Semicontinuity of the Łojasiewicz exponent

Arkadiusz Płoski

January 2011

Abstract. We prove that the Łojasiewicz exponent $l_0(f)$ of a finite holomorphic germ $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ is lower semicontinuous in any multiplicity-constant deformation of $f$.

1 Introduction

Let $\mathbb{C}\{z\}$ denote the ring of convergent power series in $n$ variables $z = (z_1, \ldots, z_n)$. Any sequence of convergent power series $h = (h_1, \ldots, h_p) \in \mathbb{C}\{z\}^p$ without constant term defines the germ of a holomorphic mapping $h : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$. We put $\text{ord } h = \inf_k \{\text{ord } h_k\}$, where $\text{ord } h_k$ denotes the order of vanishing of $h_k$ at 0 (by convention $\text{ord } 0 = +\infty$). If $|z| = \max_{j=1}^n |z_j|$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ then $\text{ord } h$ for $h \neq 0$ is the largest $\alpha > 0$ such that $|h(z)| \leq c|z|^\alpha$ with a constant $c > 0$ for $z \in \mathbb{C}^n$ close to 0 in $\mathbb{C}^n$.

Let $f = (f_1, \ldots, f_n) \in \mathbb{C}\{z\}^n$, $f(0) = 0$, define a finite holomorphic germ $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$; i.e. such that $f$ has an isolated zero at the origin 0 in $\mathbb{C}^n$ and let $I(f)$ be the ideal of $\mathbb{C}\{z\}$ generated by $f_1, \ldots, f_n$. Then $I(f)$ is of finite codimension in $\mathbb{C}\{z\}$ and the multiplicity $m_0(f)$ of $f$ is equal by definition to $\dim_{\mathbb{C}} \mathbb{C}\{z\}/I(f)$. There exist arbitrary small neighbourhoods $U$ and $V$ of 0 in $\mathbb{C}^n$ such that the mapping $U \ni z \mapsto f(z) \in V$ is an $m_0(f)$-sheeted branched covering, see [4], chapter 5, §2.

Another important characteristic of a finite germ $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ introduced and studied by M. Lejeune-Jalabert and B. Teissier in 1973–1974 seminar at the Ecole Polytechnique (in a very general setting), see [3], is the Łojasiewicz exponent $l_0(f)$ defined to be the smallest $\theta > 0$ such that there exist a neighbourhood $U$ of 0 in $\mathbb{C}^n$ and a constant $c > 0$ such that

$$|f(z)| \geq c|z|^\theta \quad \text{for all } z \in U.$$

The Łojasiewicz exponent can be calculated by means of analytic arcs (see [3].

2010 Mathematics Subject Classification. Primary: 32S05; Secondary 14B05.

Keywords and phrases: Łojasiewicz exponent, multiplicity-constant deformation, Newton polygon.
\[\phi(s) = (\phi_1(s), \ldots, \phi_n(s)) \in \mathbb{C}\{s\}^n, \phi(0) = 0, \phi(s) \neq 0 \text{ in } \mathbb{C}\{s\}^n:\]

\[l_0(f) = \sup_{\phi} \left\{ \frac{\text{ord } f \circ \phi}{\text{ord } \phi} \right\}.\]

The following lemma \cite{7}, Corollary 1.4 will be useful for us.

**Lemma 1.1** Let \(f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)\) be a finite holomorphic germ. Then \(l_0(f) \leq m_0(f)\) with equality if and only if \(\text{rank } (\frac{\partial f}{\partial z_j}(0)) \geq n - 1\).

Now, let \(h \in \mathbb{C}\{z\}, h(0) = 0\), be a convergent power series defining an isolated singularity at \(0 \in \mathbb{C}^n\) i.e. such that the gradient of \(h, \nabla h = (\frac{\partial h}{\partial z_1}, \ldots, \frac{\partial h}{\partial z_n}) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)\) is finite at \(0 \in \mathbb{C}^n\). Then \(\mu_0 := m_0(\nabla h)\) is the Milnor number of the singularity \(h = 0\). Teissier calculated in \cite{9} \(L_0(h) := l_0(\nabla h)\) in terms of polar invariants of the singularity and proved that the Lojasiewicz exponent \(L_0(h)\) is lower semicontinuous in any \(\mu\)-constant deformation of the singularity \(h = 0\). He showed also that if we don’t assume \(\mu = \text{constant}\) that \(L_0(h)\) is neither upper or lower semicontinuous, see \cite{10}. The “jump phenomena” of the Lojasiewicz exponent was rediscovered by some authors, see \cite{5}. The aim of this note is to prove that the Lojasiewicz exponent is lower semicontinuous in any multiplicity-constant deformation of the finite holomorphic germ. The proof is based on the formula for the Lojasiewicz exponent given by the author in \cite{8} (see also Lemma 3.3 in Section 5).

## 2 Result

Let \(f = (f_1, \ldots, f_n) \in \mathbb{C}\{z\}^n, f(0) = 0\), define a finite holomorphic germ. A sequence \(F = (F_1, \ldots, F_n) \in \mathbb{C}\{t, z\}^n\) of convergent power series in \(k + n\) variables \((t, z) = (t_1, \ldots, t_k, z_1, \ldots, z_n)\) is a deformation of \(f\) if \(F(0, z) = f(z)\) in \(\mathbb{C}\{z\}\) and \(F(t, 0) = 0\) in \(\mathbb{C}\{t\}\). Then the sequence \((t, F(t, z)) \in \mathbb{C}\{t, z\}^{k+n}\) defines a holomorphic germ \((\mathbb{C}^{k+n}, 0) \to (\mathbb{C}^{k+n}, 0)\) of multiplicity \(m_0(f)\). Indeed, it is easy to check that the algebras \(\mathbb{C}\{z\}/I(t)\) and \(\mathbb{C}\{t, z\}/I(t, F)\) are \(\mathbb{C}\)-isomorphic.

We put \(F_\xi = F(t_\xi, z) \in \mathbb{C}\{z\}^n\) for \(\xi \in \mathbb{C}^k\) close to \(0\). Then \(F_\xi(0) = 0\) and \(m_0(F_\xi) \leq m_0(F_0) = m_0(f)\) for \(\xi \in \mathbb{C}^k\) close to \(0\), see \cite{13}, chapter 2, \S 5. We say that \(F\) is a multiplicity-constant deformation of the germ \(f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)\) if \(m_0(F_\xi) = m_0(F_0)\) for \(\xi \in \mathbb{C}^k\) close to \(0\).

The main result of this note is

**Theorem 2.1** Let \(F \in \mathbb{C}\{t, z\}^n\) be a multiplicity-constant deformation of the germ \(f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)\). Then

\[l_0(F_0) \leq l_0(F_\xi) \quad \text{for } \xi \in \mathbb{C}^k \text{ close to } 0 \in \mathbb{C}^k.\]

Moreover, if \(F\) is a one-parameter deformation \((k = 1)\), then \(l_0(F_\xi)\) is constant for \(\xi \neq 0\) close to \(0 \in \mathbb{C}^k\).
The proof of the theorem is given in Section 4 of this note. The inequality stated above may be strict:

**Example 2.2** (see [5], §5).

Let \( F(t, z_1, z_2) = (t z_1 + z_1^a + z_2^b + z_1^c - z_2^d) \in \mathbb{C}\{t, z_1, z_2\}\) be a one-parameter deformation of \( f(z_1, z_2) = (z_1^a + z_2^b - z_1^c) \). Assume that \( a, b, c > 1 \) are integers such that \( \text{GCD}(p, q) = 1 \) and \( bp < q \). Then \( m_0(F_t) = bp \) for all \( t \in \mathbb{C} \), i.e., \( F \) is a multiplicity-constant deformation. If \( \underline{t} \neq 0 \) then \( \text{ord} F_{\underline{t}} = 1 \) and we get \( l_0(F_{\underline{t}}) = m_0(F_{\underline{t}}) = bp \) by Lemma 1.1. Since \( \text{ord} F_0 > 1 \) we get \( \frac{l_0(F_0) < m_0(F_0) = bp} \) by another part of Lemma 1.1.

Note that C. Bivià-Ausina, see [2], Corollary 2.5 proved a result on the semicontinuity of the Lojasiewicz exponent which however, does not imply our Theorem 2.1.

One can also indicate the deformations for which the Lojasiewicz exponent is upper semicontinuous like multiplicity.

**Proposition 2.3** Let \( F \in \mathbb{C}\{t, z\}^n \) be a deformation of \( f \in \mathbb{C}\{z\}^n \) such that \( \text{rank} \left( \frac{\partial F}{\partial z} (t, 0) \right) \geq n - 1 \) for \( t \in \mathbb{C}^k \) close to \( 0 \in \mathbb{C}^k \). Then \( l_0(F_t) \leq l_0(F_0) \) for \( t \in \mathbb{C}^k \) close to \( 0 \).

**Proof.** By Lemma 1.1 we get \( l_0(F_{\underline{t}}) = m_0(F_{\underline{t}}) \) for \( \underline{t} \in \mathbb{C}^k \) close to \( 0 \) and the proposition follows from the upper semicontinuity of the multiplicity.

**Example 2.4** Let \( f(z) = (z_1^m, z_2, \ldots, z_n) \) with \( m > 1 \) and let \( F(t, z) = f(z_1 + t, z_2, \ldots, z_n) - f(t, 0, \ldots, 0) = \left( (z_1 + t)^m - t^m, z_2, \ldots, z_n \right) \) be a one-parameter deformation of \( f \). Then \( F(t, z) \) satisfies the assumption of Proposition 2.3. Using Lemma 1.1 we check that \( l_0(F_{\underline{t}}) = m_0(F_{\underline{t}}) = 1 \) for \( \underline{t} \neq 0 \) and \( l_0(F_0) = m_0(F_0) = m \).

In the example above the deformation of \( f \) is given by the translation of coordinates. Even for such a deformation the Lojasiewicz exponent may be not upper semicontinuous:

**Example 2.5** Let \( f(z_1, z_2, z_3) = (z_1^2, z_2^3, z_3^4 - z_1 z_2) \in \mathbb{C}\{z_1, z_2, z_3\}^3 \) and let \( F(t, z_1, z_2, z_3) = f(t + z_1, z_2, z_3) - f(t, 0, 0) = (2t z_1 + z_1^2, z_2^3, -t z_2 + z_3^4 - z_1 z_2) \). Then by Lemma 1.1 we get \( l_0(F_{\underline{t}}) = m_0(F_{\underline{t}}) = 9 \) for \( \underline{t} \neq 0 \). On the other hand \( m_0(F_0) = 18 \) and \( l_0(F_0) = \frac{18}{3} \) (see Example 3.3 of this note). The exponent \( l_0(F_0) \) is attained on the arc \( \phi(s) = (s^9, s^6, s^3) \).

**Remark 2.6** The case of \( \mu \)-constant deformations of isolated hypersurface singularities is much more subtle. The Teissier’s conjecture that “\( \mu \)-constant implies the constancy of the Lojasiewicz exponent” [9] Question on p. 278 is still open.
3 Characteristic polynomial and the Łojasiewicz exponent

Let $f = (f_1, \ldots, f_n) \in \mathbb{C}\{z\}^n$ be a sequence of convergent power series defining a finite holomorphic germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. Then the extension $\mathbb{C}\{z\} \supset \mathbb{C}\{f\}$ is a finite $\mathbb{C}\{f\}$-module. For any $h \in \mathbb{C}\{z\}$ there is a unique irreducible polynomial $Q_{f,h} = s^{m_h} + c_1(w)s^{m_h-1} + \cdots + c_m(w) \in \mathbb{C}\{w\}[s]$ in $n+1$ variables $(w, s) = (w_1, \ldots, w_n, s)$ such that $Q_{f,h}(f, h) = 0$. It is called the minimal polynomial of $h$ relative to $f$. Its degree $m_{f,h} := \deg_s Q_{f,h}$ divides the multiplicity $m(f)$; we put $P_{f,h} = Q_{f,h}^r$, where $r = \frac{m(f)}{m_{f,h}}$ and call $P_{f,h}$ the characteristic polynomial of $h$ relative to $f$. If $h(0) = 0$ then $Q_{f,h}$ and consequently $P_{f,h}$ is a distinguished polynomial.

Remark 3.1 Let $L = \mathbb{C}\{z\}_{(0)}$ and $K = \mathbb{C}\{f\}_{(0)}$ be fields of fractions of the ring $\mathbb{C}\{z\}$ and $\mathbb{C}\{f\}$, respectively. Then $Q_{f,h}(f, s) \in K[s]$ is the monic minimal polynomial of $h$ relative to the field extension $L/K$ and $P_{f,h}(f, s)$ is the characteristic polynomial of $h$ relative to $L/K$. For the various equivalent definitions of the characteristic polynomial, see Zariski-Samuel [14], Chapter II, §10.

The lemma below follows immediately from the Rückert-Weierstrass parametrization theorem, see [1], §31, (31.23).

Lemma 3.2 Let $P(w, s) = s^m + a_1(w)s^{m-1} + \cdots + a_m(w) \in \mathbb{C}\{w\}[s]$ be a distinguished polynomial of degree $m = m_0(f)$ and let $h \in \mathbb{C}\{z\}$, $h(0) = 0$. Then the two conditions are equivalent

(i) $P(w, s)$ is the characteristic polynomial of $h$ relative to $f$,

(ii) Let $U$ and $V$ be neighbourhoods of $0 \in \mathbb{C}^n$ such that the mapping $U \ni \underline{z} \mapsto f(\underline{z}) \in V$ is a $m_0(f)$-sheeted branched covering and $h = h(\underline{z})$ is convergent in $V$. Then the set $\{(w, s) \in V \times \mathbb{C} : P(w, s) = 0\}$ is the image of $U$ by the mapping $U \ni \underline{z} \mapsto (f(\underline{z}), h(\underline{z})) \in V \times \mathbb{C}$, provided that $U$, $V$ are small enough.

To study the Łojasiewicz exponent $l_0(f)$ it is useful to consider the inequalities of the type

(L) \[ |h(\underline{z})| \leq c|f(\underline{z})|^\theta \] near the origin $0 \in \mathbb{C}^n$.

The least upper bound of the set of all $\theta > 0$ for which (L) holds for some constant $c > 0$ in a neighbourhood $U \subset \mathbb{C}^n$ of $0$ will be denoted $o_f(h)$ and called the Łojasiewicz exponent of $h$ relative to $f$.

Lemma 3.3 Let $P_{f,h}(w, s) = s^m + a_1(w)s^{m-1} + \cdots + a_m(w) \in \mathbb{C}\{w, s\}$ be the characteristic polynomial of $h \in \mathbb{C}\{z\}$, $h \neq 0$, relative to $f$. Let $I = \{i \in \{1, \ldots, m\} : a_i \neq 0\}$. Then

\[ o_f(h) = \min_{i \in I} \left\{ \frac{1}{i} \ord_a \right\} . \]
Proof. (after [8], proof of Theorem 2.3). Let \( U \) and \( V \) be neighbourhoods of \( 0 \in \mathbb{C}^n \) such that the mapping \( U \ni z \to f(z) \in V \) is an \( m_0(f) \)-sheeted branched covering and \( h = h(z) \) is convergent in \( V \). Let \( P(w, s) \) be the characteristic polynomial of \( h \) relative to \( f \). Then by Lemma 3.2 we have that the inequality \( |h(z)| < |f(z)|^{\theta} \), \( z \in U \), is equivalent to the estimate

\[
\{(w, s) \in V \times \mathbb{C} : P(w, s) = 0\} \subset \{(w, s) \in V \times \mathbb{C} : |s| \leq |w|^\theta\}
\]

for \( U, V \) small enough.

Let \( \Theta_0 = \min_{i \in I} \left\{ \frac{1}{l_i} \ord a_i \right\} \). It is easy to check (see [8], Proposition 2.2) that \( \Theta_0 \) is the largest number \( \theta > 0 \) for which (\( * \)) holds. This proves the lemma. \( \square \)

**Lemma 3.4** \( l_0(f) = \left( \min_{i=1}^n \{\alpha_i(z_i)\} \right)^{-1} \).

Proof. Obvious. \( \square \)

**Example 3.5** Let us get back to Example 2.5. Let \( f = (f_1, f_2, f_3) = (z_1^2, z_2^3, z_3^4 - z_1 z_2) \). We have \( m_0(f) = 18 \). The characteristic polynomials of \( z_1 \) and \( z_2 \) are \( (s_1^2 - w_1)^9 \) and \( (s_2^2 - w_2)^6 \) respectively, hence \( \alpha_f(z_1) = \frac{1}{2}, \alpha_f(z_2) = \frac{1}{2} \). To calculate \( \alpha_f(z_3) \) let us observe that

\[
P(w, s) = (s^3 - w_3)^6 - w_1^3 w_2^2
\]

is the characteristic polynomial of \( h = z_3 \) relative to \( f \). Indeed, we have \( P(f, z_3) = 0 \) in \( \mathbb{C}\{z\} \) and \( P(w, s) \) is irreducible: if \( u \) is a variable then \( P(u, u, 0, s) = s^{18} - u^9 \) is irreducible, whence \( P(w, s) \) is irreducible.

Write \( P(w, s) = 18 \cdot (w_3 - w_1)^{15} + \cdots + (w_3^5 - w_1^3 w_2^2) \). Using Lemma 3.3 we check that \( \alpha_f(z_3) = \frac{1}{18} \). Then we get \( l_0(f) = \left( \min \left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{18} \right\} \right)^{-1} = \frac{18}{5} \).

**Lemma 3.6** Let \( F = F(t, z) \in \mathbb{C}\{t, z\}^n \) be a multiplicity-constant deformation of a finite germ \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0) \) and let \( h \in \mathbb{C}\{z\} \). \( h(0) = 0 \). Let \( P_h(t, w, s) = s^m + a_1(t, w)s^{m-1} + \cdots + a_m(t, w) \in \mathbb{C}\{t, w\}\{s\} \) be the characteristic polynomial of \( h \) relative to \( f(t, z) \). Then for \( t \in \mathbb{C}^k \) close enough to \( 0 \in \mathbb{C}^k \) the polynomial \( P_h(t, w, s) = s^m + a_1(t, w)s^{m-1} + \cdots + a_m(t, w) \in \mathbb{C}\{w\}\{s\} \) is the characteristic polynomial of \( h \) relative to \( F(t, z) \in \mathbb{C}\{z\}^n \).

Proof. There exist arbitrary small neighbourhoods \( U \) and \( V \) of \( 0 \in \mathbb{C}^n \) and \( W \) of \( 0 \in \mathbb{C}^k \) such that mapping \( W \times U \ni (t, z) \to (F(t, z), h(z)) \in W \times V \) is \( m_0(f) \)-sheeted branched covering. Since \( F = F(t, z) \) is a multiplicity-constant deformation the mappings \( U \ni z \to F(t, z) \in V \) we are also \( m_0(f) \)-sheeted branched coverings. Fix \( h = h(z) \in \mathbb{C}\{z\} \), \( \alpha_f(0) = 0 \). Shrinking the neighbourhoods \( W \times U \) and \( W \times V \) we get by Lemma 3.2 that the image of \( W \times U \) under the mapping \( W \times U \ni (t, z) \to (F(t, z), h(z)) \in W \times V \times \mathbb{C} \) has the equation \( P_h(t, w, s) = 0 \) in \( W \times V \times \mathbb{C} \). Therefore the image of \( U \) under the mapping \( U \ni z \to (F(t, z), h(z)) \in V \times \mathbb{C} \) has the equation \( P_h(t, w, s) = 0 \) in \( V \times \mathbb{C} \). Using again Lemma 3.2 we have that \( P_h(t, w, s) \) is the characteristic polynomial of \( h \) relative to \( F(t, z) \). \( \square \)

5
4 Proof of the main result

Let us begin with

**Theorem 4.1** Let $F = F(t, z) \in \mathbb{C}\{t, z\}^{n}$ be a multiplicity-constant deformation of a finite germ $f : (\mathbb{C}^{n}, 0) \to (\mathbb{C}^{n}, 0)$. Let $h \in \mathbb{C}\{z\}$, $h \neq 0$. Then

$$o_{F}(h) \leq o_{F_{\zeta}}(h) \quad \text{for } \zeta \in \mathbb{C}^{k} \text{ close to } 0 \in \mathbb{C}^{k}.$$ 

Moreover, if $F$ is a one-parameter deformation ($k = 1$), then $o_{F}(h)$ is constant for $t \neq 0$ close to $0 \in \mathbb{C}$.

**Proof.** Let $P_{h}(t, w, s) = s^{m} + a_{1}(t, w)s^{m-1} + \cdots + a_{m}(t, w) \in \mathbb{C}\{t, w\}[s]$ be the characteristic polynomial of $h$ relative to $(t, F(t, z)) \in \mathbb{C}\{t, z\}^{k+n}$. Then by Lemma 3.2 for $\zeta \in \mathbb{C}^{k}$ close to $0 \in \mathbb{C}^{k}$ we have that $P_{h}(\zeta, w, s) = s^{m} + a_{1}(\zeta, w)s^{m-1} + \cdots + a_{m}(\zeta, w) \in \mathbb{C}\{w\}[s]$ is the characteristic polynomial of $h$ relative to $F_{\zeta}$. By Lemma 3.3 $o_{F_{\zeta}}(h) = \inf_{i} \left\{ \frac{\text{ord } a_{i}(\zeta, w)}{i} \right\} \leq \inf_{i} \left\{ \frac{\text{ord } a_{i}(0, w)}{i} \right\} = o_{F_{0}}(h)$ for $\zeta \in \mathbb{C}^{k}$ close to $0 \in \mathbb{C}^{k}$ since $\text{ord } a_{i}(\zeta, w) \leq \text{ord } a_{i}(0, w)$ if $|\zeta|$ is small. If $k = 1$ then $\text{ord } a_{i}(\zeta, w) \equiv \text{const}$ for $\zeta \neq 0$ close to $0 \in \mathbb{C}$ and $o_{F_{\zeta}}(h) = \text{const}$. □

**Proof of Theorem 2.1** Use Theorem 4.1 and Lemma 3.3. □

5 Łojasiewicz exponent and the Newton polygon

Let $P(w, s) = s^{m} + a_{1}(w)s^{m-1} + \cdots + a_{m}(w) \in \mathbb{C}\{w, s\}$ be a distinguished polynomial in variables $(w, s) = (w_{1}, \ldots, w_{m}, s)$. Put $a_{0}(w) = 1$ and $I = \{i : a_{i} \neq 0\}$. The Newton polygon $\mathcal{N}(P)$ of $P$ is defined to be

$$\mathcal{N}(P) = \text{convex} \bigcup_{i \in I} \left( (\text{ord } a_{i}, m-i) + \mathbb{R}^{2}_{+} \right), \quad \text{where } \mathbb{R}_{+} = \{a \in \mathbb{R} : a \geq 0\}.$$ 

Then $\mathcal{N}(P)$ intersects the vertical axis at point $(0, m)$ and the horizontal axis at point $(\text{ord } a_{m}, 0)$ provided that $a_{m} \neq 0$. Note that $\theta(P) := \inf_{i} \left\{ \frac{\text{ord } a_{i}}{i} \right\}$ is equal to the inclination of the first side of the Newton polygon $\mathcal{N}(P)$, see [12].

Let $f : (\mathbb{C}^{n}, 0) \to (\mathbb{C}^{n}, 0)$ be a finite holomorphic germ and let $h \in \mathbb{C}\{z\}$, $h(0) = 0$, $h \neq 0$ in $\mathbb{C}\{z\}$. We put

$$\mathcal{N}(f, h) = \sigma(\mathcal{N}(P_{f, h})), $$ 

where $\sigma$ is the symmetry of $\mathbb{R}^{2}_{+}$ given by $\sigma(\alpha, \beta) = (\beta, \alpha)$, and call $\mathcal{N}(f, h)$ the Newton polygon of $h$ relative to $f$.

> From the proof of Theorem 4.1 it follows the semicontinuity of the Newton polygon in Teissier’s sens, see [11], pp. and [9].
Theorem 5.1 Let $F = F(t, z) \in \mathbb{C}[t, z]^n$ be a multiplicity-constant deformation of $f$. Then

$$\mathcal{N}(F_t, h) \subset \mathcal{N}(F_0, h) \quad \text{for } t \in \mathbb{C}^k \text{ close to } 0.$$ 

If $k = 1$ then $\mathcal{N}(F_t, h)$ does not depend on $t$ provided that $t \neq 0$ is close to $0 \in \mathbb{C}$.

Observe that $\mathcal{N}(f, h)$ intersects the horizontal axis at point $(m_0(f), 0)$. The intersection of the last edge (with vertex at $(m_0(f), 0)$) of $\mathcal{N}(f, h)$ is equal to $\frac{1}{\eta(f)}$. We will prove elsewhere that $\mathcal{N}(f, h)$ is identical to the Newton polygon of the pair of ideals $I(f), I(h) = (h)\mathbb{C}\{z\}$ introduced by Teissier in [10]. In the notation of [3], Complément 2 we have $\mathcal{N}(f, h) = N_{I(f)}(h)$.

References

[1] S. S. Abhyankar, Local Analytic Geometry, Academic Press 1964.

[2] C. Bivià-Ausina, Local Lojasiewicz exponents, Milnor numbers and mixed multiplicities of ideals, Math. Z. (2009) 262: 389-409.

[3] M. Lejeune-Jalabert, B. Teissier, Clôture intégrale des idéaux et équisingularité, Centre Mathématiques, Université Scientifique et Medical de Grenoble (1974). See also Ann. Fac. Sci. Toulouse Math. (6) 17, No.4 (2008), 781-859.

[4] S. Lojasiewicz, Introduction to Complex Analytic Geometry, Birkhäuser Verlag 1991.

[5] J. Mc Neal, A. Némethi, The order of contact of a holomorphic ideal in $\mathbb{C}^2$, Math. Z. 250(4) (2005), 873-883.

[6] A. Płoski, Une évaluation pour les sous-ensembles analytiques complexes, Bull. Pol. Acad. Sci. Math. 31 (1983), 259-262.

[7] A. Płoski, Sur l’exposant d’une application analytique I, Bull. Ac. Pol.: Math., 32 (1984), 669-673.

[8] A. Płoski, Multiplicity and the Lojasiewicz exponent, Singularities (Warsaw, 1985), 353-364, Banach Center Publications, 20, PWN, Warsaw, (1985).

[9] B. Teissier, Variétés polaires I – Invariant polaires de singularités d’hypersurfaces, Invent. Math. 40 (1977), 267-292.

[10] B. Teissier, Jacobian Newton polyhedra and equisingularity, Preceedings R.I.M.S. Conference on singularities, Kyoto, April 1978, (Publ. R.I.M.S. 1978).

[11] B. Teissier, The hunting of invariants in the geometry of discriminants, in: Real and Complex Singularities, Oslo 1976, Per Holm (1977), 565-677.
[12] **B. Teissier**, *Complex Curve Singularities: a biased introduction*, Singularities in geometry and topology, 825-887, World Scientific Publishing, Hackensack, NJ, 2007.

[13] **J. C. Tougeron**, *Idéaux de fonctions différentiables*, Springer-Verlag 1972.

[14] **O. Zariski, P. Samuel**, *Commutative Algebra Vol I*, Van Nostrand Company 1958.

Kielce University of Technology  
Department of Mathematics  
Al. 1000 L PP 7  
25-314 Kielce, Poland  
E:mail: matap@tu.kielce.pl