Hamiltonian formulation of unimodular gravity in the teleparallel geometry

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Abstract

In the context of the teleparallel equivalent of general relativity we establish the Hamiltonian formulation of the unimodular theory of gravity. Here we do not carry out the usual 3 + 1 decomposition of the field quantities in terms of the lapse and shift functions, as in the ADM formalism. The corresponding Lagrange multiplier is the timelike component of the tetrad field. The dynamics is determined by the Hamiltonian constraint $\mathcal{H}_0'$ and a set of primary constraints. The constraints are first class and satisfy an algebra that is similar to the algebra of the Poincaré group.

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1 Introduction

The unimodular theory of gravity, or simply unimodular relativity, is an alternative theory of gravity considered by Einstein in 1919 [1] in the cosmological context, in order to allow homogeneous, static solutions of the fields equations. It turned out that it is equivalent to general relativity with the cosmological constant appearing as an integration constant. Anderson and Finkelstein in 1971 placed this theory in the Lagrangian form [2]. The unimodular theory of gravity is a modification of general relativity in the sense that now it is introduced a condition that requires the determinant of the space-time metric to have a fixed value ($g = -1$) [2]. This condition has the effect of reducing the symmetry group from the full space-time diffeomorphism invariance to invariance under only diffeomorphisms that preserve the nondynamical fixed volume element. When we introduce this condition in the Hilbert-Einstein action, the field equations that arise are equivalent to those obtained from Einstein’s theory in the presence of a cosmological term.

In view of the relation between the unimodular theory of gravity and the emergence of a cosmological constant, recently this theory has been considered from several points of view [3, 4, 5, 6, 7, 8, 9] in order to attempt a solution to the cosmological constant problems both at the classical and quantum levels. These approaches reveal two important aspects of the theory. First, since this theory has a fixed determinant of the metric tensor, contributions to the energy-momentum tensor of the form $C g_{\mu \nu}$, where $C$ is a constant, are not sources of curvature in the field equations [7]. This seems to solve one of the cosmological constant problems, which is suppressing the huge contribution to the cosmological constant that arises from quantum corrections [7].

Second, in the ordinary (ADM type) canonical formulation of the unimodular theory the lapse function $N$ is no longer an independent variable, since now it is given by $N = [g^{(3)}]^{-1/2}$. A consequence of this change of status of $N$ is that the primary Hamiltonian constraint of the ordinary canonical formulation of general relativity, $\mathcal{H}_\perp = 0$, obtained by independent variation of the total Hamiltonian with respect to $N$, no longer emerges in the unimodular relativity as a secondary constraint, hence the total Hamiltonian of the theory does not vanish. The Hamiltonian constraint equations $\mathcal{H}_i = 0$ do remain present, because they are obtained from variation of the total Hamiltonian with respect to the shift function $N^i$, which remains an independent variable. Because of this feature of the Hamiltonian formulation of the uni-
modular theory of gravity, in the procedure of canonical quantization it is possible to unfreeze the time-dependent Schrödinger equation \[4, 5, 7, 9\].

It is well known that for any physical theory the Hamiltonian formulation reveals important aspects of the theory, and serve as a starting point for the process of canonical quantization. The Hamiltonian formulation distinguishes the hyperbolic field equations (evolution equations) from the elliptic field equations (constraints). In the work of Arnowitt, Deser and Misner (ADM) \[10\] the Hamiltonian analysis of Einstein’s general relativity reveals that the time evolution of the field quantities is determined by the Hamiltonian and vector constraints. Thus four of the ten Einstein’s equations acquire a well defined meaning. This is an essential feature of the canonical quantization program.

The theory of general relativity can also be formulated in the teleparallel (Weitzenböck) geometry \[11\]. In this framework the dynamical field quantities are the tetrad fields \(e_{a\mu}\), where \(a\) and \(\mu\) are \(SO(3,1)\) and space-time indices, respectively. By using these fields it is possible to construct the Lagrangian density of the teleparallel equivalent of general relativity (TEGR) \[12, 13, 14, 15, 16\] which generates Einstein’s equations in terms of the tetrad fields. The Lagrangian density, in the TEGR, is given in terms of a quadratic combination in the torsion tensor \(T_{a\mu\nu} = \partial_{\mu}e_{a\nu} - \partial_{\nu}e_{a\mu} \). This connection describes the space-time endowed with absolute parallelism \[18\].

In the Weitzenböck space-time two vectors located at \(x^\mu\) and \(x^\mu + dx^\mu\), \(V^\mu(x)\) and \(V^\mu(x + dx)\), are said to be parallel if their projections on the tangent space by means of the tetrad field are identical \[12\]. The vectors \(V^a(x) = e^a_\mu(x)V^\mu(x)\) and \(V^a(x + dx) = e^a_\mu V^\mu(x) + (e^a_\mu \partial_\lambda V^\mu + V^\mu \partial_\lambda e^a_\mu)dx^\lambda = V^a(x) + e^a_\mu(\nabla_\lambda V^\mu)dx^\lambda\), where the covariant derivative \(\nabla\) is constructed out of the Weitzenböck connection \(\Gamma^\lambda_{\beta\gamma} = e^a_\lambda \partial_\beta e_{a\gamma}\), are projected at \(x^\mu\) and \(x^\mu + dx^\mu\), respectively. The condition of absolute parallelism, \(V^a(x) = V^a(x + dx)\), holds if the covariant derivative \(\nabla_\lambda V^\mu\) vanishes. Given that \(\nabla_\lambda e_{a\mu} \equiv 0\), the tetrad fields \(e_{a\mu}\) constitute a set of autoparallel fields.

In the Hamiltonian formulation of the TEGR it is possible to address the notion of energy-momentum and angular momentum of the gravitational field \[17\]. Here, the total Hamiltonian is given by a combination of first class constraints. The field equations of the theory, either in Lagrangian or in Hamiltonian form, suggest definitions for the gravitational energy-momentum and angular momentum. The Lagrangian field equations also allow the definition
of the gravitational energy-momentum tensor, as well as the balance equations for the energy and momentum of the field. These important aspects of TEGR serve as motivation to consider the unimodular theory of gravity in this geometric framework. The possible relation between the cosmological constant and dark energy constitutes a further motivation for the present investigation. Within the context of the TEGR it will be possible to analyze whether dark energy is an unexpected form of gravitational energy that arises as a consequence of the cosmological constant.

In this work we present the Hamiltonian formulation of the unimodular theory of gravity in the context of the TEGR. We perform the 3+1 decomposition and obtain the total Hamiltonian as a combination of first class constraints. The analysis presented here is similar to that obtained in [19], the difference residing in the fact that here we introduce the unimodular condition $\sqrt{-g} - 1 = 0$ in the total Lagrangian density. As a consequence, the theory and in particular the Hamiltonian density depend on a cosmological constant. If we ultimately require the cosmological constant to vanish, we recover the same results presented in [19]. In addition, we will present the constraint algebra in a much more simple form than that presented in [19]. We consider this latter result as a major achievement of the present analysis. The simplification of the Hamiltonian formulation is crucial for a better understanding of the theory.

Notation: space-time indices $\mu, \nu, \ldots$ and SO(3,1) indices $a, b, \ldots$ run from 0 to 3. Time and space indices are indicated according to $\mu = 0, i$, $a = (0), (i)$. The tetrad field is denoted by $e^a_\mu$, and the flat, Minkowski space-time metric tensor raises and lowers tetrad indices and is fixed by $\eta_{ab} = e_{\alpha\mu}e_{\beta\nu}g^{\mu\nu} = (-1, +1, +1, +1)$. The determinant of the tetrad field is represented by $e = \det(e^a_\mu) = \sqrt{-g}$ and we use the constants $G = c = 1$.

2 Lagrangian Formulation

In this section we will first demonstrate the equivalence of the TEGR with Einstein’s general relativity. It is well known that in the Riemannian geometry the Christoffel symbols $^0\Gamma^\lambda_{\mu\nu}$ are symmetric in the lower indices and therefore the corresponding torsion tensor vanishes. However, in the TEGR the field equations are constructed out of the torsion tensor $T^\lambda_{\mu\nu}$, related to the anti-symmetric part of the Weitzenböck connection, $\Gamma^\lambda_{\mu\nu} = e^{a\lambda}\partial_\mu e_a_{\nu}$, where $T^\lambda_{\mu\nu} = e^a_\lambda T^a_{\mu\nu}$ and
The torsion-free Levi-Civita connection is given by

\[ T^a_{\mu\nu} = \partial_{\mu} e^a_{\nu} - \partial_{\nu} e^a_{\mu}, \]  

(1)

The Christoffel symbolos \( \Gamma^{a}_{\lambda \mu \nu} \) and the Levi-Civita connection \( \omega^{a}_{\mu a b} \) are identically related by

\[ \omega^{a}_{\mu a b} = e_{\lambda a} \partial_{\mu} e^a_{\nu} + e_{\lambda a} (\omega^{a}_{\mu a b} b^a_{\nu}). \]  

(3)

Using the above equation it is possible to obtain the identity

\[ \omega^{a}_{\mu a b} = -K^{a}_{\mu a b}, \]  

(4)

where \( K^{a}_{\mu a b} = \frac{1}{2} e^a_{\lambda b} \partial_{\mu} e^a_{\nu} (T_{\lambda \mu \nu} + T_{\nu \lambda \mu} + T_{\mu \lambda \nu}) \) is the contortion tensor. This identity is important in the construction of the Lagrangian density of the TEGR. From Eq. (4) it is possible to obtain the scalar curvature \( R(\omega) \), from which we can build the following identity,

\[ e R(\omega) = -e \Sigma^{abc} T_{abc} + 2 \partial_{\mu} (e T^\mu), \]  

(5)

where \( e \) is the determinant of the tetrad field \( e^a_{\mu} \) and \( T^a = T^b_{ba} \). \( \Sigma^{abc} \) is defined by [16]

\[ \Sigma^{abc} = \frac{1}{4} \left( T^{abc} + T^{bac} - T^{cab} \right) + \frac{1}{2} \left( \eta^{ac} T^b - \eta^{ab} T^c \right), \]  

(6)

In Eq. (5) both sides are invariant under Lorentz transformations. By eliminating the divergence term in Eq. (5) we can define the Lagrangian density of the TEGR as

\[ \mathcal{L}(e_{\mu a}) = -k e \Sigma^{abc} T_{abc} - \mathcal{L}_M, \]  

(7)

where \( k = 1/(16\pi) \) and \( \mathcal{L}_M \) represent the Lagrangian density for the matter fields.

Since the sum of both terms on the right hand side of Eq. (5) is invariant under local Lorentz transformations, the term \( -k e \Sigma^{abc} T_{abc} \) alone does
not display the invariance, unless the coefficients of the local Lorentz transformations fall off to zero sufficiently fast at spacelike infinity, so that the divergence term $\partial_\mu (e T^\mu)$ plays no role to the local Lorentz invariance of the action integral [20]. In general, under an arbitrary local Lorentz transformation the term $-k e \Sigma^{abc} T_{abc}$ is invariant up to a total divergence.

The variation of $L(e_{a\mu})$ with respect to $e^{a\mu}$ yields the fields equations. They read

$$e_{a\lambda} e_{b\nu} \partial_\nu (e \Sigma^{b\lambda\nu}) - e (\Sigma^{b\nu} e T_{b\nu\mu} - \frac{1}{4} e_{a\mu} T^{bcd} \Sigma_{bcd}) = \frac{1}{4k} e T_{a\mu} ,$$  \hspace{1cm} (8)

where $\delta L_M/\delta e^{a\mu} \equiv e T_{a\mu}$. It is possible to show that these field equations are equivalent to Einstein’s equations. After some algebraic manipulations we verify that the left hand side of the field equations above are identically equal to

$$\frac{1}{2} [R_{a\mu}(e) - \frac{1}{2} e_{a\mu} R(e)] .$$  \hspace{1cm} (9)

From now on we will consider the Lagrangian density in (7) subject to the unimodular condition. If we want to arrive at the field equations for the unimodular theory of gravity, we have to vary the Lagrangian density in (7) subject to the unimodular condition $e - 1 = 0$. This can be done by using the Lagrange multipliers method. For this purpose we add to the Lagrangian density the field $\Lambda(x)$ that yields a field equation that is precisely the unimodular condition. Therefore the unimodular Lagrangian density is written as

$$L'(e_{a\mu}, \Lambda(x)) = -k e \Sigma^{abc} T_{abc} - L_M + \Lambda (e - 1) ,$$  \hspace{1cm} (10)

Except for the unimodular condition, the tetrad field $e_{a\mu}$ is a priori unconstrained. The field equations are obtained by varying $L'(e_{a\mu}, \Lambda(x))$ with respect to $e^{a\mu}$ and $\Lambda(x)$, respectively. They are given by

$$R_{a\mu}(e) - \frac{1}{2} e_{a\mu} R(e) - \frac{1}{2k} e_{a\mu} \Lambda(x) = \frac{1}{2k} T_{a\mu} ,$$  \hspace{1cm} (11)

$$e - 1 = 0 .$$  \hspace{1cm} (12)

Taking the trace of (11), we obtain $\Lambda(x)$ as

$$\Lambda(x) = -\frac{1}{8} \left( k R(e) + \frac{1}{2} T \right) ,$$  \hspace{1cm} (13)
which allows us to rewrite the field equations (11) as
\[ R_{\alpha\mu}(e) - \frac{1}{4} e_{\alpha\mu} R(e) = \frac{1}{2k} \left( T_{\alpha\mu} - \frac{1}{4} e_{\alpha\mu} T \right). \] (14)

Since the covariant derivative of (9) vanishes, it is possible to show, with the help of (13), that \( \Lambda(x) \) is a space-time independent quantity,
\[ \frac{1}{8} \partial_\mu \left( kR(e) + \frac{1}{2} T \right) = \partial_\mu \Lambda(x) = 0 . \] (15)

The right hand side of Eq. (14) is invariant under the transformation
\[ T_{\alpha\mu} \to T_{\alpha\mu} + e_{\alpha\mu} C , \] (16)
where \( C \) is a space-time constant. These transformations may be interpreted as corrections to the energy-momentum tensor. Therefore the tensor \( R_{\alpha\mu}(e) \) on the left hand side of (14) is not affected by the transformations above. In addition, under this transformation Eq. (13) yields
\[ \Lambda \to \Lambda - \frac{1}{4} C . \] (17)
Thus, by combining Eqs. (16) and (17), the field equations (11) are unchanged under the transformations (16). A similar result was observed in [7] in terms of metric tensor.

### 3 The Legendre transform

In order to obtain the Hamiltonian density we rewrite the Lagrangian density \( \mathcal{L}'(e_{\alpha\mu}, \Lambda(x)) \) in the form \( \mathcal{L}' = p\dot{q} - \mathcal{H}_0 \), and identify the primary constraints. To do this, we will not carry out the 3 + 1 decomposition of the field quantities in terms of the lapse and shift functions. Therefore in the following both \( e_{\alpha\mu} \) and \( g_{\mu\nu} \) are space-time fields. The procedure adopted here is similar to that presented in [19].

From the Lagrangian density in (10) we obtain the momentum canonically conjugated to \( e_{\alpha\mu} \). It is given by
\[ \Pi^{\alpha\mu} = 4k e \Sigma^{\alpha\mu 0} . \] (18)
Given that \( \Sigma^{abc} = -\Sigma^{acb} \), we have \( \Pi^{\alpha 0} \equiv 0 \). In terms of (18), the Lagrangian density (10) can be rewritten as
\[ L'(e_{\alpha\mu}, \Lambda(x)) = \Pi^{ai} \dot{e}_{ai} - \Pi^{ai} \partial_{\alpha} e_{a0} - \frac{1}{2} \Pi^{ai} T_{a0i} - \]

\[ - ke{\Sigma}^{aij} T_{aij} + \Lambda(e - 1) , \]

where the dot over \( e_{ai} \) represents the time derivative. Also, we are assuming that \( L_M = 0 \).

Before we proceed, let us consider the full expression of \( \Pi^{ai} \) in terms of the torsion tensor. From Eqs. (18) and (6) it can be written as

\[ \Pi^{ai} = ke\{g^{00}(-g^{ij}T^0_{0j} - e^{ai}T^i_{0j} + 2e^{ai}T^j_{0j}) + \]

\[ + g^{0k}(g^{0j}T^a_{0j} + e^{ai}T^0_{0j}) + e^{ai}(g^{0j}T^i_{0j} + g^{ij}T^0_{0j}) - \]

\[ - 2(e^{ai}g^{0i}T^0_{0j} + e^{ai}g^{0j}T^i_{0j}) - g^{0k}g^{ij}T^a_{kj} + \]

\[ + e^{ak}(g^{0j}T^i_{kj} - g^{ij}T^0_{kj}) - 2(g^{k0}e^{ai} - g^{ki}e^{a0})T^j_{ji} \} . \]

Denoting \((...)\) and \([...\)] as the symmetric and antisymmetric parts of the field quantities, respectively, we decompose \( \Pi^{ai} \) into irreducible components,

\[ \Pi^{ai} = e^a_k \Pi^{(ki)} + e^a_k \Pi^{[ki]} + e^a_0 \Pi^{0i}, \]

where

\[ \Pi^{(ki)} = ke\{g^{00}(-g^{kj}T^i_{mj} + g^{ik}g^{il}) + g^{0k}(g^{0j}g^{il} - g^{0l}g^{ji}) \]

\[ + g^{0j}(g^{0i}g^{kl} - g^{0l}g^{ik})\} (T_{0ij} + T_{0j0}) + ke\Delta^{ki} , \]

\[ \Delta^{ki} = -g^{0m}(g^{kj}T^i_{mj} + g^{ij}T^k_{mj} - 2g^{ik}T^j_{mj}) - \]

\[ -(g^{km}g^{0i} + g^{im}g^{0k})T^j_{mj} , \]

\[ \Pi^{[ki]} = -ke\{g^{km}g^{ij}T^0_{mj} - (g^{km}g^{0i} - g^{im}g^{0k})T^j_{mj} \} , \]

\[ \Pi^{0i} = -2ke\{g^{ij}g^{0m}T^0_{mj} - (g^{0i}g^{m} - g^{00}g^{im})T^j_{mj} \} . \]

An important point in this analysis is that only the symmetric components \( \Pi^{(ki)} \) depend on \( T_{a0j} \), which contains the time derivative of the tetrad field. The other six components \( \Pi^{[ki]} \) and \( \Pi^{0k} \) depend solely on \( T_{aij} \). Therefore we can express only six components of the “velocity” fields \( T_{a0j} \) in terms of the
six components $\Pi^{(ki)}$. To do this we note from Eq. (22) that $\Pi^{(ki)}$ depends only on the symmetric components of $T_{a0j}$. We define

$$\psi_{ij} = T_{i0j} + T_{j0i},$$

and substitute the above definition into Eq. (22). We also define

$$P^{ki} = \frac{1}{ke} \Pi^{(ki)} - \Delta^{ki},$$

and find that $P^{ki}$ depend only on $\psi_{ij}$,

$$P^{ki} = -g^{00}(g^{km}g^{ij}\dot{\psi}_{mj} - g^{ki}\dot{\psi}) + (g^{0k}g^{im}g^{0j} + g^{0i}g^{km}g^{0j})\psi_{mj} - (g^{ik}g^{0m}g^{0j}\psi_{mj} + g^{0k}g^{0i}\psi),$$

where $\psi = g^{ij}\dot{\psi}_{ij}$.

We can now invert $\psi_{ij}$ in terms of $P^{ki}$. After a number of manipulations we arrive at

$$\psi_{ij} = -\frac{1}{g^{kl}} \left( P^{ki} g_{kl} g_{ij} - \frac{1}{2} g_{ij} P \right),$$

where $P = g_{ik}P^{ik}$.

By using the definition of $\Sigma^{abc}$ in terms of the torsion tensor, and using Eqs. (20), (25) and (28), we conclude that the third and fourth terms on the right hand side of Eq. (19) can be rewritten as

$$-\frac{1}{2} \Pi^{ai} T_{aoi} - ke \Sigma^{aij} T_{aij} = \frac{ke}{4g^{00}} \left( g_{ik}g_{jl} P^{ik} P^{kl} - \frac{1}{2} P^2 \right)$$

$$- \left( \frac{1}{4} g^{ik} g^{jl} T^{ra} i T^{akl} + ke \frac{1}{2} g^{il} T^{k} i T^{kl} - g^{il} T^{j} i T^{kl} \right).$$

Thus, finally we obtain the primary Hamiltonian density, $\mathcal{H}'_0 = \Pi^{ai} \dot{e}_{ai} - \mathcal{L}'$, as

$$\mathcal{H}'_0 = \mathcal{H}_0 - \Lambda(e - 1),$$

where

$$\mathcal{H}_0(e_{ai}, \Pi^{ai}, e_{a0}) = -e_{a0} \partial_i \Pi^{ai} - \frac{ke}{4g^{00}} \left( g_{ik}g_{jl} P^{ik} P^{kl} - \frac{1}{2} P^2 \right)$$

$$+ ke \left( \frac{1}{4} g^{ik} g^{jl} T^{ra} i T^{akl} + \frac{1}{2} g^{il} T^{k} i T^{kl} - g^{il} T^{j} i T^{kl} \right).$$
Now we can write the total Hamiltonian density. For this purpose we have to indentify the primary constraints. They are related to expressions (23) and (24), which represent relations between $e_{ai}$ and the momenta $\Pi_{ai}$. Thus we define

$$
\Gamma^{ik} = -\Gamma^{ki} = (\Pi^{ik} - \Pi^{ki}) + 2ke \{ g^{im}g^{kj}\mathcal{T}^{0}_{mj} - (g^{im}g^{0k} - g^{km}g^{0i})\mathcal{T}^{j}_{mj} \},
$$

$$
\Gamma^{0k} = \Pi^{0k} + 2ke \{ g^{kj}g^{0m}\mathcal{T}^{0}_{mj} - (g^{0k}g^{0m} - g^{00}g^{km})\mathcal{T}^{j}_{mj} \}.
$$

Before we write the total Hamiltonian density, we will simplify the constraints above. Since $\Pi_{a0} \equiv 0$, we can write the constraints above as a single constraint $\Gamma^{ab} = -\Gamma^{ba}$, where $\Gamma^{ik} = e_{ai}^{~i}e_{bj}^{~k}\Gamma^{ab}$ and $\Gamma^{0k} = e_{a0}^{~i}e_{bj}^{~k}\Gamma^{ab}$. Thus in view of Eq. (18) $\Gamma^{ab}$ can be written as

$$
\Gamma^{ab} = 2\Pi^{[ab]} + 4ke(\Sigma^{a0b} - \Sigma^{b0a}) .
$$

Therefore the total Hamiltonian density is given by

$$
\mathcal{H}' = \mathcal{H}'_{0} + \lambda_{ab}\Gamma^{ab} + \lambda_{a}\Pi^{a0} ,
$$

where $\lambda_{ab} = -\lambda_{ba}$ and $\lambda_{a}$ are Lagrange multipliers to be determined. Although in the usual Hamiltonian formalism of the TEGR the term that involves the constraint $\Pi^{a0} \equiv 0$ does not generate any additional information, here we have to add it to the total Hamiltonian density because it will be important to analyze the time evolution of the unimodular condition.

## 4 Secondary constraints

Considering Eq. (18) we notice that the momenta $\Pi^{a0}$ vanish identically, and so they constitute primary constraints whose time evolution induces secondary constraints,

$$
C^{a} \equiv \frac{\delta\mathcal{H}'}{\delta e_{a0}} = 0 .
$$

According to the terminology of Dirac, secondary constraints are relations between the fields and momenta which must be independent of the primary constraints, otherwise these relations will be equivalent to primary constraints.
In what follows, in order to obtain the expression of $C^a$ we have to vary only $H'_0$ with respect to $e_{a0}$, because the variation of $\Gamma^{bc}$ with respect to $e_{a0}$ vanishes identically,

$$\frac{\delta \Gamma^{ab}}{\delta e_{c0}} \equiv 0.$$  \hspace{1cm} (36)

To obtain the expression of $C^a$ we make use of the variation $\delta e^{c\mu}/\delta e_{a0} = -e^{a\mu} e^{c0}$. In addition, we need of the variation of $P^{ij}$ with respect to $e_{a0}$. It is given by

$$\frac{\delta P^{ij}}{\delta e_{a0}} = -e_{a0} P^{ij} + \gamma^{aij},$$

where $\gamma^{aij}$ is defined as

$$\gamma^{aij} = -e^{ak} [g^{00}(g^{im} T^{ki}_{\text{km}} + g^{im} T^{lj}_{\text{km}} + 2g^{ij} T^{m}_{\text{mk}}) + g^{0m}(g^{0j} T^{i}_{\text{mk}} + g^{0i} T^{j}_{\text{mk}}) - 2g^{0i} g^{0j} T^{m}_{\text{mk}} + (g^{jm} g^{0i} + g^{im} g^{0j} - 2g^{ij} g^{0m}) T^{0}_{\text{mk}}],$$

which satisfies $e_{a0} \gamma^{aij} = 0$. With these considerations we can now calculate $C^a$. After a long calculation we arrive at the expression for $C^a$, which is given by

$$C^a = -\partial_i \Pi^{ai} + e^{a0} \left[ -\frac{1}{4g^{00}} k e (g^{ik} g^{jl} P^{ij} P^{kl} - \frac{1}{2} P^2) + ke \left( \frac{1}{4} g^{im} g^{nj} T^{b}_{\text{mn}} T^{bij} + \frac{1}{2} g^{mi} T^{m}_{\text{ij}} - g^{ik} T^{m}_{\text{mn}} T^{n}_{\text{nk}} \right) - \frac{1}{2g^{00}} k e (g^{ik} g^{jl} \gamma^{aij} P^{kl} - \frac{1}{2} g_{ij} \gamma^{aij} P) - k e e^{aij} (g^{0m} g^{nj} T^{b}_{\text{ij}} T^{mn}_{\text{bn}} + g^{ij} T^{0}_{\text{mn}} T^{m}_{\text{ij}} + g^{0j} T^{m}_{\text{mj}} T^{m}_{\text{ni}} - 2g^{0k} T^{m}_{\text{mk}} T^{n}_{\text{ni}} - 2g^{ik} T^{0}_{\text{ij}} T^{m}_{\text{nk}}) - e^{a0} \Lambda e \right].$$  \hspace{1cm} (37)

The constraint above admits a simplification. After a number of manipulations we can show that the expression above can be written as

$$C^a = \frac{\delta H'_0}{\delta e_{a0}} = e^{a0} (H_0 - \Lambda e) + e^{ai} H_i,$$

where $H_i$ is defined as

$$H_i = -e_{ci} \partial_k \Pi^{ck} - \Pi^{ck} T_{cki}.$$  \hspace{1cm} (38)
From Eq. (38) we note that \( C' a \) satisfies the following relation
\[
e_{a0} C'^a = \mathcal{H}_0 - \Lambda e .
\]

In addition, because the variation of \( \mathcal{H}_i \) with respect to \( e_{a0} \) is identically null, it follows from Eqs. (30) and (38) that
\[
\frac{\delta C'^a}{\delta e_{c0}} = e^{a0} C'^c - e^{a0} C'^c \equiv 0 ,
\]

Therefore, in view of Eqs. (30), (34) and (39) we can write the total Hamiltonian density as
\[
\mathcal{H}'(e_{ai}, \Pi^{ai}, e_{a0}, \lambda_{ab}, \Lambda) = e_{a0} C'^a + \lambda_{ab} \Gamma^{ab} + \lambda_a \Pi^{a0} + \Lambda ,
\]
in terms of the constraints \( C'^a, \Gamma^{ab} \) and \( \Pi^{a0} \). We note that the vanishing of the constraints \( C'^a, \Gamma^{ab} \) and \( \Pi^{a0} \) does not imply the vanishing of \( \mathcal{H}' \), which depend on \( \Lambda \).

The variation of \( \mathcal{H}' \) with respect to \( e_{a0} \) yields the constraints \( C'^a \). Therefore we observe that \( e_{a0} \) in the total Hamiltonian density \( \mathcal{H}' \) arises as Lagrange multipliers, together with \( \lambda_{ab} \) and \( \lambda_a \). Moreover, as we will see in the next section, no new constraint appears in the formalism by time evolution of the secondary constraints \( C'^a \). The main difference between the formalism presented here and the Hamiltonian formulation presented in Ref. [19] is that in the present case the canonical Hamiltonian density \( \mathcal{H}'_0 \) does not vanish as a consequence of the secondary constraint \( C'^a = 0 \) (see Eq. (39)). This feature takes place here because of the unimodular condition \( e - 1 = 0 \), which implies that not all components of \( e_{a\mu} \) are independent. We remark that we have not explicitly implemented in the expressions above the condition \( e - 1 = 0 \). The variation \( \delta \mathcal{H}'/\delta \Lambda = 0 \) yields the unimodular condition \( e - 1 = 0 \).

We remark that the structure of the Hamiltonian density given by Eq. (41) is very much different from the Hamiltonian formulation of tetrad gravity constructed out of the scalar curvature density \( eR(\omega) \), in terms of the tetrad field and the spin connection as given by Eq. (2) (see, for instance, Ref. [21]). The essential difference between the Hamiltonian formulation derived from Eq. (7) and those obtained out of invariants of the curvature tensor (typically, the scalar curvature density) is that the Hamiltonian constraint in the present case naturally emerges with a total divergence of the type.
−∂_i Π^{ai} (the first term on the right hand side of of Eq. (37)), that gives rise
to the total energy-momentum four-vector (see Eq. (59) below). In contrast,
the Hamiltonian constraint in the ADM type formulation of tetrad theories
of gravity does not display any nontrivial, total divergence (see Eq. (22) of
[21], which is very much similar to the Hamiltonian constraint of the ADM
formulation). It is possible to establish total divergences, in the form of scalar
or SO(3,1) vector densities, in theories constructed out of the torsion tensor,
but not in metrical theories of gravity.

We finally observe that the timelike component \( e^a_0 \) of the tetrad field,
which stands as a Lagrange multiplier in Eq. (41), may be expressed in
terms of the lapse and shift functions as \[ e^a_0 = \eta^a N + N^i e^a_i, \]
where \( \eta^a = -Ne^{a0} \) is a timelike vector that satisfies
\[ \eta_a e^a_i = 0, \quad \eta_a \eta^a = -1, \]
and whose direction may be fixed by means of a local Lorentz rotation.
Therefore the Lagrange multiplier \( e^a_0 \) encompasses both the lapse and shift
functions, according to (42). However, the lapse function does not appear in
the contraction \( e^a_0 C'^a = N(\eta_a C'^a) + N^i(e^a_i C'^a). \) Considering the expression
of \( C'^a \) we easily find
\[ \begin{aligned}
N(\eta_a C'^a) &= (\mathcal{H}_0 - \Lambda e) - N^i \mathcal{H}_i, \\
N^i(e^a_i C'^a) &= N^i \mathcal{H}_i,
\end{aligned} \]
in agreement with (38). In the expression above we have considered \( N = (-g^{00})^{-1/2} \) and \( N^i = g^{0i}/N^2. \) Thus the lapse function does not arise as a
Lagrange multiplier in the Hamiltonian density.

5 Lagrange multipliers and Poisson brackets

Before we obtain the Poisson brackets of the constraints of the theory, we
will determine the expressions for the Lagrange multipliers \( \lambda_{ab} \) and \( \lambda_a \) that
arise in \( \mathcal{H}' \). The Poisson brackets between two quantities \( A \) and \( B \) is defined as
\[ \{A, B\} = \int d^3 z \left( \frac{\delta A}{\delta e_{a\mu}(z)} \frac{\delta B}{\delta \Pi^{a\mu}(z)} - \frac{\delta A}{\delta \Pi^{a\mu}(z)} \frac{\delta B}{\delta e_{a\mu}(z)} \right), \]

13
from what we can write down the time evolution equations. The first set of Hamilton’s equations is given by

\begin{align}
\dot{e}_{a\mu}(x) &= \{e_{a\mu}(x), \int d^3 y \mathcal{H}'(y)\} \\
&= \int d^3 y \frac{\delta}{\delta \Pi^{a\mu}(x)} [\mathcal{H}'_0(y) + \lambda_{bc}(y) \Gamma^{bc}(y) + \\
&+ \lambda_a(y) \Pi^{a0}(y)].
\end{align}

In the equation above the dot over $e_{a\mu}$ represents the time derivative. This equation can be worked out so that for $\mu = 0$ we obtain

$$
\dot{e}_{a0} = \lambda_a,
$$

and for $\mu = j$,

$$
T_{a0j} = -\frac{1}{2g^{00}} e_a^k \left( g_{ik} g_{jm} P^{lm} - \frac{1}{2} g_{kj} P \right) + 2\lambda_{aj},
$$

from what follows

$$
T_{i0j} + T_{j0i} = \psi_{ij} = -\frac{1}{g^{00}} \left( g_{ii} g_{jm} P^{lm} - \frac{1}{2} g_{ij} P \right),
$$

and

$$
\lambda_{ab} = \frac{1}{4} (T_{a0b} - T_{b0a} + e_a^0 T_{00b} - e_b^0 T_{00a}).
$$

Therefore the Lagrange multipliers acquire a well-defined meaning in terms of the time derivatives of the field quantities and consequently we can obtain an expression for $\Pi^{ij}$ in terms of $\psi_{ij}$ by using equation (25). The dynamical evolution of the fields quantities is completed with the second set of Hamilton’s equations for $\Pi^{a\mu}$,

$$
\dot{\Pi}^{a\mu}(x) = \{\Pi^{a\mu}(x), \int d^3 y \mathcal{H}'(y)\} = -\int d^3 y \left( \frac{\delta \mathcal{H}'(y)}{\delta e_{a\mu}(x)} \right).
$$

The calculations of the Poisson brackets of the constraints are very long, tedious and intricate. Here we will just present the results. We first calculate the Poisson brackets between $\mathcal{H}'_0(x)$ and $\mathcal{H}'_0(y)$, and then the Poisson brackets between $\mathcal{H}'_0(x)$ and $\Gamma^{bc}(y)$. They are given by, respectively,

$$
\{\mathcal{H}'_0(x), \mathcal{H}'_0(y)\} = 0,
$$


\textbf{14}
\[ \{ \mathcal{H}_0(x), \Gamma^{bc}(y) \} = (e^b_0 C^{dc} - e^c_0 C^{db}) \delta(x-y). \] (49)

By using the definition of \( C^{ra} \) in Eq. (38), and the relation given by Eq. (40), together with Eq. (48), it follows that
\[ \{ C^{ra}(x), C^{rb}(y) \} = 0. \] (50)

For the calculation of the second Poisson bracket we again use the definition of \( C^{ra} \) in Eq. (38) and the fact that the variation of \( \Gamma^{ab} \) and \( C^{ra} \) with respect to \( e_{c0} \) is identically zero (see Eqs. (36) and (38)). So, taking the variation of equation (49) with respect to \( e_{a0} \) on both sides we obtain
\[ \{ C^{ra}(x), \Gamma^{bc}(y) \} = \left( \eta^{ab} C^{bc} - \eta^{ac} C^{db} \right) \delta(x-y). \] (51)

And finally, by means of explicit calculations we obtain the third Poisson bracket, which is given by
\[ \{ \Gamma^{ab}(x), \Gamma^{cd}(y) \} = \left( \eta^{ac} \Gamma^{bd} + \eta^{bd} \Gamma^{ac} - \eta^{ac} \Gamma^{bd} - \eta^{bd} \Gamma^{ac} \right) \delta(x-y). \] (52)

We remark here that the Poisson brackets of the constraints \( \Pi^{a0} \) with \( C^{ra} \) and \( \Gamma^{ab} \) vanish identically.

Let us now analyze the time evolution of the unimodular condition, which amounts to calculating the time evolution of the determinant \( e \), namely,
\[ \dot{e}(x) = \{ e(x), \int \mathcal{H}'(y) d^3y \}. \]

Working out both sides of this equation and using that \( \dot{e}_{a0} = \lambda_a \) and \( e^{aj} \lambda_{aj} = 0 \) we obtain the following relation,
\[ \dot{e} = ee^{aj} \dot{e}_{aj} = ee^{aj} \left[ \partial_j e_{a0} - \frac{1}{2} g^{ak} e_a k \left( g_{l[m} P^{lm} - \frac{1}{2} g_{k} P \right) \right]. \]

Assuming that \( e \) is not null and that \( e^{aj} \) are arbitrary field quantities, this relation is equivalent to the relation shown in Eq. (45), that is obtained from the first set of Hamilton’s equations. Thus we see that the unimodular condition does not generate any additional constraint in the formalism.

Therefore, in view of the constraint algebra above for \( C^{ra} \) and \( \Gamma^{ab} \), we see that these constraints constitute a set of first class constraints. The algebra is very much similar to the algebra of the Poincaré group. As asserted at the end of the previous section, given that the total Hamiltonian density
is a combination of the constraints $C^a$, $\Gamma^{ab}$ and $\Pi^{a0}$, plus the cosmological constant $\Lambda$, no new constraint arises in the formalism by means of the time evolution of $C^a$ and $\Gamma^{ab}$, as the Poisson brackets (50), (51) and (52) vanish weakly. It is important to note that if we make $\Lambda = 0$ in this theory, the Hamiltonian formalism presented here reduces to the Hamiltonian formalism of the TEGR presented in Ref. [19] and as a consequence all Poisson brackets presented in Ref. [19] can be obtained from the Poisson brackets shown in Eqs. (50 - 52).

6 Summary of the results of the paper

The results of the paper can be summarized as follows.

1. The configuration space is described by the tetrad field $e^a_\mu$ and the function $\Lambda(x)$ which, in view of Eq. (15), turns out to be a constant. The Lagrangian field equations for $e^a_\mu$ are given by (11) or by Eq. (60) below. Note that Eq. (15) is obtained by taking the covariant derivative of the field equations.

2. The total Hamiltonian density is given by Eq. (41). The phase space of the theory is constructed out of the pairs of canonically conjugated field quantities $(e_{ai}, \Pi^{ai})$ and $(e_{a0}, \Pi^{a0})$, and $\Lambda$. We found that it was not necessary to introduced the momentum $\Pi_{\Lambda}$ canonically conjugated to $\Lambda$, since we would have $\Pi_{\Lambda} = 0$. In view of the fact that Lagrangian density does not contain the time derivative of $e_{a0}$, we have $\Pi^{a0} = 0$. The complete set of Lagrange multipliers is $(e_{a0}, \lambda_{ab}, \lambda_a)$.

3. All constraints are first class. They are given by $C^a$ (Eq. (38)), $\Gamma^{ab}$ (Eq. (33)), and by the trivial constraint $\Pi^{a0} = 0$. In view of Eqs. (30) and (34) the condition $e - 1 = 0$ follows from the equation $\delta \mathcal{H}'/\delta \Lambda = 0$.

4. The constraint algebra is given by Eqs. (50), (51) and (52). The Poisson bracket of the quantity $(e - 1)$ with the total Hamiltonian yields ultimately the evolution equation for the tetrad field $e_{a\mu}$, and therefore it does not generate additional constraints. Moreover, the Poisson brackets between $\Pi^{a0}$ and $C^a$ and $\Gamma^{ab}$ vanish strongly in view of Eqs. (36) and (40).

5. The physical degrees of freedom of the theory may be counted in the following way. The pair of dynamical field quantities $(e_{ai}, \Pi^{ai})$ displays $12 +$
12 = 24 degrees of freedom. The 4+6 first class constraints \((C^a, \Gamma^{ab})\) generate symmetries of the action, and thus they reduce 10+10=20 degrees of freedom. Therefore in the phase space of the theory there are 4 degrees of freedom, as expected. The unimodular condition \(e - 1 = 0\) enforces the diffeomorphisms of the theory to be transverse diffeomorphisms \(x^\mu = x^\mu + \xi^\mu(x)\), defined by the condition \(\partial_\mu \xi^\mu = 0\) \[8\]. Thus the unimodular condition \(e - 1 = 0\) reduces one degree of freedom of the tetrad field, and at the same time reduces the symmetry under diffeomorphisms. Therefore it does not alter the counting of physical degrees of freedom.

The action of the constraints \(C^a\) and \(\Gamma^{ab}\) on the tetrad field may be computed by means of the Poisson brackets defined in section 5. We find it more convenient to analyse separately the action of \(H^0_i\) and \(H_i\), instead of \(C^a\). Let \(\varepsilon_{ab}(x) = -\varepsilon_{ba}(x)\), \(\varepsilon^i(x)\) and \(\varepsilon^0(x)\) represent arbitrary infinitesimal functions. After some calculations we find

\[
\delta e_{a\mu}(x) \equiv \varepsilon_{bc}(x) \int d^3y \{e_{a\mu}(x), \Gamma^{bc}(y)\} = 2\varepsilon_{ab}(x)e^b_{\mu}, \tag{53}
\]

\[
\delta e_{a\mu}(x) \equiv \varepsilon^i(x) \int d^3y \{e_{a\mu}(x), H_i(y)\} = -\varepsilon^i(x)\delta^k_\mu \partial_\mu \varepsilon_{ak}, \tag{54}
\]

\[
\delta e_{a\mu} \equiv \varepsilon^0(x) \int d^3y \{e_{a\mu}(x), H_0(y)\} = \varepsilon^0(x)\delta^k_\mu [\varepsilon_{ak}(x) - 2\lambda_{ak}(x)], \tag{55}
\]

where \(\lambda_{ak}\) is defined by \[10\],

\[
\lambda_{ak} = \frac{1}{4}(T_{a0k} - T_{k0a} + e_a^0 T_{00k}),
\]

and

\[
\dot{e}_{ak} = \int d^3y \{e_{ak}, H'(y)\}.
\]

Equations \[(53)\] and \[(54)\] indicate that \(\Gamma^{bc}\) and \(H_i\) have a clear interpretation as generators of local Lorentz transformations and spatial diffeomorphisms, respectively. Equation \[(55)\] tells us that \(H'_0\) generates the time evolution of \(e_{ak}\) provided the constraint \(\Gamma^{bc}\) vanishes strongly, so that the Hamiltonian density does not contain the Lagrange multipliers \(\lambda_{ab} = -\lambda_{ba}\). However, in the general case, when \(\Gamma^{bc}\) is not required to vanish, we have
\[
\delta g_{ij} = \delta(e^a_i e_{aj}) = \varepsilon^0(x)\{g_{ij}(x), \int d^3y \mathcal{H}_0'(y)\} = \varepsilon^0(x)\dot{g}_{ij}, \quad (56)
\]
and
\[
\delta g_{\mu\nu} = \varepsilon_{ab}(x)\{g_{\mu\nu}(x), \int d^3y \Gamma^{ab}(y)\} = 0. \quad (57)
\]

7 Concluding remarks

In this paper we have obtained the Hamiltonian formulation and constraint algebra of the unimodular theory of gravity in the framework of the TEGR. The constraints are first class, and the constraint algebra is presented in a more simple form, as compared to the formulation obtained in Ref. [19]. The simplification achieved is significant. Although in the unimodular theory of gravity the condition \( e = 1 \) holds, we have kept the determinant \( e \) in all expressions. The transition to the ordinary formulation of the TEGR is easily obtained by just making \( \Lambda = 0 \) and dropping the condition \( e = 1 \).

In the constraint algebra determined by Eqs. (50), (51) and (52) the structure functions are space-time independent functions. This feature may be relevant to a possible approach to the quantization of the gravitational field. We recall that the master constraint programme for loop quantum gravity \[25\] is an approach to the canonical quantization of the gravitational field whose idea is to replace the infinity of constraints of the theory (one at each space-time event) by a single master equation. The difficulty in applying the master constraint programme to the known formulations of canonical gravity is that the representation theory of the usual Dirac algebra of constraints (the hypersurface deformation algebra) is very intricate due to the space-time dependent structure functions that arise in the Poisson brackets of the constraints. On the other hand, the structure of the algebra given by Eqs. (50), (51) and (52) is very simple. The representation of this algebra may lead to a viable approach to the quantization of gravity.

The field equations for the gravitational field in the Hamiltonian or Lagrangian form of the TEGR allow definitions of the energy-momentum and angular momentum of the gravitational field. These definitions are not obtained out of the action integral or the total Hamiltonian. In the framework of unimodular relativity we establish these definitions in similarity with the previous approach \[17\]. We consider first Eq. (37). The equation \( C'^a = 0 \) may be written in a simplified form as
− \partial_i \Pi^{ai} = h^a + e^{a0} \Lambda e \ , \tag{58}

where the intricate definition for $h^a$ may be obtained directly from Eq. (37).

In similarity with the procedure of Ref. [17], the integral form of the equation above yields the definition of the total energy, which includes now the contribution of the cosmological constant,

$$P^a = - \int_V d^3 x \partial_i \Pi^{ai} . \tag{59}$$

$V$ is a finite volume of the three-dimensional space. This definition may also be obtained in the Lagrangian framework. The field equation (11) may be written as

$$e_a \lambda^b \partial_\nu (e^{b \lambda \nu} - e(\Sigma^{b \nu} a T_{b \nu \mu} - \frac{1}{4} e_{a \mu} T^{bcd} \Sigma_{bcd}) = \frac{1}{4k} e(T_{a \mu} + e_{a \mu} \Lambda) . \tag{60}$$

By following the same procedure of Ref. [23], we find that the equation above may be expressed in terms of $\Pi^{ai}$ according to

$$\partial_i \Pi^{ai} = - k e e^{a \mu} (4 \Sigma^{b j \mu} T_{b j \mu} - \delta^{a \mu} \Sigma^{bcd} T_{bcd}) - e e^{a \mu} (T^0 \mu + g^0 \mu) . \tag{61}$$

The integral form of this equation yields

$$P^a = \int_V d^3 x e e^a \mu (t^0 \mu + T^0 \mu + g^0 \mu) = - \int_V d^3 x \partial_i \Pi^{ai} . \tag{62}$$

The quantity $t^{\lambda \mu} = k(4 \Sigma^{bc \mu} T_{bc \mu} - g^{\lambda \mu} \Sigma^{bcd} T_{bcd})$ is a tensor under general coordinate transformations, and is interpreted as the gravitational energy-momentum tensor [23, 24]. In the absence of the energy-momentum $T^{\mu \nu}$ for the matter fields, $P^a$ does represent the gravitational energy-momentum four-vector, again including the contribution of the cosmological constant. We emphasize that the definition of the energy-momentum four-vector $P^a$ is obtained directly from the field equations, not from the action integral. The tetrad field $e^a \mu$ yields the space-time metric tensor, and at the same time establishes the frame for a given observer in space-time endowed with the four-velocity $u^\mu = e_{(0)}^\mu$.
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