Well-Posedness of some nonlinear Volterra-Fredholm integral and integro-dynamic equations on time scales

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Abstract

In this paper we study well posedness of a certain nonlinear Volterra-Fredholm dynamic integral and integro-dynamic equations on unbounded interval from arbitrary time scale. We derive the time scale analogue of certain integral inequalities of Pachpatte type and using them with Banach’s fixed-point theorem to establish the results.

Keywords: Pachpatte inequalities, time scales, integral equations, well posedness, Banach’s fixed-point theorem.

1 Introduction

Recently, numerous mathematicians have investigated some of the qualitative and the quantitative properties of different types of integral equations on time scales by using different techniques, see [1, 4, 5, 7, 8, 9]. Integral equations on time scales are thought to have a great potential for many applications in various areas of natural science and to give a deeper understanding than the traditional integral and summation equations to many phenomena, especially those that occur mutually in continuous-discrete manner with the flow of time, see [12].

The results of this article appeared in a preliminary form as a scientific poster in the proceedings of the first international conference, new horizons in basic and applied science (ICNHBAS), 21-23 September, 2013, Hurghada, Egypt.
In the present paper we shall consider the general nonlinear dynamic integral equation
\[
x(t) = f(t, x(t), \int_a^t h(t, s, x(s))\,\Delta s, \int_a^b g(t, s, x(s))\,\Delta s), \quad t \in I_T, \quad (1.1)
\]
and the integro-dynamic equation
\[
x^\Delta(t) = f(t, x(t), \int_a^t h(t, s, x(s))\,\Delta s, \int_a^b g(t, s, x(s))\,\Delta s), \quad t \in I_T, \quad (1.2)
\]
where \(a < b\), and \(I_T = [a, \infty) \cap T\). A time scale \(T\) is a nonempty closed subset from the reals. We assume that \(f : I_T \times X \times X \times X \to X\) is rd-continuous in the first variable, while the functions \(h : I_T^2 \times X \to X\), \(g : I_T^2 \times X \to X\) are assumed to be rd-continuous in the second argument, and \(x\) is the unknown function. Here \(X\) is a Banach space. The integral is the delta integral and \(\Delta\) denotes the delta derivative, for more details see [2, 3]. By a solution of (1.1) (resp. Eq. (1.2)) we mean rd-continuous function \(x : I_T \to X\) that satisfies Eq. (1.1) (resp. Eq. (1.2)).

In this paper, we shall investigate the well-posedness of equations (1.1) and (1.2). We shall study existence and uniqueness of the solutions, besides continuous dependence on data and parameters.

Our methods involve using Banach’s fixed-point theorem on an appropriate metric space and deriving the time scale analogue of certain integral inequalities of Pachpatte type to study some qualitative properties of equations (1.1) and (1.2). The results of this paper generalize some of the results in articles [6, 7, 8, 9] on \(T = \mathbb{R}\).

The work in this paper is organized as follows. Section 3 establishes the time scale analogue of some integral inequalities of Pachpatte type. In Section 4, we prove existence and uniqueness of solutions of equations (1.1) and (1.2), respectively. In Section 5, we establish estimates on the solutions of equations (1.1) and (1.2), respectively. Section 6 investigates the continuous dependence of the solutions of equations (1.1) and (1.2) on the functions involved.

# 2 Preliminaries

In this section we introduce some definitions, notations, and preliminary results which will be used throughout this paper.

**Definition 2.1.** A time scale \(T\) is a nonempty closed subset of the real numbers \(\mathbb{R}\).

**Definition 2.2.** The mappings \(\sigma, \rho : T \to T\) defined by \(\sigma(t) = \inf\{s \in T : s > t\}\), and \(\rho(t) = \sup\{s \in T : s < t\}\) are called the jump operators.

If \(T\) has a left scattered maximum \(m\), then define \(T^\kappa = T - \{m\}\), otherwise \(T^\kappa = T\).
Definition 2.3. A function \( f : T \rightarrow X \) is said to be delta differentiable at the point \( t \in T \) if there exist an element \( f^\Delta(t) \in X \) with the property that given any \( \varepsilon > 0 \) there is a neighborhood \( U \) of \( t \) with \( \|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]\| \leq \varepsilon|\sigma(t) - s| \) for all \( s \in U \). The function \( f^\Delta(t) \) is the delta derivative of \( f \) at \( t \).

For \( T = \mathbb{R} \), we have \( f^\Delta(t) = f'(t) \), and for \( T = \mathbb{Z} \), we have \( f^\Delta(t) = \Delta f(t) = f(t+1) - f(t) \).

Definition 2.4. A function \( F : T \rightarrow X \) is called an antiderivative of \( f : T \rightarrow X \) provided \( F^\Delta(t) = f(t) \) for all \( t \in T \). The \( \Delta \)-integral of \( f \) is defined by

\[
\int_r^s f(t)\Delta t = F(s) - F(r), \quad \text{for all } r, s \in T.
\]

If \( T = \mathbb{R} \), then \( \int_a^b f(s)\Delta s = \int_a^b f(s)ds \), while if \( T = \mathbb{Z} \), then \( \int_a^b f(s)\Delta s = \sum_{s=a}^{b-1} f(s) \).

Definition 2.5. A function \( f : T \rightarrow X \) is called right-dense continuous (rd-continuous) if \( f \) is continuous at every right-dense point \( t \in T \) and the left-sided limits exist (i.e. finite) at every left-dense point \( t \in T \). The family of all rd-continuous functions from \( T \) to \( X \) is denoted by \( C_{rd}(T; X) \).

The family of all regressive functions is denoted by

\[
\mathcal{R} := \left\{ p \in C_{rd}(T; \mathbb{R}) \text{ and } 1 + p(t)\mu(t) \neq 0, \forall t \in T \right\},
\]

and the set of positively regressive functions is denoted by

\[
\mathcal{R}^+ := \left\{ p \in C_{rd}(T; \mathbb{R}) \text{ and } 1 + p(t)\mu(t) > 0, \forall t \in T \right\}.
\]

Definition 2.6. If \( p \in \mathcal{R} \), then we define the generalized exponential function by

\[
e_p(t, s) = \exp \left( \int_s^t \xi_{p(t)}(p(\tau))\Delta \tau \right) \quad \text{for } t, s \in T,
\]

with the cylinder transformation

\[
\xi_h(z) = \begin{cases} \frac{\log(1+hz)}{h} & \text{if } h \neq 0 \\ z & \text{if } h = 0. \end{cases}
\]

In the case \( T = \mathbb{R} \), the exponential function is given by

\[
e_p(t, s) = \exp \left( \int_s^t p(\tau)d\tau \right),
\]
for \( s, t \in \mathbb{R} \), where \( p : \mathbb{R} \to \mathbb{R} \) is a continuous function. In the case \( T = \mathbb{Z} \), the exponential is given by
\[
e_p(t, s) = \prod_{\tau = s}^{t-1} [1 + p(\tau)],
\]
for \( s, t \in \mathbb{Z} \), where \( p : \mathbb{Z} \to \mathbb{R} \), \( p(t) \neq -1 \) for all \( t \in \mathbb{Z} \).
For more basic properties of the generalized exponential function, see [2].

**Definition 2.7.** For \( k(x, y) \in \mathcal{R} \) with respect to the \( y \), the generalized exponential function is defined by
\[
e_{k(x, \cdot)}(t, s) = \exp\left( \int_{s}^{t} \xi_{\mu(\tau)}(k(x, \tau)) \Delta \tau \right),
\]
for \( s, t \in T \).

Let \( \beta > 0 \) be a constant and let \( || \cdot || \) denotes a norm on \( X \). We consider the space \( C_\beta(I_T; X) \) of all rd-continuous functions such that
\[
\sup_{t \in I_T} ||x(t)|| e^{\beta(t, a)} < \infty,
\]
coupled with a suitable norm, namely
\[
||x||_\beta^\infty = \sup_{t \in I_T} \frac{||x(t)||}{e^{\beta(t, a)}}.
\]
We can follow the proof of Lemma 3.3 in [11] to obtain the following result.

**Lemma 2.8.** If \( \beta > 0 \) is a constant, then \( (C_\beta(I_T; X), || \cdot ||_\beta^\infty) \) is a Banach space.

We use the following comparison lemma in our study, see [2].

**Lemma 2.9.** Suppose \( u, \beta \in C_{rd}(I_T, \mathbb{R}) \) and \( \alpha \in \mathbb{R}^+ \). If
\[
u^\Delta(t) \leq \alpha(t)u(t) + \beta(t), \quad \forall \ t \in I_T,
\]
then
\[
u(t) \leq u(a)e_\alpha(t, a) + \int_{a}^{t} e_\alpha(t, \sigma(\tau))\beta(\tau) \Delta \tau, \quad \forall \ t \in I_T.
\]

### 3 New Pachpatte type inequalities on time scales

In this section, we establish the time scale analogue of the integral inequalities of Pachpatte type given in [10].

**Theorem 3.1.** Let the functions \( k_i(\cdot, \cdot) : I_T \times I_T \to \mathbb{R}_+ \) (for \( i = 1, 2 \)) be nondecreasing in the first variable and rd-continuous in the second variable. Also, assume that \( u \in C_{rd}(I_T, \mathbb{R}_+) \) and
\[
u(t) \leq c + \int_{a}^{t} k_1(t, s)u(s) \Delta s + \int_{a}^{b} k_2(t, s)u(s) \Delta s, \quad t \in I_T,
\]
(3.1)
where \( c \geq 0 \) is a real constant. If

\[
p = \int_a^b k_2(b, s)e_{k_2(s, .)}(s, a)\Delta s < 1, \quad t \in I_T,
\]

then

\[
u(t) \leq \frac{c}{1 - p} e_{k_1(t, .)}(t, a), \quad t \in I_T.
\]

**Proof.** Fix any \( T \in I_T \). Then for \( t \in [a, T]_T \) we have

\[
u(t) \leq c + \int_a^t k_1(T, s)u(s)\Delta s + \int_t^b k_2(T, s)u(s)\Delta s.
\]

Define the function \( z \) by

\[
z(t, T) = c + \int_a^t k_1(T, s)u(s)\Delta s + \int_t^b k_2(T, s)u(s)\Delta s, \quad t \in [a, T]_T.
\]

Then

\[
u(t) \leq z(t, T), \quad t \in [a, T]_T,
\]

and

\[
z(a, T) = c + \int_a^b k_2(T, s)u(s)\Delta s.
\]

By differentiating \( z(t, T) \) with respect to \( t \), we obtain

\[
z'(t, T) = k_1(T, t)u(t) \leq k_1(T, t)z(t, T), \quad t \in [a, T]_T.
\]

Since \( k_1(T, t) \in \mathbb{R}^+ \), we apply Lemma 2.9 with \( \beta = 0 \) to obtain

\[
z(t, T) \leq z(a, T)e_{k_1(T, .)}(t, a), \quad t \in [a, T]_T.
\]

Since \( T \) is selected from \( I_T \) arbitrarily, we replace \( T \) by \( t \) in \( 3.6 \), \( 3.7 \), and \( 3.9 \). So, we have

\[
u(t) \leq z(t, t) \leq z(a, t)e_{k_1(t, .)}(t, a), \quad t \in I_T,
\]

where

\[
z(a, t) = c + \int_a^b k_2(t, s)u(s)\Delta s, \quad t \in I_T.
\]

Since \( z(a, t) \) is nondecreasing in \( t \), then

\[
z(a, t) \leq z(a, b), \quad t \in I_T,
\]

and

\[
u(t) \leq z(a, b)e_{k_1(t, .)}(t, a), \quad t \in I_T.
\]
From (3.11) and (3.12), we have
\[
z(a, b) = c + \int_a^b k_2(b, s)u(s)\Delta s \\
\leq c + \int_a^b k_2(b, s)z(a, b)e_{k_1(s, \ldots)}(s, a)\Delta s \\
= c + z(a, b)\int_a^b k_2(b, s)e_{k_1(s, \ldots)}(s, a)\Delta s.
\]

In view of condition (3.2) it is easy to observe that
\[
z(a, b) \leq c + \frac{c}{1 - p}, \quad t \in I_T. \tag{3.13}
\]
Applying (3.13) in (3.12) we obtain the desired inequality. \Box

**Theorem 3.2.** Let \( u, f, g, h \in C_{rd}(I_T, \mathbb{R}^+) \). Assume
\[
u(t) \leq k + \int_a^t f(s)\left[ u(s) + \int_a^s g(\tau)u(\tau)\Delta \tau + \int_s^b h(\tau)u(\tau)\Delta \tau \right] \Delta s, \quad t \in I_T, \tag{3.14}
\]
where \( k \geq 0 \) is a real constant. If
\[
r = \int_a^b h(\tau)e_{f + g}(\tau, a)\Delta \tau < 1, \tag{3.15}
\]
then
\[
u(t) \leq \frac{k}{1 - r}e_{f + g}(t, a), \quad t \in I_T. \tag{3.16}
\]

**Proof.** Define a function \( z(t) \) by the right hand side of (3.14). Then \( z(a) = k \), \( u(t) \leq z(t) \) and
\[
z^\Delta = f(t)\left[ u(t) + \int_a^t g(\tau)u(\tau)\Delta \tau + \int_a^b h(\tau)u(\tau)\Delta \tau \right] \\
\leq f(t)\left[ z(t) + \int_a^t g(\tau)z(\tau)\Delta \tau + \int_a^b h(\tau)z(\tau)\Delta \tau \right]
\]
for \( t \in I_T \). Define \( w(t) \) by
\[
w(t) = z(t) + \int_a^t g(\tau)z(\tau)\Delta \tau + \int_a^b h(\tau)z(\tau)\Delta \tau, \quad t \in I_T.
\]
Then \( z(t) \leq w(t) \), \( z^\Delta \leq f(t)w(t) \). We have
\[
w(a) = k + \int_a^b h(\tau)z(\tau)\Delta \tau, \tag{3.17}
\]
and

\[ w^\Delta(t) = z^\Delta(t) + g(t)z(t) \leq f(t)w(t) + g(t)z(t) \leq [f(t) + g(t)]w(t). \]

This implies, by Lemma 2.9, that

\[ w(t) \leq w(a)e_{f+g}(t, a), \quad t \in I. \]

Since \( z(t) \leq w(t) \), we get

\[ z(t) \leq w(a)e_{f+g}(t, a), \quad t \in I_T. \] (3.18)

Using (3.18) on the right hand side of (3.17), we obtain

\[ w(a) \leq k + w(a) \int_a^b h(\tau)e_{f+g}(\tau, a)\Delta \tau. \]

In view of (3.15) we get

\[ w(a) \leq \frac{k}{1-r}. \] (3.19)

Combining (3.19), (3.18) with the inequality \( u(t) \leq z(t) \), we get the desired inequality. \( \square \)

### 4 Existence and Uniqueness of Solutions

Banach’s fixed point theorem is a powerful and important tool in providing sufficient conditions for existence and uniqueness of solutions of dynamic equations as well as integral equations on time scales, see \([5, 11]\). In this section, we use Banach’s fixed point theorem to prove the existence and uniqueness of solutions of equations (1.1) and (1.2), respectively.

**Theorem 4.1.** Consider the integral equation (1.1). Suppose that there exist non-negative constants \( M, L, N \) and \( \gamma > 1 \) such that the following conditions are satisfied

\[ ||f(t, u, v, w) - f(t, \bar{u}, \bar{v}, \bar{w})|| \leq M\{||u - \bar{u}|| + ||v - \bar{v}|| + ||w - \bar{w}||\}, \] (4.1)

\[ ||h(t, s, v) - h(t, s, \bar{v})|| \leq L||v - \bar{v}||, \] (4.2)

\[ ||g(t, s, u) - g(t, s, \bar{u})|| \leq N||u - \bar{u}||, \] (4.3)

\[ M(1 + \frac{1}{\gamma}) < 1. \] (4.4)

In addition, assume that

\[ A_1 := \sup_{t \in I_T} e_{\beta}(t, a) \left[ ||f(t, 0, \int_0^t h(t, s, 0)\Delta s, \int_a^b g(t, s, 0)\Delta s)|| \right] < \infty, \] (4.5)

where \( \beta \) is the solution of the equation \( \beta := \gamma(L + Ne_{\beta}(b, a)) \). Then the integral equation (1.1) has a unique solution \( x \in C_\beta(I_T; \mathbb{X}) \).
Proof. Consider the Banach space \((C_\beta(I_T; X), \| \cdot \|_\beta)\). Define the operator \(F : C_\beta(I_T; X) \to C_\beta(I_T; X)\) by
\[
[Fx](t) := f(t, x(t), \int_a^t h(t, s, x(s))\,ds, \int_a^b g(t, s, x(s))\,ds),
\]
for \(t \in I_T\). Fixed points of \(F\) will be solutions to (1.1). Now, we prove that \(F\) maps \(C_\beta(I_T; X)\) into itself. Let \(x \in C_\beta(I_T; X)\) and using the hypotheses, we have
\[
\|Fx\|_\beta = \sup_{t \in I_T} \frac{\|F_x(t)\|}{e_\beta(t, a)}
\]
\[
\leq \sup_{t \in I_T} \frac{1}{e_\beta(t, a)} \left\| f(t, x(t), \int_a^t h(t, s, x(s))\,ds, \int_a^b g(t, s, x(s))\,ds) - f(t, 0, \int_a^t h(t, s, 0)\,ds, \int_a^b g(t, s, 0)\,ds) \right\|
\]
\[
+ \sup_{t \in I_T} \frac{1}{e_\beta(t, a)} \left\| f(t, 0, \int_a^t h(t, s, 0)\,ds, \int_a^b g(t, s, 0)\,ds) \right\|
\]
\[
\leq A_1 + \sup_{t \in I_T} \frac{1}{e_\beta(t, a)} M \left\{ \|x(t)\| + L \int_a^t \|x(s)\|\,ds + N \int_a^b \|x(s)\|\,ds \right\}
\]
\[
= A_1 + M \left\{ \sup_{t \in I_T} \frac{\|x(t)\|}{e_\beta(t, a)} + L \sup_{t \in I_T} \frac{1}{e_\beta(t, a)} \int_a^t \|x(s)\|\,ds \frac{e_\beta(s, a)}{e_\beta(s, a)} \Delta s
\]
\[
+ N \sup_{t \in I_T} \frac{1}{e_\beta(t, a)} \int_a^b \frac{e_\beta(s, a)}{e_\beta(s, a)} \|x(s)\|\,ds \right\}
\]
\[
\leq A_1 + M \left\{ \|x\|_\beta + L \|x\|_\beta \sup_{t \in I_T} \frac{1}{e_\beta(t, a)} \int_a^t \frac{e_\beta(s, a)}{e_\beta(s, a)} \Delta s
\]
\[
+ N \|x\|_\beta \sup_{t \in I_T} \frac{1}{e_\beta(t, a)} \int_a^b \frac{e_\beta(s, a)}{e_\beta(s, a)} \Delta s \}
\]
\[
\leq A_1 + M \|x\|_\beta \left\{ 1 + L \sup_{t \in I_T} \frac{1}{e_\beta(t, a)} \left( \frac{e_\beta(t, a)}{e_\beta(t, a)} - 1 \right) + N \sup_{t \in I_T} \frac{1}{e_\beta(t, a)} \left( \frac{e_\beta(b, a)}{e_\beta(b, a)} - 1 \right) \}
\]
\[
\leq A_1 + M \|x\|_\beta \left\{ 1 + \frac{L}{\beta} \sup_{t \in I_T} \left( 1 - \frac{1}{e_\beta(t, a)} \right) + \frac{N}{\beta} \frac{1}{e_\beta(a, a)} (e_\beta(b, a) - 1) \}
\]
\[
= A_1 + M \|x\|_\beta \left\{ 1 + \frac{1}{\beta} (L + Ne_\beta(b, a)) \right\}
\]
\[
< A_1 + M \|x\|_\beta \left\{ 1 + \frac{1}{\gamma} \right\} < \infty.
\]

In addition to the assumptions of equation (1.1).

Proof. The corresponding integral equation to (1.2) is

\[ x(t) = x(a) + \int_a^t f(\tau, x(\tau), \int_a^\tau h(\tau, s, x(s)) \Delta s, \int_a^\tau g(\tau, s, x(s)) \Delta s) \Delta \tau. \]

This proves that the operator \( F \) maps \( C_\beta(I_T; X) \) into itself. Now let \( u, v \in C_\beta(I_T; X) \). From the hypotheses, we have

\[
\| F u - F v \|_\beta = \sup_{t \in I_T} \frac{1}{\beta(t, a)} \left\| (F u)(t) - (F v)(t) \right\|
\]

\[
= \sup_{t \in I_T} \frac{1}{\beta(t, a)} \left\| f(t, u(t), \int_a^t h(t, s, u(s)) \Delta s, \int_a^b g(t, s, u(s)) \Delta s \right\|
\]

\[
- f(t, v(t), \int_a^t h(t, s, v(s)) \Delta s, \int_a^b g(t, s, v(s)) \Delta s) \right\|
\]

\[
\leq \sup_{t \in I_T} \frac{1}{\beta(t, a)} \left\{ \| u(t) - v(t) \| + L \int_a^t \| u(s) - v(s) \| \Delta s + N \int_a^b \| u(s) - v(s) \| \Delta s \right\}
\]

\[
= M \left\{ \sup_{t \in I_T} \frac{\| u(t) - v(t) \|}{\beta(t, a)} + \sup_{t \in I_T} \frac{1}{\beta(t, a)} L \int_a^t \frac{\| u(s) - v(s) \|}{\beta(s, a)} \Delta s 
\right. 
\]

\[
+ N \sup_{t \in I_T} \frac{\| u(t) - v(t) \|}{\beta(t, a)} \int_a^b \frac{\| u(s) - v(s) \|}{\beta(s, a)} \Delta s \right\}
\]

\[
\leq M \left\{ 1 + \frac{M}{\beta(t, a)} \right\} \sup_{t \in I_T} \frac{1}{\beta(t, a)} L \int_a^t \frac{\| u(s) - v(s) \|}{\beta(s, a)} \Delta s 
\]

\[
+ \frac{M}{\beta(t, a)} L \int_a^t \frac{\beta(t, a)}{\beta(s, a)} \Delta s \right\}
\]

\[
= M \left\{ 1 + \frac{1}{\beta(t, a)} \right\} \left\{ 1 + \frac{M}{\beta(t, a)} \right\} \sup_{t \in I_T} \frac{1}{\beta(t, a)} L \int_a^t \frac{\| u(s) - v(s) \|}{\beta(s, a)} \Delta s 
\]

\[
+ \frac{M}{\beta(t, a)} L \int_a^t \frac{\beta(t, a)}{\beta(s, a)} \Delta s \right\}
\]

\[
= M \left\{ 1 + \frac{1}{\beta(t, a)} \right\} \left\{ 1 + \frac{M}{\beta(t, a)} \right\} \sup_{t \in I_T} \frac{1}{\beta(t, a)} L \int_a^t \frac{\| u(s) - v(s) \|}{\beta(s, a)} \Delta s 
\]

Since \( M(1 + \frac{1}{\gamma}) < 1 \), it follows from the Banach’s fixed point theorem that \( F \) has a unique fixed point in \( C_\beta(I_T; X) \). The fixed point of \( F \) is the unique solution \( x \) of equation (1.1). \( \square \)

**Theorem 4.2.** In addition to the assumptions of (1.1)–(1.4) from Theorem 4.1, assume that

\[
A_2 := \sup_{t \in I_T} \frac{1}{\beta(t, a)} \left\| \int_a^t f(\tau, 0, \int_a^\tau h(\tau, s, 0) \Delta s, \int_a^\tau g(\tau, s, 0) \Delta s) \Delta \tau \right\| < \infty.
\]

Then the integro-dynamic equation (1.2) has a unique solution \( x \in C_\beta(I_T; X) \).

**Proof.** The corresponding integral equation to (1.2) is

\[
x(t) = x(a) + \int_a^t f(\tau, x(\tau), \int_a^\tau h(\tau, s, x(s)) \Delta s, \int_a^\tau g(\tau, s, x(s)) \Delta s) \Delta \tau.
\]
Define the operator $F : C_{\beta}(I; X) \rightarrow C_{\beta}(I; X)$ by

$$[Fx](t) := x(a) + \int_a^t f(\tau, x(\tau), \int_a^\tau h(\tau, s, x(s)) \Delta s, \int_a^b g(\tau, s, x(s)) \Delta s) \Delta \tau,$$

for $t \in I_T$. By following the same argument as in proof of Theorem 4.1, we can similarly prove the existence and uniqueness of solutions of equation (1.2) on $I_T$. □

# 5 Estimate on solutions

The following two theorems provide a certain estimate on the solutions of equations (1.1) and (1.2), respectively.

**Theorem 5.1.** Consider the nonlinear dynamic integral equation (1.1). Suppose that there exist positive constants $\alpha, B, C$, and $0 \leq N < 1$ such that

$$\|f(t, u, v, w) - f(t, \bar{u}, \bar{v}, \bar{w})\| \leq N\{\|u - \bar{u}\| + \|v - \bar{v}\| + \|w - \bar{w}\|\}, \quad (5.1)$$

$$\|h(t, s, v) - h(t, s, \bar{v})\| \leq Be^\alpha(s, t)\|v - \bar{v}\|, \quad (5.2)$$

$$\|g(t, s, u) - g(t, s, \bar{u})\| \leq Ce^\alpha(s, t)\|u - \bar{u}\|, \quad (5.3)$$

$$q := \int_a^b CN^*e^{BN^*}(s, a)\Delta s < 1, \quad N^* = \frac{N}{1 - N}. \quad (5.4)$$

hold. Moreover, assume that there exists a nonnegative constant $d$ such that

$$\left\| f(t, 0, \int_a^t h(t, s, 0)\Delta s, \int_a^b g(t, s, 0)\Delta s) \right\| \leq \frac{d}{e_{\alpha}(t, a)}, \quad t \in I_T. \quad (5.5)$$

If $x(t)$ is any solution of equation (1.1), then

$$\|x(t)\| \leq \frac{d}{(1 - N)(1 - q)}e_{BN^*}(t, a), \quad t \in I_T. \quad (5.6)$$

**Proof.** For any solution $x(t)$ of (1.1) on $I_T$, we have

$$\|x(t)\| \leq \left\| f\left(t, 0, \int_a^t h(t, s, 0)\Delta s, \int_a^b g(t, s, 0)\Delta s\right) \right\|$$

$$+ \left\| f\left(t, x(t), \int_a^t h(t, s, x(s))\Delta s, \int_a^b g(t, s, x(s))\Delta s\right) \right\|$$

$$- f\left(t, 0, \int_a^t h(t, s, 0)\Delta s, \int_a^b g(t, s, 0)\Delta s\right).$$

From the assumptions we see that

$$\|x(t)\| \leq \frac{d}{e_{\alpha}(t, a)} + N\|x(t)\| + \int_a^t Be_{\alpha}(s, t)\|x(s)\|\Delta s + \int_a^b Ce_{\alpha}(s, t)\|x(s)\|\Delta s, \quad t \in I_T.$$
Multiplying both sides of the above inequality by \(e_{\alpha}(t, a)\) and rearranging the terms we observe that
\[
e_{\alpha}(t, a)\|x(t)\| \leq \frac{d}{1 - N} + \int_{a}^{t} BN^*e_{\alpha}(s, a)\|x(s)\|\Delta s + \int_{a}^{b} CN^*e_{\alpha}(s, a)\|x(s)\|\Delta s,
\]
for \(t \in I_{\tau}\), where we have used the semigroup property
\[
e_{\alpha}(s, a) = e_{\alpha}(s, t)e_{\alpha}(t, a).
\]
Now, in view of condition (5.4) and applying Theorem 3.1 we obtain
\[
e_{\alpha}(t, a)\|x(t)\| \leq \frac{d}{(1 - N)(1 - q)} e_{BN^*}(t, a), \quad t \in I_{\tau}.
\]
By using the identity [2, Theorem 2.36], we deduce that
\[
\|x(t)\| \leq \frac{d}{(1 - N)(1 - q)} e_{BN^* \circ \alpha}(t, a), \quad t \in I_{\tau}.
\]

\[\square\]

**Theorem 5.2.** Consider the integro-dynamic equation (1.2). Assume that there are functions \(k_1, k_2, k_3 \in C_{rd}(I_{\tau}; \mathbb{R}^+)\) such that
\[
\|f(t, u, v, w) - f(t, \bar{u}, \bar{v}, \bar{w})\| \leq k_1(t)\{\|u - \bar{u}\| + \|v - \bar{v}\| + \|w - \bar{w}\|\}, \tag{5.7}
\]
\[
\|h(t, s, v) - h(t, s, \bar{v})\| \leq k_2(t)\|v - \bar{v}\|, \tag{5.8}
\]
\[
\|g(t, s, u) - g(t, s, \bar{u})\| \leq k_3(t)\|u - \bar{u}\|, \tag{5.9}
\]
\[
r := \int_{a}^{b} k_3(\tau)e_{k_1+k_2}(\tau, a)\Delta \tau < 1, \tag{5.10}
\]
hold. If \(x(t)\) is any solution of (1.2) on \(I_{\tau}\), then
\[
\|x(t)\| \leq \frac{m}{1 - r} e_{k_1+k_2}(t, a), \quad t \in I_{\tau}. \tag{5.11}
\]
where
\[
m := \sup_{t \in I_{\tau}}\|x(a)\| + \int_{a}^{t} f \left(\tau, 0, \int_{a}^{\tau} h(\tau, s, 0)\Delta s, \int_{a}^{b} g(\tau, s, 0)\Delta s\right)\Delta \tau < \infty. \tag{5.12}
\]

**Proof.** For any solution \(x(t)\) of (1.2) on \(I_{\tau}\), we have
\[
\|x(t)\| \leq \|x(a)\| + \int_{a}^{t} f \left(\tau, 0, \int_{a}^{\tau} h(\tau, s, 0)\Delta s, \int_{a}^{b} g(\tau, s, 0)\Delta s\right)\Delta \tau
\]
\[+ \right| \int_{a}^{t} f \left(\tau, x(\tau), \int_{a}^{\tau} h(\tau, s, x(s))\Delta s, \int_{a}^{b} g(\tau, s, x(s))\Delta s\right)
\]
\[+ f \left(\tau, 0, \int_{a}^{\tau} h(\tau, s, 0)\Delta s, \int_{a}^{b} g(\tau, s, 0)\Delta s\right)\\Delta \tau\|, \quad t \in I_{\tau}.
\]
From the assumptions we get

\[ \|x(t)\| \leq m + \int_a^t k_1(\tau) \left[ \|x(\tau)\| + \int_a^\tau k_2(\tau)\|x(s)\|\Delta s + \int_a^b k_3(\tau)\|x(s)\|\Delta s \right] \Delta \tau, \quad t \in I_T. \]

Now applying Theorem 3.2 to the above inequality, we get the desired estimate. \[ \square \]

6 Continuous Dependence of Solutions

In this section, we are interested in estimating the change in the solution for equations (1.1) and (1.2) respectively, when the function \( f, h, g \) are allowed to change. Besides equation (1.1) we consider the perturbed equation

\[ y(t) = \tilde{f}(t, y(t), \int_a^t \tilde{h}(t, s, y(s))\Delta s, \int_a^b \tilde{g}(t, s, y(s))\Delta s), \quad (6.1) \]

for \( t \in I_T := [a, \infty)_T \), where \( \tilde{f} : I_T^2 \times X \to X \) is rd-continuous in its first variable, while the functions \( \tilde{h} : I_T^2 \times X \to X \) and \( \tilde{g} : I_T^2 \times X \to X \), are rd-continuous in its second variable. Let \( x(t) \) and \( y(t) \) be the solutions of equations (1.1) and (6.1) respectively. We answer the following question:

Under what conditions does the solution \( x(t) \) of equation (1.1) depend continuously on the functions involved \( f, h, \) and \( g \)?

**Theorem 6.1.** Consider the integral equation (1.1). Suppose that there are two functions \( n, m : I_T^2 \to \mathbb{R} \) which are rd-continuous in its second variable and a constant \( 0 \leq N < 1 \) such that

\[ \|f(t, u, v, w) - f(t, \bar{u}, \bar{v}, \bar{w})\| \leq N\{\|u - \bar{u}\| + \|v - \bar{v}\| + \|w - \bar{w}\|\}, \quad (6.2) \]

\[ \|h(t, s, v) - h(t, s, \bar{v})\| \leq m(t, s)\|v - \bar{v}\|, \quad (6.3) \]

\[ \|g(t, s, u) - g(t, s, \bar{u})\| \leq n(t, s)\|u - \bar{u}\|, \quad (6.4) \]

\[ p(t) := N^* \int_a^b n(t, s)e_{m(s, \cdot)}(s, a)\Delta s < 1, \quad N^* = \frac{N}{1 - N}, \quad (6.5) \]

hold. In addition, assume that \( \varepsilon \) is arbitrary positive number, and \( y(t) \) is a solution of (6.1) such that

\[ R(t) = \left\| f(t, y(t), \int_a^t h(t, s, y(s))\Delta s, \int_a^b g(t, s, y(s))\Delta s \right\| - \tilde{f}(t, y(t), \int_a^t \tilde{h}(t, s, y(s))\Delta s, \int_a^b \tilde{g}(t, s, y(s))\Delta s) < \varepsilon, \quad t \in I_T, \]

where \( f, h, g \) and \( \tilde{f}, \tilde{h}, \tilde{g} \) are the functions involved in (1.1) and (6.1) respectively. Then every solution \( x(t) \) of (1.1) satisfies

\[ \|x(t) - y(t)\| \leq \frac{\varepsilon}{(1 - N)(1 - p(t))} e_{N^*m(t, \cdot)}(t, a), \quad t \in I_T. \quad (6.6) \]
Furthermore if $I_{\tau} = [a, b]_{\tau}$, then every solution of equation \( (1.1) \) depends continuously on the functions $f, g, h$ involved.

**Proof.** Let $x(t)$ be a solution of \( (1.1) \), and $y(t)$ be a solution of the perturbed equation \( (6.1) \). Put $u(t) := \|x(t) - y(t)\|$, $t \in I_{\tau}$. We obtain

$$u(t) \leq \left\| f(t, x(t), \int_{a}^{t} h(t, s, x(s)) \Delta s, \int_{a}^{b} g(t, s, x(s)) \Delta s) \right\|$$

$$- f(t, y(t), \int_{a}^{t} h(t, s, y(s)) \Delta s, \int_{a}^{b} g(t, s, y(s)) \Delta s)\right\|$$

$$+ \left\| f(t, y(t), \int_{a}^{t} h(t, s, y(s)) \Delta s, \int_{a}^{b} g(t, s, y(s)) \Delta s) \right\|$$

$$- f(t, y(t), \int_{a}^{t} h(t, s, y(s)) \Delta s, \int_{a}^{b} \tilde{g}(t, s, y(s)) \Delta s)\right\|$$

$$\leq \varepsilon + N\left[ u(t) + \int_{a}^{t} m(t, s) u(s) \Delta s + \int_{a}^{b} n(t, s) u(s) \Delta s \right].$$

Using the assumption $0 \leq N < 1$, we get

$$u(t) \leq \frac{\varepsilon}{1 - N} + \int_{a}^{t} N^{*} m(t, s) u(s) \Delta s + \int_{a}^{b} N^{*} n(t, s) u(s) \Delta s.$$

Now in view of Theorem \( 3.1 \) the previous inequality yields

$$u(t) := \|x(t) - y(t)\| \leq \frac{\varepsilon}{(1 - N)(1 - p(t))} e^{N^{*} m(t, .)}(t, a), \quad t \in I_{\tau}.$$

The function $e^{N^{*} m(t, .)}(t, a)$ is rd-continuous function on the compact interval $I_{\tau} = [a, b]_{\tau}$, so it is a bounded function. Therefore, the solution $x(t)$ of equation \( (1.1) \) depends continuously on the functions involved $f, g, h$.

Next, we introduce a result on continuous dependence to solutions of equation \( (1.2) \) on the functions involved $f, g, h$. Consider the perturbed equation

$$y(t) = y(a) + \int_{a}^{t} \tilde{f} \left( \tau, y(\tau), \int_{a}^{\tau} \tilde{h}(\tau, s, y(s)) \Delta s, \int_{a}^{b} \tilde{g}(\tau, s, y(s)) \Delta s \right) \Delta \tau,$$  \hspace{1cm} (6.7)

for $t \in I := [a, \infty]_{\tau}$ where $\tilde{f} : I_{\tau} \times \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is rd-continuous in its first variable, while the functions $\tilde{h} : I_{\tau}^{2} \times \mathbb{X} \to \mathbb{X}$ and $\tilde{g} : I_{\tau}^{2} \times \mathbb{X} \to \mathbb{X}$, are rd-continuous in its second variable.

**Theorem 6.2.** Assume that there are $k_{1}, k_{2}, k_{3} \in C_{rd}(I_{\tau}; \mathbb{R}_{+})$ such that conditions \( (5.7) - (5.10) \) hold. If $x$ and $y$ are solutions of equations \( (1.2) \) and \( (6.7) \) respectively that satisfy

$$P(t) := \|x(a) - y(a)\| + \int_{a}^{t} \left\| \tilde{f} \left( \tau, x(\tau), \int_{a}^{\tau} \tilde{h}(\tau, s, x(s)) \Delta s, \int_{a}^{b} \tilde{g}(\tau, s, x(s)) \Delta s \right) \right\| \Delta \tau < \varepsilon,$$
then
\[ \| x(t) - y(t) \| \leq \frac{\varepsilon}{1 - r} e^{k_1 + k_2}(t, a), \quad t \in I_T. \]

**Proof.** Let \( x(t) \) be a solution of (1.2), and \( y(t) \) be a solution of the perturbed equation (6.7). Put \( u(t) := \| x(t) - y(t) \|, \quad t \in I_T. \) We have
\[
\begin{align*}
\int_{a}^{t} \left[ u(\tau) + \int_{a}^{\tau} k_2(\tau) u(s) \Delta s + \int_{a}^{b} k_3(\tau) u(s) \Delta s \right] \Delta \tau
\end{align*}
\]
Applying Theorem 3.2, we obtain
\[ u(t) \leq \frac{\varepsilon}{1 - r} e^{k_1 + k_2}(t, a), \quad t \in I_T. \]

**Corollary 6.3.** Assume that there are \( k_1, k_2, k_3 \in C_{rd}(I_T; \mathbb{R}^+) \) such that conditions (5.7) - (5.10) hold. If \( I_T = [a, b]T \) is a compact interval, then every solution of equation (1.2) depends continuously on the functions involved.

**Proof.** Since \( e^{k_1 + k_2}(t, a) \) is rd-continuous function on the compact interval \( I_T = [a, b]T \), so it is bounded function. Therefore, every solution \( x(t) \) of equation (1.2) depends continuously on the functions involved \( f, g, h \).}

\[
\begin{align*}
\text{References}
\end{align*}
\]
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