Properties of an Arithmetic Code for Geodesic Flows

Daniel P. B. Chaves¹, Reginaldo Palazzo Jr.¹ and José R. Rios Leite²

¹ Department of Telematics, School of Electrical and Computer Engineering, State University of Campinas, 13081-970 Campinas-SP, Brazil
² Department of Physics, Federal University of Pernambuco, 50670-901 Recife-PE, Brazil
E-mail: dpb.chaves@gmail.com, palazzo@dt.fee.unicamp.br, rios@df.ufpe.br

Abstract. Topological analysis of chaotic dynamical systems emerged in the nineties as a powerful tool in the study of strange attractors in low-dimensional dynamical systems. It is based on identifying the stretching and squeezing mechanisms responsible for creating a strange attractor and organize all the unstable periodic orbits in this attractor. This method is concerned with the manifold generated by the chaotic system. Furthermore, as a mathematical object, the manifolds have a well studied geometric and algebraic structure, particularly for the case of compact surfaces. Intending to use this structure in the analysis and application of chaotic systems through their topological characteristics, we determine properties of geodesic codes for compact surfaces necessary for the construction of encoders from the symbolic sequences of experimental data generated by the unstable periodic orbits of the strange attractor (related to the behavior changes of the system with the variation of control parameters) to the geodesic code sequences, which permits to use the surface structure to study the system orbits.

1. Introduction
In the works of Gilmore, Lefranc, Plumecoq and others [1–6], topological methods have been applied to analyze the chaos behavior of dissipative dynamical systems. The topological approach is appropriate to analyze low-dimension dynamical systems, that is, those with Lyapunov dimension less than three or, in other words, whose flow rapidly relaxes to a three-dimensional subspace of the phase space. It intends to complement other methods of analysis based on values of metrics and dynamical invariants. However, it goes beyond by describing how to model the dynamics, allowing the determination and validation of models for dynamical mechanisms that generate a specific chaotic data, and being robust under changes in control-parameter values. The topological analysis relies on representing the mechanisms that generate the strange attractor and organize its unstable periodic orbits (UPO’s) through a branched manifold, that is, a simpler two-dimensional structure that preserves the topological properties of the strange attractor. The branched manifold is determined by extracting topological-invariant properties from the chaotic time series which need not be exceptionally clean nor long. Thus we may consider, up to some extent, the topological approach tolerates noise and inaccurate sampling for a fairly practical amount of data to set up the conclusions about the system.

The topological characterization of a strange attractor is divided into a gross and a fine structure. Both are determined by computing the linking numbers and relative rotation rates of the UPO’s [1, 2], which demonstrate the central matter of the UPO’s for topological analysis. The gross structure is provided by a set of integers derived from the underlying branched manifold, and it is robust under control-parameter variation. The fine structure relies on unfolding the
branched manifold by changing the control parameters in such a way that some subsets of orbits are removed and the remaining orbits can be determined by a basis set of orbits that forces the presence of all remaining ones. These elements may become clearer if compared with a typical communication process. In this case, the gross structure can be compared with the subset of the Euclidean space that contains the signal constellation in a typical modulation scheme; and the fine structure specifies the constraints that have to be satisfied and how information is conveyed when using the branched manifold in a communication process.

The study of physical systems by applying geometric and topological concepts have already been proved to be successful in studying standard Hamiltonian systems \cite{7, 8}. The trajectories of these systems can be seen as geodesics of the phase-space manifold equipped with the standard Jacobi metric, and therefore questions about the stability/instability properties of the dynamics can be tackled by studying the corresponding properties of the geodesics. Furthermore, this approach has provided an alternative point of view about Hamiltonian chaos with respect to previous explanations. In this work we follow a similar approach by proposing the application of the well known algebraic and geometric structure of the canonical surfaces to study streams of data obtained from experiments or simulations of low-dimensional dissipative dynamical systems. Any compact orientable surface can be deformed in a canonical surface by a homotopy deformation, that is, preserving its topological properties. The canonical surfaces are informally distinguished by the number of holes or handles as shown in Figure 1. This number is called the genus of the surface and can be rigorously specified \cite{9, 10}. Thus, we can specify an embedding from the orbit flow in a branched manifold to one of the canonical surfaces, so that the topological properties of the flow in the branched manifold are matched to the topological properties of the geodesic flow in the canonical surface. Since we are working with symbol sequences, a natural way to achieve this task is to code the symbol sequences of the orbits, obtained from experiments or simulations, to symbol sequences coding the geodesics of the canonical surfaces. In this work we consider the geodesic code proposed by Adler and Flatto \cite{11}, since it has a well defined direct-graph representation of the code sequences and defines an easy-to-calculate hyperbolic measure determined from the code symbols. As additional information to specify an encoder, we should determine the entropy of the code sequences (that form a shift-space) and the intrinsic properties of the code sequences that can be helpful to the construction of an encoder. In Section 4 we determine this additional information, by specifying the maximum full-shift (set of all possible bi-infinite symbol sequences over a finite alphabet) embedded in a geodesic code and its topological entropy, as a function of the genus of the canonical surface.

Additionally, in Section 2 we give an overview of how topological analysis can be applied to study the chaotic behavior of low-dimensional dissipative dynamical systems. In Section 3 basic concepts in hyperbolic geometry, geodesic flow, and the considered geodesic code are presented. Finally, in Section 5 we draw the conclusions.

2. Topological elements

In spite of the simplicity of some nonlinear dynamical systems, exact analytical solutions are not known for the differential equations describing their behaviors. On the other hand, approximate solutions do not grasp the general behavior of the system. A breakthrough to this problem was given by the French mathematician Henry Poincaré, based on the study of how a set of nearby initial conditions (an entire neighborhood in phase space) evolves, which gave rise to the now called topological theory. His intent was to study conservative dynamical systems, the Hamiltonian ones. However, the Birman-Williams Theorem generalizes the results considered
from the topological approach for low-dimensional dissipative dynamical systems whenever the Lyapunov dimension is less than three [1, 2].

2.1. Chaos mechanisms
A chaotic behavior is in essence recurrent, non-periodic and sensitive to initial conditions. By sensitive to initial conditions we mean the distance between any two nearby points in the phase space tends to increase exponentially, at least for a sufficiently small time interval, in other words, the points are stretched apart. Considering a motion in a bounded phase space, these points should eventually reach a maximum separation and then begin to approach each other again, in other words, the initial conditions are squeezed together. These behaviors are represented in Figure 2 (a), where we emphasize the relation between the stretching and squeezing mechanisms and the flow direction. The branched line determines the one-dimensional region of the branched manifold where the branch flows get so squeezed together that become indistinguishable from each other, however still different. This is the essential feature to justify irreversibility.

Furthermore, in a dissipative dynamical system all volumes in the phase space shrink asymptotically to zero overtime. In this case, the stretching and squeezing mechanisms imply that almost all initial conditions will asymptotically gravitate to a strange attractor, and that the flow takes place in a two-dimensional manifold almost everywhere but at the tear point of the stretching process and at squeezed regions meeting at the branch line. These are the two possible singularities that keep the region determined by the flow from being a two-dimensional manifold. The stretching and squeezing mechanisms form the fundamental blocks that build up the resulting two-dimensional structure (the branched manifold), by connecting every outflow to some inflow and vice-versa. The Birman-Williams Theorem makes rigorous the existence of such a branched manifold whenever the flow rapidly relaxes to a three-dimensional subspace of the phase space, in other words, the associated attractor has Lyapunov dimension less than three. From a finite length of discretely sampled scalar time-series data, the topological approach permits to determine the stretching and squeezing mechanisms that build up the attractor, and so to specify a dynamical system model that reproduces an experimental data set to an acceptable level. This is obtained by determining the organization of the UPO’s within the strange attractor through the topological invariants: Linking numbers and relative rotation rates. As an advantage of the topological approach, such invariants are unaltered under smooth coordinate transformations, control-parameters changes, and do not depend on the orbits’ stability provided that they exist. Thus, the topological approach is robust in identifying the chaos mechanisms, that is, the branched manifold. All necessary information to proceed with a topological analysis is represented in a template, a third topological invariant, that also exhibits the previously stated robustness properties. Details about its construction and properties can be found in [1, 2, 6].

2.2. Linking numbers and templates
The linking number amounts to the signed numbers of crossings between orbits, from an orbit with itself, or between branches of the branched manifold or template. The signs +1 and −1 are given for each cross as specified in Figure 2 (b). The linking numbers can be extracted from the template, that are a standard form of representing the branched manifold for purpose of computing the topological invariants. The template is a projection of the two-dimensional
branched manifold, which is embedded in $\mathbb{R}^3$, into a two-dimensional subspace, in such a way that the crossing information is preserved. Since the Birman-Williams Theorem guarantees that the strange attractor projects to a two-dimensional branched manifold, maintaining its topological organization invariant under the projection, thus the template is a standard form of summarizing the stretching and squeezing mechanisms responsible for generating the strange attractor. These mechanisms are represented in the segments $A_1$ and $A_2$ in the template, as shown in Figure 3. The segment $A_1$ summarizes the signed number of twists in each branch of the flow. The segment $A_2$ describes the branch crossings, such that, each branch crossing is assigned an integer in the same way as shown in Figure 2 (b). In the region labeled $B$ the branches are squeezed together, and the order in which they are squeeze is represented in the template. The algebraic counterpart of this description has three components, for a branched manifold with $n$ branches: (Topological Matrix - $T$) An $n \times n$ symmetric matrix that describes the topological organization of the branches, where the diagonal element $T_{ii}$ describes the branch twist of branch $i$, and the off-diagonal elements $T_{ij} = T_{ji}$ describe how branches cross each other; (Joining Array - $A$) A $1 \times n$ array that describes the order in which branches are joined at branch lines, such that, the closer a branch is to the observer, the lower the branch number; (Transition Matrix - $I$) An $n \times n$ matrix that describes which branches flow into which branches, following the convention that the element $I_{ij}$ is one if branch $i$ flows into branch $j$, and zero otherwise. Thus, matrix $I$ gives a description of the symbolic dynamics of allowed periodic orbits of the associated Markov shift. The integers that appear in the algebraic description are extracted from the experimental data, and form the sufficient information to identify the stretching and squeezing mechanisms that generate the chaotic behavior.

Every usual chaotic system studied is non-hyperbolic. As a consequence of this, the experimental symbol sequences that encodes the UPO’s of its attractor will generate forbidden finite symbol sequences that vary with the control parameters. However, since the template is determined for a considered range of control parameters its structure remains invariant under the variation of them. Thus, once a template satisfying the attractor topology is identified, it remains invariant in the considered range of control parameters while a fixed setting of the control parameters restricts the flow to a subset of the template (sub-template). A physical interpretation of this phenomenon is that orbits are created, annihilated or even both as the control parameters vary, and their topological organization does not change with these variations. Thus, the template corresponds to the gross structure and the flow is pushed around on the template as the control parameters vary. Accordingly, the template establishes a communication framework that is determined only once and remains invariant as long as the orbits remain embedded in the strange attractor as the control parameters vary. Matrix $I$ determines the restrictions that should be satisfied by every orbit-code sequence, and while the control parameters vary the set of forbidden sequences changes. This can be used to specify a relationship between the basis sets of orbits and the control parameters, and so encode information to be transmitted.
3. Geodesic code

A canonical geometric surface originates from a regular tessellation of one of the three possible geometric planes: Euclidean ($\mathbb{R}^2$), spherical ($S^2$), and hyperbolic ($H^2$). Regarding the objective of this paper, a regular tessellation of $H^2$ is a covering of the entire plane by non-overlapping regular $H^2$-polygons meeting only along complete edges, or at vertices. We also consider the existence of a group of isometries acting on polygon $\Pi$. Furthermore, since the hyperbolic isometries are conformal mappings, they preserve area and angle. Thus, a tessellation can be seen as a mosaic, of equal polygons, of $H^2$.

The set of all isometries that map $\Pi$ to any other polygon of the tessellation forms a group $\Gamma$. The group $\Gamma$ is called a Fuchsian group. Thus, the tessellation is generated by the action of $\Gamma$ on $\Pi$ called fundamental region of $\Gamma$. $\Gamma$ is infinity, but it has a finite number of generators given by a subset $\Gamma_0 \subset \Gamma$ of isometries that maps $\Pi$ to its adjacent images in the tessellation which share an edge with $\Pi$. In practice, $\Gamma_0$ is specified by a set of isometries that map each edge of $\Pi$ to a non-adjacent edge. Thus, if $h \in \Gamma_0$ maps the edge $e$ to the edge $f$ of $\Pi$, we say that $e$ and $f$ are "glued" by $h$. Note that $h^{-1}$ maps $f$ to $e$ by the inverse mapping. As a consequence, the action of the maps in $\Gamma_0$, are called pairing maps or transformations. By gluing together the edges of $\Pi$ by the elements of $\Gamma_0$, we obtain a geometric surface that inherit the local metric structure of the covering plane, that is, the differential metric of the surface is equal to the differential metric of the hyperbolic plane. There exist more than one model for the hyperbolic plane, in this work we consider the disk model or the Poincaré disk model, represented by $D^2$. Let $\Pi^o$ be the interior of a polygon $\Pi$, then $\Pi$ is a fundamental region if it satisfies: (i) $h_1 \Pi^o \cap h_2 \Pi^o = \emptyset$ for $h_1 \neq h_2$, $h_1, h_2 \in \Gamma$; (ii) $\bigcup_{h \in \Gamma} h\Pi = D^2$; (iii) $\lambda(\Pi - \Pi^o) = 0$, where $\lambda$ denotes the hyperbolic area. These concepts are illustrated in Figure 4: The hyperbolic polygon $\Pi$ is the closed region formed by the solid lines; the straight dashed lines indicate the edges of $\Pi$ that are glued by side-pairing maps in $\Gamma_0$; $h_1$ is the side-pairing map that glues edge 1 to the edge 7; the dashed lines extending from the edges of $\Pi$ to the boundary circle of $D^2$ are geodesics whose segments form the edges of $\Pi$ (a hyperbolic geodesic in $D^2$ is a circular arc orthogonal to the boundary circle of $D^2$); the closed curves $\gamma_i$ in $S$ are obtained by gluing the edges of $\Pi$ together, the numbers above and below the curve indicate the edges that form it.

The process of determining the geometric surfaces is similar to the three possible planar geometries. In addition to the metric, they can also differ in their topological properties. It is well known that the sphere is the only orientable spherical surface, and that all compact orientable Euclidean surfaces are tori, hence a surface of genus one. All compact orientable surfaces of genus greater than one are hyperbolic surfaces. Furthermore, there exists a hyperbolic surface for any genus greater than one. This property justifies the choice of hyperbolic surfaces to
embed the flow dynamics associated with a branched manifold, since they provide the necessary topological structure to embed the great amount of possible branched-manifold topologies.

3.1. Coding geodesics

The geodesic code in consideration is the one proposed by Adler and Flatto in [11] which is a special case of the more general coding process proposed by Series in [12]. It is based on the relationship identified by Bowen and Series between the geodesic flow on a compact surface of constant negative curvature and the interval map defined by the Markov partition of the boundary \( \partial \mathbb{D}^2 \) of the Poincaré disk, providing the relation between their symbolic dynamics [13]. The previous works explore the properties and existing interrelationships between these systems as for instance the use of ergodicity of the interval maps to prove ergodicity of geodesic flows. Conversely, explicit formulas for invariant measures of interval maps from invariant measures for flows were also developed in these works.

The coding process is similar to the binary expansion of the points in the unit interval \([0, 1)\) by the piecewise linear expansion map \( f(x) = 2x \), where \((\alpha)\) denotes the fractional part of \(\alpha\). For this simple case, the subintervals \([0, \frac{1}{2})\) and \([\frac{1}{2}, 1)\) form a Markov partition of \([0, 1)\). The \(n\)-th digit of the binary expansion of \(x \in [0, 1)\) is equal to 0 if \(f^n(x) \in [0, \frac{1}{2})\), and 1 otherwise.

The idea of the geodesic coding is similar, but the function \(f\) is substituted by the side-pairing transformations of a hyperbolic polygon \(\Pi\) with \((8g - 4)\) edges that is the fundamental region of a Fuchsian group \(\Gamma\), and the Markov partition is formed by the arcs in \(\partial \mathbb{D}^2\) with endpoints determined by the geodesics whose segments form the boundary of \(\Pi\). This is shown in Figure 4 where the geodesics are formed by the extension of the edges of \(\Pi\) as illustrated by the dashed arcs, and the endpoints specified by the geodesic obtained from the extension of the \(i\)-th edge of \(\Pi\) are \(a_i, b_i\). By considering hyperbolic polygons with \(8g - 4\) edges as fundamental regions the proposed analytical method is not restricted, since it is well known that for any surface of genus \(g \geq 2\) there exists a fundamental polygon \(\Pi\) whose edges are the \(8g - 4\) consecutive geodesics segments \(s_1, s_2, \ldots s_{8g-4}\) with counterclockwise orientation. The side-pairing maps \(\Gamma_0\) are given by the \(\sigma(i)\) permutation of order 2 of the set \([1, \ldots, 8g - 4]\), defined by

\[
\sigma(i) = \begin{cases} 
4g - i \mod (8g - 4), & \text{if } i \text{ odd}, \\
2 - i \mod (8g - 4), & \text{if } i \text{ even},
\end{cases}
\]

such that, there exists a unique element \(h_i \in \Gamma_0\) that maps \(s_i\) to \(s_{\sigma(i)}\). This is shown in Figure 4 for \(g = 2\). Thus, the Markov partition is given by the intervals \(i_1 = [a_i, b_{i-1})\) and \(i_2 = [b_{i-1}, a_{i+1})\), and the expansion map is defined as \(f(\xi) = h_i(\xi)\) whenever \(\xi \in [a_i, a_{i+1})\), for \(1 \leq i \leq 8g - 4\). All operations are modulo \(8g - 4\), then \(i + 1 = 1\) if \(i = 8g - 4\) and \(i - 1 = 8g - 4\) if \(i = 1\).

The coding process is similar to the binary expansion of points in the unit interval. An example is shown in Figure 5, where part of \(\Pi\) is shown, and adjacent to \(\Pi\) the image of \(\Pi\) by the application of an isometry \(h_i\) on \(\Pi\), which is indicated with the label \(h_i^{-1}\Pi\). The regions \(\Pi\) and \(h_i^{-1}\Pi\) have the same hyperbolic metric properties, however under a Euclidean perspective an exponential reduction of lengths and areas are observed. This fact is used to coding the extreme points \(\xi\) (forward) and \(\eta\) (backward) of a geodesic \(\gamma\) in \(\mathbb{D}^2\). The coding process is recursive, let the region \(\Pi\) be the reference after the \(n\)-th application of the expansion map \(f\), \(n \in \mathbb{Z}\). Since the partition interval where the forward point \(\xi\) of the geodesic \(\gamma\) defined by \(\xi, \eta\) lies is \(i_2\), it follows that if \(w = \ldots w_{-1}w_0w_1w_2\ldots\) is the bi-infinite codeword of \(\gamma\), then \(w_n = i_2\). For the next step consider Figure 5, the region of the tessellation of \(\mathbb{D}^2\) defining the partition of the interval \(i_2\) in \(\mathbb{D}^2\) is \(h_i^{-1}\Pi\), due to the fact that this is the hyperbolic polygon adjacent to \(\Pi\) at the edge \(i\) determining the partition intervals \(i_1\) and \(i_2\). Now, either \(\xi\) belongs to the interval \(j_1\) or \(j_2\) or \((j + 1)\) of the partition defined by \(h_i^{-1}\Pi\) in the previous defined region \(i_2\), thus \(w_{n+1}\) is equal to \(j_1\), \(j_2\) or \((j + 1)\) depending on the interval that contains \(\xi\). The same idea follows for the
only happens when the elements \{over a hyperbolic surface of any genus \} generated by the orbits of a branched manifold obtained from a dynamical system to geodesics generate the bi-infinite codewords of the geodesics. Thus, we may encode the symbolic sequences the geodesic flow, where bi-infinite sequences obtained by following bi-infinite paths in the graph generate the bi-infinite codewords of the geodesics. Thus, we may encode the symbolic sequences generated by the orbits of a branched manifold obtained from a dynamical system to geodesics over a hyperbolic surface of any genus \(g \geq 2\) [14]. The symbolic dynamics associated with the orbits code is determined by the transition matrix \(I\) obtained from the algebraic description of the template.

4. Maximum full-shift and topological entropy

In this section we present the main results of the paper. Theorem 2 is concerned with a structural property of the geodesic code, and it establishes the maximum full-shift embedded in the geodesic-code sequences as a function of the genus of the surface associated with the hyperbolic polygon \(\Pi\) with \((8g - 4)\)-edges applied in the geodesic-coding process. Theorem 6 specifies the topological entropy of the geodesic-code sequences as a function of the genus of the surface, this sets the capacity of the set of sequences for conveying information [14].

These results are related with concepts of symbolic dynamics, so we begin with some definitions and concepts about it. Thus, the set of all possible finite sequences over an alphabet \(A\) is denoted by \(A^*\), this include the empty sequence \(\varepsilon\), such that, \(w\varepsilon = \varepsilon w = w\) for every \(w \in A^*\). The set \(A^n\) is called the free monoid of \(A\), with concatenation of words as the considered operation. Let \(F \subseteq A^n\) and \(A^\infty\) be the full-shift formed by the set of all bi-infinite sequences over \(A\). A shift-space \(X_F\) is the set of all elements in \(A^\infty\) having no factors in the forbidden set \(F\), equivalently, \(X_F = \{x \in A^\infty \mid x[i,j] \neq w \text{ for every } w \in F \text{ and } i,j \in \mathbb{Z}, i \leq j\}\) where \(x[i,j] = x_i \ldots x_j\) for \(x = \ldots x_{-1}x_0x_1\ldots \in A^\infty\). The language of a shift-space \(X_F\) is the set \(B(X_F) = \{w \in A^* \mid w = x[i,j] \text{ for some } x \in X_F\}\) and includes \(\varepsilon\).

4.1. Maximum embedded full-shift

Lemma 1. Let \(L = \{1, \ldots, k\}\), where \(k \geq 4\) and \(k\) even, and \(\sigma : L \rightarrow L\) be an arbitrary permutation. Suppose a partition \(\{\mathcal{C}, \mathcal{P}\}\) of \(L\), such that, \(\mathcal{P} = \bigcup_{i \in \mathcal{C}} \mathcal{P}(i)\) where \(\mathcal{P}(i) = \{(\sigma(i) - 1), \sigma(i), (\sigma(i) + 1)\}\). Then, the greatest value of \(|\mathcal{C}|\) is reached for \(|\mathcal{P}| = |\mathcal{C}| + 2\), and it only happens when the elements \(\{\sigma(i)\}_{i \in \mathcal{C}}\) are consecutive.

Proof: Initially, \(i, j \in L\) are consecutive if \(|i - j| \mod k| = 1\), observing that \((k - 1) \equiv -1 \mod k\) and thus \(|(k - 1) \mod k| = 1\). We have to show there exist no \(\sigma(\cdot)\) satisfying \(|\mathcal{P}| = |\mathcal{C}|\)
4.2. Topological entropy

A graph consists of a pair of finite sets $G = (V, E)$ where $V$ is the set of states and $E$ is the set of edges, such that, to each edge $e \in E$ is associated with an initial state $i(e)$ and a terminal state $t(e)$. A path $\pi$ in $G$ is a sequence of edges, that is, if $\pi = e_1 \ldots e_n$, then $t(e_i) = i(e_{i+1})$ for $1 \leq i < n$. A labeled graph is a pair $\mathcal{G} = (G, L)$ such that $G$ is an underlying graph and $L : E \rightarrow A$ is the label function where $A$ is a finite set called alphabet. A labeled graph $\mathcal{G}$ is right-resolving if for any $e_1, e_2 \in E$ such that $i(e_1) = i(e_2)$ then $L(e_1) \neq L(e_2)$. Observe that $G$ can be considered a labeled graph by taking $L$ as the identity function. Let $A_{ij}$ denote the number of edges in $G$ with initial state $i$ and terminal state $j$, for every $i, j \in V$. The adjacency matrix of $G$ is $A = [A_{ij}]$, and its construction from $G$ is denoted by $A_G$. For a labeled graph $\mathcal{G} = (G, L)$, we denote its adjacency matrix by $A_{\mathcal{G}} = A_G$. A sofic-shift $X_\mathcal{G}$ is a shift-space obtained by reading off the labels of consecutive edges in a labeled graph $\mathcal{G}$. In this case $\mathcal{G}$ is a presentation of $X_\mathcal{G}$. Our interest is on sofic-shifts known as topological Markov shifts (TMS), characterized by a forbidden set whose
Proposition 4. Let \( \mathcal{G} = (G, \mathcal{L}) \) be a right-resolving labeled graph. Then \( h(X_{\mathcal{G}}) = h(X_G) \).

Proposition 5. Let \( G \) be a graph having adjacency matrix \( A \). Then \( h(X_G) = \log \lambda_A \), where \( \lambda_A \) is the largest eigenvalue of \( A \).

Theorem 6. The topological entropy of the geodesic code sofic-shift \( X \) is given as a function of the coding region \( \Pi \) by

\[
h(X) = \log \left[ (4g - 3) + \sqrt{(4g - 3)^2 - 1} \right].
\]
Proof: Since the labeled graphs $\mathcal{G}_X$ and $\mathcal{G}_Y$ have the same underlying graph and are both right-resolving, it follows from Proposition 4 that $h(X) = h(Y)$. From Lemma 3, it follows that $h(Y) = h(Z)$. Thus, from Proposition 5 we may apply the adjacency matrix $A_{\mathcal{G}_Z}$ of $\mathcal{G}_Z$ to determine the topological entropy of $X$. Since $A_{\mathcal{G}_Z}$ has order two independent of the genus of the coding region $\Pi$ and its entries are only dependent on the genus, we obtain an algebraic expression for $h(X)$ by determining $h(Z)$. From (5) and (6), $A_{\mathcal{G}_Z}$ is given by

$$A_{\mathcal{G}_Z} = \begin{bmatrix} 1 & 1 \\ 8g - 8 & 8g - 7 \end{bmatrix}.$$ \hfill (8)

The largest eigenvalue $\lambda_{A_{\mathcal{G}_Z}}$ of $A_{\mathcal{G}_Z}$ is the largest root of the characteristic polynomial $\lambda^2 + \lambda (6 - 8g) + 1$ of $A_{\mathcal{G}_Z}$. Thus,

$$\lambda_{A_{\mathcal{G}_Z}} = \frac{(8g - 6) + \sqrt{(6 - 8g)^2 - 4}}{2} = (4g - 3) + \sqrt{(4g - 3)^2 - 1}.$$ 

Hence, $h(X) = h(Y) = \log \lambda_{A_{\mathcal{G}_Z}}$. \hfill □

5. Discussion
There exists a well defined geodesic coding process that share similarities with the UPO’s coding by defining a Poincaré section on the edges of $\Pi$ [11], and there exists an easy-coding procedure from the obtained symbol sequences and those obtained by the method described in Section 3. These geodesic codes are a two-fold analytical tool: allows an association of the UPO’s sequences with the generators of the Fuchsian group of $\Gamma_0$ from the first coding method, and a hyperbolic measure from the second. These elements are considered in usual encoding and decoding procedures, and they may also be employed in chaotic communication systems.

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References
[1] Gilmore R and Lefranc M 2002 The Topology of Chaos: Alice in Stretch and Squeezeland (New York, United States: John Wiley & Sons)
[2] Gilmore R 1998 Rev. Mod. Phys. 70 1455–529
[3] Plumeçon J and Lefranc M 2000 Physica D 144 231–58
[4] Plumeçon J and Lefranc M 2000 Physica D 144 259–78
[5] Lefranc M, Glorieux P, Papoff F, Molesti F and Arimondo E 1994 Phys. Rev. Lett. 73 1364–67
[6] Lefranc M 2007 The topology of deterministic chaos: Stretching, squeezing and linking Physics and Theoretical Computer Science: From Numbers and Languages to (Quantum) Cryptography ed Gazeau J, Nesetril J and Rovan B (Amsterdam, The Netherlands: IOS Press) pp 17–29
[7] Casetti L, Pettini M and Cohen E G D 2000 Physica D 144 231–58
[8] Firby P A and Gardiner C F 1992 Amer. Math. Soc. 25 229–334
[9] Series C 1981 Acta Math. 146 103–28
[10] Series C 1997 Inst. Hautes Études Sci. Publ. Math. 50 153–70
[11] Lind D and Marcus B H 1995 An Introduction to Symbolic Dynamics and Coding (Cambridge, United Kingdom: Cambridge University Press)