Secure Multiplex Coding with Dependent and Non-Uniform Multiple Messages

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Abstract

The secure multiplex coding (SMC) is a technique to remove rate loss in the coding for wire-tap channels and broadcast channels with confidential messages caused by the inclusion of random bits into transmitted signals. SMC replaces the random bits by other meaningful secret messages, and a collection of secret messages serves as the random bits to hide the rest of messages. In the previous researches, multiple secret messages were assumed to have independent and uniform distributions, which is difficult to be ensured in practice. We remove this restrictive assumption by a generalization of the channel resolvability technique.

We also give practical construction techniques for SMC by using an arbitrary given error-correcting code as an ingredient, and channel-universal coding of SMC. By using the same principle as the channel-universal SMC, we give coding for the broadcast channel with confidential messages universal to both channel and source distributions.

Index Terms

broadcast channel with confidential messages, information theoretic security, multiuser information theory, universal coding, the secure multiplex coding

I. INTRODUCTION

A. Overview

Recently, the security of personal information is demanded much more. The wire-tap model is a typical secure message transmission model with the presence of an eavesdropper. Specially, there are the legitimate sender called Alice, the legitimate receiver called Bob, and the eavesdropper Eve. There is also a noisy broadcast channel from Alice to Bob and Eve. Alice wants to send secret messages reliably to Bob and secretly from Eve. This problem was first formulated by Wyner [35]. Csiszár and Körner generalized Wyner’s original problem to include common messages from Alice to both Bob and Eve, and determined the optimal information rate tuples of the secret message and the common message, and the information leakage rate of the secret message to Eve, which is measured by the conditional entropy of the secret message given Eve’s received signal [9]. They called their generalized problem as the broadcast channel with confidential messages, hereafter abbreviated as BCC. The secrecy of messages over the wire-tap channel and the BCC is realized by including meaningless random variable, which is called the dummy message, into Alice’s transmitted signal. This decreases the information rate.

In order to get rid of this information rate loss, Yamamoto et al. [22] proposed the secure multiplex coding, hereafter abbreviated as SMC, as a generalization of the wire-tap channel coding. The SMC can be used, for example, in the following case. When a company treats a collection of personal information,
it is required to keep the secrecy of the respective personal information. However, it may not be required to keep the secrecy of the relation among several personal information. For example, when all of personal information are subject to the uniform distribution of the same length bit sequence, the secrecy of their exclusive OR may not be required. Consider the case when the sender Alice sends the collection of $T$ persons’ personal information $S_1, \ldots, S_T$ via the channel partially leaked to Eve. It is required that the receiver Bob can decode all of $S_1, \ldots, S_T$, and that Eve cannot obtain any information of the respective personal information. In order to keep the secrecy of the message $S_i$ from Eve, Yamamoto et al. [22] proposed to use the remaining information $S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_T$ as the dummy message for the message $S_i$. Then, they realized the secrecy of the message $S_i$ without loss of the information rate. This type of coding problem is called the SMC. It is known that the application of the channel resolvability [13] yields the security of the wire-tap channel model [15]. Hence, employing this method, Yamamoto et al. [22] proved the security of SMC.

On the other hand, since $S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_T$ are personal information, they are not necessarily uniform random bits and might be dependent, while the existing papers [27], [22] assumed their uniformity and independence. Such assumption is difficult to be ensured in practice. Unfortunately, the application of the original channel resolvability can prove the security only when the messages $S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_T$ are conditionally uniform and independent of $S_i$ because it treats the approximation of the channel output distribution with the uniform input random variable. One may consider that the compressed data satisfies that assumption so that the removal of that assumption is not needed. However, as is shown in [14], [16], the compressed data is not uniform in the sense of the variational distance nor the divergence. That is, the uniformity assumption does not hold for such compressed data. Hence, the removal of the assumption is essential for non-uniform information source.

The reader might also conceive that this problem could be solved by a straightforward combination of the coding for intrinsic randomness [33] and that for the original secure multiplex coding [22], [27]. We emphasize that this is false. We cannot recover the original secret messages from a codeword from an intrinsic randomness encoder, and a new technique must be deployed to remove the independence and uniform assumption on the multiple secret messages. One of the main contributions of this paper is to remove that assumption. In order to treat the non-uniform and dependent case, we need a generalization of the channel resolvability. Hence, this paper also studies a generalization of the channel resolvability problem [13], [15].

Even after we solve the above problem by a generalization of the channel resolvability problem, the security of $S_i$ depends on the randomness and the dependence of the remaining messages $S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_T$ on $S_i$. This dependence causes another difficulty in the asymptotic formulation of SMC. That is, we need to characterize the randomness and the dependence in the asymptotic setting. For this purpose, we introduce several kinds of asymptotic conditional uniformity conditions and study their properties. In addition to this, for the case when the channel is unknown, we also treat universal coding for the secure multiplex coding [22]. Further, as a byproduct, we obtain source-channel universal coding for the broadcast channel with confidential messages [9]. We divide the introductory section to six subsections.

### B. Generalization of the Channel Resolvability

For a given channel $W$ with input alphabet $X$ and output alphabet $Y$, and given information source $X$ on $X$, Han and Verdú [13] considered to find a coding $f: \mathcal{A} \to X$ and a random variable $A$ such that the distributions of $W(f(A))$ is close to $W(X)$ with respect to the variational distance or the normalized divergence, and evaluated the minimum resolution of $A$ to make the variational distance or the normalized divergence asymptotically zero. In their problem formulation, one can choose the randomness $A$ used to simulate the channel output distribution.

In this paper, we shall consider the situation in which we are given a channel $W$, an information source $X$, and randomness $A$ and asked to find coding $f: \mathcal{A} \to X$ such that $W(f(A))$ is as close as possible to $W(X)$ with respect to unnormalized divergence. We shall study how close $W(f(A))$ can be to $W(X)$ in
Theorems 9 and 11 in Section V. Hence, this problem can be regarded as a generalization of channel resolvability because this problem contains the original channel resolvability as a special case in the above sense.

C. Asymptotic Conditional Uniformity

In Subsection VII-A in order to characterize the randomness and the dependence of the messages $S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_T$ on the other message $S_i$ asymptotically, we introduce three asymptotic conditional uniformity conditions. Then, we can characterize what a conditional distribution of the messages $S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_T$ has a similar performance to the conditionally uniform distribution when we apply SMC. We summarize the relations among those conditions as Theorem 24. In particular, in Appendix C we show that two introduced asymptotic conditional uniformity conditions are equivalent. Hence, we essentially have two different conditional uniformity conditions, namely, the weaker and the stronger asymptotic conditional uniformity conditions.

In Subsection VII-B we give sufficient conditions for the Slepian-Wolf compression so that the compressed data satisfies these asymptotic conditional uniformity conditions. For the stationary ergodic sources, we show the existence of a sequence of Slepian-Wolf codes whose compressed data satisfies the weaker asymptotic conditional uniformity conditions (Theorem 25 and Remark 26). Also for the i.i.d. sources, we show the existence of a sequence of Slepian-Wolf codes whose compressed data satisfies the stronger asymptotic conditional uniformity conditions (Theorem 27 and Remark 28).

D. Secure Multiplex Coding

Here, we explain the detail of our contributions to SMC. As is explained above, we have to realize the security of $S_i$ when the remaining messages $S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_T$ are not uniform and are dependent on the message $S_i$. In order to solve this problem, we employ our generalized channel resolvability coding in Theorems 9 and 11. Then, we can construct coding for a wire-tap channel that can ensure the secrecy of message against the eavesdropper Eve when the dummy message used by the encoder is non-uniform and statistically dependent on the secret message that has to be kept secret from Eve. We apply our generalized channel resolvability coding to the above SMC case. Hence, we can remove the independence and uniform assumption on the multiple secret messages while the original paper [22] by Yamamoto et al. and the previous paper [27] by the present authors assumed the independence and uniformity of the multiple secret messages.

Indeed, Yamamoto et al. [22] treated only the secrecy of each message $S_i$, and did not evaluated the information leakage of multiple messages $S_{i_1}, \ldots, S_{i_n}$ to Eve, and the present authors analyzed such information leakage in [27]. The present authors also generalized coding in [27] so that Alice’s encoder can support the common message $S_0$ to both Bob and Eve. The present authors also characterized the achievable information leakage rate in [27]. Those enhancements are retained in this paper.

In Section VI we shall give two code constructions for SMC. The first one in Subsection VI-B is based on the channel resolvability coding in Theorem 11, while the second one is based on Theorem 9 in Subsection VI-C. The second construction is insufficient to prove the capacity region, but can provide a better exponential decreasing rate of the mutual information of secret messages to Eve under certain situations. By using these constructions, we shall evaluate the decoding error probability and the mutual information to Eve in Section VI in single-shot setting in the sense of [34].

In Section VIII we formulate the capacity region of SMC, analyze the asymptotic performance of two constructions, and prove that the first construction achieves the capacity region of SMC. The capacity region is defined based on the weaker asymptotic conditional uniformity condition given in Definition 31. In Section IX, we shall prove that the mutual information to Eve converges to zero when the normalized mutual information to Eve converges to zero under the stronger asymptotic conditional uniformity given in Definition 23. The convergence is so-called the strong security [28]. In Subsection IX-B we also derive the exponent of the mutual information to Eve.
In Theorem 16 of Section VI, we show that we can have an upper bound of mutual information between multiple secret messages and Eve’s received signal, by attaching randomly chosen group homomorphisms satisfying Condition 10 to any given error-correcting code for channels with single sender and single receiver or the broadcast channel with degraded message sets [23]. However, the upper bound in Theorem 16 becomes difficult to be computed when the error-correcting code is not given by the standard random coding in information theory. In Section X, we shall give two upper bounds on the mutual information that can be computed easily in practice when the construction of Section VI is combined with an arbitrary given error-correcting code. Section X gives is enhancement of our earlier proceeding paper [18].

E. Universal Coding

Universal coding is construction of encoder and decoder that do not use the statistical knowledge on the underlying information system (usually channel and/or source) [8]. In Section XI we shall give a construction of SMC universal to channel. The basic idea in Section XI is to combine the construction in Section VI with the universal coding using constant-type codes for the broadcast channel with degraded messages sets (BCD) in [24], while in Sections VII–IX the superposition random coding in [23] is used as their error-correcting mechanism. The exponent given in Section XI is better than that given in our earlier proceeding paper [18].

Channel-universal coding for BCC had not been studied before [19], and coding for BCC can be regarded as a special case of SMC while Muramatsu et al. [29] treat channel-universal coding for wire-tap channel independently of [19]. In Section XI and [19] we consider SMC universal to channel, but its universality to the source is not considered. In Section XII we give a coding for BCC universal to both channel and source. Its channel-universality is realized by the same principle as Section XI and [19]. The exponent given in Section XI is also greater than that given in our earlier proceeding paper [19].

In Section XIII we compare the exponent of leaked information given in Sections XI and XII and that given in Subsection IX-B. As a result, we show that the exponent in Sections XI and XII is greater than one of exponents in Subsection IX-B which is the same as that in [19]. We also derive the equality condition.

F. Organization of This Paper

The outline of this paper is given as follows. First, we prepare notations used in this paper in Section II. Then, we review the formulation and existing results of BCC in Subsection III-A. We give its reformulation for the dependent and non-uniform messages case in Subsection III-B. This new formulation is essential in the later discussion for SMC with dependent and non-uniform multiple messages. In Subsection IV-A, we review the formulation and existing results of BCD as a special case of BCC, which will be used for our codes of SMC. In Subsection IV-B, we review Körner and Sgarro [24]’s result for universal code for BCD, which will be used for our construction of universal codes for SMC and BCC. In Section V, we proceed to generalization of channel resolvability, which is a key idea of the paper and is used for codes of SMC and universal codes for SMC and BCC. Section VI introduces SMC with the single-shot setting. Section VII introduces three asymptotic conditional uniformity conditions. Based on these conditions, Sections VIII–X treats SMC with the asymptotic setting, as is explained in Subsection I-D. In Section XI, combining the discussion of Subsections IV-A and VI-D, we propose universal coding for SMC by using Körner and Sgarro [24]’s universal coding for BCD. In Section XII, we propose source-channel universal coding for BCC. Appendices are devoted for several additionally required discussions for asymptotic conditional uniformity conditions. This paper contains two types of descriptions for each topics, i.e., the single-shot description [34] and the n-fold description. Formulations and many coding theorems are given with the single-shot decrition. The definitions of capacity regions are given in the n-fold description.
II. Notation in This Paper

$X$ denotes the channel input alphabet and $Y$ (resp. $Z$) denotes the channel output alphabet to Bob (resp. Eve). We assume that $X$, $Y$, and $Z$ are finite unless otherwise stated. We denote the conditional probability of the channel to Bob (resp. Eve) by $P_{Y|X}$ (resp. $P_{Z|X}$). Also, we denote the distribution of the random variable $X$ by $P_X$.

We denote the uniform distribution on $\Omega \subset X$ by $P_{\text{mix},\Omega}$. When $\Omega$ is a subset of $X \times Y$, $P_{\text{mix},\Omega}$ is a joint distribution for the random variables $X$ and $Y$. We denote the marginal distribution of $P_{\text{mix},\Omega}$ for the random variable $X$ and the random variable $Y$ by $P_{X,\text{mix},\Omega}$ and $P_{Y,\text{mix},\Omega}$, respectively. Further, the conditional distribution on the random variable $X$ conditioned to the other random variable $Y$ is denoted by $P_{X|Y,\text{mix},\Omega}$, i.e.,

$$P_{X|Y,\text{mix},\Omega}(x|y) = P_{X|Y=y,\text{mix},\Omega}(x) = \frac{P_{\text{mix},\Omega}(x,y)}{P_{Y,\text{mix},\Omega}(y)}$$

for $x \in X$ and $y \in Y$. We denote the support of the distribution $P_X$ by $\text{supp}(P_X)$. Given a joint distribution $P_{XY}$, we define the distribution $P_{X|Y=x}$ on $X$ by $P_{X|Y=x}(x) := P_{X|Y=x}(y)$. If we need to treat another distribution of the same random variables $X$ and $Y$, we denote it by $Q_{XY}$. In this case, we denote the marginal distribution over $X$ by $Q_X$, and the conditional distribution by $Q_{X|Y}$. We also define the distribution $Q_{X|Y=y}$ on $X$ by $Q_{X|Y=y}(x) := Q_{X|Y=y}(x)$. When we have to treat more than two distributions on $X$, $Y$, and $Z$, the above notation is not useful. In this case, we consider the set $P(X)$ of probability distributions on $X$ or the set $W(X, Y)$ of conditional probability distributions from $X$ to $Y$, which are mathematically equivalent to probability transition matrices. When the output alphabet of the channel is given as a product set $Y \times Z$, the alphabet is written by $W(X, Y \times Z)$. For any probability transition matrix $W \in W(X, Y \times Z)$, $W_x$ expresses the output distribution when the input $X$ is $x$. When we focus on the random variable $Y$, we use the notation $W_{Y|x} := \sum_{z \in Z} W_x(y, z)$.

In the following, we treat an arbitrary probability transition matrix $W \in W(X, Y)$. Given a subset $\Omega \subset X$, we define the restriction $W|_{\Omega} \in W(\Omega, Y)$ by $W|_{\Omega}(y|x) = W(y|x)$ for $x \in \Omega$ and $y \in Y$. We often employ another probability transition matrix $\Xi$ from $V$ to $X$. We define the probability transition matrix from $V$ to $Y$ by $W \circ \Xi(y) := \sum_{x \in X} W_x(y) \Xi_x(x)$ for $v \in V$ and $y \in Y$. When a probability distribution $P$ on $X$ is given, we define the distribution on $Y$ by $W \circ P(y) := \sum_{x \in X} W_x(y) P(x)$ for $y \in Y$. When we need the joint distribution on $X \times Y$, we use the notation $W \times P(x,y) := W_x(y) P(x)$ for $x \in X$ and $y \in Y$ as $[c]$. Similarly, when a distribution $P_{XY}$ on $X \times Y$ is given, we use the notation $W \times P_{XY}(v,x,y) := W_x(y) P_{XY}(v,x,y)$ for $v \in V$, $x \in X$, and $y \in Y$.

When a function $f : V \rightarrow X$ is given and a random variable $V$ taking the values in $V$ obeys the distribution $P_V$, we can define the random variable $f(V)$ taking the values in $X$. The random variable $f(V)$ takes the value $x$ with probability $\sum_{v \in f^{-1}(x)} P_V(v)$. We also use the same symbol $f : V \rightarrow X$ to denote the probability transition matrix from $V$ to $X$, in which, the output value is deterministically determined by the input. Then, $W \circ f$ is a stochastic mapping $V$ onto $Y$, and we have

$$(W \circ f)(y|v) = W(y|f(v))$$

for $v \in V$ and $y \in Y$. Given a probability transition matrix $W' \in W(U, V)$, we define $f \circ W' \in W(U, X)$ by

$$(f \circ W')(x|u) := \sum_{v \in f^{-1}(x)} W'(v|u)$$

for $x \in X$ and $u \in U$. As a special case, given a distribution $Q$ on $V$, $f \circ Q$ is defined as a distribution on $X$ in the following way.

$$(f \circ Q)(x) := \sum_{v \in f^{-1}(x)} Q(v).$$
Remember that \( W_x \) denotes the output distribution on the output alphabet \( Y \) with input \( x \). Then, \( W_x \) is the random variable taking its values on the output distributions on \( Y \). Given a real valued function \( g \) of distributions on \( Y \), we regard \( g(W_x) \) as a random variable taking the value \( g(W_x) \) with the probability \( P_X(x) \). Hence, we obtain

\[
E_X g(W_X) = \sum_x P_X(x) g(W_x),
\]

where \( E_X \) denotes the expectation concerning \( X \).

Given two random variables \( X \) and \( Y \), for a real valued function \( h \) on \( X \times Y \), we regard \( E_{X|Y} h(X, Y) \) as a random variable taking the value \( E_{X|Y} h(X, y) \) with the probability \( P_Y(y) \). In order to identify an information quantity, e.g., mutual information \( I(X; Y) \) and the Shannon entropy \( H(X) \), we sometimes need to specify the distribution \( P \) of interest. In such a case, we use the notations \( I(X; Y)[P] \) and \( H(X)[P] \) for identifying what distribution is considered.

Further, in this paper, we discuss our codes and their performances in the single-shot setting when their descriptions do not require their asymptotic discussions. However, in several parts, we need to treat \( n \)-fold memoryless extensions when we discuss their asymptotic performances. Hence, we need to prepare the notations for \( n \)-fold independent and identical distributions and \( n \)-fold memoryless extensions of given channels. For a given probability distributions \( Q \) and \( P_X \) of the random variable \( X \) on \( X \), we denote their \( n \)-fold independent and identical distributions by \( Q^n \) and \( P_X^n \).

When we consider the random variables on \( X^n \), even if they do not obey the independent and identical distributions, we denote the random variables by \( X^n \) and denote their distributions by \( P_{X^n} \). However, when we consider a general sequence of random variables those take values not in the product sets \( X^n \) but in general sets \( X_n \), we denote the random variables by \( X_n \) and denote their distributions by \( P_{X_n} \). Similarly, for a given probability transition matrices \( W \) and \( P_{Y|X} \) from \( X \) to \( Y \), we denote their \( n \)-fold memoryless extensions by \( W^n \) and \( P_{Y|X}^n \).

We also denote the set of positive real numbers by \( \mathbf{R}^+ \), and denote the set of non-negative real numbers by \( \mathbf{R}_{\geq 0} \).

### III. Broadcast Channels with Confidential Messages

#### A. Review of Existing Results

First, we give a formulation of broadcast channels with confidential messages with single shot setting. Let Alice, Bob, and Eve be as defined in Section I. \( X \) denotes the channel input alphabet and \( Y \) (resp. \( Z \)) denotes the channel output alphabet to Bob (resp. Eve). We assume that \( X, Y, \) and \( Z \) are finite unless otherwise stated.

We denote the conditional probability of the channel to Bob (resp. Eve) by \( P_{Y|X} \) (resp. \( P_{Z|X} \)). The purpose of broadcast channels with confidential messages is the following. (1) Alice reliably sends the common message \( E \) to Bob and Eve. (2) Alice confidentially and reliably sends the secret message \( S \) to Bob. Here, we denote the sets of the common messages and the secret messages by \( E \) and \( S \). Our code is given by Alice’s stochastic encoder \( \varphi_a \) from \( S \times E \) to \( X \). Bob’s deterministic decoder \( \varphi_b : Y \to S \times E \) and Eve’s deterministic decoder \( \varphi_e : Z \to \hat{E} \). The triple \( \varphi = (\varphi_a, \varphi_b, \varphi_e) \) is called a code for broadcast channels with confidential messages. Then, when the common message \( E \) and the secret message \( S \) obey the distribution \( P_{S,E} \), the performance is evaluated by the following quantities. (1) The sizes of the sets of the common messages and the secret messages, i.e., \( |E| \) and \( |S| \). (2) Bob’s decoding error probability \( P_{b}[P_{Y|X}, \varphi, P_{S,E}] \), which is the probability \( \Pr[(S, E) \neq (\hat{S}, \hat{E})] \) under the distribution \( (P_{Y|X} \circ \varphi_a, P_{S,E}) \). (3) Eve’s decoding error probability \( P_e[P_{Y|X}, \varphi, P_{S,E}] \), which is the probability \( \Pr[E \neq \varphi_e(Z)] \) under the distribution \( (P_{Z|X} \circ \varphi_a) \times P_{S,E} \). (4) Eve’s uncertainty \( H(S|Z)[P_{Z|X}, \varphi_a, P_{S,E}] \), which is the conditional entropy \( H(S|Z) \) under the distribution \( (P_{Z|X} \circ \varphi_a) \times P_{S,E} \). Since these quantities are functions of the channel and the code, such dependencies are denoted by the symbol \( [P_{Y|X}, \varphi, P_{S,E}] \) in the above notation. Instead of \( H(S|Z)[P_{Z|X}, \varphi_a, P_{S,E}] \), we sometimes treat (5) leaked information \( I(S; Z)[P_{Z|X}, \varphi_a, P_{S,E}] \), which is the mutual information \( I(S; Z) \) under the distribution \( (P_{Z|X} \circ \varphi_a) \times P_{S,E} \).
We sometimes need to evaluate the error probability when $S$ and/or $E$ is fixed. In such a case, we denote it by $P_b[P_{Y|X}, \varphi, P_{S\pi=S_e}]$, $P_b[P_{Y|X}, \varphi, S = s, E = e]$, and $P_e[P_{Y|X}, \varphi, P_{S\pi=E=e}]$.

Now, we review the asymptotic formulation of broadcast channels with confidential messages with the $n$-fold discrete memoryless extension when both of the common messages and the secret messages are subject to uniform distributions. The set $S_n$ denotes the set of the confidential message and $E_n$ does the set of the common message when the block coding of length $n$ is used. We shall define the achievability of a rate triple $(R_1, R_e, R_0)$. For the notational convenience, we fix the base of logarithm, including one used in entropy and mutual information, to the base of natural logarithm.

**Definition 1**: [9] The rate triple $(R_1, R_e, R_0)$ is said to be achievable for the information leakage rate criterion if the following conditions hold. The size of the sets of the common and confidential messages are $|E_n| = e^{nR_0}$ and $|S_n| = e^{nR_1}$. The common and confidential messages are subject to the uniform and independent distribution on $S_n$ and $E_n$. There exists a sequence of the codes $\varphi_n = (\varphi_{a,n}, \varphi_{b,n}, \varphi_{c,n})$, i.e., Alice’s stochastic encoder $\varphi_{a,n}$ from $S_n \times E_n$ to $X^n$, Bob’s deterministic decoder $\varphi_{b,n} : Y^n \rightarrow S_n \times E_n$ and Eve’s deterministic decoder $\varphi_{c,n} : Z^n \rightarrow E_n$ such that

$$\lim_{n \to \infty} P_b[P_{Y|X}, \varphi_n, P_{mix,S_n,E_n}] = 0$$

$$\lim_{n \to \infty} P_e[P_{Z|X}, \varphi_n, P_{mix,S_n,E_n}] = 0$$

$$\lim_{n \to \infty} \frac{H(S_n|Z^n)[P_{Y|X}, \varphi_{a,n}, P_{mix,S_n,E_n}]}{n} \geq R_e.$$

The capacity region with the information leakage rate criterion of the BCC is the closure of the achievable rate triples for the information leakage rate criterion.

**Theorem 2**: [9] The capacity region with the information leakage rate criterion of the BCC is given by the set of $R_0, R_1$ and $R_e$ such that there exists a Markov chain $U \rightarrow V \rightarrow X \rightarrow YZ$ and

$$R_1 + R_0 \leq I(V; Y|U) + \min[I(U; Y), I(U; Z)],$$

$$R_0 \leq \min[I(U; Y), I(U; Z)],$$

$$R_e \leq I(V; Y|U) - I(V; Z|U),$$

$$R_e \leq R_1.$$

As described in [25], $U$ can be regarded as the common message, $V$ the combination of the common and the confidential messages, and $X$ the transmitted signal.

In this paper, we treat the source-channel universal coding for BCC, in which, we guarantee the security independently of the choice of the source distribution. While the lower bound of the above conditional entropy $H(S_n|Z^n)[P_{Y|X}, \varphi_{a,n}, P_{S_n,E_n}]$ depends on the the source distribution $P_{S_n,E_n}$, we can find an upper bound of mutual information that does not depend on the source distribution, as is shown in Section XII. As a preparation for the above source-channel universal coding for BCC, we propose another type of capacity region for the uniform and independent distributed case while the non-uniform and dependent case will be treated latter.

**Definition 3**: The rate triple $(R_1, R_e, R_0)$ is said to be achievable for the leaked information criterion if the following conditions hold. In this notation, $R_1, R_e, R_0$ denote the rates of the confidential message, the leaked information, and the common message, respectively. The size of the sets of the common and confidential messages are $|E_n| = e^{nR_0}$ and $|S_n| = e^{nR_1}$, and the common and confidential messages are subject to the uniform and independent distribution on $S_n$ and $E_n$. There exists a sequence of the codes $\varphi_n = (\varphi_{a,n}, \varphi_{b,n}, \varphi_{c,n})$, i.e., Alice’s stochastic encoder $\varphi_{a,n}$ from $S_n \times E_n$ to $X^n$, Bob’s deterministic decoder
\( \varphi_{b,n} : Y^n \rightarrow S_n \times \mathcal{E}_n \) and Eve’s deterministic decoder \( \varphi_{e,n} : Z^n \rightarrow \mathcal{E}_n \) such that
\[
\lim_{n \to \infty} P_b[\{P_{Y|X}^n, \varphi_n, P_{\text{mix},S,\mathcal{E}_n}\}] = 0
\]
\[
\lim_{n \to \infty} P_e[\{P_{Z|X}^n, \varphi_n, P_{\text{mix},S,\mathcal{E}_n}\}] = 0
\]
\[
\limsup_{n \to \infty} \frac{I(S^n; Z^n)[\{P_{Y|X}^n, \varphi_{a,n}, P_{\text{mix},S,\mathcal{E}_n}\}]}{n} \leq R_l.
\]

The capacity region with the leaked information criterion of the BCC is the closure of the achievable rate triples.

The capacity region with the leaked information criterion of the BCC is characterized as a corollary of Theorem 2.

**Corollary 4:** The capacity region with the leaked information criterion of the BCC is given by the set of \( R_0, R_1 \) and \( R_l \), such that there exists a Markov chain \( U \rightarrow V \rightarrow X \rightarrow YZ \) and

\[
R_1 + R_0 \leq I(V; Y|U) + \min[I(U; Y), I(U; Z)],
\]
\[
R_0 \leq \min[I(U; Y), I(U; Z)],
\]
\[
R_l \geq R_1 - [I(V; Y|U) - I(V; Z|U)]_+.
\]

That is, when \( R_1 + R_0 < I(V; Y|U) + \min[I(U; Y), I(U; Z)] \) and \( R_0 < \min[I(U; Y), I(U; Z)] \), there exists a sequence of the codes \( \varphi_n = (\varphi_{a,n}, \varphi_{b,n}, \varphi_{e,n}) \), i.e., Alice’s stochastic encoder \( \varphi_{a,n} \) from \( S_n \times \mathcal{E}_n \) to \( X^n \), Bob’s deterministic decoder \( \varphi_{b,n} : Y^n \rightarrow S_n \times \mathcal{E}_n \) and Eve’s deterministic decoder \( \varphi_{e,n} : Z^n \rightarrow \mathcal{E}_n \) such that
\[
\lim_{n \to \infty} P_b[\{P_{Y|X}^n, \varphi_n, P_{\text{mix},S,\mathcal{E}_n}\}] = 0
\]
\[
\lim_{n \to \infty} P_e[\{P_{Z|X}^n, \varphi_n, P_{\text{mix},S,\mathcal{E}_n}\}] = 0
\]

and
\[
\limsup_{n \to \infty} \frac{I(S^n; Z^n)[\{P_{Y|X}^n, \varphi_{a,n}, P_{\text{mix},S,\mathcal{E}_n}\}]}{n} \leq R_1 - I[(V; Y|U) - I(V; Z|U)]_+.
\]

**B. Our Approach to BCC**

Next, we consider the BCC with the single-shot setting when the common and confidential messages do not obey the uniform and independent distributions on \( S \) and \( E \), i.e., the confidential message \( S \) may have a correlation with the common messages \( E \). When the confidential message \( S \) is independent of the common messages \( E \),
\[
I(S; Z) \leq I(S; ZE) = I(S; Z|E) + I(S; E) = I(S; Z|E),
\]
\[
I(S; Z) = H(S) - H(S|Z) \geq H(S|E) - H(S|Z)
\]
\[
= H(S|E) - (H(S|ZE) + I(S; E|Z)) = I(S; Z|E) - I(S; E|Z)
\]
\[
\geq I(S; Z|E) - H(E|Z) \geq I(S; Z|E) - H(E|\varphi_e(Z)).
\]

When the error probability goes to zero, Fano’s inequality guarantees that \( H(E|Z) \) goes to zero. Hence, \( I(S; Z) \) and \( I(S; Z|E) \) have the same asymptotic behaviors. So, even if we replace \( I(S; Z) \) by \( I(S; Z|E) \) in Definition 3, we obtain the same capacity region. However, when the confidential message \( S \) is dependent on the common messages \( E \), \( I(S; Z) \) and \( I(S; Z|E) \) have the different asymptotic behavior as follows. Since
\[
I(S; Z) = I(S; ZE) - I(S; E|Z)
\]
\[
\geq I(S; E) - H(E|Z) \geq I(S; E) - H(E|\varphi_e(Z)),
\]
$I(S;Z)$ is asymptotically lower bounded by $I(S;E)$ when the error probability goes to zero. That is, when the mutual information $I(S;E)$ is positive, the mutual information $I(S;Z)$ cannot go to zero because Eve can infer the secret message from the common message. Thus, it is not suitable to treat the mutual information $I(S;Z)$ as leaked information from $Z$. Hence, we adopt the conditional mutual information $I(S;Z|E)$ as leaked information from $Z$.

Remark 5: Csiszár and Körner [9] treated BCC with non-uniform information source. However, their formulation was different from our formulation in the following point. In their formulation, they fixed a correlated non-uniform distribution $P_{S,E}$ on $S \times E$ and assumed that the information source $S_n$ and $E_n$ obey its $n$-fold independent and identical distribution $P_{S,E}$. In addition to this, their code depends on the distribution $P_{S,E}$. However, in our formulation, we do not assume the independent and identical distributed condition for the distribution $P_{S_n,E_n}$ of the information source $S_n$ and $E_n$. This is because information source is not given as an independent and identical distribution or known, in general. Hence, we study a universal code independent of the distribution $P_{S_n,E_n}$ of sources in Section XII. Thus, our code is useful for a realistic case.

IV. BROADCAST CHANNELS WITH DEGRADED MESSAGE SETS

A. Capacity Region

Next, we review the broadcast channel with degraded message sets (abbreviated as BCD) considered by Körner and Marton [23] in the single-shot setting. If we set $R_e = 0$ in the BCC, the secrecy requirement is removed from BCC, and the coding problem is equivalent to BCD. In this problem, we treat the private message $S_p$ taking values in $S_p$ and the common message $S_c$ taking values in $S_c$.

Corollary 6: [23] The capacity region of the BCD is given by the pair of the rate $R_c$ of common message and the rate $R_p$ of private message such that there exists a Markov chain $U \rightarrow V = X \rightarrow YZ$ and

$$R_c \leq \min[I(U;Y), I(U;Z)],$$

$$R_c + R_p \leq I(V;Y|U) + \min[I(U;Y), I(U;Z)].$$

Note that the statement of our Corollary 6 is the same as [9, Corollary 5] and different from [23]. However, as is stated in [9, Remark 5], the equivalence between the two statements can be easily shown by some algebra.

Here, we only consider a sequence of codes that achieves the rate pair $(R_c, R_p)$ satisfying

$$R_c < \min[I(U;Y), I(U;Z)], \quad R_p < I(V;Y|U).$$

(5)

For a given Markov chain $U \rightarrow V = X \rightarrow YZ$, we construct an ensemble of codes by the following random coding with the single-shot setting, which is mathematically equivalent to the construction by Kaspi and Merhav [21].

Code Ensemble 1 (Kaspi and Merhav [21, Section III]): For an arbitrary element $s_c \in S_c$, $\Phi_c(s_c)$ is the random variable taking values in $U$ and is subject to the distribution $P_U$, and is independent of $\Phi_c(s_c')$ with $s_c' \neq s_c \in S_c$. For an arbitrary element $s_p \in S_p$, $\Phi_p(s_c, s_p)$ is the random variable taking values in $V$, is independent of $\Phi_p(s_c', s_p')$ with $s_c' \neq s_c$, and depends on the random variable $\Phi_c(s_c)$. Under the condition $\Phi_c(s_c) = u$, the random variable $\Phi_p(s_c, s_p)$ is subject to the distribution $P_{V|U=u}$ and is conditionally independent of $\Phi_p(s_c, s_p')$ with $s_p' \neq s_p$. Bob’s decoder $\Phi_b$ and Eve’s decoder $\Phi_e$ are defined as the maximum likelihood decoders. The quartet $(\Phi_p, \Phi_c, \Phi_b, \Phi_e)$ is abbreviated as $\Phi$. 
In order to describe upper bounds of error probabilities, in terms of given conditional distributions \( P_{Z|L} \), \( P_{Z|V} \), \( P_{V|U} \) and distributions \( P_L \), \( P_U \), we introduce the following functions as in [17].

\[
E_0(\rho|P_{Z|L}, P_L) := \log \sum_z \left( \sum_{\ell} P_L(\ell)(P_{Z|L}(z|\ell)^{1/(1-\rho)}) \right)^{1-\rho}, \quad (6)
\]

\[
E_0(\rho|P_{Z|V}, P_{V|U}, P_U) := \log \sum_u \sum_z P_{U}(u) \sum_v \left( \sum_y P_{V|U}(v|u)(P_{Z|V}(z|v)^{1/(1-\rho)}) \right)^{1-\rho}.
\]

Observe that \( E_0 \) is essentially Gallager’s function \( E_0 \) [12].

**Lemma 7:** [21 Theorem 1 and Section IV] The above ensemble of codes \( \Phi \) satisfies the following inequalities.

\[
E_0 P_b[P_{Y|V}, \Phi] \leq |S_p|^\rho e^{E_0(-\rho P_{Y|V}, P_{V|U}, P_U)}
\]

\[
+ (|S_p||S_p|)^\rho e^{E_0(-\rho P_{Y|U}, P_{V|U})}
\]

\[
E_0 P_c[P_{Z|V}, \Phi] \leq |S_c|^\rho e^{E_0(-\rho P_{Z|U}, P_U)}.
\]

Here, we should remark that Inequalities (7) and (8) hold for any distribution over the messages because the proof by [21] does not make any assumption for the distribution over the messages.

Due to Lemma 7, Markov inequality guarantees that

\[
\Pr \Omega_1 < \frac{1}{2}, \quad \Pr \Omega_2 < \frac{1}{2}
\]

\[
\Omega_1 := \left\{ P_b[P_{Y|V}, \Phi, P_{\text{mix}, S_p, S_c}] > 2|S_p|^\rho e^{E_0(-\rho P_{Y|V}, P_{V|U}, P_U)} + 2(|S_c||S_p|)^\rho e^{E_0(-\rho P_{Y|U}, P_{V|U})} \right\}
\]

\[
\Omega_2 := \{ P_c[P_{Z|V}, \varphi, P_{\text{mix}, S_p, S_c}] > 2|S_c|^\rho e^{E_0(-\rho P_{Z|U}, P_U)} \}.
\]

Since \( \Pr(\Omega_1 \cup \Omega_2) < 1 \), we have \( \Pr(\Omega_1^c \cap \Omega_2^c) > 0 \). That is, for an arbitrary distribution \( P_{S_p, S_c} \) over the messages, there exists a code \( \varphi \) such that

\[
P_b[P_{Y|V}, \varphi, P_{S_p, S_c}] \leq 2|S_p|^\rho e^{E_0(-\rho P_{Y|V}, P_{V|U}, P_U)} + 2(|S_c||S_p|)^\rho e^{E_0(-\rho P_{Y|U}, P_{V|U})}
\]

\[
P_c[P_{Z|V}, \varphi, P_{S_p, S_c}] \leq 2|S_c|^\rho e^{E_0(-\rho P_{Z|U}, P_U)}.
\]

Now, we apply the above inequalities to the \( n \)-fold discrete memoryless extension. Then, for an arbitrary distribution \( P_{S_{p,n}, S_{c,n}} \) over the messages, there exists a sequence of codes \( \varphi_n \) with the rate of common message \( R_c \) and the rate of private message \( R_p \) of length \( n \) such that

\[
P_b[P_{Y|V}, \varphi_n, P_{S_{p,n}, S_{c,n}}] \leq 2e^{n(R_p + E_0(-\rho P_{Y|V}, P_{V|U}, P_U))} + 2e^{n(R_p + R_c + E_0(-\rho P_{Y|U}, P_{V|U}))}
\]

\[
P_c[P_{Z|V}, \varphi_n, P_{S_{p,n}, S_{c,n}}] \leq 2e^{n(R_c + E_0(-\rho P_{Z|U}, P_U))}.
\]

The above values go to zero under the condition (5), because the condition (5) guarantees that both exponents are positive with sufficiently small \( \rho > 0 \).

Indeed, Kaspi and Merhav [21] derived a better bound than (8) by employing four parameters even in the single-shot setting. The bound (8) can be seen as a special case of Kaspi and Merhav [21]’s bound. Since the bound (8) can derive the capacity region of SMC, we only use the bound (8) for simplicity.
B. Universal Code for BCD

Körner and Sgarro [24] provided the code that attains the above rate region universally for source and channel in the following sense. In order to describe their result, we introduce the following exponents for $W_Y \in \mathcal{W}(\mathcal{V}, \mathcal{Y})$ and $W_Z \in \mathcal{W}(\mathcal{V}, \mathcal{Z})$

$$\hat{E}^b(R_p, R_c, W_Y \times Q_{VU}) := \min_{\hat{W}_Y \in \mathcal{W}(U \times V, Y)} D(\hat{W}_Y || W_Y | Q_{VU}) + \hat{E}^b(R_p, R_c, \hat{W}_Y \times Q_{VU})$$

(13)

$$\hat{E}^e(R_c, W_Z \times Q_U) := \min_{\hat{W}_Z \in \mathcal{W}(U \times V, Z)} D(\hat{W}_Z || W_Z | Q_{VU}) + [I(U; Z][\hat{W}_Z \times Q_{VU}] - R_c]_+$$

(14)

where

$$D(\hat{W}_Y || W_Y | Q_{VU}) := \sum_{u,v} Q_{VU}(u,v)D(\hat{W}_{Yu,v} || W_{Yu,v})$$

$$\hat{E}^b(R_p, R_c, \hat{W}_Y \times Q_{VU}) := \min([I(VU; Y][\hat{W}_Y \times Q_{U,V}] - R_p - R_c]_+,$$

$$[I(V; Y|U][\hat{W}_Y \times Q_{U,V}] - R_p]_+$$.

(15)

In the above definition, $W_Y$ and $W_Z$ are treated as elements of $\mathcal{W}(U \times V, Y)$ and $\mathcal{W}(U \times V, Z)$, respectively. Then, Körner and Sgarro [24] provided the code that attains the above exponent universally for channel and source in the following way.

Theorem 8: [24] For an arbitrary real number $\epsilon > 0$, there exists an integer $N$ satisfying the following. For an arbitrary integer $n \geq N$, a given joint type $Q_{VU}$ of length $n$ on the sets $\mathcal{V} \times \mathcal{U}$, and rates $R_p$ and $R_c$, there exists a code $\varphi_n$ with the rates $R_p$ and $R_c$ such that

$$P[p^m, \varphi_n, S_{p,n} = s_{p,n}, S_{c,n} = s_{c,n}] \leq \exp(-n[\hat{E}^b(R_p, R_c, W_Y \times Q_{U,V}) - \epsilon]),$$

(16)

$$P[e^m, \varphi_n, S_{p,n} = s_{p,n}, S_{c,n} = s_{c,n}] \leq \exp(-n[\hat{E}^e(R_c, W_Z \times Q_{U,V}) - \epsilon])$$

(17)

for any $s_{p,n} \in S_{p,n}$, $s_{c,n} \in S_{c,n}$ and any $W \in \mathcal{W}(\mathcal{V}, \mathcal{Y} \times \mathcal{Z})$.

V. General Channel Resolovability

In the wire-tap channel model, when the dummy message obeys the uniform distribution, channel resolovability [13] can be used for guaranteeing the security [15]. In this paper, we consider the security of SMC with non-uniform and dependent secret messages. For the analysis of this case, we have to consider the secrecy when the dummy message does not necessarily obey the uniform distribution. Hence, the security evaluation [15] based on the original channel resolovability cannot be extended to the security of SMC with non-uniform and dependent secret messages. Thus, we need a generalization of channel resolovability. In this section, we propose a generalization of channel resolovability in the single-shot setting.

First, we fix a channel $W$ from the alphabet $X$ to the alphabet $\mathcal{Y}$. For a fixed distribution $P_X$ on $X$, we focus on an encoder $\Lambda$ from the message set $\mathcal{A}$ to the alphabet $X$. The purpose of the encoder $\Lambda$ is approximation of the average output distribution $W \circ P_X$ by the output distribution with input $\Lambda(A)$. The original channel resolovability [13] treats the minimum asymptotic rate of $|\mathcal{A}|$ such that the output distribution $W \circ \Lambda \circ P_{\text{mix,}A}$ can approximate the average output distribution $W \circ P_X$ with a suitable choice of $\Lambda$ in the sense that the variational distance goes to zero. In the single-shot setting, the problem can be converted to the following way: How well the given average output distribution $W \circ P_X$ can be approximated by the output distribution $W \circ \Lambda \circ P_{\text{mix,}A}$ when the cardinality $|\mathcal{A}|$ is less than a given amount. In this
paper, we consider this approximation problem when the message $A$ does not obey the uniform distribution $P_{\text{mix,}\mathcal{A}}$. Since our problem can be regarded as a generalization of channel resolvability, it is called general channel resolvability, which is essential for the secure multiplex coding with common messages with dependent and non-uniform secret messages.

Now, we apply the random coding on the alphabet $A$ with the probability distribution $P_X$. For an arbitrary $a \in \mathcal{A}$, $\Lambda(a)$ is the random variable subject to the distribution $P_X$ on $\mathcal{X}$. For $a \neq a' \in \mathcal{A}$, $\Lambda(a)$ is independent of $\Lambda(a')$. Then, the random encoder $\Lambda := \{\Lambda(a)\}_{a \in \mathcal{A}}$ gives the map from $\mathcal{A}$ to $\mathcal{X}$ as $a \mapsto \Lambda(a)$. For distributions $P_A$ on $\mathcal{A}$ and $P_{AB}$ on $\mathcal{A} \times \mathcal{B}$, we can define Rényi entropy and conditional Rényi entropy

\[
H_{1+p}(A) := -\frac{1}{p} \log \sum_a P_A(a)^{1+p},
\]

\[
H_{1+p}(A|B) := -\frac{1}{p} \log \sum_{a,b} P_B(b)P_{A|B=a}(a)^{1+p}.
\]

$H_1(A)$ and $H_1(A|B)$ are defined to be $H(A)$ and $H(A|B)$. We also define the functions [17]

\[
\psi(\rho\|Q) := \log \sum_a Q(a)^{1+p}P(a)^{-\rho},
\]

\[
\psi(\rho|W, P_X) := \log \sum_x P_X(x)e^{\rho (|W\circ P_X)}.
\]

Now, we prepare several important properties of the above quantities: Since $\rho \mapsto \rho H_{1+p}(A)$, $\rho \mapsto \rho H_{1+p}(A|B)$ are concave and $0 H_1(A) = 0 H_{1+0}(A|B) = 0$, we have

\[
H_{1+p'}(A) \leq H_{1+p}(A), \quad H_{1+p'}(A|B) \leq H_{1+p}(A|B)
\]

for $0 \leq \rho \leq \rho'$. Similarly, as is shown in [17], the function $\psi(\rho\|P)$ satisfies the following properties:

1. $\rho \mapsto \psi(\rho\|P)$ is convex.
2. $\psi(0\|P) = 0$.
3. $\frac{d}{d\rho} \psi(\rho\|P)|_{\rho=0} = D(\rho\|P)$. Hence,

\[
D(\rho\|P) = \lim_{\rho \to 0} \frac{\psi(\rho\|P)}{\rho} \leq \frac{\psi(\rho\|P)}{\rho} \leq \frac{\psi(\rho'\|P)}{\rho'}
\]

for $0 < \rho \leq \rho'$.

Then, we have the following theorem:

**Theorem 9 (General channel resolvability):** For $\rho \in (0, 1]$, we have

\[
\mathbf{E}_\Lambda e^{\rho D(W\circ \Lambda \circ P_A\|W \circ P_X)} \leq \mathbf{E}_\Lambda e^{\rho (|W\circ P_X)} \leq 1 + e^{-\rho H_{1+p}(A)} e^{\rho (|W \circ P_X)}.
\]

By applying Jensen inequality to the function $x \mapsto e^x$, Theorem 9 yields

\[
\mathbf{E}_\Lambda D(W \circ \Lambda \circ P_A\|W \circ P_X) \leq \frac{1}{\rho} \log \mathbf{E}_\Lambda e^{\rho D(W\circ \Lambda \circ P_A\|W \circ P_X)}
\]

\[
\leq \frac{1}{\rho} \log (1 + e^{-\rho H_{1+p}(A)} e^{\rho (|W \circ P_X)}),
\]

which is non-uniform generalization of [15 Lemma 2]. This theorem will be used for the proof of Theorem 17.

**Proof:** Due to (20), we have

\[
\rho D(W \circ \Lambda \circ P_A\|W \circ P_X) \leq \psi(\rho\|W \circ \Lambda \circ P_A\|W \circ P_X).
\]
The average of $e^{\phi(\rho|W_X \circ P_A|W_X P_X)}$ is evaluated as

$$E_A e^{\phi(\rho|W_X \circ P_A|W_X P_X)}$$

$$= E_A \sum_y \left( \sum_a P_A(a) W_{\Lambda(a)}(y) \right) 1 + \rho (W \circ P_X)(y)^{-\rho}$$

$$= E_A \sum_y \left( \sum_a P_A(a) W_{\Lambda(a)}(y) \right) 1 + \sum_{a' \neq a} P_A(a') W_{\Lambda(a')}(y)^{\rho} (W \circ P_X)(y)^{-\rho}$$

$$\leq \sum_y \sum_a \left( E_{\Lambda(a)} P_A(a) W_{\Lambda(a)}(y) \right) 1 + \sum_{a' \neq a} P_A(a') W_{\Lambda(a')}(y)^{\rho} (W \circ P_X)(y)^{-\rho}$$

$$= \sum_y \sum_a \left( E_{\Lambda(a)} P_A(a) W_{\Lambda(a)}(y) \right) 1 + \sum_{a' \neq a} P_A(a') W_{\Lambda(a')}(y)^{\rho} (W \circ P_X)(y)^{-\rho}$$

$$= 1 + \sum_y \sum_a E_{\Lambda(a)} P_A(a) 1 + \rho W_{\Lambda(a)}(y)^{1 + \rho} (W \circ P_X)(y)^{-\rho}$$

$$= 1 + \sum_a P_A(a)^{1 + \rho} \sum_y P_X(y) W_{\Lambda(a)}(y)^{1 + \rho} (W \circ P_X)(y)^{-\rho}$$

$$= 1 + \sum_a P_A(a)^{1 + \rho} e^{\phi(\rho|W_X P_X)}.$$ (21)

In the above derivation, (21) follows from the concavity of $x \mapsto x^\rho$, (22) follows from $\sum_{a' \neq a} P_A(a') \leq 1$, (23) follows from the inequality $(x + y)^\rho \leq x^\rho + y^\rho$. ■

Next, in order to reduce the complexity of encoding, we consider the case when $X$ and $\mathcal{A}$ are Abelian groups. We introduce the following condition for the ensemble for injective homomorphisms $F$ from $\mathcal{A}$ to $X$. Namely, the following is a condition for a random variable $F$ that takes its values on injective homomorphisms from $\mathcal{A}$ to $X$.

**Condition 10:** For arbitrary elements $x \neq 0 \in X$ and $a \neq 0 \in \mathcal{A}$, the relation $F(a) = x$ holds with probability at most $\frac{1}{|\mathcal{A}| - 1}$.

When $X$ and $\mathcal{A}$ are vector spaces over a finite field $\mathbb{F}_q$, the set of all injective homomorphisms from $\mathcal{A}$ to $X$ satisfies Condition 10.

We choose another random variable $G$ in $X$ that obeys the uniform distribution on $X$ and is independent of the choice of $F$. Then, we define a map $\Lambda_{F,G}(a) := F(a) + G$ and have the following theorem:
Theorem 11 (Algebraic channel resolvability): Under the above choice, we obtain

\[
E_{F,G}e^{\phi(W_{\Lambda_{F,G}\circ P_A}\parallel W_{P_{\text{mix},X}})} \leq E_{F,G}e^{\phi(W_{\Lambda_{F,G}\circ P_A}\parallel W_{P_{\text{mix},X}})} \\
\leq 1 + e^{-\rho H_1(Y)} \leq 1 + e^{-\rho H_1(Y)}.
\]

This theorem will be used for the proof of Lemma 15, which is essential for the proof of Theorem 16.

Proof: We introduce the random variable \( Z_a := \Lambda_{F,G}(a) = F(a) + G \). The random variable \( Z_a \) is independent of the choice of \( F \). For \( a' \in A \), \( \Lambda_{F,G}(a') = F(a' - a) + Z_a \). Since \(|\mathcal{X}| - 1)E_{F,Z_a}W_{\Lambda_{F,G}(a)}(y) = (|\mathcal{X}| - 1)E_{F}W_{F(a' - a) + Z_a}(y) \leq \sum_{a} W_a(y) = |\mathcal{X}|W \circ P_{\text{mix},X}(y) \) for \( a \in A \) and \( y \in Y \), we obtain \( E_{F,Z_a}W_{\Lambda_{F,G}(a)}(y) \leq \frac{|\mathcal{X}|}{|\mathcal{X}| - 1} W \circ P_{\text{mix},X}(y) \) for \( a \in A \) and \( y \in Y \). Further, since \( F \) is injective, we have \(|A| \leq |\mathcal{X}| \), which implies \( \sum_{a} P_A(a)^2 \geq \frac{1}{|\mathcal{X}|} \geq \frac{1}{|\mathcal{X}|} \). Hence, since \( x \mapsto x^\rho \) is concave, we obtain

\[
\sum_a P_A(a)^2 \leq \frac{1}{|\mathcal{X}|} \leq \frac{1}{|\mathcal{X}|} \leq \frac{1}{|\mathcal{X}|}.
\]

Our proof of Theorem 9 can be applied to our proof of Theorem 11 by replacing \( \Lambda(a) \), \( \Lambda|\Lambda(a) \), and \( P_X \).
by $Z_a$, $F|Z_a$ and $P_{\text{mix},X}$. Then, we obtain

$$
\mathbb{E}_{F,G}\sum_{a'} P(a') W_{\Lambda_{F,G}(a')} (y) (P_A(a) W_{\Lambda_{F,G}(a)}(y)) \leq \sum_{a} \sum_{y} \left( \mathbb{E}_{Z_a} P_A(a) W_{\Lambda_{F,G}(a)}(y) (P_A(a) W_{\Lambda_{F,G}(a)}(y)) + \mathbb{E}_{F|Z_a} \sum_{a' \neq a} P_A(a') W_{\Lambda_{F,G}(a')}(y)^\rho W \circ P_{\text{mix},X}(y)^{-\rho} \right)
$$

$$
= \sum_{a} \sum_{y} \left( \mathbb{E}_{Z_a} P_A(a) W_{Z_a}(y) (P_A(a) W_{Z_a}(y)) + \frac{1}{|X|} \sum_{a' \neq a} P_A(a') W \circ P_{\text{mix},X}(y)^\rho W \circ P_{\text{mix},X}(y)^{-\rho} \right)
$$

$$
\leq \sum_{a} \sum_{y} \left( \mathbb{E}_{Z_a} P_A(a) W_{Z_a}(y) (P_A(a) W_{Z_a}(y)) + \frac{1}{1/|X|} \sum_{a' \neq a} P_A(a') W \circ P_{\text{mix},X}(y)^\rho W \circ P_{\text{mix},X}(y)^{-\rho} \right)
$$

$$
= \sum_{a} P_A(a) \left( \frac{1}{1-1/|X|} \sum_{y} \mathbb{E}_{Z_a} W_{Z_a}(y) (P_A(a) W_{Z_a}(y)) \right)
$$

$$
= \sum_{a} P_A(a) \left( \frac{1}{1-1/|X|} \sum_{y} \mathbb{E}_{Z_a} W_{Z_a}(y)^{1+\rho} W \circ P_{\text{mix},X}(y)^{-\rho} \right)
$$

$$
\leq 1 + \sum_{a} P_A(a) \left( \frac{1}{1-1/|X|} \mathbb{E}_{Z_a} \sum_{y} W_{Z_a}(y)^{1+\rho} W \circ P_{\text{mix},X}(y)^{-\rho} \right).
$$

In the above derivation, (22) follows from $\sum_{a' \neq a} P_A(a') \leq 1$, (27) follows from the inequality $(x + y)^\rho \leq x^\rho + y^\rho$. The final inequality follows from (25).

In the following, we assume that the input alphabet $X$ is an Abelian group, and an action of $X$ on the output alphabet $Y$ is given as $x \cdot y$ for $x \in X$ and $y \in Y$. A channel $W$ from $X$ to $Y$ is regular in the sense of Delsarte-Piret [10], if there is a probability distribution $P_Y$ such that

$$
W_A(y) = P_Y(x \cdot y).
$$

Since a regular channel $W$ satisfies

$$
D(W \circ \Lambda_{F,g} \circ P_A | W \circ P_{\text{mix},X}) = D(W \circ \Lambda_{F,g'} \circ P_A | W \circ P_{\text{mix},X})
$$

for any $g, g' \in X$, we obtain the following corollary. This corollary implies that we do not need the additional random variable $G$ in the regular channel case.
Corollary 12: When the channel $W$ is a regular channel given by a distribution $P_Y$ on $Y$, we obtain
\[
E_F e^{\rho(D(W \circ \Lambda_F, g) \circ P_X || W \circ P_{mix,X})} \leq E_F e^{\phi(\rho | W \circ \Lambda_F, g, P_X || W \circ P_{mix,X})}
\leq 1 + e^{-\rho H_{1+y}(A)} e^{\phi(\rho | W, P_{mix,X})} = 1 + e^{-\rho H_{1+y}(A)} e^{\phi(\rho | P_Y || P_Y)}
\]
for any $g \in X$, where $P_Y(y) := \sum_x P_{mix,X}(x) P_Y(x \cdot y)$.

Proof: Due to Theorem 9 it is enough to show $\psi(\rho | W, P_{mix,X}) = \psi(\rho | P_Y || P_Y)$. Since $P_Y(y) = W \circ$
\[
P_{\text{mix},X}(y) = W \circ P_{\text{mix},X}(x \cdot y),
\]
we have
\[
e^{\phi \rho W P_{\text{mix},X}} = \sum_x P_{\text{mix},X}(x) \sum_y P_Y(x \cdot y)^{1+\rho} \overline{P}_Y(y)^{-\rho}
\]
\[
= \sum_x P_{\text{mix},X}(x) \sum_y P_Y(y)^{1+\rho} \overline{P}_Y(x^{-1} \cdot y)^{-\rho}
\]
\[
= \sum_x P_{\text{mix},X}(x) \sum_y P_Y(y)^{1+\rho} \overline{P}_Y(y)^{-\rho}
\]
\[
= \sum_y P_Y(y)^{1+\rho} \overline{P}_Y(y)^{-\rho} = e^{\phi \rho W | P_Y}. \]

VI. SECURE MULTIPLEX CODING WITH COMMON MESSAGES: SINGLE-SHOT SETTING

In this section, we give the formulation of the secure multiplex coding with common messages. After the formulation, we give two kinds of random construction of codes for the secure multiplex coding with common messages and evaluate their performance in the single-shot setting.

A. Formulation and Preparation

In the secure multiplex coding with common messages, Alice sends the common message \( S_0 \) to Bob and Eve, and \( T \) secret messages \( S_1, \ldots, S_T \) to Bob. We do not necessarily assume the uniformity or independence for the distributions of messages \( S_0, S_1, \ldots, S_T \). Hence, there might exist statistical correlations among messages \( S_0, S_1, \ldots, S_T \). Even in this scenario, Alice and Bob can use \( S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_T \) as random bits making \( S_i \) ambiguous to Eve. When we focus on \( S_I = (S_i; i \in I) \) for a non-empty proper subset \( I \subseteq \{1, \ldots, T\} \), the remaining information \( S_I \) serves as random bits making \( S_I \) ambiguous to Eve. The messages \( S_0, S_1, \ldots, S_T \) are assumed to belong to the sets \( S_0, S_1, \ldots, S_T \). The set \( S_1 \times \ldots \times S_T \) of all secret messages is denoted by \( S \). In order to explain the SMC model without \( S_0 \), we consider the following example. Consider the case when \( S_1, \ldots, S_T \) are personal information for \( T \) persons. That is, \( S_i \) corresponds to the personal information of the \( i \)-th person. Assume that it is required only to keep the secrecy of the respective personal information \( S_1, \ldots, S_T \) from the third party. The secrecy of the relation among respective personal informations is not required. For example, when \( S_1, \ldots, S_T \) are the uniform random bits with the same size, the secrecy of the sum \( S_1 \oplus \ldots \oplus S_T \) is not required, where \( \oplus \) is exclusive OR. In order to treat this secrecy problem, we give a formulation of the SMC model as follows.

The purpose of the coding in the SMC model is to reliably send the messages \( S_0, S_1, \ldots, S_T \) to Bob, and to make \( S_I \) ambiguous to Eve by using the remaining information \( S_I \) for several non-empty proper subsets \( I \subseteq \{1, \ldots, T\} \). Our code is given by Alice’s stochastic encoder \( \varphi_a \) from \( S \times S_0 \) to \( X \), Bob’s deterministic decoder \( \varphi_b : Y \rightarrow S \times S_0 \) and Eve’s deterministic decoder \( \varphi_e : Z \rightarrow S_0 \). The triple \( \varphi = (\varphi_a, \varphi_b, \varphi_e) \) is called a code for the secure multiplex coding with common messages. Then, the performance is evaluated by the following quantities: (1) The sizes of the sets of the common messages and all of the secret messages, i.e., \( |S_0|, |S_1|, \ldots, |S_T| \). (2) Bob’s decoding error probability \( P_b[P_{Y|X, \varphi, P_{S_T}}] \), which is the probability \( \Pr((S_0, S_1, \ldots, S_T) \neq \varphi_b(Y)) \) under the distribution \( P_{Y|X, \varphi_a} \times P_{S_T} \) with \( T := \{0, \ldots, T\} \). (3) Eve’s decoding error probability \( P_e[P_{Z|X, \varphi, P_{S_T}}] \), which is the probability \( \Pr(S_0 \neq \varphi_e(Z)) \) under the distribution \( P_{Z|X, \varphi_a} \times P_{S_T} \). (4) Leaked information \( I(S_I; Z|S_0)[P_{Z|X, \varphi_a, P_{S_T}}] \) for non-empty proper subset \( I \subseteq \{1, \ldots, T\} \), which is the mutual information \( I(S_I; Z|S_0) \) under the distribution \( P_{Z|X, \varphi_a} \times P_{S_T} \). Instead of \( I(S_I; Z|S_0)[P_{Z|X, \varphi_a}] \), other researchers sometimes treat (5) Eve’s uncertainty \( H(S_I|Z, S_0)[P_{Z|X, \varphi_a, P_{S_T}}] \), which is the conditional entropy \( H(S_I|Z, S_0) \) under the distribution \( P_{Z|X, \varphi_a} \times P_{S_T} \). However, when we treat the universality of our code, leaked information \( I(S_I; Z|S_0)[P_{Z|X, \varphi_a, P_{S_T}}] \) is used as criterion for
performance of our code. That is, we adopt leaked information \( I(S_I; Z|S_0)[P_{ZX}, \varphi_a, P_{S_T}] \) rather than Eve’s uncertainty \( H(S_I|Z, S_0)[P_{ZX}, \varphi_a, P_{S_T}] \).

In the above formulation, we treat the leaked information \( I(S_I; Z|S_0)[P_{ZX}, \varphi_a] \) for several non-empty proper subsets \( I \subseteq \{1, \ldots, T\} \). Depending on the situation, we decide which non-empty proper subset \( I \) is considered. Hence, in that case, we can fix a family \( J \) of non-empty proper subsets \( I \) of \( \{1, \ldots, T\} \) for which we discuss the leaked information \( I(S_I; Z|S_0)[P_{ZX}, \varphi_a] \). For example, in the case of the above personal information, we consider the subsets \( \{1\}, \{2\}, \ldots, \{T\} \). Hence, we choose \( J \) as \( J := \{\{1\}, \{2\}, \ldots, \{T\}\} \). When we do not specify the family \( J \), we treat the leaked information \( I(S_I; Z|S_0)[P_{ZX}, \varphi_a] \) for all non-empty proper subsets \( I \) of \( \{1, \ldots, T\} \).

This model can be regarded as a generalization of the wire-tap model in the following way. When there is no common messages and \( T = 2 \), there exist only two messages \( S_1 \) and \( S_2 \) in the secure multiplex coding. In the wire-tap channel model, \( S_1 \) corresponds to the message to be secretly sent to Bob, and \( S_2 \) does to the dummy message making \( S_1 \) ambiguous to Eve. As a special case of our code, a wire-tap code is given by Alice’s stochastic encoder \( \varphi_a \) from \( S_1 \times S_2 \) to \( X \) and Bob’s deterministic decoder \( \varphi_b : Y \rightarrow S_1 \). Then, the performance is evaluated by the following quantities. (1) The size of the secret message \( |S_1| \). (2) Bob’s decoding error probability \( P_L[|Y_X, \varphi, P_{S_{1,2}}] \). (4) Leaked information \( I(S_1; Z|S_0)[P_{ZX}, \varphi_a, P_{S_{1,2}}] \).

In order to guarantee that the leaked information is small, we employ the method of generalized channel resolvability given in Section \( \nabla \). In order to employ this method, we have to use the random coding method to construct a code \( \varphi \). In this section, we propose two kinds of random construction for our code. In Subsection \( \text{VI-B} \), we give the first construction, which attains the capacity region that will be defined in Section \( \text{VIII-B} \). This construction has two steps. In the first step, similar to the BCD encoder, we use the superposition random coding. In the second step, as illustrated in Fig. \( \dagger \) we split the confidential message into the private message \( B_2 \) and a part \( B_1 \) of the common message encoded by the BCD encoder. The coding scheme for BCC in \( [9] \) uses this kind of message splitting. The average leaked information under this kind of construction is evaluated by Theorem \( \text{III} \) which is an algebraic version of channel resolvability. In Subsection \( \text{VI-C} \), we give the second construction, which realizes a better exponential decreasing rate for leaked information, in specific cases. However, it cannot fully achieve the capacity region. The average leaked information with this kind of construction is evaluated by Theorem \( \text{II} \) which is a simple generalization of channel resolvability.

Before giving two kinds of random construction of codes for the secure multiplex coding with common messages, we prepare several useful facts. As in \( [12] \), we introduce the following function, which is an extended version of \( \psi(\rho|W, P) \) defined in \( (18) \)

\[
\psi(\rho|P_{ZIV}, P_{IUV}, P_U) = \log \sum_u P_U(u) \sum_v P_{IUV}(v|u) \sum_z P_{ZIV}(z|v)^{1+\rho} P_{ZUI}(z|u)^{-\rho}.
\]  

In the above notation, the definition \( (18) \) of \( \psi(\rho|W, P) \) can be rewritten as follows.

\[
\psi(\rho|P_{ZIL}, P_L) = \log \sum_{\ell} \sum_z P_L(\ell) P_{ZIL}(z|\ell)^{1+\rho} P_{ZL}(z)^{-\rho}.
\]  

\text{Proposition 13:} \( [12], [17] \) \exp(E_0(\rho|P_{ZIL}, P_L)) \) is concave with respect to \( P_L \) with fixed \( 0 < \rho < 1 \) and \( P_{ZIL} \). For fixed \( 0 < \rho < 1 \), \( P_L \) and \( P_{ZIL} \) we have

\[
\exp(\psi(\rho|P_{ZIL}, P_L)) \leq \exp(E_0(\rho|P_{ZIL}, P_L))
\]  

\[
\exp(E_0(\rho|P_{ZIL}, P_L)) \leq C_1 \exp(E_0(\rho|P_{ZIL}, \tilde{P}_L))
\]  

when another distribution \( \tilde{P}_L \) of \( L \) satisfies \( P_L(\ell) \leq C_1 \tilde{P}_L(\ell) \) holds for any \( \ell \).
When we fix a code \( \varphi \), we obtain the following observations. Any distribution \( \tilde{P}_Z \) on \( Z \) and any non-empty proper subset \( I \subseteq \{1, \ldots, T\} \) satisfy
\[
\rho I(S_I;Z|S_0)[P_{Z|V}, \varphi, P_{S_T}]
\leq \rho \sum_{s_0} P_{S_0}(s_0)D(P_{Z|S_I=S_0, \varphi}||P_{Z|S_0=S_0, \varphi} \times P_{S_I|S_0=S_0, \varphi})
\leq \rho \sum_{s_0} P_{S_0}(s_0)D(P_{Z|S_I=S_0, \varphi}||\tilde{P}_Z \times P_{S_I|S_0=S_0, \varphi})
\leq \sum_{s_0} P_{S_0}(s_0) \sum_{s_I} P_{S_I|S_0}(s_I|s_0) \rho D(D(P_{Z|S_I=S_0, \varphi}||\tilde{P}_Z), (34)
\]
where (34) follows from the following general inequality
\[
D(P_{X,Y}||P_X \times P_Y) \leq D(P_{X,Y}||Q_X \times P_Y)
\]
for any distribution \( Q_X \) over \( X \). Due to (20), we have
\[
\rho D(D(P_{Z|S_I=S_0, \varphi}||\tilde{P}_Z)) \leq \psi(\rho|P_{Z|S_I=S_0, \varphi}||\tilde{P}_Z).
\]
Thus, combining Jensen inequality and the above observations, we obtain the following lemma.

**Lemma 14:** Any distribution \( \tilde{P}_Z \) on \( Z \) and any non-empty proper subset \( I \subseteq \{1, \ldots, T\} \) satisfy
\[
e^\rho I(S_I;Z|S_0)[P_{Z|V, \varphi, P_{S_T}}] \leq e^{\sum_{s_0} P_{S_0}(s_0) \sum_{s_I} P_{S_I|S_0}(s_I|s_0) \rho D(D(P_{Z|S_I=S_0, \varphi}||\tilde{P}_Z))}
\leq \sum_{s_0} P_{S_0}(s_0) \sum_{s_I} P_{S_I|S_0}(s_I|s_0) e^{\rho D(D(P_{Z|S_I=S_0, \varphi}||\tilde{P}_Z))}
\leq \sum_{s_0} P_{S_0}(s_0) \sum_{s_I} P_{S_I|S_0}(s_I|s_0) e^{\rho D(D(P_{Z|S_I=S_0, \varphi}||\tilde{P}_Z))}
\]

**B. First Construction**

First, we give the first kind of random coding for SMC as follows.

**Code Ensemble 2:** First Step: For a given Markov chain \( U \rightarrow V \rightarrow X \rightarrow YZ \), we introduce two random variables \( B_1 \) and \( B_2 \) that take values in Abelian groups \( B_1 \) and \( B_2 \) and are subject to the uniform distributions. The pair of random variables \( (B_1, B_2) \) is used for sending all of the secret messages in \( S_1 \times \cdots \times S_T \). Assuming that \( S_1 \times \cdots \times S_T \) has an Abelian group structure, we give the random coding \( \Phi_r \) and \( \Phi_p \) in the same way as Code Ensemble [1] with \( S_c = S_0 \times B_1 \) and \( S_p = B_2 \).

Second Step: We choose an ensemble satisfying Condition [10] of isomorphisms \( F' \) from \( S_1 \times \cdots \times S_T \) to \( B_1 \times B_2 \) as Abelian groups. We choose the random variable \( G' \in B_1 \times B_2 \) that obeys the uniform distribution on \( B_1 \times B_2 \) and is independent of the choice of \( F' \) and anything else. Then, we define a map \( \Phi_{F',G'}(s) := F'(s) + G' \). Combining the above codes, we construct the code \( \Phi_a = \Phi_a \circ \Lambda_{F',G'} : S_0 \times S_1 \times \cdots \times S_T \rightarrow V \) as \( (s_0, s_1, \ldots, s_T) \mapsto \Phi_p(s_0, \Lambda_{F',G'}(s_1, \ldots, s_T)) \). Similar to the case of BCD, Bob’s decoder \( \Phi_b \) and Eve’s decoder \( \Phi_e \) are defined as the maximum likelihood decoders. Hence, our code is written by the triple \((\Phi_a, \Phi_b, \Phi_e)\). The structure of encoder is illustrated in Fig. [1].

As a special case of Code Ensemble [2] a wire-tap code is given as the case when \( T = 2 \) and we do not have the random variables \( S_0 \). For a fixed code \( \varphi_p \), \( P_{Z|S_0=s_0, \Phi_p=\varphi_p} \) denotes the average output distribution of the channel of the transmitted codeword \( \varphi_p(s_0, B_1, B_2) \) averaged over \( B_1, B_2 \). In order to evaluate the averaged performance of the above code \( (\Phi_a, \Phi_b, \Phi_e) \), we prepare the following lemma.
Lemma 15: When the code $\Phi_p$ is fixed to $\varphi_p$ in the BCD part, we have the following average performance.

$$E_{F',G'} \exp(pI(S_f; Z|s_0)[P_{Z|V}, \varphi_p \circ \Lambda_{F',G'}, P_{S_T}])$$

$$\leq E_{F',G'} \sum_{s_0} P_{S_0}(s_0) \sum_{s_f} P_{S_f|S_0}(s_f|s_0)$$

$$\cdot e^{pD(P_{Z|S_f=s_f,s_0=0},P_{Z|S_f=s_f,s_0=0})}$$

$$\leq 1 + \sum_{s_0} P_{S_0}(s_0) \sum_{s_f} P_{S_f|S_0}(s_f|s_0) e^{-pH_1(\cdot|S_f=s_f,s_0=0)}$$

$$\cdot e^{\phi(pP_{Z|S_f=s_f,s_0=0},P_{mix}|b_1,b_2)}.$$  \hspace{1cm} (40)

Further, when $P_{Z|V}$ is a regular channel and the map $\varphi_p|S_0=s_0 : (b_1, b_2) \mapsto \varphi_p(b_1, b_2, s_0)$ is a homomorphism from an Abelian group $B_1 \times B_2$ to an Abelian group $V$ for any $s_0 \in S_0$, the inequalities (40) hold even when $G'$ is a constant $g'$.

Lemma 15 will be applied for the evaluation of the performance of Code Ensemble 2. However, it will be also used for the evaluation of the performance of another type of codes without common messages based on a specific error correcting code in Section X. Hence, Lemma 15 addresses the case when the map $\varphi_p|S_0=s_0$ is a homomorphism.

Lemma 15 yields the following observation. Applying Jensen’s inequality for the convex function $x \mapsto e^x$ and the inequality $\log(1 + x) \leq x$, we obtain

$$E_{F',G'} \exp(pI(S_f; Z|s_0)[P_{Z|V}, \varphi_p \circ \Lambda_{F',G'}, P_{S_T}])$$

$$\leq \log\left(1 + \sum_{s_0} P_{S_0}(s_0) \sum_{s_f} P_{S_f|S_0}(s_f|s_0) e^{-pH_1(\cdot|S_f=s_f,s_0=0)}
\cdot e^{\phi(pP_{Z|S_f=s_f,s_0=0},P_{mix}|b_1,b_2)}\right)$$

$$\leq \sum_{s_0} P_{S_0}(s_0) \sum_{s_f} P_{S_f|S_0}(s_f|s_0) e^{-pH_1(\cdot|S_f=s_f,s_0=0)}
\cdot e^{\phi(pP_{Z|S_f=s_f,s_0=0},P_{mix}|b_1,b_2)}. \hspace{1cm} (41)$$

Proof: Applying (38) to the case when $P_Z = P_{Z|S_0=s_0}$, we obtain

$$E_{F',G'} e^{pI(S_f; Z|s_0)[P_{Z|V}, \varphi_p \circ \Lambda_{F',G'}, P_{S_T}]}$$

$$\leq E_{F',G'} \sum_{s_0} P_{S_0}(s_0) \sum_{s_f} P_{S_f|S_0}(s_f|s_0)$$

$$\cdot e^{pD(P_{Z|S_f=s_f,s_0=0},P_{Z|S_f=s_f,s_0=0})}$$

$$\leq E_{F',G'} |\varphi_p| \sum_{s_0} P_{S_0}(s_0) \sum_{s_f} P_{S_f|S_0}(s_f|s_0)$$

$$\cdot e^{\phi(pP_{Z|S_f=s_f,s_0=0},P_{mix}|b_1,b_2)}. \hspace{1cm} (42)$$

For a fixed $s_T$, we apply Theorem 11 to the case when $A$ is $S_T$, $X$ is $B_1 \times B_2$, $G$ is $G' + F'(s_T, 0)$, which is independent of $F'$, and $F$ is the map $s_T \mapsto F'(0, s_T)$ that satisfies Condition 10. Then, $\Lambda_{F',G'}(s_T, s_T') = F'(s_T, s_T') + G' = F'(0, s_T') + Z_{s_T}$. Thus, we obtain

$$E_{F',G'} e^{\phi(pP_{Z|S_f=s_f,s_0=0},P_{mix}|b_1,b_2)}$$

$$\leq 1 + e^{-pH_1(\cdot|S_f=s_f,s_0=0)} \cdot e^{\phi(pP_{Z|S_f=s_f,s_0=0},P_{mix}|b_1,b_2)}. \hspace{1cm} (43)$$

Thus, we obtain (40).

Further, when $P_{Z|V}$ is a regular channel and the map $\varphi_p|S_0=s_0 : (b_1, b_2) \mapsto \varphi_p(b_1, b_2, s_0)$ is a homomorphism from an Abelian group $B_1 \times B_2$ to an Abelian group $V$ for any $s_0 \in S_0$, the channel $P_{Z|V} \circ \varphi_p|S_0=s_0$ is a
regular channel from $\mathcal{B}_1 \times \mathcal{B}_2$ to $\mathcal{V}$. Hence, due to Corollary [12] the inequalities (40) hold even when $G'$ is a constant $g'$.

Using the above lemma, we obtain the following theorem, which gives the averaged performance of the above code $(\Phi_a, \Phi_b, \Phi_e)$. By using this theorem, we will give the capacity region in Subsection VIII-B.

**Theorem 16:** Assume that the code $\Phi = (\Phi_a, \Phi_b, \Phi_e)$ is the ensemble given in Code Ensemble [2]. Then, the inequalities

\[
\mathbb{E}_{\Phi_a} \exp(\rho I(S_f; Z|S_0)[P_{Z|V}, \Phi_a, P_{S_f}])
\leq \mathbb{E}_{\Phi_a} \sum_{s_0} P_{S_0}(s_0) \sum_{s_f} P_{S_f|S_0}(s_f|s_0)e^{\rho(D(P_{Z|V} - \mathcal{I}_f|s_f, s_0 - \mathcal{I}_a, \mathcal{I}_b)P_{Z|V} - \mathcal{I}_a, \mathcal{I}_b))}
\leq 1 + |B_1|^\rho e^{-\rho H_1(s_f|s_0) + E_0(\rho P_{Z|V}, \rho P_{V|U}, P_U)},
\]

and

\[
\mathbb{E}_{\Phi} P_b[P_{Y|V}, \Phi, P_{S_f}] \leq |B_2|^\rho e^{E_0(\rho P_{Y|V}, \rho P_{V|U}, P_U)}
+ (|S_0||S|)^\rho e^{E_0(\rho P_{Y|V}, \rho P_{V|U}, P_U)},
\]

\[
\mathbb{E}_{\Phi} P_e[P_{Z|V}, \Phi, P_{S_f}] \leq |S_0|^\rho e^{E_0(\rho P_{Z|V}, \rho P_{V|U}, P_U)},
\]

hold.

**Theorem 16** yields the following observation. Applying Jensen’s inequality to the convex function $x \mapsto e^x$, we obtain

\[
\mathbb{E}_{\Phi_a, \rho} I(S_f; Z|S_0)[P_{Z|V}, \Phi_a, P_{S_f}]
\leq \log(1 + |B_1|^\rho e^{-\rho H_1(s_f|s_0) + E_0(\rho P_{Z|V}, \rho P_{V|U}, P_U)})
\leq |B_1|^\rho e^{-\rho H_1(s_f|s_0) + E_0(\rho P_{Z|V}, \rho P_{V|U}, P_U)},
\]

The number of non-empty proper subsets $I \subset \{1, \ldots, T\}$ is $2^T - 2$. Similar to (9) and (10), since $2(2^T - 2) + 2 = 2^{T+1} - 2 < 2^{T+1}$, Markov inequality guarantees that there exists a code $\varphi = (\varphi_a, \varphi_b, \varphi_e)$ such that

\[
\exp(\rho I(S_f; Z|S_0)[P_{Z|V}, \varphi_a, P_{S_f}])
\leq 2^{T+1} \inf_{0 \leq \rho \leq 1} (1 + |B_1|^\rho e^{-\rho H_1(s_f|s_0) + E_0(\rho P_{Z|V}, \rho P_{V|U}, P_U)})
\leq 2^{T+1} e^{\rho \log |B_1|^\rho e^{-\rho H_1(s_f|s_0) + E_0(\rho P_{Z|V}, \rho P_{V|U}, P_U)}},
\]

\[
I(S_f; Z|S_0)[P_{Z|V}, \varphi_a, P_{S_f}]
\leq \inf_{0 \leq \rho \leq 1} \frac{2^{T+1}}{|B_1|^\rho e^{-\rho H_1(s_f|s_0) + E_0(\rho P_{Z|V}, \rho P_{V|U}, P_U)}},
\]

for any non-empty proper subset $I \subset \{1, \ldots, T\}$. Taking the logarithm in (48), we obtain

\[
I(S_f; Z|S_0)[P_{Z|V}, \Phi_a, P_{S_f}]
\leq \left[ \log |B_1| + \frac{1}{\rho} \varphi(\rho P_{Z|V}, P_{V|U}, P_U) H_1(s_f|s_0) \right] + (T + 3) \frac{\log 2}{\rho}.
\]

**Proof of Theorem 16**
We show (44). Using (32), we obtain
\[E_{\Phi_0, \Phi_c} e^{\rho (\rho P_{Z|B_1, B_2, S_0|s_0, \Phi_0, \Phi_p})} \leq E_{\Phi_0, \Phi_c} e^{E_0 (\rho P_{Z|B_1, B_2, S_0|s_0, \Phi_0, \Phi_p})} \]
\[= E_{\Phi_0, \Phi_c} \sum_z \sum_{b_1, b_2} P_{B_1, b_1} (b_1, b_2) P_{Z|B_1, B_2, S_0, \Phi_0} (z|b_1, b_2) e^{\rho (\rho P_{Z|B_1, B_2, S_0, \Phi_0, \Phi_p})} \]
\[= E_{\Phi_0, \Phi_c} \sum_z \sum_{b_1, b_2} \frac{1}{|B_1||B_2|} P_{Z|V} (z|\Phi_p (s_0, b_1, b_2)) e^{\rho (\rho P_{Z|V} (z|\Phi_p (s_0, b_1, b_2)))} \]
\[= E_{\Phi_0, \Phi_c} \sum_z \sum_{b_1} \frac{1}{|B_1|} \sum_{b_2} \frac{1}{|B_2|} P_{Z|V} (z|\Phi_p (s_0, b_1, b_2)) e^{\rho (\rho P_{Z|V} (z|\Phi_p (s_0, b_1, b_2)))} \]
\[= E_{\Phi_0, \Phi_c} \sum_z \sum_{b_1} \frac{|B_1|}{|B_1|} \sum_{b_2} \frac{1}{|B_2|} P_{Z|V} (z|\Phi_p (s_0, b_1, b_2)) e^{\rho (\rho P_{Z|V} (z|\Phi_p (s_0, b_1, b_2)))} \]
\[= E_{\Phi_0, \Phi_c} \sum_z \sum_{b_1} \frac{|B_1|}{|B_1|} \sum_{b_2} \frac{1}{|B_2|} P_{Z|V} (z|\Phi_p (s_0, b_1, b_2)) e^{\rho (\rho P_{Z|V} (z|\Phi_p (s_0, b_1, b_2)))} \]
\[= E_{\Phi_0, \Phi_c} e^{E_0 (\rho P_{Z|V, P_{V|U}})} \]
where (53), (54), (55), and (57) follow from (32), the inequality \((x + y)^{1-p} \leq x^{1-p} + y^{1-p}\), the concavity of \(x \mapsto x^{1-p}\), and the definition of the ensemble of the code \(\Phi_p\), respectively.

Summarizing the above discussion, we obtain
\[E_{\Phi_0, \Phi_c} e^{\rho (S I|Z|S_0) (P_{Z|V, \Phi_0, \Phi_p})} \]
\[\leq E_{\Phi_0, \Phi_c} \sum_{s_0, s_f} P_{S_0} (s_0, s_f) P_{S_I|S_0} (s_I|s_0) e^{\rho (P_{Z|B_1, B_2, S_0, \Phi_0, \Phi_p})} \]
\[= E_{\Phi_0, \Phi_c} \sum_{s_0, s_f} P_{S_0} (s_0, s_f) P_{S_I|S_0} (s_I|s_0) \sum_{s_f} e^{\rho (P_{Z|B_1, B_2, S_0, \Phi_0, \Phi_p})} \]
\[\leq \sum_{s_0} P_{S_0} (s_0) \sum_{s_f} P_{S_I|S_0} (s_I|s_0) \sum_{s_f} e^{\rho (P_{Z|B_1, B_2, S_0, \Phi_0, \Phi_p})} \]
\[\leq \sum_{s_0} P_{S_0} (s_0) \sum_{s_f} P_{S_I|S_0} (s_I|s_0) \sum_{s_f} e^{\rho (P_{Z|B_1, B_2, S_0, \Phi_0, \Phi_p})} \]
\[= e^{\rho (P_{Z|B_1, B_2, S_0, \Phi_0, \Phi_p})} \]
where (59), (60), and (61) follow from (38), the second inequality in Lemma 15, and (58), respectively. Then, we obtain (44).

Further, (45) and (46) follow from Lemma 7.
C. Second Construction

Now, we introduce the second kind of random coding for SMC.

**Code Ensemble 3:** For a given Markov chain \( U \to V \to X \to YZ \), we give the random coding \( \Phi_c \) and \( \Phi_p \) in the same way as Code Ensemble \([1]\) with \( S_c = S_0 \) and \( S_p = S_1 \times \cdots \times S_T \). Similar to the case of BCD, Bob’s decoder \( \Phi_b \) and Eve’s decoder \( \Phi_e \) are defined as the maximum likelihood decoders. Hence, our code is written by the quartet(\( \Phi_c, \Phi_p, \Phi_b, \Phi_e \)).

The averaged performance of the above code is evaluated by the following theorem. Indeed, we cannot derive the capacity region from the following theorem. However, the following theorem has an advantage when the conditional mutual information goes to zero. As is explained in Section IX, the following theorem yields a better bound for the exponential decreasing rate of the conditional mutual information than Theorem \([16]\) in a specific case.

**Theorem 17:** The above ensemble of codes \( \Phi = (\Phi_c, \Phi_p, \Phi_b, \Phi_e) \) satisfies the following inequalities.

\[
E_\Phi \exp(\rho I(S_T; Z|S_0)[P_{Z|V}, \Phi, P_{S_T}]) \\
\leq 1 + e^{-\rho H_{\psi_0}(S_T|S_0)+\psi(\rho P_{Z|V}, P_{V|U}, P_U)},
\]

\[
E_\Phi P_p[P_{Y|V}, \Phi, P_{S_T}] \\
\leq |S_0| e^{E_\Phi(-\rho P_{Z|V}, P_{V|U}, P_U)},
\]

\[
E_\Phi P_e[P_{Z|V}, \Phi, P_{S_T}] \leq |S_0| e^{E_\Phi(-\rho P_{Z|V}, P_U)}.
\]

Theorem \([17]\) yields the following observation. Applying Jensen’s inequality to the convex function \( x \mapsto e^x \), we obtain

\[
E_\Phi \rho I(S_T; Z|S_0)[P_{Z|V}, \Phi, P_{S_T}] \\
\leq \log(1 + e^{-\rho H_{\psi_0}(S_T|S_0)+\psi(\rho P_{Z|V}, P_{V|U}, P_U)}) \\
\leq e^{-\rho H_{\psi_0}(S_T|S_0)+\psi(\rho P_{Z|V}, P_{V|U}, P_U)}.
\]

(65)

Similar to (48), (49), (50), and (51), Markov inequality guarantees that there exists a code \( \varphi \) such that

\[
\rho I(S_T; Z|S_0)[P_{Z|V}, \varphi, P_{S_T}] \\
\leq 2^{T+1}(1 + e^{-\rho H_{\psi_0}(S_T|S_0)+\psi(\rho P_{Z|V}, P_{V|U}, P_U)}) \\
\leq 2^{T+2} e^{-\rho H_{\psi_0}(S_T|S_0)+\psi(\rho P_{Z|V}, P_{V|U}, P_U)},
\]

(66)

\[
\rho I(S_T; Z|S_0)[P_{Z|V}, \varphi, P_{S_T}] \\
\leq 2^{T+1} e^{-\rho H_{\psi_0}(S_T|S_0)+\psi(\rho P_{Z|V}, P_{V|U}, P_U)},
\]

(67)

\[
P_p[P_{Y|V}, \varphi, P_{S_T}] \\
\leq 2^{T+1} |S_0| e^{E_\Phi(-\rho P_{Z|V}, P_{V|U}, P_U)} + 2^{T+1} |S_0| e^{E_\Phi(-\rho P_{Z|V}, P_U)},
\]

(68)

\[
P_e[P_{Z|V}, \varphi, P_{S_T}] \\
\leq 2^{T+1} |S_0| e^{E_\Phi(-\rho P_{Z|V}, P_U)}.
\]

(69)

Taking the logarithm in (66), we obtain

\[
I(S_T; Z|S_0)[P_{Z|V}, \Phi, P_{S_T}] \\
\leq (T + 3) \frac{\log 2}{\rho} + \frac{1}{\rho} \psi(\rho P_{Z|V}, P_{V|U}, P_U) - H_{\psi_0}(S_T|S_0).\]

(70)

**Proof of Theorem 17**
Inequalities (63) and (64) can be shown by Lemma 7. Then, we show (62). Then, applying (39) to the case with $P_Z = P_{Z|U=\Phi(o)}$, we obtain
\[
E_\Phi e^{\rho(I(S_f Z|S_0)\Phi)} \\
\leq E_\Phi \sum_{s_0} P_{S_0}(s_0) \sum_{s_f} P_{S_f|S_0}(s_f|s_0) e^{\rho(P_Z Z|F_{s_f s_0 \Phi})} \\
= \sum_{s_0} P_{S_0}(s_0) \sum_{s_f} P_{S_f|S_0}(s_f|s_0) \\
\cdot E_{\Phi_o} E_{\Phi_o}(1 + e^{-\rho H_{1,\rho} Z|S_f \Phi(o)} e^{\rho(P_Z Z|F_{s_f s_0 \Phi})}) \\
= \sum_{s_0} P_{S_0}(s_0) \sum_{s_f} P_{S_f|S_0}(s_f|s_0) \\
\cdot (1 + e^{-\rho H_{1,\rho} Z|S_f \Phi(o)} e^{\rho(P_Z Z|F_{s_f s_0 \Phi})}) \\
= 1 + e^{-\rho H_{1,\rho} Z|S_f \Phi(o)} e^{\rho(P_Z Z|F_{s_f s_0 \Phi})},
\]
which implies (62).

D. Group Symmetry

Next, when the channel has a nice property with respect to group action, we treat the upper bound of the leaked information with a fixed BCD code $\varphi_p$. That is, we discuss the upper bound given in Lemma 15 under an assumption for group action, which will be given latter. The following analysis is required for evaluation of universal coding in Sections XI and XII and a practical code construction in Subsection X-B.

For simplicity, we first discuss the case with no common message, i.e., $|S_0| = 1$ and $|B_1| = 1$. Assume that a group $G$ acts on $\mathcal{V}$ and $\mathcal{Z}$. The action of $g \in G$ is written as $g \cdot v$ and $g \cdot z$ for $v \in \mathcal{V}$ and $z \in \mathcal{Z}$. Then, due to Eqs. (2), (3), and (4), we have
\[
(g^{-1} \circ P_Z V \circ g)(z|v) = P_Z V (g \cdot z|g \cdot v) \\
(g^{-1} \circ P_V V)(v) = P_V (g \cdot v).
\]
Then, the set $\mathcal{V}$ can be divided to orbits $\{\mathcal{V}_o\}_{o \in O}$ by the action of $G$. The set $O$ of indexes of the orbits is called the orbit space. Given a code $\varphi_p$ as an injective map from $B_2$ to $\mathcal{V}$, Recall that we denote the uniform distribution on the image $\text{Im} \varphi_p$ by $P_{\text{mix,Im} \varphi_p}$, and we define the distribution $P_{\varphi_p}(o) := |\text{Im} \varphi_p \cap \mathcal{V}_o|/|\text{Im} \varphi_p|$ on the orbit space $O$ and the distribution $\overline{P}_{\varphi_p}$ on $\mathcal{V}$ by $\overline{P}_{\varphi_p}(v) := \frac{P_{\varphi_p}(o)}{|\mathcal{V}_o|}$ when the element $v$ belongs to the subset $\mathcal{V}_o$. Then, we obtain the following lemma.

**Lemma 18:** When the relation $g^{-1} \circ P_Z V \circ g = P_Z|V$ holds for any $g \in G, v \in \mathcal{Z}$, and $v \in \mathcal{V}$,
\[
\psi(\rho|P_Z|B_2, \varphi_p = \varphi_p, P_{\text{mix,B}_2}) = \psi(\rho|P_Z|V, P_{\text{mix,Im} \varphi_p}) \\
\leq E_0(\rho|P_Z|V, P_{\text{mix,Im} \varphi_p}) \leq E_0(\rho|P_Z|V, \overline{P}_{\varphi_p}).
\]
In particular, when the image $\text{Im} \varphi_p$ is included in one orbit $\mathcal{V}_o$, $\overline{P}_{\varphi_p}$ is the uniform distribution on the orbit $\mathcal{V}_o$. 
Proof: Since \( e^{E_0(p|g^{-1} o P_{Z|V} o g\cdot g^{-1} o P_{\text{mix}, \varphi})} = e^{E_0(p|P_{Z|V} o g\cdot g^{-1} o P_{\text{mix}, \varphi})} \), we have
\[
\begin{align*}
e^{\phi(p|P_{Z|V} o P_{\text{mix}, \varphi})} & \leq e^{E_0(p|P_{Z|V} o P_{\text{mix}, \varphi})} = \sum_{S \in G} \frac{1}{|G|} e^{E_0(p|P_{Z|V} o g\cdot g^{-1} o P_{\text{mix}, \varphi})} \\
& = \sum_{g \in G} \frac{1}{|G|} e^{E_0(p|P_{Z|V} o g\cdot g^{-1} o P_{\text{mix}, \varphi})} \\
& \leq e^{E_0(p|P_{Z|V} o \sum_{g \in G} \frac{1}{|G|} g\cdot g^{-1} o P_{\text{mix}, \varphi})} = e^{E_0(p|P_{Z|V} o \varphi)}.
\end{align*}
\] (72)

Next, we consider the general case. Assume that a group \( G \) acts on \( U, V, \) and \( \mathcal{Z} \). The code pair code \((\varphi_c, \varphi_p)\) is a map from \( S_0 \times B_1 \times B_2 \) to \( U \times V \). For a given \( s_0 \in S_0 \), we define the maps \( \varphi_c|_{S_0=s_0} \) and \( \varphi_p|_{S_0=s_0} \) by
\[
\varphi_c|_{S_0=s_0}(b_1) := \varphi_c(s_0, b_1) \in U \\
(\varphi_c, \varphi_p)|_{S_0=s_0}(b_1, b_2) := (\varphi_c(s_0, b_1), \varphi_p(s_0, b_1, b_2)) \in U \times V.
\]

For simplicity, we assume that the image of \((\varphi_c, \varphi_p)|_{S_0=s_0}\) is included in one orbit in \( U \times V \), which is denoted by \((V \times U)_o\). Hence, the image of \( \varphi_c|_{S_0=s_0}\) is included in one orbit in \( U \), which is denoted by \( U_o \).

Lemma 19: Assume that the image of \((\varphi_c, \varphi_p)|_{S_0=s_0}\) is included in an orbit \((V \times U)_o\) in \( U \times V \). When the relation \( g^{-1} o P_{Z|V} o g = P_{Z|V} \) holds for any \( g \in G \), the relation
\[
\begin{align*}
e^{\phi(p|P_{Z|V} o P_{\text{mix}, \varphi})} & \leq e^{E_0(p|P_{Z|V} o P_{\text{mix}, \varphi})} \\
& \leq |B| \cdot e^{E_0(p|P_{Z|V} o P_{\text{mix}, \varphi})} \\
& \leq e^{E_0(p|P_{Z|V} o \sum_{g \in G} \frac{1}{|G|} g\cdot g^{-1} o P_{\text{mix}, \varphi})} = e^{E_0(p|P_{Z|V} o \varphi)}.
\end{align*}
\] (73)

holds for any \( s_0 \in S_0 \).

Proof: For a given \( u \in U_o \), we define the stabilizer of \( u \) by \( H_u := \{ g \in G : g \cdot u = u \} \), which is a subgroup of \( G \). For arbitrary \( u \in U_o \), we define the two subsets \( V'_u, V_u \subset V \) by \( (u) \times V'_u = \text{Im}(\varphi_c, \varphi_p)|_{S_0=s_0} \cap ((u) \times V) \) and \( (u) \times V_u = (V \times U)_o \cap ((u) \times V) \). Then, we obtain the relations
\[
\begin{align*}
P_{V|u=\text{mix}, \varphi_c, \varphi_p}_{|S_0=s_0} &= P_{V|\text{mix}, V'_u} \quad (74) \\
P_{V|u=\text{mix}, \varphi_c, \varphi_p}_{|S_0=s_0} &= P_{V|\text{mix}, V_u} \quad (75)
\end{align*}
\]

For the definitions of the left hand sides, see [1]. We can also show that
\[
\cup_{g \in H_u} \{ g \cdot v : v \in V'_u \} = V_u.
\]

Since \( g^{-1} o P_{V|u=\text{mix}, \varphi_c, \varphi_p}_{|S_0=s_0} = P_{V|u=\text{mix}, \varphi_c, \varphi_p}_{|S_0=s_0} \), the condition \( g^{-1} o P_{Z|V} o g = P_{Z|V} \) implies that
\[
\begin{align*}
e^{E_0(p|g^{-1} o P_{Z|V} o g\cdot g^{-1} o P_{\text{mix}, \varphi})} & = e^{E_0(p|P_{Z|V} o \varphi)}.
\end{align*}
\] (76)

We obtain the following relations. In the following derivation, (77) and (79) follow from (55) and (76), respectively. Applying Lemma 18 to the case of \( G = H_u \), we obtain the inequality (78) from (74) and
totically conditionally uniform
the message sets
asymptotic setting. In order to treat the capacity region and the strong security, we introduce several kinds
performance without fixing the conditional entropy rate of the dummy message
the conditional entropy of the dummy message
above modification. These extensions to the channel
All of the discussions in this section are still valid even if we replace
Definition 21:
Section VI deals with the security when a channel $P_{Z|V}$ from $V$ to $Z$ is given. The
discussion of Section VI can be extended to the case with a channel $P_{Z|V,U}$ from $V \times U$ to $Z$. In this
case, $\psi(\rho|P_{Z|V}, P_{V|U}, P_U)$ and $E_0(\rho|P_{Z|V}, P_{V|U}, P_U)$ are modified to
\[
\begin{align*}
\psi(\rho|P_{Z|V,U}, P_{V|U}, P_U) & := \log \sum_u P_U(u) \sum_v P_{V|U}(v|u) \sum_z P_{Z|V,U}(z|v,u)^{1+\rho} P_{Z|U}(z|u)^{-\rho} \\
E_0(\rho|P_{Z|V,U}, P_{V|U}, P_U) & := \log \sum_u P_U(u) \sum_z \left( \sum_v P_{V|U}(v|u)(P_{Z|V,U}(z|v,u)^{1/(1-\rho)}) \right)^{1-\rho}.
\end{align*}
\]
All of the discussions in this section are still valid even if we replace $P_{Z|V}(z|v)$ by $P_{Z|V,U}(z|v,u)$ with the
above modification. These extensions to the channel $P_{Z|V,U}$ will be used in Section XI as a mathematical
tool for our proof.

VII. ASYMPTOTIC CONDITIONAL UNIFORMITY

A. Three Kinds of Asymptotic Conditional Uniformity Conditions

In SMC, we use the message $S_{I^c}$ as a dummy message. The secrecy of the message $S_I$ depends on
the conditional entropy of the dummy message $S_{I^c}$ given $S_I$. Then, it is not easy to treat the asymptotic
performance without fixing the conditional entropy rate of the dummy message $S_{I^c}$. Hence, we need to
characterize the randomness of the dummy message $S_{I^c}$ under the condition with respect to $S_I$ in the
asymptotic setting. In order to treat the capacity region and the strong security, we introduce several kinds
of asymptotic conditional uniformity conditions for a general sequence of source distributions $P_{S_{I^c,n}}$
on the message sets $S_i$ for $i = 0, 1, \ldots, T$ satisfying the relations $|S_i| := e^{nR_i}$ for $i = 0, 1, \ldots, T$.

Definition 21: The sequence of distributions $P_{S_{I^c,n}}$ of the dummy message $S_{I^c,n}$ is called weak asymptotically conditionally uniform (WACU) for a non-empty proper subset $I \subseteq \{1, \ldots, T\}$ when
\[
\lim_{n \to \infty} \frac{1}{n} H(S_{I^c,n}|S_{I,n}, S_{0,n}) = \sum_{i \in I} R_i.
\]
**Definition 22:** The sequence of distributions $P_{S_{I^c,n}}$ of the dummy message $S_{I^c,n}$ is called *semi-weak asymptotically conditionally uniform* (SW ACU) for a non-empty proper subset $I \subseteq \{1, \ldots, T\}$ when the relation

$$
\lim_{n \to \infty} \frac{1}{n} H_{1/2} (S_{I^c,n} \mid S_{I,n}, S_{0,n}) = \sum_{i \in I^c} R_i
$$

(81)

holds for any $\delta > 0$.

**Definition 23:** Fix an arbitrary fixed real number $\epsilon \geq 0$. The sequence of distributions $P_{S_{I^c,n}}$ of the dummy message $S_{I^c,n}$ is called *$\epsilon$-strong asymptotically conditionally uniform* ($\epsilon$-SACU) for a non-empty proper subset $I \subseteq \{1, \ldots, T\}$ when the relation

$$
H_{\log}(I^c) \geq \sum_{i \in I^c} (R_i - \epsilon),
$$

(82)

where

$$
H_{\log}(I^c) := \lim_{\delta \to \infty} \liminf_{n \to \infty} \frac{1}{n} H_{1/2} (S_{I^c,n} \mid S_{I,n}, S_{0,n}).
$$

(83)

Since $\rho - 1$ behaves as $\delta \log n$ in (83), we use the subscript log in (83). In the case of $\epsilon = 0$, it is simply called *strong asymptotically conditionally uniform* (SACU) for a non-empty proper subset $I \subseteq \{1, \ldots, T\}$. In this case, the condition (82) is equivalent with

$$
H_{\log}(I^c) = \sum_{i \in I^c} R_i
$$

(84)

because the opposite inequality holds due to the cardinalities of respective message sets.

In particular, when the sequence of distributions $P_{S_{I^c,n}}$ of the dummy message $S_{I^c,n}$ is WACU for any non-empty proper subset $I \subseteq \{1, \ldots, T\}$, it is simply called WACU. We sometimes fix a family $J$ of non-empty proper subsets $I$ of $\{1, \ldots, T\}$, and treat only non-empty proper subsets $I \in J$. In this case, we call the sequence of distributions $P_{S_{I^c,n}}$ WACU for a family $J$ when it is WACU for any non-empty proper subset $I \in J$. We also apply these conventions to SW ACU, SACU, and $\epsilon$-SACU. The relations among the above conditions are summarized as follows.

**Theorem 24:** The following relations hold.

$$
\text{SACU} \Rightarrow \text{SW ACU} \iff \text{WACU}
$$

$$
\Downarrow
$$

$$
\epsilon\text{-SACU}
$$

**Proof:** The equivalence between SW ACU and WACU will be shown as Lemma 83 in Appendix C. Other relations are trivial from their definitions.

In fact, as is shown in Subsection VII-B even if the original information does not satisfy the WACU condition (80) or the SACU condition (80) with $\epsilon = 0$, if we apply Slepian-Wolf data compression [30] to the original sources so that the total compressed rate of the whole data attains the entropy rate of the whole sources, the compressed data satisfies the WACU condition (80) and/or the SACU condition (84). Similarly, as is shown in Subsection VII-B, even if the original information does not satisfy the $\epsilon$-SACU condition (82), if we apply Slepian-Wolf data compression [30] to the original sources so that the error probability goes to zero exponentially and the difference between the entropy rate of the whole system and the total compressed rate is less than $\epsilon$, the compressed data satisfies the $\epsilon$-SACU condition (82).
B. Asymptotic Conditional Uniformity Conditions and Slepian-Wolf Data Compression

In Subsection IX-A, we have introduced several asymptotic conditional uniformity conditions. In this subsection, we clarify which kind of data compressed by Slepian-Wolf compression satisfies asymptotic conditional uniformity conditions. For this purpose, we assume that the random variables $S_0, S_1, \ldots, S_T$ are subject to the $n$-fold stationary ergodic joint distribution $P_n^{1T}$ over $S_0 \times S_1 \times \cdots \times S_T$. The symbols $H(S_0, \ldots, S_T), H(S_I),$ and $H(S_0, S_I)$ describe the entropy rates of the respective random variables for any non-empty proper subset $I \subseteq \{1, \ldots, T\}.$ The following theorem treats the WACU condition for the compressed data.

**Theorem 25:** We choose the asymptotic compression rates $R_0, \ldots, R_T$ such that $\sum_{i=0}^T R_i = H(S_0, \ldots, S_T)$ and $\sum_{i \in I} R_i \leq H(S_I),$ and $R_0 + \sum_{i \in I} R_i \leq H(S_0, S_I)$ for any non-empty proper subset $I \subseteq \{1, \ldots, T\}.$ Choose a sequence $m_n$ such that $\frac{m_n}{n} \rightarrow 1.$ Let $\varphi_n^i : S_i^{m_n} \rightarrow \{1, \ldots, [e^{nR_i}]\}$ be Slepian-Wolf encoders and $\hat{\varphi}_n^i : \{1, \ldots, [e^{nR_i}]\} \times \cdots \times \{1, \ldots, [e^{nR_T}]\} \rightarrow S_i^{m_n} \times \cdots \times S_T^{m_n}$ be its Slepian-Wolf decoder for any positive integer $n$ such that

$$
\epsilon(\varphi_n^i, \hat{\varphi}_n^i) := \Pr\{\varphi_n^i(S_0^{m_n}, \ldots, S_T^{m_n}) \neq \hat{\varphi}_n^i(\varphi_n^i(S_0^{m_n}), \ldots, \varphi_n^i(S_T^{m_n}))\} \rightarrow 0,
$$

where $\varphi^n = (\varphi_0^n, \ldots, \varphi_T^n).$ Then, we have

$$
\lim_{n \rightarrow \infty} \frac{1}{n} H((\varphi_n^i(S_i^{m_n}))_{i \in I}, (\varphi_n^i(S_i^{m_n}))_{i \in I}, \varphi_0^n(S_0^{m_n})) = \sum_{i \in I} R_i
$$

for any non-empty proper subset $I \subseteq \{1, \ldots, T\}.$ That is, the compressed data satisfies the WACU condition (80).

**Remark 26:** Theorem 25 gives only a sufficient condition (85) for the compressed data satisfying the WACU condition. For construction of the compressed data satisfying the WACU condition, it is needed to clarify the existence of a code whose the compressed data satisfying the condition (85).

In the single terminal Markovian case, under the condition $\frac{m_n}{n} \rightarrow 1,$ the second order asymptotic analysis in [16, Section VII] guarantees that there exists sequence of the pairs of an encoder and a decoder satisfying (85) if and only if $\frac{n-m_n}{\sqrt{n}} \rightarrow \infty.$ The extension to the Slepian-Wolf coding has been done with the i.i.d. case [32]. For the boundary of the attainable rate region of Slepian-Wolf data compression in the stationary ergodic case [5], we can show the existence of the pair of an encoder and a decoder satisfying (85) with a suitable choice of the sequence $m_n$ under the condition $\frac{m_n}{n} \rightarrow 1$ in the following way.

Choose the rates $R_i + \delta$ for any $\delta > 0.$ Let $\varphi_n^{i, \delta} : S_i^{m_n} \rightarrow \{1, \ldots, [e^{nR_i(1+\delta)}]\}$ be Slepian-Wolf encoders and $\hat{\varphi}_n^{i, \delta} : \{1, \ldots, [e^{nR_i(1+\delta)}]\} \times \cdots \times \{1, \ldots, [e^{nR_T(1+\delta)}]\} \rightarrow S_i^{m_n} \times \cdots \times S_T^{m_n}$ be its Slepian-Wolf decoder such that $\epsilon(\varphi_n^{i, \delta}, \hat{\varphi}_n^{i, \delta}) \rightarrow 0$ with $\varphi_n^{i, \delta} := (\varphi_0^{i, \delta}, \ldots, \varphi_T^{i, \delta}).$ For an arbitrary integer $l,$ we choose an integer $n_l$ such that the inequality $\epsilon(\varphi_n^{i, \delta}, \hat{\varphi}_n^{i, \delta}) \leq \frac{1}{l}$ holds for any $n \geq n_l.$ We define $m_n$ to be $m_n := \lceil \frac{n}{1+1/l} \rceil,$ where we choose $l$ such that $n_l \leq n < n_{l+1}.$ Here, we can choose the integer $l$ for any positive integer $n.$ The construction guarantees that $R_l(1+1/l)(m_n + 1) \geq R_n \geq R_l(1+1/l)m_n.$ We define the pair of an encoder and a decoder $(\varphi_n^i, \hat{\varphi}_n^i)$ to be $(\varphi_n^{i, \delta}, \hat{\varphi}_n^{i, \delta}).$ That is, $\varphi_n^i$ is chosen to be $\varphi_n^{i, \delta}.\text{ Our choices guarantee that } m_n \approx \frac{n}{1+1/l} \rightarrow 1, \text{ and } \epsilon(\varphi_n^i, \hat{\varphi}_n^i) \approx \epsilon(\varphi_n^{i, \delta}, \hat{\varphi}_n^{i, \delta}) \leq \frac{1}{l} \rightarrow 0.\text{ In this construction, the encoder } \varphi_n^i \text{ is a map from } S_i^{m_n} \text{ to } \{1, \ldots, [e^{nR_i(1+\delta)}]\} \in \{1, \ldots, [e^{nR_i}]\} \text{ because } R_l \geq m_n R_l(1+1/l).\text{ Hence, the pair of an encoder and a decoder } (\varphi_n^i, \hat{\varphi}_n^i) \text{ satisfies the assumption of Theorem 25.}

**Proof of Theorem 25** Asssume that the code $\varphi_n = (\varphi_0^n, \ldots, \varphi_T^n)$ satisfies (85). Since the stationary ergodic source satisfies the strong converse property for the data compression, due to folklore source coding theorem [14, Theorem 3.1], the code $\varphi_n$ satisfies

$$
\lim_{n \rightarrow \infty} \frac{1}{n} H(\varphi_0^n(S_0^{m_n}), \ldots, \varphi_T^n(S_T^{m_n})) = \sum_{i=0}^T R_i.
$$

1 The following discussion does not require any property for source distribution. That is, it can be extended to Slepian-Wolf data compression for the general information source [42] in the sense of Han-Verdú [13].
Since \( \frac{1}{n}H((\varphi^n_i(S_{m_i}^n))_{i \in I}, (\varphi^n_0(S_{m_0}^n))_{i \in I}, \varphi^n_0(S_{m_0}^n)) \leq \sum_{i \in I} R_i \) and \( \frac{1}{n}H((\varphi^n_i(S_{m_i}^n))_{i \in I}, \varphi^n_0(S_{m_0}^n)) \leq R_0 + \sum_{i \in I} R_i \), we obtain (86).

In Subsection X-A we have introduced the \( \epsilon \)-strong asymptotic conditional uniformity (82) as another kind of asymptotic conditional uniformity. The following theorem shows the \( \epsilon \)-strong asymptotic conditional uniformity for the compressed data.

**Theorem 27:** We fix a sequence \( m_n \) such that \( m_n^\epsilon / n \to 1 \). We also fix an arbitrary \( \epsilon \geq 0 \) and an arbitrary non-empty proper subset \( I \subseteq \{1, \ldots, T\} \). Then, we choose the asymptotic compression rates \( R_0, \ldots, R_T \) such that \( \sum_{i \in I} R_i = H(S_0, \ldots, S_T) + \epsilon \) and

\[
\sum_{i \in I} R_i \leq H(S_T), \quad R_0 + \sum_{i \in I} R_i \leq H(S_0, S_T) \tag{87}
\]

We choose a Slepian-Wolf encoder \( \varphi^n = (\varphi^n_0, \ldots, \varphi^n_T) \) and a Slepian-Wolf decoder \( \hat{\varphi}^n \) as a map \( \varphi^n_i : S_{m_i}^n \to \{1, \ldots, [e^{nR_i}]\} \) and a map \( \hat{\varphi}^n : \{1, \ldots, [e^{nR_0}]\} \times \cdots \times \{1, \ldots, [e^{nR_T}]\} \to S_{m_0}^n \times \cdots \times S_{m_T}^n \). When the decoding error probability \( \varepsilon(\varphi^n, \hat{\varphi}^n) \) satisfies

\[
\varepsilon(\varphi^n, \hat{\varphi}^n)p(n) \to 0 \tag{88}
\]

for any polynomial \( p(n) \), the relation

\[
\lim_{n \to \infty} \frac{1}{n}H_{1+\rho_0}(((\varphi^n_i(S_{m_i}^n))_{i \in I}, (\varphi^n_0(S_{m_0}^n))_{i \in I}, \varphi^n_0(S_{m_0}^n))) \\
\geq (\sum_{i \in I} R_i) - \epsilon \geq \sum_{i \in I} (R_i - \epsilon) \tag{89}
\]

holds with \( \rho_0 = \delta \log n \) for any \( \delta > 0 \). That is, the compressed data \( (\varphi^n_0(S_{m_0}^n), \ldots, \varphi^n_T(S_{m_T}^n)) \) satisfies the \( \epsilon \)-SACU condition (82) for the non-empty proper subset \( I \subseteq \{1, \ldots, T\} \). In particular, in the case of \( \epsilon = 0 \), the compressed data \( (\varphi^n_0(S_{m_0}^n), \ldots, \varphi^n_T(S_{m_T}^n)) \) satisfies the SACU condition for the non-empty proper subset \( I \subseteq \{1, \ldots, T\} \).

Hence, if the relation (87) holds for any non-empty proper subset \( I \subseteq \{1, \ldots, T\} \), the compressed data \( (\varphi^n_0(S_{m_0}^n), \ldots, \varphi^n_T(S_{m_T}^n)) \) satisfies the \( \epsilon \)-SACU condition (82).

**Remark 28:** Theorem 27 gives only a sufficient condition (88) for the compressed data satisfying the \( \epsilon \)-SACU condition (82). Hence, it is necessary to clarify the existence of a code whose compressed data satisfying the condition (88).

In the i.i.d. case, for an arbitrary \( \epsilon > 0 \) and an arbitrary sequence \( m_n \) satisfying \( \lim_{n \to \infty} m_n^\epsilon / n = 1 \), there exists a sequence of Slepian-Wolf codes \( (\varphi^n, \hat{\varphi}^n) \) with any rate tuples given in Theorem 27 such that the decoding error probability \( \varepsilon(\varphi^n, \hat{\varphi}^n) \) goes to zero exponentially with respect to \( n \) (39). That is, there exists a Slepian-Wolf code satisfying the condition (88) in Theorem 27. However, it is not so easy to give a required code in the case of \( \epsilon = 0 \). In Appendix B we give such a code when \( m_n := \frac{n}{1+\frac{c}{n}} \) with \( t > 1/2 \) and \( c > 0 \).

**C. Proof of Theorem 27**

For the proof of Theorem 27 we prepare the following lemma for treating the relation between the conditional Rényi entropy of the compressed data and the decoding error probability. The following lemma treats the single terminal data compression for a random variable \( S \) on a set \( S \) in the single-shot setting.

**Lemma 29:** Any encoder \( \varphi : S \to \{1, \ldots, M\} \) and any decoder \( \hat{\varphi} : \{1, \ldots, M\} \to S \) for a random variable \( S \) satisfy

\[
e^{-pH_{1+\rho}(\varphi(S))} \leq e^{-pH_{1+\rho}(\varphi(S))} \leq 2^p e^{-pH_{1+\rho}(\varphi(S))} + 2^p e(\varphi, \hat{\varphi})^1 + p, \tag{90}
\]

where \( e(\varphi, \hat{\varphi}) \) is the decoding error probability \( \Pr\{S \neq \hat{\varphi}(\varphi(S))\} \).
Proof: First, we show the first inequality. Using the inequality \(x^{1+p} + y^{1+p} \leq (x + y)^{1+p}\) for \(x, y \geq 0\), we obtain

\[
\left( \sum_{s \in \varphi^{-1}(i)} P_S(s) \right)^{1+p} \geq \sum_{s \in \varphi^{-1}(i)} P_S(s)^{1+p}
\]

for any \(i = 1, \ldots, M\). Hence,

\[
e^{-pH_{1+p}(\varphi(S))} = \sum_{i=1}^{M} \left( \sum_{s \in \varphi^{-1}(i)} P_S(s) \right)^{1+p}
\]

\[
\geq \sum_{i=1}^{M} \sum_{s \in \varphi^{-1}(i)} P_S(s)^{1+p} = \sum_{s} P_S(s)^{1+p} = e^{-pH_{1+p}(S)},
\]

which implies the first inequality of (90).

Next, we show the second inequality of (90). Given an arbitrary element \(i\) in the codebook, we have two cases: (1) The element \(s_i := \hat{\varphi}(i)\) belongs to \(\varphi^{-1}(i)\), i.e., there exists exact one element \(s_i \in \varphi^{-1}(i)\) such that \(\hat{\varphi}(\varphi(s_i)) = s_i\). (2) There exists no element \(s_i \in \varphi^{-1}(i)\) such that \(\hat{\varphi}(\varphi(s_i)) = s_i\). In case (1),

\[
\left( \sum_{s \in \varphi^{-1}(i)} P_S(s) \right)^{1+p} = \left( P_S(s_i) + \sum_{s \in \varphi^{-1}(i) \setminus \hat{\varphi}(\varphi(s)) \neq s} P_S(s) \right)^{1+p}
\]

\[
= 2^{1+p} \left( \frac{1}{2} P_S(s_i) + \frac{1}{2} \sum_{s \in \varphi^{-1}(i) \setminus \hat{\varphi}(\varphi(s)) \neq s} P_S(s) \right)^{1+p}
\]

\[
\leq 2^{1+p} \left( \frac{1}{2} P_S(s_i)^{1+p} + \frac{1}{2} \left( \sum_{s \in \varphi^{-1}(i) \setminus \hat{\varphi}(\varphi(s)) \neq s} P_S(s) \right)^{1+p} \right)
\]

\[
= 2^p P_S(s_i)^{1+p} + 2^p \left( \sum_{s \in \varphi^{-1}(i) \setminus \hat{\varphi}(\varphi(s)) \neq s} P_S(s) \right)^{1+p}.
\]

In case (2),

\[
\left( \sum_{s \in \varphi^{-1}(i)} P_S(s) \right)^{1+p} = \left( \sum_{s \in \varphi^{-1}(i) \setminus \hat{\varphi}(\varphi(s)) \neq s} P_S(s) \right)^{1+p}.
\]

Hence, we obtain

\[
e^{-pH_{1+p}(\varphi(S))} = \sum_{i} \left( \sum_{s \in \varphi^{-1}(i)} P_S(s) \right)^{1+p}
\]

\[
\leq 2^p \sum_{i} P_S(s_i)^{1+p} + 2^p \left( \sum_{s \in \varphi^{-1}(i) \setminus \hat{\varphi}(\varphi(s)) \neq s} P_S(s) \right)^{1+p}
\]

\[
\leq 2^p \sum_{s} P_S(s)^{1+p} + 2^p \left( \sum_{s \in \varphi^{-1}(i) \setminus \hat{\varphi}(\varphi(s)) \neq s} P_S(s) \right)^{1+p} \tag{91}
\]

\[
= 2^p P_S(s_i)^{1+p} + 2^p \left( \sum_{s \in \varphi^{-1}(i) \setminus \hat{\varphi}(\varphi(s)) \neq s} P_S(s) \right)^{1+p}
\]

\[
= 2^p e^{-pH_{1+p}(S)} + 2^p e(\varphi, \hat{\varphi})^{1+p},
\]

where (91) follow from the inequality \(x^{1+p} + y^{1+p} \leq (x + y)^{1+p}\) for \(x, y \geq 0\). Hence, we obtain the second inequality.

Then, we obtain the following corollary of Lemma 29. The following corollary treats the single terminal data compression for a general sequence of random variables \(S_n\).
Corollary 30: Let $\varphi^n$ be an encoder and $\hat{\varphi}^n$ be a decoder for a general sequence of random variables $S_n$. When the decoding error probabilities $\epsilon(\varphi^n, \hat{\varphi}^n)$ and the sequence $\{\rho_n\}$ of positive real numbers satisfy
\[
\lim_{n \to \infty} \epsilon(\varphi^n, \hat{\varphi}^n)^{1/\rho_n} e^{H_1(S_n)} = 0,
\] (92)
we have
\[
\lim_{n \to \infty} \frac{1}{n} H_{1+\rho_n}(\varphi^n(S_n)) = \lim_{n \to \infty} \frac{1}{n} H_{1+\rho_n}(S_n).
\] (93)

Proof of Corollary 30: The inequality $\lim_{n \to \infty} \frac{1}{n} H_{1+\rho_n}(\varphi^n(S_n)) \leq \lim_{n \to \infty} \frac{1}{n} H_{1+\rho_n}(S_n)$ follows from the first inequality (90). We show only the inequality $\lim_{n \to \infty} \frac{1}{n} H_{1+\rho_n}(\varphi^n(S_n)) \geq \lim_{n \to \infty} \frac{1}{n} H_{1+\rho_n}(S_n)$. Using the second inequality in (90), we have
\[
\lim_{n \to \infty} \frac{1}{n} H_{1+\rho_n}(\varphi^n(S_n)) = \lim_{n \to \infty} \frac{1}{n} \log e^{-\rho_n H_{1+\rho_n}(\varphi^n(S_n))}
\]
\[
\geq \lim_{n \to \infty} \frac{1}{n} \log(2^{\rho_n} e^{-\rho_n H_{1+\rho_n}(S_n)} + 2^{\rho_n} \epsilon(\varphi^n, \hat{\varphi}^n)^{1/\rho_n})
\]
\[
= \lim_{n \to \infty} \frac{1}{n} \log(2^{\rho_n} e^{-\rho_n H_{1+\rho_n}(S_n)})
\]
\[
= \lim_{n \to \infty} \frac{1}{n} (H_{1+\rho_n}(S_n) - \log 2) = \lim_{n \to \infty} \frac{1}{n} H_{1+\rho_n}(S_n),
\] (94)
where (94) follows from the assumption (92).

Now, we show Theorem 27.

Proof of Theorem 27: For the proof of Theorem 27 we choose $\rho'_n$ so that $\rho'_n(1-\rho'_n) = \rho_n$. Since $\lim_{n \to \infty} \frac{m_n}{n} = 1$ and $\rho \geq \rho'_n$ for all $n$, we have
\[
H_{1+\rho}(S_0, \ldots, S_T) \leq \liminf_{n \to \infty} \frac{1}{n} H_{1+\rho_n}(S_0^{m_n}, \ldots, S_T^{m_n}) \leq \limsup_{n \to \infty} \frac{1}{n} H_{1+\rho_n}(S_0^{m_n}, \ldots, S_T^{m_n}) \leq H(S_0, \ldots, S_T).
\] Since $\rho'_n \to 0$ and $\lim_{\rho' \to 0} H_{1+\rho}(S_0, \ldots, S_T) = H(S_0, \ldots, S_T),
\[
\lim_{n \to \infty} \frac{1}{n} H_{1+\rho_n}(S_0^{m_n}, \ldots, S_T^{m_n}) = H(S_0, \ldots, S_T).
\] (95)

Since $\rho'_n$ behaves as $\frac{\delta \log n}{n}$, due to the relation (95), the quantity $\epsilon(\varphi^n, \hat{\varphi}^n)^{1/\rho'_n}$ behaves as $\epsilon(\varphi^n, \hat{\varphi}^n)^{1/\rho'_n} = n^{\delta H(S_0, \ldots, S_T)}$. Since $\epsilon(\varphi^n, \hat{\varphi}^n)^{1/\rho'_n} \leq \epsilon(\varphi^n, \hat{\varphi}^n)$, the condition (88) guarantees the condition (92). Hence, Corollary 30 guarantees that
\[
\lim_{n \to \infty} \frac{1}{n} H_{1+\rho_n}(\varphi_0^n(S_0^{m_n}), \ldots, \varphi_T^n(S_T^{m_n})) = (\sum_{i=0}^{T} R_i) - \epsilon.
\]
Since $\log \varphi_0^n(S_0^{m_n}) \times \prod_{i \in I} \varphi_i^n(S_i^{m_i}) = n(R_0 + \sum_{i \in I} R_i)$, Corollary 27 in Appendix A implies (89).

VIII. Secure Multiplex Coding with Common Messages: Asymptotic Performance

In this section, we treat the asymptotic asymptotic performance for the secure multiplex coding with common messages when the channel is given as the $n$-fold discrete memoryless channel of a given broadcast channel $P_{YZ|X}$. First, we treat what performance can be achieved by using Code Ensemble 2 and Theorem 16 in Subsection VI-B without any assumption for the distribution of sources. In the next step, we define the capacity region under the asymptotic uniformity of information sources. In SMC, this restriction for the sources is essential for our definition of the capacity region. After this definition, we concretely give the capacity region.
A. General Sequence of Information Sources

First, we treat the secure multiplex coding with common messages with general sequence of information sources. For a given set of rates \( (R_i)_{i=0}^T \), we give a general sequence of source distributions \( P_{S,T,i} \) on the message sets \( S_{i,n} \) for \( i = 0, 1, \ldots, T \) satisfying the relations \( |S_{i,n}| = e^{nR_i} \) for \( i = 0, 1, \ldots, T \). For a given Markov chains \( U \rightarrow V \rightarrow X \rightarrow YZ \), we give an asymptotic code construction in the following way.

**Code Construction 4:** For a given set of rates \( (R_i)_{i=0}^T \), we introduce other parameters \( R_p \) and \( R_c \) satisfying

\[
R_c + R_p = \sum_{i=0}^T R_i, \quad R_c \geq R_0.
\]

In the following, we denote the set of \( ((R_i)_{i=0}^T, R_p, R_c) \) satisfying the above condition by \( \mathcal{R}_T \). In order to apply Code Ensemble 2 in Subsection VI-B, we fix Abelian groups \( |B_{1,n}| \) and \( |B_{2,n}| \) satisfying \( |\mathcal{B}_{1,n}| = e^{n(R_c - R_0)} \) and \( |\mathcal{B}_{2,n}| = e^{nR_p} \). Applying Code Ensemble 2 and Theorem 16 to the \( n \)-fold discrete memoryless extension \( U^n \rightarrow V^n \rightarrow X^n \rightarrow Y^nZ^n \) of the above Markov chain and the Abelian groups \( \mathcal{B}_{1,n} \) and \( \mathcal{B}_{2,n} \), we find the code \( \varphi_n = (\varphi_{a,n}, \varphi_{b,n}, \varphi_{c,n}) \) with the message sets \( S_{i,n} \) for \( i = 0, 1, \ldots, T \) satisfying (48), (49), (50), and (51).

The performance of the code \( \varphi_n \) is characterized as follows. The relations (50) and (51) guarantee that

\[
\lim \inf_{n \to \infty} \frac{1}{n} \log P_e[P_{Z|V,V_{S_{T,n}}, \varphi_n, P_{S_{T,n}}}] 
\]

\[
\geq -\rho R_p - \max[E_0(-\rho)|P_{Y|V,V}, P_U], E_0(-\rho)|P_{Y|U,V}, P_U],
\]

\[
\geq -\rho R_c - E_0(-\rho)|P_{Z|U,V}, P_U],
\]

for any \( \rho \in (0, 1] \). When

\[
R_c < \min[I(Y; U), I(Z; U)], \quad R_p < I(Y; V|U),
\]

the above both exponents (97) and (98) are positive, i.e., the both error probabilities go to zero exponentially. Further, due to (52), the leaked information for \( S_{i,n} \) can be evaluated as

\[
\frac{1}{n} I(S_{i,n}; Z^n|S_{0,n})[P^n_{Z|V,V_{S_{T,n}}, \varphi_{a,n}, P_{S_{T,n}}}] 
\]

\[
\leq \left[(R_c - R_0) + \frac{1}{n} H_1(S_{i,n}|S_{0,n}, S_{i,n})\right] + (T + 3) \frac{\log 2}{n \rho},
\]

Substituting \( \rho = a/n \) with an arbitrary real \( a > 0 \) and taking the limits \( n \to \infty \), we obtain

\[
\lim \sup_{n \to \infty} \frac{1}{n} I(S_{i,n}; Z^n|S_{0,n})[P^n_{Z|V,V_{S_{T,n}}, \varphi_{a,n}, P_{S_{T,n}}}] 
\]

\[
\leq \left[(R_c - R_0) + I(Z; V|U) - \lim \inf_{n \to \infty} \frac{1}{n} H_1(S_{i,n}|S_{i,n}, S_{0,n})\right] + (T + 3) \frac{\log 2}{a},
\]

Taking the limits \( a \to \infty \), we obtain

\[
\lim \sup_{n \to \infty} \frac{1}{n} I(S_{i,n}; Z^n|S_{0,n})[P^n_{Z|V,V_{S_{T,n}}, \varphi_{a,n}, P_{S_{T,n}}}] 
\]

\[
\leq \left[(R_c - R_0) + I(Z; V|U) - \lim \inf_{a \to \infty} \frac{1}{n} H_1(S_{i,n}|S_{i,n}, S_{0,n})\right].
\]

So, the asymptotic performance of our code is characterized in (97), (98), and (100).
B. Capacity Region

Next, in order to characterize the limit of the asymptotic performance of the secure multiplex coding with common messages, we define the capacity region based on the WACU condition (80). For this purpose, we treat the transmission rate tuple \((R_i)_{i=0,...,T} = (R_0, R_1, \ldots, R_T)\) and the information leakage rate tuple \((R_{l,I})_{\emptyset \neq I \subseteq \{1,...,T\}}\). The latter describes the rate of the leaked information for the message \(S_{I,n}\). Combining both tuples, we call \(((R_i)_{i=0,...,T}, (R_{l,I})_{\emptyset \neq I \subseteq \{1,...,T\}})\) the rate tuple.

**Definition 31:** The rate tuple \(((R_i)_{i=0,...,T}, (R_{l,I})_{\emptyset \neq I \subseteq \{1,...,T\}})\) is said to be achievable for the secure multiplex coding with \(T\) secret messages for the channel \(P_{Y|Z,X}\) if there exist a sequence of codes \(\varphi_n = (\varphi_{a,n}, \varphi_{e,n}, \varphi_{e,n})\), i.e., Alice’s stochastic encoder \(\varphi_{a,n}\) from \(S_{0,n} \times S_{1,n} \times \cdots \times S_{T,n}\) to \(X^n\), Bob’s deterministic decoder \(\varphi_{e,n}: Y^n \rightarrow S_{0,n} \times S_{1,n} \times \cdots \times S_{T,n}\) and Eve’s deterministic decoder \(\varphi_{e,n}: Z^n \rightarrow S_{0,n}\) satisfying the following conditions: (1) The \(i\)-th secret message set \(S_{i,n}\) has cardinality \(e^{R_i}\); for \(i = 1, \ldots, T\), and the common message set \(S_{0,n}\) has cardinality \(e^{R_0}\). (2) When a sequence of joint distributions \(P_{S_{T,n}}\) on the message sets \(S_{i,n}\) for \(T = 0, 1, \ldots, T\) satisfies the WACU condition (80) for a non-empty proper subset \(I \subset \{1, \ldots, T\}\), the relations

\[
\lim_{n \to \infty} P_b[P_{Y|X}^{n}, \varphi_n, P_{S_{T,n}}] = 0 \\
\lim_{n \to \infty} P_e[P_{Z|X}^{n}, \varphi_n, P_{S_{T,n}}] = 0 \\
\lim_{n \to \infty} I(S_{I,n}; Z^n|S_0^n)[P_{Z|X}^{n}, \varphi_n, P_{S_{T,n}}] \leq R_{l,I}
\]

hold. The capacity region \(C\) of the secure multiplex coding is the closure of the achievable rate tuples \(((R_i)_{i=0,...,T}, (R_{l,I})_{\emptyset \neq I \subseteq \{1,...,T\}})\).

**Theorem 32:** The capacity region of the secure multiplex coding with common messages is given by the set of rate tuples \(((R_i)_{i=0,...,T}, (R_{l,I})_{\emptyset \neq I \subseteq \{1,...,T\}})\) such that there exist a Markov chain \(U \rightarrow V \rightarrow X \rightarrow YZ\) and

\[
R_0 \leq \min[I(U; Y), I(U; Z)] \\
\sum_{i=0}^{T} R_i \leq I(V; Y|U) + \min[I(U; Y), I(U; Z)] \\
R_{l,I} \geq \sum_{i \in I} R_i - [I(Y; V|U) - I(Z; V|U)]_+
\]

for any non-empty proper subset \(I \subset \{1, \ldots, T\}\).

**Proof:** The converse part of this coding theorem follows from that for Corollary 4 with the uniform distribution on the whole message sets. The direct part can be shown by the following lemma.

**Lemma 33:** Choose a sufficiently small real number \(\epsilon > 0\) and \((R_i)_{i=0}^{T}\) for \(i = 0, 1, \ldots, T\) satisfying

\[
R_0 < \min[I(U; Y), I(U; Z)], \\
\sum_{i=0}^{T} R_i + \epsilon < I(V; Y|U) + \min[I(U; Y), I(U; Z)]
\]

Then, the code \(\varphi_n\) given by Code Construction 4 with the choices \(R_p := I(Y; V|U) - \epsilon\) and \(R_c := \sum_{i=0}^{T} R_i - R_p\) satisfies

\[
\lim_{n \to \infty} P_b[P_{Y|V}^{n}, \varphi_n, P_{S_{T,n}}] = 0 \\
\lim_{n \to \infty} P_e[P_{Z|V}^{n}, \varphi_n, P_{S_{T,n}}] = 0
\]

and

\[
\limsup_{n \to \infty} \frac{1}{n} I(S_{I,n}; Z^n|S_0^n)[P_{Z|V}^{n}, \varphi_n, P_{S_{T,n}}] \\
\leq \sum_{i \in I} R_i - [I(Y; V|U) - I(Z; V|U)]_+ + \epsilon
\]
when the sequence of the joint distributions $P_{S_{T,n}}$ of information source satisfies the WACU condition for any non-empty proper subset $I \subseteq \{1, \ldots, T\}$.

**Proof:** Since the conditions (105) and (106) guarantee the conditions (99), we obtain (107) and (108). We need to show only (109). Assume that $I(Y; V|U) \leq I(Z; V|U)$. Since $|S_{I,n}| = e^n \sum_{i \in I} R_i$, we obtain

$$\frac{1}{n} I(S_{I,n}; Z_n|S_0,n)[P^n_{Z|V}, \varphi_n, P_{S_{T,n}}] \leq \sum_{i \in I} R_i,$$

which implies (109). Hence, it is enough to consider the case $I(Y; V|U) > I(Z; V|U)$. Since, as is shown in Lemma 83 in Appendix C, the equivalence between the SWACU condition (81) and the WACU condition (80) holds, we obtain

$$\lim_{a \to \infty} \lim_{n \to \infty} \frac{1}{n} H_{1+a/n}(S_{I,n}|S_0,n) = \sum_{i \in I} R_i. \quad (110)$$

The relations (100) and (110) yield

$$\lim_{n \to \infty} \frac{1}{n} I(S_{I,n}; Z_n|S_0,n)[P^n_{Z|V}, \varphi_n, P_{S_{T,n}}] \leq (R_c - R_0) + I(Z; V|U) - \sum_{i \in I} R_i \quad \text{(111)}$$

Therefore, since $R_p = I(Y; V|U) - \epsilon$, (111) implies (109) when $I(Y; V|U) > I(Z; V|U)$.

**IX. Secure Multiplex Coding with Common Messages: Strong Security**

**A. Strong Security**

In this section, we treat the strong security. A sequence of codes $\varphi_n$ is called strongly secure for a subset $I \subseteq \{1, \ldots, T\}$ and a sequence of distributions $P_{S_{T,n}}$ when the relation

$$\lim_{n \to \infty} I(S_{I,n}; Z_n|S_0,n)[P^n_{Z|X}, \varphi_n, P_{S_{T,n}}] = 0 \quad (112)$$

holds. Now, we fix a family $J$ of non-empty proper subsets $I$ of $\{1, \ldots, T\}$, and consider only the security of the messages $S_{I,n}$ for all $I \in J$.

**Theorem 34:** Assume that the transmission rate tuple $(R_i)_{i=0,\ldots,T} = (R_0, R_1, \ldots, R_T)$ belongs to the inner of the capacity region with $R_{i,T} = 0$ for any subset $I \in J$, i.e., there exist an information leakage rate rate tuple $(R_{i,I})_{\emptyset \neq I \in J}$ such that

$$((R_i)_{i=0,\ldots,T}, (0)_{I \in J}\setminus I, (R_{i,I})_{\emptyset \neq I \in J}) \in \text{inn}(C), \quad (113)$$

where $\text{inn}(C)$ denotes the inner of the set $C$. Then, there exists a Markov chain $U \to V \to X$ such that

$$\epsilon := \min_{I \in J} \frac{I(Y; V|U) - I(Z; V|U) - \sum_{i \in I} R_i}{|I'|} > 0, \quad (114)$$

$$R_0 < \min [I(U; Y), I(U; Z)],$$

$$\sum_{i=0}^T R_i < I(V; Y|U) + \min [I(U; Y), I(U; Z)].$$

Next, we choose a small real $\epsilon' > 0$ such that $\epsilon' < \frac{\epsilon}{2}$ and $\epsilon' < I(V; Y|U) + \min [I(U; Y), I(U; Z)] - \sum_{i=0}^T R_i$. The code $\varphi_n$ given by Code Construction 4 with the choices $R_p := I(Y; V|U) - \epsilon$ and $R_c := \sum_{i=0}^T R_i - R_p$ satisfies (107), (108), and the strong security

$$\lim_{n \to \infty} I(S_{I,n}; Z_n|S_0,n)[P^n_{Z|V}, \varphi_n, P_{S_{T,n}}] = 0 \quad (115)$$
for any subset $I \in J$ when the sequence of distributions $P_{ST,n}$ satisfies the $(\epsilon - 2\epsilon')$-SACU condition \eqref{eq:116} for the subset $I$.

Thanks to Theorem 34, the strong security holds at all inner points of the capacity region $C$ with $R_{I,J} = 0$ for any subset $I \in J$ under the $\epsilon$-SACU condition \eqref{eq:116} for any subset $I \in J$. In order to show Theorem 34 we prepare the following lemma.

**Lemma 35:** We fix a subset $I \subseteq \{1, \ldots, T\}$. Assume that the transmission rate tuple $(R_i)_{i=0,\ldots,T}$, the sequence of distributions $P_{ST,n}$, and a Markov chain $U \rightarrow V \rightarrow X$ satisfy that

$$
\delta' := \frac{1}{2}\left( H_{\text{log}}(I^c) \right.
- \left( \sum_{i=1}^{T} R_i - I(Y; V|U) + I(Z; V|U) \right) > 0,
\tag{116}
$$

$$
R_0 < \min[I(U; Y), I(U; Z)],
$$

$$
\sum_{i=0}^{T} R_i < I(V; Y|U) + \min[I(U; Y), I(U; Z)].
$$

When we choose a small real $\epsilon' > 0$ such that $\epsilon' \leq \delta'$ and $\epsilon' < I(V; Y|U) + \min[I(U; Y), I(U; Z)] - \sum_{i=0}^{T} R_i$, the code $\varphi_n$ given by Code Construction 4 with the choices $R_p := I(Y; V|U) - \epsilon'$ and $R_c := \sum_{i=0}^{T} R_i - R_p$ satisfies \eqref{eq:107}, \eqref{eq:108}, and the strong security

$$
\lim_{n \rightarrow \infty} I(S_{I,n}; Z_n|S_{0,n})[P_{Z|V}^n, \varphi_n, P_{ST,n}] = 0.
\tag{117}
$$

**Proof of Theorem 34:** First, we fix an arbitrary subset $I \in J$. Hence,

$$
\sum_{i \in I^c} (R_i - (\epsilon - 2\epsilon')) - \left( \sum_{i=1}^{T} R_i - I(Y; V|U) + I(Z; V|U) \right)
\geq \left( \sum_{i \in I^c} R_i \right) - |I^c| (\epsilon - 2\epsilon') - \left( \sum_{i=1}^{T} R_i - I(Y; V|U) + I(Z; V|U) \right)
= I(Y; V|U) - I(Z; V|U) - \sum_{i \in I^c} R_i - |I^c| (\epsilon - 2\epsilon')
\geq |I^c| (\epsilon - |I^c| (\epsilon - 2\epsilon')) = 2|I^c| \epsilon' \geq 2\epsilon'.
$$

Thus, since the sequence of distributions $P_{ST,n}$ satisfies the $\epsilon - 2\epsilon'$-SACU condition \eqref{eq:116} for the subset $I$,

$$
\delta' := \frac{1}{2}\left( H_{\text{log}}(I^c) \right.
- \left( \sum_{i=1}^{T} R_i - I(Y; V|U) + I(Z; V|U) \right)
\geq \frac{1}{2}\left( \sum_{i \in I^c} (R_i - (\epsilon - 2\epsilon')) - \left( \sum_{i=1}^{T} R_i - I(Y; V|U) + I(Z; V|U) \right) \right) \geq \epsilon'.
$$

Hence, any real number $\epsilon' > 0$ given in Theorem 34 satisfies the condition for $\epsilon' > 0$ in Lemma 35. Hence, applying Lemma 35 we obtain \eqref{eq:115} for the subset $I$. Since the subset $I$ is an arbitrary element
of $J$, we obtain Theorem 34.

**Proof of Lemma 35.** Since $\varepsilon' > 0$, we have the second condition of (99). Due to the choice of $\varepsilon' > 0$,  
\[ 0 = I(Y; V|U) - \varepsilon' - R_p \]
\[ > I(Y; V|U) - \left( I(V; Y|U) + \min[I(U; Y), I(U; Z)] - \sum_{i=0}^{T} R_i \right) - R_p \]
\[ = \sum_{i=0}^{T} R_i - \min[I(U; Y), I(U; Z)] - R_p \]
\[ = R_c - \min[I(U; Y), I(U; Z)], \]

which implies the first condition of (99). Hence, we obtain (107) and (108).

Next, we define
\[ \rho_n := \frac{2 \log n}{n \delta'}, \]
\[ C_n := \left( -\rho_n n(R_c - R_0) + \rho_n H_{1+n}(S_{I,S,n} | S_{I,n}, S_{0,n}) \right. \]
\[ \left. \quad - n E_0(\rho_n | P_{Z|V}, P_{V|U}, P_U) \right). \]

The condition (116) and $\varepsilon' \leq \delta'$ imply that
\[
\liminf_{n \to \infty} \frac{C_n}{n \rho_n} \geq \frac{\delta'}{2}
\]

That is, we can choose a sufficiently large integer $N$ such that
\[
\frac{C_n}{n \rho_n} \geq \frac{\delta'}{2}
\]

for $n \geq N$. Due to (99), the leaked information for $S_{I,n}$ can be evaluated as
\[ I(S_{I,n}; Z_n | S_{0,n}) [P_{Z|V}^n, \varphi_n, P_{S_{T,n}}] \leq \frac{2^{T+2} e^{-C_n}}{\rho_n}. \]

Since (119) implies that
\[
- \log(\frac{2^{T+2} e^{-C_n}}{\rho_n}) = -(T + 2) \log 2 + C_n + \log \rho_n \]
\[ \geq -(T + 2) \log 2 + \frac{\delta'}{2} n \rho_n + \log \rho_n \]
\[ = -(T + 2) \log 2 + \log n - \log \frac{\delta'}{2} \to \infty, \]

we obtain (117).
B. Exponential Decreasing Rate

In this subsection, we treat the exponential decreasing rate of leaked information. Unfortunately, the \( \epsilon \)-SACU condition \( (82) \) is not sufficient for deriving a good exponential decreasing rate of leaked information. Hence, in this subsection, given a sequence of distributions \( P_{S,T,n} \), we introduce the following quantity

\[
H_{1+p}(I^c) := \liminf_{n \to \infty} \frac{1}{n} H_{1+p}(S_{I,n}|S_{I,n}, S_{0,n})
\]

for any subset \( I \subset \{1, \ldots, T\} \) and any \( \rho \in (0, 1) \).

Theorem 36: For given \( (R_i)_{i=0}^T \), we choose \( R_p \) and \( R_c \) as follows.

\[
R_c \geq R_0, \quad R_c + R_p = \sum_{i=0}^T R_i.
\]

We fix a real number \( \epsilon > 0 \). We choose a code \( \varphi_n \) given by Code Construction 4 with the above choices \( R_p \) and \( R_c \) and a given Markov chain \( U \to V \to X \). When the sequence of distributions \( P_{S,T,n} \) satisfies the \( \epsilon \)-SACU condition \( (82) \) for a non-empty proper subset \( I \subset \{1, \ldots, T\} \), the sequence of codes \( \varphi_n \) satisfies \( (97), (98) \), and

\[
\liminf_{n \to \infty} \frac{1}{n} \log I(S_{I,n}; Z_n|S_{0,n})[P_{Z,V}\phi, \Phi_n, P_{S,T,n}] \\
\geq \sup_{0 < \rho < 1} \rho(H_{1+p}(I^c) - R_c + R_0) - E_0(\rho|P_{Z,V}, P_{V|U}, P_U).
\]

In particular, when the distribution \( P_{S,T,n} \) is uniform, we obtain

\[
\liminf_{n \to \infty} \frac{1}{n} \log I(S_{I,n}; Z_n|S_{0,n})[P_{Z,V}\phi, \Phi_n, P_{S,T,n}] \\
\geq E_0^E(R_p - \sum_{i \in I} R_i, P_{Z,V,U}),
\]

where

\[
E_0^E(R, P_{Z,V,U}) := \sup_{0 < \rho < 1} \rho R - E_0(\rho|P_{Z,V}, P_{V|U}, P_U).
\]

Theorem 36 yields the following observation. When \( R_p - \epsilon - \sum_{i \in I} R_i > I(Z; V|U) \) and \( H_{1+p}(I^c) \geq (\sum_{i \in I} R_i) - \epsilon \) holds with a small \( \rho > 0 \), the exponent \( (121) \) is positive, i.e., the leaked information goes to zero exponentially. In particular, when

\[
\sum_{i=1}^T R_i < I(Y; V|U), \quad R_0 < \min[I(U; Y), I(U; Z)],
\]

we can choose \( R_p \) and \( R_c \) by

\[
R_p := \sum_{i=1}^T R_i, \quad R_c := R_0.
\]

Then, the inequalities \( (97) \) and \( (98) \) can be simplified to

\[
\liminf_{n \to \infty} \frac{1}{n} \log P_b[P_{Y|V}, \Phi_n, P_{S,T,n}] \\
\geq - \rho \sum_{i=1}^T R_i - \max[E_0(-\rho|P_{Y|V}, P_{V|U}, P_U), E_0(-\rho|P_{Y|U,V}, P_{V,U})],
\]

\[
\liminf_{n \to \infty} \frac{1}{n} \log P_e[P_{Z|V}, \Phi_n, P_{S,T,n}] \geq -\rho R_0 - E_0(-\rho|P_{Z|U}, P_U)
\]

(127)
with any $\rho \in (0, 1]$. Then, the both decoding error probabilities goes zero exponentially. Further, the inequality (121) can be simplified to
\[
\liminf_{n \to \infty} \frac{-1}{n} \log I(S_{I,n}; Z_n | S_{0,n}) [P^n_{Z,V}, \Phi_n, P_{S,T,n}] 
\geq \sup_{0 < \rho < 1} \rho H_{1+\rho}(I^c) - E_0(\rho | P_{Z,V}, P_{V|U}, P_U).
\] (128)

Further, in the case of (124) and (125), when the WACU condition holds for $I$, the inequality (100) can be simplified to
\[
\limsup_{n \to \infty} \frac{1}{n} I(S_{I,n}; Z_n | S_{0,n}) [P^n_{Z,V}, \Phi_n, P_{S,T,n}] 
\leq R_c - R_0 + I(Z; V|U) - \sum_{i \in I^c} R_i = I(Z; V|U) - \sum_{i \in I} R_i.
\] (129)

**Proof of Theorem 36**: In Subsection VIII-A we have already shown (97) and (98). Hence, we need to only show (121). Due to (49), the leaked information for $S_{I,n}$ can be evaluated as
\[
I(S_{I,n}; Z_n | S_{0,n}) [P^n_{Z,V}, \Phi_n, P_{S,T,n}] 
\leq 2^{T+2} e^{\epsilon n(R_c - R_0) - \rho H_{1+\rho}(I_{S_{I,n}, S_{0,n}}) + n E_0(\rho | P_{Z,V}, P_{V|U}, P_U)}.
\]

Hence,
\[
\liminf_{n \to \infty} \frac{-1}{n} \log I(S_{I,n}; Z_n | S_{0,n}) 
\geq \rho \liminf_{n \to \infty} \frac{1}{n} I(S_{I,n}; S_{I,n}, S_{0,n}) 
- \rho(R_c - R_0) - E_0(\rho | P_{Z,V}, P_{V|U}, P_U) 
\geq \sup_{0 < \rho < 1} \rho(H_{1+\rho}(I^c) - R_c + R_0) - E_0(\rho | P_{Z,V}, P_{V|U}, P_U).
\]

Hence, we obtain (121).

When the condition (124) holds, the exponent (128) can be improved by using Theorem 17 with Code Ensemble 3 in the following way.

**Theorem 37**: We fix a real number $\epsilon \geq 0$. Let $\varphi_n$ be a code given in Code Ensemble 3 in Subsection VI-C satisfying (70), (67), (68), and (69) of length $n$ with $|S_{i,n}| := e^{\epsilon n R}$ for $i = 0, 1, \ldots, T$ and a given Markov chain $U \to V \to X$. The sequence of codes $\varphi_n$ satisfies (126), (127), (129), and
\[
\liminf_{n \to \infty} \frac{-1}{n} \log I(S_{I,n}; Z_n | S_{0,n}) [P^n_{Z,V}, \Phi_n, P_{S,T,n}] \geq \max_{0 \leq \rho \leq 1} \rho H_{1+\rho}(I^c) - \psi(\rho | P_{Z,V}, P_{V|U}, P_U).
\] (130)

In particular, when the distribution $P_{S,T,n}$ are uniform, we obtain
\[
\liminf_{n \to \infty} \frac{-1}{n} \log I(S_{I,n}; Z_n | S_{0,n}) [P^n_{Z,V}, \Phi_n, P_{S,T,n}] \geq \tilde{E}_\psi(\sum_{i \in I^c} R_i, P_{Z,V,U}),
\]

where
\[
\tilde{E}_\psi(R, P_{Z,V,U}) := \max_{0 \leq \rho \leq 1} \rho R - \psi(\rho | P_{Z,V}, P_{V|U}, P_U).
\]

Now, we compare Theorems 36 and 37. Since (130) is larger than (128) due to (32), Theorem 37 is better than Theorem 36 when the relation (124) holds. Otherwise, the error exponent of (126) and/or (127) is not positive. That is, Theorem 37 cannot yield a reliable communication. In summary, Theorem 36 has a wider applicability than Theorem 37. In the special case (124), Theorem 37 is better than Theorem 36.
Proof: The conditions (68) and (69) guarantee (126) and (127). Due to the $\epsilon$-SACU condition, the condition (70) guarantees (129). Using (67) and the $\epsilon$-SACU condition, we obtain

$$I(S_{\mathcal{I},n}; Z_0|S_{0,n})[P_{Z|V}^n, \Phi_n, P_{S_{\mathcal{T},n}}] \leq \frac{2T+2}{\rho} e^{-\rho H_{1,\psi}(S_{\mathcal{F},n}|S_{\mathcal{I},n}, S_{0,n}) + n\psi(P_{Z|V}, P_{V|U}, P_U)}.$$  

Then,

$$\lim_{n\to\infty} \frac{1}{n} \log I(S_{\mathcal{I},n}; Z_0|S_{0,n})[P_{Z|V}^n, \Phi_n, P_{S_{\mathcal{T},n}}] \geq \rho H_{1,\psi}(I^c) - \psi(P_{Z|V}, P_{V|U}, P_U).$$  

Hence, we obtain (130).

When the above discussion is applied to the wire-tap channel model, we obtain an extension of existing results to the case of the asymptotic uniform dummy message. That is, we consider the case with no common messages and $T = 2$ when $S_1$ corresponds to the message to be secretly sent to Bob, and $S_2$ does to the dummy message making $S_1$ ambiguous to Eve. For a given rate $R_1$ of secret message and a given rate $R_2$ of dummy message, the RHS of (126) coincides with the Gallager exponents, the RHS of (128) coincides with the RHS of (59) in [15], and the RHS of (130) coincides with the exponents of the RHS of (15) in [17].

X. Practical Code Construction

In Section X, we consider how we can construct practically usable encoder and decoder for the secure multiplex coding. In this section, we treat practical code construction in the single-shot setting unless otherwise stated.

It is a common practice to assume the uniform distribution of messages when one evaluates the decoding error probability, and decoding error probabilities with non-uniform message distributions are rarely considered in practice. Thus, we always assume the uniform message distribution when we treat the decoding error probability in this section. On the other hand, the message distribution can be arbitrary when we treat the leaked information to Eve.

A. First Practical Code Construction: First Type Evaluation

We construct a code for the secure multiplex coding based on a given code $\varphi_p$ for BCD with the common message in $S_c$ and the private message in $S_p$. We assume that encoding and decoding of $\varphi_p$ can be efficiently executed. We shall attach $F'$ and $G'$ in the second step of Code Ensemble 2 to $\varphi_p$ so that the resulting code for SMC enables efficient encoding and decoding. This type of construction is much more practical than Code Ensemble 2 because Code Ensemble 2 uses the random coding for the error correcting code $\varphi_p$, which does not enable efficient encoding nor decoding. To use the code with $F'$ and $G'$ attached, we have to evaluate decoding error probability and the amount of information leaked to Eve. The former is less than or equal to that of the underlying error correcting code $\varphi_p$, and the average of the latter over the ensemble of $F'$ and $G'$ can be evaluated by Lemma 15 with a fixed error correcting code $\varphi_p$. Now, we present a code construction.

Code Construction 5: First, in order to apply Lemma 15 we divide the common message set $S_c$ of the BCD code $\varphi_p$ to $S_0 \times B_1$, and denote the private message set $S_p$ of $\varphi_p$ by $B_2$. That is, the code $\varphi_p$ is regarded as a map from $S_0 \times B_1 \times B_2$ to $X$. Then, based on the code $\varphi_p$, assuming the Abelian group structures in $B_1$ and $B_2$, we choose an ensemble of isomorphisms $F'$ from $S_1 \times \cdots \times S_T$ to $B_1 \times B_2$ as Abelian groups satisfying Condition 10 while we do not assume any algebraic assumption for the code $\varphi_p$. We choose the random variable $G' \in B_1 \times B_2$ that obeys the uniform distribution on $B_1 \times B_2$ and is independent of the choice of $F'$ and anything else. Then, by defining a map $\Lambda_{F',G'}(s) := F'(s) + G'$,
we obtain our encoder \( \varphi_p \circ \Lambda_{F',G'}(s_0, s_1, \ldots, s_T) = \varphi_p(s_0, \Lambda_{F',G'}(s_1, \ldots, s_T)) \). The decoder is constructed by applying the inverse \( \Lambda_{F',G'}^{-1}(b_1, b_2) = F'^{-1}(b_1, b_2) - G' \) to the decoded message of the code \( \varphi_p \).

The average of the leaked information of the above constructed code is evaluated as follows.

**Lemma 38:**

\[
E_{F',G'} I(S_I; Z|S_0)[P_{Z|V}, \varphi_p \circ \Lambda_{F',G'}, P_{S_T}] \\
\leq e^{E_{0,\max}(\rho|P_{Z|V}) - H_{1_{\rho}}(S_F|S_I S_0)}.
\]  \( (132) \)

**Proof:** Applying Lemma 15, we obtain

\[
E_{F',G'} \exp(\rho I(S_I; Z|S_0)[P_{Z|V}, \varphi_p \circ \Lambda_{F',G'}, P_{S_T}]) \leq 1 + \sum_{s_0} P_{S_0}(s_0) \sum_{s_I} P_{S_I|S_0}(s_I|s_0)e^{-\rho H_{1_{\rho}}(S_F|S_I S_0 = s_0)} \\
\cdot e^{\rho|P_{Z|V}(Z|\varphi_p(s_0, b_1, b_2))}.
\]  \( (133) \)

Since

\[
e^{\rho|P_{Z|V}(Z|\varphi_p(s_0, b_1, b_2))} \leq e^{E_0(\rho|P_{Z|V}) - H_{1_{\rho}}(S_F|S_I S_0 = s_0)},
\]

we obtain

\[
E_{F',G'} \exp(\rho I(S_I; Z|S_0)[P_{Z|V}, \varphi_p \circ \Lambda_{F',G'}, P_{S_T}]) \leq 1 + \sum_{s_0} P_{S_0}(s_0) e^{-\rho H_{1_{\rho}}(S_F|S_I S_0 = s_0)} \cdot \sum Z \left( \sum_{b_1, b_2} \frac{1}{|B_1||B_2|} P_{Z|V}(Z|\varphi_p(s_0, b_1, b_2))^\frac{1}{1-\rho} \right)^{1-\rho}.
\]  \( (134) \)

It can be simplified as follows.

\[
\sum Z \left( \sum_{b_1, b_2} \frac{1}{|B_1||B_2|} P_{Z|V}(Z|\varphi_p(s_0, b_1, b_2))^\frac{1}{1-\rho} \right)^{1-\rho} \leq \max_{P_V} \sum_{Z} \left( \sum_{V} P_V(V) P_{Z|V}(Z|V)^\frac{1}{1-\rho} \right)^{1-\rho}
\]

\[
= \max_{P_V} e^{E_0(\rho|P_{Z|V}) - H_{1_{\rho}}(S_F|S_I S_0)} = e^{E_{0,\max}(\rho|P_{Z|V})},
\]

where

\[
E_{0,\max}(\rho|P_{Z|V}) := \log \max_{P_V} \sum_{Z} \left( \sum_{V} P_V(V) P_{Z|V}(Z|V)^\frac{1}{1-\rho} \right)^{1-\rho}.
\]  \( (135) \)

That is, using the relation \( \sum_{s_0} P_{S_0}(s_0)e^{-\rho H_{1_{\rho}}(S_F|S_I S_0 = s_0)} = e^{-\rho H_{1_{\rho}}(S_F|S_I S_0)} \), we have

\[
E_{F',G'} \exp(\rho I(S_I; Z|S_0)[P_{Z|V}, \varphi_p \circ \Lambda_{F',G'}, P_{S_T}]) \leq 1 + e^{-\rho H_{1_{\rho}}(S_F|S_I S_0)} e^{E_{0,\max}(\rho|P_{Z|V})}.
\]  \( (136) \)

Combining the Jensen inequality for \( x \mapsto e^x \), we obtain the desired upper bound \( (132) \).

The logarithm of the RHS of \( (132) \) has the following property.
Lemma 39: The functions $\rho \mapsto E_0(\rho|P_{Z|V}, Q_V) - \rho H_{1+\rho}(S_{I'|}S_I,S_0) - \log \rho$ and $\rho \mapsto E_{0,\max}(\rho|P_{Z|V}, Q_V) - \rho H_{1+\rho}(S_{I'|}S_I,S_0) - \log \rho$ are convex.

Proof: The function $\rho \mapsto E_0(\rho|W_Z, Q_V)$ is convex [12]. Also the function $\rho \mapsto \rho H_{1+\rho}(S_{I'|}S_I,S_0)$ is concave. Hence, $E_0(\rho|P_{Z|V}, Q_V) - \rho H_{1+\rho}(S_{I'|}S_I,S_0) - \log \rho$ is convex.

Further, given convex functions $x \mapsto f_i(x)$, the function $x \mapsto \max_i f_i(x)$ is also convex. Hence, the function $\rho \mapsto \max_{Q_V} E_0(\rho|P_{Z|V}, Q_V) - \rho H_{1+\rho}(S_{I'|}S_I,S_0) - \log \rho = E_{0,\max}(\rho|P_{Z|V}, Q_V) - \rho H_{1+\rho}(S_{I'|}S_I,S_0) - \log \rho$ is convex.

As is explained latter, the bound $e^{E_{0,\max}(\rho|P_{Z|V})}$ is computable in the discrete memoryless case. On the other hand, the error probabilities can be upper bounded by the average error probabilities of the code $\varphi_p$. Suppose that we are given arbitrary error-correcting code $\varphi_p$ for the broadcast channel $P_{Y_{Z|V}}$. The code $\varphi_p$ can be, for example, an LDPC code [40] or a Turbo code [41] when there is no common message. We have to determine the necessary amount of dummy randomness so that the amounts of leaked information is below specified levels. In order to do so, we rewrite Lemma 38 so that it explicitly takes the amount of dummy randomness into account. Suppose that $S_1, \ldots, S_{T-1}$ are secret messages, and $S_T$ is the dummy randomness whose secrecy is not required. We assume that $S_T$ has the uniform distribution on its alphabet $S_T$ and is statistically independent of all other random variables. Denote $\{1, \ldots, T-1\} \setminus \mathcal{I}$ by $\mathcal{I}'$. As a corollary to Lemma 38, we have:

Lemma 40: If $T \notin \mathcal{I}$ then we have

$$E_{F',G'}I(S_{I';}Z|S_0)[P_{Z|V}, \varphi_p \circ A_{F',G'}, P_{S_T}] \leq e^{E_{0,\max}(\rho|P_{Z|V}) - \rho \log \max_{|S_T|} H_{1+\rho}(S_{I'|}S_I,S_0))}.$$

By using Eq. (137), from $\varphi_p$ we can construct a code for the secure multiplex coding as follows. For each proper nonempty set $\mathcal{I} \subseteq \{1, \ldots, T-1\}$, $\epsilon_\mathcal{I}$ denotes the maximum acceptable information leakage for $I(S_{I';}Z)$. Denote by $\epsilon_2$ the maximum acceptable probability for a chosen $F'$, $G'$ not making $I(S_{I';}Z|S_0)$ below $\epsilon_\mathcal{I}$ for some $\mathcal{I}$.

Adjust the size $|S_T|$ of the dummy randomness so that

$$\epsilon_\mathcal{I} := \frac{2^{T-1}}{\epsilon_2} \log \inf_{\rho \in (0,1)} \frac{e^{E_{0,\max}(\rho|P_{Z|V}) - \rho \log \max_{|S_T|} H_{1+\rho}(S_{I'|}S_I,S_0))}}{\rho}.$$

Then, to (137), we obtain

$$E_{F',G'}I(S_{I';}Z|S_0)[P_{Z|V}, \varphi_p \circ A_{F',G'}, P_{S_T}] \leq \epsilon_2 \epsilon_\mathcal{I} / 2^{T-1}.$$

Then, by the Markov inequality the probability of choosing $F'$ and $G'$ making $I(S_{I';}Z|S_0) \leq \epsilon_\mathcal{I}$ simultaneously for all $\mathcal{I} \subseteq \{1, \ldots, T-1\}$ is $\geq 1 - \epsilon_2$.

When the channel is a regular channel in the sense of Delsarte-Piret [10], the value $E_{0,\max}(\rho|P_{Z|V})$ can be calculated as follows:

Lemma 41: When the channel $P_{Z|V}$ is regular in the sense of Delsarte-Piret [10],

$$E_{0,\max}(\rho|P_{Z|V}) = E_0(\rho|P_{Z|V}, P_{\text{mix},V}).$$

Further, when the code $\varphi_p$ is a homomorphism as Abelian group, the inequality

$$E_{F'|G' \rightarrow g} I(S_{I';}Z|S_0)[P_{Z|V}, \varphi_p \circ A_{F',g'}, P_{S_T}] \leq e^{E_{0}(\rho|P_{Z|V}, P_{\text{mix},V}) - \rho H_{1+\rho}(S_{I'|}S_I,S_0))}.$$

holds for any $g' \in G'$.

Thanks to Lemma 41 in the regular case, when the code $\varphi_p$ is a homomorphism as Abelian group, the above procedure for the construction of our code (Code Construction 5) can be simplified to the following way. It is enough to choose $F'$ and to fix $G'$ to be 0, and we can replace $E_{0,\max}(\rho|P_{Z|V})$ by $E_0(\rho|P_{Z|V}, P_{\text{mix},V})$. 

That is, it is enough to calculate $\inf_{\rho \in (0,1)} E_0(\rho)_{PZ|V, P_{\text{mix}, V}} - \rho(\log |S_T| + H_{1+p}(S_T;|S_T, S_0)) - \log \rho$. Due to Lemma 39, $E_0(\rho)_{PZ|V, P_{\text{mix}, V}} - \rho(\log |S_T| + H_{1+p}(S_T;|S_T, S_0)) - \log \rho$ is convex with respect to $\rho$, and the infimum is computable by the bisection method [4, Algorithm 4.1].

**Proof of Lemma 41**: First, we choose $P_V'$ such that

$$E_{0,\text{max}}(\rho)_{PZ|V} = E_0(\rho)_{P'_{Z|V}, P_V'}.$$  

Define $P'_{V,v_0}$ for $v_0 \in V$ by

$$P'_{V,v_0}(v) = P'_V(v + v_0).$$

Then,

$$e^{E_0(\rho)_{PZ|V, P_V'}} = e^{E_0(\rho)_{PZ|V, P_{V,v_0}}},$$

The concavity $P_V \mapsto e^{E_0(\rho)_{PZ|V, P_V}}$ implies that

$$\sum_{v_0 \in V} \frac{1}{|V|} e^{E_0(\rho)_{PZ|V, P'_{V,v_0}}} \leq e^{E_0(\rho)_{PZ|V, P_{V,v_0}}},$$

which implies (138).

Next, we show (139). When the code $\varphi_p$ is a homomorphism as Abelian group, as is mentioned in Lemma 15, we have $E_{F;G'}(S_T; Z|S_0)_{P_{Z|V}, \varphi_p \circ \Lambda_{F', G'}}, P_S_T = E_{F;G'}(S_T; Z|S_0)_{P_{Z|V}, \varphi_p \circ \Lambda_{F', G'}, P_{S_T}}$. Hence, combining (138), we obtain (139).

When the channel is given as the $n$-fold discrete memoryless extension $P_{Z|V}$ of $P_{Z|V}$, $E_{0,\text{max}}(\rho)_{P^n_{Z|V}}$ has the following characterization. Using (11), we obtain

$$\max_{P,v^n} \left( \sum_{x^n} \sum_{v^n} P_{V^n}(v^n)_{P_{Z|V^n}(x^n|v^n)^{1/\rho}} \right)^{1-\rho} = e^{nE_{0,\text{max}}(\rho)_{P_{Z|V}}}. $$

Thus, we can apply the above discussion to the $n$-fold memoryless case by replacing $E_{0,\text{max}}(\rho)_{P_{Z|V}}$ and $P_{Z|V}$ by $nE_{0,\text{max}}(\rho)_{P_{Z|V}}$ and $P_{Z|V}$. That is, it is enough to calculate $\inf_{\rho \in (0,1)} nE_{0,\text{max}}(\rho)_{P_{Z|V}} - \rho(\log |S_T| + H_{1+p}(S_T;|S_T, S_0)) - \log \rho$. Since, as is mentioned in Proposition 13, $Q_V \mapsto e^{E_0(\rho)_{P^n_{Z|V}, Q_V}}$ is concave and $x \mapsto \log x$ is monotone increasing and concave, $Q_V \mapsto E_0(\rho)_{P^n_{Z|V}, Q_V}$ is concave. Hence, $E_{0,\text{max}}(\rho)_{P_{Z|V}}$, $Q_V = \max_{Q_V} E_0(\rho)_{P_{Z|V}, Q_V}$ can be easily computed. Due to Lemma 39, $E_{0,\text{max}}(\rho)_{P_{Z|V}} - \rho(\log |S_T| + H_{1+p}(S_T;|S_T, S_0)) - \log \rho$ is convex concerning with respect to $\rho$, the infimum is computable by the bisection method [4, Algorithm 4.1].

### B. First Practical Construction: Second Type Evaluation

In the above discussion, we have to consider the maximum value $E_{0,\text{max}}(\rho)_{P_{Z|V}}$. However, when there is no common message and the channel $P_{Z|V}$ is not regular, one can improve the bound (132) in the $n$-fold memoryless case under the same code construction (Code Construction 5) as the following way. In the following, we treat the $n$-fold memoryless extension $P_{Z|V}$. Given an encoder $\varphi_p : B_2 \to V^n$, we define the weight distribution $P_{\varphi_p}$ over the set $T_n(V)$ of types of length $n$ of the set $V$ by

$$P_{\varphi_p}(Q_V) := \frac{[\{v^n \in \text{Im } \varphi_p \text{[the type of } v^n \text{ is } Q_V.]\}}{|\text{Im } \varphi_p|} \quad (140)$$

for $Q_V \in T_n(V)$. Using the above weight distribution $P_{\varphi_p}$, we define the distribution

$$\overline{P}_{\varphi_p}(v^n) := \frac{P_{\varphi_p}(Q_V)}{|T_n(Q_V)|}$$
for $v^n \in \mathcal{V}^n$, where $Q_V$ is the type of $v^n$ and

$$T_n(Q_V) := \{v^n \in \mathcal{U}^n | \text{the type of } v^n \text{ is } Q_V\}.$$ 

We construct our code by the same way as Subsection X-A. We apply Lemma 18 to the case when $G$ is the $n$-th permutation group, $\mathcal{V}$ is $\mathcal{V}^n$, and $P_{Z|V}$ is $P^n_{Z|V}$. Then,

$$e^{\psi(\rho)}P_{Z|V}^n \leq e^{E_0(\rho)P_{Z|V}^nT_{\phi_p}}.$$ 

Hence, combining (133), we obtain

$$E_{F', G'} \exp(\rho I(S; Z)|P_{Z|V}^n, \phi_p \circ \Lambda_{F', G'}, P_{S_T}) \leq 1 + e^{E_0(\rho)P_{Z|V}^nT_{\phi_p} - \rho H_{1+\rho}(S; I_S)}.$$ 

Since $e^x$ is convex, we obtain

$$E_{F', G'} I(S; Z)|P_{Z|V}^n, \phi_p \circ \Lambda_{F', G'}, P_{S_T} \leq e^{E_0(\rho)P_{Z|V}^nT_{\phi_p} - \rho H_{1+\rho}(S; I_S)}.$$ 

Therefore, we obtain

$$E_{F', G'} I(S; Z)|P_{Z|V}^n, \phi_p \circ \Lambda_{F', G'}, P_{S_T} \leq e^{E_0(\rho)P_{Z|V}^nT_{\phi_p} - \rho H_{1+\rho}(S; I_S)}.$$ 

When $C_1$ is sufficiently small and $Q_{\phi_p}$ does not give the maximum $E_{0, \max}(\rho|P^n_{Z|V})$, the RHS of (141) is smaller than the RHS of (132). Similar to the regular case of Subsection X-A we can calculate

$$\inf_{\rho \in (0, 1)} E_0(\rho|P_{Z|V}^n, Q_{\phi_p}^n) - \rho \log |S_T| + H_{1+\rho}(S_T; S_0) - \log \rho + \log C_1$$

by the bisection method [4, Algorithm 4.1]. Therefore, in the above case, the method in this subsection improves that in Subsection X-A.
C. Second Practical Construction

In the previous construction, when the channel is not a regular channel, we have to use an upper bound \( \frac{1}{\rho} \sum_{b_1 \in B_1} P_{Z|V}(b_1) \). In order to use a smaller upper bound \( \frac{1}{\rho} \sum_{b_1 \in B_1} P_{Z|V}(b_1) \) even for a non-regular channel, we introduce another practical construction when there is no common message.

Assume that \( \mathcal{V} \) has an Abelian group structure. Now, we give a code ensemble from an arbitrary Abelian group \( \mathcal{B} \) and an arbitrary encoder \( \varphi : \mathcal{B}_2 \rightarrow \mathcal{V} \) satisfying that the map \( \varphi \) is an injective homomorphism. In particular, when \( \mathcal{B}_2 \) and \( \mathcal{V} \) are vector spaces over the finite field \( \mathbb{F}_2 \), the map \( \varphi \) can be given as a linear code, such as an LDPC code \([40]\) or a Turbo code \([41]\). However, we do not necessarily need to assume any algebraic structure in the channel \( P_{Z|V|Y} \).

In the previous construction, when the channel is not a regular channel, we have to use an upper bound \( \frac{1}{\rho} \sum_{b_1 \in B_1} P_{Z|V}(b_1) \). Then, we stress that in Code Ensemble \([6]\) we use single encoder \( \varphi \), while in Code Construction \([7]\) we use multiple encoders with the same code length and different information rates.

**Code Ensemble 6:** We modify the random code given in Lemma \([15]\) as follows. We choose an ensemble of isomorphisms \( F' \) from \( S_1 \times \cdots \times S_T \) to \( \mathcal{B}_2 \) satisfying Condition \([10]\). We choose the random variable \( G'' \in \mathcal{V} \) that obeys the uniform distribution on \( \mathcal{V} \) statistically independent of the choice of \( F' \). Then, we define the encoder \( \tilde{F}_{F',G''}(s) := (\varphi \circ F')(s) + G'' \). The decoder is given by \( \tilde{A}_{F',G''}(\cdot) = F'^{-1}(\varphi(\cdot - G'') + y_H) \) by using the decoder \( \tilde{\varphi} \) of \( \varphi \).

This code ensemble can be understood in the following way. We define the random variable \( H \) in the quotient group \( \mathcal{V}/\varphi(\mathcal{B}_2) \) that obeys the uniform distribution. Let \( \{y_h\} \) be the set of coset representatives. Let \( G' \) be the random variable subject to the uniform distribution on \( \mathcal{B}_2 \). Then, \( G' \) is given as \( \varphi(G') + y_H \). That is, the encoder and the decoder can be given as follows. \( \tilde{A}_{F',G',H}(s) := (\varphi \circ F')(s) + G' + y_H \) and \( \tilde{A}_{F',G',H}(\cdot) = F'^{-1}(\varphi(\cdot - G' - y_H)) \).

In Code Ensemble \([6]\) the random variable \( H \) corresponds to the choice of the codebook for error correction. Let \( \varepsilon_H \) be the decoding error probability when we use \( H \) as the codebook and the message obeys the uniform distribution. Hence, we consider that \( \varepsilon_H \) expresses the decoding error probability when we use \( H \) as the codebook in the following code construction.

For Code Ensemble \([6]\), we have the following lemma:

**Lemma 42:** The inequality

\[
E_{F',G',H} e^{H(s_I|Z)[P_{Z|V},\tilde{A}_{F',G',H},P_{S_T}]} \leq 1 + e^{-\rho H_{1|P}(S_I|Z)} e^{E_{0}(\rho|P_{Z|V},P_{mix,V})}
\]

holds for each subset \( I \subseteq \{1, \ldots, T-1\} \). Thus, applying Jensen inequality to \( x \mapsto e^x \), we have

\[
E_{F',G',H} I(s_I;Z)[P_{Z|V},\tilde{A}_{F',G',H},P_{S_T}] \leq e^{E_{0}(\rho|P_{Z|V},P_{mix,V})-\rho H_{1|P}(S_I|Z)}
\]

(143)

**Proof:** We apply \([134]\) to the case when \( |S_0| = 1 \), \( S_0 = \{s_0\} \), \( |B_1| = 1 \), \( B_1 = \{b_1\} \), and the map \( \varphi_0 \) is given as \( \varphi_0(b_0, b_1, b_2) = \varphi(b_2) + y_h \) for any \( b_2 \in B_2 \). Then, we obtain

\[
E_{F',G'} e^{H(s_I|Z)[P_{Z|V},\tilde{A}_{F',G',H},P_{S_T}]} \leq 1 + e^{-\rho H_{1|P}(S_I|Z)} \sum_{z} \left( \sum_{b_2} \frac{1}{|B_2|} P_{Z|V}(z|\varphi(b_2) + y_h) \right)^{1-\rho}.
\]
Hence, we obtain
\[
E_{F',G',H}e^{\rho H(S_I;Z)[P_{Z|V},\tilde{\Lambda}_{F',G',H},P_{S_T}]}
= E_{H}E_{F',G'}|He^{\rho H(S_I;Z)[P_{Z|V},\tilde{\Lambda}_{F',G',H},P_{S_T}]}
\leq 1 + e^{-\rho H_{1 IV}(S_{I|V}|S_I)}E_{H} \sum_{b_2} \left\{ \frac{1}{|B_2|} P_{Z|V}(z|\varphi(b_2) + y_H)\right\}^{1-\rho}
\leq 1 + e^{-\rho H_{1 IV}(S_{I|V}|S_I)} \sum_{b_2} \left\{ \frac{1}{|B_2|} P_{Z|V}(z|\varphi(b_2) + y_H)\right\}^{1-\rho}
= 1 + e^{-\rho H_{1 IV}(S_{I|V}|S_I)} e^{E_{0}\rho P_{Z|V,P_{mix,V}}}
\]
which implies (142).

In order to construct a code for the secure multiplex coding (with no common message), we define the notations as follows. Let \( \epsilon_I \) be the maximum acceptable information leakage for \( I(S_I;Z) \) for each \( I \subseteq \{1, \ldots, T - 1\} \). Let \( \epsilon_b \) be the maximum acceptable error probability. Let \( \epsilon_2 \) be the maximum acceptable probability a chosen \( F', G'' \) not making \( I(S_I;Z) \) below \( \epsilon_I \). These parameters \( \epsilon_b, \epsilon_I, \) and \( \epsilon_2 \) are the requirements for our code construction.

**Code Construction 7:** In this construction, in contrast to Subsections X-A and X-B, we assume that we are given multiple error-correcting codes with the same code length \( n \) and different information rates. Using (143), we construct a code for the secure multiplex coding (with no common message) as follows:

1. We choose a suitable Abelian group \( G_B \), a suitable code \( \varphi \), a suitable sacrifice bit length (the size of \( T \)-th message), and a suitable real value \( \epsilon_1 \in (0, 1) \) satisfying that
   \[
   \epsilon_b \geq \frac{E_{H}e_{H}}{\epsilon_1} \quad \quad (144)
   \]
   \[
   \epsilon_I \geq 2^{T-1} \inf_{\rho \in (0,1)} \frac{e^{E_{0}\rho P_{Z|V,P_{mix,V}}}-\rho H_{1 IV}(S_{I|V}|S_I)}{\rho \epsilon_2 (1-\epsilon_1)} \quad \quad (145)
   \]
2. We choose \( H \) randomly. Then, we check that \( \epsilon_H \) is less than \( \epsilon_b \). If not, we choose another \( H \). We repeat this process until it is successful. We denote the final choice of \( H \) by \( H' \). Thanks to Markov inequality and (144), the successful probability for one trial is at least \( 1 - \epsilon_1 \).
3. We choose \( F' \) and \( G' \) randomly. Then, we pair the encoder \( \Lambda_{F',G',H}(s) := (\varphi \circ F')(s) + G' + y_{H'} \) and the decoder \( \Lambda_{F',G',H}(v) := F'^{-1}(\hat{\varphi}(v - G' - y_{H'})) \).

**Theorem 43:** Under the above construction, the inequality
\[
I(S_I;Z)[P_{Z|V},\tilde{\Lambda}_{F',G',H},P_{S_T}] \leq \epsilon_I
\]
holds for all subsets \( I \subseteq \{1, \ldots, T - 1\} \) with at least probability \( 1 - \epsilon_2 \).

**Proof:** Markov inequality guarantees that \( \Pr[\epsilon_H \leq \epsilon_b] \geq 1 - \epsilon_1 \). Hence, we obtain
\[
E_{F',G',H}I(S_I;Z)[P_{Z|V},\tilde{\Lambda}_{F',G',H},P_{S_T}]
= E_{F',G',H}[\Pr[\epsilon_H \leq \epsilon_b]I(S_I;Z)[P_{Z|V},\tilde{\Lambda}_{F',G',H},P_{S_T}]
\leq \Pr[\epsilon_H \leq \epsilon_b] \frac{1}{\Pr[\epsilon_H \leq \epsilon_b]} E_{F',G',H}[\Pr[\epsilon_H \geq \epsilon_b]I(S_I;Z)[P_{Z|V},\tilde{\Lambda}_{F',G',H},P_{S_T}]
= \frac{1}{\Pr[\epsilon_H \leq \epsilon_b]} E_{F',G',H}[I(S_I;Z)[P_{Z|V},\tilde{\Lambda}_{F',G',H},P_{S_T}]
\leq \frac{1}{\Pr[\epsilon_H \leq \epsilon_b]} E_{F',G',H}[I(S_I;Z)[P_{Z|V},\tilde{\Lambda}_{F',G',H},P_{S_T}]
\leq \epsilon_2 \epsilon_I / 2^{T-1}
\]
for every \( I \), where \( E_{F',G',H}[\epsilon_H \leq \epsilon_b] \) denotes the expectation under the condition \( \epsilon_H \leq \epsilon_b \). The final inequality follows from (143). Since the above choice of \( F', G' \) and \( H' \) is restricted to the set \( \{(f', g', h')|\epsilon_h \leq \epsilon_b\} \),
due to Markov inequality, the probability of choosing $F'$, $G'$ and $H'$ making (146) simultaneously for all $I \subset \{1, \ldots, T-1\}$ is not less than $1 - \varepsilon_2$.

Further, when the channel is given as the $n$-fold discrete memoryless extension $P_{Z|V}^n$ of $P_{Z|V}$, the quantity $E_0(\rho|P_{Z|V}^n, P_{\text{mix},V}^n)$ is simplified to $nE_0(\rho|P_{Z|V}, P_{\text{mix},V})$. Hence, similar to the regular case of Subsection X-A, we can calculate the right hand side of (145) by the bisection method [4 Algorithm 4.1].

XI. Channel-Universal Coding for Secure Multiplex Coding with Common Messages

In order to treat universal coding for the multiplex coding with common messages, we introduce the universally attainable exponents of the multiplex coding with common messages in the $n$-fold discrete memoryless setting by adjusting the original definition for the BCD given by Körner and Sgarro [24].

In order to treat universal coding for secure multiplex coding with common messages, we focus on $2^{T+1-2}$ functions to express the evaluations of the exponential decreasing rates of decoding error probabilities and the asymptotic evaluations of leaked information. For describing bounds of the exponential decreasing rates of both decoding error probabilities, we need two functions. For treating the asymptotic evaluations of leaked information, we need $2^{T+1}-4$ functions because the number of non-empty proper subsets $I \neq \emptyset \subset \{1, \ldots, T\}$ is $2^{T-2}$ and we treat the exponential decreasing rates and the information leakage rates of leaked information for respective non-empty proper subsets $I \neq \emptyset \subset \{1, \ldots, T\}$. Then, we need to treat $2^{T+1-2}$ functions. Since we do not assume the uniformity, we cannot describe our bounds of the exponential decreasing rate and the information leakage rate of leaked information as functions of the rate tuples $(R_p, R_c, (R_i)_{i=0,1,...,T})$. In the following discussion, we treat our bound of the exponential decreasing rate of leaked information for a non-empty proper subset $I \neq \emptyset \subset \{1, \ldots, T\}$ as a function of $H_c(I^c)$, $R_c$, and $R_0$ as well as the channel $W$. Similarly, we treat our bound of the information leakage rate of leaked information for a non-empty proper subset $I \neq \emptyset \subset \{1, \ldots, T\}$ as a function of $H_{\log}(I^c)$, $R_c$, and $R_0$ as well as the channel $W$. Our bounds of the exponential decreasing rates of both decoding error probabilities are described as functions of $R_p$, $R_c$, and the channel $W$. Hence, the outcomes of the above $2^{T+1-2}$ functions are decided by $2^{T+1}-1$ real numbers $R_p$, $R_c$, $R_0$, and $(H_c(I^c), H_{\log}(I^c))_{I \neq \emptyset \subset \{1, \ldots, T\}}$ as well as the channel $W$.

Definition 44: A set of functions $(E^b, E^e, (E^T, E^I)_{I \subseteq \{1, \ldots, T\}})$ from $\mathbb{R}_{\geq 0}^{2^{T+1}-1} \times W(X, Y \times Z)$ to $\mathbb{R}_{\geq 0}^{2^{T+1}-2}$ is said to be a universally attainable set of exponents and information leakage rate for the family $W(X, Y \times Z)$ if for any $\varepsilon > 0$ and any rate tuples $(R_p, R_c, (R_i)_{i=0,1,...,T})$, there exist a sufficiently large integer $N$ and a sequence of codes $\varphi_n$ of length $n$ satisfying the following conditions: (1) The $i$-th secret message set $S_{i,n}$ of the code $\varphi_n$ has cardinality $e^{nR_i}$ for $i = 1, \ldots, T$, and the common message sets $S_{0,n}$ has cardinality $e^{nR_0}$. (2) Any sequence of joint distributions $P_{S_{T,n}}$ for all of the $i$-th secret $S_{i,n}$ on $S_{i,n}$ and the common message $S_{0,n}$ on $S_{0,n}$ satisfies the inequalities

\begin{align}
P_b[W^n, \varphi_n, P_{S_{T,n}}] &\leq \exp(-n[E^b(R_p, R_c, R_0, W) - \varepsilon]), \\
&\leq \exp(-n[E^b(R_p, R_c, W) - \varepsilon]),
\end{align}

and

\begin{align}
&\liminf_{n \to \infty} -\frac{1}{n} \log I(S_{I,n}; Z^n|S_{0,n})[W^n, \varphi_n, P_{S_{T,n}}] \\
&\geq E^c(R_p, R_c, R_0, (H_c(I^c), H_{\log}(I^c))_{I \neq \emptyset \subseteq \{1, \ldots, T\}}, W),
\end{align}

\begin{align}
&\limsup_{n \to \infty} -\frac{1}{n} \log I(S_{I,n}; Z^n|S_{0,n})[W^n, \varphi_n, P_{S_{T,n}}] \\
&\leq E^c(R_p, R_c, R_0, (H_c(I^c), H_{\log}(I^c))_{I \neq \emptyset \subseteq \{1, \ldots, T\}}, W),
\end{align}

hold for any channel $W \in W(X, Y \times Z)$, any non-empty proper subset $I \neq \emptyset \subset \{1, \ldots, T\}$, and any $n \geq N$. Here, $E^b(R_p, R_c, R_0, (H_c(I^c), H_{\log}(I^c))_{I \neq \emptyset \subseteq \{1, \ldots, T\}}, W)$ and $E^c(R_p, R_c, R_0, (H_c(I^c), H_{\log}(I^c))_{I \neq \emptyset \subseteq \{1, \ldots, T\}}, W)$ are abbreviated to $E^b(R_p, R_c, R_0, W)$ and $E^c(R_p, R_c, R_0, W)$ because they do not depend on $(H_c(I^c), H_{\log}(I^c))_{I \neq \emptyset \subseteq \{1, \ldots, T\}}$.
For the reason why we employ the limiting forms in (149) and (150), see Remark 51. Note that we do not consider here the universality for source while Körner and Sgarro [24] show the universality for source as well as that for channel, as reviewed in Theorem 8 of this paper. In order to guarantee the secrecy for $S_{I,n}$, we need sufficient randomness of $S_{I,n}$. That is, the secrecy of $S_{I,n}$ depends on $H_2(I^c)$ and $H_{\log}(I^c)$, which depends on the source distribution. Hence, it is impossible to show the universality for source in SMC.

We fix a distribution $Q_{VU}$ on $\mathcal{U} \times \mathcal{V}$ and a channel $\Xi : \mathcal{V} \rightarrow \mathcal{X}$. Then, we present a universally attainable set of exponents and leaked information rate in terms of $Q_{VU}$ and $\Xi$ in the following way. Given a broadcast $W : \mathcal{X} \rightarrow \mathcal{Y} \times \mathcal{Z}$ and the real numbers $(R_p, R_c, R_0, (H_2(I^c), H_{\log}(I^c)))_{I^c \neq \emptyset \subseteq [1, \ldots, T]}$, the tuple of exponents and information leakage rate are given as

$$E^b = E^b(R_p, R_c, R_0, W)$$

$$:= E^b(R_p, R_c, (W_Y \circ \Xi) \times Q_{VU}),  \quad (151)$$

$$E^e = E^e(R_p, R_c, R_0, W)$$

$$:= E^e(R_c, (W_Z \circ \Xi) \times Q_{VU}),  \quad (152)$$

$$E^I_+ = E^I_+(R_p, R_c, R_0, (H_2(I^c), H_{\log}(I^c)))_{I^c \neq \emptyset \subseteq [1, \ldots, T]}, W$$

$$:= E^I_+(H_2(I^c) - R_c + R_0, (W_Z \circ \Xi) \times Q_{VU}),  \quad (153)$$

$$E^I_- = E^I_-(R_p, R_c, R_0, (H_2(I^c), H_{\log}(I^c)))_{I^c \neq \emptyset \subseteq [1, \ldots, T]}, W$$

$$:= I(V; Z|U)[(W_Z \circ \Xi) \times Q_{VU}] - H_{\log}(I^c) + R_c - R_0  \quad (154)$$

for a non-empty proper subset $I^c \subseteq [1, \ldots, T]$, where $E^b$, $E^e$, and $E^I_0$ are given by (13), (14), and (123), and $E^I$ is given by

$$E^I(R, \overline{W}_Z \times Q_{VU})$$

$$:= \min_{W_Z \in \mathcal{W} \subseteq \mathcal{U} \times \mathcal{V} \times \mathcal{Z}} D(W_Z || \overline{W}_Z|Q_{VU}) + [R - I(V; Z|U)[W_Z \times Q_{VU}]]  \quad (155)$$

for $\overline{W}_Z \in \mathcal{W}(\mathcal{V}, \mathcal{Z})$. Hence, our quadruple of exponents and information leakage rate depends on $Q_{VU}$ and $\Xi$.

**Theorem 45 (Extension of [24 Theorem 1, part (a)]):** Eqs. (151)–(154) are universally attainable rates of exponents and information leakage rate in the sense of Definition 44.

**Proof:** In the proof, since we treat the channel $W_Z \circ \Xi : \mathcal{V} \rightarrow \mathcal{Z}$, we abbreviate it as $\overline{W}_Z$. First, we give the outline of our proof. We shall modify the constant composition code used by Körner and Sgarro [24]. We do not evaluate the decoding error probability, because that of our code is not larger than that given in [24]. Observe that our exponents in Eqs. (151) and (152) are the same as [24] with the channel $\overline{W}_Z = W_Z \circ \Xi$. We shall evaluate only the mutual information. For this purpose, we prepare general notations and properties of type and conditional type in Step (1). Next, in Steps (2) and (3), we prepare several notations and properties of type and conditional type that are specific to our proof. In Step (4), we apply the random coding and evaluate the leaked information when the channel is given by the conditional types. Then, we choose a code whose leaked information is evaluated for all conditional types and whose error is evaluated for all discrete memoryless channels. In Step (5), we evaluate the leaked information under the above chosen code for all discrete memoryless channels.

**Step (1): Preparation of general notations and properties of type and conditional type:**

For the following construction of our code, we prepare general notations for types. These notations will be used also in the next section. For a given type $Q_U$ of length $n$ on a set $\mathcal{U}$, we define the set $T_n(Q_U)$ as

$$T_n(Q_U) := \{u^n \in \mathcal{U}^n | \text{the type of } u^n \text{ is } Q_U\}.$$

Hence, for a given type $Q_{VU}$ of length $n$ on a set $\mathcal{V} \times \mathcal{U}$, the set $T_n(Q_{VU})$ is written as

$$T_n(Q_{VU}) = \{(u^n, v^n) \in \mathcal{V}^n \times \mathcal{U}^n | \text{the type of } (v^n, u^n) \text{ is } Q_{VU}\}.$$
The marginal distribution $Q_U$ over $\mathcal{U}$ of the type $Q_{VU}$ of length $n$ on the set $\mathcal{V} \times \mathcal{U}$ is a type of length $n$ on the set $\mathcal{U}$. Given a type $Q_V$ of length $n$ on the set $\mathcal{U}$, we define the set of conditional types on the set $\mathcal{V}$ with respect to $Q_V$ as

$$\mathcal{T}_{n,V}(Q_U) := \{ \text{probability transition matrix } W \text{ from } \mathcal{U} \text{ to } \mathcal{V} \mid W \times Q_U \text{ is a type of length } n \text{ on a set } \mathcal{V} \times \mathcal{U} \}. $$

The cardinality $|\mathcal{T}_{n,V}(Q_U)|$ is upper bounded as

$$|\mathcal{T}_{n,V}(Q_U)| \leq (n+1)^{|\mathcal{V} \times \mathcal{U}|}. \tag{156}$$

In particular, given a type $Q_{VU}$ of length $n$ on the set $\mathcal{V} \times \mathcal{U}$, we define the conditional type $Q_{V|U}$ such that $Q_{VU} = Q_{V|U} \times Q_{U}$. We also define the set $T_n(Q_{V|U})_{U^n} := \{ v^n \in \mathcal{V}^n \mid \text{the type of } (v^n, u^n) \text{ is } Q_{VU} \}$.

We denote the uniform distribution $P_{\text{mix}, T_n(Q_U)}$ on $T_n(Q_U)$ by $\Upsilon_n(Q_U)$. Then, for a given type $Q_{VU}$ of length $n$ on a set $\mathcal{V} \times \mathcal{U}$, $\Upsilon_n(Q_{VU})$ represents the uniform distribution $P_{\text{mix}, T_n(Q_{VU})}$ on $T_n(Q_{VU})$. Further, for an arbitrary $W \in T_{n,V}(Q_{VU})$, $\Upsilon_n(W \times Q_U)$ represents the uniform distribution on $T_n(W \times Q_U)$. Then, we define the probability transition matrix $\Upsilon_n(W)$ from $\mathcal{V}^n$ to $\mathcal{U}^n$ such that $\Upsilon_n(W) \times \Upsilon_n(Q_U) = \Upsilon_n(W \times Q_U)$.

When $P_{V^n|U^n}$ is a distribution over $\mathcal{V}^n \times \mathcal{U}^n$ and invariant under the permutation of the indices, the distribution $P_{V^n|U^n}$ can be written as

$$P_{V^n|U^n} = \sum_{Q_{VU} \in T_{n,V}(Q_U)} \lambda_{P_{V^n|U^n}}(Q_{VU}) \Upsilon_n(Q_{VU}) \tag{157}$$

with non-negative constants $\lambda(Q_{VU})$. In particular, the independent and identical distribution $P^n_v$ of $P_V$ can be written as

$$P^n_v = \sum_{Q_V} \lambda_{P^n_v}(Q_V) \Upsilon_n(Q_V) \tag{158}$$

with

$$\lambda_{P^n_v}(Q_V) = P^n_v(T_n(Q_V)) \leq e^{-nD(Q_V \mid P_V)}. \tag{159}$$

When the marginal distribution over $\mathcal{U}^n$ of $P_{V^n|U^n}$ can be written as $P_{\text{mix}, T_n(Q_U)} = \Upsilon_n(Q_U)$ with a type $Q_U$ on the set $\mathcal{U}$, we have

$$P_{V^n|U^n} = \sum_{Q_{VU} \in T_{n,V}(Q_U)} \lambda_{P_{V^n|U^n}}(Q_{V|U} \times Q_U) \Upsilon_n(Q_{V|U} \times Q_U)$$

$$= \sum_{Q_{VU} \in T_{n,V}(Q_U)} \lambda_{P_{V^n|U^n}}(Q_{V|U} \times Q_U) (\Upsilon_n(Q_{V|U}) \times \Upsilon_n(Q_U))$$

$$= \left( \sum_{Q_{VU} \in T_{n,V}(Q_U)} \lambda_{P_{V^n|U^n}}(Q_{V|U} \times Q_U) \Upsilon_n(Q_{V|U}) \right) \Upsilon_n(Q_U). \tag{160}$$

We define the channel $P_{V^n|U^n}$ by $P_{V^n|U^n} = P_{V^n|U^n} \times \Upsilon_n(Q_U)$ and the real number $\lambda_{P_{V^n|U^n}}(Q_{V|U}) := \lambda_{P_{V^n|U^n}}(Q_{V|U} \times Q_U)$ for $Q_{V|U} \in T_{n,V}(Q_U)$. Then, we obtain

$$P_{V^n|U^n} = \sum_{Q_{V|U} \in T_{n,V}(Q_U)} \lambda_{P_{V^n|U^n}}(Q_{V|U}) \Upsilon_n(Q_{V|U}). \tag{161}$$

Now, we consider the $n$-fold discrete memoryless channel $P^n_{V|U}$. For a given type $Q_U$ on the set $\mathcal{U}$, we apply the relation $\lambda_{P_{V^n|U^n}}(Q_{V|U})$ to the joint distribution $P^n_{V|U|T_n(Q_U)} \times \Upsilon_n(Q_U)$. Then, (161) implies that

$$P^n_{V|U|T_n(Q_U)} = \sum_{Q_{V|U} \in T_{n,V}(Q_U)} \lambda_{P^n_{V|U}}(Q_{V|U}) \Upsilon_n(Q_{V|U}). \tag{162}$$
Choosing $u^n \in T_n(Q_U)$, we have
\[
\mathcal{T}_n(\mathcal{Q}_V(U))(T_n(Q_V|U)U^n = u^n) = \begin{cases} 
1 & \text{if } Q_{V|U} = Q_{V|U} \\
0 & \text{otherwise.} 
\end{cases} \tag{163}
\]
Combining (162) and (163), we obtain
\[
\lambda_{P_{Q|U}}(Q_{V|U}) = P_{Q|U|T_n(Q_U)}(T_n(Q_{V|U})U^n = u^n) \\
= \prod_{u \in \mathcal{U}} (P_{V|U=n})^{nQ(U|u)}(T_n(Q_{V|U} = n)) \\
\leq e^{-\sum_{u \in \mathcal{U}} nQ(U|n)D(Q_{V|U=n}||P_{V|U=n})} \\
= e^{-nD(Q_{V|U}||P_{V|U}|Q_U)}, \tag{164}
\]
where (164) follows from (159).

Step (2): Preparation of notations and properties of conditional types based on a joint type on $\mathcal{U} \times \mathcal{V}$.

In this step, we prepare several important properties based on a type of length $n$ on the set $\mathcal{U} \times \mathcal{V} \times \mathcal{Z}$. Now, we focus on a conditional type $W_Z \in \mathcal{T}_{n,Z}(Q_{VU})$, which gives a type $W_Z \times Q_{VU}$ of length $n$ on the set $\mathcal{U} \times \mathcal{V} \times \mathcal{Z}$. Note that in order to make a type of length $n$ on the set $\mathcal{U} \times \mathcal{V} \times \mathcal{Z}$, we need to choose $W_Z$ not from $\mathcal{T}_{n,Z}(Q_U)$ but from $\mathcal{T}_{n,Z}(Q_{VU})$. Now, we treat the channel $W_Z$ as a channel from $\mathcal{V} \times \mathcal{U}$ to $\mathcal{Z}$ while the output distribution of the channel $W_Z$ does not depend on the choice of $u \in \mathcal{U}$. In our code $\varphi_{a,n}$, the random variable $V^nU^n$ takes values in the subset $T_n(Q_{VU})$. Hence, it is sufficient to treat the channel whose input alphabet is the subset $T_n(Q_{VU})$ of $\mathcal{V} \times \mathcal{U}$. Based on (162), we make a convex decomposition
\[
\overline{W}_Z^{n}\left|T_n(Q_{VU}) = \sum_{W_Z \in T_{n,Z}(Q_{VU})} \lambda_{n,T}(W_Z)\mathcal{T}_n(W_Z), \tag{166}
\right.
\]
with non-negative constants $\lambda_{n,T}(W_Z)$. Then, due to (165), we have
\[
\lambda_{n,T}(W_Z) \leq e^{-nD(W_Z||\overline{W}_Z(\mathcal{VU})). \tag{167}
\]
For an arbitrary code $\varphi_{a,n}$, the joint convexity of the conditional relative entropy yields that
\[
I(S_{I,a};Z^n|S_{0,a},[\overline{W}_Z^{n},\varphi_{a,n},P_{S,T,a}]} \\
\leq \sum_{\mathcal{W}_Z \in T_{n,Z}(Q_{\mathcal{VU}})} \lambda_{n,T}(W_Z)I(S_{I,a};Z^n|S_{0,a},[\mathcal{T}_n(W_Z),\varphi_{a,n},P_{S,T,a}]. \tag{168}
\]
Next, in order to treat each channel $\mathcal{T}_n(W_Z)$, we fix a conditional type $W_Z \in T_{n,Z}(Q_{VU})$ and study the properties of the channel $\mathcal{T}_n(W_Z)$. Under the joint type $Q_{VU} := W_Z \times Q_{VU}$, we define the numbers
\[
N(U) := |T_n(Q_U)|, \quad N(UZ) := |T_n((W_Z \times Q_{V|U}) \times Q_U)|, \\
N(VU) := |T_n(Q_{VU})|, \quad N(VUZ) := |T_n(W_Z \times Q_{VU})|
\]
and
\[
N(Z|U) := N(UZ)/N(U), \quad N(V|UZ) := N(VUZ)/N(UZ), \\
N(V|U) := N(VU)/N(U), \quad N(Z|VU) := N(VUZ)/N(VU).
\]
Then, due to (8), we have
\[
|T_{n,Z}(Q_{VU})|^{-1}e^{nH(Z|U)||W_Z \times Q_{VU}} \leq N(Z|U) \leq e^{nH(Z|U)||W_Z \times Q_{VU}} \tag{169}
\]
\[
|T_{n,Z}(Q_{VU})|^{-1}e^{nH(Z|V)||W_Z \times Q_{VU}} \leq N(Z|VU) \leq e^{nH(Z|V)||W_Z \times Q_{VU}}. \tag{170}
\]
Then, we obtain the following lemma.
Lemma 46: Any conditional type \( W_Z \in \mathcal{T}_{n,Z}(Q_{VU}) \) satisfies
\[
E_0(\rho | \Upsilon_n(W_Z), P_{V^n|U^n, \text{mix}, T_n(Q_{VU})}, P_{\text{mix}, T_n(Q_{VU})})
= \rho \log \frac{N(Z|U)}{N(Z|VU)}\tag{171}
\]
\[
= \rho I(V; Z|U)[\Upsilon_n(W_Z) \times P_{\text{mix}, T_n(Q_{VU})}]
\leq n \rho I(V; Z|U)[W_Z \times Q_{VU}] + \rho \log |\mathcal{T}_{n,Z}(Q_{VU})|\tag{173}
\]
for any \( \rho \in (0, 1) \). Here \( P_{V^n|U^n, \text{mix}, T_n(Q_{VU})} \) is defined as a special case of Eq. (1).

Proof: Under the joint type \( Q_{VZU} := W_Z \times Q_{VU}, \) since \( \Upsilon_n(W_Z) = P_{Z^n|V^n U^n, \text{mix}, T_n(Q_{VU})} \), we obtain
\[
= e^{E(\rho | \Upsilon_n(W_Z), P_{V^n|U^n, \text{mix}, T_n(Q_{VU})}, P_{\text{mix}, T_n(Q_{VU})})}
\]
\[
= \sum_{u^n \in T_n(Q_{VU})} \frac{1}{N(U)} \sum_{v^n \in T_n(Q_{ZU})} \left( \sum_{v^n \in T_n(Q_{ZU})} P_{V^n|U^n, \text{mix}, T_n(Q_{VU})}(v^n|u^n)(P_{Z^n|V^n U^n, \text{mix}, T_n(Q_{VU})}(z^n|v^n, u^n)) \right)^{1-\rho}
\]
\[
= \frac{N(Z|U)^{1-\rho}}{N(V|U)^{1-\rho}} = \frac{N(Z|U)^{1-\rho}}{N(V|U)^{1-\rho}} \cdot \frac{N(V|U)^{1-\rho}}{N(Z|VU)^{1-\rho}}
\]
which implies (171). Since
\[
\log N(Z|U) - \log N(Z|VU)
= H(Z|U)[\Upsilon_n(W_Z) \times P_{\text{mix}, T_n(Q_{VU})}]
- H(Z|VU)[\Upsilon_n(W_Z) \times P_{\text{mix}, T_n(Q_{VU})}]
= I(V; Z|U)[\Upsilon_n(W_Z) \times P_{\text{mix}, T_n(Q_{VU})}],
\]
we obtain (172). Combining (169) and (170), we obtain (173).

Step (3): Preparation of notations and properties concerning conditional types based on a type on \( V):\n
In this step, we focus only on a convex decomposition different from (166). For a given type \( Q_{V} \) of length \( n \) on a set \( V \), we focus on the set
\[
\mathcal{W}_{n,Z}(Q_V) := \{ \Upsilon_n(W_Z) | W_Z \in \mathcal{T}_{n,Z}(Q_V) \}.
\]
In our code \( \varphi_{a,n} \), the random variable \( V^n \) takes values in the subset \( T_n(Q_V) \). Hence, if we focus on the set \( V^n \) as inputs, it is sufficient to treat the channel whose input alphabet is the subset \( T_n(Q_V) \) of \( V^n \). Then, due to (162), we have another type of convex combination:
\[
\overline{W}_Z|T_n(Q_V) = \sum_{\Theta_n \in \mathcal{W}_{n,Z}(Q_V)} \lambda_n, W(\Theta_n) \Theta_n, \tag{174}
\]
where \( \lambda_n, W(\Theta_n) \) is a non-negative constant. Then, for an arbitrary code \( \varphi_{a,n} \), the joint convexity of the conditional relative entropy yields that
\[
I(S_{T,n}; Z^n|S_{0,n}, \overline{W}_Z^n, \varphi_{a,n}, P_{S_{T,n}}) \leq \sum_{\Theta_n \in \mathcal{W}_{n,Z}(Q_V)} \lambda_n, W(\Theta_n) I(S_{T,n}; Z^n|S_{0,n}, \Theta_n, \varphi_{a,n}, P_{S_{T,n}})\tag{175}
\]
Next, we introduce the quantity
\[\varepsilon_{n,p,I}(W^n_z, Q^n_{V,U}) := \exp(n\rho(R_c - R_0) - \rho H_{1+\rho}(S_{I,n}, S_{0,n}) + E_0(\rho|W^n_z, Q^n_{V,U}, Q^0_{U}))\] (176)
for any channel \(W^n_z\) from \(V^n\) to \(Z^n\) and any distribution \(Q^n_{V,U}\) on \(V^n \times U^n\).

Then, we have the following lemma.

**Lemma 47:** Any joint type \(Q_{V,U}\) of length \(n\) on a set \(V \times U\) and any channel \(\Theta_n \in W_{n,Z}(Q_V)\) satisfy
\[\exp(E_0(\rho|W^n_z, P^{n|U^n,\text{mix},T_n(Q_{V,U})}, P_{\text{mix},T_n(Q_{U}))}) \leq (n + 1)^{\|U\|+\|V\|} \exp(E_0(\rho|W^n_z, Q^n_{V,U})\), \]
(177)

\[\lambda_{n,W}(\Theta_n)e_{n,p,I}(\Theta_n, P_{\text{mix},T_n(Q_{V,U}))} \leq (n + 1)^{\|U\|+\|V\|} e_{n,p,I}(W^n_z, Q^n_{V,U}).\] (178)

We obtain
\[\limsup_{n \to \infty} \frac{1}{n\rho_n} \log \varepsilon_{n,p,I}(W^n_z, Q^n_{V,U}) \leq I(V; Z|U)|W^n_z 	imes Q^n_{V,U} - H_{\log}(I^c) + R_c - R_0 = E^{I}_F\] (179)
with \(\rho_n = \frac{\delta \log n}{n}\) for any \(\delta > 0\). Further, when \(S_{I,n}\) is the uniform random number and independent of \(S_{I,n}\) and \(S_{0,n}\), we have
\[\varepsilon_{n,p,I}(W^n_z, Q^n_{V,U}) = \varepsilon_{1,p,I}(W^n_z, Q^n_{V,U})^n\] (180)
and
\[\lim_{\rho \to 0} \frac{\left[ \log \varepsilon_{1,p,I}(W^n_z, Q^n_{V,U}) \right]_+}{\rho} = I(V; Z|U) - R_p + \sum R_i.\] (181)

The convergence in (181) is uniform.

**Proof:** First, we show (177). For arbitrary \(u \in U\) and \(v \in V\), the distribution \(P_{\text{mix},T_n(Q_{V,U})}\) satisfies
\[P^{n|U^n,\text{mix},T_n(Q_{V,U})}(v|u) \leq (n + 1)^{\|U\|+\|V\|} Q^n_{V,U}(v|u)\] (182)
by [8] Lemma 2.5, Chapter 1, and
\[P_{\text{mix},T_n(Q_{U})}(u) \leq (n + 1)^{\|U\|} Q^n_{U}(u),\] (183)
by [8] Lemma 2.3, Chapter 1. Then, the relations (183), (182), and (183) yield (177). Next, we show (178). We can also show that
\[\lambda_{n,W}(\Theta_n)e^{E_0(\rho|\Theta_n,P^{n|U^n,\text{mix},T_n(Q_{V,U})},P_{\text{mix},T_n(Q_{U}))})\]
\[= \sum_u P_{\text{mix},T_n(Q_{U})}(u) \sum_v P^{n|U^n,\text{mix},T_n(Q_{V,U})}(v|u)(\lambda_{n,W}(\Theta_n)\Omega_{n}(v))^{1+\rho} \leq \sum_u P_{\text{mix},T_n(Q_{U})}(u) \left( \sum_v P^{n|U^n,\text{mix},T_n(Q_{V,U})}(v|u) \left( \sum_{\Theta'_n \in W_{n,Z}(Q_{V})) \lambda_{n,W}(\Theta'_n)\Omega'_n(v) \right)^{1+\rho} \right)^{-\rho} \]
\[= e^{E_0(\rho|W^n_z,P^{n|U^n,\text{mix},T_n(Q_{V,U})},P_{\text{mix},T_n(Q_{U})))}.\] (184)

Combining (177) and (184), we obtain
\[(n + 1)^{\|U\|+\|V\|} e^{E_0(\rho|W^n_z,Q^n_{V,U},Q^n_{U})} \geq \lambda_{n,W}(\Theta_n)e^{E_0(\rho|\Theta_n,P^{n|U^n,\text{mix},T_n(Q_{V,U})},P_{\text{mix},T_n(Q_{U}))}).\] (185)
Due to the definition of $\varepsilon_{n,p}(W^Z, Q_{V|U})$, the relation (185) is equivalent with the relation (178).

The relation (179) can be shown as follows.

$$\lim_{n \to \infty} \frac{1}{n\rho_n} \log \varepsilon_{n,p,n,p}(W^Z, Q^n_{V|U})$$

$$= \lim_{n \to \infty} (R_c - R_0) - \frac{1}{n} H_1(\varepsilon_{n,p,n,p}(S_{T,n}, S_{0,n}) + \frac{n}{\delta} \log n E_0(\frac{\delta \log n}{n} W^Z, Q^n_{V|U}, Q^n_U))$$

$$\leq R_c - R_0 - \frac{H(\log I^c)}{n} + I(V; Z | U) = E_\pi^T.$$ 

The relations (180) and (181) are trivial.

**Step (4): Evaluation of the leaked information when the channel is given by the uniform distribution on a fixed conditional type:**

Recall the fixed code $\varphi_{b,n}$ for BCD given in Theorem 8. The message sets of the code $\varphi_{p,n}$ are $S_{0,n} \times B_{1,n}$ and $B_{2,n}$ with $|B_{1,n}| = e^{n(R_c - R_0)}$ and $|B_{2,n}| = e^{n R_p}$. We attach the other random coding $\Lambda_{F,G,n}$ for message $S_{1,n}, \ldots, S_{T,n}$ given as Second Step of Code Ensemble 2 in Subsection VI-B to the code $\varphi_{p,n}$. That is, the encoder is given by $\Phi_{e,n} = (\varphi_{p,n}, \Lambda_{F,G,n})$. In the following, Bob’s decoder $\Phi_{b,n}$ and Eve’s decoder $\Phi_{e,n}$ are given as the maximum mutual information decoder. We treat the ensemble of codes $\Phi_n := (\Phi_{a,n}, \Phi_{b,n}, \Phi_{e,n})$.

First, related to the decomposition (166), we focus on a fixed arbitrary element $x \in T_{n, Z}$. We recall the discussion in Subsection VI-D. As is mentioned in Remark 20, the discussion in Section VI can be applied the channel $W^Z$, whose output distribution depends on the element of $U$ as well as the element of $V$. Then, we apply Lemma 19 to the case when $P_{Z|V} = W^Z, G$ is the $n$-th permutation group, $(U \times V)_o$ is $T_n(Q_{UV})$, and $P_{V|U}$ is $T_n(W_Z)$. Note that the $n$-th permutation group acts on $T_n(Q_{UV})$ transitively. We obtain

$$e^{\rho \phi(T_n(W_Z), P_{V|U}(n, \max T_n(Q_{UV})); P_{\max T_n(Q_{UV}))}}$$

Combining Lemma 15 and the above inequality, we obtain

$$E_0^{\phi_{o,a}} \exp(\rho I(S_{I,n}; Z^n|S_{0,n}))[T_n(W_Z), \Phi_{a,n}, P_{S_{T,n}}]$$

$$\leq 1 + e^{n p(R_c - R_0) - \rho H_1 + \rho \exp(\rho I(S_{I,n}; Z^n|S_{0,n}))[T_n(W_Z), \Phi_{a,n}, P_{S_{T,n}}]} e^{E_0(T_n(W_Z), P_{V|U}(n, \max T_n(Q_{UV})); P_{\max T_n(Q_{UV}))}}.$$ 

(186)

Hence, we obtain the following relations. In the following derivation, the first inequality follows from the convexity of $x \mapsto e^x$. The third inequality follows from (173).

$$\exp(\rho E_0^{\phi_{o,a}} I(S_{I,n}; Z^n|S_{0,n})[T_n(W_Z), \Phi_{a,n}, P_{S_{T,n}}])$$

$$\leq E_0^{\phi_{o,a}} \exp(\rho I(S_{I,n}; Z^n|S_{0,n})[T_n(W_Z), \Phi_{a,n}, P_{S_{T,n}}])$$

$$\leq 1 + e^{n p(R_c - R_0) - \rho H_1 + \rho \exp(\rho I(S_{I,n}; Z^n|S_{0,n}))[T_n(W_Z), \Phi_{a,n}, P_{S_{T,n}}]} e^{E_0(T_n(W_Z), P_{V|U}(n, \max T_n(Q_{UV})); P_{\max T_n(Q_{UV}))}}$$

$$\leq 1 + |T_{n, Z}(Q_{UV})|^p e^{n p(R_c - R_0) - \rho H_1 + \rho \exp(\rho I(V; Z|U)|W_Z) W_{Z\times Q_{UV}}.$$ 

for any $\rho \in (0, 1)$. Taking the limit $\rho \to 1 - 0$, we have

$$\exp(\rho E_0^{\phi_{o,a}} I(S_{I,n}; Z^n|S_{0,n})[T_n(W_Z), \Phi_{a,n}, P_{S_{T,n}}])$$

$$\leq 1 + |T_{n, Z}(Q_{UV})|^p e^{n p(R_c - R_0) - H_2(S_{I,n}; S_{0,n})} e^{\rho II(V; Z|U)|W_Z\times Q_{UV}}.$$ 

(187)

Since $\log(1 + x) \leq x$, taking the logarithm in (187), we have

$$E_0^{\phi_{o,a}} I(S_{I,n}; Z^n|S_{0,n})[T_n(W_Z), \Phi_{a,n}, P_{S_{T,n}}]$$

$$\leq \log(1 + |T_{n, Z}(Q_{UV})|^p e^{n p(R_c - R_0) - H_2(S_{I,n}; S_{0,n})} e^{\rho II(V; Z|U)|W_Z\times Q_{UV}})$$

$$\leq |T_{n, Z}(Q_{UV})|^p e^{n p(R_c - R_0) - H_2(S_{I,n}; S_{0,n})} e^{\rho II(V; Z|U)|W_Z\times Q_{UV}}.$$
Since \( \log |Z^n| = n \log |Z| \leq |\mathcal{T}_{n,Z}(Q_VU)| \), we have
\[
E_{\Phi_{a,n}} I(S_{I,n}; Z^n|S_{0,n})[\gamma_n(W_Z), \Phi_{a,n}, P_{S_{T,n}}] \leq |\mathcal{T}_{n,Z}(Q_VU)|. \tag{188}
\]
Hence,
\[
E_{\Phi_{a,n}} I(S_{I,n}; Z^n|S_{0,n})[\gamma_n(W_Z), \Phi_{a,n}, P_{S_{T,n}}] \\
\leq |\mathcal{T}_{n,Z}(Q_VU)| e^{-H_2(S_{I,n}|S_{T,n}, S_{0,n}) - n(R_c - R_0 + I(V; Z|U)[W_Z \times Q_V U])_+}. \tag{189}
\]

Next, related to the decomposition (174), we focus on a fixed arbitrary \( \Theta_n \in \mathcal{W}_{n,Z}(Q_V) \). Similar to (186), Lemmas 15 and 19 yield that
\[
E_{\Phi_{a,n}} \exp(\rho I(S_{I,n}; Z^n|S_{0,n})[\Theta_n, \Phi_{a,n}, P_{S_{T,n}}]) \\
\leq 1 + e^{\rho (R_c - R_0 - H_1(\Phi_{a,n}|S_{I,n}, S_{0,n}))} e^{E_0(\Theta_n, P_{\min}(\gamma_n(\Theta_n)|Q_V))} P_{\max}(Q_V) \tag{190}
\]
Observe that we have shown that the averages over \( \Phi_{a,n} \) of \( \exp(\rho I(S_{I,n}; Z^n|S_{0,n})[\Theta_n, \Phi_{a,n}, P_{S_{T,n}}]) \) and \( I(S_{I,n}; Z^n|S_{0,n})[\Theta_n, \Phi_{a,n}, P_{S_{T,n}}] \) are smaller than Eqs. (190) and (189).

Choosing \( p_1(n) := 2^l |\mathcal{T}_{n,Z}(Q_V U)| + |\mathcal{W}_{n,Z}(Q_V)| + 1 \), thanks to the Markov inequality in the same as (9) and (10), given a fixed \( \rho \in (0, 1) \), we can see that there exists at least one code \( \varphi_n \) such that the relations
\[
I(S_{I,n}; Z^n|S_{0,n})[\gamma_n(W_Z), \varphi_{a,n}, P_{S_{T,n}}] \\
\leq p_1(n)E_{\Phi_{a,n}} I(S_{I,n}; Z^n|S_{0,n})[\gamma_n(W_Z), \Phi_{a,n}, P_{S_{T,n}}] \leq p_1(n)|\mathcal{T}_{n,Z}(Q_V U)| e^{-H_2(S_{I,n}|S_{T,n}, S_{0,n}) - n(R_c - R_0 + I(V; Z|U)[W_Z \times Q_V U])_+} \exp(\rho I(S_{I,n}; Z^n|S_{0,n})[\Theta_n, \Phi_{a,n}, P_{S_{T,n}}]) \leq p_1(n)E_{\Phi_{a,n}} \exp(\rho I(S_{I,n}; Z^n|S_{0,n})[\Theta_n, \Phi_{a,n}, P_{S_{T,n}}]) \leq p_1(n)(1 + e^{\rho (R_c - R_0 - H_1(\Phi_{a,n}|S_{I,n}, S_{0,n}))} e^{E_0(\Theta_n, P_{\min}(\gamma_n(\Theta_n)|Q_V))}) \tag{191}
\]
hold for any \( W_Z \in \mathcal{T}_{n,Z}(Q_V U) \) and \( \Theta_n \in \mathcal{W}_{n,Z}(Q_V) \).

**Step 5:** Evaluation of the leaked information when the channel is given by discrete memoryless channel:

Using (191), we obtain
\[
I(S_{I,n}; Z^n|S_{0,n})[\gamma_n(W_Z), \varphi_{a,n}, P_{S_{T,n}}] \\
\leq \sum_{W_Z \in \mathcal{T}_{n,Z}(Q_V U)} \lambda_{n,T}(W_Z) I(S_{I,n}; Z^n|S_{0,n})[\gamma_n(W_Z), \varphi_{a,n}, P_{S_{T,n}}] \leq \sum_{W_Z \in \mathcal{T}_{n,Z}(Q_V U)} \lambda_{n,T}(W_Z) p_1(n)|\mathcal{T}_{n,Z}(Q_V U)| e^{-H_2(S_{I,n}|S_{T,n}, S_{0,n}) - n(R_c - R_0 + I(V; Z|U)[W_Z \times Q_V U])_+} \leq \sum_{W_Z \in \mathcal{T}_{n,Z}(Q_V U)} p_1(n)|\mathcal{T}_{n,Z}(Q_V U)| e^{-D(W_Z||\gamma_n(W_Z)) + H_2(S_{I,n}|S_{T,n}, S_{0,n}) - n(R_c - R_0 + I(V; Z|U)[W_Z \times Q_V U])_+} \leq \sum_{W_Z \in \mathcal{T}_{n,Z}(Q_V U)} p_1(n)|\mathcal{T}_{n,Z}(Q_V U)| e^{-D(W_Z||\gamma_n(W_Z)) + H_2(S_{I,n}|S_{T,n}, S_{0,n}) - n(R_c - R_0 + I(V; Z|U)[W_Z \times Q_V U])_+} = p_1(n)|\mathcal{T}_{n,Z}(Q_V U)|^2 e^{-D(W_Z||\gamma_n(W_Z)) + H_2(S_{I,n}|S_{T,n}, S_{0,n}) - n(R_c - R_0 + I(V; Z|U)[W_Z \times Q_V U])_+}. \tag{192}
\]
where (193), (194), and (195) follow from (168), (191), and (177) and (178), respectively. Hence,
\[
\lim_{n \to \infty} \frac{1}{n} \log I(S_{I,n}; Z^n|S_{0,n})[\gamma_n(W_Z), \varphi_{a,n}, P_{S_{T,n}}] \\
\geq \lim_{n \to \infty} \frac{1}{n} \min_{W_Z} nD(W_Z||\gamma_n(W_Z)) + H_2(S_{I,n}|S_{T,n}, S_{0,n}) - n(R_c - R_0 + I(V; Z|U)[W_Z \times Q_V U])_+ \\
= \min_{W_Z} D(W_Z||\gamma_n(W_Z)) + H_2(S_{I,n}|S_{T,n}, S_{0,n}) - n(R_c - R_0 + I(V; Z|U)[W_Z \times Q_V U])_+ = E^T_+ \tag{198}
\]
Next, defining \( p_2(n) := p_1(n)(n+1)^{\mathcal{U}_T^2|Y|}W_{n, \mathcal{Z}}(Q_V) \), we obtain the following inequalities, in which, the first, second, and third inequalities follow from the convexity of function \( x \mapsto \exp(x) \) and (175), (192), and (178), respectively.

\[
\begin{align*}
\exp(\rho I(S_{I,n}; Z^n|S_{0,n})(W^n_Z, \varphi_{a,n}, P_{S_{T,n}})) & \\
\leq \sum_{\Theta_n \in W_{n, \mathcal{Z}}(Q_V)} \lambda_{n,W}(\Theta_n) \exp(\rho I(S_{I,n}; Z^n|S_{0,n})(W^n_n, \varphi_{a,n}, P_{S_{T,n}})) & \\
\leq \sum_{\Theta_n \in W_{n, \mathcal{Z}}(Q_V)} \lambda_{n,W}(\Theta_n) p_1(n)(1 + \varepsilon_{n,p,T}(\Theta_n, P_{\text{mix},T_n}(Q_{V,U}))) & \\
\leq \sum_{\Theta_n \in W_{n, \mathcal{Z}}(Q_V)} p_1(n)(n+1)^{\mathcal{U}_T^2|Y|}(1 + \varepsilon_{n,p,T}(\overline{W}_n^a, Q_{V,U}))) & \\
=p_1(n)|W_{n, \mathcal{Z}}(Q_V)|(n+1)^{\mathcal{U}_T^2|Y|}(1 + \varepsilon_{n,p,T}(\overline{W}_n^a, Q_{V,U}))) & \\
=p_2(n)(1 + \varepsilon_{n,p,T}(\overline{W}_n^a, Q_{V,U})). & \tag{199}
\end{align*}
\]

Taking the logarithm, we have

\[
\begin{align*}
I(S_{I,n}; Z^n|S_{0,n})(W^n_Z, \varphi_{a,n}, P_{S_{T,n}}) & \\
\leq \frac{\log p_2(n)(1 + \varepsilon_{n,p,T}(\overline{W}_n^a, Q_{V,U})))}{\rho} & \\
\leq \frac{\log(2p_2(n))}{\rho} + \frac{[\log \varepsilon_{n,p,T}(\overline{W}_n^a, Q_{V,U})]}{\rho}. & \tag{200}
\end{align*}
\]

Now, we choose sufficiently large real \( \delta > 0 \) such that

\[
\lim_{n \to \infty} \frac{\log(2p_2(n))}{n \cdot \frac{\delta \log n}{n}} = \lim_{n \to \infty} \frac{\log(2p_2(n))}{\delta \log n} = 0. & \tag{201}
\]

Due to (179) in Lemma 47, (200), and (201), choosing \( \rho = \frac{\delta \log n}{n} \), we obtain

\[
\limsup_{n \to \infty} \frac{1}{n} I(S_{I,n}; Z^n|S_{0,n})(W^n_Z, \varphi_{a,n}, P_{S_{T,n}}) \leq E^T_\lim. & \tag{202}
\]

Therefore, using (198) and (202), we can see that \( (E^b, E^*, E^T_+, E^T_\lim) \) is a universally attainable quadruple of exponents in the sense of Definition 44.

Remark 48: One might consider that if we apply the random coding of Theorem 17 to the uniform distribution \( P_{\text{mix},T_n}(Q_{V,U}) \), we obtain a better exponent. However, this method yields the same exponent because \( \psi(\rho)|T_n(W_Z), P_{V|T_n,\text{mix},T_n(Q_{V,U})}, P_{\text{mix},T_n(Q_{U})} \) is the same as \( E_0(\rho|T_n(W_Z), P_{V|T_n,\text{mix},T_n(Q_{V,U})}, P_{\text{mix},T_n(Q_{U})}) \), which is shown as

\[
\begin{align*}
E_0(\rho|T_n(W_Z), P_{V|T_n,\text{mix},T_n(Q_{V,U})}, P_{\text{mix},T_n(Q_{U})}) & \\
= \sum_{u \in T_n(Q_{U})} \frac{1}{N(U)} \sum_{v \in T_n(Q_{V,U=}u)} \frac{1}{N(V|U)} \sum_{z \in T_n(Q_{Z|V,U=}u)} \frac{1}{N(Z|V,U)}(1 + \rho)(\frac{1}{N(Z|U)})^{-\rho} & \\
= \frac{N(Z|U)^\rho}{N(Z|V,U)^\rho}.
\end{align*}
\]
XII. Source-Channel Universal Coding for BCC

Now, we introduce the concept of “source-channel universal code for BCC” for the n-fold discrete memoryless extension of a discrete channel. In a realistic setting, we do not have statistical knowledge of the sources and the channel, precisely. In order to treat such a case, we have to make a code whose performance is guaranteed independently of the statistical properties of the sources and the channel. Such a kind of universality is called source-channel universality, and studied for the case of wire-tap channel [26] and for the case of BCD [24]. Although the transmission rates are characterized by the pair \((R_0, R_1)\), in order to make a code achieving the capacity region of BCC, we employ other two parameters \(R_c\) and \(R_p\) that satisfy \(R_0 \leq R_c\) and \(R_0 + R_1 \leq R_c + R_p\). Hence, in the following definition of a universally attainable quadruple of exponents and leaked information rate, we focus on the set \(\mathbf{R}_{\text{BCC}}^4 := \{(R_0, R_c, R_0, R_1) \in (R^+)^4 | R_0 \leq R_c, R_0 + R_1 \leq R_c + R_p\} \).

Definition 49: A set of functions \((E^b, E^e, E^l_+ , E^l_-)\) from \(\mathbf{R}_{\text{BCC}}^4 \times \mathcal{W}(X, Y \times Z)\) to \(\mathbf{R}_{\text{BCC}}^{4}\) is said to be a universally attainable quadruple of exponents and leaked information rate for the family of channels \(\mathcal{W}(X, Y \times Z)\) and for sources if for \(\epsilon > 0\) and \((R_0, R_c, R_0, R_1) \in \mathbf{R}_{\text{BCC}}^4\), there exist a sufficiently large integer \(N\) and a sequence of codes \(\Phi_n\) of length \(n\) satisfying the following conditions. (1) The confidential message set \(S_n\) of the code \(\Phi_n\) has cardinality \(e^{nR_1}\) and the common message set \(E_n\) of the code \(\Phi_n\) has cardinality \(e^{nR_0}\). (2) The inequalities

\[
P_b[W^n, \Phi_n, P_{S_n,E_n}] \leq \exp(-n[E^b(R_p, R_c, R_0, R_1, W) - \epsilon]),
\]

\[
P_c[W^n, \Phi_n, P_{S_n,E_n}] \leq \exp(-n[E^c(R_p, R_c, R_0, R_1, W) - \epsilon]),
\]

and

\[
I(S_n; Z^n|E_n)[W^n, \Phi_n, P_{S_n,E_n}]
\leq \max[\exp(-n[E^l_+(R_p, R_c, R_0, R_1, W) - \epsilon]),
\]

\[
n[E^l_-(R_p, R_c, R_0, R_1, W) + \epsilon]]
\]

hold for any sequence of joint distributions \(P_{S_n,E_n}\) for the confidential message \(S_n\) on \(S_n\) and the common message \(E_n\) on \(E_n\), and the \(n\)-th memoryless extension \(W^n\) of any channel \(W \in \mathcal{W}(X, Y \times Z)\) and \(n \geq N\).

Then, given a distribution \(Q_{VU}\) on \(U \times V\) and a channel (probability transition matrix) \(\Xi : V \rightarrow X\), we present a universally attainable quadruple of exponents and leaked information rate as follows. Given rates \((R_0, R_c, R_0, R_1) \in (R^+)^4\) and a broadcast \(W \in \mathcal{W}(X, Y \times Z)\), the quadruple \(E^b, E^c, E^l_+\) and \(E^l_-\) are given as

\[
E^b = E^b(R_p, R_c, R_0, R_1, W) := \hat{E}^b(R_p, R_c, (W \circ \Xi) \circ Q_{VU}),
\]

\[
E^c = E^c(R_p, R_c, R_0, R_1, W) := \hat{E}^c(R_c, (W \circ \Xi) \circ Q_{VU}),
\]

\[
E^l_+ = E^l_+(R_p, R_c, R_0, R_1, W) := \hat{E}^l_+(R_p - R_1, (W \circ \Xi) \circ Q_{VU}),
\]

\[
E^l_- = E^l_-(R_p, R_c, R_0, R_1, W) := I(V; Z|U) - R_p + R_1.
\]

Theorem 50 (Extension of [24, Theorem 1, part (a)]): Eqs. \((206)-(209)\) are source-channel universally attainable rates of exponents and information leakage rate in the sense of Definition 49.

Therefore, our source-channel universal code attaining Eqs. \((206)-(209)\) depends on \(R_p, R_c\), the distribution \(Q_{VU}\) on \(U \times V\), and the channel \(\Xi : V \rightarrow X\).

We prove Theorem 50 by expurgating the messages in the code given in Theorem 45. The outline of the proof is as follows: First, in Step (1), similar to Theorem 45 we evaluate the leaked information when the channel is given by the conditional types and the source obeys the uniform distribution. Then, for a given code in Step (1), we expurgate the common message \(E_n\) in Step (2) and the secret message \(S_n\) in Step (3). We evaluate the leaked information of the expurgated code for an arbitrary source distribution and an arbitrary conditional type in Step (4). Based on this evaluation, we evaluate the leaked information of the expurgated code for an arbitrary source distribution and an arbitrary discrete memoryless channel in Step (5).
In the following proof, we assume that the secret message $S_n$ and the common message $E_n$ obey the uniform distributions on $S_n$ and $E_n$. However, expurgations $S_n'$ and $E_n'$ of the secret message $S_n$ and the common message $E_n$ are allowed to obey arbitrary distributions.

**Step (1): Evaluation of the leaked information when the channel is given as the uniform distribution on a fixed conditional type:**

Recall the fixed code $\varphi_{p,n}$ for BCD given in Theorem 8. The code $\varphi_{p,n}$ has the private message set $S_{0,n} \times \mathcal{B}_{1,n}$ and the common message set $\mathcal{B}_{2,n}$. We attach the random coding $\Lambda_{F,G,n}$ for message $S_{1,n}, \ldots, S_{T,n}$ given as Second Step of Code Ensemble 2 in Subsection VI-B to the code $\varphi_{p,n}$ when $T = 2$, $S_{1,n} = S_{n}$, $S_{0,n} = E_n$, and $S_{2,n}$ is the random number subject to the uniform distribution, which is used as the dummy for making $S_n$ secret for Eve. The uniformity of the distribution guarantees that

$$H_{1+p}(S_{2,n}|S_{1,n}, S_{0,n}) = n(R_c + R_p - R_1 - R_2) \quad (210)$$

for any $\rho \in (0, 1]$. Then, the encoder is given by $\Phi_{a,n} = (\varphi_{p,n}, \Lambda_{F,G,n})$. In the following, Bob’s decoder $\Phi_{b,n}$ and Eve’s decoder $\Phi_{e,n}$ are given as the maximum mutual information decoder. We treat the ensemble of codes $\Phi_n := (\Phi_{a,n}, \Phi_{b,n}, \Phi_{e,n})$.

For an arbitrary $\Theta_n \in \mathcal{W}_{n,Z}(Q_V)$ and an arbitrary $\rho \in (0, 1)$, the combination of Lemmas 15 and 19 yields that

$$\mathbf{E}_{\Phi_{a,n}} \sum_e P_{E_n}(e) \sum_s P_{S_n|E_n}(s|e) \cdot \exp(\rho D(P_{Z^n|S_n=\epsilon,E_n=\epsilon,\varphi_{a,n}}||P_{Z^n|E_n=\epsilon,\varphi_{a,n}})(\Theta_n))$$

$$\leq 1 + \epsilon_n \rho, \Theta_n(1)(\Theta_n, P_{\text{mix},T_n(Q_V)}),$$

$$= 1 + \epsilon_n \rho, \Theta_n(1)(\Theta_n, P_{\text{mix},T_n(Q_V)}),$$

(211)

where $D(P_{Z^n|S_n=\epsilon,E_n=\epsilon,\varphi_{a,n}}||P_{Z^n|E_n=\epsilon,\varphi_{a,n}})$ denotes the relative entropy $D(P_{Z^n|S_n=\epsilon,E_n=\epsilon,\varphi_{a,n}}||P_{Z^n|E_n=\epsilon,\varphi_{a,n}})$ when the channel is $\Theta_n \in \mathcal{W}_{n,Z}(Q_V)$.

The relations (210) and (189) with $T = 2$ yield

$$\mathbf{E}_{\Phi_{a,n}} I(S_{I,n}; Z^n|S_{0,n})[T_n(W_Z), \Phi_{a,n}, P_{S_{T,n}}]$$

$$\leq |T_{n,Z}(Q_V)| e^{-n[R_0 - 2 \log(n) + 1]}.$$

(212)

Thanks to the Markov inequality in the same way as (9) and (10), given a fixed $\rho \in (0, 1)$, due to (211) and (212), we can see that there exists at least one code $\varphi_{a,n}$ such that the relations

$$I(S_{I,n}; Z^n|S_{0,n})[T_n(W_Z), \varphi_{a,n}, P_{S_{T,n}}]$$

$$\leq p_1(n)|T_{n,Z}(Q_V)| e^{-n[R_0 - R_1 - K[Z]|W_Z \times Q_V]|},$$

$$\sum_e P_{E_n}(e) \sum_s P_{S_n|E_n}(s|e) \cdot \exp(\rho D(P_{Z^n|S_n=\epsilon,E_n=\epsilon,\varphi_{a,n}}||P_{Z^n|E_n=\epsilon,\varphi_{a,n}})(\Theta_n))$$

$$\leq p_1(n)(1 + \epsilon_n \rho, \Theta_n(1)(\Theta_n, P_{\text{mix},T_n(Q_V)}))$$

(214)

hold for any $W_Z \in T_{n,Z}(Q_V)$ and $\Theta_n \in \mathcal{W}_{n,Z}(Q_V)$.

**Step (2): Expurgation for common message $E_n$:**

We choose $p_3(n) := 2p_1(n)$. When $e$ is randomly chosen from $E_n$ subject to the uniform distribution, the element $e$ satisfies all of the following conditions at least with probability of $1 - p_1(n)/p_3(n) = \frac{1}{2}$. The
relations

\[
\sum_s P_{S_n|E_n}(s|e_0) \exp(\rho D(P_{Z^n|S_n=s,E_n=e,\varphi_{a,n}}||P_{Z^n|E_n=e,\varphi_{a,n}})(\Theta_n))
\leq p_1(n)p_3(n)(1 + \varepsilon_{n,\rho,1}) (\Theta_n, P_{\text{mix},T_n(Q_V)}) \times P_{S_n|E_n}(s|e_0) \exp(\rho D(P_{Z^n|S_n=s,E_n=e,\varphi_{a,n}}||P_{Z^n|E_n=e,\varphi_{a,n}})(\Theta_n))
\leq p_1(n)p_3(n)(1 + \varepsilon_{n,\rho,1}) (\Theta_n, P_{\text{mix},T_n(Q_V)}),
\]

(216)

hold for any elements \(W_Z \in T_n.Z(Q_V)\) and \(\Theta_n \in W_n.Z(Q_V)\), and \(n \geq N\). Thus, there exist \(|E_n|/2\) elements \(e \in E_n\) satisfies the above conditions. So, we denote the set of such elements by \(E'_n\).

Step (3): Expurgation for secret message \(S_n\):

Then, when \(s\) is randomly chosen from \(S_n\) subject to the uniform distribution, the element \(s\) satisfies all of the following conditions at least with probability of \(1 - p_1(n)/p_3(n) \geq \frac{1}{2}\): The relations

\[
\sum_s P_{S_n|E_n}(s|e_0) \exp(\rho D(P_{Z^n|S_n=s,E_n=e,\varphi_{a,n}}||P_{Z^n|E_n=e,\varphi_{a,n}})(\Theta_n))
\leq p_1(n)p_3(n)(1 + \varepsilon_{n,\rho,1}) (\Theta_n), P_{\text{mix},T_n(Q_V)}), \times P_{S_n|E_n}(s|e_0) \exp(\rho D(P_{Z^n|S_n=s,E_n=e,\varphi_{a,n}}||P_{Z^n|E_n=e,\varphi_{a,n}})(\Theta_n))
\leq p_1(n)p_3(n)(1 + \varepsilon_{n,\rho,1}) (\Theta_n), P_{\text{mix},T_n(Q_V)}),
\]

(217)

hold for any elements \(e' \in E'_n\), \(W_Z \in T_n.Z(Q_V)\), \(\Theta_n \in W_n.Z(Q_V)\), and \(n \geq N\). Thus, there exist \(|S_n|/2\) elements \(s \in S_n\) satisfies the above conditions. So, we denote the set of such elements by \(S'_n\).

Step (4): Universal code that works for all sources when the channel is given as the uniform distribution on a fixed conditional type:

In the following discussion, \(P_{S'_n,E'_n}\) is an arbitrary joint distribution of the random variables \(S'_n\) and \(E'_n\) on \(S'_n \times E'_n\). For a given \(e \in E'_n\), we consider two kinds of marginal distributions of \(Z^n\) as follows.

\[
P_{Z^n|E'_n=e,\varphi_{a,n}} = \sum_{s \in S_n} P_{S_n}(s) P_{Z^n|S_n=s,E'_n=e,\varphi_{a,n}}
\]

\[
P'_{Z^n|E'_n=e,\varphi_{a,n}} = \sum_{s' \in S_n} P_{S_n}(s') P_{Z^n|S_n=s',E'_n=e,\varphi_{a,n}}
\]

The former marginal distribution is discussed in Steps (1), (2), and (3). Hence, using (36), (216), and (217), we obtain

\[
I(S'_n,Z^n|E'_n)[\Theta_n(W_Z), \varphi_{a,n}, P_{S'_n,E'_n}]
\]

\[
= \sum_{e \in E'_n} P_{E'_n}(e) D(P_{Z^n|S'_n=E'_n=e,\varphi_{a,n}} || P'_{Z^n|E'_n=e,\varphi_{a,n}}) \times P_{S'_n|E'_n=e}[\Theta_n(W_Z)]
\]

\[
\leq \sum_{e \in E'_n} P_{E'_n}(e) D(P_{Z^n|S'_n=E'_n=e,\varphi_{a,n}} || P'_{Z^n|E'_n=e,\varphi_{a,n}}) \times P_{S'_n|E'_n=e}[\Theta_n(W_Z)]
\]

\[
= \sum_{e \in E'_n} P_{E'_n}(e) \sum_{s \in S_n} P_{S_n}(s) D(P_{Z^n|S'_n=s,E'_n=e,\varphi_{a,n}} || P'_{Z^n|E'_n=e,\varphi_{a,n}})[\Theta_n(W_Z)]
\]

\[
\leq p_1(n)p_3(n)^2 |T_n.Z(Q_V)| e^{-n[R_p-R_1-I(V;Z')(U)[W_Z \times Q_V]]},
\]

(218)
for any elements \( W_z \in \mathcal{T}_{n,z}(Q_{VU}) \), \( \Theta_n \in \mathcal{W}_{n,z}(Q_V) \), and \( n \geq N \). Similarly, using the convexity of \( x \mapsto e^x \), (16), (216), and (217), we obtain

\[
\exp(\rho I(S'_n; Z^n|E'_n)[W_Z, \varphi_{a,n}, P_{S'_n,E'_n}]) \leq p_4(n)(1 + \epsilon_{n,p,1}(W_Z, Q_{VU})^n)
\]

(220)

for any sequence of joint distributions \( P_{S'_n,E'_n} \) and \( n \geq N \).

Using (220), for an arbitrary \( \epsilon > 0 \), we can choose an integer \( N_1 \) such that

\[
\log I(S'_n; Z^n|E'_n)[W_Z, \varphi_{a,n}, P_{S'_n,E'_n}] \leq -n(E'_+ - nR_0, R_1, W) - \epsilon
\]

(222)

for \( n \geq N_1 \). Due to (221), we obtain

\[
\frac{1}{n} I(S'_n; Z^n|E'_n)[W_Z, \varphi_{a,n}, P_{S'_n,E'_n}] \leq \log p_5(n) + \log(1 + \epsilon_{1,p,1}(W_Z, Q_{VU})^n)
\]

\[
\leq \frac{\log p_5(n) + \log 2 + \log \epsilon_{1,p,1}(W_Z, Q_{VU})^n}{np} \leq \frac{\epsilon_{1,p,1}(W_Z, Q_{VU})^n}{np}
\]

(223)

When \( \rho = \frac{1}{\sqrt{n}} \), as is mentioned in Lemma 47, the RHS of (223) converges \( E'_+ - nR_0, R_1, W \) uniformly. Hence, for an arbitrary \( \epsilon > 0 \), we can choose an integer \( N_2 \) such that

\[
I(S'_n; Z^n|E'_n)[W_Z, \varphi_{a,n}, P_{S'_n,E'_n}] \leq n(E'_+ - nR_0, R_1, W) + \epsilon
\]

(224)

for \( n \geq N_2 \).

Therefore, since the original code \( \varphi_{0,n} \) satisfies (16) and (17), using (222) and (224), we can see that \( (E^b, E^r, E'_+, E'_1) \) is a universally attainable quadruple of exponents in the sense of Definition 49.

Remark 51: In this section, we treat the leaked information asymptotically as (205). However, in Section XI, we have treated it non-asymptotically as (149) and (150). The difference is caused by the condition for the sequence of joint distributions \( P_{S_{z,n}} \). In Section XI, we do not assume the uniformity. However, in this section, we can use uniform distribution of \( S_{z,n} \). Hence, we can calculate the relative Rényi entropy as (210) non-asymptotically.
XIII. COMPARISON OF EXPONENTS OF LEAKED INFORMATION

In this section, we compare the exponent of leaked information given in Sections XI and XII and the exponents of leaked information given in Subsection [IX-B] when the source distribution $P_{S_T}$ is uniform. First, in Subsection XIII-A we compare the exponent given in Sections XI and XII with the above mentioned exponent. Then, we clarify that the exponent in Sections XI and XII is greater than one of exponents in Subsection [IX-B] which is the same as that in [I9]. Next, in Subsection XIII-B we give equality conditions between two exponents. In the remaining subsections, we give proofs of Lemmas used in Subsections XIII-A and XIII-B.

A. Comparison between Two Exponents $\tilde{E}(R, \overline{W}_Z \times Q_{\nu})$ and $\tilde{E}^{E_0}(R, \overline{W}_Z \times Q_{\nu})$

First, we characterize the exponent $\tilde{E}^{E_0}(R, \overline{W}_Z \times Q_{\nu}) = \sup_{\rho \in (0, 1)} E_0(\rho | \overline{W}_Z, Q_{\nu})$, which describes the exponent of leaked information when $R$ is $\rho R - \sum_{i \in I} R_i$ and the source distribution $P_{S_T}$ is uniform, as is shown in Subsection [IX-B]. The exponent can be attained by the code constructed in the first construction (Subsection [VI-B]). Since $E_0(\rho | \overline{W}_Z, Q_{\nu})$ is convex with respect to $\rho$ [I2], $F_{\rho}(Q_{\nu}, Q) := \frac{d}{d\rho} E_0(\rho | \overline{W}_Z, Q_{\nu}, Q)$ is monotonically increasing with respect to $\rho$. As limits, we define

$$F_1(Q_{\nu}, Q) := \lim_{\rho \to 1^-} F_{\rho}(Q_{\nu}, Q)$$

$$E_0(1 | \overline{W}_Z, Q_{\nu}, Q) := \lim_{\rho \to 1^-} E_0(\rho | \overline{W}_Z, Q_{\nu}, Q).$$

In particular, when $Q_{\nu}$ equal $Q_{\nu} = E_{\nu} = \tilde{E}(R, \overline{W}_Z \times Q_{\nu})$, $\tilde{E}^{E_0}(R, \overline{W}_Z \times Q_{\nu})$, and the above values depend only on $Q_{\nu}$. Then, $\tilde{E}(R, \overline{W}_Z \times Q_{\nu})$, $\tilde{E}^{E_0}(R, \overline{W}_Z \times Q_{\nu})$, $E_0(1 | \overline{W}_Z, Q_{\nu}, Q)$, $F_{\rho}(Q_{\nu}, Q)$, and $F_{\rho}(Q_{\nu}, Q)$ are simplified to $\tilde{E}(R, \overline{W}_Z \times Q_{\nu})$, $\tilde{E}^{E_0}(R, \overline{W}_Z \times Q_{\nu})$, $E_0(1 | \overline{W}_Z, Q_{\nu}, Q)$, $F_{\rho}(Q_{\nu}, Q)$, and $F_{\rho}(Q_{\nu}, Q)$. Then, we obtain the following lemma.

**Lemma 52:** (1) Case of $R < F_1(Q_{\nu}, Q)$. There uniquely exists $\rho \in (0, 1)$ such that $R = F_{\rho}(Q_{\nu}, Q)$. Then, the exponent $\tilde{E}^{E_0}(R, \overline{W}_Z \times Q_{\nu})$ can be characterized as

$$\tilde{E}^{E_0}(R, \overline{W}_Z \times Q_{\nu}) = \rho_0 R - E_0(\rho_0 | \overline{W}_Z, Q_{\nu}, Q).$$

(2) Case of $R \geq F_1(Q_{\nu}, Q)$. The exponent $\tilde{E}^{E_0}(R, \overline{W}_Z \times Q_{\nu})$ can be characterized as

$$\tilde{E}^{E_0}(R, \overline{W}_Z \times Q_{\nu}) = R - E_0(1 | \overline{W}_Z, Q_{\nu}, Q).$$

The quantities appearing in Lemma 52 can be characterized as follows.

**Lemma 53:** The quantities $F_{\rho}(Q_{\nu}, Q)$, $F_1(Q_{\nu}, Q)$, and $E_0(1 | \overline{W}_Z, Q_{\nu}, Q)$ are calculated as

$$F_{\rho}(Q_{\nu}, Q) = \frac{\sum_u Q_{\nu}(u) \sum_i (\sum_v \frac{1}{1 - \rho} (\log \overline{W}_Z(z|v)) Q_{\nu}(v|u) \overline{W}_Z(z|v)^{1 - \rho})(\sum_v Q_{\nu}(v|u) \overline{W}_Z(z|v)^{1 - \rho})}{\sum_u Q_{\nu}(u) \sum_i \sum_v Q_{\nu}(v|u) \overline{W}_Z(z|v)^{1 - \rho} - \sum_u Q_{\nu}(u) \sum_i \log(\sum_v Q_{\nu}(v|u) \overline{W}_Z(z|v)^{1 - \rho})(\sum_v Q_{\nu}(v|u) \overline{W}_Z(z|v)^{1 - \rho}) - \sum_u Q_{\nu}(u) \sum_v Q_{\nu}(v|u) \overline{W}_Z(z|v)^{1 - \rho}).$$

$$F_1(Q_{\nu}, Q) = -\frac{\sum_u Q_{\nu}(u) \sum_i \log(\sum_v Q_{\nu}(v|u) \max_{\nu'} \overline{W}_Z(z|\nu'))}{\sum_i \max_{\nu'} \overline{W}_Z(z|\nu')}$$

$$E_0(1 | \overline{W}_Z, Q_{\nu}, Q) = \log \sum_u Q_{\nu}(u) \sum_{z \in \text{supp}(Q_{\nu}|u)} \max_{\nu'} \overline{W}_Z(z|\nu').$$
In particular, $F_\rho(Q_V)$, $F_1(Q_V)$, and $E_0(1|\overline{W}_Z, Q_V)$ are simplified to

$$F_\rho(Q_V) = \frac{\sum_z (\sum_{v} \frac{1}{1 - \rho} (\log \overline{W}_Z(z|v)) Q_V(v) \overline{W}_Z(z|v) \frac{1}{1 - \rho} ) (\sum_{v} Q_V(v) \overline{W}_Z(z|v) \frac{1}{1 - \rho} )^{-\rho} } {\sum_z (\sum_{v} Q_V(v) \overline{W}_Z(z|v) \frac{1}{1 - \rho} )^{-1} - \sum_z \log(\sum_{v} Q_V(v) \overline{W}_Z(z|v) \frac{1}{1 - \rho} ) (\sum_{v} Q_V(v) \overline{W}_Z(z|v) \frac{1}{1 - \rho} )^{-1} }.$$

(232)

$$F_1(Q_V) = -\frac{\sum_z \log(\sum_{v \in \mathcal{V}} Q_V(v)) \max_{v'} \overline{W}_Z(z|v')}{\sum_{v} Q_V(U) \sum_z \max_{v'} \overline{W}_Z(z|v')}.$$

(233)

$$E_0(1|\overline{W}_Z, Q_V) = \log \sum_z \max_{v \in \text{supp}(Q_v)} \overline{W}_Z(z|v).$$

(234)

Further, the map $Q_V \mapsto F_1(Q_V)$ is concave.

The proof of Lemma 54 will be given in Subsection XIII-D. For a detail analysis for the exponent $E_{E_0}(R, \overline{W}_Z \times Q_V)$, we define

$$E_0(\rho) := \max_{Q'_{V|U}} E_0(\rho|\overline{W}_Z, Q'_{V|U}, Q_U) = \max_{Q'_V} E_0(\rho|\overline{W}_Z, Q'_V)$$

$$E_0(1) := \max_{Q'_{V|U}} E_0(1|\overline{W}_Z, Q'_{V|U}, Q_U) = \max_{Q'_V} E_0(1|\overline{W}_Z, Q'_V)$$

$$F_\rho := \frac{d}{d\rho} E_0(\rho), \quad F_1 := \lim_{\rho \to 1^{-}} F_\rho,$$

and

$$\mathcal{K} := \{(z, v) \in \mathcal{Z} \times \mathcal{V}| \overline{W}_Z(z|v) = \max_{v'} \overline{W}_Z(z|v')\}$$

$$\mathcal{Z}_v := \{z \in \mathcal{Z}|(z, v) \in \mathcal{K}\}, \quad \mathcal{V}_z := \{v \in \mathcal{V}|(z, v) \in \mathcal{K}\}.$$  

(235)

Due to the compactness of the set $\mathcal{P}(\mathcal{U})$, we have

$$\lim_{\rho \to 1^{-}} \max_{Q'_V} E_0(1|\overline{W}_Z, Q'_{V|U}) = \max_{Q'_V} \lim_{\rho \to 1^{-}} E_0(1|\overline{W}_Z, Q'_V).$$

Hence, we obtain the following lemma for characterization of the quantity $E_0(1)$.

**Lemma 54:**

$$E_0(1) = \log \sum_z \max_v \overline{W}_Z(z|v) = \lim_{\rho \to 1^{-}} E_0(\rho).$$

(236)

Then, we have the following characterization for a special case of Case (2) of Lemma 52.

**Lemma 55:** Assume that $\bigcup_{v \in \text{supp}(Q_u)} \mathcal{Z}_v = \mathcal{Z}$ for any $u \in \text{supp}(Q_U)$. When $R \geq F_1(Q_{V|U}, Q_U)$, we have

$$E_0(1) = E_0(1|\overline{W}_Z, Q_{V|U}, Q_U)$$

(237)

and

$$\tilde{E}^{E_0}(R, \overline{W}_Z \times Q_V) = R - E_0(1).$$

(238)

The proof of Lemma 55 will be given in Subsection XIII-E.

For comparison between two exponential decreasing rates $\tilde{E}^{E_0}(R, \overline{W}_Z \times Q_V)$ and $\tilde{E}(R, \overline{W}_Z \times Q_V)$, we prepare the following lemma.
Lemma 56: Any channel $\overline{W}_Z \in \mathcal{W}(\mathcal{V}, \mathcal{Z})$ satisfies
\[
\min_{w_z \in \mathcal{W}(U \times V, Z)} D(W_Z|\overline{W}_Z(Q_{VU}) - \rho I(V; Z|U)[W_Z \times Q_{VU}]
\geq - E_0(\rho|\overline{W}_Z, Q_{VU}, Q_U)
\]
for any $\rho \in (0, 1)$.
The proof of Lemma 56 will be given in Subsection XIII-F. Since the inequalities
\[
\tilde{E}'(R, \overline{W}_Z \times Q_{VU})
= \min_{w_z \in \mathcal{W}(U \times V, Z)} D(W_Z|\overline{W}_Z(Q_{VU}) + [R - I(V; Z|U)[W_Z \times Q_{VU}]]
\geq \min_{w_z \in \mathcal{W}(U \times V, Z)} D(W_Z|\overline{W}_Z(Q_{VU}) + \rho[R - I(V; Z|U)[W_Z \times Q_{VU}]]
\geq \min_{w_z \in \mathcal{W}(U \times V, Z)} D(W_Z|\overline{W}_Z(Q_{VU}) + \rho(R - I(V; Z|U)[W_Z \times Q_{VU}])
\]
hold for any $\rho \in (0, 1)$, we obtain the following theorem.

Theorem 57:
\[
\tilde{E}'(R, \overline{W}_Z \times Q_{VU})
\geq \sup_{\rho \in (0, 1)} \rho R - E_0(\rho|\overline{W}_Z, Q_{VU}, Q_U) = \tilde{E}_0(R, \overline{W}_Z \times Q_{VU}).
\]

B. Equality Conditions of (241)
In this subsection, we derive equality conditions of (241). For this purpose, we prepare two lemmas.

Lemma 58: For a fixed $\rho \in (0, 1)$, the following three conditions for a distribution $Q_V$ are equivalent.
(i) The following value does not depend on $v \in \mathcal{V}$.
\[
\sum_z \overline{W}_Z(z|v)^{1/\rho}(\sum_{v'} Q_V(v')\overline{W}_Z(z|v')^{1/\rho})^{-\rho}
\]
(ii) The following relation holds.
\[
E_0(\rho|\overline{W}_Z, Q_V) = E_0(\rho) = \max_{Q_V'} E_0(\rho|\overline{W}_Z, Q_V').
\]
(iii) The following relations hold for any $v \in \mathcal{V}$.
\[
\sum_z \overline{W}_Z(z|v)^{1/\rho}(\sum_{v'} Q_V(v')\overline{W}_Z(z|v')^{1/\rho})^{-\rho} = \max_{Q_V'} \sum_z (\sum_{v'} Q_V'(v')\overline{W}_Z(z|v')^{1/\rho})^{1-\rho} = \max_{Q_V'} e^{E_0(\rho|\overline{W}_Z, Q_V')} = e^{E_0(\rho)}.
\]
The proof of Lemma 58 will be given in Subsection XIII-F.

Lemma 59: The following three conditions for a distribution $Q_V$ are equivalent.
(i) The following value does not depend on $v \in \mathcal{V}$.
\[
\sum_{z \in Z_v} \frac{\max_{v' \in V_z} \overline{W}_Z(z|v')}{\sum_{v'' \in V_z} Q_V(v'')} = \sum_{z \in Z_v} \frac{\overline{W}_Z(z|v)}{\sum_{v'' \in V_z} Q_V(v'')}.
\]
(ii) The following relation holds.
\[
F_1(Q_V) = \min_{Q_V'} F_1(Q_V').
\]
(iii) The following relations hold for any $v \in \mathcal{V}$.
\[
\sum_{z \in Z_v} \max_{v' \in V_z} \overline{W}_Z(z|v') = \sum_{z \in Z_v} \overline{W}_Z(z|v) = \sum_{v'' \in V_v} Q_V(v'') = \sum_{z} \max_{v'} \overline{W}_Z(z|v').
\]
The proof of Lemma 58 will be given in Subsection XIII-G.

Then, we introduce two conditions for a distribution $Q_V$.

**Condition 60:** Given a fixed $\rho \in (0, 1)$, the distribution $Q_V$ satisfies the condition given in Lemma 58.

**Condition 61:** The distribution $Q_V$ satisfies the condition given in Lemma 59.

Since Condition 60 depends on $\rho$, we describe it by “Condition 60 with $\rho$” when we need to clarify the dependence on $\rho$.

**Lemma 62:** When distribution $Q_V$ and $Q_V'$ satisfy Condition 60 with $\rho$, the relation $\sum_v Q_{V'}(v) \overline{W}_Z(z|v)^{1/\rho} = \sum_v Q_{V'}(v) \overline{W}_Z(z|v)^{1/\rho'}$ holds for any $z \in Z$. That is the value $\sum_v Q_{V'}(v) \overline{W}_Z(z|v)^{1/\rho'}$ does not depend on the choice of $Q_V$ as long as the distribution $Q_V$ satisfies Condition 60 with $\rho$.

The proof of Lemma 62 will be given in Subsection XIII-E.

**Lemma 63:** When distribution $Q_V$ and $Q_V'$ satisfy Condition 61 with $\rho$, the relation $\sum_{v''} Q_{V'}(v'') = \sum_{v''} Q_{V'}(v'')$ holds for any $z \in Z$. That is the value $\sum_v Q_{V'}(v) \overline{W}_Z(z|v)^{1/\rho'}$ does not depend on the choice of $Q_V$ as long as the distribution $Q_V$ satisfies Condition 61.

The proof of Lemma 63 will be given in Subsection XIII-G. Hence, we can define the transition matrices $W_{Z,\rho}$ and $W_{Z,1}$ from $V$ to $Z$ by

$$
W_{Z,\rho}(z|v) := \frac{\overline{W}_Z(z|v)^{1/\rho} (\sum_v Q_{V\rho}(v) \overline{W}_Z(z|v)^{1/\rho})^{-\rho}}{\sum_z \overline{W}_Z(z|v)^{1/\rho} (\sum_v Q_{V\rho}(v) \overline{W}_Z(z|v)^{1/\rho})^{-\rho}},
$$

$$
W_{Z,1}(z|v) := \left\{ \begin{array}{ll} \\
\frac{\sum_{v''} Q_{V1}(v'') \overline{W}_Z(z|v'')}{\sum_{z''} \overline{W}_Z(z''|v'')}, & z \in Z_V \\
0, & z \in Z'_V,
\end{array} \right.
$$

where the distributions $Q_{V\rho}$ and $Q_{V1}$ satisfy Condition 60 with $\rho$ and Condition 61 respectively. These definitions do not depend on the choices of $Q_{V\rho}$ and $Q_{V1}$.

**Lemma 64:** When $Q_{V\rho}$ satisfies Condition 60 with $\rho$, we have

$$
F_{\rho} = F_{\rho}(Q_{V\rho}) = I(V; Z)[W_{Z,\rho} \times Q_{V\rho}] \tag{244}
$$

$$
D(W_{Z,\rho}||\overline{W}_Z|Q_{V\rho}) = \rho F_{\rho} - E_0(\rho) \tag{245}
$$

The proof of Lemma 64 will be given in Subsection XIII-F.

**Lemma 65:** When $Q_{V1}$ satisfies Condition 61, we have

$$
F_1 = F_1(Q_{V1}) = I(V; Z)[W_{Z,1} \times Q_{V1}] \tag{246}
$$

$$
D(W_{Z,1}||\overline{W}_Z|Q_{V1}) = F_1 - E_0(1) \tag{247}
$$

The proof of Lemma 65 will be given in Subsection XIII-G.

**Lemma 66:** For any $\rho \in (0, 1)$, we choose the distribution $Q_{V\rho}$ satisfying Condition 60 with $\rho$. We choose a sequence $\rho_n$ such that $\rho_n \to 0$ as $n \to \infty$ and the limit distribution $\lim_{n \to \infty} Q_{V\rho_n}$ exists. (Since the set of distributions over $V$ is compact, such a sequence $\rho_n$ exists.) Then, the limit distribution $\lim_{n \to \infty} Q_{V\rho_n}$ satisfies Condition 61.

The proof of Lemma 66 will be given in Subsection XIII-H.

Then, using the above lemmas, we can characterize equality conditions of (241) for the case $Q_{UV} = Q_U \times Q_V$ in the following way.

**Theorem 67:** (1) Case of $R < F_1$. We choose $\rho \in (0, 1)$ such that $R = F_\rho$. When $Q_{V\rho}$ satisfies Condition 60 with $\rho$ the relations

$$
\min_{\tilde{E}_v} \tilde{E}_v(R, \overline{W}_Z \times Q_{V\rho}) = \min_{\tilde{E}_v} \tilde{E}_v(R, \overline{W}_Z \times Q_{V\rho}) = \min_{\tilde{E}_v} \tilde{E}_v(R, \overline{W}_Z \times Q_{V\rho}) = \rho R - E_0(\rho) \tag{248}
$$

hold, which implies the equality in (241).
(2) Case of $R \geq F_1$. When $Q_{V,1}$ satisfies Condition \[61\] the relations
\[
\min_{Q_v} \tilde{E}^i(R, W_Z \times Q_v) = \min_{Q_v} \tilde{E}^E_0(R, W_Z \times Q_v) \\
= \tilde{E}^i(R, W_Z \times Q_{V,1}) = \tilde{E}^E_0(R, W_Z \times Q_{V,1}) = R - E_0(1)
\]
hold, which implies the equality in (241).

Combining the above relations and Lemma 57, we obtain
\[
\min_{Q_v} \tilde{E}^i(R, W_Z \times Q_v) = \min_{Q_v} \tilde{E}^E_0(R, W_Z \times Q_v) = \max_{\rho \in [0,1]} \rho R - E_0(\rho).
\]

Proof of Theorem 67. First, we show (248). Since $I(Z; V)[W_{Z,\rho} \times Q_{V,\rho}] = F_{\rho} = R$ follows from (244), the relations (245) and (242) imply
\[
\tilde{E}^i(R, W_Z \times Q_{V,\rho}) \leq D(W_{Z,\rho}||W_Z|Q_{V,\rho}) + [R - I(Z; V)[W_{Z,\rho} \times Q_{V,\rho}]]_+ \\
= \rho F_{\rho} - E_0(\rho) = \rho R - E_0(\rho|W_Z, Q_{V,\rho}) = \tilde{E}^E_0(R, W_Z \times Q_{V,\rho}).
\]

Any distribution $Q_v$ satisfies
\[
\rho R - E_0(\rho) \leq \rho R - E_0(\rho|W_Z, Q_v) \leq \tilde{E}^E_0(R, W_Z \times Q_v),
\]
which implies
\[
\rho R - E_0(\rho) \leq \min_{Q_v} \tilde{E}^E_0(R, W_Z \times Q_{V,\rho}).
\]

Combining the above relations and Lemma 57, we obtain
\[
\tilde{E}^i(R, W_Z \times Q_{V,\rho}) \leq \rho R - E_0(\rho) = \tilde{E}^E_0(R, W_Z \times Q_{V,\rho}) \\
\leq \min_{Q_v} \tilde{E}^E_0(R, W_Z \times Q_v) \leq \min_{Q_v} \tilde{E}^i(R, W_Z \times Q_v),
\]
which implies (248).

Next, we show (249). The relations (246) and (247) imply
\[
\tilde{E}^i(R, W_Z \times Q_{V,1}) \leq D(W_{Z,1}||W_Z|Q_{V,1}) + [R - I(Z; V)[W_{Z,1} \times Q_{V,1}]]_+ \\
= F_1 - E_0(1) + [R - F_1]_+ = F_1 - E_0(1) + R - F_1 \\
= R - E_0(1) = R - E_0(1|W_Z, Q_{V,1}) = \tilde{E}^E_0(R, W_Z \times Q_{V,1}).
\]

Any distribution $Q_v$ satisfies
\[
R - E_0(1) \leq R - E_0(1|W_Z, Q_v) \leq \tilde{E}^E_0(R, W_Z \times Q_v),
\]
which implies
\[
R - E_0(1) \leq \min_{Q_v} \tilde{E}^E_0(R, W_Z \times Q_{V,\rho}).
\]

Combining the above relations and Lemma 57, we obtain
\[
\tilde{E}^i(R, W_Z \times Q_{V,\rho}) \leq R - E_0(1) = \tilde{E}^E_0(R, W_Z \times Q_{V,\rho}) \\
\leq \min_{Q_v} \tilde{E}^E_0(R, W_Z \times Q_v) \leq \min_{Q_v} \tilde{E}^i(R, W_Z \times Q_v),
\]
which implies (249).

For the general case, we prepare the generalizations of Lemmas 64 and 65. The following lemmas follow from Lemmas 64 and 65.
Lemma 68: When $Q_{V|U=u}$ satisfies Condition 60 with $\rho$ for any $u \in \text{supp}(Q_U)$,

$$F_\rho = F_\rho(Q_{V|U}, Q_U) = I(V; Z|U)[W_{Z,\rho} \times Q_{VU}]$$

$$D(W_{Z,\rho}||\overline{W}_{Z|Q_{VU}}(Q_{VU})) = F_\rho - E_0(\rho).$$

Lemma 69: When $Q_{V|U=u}$ satisfies Condition 61 for any $u \in \text{supp}(Q_U)$,

$$F_1 = F_1(Q_{V|U}, Q_U) = I(V; Z|U)[W_{Z,1} \times Q_{VU}]$$

$$D(W_{Z,1}||\overline{W}_{Z|Q_{VU}}(Q_{VU})) = F_1 - E_0(1).$$

Then, we can characterize equality conditions for (241) in the general case. That is, similar to Theorem 67 using Lemmas 68 and 69 we can show the following theorem.

**Theorem 70:** (1) Case of $R < F_1$. We choose $\rho \in (0, 1)$ such that $R = F_\rho$. When $Q_{V|U=u}$ satisfies Condition 60 with $\rho$ for any $u \in \text{supp}(Q_U)$, the relations

$$\min \tilde{E}^l(R, \overline{W}_Z \times Q_{VU}) = \min \tilde{E}^l(R, \overline{W}_Z \times Q'_V)$$

$$= \min \tilde{E}^{E_0}(R, \overline{W}_Z \times Q'_{VU}) = \min \tilde{E}^{E_0}(R, \overline{W}_Z \times Q'_V)$$

$$= \tilde{E}^l(R, \overline{W}_Z \times Q_{VU}) = \tilde{E}^{E_0}(R, \overline{W}_Z \times Q_{VU}) = \rho R - E_0(\rho)$$

(250)

hold, which implies the equality in (241).

(2) Case of $R \geq F_1$. When $Q_{V|U=u}$ satisfies Condition 61 for any $u \in \text{supp}(Q_U)$, the relations

$$\min \tilde{E}^l(R, \overline{W}_Z \times Q'_{VU}) = \min \tilde{E}^l(R, \overline{W}_Z \times Q'_V)$$

$$= \min \tilde{E}^{E_0}(R, \overline{W}_Z \times Q'_{VU}) = \min \tilde{E}^{E_0}(R, \overline{W}_Z \times Q'_V)$$

$$= \tilde{E}^l(R, \overline{W}_Z \times Q_{VU}) = \tilde{E}^{E_0}(R, \overline{W}_Z \times Q_{VU}) = R - E_0(1)$$

(251)

hold, which implies the equality in (241).

Then, we obtain the following two corollaries.

**Corollary 71:** When the channel $W_Z$ is regular and $Q_V$ is the uniform distribution, the equality in (241) holds.

**Proof:** When the channel $W_Z$ is regular, the uniform distribution over $\mathcal{V}$ satisfies Condition 60 with $\rho$. Hence, when $Q_V$ is the uniform distribution, the equality in (241) holds.

**Corollary 72:** When $R = F_\rho$ and $Q_{V|U=u}$ satisfies Condition 61 for any $u \in \text{supp}(Q_U)$, we have

$$\tilde{E}^l(R, \overline{W}_Z \times Q_{VU}) \leq \tilde{E}^{E_0}(R, \overline{W}_Z \times Q_{VU})$$

In the above case of Corollary 72, the exponent $\tilde{E}^l(R, \overline{W}_Z \times Q_{VU})$ cannot improve the exponent $\tilde{E}^{E_0}(R, \overline{W}_Z \times Q_{VU})$, which is the exponent of the code constructed in the second construction (Subsection VI-C) and is given in Subsection IX-B. However, the relation between $\tilde{E}^l(R, \overline{W}_Z \times Q_{VU})$ and $\tilde{E}^{E_0}(R, \overline{W}_Z \times Q_{VU})$ remains unknown up to now.

**C. Examples**

In this subsection, we numerically compare

$$\tilde{E}^l(R, \overline{W}_Z \times Q_{V}) = \min_{z \in \text{supp}(Q_{V})} D(W_Z||\overline{W}_{Z|Q_{V}}) + \left[R - I(V; Z)[W_Z \times Q_V]\right]_+$$

$$\tilde{E}^{E_0}(R, \overline{W}_Z \times Q_{V}) = \max_{0 \leq \rho \leq 1} \rho R - E_0(\rho)W_Z, Q_V)$$

$$\tilde{E}^{\psi}(R, \overline{W}_Z \times Q_{V}) = \max_{0 \leq \rho \leq 1} \rho R - \psi(\rho)W_Z, Q_V)$$
in two examples.

Example 73: In this example, we address the channel given by a $2 \times 2$ general transition matrix. Consider the case when $\mathcal{Z} = \mathcal{V} = \{1, 2\}$. Define the transition matrix $\overline{W}_Z$ by

$$
\overline{W}_Z := \begin{pmatrix}
1 - p & q \\
p & 1 - q
\end{pmatrix}
$$

(252)

with $p > q \in (0, 1/2)$. When $Q_V(1) = 1/2$ and $Q_V(2) = 1/2$, we have

$$
E_0(\rho|\overline{W}_Z, Q_V) = \log\left(\frac{1}{2}(1 - p)^{1/\rho} + \frac{1}{2}q^{1/\rho}\right)^{1-\rho} + \frac{1}{2}p^{1/\rho} + \frac{1}{2}(1 - q)^{1/\rho}
$$

(253)

and

$$
\psi(\rho|\overline{W}_Z, Q_V) = \log\left(\frac{1}{2}(1 - p)^{1/\rho} + \frac{1}{2}q^{1/\rho}\right)^{1-\rho} + \frac{1}{2}p^{1/\rho} + \frac{1}{2}(1 - q)^{1/\rho}
$$

(254)

Fig. 2 suggests that $\tilde{E}(R, \overline{W}_Z \times Q_V)$ is larger than $\tilde{E}(R, \overline{W}_Z \times Q_V)$. In Fig. 3 we numerically calculate $\text{argmax}_{0 \leq \rho \leq 1} \rho - E_0(\rho|\overline{W}_Z, Q_V)$ and $\text{argmax}_{0 \leq \rho \leq 1} \rho - \psi(\rho|\overline{W}_Z, Q_V)$ which realize $\tilde{E}(R, \overline{W}_Z \times Q_V)$ and $\tilde{E}(R, \overline{W}_Z \times Q_V)$, respectively.

---

Fig. 2. Lower bounds of exponent in Example 73 with $p = 0.01$ and $q = 0.3$. In this case, $I(V;Z|\overline{W}_Z \times Q_V) = 0.317054$. Thick line, Dashed line, and Normal line plot $\tilde{E}(R, \overline{W}_Z \times Q_V)$, $\tilde{E}(R, \overline{W}_Z \times Q_V)$, and $\tilde{E}(R, \overline{W}_Z \times Q_V)$ as functions of $R$ from $R = 0.317054$ to $R = \log 2 = 0.693147$ with the origin $(0.3, 0)$.

Example 74: In this example, we consider the case when states satisfying Conditions (60) and (61) are not unique. Consider the case when $\mathcal{Z} = \mathcal{V} = \{1, 2, 3, 4\}$. Define the transition matrix $\overline{W}_Z$ by

$$
\overline{W}_Z := \begin{pmatrix}
\frac{1}{2} - p & p & \frac{1}{2} - p & p \\
p & \frac{1}{2} - p & p & \frac{1}{2} - p \\
\frac{1}{2} - p & p & \frac{1}{2} - p & p \\
p & \frac{1}{2} - p & p & \frac{1}{2} - p
\end{pmatrix}
$$

(255)

with $p \in (0, 1/4)$. When $Q_V(1) = q$, $Q_V(2) = q$, $Q_V(3) = 1 - q$, and $Q_V(4) = 1 - q$, we have

$$
\sum_{z} \overline{W}_Z(z|v) \frac{1}{\rho} \left(\sum_{v'} Q_V(v') \overline{W}_Z(z|v')\right)^{1/\rho} = 4\left(\frac{1}{2}(1 - p)^{1/\rho} + \frac{1}{2}p^{1/\rho}\right) = 2^{1/\rho}(\frac{1}{2}(1 - p)^{1/\rho} + \frac{1}{2}p^{1/\rho})^{1-\rho}
$$

(256)

and

$$
= 2^{1/\rho}(\frac{1}{2}(1 - p)^{1/\rho} + \frac{1}{2}p^{1/\rho})^{1-\rho}
$$

(257)
Given in (236). Since \( \tilde{E}^\psi (R, \overline{W}_Z \times Q_V) \) realizes \( E^\psi (R, \overline{W}_Z \times Q_V) \). There is no graph corresponding to \( \tilde{E}^\psi (R, \overline{W}_Z \times Q_V) \) because \( \tilde{E}^\psi (R, \overline{W}_Z \times Q_V) \) is not given as maximization with respect to \( \rho \). The origin is \((0.3, 0)\).

for all \( v \in V \), which implies Condition [60] Hence,

\[
E_0(\rho) = E_0(\rho | W_Z, Q_V) = (1 + \rho) \log 2 + (1 - \rho) \log((\frac{1}{2} - p)^{1+n} + p^{1+n})
\]

(258)

\[
F_\rho = F_\rho (Q_V) = \log 2 - \log((\frac{1}{2} - p)^{1+n} + p^{1+n}) - \frac{1}{1 - \rho} \frac{1}{(\frac{1}{2} - p)^{1+n} + p^{1+n}} \log p
\]

(259)

\[
\psi(\rho | W_Z, Q_V) = (2\rho + 1) \log 2 + \log((\frac{1}{2} - p)^{1+n} + p^{1+n}).
\]

(260)

Next, we check Condition [61] For this purpose, we check Condition (i) in Lemma [59] by treating \( V_z \) given in (236). Since \( V_1 = \{1, 3\} \), \( V_2 = \{2, 4\} \), \( V_3 = \{1, 4\} \), and \( V_4 = \{2, 3\} \), in the above choice of \( Q_v \), we have \( \sum_{v' \in V_z} Q_v(v') = \frac{1}{2} \), which implies

\[
\sum_{z \in Z_v} \frac{\max_{v' \in V_z} W_Z(z | v')}{\sum_{v'' \in V_z} Q_v(v'')} = 2 \frac{1}{2} - p = 4(\frac{1}{2} - p)
\]

(261)

for all \( v \in V \). Thus, Condition [61] holds. Hence,

\[
E_0(1) = \log 4(\frac{1}{2} - p)
\]

(262)

\[
F_1 = \log 2.
\]

(263)

Further, Theorem [70] guarantees that \( \tilde{E}^{E_0}(R, \overline{W}_Z \times Q_V) = \tilde{E}^{E_0}_f (R, \overline{W}_Z \times Q_V) \). So, we numerically compare only \( \tilde{E}^\psi (R, \overline{W}_Z \times Q_V) \) and \( \tilde{E}^{E_0}(R, \overline{W}_Z \times Q_V) \) in Fig. [4]. Since \( \tilde{E}^{E_0}(R, \overline{W}_Z \times Q_V) \) attains the minimum value due to Theorem [70], \( \tilde{E}^{E_0}(R, \overline{W}_Z \times Q_V) \) does not depend on \( q \). Further, \( \tilde{E}^\psi (R, \overline{W}_Z \times Q_V) \) also does not depend on \( q \) due to the form of \( \tilde{E}^\psi (R, \overline{W}_Z \times Q_V) \). Similar to Fig. [3], Fig. [5] suggests that the parameter \( \rho \) realizing \( \tilde{E}^{E_0}(R, \overline{W}_Z \times Q_V) \) has a behavior different from the parameter \( \rho \) realizing \( \tilde{E}^\psi (R, \overline{W}_Z \times Q_V) \).
Fig. 4. Lower bounds of exponent in Example 74 with \( p = 0.1 \). In this case, \( I(V;Z)[\overline{W}_Z \times Q_V]\) = 0.192745. Thick line and Normal line express \( \tilde{E}^\psi(R,\overline{W}_Z \times Q_V) \) and \( \tilde{E}^E(R,\overline{W}_Z \times Q_V) \) as functions of \( R \) from \( R = 0.192745 \) to \( R = 1 \) with the origin \((0.1,0)\). Thick line is straight when \( R \geq 0.4 \) because \( \argmax_{0 \leq \rho \leq 1} \rho R - \psi(\rho|\overline{W}_Z, Q_V) \) is 1 when \( R \geq 0.4 \), as in Fig 5. Normal line is straight when \( R \geq 0.7 \) because \( \argmax_{0 \leq \rho \leq 1} \rho R - E_0(\rho|\overline{W}_Z, Q_V) \) is 1 when \( R \geq 0.7 \), as in Fig 5.

D. Proof of Lemma 53

**Proof**: We can show (229) and (231) by direct calculations. Now, we show (231). In general, when \( b_1 > 0 \) and \( a_1 = a_2 = \ldots = a_l > a_{l+1} > \ldots > a_k > 0 \) for \( i = l + 1, \ldots, k \), the relation

\[
\lim_{\rho \to 1-0} \left( \sum_{i=1}^{k} b_i a_i^{\frac{1}{1-p}} \right)^{-\rho} = \lim_{\rho \to 1-0} \left( \sum_{i=1}^{l} b_i a_i^{\frac{1}{1-p}} \right)^{-\rho} (1 + \sum_{i=l+1}^{k} \frac{b_i}{\sum_{j=1}^{l} b_j a_j^{\frac{1}{1-p}}})^{-\rho}
\]

\[
= \lim_{\rho \to 1-0} \left( \sum_{i=1}^{l} b_i a_i^{\frac{1}{1-p}} \right)^{-\rho} = a_1
\]
holds. That is, the difference \((\sum_{i=1}^{k} b_i a_i^{1-p})^{1-p} - ((\sum_{i=1}^{l} b_i a_i^{1-p})^{1-p})\) behaves as \(O(\exp(-\frac{a}{1-p}))\) with a constant \(a\). Applying the above general discussion, we have

\[
\lim_{\rho \to 1^{-0}} \sum_{u} Q_U(u) \sum_{z} \left( \sum_{v} Q_{V|U}(v|u) \overline{W}_Z(z|v)^{1-\rho} \right)
= \lim_{\rho \to 1^{-0}} \sum_{u} Q_U(u) \sum_{z} \left( \sum_{v \in \mathcal{V}_z(\mathcal{Q}_{V|U})} Q_{V|U}(v|u) (\max_{v \in \text{supp}(\mathcal{Q}_{V|U})} \overline{W}_Z(z|v))^{1-\rho} \right)
= \lim_{\rho \to 1^{-0}} \sum_{u} Q_U(u) \sum_{z} \left( \max_{v \in \text{supp}(\mathcal{Q}_{V|U})} \overline{W}_Z(z|v) \right)
= \sum_{u} Q_U(u) \sum_{z} \left( \max_{v \in \text{supp}(\mathcal{Q}_{V|U})} \overline{W}_Z(z|v) \right).
\]

where \(\mathcal{V}_z(\mathcal{Q}_{V|U}) := \{v \in \text{supp}(\mathcal{Q}_{V|U}) \mid \max_{v \in \text{supp}(\mathcal{Q}_{V|U})} \overline{W}_Z(z|v)\}\). Hence, we obtain (231). Next, we show (230). We have

\[
\frac{d}{d\rho} E_0(\rho \overline{W}_Z, \mathcal{Q}_{V|U}, \mathcal{Q}_U)
= \frac{\sum_{u} Q_U(u) \sum_{z} (\sum_{v} \frac{1}{1-p} (\log \overline{W}_Z(z|v)) \mathcal{Q}_{V|U}(v|u) \overline{W}_Z(z|v)^{1-\rho}) (\sum_{v} Q_{V|U}(v|u) \overline{W}_Z(z|v)^{1-\rho})}{\sum_{u} Q_U(u) \sum_{z} (\sum_{v} Q_{V|U}(v|u) \overline{W}_Z(z|v)^{1-\rho})^{1-\rho}} - \frac{\sum_{u} Q_U(u) \sum_{z} (\sum_{v} \log (\max_{v \in \mathcal{V}_z(\mathcal{Q}_{V|U})} \mathcal{Q}_{V|U}(v|u)) (\sum_{v} Q_{V|U}(v|u) \overline{W}_Z(z|v)^{1-\rho})^{1-\rho})}{\sum_{u} Q_U(u) \sum_{z} (\sum_{v} Q_{V|U}(v|u) \overline{W}_Z(z|v)^{1-\rho})^{1-\rho}}.
\]

When \(\rho\) approaches 1, \(\sum_{v \in \mathcal{V}_z(\mathcal{Q}_{V|U})} Q_{V|U}(v|u) \overline{W}_Z(z|v)^{1-\rho}\) approaches \((\sum_{v \in \mathcal{V}_z(\mathcal{Q}_{V|U})} (\sum_{v} Q_{V|U}(v|u)) (\max_v \overline{W}_Z(z|v))^{1-\rho})^{1-\rho}\). Hence,

\[
\lim_{\rho \to 1^{-0}} \frac{d}{d\rho} E_0(\rho \overline{W}_Z, \mathcal{Q}_{V|U}, \mathcal{Q}_U)
= \lim_{\rho \to 1^{-0}} \left( \frac{\sum_{u} Q_U(u) \sum_{z} (\frac{1}{1-p} \log \max_v \overline{W}_Z(z|v')) (\sum_{v \in \mathcal{V}_z(\mathcal{Q}_{V|U})} Q_{V|U}(v|u))^{1-\rho} \max_v \overline{W}_Z(z|v'))}{\sum_{u} Q_U(u) \sum_{z} (\sum_{v \in \mathcal{V}_z(\mathcal{Q}_{V|U})} Q_{V|U}(v|u))^{1-\rho} \max_v \overline{W}_Z(z|v')) - \frac{\sum_{u} Q_U(u) \sum_{z} (\sum_{v \in \mathcal{V}_z(\mathcal{Q}_{V|U})} \log (\max_v \overline{W}_Z(z|v')) (\sum_{v} Q_{V|U}(v|u))^{1-\rho} \max_v \overline{W}_Z(z|v'))}{\sum_{u} Q_U(u) \sum_{z} (\sum_{v} Q_{V|U}(v|u))^{1-\rho} \max_v \overline{W}_Z(z|v'))} \right)
= \lim_{\rho \to 1^{-0}} \left( - \frac{\sum_{u} Q_U(u) \sum_{z} (\sum_{v \in \mathcal{V}_z(\mathcal{Q}_{V|U})} Q_{V|U}(v|u))^{1-\rho} \max_v \overline{W}_Z(z|v'))}{\sum_{u} Q_U(u) \sum_{z} (\sum_{v} Q_{V|U}(v|u))^{1-\rho} \max_v \overline{W}_Z(z|v'))} \right)
= \lim_{\rho \to 1^{-0}} - \frac{\sum_{u} Q_U(u) \sum_{z} (\sum_{v \in \mathcal{V}_z(\mathcal{Q}_{V|U})} Q_{V|U}(v|u)) \max_v \overline{W}_Z(z|v'))}{\sum_{u} Q_U(u) \sum_{z} \max_v \overline{W}_Z(z|v'))},
\]

which implies (230). Finally, since \(x \mapsto -\log x\) is concave, the map \(Q_V \mapsto F_1(Q_V)\) is concave.

\[\tag{265}\]

**E. Proof of Lemma 55**

Due to (231), we have

\[
E_0(1) = \max_{Q_{V|U}} \lim_{\rho \to 1^{-0}} E_0(\rho \overline{W}_Z, \mathcal{Q}_{V|U}, \mathcal{Q}_U)
= \max_{Q_{V|U}} \log \sum_{u} Q_U(u) \sum_{z} \max_{v \in \text{supp}(\mathcal{Q}_{V|U})} \overline{W}_Z(z|v)
= \log \sum_{z} \max_{v} \overline{W}_Z(z|v),
\]
which implies (237).

Assume that the support of \( Q_{v[U=u]} \) contains \( \{v \in \mathcal{V} | \min \frac{\max_{y} \frac{W_{Z}(v'y')}{W_{Z}(v)}}{W_{Z}(v)} = 1 \} \) for any \( u \in \text{supp}(Q_{U}) \). Due to (231), we have

\[
E_{0}(1|W_{Z}, Q_{v[U], Q_{U}}) = \log \sum_{z} \max_{y} W_{Z}(z|v).
\]  

(266)

Combining (237), we obtain (238). Hence, as a special case of (228), we obtain (239).

**F. Proofs of Lemmas 58, 62, and 64**

**Lemma 75:** Let \( f \) be a concave \( C^{1} \) function from \( \mathbb{R}^{d} \) to \( \mathbb{R} \) and \( \mathcal{P}(d) \) be the subset \( \{(x_{1}, \ldots, x_{d}) \in \mathbb{R}^{d} | x_{i} \geq 0, \sum_{i=1}^{d} x_{i} = 1 \} \). The following two conditions for \( x = (x_{1}, \ldots, x_{d}) \in \mathcal{P}(d) \) are equivalent.

(i) \[
f(x) = \max_{x' \in \mathcal{P}(d)} f(x).
\]  

(267)

(ii) The following relation holds for any \( i \neq j \).

\[
\frac{\partial}{\partial x_{i}} f(x) = \frac{\partial}{\partial x_{j}} f(x).
\]  

(268)

**Proof of Lemma 75** We choose variable \( y = (y_{1}, \ldots, y_{d-1}) \in \mathbb{R}^{d-1} \), and define a function \( \tilde{f}(y) := f(y_{1}, \ldots, y_{d-1}, 1 - \sum_{i=1}^{d-1} y_{i}) \). Due to the concavity, the condition (i) holds if and only if \( \frac{\partial}{\partial y_{i}} \tilde{f}(y) = 0 \) for \( i = 1, \ldots, d-1 \). This condition is equivalent to the condition (ii) because \( \frac{\partial}{\partial y_{i}} \tilde{f}(y_{1}, \ldots, y_{d-1}, 1 - \sum_{i=1}^{d-1} y_{i}) = \frac{\partial}{\partial x_{i}} f(y_{1}, \ldots, y_{d-1}, 1 - \sum_{i=1}^{d-1} y_{i}) \).

**Proof of Lemma 58** In order to apply Lemma 75, we regard all of probabilities \( Q_{v}(v) \) as independent parameters by removing the constraint \( \sum_{v} Q_{v}(v) = 1 \). The partial derivatives are calculated as

\[
\frac{\partial}{\partial Q_{v}(v)} \sum_{z} \sum_{v'} Q_{v}(v') W_{Z}(z|v')^{\frac{1}{1-\rho}} = \sum_{z} (1 - \rho) \sum_{v'} Q_{v}(v') W_{Z}(z|v')^{-\rho} W_{Z}(z|v')^{\frac{1}{1-\rho}}.
\]

Hence, Lemma 75 guarantees the equivalence between (i) and (ii). Condition (iii) trivially implies Condition (i).

The remaining task is showing Condition (i) implies Condition (iii). Assume Condition (i). Since \( \sum_{z} W_{Z}(z|v)^{\frac{1}{1-\rho}} (\sum_{v'} Q_{v}(v') W_{Z}(z|v')^{\frac{1}{1-\rho}})^{-\rho} \) does not depend on \( v \) and Condition (ii) holds,

\[
\sum_{z} W_{Z}(z|v)^{\frac{1}{1-\rho}} (\sum_{v'} Q_{v}(v') W_{Z}(z|v')^{\frac{-\rho}{1-\rho}}) = \sum_{v} Q_{v}(v) \sum_{z} W_{Z}(z|v)^{\frac{1}{1-\rho}} (\sum_{v'} Q_{v}(v') W_{Z}(z|v')^{\frac{1}{1-\rho}})^{-\rho}
\]  

\[
= \sum_{v} (\sum_{v'} Q_{v}(v') W_{Z}(z|v')^{\frac{1}{1-\rho}})^{1-\rho} = e^{E_{0}(\rho | W_{Z}, Q_{v})} \max_{Q_{v}} e^{E_{0}(\rho | W_{Z}, Q_{v})} = e^{E_{0}(\rho)}.
\]

**Proof of Lemma 62** Assume that

\[
\sum_{v} Q_{v}(v) W_{Z}(z|v)^{\frac{1}{1-\rho}} \neq \sum_{v} Q'_{v}(v) W_{Z}(z|v)^{\frac{1}{1-\rho}}
\]  

(269)
for any \( z \in \mathbb{Z} \). Due to the strict concavity of \( x \mapsto x^{1-\rho} \), we have
\[
\frac{1}{2} \left( \sum_v Q_v(v) \overline{W}_Z(z|v) \right)^{1-\rho} + \frac{1}{2} \left( \sum_v Q'_v(v) \overline{W}_Z(z|v) \right)^{1-\rho} < \left( \sum_v \left( \frac{1}{2} Q_v(v) + \frac{1}{2} Q'_v(v) \right) \overline{W}_Z(z|v) \right)^{1-\rho}. \tag{270}
\]
Hence,
\[
\frac{1}{2} \sum_z \left( \sum_v Q_v(v) \overline{W}_Z(z|v) \right)^{1-\rho} + \frac{1}{2} \sum_z \left( \sum_v Q'_v(v) \overline{W}_Z(z|v) \right)^{1-\rho} < \sum_v \left( \frac{1}{2} Q_v(v) + \frac{1}{2} Q'_v(v) \right) \overline{W}_Z(z|v) \right)^{1-\rho}. \tag{271}
\]
However, Lemma 58 guarantees that
\[
\sum_z \left( \sum_v Q_v(v) \overline{W}_Z(z|v) \right)^{1-\rho} = \sum_z \left( \sum_v Q'_v(v) \overline{W}_Z(z|v) \right)^{1-\rho} = \max_{Q'_v} e^{E_0(\rho|\overline{W}_Z, Q'_v)}.
\tag{272}
\]
Since (271) contradicts (272), we obtain the desired argument. 

Proof of Lemma 64: As
\[
W_{Z|\rho} \circ Q_{V|\rho}(z) = \frac{\left( \sum_v Q_{V|\rho}(v) \overline{W}_Z(z|v) \right)^{1-\rho}}{\sum_v \left( \sum_v Q_{V|\rho}(v) \overline{W}_Z(z|v) \right)^{1-\rho}},
\]
we can calculate the mutual information \( I(Z; V)[W_{Z|\rho} \times Q_{V|\rho}] \) as
\[
I(Z; V) = \sum_{v,z} \frac{Q_{V|\rho}(v) \overline{W}_Z(z|v)^{1-\rho}}{\sum_v \overline{W}_Z(z|v)^{1-\rho}} \cdot \left( \log \overline{W}_Z(z|v)^{1-\rho} - \log \sum_v Q_{V|\rho}(v) \overline{W}_Z(z|v)^{1-\rho} \right) \]
\[
= \sum_{v,z} \frac{Q_{V|\rho}(v) \overline{W}_Z(z|v)^{1-\rho}}{\sum_v \overline{W}_Z(z|v)^{1-\rho}} \cdot \left( 1 - \rho \right) \log \overline{W}_Z(z|v)^{1-\rho} \]
\[
= F_{\rho}(Q_{V|\rho}). \tag{273}
\]
where the final equation follows from (233). We obtain the second equation of (244).

Since the constraint (i) in Lemma 58 for \( Q_{V|\rho} \) is differentiable with respect to \( \rho \), for a given \( \rho_0 \in (0, 1) \), we can choose \( Q_{V|\rho} \) such that the map \( \rho \mapsto Q_{V|\rho} \) is differentiable at least in an enough small neighborhood of \( \rho_0 \). Since
\[
\frac{d}{d\rho} E_0(\rho|\overline{W}_Z, Q_{V|\rho})|_{\rho=\rho_0} = 0, \tag{274}
\]
we have
\[
F_{\rho_0} = \frac{d}{d\rho} E_0(\rho|\overline{W}_Z, Q_{V|\rho})|_{\rho=\rho_0}
\]
\[
= \frac{d}{d\rho} E_0(\rho|\overline{W}_Z, Q_{V_0})|_{\rho=\rho_0} + \frac{d}{d\rho} E_0(\rho|\overline{W}_Z, Q_{V|\rho})|_{\rho=\rho_0}
\]
\[
= \frac{d}{d\rho} E_0(\rho|\overline{W}_Z, Q_{V_0})|_{\rho=\rho_0} = F_{\rho_0}(Q_{V_0}). \tag{275}
\]
Hence, we obtain the first equation of (244).
The conditional divergence $D(W_Z|\bar{W}_Z|Q_{V\rho})$ is calculated to
\[
D(W_{V\rho}\|\bar{W}_Z|Q_{V\rho}) = \sum_{v,z} \frac{Q_{V\rho}(v)\bar{W}_Z(z|v)}{\sum_z \bar{W}_Z(z|v)^{1-\rho}} \left( \sum_{v'} Q_{V\rho}(v') \bar{W}_Z(z|v')^{1-\rho} \log \frac{\bar{W}_Z(z|v)^{1-\rho}}{\sum_{v'} Q_{V\rho}(v') \bar{W}_Z(z|v')^{1-\rho}} - \log \bar{W}_Z(z|v) \right)
- \sum_v Q_{V\rho}(v) \log \left( \sum_z \bar{W}_Z(z|v)^{1-\rho} \left( \sum_{v'} Q_{V\rho}(v') \bar{W}_Z(z|v')^{1-\rho} \right)^{-\rho} \right)
\]
\[
= \sum_{v,z} \frac{Q_{V\rho}(v)\bar{W}_Z(z|v)}{\sum_z \bar{W}_Z(z|v)^{1-\rho}} \left( \sum_{v'} Q_{V\rho}(v') \bar{W}_Z(z|v')^{1-\rho} \left( \frac{\rho}{1-\rho} \log \bar{W}_Z(z|v) - \rho \log \left( \sum_v Q_{V\rho}(v) \bar{W}_Z(z|v)^{1-\rho} \right) \right) \right)
- \sum_v Q_{V\rho}(v) \log \left( \sum_z \left( \sum_{v'} Q_{V\rho}(v') \bar{W}_Z(z|v')^{1-\rho} \right)^{-\rho} \right)
\]
\[
= \rho F_{\rho}(Q_{V\rho}) - \sum_v Q_{V\rho}(v) \log \left( \sum_z \left( \sum_{v'} Q_{V\rho}(v') \bar{W}_Z(z|v')^{1-\rho} \right) \right)
= \rho F_{\rho} - E(\rho|\bar{W}_Z, Q_{V\rho}).
\]
We obtain (245).

G. Proofs of Lemmas 59, 63 and 65

Proof of Lemma 59: In order to apply Lemma 75, we regard all of probabilities $Q_v(v)$ as independent parameters by removing the constraint $\sum_v Q_v(v) = 1$. The partial derivatives are calculated as
\[
\frac{\partial}{\partial Q_v(v)} = \frac{\sum_{v'\in V^c} \max_{v'} Q_v(v') \bar{W}_Z(z|v')}{\sum_{z} \sum_{v'\in V^c} \max_{v'} Q_v(v') \bar{W}_Z(z|v')} = -\sum_{z} \frac{\sum_{v'\in V^c} \max_{v'} Q_v(v') \bar{W}_Z(z|v')}{\sum_{v'\in V^c} \max_{v'} Q_v(v') \bar{W}_Z(z|v')}
\]
Hence, Lemma 75 guarantees the equivalence between (i) and (ii). Condition (iii) trivially implies Condition (i).

The remaining task is showing Condition (i) implies Condition (iii). Assume Condition (i). Since $\sum_z \bar{W}_Z(z|v)^{1-\rho} \left( \sum_{v'} Q_v(v') \bar{W}_Z(z|v')^{1-\rho} \right)^{-\rho}$ does not depend on $v$ and Condition (ii) holds, we have
\[
\sum_{z} \frac{\bar{W}_Z(z|v)}{\sum_{v'} \max_{v'} Q_v(v')} = \sum_{z} \sum_{v'\in V^c} \max_{v'} Q_v(v') \bar{W}_Z(z|v') = \sum_v Q_v(v) \sum_{z} \sum_{v'\in V^c} \max_{v'} Q_v(v') \bar{W}_Z(z|v')
\]
\[
= \sum_{v} Q_v(v) \frac{\max_{v'} \bar{W}_Z(z|v')}{\max_{v'} Q_v(v')} = \sum_{z} \sum_{v'\in V^c} Q_v(v') \frac{\max_{v'} \bar{W}_Z(z|v')}{\max_{v'} Q_v(v')} = \sum_{z} \max_{v'} \bar{W}_Z(z|v')
\]

Proof of Lemma 63: We focus on the function $\{ \sum_{v'\in V^c} Q_v(v') \} \mapsto \frac{\sum_{v'\in V^c} \max_{v'} Q_v(v') \bar{W}_Z(z|v')}{\sum_{v'\in V^c} \max_{v'} \bar{W}_Z(z|v')}$, which is strictly concave. Hence, when there exists an element $z \in \mathcal{Z}$ such that $\sum_{v'\in V^c} Q_v(v') \neq \sum_{v'\in V^c} Q_v'(v')$ for two distributions $Q_v$ and $Q'_v$, the convex combination $\frac{Q_v + Q'_v}{2}$ gives a strictly greater value for the above function, which contradicts (ii) of Lemma 59. Hence, $\sum_{v'\in V^c} Q_v(v') = \sum_{v'\in V^c} Q'_v(v')$ for all $z \in \mathcal{Z}$.

Proof of Lemma 65: Since
\[
W_{Z,1} \times Q_{V,1}(v, z) = \begin{cases} 
Q_{V,1}(v)\bar{W}_Z(z|v) & z \in \mathcal{Z}_v \\
0 & \text{else} 
\end{cases} \quad z \in \mathcal{Z}_v
\]
(276)
the mutual information $I(V; Z)[W_{Z,1} \times Q_{V,1}]$ is calculated as
\[
I(V; Z)[W_{Z,1} \times Q_{V,1}] = -\frac{\sum_z \log(\sum_{v' \in V_z} Q_{V,1}(v)) \max_{v'} W_Z(z|v')}{\sum_{v'} \max_{v'} W_Z(z|v')}
\]
\[
= F_1(Q_{V,1}),
\]
(277)
where the final equation follows from (234). Hence, we obtain the second equation in (246). The first equation in (246) follows from the limit $\rho \to 1 - 0$ at (275).

When $Q_V$ satisfies Condition 61, we have
\[
D(W_{Z,1}||W_Z|Q_V) = -\sum_{z,v} W_{Z,1} \times Q_{V,1}(v, z) \log(\sum_{v'' \in V_z} Q_V(v'')) \sum_{v'} \max_{v'} W_Z(z|v')
\]
\[
= -\log(\sum_{v'} \max_{v'} W_Z(z'|v')) - \sum_{z} \log(\sum_{v'' \in V_z} Q_V(v'')) W_{Z,1} \circ Q_V(z)
\]
\[
= -\log(\sum_{v'} \max_{v'} W_Z(z'|v')) - \frac{\sum_z \log(\sum_{v'' \in V_z} Q_V(v')) \max_{v'} W_Z(z|v')} {\sum_{v'} \max_{v'} W_Z(z|v')} = F_1 - E_0(1),
\]
which implies (247).

H. Proof of Lemma 66

Proof of Lemma 66: Due to Condition 60 with $\rho$, we can choose a constant $C_\rho$ in the following way: the relation
\[
C_\rho = \sum_z W_Z(z|v) \frac{1}{\gamma} (\sum_{v'} Q_{V,1}(v') W_Z(z|v') \frac{1}{\gamma})^{-\rho}
\]
(278)
holds for all $v$. Due to the general relation as (264), we have
\[
C := \lim_{\rho \to 1 - 0} C_\rho
\]
\[
= \lim_{\rho \to 1 - 0} \sum_z W_Z(z|v) \frac{1}{\gamma} (\sum_{v'} Q_{V,1}(v') W_Z(z|v') \frac{1}{\gamma})^{-\rho}
\]
\[
= \lim_{\rho \to 1 - 0} \sum_{z \in Z, v' \in V_z} (\sum_{v''} Q_{V,1}(v''))^{-\rho} \max_{v'} W_Z(z|v')
\]
\[
= \sum_{z \in Z} \max_{v'} \frac{W_Z(z|v')}{\sum_{v'' \in V_z} (\lim_{n \to \infty} Q_{V,1}(v''))}.
\]
Since $C$ does not depend on $v$, the distribution $\lim_{n \to \infty} Q_{V,1}$ satisfies Condition 61.
I. Proof of Lemma \[56\]

We show the inequality in (240).

\[
\begin{align*}
\min_{w_z \in W(\mathcal{U} \times \mathcal{V}, \mathcal{Z})} & \quad D(W_Z||W_Z|Q_{UV}) - \rho I(V; Z|U) [W_Z \times Q_{UV}] \\
= & \quad \min_{w_z \in W(\mathcal{U} \times \mathcal{V}, \mathcal{Z})} \left( \sum_u Q_U(u) \left( \sum_v Q_{V|U}(v|u) \sum_z W_Z(z|u, v) \log \frac{W_Z(z|u, v)}{W_Z(z|v)} \right) \\
- & \quad \rho \min_{Q \in P(\mathcal{Z})} \sum_v Q_{V|U}(v|u) \sum_z W_Z(z|u, v) \log \frac{W_Z(z|u, v)}{Q(z)} \right) \\
= & \quad \min_{w_z \in W(\mathcal{U} \times \mathcal{V}, \mathcal{Z})} \max_{\tilde{w}_z \in W(\mathcal{U}, \mathcal{Z})} \sum_u Q_U(u) \sum_v Q_{V|U}(v|u) \sum_z W_Z(z|u, v) \log \frac{W_Z(z|u, v)}{W_Z(z|v)} - \rho \sum_z W_Z(z|u, v) \log \frac{W_Z(z|u, v)}{W_Z(z|u)} \\
= & \quad \min_{w_z \in W(\mathcal{U} \times \mathcal{V}, \mathcal{Z})} \max_{\tilde{w}_z \in W(\mathcal{U}, \mathcal{Z})} \sum_u Q_U(u) \sum_v Q_{V|U}(v|u) \sum_z W_Z(z|u, v) \log \frac{W_Z(z|u, v)^{1-\rho} \tilde{W}_Z(z|u)^{\rho}}{W_Z(z|v)} \\
= & \quad \max_{\tilde{w}_z \in W(\mathcal{U} \times \mathcal{V}, \mathcal{Z})} \min_{w_z \in W(\mathcal{U} \times \mathcal{V}, \mathcal{Z})} \sum_u Q_U(u) \sum_v Q_{V|U}(v|u) \sum_z W_Z(z|u, v) \log \frac{W_Z(z|u, v)^{1-\rho} \tilde{W}_Z(z|u)^{\rho}}{\tilde{W}_Z(z|v)} \\
= & \quad (1 - \rho) \max_{\tilde{w}_z \in W(\mathcal{U}, \mathcal{Z})} \sum_u Q_U(u) \sum_v Q_{V|U}(v|u) \min_{\tilde{P}_Z(\mathcal{Z})} \tilde{P}_Z(\mathcal{Z}) \log \frac{\tilde{P}_Z(\mathcal{Z})}{W_Z(z|v)^{1-\rho} \tilde{W}_Z(z|u)^{\rho}} \\
= & \quad (1 - \rho) \sum_u Q_U(u) \sum_v Q_{V|U}(v|u) \log \sum_z W_Z(z|v)^{1-\rho} \tilde{W}_Z(z|u)^{\rho} \\
\geq & \quad (1 - \rho) \sum_u Q_U(u) \log \sum_v Q_{V|U}(v|u) \sum_z W_Z(z|v)^{1-\rho} \tilde{W}_Z(z|u)^{\rho} \\
= & \quad (1 - \rho) \sum_u Q_U(u) \log \sum_v Q_{V|U}(v|u) \sum_z W_Z(z|v)^{1-\rho} \tilde{Q}_Z(z)^{\rho}.
\end{align*}
\]

The above derivation can be shown in the following way. The equality (279) follows from the minimax theorem [11, Chap. IV Prop. 2.3] because the function is concave for \(\tilde{W}_Z\) and is convex for \(W_Z\). The equality (280) holds because the minimum is attained with \(\tilde{P}_Z(\mathcal{Z}) = \frac{W_Z(z|v)^{1-\rho} \tilde{W}_Z(z|u)^{\rho}}{\sum_z W_Z(z|v)^{1-\rho} \tilde{W}_Z(z|u)^{\rho}}\). The inequality (281) follows from the concavity of \(x \mapsto \log x\).

Since \(\frac{1}{1-\rho} + \frac{\rho}{1-\rho} = 1\), the reverse Hölder inequality yields that

\[
\begin{align*}
\sum_z \left( \sum_v Q_{V|U}(v|u) \tilde{W}_Z(z|v)^{1-\rho} \tilde{Q}_Z(z)^{\rho} \right) \tilde{Q}_Z(z)^{\rho} & \geq \left( \sum_z \left( \sum_v Q_{V|U}(v|u) \tilde{W}_Z(z|v)^{1-\rho} \right) \tilde{Q}_Z(z)^{\rho} \right) \tilde{Q}_Z(z)^{\rho} \\
\geq & \quad \min_{\tilde{Q}_Z(\mathcal{Z})} \left( \sum_z \left( \sum_v Q_{V|U}(v|u) \tilde{W}_Z(z|v)^{1-\rho} \right) \tilde{Q}_Z(z)^{\rho} \right) \tilde{Q}_Z(z)^{\rho} = \left( \sum_z \left( \sum_v Q_{V|U}(v|u) \tilde{W}_Z(z|v)^{1-\rho} \right) \tilde{Q}_Z(z)^{\rho} \right) \tilde{Q}_Z(z)^{\rho}.
\end{align*}
\]

The equality holds only when \(\left( \sum_v Q_{V|U}(v|u) \tilde{W}_Z(z|v)^{1-\rho} \right)^{1-\rho} = C \tilde{Q}_Z(z)\) with a constant \(C\). Hence,

\[
\min_{\tilde{Q}_Z(\mathcal{Z})} \sum_z \left( \sum_v Q_{V|U}(v|u) \tilde{W}_Z(z|v)^{1-\rho} \right) \tilde{Q}_Z(z)^{\rho} = \left( \sum_z \left( \sum_v Q_{V|U}(v|u) \tilde{W}_Z(z|v)^{1-\rho} \right) \tilde{Q}_Z(z)^{\rho} \right) \tilde{Q}_Z(z)^{\rho}.
\]
Thus,

\[-(1 - \rho) \sum_u Q_U(u) \log \min_{Q_z \in \mathcal{P}(Z)} \left( \sum_z \left( \sum_v Q_{V|U}(v|u) \bar{W}_Z(z|v) \right) \bar{Q}_Z(z)^{1 - \rho} \right) \]

\[= - (1 - \rho) \sum_u Q_U(u) \log \left( \sum_z \left( \sum_v Q_{V|U}(v|u) \bar{W}_Z(z|v) \right)^{1 - \rho} \right) \]

\[= \sum_u Q_U(u) \log \left( \sum_z \left( \sum_v Q_{V|U}(v|u) \bar{W}_Z(z|v) \right)^{1 - \rho} \right) \]

\[\geq \log \sum_u Q_U(u) \left( \sum_v Q_{V|U}(v|u) \bar{W}_Z(z|v) \right)^{1 - \rho} \]

\[= \rho E(\rho|\bar{W}_Z, Q_{V|U}, Q_U), \]  

(283)

where (283) follows from the concavity of \( x \mapsto \log x \). The combination of (282) and (284) yields (240).

The equality in (281) holds if and only if for an arbitrary fixed \( u \), \( \sum_w \bar{W}_Z(z|v) \bar{W}_Z(z|u)^{1 - \rho} \) does not depend on \( v \) with \( \bar{W}_Z(z|u) = (\sum_v Q_{V|U}(v|u) \bar{W}_Z(z|v)^{1 - \rho} / \sum_v Q_{V|U}(v|u) \bar{W}_Z(z|v)^{1 - \rho})^{1 - \rho} \), i.e., the quantity \( \sum_w \bar{W}_Z(z|v) \bar{W}_Z(z|u)^{1 - \rho} (\sum_v Q_{V|U}(v|u) \bar{W}_Z(z|v)^{1 - \rho})^{1 - \rho} \) does not depend on \( v \) for an arbitrary fixed \( u \). The condition holds when \( Q_{V|U=\tilde{u}} \) is argmin \( Q_v, E(\rho|\bar{W}_Z, Q_V) \) because of Lemma 58. Further, the equality in (283) holds in this case. Hence, when \( Q_{V|U=\tilde{u}} \) is argmin \( Q_v, E(\rho|\bar{W}_Z, Q_V) \), the equality holds in the inequality (240).

XIV. Conclusion

In order to treat the secure multiplex coding with dependent and non-uniform multiple messages and common messages, we have generalized resolvability to the case when input random variable is subject to a non-uniform distribution. Two kinds of generalization have been given. The first one (Theorem 9) is a simple extension of Han–Verdú’s channel resolvability coding [13] with the non-uniform inputs. The second one (Theorem 11) uses randomly chosen affine mapping satisfying Condition 10 with the non-uniform inputs.

We have constructed two kinds of codes for the above type of SMC. Similar to BCC in [9], the first construction has two steps. In the first step, similar to the BCD encoder, we apply superposition random coding. In the second step, as is illustrated in Fig. 1, we split the confidential message into the private message \( B_2 \) and a part \( B_1 \) of the common message encoded by the BCD encoder. Employing the second type of channel resolvability, we have derived a non-asymptotic formula for the average leaked information under this kind of code construction. On the other hand, in the second construction, the confidential message is simply sent as the private message encoded by the BCD encoder. Hence, it has only one step. Employing the first type of channel resolvability, we have derived a non-asymptotic formula for the average leaked information under this kind of code construction.

For asymptotic treatment for the non-uniform and dependent sources, we have introduced three kinds of asymptotic conditional uniformity conditions. Then, we have clarified the relation among three conditions, especially, that two of them are equivalent. Further, we have shown that these conditions can be satisfied by data compressed by Slepian-Wolf compression, in the respective senses. Extending the above formula for the first construction to the asymptotic case, we have derived the capacity region of SMC defined in our general setting, in which, the message is allowed to be dependent and non-uniform while it has to satisfy the weaker asymptotic conditional uniformity condition. We have shown the strong security when the leaked information rate is zero and the message satisfies the stronger asymptotic conditional uniformity condition. Using the both formulas, we have also derived the exponential decreasing rate of leaked information. While the first formula gives an upper bound in any case, the second one gives a better upper bound in some specific cases.

We have also given two kinds of practical constructions for SMC by using ordinary linear codes. Following our constructions, we can make a code satisfying a required security level. Further, we have given a universal code for SMC, which does not depend on the channel. Extending this result, we have
derived a source-channel universal code for BCC, which does not depend on the channel or the source distribution.

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APPENDIX A

INEQUALITY BETWEEN RÉNYI ENTROPY AND CONDITIONAL RÉNYI ENTROPY

In this appendix, we derive a useful inequality between Rényi entropy and conditional Rényi entropy, which was used in Subsection VII-B. For this purpose, we prepare the following lemma.

Lemma 76: Any distribution $P_{XY}$ over $X \times Y$ and any conditional distribution $Q_{Y|X}$ over $Y$ conditioned with $X$ satisfy

$$\psi(\rho|P_{X,Y}|Q_{X,Y}) \geq \frac{1}{1-\rho}\psi(\rho(1-\rho)|P_{X,Y}|Q_{Y|X} \times P_X)$$

(285)

for $\rho > 0$, where $P_X$ is the marginal distribution of $P_{X,Y}$ on $X$.

Since $\frac{1}{\rho}\psi(\rho|P_{X,Y}|Q_{X,Y}) = \log(|X||Y|) - H_{1+\rho}(X,Y)$ and $\frac{1}{\rho(1-\rho)}\psi(\rho(1-\rho)|P_{X,Y}|Q_{Y|X} \times P_X) = \log |Y| - H_{1+\rho(1-\rho)}(Y|X)$, we obtain the following corollary of the above lemma as an inequality between Rényi entropy and conditional Rényi entropy.

Corollary 77: For $\rho > 0$, arbitrary random variables $X$ and $Y$ over $X$ and $Y$ satisfy

$$\log(|X||Y|) - H_{1+\rho}(X,Y) \geq \log |Y| - H_{1+\rho(1-\rho)}(Y|X),$$

(286)

which implies

$$\log |X| + H_{1+\rho(1-\rho)}(Y|X) \geq H_{1+\rho}(X,Y).$$

(287)

Proof of Lemma 76. Applying Hölder inequality $\sum_x P_X(x)|A(x)|B(x) \leq (\sum_x P_X(x)|A(x)|^{\frac{1}{\tau}})^{1-\rho}(\sum_x P_X(x)|B(x)|^{\frac{1}{\rho}})^{\frac{1}{\tau}}$, to the case $A(x) = P_X(x)^\rho Q_X(x)^{-\rho}(\sum_y P_{Y|X}(y|x)^{1+\rho(1-\rho)}Q_{Y|X}(y|x)^{-\rho(1-\rho)})^{\frac{1}{\tau}}$ and $B(x) = P_X(x)^{\frac{1}{\tau}}Q_X(x)^{\frac{1}{\tau}}$, we obtain the following. In the following derivation, we employ the above Hölder inequality in (289), and
the Jensen inequality for the convex function $x \mapsto x^\frac{1}{\rho}$ in (288), (290), and (291).

$$e^{\frac{\phi}{\rho}(P_{X,Y}\|Q_{X,Y})} = \left(\sum_x P_X(x) \sum_y P_{Y|X}(y|x)^{1+\rho(1-\rho)} Q_{Y|X}(y|x)^{-\rho(1-\rho)}\right)^{\frac{1}{\rho}}$$

$$\leq \sum_x P_X(x) \left(\sum_y P_{Y|X}(y|x)^{1+\rho(1-\rho)} Q_{Y|X}(y|x)^{-\rho(1-\rho)}\right)^{\frac{1}{\rho}}$$

$$= \sum_x P_X(x) P_X(x)^\rho Q_X(x)^{-\rho} \sum_y P_{Y|X}(y|x)^{1+\rho(1-\rho)} Q_{Y|X}(y|x)^{-\rho(1-\rho)} \left(P_X(x)^{-\rho} Q_X(x)^{\rho}\right)$$

$$\leq \left(\sum_x P_X(x) P_X(x)^\rho Q_X(x)^{-\rho} \left(\sum_y P_{Y|X}(y|x)^{1+\rho(1-\rho)} Q_{Y|X}(y|x)^{-\rho(1-\rho)}\right)\right)^{\frac{1}{\rho}} \cdot (\sum_x P_X(x) P_X(x)^{-1} Q_X(x)^{\rho})$$

$$= \left(\sum_x P_X(x) P_X(x)^\rho Q_X(x)^{-\rho} \left(\sum_y P_{Y|X}(y|x)(P_{Y|X}(y|x))^{1-\rho(1-\rho)} Q_{Y|X}(y|x)^{-\rho(1-\rho)}\right)\right)^{\frac{1}{\rho}} \cdot 1^\rho$$

$$\leq \sum_x P_X(x) P_X(x)^\rho Q_X(x)^{-\rho} \left(\sum_y P_{Y|X}(y|x)(P_{Y|X}(y|x))^{\rho(1-\rho)} Q_{Y|X}(y|x)^{-\rho(1-\rho)}\right)$$

$$\leq \sum_x P_X(x) P_X(x)^\rho Q_X(x)^{-\rho} \left(\sum_y P_{Y|X}(y|x)(P_{Y|X}(y|x))^{\rho(1-\rho)} Q_{Y|X}(y|x)^{-\rho(1-\rho)}\right)$$

$$= \sum_x P_X(x) P_X(x)^\rho Q_X(x)^{-\rho} \left(\sum_y P_{Y|X}(y|x)(P_{Y|X}(y|x))^\rho Q_{Y|X}(y|x)^{-\rho}\right)$$

$$= \sum_{x,y} P_{X,Y}(x,y) Q_{X,Y}(x,y)^{-\rho} = e^{\frac{\phi}{\rho}(P_{X,Y}\|Q_{X,Y})}.$$

**Appendix B**

Existence of Code Required in Theorem 27 with $\epsilon = 0$

In this appendix, we show the existence of Slepian-Wolf data compression code satisfying the condition required in Theorem 27 with $\epsilon = 0$ in the two-terminal and i.i.d. case. For this purpose, we assume that the random variables $(S_1^n, S_2^n)$ are subject to the $n$-fold i.i.d. distribution of a given non-uniform joint distribution of $S_1$ and $S_2$. For this purpose, we recall the definition of achievable rate pair for Slepian-Wolf compression.

**Definition 78:** A rate pair $(R_1, R_2)$ is called *achievable* when there exists a sequence of encoders $\varphi^n = (\varphi^n_1, \varphi^n_2)$ ($\varphi^n_i : S^n_i \to \{1, \ldots, [2^{nR_i}]\}$) and decoders $\hat{\varphi}^n$ ($\hat{\varphi}^n : \{1, \ldots, [2^{nR_1}]\} \times \{1, \ldots, [2^{nR_2}]\} \to S^n_1 \times S^n_2$) such that the decoding error probability $\varepsilon(\varphi^n, \hat{\varphi}^n)$ satisfies

$$\lim_{n \to \infty} \varepsilon(\varphi^n, \hat{\varphi}^n) = 0.$$  

(293)

Then, we prepare the following lemma.

**Lemma 79:** Let $(R_1, R_2)$ be a pair of achievable rates for Slepian-Wolf compression satisfying $R_1 + R_2 = H(S_1, S_2)$. When the compression rate pair $(R_{1,n}, R_{2,n})$ behaves as $R_{1,n} = R_1 + \frac{c_1}{n}$ and $R_{2,n} = R_2 + \frac{c_2}{n}$ with $t > 1/2$ and $c_1 > c_2 > 0$, there exists a sequence of Slepian-Wolf codes $(\varphi^n, \hat{\varphi}^n) = ((\varphi^n_1, \varphi^n_2), \hat{\varphi}^n)$ for any positive integer $n$ such that $\varphi^n_i$ is a map from $S^n_i$ to $\{1, \ldots, [2^{nR_i}]\}$ for $i = 1, 2$ and the decoding error probability $\varepsilon(\varphi^n, \hat{\varphi}^n)$ satisfies

$$\lim_{n \to \infty} \frac{-1}{H_{R_1,R_2}} \log \varepsilon(\varphi^n, \hat{\varphi}^n) \geq \min \left(\frac{c_1^2}{4V(S_1)}, \frac{c_2^2}{4V(S_2|S_1)}\right) \left(1 - \frac{c_2}{4V(S_2)}\right) \left(1 - \frac{c_1}{4V(S_1|S_2)}\right),$$

where $V(S_1|S_2) := \sum_{s_1,s_2} P_{S_1,S_2}(s_1, s_2)(\log P_{S_2|S_1}(s_2|s_1) - H(S_2|S_1))^2$ and $\lambda \in [0, 1]$ is the real number satisfying that

$$\lambda(H(S_1), H(S_2|S_1)) + (1 - \lambda)(H(S_1|S_2), H(S_2)).$$

(295)
Further, when $R_1 = H(S_1)$ and $R_2 = H(S_2|S_1)$ and the compression rates $(R_{1,n}, R_{2,n})$ behaves as $R_{1,n} = H(S_1) + \frac{c_1}{n}$ and $R_{2,n} = H(S_2|S_1) + \frac{c_2}{n}$ with $r > 1/2$ and $c_1 > c_2 > 0$, there exists a sequence of Slepian-Wolf codes $(\varphi_n, \hat{\varphi}_n)$ such that the decoding error probability $e(\varphi_n, \hat{\varphi}_n)$ satisfies
\[
\liminf_{n \to \infty} -\frac{1}{n^{2-1}} \log e(\varphi_n, \hat{\varphi}_n) \geq \min(\frac{c_1^2}{4V(S_1)}, \frac{c_2^2}{4V(S_2|S_1)}).
\] (296)

We will prove Lemma 79 after preparing several lemmas. Using Lemma 79 we make a Slepian-Wolf compression whose compressed data satisfies the SACU condition. Let $(R_1, R_2)$ be a pair of achievable rates for Slepian-Wolf compression satisfying $R_1 + R_2 = H(S_1, S_2)$. Then, let $\varphi_n = (\varphi_1^n, \varphi_2^n)$ and $\hat{\varphi}_n$ be the Slepian-Wolf encoders and the Slepian-Wolf decoder given in Lemma 79 with the case of $c_1 = R_1$ and $c_2 = R_2$. We choose the integer $m_n := \lfloor \frac{n}{R_1 + R_2} \rfloor \leq \lfloor \frac{R_1 n}{R_1 + R_2} \rfloor = \lfloor \frac{R_2 n}{R_1 + R_2} \rfloor$ for $t > 1/2$ and $c > 0$.

Then, we obtain the Slepian-Wolf encoders $\varphi_i^{m_n} : S_i^{m_n} \to \{1, \ldots, [e^{nR_i}]\}$ and the Slepian-Wolf decoder $\hat{\varphi}_i^{m_n} : \{1, \ldots, [e^{nR_i}]\} \times \{1, \ldots, [e^{nR_i}]\} \to S_1^{m_n} \times S_2^{m_n}$. Using the code, we define the Slepian-Wolf encoders $\varphi_i^{1, 2, i} : S_i^{m_n} \to \{1, \ldots, [e^{nR_i}]\}$ and the Slepian-Wolf decoder $\hat{\varphi}_i^{1, 2, i} : \{1, \ldots, [e^{nR_i}]\} \times \{1, \ldots, [e^{nR_i}]\} \to S_1^{m_n} \times S_2^{m_n}$ by
\[
\varphi_i^{1, 2, i}(s_i^{m_n}) := \varphi_i^{m_n}(s_i^{m_n})
\] (297)
\[
\hat{\varphi}_i^{1, 2, i}(x_1, x_2) := \hat{\varphi}_i^{m_n}(x_1, x_2).
\] (298)

Then, due to Lemma 79, since $m_n(R_1 + R_1) = nR_1$ and $m_n(R_2 + R_2) = nR_2$, the code $((\varphi_1^{1, 2, i}, \varphi_2^{1, 2, i}), \hat{\varphi}_i^{1, 2, i})$ satisfies the condition (298) in Theorem 27 with $\epsilon = 0$. Theorem 27 guarantees that the compressed data satisfies the SACU condition.

Now, in order to show Lemma 79 we prepare several lemmas.

Lemma 80 ([36], [37], [38]): For a given compression rate $R_2 > 0$, there exists a pair of the encoder $\varphi_n$ and the decoder $\hat{\varphi}_n$ of the random variable $S_2^n$ with the side information $S_1^n$ such that the decoding error probability $e(\varphi_n, \hat{\varphi}_n)$ satisfies
\[
e(\varphi_n, \hat{\varphi}_n) \leq e^{-n(pR_2 - E_0(-p|S_2^n|S_1^n))}
\] (299)
for any $p \in (0, 1)$, where
\[
E_0(p|S_2^n|S_1^n) := \log \sum_{s_1} \left( \sum_{s_2} P_{S_1, S_2}(s_1, s_2) \frac{1}{1-p} \right)^{1-p}.
\] (300)

Note that when there is no side information, we have
\[
E_0(-p|S_2^n) = \rho H_\frac{1}{1-p}(S_2).
\] (301)

Lemma 81: The quantity $E_0(-p|S_2^n|S_1^n)$ has the expansion
\[
E_0(-p|S_2^n|S_1^n) = \rho H(S_2|S_1) + \rho^2 V(S_2|S_1)
\] (302)
with small $\rho$. In particular, the quantity $\rho H_\frac{1}{1-p}(S_1)$ has the expansion
\[
\rho H_\frac{1}{1-p}(S_1) = \rho H(S_1) + \rho^2 V(S_1)
\] (303)
with small $\rho$ and $V(S_1) := \sum_{s_1} P_{S_1}(s_1)(\log P_{S_1}(s_1) - H(S_1))^2$.

Proof: Take the Taylor expansion of $e^{E_0(-p|S_2^n|S_1^n)}$ as
\[
e^{E_0(-p|S_2^n|S_1^n)} = 1 + \rho H(S_2|S_1) + \frac{\rho^2}{2} \sum_{s_1, s_2} P_{S_1, S_2}(s_1, s_2)(\log P_{S_2|S_1}(s_2|s_1))^2 + o(\rho^2).
\] (304)
Taking the logarithm, we obtain (302).
Lemma 82: Let \((R_1, R_2)\) belong to the Slepian-Wolf compression region of \((S_1^n, S_2^n)\). We choose the rates \(R_1', R_2', R_1'', \text{ and } R_2''\) and the real number \(\lambda \in [0, 1]\) such that

\[
(R_1, R_2) = \lambda (R_1', R_2') + (1 - \lambda) (R_1'', R_2''). \tag{305}
\]

Then, there exists a pair of the Slepian-Wolf encoder \(\varphi^n\) and the decoder \(\hat{\varphi}^n\) such that the decoding error probability \(\varepsilon(\varphi^n, \hat{\varphi}^n)\) satisfies

\[
\varepsilon(\varphi^n, \hat{\varphi}^n) \leq \inf_{\rho \in (0, 1]} e^{-\lambda n \rho R_1' - \rho H_{\frac{1}{\rho}}(S_1)} + \inf_{\rho \in (0, 1]} e^{-\lambda n \rho R_2' - \rho H_{\frac{1}{\rho}}(S_2)} + \inf_{\rho \in (0, 1]} e^{-(1-\lambda)n \rho R_1'' - \rho H_{\frac{1}{\rho}}(S_1)} + \inf_{\rho \in (0, 1]} e^{-(1-\lambda)n \rho R_2'' - \rho H_{\frac{1}{\rho}}(S_2)}, \tag{306}
\]

Also, there exists a pair of the Slepian-Wolf encoder \(\varphi^n\) and the decoder \(\hat{\varphi}^n\) such that the decoding error probability \(\varepsilon(\varphi^n, \hat{\varphi}^n)\) satisfies

\[
\varepsilon(\varphi^n, \hat{\varphi}^n) \leq \inf_{\rho \in (0, 1]} e^{-n \rho R_1' - \rho H_{\frac{1}{\rho}}(S_1)} + \inf_{\rho \in (0, 1]} e^{-n \rho R_2' - \rho H_{\frac{1}{\rho}}(S_2)}, \tag{307}
\]

**Proof:** First, we show the existence of a sequence of codes satisfying (307). We apply the usual data compression for \(S_1^n\), and the data compression given in Lemma 80 for \(S_2^n\). The decoder is given by combination of the respective decoders. Since the decoding error probability is bounded by the sum of the decoding error probabilities of \(S_1^n\) and \(S_2^n\), we obtain (307).

Next, we show the existence of a sequence of codes satisfying (306). We divide \(n\) symbols into two parts, \(\lambda n\) symbols and \((1 - \lambda) n\) symbols. We apply the construction given in the previous paragraph with the rates \((R_1', R_2')\) to the first part, and apply the same construction with the rates \((R_1'', R_2'')\) to the second part. Due to Lemma 80 the decoding error probability of the first part is less than \(\inf_{\rho \in (0, 1]} e^{-\lambda n \rho R_1' - \rho H_{\frac{1}{\rho}}(S_1)} + \inf_{\rho \in (0, 1]} e^{-(1-\lambda)n \rho R_1'' - \rho H_{\frac{1}{\rho}}(S_1)}\), and the decoding error probability of the second part is less than \(\inf_{\rho \in (0, 1]} e^{-(1-\lambda)n \rho R_2' - \rho H_{\frac{1}{\rho}}(S_2)} + \inf_{\rho \in (0, 1]} e^{-(1-\lambda)n \rho R_2'' - \rho H_{\frac{1}{\rho}}(S_2)}\). Then, we obtain (306).

**Proof of Lemma 79:** First, we consider the case when \(R_1 = H(S_1)\) and \(R_2 = H(S_2 | S_1)\). Since \(R_{1,n} = H(S_1) + \frac{c}{n}\) and \(R_{2,n} := H(S_2 | S_1) + \frac{c}{n}\), we can show that

\[
\lim_{n \to \infty} \frac{-1}{2^{l-1}} \log \inf_{\rho \in [0, 1]} e^{-n \rho R_{1,n} - \rho H_{\frac{1}{\rho}}(S_1)} = \frac{c^2}{4V(S_1)}. \tag{308}
\]

\[
\lim_{n \to \infty} \frac{-1}{2^{l-1}} \log \inf_{\rho \in [0, 1]} e^{-n \rho R_{2,n} - \rho H_{\frac{1}{\rho}}(S_2 | S_1)} = \frac{c^2}{4V(S_2 | S_1)}. \tag{309}
\]

Since the proof of (308) is similar to those of (309), we show only (308). When \(\rho\) is sufficiently small, due to Lemma 81, we have

\[
\rho R_{1,n} - \rho H_{\frac{1}{\rho}}(S_1) \approx \rho \frac{c}{n} - \rho^2 V(S_1) = -V(S_1)(\rho - \frac{c}{2V(S_1)n})^2 + \frac{c^2}{4V(S_1)n^2}. \tag{310}
\]

Hence, \(\inf_{\rho \in [0, 1]} e^{-n \rho R_{1,n} - \rho H_{\frac{1}{\rho}}(S_1)} \approx e^{-n \frac{c^2}{4V(S_1)n^2}}\), which implies (308). Then, we apply the evaluation (307) for the decoding error probability in Lemma 82 to the case when \(R_1, R_2\) are \(R_{1,n}, R_{2,n}\). Combining the relations (308) and (309), we obtain (296).

Next, we show the general case. We choose \(R_1', R_2' := H(S_1) + \frac{c}{n}, R_2'' := H(S_2 | S_1) + \frac{c}{n}, R_1'' := H(S_1 | S_2) + \frac{c}{2n}, R_2'' := H(S_2) + \frac{c}{2n}\). Then, we obtain

\[
(R_{1,n}, R_{2,n}) = \lambda (R_1', R_2') + (1 - \lambda)(R_1'', R_2''). \tag{311}
\]
Then, similar to (308) and (309), we can show that

$$\lim_{n \to \infty} \frac{-1}{n^{2r-1}} \log \inf_{\rho \in (0,1]} e^{-\lambda n(\rho R_{1,n}' - \rho H_{\infty}(S_1))} = \frac{c_1^2}{4V(S_1)}$$  \hspace{1cm} (312)

$$\lim_{n \to \infty} \frac{-1}{n^{2r-1}} \log \inf_{\rho \in (0,1]} e^{-\lambda n(\rho R_{2,n}' - \rho H_{\infty}(S_1))} = \frac{c_2^2}{4V(S_1 | S_2)}$$  \hspace{1cm} (313)

$$\lim_{n \to \infty} \frac{-1}{n^{2r-1}} \log \inf_{\rho \in (0,1]} e^{-(1-\lambda)n(\rho R_{1,n}' - E_{\omega}(\rho S_1 | S_2))} = (1 - \lambda) \frac{c_2^2}{4V(S_2)}$$  \hspace{1cm} (314)

$$\lim_{n \to \infty} \frac{-1}{n^{2r-1}} \log \inf_{\rho \in (0,1]} e^{-(1-\lambda)n(\rho R_{2,n}' - \rho H_{\infty}(S_2))} = (1 - \lambda) \frac{c_1^2}{4V(S_1 | S_2)}$$  \hspace{1cm} (315)

We apply the evaluation (306) for the decoding error probability in Lemma 82 to the case when $R_1', R_2', R_1''$, $R_2''$, $R_1'', R_2''$ are $R_{1,n}', R_{2,n}', R_{1,n}'', R_{2,n}''$. Combining the relations (312), (313), (314) and (315), we obtain (294).

\[\square\]

**Appendix C**

**Equivalence Between the SWACU Condition and the WACU Condition**

In Subsection [VII-A] we have introduced three asymptotic conditional uniformity conditions. The aim of this appendix is to show the equivalence between the SWACU condition and the WACU condition, which was used in our proof of Theorem 32.

**Lemma 83:** Let $A_n$ be a random variable on the set $\mathcal{A}_n$ with the cardinality $e^{\alpha R}$ and $B_n$ be another random variable for any positive integer $n$. Then, the relation

$$\lim_{n \to \infty} \frac{1}{n} H(A_n | B_n) = R$$  \hspace{1cm} (316)

holds, if and only if

$$\lim_{n \to \infty} \frac{1}{n} H_{1+\alpha/n}(A_n | B_n) = R$$  \hspace{1cm} (317)

for any $\alpha > 0$.

Lemma 83 will be shown after Lemma 84, which is used in the proof of Lemma 83. Thanks to Lemma 83, we can replace the WACU condition (80) by the SWACU condition (81). Indeed, in order to apply our results in Section [VI] to the proof of Theorem 32, we need evaluation conditional Rényi entropy instead of conditional entropy, as is discussed around (100). Lemma 83 provides the evaluation of conditional Rényi entropy (317) from the evaluation of conditional entropy (316). Hence, Lemma 83 is useful for the application of our results in Section [VI] to the asymptotic setting.

**Lemma 84:** Let $A$ be a random variable on the set $\mathcal{A}$ with the cardinality $M$ and $B$ be another random variable. For arbitrary $\epsilon_1 > 0$ and $1 \geq \epsilon_2 > 0$, we define the subset of joint distributions for $A$ and $B$ as

$$\mathcal{P}_{\epsilon_1,\epsilon_2,M}^{A|B} := \{P_{A,B} | P_{A,B}((a,b)) - \log P_{A|B}(a|b) \leq \log M - \epsilon_1 \leq \epsilon_2\}.$$  \hspace{1cm} (318)

Then,

$$\max_{P_{A,B} \in \mathcal{P}_{\epsilon_1,\epsilon_2,M}^{A|B}} H(A|B) \leq \log M - \epsilon_2(e^{-\epsilon_1} - 1 + \epsilon_1)$$  \hspace{1cm} (319)

$$\min_{P_{A,B} \in \mathcal{P}_{\epsilon_1,\epsilon_2,M}^{A|B}} H_{1+\rho}(A|B) \geq -\frac{1}{\rho} \log((1 - \epsilon_2)^{\epsilon_2 \rho} + \epsilon_2).$$  \hspace{1cm} (320)

Here, since the region $\mathcal{P}_{\epsilon_1,\epsilon_2,M}^{A|B}$ is compact, the above maximum and the above minimum exist.
Proof of Lemma 84: For an arbitrary integer \( k \), we define the set
\[
P^A_{\epsilon_1, \epsilon_2, M, k} := \left\{ P_A \mid P_A[a \mid \log P_A(a) \leq \log M - \epsilon_1] \leq \epsilon_2, \right\}
\]
and define the function
\[
f(x) := \epsilon_2(\log x - \log \epsilon_2) + (1 - \epsilon_2)(\log(M - x) - \log(1 - \epsilon_2))
\]
for \( \epsilon_2 \in (0, 1) \). The set \( P^A_{\epsilon_1, \epsilon_2, M, k} \) is a non-empty set only when the integer \( k \) belongs to \( [0, \epsilon_2 Me^{-\epsilon_1}] \). Under the above choice of \( k \), we have
\[
\max_{P_A \in P^A_{\epsilon_1, \epsilon_2, M, k}} H(A) = f(k)
\]
and
\[
\max_{P_A \in P^A_{\epsilon_1, \epsilon_2, M}} H(A) = \max_{k \in [0, \epsilon_2 Me^{-\epsilon_1}]} f(k),
\]
where \( k \) is restricted to an integer in the maximum. Taking the derivative, we have
\[
f'(x) = \frac{\epsilon_2}{x} - \frac{1 - \epsilon_2}{M - x},
\]
which is positive when \( x < M \epsilon_2 \). Hence,
\[
\max_{P_A \in P^A_{\epsilon_1, \epsilon_2, M}} H(A)
\leq f(\epsilon_2 Me^{-\epsilon_1})
\leq \epsilon_2(\log(M - \epsilon_1) + (1 - \epsilon_2)(\log M + \log(1 - \epsilon_2 e^{-\epsilon_1}) - \log(1 - \epsilon_2))
\leq \log M - \epsilon_2 \epsilon_1 + (1 - \epsilon_2) \log(1 + \frac{\epsilon_2(1 - e^{-\epsilon_1})}{1 - \epsilon_2})
\leq \log M - \epsilon_2 \epsilon_1 + (1 - \epsilon_2) \frac{\epsilon_2(1 - e^{-\epsilon_1})}{1 - \epsilon_2}
\leq \log M - \epsilon_2 e^{-\epsilon_1} - 1 + \epsilon_1.
\]
Since \( \log M - \epsilon_2 e^{-\epsilon_1} - 1 + \epsilon_1 \) is an affine function of \( \epsilon_2 \), we obtain (319).

On the other hand, using the set \( \Omega := |a - \log P_A(a) \leq \log M - \epsilon_1| \), we have
\[
\max_{P_A \in P^A_{\epsilon_1, \epsilon_2, M}} e^{-\rho H_{1+\rho}(A)} = \sum_{a \in \Omega} (P_A(a))^{1+\rho} + \sum_{a \in \Omega^c} (P_A(a))^{1+\rho}
\leq (1 - \epsilon_2)^{\epsilon_2 \rho_1} + \epsilon_2^{1+\rho} \leq (1 - \epsilon_2) \frac{\epsilon_2^2}{M^\rho} + \epsilon_2.
\]
Since \( (1 - \epsilon_2) \frac{\epsilon_2^2}{M^\rho} + \epsilon_2 \) is a linear function of \( \epsilon_2 \), we obtain
\[
\max_{P_A \in P^A_{\epsilon_1, \epsilon_2, M}} e^{-\rho H_{1+\rho}(A)} \leq (1 - \epsilon_2) \frac{\epsilon_2^2}{M^\rho} + \epsilon_2,
\]
which implies (320).

Proof of Lemma 83: Since (317) implies (316), we only show (317) from (316). For an arbitrary small number \( \epsilon > 0 \), we define the probability
\[
\delta_n := P_{A^n, B^n}((a, b)) - \frac{1}{n} \log P_{A^n|B^n}(a) \leq R - \epsilon.
\]
Applying Eq. (319) of Lemma 84 to the case when $\epsilon_1 = n\epsilon$ and $\epsilon_2 = \delta_n$, we obtain

$$H(A_n|B_n) \leq nR - \delta_n(e^{-n\epsilon} - 1 + n\epsilon).$$

That is,

$$\delta_n \leq \frac{R - \frac{1}{n}H(A_n|B_n)}{e^{-\alpha/n} + \epsilon}.$$ (321)

Thus, $\lim_{n \to \infty} \delta_n = 0$. Hence, Eq. (320) of Lemma 84 guarantees that

$$H_{1+\alpha/n}(A_n|B_n) \geq -\frac{n}{\alpha} \log((1 - \delta_n)e^{\alpha(e-R)} + \delta_n).$$ (322)

Thus,

$$\liminf_{n \to \infty} \frac{1}{n}H_{1+\alpha/n}(A_n|B_n) \geq \liminf_{n \to \infty} -\frac{1}{\alpha} \log((1 - \delta_n)e^{\alpha(e-R)} + \delta_n)$$

$$= R - \epsilon.$$

Since $\epsilon > 0$ is arbitrary,

$$\liminf_{n \to \infty} \frac{1}{n}H_{1+\alpha/n}(A_n|B_n) \geq R.$$

Since the cardinality of $\mathcal{A}_n$ is $e^{\alpha R}$, we have $\frac{1}{n}H_{1+\alpha/n}(A_n|B_n) \leq R$. Hence,

$$\lim_{n \to \infty} \frac{1}{n}H_{1+\alpha/n}(A_n|B_n) = R.$$

Combining relation (19), we obtain the desired argument.

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