Ritus functions for graphene-like systems with magnetic fields generated by first-order intertwining operators

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Abstract

In this work, we construct the exact propagator for Dirac fermions in graphene-like systems immersed in external static magnetic fields with non-trivial spatial dependence. Such field profiles are generated within a first-order supersymmetric framework departing from much simpler (seed) magnetic field examples. The propagator is spanned on the basis of the Ritus eigenfunctions, corresponding to the Dirac fermion asymptotic states in the non-trivial magnetic field background which nevertheless admits a simple diagonal form in momentum space. This strategy enlarges the number of magnetic field profiles in which the fermion propagator can be expressed in a closed-form. Electric charge and current densities are found directly from the corresponding propagator and compared against similar findings derived from other methods.

1. Introduction

Physics of pseudo-relativistic Dirac fermions in two spatial dimensions continues to attract the attention of a vast community around the globe that considers these entities as fundamental in importance as the building blocks of the Universe [1]. From the seminal work of Wallace [2], the interest on this kind of excitations in condensed matter realms (see, for instance, [3, 4] and references therein) has been put forward in quantum Hall [5–7], high-Tc superconductivity [8] and other bidimensional systems [9, 10]. In recent years, graphene [5, 11] and the plethora of new 2D materials (see [12–15] for recent reviews) have increased the interest in these systems not only because of the potential technological applications, but also because of the fundamental physics that can be explored in a condensed matter physics environment [1, 3, 4]. The dynamics of pseudo-relativistic quasiparticle states has been explored under the influence of different external agents like under strain, curvature effects and in the presence of (external or induced) electric and magnetic fields [1, 10, 16–19].

For the dynamics of Dirac fermions influenced by external electromagnetic fields a lot of attention has been paid to understand the electronic states in background fields configurations related to uniform magnetic field, crossed electric and magnetic fields, parallel electric and magnetic fields and the plane wave electromagnetic field cases. Further configurations of static magnetic fields with spatially varying profile have also been considered from the supersymmetric quantum mechanical structure of the Dirac equation in fields of this type [20, 21]. Examples include the uniform magnetic field case (and variations including an electric field), the Scarf potential (both hyperbolic and trigonometric), and the Morse potential along one spatial dimension [16, 22]. Being more precise, supersymmetry in quantum mechanics is a theoretical framework that allows to map the solutions from a stationary Schrödinger problem in a static one-dimensional potential to another stationary Schrödinger problem with a different potential that is called the supersymmetric partner of the former. Supersymmetry is realized in different manners, such as the factorization method [23, 24] and the Darboux transformation...
[25, 26], which are equivalent. An interesting variant of the supersymmetric framework was developed in [27, 28] in which rather than starting from the solutions to a Ricatti equation, new potentials are generated departing from the solutions of an initial wave equation.

In many physical situations, nevertheless, it is equally useful to know the corresponding propagator for these electronic states. However, because the asymptotic states do not correspond to plane waves, the representation of the two-point function is cumbersome rendering almost impossible to write the propagator in a closed form except for a handful of examples related to the uniform electric/magnetic field either parallel or perpendicular and plane wave electromagnetic field. Alternative representations have been developed for this purpose. Among several others, the Schwinger method [29], the spectral representation [30] and the Ritus method [31–33] allow to write a closed form of the propagator.

In this article, we revisit the construction of the propagator of 2D Dirac fermions in a background static magnetic field, which is relevant to monolayer graphene and related systems. For this purpose, we expand the propagator in the basis of Ritus functions, namely, the eigenfunctions of the operator \((\gamma \cdot \Pi)^2\) where \(\Pi_\mu = p_\mu + eA_\mu\) is the canonical momentum operator that includes the effect of the external magnetic field through minimal coupling (with \(A_\mu\) denoting the corresponding vector potential and \(e\) is the elementary charge) and \(\gamma^a\) denote the \(2 \times 2\) covariant Dirac matrices. We consider non-trivial magnetic background fields derived within a generalization of the first order intertwining formalism of [27, 28] in which, starting from seed solutions corresponding to the Ritus eigenfunctions for the uniform and an exponentially decaying magnetic fields [16, 20–22], we construct the new Ritus eigenfunctions corresponding to more intricate magnetic field profiles written in terms of highly transcendental functions. In doing so, we extend the number of cases in which the propagator for Dirac fermions in non-trivial magnetic field backgrounds can be expressed in a closed form. To achieve that goal and aiming a self-contained presentation of our findings, we have organized the remaining of the article as follows: In the next section we briefly present the Ritus method to derive the Dirac fermion propagator in a general static external magnetic field. In Section 3 we present the first-order intertwining framework to generate further inhomogeneous magnetic field profiles from seed (known) solutions to the Ritus eigenfunctions. We work out the explicit examples of non-trivial magnetic fields derived from the seed uniform and the exponentially decaying magnetic field Ritus eigenfunctions in detail. In Section 4 we derive the electric charge and current densities from the constructed propagator. Finally, we conclude in Section 5.

2. Fermion propagator in external magnetic fields

We start our discussion of the construction of the fermion propagator in external magnetic fields within the Ritus formalism (see [21] for a pedagogical presentation of the framework). Such a construction is relevant for monolayer graphene and other 2D materials for which the charge carriers behave as Dirac fermions. Let us consider a magnetic field pointing perpendicularly to the plane of motion of Dirac fermions, in such a way that, working in a Landau-like gauge, we introduce an electromagnetic potential \(A^\mu = (0, 0, \mathcal{V}_0(x))\), where \(\mathcal{V}_0(x)\) is a scalar function such that \(\mathcal{V}_0(x) = \partial_x \mathcal{V}_0(x)\) defines the profile of the field. In these circumstances, the fermion propagator cannot be diagonalized on the basis of the kinetic momentum eigenfunctions, because the asymptotic states of these fermions in a background magnetic fields do not correspond to plane waves. Motivated by this observation, we notice that the Green function for Dirac particles, \(G(z, z')\), satisfies

\[
((\gamma \cdot \Pi) - m) G(z, z') = \delta^D(z - z'),
\]

with \(z'' = (t, x, y), \gamma^a\) denoting the Dirac matrices (we consider the representation \(\gamma^0 = \sigma_3, \gamma_1 = i\sigma_1, \) and \(\gamma^2 = i\sigma_2\)), where \(\sigma_i\) are the Pauli matrices, and \(\Pi_\mu\) is the canonical momentum. We omit the Lorentz index in the vectors to keep a shorthand notation when necessary. Moreover, although for monolayer graphene the mass gap \(m\) vanishes, it becomes a relevant parameter in other systems, and that is why we keep it finite. Eventually, we discuss the limit \(m \to 0\). Since \(G(z, z')\) commutes with \((\gamma \cdot \Pi)^2\), we expand the propagator on the basis of the eigenfunctions of the later, namely, the functions \(\mathcal{E}_p(z)\) satisfying

\[
(\gamma \cdot \Pi)^2 \mathcal{E}_p(z) = p^2 \mathcal{E}_p(z),
\]

where the eigenvalue \(p^2\) can be any real number corresponding, as we shortly will see, to the magnitude squared of the vector \(p^\mu\) (or simply \(p\) to avoid cumbersome notation) that labels the functions \(\mathcal{E}_p(z)\). We refer to the functions \(\mathcal{E}_p(z)\) as the Ritus eigenfunctions [31–33]. It can be directly verified that these functions \(\mathcal{E}_p(z)\) fulfill the closure and completeness relations

\[
\int \mathrm{d}^2 \mathbf{z} \, \mathcal{E}_p(z) \mathcal{E}_p(z) = \frac{1}{2} \delta(p - p'),
\]

5In our conventions, Greek indices \(\mu, \nu, \ldots = 0, 1, 2\), whereas Latin indices \(i, j, \ldots = 1, 2\).
\[ \int d^4 p \, E_p(z') \overline{E}_p(z) = \delta(z-z'), \]

with \( \overline{E}_p(z) = \gamma^0 E_p^\mu(z) \gamma^0 \) and \( \mathbb{I} \) is the 2 \( \times \) 2 unit matrix.

In order to construct the Ritus eigenfunctions, we notice that the operator

\[ (\gamma \cdot \Pi)^2 = \gamma^\mu \gamma^\nu \Pi_\mu \Pi_\nu = \Pi^2 + \frac{1}{2} \sigma^{\mu \nu} F_{\mu \nu}, \]

where \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic field strength tensor and \( \sigma^{\mu \nu} = i [\gamma^\mu, \gamma^\nu] / 2 \). For a static magnetic field pointing perpendicularly to the plane, the only non-vanishing components of these tensors are

\[ F_{12} = - F_{21} = W_0'(x), \quad \sigma^{12} = \sigma_3. \]

Then, the eigenvalue equation (2) becomes

\[ (\Pi^2 + \sigma_3 W_0'(x)) \overline{E}_p(z) = p^2 \overline{E}_p(z), \]

from where we observe that the Ritus eigenfunctions are actually matrices, whose explicit form is

\[ E_p(z) = \begin{pmatrix} E_{p,+1}(z) & 0 \\ 0 & E_{p,-1}(z) \end{pmatrix}. \]

Notice that the subscript \( p \), which is the shorthand notation of the vector \( \rho_\mu = (p_\mu, p_3, k) \) is a vector that contains the eigenvalues of the operators \( i \partial_\mu - i \partial_3 \) and \( H_\sigma \), respectively, and whose norm squared corresponds to the eigenvalue in equation (2). That is, the components of the vector \( \rho_\mu \) are the numbers such that

\[ i \partial_\mu E_p(z) = p_\mu E_p(z), \quad i \partial_3 E_p(z) = - p_3 E_p(z), \quad H_\sigma E_p(z) = k E_p(z), \]

with \( H_\sigma = - (\gamma \cdot \Pi)^2 + \Pi_3^2 \). These eigenvalues allow us to write the scalar functions as

\[ E_{p,\sigma}(z) = e^{-i(p_3 z - p_3 z')} F_{k, p, \sigma}(x), \]

where \( \sigma = \pm 1 \) are the eigenvalues of \( \sigma_3 \) and the functions \( F_{k, p, \sigma}(x) \) satisfy

\[ [- \partial_k^2 + (p_2 + e W_0(x))^2 - \sigma e W_0'(x)] F_{k, p, \sigma} = k F_{k, p, \sigma}, \]

which corresponds to a Pauli equation for a particle with mass \( m = 1/2 \) and gyromagnetic factor \( g = 2 \). This equation possesses a supersymmetric structure as we will briefly discuss below. Thus, \( F_{k, p, \sigma}(x) \) are the solutions of the equations in (10) associated to each of the supersymmetric-partner potentials

\[ V_0^\sigma(x) = (p_2 + e W_0(x))^2 - \sigma e W_0'(x). \]

From now on, we fix the value \( \sigma = 1 \). Then, we have the required ingredients to construct the Ritus eigenfunctions from a first-order supersymmetric formalism.

### 3. Supersymmetric framework for the Ritus eigenfunctions

Similar to the case of the standard harmonic oscillator, the formalism of first-order supersymmetric quantum mechanics (1-SUSY QM) introduces two first-order differential operators \( L_0^\pm \) explicitly given by

\[ L_0^\pm = \pm \frac{d}{dx} + W_0(x), \]

where \( W_0(x) \) is known as the superpotential. Here, \( L_0^+ \) and \( L_0^- \) are adjoint operators to each other. With them, a pair of Hamiltonians \( \mathcal{H}_+ \) and \( \mathcal{H}_- \), whose respectively spectra are \( k_0^+ \) and \( k_0^- \), can be factorized as

\[ \mathcal{H}_- = L_0^- L_0^+. \]

Here, the so-called intertwining operators \( L_0^\pm \) satisfy the relations

\[ \mathcal{H}_+ L_0^\pm = L_0^\mp \mathcal{H}_-. \]

By a simple inspection, one can recognize that in the construction presented in the previous section, the functions \( F_{k, p, \sigma} \) are related by a supersymmetric transformation. Indeed, the action of the intertwining operators \( L_0^\pm \) on the solutions of the Hamiltonians (10) is (see the left panel in figure 1)

\[ F_{p, \sigma - 1}(x) = \frac{L_0^- F_{p+1, p, \sigma}(x)}{\sqrt{k_{n+1}^+}}, \quad F_{p+1, p, \sigma}(x) = \frac{L_0^+ F_{p, p-1}(x)}{\sqrt{k_n^-}}, \]

where \( \sigma = \pm 1 \).
where the ground state, which is annihilated by the operator $L_0^-$, behaves as

$$F_0(x) \sim e^{-\int W_0(x) \, dx}.$$  \hspace{1cm} (16)

This observation implies that we can write

$$W_0(x) = -\frac{(F_{0,p_x+1}(x))'}{F_{0,p_x+1}(x)}.$$  \hspace{1cm} (17)

Furthermore, the energy levels of $\mathcal{H}_+$ turn out to be

$$k^-_n = k^+_n, \quad k^+_0 = 0.$$  \hspace{1cm} (18)

These expressions indicate that the eigenfunctions and eigenvalues of the problem can be found through the operators $L_\sigma^\pm$, which simplifies the calculations since this involves just first-order derivatives. Also, the magnetic field profile $B_0(x)$ can be related to the electromagnetic potential $A_\mu(x)$, the superpotential $W_0(x)$, and the ground state of $H_+$ as follows:

$$B_0(x) = A_1^\prime(x) = \frac{1}{e} W_0^\prime(x) = -\frac{1}{e} \frac{d^2}{dx^2} \{ \ln[F_{0,p_x+1}(x)] \},$$  \hspace{1cm} (19)

which implies that is valid to make $V_0(x) \equiv W_0(x)$.

### 3.1. Generalized first order intertwining

In this section we introduce the first order supersymmetric formalism to generate inhomogeneous magnetic fields from intertwining operators. We follow closely the discussion of [28]. Taking as starting Hamiltonian the one with $\sigma = 1$ in (10), the first step of the method consists in displacing the energy of the Hamiltonian $\tilde{\mathcal{H}}_0$, as follows:

$$\tilde{\mathcal{H}}_0 \equiv \mathcal{H}_+ - \epsilon_1 = -\frac{d^2}{dx^2} + V_i(x) - \epsilon_i,$$  \hspace{1cm} (20)

so that $V_0(x) = V_i(x) - \epsilon_1$, where $\epsilon_1 \leq k^+_0 = 0$. Here $\tilde{\mathcal{H}}_0$ is the Hamiltonian upon which the 1-SUSY QM formalism will be applied.

The second step is to build a new Hamiltonian $\tilde{\mathcal{H}}_i$ departing from $\tilde{\mathcal{H}}_0$ through the intertwining relation (see the right panel in figure 1):

$$\tilde{\mathcal{H}}_i L_i^+ = L_i^+ \tilde{\mathcal{H}}_0,$$  \hspace{1cm} (21)

where $\tilde{\mathcal{H}}_i$ and $L_i^\pm$ are given by

$$\tilde{\mathcal{H}}_i = -\frac{d^2}{dx^2} + V_i(x, \epsilon_1), \quad L_i^\pm = \mp \frac{d}{dx} + W_i(x, \epsilon_1),$$  \hspace{1cm} (22)

respectively, which implies $\tilde{\mathcal{H}}_0 = L_1^+ L_1^-$ and $\tilde{\mathcal{H}}_1 = L_1^+ L_1^-$. This leads to the following relations for $W_i$ and $V_i$ derived from $V_0$,

$$W_i^2(x, \epsilon_1) + W_i^2(x, \epsilon_1) = V_0(x),$$  \hspace{1cm} (23a)
\[ V_1(x, \epsilon_1) = \tilde{V}_0(x) - 2W'(x, \epsilon_1). \]  

(23b)

Let us suppose now that we can write \( W_1(x, \epsilon_1) = u'_{1}/u_{1} \). The above relations lead us to the following expression for \( u_{1} \):

\[ -u''_{1} + \tilde{V}_0(x)u_{1} = 0. \]  

(24)

The corresponding magnetic field giving place to \( V_1(x, \epsilon_1) \) is obtained from

\[ B_1(x, \epsilon_1) = \frac{1}{e} \frac{d}{dx} [W_1(x, \epsilon_1)] = -B_0(x) - \frac{1}{e} \frac{d^2}{dx^2} \left[ \ln \left( \frac{F_{0;p_{2};+1}(x)}{u_{1}} \right) \right]. \]  

(25)

The third step of the method is to identify the eigenfunctions and eigenvalues of the new system. The energy levels for \( \tilde{\mathcal{H}}_0 \) and \( \mathcal{H}_0 \) are those of \( \mathcal{H}_+ \), displaced by the quantity \( -\epsilon_1 \), plus the ground state of \( \mathcal{H}_0 \) at zero energy:

\[ k_n^{(0)} = k_n^{+} - \epsilon_1, \]  

(26a)

\[ k_0^{(1)} = 0, \quad k_{n+1}^{(1)} = \sqrt{k_n^{(0)}}, \quad n = 0, 1, \ldots, \]  

(26b)

with \( \epsilon_1 \leq k_0^{+} = 0 \). The unknown eigenfunctions associated with these energies are given by:

\[ F_{0;p_{2};+1}(x) \sim \frac{1}{u_{1}}, \quad F_{n+1,p_{2};+1}(x) = \frac{1}{\sqrt{k_n^{(0)}}} L_1^{-1} F_{n,p_{2};+1}(x). \]  

(27)

where the eigenfunctions \( F_{n,p_{2};+1}(x) \) of \( \mathcal{H}_+ \), and consequently those of \( \tilde{\mathcal{H}}_0 \), are assumed to be known. In addition, the ground state of \( \mathcal{H}_0 \) fulfills the condition \( L_1^{-1} F_{0;p_{2};+1}(x) = 0 \).

It is worth noting that, according to the 1-SUSY QM formalism, since \( \epsilon_1 \leq k_0^{+} \) and depending on the choice of the function \( u_{1} \), three different cases can arise for the spectrum of the Hamiltonian \( \mathcal{H}_0 \): that it does not include the ground state of \( \tilde{\mathcal{H}}_0 \), or that it has an extra energy level, or that it is isospectral to \( \tilde{\mathcal{H}}_0 \). Below we discuss in detail two examples of magnetic field profiles, namely, the homogeneous field and the exponentially decaying magnetic field, only for one such case.

3.1.1. Uniform magnetic field

First, let us consider a uniform magnetic field, for which the vector potential is

\[ A(x) = B_0 x \hat{y} \implies B_0(x) = B_0 \hat{z}, \]  

(28)

and the corresponding superpotential reads as

\[ W_0(x) = \frac{\omega}{2} x + p_2, \quad \omega = 2eB_0. \]  

(29)

From this function, we obtain the superpartner potential which give explicitly that

\[ \mathcal{H}_+ = -\frac{d^2}{dx^2} + V_+(x) = -\frac{d^2}{dx^2} + \frac{\omega^2}{4} \left( x + \frac{2p_2}{\omega} \right)^2 - \frac{\omega}{2}, \]  

(30)

and its eigenenergies that correspond to those of a shifted quantum harmonic oscillator

\[ k_n^{+} = \frac{\omega}{4} n, \quad n = 0, 1, 2, \ldots, \]  

(31)

while the corresponding eigenfunctions can be expressed as

\[ F_{n,p_{2};+1}(x) = N_n e^{-\frac{1}{4}(x+\frac{\omega}{2})^2} \text{H}_n \left[ \sqrt{\frac{\omega}{2}} \left( x + \frac{2p_2}{\omega} \right) \right], \]  

(32)

with \( N_n = \left[ \frac{1}{2 \pi n! \left( \frac{\omega}{2} \right)^{1/2}} \right] \) being the normalization constant and \( \text{H}_n(x) \) are the Hermite polynomials. By defining the dimensionless quantity

\[ \eta(x) = \sqrt{\frac{\omega}{2}} \left( x + \frac{2p_2}{\omega} \right), \]  

(33)

we simplify the eigenfunctions as

\[ F_{n,p_{2};+1}(\eta) = N_n e^{-\eta^2/2} \text{H}_n(\eta). \]  

(34)

This expression corresponds to our seed solution. Next, we want to construct a non-trivial magnetic field profile starting from the uniform case by applying the 1-SUSY QM formalism. As stated earlier, the first step is to shift the energy of \( \mathcal{H}_+ \) as follows:
Thus, the potential \( \tilde{\mathcal{H}}_0 = \mathcal{H}_+ - \epsilon_1 = -\frac{\omega}{2} \frac{d^2}{d\eta^2} + \frac{\omega}{2} \eta^2 - \frac{\omega}{2} - \epsilon_1 \),

with \( \epsilon_1 \leq k_0^2 = 0 \). Thus, the potential \( \tilde{V}_0 \) reads

\[
\tilde{V}_0(\eta) = \frac{\omega}{2} \eta^2 - \frac{\omega}{2} - \epsilon_1,
\]

From here, we can readily obtain \( W_1(x, \epsilon_1) \) and correspondingly, \( V_1(x, \epsilon_1) \). Then, from the replacement \( W_1(x, \epsilon_1) = u'/u_0 \) in equation (24), we easily infer that

\[
u_1 = e^{-\eta^2/2} \left[ iF_1\left[n + \frac{3}{2}, \eta^2]\right] + 2\nu_1 \frac{\Gamma(a + 1/2)}{\Gamma(a)} \eta F_1\left[a + \frac{1}{2}, \frac{3}{2}, \eta^2]\right],
\]

with \( a = -\epsilon_1/(2\omega), \nu_1 \in (-1, 1) \). For definitiveness and comparison with the findings of [28], by choosing the parameters \( \epsilon_1 = -k_0^2/5 = -\omega/5 \) and \( \nu_1 = 0 \), we have \( a = 1/10 \) and

\[
W_1(\eta, \epsilon_1) = \sqrt{\frac{\omega}{2}} \eta \left[-1 + \frac{2}{5} \frac{iF_1\left[n + \frac{3}{2}, \eta^2]\right]}{iF_1\left[1, \frac{1}{2}, \eta^2\right]} \right],
\]

\[
V_1(\eta, \epsilon_1) = \tilde{V}_0(\eta) - \sqrt{2\omega} \frac{d}{d\eta} \left[ \sqrt{\frac{\omega}{2}} \eta \left[-1 + \frac{2}{5} \frac{iF_1\left[n + \frac{3}{2}, \eta^2\right]}{iF_1\left[1, \frac{1}{2}, \eta^2\right]} \right] \right],
\]

\[
B_1(\eta, \epsilon_1) = -B_0 + \frac{2B_0}{5} \frac{d}{d\eta} \left[ \sqrt{\frac{\omega}{2}} \eta \frac{iF_1\left[n + \frac{3}{2}, \eta^2\right]}{iF_1\left[1, \frac{1}{2}, \eta^2\right]} \right].
\]

A plot of the generated potential \( V_1(x, \epsilon_1) \) and the magnetic field profile \( B_1(x, \epsilon_1) \) in this case is shown in figure 2.

Then, the eigenenergies of the system are explicitly

\[
k_{n+1}^{(1)} = 0, \quad k_{n+1}^{(i)} = \omega (n + \frac{1}{5}), \quad n = 0, 1, 2, \ldots,
\]

while the corresponding Ritus eigenfunctions, taking into account (9), are given by:

\[
E_{0,p}^{(1)}(\eta, y, t) \sim e^{-i(p_0^2 - \omega) t} e^{\eta^2/2} iF_1\left[\frac{n}{10}, \frac{1}{2}, \eta^2\right],
\]

\[
E_{n+1,p}^{(1)}(x, y, t) = e^{-i(p_0^2 - \omega) t} F_{n+1,p}^{(1)}(x)
\]

\[
= e^{-i(p_0^2 - \omega) t} \frac{1}{\sqrt{\omega(n + 1/5)}} R_{\eta, p_{n+1}} + \frac{2\eta}{5} \frac{iF_1\left[n + \frac{3}{2}, \eta^2\right]}{iF_1\left[1, \frac{1}{2}, \eta^2\right]} F_{h, p_{n+1}} = \sqrt{2n} F_{n-1,p_{n+1}}
\]

Figure 2. (a) Generated potential \( V_1(x, \epsilon_1) \) (red, \( \cdots \)) and the initial one \( V_0(x) \) (dark blue, \( \cdots \)). (b) Generated magnetic field \( B_1(x, \epsilon_1) \) (red, \( \cdots \)) and the constant initial one \( B_0 \) (dark blue, \( \cdots \)). In both cases \( B_0 = \frac{\pi}{2}, p_1 = 1, \epsilon_1 = -\frac{\omega}{2} \) and \( \omega = 1 \).
for $n = 0, 1, 2, \ldots$. The joint choice of $\epsilon_1$ and the function $u_1$ allows that the energy spectrum of $\mathcal{H}_0$ has an extra level in comparison with that of $\mathcal{H}_0$.

Inserting these expressions into equation (7), we obtain the Ritus eigenfunctions for a seed constant magnetic field for the graphene to first-order intertwining which gives raise to the highly non-trivial magnetic field profile in equation (38c).

3.1.2. Exponentially decaying magnetic field

Let us now consider the vector potential

$$\mathbf{A}(x) = - \frac{B_0}{\alpha} (e^{-\alpha x} - 1) \hat{y}, \quad B_0 > 1, \quad \alpha \geq 0,$$

where we refer to $\alpha$ as the inhomogeneity term.

Thus, we have

$$\mathbf{B}_0(x) = B_0 e^{-\alpha x} \hat{z} \implies W_0(x) = p_2 - D(e^{-\alpha x} - 1), \quad D = \frac{e B_0}{\alpha},$$

which leads to the Morse potentials:

$$V_0^m(x) = q_2^2 + D^2 e^{-2\alpha x} - 2D \left( q_2 + \frac{\alpha}{2} \right) e^{-\alpha x},$$

where $q_2 = p_2 + D$. Note that our results coincide with those in [28, 34] making the replacement $q_2 \rightarrow k_2$.

By defining the quantity

$$\rho(x) \equiv \frac{2D}{\alpha} e^{-\alpha x},$$

the eigenfunctions $F_{n, p_2 + 1}(\rho)$ of $\mathcal{H}_0$ are given by [28, 34, 35]

$$F_{n, p_2 + 1}(\rho) = N_n e^{-\rho/2} (q_2 + \alpha n) F_n^{q_2 + \alpha n}(\rho), \quad n = 0, 1, 2, \ldots \leq q_2/\alpha,$$

where $N_n$ is the corresponding normalization constant and $L_n^\alpha(x)$ are the Laguerre polynomials, and its eigenenergies turn out to be

$$k_n^+ = \alpha \left( n q_2 - \alpha n \right), \quad n = 0, 1, \ldots.$$

We chose $V_0^m(\rho)$ and displace it by $-\epsilon_1$ to produce $\tilde{V}_0(\rho)$, namely,

$$\tilde{V}_0(\rho) = q_2^2 + \alpha^2 \rho^2 - \alpha \rho \left( q_2 + \frac{\alpha}{2} \right) - \epsilon_1.$$

Again, the new potential $V_1(\rho, \epsilon_1)$ depends on $W_1(\rho, \epsilon_1)$, which is a solution of the Riccati equation:

$$W_1^2(\rho, \epsilon_1) + W_1'(\rho, \epsilon_1) = \tilde{V}_0(\rho),$$

$$V_1(\rho, \epsilon_1) = \tilde{V}_0(\rho) - 2W_1'(\rho, \epsilon_1).$$

The new superpotential is written as $W_1(\rho, \epsilon_1) = u'/u_1$, with $u_1$ being the general solution of the Schrödinger equation

$$-u''_1 + \tilde{V}_0(\rho) u_1 = 0,$$

$$u_1 = e^{-\rho/2} \left( \frac{\alpha \rho}{2D} \right)^{\epsilon_1/\alpha} \left[ F_1[a, b, \rho] + \left( \frac{2q_2}{\alpha} \right) \left( 1 + \frac{1}{\nu_1} \right) U[a, b, \rho] \right].$$

where $\nu_1$ obeys the restriction $\nu_1 \in \mathbb{R} - \{-1, 0\}$ and the parameters $a$ and $b$ are defined as:

$$a = -\frac{q_2}{\alpha} + \frac{\sqrt{q_2^2 - \epsilon_1}}{\alpha}, \quad b = 1 + \frac{2\sqrt{q_2^2 - \epsilon_1}}{\alpha}.$$

Therefore, the superpotential turns out to be:

$$W_1(\rho, \epsilon_1) = \frac{\alpha \rho}{2} - \sqrt{q_2^2 - \epsilon_1} + \mathcal{F}(\rho),$$

where the function $\mathcal{F}(\rho)$ reads

$$\mathcal{F}(\rho) = -\left( \frac{a}{b} \right) \left[ F_1[1 + a, 1 + b, \rho] - \frac{2q_2}{\alpha} \left( 1 + \frac{1}{\nu_1} \right) U[1 + a, 1 + b, \rho] \right]$$

$$+ \left( \frac{a}{b} \right) \left[ F_1[a, b, \rho] + \frac{2q_2}{\alpha} \left( 1 + \frac{1}{\nu_1} \right) U[a, b, \rho] \right].$$
Thus, the new potential and associated magnetic field are given by (see equations (25) and (48b)):

\[
V_1(\rho, \epsilon_1) = \tilde{V}_0(\rho) + 2\alpha\rho \left[ \mathcal{F}(\rho) + \frac{\alpha\rho}{2} \right],
\]

\[= -\frac{\alpha^2 \rho^2}{2e} - \frac{\alpha \rho}{e} \frac{d}{d\rho} \mathcal{F}(\rho).
\]

A plot of the generated potential \(V_1(x, \epsilon_1)\) and the magnetic field profile \(B_1(x, \epsilon_1)\) in this case is shown in figure 3.

In order to compare our results, we set the factorization energy as

\[
k^2 = -\frac{1}{\epsilon_1^2} = -\frac{2}{\alpha^2} q_0^2 - \frac{\alpha}{2} (2q_2 - \alpha),
\]

so the eigenenergies for the problem are given by:

\[
k^{(1)}_n = 0, \quad k^{(1)}_{n+1} = \frac{q_0^2}{\epsilon_1^2} = \alpha n(2q_2 - \alpha) + \frac{\alpha}{2}(2q_2 - \alpha), \quad n = 0, 1, \ldots
\]

The eigenfunctions corresponding to \(\mathcal{H}_0\) take the form

\[
F^{(1)}_{n,\rho}(\rho) \sim e^{\rho/2} \frac{2n!}{\alpha^n} \mathcal{F}_1[a, b, \rho] + \frac{2n}{\alpha^n} \left(1 + \frac{1}{\alpha} \right) U[a, b, -\rho],
\]

\[
\times \left(\epsilon_1^2 - \frac{q_0^2}{\epsilon_1^2} + \mathcal{F}(\rho) - A^-\right) E_{n\rho+1}(\rho),
\]

for \(n = 0, 1, 2, \ldots\), where

\[
A^- = -\frac{\alpha \rho}{d\rho} + \left(\frac{q_0^2}{\epsilon_1^2} - \frac{\alpha \rho}{2}\right).
\]

Therefore, the corresponding Ritus eigenfunctions, taking into account (9), are given by:

\[
E^{(1)}_{0,\rho}(\rho, t) = e^{-i(p_1\rho - P_2 \rho^2)} F^{(1)}_{0,\rho}(\rho)
\]

\[
\sim e^{-i(p_1\rho - P_2 \rho^2)} e^{\rho/2} \frac{2n!}{\alpha^n} \mathcal{F}_1[a, b, \rho] + \left(\frac{2n}{\alpha^n}\right) \left(1 + \frac{1}{\alpha} \right) U[a, b, -\rho],
\]

\[
\times \left(\epsilon_1^2 - \frac{q_0^2}{\epsilon_1^2} + \mathcal{F}(\rho)\right) E_{n\rho+1}(\rho)
\]

\[
\times \left(\epsilon_1^2 - \frac{q_0^2}{\epsilon_1^2} + \mathcal{F}(\rho) - A^-\right) E_{n\rho+1}(\rho),
\]

\[
\times \left(\epsilon_1^2 - \frac{q_0^2}{\epsilon_1^2} + \mathcal{F}(\rho) - A^-\right) E_{n\rho+1}(\rho).
\]
\[ E^{(1)}_{n+1,1,p}(x, y, t) = e^{-i(p_d - p_j)\cdot p_j} E^{(1)}_{n+1,1,p}(x) \]
\[ = e^{-i(p_d - p_j)\cdot p_j} \left( (q_2 - \sqrt{q_2^2 - \epsilon_1 + F(\rho)}) - (A) \right) F_{n+1,1}(\rho) \sqrt{\alpha \left[ n(2q_2 - \alpha n) + \frac{1}{2}(2q_2 - \alpha) \right]}, \] (57b)

for \( n = 0, 1, 2, \ldots \). Once again, the joint selection of \( \epsilon_1 \) and \( u_1 \) allows that \( \mathcal{H}_f \) has an extra energy level than \( \mathcal{H}_0 \).

Inserting these expressions into equation (7), we obtain the Ritus eigenfunctions for a seed exponentially decaying static magnetic field for the graphene to first-order intertwining in the magnetic field in equation (53b).

4. Charge and current density

Physically, Ritus eigenfunctions \( \mathcal{E}_p(z) \) correspond to the asymptotic states of electrons in graphene with momentum \( \vec{p} \) in the external field. Therefore, we can use these functions to diagonalize the fermion propagator \( S(z, z') \) in momentum space in the same way plane waves are used to define the Fourier transform,

\[ S(z, z') = \int d^3p \, \mathcal{E}_p(z) S_F(p) \mathcal{E}_p(z'). \] (58)

Inserting this Green’s functions in equation (1), using the property [36]

\[ (\gamma \cdot \Pi) \mathcal{E}_p(z) = \mathcal{E}_p(z) (\gamma \cdot \vec{p}), \] (59)

where \( \vec{p} \) is the shorthand notation to define the three-momentum vector \( \vec{p}^0 = (p_0, 0, \sqrt{k}) \) that satisfies \( p^2 = p_0^2 - k \) [21] and the properties (3a), the propagator in momentum space takes the form

\[ S_F(p) = \frac{1}{\gamma \cdot \vec{p} - m}, \] (60)

similar to the free-particle propagator, but the momentum \( \vec{p} \), which carries the quantum numbers induced on the dynamics of Dirac fermions by the presence of the external field. In the configuration space, we write the propagator as

\[ S(z, z') = \int d^3p \, \mathcal{E}_p(z) \left[ \frac{1}{\gamma \cdot \vec{p} - m} \right] \mathcal{E}_p(z'). \] (61)

From this expression we can find the value of the electric charge and induced vacuum current densities. First, we notice that we can write

\[ \mathcal{E}_p = E_{p,+,1} P_+ + E_{p,1} P_-, \quad \mathcal{E}_p = E_{p,+,1}^{*} P_+ + E_{p,1}^{*} P_-, \] (62)

where we use projection operators \( P_{\pm} = \frac{1}{2}(1 \pm \gamma^0) = \frac{1}{2}(1 \pm i\gamma^\gamma^0) \), or more explicitly:

\[ P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \] (63)

which satisfy \( P_{\pm} P_{\pm} = P_{\pm} \) and \( P_{\pm} P_{\pm} = 0 \). Hence, from the definition

\[ j^\mu = \text{Tr}\{ \gamma^\mu S(z, z') \}, \] (64)

we have that

\[ \text{Tr}\{ \gamma^0 S(z, z') \} = \text{Tr}\left\{ 0 \int d^3p \left[ \frac{\gamma^0 p_{\mu} + m}{p^2 - m^2} \right] \left[ |E_{p,+,1}|^2 P_+ + |E_{p,1}|^2 P_- \right] \right\} \]
\[ = \int d^3p \left[ \frac{\gamma^0 p_{\mu}}{2} \left( |E_{p,+,1}|^2 + |E_{p,1}|^2 \right) + \frac{\gamma^0 \gamma^0}{2} \left( |E_{p,+,1}|^2 + |E_{p,1}|^2 \right) \right] \]
\[ + m \text{Tr}\left\{ \frac{\gamma^0}{2} \left( |E_{p,+,1}|^2 + |E_{p,1}|^2 \right) + \frac{\gamma^0 \gamma^0}{2} \left( |E_{p,+,1}|^2 + |E_{p,1}|^2 \right) \right\}. \] (65)

Upon taking traces, we obtain

\[ \text{Tr}\{ \gamma^0 S(z, z') \} = \int \frac{d^3p}{p^2 - m^2} \left( \rho_0 (|E_{p,+,1}|^2 + |E_{p,1}|^2) + m (|E_{p,+,1}|^2 + |E_{p,1}|^2) \right). \] (66)
Therefore:

\[
j^0 = -ie \text{Tr}\{\gamma^0 S(z, z')\} = -ie \int \frac{d^3p}{p^2 - m^2} (p_0(|E_{p,z}^+|^2 + |E_{p,z}^-|^2) + m(|E_{p,z}^+|^2 + |E_{p,z}^-|^2)).
\]  \hspace{1cm} (67)

Since the first integral is odd respect to \(p_0\), we have that

\[
j^0 = -ie \int \frac{d^3p}{p^2 - m^2} m(|E_{p,z}^+|^2 + |E_{p,z}^-|^2).
\]  \hspace{1cm} (68)

On the other hand,

\[
j^\ell = -ie \text{Tr}\{\gamma^\ell S(z, z')\}, \quad \ell = 1, 2,
\]  \hspace{1cm} (69)

where

\[
\text{Tr}\{\gamma^\ell S(z, z')\} = \int \frac{d^3p}{p^2 - m^2} \left[ p_0 \text{Tr}\{\gamma^\ell (E_{p,z}^+P_0 + E_{p,z}^-P_0) \gamma^\mu (E_{p,z}^+P_0 + E_{p,z}^-P_0)\} + m \text{Tr}\{\gamma^\ell (E_{p,z}^+P_0 + E_{p,z}^-P_0)(E_{p,z}^+P_0 + E_{p,z}^-P_0)\}\right]
\]

\[
= \int \frac{d^3p}{p^2 - m^2} \left[ p_0 \left\{ \frac{|E_{p,z}^+|^2}{4} (\gamma^\mu + \gamma_\mu^0 + \gamma_0^0\gamma^\mu) + \frac{|E_{p,z}^-|^2}{4} (\gamma^\mu + \gamma_\mu^0 + \gamma_0^0\gamma^\mu) \right\} + \frac{E_{p,z}^+E_{p,z}^-}{4} (\gamma^\mu - \gamma_\mu^0 - \gamma_0^0\gamma^\mu) \right].
\]  \hspace{1cm} (70)

Then, performing the traces with the aid of the identities

\[
\text{Tr}\{\gamma^\ell \gamma^\mu\} = -2\delta^{\ell\mu}, \quad \text{Tr}\{\gamma^\ell \gamma_0^0\gamma^\mu \} = \text{Tr}\{\gamma^\ell (2\gamma^\mu\gamma^0 - \gamma_\mu^0\gamma^0)\gamma^0\} = 2\delta^{\ell\mu},
\]  \hspace{1cm} (71)

we have that

\[
\text{Tr}\{\gamma^\ell S(z, z')\} = \int \frac{d^3p}{p^2 - m^2} \left[ \frac{|E_{p,z}^+|^2}{4} (2\delta^{\ell\mu} - 2\delta^{0\mu}) + \frac{E_{p,z}^+E_{p,z}^-}{4} (2\delta^{\ell\mu} - 4i\varepsilon^{0\mu\nu\rho} - 2\delta^{\ell\mu}) \right].
\]  \hspace{1cm} (72)

Now, by taking \(p_0 = (p_0, 0, \sqrt{k})\), it follows that

\[
\text{Tr}\{\gamma^\ell S(z, z')\} = \int \frac{d^3p}{p^2 - m^2} \left[ \alpha^{\ell}(p_0 + P_{p,z}) + E_{p,z}^-P_{p,z} \right],
\]  \hspace{1cm} (73)

where \(\alpha^{\ell} = i\ell\sqrt{k}\) for \(\ell = 1, 2\). Therefore,

\[
j^\ell = -ie \int \frac{d^3p}{p^2 - m^2} \alpha^{\ell}(p_0 + P_{p,z}) + E_{p,z}^-P_{p,z}, \quad \ell = 1, 2.
\]  \hspace{1cm} (74)

In the next subsection we obtain the charge and current densities for the magnetic field profiles obtained in the previous section.

4.1. Charge and current density for a seed constant magnetic field

Inserting the explicit solutions of the equations (40a) and (40b) into the equation (68), we obtain that the charge density to first order intertwining is
\[ j^0 = -ie \int dp_0 dp_1 \sum_{n=0}^{\infty} \frac{m}{p_0^2 - k_n^{(1)} - m^2} \left( |E_{p,+1}|^2 + |E_{p,-1}|^2 \right) \]
\[ = \pi e \int dp_2 \left[ \text{sgn}(m) \rho_0(x, p_2) + \sum_{n=0}^{\infty} \frac{m}{\sqrt{m^2 + (n + 1/5)\omega}} \rho_{n+1}(x, p_2) \right] \]
\[ = \pi e \int dp_2 \left[ \text{sgn}(m) |\Lambda_n^{(1)}|^2 \left( F_1 \left[ \frac{1}{10}, \frac{3}{2}, \eta^2 \right] \right)^2 + \sum_{n=0}^{\infty} \frac{m}{\sqrt{m^2 + (n + 1/5)\omega}} \right] \]
\[ \times \left( \frac{|\Lambda_n^{(1)}|^2}{2(n + 1/5)} \left( \frac{2n + 1}{5} F_1 \left[ \frac{11}{10}, \frac{3}{2}, \eta^2 \right] F_{n-1} - \sqrt{2n} E_{n-1,p_0+1}^2 \right) + E_{n,p_0+1}^2 \right), \tag{75} \]

where \( \Lambda_n^{(1)} \) denotes the corresponding normalization constants of the functions \( E_{n,p}(x, y, t) \), and we have used the following result

\[ \int_{-\infty}^{\infty} \frac{dp_0}{p_0^2 + b} = \frac{\pi}{\sqrt{b}}. \tag{76} \]

Similarly, inserting the explicit solutions of the equations (40a) and (40b) into the equation (74), we obtain that the current density to first order intertwining is

\[ j^0 = -ie \text{Tr} \{ \gamma^5 S(z, z') \} = -2i e \int dp_0 dp_2 \sum_{n=0}^{\infty} \frac{\sqrt{k_n^{(1)}}}{m^2 - k_n^{(1)} - m^2} E_{n,p,+1}^2 E_{n,p,-1} \]
\[ = -2i e \pi \int dp_2 \sum_{n=0}^{\infty} \frac{\sqrt{(n + 1/5)\omega}}{m^2 + (n + 1/5)\omega} j_{n+1}(x, p_2) \]
\[ = -2i e \pi \int dp_2 \sum_{n=0}^{\infty} \frac{\sqrt{(n + 1/5)\omega}}{m^2 + (n + 1/5)\omega} \Lambda_n^{(1)} \]
\[ \times \frac{E_{n,p_0+1}}{2(n + 1/5)} \left( \frac{2n + 1}{5} F_1 \left[ \frac{11}{10}, \frac{3}{2}, \eta^2 \right] F_{n-1} - \sqrt{2n} E_{n-1,p_0+1} \right) \] \tag{77}
4.2. Charge and current density for a seed exponentially decaying magnetic field

Inserting the explicit solutions of the equations (57a) and (57b) into the equation (68), we obtain that the charge density to first order intertwining is

\[
j^0 = -ie \int dp_0 dp_2 \sum_{n=0}^{\infty} \frac{m}{p_0^2 - k_n^{(1)} - m^2} (|E_{p,+}|^2 + |E_{p,-}|^2)
= \pi e \int dp_2 \left[ \text{sgn}(m) \rho_0(x, p_2) + \sum_{n=0}^{\infty} \frac{m}{\sqrt{m^2 + k_{n+1}^{(1)}}} \rho_{n+1}(x, p_2) \right]
= \pi e \int dp_2 \left[ \text{sgn}(m) \|\mathcal{A}_0^{(1)} F_{0,p_2}^{(1)}\|^2 + \sum_{n=0}^{\infty} \frac{m}{\sqrt{m^2 + k_{n+1}^{(1)}}} (|\mathcal{N}_{n+1}^{(1)} F_{n+1,p_2}^{(1)}|^2 + |F_{n+1,p_2}^{(1)}|^2) \right].
\]  

(78)

Once again, inserting the explicit solutions of the equations (57a) and (57b) into the equation (74), we obtain that the charge density to first order intertwining is

\[
j^f = -ie \text{Tr} \{\gamma^f S(x, z')\} = -2i^{f+1}e \int dp_0 dp_2 \sum_{n=0}^{\infty} \frac{\sqrt{k_n^{(1)}}}{m^2 + k_{n+1}^{(1)}} j_{n+1}(x, p_2)
= -2i^{f+1}e \int dp_2 \sum_{n=0}^{\infty} \frac{\sqrt{k_{n+1}^{(1)}}}{m^2 + k_{n+1}^{(1)}} \mathcal{N}_{n+1}^{(1)} F_{n+1,p_2}^{(1)}(\rho) F_{n+1,p_2}^{(1)}(\rho).
\]

(79)

Thus, we calculate the probability density and probability current with these expressions, which are plotted in figures 6 and 7.

4.3. Behavior for small inhomogeneity \( \alpha \)

Let us consider the asymptotic behavior of the previous results for small values of the parameter \( \alpha \) in order to compare them with those for the constant magnetic field case, since in the limit \( \alpha \to 0 \) the exponentially decaying magnetic field tends to the constant magnetic field.

Indeed, rewriting \( D = \omega^2/(2\alpha) \), being \( \omega \) as in equation (29), the superpotential \( W_0(x) \), the Morse potential \( V_0(x) \) and the eigenenergies \( k_n^{(1)} \) in equations (42), (43) and (46), respectively, turn into:

\[
\lim_{\alpha \to 0} W_0(x) = \lim_{\alpha \to 0} p_2 = \frac{\omega}{2\alpha} (e^{-\alpha x} - 1) = \frac{\omega}{2} x + p_2,
\]

\[
\lim_{\alpha \to 0} V_0(x) = \lim_{\alpha \to 0} \left( p_2 + \frac{\alpha}{2\alpha} \right)^2 + \frac{\omega^2}{4\alpha} e^{-2\alpha x} - \frac{\omega}{\alpha} p_2 + \frac{\alpha}{2} + \frac{\omega}{2} e^{-\alpha x}
= \frac{\omega^2}{4} \left( x + \frac{2p_2}{\omega} \right)^2 - \frac{\omega}{2}.
\]

(80a)

(80b)
Figure 6. Probability density $\rho_0(x)$ for the ground state $n=0$ (dark blue, ---) and the excited states $\rho_{n+1}(x)$: $n=0$ (red, - - -), $n=1$ (blue, ---) and $n=2$ (green, - - -). In all the cases $B_0 = 1$, $\nu_1 = -\frac{3}{2}$, $\nu_2 = 5\alpha$, $\epsilon_1 = -\frac{11\alpha}{2}$ and $\alpha = 1$.

Figure 7. Current densities $j_{n+1}(x)$ for the excited states: $n=0$ (red, - - -), $n=1$ (blue, ---) and $n=2$ (green, - - -). In all the cases $B_0 = 1$, $\nu_1 = -\frac{3}{2}$, $\nu_2 = 5\alpha$, $\epsilon_1 = -\frac{11\alpha}{2}$ and $\alpha = 1$.

Figure 8. (a) Generated potential $V_1(x, \epsilon_1)$ in (53a) and (b) generated magnetic field $B_1(x, \epsilon_1)$ in (53b) for small inhomogeneity ($\alpha < 1$): $\alpha = 0.11$ (dark blue, ---), $\alpha = 0.09$ (red, - - -), $\alpha = 0.07$ (blue, ---) and $\alpha = 0.05$ (green, - - -). In all the cases $B_0 = 1$, $\nu_1 = -\frac{3}{2}$, $\nu_2 = 1$ and $\epsilon_1 = -\frac{11\alpha}{2}$.
and derived from the selection of the parameter implemented to the exponentially decaying magnetic has one more level than seed solutions corresponding to the Ritus eigenfunctions for a constant magnetic trivial and inhomogeneous external magnetic fields. We have constructed the Dirac fermion propagator for graphene-like systems in the presence of non-trivial examples of inhomogeneous propogator in the non-trivial cases of inhomogeneous densities in the literature. On the other hand, we have found the charge and current densities from the constructed Dirac fermion propagator whose coincide with equations (29), (30) and (31), respectively. In figure 8, we show the behavior of the potential $V_\gamma(x, \epsilon_1)$ and the magnetic field profile $B_\xi(x, \epsilon_1)$ for small values of $\alpha$. According to the plots, for small inhomogeneity the potential and the magnetic field generated by the supersymmetric transformation have a similar behavior to their counterparts in the constant magnetic field case for $|x| \to \infty$, while for values of $x$ in the inner region of $V_\gamma(x, \epsilon_1)$ and $B_\xi(x, \epsilon_1)$, such functions do not look like those in figure 2.

On the other hand, the probability density $\rho_0(x)$ does not look like the corresponding one for the constant magnetic field case, as shown in figure 9. This suggests us that after the supersymmetric transformation is implemented to the exponentially decaying magnetic field case, we are not able to recover the eigenfunction $F^{(1)}_{\alpha, p_2} \sim 1/\nu_1$ for the ground state of the Hamiltonian $\mathcal{H}_0$ of the constant magnetic field case. Likewise, the corresponding probability density and probability current for small values of $\alpha$ and $\epsilon_1 = -k_0^+/5$ are shown in figure 10. As we can see, the probability densities $\rho_0(x)$ and the current densities $j_0(x)$ for the excited states with $n > 1$ exhibit a subtle resemblance to the plots shown in figures 4 and 5 for a constant magnetic field $B_0 = 1/2$ and $p_2 = 1$. However, such functions are asymmetric respect to $x = -2$ in comparison with those that correspond to the eigenfunctions in equations (40a) and (40b).

5. Final remarks

In this work, we have studied the Dirac fermion propagator for graphene-like systems in external magnetic fields. We have constructed the Dirac fermion propagator for graphene-like systems in the presence of non-trivial and inhomogeneous external magnetic fields generated by first-order intertwining operators from the seed solutions corresponding to the Ritus eigenfunctions for a constant magnetic field and an exponentially decaying magnetic field [22, 34, 37–40] already known in literature. We constructed the propagator in the basis of the eigenfunctions of the operator $(\gamma \cdot \Pi)^2$ for new non-trivial magnetic field profiles, hence extending the number of cases in which the propagator admits a closed form representation. The generalized first-order intertwining method presented here have been followed of the discussion in [27, 28]. By choosing the parameters $\epsilon_1 = -k_0^+/5 = -\omega/5$ and $\nu_1 = 0$, we have obtained the generated potential $V_\gamma(x, \epsilon_1)$ and the magnetic field profile $B_\xi(x, \epsilon_1)$ for the case of the seed uniform magnetic field whose graphs have been plotted in figure 2 and that agree with [28]. Similarly, taking $\epsilon_1 = -k_0^+/2 = -\alpha(2q_1 + \alpha)/2$ and the restriction $\nu_1 \in \mathbb{R} - \{-1, 0\}$, we obtained the generated potential $V_\gamma(x, \epsilon_1)$ and the magnetic field profile $B_\xi(x, \epsilon_1)$ for the case of the seed exponentially decaying magnetic field whose graphs have been plotted in figure 3 and that agree with [28] too. In both cases, the energy spectrum of the corresponding Hamiltonian $\mathcal{H}_0$ has one more level than $\mathcal{H}_0$ derived from the selection of the parameter $\epsilon_1$ and the function $\nu_1$.

On the other hand, we have found the charge and current densities from the constructed Dirac fermion propagator in the non-trivial examples of inhomogeneous fields derived from the intertwining framework. From such densities, the probability density and the probability current for both cases of fields have been obtained and plotted in figures 4, 5, 6 and 7 for the ground state $n = 0$ and some excited states that reproduce the densities in the literature [28] from the direct solutions of the wave equation in the said background fields. About
the number of nodes observed in the plots of the probability densities $\rho_{n+1}(x)$ and current densities $j_{n+1}(x)$, and its relation with index $n$, let us point out that the wave functions $F_{n,\mu_{p}+1}$ and $F_{n,\mu_{p}-1}$ satisfy, individually, the node theorem but the probability density $\rho_{0}$ not necessary does it, since it depends how such functions overlap. The current density $j_{\ell}$ also mixes functions $F_{n,\mu_{p}+1}$ and $F_{n,\mu_{p}-1}$, so that node pattern in such a current can be traced back to the individual behavior of these functions, but the product and sum of individual contributions might not follow an obvious node pattern. Something similar occurs when the 1-SUSY QM formalism is applied to generate the functions $F_{n,\mu_{p}}^{(1)}$, which no necessary satisfy the node theorem.

Figure 10. Probability density $\rho_{0}(x)$ (left-hand) and current densities $j_{n+1}(x)$ (right-hand) for the excited states in (57): $n = 0$ (red, --), $n = 1$ (blue, - -) and $n = 2$ (green, - - -). In all the cases $B_{0} = \frac{1}{2}$, $\nu_{1} = \frac{1}{2}$, $p_{2} = 1$, $\epsilon_{1} = -\frac{\hbar^{2}}{4}$ and small inhomogeneity ($\alpha < 1$).
Additionally, we have shown that for the limit $\alpha \to 0$ the exponentially decaying magnetic field tends to the constant magnetic field. Thus, taking the limit $\alpha \to 0$ in the exponentially decaying magnetic field Morse potentials (equations (42), (43)) and eigenenergies (equation (46)), respectively, we have obtained the Morse potential $V_0^+ (x)$ (equations (80b), (80a)) and the eigenenergies $k_n^+$ (equation (80c)) that coincide with the superpotential (equations (29), (30)) and eigenenergies (equation (31)) of the uniform magnetic field case, respectively. Also, we have shown that for small inhomogeneity $\alpha$ the behavior of the potential $V_1(x, \epsilon_1)$ and the magnetic field profile $B(x, \epsilon_1)$ in general does not coincide with the uniform magnetic field case, except in the asymptotic limit $|x| \to \infty$.

Moreover, we have plotted the probability density taking the limit $\alpha \to 0$ in order to recover the probability density of the uniform magnetic field case, however this does not occur with our results as the figure 9 shows because after the supersymmetric transformation is implemented to the exponentially decaying magnetic field case, we have not recovered the eigenfunction $B_0^{(1)} \sim 1/u_0$ for the ground state of the Hamiltonian $H_0$ of the constant magnetic field case. For small values of $\alpha$ and $\epsilon_1 \to -k_1^+ / 5$ we have obtained the probability density and probability current as shown in figure 10 where the plots are not symmetric respect to $x = -2$, as occurs in figures 4 and 5 for the excited states.

For the future, we are planning to obtain the Ritus functions for graphene-like systems for the fields studied in this work by second-order intertwining operators. Results will be reported elsewhere. This work is expected to become a guide for colleagues interested in the theoretical developments of graphene-like systems.

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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