Bifurcation set for a disregarded Bogdanov-Takens unfolding: Application to 3D cubic memristor oscillators

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Received: 8 September 2020 / Accepted: 5 March 2021 / Published online: 5 April 2021
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Abstract Motivated by the dynamical analysis of certain memristor-based oscillators, in this paper we derive the bifurcation set for a three-parametric Bogdanov-Takens unfolding that has not been previously considered in the literature (the saddle-focus-saddle case). By using several first-order Melnikov functions, we obtain for the first time analytical approximations for the bifurcation curves corresponding to homoclinic and heteroclinic connections, which along with the curves associated to local bifurcations organize the parametric regions with different qualitative phase planes. Our interest is the study of a family of 3D memristor oscillators, whose memristor characteristic function is a cubic polynomial. We show that these systems have a first integral; thus, after reducing the problem in one dimension, we can take advantage of the bifurcation set previously obtained. For a certain parameter region, the existence of closed surfaces completely foliated by periodic orbits in the original three-dimensional setting is shown. Additionally, we clarify some misconceptions that arise from the numerical simulations of these systems, emphasizing the important role played by the invariant manifolds associated to the involved first integral.

Keywords Bifurcation analysis · Bogdanov-Takens singularity · Homoclinic orbit · Heteroclinic connection · Melnikov function · Memristor oscillators

1 Introduction

From some decades ago, bifurcation analysis has become a crucial tool to understand the richness of different dynamical behaviors that one can find in a given dynamical system, see [26]. In planar systems, the existence of some local high-codimension bifurcations may reveal the presence of other non-local bifurcations [10, 14] that define in the parameter space new bifurcation curves whose global character makes them difficult to determine. This is, for instance, the case regarding the appearance of homoclinic or heteroclinic connections.

A homoclinic connection is an orbit of the system that emerging from a saddle equilibrium point returns to itself, generically creating or destroying a periodic orbit. A heteroclinic connection joins two different equilibrium points of the system so that the appearance or disappearance of this connection can determine changes in the basin of attraction of a positively invariant set, see for instance [37].

Following [17], the techniques to study homoclinic orbits in planar vector fields were well developed dur-
ing the 1920s in the works of Dulac. The fundamental idea is that the recurrent behavior near a connecting orbit should be studied in a fashion similar to that used in studying periodic orbits via Poincaré return maps. There are however some additional difficulties in the study of homoclinic orbits, which significantly complicate the analysis.

Although in most cases the bifurcation curves associated to homoclinic and heteroclinic connections are studied by means of numerical continuation techniques [9, 16, 17, 38], there appear situations where some analytical tools are useful. Thus, when a planar system can be written as a perturbed Hamiltonian system, we can calculate some Melnikov functions, introduced by Melnikov in [29], and under certain hypotheses, the zeros of the associated Melnikov function determine the existence of periodic orbits, homoclinic loops or heteroclinic connections [3, 8, 20].

According to the classification proposed in [10], the codimension three deformation of the Bogdanov-Takens normal form is given by the unfolding

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= ?
\end{align*}
\]  

(1)

a family of systems whose presence has been widely reported in different applications, see for instance [2, 18, 24]. Many contributions have been made through the study of bifurcation phenomena in these systems. In [15], the authors studied the global bifurcation diagram of the three-parameter family

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= ?
\end{align*}
\]  

(2)

and fixing \( \mu_3 > 0 \), they obtained analytical approximations for the bifurcation curves of the homoclinic orbits, by using Melnikov functions. Later on, the above work was quoted in [23], where a numerical analysis of the same model was performed. In [11–13], it was considered the system

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= ?
\end{align*}
\]  

(3)

which up to the best of our knowledge, seems to be a disregarded situation that is able to exhibit however a rather interesting dynamical behavior, see Figs. 1 and 2. Here, the maximal dynamical richness appears for an equilibrium distribution of type saddle-focus-saddle on the \( x \)-axis. As shown later, system (3) naturally appears in the analysis of certain 3D and 4D memristor oscillators [1, 22, 31, 32] under specific hypotheses on the memristor characteristics, once achieved a dimensional reduction thanks to the existence of a first integral.

The paper is organized in the following way. First, in Sect. 2 we present our main results. We review the information that can be gained by means of a local analysis of the system. Next, Melnikov theory is applied to approximate the homoclinic and heteroclinic curves on a convenient parameter plane, completing the bifurcation set of system (3), so splitting such a plane in regions with different qualitative dynamical behavior, see again Figs. 1 and 2. Later, in Sect. 3, we show how the above analysis is useful to derive all the possible responses of certain 3D canonical memristor oscillators, when the flux-charge characteristics function is a specific cubic polynomial, generalizing some results given in [1]. As one of the possible dynamical behaviors, we focus our attention in justifying, thanks to the previous analysis, the existence of a topological sphere in the 3D phase-space completely foliated by periodic orbits. Thus, we confirm previous numerical results reported in [25, 30]. The necessity of incorporating rigorous techniques in the analysis of memristor oscillators is emphasized with the material of Sect. 4, where following a similar procedure for the dimensional reduction of Sect. 3, we can refute several recently published studies that report the existence of an infinite number of hidden attractors in a three-dimensional memristor-based autonomous Duffing oscillator. Proofs of main results appear in
Fig. 1 From left to right, the bifurcation set of system (3) and two successive magnifications, taking \( \mu_3 > 0 \) sufficiently small. It is symmetric with respect to the \( \mu_2 \)-axis. Curves labelled \( \psi_{sn}, \psi_H, \psi_{ht} \) and \( \psi_h \) represent saddle-node bifurcation, Andronov-Hopf bifurcation, heteroclinic connection and homoclinic connection points respectively, and they are defined in (6), (7), (17), and (23). Labels BT, DHT and \( S_±^1 - S_±^2 \) stand for Bogdanov-Takens bifurcation, double heteroclinic connection and Schecter’s points, respectively and they are defined in (8), (18) and (19). Point \( E \), being a heteroclinic connection point and a Andronov-Hopf bifurcation point, is not a higher co-dimension bifurcation point, since such bifurcations involve different equilibrium points. See Sect. 2 for the definition of the different bifurcation curves and higher co-dimension bifurcation points described above.

Fig. 2 Phase portraits of system (3) in the parameter regions of Fig. 1 (labeled with digits in bold). The thick lines are the boundaries of the basin of attraction of a limit cycle, formed by some stable and unstable manifolds of the saddle points. The green lines are the heteroclinic connections. In magnifications, the stable and unstable manifolds of the saddle points are denoted by \( W^s \) and \( W^u \).

Sect. 5, while some technical results are relegated to Appendix.

2 Main results

We start by studying the local and global bifurcations that occur in system (3). For ease of reading, all the proofs for the results of this section appear in sect. 5. First, we note that the system is invariant under the transformation
\[
(x, y, \mu_1, \mu_2, \mu_3) \rightarrow (-x, -y, -\mu_1, \mu_2, \mu_3),
\]
and therefore, it suffices to study the bifurcation set for \( \mu_1 \geq 0 \).

The following result is a direct consequence of Bendixson’s criterion [21], after computing the Jaco-
bifurcation of system (3) on the parametric plane

\[ J(x, y) = \begin{pmatrix} 0 & 1 \\ \mu_2 + 3x^2 - 6yx & \mu_3 - 3x^2 \end{pmatrix}, \]

since for \( \mu_3 \leq 0 \) the divergence of system (3) does not change its sign.

**Lemma 1** System (3) does not have closed orbits for \( \mu_3 \leq 0 \).

The equilibrium points of the system are of the form \( (x, y) = (\bar{x}, 0) \), being \( \bar{x} \) a solution of the cubic \( \mu_1 + \mu_2 x + x^3 = 0 \). It follows a technical result that provides a study of the number of equilibria in system (3) and their topological type.

**Lemma 2** Considering system (3), the following statements hold.

(a) If \( \mu_2 \geq 0 \) or we have \( \mu_2 < 0 \) with \( 27\mu_1^2 + 4\mu_2^3 > 0 \), then the system has only one equilibrium point.

(b) If \( \mu_2 < 0 \) and \( 27\mu_1^2 + 4\mu_2^3 = 0 \) we have two equilibrium points.

(c) If \( \mu_2 < 0 \) and \( 27\mu_1^2 + 4\mu_2^3 < 0 \), then the system has the three equilibrium points \( x_i = (s_i, 0) \) with \( i \in \{L, C, R\} \) such that

\[
s_L < -(-\mu_2/3)^{1/2} < s_C < (-\mu_2/3)^{1/2} < s_R,
\]

and \( s_L + s_C + s_R = 0 \). Furthermore, \( x_L \) and \( x_R \) are saddles while \( x_C \) is an anti-saddle (node or focus).

In the next result, we give a characterization of the local bifurcations of system (3) on the parametric plane \((\mu_2, \mu_1)\), assuming a fixed value of the parameter \( \mu_3 \). Note that we have put in the plane \((\mu_2, \mu_1)\) the \( \mu_1 \)-axis as the vertical, being the \( \mu_2 \)-axis the horizontal one, see Fig. 1.

**Proposition 1** The following statements hold for system (3).

(a) Given \( \mu_3 \in \mathbb{R} \) the parameter values in the set

\[ \varphi_{sn} = \{(\mu_2, \mu_1) : 27\mu_1^2 + 4\mu_2^3 = 0\}, \]

correspond with saddle-node bifurcations of equilibria. In particular, the system has a cusp bifurcation of equilibria at \( C = (0, 0) \), the origin of plane \((\mu_2, \mu_1)\).

(b) Given \( \mu_3 > 0 \), the parameter values in the set

\[ \varphi_H = \{(\mu_2, \mu_1) : \mu_1 = \pm (\mu_3/3)^{3/2} \pm (\mu_3/3)^{1/2} \mu_2, \ \mu_2 < -\mu_3\}, \]

represent Andronov-Hopf bifurcation points of codimension one for the central equilibrium point \( x_C \), see Lemma 2(c) and Fig. 1(a).

(c) The set defined in (7) determines a symmetric pair of straight half lines emanating from two points corresponding to Bogdanov-Takens bifurcation points, namely

\[ BT \equiv \left(-\mu_3, \pm 2(\mu_3/3)^{3/2}\right), \]

see Fig. 1(b).

Now, by fixing one parameter and resorting to Melnikov theory, we will approximate the global bifurcation curves of system (3). Before that, we must write the system as a perturbed Hamiltonian system. This can be done in different ways, as indicated in the next result. The possibility of resorting to one of the next two reparameterizations will be helpful later.

**Proposition 2** System (3) can be written as two different perturbed Hamiltonian systems, as follows.

(a) Taking

\[ \mu_1 = \varepsilon^4 v_1, \ \mu_2 = -\varepsilon^2 v_2, \ \mu_3 = \varepsilon^2 v_3, \]

the system can be rewritten as

\[ \dot{x} = y, \]
\[ \dot{y} = -v_2 x + x^3 + \varepsilon (v_1 + v_3 y - 3x^2 y), \]

which for \( \varepsilon = 0 \) corresponds to the Hamiltonian function

\[ H_1(x, y) = \frac{y^2}{2} + \frac{x^2}{2} - \frac{x^4}{4}. \]

(b) Taking

\[ \mu_1 = \varepsilon^3 v_1, \ \mu_2 = -\varepsilon^2 v_2, \ \mu_3 = \varepsilon^2 v_3, \]

the system can be rewritten as

\[ \dot{x} = y, \]
\[ \dot{y} = v_1 - v_2 x + x^3 + \varepsilon (v_3 y - 3x^2 y), \]

which for \( \varepsilon = 0 \) corresponds to the Hamiltonian function

\[ H_2(x, y) = \frac{y^2}{2} - v_1 x + \frac{x^2}{2} - \frac{x^4}{4}. \]
The phase portrait for the unperturbed ($\varepsilon = 0$) Hamiltonian systems (10) and (13) are shown in Fig. 3. Note that when $v_1 = 0$ we obtain $H_1(x, y) = H_2(x, y)$, and so in that case it suffices to study the properties of the Hamiltonian $H_1$.

For $\varepsilon = 0$, the Hamiltonian system (10) has a pair of heteroclinic connections $\Gamma_\pm(t) = (x(t), \pm y(t))$, parameterized by

$$
\begin{align*}
    x(t) &= \sqrt{v_2} \tanh \left( \sqrt{v_2/2} t \right), \\
    y(t) &= \frac{v_2}{\sqrt{2}} \sech^2 \left( \sqrt{v_2/2} t \right),
\end{align*}
$$

where $-\infty < t < \infty$ and $v_2 > 0$. Note that the extra condition of Lemma 2(c) for the existence of three equilbria becomes after (9) the inequality $27 \varepsilon^2 v_1^2 < 4 v_2^3$, which is always satisfied if $\varepsilon$ is sufficiently small. In the next two results, we compute the Melnikov function along the heteroclinic connection $\Gamma_+$ for the perturbed Hamiltonian system (10), and recalling the reparameterization (9), we obtain valuable approximations for the heteroclinic bifurcation curves of system (3).

**Proposition 3** If we consider the perturbed Hamiltonian system (10) for parameters $V = (v_1, v_2, v_3)$ with $v_2 > 0$, then the Melnikov function associated to the heteroclinic connections $\Gamma_\pm$ is given by

$$
M_{ht}(V) = \frac{2}{15} \sqrt{v_2} \left( 15 v_1 + 5 \sqrt{2} v_2 v_3 - 3 \sqrt{2} v_3^2 \right),
$$

After fixing one parameter of system (3), we can now give the promised approximation for the heteroclinic connection curves in the corresponding section of the parameter space.

**Proposition 4** Consider system (3) with $\mu_3 > 0$ sufficiently small. Then the system has a unique hyperbolic heteroclinic connection for parameter values on the parametric plane $(\mu_2, \mu_1)$ in a neighborhood of the curve

$$
\varphi_{ht} = \{(\mu_2, \mu_1) : \mu_1 = \pm \sqrt{\frac{2}{15}} \mu_2 \left( 3 \mu_2 + 5 \mu_3 \right) + 27 \mu_1^2 \equiv - \mu_2 - 4 \mu_3^2 < 0, \mu_2 \neq - \frac{5}{3} \mu_3 \}.
$$

When $\mu_1 = 0$ and $\mu_3 > 0$, we obtain on the parameter plane $(\mu_2, \mu_1)$ the double heteroclinic connection point

$$
DHT \equiv (- \frac{5}{3} \mu_3, 0).
$$

Recall that the so-called Schecter’s points are codimension two points defined by the intersection of a saddle-node curve and a homoclinic or heteroclinic curve, for more details see [34]. Taking the intersection points of the saddle-node bifurcation curve and the heteroclinic curves given in (6) and (17) respectively, we obtain a first-order approximation for the Schecter’s points of the system. Since the system is symmetric with respect to the parameter $\mu_1$, the system has four Schecter’s points (see Fig. 1), namely

$$
\begin{align*}
    S_1^+ &= \rho \left( (5/27), \pm (5 \sqrt{10} / 729) \left( \sqrt{18 \mu_3 + 5} + 5 \sqrt{5} \right) \right), \\
    S_2^+ &= - \rho \left( (5/27), \pm (5 \sqrt{10} / 729) \left( \sqrt{18 \mu_3 + 5} - 5 \sqrt{5} \right) \right),
\end{align*}
$$

where

$$
\rho = 9 \mu_3 + 5 - \sqrt{5} \sqrt{18 \mu_3 + 5}.
$$

Now, by using the homoclinic connection of Hamiltonian system (14), we compute the associated Melnikov function for system (3) when $v_3 = 1$. 

\[ \square \]
Proposition 5 If we consider system (13) and $\mathbf{v} = (v_1, v_2, v_3)$ with $v_1 > 0$, $v_2 > 0$ and $v_3 = 1$, then the Melnikov function associated to the homoclinic orbit with connection point $(0, s_R)$, it is given by

$$M(\mathbf{v}) = \sqrt{2} \frac{\cosh^2(\theta)}{\cosh^2(\theta) + 2} \left( F_1(\theta) + v_2 F_2(\theta) \right),$$

where

$$F_1(\theta) = 72(\cosh^5 + 1) + 240\cosh^3 \theta$$

$$- 320 \cosh^2 \theta \sinh \theta - 80 \cosh^4 \theta \sinh \theta + 480 \cosh \theta, \quad F_2(\theta) = 1440 \cosh \theta - 768 \sinh \theta - \cosh^3 \theta$$

$$- 1344 \cosh^2 \theta \sinh \theta - 48 \cosh^4 \theta \sinh \theta$$

and $0 < \theta < \infty$, with

$$\cosh \theta = \frac{2s_R}{\omega}, \quad \omega^2 = 2(v_2 - s_R^2) > 0, \quad v_1 = v_2 s_R - s_R^3,$$

being $s_R$ the biggest positive root of the equation $v_1 - v_2 x + x^3 = 0$, see Fig. 10.

As a direct consequence of the above result, we give an analytical approximation of the bifurcation curves for homoclinic connections of system (3).

Proposition 6 Consider system (3) with $\mu_3 > 0$ sufficiently small. Then the system has a unique homoclinic orbit for parameter values $(\mu_2, \mu_1)$ in a neighborhood of the curve

$$\varphi_h = \{(\mu_2, \mu_1) \in \mathbb{R}^2 : \mu_2 = -\mu_3 v_2(\theta), \quad \mu_1 = \pm \mu_3^{3/2} v_1(\theta), \quad 0 < \theta < \infty \},$$

where

$$v_2(\theta) = \frac{10(\cosh 2\theta + 5)(\sinh \theta + \sinh 3\theta - 12\cosh \theta)}{3(370 \sinh \theta + 115 \sinh 3\theta + 360 \sinh 3\theta)}, \quad v_1(\theta) = v_2(\theta) s - s^3, \quad s^2 = \frac{\cosh^2 \theta}{2 + \cosh^2 \theta} v_2(\theta),$$

see Fig. 4(a). Moreover, for the points $(\mu_2(\theta), \mu_1(\theta))$ at the curve $\varphi_h$ we have

$$\lim_{\theta \to 0^+} (\mu_2, \mu_1) = \left(-\mu_3, \frac{\mu_3}{3} \sqrt{3/3} \right), \quad \lim_{\theta \to \infty} (\mu_2, \mu_1) = (-5\mu_3/3, 0).$$

Remark 1 Note that from the previous result we obtain the two points

$$\lim_{\theta \to 0^+} (\mu_2, \mu_1) \equiv BT, \quad \lim_{\theta \to \infty} (\mu_2, \mu_1) \equiv DHT,$$

where the points $BT$ and $DHT$ are given in (8) and (18), respectively.

Remark 2 After some algebra, to find the maximum values for the function $v_2$ defined in (23) amounts to determine the zeros of the function

$$h_1(x) = 2x (6 \cosh 2x + \cosh 4x + 33) - 5(10 \sinh 2x + \sinh 4x),$$

where $h_1(0) = 0$, $h_1(1) < 0$ and $h_1(2) > 0$, see Fig. 4(c). Accordingly, at $\theta^* \approx 1.8630981$ the function $v_2$ has a maximum given by $v_2(\theta^*) \approx 2.454887$, see Fig. 4(b), so that

$$-\frac{5}{2} \mu_3 < -v_2(\theta^*) \mu_3 < -\frac{5}{3} \mu_3.$$
Now, if we consider system (3) with $\mu_3 > 0$ sufficiently small, $\mu_2 < -(5/2)\mu_3$ and $\mu_1$ such that

$$|\mu_1| < \left( \frac{\mu_3}{3} \right)^{3/2} - \left( \frac{\mu_3}{3} \right)^{1/2} \mu_2,$$

then the system has at least one stable limit cycle, see regions and panels labeled 2, 3, 4 of Figs. 1 and 2. This assertion is a direct consequence of Andronov-Hopf bifurcation lines given in (7) and Poincaré-Bendixson theorem [37], since the sign of the Melnikov function guarantees the existence of a compact positive invariant set with only one unstable equilibrium point in its interior.

Assembling all the above results, we show in Fig. 1, the complete bifurcation set of system (3). The panels of Fig. 2 give the different phase portraits for the curves and parameter regions labeled with digits, showing all the possible configurations. Thereby, we achieve a global perspective of the different possible dynamical behaviors for the more relevant case, which from Lemma 1 requires $\mu_3 > 0$. Even our analytical predictions for the global bifurcations require a small value of such parameter, we should not to underestimate such approximations. For instance, if we take system (3) with $\mu_3 = 0.1$ and compare the prediction for the homoclinic bifurcation curve with the obtained by using the shooting method [33], see Fig. 5, the agreement is really excellent.

From the point of view of applications, we arrive at the family of systems (3) coming from the study of some memristor-based 3D electronic oscillators, see next section. It should be emphasized that, from our analysis and particularly from Remark 2, only the regions labelled with digits 2, 3 and 4 in Fig. 1 are significant to get stable oscillations. Furthermore, it is also evident that the stable limit cycles that appear in such parameter regions are not globally stable, since the unstable manifolds of external saddles preclude the existence of global stability at all. Nevertheless, we can take advantage of our study to advice practitioners in choosing parameters belonging to region 2, where the attraction basin of the limit cycle is somehow bigger.

### 3 Application to a 3D Canonical Memristor Oscillator

As a relevant application of the above study, in this section we will show, for a 3D canonical memristor oscillator whose flux-charge characteristics is a monotone cubic polynomial, the existence of a topological sphere completely foliated by periodic orbits organized around two fixed points. The existence of this sphere had been reported numerically in [25, 30].

We start by considering the modeling of an elementary oscillator endowed with one flux-controlled memristor $M$, see Fig. 6 and [22]. In the shown circuit the values of $L$ and $C$ for the impedance and capacitance are positive constants, while the resistor has a negative value $-R$. From Kirchhoff’s laws we see that

$$i_R(\tau) - i_L(\tau) = 0,$$

$$i_L(\tau) - i_C(\tau) - i_M(\tau) = 0,$$

$$-v_R(\tau) + v_L(\tau) + v_C(\tau) = 0,$$

$$v_C(\tau) - v_M(\tau) = 0,$$
where \( v, i \) stand for the voltage and current, respectively, across the corresponding element of the circuit as indicated by the subscript.

In Section 3.2 of [22], this circuit is proposed as a third-order canonical memristor oscillator but the notation is slightly different as follows. They take \( i_1 = i_C, i_3 = i_L = i_R, i = i_M, v_1 = v_C = v_M, v_3 = v_L, v_4 = v_R, q_1 = q_C, q_3 = q_L, q_4 = q_R \) and \( \varphi = \varphi_M \). Thus, they write the two equations

\[
\dot{i}_1 = i_3 - i, \quad \dot{v}_3 = v_4 - v_1, \tag{25}
\]

and, after integrating respect to time, they arrive to the corresponding expressions for charges \((q)\) and fluxes \((\varphi)\), namely

\[
q_1 = q_3 - q(\varphi), \quad q_3 = q_4 - q_1, \tag{26}
\]

where \( q(\varphi) \) stands for the nonlinear flux-charge characteristics of the flux-controlled memristor. Solving now for \((q_3, q_4)\) and taking into account that \( q_1 = \varphi \) since \( v_1 = v_M \), it is immediate to obtain

\[
q_3 = q_1 + q(\varphi), \quad q_4 = \varphi + q_3, \tag{27}
\]

so that authors conclude that a good choice for independent variables is the triple \((q_1, q_3, q_4)\), that is, the charge of capacitor \( C \), the flux of the inductor \( L \) and the flux of the memristor, respectively. Accordingly, by taking derivatives in \((25)\), the following set of differential equations is proposed,

\[
\begin{align*}
C \dot{q}_1 &= i_1 - W(\varphi) v_1, \\
L \dot{q}_3 &= R i_3 - v_1, \\
\dot{\varphi} &= v_1,
\end{align*}
\]

\(\dot{q}_1 = i_C, \dot{q}_3 = i_L = i_R, i = i_M, \dot{v}_1 = v_C = v_M, \dot{v}_3 = v_L, v_4 = v_R, q_1 = q_C, q_3 = q_L, q_4 = q_R \) and \( \varphi = \varphi_M \).

Finally, they rewrite the system as follows,

\[
\begin{align*}
\dot{x} &= \alpha (y - W(z)x), \\
\dot{y} &= -\xi x + \beta y, \\
\dot{z} &= x, \tag{29}
\end{align*}
\]

where \( x = v_1, y = i_3, z = \varphi \), and the parameters used are \( \alpha = 1/C, \xi = 1/L, \beta = R/L \), so that, \( \alpha, \xi, \beta > 0 \). An important observation is that the parameter \( \alpha \) is not essential so that it can be removed with the change of variables and parameters

\[
\begin{align*}
\tilde{x} &= x, \\
\tilde{y} &= a y, \\
\tilde{z} &= z, \\
\tilde{\xi} &= \alpha \xi, \\
\tilde{a} &= \alpha a, \\
\tilde{b} &= \alpha b, \\
\tilde{W} &= \alpha W,
\end{align*}
\]

\(\tilde{a} = a, \tilde{b} = b \) to be assumed in the sequel, omitting also tildes to alleviate the notation. Therefore, we need to study the system

\[
\begin{align*}
\dot{x} &= -W(z)x + y, \\
\dot{y} &= -\xi x + \beta y, \\
\dot{z} &= x, \tag{29}
\end{align*}
\]

where \( W(z) = q'(z) = 3z^2 + 2az + b \), and \( q(z) = z^3 + az^2 + bz \), \( a^2 - 3b < 0 \), which assumes that the memristor is passive \((q'(z) > 0 \) for all \( z \). Note that in a symmetric memristor we should have \( a = 0 \), which is a much easier case to study.

System \((29)\) belongs to a class of systems whose dimensional reduction is possible thanks to the existence of a first integral, as shown in Appendix. Like other models of memristor oscillators, system \((29)\) has some special features. For instance, it has a continuum of equilibria on the \( z \)-axis, that is, every point \((\tilde{x}, \tilde{y}, \tilde{z})\) with \(\tilde{x} = \tilde{y} = 0\) is an equilibrium point for the system. Accordingly, the Jacobian matrix at any of these points has a zero eigenvalue. Regarding only the other two eigenvalues, we will assume other natural conditions, namely

\[
a^2 - 3b + 3\frac{\xi}{\beta} > 0, \tag{31}
\]

to have equilibrium points of anti-saddle type, and

\[
a^2 - 3b + 3\beta > 0,
\]

which allows to have equilibria with positive trace (otherwise, all the equilibria have negative trace).

Taking the parameters \( a_1 = -1, a_2 = 1, a_2 = -\xi \) and \( a_2 = \beta \) in Proposition 12 of Appendix, we obtain that for all \( h \in \mathbb{R} \), system \((29)\) has an invariant manifold \( S_h \) defined by

\[
S_h = \{(x, y, z) \in \mathbb{R}^3 : -\beta x + y - \beta z^3 - a\beta z^2 + (\xi - b\beta)z = h \}.
\]

Moreover, assuming \( c = 1 \) in Corollary 1 of the Appendix, we obtain that on each invariant manifold
\(S_h\), the system is topologically equivalent to the Liénard system
\[
\begin{align*}
\dot{x} &= y - x^3 - x^2 - (b - \beta)x, \\
\dot{y} &= \beta x^3 + a\beta x^2 + (b\beta - \xi) x + h.
\end{align*}
\] (33)
In Figure 7, we show the invariant manifold (32) corresponding to \(h = 0.3\) and the set of parameters \(\xi = 80, a = b = 1, \beta = 5\), along with the phase space of the equivalent Liénard system (33). Parameters are taken like in Fig. 14 of [30].

From Proposition 14(a) in Appendix, the system can be rewritten as
\[
\dot{x} = y, \quad \dot{y} = \mu_1 + \mu_2 x + \mu_3 y + x^3 - 3x^2 y,
\] (34)
where the new parameter are
\[
\mu_1 = \frac{1}{27\beta^{3/2}} (27h + 9a\xi + 2a^3\beta - 9ab\beta),
\]
\[
\mu_2 = -\frac{1}{3\beta} \left( a^2 - 3b + 3\frac{\xi}{\beta} \right) < 0,
\]
\[
\mu_3 = \frac{1}{3\beta} \left( a^2 - 3b + 3\beta \right) > 0.
\]

From Remark 2, given \(\mu_3\) sufficiently small, for all \(\mu_2 < -5\mu_3/2 < 0\) and \(\mu_1\) such that
\[
|\mu_1| < \left( \frac{\mu_3}{3} \right)^{3/2} - \left( \frac{\mu_3}{3} \right)^{3/2} \mu_2,
\] (36)
we have that the system has a stable limit cycle. Therefore, we can give the following result in terms of the parameter \(h\), which determines the planar system (33) that controls the dynamics in the corresponding manifold \(S_h\) of 3D system (29). The result guarantees the existence of a topological sphere in the 3D phase-space completely foliated by periodic orbits, which appear organized around two fixed points.

**Proposition 7** Consider system (29) with \(\beta, \xi > 0\) and the function \(q\) defined as in (30), satisfying \(a^2 - 3b < 0\) (passive memristor) and the condition for existence of anti-saddles (31). Furthermore, assume that
\[
a^2 - 3b + 3\beta > 0
\] (37)
is sufficiently small and that the following inequality
\[
\beta (3b - a^2) - 3\xi < (5/2) \left( 3b - a^2 - 3\beta \right) \beta < 0
\] (38)
holds. Then for all \(h \in \mathbb{R}\) with
\[
-\frac{A}{27} < h < \frac{B}{27},
\]
where
\[
A = \left( 4a^2\beta + 3\beta^2 - 12b\beta + 9\xi \right) \sqrt{a^2 - 3b + 3\beta} + 9a\xi + 2a^3\beta - 9ab\beta,
\]
\[
B = \left( 4a^2\beta + 3\beta^2 - 12b\beta + 9\xi \right) \sqrt{a^2 - 3b + 3\beta} - 9a\xi - 2a^3\beta + 9ab\beta,
\]
the system has a stable periodic orbit. Moreover, there exists a topological sphere \(\mathcal{Q}\) foliated by such periodic orbits, which appear organized around two fixed points, see Fig. 8.

**Proof** From Remark 2, and after substituting the values of \(\mu_1, \mu_2\) and \(\mu_3\) given in (35), we obtain as sufficient conditions for the existence of limit cycles the inequalities (37)-(38). Now, from (36) we obtain the condition \(|\mu_1| < (1/3) (\mu_3/3)^{1/2} (\mu_3 - 3\mu_2)\), so that \(\mu_3 - 3\mu_2 > 0\), since from hypotheses we have \(\mu_2 < -(5/2)\mu_3 < 0\). Now after some algebra, the condition becomes
\[
|27h + 9a\xi + 2a^3\beta - 9ab\beta| < \left( 4a^2\beta + 3\beta^2 - 12b\beta + 9\xi \right) \sqrt{a^2 - 3b + 3\beta}.
\]

Taking into account the absolute value, and grouping terms, we obtain the values of \(A\) and \(B\) defined in (39). Finally, from Remark 2 system (29) has a stable periodic orbit on each \(S_h\) defined in (32). By varying continuously the parameter \(h\), the periodic orbits locally define a cylindrical surface that turns globally to be a sphere foliated by such periodic orbits, which appear surrounding the two fixed points with zero trace, namely \((\tilde{x}, \tilde{y}, \tilde{z}) = (0, 0, \tilde{z}_\pm)\) with
\[
3\tilde{z}_\pm = -a \pm \sqrt{a^2 - 3b + 3\beta}.
\]

We remark that to avoid the accuracy problems in numerically integrating system (29), which should respect the invariant manifolds \(S_h\) defined in (32), Fig. 8 has been obtained by using expression (63) from Corollary 1 in Appendix and confirms previous numerical results reported in Figure 14 of [30].

### 4 False Hidden Attractors in Memristor-Based Autonomous Duffing Oscillators

In this section, we give another application of the previous analysis. An attractor is called a hidden attractor...
Fig. 7  (a) The invariant manifold (32) corresponding to the set of parameters $\xi = 80, a = b = 1, \beta = 5$ and $h = 0.3$ is shown. Parameters are taken like in Fig. 14 of [30]. In black the infinite number of equilibrium points of system (29) and in blue a periodic orbit of the system contained in the invariant manifold.

(b) The phase plane of the equivalent Liénard system (33) corresponding to parameters given in (a), showing a limit cycle in blue; regarding Proposition 13, we also draw the function $g(X)$ (in red) and the function $F(X)$ (in black). (c) A partial zoom of figure (b)

Fig. 8  For system (29) with parameters $a = 1, b = 4.8, \beta = 5$ and $\xi = 80$, drawing the periodic orbits on each invariant manifold $S_h$ defined in (32), we obtain some slices of the surface $\Omega$ predicted in Proposition 7. For this set of parameters we get $\mu_3 = 0.86 > 0$, $\mu_2 = -3.86 < -(5/2)\mu_3$, $A = 4236.1$ and $B = 2506.1$

if its basin of attraction does not intersect any neighborhood of equilibria; otherwise, it is called a self-excited attractor, for more details see [27,28]. Recently in [19,35,36] it was reported the existence of an infinite number of hidden attractors in a memristor-based autonomous Duffing oscillators, whose memristance function is a cubic polynomial. Here, by using a similar approach to the followed in the previous section, we will show that such hidden attractors are not possible, so that the numerical simulations included in [19,35,36] are misleading.

The quoted memristor-based autonomous Duffing oscillator is modeled by the dynamical system
\[ \dot{x} = y, \]
\[ \dot{y} = z, \]
\[ \dot{z} = -\alpha z - M(x)y, \]

where the memristance function \( M \) (possibly discontinuous) is defined as
\[ M(x) = \frac{d\phi(x)}{dx} \]
and \( \phi \) is a continuous function. In fact, the quoted authors consider system (40) with the function \( \phi(x) = \omega x + \beta x^3 \), for some concrete values of parameters, see later, but we will try to make an analysis of the model as much general as possible, without fixing for the moment the function \( \phi \). System (40) has a continuum of equilibria, since any point of the \( x \)-axis is an equilibrium point. In the next result, we show that although system (40) does not belong to the family (47) of Appendix, the system also has a first integral, so possessing an infinite number of invariant manifolds.

**Proposition 8** Consider system (40) with the function \( M \) defined as in (41). For any \( h \in \mathbb{R} \) the set
\[ S_h = \{(x, y, z) \in \mathbb{R}^3 : H_1(x, y, z) = h\} \]
is an invariant manifold for the system, where we have introduced the continuous function
\[ H_1(x, y, z) = \phi(x) + \alpha y + z. \]

Therefore, the system has an infinite number of invariant manifolds foliating the whole \( \mathbb{R}^3 \), and so the dynamics is essentially two-dimensional.

**Proof** Taking \( H_1 \) as in (43), define for any solution \((x(\tau), y(\tau), z(\tau))\) of (40) the auxiliary continuous function
\[ h(\tau) = H_1(x(\tau), y(\tau), z(\tau)) \]
Now, a direct computation gives, excepting the points where the function could be not differentiable,
\[
\begin{align*}
h'(\tau) &= \frac{d\phi(x)}{dx} \dot{x} + \alpha \dot{y} + \dot{z} \\
&= M(x)y + \alpha z - \alpha z - M(x)y = 0.
\end{align*}
\]
Then \( h \) is piecewise constant along the orbits of (40), but as \( h \) is continuous by definition, it should be globally constant. In short, the level sets of \( H_1 \) are invariant for the flow. \( \square \)

Now, by using the above result, for any initial condition \((x_0, y_0, z_0)\) we can compute the value \( h = H_1(x_0, y_0, z_0) \) and conclude that the complete trajectory should remain in the set \( S_h \). The orbits on this set \( S_h \) can be easily projected onto the plane \( z = 0 \), and so we can deduce the dynamical behavior of the system by studying the corresponding planar system, as follows.

**Proposition 9** Consider system (40) with the function \( M \) defined as in (41). Then on each invariant set \( S_h \) defined in (42) the system is topologically equivalent to the planar system
\[ \dot{x} = y, \]
\[ \dot{y} = -\phi(x) - \alpha y + h. \]

Moreover, \((x(\tau), y(\tau)) \in \mathbb{R}^2\) is a solution of the above system if and only if
\[ E_h(x(\tau), y(\tau)) = \begin{pmatrix} x(\tau) \\ y(\tau) \end{pmatrix} \]
is a solution of system (40).

**Proof** From Proposition 8 we can solve for \( z \) in the equation \( H_1(x, y, z) = h \), and write
\[ z = h - \phi(x) - \alpha y. \]
Replacing this expression into the first and second equation of (40) we obtain system (44). Suppose that \((x(\tau), y(\tau)) \in \mathbb{R}^2\) is a solution of system (44). Taking \( z(\tau) = h - \alpha y(\tau) - \phi(x(\tau)) \)
we obtain
\[
\begin{align*}
\dot{z}(\tau) &= -\alpha \dot{y}(\tau) - \frac{d\phi(x(\tau))}{dx} \dot{x}(\tau) \\
&= -\alpha (h - \phi(x(\tau)) - \alpha y(\tau)) - M(x(\tau))y(\tau) = \\
&= -\alpha (z(\tau)) - M(x(\tau))y(\tau).
\end{align*}
\]
and the proposition follows. \( \square \)

Thanks to the following result, we show that for \( \alpha \neq 0 \) system (40) cannot have periodic solutions, see Proposition 11 below.

**Proposition 10** Consider system (44). The following statements hold.
(a) For \( \alpha = 0 \) the system is conservative, since it has the Hamiltonian function
\[ H_2(x, y) = \frac{y^2}{2} - hx + \Phi(x), \]
where \( \Phi(x) \) is such that \( \Phi'(x) = \phi(x) \).
(b) For $\alpha \neq 0$ the system does not have periodic solutions.

Proof When $\alpha = 0$, we get

$$\frac{\partial H_2}{\partial x} \dot{x} + \frac{\partial H_2}{\partial y} \dot{y} = (-h + \phi(x)) y + y (-\phi(x) + h) = 0,$$

as stated.

The divergence of the system is constant, namely $\Delta = -\alpha$. Therefore, for $\alpha \neq 0$ the divergence of system (44) does not vanish and has a constant sign, so that from Bendixson’s criterion [21], there cannot be periodic solutions. \qed

Thus, as a consequence of Propositions 9 and 10, we state the main result of this section.

**Proposition 11** For any continuous function $\phi$ and any value $\alpha \neq 0$, the 3D system (40) cannot have periodic orbits.

Proof From Proposition 8 and 9, every periodic orbit, if any, would be contained in a certain set $S_h$ as defined in (42), and so it should project onto a periodic orbit for the corresponding system (44). Since from Proposition 10, such a planar system cannot have periodic orbits for $\alpha \neq 0$, we get a contradiction, and the proposition follows. Note however that for $\alpha = 0$ the planar systems (44) become Hamiltonian and so they could have an infinite number of periodic orbits on each invariant set $S_h$. \qed

Regarding [19,35,36] authors consider system (40) with the function $\phi(x) = \omega x + \beta x^3$ and the set of parameters $\alpha = 0.0001$, $\omega = 0.35$, $\beta = 0.85$. In all the quoted references, authors reported the existence of an infinite number of stable periodic orbits coexisting with an infinite number of stable equilibria, what is not possible at all, from Proposition 11. Their wrong conclusions on the existence of hidden attractors come after taking into account several numerical simulations where they detect seemingly periodic orbits, the periodicity of these attractors being confirmed by Lyapunov exponents (the first largest Lyapunov exponent is zero, they say). Clearly, this kind of numerical criterium is very prone to errors, even more for a vector field with such a small value ($-\alpha$) for the divergence.

For instance, they consider the orbit with initial conditions $(x_0, y_0, z_0) = (0.8, 0, 6, 0, 0)$, showing its three-dimensional phase plot and the time series for $(x(t), y(t), z(t))$, see Figs. 2 and 3 in [35]. From Proposition 8, using the parameter values and the initial conditions, we obtain that the complete orbit must be in the invariant manifold

$$S_h = \{ (x, y, z) \in \mathbb{R}^3 : \omega x + \beta x^3 + \alpha y + z = h \},$$

where $h = 0.35 \times 0.85 + 0.85 \times 0.85 \times 10^{-4} \times 0.6 + 0.0 = 0.71526$, see Fig. 9(a). Furthermore, from Proposition 9, the dynamics for the variables $x$ and $y$ on such manifold is ruled by the planar system

$$\dot{x} = y, \quad \dot{y} = -\omega x - \beta x^3 - \alpha y + h,$$

where $h$ is as computed and the parameters $\alpha$, $\beta$ and $\omega$ are as before. The only fixed point for this planar system is $(x^*, y^*)$, where $y^* = 0$ and $x^* \approx 0.80003$, the only real root of the equation $\omega x + \beta x^3 - h = 0$. Thus, the only equilibrium in $S_{0.71526}$ is $(x^*, y^*, z^*)$, where $z^* = 0$, acting as a stable focus with a rather small contraction rate, since the real part of its complex eigenvalues is $-\alpha/2 = -5 \times 10^{-5}$. Probably, this fact is the reason for the failure of the numerical criteria employed for assuring periodicity.

Effectively, it is not difficult to repeat the simulation for these values, see Fig. 9(b)-(d), and to conclude that, as Proposition 11 warranties, the system cannot have periodic orbits; what happens is that one must wait a very long time to see how the orbit approaches little by little the stable focus. Thus, the statements made in the quoted papers about the existence of an infinity of hidden attractors are clearly wrong, probably after giving too much credit to short numerical simulations. This emphasizes the relevance of the approach followed in this work which allows to avoid misconceptions coming just from numerical experiments.

5 Proofs of main results

We start by giving the proof of Lemma 2.

**Proof of Lemma 2** We study the roots of the polynomial $p(x) = \mu_1 + \mu_2 x + x^3$. Since $p'(x) = \mu_2 + 3x^2$, if $\mu_2 \geq 0$ we obtain $p'(x) \geq 0$ and so the polynomial has only one root.

In the rest of the proof we assume $\mu_2 < 0$. The derivative $p'(x)$ vanishes at the points $x_\pm = \pm (\mu_2/3)^{1/2}$ being a maximum and minimum local respectively, also a direct computation gives $p(x_\pm) = \mu_1 \mp 2 (\mu_2/3)^{3/2}$. When $p(x_-) < 0$ or $p(x_+) > 0$ the graph of $p(x)$ only crosses once the $x$-axis and so these
inequalities provides the condition $27\mu_1^2 + 4\mu_2^3 > 0$, and the statement (a) follows. Assuming $p(x_+) = 0$ or $p(x_-) = 0$, the statement (b) follows.

Finally, if $p(x_+) < 0 < p(x_-)$ then we have three roots as indicated in (5). Moreover, using the relation between roots and coefficients of polynomials, we get

\[ \mu_1 = -s_Ls_Cs_R, \quad \mu_2 = s_Cs_L + s_Cs_R + s_Ls_R, \quad s_L + s_C + s_R = 0. \]

For the Jacobian matrix given in (4), we get 

\[ J(s_1, 0) = -((\mu_2 + 3s_1^2)) = -p'(s_1). \]

As we know that $p'(s_L) > 0, p'(s_C) < 0$ and $p'(s_R) > 0$, the conclusion follows and the proof is complete. □

Next, we give the proofs for Propositions 1 to 6.

**Proof of Proposition 1** Statement (a) is a direct consequence of the equations

\[ x^3 + \mu_2x + \mu_1 = 0, \quad \mu_2 + 3x^2 = 0, \]

to be fulfilled for any non-hyperbolic equilibrium $(x, 0)$ at a saddle-node bifurcation.

Let $(\tilde{x}, 0)$ an equilibrium point of system (3). Considering the Jacobian matrix $J$ given in (4), then $J(\tilde{x}, 0)$ has two purely imaginary eigenvalues when taking $\mu_3 > 0$, the value $\tilde{x}$ satisfies $\tilde{x} = \pm\sqrt{\mu_3/3}$ with $\mu_2 < -\mu_3 < 0$, because then $\mu_2 + 3\tilde{x}^2 < 0$. The last inequality is fulfilled only for the equilibrium point $x_C$, see Lemma 2(c). Since $(\tilde{x}, 0)$ is an equilibrium point we have

\[ \mu_1 + \mu_2 (\pm\sqrt{\mu_3/3}) + (\pm\sqrt{\mu_3/3})^3 = 0, \]

and statement (b) follows. To show statement (c) is sufficient to consider the equations trace $(J(\tilde{x}, 0)) = \det(J(\tilde{x}, 0)) = 0$. □

**Proof of Proposition 2** The blow-up transformation $x_1 = (1/\epsilon)x, y_1 = (1/\epsilon^2)y$, and $\tilde{t} = \epsilon t$, allows to rewrite system (3) as

\[ x_1' = y_1, \quad y_1' = x_1^3 + \frac{\mu_2}{\epsilon^2}x_1 + \frac{\mu_1}{\epsilon} + \frac{\mu_3}{\epsilon} y_1 - 3\epsilon x_1^2 y_1, \]

where the prime denotes derivatives with respect to the new time $\tilde{t}$. Now, using (9) and (12), after some elementary algebra we obtain systems (10) and (13), respectively. □

**Proof of Proposition 3** The system can be written as

\[ (\dot{x}, \dot{y})^T = f(x, y) + \epsilon g(x, y), \]

where $f(x, y) = (y, -v_2x + x^3)^T$ and $g(x, y) = (0, v_1 + v_3y - 3x^2y)^T$. Thus, we have $f \wedge g = y(v_1 + v_3y - 3x^2y)$. Accordingly, the Melnikov function is defined by

\[ M_{ht}(\nu) = \int_{-\infty}^{\infty} f(x(t), \pm y(t)) \wedge g(x(t), \pm y(t)) dt = \int_{-\infty}^{\infty} \pm y(t) \left[ v_1 \pm (v_3 - 3x^2(t))y(t) \right] dt, \]

where $x(t)$ and $y(t)$ are defined as in (15). After a direct computation we obtain (16).

**Proof of Proposition 4** Fixing $v_3 = 1$ in the Melnikov function given in (16), and imposing the condition $M_{ht}(v_1, v_2, 1) = 0$, we obtain

\[ v_1 = \frac{\sqrt{2}}{15}v_2(3v_2 - 5). \]

From (9) we get $\epsilon = \sqrt{\mu_5}, \mu_1 = \mu_3^2v_1, \text{and} \mu_2 = -\mu_3v_2, \text{so that}$

\[ \mu_1 = \mu_3^2v_1 = -\mu_3^2\frac{\sqrt{2}}{15} \mu_3 \left( 3 \left( -\frac{\mu_2}{\mu_3} \right) - 5 \right), \]

and the conclusion follows. □

**Proof of Proposition 5** We consider the unperturbed Hamiltonian system given in (13) with $v_2 > 0$. From Lemma 2(c), the system has 3 equilibrium points $x_i = (s_i, 0)$, where $x_L$ and $x_R$ are saddle points and $x_C$ is a focus or node and

\[ s_L < s_C < s_R, \quad s_L + s_C + s_R = 0, \quad s_Ls_Cs_R = -v_1. \]

We study only the case $v_1 > 0$, for the case $v_1 < 0$ is analogous.

System (13) can written as

\[ (\dot{x}, \dot{y})^T = f(x, y) + \epsilon g(x, y). \]

Now, assuming $v_1 > 0$, by Green’s Theorem, the homoclinic Melnikov function of the system can rewritten as

\[ M_h(\nu) = \int \int_D (\frac{-\partial g(x, y)}{\partial y}) dA, \]

where $D$ is the region bounded by the homoclinic orbit that joins the equilibrium point $(s_R, 0)$ with itself. By fixing $v_3 = 1$ (that is $\mu_3 > 0$), and taking $p(x) = v_1 - v_2x + x^3$, we get $p(s_R) = v_1 - v_2s_R + s_R^3 = 0$, that is

\[ v_1 = s_R(v_2 - s_R^2), \]

(46)
Fig. 9 (a) The invariant manifold $S_h$ defined in (42), where $h = 0.71526$, which contains the orbit with initial conditions $(x_0, y_0, z_0) = (0.8, 0, 6, 0.0)$. Since the orbit is a spiral whose windings are so close to form a big spot, we only draw the fixed point $(x^*, y^*, z^*)$ with $y^* = z^* = 0$ and $x^* \approx 0.80003$ that determines the stable focus dynamics on the manifold. In (b),(c) and (d) the value of $x(t)$ for such an orbit, when we wait a sufficiently long time and so $v_2 - s_R^2 > 0$. Taking the auxiliary function

$$q(x) = \int_0^x p(x)dx = v_1x - v_2x^2 + \frac{x^4}{4},$$

and using (14), the homoclinic loop is given by the points $(x, y^\pm(x))$ where $\bar{x} \leq x \leq s_R$, $y^\pm(x) = \pm\sqrt{2}\sqrt{q(x) - q(s_R)}$, and $y^\pm(\bar{x}) = y^\pm(s_R) = 0$, see Fig. 10. Now, thanks to the symmetry of the loop, the Melnikov function is

$$M_h(\bar{x}) = 2\int_0^{s_R} \frac{(x^2-1)dx}{\sqrt{(x+s_R)^2 - 2(xv_2 - s_R^2)}} = \sqrt{2}\int_0^{s_R} \frac{(x^2-1)(s_R-x)}{\sqrt{(x+s_R)^2 - 2(xv_2 - s_R^2)}dx}$$

where from (46) $\omega^2 = 2(v_2 - s_R^2)$, and we have used that

$$q(x) - q(s_R) = \frac{1}{4}(x - s_R)^2\left[(x + s_R)^2 - \omega^2\right].$$

Making the change of variable

$$x + s_R = \omega \cosh \theta,$$

and noting that $q(s_R) - q(\bar{x}) = 0$ we see that $\bar{x} + s_R = \omega$, which corresponds to $\theta = 0$, while for $x = s_R$ the corresponding value of $\theta = \theta_{s_R}$ satisfies $\cosh \theta_{s_R} = 2s_R/\omega$, or also $\omega^2 \cosh^2 \theta_{s_R} = 4s_R^2$, that is, $(s_R^2 + v_2) \cosh^2 \theta_{s_R} = 2s_R^2$. Thus, we get

$$s_R^2 = \frac{\cosh^2 \theta_{s_R}}{2 + \cosh^2 \theta_{s_R}}v_2.$$

Now we arrived to

$$M_h(\bar{x}) = \sqrt{2}\omega^2 \int_0^{\theta_{s_R}} (1 - 3(\omega \cosh \theta - s_R^2))$$

$$(2s_R - \omega \cosh \theta) \sinh^2 \theta d\theta,$$

and after some computations, we obtain (20) and (21), where $\theta_{s_R}$ has been denoted by $\theta$ to alleviate notation. The proof is complete. \hfill \Box

**Proof of Proposition 6** The Melnikov function given in (20)-(21) vanishes at the points $(v_1(\theta), v_2(\theta))$ defined in (23). Taking $v_3 = 1$ in (12) we obtain $\mu_1 = \mu_3 v_1$ and $\mu_2 = -\mu_3 v_2$, and after some computations the conclusion follows. \hfill \Box

6 Conclusions

Motivated by the dynamical analysis of 3D memristor oscillators whose nonlinear characteristics is a cubic polynomial, and after showing that their dynamics is
essentially two-dimensional, the need to consider a disregarded unfolding of the Bogdanov-Takens singularity naturally arose. The corresponding bifurcation set, including both local and global bifurcations has been described. While local bifurcations can be easily detected, the characterization of global bifurcations parameters curves is much more involved; after resorting to Melnikov’s theory it was possible to obtain good approximations of such curves providing a complete description of the bifurcation set.

Regarding the considered 3D memristor oscillators, and by working within some parameters regions of the above bifurcation set, it has been possible to show rigorously the existence of multiple periodic orbits organized around two fixed points that form a topological sphere.

When the same approach is applied to a different family of 3D memristor oscillators, it has been shown that oscillations are not possible, contrarily to what had been recently claimed in some specific cases.

**Appendix: Dimensional reduction in 3D memristor oscillators**

We consider a family of three-dimensional systems, which is general enough to capture all the mathematical models of memristor oscillators given in (29). Such family was been studied in [1] and [31], where the authors showed that the dynamics of such a family of three-dimensional systems is essentially ruled by a one parameter set of two-dimensional systems. We consider the system

\[
\begin{align*}
\dot{x} &= a_{11}W(z)x + a_{12}y, \\
\dot{y} &= a_{21}x + a_{22}y, \\
\dot{z} &= x,
\end{align*}
\]

(47)

where the constants \(a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}\) and the function \(W\) allows to define a continuous function

\[
q(z) = \int_0^z W(s)ds.
\]

(48)

The next result guarantees that the dynamics of system (47) is essentially two-dimensional, see [1] for a proof.

**Proposition 12** Consider system (47) where the functions \(W\) and \(q\) are related as in (48). For any \(h \in \mathbb{R}\), the set

\[
S_h = \{(x, y, z) \in \mathbb{R}^3 : -a_{22}x + a_{12}y - a_{12}a_{21}z + a_{11}a_{22}q(z) = h\}
\]

(49)

is an invariant manifold for the system. Therefore, the system has an infinite family of invariant manifolds foliating the whole \(\mathbb{R}^3\), and so the dynamics is essentially two-dimensional.

In the following result we show that on each invariant set \(S_h\) given in (49), and for any continuous function \(q\) defined as in (48), the dynamics is topologically equivalent to a Liénard system. Furthermore, we give for any solution of the Liénard system with a given value of \(h\), the corresponding solution of the 3D canonical model (47). This result is a generalization of Theorem 3 given in [1], where the function \(q\) was considered to be a continuous piecewise linear function.

**Proposition 13** Consider system (47) with the function \(q\) defined as in (48). If \(a_{12} \neq 0\), then on each invariant set \(S_h\) given in (49), the dynamics is topologically equivalent to the Liénard system

\[
\dot{X} = Y - F(X), \quad \dot{Y} = -g(X) + h,
\]

(50)

where \(F\) and \(g\) are given by

\[
\begin{align*}
F(X) &= -a_{11}q(X) - a_{22}X, \\
g(X) &= a_{11}a_{22}q(X) - a_{12}a_{21}X
\end{align*}
\]

(51)

Moreover, \((X(\tau), Y(\tau)) \in \mathbb{R}^2\) is a solution of the Liénard system (50) for a given \(h \in \mathbb{R}\), if and only if \(E_h(X(\tau), Y(\tau)) \in \mathbb{R}^3\) is a solution of system (47) on \(S_h\), where
where the functions $f_1$ and $f_2$ are defined as

$$f_1(\tilde{z}) = a_{11} W(\tilde{z}) + a_{22},$$
$$f_2(\tilde{z}) = a_{22} a_{11} W(\tilde{z}) - a_{12} a_{21}.$$

From Proposition 12, the invariant manifolds (49) for system (54)-(55) can be written in the new variables as

$$\tilde{S}_h = \{(x, y, z) \in \mathbb{R}^3 : -\tilde{y} + g(\tilde{z}) = h\}.$$

Now, replacing the condition given in (56) in the first equation of (54) and removing the unnecessary second equation, we obtain the system

$$\color{#6B8E23}{\begin{align*}
\dot{x} &= f_1(\tilde{z}) x - y, \\
\dot{y} &= f_2(\tilde{z}) x, \\
\dot{z} &= x,
\end{align*}}$$

where the function $g$ is defined by

$$g(u) = a_{11} a_{22} q(u) - a_{12} a_{21} u.$$

After the change of variables

$$X = \tilde{z},$$
$$Y = -\tilde{F}(\tilde{z}) + x,$$

where $F$ is

$$\tilde{F}(z) = a_{11} q(z) + a_{22} z,$$

we obtain

$$\color{#6B8E23}{\dot{x} = f_1(\tilde{z}) x - y,}$$
$$\color{#6B8E23}{\dot{y} = f_2(\tilde{z}) x - g(\tilde{z}) + h}$$

and taking $F(X) = -\tilde{F}(X)$ we obtain system (50)-(51).

If $(X(\tau), Y(\tau)) \in \mathbb{R}^2$ is a solution of system (50)-(51) for a given $h \in \mathbb{R}$, we have from (59) that

$$\left(\begin{array}{c}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{array}\right) = \left(\begin{array}{c}
Y(\tau) - F(\tilde{z}(\tau)) \\
F(\tilde{z}(\tau)) - g(\tilde{z}(\tau)) + h
\end{array}\right).$$

is a solution of system (57). From (56), we obtain on $\tilde{S}_h$ that $y = g(\tilde{z}) - h$, with $g$ as in (58). Thus,

$$\left(\begin{array}{c}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{array}\right) = \left(\begin{array}{c}
Y(\tau) - F(\tilde{z}(\tau)) \\
g(\tilde{z}(\tau)) - h
\end{array}\right),$$

is a solution of system (54) on $\tilde{S}_h$. Finally, from (53) we obtain for system (47) the solution $x(\tau) = \tilde{x}(\tau)$,

$$y(\tau) = \frac{1}{a_{12}} [a_{22} \tilde{x}(\tau) - \tilde{y}(\tau)] = \frac{1}{a_{12}} \left[a_{22} Y(\tau) - a_{22} F(\tilde{z}(\tau)) - g(X(\tau)) + h\right] = \frac{1}{a_{12}} \left[a_{22} Y(\tau) + a_{22} \tilde{F}(\tilde{z}(\tau)) - g(X(\tau)) + h\right],$$

and $z(\tau) = \tilde{z}(\tau)$. The conclusion follows from the fact that for all $X$ we have

$$a_{22} \tilde{F}(X) - g(X) = (a_{22}^2 + a_{12} a_{21}) X.$$

In order to apply the analysis performed to system (3), in what follows we consider the function $q$ defined by a cubic polynomial, that is, we assume

$$W(z) = 3c z^2 + 2 a z + b, \quad q(z) = c z^3 + a z^2 + b z,$$

(61)

with $c \neq 0$. As a direct consequence of Propositions 12 and 13, we obtain the next result.

**Corollary 1** Consider system (47) with the functions $q$ and $W$ defined as in (61). If $a_{12} \neq 0$, then on each invariant set $S_h$ given by

$$S_h = \{(x, y, z) \in \mathbb{R}^3 : -a_{22} x + a_{12} y + a_{11} a_{22} c z^3 + a a_{11} a_{22} z^2 + (b a_{11} a_{22} - a_{12} a_{21}) z = h\}$$

the dynamics is topologically equivalent to the planar Liénard system

$$\dot{x} = y + c a_{11} x^3 + a a_{11} x^2 + (b a_{11} + a_{22}) x.$$
Moreover, \( (x(t), y(t)) \in \mathbb{R}^2 \) is a solution of the Liénard system (62) for a given \( h \in \mathbb{R} \), if and only if \( E_h(x(t), y(t)) \in \mathbb{R}^3 \) is a solution of system (47) on \( S_h \), where

\[
E_h(x(t), y(t)) = \left( y(t) + c_{a1} x(t)^3 + a_{21} x(t)^2 + (a_{11} + a_{22}) x(t) \right) \left( \frac{1}{\gamma(t)} [a_{22}^2 + a_{12} a_{21} y(t) - a_{22} y(t) + h] \right).
\]

(63)

In the next Proposition, we show that system (62) can be written into the form (1).

**Proposition 14** The following statements hold for system (62).

(a) If \( a_{22} \neq 0 \) and \( a_{11} a_{22} < 0 \) then the system can be written into the form

\[
\dot{x} = y, \quad \dot{y} = \mu_1 + \mu_2 x + c x^3 + \mu_3 y + 3c a_{11} x^2 y.
\]

(64)

where the new parameters \( \mu_1, \mu_2 \) and \( \mu_3 \) are given by

\[
\mu_1 = \frac{27 c h + a_{11} a_{22} a (9 c b - 2 a^2) - 9 c a_{11} a_{22} a_{21}}{27 c^2 (-a_{11} a_{22})^{5/2}},
\]

\[
\mu_2 = \frac{a_{11} a_{22} (a^2 - 3 c b) + 3 c a_{12} a_{21}}{3c (a_{11} a_{22})^{3/2}},
\]

\[
\mu_3 = \frac{a_{11} (a^2 - 3 c b) - 3 c a_{22}}{3c a_{11} a_{22}}.
\]

(b) If \( a_{22} = 0 \) then the system can be written into the form

\[
\dot{x} = y, \quad \dot{y} = \mu_1 + \mu_2 x + \mu_3 y + 3c a_{11} x^2 y,
\]

(66)

where the new parameters \( \mu_1, \mu_2 \) and \( \mu_3 \) are defined by

\[
\mu_1 = h + \frac{a a_{12} a_{21}}{3c}, \quad \mu_2 = a_{12} a_{21},
\]

\[
\mu_3 = b a_{11} - \frac{a^2 a_{11}}{3c}.
\]

(67)

**Proof** First, the change of variables

\[
\dot{u} = v + c_{a1} u^3 + \lambda_1 u,
\]

\[
\dot{v} = -c_{a1} a_{22} u^3 + \lambda_2 u + \lambda_3,
\]

(68)

where the new parameters are

\[
\lambda_1 = a_{22} + b a_{11} - \frac{1}{3} a^2 a_{11},
\]

\[
\lambda_2 = a_{12} a_{21} - b a_{11} a_{22} + \frac{1}{3} c a_{11} a_{22},
\]

\[
\lambda_3 = h + \frac{b}{3} - a_{11} a_{22} - \frac{1}{3} c a_{12} a_{21} - \frac{2}{27} c^2 a_{11} a_{22}.
\]

If \( a_{11} a_{22} < 0 \), the change of variable

\[
x = \frac{1}{(-a_{11} a_{22})^{1/2}} u, \quad y = v, \quad \tau = \frac{1}{-a_{11} a_{22}} t,
\]

transforms system (68)-(69) into

\[
\dot{x} = \frac{1}{(-a_{11} a_{22})^{3/2}} y + c_{a1} x^3 - \frac{\lambda_1}{a_{11} a_{22}} x,
\]

\[
\dot{y} = \frac{\lambda_3}{-a_{11} a_{22}} + \frac{\lambda_2}{(-a_{11} a_{22})^{1/2}} x + c (-a_{11} a_{22})^{3/2} x^3
\]

and taking into account that

\[
\ddot{x} = \frac{1}{(-a_{11} a_{22})^{3/2}} \dot{y} + 3 c_{a1} x^2 \dot{x} - \frac{\lambda_1}{a_{11} a_{22}} \dot{x},
\]

and after some algebra, statement (a) follows.

If \( a_{22} = 0 \), then from system (68), we obtain statement (b) after a direct computation.

**Acknowlegements** Andrés Amador is supported by Pontificia Universidad Javeriana Cali-Colombia, Enrique Ponce and Emilio Freire are partially supported by the Spanish Ministry of Economy and Competitiveness, in the frame of projects MTM2015-65608-P and PGC2018-096265-B-I00, and by the Consejería de Economía y Conocimiento de la Junta de Andalucía under Grant P12-FQM-1658.

**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

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