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ON THE STABILITY OF MAPPINGS AND AN ANSWER TO A PROBLEM OF TH. M. RASSIAS

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Theorem 1 Let \((G, +)\) be an abelian group, \(k\) an integer, \(k \geq 2\). \((X, \|\|)\) a Banach space, \(\varphi : G \times G \to [0, \infty)\) a mapping such that
\[
\phi_k(x, y) = \sum_{n=0}^{\infty} \frac{1}{k^{n+1}} \varphi(k^n x, k^n y) < \infty, \quad \forall x, y \in G
\]
and \(f : G \to X\) a mapping with the property
\[
\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y), \quad \forall x, y \in G.
\]
Then there exists a unique additive mapping \(T : G \to X\) such that
\[
\|f(x) - T(x)\| \leq \sum_{m=1}^{k-1} \phi_k(x, mx), \quad \forall x \in G.
\]
Proof. Setting \( y = x \) and \( y = 2x \) in relation (2) we obtain

\[
\|f(2x) - 2f(x)\| \leq \varphi(x, x), \quad \forall x \in G
\]

and respectively,

\[
\|f(3x) - f(x) - f(2x)\| \leq \varphi(x, 2x), \quad \forall x \in G.
\]

Using the triangle inequality and the last two relations, it follows:

\[
\|f(3x) - 3f(x)\| \leq \|f(3x) - f(x) - f(2x)\| + \|f(2x) - 2f(x)\| \leq \varphi(x, 2x) + \varphi(x, x) = \sum_{m=1}^{2} \varphi(x, mx)
\]

hence

\[
\|f(3x) - 3f(x)\| \leq \sum_{m=1}^{2} \varphi(x, mx).
\]

We will prove by mathematical induction after \( k \) the following inequality:

\[
\|f(kx) - kf(x)\| \leq \sum_{m=1}^{k-1} \varphi(x, mx).
\]

Indeed, for \( k = 2 \) and \( k = 3 \) we have the relation (4) and, respectively (5). Suppose (6) true for \( k \) and let us prove it for \( k + 1 \). We replace \( y \) by \( kx \) in (2) and we obtain:

\[
\|f((k + 1)x) - f(x) - f(kx)\| \leq \varphi(x, kx).
\]

Hence, it follows

\[
\|f((k + 1)x) - (k + 1)f(x)\| \leq \|f((k + 1)x) - f(x) - f(kx)\| + \|f(kx) - kf(x)\| \leq \varphi(x, kx) + \sum_{m=1}^{k-1} \varphi(x, mx) = \sum_{m=1}^{k} \varphi(x, mx),
\]

using (6) for the last inequality.

So, relation (6) is true for any \( k \geq 2 \), integer.

Dividing (6) by \( k \) we obtain:

\[
\left\| \frac{f(kx)}{k} - f(x) \right\| \leq \sum_{m=1}^{k-1} \frac{\varphi(x, mx)}{k}.
\]

We claim that

\[
\left\| \frac{f(k^n x)}{k^n} - f(x) \right\| \leq \sum_{m=1}^{k-1} \sum_{p=0}^{n-1} \frac{1}{k^{p+1}} \varphi(k^p x, mk^p x), \quad \forall x \in G.
\]
We see that for \( n = 1 \) we have (7). We suppose (8) true for \( n \) and we will prove it for \( n + 1 \). We replace \( x \) by \( kx \) in (8) and we have

\[
\left\| \frac{f(k^n \cdot kx)}{k^n} - f(kx) \right\| \leq \sum_{m=1}^{k-1} \sum_{p=0}^{n-1} \frac{1}{k^{p+1}} \varphi(k^p \cdot kx, mk^p \cdot kx), \quad \forall x \in G.
\]

Dividing this relation by \( k \), it follows:

\[
\left\| \frac{f(k^{n+1}x)}{k^{n+1}} - \frac{f(kx)}{k} \right\| \leq \sum_{m=1}^{k-1} \sum_{p=0}^{n-1} \frac{1}{k^{p+1}} \varphi(k^{p+1}x, mk^{p+1}x)
\]

and further,

\[
\left\| \frac{f(k^{n+1}x)}{k^{n+1}} - f(x) \right\| \leq \left\| \frac{f(k^{n+1}x)}{k^{n+1}} - \frac{f(kx)}{k} \right\| + \left\| \frac{f(kx)}{k} - f(x) \right\| \leq
\]

\[
\sum_{m=1}^{k-1} \sum_{p=0}^{n-1} \frac{1}{k^{p+1}} \varphi(k^{p+1}x, mk^{p+1}x) + \sum_{m=1}^{k-1} \frac{1}{k} \varphi(x, mx) =
\]

\[
= \sum_{m=1}^{k-1} \left[ \sum_{p=1}^{n} \frac{1}{k^{p+1}} \varphi(k^{p}x, mk^{p}x) + \frac{1}{k} \varphi(x, mx) \right] =
\]

\[
= \sum_{m=1}^{k-1} \sum_{p=0}^{n} \frac{1}{k^{p+1}} \varphi(k^{p}x, mk^{p}x), \quad \forall x \in G,
\]

so (8) is true for each \( n \in \mathbb{N}^* \), by mathematical induction.

Then, for \( 0 < n_1 < n \), we have

\[
\left\| \frac{f(k^n x)}{k^n} - \frac{f(k^{n_1} x)}{k^{n_1}} \right\| = \frac{1}{k^{n_1}} \left\| \frac{f(k^{n-n_1}(k^{n_1} x))}{k^{n-n_1}} - f(k^{n_1} x) \right\| \leq
\]

\[
\leq \frac{1}{k^{n_1}} \sum_{m=1}^{k-1} \sum_{p=0}^{n-n_1-1} \frac{1}{k^{p+1}} \varphi(k^{p+n_1}x, mk^{p+n_1}x) =
\]

\[
= \sum_{m=1}^{k-1} \sum_{p=0}^{n-n_1} \frac{1}{k^{p+1}} \varphi(k^{p}x, mk^{p}x) \to 0 \text{ as } n_1 \to \infty.
\]

Therefore, the sequence \( \left\{ \frac{f(k^n x)}{k^n} \right\}_{n \in \mathbb{N}^*} \) is a fundamental sequence. Because \( X \) is a \textbf{BANACH} space it follows that there exists \( \lim_{n \to \infty} \frac{f(k^n x)}{k^n} \), \( \forall x \in G \), denoted by \( T(x) \), so \( T : G \to X \) and we claim that \( T \) is an additive mapping.
From (2) we have
\[ \|f(k^n x + k^n y) - f(k^n x) - f(k^n y)\| \leq \varphi(k^n x, k^n y), \quad \forall x, y \in G. \]

Hence,
\[ \left\| \frac{f(k^n(x+y))}{k^n} - \frac{f(k^n x)}{k^n} - \frac{f(k^n y)}{k^n} \right\| \leq \frac{1}{k^n} \varphi(k^n x, k^n y), \quad \forall x, y \in G. \]

Taking the limit as \( n \to \infty \) we obtain:
\[ \|T(x+y) - T(x) - T(y)\| \leq \lim_{n \to \infty} \frac{1}{k^n} \varphi(k^n x, k^n y) = 0 \]

using the relation (1). This implies \( T(x+y) = T(x) + T(y), \quad \forall x, y \in G. \)

To prove that (3) holds, we take the limit as \( n \to \infty \) in (8) and we obtain
\[ \|T(x) - f(x)\| \leq \lim_{n \to \infty} \sum_{m=1}^{k-1} \sum_{p=0}^{n-1} \frac{1}{k^{p+1}} \varphi(k^p x, mk^p x) = \]
\[ = \sum_{m=1}^{k-1} \sum_{p=0}^{\infty} \frac{1}{k^{p+1}} \varphi(k^p x, mk^p x) = \sum_{m=1}^{k-1} \phi_k(x, mx), \quad \forall x \in G. \]

Supposing now that there exists another additive mapping \( T_1 : G \to \mathbb{X} \) with the property (3). Then
\[ \|T_1(x) - T(x)\| = \left\| \frac{T_1(k^n x)}{k^n} - \frac{T(k^n x)}{k^n} \right\| \leq \]
\[ \leq \frac{1}{k^n} (\|T_1(k^n x) - f(k^n x)\| + \|f(k^n x) - T(k^n x)\|) \leq \]
\[ \leq \frac{2}{k^n} \sum_{m=1}^{k-1} \phi_k(k^n x, mk^n x) = \frac{2}{k^n} \sum_{m=1}^{k-1} \sum_{p=0}^{\infty} \frac{1}{k^{p+1}} \varphi(k^{p+n} x, mk^{p+n} x) = \]
\[ = \sum_{m=1}^{k-1} \sum_{p=0}^{\infty} \frac{1}{k^{p+1}} \varphi(k^p x, mk^p x). \]

Thus, \( \lim_{n \to \infty} \|T_1(x) - T(x)\| = 0 \), for any \( x \in G \), which implies \( T_1(x) = T(x), \quad \forall x \in G. \)

\[ \text{Q.E.D.} \]

**Remarks:**
1. For \( k = 2 \) we obtain for the first relation:
\[ \phi_2(x, y) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \varphi(2^n x, 2^n y) < \infty, \quad \forall x, y \in G \]
and for the third relation
\[ \|f(x) - T(x)\| \leq \phi_2(x, x), \quad \forall x \in G \]
which is the main theorem from [1].

2. If we take \( \varphi(x, y) = \theta(||x||^p + ||y||^p) \) with \( \theta \geq 0 \) and \( p \in [0, 1) \) we have
\[
\phi_k(x, y) = \sum_{n=0}^{\infty} \frac{1}{k^{n+1}} \theta(k^n(||x||^p + ||y||^p)) = \\
= \frac{\theta}{k} (||x||^p + ||y||^p) \sum_{n=0}^{\infty} k^{n(p-1)} = \\
= \frac{\theta}{k} (||x||^p + ||y||^p) \cdot \frac{k}{k-k^p}.
\]
Then \( \phi_k(x, mx) = \frac{k\theta}{k-k^p} \cdot \frac{1}{k} \cdot ||x||^p(1 + m^p) \) and
\[
\sum_{m=1}^{k-1} \phi_k(x, mx) = \frac{k\theta}{k-k^p} ||x||^p \frac{1}{k} \sum_{m=1}^{k-1} (1 + m^p) = \\
= \frac{k\theta}{k-k^p} ||x||^p \frac{1}{k} (k + \sum_{m=2}^{k-1} m^p) = \frac{k\theta}{k-k^p} ||x||^p s(k, p).
\]
where \( s(k, p) = 1 + \frac{1}{k} \sum_{m=2}^{k-1} m^p \) which implies the theorem of Th.M.Rassias proved in [3].

We prove that the best possible value of \( k \) is 2. Set
\[ R(p) = \frac{k}{2 - 2^p} \quad \text{and} \quad Q(k, p) = \frac{k \cdot s(k, p)}{k - k^p}, k > 2. \]
We prove that
\[ R(p) < Q(k, p) \quad \text{for all} \quad k \geq 3. \quad (9) \]
The verification of (9) follows by mathematical induction on \( k \).

The case \( k = 3 \) is true, because
\[
Q(3, p) - R(p) = \frac{2 \cdot 3^p - 2^p - 4^p}{(2 - 2^p)(3 - 3^p)} > 0,
\]
where we use the Jensen inequality for the concave function
\[
f : (0, \infty) \to \mathbb{R}, \quad f(x) = x^p, p \in [0, 1) : \\
(\frac{x_1 + x_2}{2})^p > \frac{x_1^p + x_2^p}{2} \quad \text{for} \quad x_1, x_2 \in (0, \infty) \quad (10)
\]
Assume now that (9) is true and we prove that
\[ Q(k+1, p) > R(p). \] (11)

We have from (9)
\[
Q(k+1) - R(p) > R(p) \left( \frac{k - k^p}{k + 1 - (k + 1)^p} + \frac{k^p + 1}{k + 1 - (k + 1)^p} - \frac{2}{2 - 2^p} \right) = \frac{2(k + 1)^p - 2^p - k^p \cdot 2^p}{(2 - 2^p)(k + 1 - (k + 1)^p)} > 0,
\]
where we use the inequality (10) with \( x_1 = 2k, x_2 = 2 \).

Thus, (9) is proved.
This last result gives an answer to a problem that was posed by TH.M. RASSIAS in 1991.

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