SIMPLICITY OF ALGEBRAS ASSOCIATED TO ÉTALE GROUPOIDS

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Abstract. We prove that the full $C^*$-algebra of a second-countable, Hausdorff, étale, amenable groupoid is simple if and only if the groupoid is both topologically principal and minimal. We also show that if $G$ has totally disconnected unit space, then the complex $^*$-algebra of its inverse semigroup of compact open bisections, as introduced by Steinberg, is simple if and only if $G$ is both effective and minimal.

1. Introduction

Let $G$ be a groupoid which is étale in the sense that $r, s : G \to G(0)$ are local homeomorphisms. Complex algebras $A(G)$ associated to locally compact, Hausdorff, étale groupoids $G$ with totally disconnected unit spaces were introduced in [33]. There, Steinberg shows that $A(G)$ can be used to describe inverse-semigroup algebras. These algebras, which we call Steinberg algebras, were also examined in [7], where they are shown to include the complex Kumjian-Pask algebras of higher-rank graphs [4], and hence the complex Leavitt path algebras of directed graphs [1]. In general, $A(G)$ is dense in $C^*(G)$, the $C^*$-algebra associated to $G$. The criteria of [33] which characterise simplicity of a higher-rank graph $C^*$-algebra also characterise simplicity of the associated Kumjian-Pask algebra [4, Theorem 5.14]. Encouraged by this, we set out to investigate the simplicity of $A(G)$.

Translating from the higher-rank graph setting, we hoped to prove that $G$ is topologically principal in the sense that the units with trivial isotropy are dense in the unit space, and minimal in the sense that the unit space has no nontrivial open invariant subsets, if and only if $A(G)$ is simple. Although the “if” implication was not known in the $C^*$-algebra setting, we hoped that in the situation of algebras, where there are no continuity hypotheses to check when constructing representations, we could adapt the ideas of [33, Proposition 3.5]. Our initial attempts to prove the result failed. We eventually realised that the natural necessary condition is not that $G$ be topologically principal, instead it is that $G$ be effective: every open subset of $G \setminus G(0)$ contains an element $\gamma$ such that $r(\gamma)$ and $s(\gamma)$ are distinct. For if an étale groupoid $G$ with totally disconnected unit space is not effective, then there exists a compact open set $B \subseteq G \setminus G(0)$ consisting purely of isotropy on which the range and source maps are homeomorphisms. It follows that $1_{r(B)} - 1_B$ belongs to $A(G)$ and vanishes under a natural homomorphism from $A(G)$ to the algebra of endomorphisms of the free complex module $F(G(0))$ with basis $G(0)$ (see Proposition 4.4). That $G$ is effective is, in general, a strictly weaker condition than that it is topologically principal (see Examples 6.3 and 6.4), though they are equivalent in

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1See Remark 2.3 regarding terminology.
the higher-rank graph setting. We show that effectiveness, together with minimality, is necessary and sufficient for simplicity of $A(G)$ (see Theorem 4.1).

It came as a surprise to discover that the arguments we had developed for $A(G)$ could be adapted to give new results in the $C^*$-algebraic setting provided that $G$ is second-countable; this amounts to restricting our attention to separable $C^*$-algebras. In this setting we can also drop the requirement that $G^{(0)}$ is totally disconnected. A Baire-category argument [31, Proposition 3.6] shows that a second-countable, Hausdorff and étale groupoid $G$ is effective if and only if it is topologically principal. Combining all of this, we fill in the missing piece of the simplicity puzzle for étale groupoid $C^*$-algebras. That is, we show that if $C^*(G)$ is simple, then $G$ must be topologically principal. Hence we are able to give necessary and sufficient conditions for the simplicity of $C^*(G)$ as well. Though some parts of what we have done can be found in the literature, we have taken pains to make our results self-contained and to take the most elementary path possible. There are many classes of $C^*$-algebras with étale groupoid models (see for example [8, 11, 13, 14, 16, 19, 26, 28, 31, 36]), so we expect that our results will find numerous applications.

After a short preliminaries section, we describe in Section 3 a number of equivalent conditions to a locally compact, Hausdorff, étale groupoid $G$ being effective. We show that these equivalent conditions are formally weaker than $G$ being topologically principal, but are equivalent to $G$ being topologically principal if $G$ is second-countable. We present our structure theorems for the Steinberg algebra $A(G)$ in Section 3. In Section 5 we prove $C^*$-algebraic versions of these results. We choose to pay the price of more-technical statements in order to describe how our techniques apply to non-amenable groupoids. In a short examples section we indicate why our techniques cannot be adapted to characterise simplicity of the reduced $C^*$-algebra of an étale groupoid and why our results do not extend readily to twisted groupoid $C^*$-algebras. We also provide an example of a non-étale groupoid in which every unit has infinite isotropy but no open set consists entirely of isotropy. By changing the topology, we also construct an étale groupoid with totally disconnected unit space (which is not second-countable) with the same property. We finish by relating our results to those of Exel-Vershik [12] and of Exel-Renault [11].

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2. Preliminaries

If $X$ is a topological space and $D \subseteq X$, then we shall write $D^0$ for the interior $\bigcup\{U \subseteq D : U$ is open in $X\}$ of $D$.

A groupoid $G$ is a small category in which every morphism has an inverse. When $G$ is endowed with a topology under which the range, source, and composition maps are continuous, $G$ is called a topological groupoid. We say $G$ is étale if $r$ and $s$ are local homeomorphisms. It then follows that $G^{(0)} := \{\gamma \gamma^{-1} : \gamma \in G\}$ is open in $G$. If $G$ is Hausdorff, then $G^{(0)}$ is also closed in $G$. For a more detailed description of étale groupoids, see [24].

A subset $B$ of $G$ such that $r$ and $s$ both restrict to homeomorphisms of $B$ is called a bisection of $G$. If $G$ is a locally compact, Hausdorff, étale groupoid, then there is a base
for the topology on $G$ consisting of open bisections with compact closure (we call such sets precompact in this paper). As demonstrated in [31, 32], if $G(0)$ is totally disconnected and $G$ is locally compact, Hausdorff, and étale, then there is base for the topology on $G$ consisting of compact open bisections.

For subsets $D, E$ of $G(0)$, define

$$G_D := \{ \gamma \in G : s(\gamma) \in D \}, \quad G_E := \{ \gamma \in G : r(\gamma) \in E \} \quad \text{and} \quad G_{D,E} := G_E \cap G_D.$$  

In a slight abuse of notation, for $u, v \in G(0)$ we denote $G_u := G\{u\}$, $G^u := G(0)^v$ and $G^v := G(0)^u$. The isotropy group at a unit $u$ of $G$ is the group $G_u = \{ \gamma \in G : r(\gamma) = s(\gamma) = u \}$. We say $u$ has trivial isotropy if $G_u = \{ u \}$. The isotropy subgroupoid of a groupoid $G$ is $\text{Iso}(G) := \bigcup_{u \in G(0)} G_u$. Since $r$ and $s$ are continuous, the isotropy subgroupoid of $G$ is a closed subset of $G$.

A subset $D$ of $G(0)$ is called invariant if $s(\gamma) \in D \implies r(\gamma) \in D$ for all $\gamma \in G$. Since $G$ contains inverses, this is equivalent to saying that $D = \{ r(\gamma) : s(\gamma) \in D \} = \{ s(\gamma) : r(\gamma) \in D \}$; hence $G_D = G_{D,D}$ and $G_D$ is a groupoid with unit space $D$. Also, $D$ is invariant if and only if its complement is invariant.

For subsets $S$ and $T$ of $G$, define $ST = \{ s(\gamma) : \gamma \in S, \alpha \in T, \quad s(\gamma) = r(\alpha) \}$.

**Definition 2.1.** Let $G$ be a locally compact, Hausdorff groupoid. We say that $G$ is topologically principal if $\{ u \in G(0) : G_u = \{ u \} \}$ is dense in $G(0)$. We say that $G$ is minimal if $G(0)$ has no nontrivial open invariant subsets. We say $G$ is effective if the interior of $\text{Iso}(G) \setminus G(0)$ is empty.

**Remark 2.2.** To relate our later results to those of Thomsen [35], we observe that a Hausdorff, étale groupoid $G$ is topologically principal if and only if each open invariant subset of $G(0)$ contains a point with trivial isotropy. To see this, note that the “only if” implication is trivial. So suppose that every open invariant set contains a point with trivial isotropy, and fix an open subset $U$ of $G(0)$. Then $r(G_U)$ is an open invariant set, so contains a point $u$ with trivial isotropy. Fix $\gamma \in G_U$ with $r(\gamma) = u$. Since $G_{s(\gamma)}^{s(\gamma)} = \gamma^{-1} G_u \gamma = \gamma^{-1} \{ u \} \gamma = r(\gamma)$, we see that $s(\gamma)$ has trivial isotropy. That is, the set $U$ contains a point with trivial isotropy. So $G$ is topologically principal.

It follows immediately from this that if a minimal groupoid $G$ has a unit with trivial isotropy then it is topologically principal.

**Remark 2.3.** In groupoid literature, the condition which we are calling topologically principal has gone under this name and a number of others, including “essentially free,” “topologically free,” and “essentially principal.” We have chosen the one we believe to be least open to misinterpretation: The usage of the term “principal” for groupoids with everywhere-trivial isotropy seems uncontroversial, so “topologically principal” is suggestive. Our choice also seems to match what Renault himself has settled on [31, 32].

Similarly, our usage of the terms minimal and effective seem to be standard (see, for example, [28, Definition I.4.1] and [31, Definition 3.4]) but are possibly not universal.

### 3. Topologically Principal Groupoids

The following lemma establishes the equivalent conditions that we use in Theorem 4.1 to characterise simplicity of $A(G)$.

**Lemma 3.1.** Let $G$ be a locally compact, Hausdorff, étale groupoid. The following are equivalent:
(1) $G$ is effective;
(2) the interior of $\text{Iso}(G)$ is $G^{(0)}$;
(3) for every nonempty open bisection $B \subseteq G \setminus G^{(0)}$, there exists $\gamma \in B$ such that $s(\gamma) \neq r(\gamma)$;
(4) for every compact $K \subseteq G \setminus G^{(0)}$ and every nonempty open $U \subseteq G^{(0)}$, there exists a nonempty open subset $V \subseteq U$ such that $VKV = \emptyset$.

If $G$ is topologically principal, then $G$ is effective. If $G$ is second-countable and effective, then $G$ is topologically principal.

Proof of Lemma 3.2. Since $G$ is Hausdorff and étale, $G^{(0)}$ is both open and closed in $G$. So the interior $S^\circ$ of any subset $S$ of $G$ is equal to the disjoint union $(S \cap G^{(0)})^\circ \cup (S \setminus G^{(0)})^\circ$. Thus (3) is equivalent to (1)

We have (1) $\implies$ (3) because open bisections are in particular open sets. That $G$ is étale also implies that the collection of all open bisections of $G$ form a base for the topology on $G$. In particular, every open set contains an open bisection, giving (3) $\implies$ (1)

To see (1) implies (3), we prove the contrapositive. Suppose that (3) does not hold, and fix an open bisection $B_0 \subseteq G \setminus G^{(0)}$ such that $r(\gamma) = s(\gamma)$ for all $\gamma \in B_0$. That is, $B_0 \subseteq \text{Iso}(G)$. By shrinking if necessary, we may assume that $B_0$ is precompact. Since $G$ is locally compact and Hausdorff, it is a regular topological space (that is, points can be separated from compact sets by disjoint open sets). Thus, there is an open subset $B$ of $B_0$ whose closure $K$ is compact and contained in $B_0$. Let $U = r(B)$, and fix a nonempty open subset $V$ of $U$. Since $K \subseteq \text{Iso}(G)$, we have $VK = KV$, and in particular $VK \neq \emptyset$. Hence (3) does not hold.

To show that (3) implies (1), we begin with a claim.

Claim 3.2. Suppose that $B \subseteq G \setminus G^{(0)}$ is an open bisection and that $\gamma \in B \setminus \text{Iso}(G)$. Then there is an open set $V \subseteq r(B)$ such that $\gamma \in VB$ and $s(VB) \cap V = \emptyset$.

Proof of Claim 3.2. Since $r(\gamma) \neq s(\gamma)$ and $G$ is Hausdorff, there exist open neighbourhoods $W$ of $r(\gamma)$ and $W'$ of $s(\gamma)$ such that $W \cap W' = \emptyset$. Let $V := W \cap r(BW')$. Notice that $r(\gamma) \in V$ so $V$ is not empty. Then $\gamma \in VB$, and since $B$ is a bisection, $s(VB) = s(WB \cap BW') \subseteq W'$ and hence is disjoint from $V \subseteq W$. \hspace{1cm} \Box

Now suppose (3), and fix a compact $K \subseteq G \setminus G^{(0)}$ and an open $U \subseteq G^{(0)}$. We construct a nonempty open set $V \subseteq U$ such that $VKV = \emptyset$. If $U$ is not a subset of $r(K)$, then $V = U \setminus r(K)$ will suffice, so suppose that $U \subseteq r(K)$. Because $G$ is regular and $G^{(0)}$ is open, there is a base for the topology on $G \setminus G^{(0)}$ consisting of precompact open bisections whose closures are themselves contained in open bisections which do not intersect $G^{(0)}$. Since $K$ is compact, we may cover $K$ by a finite set $B$ of such precompact open bisections.

For each $B \in B$, fix an open bisection $C_B$ such that $B \subseteq C_B \subseteq G \setminus G^{(0)}$. For each $B \in B$ the set $UC_B$ is an open bisection which does not intersect $G^{(0)}$, and the $r(B)$ cover $U$ so at least one $UC_B$ is nonempty. So (3) implies that there exists $\gamma \in \bigcup_{B \in B} UC_B \setminus \text{Iso}(G)$. Let $F := \{ B \in B : \gamma \in UC_B \}$. For each $B \in F$, Claim 3.2 yields an open set $V_B \subseteq r(UC_B)$ such that $s(V_BC_B) \cap V_B = \emptyset$. Let

$$V := U \cap \left( \bigcap\{ V_B : B \in F \} \right) \setminus \left( \bigcup \{ r(B') : B' \in B \setminus F \} \right).$$

Then $V$ is open by definition, and nonempty because it contains $r(\gamma)$. 

Fix \( \alpha \in VK \); we must show \( s(\alpha) \notin V \). Since \( \alpha \in K \) and \( B \) is a cover of \( K \), we have \( \alpha \in B \) for some \( B \in B \). Also, since \( r(\alpha) \in V \), we have \( B \in \mathcal{F} \). Hence

\[
s(\alpha) \in s(VB) \subseteq s(\overline{VB}),
\]
and \( s(VB) \cap V \subseteq s(VB) \cap V = \emptyset \). Therefore \( s(\alpha) \notin V \). Thus (3) implies (4).

The final two statements follow from [31, Proposition 3.6] since every locally compact Hausdorff space has the Baire property.

\[\square\]

**Remark 3.3.** The final assertion of Lemma 3.1 need not hold if \( G \) is not second-countable (see Example 6.4) or if \( G \) is not étale (see Example 6.3).

### 4. Simplicity of Steinberg algebras

In this section, we consider locally compact, Hausdorff, étale groupoids with totally disconnected unit spaces. This puts us in the setting of [7]. For such a groupoid \( G \), let

\[
A(G) := \text{span}\{1_B : B \text{ is a compact open bisection}\}
\]
as in [7]. For \( f, g \in A(G) \subseteq C_c(G) \), define

\[
(f \ast g)(\gamma) = \sum_{r(\alpha) = r(\gamma)} f(\alpha) g(\alpha^{-1} \gamma) = \sum_{\alpha \beta = \gamma} f(\alpha) g(\beta).
\]

Under these operations and pointwise addition and scalar multiplication, \( A(G) \) is a *-subalgebra of \( C_c(G) \). It coincides with the complex inverse semigroup algebra \( \mathbb{C}G \) introduced in [34].

We call \( A(G) \) the Steinberg algebra of \( G \).

**Theorem 4.1.** Let \( G \) be a locally compact, Hausdorff, étale groupoid such that \( G^{(0)} \) is totally disconnected. Then \( A(G) \) is simple if and only if \( G \) is both effective and minimal.

Our proof was guided by that of Theorem 5.14 in [4]. However, their arguments rely heavily on the underlying higher-rank graph structure so our approach looks very different. The first step is to prove that the Cuntz-Krieger uniqueness theorem for \( A(G) \) [7, Theorem 5.2] still holds if we replace the hypothesis that \( G \) is topologically principal with the hypothesis that \( G \) is effective.

**Lemma 4.2.** Let \( G \) be a locally compact, Hausdorff, étale groupoid with totally disconnected unit space. Suppose that \( G \) is effective and that \( I \) is a nontrivial ideal of \( A(G) \). Then there is a compact open subset \( V \subseteq G^{(0)} \) such that \( 1_V \in I \).

**Proof.** Fix \( b \in I \setminus \{0\} \). Let \( c := b^* \ast b \). For \( u \in G^{(0)} \) we have

\[
c(u) = \sum_{\gamma \in G_u} b^*(\gamma^{-1})b(\gamma) = \sum_{\gamma \in G_u} b(\gamma)b(\gamma) \geq \max_{\gamma \in G_u} |b(\gamma)|^2.
\]

In particular, the function

\[
c_0 := \begin{cases} c(\gamma) & \text{if } \gamma \in G^{(0)}; \\ 0 & \text{otherwise} \end{cases}
\]
is nonzero. Because \( G^{(0)} \) is both open and closed, \( c_0 \in A(G) \).

\[\footnote{\text{We prefer the notation } A(G) \text{ because Steinberg’s notation } \mathbb{C}G \text{ suggests the free } \mathbb{C}-\text{module with basis } G, \text{ which is substantially larger. To avoid clashing with Steinberg’s notation, we use } F(W) \text{ for the free complex module with basis } W.} \]

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Proposition 4.3. Let $A$ be a connected unit space and let $W$ be a collection of compact open subsets of $G^{(0)}$, and each $a_U$ is nonzero. Let $K$ be the support of $c - c_0$. Notice that $K \subseteq G \setminus G^{(0)}$.

Fix $U \in \mathcal{U}$. Since $\text{Iso}(G)^0 = G^{(0)}$, the implication $(2) \implies (4)$ of Lemma 3.1 implies that there exists a nonempty open subset $V \subseteq U$ such that $V \cap K = \emptyset$. Since $G$ has a basis of compact open sets, we can assume $V$ is also compact.

For $\gamma \in G$ we have

$$1_V(c - c_0)(1_V)(\gamma) = 1_V(r(\gamma))(c - c_0)(\gamma)1_V(s(\gamma)) = 0.$$ 

So $1_Vc_0 = 1_Vc_1 = a_U1_V$. Hence $1_V \in I$. \hfill \Box

Another key ingredient in our proof of Theorem 4.1 is the following generalisation of the infinite-path representation of a Kumjian-Pask algebra as defined on page 9 of [1]. In our setting, the infinite-path space becomes the unit space of $G$. In fact, the construction of [4] works for any invariant subset $W$ of $G^{(0)}$. Given such a set $W$, we write $\mathbb{F}(W)$ for the free (complex) module with basis $W$. We use these representations to construct nontrivial ideals of $A(G)$ when there exists either a nontrivial open invariant subset of $G^{(0)}$ or a nonempty open subset of $\text{Iso}(G) \setminus G^{(0)}$.

Proposition 4.3. Let $G$ be a locally compact, Hausdorff, étale groupoid with totally disconnected unit space and let $W$ be an invariant subset of $G^{(0)}$.

(1) For every compact open bisection $B \subseteq G$, there is a unique function $f_B : G^{(0)} \to \mathbb{F}(W)$ that has support contained in $s(B)$ and satisfies $f_B(s(\gamma)) = r(\gamma)$ for all $\gamma \in B$.

(2) There is a unique representation $\pi_W : A(G) \to \text{End}(\mathbb{F}(W))$ such that $\pi_W(1_B)u = f_B(u)$ for every compact open bisection $B$ and all $u \in W$.

Proof. Let $B$ be a compact, open bisection in $G$. The formula $s(\gamma) \mapsto r(\gamma)$ for $\gamma$ in $B$ specifies a well-defined homeomorphism from $s(B)$ to $r(B)$. Thus, the function $f_B$ can be defined as stated in [1]. To prove (2), first notice that the universal property of the free module $\mathbb{F}(W)$ implies that there is an element $t_B \in \text{End}(\mathbb{F}(W))$ extending $f_B|W$. Let $c : G \to \{e\}$ be the trivial cocycle. Then every bisection of $G$ is $c$-graded under $c$, so the set $B^{co}_c(G)$ of [4, Definition 3.10] is the set of all compact open bisections of $G$. We claim that the collection $\{t_B : B \in B^{co}_c(G)\}$ gives a representation of $B^{co}_c(G)$ in $\text{End}(\mathbb{F}(W))$ as defined in Definition 3.10 of [4].

To prove our claim, we must verify that:

(R1) $t_0 = 0$;
(R2) $t_B|D = t_{BD}$ for all compact open bisections $B$ and $D$; and
(R3) $t_B + t_D = t_{B \cup D}$ whenever $B$ and $D$ are disjoint compact open bisections such that $B \cup D$ is a bisection.

It is straightforward to check that each of these conditions holds for the functions $f_B$, and hence for the endomorphisms $t_B$ as well.

Now, the universal property of $A(G)$, stated in Theorem 3.11 of [4], gives a unique homomorphism $\pi_W : A(G) \to \text{End}(\mathbb{F}(W))$ such that $\pi_W(1_B) = t_B$ for all $B \in B^{co}_c(G)$.
Proof of Theorem 4.1. Suppose $G$ is an involution on $A$, a locally compact, Hausdorff, etale groupoid. The formulas (1) and (2) for convolution and $V$-Lemma 4.2 implies that there is a compact open subset $B$. To see this, let $B$ be the homomorphism of Proposition 3.3. Then $\pi$ is injective if and only if $G$ is effective.

Proof. First suppose that $G$ is effective. Since $\pi(1_V) \neq 0$ for all compact open $V \subseteq G^{(0)}$, $\pi$ is injective by the contrapositive of Lemma 4.2 applied to $I = \ker(\pi)$.

Now suppose that $G$ is not effective. By (1) $\iff$ (3) of Lemma 3.1, there exists a nonempty compact open bisection $B \subseteq G \setminus G^{(0)}$ so that for every $\gamma \in B$, $r(\gamma) = s(\gamma)$. Hence $B \neq s(B)$ but $f_B = f_{s(B)}$, where $f_B$ is defined in Proposition 4.3. Thus $\pi_{G^{(0)}}(1_B) = \pi_{G^{(0)}}(1_s(B))$ giving $1_B - 1_{s(B)} \in \ker(\pi_{G^{(0)})}$. Since $B \neq s(B)$ we have $1_B - 1_{s(B)} \neq 0$, so $\ker(\pi_{G^{(0)})} \neq \{0\}$.

Proposition 4.5. Let $G$ be a locally compact, Hausdorff, etale groupoid with totally disconnected unit space. Then $G$ is minimal if and only if every nonzero $f \in A(G)$ such that $\operatorname{supp} f \subseteq G^{(0)}$ generates $A(G)$ as an ideal.

Proof. Suppose $G$ is minimal. Fix $f \in A(G) \setminus \{0\}$ such that $\operatorname{supp} f \subseteq G^{(0)}$. Let $I$ be the ideal of $A(G)$ generated by $f$. Fix $g \in A(G)$; we must show that $g \in I$. Since $f$ is nonzero and locally constant [7, Lemma 3.4], there exist $c \in C \setminus \{0\}$ and a compact open $U \subseteq G^{(0)}$ so that $f|_U \equiv c$. Then $1_U = \frac{1}{2} (1_U * f) \in I$. Let $K := r(\operatorname{supp}(g)) \subseteq G^{(0)}$. Then $K$ is compact and open by [7, Lemma 3.2]. Since $s(G^U)$ is a nonempty open invariant set, it is all of $G^{(0)}$. Therefore $K \subseteq s(G^U)$. So for each $u \in K$, there exists $g_u$ with $r(\gamma_u) \in U$ and $s(\gamma_u) = u$. For each $u$, let $B_u$ be a compact open bisection containing $\gamma_u$ such that $r(B_u) \subseteq U$ and $s(B_u) \subseteq K$. Then $1_s(B_u) = 1_{B_u} * 1_U * 1_{B_u}$ belongs to $I$. Since $K$ is compact, there is a finite subset $\{u_1, \ldots, u_n\}$ of $K$ such that $\{ s(B_{u_i}) : 1 \leq i \leq n \}$ covers $K$. By disjointification of the collection $\{ s(B_{u_i}) : 1 \leq i \leq n \}$ (see [7, Remark 2.5]), we may assume that the $s(B_{u_i})$ are mutually disjoint. For each $i$, the function $k_i := 1_{s(B_{u_i})} \leq 1_{s(B_{u_i})}$ belongs to $I$, so $1_K = \sum k_i \in I$. Hence $g = 1_k * g \in I$.

Conversely, suppose $G$ is not minimal. Let $U$ be a nontrivial open invariant subset of $G^{(0)}$. Then the complement $W := G^{(0)} \setminus U$ is itself an invariant subset of $G^{(0)}$. Let $\pi_W : A(G) \to \operatorname{End}(F(W))$ be the nonzero homomorphism of Proposition 1.3. The kernel of $\pi_W$ is a proper ideal of $A(G)$. To complete the proof, it suffices to show that $\ker(\pi_W) \neq \{0\}$. To see this, let $B \subseteq U$ be a compact open set. Then $1_B \in \ker(\pi_W) \setminus \{0\}$.

Proof of Theorem 4.4. Suppose $A(G)$ is simple. Then $\pi_{G^{(0)}}$ is injective so Proposition 4.4 implies that $G$ is effective. Since $A(G)$ is simple, every function with support contained in $G^{(0)}$ generates $A(G)$ as an ideal. Hence, $G$ is minimal by Proposition 4.5.

Conversely, suppose that $G$ is effective and minimal. Fix a nonzero ideal $I$ in $A(G)$. Lemma 4.2 implies that there is a compact open subset $V \subseteq G^{(0)}$ such that $1_V \in I$. Proposition 4.5 implies that the ideal generated by $1_V$ is all of $A(G)$, so $I = A(G)$.

5. Simplicity of groupoid C*-algebras

For details of the following, see, for example, [28] or [24]. Let $G$ be a second-countable, locally compact, Hausdorff, etale groupoid. The formulas (1) and (2) for convolution and involution on $A(G)$ described in the preceding section also define a convolution and involution on $C_*(G)$. With these operations, and pointwise addition and scalar multiplication,
$C_c(G)$ is a complex $*$-algebra. The $I$-norm on $C_c(G)$ defined by

$$
\|f\|_I = \sup_{u \in G^{(0)}} \max \left\{ \sum_{\gamma \in G_u} |f(\gamma)|, \sum_{\gamma \in G_u} |f(\gamma)| \right\}
$$

is a $*$-algebra norm (see Proposition II.1.4 of [28]) but not typically a $C^*$-norm. The full norm on $C_c(G)$ is defined by

$$
\|f\| := \sup\{\|\pi(f)\| : \pi \text{ is an } I\text{-norm-bounded } *\text{-representation of } C_c(G)\},
$$

and $C^*_r(G)$ is defined to be the completion of $C_c(G)$ in the full norm.

There is a distinguished family of $I$-norm-bounded representations of $C_c(G)$, called the regular representations; each is indexed by a $u \in G^{(0)}$ and denoted Ind$_u$. Specifically, the regular representation Ind$_u$ is the representation of $C_c(G)$ on $\ell^2(G_u)$ implemented by convolution. That is, Ind$_u(f)\delta_v = \sum_{\beta \in G(v)} f(\beta^{-1}\gamma)\delta_\beta$. The reduced $C^*$-algebra $C^*_r(G)$ is the completion of $C_c(G)$ in the reduced norm $\|f\|_r = \sup_{u \in G^{(0)}} \|\text{Ind}_u(f)\|$. The reduced norm is dominated by the full norm, so $C^*_r(G)$ is a quotient of $C^*(G)$.

We can now state our main theorem.

**Theorem 5.1.** Let $G$ be a second-countable, locally compact, Hausdorff, étale groupoid. Then $C^*(G)$ is simple if and only if all of the following conditions are satisfied:

1. $C^*_r(G) = C^*_r(G)$;
2. $G$ is topologically principal; and
3. $G$ is minimal.

Our proof of Theorem 5.1 relies on the following adaptation of the augmentation representation of a discrete group. Let $G$ be a groupoid as in Theorem 5.1. For each $u \in G^{(0)}$, let $[u]$ denote the orbit of $u$ under $G$; that is $[u] = r(G_u)$.

**Proposition 5.2.** Let $G$ be a second-countable, locally compact, Hausdorff, étale groupoid. Fix $u \in G^{(0)}$. There is a unique representation $\pi_{[u]}$ of $C^*_r(G)$ on $\ell^2([u]) = \overline{\text{span}} \{\delta_v : v \in [u]\}$ such that for each $f \in C_c(G)$ and $v \in [u]$,

$$
\pi_{[u]}(f)\delta_v := \sum_{\gamma \in G_v} f(\gamma)\delta_{r(\gamma)}.
$$

**Remark 5.3.** In equation 3, we described $\pi_{[u]}$ in terms of the canonical orthonormal basis for $\ell^2([u])$. For an alternative description, let $\mu$ be the measure $\mu(\gamma) = |V \cap [u]|$ on $G^{(0)}$. Then $\pi_{[u]}$ is the representation on $L^2(G^{(0)}, \mu)$ obtained from the usual left-action of $C_c(G)$ on $C_c(G^{(0)})$ — namely $f \cdot \phi(u) = \sum_{\gamma \in G_u} f(\gamma)\phi(s(\gamma))$.

**Proof of Proposition 5.2.** For $f \in C_c(G)$ and a finite linear combination $h = \sum_{v \in [u]} h_v\delta_v$, let $f \cdot h$ be the vector $\sum_{v \in [u]} \sum_{\gamma \in G_v} f(\gamma)h_v\delta_{r(\gamma)}$. Then $h \mapsto f \cdot h$ is linear, and $f \cdot \delta_v$ is equal to the right-hand side of 3. The following is adapted directly from the proof of [28] Proposition II.1.7. Fix $f \in C_c(G)$. For $v, w \in [u]$, we have

$$
(f \cdot \delta_v|\delta_w) = \sum_{\gamma \in G_v} (f(\gamma)|\delta_{r(\gamma)})|\delta_w) = \sum_{\gamma \in G_v} f(\gamma) = \sum_{\gamma \in G_w} (\delta_v|f(\gamma^{-1})\delta_w) = (\delta_v|f^* \cdot \delta_w).
$$
Since $k \mapsto f \cdot k$ is linear on $\text{span}\{\delta_v : v \in [u]\}$, it follows that $(f \cdot k|k') = (k)f^* \cdot k'$ for all $k, k' \in C_c([u])$. In particular, for a finite linear combination $h = \sum_{v \in [u]} h_v \delta_v$,

$$\|f \cdot h\|^2 = \left(\sum_{\gamma \in G_{[u]}} \left|(f^* f)\gamma\right| h_{s(\gamma)} h_{r(\gamma)}\right)^2 = \left(\sum_{\gamma \in G_{[u]}} \left|(f^* f)\gamma\right| h_{s(\gamma)}\right) \left(\sum_{\gamma \in G_{[u]}} \left|(f^* f)\gamma\right| h_{r(\gamma)}\right).$$

So the Cauchy-Schwarz inequality gives

$$\|f \cdot h\|^2 \leq \left(\sum_{\gamma \in G_{[u]}} \left|(f^* f)\gamma\right| h_{s(\gamma)}\right)^2 \left(\sum_{\beta \in G_{[u]}} \left|(f^* f)\beta\right| h_{r(\beta)}\right)^2 = \left(\sum_{\gamma \in G_{[u]}} \left(\sum_{\gamma \in G_{[u]}} \left|(f^* f)\gamma\right| h_{\gamma}\right)^2\right)^{1/2} \left(\sum_{\beta \in G_{[u]}} \left(\sum_{\beta \in G_{[u]}} \left|(f^* f)\beta\right| h_{\beta}\right)^2\right)^{1/2} \left(\sum_{\gamma \in G_{[u]}} \left(\sum_{\gamma \in G_{[u]}} \left|(f^* f)\gamma\right| h_{\gamma}\right)^2\right)^{1/2} \left(\sum_{\beta \in G_{[u]}} \left(\sum_{\beta \in G_{[u]}} \left|(f^* f)\beta\right| h_{\beta}\right)^2\right)^{1/2} \leq \|(f^* f)\|_\|h\|^2.$$

Proposition II.1.4 of [28] (or direct calculation) shows that $\|f^* f\|_I \leq \|f\|_I^2$, and it follows that $\|f \cdot h\| = \|f\|_I \|h\|$. Thus, for each $f \in C_c(G)$ the formula [3] determines a bounded linear operator $\pi_{[u]}(f)$ on $L^2([u])$, and the map $f \mapsto \pi_{[u]}(f)$ is bounded with respect to the $I$-norm. By definition of the norm on $C^*(G)$, it therefore remains only to show that $\pi_{[u]}$ is a *-homomorphism from $C_c(G)$ to $B(L^2([u]))$. The calculation [1] shows that $\pi_{[u]}(f)^* = \pi_{[u]}(f^*)$. For $f, g \in C_c(G)$ and $v \in [u]$, 

$$\pi_{[u]}(f \ast g)\delta_v = \sum_{\gamma \in G_v} (f \ast g)(\gamma) \delta_{r(\gamma)} = \sum_{\alpha \in G_v} f(\alpha)g(\beta)\delta_{r(\alpha)} = \sum_{\beta \in G_v} \sum_{\alpha \in G_v} f(\alpha)g(\beta)\delta_{r(\alpha)} = \sum_{\beta \in G_v} \pi_{[u]}(f)(g(\beta)\delta_{r(\beta)}) = \pi_{[u]}(f)\pi_{[u]}(g)\delta_v.$$ 

Hence $\pi_{[u]}$ is a *-homomorphism as required. 

**Remark 5.4.** The direct sum $\epsilon_G := \bigoplus_{[u] \in G^{(0)}} \pi_{[u]}$ of $G$ is faithful on $C_0(G^{(0)})$. To see this, fix $f \in C_c(G^{(0)}) \setminus \{0\}$ and $u \in G^{(0)}$ such that $f(u) \neq 0$. Then $\left\|\epsilon_G(f)\right\| \geq \left\|\pi_{[u]}(f)\delta_u\right\| = \left\|f(u)\delta_u\right\| = 0$. If $G$ is a (discrete) group, then $\epsilon_G$ is just the 1-dimensional representation of $C^*(G)$ induced by the unitary representation $\epsilon : g \mapsto 1$ of $G$, sometimes called the augmentation representation of $G$.

**Proposition 5.5.** Let $G$ be a second-countable, locally compact, Hausdorff, étale groupoid.

1. Suppose that $G$ is topologically principal. Then every ideal $I$ of the reduced $C^*$-algebra $C_r^*(G)$ satisfies $I \cap C_c(G^{(0)}) \neq \{0\}$.

2. Suppose that every ideal of the full $C^*$-algebra $C^*(G)$ satisfies $I \cap C_0(G^{(0)}) \neq \{0\}$. Then $G$ is topologically principal.
Proof.\footnote{Exel uses the term \textit{essentially principal} for what we call effective (see \cite{10} p. 897)} Since $G$ is topologically principal, Lemma \ref{3.3} implies that it is effective. The result then follows from \cite{10} Theorem 4.4\footnote{We prove the contrapositive. Suppose that $G$ is not topologically principal. Then Lemma \ref{3.3} implies that there is an open bisection $B$ in $G \setminus G(0)$ consisting entirely of isotropy. Let $\epsilon_G$ be the direct sum representation defined in Remark \ref{5.4}. We show that $\ker(\epsilon_G)$ is a nontrivial ideal in $C^*(G)$ that does not intersect $C_0(G(0))$. By Remark \ref{5.4} $\ker(\epsilon_G) \cap C_0(G(0)) = \{0\}$ so it suffices to construct a nonzero element of $\ker \epsilon_G$.} (see also \cite{30} Corollary 4.9).

We prove the contrapositive. Suppose that $G$ is not topologically principal. Then Lemma \ref{3.3} implies that there is an open bisection $B$ in $G \setminus G(0)$ consisting entirely of isotropy. Let $\epsilon_G$ be the direct sum representation defined in Remark \ref{5.4}. We show that $\ker(\epsilon_G)$ is a nontrivial ideal in $C^*(G)$ that does not intersect $C_0(G(0))$. By Remark \ref{5.4} $\ker(\epsilon_G) \cap C_0(G(0)) = \{0\}$ so it suffices to construct a nonzero element of $\ker \epsilon_G$.

For each $u \in s(B)$, let $\gamma_u$ be the unique element in $B$ such that $s(\gamma_u) = u$. Fix a nonzero function $f \in C_c(G)$ such that $\text{supp}(f) \subseteq B$, and define $f_0 \in C_c(G(0))$ by

$$f_0(u) := \begin{cases} f(\gamma_u) & \text{if } u \in s(B), \\ 0 & \text{otherwise.} \end{cases}$$

Since $B \cap G(0) = \emptyset$ and since $f \neq 0$, we have $f - f_0 \neq 0$. We claim that $\epsilon_G(f - f_0) = 0$; that is, $\pi[u](f - f_0) = 0$ for all $u \in G(0)$. To see this, fix $u \in G(0)$ and $v \in [u]$. Then

$$\pi[u](f - f_0)\delta_v = \sum_{\gamma \in G_v} f(\gamma)\delta_{r(\gamma)} - \sum_{\alpha \in G_v} f_0(\alpha)\delta_{r(\alpha)}.$$  

If $v \notin s(B)$, then $f(\gamma) = f_0(\alpha) = 0$ for all $\gamma, \alpha \in G_v$, so $\pi[u](f - f_0)\delta_v = 0$. Suppose that $v \in s(B)$. Since $f_0$ is supported on units and $f$ is supported on $B$,

$$\sum_{\gamma \in G_v} f(\gamma)\delta_{r(\gamma)} - \sum_{\alpha \in G_v} f_0(\alpha)\delta_{r(\alpha)} = f(\gamma_v)\delta_{r(\gamma_v)} - f_0(v)\delta_v = f(\gamma_v)\delta_{r(\gamma_v)} - f(\gamma_v)\delta_v.$$ 

Since $B \subseteq \text{Iso}(G)$, we have $r(\gamma_v) = s(\gamma_v) = v$, and it follows that $\pi[u](f - f_0)\delta_v = 0$. \qed

The following standard lemma is used in the proofs of Proposition \ref{5.7} and Corollary \ref{5.9}.

\begin{lemma}
Let $G$ be a locally compact, Hausdorff, étale groupoid. Suppose that $h \in C_c(G)$ is supported on a bisection $B$ and that $f \in C_c(G(0))$. Then $h \ast f \ast h^* \in C_c(G(0))$ with support contained in $r(B) \subseteq G(0)$ and satisfies

$$\langle h \ast f \ast h^*(r(\gamma)) \rangle = |h(\gamma)|^2f(s(\gamma)) \quad \text{for all } \gamma \in B.$$

\end{lemma}

\begin{proof}
For $\alpha \in G$, we have

$$h \ast f \ast h^*(\alpha) = \sum_{\gamma \beta^{-1} = \alpha} h(\gamma)f(\eta)h(\beta).$$

Fix $\gamma \beta^{-1} \in G$ with $h(\gamma)f(\eta)h(\beta) \neq 0$. Since $\text{supp}(f) \subseteq G(0)$, we have $\eta = s(\gamma) = s(\beta)$. Since $h$ is supported on the bisection $B$, it follows that $\gamma, \beta \in B$ and $\beta = \gamma$. Hence $\gamma \beta^{-1} = \gamma s(\gamma)\gamma^{-1} = r(\gamma) \in r(B)$. Thus the sum on the right of (5) is zero if $\alpha \notin r(B)$, and has only one nonzero term $h(\gamma)f(s(\gamma))h(\beta) = |h(\gamma)|^2f(s(\gamma))$ if $\alpha = r(\gamma) \in r(B)$. \qed

\end{proof}

\begin{proposition}
Let $G$ be a second-countable, locally compact, Hausdorff, étale groupoid. The following are equivalent:

\begin{enumerate}
\item $G$ is minimal;
\item the ideal of $C^*(G)$ generated by any nonzero $f \in C_c(G(0))$ is $C^*(G)$; and
\item the ideal of $C^*_r(G)$ generated by any nonzero $f \in C_c(G(0))$ is $C^*_r(G)$.
\end{enumerate}

\end{proposition}

\begin{proof}

\end{proof}
Proof. (1) $\implies$ (2) and (1) $\implies$ (3). Let $f \in C_c(G^{(0)} \setminus \{0\})$ and let $I$ be the ideal of $C^*(G)$ generated by $f$. We claim that $C_c(G^{(0)}) \subseteq I$. Since $I \cap C_0(G^{(0)})$ is an ideal of $C_0(G^{(0)})$, it suffices to show that for each $u \in G^{(0)}$, there exists $g \in I \cap C_0(G^{(0)})$ such that $g(u) \neq 0$. Fix $u \in G^{(0)}$. Let $U := \{ v \in G^{(0)} : f(v) \neq 0 \}$. Then $U$ is nonempty and open, and hence $r(G_U)$ is open because $s$ is continuous and the local homeomorphism $r$ is an open map. So $r(G_U)$ is a nonempty open invariant set, and hence is equal to $G^{(0)}$ because $G$ is minimal. In particular, there exists $\gamma \in G$ such that $s(\gamma) \in U$ and $r(\gamma) = u$. Fix $h \in C_c(G)$ such that $\text{supp}(h)$ is contained in a bisection and $h(\gamma) = 1$. Lemma 5.6 implies that $(h * f * h^*)(u) = |h(\gamma)|^2 f(s(\gamma)) = f(s(\gamma)) = 0$. So $g := h * f * h^*$ belongs to $I \cap C_0(G^{(0)})$ with $g(u) = 1$. This proves the claim.

Fix $F \in C_c(G)$. Then any $g \in C_c(G^{(0)})$ such that $g|_{r(\text{supp}(F))} \equiv 1$ satisfies $g * F = F$. Hence $C_c(G) \subseteq I$, and so $I = C^*(G)$. Let $q : C^*(G) \rightarrow C^*_r(G)$ be the quotient map. Then the ideal $I_r$ of $C^*_r(G)$ generated by $f$ is $q(I)$. Since $q$ restricts to the identity map on $C_c(G)$, we have $C_c(G) \subseteq I_r$ as well, and hence $I_r = C^*_r(G)$.

(2) $\implies$ (1) and (3) $\implies$ (1). We prove the contrapositive. Suppose that $U$ is a nonempty proper open invariant subset of $G^{(0)}$. Fix $f \in C_c(G) \setminus \{0\}$ such that $\text{supp}(f) \subseteq U$. Then $f \in C_c(G^{(0)})$. Fix $u \in G^{(0)} \setminus U$. Since $G^{(0)} \setminus U$ is invariant, $[u] \subseteq G^{(0)} \setminus U$, so $f(v) = 0$ for all $v \in [u]$. It follows that the image of $f$ under the regular representation $\text{Ind}_u$ is zero. On the other hand, for any $g \in C_c(G^{(0)})$ such that $g(u) = 1$, we have $\text{Ind}_u(g)\delta_u = g(u)\delta_u \neq 0$. So $\text{Ind}_u$ is a nonzero representation of $C_c(G)$ with nontrivial kernel. Since $\text{Ind}_u$ extends to each of $C^*_r(G)$ and $C^*(G)$ it follows that the ideals of each of $C^*(G)$ and $C^*_r(G)$ generated by $f$ are proper ideals.

Remark 5.8. Suppose that $G$ is locally compact, Hausdorff and étale. Thomsen observes in [33] that if $G$ has a unit with trivial isotropy, then $G$ is topologically principal whenever it is minimal (see Remark 2.2). He then deduces that if $G$ has a unit with trivial isotropy, then $C^*(G)$ is simple if and only if $G$ is minimal. We recover this result from Proposition 5.7 together with (1) $\iff$ (3) of Proposition 5.7.

Proof of Theorem 5.7. Suppose $C^*(G)$ is simple. Then the quotient map from $C^*(G) \rightarrow C^*_r(G)$ has trivial kernel and hence the two coincide. Moreover, $C^*(G)$ is the only nonzero ideal of $C^*(G)$ and $C^*(G) \cap C_0(G^{(0)}) \neq \{0\}$ so Proposition 5.5 implies that $G$ is topologically principal. The simplicity of $C^*(G)$ implies that every $f \in C_c(G^{(0)})$ generates $C^*(G)$ as an ideal and so Proposition 5.7 implies that $G$ is minimal.

Now suppose that $C^*(G) = C^*_r(G)$ and that $G$ is topologically principal and minimal. Fix a nonzero ideal $I$ in $C^*(G)$. Since $C^*(G) = C^*_r(G)$, Proposition 5.5 implies there exists a nonzero $f \in C_c(G^{(0)}) \cap I$; and then (1) $\implies$ (3) of Proposition 5.7 implies that the ideal generated by $f$ is $C^*(G)$. Thus $I = C^*(G)$.

Corollary 5.9 below characterises the measurewise-amenable, étale groupoids for which the ideal structure of $C^*(G)$ coincides with the $G$-invariant ideal structure of $C_0(G^{(0)})$. The argument for the "if" implication is standard (see, for example, [28] Proposition 4.6), but we include it for completeness.

The notion of amenability for groupoids is somewhat technical; for a detailed discussion, see [3]. For our purposes, we only need the following two facts. First, if $G$ is measurewise amenable, then $C^*(G) = C^*_r(G)$ [3 Proposition 3.3.5]. Second, suppose that $U \subseteq G^{(0)}$
is open and invariant. If \( G \) is measurewise amenable then each of \( G_U \) and \( G_{G^0 \setminus U} \) is measurewise amenable [8, Corollary 5.3.21] \(^4\).

If \( D \subseteq G^0 \) is a closed invariant set, then \( \{ f \in C_e(G) : f|_{G_D} \equiv 0 \} \) is an ideal of \( C^*(G) \) isomorphic to \( C^*(G_{G^0 \setminus D}) \), and the quotient is isomorphic to \( C^*(G_D) \) (see [22, Lemma 2.10]). This decomposition fails in general for reduced \( C^*\)-algebras.

Corollary 5.9. Let \( G \) be a second-countable, locally compact, Hausdorff groupoid. Suppose that \( G \) is measurewise amenable and étale. Then \( D \mapsto \{ f \in C_e(G) : f|_{G_D} \equiv 0 \} \) is a bijection between closed invariant subsets of \( G^0 \) and ideals of \( C^*(G) \) if and only if, for every closed invariant \( D \subseteq G^0 \), \( G_D \) is topologically principal.

**Proof.** First, we claim that there is a bijection between closed invariant subsets \( D \) and ideals of the form \( I \cap C_0(G^0) \), where \( I \) is an ideal in \( C^*(G) \). Let \( D \) be a closed invariant subset. Then the map that sends \( D \) to the ideal \( \{ f \in C_0(G^0) : f|_D \equiv 0 \} \subseteq C_0(G^0) \) is a well defined injection. To see that this map is a surjection onto the set of ideals of the form \( I \cap C_0(G^0) \), let \( I \) be an ideal of \( C^*(G) \). Since the multiplication in \( C_0(G^0) \) is pointwise, the ideal \( I \cap C_0(G^0) \) has the form \( \{ f \in C_0(G^0) : f|_D \equiv 0 \} \) for some closed \( D \subseteq G^0 \). We show that \( D \) is invariant by establishing that its complement is invariant. Fix \( \gamma \in G \) such that \( s(\gamma) \notin D \), and \( f \in I \cap C_0(G^0) \) such that \( f(s(\gamma)) = 1 \). We must show that \( r(\gamma) \notin D \). Let \( B \) be an open bisection of \( G \) containing \( \gamma \), and \( h \) be a function supported on \( B \) such that \( h(\gamma) = 1 \). By Lemma 5.6, \( (h * f * h^*)(r(\gamma)) = |h(\gamma)|^2f(s(\gamma)) = 1 \), so \( r(\gamma) \notin D \). This proves our claim.

Now, it suffices to show that \( I \mapsto I \cap C_0(G^0) \) is a bijection if and only if \( G_D \) is topologically principal for each closed invariant \( D \subseteq G^0 \).

First, suppose that \( G_D \) is topologically principal for every closed invariant \( D \subseteq G^0 \). Fix an ideal \( I \) of \( C^*(G) \). Let \( J \) be the ideal of \( C^*(G) \) generated by \( I \cap C_0(G^0) \). Then \( J \subseteq I \). Since \( G \) is measurewise amenable, \( C^*(G) = C^*_e(G) \). Hence \( J \neq \{0\} \) by Proposition 5.5. We must show that \( J = I \).

Let \( J_0 := J \cap C_0(G^0) \), and let \( D := \{ u \in G^0 : f(u) = 0 \text{ for all } f \in J_0 \} \); so \( J_0 \) is the ideal \( \{ f \in C_0(G^0) : f(u) = 0 \text{ for all } u \in D \} \) of \( C_0(G^0) \). As above, \( D \) is a closed invariant subset of \( G^0 \). So [8, Remark 4.10] implies that restriction of functions induces an isomorphism \( C^*(G)/J \cong C^*(G_D) \), and this isomorphism carries \( I/J \) to an ideal of \( C^*(G_D) \) that has trivial intersection with \( C_0(D) \) by construction of \( J \). Corollary 5.3.21 of [8] implies that \( G_D \) is measurewise amenable, so Proposition 5.5 implies that \( I/J \) is trivial and hence \( I = J \) as required.

We prove the reverse implication by contrapositive. Suppose that there exists a closed invariant subset \( D \) of \( G^0 \) such that \( G_D \) is not topologically principal. Lemma 5.1 shows that \( \text{Iso}(G_D)^0 \neq D \). Let \( I(D) \) be the ideal of \( C^*(G) \) generated by \( \{ f \in C_e(G^0) : f|_D \equiv 0 \} \). Again by [8, Remark 4.10], restriction of functions induces an isomorphism \( \phi : C^*(G)/I(D) \to C^*(G_D) \). Proposition 5.5 applied to the groupoid \( G_D \) gives a nontrivial ideal \( J \) of \( C^*(G_D) \) such that \( J \cap C_e(D) = \{0\} \). Let \( q_D : C^*(G) \to C^*(G)/I(D) \) and \( q_J : C^*(G_D) \to C^*(G_D)/J \) be the quotient maps. Let \( K := \ker(q_J \circ \phi \circ q_D) \). That \( J \cap C_e(D) = \{0\} \) forces \( K \cap C_e(G^0) = C_0(D) \). That \( J \) is nontrivial implies that \( K \neq I(D) \). Since \( K \cap C_0(G^0) = I(D) \cap C_0(G^0) \), the result follows. \( \square \)
Remark 5.10. The hypothesis of measurewise amenability in Corollary 5.9 is required only to guarantee that $C^*\left(G_D\right) = C^*_r\left(G_D\right)$ for every closed invariant subset of $G^{(0)}$. So the theorem also holds under this formally weaker (but less checkable) hypothesis.

Recall that an étale groupoid $G$ is locally contracting if for every nonempty open subset $U$ of $G^{(0)}$, there exists an open subset $V$ of $U$ and an open bisection $B$ such that $\overline{V} \subseteq s(B)$ and $r(BV) \subseteq V$. [2] Definition 2.1. In the following corollary, we use Theorem 5.1 and Lemma 5.1 to strengthen [2] Proposition 2.4.

Corollary 5.11. Let $G$ be a second-countable, locally compact, Hausdorff groupoid. Suppose that $G$ is also locally contracting and étale, and that $C^*(G)$ is simple. Then $C^*(G)$ is purely infinite.

Proof. Theorem 5.1 implies that $G$ is topologically principal, so [2] Proposition 2.4 implies that every nonzero hereditary $*$-subalgebra of $C^*(G)$ contains an infinite projection. □

6. Examples

In this section, we present some examples to indicate why the hypotheses on our main theorem are needed. We also demonstrate that the final assertion of Lemma 5.1 fails if $G$ is either not second-countable or not étale.

Example 6.1 (Amenability). Theorem 5.1 cannot be strengthened to a characterisation of simplicity for $C^*_r(G)$ for locally compact, Hausdorff, étale groupoids: the free group $F_2$ on two generators, regarded as a discrete groupoid with just one unit, is a second countable, locally compact, Hausdorff, étale groupoid that is not topologically principal. However, Powers proved in [25] that $C^*_r(F_2)$ is simple.

Example 6.2 (Twisted groupoid algebras). Our characterisation of simplicity does not extend to groupoid $C^*$-algebras that are ‘twisted’ by a 2-cocycle, as defined in [28]. To see why, consider the group $\mathbb{Z}^2$ regarded as a discrete groupoid with one unit. This is a locally compact, Hausdorff, étale, amenable groupoid with $\text{Iso}(\mathbb{Z}^2) = \mathbb{Z}^2$, so our theorem reduces to the observation that $C^*(\mathbb{Z}^2) \cong C(\mathbb{T}^2)$ is not simple. To see that this does not extend to twisted algebras, fix $\theta \in [0,1] \setminus \mathbb{Q}$ and let $\phi_\theta : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{T}$ be the $\mathbb{T}$-valued 2-cocycle $\theta((m_1,m_2),(n_1,n_2)) = e^{i\theta(m_1n_2)}$. It is well known that the twisted groupoid $C^*$-algebra $C^*(\mathbb{Z}^2,\phi_\theta)$ is the irrational rotation algebra $A_\theta$ and hence simple.

In Section 5 we were able to replace the hypothesis that $G$ is effective, used in Section 4, with the more familiar hypothesis that it is topologically principal. The justification for this is [31] Proposition 3.6, which tells us that for second-countable, Hausdorff and étale groupoids, the two hypotheses are equivalent. One might ask whether the conditions are equivalent in general. The next example shows that for non-étale $G$, effectiveness does not entail being topologically principal.

Example 6.3. Let $X := (0,1) \times \mathbb{T}$. Define a continuous right action of $\mathbb{R}$ on $X$ by

$$(s, e^{i\theta}) \cdot t = (s, e^{i(\theta + 2\pi t)}).$$

Let $G$ be the transformation-group groupoid $X \rtimes \mathbb{R}$. For each $u = (s, e^{i\theta}) \in G^{(0)}$, the isotropy group is $G^u = \{u\} \times s\mathbb{Z}$, so no point in $G^{(0)}$ has trivial isotropy. Fix an open set $U$ in $G$. We must show that $U \setminus \text{Iso}(G) \neq \emptyset$. Since $U$ is open, there exist $0 < a < b < 1$, $\theta \in (0,2\pi)$, and $t \in \mathbb{R} \setminus \{0\}$ such that $((a,b) \times \{e^{i\theta}\}) \times \{t\} \subseteq U$. Fix $s \in (a,b)$. If
st \not\in \mathbb{Z} \text{ then } ((s, e^{i\theta}), t) \in U \setminus \text{Iso}(G). \text{ So suppose that } st \in \mathbb{Z}. \text{ Choose } \varepsilon \in (0, \frac{1}{t}) \text{ such that } s + \varepsilon \in (a, b). \text{ Then } st < (s + \varepsilon)t < st + 1, \text{ so } (s + \varepsilon)t \not\in \mathbb{Z}. \text{ Hence } ((s + \varepsilon, e^{i\theta}), t) \in U \setminus \text{Iso}(G).

Our next example is also effective without being topologically principal. This time \( G \) is étale and has totally disconnected unit space, but is not second-countable. This shows that Lemma 4.2 is strictly stronger than [7, Theorem 5.2].

**Example 6.4.** Let \( K \) denote the Cantor set and give \( \mathbb{T} \) the discrete topology. Let \( X \) be the topological product space \((K \cap (0, 1)) \times \mathbb{T}\). Define an (algebraic) action of \( \mathbb{R} \) on \( X \) by restriction of the action of Example 6.3. Endow the acting copy of \( \mathbb{R} \) with the discrete topology. Then the action is continuous and the transformation groupoid \( G \) is étale (but not second-countable). Moreover, every open subset of \( G \) which does not intersect \( G(0) \) contains a subset of the form \(( (K \cap (a, b)) \times \{e^{i\theta}\}) \times \{t\} \) as in Example 6.3 so arguing as in that example (using that \( K \cap (a, b) \) has no isolated points), we see that the interior of the isotropy subgroupoid is \( G(0) \).

In Examples 6.3 and 6.4, \( G(0) \) admits many nontrivial closed proper invariant subsets. We do not have an example of a locally compact, Hausdorff, étale, minimal, effective groupoid that is not topologically principal.

### 7. Exel-Vershik systems

When we first began trying to prove that \( A(G) \) is simple if and only if \( G \) is minimal and topologically principal, we went looking for examples — other than higher-rank graph groupoids — of étale groupoids with totally disconnected unit spaces to test the hypothesis. We were led to the work of Exel and Vershik in [12]. Their characterisation of simplicity [12, Theorem 11.2] led us to condition (3) of Lemma 3.1 and from there to our main simplicity theorems. In this section, we investigate the relationship between our result and that of Exel and Vershik. We obtain a generalisation of their simplicity theorem to a very broad class of dynamical systems.

Recall that an Ore semigroup is a monoid \( M \) which is cancellative and satisfies:

\[
\text{(6) for all } m, n \in M, \text{ there exist } p, q \in M \text{ such that } pm = qn.
\]

**Definition 7.1.** An Exel-Vershik system is a triple \((X, M, T)\) consisting of a second-countable, locally compact, Hausdorff space \( X \), a countable discrete Ore semigroup \( M \), and an action \( T \) of \( M \) on \( X \) by local homeomorphisms; we write \( T^m \) for the local homeomorphism associated to \( m \in M \).

**Remark 7.2.** Every commutative monoid and every group is an Ore semigroup: if \( M \) is commutative then \( nm = mn \) and if \( M \) is a group then \( m^{-1}m = n^{-1}n \). Indeed, a monoid \( M \) is an Ore semigroup if and only if there is an embedding of \( M \) in a group \( \Gamma = \Gamma(M) \) such that \( \Gamma = M^{-1}M \) (see, for example, [20, Theorem 1.2]). The group \( \Gamma \) is unique up to isomorphism and we call it the Grothendieck group of \( M \).

Let \((X, M, T)\) be an Exel-Vershik system and let \( \Gamma(M) \) be the Grothendieck group of \( M \). Consider the set

\[
G(X, T) := \{(x, m^{-1}n, y) \in X \times \Gamma(M) \times X : m, n \in M, T^m(x) = T^n(y)\}.
\]
Remark 7.2 and [11, Proposition 3.1] imply that the formulas
\[ r(x, m^{-1}n, y) = x \quad \text{and} \quad s(x, m^{-1}n, y) = y \]
\[ (x, m^{-1}n, y) \cdot (y, p^{-1}q, z) := (x, m^{-1}np^{-1}q, z) \quad \text{and} \quad (x, m^{-1}n, y)^{-1} = (y, n^{-1}m, x) \]
make \( G(X, T) \) into a groupoid.\footnote{Our convention for \( G(X, T) \) is slightly different than in [11] for compatibility with Example 7.3.2.}

For precompact open subsets \( U, V \) of \( X \) and \( m, n \in M \) such that \( T^m|_U \) and \( T^n|_V \) are homeomorphisms and \( T^m(U) = T^n(V) \), let
\[ Z(U, V, m, n) := \{(x, m^{-1}n, y) : x \in U, y \in V, T^m(x) = T^n(y)\}. \]
Then the sets \( Z(U, V, m, n) \) form a base of precompact open bisections for a second-countable topology on \( G(X, T) \). Under this topology, \( G(X, T) \) is a locally compact, Hausdorff, étale groupoid [11, Proposition 3.2].

Examples 7.3.
\( (1) \) If \( M = \mathbb{N} \) then \( G(X, T) \) is the Deaconu-Renault groupoid of the local homeomorphism \( T \) [8].
\( (2) \) Let \( M \) be a discrete group and suppose \( T \) is an action of \( M \) on \( X \). Then \( T^g \) is a homeomorphism for all \( g \in M \) so in particular a local homeomorphism. The Grothendieck group of \( M \) is \( M \). Further if \( T^m(x) = T^n(y) \) then \( x = T^{-1}m(n(y)) \). So
\[ G(X, T) = \{(T^g(y), g, y) : y \in X, g \in M\} \]
and for each basic open set \( Z(U, V, m, n) \), we have
\[ Z(U, V, m, n) = \{(T^{-1}n(y), m^{-1}n, y) : y \in V, T^{-1}n(y) \in U\} \]
\[ = \{(T^{-1}n(y), m^{-1}n, y) : y \in V \cap T^{-1}n(U)\}. \]
Thus the map \( (T^g(y), g, y) \mapsto (y, g) \) induces an isomorphism of \( G(X, T) \) with the transformation-group groupoid \( X \rtimes_T M \).

\( (3) \) Let \( \Lambda \) be a row-finite higher-rank graph with no sources as in [18]. Recall that \( \Lambda^\infty \) denotes the infinite-path space of \( \Lambda \) and that for \( n \in \mathbb{N}^k \) we write \( \sigma^n \) for the shift map \( \sigma^n(x)(p, q) = x(p+q, q+n) \) on \( \Lambda^\infty \). The groupoid \( G_\Lambda \) of [18] is then identical to the groupoid corresponding to the Exel-Vershik system \((\Lambda^\infty, \mathbb{N}^k, \sigma)\). Kumjian and Pask show that \( G_\Lambda \) is amenable in [18, Theorem 5.5].

The next definition is an extrapolation of [12, Definition 10.1] to arbitrary Exel-Vershik systems. This notion of topological freeness of \((X, M, T)\) is formally weaker than that of [1, Definition 1] when \( M \) is a countable discrete abelian group.

Definition 7.4. We say an Exel-Vershik system \((X, M, T)\) is topologically free if for every pair \( m \neq n \in M \) the set \( \{x \in X : T^m(x) = T^n(x)\} \) has empty interior.

Proposition 7.5. An Exel-Vershik system \((X, M, T)\) is topologically free if and only if the associated groupoid \( G(X, T) \) is topologically principal.

Proof. Suppose that \((X, M, T)\) is not topologically free. Then there exist \( m \neq n \in M \) and an open set \( U \subseteq X \) such \( T^m(x) = T^n(x) \) for all \( x \in U \). Fix \( z \in U \) and neighbourhoods \( W_m \) and \( W_n \) of \( z \) in \( U \) such that \( T^l|_{W_l} \) is a homeomorphism for \( l = m, n \). Define \( V = W_m \cap W_n \). Then \( z \in V \) and \( T^l|_V \) is a homeomorphism for both \( l = m, n \). Since \( T^m(x) = T^n(x) \) for all \( x \in V \subseteq U \) we have \( T^m(V) = T^n(V) \), so the set \( Z(V, V, m, n) \) is an open subset of \( \text{Iso}(G(X, T)) \setminus X \). Thus Lemma 3.3 implies that \( G(X, T) \) is not topologically principal.
Conversely, suppose that $G$ is not topologically principal. By Lemma 5.1 there exists an open bisection $B \subseteq G(X, T) \setminus X$ such that $r(\gamma) = s(\gamma)$ for all $\gamma \in B$. So there is a basic open set $Z(U, V, m, n)$ contained in $B$. That $B \subseteq G(X, T) \setminus X$ forces $m \neq n$. Since $Z(U, V, m, n) \subseteq B$ and $r(\gamma) = s(\gamma)$ for all $\gamma \in B$, we have $U = V$ and $T^m x = T^n x$ for all $x \in U$. So $(X, M, T)$ is not topologically free.

Remark 7.6. The special case of Example 7.3(1) where $X$ is a compact Hausdorff space and $T : X \to X$ a covering map was considered in [6]. Proposition 7.5 implies that $(X, M, T)$ is topologically principal, and so Proposition 5.5 recovers [6, Theorem 6 ((1) ⇔ (2))].

Remark 7.7. Recall from [11, Definition 4.3] that a row-finite higher-rank graph $\Lambda$ with no sources is aperiodic if for any $v \in \Lambda^0$ there exists $x \in \Lambda^\infty$ such that $r(x) = v$ and $\sigma^n(x) \neq \sigma^m(x)$ for all $m \neq n \in \mathbb{N}^k$. Recall also from [33, Definition 1] that $\Lambda$ has no local periodicity if for any $n \neq m \in \mathbb{N}^k$ and $v \in \Lambda^0$ there exists $x \in \Lambda^\infty$ such that $r(x) = v$ and $\sigma^n(x) \neq \sigma^m(x)$. Kumjian and Pask show that $\Lambda$ is aperiodic if and only if $G_\Lambda$ is topologically principal [11, Proposition 4.5]. A similar argument shows that $\Lambda$ has no local periodicity if and only if the Exel-Vershik system $(\Lambda^\infty, \mathbb{N}^k, \sigma)$ is topologically free. Thus Proposition 7.5 can be viewed as a generalisation of [33, Lemma 3.2].

Theorem 5.1 and Proposition 7.5 imply that if the full and reduced $C^*$-algebras of the groupoid $G(X, T)$ of an Exel-Vershik system $(X, M, T)$ coincide, then the associated $C^*$-algebra $C(X \rtimes_T \Gamma(M))$ is simple and if and only if the system is topologically free and for each $x \in X$ the orbit

$$[x]_T := \{y \in X : T^m y = T^n x \text{ for some } m, n \in M\}$$

is dense in $X$. It is therefore an interesting question whether $C^*(G(X, T)) = C^*_r(G(X, T))$ whenever $\Gamma(M)$ is amenable. We give a partial answer which applies to all systems for which Exel and Renault’s results guarantee that the Exel crossed product $C(X \rtimes_T \Gamma(M))$ of [9] coincides with $C^*(G(X, T))$.

Corollary 7.8. Suppose $M$ is an Ore semigroup such that $\Gamma(M)$ is amenable. Suppose that $(X, M, T)$ is an Exel-Vershik system satisfying the standing hypotheses 4.1 of [11]. Then $C^*(G(X, T)) = C^*_r(G(X, T))$. Moreover, $C(X \rtimes_T \Gamma(M))$ is simple if and only if the system is topologically free and $[x]_T = X$ for each $x \in X$.

Proof. The second assertion follows from Theorem 5.1 once we show that $C^*(G(X, T)) = C^*_r(G(X, T))$. For this let $\pi$ be the isomorphism $\pi : C(X \rtimes_T \Gamma(M)) \cong C^*(G(X, T))$ of [11, Theorem 6.6], and let $q : C^*(G(X, T)) \to C^*_r(G(X, T))$ be the quotient map. It suffices to show that $q \circ \pi$ is injective. For this, just run the proof of [11, Theorem 6.6] replacing $C^*(G(X, T))$ with $C^*_r(G(X, T))$. It is only necessary to check that $q \circ (\pi \times \sigma)$ is injective on each graded subspace, and for this the argument of [11, Proposition 6.5] suffices because the calculations in that proof involve elements of $C_*(G(X, T))$.

Amenability is irrelevant to the Steinberg algebras of Section 4.4. So Exel-Vershik systems $(X, M, T)$ where $X$ is a Cantor set should provide interesting examples of Steinberg algebras $A(G(X, T))$ for which simplicity is characterised by Theorem 4.4.

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