Abstract

We consider a scattering map that arises in the $\bar{\partial}$ approach to the scattering theory for the Davey-Stewartson II equation and show that the map is an invertible map between certain weighted $L^2$ Sobolev spaces.
1 Introduction

This note considers a map $R$ that arises in the scattering theory for a first-order system (1.1) in the plane. The map $R$ takes a potential $q$ to scattering data that appears as a coefficient in the asymptotic expansion of solutions to the first-order system. The map $R$ also arises in the so-called $\bar{\partial}$ approach to scattering theory for the Davey-Stewartson II equation. The scattering transform we discuss below first arose in work of Fokas and Ablowitz [FA84] as a transformation to convert one of the Davey-Stewartson equations to a linear evolution equation. This same transform also proved to be useful in the work of Brown and Uhlmann [BU97] and Barceló, Barceló and Ruiz [BBR01] who investigated the inverse conductivity problem with less regular coefficients in planar domains.

The formal theory and the beginning of a rigorous treatment of the scattering transform may be found in work of Fokas and Ablowitz [FA84] and Beals and Coifman [BC92].
One remarkable feature of Beals and Coifman is that the map $\mathcal{R}$ satisfies a Plancherel identity,

$$\int_{\mathbb{C}} |q|^2 \, dx = \int_{\mathbb{C}} |\mathcal{R}(q)|^2 \, dk,$$

at least for potentials $q$ that are sufficiently regular. Since the map $\mathcal{R}$ is not linear, this identity does not imply the continuity of the map $\mathcal{R}$ or even that $\mathcal{R}(q)$ is defined for all $q$ in $L^2$. There are several authors who have established continuity of the transform on other spaces. Sung [Sun94a, Sun94b, Sun94c] develops estimates for the scattering transform in the Schwartz space. Brown [Bro01] establishes that the map $\mathcal{R}$ is continuous in a neighborhood of the origin in $L^2$. Perry [Per] considers the map on a weighted Sobolev space $H^{1,1}$ and shows that $\mathcal{R}$ is locally Lipschitz continuous on this space. The weighted Sobolev spaces $H^{\alpha,\beta}$ are defined as

$$H^{\alpha,\beta} = \{ f : \langle D \rangle^\alpha f \in L^2 \text{ and } \langle \cdot \rangle^\beta f \in L^2 \}.$$

We use $\langle x \rangle$ to denote $\langle x \rangle = 1 + |x|^2/2$ and then $\langle D \rangle^\beta$ is the Fourier multiplier operator $\langle D \rangle^\beta f = (\langle \cdot \rangle^\beta f)$. Astala, Faraco, and Rogers [AFR15] consider the map $\mathcal{R}$ on the space $H^{\alpha,\alpha}$ with $\alpha > 0$ and show that $\mathcal{R} : H^{\alpha,\alpha} \to L^2$ is locally Lipschitz continuous. The main result of the current work is to show that $\mathcal{R} : H^{\alpha,\beta} \to H^{\beta,\alpha}$ when $0 < \alpha, \beta < 1$ and that the map $\mathcal{R}$ is locally Lipschitz continuous. We also give a result which shows that in the spaces $H^{\alpha,\beta}$, the difference $\mathcal{R}(q) - \hat{q}$ is better behaved than $\hat{q}$. In particular, if $q$ is in $H^{\alpha,\beta}$, then $\mathcal{R}(q) - \hat{q}$ will lie in $H^{2\beta,2\alpha}$, at least for $\alpha, \beta < 1/2$.

Our result in this paper is a two-dimensional analogue of the results of X. Zhou [Zho98] which give mapping properties of a scattering transform for the ZS-AKNS system on weighted Sobolev spaces on the real line.

We begin our development by sketching the definition of the map $\mathcal{R}$. We let $q$ be a function on the complex plane and for much of the argument we will assume that $q$ is in the Schwartz class $\mathcal{S}(\mathbb{C})$. We will establish estimates on the map $\mathcal{R}$ with constants that depend only on the norm of $q$ in a weighted Sobolev space. With these estimates it will be possible to extend the map $\mathcal{R}$ from the Schwartz space to $H^{\alpha,\beta}$ with $\alpha > 0$ and $\beta > 0$. Throughout the paper, we will use $e_k(x) = e_x(k) = \exp(k\bar{x} - kx)$. We consider the system

$$\begin{cases}
\bar{\partial}_1 \mu_1(x, k) = \frac{1}{2} e_k(x)q(x)\bar{\mu}_2(x, k) \\
\bar{\partial}_2 \mu_2(x, k) = \frac{1}{2} e_k(x)q(x)\bar{\mu}_1(x, k) \\
\lim_{|x| \to \infty} (\mu_1(x, k), \mu_2(x, k)) = (1, 0).
\end{cases} \tag{1.1}$$

Above, $\bar{\partial} = \frac{1}{2}(\partial_{\bar{x}_1} + i \partial_{\bar{x}_2})$ and $\partial = \frac{1}{2}(\partial_{x_1} - i \partial_{x_2})$ denote the standard derivatives with respect to the complex variables $\bar{x}$ and $x$. When we need to differentiate with respect to $k$ and $\bar{k}$, we will write $\partial/\partial k$ and $\bar{\partial}/\partial \bar{k}$.
We define our scattering transform $\mathcal{R}$ by

$$\mathcal{R}(q)(k) = \frac{1}{\pi} \int_\mathbb{C} e_k(x)q(x)\mu_1(x,k)\,dx.$$  

Since the function $\mu_1$ approaches one at infinity, it is plausible that $\mathcal{R}$ is a non-linear generalization of the Fourier transform. To make this more precise, we introduce a variant of the Fourier transform that we will use throughout the paper. For a function $\psi$ in $L^1$, we define our Fourier transform by

$$\hat{\psi}(k) = \frac{1}{\pi} \int_\mathbb{C} e_k(x)\psi(x)\,dx$$

and

$$\hat{\psi}(k) = \frac{1}{\pi} \int_\mathbb{C} e^{-2i(k_1x_2+k_2x_1)}\psi(x)\,dx.$$  

From the second expression it is clear how our Fourier transform is related to more common normalizations of the Fourier transform. With our definition, the Fourier transform is the linearization at 0 of the scattering transform. For convenience, we list several standard properties of the Fourier transform translated to our normalization.

If we put

$$\hat{u}(x) = \frac{1}{\pi} \int_\mathbb{C} e_k(-x)u(k)\,dk,$$

the Fourier inversion formula reads

$$\hat{\check{u}} = \hat{\check{u}} = u,$$

at least for $u$ in the Schwartz class. If we let $f*g$ denote the convolution, we have that

$$(f*g)(k) = \pi \hat{f}(k)\hat{g}(k)$$  \hspace{1cm} (1.2)

and

$$(fg)(k) = \frac{1}{\pi} (\hat{f} \ast \hat{g})(k).$$  \hspace{1cm} (1.3)

We recall that the Cauchy transform, $\mathcal{C}$, defined by

$$\mathcal{C}f(x) = \frac{1}{\pi} \int_\mathbb{C} \frac{f(y)}{x-y}\,dy$$

gives a right inverse to the operator $\partial$. Using that $(\hat{\partial f})(k) = k\hat{f}(k), (\hat{\partial f})(k) = -k\hat{f}(k)$ and the representation of $\hat{\partial}^{-1}$ and a similar representation of $\hat{\partial}^{-1}$, we obtain that $(1/z) = 1/k$ and $(1/z) = -1/k$. From these observations, we obtain

$$\frac{1}{\pi} \int_\mathbb{C} \frac{e_k(y)q(y)}{\bar{y} - x}\,dy = \frac{1}{\pi} \int_\mathbb{C} \frac{\hat{q}(\ell)e_x(k-\ell)}{k-\ell}\,d\ell$$  \hspace{1cm} (1.4)

and

$$\frac{1}{\pi} \int_\mathbb{C} \frac{e_k(-y)\bar{q}(y)}{y - x}\,dy = \frac{1}{\pi} \int_\mathbb{C} \frac{\bar{q}(\ell)e_x(\ell - k)}{k-\ell}\,d\ell.$$  \hspace{1cm} (1.5)
These formulae will be used in section 3 below.

We are ready to outline the construction of the solutions of the system (1.1). We let $T_k$ be the operator given by

$$T_k f(x) = \frac{1}{2} C(e_k(\cdot)q\bar{f})(x)$$

where $q$ is the potential. Throughout this paper, we assume that $q$ is in the Schwartz class in order to simplify the argument. As a last step, we will use the local Lipschitz continuity of $R$ from $H^{\alpha,\beta}$ to $H^{\beta,\alpha}$ to extend the map to a weighted Sobolev space. It is clear that if $q$ is in $H^{\alpha,\beta}$ with $\alpha > 0$ and $\beta > 0$, $\mu_1$ and $\mu_2$ are in $L^\infty(C^2)$ and are solutions of the integral equations

$$\mu_1 = 1 + T_k(\mu_2) \quad (1.6)$$
$$\mu_2 = T_k(\mu_1) \quad (1.7)$$

then $(\mu_1, \mu_2)$ are solutions of (1.1). Furthermore, if we substitute (1.7) into (1.6) we obtain

$$\mu_1 = 1 + T_k^2(\mu_1) \quad (1.8)$$

and if $\mu_1$ is a solution of (1.8), then $(\mu_1, T_k(\mu_1)) = (\mu_1, \mu_2)$ is a solution of (1.1).

Finally, we observe that the map $R$ is invertible and the inverse map $I$ can be given by

$$I(f) = \overline{R(f)} \quad (1.9)$$

The invertibility of $R$ on the Schwartz space is in the work of Sung [Sun94a, Sun94b, Sun94c] and Perry [Per] gives invertibility on $H^{1.1}$. The estimates of this paper will allow us to extend the invertibility to the family of spaces $H^{\alpha,\beta}$. The formula (1.9) can be found in the work of Astala, Faraco, and Rogers [AFR15] and Perry.

Our main result is the following theorem which gives the properties of the scattering map.

**Theorem 1.10** The map $R$ maps $H^{\alpha,\beta}$ to $H^{\beta,\alpha}$ and is locally Lipschitz continuous, provided $0 < \alpha, \beta < 1$.

More precisely, if we fix $\alpha$ and $\beta$ in $(0,1)$, then there exists an increasing function in $M_0$, $C = C(M_0) = C(M_0, \alpha, \beta)$ so that if $\|q\|_{H^{\alpha,\beta}} \leq M_0$ and $\|q'\|_{H^{\alpha,\beta}} \leq M_0$ then

$$\|R(q) - R(q')\|_{H^{\beta,\alpha}} \leq C\|q - q'\|_{H^{\alpha,\beta}}.$$

In addition, if $\alpha, \beta$ are in the interval $(0, 1/2)$, then

$$\|R(q) - \hat{q}\|_{H^{2\beta,2\alpha}} \leq C(M_0), \quad \text{if} \quad \|q\|_{H^{\alpha,\beta}} \leq M_0.$$

Thanks to the identity (1.9), the same results hold for the inverse scattering map.
We begin the proof of Theorem 1.10 by iterating the integral equation (1.8) for $\mu_1$ to obtain finite expansions for $\mu_1$,

$$\mu_1(x, k) = \sum_{j=0}^{N-1} T_k^{2j}(1)(x) + T_k^{2N}(\mu_1)(x), \quad N = 1, 2, 3, \ldots \quad (1.11)$$

We substitute this expression for $\mu_1$ into the definition of $R(q)$ to obtain

$$R(q)(k) = \sum_{j=0}^{N-1} r_j(k) + r(N)(k), \quad N = 1, 2, 3, \ldots, \quad (1.12)$$

where

$$r_j(k) = \frac{1}{\pi} \int_C q(y) e_k(y) T_k^{2j}(1)(y) dy \quad (1.13)$$

and the remainder term is given by

$$r(N)(k) = \frac{1}{\pi} \int_C q(y) e_k(y) T_k^{2N}(\mu_1(\cdot, k))(y) dy. \quad (1.14)$$

The notation $\overline{T_k^{2N}(f)}$ is ambiguous and we intend $\overline{T_k^{2N}(f)}$ to mean the complex conjugate of $T_k^{2N}(f)$, $\overline{T_k^{2N}(f)} = \left(\overline{T_k^{2N}(f)}\right)$. The term $r_0 = \hat{q}$ is just the Fourier transform. We will use duality and estimates for certain Brascamp-Lieb forms to estimate the terms $r_j$ for $j \geq 1$. For $N$ sufficiently large (depending on $\alpha$ and $\beta$) we will be able to show that the remainder term is in $H^{1.1}$. This second step is where we require that $\alpha$ and $\beta$ be positive, while the estimates for the terms $r_j$ hold for $\alpha = 0$ or $\beta = 0$. The estimates in $L^2$ can be found in the work of one of the authors [Bro01]. Work of Nie [NB11] and M. Christ’s appendix to Perry’s paper [Per] give different proofs of these estimates for the terms $r_j$.

Our argument follows the argument of Perry when the potentials are in $H^{1.1}$. The innovations in this paper are new estimates for multi-linear forms and a certain amount of persistence that is needed to estimate the remainder term.

The outline of this paper is as follows. In section 2, we give more details on the construction of the special solutions ($\mu_1, \mu_2$). In section 3, we prove the estimates for the multi-linear forms in (1.13) and in section 4, we study the remainder term (1.14).

Acknowledgment. We thank K. Astala, D. Faraco, and K. Rogers for showing us a preliminary version of their work [AFR15]. This was helpful in carrying the research reported below. In particular, their decay estimate (2.5) below is a main step in our work.

2 Constructing the scattering solutions

In this section, we collect several estimates that will be needed in section 4 and give a construction of the solutions ($\mu_1, \mu_2$) to the equations (1.1).
 Much of our analysis will take place in $L^p$-spaces and we begin by observing that for $\beta > 0$ and $1 > \alpha > 0$, we have the inclusion

$$H^{\alpha,\beta} \subset L^p,$$

$$\frac{1}{2} - \frac{\alpha}{2} \leq \frac{1}{p} < \frac{1}{2} + \frac{\beta}{2}.$$  \hfill (2.1)

The estimate for $p > 2$ follows from the Sobolev embedding theorem and the estimate for $p < 2$ follows from the inequality of Hölder. We let $I_1$ denote the fractional integral given by

$$I_1(f)(x) = \frac{1}{\pi} \int_C \frac{f(y)}{|x - y|} dy.$$ 

For estimates involving the size of $C(f)$, the elementary inequality $|C(f)(x)| \leq I_1(|f|)(x)$ means that it is sufficient to give estimates for $I_1(f)$. For $p \neq 2$, it is well-known (see Vekua [Vek62]) that

$$\|I_1(f)\|_{L^\infty} \leq C_p \|f\|_{L^p}^{1/2} \|f\|_{L^{p'}}^{1/2}. \hfill (2.2)$$

Above, $p'$ is the usual conjugate exponent. We recall the well-known Hardy-Littlewood-Sobolev estimate for fractional integration which may be found in [Ste70], for example. If $1 < p < 2$, we have

$$\|I_1(f)\|_{L^p} \leq C_p \|f\|_{L^p}, \quad \frac{1}{p} = \frac{1}{p} - \frac{1}{2}, \hfill (2.3)$$

We will find it useful to work in the weighted $L^p$-spaces, $L^p_\alpha = \{f : \langle \cdot \rangle^\alpha f \in L^p\}$. Occasionally, we will also use the scale-invariant or homogeneous version of these spaces $\tilde{L}_\alpha^p = \{f : |\cdot|^{\alpha} f \in L^p\}$. For $1 < p \leq \tilde{p} < \infty$ and $\alpha$, $\beta$ satisfying $-\frac{2}{\tilde{p}} < \beta 

\leq \alpha < \frac{2}{p'}$ and $\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{1-\alpha+\beta}{2}$, we have

$$\|I_1(f)\|_{\tilde{L}_\beta^p} \leq C(p, \tilde{p}, \alpha, \beta) \|f\|_{\tilde{L}_\alpha^p}. \hfill (2.4)$$

The estimate (2.4) follows easily from the work of Sawyer and Wheeden [SW92, Theorem 1]. The corresponding estimate in the homogeneous spaces $\tilde{L}_\alpha^p$ is due to Stein and Weiss [SW58]. Finally, we recall a result of Astala, Faraco and Rogers who show that for $\alpha \geq 0$ and $2 < \tilde{p} < \infty$, we have

$$\|T_k^2(f)\|_{\tilde{L}^p} \leq C\langle k \rangle^{-\alpha} \|q\|^2_{H^{\alpha,0}} \|f\|_{L^p}. \hfill (2.5)$$

Occasionally in the sequel, we will want to display the dependence of $T_k$ and $\mu_j$ on the potential $q$. We will do this by writing $\mu_j(q; \cdot, \cdot)$ and $T_{k,q}$. 

**Proposition 2.6** Let $\epsilon > 0$ and suppose that $q \in H^{\epsilon,\epsilon}$. Fix $\tilde{p}_0$ with $1/\tilde{p}_0 \in (0, \epsilon/2)$. We may construct solutions $(\mu_1, \mu_2)$ of (1.1) with $(\mu_1 - 1, \mu_2)$ in $L^\infty(L^\infty_\alpha(C^2))$. Furthermore, there exists a constant $C = C(M_0)$ so that if $\|q\|_{H^{\epsilon,\epsilon}} \leq M_0$ and $\|q'\|_{H^{\epsilon,\epsilon}} \leq M_0$, then

$$\sup_k \|\mu_1(q; \cdot, k) - \mu_1(q'; \cdot, k)\|_{L^{\tilde{p}_0}} \leq C(M_0) \|q - q'\|_{H^{\epsilon,\epsilon}}.$$
Proof. We may assume that $\varepsilon \in (0, 1)$. From Hölder’s inequality and the Hardy-Littlewood-Sobolev inequality (2.3) it follows that if $q$ is in $L^2$ and $\tilde{p}$ is in $(2, \infty)$, the map $f \rightarrow T_k f$ is bounded on $L^p$ and we have the continuity result

$$
\|T_{k,q}(f) - T_{k,q'}(f)\|_{L^{\tilde{p}}} \leq C\|q - q'\|_{L^2}\|f\|_{L^{\tilde{p}}}.
$$

Furthermore, by approximating $q$ in $L^2$ by functions that are bounded and compactly supported, we can see that the map $T_k$ is compact on $L^\tilde{p}$. If we also have that $\langle \cdot \rangle^\theta q \in L^2$, then

$$
\|T_{k,q} - T_{k',q}\|_{L(L^p)} \leq C|k - k'|^\theta\|q\|_{H^0,\theta}, \quad \tilde{p} \in (2, \infty).
$$

(2.7)

To establish (2.7), observe that

$$
|e_k(y) - e_{k'}(y)| \leq 2^\theta|k - k'|^\theta y^\theta
$$

and thus

$$
|T_k(f)(x) - T_{k'}(f)(x)| \leq \frac{1}{\pi} \int_C \frac{|f(y)||q(y)|}{|x - y|}|e_k(y) - e_{k'}(y)|\,dy
$$

$$
\leq C|k - k'|^\theta I_1(\langle |\cdot|^\theta |q|f\rangle)(x).
$$

Thus by the inequality of Hardy-Littlewood-Sobolev (2.3) and Hölder’s inequality, we have

$$
\|T_k(f) - T_{k'}(f)\|_{L^\tilde{p}} \leq C|k - k'|^\theta\|f\|_{L^\tilde{p}}\|q\|_{H^0,\theta}.
$$

In the argument below, we will apply this estimate with $\theta = \varepsilon/2$.

We observe that a solution of (1.1) should also solve the integral equations (1.8). According to a standard argument using the Liouville theorem for pseudo-analytic functions (see [BU97, Section 3] or [Sun94a]), the operator $(I - T_k^2)$ is injective on $L^\tilde{p}$ for $2 < \tilde{p} < \infty$. Since $T_k^2$ is compact, the Fredholm theory tells us that $(I - T_k^2)^{-1}$ exists as an operator in $L(L^\tilde{p})$. The continuity of the map $(k, q) \rightarrow T_{k,q}$ implies that $\|(I - T_{k,q}^2)^{-1}\|_{L(L^p)}$ is a continuous function for $(k, q)$ in $C \times H^{0,\varepsilon/2}$. According to the decay estimate of [APR15] (restated as (2.5)), given $M_0$, we may find $R_0$ so that $\|T_k^2\|_{L(L^p)} \leq 1/2$ if $|k| \geq R_0$ and $\|q\|_{H^{0,\varepsilon}} \leq M_0$. The embedding $H^{\varepsilon,\varepsilon} \subset H^{0,\varepsilon/2}$ is compact and thus the continuous function $\|(I - T_k^2)^{-1}\|_{L(L^p)}$ will be bounded on the compact set, $\{(k, q) : |k| < R_0, \|q\|_{H^{0,\varepsilon}} \leq M_0\} \subset C \times H^{0,\varepsilon/2}$.

If we use the embedding of $H^{\varepsilon,\varepsilon}$, (2.1), and (2.3) we can see that $T_k(1)$ lies in $L^\infty(L^\tilde{p}_x)$. Thus we may set

$$
\mu_1(\cdot, k) = 1 + (I - T_k^2)^{-1}(T_k(1)), \quad \mu_2(\cdot, k) = T_k(\mu_1)
$$

and then $(\mu_1, \mu_2)$ is a solution of (1.1).
Next we observe that thanks to the continuity of the map \((k, q) \mapsto T_{k,q}(1)\) from \(H^{t,\epsilon} \) into \(L^\tilde{p}_0\), it follows that the map \(q \mapsto \mu_1(q; \cdot, k) - 1\) is continuous from \(H^{t,\epsilon} \) into \(L^\tilde{p}_0\). In fact, we can write

\[
\mu_1(q; \cdot, k) - \mu_1(q'; \cdot, k) = (I - T_{k,q}^2)^{-1}(T_{k,q}^2 - T_{k,q'}^2)(1) + (((I - T_{k,q}^2)^{-1} - (I - T_{k,q'}^2)^{-1})(T_{k,q}(1)),
\]

and conclude that

\[
\|\mu_1(q; \cdot, k) - \mu_1(q'; \cdot, k)\|_{L^\tilde{p}_0} \leq C(R_0)\|q - q'\|_{H^{t,\epsilon}}, \quad \text{if } \|q\|_{H^{t,\epsilon}} \leq M_0 \text{ and } \|q'\|_{H^{t,\epsilon}} \leq M_0.
\]

Now that we have \(\mu_1\) with \(\mu_1 - 1 \in L^\infty_k(L^\tilde{p}_0)\), our next step is to show that we have \(\mu_1 - 1\) in an interval of \(L^\tilde{p}\) spaces. However, before we do this, we give a simple lemma that will be used to obtain the local Lipschitz continuity of multi-linear expressions.

**Lemma 2.8** Let \(\Lambda : X_0 \times X_1 \times \cdots \times X_N \to Y\) be a bounded multi-linear operator with \(X_j\) and \(Y\) normed vector spaces. More precisely, assume that for some constant \(C_0\), we have

\[
|\Lambda(q_1, \ldots, q_N)| \leq C_0 \prod_{j=1}^N \|q_j\|_{X_j}.
\]

Then we obtain the following local Lipschitz continuity result. If \(\|q_j\|_{X_j} \leq M_0\) and \(\|q'_j\|_{X_j} \leq M_0\), then

\[
\|\Lambda(q_1, \ldots, q_N) - \Lambda(q'_1, \ldots, q'_N)\|_Y \leq C_0 M_0^n \sum_{j=1}^N \|q_j - q'_j\|_{X_j}.
\]

**Proof.** We write

\[
\Lambda(q_1, \ldots, q_N) - \Lambda(q'_1, \ldots, q'_N) = \sum_{j=1}^N \Lambda(q_1, \ldots, q_{i-1}, q_i - q'_i, q'_{i+1}, \ldots, q'_N)
\]

and use the boundedness of \(\Lambda\).

**Proposition 2.9** Let \(\epsilon\) lie in the interval \((0, 1)\). If \(q, q' \in H^{t,\epsilon}\) with \(\|q\|_{H^{t,\epsilon}} \leq M_0\) and \(\|q'\|_{H^{t,\epsilon}} \leq M_0\), then \((\mu_1 - 1) \in L^\tilde{p}\) when \(1/\tilde{p}\) in \([0, \epsilon) \cap [0, 1/2)\) and

\[
\|\mu_1(q; \cdot, k) - \mu_1(q'; \cdot, k)\|_{L^\tilde{p}} \leq C(M_0, \tilde{p})\|q - q'\|_{H^{t,\epsilon}}.
\]
Proof. Let $\mu_1 = \mu_1(q; \cdot, \cdot)$. We have $\mu_1 - 1 \in L^{\tilde{p}_0}$ with $\tilde{p}_0$ as in Proposition 2.6 and then (1.8) implies that $\mu_1 - 1 = T_k^2(\mu_1 - 1) + T_k^2(1)$. Since $H^{\epsilon, \epsilon} \subset L^p$, for $1/p \in [1/2 - \epsilon/2, 1/2 + \epsilon/2)$, it follows from the inequalities of Hölder and Hardy-Littlewood-Sobolev (2.3) that $T_k^2(\mu_1 - 1) \in L^{\tilde{p}}$ for $1/\tilde{p} \in (0, \epsilon) \cap (0, 1/2)$, $T_k^2(1) \in L^{\tilde{p}}$ for $1/\tilde{p} \in (0, \epsilon)$ and we have the estimates

$$
\|T_k^2(\mu_1 - 1)\|_{L^{\tilde{p}}} \leq C\|q\|_{H^{\epsilon, \epsilon}}^2 \|\mu_1 - 1\|_{L^{\tilde{p}_0}}, \quad \|T_k^2(1)\|_{L^{\tilde{p}}} \leq C\|q\|_{H^{\epsilon, \epsilon}}^2.
$$

(2.10)

Thus $\mu_1 - 1 \in L^{\tilde{p}}$ for $1/\tilde{p} \in (0, \epsilon)$. Similar considerations and the estimate (2.2) give that $\mu_1$ is bounded.

Once we recognize that $T_k^2(f)$ is a multi-linear expression in $q$ and $\tilde{f}$, the estimate for the differences follows from (2.10), Proposition 2.6, and Lemma 2.8.

\section{Brascamp-Lieb Forms}

We consider a family of Brascamp-Lieb forms. A criterion for the finiteness of these forms on families of $L^p$-spaces was given by Barthe [Bar98, Proposition 3] and simpler proofs of his criterion were given by Carlen, Lieb, and Loss [CLL04, Theorem 4.2] and Bennett, Carbery, Christ, and Tao [BCCT10, Remark 2.1]. We note that Barthe and Carlen, Lieb, and Loss also give information about the best constant in these inequalities. We are not able to make use of this information in our work. A simple approach to the finiteness of these forms can be found in the dissertation of Z. Nie, see [NB11] (and may be well-known).

To describe the forms we will consider, fix $N$ and let $E \subset \mathbb{R}^{N+1}$ be a finite collection of non-zero vectors. The set $E$ carries the structure of a matroid (see [Lee04] for the definition). Using $E$, we will define a closed convex subset of $\mathbb{R}^{N+1}$ which is called the matroid polytope for $E$. We denote elements of $[0, 1]^E$ as functions $\theta : E \to [0, 1]$. If $A \subset E$ is a set, then we let $\chi_A$ be the indicator function of the set $A$. Thus

$$
\chi_A(v) = \begin{cases} 
1, & v \in A \\
0, & v \notin A.
\end{cases}
$$

The matroid polytope of $E$, $\mathcal{P}(E)$, is defined to be the convex hull of the set $\{\chi_B : B \subset E \text{ and } B \text{ is a basis for } \mathbb{R}^{N+1}\}$. Given a set of vectors $E$, we define a multi-linear form

$$
\Lambda(f_v|v \in E) = \int_{\mathbb{C}^{N+1}} \prod_{v \in E} f_v(v \cdot x) \, dx
$$

where $x = (x_0, x_1, \ldots, x_N)$ is a point in $\mathbb{C}^{N+1}$ and $v \cdot x = \sum_{i=0}^N v_i x_i$ is the standard bilinear inner product. We initially assume that the functions $f_v$ are non-negative so that the integral defining $\Lambda(f_v|v \in E)$ will exist, though it may be infinite. The estimates of Theorem 3.1 will give us estimates for the form when the functions $\{f_v : v \in E\}$ belong to certain Lorentz spaces. These estimates will allow us to extend the form to appropriate products of Lorentz spaces.
We will need to consider these forms when the functions $f_v$ are in Lorentz spaces $L^{p,r}(\mathbb{C})$, $1 \leq p, r \leq \infty$, and refer the reader to the monograph of Bergh and Lofström for the definition of these spaces [BL76, p. 8]. Our main estimate for Brascamp-Lieb forms is the following theorem.

**Theorem 3.1** Suppose that the function $(1/p_v)v \in E$ lies in the interior of the set $\mathcal{P}(E_1)$ and $\sum_{v \in E} 1/r_v \geq 1$. Then there is a finite constant $C$ such that

$$\Lambda(f_v|v \in E) \leq C \prod_{v \in E} \|f_v\|_{L^{p_v,r_v}}.$$

**Proof.** As noted above, this is essentially a result of Barthe. To connect our statement to the result of Barthe, we first observe that the finiteness of the form on $\mathbb{C}^{N+1}$ is equivalent to the finiteness of the form on $\mathbb{R}^{N+1}$, i.e. the form obtained when $x \in \mathbb{R}^{N+1}$ instead of $\mathbb{C}^{N+1}$. This follows from the theorem of Fubini.

Next we observe that Barthe gives a description of the family of $L^p$-spaces for which the form is finite in terms of a family of inequalities satisfied by the reciprocals, $(1/p_v)v \in E$. It is known that the inequalities of Barthe describe the matroid polytope, see the excellent monograph of J. Lee [Lee04], for example. Finally, a multi-linear version of the real method of interpolation allows us to pass from $L^p$ estimates in $\mathcal{P}(E)$ to Lorentz space estimates in the interior of $\mathcal{P}(E)$. See work of Christ [Chr85] or Janson [Jan86] for the multi-linear interpolation results.

We will need to consider two sets of vectors in this section. The first we will denote by $E_1 \subset \mathbb{R}^{N+1}$ for $N \geq 2$ and is given by

$$E_1 = E_1^{N+1} = \{e_0, e_1, \ldots, e_N, e_0 - e_1, \ldots, e_{N-1} - e_N, \zeta = \sum_{j=0}^{N} (-1)^j e_j\}. \quad (3.2)$$

Note that the condition $N \geq 2$ guarantees that $E_1$ contains $2N + 2$ distinct vectors. The second set of vectors will only be needed in odd dimensions. We define $E_2 \subset \mathbb{R}^{2N+1}$ for $N \geq 1$ by

$$E_2 = E_2^{2N+1} =$$

$$\{e_0, e_1, \ldots e_{2N}, \sum_{j=0}^{2k-1} (-1)^j e_j, \sum_{j=0}^{2k-1} (-1)^j e_{2N-j}, k = 1, \ldots, N, \zeta = \sum_{j=0}^{2N} (-1)^j e_j\}. \quad (3.3)$$

The following lemma shows that the constant function $\theta(v) = 1/2$ lies in the interior of the matroid polytopes for $E_1$ and $E_2$.

**Lemma 3.4** For $j = 1$ or 2, let $A_{N+1} = \{\theta : \sum_{v \in E_j} |\theta(v) - 1/2| \leq 1, \sum_{v \in E_j} \theta(v) = N + 1\}$. Then $A_{N+1} \subset \mathcal{P}(E_1)$ for $N = 2, 3, \ldots$ and $A_{2N+1} \subset \mathcal{P}(E_2)$ for $N = 1, 2, \ldots$.  

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See Appendix A for the proof.

Our main goal for this section is to establish estimates for the following two forms,

\[ \Lambda_1(q_0, \ldots, q_N, t) = \int_{C^{N+1}} \frac{q_0(x_0) \cdots q_N(x_N) t(\zeta \cdot x)}{|x_0 - x_1| \cdots |x_{N-1} - x_N|} \, dx \]

and

\[ \Lambda_2(q_0, \ldots, q_{2N}, t) = \int_{C^{2N+1}} \frac{q_0(x_0) \cdots q_{2N}(x_{2N}) t(\zeta \cdot x)}{|x_0 - x_1| \cdots |x_0 - x_1 \cdots - x_{2N-1}| \cdots |x_{2N} - x_{2N-1} \cdots - x_1|} \, dx. \]

The next lemma gives several estimates for these forms that are the main step in obtaining estimates for the terms \( r_j \) in the expansion (1.12) of the scattering map.

**Lemma 3.5** Suppose \( 0 \leq \alpha < 1 \). The following estimates for the forms \( \Lambda_1 \) and \( \Lambda_2 \) hold:

\[ \Lambda_1(q_0, \ldots, q_{2N}, t) \leq C \|t\|_{H^0, -\alpha} \prod_{j=0}^{2N} \|q_j\|_{H^0, \alpha/(N+1)}, \quad N \geq 1 \]  

(3.6)

\[ \Lambda_1(q_0, \ldots, q_N, t) \leq C \|t\|_{L^2} \prod_{j=0}^{N} \|q_j\|_{L^2}, \quad N \geq 2 \]  

(3.7)

\[ \Lambda_2(q_0, \ldots, q_{2N}, t) \leq C \|t\|_{H^0, -\alpha} \prod_{j=0}^{2N} \|q_j\|_{H^0, \alpha/(N+1)}, \quad N \geq 1. \]  

(3.8)

**Proof.** We begin by proving the estimate (3.6) for the form \( \Lambda_1 \). We start with the elementary observation that for \( 0 \leq \alpha \leq 1 \), we have

\[ 1 \leq \frac{|x_0|^{\alpha} + \sum_{j=1}^N |x_{2j-1} - x_{2j}|^\alpha}{|x_0 - x_1 + \cdots + x_{2N}|^\alpha}. \]

To establish a relation between \( \Lambda_1 \) and \( \Lambda(f_v \mid v \in E_1) \), we define \( f^k_v \) for \( k = 0, \ldots, N \) and \( v \in E_1 \) by

\[ f^0_{e_0} = |\cdot|^\alpha |q_0| \]

\[ f^k_{e_j} = |q_j|, \quad (j, k) \neq (0, 0) \]

\[ f^k_{e_{2k-1}e_{2k}} = 1/|\cdot|^{1-\alpha}, \quad k = 1, \ldots, N \]

\[ f^k_{e_{j-1}e_j} = 1/|\cdot|, \quad j \neq 2k \]

\[ f^k_{\zeta} = t/|\cdot|^{\alpha}, \quad k = 0, \ldots, N. \]
With these definitions we have
\[
|\Lambda_1(q_0, \ldots, q_{2N}, t)| \leq \sum_{k=0}^{N} \Lambda(f^k_v | v \in E_1). \tag{3.9}
\]

Apply Theorem 3.1 to the first term in the sum on the right-hand side of (3.9) and observe that Lemma 3.4 tells us that the constant function \( \theta = 1/2 \) is in the interior of \( P(E_1) \). Now we are able to conclude that
\[
\Lambda(f^0_v | v \in E_1) \leq C \|t\|_{L^2} \|q_0\|_{L^2} \prod_{j=1}^{2N} \|q_j\|_{L^2}.
\]

Here we have used that \( \cdot |^{-1} \) is in the Lorentz space \( L^{2,\infty} \). For the terms \( \Lambda(f^k_v | v \in E_1), \ k \geq 1 \), we let \( 1/p_{e0} = (1 + \alpha)/2, \ 1/p_{e2k-1-e2k} = (1 - \alpha)/2 \) and use Theorem 3.1 to obtain that
\[
\Lambda(f^k_v | v \in E_1)
\leq C \|t\|_{L^2} \|q_0\|_{L^2} \prod_{j=1}^{2N} \|q_j\|_{L^2} \cdot \|\cdot |^{-1}\|_{L^{2,\infty}} \cdot \|\cdot |^{-1}\|_{L^{2,\infty}}^{2N-1}. \tag{3.10}
\]

The generalization of Hölder’s inequality to Lorentz spaces (which may be proven by multilinear interpolation) implies that
\[
\|q_0\|_{L^{2,1+\alpha,2}} \leq C \|q_0\|_{L^2} \|\cdot |^{-1}\|_{L^{2,\infty}} \cdot \|\cdot |^{-1}\|_{L^{2,\infty}}. \tag{3.11}
\]

Since \( \|\cdot |^{-1}\|_{L^{2,\infty}} \leq C_\alpha \), from (3.10) and (3.11), we may conclude that
\[
\Lambda(f^k_v | v \in E_1) \leq C \|t\|_{L^2} \|q_0\|_{L^2} \prod_{j=1}^{2N} \|q_j\|_{L^2}.
\]

A similar argument replacing 0 by another even index \( 2k \) implies that we have the estimates
\[
\Lambda_1(q_0, \ldots, q_{2N}, t) \leq C \|t\|_{L^2} \|q_{2k}\|_{L^2} \prod_{j \neq 2k} \|q_j\|_{L^2}, \quad k = 1, \ldots, N.
\]

Combining the cases \( \alpha = 0 \) and \( \alpha > 0 \) and multi-linear interpolation by the complex method gives the estimate (3.6). The estimate (3.7) follows directly from Theorem 3.1 and Lemma 3.4.

Finally, the estimate (3.8) can be obtained by beginning with the elementary estimate
\[
1 \leq \frac{|x_0 - x_1 + \cdots - x_{2k-1}|^\alpha + |x_{2k}|^\alpha + |x_{2k+1} - x_{2k+2} \cdots - x_{2N}|^\alpha}{|x_0 - x_1 + \cdots + x_{2N}|^\alpha}
\]
and arguing as in the case of (3.6).
We now turn to the estimates for the multi-linear expressions $r_j$ defined in (1.13). Using the estimates (3.6) and (3.8) we prove the following result.

**Proposition 3.12** If $\alpha \in [0, 1)$ and $r_j$ is as defined in (1.13), then

$$
\|r_j\|_{H^\alpha,0} \leq C_j \|q\|_{H^{\alpha/(j+1)},0}^{2j+1}, \tag{3.13}
$$

$$
\|r_j\|_{H^\alpha,\alpha} \leq C_j \|q\|_{H^{\alpha/(j+1),0},0}^{2j+1}. \tag{3.14}
$$

**Proof.** Using the definition of $r_j$, (1.13) and the theorem of Fubini, we have

$$
\int_{\mathbb{C}} r_j(k)t(k) \, dk = \frac{2}{(2\pi)^{2j+1}} \int_{C^{2j+1}} \frac{q(y_0)\bar{q}(y_1) \cdots q(y_{2j})\hat{t}(y_0 - y_1 + y_2 - \cdots + y_{2j})}{(y_0 - \bar{y}_1)(y_1 - y_2) \cdots (y_{2j} - y_1)} \, dy.
$$

The use of Fubini can be justified since we assume that $q$ and $t$ are in the Schwartz class. From the above displayed equation, it is easy to see that

$$
|\int_{\mathbb{C}} r_j(k)t(k) \, dk| \leq \frac{2}{(2\pi)^{2j+1}} \Lambda_1(|q|, \ldots, |q|, |\hat{t}|)
$$

where the form $\Lambda_1$ acts on $2j + 1$ copies of $|q|$. The estimate (3.6) and duality implies that for $0 \leq \alpha < 1$,

$$
\|D^\alpha r_j\|_{L^2} \leq C(\alpha, j)\|q\|_{H^{\alpha/(j+1),0}}^{2j+1}.
$$

Combining the cases $\alpha = 0$ and $\alpha > 0$ gives the estimate (3.13).

To obtain the decay of $r_j$ in estimate (3.14), we again start with the definition of $r_j$ in (1.13) and use (1.4) and (1.5) alternately to obtain

$$
\int_{\mathbb{C}} t(k)r_j(k) \, dk = \frac{2}{(2\pi)^{2j+1}}
\times \int_{C^{2j+1}} \frac{\hat{q}(k_0)\bar{q}(k_1) \cdots \hat{q}(k - k_0 + k_1 - \cdots - k_{2j-1})t(k)}{(k_0 - \bar{q}_1)(k_0 - k_1 + \cdots - k_{2j-1})(k_0 - k_1 + \cdots + k_{2j-2} - k)}
\, dk_0 \cdots dk_{2j-1} dk.
$$

We make the change of variables $k = k_0 - k_1 + \cdots + k_{2j}$, $dk = dk_{2j}$ and obtain

$$
\int_{\mathbb{C}} t(k)r_j(k) \, dk = \frac{2}{(2\pi)^{2j+1}}
\times \int_{C^{2j+1}} \frac{\hat{q}(k_0)\bar{q}(k_1) \cdots \hat{q}(k_{2j})t(k_0 - k_1 + \cdots + k_{2j})}{(k_0 - \bar{q}_1)(k_0 - k_1 + \cdots - k_{2j-1})(k_2 - k_{2j-1}) \cdots (k_{2j} - k_{2j-2} + \cdots + k_1)}
\, dk.
$$

On the right, $dk = dk_0 \, dk_1 \ldots dk_{2j}$. Now the estimate (3.8) quickly leads to the result (3.14). 

\hspace{1cm} ■
4 The remainder term

The final section gives estimates for the remainder term. The moral of this section is that when \( q \in H^{\varepsilon,\varepsilon} \) with \( \varepsilon > 0 \), then the operator \( T_k^N(\mu_1) \) becomes smoother and decays more rapidly as \( N \) increases. This allows us to estimate the remainder \( r^{(N)} \) (defined in [1.14]) for \( N \) large. The details are a bit tedious. We begin by listing several properties of the operator \( T_k \) that start to make the previous sentences precise.

**Proposition 4.1** Let \( q \in H^{\varepsilon,\varepsilon} \) with \( 0 < \varepsilon < 1 \) and suppose that \( \|q\|_{H^{\varepsilon,\varepsilon}} \leq M_0 \). If \( \mu_1 = \mu_1(q; \cdot, \cdot) \) is the solution constructed in Proposition 2.6, we have the following estimates for \( T_k^N(\mu_1) \). Assume that \( p \) and \( \tilde{p} \) are related by \( 1/p = 1/2 + 1/\tilde{p} \).

a) We have \( T_k^N(\mu_1) \in L^{\tilde{p}} \) if \( 1/\tilde{p} \in [0, \rho) \cap [0, \frac{1}{2}) \) and

\[
\|T_k^N(\mu_1)\|_{L^{\tilde{p}}} \leq C\|q\|_{H^{\varepsilon,\varepsilon}}\|\mu_1\|_{L^{\infty}}. \tag{4.2}
\]

b) For \( 1/\tilde{p} \in (0, 1/2) \) and \( j \geq 0 \), we have

\[
\|T_k^{N+2j}(\mu_1)\|_{L^{\tilde{p}}} \leq C\langle k \rangle^{-j}\|q\|_{H^{\varepsilon,\varepsilon}}^{2j}\|T_k^N(\mu_1)\|_{L^{\tilde{p}}}. \tag{4.3}
\]

c) Provided \( j \geq 0 \) satisfies \( j\varepsilon < 2/p' \), we have the estimate

\[
\|T_k^{N+j}(\mu_1)\|_{L^{\tilde{p}}_\varepsilon} \leq C\|q\|_{H^{\varepsilon,\varepsilon}}\|T_k^N(\mu_1)\|_{L^{\tilde{p}}}. \tag{4.4}
\]

If in addition, \( q' \in H^{\varepsilon,\varepsilon} \) with \( \|q'\|_{H^{\varepsilon,\varepsilon}} \leq M_0 \), we have the following estimates for differences. Given \( 1/\tilde{p} \in [0, 1/2) \), there exist \( N \) such that

a') \( \|T_k^N(\mu_1(q; \cdot, k)) - T_k^N(\mu_1(q'; \cdot, k))\|_{L^{\tilde{p}}} \leq C(M_0)\|q - q'\|_{H^{\varepsilon,\varepsilon}}. \)

b') For \( 1/\tilde{p} \) in \( (0, 1/2) \) and \( j \geq 0 \), we have

\[
\|T_k^{N+2j}(\mu_1(q; \cdot, k)) - T_k^{N+2j}(\mu_1(q'; \cdot, k))\|_{L^{\tilde{p}}} \leq C(M_0)\langle k \rangle^{-j}\|q - q'\|_{H^{\varepsilon,\varepsilon}}. \]

c') Given \( \tilde{p} \) in \( [2, \infty) \) and \( j \) and \( \varepsilon \) with \( 0 \leq j\varepsilon < 2/p' \), we have

\[
\|T_k^{N+j}(\mu_1(q; \cdot, k)) - T_k^{N+j}(\mu_1(q'; \cdot, k))\|_{L^{\tilde{p}}_\varepsilon} \leq C(M_0)\|q - q'\|_{H^{\varepsilon,\varepsilon}}. \]

**Proof.** Note that by Proposition 2.9, we have that \( \mu_1 \) lies in \( L^\infty(C^2) \). The first estimate is a consequence of the Hardy-Littlewood-Sobolev inequality (2.3), the estimate (2.2), Hölder’s inequality, and the observation (2.1) that \( H^{\varepsilon,\varepsilon} \subset L^p \), if \( 1/p \in [1/2 - \varepsilon/2, 1/2 + \varepsilon/2] \). Thus the maps \( f \to qf \to T_k(f) \) take \( L^{\tilde{p}} \to L^t \to L^{\tilde{p}_1} \) where \( 1/t \in [1/\tilde{p} + 1/2 - \varepsilon/2, 1/\tilde{p} + 1/2 + \varepsilon/2) \cap [0, 1/2) \) and \( 1/\tilde{p}_1 \in [0, 1/\tilde{p} + \varepsilon/2) \cap [0, 1/2) \), and

\[
\|T_k(f)\|_{L^{\tilde{p}_1}} \leq C\|q\|_{H^{\varepsilon,\varepsilon}}\|f\|_{L^{\tilde{p}}}. \]

Iterating this estimate gives a).
The estimate b) follows quickly from (2.5) and can be found in the work of Astala, Faraco, and Rogers [AFR15].

The third estimate (4.4) depends on the result (2.4) on fractional integration in weighted Lebesgue spaces. Using the estimate (2.4) and Hölder’s inequality, the maps $f \rightarrow qf \rightarrow T_k(f)$ will map $L^p_\alpha \rightarrow L^p_{\alpha+\epsilon} \rightarrow L^p_{\alpha+\epsilon}$ where as usual $1/p = 1/\tilde p + 1/2$ and the second step requires the condition that $-2/\tilde p < \alpha + \epsilon < 2/p'$ in order to use our result on fractional integration (2.4).

The estimates for the differences follow by recognizing that each term is a multi-linear expression in several copies of $q$ and $\mu_1$, the continuity of $\mu_1$ with respect to $q$ given in Proposition 2.6 and the result of Lemma 2.8.

We begin by showing that if $q \in H^{\epsilon,\epsilon}$ then given $\alpha$ we may choose $N = N(\alpha)$ so that $r^{(N)}$ lies in $H^{0,\alpha}$.

**Lemma 4.5** Let $q \in H^{\epsilon,\epsilon}$ with $0 < \epsilon < 1$. Given $\alpha \in \mathbb{R}$, there exist $N$ such that $r^{(N)}$ lies in $H^{0,\alpha}$. If $q$ and $q'$ lie in $\{q \in H^{\epsilon,\epsilon} : \|q\|_{H^{\epsilon,\epsilon}} \leq M_0\}$, then

$$\|r^{(N)}\|_{H^{0,\alpha}} \leq C(M_0, \alpha)$$

$$\|r^{(N)}(q; \cdot) - r^{(N)}(q'; \cdot)\|_{H^{0,\alpha}} \leq C(M_0, \alpha)\|q - q'\|_{H^{\epsilon,\epsilon}}.$$  

**Proof.** This is straightforward and follows an argument in Perry [Per]. Since $q \in H^{\epsilon,\epsilon}$, according to (2.1) we have $q \in L^p$ with $p$ defined by $1/p = 1/2 + \epsilon/4$, say. We let $p'$ be the conjugate exponent as usual. According to parts a) and b) of Proposition 4.1, we may choose $N$ so that

$$\|T_k^{2N}(\mu_1)\|_{L^{p'}} \leq C(M_0, \alpha)\langle k \rangle^{-2-\alpha}.$$

Thus by Hölder’s inequality, we have

$$|r^{(N)}(k)| \leq \frac{1}{\pi} \int_C |e_k(x)q(x)T_k^{2N}(\mu_1)| \, dx \leq \frac{1}{\pi} \|q\|_{L^p} \|T_k^{2N}(\mu_1)\|_{L^{p'}} \leq C\langle k \rangle^{-2-\alpha}.$$

This gives the first conclusion (4.6).

To estimate the differences, we observe that $r^{(N)}$ is a multi-linear operator in $q, \bar q$ and $\mu_1$ and use Lemma 2.8 and Proposition 2.6 to estimate $\mu_1(q; \cdot) - \mu_1(q'; \cdot)$ in terms of $q - q'$. With these observations the continuity (4.7) follows from Lemma 2.8.

We employ a similar strategy to estimate $\|r\|_{H^{0,0}}$. As part of this we will need a lemma to show that $r^{(N)}$ is smooth. The proof of this result is more involved.

To estimate $r^{(N)}(k)$, we begin by computing

$$\frac{\partial}{\partial k} r^{(N)}(k) = -\frac{1}{\pi} \int_C yq(y)e_k(y)T_k^{2N}(\mu_1)(y) \, dy + \frac{1}{\pi} \int_C q(y)e_k(y) \frac{\partial}{\partial k} T_k^{2N}(\mu_1)(y) \, dy.$$
Thus we need to study the expressions $\frac{\partial}{\partial \bar{k}} T^2 N_k(\mu_1)$ and $\bar{y} T^2 N_k(\mu_1)$.

Our first step is to derive an expression for the $\partial/\partial \bar{k}$-derivative of the function $T^2 N_k(\mu_1)$. This generalizes the $\partial/\partial k$ equation for $\mu_1$. For this exercise, we assume that $q$ is in the Schwartz space, which will allow us to concentrate on the formulae rather than convergence.

**Lemma 4.8** If $q$ is in $\mathcal{S}(\mathbb{C})$, then

$$\frac{\partial}{\partial \bar{k}} T^2 N_k(\mu_1) = \frac{1}{2} \bar{r}(k) T^2 N_k(\mu_2) + \frac{1}{2\pi} \sum_{j=1}^{N} T_{kj}^{-1}(1) \int_{\mathbb{C}} \bar{q}(x)e_k(-x)T^{2N-2j}(\mu_1)(x) dx.$$  

**Proof.** Taking the derivative with respect to $\bar{k}$ gives

$$\frac{\partial}{\partial \bar{k}} T^2 N_k(\mu_1)(x) = T^2 N_k(\frac{\partial \mu_1}{\partial \bar{k}})(x) + \frac{1}{2\pi} \sum_{j=1}^{N} T_{kj}^{-1}(1)(x) \cdot \int_{\mathbb{C}} \bar{q}(x_{2j})e_k(-x_{2j})T^{2N-2j}(\mu_1)(x) dx_{2j}.$$  

If we recall the $\partial/\partial \bar{k}$-equation for $\mu$ (see Perry [Per], for example), $\frac{\partial}{\partial \bar{k}} \mu_1 = \frac{1}{2} \bar{r} \mu_2$, we obtain the Lemma.

**Lemma 4.9** If $q \in \mathcal{S}(\mathbb{C})$ and $1 \leq n \leq N$, then

$$\bar{x} T^N_k(f)(x) = \eta(x) \sum_{j=1}^{n} \bar{T}^{-j}_{kj}(1)(x) \left( \frac{1}{2\pi} \int_{\mathbb{C}} q(y)e_k(y)T^{N-j}_{kj}(f)(y) dy \right)^{s_{j-1}} + \bar{q} T^{-n}_{kj}(\bar{\cdot})T^{N-n}(f)(x).$$  

Here $\eta(x) = \bar{x}/x$ and $\bar{T} = T_{k,\bar{q}}$ and $\bar{q} = \eta q$. We use $x^{s_{j}}$ to denote $j$ applications of the map $x \rightarrow \bar{x}$. Thus $x^{s_{j}} = x$ if $j$ is even and $x^{s_{j}} = \bar{x}$ if $j$ is odd.

**Proof.** We begin with the identity,

$$\bar{x} T_k(f)(x) = \frac{\bar{x}}{x} \frac{1}{2\pi} \int_{\mathbb{C}} q(y)e_k(y)\bar{f}(y) dy + \frac{\bar{x}}{x} \frac{1}{2\pi} \int_{\mathbb{C}} \frac{q(y)e_k(y)y\bar{f}(y)}{x - y} dy$$

$$= \frac{\bar{x}}{x} \frac{1}{2\pi} \int_{\mathbb{C}} q(y)e_k(y)\bar{f}(y) dy + \frac{\bar{x}}{x} \bar{T}_{k}(\bar{\cdot})f(x).$$  

Iterating this result gives the Lemma.
We are ready to give an estimate on the smoothness of the function $r^{(N)}$.

**Lemma 4.10** Let $q \in H^{\epsilon, \kappa}$ with $\|q\|_{H^{\epsilon, \kappa}} \leq M_0$. Then there exist $N = N(\epsilon)$ such that

$$\| \frac{\partial}{\partial k} r^{(N)} \|_{L^2} \leq C(M_0).$$

If $q, q'$ are both in $\{ q : \|q\|_{H^{\epsilon, \kappa}} \leq M_0 \}$, then

$$\| \frac{\partial}{\partial k} r^{(N)}(q; \cdot) - \frac{\partial}{\partial k} r^{(N)}(q'; \cdot) \|_{L^2} \leq C(M_0, \epsilon) \| q - q' \|_{H^{\epsilon, \kappa}}.$$

**Proof.** We differentiate $r^{(N)}$ and obtain

$$\frac{\partial}{\partial k} r^{(N)}(k) = \frac{1}{\pi} \int \frac{q(y) e_k(y) T_{k}^{2N}(\mu_1)(y)}{y^2} \; dy + \frac{1}{\pi} \int \frac{q(y) e_k(y) \frac{\partial}{\partial k} T_{k}^{2N}(\mu_1)(y)}{y} \; dy$$

$$= \sum_{j=1}^{N} I_j + II + \sum_{j=1}^{N} III_j + IV$$

with

$$I_j = -\frac{1}{\pi} \int q(y) \tilde{\eta}(y) e_k(y) (\tilde{T}_k^{j-1}(1)(y))^* \; dy$$

$$\times (\frac{1}{2\pi} \int q(y) e_k(y) T_k^{2N-j}(\mu_1)(y) \; dy)^j = A_j B_j$$

$$II = -\frac{1}{\pi} \int q(y) \tilde{\eta}(y) e_k(y) (\tilde{T}_k^{N}(\cdot) T_k^{N}(\mu_1)) (y) \; dy.$$ 

$$III_j = \frac{1}{2\pi^2} \int q(y) e_k(y) (T_k^{2j-1}(1)(y))^* \; dy \cdot \int q(y) e_k(y) T_k^{2N-2j}(\mu_1)(y) \; dy = C_j D_j.$$ 

Finally, we define $IV$ by

$$IV = \frac{1}{2\pi^2} r(k) \int q(y) e_k(y) (T_k^{2N}(\mu_2)(y))^* \; dy.$$

To obtain this representation we use Lemma 4.8 and then Lemma 4.9 to commute $y$ through $N$ applications of $T_k$.

We proceed to show that each of the terms $I_j, II, III_j, IV$ are in $L^2$. To estimate the terms $I_j = A_j \cdot B_j$, we first suppose that $j \geq 3$. Then estimate (3.6) or (3.7) gives that $A_j$ is in $L^2$. Since $2N - j \geq N$, we may choose $N$ large so that part a) of Proposition 4.1 gives $T_k^{2N-j}(\mu_1) \in L^{p'}$ for $1/p' = 1/2 - \epsilon/4$, say. Since we also have $q \in L^p$, we conclude that $B_j$ is in $L^\infty$. For the case $j = 2$, we use Lemma 4.12 to conclude that $A_2$ is in $L^p$ for some $p > 2$ while by (2.5), we may choose $N = N(\epsilon)$ so that $|B_2(k)| \leq \langle k \rangle^{-2}$ and thus the product $A_2 \cdot B_2$ is in $L^2$. Finally, $A_1 = \frac{1}{2} (q \tilde{\eta})^*$ and hence lies in $L^2$ by Plancherel's theorem.
For the term $II$, we begin by using (2.5) and Proposition 2.9 to conclude that for $N$ large and $1/\tilde{p} = \varepsilon/4$, say,

$$\|\langle \cdot \rangle^{1-\varepsilon} T_k^N(\mu_1)\|_{L^p} \leq C(\|q\|_{L^{s\cdot}}) \langle k \rangle^{-2}.$$ 

Then since $\langle \cdot \rangle^\varepsilon q \in L^2$, we have $\|\langle \cdot \rangle^\varepsilon T_k^N(\mu_1)\|_{L^p} \leq C(\|q\|_{L^{s\cdot}}) \langle k \rangle^{-2}$ for $1/p = 1/2 + \varepsilon/4$. Now applying $\tilde{T}_k^N$ (and possibly choosing $N$ larger) we have $\|\tilde{T}_k^N(\langle \cdot \rangle^\varepsilon T_k^N(\mu_1))\|_{L^{\tilde{p}1}} \leq C(\|q\|_{L^{s\cdot}}) \langle k \rangle^{-2}$ for $1/\tilde{p}1 = 1/2 + \varepsilon/4$. Then Hölder’s inequality gives the estimate $|II(k)| \leq \langle k \rangle^{-2}$ and hence that $II$ lies in $L^2$. A similar strategy handles the term $III_j = C_j \cdot D_j$ for $2j \leq N$. Here $C_j$ is in $L^{2}$ by (3.6) as long as $2j \geq 4$. The estimate (4.2), estimate (2.1), and Hölder’s inequality imply that $D_j$ is bounded. When $2j = 2$, we use Lemma 4.12 and (2.5) as in the estimate for $I_2$.

Next we consider $III_j$ when $2j \geq N$. Here we argue as in Lemma 4.5 to conclude that $D_j$ is in $L^2$ and since $2j \geq N$, we may choose $N$ large so that (4.2) and Hölder’s inequality imply $C_j$ is bounded.

Finally, we show the term $IV$ is in $L^2$. We first recall that Astala, Faraco, and Rogers [AFR15, Theorem 3.4] show that the map $q \rightarrow R(q)$ maps $H^{s\cdot}$ to $L^2$ and is locally Lipschitz continuous. With this result, we only need to show that the map

$$q \rightarrow \int_{C} q(y)e_k(y)(T_k^{2N}(\mu_2))(y)^* dy$$

(4.11)

takes $H^{s\cdot}$ into $L^\infty$. From the estimate (2.1), we have that $q \in L^p$ for some $p > 2$ while part a) of Proposition 4.1 and (1.7) imply that for $N$ large, we have $T_k^{2N}(\mu_2)$ in the dual space, $L^{p'}$. It follows that $\int_{C} q(y)e_k(y)(T_k^{2N}(\mu_2))(y)^* dy$ lies in $L^\infty$.

To complete the proof, we need to show that each of these terms is a locally Lipschitz continuous function of $q$. This follows from similar arguments using Proposition 4.1 a’-c’) and Lemma 2.8.

Lemma 4.12 If $q \in L^{s_1}$ with $4/3 < s_2 < 2$, then $r(k) = \int_{C} q(y)e_k(y)T_k(1)(y) dy$ lies in $L^{s_r}$ where $1/s_r = 3/2 - 2/s_q$ and we have the estimate

$$\|r\|_{L^{s_r}} \leq C\|q\|_{L^{s_q}}^2.$$ 

Proof. We will estimate $r$ by duality and thus we choose $t$ a nice function in $L^{s_2'}$. We write

$$\int_{C} t(k)r(k) dk = \frac{1}{2\pi^2} \int_{C} \tilde{g}(z)q(y)\tilde{t}(y-z) dy dz.$$ (4.13)

We may use the Hausdorff-Young inequality and the interpolation result of Christ [Chr85] or Janson [Jan86] to find that the right-hand side of (4.13) is finite when $2/s_q + 1/s_r = 3/2$. Note that the condition on the second index in the Lorentz spaces will always hold true since $2/s_q \geq 2$. 


We are ready to give the proof of Theorem 1.10.

**Proof of Theorem 1.10.** We write \( r(k) = R(q)(k) = \sum_{j=0}^N r_j(k) + r^{(N)}(k) \) where \( r_j \) and \( r^{(N)} \) are as in (1.13) and (1.14). According to Proposition 3.12 we have \( r_j \) is in \( H^{\beta,\alpha} \) if \( q \) is in \( H^{\alpha,\beta} \). By Lemma 4.5 and Lemma 4.10, given \( \epsilon > 0 \), we may find \( N = N(\epsilon) \) so that \( r^{(N)} \) is in \( H^{\beta,\alpha} \) if \( q \) is in \( H^{\epsilon,\epsilon} \).

To estimate \( r - \hat{q} \), we note that \( r_0 = \hat{q} \) and then use Lemma 3.1 to conclude that \( r_j \in H^{2\beta,2\alpha} \) when \( q \in H^{\alpha,\beta} \) and \( j \geq 1 \). Finally, we use Lemma 4.5 and Lemma 4.10 to conclude that the remainder \( r^{(N)} \) lies in \( H^{2\beta,2\alpha} \). This gives the second estimate of the Theorem.

### Appendix: The matroid polytope

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We give the proof of Lemma 3.4 in this section. We let \( E \) denote one of the matroids \( E_1 \) or \( E_2 \) defined in section 3. Our strategy is to show that for any pair of vectors \((v,w)\) from \( E \), we may find two bases \( B_1 \) and \( B_2 \) so that \( \{v\} = B_1 \cap B_2 \) and \( \{w\} = E \setminus (B_1 \cup B_2) \). Since \( \chi_{B_1} \in \mathcal{P}(E) \) and \( \mathcal{P}(E) \) is convex, it follows that \( \Phi_{v,w} = \frac{1}{2}(\chi_{B_1} + \chi_{B_2}) \) lies in \( \mathcal{P}(E) \). We have that

\[
\Phi_{v,w}(u) = \begin{cases} 
1, & u = v \\
0, & u = w \\
1/2, & \text{else.}
\end{cases}
\]

It is not hard to show that \( A \), the set in Lemma 3.4, is the convex hull of \( \Phi_{v,w} \) for all pairs \((v,w)\) from \( E \).

We first consider the matroid \( E_1 \) with \( N \geq 2 \).

**Lemma A.1** Consider a pair \((v,w)\) with elements from \( E_1 \). We may find two bases \( B_1 \) and \( B_2 \) of \( \mathbb{R}^{N+1} \) so that \( B_1 \cap B_2 = \{v\} \) and \( E_1 \setminus (B_1 \cup B_2) = \{w\} \).

**Proof.** Our proof is not particularly clever. We consider a number of cases and list the bases in each case. In several of the cases, we will make use of the map \( e_k \to e_{N-k} \) which preserves the elements of \( E_1 \), at least up to a sign. This will allow us to assume that the vectors satisfy certain extra conditions. We let \( S = \{e_0, e_1, \ldots, e_N\} \), \( D = \{e_0 - e_1, \ldots, e_{N-1} - e_N\} \), and \( \zeta = e_0 - e_1 + \cdots + (-1)^N e_N \).

**Case 1:** \( v = e_k \) and \( w = e_\ell \). In this case we may let \( B_1 = (S \setminus \{e_\ell\}) \cup \{\zeta\} \) and \( B_2 = D \cup \{e_k\} \).
Lemma A.2

Consider a pair of vectors $(v, w)$ from $E_2$. We may find a pair of bases $B_j$, $j = 1, 2$ so that $B_1 \cap B_2 = \{v\}$ and $(E_2 \setminus B_1) \cap (E_2 \setminus B_2) = \{w\}$.

Proof. Again the proof is by cases. We let $S = \{e_0, e_1, e_2, \ldots e_{2N}\}$, $D = \{e_0 - e_1, \ldots, e_{2N-1} - e_{2N}, e_{2N} - e_{2N-1}, \ldots, e_{2N} - e_{2N-1} - e_{1}\}$, and set $\zeta = e_0 - e_1 + \cdots + e_{2N}$.

Case 1: $v = e_j$ and $w = e_k$. Let $B_1 = (S \setminus \{e_k\}) \cup \{\zeta\}$ and $B_2 = D \cup \{e_j\}$.

Case 2: $v = e_j$ and $w = e_k$. Let $B_1 = (S \setminus \{e_j\}) \cup \{\zeta\}$ and $B_2 = D \cup \{\zeta\}$.

Case 3: $v = e_j$ and $w = e_k$. Let $B_1 = (S \setminus \{e_j\}) \cup \{\zeta\}$ and $B_2 = D \cup \{\zeta\}$.

Case 4: $v = e_j$ and $w = d \in D$. If necessary, we may apply the transformation, $e_k \rightarrow e_{2N-k}$ so that the vector $d$ has the form $d = e_0 - e_1 + \cdots - e_{k}$. Now we choose a vector $e_{\ell}$ which depends on $e_j$ with $\ell$ defined as follows:

$$e_{\ell} = \begin{cases} e_{2N}, & j = 0, \\ e_0, & \text{else}. \end{cases}$$

Then we let the bases be $B_1 = (S \setminus \{e_{\ell}\}) \cup \{\zeta\}$ and $B_2 = (D \setminus \{d\}) \cup \{e_j, e_{\ell}\}$.
Case 5: \( v = d \in D \) and \( w = e_j \). We may assume that \( d = e_0 - e_1 + \cdots - e_k \), as above. We choose a vector \( e_\ell \) to be \( e_\ell = e_{2N} \) if \( j \leq k \) and \( e_\ell = e_0 \) if \( j > k \). Then we put \( B_1 = (S \setminus \{e_j, e_\ell\}) \cup \{\zeta, d\} \) and \( B_2 = D \cup \{e_\ell\} \).

Case 6: \( v = d_1 \in D \) and \( w = d_2 \in D \). If necessary, we may apply the transformation, \( e_j \rightarrow e_{2N-j} \) and assume that \( d_2 = e_0 - e_1 + \cdots - e_k \). We put \( B_1 = (S \setminus \{e_0, e_{2N}\}) \cup \{\zeta, d_1\} \) and \( B_2 = (D \setminus \{d_2\}) \cup \{e_0, e_{2N}\} \).

Case 7: \( v = d \in D \) and \( w = \zeta \). By flipping the vectors, we may assume that \( d = e_0 - \cdots - e_k \). We put \( B_1 = (S \setminus \{e_k\}) \cup \{d\} \) and \( B_2 = D \cup \{e_k\} \).

Case 8: \( v = \zeta \) and \( w = d \in D \). Again, we may assume that \( d = e_0 - e_1 + \cdots - e_k \). We let \( B_1 = (S \setminus \{e_k\}) \cup \{\zeta\} \) and \( B_2 = (D \setminus \{d\}) \cup \{\zeta\} \cup \{e_k\} \).

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