EXISTENCE AND ASYMPTOTICAL BEHAVIOR OF POSITIVE SOLUTIONS FOR THE SCHRÖDINGER-POISSON SYSTEM WITH DOUBLE QUASI-LINEAR TERMS

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Abstract. In this paper, we consider the following Schrödinger-Poisson system with double quasi-linear terms
\begin{align*}
-\Delta u + V(x)u + \varphi u - \frac{1}{2}u\Delta u^2 &= \lambda f(x, u), &\text{in } \mathbb{R}^3, \\
-\Delta \varphi - \varepsilon^4 \Delta_4 \varphi &= u^2, &\text{in } \mathbb{R}^3,
\end{align*}
where \(\lambda, \varepsilon > 0\) are parameters. Under suitable assumptions on \(V\) and \(f\), we prove that the above system admits at least one pair of positive solutions for \(\lambda\) large by using perturbation method and truncation technique. Furthermore, we research the asymptotical behavior of solutions with respect to the parameters \(\lambda\) and \(\varepsilon\) respectively. These results extend and improve some existing results in the literature.

1. Introduction. In this article, we are devoted to studying the following Schrödinger-Poisson system
\begin{align}
-\Delta u + V(x)u + \varphi u - \frac{1}{2}u\Delta u^2 &= \lambda f(x, u), &\text{in } \mathbb{R}^3, \\
-\Delta \varphi - \varepsilon^4 \Delta_4 \varphi &= u^2, &\text{in } \mathbb{R}^3,
\end{align}
where \(\lambda, \varepsilon > 0\) are parameters, \(V\) and \(f\) satisfy the following assumptions respectively
\((V)\) \(V \in C(\mathbb{R}^3, \mathbb{R}), V_0 := \inf_{x \in \mathbb{R}^3} V(x) > 0\) and \(\lim_{|x| \to \infty} V(x) = \infty;\)
\((f_1)\) \(f(x, s) \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})\) and \(f(x, s) \equiv 0\) for \(x \in \mathbb{R}^3, s \leq 0;\)
\((f_2)\) there exists \(p \in (4, 12)\) such that \(\lim_{s \to 0} \frac{f(x, s)}{s} = \lim_{s \to \infty} \frac{f(x, s)}{s^p} = 0\) for all \(x \in \mathbb{R}^3;\)
\((f_3)\) there exists \(\nu > 4\) such that
\(0 < F(x, s) := \int_0^s f(x, t)dt \leq \frac{1}{\nu}sf(x, s)\) for all \(x \in \mathbb{R}^3\) and \(s > 0;\)
\((f_4)\) there exists \(\theta > 4\) such that
\(f(x, s) \geq s^{\theta-1}\) for all \(x \in \mathbb{R}^3\) and \(s > 0.\)

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We establish the existence and asymptotical behavior of the positive solutions by using variational methods, perturbation scheme and truncation technique.

In the last few years, many mathematicians are concerned with the following quasi-linear Schrödinger-Poisson system

\[
\begin{aligned}
-\Delta u + V(x)u + K(x)\phi u &= a(x)f(x,u), & \text{in } \mathbb{R}^3, \\
-\Delta \phi - \varepsilon^4 \Delta_4 \phi &= K(x)u^2, & \text{in } \mathbb{R}^3,
\end{aligned}
\]  

(2)

which appears when studying a quantum mechanical model of extremely small devices in semiconductor nanostructures as considering quantum structure and the longitudinal filed oscillations during the beam propagation. To the best of our knowledge, System (2) is a variant form of the following more general Schrödinger-Poisson system

\[
\begin{aligned}
i\partial_t u &= -\frac{1}{2}\Delta u + (V + \phi(x))u, & \text{in } \mathbb{R}^3, \\
-\text{div}[\varepsilon |\nabla \phi| \nabla \phi] &= |u|^2 - n^*, & \text{in } \mathbb{R}^3, \\
u(x,0) &= u(x), & \text{in } \mathbb{R}^3,
\end{aligned}
\]  

(3)

where \( \varepsilon \geq 0, V \) denotes a real effective potential function which does not depend on time \( t \), \( n^* \) represents a dopant-density and the charge density \( n(x,t) \) is arising from the Schrödinger wave function \( u(x,t) \) by \( n(x,t) = |u(x,u)|^2 \). This system corresponds to a quantum mechanical model where the quantum has an important effect. For more physical background about this system, readers can refer to [24, 25, 32] and the references within.

When \( \varepsilon = 0 \) in (2), it becomes the well-known Schrödinger-Poisson system

\[
\begin{aligned}
-\Delta u + V(x)u + K(x)\phi u &= a(x)f(x,u), & \text{in } \mathbb{R}^3, \\
-\Delta \phi &= K(x)u^2, & \text{in } \mathbb{R}^3,
\end{aligned}
\]  

(4)

which has a strong physical meaning because of the influence in quantum mechanics models and in semiconductor theory (see e.g. [6, 7, 8, 9, 11, 22]). From a physical point of view, it describes system of identical charged particles interacting each other in the case that magnetic influence could be ignored and its solution is a standing wave for such a system. System (4) was firstly introduced by Benci and Fortunato [6] as a model which describes standing waves interacting with an unknown electrostatic field. There are a large amount of works have been devoted to the existence of solutions for (4) under distinct assumptions on the potential function \( V \) and the nonlinearity term \( f \) by using Variational Methods (see [1, 34, 40]). Let us recall some rough results of this study. If \( V(x) = K(x) = a(x) = 1, \) D’Aprile and Mugnai [3] obtained the existence of radially symmetric solitary waves of (4) with \( f(x,u) = |u|^{p-1}u \) for \( 3 \leq p < 5 \). Cerami and Vaira [12] obtained the existence of positive solutions for (4) where \( f(x,u) = a(x)|u|^{p-1}u \) without any symmetry assumptions on the potential \( a(x) \). In [35], by using the concentration compactness principle and Nehari manifold method, Ruiz proved some existence and non-existence results of positive solutions for the following system

\[
\begin{aligned}
-\Delta u + u + \lambda \phi u &= |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\
-\Delta \phi &= 4\pi u^2, & \text{in } \mathbb{R}^3,
\end{aligned}
\]  

(5)

where \( \lambda > 0 \) and \( 2 < p < 6 \). Azzollini and Pomponio [4] studied the existence of a ground state solution of (4) for \( f(x,u) = |u|^{p-2}u \) with \( 3 < p < 6 \) when \( V \) is a positive constant or the case \( 4 < p < 6 \) when \( V \) is non-constant. In [41], Zhao and
Zhao considered the existence and multiplicity of solutions for sublinear problem of (4) with $f(x,u) = |u|^{p-1}u$ and $2 < p \leq 3$ by applying Variational Methods. Later, in [42], they extended the existing results to the critical case and proved the following problem

$$
\begin{align*}
-\Delta u + u &= \mu |u|^{p-2}u + u^5, \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= u^2, \quad \text{in } \mathbb{R}^3
\end{align*}
$$

admits one positive radial solution for all $\mu > 0$ if $4 < p < 6$ and for $\mu$ large enough if $2 < p \leq 4$. After that, combining the constraint variational method with the Brouwer degree theory, Wang and Zhou in [39] obtained one least energy sign-changing solution of (4) with $f(x,u) = |u|^{p-2}u$ and $p \in (4,6)$. With the help of the method of invariant sets of descending flow introduced in [30], Liu, Wang and Zhang [31] proved (4) possesses the infinitely many sign-changing solutions. For more existing results of this system, readers can refer to ([2, 5, 12, 13, 14, 18, 19, 23, 27, 36, 37, 38]) and the references therein.

Another interesting point which attracts many researchers in last several years is the quasi-linear (also called modified) Schrödinger-Poisson system. There have been some existing results of positive solutions and sign-changing solutions of this type equation. For instance, Nie and Wu in [33] obtained the two non-trivial solutions for the following system

$$
\begin{align*}
-\Delta u + V(x)u + \phi u - \frac{1}{2}u\Delta u^2 &= |u|^{p-1}u + h(x), \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= u^2, \quad \text{in } \mathbb{R}^3
\end{align*}
$$

In [19], by using a perturbation method introduced in [29], Feng and Zhang proved the existence of non-trivial solutions with general nonlinearity

$$
\begin{align*}
-\Delta u + V(x)u + \phi u - \frac{1}{2}u\Delta u^2 &= f(x,u), \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= u^2, \quad \text{in } \mathbb{R}^3
\end{align*}
$$

After that, Chen, Tang and Gao [13] obtained positive solutions, negative solutions and a sequence of high energy solutions for quasi-linear Schrödinger-Kirchhoff-Poisson type system.

Alternatively, when $\varepsilon \neq 0$ in (2), this kind of system has received more and more attention from mathematical community. In [17], Ding et al. studied the existence and asymptotical behavior of solutions for (2) when possessing an asymptotically linear nonlinearity. Later, Li and Yang [26] researched the existence and uniqueness of a globally mild solution to the initial boundary value problem under the one-dimension case. Figueiredo and Siciliano [21] were concerned with the following problem in the two-dimensional case

$$
\begin{align*}
-\Delta u + \phi u &= f(u), \quad \text{in } \Omega, \\
-\Delta \phi - \varepsilon^2 \Delta_4 \phi &= u^2, \quad \text{in } \Omega, \\
u = \phi &= 0, \quad \text{on } \partial \Omega,
\end{align*}
$$

where $f$ is assumed to be an exponential critical nonlinearity, they obtained the existence of the solutions as well as the asymptotical behavior with respect to the parameter $\varepsilon$. Very recently, Figueiredo and Siciliano in [20] extended the bounded domain case to $\mathbb{R}^3$ and proved the existence of solutions for the following quasi-linear
Schrödinger-Poisson system

\[
\begin{cases}
-\Delta u + u + \phi u = \lambda f(x, u) + |u|^4 u, & \text{in } \mathbb{R}^3, \\
-\Delta \phi - \varepsilon^2 \Delta_4 \phi = u^2, & \text{in } \mathbb{R}^3.
\end{cases}
\]

(10)

It is worth pointing that these aforementioned papers considered the existence of solutions for the Schrödinger-Poisson system with either \( u\Delta u^2 \) or \( \Delta_4 \phi \). In that direction, it is very natural for us to pose the following question: Can we find solutions for the Schrödinger-Poisson system with double quasi-linear terms?

Motivated by above works, we will give an affirmative answer in this paper. However, if we apply the standard critical points theory to study Schrödinger-Poisson system (1), we have to manage with several difficulties. Firstly, on account of the existence of the term \( u\Delta u^2 \), it is not easy to find an appropriate working space in which the Euler-Lagrange functional corresponding to (1) possesses both smoothness and compactness properties. There are several methods dealing with these problems. One is dual method introduced in [15], which cannot be used here since (1) possesses the non-local term. Another is variable substitution, however, it is invalid for our problem due to the influence of the second equation in (1). Hence we need to give more delicate analysis when dealing with truncating process. To the best of our knowledge, it seems that there are no results concerning the Schrödinger-Poisson system with double quasi-linear terms?

Now, we are in the position to state our main results. The definitions of the spaces \( H_V(\mathbb{R}^3) \), \( D^{1,2}(\mathbb{R}^3) \) and \( X \) will be given below.

**Theorem 1.1.** Assume that (V) and \((f_1) - (f_4)\) hold. Then there exists \( \Lambda > 0 \) such that for all \( \lambda > \Lambda \), Problem (1) admits at least a pair of positive solutions \((u_{\lambda,\varepsilon}, \phi_{\lambda,\varepsilon})\) in \( H_V(\mathbb{R}^3) \times X \). Furthermore, for every \( \varepsilon > 0 \), the following limits hold:

(i): \( \lim_{\lambda \to \infty} \| u_{\lambda,\varepsilon} \|_V = 0 \),

(ii): \( \lim_{\lambda \to \infty} \| \phi_{\lambda,\varepsilon} \|_X = 0 \),

(iii): \( \lim_{\lambda \to \infty} | \phi_{\lambda,\varepsilon} |_\infty = 0 \).

**Theorem 1.2.** Assume that (V) and \((f_1) - (f_4)\) hold. Let \( \tilde{\lambda} \geq \Lambda \) be fixed in (1) and \((u_{\tilde{\lambda},\varepsilon}, \phi_{\tilde{\lambda},\varepsilon})\) be the pair of positive solutions given by Theorem 1.1. Then

(i): \( \lim_{\varepsilon \to 0^+} u_{\tilde{\lambda},\varepsilon} = u_{\tilde{\lambda},0} \) in \( H_V(\mathbb{R}^3) \),

(ii): \( \lim_{\varepsilon \to 0^+} \phi_{\tilde{\lambda},\varepsilon} = \phi_{\tilde{\lambda},0} \) in \( D^{1,2}(\mathbb{R}^3) \),

where \((u_{\tilde{\lambda},0}, \phi_{\tilde{\lambda},0}) \in H_V(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\) is a pair of positive solutions of the following Schrödinger-Poisson system

\[
\begin{cases}
-\Delta u + V(x)u + \phi u - \frac{1}{2} u\Delta u^2 = \lambda f(x, u), & \text{in } \mathbb{R}^3, \\
-\Delta \phi = u^2, & \text{in } \mathbb{R}^3.
\end{cases}
\]

(11)
Remark 1. Theorem 1.1 and Theorem 1.2 complement the previous results on Schrödinger-Poisson system in the existing literature. Compared with [19] and [20], our problem is more complicated on account of the interactive influence of the two quasi-linear terms. It is noteworthy that there exist two gaps for $p \in (2, 4]$ and $p = 12$, we shall take into consideration the two cases in our future work. Obviously, one of the two problems is more difficult due to the lack of compactness.

Below we give a sketch of the proofs of our main results:

1) Firstly, we define a perturbation functional of class $C^1$ by adding a 4-Laplacian operator (see (15)), then we can prove the perturbation functional satisfies the Mountain Pass geometry. By Mountain Pass Theorem without $(PS)_c$ condition, we can obtain a $(PS)_c$ sequence.

2) Furthermore, to prove the boundedness of the $(PS)_c$ sequence, we adopt the truncation scheme to deal with the “bad” part (see (21)). Under suitable assumptions, we prove the perturbation functional satisfies the $(PS)_c$ condition. Thus the existence of the nontrivial critical point of the perturbation functional is proved.

3) Finally, by using Moser iteration method and $L^\infty$-estimation, we prove that the critical point of the perturbation functional is actually the critical point of the original functional, which implies the existence of the nontrivial solution of Problem (1). Besides, we consider the asymptotical behavior of the nontrivial solution with respect to the parameters $\lambda$ and $\varepsilon$.

The rest of this paper is organized as follows. In section 2, we show some technical lemmas and introduce the perturbation functional. In section 3, by using the truncation method, we prove the existence of nontrivial critical points of the perturbation functional. Section 4 is devoted to proving Theorem 1.1 by applying Moser iteration technique. The proof of Theorem 1.2 will be finished in section 5. Throughout this paper, we make use of the following notations:

- $H^1(\mathbb{R}^3)$ denotes the usual Sobolev space endowed with the scalar product and norm given by
  \[
  (u, v) = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv) dx, \quad ||u||_{H^1} = (u, u)^{1/2}, \quad \text{for any } u, v \in H^1(\mathbb{R}^3);
  \]
- For $1 \leq s < \infty$, $L^s(\mathbb{R}^3)$ is the usual Lebesgue space with the norm $||u||_s = \left( \int_{\mathbb{R}^3} |u|^s dx \right)^{1/s}$ and $|\cdot|_{\infty}$ denotes the $L^\infty$-norm;
- $D^{1,p}(\mathbb{R}^3)$ ($p \geq 2$) is the Banach space defined as the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $||u||_{D^{1,p}} = ||\nabla u||_p$;
- We use “$\rightarrow$” and “$\rightharpoonup$” to denote the strong and weak convergence in the related function space respectively;
- For any $x_0 \in \mathbb{R}^3$ and $R > 0$, $B_R(x_0)$ denotes the ball centered at $x_0$ with radius $R$;
- $C, C_1, C_2, \cdots$ represent positive constants which may change from lines to lines and $o_n(1)$ denotes the quantity that tends to 0 as $n \to \infty$.

2. Variational framework and preliminaries. In this section, we introduce the variational setting and some technical lemmas. As mentioned above, we will apply the perturbation method to deal with our problem. Firstly, we describe the working space as follows:

Define
\[
E = W^{1,4}(\mathbb{R}^3) \cap H_V(\mathbb{R}^3),
\]
where
\[ H_V(\mathbb{R}^3) := \{ u \in H^1(\mathbb{R}^3) | \int_{\mathbb{R}^3} V(x)u^2dx < +\infty \}, \]
which is a Hilbert space endowed with the inner product and norm
\[ (u, v)_V = \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv)dx, \quad ||u||_V = (u, u)_V^{1/2}, \]

\[ W^{1,4}(\mathbb{R}^3) \]

endowed with the inner product and norm
\[ (u, v)_W = \int_{\mathbb{R}^3} (|\nabla u|^2 \nabla u \nabla v + u^3v)dx, \quad ||u||_W = (u, u)_W^{1/4}. \]
The norm of \( E \) is given by
\[ ||u|| = (||u||_V^2 + ||u||_W^4)^{1/2}. \]
We define
\[ X = D^{1,2}(\mathbb{R}^3) \cap D^{1,4}(\mathbb{R}^3), \]
which is a Banach space endowed with the norm
\[ ||\phi||_X = ||\nabla \phi||_2 + ||\nabla \phi||_4. \]
Furthermore, it is noteworthy that the embedding from \( H_V(\mathbb{R}^3) \) into \( L^2(\mathbb{R}^3) \) is compact (see [16]). Applying the interpolation inequality, we get that the embedding from \( E \) into \( L^p(\mathbb{R}^3) \) is compact for \( 2 \leq p < 12 \). Moreover, \( X \hookrightarrow L^\infty(\mathbb{R}^3) \) is continuous (see [22]).

We want to find \((u_{\lambda, \varepsilon}, \phi_{\lambda, \varepsilon}) \in H_V(\mathbb{R}^3) \times X\) satisfying

\[ \int_{\mathbb{R}^3} (\nabla u_{\lambda, \varepsilon} \nabla v + V(x)u_{\lambda, \varepsilon}v)dx + \int_{\mathbb{R}^3} u_{\lambda, \varepsilon}^2 \nabla u_{\lambda, \varepsilon} \nabla vdx + \int_{\mathbb{R}^3} |\nabla u_{\lambda, \varepsilon}|^2 u_{\lambda, \varepsilon}vdx \]
\[ + \int_{\mathbb{R}^3} \phi_{\lambda, \varepsilon} u_{\lambda, \varepsilon}vdx = \lambda \int_{\mathbb{R}^3} f(x, u_{\lambda, \varepsilon})vdx, \quad \text{for all } v \in C_0^\infty(\mathbb{R}^3) \]
\[ \int_{\mathbb{R}^3} \nabla \phi_{\lambda, \varepsilon} \nabla \phi dx + \varepsilon^4 \int_{\mathbb{R}^3} |\nabla \phi_{\lambda, \varepsilon}|^2 \nabla \phi_{\lambda, \varepsilon} \nabla \phi dx = \int_{\mathbb{R}^3} u_{\lambda, \varepsilon}^2 \phi dx, \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}^3). \]

The associated energy functional is given by
\[ I_{\lambda, \varepsilon}(u, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2 + u^2|\nabla u|^2)dx + \frac{1}{2} \int_{\mathbb{R}^3} \nabla u^2 dx \]
\[ - \lambda \int_{\mathbb{R}^3} F(x, u)dx - \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi|^2 dx - \frac{\varepsilon^4}{8} \int_{\mathbb{R}^3} |\nabla \phi|^4 dx. \]

As mentioned above, there are some technical obstacles in applying Variational Methods directly to the functional \( I_{\lambda, \varepsilon} \) due to the lack of an appropriate space. It seems that there is no natural space to ensure \( I_{\lambda, \varepsilon} \in C^1(E \times X, \mathbb{R}^3) \). On account of the term \( \int_{\mathbb{R}^3} u^2|\nabla u|^2dx \), Nie and Wu in [33] considered the Banach space \( \tilde{E} = \{ u \in H^1(\mathbb{R}^3) | \int_{\mathbb{R}^3} u^2|\nabla u|^2dx < +\infty \} \). However, it is hard to obtain the boundedness of \((PS)\) sequence in \( \tilde{E} \); thus the usual minimax techniques cannot be applied. In order to overcome the above difficulties, we will employ the perturbation method introduced in [29] and define the perturbation functional of \( I_{\lambda, \varepsilon} \) by adding a 4-Laplacian operator. To be more precise, for \( \mu \in (0, 1) \), we define
\[ I_{\lambda, \varepsilon, \mu}(u, \phi) = \frac{\mu}{4} \int_{\mathbb{R}^3} (|\nabla u|^4 + u^4)dx + I_{\lambda, \varepsilon}(u, \phi). \]
In this direction, by standard argument, we can prove that $I_{\lambda, \varepsilon, \mu}$ is well-defined and $I_{\lambda, \varepsilon, \mu} \in C^1(E \times X, \mathbb{R})$. The idea of perturbation method is to obtain the existence of critical points of $I_{\lambda, \varepsilon, \mu}$ for $\mu > 0$ small and to establish suitable estimates for the critical points as $\mu \to 0$, so that we may pass to the limit to get solutions of the original problem.

Similar to the analysis in [20], for each $u \in H^1(\mathbb{R}^3)$ and $u^2 \in X'$ (the dual space of $X$) in the sense that the map

$$u^2 : \phi \in X \to \int_{\mathbb{R}^3} \phi u^2 dx \in \mathbb{R}$$

is linear and continuous. Then for any $u \in H^1(\mathbb{R}^3)$ fixed, there exists a unique $\phi_\varepsilon(u)$ in $X$ satisfying

$$-\Delta \phi_\varepsilon(u) - \varepsilon^4 \Delta_4 \phi_\varepsilon(u) = u^2, \quad \text{in } \mathbb{R}^3. \quad (16)$$

In the following, we shall use $\phi_\varepsilon(u)$ to denote the unique solution of (16) and we have

$$\int_{\mathbb{R}^3} |\nabla \phi_\varepsilon(u)|^2 dx + \varepsilon^4 \int_{\mathbb{R}^3} |\nabla \phi_\varepsilon(u)|^4 dx = \int_{\mathbb{R}^3} \phi_\varepsilon(u) u^2 dx. \quad (17)$$

Now we conclude some properties of $\phi_\varepsilon(u)$ which are useful for studying our problem.

**Lemma 2.1.** ([20]) If $u_n \to u$ in $L^{\frac{4}{3}}(\mathbb{R}^3)$ as $n \to \infty$, then for any $\varepsilon > 0$ fixed, there hold

(i): $\lim_{n \to +\infty} \int_{\mathbb{R}^3} |\nabla \phi_\varepsilon(u_n)|^2 dx = \int_{\mathbb{R}^3} |\nabla \phi_\varepsilon(u)|^2 dx$,

(ii): $\lim_{n \to +\infty} \int_{\mathbb{R}^3} |\nabla \phi_\varepsilon(u_n)|^4 dx = \int_{\mathbb{R}^3} |\nabla \phi_\varepsilon(u)|^4 dx$,

(iii): $\lim_{n \to +\infty} \phi_\varepsilon(u_n) = \phi_\varepsilon(u)$ in $L^\infty(\mathbb{R}^3)$,

(iv): $\lim_{n \to +\infty} \int_{\mathbb{R}^3} \phi_\varepsilon(u_n) u_n^2 dx = \int_{\mathbb{R}^3} \phi_\varepsilon(u) u^2 dx$.

**Lemma 2.2.** ([10]) If $u_\varepsilon \to u$ in $L^{\frac{4}{3}}(\mathbb{R}^3)$ as $\varepsilon \to 0^+$, then

$\phi_\varepsilon(u_\varepsilon) \to \phi_0(u)$ in $D^{1,2}(\mathbb{R}^3)$ and $\varepsilon \phi_\varepsilon(u_\varepsilon) \to 0$ in $D^{1,4}(\mathbb{R}^3)$.

**Lemma 2.3.** ([20]) For all $\varepsilon > 0$, let $G_{\Phi_\varepsilon}$ be the graph of the map $\Phi_\varepsilon$ defined by

$\Phi_\varepsilon : u \in H^1(\mathbb{R}^3) \to \phi_\varepsilon(u) \in X$.

Then $\Phi_\varepsilon \in C^1(H^1(\mathbb{R}^3), X)$ and

$G_{\Phi_\varepsilon} = \{(u, \phi) \in H^1(\mathbb{R}^3) \times X \mid \partial_u I_{\lambda, \varepsilon, \mu}(u, \phi) = 0\}$.

Define

$I_{\lambda, \varepsilon, \mu}(u) : = I_{\lambda, \varepsilon, \mu}(u, \phi_\varepsilon(u))$

$$= \frac{\mu}{4} \|u\|_W^4 + \frac{1}{2} \|u\|_V^2 + \frac{1}{2} \int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi_\varepsilon(u)|^2 dx + \frac{3\varepsilon^4}{8} \int_{\mathbb{R}^3} |\nabla \phi_\varepsilon(u)|^4 dx - \lambda \int_{\mathbb{R}^3} F(x, u) dx. \quad (18)$$

Moreover, one can see that $I_{\lambda, \varepsilon, \mu} \in C^1(E, \mathbb{R})$ and for $v \in C_0^\infty(\mathbb{R}^3)$, by lemma 2.3, there holds

$I'_{\lambda, \varepsilon, \mu}(u)v = \partial_u I_{\lambda, \varepsilon, \mu}(u, \phi_\varepsilon(u))v + \partial_\varepsilon I_{\lambda, \varepsilon, \mu}(u, \phi_\varepsilon(u)) \phi_\varepsilon'(u)v$

$$= \partial_u I_{\lambda, \varepsilon, \mu}(u, \phi_\varepsilon(u))v.$$

Then by (15), one yields
\[
I'_{\lambda, \varepsilon, \mu}(u) = \mu \int_{\mathbb{R}^3} (|\nabla u|^2 \nabla u \nabla v + u^3 v) \, dx + \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x) uv) \, dx
+ \int_{\mathbb{R}^3} u^2 \nabla u \nabla v \, dx + \int_{\mathbb{R}^3} |\nabla u|^2 uv \, dx + \int_{\mathbb{R}^3} \phi_\varepsilon(u) uv \, dx - \lambda \int_{\mathbb{R}^3} f(x, u) \, dv. \tag{19}
\]

**Remark 2.** Let \( \lambda, \varepsilon \) and \( \mu \) be fixed. Then the following statements are equivalent:

(i): \((u_{\lambda, \varepsilon, \mu}, \phi_{\lambda, \varepsilon, \mu}) \in E \times X\) is a pair of critical points of \( I_{\lambda, \varepsilon, \mu} \),

(ii): \( u_{\lambda, \varepsilon, \mu} \) is a critical point of \( I_{\lambda, \varepsilon, \mu} \) and \( \phi_\varepsilon(u_{\lambda, \varepsilon, \mu}) \).

For the sake of simplicity in the presentation, we introduce the following functional
\[
J_\varepsilon : u \in H^1(\mathbb{R}^3) \to \frac{\mu}{4} \int_{\mathbb{R}^3} |\nabla \phi_\varepsilon(u)|^2 \, dx + \frac{3\varepsilon^4}{8} \int_{\mathbb{R}^3} |\nabla \phi_\varepsilon(u)|^4 \, dx.
\]

Thus we get
\[
I_{\lambda, \varepsilon, \mu}(u) = \frac{\mu}{4} \|u\|_W^4 + \frac{1}{2} \|u\|_V^2 + \frac{1}{2} \int_{\mathbb{R}^3} u^2 |\nabla u|^2 \, dx + J_\varepsilon(u) - \lambda \int_{\mathbb{R}^3} F(x, u) \, dx.
\]

**Remark 3.** By above analysis, we know that \( J_\varepsilon \in C^1(H^1(\mathbb{R}^3), \mathbb{R}) \) and for any \( v \in H^3(\mathbb{R}^3) \),
\[
\frac{d}{dt} J_\varepsilon(tv) = J'_\varepsilon(tv) v = t \int_{\mathbb{R}^3} \phi_\varepsilon(tv) v^2 \, dx.
\]

3. **Compactness results for perturbation functional.** In this section, we study the perturbation functional \( I_{\lambda, \varepsilon, \mu} \). For convenience, we replace \( I_{\lambda, \varepsilon, \mu}(u) \) with \( I_\lambda(u) \). Indeed, the following results hold uniformly in \( \varepsilon \) and \( \mu \).

3.1. **Truncated functional.** Due to the “growth” of order 4 in \( J_\varepsilon \), it is not easy to prove the boundedness of the \((PS)_c\) sequence. We follow the strategy introduced in [20] and define a truncated functional of \( I_\lambda \).

Let \( \eta \in C_0^\infty(\mathbb{R}^+, [0, 1]) \) be a cut-off function defined by
\[
\begin{cases}
\eta(t) = 1, & \text{if } t \in [0, 2], \\
0 \leq \eta(t) \leq 1, & \text{if } t \in (2, 4), \\
\eta(t) = 0, & \text{if } t \in [4, +\infty),
\end{cases}
\]
\[
\eta' \leq 0, \quad |\eta'|_\infty \leq 2. \tag{20}
\]

For each \( T > 0 \), we define \( k_T(u) = \eta(\|u\|^2_W + \|u\|^4_V) \) and the truncated functional \( I^T_\lambda : E \to \mathbb{R} \) is given by
\[
I^T_\lambda(u) = \frac{\mu}{4} \|u\|_W^4 + \frac{1}{2} \|u\|_V^2 + \frac{1}{2} \int_{\mathbb{R}^3} u^2 |\nabla u|^2 \, dx + k_T(u) J_\varepsilon(u) - \lambda \int_{\mathbb{R}^3} F(x, u) \, dx. \tag{21}
\]

Then we can see that \( I^T_\lambda(u) \in C^1(E, \mathbb{R}) \) and for any \( v \in C_0^\infty(\mathbb{R}^3) \),
\[
(I^T_\lambda)'(u)v = \mu \int_{\mathbb{R}^3} (|\nabla u|^2 \nabla u \nabla v + u^3 v) \, dx + \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x) uv) \, dx
+ \int_{\mathbb{R}^3} u^2 \nabla u \nabla v \, dx + \int_{\mathbb{R}^3} |\nabla u|^2 uv \, dx + k_T(u) \int_{\mathbb{R}^3} \phi_\varepsilon(u) uv \, dx \tag{22}
\]
+ \frac{2}{T} \eta' \left( \frac{|v|^2}{T} + \frac{|u|}{T} \right) J_\varepsilon(u) \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv)dx \\
+ \frac{4}{T} \eta' \left( \frac{|v|^2}{T} + \frac{|u|}{T} \right) J_\varepsilon(u) \int_{\mathbb{R}^3} (|\nabla u|^2 \nabla u \nabla v + u^3v)dx \\
- \lambda \int_{\mathbb{R}^3} f(x,u)vdx.

In the sequel, we prove that the functional $I_A^T$ satisfies the Mountain Pass geometry uniformly in $\varepsilon$ and $\mu$.

**Lemma 3.1.** Assume that $(V)$ and $(f_1)-(f_4)$ hold. Then

(i): for every $\lambda > 1$ fixed, there exist $\delta_\lambda, \rho_\lambda > 0$ such that for any $T > 0$, $\varepsilon > 0$ and $\mu \in (0,1)$,

$$\inf \{ I_A^T(u) : u \in E \text{ with } ||u||_V = \delta_\lambda \} \geq \rho_\lambda,$$

(ii): for every $T > 0$ fixed, there exists $e_T \in E$ with $||e_T||_V > \delta_\lambda$ (given in (i)) such that for any $\lambda > 1$, $\varepsilon > 0$ and $\mu \in (0,1)$,

$$I_A^T(e_T) < 0.$$

**Proof.** (i) Let $\lambda > 1$ be fixed. From $(f_1)$ and $(f_2)$, for each $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$F(x,s) \leq \varepsilon |s|^2 + C_\varepsilon |s|^p, \ x \in \mathbb{R}^3, s \in \mathbb{R}.$$ \hfill (23)

By Sobolev embedding theorem, we obtain

$$I_A^T(u) \geq \frac{1}{2} ||u||_V^2 - \lambda \varepsilon \int_{\mathbb{R}^3} |u|^2dx - \lambda C_\varepsilon \int_{\mathbb{R}^3} |u|^pdx$$

$$\geq \frac{1}{2} ||u||_V^2 - C_1 \lambda \varepsilon ||u||_V^2 - C_2 \lambda ||u||_V^p.$$ \hfill (24)

Since $4 < p < 12$, choose $\varepsilon$ and $\delta_\lambda$ small enough such that for some $\rho_\lambda > 0$,

$$I_A^T(u) \geq \rho_\lambda \text{ for } u \in \partial \Sigma_\delta_\lambda,$$

where

$$\Sigma_\delta_\lambda = \{ u \in E \setminus \{0\} : ||u||_V \leq \delta_\lambda \}.$$

(ii) Let $T > 0$ be fixed. For any $v \in E$ with $v \neq 0$ and $t \geq 0$, by the definition of $k_T$, we know that

$$\lim_{t \to +\infty} k_T(tv) = 0.$$ \hfill (25)

And from $(f_4)$, one can deduce that as $t \to +\infty$,

$$I_A^T(tv) = \frac{\mu t^4}{4} ||v||_W^2 + \frac{t^2}{2} ||v||_V^2 + \frac{t^4}{2} \int_{\mathbb{R}^3} v^2 |\nabla v|^2 dx + k_T(tv)J_\varepsilon(tv) - \lambda \int_{\mathbb{R}^3} F(x,tv)dx$$

$$\leq \frac{\mu t^4}{4} ||v||_W^2 + \frac{t^2}{2} ||v||_V^2 + C_3 t^4 ||v||_H^4 - C_4 t^\theta ||v||_\theta^\theta \to -\infty,$$

where we use the fact that $\theta < 4$. Let $t^* > 0$ and define a path $h : [0,1] \to E$ by $h(t) = t(t^*v)$. For $t^* > 0$ large enough, we can have

$$\int_{\mathbb{R}^3} |\nabla h(1)|^2 dx + \int_{\mathbb{R}^3} V(x)h^2(1)dx > \delta_\lambda^2 \text{ and } I_A^T(h(1)) < 0,$$

where $\delta_\lambda$ is given in (i). Setting $e_T = h(1)$, then the result follows. \hfill $\square$

**Remark 4.** Here we want to emphasize that $\delta_\lambda$ and $\rho_\lambda$ in (i) do not depend on $T, \varepsilon, \mu$ and $e_T$ in (ii) does not depend on $\lambda, \varepsilon, \mu$. 
Applying the Mountain Pass Theorem ([40]), we know that for every \( T > 0, \lambda > 1, \varepsilon > 0 \) and \( \mu \in (0,1) \), there exists a sequence \( \{u_n\} \subset E \), which depends on \( T, \lambda, \varepsilon \) and \( \mu \), such that as \( n \to \infty \),
\[
I_T^T(u_n) \to c_{\lambda,\varepsilon,\mu}^T \quad \text{and} \quad (I_T^T)'(u_n) \to 0,
\]
where
\[
c_{\lambda,\varepsilon,\mu}^T := \inf_{\gamma \in \Gamma_T^T} \max_{t \in [0,1]} I_T^T(\gamma(t)) > 0
\]
and
\[
\Gamma_T^T := \{ \gamma \in C([0,1], E) | \gamma(0) = 0, \gamma(1) = e_T \}.
\]

Define the path
\[
\gamma_*: t \in [0,1] \to te_t \in E.
\]
It is easy to see that \( \gamma_* \in \bigcap_{\lambda > 1, \varepsilon > 0, \mu \in (0,1)} \Gamma_T^T \).

3.2. Estimation of \( c_{\lambda,\varepsilon,\mu}^T \) and Boundedness of \((PS)_{c_{\lambda,\varepsilon,\mu}^T}\) sequence. In this subsection, we shall prove that every \((PS)_{c_{\lambda,\varepsilon,\mu}^T}\) sequence of \( I_T^T \) is bounded in \( E \).

To do this, we need to estimate the value of \( c_{\lambda,\varepsilon,\mu}^T \). Firstly, we give the following crucial lemma.

**Lemma 3.2.** Suppose that \((V)\) and \((f_1)-(f_4)\) hold. Let \( T > 0, \lambda > 1, \varepsilon > 0 \) and \( \mu \in (0,1) \) be fixed, then for every \( w \in E \setminus \{0\} \), the function
\[
t \in [0, +\infty) \to I_T^T(tw) \in \mathbb{R}
\]
has a positive global maximum point, denoted by \( t_T^T(w) \), which is independent of \( \varepsilon \) and \( \mu \). Moreover, for any \( T > 0 \), there holds
\[
\lim_{\lambda \to +\infty} t_T^T(w) = 0.
\]

**Proof.** It follows from \((f_1)\) and \((f_2)\) that for each \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that
\[
F(x,s) \leq \varepsilon |s|^2 + C_\varepsilon |s|^p, \quad x \in \mathbb{R}^3, s \in \mathbb{R}.
\]
Then for \( w \in E \setminus \{0\} \) fixed, by Sobolev embedding theorem, we have
\[
I_T^T(tw) \geq \frac{t^2}{2} ||w||_V^2 - \lambda \varepsilon t^2 \int_{\mathbb{R}^3} |w|^2 dx - \lambda C_\varepsilon t^p \int_{\mathbb{R}^3} |w|^p dx
\]
\[
\geq \frac{t^2}{2} ||w||_V^2 - C_5 \lambda \varepsilon t^2 ||w||_V^2 - C_6 C_\varepsilon \lambda t^p ||w||_V^p
\]
\[
= t^2 \left( \frac{1}{2} - C_5 \lambda \varepsilon ||w||_V^2 - C_6 C_\varepsilon \lambda t^p ||w||_V^p \right).
\]

Since \( p > 4 \), choosing \( \varepsilon \) small enough such that \( C_5 \lambda \varepsilon < \frac{1}{4} \), then one yields that \( I_T^T(tw) > 0 \) for \( t \) sufficiently small.

From \((f_3)\), there exists \( C_7 > 0 \) such that
\[
F(x,s) \geq C_7 |s|'' , \quad x \in \mathbb{R}^3, s \in \mathbb{R}.
\]
Then, pass to the limit as $t \to \infty$, we obtain
\[
I^T_t(tw) = \left(\frac{\mu t^4}{4} ||w||_{W}^4 + \frac{t^2}{2} ||w||_{V}^2 + \frac{t^4}{2} \int_{\mathbb{R}^3} w^2 |\nabla w|^2 dx \right) \\
+ kT(tw)J_{t}(tw) - \lambda \int_{\mathbb{R}^3} F(x, tw)dx \leq \left(\frac{t^4}{4} ||w||_{W}^4 + \frac{t^2}{2} ||w||_{V}^2 + \frac{t^4}{2} \int_{\mathbb{R}^3} w^2 |\nabla w|^2 dx \right) \\
- C_\gamma \lambda t^\nu \int_{\mathbb{R}^3} w^\nu dx \to -\infty,
\]
where we make use of the fact $\nu > 4$. Therefore we obtain the existence of $t^\gamma T_{\lambda}(w)$ and one can see that $t^\gamma T_{\lambda}(w) > 0$.

Now we prove $\lim_{\lambda \to +\infty} t^\lambda T_{\lambda}(w) = 0$. For the sake of simplicity, we set $t^\lambda \lambda = t^\gamma T_{\lambda}(w)$.

Without loss of generality, we set $||w|| = 1$, thus $||w||_{W} \leq 1$ and $||w||_{V} \leq 1$. Moreover, we know that $t^\lambda \lambda$ satisfies $(22)$. Taking $v = t^\lambda w$ in $(22)$, then

\[
\lambda \int_{\mathbb{R}^3} f(x, t^\lambda \lambda) t^\lambda \lambda wdx = \mu t^4 ||w||_{W}^4 + t^2 ||w||_{V}^2 + 2t^4 \int_{\mathbb{R}^3} w^2 |\nabla w|^2 dx \\
+ t^2 \eta \left(\frac{t^2 ||w||_{V}^2}{T^2} + \frac{t^4 ||w||_{W}^4}{T^4}\right) \int_{\mathbb{R}^3} \phi \phi_{t^\lambda w} w^2 dx \\
+ 2t^2 \frac{\eta'}{T^2} \left(\frac{t^2 ||w||_{V}^2}{T^2} + \frac{t^4 ||w||_{W}^4}{T^4}\right) ||w||_{V}^2 J_{t^\lambda w} \\
+ 4t^4 \frac{\eta'}{T^4} \left(\frac{t^2 ||w||_{V}^2}{T^2} + \frac{t^4 ||w||_{W}^4}{T^4}\right) ||w||_{W}^4 J_{t^\lambda w}.
\]

It follows from $\eta' < 0$ and $(f_4)$ that
\[
\lambda t^\theta \theta ||w||_{\theta}^\theta \leq t^4 + t^2 + 2t^4 \int_{\mathbb{R}^3} w^2 |\nabla w|^2 dx \\
+ t^2 \eta \left(\frac{t^2 ||w||_{V}^2}{T^2} + \frac{t^4 ||w||_{W}^4}{T^4}\right) \int_{\mathbb{R}^3} \phi \phi_{t^\lambda w} w^2 dx.
\]

Supposing that $t^\lambda$ is unbounded, then there exists $\lambda_n \to \infty$ as $n \to \infty$ such that $\lim_{n \to \infty} \lambda_n = \infty$. Thus, for $n$ large enough, in view of $(25)$, one can get
\[
\lambda_n t^\theta \theta ||w||_{\theta}^\theta \leq t^4 + t^2 + 2t^4 \int_{\mathbb{R}^3} w^2 |\nabla w|^2 dx,
\]
which is a contradiction on account of $\theta > 4$. Therefore, up to a subsequence, there exists $\bar{t} \geq 0$ such that
\[
t^\lambda \to \bar{t} as \lambda \to \infty.
\]

We claim $\bar{t} = 0$. Arguing by contradiction that $\bar{t} > 0$, from Lemma 2.1, we deduce that $\phi \phi_{t^\lambda w} \to \phi \phi_{\bar{t} w}$ in $L^\infty(\mathbb{R}^3)$ as $\lambda \to +\infty$.

Passing to the limit as $\lambda \to \infty$ in $(30)$, one yields that
\[
\bar{t}^4 + \bar{t}^2 + 2\bar{t}^4 \int_{\mathbb{R}^3} w^2 |\nabla w|^2 dx + \bar{C} \bar{t}^2 ||w||_{\theta} \geq \lambda \bar{t}^\theta ||w||_{\theta} \to +\infty,
\]
which is impossible. Hence $\bar{t} = 0$ and the proof is completed. \(\square\)
Now we are in the position to estimate the value of $c_{\lambda,\varepsilon,\mu}^T$.

**Lemma 3.3.** Assume that $(V)$ and $(f_1)-(f_4)$ hold. Then for any $T > 0$, there holds

$$\lim_{\lambda \to \infty} \sup_{\varepsilon > 0, \mu > 0} c_{\lambda,\varepsilon,\mu}^T = 0.$$  

**Proof.** Let $T > 0$ be fixed. It is sufficient to prove that for any $\tau > 0$, there exists $\lambda_0 > 1$ such that for all $\lambda > \lambda_0$,

$$0 < \max_{t \in [0,1]} I_{\lambda}^T(\gamma_*(t)) < \tau,$$

where $\gamma_*$ is defined in (26).

Let $w \in C_0^\infty(\mathbb{R}^3)$ and $w \geq 0$ with $||w|| = 1$. By Lemma 3.2, there exists $t_\lambda := t_\lambda^T(w) > 0$ (independent of $\varepsilon, \mu$) such that $I_{\lambda}^T(t_\lambda w) = \max_{t \geq 0} I_{\lambda}^T(tw)$ and $\lim_{\lambda \to \infty} t_\lambda = 0$.

In this perspective, we can conclude that

$$\lim_{\lambda \to \infty} \sup_{\varepsilon > 0, \mu > 0} t_\lambda = 0.$$

On the other hand, since $k_T$ and $J_\varepsilon$ are both continuous, then for any $\varepsilon \geq 0$, we have

$$\lim_{\lambda \to \infty} k_T(t_\lambda w) = 1 \quad \text{and} \quad \lim_{\lambda \to \infty} J_\varepsilon(t_\lambda w) = 0.$$

Therefore for any $\tau > 0$, there exists $\lambda_0 > 1$ large such that for all $\lambda > \lambda_0$,

$$0 < \max_{t \in [0,1]} I_{\lambda}^T(\gamma_*(t)) = I_{\lambda}^T(t_\lambda w) \leq \frac{t_\lambda^4}{4} + \frac{t_\lambda^2}{2} + \frac{t_\lambda^4}{2} \int_{\mathbb{R}^3} w^2 |\nabla w|^2 dx + k_T(t_\lambda w)J_\varepsilon(t_\lambda w) < \tau.$$

(31)

Thus the proof of this lemma is finished. \qed

Next we will show that every $(PS)_{c_{\lambda,\varepsilon,\mu}^T}$ sequence of $I_{\lambda}^T$ is bounded in $E$.

**Lemma 3.4.** Assume that $(V)$ and $(f_1)-(f_4)$ hold. For fixed $T > 0$, then there exists $\lambda(T)$ large enough such that for all $\lambda > \lambda(T)$,

$$\sup_{\varepsilon > 0, \mu > 0} c_{\lambda,\varepsilon,\mu}^T \leq \frac{\nu - 2}{2\nu} T^2. \quad (32)$$

Furthermore, for given $\varepsilon, \mu > 0$, if $\{u_n\}$ is a $(PS)_{c_{\lambda,\varepsilon,\mu}^T}$ sequence of $I_{\lambda}^T$, then

$$||u_n||_V \leq T \quad \text{and} \quad ||u_n||_W \leq T.$$

**Proof.** In view of Lemma 3.3, we can easily deduce that (32) holds. Since $\{u_n\}$ is a $(PS)_{c_{\lambda,\varepsilon,\mu}^T}$ sequence of $I_{\lambda}^T$, that is

$$I_{\lambda}^T(u_n) \to c_{\lambda,\varepsilon,\mu}^T, \quad (I_{\lambda}^T)'(u_n) \to 0. \quad (33)$$

Firstly, we show that $||u_n||_V \leq 2T$ and $||u_n||_W \leq \sqrt{2}T$. If the conclusions do not hold, then there exists a subsequence of $\{u_n\} \subset E$, still denoted by $\{u_n\}$, satisfying

$$||u_n||_V > 2T \quad \text{or} \quad ||u_n||_W > \sqrt{2}T. \quad (34)$$
From (33) and (f3), we have
\[ c_{\lambda,\varepsilon,\mu}^T + o_n(1) = I_{\lambda}^T(u_n) - \frac{1}{\nu} (I_{\lambda}^T)'(u_n)u_n \]
\[ \geq \frac{(\nu - 4)\mu}{4\nu} ||u_n||_W^4 + \frac{\nu - 2}{2\nu} ||u_n||_V^2 + \frac{\nu - 4}{2\nu} \int_{\mathbb{R}^3} u_n^2 |\nabla u_n|^2 dx \]
\[ + \eta \left( \frac{||u_n||_V^2}{T^2} + \frac{||u||_W^4}{T^4} \right) \left[ J\varepsilon(u_n) - \frac{1}{\nu} \int_{\mathbb{R}^3} \phi\varepsilon(u_n) u_n^2 dx \right] \]
\[ - \frac{2}{\nu T^2} \eta' \left( \frac{||u_n||_V^2}{T^2} + \frac{||u||_W^4}{T^4} \right) ||u_n||_V^2 J\varepsilon(u_n) \]
\[ - \frac{4}{\nu T^4} \eta' \left( \frac{||u_n||_V^2}{T^2} + \frac{||u||_W^4}{T^4} \right) ||u_n||_W^4 J\varepsilon(u_n) \]
\[ \geq \frac{(\nu - 4)\mu}{4\nu} ||u_n||_W^4 + \frac{\nu - 2}{2\nu} ||u_n||_V^2 \]
\[ \geq \frac{2(\nu - 2)}{\nu} T^2, \]
where we use the fact that \( \nu > 4, \eta' \leq 0 \) and (34), which is a contradiction to (32). Now we prove \( ||u_n||_V \leq T \) and \( ||u_n||_W \leq T \). Assuming by contradiction that \( T < ||u_n||_V \leq 2T \) or \( T < ||u_n||_W \leq \sqrt{2T} \), similar above, we have
\[ c_{\lambda,\varepsilon,\mu}^T + o_n(1) = I_{\lambda}^T(u_n) - \frac{1}{\nu} (I_{\lambda}^T)'(u_n)u_n \]
\[ \geq \frac{(\nu - 4)\mu}{4\nu} ||u_n||_W^4 + \frac{\nu - 2}{2\nu} ||u_n||_V^2 + \frac{\nu - 4}{2\nu} \int_{\mathbb{R}^3} u_n^2 |\nabla u_n|^2 dx \]
\[ + \eta \left( \frac{||u_n||_V^2}{T^2} + \frac{||u||_W^4}{T^4} \right) \left[ J\varepsilon(u_n) - \frac{1}{\nu} \int_{\mathbb{R}^3} \phi\varepsilon(u_n) u_n^2 dx \right] \]
\[ - \frac{2}{\nu T^2} \eta' \left( \frac{||u_n||_V^2}{T^2} + \frac{||u||_W^4}{T^4} \right) ||u_n||_V^2 J\varepsilon(u_n) \]
\[ - \frac{4}{\nu T^4} \eta' \left( \frac{||u_n||_V^2}{T^2} + \frac{||u||_W^4}{T^4} \right) ||u_n||_W^4 J\varepsilon(u_n) \]
\[ \geq \frac{(\nu - 4)\mu}{4\nu} ||u_n||_W^4 + \frac{\nu - 2}{2\nu} T^2 + \eta(\varepsilon) \left[ J\varepsilon(u_n) - \frac{1}{\nu} \int_{\mathbb{R}^3} \phi\varepsilon(u_n) u_n^2 dx \right] \]
\[ \geq \frac{(\nu - 2)}{2\nu} T^2, \]
where we use the fact that \( \eta \) is a decreasing function, which contradicts to (32). At this point, we conclude that \( ||u||_V < T \) and \( ||u||_W < T \). \( \square \)

**Remark 5.** It follows from Lemma 3.4 that for any \( T > 0 \) and \( \varepsilon, \mu > 0 \) given, there exists \( \lambda(T) > 0 \) such that for all \( \lambda > \lambda(T) \), every \((PS)_{\epsilon,\mu}\) sequence of \( I_{\lambda}^T \) is bounded in \( E \) and it is actually a \((PS)\) sequence of \( I_{\lambda} \) at level \( c_{\lambda,\varepsilon,\mu} = c_{\lambda,\varepsilon,\mu}^T \).

### 3.3. Compactness results of the functional \( I_{\lambda} \).

**Lemma 3.5.** Assume that \((V)\) and \((f_1) - (f_4)\) hold. For given \( T > 0, \mu > 0 \) and \( \varepsilon > 0 \), there exists \( \lambda(T) \) large enough such that for \( \lambda > \lambda(T) \), the functional \( I_{\lambda} \) satisfies the \((PS)\) condition at the level \( c_{\lambda,\varepsilon,\mu} \).

**Proof.** Let \( \{u_n\} \) be a \((PS)_{\epsilon,\mu}\) sequence of \( I_{\lambda} \), that is
\[ I_{\lambda}(u_n) \rightarrow c_{\lambda,\varepsilon,\mu}, \ I_{\lambda}'(u_n) \rightarrow 0 \text{ as } n \rightarrow \infty. \]  
\[ (35) \]
We shall prove that \( \{u_n\} \) admits a convergent subsequence in \( E \). In view of Remark 5, we know that \( \{u_n\} \) is bounded in \( E \) and \( ||u_n||^2 = ||u_n||_V^2 + ||u_n||_W^4 \leq T^2 + T^4 \) uniformly in \( \varepsilon \) and \( \mu \). Up to a subsequence, there exists \( u := u^T \in E \) satisfying \( ||u||^2 \leq T^2 + T^4 \) such that \( u_n \to u \) in \( E, u_n \to u \) in \( L^s(\mathbb{R}^3) \) \( (2 \leq s < 12 \) and \( u_n \to u \) a.e. in \( \mathbb{R}^3 \). Furthermore, since \( I'_\lambda(u_n) \to 0 \) as \( n \to \infty \), then for all \( v \in C_0^\infty(\mathbb{R}^3) \),

\[
\mu \int_{\mathbb{R}^3} (|\nabla u_n|^2 \nabla u_n \nabla v + u_n^3 v)dx + \int_{\mathbb{R}^3} (\nabla u_n \nabla v + V(x)u_n v)dx + \int_{\mathbb{R}^3} u_n^2 \nabla u_n \nabla vdx \\
+ \int_{\mathbb{R}^3} |\nabla u_n|^2 u_n vdx + \int_{\mathbb{R}^3} \phi_\varepsilon(u_n) u_n vdx = \lambda \int_{\mathbb{R}^3} f(x, u_n) vdx + o_n(1). \tag{36}
\]

Passing to the limit as \( n \to \infty \) in (36), we obtain

\[
\mu \int_{\mathbb{R}^3} (|\nabla u|^2 \nabla u \nabla v + u^3 v)dx + \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv)dx + \int_{\mathbb{R}^3} u^2 \nabla u \nabla vdx \\
+ \int_{\mathbb{R}^3} |\nabla u|^2 uvdx + \int_{\mathbb{R}^3} \phi_\varepsilon(u) uvdx = \lambda \int_{\mathbb{R}^3} f(x, u) vdx,
\]

which implies \( u \) is a critical point of \( I_\lambda \). From Remark 2, we know that \( (u, \phi_\varepsilon(u)) \in E \times X \) gives rise to a pair of critical points of \( I_{\lambda, \varepsilon, \mu} \).

On the other hand, we note that

\[
\mu C||u_n - u||_W^4 + ||u_n - u||_V^2 \\
\leq \mu \int_{\mathbb{R}^3} (|\nabla u_n|^2 \nabla u_n - |\nabla u|^2 \nabla u) \nabla (u_n - u) + (u_n^3 - u^3)(u_n - u))dx + ||u_n - u||_V^2 \\
= (I'_\lambda(u_n) - I'_\lambda(u))(u_n - u) - \int_{\mathbb{R}^3} [\phi_\varepsilon(u_n)]u_n - \phi_\varepsilon(u)u(u_n - u)dx \\
- \int_{\mathbb{R}^3} (|\nabla u_n|^2 u_n - |\nabla u|^2 u)(u_n - u)dx - \int_{\mathbb{R}^3} (u_n^2 \nabla u_n - u^2 \nabla u) \nabla (u_n - u)dx \\
+ \int_{\mathbb{R}^3} (f(x, u_n) - f(x, u))(u_n - u)dx \\
=: I_1 + I_2 + I_3 + I_4 + I_5. \tag{37}
\]

It is obvious that \( I_1 \to 0 \) as \( n \to \infty \). In the following, we estimate \( I_i \) \( (i = 2, 3, 4, 5) \) respectively. Firstly, by Lemma 2.1, pass to the limit as \( n \to \infty \), we have

\[
|I_2| = \left| \int_{\mathbb{R}^3} [\phi_\varepsilon(u_n)]u_n - \phi_\varepsilon(u)u(u_n - u)dx \right| \\
\leq \int_{\mathbb{R}^3} [\phi_\varepsilon(u_n)](u_n - u)(u_n - u)dx + \int_{\mathbb{R}^3} [\phi_\varepsilon(u_n) - \phi_\varepsilon(u)]u(u_n - u)dx \tag{38}
\]

\[
\leq |\phi_\varepsilon(u_n)|_6 ||u_n - u||_{L^4}^2 + |\phi_\varepsilon(u_n) - \phi_\varepsilon(u)|_6 |u|_{L^4} ||u_n - u||_{L^4} \to 0.
\]

Moreover, for \( I_3 \), using Hölder inequality, one can see that as \( n \to \infty \),

\[
|I_3| = \left| \int_{\mathbb{R}^3} (|\nabla u_n|^2 u_n - |\nabla u|^2 u)(u_n - u)dx \right| \\
\leq |\nabla u_n|^2 ||u_n||_4 |u_n - u|_4 + |\nabla u|^2 ||u||_4 |u_n - u|_4 \to 0. \tag{39}
\]
Finally, we give the estimations of $I_4$ and $I_5$. Since $u_n \to u$ in $L^s(\mathbb{R}^3)$ ($2 \leq s < 12$), by Hölder inequality, we get

$$
I_4 = -\left( \int_{\mathbb{R}^3} (u_n^2 \nabla u_n - u^2 \nabla u) \nabla (u_n - u) dx \right)
= -\int_{\mathbb{R}^3} u_n^2 \nabla (u_n - u) dx - \int_{\mathbb{R}^3} (u_n^2 - u^2) \nabla u \nabla (u_n - u) dx
\leq \int_{\mathbb{R}^3} |u_n - u||u_n + u||\nabla u||\nabla (u_n - u)| dx
\leq |u_n - u|_4 |\nabla (u_n - u)|_4 |\nabla u|_4 |u_n + u|_4 \to 0, \text{ as } n \to \infty
$$

and

$$
|I_5| = \left| \int_{\mathbb{R}^3} (f(x, u_n) - f(x, u)) (u_n - u) dx \right|
\leq \int_{\mathbb{R}^3} |\epsilon||u_n|^2 + C_\epsilon \int_{\mathbb{R}^3} |u_n|^p - 1| |u|_p dx
\leq 4\epsilon |(u_n|^2 + |u|^2)| + C_\epsilon \int_{\mathbb{R}^3} |u_n|^p - 1 |u|_p
\leq C(\epsilon + C_\epsilon |u_n - u|_p) \to 0, \text{ as } n \to \infty.
$$

By (37)-(41), we claim $I_4 \to 0$. In fact, if $I_4 < 0$, then we can get

$$
\mu C |u_n - u|_W^4 + |u_n - u|_V^2 < 0,
$$

which is impossible. At this point, we can conclude that $|u_n - u|_W \to 0$ and $|u_n - u|_V \to 0$ as $n \to \infty$, which imply that $|u_n - u| \to 0$ as $n \to \infty$. Thus we finish the proof of this lemma.

**Remark 6.** Lemma 3.5 implies the existence of the critical points for the functional $I_\lambda$. Furthermore, we shall show the asymptotical behavior of the critical points of $I_\lambda$ with respect to $\epsilon$. Indeed, this property is crucial for proving Theorem 1.2.

In order to emphasize the relations between the value of $I_\lambda$ and the parameters $\epsilon$, $\mu$, in what follows, we will reuse the notation $I_{\lambda,\epsilon,\mu}$.

**Lemma 3.6.** Assume that (V) and (f_1) - (f_4) hold. For any $T, \mu > 0$ fixed, let $\epsilon_n \to 0$ as $n \to \infty$ and $\{u_n\} \subset E$ be a sequence of critical points of $I_{\lambda,\epsilon,\mu}$ at the level $c_{\lambda,\epsilon,\mu}$. Then there exists $\lambda(T)$ large enough such that

$$
\sup_{\epsilon_n > 0, \mu > 0} c_{\lambda,\epsilon,\mu} \leq \frac{\nu - 2}{2\nu} T^2, \text{ for all } \lambda > \lambda(T).
$$

Moreover, up to a subsequence, there exists $u_{\lambda,\mu} \in E$ such that as $n \to \infty$,

$$
u_n \to u_{\lambda,\mu} \text{ in } E \text{ and } I_{\lambda,\epsilon,\mu}(u_n) \to I_{\lambda,\epsilon,\mu}(u_{\lambda,\mu}).
$$

**Proof.** Recording Lemma 3.3, we can easily infer that (42) holds. Next we want to show that $\{u_n\}$ is bounded in $E$. Note that

$$
\frac{\nu - 2}{2\nu} T^2 \geq I_{\lambda,\epsilon,\mu}(u_n) - \frac{1}{\nu} I_{\lambda,\epsilon,\mu}'(u_n) u_n
\geq \frac{(\nu - 4)\mu}{4\nu} |u_n|_W^4 - \frac{\nu - 2}{2\nu} |u_n|_V^2 + \frac{\nu - 4}{2\nu} \int_{\mathbb{R}^3} u_n^2 |\nabla u_n|^2 dx,
$$

which implies that $|u_n|_W$ and $|u_n|_V$ are bounded. Then there exists $u_{\lambda,\mu} \in E$ such that $u_n \to u_{\lambda,\mu}$ in $E$ as $n \to \infty$. Then by repeating the argument used in
Lemma 3.5, we can prove that
\[ u_n \to u_{\lambda,\mu} \text{ in } E \text{ as } n \to \infty. \]

On the other hand, since
\[ u_n^2 \to u_{\lambda,\mu}^2 \text{ in } L^\frac{8}{3}(\mathbb{R}^3) \text{ as } n \to \infty, \]
in view of Lemma 2.2, we can obtain
\[ \phi_{\varepsilon_n}(u_n) \to \phi_0(u_{\lambda,\mu}) \text{ in } D^{1,2}(\mathbb{R}^3), \quad \varepsilon_n\phi_{\varepsilon_n}(u_n) \to 0 \text{ in } D^{1,4}(\mathbb{R}^3). \]

Then as \( n \to \infty, \)
\[
I_{\lambda,\varepsilon_n,\mu}(u_n) = \frac{\mu}{4} ||u_n||_W^4 + \frac{1}{2} ||u_n||_V^2 + \frac{1}{2} \int_{\mathbb{R}^3} u_n^2 |\nabla u_n|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi_{\varepsilon_n}(u_n)|^2 \, dx \\
+ \frac{3\varepsilon_n^4}{8} \int_{\mathbb{R}^3} |\nabla \phi_{\varepsilon_n}(u_n)|^4 \, dx - \lambda \int_{\mathbb{R}^3} F(x, u_n) \, dx \\
\to \frac{\mu}{4} ||u_{\lambda,\mu}||_W^4 + \frac{1}{2} ||u_{\lambda,\mu}||_V^2 + \frac{1}{2} \int_{\mathbb{R}^3} u_{\lambda,\mu}^2 |\nabla u_{\lambda,\mu}|^2 \, dx \\
+ \frac{3\varepsilon_n^4}{8} \int_{\mathbb{R}^3} |\nabla \phi_0(u_{\lambda,\mu})|^4 \, dx - \lambda \int_{\mathbb{R}^3} F(x, u_{\lambda,\mu}) \, dx \\
= I_{\lambda,0,\mu}(u_{\lambda,\mu}).
\]

Thus the proof of this lemma is finished. \( \square \)

4. Proof of Theorem 1.1. Now we will complete the proof of Theorem 1.1 by using Moser iteration technique and \( L^\infty \)-estimation.

Lemma 4.1. Assume that \((V)\) and \((f_1) - (f_2)\) hold. For any \( T, \varepsilon > 0 \) fixed, let \( \mu_n \to 0 \) as \( n \to \infty \) and \( \{u_n\} \subset H_V(\mathbb{R}^3) \) be a sequence of critical points of \( I_{\lambda,\varepsilon,\mu_n} \) at the level \( c_{\lambda,\varepsilon,\mu_n} \). Then there exists \( \Lambda > 0 \) such that for \( \lambda > \Lambda, \)
\[
\sup_{\varepsilon > 0, \mu_n > 0} c_{\lambda,\varepsilon,\mu_n} \leq \frac{\nu - 2}{2\nu} T^2. \quad (44)
\]

Moreover, there exists \( u_{\lambda,\varepsilon} \in H_V(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \) such that as \( n \to \infty, \)
\[ u_n \to u_{\lambda,\varepsilon} \text{ in } H_V(\mathbb{R}^3), \quad u_n \nabla u_n \to u_{\lambda,\varepsilon} \nabla u_{\lambda,\varepsilon} \text{ in } L^2(\mathbb{R}^3), \]
\[ \mu_n \int_{\mathbb{R}^3} (|\nabla u_n|^4 + |u_n|^4) \, dx \to 0, \quad I_{\lambda,\varepsilon,\mu_n}(u_n) \to I_{\lambda,\varepsilon,0}(u_{\lambda,\varepsilon}) \]
and \( u_{\lambda,\varepsilon} \) is a critical point of the original functional \( I_{\lambda,\varepsilon,0} \) at the level \( c_{\lambda,\varepsilon,0} \).

Proof. Recording from Lemma 3.3, for any \( T > 0 \), there exists \( \Lambda := \Lambda(T) \) large enough such that for all \( \lambda > \Lambda, \)
\[
\sup_{\varepsilon > 0, \mu_n > 0} c_{\lambda,\varepsilon,\mu_n} \leq \frac{\nu - 2}{2\nu} T^2. \quad (45)
\]

In the sequel, we divide the proof into the following several steps:

(i) \( \{u_n\} \) is bounded in \( H_V(\mathbb{R}^3) \).

In fact, similar to (43), we have
\[
\frac{\nu - 2}{2\nu} T^2 \geq I_{\lambda,\varepsilon,\mu_n}(u_n) - \frac{1}{\nu} I'_{\lambda,\varepsilon,\mu_n}(u_n) u_n \\
\geq (\nu - 4) \frac{\mu_n}{4\nu} ||u_n||_W^4 + \frac{\nu - 2}{2\nu} ||u_n||_V^2 + \frac{\nu - 4}{2\nu} \int_{\mathbb{R}^3} u_n^2 |\nabla u_n|^2 \, dx,
\]
which yields that \( \{u_n\} \) is bounded in \( H_V(\mathbb{R}^3) \) and \( \{u_n \nabla u_n\} \) is bounded in \( L^2(\mathbb{R}^3) \) since \( \nu > 4 \). Then there exists \( u_{\lambda,\epsilon} \in E \) such that \( u_n \rightharpoonup u_{\lambda,\epsilon} \) in \( H_V(\mathbb{R}^3) \), \( u_n \nabla u_n \rightharpoonup u_{\lambda,\epsilon} \nabla u_{\lambda,\epsilon} \) in \( L^2(\mathbb{R}^3) \) and \( u_n \rightarrow u_{\lambda,\epsilon} \) a.e. in \( \mathbb{R}^3 \) as \( n \rightarrow \infty \).

(ii) \( \{u_n\} \) is bounded in \( L^\infty(\mathbb{R}^3) \) and \( |u_n|_\infty \leq C \) (\( C \) is independent of \( n \)). Since \( \{u_n\} \) is a sequence of critical points of \( I_{\lambda,\epsilon,\mu_n} \), then for any \( v \in C^0(\mathbb{R}^3) \), there holds

\[
\mu_n \int_{\mathbb{R}^3} (|\nabla u_n|^2 \nabla u_n \nabla v + u_n^2 v)dx + \int_{\mathbb{R}^3} (\nabla u_n \nabla v + V(x)u_n v)dx
\]

\[
+ \int_{\mathbb{R}^3} u_n^2 \nabla u_n \nabla vdx + \int_{\mathbb{R}^3} |\nabla u_n|^2 u_n vdx + \int_{\mathbb{R}^3} \phi_\epsilon(u_n)u_n vdx
\]

\[
= \lambda \int_{\mathbb{R}^3} f(x, u_n) vdx.
\]

Define

\[
u^M = \begin{cases} u_n & \text{if } |u| \leq M, \\ M & \text{if } |u| > M. \end{cases}
\]

Let \( v = u_n|u_n|^2|v|^{s-1} \) be a testing function in (46), where \( s > 1 \) will be chosen later. Similar to Lemma 3.2-(ii) in [19], using Moser iteration argument (see [28]), we can prove that \( |u_n|_\infty \leq C \) and \( |u_{\lambda,\epsilon}|_\infty \leq C \).

(iii) \( u_{\lambda,\epsilon} \in H_V(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \) is a critical point of \( I_{\lambda,\epsilon,0} \).

Take \( \xi_1 = \psi e^{-u_n}, \xi_2 = \psi e^{u_n}, \) where \( \psi \in C^0(\mathbb{R}^3), \psi \geq 0 \). Substituting \( \xi_1 \) and \( \xi_2 \) into (46) respectively and repeating the method in Lemma 3.2-(iii) in [19], we obtain

\[
\int_{\mathbb{R}^3} (\nabla u_{\lambda,\epsilon} \nabla \psi + V(x)u_{\lambda,\epsilon} \psi)dx + \int_{\mathbb{R}^3} u_{\lambda,\epsilon}^2 \nabla u_{\lambda,\epsilon} \nabla \psi dx
\]

\[
+ \int_{\mathbb{R}^3} |\nabla u_{\lambda,\epsilon}|^2 u_{\lambda,\epsilon} \psi dx + \int_{\mathbb{R}^3} \phi_\epsilon(u_{\lambda,\epsilon})u_{\lambda,\epsilon} \psi dx = \lambda \int_{\mathbb{R}^3} f(x, u_{\lambda,\epsilon}) \psi dx,
\]

which implies that \( u_{\lambda,\epsilon} \) is a critical point of the original functional \( I_{\lambda,\epsilon,0} \).

Furthermore, by standard argument, we can prove that

\[
||u_n||_V \rightarrow ||u_{\lambda,\epsilon}||_V, \quad u_n \nabla u_n \rightarrow u_{\lambda,\epsilon} \nabla u_{\lambda,\epsilon} \quad \text{in} \quad L^2(\mathbb{R}^3),
\]

\[
\mu_n \int_{\mathbb{R}^3} (|\nabla u_n|^4 + |u_n|^4)dx \rightarrow 0, \quad I_{\lambda,\epsilon,\mu_n}(u_n) \rightarrow I_{\lambda,\epsilon,0}(u_{\lambda,\epsilon}).
\]

Thus, Lemma 4.1 will follow directly from the above three steps. \( \square \)

**Proof of Theorem 1.1.** From Lemma 4.1, we know that there exists \( \Lambda > 0 \) large enough such that for all \( \lambda > \Lambda, (1) \) admits a pair of nontrivial solutions \( (u_{\lambda,\epsilon}, \phi_{\lambda,\epsilon}(u_{\lambda,\epsilon})) \). Now, we show that \( (u_{\lambda,\epsilon}, \phi_{\lambda,\epsilon}(u_{\lambda,\epsilon})) \) is positive. In fact, by multiplying the second equation of (1) by \( \phi_{\lambda,\epsilon}(u_{\lambda,\epsilon})^- := \max\{-\phi_{\lambda,\epsilon}(u_{\lambda,\epsilon}), 0\} \) and integrating, we arrive at

\[
\int_{\mathbb{R}^3} |\phi_{\lambda,\epsilon}(u_{\lambda,\epsilon})^-|^2 dx + \int_{\mathbb{R}^3} |\phi_{\lambda,\epsilon}(u_{\lambda,\epsilon})^-|^4 dx \leq 0,
\]

which implies that \( \phi_{\lambda,\epsilon}(u_{\lambda,\epsilon})^- \equiv 0 \). Similarly, we can also prove \( u_{\lambda,\epsilon}^- \equiv 0 \).

Furthermore, for fixed \( \epsilon > 0 \), from Lemma 3.3, we have that

\[
0 = \lim_{\lambda \rightarrow \infty} c_{\lambda,\epsilon,0} \quad \lim_{\lambda \rightarrow \infty} \left( I_{\lambda,\epsilon,0}(u_{\lambda,\epsilon}) - \frac{1}{\nu} I_{\lambda,\epsilon,0}(u_{\lambda,\epsilon}) \right)
\]

\[
\geq \lim_{\lambda \rightarrow \infty} \frac{\nu - 2}{\nu} ||u_{\lambda,\epsilon}||_V^2,
\]
which implies that \( \lim_{\lambda \to \infty} ||u_{\lambda,\varepsilon}||_V = 0 \). Then by the continuity of the map \( \Phi_{\lambda,\varepsilon} \) (given in Lemma 2.3) and the fact that \( X \to L^\infty(\mathbb{R}^3) \), we get

\[
\lim_{\lambda \to \infty} ||\phi_{\lambda,\varepsilon}||_X = 0 \quad \text{and} \quad \lim_{\lambda \to \infty} ||\phi_{\lambda,\varepsilon}||_\infty = 0.
\]

\[
\square
\]

5. Proof of Theorem 1.2. In this section, we consider the asymptotical behavior of the pair of positive solutions \((u_{\lambda,\varepsilon}, \phi_{\lambda,\varepsilon}(u_{\lambda,\varepsilon}))\) with respect to the parameter \(\varepsilon\). From now on, we fix the parameter \(\lambda > \Lambda\), where \(\Lambda\) is given above. Our proof relies on Lemma 3.6 and Lemma 4.1.

Let \(\varepsilon \to 0\) and \(\{u_{\lambda,\varepsilon}\} \subset H_V(\mathbb{R}^3)\) be a sequence of solutions of (1). By using similar method in Lemma 3.4, we can prove that \(\{u_{\lambda,\varepsilon}\}\) is bounded in \(H_V(\mathbb{R}^3)\). Then there exists \(u_{\lambda,0} \in H_V(\mathbb{R}^3)\) such that as \(\varepsilon \to 0\)

\[
u_{\lambda,\varepsilon} \rightharpoonup u_{\lambda,0} \quad \text{in} \quad H_V(\mathbb{R}^3).
\]

We claim that \(u_{\lambda,\varepsilon} \to u_{\lambda,0} \) in \(H_V(\mathbb{R}^3)\). In fact, let \(\mu \to 0\) and \(\varepsilon \to 0\), then

\[
||u_{\lambda,\varepsilon} - u_{\lambda,0}||_V \leq ||u_{\lambda,\varepsilon} - u_{\lambda,0,\mu}||_V + ||u_{\lambda,0,\mu} - u_{\lambda,0}||_V \to 0,
\]

where we make use of Lemma 3.6 and Lemma 4.1. Therefore, we conclude as \(\varepsilon \to 0\)

\[
u_{\lambda,\varepsilon} \to u_{\lambda,0} \quad \text{in} \quad H_V(\mathbb{R}^3).
\]

By repeating the argument in Lemma 4.1, we know that \(u_{\lambda,\varepsilon}, u_{\lambda,0} \in L^\infty(\mathbb{R}^3)\).

Next, we will show that \(u_{\lambda,0} \in H_V(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)\) is a solution of (11). On one hand, the compact embedding from \(H_V(\mathbb{R}^3)\) into \(L^s(\mathbb{R}^3)\) \((2 \leq s < 12)\) implies that

\[
u_{\lambda,\varepsilon}^2 \to u_{\lambda,0}^2 \quad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad L^{\frac{s}{2}}(\mathbb{R}^3).
\]

In view of Lemma 2.2, we can obtain as \(\varepsilon \to 0\),

\[
\phi_{\varepsilon}(u_{\lambda,\varepsilon}) \to \phi_0(u_{\lambda,0}) \quad \text{in} \quad D^{1,2}(\mathbb{R}^3), \quad \varepsilon \phi_{\varepsilon}(u_{\lambda,\varepsilon}) \to 0 \quad \text{in} \quad D^{1,4}(\mathbb{R}^3).
\]

On the other hand, since \(u_{\lambda,\varepsilon}\) is a solution of (1), then for any \(v \in C_0^\infty(\mathbb{R}^3)\), we have

\[
\int_{\mathbb{R}^3} (\nabla u_{\lambda,\varepsilon} \cdot \nabla v + V(x)u_{\lambda,\varepsilon}v)dx + \int_{\mathbb{R}^3} u_{\lambda,\varepsilon}^2 \nabla u_{\lambda,\varepsilon} \cdot \nabla vdx + \int_{\mathbb{R}^3} |\nabla u_{\lambda,\varepsilon}|^2 u_{\lambda,\varepsilon}vdx
\]

\[+ \int_{\mathbb{R}^3} \phi_{\varepsilon}(u_{\lambda,\varepsilon}) u_{\lambda,\varepsilon} vdx = \lambda \int_{\mathbb{R}^3} f(x, u_{\lambda,\varepsilon})vdx.
\]

Let \(\psi \in C_0^\infty(\mathbb{R}^3)\) with \(\text{supp}(\psi) \subset K\), where \(K\) is a compact set. For any \(\psi \geq 0\), set \(\xi = \psi e^{-u_{\lambda,\varepsilon}}\). Substituting \(\xi\) into (48), similar to the analysis in [13], we can obtain

\[
\int_{\mathbb{R}^3} (\nabla u_{\lambda,0} \cdot \nabla \psi + V(x)u_{\lambda,0}\psi)dx + \int_{\mathbb{R}^3} u_{\lambda,0}^2 \nabla u_{\lambda,0} \cdot \nabla \psi dx + \int_{\mathbb{R}^3} |\nabla u_{\lambda,0}|^2 u_{\lambda,0}\psi dx
\]

\[+ \int_{\mathbb{R}^3} \phi_{\varepsilon}(u_{\lambda,0}) u_{\lambda,0} \psi dx \geq \lambda \int_{\mathbb{R}^3} f(x, u_{\lambda,0})\psi dx.
\]

Setting \(\xi = \psi e^{u_{\lambda,\varepsilon}}\), then we can obtain an opposite inequality. Therefore, for any \(\psi \in C_0^\infty(\mathbb{R}^3)\), we can conclude that

\[
\int_{\mathbb{R}^3} (\nabla u_{\lambda,0} \cdot \nabla \psi + V(x)u_{\lambda,0}\psi)dx + \int_{\mathbb{R}^3} u_{\lambda,0}^2 \nabla u_{\lambda,0} \cdot \nabla \psi dx + \int_{\mathbb{R}^3} |\nabla u_{\lambda,0}|^2 u_{\lambda,0}\psi dx
\]

\[+ \int_{\mathbb{R}^3} \phi_{\varepsilon}(u_{\lambda,0}) u_{\lambda,0} \psi dx = \lambda \int_{\mathbb{R}^3} f(x, u_{\lambda,0})\psi dx.
\]

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which implies that \( u_{\lambda,0} \) gives rise to a pair of solutions \((u_{\lambda,0}, \phi_0(u_{\lambda,0}))\) of the Schrödinger-Poisson system (11). Thus the proof of Theorem 1.2 is finished.

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