Approximation properties by some modified Szász-Mirakjan-Kantorovich operators

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Abstract. The present article deals with the local approximation results by means of Lipschitz maximal function, Ditzian-Totik modulus of smoothness and Lipschitz type space having two parameters for the summation-integral type operators defined by Mishra and Yadav (Tbilisi Mathematical Journal. 11(3), (2018), 175-91). Further, we determine the rate of convergence in the term of the the with derivative of bounded variation and for the quantitative means of the defined operators, we establish the quantitative Voronovskaya type and Grüss type theorems. Moreover the examples are given with graphical representation to support the main results.

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1. Introduction

To study the approximations properties on unbounded interval, Szász \cite{39} and Mirakjan \cite{31} introduced the operators known as Szász-Mirakjan operators. In 1954, Butzer \cite{6} generalized into integral modification of the Szász-Mirakjan operators known as Szász-Mirakjan-Kantorovich operators. Totik \cite{40} studied the approximations properties of the Szász-Mirakjan-Kantorovich operators. Some modifications regarding Kantorovich variant can be seen in various papers such as \cite{11,16,43}. Using Brenke-type polynomials, Taşdelen et al. \cite{41} presented Kantorovich variant operators introduced by Verma et al. \cite{42}. The approximation problems are discussed in many research articles for Kantorovich type operators, such as \cite{5,13,32}.

Here, the rate of convergence will be discussed by means of the function with derivative of bounded variation. First of all, in 1979, Bojanic \cite{7} estimated the rate of convergence for Fourier series while in 1983, this property has been discussed for linear positive operators by Cheng \cite{10}. Guo and Khan \cite{17} obtained the rate of convergence for some operators using function of bounded variation. Also, Guo \cite{19} established the rate of convergence for the Durrmeyer operator independently. Later on the significant contributions have been seen in \cite{8,9}. After two years, an important discussion was occurred regarding convergence rates of approximation for functions of bounded variation and for functions with derivatives of bounded variation in \cite{38} by Shaw et al. In this direction, many researchers, authors played the significant role to establish the approximations results regarding rate of convergence by means of function of bounded variation and now a days this type of research is being done with much better quality. We refer some important contributions for the reader \cite{4,25,26,28,36,37}.

Also, one of the discussing area of research is quantitative means of Voronovskaya type theorem. In 2006, Gonska et al. \cite{24}, established the quantitatively Voronovskaya type theorem for any linear positive operators on any compact interval using Taylor’s formula for the \textsuperscript{n}th continuously differentiable function and obtained an estimate in terms of the least concave majorant of the modulus of continuity. In 1935, an equality was developed by Grüss, known as Grüss inequality on his name, this inequality shows a relation
between the integral of product and product of integrals of two functions. An application of this inequality has been seen in approximation theory in 2011 by Acu et al. [1] and using the Grüss inequality for the Bernstein polynomials, Gal and Gonska [25] proved the Grüss Voronovskaya type theorem. Gonska and Tachev [22] obtained a new approach of Bernstein’s operators on applying Grüss type inequalities using the least concave majorant of the first order modulus of continuity. Now, this has been become broad area of research. Recently, Acar [2] obtained the quantitative Voronokskaya and Grüss Voronokskaya type theorems for the Szász operators in quantum calculus. We refer some papers which have significant contributions in this regard as [3, 14, 15, 18, 34].

Motivated by the above works, we study the approximation properties for the operators defined by Mishra and Yadav [33]. They introduced some modified Szász-Mirakjan-Kantorovich operators. Direct results and weighted approximation properties have been discussed as well as they determined the rate of convergence and the comparison took place with the Szász-Mirakjan-Kantorovich operators by graphical analysis. The modified operators are as given below:

\[
R_{m,a}(f; x) = m \sum_{k=0}^{\infty} s_m^a(x) \int_{R} f(t) \, dt,
\]

where \( s_m^a(x) = a^{\frac{-x}{1+a^{-1}}} \frac{x^k (\log a)^k}{(-1+a^{-1})^k k!} \), \( m \in \mathbb{N}, x \in X \) and \( a > 1 \) (fixed).

In this regard, we shall further investigate other properties of above operators (1.1) for approximations point of view. The main aim of this article is to investigate the approximation properties like as rate of convergence in the term of function with derivative of bounded variations, local approximations properties including order of approximation in terms of Lipschitz Maximal function, Ditzian-Totik modulus of smoothness, Peetre’s \( K \)-functional and in a new type of Lipschitz-space having two parameters. Next section consists a quantitative approximation and additionally quantitative Voronovskaya type, Grüss Voronovskaya type theorems are established. Also, we study the graphical analysis of proposed operators in last section.

Here, we point out some basic lemmas, which are used to prove our main theorem. Let us define the function \( e_i = x^i \), where \( i = 0, 1, 2, 3 \), then we have the following lemma.

**Lemma 1.1.** [33] For every \( x \in [0, \infty) \) and \( a > 1 \) fixed, it holds

1. \( \hat{R}_{m,a}(e_0; x) = 1 \),
2. \( \hat{R}_{m,a}(e_1; x) = \frac{1}{2m} + \frac{x \log a}{(-1 + a^{-1}) m} \),
3. \( \hat{R}_{m,a}(e_2; x) = \frac{1}{3m^2} + \frac{2x \log a}{(-1 + a^{-1}) m^2} + \frac{x^2 (\log a)^2}{(-1 + a^{-1})^2 m^2} \),
4. \( \hat{R}_{m,a}(e_3; x) = \frac{1}{4m^3} + \frac{7x \log a}{(-1 + a^{-1}) m^3} + \frac{9x^2 (\log a)^2}{(-1 + a^{-1})^2 m^3} + \frac{x^3 (\log a)^3}{(-1 + a^{-1})^3 m^3} \).

Consider, \( \Lambda_m^n(x) = \hat{R}_{m,a}(\xi_m^n(t); x) \) are known as central moments, where \( \xi_m^n(t) = (t-x)^m, m = 1, 2, 3, 4 \) then by Lemma 1.1, following results are obtained.

**Lemma 1.2.** [33] For every \( x \geq 0 \), we have

1. \( \Lambda_1(x) = -\frac{(-1 + 2mx)}{2m} + \frac{\log a}{m(-1 + a^{-1})} \),
2. \( \Lambda_2^2(x) = \frac{(1 - 3mx + 3m^2 x^2)}{3m^2} + \frac{-2(-1 + a^{-1})(-1 + mx)?x \log a}{(-1 + a^{-1})^2 m^2} + \frac{x^2 (\log a)^2}{(-1 + a^{-1})^2 m^2} \).
3. \( \Lambda_i^3(x) = -\frac{(-1 + 4mx - 6m^2x^2 + 4m^3x^3)}{4m^3} + \frac{x(7 - 12mx + 6m^2x^2) \log a}{2 (-1 + a^\frac{1}{m})^2 m^3} \)

\[ -\frac{3x^2(-3 + 2mx)(\log a)^2 + 4x^3(\log a)^3}{2 (-1 + a^\frac{1}{m})^2 m^3}, \]

4. \( \Lambda_i^4(x) = \frac{1}{5 (-1 + a^\frac{1}{m})^4 m^4} \left( (-1 + a^\frac{1}{m})^4 (1 - 5mx + 10m^2x^2 - 10m^3x^3 + 5m^4x^4) \right) \)

\[ -10 (-1 + a^\frac{1}{m})^3 x(-3 + 7mx - 6m^2x^2 + 2m^3x^3) \log a \]

\[ +15 (-1 + a^\frac{1}{m})^2 x^2(5 - 6mx + 2m^2x^2)(\log a)^2 \]

\[ -20 (-1 + a^\frac{1}{m})^3 x^3(-2 + mx)(\log a)^3 + 5x^4(\log a)^4). \]

**Lemma 1.3.** For all \( x \geq 0 \), then there exist a positive \( C \) for which, we have following inequalities:

\[ \Lambda_i^1(x) \leq \frac{1}{2m}, \]

\[ \Lambda_i^2(x) \leq \frac{C}{m}x(x+1). \]

**Proof.** For \( m \in \mathbb{N} \), we have

\[ \Lambda_i^1(x) = \frac{1}{2m} - x + \frac{x \log a}{(-1 + a^\frac{1}{m}) m} \]

\[ \leq \frac{1}{2m} - x + x = \frac{1}{2m} \]

\[ \Lambda_i^2(x) = \frac{1 - 3mx + 3m^2x^2}{3m^2} - \frac{2(mx - 1)x \log a}{(-1 + a^\frac{1}{m}) m^2} + x^2 \left( \frac{\log a}{(-1 + a^\frac{1}{m}) m} \right)^2 \]

\[ = \frac{1}{3m^2} \left( -\frac{x}{m} + \left( \frac{\log a}{(-1 + a^\frac{1}{m}) m} - 1 \right)^2 x^2 + \frac{2x}{m} \left( \frac{\log a}{(-1 + a^\frac{1}{m}) m} \right) \right) \]

\[ \leq \frac{1}{3m^2} \left( -\frac{x}{m} + \frac{x^2}{m} + \frac{2x}{m} \right) \]

\[ \leq \frac{1}{3m^2} + \frac{x(x+1)}{m} \leq \frac{C}{m}x(x+1). \]

**Lemma 1.4.** For each \( x \geq 0 \), one can obtain

1. \( \lim_{m \to \infty} m\Lambda_i^2(x) = x \)
2. \( \lim_{m \to \infty} m^2\Lambda_i^3(x) = -\frac{1}{2}x(3x \log a - 5) \)
3. \( \lim_{m \to \infty} m^3\Lambda_i^4(x) = 15x^3. \)

**Proof.** Using the Lemma 1.2, we can write as:

\[ \lim_{m \to \infty} m\Lambda_i^2(x) = \lim_{m \to \infty} \left( \frac{1 - 3mx + 3m^2x^2}{3m} - \frac{2(-1 + a^\frac{1}{m})(-1 + mx)x \log a}{(-1 + a^\frac{1}{m})^2 m} + \frac{x^2(\log a)^2}{(-1 + a^\frac{1}{m})^2 m} \right) \]

\[ = \lim_{m \to \infty} \left( \frac{(1 - 3mx + 3m^2x^2)(-1 + a^\frac{1}{m})^2}{3} - 6(-1 + a^\frac{1}{m})(-1 + mx)x \log a + 3x^2(\log a)^2 \right) \]

\[ \frac{1}{3} (-1 + a^\frac{1}{m})^2 m \]
\[ I = \lim_{l \to 0} \left( \frac{(a^l - 1)^2 (l^2 - 3lx + 3x^2) - 6x (a^l - 1) \log a (lx - l^2) + 3l^2x^2(\log a)^2}{3l(1 + a^l)^2} \right), \quad \left( \frac{0}{0} \right) \text{ form}. \]

Using three times L'Hospital rule for the above limit, we obtain
\[ I = \lim_{l \to 0} \frac{P}{Q}. \]

where
\[ P = 2a^l \log(a) \left( (\log a)^2 (l^2 (4a^l - 1) + 12x^2 (a^l - 1) - 3lx (4a^l - 7)) + 3 \log(a) (-6xa^l + l (4a^l - 2) + 9x) \right) + 6 (a^l - 1) + 3lx(\log a)^2(l - x) \]
\[ Q = 6a^l(\log a)^2 (6a^l + l (4a^l - 1) \log(a) - 3), \]
and then
\[ \lim_{l \to 0} P = 18x(\log a)^2 \]
\[ \lim_{l \to 0} Q = 18(\log a)^2, \]
and hence \( I = x. \) Similarly, we can prove other parts.

**Theorem 1.1.** If \( g \in C_B[0, \infty) \) then for all \( m \in \mathbb{N}, \) it holds
\[ \lim_{m \to \infty} \hat{R}_{m,a}(g; x) = g(x), \]
uniformly on every compact subset of \([0, \infty).\)

**Remark 1.1.** If \( g \) be a continuous and bounded function on \([0, \infty)\) with supremum norm as \( \|g\| = \sup_{x \geq 0}|g(x)| \) then
\[ |\hat{R}_{m,a}(g; x)| \leq \|g\|. \]

**Remark 1.2.** One may write the above operators into integral representation as
\[ \hat{R}_{m,a}(g; x) = \int_{0}^{\infty} \mathcal{R}(x,t)g(t) \, dt, \]
where \( \mathcal{R}(x,u) = m \sum_{k=0}^{\infty} s_{m}^{a}(x)\chi_{m,k}(x,t), \) where \( \chi_{m,k}(x,t) \) is the characteristic function of the interval \( \left[ \frac{k}{m}, \frac{k+1}{m} \right] \) with respect to \([0, \infty).\)

## 2. Local results

This section deals with the local approximation properties for the defined operators. Here, we determine the rate of convergence by means of some spaces known as Lipschitz Maximal space defined by Lenze [29] in 1988, with order \( \alpha \in (0, 1] \) and it can be defined as follows:
\[ \eta_{\alpha}(f; x) = \sup_{t,x \geq 0} \frac{|f(t) - f(x)|}{|t - x|^\alpha}, \quad t \neq x. \]

Here, an upper bound can be obtained for the defined operators (1.1) with the function in the terms of Lipschitz Maximal function.
THEOREM 2.1. For \( g \in C_B[0, \infty) \) and for every \( \geq 0 \), we obtain
\[
|\hat{R}_{m,a}(g; x) - g(x)| \leq \eta_s(g; x)\sqrt{\Lambda_2^p(x)},
\]
where \( \Lambda_2^p(x) = \frac{(1-3m+3m^2)x^2}{3m^4} - \frac{2(1+\alpha\frac{1}{m})(1-3m+3m^2)x\log a}{(1+\alpha\frac{1}{m})m^2} + \frac{x^2(\log a)^2}{(1+\alpha\frac{1}{m})m^2}. \)

PROOF. Using the definition of Lipschitz Maximal space and applying the defined operators (1.1), we get
\[
|\hat{R}_{m,a}(g; x) - g(x)| \leq \eta_s(g; x)\hat{R}_{m,a}(\vert t - x \vert^{\alpha}; x).
\]
Using Lemma 1.2 and applying H"older inequality with \( p = \frac{2}{\alpha} \) and \( p = \frac{2}{1-\alpha} \), we get
\[
|\hat{R}_{m,a}(g; x) - g(x)| \leq \eta_s(g; x)\left(\hat{R}_{m,a}(\vert t - x \vert^{2}; x)\right)^\frac{1}{2} = \eta_s(g; x)\sqrt{\Lambda_2^p(x)}.
\]

Hence, the proof is completed. \( \square \)

Now, we find the order of approximation for the defined operators (1.1) in terms of Ditzian-Totik modulus of smoothness. So, consider the function \( g \in C_B[0, \infty) \) for which, the Ditzian-Totik modulus of smoothness is defined by
\[
\omega_\psi(g; \epsilon) = \sup_{h \in (0, \epsilon)} \left\{ \left| g\left(x + \frac{h\psi(x)}{2}\right) - g\left(x - \frac{h\psi(x)}{2}\right) \right| : x \pm \frac{h\psi(x)}{2} \in (0, \infty) \right\},
\]
where \( \psi(x) = (x(1 + x))^\frac{1}{\alpha} \) and the appropriate \( K \)-functional can be defined by
\[
K_\psi(g; \epsilon) = \inf_{f \in \mathcal{W}_\psi[0, \infty)} \{ \| f - g \| + \epsilon \| \psi f' \| , \epsilon > 0 \},
\]
where \( \mathcal{W}_\psi[0, \infty) = \{ f : f \in AC_{loc}[0, \infty); \| \psi f' \| < \infty \} \), here \( AC_{loc}[0, \infty) \) is the space of absolutely continuous and differentiable function on every compact interval of \([0, \infty)\). A relation is obtained between Ditzian-Totik modulus of smoothness and the appropriated \( K \)-functional from [12], according to that, there exist a positive constant \( M \), such that
\[
M^{-1} \omega_\psi(g; \epsilon) \leq K_\psi(g; \epsilon) \leq M \omega_\psi(g; \epsilon).
\]

THEOREM 2.2. Consider \( g \in C_B[0, \infty) \), \( x \geq 0 \), it holds
\[
|\hat{R}_{m,a}(g; x) - g(x)| \leq 2K_\psi\left( g; \frac{u(x)\sqrt{\Lambda_2^2(x)}}{\psi(x)} \right),
\]
where \( u(x) = \sqrt{x} + \sqrt{1 + x} \) and \( \Lambda_2^2(x) \) can be obtained by the Lemma 1.2.

PROOF. Using Taylors theorem by considering \( f \in \mathcal{W}_\psi[0, \infty) \), we have
\[
f(t) - f(x) = \int_x^t f'(l)dl = \int_x^t \frac{f'(l)f(l)}{\psi(l)}dl
\]

\[
f(t) - f(x) \leq \| f' \| \int_x^t \frac{1}{\psi(l)}dl
\]

\[
= \| f' \| \left[ \int_x^t \frac{1}{\sqrt{l(l+1)}}dl \right] \leq \| f' \| \left[ \int_x^t \left( \frac{1}{\sqrt{l}} + \frac{1}{\sqrt{l+1}} \right)dl \right]
\]

\[
= 2\| f' \| \left[ \sqrt{l} + \sqrt{l+1} \right]_x
\]

\[
= 2\| f' \| \left[ \sqrt{x} - \sqrt{x + 1} \right]_x
\]

\[
\]
Using Lemma 1.1, Remark 1.1 and by the above inequality, we can write as

\[ |\hat{R}_{m,a}(g; x) - g(x)| \leq |\hat{R}_{m,a}(g-f; x)| + |\hat{R}_{m,a}(f; x) - f(x)| + |f(x) - g(x)| \]

\[ \leq \hat{R}_{m,a}(|g-f; x|) + \hat{R}_{m,a}(|f(t) - f(x)|; x) + ||g-f|| \]

\[ \leq 2||g-f|| + 2||f'|| \frac{u(x)}{\psi(x)} \hat{R}_{m,a}((t - x); x) \]

\[ \leq 2||g-f|| + 2||f'|| \frac{u(x)}{\psi(x)} \left( \hat{R}_{m,a}((t - x)^2; x) \right)^{\frac{1}{2}} \]

\[ = 2||g-f|| + 2||f'|| \frac{u(x)}{\psi(x)} \left( \Lambda^2_t(x) \right)^{\frac{1}{2}}, \]

taking infimum on right side over all \( f \in W_x[0, \infty) \), we get

\[ (2.8) \quad |\hat{R}_{m,a}(g; x) - g(x)| \leq 2K_{\psi} \left( g; \frac{u(x)\sqrt{\Lambda^2_t(x)}}{\psi(x)} \right). \]

Thus, the proof is completed. \( \square \)

Özarslan and Aktuğlu [35] defined a new type of Lipschitz-space having two parameters. Let \( u, v > 0 \) be fixed numbers, then Lipschitz-type-space is defined by:

\[ Lip^u,v_M(a) = \left\{ g \in C[0, \infty) : |g(y) - g(x)| \leq M \frac{|y-x|^a}{(y+ux^2 + vx)^\frac{1}{2}} : x, y \in [0, \infty) \right\}, a \in (0,1]. \]

Using the above definition, we have the local approximation result:

**Theorem 2.3.** Let \( g \in Lip^u,v_M(a) \) with \( a \in (0,1] \) then for every \( x \geq 0 \), it holds:

\[ (2.10) \quad |\hat{R}_{m,a}(g; x) - g(x)| \leq M \left( \frac{\Lambda^2_t(x)}{(ux^2 + vx)^\frac{1}{2}} \right)^{\frac{1}{2}}. \]

**Proof.** We prove the above theorem within case for \( a \in (0,1] \). So, far that, consider

**Case 1.** when \( a = 1 \), then

\[ |\hat{R}_{m,a}(g; x) - g(x)| \leq \hat{R}_{m,a}(|g(t) - g(x)|; x) \]

\[ \leq M\hat{R}_{m,a} \left( \frac{|t-x|}{(y+ux^2 + vx)^\frac{1}{2}} \right) \]

\[ \leq \frac{M}{(ux^2 + vx)^\frac{1}{2}} \hat{R}_{m,a}((t - x); x) \]

\[ \leq \frac{M}{(ux^2 + vx)^\frac{1}{2}} \left( \hat{R}_{m,a}((t - x)^2; x) \right)^{\frac{1}{2}} \]

\[ = \frac{M\sqrt{\Lambda^2_t(x)}}{(ux^2 + vx)^\frac{1}{2}}. \]

**Case 2.** when \( a \in (0,1) \) the

\[ |\hat{R}_{m,a}(g; x) - g(x)| \leq \hat{R}_{m,a}(|g(t) - g(x)|; x) \]

\[ \leq M\hat{R}_{m,a} \left( \frac{|t-x|^a}{(y+ux^2 + vx)^\frac{1}{2}} \right) \]

\[ \leq \frac{M}{(ux^2 + vx)^\frac{1}{2}} \hat{R}_{m,a}((t - x)^a; x). \]
Let \( p = \frac{1}{a} \), \( q = \frac{1}{b} \) and applying Hölder inequality, we get
\[
|\hat{R}_{m,a}(g; x) - g(x)| \leq \frac{M}{(ux^2 + vx)^{\frac{a}{2}}} \left(\hat{R}_{m,a}(|t - x|; x)\right)^a \\
\leq \frac{M}{(ux^2 + vx)^{\frac{a}{2}}} \left(\hat{R}_{m,a}(|t - x|^2; x)\right)^{\frac{a}{2}} \\
= M \left(\frac{\Lambda^2(x)}{ux^2 + vx}\right)^{\frac{a}{2}}
\]
hence, the required result is obtained.

3. Rate of convergence in term of the derivative of bounded variation

Now we determine the rate of convergence of the said operators in the space of the function of bounded variation by considering \( DBV[0, \infty) \), the set of all continuous function having derivative of bounded variation on every finite sub-interval of \([0, \infty)\). One can observe that for each \( g \in DBV[0, \infty) \), it can be written
\[
g(x) = \int_0^x h(s) \, ds + g(0),
\]
where \( h \) is a function bounded of variation on each finite sub-interval of \([0, \infty)\). To determine the rate of convergence in the terms of function of bounded variation, for which the function \( g \in DBV[0, \infty) \), we use an auxiliary operators \( g_x \) such that
\[
g_x(t) = \begin{cases} 
  g(t) - g(x), & 0 \leq t < x, \\
  0, & t = x, \\
  g(t) - g(x+), & x < t < \infty.
\end{cases}
\]
Moreover, we denote \( V^b_a \) as total variation of a real valued function \( g \) defined on \([a, b] \subset [0, \infty)\) with the quantity
\[
V^b_a = \sup_S \left( \sum_{j=0}^{n-1} |g(x_{j+1}) - g(x_j)| \right),
\]
where, \( S \) is the set of all partition \( P = \{a = x_0, \cdots, x_n = b\} \) of the interval \([a, b]\).

**Lemma 3.1.** For all \( x \in [0, \infty) \), \( n \in \mathbb{N} \) there exist a positive constant \( M > 0 \), it can be written as

1. \( J(x, y) = \int_0^y \mathcal{R}(x, t) \, dt \leq \frac{C}{m(x-y)^2}(x(x+1)), \ 0 \leq y < x, \)
2. \( 1 - J_n(x, z) = \int_z^\infty \mathcal{R}(x, t) \, dt \leq \frac{C}{m(z-x)^2}(x(x+1)), \ x < z < \infty. \)

**Proof.** Here, \( 0 \leq y < x \) and \( x \geq 0 \) then we have
\[
\int_0^y \mathcal{R}(x, t) \, dt \leq \int_0^y \left( \frac{x-t}{x-y} \right)^2 \mathcal{R}(x, t) \, dt \\
\leq \frac{\Lambda^2(x)y}{(x-y)^2} \\
\leq \frac{C}{(x-y)^2} \frac{x(x+1)}{m}. 
\]
Similarly, other inequality can be proved.

**Theorem 3.1.** Let \( f \in DBV[0, \infty) \) and \( \forall \ x \in [0, \infty) \), it holds
\[
|\hat{R}_{m,a}(f; x) - f(x)| \leq \frac{1}{4m} |f'(x+) + f'(x-)| + \left( \frac{C}{4m} x(x+1) \right)^{\frac{1}{2}} |f'(x+) - f'(x-)| + \frac{C(x+1)}{m} \sum_{k=0}^{\lfloor \sqrt{m} \rfloor} V^e_{x - \frac{k}{m}} f'_x
\]
\[
\frac{x}{\sqrt{m}} f'_x + \frac{x}{\sqrt{m}} V_{+}^{x} \widehat{m} (f'_x) + \frac{C(x+1)}{m} \sum_{k=0}^{\lfloor m \rfloor} V_{+}^{x} \Lambda_k (f'_x).
\]

**Proof.** Using the hypothesis (3.2), we have

\[
f'(s) = f'_x(s) + \frac{1}{2} (f'(x^+) + f'(x^-)) + \frac{1}{2} (f'(x^+) - f'(x^-)) \text{sgn}(s - x)
\]

where \( \sigma_x(s) \)

\[
\sigma_x(s) = \begin{cases} 
1 & s = x \\
0 & s \neq x.
\end{cases}
\]

By equation (1.5), it can be written as

\[
\hat{R}_{m,a}(f;x) - f(x) = \int_0^{\infty} \mathcal{R}(x,s)(f(s) - f(x)) \, ds
\]

\[
= \int_0^{\infty} \mathcal{R}(x,s) \left( \int_x^s f'(t) \, dt \right) \, ds.
\]

Here, it is clear that

\[
\int_x^s \sigma_x(t) \, dt = 0,
\]

therefore

\[
\int_0^{\infty} \mathcal{R}(x,s) \left( \int_x^s \{ f'(t) - \frac{1}{2} (f'(x^+) + f'(x^-)) \} \, dt \right) \, ds = 0.
\]

By (1.5), we yield

\[
\int_0^{\infty} \mathcal{R}(x,s) \left( \int_x^s \frac{1}{2} (f'(x^+) + f'(x^-)) \, dt \right) \, ds = \frac{1}{2} (f'(x^+) + f'(x^-)) \int_0^{\infty} \mathcal{R}(x,s)(s - x) \, ds
\]

\[
= \frac{1}{2} (f'(x^+) + f'(x^-)) \Lambda_k(x).
\]

And

\[
\left| \int_0^{\infty} \mathcal{R}(x,s) \left( \frac{1}{2} \int_x^s (f'(x^+) - f'(x^-)) \text{sgn}(t - x) \, dt \right) \, ds \right| \leq \frac{1}{2} |(f'(x^+) - f'(x^-))| \int_0^{\infty} \mathcal{R}(x,s)(s - x) \, ds
\]

\[
\leq \frac{1}{2} |(f'(x^+) - f'(x^-))| \hat{R}_{m,a}(|s - x|; x)
\]

\[
\leq \frac{1}{2} |(f'(x^+) - f'(x^-))| \left( \Lambda_k^2(x) \right)^{1/2}
\]

Using Lemmas (1.4, 1.3), we have following inequality holds:

\[
|R_{m,a}(f;x) - f(x)| \leq \frac{1}{2} |f'(x^+) + f'(x^-)| \Lambda_k^1(x) + \frac{1}{2} |f'(x^+) - f'(x^-)| \left( \Lambda_k^2(x) \right)^{1/2}
\]
\[
\int_0^\infty \mathcal{R}(x,s) \left( \frac{1}{2} \int_x^s (f'_x(t)) \, dt \right) \, ds
\]

Here,

\[
\int_0^\infty \mathcal{R}(x,s) \left( \int_x^s (f'_x(u)) \, du \right) \, ds = \int_0^x \mathcal{R}(x,s) \left( \int_x^s (f'_x(t)) \, dt \right) \, ds + \int_x^\infty \mathcal{R}(x,s) \left( \int_x^s (f'_x(t)) \, dt \right) \, ds
\]

(3.11)

\[
= L_1 + L_2,
\]

where

\[
L_1 = \int_0^x \left( \int_x^s (f'_x(t)) \, dt \right) \frac{\partial}{\partial s} (J(x,s)) \, ds
\]

\[
= \int_0^x f'_x(s) J(x,s) \, ds
\]

(3.12)

\[
= \int_y^0 f'_x(s) J(x,s) \, ds + \int_0^y f'_x(s) J(x,s) \, ds
\]

Here, we consider \( y = x - \frac{x}{\sqrt{m}} \) then by above equality, one can write

\[
\left| \int_x^{x - \frac{x}{\sqrt{m}}} f'_x(s) J(x,s) \, ds \right| \leq \int_{x - \frac{x}{\sqrt{m}}}^x |f'_x(s)| |J(x,s)| \, ds
\]

\[
\leq \int_{x - \frac{x}{\sqrt{m}}}^x |f'_x(s) - f'_x(x)| \, ds, \quad f'_x(x) = 0 \text{ (where } |J(x,s)| \leq 1)\]

\[
\leq \int_{x - \frac{x}{\sqrt{m}}}^x V_x f'_x \, ds
\]

\[
\leq V_x \int_{x - \frac{x}{\sqrt{m}}}^x f'_x \, ds\]

(3.13)

\[
= \frac{x}{\sqrt{m}} \left( V_x f'_x \right)
\]

Using Lemma 3.1 for solving second term, we get

\[
\int_x^{x - \frac{x}{\sqrt{m}}} |f'_x(s)| J(x,s) \, ds \leq C \frac{x(x + 1)}{m} \int_{x - \frac{x}{\sqrt{m}}}^x \frac{|f'_x(s)|}{(x - s)^2} \, ds
\]

\[
\leq C \frac{x(x + 1)}{m} \int_{x - \frac{x}{\sqrt{m}}}^x V_x f'_x \frac{1}{(x - s)^2} \, ds
\]

\[
= C \frac{x(x + 1)}{x m} \int_{x - \frac{x}{\sqrt{m}}}^x V_x f'_x \, dp
\]
\[ L \cdot \] Using the value of \[ H \]. Hence,

\begin{equation}
|L_1| \leq \frac{C(x+1)}{m} \sum_{k=0}^{\sqrt{m}} \left( V_{x-\frac{m}{x}} f'_x \right).
\end{equation}

To solve \( L_2 \), we reform \( L_2 \) and integrating by parts, we have

\[ |L_2| = \int_x^z \left( \int_x^s f'_x(t)dt \right) \frac{\partial}{\partial s}(1-J(x,s))ds + \int_x^z \left( \int_x^s f'_x(t)dt \right) \frac{\partial}{\partial s}(1-J(x,s))ds \]

\[ \leq \int_x^z \left( \int_x^s f'_x(t)dt \right) \frac{\partial}{\partial s}(1-J(x,s))ds \]

\[ = \int_x^z f'_x(s)(1-J(x,s))ds \]

\[ \leq \int_x^z V_s f'_x ds + \frac{C(x+1)}{m} \int_x^z V_s f'_x \frac{1}{(s-x)^2} ds \]

On substituting \( s = x \left( 1 + \frac{1}{\eta} \right) \), we obtain

\[ |L_2| \leq \frac{x}{\sqrt{m}} V_x^x \left( f'_x \right) + \frac{C(x+1)}{m} \int_0^{\sqrt{m}} V_x^{x+\frac{m}{x}} f'_x d\eta \]

\[ \leq \frac{x}{\sqrt{m}} V_x^x \left( f'_x \right) + \frac{C(x+1)}{m} \sum_{k=0}^{\sqrt{m}} \int_k^{V_x^{x+\frac{m}{x}}} f'_x d\eta \]

\[ = \frac{x}{\sqrt{m}} V_x^x \left( f'_x \right) + \frac{C(x+1)}{m} \sum_{k=0}^{\sqrt{m}} V_x^{x+\frac{m}{x}} \left( f'_x \right). \]

Using the value of \( L_1, L_2 \) in (3.11) and with the help of (3.10), we obtain required result.
4. Quantitative Approximation

In 2007, Ispir [26] proposed the weighted modulus of continuity $\Delta(g; \xi)$ for any $\xi > 0$, in the weighted space $C_w^k[0, \infty)$ to estimate the degree of approximation, which is as follows:

\[
\Delta(g; \xi) = \sup_{0 \leq h \leq \xi, \; 0 \leq x \leq \infty} \frac{|g(x + h) - g(x)|}{(1 + h^2)(1 + x^2)}, \quad g \in C_w^k[0, \infty).
\]

where the weighted space is defined as $C_w^k[0, \infty) = \{g \in C_w[0, \infty), \lim_{x \to \infty} \frac{|g(x)|}{w(x)} < +\infty\}$, $C_w[0, \infty) = \{g \in B_w[0, \infty), g \text{ is continuous}\}$, $B_w[0, \infty) = \{g : [0, \infty) \to \mathbb{R} | (g(x)(Mw(x))\}$, here $M$ (depending on the function) is a positive constant and $w(x) = 1 + x^2$ is a weight function.

**Remark 4.1.** For $g \in C_w^k[0, \infty)$

\[
\lim_{\xi \to 0} \Delta(g; \xi) = 0.
\]

On can obtains as, $\Delta(f; \lambda \xi) \leq 2(1 + \xi^2)(1 + \lambda)\Delta(f; \xi)$. $\lambda > 0$.

Using the weighted modulus of continuity and defined inequality, one can show that

\[
|g(t) - g(x)| \leq (1 + x^2)(1 + (t - x)^2)\Delta(g; |t - x|)
\]

\[
\Delta(g; \xi) \leq 2 \left(1 + \frac{|t - x|}{\xi}\right)(1 + \xi^2)(1 + (t - x)^2)(1 + x^2)\Delta(f; |t - x|).
\]

As the consequence of the weighted modulus of continuity, we determine the degree of approximation of the operators $U_{k}^{[\xi]}(g; x)$ in the weighted space $C_w^k[0, \infty)$.

4.1. Quantitative Voronovskaya type theorem.

**Theorem 4.1.** Let $g', g'' \in C_w^k[0, \infty)$ and for sufficiently large value of $m \in \mathbb{N}$, then for each $x \geq 0$, it holds

\[
m \left| \hat{R}_{m,a}(g;x) - g(x) - g'(x)\Lambda_1(x) - \frac{g''(x)}{2!}\Lambda_2(x) \right| = O(1)\Delta \left( g; \sqrt{\frac{1}{m}} \right).
\]

**Proof.** By Taylor’s expansion, it can be written as:

\[
g(t) - g(x) = g'(x)\xi(t) + \frac{g''(x)}{2!}\xi^2(t) + \zeta(t, x),
\]

where $\zeta(t, x) = \frac{g'''(\theta - s) - g''(s)}{2!}(\theta - x)^2$ and $\zeta \in (t, x)$. Applying operators (1.1) on both sides to the above expansion, then one can obtains

\[
m \left| \hat{R}_{m,a}(g;x) - g(x) - g'(x)\Lambda_1(x) - \frac{g''(x)}{2!}\Lambda_2(x) \right| \leq m\hat{R}_{m,a}(|\zeta(t, x); x|).
\]

Now using the property of weighted modulus of continuity, we get

\[
\frac{g''(\theta) - g''(s)}{2!} \leq \left(1 + \frac{|s - x|}{\xi}\right)(1 + \xi^2)(1 + (t - x)^2)(1 + x^2)\Delta(f''; \xi)
\]

and also

\[
\left| \frac{g''(\theta) - g''(s)}{2!} \right| \leq \begin{cases} 
2(1 + \xi^2)^2(1 + x^2)\Delta(g''; \xi), & |s - x| < \xi, \\
2(1 + \xi^2)^2(1 + x^2)\frac{(s - x)^4}{\xi^4} \Delta(g''; \xi), & |s - x| \geq \xi.
\end{cases}
\]

Now for $\xi \in (0, 1)$, we get

\[
\left| \frac{g''(\theta) - g''(s)}{2} \right| \leq 8(1 + x^2) \left(1 + \frac{(s - x)^4}{\xi^4}\right)\Delta(g''; \xi).
\]
Hence,

\[ (|\zeta(s, x)|; x) \leq 8(1 + x^2) \left( (s - x)^2 + \frac{(s - x)^6}{\xi^4} \right) \Delta(g'', \xi). \]

Thus, applying the Lemma 1.4, one can obtain

\[
\hat{R}_{m,a}(\zeta(t, x); x) \leq 8(1 + x^2)\Delta(g'', \xi) \left( \hat{R}_{m,a}((s - x)^2; x) + \frac{\hat{R}_{m,a}((s - x)^6; x)}{\xi^4} \right)
\leq 8(1 + x^2)\Delta(g'', \xi) \left( O\left(\frac{1}{m}\right) + \frac{1}{\xi^4} O\left(\frac{1}{m^3}\right) \right), \text{ as } m \to \infty.
\]

Choose, \( \xi = \sqrt{\frac{1}{n}} \), then

\[
\hat{R}_{m,a}(\zeta(t, x); x) \leq 8O\left(\sqrt{\frac{1}{m}}\right) \Delta\left(g'', \sqrt{\frac{1}{m}}\right) (1 + x^2).
\]

Hence, we reach on

\[
m\hat{R}_{m,a}(\zeta(s, x); x) = O(1) \Delta\left(g'', \sqrt{\frac{1}{m}}\right).
\]

By (4.4) and (4.8), we obtain the required result. \( \square \)

Using the above theorem, we obtain a new asymptotic type formula for the defined operators using two functions from the weighted space \( C_w^k[0, \infty) \).

4.2. Grüss Voronovskaya type theorem.

**Theorem 4.2.** Let \( \mu, \nu \in C_w^k[0, \infty) \) then for \( \mu', \mu'', \nu', \nu'' \in C_w^k[0, \infty) \), it holds

\[
\lim_{m \to \infty} m \left( \hat{R}_{m,a}(\mu\nu; x) - \hat{R}_{m,a}(\mu; x)\hat{R}_{m,a}(\nu; x) \right) = x\mu'(x)\nu'(x).
\]

**Proof.** By making suitable arrangement and using well known properties of derivative of multiplication of two functions, we get

\[
m \left( \hat{R}_{m,a}(\mu\nu; x) - \hat{R}_{m,a}(\mu; x)\hat{R}_{m,a}(\nu; x) \right) = m \left\{ \left( \hat{R}_{m,a}(\mu\nu; x) - \mu(x)\nu(x) - (\mu\nu)'\Lambda_1^1(x) - \frac{(\mu\nu)''}{2!}\Lambda_1^2(x) \right) \\
- g(x) \left( \hat{R}_{m,a}(\mu; x) - \mu(x) - \mu'(x)\Lambda_1^1(x) - \frac{\mu''(x)}{2!}\Lambda_1^2(x) \right) \\
- \hat{R}_{m,a}(\mu; x) \left( \hat{R}_{m,a}(\nu; x) - \nu(x) - \nu'(x)\Lambda_1^1(x) - \frac{\nu''(x)}{2!}\Lambda_1^2(x) \right) \\
+ \frac{\nu'(x)}{2!} \hat{R}_{m,a}((t - x)^2; x) \left( \mu - \hat{R}_{m,a}(\mu; x) \right) + \mu'(x)\nu'(x)\Lambda_1^2(x) \\
+ \nu'(x)\Lambda_1^1(x) \left( \mu - \hat{R}_{m,a}(\mu; x) \right) \right\}.
\]

For sufficiently large value of \( m \), i.e. for \( m \to \infty \), with the help of Theorems 1.1 and 4.1, Lemma 1.4, the proof completed after talking the limit on both sides to the above equality. \( \square \)
5. Graphical Representation

In this segment, the graphical approach are shown for the said operators (1.1) regarding convergence to the given function.

**Example 5.1.** Consider the function $f(x) = x^2e^x$ with $x \in [0, 5]$. Here we take the value of $m = 10, 25, 100$ for the given operators $\tilde{R}_{m,a}(f; x)$ and fixed value of $a = 2$. Then we obtain, the better approximation by the said operators as we increase the vale of $m$.

![Figure 1](image1.png)

**Figure 1.** The convergence of the operators $\tilde{R}_{m,a}(f; x)$ to the function $f(x)$(green).

**Example 5.2.** Here, we take the function $f(x) = x \cos(2x + 1)$ and $x \in [0, 5]$, then the approximation of the given function by the said operators take place by graphical representation. Here errors decrease, as the value of $m$ is increased.

![Figure 2](image2.png)

**Figure 2.** The convergence of the operators $\tilde{R}_{m,a}(f; x)$ to the function $f(x)$(green).

5.1. **Concluding Remark.** As we increase the value of $m$, the approximation is good, i.e. for the large value of $m$ the error is minimum.

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