COHEN-MACAULAY BINOMIAL EDGE IDEALS IN TERMS OF BLOCKS WITH WHISKERS

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ABSTRACT. For a graph $G$, Bolognini et al. have shown $J_G$ is strongly unmixed $\Rightarrow$ $J_G$ is Cohen-Macaulay $\Rightarrow$ $G$ is accessible, where $J_G$ denotes the binomial edge ideals of $G$. Accessible and strongly unmixed properties are purely combinatorial. We give some motivations to focus only on blocks with whiskers for the characterization of all $G$ with Cohen-Macaulay $J_G$. We show that accessible and strongly unmixed properties of $G$ depend only on the corresponding properties of its blocks with whiskers and vice versa. Also, we give an infinite class of graphs whose binomial edge ideals are Cohen-Macaulay, and from that, we classify all $r$-regular $r$-connected graphs such that attaching some special whiskers to it, the binomial edge ideals become Cohen-Macaulay. Finally, we define a new class of graphs, called strongly $r$-cut-connected and prove that the binomial edge ideal of any strongly $r$-cut-connected accessible graph having at most three cut vertices is Cohen-Macaulay.

1. Introduction

Let $G$ be a simple graph (i.e., a finite undirected graph without multiple edges and loops) on the vertex set $V(G) = [n] = \{1, \ldots, n\}$. Consider the ring $R = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$, where $K$ is a field. The binomial edge ideal of $G$, denoted by $J_G$, is the ideal of $R$ defined as

$$J_G = \langle f_{ij} = x_iy_j - x_jy_i \mid \{i, j\} \in E(G) \text{ with } i < j \rangle.$$

The study of binomial edge ideals have been started in 2010 through the articles [12] and [19] independently. Concept of binomial edge ideals arises from the study of the ideal generated by $2$-minors of a $2 \times n$-generic matrix.

Throughout the previous eleven years, many works related to the algebraic properties and invariants of these ideals have been done (see [2], [6], [10], [11], [14], [13], [16], [18], [23], [24]). Generally, people try to...
see these things in terms of the combinatorial properties of the underlying graph. We are interested to classify those $G$ for which $J_G$ is Cohen-Macaulay. Although, several works have been done in this direction (see [1], [3], [4], [5], [8], [9], [12], [17], [15], [21], [22]), but full characterization of Cohen-Macaulay binomial edge ideals is still widely open.

To give a combinatorial characterization of Cohen-Macaulay binomial edge ideals, Bolognini et al., in [5], have introduced two combinatorial properties of graphs: accessible (Definition 2.1) and strongly unmixed (Definition 2.2) property. Specifically, they have proved that $J_G$ is strongly unmixed $\Rightarrow$ $J_G$ is Cohen-Macaulay $\Rightarrow$ $G$ is accessible. Moreover, they conjectured [5, Conjecture 1.1] on the equivalency of these three properties. In [17], the authors showed if $R/J_G$ satisfies Serre’s condition $S_2$, then $G$ is accessible. For any ideal $I \subseteq R$, it is an well known result that $R/I$ is Cohen-Macaulay if and only if $R/I$ satisfies Serre’s condition $S_r$ for all $r \geq 1$. Therefore, combining the above results we get for any graph $G$,

$J_G$ strongly unmixed $\Rightarrow$ $J_G$ Cohen-Macaulay $\Rightarrow$ $R/J_G$ is $S_2$ $\Rightarrow$ $G$ accessible.

Merging [5, Conjecture 1.1] and [17, Conjecture 0.1], we get the following.

**Conjecture 1.1.** Let $G$ be a graph. Then

$J_G$ strongly unmixed $\iff$ $J_G$ Cohen-Macaulay $\iff$ $R/J_G$ is $S_2$ $\iff$ $G$ accessible.

To prove the above Conjecture 1.1, it is enough to show that $G$ is accessible implies $J_G$ is strongly unmixed.

A block of a connected graph $G$ is a maximal induced subgraph of $G$ which has no cut vertex. In many papers, we have seen examples of Cohen-Macaulay $J_G$ are some blocks with whiskers (see [4], [5], [17], [21], [22]). We give the motivation to study only blocks with some whiskers to characterize all Cohen-Macaulay $J_G$. Also, to prove the Conjecture 1.1 we need to focus only on blocks with whiskers. We give some classes of binomial edge ideals in support of Conjecture 1.1. The paper is arranged in the following manner.

In Section 2 we recall some definitions, concept, notations related to graph theory and commutative algebra. Then we mention some results from [5], [17], [20] which have been used frequently in our work.

In Section 3 we first prove some results regarding primary decomposition and unmixedness of $J_G$ and accessibility of $G$ for the sake of the rest of the paper. The main results of this section are the following.

**Theorem 1.2** (Theorem 3.9, 3.10, 3.15 and 3.17). Let $G$ be a graph. Then the following are equivalent.

(i) $J_G$ is strongly unmixed (resp. $G$ is accessible).

(ii) $J_G$ is unmixed and $J_B$ is strongly unmixed (resp. $B$ is accessible) for each block $B$ of $G$. 

The above Theorem 1.2 ensures that it is enough to study only blocks with whiskers for characterization of accessible graphs, strongly unmixed and Cohen-Macaulay binomial edge ideals. Also, due to the Theorem 1.2, the Conjecture 1.1 boils down to only blocks with whiskers instead of any graph $G$. At the end, in Theorem 3.19, we settle down an open problem (see Question 3.18) given in [5, Problem 7.2] for the case of accessibility of $G$ and strongly unmixedness of $J_G$.

In Section 4, motivated from [17, Question 5.4], we started to find $r$-connected planar accessible graphs. Finally, we manage to find all $r$-regular $r$-connected non-complete blocks with whiskers, denoted by $K_r \star K_r$, $r \geq 2$ (see 4), for which the Conjecture 1.1 hold (see Theorem 4.6). But, among them the only planar graphs are $K_2 \star K_2$ and $K_3 \star K_3$.

In Section 5, we define new classes of graphs called $r$-cut-connected and strongly $r$-cut-connected (Definition 5.4 and 5.5). We give the following theorem and Example 5.10 compiling those blocks with whiskers for which the Conjecture 1.1 hold.

**Theorem 1.3 (Theorem 5.9).** Let $G$ be a graph such that every block $B$ of $G$ satisfies any of the following conditions:

(a) $B$ is chordal; (b) $\overline{B}$ is traceable; (c) $B$ is a chain of cycles (see [17, Definition 4.2]); (d) $B = K_m \star_r K_n$; (e) $\overline{B}$ is strongly 3-cut-connected containing at most 3 cut vertices of $G$. Then the following are equivalent:

(i) $J_G$ is Cohen-Macaulay;
(ii) $R/J_G$ is $S_2$;
(iii) $G$ is accessible;
(iv) $J_G$ is unmixed and each $\overline{B}$ is accessible.
(v) $J_G$ is strongly unmixed.

At the end, we conclude by putting Question 5.11 and 5.12 keeping similarities with our key results.

2. Preliminaries

In this article, we assume all graphs are simple. For a graph $G$, we denote the vertex set by $V(G)$ and edge set by $E(G)$. For a subset $W \subseteq V(G)$, the induced subgraph of $G$ on the vertex set $W$ is denoted by $G[W]$ and for $T \subseteq V(G)$, we mean by $G \setminus T$ as the graph $G[V(G) \setminus T]$. If $\{u, v\} \in E(G)$, then we say $u$ is adjacent to $v$ or vice versa. Similarly, we say $v$ is adjacent to $A$ (or $A$ is adjacent to $v$) if $v$ is adjacent to a vertex in $A$, where $A \subseteq V(G)$ and $v \in V(G)$.

For a vertex $v \in V(G)$, we call $\mathcal{N}_G(v) = \{u \in V(G) \mid \{u, v\} \in E(G)\}$ the neighbor set of $v$ in $G$. We denote by $\deg_G(v) = |\mathcal{N}_G(v)|$ the degree of a vertex $v$ in $G$. If $\mathcal{N}_G(v) = \{u\}$, then $\{u, v\} \in E(G)$ is called a
whisker attached to u. A path from u to v of length n in G is a sequence of vertices \( u = v_0, \ldots, v_n = v \in V(G) \) such that \( \{v_{i-1}, v_i\} \in E(G) \) for each \( 1 \leq i, j \leq n \) and \( v_i \neq v_j \) if \( i \neq j \). A chordless path of length n, denoted by \( P_n \), is a path of length n without any induced cycle in it.

A graph is called complete if there is an edge between every pair of vertices and \( K_n \) denotes the complete graph on \( n \) vertices. A vertex \( v \in V(G) \) is said to be a free vertex of \( G \) if the induced subgraph \( G[N_G(v) \cup \{v\}] \) is complete. A graph \( G \) is called decomposable into \( G_1 \) and \( G_2 \) if \( G = G_1 \cup G_2 \) with \( V(G_1) \cap V(G_2) = \{v\} \) such that \( v \) is free vertex of both \( G_1 \) and \( G_2 \).

A vertex \( v \in V(G) \) is said to be a cut vertex or cut point of \( G \) if removal of \( v \) from \( G \) increases the number of connected components. A connected graph \( G \) is called \( r \)-connected if removal of any set of vertices with cardinality less than \( r \) keeps the graph connected. A graph \( G \) is called \( r \)-regular if \( \deg_G(v) = r \) for all \( v \in V(G) \).

Let \( G \) be a graph on the vertex set \( V(G) = [n] \). A set \( T \subseteq [n] \) is said to be a cutset of \( G \) (or we said \( T \) has a cut point property) if each \( t \in T \) is a cut vertex of \( G \setminus (T \setminus \{t\}) \). We denote by \( \mathcal{C}(G) \) the set of all cutsets of \( G \).

For \( T \subseteq [n] \), we denote the number of connected components of the graph \( G \setminus T \) by \( c_G(T) \) (or sometimes by \( c(T) \) if the graph is clearly understood from the context). Let \( G_1, \ldots, G_{c(G)} \) be the connected components of \( G \setminus T \). For each \( G_i \), we denote by \( \tilde{G}_i \), the complete graph on the vertex set \( V(G_i) \).

We set

\[
\mathcal{P}_T(G) = \left\langle \bigcup_{i \in T} \{x_i, y_i\}, J_{\tilde{G}_1}, \ldots, J_{\tilde{G}_{c(G)}} \right\rangle.
\]

Then \( \mathcal{P}_T(G) \) is a prime ideal and \( \mathcal{J}_G = \cap_{T \subseteq [n]} \mathcal{P}_T(G) \). By \cite{12}, \( \mathcal{P}_T(G) \) is a minimal prime ideal of \( \mathcal{J}_G \) if and only if \( T \in \mathcal{C}(G) \) i.e.,

\[
\mathcal{J}_G = \bigcap_{T \in \mathcal{C}(G)} \mathcal{P}_T(G),
\]

is the minimal primary decomposition of \( \mathcal{J}_G \).

From \cite{12} Lemma 3.1, height \( \phi(T) = n + |T| - c(T) \). Since \( \phi \in \mathcal{C}(G) \), \( \mathcal{J}_G \) is unmixed (i.e., heights of all minimal primes of \( \mathcal{J}_G \) are same) if and only if \( c(T) = |T| + c \) for any \( T \in \mathcal{C}(G) \), where \( c \) denotes the number of connected components of \( G \). It follows from \cite{20} Proposition 2.1, that if \( v \) is a free vertex of \( G \), then \( v \notin T \) for all \( T \in \mathcal{C}(G) \).

For a vertex \( v \) of a graph \( G \), we denote by \( G_v \) the following graph:

\[
V(G_v) = V(G) \quad \text{and} \quad E(G_v) = E(G) \cup \{\{i, j\} \mid i, j \in N_G(v), i \neq j\}.
\]

Note that, \( v \) is a free vertex of \( G_v \).

Let us recall some definitions and results from \cite{5}, \cite{17}, \cite{20}.
Definition 2.1 ([5], Definition 2.2). Let $G$ be a graph. A cutset $T \in \mathcal{C}(G)$ is said to be accessible if there exists $t \in T$ such that $T \setminus \{t\} \in \mathcal{C}(G)$. We say $\mathcal{C}(G)$ is an accessible set system if every non-empty cutset $T \in \mathcal{C}(G)$ is accessible. The graph $G$ is said to be accessible if $J_G$ is unmixed and $\mathcal{C}(G)$ is an accessible set system.

Definition 2.2 ([5], Definition 5.6). Let $G$ be a graph. We say $J_G$ is strongly unmixed if the connected components of $G$ are complete graphs or if $J_G$ is unmixed and there exists a cut vertex $v$ of $G$ for which $J_{G \setminus \{v\}}$, $J_{G \setminus \{v\}}$, and $J_{G \setminus \{v\}}$ are strongly unmixed. Sometimes we will say $G$ is strongly unmixed instead of saying $J_G$ is strongly unmixed.

The following results we have used extensively in our work.

Theorem 2.3 ([5], Theorem 4.12). Let $G$ be a connected non-complete accessible graph. Then

(i) every non-empty cutset of $G$ contains a cut vertex;
(ii) the induced subgraph on the cut vertices of $G$ is connected;
(iii) every vertex of $G$ is adjacent to a cut vertex.

Proposition 2.4 ([5], Proposition 4.18). Let $G$ be an accessible graph and $T \in \mathcal{C}(G)$. If $T$ contains some non-cut vertices, then $T \setminus \{t\} \in \mathcal{C}(G)$ for some non-cut vertex $t \in T$.

Proposition 2.5 ([5], Proposition 5.2). Let $G$ be a connected graph such that $J_G$ is unmixed and $v$ be a cut vertex of $G$. If $H_1$ and $H_2$ are the connected components of $H = G \setminus \{v\}$, then the following are equivalent:

(i) $J_H$ is unmixed;
(ii) $T \in \mathcal{C}(H)$ implies $N_{H_1}(v) \not\subseteq T$ and $N_{H_2}(v) \not\subseteq T$;
(iii) $\mathcal{C}(H) = \{T \subseteq V(H) \mid T \cup \{v\} \in \mathcal{C}(G)\}$.

Lemma 2.6 ([17], Lemma 4.10). Let $G$ be a graph and $v$ be a free vertex of $G$. If $J_G$ is unmixed and $J_{G \setminus \{v\}}$ is strongly unmixed, then $J_G$ is strongly unmixed.

Remark 2.7. For a graph $G$ with connected components $G_1, \ldots, G_r$, $J_G$ is unmixed (resp. Cohen-Macaulay, strongly unmixed) if and only if $J_{G_i}$ is unmixed (resp. Cohen-Macaulay, strongly unmixed) for each $i = 1, \ldots, r$. The same holds for the accessible property of $G$. So, we will assume any graph is connected if it is not clear from the context.

Remark 2.8. Let $G = G_1 \cup G_2$ with $V(G_1) \cap V(G_2) = \{v\}$ be a decomposable graph, where $v$ is a free vertex in both $G_1$ and $G_2$. Then

(i) $J_G$ is unmixed (resp. Cohen-Macaulay) if and only if $J_{G_i}$ is unmixed (resp. Cohen-Macaulay) for each $i = 1, 2$ (see [20]).
(ii) $J_G$ is accessible (resp. strongly unmixed) if and only if $J_{G_i}$ is accessible (resp. strongly unmixed) for each $i = 1, 2$ (see [17]).
3. Block-Wise Strongly Unmixed, Accessible and Cohen-Macaulay properties

The main aim of this section is to establish the connection between accessible, strongly unmixed properties of a graph and its corresponding blocks with whiskers. We start by proving some results regarding primary decomposition and unmixedness of $J_G$ as well as the accessibility of $G$.

**Proposition 3.1.** Let $G = G_1 \cup G_2$ be a graph such that $V(G_1) \cap V(G_2) = \{v\}$. Then $\mathcal{C}(G) = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$, where

$\mathcal{A} = \{S_1 \cup S_2 \mid S_1 \in \mathcal{C}(G_1), S_2 \in \mathcal{C}(G_2) \text{ and } v \not\in S_1 \cup S_2\}$

$\mathcal{B} = \{T_1 \cup S_1 \mid v \in S_2 \in \mathcal{C}(G_2) \text{ and } T_1 \in \mathcal{C}(G_1 \setminus \{v\})\}$

$\mathcal{C} = \{S_1 \cup T_2 \mid v \in S_1 \in \mathcal{C}(G_1) \text{ and } T_2 \in \mathcal{C}(G_2 \setminus \{v\})\}$

$\mathcal{D} = \{T_1 \cup T_2 \cup \{v\} \mid N_{G_i}(v) \not\subseteq T_i \in \mathcal{C}(G_i \setminus \{v\}), \text{ for } i \in \{1, 2\}\}.$

**Proof.** Let $S \in \mathcal{C}(G)$ be such that $v \not\in S$. Then $S = S_1 \cup S_2$, where $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$. For any $s \in S_1$, $s$ is a cut vertex in $G \setminus (S \setminus \{s\})$. Now $v \not\in S$ implies $v \in V(G \setminus (S \setminus \{s\}))$ and therefore, $s$ is also a cut vertex in $G_1 \setminus (S_1 \setminus \{s\})$. Hence $S_1 \in \mathcal{C}(G_1)$ and by similar argument we get $S_2 \in \mathcal{C}(G_2)$. Thus, $S \in \mathcal{A}$. Now assume $S \in \mathcal{C}(G)$ and $v \in S$. Then we can write $S = T_1 \cup T_2 \cup \{v\}$, where $T_i \subseteq V(G_i \setminus \{v\})$ for $i = 1, 2$. Note that any $t \in T_1$ is a cut vertex of $G \setminus (S \setminus \{t\})$ which imply $t$ is a cut point of $G_1 \setminus (T_1 \cup \{v\} \setminus \{t\}) = (G_1 \setminus \{v\}) \setminus (T_1 \setminus \{t\})$. Therefore $T_1 \in \mathcal{C}(G_1 \setminus \{v\})$ and similarly, we have $T_2 \in \mathcal{C}(G_2 \setminus \{v\})$. Now we will consider few cases. Suppose $N_{G_i}(v) \subseteq T_i$ for $i = 1, 2$. Then $v$ can not be a cut vertex in $G \setminus (S \setminus \{v\})$ which is a contradiction to the fact that $S \in \mathcal{C}(G)$. Now assume $N_{G_1}(v) \subseteq T_1$ but $N_{G_2}(v) \not\subseteq T_2$. Set $S_2 = T_2 \cup \{v\}$. Since $v$ is a cut vertex in $G \setminus (T_1 \cup T_2)$ and $N_{G_1}(v) \subseteq T_1$, it is easy to observe that $v$ is a cut vertex in $G_2 \setminus (S_2 \setminus \{v\})$. Also, $T_2 \in \mathcal{C}(G_2 \setminus \{v\})$ implies any $t \in T_2$ is a cut point of $(G_2 \setminus \{v\}) \setminus (T_2 \setminus \{t\}) = G_2 \setminus (S_2 \setminus \{t\})$. Therefore, we have $T_1 \in \mathcal{C}(G_1 \setminus \{v\})$ and $S_2 \in \mathcal{C}(G_2)$. Hence $S = T_1 \cup S_2 \in \mathcal{B}$. Similarly, if $N_{G_1}(v) \not\subseteq T_1$ and $N_{G_2}(v) \subseteq T_2$, then we get $S \in \mathcal{C}$. Consider $N_{G_1}(v) \not\subseteq T_i$ for $i = 1, 2$. Then $S \in \mathcal{D}$ is clear from definition of $\mathcal{D}$.

Conversely, let $S \in \mathcal{A}$. Then $S = S_1 \cup S_2$ with $S_1 \in \mathcal{C}(G_1), S_2 \in \mathcal{C}(G_2)$, and $v \not\in S$. Now, any $s_i \in S_i$ is a cut vertex in $G_i \setminus (S_i \setminus \{s_i\})$ which imply $s_i$ is a cut vertex in $G \setminus (S \setminus \{s\})$ as $v \not\in S$, where $i \in \{1, 2\}$. Thus, $S \in \mathcal{C}(G)$. Let $S \in \mathcal{B}$. Then $S = T_1 \cup S_2$ such that $T_1 \in \mathcal{C}(G_1 \setminus \{v\})$ and $v \in S_2 \in \mathcal{C}(G_2)$. Let $t \in T_1$ be any vertex. Then $t$ is a cut vertex in $(G_1 \setminus \{v\}) \setminus (T_1 \setminus \{t\})$ and since $v \in S_2$, $t$ is also a cut point of $G \setminus (S \setminus \{t\})$. Again, any $s \in S_2$ is a cut vertex in $G_2 \setminus (S_2 \setminus \{s\})$. Since $T_1 \cap V(G_2) = \emptyset$, $s$ is a cut point of $G \setminus (S \setminus \{s\})$ too. Hence, $S \in \mathcal{C}(G)$. Similarly, we have $\mathcal{C} \subseteq \mathcal{C}(G)$. For $S \in \mathcal{D}$, we have $S = T_1 \cup T_2 \cup \{v\}$, where $N_{G_i}(v) \not\subseteq T_i \in \mathcal{C}(G_1 \setminus \{v\})$
Corollary 3.2. Let $G = G_1 \cup G_2$ be such that $V(G_1) \cap V(G_2) = \{v\}$. If the following conditions hold:

(i) $J_{G \setminus \{v\}}$ is unmixed;
(ii) for $S_i \in \mathcal{E}(G_i)$ with $v \notin S_i$, we have $c_{G_i}(S_i) = |S_i| + 1$ and for $S_i \in \mathcal{E}(G_i)$ with $v \in S_i$, we have $c_{G_i}(S_i) = |S_i|$, where $i = 1, 2$;

then $J_G$ is unmixed.

Proof. We have $\mathcal{E}(G) = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ as described in Proposition 3.1.

Let $T \in \mathcal{A}$. Then $T = S_1 \cup S_2$, where $v \notin S_i \in \mathcal{E}(G_i)$. Therefore by given condition (ii), we have $c_{G_i}(S_i) = |S_i| + 1$ and one component of $G_i \setminus S_i$ contains $v$ for $i = 1, 2$. Thus, $c_{G}(T) = |T| + 1$ is clear. Let $T \in \mathcal{B}$. Then $T = T_1 \cup T_2$, where $v \in S_2 \in \mathcal{E}(G_2)$ and $T_1 \in \mathcal{E}(G_1 \setminus \{v\})$. By condition (i) and (ii), we have $c_{G_1 \setminus \{v\}}(T_1) = |T_1| + 1$ and $c_{G_2}(T_2) = |S_2|$. So it is clear that $c_{G}(T) = |T| + 1$. The case of $T \in \mathcal{C}$ is similar. If $T \in \mathcal{D}$, then $T = T_1 \cup T_2 \cup \{v\}$ with $N_{G_i}(v) \subseteq T_i \in \mathcal{E}(G_i \setminus \{v\})$. By condition (i), $c_G(T) = |T_1| + 1 + |T_2| + 1 = |T| + 1$. Hence we can conclude that $J_G$ is unmixed.

Proposition 3.3. Let $G$ be an accessible graph and $v_1, v_2$ be two cut vertices of $G$ such that $\{v_1, v_2\} \notin E(G)$. If $G \setminus \{v_1\}$ and $G \setminus \{v_2\}$ are accessible, then $G \setminus \{v_1, v_2\}$ is accessible.

Proof. If $G$ is disconnected and $v_1, v_2$ belong to two different connected components of $G$, then by definition of accessibility $G \setminus \{v_1, v_2\}$ is accessible. So we may assume $G$ is connected. Since $v_1$ is not adjacent to $v_2$, $\{v_1, v_2\} \in \mathcal{E}(G)$ and $J_G$ being unmixed, $c_G(\{v_1, v_2\}) = 3$. Let $H_1, H'$ be two connected components of $G \setminus \{v_1\}$ with $v_2 \in V(H_1)$ and $H_2, H''$ be two connected components of $G \setminus \{v_2\}$ with $v_1 \in V(H_2)$. Then $H_1 \setminus (V(H') \cup \{v_2\}) = H_2 \setminus (V(H') \cup \{v_1\}) = H$ (say) and $H, H', H''$ are three connected components of $G \setminus \{v_1, v_2\}$. Since $G \setminus \{v_i\}$ is accessible for $i = 1, 2$, we have $H_1, H', H_2, H''$ are accessible. So, it is enough to show that $H$ is accessible. Let $T \in \mathcal{E}(H)$. Suppose $N_{H_1}(v_1) \subseteq T$. Since $G$ and $G \setminus \{v_1\}$ are accessible, by Proposition 2.5, $T \notin \mathcal{E}(G \setminus \{v_1\})$ i.e., $T \notin \mathcal{E}(H_1)$. But $T \in \mathcal{E}(H)$ which imply there must exists $w \in N_{H_2}(v_2)$ such that $w \notin T$. Then $v_2$ is a cut point in $H_1 \setminus (T \cup \{v_2\} \setminus \{t\})$ and obviously, every $t \in T$ is a cut vertex in $H_1 \setminus (T \cup \{v_2\} \setminus \{t\})$. Therefore, $T \cup \{v_2\} \in \mathcal{E}(H_1)$ and $H_1$ being unmixed, we have $c_{H_1}(T \cup \{v_2\}) = |T| + 2$. Thus, $c_H(T) = |T| + 1$. Similarly, if $N_{H_2}(v_2) \subseteq T$, then also we get $c_H(T) = |T| + 1$. Now, assume $N_{H_1}(v_1) \notin T$ and $N_{H_2}(v_2) \notin T$. Since
\{v_1, v_2\} \not\in E(G), v_1 \text{ is a cut vertex in } G \setminus (T \cup \{v_2\}) \text{ and } v_2 \text{ is a cut vertex in } G \setminus (T \cup \{v_1\}). \text{ Thus, } T \cup \{v_1, v_2\} \in \mathcal{C}(G) \text{ and } c_G(T \cup \{v_1, v_2\}) = |T| + 3 \text{ as } J_G \text{ is unmixed. Since } c_G(\{v_1, v_2\}) = 3 \text{ and } T \subseteq V(H), \text{ we have } c_H(T) = |T| + 1. \text{ Hence } J_H \text{ is unmixed and so, } J_{G \setminus \{v_1, v_2\}} \text{ is unmixed. Since } H_1 \text{ is connected accessible graph and } v_2 \text{ is a cut vertex in } H_1, \text{ by } [5, \text{ Proposition 5.14}], H_1 \setminus \{v_2\} \text{ is accessible. Therefore we get } H \text{ is accessible and so is } G \setminus \{v_1, v_2\}.

Proposition 3.4. Let } G \text{ be a simple graph such that } J_G \text{ is unmixed and } G \setminus \{v\} \text{ is accessible for a free vertex } v \in V(G). \text{ Then } G \text{ is accessible.}

Proof. From the proof of [20, Lemma 2.2], we have } \mathcal{C}(G \setminus \{v\}) \subseteq \mathcal{C}(G) \text{ for a free vertex } v \text{ of } G. \text{ Let } T \in \mathcal{C}(G). \text{ Then } v \not\in T \text{ as } v \text{ is a free vertex of } G. \text{ Set } H = G \setminus \{v\}. \text{ We will consider two cases:}

Case-I: \text{ Let } \mathcal{N}_G(v) \not\subseteq T. \text{ Then } T \in \mathcal{C}(H) \text{ by } [20, \text{ Lemma 2.2}]. \text{ Since } H \text{ is accessible, } T \text{ is accessible as a cutset of } H \text{ as well as of } G.

Case-II: \text{ Assume } \mathcal{N}_G(v) \subseteq T. \text{ Then } S = \mathcal{N}_G(v) \in \mathcal{C}(G) \text{ is clear. As } J_G \text{ is unmixed } c_G(S) = |S| + 1. \text{ Then } H \setminus S \text{ has } |S| \text{ connected components. So, } S \not\in \mathcal{C}(H) \text{ as } J_H \text{ is unmixed. Let } S = \{s_1, \ldots, s_k\} \text{ and connected components of } G \setminus S \text{ are } A_{s_1}, \ldots, A_{s_k}, \{v\}. \text{ Assume } k > 1. \text{ Since } S \not\in \mathcal{C}(H), \text{ there exists an } s_i \text{ which is not a cut vertex in } H \setminus (S \setminus \{s_i\}), \text{ but } s_i \text{ is a cut vertex in } G \setminus (S \setminus \{s_i\}). \text{ Therefore, } s_i \text{ is adjacent to only one connected components of } H \setminus S, \text{ say } A_{s_i}. \text{ For simplicity of notations, let } i = 1 \text{ i.e., } s_1 \text{ is a cut vertex in } G \text{ and it is adjacent to only } A_{s_1} \text{ in } H \setminus (S \setminus \{s_1\}). \text{ Then clearly } S \setminus \{s_1\} \in \mathcal{C}(H) \text{ and thus } S \text{ is accessible. Let } V(A_{s_1}) \cap T = T'. \text{ Since } s_1 \text{ and } A_{s_1} \text{ are not adjacent to } A_{s_2}, \ldots, A_{s_k}, \text{ we have } T'' = T \setminus (T' \cup \{s_1\}) \in \mathcal{C}(H) \text{ and } T' \cup \{s_1\} \in \mathcal{C}(G). \text{ We have } V(A_{s_1}) \setminus T' \neq \phi \text{ and } \mathcal{N}_{A_{s_1}}(s_1) \not\subseteq T'' \text{ otherwise, } s_1 \text{ can not be a cut vertex of } G \setminus T'. \text{ Since } k > 1, s_1 \text{ is a cut point in } H \setminus T'. \text{ Let } T' \neq \phi. \text{ Then every } t \in T' \text{ is a cut point of } H \setminus (T' \cup \{s_1\}) \{\{\}. \text{ Hence } T' \cup \{s_1\} \in \mathcal{C}(H). \text{ By accessibility of } H \text{ and Proposition 2.4,}

\begin{equation}
(T' \cup \{s_1\}) \setminus \{t'\} \in \mathcal{C}(H)
\end{equation}

for some } t' \in T'. \text{ Now } T'' \in \mathcal{C}(H) \text{ and } s_1 \not\in T'' \text{ is a cut vertex of } G \text{ with } \mathcal{N}_H(s_1) \not\subseteq T'' \text{ together imply } T'' \cup \{s_1\} \in \mathcal{C}(G). \text{ So, we have by } [3.1]

\begin{equation}
(T'' \cup \{s_1\} \cup T') \setminus \{t'\} = T \setminus \{t'\} \in \mathcal{C}(G).
\end{equation}

Thus, } T \text{ is accessible. If } T' = \phi, \text{ then } T'' \in \mathcal{C}(H) \text{ implies } T \setminus \{s_1\} = T'' \in \mathcal{C}(G) \text{ and so, } T \text{ is accessible. Now assume } k = 1. \text{ Then any } T \in \mathcal{C}(G) \text{ with } s_1 \not\in T \text{ implies } T \in \mathcal{C}(H) \text{ and we are done. Let } T \in \mathcal{C}(G) \text{ such that } s_1 \in T. \text{ Then } \mathcal{N}_G(s_1) \setminus \{v\} \not\subseteq T. \text{ Suppose } T \setminus \{s_1\} \not\in \mathcal{C}(H). \text{ Then there exists } t \in T \setminus \{s_1\} \text{ such that } t \text{ is not a cut point of } (G \setminus \{v\}) \setminus (T \cup \{s_1\}) \{t\} = \mathcal{H}_t. \text{ Now } s_1, t \in V(H_t) \text{ is clear and } t \text{ is a cut point of } G \setminus (T \setminus \{t\}). \text{ So, there exists } x, y \in \mathcal{N}_H(s_1) \text{ such that } x, y \text{ are not connected in } H \setminus T. \text{ Therefore,
Lemma 3.5. Let \( G = G_1 \cup G_2 \) be a graph such that \( V(G_1) \cap V(G_2) = \{v\} \). Then \( \mathcal{C}(G_v) = \mathcal{C}((G_1)_v) \cup \mathcal{C}((G_2)_v) \). In particular, \( J_{(G_1)_v} \) and \( J_{(G_2)_v} \) are unmixed (resp, accessible) if and only if \( J_{G_v} \) is unmixed (resp, accessible).

Proof. Let \( T \in \mathcal{C}(G_v) \). Set \( T \cap V(G_i) = T_i \) for \( i = 1, 2 \). Then \( T = T_1 \cup T_2 \) and \( v \) being a free vertex \( v \not\in T \). Now, any \( t \in T_1 \) is a cut point of \( G_v \setminus (T_1 \setminus \{t\}) \) imply \( t \) is also a cut point in \( (G_i)_v \setminus (T_i \setminus \{t\}) \), where \( i = 1, 2 \). Thus, \( T_i \in \mathcal{C}((G_i)_v) \) for \( i = 1, 2 \) and \( \mathcal{C}(G_v) \subseteq \mathcal{C}((G_1)_v) \cup \mathcal{C}((G_2)_v) \).

Let \( T_i \in \mathcal{C}((G_i)_v) \) for \( i = 1, 2 \). Since \( v \) is a free vertex in \( (G_i)_v \), \( v \not\in T_i \) and each \( t \in T_i \cup T_2 \) is a cut vertex in \( G_v \setminus (T_1 \cup T_2 \setminus \{t\}) \), where \( i \in \{1, 2\} \). Therefore, \( T_1 \cup T_2 \in \mathcal{C}(G_v) \) and \( \mathcal{C}(G_v) = \mathcal{C}((G_1)_v) \cup \mathcal{C}((G_2)_v) \).

Note that, \( (G_1)_v \setminus T_1 \) contains a connected component containing \( v \), where \( T_i \in \mathcal{C}((G_i)_v) \) and \( i = 1, 2 \). Thus, we have

\[
c_{G_v}(T_1 \cup T_2) = c_{(G_1)_v}(T_1) + c_{(G_2)_v}(T_2) - 1.
\]

Hence, \( J_{(G_1)_v} \) and \( J_{(G_2)_v} \) are unmixed (resp, accessible) if and only if \( J_{G_v} \) is unmixed (resp, accessible).

Proposition 3.6. Let \( G = G_1 \cup G_2 \) be a graph such that \( V(G_1) \cap V(G_2) = \{v\} \). If \( J_{G_v} \) is strongly unmixed, then \( J_{(G_1)_v} \) and \( J_{(G_2)_v} \) are strongly unmixed.

Proof. Let \( |V(G_v)| = n \) and we will proceed by induction on \( n \). If one of \( G_1 \) or \( G_2 \) is a graph with only one vertex which is \( v \), then the result holds trivially. So assume \( G_1, G_2 \) are not empty (i.e. a graph without edges).

Then the base case will be \( n = 3 \) and \( G \) is a \( P_2 \) with \( (G_i)_v = K_2 \) for each \( i \in \{1, 2\} \). Therefore the result follows by definition of strongly unmixed.

Assume \( n > 3 \). If \( G_v \) is complete, then \( (G_1)_v, (G_2)_v \) are both complete and the result follows by definition. If \( G_v \) is non-complete, then there exists a cut vertex \( u \) of \( G_v \) such that \( G_v \setminus \{u\}, (G_v)_u, (G_v)_u \setminus \{u\} \) are strongly unmixed. Note that \( v \) being a free vertex of \( G_v \), it can not be a cut vertex of \( G_v \). Let us suppose \( u \in V(G_1) \). By Lemma 3.5, \( J_{(G_1)_u} \) is unmixed as \( J_{G_v} \) is unmixed for \( i = 1, 2 \). Let \( G_v \setminus \{u\} = H \cup H_2 \), where \( V(G_2) \subseteq V(H) \).

Then \( (G_1)_v \setminus \{u\} = H_1 \cup H_2 \), where \( H_1 = H \setminus V(G_2 \setminus \{v\}) \). Now \( H, H_2 \) are strongly unmixed and \( H = (H_1 \cup G_2)_v \). Since \( H \) has less than \( n \) vertices, by induction hypothesis the binomial edge ideals of \( (G_2)_v \) and \( (H_1)_v \) are strongly unmixed. Thus, \( (G_1)_v \setminus \{u\} \) is strongly unmixed.

Also, we observe that

\[
(G_v)_u \setminus \{u\} = \left( ((G_1)_u \setminus \{u\}) \cup (G_2)_v \right)_u.
\]

Since \( V((G_1)_u \setminus \{u\}) \cap V(G_2) = \{v\} \), using induction hypothesis, we get the binomial edge ideals of \( ((G_1)_u \setminus \{u\})_v \) and \( (G_2)_v \) are strongly unmixed.
Note that $((G_1)_u \setminus \{u\})_v = ((G_1)_v)_u \setminus \{u\}$. Therefore, $J_{((G_1)_v)_u \setminus \{u\}}$ is strongly unmixed and by Lemma 2.6, $J_{((G_1)_v)_u}$ is also strongly unmixed. Hence $J_{(G_1)_u}$ and $J_{(G_2)_v}$ are strongly unmixed. The case of $u \in V(G_2)$ is similar. \hfill $\Box$

**Proposition 3.7.** Let $G = G_1 \cup G_2$ be a graph with $V(G_1) \cap V(G_2) = \{v\}$. If $J_{(G_1)_u}$ and $J_{(G_2)_v}$ are strongly unmixed, then $J_{G_v}$ is strongly unmixed.

**Proof.** We proceed by induction on $n$, where $n = |V(G_v)|$. If one of $G_1$ and $G_2$ is empty, then the result follows trivially. So $n = 3$ is the base case for which $G_v$ is complete and hence $J_{G_v}$ is strongly unmixed. If $(G_1)_v$ and $(G_2)_v$ are complete, then $G_v$ is also complete and so $J_{G_v}$ is strongly unmixed. Assume at least one of $(G_1)_v$ and $(G_2)_v$ is non-complete, say $(G_1)_v$. By Lemma 3.5, $J_{G_v}$ is unmixed. For simplicity of notation we set $H^i = (G_i)_v$, where $i = 1, 2$. Since $H^1$ is non-complete and $J_{H^1}$ is strongly unmixed, there exists a cut vertex $u$ of $H^1$ for which $J_{H^1 \setminus \{u\}}$, $J_{H^1_u}$, and $J_{H^1 \setminus \{u\}}$ are strongly unmixed. Let $H^{1'} \setminus \{u\} = H' \cup H''$, where $v \in V(H')$. Then $G_v \setminus \{u\} = H \cup H''$, where $V(G_2) \subseteq V(H)$. Note that,

$$H = (H' \cup G_2)_v \text{ if } N_{G_1}(v) \neq \{u\},$$

$$= (G_2)_v \quad \text{ if } N_{G_1}(v) = \{u\}.$$

Now $J_{H'} = J_{H_u}$ and $J_{(G_2)_v}$ are strongly unmixed. Thus, using induction hypothesis, we get $J_H$ is strongly unmixed. $J_{H''}$ being also strongly unmixed, $J_{G_v \setminus \{u\}}$ is strongly unmixed. Consider $(G_v)_u \setminus \{u\}$ and note that

$$(G_v)_u \setminus \{u\} = (H^1_u \setminus \{u\} \cup G_2)_v.$$

Now $(H^1_u \setminus \{u\})_v = H^1_u \setminus \{u\}$ and hence by induction hypothesis, $J_{(G_v)_u \setminus \{u\}}$ is strongly unmixed. Since $u$ is a free vertex in $(G_v)_u$, by Lemma 2.6, we have $J_{(G_v)_u}$ is strongly unmixed. \hfill $\Box$

**Definition 3.8.** Let $G$ be a connected graph such that $J_G$ is unmixed and $B$ be a block of $G$. Let $V = \{v_1, \ldots, v_k\}$ be the set of cut vertices of $G$ belonging to $V(B)$. Then we can write

$$G = B \cup \left( \bigcup_{i=1}^k G_i \right),$$

where $V(G_i) \cap V(B) = \{v_i\}$ for each $1 \leq i \leq k$, and the connected components of $G \setminus V$ are $B \setminus V$ (may be empty), $G_1 \setminus \{v_1\}, \ldots, G_k \setminus \{v_k\}$.

Considering the decomposition 3.2, we define a new graph $\overline{B^W}$, where $W = \{v_{s_1}, \ldots, v_{s_r}\} \subseteq V$ such that

- $V(\overline{B^W}) = V(B) \cup \{f_{v_{s_1}}, \ldots, f_{v_{s_r}}\}$,
- $E(\overline{B^W}) = E(B) \cup (\bigcup_{v \in W} E(G_i)) \cup \{\{v_{s_i}, f_{v_{s_i}}\} \mid i = 1, \ldots, r\}$. 


By \( \overline{B} \) with respect to \( G \), we mean \( \overline{B}^l \) and call it the block with whiskers of \( G \) (Sometimes we write only \( \overline{B} \) if the graph is clear from the context or sometimes by \( \overline{B} \) we mean a block attaching with some whiskers). In simple words, \( \overline{B} \) is the graph attaching whiskers to all the cut vertices \( v_i \) of \( G \) belong to \( V(B) \) replacing \( G_i \)’s.

**Theorem 3.9.** Let \( G \) be an accessible graph and \( B \) be any block of \( G \). Let \( V = \{v_1, \ldots, v_k\} \) be the set of cut vertices of \( G \) belong to \( V(B) \). Then for any \( W \subseteq V \), \( \overline{B}^W \) is accessible. In particular, \( \overline{B} \) is accessible.

**Proof.** Let \( T \in \mathcal{E}(\overline{B}^W) \) and without loss of generality \( W = \{v_1, \ldots, v_r\} \), where \( r \leq k \). Assume the decomposition of \( G \) with respect to \( B \) is

\[
G = B \cup \left( \bigcup_{i=1}^{k} G_i \right),
\]

where \( V(G_i) \cap V(B) = \{v_j\} \) for each \( i \in [k] \), and the connected components of \( G \) \( \setminus V \) are \( B \setminus V \) (may be empty), \( G_1 \setminus \{v_1\}, \ldots, G_k \setminus \{v_k\} \).

Then the connected components of \( \overline{B}^W \setminus V \) are \( B \setminus V \) (may be empty), \( \{f_{v_1}\}, \ldots, \{f_{v_r}\}, G_{r+1} \setminus \{v_{r+1}\}, \ldots, G_k \setminus \{v_k\} \). Let \( A_1, \ldots, A_r \) be the connected components of \( \overline{B}^W \setminus T \). Now for each \( A_i \) we consider an induced subgraph \( A'_i \) of \( G \) as follows

- \( V(A'_i) = (V(A_i) \setminus F) \cup \left( \bigcup_{f_{v_j} \in V(A)} (V(G_j) \setminus \{v_j\}) \right) \);

- \( E(A'_i) = E(A_i) \setminus F \cup \left( \bigcup_{\{v_j, f_{v_j}\} \in E(A)} E(G_j) \right) \) if \( V(A) \not\subseteq F \);

\[
= E(G_j \setminus \{v_j\}) \quad \text{if} \quad V(A_i) = \{f_{v_j}\};
\]

where \( F = \{f_{v_1}, \ldots, f_{v_r}\} \). Then it is easy to see that \( A'_1, \ldots, A'_r \) are the only connected components of \( G \setminus T \). Now \( T \in \mathcal{E}(\overline{B}^W) \) implies \( T \in \mathcal{E}(G) \) and also, we have shown that \( c_{\overline{B}^W}(T) = c_G(T) \). Therefore, \( J_{\overline{B}^W} \) is unmixed as \( J_G \) is unmixed. Note that if \( T \in \mathcal{E}(G) \) and \( T \subseteq V(\overline{B}^W) \setminus F \), then \( T \in \mathcal{E}(\overline{B}^W) \) also. Since each \( f_{v_i} \) for \( i = 1, \ldots, r \) is a free vertex of \( \overline{B}^W \), \( T \in \mathcal{E}(\overline{B}^W) \) gives \( T \subseteq V(\overline{B}^W) \setminus F \). Hence \( \overline{B}^W \) is accessible as \( G \) is so. \( \square \)

**Theorem 3.10.** Let \( G \) be a graph such that \( J_G \) is strongly unmixed. Let \( B \) be a block of \( G \) and \( V \) be the set of cut vertices of \( G \) belonging to \( V(B) \). Then for any \( W \subseteq V \), \( J_{\overline{B}^W} \) is strongly unmixed. In particular, \( J_{\overline{B}} \) is strongly unmixed.
Suppose Case-I: \( J \) decomposition of \( H \). Then by Proposition 3.6, \( J \) is strongly unmixed and by Lemma 2.6, \( J \) unmixed implies that \( J \) is strongly unmixed by induction. Again by induction hypothesis, \( J \) is strongly unmixed imply \( J \) is strongly unmixed and by Lemma 2.6, \( J \) is also strongly unmixed. Hence \( J \) is strongly unmixed.

Case-II: Let \( v = v_i \in V(B) \). Let \( B = H' \cup H'' \), where \( V(B) \subseteq H'' \). Then \( H' = G_i \setminus \{v\} \) or \( H' = \{v\} \). Since, \( J \) is strongly unmixed, \( J \) is strongly unmixed and by induction hypothesis \( J \) is strongly unmixed. Therefore \( J \) is strongly unmixed. If \( H' = G_i \setminus \{v\} \), then by induction we have \( J \) is strongly unmixed and thus, by Lemma 2.6, \( J \) is strongly unmixed. Set \( B = H \setminus (V(G_i) \setminus \{v\}) = H \). Then \( H \setminus \{v\} = H'' \). In this case observe that,

\[
(G)_v = (H \cup G_i)_v.
\]

Then by Proposition 3.6, \( J \) is strongly unmixed. Now suppose \( H' = \{v\} \). Then observe that \( B \setminus \{v\} = H'' \cup \{v\} \) and \( (B \setminus \{v\}) \cup \{v\} \approx H_v \) and so \( J \) is strongly unmixed. By Lemma 2.6, \( J \) is strongly unmixed and hence from definition, \( J \) is strongly unmixed. \( \square \)

Lemma 3.11. Let \( G \) be a graph and \( v \in V(G) \) is a free vertex of \( G \). Then the following hold.

(i) If \( N_G(v) \not\subseteq \mathcal{E}(G) \), then \( J_G \) is unmixed if and only if \( J_G \setminus \{v\} \) is unmixed.

(ii) If for all \( T \in \mathcal{E}(G) \) with \( N_G(v) \subseteq T \) we have \( c_G(T) = |T| + 1 \) and \( J_G \setminus \{v\} \) is unmixed, then \( J_G \) is unmixed.

Proof. (i): Suppose \( N_G(v) \subseteq T \) for some \( T \in \mathcal{E}(G) \). Then each \( s \in N_G(v) \) is a cut point of \( G \setminus (T \setminus \{s\}) \). So, there exists \( w \in N_G(s) \setminus \{v\} \) such that \( w \not\in T \). Therefore \( s \) is a cut point of \( G \setminus (N_G(v) \setminus \{s\}) \) for every \( s \in N_G(v) \). Hence, \( N_G(v) \subseteq \mathcal{E}(G) \). So, \( N_G(v) \not\subseteq \mathcal{E}(G) \) implies \( N_G(v) \not\subseteq T \) for all \( T \in \mathcal{E}(G) \). Then by [20, Lemma 2.2], we have \( T \in \mathcal{E}(G) \) if and only if \( T = \mathcal{E}(G) \setminus \{v\} \). Also it is easy to verify that \( c_G(T) = c_G(v)(T) \) for all \( T \in \mathcal{E}(G) \setminus \{v\} \). Hence \( J_G \) is unmixed if and only if \( J_G \setminus \{v\} \) is unmixed.
(ii): Let \( T \in \mathcal{C}(G) \) with \( \mathcal{N}_G(v) \not\subseteq T \). Then by \([20]\) Lemma 2.2, \( T \in \mathcal{C}(G \setminus \{v\}) \) and notice that a connected component of \((G \setminus \{v\}) \setminus T\) contains a vertex \( w \in \mathcal{N}_G(v) \). Thus, we have \( c_G(T) = c_{G \setminus \{v\}}(T) \). Since \( J_{G \setminus \{v\}} \) is unmixed, by the given hypothesis, for all \( T \in \mathcal{C}(G) \) we get \( c_G(T) = |T| + 1 \) and hence, \( J_G \) is unmixed.

**Proposition 3.12.** Let \( G \) be a graph and \( v \in V(G) \) be a free vertex of \( G \). If \( J_G \) is strongly unmixed and there exists no cutset \( T \in \mathcal{C}(G) \) such that \( \mathcal{N}_G(v) \subseteq T \), then \( J_{G \setminus \{v\}} \) is strongly unmixed.

**Proof.** Let \( G \setminus \{v\} = H \). Since \( \mathcal{N}_G(v) \not\subseteq \mathcal{C}(G) \) and \( v \) is a free vertex of \( G \), by Lemma 3.11. \( J_H \) is unmixed. We proceed by induction on the number of vertices \( n \) of \( G \). For \( n = 1, 2 \), the result holds trivially. If \( G \) is complete, then \( H \) is also complete and we are done. Suppose \( G \) is not complete. Then there exists a cut vertex \( u \in V(G) \) of \( G \) for which \( J_{G \setminus \{u\}}, J_{G_u}, \) and \( J_{G_u \setminus \{u\}} \) are strongly unmixed. Since \( v \) is a free vertex in \( G \), \( v \in V(G \setminus \{u\}) \) is a free vertex of \( G \setminus \{v\} \). Note that \((G \setminus \{u\}) \setminus \{v\} = H \setminus \{u\} \). If there exists \( T \in \mathcal{C}(G \setminus \{u\}) \) such that \( \mathcal{N}_{G \setminus \{u\}}(v) \subseteq T \), then by Proposition 2.5, \( T \cup \{u\} \in \mathcal{C}(G) \) which leads to a contradiction as \( \mathcal{N}_G(v) \subseteq T \cup \{u\} \). Thus, \( G \setminus \{u\} \) satisfies the given conditions and has less than \( n \) vertices. Therefore, by induction hypothesis, \( J_{H \setminus \{u\}} \) is strongly unmixed. Now, from ([5], Lemma 4.5 and Lemma 5.5), we have

\[
\mathcal{C}(G_u \setminus \{u\}) = \mathcal{C}(G_u) \setminus \{T \in \mathcal{C}(G_u) \mid \mathcal{N}_G(u) \subseteq T \} = \mathcal{C}(G_u) = \{T \in \mathcal{C}(G) \mid u \not\in T\}.
\]

Suppose there exists \( T \in \mathcal{C}(G_u \setminus \{u\}) \) such that \( \mathcal{N}_{G_u \setminus \{u\}}(v) \subseteq T \). Then \( T \in \mathcal{C}(G \setminus \{u\}) \) is clear and by Proposition 2.5 \( T \cup \{u\} \in \mathcal{C}(G) \). Note that \( \mathcal{N}_G(v) \setminus \{u\} \subseteq \mathcal{N}_{G_u \setminus \{u\}}(v) \subseteq T \) which imply \( \mathcal{N}_G(v) \subseteq T \cup \{u\} \in \mathcal{C}(G) \), a contradiction. Therefore, for all \( T \in \mathcal{C}(G_u \setminus \{u\}) \) we have \( \mathcal{N}_{G_u \setminus \{u\}}(v) \not\subseteq T \). Since \((G_u \setminus \{u\}) \setminus \{v\} = H_u \setminus \{u\} \), by induction hypothesis \( J_{H_u \setminus \{u\}} \) is strongly unmixed and by Lemma 2.6, \( J_{H_u} \) is strongly unmixed.

**Lemma 3.13.** Let \( G = G_1 \cup G_2 \) be such that \( V(G_1) \cap V(G_2) = \{v\} \). Consider the graph \( \overline{G_i} \) by attaching a whisker \( \{v, f_v\} \) to \( G_i \) at \( v \). If \( J_{G_i} \) is unmixed, then \( J_{\overline{G_i}} \) is unmixed for \( i = 1, 2 \).

**Proof.** Without loss of generality, we assume \( G \) is connected. Then \( \overline{G_i} \) is connected for \( i = 1, 2 \). Let \( T \in \mathcal{C}(\overline{G_1}) \). Then it is clear that \( T \in \mathcal{C}(G) \). Let \( A_1, \ldots, A_k \) be the connected components of \( \overline{G_1} \setminus T \). One of components, say \( A_1 \), will contain the vertex \( f_v \). We consider the graph \( A'_1 \) as follows

- \( V(A'_1) = (V(A_1) \setminus \{f_v\}) \cup V(G_2 \setminus \{v\}) \);
- \( E(A'_1) = E(A_1 \setminus \{f_v\}) \cup E(G_2) \) if \( \{v, f_v\} \in E(A_1) \),
  \[= E(G_2 \setminus \{v\}) \] if \( \{v, f_v\} \not\in E(A_1) \) i.e., \( V(A_1) = \{f_v\} \).
Then $A'_i$ is connected and $A_1, A_2, \ldots, A_k$ are the only connected components of $G \setminus T$. Therefore, \( c_{G_i}(T) = c_G(T) \) for all $T \in \mathcal{C}(G_i)$ and so, $J_{G_i}$ is unmixed as $J_G$ is so. Similarly, $J_{G_2}$ is unmixed.

**Lemma 3.14.** Let $G = G_1 \cup G_2$ be such that $V(G_1) \cap V(G_2) = \{v\}$. Consider the graph $G_i$ by attaching a whisker $\{v, f_v\}$ to $G_i$ at $v$. If $J_G$ is unmixed and $G_1, G_2$ are accessible, then $G$ is accessible.

**Proof.** Without loss of generality, we can assume $G$ is connected. Let $\mathcal{C}(G) = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ as described in Proposition 3.1. Let $T \in \mathcal{A}$. Then $T = S_1 \cup S_2$, where $v \notin S_i \in \mathcal{C}(G_i)$ for $i = 1, 2$. Clearly, $S_i \in \mathcal{C}(G_i)$ and so, there exists $s_i \in S_i$ such that $S_i \setminus \{s_i\} \in \mathcal{C}(G_i)$, where $i \in \{1, 2\}$. Now, $v \notin S_1 \setminus \{s_1\}$ imply $S_1 \setminus \{s_1\} \in \mathcal{C}(G_1)$ by [20, Lemma 2.2]. Therefore, $T \setminus \{s_1\} \in \mathcal{A}$ and so, $T$ is accessible. Let $T = T_1 \cup T_2 \in \mathcal{B}$, where $T_i \in \mathcal{C}(G_i \setminus \{v\})$ and $v \notin S_2 \in \mathcal{C}(G_2)$. If $\mathcal{N}_{G_1}(v) \subseteq T_1$, then $T_i \in \mathcal{C}(G_i)$ and one connected component of $G_1 \setminus T_1$ consists of only the edge $\{v, f_v\}$. Again, $S_2 \in \mathcal{C}(G_2)$. Since $J_{G_i}$ is unmixed for $i = 1, 2$, we have

\[
c_G(T) = |T_1| + 1 + |S_2| + 1 - 2 = |T|,
\]

which gives a contradiction to the fact that $J_G$ is unmixed. Thus, $\mathcal{N}_{G_1}(v) \not\subseteq T_1$ and so, $T \in \mathcal{D}$ also. Similarly, $T \in \mathcal{C}$ implies $T \in \mathcal{D}$. Now pick $T \in \mathcal{D}$. Then $T = T_1 \cup T_2 \cup \{v\}$, where $\mathcal{N}_{G_1}(v) \not\subseteq T_i \in \mathcal{C}(G_i \setminus \{v\})$ for $i = 1, 2$. Then $T_i \cup \{v\} \in \mathcal{C}(G_i)$ and by [5, Lemma 4.16] and Proposition 2.4, we have $v \neq t_i \in T_i$ such that $T_i \cup \{v\} \setminus \{t_i\} \in \mathcal{C}(G_i)$ for $i = 1, 2$. Then $T \setminus \{t_1\} \in \mathcal{D}$ which implies $T$ is accessible. So, $\mathcal{C}(G)$ is an accessible set system and unmixedness of $J_G$ is given. Hence $G$ is accessible. \[
\]

**Theorem 3.15.** Let $G$ be a simple connected graph such that $J_G$ is unmixed, and for each block $B$ of $G$, $\overline{B}$ is accessible. Then $G$ is accessible.

**Proof.** We will use induction on $n$, where $|V(G)| = n$. For, $n = 1$ and $n = 2$, $G$ is complete and we are done. If all the blocks of $G$ is complete then there is nothing to prove. Let $n > 2$. If there is a block $B$ of $G$ for which $\overline{B} = G$, then $G$ is accessible as $\overline{B}$ is so. Now consider a block $B$ of $G$ which is not complete and take a cut vertex $v$ of $G$ belong to $V(B)$ such that $G \setminus \{v\}$ does not contain a $K_1$. Let $G = G_1 \cup G_2$ be such that $V(G_1) \cap V(G_2) = \{v\}$ and $V(B) \subseteq V(G_1)$. Now, consider the graph $G_i$ as described in Lemma 3.14, where $i = 1, 2$. Then by Lemma 3.13, $J_{G_i}$ and $J_{G_2}$ are unmixed as $J_G$ is unmixed. Thus, by induction hypothesis, $G_i$ is accessible for $i = 1, 2$. Hence by Lemma 3.14 $G$ is accessible. \[
\]

**Lemma 3.16.** Let $G = G_1 \cup G_2$ be a simple connected graph such that $V(G_1) \cap V(G_2) = \{v\}$, where $v$ is a cut vertex of $G$. Let $J_G$ and $J_{G \setminus \{v\}}$ be unmixed. Then $J_{G,v}$ is strongly unmixed implies $J_{G \setminus \{v\}}$ is strongly unmixed.
Proof. By Proposition 3.6 and Proposition 5.7, we have $J_{G_1}$ is strongly unmixed if and only if $J_{(G_1)_v}$ and $J_{(G_2)_v}$ are strongly unmixed. First we will show that if one of $\mathcal{N}_{G_1}(v)$ or $\mathcal{N}_{G_2}(v)$ is singleton then the result holds. Let $\mathcal{N}_{G_1}(v) = \{u\}$. Then $(G_1)_v = G_1$. By Proposition 2.5, $J_G$ and $J_{G \setminus \{v\}}$ are unmixed imply $u \not\in T$ for any $T \in \mathcal{C}(G_1 \setminus \{v\})$ i.e., $u$ is a free vertex in $G_1 \setminus \{v\}$. In this case, $G_1$ is a decomposable graph as

$$G_1 = (G_1)_v = (G_1 \setminus \{v\}) \cup K_2,$$

where $V(G_1 \setminus \{v\}) \cap V(K_2) = \{u\}$. Therefore $J_{G_1 \setminus \{v\}}$ is strongly unmixed by Remark 2.8. Then we can see that

$$G_v \setminus \{v\} = G_1 \setminus \{v\} \cup (G_2)_v',$$

where $V(G_1 \setminus \{v\}) \cap V((G_2)_v') = \{u\}$ and $(G_2)_v' \simeq (G_2)_v$, just relabeling the vertex $v$ of $(G_2)_v$ by the vertex $u$. Again, using Remark 2.8 we get $J_{G_v \setminus \{v\}}$ is strongly unmixed.

Now we will use induction on the number of vertices $n$ of $G_v \setminus \{v\}$. Since $v$ is a cut vertex of $G$, the base case is $n = 2$ and in this case, $G_v \setminus \{v\}$ being a $K_2$, we are done. Let $n > 2$ and the hypothesis is true for all such graphs with less than $n$ vertices. We can assume $|\mathcal{N}_{G_i}(v)| > 1$ for $i = 1, 2$ and in this case, $(G_i)_v$ is complete if and only if $(G_i)_v \setminus \{v\}$ is complete as $J_{(G_i)_v}$ is strongly unmixed, where $i = 1, 2$. Then we may assume one of $(G_i)_v$ is not complete, say $(G_1)_v$, otherwise, $G_v \setminus \{v\}$ will be complete and the result follows. Now there exists a cut vertex $x \in V(G_1)$ for which the binomial edge ideals of $(G_1)_v \setminus \{x\}$, $((G_1)_v)_x$, and $((G_1)_v)_x \setminus \{x\}$ are strongly unmixed. Set $D = G_v \setminus \{v\}$. Let $(G_1)_v \setminus \{x\} = H' \sqcup H''$, where $v \in V(H'')$. Then $D \setminus \{x\} = H \cup H''$, where $V(G_2 \setminus \{v\}) \subseteq V(H)$. Note that $H = (G_2)_v \setminus \{v\}$, where $G_2 = (G_2)_v \cup H'$. Then $V((G_2)_v) \cap V(H') = \{v\}$ and $v$ is a free vertex in both of $(G_2)_v$ and $H'$. Since $J_{(G_2)_v}$ and $J_{H'}$ are unmixed, by Remark 2.8, $J_{G_2}$ is unmixed. By Proposition 2.5, $J_{G_2 \setminus \{v\}}$ is unmixed implies there exists no $S \in \mathcal{C}(G_2 \setminus \{v\})$ containing $\mathcal{N}_{(G_2)_v}$ and so, $\mathcal{N}_{(G_2)_v}(v) \not\subseteq T$ for any $T \in \mathcal{C}(G_2 \setminus \{v\})$. If there exists $T' \in \mathcal{C}(H'' \setminus \{v\})$ such that $\mathcal{N}_{H'}(v) \subseteq T'$, then $\mathcal{N}_{G_1}(v) \subseteq T' \cup \{x\} \in \mathcal{C}(G_1 \setminus \{v\})$ which is a contradiction to the fact that $J_{G_1 \setminus \{v\}}$ is unmixed. Therefore, by Proposition 2.5, $G_2 \setminus \{v\}$ is unmixed. Now, $J_{H'} = J_{H''}$ and $J_{(G_2)_v}$ are strongly unmixed and hence by Proposition 3.7, $J_{(G_2)_v}$ is strongly unmixed. Therefore, by induction hypothesis we have $J_H$ is strongly unmixed and so is $J_{G_v \setminus \{x\}}$.

Also, we can observe that

$$D_x \setminus \{x\} = L_v \setminus \{v\} \text{ with } L = (((G_1)_v)_x \setminus \{x\}) \cup (G_2)_v.$$ 

Then $L$ is a decomposable graph and so, $J_L$ is unmixed as $J_{((G_1)_v)_x \setminus \{x\}}$ and $J_{(G_2)_v}$ are unmixed. We have already proved the binomial edge ideals of $(G_2)_v \setminus \{v\}$ and $((G_1)_v \setminus \{v\}) \setminus \{x\}$ are unmixed. Therefore, the binomial
edge ideal of \(((G_1)_{x})_x \setminus \{x\}\setminus \{v\}\) is unmixed and so is \(J_{L\setminus \{v\}}\). Again using Proposition 5.7, we see that \(J_{L\setminus \{v\}}\) is strongly unmixed and thus, by induction, \(J_{D_x \setminus \{x\}}\) is strongly unmixed. By Lemma 2.6, \(J_{D_x}\) is also strongly unmixed. Hence \(J_{G_r \setminus \{v\}}\) is strongly unmixed. \(\square\)

Recall the notion of block graph of a given graph. Let \(G\) be a graph with blocks \(B_1, \ldots, B_m\). We denote the block graph of \(G\) by \(\mathcal{B}(G)\), which is defined as follows:

- \(V(\mathcal{B}(G)) = \{B_1, \ldots, B_m\}\).
- \(E(\mathcal{B}(G)) = \{\{B_i, B_j\} \mid V(B_i) \cap V(B_j) \neq \emptyset\}\).

**Theorem 3.17.** Let \(G\) be a simple graph such that \(J_G\) is unmixed. If \(J_{\mathcal{B}(G)}\) is strongly unmixed for each block \(B\) of \(G\), then \(J_G\) is strongly unmixed.

**Proof.** If \(G\) has no cut vertex, then the result follows trivially. So assume \(G\) has some cut vertices. First we will prove the following claim:

**Claim:** There exist two blocks \(B_p\) and \(B_q\) of \(G\) such that \(V(B_p) \cap V(B_q) = \{v\}\) and the binomial edge ideals of \((\overline{B}_i)_{v} \setminus \{v\}\), \((\overline{B}_i)_{v}\), and \((\overline{B}_i)_{v} \setminus \{v\}\) are strongly unmixed, where \(i \in \{p, q\}\).

**Proof of claim:** Since \(J_G\) is unmixed, by [21], Proposition 1.3, \(\mathcal{B}(G)\) is a tree and this implies each cut vertex of \(G\) lies exactly in two blocks of \(G\). Take a block \(B_1\) which contains only one cut vertex \(v_1\) of \(G\). Then \(J_{\overline{B}_1}\) is strongly unmixed gives \(J_{\overline{B}_1 \setminus \{v_1\}}, J_{\overline{B}_1 \setminus \{v_1\}}, \) and \(J_{\overline{B}_1 \setminus \{v_1\}}\) are strongly unmixed. Let \(B_2\) be the other block of \(G\) containing \(v_1\). Now, \(J_{\overline{B}_2}\) is strongly unmixed and so, there exists a cut vertex \(v_2\) of \(\overline{B}_2\) for which \(J_{\overline{B}_2 \setminus \{v_2\}}, J_{\overline{B}_2 \setminus \{v_2\}}, \) and \(J_{\overline{B}_2 \setminus \{v_2\}}\) are strongly unmixed. If \(v_1 = v_2\), then we are done. Otherwise, consider the other block \(B_3\) which contains the cut vertex \(v_2\). Continuing this process, if we do not get a cut vertex \(v_j\) for which the binomial edge ideals of \((\overline{B}_j)_{v_j} \setminus \{v_j\}\), \((\overline{B}_j)_{v_j}\), and \((\overline{B}_j)_{v_j} \setminus \{v_j\}\) are strongly unmixed, then after finite number of step we will reach to a block \(B_k\) which has only one cut vertex. Therefore, by the given conditions, the binomial edge ideals of \((\overline{B}_k)_{v_{k-1}} \setminus \{v_{k-1}\}\), \((\overline{B}_k)_{v_{k-1}}\), \((\overline{B}_k)_{v_{k-1}} \setminus \{v_{k-1}\}\) are strongly unmixed and also, the binomial edge ideals of \((\overline{B}_k)_{v_{k-1}} \setminus \{v_{k-1}\}\), \((\overline{B}_k)_{v_{k-1}}\), \((\overline{B}_k)_{v_{k-1}} \setminus \{v_{k-1}\}\) are strongly unmixed. Thus, in \(G\) we will get two blocks \(B_p\) and \(B_q\) such that \(V(B_p) \cap V(B_q) = \{v\}\) and the binomial edge ideals of \((\overline{B}_i)_{v} \setminus \{v\}\), \((\overline{B}_i)_{v}\), and \((\overline{B}_i)_{v} \setminus \{v\}\) are strongly unmixed, where \(i \in \{p, q\}\).

Due to Remark 2.8 we may assume \(G\) is indecomposable. We proceed by induction on the number of vertices \(n\) of \(G\). Let \(G \setminus \{v\} = H_1 \cup H_2\), where \(V(B_p \setminus \{v\}) \subseteq V(H_1)\) and \(V(B_q \setminus \{v\}) \subseteq V(H_2)\). Let \(B\) be a block of \(H_1\). Suppose \(B_p\) is \(K_2\), then we can write \(G = G_1 \cup B_p \cup G_2\) such that \(V(G_1) \cap V(B_p) = \{v\}\) and \(V(G_2) \cap V(B_p) = \{w\}\). Since \(G\) is
indecomposable, \(v\) and \(w\) can not be free vertices in \(G_1\) and \(G_2\), respectively. So, there exists \(T_i \in \mathcal{E}(G_i)\) for \(i = 1, 2\) such that \(v \in T_1\) and \(w \in T_2\). Note that \(T_1, T_2, T_1 \cup T_2 \in \mathcal{E}(G)\). Since \(J_G\) is unmixed, \(c_{G_i}(T_i) = |T_i|\) for \(i = 1, 2\). Therefore, \(c_G(T_1 \cup T_2) = |T_1 \cup T_2|\) which gives a contradiction as \(J_G\) is unmixed. Hence \(B_p\) can not be \(K_2\) and similarly, \(B_q\) can not be \(K_2\). In this situation, if \(B\) is also a block of \(G\), then \(\overline{B}\) with respect to \(H_1\) and \(\overline{B}\) with respect to \(G\) are same. So, \(J_{\overline{B}}\) is strongly unmixed. If \(B\) is not a block of \(G\), then \(V(B) \subseteq V(B_p \setminus \{v\})\) and in this case, \(\overline{B}\) with respect to \(H_1\) is equal to \(\overline{B}\) with respect to \(\overline{B_p} \setminus \{v\}\). Since \(J_{\overline{B_p} \setminus \{v\}}\) is strongly unmixed, by Theorem 3.10, \(J_{\overline{B}}\) is strongly unmixed. Also, \(J_{H_1}\) is unmixed by Proposition 2.5. Therefore, by induction hypothesis, \(J_{H_1}\) is strongly unmixed. Similarly, \(J_{H_2}\) is strongly unmixed and so is \(J_{G \setminus \{v\}}\). For a fix \(n\), we will use induction on the number of blocks \(k\) of \(G\). If \(k = 1\), then by the given condition \(G\) is complete and so \(J_G\) is strongly unmixed. Now, consider \(G_v\) and \(G_v \setminus \{v\}\). Then for any block \(B\) of \(G_v\) and \(G_v \setminus \{v\}\) such that \(V(B_p \cup B_q) \setminus \{v\} \not\subseteq B\), we have \(\overline{B}\) with respect to \(G_v\) or \(G_v \setminus \{v\}\) is same as \(\overline{B}\) with respect to \(G\). Thus, \(J_{\overline{B}}\) is strongly unmixed. Let \(D = B_p \cup B_q\). Then \(D_v\) is the block of \(G_v\) containing \(V(B_p \cup B_q)\) and \(D_v \setminus \{v\}\) is the block of \(G_v \setminus \{v\}\) containing \(V(B_p \cup B_q) \setminus \{v\}\). Since the binomial edge ideals of \(\overline{B_p} \setminus \{v\}\) and \(\overline{B_q} \setminus \{v\}\) are strongly unmixed, by Proposition 3.7, \(J_{\overline{B_v}}\) is strongly unmixed and by Lemma 3.16, \(J_{\overline{B_v} \setminus \{v\}}\) is strongly unmixed. Number of blocks in \(G_v\) and \(G_v \setminus \{v\}\) is \(k - 1\). Thus, by induction hypothesis, we have \(J_{G_v}\) and \(J_{G_v \setminus \{v\}}\) are strongly unmixed and hence, \(J_G\) is strongly unmixed.

In [5] Problem 7.2, the authors proposed the following question:

**Question 3.18.** Let \(G\) and \(H\) be two disjoint connected graphs such that \(J_G\) and \(J_H\) are unmixed. Let \(v, w\) be the cut vertices of \(G, H\), respectively, for which \(J_{G \setminus \{v\}}\) and \(J_{H \setminus \{w\}}\) are unmixed. Set \(G \setminus \{v\} = G_1 \cup G_2, H \setminus \{w\} = H_1 \cup H_2\). Let \(F_{ij}\) be the graph obtained by gluing \(G[V(G_i) \cup \{v\}]\) and \(H[V(H_j) \cup \{w\}]\) identifying \(v\) and \(w\), where \(i, j = 1, 2\). If \(J_G, J_H\) are Cohen-Macaulay, is it true that \(J_{F_{ij}}\) is Cohen-Macaulay? If \(G\) and \(H\) are accessible, is it true that \(F_{ij}\) is accessible?

Using some previous results of this section, we manage to give a partial answer to Question 3.18 through the following Theorem 3.19:

**Theorem 3.19.** Let \(G = G_1 \cup G_2\) with \(V(G_1) \cap V(G_2) = \{v\}\) and \(H = H_1 \cup H_2\) with \(V(H_1) \cap V(H_2) = \{w\}\) be two distinct connected graphs such that \(J_G, J_H, J_{G \setminus \{v\}}, J_{H \setminus \{w\}}\) are unmixed. Consider the graph \(F_{ij} = G_i \cup H_j\) identifying the vertices \(v\) and \(w\), labeling as \(v\) i.e., \(V(G_i) \cap V(H_j) = \{v\}\), for \(i, j \in \{1, 2\}\). Then the following hold for each \(i, j \in \{1, 2\}\).
(i) If $G$ and $H$ are accessible, then $F_{ij}$ is accessible.

(ii) If $J_G$ and $J_H$ are strongly unmixed, then $J_{F_{ij}}$ is strongly unmixed.

Proof. Choose an $F_{ij}$ for $i, j \in \{1, 2\}$. Since $J_{G \setminus \{v\}}$ and $J_{H \setminus \{v\}}$ are unmixed, $J_{F_{ij} \setminus \{v\}}$ is unmixed. Let $S_i \in \mathcal{C}(G_i)$ with $v \not\in S_i$. Then $S_i \in \mathcal{C}(G)$ and $c_G(S_i) = c_{G_i}(S_i) = |S_i| + 1$ as $J_G$ is unmixed. Suppose $S_i \in \mathcal{C}(G_i)$ with $v \in S_i$. Then $S_i \in \mathcal{C}(G)$ and $c_G(S_i) = |S_i| + 1$ as $J_G$ is unmixed. In this case, one connected component of $G \setminus S_i$ is $G_k \setminus \{v\}$, where $i \neq k \in \{1, 2\}$. Therefore, $c_G(S_i) = |S_i|$. Thus, $G_i$ satisfy the condition (ii) of Corollary 3.2. Similarly, $F_j$ also satisfy the condition (ii) of Corollary 3.2. Hence by Corollary 3.2 $J_{F_{ij}}$ is unmixed. Since, $G$ and $H$ are accessible, by Theorem 3.9 and Theorem 3.10, for each block $B$ of $G$, $\overline{B}$ with respect to $G$ is accessible and strongly unmixed. Same holds for $H$. Note that for any block $B$ of $F_{ij}$, $\overline{B}$ with respect to $F_{ij}$ is either $\overline{B}$ with respect to $G$ or $\overline{B}$ with respect to $H$. Therefore, by Theorem 3.15 $F_{ij}$ is accessible and by Theorem 3.17 $J_{F_{ij}}$ is strongly unmixed.

![Figure 1](image1.png)

**Figure 1.** $J_G$ and $J_H$ are Cohen-Macaulay.

![Figure 2](image2.png)

**Figure 2.** $J_F$ is Cohen-Macaulay but $J_L$ is not.
Lemma 4.1. Let \( W \) be in a special manner such that we will first prove

\[
\text{Proof.}
\]

Example 6.11]. The graph \( H \) in Figure 1 is bipartite accessible graph and so, \( J_H \) is strongly unmixed by [5 Corollary 6.9]. In Figure 2, \( L \) is the graph obtained by attaching \( G \) in Figure 1b, \( H \) in Figure 1a is strongly unmixed (proved in [5, Example 6.11]). The graph \( H \) in Figure 1b is bipartite accessible graph and hence, \( J_H \) is strongly unmixed by [5 Corollary 6.9]. In Figure 2, \( L \) is the graph obtained by attaching \( G \) in Figure 1a, \( H \) in Figure 1a is strongly unmixed proceeding by induction.

Note that \( G \) \( \{1, 7\} \) is not unmixed as \( J_G \) is unmixed and \( \{3, 4, 6\} \in \mathcal{C}(G \{1\}) \). Therefore, by Proposition 3.1, \( J_L \) is not unmixed and so \( J_L \) can not be Cohen-Macaulay. On the other hand, \( G \{3, 9\} \) and \( H \{4, 6\} \) is unmixed by Proposition 2.5. Therefore, by Corollary 3.2, \( J_F \) is unmixed and hence, by Theorem 3.17, \( J_F \) is strongly unmixed as well as Cohen-Macaulay.

4. COHEN-MACULA Y BINOMIAL EDGE IDEALS OF \( r \)-REGULAR \( r \)-CONNECTED BLOCKS WITH WHISKERS

In this part, we define a new class of indecomposable graphs by attaching two complete graphs in a particular manner and adding some whiskers. We show the Conjecture 1.1 holds for these graphs and classify all \( r \)-regular \( r \)-connected graphs with whiskers whose binomial edge ideals are Cohen-Macaulay.

Construction: Take two complete graphs \( K_m \) and \( K_n \) with \( V(K_m) = \{x_1, \ldots, x_m\} \) and \( V(K_n) = \{y_1, \ldots, y_n\} \). Let \( r \leq m \) and \( r \leq n \) be a positive integer. The star product of \( K_m \) and \( K_n \) with respect to \( r \), denoted by \( K_m *_r K_n \), is defined in such a way that

\[
\begin{align*}
\bullet \quad V(K_m *_r K_n) &= V(K_m) \sqcup V(K_n), \\
\bullet \quad E(K_m *_r K_n) &= E(K_m) \cup E(K_n) \cup \{x_i, y_i \mid 1 \leq i \leq r\}.
\end{align*}
\]

Note that \( K_m *_r K_n = K_n *_r K_m \) i.e., the star product is commutative. For \( m = n = r \), we write \( K_r * K_r \) instead of \( K_r *_r K_r \).

Now we will consider the graph \( \overline{K_m *_r K_n} \), by adding some whiskers to it in a special manner such that

\[
\begin{align*}
\bullet \quad V(\overline{K_m *_r K_n}) &= V(K_m *_r K_n) \sqcup \{f_{x_i} : f_{y_i} \mid 2 \leq i \leq r\}; \\
\bullet \quad E(\overline{K_m *_r K_n}) &= E(K_m *_r K_n) \cup \{x_i, f_{x_i} \}, \{y_i, f_{y_i} \} \mid 2 \leq i \leq r\}.
\end{align*}
\]

Lemma 4.1. Let \( G = K_m *_r K_n \). Then \( J_G \) is unmixed.

Proof. We will first prove \( G_r = \overline{K_r * K_r} \) is unmixed proceeding by induction on \( r \).
For $r = 1$, $G_1$ is $K_2$ and $\phi$ being the only cutset of $G_1$, $J_{G_1}$ is unmixed. When $r = 2$, we can see that $J_{G_2}$ is unmixed by [21] Lemma 2.9. Now consider the graph $G_r \setminus \{x_r\} = \{f_{x_r}\} \cup H$. Clearly, $H$ is a decomposable graph such that $H = H_1 \cup H_2$ with $H_1 = H[\{y_r, f_{y_r}\}]$ and $H_2 = H \setminus \{f_{y_r}\}$. Notice that $G_{r-1} = H_2 \setminus \{y_r\}$. Since $y_r$ is a free vertex in $H_2$, $y_r \notin S$ for all $S \in \mathcal{C}(H_2)$. Also, $N_{H_2}(y_r) \not\subseteq T$ for any $T \in \mathcal{C}(G_{r-1})$. By induction hypothesis, $J_{G_{r-1}}$ is unmixed and therefore, by Corollary [3.2] and [20] Lemma 2.2, we have $J_H$ is unmixed. Again, $G_r \setminus \{f_{x_r}\}$ and $G_r[\{x_r, f_{x_r}\}]$ satisfy the condition (ii) of Corollary [3.2]. Hence using Corollary [3.2] we get $J_{G_r}$ is unmixed as $J_{G_r \setminus \{x_r\}}$ is unmixed.

If $m > r + 1$ or $m = n + 1$ or $m = r + 1$, then we can observe that $c_{G_r}(T) = |T| + 1$ for all $T \in \mathcal{C}(G(r))$ with $N_{G_r}(x_{r+1}) \subseteq T$ or $N_{G_r}(y_{r+1}) \subseteq T$. Therefore, using Lemma [3.11] we get $J_T$ is unmixed. For $m > r + 1$ or $n > r + 1$, $x_{r+1}$, $x_r$, $y_n$, $y_{r+1}$, $y_m$, and $y_{r+1}$, $y_n$ are free vertex of $G$. Note that $N_{G_r}(x_i) \not\subseteq \mathcal{C}(G)$ for all $1 \leq i \leq m$ if $m > r + 1$ and $N_{G_r}(y_i) \not\subseteq \mathcal{C}(G)$ for all $1 \leq i \leq n$ if $n > r + 1$. Therefore, by repeating application of Lemma [3.11] we get $J_T$ is unmixed.

**Lemma 4.2.** Let $G = K_m \ast K_n$. Then $\overline{G}$ is accessible.

**Proof.** We first prove $G_r = \overline{K_r \ast K_r}$ is accessible by induction on $r$.

For $r = 1$, $G_1 = K_2$ and by [5] Remark 4.2, it is accessible. For $r = 2$, $J_{G_2}$ is Cohen-Macaulay by [21] Lemma 2.9 and hence $G_2$ is accessible by [5] Theorem 3.5. Let us consider the graph $G_r \setminus \{x_r\} = \{f_{x_r}\} \cup H$. Now $H$ is a decomposable graph with the decomposition $H = H_1 \cup H_2$, where $H_1 = H[\{y_r, f_{y_r}\}]$ and $H_2 = H \setminus \{f_{y_r}\}$. $y_r$ is a free vertex of $H_2$ and $H_2 \setminus \{y_r\} = G_{r-1}$, which is accessible by induction hypothesis. Also, unmixedness of $J_{H_2}$ is clear from proof of Lemma [4.1]. Therefore, by Lemma [3.11] $H_2$ is accessible. $H_1$ being $K_2$ is accessible. Thus, by Remark [2.8] $H$ is accessible. Now $G_r = G^1 \cup G^2$, where $V(G^1) \cap V(G^2) = \{x_r\}$. Then $G^1$ is $K_2$ with $G^1 \setminus \{x_r\} = \{f_{x_r}\}$ and $G^2 \setminus \{x_r\} = H$. Clearly, $\mathcal{C}(G^1) = \mathcal{C}(G^1 \setminus \{x_r\}) = \{\phi\}$. Considering $G_r = G^1 \cup G^2$, we have $\mathcal{C}(G_r) = A \cup B \cup C \cup D$ as described in Proposition [3.1]. Let $T \in A$. Then $T = S_2$, where $x_r \not\subseteq S_2 \in \mathcal{C}(G^2)$. If $x_i \in \{x_1, \ldots, x_{r-1}\} \cap S_2$, then $S_2 \setminus \{x_i\} \in \mathcal{C}(H)$ and so, $S_2$ is accessible. If $\{x_1, \ldots, x_{r-1}\} \cap S_2 = \phi$, then $S_2 \subseteq \mathcal{C}(H)$ and so is accessible. Let $T \in B$. Then $T = \phi \cup S_2 = S_2$, where $x_r \subseteq S_2 \in \mathcal{C}(G^2)$. Then from the construction of $G_r$ it is easy to observe that $T \setminus \{x_r\} \in \mathcal{C}(H)$ and hence $T$ is accessible. Since $H$ is accessible and $\mathcal{C}(G^1) = \mathcal{C}(G^1 \setminus \{x_r\}) = \phi$, any $T \in C \cup D$ is accessible. Thus $G_r$ is accessible. Now for $m > r$ or $n > r$, $x_{r+1}$, $x_m$, and $y_{r+1}$, $y_m$ are free vertex of $G$. From Lemma [4.1] $J_{G_r}$ is unmixed and hence by repeating application of Proposition [3.4] we get $G_r$ is accessible. \qed
**Theorem 4.3.** Let \( G = K_m \ast_r K_n \). Then \( J_G \) is strongly unmixed.

**Proof.** Let \( G_r = \overline{K_r} \ast \overline{K_r} \). For \( m > r \) or \( n > r \), \( x_{r+1}, \ldots, x_m \) and \( y_{r+1}, \ldots, y_n \) are free vertex of \( \overline{G} \). Using repeating applications of Lemma 4.1 and Lemma 2.6, it is enough to prove \( J_{G_r} \) is strongly unmixed to show \( J_G \) is strongly unmixed.

Note that \( G_1 = K_2 \) and so is \( J_{G_1} \) is strongly unmixed by definition. Also, \( G_2 \) is a bipartite accessible graph and hence \( J_{G_2} \) is strongly unmixed by \([5, \text{Corollary 6.9}]\). We proceed by induction on \( r \). By Lemma 4.1, \( J_G \) is unmixed. Consider \( G_r \setminus \{x_r\} = \{f_{x_r}\} \cup H \). Then by proof of Lemma 4.2, we have \( G_r \) is accessible and \( H \) is accessible. Therefore by \([5, \text{Corollary 5.16}]\), \( G_r \setminus \{x_r\}, (G_r)_x, \) and \( (G_r)_y \setminus \{x_r\} \) are accessible and so, the corresponding binomial edge ideals are unmixed. Now \( H \) is decomposable as \( H = H_1 \cup H_2 \), where \( H_1 = H[\{y_r, f_{y_r}\}] \) and \( H_2 = H \setminus \{f_{y_r}\} \). Note that \( H_2 \setminus \{y_r\} = G_{r-1} \) and so \( J_{H_2 \setminus \{y_r\}} \) is strongly unmixed by induction hypothesis. Since \( y_r \) is a free vertex of \( H_2 \) and \( J_{H_2} \) is unmixed, by Lemma 2.6 \( J_{H_2} \) is strongly unmixed. Thus, \( J_H \) is strongly unmixed by Remark 2.8 and so is \( J_{G_r \setminus \{x_r\}} \). Now consider the graph \( (G_r)_x \setminus \{x_r\} = D \) and set \( D' = D \setminus \{f_{x_r}\} \). Observe that \( D' \setminus \{y_r\} = \{f_{y_r}\} \cup G_{r-1} \). By induction hypothesis \( J_{G_{r-1}} \) is strongly unmixed and so is \( J_{D' \setminus \{y_r\}} \). Also, \( D'_{y_r} \) and \( D'_{y_r} \setminus \{y_r\} \) are complete graph with some whiskers and hence \( J_{D'_{y_r}} \) and \( J_{D'_{y_r} \setminus \{y_r\}} \) are strongly unmixed. Now

\[
\mathcal{N}_D(f_{x_r}) = \{x_1, \ldots, x_{r-1}, y_r\} \notin \mathcal{C}(D)
\]

as \( x_1 \) can not be a cut point in \( D \setminus \{x_1, \ldots, x_{r-1}, y_r\} \). Since \( f_{x_r} \) is a free vertex in \( D \), from Lemma 3.11 we have \( J_{D'} \) is unmixed. Therefore by definition \( J_{D'} \) is strongly unmixed. Using Lemma 2.6 we get \( J_D \) is strongly unmixed. \( (G_r)_x \) is unmixed, \( x_r \) is a free vertex of \( (G_r)_x \), and \( J_D \) is strongly unmixed together imply \( J_{(G_r)_x} \) is strongly unmixed (using Lemma 2.6). Hence by definition \( J_G \) is strongly unmixed. \( \square \)

**Lemma 4.4.** Let \( B \) be a non-complete block such that \( B \) is \( r \)-regular and \( r \)-connected. If \( \overline{B} \) is accessible, then for any \( v \in V(B) \), \( B \setminus \mathcal{N}_B(v) \) contains two connected components.

**Proof.** Let \( v \in V(B) \) be such that \( B \setminus \mathcal{N}_B(v) \) contains the connected components \( A_1, \ldots, A_s \) and \( \{v\} \). Since \( B \) is non-complete, \( r \)-regular and \( r \)-connected, \( \mathcal{N}_B(v) \in \mathcal{C}(\overline{B}) \) is clear. By accessibility of \( \overline{B} \), there exists \( w \in \mathcal{N}_B(v) \) such that \( \mathcal{N}_B(v) \setminus \{w\} \in \mathcal{C}(\overline{B}) \). Now any \( A_i \) is adjacent to all vertices belong to \( \mathcal{N}_B(v) \), otherwise we will get less than \( r \) vertices to disconnect the block \( B \). Since \( \mathcal{N}_B(v) \setminus \{w\} \in \mathcal{C}(\overline{B}) \), \( w \) is adjacent to exactly two connected components of \( \overline{B} \setminus \mathcal{N}_B(v) \) one of which contains \( v \). Therefore \( s \) must be 1 and \( w \) can not be a cut vertex of \( \overline{B} \)
or \( s = 0 \) and \( w \) is a cut vertex of \( \overline{B} \). But, \( s = 0 \) implies \( B \) is complete and so \( s = 1 \) is the only possibility. □

**Proposition 4.5.** Let \( B \) be a non-complete \( r \)-regular \( r \)-connected block. If \( \overline{B} \) is accessible, then \( \overline{B} = G_r = K_r \ast K_r \), where \( r \geq 2 \).

**Proof.** For each \( v \in V(B) \), we have \( N_B(v) \in ℂ(\overline{B}) \) as \( B \) is non-complete \( r \)-regular \( r \)-connected block. Unmixed property of \( J_{\overline{B}} \) implies

\[
c_{\overline{B}}(N_B(v)) = |N_B(v)| + 1 = r + 1.
\]

From Lemma 4.4 it follows that \( B \setminus N_B(v) \) contains two connected components and therefore \( N_B(v) \) should have \( r - 1 \) cut vertices of \( \overline{B} \). Let \( L \) be the set of cut vertices of \( \overline{B} \) and \( |V(B)| = n \). Since each vertex of \( B \) add \( r - 1 \) degree to cut vertices, we have

\[
\sum_{v \in L} deg_B(v) = r |L| = (r - 1)n \Rightarrow |L| = \frac{(r - 1)n}{r}.
\]

Therefore, the number of non-cut vertices of \( \overline{B} \) in \( B \) is \( n - |L| = n \frac{r}{r} \). Since each cut vertex of \( \overline{B} \) is adjacent to a non-cut vertex in \( B \), we have \( L \in ℂ(\overline{B}) \). \( J_{\overline{B}} \) being unmixed, \( c_{\overline{B}}(L) = |L| + 1 \) and so, the induced subgraph on the set of non-cut vertices of \( B \) is connected. But, the degree of any non-cut vertex in the induced subgraph \( B \setminus L \) will be one. So the only possibility is \( B \setminus L \) is an edge i.e., \( \frac{n}{r} = 2 \) and the two non-cut vertices are adjacent. Now \( n = 2r \) and \( |L| = 2(r - 1) \). Let \( V(B) = \{u_1, \ldots, u_r, v_1, \ldots, v_r\} \) be such that \( u_1 \) and \( v_1 \) are the only non-cut vertices of \( \overline{B} \) belonging to \( B \). Then \( \{u_1, v_1\} \in E(B) \). Without loss of generality we can assume \( N_B(u_1) = \{u_2, \ldots, u_r\} \) and \( N_B(v_1) = \{v_2, \ldots, v_r\} \). Suppose \( \{u_i, u_j\} \not\in E(B) \) for some \( 2 \leq i, j \leq r \). Now consider \( T = N_B(v_1) \cup N_B(u_i) \cup N_B(u_j) \).

Now \( T \) contains \(|T| - 1 \) cut vertices of \( \overline{B} \) and \( B \setminus T \) contains at least three connected components which include \( \{u_i\}, \{u_j\}, \{v_1\} \). Then it is easy to see that \( T \in ℂ(\overline{B}) \) and \( c_{\overline{B}}(T) \geq |T| - 1 + 3 = |T| + 2 \), a contradiction to the fact that \( J_{\overline{B}} \) is unmixed. Therefore \( N_B(u_1) \setminus \{v_1\} \) is complete and similarly, \( N_B(v_1) \setminus \{u_1\} \) is also complete. Since, \( B \) is \( r \)-regular, the only possibility is \( B = K_r \ast K_r \) (doing some relabeling). Hence \( \overline{B} = G_r = K_r \ast K_r \). □

**Theorem 4.6.** Let \( G = \overline{B} \) be a connected graph such that \( B \) is a non-complete \( r \)-regular \( r \)-connected. Then the following are equivalent:

(i) \( G = K_r \ast K_r \) with \( r \geq 2 \);
(ii) \( J_G \) is Cohen-Macaulay;
(iii) \( R/J_G \) is \( S_2 \);
(iv) \( G \) is accessible;
(v) \( J_G \) is strongly unmixed.
Proof. (i) ⇒ (v) follows from Theorem 4.3. We know (v) ⇒ (ii) ⇒ (iii) ⇒ (iv).

Also, by Proposition 4.5, (iv) ⇒ (i). Therefore we have (i) ⇒ (v) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (i).

□

Figure 3. Planar graph $K_3 \times K_3$ with $J_{K_3 \times K_3}$ Cohen-Macaulay.

Example 4.7. In the Figure 3, the graph $K_3 \times K_3$ is a planar accessible graph such that $K_3 \times K_3$ is 3-regular 3-connected. Note that for $r > 3$, $K_r \times K_r$ is non-planar graph. By Theorem 4.6, $J_{K_3 \times K_3}$ is strongly unmixed (so is Cohen-Macaulay) and $K_3 \times K_3$ is the largest $r$-regular $r$-connected non-complete planar graph with $K_3 \times K_3$ is accessible.

5. COHEN-MACaulay BINOMIAL EDGE IDEALS OF GRAPHS CONTAINING UP TO THREE CUT Vertices

In this section, we introduce a new family of graphs inductively. Then for these graphs having at most three cut vertices, we show that the equvalency of accessibility, Cohen-Macaulayness, and strongly unmixedness of binomial edge ideals holds.

Lemma 5.1. Let $B$ be a block of a graph such that $B$ is accessible with three cut vertex $\{v_1, v_2, v_3\}$ and $\{v_2, v_3\} \in E(B)$ but $\{v_1, v_3\} \not\in E(B)$. If there exists no path between $v_1$ and $v_3$ in $B \setminus \{v_2\}$ and $B \setminus \{v_2, y_1\}$, for some $x_1 \in N_B(v_1) \setminus \{v_2\}$ and $y_1 \in N_B(v_3) \setminus \{v_2\}$, then $J_{B \setminus \{v_1\}}$ is unmixed for some $i \in \{1, 2, 3\}$.

Proof. $B$ is accessible implies $J_B$ is unmixed and so $c(\{v_2\}) = 2$. Therefore, in $B \setminus \{v_2\}$, there is a path between $v_1$ and $v_3$. By the given condition, $v_1, x_1, \ldots, y_1, v_3$ is a path in $B \setminus \{v_2\}$. Suppose there exists no cut vertex $v_i$ of $B$ such that $J_{B \setminus \{v_i\}}$ is unmixed. Then by [5, Proposition 6.1], for each cut vertex $v_i \in V(B)$, where $1 \leq i \leq 3$, there exists a cut set $S_i$ of
Let $\mathcal{N}_B(v_i) = \{v_2, x_1, \ldots, x_s\}$ and assume $s \geq 2$. It is clear that 
\[ \{v_2, x_1\} \in \mathcal{C}(B) \] and also, we have $\mathcal{N}_B(v_1) \in \mathcal{C}(B)$. Then from Proposition 2.4 we can conclude \[ \{v_2, x_j\} \in \mathcal{C}(B) \] and \[ \{v_2, x_j, x_i\} \in \mathcal{C}(B) \] for some $i, j \in \{1, \ldots, s\}$. Let $j \neq 1$. Then there is a connected component $A_j$ in $B \setminus \{v_2, x_j\}$ other than the component containing \[ \{v_1, v_3\} \] and the component \(f_{v_2}\). Also, $A_j$ is adjacent to only $x_j$ and $v_2$ in $B$. Thus, Theorem 2.3 implies that every vertex in $A_j$ is adjacent to the cut vertex $v_2$. Choose $y \in V(A_j)$. In $(\mathcal{B} \setminus \{v_2\}) \setminus (S_2 \setminus \{y\})$, $y$ cannot be a cut vertex and this contradicts the fact $\mathcal{N}_B(v_2) \subseteq S_2 \in \mathcal{C}(B \setminus \{v_2\})$. Therefore $j = 1$ is the only possibility and without loss of generality taking $i = 2$ we get 
\[ \{v_3, x_1, x_2\} \in \mathcal{C}(B). \] As $J_B$ is unmixed, we have $c_B(\{v_2, x_1, x_2\}) = 4$. Let $A_1, A_2, A_3, \{f_{v_2}\}$ be the four connected components of $B \setminus \{v_2, x_1, x_2\}$ such that $v_1 \in V(A_1)$, $v_3 \in V(A_3)$. Now $A_2$ contains no cut vertex of $B$ and vertices belong to $A_2$ can be adjacent to only $x_1, x_2$ and $v_2$ in $B$ outside $A_2$. Let $w \in V(A_2)$ be any vertex. Then $w$ should be adjacent to a cut vertex in $B$ as $B$ is accessible and the only choice is $v_2$. Thus, $V(A_2) \subseteq \mathcal{N}_B(v_2)$. Again we have $\mathcal{N}_B(v_2) \subseteq S_2 \in \mathcal{C}(B \setminus \{v_2\})$. Therefore, in $(\mathcal{B} \setminus \{v_2\}) \setminus (S_2 \setminus \{w\})$, $w$ should be a cut vertex and so we should have \(\{x_1, w\}, \{x_2, w\} \in E(B)\) and \(\{x_1, x_2\} \notin E(B)\) with $x_1, x_2 \notin \mathcal{N}_B(v_2)$. These conditions hold for any vertex in $A_2$.

**Case-I:** Let $v_1, x_1, z, \ldots, y_1, v_3$ be a chordless path between $v_1$ and $v_3$ in $G \setminus \{v_2\}$, where $z, x_1, y_1$ are distinct. Then $w \neq z$ otherwise, $v_1, x_2, w = z, \ldots, y_1, v_3$ would be a path. Since the path is chordless, \{z, v_2\} $\in E(B)$ follows from Theorem 2.3. Now $\mathcal{N}_B(v_2) \in \mathcal{C}(B)$ implies there exists $w' \in \mathcal{N}_B(v_2)$ such that \{v_1, v_3, w'\} $\in \mathcal{C}(B)$ by Proposition 2.4. Suppose $w' = z$ and \{f_{v_1}\}, \{f_{v_3}\}, $A_{v_2}, A_z$ are four connected components of $B \setminus \{v_1, v_3, z\}$, where $v_2 \in A_{v_2}$. Then any vertex in $A_z$ is adjacent to one of $v_1$ or $v_3$. As there is no path between $v_1$ and $v_3$ in $B \setminus \{v_2, x_1\}$ and $B \setminus \{v_2, y_1\}$, the only possibility is $A_z \subseteq \{x_1, y_1\}$. But \{x_1, w\}, \{w, v_2\} $\in E(B)$ implies $x_1 \notin A_z$ and similarly, $y_1 \notin A_z$. Therefore $w' \neq z$. Let \{f_{v_1}\}, \{f_{v_3}\}, $A_{w'}, A_{w'}$ are four connected components of $B \setminus \{v_1, v_3, w'\}$, where $v_2 \in A_{w'}$. By Theorem 2.3 every vertex in $A_{w'}$ is adjacent to $v_1$ or $v_3$. Note that $x_1, y_1 \in A_{w'}$. If there exists $x', y' \in V(A_{w'})$ such that \{x', v_1\} $\in E(B)$ and \{y', v_3\} $\in E(B)$, then $v_1, x', \ldots, y', v_3$ or $v_1, x' = y', v_3$ will be a path and it contradicts our given hypothesis. So $A_{w'}$ is adjacent to exactly one of $v_1$ or $v_3$, without loss of generality, say $v_1$. Take $u \in V(A_{w'})$. Now $\mathcal{N}_B(v_1) \subseteq S_1 \in \mathcal{C}(B \setminus \{v_1\})$ but in $(\mathcal{B} \setminus \{v_1\}) \setminus (S_1 \setminus \{u\})$, $u$ cannot be a cut vertex and this gives a contradiction.
**Case-II:** Let $v_1, x_1, y_1, v_3$ be the only chordless path between $v_1$ and $v_3$ in $G \setminus \{v_2\}$. Then there exists $\{v_2, x_1, x_2\} \in \mathcal{C}(\overline{B})$ and $\{v_2, y_1, y_2\} \in \mathcal{C}(\overline{B})$, where $x_1, x_2 \in N_B(v_1), y_1, y_2 \in N_B(v_3)$ such that $y_1, y_2$ are similar as $x_1, x_2$ as mentioned in the beginning. In this set up, it is easy to observe that, $S = \{v_2, x_1, x_2, y_1, y_2\} \in \mathcal{C}(\overline{B})$. But $S \setminus \{s\} \not\in \mathcal{C}(\overline{B})$ for any $s \in \{x_1, x_2, y_1, y_2\}$ which contradicts the accessibility of $\overline{B}$.

**Case-III:** Let $v_1, x_1, v_3$ be the path between $v_1$ and $v_3$ in $G \setminus \{v_2\}$. Now $N_B(v_2) \in \mathcal{C}(\overline{B})$ implies there is $t \in N_B(v_2)$ such that $\{v_1, v_3, t\} \in \mathcal{C}(\overline{B})$ by Proposition 2.4. Note that $\{v_2, x_1\} \in \mathcal{C}(\overline{B})$. Let $H_1, H_3, \{f_{v_2}\}$ be three connected components of $B \setminus \{v_2, x_1\}$ such that $v_1 \in H_1$ and $v_3 \in H_3$. We have $t \in H_1$ or $t \in H_3$, say $t \in H_1$. Suppose $\{f_{v_1}\}, \{f_{v_3}\}, A_{v_1}, A_{v_3}$ are the four connected components of $B \setminus \{v_1, v_3, t\}$ such that $v_2 \in A_{v_2}$. Observe that $x_1 \in A_{v_1}$. Therefore, any $u \in A_t$ is adjacent to $v_1$, otherwise there will be a path between $v_1$ and $v_3$ in $\overline{B} \setminus \{v_2, x_1\}$. As $u$ can be adjacent to only $v_1$ and $t$ other than the vertices of $V(A_t)$, in $\overline{(B \setminus \{v_1\}) \setminus \{S_1 \setminus \{u\}\}}$, $u$ cannot be a cut vertex. Thus, $N_B(v_1) \subseteq S_1 \subseteq \mathcal{C}(\overline{B})$ gives a contradiction.

So our assumption $s \geq 2$ was wrong and we will examine the case $s = 1$. Now $N_B(v_1) = \{v_2, x_1\}$ and similarly, we have $N_B(v_3) = \{v_2, y_1\}$. Then by Lemma 6.2, we get $\{v_2, x_1\} \not\in E(B)$ and $\{v_2, y_1\} \not\in E(B)$.

**Case-A:** Let $x_1 \neq y_1$. Note that $\{v_1, v_3\} \in \mathcal{C}(\overline{B})$. So, $c(\{v_1, v_3\}) = 3$ as $J_{\overline{B}}$ is unmixed. Therefore, $x_1, y_1, v_2$ are connected in $\overline{B} \setminus \{v_1, v_3\}$. Also, note that any vertex in $V(B) \setminus \{x_1, y_1\}$ is adjacent to $v_2$ by Theorem 2.3. So $N_B(v_2) \setminus \{v_1, v_3\} \neq \emptyset$. Since $N_B(v_2) \subseteq S_2 \subseteq \mathcal{C}(\overline{B} \setminus \{v_2\})$, for any $w \in N_B(v_2) \setminus \{v_1, v_3\}$, $w$ should be adjacent to both $x_1, y_1$ and also, $\{x_1, y_1\} \not\in E(B)$. We have $N_B(v_2) \in \mathcal{C}(\overline{B})$ and $B \setminus N_B(v_2)$ has five connected components, namely, $\{f_{v_1}\}, \{f_{v_3}\}, \overline{B} \setminus \{f_{v_2}, v_2\}, \{x_1\}, \{y_1\}$. Thus, by unmixed property of $J_{\overline{B}}$ we have $N_B(v_2) = \{v_1, v_3, w_1, w_2\}$. But, we do not have $\{v_1, v_3, w_1\}$ or $\{v_1, v_3, w_2\}$ as a cut set of $\overline{B}$ which gives a contradiction by Proposition 2.4 to the fact that $\overline{B}$ is accessible.

**Case-B:** Let $x_1 = y_1$. Again $\{v_1, v_3\} \in \mathcal{C}(\overline{B})$ and So, $c(\{v_1, v_3\}) = 3$. Then there exists $w \in N_B(v_2) \setminus \{v_1, v_3\}$ such that $\{w, x_1\} \in E(B)$. Now it is easy to observe that $\{v_2, x_1\} \in \mathcal{C}(\overline{B})$ but, $c(\{v_2, x_1\}) > 3$ which is a contradiction.

Hence, our initial assumption was wrong and we can conclude that $J_{\overline{B} \setminus \{v_i\}}$ is unmixed for some $i \in \{1, 2, 3\}$. \hfill $\square$

**Lemma 5.2.** Let $B$ be a block of a graph such that $\overline{B}$ is accessible with three cut vertex $\{v_1, v_2, v_3\}$ and $\{v_1, v_2\}, \{v_2, v_3\} \in E(B)$ but $\{v_1, v_3\} \not\in E(B)$. If there exists a path between $v_1$ and $v_3$ in $B \setminus \{v_2\}$ for each $x \in N_B(v_1)$, then $J_{\overline{B} \setminus \{v_i\}}$ is unmixed for $i \in \{1, 2, 3\}$. 
Proof. Suppose $J_{\overline{B}(v_i)}$ is not unmixed for all $1 \leq i \leq 3$. Then by \cite[Proposition 6.1]{5}, for each cut vertex $v_i \in V(B)$, $1 \leq i \leq 3$, there is a cut set $S_i \in \mathcal{C}(\overline{B} \setminus \{v_i\})$ such that $\mathcal{N}_B(v_i) \subseteq S_i$ and so, by \cite[Remark 5.4]{5}, $\mathcal{N}_B(v_i) \in \mathcal{C}(\overline{B})$ for all $1 \leq i \leq 3$. As $\overline{B}$ is accessible by Proposition 2.4, $\mathcal{N}_B(v_1) \in \mathcal{C}(\overline{B})$ implies there exists $x_1 \in \mathcal{N}_B(v_1)$ such that $\{v_2, x_1\} \in \mathcal{C}(\overline{B})$. Since $J_{\overline{B}}$ is unmixed, $c(\{v_2, x_1\}) = 3$. After removing $v_2$ and $x_1$, there will be a path between $v_1$ and $v_3$. Let $\{f_{v_3}\}, A^{x_1}, A_{v_1v_3}$ be the three components of $\overline{B} \setminus \{v_2, x_1\}$ such that $v_1, v_3 \in A_{v_1v_3}$. By Theorem 2.3, any vertex of $A^{x_1}$ is adjacent to $v_2$. Also, any vertex of $A^{x_1}$ can be adjacent to only $x_1$ and $v_2$ outside $A^{x_1}$. Choose a vertex $w \in A^{x_1}$. Then $w$ cannot be a cut vertex in $(\overline{B} \setminus \{v_2\}) \setminus (S_2 \setminus \{w\})$ and this contradicts the fact that $S_2 \in \mathcal{C}(\overline{B} \setminus \{v_2\})$. Hence, our assumption was wrong and $J_{\overline{B}(v_1)}$ is unmixed for $i \in \{1, 2, 3\}$. \qed

**Proposition 5.3.** Let $B$ be a block such that $\overline{B}$ is accessible with three cut vertices. Then there exists a cut vertex $v$ of $\overline{B}$ for which $J_{\overline{B}(v)}$ is unmixed.

**Proof.** If the induced subgraph on the three cut vertex of $\overline{B}$ is complete then by \cite[Proposition 6.6]{5}, there is a cut vertex $v \in V(B)$ of $\overline{B}$ such that $J_{\overline{B}(v)}$ is unmixed. Let $v_1, v_2, v_3$ be the cut vertices belonging to $V(B)$ and assume $B[\{v_1, v_2, v_3\}]$ is not complete. By Theorem 2.3, $B[\{v_1, v_2, v_3\}]$ is connected and so without loss of generality let $E(B[\{v_1, v_2, v_3\}]) = \{\{v_1, v_2\}, \{v_2, v_3\}\}$. Note that $\{v_2\} \in \mathcal{C}(\overline{B})$ and thus, in $\overline{B} \setminus \{v_2\}$ there will be a path between $v_1$ and $v_3$ as $J_{\overline{B}}$ is unmixed. Suppose there exists vertices $x \in \mathcal{N}_B(v_1) \setminus \{v_2\}$ and $y \in \mathcal{N}_B(v_3) \setminus \{v_2\}$ such that there is no path between $v_1$ and $v_3$ in $\overline{B} \setminus \{v_2, x\}$ and also, in $\overline{B} \setminus \{v_2, y\}$. Then by Lemma 5.1, $J_{\overline{B}(v_i)}$ is unmixed for some $i \in \{1, 2, 3\}$. Now assume for each vertex $x \in \mathcal{N}_B(v_1)$ there is a path between $v_1$ and $v_3$ in $\overline{B} \setminus \{v_1, x\}$. Then by Lemma 5.2, $J_{\overline{B}(v_i)}$ is unmixed for some $i \in \{1, 2, 3\}$. Similarly, if there is a path between $v_1$ and $v_3$ in $\overline{B} \setminus \{v_3, y\}$ for each vertex $y \in \mathcal{N}_B(v_3)$, then again by Lemma 5.2, $J_{\overline{B}(v_i)}$ is unmixed for some $i \in \{1, 2, 3\}$. \qed

**Definition 5.4.** A connected graph $G$ is said to be $r$-cut-connected if $G$ has no cut vertex or for any cut vertex $v$ of $G$, the number of cut vertices in any connected component of $G \setminus \{v\}$ is less than or equal to $r$.

For a disconnected graph $G$, if every connected components of $G$ is $r$-cut-connected, then we call $G$ is $r$-cut-connected.

**Definition 5.5.** A graph $G$ is called strongly $r$-cut-connected if $G$ is $r$-cut-connected and for any cut vertex $v$ of $G$, $G \setminus \{v\}$ is strongly $r$-cut-connected.
Lemma 5.6. If \( G \) is strongly \( r \)-cut-connected, then for any cut vertex \( v \) of \( G \), \( G_v \) and \( G_v \setminus \{v\} \) are strongly \( r \)-cut-connected.

Proof. It is clear that \( G \) is \( r \)-cut-connected implies \( G_v \) and \( G_v \setminus \{v\} \) are also \( r \)-cut-connected. We use induction on the number of vertices of \( G_v \). If \( G_v \) has no cut vertex, then we are done and note that for the base case \( G_v \) has no cut vertex. Let \( u \) be a cut vertex of \( G_v \). Then \( u \) is also a cut vertex of \( G \). Now \( G \setminus \{u\} \) is strongly \( r \)-cut-connected and so is \( (G \setminus \{u\})_v \) by induction hypothesis. Note that \( (G \setminus \{u\})_v = G_v \setminus \{u\} \) and hence \( G_v \) is strongly \( r \)-cut-connected. In a similar way, \( G_v \setminus \{v\} \) is also strongly \( r \)-cut-connected. \( \square \)

Theorem 5.7. Let \( G \) be a strongly 3-cut-connected graph having at most three cut vertices in any connected component. Then the following properties of \( G \) are equivalent:

(i) \( J_G \) is Cohen-Macaulay;
(ii) \( R/J_G \) is \( S_2 \);
(iii) \( G \) is accessible;
(iv) \( J_G \) is strongly unmixed.

Proof. (iv) \( \Rightarrow \) (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) is known.

(iii) \( \Rightarrow \) (iv): Let \( \mathcal{G} \) be the class of strongly 3-cut-connected accessible graphs having at most three cut vertices in any connected component. Let \( G \in \mathcal{G} \). If for every block \( B \) of \( G \), the induced subgraph on the cut vertices of \( G \) belong to \( V(B) \) is complete, then by [5, Proposition 6.6], there exists a cut vertex \( v \) of \( G \) for which \( G \setminus \{v\} \) is unmixed. Now assume the induced subgraph on the cut vertices of \( G \) belong to a block is not complete. Since \( G \) is accessible, by Theorem 2.3, the only possibility is there is a block of \( G \) containing all three cut vertices and the induced subgraph on the cut vertices is a \( P_2 \). In this case, using Proposition 5.3, we get a cut vertex \( v \) such that \( G \setminus \{v\} \) is unmixed. Therefore, by [5, Corollary 5.16], \( G \setminus \{v\}, G_v, G_v \setminus \{v\} \) are accessible. By definition \( G \setminus \{v\} \) is strongly 3-cut-connected and by Lemma 5.6, \( G_v, G_v \setminus \{v\} \) are strongly 3-cut-connected. Since \( G \) is 3-cut-connected,
number of cut vertices in any connected components of \( G \setminus \{v\}, G_v \) and \( G_v \setminus \{v\} \) are less than or equal to three. Thus, \( G \setminus \{v\}, G_v, G_v \setminus \{v\} \in \mathcal{G} \) and hence by [5, Proposition 5.13], \( J_G \) is strongly unmixed. □

**Example 5.8.** Consider the graph \( G \) in Figure 4. By computing the primary decomposition of \( J_G \) using Singular, we get

\[
\mathcal{C}(G) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 5\}, \{3, 7\},
\{1, 2, 3\}, \{1, 2, 5\}, \{1, 3, 7\}, \{2, 3, 5\}, \{2, 3, 7\},
\{1, 2, 3, 5\}, \{1, 2, 3, 7\}, \{1, 3, 4, 7\}\}.
\]

Note that \( G \) is accessible. Now \( G \setminus \{1\} \) contains two cut vertices 2 and 3. The sets of cut vertices in \( G \setminus \{2\} \) and \( G \setminus \{3\} \) are \( \{1, 3, 5\} \) and \( \{1, 2, 7\} \), respectively. Again, the sets of cut vertices of \( G \setminus \{1, 2\}, G \setminus \{1, 3\}, G \setminus \{2, 3\}, G \setminus \{2, 5\}, G \setminus \{3, 7\} \) are \( \{3, 5\}, \{2, 7\}, \{1, 5, 7\}, \{1, 3\}, \{1, 2\} \), respectively. Also, \( G \setminus \{1, 2, 3\}, G \setminus \{1, 2, 5\}, G \setminus \{1, 3, 7\}, G \setminus \{2, 3, 5\}, G \setminus \{2, 3, 7\} \) contains \( \{5, 7\}, \{3\}, \{2, 4\}, \{1\}, \{1\} \), respectively as a set of cut vertices. If we go further, then there will be no cut vertices left. From this observation, it is clear that \( G \) is strongly 3-cut-connected graph with three cut vertices. Hence by Theorem 5.7, \( J_G \) is strongly unmixed and so is Cohen-Macaulay.

**Theorem 5.9.** Let \( G \) be a graph such that every block \( B \) of \( G \) satisfies any of the following conditions:

(a) \( B \) is chordal; (b) \( \overline{B} \) is traceable; (c) \( B \) is a chain of cycles (see [17, Definition 4.2]); (d) \( B = K_m \ast_r K_n \); (e) \( \overline{B} \) is strongly 3-cut-connected containing at most 3 cut vertices of \( G \). Then the following are equivalent:

(i) \( J_G \) is Cohen-Macaulay;
(ii) \( R/J_G \) is \( S_2 \);
(iii) \( G \) is accessible;
(iv) \( J_G \) is unmixed and each \( \overline{B} \) is accessible.
(v) \( J_G \) is strongly unmixed.

**Proof.** (v) ⇒ (i) ⇒ (ii) ⇒ (iii) is clear.

(iii) ⇔ (iv): Follows from Theorem 5.9 and Theorem 3.15.

(iv) ⇒ (v): If \( B \) is chordal, then \( J_{\overline{B}} \) is strongly unmixed by [5, Theorem 6.4] and if \( \overline{B} \) is traceable, then by [5, Theorem 6.8], \( J_{\overline{B}} \) is strongly unmixed. For a block \( B \) which is a block of chains, strongly unmixed property of \( J_{\overline{B}} \) follows from [17, Theorem 4.17]. By Lemma 4.2, Theorem 4.3 and the structure of \( K_m \ast_r K_n \), it follows that \( J_{K_m \ast_r K_n} \) is strongly unmixed if and only if \( K_m \ast_r K_n \) is accessible. If \( B \) belongs to the category (e), then \( J_{\overline{B}} \) is strongly unmixed by Theorem 5.7. Now \( J_G \) is unmixed and we see for each block \( B \) of \( G \), \( J_{\overline{B}} \) is strongly unmixed. Hence by Theorem 3.17, \( J_G \) is strongly unmixed. □
Example 5.10. Consider the graph $G$ in Figure 5. We see $G$ has 5 blocks $B_1, \ldots, B_5$ and for each $B_i$ we consider the graph $\overline{B_i}$ with respect to $G$. Using Singular ([7]), we check that each $\overline{B_i}$ is accessible. Therefore, by [5, Theorem 6.8], $J_{\overline{B_1}}$ is strongly unmixed and by [5, Theorem 6.4], $J_{\overline{B_2}}$ is strongly unmixed. We proved in Example $J_{\overline{B_3}}$ is strongly unmixed. Also, strongly unmixed property of $J_{\overline{B_4}}$ follows from [17, Theorem 4.17] and $J_{\overline{B_5}}$ is strongly unmixed by Theorem 4.6. Using repeating application of Corollary 3.2 we see that $J_G$ is unmixed. Thus, by Theorem 3.17, $J_G$ is strongly unmixed. (or by Theorem 5.9.)

As a consequence of our Theorem 3.9, 3.10, 3.15 and 3.17, we are proposing the following Question 5.11.

Question 5.11. For a connected graph $G$, is it true that $J_G$ is Cohen-Macaulay if and only if $J_G$ is unmixed and $J_B$ is Cohen-Macaulay for each block $B$ of $G$?

We answer the Question 3.18 arise in [5], for the case of accessibility and strongly unmixedness in our Theorem 3.19. So we are repeating the Question 3.18 with the unproven part as follows.

Question 5.12. Let $G$ and $H$ be two disjoint connected graphs such that $J_G$ and $J_H$ are unmixed. Let $v, w$ be the cut vertices of $G, H$, respectively, for which $J_{G \setminus \{v\}}$ and $J_{H \setminus \{w\}}$ are unmixed. Set $G \setminus \{v\} = G_1 \cup G_2, H \setminus \{w\} = H_1 \cup H_2$. Let $F_{ij}$ be the graph obtained by gluing $G[V(G_i) \cup \{v\}]$ and $H[V(H_j) \cup \{w\}]$ identifying $v$ and $w$, where $i, j = 1, 2$. If $J_G$ and $J_H$ are Cohen-Macaulay, is it true that $J_{F_{ij}}$ is Cohen-Macaulay?
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