\(\theta\)-continuous for every \(\sigma_i\)-regular closed set \(A\) of \(Y\);
(4) \( (i, j) \)-b \( \text{Cl}(f^{-1}(V)) \subset f^{-1}(i \text{ Cl}(V)) \) for every \( \sigma_i \)-open set \( V \subset Y \);

(5) \( f^{-1}(V) \subset (i, j) \)-b \( \text{Int}(f^{-1}(j \text{ Cl}(V))) \) for every \( \sigma_i \)-open set \( V \subset Y \), for all \( i, j = 1, 2 \).

**Proof:** (1) \( \Rightarrow \) (2) Let \( B \subset Y \) and \( x \in X \cap f^{-1}(i \text{ Cl}(B)) \). Then \( f(x) \in Y \cap f \text{ Cl}(B) \). This implies there exists a \( \sigma_i \)-open set \( A \) of \( Y \) containing \( f(x) \) such that \( A \cap B = \emptyset \). Therefore \( A \cap j \text{ Int}(i \text{ Cl}(B)) = \emptyset \) and hence \( j \text{ Cl}(A) \cap j \text{ Int}(i \text{ Cl}(B)) = \emptyset \). Since \( f \) is \( (i, j) \)-weakly b-continuous, so there exists an \( (i, j) \)-b-open set \( C \) such that \( x \in C \) and \( f(C) \subset j \text{ Cl}(A) \). Thus \( f(C) \cap j \text{ Int}(i \text{ Cl}(B)) = \emptyset \). This implies \( C \cap f^{-1}(j \text{ Int}(i \text{ Cl}(B))) = \emptyset \). Therefore by Lemma 2.4, we have \( x \in X \setminus (i, j) \)-b \( \text{Cl}(f^{-1}(j \text{ Int}(i \text{ Cl}(B)))) \). Hence \( (i, j) \)-b \( f^{-1}(j \text{ Int}(i \text{ Cl}(B))) \subset f^{-1}(i \text{ Cl}(B)) \).

(2) \( \Rightarrow \) (3) Let \( B \) be any \( (i, j) \)-regular closed set in \( Y \). Therefore \( A = i \text{ Cl}(j \text{ Int}(A)) \). Now \( (i, j) \)-b \( \text{Cl}(f^{-1}(j \text{ Int}(A))) = (i, j) \)-b \( \text{Cl}(f^{-1}(j \text{ Int}(i \text{ Cl}(j \text{ Int}(A))))) \subset f^{-1}(i \text{ Cl}(B)) \).

(3) \( \Rightarrow \) (4) Let \( V \) be \( \sigma_i \)-open subset of \( Y \). Since \( i \text{ Cl}(V) \) is \( (i, j) \)-regular closed in \( Y \), \( (i, j) \)-b \( \text{Cl}(f^{-1}(V)) \subset (i, j) \)-b \( \text{Cl}(f^{-1}(j \text{ Int}(i \text{ Cl}(V)))) \subset f^{-1}(i \text{ Cl}(V)) \).

(4) \( \Rightarrow \) (5) Let \( V \) be \( \sigma_i \)-open subset of \( Y \). Since \( j \text{ Cl}(V) \) is \( \sigma_i \)-open in \( Y \), so by hypothesis we have \( (i, j) \)-b \( \text{Cl}(f^{-1}(V) \setminus j \text{ Cl}(V))) \subset f^{-1}(i \text{ Cl}(V)) \).

\[
(i, j) \text{-b Cl}((X \setminus f^{-1}(j \text{ Cl}(V))) \cap X \cap f^{-1}(i \text{ Cl}(V)))
\]

\[
\Rightarrow (i, j) \text{-b Cl}(X \setminus f^{-1}(j \text{ Cl}(V))) \subset X \setminus f^{-1}(i \text{ Cl}(V))
\]

Therefore \( f^{-1}(V) \subset (i, j) \)-b \( \text{Int}(f^{-1}(j \text{ Cl}(V))) \).

(5) \( \Rightarrow \) (1) Follows from Theorem 1.

**Theorem 4.** Let \( f : (X, \tau, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a function. Then the following properties are equivalent:

1. \( f \) is \( (i, j) \)-weakly b-continuous.
2. \( x \in (i, j) \)-b \( \text{Int}(f^{-1}(j \text{ Cl}(A))) \) for each \( \sigma_i \)-neighbourhood \( A \) of \( f(x) \), for all \( i, j = 1, 2 \).

**Proof:** (1) \( \Rightarrow \) (2) Let \( A \) be a \( \sigma_i \)-neighbourhood of \( f(x) \) and \( x \subset X \). Since \( f \) is \( (i, j) \)-weakly b-continuous, so there exists an \( (i, j) \)-b-open set \( B \) containing \( x \) such that \( f(B) \subset j \text{ Cl}(A) \). Further \( B \subset f^{-1}(j \text{ Cl}(A)) \) and \( B \cap (i, j) \)-b \( \text{Int}(f^{-1}(j \text{ Cl}(A))) \). Hence \( f \) is \( (i, j) \)-weakly b-continuous.
An open subset of a bitopological space is said to be pairwise Urysohn if for each distinct points \( x, y \) of \( X \) there exists a \( \tau_i \)-open set \( U \) and a \( \tau_j \)-open set \( V \) such that \( x \in U, y \in V \) and \( j Cl(U) \cap i Cl(V) = \emptyset \) for \( i \neq j \) and \( i, j = 1, 2 \).

**Definition 4.** A bitopological space \((X, \tau_1, \tau_2)\) is said to be pairwise Urysohn if for each distinct points \( x, y \) of \( X \) there exists a \( \tau_i \)-open set \( U \) and a \( \tau_j \)-open set \( V \) such that \( x \in U, y \in V \) and \( j Cl(U) \cap i Cl(V) = \emptyset \) for \( i \neq j \) and \( i, j = 1, 2 \).

**Theorem 3.** If \((Y, \sigma_1, \sigma_2)\) is a pairwise Urysohn and \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is \((i, j)\)-weakly b-continuous injection, then \((X, \tau_1, \tau_2)\) is pairwise b-\( T_2 \), for all \( i, j = 1, 2 \).

**Proof:** Let \( x \) and \( y \) be any two distinct points of \( X \), then \( f(x) \neq f(y) \). Since \( Y \) is pairwise Urysohn, there exists a \( \tau_i \)-open set \( U \) and a \( \tau_j \)-open set \( V \) such that \( f(x) \in U, f(y) \notin V \) and \( j Cl(U) \cap i Cl(V) = \emptyset \).

By Theorem 3.1 we have \( x \in f^{-1}(U) \cap (f^{-1}(j Cl(V))) = \emptyset \). Hence \( f^{-1}(j Cl(V)) \cap f^{-1}(i Cl(V)) = \emptyset \) for all \( i, j = 1, 2 \). This implies \((X, \tau_1, \tau_2)\) is pairwise b-\( T_2 \).

**Theorem 4.** If a function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is \((i, j)\)-weakly b-continuous and

\( f \) is \((i, j)\)-semi-continuous then \( f \) is \((i, j)\)-weakly b-continuous, for all \( i, j = 1, 2 \).

**Proof:** Let \( V \) be a \( \sigma_i \)-open subset of \((Y, \sigma_1, \sigma_2)\). Since \( f \) is \((i, j)\)-weakly b-continuous, then by Theorem 3.1 we have \( f^{-1}(V) \subseteq (i, j)-\)\(b Cl(f^{-1}(j Cl(V))) \). By Lemma 2.2, \( f^{-1}(x) \) is \((i, j)\)-b-open in \( X \) and hence \( f \) is \((i, j)\)-b-continuous.

**Definition 3.** A bitopological space \((X, \tau_1, \tau_2)\) is said to be pairwise b-\( T_2 \) if for each pair of distinct points \( x \) and \( y \) of \( X \), there exists a \((i, j)\)-b-open set \( U \) containing \( x \) and a \((j, i)\)-b-open set \( V \) containing \( y \) such that \( U \cap V = \emptyset \) for \( i \neq j \) and \( i, j = 1, 2 \).

**Conclusion**

The notion of b-continuous functions in a bitopological space has been generalized and the notion of weakly b-continuous functions has been further studied.
introduced. The notion of b-frontier of a subset in a bitopological space has been introduced. It is shown, if $(Y, \sigma_1, \sigma_2)$ is a pairwise Urysohn and $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $(i, j)$- weakly b-continuous injection, then $(X, \tau_1, \tau_2)$ is pairwise b-$T_2$. These notions can be applied for investigating many other properties and some properties relative to separation axioms.

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