PT and non-PT-Symmetric Solutions of the Schrödinger Equation for the Generalized Woods-Saxon Potential

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Abstract

We investigate complex PT and non-PT-symmetric forms of the generalized Woods-Saxon potential. We also look for exact solutions of the Schrödinger equation for the PT and/or non-PT-symmetric potentials of the kind mentioned above. Nikiforov-Uvarov method is used to obtain their energy eigenvalues and associated eigenfunctions.

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1 Introduction

A large variety of potentials with the real or complex forms are encountered in various fields of the physics. A consistent physical theory of quantum mechanics in terms of Hermitian Hamiltonians is constructed on a complex Hamiltonian. In this case, its energy levels are real and positive as a consequence of PT-symmetry. Where \( P \) and \( T \) stand for the parity (or space) and time reversal operators, respectively. It is also well-known that PT-symmetry does not lead to completely real spectrums, because there are several potentials where part or all of the energy spectrums are complex. Exact solution of the Schrödinger equation for these potentials are generally of interest \[1\ 2\ 3\]. Recently, Bender and his co-workers have studied a number of complex potentials on PT-symmetric quantum mechanics. They have

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showed that the energy eigenvalues of the Schrödinger equation are real when $PT$-symmetry is unbroken, whereas they come in the shape of complex conjugate pairs when $PT$-symmetry is spontaneously broken [4]. In these studies, some numerical and analytical techniques have been used to investigate non-Hermitian Hamiltonians with real or complex spectra [5, 6].

$PT$-invariant operators have been analyzed for real and complex spectra by using a variety of techniques such as variational methods [7], numerical approaches [8], semiclassical estimates [9], Fourier analysis [10] and group theoretical approach with the Lie algebra [11]. It is pointed out that $PT$-invariant complex-valued operators may have real or complex energy eigenvalues. Many authors have studied on $PT$-symmetric and non-$PT$-symmetric non-Hermitian potential cases such as flat and step potentials with the framework of SUSYQM [12], exponential type potentials [13], quasi exactly solvable potentials [14], complex Hénon-Heiles potential [15] and deep potential to describe optic-model analysis of elastic and inelastic scattering processes [16].

Recently, an alternative method which is known as the Nikiforov-Uvarov (NU) has been introduced in solving the Schrödinger equation. The solution of the Schrödinger equation for well-known potentials and Schrödinger-like (i.e. Dirac and Klein-Gordon) equations for a Coulomb potential have been obtained by using this method [17]. The increasing interest on this solution method shows that $PT$-symmetric potentials are also suitable in solving the Schrödinger equation for a exponential-type potential. This type potential has been used in the work of Berkdemir et.al., [18] for the nuclear scattering applications. The potential known as Woods-Saxon has been generalized with an additional derivative term and solved by using the NU method to obtain eigenvalue equations. However, $PT$ and non $PT$-symmetric versions of the generalized Woods-Saxon potential have not been introduced in the literature and solved analytically for the Schrödinger equation. From this point of view, this method avoids direct solving of Schrödinger equation for these potential forms and makes the problem more interesting.

The organization of the paper is as follows. In Sec. II, we introduced a short summarization of the NU method, since the detailed description has been already given in Ref.[17] and also used in Ref.[18]. In Sec. III, the method utilized to solve the Schrödinger equation for $PT$ and non-$PT$-symmetric non-Hermitian forms of the generalized Woods-Saxon potential. Finally, in Sec. IV, we concluded the article with a summary of our main remarks.
2 Overview of the Nikiforov-Uvarov Method

The method of solving second-order differential equations developed by Nikiforov-Uvarov [17] has been used in appearing many problems of quantum mechanics. In principle, the method makes it possible to present the theory of special orthogonal functions [19] by following a differential equation of the form

\[
\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)} \psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} \psi(s) = 0,
\]

where \(\sigma(s)\) and \(\tilde{\sigma}(s)\) are polynomials, at most second-degree, and \(\tilde{\tau}(s)\) is a first-degree polynomial. Following the method used in Ref. [18], we transform the equation for \(\psi(s)\) to an equation of hypergeometric type

\[
\sigma(s)y'' + \tau(s)y' + \lambda y = 0,
\]

by inserting \(\psi(s) = \phi(s)y(s)\), where \(\phi(s)\) satisfies the equation \(\phi(s)'/\phi(s) = \pi(s)/\sigma(s)\). \(\tau(s) = \tilde{\tau}(s) + 2\pi(s)\) and its derivative has to be negative. In the present case, the polynomial \(\pi(s)\) is

\[
\pi = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)}
\]

where the parameter \(k\) is a constant \((k = \lambda - \pi')\). \(y(s)\) is the hypergeometric type function whose polynomial solutions are given by Rodrigues relation

\[
y_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n} \left[\sigma^n(s)\rho(s)\right],
\]

where \(B_n\) is a normalizing constant and the weight function \(\rho(s)\) must be satisfied the case

\[
\frac{d}{ds} [\sigma(s)\rho(s)] = \tau(s)\rho(s).
\]

On the other hand, the energy eigenvalues of Eq. (4) are determined by

\[
\lambda = \lambda_n = -n\tau' - \frac{n(n-1)}{2}\sigma'', \quad (n = 0, 1, 2, ...).
\]

Hence, the solution of second-order differential equations can be solved by means of this method and found their energy eigenvalues and associated wavefunctions analytically.
3 Calculations for the Generalized Woods-Saxon Potential

The interactions between nuclei are commonly described by using a potential that consist of the Coulomb and the nuclear potentials. The nuclear potential is usually taken in the Woods-Saxon potential form, which is one of the important potentials in nuclear physics. Moreover, the Woods-Saxon potential can be also used to describe the interaction of a nucleon with a heavy nucleus. In this paper, we selected a most general form of the Woods-Saxon potential given by \[18\]:

\[ V(x) = -\frac{V_0}{1+z} - \frac{(V_0/a)z}{(1+z)^2}, \] (7)

where \(z = \exp(\frac{x-R_0}{a})\), \(V_0\) is the potential depth, \(R_0\) is the width of the potential or the nuclear radius and the parameter \(a\) is the thickness of a surface layer in which the potential falls off from \(V = 0\) outside to \(V = -V_0\) inside the nucleus. The second term of the right-side of Eq.(7) denotes the derivative of Woods-Saxon potential responsible for the generalization while the first term of the right-side of the same equation represents the Woods-Saxon potential. In order to calculate the energy eigenvalues and the corresponding eigenfunctions, the potential function given by Eq.(7) is substituted into the one-dimensional form of the Schrödinger equation:

\[ \frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2} \left[ E + \frac{V_0}{1+qe^{2\alpha x}} + \frac{Ce^{2\alpha x}}{(1+qe^{2\alpha x})^2} \right] \psi(x) = 0. \] (8)

Here, some assignments are made in the Schrödinger equation such as \(1/a \equiv 2\alpha\), \(q = \exp(-2\alpha R_0)\) and \(C = 2\alpha V_0 q\).

We used the NU method to obtain the exact solutions of the SE with the PT-/non-PT-symmetric potentials only for the s-states. The energy eigenvalues and eigenfunctions are found in the real or complex forms and in terms of Jacobi polynomials, respectively.

3.1 PT-symmetric and non-Hermitian Woods-Saxon case

We are going to consider different forms of the generalized Woods-Saxon potential, namely at least one of the parameters is imaginary. For a special case, we take the potential parameters in Eq.(7) as \(V_0 \rightarrow V_0\) and \(\alpha \rightarrow i\alpha_I\), where \(\alpha_I\) is a real parameter of the imaginary part. Such
a potential is called as PT-symmetric and also non-Hermitian, since the property \( V(-x)^* = V(x) \) is exist. Hence, the concept of PT-symmetry can be also used in one-dimensional quantum mechanical problems as many problems exhibiting PT-symmetry \([21, 22]\). In this case, the new shape of the potential in one-dimensional space becomes

\[
V(x) = -\frac{V_0}{1 + qe^{2i\alpha_I x}} - \frac{Ce^{2i\alpha_I x}}{(1 + qe^{2i\alpha_I x})^2}
\]

or it can be written as a complex function

\[
V(x) = -V_0 \left(\frac{1 + q \cos 2\alpha_I x - iq \sin 2\alpha_I x}{1 + q^2 + 2q \cos 2\alpha_I x}\right)
- C \left(\frac{2q + (1 + q^2) \cos 2\alpha_I x - i(1) \sin 2\alpha_I x}{1 + q^2 + 2q \cos 2\alpha_I x}\right).
\]

The type of this potential is known as a complex periodic potential having PT-symmetric and its form is given by \( V(x) = i\sin^{2n+1}(x), \ n = 0, 1, 2, ... \) \([23]\). A detailed discussion exhibits for this potential that it has real band spectra from Ref.[25]. In our case, we will consider the form given in Eq.(9), following a procedure similar to the previous section.

Now, in order to apply the NU-method, we rewrite Eq.(8) by using a new variable of the form \( s = -e^{2i\alpha_I x} \),

\[
\frac{d^2\psi(s)}{ds^2} + \frac{1}{s} \frac{d\psi(s)}{ds} - \frac{m}{2\hbar^2 \alpha_I^2 s^2} \left[ E + \frac{V_0}{(1 - qs)} - \frac{Cs}{(1 - qs)^2} \right] \psi(s) = 0.
\]

By introducing the following dimensionless parameters

\[
e = -\frac{mE}{2\hbar^2 \alpha_I^2} > 0 \quad (E < 0), \quad \beta = \frac{mV_0}{2\hbar^2 \alpha_I^2} \quad (\beta > 0), \quad \gamma = \frac{mC}{2\hbar^2 \alpha_I^2} \quad (\gamma > 0)
\]

which leads to a hypergeometric type equation defined in Eq.\(\text{(11)}\):

\[
\frac{d^2\psi(s)}{ds^2} + \frac{1 - qs}{s(1 - qs)} \frac{d\psi(s)}{ds} + \frac{1}{s^2(1 - qs)^2} \left[ e(1 - qs)^2 - \beta(1 - qs) + \gamma s \right] \psi(s) = 0.
\]

After the comparison of Eq.\(\text{(13)}\) with Eq.\(\text{(11)}\), we obtain the corresponding polynomials as

\[
\tilde{\tau}(s) = 1 - qs, \quad \sigma(s) = s(1 - qs), \quad \tilde{\sigma}(s) = \varepsilon q^2 s^2 - (2\varepsilon q - \beta q - \gamma) s - \beta + \varepsilon.
\]

Substituting these polynomials into Eq.\(\text{(11)}\), we obtain \(\pi(s)\) function as

\[
\pi(s) = -\frac{qs}{2} \pm \frac{1}{2} \sqrt{(q^2 - 4\varepsilon q^2 - 4kq) s^2 + 4(-\beta q - \gamma + 2\varepsilon q + k) s + 4(\beta - \varepsilon)}
\]
taking $\sigma'(s) = 1 - 2qs$. The discriminant of the upper expression under the square root has to be zero. Hence, the expression becomes the square of a polynomial of first degree;

$$(-\beta q - \gamma + 2\varepsilon q + k)^2 - (\beta - \varepsilon) \left(q^2 - 4\varepsilon q^2 - 4kq\right) = 0.$$  \hspace{1cm} (16)

When the required arrangements are done with respect to the constant $k$, its double roots are derived as $k_{1,2} = (\gamma - \beta q) \pm q\sqrt{(\beta - \varepsilon)(1 - \frac{4}{q})}$.

Thus substituting, these values for each $k$ into Eq.(15) following possible solution is obtained for $\pi(s)$

$$\pi(s) = -\frac{qs}{2} \pm \frac{1}{2} \begin{cases} 
[(2\sqrt{\beta - \varepsilon} + \sqrt{1 - \frac{4}{q}})qs - 2\sqrt{\beta - \varepsilon}], \\
\text{for } k = (\gamma - \beta q) - q\sqrt{(\beta - \varepsilon)(1 - \frac{4}{q})} \\
[(2\sqrt{\beta - \varepsilon} - \sqrt{1 - \frac{4}{q}})qs - 2\sqrt{\beta - \varepsilon}], \\
\text{for } k = (\gamma - \beta q) + q\sqrt{(\beta - \varepsilon)(1 - \frac{4}{q})} 
\end{cases} \hspace{1cm} (17)$$

After appropriate choice of the polynomial $\pi(s)$ and $k$, we can write the function $\tau(s)$ which has a negative derivative as follows

$$\tau(s) = 1 + 2\sqrt{\beta - \varepsilon} - qs \left(2 + 2\sqrt{\beta - \varepsilon} + \sqrt{1 - 4\gamma/q}\right), \hspace{1cm} (18)$$

and then its negative derivatives become

$$\tau'(s) = -q \left(2 + 2\sqrt{\beta - \varepsilon} + \sqrt{1 - 4\gamma/q}\right). \hspace{1cm} (19)$$

A particularly interesting result of Eq.(19) is that the polynomial $\tau(s)$ is a generalization of the NU method to the complex quantum mechanics. Therefore, from Eq.(6) and Eq.(19), we write

$$\lambda = \lambda_n = nq \left(2 + 2\sqrt{\beta - \varepsilon} + \sqrt{1 - 4\gamma/q}\right) + n(n-1)q, \hspace{1cm} (20)$$

and also obtain

$$\lambda = (\gamma - \beta q) - q\sqrt{(\beta - \varepsilon)(1 - 4\gamma/q)} - \frac{q}{2} - q \left(\sqrt{\beta - \varepsilon} + \frac{1}{2}\sqrt{1 - 4\gamma/q}\right). \hspace{1cm} (21)$$

With the comparison of $\lambda$ values in Eq.(20) and Eq.(21), we found the energy eigenvalues as follows

$$E_{nq} = \frac{2\hbar^2}{m} \left(\frac{\alpha I}{4}\right)^2 \left(1 + 2n + \sqrt{1 - 2mc^2/q^2}\right)^2 + \left(\frac{mV_0/2\hbar^2\alpha I}{1 + 2n + \sqrt{1 - 2mc^2/q^2}}\right)^2 - \frac{mV_0}{4\hbar^2} \right) \hspace{1cm} (22)$$
\[ \varepsilon_{nq} = \frac{\gamma}{2} - \frac{1}{16} \left( 1 + 2n + \sqrt{1 - 4\gamma/q} \right)^2 - \left( \frac{\beta}{1 + 2n + \sqrt{1 - 4\gamma/q}} \right)^2. \] (23)

It is clear that the energy eigenvalues have a real part and then we can say that the real part of energy eigenvalues in Eq. (23) determines the energy spectrum in frame of PT-symmetric quantum mechanics. This situation can be also seen from the condition \( \varepsilon_{nq} > 0 \), since the energy eigenvalues of the generalized Woods-Saxon potential are negative. In this sense, the number of discrete levels is finite and determined by the inequality \( (b^4)^2 + (\beta b)^2 < \beta^2 \), where \( b = 1 + 2n + \sqrt{1 - 4\gamma/q} \), \( \beta = \frac{mV_0}{2\hbar^2 a_i^2} \) and \( \gamma = \frac{mc^2}{2\hbar^2 a_i^2} \). Hence, we can write a condition on the discrete levels for the negative energy spectrum of the complex parameter generalized Woods-Saxon potential if \( n < \sqrt{mV_0/2\hbar^2 a_i^2} - \sqrt{1/4 - \gamma/q - 1/2} \).

Let us now find the corresponding eigenfunctions. The polynomial solutions of the hypergeometric function \( y(s) \) depend on the determination of weight function \( \rho(s) \) which is satisfies the differential equation \[ \sigma(s) \rho(s) = \tau(s) \rho(s) \] where \( \sigma(s) = (1 - qs)^{-\nu - 1} s^{2\sqrt{\beta - \varepsilon}} \), \( \nu = 1 + \frac{\sqrt{1 - 4\gamma/q}}{2} \). Substituting into the Rodrigues relation given in Eq. (4), the eigenfunctions are obtained in the following form

\[ y_{nq}(s) = A_n (1 - qs)^{-\nu - 1} s^{2\sqrt{\beta - \varepsilon}} \frac{d^n}{ds^n} \left[ (1 - qs)^{n + \nu - 1} s^{n + 2\sqrt{\beta - \varepsilon}} \right], \] (25)

where \( A_n = 1/n! \). Choosing \( q = 1 \), the polynomial solutions of \( y_n(s) \) are expressed in terms of Jacobi Polynomials \([\text{constant}]P_n^{(2\sqrt{\beta - \varepsilon}, \nu - 1)} (1 - 2s) \) [19], which is one of the orthogonal polynomials. By substituting \( \pi(s) \) and \( \sigma(s) \) into the expression \( \phi(s)'/\phi(s) = \pi(s)/\sigma(s) \), the other part of the wave function is found as

\[ \phi(s) = (1 - s)^{\mu} s^{2\sqrt{\beta - \varepsilon}}, \] (26)

where \( \mu = \nu/2 \). Combining the Jacobi polynomials with \( \phi(s) \), the s-wave complex functions are constructed as

\[ \psi_n(s) = B_n s^{2\sqrt{\beta - \varepsilon}} (1 - s)^{\mu} P_n^{(2\sqrt{\beta - \varepsilon}, \nu - 1)} (1 - 2s), \] (27)

where \( B_n \) is the normalizing constant.
3.2 Non-PT symmetric and non-Hermitian Woods-Saxon case

In order to be more specific, we are going to take the potential parameters given in Eq.(7) as
\[ V_0 \rightarrow V_0R + iV_0I \] and \( \alpha \rightarrow \alpha \), where \( V_0R \) and \( V_0I \) are the real parameters of the complex part. Such a potential is called as non-PT-symmetric but non-Hermitian. In this case, the condition of PT-symmetry is not occurred due to the fact that \( V(-x)^* \neq V(x) \). Hence, its shape becomes
\[ V(x) = -\left[ \frac{V_0R}{1+z} + \frac{Cz}{(1+z)^2} + i \frac{V_0I}{1+z} \right], \] (28)
and these types potentials are called as the pseudo-Hermitian. To simplify the form of the above equation, we take again an independent variable by choosing the form \( s = -e^{2\alpha x} \), and then we obtain the generalized equation of hypergeometric type,
\[ \frac{d^2 \psi(s)}{ds^2} + \frac{1-qs}{s(1-q)s} \frac{d\psi(s)}{ds} + \frac{1}{s^2(1-q)s^2} \left[ -\varepsilon(1-q)s^2 + (\beta + i\delta)(1-q)s - \gamma s \right] \psi(s) = 0, \] (29)
for which
\[ \tilde{\tau}(s) = 1 - qs, \quad \sigma(s) = s(1 - qs), \quad \tilde{\sigma}(s) = -\varepsilon q^2 s^2 + (2\varepsilon q - \beta q - i\delta q - \gamma) s^2 + \beta + i\delta - \varepsilon, \]
\[ \varepsilon = -\frac{mE}{2\hbar^2 \alpha^2} > 0, \quad \beta = \frac{mV_0R}{2\hbar^2 \alpha^2}, \quad \gamma = \frac{mC}{2\hbar^2 \alpha^2}, \quad \delta = \frac{mV_0I}{2\hbar^2 \alpha^2} \quad (\beta, \gamma, \delta > 0). \] (30)

Following the solution procedures of the NU–method, we can derive the possible solutions for \( \pi(s) \) as below
\[ \pi(s) = \frac{-qs}{2} \pm \frac{1}{2} \left[ \left( 2\sqrt{\varepsilon - (\beta + i\delta)} \right) \mp \sqrt{1 + \frac{4\gamma}{q}} \right] q s \pm 2\sqrt{\varepsilon - (\beta + i\delta)} , \] (31)
for \( k = (\beta q - \gamma + i\delta q) \pm q \sqrt{(\varepsilon - \beta - i\delta)(1 + \frac{4\gamma}{q})} \). After performing an appropriate choice for \( k \) and \( \pi(s) \), we can write \( \tau(s) \) and \( \tau' \) as
\[ \tau(s) = 1 + 2\sqrt{\varepsilon - (\beta + i\delta)} - qs \left( 2 + 2\sqrt{\varepsilon - (\beta + i\delta)} + \sqrt{1 + 4\gamma / q} \right) , \]
\[ \tau'(s) = -q \left( 2 + 2\sqrt{\varepsilon - (\beta + i\delta)} + \sqrt{1 + 4\gamma / q} \right) . \] (32)

In the present case, the eigenvalue equation given in Eq.(6) is established as follows
\[ \lambda = \lambda_n = nq \left( 2(1 + \sqrt{\varepsilon - (\beta + i\delta)} ) + \sqrt{1 + \frac{4\gamma}{q}} \right) + n(n-1)q, \] (33)
and then if it is compared with the another form of $\lambda$,

$$
\lambda = (\beta q - \gamma + i\delta q) - q\sqrt{(\varepsilon - \beta - i\delta) \left(1 + \frac{4\gamma}{q}\right)} - \frac{q}{2}
- \frac{q}{2} \left(2\sqrt{\varepsilon - \beta - i\delta} + \sqrt{1 + \frac{4\gamma}{q}}\right).
$$

(34)

the energy eigenvalues are reduced to the following form

$$
E_{nq} = -\frac{\hbar^2}{2ma^2} \left\{ \frac{1}{16} \left[ \sqrt{1 + \frac{8mCa^2}{\hbar^2 q}} + (1 + 2n)^2 \right]^2 + \frac{4 \left( \frac{ma^2}{\hbar^2} \right)^2 (V_{0R}^2 - V_{0I}^2)}{\sqrt{1 + \frac{8mCa^2}{\hbar^2 q}} + (1 + 2n)^2} \right\}
- \frac{i\hbar^2}{2ma^2} \left\{ \frac{4 \left( \frac{ma^2}{\hbar^2} \right)^2 V_{0R} V_{0I}}{\sqrt{1 + \frac{8mCa^2}{\hbar^2 q}} + (1 + 2n)^2} + \frac{mV_{0I}a^2}{\hbar^2} \right\}.
$$

(35)

In order to mention from an acceptable energy spectra in Eq.(35), we have to provide the visible condition given by if and only if $n > 8 (\delta^2 - \beta^2)^{1/4} - \kappa/2$ for both $V_{0R} > V_{0I}$ and $V_{0R} < V_{0I}$, where $\kappa = \sqrt{1 + \frac{4\gamma}{q}} + 1$.

Now, our procedure is that the corresponding unnormalized wave functions should be determined in terms of Jacobi polynomials. By considering the Eq.(5) and using Eq.(32), we obtain

$$
\rho(s) = (1 - qs)^{\kappa-1} s^2 \sqrt{\varepsilon - \beta - i\delta}.
$$

(36)

After that, we derive the polynomial $\phi(s)$ from the equality $\phi(s)/\phi(s) = \pi(s)/\sigma(s)$ as follows

$$
\phi(s) = s \sqrt{\varepsilon - \beta - i\delta} (1 - qs)^{\kappa/2}.
$$

(37)

Hence, we found the relevant wave functions in terms of Jacobi polynomials

$$
\psi_n(s) = D_n s \sqrt{\varepsilon - \beta - i\delta} (1 - s)^{\kappa/2} P_n^{(2\sqrt{\varepsilon - \beta - i\delta}, \kappa-1)} (1 - 2s),
$$

(38)

where $D_n$ is the another normalizing constant and the parameter $q$ is chosen as 1.

### 4 Conclusions

The exact solutions of the Schrödinger equation for the generalized Woods-Saxon potential with zero angular momentum are obtained to found the energy eigenvalues and eigenfunctions. Nikiforov-Uvarov method is used to solve the Schrödinger equation. The corresponding wave functions are expressed in terms of Jacobi polynomials. It is pointed out that
the NU method can be generalized by the complex polynomials $\pi(s)$ and $\tau(s)$ with the framework of PT-symmetric quantum mechanics. When the potential parameter $\alpha$ is purely complex, it is seen that the energy eigenvalues are in the real form. In this point, the shape of potential takes the complex $PT$-symmetric form. It is not difficult to see that the energy eigenvalues determines the discrete levels or energy spectrum under the condition of $n < \sqrt{mV_0/2\hbar^2\alpha_I^2} - \sqrt{1/4 - \gamma/q - 1/2}$. Therefore, if the parameter of potential $V_0$ is chosen as complex, it is clear that the energy eigenvalues represent the complex energy spectrum. We realize that the energy eigenvalues consist of essentially a real component. Consequently, we find that the energy eigenvalues of a single particle within the complex PT and non-PT-symmetric potentials are given in Eq.(22) and Eq.(35), respectively.
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