FACTORIZING IN A DISSIPATIVE QUANTUM COMPUTER

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ABSTRACT: We describe an array of quantum gates implementing Shor’s algorithm for prime factorization in a quantum computer. The array includes a circuit for modular exponentiation with several subcomponents (such as controlled multipliers, adders, etc) which are described in terms of elementary Toffoli gates. We present a simple analysis of the impact of losses and decoherence on the performance of this quantum factoring circuit. For that purpose, we simulate a quantum computer which is running the program to factor $N = 15$ while interacting with a dissipative environment. As a consequence of this interaction randomly selected qubits may spontaneously decay. Using the results of our numerical simulations we analyze the efficiency of some simple error correction techniques.

1. Introduction

In recent years there has been an explosion of activity in the area of quantum computation (see Ref. 1,2). In part, this was a consequence of a very important discovery made in 1994 by Peter Shor, who demonstrated that two problems which are thought to be classically intractable (finding prime factors and discrete logarithms of integer numbers) could be efficiently solved in a quantum computer [3, 4]. Shor’s results added a practical motivation for the study of quantum computation which, until that time had received the attention of a smaller community of people interested in fundamental aspects of quantum mechanics, the physics of information, algorithmic complexity theory, etc. By now, quantum computation is a growing field which is developing not only due to the work of theorists but,
fortunately, also due to recent advances in experimental techniques. In fact, in the last two years there have been a few interesting experiments aiming at constructing quantum gate prototypes (see Ref. 5,6,7).

There are many open questions concerning the mathematics and also the physics of quantum computers. In fact, we still don’t know what is the real power of quantum computation from the algorithmic complexity point of view. (Until now, attempts towards demonstrating their usefulness to solve NP-complete problems were not successful.) On the other hand, the physics of quantum computers also presents many important challenges. Among the most important open questions is the understanding of the impact of the process of decoherence (an issue that attracted some attention over the last two years [8,9,10,11,12]). Decoherence [13] is a physical process by which the quantum interference effects, essential for the proper functioning of the quantum computer, are destroyed by the interaction between the computer and its environment. This interaction creates irreducible computer–environment correlations which, as the environment is unobserved, induce the dynamical collapse of the computer’s wave function. Decoherence may be potentially devastating but, as recent studies suggest, there may be ways in which one can reduce the problem. For that purpose, a few ideas have been advanced. Shor proposed a procedure for recovering quantum coherence by using coding [14] (see also Ref. 12), and similar methods have been proposed for “purifying” entangled pairs before using them for transmitting quantum information through noisy channels [15]. This, combined with the possibility of building error correction schemes based on the “watchdog” effect [16] are promising ideas that are currently under investigation.

However, to give a specific answer to the question of how important is decoherence for factoring one needs to be rather specific. The answer will depend upon the computer implementation (hardware) and also on the particular algorithm (software) used. For example, the possibility of implementing error correction schemes based on watchdog effect depends upon having a computer evolving in such a way that at some known instants it is in a known state (or at least some qubits are in a known state, so that we can measure them without disturbing the computer). The aim of this paper is to begin a study on the impact of dissipation and decoherence on a quantum factoring computer. For this purpose we design a quantum factoring circuit analyzing how its performance is affected when the interaction with an
environment is included.

Several recent papers are related to ours: Chuang et al. [10] described on general grounds the potentially devastating effects that decoherence may have upon a factoring computer. Their results, which were obtained using a simple description of the quantum computer, which makes no reference to a specific quantum circuit, suggest that by having a low enough decay rate and using appropriate error correction techniques one may be able to implement factoring in a quantum computer. Cirac and Zoller [5] presented a numerical study of the effects of errors on the quantum Fourier Transform (FT) subroutine, which plays a central role on the factoring program. Their simulation was done by considering the effect of spontaneous decay while a computer made of cold trapped ions runs the FT program (previously designed by Coppersmith and others [4,17]). Other studies of decoherence on quantum computers have been presented which are not directly related to the issue of factoring. For example, the importance of losses and decoherence have been analyzed [18] for the optical quantum computer designed by Chuang and Yamamoto [19] to solve Deutsh’s Oracle problem [20] for a one bit function. The effect of decoherence upon a static quantum computer was also analyzed in Ref. 8,11.

The paper is divided in two parts: We first present an array of reversible quantum logic gates which implements Shor’s algorithm for factoring integer numbers in a quantum computer. To do that we first created subcomponents which some specific tasks such as controlled multiplication, controlled sums, mod(N), etc. Then, we combined these subcomponents in the precise way required to run Shor’s algorithm. The existence of work–qubits (required to handle the reversible logic) makes the design of the quantum circuit a rather nontrivial task. In fact, for the quantum computer to work properly, it is necessary to reversibly erase the records created along the computational path (stored in the work–qubits). As an example, we present the gate array that could be used to factor $N = 15$ in a quantum computer.

Designing the factoring circuit is the first step required for studying the impact of decoherence and the possibility of implementing error correction schemes. This is the purpose of the second part of the paper where we study how the coupling to an environment affects the functioning of the quantum factoring circuit. For this, we use an oversimplified model of the system–environment interaction. We assume
that this interaction takes place only at certain (randomly chosen) moments of time affecting only a few (randomly chosen) qubits which may spontaneously decay.

After completing the design of the factoring circuit, and while we were working on the numerical simulations to model dissipation, we became aware that a very similar gate array was recently developed by Vedral, Barenco and Ekert [21]. Our circuit produces the same final quantum state and has roughly the same requirements (in number of qubits and time steps) than the one described in Ref. 21 (in that paper the authors did not attempt to analyze the impact of losses and decoherence on the performance of their quantum circuit, an issue which we analyze here). More recently Plenio and Knight [22] used some of the conclusions of Ref. 21 (the number of required qubits and time steps) to discuss some of the limitations imposed by decoherence on the size of the numbers one could factorize using various physical setups.

In Section 2 we briefly describe both the mathematical basis for Shor’s algorithm and the basic steps a quantum computer would need to follow in order to implement it. In Section 3 we describe the quantum network for implementing modular exponentiation. We go from the coarser description where the circuit is just a black box to the fine grained picture where every component is dissected and built from elementary Toffoli gates. We analyze the architecture required to factor numbers of \( L \) bits and explicitly exhibit the circuit to factor \( N = 15 \), which requires 28 qubits (the circuit to factor \( L \) bit numbers needs \( 5L + 8 \) qubits and involves a number of elementary gates which, for large \( L \) is close to \( 240L^3 \)). In Section 4 we address the importance of decoherence and the possible strategies for error correction. We summarize our results in Section 5.

2. Shor’s algorithm

In 1994, Peter Shor invented an algorithm for a quantum computer that could be used to find the prime factors of integer numbers in time. We will now briefly review the most important aspects of Shor’s algorithm and later consider the way to implement it in a quantum computer.

The mathematical basis for Shor’s algorithm is the following: (see Ref. 3,4,23): The goal is to find the prime factors of an integer number \( N \). Instead of doing this
directly, the algorithm finds the *order*, \( r \), of a number \( x \). The order of \( x \) is defined as the least integer \( r \) such that \( x^r \equiv 1 \pmod{N} \). Knowing \( r \) one can find the prime factors of \( N \) by using some results proved in Number Theory. Factorization reduces to finding \( r \) if one uses a randomized algorithm: as Shor shows in Ref. 4, choosing \( x \) at random and finding its order \( r \), one can find a nontrivial factor by computing the greatest common divisor \( a = \gcd(x^{r/2} - 1, N) \). In fact, \( a \) is a nontrivial factor of \( N \) unless \( r \) is odd or \( x^{r/2} = -1 \pmod{N} \). As \( x \) is chosen at random, the probability for the method yielding a nontrivial prime factor of \( N \) is \( 1 - 1/2^{k-1} \), where \( k \) is the number of distinct prime factors of \( N \).

In his seminal paper [3, 4], Shor showed that a quantum computer could efficiently find the order \( r \) of the number \( x \) and, therefore, factorize \( N \) in polynomial time. Let us now describe the basic operation of this quantum computer. This requires two quantum registers, which hold integers represented in binary notation. There should also be a number of work–qubits, which are required along the calculation but should be in a standard state (say \( |0\rangle \)) both at the beginning and at the end of the calculation. The role of these work–qubits is very important and will be described in detail in the next section. For the moment, we will concentrate on describing the state of the computer before and after every major step of the program. For that purpose, we can forget for the moment these qubits. Apart from the quantum registers, there is also some classical information we should provide for operating the quantum computer. Thus, we will assume that the numbers \( N \) (the one we want to factor), \( x \) (chosen randomly \( \pmod{N} \)) and a randomly chosen \( q \), which is such that \( N^2 \leq q \leq 2N^2 \) are part of the classical information available to the quantum computer.

We start the process by preparing the first register in a uniform superposition of the states representing all numbers \( a \leq q - 1 \) (this can be done by a standard technique, i.e. rotating each individual qubit putting it in a superposition \( \frac{1}{\sqrt{q}}(|0\rangle + |1\rangle) \)). The state of the computer is then

\[
|\Psi_0\rangle = \frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} |a\rangle |0\rangle
\]  

(1)

The next step is to unitarily evolve the computer into the state

\[
|\Psi_1\rangle = \frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} |a\rangle |x^a \pmod{N}\rangle.
\]  

(2)
Then, we Fourier transform the first register. That is, we apply a unitary operator that maps the state \( |\Psi_1\rangle \) into

\[
|\Psi_2\rangle = \frac{1}{q} \sum_{a=0}^{q-1} \sum_{c=0}^{q-1} \exp(2\pi i ac/q) |c\rangle |x^a (\text{mod} N)\rangle.
\] (3)

The final step is to observe both registers (the method could be implemented observing just the first register but, following Shor [4], for clarity we assume both registers are observed). The probability for finding the state \( |c\rangle |x^k (\text{mod} N)\rangle \) is:

\[
P(c, x^k (\text{mod} N)) = \left| \frac{1}{q} \sum_{a/x^a \equiv x^k} \exp(2\pi i ac/q) \right|^2, \tag{4}
\]

where the sum is over all numbers \( 0 \leq a \leq q - 1 \) such that \( x^a = x^k (\text{mod} N) \). This sum can be transformed into

\[
P(c, x^k (\text{mod} N)) = \left| \frac{1}{q} \sum_{b=0}^{[q-1-k]/r} \exp(2\pi i b\{rc\}q/q) \right|^2, \tag{5}
\]

where \( \{rc\}q \) is an integer in the interval \(-q/2 < \{rc\}q \leq q/2\) which is congruent to \( rc \) (mod \( q \)). As shown by Shor, the above probability has well defined peaks if \( \{rc\}q \) is small (less than \( r \)), i.e., if \( rc \) is a multiple of \( q \) (\( rc = dq \) for some \( d < N \)). Thus, knowing \( q \) and the fact that the position of the peaks \( c \) will be close to numbers of the form \( dq/r \), we can find the order \( r \) (using well established continuous fraction techniques).

There is no doubt that Shor’s algorithm would work if a quantum computer could be built. However, to implement Shor’s algorithm in a quantum computer one needs to explicitly construct the program. The procedure for Fourier transforming is well known and has been extensively discussed in several recent papers (see Ref. 4,17,23). To explicitly construct the unitary evolution that takes the state \( |\Psi_0\rangle \) into the state \( |\Psi_1\rangle \) is a rather nontrivial task which we will describe in the next section[21].

3. Quantum network for modular exponentiation.

We will present an array of quantum gates which maps the state \( |a\rangle \otimes |0\rangle \) into \( |a\rangle \otimes |x^a (\text{mod} N)\rangle \) transforming the state \( |\Psi_0\rangle \) into \( |\Psi_1\rangle \). We describe the quantum
circuit using diagrams such as the one in Figure 1 which must be interpreted as representing the time evolution of the system with time flowing from left to right. Each line represents a single qubit, i.e. a two level system (a thick line will represent a bundle of qubits). In describing the circuit we will go in steps from the coarse description of Figure 1a (where the computer is a black box) to a fine grained description where the computer consists of a complex array of interconnected elementary gates.

We will use Toffoli gates as “elementary” components and follow the notation of Ref. 24 denoting a gate acting on three qubits as \( \Lambda_2 \). The action of a Toffoli gate on a computational state \( |x_1, x_2, x_3\rangle \) (where \( x_i \in \{0, 1\} \)) is \( \Lambda_2|x_1, x_2, x_3\rangle = |x_1, x_2, x_3 \oplus (x_1 \land x_2)\rangle \) where \( \oplus \) denotes the exclusive OR and \( \land \) the AND operation between the Boolean variables \( x_i \). Thus, Toffoli gates are just controlled–NOT gates where the last qubit changes its state only if the two control qubits are set to 1. It will also be convenient to use generalized Toffoli gates, with \( n \) control–qubits, which are denoted as \( \Lambda_n \). Of course, all these gates can be constructed in terms of one and two–qubit operations, as explained in Ref. 24. The diagram representing the gate \( \Lambda_n \) is shown in Figure 1b.

To design a quantum circuit for modular exponentiation we should first notice that if the binary representation of \( a \) is \( a = \sum_{i=0}^{n} a_i 2^i \), then

\[
y^a \pmod{N} = \prod_{i=0}^{n} \left( (y^{2^i})^{a_i} \pmod{N} \right).
\]

(6)

Thus, modular exponentiation is just a chain of products where each factor is either equal to 1 if \( a_i = 0 \) or equal to \( y^{2^i} \) if \( a_i = 1 \). Therefore, the circuit is easily constructed if one is allowed to use a controlled multiplier as an auxiliary unit (which at this level, acts as a new black box). In Figure 2 we show the basic architecture of the array of controlled multipliers required for modular exponentiation. For the first multiplication the control qubit is \( a_0 \) and after each multiplication the control is moved to the next qubit. For this array to work we need to know all the numerical factors entering in (6) (thus, we must classically compute the numbers \( y^{2^i} \pmod{N} \)).

Our next step is to analyze the controlled multiplier. Given an input \( |I\rangle \), this circuit, which we denote as \( \Pi_N(C) \), produces an output \( |I \ast C\pmod{N}\rangle \). The controlled multiplier is constructed using a smaller black box: a controlled mod\( N \)
adder. In fact, multiplication of two numbers \( I = \sum_{i=0}^{n} I_i 2^i \) and \( C \) reduces to a sum of the form \( \sum_{i=0}^{n} I_i * (2^i C) \). Thus, we just need to use \( I_i \) as the control qubit in a controlled mod\( N \) adder adding the number \( (2^i C) \) (a circuit which we denote as \( S_N(2^i C) \)). The numbers involved in the sum must also be provided as classical information (we need to classically compute all numbers \( 2^i y 2^j \), with \( i, j \leq L \) where \( L \) is the number of bits of \( N \)). In Figure 3 we show a controlled multiplier for 4–bit numbers. The same architecture can be used to multiply \( L \)–bit numbers. In that case, the controlled multiplier requires \( L+1 \) work–qubits, whose state is set to zero before and after its operation. As we will see below, the controlled adder itself also requires some work–space which must be independent of the one used specifically for multiplication.

As shown in Figure 3, \( \Pi_N(C) \) is schematically divided into three pieces. In all of them the work–qubits play an important role. The quantum state entering the circuit is \( |\chi_0\rangle = |I\rangle \otimes |0\rangle_{wb} \), where \( I \) is the number stored in the input register and \( |0\rangle_{wb} \) is the state of the work–qubits. The qubits \( |I_i\rangle \) are used as control for the \( S_N(2^i C \mod N) \) adders and the result of the sum is temporarily written in the work–qubits. After this, the state is \( |\chi_1\rangle = |I\rangle \otimes |I^* C\rangle_{wb} \): almost what we need, except for the fact that the input \( |I\rangle \) also appears in the output state. Erasing this extra copy of the input is essential: Otherwise we would be keeping a record of the computational path affecting the interference pattern of the quantum computer (apart from forcing us to use an enormous amount of space). The reversible erasure of the input is the purpose of the second part of the circuit. In designing this we followed well known techniques developed by Bennett [25] and described by Shor [4]. The procedure is as follows: We first consider the evolution operator \( \tilde{U} \) mapping the input \( |0\rangle \otimes |I_i\rangle_{wb} \) into \( |I^* C^{-1}\rangle \otimes |I_i\rangle_{wb} \), where \( C^{-1} \) is the multiplicative inverse of \( C \) (mod\( N \)) (the number satisfying \( C \cdot C^{-1} = 1 \) (mod\( N \))). The operator needed in the second part of the multiplier is \( \tilde{U}^{-1} \). To convince ourselves that this is the case, we should notice that, as the input to the second part of the multiplier is \( |\chi_1\rangle = |I\rangle \otimes |I^* C\rangle_{wb} \), the output will be \( |\chi_2\rangle = \tilde{U}^{-1} |\chi_1\rangle = |0\rangle \otimes |I^* C\rangle_{wb} \) (because, by construction, \( \tilde{U} \) satisfies \( \tilde{U}(|0\rangle \otimes |I^* C\rangle_{wb}) = |I\rangle \otimes |I^* C\rangle_{wb} = |\chi_1\rangle \)). The circuit for \( \tilde{U}^{-1} \), shown in the figure, is just the specular image of the one used for the first part of the multiplier (switching the role of register and work–qubits). Finally, the multiplier is completed with a controlled swap that interchanges once more the register and work–qubits so that the final state of the work–qubits is
always $|0\rangle_{wb}$.

The circuit for doing controlled mod $N$ sums of a number $X$, which is stored in a quantum register, and a number $Y$, stored in a classical register, is called $S_N(Y)$. This circuit, for 5–bit numbers, is shown in Figure 4 (generalization to $L$ bit numbers is straightforward). The circuit for $S_N(Y)$ is built using a simple controlled adder, which we denote as $S(Y)$ whose functioning will be explained below. The only difference between $S_N(Y)$ and $S(Y)$ is that the former gives the output modulo $N$. Constructing a reversible circuit for computing the sum mod $N$ is not a trivial task which is only possible because we know that the two numbers being added ($X$ and $Y$) are both less than $N$ (and therefore $X+Y \leq 2N-2$). Without this information it would not be possible to compute mod $N$ reversibly without keeping unwanted records of the computation (since mod $N$ is not a one to one function). The input to the circuit is $|\bar{\chi}_0\rangle = |X\rangle \otimes |0\rangle_{wb}$. After the first adder, this is transformed to $|\bar{\chi}_1\rangle = |X+Y\rangle \otimes |0\rangle_{wb}$. We then apply another simple adder adding the positive number $2^{L+1} - N$ producing an output $|\bar{\chi}_2\rangle = |2^{L+1} + X + Y - N\rangle \otimes |0\rangle_{wb}$. The most significant bit (MSB) of $2^{L+1} + X + Y - N$ is one (zero) if $X + Y \geq N$ ($X + Y < N$). It is easy to realize that the opposite is true for the second MSB of the output. Thus if we use this qubit to control the inverse operation, we will add $N$ only if $X + Y < N$. Therefore, after the third gate of the circuit shown in Figure 4, the first $L$ qubits of the output always store the number $A + C \mod N$. However, the $L + 1$– and $L + 2$–qubits, which are used to control the third gate, keep a record of the first result. As usual, this record must be reversibly erased and this can be done by using the following simple trick: We first add the positive number $2^L - Y$ and notice that the MSB of the result $2^L - Y + (X \mod N)$ is always identical to the qubit used to control the third gate. Thus, we are done: we apply a control–NOT gate and then we undo the first sum (by adding $Y$).

So far, we first explained modular exponentiation in terms of controlled multiplication $\Pi_N(C)$. Later, we explained $\Pi_N(C)$ in terms of controlled mod $N$ sums $S_N(Y))$ and this circuit in terms of a simple adder $S(Y)$. We will now present the gate array for the simple controlled adder $S(X)$ which is best explained in terms of a smaller gate: a controlled two–qubit adder. This will be our smallest black box and, for clarity, we will explain here how it works. The two–qubit adder, denoted as $\Sigma(\sigma)$ has four input qubits and a classical input bit $\sigma$ (i.e., there are two types of two–qubit adders, one for $\sigma = 0$ and another for $\sigma = 1$). The first input qubit
is the control, the second qubit is $i_1$, the third one is $i_2$ and the fourth one is a work–qubit which is always set to 0 at the input. At the output, the control qubit is unchanged, the first qubit changes into the Least Significant Bit (LSB) of the sum $(i_1 + i_2 + \sigma)$, the third one stores $i_2$ and the fourth stores the MSB of the sum. In Figure 5 we can see how to build the gates $\Sigma(0)$ and $\Sigma(1)$ (and other useful simple gates) in terms of Toffoli gates.

Using $\Sigma(\sigma)$ it is possible to construct a circuit mapping an input $|X\rangle$ into $|X + Y\rangle$. This is displayed in Figure 6 where, for simplicity, we assumed that both $X$ and $Y$ have 5 bits. For numbers of $L$ bits the number of work–qubits required is $L + 3$. The quantum state entering the adder is $|\tilde{X}_0\rangle = |X\rangle \otimes |0\rangle_{wb}$. This goes through the sequence of two–qubit adders $\Sigma(Y_i)$ (we use $X_i, Y_i \in \{0, 1\}$ for the binary representation of $X$ and $Y$). After this chain of $\Sigma$–gates, the state is $|\tilde{X}_1\rangle = |X\rangle \otimes |X + Y\rangle_{wb}$, which has an unwanted copy of the input. To reversibly erase this extra copy we apply the same method used in the multiplication: We first consider an auxiliary operator $W$ that adds the positive number $\bar{Y} \equiv 2^L - Y$ ($\bar{Y}$ is known as the two’s complement of $Y$ and its binary representation is simply obtained from that of $Y$ by interchanging zeros and ones and adding 1). The operator $W$ satisfies $W(|R\rangle \otimes |0\rangle_{wb}) = |R\rangle \otimes |R + 2^L - Y\rangle_{wb}$. Therefore, its inverse is such that $W^{-1}|X + Y\rangle \otimes |2^L + X\rangle_{wb} = |X + Y\rangle \otimes |0\rangle_{wb}$ which is precisely what we need as the output of our circuit (the properties of $W^{-1}$ simply follow from that of $W$ which, by construction satisfies $W|X + Y\rangle \otimes |0\rangle_{wb} = |X + Y\rangle \otimes |2^L + X\rangle_{wb}$). Therefore, using $W^{-1}$ after appropriately interchanging the role of the register and the work–qubits (and adding an extra work–qubit to store the qubit representing $2^L$) we complete the controlled adder. The circuit for $W^{-1}$ which is shown in Figure 6 is almost the specular image from the one used as the first part of the adder. The only difference is that instead of the first two qubit adder we can use a smaller circuit which only stores the LSB of the first sum (this circuit is shown in Figure 5).

Having explained the essential pieces of the quantum computer, let us now summarize what are its space and time requirements (i.e., the number of qubits and the number of elementary operations). As explained above, to factor an $L$ bit number we need: $L + 1$ qubits as work–space for the controlled multiplier and $L + 4$ for controlled sums. The mod$N$ circuit as well as the controlled swap require an extra work–qubit each. Adding the qubits required to store the two quantum
registers ($2L + 1$ qubits to store $a$ in the first register and $L$ qubits for the second register) we get a total of $5L + 8$ qubits. Computing the number of elementary operations is also possible. By inspecting our controlled adder one realizes that the number of elementary gates is $\alpha L + \beta (L + 1) + (L + 2) \gamma$ where $\alpha$, $\beta$ and $\gamma$ are, respectively, the number of gates in a two–qubit adder, its inverse and the one in a swap circuit. Using the estimate $\alpha = \beta = 3$ one gets $12n + 17$ operations for the sum. Using similar arguments to analyze the multipliers one finally concludes that the complete modular exponentiation circuit requires $240n^3 + 484n^2 + 182n$ elementary operations. For $L = 4$ this is about $2.5 \times 10^4$.

4. Losses and decoherence in a factoring computer

Before analyzing the impact of dissipative effects on the quantum circuit it is convenient to introduce some notation. The quantum computer has a Hilbert space with a computational basis with states $|r_1, r_2, wb\rangle$ (where $r_1$, $r_2$ and $wb$ are the bit–strings determining the states of the first register, the second register and the work–qubits respectively). We assume that the environment $E$ has a Hilbert space spanned by a basis of states $|e\rangle_E$. The quantum state of the computer–environment ensemble can always be written as

$$|\Psi(t)\rangle = \sum_{r_1, r_2, wb, e} A(r_1, r_2, wb, e, t) |r_1, r_2, wb, e\rangle.$$ (7)

The temporal evolution of the probability amplitude $A(r_1, r_2, wb, e, t)$ is governed by the interplay between the quantum circuit described in Section 3 and the computer–environment interaction. At the initial time, when the computer is in state (1), the amplitudes are given by:

$$A(r_1, r_2, wb, e, t = 0) = \frac{1}{\sqrt{q}} \delta(r_2, 0) \delta(wb, 0) \delta(e, 0).$$ (8)

Here we assumed that the computer is initially uncorrelated with the environment which is taken to be in an unexcited state $|0\rangle_E$ (we use $\delta(a, b)$ to denote Kronecker’s delta function). If the computer evolves without interaction with the environment the amplitudes after the modular exponentiation circuit are:

$$A_{exact}(r_1, r_2, wb, e, t = t_f) = \frac{1}{\sqrt{q}} \delta(r_2, y^{r_1}(modN)) \delta(wb, 0) \delta(e, 0).$$ (9)
However, when the computer interacts with the environment, the actual amplitudes will deviate from the exact expression (9). To model this interaction we will use a very simple approach which incorporates the losses induced by the spontaneous decay of the computer’s qubits: The environment consists of a collection of two level systems $E_i$, i.e. a collection of “environmental qubits” (each $E_i$-qubit has an excited state $|1\rangle_{E_i}$ and a ground state $|0\rangle_{E_i}$). For simplicity we will assume that at a given time, a randomly selected computer qubit $q_i$ interacts with one of environmental qubits $E_i$. As a result of this sudden interaction correlations are established according to:

$$
|1\rangle_{q_i} |0\rangle_{E_i} \rightarrow p_1^{1/2} |1\rangle_{q_i} |0\rangle_{E_i} + p_2^{1/2} |0\rangle_{q_i} |1\rangle_{E_i}
$$

(10)

where $p_2 = 1 - p_1$. The interpretation of the evolution (10) is quite clear: If the computer qubit is in the state $|1\rangle_{q_i}$, it has a probability $p_1$ to persist and a probability $p_2$ to decay into $|0\rangle_{q_i}$ creating an excitation in the environment. On the other hand, if the computer qubit is in the state $|0\rangle_{q_i}$ nothing happens. It is worth mentioning that the decay rules (10) implicitly assume that the state used to represent the computational 0 is the ground state (or, at least, has lower energy than the one used to represent the computational 1). In fact, the situation may be exactly the opposite in which case the rules (10) must be trivially modified by interchanging the roles of $|1\rangle_{q_i}$ and $|0\rangle_{q_i}$ (see below). More general evolution rules (such as the ones used in Ref. 14, which are best suited to analyze a noisy but almost losseless computer) will be studied elsewhere [26].

Thus, we can summarize the basic ingredients of our computer–environment model: i) It is characterized by a randomly chosen sequence of times $(t_1, \ldots, t_n)$ which define the instants where the computer interacts with the environment (in between these times the computer evolves according to the unitary operators associated with the quantum circuit described in the previous section). ii) At each time $t_i$ we randomly choose a computer qubit $q_i$ which is involved in a sudden interaction with an environmental qubit $E_i$. iii) As a consequence of this interaction the computer–environment ensemble evolves according to the rules (10). Implicit in our assumptions is the validity of the simplifying Markovian approximation which assures that at every instant $t_i$ a different (and independent) environmental qubit $E_i$ is involved in the interaction. A simple way of visualizing this computer–environment model is by thinking of the times $t_i$ as the instants where there may
be a “branching” of the computational trajectory. Every time an environmental qubit is excited an “erroneous” computational trajectory emerges. At the end of the modular exponentiation circuit, the state vector of the computer–environment ensemble is written as in (7) with an amplitude which will not be given by (9). We already admitted that this is an oversimplification of reality (which has been used before to model losses in quantum computation [18]).

We computed the amplitudes from the output state of the Fourier Transform circuit which follows modular exponentiation (the discrete FT circuit is described in the literature [4, 5, 17]). In Figure 7 results are presented for the probability for finding \( r_1 \) in the first register and \( r_2 = 7 \) in the second register. The ideal result, plotted in Fig. 7(a), is obtained from eq. (5). This error–free curve has three sharp peaks, with a separation approximately equal to \( q/r = 130/4 \) (we deliberately choose a rather small value for \( q \) so that the small structure in the plots can be seen using a reasonable scale). Provided we don’t know the final state of the environment and the work–qubits (see below) the probability is

\[
P_{\text{NED}}(r_1, r_2) = \sum_{wb,e} |A(r_1, r_2, wb, e, t)|^2. \tag{11}
\]

(the suffix stands for “no error detection”, see below). This probability is shown in Figure 7(b) where we can see that the errors slightly widen the peaks and notably decrease their amplitudes. As the number of errors is increased it will be less and less likely to measure a value of \( r_1 \) located near a peak making the identification of the order \( r \) (obtained from the separation between peaks, as explained in Ref. 4) more and more difficult. The appearance of intermediate peaks is also evident in Fig. 7(b). Apart from the above probability we also calculated the probability for finding \( r_1 \) in the first register, \( r_2 = 7 \) in the second and the work–qubits in the state \( |0\rangle_{wb} \), i.e.: 

\[
P_{\text{ED}}(r_1, r_2) = \sum_{e} |A(r_1, r_2, wb = 0, e, t)|^2. \tag{12}
\]

This is plotted in Fig. 7(c) where we see that while a noisy dc component (present in (b)) is supressed, the amplitude ratio between the misleading and correct peaks is increased. These plots correspond to simulations of the quantum computer running the program to factor \( N = 15 \) while coupled to an environment at a randomly chosen set of ten instants \( t_i \) (we use \( p_1 = p_2 = 1/2 \)). The modular exponentiation circuit requires about \( 2.5 \times 10^4 \) elementary (Toffoli) gates. This roughly correspond
to $10^5$ one bit operations for Cirac and Zoller’s cold ions computer [5]. Thus, in that case we are considering an error rate of the order of $10^{-4}$, which is a rather optimistic figure.

Our simulations not only can be used to visualize the importance of the environmental interaction on the quantum algorithm but also to test simple error detection (and correction) schemes. The simplest of such schemes is probably the one based on checking the state of the qubits which are supposed to be in a known state. Our factoring program is suited for this purpose since the work–qubits must start and end in the state representing the computational 0. Two comments concerning error detection (and correction) are in order: First, by checking the final state of the work–qubits we are not able to detect a special class of errors which are produced by the decay of the qubits representing the first and second registers of the computer ($r_1$ and $r_2$). Errors of that kind leave (most of the time) the work–qubits untouched but generate a misleading output (they are responsible for the intermediate peaks seen in Figure 7(c) which make the measurement of the order $r$ a much more difficult task). Second, and more important, by measuring the final state of the work–qubits we are only able to the detect errors but not to correct (or prevent) them.

Of course, it would be much better to have a method enabling us to prevent the errors from occurring. For this, the use of the watchdog effect [16] has been proposed. Thus, if some of the computer’s qubits are supposed to be in a known state at some time, one could inhibit their decay by making a measurement on the known state. This method can indeed be applied here since the work–qubits are supposed to be in the state representing the computational 0 at many intermediate instants of the computation. In fact, this is what happens after the action of each $\Pi_N(C)$ circuit and after the action of each controlled adder $S_N(C)$. For large $L$, the number of times one could measure the state of some of the work–qubits grows as $L^2$.

To test the efficiency of the watchdog effect as an error correction technique we slightly changed our computer–environment interaction model. In fact, we now assume that the decay rules are of the form (10) but with time dependent coefficients given by:

\[
p_1(t) = \exp(-\gamma t), \quad p_2(t) = 1 - p_1(t).
\]
In this way the decay probability for a qubit increases with time (measured from the start of the computation and, by convention, expressed in units of the total time required to run the program, i.e. \( t = 1 \) corresponds to the end of the computation). The decay rate \( \gamma \) is taken to be \( \gamma = 2.5 \) so that towards the end of the computation a qubit will have a high decay probability \( (p_2(t = 1) \approx 9/10) \). The assumption of an exponential decay is not essential (it is just a reasonable approximation which we adopt here for simplicity).

To implement the watchdog we measure the state of the work–qubits at every instant when they are supposed to be in the computational 0. Every time we do this we reset the time in (13). Thus, a work–qubit will decay with probabilities given by (13) where the time will effectively be measured from the last instant in which the work–qubit was supposed to be in the computational 0 state. On the other hand, the qubits involved in the first or second registers of the computer will have decay probabilities given by (13) with time counting from the beginning of the computation.

The effectiveness of the watchdog effect as an error prevention technique can be seen in Figure 8 where the exact probability is plotted together with the ones obtained with and without watchdog. Without using this method we get a very noisy probability with a substantial widening of the principal peaks. The amplitude of the central peak, which is about 0.1, is of the same order as the one shown in Fig. 7(b) (but the decay rules we are using here are more damaging than the ones we used before). Using the watchdog technique we substantially increase the amplitude of the main peaks (by a factor of four) and also eliminate almost all the noise. The only remaining spurious peaks are those produced by the decay of qubits involved in the first and second registers. They cannot be eliminated using watchdog since their existence is not a consequence of a process affecting the work–qubits.

5. Summary and outlook.

The factoring circuit we presented is by no means optimal. Several improvements are possible to reduce the number of work–qubits. However, when designing a circuit for practical purposes one has to have in mind that the existence of work–qubits is not necessarily a burden. Our results show they can play a very useful role
allowing the use of the watchdog effect as an error prevention technique. It would be important to find the optimal balance minimizing the number of work–qubits but still allowing an efficient use of the watchdog method.

The simulations we performed are rather simple and do not allow us to test the importance of other sources of problems for quantum computers. One of the most important sources of errors we excluded here is related with the fact that the elementary quantum gates are never 100% efficient. If we think of Cirac and Zoller’s [5] cold ions hardware, the elementary gates are built by applying a sequence of laser pulses on individual ions. If these pulses are not exact π–pulses (or π/2–pulses) the quantum gate will not be exactly the one we want. The corresponding unitary evolution operator $U_{\text{real}}$ will have nonzero matrix elements in places where the exact quantum gate operator $U_{\text{ideal}}$ has zero matrix elements. These imperfections may be rather important since their effects accumulate in time. To include this effects in our model one needs to follow the evolution of the computer’s state vector in the $2^{28}$–dimensional Hilbert space. Even though our work enables us to explicitly write down the matrix $U_{\text{real}}$ at every step of the calculation, we are not able to numerically simulate this because of space limitations (thus, simulating a quantum computer with $N$ qubits needs an exponentially large amount of space in a classical computer). Simulations of smaller versions of our circuit for modular exponentiation will be presented elsewhere [26].

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Figure 1: a) Black box description of the circuit for modular exponentiation. When $N$ has four bits one needs nine qubits to represent $a$ and fifteen extra qubits to be used as workspace. b) A $\Lambda_4$ Toffoli gate with 4-control bits: $x_1, x_2, x_3, x_4$. $x_5 \rightarrow x_5 \oplus (x_1 \land x_2 \land x_3 \land x_4)$

Figure 2: The gate array used for modular exponentiation. $Y^a \mod N$ is calculated by repeatedly multiplying the second register by $Y^{2^m \mod N}$ only if $a_m = 1$. Each box multiplies its input by $Y^{2^m \mod N}$ only if the control bit $a_m$ is 1.

Figure 3: a) The 3 stages of the controlled multiplier (mod N) $\Pi_N(C)$: first the input $I$ is multiplied by $C$. Then $I$ is reversibly erased and finally the result is swapped with the upper register. b) Multiplication by $C$ is achieved by repeated addition of $2^m C \mod N$ controlled by $I_m$. This is done using the controlled mod N adders $S_N(2^m C \mod N)$. In the Figure we denote mod N as $\%N$.

Figure 4: Addition mod N is achieved with 5 controlled adders: The first adds $C$ to the input. The second “subtracts” $N$ from $a + C$. The third operation adds $N$ only if $a + C$ is smaller than $N$. At this stage the first 4 bits have $a + C \mod N$. The last two stages erase the record left in the 7th bit, whose state depends on the sign of $a + C - N$.

Figure 5: The two-qubit adders $\Sigma(\sigma)$ are shown in terms of Toffoli gates. They have four input and four output qubits. If the inputs are $ctl$, $i_1$, $i_2$ and 0, the outputs are $ctl$, the least significant bit (LSB) of $i_1 + i_2 + \sigma$, $i_2$ and the most significant bit (MSB) of the sum. A swap gate is also shown that interchanges its two input qubits: $i_1$ and $i_2$.

Figure 6: a) Addition is performed in 3 stages: The first adds $Y$ to the input $X$, the second interchanges $X$ with $X + Y$ and the last reversibly erases the input $X$. b) The first and last stages are shown in terms of the individual qubits and two-qubit adders $\Sigma(\sigma)$. $Y_0 \ldots Y_4$ are the bits in the binary representation of $Y$. $\bar{Y} \equiv 2^L - Y$ is used to erase $X$.

Figure 7: Probability distribution for $r_1$ and $r_2 = 7$. In the simulations $N = 15$, $q = 130$ and $p_1 = p_2 = 1/2 \forall t \in [0,1]$. a) Exact result. b) Result with ten decaying qubits at randomly chosen instants of time $t_1 \ldots t_{10}$. c) Probability distribution for $r_1$, $r_2 = 7$ and all work-qubits in their zero state.

Figure 8: Probability distribution for $r_1$ and $r_2 = 7$. In the simulation $N = 15$, $q = 130$ and the decay rate $\gamma$ is chosen so that $p_2(t = 1) \approx 9/10$. a) Exact result. b) Result with ten decaying qubits and using the watchdog effect on every work–qubit. c) Result with ten decaying qubits without using the watchdog effect.
Quantum Computer

Figure 1 (Miquel, Paz, Perazzo)
Modular Exponentiation

Figure 2 (Miquel, Paz, Perazzo)
Figure 3 (Miquel, Paz, Perazzo)
Adder (mod N) \( S_N(Y) \)

\[
\begin{align*}
S(Y) & \quad S(32-N) & \quad S(N) & \quad S(16-Y) & \quad S(16-Y)^{-1} \\
X_0 & \quad X_1 & \quad X_2 & \quad X_3
\end{align*}
\]

\[X + Y \mod N\]

Figure 4 (Miquel, Paz, Perazzo)
2-bit adders

\[ \sum(1) = i_1 \oplus i_2 \equiv 1 \]

\[ \sum(0) = i_1 \oplus i_2 \equiv 0 \]

\[ \sum'(1) = i_1 \oplus i_2 \equiv 1 \]

\[ \sum'(0) = i_1 \oplus i_2 \equiv 0 \]

\[ \text{swap gate} \]

\[ i_1 \rightarrow i_2 \]

\[ i_2 \rightarrow i_1 \]

Figure 5 (Miquel, Paz, Perazzo)
Figure 6 (Miquel, Paz, Perazzo)
Figure 7 (Miquel, Paz, Perazzo)
Figure 8 (Miquel, Paz, Perazzo)