EMBEDDABILITY OF RIGHT-ANGLED ARTIN GROUPS ON THE COMPLEMENTS OF LINEAR FORESTS

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Abstract. In this article, we prove that embeddings of right-angled Artin group $A_1$ on the complement of a linear forest into another right-angled Artin group $A_2$ can be reduced to full embeddings of the defining graph of $A_1$ into the extension graph of the defining graph of $A_2$.

1. Introduction

Let $\Gamma$ be a simple graph (abbreviated a graph). We denote the vertex set and the edge set of $\Gamma$ by $V(\Gamma)$ and $E(\Gamma)$, respectively. The right-angled Artin group on $\Gamma$ is the group given by the following presentation:

$$A(\Gamma) = \langle V(\Gamma) \mid v_i v_j v_i^{-1} v_j^{-1} = 1 \text{ if } \{v_i, v_j\} \in E(\Gamma) \rangle.$$

A graph homomorphism is a map between the vertex sets of two graphs, which maps adjacent vertices to adjacent vertices. An injective graph homomorphism (abbreviated an embedding) $\iota : \Lambda \to \Gamma$, then we denote by $\Lambda \leq \Gamma$ and $\Lambda$ is called a full subgraph of $\Gamma$. If there is a full embedding $\iota : \Lambda \to \Gamma$, then we denote by $\Lambda(\iota)$ isomorphic to $A(\Gamma)$ as a group if and only if $\Lambda$ is isomorphic to $A(\Gamma)$ as a graph.

A celebrated theorem due to Kim–Koberda states that, if a finite graph $\Lambda$ is a full subgraph of the extension graph $\Gamma^e$ of a finite graph $\Gamma$, then we have an injective homomorphism (abbreviated an embedding) $A(\Lambda) \to A(\Gamma)$. In this article, a path graph $P_n$ on $n$ $(\geq 1)$ vertices is the graph whose underlying space is homeomorphic to the origin $\{0\}$ or unit interval $[0, 1]$ in the 1-dimensional Euclidean space. A linear forest is the disjoint union of finitely many path graphs. The complement $\Lambda^c$ of a graph $\Lambda$ is the graph consisting of the vertex set $V(\Lambda^c) = V(\Lambda)$ and the edge set $E(\Lambda^c) = \{\{u, v\} \mid u, v \in V(\Lambda), \{u, v\} \notin E(\Lambda)\}$.

In [3] the author “proved” the following theorem.

Theorem 1.1. Let $\Lambda$ be the complement of a linear forest and $\Gamma$ a finite graph. If $A(\Lambda) \to A(\Gamma)$, then $\Lambda \leq \Gamma$.

We remark that Theorem 1.1 is equivalent to [3, Theorem 1.3(1)]. In the proof of [3, Theorem 1.3(1)], the author used the following “Theorem” (see the second line of the proof of Theorem 3.6 in [3]).

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Theorem 1.2 ([1] Theorem 3.14]). Let $\Lambda$ be the complement of a forest and $\Gamma$ a finite graph. If $A(\Lambda) \hookrightarrow A(\Gamma)$, then $\Lambda \leq \Gamma^e$.

However, E. Lee and S. Lee [5] pointed out that the above “Theorem” is incorrect by giving a counter-example. Thus the author’s proof of Theorem 1.1 in [3] is not valid.

The purpose of this article is to give a complete proof of Theorem 1.1 by establishing the following theorem which shows that “Theorem 1.2” holds when $\Lambda$ is the complement of a linear forest.

Theorem 1.3. Let $\Lambda$ be the complement of a linear forest and $\Gamma$ a finite graph. If $A(\Lambda) \hookrightarrow A(\Gamma)$, then $\Lambda \leq \Gamma^e$.

In fact, the author applied “Theorem 1.2” only for the complement of linear forests in the proof of Theorem 1.3(1) in [3].

We note that this theorem gives a partial positive answer to the following question.

Question 1.4 ([4] Question 1.5]). For which graphs $\Lambda$ and $\Gamma$ do we have $A(\Lambda) \hookrightarrow A(\Gamma)$ only if $\Lambda \leq \Gamma^e$?

With regard to this question, the reader is referred to the introduction of the paper [5] Question 1.5] due to Lee–Lee.

This article is organized as follows. In Section 2, we introduce terminology and known results. For the sake of convenience, we discuss relation between graph-join (a certain graph operation) and embedding problems in Section 3. Section 4 is devoted to the proof of Theorem 1.3.

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2. Preliminaries

Suppose that $\Gamma$ is a graph. An element of $V(\Gamma) \cup V(\Gamma)^{-1}$ is called a letter. Any element in $A(\Gamma)$ can be expressed as a word, which is a finite multiplication of letters. Let $w = a_1 \cdots a_l$ be a word in $A(\Gamma)$ where $a_1, \ldots, a_l$ are letters. We say $w$ is reduced if any other word representing the same element as $w$ in $A(\Gamma)$ has at least $l$ letters. The following lemma is useful for checking whether a given word is reduced or not (cf. [2] Section 5).

Lemma 2.1. Let $w$ be a word in $A(\Gamma)$. Then $w$ is reduced if and only if $w$ does not contain a word of the form $v^\epsilon x v^{-\epsilon}$, where $v$ is a vertex of $\Gamma$, $\epsilon = \pm 1$ and $x$ is a word such that $v$ is commutative with all of the letters in $x$.

The support of a reduced word $w$ is the smallest subset $S$ of $V(\Gamma)$ such that each letter of $w$ is in $S$ or $S^{-1}$; we write $S = \text{supp}(w)$. It is well-known that the support does not depend on the choice of a reduced word, and so we can define the support of an element of $A(\Gamma)$. By a clique, we mean a complete subgraph of a graph. We rephrase a special case of Servatius’ Centralizer Theorem [6] The Centralizer Theorem in Section III] as follows.

Lemma 2.2. Let $w_1$ and $w_2$ be reduced words in $A(\Gamma)$ whose supports span cliques in $\Gamma$. Then the words $w_1$ and $w_2$ are commutative if and only if $\text{supp}(w_1)$ and $\text{supp}(w_2)$ are contained in a single clique in $\Gamma$. 
In this article, we say that a homomorphism $\psi: \Lambda(\Lambda) \to \Lambda(\Gamma)$ between two right-angled Artin groups satisfies (KK) condition if $\text{supp}(\psi(v))$ consists of mutually adjacent vertices in $\Gamma$ for all $v \in V(\Lambda)$ (i.e., $\text{supp}(\psi(v))$ spans a clique in $\Gamma$).

**Theorem 2.3** ([3 Theorem 4.3]). Suppose that $\Lambda$ and $\Gamma$ are finite graphs and $A(\Lambda) \hookrightarrow A(\Gamma)$. Then there is an embedding $\psi: A(\Lambda) \hookrightarrow A(\Gamma^c)$ such that $\psi$ satisfies (KK) condition. Namely, for all $v \in V(\Lambda)$, $\text{supp}(\psi(v))$ consists of mutually adjacent vertices in $\Gamma^c$.

### 3. Graph-join

In this section, we prove Proposition 3.2, which says that, for two finite graphs $\Lambda$ and $\Gamma$ such that there is an embedding $A(\Lambda) \hookrightarrow A(\Gamma)$ satisfying condition (KK), the problem of finding a full embedding $\Lambda \to \Gamma$, with a certain restriction, is reduced to the corresponding problems for the “join-components” of $\Lambda$. The (graph-)join $\Lambda_1 \ast \cdots \ast \Lambda_m$ of graphs $\Lambda_1, \ldots, \Lambda_m$ is the graph obtained from the disjoint union $\Lambda_1 \sqcup \cdots \sqcup \Lambda_m$ by joining the vertices $u$ and $v$ for all $u \in V(\Lambda_i), v \in V(\Lambda_j)$ ($i \neq j$).

In this article, we say that a graph $\Lambda$ is irreducible (with respect to join) if $\Lambda$ cannot be the join of two non-empty graphs. Any finite graph $\Lambda$ is the join of finitely many irreducible graphs. Indeed, this follows from the fact that $\Lambda = \Lambda_1 \ast \cdots \ast \Lambda_m$ if and only if $\Lambda^c = \Lambda_1^c \sqcup \cdots \sqcup \Lambda_m^c$. This fact also implies the following lemma.

**Lemma 3.1.** A finite graph $\Lambda$ is irreducible if and only if $\Lambda^c$ is connected. In particular, if $\Lambda$ is an irreducible graph containing at least two vertices, then for any vertex $u \in V(\Lambda)$, there is a vertex $u' \in V(\Lambda)$ such that $u$ and $u'$ are non-adjacent.

**Proof.** Suppose that $\Lambda$ is an irreducible graph containing at least two vertices. Pick a vertex $u \in V(\Lambda)$. If $u$ does not have a non-adjacent vertex, then we have a decomposition $\Lambda = \{u\} \ast \hat{\Lambda}$, where $\hat{\Lambda}$ is a full subgraph spanned by $V(\Lambda) \setminus \{u\}$, a contradiction. \hfill $\square$

Besides, the right-angled Artin group on the join, $A(\Lambda_1 \ast \cdots \ast \Lambda_m)$, is isomorphic to the direct product $A(\Lambda_1) \times \cdots \times A(\Lambda_m)$. For simplicity, if $\psi: A(\Lambda) \to A(\Gamma)$ is a homomorphism, then by $\text{supp}(\psi)$ we denote $\cup_{v \in V(\Lambda)} \text{supp}(\psi(v))$.

**Proposition 3.2.** Let $\Lambda$ be the join $\Lambda_1 \ast \cdots \ast \Lambda_m$ of finite irreducible graphs $\Lambda_1, \ldots, \Lambda_m$, and let $\Gamma$ be a finite graph. Suppose that the following conditions hold:

1. There is an embedding $\psi: A(\Lambda) \hookrightarrow A(\Gamma)$ satisfying condition (KK).
2. For each $1 \leq i \leq m$, there is a full embedding $\iota_i: \Lambda_i \to \Gamma$ with $\iota_i(\Lambda_i) \subset \text{supp}(\psi_i)$, where $\psi_i$ is the restriction of $\psi$ to $A(\Lambda_i)$.

Then there is a full embedding $\iota: \Lambda \to \Gamma$ with $\iota(\Lambda) \subset \text{supp}(\psi)$.

We first prove this proposition in a special case.

**Lemma 3.3.** Let $\Lambda_1$ be a finite irreducible graph containing at least two vertices, and let $\Lambda_2$ and $\Gamma$ be finite graphs. Suppose that the following conditions hold:

1. There is an embedding $\psi: A(\Lambda_1 \ast \Lambda_2) \hookrightarrow A(\Gamma)$ satisfying condition (KK).
2. For $i = 1, 2$, there are full embeddings $\iota_i: \Lambda_i \to \Gamma$ with $\iota_i(\Lambda_i) \subset \text{supp}(\psi_i)$, where $\psi_i$ is the restriction of $\psi$ to $A(\Lambda_i)$.

Then the map $\iota: \Lambda_1 \ast \Lambda_2 \to \Gamma$, defined by $\iota(v) = \iota_1(v)$ or $\iota_2(v)$ according to whether $v \in V(\Lambda_1)$ or $v \in V(\Lambda_2)$, is a full embedding with $\iota(\Lambda_1 \ast \Lambda_2) \subset \text{supp}(\psi)$.
Proof. We have only to prove: (i) \( \iota_1(A_1) \cap \iota_2(A_2) = \emptyset \) and (ii) \( \forall u \in V(A_1), \forall v \in V(A_2), \iota_1(u) \) and \( \iota_2(v) \) are adjacent in \( \Gamma \). In fact (i) and (ii) imply that the map \( \iota : A_1 \ast A_2 \rightarrow \Gamma \) is a full embedding. Moreover, the assumptions \( \iota_i(A_i) \subset \text{supp}(\psi_i) \) imply that the full embedding \( \iota \) satisfies the desired property that \( \iota(A_1 \ast A_2) \subset \text{supp}(\psi) \).

(i) Pick \( u_1 \in V(A_1) \) and \( u_2 \in V(A_2) \). Since \( A_1 \) is irreducible and has at least two vertices, \( A_1 \) has a vertex \( u_1' \) which is non-adjacent to \( u_1 \) in \( A_1 \) by Lemma 3.1. Since \( u_1 \) is not adjacent to \( u_1' \) in \( A_1 \), and since \( \iota_1 : A_1 \rightarrow \Gamma \) is full, \( \iota_1(u_1) \) is not adjacent to \( \iota_1(u_1') \) in \( \Gamma \). On the other hand, we can prove that \( \iota_2(u_2) \) is either identical with \( \iota_1(u_1') \) or adjacent to \( \iota_1(u_1') \) in \( \Gamma \) as follows (and so \( \iota_1(u_1) \neq \iota_2(u_2) \) in any case). By the assumptions that \( \iota_i(A_i) \subset \text{supp}(\psi_i) \) \( (i = 1, 2) \), there are vertices \( u_1' \in V(A_1) \) and \( u_2 \in V(A_2) \) such that \( \iota_1(u_1') \in \text{supp}(\psi(u_1')) \) and \( \iota_2(u_2) \in \text{supp}(\psi(u_2)) \). Moreover, since \( A_1 \) and \( A_2 \) are joined in \( A_1 \ast A_2 \), the image \( \psi(u_2) \) is commutative with \( \psi(u_1') \), and therefore \( \text{supp}(\psi(u_2)) \) and \( \text{supp}(\psi(u_1')) \) are contained in a single clique by (KK) condition and Lemma 2.2. Thus \( \iota_2(u_2) \) is either adjacent to \( \iota_1(u_1) \) or identical with \( \iota_1(u_1') \) in \( \Gamma \).

(ii) Pick \( u_1 \in V(A_1) \) and \( u_2 \in V(A_2) \). There are vertices \( \bar{u}_1 \in V(A_1) \) and \( \bar{u}_2 \in V(A_2) \) such that \( \iota_1(u_1) \in \text{supp}(\psi(\bar{u}_1)) \) and \( \iota_2(u_2) \in \text{supp}(\psi(\bar{u}_2)) \). Since \( A_1 \) and \( A_2 \) are joined in \( A_1 \ast A_2 \), \( \psi(\bar{u}_1) \) and \( \psi(\bar{u}_2) \) are commutative. Hence, \( \text{supp}(\bar{u}_1) \) and \( \text{supp}(\bar{u}_2) \) are contained in a single clique in \( \Gamma \) by Lemma 2.2. Thus, \( \iota_1(u_1) \) and \( \iota_2(u_2) \) are adjacent in \( \Gamma \).

By \( K_n \), we denote the complete graph on \( n \) vertices. The right-angled Artin group on \( K_n \), \( A(K_n) \), is isomorphic to \( \mathbb{Z}^n \).

Lemma 3.4. Let \( \Gamma \) be a finite graph. Suppose that \( \psi : A(K_n) \rightarrow A(\Gamma) \) is an embedding satisfying condition (KK). Then there is a full embedding \( \iota : K_n \rightarrow \Gamma \) with \( \iota(K_n) \subset \text{supp}(\psi) \).

Proof. Since \( \psi \) satisfies condition (KK) and since \( K_n \) is complete, \( \text{supp}(\psi) \) spans a clique on \( l \) vertices of \( \Gamma \) by Lemma 2.2. Hence, we have an embedding \( A(K_n) \cong \mathbb{Z}^n \rightarrow \mathbb{Z}^l \rightarrow A(\Gamma) \). This implies \( n \leq l \), and so we obtain an injective map \( V(K_n) \rightarrow \text{supp}(\psi) \), which induces a full embedding \( \iota : K_n \rightarrow \Gamma \) with \( \iota(K_n) \subset \text{supp}(\psi) \).

Proof of Proposition 3.2. We may assume that each of \( A_1, \ldots, A_n \) is a singleton graph and each of \( A_{n+1}, \ldots, A_m \) has at least two vertices. Put \( A_0 := A_1 \ast \cdots \ast A_n \). Then \( A_0 \) is isomorphic to the complete graph on \( n \) vertices, \( K_n \). In addition, we can decompose \( A \) into \( A_0 \ast \ast \ast \ast A_n \). Hence, we have \( A(A) = A(A_0) \times A(A_{n+1}) \times \cdots \times A(A_m) \). By restricting \( \psi \) to the abelian factor \( A(A_0) \), we obtain an embedding \( \psi_0 : A(A_0) \rightarrow A(\Gamma) \) satisfying condition (KK). Therefore, by Lemma 3.4 we obtain a full embedding \( \iota_0 : A_0 \rightarrow \Gamma \) with \( \iota_0(A_0) \subset \text{supp}(\psi_0) \). Consider the family of full embeddings \( \iota_0, \iota_{n+1}, \ldots, \iota_m \). Since each of \( A_{n+1}, \ldots, A_m \) is irreducible and has at least two vertices, by repeatedly applying Lemma 3.3 we obtain the desired full embedding \( \iota : A_0 \ast A_{n+1} \ast \cdots \ast A_m \rightarrow \Gamma \).

4. Proof of Theorem 1.3

In this section we prove Theorem 1.3. We first rephrase Theorem 1.3 in terms of join. Recall that \( (A_1 \sqcup \cdots \sqcup A_m)^c = A_1^c \ast \cdots \ast A_m^c \). Hence, if \( \Lambda \) is the complement of a linear forest, namely, if \( \Lambda \) is the complement of the disjoin union of finitely many path graphs, then \( \Lambda \) is the join of the complements of finitely many path graphs.
Theorem 4.1 (rephrased). Let $\Lambda$ be the join of the complements of finitely many path graphs and $\Gamma$ a finite graph. If $A(\Lambda) \hookrightarrow A(\Gamma)$, then $\Lambda \leq \Gamma^c$.

To obtain a full embedding $\Lambda \to \Gamma^c$ in the assertion above, we consider the join-component, the complement $P_n^c$ of the path graph $P_n$ on $n$ vertices.

Lemma 4.2. Let $n$ be a positive integer other than 3 and $\Gamma$ a finite graph. Suppose that $\psi: A(P_n^c) \to A(\Gamma)$ is an embedding satisfying condition $(KK)$. Then there is a full embedding $\iota: P_n^c \to \Gamma$ with $\iota(v) \in \text{supp}(\psi(v))$ ($\forall v \in V(P_n^c)$). In particular, $\iota(P_n^c) \subset \text{supp}(\psi)$.

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of $P_n^c$ labelled as illustrated in Figure 1.

The assertion is trivial in the case where $n = 1$. Therefore, we may assume $n = 2$ or $n \geq 4$. Suppose $n = 2$. Then $P_2^c$ consists of two vertices $v_1, v_2$. If there is no full embedding $\iota: P_2^c \to \Gamma$ with $\iota(v_1) \in \text{supp}(\psi(v_1))$ and $\iota(v_2) \in \text{supp}(\psi(v_2))$, then $\text{supp}(\psi(v_1))$ and $\text{supp}(\psi(v_2))$ do not have distinct vertices $u_1 \in \text{supp}(\psi(v_1))$ and $u_2 \in \text{supp}(\psi(v_2))$ such that $\{u_1, u_2\} \not\in E(\Gamma)$. Hence the supports, supp($\psi(v_1)$) and supp($\psi(v_2)$), are contained in a single clique in $\Gamma$, and so $\psi(v_1)$ and $\psi(v_2)$ are commutative in $A(\Gamma)$. This implies that a non-trivial element $[v_1, v_2] := v_1 v_2^{-1} v_2^{-1}$ of $A(P_2^c) = F_2$ is an element of the kernel of $\psi$, a contradiction.

We now assume $n \geq 4$. By $C_4$, we denote the clique in $\Gamma$ spanned by supp($\psi(v_1)$). Since $\psi(v_1)$ and $\psi(v_2)$ are commutative when $|i-j| > 1$, we obtain the following claim by Lemma 4.2.

Claim 4.3. If $|i-j| > 1$, then any vertex of $C_i$ and any vertex of $C_j$ are either identical or adjacent in $\Gamma$.

If $\Gamma$ has a sequence of mutually distinct vertices $y^{(1)}, y^{(2)}, \ldots, y^{(n)}$ such that $y^{(i)} \in V(C_i)$ and that $y^{(i-1)}$ and $y^{(i)}$ are non-adjacent, then the map $\iota: P_n^c \to \Gamma$ defined by $\iota(v_i) := y^{(i)}$ ($1 \leq i \leq n$) determines an embedding $P_n^c \to \Gamma$ by Claim 4.3. Since $y^{(i-1)}$ and $y^{(i)}$ are non-adjacent, $\iota: P_n^c \to \Gamma$ is a full embedding. Therefore we have only to prove that $\Gamma$ has a sequence of mutually distinct vertices $y^{(1)}, y^{(2)}, \ldots, y^{(n)}$ such that $y^{(i)} \in V(C_i)$ and that $y^{(i-1)}$ and $y^{(i)}$ are non-adjacent.

\begin{figure}[h]
\centering
\includegraphics[scale=0.5]{figure1.png}
\caption{This picture illustrates $P_n^c$. Real lines with the characters $c$ represent the edges in $P_n = (P_n^c)^c$, each of which joins non-adjacent vertices in $P_n^c$. In this picture, any two distinct vertices not joined by a line are adjacent in $P_n^c$.}
\end{figure}
Suppose, on the contrary, that

\[ (**\) the graph \( \Gamma \) does not have a sequence of mutually distinct vertices \( y^{(1)}, y^{(2)}, \ldots, y^{(n)} \) such that \( y^{(i)} \in V(C_i) \) and that \( y^{(i-1)} \) and \( y^{(i)} \) are non-adjacent.\]

To deduce a contradiction, we will prove that the commutator \([v_1, v_n]\) is a non-trivial element of the kernel of \( \psi \). We first observe that \([v_1, v_n] \) is non-trivial in \( A(P_n) \). Since the preceding letter and succeeding letter of each letter \( v^e \) in the word \([v_1, v_n] \) are not commutative with \( v^e \), the word \([(v_1)^{v_2 v_3 \cdots v_{n-1}}, v_n] \) does not admit reduction in the sense of Lemma 2.1 This implies that the word \([(v_1)^{v_2 v_3 \cdots v_{n-1}}, v_n] \) is reduced and a non-trivial element in \( A(P_n) \). Thus the remaining task is to show that \([(v_1)^{v_2 v_3 \cdots v_{n-1}}, v_n] \in \ker \psi \). To this end, it is enough to prove that the element \( \psi(v_1)\psi(v_2)\psi(v_3)\cdots\psi(v_{n-1}) \) can be represented as a word consisting of vertices of \( C_1, \ldots, C_{n-1} \), each of which is commutative with all of the vertices of \( C_n \). We first inductively define the family \( Y^{(1)}, Y^{(2)}, \ldots, Y^{(n-1)} \) of (possibly empty) subsets of \( V(\Gamma) \) as follows:

(i) \( Y^{(1)} := V(C_1) \).

(ii) Suppose that \( Y^{(i-1)} \) is defined. Then we set

\[ Y^{(i)} := \{ y \in V(C_i) \mid \exists x \in Y^{(i-1)} \text{ such that } \{x, y\} \notin E(\Gamma) \}. \]

With regard to this family \( Y^{(1)}, Y^{(2)}, \ldots, Y^{(n-1)} \), we claim that:

**Claim 4.4.** Any vertex of \( C_n \) and any vertex in \( \cup_{i=1}^{n-1} Y^{(i)} \) are either identical or adjacent in \( \Gamma \).

**Proof of Claim 4.4.** Note that any vertex of \( C_n \) and any vertex in \( \cup_{i=1}^{n-2} Y^{(i)} \) are either identical or adjacent in \( \Gamma \) by Claim 4.3. So we have to show that if \( Y^{(n-1)} \neq \emptyset \), any element \( y^{(n-1)} \in Y^{(n-1)} \) is commutative with all vertices of \( C_n \). By the construction of \( Y^{(n-1)} \), there is an element \( y^{(n-2)} \in Y^{(n-2)} \) which is not commutative with \( y^{(n-1)} \). By repeating this argument, we can find a sequence \( y^{(1)}, \ldots, y^{(n-1)} \) such that \( y^{(i)} \in Y^{(i)} \), and that \( y^{(i-1)} \) and \( y^{(i)} \) are not commutative (\( 2 \leq i \leq n-1 \)).

Suppose, on the contrary, that there is an element \( y^{(n)} \in V(C_n) \), which is not commutative with \( y^{(n-1)} \). Then we can observe that \( y^{(i)} \neq y^{(j)} \) if \( i < j \) as follows. We first consider the case where \( i = 1 \) and \( 2 \leq j \leq n-1 \). Notice that the element \( y^{(1)} \) is either identical with \( y^{(j+1)} \) or adjacent to \( y^{(j+1)} \) by Claim 4.3. On the other hand, the element \( y^{(j)} \) is non-adjacent to \( y^{(j+1)} \). Hence, we obtain that \( y^{(1)} \neq y^{(j)} \).

We next consider the case where \( i = 1 \) and \( j = n \). Since \( n \geq 4 \), we have \( n-2 > 2 \). Although the element \( y^{(n)} \) is non-adjacent to \( y^{(n-2)} \), the element \( y^{(1)} \) must be adjacent to \( y^{(n-1)} \) by Claim 4.3. Hence, we obtain that \( y^{(i)} \neq y^{(j)} \).

If \( i \geq 2 \), the element \( y^{(i)} \) is non-adjacent to \( y^{(i-1)} \), but the element \( y^{(i)} \) is either adjacent to \( y^{(i-1)} \) or identical with \( y^{(i-1)} \). So \( y^{(i)} \neq y^{(j)} \). Thus \( y^{(i)} \neq y^{(j)} \) if \( i < j \), and therefore the sequence \( y^{(1)}, \ldots, y^{(n)} \) violates our assumption (**). \( \Box \)

**Fix reduced words** \( W_i \) in \( V(C_i) \) representing \( \psi(v_i) \) (\( 1 \leq i \leq n \)). If two given words \( w_1, w_2 \) represent the same element in \( A(\Gamma) \), then we denote by \( w_1 = w_2 \). If \( w_1, w_2 \) are identical as words, then we denote by \( w_1 \equiv w_2 \). We inductively construct words \( \tilde{W}_1, \ldots, \tilde{W}_{n-1} \) satisfying the following conditions.

(W-1) \( \tilde{W}_1 \) is a word in \( Y^{(1)} \). Namely, the word \( \tilde{W}_1 \) consists of the vertices in \( Y^{(1)} \).
(W-i) \( \bar{W}_i \) is a word in \( Y^{(i)} \). Moreover,
\[
\bar{W}_i^{-1} \cdots \bar{W}_2^{-1} \bar{W}_1 W_2 \cdots \bar{W}_i = W_i^{-1} \cdots W_2^{-1} W_1 W_2 \cdots W_i
\]
as elements in \( A(\Gamma) \) (\( i \geq 2 \)).

Let us start the construction of the words \( \bar{W}_1, \ldots, \bar{W}_{n-1} \).

(Step 1) \( \bar{W}_1 \equiv W_1 \). Obviously \( \bar{W}_1 \) satisfies (W-1).

(Step 2) If \( W_2 \) is a word in \( Y^{(2)} \), set \( \bar{W}_2 \equiv W_2 \). Then the word \( \bar{W}_2 \) satisfies (W-2). We now suppose that \( W_2 \) is not a word in \( Y^{(2)} \), i.e., there is a vertex \( v \in V(C_2) \setminus Y^{(2)} \) such that the letter \( v^\epsilon \) \( (\epsilon = \pm 1) \) is contained in \( W_2 \). We write \( W_2 \equiv w_2 v^\epsilon w'_2 \). Then \( \bar{W}_2^{-1} W_1 W_2 \equiv (w_2')^{-1}(v^\epsilon)^{-1} w_2 W_1 v^\epsilon w'_2 \). Noting that the letter \( v^\epsilon \) is commutative with \( w_2 \), because \( C_2 \) is a clique containing \( v \). By the definition of \( Y^{(2)} \), the letter \( v^\epsilon \) is commutative with \( \bar{W}_1 \). Hence, \( \bar{W}_2^{-1} W_1 W_2 \equiv (w_2')^{-1}(w_2)^{-1} W_1 w_2 w'_2 \) in \( A(\Gamma) \). If \( w_2 w'_2 \) is a word in \( Y^{(2)} \), we set \( \bar{W}_2 \equiv w_2 w'_2 \). If not, then applying the same reduction to \( w_2 w'_2 \) until we obtain a word \( \bar{W}_2 \) in \( Y^{(2)} \). Then \( \bar{W}_2 \) satisfies the condition (W-2).

(Step i) Assume that \( \bar{W}_1, \ldots, \bar{W}_{i-1} \) satisfy the conditions (W-1), (W-2), \ldots, (W-(i-1)), respectively. If \( W_i \) is a word in \( Y^{(i)} \), set \( \bar{W}_i \equiv W_i \). Then the word \( \bar{W}_i \) satisfies (W-i). We now suppose that \( W_i \) is not a word in \( Y^{(i)} \). Since \( W_i \) is not a word in \( Y^{(i)} \), there is a vertex \( v \in V(C_i) \setminus Y^{(i)} \) such that the letter \( v^\epsilon \) \( (\epsilon = \pm 1) \) is contained in \( W_i \). So we write \( W_i \equiv w_i v^\epsilon w'_i \). Then we have the following equality:
\[
\bar{W}_i^{-1} \bar{W}_{i-1}^{-1} \cdots \bar{W}_2^{-1} \bar{W}_1 W_2 \cdots \bar{W}_i = (w'_i)^{-1}(v^\epsilon)^{-1}(w_i)^{-1} \bar{W}_{i-1}^{-1} \cdots \bar{W}_2^{-1} \bar{W}_1 W_2 \cdots \bar{W}_{i-1} w_i v^\epsilon w'_i.
\]
Since \( \bar{W}_1, \ldots, \bar{W}_{i-2} \) are words in \( Y^{(1)} \), \ldots, \( Y^{(i-2)} \), respectively, the letter \( v^\epsilon \) is commutative with each of \( \bar{W}_1, \ldots, \bar{W}_{i-2} \) by Claim 2.2. In addition, since \( v \in V(C_i) \setminus Y^{(i)} \) and since \( \bar{W}_{i-1} \) is a word in \( Y^{(i-1)} \), the letter \( v^\epsilon \) is commutative with \( \bar{W}_{i-1} \). Furthermore, \( v^\epsilon \) is commutative with \( w_i \), because \( C_i \) is a clique. Thus we have:
\[
(w'_i)^{-1}(v^\epsilon)^{-1}(w_i)^{-1} \bar{W}_{i-1}^{-1} \cdots \bar{W}_2^{-1} \bar{W}_1 W_2 \cdots \bar{W}_{i-1} w_i v^\epsilon w'_i \equiv (w'_i)^{-1}(w_i)^{-1} \bar{W}_{i-1}^{-1} \cdots \bar{W}_2^{-1} \bar{W}_1 W_2 \cdots \bar{W}_{i-1} w_i v^\epsilon w'_i.
\]
If \( w_i w'_i \) is a word in \( Y^{(i)} \), we set \( \bar{W}_i \equiv w_i w'_i \). If not, then applying the same reduction to \( w_i w'_i \) until we obtain a word \( \bar{W}_i \) in \( Y^{(i)} \). In the end, \( \bar{W}_i \) obviously satisfies the condition (W-i).

By Claim 4.4 \( \bar{W}_1, \ldots, \bar{W}_{n-1} \) are commutative with \( \bar{W}_n \), which is a representative of \( \psi(v_n) \). Since \( \psi(v_1)^{\psi(v_2)^{\psi(v_3)\cdots \psi(v_{n-1})}} \) is a multiplication of \( \bar{W}_1, \ldots, \bar{W}_{n-1} \), it is commutative with \( \psi(v_n) \). Thus the commutator \([\bar{W}_1, \ldots, \bar{W}_{n-1}, v_n] \) is an element of \( \ker \psi \), as desired.

To treat the case where \( n = 3 \), we use the following lemma due to Kim–Koberda.

**Lemma 4.5** ([Theorem 5.4]). Let \( \Lambda \) be a finite graph whose right-angled Artin group \( A(\Lambda) \) has no center. Suppose that \( A(\Lambda) \) has an embedding into the direct product \( G_1 \times G_2 \) of (non-trivial) groups \( G_1, G_2 \). If the natural projections \( A(\Lambda) \rightarrow G_i \) \( (i = 1, 2) \) have non-trivial kernels, then \( \Lambda \) contains a full subgraph isomorphic to the cyclic graph of length 4.
Lemma 4.6. Let $\Gamma$ be a finite graph. Suppose that $\psi: A(P^c_3) \to A(\Gamma)$. Then there is a full embedding $\nu: P^c_3 \to \Gamma$ with $\nu(P^c_3) \subset \text{supp}(\psi)$.

Proof. For simplicity, we assume that $\text{supp}(\psi) = V(\Gamma)$. Suppose, on the contrary, that

(A) $P^c_3$ is not a full subgraph of $\Gamma$. Namely, $P_3$ is not a full subgraph of $\Gamma^c$.

We first prove that $\Gamma^c$ is the disjoint union of finitely many complete graphs. Let $C$ be a connected component of $\Gamma^c$. If $\#V(C) \leq 2$, then the connectedness of $C$ obviously implies that $C$ is complete. So we may assume $\#V(C) \geq 3$. Pick two edges $e_1^c$ and $e_2^c$ of $C \subseteq \Gamma^c$ that share a vertex. Then, by our assumption (A), the set $e_1^c \cup e_2^c$ of vertices spans a triangle in $\Gamma^c$. In other words, the initial vertex and terminal vertex of any edge-path consisting of three vertices in $C$ is adjacent. By repeatedly using this fact, we can verify that, for any edge-path in $C$, the initial vertex is adjacent to the terminal vertex. Therefore the connected component $C$ must be complete. Thus, $\Gamma^c$ is the disjoint union of finitely many complete graphs, and so $\Gamma$ is the join of finitely many edgeless graphs. Hence, $A(\Gamma)$ is the direct product $A_1 \times \cdots \times A_m$ of free groups $A_1, \ldots, A_m$. Since $A(P^c_3)$ is not free and since $A_1 \times \cdots \times A_m$ contains an embedded $A(P^c_3)$, the integer $m$ is greater than 1. We now regard $A(\Gamma)$ as the direct product $(A_1 \times \cdots \times A_{m-1}) \times A_m$ of two direct factors, $A_1 \times \cdots \times A_{m-1}$ and $A_m$. Let $\pi_{m-1}, \pi_m$ denote the projections $A(P^c_3) \to A_1 \times \cdots \times A_{m-1}$ and $A(P^c_3) \to A_m$, respectively. Then since $A(P^c_3)$ is not free, the projection $\ker(\pi_m)$ must be non-trivial. Note that $A(P^c_3) \cong \mathbb{Z} \rtimes \mathbb{Z}[2]$ has no center. If $\ker(\pi_{m-1})$ is non-trivial, then by Lemma 4.5, the defining graph $P^c_3$ must have a full subgraph isomorphic to the cyclic graph of length 4, a contradiction. So we may assume that $\pi_{m-1}$ is injective. In other words, $A(P^c_3) \hookrightarrow A_1 \times \cdots \times A_{m-1}$. Hence, by repeating this argument, we can reduce the number of the direct factors in the target group. Finally, we have that $A(P^c_3) \hookrightarrow A_1$, which is impossible. □

Lemma 4.7. Let $\Lambda$ be the join of the complements of finitely many path graphs and $\Gamma$ a finite graph. Suppose that $\psi: A(\Lambda) \to A(\Gamma)$ is an embedding satisfying condition (KK). Then there is a full embedding $\nu: \Lambda \to \Gamma$ with $\nu(\Lambda) \subset \text{supp}(\psi)$.

Proof. Suppose that $\Lambda_1, \ldots, \Lambda_m$ is the irreducible graphs such that $\Lambda = \Lambda_1 \ast \cdots \ast \Lambda_m$ and $\Lambda_i \cong P^c_{\mu_i}$ ($1 \leq i \leq m$). Then by restricting $\psi$ to each direct factor $A(\Lambda_i)$, we obtain $\psi: A(\Lambda_i) \to A(\Gamma)$ with condition (KK). Now by Lemma 4.2 and 4.6, we obtain full embeddings $\nu_i: \Lambda_i \to \Gamma$ with $\nu_i(\Lambda_i) \subset \text{supp}(\nu_i)$ (for $1 \leq i \leq m$). Since $\Lambda_1, \ldots, \Lambda_m$ are path graphs, their connectedness together with Lemma 3.1 implies that $\Lambda_1, \ldots, \Lambda_m$ are irreducible. Thus, by applying Proposition 3.2 to $\Lambda = \Lambda_1 \ast \cdots \ast \Lambda_m$, we obtain the result that there is a full embedding $\nu: \Lambda \to \Gamma$ with $\nu(\Lambda) \subset \text{supp}(\psi)$. □

Proof of Theorem 1.3 (Theorem 4.1). Suppose that there is an embedding of the right-angled Artin group $A(\Lambda)$ on the join $\Lambda$ of the complements of finitely many path graphs into the right-angled Artin group $A(\Gamma)$ on a finite graph $\Gamma$. By Theorem 2.3 due to Kim–Koberda, we have an embedding $\psi: A(\Lambda) \hookrightarrow A(\Gamma^c)$ satisfying condition (KK), where $\Gamma^c$ is the extension graph of $\Gamma$. Consider the full subgraph $\Gamma'$ of $\Gamma^c$, which is spanned by $\text{supp}(\psi) = \bigcup_{v \in V(\Lambda)} \text{supp}(\nu(v))$. Then we have an embedding $\psi: A(\Lambda) \hookrightarrow A(\Gamma')$ satisfying condition (KK). Now, by Lemma 4.7, we have $\Lambda \leq \Gamma'$. Thus $\Lambda \leq \Gamma' \leq \Gamma^c$, as desired. □
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