Deconvolution of linear systems with quantized input: an information theoretic viewpoint.

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Abstract

In spite of the huge literature on deconvolution problems, very little is done for hybrid contexts where signals are quantized. In this paper we undertake an information theoretic approach to the deconvolution problem of a simple integrator with quantized binary input and sampled noisy output. We recast it into a decoding problem and we propose and analyze (theoretically and numerically) some low complexity on-line algorithms to achieve deconvolution.

Keywords: Hybrid deconvolution systems, Input estimation, Bit-MAP decoding.

1 Introduction

The deconvolution problem is ubiquitous in many scientific and technological areas such as seismology, astrophysics, image processing and medical applications (see e.g. [2, 3, 4, 10, 18, 19]). Its most general formulation is as follows. We consider a time horizon $T$ (possibly infinite), a convolution kernel $K(t)$ and the input/output system

$$ x(t) = \int_0^t K(t-s)u(s)ds $$

(1)

(we implicitly assume that $K$ and $u$ are s.t. the above integral makes sense). The problem is to estimate the input $u$ from some noisy version $y$ of the output $x$.

This is an instance of inverse problem: to see why the problem is difficult we focus on the special case $K = 1$ which will be the case considered throughout this paper. In this context, (1) can be written as

$$ \dot{x}(t) = u(t), \quad x(0) = 0. $$

(2)

Since the operation of differentiation is not robust with respect to noise perturbation, the reconstruction of $u$ from $y$ cannot be simply done by differentiation. The goal is then to estimate $u$, using the available information on $x$ and any a priori information on $u$. Several procedures can be exploited to accomplish this task and the choice is in general motivated by a suitable trade-off between precision of the solution and complexity of the algorithm.
Classical algorithms due to Tikhonov \cite{21,22} are based on a penalization technique and work off-line: the estimation \( \hat{u} \) at any time depends on the whole signal \( y(t) \) with \( t \in [0,T] \). This is a significant drawback in on-line or interactive data flows application where the delay in estimation is required to remain bounded. Causal algorithms have been studied in \cite{7,8}, where bounds on the error have been obtained for the case of bounded noises and regularity assumptions on the input signals \( u \).

An outstanding problem is how to use possible side information available on the input signal \( u(t) \) on the above algorithms: indeed, while functional, and more generally convex, constraints can be incorporated in the above algorithms, things are quite less clear for more general constraints. In this paper we focus on the case when \( u \) is known to be a piecewise constant signal with values restricted to a fixed known finite discrete alphabet. This turns out to be a significant issue in the context of hybrid systems where continuous-time systems are driven by discrete digital signals. Such constraints are clearly of a non-convex type and is not obvious how to include them in classical deconvolutional algorithms.

In this work we will undertake an information-theoretic approach to causal deconvolution problems with sampled quantized inputs introducing algorithms which reconstruct \( u \) through a decoding procedure. A key feature of these algorithms is that they present very low complexity structure, while they exhibit performance quite close to the information theoretical limit. The main mathematical results consist in a rigorous analysis of the asymptotic performance of the proposed algorithms employing tools from the ergodic theory of Markov Processes.

In Section 2 we will give all the mathematical details regarding the deconvolution problem with quantized input signals. In particular, we will link it to classical decoding problems and we will study the possibility to use classical decoding techniques for our purpose. In Section 3 we will develop a couple of low complexity deconvolution algorithms comparing their performance. Section 4 is the core of our paper: it is devoted to a deep analysis of the proposed algorithms. Using Markov Processes ergodic theorems we will be able to give theoretical results on their behavior in the asymptotic regime (time range going to \( \infty \)).

We conclude now the introduction with notation and terminology to be used throughout the paper.

1.1 Notation

Given a subset \( A \) of a set \( \Omega \), \( \mathbb{1}_A : \Omega \rightarrow \{0,1\} \) is the indicator function, defined by \( \mathbb{1}_A(x) = 1 \) if \( x \in A \) and \( \mathbb{1}_A(x) = 0 \) otherwise. Erfc indicates the complementary error function, defined by \( \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-s} ds \) for any \( x \in \mathbb{R} \). \( B(\Omega) \) indicates the Borel \( \sigma \)-algebra of \( \Omega \).

Capital letters will be used to name random variables (r.v.’s for short), while boldface capital letters will be vectors whose components are r.v.’s. \( P \) will be the probability on discrete r.v.’s, while \( f(\cdot) \) the probability density function of continuous or hybrid (that is, involving both continuous and discrete events) r.v.’s. Instead, \( \mathbf{P} \) will denote the transition probability matrix of a Markov Chain (Section 4.1.1) and \( \mathbf{P}(\cdot,\cdot) \) the transition probability kernel of a Markov Process (Section 4.2.1). Finally, \( E \) will be the mean operator.
2 Statement of the problem

2.1 The deconvolution problem

In the following we stick to the system (1) under the assumptions we make throughout this paragraph.

Assumption 1 The available output signal is a noisy, sampled version of $x(t)$:

$$y_k = x_k + n_k$$

where $x_k = x(\tau k)$, $\tau > 0$ being the constant sampling time, and $n_k$’s are realizations of independent, identically distributed Gaussian variables $N_k$’s of 0 mean and variance $\sigma^2$.

We will denote by $y = (y_1, \ldots, y_K) \in \mathbb{R}^K$ the vector of all available measures ($K = T/\tau$ is assumed to be an integer) and by $y_{ab} = (y_a, y_{a+1}, \ldots, y_b)$ the available measures from time $a$ to time $b$, with $a, b \in \{1, \ldots, K\}$, $a < b$.

A deconvolution algorithm consists in a function $\Gamma : \mathbb{R}^K \to \mathbb{R}^{[0,T]}$.

$\hat{u} = \Gamma(y)$ is the estimated input and in general it will not coincide with the true input $u$. What in general we request is a bound on the error $u - \hat{u}$ and some consistency property: when the variance of the noise and the sampling time go to 0, the error should converge (in some suitable sense) to 0.

We say that a deconvolution algorithm $\Gamma$ is causal (with delay $k_0$ $\tau$, $k_0 \in \mathbb{N}$) if there exists a sequence of functions $\Gamma_k : \mathbb{R}^{k+k_0} \to \mathbb{R}^{[(k-1)\tau,k\tau]}$, where $k = 1, 2, \ldots$, such that

$$\Gamma(y)|_{t \in [(k-1)\tau,k\tau]} = \Gamma_k(y_{1+k_0}).$$

Such an algorithm estimates the unknown signal in the current time interval $[(k-1)\tau,k\tau]$ exploiting the past and present information $y_1, \ldots, y_k$ along with a possible bounded future information $y_{k+1}, \ldots, y_{k+k_0}$.

We now come to the assumptions on the input signals.

Assumption 2 There is a finite alphabet $\mathcal{U} \subset \mathbb{R}$ and we consider signals of type

$$u(t) = \sum_{k=0}^{K-1} u_k \mathbb{1}_{[k\tau,(k+1)\tau]}(t) \quad u_k \in \mathcal{U}. \quad (3)$$

$u(t)$, with $t \in [0,T]$, is then completely determined by the sequence of samples $u_0, u_1, \ldots, u_{K-1}$. For simplicity we assume the sampling time $\tau$ to be the same as in the output and to have an exact synchronization in the sampling instants. The output signals are now identified by samples $x_1, x_2, \ldots, x_K \in \mathcal{X}$, where $\mathcal{X} \subset \mathbb{R}$ is a suitable alphabet (recall that we have fixed $x_0 = 0$). Of course, in principle, one could still use the deconvolution algorithms in [7, 8] or [21, 22], however, there would be no way to use inside the algorithm the a priori information on the quantization of $u$. Instead we now show that, in this case, our deconvolution problem can completely be recasted into a discrete decoding problem. Notice indeed that the input/output system is simply described by

$$\begin{cases}
  x_0 = 0 \\
  x_{k+1} = x_k + \tau u_k, \quad k = 0, \ldots, K - 1.
\end{cases} \quad (4)$$
The vector \( \mathbf{x} = (x_1, \ldots, x_K) \) can thus be seen as a coded version of \( \mathbf{u} = (u_0, \ldots, u_{K-1}) \): we can write \( \mathbf{x} = \mathcal{E}(\mathbf{u}) \) where \( \mathcal{E} \) denotes the encoder given by (4). Afterwards, \( \mathbf{x} \) is transformed as it was transmitted through a classical Additive White Gaussian Noise (AWGN) channel, the received output being given by \( y_k = x_k + n_k \).

It is on the basis of these measures that we have to estimate the ‘information signal’ \( \mathbf{u} \). Notice that the real time \( t \) is completely out of the problem at this point and everything can be considered at the discrete sampling clock time. In the coding theory language, a decoder is exactly a function \( \mathcal{D} : \mathbb{R}^K \to \mathcal{U}^K \) which allows to construct an estimation of the input signal: \( \hat{\mathbf{u}} = \mathcal{D}(\mathbf{y}) \). Even in this context we can talk about causal algorithm if there exists a sequence of functions \( \mathcal{D}_k : \mathbb{R}^{k+k_0} \to \mathcal{U} \) such that

\[
\mathcal{D}(\mathbf{y})_{k-1} = \mathcal{D}_k(\mathbf{y}^{k+k_0}) \quad k = 1, \ldots, K.
\]

Finally,

**Assumption 3** The unknown input is assumed to be generated by a stochastic source with a known distribution, independent from the noise source.

The particular source distribution considered in this work will be introduced in Section 2.4.

According to the notation given in Section 1.1, in the sequel \( U_k \) will identify the input r.v. at time \( k \), \( X_k \) the corresponding system output given by expression (4) \( Y_k = X_k + N_k \) the measured output, \( N_k \) being the Gaussian noise. Furthermore, \( \hat{X}_k = \mathcal{D}(Y)_k \) and \( \bar{X}_k = \tilde{X}_{k-1} + \tau \tilde{U}_{k-1} \) (\( \tilde{X}_0 = 0 \)) will be respectively the estimated input and the estimated state. Finally, \( U = (U_0, \ldots, U_{K-1}) \), \( \hat{U} = (\hat{U}_0, \ldots, \hat{U}_{K-1}) \), \( Y = (Y_1, \ldots, Y_K) \), \( Y^b_a = (Y_a, \ldots, Y_b) \), \( a, b \in \{1, \ldots, K\} \), \( a < b \).

### 2.2 Error Evaluation: The Mean Square Cost

A fundamental issue in the deconvolution problem is the choice of the norm with respect to which errors are evaluated. In this context, we consider the mean square cost:

\[
\mathcal{D}(\mathbf{D}) = \tau \mathbb{E} \left( \| \mathbf{U} - \tilde{\mathbf{U}} \|^2 \right) = \tau \sum_{k=0}^{K-1} \mathbb{E} \left( |U_k - \tilde{U}_k|^2 \right).
\]

We now define \( \mathcal{D}^* \) as the decoder minimizing \( \mathcal{D}(\mathbf{D}) \) among all the possible decoders. It can be constructed as follows: given the density \( f_{\mathbf{Y}}(\mathbf{y}) \) of \( \mathbf{Y} \), notice that

\[
\mathcal{D}(\mathbf{D}) = \tau \sum_{k=0}^{K-1} \int_{\mathbb{R}^K} \mathbb{E}(|U_k - \mathcal{D}(\mathbf{y})_k|^2|\mathbf{Y} = \mathbf{y}) f_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y}.
\]

Hence, for any \( \mathbf{y} \in \mathbb{R}^K \),

\[
\mathcal{D}^*(\mathbf{y})_k = \arg\min_{v \in \mathcal{U}} \mathbb{E}(|U_k - v|^2|\mathbf{Y} = \mathbf{y}) = \arg\min_{v \in \mathcal{U}} \sum_{u \in \mathcal{U}} |u - v|^2 P(U_k = u|\mathbf{Y} = \mathbf{y}).
\]
This turns out to be a finite optimization problem which can be solved by means of a marginalization procedure and a Bayesian inversion:

\[ P(U_k = u | Y = y) = \sum_{u \in U^k : u_k = u} \frac{f(y; x_k)(y | E(u))P(U = u)}{f(y)}. \]

Analogously, we can define \( D^{*k_0} \) as the decoder minimizing \( \bar{d}(D) \) among all the possible causal decoders with delay \( k_0 \):

\[ D^{*k_0}(y)_{k-1} = D^{*k_0}(y^{k+k_0}) = \arg\min_{v \in U} \sum_{u \in U} |u - v|^2 P(U_k = u | Y^{k+k_0} = y^{k+k_0}). \]

### 2.3 The BCJR algorithm

In practice, the decoder \( D^* \) can be implemented with the well-known BCJR algorithm [1]. This algorithm computes the probabilities of states and transitions of a Markov source, given the observed channel outputs; in other words, it provides the so-called APP (a posteriori probabilities) on states and transitions, therefore on coded and information symbols.

Let us briefly remind the BCJR procedure. For \( i, j \in X \), we define the following probability density functions:

\[ \alpha_k(i) = f(x_k, y^k_i)(i, y^k_i) \quad k = 1, \ldots, K \]
\[ \beta_k(i) = f(y^{k+1}_K | x_k)(y^K_i | i) \quad k = 0, \ldots, K - 1 \]
\[ \Gamma_k(i, j) = f(x_k, x_{k-1}, y)(j, y | i) \quad k = 1, \ldots, K. \]

For any \( k = 1, \ldots, K \), the APP on states and on transitions respectively are:

\[ \lambda_k(i) = f(x_k, y)(i, y) \]
\[ \sigma_k(i, j) = f(x_k, x_{k-1}, y)(j, i, y). \]

Given the following initial and final conditions:

\[ \alpha_0(i) = P(X_0 = i) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases} \]
\[ \beta_K(i) = 1 \text{ for any } i \in X \]

for \( k = 1, \ldots, K \) we have

\[ \lambda_k(i) = \alpha_k(i)\beta_k(i) \]
\[ \sigma_k(i, j) = \alpha_{k-1}(i)\Gamma_k(i, j)\beta_k(j) \]

where \( \alpha_k(i) \) and \( \beta_k(i), i \in X \), can be respectively computed with a forward and a backward recursions:

\[ \alpha_k(i) = \sum_{h \in X} \alpha_{k-1}(h)\Gamma_k(h, i) \quad \beta_k(i) = \sum_{h \in X} \Gamma_{k+1}(i, h)\beta_{k+1}(h). \]

The APP are then recursively computed and finally used to decide on the transmitted input sequence.
Analogous causal versions of the BCJR algorithm can be used to implement the decoder (5) with delay \( k_0 \). For \( k = 1, \ldots, K - k_0 \), the APP on the transitions becomes

\[
\tilde{\sigma}_k(i, j) = f_{(X_k, X_{k-1}, Y_{k+k_0}^{k+1})}(j, i, Y_1^{k}) = \alpha_{k-1}(i)\Gamma_k(i, j)\tilde{\beta}_k(j)
\]

where \( \alpha_k \) and \( \Gamma_k \) are defined as above, while \( \tilde{\beta}_k(j) = f_{(X_{k+1}^{k+k_0}, Y_{k+1})}(X_{k+1}^{k+k_0}, Y_{k+1}) \). For \( k > K - k_0 \), we recast into the classical formulation (7). For brevity, we will refer to the causal BCJR as CBCJR.

### 2.4 Further Assumptions

In the sequel of this work, we will make two further assumptions on the input:

**Assumption 4** The input alphabet is binary: \( \mathcal{U} = \{0, 1\} \).

**Assumption 5** For \( k = 0 \ldots K - 1 \), the \( U_k \)'s are independent and uniformly distributed: \( P(U_k = 0) = P(U_k = 1) = \frac{1}{2} \). In particular, the \( U_k \)'s are independent from the Gaussian noises \( N_k \).

Now the probabilistic setting introduced at the end of Section 2.1 is complete and we can resume the system as follows: given \( X_0 = \hat{X}_0 = 0 \), for \( k = 1, \ldots, K \),

\[
\begin{align*}
U_{k-1} &\sim \text{Bernoulli} \left( \frac{1}{2} \right); \\
X_k &\sim X_{k-1} + \tau U_{k-1}; \\
N_k &\sim \mathcal{N}(0, \sigma^2); \\
Y_k &\sim X_k + N_k; \\
\hat{U}_{k-1} &\sim \mathcal{D}(Y)_{k-1}; \\
\hat{X}_k &\sim \hat{X}_{k-1} + \tau \hat{U}_{k-1}.
\end{align*}
\]

Notice that also \( X_k \)'s are independent from \( N_k \)'s.

Under Assumption 4,

\[
\bar{d}(\mathcal{D}) = \tau \sum_{k=0}^{K-1} \mathbb{E} \left( |U_k - \hat{U}_k| \right) = \tau KP_0(e)
\]

where

\[
P_0(e) = \frac{1}{K} \sum_{k=0}^{K-1} P(\hat{U}_k \neq U_k) = \frac{1}{K} \mathbb{E}(|\mathcal{U} - \hat{\mathcal{U}}|)
\]

is the so-called Bit Error Rate (also denoted by BER), a very common performance measure in digital transmissions that expresses the average number of bits in error. In our context, minimizing \( \bar{d}(\mathcal{D}) \) is equivalent to minimizing the BER and, therefore, the optimal decoder \( \mathcal{D}^* \) that performs this minimization coincides with the well-known Bit-MAP (Maximum a posteriori) decoder (see [15, 1]):

\[
\mathcal{D}^*(y)_k = \arg\max_{u \in \{0, 1\}} P(U_k = u | Y = y).
\]
At step $k$:  
| Computations | Storage Locations | Decoding Delay |
|---------------|-------------------|----------------|
| $O(k)$        | $O(k)$            | $K - k$        |
| $\text{BCJR}$| $O(k)$            | $k_0 = 0$      |

Table 1:

Its causal version is given by

$$D^{*k_0}(y)_k = \arg\max_{u \in \{0, 1\}} P(U_k = u|Y_{k+1+k_0} = y_{k+1+k_0}).$$  \hspace{1cm} (13)

We introduce here also the Conditional Bit Error Rate, CBER for short:

$$P_b(\epsilon|U) = \frac{1}{K} \sum_{k=0}^{K-1} P(\hat{U}_k \neq U_k|U) = \frac{1}{K} \mathbb{E}(|U - \hat{U}| |U).$$  \hspace{1cm} (14)

While the BER is a parameter that evaluates the mean performance of the transmission model, the CBER describes its behavior for each possible sent sequence. The CBER is then a relevant parameter for our system, whose decoding performance changes in function of the transmitted input.

For computational simplicity, from now onwards let

$$\tau = 1$$  \hspace{1cm} (15)

so that $X = \{0, \ldots, K\}$ and in particular, if $X_0 = 0$, $X_k \in \{0, \ldots, k\}$. In the BCJR implementation of decoders \cite{12} and \cite{13}, we obtain that $\alpha_k(i)$, $i = 0, 1, \ldots, K$, is null for any $i > k$, while matrices $\Gamma_k$ and $\sigma_k$ are non-null only on diagonal and superdiagonal. By Assumption 5 $P(X_k = j|X_{k-1} = i) = 1/2$ if $j = i, i + 1$ and 0 otherwise. Recalling that the transition between $X_k$ and $Y_k$ is modeled by an AWGN channel, $f(y_k|X_k)(y_k|j) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left( -\frac{(y_k - j)^2}{2\sigma^2} \right)$, we obtain

$$\Gamma_k(i, j) = f(y_k|X_k)(y_k|j)P(X_k = j|X_{k-1} = i)$$

$$= \frac{1}{2\sigma\sqrt{2\pi}} \exp \left( -\frac{(y_k - j)^2}{2\sigma^2} \right) \text{ for } j = i, i + 1.$$  \hspace{1cm} (16)

Given $\Gamma_k$, $\sigma_k$ or its causal version $\tilde{\sigma}_k$ can be recursively computed and the corresponding decoding rules are:

$$\text{BCJR} \quad D^*(y)_{k-1} = \begin{cases} 0 & \text{if } \sum_{i=0}^{k-1} \sigma_k(i, i + 1) \leq \sum_{i=0}^{k-1} \sigma_k(i, i) \\ 1 & \text{otherwise.} \end{cases}$$  \hspace{1cm} (17)

$$\text{CBCJR} \quad D^{*k_0}(y)_{k-1} = \begin{cases} 0 & \text{if } \sum_{i=0}^{k-1} \tilde{\sigma}_k(i, i + 1) \leq \sum_{i=0}^{k-1} \tilde{\sigma}_k(i, i) \\ 1 & \text{otherwise.} \end{cases}$$  \hspace{1cm} (18)

3 Suboptimal Causal Decoding Algorithms

Causality has a price and the CBCJR algorithm has clearly a worse performance than BCJR.
By simulating our system, we quantify the performance gap between BCJR and CBCJR ($k_0 = 0$) as we can appreciate in Figure 1: the two curves represent the corresponding BER’s in function of the Signal-to-Noise Ratio (SNR), here defined as $\tau^2/\sigma^2 = 1/\sigma^2$. These outcomes are the averages over 5000 transmissions, each of which being a 100 bit message. Avoid unacceptable delays and complexity problems in the BCJR and CBCJR implementation). We remark that CBCJR has the best performance among causal deconvolution algorithms.

Moreover, by comparing the efficiency of the two procedures (the results are reported in Table 1), we gather that for both BCJR and CBCJR the required computations and storage locations linearly increase with the number of transmitted bits, which is a drawback in case of long transmission.

This fact motivates the development of new suboptimal causal algorithms that improve the efficiency without substantial loss of reliability. To achieve that, we implement the CBCJR fixing the number of states, that is, at each step we save the $n$ states with largest probability (where $n$ is arbitrarily chosen) and we discard the others.

We now introduce the algorithms in the cases $n = 1$ and $n = 2$, which are of great interest for their low complexity, and we show some simulations’ outcomes.

### 3.1 One State Algorithm

A suboptimal causal decoder $D^{(1)} : \mathbb{R}^K \to \{0, 1\}^K$ can be derived from the CBCJR by assuming the most probable state to be the correct one. At any step $k = 0, 1, \ldots$, $D^{(1)}$ decides on the current bit by a single MAP procedure and upgrades the estimated state, which is the only one value that requires to be stored.

Consider (6), (9) and (18). Given the estimated state $\hat{x}_{k-1}$, the decoding rule of $D^{(1)}$ at time step $k$ is given by (18) with no backward recursion $\tilde{\beta}_k(j)$ and $\alpha_{k-1}(\hat{x}_{k-1}) = 1$, $\alpha_{k-1}(j) = 0$ for any $j \neq \hat{x}_{k-1}$. This reduces the decoding task to the comparison between two distances; in fact, the One State algorithm that implements $D^{(1)}$ is as follows:

1. Initialization: $\hat{x}_0 = 0$;
2. For $k = 1, \ldots, K$, given the received symbol $y_k \in \mathbb{R}$, 
\[ \hat{\alpha}_{k-1} = D^{(1)}(y)_{k-1} = \arg\max_{u \in \{0,1\}} P(U_{k-1} = u | Y_k = y_k, X_{k-1} = \hat{x}_{k-1}) \]
\[ = \begin{cases} 
0 & \text{if } \Gamma_k(\hat{x}_{k-1}, \hat{x}_{k-1}) \geq \Gamma_k(\hat{x}_{k-1}, \hat{x}_{k-1} + 1) \\
1 & \text{otherwise} \end{cases} \] (19)
\[ \hat{x}_k = \hat{x}_{k-1} + \hat{\alpha}_{k-1} \]
and given the equality [16] in the AWGN case,
\[ \Gamma_k(\hat{x}_{k-1}, \hat{x}_{k-1}) \geq \Gamma_k(\hat{x}_{k-1}, \hat{x}_{k-1} + 1) \iff |y_k - \hat{x}_{k-1}| \leq |y_k - (\hat{x}_{k-1} + 1)|. \] (20)

3.2 Two States Algorithm

By fixing $n = 2$, we derive a decoder $D^{(2)} : \mathbb{R}^K \to \{0,1\}^K$ that, at each step, estimates the current input bit and computes and stores the two most likely states along with the corresponding probabilities $\alpha_k(i)$ (defined by [6]). As for the One State Algorithm, the estimation of the input bit is performed by a MAP decoding rule (18) with no backward recursion and summing over the two “surviving” states. In detail, the recursive Two States algorithm that implements $D^{(2)}$ is the following:

1. For $k = 1$, given the unique starting state $\hat{x}_0 = 0$, we estimate the first bit by a One State procedure:
\[ \hat{u}_0 = D^{(2)}(y)_{0} = \arg\max_{u \in \{0,1\}} P(U_0 = u | Y_1 = y_1, X_0 = 0) \]
\[ = \begin{cases} 
0 & \text{if } |y_1| \leq |y_k - 1| \\
1 & \text{otherwise.} \end{cases} \] (21)

Afterwards, the possible states are two: $\hat{x}_1(0) = 0$ and $\hat{x}_1(1) = 1$ and the corresponding probabilities $\alpha_1(0)$ and $\alpha_1(1)$ in our framework are given by
\[ \alpha_1(j) = f(X_1, Y_1)(j, y_1) = f(Y_1 | X_1)(y_1 | j) P(X_1 = j) \]
\[ = f(Y_1 | X_1)(y_1 | j) P(U_0 = j) = \frac{1}{2} f(Y_1 | X_1)(y_1 | j), \quad j \in \{0,1\}. \]
We then normalize these probabilities so that \( \alpha_1(0) + \alpha_1(1) = 1 \) and we just store the couple of values \((\alpha_1(0), \hat{x}_1(0))\), as this is sufficient to retrieve also \((\alpha_1(1), \hat{x}_1(1)) = (1 - \alpha_1(0), \hat{x}_1(0) + 1)\). For notational simplicity we rename the stored vector \((\alpha_1(0), \hat{x}_1(0))\) as \((\alpha_1, \hat{x}_1)\).

2. For \( k = 2, 3, \ldots, K \), given \((\alpha_{k-1}, \hat{x}_{k-1})\) and \( F_k = f_{(X_k|Y_k)}(\hat{x}_k, y_k^k) \)

\[
\hat{u}_{k-1} = D^{(2)}(y)_{k-1} = \arg\max_{u \in \{0, 1\}} P(U_{k-1} = u|Y_k = y_k, X_{k-1} = \hat{x}_{k-1}, F_{k-1} = \alpha_{k-1}) = \begin{cases} 
0 & \text{if } \alpha_{k-1} \Gamma_k(\hat{x}_{k-1}, \hat{x}_{k-1}) + (1 - \alpha_{k-1}) \Gamma_k(\hat{x}_{k-1} + 1, \hat{x}_{k-1} + 1) \geq \alpha_{k-1} \Gamma_k(\hat{x}_{k-1}, \hat{x}_{k-1} + 1) + (1 - \alpha_{k-1}) \Gamma_k(\hat{x}_{k-1} + 1, \hat{x}_{k-1} + 2) \\
1 & \text{otherwise.} 
\end{cases}
\]

From step \( k - 1 \), three possible states arise: \( \hat{x}_{k-1}, \hat{x}_{k-1} + 1 \) and \( \hat{x}_{k-1} + 2 \), whose probabilities are given by the forward recursion in [8]:

\[
\begin{align*}
\alpha_k(\hat{x}_{k-1}) &= \alpha_{k-1} \Gamma_k(\hat{x}_{k-1}, \hat{x}_{k-1}) \\
\alpha_k(\hat{x}_{k-1} + 1) &= \alpha_{k-1} \Gamma_k(\hat{x}_{k-1}, \hat{x}_{k-1} + 1) + (1 - \alpha_{k-1}) \Gamma_k(\hat{x}_{k-1} + 1, \hat{x}_{k-1} + 1) \\
\alpha_k(\hat{x}_{k-1} + 2) &= (1 - \alpha_{k-1}) \Gamma_k(\hat{x}_{k-1} + 1, \hat{x}_{k-1} + 2).
\end{align*}
\]

(22)

which can be reduced as follows in the case [16]:

\[
\begin{align*}
\alpha_k(\hat{x}_{k-1}) &= \alpha_{k-1} \frac{1}{2\sigma\sqrt{2\pi}} \exp \left( -\frac{(y_k - \hat{x}_{k-1})^2}{2\sigma^2} \right) \\
\alpha_k(\hat{x}_{k-1} + 1) &= \alpha_{k-1} \frac{1}{2\sigma\sqrt{2\pi}} \exp \left( -\frac{(y_k - (\hat{x}_{k-1} + 1))^2}{2\sigma^2} \right) \\
\alpha_k(\hat{x}_{k-1} + 2) &= (1 - \alpha_{k-1}) \frac{1}{2\sigma\sqrt{2\pi}} \exp \left( -\frac{(y_k - (\hat{x}_{k-1} + 2))^2}{2\sigma^2} \right).
\end{align*}
\]

Since \( |y_k - (\hat{x}_{k-1} + 1)| \neq \max\{|y_k - (\hat{x}_{k-1} + j)|, j = 0, 1, 2\} \), in the AWGN case \( \alpha_k(\hat{x}_{k-1} + 1) \neq \min\{\alpha_k(\hat{x}_{k-1} + j), j = 0, 1, 2\} \). Hence, the state \( \hat{x}_{k-1} + 1 \) is never discarded and also the two “surviving” states are always adjacent. Therefore,

- we calculate \( \alpha_{\min} = \min\{\alpha_k(\hat{x}_{k-1}), \alpha_k(\hat{x}_{k-1} + 2)\} \).
- If \( \alpha_{\min} = \alpha_k(\hat{x}_{k-1}) \), the surviving states are \((\hat{x}_{k-1} + 1, \hat{x}_{k-1} + 2)\) with probabilities \((\alpha_k(\hat{x}_{k-1} + 1), \alpha_k(\hat{x}_{k-1} + 2))\). We then store the lowest state along with the corresponding normalized probability: \((\alpha_k, \hat{x}_k) = \frac{\alpha_k(\hat{x}_{k-1} + 1)}{\alpha_k(\hat{x}_{k-1} + 1) + \alpha_k(\hat{x}_{k-1} + 2)}, \hat{x}_{k-1} + 1\).
- Similarly, if \( \alpha_{\min} = \alpha_k(\hat{x}_{k-1} + 2) \), \((\alpha_k, \hat{x}_k) = \frac{\alpha_k(\hat{x}_{k-1} + 2)}{\alpha_k(\hat{x}_{k-1} + 1) + \alpha_k(\hat{x}_{k-1} + 2)}, \hat{x}_{k-1} + 1\).
At step $k$:

|                | Computations | Storage Locations | Decoding Delay |
|----------------|--------------|-------------------|----------------|
| BCJR           | $O(k)$       | $O(k)$            | $K - k$        |
| CBCJR          | $O(k)$       | $O(k)$            | 0              |
| ONE STATE      | $O(1)$       | 1                 | 0              |
| TWO STATES     | $O(1)$       | 2                 | 0              |

Table 2:

3.3 Simulations and comparisons

Figure 3: Performance comparison of different causal decoders.

We report now the simulations’ outcomes concerning the decoders $\mathcal{D}^*$, $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$, respectively implemented with CBCJR, One State and Two States algorithms. The simulations have been performed considering 5000 different transmissions, each of which being a 100 bit message. The obtained results are then the averages overall transmissions.

In Figure 3 we compare the efficiency of the three decoding schemes, in terms of BER: we evidence that two states are sufficient to achieve performance very close to the causal optimum: we observe that the gain between $\mathcal{D}^{(2)}$ and $\mathcal{D}^*$ never exceeds 0.15 dB, while it achieves 0.8 dB between $\mathcal{D}^{(1)}$ and $\mathcal{D}^*$ for BER’s values between 0.2 and 0.3. Moreover, as we report in Table 2 the complexity of One State and Two States algorithms is constant when the number is constant and no delay is produced in the decoding: this makes them efficient even for long-time transmissions, i.e., for a large number of states.

4 Suboptimal Causal Decoding Algorithms: Theoretic Analysis

In this section, we propose an exhaustive theoretic analysis of One State and Two States algorithms and we provide a formal setting for the analytical computation of their performance. According to Definitions 11 and 14 in Section 1,
we will compute both the BER and the CBER, which respectively describe the decoding for the “mean input” and for each possible input.

The natural setting of this analysis is the theory of Markov Processes, in countably infinite or not countable spaces (we will talk about Markov Chains when the space is countably infinite).

4.1 Theoretic Analysis of the One State Algorithm

Suppose to transmit $K$ (possibly infinite) bits and to decode by the One State method. The starting point of our analysis is the definition, at any step $k = 1, 2, 3, \ldots$, of the r.v.

$$D_k = \tilde{X}_k - X_k \in \mathbb{Z}$$

Equation (23)

$\tilde{X}_k$ being defined [10]. $D_k$ actually represents the difference between the actual and the estimated state values. Since $D_0 = 0$, the following recursive relationship holds:

$$D_{k+1} = D_k + \hat{U}_k - U_k$$

Equation (24)

where $\hat{U}_{k-1} = \mathcal{D}^{(1)}(y)_{k-1}$ (see the algorithm [19]). While $U_k$’s are independent, $\hat{U}_k$ is function of $U_k$ and $D_k$. Then, the stochastic process $(D_k)_{k \in \mathbb{Z}}$ is a Markov Chain (whose definition is formally given in the next section), which can be exploited to carry on our analysis; in order to do that, let us first review some basic elements of Markov theory.

4.1.1 Markov Chains

The definitions and results introduced in this Section can be retrieved in the Chapter 3 of [20] or in the Chapter 3 of [11].

By Markov Chain we intend any sequence of random variables $(X_n)_{n=0,1,\ldots}$ assuming values in a countable set $X$ and satisfying the Markov property:

$$P(X_{n+1} = y|X_n = x, X_{n-1}, \ldots, X_0) = P(X_{n+1} = y|X_n = x).$$

If the chain is time-homogeneous, that is

$$P(X_{n+1} = y|X_n = x) = P(X_{n+m+1} = y|X_{n+m} = x),$$

the transition probabilities $P_{x,y} = P(X_{n+1} = y|X_n = x)$ are the entries of the stochastic transition probability matrix $P \in [0,1]^{X \times X}$.

We review some important properties of a Markov Chain $(X_n)_{n=0,1,\ldots}$ on $X = \mathbb{Z}$:

Definition 1 [20]. Section 3.1] Two states $x, y \in \mathbb{Z}$ communicate if there exist $n, m \in \mathbb{N}$ s.t. $(P^n)_{y,x} > 0$ and $(P^m)_{x,y} > 0$. If all the states communicate, the Markov Chain is said to be irreducible.

Definition 2 [20]. Section 3.2.3] Let $\tau_j = \min\{n > 0 : X_n = j\}$: a state $j$ is said to be positive recurrent if $E(\tau_j|X_0 = j) < \infty$. The Markov Chain itself is said to be positive recurrent if all its states are so.

Proposition 3 [20]. Last part of Section 3.2.3] If a Markov Chain is irreducible and has one positive recurrent state, then all the states are so, that is the chain is positive recurrent.

Definition 4 [20]. Section 3.2.3] A invariant (or stationary) probability vector is a probability vector $\Phi$ (that is, $\Phi \in [0,1]^X$ and $\sum_{x \in X} \Phi_x = 1$) such that $\Phi^T P = \Phi^T$. 

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The existence of an invariant probability vector, assured under some conditions, gives an important convergence result, as stated in the following

**Proposition 5** [20, Sections 3.2.3-3.2.4] An irreducible, positive recurrent Markov Chain admits a unique invariant probability vector $\Phi$. Moreover, $\Phi$ is the limit of the so-called Cesàro sum, that is

$$
\lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} (P^k)_{x,d} = \Phi_d \quad \forall x \in \mathbb{Z}.
$$

### 4.1.2 The mean BER

Let us go back to the One State algorithm. According to 24, $(D_k)_{k \in \mathbb{N}}$ is a countable homogeneous Markov Chain on $\mathbb{Z}$, with transition probabilities

$$
P_{x,y} = P(D_{k+1} = y | D_k = x) = \frac{1}{2} [P_{x,y}(0) + P_{x,y}(1)]
$$

where $P_{x,y}(u) = P(D_{k+1} = y | D_k = x, U_k = u), u \in \{0, 1\}$. Notice that the only non-null entries of $P(u)$ are the following:

- $P_{d,d+1}(0) = \frac{1}{2} \text{erfc} \left( \frac{d + \frac{1}{2}}{\sqrt{2} \sigma} \right)$
- $P_{d,d}(0) = 1 - P_{d,d+1}(0)$
- $P_{d,d}(1) = \frac{1}{2} \text{erfc} \left( \frac{d - \frac{1}{2}}{\sqrt{2} \sigma} \right)$
- $P_{d,d-1}(1) = 1 - P_{d,d}(1)$

$P$ is tridiagonal and, for any $x, y \in \mathbb{Z}$, $P_{x,y} = P_{-x,-y}$ and $P_{x,y} > 0$ if and only if $|x - y| \leq 1$; by iteration, for any $n \in \mathbb{N}$, $(P^n)_{x,y} > 0$ if and only if $|x - y| \leq n$. Hence, given any couple of states $x, y \in \mathbb{Z}$ with distance $|x - y| = m$, $(P^m)_{x,y} > 0$ and $(P^n)_{x,y} > 0$, that is, $(D_k)_{k \in \mathbb{N}}$ is irreducible. Moreover,

**Lemma 6** $(D_k)_{k \in \mathbb{N}}$ is positive recurrent.

**Proof** It suffices to apply the following criterion proposed in [20]: if there exists a function $g \in \mathbb{R}^{+\mathbb{Z}}$ so that $g_x \geq (P^1 g)_x + \varepsilon$ for any $x \in \mathbb{Z} \setminus \{y\}$ and for some $\varepsilon > 0$, then $y$ is a positive recurrent state.

In our case, it is easy to prove that $y = 0$ is a positive recurrent state considering $g_x = |x|$. Moreover, given that the chain is irreducible, if one state is positive recurrent, all states are so. $\blacksquare$

**Proposition 7** The following statements hold:

1. $(D_k)_{k \in \mathbb{N}}$ admits a unique invariant probability vector $\Phi$;
2. $\Phi$ is defined by

$$
\Phi_d = \Phi_0 \prod_{i=1}^{d} \frac{P_{i-1,i}}{P_{i,i-1}}
$$

where

$$
\Phi_0 = \left[ 1 + 2 \sum_{d=1}^{\infty} \prod_{i=1}^{d} \frac{P_{i-1,i}}{P_{i,i-1}} \right]^{-1}.
$$
Proof. (1) It follows from Proposition 5.
(2) By \((\Phi^T_P)_d = \Phi^d_T\), for any \(d \in \mathbb{Z}\), it follows that
\[\Phi_{d-1}P_{d-1,d} - \Phi_dP_{d,d-1} = c \quad (c \text{ constant}). \tag{26}\]
In particular, as \(\Phi_d = \Phi_{-d}\) for any \(d \in \mathbb{Z}\) (this is due to the uniqueness of the invariant measure and to the symmetry of \(P\)), it suffices to substitute values \(d = 0\) and \(d = 1\) in (26) to conclude that \(c = 0\); hence, relation (26) holds.

Notice that \(c = 0\) corresponds to the property of time-reversibility of a Markov Chain (see Section 4.8 of [16]), hence one could even prove it by Theorem 4.2 in [16], after having introduced the concepts of aperiodicity and ergodicity of a Markov Chain.

From Proposition 7 we deduce in particular that for any \(d \in \mathbb{Z}\), \(\Phi_d > 0\).

Moreover, since \(P_{i-1,i}/P_{i,i-1} < 1\) for \(i \geq 1\), \(\Phi_d\) has a maximum at \(d = 0\) and it is monotone decreasing for \(d > 0\).

As a consequence of Proposition 5,

**Corollary 8** Let \(q_d = \mathbb{P}[\hat{U}_k \neq U_k | D_k = d] = P_{d,d+1} + P_{d,d-1}\), then
\[
\lim_{K \to \infty} P_b(e) = \lim_{K \to \infty} K^{-1} \sum_{k=0}^{K-1} q_d \Phi_d.
\]

Proof. Since

\[
P_b(e) = \frac{1}{K} \sum_{k=0}^{K-1} \sum_{d \in \mathbb{Z}} q_d P(D_k = d) = \frac{1}{K} \sum_{k=0}^{K-1} \sum_{d \in \mathbb{Z}} q_d (P^k)_{0,d}
\]
the result follows from Proposition 5 and by the Lebesgue’s Dominated Convergence Theorem. Indeed,

\[
\frac{1}{K} \sum_{k=0}^{K-1} \sum_{d \in \mathbb{Z}} q_d (P^k)_{0,d} = \sum_{d \in \mathbb{Z}} q_d \left( \frac{1}{K} \sum_{k=0}^{K-1} (P^k)_{0,d} \right)
\]
where \(\frac{1}{K} \sum_{k=0}^{K-1} (P^k)_{0,d} \leq 1\).

This concludes the computation of the BER in case of long-time transmission, given the distribution of the input source. In the next paragraph we study how the performance depends on the transmitted input sequence.

4.1.3 The Conditional BER

In the asymptotic case, the CBER converges to the same limit of the BER for almost all the possible inputs:

**Theorem 9** Let \(\pi\) be the uniform Bernoulli probability measure over \(\{0, 1\}^\mathbb{N}\). Then, for the One State algorithm,
\[
\lim_{K \to \infty} P_b(e | U) = \lim_{K \to \infty} P_b(e) \quad \text{for } \pi\text{-a.e. } U.
\]

Theorem 9 gives a stronger result than Corollary 8: the mean behavior of the One State algorithm is stated to be the behavior for each possible input occurrence, except for a \(\pi\)-negligible set. To prove Theorem 9 we will refer to the theory of Markov Chains in Random Environments (see Sections 5.1 and 5.2 in the Appendix).
4.2 Theoretic Analysis of the Two States Algorithm

Similar to the One State algorithm, the Two States procedure can be studied through the Markov Theory, which provides the instruments to compute both BER and CBER. As shown in Section 3.2, the Two States procedure stores, at each step, a state and its normalized probability, this information being sufficient to individuate also the second state and probability. Let $X_k$ be the r.v. representing the stored state, $Y_k$ the current correct state, $D_k = X_k - X_{k-1}$ and $A_k$ the r.v corresponding to the probability of $X_k$: now, the stochastic process $(A_k, D_k)_{k \in \mathbb{N}}$ in $[0, 1] \times \mathbb{Z}$ is a Markov Process, whose definition (which actually extends the definition of Markov Chain from a denumerable to a continuous set) is now given.

4.2.1 Markov Processes

The definitions and results introduced in this Section can be retrieved in [12] or in the Chapter 2 of [11]. Consider a set $X$ endowed with a countably generated $\sigma$-field $F$. A transition probability kernel (or Markov probability kernel, see, e.g., [12] Section 3.4.1) on $(X, F)$ is an application $P : X \times F \to [0, 1]$ such that

(i) for each $F \in F$, $P(\cdot, F)$ is a non-negative measurable function;

(ii) for each $x \in X$, $P(x, \cdot)$ is a probability measure (p.m. for short) on $(X, F)$.

Given a bounded measurable function $v$ on $(X, F)$, we denote by $Pv$ the bounded measurable function on $(X, F)$ defined as

$$(Pv)(x) = \int_x v(y)P(x, dy).$$

Further, let $\mu$ be a measure on $(X, F)$: we define the measure $\mu P$

$$(\mu P)(F) = \int_X P(x, F)\mu(dx) \quad F \in F. \tag{28}$$

We define the $n$-th power of the transition kernel $P$ simply putting $P^1(x, F) = P(x, F)$ and $P^n(x, F) = \int_x P(x, dy)P^{n-1}(y, F)$. It is easy to see that $P^n(x, F)$ are transition kernels, too. Corresponding actions on bounded functions and on measures will be respectively denoted by $P^n v$ and $\mu P^n$.

**Definition 10** [12 (10.1)] A measure $\psi$ on $(X, F)$ is said to be invariant for the transition kernel $P$ if $\psi P = \psi$.

We define a homogeneous Markov Process on space $(X, F)$ with transition kernel $P$ as a sequence of $X$-valued random variables $(X_n)_{n \in \mathbb{N}}$ such that, for any $x \in X$ and $F \in F$,

$\text{Prob}(X_{n+1} \in F | X_n = x, X_{n-1}, \ldots, X_0) = \text{Prob}(X_{n+1} \in F | X_n = x) = P(x, F)$

for any $n \in \mathbb{N}$. The evolution of $(X_n)_{n \in \mathbb{N}}$ is completely described once we fix a probability law $\mu$ of $X_0$ on $(X, F)$: if $\mu$ is invariant, then the Markov Process is said to be stationary: all the r.v.’s $X_n$ are distributed according to $\mu$. Notice also that for any $x \in X$ and $F \in F$, $\text{Prob}(X_{m+n} \in F | X_m = x) = P^n(x, F)$ for any $m, n \in \mathbb{N}$.

From now onwards, we will assume that $X$ is a locally compact separable metric space: under this topological condition we can easily prove the existence of an invariant measure (see [12] Section 12.3). Let $\mathcal{B}(X)$ be the Borel $\sigma$-algebra of $X$. 

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Definition 11  [12] Sections 6.1.1, 11.3.1] Let $P$ be a transition kernel on $(X, B(X))$. If $P(\cdot, O)$ is a lower semicontinuous function for any open set $O \in B(X)$, then $P$ is said to be weak Feller. Moreover, we say that $P$ verifies the Drift Condition if there exist a compact set $C \subset X$, a constant $b < \infty$ and a function $V : X \to [0, \infty]$ not always infinite such that
\begin{equation}
\Delta V(x) := \int_X P(x, dy)V(y) - V(x) \leq -1 + b1_C(x)
\end{equation}
for every $x \in X$.

Proposition 12  [12] Theorem 12.3.4] If a transition kernel $P$ is weak Feller and verifies the Drift Condition, then it admits an invariant p.m..

Under some further conditions, also the uniqueness of the invariant measure can be proved.

Definition 13  [12] Section 4.2.1] For any $B \in B(X)$, let $\tau_B = \min\{n > 0 : X_n \in B\}$. $(X_n)_{n \in \mathbb{N}}$ is said to be $\mu$-irreducible if there exists a measure $\mu$ on $B(X)$ such that for every $x \in X$, $\mu(B) > 0$ implies $P(\tau_B < +\infty | X_0 = x) > 0$.

A $\mu$-irreducible Markov Process whose kernel admits an invariant p.m. is said to be positive recurrent.

Proposition 14  [12] Theorem 10.0.1, Proposition 10.1.1] The kernel of a positive recurrent Markov Process admits a unique invariant p.m..

Furthermore,

Definition 15  [14] Definitions 2.2.2, 2.4.1] A set $B \in B(X)$ is said to be invariant if $P(x, B) \geq 1_B(x)$ for every $x \in X$.

A p.m. $\mu$ on $B(X)$ is said to be ergodic if $\mu(B) = 0$ or $\mu(B) = 1$ for every invariant set $B \in B(X)$.

Proposition 16  [11] Proposition 2.4.3] If a Markov Process admits a unique invariant p.m. $\mu$, then $\mu$ is ergodic.

A fundamental issue for our analysis is the Ergodic Theorem of Markov Processes, which is the transposition into stochastic terms of the Birkhoff’s Individual Ergodic Theorem ([24, Theorem 1.14]). Here we report its version under the ergodicity condition for an invariant p.m.; for a more general treatise, see [9, 11].

Theorem 17 (Ergodic Theorem)  [11] Theorem 2.3.4 - Proposition 2.4.2] Assume that a kernel $P$ on $(X, B(X))$ admits an ergodic invariant p.m. $\mu$. Then, for any non-negative function $v \in L_1(X, B(X), \mu)$,
\[\lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} (P^k v)(x) = \int_X v \, d\mu \text{ for } \mu\text{-a.e. } x \in X.\]

Finally, we report a result of direct convergence for the iterates of the kernel, in the case of no periodic behavior.
Appendix A

12. First, we check the Drift Condition. By equations (49)- (51) in the Property and the Drift Condition; the result will then follow from Proposition 19.

**Proposition 19** [23, Proposition 3.8] For a positive recurrent, aperiodic Markov Process with invariant p.m. \(\mu\), \(\|P^n(x, \cdot) - \mu\| \to 0\) as \(n \to \infty\) for \(\mu\)-a.e. \(x \in X\).

**4.2.2 The Mean BER**

Let \(A_k\) be the r.v. representing the normalized probability of the stored state in the Two States algorithm. We observe that \((A_k, D_k)_{k \in \mathbb{N}}\) is a Markov Process in \(([0, 1] \times \mathbb{Z}, \mathcal{B}([0, 1]) \times \mathcal{P}(\mathbb{Z}))\) where \(\mathcal{B}([0, 1])\) is the Borel \(\sigma\)-field on \([0, 1]\) and \(\mathcal{P}(\mathbb{Z})\) is the discrete \(\sigma\)-field of \(\mathbb{Z}\). In order to completely define the process, we provide also an initial distribution \(\mathcal{L} \times \kappa\), \(\mathcal{L}\) and \(\kappa\) respectively being the usual Lebesgue measure on \([0, 1]\) and the counting measure on \(\mathbb{Z}\).

The transition probability kernels will be explicitly computed in the Appendix 5.3.

**Proposition 20** The kernel of \((A_k, D_k)_{k \in \mathbb{N}}\) admits an invariant p.m. \(\tilde{\nu}\).

**Proof** We prove that the kernel of \((A_k, D_k)\) satisfies both the Weak Feller Property and the Drift Condition; the result will then follow from Proposition 12. First, we check the Drift Condition. By equations (49) - (51) in the Appendix,

\[
P((\alpha, d), [0, 1] \times \{d + 1\}) = \frac{1}{4} \text{erfc} \left( \frac{\sigma^2 \log \sqrt{\frac{\alpha}{1 - \alpha}} + d + 1}{\sigma \sqrt{2}} \right) \tag{30}
\]

\[
P((\alpha, d), [0, 1] \times \{d - 1\}) = \frac{1}{2} - \frac{1}{4} \text{erfc} \left( \frac{\sigma^2 \log \sqrt{\frac{\alpha}{1 - \alpha}} + d}{\sigma \sqrt{2}} \right).
\]

In particular, \(P((\alpha, d), [0, 1] \times \{d + 1\})\) and \(P((\alpha, d), [0, 1] \times \{d - 1\})\) have values in \([0, 1/2]\) and are monotone respectively decreasing and increasing with respect to \(\alpha\). Now, let us define

\[
\delta_d = \frac{1}{2(|d| + 10)} \tag{31}
\]

and

\[
V(\alpha, d) = \begin{cases} 
\frac{d^2}{d^2 + 2|d|} & \text{if } d \geq 0, \alpha \geq \delta_d \text{ or if } d < 0, \alpha \leq 1 - \delta_d; \\
\frac{d^2}{d^2 + 2|d|} & \text{otherwise.}
\end{cases} \tag{32}
\]

We are going to prove that \(V\) fulfills the Drift inequality for some compact \(C\):

\[
\Delta V(\alpha, d) = \int_{[0, 1] \times \mathbb{Z}} P((\alpha, d), d(\alpha', d')) V(\alpha', d') - V(\alpha, d) \leq -1 + b\mathbb{1}_C(\alpha, d) \tag{33}
\]

for every \((\alpha, d) \in [0, 1] \times \mathbb{Z}\). In order to individuate \(C\), let us find out the values of \((\alpha, d)\) such that (33) holds with \(\mathbb{1}_C(\alpha, d) = 0\). Recall that \(P((\alpha, d),\)
$A \times \{d'\} > 0 \Rightarrow d' \in \{d-1, d, d+1\}$ for any $\alpha \in [0, 1], A \in \mathcal{B}([0, 1])$.

In the next, let us use the notation $\omega = (\alpha, d), \omega' = (\alpha', d')$.

If $d \geq 0$,

$$\Delta V(\omega) = \int_0^1 \sum_{d'=d-1}^{d+1} P(\omega, (\alpha, d))V(\omega') - V(\omega)$$

$$= \sum_{d'=d-1}^{d+1} \left[ \int_0^{\delta_d} P(\omega, (\alpha, d'))(2d' + d^2) + \int_0^{\delta_d} P(\omega, (\alpha, d'))d^2 \right] - V(\omega)$$

$$= \sum_{d'=d-1}^{d+1} \left[ \int_0^1 P(\omega, (\alpha, d'))d^2 + \int_0^1 P(\omega, (\alpha, d'))2d' \right] - V(\omega)$$

$$= \sum_{d'=d-1}^{d+1} \left[ P(\omega, [0, 1] \times \{d'\})d^2 + P(\omega, [0, \delta_d] \times \{d'\})2d' \right] - V(\omega)$$

$$= d^2 + 2d[P(\omega, [0, 1] \times \{d+1\}) - P(\omega, [0, 1] \times \{d-1\}) + P(\omega, [0, 1] \times \{d+1\}) + P(\omega, [0, 1] \times \{d-1\}) + 2dP(\omega, [0, \delta_d] \times Z)$$

As $P(\omega, [0, 1] \times \{d+1\}) + P(\omega, [0, 1] \times \{d-1\}) \leq \frac{1}{2}$ (see equations [30]) and $P(\omega, [\beta_1, \beta_2] \times Z) \leq G(\beta_2 - \beta_1)$ (see Lemma 28 in the Appendix 5.6),

$$\Delta V(\omega) \leq d^2 + 2d[P(\omega, [0, 1] \times \{d+1\}) - P(\omega, [0, 1] \times \{d-1\})] + \frac{1}{2}$$

$$+ 2(d + 1)G\delta_d - V(\omega)$$

$$\leq d^2 + 2d[P(\omega, [0, 1] \times \{d+1\}) - P(\omega, [0, 1] \times \{d-1\})] + \frac{1}{2} + G - V(\omega)$$

(34)

where we exploited that $2(d + 1)G\delta_d < G$ by the definition [51] of $\delta_d$.

If $d < 0$, by analogous computation we obtain again the inequality [34]. Let us study the behavior of this bound for every $\omega \in [0, 1] \times Z$, according to the partition of $[0, 1] \times Z$ into four subsets given by the definition of $V$.

**Subset 1**: If $d \geq 0$ and $\alpha \geq \delta_d$, $V(\omega) = d^2$ and

$$P(\omega, [0, 1] \times \{d+1\}) \leq \frac{1}{4} \text{erfc} \left( \frac{\sigma^2 \log \sqrt{1 - \delta_d} + d}{\sqrt{2}} \right)$$

$$P(\omega, [0, 1] \times \{d-1\}) \geq \frac{1}{2} - \frac{1}{4} \text{erfc} \left( \frac{\sigma^2 \log \sqrt{1 - \delta_d} + d}{\sqrt{2}} \right)$$

hence inequality [34] becomes

$$\Delta V(\omega) \leq G + d \left[ \text{erfc} \left( \frac{\sigma^2 \log \sqrt{1 - \delta_d} + d}{\sqrt{2}} \right) - 1 \right] + \frac{1}{2}$$

$$= G + d \left[ \text{erfc} \left( \frac{-\frac{\sigma^2}{2} \log(2d + 19) + d}{\sqrt{2}} \right) - 1 \right] + \frac{1}{2}.$$
As \( \text{erfc}(x) \in (1, 2) \) when the argument \( x \) is negative, then for \( d \) is sufficiently large the quantity in the square bracket is negative. Moreover, this quantity is multiplied by \( d \); hence, there necessarily exists an integer \( d_0^+ > 0 \), depending on the noise \( \sigma \), such that for any \( d > d_0^+ \), \( \Delta V(\omega) \leq -1 \).

**Subset 2:** If \( d < 0 \) and \( \alpha \leq 1 - \delta_d \),
\[
P(\omega, [0, 1] \times \{d + 1\}) \geq \frac{1}{4} \text{erfc} \left( \frac{-\sigma^2 \log \sqrt{\frac{\delta_d}{1-\delta_d}} + d + 1}{\sigma \sqrt{2}} \right)
\]
\[
P(\omega, [0, 1] \times \{d - 1\}) \leq \frac{1}{2} - \frac{1}{4} \text{erfc} \left( \frac{-\sigma^2 \log \sqrt{\frac{\delta_d}{1-\delta_d}} + d + 1}{\sigma \sqrt{2}} \right)
\]

hence inequality \((34)\) becomes
\[
\Delta V(\omega) \leq G + d \left[ \text{erfc} \left( \frac{-\sigma^2 \log \sqrt{\frac{\delta_d}{1-\delta_d}} + d + 1}{\sigma \sqrt{2}} \right) - 1 \right] + \frac{1}{2}
\]
\[
= G + d \left[ \text{erfc} \left( \frac{-\sigma^2 \log(-2d + 19) + d + 1}{\sigma \sqrt{2}} \right) - 1 \right] + \frac{1}{2}
\]
The computation is now analogous to the previous case and we conclude that there necessarily exists an integer \( d_0^- < 0 \), depending on the noise, such that for any \( d < d_0^- \), \( \Delta V(\omega) \leq -1 \).

**Subset 3:** If \( d \geq 0 \) and \( \alpha < \delta_d \), \( V(\omega) = d^2 + 2d \); moreover, we have no tight bounds for \( P(\omega, [0, 1] \times \{d + 1\}) \) and \( P(\omega, [0, 1] \times \{d - 1\}) \): we can just notice that their difference is smaller than \( \frac{1}{2} \). Substituting it in \((34)\) we obtain
\[
\Delta V(\omega) \leq d^2 + G + \frac{1}{2} + d - d^2 - 2d = G + \frac{1}{2} - d
\]

hence \( \Delta V(\omega) \leq -1 \) if \( d > d_1 \equiv G + \frac{3}{2} \).

**Subset 4:** If \( d < 0 \) and \( \alpha > 1 - \delta_d \), \( V(\omega) = d^2 - 2d \); as \( P(\omega, [0, 1] \times \{d + 1\}) - P(\omega, [0, 1] \times \{d - 1\}) \geq -\frac{1}{2} \),
\[
\Delta V(\omega) \leq G + \frac{1}{2} + d
\]

and \( \Delta V \leq -1 \) if \( d < -d_1 \).

Now, it is easy to verify that the subsets of \([0, 1] \times Z\) not yet considered form the compact set \([(0, \delta_d] \times \{0, \ldots, d_1\}) \cup ([\delta_d, 1] \times \{0, \ldots, d_1^+\}) \cup ([0, 1 - \delta_d] \times \{d_0^-, \ldots, -1, 0\}) \cup ([1 - \delta_d, 1] \times \{-d^3, \ldots, -1, 0\}) \). For simplicity, we can consider the bigger compact set \( C = [0, 1] \times \{-d_C, \ldots, d_C\} \), where \( d_C = \max\{d_0^+, -d_0^-, d_1\} \); now, it is easy to check that for any \( \omega \in C \) the Drift Condition is satisfied whenever \( b \geq G + d_C + \frac{1}{2} \).

We now check the Weak Feller Property. Given any open interval \( I \subset [0, 1] \) and \( d' \in Z \), the continuity of \( P(\cdot, I \times \{d'\}) \) can be easily verified by the equations
Given $q$ and ergodic by Propositions 14 and 16. Therefore, by the Ergodic Theorem 17, the initial state (1) technical computation and is postponed in the Appendix 5.5), then

This result cannot be immediately applied to evaluate the BER since the convergence is not assured for all the initial states. In particular, let call $\mathcal{N} \subset [0,1] \times \mathbb{Z}$ the negligible set for which there is no convergence and let $N_0 = \{\alpha \in [0,1] : (\alpha,0) \in \mathcal{N}\}$. Now, recalling the Remark 1

$$P_b(e) = \frac{1}{K} \bar{\eta}(1,0) + \frac{1}{K} \sum_{k=1}^{K-1} \int_{\alpha_1 \in [0,1]} \sum_{d_1 \in \mathbb{Z}} P((1,0),(d\alpha_1,d_1))(P^k\bar{\eta})(\alpha_1,d_1)$$

$$= \frac{1}{K} \bar{\eta}(1,0) + \frac{1}{K} \sum_{k=1}^{K-1} \int_{\alpha_1 \in [0,1]} P((1,0),(d\alpha_1,0))(P^k\bar{\eta})(\alpha_1,0).$$

### Corollary 21

**Given** the invariant p.m. $\tilde{\varphi}$,

$$\lim_{K \to \infty} P_b(e) = \int_{[0,1] \times \mathbb{Z}} \bar{\eta} \, d\tilde{\varphi}.$$

**Proof** (A$_k$, D$_k$)$_{k \in \mathbb{N}}$ is $(\mathcal{L} \times \kappa)$-irreducible (the proof of this fact requires some technical computation and is postponed in the Appendix 5.5), then $\tilde{\varphi}$ is unique and ergodic by Propositions 14 and 16. Therefore, by the Ergodic Theorem 17

$$\lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} (P^k\bar{\eta})(\alpha,d) = \int_{[0,1] \times \mathbb{Z}} \bar{\eta} \, d\tilde{\varphi} \text{ a.e. } (\alpha,d).$$

This result cannot be immediately applied to evaluate the BER since the convergence is not assured for all the initial states. In particular, let call $\mathcal{N} \subset [0,1] \times \mathbb{Z}$ the negligible set for which there is no convergence and let $N_0 = \{\alpha \in [0,1] : (\alpha,0) \in \mathcal{N}\}$. Now, recalling the Remark 1

$$P_b(e) = \frac{1}{K} \bar{\eta}(1,0) + \frac{1}{K} \sum_{k=1}^{K-1} \int_{\alpha_1 \in [0,1]} \sum_{d_1 \in \mathbb{Z}} P((1,0),(d\alpha_1,d_1))(P^k\bar{\eta})(\alpha_1,d_1)$$

$$= \frac{1}{K} \bar{\eta}(1,0) + \frac{1}{K} \sum_{k=1}^{K-1} \int_{\alpha_1 \in [0,1]} P((1,0),(d\alpha_1,0))(P^k\bar{\eta})(\alpha_1,0).$$
By the Lebesgue’s Dominated Convergence Theorem,

\[ \lim_{K \to \infty} P_b(e) = \int_{\alpha_1 \in [0,1]} P((1,0), (d\alpha_1, 0)) \lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K-1} (P^{k-1}\eta)(\alpha_1, 0). \]

Notice that \( L(N_0) = 0 \), otherwise \( \tilde{\phi}(N_0 \times \{0\}) = \int_{[0,1] \times \mathbb{Z}} P(\omega, N_0 \times \{0\}) \tilde{\phi}(d\omega) > C_{\varepsilon,d} L(N_0) > 0 \) by Proposition 27. By Proposition 28, this implies that \( P((1,0), N_0 \times \{0\}) = 0 \). Finally,

\[ \lim_{K \to \infty} P_b(e) = \int_{\alpha_1 \in [0,1] \setminus N_0} P((1,0), (d\alpha_1, 0)) \lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K-1} (P^{k-1}\eta)(\alpha_1, 0) \]

\[ = \int_{\alpha_1 \in [0,1] \setminus N_0} P((1,0), (d\alpha_1, 0)) \int_{[0,1] \times \mathbb{Z}} \eta \, d\tilde{\phi} = \int_{[0,1] \times \mathbb{Z}} \eta \, d\tilde{\phi} \]

as \((\alpha_1, 0) \notin N\). ■ The function \( \eta(\alpha, d) \) is explicitly computed in the Appendix 5.4.

### 4.2.3 The Conditional BER

The CBER for the Two States algorithm can be derived just as we computed it for the One State case, in fact it holds the following

**Theorem 22** Let \( \pi \) be the uniform Bernoulli probability measure over \( \{0,1\}^\mathbb{N} \). Then, for the Two States algorithm,

\[ \lim_{K \to \infty} P_b(e|U) = \lim_{K \to \infty} P_b(e) \quad \text{for } \pi\text{-a.e. } U. \]

We refer the reader to the Appendix 5.7 for the proof.

### 4.3 Direct Convergence to \( \tilde{\phi} \)

The explicit construction of an invariant p.m. is an intricate issue in the not countable framework. When ergodic results are available, one can approximate it by several procedures (see, e.g., [11, Chapter 12]). In our framework, we can obtain an approximation by Proposition 19 which states the direct convergence of the iterates \( P^n(\cdot, \cdot) \) to the invariant p.m.. Before illustrating that, let us prove that the hypotheses of Proposition 19 hold.

**Proposition 23** The Markov Process \( (A_k, D_k)_{k \in \mathbb{N}} \) is strongly aperiodic.

**Proof** Let us consider the probability measure \( L \times \delta_d \) on \( ([0,1] \times \mathbb{Z}, \mathcal{B}([0,1]) \times \mathcal{P}(\mathbb{Z})) \), where \( L \) is the Lebesgue measure and \( \delta_d(d) = 1 \) if \( d = \bar{d} \), 0 otherwise. By Proposition 27, \( P((\alpha, d), M \times \{d\}) > \frac{1}{2} C_{\varepsilon,d} L(M) \), \( C_{\varepsilon,d} > 0 \). Then, considering the Definition 18 with \( \nu = L \times \delta_d \), \( c = \frac{1}{2} C_{\varepsilon,d} \) and \( A = [0,1] \times \{d\} \), the proposition is proved. ■ This result along with Proposition 19 yields:

**Corollary 24 (Direct Convergence)** \( \|P^n((\alpha, d), \cdot) - \tilde{\phi}\| \to 0 \) as \( n \to \infty \) for \( \phi\text{-a.e. } (\alpha, d) \in [0,1] \times \mathbb{Z} \).
4.3.1 Analytic vs Simulations’ outcomes

To conclude our analysis of One State and Two States algorithms, we compare the simulations’ outcomes with the theoretic results: we expect the BER’s obtained by the simulations of sufficiently long transmissions to be consistent to the analytic computations.

By Corollaries 8 and 21, the BER’s can be computed once we know the corresponding invariant distributions. While for the One State algorithm the invariant measure is explicitly given by (25), for the Two States algorithm we have approximated it using the Corollary 24. In particular, we have discretized the kernel $P$ into a matrix, afterwards we have computed the iterates $P^n$ for a sufficiently large $n$, so that to obtain an equilibrium condition, that is, a matrix whose rows are all equal up to numerical roundoff. At this point, any row of the matrix is a discretized, approximated version of the invariant p.m.
In Figures 4 and 5 we compare analytic and simulations’ outcomes: as expected, they do not present substantial differences.

5 Appendix

5.1 Markov Chains in Random Environments

Consider a countable set $\Theta$ and a family of transition probability kernels \{\(P_\theta, \theta \in \Theta\)\} on a space \((X, \mathcal{F})\). Given a \(\sigma\)-field \(\mathcal{B}\) of \(\Theta\), let \((\theta_n)_{n \in \mathbb{N}}\) and \((X_k)_{k \in \mathbb{N}}\) respectively be sequences of \(\Theta\)-valued and \(X\)-valued r.v.’s. \(P_{\theta_0}(X_k, F)\) can now be interpreted as the transition probability of \(X_k\) to set \(F\) depending on the r.v \(\theta_k\), which represents to so-called random environment.

We say that \((X_k)_{k \in \mathbb{N}}\) with \((\theta_n)_{n \in \mathbb{N}}\) is a Markov Chain in Random Environment (or MCRE) if

\[
P(X_{k+1} \in F | X_k, \ldots, X_0, (\theta_n)_{n \in \mathbb{N}}) = P_{\theta_k}(X_k, F) \quad \text{a.s.}
\]

for all \(F \in \mathcal{F}\) and \(k = 0, 1, \ldots\)

Let us define \(\Theta^N = \prod_{0}^{\infty} \Theta\) and \(\mathcal{B}^N = \prod_{0}^{\infty} \mathcal{B}\). An important feature of a MCRE is that we can always associate to it a classical Markov Process. In fact, given any \(x \in X\) and \(\bar{\theta} = (\theta_0, \theta_1, \ldots) \in \Theta^\mathbb{N}\) and denoting by \(T\) the left sequence shift on \(\Theta^\mathbb{N}\) (that is, \(T\bar{\theta} = \tilde{\theta}\) with \(\tilde{\theta}_n = \theta_{n+1}\) for any \(n \in \mathbb{N}\)), we can introduce the following transition probability kernel on \((X \times \Theta^\mathbb{N}, \mathcal{F} \times \mathcal{B}^\mathbb{N})\):

\[
P((x, \theta), F \times B) = P_{\theta_0}(x, F) \mathbb{1}_B(T\theta)
\]

which determines a Markov Process \((X_k, T^k(\theta_n)_{n \in \mathbb{N}})_{k \in \mathbb{N}}\) on \((X \times \Theta^\mathbb{N}, \mathcal{F} \times \mathcal{B}^\mathbb{N})\). From now onwards, we will refer to it as to the Extended Markov Process, EMP for short.

Remark 3 : As noted in the Section 1 of \cite{5}, if the random environments \(\theta_n\)’s are independent, then \((X_k)_{k \in \mathbb{N}}\) is a Markov Process with transition probability kernel \(P(x, F) = \mathbb{E}_{\theta \in \Theta^\mathbb{N}}[P_{\theta}(x, F)]\). In other terms, \((X_k)_{k \in \mathbb{N}}\) is the Markov Process moving in the average environment.

In this framework, we prove the following

Proposition 25 Let \((X_k)_{k \in \mathbb{N}}\) with \((\theta_n)_{n \in \mathbb{N}}\) be a MCRE on \(X \times \Theta^\mathbb{N}\). Suppose that the random environments \(\theta_n\)’s are independent, identically distributed with distribution \(\pi_0\) on \((\Theta, \mathcal{B})\) and that the kernel of the Markov Process \((X_k)_{k \in \mathbb{N}}\) admits an invariant p.m. \(\phi\); given the distribution \(\pi = \times_{0}^{\infty} \pi_0\) over \((\Theta^\mathbb{N}, \mathcal{B}^\mathbb{N})\),

\[
\psi = \phi \times \pi
\]

is an invariant p.m. for the EMP \((X_k, T^k(\theta_n)_{n \in \mathbb{N}})_{k \in \mathbb{N}}\) over \((X \times \Theta^\mathbb{N}, \mathcal{F} \times \mathcal{B}^\mathbb{N})\).

Proof Let \(\omega = (x, \theta) \in X \times \Theta^\mathbb{N}\). \(\psi\) is an invariant for \((X_k, \theta_k)_{k \in \mathbb{N}}\) if

\[
\int_{X \times \Theta^\mathbb{N}} P(\omega, F \times B) \psi(d\omega) = \psi(F \times B)
\]
for any \( F \times B \) such that \( F \in \mathcal{F}, B \in \mathcal{B}^\mathbb{N} \). Now,
\[
\int_{\mathbf{X} \times \Theta^\mathbb{N}} P(\omega, F \times B) \psi(d\omega) = \int_{\mathbf{X}} \int_{\Theta^\mathbb{N}} P_{\theta_0}(x, F) 1_B(\theta_1, \theta_2, \ldots) \pi(d\theta) \phi(dx)
\]
\[
= \pi(B) \int_{\mathbf{X}} \sum_{\theta_0 \in \Theta} P_{\theta_0}(x, F) \pi_0(\theta_0) \phi(dx)
\]
\[
= \pi(B) \int_{\mathbf{X}} P(x, F) \phi(dx) = \pi(B) \phi(F) = \psi(F \times B)
\]
where we have exploited the fact that \( \phi \) is invariant. This Proposition is a partial extension of the Theorem 5 in [13], which states the same result in the case of denumerable state space \( X \) and attests also the inverse implication (that is, all the invariant p.m.’s are product measures of kind (37) still in the denumerable framework.

For a more detailed treatise on MCRE’s, we refer the reader to [5, 6, 13, 14].

5.2 Proof of Theorem 9

From equation (24), \((D_k)_{k \in \mathbb{N}}\) with \((U_k)_{k \in \mathbb{N}}\) turns out to be a countable MCRE. This is the right way to look at \((D_k)_{k \in \mathbb{N}}\) if we want to understand its behavior with respect to typical instances of the input \( U = (U_0, U_1, \ldots) \). For any \( x, y \in \mathbb{Z} \), we have
\[
P(D_{k+1} = y|D_k = x, D_{k-1}, \ldots, D_0; U) = P_{x,y}(U_k).
\]
Consider the space \((\mathbb{Z} \times \{0,1\}^\mathbb{N}, \mathcal{P}(\mathbb{Z}) \times \prod_0^\infty \mathcal{P}([0,1]))\) endowed with the initial distribution \( \kappa \times \pi \), where \( \kappa \) is the counting measure on \( \mathbb{Z} \) and \( \pi \) is the usual Bernoulli measure on \( \{0,1\}^\mathbb{N} \). Given \( x, y \in \mathbb{Z} \), \( u = (u_0, u_1, \ldots) \in \{0,1\}^\mathbb{N} \) and \( B \in \prod_0^\infty \mathcal{P}([0,1]) \), the EMP is defined by the transition probability kernel
\[
P((x, u); (y) \times B) = P_{x,y}(u_0) 1_B(Tu).
\]
By Proposition 25, an invariant probability measure exists for our EMP and we explicitly compute it: in fact, let \( \phi \) be a p.m. on \((\mathbb{Z}, \mathcal{P}(\mathbb{Z}))\) given by \( \phi((d)) = \Phi_d \), \( \Phi_d \) being the invariant probability vector defined in the Proposition 7 for any integer \( d \). Then, \( \psi = \phi \times \pi \) is an invariant p.m. for the EMP.

We can verify that \( \psi \) is ergodic by the following criterion (see Chapter 3 of [5]). Let \( P(U_0, U_1, \ldots, U_{n-1}) \) the transition matrix whose entries are
\[
P_{x,y}(U_0, \ldots, U_{n-1}) = P(D_n = y|D_0 = x, U_0, \ldots, U_{n-1}).
\]
If for each \( x, y \in \mathbb{Z} \) and \( \pi \)-a.e. \( U \) there exist \( n = n(x, y, U) \in \mathbb{N} \) and \( z = z(x, y, U, n) \in \mathbb{Z} \) such that \( P_{x,z}(U_0, \ldots, U_{n-1})P_{y,z}(U_0, \ldots, U_{n-1}) > 0 \), then \( \psi \) is ergodic. In our context it is easy to check that given any couple of starting states \( x \) and \( y \), after \( n > |x - y| \) steps we have a non-null probability of having joined a common state \( z \).

Define \( q_d(U_k) = P(U_k \neq U_k|D_k = d, U_k = P_{d,d+1}(U_k) + P_{d,d-1}(U_k) \) \( \pi_{d} \) is actually the mean of \( q_d \). For any \( K \in \mathbb{N} \) and given \( D_0 = 0 \), the CBER can be expressed as follows:
\[
P_0(e|U) = \frac{1}{K} \sum_{k=0}^{K-1} \sum_{d \in \mathbb{Z}} q_d(U_k) = P_{0,0}(U_0, U_1, \ldots, U_{k-1})
\]
Notice that, since the $U_k$'s $k \in \mathbb{N}$ are independent, $P(U_0, U_1, \ldots, U_{k-1}) = P(U_0)P(U_1)\cdots P(U_{k-1})$.

Consider $\omega = (x, U)$ and the function $g(\omega) = q_\omega(U_0)$: we have that
\[
\sum_{d \in \mathbb{Z}} q_d(U_k)P_{x,d}(U_0, \ldots, U_{k-1}) = P^k g(x, U)
\]
and notice that
\[
P_b(e|U) = \frac{1}{K} \sum_{k=0}^{K-1} P^k g(0, U).
\]  
(41)

Now, by the Ergodic Theorem \[17\]
\[
\lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} P^k g(\omega) = \int_{Z \times \{0,1\}^K} g(\omega)\psi(d\omega) \text{ for } \psi\text{-a.e. } \omega.
\]  
(42)

Notice that, as pointed out after Proposition \[7\] $\phi((d)) > 0$ for any $d \in \mathbb{Z}$; then, a set $\{d\} \times B, d \in \mathbb{Z}, B \subset \{0,1\}^K$, is $\psi$-negligible if and only if $\pi(B) = 0$. Hence, in (42), “$\psi$-a.e. $\omega$” is equivalent to “for any $d \in \mathbb{Z}$ and $\pi$-a.e. $U$”.

This, along with the equality (11), implies that
\[
\lim_{K \to \infty} P_b(e|U) = \int_{Z \times \{0,1\}^K} g(\omega)\psi(d\omega) \text{ for } \pi\text{-a.e. } U.
\]  
(43)

Finally, recalling that $\psi = \phi \times \pi$,
\[
\int_{Z \times \{0,1\}^K} g(\omega)\psi(d\omega) = \sum_{d \in \mathbb{Z}} \sum_{U_0=0,1} q_d(U_0)\pi(U_0)\Phi_d = \sum_{d \in \mathbb{Z}} \Phi_d.
\]

5.3 Two States Algorithm: Computation of the Transition Probabilities

In the next pages, we compute the probability of moving from a state $(\alpha, d) \in [0,1] \times \mathbb{Z}$ to a set of type $(0, \beta) \times \{d'\}, \beta \in [0,1], d' \in \mathbb{Z}$, for the Markov Process $(A_k, D_k)_{k \in \mathbb{N}}$ defined in Section 3.2. Let $P_u((\alpha, d), (0, \beta) \times \{d'\})$ be the transition probability given the transmitted bit $u$: $P((\alpha, d), (0, \beta) \times \{d'\}) = \frac{1}{2} P_0((\alpha, d), (0, \beta) \times \{d'\}) + \frac{1}{2} P_1((\alpha, d), (0, \beta) \times \{d'\})$ are null if $d' \notin \{d-1, d, d+1\}$, if $d' = d+1$ and $u = 1$ or if $d' = d-1$ and $u = 0$; we now compute the non-null instances. Given $(\alpha, d) \in (0,1) \times \mathbb{Z}$ and $x \in \{\alpha, (1-\alpha)^{-1}, 1\}, y \in \{d-1, d, d+1\}, z \in (0,1)$, we define:
\[
c_{\alpha} = \frac{\exp(1/\sigma^2)}{\alpha(1-\alpha)}
\]
\[
h_{x,y}(z) = \frac{\sigma^2 \log \left(\frac{x-1}{z}\right)}{\sigma \sqrt{2}} + y + \frac{1}{2}
\]  
(44)
\[
H_{x,y}(z) = \frac{1}{2} \erfc(h_{x,y}(z)).
\]

Notice that these quantities depend on the noise variance $\sigma^2$, even if the notation does not emphasize that. Remind also Definition \[22\].
Case 1: $d' = d, u = 0$.

\[ P_0((\alpha, d), (0, \beta) \times \{d\}) = \text{Prob}(\zeta_3 \leq \zeta_1 \leq \beta(\zeta_1 + \zeta_2) | A_k = \alpha, D_k = d, U_k = 0) = \begin{cases} 0 & \text{if } \alpha = 0 \text{ or if } \alpha \in (0,1) \text{ and } \beta \leq \frac{\epsilon_{\alpha}}{1 + \beta_{\epsilon_{\alpha}}} \\ H_{\alpha,d} (\beta) - H_{\alpha,d} \left( \frac{1}{1 + \beta_{\epsilon_{\alpha}}} \right) & \text{if } \alpha \in (0,1) \text{ and } \beta > \frac{\epsilon_{\alpha}}{1 + \beta_{\epsilon_{\alpha}}} \\ H_{1,d} (\beta) & \text{if } \alpha = 1. \end{cases} \] (45)

Case 2: $d' = d, u = 1$.

\[ P_1((\alpha, d), (0, \beta) \times \{d\}) = \text{Prob}(\zeta_3 \geq \zeta_1 \cap (\beta \zeta_3 \geq (1 - \beta) \zeta_2) | A_k = \alpha, D_k = d, U_k = 1) = \begin{cases} H_{\frac{\epsilon_{\alpha}}{1 + \beta_{\epsilon_{\alpha}}},d} (\beta) & \text{if } \alpha = 0 \text{ or if } \alpha \in (0,1) \text{ and } \beta \leq \frac{\epsilon_{\alpha}}{1 + \beta_{\epsilon_{\alpha}}} \\ H_{\frac{\epsilon_{\alpha}}{1 + \beta_{\epsilon_{\alpha}}},d} \left( \frac{\epsilon_{\alpha}}{1 + \beta_{\epsilon_{\alpha}}} \right) & \text{if } \alpha \in (0,1) \text{ and } \beta > \frac{\epsilon_{\alpha}}{1 + \beta_{\epsilon_{\alpha}}} \\ 0 & \text{if } \alpha = 1. \end{cases} \] (46)

Case 3: $d' = d + 1, u = 0$.

\[ P_0((\alpha, d), (0, \beta) \times \{d + 1\}) = \text{Prob}(\zeta_3 \geq \zeta_1 \cap (\beta \zeta_3 \geq (1 - \beta) \zeta_2) | A_k = \alpha, D_k = d, U_k = 0) = \begin{cases} H_{\frac{\epsilon_{\alpha}}{1 + \beta_{\epsilon_{\alpha}}},d+1} (\beta) & \text{if } \alpha = 0 \text{ or if } \alpha \in (0,1) \text{ and } \beta \leq \frac{\epsilon_{\alpha}}{1 + \beta_{\epsilon_{\alpha}}} \\ H_{\frac{\epsilon_{\alpha}}{1 + \beta_{\epsilon_{\alpha}}},d+1} \left( \frac{\epsilon_{\alpha}}{1 + \beta_{\epsilon_{\alpha}}} \right) & \text{if } \alpha \in (0,1) \text{ and } \beta > \frac{\epsilon_{\alpha}}{1 + \beta_{\epsilon_{\alpha}}} \\ 0 & \text{if } \alpha = 1. \end{cases} \] (47)

Case 4: $d' = d - 1, u = 1$.

\[ P_1((\alpha, d), (0, \beta) \times \{d - 1\}) = \text{Prob}(\zeta_3 \leq \zeta_1 \leq \beta(\zeta_1 + \zeta_2) | A_k = \alpha, D_k = d, U_k = 1) = \begin{cases} 0 & \text{if } \alpha = 0 \text{ or if } \alpha \in (0,1) \text{ and } \beta \leq \frac{1}{1 + \beta_{\epsilon_{\alpha}}} \\ H_{\alpha,d-1} (\beta) - H_{\alpha,d-1} \left( \frac{1}{1 + \beta_{\epsilon_{\alpha}}} \right) & \text{if } \alpha \in (0,1) \text{ and } \beta > \frac{1}{1 + \beta_{\epsilon_{\alpha}}} \\ H_{1,d-1} (\beta) & \text{if } \alpha = 1. \end{cases} \] (48)

Remark 4: As $\epsilon_{\alpha} > 2, \frac{1}{1 + \beta_{\epsilon_{\alpha}}} < \frac{1}{3} < \frac{2}{3} < \frac{\epsilon_{\alpha}}{1 + \beta_{\epsilon_{\alpha}}}$.

Summing up:

\[ P((\alpha, d), (0, \beta) \times \{d\}) = \begin{cases} \frac{1}{2} \begin{cases} H_{1,d} (\beta) & \text{if } \alpha = 0 \text{ or if } \alpha = 1 \\ H_{\frac{\epsilon_{\alpha}}{1 + \beta_{\epsilon_{\alpha}}},d} (\beta) & \text{if } \alpha \in (0,1) \text{ and } \beta \leq \frac{1}{1 + \beta_{\epsilon_{\alpha}}} \\ H_{\frac{\epsilon_{\alpha}}{1 + \beta_{\epsilon_{\alpha}}},d} (\beta) + H_{\frac{\epsilon_{\alpha}}{1 + \beta_{\epsilon_{\alpha}}},d} \left( \frac{1}{1 + \beta_{\epsilon_{\alpha}}} \right) & \text{if } \alpha \in (0,1) \text{ and } \frac{1}{1 + \beta_{\epsilon_{\alpha}}} < \beta \leq \frac{\epsilon_{\alpha}}{1 + \beta_{\epsilon_{\alpha}}} \\ H_{\frac{\epsilon_{\alpha}}{1 + \beta_{\epsilon_{\alpha}}},d} (\beta) + H_{\frac{\epsilon_{\alpha}}{1 + \beta_{\epsilon_{\alpha}}},d} \left( \frac{\epsilon_{\alpha}}{1 + \beta_{\epsilon_{\alpha}}} \right) & \text{if } \alpha \in (0,1) \text{ and } \beta > \frac{\epsilon_{\alpha}}{1 + \beta_{\epsilon_{\alpha}}} \\ \end{cases} \end{cases} \] (49)
\[ P((\alpha, d), (0, \beta) \times \{d + 1\}) = \]
\[
\begin{cases}
    H_{\frac{1}{1+\epsilon\alpha},d+1}(\beta) & \text{if } \alpha = 0 \text{ or } \alpha \in (0,1) \text{ and } \beta \leq \frac{\epsilon\alpha}{1+\epsilon\alpha} \\
    H_{\frac{1}{1+\epsilon\alpha},d+1}(\frac{\epsilon\alpha}{1+\epsilon\alpha}) & \text{if } \alpha \in (0,1) \text{ and } \beta > \frac{\epsilon\alpha}{1+\epsilon\alpha} \\
    0 & \text{if } \alpha = 1
\end{cases}
\]
(50)

\[ P((\alpha, d), (0, \beta) \times \{d - 1\}) = \]
\[
\begin{cases}
    H_{\alpha,d-1}(\beta) - H_{\alpha,d-1}(\frac{1}{1+\epsilon\alpha}) & \text{if } \alpha = 0 \text{ or } \alpha \in (0,1) \text{ and } \beta \leq \frac{1}{1+\epsilon\alpha} \\
    H_{\alpha,d-1}(\beta) & \text{if } \alpha \in (0,1) \text{ and } \beta > \frac{1}{1+\epsilon\alpha} \\
    0 & \text{if } \alpha = 1
\end{cases}
\]
(51)

5.4 Two States Algorithm: Computation of \( \overline{\eta}(\alpha, d) \)

The function \( \overline{\eta} \) on \([0,1] \times \mathbb{Z} \) defined in the Corollary 21 is given by \( \overline{\eta}(\alpha, d) = \frac{1}{2} P(\hat{U}_k = 1|U_k = 0, A_k = \alpha, D_k = d) + \frac{1}{2} P(\hat{U}_k = 0|U_k = 1, A_k = \alpha, D_k = d) \).

Note that
\[
P(\hat{U}_k = 1|U_k = 0, A_k = \alpha, D_k = d) = \]
\[
\text{Prob}(\alpha f_{Y_k+1|X_k+1}(y_{k+1}|\hat{x}_k + 1) + (1 - \alpha) f_{Y_k+1|X_k+1}(y_{k+1}|\hat{x}_k + 2)
\>
\alpha f_{Y_k+1|X_k+1}(y_{k+1}|\hat{x}_k + 1) + (1 - \alpha) f_{Y_k+1|X_k+1}(y_{k+1}|\hat{x}_k + 1)) = \frac{1}{2} \text{erfc} \left( \frac{\sigma^2 \log z_1 + d + \frac{1}{2}}{\sqrt{2}\sigma} \right)
\]
where \( z_1 \) is the positive solution of the equation \( (1 - \alpha) e^{-\frac{z^2}{2\sigma}} z^2 + (2\alpha - 1) z - \alpha = 0 \).

Similarly,
\[
P(\hat{U}_k = 0|U_k = 1, A_k = \alpha, D_k = d) = 1 - \frac{1}{2} \text{erfc} \left( \frac{\sigma^2 \log z_1 + d - \frac{1}{2}}{\sqrt{2}\sigma} \right)
\]

hence
\[
\overline{\eta}(\alpha, d) = \frac{1}{2} \left[ \frac{1}{2} \text{erfc} \left( \frac{\sigma^2 \log z_1 + d + \frac{1}{2}}{\sqrt{2}\sigma} \right) + 1 - \frac{1}{2} \text{erfc} \left( \frac{\sigma^2 \log z_1 + d - \frac{1}{2}}{\sqrt{2}\sigma} \right) \right].
\]

Naturally, if \( \alpha = 1 \), then \( \overline{\eta}(\alpha, d) = \frac{1}{2} \left[ \frac{1}{2} \text{erfc} \left( \frac{d+\frac{1}{2}}{\sqrt{2}\sigma} \right) + 1 - \frac{1}{2} \text{erfc} \left( \frac{d-\frac{1}{2}}{\sqrt{2}\sigma} \right) \right] = \overline{\eta}_d \) and we recast into the One State case.

5.5 Two States Algorithm: Proof of the \((\mathcal{L} \times \kappa)\)-irreducibility of \((A_k, D_k)_{k \in \mathbb{N}}\)

In this paragraph, we complete the proof of the Corollary 21 showing the \((\mathcal{L} \times \kappa)\)-irreducibility of \((A_k, D_k)_{k \in \mathbb{N}}\) in the space \([0,1] \times \mathbb{Z}, B([0,1]) \times \mathcal{P}(\mathbb{Z})\). For this purpose, we first prove that any non-negligible Borel subset of kind \( M \times \{d'\} \subset [0,1] \times \mathbb{Z} \) is achievable with positive probability from any \((\alpha, d)\), in one or two steps, if \( d' \in \{d-1, d, d+1\} \) and \( M \) is sufficiently far from the extreme points of \([0,1] \times \mathbb{Z} \).
Lemma 26 For any $\varepsilon > 0$, $d \in \mathbb{Z}$, there exists a constant $C_{\varepsilon,d} > 0$ such that the following inequalities hold for every $(\alpha, d) \in [0, 1] \times \mathbb{Z}$ and $M \in B([\varepsilon, 1 - \varepsilon])$:

\[
P((\alpha, d), M \times \{d\}) \geq C_{\varepsilon,d}L(M)
\]

\[
P^2((\alpha, d), M \times \{d + 1\}) \geq C_{\varepsilon,d}L(M)
\]

\[
P^2((\alpha, d), M \times \{d - 1\}) \geq C_{\varepsilon,d}L(M)
\]

where $L$ is the Lebesgue measure.

Proof First, we prove the lemma on the open intervals $(\beta_1, \beta_2) \subset [\varepsilon, 1 - \varepsilon]$. For shortness of notation, let $\bar{\alpha} = \frac{1}{1 + c_\alpha}$.

Consider the first inequality. On the basis of the equations (49) and Remark 4 the following cases may occur:

1. If $\alpha = 0$, $(\beta_1, \beta_2) \subset [\varepsilon, 1 - \varepsilon]$ or if $\alpha \in (0, 1)$, $(\beta_1, \beta_2) \subset [\varepsilon, \frac{1}{1 + c_\alpha}] \subset [\varepsilon, \frac{1}{3}]$:

\[
P((\alpha, d), (\beta_1, \beta_2) \times \{d\}) = \frac{1}{2}H_{\alpha,d}(\beta_2) - \frac{1}{2}H_{\alpha,d}(\beta_1)
\]

\[
= \frac{1}{2\sqrt{\pi}} \int_{h_{\alpha,d}(\beta_2)}^{h_{\alpha,d}(\beta_1)} e^{-t^2} dt
\]

\[
= -\frac{1}{2\sqrt{\pi}} \int_{\beta_1}^{\beta_2} e^{-h_{\alpha,d}(z)^2} \frac{\partial}{\partial z} h_{\alpha,d}(z) dz
\]

\[
\geq \frac{1}{2\sqrt{\pi}} (\beta_2 - \beta_1) \min_{z \in (\beta_1, \beta_2)} (-e^{-h_{\alpha,d}(z)^2} \frac{\partial}{\partial z} h_{\alpha,d}(z))
\]

\[
\geq \frac{1}{2\sqrt{\pi}} (\beta_2 - \beta_1) \min_{z \in (\beta_1, \beta_2)} \left(-\frac{\partial}{\partial z} h_{\alpha,d}(z)\right) \min \left\{e^{-h_{\alpha,d}(\beta_1)^2}, e^{-h_{\alpha,d}(\beta_2)^2}\right\}.
\]

By definition (44), for any $x, y$, $\frac{\partial}{\partial x} h_{x,y}(z) = \frac{\sigma}{\sqrt{(z-x)^2 + \varepsilon}} \leq -\sigma 2\sqrt{2}$; moreover,

\[
\min \left\{e^{-h_{\alpha,d}(\beta_1)^2}, e^{-h_{\alpha,d}(\beta_2)^2}\right\} \geq \min \left\{e^{-h_{\alpha,d}(\varepsilon)^2}, e^{-h_{\alpha,d}(1-\varepsilon)^2}\right\} = m_{\alpha,d}.
\]

Notice now that for any $d \in \mathbb{Z}$, $m_{\alpha,d} \to 0$ if and only if $\alpha \to 1$; nevertheless, if $\alpha \to 1$, also $(1 + c_\alpha)^{-1} \to 0$ and in particular there will be some $\alpha$ such that $(1 + c_\alpha)^{-1} < \varepsilon$, which contradicts the hypothesis $\beta_1 \geq \varepsilon$. Hence, can we conclude that

\[
P((\alpha, d), (\beta_1, \beta_2) \times \{d\}) \geq \sigma \sqrt{2/\pi} \min_{\alpha} m_{\alpha,d}(\beta_2 - \beta_1) > 0
\]

where the minimum has to be computed for $\alpha$ satisfying the initial hypotheses.

2. If $\alpha = 1, (\beta_1, \beta_2) \subset [\varepsilon, 1 - \varepsilon]$ or if $\alpha \in (0, 1)$, $(\beta_1, \beta_2) \subset [\frac{\varepsilon}{1 + c_\alpha}, 1 - \varepsilon] \subset [\frac{1}{3}, 1 - \varepsilon]$; by analogous procedure, we obtain

\[
P((\alpha, d), (\beta_1, \beta_2) \times \{d\}) \geq \sigma \sqrt{2/\pi} \min_{\alpha} m_{\alpha,d}(\beta_2 - \beta_1) > 0
\]

where $m_{\alpha,d} = \min \left\{e^{-h_{\alpha,d}(\varepsilon)^2}, e^{-h_{\alpha,d}(1-\varepsilon)^2}\right\} > 0$ and its minimum is computed for $\alpha$ satisfying the above hypotheses. The positiveness holds since for any $d \in \mathbb{Z}$, $m_{\alpha,d} \to 0$ if and only if $\alpha \to 0$, which implies $\frac{c_\alpha}{1 + c_\alpha} \to 1$ and contradicts $\beta_2 \leq 1 - \varepsilon$. 28
3. Otherwise: it is straightforward to verify that
\[
P((\alpha, d), (\beta_1, \beta_2) \times \{d\}) \geq \sigma \sqrt{2/\pi} \ (m_{\alpha,d} + m_{\alpha,d})(\beta_2 - \beta_1).
\]
Finally, if we consider
\[
\bar{m}(\alpha, d, \beta_1, \beta_2) = \begin{cases} 
  m_{\alpha,d} & \text{if } \alpha = 0 \text{ or if } \alpha \in (0, 1) \text{ and } \varepsilon < 1 - \varepsilon \leq \frac{1}{1+c_\alpha}; \\
  m_{\alpha,d} & \text{if } \alpha = 1 \text{ or if } \alpha \in (0, 1) \text{ and } \frac{c_\alpha}{1+c_\alpha} < 1 - \varepsilon \leq 1 - \varepsilon; \\
  m_{\alpha,d} + m_{\alpha,d} & \text{otherwise.}
\end{cases}
\]
and
\[
C_{\varepsilon,d}^{(1)} = \sigma \sqrt{2/\pi} \ \min_{\alpha \in [0,1]} \ \bar{m}(\alpha, d, \beta_1, \beta_2)
\]
we conclude that for any \(\varepsilon > 0, d \in \mathbb{Z},
\]
\[
P((\alpha, d), (\beta_1, \beta_2) \times \{d\}) \geq C_{\varepsilon,d}^{(1)}(\beta_2 - \beta_1) \quad C_{\varepsilon,d}^{(1)} > 0.
\]

Let us prove the second inequality, on the basis of equations [50]. In this case, the component \(d\) of the state moves to \(d + 1\), which is not always possible in one step. In particular, there are two situations in which the transition probability is null: \(\alpha = 1\) and when \(\beta_1 = \frac{c_\alpha}{1+c_\alpha}\) (and given the continuity of [50] problems occur whenever \(\alpha \to 1\) or \(\beta_1 \to \frac{c_\alpha}{1+c_\alpha}\).

Both issues can be solved considering two-step transition: roughly speaking, if \(\alpha\) is close to 1, a first step is used to move \(\alpha\) away from 1 (and \(d\) remains constant); at this point, the probability to move \(d\) to \(d + 1\) is positive. On the other hand, when \(\beta_1\) is close to \(\frac{c_\alpha}{1+c_\alpha}\) a first step is used to move \(d\) to \(d + 1\) and a second one to move the component \(\alpha\) to the desired interval (and now this is possible since we recast in the case in which \(d\) remains constant, previously studied).

Let us assess this qualitative argumentation.

1. If \(\alpha = 0, (\beta_1, \beta_2) \in [\varepsilon, 1 - \varepsilon]\) or if \(\alpha \in (0, 1 - \delta_1]\) for some small \(\delta_1 > 0, (\beta_1, \beta_2) \subset [\varepsilon, \frac{c_\alpha}{1+c_\alpha}];
\]
\[
P((\alpha, d), (\beta_1, \beta_2) \times \{d+1\}) \geq \sigma \sqrt{2/\pi} \ \min_{\alpha \in [0,1-\delta_1]} \ m_{\alpha,d+1}(\beta_2 - \beta_1) > 0 \tag{55}
\]
where the positiveness of \(\min_{\alpha \in [0,1-\delta_1]} m_{\alpha,d+1} > 0\) as been discussed above.

2. If \(\alpha \in (0, 1 - \delta_1], \beta_1 \in [\varepsilon, \frac{c_\alpha}{1+c_\alpha} - \delta_2]\) for some small \(\delta_1, \delta_2 > 0\) and \(\beta_2 \in [\frac{c_\alpha}{1+c_\alpha} - \delta_2]\); the transition probability depends on \(\beta_1\), not on \(\beta_2\), and
\[
P((\alpha, d), (\beta_1, \beta_2) \times \{d+1\}) \geq \sigma \sqrt{2/\pi} \ \min_{\alpha \in (0,1-\delta_1]} \ m_{\alpha,d+1} \left(\frac{c_\alpha}{1+c_\alpha} - \beta_1\right)
\]
where \(\frac{c_\alpha}{1+c_\alpha} - \beta_1 \geq \delta_2 \geq \delta_2(\beta_2 - \beta_1).

Let us now consider the cases that require two steps to move with non-null probability into the desired set. For this purpose, notice that
\[
P^2((\alpha, d), (\beta_1, \beta_2) \times \{d + 1\}) = \\
= \int_0^1 \sum_{d'=d,d+1} \ P((\alpha, d), (d\alpha', d')) P((\alpha', d'), (\beta_1, \beta_2) \times \{d + 1\}) \tag{56}
\]
3. If \( \alpha \in (0, 1 - \delta_1] \), \( \beta_1 \in \left( \frac{c_\alpha}{1 + c_\alpha}, \delta_2, \beta_2 \right) \) and \( \beta_2 \in [\frac{c_\alpha}{1 + c_\alpha}, 1 - \varepsilon] \), we exploit that

\[
P^2((\alpha, d), (\beta_1, \beta_2) \times \{d + 1\}) \geq \\
\geq \int_{0}^{1} P((\alpha, d), (\alpha', d + 1)) P((\alpha', d + 1), (\beta_1, \beta_2) \times \{d + 1\})
\]

As \( P((\alpha', d + 1), (\beta_1, \beta_2) \times \{d + 1\}) \geq C^{(1)}_{\varepsilon,d+1}(\beta_2 - \beta_1) \) by (54),

\[
P^2((\alpha, d), (\beta_1, \beta_2) \times \{d + 1\}) \geq C^{(1)}_{\varepsilon,d+1}(\beta_2 - \beta_1) P((\alpha, d), ([0, 1), d + 1))
\]

\[
\geq C^{(1)}_{\varepsilon,d+1}(\beta_2 - \beta_1) P((\alpha, d), ([\varepsilon, 1 - \varepsilon], d + 1)) \geq \\
\geq C^{(1)}_{\varepsilon,d+1}(\beta_2 - \beta_1) \sigma \sqrt{2/\pi}(1 - 2\varepsilon) \min_{\alpha \in (0, 1 - \delta_1]} m_{\alpha,d+1}.
\]

(58)

4. If \( \alpha \in (1 - \delta_1, 1] \), we exploit that

\[
P^2((\alpha, d), (\beta_1, \beta_2) \times \{d + 1\}) \geq \\
\geq \int_{0}^{1} P((\alpha, d), (\alpha', d)) P((\alpha', d), (\beta_1, \beta_2) \times \{d + 1\}).
\]

A sufficient condition to have \( P((\alpha', d), (\beta_1, \beta_2) \times \{d + 1\}) > 0 \) is \( \beta_2 \leq \frac{c_{\alpha'}}{1 + c_{\alpha'}} \)

(see 55) which corresponds to \( \alpha'' - \alpha' + \exp \left( \frac{1}{\alpha'} \right) \left( \frac{1 - \beta_2}{\beta_2} \right)^2 \geq 0 \). This holds for any \( \alpha' \) if \( 4 \exp \left( \frac{1}{\alpha'} \right) \left( \frac{1 - \beta_2}{\beta_2} \right)^2 > 1 \), otherwise for \( \alpha' \in [0, \tilde{\alpha}] \cup [1 - \tilde{\alpha}, 1] \)

where \( \tilde{\alpha} = \frac{1}{\sqrt{1 - 4 \exp \left( \frac{1}{\alpha'} \right) \left( \frac{1 - \beta_2}{\beta_2} \right)^2}} \).

Reducing the domain of integration to \([0, \tilde{\alpha}]\), we obtain

\[
P^2((\alpha, d), (\beta_1, \beta_2) \times \{d + 1\}) \geq \\
\geq \int_{0}^{\tilde{\alpha}} P((\alpha, d), (\alpha', d)) P((\alpha', d), (\beta_1, \beta_2) \times \{d + 1\})
\]

\[
\geq \int_{0}^{\tilde{\alpha}} P((\alpha, d), (\alpha', d)) \sigma \sqrt{2/\pi} m_{\alpha',d+1}(\beta_2 - \beta_1)
\]

\[
\geq \sigma \sqrt{2/\pi} \min_{\alpha' \in [0, \tilde{\alpha}]} m_{\alpha',d+1}(\beta_2 - \beta_1) P((\alpha, d), ([0, \tilde{\alpha}], d))
\]

\[
\geq \sigma \sqrt{2/\pi} \min_{\alpha' \in [0, \tilde{\alpha}]} m_{\alpha',d+1}(\beta_2 - \beta_1) C^{(1)}_{\varepsilon,d} \alpha.
\]

(60)

Finally, gathering the bounds obtained in the previous four cases, we obtain

\[
P^2((\alpha, d), (\beta_1, \beta_2) \times \{d + 1\}) \geq C^{(2)}_{\varepsilon,d}(\beta_2 - \beta_1).
\]

(61)

where \( C^{(2)}_{\varepsilon,d} = \delta_2(1 - 2\varepsilon) \tilde{\alpha} \sigma \sqrt{2/\pi} \min_{\alpha \in [0, 1 - \delta_1]} m_{\alpha,d+1} \min \{ C^{(1)}_{\varepsilon,d}, C^{(1)}_{\varepsilon,d+1} \} > 0 \).

We omit the proof of the third inequality as it is analogous to the second one: by the same argumentation, we obtain a suitable constant \( C^{(3)}_{\varepsilon,d} \). Finally, for any small \( \varepsilon > 0 \) and \( d \in \mathbb{Z} \), \( C_{\varepsilon,d} = \min \{ C^{(1)}_{\varepsilon,d}, C^{(2)}_{\varepsilon,d}, C^{(3)}_{\varepsilon,d} \} \).
The thesis is now proved for any open interval in \([\varepsilon, 1 - \varepsilon]\). The generalization to all the open sets in \([\varepsilon, 1 - \varepsilon]\) is straightforward since any open set on the real line is countable union of disjoint open intervals. Finally, we can extend the result to all the Borelians in \([\varepsilon, 1 - \varepsilon]\). Remind that for any Lebesgue measurable set \(M\) (in particular, for any Borelian) in \(R\) there exists a sequence of open sets \(O_n\) such that \(M \subset \bigcap_{n=1}^{\infty} O_n\) and \(\mathcal{L}(M) = \mathcal{L}(\bigcap_{n=1}^{\infty} O_n)\), see [17]. As any finite intersection of open sets is open, we have

\[
P^\ast((\alpha, d), \bigcap_{n=1}^{N} O_n \times \{d'\}) \geq C_\varepsilon \mathcal{L}(\bigcap_{n=1}^{N} O_n) \geq C_\varepsilon \mathcal{L}(\bigcap_{n=1}^{\infty} O_n) = C_\varepsilon \mathcal{L}(M)
\]

for any \(d' \in \{d - 1, d, d + 1\}\) and \(r = 1, 2\) according to the value of \(d'\). This inequality holds for any \(N \in \mathbb{N}\), hence

\[
\lim_{N \to \infty} P^\ast((\alpha, d), \bigcap_{n=1}^{N} O_n \times \{d'\}) = P^\ast((\alpha, d), \bigcap_{n=1}^{\infty} O_n \times \{d'\}) \geq C_\varepsilon \mathcal{L}(M).
\]

By this lemma, it follows in particular that for any \(M \in \mathcal{B}([\varepsilon, 1 - \varepsilon])\),

\[
\left\{\begin{array}{ll}
P^{2|d-d'|}((\alpha, d), M \times \{d'\}) & \geq C_{\varepsilon,d}^{|d-d'|} \mathcal{L}(M) \quad \text{if } d \neq d'; \\
P((\alpha, d), M \times \{d\}) & \geq C_{\varepsilon,d} \mathcal{L}(M).
\end{array}\right.
\]

Moreover,

**Proposition 27** For any \(M \in \mathcal{B}([0,1])\) with \(\mathcal{L}(M) > 0\),

\[
\left\{\begin{array}{ll}
P^{2|d-d'|}((\alpha, d), M \times \{d'\}) & > \frac{1}{2} C_{\varepsilon,d}^{|d-d'|} \mathcal{L}(M) \quad \text{if } d \neq d'; \\
P((\alpha, d), M \times \{d\}) & > \frac{1}{2} C_{\varepsilon,d} \mathcal{L}(M).
\end{array}\right.
\]

In particular, \((A_k, D_k)_{k \in \mathbb{N}}\) is \((\mathcal{L} \times \kappa)\)-irreducible, \(\kappa\) being the counting measure.

**Proof** By the previous lemma, this result holds when \(M \in \mathcal{B}([\varepsilon, 1 - \varepsilon])\) given any \(\varepsilon > 0\). Now, if we consider any \(M \in \mathcal{B}([0,1])\) with \(\mathcal{L}(M) = \lambda > 0\), we have \(\mathcal{L}(M \cap [\varepsilon, 1 - \varepsilon]) = \mathcal{L}(M) - \mathcal{L}(M \cap [\varepsilon, 1 - \varepsilon]) \geq \lambda - 2\varepsilon\) and we can always choose \(\varepsilon = \varepsilon(\lambda)\) such that \(\lambda > 2\varepsilon\). For instance, let us choose \(\varepsilon = \frac{\lambda}{4}\), so that \(\lambda - 2\varepsilon = \frac{\lambda}{2}\). Therefore,

\[
P^{2|d-d'|}((\alpha, d), M \times \{d'\}) \geq P^{2|d-d'|}((\alpha, d), (M \cap [\varepsilon, 1 - \varepsilon]) \times \{d'\}) \geq C_{\varepsilon,d} \mathcal{L}(M \cap [\varepsilon, 1 - \varepsilon]) > \frac{\lambda}{2} C_{\varepsilon,d}
\]

when \(d \neq d'\), and similarly when \(d = d'\).

### 5.6 Two States Algorithm: an upper bound for the transition probability kernel

**Lemma 28** There exists a real positive constant \(G\) such that

\[
P((\alpha, d), M \times \mathbb{Z}) \leq G \mathcal{L}(M)
\]

for any \((\alpha, d) \in [0,1] \times \mathbb{Z}\) and \(M \in \mathcal{B}([0,1])\).
**Proof** First, we prove the lemma when $M$ is an open interval. Consider the equations (49) - (51): given $(\alpha, d)$, $P((\alpha, d), (\beta_1, \beta_2) \times Z)$ is equal to a sum of integrals of type $\int_{\beta_1}^{\beta_2} e^{-h_{x,y}^z(z)}(-h_{z,y}^x(z))dz$ with $x = \alpha, 1/(1 - \alpha)$ and $y = d - 1, d, d + 1$ according to the instance. As we have shown in the Proof of Lemma 2, $h_{x,y}^z(z) = \frac{\sigma}{z(z-1)^{1/2}}$, hence $g(z) = -e^{-h_{x,y}^z(z)}h_{z,y}^x(z) > 0$ for every $z \in (0, 1)$. Furthermore, $g'(z)$ is monotone decreasing over $(0, 1)$ and null in one point $z_0 \in (0, 1)$ corresponding to the unique solution of the equation $h_{x,y}^z(z) = \frac{\sigma}{z(z-1)^{1/2}}$; hence $g(z)$ is increasing in $(0, z_0)$, decreasing in $(z_0, 1)$ and admits a maximum in $z_0 \in (0, 1)$. In conclusion, $\int_{\beta_1}^{\beta_2} g(z)dz \leq G(\beta_2 - \beta_1), G = g(z_0).

The extension to all the open sets is trivial as any open set is countable union of disjoint intervals. Finally, as for any $M \in \mathcal{B}([0, 1])$ there exists a sequence of open sets $O_n$ such that $M \subset \cap_{n=1}^{\infty} O_n$ and $\mathcal{L}(M) = \mathcal{L}(\cap_{n=1}^{\infty} O_n)$ (see (17)), for any $n \in \mathbb{N}$ we can write

$$P((\alpha, d), \cap_{n=1}^{\infty} O_n \times Z) \leq P((\alpha, d), \cap_{n=1}^{\infty} O_n \times Z) \leq G\mathcal{L}(\cap_{n=1}^{\infty} O_n)$$

as any finite intersection of open sets is open. The result follows from the arbitrariness of $N$. ■

### 5.7 Proof of Theorem 22

The process $(A_k, D_k)_{k \in \mathbb{N}}$ with $(U_k)_{k \in \mathbb{N}}$ is an instance of MCRE. The corresponding EMP in $\Omega = [0, 1] \times \mathbb{Z} \times \{0, 1\}^\mathbb{N}$ is defined by the following transition probability kernel:

$$P((\alpha, d, u), A \times \{d'\} \times B) = P_{u_0}((\alpha, d, A \times \{d'\})1_B(Tu))$$

(62)

where $u = (u_0, u_1, \ldots) \in \{0, 1\}^\mathbb{N}$, $A \in \mathcal{B}([0, 1])$, $d' \in \mathbb{Z}$, $B \in \mathcal{P}([0, 1]^\mathbb{N})$.

$P_{u_0}((\alpha, d, A \times \{d'\})$ can be assessed by equations (49) - (51). Moreover, we denote by $P_{u_0,\ldots,u_{k-1}}((\alpha, d), A \times \{d'\})$ the probability of moving from $(\alpha, d) \in [0, 1] \times \mathbb{Z}$ to the set $A \times \{d'\}$ in $k$-steps, given the input sequence $(u_0, \ldots, u_{k-1}) \in \{0, 1\}^k$. By Proposition (25), $\tilde{\psi} = \phi \times \pi$ ($\phi$ being defined in Proposition 20), is an invariant p.m. for the EMP. Moreover, $\tilde{\psi}$ is ergodic.

**Lemma 29** $\tilde{\psi}$ is ergodic.

**Proof** Let $F \subset \Omega$ be an invariant set: by Definition 15, to prove the ergodicity of $\tilde{\psi}$ is sufficient to show that $\tilde{\psi}(F) > 0$ implies $\tilde{\psi}(F) = 1$.

Then, let us suppose $\tilde{\psi}(F) > 0$. We name

$$U_F = \{u \in \{0, 1\}^\mathbb{N} : (\alpha, d, u) \in F \text{ for some } (\alpha, d) \in [0, 1] \times \mathbb{Z}\};$$

$$U_0 = \{u \in \{0, 1\}^\mathbb{N} : u \text{ contains infinitely many 0's and 1's}\};$$

$$U_0^n = \{u \in U_0 : u \text{ contains at least a } n \text{ and a 1 in its first } n \text{ bits }\}, \ n \geq 2.$$ 

Given the transition probability kernel (62), if $u \in U_F$ then also $Tu \in U_F$ and since $\pi$ is an ergodic measure with respect to the shift operator $T$ (see (24) Section 1.5) and $\pi(U_F) > 0$ (otherwise $\tilde{\psi}(F) = 0$), we have that $\pi(U_F) = 1$ by the Birkhoff’s Individual Ergodic Theorem (24 Theorem 1.14).

By analogous reasoning, $\pi(U_0) = 1$. Furthermore, $U_0^{n+1} \subset U_0^n$, then $U_0^n \uparrow U_0$. This implies the existence of an $n_0 \geq 2$ such that $\pi(U_0^{n_0}) > 0$. 32
At this point, let us consider the equations (45)-(48): by applying the procedure used to prove Lemma 26 and Proposition 27, it is easy to verify that for any \((\alpha, d) \in (0, 1) \times \mathbb{Z}\),

\[
P_0((\alpha, d), M \times \{d\}) > 0 \quad \text{for any } M \in \mathcal{B}(\{1/3, 1\}), \mathcal{L}(M) > 0;
\]

\[
P_1((\alpha, d), M \times \{d\}) > 0 \quad \text{for any } M \in \mathcal{B}(\{0, 2/3\}), \mathcal{L}(M) > 0;
\]

\[
P_0((\alpha, d), M \times \{d + 1\}) > 0 \quad \text{for any } M \in \mathcal{B}(\{1/3, 1\}), \mathcal{L}(M) > 0;
\]

\[
P_1((\alpha, d), M \times \{d - 1\}) > 0 \quad \text{for any } M \in \mathcal{B}(\{1, 2/3\}), \mathcal{L}(M) > 0.
\]

(63)

where \(\frac{1}{3}\) and \(\frac{2}{3}\) are sufficient, not necessary bounds derived from Remark (4).

These inequalities yield to

\[
P_{01}((\alpha, d), M \times \{d\}) > 0 \quad \text{for any } M \in \mathcal{B}(\{0, 1\}), \mathcal{L}(M) > 0;
\]

\[
P_{10}((\alpha, d), M \times \{d\}) > 0 \quad \text{for any } M \in \mathcal{B}(\{0, 1\}), \mathcal{L}(M) > 0.
\]

(64)

Notice also that we are not considering the negligible cases \(\alpha = 0\) and \(\alpha = 1\), which may prevent the one-step transition (see (45)-(48)). Maintaining this hypothesis, consider \((\alpha, d, u) \in \mathcal{F}\) such that \(u \in \mathcal{U}_{00}\) (this is always possible since \(\mathcal{U}_{00} \subset \mathcal{U}_{F, \psi}\)-a.e.). By the invariance of \(\mathcal{F}\) and (64), we obtain that

\[
[0, 1] \times \{d\} \times \mathcal{V}_u \subset \mathcal{F}
\]

(65)

since \(u\) contains at least a 0 and a 1 in its first \(n_0\) bits. Moreover, the fact that \(\mathcal{U}_{00}^{n_0}\) is not negligible implies that we can always choose \(u \in \mathcal{U}_{00}^{n_0}\) such that \(\mathcal{V}_u = \{T^n u, n \in \mathbb{N}\}\) has measure \(\pi(\mathcal{V}_u) = 1\), as a consequence of \([24, \text{Theorem 1.14}].\) Hence,

\[
[0, 1] \times \{d\} \times \mathcal{V}_u \subset \mathcal{F}
\]

(66)

Birkhoff.

Furthermore, consider the evolution of the component \(d \in \mathbb{Z}\): from equations (63) we deduce that any \(d\) has non-null probability to achieve, in \(n\) steps, any integer belonging to

\[
D_n = \{d - m_1, d - m_1 + 2, \ldots, d + n - m_1\}
\]

\(m_1\) being the number of 1’s in the corresponding \(n\)-bit input sequence. Hence,

\[
[0, 1] \times D_n \times T^n \mathcal{V}_u \subset \mathcal{F}
\]

(67)

where \(T^n \mathcal{V}_u = \mathcal{V}_u\) \(\mathcal{V}_a\)-a.e.. Given that for any \(n, D_n \subset D_{n+1}\), in particular, \(D_{n+1}\) has one more element than \(D_n\), then \(D_n \uparrow \mathbb{Z}\). This finally proves that

\[
[0, 1] \times \mathbb{Z} \times \mathcal{V}_u \subset \mathcal{F} \quad \text{\(\mathcal{V}_a\)-a.e.}
\]

(68)

\(\tilde{\phi}(\mathcal{F}_w) = 1\). But now also \([0, 1] \times \mathbb{Z} \times \{T^w\} \subset \mathcal{F}\). which implies

\[
\tilde{\psi}(\mathcal{F}) = \tilde{\phi}([0, 1] \times \mathbb{Z}) \pi(\mathcal{V}_u) = 1.
\]

(69)

\(\blacksquare\)
Given \( q(\alpha, d, U_k) = P(\hat{U}_k \neq U_k | U_k, A_k = \alpha, D_k = d) \),

\[
P_b(e | U) = \frac{1}{K} \sum_{k=0}^{K-1} P(\hat{U}_k \neq U_k | U) = \int_0^1 \sum_{d \in \mathbb{Z}} \sum_{u \in \{0, 1\}} q(\alpha, d, U_k) P(u_0, \ldots u_{k-1}) \{ (1, 0), (d_0, d) \}. \tag{70}
\]

Now, let \( g(\alpha, d, U) = q(\alpha, d, U_0) \): it is easy to verify that

\[
P^k g(\alpha, d, U) = \int_0^1 \sum_{d' \in \mathbb{Z}} q(\alpha', d', U_k) P(u_0, \ldots u_{k-1}) \{ (\alpha, d), (d', d') \}
\]

then

\[
P_b(e | U) = \frac{1}{K} \sum_{k=0}^{K-1} P^k g(1, 0, U). \tag{71}
\]

By the Ergodic Theorem \[17\]

\[
\lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} (P^k g)(\omega) = \int_\Omega g \, d\tilde{\psi} \quad \text{for } \tilde{\psi}-\text{a.e. } \omega \in \Omega
\]

Let \( N \subset \Omega \) be the negligible set for which there is no convergence and let \( N_{0, U} = \{ \alpha \in [0, 1] : (\alpha, 0, U) \in N \} \). By the same argumentation used in Corollary \[21\] \( P_U(1, 0), N_{0, U} \times \{0\} = 0 \) and

\[
P_b(e | U) = \frac{1}{K} g(1, 0, U) + \frac{1}{K} \sum_{k=1}^{K-1} \int_{\alpha_1 \in [0, 1]} P_U((1, 0), (d_1, 0))(P^{k-1} g)(\alpha_1, 0, TU)
\]

\[
\lim_{K \to \infty} \int_{\alpha_1 \in [0, 1] \setminus N_{0, U}} P_U((1, 0), (d_1, 0)) \int_\Omega g \, d\tilde{\psi} = \int_\Omega g \, d\tilde{\psi} \quad \pi\text{-a.e. } U \in \{0, 1\}^K.
\]

which proves the thesis, as

\[
\int_\Omega g \, d\tilde{\psi} = \int_{[0, 1]} \sum_{d \in \mathbb{Z}} \sum_{u \in \{0, 1\}} q(\alpha, d, u) \tilde{\phi}(d_0, d) \pi_0(u) = \int_{[0, 1] \times \mathbb{Z}} \eta \, d\tilde{\phi}.
\]

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