Presentation of Penrose tiling as set of overlapping pentagonal stars

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Abstract. The layer structure of Penrose tiling in the form of Wieringa roof is considered. It is found that each layer can be described by set of overlapped regular pentagonal stars. Local and global pentagonal symmetry of layers can be vividly presented if the stars will be chosen of less size. Linear size ratio of stars in different layers is equal to golden mean or one. It is offered to use four types of star clusters for layer structure description. List of noticed rules of correlation of star location in different layers is proposed.

Two-dimensional Penrose tiling is one of fundamental models of quasiperiodic lattices [1]. De Bruijn was the first, who offered to construct the tiling by means of projection method [2]. The colleague of de Bruijn – R.Wieringa marked that two dimensional Penrose tiling will have look of three dimensional lattice, which consists of the same rhombuses with different orientation, if it will be used 6-dimentional cubic lattice instead of the 5-dimentional one for projection. This variant of Penrose tiling is called Wieringa roof. The Wieringa roof can be built if five central vectors – edges of Penrose tiling will be directed parallel to vectors, connecting the center of regular icosahedron and five its vertices.

If we consider the layers of Wieringa roof separately, then tiles, as parts of structure, disappear. It is necessary to take other geometrical models, such as overlapping polygons, in this case. Earlier it was offered to use overlapped regular decagons of the same size for the building of decagonal quasicrystals structure [3] and Penrose tiling [4]. This approach brought closer tile and cluster principles of quasicrystal structure describing. The icosahedral quasiperiodic lattice can be constructed from overlapped polyhedra – three dimensional stars – great stellated dodecahedra of two sizes [5]. The stars have property of mutual transparency – petals of stars can cross the stars of the same size.

Penrose tiling which used for the study of Wieringa roof was made by projection method; acceptance domain [6] was shifted by 2/5 of diagonal. The vertices of the Wieringa roof lye in four layers. The aim of the paper is to study the structure of Wieringa roof layers.

The most noticeable property of each layer is that all points (vertices of tiles) can be connected by lines, which have form of regular pentagonal star (Pythagorean star). The strict rule is working: every pentagon is nucleus of a star, i.e. if we find the regular pentagon, then there will be five points lying in tops of petals of regular star, based upon the pentagon. The rule resembles stellation of a polygon as it was defined by Kepler [7].

It has been chosen stars of less size for the description of the layers. Linear size of the stars in the first and forth layers is in golden mean $\tau = 1+\sqrt{5}/2$ more than the size of stars in second and third layers. Stars have two orientations; the orientations are connected by the inversion with center.
Figure 1. The first and second layers of Wieringa roof.

We have labeled types of less stars in second and third layers by numbers 1 and 2 depending on the orientation, and types of big stars in the first and forth layers by numbers 3 and 4. The same orientation has stars of the first and third types and stars of the second and forth types accordingly. The first and third orientation types agree with orientation of big tiles in the center of Penrose tiling. Layers of Wieringa roof are shown in Figures 1 and 2, each layer points set was divided between the crossing sets of points generating stars with different orientation (layer 1-4 means that we consider all points generating forth type stars in the first layer).

We can find three sorts of star connection in one layer with stars of the same orientation: $\alpha$) overlapped stars (the petals only are overlapping); $\beta$) contacting stars (contacting each other by the ends of petals); $\gamma$) separate stars (there are no common star vertices). Layers with $\alpha$ and $\beta$ connection sort are: 1-4, 2-2, 3-1, 4-3; layers, in which stars are contacting only each other or standing separately ($\beta$ and $\gamma$ connection sorts), are: 1-3, 2-1, 3-2, 4-4.

There are four kind of star clusters, which we shall call constellations: 'Button' (B), 'Flower' (F), 'Ring' (R) and 'Bouquet' (Q) (Figure 3 a). The analysis of layer drawings has shown that there is strict
rule: every constellation appears to be a star of star (Figure 3 b). It means that constellation is a
nucleus of star of second order; therefore we shall call the stars of stars by the same name as
constellations. One can assume that generalization of rule: 'every pentagon is a nucleus of a star' will
allow describe the mechanism of Penrose tiling building.

**Figure 2.** The third and forth layers of Wieringa roof.

Wieringa roof layers have geometrical correlation obviously. There is list of noticed rules of star
location in different layers.

1. Every 'Bouquet' consists of six stars – there is a star of opposite orientation in the same layer in
the center of the 'Bouquet' (Figure 3 a). This rule reminds of 'opening of a flower of pentagons'
operation which is acting in structure of pentagons and stars constructed by symmetry operations [8].

2. All the centers of stars coincide in two pairs of layers: in the pair 1-4 and 3-2 and in the pair 4-3
and 2-1.
3. All the star centers in 4-4 layer coincide with stars in layer 3-1 which contact the neighbors (connection sort $\beta$). The layers 1-3 and 2-2 have analogous properties.
4. There is mutual accordance between the layers 3-1 and 2-2: the center of 'Button' in one layer corresponds to $\beta$-star in the other layer.

![Figure 3](image)

**Figure 3.** a) Constellations 'Button' (B), 'Flower' (F), 'Ring' (R) and 'Bouquet' (Q); b) Stars of stars.

5. There is interconnection: any 'Ring' generates in the same layer the 'Flower' of oppositely oriented stars.

These rules let bind structures of different layers, and with the rule: 'every pentagon is a nucleus of a star' will allow to built Penrose tiling.

Five-dimensional cubic lattice is needed for building of Penrose tiling by projection method. For an explanation of the role which regular pentagonal star is playing in Penrose tiling we can mark that in projection of 5-dimensional cube into three dimensional space (acceptance domain [6]) there are two cube sections with vertices of the cube situated in vertices of regular pentagonal stars. If we denote vertices of the cube projection by the indices, then the vectors with all permutations of indices (11000) will generate a star, the oppositely oriented star in other section will be generated by all permutations of indices (11000).

In conclusion we should note that this work emphasizes latent cluster structure of Penrose tiling.

References

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