Research Article

Finitely Generated Modules over Group Rings of a Direct Product of Two Cyclic Groups

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Let $K$ be a commutative field of characteristic $p > 0$ and let $G = G_1 \times G_2$, where $G_1$ and $G_2$ are two finite cyclic groups. We give some structure results of finitely generated $K[G]$-modules in the case where the order of $G$ is divisible by $p$. Extensions of modules are also investigated. Based on these extensions and in the same previous case, we show that $K[G]$-modules satisfying some conditions have a fairly simple form.

1. Introduction

Let $K$ be a field of characteristic $p > 0$ and let $G$ be a finite group. The study of $K[G]$-modules in the case where the order of $G$ is divisible by $p$ is a very difficult task. When $G$ is a finite abelian $p$-group, we find in [1] the following statement: a complete classification of finitely generated $K[G]$-modules is available only when $G$ is cyclic or equal to $C_2 \times C_2$, where $C_2$ is the cyclic group of order 2. In [2] we find this classification in these two cases. Still more, in the case where the Sylow $p$-subgroup $P$ of $G$ is not cyclic, the groups $G$ such that $p = 2$ and $P$ is dihedral, semidihedral, or generalized quaternion are the only groups for which we can (in principle) classify the indecomposable $K[G]$-modules (see [2]). These reasons just cited show the importance of the study of $K[G]$-modules when $G$ is of order divisible by $p$ and equal to a direct product of two cyclic groups.

Now, let $K$ be a commutative field of characteristic $p > 0$ and let $G = G_1 \times G_2$, where $G_1$ and $G_2$ are two finite cyclic groups. Let $M$ be a finitely generated $K[G]$-module. When $M$ is considered as a module over a subalgebra $K[H]$ of $K[G]$ for a subgroup $H$ of the group $G$, we write $M_{\downarrow H}$.

In Section 2, we show that if $G_1$ is a cyclic $p$-group and the characteristic of $K$ does not divide the order of $G_2$, then we can have a complete system of indecomposable pairwise nonisomorphic $K[G]$-modules. In the rest, we assume that $G_1 = \langle \sigma_1 \rangle$ and $G_2 = \langle \sigma_2 \rangle$ are cyclic $p$-groups. Under conditions that $M_{\downarrow G_1}$ is a free $K[G_1]$-module and that $M/(\sigma_1 - 1)M$ is a free $K[G_2]$-module, we show that $M$ is a free $K[G]$-module. We also show that if $\sigma_2$ is of order $p^n$, $n \neq 0$, and $H_2$ is the subgroup of $G_2$ generated by $a_2^{p^n}$, with $0 < r \leq n$, then under certain conditions $M$ is a free $K[G_1 \times H_2]$-module. The fact that $M_{\downarrow G_1}$ must be a free $K[G_1]$-module is one of these conditions, and exactly in the end of this section we give a result that shows when this condition is satisfied. In Section 3 and always in the case where $G_1$ and $G_2$ are cyclic $p$-groups, we show that under some conditions $K[G]$-modules have a fairly simple form. But in case $p = 2$, $G_1$ and $G_2$ are two cyclic groups of respective orders 2 and $2^n$, $n \neq 0$; these modules have this simple form without any other assumptions other than that they must be finitely generated over $K[G]$.

2. Free $K[C_{p^m} \times C_{p^n}]$-Modules of Finite Rank

Throughout this paper, rings are assumed to be commutative with unity. We begin this section by giving a weak version of Nakayama’s lemma with an elementary proof.

Lemma 1 (Nakayama). Let $G$ be a $p$-group with $p$ odd, $R$ a ring of characteristic $p^k$ where $k$ is a natural number,
Let $M$ be an $R[G]$-module (not necessarily finitely generated), and $N$ a submodule of $M$ and $\sigma \in G$. Then, one has the following:

1. if $(\sigma - 1)M = M$, then $M = 0$;
2. if $M = (\sigma - 1)M + N$, then $M = N$;
3. if $x_i, i \in I$, are representatives in $M$ of a generating family of $M/(\sigma - 1)M$, then $(x_i)_{i \in I}$ generate $M$.

**Proof.** (1) Let $p'$ be the order of $G$. We have

$$
(\sigma - 1)^{p'} = \sum_{i=0}^{p'} C_{p'}^i (-1)^i \sigma^{p' - i}
$$

$$
= \sigma^{p'} + (-1)^{p'} + \sum_{i=1}^{p'-1} C_{p'}^i (-1)^i \sigma^{p' - i}
$$

$$
= \sum_{i=1}^{p'-1} C_i^{p'} (-1)^i \sigma^{p' - i}
$$

$$
= p \sum_{i=1}^{p'-1} \frac{1}{i} C_i^{p'} (-1)^i \sigma^{p' - i}.
$$

For $1 \leq i \leq p' - 1$, $p \mid C_i^{p'}$, so $(1/p)C_i^{p'}$ is a natural number. So

$$
(\sigma - 1)^{kp'} = \left( (\sigma - 1)^p \right)^{k}
$$

$$
= \left( p \sum_{i=1}^{p'-1} \frac{1}{i} C_i^{p'} (-1)^i \sigma^{p' - i} \right)^k
$$

$$
= p^k \left( \sum_{i=1}^{p'-1} \frac{1}{i} C_i^{p'} (-1)^i \sigma^{p' - i} \right)^k
$$

$$
= 0,
$$

since $R$ has characteristic $p$. Now $M = (\sigma - 1)M = (\sigma - 1)^2 M = \cdots = (\sigma - 1)^{kp'} M = 0$.

(2) If $M = (\sigma - 1)M + N$, then $M/N = ((\sigma - 1)M + N)/N = (\sigma - 1)(M/N)$, and then, by (1), $M/N = 0$ and then $M = N$.

(3) If $N$ is the submodule generated by $x_i$, then $M = (\sigma - 1)M + N$, and then by (2) we have $M = N$. □

**Remark 2.** Lemma 1 remains true if $p = 2$ and $R$ is of characteristic $2$.

For a ring $R$ of prime characteristic $p$ and for a cyclic group $G$ of order $p'$ generated by an element $\sigma$, we have the following lemma.

**Lemma 3.** Let $k = p'$ with $0 < r' \leq r$ and let $H$ be a subgroup of $G$ generated by $\sigma^{p'/k}$. Then, one has $R[G]/(\sigma - 1)^k R[G] \cong R[H]$ (as $R$-algebras).

**Proof.** Define

$$
\psi : \frac{R[G]}{(\sigma - 1)^k R[G]} \rightarrow R[H],
$$

where $\psi$ is a well-defined $R$-algebra homomorphism. It is easy to see that $\psi$ is surjective. As $R[G]/(\sigma - 1)^k R[G]$ and $R[H]$ are finite free modules of the same rank $k$ over $R$, $\psi$ is an isomorphism. □

**Remark 4.** With the notation of Lemma 3, $H$ is simply the subgroup of $G$ generated by $\sigma^{p'/k}$.

Let $K$ be a commutative field of characteristic $p > 0$ and let $G$ be a direct product of two finite groups $G_1$ and $G_2$. We have $K[G] = K[G_1 \times G_2] \cong R[G_2]$, where $R = K[G_1]$.

Assume that $G_1$ is a cyclic group of order $p^m$ generated by $\sigma_1$ and $p$ does not divide the order of $G_2$, $R = K[G_2]$ is a principal Artinian local ring. Indeed $K[G_1] \cong K[X]/(X - 1)^{p^m}$; this isomorphism is induced by the homomorphism $\Psi : K[X] \rightarrow K[G]$ defined by $\Psi(X) = \sigma_1$. $K[X]/(X - 1)^{p^m}$ is a principal Artinian local ring with residue field $K$ (up to isomorphism) whose maximal ideal is generated by $X - 1$. So $R$ is a principal Artinian local ring with residue field $K$ (up to isomorphism) whose maximal ideal is generated by $\sigma_1 - 1$. We have $K[G] \cong R[G_2]$, where $R$ is a principal Artinian local ring of residue field $K$. The characteristic of $K$ does not divide the order of $G_2$. Under these conditions, we can apply [3, Theorem 3.6] to have a complete system of indecomposable pairwise nonisomorphic $K[G]$-modules.

In the remainder of this section, we assume that $G_1 = C_{p^m}$ and $G_2 = C_{p^n}$ are two cyclic groups of respective orders $p^m$ and $p^n$ and are generated, respectively, by $\sigma_1$ and $\sigma_2$. We have $K[G] \cong R[G_2]$. As $R$ is a commutative ring and local and $G_2$ is a $p$-group, by [4, Proposition 10, page 239], $R[G_2]$ is a local ring. Therefore $K[G]$ is a local ring. As $K$ is commutative ring and local and $G$ is a $p$-group, $K[G]$ is a local ring by [4, Proposition 10, page 239]. So the $K[G]$-projective modules are free $K[G]$-modules.

**Lemma 5.** Let $M$ be a $K[G]$-module. Then, $M/(\sigma_1 - 1)M$ is a $K[G_1]$-module (also $M/(\sigma_2 - 1)M$ is a $K[G_2]$-module).

**Proof.** This lemma is a particular case of a more general result (see [5, page 386]). But for this particular case, we can give the following direct proof: $M/(\sigma_1 - 1)M$ is a $(K[G_1]/(\sigma_1 - 1)K[G_1])K[G_2]$-module, and we have already seen that $(\sigma_1 - 1)K[G_1]$ is the unique maximal ideal of $K[G_1]$. $K[G_1]/(\sigma_1 - 1)K[G_1] \cong K$. So $M/(\sigma_1 - 1)M$ is a $K[G_2]$-module.

Similarly we show that $M/(\sigma_2 - 1)M$ is a $K[G_1]$-module. □

**Proposition 6.** Let $M$ be a free $K[G]$-module of rank $l$. Then, $M/(\sigma_1 - 1)M$ is a free $K[G_2]$-module and $M/(\sigma_2 - 1)M$ is a free $K[G_1]$-module of the same rank $l$. □
**Proof.** As $K[G]$ is a local ring, $K[G]$-projective modules are free $K[G]$-modules, and therefore this proposition is only a particular case of a more general result (see [5, Lemma 2.2]). But for this particular case, we can give the following specific proof: we have $M \cong (K[G])^\ell = (K[G_1 \times G_2])^\ell$. So $M \cong (K[G_1][G_2])^\ell$. Then, we have

$$\begin{align*}
(\sigma_1 - 1)M &\cong (\sigma_1 - 1)(K[G_1][G_2])^\ell \\
&\cong ((\sigma_1 - 1)K[G_1][G_2])^\ell. 
\end{align*}$$

Hence,

$$\begin{align*}
(M / (\sigma_1 - 1)M) &\cong (K[G_1][G_2])^\ell / ((\sigma_1 - 1)K[G_1][G_2])^\ell \\
&\cong \left(\begin{array}{c} K[G_1][G_2] \\ (\sigma_1 - 1)K[G_1][G_2] \end{array}\right)^\ell. 
\end{align*}$$

As $K[G_1]/(\sigma_1 - 1)K[G_1] = K$ (as we have already seen), $M / (\sigma_1 - 1)M \cong (K[G_2])^\ell$. So $M / (\sigma_1 - 1)M$ is a free $K[G_2]$-module of rank $l$.

Similarly we show that $M / (\sigma_2 - 1)M$ is a free $K[G_1]$-module of rank $l$.

**Proposition 7.** Let $M$ be a $K[G]$-module. If $M_{\mathfrak{g}}$ is a free $K[G_1]$-module and $M / (\sigma_1 - 1)M$ is a free $K[G_2]$-module, then $M$ is a free $K[G]$-module.

**Proof.** $R = K[G_1]$ is a principal Artinian local ring with residue field $K$ and $\sigma_1 - 1$ is a generator of its maximal ideal. $M_{\mathfrak{g}}$ is a free $R$-module and $Q = M / (\sigma_1 - 1)M$ is a projective $K[G_2]$-submodule of $M / (\sigma_1 - 1)M$. Then, $M = P \oplus M'$, where $P$ is a projective $R[G_2]$-module and $P / (\sigma_1 - 1)P \cong Q = M / (\sigma_1 - 1)M$ (according to [3, Proposition 4.13]). We have

$$\begin{align*}
(M / (\sigma_1 - 1)M) &\cong (P \oplus M') / (\sigma_1 - 1)(P \oplus M') \\
&\cong \left(\begin{array}{c} P \\ (\sigma_1 - 1)P \oplus (\sigma_1 - 1)M' \end{array}\right) \\
&\cong \left(\begin{array}{c} P \\ (\sigma_1 - 1)M' \end{array}\right),
\end{align*}$$

So $M' / (\sigma_1 - 1)M' = 0$. By Nakayama’s lemma and the remark following it, $M' = 0$. Therefore, $M = P$ which is projective $R[G_2]$-module. As $R[G_2] \cong K[G]$ is a local ring, $M$ is a free $K[G]$-module.

Let $J_i$ be the Jacobson radical of $K[G_i]$ for $i \in \{1, 2\}$. Note that if $K$ is of characteristic $p$ (as here) and $G'$ is a cyclic $p$-group, then the Jacobson radical of $K[G']$ is none other than $(\sigma - 1)K[G']$, where $\sigma$ is a generator of $G'$ (see [5, page 122]).

Let $M$ be a finitely generated $K[G]$-module and $k$ a natural number such that $1 \leq k \leq p^r$. As $K[G] \equiv R[G_2]$, $M$ is a $R[G_2]$-module. So $M / J_iM$ is a $K[X]/(X - 1)^k$-module. $M$ is called of type $k$ if $M / J_iM$ is a free $K[X]/(X - 1)^k$-module (terminology of [6]).

**Lemma 8.** If $M$ is a $K[G]$-module of type $k$ with $k = p^r$ and $0 < r \leq n$ and $H_2$ is the subgroup of $G_2$ generated by $\sigma_2^{p^{r-1}}$, then $M / J_iM$ is a free $K[H_2]$-module.

**Proof.** As $M$ is of type $k$, $M / J_iM$ is a free $K[X]/(X - 1)^k$-module. Define

$$\psi : \frac{K[X]}{(X - 1)^k} \to K[H_2] ,$$

where $\psi$ is a well-defined $K$-algebra homomorphism. It is not difficult to show that $\psi$ is an isomorphism (using an argument similar to that done in the proof of Lemma 3). So $M / J_iM$ is a free $K[H_2]$-module.

**Theorem 9.** Let $M$ be a $K[G]$-module of type $k$, with $J_iM = 0$, and let $H_2$ be the subgroup of $G_2$ generated by $\sigma_2^{p^{r-1}}$ with $0 < r \leq n$. If $M_{\mathfrak{g}_1}$ is $R$-free and $k = p^r$, then $M$ is a free $K[G_1 \times H_2]$-module.

**Proof.** $M$ is an $R[G_2]$-module $R$-free. We have $J_iM = 0$, so $(\sigma_2 - 1)^kM = 0$, and therefore $((\sigma_2 - 1)^kR[G_2])M = 0$. So $M$ is an $R[G_2]/((\sigma_2 - 1)^kR[G_2])$-module $R$-free. By Lemma 3, $R[G_2] / (\sigma_2 - 1)^kR[G_2] \equiv R[H_2]$; then $M$ is an $R[H_2]$-module $R$-free. $M / J_iM$ is a free $K[X]/(X - 1)^k$-module, so by Lemma 8 this is a free $K[H_2]$-module. In conclusion $M$ is a $K[G_1 \times H_2]$-module such that

$$M_{\mathfrak{g}_1} \cong K[G_1] - \text{module},$$

$$M / J_iM \text{ is a free } K[H_2] - \text{module.}$$

So by Proposition 7 $M$ is a free $K[G_1 \times H_2]$-module.

In Theorem 9 we assumed that the $k[G]$-module $M$ satisfies the following condition: $M_{\mathfrak{g}_1}$ is $R$-free. So it is useful to know when this condition is satisfied. This is the subject of the following result.

**Theorem 10.** Let $M$ be a $k[G]$-module and $\sigma$ an element of $G$ of order $p^r$. The following conditions are equivalent:

1. $M_{\mathfrak{g}_1}$ is free;
2. $\dim_k(M) = \binom{p^r}{(p^r - 1)} \dim_k((\sigma - 1)M)$;
3. $\dim_k(M) = \dim_k((\sigma - 1)M) + \dim_k((\sigma - 1)^{p^{r-1}}M)$.

**Proof.** (1) $\Rightarrow$ (2) Assume that $M_{\mathfrak{g}_1}$ is free. There exists a nonzero natural number $n$ such that $M_{\mathfrak{g}_1} \equiv (k[\langle \sigma \rangle])^n$. The endomorphism $\varphi$ of $M$ defined by $\varphi(m) = (\sigma - 1)m$ for all
be a pair of extensions of $M_2$ by $M_1$. These two extensions are equivalent if there exists an isomorphism of $R[G]$-modules \( \Phi : X \to X' \) such that $\Phi u = u'$ and $\nu' \circ \Phi = \nu$. These equivalence classes of extensions form an R-module $\text{Ext}^1_{R[G]}(M_2, M_1)$. The $R[G]$-modules sequence $O \to M_1 \rightarrow M_1 \times F M_2 \rightarrow M_2 \rightarrow O$, where $i$ and $j$ denote, respectively, the canonical injection from $M_1$ to $M_1 \times F M_2$ and the second projection from $M_1 \times F M_2$ to $M_2$, is exact. The equivalence class of this sequence is denoted by $[M_1 \times F M_2]$.

Remark II. With the previous notations, derivations $F$ and modules $M_1 \times F M_2$ play the same role as the cocycles $\alpha$ and modules $M_1 \times M_2$ defined in [8].

From Proposition 25.10 of [7] we have the following result.

**Proposition 12.** The correspondence $\theta : \text{Der}(R[G], T) \rightarrow \text{Ext}^1_{R[G]}(M_2, M_1)$ defined by $\theta(F) = [M_1 \times F M_2]$ is surjective whenever $M_2$ is finitely generated and projective as $R$-module.

From Theorems 5.2 and 5.3 of [9] we have the following result.

**Proposition 13.** Let $G$ be a cyclic group of order $p'$ generated by an element $\sigma$, $K$ a field of characteristic $p$, and $M$ an indecomposable $K[G]$-module. Then, $M$ is isomorphic to $\sigma \cap 1'K[G]$, where $s$ is a natural number strictly less than $p'$.

**Lemma 14.** Let $R$ be a ring and $G = G_1 \times G_2$ a direct product of two finite groups. Let $M$ be an $R[G]$-module such that the action of $G_1$ on $M$ is trivial and let $M'$ be an $R[G_2]$-module. If $M$ is isomorphic to $M'$ as $R[G_2]$-modules and if we extend the action of $G_2$ on $M'$ to $G$ by $\sigma \cdot m' = m'$, $\forall(\sigma, m') \in G_1 \times M'$, then $M$ is isomorphic to $M'$ as $R[G]$-modules.

**Proof.** Let $\psi : M \to M'$ be an isomorphism of $R[G_2]$-modules. We extend the action of $G_2$ on $M'$ to $G$ by $\sigma \cdot m' = m'$, $\forall(\sigma, m') \in G_1 \times M'$. We easily see that the application $\psi : M \to M'$ is an isomorphism of $R[G]$-modules.

Let $K$ be a commutative field of characteristic $p > 0$. Let $G = G_1 \times G_2$, where $G_1 = C_{p^m}$ and $G_2 = C_{p^n}$ are two cyclic groups of respective orders $p^m$ and $p^n$ and are generated, respectively, by $\sigma_1$ and $\sigma_2$, and let $I_j$ be the Jacobson radical of $K[G_1]$.

**Proposition 15.** Let $M$ be a finitely generated $K[G]$-module. If $I_j M = 0$, then there exists a nonzero natural number $n'$ such that $M \cong \phi^{n'}_{s,j} K[G_2]$, $0 \leq k_j < p$, as $K[G]$-modules, where the action of $G_1$ on $\phi^{n'}_{s,j}(\sigma_2 - 1)^k K[G_2]$ is trivial.

**Proof.** If $I_j M = 0$, then the action of $G_1$ on $M$ is trivial since $I_j = (\sigma_1 - 1)K[G_1]$. By Proposition 13, there exists a nonzero natural number $n'$ such that $M \cong \phi^{n'}_{s,j} (\sigma_2 - 1)^k K[G_2]$, $0 \leq k_j < p$, as $K[G_2]$-modules. Then, Lemma 14 allows concluding the following.\]
Theorem 16. Let $M$ be a finitely generated $K[G]$-module. If $J_2^1 M = 0$, then there exist two nonzero natural numbers $n'$ and $n''$ and two $K[G]$-modules $M_1 = \psi_{n'}^n(\sigma_2 - 1)^k K[G_2]$, $0 \leq k < p^n$, and $M_2 = \psi_{n''}^n(\sigma_2 - 1)^k K[G_2]$, $0 \leq k' < p^n$, where the action of $G_2$ on $M_1$ and $M_2$ is trivial, and there exists a derivation $F$ from $K[G]$ in Hom$_K(M_2, M_1)$ such that $M \cong M_1 \times_F M_2$.

Proof. We have the exact sequence of $K[G]$-modules $O \rightarrow J_1 M \hookrightarrow M \rightarrow J_1 J_1 M \rightarrow O$. As $J_2^1 M = 0$, $J_1 (J_1 M) = 0$. So by Proposition 15 there exists a nonzero natural number $n'$ such that $J_1 M \cong \psi_{n'}^n(\sigma_2 - 1)^k K[G_2]$, $0 \leq k < p^n$, as $K[G]$-modules, where the action of $G_1$ on $\psi_{n'}^n(\sigma_2 - 1)^k K[G_2]$ is trivial. We set $M_1 = \psi_{n'}^n(\sigma_2 - 1)^k K[G_2]$. We have $J_1 (J_1 J_1 M) = 0$. So by Proposition 15 there exists a nonzero natural number $n''$ such that $M/J_1 M \cong \psi_{n''}^n(\sigma_2 - 1)^k K[G_2]$, $0 \leq k' < p^n$, as $K[G]$-modules, where the action of $G_1$ on $\psi_{n''}^n(\sigma_2 - 1)^k K[G_2]$ is trivial. We set $M_2 = \psi_{n''}^n(\sigma_2 - 1)^k K[G_2]$. Then, Proposition 12 shows that $M \cong M_1 \times_F M_2$ for a derivation $F$ from $K[G]$ in Hom$_K(M_2, M_1)$.

If $p = 2$, $G_1 = C_2$, and $G_2 = C_{2^n}$, then we have the following corollary.

Corollary 17. For all finitely generated $K[G]$-modules there exist two nonzero natural numbers $n'$ and $n''$ and two $K[G]$-modules $M_1 = \psi_{n'}^n(\sigma_2 - 1)^k K[G_2]$, $0 \leq k < 2^n$, and $M_2 = \psi_{n''}^n(\sigma_2 - 1)^k K[G_2]$, $0 \leq k' < 2^n$, where the action of $G_1$ on $\psi_{n'}^n(\sigma_2 - 1)^k K[G_2]$ is trivial. We set $M_1 = \psi_{n'}^n(\sigma_2 - 1)^k K[G_2]$ and $M_2 = \psi_{n''}^n(\sigma_2 - 1)^k K[G_2]$, then there is a derivation $F$ from $K[G]$ in Hom$_K(M_2, M_1)$ such that $M \cong M_1 \times_F M_2$.

Proof. We have $J_1 = (\sigma_2 - 1)K[G_1]$ and as $(\sigma_2 - 1)^2 = 0$ since the field $K$ is of characteristic $p = 2$, $J_2^1 M = 0$. So $J_1 J_1 M = 0$ for $K[G]$-module of finite type $M$. The rest is a simple application of Theorem 16.

Now we return to cases $G_1 = C_{p^n}$ and $G_2 = C_{p^n}$, where $G_2$ is generated by an element $\sigma_2$, and let $J_2$ be the Jacobson radical of $K[G_2]$. For an integer $k = p^n$ with $0 < r \leq n$ and for the subgroup $H_2$ of $G_2$ generated by $\sigma_2^p$, we have the following result.

Theorem 18. Let $M$ be a finitely generated $K[G]$-module, with $J_2^{k+1} M = 0$. If $M/J_2^k M \cong K[G_2]$ is $R$-free and of type $k$, then there exist two nonzero natural numbers $n'$ and $n''$ and two $K[G]$-modules $M_1 = \psi_{n'}^n(\sigma_2 - 1)^k K[G_2]$, $0 \leq k < p^n$, and $M_2 = (K[G_2 \times H_2])^n$, where the action of $G_2$ on $M_1$ is trivial, and there exists a derivation $F$ from $K[G]$ in Hom$_K(M_2, M_1)$ such that $M \cong M_1 \times_F M_2$.

Proof. We have the following exact sequence:

$$O \rightarrow J_2^k M \hookrightarrow M \rightarrow \frac{M}{J_2^k M} \rightarrow O. \quad (13)$$

As $J_2^k (J_2^k M) = 0$, by Proposition 15, there exists a nonzero natural number $n'$ such that $J_2^k M = \psi_{n'}^n(\sigma_1 - 1)^k K[G_1]$, $0 \leq k_1 < p^n$, as $K[G]$-modules, where the action of $G_2$ on $\psi_{n'}^n(\sigma_1 - 1)^k K[G_1]$ is trivial. $M/J_2^k M$ is a $K[G]$-module of type $k$ with $J_2^k (M/J_2^k M) = 0$, more $M/J_2^k M |_{G_1}$ is $R$-free, and $k = p^n$ with $0 < r \leq n$. Then, Theorem 9 shows that $M/J_2^k M$ is a free $K[G_2 \times H_2]$-module. Therefore there exists a nonzero natural number $n''$ such that $M/J_2^k M \cong (K[G_2 \times H_2])^n$. The rest is a simple application of Proposition 12.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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