Renormalization Group Flows on Line Defects

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(Dated: January 12, 2022)

We consider line defects in d-dimensional Conformal Field Theories (CFTs). The ambient CFT places nontrivial constraints on Renormalization Group (RG) flows on such line defects. We show that the flow on line defects is consequently irreversible and furthermore a canonical decreasing entropy function exists. This construction generalizes the g theorem to line defects in arbitrary dimensions. We demonstrate our results in a flow between Wilson loops in 4 dimensions.

Introduction. In lattice systems, in order to understand the physics on different length scales, we perform block-spin transformations, eliminating degrees of freedom that live at short distances. This process obviously reduces the overall number of degrees of freedom. But one can ask whether this reduces the number of degrees of freedom per lattice site, which is much less clear. In Quantum Field Theory, the number of degrees of freedom per lattice site is roughly speaking the number of fields and this raises the question of whether the number of fields decreases as we probe physics of longer and longer distances.

To address these questions precisely one has to give a non-perturbative definition of what the “number of fields” means and provide a prescription to evaluate it even when there is no weakly-coupled description in terms of fields. Starting from the work of Zamolodchikov on the $c$-function in 2d [1], several such proposals and results were discussed in diverse dimensions [2–24].

The focus of this paper is the physics of 1 dimensional defects in a CFT. Such defects can undergo nontrivial renormalization group flows while affecting the bulk very little far away from the defect. A few known examples of this kind include Wilson or ‘t Hooft lines in 4d gauge theories [25] and holography [26, 90], symmetry defects and impurities in 3d quantum critical systems [27–30, 34, 35] etc. In 2d, line defects correspond to boundaries or interfaces and appear naturally as the low-energy limit of lattice systems with impurities (see, for instance, [31–33, 75, 76]).

There is already some extensive work on renormalization group flows on various defects [36–53]. For our purposes, it is important to highlight the conjecture of Affleck and Ludwig [36] for the decreasing entropy function on line defects in 2 dimensions and its subsequent proofs [39] and [45]. Here we will discuss the properties of line defects in arbitrary dimensions. We will define an entropy function and show that it monotonically decreases. In the Supplemental Material we show how our result applies to a nontrivial flow between two different conformal Wilson lines in super Yang-Mills (SYM) theory in 4 dimensions.

The main idea we employ is that surrounding the line defect with conformal charges leads to nontrivial identifications in theory space when the defect is non-conformal. This can be expressed in terms of constraints on the dilaton living on the line defect. We show that these constraints translate to a monotonic entropy function.

DCFTs We consider local, reflection-positive Euclidean conformal field theories (CFTs) in $d \geq 2$ dimensions. We will be interested in CFTs in the presence of a line defect which preserves unitarity and locality. We will be interested in infinite straight lines or circular defects. At the fixed point of the (defect) renormalization group flow, the straight line defect preserves the subgroup $SL(2, \mathbb{R}) \times SO(d-1)$ of the full conformal group. In this case the system is called a defect CFT (DCFT). In $d = 2$, conformal line defects additionally preserve one copy of the Virasoro algebra.

DCFTs share many of the standard properties of CFTs. However, in general, the line defect does not support a stress tensor [54, 55, 60]. This statement really means that there is no possibility to localize energy on the line defect and energy always ends up being smeared into the bulk. The bulk stress tensor $T_{\mu\nu}^b$ obeys the following Ward identity [44, 56, 57, 60]:

$$\nabla_\mu T^b_{\mu\nu} = -\delta^{d-1}_D n^b_\nu D^i,$$

where $\delta^{d-1}_D$ is a delta function localized at the defect, $\{n^b_i\}$ is a basis of $d - 1$ unit vectors normal to the defect and $D^i$ is the displacement operator [44, 60], which parametrizes the breaking of translations in the directions normal to the defect. Finally we mention that all bulk correlation functions may be systematically decomposed into defect correlators via the bulk-to-defect OPE [58, 76]. This allows to study the DCFT data via a systematic bootstrap approach [59–61], similar to the one usually adopted in standard CFTs [62–64].

It will be convenient for our purposes to consider the expectation of the $SL(2, \mathbb{R})$ charges wrapping the defect. These are obtained by integrating the stress tensor contracted with the appropriate Killing vector at a fixed distance $\varepsilon$ from the defect:

$$Q_\varepsilon(D) = \int_\varepsilon d^{d-1} \Sigma^\mu (T^b_{\mu\nu}) \xi^\nu.$$ 

1 It is convenient to consider normalized correlation functions, so $(T^b_{\mu\nu})$ really stands for $(T^b_{\mu\nu})/(D)$ where $D$ is the defect operator.
2 The defect $D$, and the displacement operator, $D^i$, are distinguished by the superscript $i$. 

By conformal invariance we expect eq. (2) to yield a vanishing result for both a straight line defect and a circular one. However, due to a subtlety with the action of conformal transformations on the point at infinity, for the straight line geometry the conformal charges vanish only when the distance between the integration surface and the defect diverges sufficiently fast as \( x^d \to \pm \infty \) [25]. This issue is presumably related to the disagreement between the expectation value of circular and linear Maldacena-Wilson loops in \( \mathcal{N} = 4 \) SYM [71–73]. We provide a detailed discussion regarding this subtlety in the Supplemental Material. No issues of this sort arise for circular defects, hence we will focus on this geometry in what follows.

Let us consider for concreteness a circular defect of radius \( R \) centered around the origin on the \((x^1, x^2)\) plane \( x^3 = \ldots = x^d = 0 \). The \( SL(2, \mathbb{R}) \) Killing vectors preserved by the circle are:

\[
\xi^\mu = \frac{1}{2} \left[ \delta^\mu_a (R + x^2/R) - 2x^\mu x_a/R \right], \\
\xi^a = \delta^a_c x^c,
\]

where \( a = 1, 2 \) and indices are raised/lowered with the Euclidean metric. Here \( \xi^\mu \) are linear combinations of translations and special conformal transformations on the defect plane, while \( \xi^a \) generates rotations in the \((x^1, x^2)\) plane. In this geometry, there is no issue with the boundary condition at infinity and, consequently, the expectation values of the \( SL(2, \mathbb{R}) \) charges on a surface wrapping the defect (see fig. 1) vanish,

\[
Q_\xi(D) = 0 \quad (\text{circular defect}).
\] (4)

The statement (4) can be checked using the explicit form of the stress-tensor one-point function in a circular geometry, which depends on a unique constant \( h_D \) (see e.g. [60, 74] for the explicit expressions). In particular, in contrast to the infinite line, every point of the surface can be brought arbitrarily close to the defect compatibly with the identity (4). For this reason in the following we will focus on circular defects.

\[3\] The coefficient \( h_D \) is a physical characteristic of the DCF and it was computed in various supersymmetric examples [65–69] (for a general approach to supersymmetric line defects see [70]).

\[4\] In spite of this, such a configuration is conformally equivalent to a straight line surrounded by a surface whose radius becomes increasingly large as the line extends to infinity.

\[5\] Here we are assuming that the defect has a trivial induced submanifold metric \( g_D = X^\mu X^\nu g_{\mu\nu} = 1 \) to simplify the notation.

\[6\] In general the conformal symmetry of the line defect is violated by a nonvanishing \( T_D \). Such a non-vanishing \( T_D \) can be split (not unambiguously) into a \( c \)-number due to conformal anomalies and a nontrivial local operator. A somewhat common convention, which we also adopt here, is to have the dilaton couplings compensate for the operatorial violation of scale invariance, but not the trace anomalies, which might generically be present in the bulk - see e.g. [44, 77] for details.

Defect RG

The main goal of this work is to study defect renormalization group (DRG) flows. A DRG may be triggered perturbing a DCFT with one or more relevant defect operators. For instance, we may consider a defect operator \( \mathcal{O} \) with \( \Delta_{\mathcal{O}} < 1 \):

\[
S_{\text{DCFT}} \to S_{\text{DCFT}} + M_0^{1-\Delta_{\mathcal{O}}} \int_D d\sigma \mathcal{O}(\sigma),
\] (5)

where \( \int_D \) stand for integration along the defect and \( M_0 \) is the mass scale of the flow. Conformal invariance (i.e., \( SL(2, \mathbb{R}) \) transformations that preserve the defect) is now explicitly broken by the scale \( M_0 \) to just translations along the defect.

Due to the locality of the bulk CFT, the bulk stress tensor remains conserved and traceless (up to possible bulk trace anomalies in curved space) away from the line. However, now a defect stress tensor \( T_D \) is allowed. In other words, energy can now be stored on the defect. Not only \( T_D \) is allowed, such an operator must always exist away from the fixed points of the defect. The existence of the operator \( T_D \) is the reason that \( SL(2, \mathbb{R}) \) charges are no longer conserved. Since \( T_D \) is localized to the defect, what we mean by saying that \( SL(2, \mathbb{R}) \) charges are no longer conserved is that, if the charges are integrated on surfaces that intersect the defect, then they are not invariant under small deformations.

Invariance under translations along the defect implies that eq. (1) in the presence of \( T_D \) is modified to:

\[
\nabla_\mu T^\mu\nu_D = -\delta^{\mu-1}_{D-1} X^\nu T_D \delta^D - \delta^{\mu-1}_{D-1} n^\nu D^\mu,
\] (6)

where \( X^\nu(\sigma) \) is the embedding function describing the defect location and the dot stands for derivatives with respect to the line coordinate \( \sigma \), so that \( X^\nu \) is a tangent vector to the defect. Equation (6) merely expresses the energy balance between the bulk and the defect.

Spurious analysis and the dilaton

As it often happens in the study of RG flows, it is useful to promote the renormalization group scale to a function of position \( M(\sigma) = M_0 e^{\Phi(\sigma)} \) [77, 78], where \( \Phi(\sigma) \) is a dimensionless background dilaton field.

To linear order, the partition function of the theory depends on the dilaton through to the defect energy-momentum tensor:

\[
\log Z|_{\Phi+\delta\Phi} = \log Z|_\Phi + \int_D d\sigma \delta\Phi(\sigma) T_D(\sigma) + \frac{1}{2} \int_D d\sigma_1 \int_D d\sigma_2 \delta\Phi(\sigma_1) \delta\Phi(\sigma_2) T_D(\sigma_1) T_D(\sigma_2) + \ldots 
\] (7)
The background dilaton field acts a source for the theory. This
in turn modifies the conservation equation (6) as follows [56,
57]
\[ \nabla_\mu T^{\mu}_b = -\delta^{d-1}_D \dot{X}^\nu \left( \tilde{T}_D - \dot{\Phi} T_D \right) - \delta^{d-1}_D n^\nu D^i. \] (8)
If one views the coordinate along the defect as time, then
a nontrivial \( \Phi(\sigma) \) renders the theory time dependent and (8)
relates the non-conservation rate of the charge associated with
translations along the defect with the derivative of the dilaton
source.
A position dependent mass scale breaks the \( SL(2, \mathbb{R}) \) sym-
metry completely. What we gain by introducing the general
background field \( \Phi(\sigma) \) is that \( SL(2, \mathbb{R}) \) allows us to relate
different theories instead of directly placing constraints on a
given theory. Indeed, we will use eq. (8) in what follows to de-
rive some non-trivial identities relating theories with different
values for the source \( \Phi(\sigma) \).
**RG flows induced by the broken charges** It is crucial to
realize that the identity (4) holds irrespectively of the breaking
of scale invariance on the defect (i.e. it holds for any \( \Phi(\sigma) \)).
This is because the charges wrapping the defect do not intersec-
t it and hence such charges are oblivious to what happens on
the defect and they remain invariant under small deformations.
They can be moved off to infinity where they annihilate the
vacuum. (To see that, one can realize the wrapping surface as
the difference between two \( S^{d-1} \) surfaces outside and inside
the loop.)
As we explained, on general grounds one expects that
\( SL(2, \mathbb{R}) \) transformation can be reabsorbed into a transfor-
mation of the dilaton \( \Phi(\sigma) \), leading to relations between differ-
ent theories. This can be made precise by using eq. (4). To that
end, consider shrinking the radius of the topological surface
enclosing the defect (see fig. 1). It is clear from Gauss’s law
that the only contribution in the integration of the stress ten-
sor arises from the right hand side in eq. (8). We therefore
conclude that eq. (4) implies the following relation:
\[ 0 = Q_{\xi}(D) = \int \sigma^{d-1} \Sigma^\mu \langle T_{\mu \nu}^D \rangle \xi^\nu = \int \sigma \langle \xi_D + \xi_D \dot{\Phi} \rangle \langle T_D \rangle, \] (9)
where in the second line we integrated by parts and we denoted
by \( \xi_{\omega} \) the projection of the Killing vectors (3) on the defect.
We crucially used the fact that the normal components of the
\( SL(2, \mathbb{R}) \) Killing vectors vanish on the defect. In fact, for
more general conformal Killing vectors which do not leave
the loop invariant, an analogous identity picks an additional
contribution from the displacement operator in eq. (8) but we
do not study these identities here.

Due to the linear coupling between the defect stress tensor
and the dilaton, we may interpret eq. (9) as an equivalence
between defects with different DRG scales \( M(\sigma) \):
\[ \Phi \sim \Phi + \alpha \left( \xi_D + \xi_D \dot{\Phi} \right) \quad |\alpha| \ll 1, \] (10)
for any \( SL(2, \mathbb{R}) \) Killing vector \( \xi \) and any infinitesimal \( \alpha \). This
observation is most useful when considering the expansion of
the partition function (7) around \( \Phi = 0 \).Demanding the equiva-
ience (10) at each order in the field expansion we then find an
infinite number of identities for the correlation functions of the
defect stress tensor. At second order in the field expansion we
obtain the following one (omitting the subscript \( \Phi = 0 \) from
now on):
\[ \int_D d\sigma \xi_D(\sigma) \langle T_D(\sigma) \rangle = - \int_D d\sigma_1 \int_D d\sigma_2 \xi_D(\sigma_1) \Phi(\sigma_2) \langle T_D(\sigma_1) T_D(\sigma_2) \rangle. \] (11)

Crucially, this identity holds for any \( \Phi(\sigma) \). Notice that the
right hand side of eq. (11) for generic choices of the dilaton
profile is naively divergent. Our arguments however ensure
that these identities must hold in any regularization scheme
which preserves the invariance of the partition function under
diffeomorphisms and defect reparametrizations.
At this point it is useful to specify a cylindrical system of
coordinates on the defect: \( x^1 = R \cos \phi, \quad x^2 = R \sin \phi \) and set
\( \sigma = R \phi \). The projection of the three Killing vectors in eq. (3)
reads, respectively,
\[ \xi_D = - \sin \phi, \quad \xi_D = \cos \phi, \quad \xi_D = -1. \] (12)
Eq. (11) is trivial for \( \xi_D = -1 \), but provides non-trivial
constraints for the other two choices, which lead to identical
constraints. A particularly useful relation is obtained choosing
\( \xi_D = - \sin \phi \) and \( \Phi \propto \cos \phi \) in eq. (11). This leads to:
\[ R \int_D d\phi \langle T_D(\phi) \rangle = R^2 \int_D d\phi_1 \int_D d\phi_2 \langle T_D(\phi_1) T_D(\phi_2) \rangle \cos(\phi_1 - \phi_2), \] (13)
where we used trigonometric identities and invariance under
translations along the defect to simplify both sides. Eq. (13)
will be very useful in providing a gradient formula for the DRG
flow of a suitably defined defect entropy.

**The defect entropy** Our discussion thus far focused on de-
fects in flat space, but all our considerations apply on all confor-
ma lly equivalent manifolds. These include the \( d \)-dimensional
sphere of radius \( R \), with the defect spanning a maximal circle,
and the cylinder \( \mathbb{R} \times S^{d-1} \), with the defect on the equator of
\( S^{d-1} \) at a fixed value of the Euclidean time \( \tau = \log x^2/R = 0 \).
We can use any of these geometries to define a defect \( g \-
function, \( g(M_0 R) \), in terms of the partition function in the

\footnote{We use conventions such that the bulk stress tensor \( T^\mu_\nu \) does not contain any
contribution proportional to \( \delta^{d-1}_D \) and therefore it is traceless everywhere
[44, 56].}
presence of the defect, normalized by the partition function without it:

$$\log g(M_0 R) = \log Z_M - \log Z_\text{CFT}^{(CFT)} ,$$

(14)

where $\log Z_\text{CFT}^{(CFT)}$ is the partition function of the theory without the defect.\footnote{This terminology differs from that of [47], where the term defect entropy referred to the defect contribution to the entanglement entropy. Note that in $d = 2$ the defect entanglement entropy and ordinary entropy coincide at fixed points because $h_D = 0$. Here we see that the correct generalization to higher dimensions involves the defect entropy and not the defect entanglement entropy which is also sensitive to $h_D$ [79].} The defect contribution $g$ depends only on the dimensionless product $M_0 R$ and it reduces to a constant at the fixed points (in a sense that we will explain below).

We must now ask to what extent is $g$ well defined at the fixed points and away from them. $\log g$ can be shifted by the addition of a cosmological constant counterterm $\int d\sigma M_0 \sim M_0 R$ with an arbitrary coefficient. All other nontrivial geometric invariants which are analytic around the flat metric have dimension larger than one and cannot appear as counterterms. Therefore no additional ambiguities exist in $d > 2$ (we will discuss $d = 2$ more in detail below). Therefore, one can obtain a scheme-independent quantity which we will refer to as the defect entropy, defined as: \footnote{The extrinsic curvature is not analytic at zero and therefore is not an allowed counterterm; see e.g. [44] for a concise review of submanifold geometry. The counterterm $\int d\sigma K$ was discussed in a different context when studying the Entanglement Entropy in 2+1 dimensions, see e.g. [80, 81].}

$$s(M_0 R) = \left(1 - R \frac{\partial}{\partial R}\right) \log g(M_0 R).$$

(15)

At the fixed points, $s(M_0 R)$ is a pure number which is scheme independent. It is equal to the perimeter-independent contribution to $\log g(M_0 R)$ at the fixed point. We will refer to these fixed point values of $g$ as $g_{UV}, g_{IR}$, respectively. We will show that $s(M_0 R)$ decreases monotonically under DRG, implying $g_{UV} > g_{IR}$.

In $d = 2$ eq. (15) coincides with the interface contribution to the thermal entropy of the theory. To make the connection with $d = 2$ precise, one needs to remember that in $d = 2$ we can also allow the counterterm $\int d\sigma K$, where $K$ is the extrinsic curvature.\footnote{Log $Z_\text{M}$ is divergent on the infinite cylinder or in flat space, moreover, in general, it is ambiguous due to various counterterms. But these bulk issues cancel from the definition of $g(M_0 R)$.} Such a term vanishes for a maximal circle in $S^2$ and on $\mathbb{R} \times S^1$ and therefore all our conclusions hold unaltered on those manifolds. Furthermore $CPT$ invariance implies that the coefficient of this counterterm should be purely imaginary. Therefore the definition in eq. (15) is meaningful also in flat space provided we focus on the real part of the defect entropy.

**The gradient formula** We now have all the ingredients to derive a gradient formula for the DRG flow of the defect entropy. Since $g$ depends on $M_0 R$ only, for constant dilaton $\Phi$, it follows that $g$ depends on the combination $RM_0 e^\Phi$. We may therefore write the variation of the defect entropy $s$ under a change in the mass scale as follows:

$$M_0 \frac{\partial}{\partial M_0} s(M_0 R) = \left[\left(\frac{d}{d\Phi} - \frac{d^2}{d\Phi^2}\right) \log g \left(RM_0 e^\Phi\right)\right]_{\Phi=0} .$$

(16)

Using the expansion (7) for constant $\Phi$ we then can write eq. (16) in terms of correlation functions of the defect stress tensor

$$M_0 \frac{\partial}{\partial M_0} s(M_0 R) = R \int_D d\phi \langle T_D(\phi) \rangle - R^2 \int_D d\phi_1 \int_D d\phi_2 \langle T_D(\phi_1) T_D(\phi_2) \rangle .$$

(17)

Eq. (17) may not seem very useful at first sight. It is not manifestly sign-definite, nor is it manifestly finite. To clarify these issues, we can rewrite the first term using eq. (13). We obtain:

$$M_0 \frac{\partial}{\partial M_0} s(M_0 R) =
\begin{align*}
&- R^2 \int_D d\phi_1 \int_D d\phi_2 \langle T_D(\phi_1) T_D(\phi_2) \rangle \left[1 - \cos (\phi_1 - \phi_2) \right].
\end{align*}
$$

(18)

The right hand side of (18) is free of divergences and ambiguities due to the double zero of $1 - \cos (\phi_1 - \phi_2)$. Furthermore, (18) is manifestly negative in a reflection positive theory (note that this also applies to a connected 2-point function, as on the right hand side of (18)). Therefore, we deduce that $s$ monotonically decreases along defect RG flows, implying that the UV and IR DCFT satisfy

$$g_{UV} > g_{IR} .$$

(19)

Eq. (18) additionally implies that $s$ does does no depend on the marginal parameters on the defect.\footnote{It is often the case that $g$ and $s$ depend on the marginal couplings of the bulk CFT [93–97]. Furthermore $g$ and $s$ do not have any obvious monotonicity property under bulk RG flows [98].}

In $d = 2$, equation (19) was originally conjectured to hold for boundaries (and therefore, using the folding trick, for interfaces) by Affleck and Ludwig [32, 36]. In $d = 2$, in the regime where the DRG flow can be described in terms of finitely many couplings and beta functions, a gradient formula equivalent to eq. (18) was proposed in the context of string field theory [82–86]. It was then established by Friedan and Konechny [39]. An alternative proof of eq. (19) in $d = 2$ was also given [45] using quantum information methods.\footnote{See, for instance, also [38, 41, 87] for a holographic setup.} Our work provides an extension of those results to line defects in an arbitrary number of dimensions. We also remark that the inequality (19) was recently conjectured in [47] for arbitrary $d$. Another remark is that the trivial line has $g = 1$. However, it may a priori be that...
\( g < 1 \) for some non-trivial lines, as sometimes happens in 2d [88, 89].

Eq. (19) was extensively checked in \( d = 2 \), see e.g. [32, 33, 36, 91]. We additionally verified our results (18) and (19) in several concrete examples, including a flow between Wilson lines in \( \mathcal{N} = 4 \) SYM previously studied in [90, 92]. Details can be found in the Supplemental Material.

Finally, we remark that the partition function of higher-dimensional defects is subject to further ambiguities besides a cosmological constant, rendering a generalization of our arguments not straightforward. For two- and four-dimensional defects irreversibility of the DRG flow was proven via different means, using Weyl anomaly matching [44, 52].

Acknowledgements We acknowledge useful discussions with Bartolomeu Fiol, Sergei Gukov, Simeon Hellerman, Márk Mezei, Luigi Tizzano, Cumrun Vafa, and Yifan Wang. GC is supported by the Simons Foundation (Simons Collaboration on the Non-perturbative Bootstrap) grants 488647 and 397411. ZK and ARM are supported in part by the Simons Foundation [36] and the weak to strong coupling flow in the Kondo effect [32, 33]. In \( d > 2 \) flows between various Wilson lines in 4 dimensions and a vast class of holographic flows was considered in [47]. The results are consistent with the monotonicity of \( g \).

Let us now discuss the gradient formula:

\[
M_0 \frac{\partial s}{\partial M_0} = - R^2 \int_D d\phi_1 \int_D d\phi_2 (T_D(\phi_1)T_D(\phi_2)) \left[ 1 - \cos (\phi_1 - \phi_2) \right],
\]

(22)

where \( s \) is the defect entropy defined in the main text. We verified eq. (22) in two concrete examples: conformal perturbation theory and the flow from a standard Wilson loop to a supersymmetric one in \( \mathcal{N} = 4 \) SYM, proposed by Polchinski and Sully [90], at weak ‘t Hooft coupling.

In conformal perturbation theory, one starts from an abstract \( \text{DCFT} \) and perturbs it with one or more weakly or marginally relevant operators; one then computes the partition function expanding in the couplings of the perturbations. It is then simple to compute the defect entropy and verify the gradient formula (22) using \( T_D = \beta_i \mathcal{O}_i \), where \( \beta_i \) are the beta functions of the defect coupling and \( \mathcal{O}_i \) the defect operators with which the \( \text{DCFT} \) is deformed. In this setup, in \( d = 2 \), eq. (22) was verified to third order in perturbation theory in [91]. A similar argument holds for any \( d \).

Let us now consider the DRG flow from the ordinary Wilson loop (WL) to the 1/2 BPS Wilson-Maldacena loop (WML) in \( \mathcal{N} = 4 \) SYM in four dimensions in the planar limit. This was proposed in [90] and studied in detail in [92]. One considers a single-parameter family of Wilson loop operators in the fundamental representation

\[
W^{(\mathcal{S})} = \frac{1}{\mathcal{N}} \text{Tr} \mathcal{P} \exp \int_C \! d\tau [i A_{\mu}(x) \dot{x}^\mu + \zeta \Phi_m(x) \theta^m |\dot{x}|],
\]

(23)

where \( \theta^2 = 1 \), \( \zeta \) is the coefficient in front of the scalar coupling (\( m = 1, \cdots, 6 \)) and we follow closely the notations of [92], focusing on circular contours. The theory admits a UV fixed point \( \zeta = 0 \) and may flow to the WML (\( |\zeta| = 1 \)), that divergent integral \( \int dx^d \). This is in sharp contrast with the expected charge conservation. Technically, this is because the \( x \to \infty \) limit of the stress tensor depends on the distance \( r \) from the defect, and it is therefore not single-valued at the point at infinity. This implies that that the conformal charges vanish only when the distance between the integration surface and the defect diverges sufficiently fast as \( x^d \to \pm \infty \). This problem with the conformal charges is presumably related to the disagreement between the expectation value of circular and linear Maldacena-Wilson loops in \( \mathcal{N} = 4 \) SYM [71–73].

B. Examples

The inequality \( g_{UV} > g_{IR} \) was extensively checked in the literature in \( d = 2 \). Early examples include flows between the free and the fixed boundary conditions in the Ising model [36] and the weak to strong coupling flow in the Kondo effect [32, 33]. In \( d > 2 \) flows between various Wilson lines in 4 dimensions and a vast class of holographic flows was considered in [47]. The results are consistent with the monotonicity of \( g \).

Let us now consider the gradient formula:

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M_0 \frac{\partial s}{\partial M_0} = - R^2 \int_D d\phi_1 \int_D d\phi_2 (T_D(\phi_1)T_D(\phi_2)) \left[ 1 - \cos (\phi_1 - \phi_2) \right],
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\]

(23)

where \( \theta^2 = 1 \), \( \zeta \) is the coefficient in front of the scalar coupling (\( m = 1, \cdots, 6 \)) and we follow closely the notations of [92], focusing on circular contours. The theory admits a UV fixed point \( \zeta = 0 \) and may flow to the WML (\( |\zeta| = 1 \)), that divergent integral \( \int dx^d \). This is in sharp contrast with the expected charge conservation. Technically, this is because the \( x \to \infty \) limit of the stress tensor depends on the distance \( r \) from the defect, and it is therefore not single-valued at the point at infinity. This implies that that the conformal charges vanish only when the distance between the integration surface and the defect diverges sufficiently fast as \( x^d \to \pm \infty \). This problem with the conformal charges is presumably related to the disagreement between the expectation value of circular and linear Maldacena-Wilson loops in \( \mathcal{N} = 4 \) SYM [71–73].

A. Subtleties for the infinite line geometry

Naively, a defect on an infinite straight line is conformally equivalent to a circular defect. However, there is a subtlety associated to the infinite extent of the straight line defect. Here we detail this issue.

To explain the subtlety with straight line defects, consider a defect \( D \) which extends in the \( dth \) direction at \( x^d = 0 \), where \( i, j, \ldots \) denote indices \( 1, \ldots, d - 1 \) transverse to the line. For \( d > 2 \), the one-point function of the stress tensor depends on a constant \( h_D \) and reads [25]:

\[
\langle T^{i d}(x) \rangle = h_D \frac{d - 2}{r^d}, \quad \langle T^{i j}(x) \rangle = - h_D \frac{2 \delta^{ij} - d x^i x^j / r^2}{r^d},
\]

(20)

where \( r^2 = x^i x^i \) is the distance from the line operator (we assume that the defect has zero transverse spin for simplicity).

Eq. (20) is covariant under conformal transformations. However, a subtlety arises when we consider the expectation values of the \( S_L(2, \mathbb{R}) \) charges. The expectation values of the charges are obtained by integrating the stress tensor contracted with the appropriate Killing vector at a fixed distance \( \varepsilon \) on a cylinder around the defect:

\[
Q_\xi(D) = \int_{r=\varepsilon} d^{d-1} \Sigma^\mu (T^h_{\mu \nu}/r^2) \xi^\nu.
\]

(21)

For the dilation and special conformal Killing vectors along the line one finds a result proportional to \( 1/\varepsilon \) times a linearly divergent integral \( \int dx^d \). This is in sharp contrast with the expected charge conservation. Technically, this is because the \( x \to \infty \) limit of the stress tensor depends on the distance \( r \) from the defect, and it is therefore not single-valued at the point at infinity. This implies that that the conformal charges vanish only when the distance between the integration surface and the defect diverges sufficiently fast as \( x^d \to \pm \infty \). This problem with the conformal charges is presumably related to the disagreement between the expectation value of circular and linear Maldacena-Wilson loops in \( \mathcal{N} = 4 \) SYM [71–73].

Supplemental Material
provides an IR stable fixed point. Generically, the expectation values of the Wilson loop operators depend on the renormalization scale through:
\[ \langle W^{(\zeta)} \rangle \equiv W(\lambda; \zeta(M_0 R), M_0 R), \]
\[ M_0 \frac{\partial}{\partial M_0} W + \beta_\zeta \frac{\partial}{\partial \zeta} W = 0, \]
where we note that the boundary coupling \( \zeta \) depends on the renormalization scale \( \zeta = \zeta(M_0 R) \). Here \( \langle W^{(\zeta)} \rangle \) stands for the \( g \)-function of the defect. At weak \( 't \) Hooft coupling, \( \lambda \ll 1 \), this dependence follows from the beta function of \( \zeta \) [90]:
\[ \beta_\zeta = M_0 \frac{\partial \zeta}{\partial M_0} = -\frac{\lambda}{8\pi^2} \zeta (1 - \zeta^2) + O(\lambda^2). \]
(25)

The partition function of the DCFT was evaluated to order O(\( \lambda^2 \)) in [92], where it was found that (in units such that \( R = 1 \)):
\[ \langle W^{(\zeta)} \rangle = 1 + \frac{1}{8} \lambda + \left[ \frac{1}{192} + \frac{1}{128\pi^2} \lambda^2 \right] + O(\lambda^2). \]
(26)
Notice that these results, though perturbative in \( \lambda \), are exact throughout the flow. From eq. (26) one clearly reads \( g_{UV} > \beta_{M_0} \). Upon taking derivatives with respect to \( M_0 \) and using eq. (25), one may then compute the DRG gradient of the defect entropy to be:
\[ M_0 \frac{\partial s}{\partial M_0} = -\frac{\lambda^3}{256\pi^4} \zeta^2 (1 - \zeta^2)^2 + O(\lambda^4). \]
(27)
Hence \( s \) monotonically decreases along the DRG flow. Furthermore, since to the order we are working we may neglect anomalous dimensions, we easily find the two-point function of the defect stress tensor [92]:
\[ \langle T_D(\phi)T_D(0) \rangle = \frac{\lambda}{8\pi^2} \left[ \frac{\beta_\zeta^2}{2 \sin^2 \frac{\phi}{2}} \right]^2 + [O(\lambda)]. \]
(28)

It is then simple to use this equation and the beta-function (25) to verify explicitly the agreement of the RHS of the gradient formula (22) with the expression (27). This provides a test of the gradient formula (22) in the small \( 't \) Hooft coupling regime.

In the strong coupling regime one expects a similar DRG flow to take place. This is supported by the holographic calculation of the DCFT partition functions at the fixed points, whose ratio satisfies \( \langle W^{(0)} \rangle / \langle W^{(1)} \rangle \sim \lambda^2 \) for \( \lambda \gg 1 \) [92].

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