Some deformations of $U[sl(2)]$
and their representations

Nguyen Anh Ky
Institute of Physics
P.O. Box 429, Bo Ho, Hanoi 10 000, Vietnam

Abstract

Some one- and two-parametric deformations of $U[sl(2)]$ and their representations are considered. Interestingly, a newly introduced two-parametric deformation admits a class of infinite-dimensional representations which have no classical (non-deformed) and one-parametric deformation analogues, even at generic deformation parameters.

PACS: 02.20Uw, 03.65.Fd, MSC: 17B37, 20G05.

1. Introduction

For over two decades, quantum groups [1] - [5] have been a subject of great interest in physics and mathematics. A number of mathematical structures and physical applications of quantum groups have been obtained and investigated in details [6] - [13]. Depending on points of view, quantum groups can be approached to in several ways. One of the approaches to quantum groups are the so-called Drinfel’d-Jimbo deformation of universal enveloping algebras [2] [3]. It is shown that the quantum groups of this kind (called also quantum algebras) are non-commutative and non-cocommutative Hopf algebras [2]. By construction, such a quantum group depends on one or more parameters which are, in general, complex. One-parametric deformations have been well investigated and understood in various aspects, while multi-parametric deformations are less understood, in spite of some research progress made lately (see [14] - [19] and references therein). In this report, we consider some one- and two-parametric deformations of the universal enveloping algebra $U[sl(2)]$ and their representations.

The quantum group $U_q[sl(2)]$ as an one-parametric deformation of the universal enveloping algebra $U[sl(2)]$ is one of the best investigated quantum groups [20] - [24].
What about two-parametric deformations of $U[sl(2)]$, they have been considered in several versions and in different aspects (see, for example, [14] - [16] and references therein) though some of them are, in fact, equivalent to one-parametric deformations (upto some rescales). In this report we consider only two of them, denoted here by $U_{pq}^{(1)}[sl(2)]$ and $U_{pq}^{(2)}[sl(2)]$ (a common notation is $U_{pq}[sl(2)]$). The first quantum group $U_{pq}^{(1)}[sl(2)]$ was already given before as a subgroup of a quantum supergroup [16] [18] [19], while the second one, $U_{pq}^{(2)}[sl(2)]$, is being introduced now. It turns out that this new quantum group $U_{pq}^{(2)}[sl(2)]$ admits a class of infinite-dimensional representations which have no analogue in either the case of the non-deformed $sl(2)$ or the cases of previously introduced one- and two-parametric deformations of $U[sl(2)]$. For consistency, we first consider the non-deformed $sl(2)$, the one-parametric deformation $U_q[sl(2)]$ and the two-parametric deformations $U_{pq}^{(1)}[sl(2)]$ in the next three sections, Sect. 2, Sect. 3 and Sect. 4, respectively. Then the two-parametric deformations $U_{pq}^{(2)}[sl(2)]$ is introduced and considered in Sect. 5. Some discussions and conclusion are made in the last section, Sect. 6. Let us start now with $sl(2)$.

2. $sl(2)$ and representations

As is well known, $sl(2)$ can be generated by three generators, say $E_+, E_-$ and $H$, satisfying the commutation relations

$$[H, E_\pm] = \pm E_\pm, \quad [E_+, E_-] = 2H. \tag{1}$$

Demanding $(H)^\dagger = H$ and $(E_\pm)^\dagger = E_{\mp}$, a representation induced from a (normalised) highest weight state $|j, j\rangle$,

$$H|j, j\rangle = j|j, j\rangle, \quad E_+|j, j\rangle = 0, \tag{2}$$

has the matrix elements

$$H |j, m\rangle = m |j, m\rangle, \quad E_\pm |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1\rangle, \tag{3}$$

where $|j, m\rangle$, $m \leq j$, is one of the orthonormalised states which can be obtained by acting an appropriate monomial of $E_-$ on the highest weight state $|j, j\rangle$,

$$|j, m\rangle = A_n(E_-)^n |j, j\rangle, \quad m = j - n, \tag{4}$$

with $A_n$ a normalised coefficient, which for a non-negative (half) integral highest weight $j$ has the form

$$A_n = \sqrt{\frac{(2j)! \, n!}{(2j - n)!}}. \tag{5}$$
In this case, the representations constructed are simultaneously highest weight and lowest weight, that is,
\[ E_+|j, j\rangle = 0, \quad E_-|j, -j\rangle = 0, \]
so, they are finite-dimensional (and also unitary and irreducible) of dimension \( 2j + 1 \) (with \( m \) taking by steps of 1 all values in the range \(-j \leq m \leq j\)). The situation is similar in the case of the one-parametric quantum group \( U_q[sl(2)] \) at a generic deformation parameter \( q \) (i.e., at \( q \) not being a root of unity).

3. One-parametric deformation \( U_q[sl(2)] \)

As mentioned above, \( U_q[sl(2)] \) is one of the best investigated quantum groups. It can be defined through three generators \( E_+, E_- \) and \( H \) subject to the deformed defining relations
\[ [H, E_{\pm}] = \pm E_{\pm}, \quad [E_+, E_-] = [2H]_q, \]
where
\[ [X]_q = \frac{q^X - q^{-X}}{q - q^{-1}}, \]
is the so-called one-parametric deformation (or \( q \)-deformation, for short) of an operator or a number \( X \), with \( q \) being a, complex in general, parameter. At a generic \( q \), the representation structure of \( U_q[sl(2)] \) is similar to that of \( sl(2) \), more precisely, a representation of \( U_q[sl(2)] \) induced in the same way (as done for \( sl(2) \)) from a highest weight state \( |j, j\rangle \) defined as in (2) has the matrix elements
\[ H |j, m\rangle = m |j, m\rangle, \]
\[ E_{\pm} |j, m\rangle = \sqrt{|j \mp m|}_q |j \pm m + 1\rangle |j, m \pm 1\rangle. \]
These representations are again of finite dimension \( 2j + 1 \) (and irreducible as \( q \) is generic) for non-negative (half) integral highest weights \( j \), since Eqs. (6) are again satisfied (i.e., the representations are again highest weight and lowest weight simultaneously).

4. A two-parametric deformation \( U^{(1)}_{pq}[sl(2)] \)

The quantum group \( U^{(1)}_{pq}[sl(2)] \) considered here is only one of the two-parametric deformations of \( U[sl(2)] \) introduced in the literature. Appeared in [16] as a subgroup of a quantum supergroup \( U_{pq}[gl(2/1)] \), this quantum group can be generated by three
generators $E_+,$ $E_-$ and $H$ satisfying somewhat more twisted defining relations

$$[H, E_\pm] = \pm E_\pm, \quad [E_+, E_-] = \left(\frac{p}{q}\right)^{J-H} [2H]_{pq},$$

where $J$ is the so-called the "maximal spin" (or the hight weight) operator and

$$[X]_{pq} = \frac{q^X - p^{-X}}{q - p^{-1}},$$

is a notation for a two-parametric deformation (or $pq$-deformation) of an operator or a number $X$, with $p$ and $q$ being deformation parameters which are in general complex. Constructed in the same way for generic $p$ and $q$, the (highest weight) representations of $U^{(1)}_{pq}[sl(2)]$ are similar to those of $U_q[sl(2)]$ and $sl(2)$,

$$H |j, m\rangle = m |j, m\rangle,$$

$$E^\pm |j, m\rangle = \sqrt{|j + m|_{pq}[j \pm m + 1]_{pq}} |j, m \pm 1\rangle.$$  

(12)

They are again finite - dimensional for non-negative (half) integral $j$. Although the quantum supergroup $U_{pq}[gl(2/1)]$ is not equivalent to an one-parameter deformation of $U[gl(2/1)]$, its subgroup $U^{(1)}_{pq}[sl(2)]$ can be shown to be equivalent to an one-parametric deformation of $U[sl(2)]$. The situation is, however, different in the next case.

5. An alternative two-parametric deformation $U^{(2)}_{pq}[sl(2)]$

Now we introduce an alternative two-parametric quantum group $U^{(2)}_{pq}[sl(2)]$ generated also by three generators $E^+, E^-$ and $H$ but the latter now satisfy defining relations simpler than those of $U^{(1)}_{pq}[sl(2)]$,

$$[H, E^\pm] = \pm E^\pm, \quad [E^+, E^-] = [H]_{pq}.$$  

(13)

Following the way of constructing highest weight representations in the previously considered cases, we find representations of $U^{(2)}_{pq}[sl(2)]$ corresponding to (3), (9) and (12),

$$H |j, m\rangle = m |j, m\rangle,$$

$$E^+ |j, m\rangle = \left(\frac{q^{j+m+1}[j - m]_q - p^{-j-m-1}[j - m]_p}{q - p^{-1}}\right)^{1/2} |j, m + 1\rangle,$$

$$E^- |j, m\rangle = \left(\frac{q^{j+m}[j - m + 1]_q - p^{-j-m}[j - m + 1]_p}{q - p^{-1}}\right)^{1/2} |j, m - 1\rangle.$$

(14)
In comparison with the case of $U_{pq}[sl(2)]$, for the sake of simplicity of the defining relations, the matrix elements in this case become a bit more complicated. Moreover, even at a non-negative (half) integral highest weights $j$, we observe that representations of the form (14) are, in general, infinite-dimensional, unlike those of $sl(2)$, $U_q[sl(2)]$ and $U^{(1)}_{pq}[sl(2)]$ in (3), (9) and (12), respectively. These representaions of $U^{(2)}_{pq}[sl(2)]$ are still highest weight by construction but they are no longer lowest weight for arbitrary $p$ and $q$. It is a new class of infinite-dimensional representations of $U_{pq}[sl(2)]$ not found before in the cases of $sl(2)$ and its previously considered deformations. Note that $U^{(2)}_{pq}[sl(2)]$ is by no means equivalent to the one-parametric $U_q[sl(2)]$ unless at $p = q$ and at, maybe, other special choices of $p$ and $q$.

6. Conclusion

We have introduced a new quantum group $U^{(2)}_{pq}[sl(2)]$ which is a two-parametric deformation of $U[sl(2)]$. Interestingly, it is showed that this quantum group admits a class of infinite-dimensional representations which have no analogues in the previously considered cases of the non-deformed $sl(2)$, the one-parametric deformation $U_q[sl(2)]$ and other two-parametric deformations of $U[sl(2)]$. It is a new phenomenon of $U^{(2)}_{pq}[sl(2)]$.

**Proposition 1:** Highest weight representations of the two-parametric quantum algebra $U^{(2)}_{pq}[sl(2)]$ defined in (10) and (11) are in general infinite-dimensional, even for non-negative (half) integral highest weights.

These representations are still unitary and, in general, irreducible. They may become reducible under certain additional conditions such as one or both of the deformation parameters to be roots of unity or at some other special choices if allowed. However, in these cases, it is possible to extract finite-dimensional irreducible representations from the infinite-dimensional reducible ones.

**Proposition 2:** The representations (14) are reducible and contain finite-dimensional subrepresentations iff the equation

$$f(x) \equiv q^{2j-x}[x+1]_q - p^{-2j+x}[x+1]_p = 0$$

has positive integral solutions.

Other classes of infinite-dimensional representations of $U_{pq}[sl(2)]$ may be found by using the method of [25].

**Acknowledgement:** I would like to thank Randjbar-Daemi for kind hospitality at
the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, where some years ago a stage of this investigation was dealt with. Inspiring discussions with Vo Thanh Cuong and Nguyen thi Hong Van are also acknowledged. This work was partially supported by the Vietnam National Research Program for Natural Sciences under Grant 411101.

References

[1] L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtajan, *Algebra and Analys*, 1, 178 (1987).

[2] V. Drinfel’d, "Quantum groups", *J. Sov. Math.*, 41, 898 (1988); Zap. Nauch. Semin., 155, 18 (1986); also in *Proceedings of the International Congress of Mathematicians, Berkeley 1986*, vol 1, The American Mathematical Society, Providence, RI, 1987, pp. 798 - 820.

[3] Yu. Manin, *Quantum groups and non-commutative geometry*, Centre des Recherchers Mathématiques, Montréal, 1988.

[4] M. Jimbo, *Lett. Math. Phys.*, 10, 63 (1985), *ibid* 11, 247 (1986).

[5] S. I. Woronowicz, *Comm. Math. Phys.*, 111, 613 (1987).

[6] M. Jimbo, ed., *Yang-Baxter equation in intergrable systems*, World Scientific, Singapore, 1989.

[7] C. Gómez, M. Ruiz-Altaba and G. Sierra, *Quantum groups in two-dimensional physics*, Cambridge University Press, Cambridge, 1996.

[8] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge, 1994.

[9] Ch. Kassel, *Quantum groups*, Springer - Verlag, New York, 1995.

[10] C. N. Yang and M. L. Ge, eds., *Braid groups, knot theory and statistical mechanics*, World Scientific, Singapore, 1989.

[11] H. D. Doebner and J. D. Hennig, eds., *Quantum groups*, Lecture Notes in Physics, Springer - Verlag, Berlin, 1990, vol. 370.

[12] P. P. Kulish, ed., *Quantum groups*, Lecture Notes in Mathematics, Springer - Verlag, Berlin, 1992, vol. 1510.
[13] S. Majid, *Foundation of quantum group theory*, Cambridge University Press, Cambridge, 1995.

[14] A. Schirrmacher, J. Wess and B. Zumino, *Z. Phys. C*, 49, 317 (1991).

[15] O. Ogievetsky, J. Wess, *Z. Phys. C*, 50, 123 (1991).

[16] Nguyen Anh Ky, *J. Phys. A: Math. Gen.*, 29, 1541 (1996) or math.QA/9909067

[17] V. K. Dobrev and E. H. Tahri, *J. Phys. A: Math. Gen.*, 32, 4209 (1999).

[18] Nguyen Anh Ky, *J. Math. Phys.*, 41, 6487 (2000); math.QA/0005122

[19] Nguyen Anh Ky, *J. Phys. A: Math. Gen.*, 34, 7881 (2001); math.QA/0104105

[20] P. Kulish and N. Reshetikhin, *J. Sov. Math.*, 23, 2435 (1983).

[21] E. Sklyanin, *Funk. Anal. Appl.*, 16, 263 (1982); ibit 17, 273 (1983).

[22] P. Roche and D. Arnaudon, *Lett. Math. Phys.*, 17, 295 (1989).

[23] V. A. Groza, I. I. Kachurik and A. U. Klimyk, *J. Math. Phys.*, 31, 2769 (1990).

[24] V. Pasquier and H. Saleur, *Nulc. Phys. B*, 330, 523 (1990).

[25] L. Hadjiivanov and D Stoyanov, *J. Phys. A: Math. Gen.*, 24, L907 (1991).