GLOBAL SOLUTIONS OF 3-D NAVIER-STOKES SYSTEM WITH SMALL UNIDIRECTIONAL DERIVATIVE

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Abstract. Given initial data \( u_0 = (u^h_0, u^3_0) \in H^{-\delta,0} \cap H^{\frac{1}{2}}(\mathbb{R}^3) \) with both \( u^h_0 \) and \( \nabla_h u^h_0 \) belonging to \( L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3; L^2(\mathbb{R}^2)) \) and \( u^3_0 \in L^\infty(\mathbb{R}^3; H^{-\delta}(\mathbb{R}^2)) \) for some \( \delta \in [0, 1] \), if in addition \( \partial_3 u_0 \) belongs to \( H^{-\frac{1}{2},0} \cap H^{\frac{1}{2},0}(\mathbb{R}^3) \), we prove that the classical 3-D Navier-Stokes system has a unique global Fujita-Kato solution provided that \( \|\partial_3 u_0\|_{H^{-\frac{1}{2},0}} \) is sufficiently small compared to a constant which depends only on the norms of the initial data. In particular, this result provides some classes of large initial data which generate unique global solutions to 3-D Navier-Stokes system.

Keywords: Navier-Stokes system, Anisotropic Littlewood-Paley theory, Slow variable

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1. Introduction

In this paper, we consider the following 3-D incompressible Navier-Stokes system:

\[
\begin{align*}
\partial_t u + u \cdot \nabla u - \Delta u &= -\nabla p, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\text{div} u &= 0, \\
u|_{t=0} &= u_0,
\end{align*}
\]

where \( u = (u^1, u^2, u^3) \) stands for the fluid velocity and \( p \) for the scalar pressure function, which guarantees the divergence free condition of the velocity field.

In 1933, Leray proved in the seminar paper [18] that given a finite energy initial data, \( u_0 \), \((NS)\) has a global in time weak solution \( u \) which verifies the energy inequality

\[
\frac{1}{2} \| u(t) \|_{L^2}^2 + \int_0^t \| \nabla u(t') \|_{L^2}^2 \, dt' \leq \frac{1}{2} \| u_0 \|_{L^2}^2.
\]

However, the regularity and uniqueness of such solutions is still one of the biggest open questions in the field of mathematical fluid mechanics except the case when the initial data have special structure. For instance, with axisymmetric initial velocity and without swirl component, Ladyzhenskaya [17] and independently Ukhovskii and Yudovich [23] proved the existence of weak solution along with the uniqueness and regularity of such solution to \((NS)\).

Fujita-Kato [13] constructed local in time unique solution to \((NS)\) with initial data in \( H^{\frac{1}{2}}(\mathbb{R}^3) \). Furthermore, if \( \| u_0 \|_{H^{\frac{1}{2}}} \) is sufficiently small, then the solution exists globally in time (see [14] for similar result with initial data in \( L^2(\mathbb{R}^3) \)). This result was extended by Cannone, Meyer and Planchon [11] for initial data belonging to \( B^{-1+\frac{1}{p}+\frac{1}{2}}_{p,\infty}(\mathbb{R}^3) \) with \( p \in ]3, \infty[ \). The end-point result in this direction is given by Koch and Tataru [16]. They proved that
given initial data being sufficiently small in BMO$^{-1}(\mathbb{R}^3)$, then (NS) has a unique global solution. We remark that for $p \in [3, \infty]$, there holds
\[ H^\frac{1}{2}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \hookrightarrow B_{p,\infty}^{-1+\frac{2}{p}}(\mathbb{R}^3) \hookrightarrow \text{BMO}^{-1}(\mathbb{R}^3) \hookrightarrow B_{\infty,\infty}^{-1}(\mathbb{R}^3), \]
and the norms to the above spaces are scaling-invariant under the following transformation:
\begin{equation}
(1.2) \quad u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x) \quad \text{and} \quad u_{0,\lambda}(x) = \lambda u_0(\lambda x).
\end{equation}
We notice that for any solution $u$ of (NS) on $[0, T]$, $u_\lambda$ determined by (1.2) is also a solution of (NS) on $[0, T/\lambda^2]$. We remark that the largest space, which belongs to $S'(\mathbb{R}^3)$ and the norm of which is scaling invariant under (1.2), is $B_{\infty,\infty}^{-1}(\mathbb{R}^3)$ (see [19]). Moreover, Bourgain and Pavlović [3] proved that (NS) is actually ill-posed with initial data in $B_{\infty,\infty}^{-1}(\mathbb{R}^3)$. That is the reason why we call such kind of initial data, which generates a unique global solution to (NS) and the $B_{\infty,\infty}^{-1}(\mathbb{R}^3)$ norm of which is large enough, as large initial data.

We now list some examples of large initial data which generate unique global solutions to (NS). First of all, for any initial data, Raugel and Sell [22] obtained the global well-posedness of (NS) in a thin enough domain. This result was extended by Ragnel, Sell and Iftimie in [12] that (NS) has a unique global periodic solution provided that the initial data $u_0$ can be split as $u_0 = v_0 + w_0$, with $v_0$ being a bi-dimensional solenoidal vector field in $L^2(T_h^2)$ and $w_0 \in H^\frac{1}{2}(T^3)$, such that
\[ \|w_0\|_{H^\frac{1}{2}(T^3)} \exp(\|u_0\|_{L^2(T_h^2)}^2) \] is sufficiently small.

Chemin and Gallagher [4] constructed another class of examples of initial data which is big in $B_{\infty,\infty}^{-1}(\mathbb{T}^3)$ and strongly oscillatory in one direction. More precisely, for any given positive integer $N_0$, there exists some $N_1$ such that for any integer $N > N_1$, (NS) has a unique global solution with initial data
\[ u_0^N = \left( N v_0^h(x_h) \cos(Nx_3), -\text{div}_h v_0^h(x_h) \sin(Nx_3) \right), \]
where $v_0^h$ is any bi-dimensional solenoidal field on $\mathbb{T}^2$ with
\[ \text{Supp} \; \hat{v}_0^h \subset [-N_0, N_0]^2 \quad \text{and} \quad \|v_0^h\|_{L^2(\mathbb{T}^2)} \leq C(\ln N)^\frac{1}{2}. \]

The other interesting class of large initial data which can generate unique global solutions to (NS) is the so-called slowly varying data, which was introduced by Chemin and Gallagher in [5] (see also [6, 9]).

**Theorem 1.1** (Theorem 3 of [5]). Let $v_0^h = (v_{01}^h, v_{02}^h)$ be a horizontal, smooth divergence free vector field on $\mathbb{R}^3$ such that it and all its derivatives belong to $L^2(\mathbb{R}^3) \cap L^2(\mathbb{R}^3; H^{-1}(\mathbb{R}^2))$; let $w_0$ be a smooth divergence free vector field on $\mathbb{R}^3$. Then there exists a positive constant $\varepsilon_0$ such that if $\varepsilon \leq \varepsilon_0$, then the initial data
\begin{equation}
(1.3) \quad u_\varepsilon(x) = (v_{01}^h + \varepsilon w_{01}^h, v_{02}^h, w_{03}^h)(x_h, \varepsilon x_3)
\end{equation}
generates a unique, global smooth solution $u_\varepsilon$ of (NS).

On the other hand, Kukavica and Ziane proved in [15] that if
\[ \int_0^{T^*} \|\partial_3 u(t, \cdot)\|_{L^q}^q dt < \infty \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} = 2 \quad \text{and} \quad p \in ]3/4, 3[, \]
then local smooth solution of (NS) can be extended beyond time $T^*$. 

Motivated by [5] and [15], we are going to study the global well-posedness of (NS) with initial data \( u_0 \) satisfying \( \partial_3 u_0 \) being sufficiently small in some critical spaces. Before we present the main result, let us recall the definition of anisotropic Sobolev space \( H^{s,s'}(\mathbb{R}^3) \).

**Definition 1.1.** For \((s,s')\) in \(\mathbb{R}^2\), \(H^{s,s'}(\mathbb{R}^3)\) denotes the space of homogeneous tempered distribution \(a\) such that
\[
\|a\|^{2}_{H^{s,s'}} \overset{def}{=} \int_{\mathbb{R}^3} |\xi_h|^{2s} |\xi_3|^{2s'} |\hat{a}(\xi)|^2 d\xi < \infty \quad \text{with} \quad \xi_h = (\xi_1, \xi_2, \text{and} \xi_3 = (\xi_h, \xi_3).}
\]

The main result of this paper states as follows.

**Theorem 1.2.** Let \( \delta \in ]0,1[\), \( u_0 = (u^h_0, u^3_0) \in H^{-\delta,0} \cap H^1(\mathbb{R}^3) \) with both \( u^h_0 \) and \( \nabla_h u^h_0 \) belonging to \( L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3; L^2(\mathbb{R}^3)) \) and \( u^h_0 \in L^\infty(\mathbb{R}^3; H^{-\delta}(\mathbb{R}^3)) \). If we assume in addition that \( \partial_3 u_0 \in H^{-\frac{3}{2},0} \cap H^\frac{3}{2,0}(\mathbb{R}^3) \), then there exists a small enough positive constant \( \epsilon_0 \) such that if
\[
\|\partial_3 u_0\|^2_{H^{-\frac{3}{2},0}} \exp \left( C(A_\delta(u^h_0) + B_\delta(u_0)) \right) \leq \epsilon_0,
\]
where
\[
A_\delta(u^h_0) \overset{def}{=} \left( \frac{\|\nabla_h u^h_0\|^2_{L^\infty(L^2)}}{\|u^h_0\|^2_{L^\infty(L^2)}} + \|u^h_0\|^2_{L^\infty(L^2)} \right) \exp \left( C(1 + \|u^h_0\|^4_{L^\infty(L^2)}) \right),
\]
and
\[
B_\delta(u_0) \overset{def}{=} \|u_0\|_{H^{-\delta,0}} \|u_0\|_{H^{\delta,0}} \|\partial_3 u_0\|_{H^{-\frac{3}{2},0}} \|\partial_3 u_0\|_{H^{-\frac{1}{2},0}} \exp \left( CA_\delta(u^h_0) \right),
\]
(\(\text{NS}\)) has a unique global solution \( u \in C(\mathbb{R}^+; H^\frac{3}{2}) \cap L^2(\mathbb{R}^+; H^\frac{3}{2}) \).

Let us present some examples of initial data the norm of which are big in \( B^{-1,\infty}_{\infty,\infty}(\mathbb{R}^3) \), yet they satisfy the smallness condition (1.4).

1. The first class of example is the slow variable data given by (1.3). It is easy to observe that for any \( \epsilon \in ]0,1[\), both \( A_\delta(u^{\epsilon, h}_0) \) and \( B_\delta(u^{\epsilon}_0) \) have uniform upper bounds which are independent of \( \epsilon \). Whereas it follows from trivial calculation that
\[
\|\partial_3 u_0^{\epsilon}\|_{H^{-\frac{3}{2},0}} \leq C\epsilon \left( \|\partial_3 u^h_0\|_{H^{-\frac{3}{2},0}} + \epsilon \|\partial_3 u^h_0\|_{H^{-\frac{1}{2},0}} + \|\partial_3 u^h_0\|_{H^{-\frac{1}{2},0}} \right).
\]
So that as long as \( \epsilon \) is sufficiently small, the slow variable data (1.3) satisfies (1.4). Hence Theorem 1.1 is a direct corollary of Theorem 1.2.

2. Motivated by (1.3), for any positive integer \( N \) and \((v_k, w_k), k = 1, \ldots, N\), satisfying the assumptions of Theorem 1.1, we construct initial data which is a linear combination of \( N \) slowly varying parts as follows
\[
u^{(\epsilon_1, \ldots, \epsilon_N)}_0(x) = \sum_{k=1}^N (v^h_k + \epsilon_k w^h_k, w^3_k)(x, x, x_3).
\]
It is easy to check that \( u^{(\epsilon_1, \ldots, \epsilon_N)}_0 \) satisfies the smallness condition (1.4) provided \( \epsilon_1, \ldots, \epsilon_N \) are sufficiently small.

3. We can also consider the initial data which is slowly varying in one variable but fast varying in another variable. For instance,
\[
u^{(\epsilon, \lambda)}_0(x) = (v^1_0 + \epsilon w^1_0, \lambda v^2_0 + \lambda \epsilon w^2_0, \lambda w^3_0)(\lambda x_1, x_2, \epsilon x_3).
\]
Then it follows from Sobolev inequality that
\[
\|\partial_3 u_0^{(\varepsilon, \lambda)}\|_{H^{-\frac{1}{2},0}} \leq C\|\partial_3 u_0^{(\varepsilon, \lambda)}\|_{L^2_0(L^2)} \\
\leq C\lambda^{\frac{1}{2}}\varepsilon\|\nabla(u_0^{h(\varepsilon)}, \partial_3 u_0)\|_{L^2_0(L^2)}.
\]

On the other hand, we have
\[
A_\delta(u_0^{(\varepsilon, \lambda), h}) \leq C\exp(C\lambda^2), \quad \text{and} \quad B_\delta(u_0^{(\varepsilon, \lambda)}) \leq C\exp(C\exp(C\lambda^2)).
\]

Therefore as long as \(\varepsilon\) and \(\lambda\) verify the condition that
\[
\varepsilon^{\frac{1}{2}}\exp(C\exp(\exp(\lambda^2)))
\]
is still sufficiently small, \(u_0^{(\varepsilon, \lambda)}(x)\) satisfies the smallness condition (1.4).

(4) The other class of examples can be obtained by cutting the horizontal frequency of the initial data. For instance, given smooth solenoidal vector field \(u_0\) with
\[
\text{Supp } \hat{u}_0 \subset \{\xi = (\xi_h, \xi_3) \in \mathbb{R}^3 : |\xi_h| > R\}
\]
for some positive constant \(R\), we find
\[
\|\partial_3 u_0\|_{H^{-\frac{1}{2},0}} \sim R^{-\frac{1}{2}}.
\]

Thus \(u_0\) satisfies the smallness condition (1.4) provided \(R\) is sufficiently large.

We refer Proposition 1.1 of [5] for the calculation of the \(B^{-1}_{\infty, \infty}(\mathbb{R}^3)\) norm to the data given by (1.3), (1.7) and (1.8).

Let us end this section with the notations we shall use in this context.

\textbf{Notations:} We designate the \(L^2\) inner product of \(f\) and \(g\) by \((f|g)_{L^2}\). We shall always denote \(C\) to be a uniform constant which may vary from line to line. \(a \leq b\) means that \(a \leq Cb\). \((d_j)_{j \in \mathbb{Z}}\) (resp. \((d_{k,\ell})_{(k,\ell) \in \mathbb{Z}^2}\) and \((e_j)_{j \in \mathbb{Z}}\) stands for a generic element on the unit sphere of \(l^1(\mathbb{Z})\) (resp. \(l^1(\mathbb{Z}^2)\) and \(l^2(\mathbb{Z})\)) so that \(\sum_{j \in \mathbb{Z}} d_j = 1\) (resp. \(\sum_{(k,\ell) \in \mathbb{Z}^2} d_{k,\ell} = 1\) and \(\sum_{j \in \mathbb{Z}} e_j^2 = 1\)).

2. Ideas and Structure of the Proof

In this section, we shall sketch the main ideas of the proof to Theorem 1.2. Given initial data \(u_0\) satisfying the assumptions of Theorem 1.2, classical Fujita-Kato theorem ([13]) ensures that \((NS)\) has a unique solution
\[
(2.1) \quad u \in \mathcal{E}_{T^*} \stackrel{\text{def}}{=} C([0, T^*]; H^{\frac{1}{2}}(\mathbb{R}^3)) \cap L^2_{\text{loc}}([0, T^*]; H^{\frac{3}{2}}(\mathbb{R}^3)),
\]
where \(T^*\) is the maximal existence time of this solution. The goal of this paper is to prove that \(T^* = \infty\) under the smallness condition (1.4).

We first remark that the key ingredient used in [5, 6, 9] is that with a slow variable for the solution of \((NS)\), one can decompose it as a sum of a large two-dimensional solution of \((NS)\) with a parameter and a small three-dimensional one. Here the assumption in (1.4) motivates us to expect that \(\partial_3 u(t, x)\) should be small in some sense and therefore the convection term, \(u \cdot \nabla u\), in \((NS)\) can be approximated by \(u^h \cdot \nabla_h u\), that is, under the smallness condition (1.4), we may approximate \((NS)\) by
\[
(2.2) \quad \partial_t u^h + \bar{u}^h \cdot \nabla_h u^h - \Delta u^h = -\nabla_h \bar{p}, \quad \text{div}_h u^h = 0.
\]
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However, we can not simply set the initial data \( \vec{u}_0 \) to be \( u_0^{\text{h}} \), which does not satisfy the horizontal divergence free condition. To overcome this difficulty, let us recall the Biot-Sarvart’s law for a 2-D vector field \( u^\text{h} \):

\[
(2.3) \quad u^\text{h} = u^\text{h}_{\text{curl}} + u^\text{h}_{\text{div}}, \quad \text{where} \quad u^\text{h}_{\text{curl}} \overset{\text{def}}{=} \nabla^\bot_h \Delta^{-1}_h (\text{curl}_h u^\text{h}), \quad u^\text{h}_{\text{div}} \overset{\text{def}}{=} \nabla_h \Delta^{-1}_h (\text{div}_h u^\text{h}).
\]

Here and in the sequel, we always denote \( \text{curl}_h u^\text{h} \overset{\text{def}}{=} \partial_1 u^2 - \partial_2 u^1 \), \( \text{div}_h u^\text{h} \overset{\text{def}}{=} \partial_1 u^1 + \partial_2 u^2 \), and \( \nabla_h \overset{\text{def}}{=} (\partial_1, \partial_2), \quad \nabla^\bot_h \overset{\text{def}}{=} - (\partial_2, \partial_1), \quad \Delta_h \overset{\text{def}}{=} \partial^2_1 + \partial^2_2 \). It is easy to observe that \( \text{div}_h u^\text{h}_{\text{curl}} = 0 \) and \( \text{curl}_h u^\text{h}_{\text{div}} = 0 \). This motivates us to define \((\vec{u}^\text{h}, \vec{p})\) via

\[
\begin{cases}
\partial_t \vec{u}^\text{h} + \vec{u}^\text{h} \cdot \nabla_h \vec{u}^\text{h} - \Delta \vec{u}^\text{h} = -\nabla_h \vec{p}, \\
\text{div}_h \vec{u}^\text{h} = 0, \\
\vec{u}^\text{h}|_{t=0} = \vec{u}_0^\text{h} = u^\text{h}_{\text{curl}}|_{t=0} = \nabla^\bot_h \Delta^{-1}_h (\text{curl}_h u^\text{h}_0).
\end{cases}
\]

(2.4) seemingly looks like \((NS2.5D)\) in [9], whose solution has been used to approximate the true solution of 3-D Navier-Stokes equations with a slow variable. We remark that the system \((2.4)\) here is simpler due to the fact that there is no small parameter in the front of \( \partial^2_3 \) in \((2.4)\). For this system, we deduce from the proof of Theorem 1.2 of [9] that:

**Theorem 2.1.** Let \( u_0^\text{h} \) and \( \nabla_h u_0^\text{h} \) be in \( L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}_v; L^2(\mathbb{R}_x^3)) \). Then \( \vec{u}_0 \) generates a unique global solution to \((2.4)\) in the space \( L^\infty(\mathbb{R}_t^+; L^2(\mathbb{R}_x^3) \cap L^\infty(\mathbb{R}_v; H^1(\mathbb{R}_x^3))) \cap L^2(\mathbb{R}_t^+; H^1(\mathbb{R}_x^3)) \).

Moreover, if in addition \( u_0^\text{h} \) belongs to \( L^\infty(\mathbb{R}_v; H^{-\delta}(\mathbb{R}_x^3)) \) for some \( \delta \in [0, 1] \), then we have

\[
(2.5) \quad \int_0^\infty \| \nabla_h u^\text{h}(t) \|_{L^\infty(L^2_x)}^2 \, dt \leq A_\delta(u_0^\text{h}),
\]

where \( A_\delta(u_0^\text{h}) \) is determined by \((1.5)\).

Concerning the system \((2.4)\), we have the following proposition:

**Proposition 2.1.** Under the assumptions of Theorem 2.1, if in addition both \( u_0^\text{h} \) and \( \partial_3 u_0^\text{h} \) belong to \( B^{\sigma_1, 0} \) for some \( \sigma_1 \in ]-1, 1[ \), \((2.4)\) has a unique global solution,

\[
(2.6) \quad \vec{u}^\text{h} \in \mathcal{E} \overset{\text{def}}{=} L^\infty(\mathbb{R}_t^+; L^2(\mathbb{R}_x^3) \cap L^\infty(\mathbb{R}_v; H^1(\mathbb{R}_x^3))) \cap L^2(\mathbb{R}_t^+; H^1(\mathbb{R}_x^3) \cap L^\infty(\mathbb{R}_v; H^1(\mathbb{R}_x^3)))
\]

so that for any \( t > 0 \) and for any \( \sigma_2 \in ]0, 1[ \), we have

\[
(2.7) \quad \| \vec{u}^\text{h} \|_{L^\infty_t(B^{\sigma_2, 0}_{2, 1} \cap B^{\sigma_2, 0}_{\infty, 2})} + \| \nabla \vec{u}^\text{h} \|_{L^2_t(B^{\sigma_2, 0}_{2, 1} \cap B^{\sigma_2, 0}_{\infty, 2})} \leq C \| u_0^\text{h} \|_{B^{\sigma_2, 0}}^{1-\sigma_2} \| \partial_3 u_0^\text{h} \|_{B^{\sigma_2, 0}}^{\sigma_2} \exp(\overline{C} A_\delta(u_0^\text{h})),
\]

and

\[
(2.8) \quad \| \partial_3 \vec{u}^\text{h} \|_{L^2(H^{\sigma_1, 0})} + \| \nabla \partial_3 \vec{u}^\text{h} \|_{L^2(H^{\sigma_1, 0})} \leq C \| \partial_3 u_0^\text{h} \|_{H^{\sigma_1, 0}} \exp(\overline{C} A_\delta(u_0^\text{h})),
\]

for \( A_\delta(u_0^\text{h}) \) being determined by \((1.5)\).

The definitions of the functional spaces will be presented in Section 3.

It is easy to check that the system satisfies the difference \( u - \vec{u} \), where \( \vec{u} = (\vec{u}^\text{h}, 0) \) contains quadric term \( \vec{u}^\text{h} \cdot \nabla_h u^3 \), which is not small. The way to round this difficulty is to introduce a correction velocity, \( \vec{u} \), to be determined by

\[
(2.9) \quad \begin{cases}
\partial_t \vec{u} + \vec{u} \cdot \nabla_h \vec{u} - \Delta \vec{u} = -\nabla_h \vec{p}, \\
\text{div} \vec{u} = 0, \\
\vec{u}_0^h = u_0^h, \quad \vec{u}^3|_{t=0} = \vec{u}_0^3 = \vec{u}_0^3 = u_0^3 = 0.
\end{cases}
\]

This is just for convenience of notations, and one should keep in mind that \( \vec{u}^3 = 0 \).
We emphasize that it is crucial to prove that under the smallness condition (1.4), \( \tilde{u}^h \) is indeed small in some critical spaces. The main difficulty lies in the estimate of the pressure term \( \nabla h \tilde{p} \). As a matter of fact, by taking space divergence to (2.9) and using the condition that \( \text{div}_h \tilde{u}^h = \text{div} \tilde{u} = 0 \), we obtain

\[
- \Delta \tilde{p} = \text{div}_h \left( \tilde{u}^h \cdot \nabla h \tilde{u}^h + \tilde{u}^3 \partial_3 \tilde{u}^h \right).
\]

We decompose the pressure \( \tilde{p} \) into \( \tilde{p}_1 + \tilde{p}_2 \) with

\[
\tilde{p}_1 \overset{\text{def}}{=} (-\Delta)^{-1} \text{div}_h \left( \tilde{u}^h \cdot \nabla h \tilde{u}^h \right) \quad \text{and} \quad \tilde{p}_2 \overset{\text{def}}{=} (-\Delta)^{-1} \text{div}_h \left( \tilde{u}^3 \partial_3 \tilde{u}^h \right).
\]

In particular, with \( \partial_3 \tilde{u}^h \) being sufficiently small, we can prove that \( \nabla h \tilde{p} \) is indeed small in the case when \( \tilde{u}^h \) is small. Therefore, we can propagate the smallness condition for \( \tilde{u}^h(t) \) for \( t > 0 \).

Concerning the linear system (2.9), we have the following \textit{a priori} estimates:

**Proposition 2.2.** Let \( \tilde{u}^h \) be the global solution of (2.4) determined by Proposition 2.1. Let \( \bar{u} \) be a smooth enough solution of (2.9). Then for any \( \sigma_1 \in [-1, 1], \sigma_2 \in [0, 1] \), and for any \( t > 0 \), we have

\[
\| \tilde{u} \|_{L^\infty_t(B_{\sigma_1}^\sigma_2)} + \| \nabla \tilde{u} \|_{L^2_t(B_{\sigma_1}^\sigma_2)} \leq C \| u_0 \|_{L^1_{B_{\sigma_1}^\sigma_2}} \| \partial_3 u_0 \|_{H^1_{B_{\sigma_1}^\sigma_2}} \exp \big( C A_{\delta} (u^h_0) \big),
\]

(2.11)

\[
\| \partial_3 \tilde{u} \|_{L^\infty_t(H^{\sigma_1,0})} + \| \nabla \partial_3 \tilde{u} \|_{L^2_t(H^{\sigma_1,0})} \leq C \| \partial_3 u_0 \|_{H^{\sigma_1,0}} \exp \big( C A_{\delta} (u^h_0) \big),
\]

(2.12)

\[
\| \tilde{u}^h \|_{L^\infty_t(H^{\sigma_2,0})} + \| \nabla \tilde{u}^h \|_{L^2_t(H^{\sigma_2,0})} \leq C \| \partial_3 u_0 \|_{H^{\sigma_2,0}} \exp \big( C A_{\delta} (u^h_0) \big).
\]

(2.13)

The proofs of the above two propositions will be presented in Section 4.

With \( \bar{u} \) and \( \tilde{u}^h \) being determined respectively by the systems (2.4) and (2.9), we can split the solution \( (u, p) \) of (NS) as

\[
u = \bar{u} + \tilde{u} + v, \quad p = \tilde{p} + \bar{p} + q,
\]

(2.14)

It is easy to verify that the remainder term \( (v, q) \) satisfies

\[
\left\{ \begin{array}{l}
\partial_t v^3 + (v + \bar{u}) \cdot \nabla v^3 + \tilde{u}^h \cdot \nabla v^3 - \Delta v^3 = -\nabla h q, \\
\partial_t v^1 + (v + \bar{u}) \cdot v^1 + \tilde{u}^h \cdot v^1 - \Delta v^1 = -\partial_3 (p - \bar{p}), \\
\text{div } v = 0, \\
v|_{t=0} = 0.
\end{array} \right.
\]

(2.15)

Let \( \omega \overset{\text{def}}{=} \text{curl}_h \tilde{u}^h \). Motivated by \cite{10}, we can equivalently reformulate (2.15) as

\[
\left\{ \begin{array}{l}
\partial_t \omega + (v + \bar{u}) \cdot \nabla \Omega^h + \tilde{u}^h \cdot \nabla \omega - \Delta \omega \\
= \Omega^h \partial_3 u^3 + \partial_3 u^3 \partial_3 u^1 - \partial_1 u^3 \partial_3 u^2 + \partial_2 u^h \cdot \nabla h \tilde{u}^1 - \partial_1 \tilde{u}^h \cdot \nabla h \tilde{u}^2,
\end{array} \right.
\]

\[
\left\{ \begin{array}{l}
\partial_t v^3 + (v + \bar{u}) \cdot v^3 + \tilde{u}^h \cdot v^3 - \Delta v^3 \\
= -\partial_3 \Delta^{-1} \left( \sum_{\ell,m=1}^{3} \partial_\ell \tilde{u}^m \partial_\ell u^\ell - \sum_{\ell,m=1}^{3} \partial_\ell \tilde{u}^m \partial_\ell \tilde{u}^\ell \right),
\end{array} \right.
\]

\[
\omega|_{t=0} = 0, \quad v^3|_{t=0} = 0,
\]

(2.16)

where \( \Omega^h \overset{\text{def}}{=} \text{curl}_h u^h = \omega + \tilde{\omega} + \bar{\omega} \) and \( \tilde{\omega} \overset{\text{def}}{=} \text{curl}_h \tilde{u}^h \).
Notice that the right-hand side of the $\omega$ equation in (2.16) contains terms either with $\partial_3 u$ or $\tilde{u}^h$, which are small according to Propositions 2.1 and 2.2. Similar observation holds for the source terms in the $v^3$ equation of (2.16). Therefore, we have reason to expect that both $\omega$ and $v^3$ can exist and keep being small in some critical spaces for all time. To rigorously justify this expectation, we deduce from the discussions at the beginning of this section and Propositions 2.1 and 2.2 that (2.15) also has a unique maximal solution $v \in \mathcal{E}_{T^*}$, which is determined by (2.1).

Let us denote

$$
(2.17) \quad M(t) \overset{\text{def}}{=} \|\nabla v^3(t)\|_{H^{-\frac{1}{2},0}}^2 + \|\omega(t)\|_{H^{-\frac{1}{2},0}}^2, \quad N(t) \overset{\text{def}}{=} \|\nabla^2 v^3(t)\|_{H^{-\frac{1}{2},0}}^2 + \|\nabla\omega(t)\|_{H^{-\frac{1}{2},0}}^2.
$$

We shall prove the following propositions in Section 5.

**Proposition 2.3.** For any $t < T^*$, the maximal solution $v \in \mathcal{E}_{T^*}$ of (2.15) satisfies

$$
(2.18) \quad \frac{d}{dt}\|\nabla v^3\|_{H^{-\frac{1}{2},0}}^2 + 2\|\nabla^2 v^3\|_{H^{-\frac{1}{2},0}}^2 \leq \left(\frac{1}{4} + CM^\frac{1}{2}\right)N + CM\left(\|\nabla \tilde{u}^h\|_{B_{2,1}^{0,\frac{1}{2}}}^2 + \|\nabla \tilde{u}^h\|_{B_{2,1}^{0,1}}^2\right)
$$

$$
+ C\|\tilde{u}^h\|_{H^\frac{1}{2},0}^2\|\nabla \tilde{v}^3\|_{B_{2,1}^{0,1}}^2 + C\left(\|\nabla_3 \tilde{u}^h\|_{H^{-\frac{1}{2},0}}^2 + \|\nabla_3 \tilde{u}^h\|_{H^{-\frac{1}{2},0}}^2\right)\left(\|\tilde{u}^h\|_{B_{2,1}^{0,\frac{1}{2}}}^2 + \|\tilde{u}^h\|_{B_{2,1}^{0,1}}^2\right),
$$

and

$$
(2.19) \quad \frac{d}{dt}\|\omega\|_{H^{-\frac{1}{2},0}}^2 + 2\|\nabla\omega\|_{H^{-\frac{1}{2},0}}^2 \leq \left(\frac{1}{4} + CM^\frac{1}{2} + M\right)N + CM\left(\|\nabla \tilde{u}^h\|_{B_{2,1}^{0,\frac{1}{2}}}^2 + \|\nabla \tilde{u}^h\|_{B_{2,1}^{0,1}}^2\right)
$$

$$
+ C\|\nabla \tilde{u}^h\|_{H^\frac{1}{2},0}^2 + C\left(\|\nabla_3 \tilde{u}^h\|_{H^{-\frac{1}{2},0}}^2 + \|\nabla_3 \tilde{u}^h\|_{H^{-\frac{1}{2},0}}^2\right)\left(1 + \|\tilde{u}^h\|_{B_{2,1}^{0,1}}^2 + \|\tilde{u}^h\|_{B_{2,1}^{0,\frac{1}{2}}}^2\right).
$$

In Section 6, we shall conclude the proof of Theorem 1.2. The strategy of the proof is as follows.

**Proposition 2.4.** Under the assumptions of Theorem 1.2, there exists some positive constant $\eta$ such that

$$
(2.20) \quad \sup_{t \in [0,T^*]} \left(\int_{0}^{t} M(t') dt' + \int_{0}^{t} N(t') dt'\right) \leq \eta.
$$

With this estimate at hand, we then appeal to the following regularity criteria for the local Fujita-Kato solution of (NS):

**Theorem 2.2** (Theorem 1.5 of [10]). Let $u$ be a solution of (NS) in the space $\mathcal{E}_{T^*}$. If the maximal existence time $T^*$ is finite, then for any $(p_{i,j})$ in $[1, \infty]^9$, one has

$$
(2.21) \quad \sum_{1 \leq i,j \leq 3} \int_{0}^{T^*} \|\partial_t u^j(t)\|_{B_{\infty, \infty}^{2,1}}^{p_{i,j}} dt = \infty.
$$

3. **Anisotropic Littlewood-Paley Theory**

In this section, we shall collect some basic facts on anisotropic Littlewood-Paley theory from [2]. Let us first recall the following anisotropic dyadic operators:

$$
(3.1) \quad \Delta_k^h a = F^{-1}(\varphi(2^{-k}|\xi_h|)\hat{a}(\xi)), \quad \Delta_k^\chi a = F^{-1}(\varphi(2^{-t} |\xi_3|)\hat{a}(\xi)),
$$

$$
S_k^h a = F^{-1}(\chi(2^{-k}|\xi_h|)\hat{a}(\xi)), \quad S_k^\chi a = F^{-1}(\chi(2^{-t} |\xi_3|)\hat{a}(\xi)),
$$
where $\xi = (\xi_h, \xi_3)$ and $\xi_h = (\xi_1, \xi_2)$, $\mathcal{F}a$ and $\hat{a}$ denote the Fourier transform of $a$, while $\mathcal{F}^{-1}a$ denotes its inverse Fourier transform, $\chi(\tau)$ and $\varphi(\tau)$ are smooth functions such that

$$\text{Supp } \varphi \subset \left\{ \tau \in \mathbb{R} : \frac{3}{4} \leq |\tau| \leq \frac{8}{3} \right\} \quad \text{and} \quad \forall \tau > 0, \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\tau) = 1,$$

$$\text{Supp } \chi \subset \left\{ \tau \in \mathbb{R} : |\tau| \leq \frac{4}{3} \right\} \quad \text{and} \quad \forall \tau \in \mathbb{R}, \chi(\tau) + \sum_{j \geq 0} \varphi(2^{-j}\tau) = 1.$$ 

Let us recall the anisotropic Bernstein inequalities from [8, 20].

**Lemma 3.1.** Let $B_h$ (resp. $B_v$) a ball of $\mathbb{R}_h^2$ (resp. $\mathbb{R}_v$), and $C_h$ (resp. $C_v$) a ring of $\mathbb{R}_h^2$ (resp. $\mathbb{R}_v$); let $1 \leq p_2 \leq p_1 \leq \infty$ and $1 \leq q_2 \leq q_1 \leq \infty$. Then there holds:

1. If $\text{Supp } \hat{a} \subset 2^k B_h \Rightarrow \|\partial^\alpha_x a\|_{L_p^1(L_t^{q_1})} \lesssim 2^{k(|\alpha|+2(1/p_2-1/p_1))}\|a\|_{L_{q_2}^2(L_t^{q_1})}$;
2. If $\text{Supp } \hat{a} \subset 2^k C_h \Rightarrow \|\partial^\alpha_x a\|_{L_p^1(L_t^{q_1})} \lesssim 2^{-(1/2)^{1/q_1}}\|\partial^\alpha_x a\|_{L_{q_2}^1(L_t^{q_1})}$;
3. If $\text{Supp } \hat{a} \subset 2^k C_v \Rightarrow \|a\|_{L_p^1(L_t^{q_1})} \lesssim 2^{-kN} \sup_{|\alpha|=N} \|\partial^\alpha_x a\|_{L_{q_2}^1(L_t^{q_1})}$;
4. If $\text{Supp } \hat{a} \subset 2^k C_v \Rightarrow \|a\|_{L_p^1(L_t^{q_1})} \lesssim 2^{-\ell N} \|\partial^\alpha x a\|_{L_{q_2}^1(L_t^{q_1})}$.

**Definition 3.1.** Let us define the anisotropic Besov space $B_{p,q}^{s_1,s_2}$ (with usual adaptation when $q$ equal $\infty$) as the space of homogenous tempered distributions $u$ so that

$$\|u\|_{B_{p,q}^{s_1,s_2}} \stackrel{\text{def}}{=} \left( \sum_{k \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} 2^{qks_1}2^{q\ell s_2}\|\Delta^h_{k}\Delta^v_{\ell}u\|_{L_p^q}^q \right)^{1/q} < \infty.$$

We remark that $B_{2,2}^{s_1,s_2}$ coincides with the classical anisotropic Sobolev space $H^{s_1,s_2}$ given by Definition 1.1.

**Definition 3.2.** Let $s \in \mathbb{R}$, $p \in [1, \infty]$ and $a \in S'_0(\mathbb{R}^3)$, we define

$$\|a\|_{B^{s,0}} \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}} 2^{ks}\|\Delta^h_k a\|_{L^2},$$

and the corresponding Chemin-Lerner type norm (see [7])

$$\|a\|_{\mathcal{F}B^{s,0}} \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}} 2^{ks}\|\Delta^h_k a\|_{L^p_{1};(L^2)}.$$

**Remark 3.1.** For any $a \in H^{s,0}$, we deduce from Fourier-Plancherel inequality that

$$\|\Delta^h_k a\|_{L^2} = \left( \sum_{\ell \in \mathbb{Z}} \|\Delta^h_k\Delta^v_{\ell}a\|_{L^2}^2 \right)^{1/2} \lesssim \left( \sum_{\ell \in \mathbb{Z}} c_{k,\ell}^22^{-2ks} \right)^{1/2}\|a\|_{H^{s,0}} \lesssim c_k 2^{-ks}\|a\|_{H^{s,0}},$$

where $(c_k,\ell)_{(k,\ell) \in \mathbb{Z}^2}$ (resp. $(c_k)_{k \in \mathbb{Z}}$) is a generic element of $\ell^2(\mathbb{Z}^2)$ (resp. $\ell^2(\mathbb{Z})$) so that

$$\sum_{(k,\ell) \in \mathbb{Z}^2} c_{k,\ell}^2 = 1 \quad \text{and} \quad \sum_{k \in \mathbb{Z}} c_k^2 = 1.$$


Then for any \( s \in [s_1, s_2] \) and for any integer \( N \), we find
\[
\|a\|_{B^{s,0}} = \sum_{k \leq N} 2^{ks} \|\Delta_k^h a\|_{L^2} + \sum_{k > N} 2^{ks} \|\Delta_k^h a\|_{L^2} \\
\leq \sum_{k \leq N} c_k 2^k(s-s_1)\|a\|_{H^{s_1,0}} + \sum_{k > N} 2^{-k(s_2-s)}\|a\|_{H^{s_2,0}} \\
\lesssim 2^{N(s_2-s_1)}\|a\|_{H^{s_1,0}} + 2^{-N(s_2-s)}\|a\|_{H^{s_2,0}}.
\]

Taking the integer \( N \) so that \( 2^{N(s_2-s_1)} \sim \frac{\|a\|_{H^{s_2,0}}}{\|a\|_{H^{s_1,0}}} \) leads to
\[
(3.2) \quad \|a\|_{B^{s,0}} \leq C \|a\|_{H^{s_1,0}}^{\frac{s_2-s}{2}} \|a\|_{H^{s_2,0}}^{\frac{s-s_1}{2}}.
\]

On the other hand, to overcome the difficulty that one cannot use Gronwall’s inequality in the Chemin-Lerner type space, we need the time-weighted Chemin-Lerner norm, which was introduced by Paicu and the second author in [21]:

**Definition 3.3.** Let \( f(t) \in L^1_{\text{loc}}(\mathbb{R}_+) \), \( f(t) \geq 0 \). We denote
\[
\|u\|_{L^2_T(B^{s,0})} = \sum_{k \in \mathbb{Z}} 2^{ks} \left( \int_0^T f(t) \|\Delta_k^h u(t)\|_{L^2}^2 \, dt \right)^{\frac{1}{2}}.
\]

We also need Bony’s decomposition from [1] for the horizontal variables to study the law of product in Besov spaces:
\[
ab = T^h_{a,b} + T^h_{b,a} + R^h(a,b) \quad \text{with}
\]
\[
(3.3) \quad T^h_{a,b} = \sum_{j \in \mathbb{Z}} S^h_{j-1} a \Delta_j^h b, \quad R^h(a,b) = \sum_{j \in \mathbb{Z}} \Delta_j^h a \tilde{\Delta}_j^h b \quad \text{where} \quad \tilde{\Delta}_j^h \overset{\text{def}}{=} \Delta_{j-1}^h + \Delta_j^h + \Delta_{j+1}^h.
\]

As an application of the above basic elements on Littlewood-Paley theory, we present the following law of product in \( B^{s,0} \).

**Lemma 3.2.** For any \( s_1, s_2 \leq 1 \) satisfying \( s_1 + s_2 > 0 \), one has
\[
(3.4) \quad \|ab\|_{B^{s_1+2-1,0}} \lesssim \|a\|_{B^{s_1,1,\frac{1}{2}}} \|b\|_{B^{s_2,0}}.
\]

**Proof.** According to (3.3), we split the product \( ab \) into three parts: \( T^h_{a,b}, T^h_{b,a} \) and \( R^h(a,b) \). Due to \( s_1 \leq 1 \), we deduce from Lemma 3.1 that
\[
\|S^h_k a\|_{L^\infty} \lesssim \sum_{k' \leq k-1} 2^{k'2^k} \|\Delta_k^h \Delta_l^v a\|_{L^2} \\
\lesssim \sum_{k' \leq k-1} d_{k',l} 2^{(1-s_1)} \|a\|_{B^{s_1,1,\frac{1}{2}}} \lesssim 2^{(1-s_1)} \|a\|_{B^{s_1,1,\frac{1}{2}}},
\]

where \( (d_{k,l})_{(k,l) \in \mathbb{Z}^2} \) (resp. \( (d_k)_{k \in \mathbb{Z}} \)) designates a generic element on the unit sphere of \( \ell^1(\mathbb{Z}^2) \) (resp. \( \ell^1(\mathbb{Z}) \)) so that \( \sum_{(k,l) \in \mathbb{Z}^2} d_{k,l} = 1 \) (resp. \( \sum_{k \in \mathbb{Z}} d_k = 1 \)). Then by virtue of the support
properties to the Fourier transform of the terms in $T^b_\alpha b$, we infer
\[
\|\Delta^h_k T^b_\alpha b\|_{L^2} \lesssim \sum_{|k'| = k} \| S^h_{k'} a\|_{L^\infty} \| \Delta^h_k b\|_{L^2}
\]
\[
\lesssim \sum_{|k'| = k} d_k 2^{k'(1-s_1-s_2)} \| a\|_{L^{s_1^*,1}_2} \| b\|_{B^{s_2,0}}
\]
\[
\lesssim d_k 2^{k(1-s_1-s_2)} \| a\|_{L^{s_1^*,1}_2} \| b\|_{B^{s_2,0}}.
\]
Whereas it follows from Lemma 3.1 that
\[
\| \Delta^h_k a\|_{L^\infty(L^2_k)} \lesssim \sum_{\ell \in \mathbb{Z}} 2^\ell \| \Delta^h_k \Delta^h_\ell a\|_{L^2} \lesssim \sum_{\ell \in \mathbb{Z}} d_k \ell 2^{k s_1} \| a\|_{L^{s_1^*,1}_2} \| b\|_{B^{s_2,0}} \lesssim d_k 2^{k s_1} \| a\|_{L^{s_1^*,1}_2},
\]
and due to $s_2 \leq 1$, we have $\| S^h_k b\|_{L^\infty(L^2_k)} \approx 2^{k(1-s_2)} \| b\|_{B^{s_2,0}}$. As a result, it comes out
\[
\| \Delta^h_k T^b_\alpha a\|_{L^2} \lesssim \sum_{|k'| = k} \| S^h_{k'} b\|_{L^\infty(L^2_k)} \| \Delta^h_k a\|_{L^\infty(L^2_k)}
\]
\[
\lesssim \sum_{|k'| = k} d_k 2^{k'(1-s_1-s_2)} \| a\|_{L^{s_1^*,1}_2} \| b\|_{B^{s_2,0}}
\]
\[
\lesssim d_k 2^{k(1-s_1-s_2)} \| a\|_{L^{s_1^*,1}_2} \| b\|_{B^{s_2,0}}.
\]
For the remainder term, in view of (3.5) and the assumption that $s_1 + s_2 > 0$, we find
\[
\| \Delta^h_k R^b(a, b)\|_{L^2} \lesssim 2^k \sum_{k' \geq k-3} \| \Delta^h_k a\|_{L^\infty(L^2_k)} \| \Delta^h_k b\|_{L^2}
\]
\[
\lesssim 2^k \sum_{k' \geq k-3} d_k 2^{-k(s_1+s_2)} \| a\|_{L^{s_1^*,1}_2} \| b\|_{B^{s_2,0}}
\]
\[
\lesssim d_k 2^{k(1-s_1-s_2)} \| a\|_{L^{s_1^*,1}_2} \| b\|_{B^{s_2,0}}.
\]
By summing up the above estimates, we conclude the proof of (3.4). \qed

The following law of product will be frequently used in the rest of this paper.

**Lemma 3.3** (A special case of Lemma 4.5 in [10]). For any $s_1, s_2 < 1$, $s_3 \leq 1$ with $s_1 + s_2 > 0$, $s_1 + s_3 > 0$, and for any $r_1, r_2 < \frac{3}{2}$, $r_3 \leq \frac{1}{2}$ with $r_1 + r_2 > 0$, $r_1 + r_3 > 0$, we have
\[
\| a b \|_{H^{s_1+s_2-1, r_1+r_2-\frac{3}{2}}} \lesssim \| a \|_{H^{s_1, r_1}} \| b \|_{H^{s_2, r_2}},
\]
(3.6)
\[
\| a b \|_{H^{s_1+s_3-1, r_1+r_3-\frac{1}{2}}} \lesssim \| a \|_{H^{s_1, r_1}} \| b \|_{B^{s_3, r_3}}.
\]
(3.7)

We remark that (3.7) can be seen as a borderline case of (3.6). However, this small generalization can be crucial in this context. Indeed, considering the initial data given by (1.3), we observe that
\[
\| v^h_0(x_h, \varepsilon x_3)\|_B = \varepsilon^{r-\frac{3}{2}} \| v^h_0(x)\|_B, \text{ where } B = H^{s_1, r_1} \text{ or } B^{s_3, r_3}.
\]
So that to keep the uniform boundedness of $\| v^h_0(x_h, \varepsilon x_3)\|_B$, we must require $r \geq \frac{1}{2}$. Thanks to the product law in the homogeneous Besov spaces for the vertical variable, we can choose $r$ to be nothing but $\frac{1}{2}$. 

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4. A priori estimates of the systems (2.4) and (2.9)

The goal of this section is to present the proof of Propositions 2.1 and 2.2.

4.1. Two useful lemmas. The key ingredients used in the proof of Propositions 2.1 and 2.2 are the subsequent two lemmas:

**Lemma 4.1.** Let \( f(t) \) be the support properties of the Fourier transform of the terms in \( T_{\tilde{u}h} \nabla h b \), we get

\[
\left\| \Delta^h_b (T_{\tilde{u}h} \nabla h b) \right\|_{L^2} = \sum_{|k' - k| \leq 4} \left( \left( [\Delta^h_b, S_{k'-1} \tilde{u}^h] \Delta^h_b \nabla h b \right)_{L^2} + \left( [\Delta^h_b, S_{k'-1} \tilde{u}^h] \Delta^h_b \nabla h b \right)_{L^2} + \left( [\Delta^h_b, S_{k'-1} \tilde{u}^h] \Delta^h_b \nabla h b \right)_{L^2} \right)
\]
On the other hand, we get, by applying Lemma 3.1, that

\[
\int_0^t \left| \frac{\Delta_k^h(R^h(\bar{u}^h, \nabla_h b) | \Delta_k^h b)}{L^2} \right| dt' \lesssim 2^k \sum_{k' \geq k-3} \int_0^t \left\| \Delta_k^h \bar{u}^h \right\|_{L^\infty(L^2_h)} \left\| \Delta_k^h \nabla_h b \right\|_{L^2} \left\| \Delta_k^h b \right\|_{L^2} dt' \\
\lesssim 2^k \sum_{k' \geq k-3} 2^{-k'} \int_0^t \left\| \nabla_h \bar{u}^h \right\|_{L^\infty(L^2_h)} \left\| \Delta_k^h \nabla_h b \right\|_{L^2} \left\| \Delta_k^h b \right\|_{L^2} dt' \\
\lesssim 2^k \sum_{k' \geq k-3} 2^{-k'} \left\| \Delta_k^h \nabla_h b \right\|_{L^2(L^2)} \left( \int_0^t f(t') \left\| \Delta_k^h b \right\|_{L^2} dt' \right)^{\frac{1}{2}} \\
\lesssim d_k 2^{-2ks} \sum_{k' \geq k-3} d_{k'} 2^{-\left(k' - k\right)(1 + s)} \left\| \nabla_h b \right\|_{L^2(B^s, \infty)} \left\| b \right\|_{L^2(B^{s}, \infty)}.
\]

Then using the fact that \( s > -1 \), we achieve

\[
(4.4) \quad \int_0^t \left| \left( \Delta_k^h \left( T^h_{\nabla_h b} \bar{u}^h \right) | \Delta_k^h b \right) \left( L^2 \right) \right| dt' \lesssim d_k 2^{-2ks} \left\| \nabla_h b \right\|_{L^2(B^s, \infty)} \left\| b \right\|_{L^2(B^{s}, \infty)}.
\]

It remains to handle the estimate of \( \int_0^t \left| \left( \Delta_k^h \left( T^h_{\nabla_h b} \bar{u}^h \right) | \Delta_k^h b \right) \left( L^2 \right) \right| dt' \). Indeed applying Lemma 3.1 yields

\[
\int_0^t \left| \left( \Delta_k^h \left( T^h_{\nabla_h b} \bar{u}^h \right) \right) \left( L^2 \right) \right| dt' \\
\lesssim 2^{-k} \sum_{|k' - k| \leq 4} 2^{-k'} \int_0^t \left\| S_{k' - 1}^h \nabla_h b \right\|_{L^2(L^\infty_h)} \left\| \Delta_k^h \nabla_h \bar{u}^h \right\|_{L^\infty(L^2_h)} \left\| \Delta_k^h \nabla_h b \right\|_{L^2} dt' \\
\lesssim 2^{-2k} \sum_{|k' - k| \leq 4} \left( \int_0^t \left\| \nabla_h \bar{u}^h \right\|_{L^\infty(L^2_h)}^2 \left\| S_{k' - 1}^h \nabla_h b \right\|_{L^2(L^\infty_h)}^2 dt' \right)^{\frac{1}{2}} \left\| \Delta_k^h \nabla_h b \right\|_{L^2(L^2)}.
\]

Yet due to \( s < 2 \), we have

\[
\left( \int_0^t f(t') \left\| S_{k' - 1}^h \nabla_h b \right\|_{L^2(L^\infty_h)}^2 dt' \right)^{\frac{1}{2}} \lesssim \sum_{k'' \leq k' - 2} 2^{2k''} \left( \int_0^t f(t') \left\| \Delta_k^h b \right\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
\lesssim \sum_{k'' \leq k' - 2} d_{k''} 2^{2k''(2 - s)} \left\| b \right\|_{L^2(B^{s}, \infty)} \\
\lesssim d_k 2^{k''(2 - s)} \left\| b \right\|_{L^2(B^{s}, \infty)}.
\]

As a result, it comes out

\[
(4.5) \quad \int_0^t \left| \left( \Delta_k^h \left( T^h_{\nabla_h b} \bar{u}^h \right) \right) \left( L^2 \right) \right| dt' \lesssim d_k 2^{-2ks} \left\| \nabla_h b \right\|_{L^2(B^s, \infty)} \left\| b \right\|_{L^2(B^{s}, \infty)}.
\]

This together with (4.3) and (4.4) ensures (4.1) for any \( s \in ]-1, 2[ \).
Remark 4.1. We notice that the condition $s < 2$ is only used in the proof of (4.6). In the case when $b = \tilde{u}^h$, we can handle the estimate (4.6) alternatively as follows

\[
\int_0^t \left| \left( \Delta^h_k(T^h_{\tilde{u}^h} \tilde{u}^h) \right)_{L^2} \right| \, dt' \\
\lesssim 2^{-k} \sum_{|k' - k| \leq 4} 2^{k'} \int_0^t \| \mathbb{S}^h_{k'} \nabla \tilde{u}^h \|_{L^\infty(L^2_v)} \| \Delta^h_{k'} \tilde{u}^h \|_{L^2} \| \Delta^h_k \nabla \tilde{u}^h \|_{L^2} \, dt' \\
\lesssim \sum_{|k' - k| \leq 4} \left( \int_0^t \| \nabla \tilde{u}^h \|_{L^2(L^2_v)} \| \Delta^h_{k'} \tilde{u}^h \|_{L^2} \, dt' \right)^{\frac{1}{2}} \| \Delta^h_k \nabla \tilde{u}^h \|_{L^2(L^2)} \\
\lesssim d_k^2 2^{-2k} \| \nabla \tilde{u}^h \|_{L^2_v(B^{s_0})} \| \nabla \tilde{u}^h \|_{L^2_v(B^{s_0})},
\]

which does not require $s < 2$. Therefore, (4.1) holds for any $s > -1$ in the case when $b = \tilde{u}^h$ (actually when $b = \tilde{u}^h \cdot g(t)$ for any function $g(t)$).

Lemma 4.2. Let $f(t) \equiv \| \nabla \tilde{u}^h(t) \|_{L^\infty(L^2_v)}^2$ and $s \in ]-1,1[$, one has

\[
\int_0^t \left| \left( \Delta^h_k(b \cdot \nabla \tilde{u}^h) \right)_{L^2} \right| \, dt' \lesssim d_k^2 2^{-2ks} \| \nabla \tilde{u}^h \|_{L^2_v(B^{s_0})} \| b \|_{L^2_v(B^{s_0})}.
\]

Proof. By applying Bony’s decomposition for the horizontal variables, (3.3), to $b \cdot \nabla \tilde{u}^h$, we write

\[
b \cdot \nabla \tilde{u}^h = T^h_b \nabla \tilde{u}^h + T^h_{\nabla \tilde{u}^h} b + R^h(b, \nabla \tilde{u}^h).
\]

It follows from Lemma 3.1 that

\[
\int_0^t \left| \left( \Delta^h_k(T^h_b \nabla \tilde{u}^h) \right)_{L^2} \right| \, dt' \\
\lesssim 2^{-k} \sum_{|k' - k| \leq 4} \int_0^t \| \mathbb{S}^h_{k'} b \|_{L^\infty(L^2_v)} \| \Delta^h_{k'} \nabla \tilde{u}^h \|_{L^\infty(L^2_v)} \| \Delta^h_k \nabla \tilde{u}^h \|_{L^2} \, dt' \\
\lesssim 2^{-k} \sum_{|k' - k| \leq 4} \left( \int_0^t \| \nabla \tilde{u}^h \|_{L^2(L^2_v)} \| \mathbb{S}^h_{k'} b \|_{L^\infty(L^2_v)} \, dt' \right)^{\frac{1}{2}} \| \Delta^h_k \nabla \tilde{u}^h \|_{L^2(L^2)},
\]

Yet it follows from the derivation of (4.5) that

\[
\left( \int_0^t \| \nabla \tilde{u}^h \|_{L^\infty(L^2_v)}^2 \| \mathbb{S}^h_{k'} b \|_{L^\infty(L^2_v)}^2 \, dt' \right)^{\frac{1}{2}} \lesssim d_k^2 2^{k(1-s)} \| b \|_{L^2_v(B^{s_0})} \quad \forall \ s < 1,
\]

which implies for any $s < 1$ that

\[
\int_0^t \left| \left( \Delta^h_k(T^h_b \nabla \tilde{u}^h) \right)_{L^2} \right| \, dt' \lesssim d_k^2 2^{-2ks} \| b \|_{L^2_v(B^{s_0})} \| \nabla \tilde{u}^h \|_{L^2_v(B^{s_0})}.
\]
Applying Lemma 3.1 twice yields
\[
\int_0^t \left| (\Delta_k^h(T_k^h u, b) | \Delta_k^h b) \right|_{L^2} \, dt' \leq \sum_{k' - k \leq 4} \int_0^t \| S_{k-1}^h \nabla_h \bar{u}^h \|_{L^\infty} \| \Delta_k^h b \|_{L^2} \frac{\Delta_k^h b}{L^2} \, dt'
\]
(4.9)

\[
\leq \sum_{k' - k \leq 4} \frac{\| \Delta_k^{h'} \nabla_h b \|_{L^2}}{L^2} \left( \int_0^t \| \nabla_h \bar{u}^h \|_{L^\infty(L_k^h)} \| \Delta_k^h b \|_{L^2} \, dt' \right)^{\frac{1}{2}}
\]
\[
\leq d_k 2^{-2k_s} \| \nabla_h b \|_{L^2(B^2) \mid B^2} \| b \|_{L^2(B^2)}
\]

And the remainder term can be handled as follows
\[
\int_0^t \left| (\Delta_k^h(R^h(b, \nabla_h \bar{u}^h)) | \Delta_k^h b) \right|_{L^2} \, dt' \leq 2^k \sum_{k' - k \geq 3} \int_0^t \| \Delta_k^{h'} \nabla_h b \|_{L^2} \| \Delta_k^{h'} \nabla_h \bar{u}^h \|_{L^\infty(L_k^h)} \| \Delta_k^h b \|_{L^2} \, dt'
\]
\[
\leq 2^k \sum_{k' - k \geq 3} 2^{-k'} \int_0^t \| \nabla_h \bar{u}^h \|_{L^\infty(L_k^h)} \| \Delta_k^{h'} \nabla_h b \|_{L^2} \| \Delta_k^h b \|_{L^2} \, dt'
\]
\[
\leq 2^k \sum_{k' - k \geq 3} 2^{-k'} \| \Delta_k^{h'} \nabla_h b \|_{L^2} \left( \int_0^t \| \Delta_k^{h'} b \|_{L^2} \, dt' \right)^{\frac{1}{2}}
\]
\[
\leq d_k 2^{-2k_s} \sum_{k' - k \geq 3} \| \nabla_h b \|_{L^2(B^2) \mid B^2} \| b \|_{L^2(B^2)}
\]

which together with the fact that \( s > -1 \) ensures that
\[
\int_0^t \left| (\Delta_k^h(R^h(\nabla_h \bar{u}^h, b)) | \Delta_k^h b) \right|_{L^2} \, dt' \leq d_k 2^{-2k_s} \| \nabla_h b \|_{L^2(B^2) \mid B^2} \| b \|_{L^2(B^2)}
\]

Along with (4.8) and (4.9), we complete the proof of (4.7). \( \square \)

4.2. The proof of Proposition 2.1. In the rest of this section, for any \( \lambda > 0 \) and function \( a \), we always denote
\[
a_\lambda(t) \overset{\text{def}}{=} a(t) \exp(-\lambda \int_0^t f(t') \, dt') \quad \text{with} \quad f(t) \overset{\text{def}}{=} \| \nabla_h \bar{u}^h(t) \|_{L^2(L_k^h)}^2.
\]

Proposition 4.1. Let \( \bar{u}^h \) be a smooth enough solution of (2.4) on \([0, T]\). Then for any \( t \in [0, T] \) and \( s \in ]-1, \infty[ \), we have
\[
\| \bar{u}^h \|_{L^\infty(B^2)} + \| \nabla \bar{u}^h \|_{L^2(B^2)} \leq 2 \| u_0^h \|_{B^2} \exp(C A_\delta(u_0^h)),
\]
where \( A_\delta(u_0^h) \) is determined by (1.5).

Proof. By multiplying \( \exp(-\lambda \int_0^t f(t') \, dt') \) to (2.4), we write
\[
\partial_t \bar{u}^h + \lambda f(t) \bar{u}^h + \bar{u}^h \cdot \nabla_h \bar{u}^h - \Delta \bar{u}^h = -\nabla_h \bar{p}_\lambda.
\]

Applying \( \Delta_k^h \bar{u}^h \) to the above equation and taking \( L^2 \) inner product of the resulting equation with \( \Delta_k^h \bar{u}^h \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \Delta_k^h \bar{u}^h(t) \|_{L^2}^2 + \lambda f(t) \| \Delta_k^h \bar{u}^h(t) \|_{L^2}^2 + \| \Delta_k^h \nabla \bar{u}^h \|_{L^2}^2 = -(\Delta_k^h (\bar{u}^h \cdot \nabla_h \bar{u}^h) | \Delta_k^h \bar{u}^h)_{L^2}.
\]
By virtue of Remark 4.1, for any $s > -1$, one has

$$
(4.14) \quad \int_0^t \left| (\Delta^h_k (\bar{u}^h \cdot \nabla_h \bar{u}^h) - \Delta^h_k \bar{u}^h) \right|_{L^2} dt' \lesssim d_k^2 2^{-2ks} \| \nabla_h \bar{u}^h \|_{L^2_t(B^{s,0})} \| \bar{u}^h \|_{L^2_t f(B^{s,0})}.
$$

Then by integrating (4.13) over $[0,t]$ and inserting (4.14) into the resulting equality, we get

$$
\| \Delta^h_k \bar{u}^h \|_{L^\infty_t(B^{s,0})}^2 + \int_0^t f(t') \| \Delta^h_k \bar{u}^h(t') \|_{L^2}^2 dt' + \| \Delta^h_k \bar{u}^h \|_{L^2_t(B^{s,0})}^2 \lesssim \| \Delta^h_k \bar{u}^h \|_{L^\infty_t(B^{s,0})}^2 + d_k^2 2^{-2ks} \| \nabla_h \bar{u}^h \|_{L^2_t(B^{s,0})} \| \bar{u}^h \|_{L^2_t f(B^{s,0})}^2.
$$

In view of Definition 3.2, by multiplying the above inequality by $2^{2ks}$ and taking square root of the resulting inequality, and then summing up the resulting inequality over $\Omega$, we achieve

$$
\| \bar{u}^h \|_{L^\infty_t(B^{s,0})} + \| \nabla \bar{u}^h \|_{L^2_t(B^{s,0})} \leq 2 \| u_0^h \|_{B^{s,0}} \quad \text{for } \forall s > -1.
$$

Yet it follows from the definition (4.10) that

$$
\left( \| \bar{u}^h \|_{L^\infty_t(B^{s,0})} + \| \nabla \bar{u}^h \|_{L^2_t(B^{s,0})} \right) \exp \left( -\lambda \int_0^t f(t') dt' \right) \leq \| \bar{u}^h \|_{L^\infty_t(B^{s,0})} + \| \nabla \bar{u}^h \|_{L^2_t(B^{s,0})},
$$

which implies that for any $s > -1$, there holds

$$
\| \bar{u}^h \|_{L^\infty_t(B^{s,0})} + \| \nabla \bar{u}^h \|_{L^2_t(B^{s,0})} \leq 2 \| u_0^h \|_{B^{s,0}} \exp \left( C \| \nabla_h \bar{u}^h \|_{L^2(B^{s,0})}^2 \right). \leq \| \nabla \bar{u}^h \|_{L^2_t(B^{s,0})}.
$$

Along with (2.5), we deduce (4.11). This completes the proof of Proposition 4.1.

\[ \square \]

**Proposition 4.2.** Under the assumptions of Proposition 4.1, for any $t \in [0, T]$ and $s \in [-1, 1]$, we have

$$
(4.15) \quad \| \partial_3 \bar{u}^h \|_{L^\infty_t(B^{s,0})} + \| \nabla \partial_3 \bar{u}^h \|_{L^2_t(B^{s,0})} \leq 2 \| \partial_3 u_0^h \|_{B^{s,0}} \exp \left( C A_{\delta}(u_0^h) \right).
$$

**Proof.** We first get, by applying $\partial_3$ to the equation (4.12), that

$$
\partial_3 \partial_3 \bar{u}^h + \lambda f(t) \partial_3 \bar{u}^h + \bar{u}^h \cdot \nabla_h \partial_3 \bar{u}^h + \partial_3 \bar{u}^h \cdot \nabla_h \bar{u}^h - \Delta \partial_3 \bar{u}^h = -\nabla_h \partial_3 \bar{u}^h.
$$

Applying $\Delta^h_k$ to the above equation and taking $L^2$ inner product of the resulting equation with $\Delta^h_k \partial_3 \bar{u}^h$, we obtain

$$
\frac{1}{2} \frac{d}{dt} \| \Delta^h_k \partial_3 \bar{u}^h(t) \|_{L^2}^2 + \lambda f(t) \| \Delta^h_k \partial_3 \bar{u}^h(t) \|_{L^2}^2 + \| \Delta^h_k \nabla \partial_3 \bar{u}^h \|_{L^2}^2
$$

$$
= -\langle \Delta^h_k (\bar{u}^h \cdot \nabla_h \partial_3 \bar{u}^h) \Delta^h_k \partial_3 \bar{u}^h \rangle_{L^2} - \langle \Delta^h_k (\partial_3 \bar{u}^h \cdot \nabla_h \bar{u}^h) \Delta^h_k \partial_3 \bar{u}^h \rangle_{L^2}.
$$

Applying Lemma 4.1 yields that for any $s \in [-1, 2]$

$$
\int_0^t \left| (\Delta^h_k (\bar{u}^h \cdot \nabla_h \partial_3 \bar{u}^h) \Delta^h_k \partial_3 \bar{u}^h) \right|_{L^2} dt' \lesssim d_k^2 2^{-2ks} \| \nabla_h \partial_3 \bar{u}^h \|_{L^2_t(B^{s,0})} \| \partial_3 \bar{u}^h \|_{L^2_t f(B^{s,0})}.
$$
Applying Lemma 4.2 gives for any $s \in [-1, 1]$ that
\[
\int_0^t \left| \left( \Delta^h_k (\partial_h \delta \tilde{u}^h \cdot \nabla_h \tilde{u}^h) \right) \left( \Delta^h_k (\partial_3 \tilde{u}^h) \right) \right|_{L^2} \ dt' \lesssim d^2 k^{-2k_s} \| \nabla_h \partial_h \delta \tilde{u}^h \|_{L^2_t (B^{s,0})} \| \partial_3 \tilde{u}^h \|_{L^2_t (B^{s,0})}.
\]
Integrating (4.16) over $[0, t]$ and inserting the above two estimates into the resulting inequality, we achieve for any $s \in [-1, 1]$ that
\[
\| \Delta^h_k \partial_3 \tilde{u}^h \|_{L^2_t (L^2)}^2 + \lambda \int_0^t f(t') \| \Delta^h_k \partial_3 \tilde{u}^h(t') \|_{L^2}^2 \ dt' + \| \Delta^h_k \nabla \partial_3 \tilde{u}^h \|_{L^2_t (L^2)}^2 \\
\lesssim \| \Delta^h_k \partial_3 \tilde{u}^h \|_{L^2_t (L^2)}^2 + C d^2 k^{-2k_s} \| \nabla_h \partial_h \delta \tilde{u}^h \|_{L^2_t (B^{s,0})} \| \partial_3 \tilde{u}^h \|_{L^2_t (B^{s,0})}.
\]
In view of Definition 3.2, by multiplying $2^{2k_s}$ to the above inequality and taking square root of the resulting inequality, and then summing up the resulting inequalities over $\mathbb{Z}$, we find
\[
\| \partial_3 \tilde{u}^h \|_{L^\infty_t (B^{s,0})} + \sqrt{\lambda} \| \partial_3 \tilde{u}^h \|_{L^2_t (B^{s,0})} + \| \nabla_3 \tilde{u}^h \|_{L^2_t (B^{s,0})} \\
\lesssim \| \partial_3 \tilde{u}^h_0 \|_{B^{s,0}} + C \| \nabla_h \partial_3 \tilde{u}^h \|_{L^2_t (B^{s,0})} \| \partial_3 \tilde{u}^h \|_{L^2_t (B^{s,0})}.
\]
Then by repeating the last step of the proof to Proposition 4.1, we deduce (4.15). This finishes the proof of the proposition. \qed

Now we are in a position to complete the proof of Proposition 2.1.

**Proof of Proposition 2.1.** It follows from Theorem 2.1 that (2.4) has a unique solution $u \in \mathcal{E}$ given by (2.6). It remains to deal with the estimates (2.7) and (2.8). Indeed for any $\sigma_1 \in [-1, 1]$ and $\sigma_2 \in [0, 1]$, and for any integer $N$, we decompose the vertical frequency of $u^h$ into the low and high frequency parts so that
\[
\| u^h \|_{L^\infty_t (B^{\sigma_1, -\sigma_2})} = \sum_{k, \ell \in \mathbb{Z}^2} 2^{k \sigma_1} 2^{\ell \sigma_2} \| \Delta^h_k \Delta^h_{\ell} u^h \|_{L^\infty_t (L^2)} \\
\lesssim \sum_{\ell \leq N} 2^{\ell \sigma_2} \sum_{k \in \mathbb{Z}} 2^{k \sigma_1} \| \Delta^h_k u^h \|_{L^\infty_t (L^2)} + \sum_{\ell > N} 2^{\ell \sigma_2} \sum_{k \in \mathbb{Z}} 2^{k \sigma_1} \| \Delta^h_k \partial_3 u^h \|_{L^\infty_t (L^2)} \\
\lesssim 2^{N \sigma_2} \| u^h \|_{L^\infty_t (B^{s_1, 0})} + 2^{N \sigma_2} \| \partial_3 u^h \|_{L^\infty_t (B^{s_1, 0})},
\]
where we used Definition 3.2 in the last step. Taking the integer $N$ so that $2^N \sim \| u^h \|_{L^\infty_t (B^{s_1, 0})}$ gives rise to
\[
\| u^h \|_{L^\infty_t (B^{\sigma_1, \sigma_2})} \lesssim \| u^h \|_{L^\infty_t (B^{s_1, 0})} \| \partial_3 u^h \|_{L^\infty_t (B^{s_1, 0})}.
\]
Similarly, we have
\[
\| \nabla u^h \|_{L^2_t (B^{\sigma_1, \sigma_2})} \lesssim \| u^h \|_{L^2_t (B^{s_1, 0})} \| \partial_3 u^h \|_{L^2_t (B^{s_1, 0})}.
\]
Inserting (4.11) and (4.15) into the above inequalities leads to
\[
\| u^h \|_{L^\infty_t (B^{\sigma_1, \sigma_2})} + \| \nabla u^h \|_{L^2_t (B^{\sigma_1, \sigma_2})} \lesssim \| u^h_0 \|_{B^{s_1, 0}} \| \partial_3 u^h_0 \|_{B^{s_1, 0}} \exp \left( A_0 (u^h_0) \right).
\]
Then (2.7) follows from (4.17) and Minkowski’s inequality.

On the other hand, along the same line to the proof of Lemmas 4.1 and 4.2, we get
\[
\int_0^t \left| \left( \Delta^h_k (u^h \cdot \nabla_h \partial_3 \tilde{u}^h) \right) \left( \Delta^h_k (\partial_3 \tilde{u}^h) \right) \right|_{L^2} \ dt' \lesssim c^2 k^{-2k_s} \| \nabla_h \partial_h \delta \tilde{u}^h \|_{L^2_t (H^{s_1, 0})} \| \partial_3 \tilde{u}^h \|_{L^2_t (H^{s_1, 0})}.
\]
where \((c_k)_{k \in \mathbb{Z}}\) is a generic element of \(\ell^2(\mathbb{Z})\) so that \(\sum_{k \in \mathbb{Z}} c_k^2 = 1\). Then by integrating (4.16) over \([0, t]\) and inserting the above inequalities into the resulting inequality, we find

\[
\|\Delta^h_k \partial_3 \tilde{u}_h^h\|_{L^\infty_t(L^2)}^2 + \lambda \int_0^t f(t') \|\Delta^h_k \partial_3 \tilde{u}_h^h(t')\|_{L^2}^2 \, dt' + \|\Delta^h_k \nabla \partial_3 \tilde{u}_h^h\|_{L^2_t(L^2)}^2 \\
\leq \|\Delta^h_k \partial_3 \tilde{u}_0^h\|_{L^2_t(L^2)}^2 + Cc_k^2 2^{-2k\sigma_1} \|\nabla_h \partial_3 \tilde{u}_h^h\|_{L^2_t(H^{s_1},0)} \|\partial_3 \tilde{u}_h^h\|_{L^2_t(L^2)}^{s_1,0}.
\]

Multiplying \(2^{2k\sigma_1}\) to the above inequality and then summing up the resulting inequalities over \(k\) gives rise to

\[
\|\partial_3 \tilde{u}_h^h\|_{L^2_t(H^{s_1},0)}^2 + \lambda \|\partial_3 \tilde{u}_h^h\|_{L^2_t(L^2)}^2 \leq \|\partial_3 \tilde{u}_0^h\|_{H^{s_1},0}^2 + C\|\nabla_h \partial_3 \tilde{u}_h^h\|_{L^2_t(H^{s_1},0)} \|\partial_3 \tilde{u}_h^h\|_{L^2_t(L^2)}^{s_1,0},
\]

from which, we deduce (2.8) by repeating the proof to the last step of Proposition 4.1. \(\square\)

### 4.3. The proof of Proposition 2.2.

**Proposition 4.3.** Let \(\tilde{u}\) be a smooth enough solution of (2.9). Then under the assumptions of Theorem 2.1, for any \(s_1 \in ]-1, 2[\) and \(s_2 \in ]-1, 1[\), we have

\[
(4.18) \quad \|\bar{u}\|_{L^\infty_t(B^{s_1} \times \Omega)} + \|\nabla \bar{u}\|_{L^2_t(B^{s_1} \times \Omega)} \leq C\|u_0\|_{B^{s_1},0} \exp\left(CA_\delta(u_0^h)\right),
\]

\[
(4.19) \quad \|\partial_3 \bar{u}\|_{L^\infty_t(B^{s_2} \times \Omega)} + \|\nabla \partial_3 \bar{u}\|_{L^2_t(B^{s_2} \times \Omega)} \leq C\|\partial_3 u_0\|_{B^{s_2},0} \exp\left(CA_\delta(u_0^h)\right),
\]

where \(A_\delta(u_0^h)\) is determined by (1.5).

**Proof.** In view of (2.9), we get, by a similar derivation of (4.13) and (4.16), that

\[
(4.20) \quad \frac{1}{2} \frac{d}{dt} \|\Delta^h_k \bar{u}_\lambda(t)\|_{L^2}^2 + \lambda f(t) \|\Delta^h_k \bar{u}_\lambda(t)\|_{L^2}^2 + \|\Delta^h_k \nabla \bar{u}_\lambda\|_{L^2}^2 = - (\Delta^h_k (\bar{u}^h \cdot \nabla \bar{u}_\lambda) | \Delta^h_k \bar{u}_\lambda)_{L^2},
\]

and

\[
(4.21) \quad \frac{1}{2} \frac{d}{dt} \|\Delta^h_k \partial_3 \bar{u}_\lambda(t)\|_{L^2}^2 + \lambda f(t) \|\Delta^h_k \partial_3 \bar{u}_\lambda(t)\|_{L^2}^2 + \|\Delta^h_k \nabla \partial_3 \bar{u}_\lambda\|_{L^2}^2 = - (\Delta^h_k (\bar{u}^h \cdot \nabla \partial_3 \bar{u}_\lambda) | \Delta^h_k \partial_3 \bar{u}_\lambda)_{L^2}.
\]

Applying Lemma 4.1 gives

\[
\int_0^t \left| (\Delta^h_k (\bar{u}^h \cdot \nabla \bar{u}_\lambda) | \Delta^h_k \bar{u}_\lambda)_{L^2} \right| \, dt' \lesssim d^2_k 2^{-2k\sigma_1} \|\nabla_h \bar{u}_\lambda\|_{L^2_t(B^{s_1},0)} \|\bar{u}_\lambda\|_{L^2_t(B^{s_1},0)},
\]

and

\[
\int_0^t \left| (\Delta^h_k (\bar{u}^h \cdot \nabla \partial_3 \bar{u}_\lambda) | \Delta^h_k \partial_3 \bar{u}_\lambda)_{L^2} \right| \, dt' \lesssim d^2_k 2^{-2k\sigma_2} \|\nabla_h \partial_3 \bar{u}_\lambda\|_{L^2_t(B^{s_2},0)} \|\partial_3 \bar{u}_\lambda\|_{L^2_t(B^{s_2},0)}.
\]

Applying Lemma 4.2 yields

\[
\int_0^t \left| (\Delta^h_k (\partial_3 \bar{u}^h \cdot \nabla \bar{u}_\lambda) | \Delta^h_k \partial_3 \bar{u}_\lambda)_{L^2} \right| \, dt' \lesssim d^2_k 2^{-2k\sigma_2} \|\nabla_h \partial_3 \bar{u}_\lambda\|_{L^2_t(B^{s_2},0)} \|\partial_3 \bar{u}_\lambda\|_{L^2_t(B^{s_2},0)}.
\]

By integrating (4.20) and (4.21) over \([0, t]\) and inserting the above estimates into the resulting inequalities, we find

\[
\|\Delta^h_k \bar{u}_\lambda\|_{L^\infty_t(L^2)} + \lambda \int_0^t f(t') \|\Delta^h_k \bar{u}_\lambda(t)\|_{L^2}^2 \, dt' + \|\Delta^h_k \nabla \bar{u}_\lambda\|_{L^2_t(L^2)}^2 \\
\leq \|\Delta^h_k \bar{u}_0\|_{L^\infty_t(L^2)} + C d^2_k 2^{-2k\sigma_1} \|\nabla_h \bar{u}_\lambda\|_{L^2_t(B^{s_1},0)} \|\bar{u}_\lambda\|_{L^2_t(B^{s_1},0)}.
\]
\[
\|\Delta_h^k \partial_3 \tilde{u}_h \|_{L^\infty_t(L^2)}^2 + \lambda \int_0^t f(t') \|\Delta_h^k \partial_3 \tilde{u}_h(t')\|_{L^2}^2 \, dt' + \|\Delta_h^k \nabla \partial_3 \tilde{u}_h\|_{L^2_t(L^2)}^2 \\
\leq \|\Delta_h^k \partial_3 \tilde{u}_0\|_{L^2_t(L^2)}^2 + C d_2^k 2^{-2k_s} \|\nabla_h \partial_3 \tilde{u}_h\|_{L^2_t(B^{s_0})} \|\partial_3 \tilde{u}_h\|_{L^2_t(J^{s_0})},
\]

With the above inequalities, by taking \( \lambda \) larger than a uniform constant, we can follow the same line as that used in the proof of Propositions 4.1 to show that

\[
\|\tilde{u}\|_{L^\infty_t(B^{s_0})} + \|\nabla \tilde{u}\|_{L^2_t(B^{s_0})} \leq 2\|\tilde{u}_0\|_{B^{s_0}} \exp \left( CA_\delta (u_0^h) \right),
\]

and

\[
\|\partial_3 \tilde{u}\|_{L^\infty_t(B^{s_0})} + \|\nabla \partial_3 \tilde{u}\|_{L^2_t(B^{s_0})} \leq 2\|\partial_3 \tilde{u}_0\|_{B^{s_0}} \exp \left( CA_\delta (u_0^h) \right).
\]

Then the estimates (4.18) and (4.19) follow once we notice that

\[
\|\tilde{u}_0\|_{B^{s_0}} = \|\nabla_h \Delta_h^{k_{-1}} (\text{div}_h u_0^b)\|_{B^{s_0}} + \|u_0^h\|_{B^{s_0}} \lesssim \|u_0\|_{B^{s_0}},
\]

and similarly \( \|\partial_3 \tilde{u}_0\|_{B^{s_0}} \lesssim \|\partial_3 u_0\|_{B^{s_0}} \). This completes the proof of the proposition.

Let us present the proof of Proposition 2.2.

**Proof of Proposition 2.2.** We first get, by a similar derivation of (4.17), that

\[
\|\tilde{u}\|_{L^\infty_t(B^{s_0})} + \|\nabla \tilde{u}\|_{L^2_t(B^{s_0})} \lesssim \|\tilde{u}\|_{L^\infty_t(B^{s_0})} \|\partial_3 \tilde{u}\|_{L^2_t(B^{s_0})}^{s_2} + \|\nabla \tilde{u}\|_{L^2_t(B^{s_0})} \|\nabla \partial_3 \tilde{u}\|_{L^2_t(B^{s_0})}^{s_2},
\]

which together with Proposition 4.3 ensures (2.11).

Whereas along the same line to the proof of Lemmas 4.1 and 4.2, we infer

\[
\int_0^t \|\Delta_h^k (\tilde{u}_h \cdot \nabla_h \partial_3 \tilde{u}_h) |\Delta_h^k \partial_3 \tilde{u}_h\|_{L^2} \, dt' \lesssim c_2^2 2^{-2k_s} \|\nabla_h \partial_3 \tilde{u}_h\|_{L^2_t(H^{s_0})} \|\partial_3 \tilde{u}_h\|_{L^2_t(H^{s_0})};
\]

\[
\int_0^t \|\Delta_h^k (\partial_3 \tilde{u}_h \cdot \nabla_h \tilde{u}_h) |\Delta_h^k \partial_3 \tilde{u}_h\|_{L^2} \, dt' \lesssim c_2^2 2^{-2k_s} \|\nabla_h \partial_3 \tilde{u}_h\|_{L^2_t(H^{s_0})} \|\partial_3 \tilde{u}_h\|_{L^2_t(H^{s_0})}.
\]

By integrating (4.21) over \([0, t]\) and inserting the above inequalities into the resulting inequality, we find

\[
\|\Delta_h^k \partial_3 \tilde{u}_h\|_{L^\infty_t(L^2)}^2 + \lambda \int_0^t f(t') \|\Delta_h^k \partial_3 \tilde{u}_h(t')\|_{L^2}^2 \, dt' + \|\Delta_h^k \nabla \partial_3 \tilde{u}_h\|_{L^2_t(L^2)}^2 \\
\lesssim \|\Delta_h^k \partial_3 \tilde{u}_0\|_{L^2_t(L^2)}^2 + c_2^2 2^{-2k_s} \|\nabla_h \partial_3 \tilde{u}_h\|_{L^2_t(H^{s_0})} \|\partial_3 \tilde{u}_h\|_{L^2_t(H^{s_0})}.
\]

Then along the same line to the derivation of (2.8), we achieve (2.12).

It remains to prove (2.13). We first get, by taking \( H^{s_0}\) inner product of the \( \tilde{u}_h \) equation in (2.9) with \( \tilde{u}_h \), that

\[
\frac{1}{2} \frac{d}{dt} \|\tilde{u}_h(t)\|_{H^{s_0}}^2 + \|\nabla \tilde{u}_h(t)\|_{H^{s_0}}^2 = - \langle \tilde{u}_h \cdot \nabla_h \tilde{u}_h, \tilde{u}_h \rangle_{H^{s_0}} - \langle \nabla_h \tilde{p}, \tilde{u}_h \rangle_{H^{s_0}}.
\]

Note that \( s_2 \in [0, 1] \), applying the law of product, Lemma 3.3, yields

\[
\|\tilde{u}_h \cdot \nabla_h \tilde{u}_h\|_{H^{s_0}} \lesssim \|\tilde{u}_h\|_{H^{s_0}} \|\nabla_h \tilde{u}_h\|_{H^{s_0}} \|\tilde{u}_h\|_{H^{s_0}} \|\tilde{u}_h\|_{H^{s_0}} \|\nabla_h \tilde{u}_h\|_{H^{s_0}} \|\tilde{u}_h\|_{H^{s_0}} \|\tilde{u}_h\|_{H^{s_0}} \|\tilde{u}_h\|_{H^{s_0}} \\
\leq C \|\nabla_h \tilde{u}_h\|_{H^{s_0}}^2 + \|\tilde{u}_h\|_{H^{s_0}}^2 + \|\nabla_h \tilde{u}_h\|_{H^{s_0}}^2 + \|\tilde{u}_h\|_{H^{s_0}}^2.
\]

Therefore, we have

\[
\frac{1}{2} \frac{d}{dt} \|\tilde{u}_h(t)\|_{H^{s_0}}^2 + \|\nabla \tilde{u}_h(t)\|_{H^{s_0}}^2 = - \langle \tilde{u}_h \cdot \nabla_h \tilde{u}_h, \tilde{u}_h \rangle_{H^{s_0}} - \langle \nabla_h \tilde{p}, \tilde{u}_h \rangle_{H^{s_0}}.
\]
To handle the pressure term, we shall use the decomposition \( \tilde{p} = \tilde{p}_1 + \tilde{p}_2 \) with \( \tilde{p}_1 \) and \( \tilde{p}_2 \) being given by (2.10). Then due to \( \sigma_2 \in [0, 1] \), by applying Lemma 3.3, we deduce that

\[
\left| \left( \nabla_h \tilde{p}_1 \mid \tilde{u}^h \right)_{H^{s_2, 0}} \right| \leq \| \tilde{u}^h \cdot \nabla_h \tilde{u}^h \|_{H^{s_2 - 1, 0}} \| \tilde{u}^h \|_{H^{1 + s_2, 0}} \\
\leq C \| \tilde{u}^h \|_{H^{s_2, 0}} \| \nabla_h \tilde{u}^h \|_{B_{2,1}^{0, \frac{1}{2}}} \| \nabla_h \tilde{u}^h \|_{H^{s_2, 0}} \\
\leq C \| \nabla_h \tilde{u}^h \|_{B_{2,1}^{0, \frac{1}{2}}}^2 \| \tilde{u}^h \|_{H^{s_2, 0}}^2 + \frac{1}{4} \| \nabla_h \tilde{u}^h \|_{H^{s_2, 0}}^2,
\]

and

\[
\left| \left( \nabla_h \tilde{p}_2 \mid \tilde{u}^h \right)_{H^{s_2, 0}} \right| \leq \| \tilde{u}^3 \partial_3 \tilde{u}^h \|_{H^{s_2, 0}} \| \tilde{u}^h \|_{H^{s_2, 0}} \\
\leq C \| \tilde{u}^3 \|_{B_{2,1}^{0, \frac{1}{2}}} \| \partial_3 \tilde{u}^h \|_{H^{s_2, 0}} \| \tilde{u}^h \|_{H^{s_2, 0}} \\
\leq C \| \nabla_h \tilde{u}^3 \|_{B_{2,1}^{0, \frac{1}{2}}}^2 \| \tilde{u}^h \|_{H^{s_2, 0}}^2 + \| \nabla_h \partial_3 \tilde{u}^h \|_{H^{s_2 - 1, 0}}^2.
\]

By inserting the above estimates into (4.22), we achieve

\[
\frac{d}{dt} \| \tilde{u}^h \|_{H^{s_2, 0}}^2 + \| \nabla \tilde{u}^h \|_{H^{s_2, 0}}^2 \leq C \left( \| \nabla_h \tilde{u}^h \|_{B_{2,1}^{0, \frac{1}{2}}}^2 + \| \nabla_h \tilde{u}^3 \|_{B_{2,1}^{0, \frac{1}{2}}}^2 \right) \| \tilde{u}^h \|_{H^{s_2, 0}}^2 + \| \nabla_h \partial_3 \tilde{u}^h \|_{H^{s_2 - 1, 0}}^2.
\]

Applying Gronwall’s inequality yields

\[
\| \tilde{u}^h \|_{L_t^\infty(H^{s_2, 0})}^2 + \| \nabla \tilde{u}^h \|_{L_t^2(H^{s_2, 0})}^2 \leq \left( \| \tilde{u}^h_0 \|_{H^{s_2, 0}}^2 + \| \nabla_h \partial_3 \tilde{u}^h \|_{L_t^2(H^{s_2 - 1, 0})}^2 \right) \times \exp \left( C \left( \| \nabla_h \tilde{u}^h \|_{B_{2,1}^{0, \frac{1}{2}}}^2 + \| \nabla_h \tilde{u}^3 \|_{B_{2,1}^{0, \frac{1}{2}}}^2 \right) \right).
\]

(4.23)

Yet it follows from Proposition 2.1 that

\[
\| \nabla_h \tilde{u}^h \|_{L_t^2(B_{2,1}^{0, \frac{1}{2}})} \leq C \| u^h_0 \|_{B_{0,0}^{\frac{1}{2}}} \| \partial_3 u^h_0 \|_{B_{0,0}^{\frac{1}{2}}} \exp \left( C A_\delta(u^h_0) \right), \\
\| \nabla_h \partial_3 \tilde{u}^h \|_{L_t^2(H^{s_2 - 1, 0})} \leq C \| \partial_3 u^h_0 \|_{H^{s_2 - 1, 0}} \exp \left( C A_\delta(u^h_0) \right).
\]

Whereas (2.11) implies that

\[
\| \nabla_h \tilde{u}^3 \|_{L_t^2(B_{2,1}^{0, \frac{1}{2}})} \leq C \| u^h_0 \|_{B_{0,0}^{\frac{1}{2}}} \| \partial_3 u^h_0 \|_{B_{0,0}^{\frac{1}{2}}} \exp \left( C A_\delta(u^h_0) \right).
\]

By inserting the above inequalities and the fact that \( \| \tilde{u}^h_0 \|_{H^{s_2, 0}} \leq \| \partial_3 u^h_0 \|_{H^{s_2 - 1, 0}} \) into (4.23), we achieve (2.13). This completes the proof of Proposition 2.2.

5. A priori estimates of the system (2.16)

5.1. The Proof of the estimate (2.18). We first observe that

\[
\| \nabla v^3 \|_{H^{\frac{1}{2}, \frac{1}{2}}}^2 = \| v^3 \|_{H^{\frac{1}{2}, \frac{1}{2}}}^2 + \| v^3 \|_{H^{\frac{1}{2}, \frac{1}{2}}}^2.
\]

Then the estimates of \( \| \nabla v^3 \|_{H^{\frac{1}{2}, \frac{1}{2}}} \) is reduced to handle the estimate of \( \| v^3 \|_{H^{\frac{1}{2}, \frac{1}{2}}} \) and \( \| v^3 \|_{H^{\frac{1}{2}, \frac{1}{2}}} \).

By taking the \( H^{\frac{1}{2}, \frac{1}{2}} \) and \( H^{\frac{1}{2}, \frac{1}{2}} \) inner product of the \( v^3 \) equation in (2.16) with \( v^3 \) respectively, we get

\[
\frac{1}{2} \frac{d}{dt} \| v^3 \|_{H^{\frac{1}{2}, \frac{1}{2}}}^2_{H^{\frac{1}{2}, \frac{1}{2}} \cap H^{\frac{1}{2}, \frac{1}{2}}} + \| \nabla v^3 \|_{H^{\frac{1}{2}, \frac{1}{2}} \cap H^{\frac{1}{2}, \frac{1}{2}}}^2 = - \left( Q \mid v^3 \right)_{H^{\frac{1}{2}, \frac{1}{2}} \cap H^{\frac{1}{2}, \frac{1}{2}}}.
\]

(5.1)
where \( Q = \sum_{i=1}^{6} Q_i \) and

\[
Q_1 = v \cdot \nabla v^3, \\
Q_2 = v \cdot \nabla \tilde{u}^3 + \tilde{u} \cdot \nabla v^3 + \tilde{u}^\delta \cdot \nabla_h v^3, \\
Q_3 = \tilde{u} \cdot \nabla \tilde{u}^3, \\
Q_4 = \partial_\delta \Delta^{-1} \sum_{\ell,m=1}^{3} \partial_\ell v^m \partial_m v^\delta, \\
Q_5 = 2\partial_\delta \Delta^{-1} \sum_{\ell,m=1}^{3} \partial_\ell v^m \partial_m (\tilde{u}^\delta + \tilde{u}), \\
Q_6 = \partial_\delta \Delta^{-1} \sum_{\ell,m=1}^{3} \left( \partial_\ell (\tilde{u}^m + \tilde{u}^m) \partial_m (\tilde{u}^\delta + \tilde{u}) - \partial_\ell \tilde{u}^m \partial_m \tilde{u}^\delta \right).
\]

(5.2)

Here and in the rest of this section, we always denote

\[
\| \cdot \|_{X \cap Y} \overset{\text{def}}{=} \| \cdot \|_X + \| \cdot \|_Y \quad \text{and} \quad (f | g)_{X \cap Y} \overset{\text{def}}{=} (f | g)_X + (f | g)_Y.
\]

Next let us handle term by term in (5.2)

- **The estimate of** \( (Q_1 | v^3)_{H^{\frac{1}{2}} \cap H^{-\frac{1}{2}}} \).

Applying Hölder’s inequality and the law of product, Lemma 3.3, we obtain

\[
|(Q_1 | v^3)_{H^{\frac{1}{2}} \cap H^{-\frac{1}{2}}} | \leq \| v \cdot \nabla v^3 \|_{H^{\frac{1}{2}}} \| v^3 \|_{H^{-\frac{1}{2}} \cap H\frac{1}{2} \cap H^{-\frac{1}{2}}} \lesssim \| v \|_{H^{\frac{1}{2}}} \| \nabla v^3 \|_{H^{-\frac{1}{2}}} \| v^3 \|_{H^{-\frac{1}{2}}} \lesssim \| v \|_{H^{\frac{1}{2}}} \| \nabla v^3 \|_{H^{-\frac{1}{2}}} \| v^3 \|_{H^{-\frac{1}{2}}}.
\]

(5.3)

While it follows from (2.3) that

\[
\| v \|_{H^{\frac{1}{2}}} \lesssim \| \nabla v^h \|_{H^{-\frac{1}{2}}} + \| v^3 \|_{H^{\frac{1}{2}}} \lesssim \| \omega \|_{H^{-\frac{1}{2}}} + \| \partial_3 v^3 \|_{H^{-\frac{1}{2}}} + \| \nabla v^3 \|_{H^{-\frac{1}{2}}}.
\]

Note that

\[
\| a \|_{H^{-\frac{1}{2}}}^{2} = \int_{\mathbb{R}^3} (|\xi|_{h}^{-1})^\frac{2}{3} (|\xi|_{h}^{-1} |\xi_3|^2)^\frac{1}{3} |\hat{a}(\xi)|^2 \, d\xi \\
\leq \left( \int_{\mathbb{R}^3} |\xi|_{h}^{-1} |\hat{a}(\xi)|^2 \, d\xi \right)^\frac{1}{3} \left( \int_{\mathbb{R}^3} |\xi|_{h}^{-1} |\xi_3|^2 |\hat{a}(\xi)|^2 \, d\xi \right)^\frac{1}{3} = \| a \|_{H^{-\frac{1}{2}}} \| \nabla a \|_{H^{-\frac{1}{2}}}.
\]

(5.4)

and recalling the definition of \( M(t) \) and \( N(t) \) given by (2.17), we infer

\[
\| v(t) \|_{H^{\frac{1}{2}}} \lesssim M^{\frac{3}{8}}(t) N^{\frac{3}{8}}(t).
\]

On the other hand, it is easy to observe that

\[
\| a \|_{H^{\frac{1}{2}}}^{2} = \int_{\mathbb{R}^3} (|\xi|_{h}^{-1})^\frac{2}{3} (|\xi|_{h}^{-1} |\xi_3|^2)^\frac{1}{3} |\hat{a}(\xi)|^2 \, d\xi \\
\leq \left( \int_{\mathbb{R}^3} |\xi|_{h}^{-1} |\hat{a}(\xi)|^2 \, d\xi \right)^\frac{1}{3} \left( \int_{\mathbb{R}^3} |\xi|_{h}^{-1} |\xi_3|^2 |\hat{a}(\xi)|^2 \, d\xi \right)^\frac{1}{3} \\
\leq \| a \|_{H^{-\frac{1}{2}}} \| \nabla a \|_{H^{-\frac{1}{2}}}.
\]

(5.5)

which together with (2.17) implies

\[
\| \nabla v^3(t) \|_{H^{\frac{1}{2}}} \lesssim M^{\frac{3}{8}}(t) N^{\frac{3}{8}}(t).
\]
Inserting the above estimates and \( \| v^3(t) \|_{H^{\frac{3}{2},0} \cap H^{-\frac{1}{2},2}} \leq N^\frac{3}{2}(t) \) into (5.3), we deduce that
\[
(5.6) \quad \left| (Q_1 \mid v^3)_{H^{\frac{3}{2},0} \cap H^{-\frac{1}{2},1}} \right| \leq CM^\frac{3}{2}(t)N(t).
\]

- The estimate of \( (Q_2 + Q_3 \mid v^3)_{H^{\frac{3}{2},0} \cap H^{-\frac{1}{2},1}} \).

By applying Hölder’s inequality and Lemma 3.3, we get
\[
\left| (Q_2 \mid v^3)_{H^{\frac{3}{2},0} \cap H^{-\frac{1}{2},1}} \right| \leq \| v \cdot \nabla u^3 + (\bar{u} + \bar{u}) \cdot \nabla v^3 \|_{H^{-\frac{1}{2},0}} \| v^3 \|_{H^{\frac{3}{2},0} \cap H^{-\frac{1}{2},2}} \lesssim (\| v \|_{H^{\frac{3}{2},0}} \| \nabla \bar{u}^3 \|_{B^{0,\frac{1}{2}}_{2,1}} + \| \bar{u} + \bar{u} \|_{B^{1,\frac{1}{2}}_{2,1}} \| \nabla v^3 \|_{H^{-\frac{1}{2},0}})N^\frac{3}{2}(t).
\]

In view of (2.3) and (2.17), we infer
\[
\| v(t) \|_{H^{\frac{3}{2},0}} \lesssim \| \nabla_h v^h(t) \|_{H^{-\frac{1}{2},0}} + \| v^3(t) \|_{H^{\frac{3}{2},0}} \lesssim \| \omega(t) \|_{H^{-\frac{1}{2},0}} + \| \partial_3 v^3(t) \|_{H^{-\frac{1}{2},0}} + \| v^3(t) \|_{H^{\frac{3}{2},0}} \leq M^\frac{3}{2}(t).
\]
(5.7)

This in turn shows that
\[
\left| (Q_2 \mid v^3)_{H^{\frac{3}{2},0} \cap H^{-\frac{1}{2},1}} \right| \leq CM^\frac{3}{2}(t)N^\frac{3}{2}(t)(\| \nabla \bar{u} \|_{B^{0,\frac{1}{2}}_{2,1}} + \| \nabla \bar{u}^h \|_{B^{0,\frac{1}{2}}_{2,1}})
\]
(5.8)

Along the same line, we have
\[
\left| (Q_3 \mid v^3)_{H^{\frac{3}{2},0} \cap H^{-\frac{1}{2},1}} \right| \leq (\| \bar{u}^h \cdot \nabla_h \bar{u}^3 \|_{H^{-\frac{1}{2},0}} + \| \bar{u}^3 \partial_3 \bar{u}^3 \|_{H^{-\frac{1}{2},0}})\| v^3 \|_{H^{\frac{3}{2},0} \cap H^{-\frac{1}{2},2}} \lesssim (\| \bar{u}^h \|_{H^{\frac{3}{2},0}} \| \nabla_h \bar{u}^3 \|_{B^{0,\frac{1}{2}}_{2,1}} + \| \bar{u}^3 \|_{B^{0,\frac{1}{2}}_{2,1}} \| \partial_3 \bar{u}^3 \|_{H^{\frac{1}{2},0}})N^\frac{3}{2}(t).
\]

Applying Young’s inequality yields
\[
(5.9) \quad \left| (Q_3 \mid v^3)_{H^{\frac{3}{2},0} \cap H^{-\frac{1}{2},1}} \right| \leq \frac{1}{100}N(t) + CM(t)(\| \nabla \bar{u}^h \|_{B^{0,\frac{1}{2}}_{2,1}} + \| \nabla \bar{u}^3 \|_{B^{0,\frac{1}{2}}_{2,1}}).
\]

- The estimate of \( (Q_4 \mid v^3)_{H^{\frac{3}{2},0} \cap H^{-\frac{1}{2},1}} \).

Let us first handle the estimate of \( (Q_4 \mid v^3)_{H^{\frac{3}{2},0}} \). Indeed it is easy to observe that
\[
\left| (Q_4 \mid v^3)_{H^{\frac{3}{2},0}} \right| \leq \left\| \partial_3 \Delta^{-1} \left( \sum_{\ell,m=1}^{2} \partial_\ell v^m \partial_m v^\ell + (\partial_3 v^3)^2 \right) \right\|_{H^{\frac{3}{2},0}} \| v^3 \|_{H^{\frac{3}{2},0}} \leq 2 \| \partial^3 \Delta^{-1}(\partial_3 v^h \cdot \nabla_h v^3) \|_{H^{\frac{3}{2},0}} \| v^3 \|_{H^{\frac{3}{2},0}} \leq \sum_{\ell,m=1}^{2} \| \partial_\ell v^m \partial_m v^\ell + (\partial_3 v^3)^2 \|_{H^{\frac{3}{2},0}} \| v^3 \|_{H^{\frac{3}{2},0}} + 2 \| \partial_3 v^h \cdot \nabla_h v^3 \|_{H^{\frac{3}{2},0}} \| v^3 \|_{H^{\frac{3}{2},0}}.
\]
(5.10)

It follows from the law of product, Lemma 3.3, and (2.3) that
\[
\left\| \sum_{\ell,m=1}^{2} \partial_\ell v^m \partial_m v^\ell + (\partial_3 v^3)^2 \right\|_{H^{-\frac{1}{2},0}} \lesssim \| \nabla_h v^h \|_{H^{\frac{3}{2},\frac{1}{4}}} + \| \partial_3 v^3 \|_{H^{\frac{3}{2},\frac{1}{4}}} \lesssim \| \omega \|_{H^{\frac{3}{4},\frac{1}{4}}} + \| \partial_3 v^3 \|_{H^{\frac{3}{2},\frac{1}{4}}}^2.
\]
and
\[
\|\partial_3 v^h \cdot \nabla_h v^3\|_{H^{-\frac{1}{2}, \frac{1}{2}}} \lesssim \|\partial_3 v^h\|_{H^{-\frac{1}{2}, 0}} \|\nabla_h v^3\|_{H^0, \frac{1}{2}}.
\]
Note that
\[
\|a\|_{L^2}^2 = \int \xi_h \left( \frac{1}{2} |\xi_h|^{-2} \right)^{\frac{1}{4}} |\partial_3 v^3|^2 \, d\xi
\]
(5.11)
\[
\lesssim \left( \int \xi_h |\partial_3 v^3|^2 \, d\xi \right)^{\frac{3}{8}} \left( \int \xi_h |\xi_3|^2 |\partial_3 v^3|^2 \, d\xi \right)^{\frac{1}{2}}
\]
\[
= \|a\|_{H^{-\frac{1}{2}, 0}}^\frac{1}{2} \|\partial_3 a\|_{H^{-\frac{1}{2}, 0}}^\frac{1}{2},
\]
and
\[
\|\nabla_h a\|_{L^2}^2 \leq \int \xi_h \left( \frac{1}{2} |\xi_h|^{-2} \right)^{\frac{1}{4}} |\partial_3 v^3|^2 \, d\xi
\]
\[
\lesssim \left( \int \xi_h |\partial_3 v^3|^2 \, d\xi \right)^{\frac{3}{8}} \left( \int \xi_h |\xi_3|^2 |\partial_3 v^3|^2 \, d\xi \right)^{\frac{1}{2}} = \|a\|_{H^{-\frac{1}{2}, 0}}^\frac{1}{2} \|\nabla a\|_{H^{-\frac{1}{2}, 0}}^\frac{1}{2},
\]
we find
\[
\|\sum_{\ell, m=1}^2 \partial_\ell v^m \partial_m v^\ell + (\partial_3 v^3)^2\|_{H^{-\frac{1}{2}, 0}} \lesssim N(t),
\]
and
\[
\|\partial_2 v^h \cdot \nabla_h v^3\|_{H^{-\frac{1}{2}, \frac{1}{2}}} \lesssim M^\frac{2}{5}(t) N^\frac{7}{5}(t).
\]
Inserting the estimates (5.4), (5.12) and (5.13) into (5.10) leads to
\[
\|[Q_4 v^3]_{H^{\frac{1}{2}, 0}}\| \leq CM^\frac{2}{5}(t) N(t).
\]
Let us turn to the estimate of \([Q_4 v^3]_{H^{\frac{1}{2}, 1}}\). Observing that
\[
\|[Q_4 v^3]_{H^{\frac{1}{2}, 1}}\| \leq \|\partial_3 \Delta^{-1} \left( \sum_{\ell, m=1}^2 \partial_\ell v^m \partial_m v^\ell + (\partial_3 v^3)^2 \right)\|_{H^{-\frac{1}{2}, 1}} \|v^3\|_{H^{-\frac{1}{2}, 1}}
\]
\[
\quad + 2 \|\partial_3 \Delta^{-1} (\partial_3 v^h \cdot \nabla_h v^3)\|_{H^{-\frac{1}{2}, \frac{1}{2}}} \|v^3\|_{H^{-\frac{1}{2}, \frac{1}{2}}}
\]
\[
\leq \left\| \sum_{\ell, m=1}^2 \partial_\ell v^m \partial_m v^\ell + (\partial_3 v^3)^2 \right\|_{H^{-\frac{1}{2}, 0}} \|\partial_3 v^3\|_{H^{-\frac{1}{2}, 0}}
\]
\[
\quad + 2 \|\partial_3 v^h \cdot \nabla_h v^3\|_{H^{-\frac{1}{2}, \frac{1}{2}}} \|v^3\|_{H^{-\frac{1}{2}, \frac{1}{2}}},
\]
from which, and (5.12), (5.13), and the fact that
\[
\|v^3\|_{H^{-\frac{1}{2}, \frac{1}{2}}} \leq \|\partial_3 v^3\|_{H^{-\frac{1}{2}, 0}}^\frac{1}{2} \|\partial_3 v^3\|_{H^{-\frac{1}{2}, \frac{1}{2}}}^\frac{1}{2} \leq M^\frac{2}{5}(t) N(t),
\]
we deduce
\[
\|[Q_4 v^3]_{H^{\frac{1}{2}, 1}}\| \leq CM^\frac{2}{5}(t) N(t).
\]
Combining (5.14) with (5.15), we achieve
\[
\|[Q_4 v^3]_{H^{\frac{1}{2}, 0} \cap H^{-\frac{1}{2}, 1}}\| \leq CM^\frac{2}{5}(t) N(t).
\]

• The estimate of \((Q_5 + Q_6 \mid v^3)_{H^{\frac{1}{2},0} \cap H^{-\frac{1}{2},1}}.\)

Due to \(\text{div} \, \bar{u} = \text{div} \, \bar{u} = 0\), we write

\[
Q_5 = 2 \sum_{\ell, m = 1}^3 \partial_3 \Delta^{-1} \partial_\ell (v^m \partial_m (\bar{u}^\ell + \bar{u}^\ell)).
\]

Applying the law of product, Lemma 3.3, yields

\[
\left| (Q_5 \mid v^3)_{H^{\frac{1}{2},0} \cap H^{-\frac{1}{2},1}} \right| \lesssim \left\| \sum_{\ell, m = 1}^3 \partial_3 \Delta^{-1} \partial_\ell (v^m \partial_m (\bar{u}^\ell + \bar{u}^\ell)) \right\|_{H^{-\frac{1}{2},0}} \|v^3\|_{H^{\frac{1}{2},0} \cap H^{-\frac{1}{2},2}}
\]

\[
\lesssim \|v\|_{H^{\frac{1}{2},0}} \left( \|\nabla \bar{u}\|_{B^{0,\frac{1}{2}}_{2,1}} + \|\nabla \bar{u}\|_{B^{0,\frac{1}{2}}_{2,1}} \right) N_{\frac{1}{4}}(t),
\]

from which and (5.7), we deduce

\[
(5.17) \quad \left| (Q_5 \mid v^3)_{H^{\frac{1}{2},0} \cap H^{-\frac{1}{2},1}} \right| \leq \frac{1}{100} N(t) + CM(t) \left( \|\nabla \bar{u}\|_{B^{0,\frac{1}{2}}_{2,1}}^2 + \|\nabla \bar{u}\|_{B^{0,\frac{1}{2}}_{2,1}}^2 \right).
\]

Similarly, by using \(\text{div} \, \bar{u} = \text{div} \, \bar{u} = 0\) once again, we write

\[
Q_6 = 2 \sum_{\ell, m = 1}^3 \partial_\ell \partial_m \Delta^{-1} \left( \partial_3 (\bar{u}^m + \bar{u}^m)(\bar{u}^\ell + \bar{u}^\ell) - \partial_3 \bar{u}^m u^\ell \right).
\]

It follows from the law of product, Lemma 3.3, that

\[
\left\| \sum_{\ell, m = 1}^3 \partial_\ell \partial_m \Delta^{-1} \left( \partial_3 (\bar{u}^m + \bar{u}^m)(\bar{u}^\ell + \bar{u}^\ell) - \partial_3 \bar{u}^m u^\ell \right) \right\|_{H^{-\frac{1}{2},0}} \|v^3\|_{H^{\frac{1}{2},0} \cap H^{-\frac{1}{2},2}}
\]

\[
\lesssim \left( \|\partial_3 \bar{u}\|_{H^{\frac{1}{2},0}} + \|\partial_3 \bar{u}\|_{H^{\frac{1}{2},0}} \right) \left( \|\bar{u}\|_{B^{0,\frac{1}{2}}_{2,1}} + \|\bar{u}\|_{B^{0,\frac{1}{2}}_{2,1}} \right) N_{\frac{1}{4}}(t).
\]

Then by applying convexity inequality, we find

\[
(5.18) \quad \left| (Q_6 \mid v^3)_{H^{\frac{1}{2},0} \cap H^{-\frac{1}{2},1}} \right| \leq \frac{1}{100} N(t) + C \left( \|\partial_3 \bar{u}\|_{H^{\frac{1}{2},0}}^2 + \|\partial_3 \bar{u}\|_{H^{\frac{1}{2},0}}^2 \right) \left( \|\bar{u}\|_{B^{0,\frac{1}{2}}_{2,1}}^2 + \|\bar{u}\|_{B^{0,\frac{1}{2}}_{2,1}}^2 \right).
\]

By inserting the estimates (5.6)-(5.9), and (5.16)-(5.18) into (5.1), we achieve (2.18).

5.2. The Proof of the estimate (2.19). By taking the \(H^{-\frac{1}{2},0}\) inner product of the \(\omega\) equation in (2.16) with \(\omega\), we get

\[
\frac{1}{2} \frac{d}{dt} \|\omega\|^2_{H^{-\frac{1}{2},0}} + \|\nabla \omega\|^2_{H^{-\frac{1}{2},0}} = - (I_1 + \cdots + I_6) \mid \omega \rangle_{H^{-\frac{1}{2},0},}
\]

with

\[
I_1 = v \cdot \nabla \omega - \omega \partial_3 u^3, \quad I_2 = \bar{u}^h \cdot \nabla (\bar{u} + \bar{w}) - \omega \partial_3 u^3, \\
I_3 = \bar{u}^3 \partial_3 (\bar{w} + \bar{w}), \quad I_4 = v \cdot \nabla (\bar{u} + \bar{w}) + (\bar{u} + \bar{u}) \cdot \nabla \omega - \omega \partial_3 \bar{u}^3, \\
I_5 = \partial_3 \bar{u}^h \cdot \nabla \bar{u}^3, \quad I_6 = - \partial_2 \bar{u}^h \cdot \nabla \bar{u}^1 + \partial_1 \bar{u}^h \cdot \nabla \bar{u}^2.
\]

Next we handle term by term above.

• The estimate of \((I_1 \mid \omega)_{H^{-\frac{1}{2},0}}.
\]

Due to \(\text{div} \, v = 0\), we get, by using integration by parts, that

\[
(v \cdot \nabla \omega \mid \omega)_{H^{-\frac{1}{2},0}} = (\text{div} \, (v \omega) \mid \omega)_{H^{-\frac{1}{2},0}} = -(v \omega \mid \nabla \omega)_{H^{-\frac{1}{2},0}}.
\]
Then it follows from Lemma 3.3 that
\[
\left|(I_1 | \omega)_{H^{-\frac{1}{2},0}} \right| \leq \|v\omega\|_{H^{-\frac{1}{2},0}} \|\nabla \omega\|_{H^{-\frac{1}{2},0}} + \|\omega \partial_3 v^3\|_{H^{-\frac{1}{2},0}} \|\omega\|_{H^{-\frac{1}{2},0}}.
\]
\[
\leq \|v\|_{L^2} \|\omega\|_{H^{\frac{1}{2},0}} \|\nabla \omega\|_{H^{-\frac{1}{2},0}} + \|\omega\|_{H^{\frac{1}{2},0}} \|\omega\|_{H^{\frac{1}{2},0}} \|\partial_3 v^3\|_{H^{\frac{1}{2},0}} \|\omega\|_{H^{-\frac{1}{2},0}},
\]
from which, (5.4), (5.5) and (5.11), we deduce that
\[
\left|(I_1 | v^3)_{H^{-\frac{1}{2},0}} \right| \lesssim M^\frac{3}{4}(t) N^\frac{3}{4}(t) \cdot M^\frac{3}{4}(t) N^\frac{3}{4}(t) \cdot N^\frac{3}{4}(t) \cdot N^\frac{3}{4}(t) \cdot M^\frac{3}{4}(t) N(t).
\]
(5.20)

• The estimate of \((I_2 | \omega)_{H^{-\frac{1}{2},0}}\).

Due to \(\text{div} \bar{u} = 0\), we have \(\text{div}_h \bar{u}^h = -\partial_3 \bar{u}\) and
\[
(\bar{u}^h \cdot \nabla_h (\bar{w} + \bar{\omega}) | \omega)_{H^{-\frac{1}{2},0}} = (\text{div}_h ((\bar{w} + \bar{\omega}) \bar{u}^h) + (\bar{w} + \bar{\omega}) \partial_3 \bar{u}^3 | \omega)_{H^{-\frac{1}{2},0}},
\]
so that we get, by integration by parts and the law of product, Lemma 3.3, that
\[
\left|(\text{div}_h (\bar{u}^h (\bar{w} + \bar{\omega})) | \omega)_{H^{-\frac{1}{2},0}} \right| \leq \|\bar{u}^h (\bar{w} + \bar{\omega})\|_{H^{-\frac{1}{2},0}} \|\nabla_h \omega\|_{H^{-\frac{1}{2},0}}
\]
\[
\leq C \|\bar{u}^h\|_{H^{\frac{1}{2},0}} \|\bar{w} + \bar{\omega}\|_{B^\frac{1}{2}} \|\nabla \omega\|_{H^{-\frac{1}{2},0}}
\]
\[
\leq \frac{1}{100} \|\nabla \omega\|_{B^\frac{1}{2}}^2 + C \|\bar{u}^h\|_{H^{\frac{1}{2},0}}^2 \|\bar{w} + \bar{\omega}\|_{B^\frac{1}{2}}^2 + \frac{1}{100} \|\partial_3 \bar{u}^3\|_{H^{\frac{1}{2},0}}^2.
\]

Applying the law of product, Lemma 3.3, once again yields
\[
\left|((\bar{w} + \bar{\omega}) \partial_3 \bar{u}^3 | \omega)_{H^{-\frac{1}{2},0}} \right| \leq \|\bar{w} + \bar{\omega}\|_{H^{\frac{1}{2},0}} \|\partial_3 \bar{u}^3\|_{H^{-\frac{1}{2},0}} \|\omega\|_{H^{-\frac{1}{2},0}}
\]
\[
\leq C \|\partial_3 \bar{u}^3\|_{H^{\frac{1}{2},0}} \|\bar{w} + \bar{\omega}\|_{B^\frac{1}{2}} \|\omega\|_{H^{-\frac{1}{2},0}}
\]
\[
\leq C \|\omega\|_{B^\frac{1}{2}}^2 \|\bar{w} + \bar{\omega}\|_{B^\frac{1}{2}}^2 + \frac{1}{100} \|\partial_3 \bar{u}^3\|_{H^{\frac{1}{2},0}}^2.
\]

And exactly along the same line, we find
\[
\left|((\bar{w} + \bar{\omega}) \partial_3 u^3 | \omega)_{H^{-\frac{1}{2},0}} \right| \leq C \|\omega\|_{B^\frac{1}{2}}^2 \|\bar{w} + \bar{\omega}\|_{B^\frac{1}{2}}^2 + \frac{1}{100} \|\partial_3 (v^3 + \bar{u}^3)\|_{H^{\frac{1}{2},0}}^2.
\]

As a result, it comes out
\[
(5.21) \quad \left|(I_2 | \omega)_{H^{-\frac{1}{2},0}} \right| \leq C M(t) \left(\|\nabla \bar{u}^h\|_{B^\frac{1}{2}}^2 + \|\nabla \bar{u}\|_{B^\frac{1}{2}}^2\right) + C \|\nabla_h \partial_3 \bar{u}^3\|_{B^\frac{1}{2}}^2 + \frac{1}{100} N(t).
\]

• The estimate of \((I_3 | \omega)_{H^{-\frac{1}{2},0}}\).

We deduce from the law of product, Lemma 3.3, that
\[
\left|(I_3 | \omega)_{H^{-\frac{1}{2},0}} \right| \leq \|\bar{u}^3 \partial_3 (\bar{w} + \bar{\omega})\|_{H^{-\frac{1}{2},0}} \|\omega\|_{H^{-\frac{1}{2},0}}
\]
\[
\lesssim \|\bar{u}^3\|_{B^\frac{1}{2}} \|\partial_3 (\bar{w} + \bar{\omega})\|_{H^{-\frac{1}{2},0}} \|\omega\|_{H^{-\frac{1}{2},0}}
\]
\[
\lesssim \|\omega\|_{H^{-\frac{1}{2},0}} \|\nabla \bar{u}^3\|_{B^\frac{1}{2}}^2 + \|\partial_3 \bar{\omega}\|_{H^{-\frac{1}{2},0}}^2 + \|\partial_3 \bar{\omega}\|_{H^{-\frac{1}{2},0}}^2
\]
\[
\lesssim \|\nabla \bar{u}^3\|_{B^\frac{1}{2}}^2 + \|\partial_3 \bar{\omega}\|_{H^{-\frac{1}{2},0}}^2 + \|\partial_3 \bar{\omega}\|_{H^{-\frac{1}{2},0}}^2.
\]

(5.22)

• The estimate of \((I_4 | \omega)_{H^{-\frac{1}{2},0}}\).
Due to \( \text{div} v = 0 \), we get, by using integration by parts and then Lemma 3.3, that
\[
\left| (v \cdot \nabla (\tilde{\omega} + \tilde{\omega}) \right|_{H^{-\frac{1}{2},0}} = \left| (v(\tilde{\omega} + \tilde{\omega}) \cdot \nabla \omega \right|_{H^{-\frac{1}{2},0}} \\
\leq \|v(\tilde{\omega} + \tilde{\omega})\|_{H^{-\frac{1}{2},0}} \|\nabla \omega\|_{H^{-\frac{1}{2},0}} \\
\leq C\|v\|_{H^{\frac{1}{2},0}} \|\tilde{\omega} + \tilde{\omega}\|_{B_{2,1}^{\frac{1}{2}}} \|\nabla \omega\|_{H^{-\frac{1}{2},0}} \\
\leq \frac{1}{100} \|\nabla \omega\|_{H^{-\frac{1}{2},0}}^2 + C\|v\|_{H^{\frac{1}{2},0}}^2 (\|\nabla h \tilde{u}^h\|_{B_{2,1}^{0,\frac{1}{2}}} + \|\nabla h \tilde{u}\|_{B_{2,1}^{0,\frac{1}{2}}}),
\]
and
\[
\left| \left( \tilde{u} + \tilde{u} \right) \cdot \nabla \omega - \omega \partial_3 \tilde{u}^3 \right|_{H^{-\frac{1}{2},0}} \\
\leq \|\tilde{u} + \tilde{u} \cdot \nabla \omega - \omega \partial_3 \tilde{u}^3\|_{H^{-\frac{1}{2},0}} \|\omega\|_{H^{-\frac{1}{2},0}} \\
\leq C\|\tilde{u} + \tilde{u}\|_{B_{2,1}^{1,\frac{1}{2}}} \|\nabla \omega\|_{H^{-\frac{1}{2},0}} + \|\omega\|_{H^{\frac{1}{2},0}} \|\partial_3 \tilde{u}^3\|_{B_{2,1}^{0,\frac{1}{2}}} \|\omega\|_{H^{-\frac{1}{2},0}} \\
\leq \frac{1}{100} \|\nabla \omega\|_{H^{-\frac{1}{2},0}}^2 + C\|\omega\|_{H^{-\frac{1}{2},0}}^2 \left( \|\nabla \tilde{u}^h\|_{B_{2,1}^{0,\frac{1}{2}}} + \|\nabla \tilde{u}\|_{B_{2,1}^{0,\frac{1}{2}}} \right).
\]
By summing up the above estimates, we achieve
\[
(5.23) \quad \left| (I_4 | \omega) \right|_{H^{-\frac{1}{2},0}} \leq CM(t) (\|\nabla \tilde{u}^h\|_{B_{2,1}^{0,\frac{1}{2}}} + \|\nabla \tilde{u}\|_{B_{2,1}^{0,\frac{1}{2}}}) + \frac{1}{50} N(t).
\]
• The estimate of \( (I_5 | \omega) \) \( H^{-\frac{1}{2},0} \).

As \( u^3 = \tilde{u}^3 + v^3 \), We shall deal with the estimate involving \( \tilde{u}^3 \) and \( v^3 \) separately. We first get, by applying the law of product, Lemma 3.3, that
\[
\left| (\partial_3 u^h \cdot \nabla \tilde{u}^3 | \omega) \right|_{H^{-\frac{1}{2},0}} \leq \|\partial_3 u^h \cdot \nabla \tilde{u}^3\|_{H^{-\frac{1}{2},0}} \|\omega\|_{H^{-\frac{1}{2},0}} \\
\leq \|\partial_3 \omega\|_{H^{-\frac{1}{2},0}} \|\nabla \tilde{u}^3\|_{H^{\frac{1}{2},0}} \|\omega\|_{H^{-\frac{1}{2},0}} \\
\leq \left( \|\partial_3 \omega\|_{H^{-\frac{1}{2},0}} \|\nabla \tilde{u}^3\|_{H^{\frac{1}{2},0}} \|\omega\|_{H^{-\frac{1}{2},0}} \right),
\]
In view of (2.3), we have
\[
\|\partial_3 u^h\|_{H^{\frac{1}{2},0}} \leq \|\nabla h \partial_3 v^3\|_{H^{-\frac{1}{2},0}} + \|\nabla h \partial_3 (u^h + \tilde{u}^h)\|_{H^{-\frac{1}{2},0}} \\
\leq \|\partial_3 \omega\|_{H^{-\frac{1}{2},0}} + \|\partial_3^2 v^3\|_{H^{-\frac{1}{2},0}} + \|\nabla \partial_3 (u^h + \tilde{u}^h)\|_{H^{-\frac{1}{2},0}} \\
\leq N^\frac{1}{2} \left( t \right) + \|\nabla \partial_3 (u^h + \tilde{u})\|_{H^{-\frac{1}{2},0}}.
\]
Whereas it follows from (5.5) that
\[
\|\nabla \tilde{u}^3\|_{H^{0,\frac{1}{2}}} \leq \|v^3\|_{H^{\frac{3}{2},0}} \|\nabla v^3\|_{H^{\frac{3}{2},0}} \leq M^\frac{3}{2} (t) N^\frac{1}{2} \left( t \right), \quad \text{and} \quad \|\omega\|_{H^{-\frac{1}{2},0}} \leq \|\omega\|_{H^{-\frac{1}{2},0}} \|\partial_3 \omega\|_{H^{-\frac{1}{2},0}} \leq M^\frac{3}{2} (t) N^\frac{1}{2} \left( t \right).
\]
Inserting the above estimates into (5.24) and using convexity inequality gives rise to
\[
\left| (\partial_3 u^h \cdot \nabla \tilde{u}^3 | \omega) \right|_{H^{-\frac{1}{2},0}} \leq CM^\frac{1}{2} (t) N(t) + M(t) N(t) + C\|\nabla \partial_3 (u^h + \tilde{u}^h)\|_{H^{-\frac{1}{2},0}}^2.
\]
Along the same line, by using (5.25), we obtain
\[ |(\partial_3 u^h \cdot \nabla_h \tilde{u}^3 | \omega)_{H^{-\frac{1}{2}, \theta}} | \leq ||\partial_3 u^h \cdot \nabla_h \tilde{u}^3||_{H^{-\frac{1}{4}, \theta}} ||\omega||_{H^{-\frac{1}{4}, \theta}} \]
\[ \leq C||\partial_3 u^h||_{H^2, \theta, 0} ||\nabla_h \tilde{u}^3||_{B^0_{2,1}, \theta} ||\omega||_{H^{-\frac{1}{4}, \theta}} \]
\[ \leq C\left(N^\frac{1}{2}(t) + ||\nabla \partial_3 (\tilde{u}^h + \tilde{u})||_{H^{-\frac{1}{4}, \theta}}\right)||\nabla \tilde{u}||_{B^0_{2,1}, \theta} M^\frac{1}{2}(t) \]
\[ \leq CM(t)||\nabla \tilde{u}||^2_{B^0_{2,1}, \theta} + C||\nabla \partial_3 (\tilde{u}^h + \tilde{u})||^2_{H^{-\frac{1}{4}, \theta}} + \frac{1}{100}N(t). \]

By combining the above two estimates, we achieve
\[ |(I_5 | \omega)_{H^{-\frac{1}{2}, \theta}} | \leq \left(\frac{1}{100} + CM^\frac{1}{2}(t) + M(t)\right)N(t) \]
\[ + C M(t)||\nabla \tilde{u}||^2_{B^0_{2,1}, \theta} + C||\nabla \partial_3 (\tilde{u}^h + \tilde{u})||^2_{H^{-\frac{1}{4}, \theta}}. \]

- The estimate of $(I_6 | \omega)_{H^{-\frac{1}{2}, \theta}}$.

We get, by applying the law of product, Lemma 3.3, that
\[ |(I_6 | \omega)_{H^{-\frac{1}{2}, \theta}} | \leq || - \partial_3 \tilde{u}^h \cdot \nabla_h \tilde{u}^3 + \partial_3 \tilde{u}^h \cdot \nabla_h \tilde{u}^2||_{H^{-\frac{1}{4}, \theta}} ||\omega||_{H^{-\frac{1}{4}, \theta}} \]
\[ \lesssim ||\nabla_h \tilde{u}^3||_{B^0_{2,1}, \theta} ||\nabla_h \tilde{u}^2||_{H^{-\frac{1}{4}, \theta}} ||\omega||_{H^{-\frac{1}{4}, \theta}} \]
\[ \lesssim M(t)||\nabla \tilde{u}||^2_{B^0_{2,1}, \theta} + ||\nabla \tilde{u}^h||^2_{H^{-\frac{1}{4}, \theta}}. \]

Substituting the estimates (5.20)-(5.23), (5.26) and (5.27) into (5.19), we obtain (2.19). This concludes the proof of Proposition 2.3.

6. The Proof of Theorem 1.2

The purpose of this section is to complete the proof of Theorem 1.2. The strategy is to verify that, the necessary condition, (2.21), for the finite time blow-up of Fujita-Kato solutions to $(NS)$ can never be satisfied. Let us first present the proof of Proposition 2.4.

Proof of Proposition 2.4. As explained in Section 2, let us consider the maximal solution $v \in \mathcal{E}_T$ of (2.15), where $T^*$ denotes the lifespan of $(NS)$ in (2.1). Let $M(t), N(t)$ be determined by (2.17), we denote
\[ T^* \overset{\text{def}}{=} \sup \left\{ T : M(t) + \int_0^T N(t') \, dt' \leq \eta \right\} \]
for some sufficiently small constant $\eta$, which will be determined later on.

We are going to prove that $T^* = T^*$ for some particular choice of $\eta$. Otherwise, if $T^* < T^*$, for any $t \in [0, T^*]$, we get, by summing up the estimates (2.18), (2.19) and using the bound in (6.1), that
\[ \frac{dM(t)}{dt} + 2N(t) \leq \left(\frac{1}{2} + C \eta^2 + \eta\right)N + CM\left(||\nabla \tilde{u}||^2_{B^0_{2,1}, \theta} + ||\tilde{u}^h||^2_{B^0_{2,1}, \theta}\right) \]
\[ + C||\nabla \tilde{u}||^2_{H^\frac{1}{2}, \theta, 0} + C||\tilde{u}^h||^2_{H^\frac{1}{2}, \theta, 0}||\nabla \tilde{u}^3||^2_{B^0_{2,1}, \theta} \]
\[ + C\left(1 + ||\tilde{u}^h||^2_{B^0_{2,1}, \theta} + ||\tilde{u}||^2_{B^0_{2,1}, \theta}\right)\left(||\nabla \partial_3 \tilde{u}^h||^2_{H^{-\frac{1}{4}, \theta}} + ||\nabla \partial_3 \tilde{u}||^2_{H^{-\frac{1}{4}, \theta}}\right). \]
In particular if we take $\eta = \min\left(\frac{1}{2}, \frac{1}{2t_0}\right)$, the first term on the right-hand side of (6.2) can be absorbed by $2N(t)$ on its left-hand side. Then by applying Gronwall’s inequality to the resulting inequality, we achieve

$$
M(t) + \int_0^t N(t') \, dt' \leq C \left\{ \| \nabla \tilde{u}^h \|_{L_t^\infty(H^{\frac{1}{2}, 0})}^2 + \| \tilde{u}^h \|_{L_t^\infty(H^{\frac{1}{2}, 0})}^2 \right\} \left( \int_0^t \nabla \partial_3 (\tilde{u}^h, \tilde{u}) \right) \| \nabla \partial_3 (\tilde{u}^h, \tilde{u}) \|_{L_t^2(B_{2,1}^{0, \frac{1}{2}})}^2 + \left( 1 + \| (\tilde{u}^h, \tilde{u}) \|_{L_t^\infty(B_{2,1}^{0, \frac{1}{2}})}^2 \right) \| \nabla \partial_3 (\tilde{u}^h, \tilde{u}) \|_{L_t^2(B_{2,1}^{0, \frac{1}{2}})}^2 \right). 
$$

(6.3)

On the other hand, it follows from Propositions 2.1 and 2.2 and (3.2) that

$$
\| (\tilde{u}^h, \tilde{u}) \|_{L_t^\infty(B_{2,1}^{0, \frac{1}{2}})}^2 + \| \nabla (\tilde{u}^h, \tilde{u}) \|_{L_t^2(B_{2,1}^{0, \frac{1}{2}})}^2 
\lesssim \| u_0^h \|_{B_0^0} \| \partial_3 u_0^h \|_{B_0^0} \exp(C A_\delta(u_0^h)) 
\lesssim \| u_0 \|_{H^{\frac{1}{2}, 0}} \| u_0 \|_{H^{\frac{1}{2}, 0}} \| \partial_3 u_0 \|_{H^{-\frac{1}{2}, 0}} \| \partial_3 u_0 \|_{H^{\frac{1}{2}, 0}} \exp(C A_\delta(u_0^h)) \equiv B_\delta(u_0), 
$$

(6.4)

and

$$
\| \nabla \partial_3 (\tilde{u}^h, \tilde{u}) \|_{L_t^2(B_{2,1}^{0, \frac{1}{2}})}^2 \lesssim \| \partial_3 u_0 \|_{H^{-\frac{1}{2}, 0}} \exp(C A_\delta(u_0^h)).
$$

Whereas it follows from (2.13) and (3.2) that

$$
\| \nabla \tilde{u}^h \|_{L_t^\infty(H^{\frac{1}{2}, 0})}^2 + \| \nabla \tilde{u}^h \|_{L_t^2(H^{\frac{1}{2}, 0})}^2 \lesssim \| \partial_3 u_0 \|_{H^{-\frac{1}{2}, 0}} \exp(C B_\delta(u_0)).
$$

Inserting the above estimates into (6.3) gives rise to

$$
M(t) + \int_0^t N(t') \, dt' \leq \| \partial_3 u_0 \|_{H^{-\frac{1}{2}, 0}}^2 \exp(C A_\delta(u_0^h) + C B_\delta(u_0)). 
$$

(6.5)

In particular, if $\epsilon_0$ in (1.4) is so small that $\epsilon_0 \leq \frac{\eta}{2}$, then we can deduce from (6.5) that

$$
\sup_{t \in [0, T^*]} \left( M(t) + \int_0^t N(t') \, dt' \right) \leq \frac{\eta}{2},
$$

which contradicts with the choice of $T^*$ given in (6.1). This in turn shows that $T^* = T^*$. \(\square\)

Let us now turn to the proof of Theorem 1.2.

**Proof of Theorem 1.2.** It is easy to observe from Lemma 3.1 that

$$
\| \Delta_j a \|_{L^\infty} \lesssim \sum_{k \leq j + N_0 \atop \ell \leq j + N_0} 2^{k^2} 2^{j^2} \| \Delta_k \Delta_j a \|_{L^2} 
\lesssim \sum_{k \leq j + N_0 \atop \ell \leq j + N_0} d_k \ell 2^k \| a \|_{B_{2,1}^{0, \frac{3}{2}}} \lesssim d_j 2^j \| a \|_{B_{2,1}^{0, \frac{3}{2}}},
$$

which implies that

$$
\| a \|_{B_{\infty, \infty}^{-1}} \lesssim \| a \|_{B_{2,1}^{0, \frac{3}{2}}}.
$$

Thanks to the above inequality and (6.4), we have

$$
\int_0^{T*} \| \nabla (\tilde{u}^h, \tilde{u}) \|_{B_{\infty, \infty}^{-1}} \, dt \lesssim \int_0^{T*} \| \nabla (\tilde{u}^h, \tilde{u}) \|_{B_{2,1}^{0, \frac{3}{2}}} \, dt \lesssim B_\delta(u_0).
$$

(6.6)
On the other hand, by virtue of (2.3) and Lemma 3.1, we infer
\[
\|\Delta_j(\nabla v^h)\|_{L^\infty} \lesssim 2^j \sum_{k \leq j+N_0} 2^k \|\Delta_k^h(\nabla v^h)\|_{L^2} \\
\lesssim 2^j \sum_{k \leq j+N_0} (\|\Delta_k^h(\nabla \omega)\|_{L^2} + \|\Delta_k^h(\nabla \partial_3 v^3)\|_{L^2}) \\
\lesssim 2^j \sum_{k \leq j+N_0} c_k(t) 2^k (\|\nabla \omega\|_{H^{-\frac{1}{2},0}} + \|\nabla \partial_3 v^3\|_{H^{-\frac{1}{2},0}}) \\
\lesssim c_j(t) 2^j (\|\nabla \omega\|_{H^{-\frac{1}{2},0}} + \|\nabla \partial_3 v^3\|_{H^{-\frac{1}{2},0}}),
\]
where \((c_j(t))_{j \in \mathbb{Z}}\) denotes a generic element of \(\ell^2(\mathbb{Z})\) so that \(\sum_{j \in \mathbb{Z}} c_j^2(t) = 1\). This together with (2.20) ensures that
\[
\int_0^{T^*} \|\nabla v^h(t)\|_{B_{\infty,\infty}^{-1}}^2 dt \lesssim \int_0^{T^*} (\|\nabla \omega\|_{H^{-\frac{1}{2},0}}^2 + \|\nabla \partial_3 v^3\|_{H^{-\frac{1}{2},0}}^2) dt \leq \eta.
\]
Similarly, by applying Lemma 3.1, we get
\[
\|\Delta_j(\nabla v^3)\|_{L^\infty} \lesssim 2^{-j} \|\Delta_j(\nabla^2 v^3)\|_{L^\infty} \lesssim 2^{-\frac{j}{2}} \sum_{k \leq j+N_0} 2^k \|\Delta_k^h \nabla^2 v^3\|_{L^2} \\
\lesssim 2^{-\frac{j}{2}} \sum_{k \leq j+N_0} c_k(t) 2^k \|\nabla^2 v^3(t)\|_{H^{-\frac{1}{2},0}} \\
\lesssim c_j(t) 2^j \|\nabla^2 v^3(t)\|_{H^{-\frac{1}{2},0}},
\]
from which and (2.20), we deduce that
\[
\int_0^{T^*} \|\nabla v^3(t)\|_{B_{\infty,\infty}^{-1}}^2 dt \lesssim \|\nabla^2 v^3\|_{L^2_t(H^{-\frac{1}{2},0})}^2 \leq \eta.
\]
This together with (6.6) and (6.7) permits us to conclude that
\[
\int_0^{T^*} \|\nabla u(t)\|_{B_{\infty,\infty}^{-1}}^2 dt < \infty.
\]
Then Theorem 2.2 ensures that \(T^* = \infty\). This completes the proof of Theorem 1.2. \(\square\)

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References

[1] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, Ann. Sci. Éc. Norm. Supér., 14 (1981), 209-246.

[2] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der mathematischen Wissenschaften, 343, Springer-Verlag Berlin Heidelberg, 2011.

[3] J. Bourgain and N. Pavlović, Ill-posedness of the Navier-Stokes equations in a critical space in 3D, J. Funct. Anal., 255 (2008), 2233-2247.

[4] J.-Y. Chemin and I. Gallagher, On the global wellposedness of the 3-D Navier-Stokes equations with large initial data, Ann. Sci. Éc. Norm. Supér., 39 (2006), 679–698.

[5] J.-Y. Chemin and I. Gallagher, Large, global solutions to the Navier-Stokes equations, slowly varying in one direction, Trans. Amer. Math. Soc., 362 (2010), 2859-2873.
[6] J.-Y. Chemin, I. Gallagher and P. Zhang, Sums of large global solutions to the incompressible Navier-Stokes equations, *J. Reine Angew. Math.*, **681** (2013), 65-82.

[7] J.-Y. Chemin and N. Lerner, Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes, *J. Differential Equations*, **121** (1995), 314-328.

[8] J.-Y. Chemin and P. Zhang, On the global wellposedness to the 3-D incompressible anisotropic Navier-Stokes equations, *Comm. Math. Phys.*, **272** (2007), 529-566.

[9] J.-Y. Chemin and P. Zhang, Remarks on the global solutions of 3-D Navier-Stokes system with one slow variable, *Comm. Partial Differential Equations*, **40** (2015), 878-896.

[10] J.-Y. Chemin and P. Zhang, On the critical one component regularity for 3-D Navier-Stokes system. *Ann. Sci. Éc. Norm. Supér.*, **49** (2016), 131-167.

[11] M. Cannone, Y. Meyer and F. Planchon, Solutions autosimilaires des équations de Navier-Stokes, Séminaire “Équations aux Dérivées Partielles” de l’École polytechnique, Exposé VIII, 1993–1994.

[12] D. Iftimie, G. Raugel and G. R. Sell, Navier-Stokes equations in thin 3D domains with Navier boundary conditions, *Indiana Univ. Math. J.*, **56** (2007), 1083–1156.

[13] H. Fujita and T. Kato, On the Navier-Stokes initial value problem I. *Arch. Ration. Mech. Anal.*, **16** (1964), 269-315.

[14] T. Kato, Strong $L^p$-solutions of the Navier-Stokes equation in $R^m$ with applications to weak solutions, *Math. Z.*, **187** (1984), 471-480.

[15] I. Kukavica and M. Ziane, Navier-Stokes equations with regularity in one direction, *J. Math. Phys.*, **48** (2007), 065-203.

[16] H. Koch and D. Tataru, Well-posedness for the Navier-Stokes equations, *Adv. Math.*, **157** (2001), 22-35.

[17] O. A. Ladyzhenskaya, Unique global solvability of the three-dimensional Cauchy problem for the Navier-Stokes equations in the presence of axial symmetry, (Russian) *Zap. Naǔčn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, **7** (1968), 155-177.

[18] J. Leray, Essai sur le mouvement d’un liquide visqueux emplissant l’espace, *Acta Math.*, **63** (1933), 193–248.

[19] Y. Meyer. *Wavelets, Paraproducts and Navier–Stokes*. Current Developments in Mathematics, International Press, Cambridge, Massachusetts, 1996.

[20] M. Paicu, Équation anisotrope de Navier-Stokes dans des espaces critiques, *Rev. Mat. Iberoam.*, **21** (2005), 179-235.

[21] M. Paicu and P. Zhang, Global solutions to the 3-D incompressible anisotropic Navier-Stokes system in the critical spaces, *Comm. Math. Phys.*, **307** (2011), 713-759.

[22] G. Raugel and G. R. Sell, Navier-Stokes equations on thin 3D domains. I. Global attractors and global regularity of solutions, *J. Amer. Math. Soc.*, **6** (1993), 503-568.

[23] M. R. Ukhovskii, and V. I. Iudovich, Axially symmetric flows of ideal and viscous fluids filling the whole space, *J. Appl. Math. Mech.*, **32** (1968) 52-61.

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