Signaling equilibria in mean-field games

Deepanshu Vasal
Department of Electrical and Computer Engineering
University of Texas, Austin
Austin, TX 78704.
dvasal@utexas.edu

Abstract

In this paper, we consider both finite and infinite horizon discounted dynamic mean-field games where there is a large population of homogeneous players sequentially making strategic decisions and each player is affected by other players through an aggregate population state. Each player has a private type that only she observes. Such games have been studied in the literature under simplifying assumption that population state dynamics are stationary. In this paper, we consider non-stationary population state dynamics and present a novel backward recursive algorithm to compute Markov perfect equilibrium (MPE) that depend on both, a player’s private type, and current (dynamic) population state. Using this algorithm, we study a security problem in cyberphysical system where infected nodes put negative externality on the system, and each node makes a decision to get vaccinated. We numerically compute MPE of the game.

I. INTRODUCTION

With increasing amount of integration of technology in our society and with recent advancements in computation and algorithmic technologies, there is an unprecedented scale of interaction among people and devices. With technologies such as ride sharing platforms and social media apps completely integrated, and new technologies such as cyber physical systems, large scale renewable energy, electric vehicles, cryptocurrencies and smart grid on the horizon, there is paramount need to design and understand the behavior of such large scale interactions and their impact on our society. In this paper, we present a new methodology to analyze such interactions through mean-field dynamic games.
Dynamic games is a powerful tool to model such sequential strategic interaction among selfish players. There are many other applications such as dynamic auctions, security, markets, traffic routing, wireless systems, social learning, oligopolies—i.e. competition among firms, and more. For instance, consider repeated online advertisement auctions, where advertisers place bids for locations on a website to sell a product. These bids are calculated based on the value of that product, which is privately observed by the advertiser and past actions of other advertisers, which are observed publicly. Each advertiser’s goal is to maximize its reward, which for an auction depends on the actions taken by others.

When players have complete information, an appropriate solution concept for such games is Markov perfect equilibria (MPE) [1], where players’ strategies depend on a coarser Markovian state of the systems, instead of the whole history of the game which grows exponentially with time and thus becomes unwieldy. In general, there exists a backward recursive methodology to compute non-signaling MPEs of a game i.e. when players’ actions do not reveal any payoff relevant private information. However, when the number of players is large, computing MPE becomes intractable.

To model the behavior of large population strategic interactions, mean-field games were introduced independently by Huang, Malhamé, and Caines [2], and Lasry and Lions [3]. In such games, there are large number of homogenous strategic players, where each player has infinitesimal affect on system dynamics and is affected by other players through a mean-field population state. There have been a number of applications such as economic growth, security in networks, oil production, volatility formation, population dynamics (see [4]–[9] and references therein).

In this paper, to model the scenarios described above we consider discounted infinite-horizon dynamic mean-field games where there is a large population of homogenous players sequentially making strategic decisions and each player is affected by other players through a mean-field population state. Each player has a private type that evolves through a controlled Markov process which only she observes and all players observe the current population state which is the distribution of other players’ types.

The work closest to ours is [10] where authors considered a model in which they studied stationary equilibria of a mean-field game. In such games, the mean-field dynamics are given by McKean Vlasov equations which are coupled together with the strategies of the players. Authors
make simplifying assumption on the model that the players are *oblivious* with respect to the mean-field statistics, and are playing in the limit such that the mean-field distribution has converged. This allows them to decouple the mean-field dynamics with that of the rest of the game.

In this paper, we remove that assumption and consider a more general and realistic model where players are *cognizant* i.e. they actively observe the current population state (which need not have converged) and act based on that population state and their own private state. In such systems, the update of population state depends on the symmetric equilibrium strategy of the players while equilibrium strategies are computed using equilibrium update. Thus equilibrium strategies and population update are interlinked together. We provide a novel backward recursive algorithm to compute *non-stationary, signaling* Markov perfect equilibrium (MPE) of that game.

Using this framework, we consider malware spread problem in a cyber-physical system where nodes get infected by an independent random process and for each node, there is a higher risk of getting infected due to negative externality imposed by other infected players. At each time $t$, each player privately observes its own state and publicly observes the population of infected nodes, based on which it has to make a decision to repair or not. Using our algorithm, we find equilibrium strategies of the players which are observed to be non-decreasing in the healthy population state.

Our algorithm is motivated by recent developments in the theory of dynamic games with asymmetric information in [11]–[15], where authors in these works have considered different models of such games and provided a sequential decomposition framework to compute Markovian perfect Bayesian equilibria of such games.

### A. Notation

We use uppercase letters for random variables and lowercase for their realizations. For any variable, subscripts represent time indices and superscripts represent player identities. We use notation $-i$ to represent all players other than player $i$ i.e.  
$-i = \{1, 2, \ldots i - 1, i + 1, \ldots, N\}$. We use notation $a_{t,t'}$ to represent the vector $(a_t, a_{t+1}, \ldots a_{t'})$ when $t' \geq t$ or an empty vector if $t' < t$. We use $a_{t}'$ to mean $(a_t^1, a_t^2, \ldots, a_t^{i-1}, a_t^{i+1}, \ldots, a_t^N)$ . We remove superscripts or subscripts if we want to represent the whole vector, for example $a_t$ represents $(a_t^1, \ldots, a_t^N)$. We denote the indicator function of any set $A$ by $\mathbb{1}\{A\}$. For any finite set $S$, $\mathcal{P}(S)$ represents space of
probability measures on $S$ and $|S|$ represents its cardinality. We denote by $P^\sigma$ (or $E^\sigma$) the probability measure generated by (or expectation with respect to) strategy profile $G$. We denote the set of real numbers by $\mathbb{R}$. For a probabilistic strategy profile of players $(\sigma^i_t)_{i \in N}$ where probability of action $a^i_t$ conditioned on $z_{1:t}, x^i_{1:t}$ is given by $\sigma^i_t(a^i_t \mid z_{1:t}, x^i_{1:t})$, we use the short hand notation $\sigma^{-i}_t(a^{-i}_t \mid z_{1:t}, x^{-i}_{1:t})$ to represent $\prod_{j \neq i} \sigma^j_t(a^j_t \mid z_{1:t}, x^j_{1:t})$. All equalities and inequalities involving random variables are to be interpreted in a.s. sense.

II. MODEL AND BACKGROUND

We consider both finite and infinite-horizon discrete-time large population sequential game as follows. There are $N$ homogenous players, where $N$ tends to $\infty$. We denote the set of homogenous players by $[N]$ and with some abuse of notation, set of time by $[T]$ for both finite and infinite time horizon. In each period $t \in [T]$, player $i \in [N]$ observes a private type $x^i_t \in X = \{1, 2, \ldots, N_x\}$ and a common observation $z_t \in Z$, takes action $a^i_t \in A = \{1, 2, \ldots, N_a\}$, and receives a reward $R(x^i_t, a^i_t, z_t)$ which is a function of its current type $x^i_t$, action $a^i_t$ and the common observation $z_t$. The common observation $z_t = (z_t(1), z_t(2), \ldots, z_t(N_x))$ be the fraction of population having type $x \in X$ at time $t$ i.e.

$$z_t(x) = \frac{1}{N} \sum_{i=1}^{N_x} \mathbb{I}\{x^i_t = x\}, \quad (1)$$

where $\sum_{i=1}^{N_x} z_t(i) = 1$. Player $i$’s type evolve as a controlled Markov process,

$$x^i_{t+1} = f_x(x^i_t, a^i_t, z_t, w^i_t). \quad (2)$$

The random variables $(w^i_{t,i})_{i,t}$ are assumed to be mutually independent across players and across time. We also write the above kernel as $x^i_{t+1} \sim Q_x(\cdot \mid x^i_t, a^i_t, z_t)$.

In any period $t$, player $i$ observes $(z_{1:t}, x^i_{1:t})$. She takes action $a^i_t$ according to a behavioral strategy $\sigma^i = (\sigma^i_t)_t$, where $\sigma^i_t : Z^t \times X^t \to \mathcal{P}(A)$. This implies $A^i_t \sim \sigma^i_t(\cdot \mid z_{1:t}, x^i_{1:t})$.

For finite time-horizon game, $G_T$, each player wants to maximize its total expected discounted reward over a time horizon $T$, discounted by discount factor $0 < \delta \leq 1$, 

$$J^{i,T} := \mathbb{E}^\sigma \left[ \sum_{t=1}^{T} \delta^{t-1} R(X^i_t, A^i_t, Z_t) \right]. \quad (3)$$
For the infinite time-horizon game, $G_\infty$, each player wants to maximize its total expected discounted reward over an infinite-time horizon discounted by discount factor $0 < \delta < 1$,

$$J_i^{\infty} := \mathbb{E}^\sigma \left[ \sum_{t=1}^{\infty} \delta^{t-1} R(X_t^i, A_t^i, Z_t) \right]. \quad (4)$$

A. Solution concept: MPE

The Nash equilibrium (NE) of $G_T$ is defined as strategies $\tilde{\sigma} = (\tilde{\sigma}_i^t)_{i \in [N], t \in [T]}$ that satisfy, for all $i \in [N]$,

$$\mathbb{E}(\tilde{\sigma}_i^t, \tilde{\sigma}_-i^t) \left[ \sum_{t=1}^{T} \delta^{t-1} R(X_t^i, A_t^i, Z_t) \right] \geq \mathbb{E}(\sigma_i^t, \tilde{\sigma}_-i^t) \left[ \sum_{t=1}^{T} \delta^{t-1} R(X_t^i, A_t^i, Z_t) \right], \quad (5)$$

For sequential games, however, a more appropriate equilibrium concept is Markov perfect equilibrium (MPE) \[1\], which we use in this paper. We note that an MPE is also a Nash equilibrium of the game, although not every Nash equilibrium is an MPE. An MPE $\tilde{\sigma}$ satisfies sequential rationality such that for $G_T$, $\forall i \in [N], t \in [T], h_i \in H_i, \sigma_i^t$,

$$\mathbb{E}(\tilde{\sigma}_i^t, \tilde{\sigma}_-i^t) \left[ \sum_{n=t}^{T} \delta^{n-t} R(X_n^i, A_n^i, Z_n) | z_{1:t}, x_{1:t}^i \right] \geq \mathbb{E}(\sigma_i^t, \tilde{\sigma}_-i^t) \left[ \sum_{n=t}^{T} \delta^{n-t} R(X_n^i, A_n^i, Z_n) | z_{1:t}, x_{1:t}^i \right], \quad (6)$$

NE and MPE for $G_\infty$ are defined in a similar way where summation in the above equations is taken such that $T$ is replaced by $\infty$.

III. A METHODOLOGY TO COMPUTE MPE

In this section, we will provide a backward recursive methodology to compute MPE for both $G_T$ and $G_\infty$. We will consider Markovian equilibrium strategies of player $i$ which depend on the common information at time $t$, $z_t$, and on its current type $x_t^i$.\[1\] Equivalently, player $i$ takes action of the form $A_t^i \sim \sigma_t^i(\cdot | z_t, x_t^i)$. Similar to the common agent approach in \[16\], an alternate way of defining the strategies of the players is as follows. We first generate partial function $\gamma_t^i : \mathcal{X} \rightarrow \mathcal{P}(\mathcal{A})$ as a function of $z_t$ through a stationary strategy function $\theta_t^i : \mathcal{Z} \rightarrow (\mathcal{X} \rightarrow \mathcal{P}(\mathcal{A}))$ such that $\gamma_t^i = \theta_t^i[z_t]$. Then action $A_t^i$ is generated by applying this prescription function $\gamma_t^i$ on player $i$’s current private information $x_t^i$, i.e. $A_t^i \sim \gamma_t^i(\cdot | x_t^i)$. Thus $A_t^i \sim \sigma_t^i(\cdot | z_t, x_t^i) = \theta_t^i[z_t](\cdot | x_t^i)$.

\[1\] Note however, that the unilateral deviations of the player are considered in the space of all strategies.
We are only interested in symmetric equilibria of such games such that \( A_i \sim \gamma_t(\cdot | x_i^t) = \theta_t[z_t](\cdot | x_i^t) \) i.e. there is no dependence of \( i \) on the strategies of the players.

For a given symmetric prescription function \( \gamma_t = \theta[z_t] \), the statistical mean-field \( z_t \) evolves according to the discrete-time McKean Vlasov equation, \( \forall y \in \mathcal{X} \):

\[
z_{t+1}(y) = \sum_{x \in \mathcal{X}} \sum_{a \in \mathcal{A}} z_t(x) \gamma_t(a|x) Q_x(y|x, a, z_t),
\]

which implies

\[
z_{t+1} = \phi(z_t, \gamma_t).
\]

### A. Backward recursive algorithm for \( G_T \)

In this subsection, we will provide a methodology to generate MPE of \( G_T \) of the form described above. We define an equilibrium generating function \( \theta_t \{ z_t \} \in \mathbb{T} \), where for each \( z_t \), we generate \( \tilde{\gamma}_t = \theta_t[z_t] \). In addition, we generate a reward-to-go function \( (V_t)_{t \in \mathbb{T}} \), where \( V_t : \mathcal{Z} \times \mathcal{X} \rightarrow \mathbb{R} \). These quantities are generated through a fixed-point equation as follows.

1. Initialize \( \forall z_{T+1}, x_{T+1}^i \in \mathcal{X}, \)

\[
V_{T+1}(z_{T+1}, x_{T+1}^i) \overset{\triangle}{=} 0.
\]

2. For \( t = T, T - 1, \ldots, 1 \), \( \forall z_t, \) let \( \theta_t[z_t] \) be generated as follows. Set \( \tilde{\gamma}_t = \theta_t[z_t] \), where \( \tilde{\gamma}_t \) is the solution of the following fixed-point equation \(^2\) \( \forall i \in [N], x_i^t \in \mathcal{X}, \)

\[
\tilde{\gamma}_t(\cdot | x_i^t) \in \arg\max_{\gamma} \mathbb{E}^{\gamma(\cdot | x_i^t)} \left[ R(x_i^t, A_i^t, z_t) + \delta V_{t+1}(\phi(z_t, \tilde{\gamma}_t), X_{t+1}^i)|z_t, x_i^t] \right],
\]

where expectation in \((10)\) is with respect to random variable \( (A_i^t, X_{t+1}^i) \) through the measure \( \gamma_t(a_i^t|x_i^t)Q_x(x_{t+1}^i|x_i^t, a_i^t, z_t) \). We note that the solution of \((10)\), \( \tilde{\gamma}_t \), appears both on the left of \((10)\) and on the right side in the update of \( z_t \), and is thus unlike the fixed-point equation found in Bayesian Nash equilibrium.

Furthermore, using the quantity \( \tilde{\gamma}_t \) found above, define

\[
V_t(z_t, x_t^i) \overset{\triangle}{=} \mathbb{E}^{\tilde{\gamma}_t(\cdot | x_i^t)} \left[ R(x_i^t, A_i^t, z_t) + \delta V_{t+1}(\phi(z_t, \tilde{\gamma}_t), X_{t+1}^i)|z_t, x_i^t] \right].
\]

\(^2\)We discuss the existence of solution of this fixed-point equation in Section \ref{sec:existence}
Then, an equilibrium strategy is defined as

$$\tilde{\sigma}_i^t(a_i^t|z_{1:t}, x_{1:t}^i) = \tilde{\gamma}_t(a_i^t|x_i^t),$$

(12)

where $\tilde{\gamma}_t = \theta[z_t].$

**Theorem 1.** A strategy $(\tilde{\sigma})$ constructed from the above algorithm is an MPE of the game i.e. $\forall t, h^i_t \in \mathcal{H}_t^i, \sigma^i,$

$$\mathbb{E}^{(\tilde{\sigma}^i\tilde{\sigma}^{-i})}\left[\sum_{n=t}^{T} \delta^{n-t} R(X_n^i, A_n^i, Z_n)|z_{1:t}, x_{1:t}^i\right] \geq \mathbb{E}^{(\sigma^i\tilde{\sigma}^{-i})}\left[\sum_{n=t}^{T} \delta^{n-t} R(X_n^i, A_n^i, Z_n)|z_{1:t}, x_{1:t}^i\right],$$

(13)

**Proof.** Please see Appendix C.

**B. Backward recursive algorithm for $G_\infty$**

For the infinite-horizon problem, we assume the reward function $R$ to be absolutely bounded.

We define an equilibrium generating function $\theta : \mathcal{Z} \to \{\mathcal{X} \to \mathcal{P}(A)\}$, where for each $z_t$, we generate $\tilde{\gamma}_t = \theta[z_t].$ In addition, we generate a reward-to-go function $V : \mathcal{Z} \times \mathcal{X} \to \mathbb{R}.$ These quantities are generated through a fixed-point equation as follows.

For all $z$, set $\tilde{\gamma} = \theta[z].$ Then $(\tilde{\gamma}, V)$ are solution of the following fixed-point equation

$$\forall z \in \mathcal{Z}, x^i \in \mathcal{X},$$

$$\tilde{\gamma}(\cdot|x^i) \in \arg \max_{\gamma(\cdot|x^i)} \mathbb{E}_{\gamma(\cdot|x^i)} \left[R(x^i, A^i, z) + \delta V(\phi(z, \tilde{\gamma}), X^i)|z, x^i\right],$$

(14a)

$$V(z, x^i) = \mathbb{E}_{\tilde{\gamma}(\cdot|x^i)} \left[R(x^i, A^i, z) + \delta V(\phi(z, \tilde{\gamma}), X^i)|z, x^i\right].$$

(14b)

where expectation in (14) is with respect to random variable $(A^i, X^i)$ through the measure $\gamma(a^i|x^i)Q_{x}(x^i'|x^i, a^i, z).

Then an equilibrium strategy is defined as

$$\tilde{\sigma}^i(a_i^t|z_{1:t}, x_{1:t}^i) = \tilde{\gamma}(a_i^t|x_i^t),$$

(15)

where $\tilde{\gamma} = \theta[z_t].$

**Theorem 2.** A strategy $(\tilde{\sigma})$ constructed from the above algorithm is an MPE of the game i.e. $\forall t, h^i_t \in \mathcal{H}_t^i, \sigma^i,$

$^3$We discuss the existence of solution of this fixed-point equation in Section III-C.
\[ E(\tilde{\sigma} | \tilde{\sigma} - i) \sum_{n=t}^{\infty} \delta^{n-t} R(X_n, A_n, Z_n) | z_{1:t}, x_{1:t}^i \geq E(\tilde{\sigma} | \tilde{\sigma} - i) \sum_{n=t}^{\infty} \delta^{n-t} R(X_n, A_n, Z_n) | z_{1:t}, x_{1:t}^i, (16) \]

**Proof.** Please see Appendix C.

**C. Existence**

In this section, we prove existence of a solution of fixed-point equations (10) and (14).

**Theorem 3.** For the finite horizon game \( \mathcal{G}_T \), and for every \( t \in \{T, T - 1, \ldots, 1\} \), there exists a solution of the fixed point equation (10).

**Proof.** Please see Appendix E.

**Theorem 4.** For the infinite horizon game \( \mathcal{G}_\infty \), there exists a solution of the fixed point equation (14).

**Proof.** Please see Appendix F.

**IV. Numerical Example: Cyber Physical Security**

We first consider a security problem in a cyber physical network with positive externalities. It is discretized version of the malware problem presented in [7]–[9], [17]. Some other applications of this model include flu vaccination, entry and exit of firms, investment, network effects. In this model, suppose there are large number of cyber physical nodes where each node has a private state \( x_i^t \in \{0, 1\} \) where \( x_i^t = 0 \) represent ‘healthy’ state and \( x_i^t = 1 \) is the infected state. Each node can take action \( a_i^t \in \{0, 1\} \), where \( a_i^t = 0 \) implies "do nothing" and \( a_i^t = 1 \) implies repair. The dynamics are given by

\[
x_{t+1}^i = \begin{cases} 
  x_t^i + (1 - x_t^i)w_t^i & \text{for } a_t^i = 0 \\
  0 & \text{for } a_t^i = 1.
\end{cases}
\] (17)

where \( w_t^i \in \{0, 1\} \) is a binary valued random variable with \( P(w_t^i = 1) = q \), which represents the probability of a node getting infected. Thus if a node doesn’t do anything, it could get infected with certain probability, however, if it takes repair action, it comes back to the healthy state. Each node gets a reward

\[
r(x_t^i, a_t^i, z_t) = -(k + z_t(1))x_t^i - \lambda a_t^i. \] (18)
where $z_t(1)$ is the mean-field population state being 1 at time $t$, $\lambda$ is the cost of repair and $(k + z_t(1))$ represents the risk of being infected. We pose it as an infinite horizon discounted dynamic game. We consider $k = 0.2$, $\lambda = 0.5$, $\delta = 0.9$, $q = 0.9$ for numerical results presented in Figures 1-4.

Fig. 1: $\gamma(1|0)$: Probability of choosing action 1, given $x^i = 0$

Fig. 2: $\gamma(1|1)$: Probability of choosing action 1, given $x^i = 1$
In this paper, we consider both finite and infinite horizon, large population dynamic game where each player is affected by others through a mean-field population state. We present a novel backward recursive algorithm to compute non-stationary, signaling Markov perfect equilibria (MPE) for such games, where each player’s strategy depends on its current private type and current

V. CONCLUSION

Fig. 3: $V(g(0), 0)$: Reward to go when state $x^i = 0$.

Fig. 4: $V(g(0), 1)$: Reward to go when state $x^i = 1$
mean-field population state. The non-triviality in the problem is that the update of population state is coupled to the strategies of the game, and is managed in the algorithm through unique construction of the fixed-point equations (10),(14). Using this algorithm, we considered a malware propagation problem where we numerically computed equilibrium strategies of the players. In general, this algorithm could be used to compute MPEs in a number of applications such as financial markets, social learning, renewable energy and more.

ACKNOWLEDGMENTS

The author would like to acknowledge the support of Simons Grant #26-7523-99 and Department of Defense grant #W911NF1510225. The author thanks Francois Baccelli and Sriram Vishwanath for encouragement and support.

APPENDIX A

Proof. We prove (13) using induction and the results in Lemma 1 and 2 proved in Appendix B.

For base case at \( t = T \), \( \forall i \in \mathcal{N} \), \((z_{1:T}, x_{1:T}^i) \in \mathcal{H}_T^i, \sigma^i \)

\[
\mathbb{E}^{\sigma_T} \{ R(X_T^i, A_T^i, Z_T) \mid z_{1:T}, x_{1:T}^i \} = V_T(z_T, x_T^i) 
\]

\[
\geq \mathbb{E}^{\sigma_T} \{ R(X_T^i, A_T^i, Z_T) \mid z_{1:T}, x_{1:T}^i \}, \quad (19a)
\]

where (19a) follows from Lemma 2 and (19b) follows from Lemma 1 in Appendix B.

Let the induction hypothesis be that for \( t + 1 \), \( \forall i \in \mathcal{N} \), \((z_{1:t+1}, x_{1:t+1}^i) \in (\mathcal{X})^{t+1}, \sigma^i \),

\[
\mathbb{E}^{\sigma_{t+1}; \sigma_{t+1}} \{ \sum_{n=t+1}^T \delta^{n-t-1} R(X_n^i, A_n^i, Z_n) \mid z_{1:t+1}, x_{1:t+1}^i \} = \sum_{n=t+1}^T \delta^{n-t} R(X_n^i, A_n^i, Z_n) \mid z_{1:t+1}, x_{1:t+1}^i \}
\]

\[
\geq \mathbb{E}^{\sigma_{t+1}; \sigma_{t+1}} \{ \sum_{n=t+1}^T \delta^{n-t-1} R(X_n^i, A_n^i, Z_n) \mid z_{1:t+1}, x_{1:t+1}^i \}, \quad (20a)
\]

Then \( \forall i \in \mathcal{N} \), \((z_{1:t}, x_{1:t}^i) \in \mathcal{H}_t^i, \sigma^i \), we have

\[
\mathbb{E}^{\sigma_t} \{ \sum_{n=t}^T \delta^{n-t-1} R(X_n^i, A_n^i, Z_n) \mid z_{1:t}, x_{1:t}^i \}
\]

\[
= V_t(z_t, x_t^i) \quad (21a)
\]

\[
\geq \mathbb{E}^{\sigma_t} \{ R(X_t^i, A_t^i, Z_t) + \delta V_{t+1}^i(Z_{t+1}, X_{t+1}^i) \mid z_{1:t}, x_{1:t}^i \} \quad (21b)
\]
We will show that this leads to a contradiction. Construct follows from induction hypothesis in $\sigma$ involved in the right conditional expectation do not depend on strategies $\sigma^i_t$.

Proof. We prove this lemma by contradiction.

Let $\delta V_t(z_t, x^i_{1:t}) \in \mathcal{H}^i_t, \sigma^i_t$, and $V_t(z_t, x^i_{1:t}) \geq \mathbb{E}^{|\sigma^i_t|^{-1}} \left\{ R(\hat{X}^i_t, A^i_t, Z_t) + \delta V_{t+1}(Z_{t+1}, X^i_{t+1}) \big| z_{1:t}, x^i_{1:t} \right\}$. (22)

Proof. We prove this lemma by contradiction.

Suppose the claim is not true for $t$. This implies $\exists \delta \tau^i_t, \tilde{z}_{1:t}, \tilde{x}_{1:t}$ such that

\[ \mathbb{E}^{\tau^i_t} \left\{ R(X^i_t, A^i_t, Z_t) + \delta V_{t+1}(Z_{t+1}, X^i_{t+1}) \big| \tilde{z}_{1:t}, \tilde{x}^i_{1:t} \right\} > V_t(\tilde{z}_t, \tilde{x}^i_t). \] (23)

We will show that this leads to a contradiction. Construct

\[ \tilde{\gamma}^i_t(a^i_t | x^i_t) = \begin{cases} \tilde{\sigma}^i_t(a^i_t | \tilde{z}_{1:t}, \tilde{x}^i_{1:t}) & x^i_t = \tilde{x}^i_t \\ \text{arbitrary} & \text{otherwise}. \end{cases} \] (24)

Then for $\tilde{z}_{1:t}, \tilde{x}^i_{1:t}$, we have

\[ V_t(\tilde{z}_t, \tilde{x}^i_t) = \max_{\gamma^i_t} \mathbb{E}^{\gamma^i_t(\tilde{z}^i_t)} \left\{ R(\tilde{x}^i_t, A^i_t, \tilde{z}_t) + \delta V_{t+1}(\phi(\tilde{z}_t, \tilde{x}^i_t), X^i_{t+1}) \big| \tilde{z}_t, \tilde{x}^i_t \right\}. \] (25a)
Then we prove the lemma by induction. For (23)

May 9, 2019 DRAFT

follows from definition of \( \hat{V}_t \) in (11). (25c) follows from definition of \( \hat{\gamma}_t \) and (25d) follows from (23). However this leads to a contradiction.

Lemma 2. \( \forall i \in \mathcal{N}, t \in \mathcal{T}, (z_{1:t}, x_{1:t}^i) \in \mathcal{H}_i \),

\[
V_i(z_t, x_t^i) = \mathbb{E}_{\hat{\sigma}_T^{i}, T} \left\{ \sum_{n=t}^{T} \delta^{n-t} R(X_n^i, A_n^i, Z_n) \left| z_{1:t}, x_{1:t}^i \right. \right\}.
\] (26)

Proof. We prove the lemma by induction. For \( t = T \),

\[
\mathbb{E}_{\hat{\sigma}_T^{i}, T} \left\{ R(X_T^i, A_T^i, Z_T) \left| z_{1:T}, x_{1:T}^i \right. \right\} = \sum_{a_T^i} R(x_T^i, a_T^i, z_T) \hat{\sigma}_T^i(a_T^i | z_T, x_T^i)
\] (27a)

\[
= V_T(z_T, x_T^i),
\] (27b)

where (27b) follows from the definition of \( V_i \) in (11). Suppose the claim is true for \( t + 1 \), i.e., \( \forall i \in \mathcal{N}, t \in \mathcal{T}, (z_{1:t+1}, x_{1:t+1}^i) \in \mathcal{H}_{i+1} \)

\[
V_{t+1}(z_{t+1}, x_{t+1}^i) = \mathbb{E}_{\hat{\sigma}_{t+1}^{i}, T} \left\{ \sum_{n=t+1}^{T} \delta^{n-t-1} R(X_n^i, A_n^i, Z_n) \left| z_{1:t+1}, x_{1:t+1}^i \right. \right\}.
\] (28)

Then \( \forall i \in \mathcal{N}, t \in \mathcal{T}, (z_{1:t}, x_{1:t}^i) \in \mathcal{H}_i \), we have

\[
\mathbb{E}_{\hat{\sigma}_T^{i}, T} \left\{ \sum_{n=t}^{T} \delta^{n-t} R(X_n^i, A_n^i, Z_n) \left| z_{1:t}, x_{1:t}^i \right. \right\} = \mathbb{E}_{\hat{\sigma}_T^{i}, T} \left\{ R(X_t^i, A_t^i, Z_t) + \delta \mathbb{E}_{\hat{\sigma}_T^{i}, T} \right\}
\]

\[
\left\{ \sum_{n=t+1}^{T} \delta^{n-t-1} R(X_n^i, A_n^i, Z_n) \left| z_{1:t}, Z_{t+1}, x_{1:t}^i, X_{t+1}^i \right. \right\} \right\} \left| z_{1:t}, x_{1:t}^i \right. \right\} \right\} \right\} \right\} \right\}
\] (29a)
\[
\left\{ \sum_{n=t+1}^{T} \delta^{n-t-1} R(X_n^i, A_n^i, Z_n) \mid z_{1:t}, Z_{t+1}, x_{1:t}, X_{t+1}^i \right\} \mid z_{1:t}, x_{1:t}^i \right\} \right) \mid z_{1:t}, x_{1:t}^i \right\} \right) \right) = E_{\sigma^i, \tilde{\sigma}^i_{t:T}} \left\{ R(X_t^i, A_t^i, Z_t) + \delta V_{t+1}(Z_{t+1}, X_{t+1}^i) \mid z_{1:t}, x_{1:t}^i \right\} \right) \right) \right) = V_t(z_t, x_t^i),
\]

(29c) follows from the induction hypothesis in (28), (29d) follows because the random variables involved in expectation, \(X_t^i, A_t^i, Z_t, Z_{t+1}, X_{t+1}^i\) do not depend on \(\tilde{\sigma}_{t+1:T}^{i} \sigma_{t+1:T}^{i} \) and (29e) follows from the definition of \(V_t\) in (11).

**APPENDIX C**

We divide the proof into two parts: first we show that the value function \(V\) is at least as big as any reward-to-go function; secondly we show that under the strategy \(\tilde{\sigma}\), reward-to-go is \(V\).

Note that \(h_t^i := (z_{1:t}, x_{1:t}^i)\).

**Part 1:** For any \(i \in \mathcal{N}, \sigma^i\) define the following reward-to-go functions

\[
W_t^{\sigma^i}(h_t^i) = E_{\sigma^i, \tilde{\sigma}^i_{t:T}} \left\{ \sum_{n=t}^{\infty} \delta^{n-t} R(X_n^i, A_n^i, Z_n) \mid h_t^i \right\}
\]

(30a)

\[
W_t^{\sigma^i,T}(h_t^i) = E_{\sigma^i, \tilde{\sigma}^i_{t:T}} \left\{ \sum_{n=t}^{T} \delta^{n-t} R(X_n^i, A_n^i, Z_n) + \delta^{T+1-t} V(Z_{T+1}, X_{T+1}^i) \mid h_t^i \right\}.
\]

(30b)

Since \(\mathcal{X}, \mathcal{A}\) are finite sets the reward \(R\) is absolutely bounded, the reward-to-go \(W_t^{\sigma^i}(h_t^i)\) is finite \(\forall i, t, \sigma^i, h_t^i\).

For any \(i \in \mathcal{N}, h_t^i \in \mathcal{H}_t^i\),

\[
V(z_t, x_t^i) - W_t^{\sigma^i}(h_t^i) = \left[ V(z_t, x_t^i) - W_t^{\sigma^i,T}(h_t^i) \right] + \left[ W_t^{\sigma^i,T}(h_t^i) - W_t^{\sigma^i}(h_t^i) \right]
\]

(31)

Combining results from Lemmas 4 and 5 in Appendix D the term in the first bracket in RHS of (31) is non-negative. Using (30), the term in the second bracket is

\[
(\delta^{T+1-t}) E_{\sigma^i, \tilde{\sigma}^i_{t:T}} \left\{ - \sum_{n=T+1}^{\infty} \delta^{n-(T+1)} R(X_n^i, A_n^i, Z_n) + V(Z_{T+1}, X_{T+1}^i) \mid h_t^i \right\}.
\]

(32)

The summation in the expression above is bounded by a convergent geometric series. Also, \(V\) is bounded. Hence the above quantity can be made arbitrarily small by choosing \(T\) appropriately large. Since the LHS of (31) does not depend on \(T\), which implies,

\[
V(z_t, x_t^i) \geq W_t^{\sigma^i}(h_t^i).
\]

(33)
Part 2: Since the strategy the equilibrium strategy $\tilde{\sigma}$ generated in (15) is such that $\tilde{\sigma}_t^i$ depends on $h_t^i$ only through $z_t$ and $x_t^i$, the reward-to-go $W_t^{\tilde{\sigma}_t^i}$ at strategy $\tilde{\sigma}$, can be written (with abuse of notation) as

$$W_t^{\tilde{\sigma}_t^i}(h_t^i) = W_t^{\tilde{\sigma}_t^i}(z_t, x_t^i) = E^{\tilde{\sigma}} \left\{ \sum_{n=t}^{\infty} \delta^{n-t} R(X_n^i, A_n^i, Z_n) \mid z_t, x_t^i \right\}.$$  (34)

For any $h_t^i \in H_t^i$,

$$W_t^{\tilde{\sigma}_t^i}(z_t, x_t^i) = E^{\tilde{\sigma}} \left\{ R(X_t^i, A_t^i, Z_t) + \delta W_{t+1}^{\tilde{\sigma}_t^i} \left( \phi(z_t, \theta[z_t]), X_{t+1}^i \right) \mid z_t, x_t^i \right\}.$$  (35a)

$$V(z_t, x_t^i) = E^{\tilde{\sigma}} \left\{ R(X_t^i, A_t^i, Z_t) + \delta V \left( \phi(z_t, \theta[z_t]), X_{t+1}^i \right) \mid z_t, x_t^i \right\}. $$  (35b)

Repeated application of the above for the first $n$ time periods gives

$$W_t^{\tilde{\sigma}_t^i}(z_t, x_t^i) = E^{\tilde{\sigma}} \left\{ \sum_{m=t}^{t+n-1} \delta^{m-t} R(X_t^i, A_t^i, Z_t) + \delta^n W_{t+n}^{\tilde{\sigma}_t^i} \left( Z_{t+n}, X_{t+n}^i \right) \mid z_t, x_t^i \right\}.$$  (36a)

$$V(z_t, x_t^i) = E^{\tilde{\sigma}} \left\{ \sum_{m=t}^{t+n-1} \delta^{m-t} R(X_t^i, A_t^i, Z_t) + \delta^n V \left( Z_{t+n}, X_{t+n}^i \right) \mid z_t, x_t^i \right\}. $$  (36b)

Taking differences results in

$$W_t^{\tilde{\sigma}_t^i}(z_t, x_t^i) - V(z_t, x_t^i) = \delta^n E^{\tilde{\sigma}} \left\{ W_{t+n}^{\tilde{\sigma}_t^i} \left( Z_{t+n}, X_{t+n}^i \right) - V \left( Z_{t+n}, X_{t+n}^i \right) \mid z_t, x_t^i \right\}. $$  (37)

Taking absolute value of both sides then using Jensen’s inequality for $f(x) = |x|$ and finally taking supremum over $h_t^i$ reduces to

$$\sup_{h_t^i} \left| W_t^{\tilde{\sigma}_t^i}(z_t, x_t^i) - V(z_t, x_t^i) \right| \leq \delta^n \sup_{h_t^i} E^{\tilde{\sigma}} \left\{ \left| W_{t+n}^{\tilde{\sigma}_t^i} \left( Z_{t+n}, X_{t+n}^i \right) - V \left( Z_{t+n}, X_{t+n}^i \right) \right| \mid z_t, x_t^i \right\}. $$  (38)

Now using the fact that $W_{t+n}, V$ are bounded and that we can choose $n$ arbitrarily large, we get

$$\sup_{h_t^i} \left| W_t^{\tilde{\sigma}_t^i}(z_t, x_t^i) - V(z_t, x_t^i) \right| = 0.$$

**APPENDIX D**

In this section, we present three lemmas. Lemma 3 is intermediate technical results needed in the proof of Lemma 4. Then the results in Lemma 4 and 5 are used in Appendix C for the proof of Theorem 2. The proof for Lemma 5 below isn’t stated as it analogous to the proof.
of Lemma [1] from Appendix B used in the proof of Theorem [1] (the only difference being a non-zero terminal reward in the finite-horizon model).

Define the reward-to-go \( W_{\sigma_i,T} \) for any agent \( i \) and strategy \( \sigma_i \) as

\[
W_{\sigma_i,T} (z_{1:t}, x_{1:t}^i) = \mathbb{E}_{\sigma_i, \tilde{\sigma}^{-i}} \left[ \sum_{n=t}^{T} \delta^{n-t} R(X_n^i, A_n^i, Z_n) + \delta^{T+1-t} G(Z_{T+1}^i, X_{T+1}^i) \mid z_{1:t}, x_{1:t}^i \right].
\]

(39)

Here agent \( i \)'s strategy is \( \sigma_i \) whereas all other agents use strategy \( \tilde{\sigma}^{-i} \) defined above. Since \( X, A \) are assumed to be finite and \( G \) absolutely bounded, the reward-to-go is finite \( \forall \, i, t, \sigma_i, z_{1:t}, x_{1:t}^i \).

In the following, any quantity with a \( T \) in the superscript refers the finite horizon model with terminal reward \( G \). For further discussion, please refer to the comments after the statement of Theorem 2.

Lemma 3. For any \( t \in T, \ i \in \mathcal{N}, \ z_{1:t}, x_{1:t}^i \) and \( \sigma_i \),

\[
V^T_t (z_t, x_t^i) \geq \mathbb{E}_{\sigma_i, \tilde{\sigma}^{-i}} \left[ R(x_t^i, A_t^i, z_t) + \delta V^T_{t+1} (\phi(z_t, \theta[z_t]), X_{t+1}^i) \mid z_{1:t}, x_{1:t}^i \right].
\]

(40)

The result below shows that the value function from the backwards recursive algorithm is higher than any reward-to-go.

Lemma 4. For any \( t \in T, \ i \in \mathcal{N}, \ z_{1:t}, x_{1:t}^i \) and \( \sigma_i \),

\[
V^T_t (z_t, x_t^i) \geq W^\sigma_{T+t} (z_{1:t}, x_{1:t}^i).
\]

(41)

Proof. We use backward induction for this. At time \( T \), using the maximization property from (10) (modified with terminal reward \( G \)),

\[
V^T_T (z_T, x_T^i) = \mathbb{E}^\gamma_T (z_T, x_T^i) \geq \mathbb{E}^{\gamma_T^i (z_T), \tilde{\gamma}_T} \left[ R(X_T^i, A_T^i, Z_T) + \delta G(\phi(z_T, \tilde{\gamma}_T^T)), X_{T+1}^i \mid z_{1:T}, x_{1:T}^i \right] \]

(42b)

\[
= W^\sigma_{T} (h_T^i)
\]

(42d)

Here the second inequality follows from (10) and (11) and the final equality is by definition in (39).

Assume that the result holds for all \( n \in \{ t+1, \ldots, T \} \), then at time \( t \) we have

\[
V^T_t (z_t, x_t^i) \]

(43a)
Here the first inequality follows from Lemma 3, the second inequality from the induction hypothesis, the third equality follows since the random variables on the right hand side do not depend on $\sigma_t$, and the final equality by definition (39).

The following result highlights the similarities between the fixed-point equation in infinite-horizon and the backwards recursion in the finite-horizon.

**Lemma 5.** Consider the finite horizon game with $G \equiv V$. Then $V^T_t = V$, $\forall i \in N$, $t \in \{1, \ldots, T\}$ satisfies the backwards recursive construction stated above (adapted from (10) and (11)).

**Proof.** Use backward induction for this. Consider the finite horizon algorithm at time $t = T$, noting that $V^T_{T+1} \equiv G \equiv V$,

$$
\gamma^T_T(\cdot | x^i_T) \in \arg \max_{\gamma \in \gamma_T(\cdot | x^i_T)} [ R(x^i_T, A^i_T, z_T) + \delta V(\phi(z_T, \gamma^T_T), X^i_{T+1}) | z_T, x^i_T ] \tag{44a}
$$

$$
V^T_T(z_T, x^i_T) = \mathbb{E}^{\gamma^T_T(\cdot | x^i_T)} [ R(x^i_T, A^i_T, z_T) + \delta V(\phi(z_T, \gamma^T_T), X^i_{T+1}) | z_T, x^i_T ] \tag{44b}
$$

Comparing the above set of equations with (14), we can see that the pair $(V, \gamma)$ arising out of (14) satisfies the above. Now assume that $V^T_n \equiv V$ for all $n \in \{t + 1, \ldots, T\}$. At time $t$, in the finite horizon construction from (10), (11), substituting $V$ in place of $V^T_{T+1}$ from the induction hypothesis, we get the same set of equations as (44). Thus $V^T_t \equiv V$ satisfies it.

**APPENDIX E**

**Theorem.** For the finite horizon game $G_T$, and for every $t \in \{T, T-1, \ldots, 1\}$, there exists a solution of the fixed point equation (10).
Proof. \( \bar{\gamma}_t \) is solution of the following fixed-point equation, \( \forall x_t^i \in \mathcal{X} \),

\[
\bar{\gamma}_t(\cdot|x_t^i) \in \arg \max_{\gamma(\cdot|x_t^i)} \mathbb{E}_{\gamma(\cdot|x_t^i)} \left[ R(x_t^i, A_t^i, z_t) + \delta V_{t+1}(\phi(z_t, \bar{\gamma}_t), X_{t+1}^i) | z_t, x_t^i \right] \\
= \arg \max_{\gamma(\cdot|x_t^i)} \sum_{a_t^i \in X_{t+1}^i} \left[ R(x_t^i, a_t^i, z_t) + \delta V_{t+1}(\phi(z_t, \bar{\gamma}_t), x_{t+1}^i) \right] \gamma_t(a_t^i|x_t^i) Q_x(x_{t+1}^i | x_t^i, a_t^i, z_t) .
\]

(i) Let \( \mathcal{B} \) be the space of \( \gamma \), where \( \gamma : \mathcal{X} \rightarrow \mathcal{P}(\mathcal{A}) \). Clearly \( \mathcal{B} \) is convex since it is a space of probability measures, and compact which is implied by the finiteness of \( \mathcal{X}, \mathcal{A} \).

(ii) From (46), we define a correspondence \( h_t : \mathcal{B} \rightrightarrows \mathcal{B} \) as follows

\[
h_t(\bar{\gamma}) = \{ \bar{\gamma}_t : \forall x_t^i \in \mathcal{X}, \bar{\gamma}_t(\cdot|x_t^i) \in \arg \max_{\gamma(\cdot|x_t^i)} \mathbb{E}_{\gamma(\cdot|x_t^i)} \left[ R(x_t^i, A_t^i, z_t) + \delta V_{t+1}(\phi(z_t, \bar{\gamma}_t), X_{t+1}^i) | z_t, x_t^i \right] \}
\]

For all \( \bar{\gamma} \in \mathcal{B} \), \( h_t(\bar{\gamma}) \) is non-empty, closed and convex since the optimization in (46) is a linear program in variables \( \gamma_t(\cdot|x_t^i) \), which lie in a compact space that is the probability simplex.

(iii) Since the optimization in (46) is linear and thus continuous in \( \gamma_t(\cdot|x_t^i) \), therefore \( h_t \) has closed graph property (from Maximum theorem [18]).

Thus using Kakutani’s fixed-point theorem [19][Lemma 20.1], there exists a solution to (10).

APPENDIX F

Theorem. For the infinite horizon game \( G_\infty \), there exists a solution of the fixed point equation (14).

Proof. The fixed point equation in (14) is given as follows.

\( (\bar{\gamma}, V) \) are solution of the following fixed-point equation, \( \forall z \in \mathcal{Z}, \bar{\gamma} = \theta(z), x^i \in \mathcal{X} \),

\[
\bar{\gamma}(\cdot|x^i) \in \arg \max_{\gamma(\cdot|x^i)} \mathbb{E}_{\gamma(\cdot|x^i)} \left[ R(x^i, A^i, z) + \delta V(\phi(z, \bar{\gamma}), X^{i'}) | z, x^i \right],
\]

\[
V(z, x^i) = \mathbb{E}_{\bar{\gamma}(\cdot|x^i)} \left[ R(x^i, A^i, z) + \delta V(\phi(z, \bar{\gamma}), X^{i'}) | z, x^i \right] .
\]

where expectation in (48a) is with respect to random variable \( (A^i, X^{i'}) \) through the measure \( \gamma(a^i|x^i)Q_x(x^{i'} | x^i, a^i, z) \).

We will prove this theorem in two parts.

Part 1 Recall that \( \theta : \mathcal{Z} \rightarrow (\mathcal{X} \rightarrow \mathcal{P}(\mathcal{A})) \) is an equilibrium generating function which defines a homogenous Markovian strategy of players \( \sigma \) as \( \sigma(a^i|z, x^i) = \theta(z)(a^i|x^i) \).

We note that \( V \) is a bounded function since \( \|V\|_\infty \leq \frac{1}{1-\delta} \sup_{x^i, a^i, z} \{ R(x^i, A^i, z) \} \). Let \( \mathcal{V} \) be the space of all such bounded functions. We consider it with supremum norm \( \| \cdot \|_\infty \).
For any $V \in \mathcal{V}$, we define an equilibrium generating function $\theta^{(V)}$ as follows. For any $z \in \mathcal{Z}$, let $\tilde{\gamma} = \theta^{(V)}[z]$, where $\tilde{\gamma}$ satisfies the following fixed-point equation,

$$
\tilde{\gamma}(\cdot|x^i) \in \arg \max_{\gamma(\cdot|x^i)} \mathbb{E}^{\gamma(\cdot|x^i)} \left[ R(x^i, A^i, z) + \delta V(\phi(z, \tilde{\gamma}), X^i') | z, x^i \right],
$$

(49)

There exists a fixed-point solution of the above equation by the same arguments as in the proof of Theorem 3 in Appendix E.

Part 2

We define the following fixed-point equation in $V$. For $z \in \mathcal{Z}, x^i \in \mathcal{X}^i$. Then

$$
V(z, x^i) = \max_{\gamma(\cdot|x^i)} \mathbb{E}^{\gamma(\cdot|x^i)} \left[ R(x^i, A^i, z) + \delta V(\phi(z, \theta^{(V)}[z]), X^i') | z, x^i \right]
$$

(50)

$$
= \max_{\gamma(\cdot|x^i)} \sum_{a^i, x^i'} \left[ R(x^i, a^i, z) + \delta V(\phi(z, \theta^{(V)}[z]), x^i') \right] \gamma(a^i|x^i)Q(x^i'|x^i, a^i, z)
$$

(51)

We define an operator $T : \mathcal{V} \to \mathcal{V}$ as follows:

$$
TV(z, x^i) = \max_{\gamma(\cdot|x^i)} \mathbb{E}^{\gamma(\cdot|x^i)} \left[ R(x^i, A^i, z) + \delta V(\phi(z, \theta^{(V)}[z]), X^i') | z, x^i \right]
$$

(52)

Let

$$
\gamma^* \in \arg \max_{\tilde{\gamma}(\cdot|x^i)} \mathbb{E}^{\tilde{\gamma}(\cdot|x^i)} \left[ R(x^i, A^i, z) + \delta V_1(\phi(z, \theta^{(V_1)}[z]), X^i') | z, x^i \right]
$$

(53)

Then,

$$
TV_1(z, x^i) - TV_2(z, x^i)
$$

$$
= \max_{\tilde{\gamma}(\cdot|x^i)} \mathbb{E}^{\tilde{\gamma}(\cdot|x^i)} \left[ R(x^i, A^i, z) + \delta V_1(\phi(z, \theta^{(V_1)}[z]), X^i') | z, x^i \right]
$$

$$
- \max_{\gamma(\cdot|x^i)} \mathbb{E}^{\gamma(\cdot|x^i)} \left[ R(x^i, A^i, z) + \delta V_2(\phi(z, \theta^{(V_2)}[z]), X^i') | z, x^i \right]
$$

(54)

$$
\leq \mathbb{E}^{\gamma^*(\cdot|x^i)} \left[ R(x^i, A^i, z) + \delta V_1(\phi(z, \theta^{(V_1)}[z]), X^i') | z, x^i \right]
$$

$$
- \mathbb{E}^{\gamma^*(\cdot|x^i)} \left[ R(x^i, A^i, z) + \delta V_2(\phi(z, \theta^{(V_2)}[z]), X^i') | z, x^i \right]
$$

(55)

$$
\leq \delta \mathbb{E}^{\gamma^*(\cdot|x^i)} \left[ V_1(\phi(z, \theta^{(V_1)}[z]), X^i') - V_2(\phi(z, \theta^{(V_2)}[z]), X^i') | z, x^i \right]
$$

(56)

$$
\leq \delta \| V_1 - V_2 \|_{\infty}.
$$

(57)

where $\| \cdot \|_{\infty}$ is the sup norm. By a similar argument, $TV_2(z, x^i) - TV_1(z, x^i) \leq \delta \| V_1 - V_2 \|_{\infty}$. Thus $T$ is a contraction mapping. Therefore from Banach fixed-point theorem [18], there exists a unique fixed point to $T$, $TV = V$. Correspondingly, there exists $\theta^{(V)}$ such that $\tilde{\gamma} = \theta^{(V)}[z]$ together with $V$ is a solution to (14).
REFERENCES

[1] E. Maskin and J. Tirole, “Markov perfect equilibrium: I. observable actions,” *Journal of Economic Theory*, vol. 100, no. 2, pp. 191–219, 2001.

[2] M. Huang, R. P. Malhamé, and P. E. Caines, “Large population stochastic dynamic games: closed-loop mckean-vlasov systems and the nash certainty equivalence principle,” *Communications in Information & Systems*, vol. 6, no. 3, pp. 221–252, 2006.

[3] J.-M. Lasry and P.-L. Lions, “Mean field games,” *Japanese Journal of Mathematics*, vol. 2, no. 1, pp. 229–260, 2007.

[4] J.-M. Lasry, P.-L. Lions, and O. Guéant, “Application of mean field games to growth theory,” 2008.

[5] O. Guéant, J.-M. Lasry, and P.-L. Lions, “Mean field games and applications,” in *Paris-Princeton lectures on mathematical finance 2010*. Springer, 2011, pp. 205–266.

[6] J. Subramanian and A. Mahajan, “Reinforcement learning in stationary mean-field games,” in *International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, 2019.

[7] M. Huang and Y. Ma, “Mean field stochastic games: Monotone costs and threshold policies,” in *2016 IEEE 55th Conference on Decision and Control (CDC)*. IEEE, 2016, pp. 7105–7110.

[8] ——, “Mean field stochastic games with binary action spaces and monotone costs,” *arXiv preprint arXiv:1701.06661*, 2017.

[9] ——, “Mean field stochastic games with binary actions: Stationary threshold policies,” in *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*. IEEE, 2017, pp. 27–32.

[10] S. Adlakha, R. Johari, and G. Y. Weintraub, “Equilibria of dynamic games with many players: Existence, approximation, and market structure,” *Journal of Economic Theory*, vol. 156, pp. 269–316, 2015.

[11] D. Vasal, A. Sinha, and A. Anastasopoulos, “A systematic process for evaluating structured perfect bayesian equilibria in dynamic games with asymmetric information,” *IEEE Transactions on Automatic Control*, 2018.

[12] D. Vasal and A. Anastasopoulos, “Decentralized Bayesian learning in dynamic games,” in *Allerton Conference on Communication, Control, and Computing*, 2016. [Online]. Available: [https://arxiv.org/abs/1607.06847](https://arxiv.org/abs/1607.06847)

[13] ——, “Signaling equilibria of dynamic LQG games with asymmetric information,” in *Conference on Decision and Control*, 2016.

[14] Y. Ouyang, H. Tavafoghi, and D. Teneketzis, “Dynamic games with asymmetric information: Common information based perfect bayesian equilibria and sequential decomposition,” *IEEE Transactions on Automatic Control*, vol. 62, no. 1, pp. 222–237, 2017.

[15] H. T. Jahormi, “On design and analysis of cyber-physical systems with strategic agents,” Ph.D. dissertation, University of Michigan, Ann Arbor, 2017.

[16] A. Nayyar, A. Mahajan, and D. Teneketzis, “Decentralized stochastic control with partial history sharing: A common information approach,” *Automatic Control, IEEE Transactions on*, vol. 58, no. 7, pp. 1644–1658, 2013.

[17] L. Jiang, V. Anantharam, and J. Walrand, “How bad are selfish investments in network security?” *IEEE/ACM Transactions on Networking*, vol. 19, no. 2, pp. 549–560, 2011.

[18] E. A. Ok, *Real analysis with economic applications*. Princeton University Press, 2007, vol. 10.

[19] M. J. Osborne and A. Rubinstein, *A Course in Game Theory*, ser. MIT Press Books. The MIT Press, 1994, vol. 1.