Estimation of execution time for multi-stage control operations

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Abstract. The paper considers issues related to such a multiple measurements approach where the measurement results affect the number of measurements taken. This may result in a significant increase in the measurement time. The work takes into account the change in the probability that the product will be suitable after a specific measurement. These probabilities depend on the number of measurements. The given algorithm for controlling the measurement process shows the random nature of the duration of its execution, which in turn leads to the need to study the duration of multiple measurements. In this paper, a semi-Markov model is constructed that allows us to determine the functions of time distribution between events in the output flow formed after measurements. The resulting model makes it possible to connect it with models of elements of the highest levels of the hierarchy of the production structure. After a specified maximum number of measurements, the model assumes that a part of the measured product is defective and thus shall be rejected. The constructed semi-Markov model may be used to predict the performance of a technical system that uses multiple measurements to improve accuracy.

1. Introduction

One of the methods for improving the accuracy of measurements is the method of multiple measurements. The disadvantage of this method is an increase in the time of control operations and, as a result, an increase in the time of production. Thus, it is necessary to manage the process of multiple measurements. The management problem is as follows. At the first measurement, an initial estimate of the value of the controlled parameter is obtained and the number of multiple measurements is determined depending on the proximity to the tolerance boundary [1-5]. In subsequent measurements, the estimate of the value of the controlled parameter is refined and, accordingly, the number of multiple measurements is refined. Since the errors of the measuring device are random, there may be cases in which each subsequent measurement may give a worse result than the previous one relative to the proximity to the tolerance limit, which leads, accordingly, to an increase in the number of multiple measurements. Therefore, the measurement time can be significantly increased. In this case, it is advisable to interrupt the measurement process of a particular product and consider it defective for the controlled parameter. Thus, it is advisable to set a control time for performing measurements or a control number of measurements after which the product is sent to the reject and the measurement process for the next product begins.

This algorithm for controlling the measurement process determines the random nature of the duration of its execution, which in turn leads to a need to study the duration of multiple measurements. This is the purpose of this work.
2. Results

Consider the case given in [6], when the number of repeated measurements is limited at \( k \)-each stage and cannot exceed a number \( m \), and the times of repeated measurements are different and are described by distribution functions (DF) \( F^g_k(t) \), where \( k=(1,n) \) is the number of stages, and \( g=(1,m) \) is the number of repeated measurements, but the probability of transition from one state to another will differ.

To determine the DF of time spent for \( m \) repeated measurements, we construct a semi-Markov process in a discrete phase space using the state graph shown in Figure 1.

![Figure 1. State graph of the \( k \)-st stage with a limited number \( m \) of repeated measurements on it](image)

Assume that the Markov recovery process is defined by a two-component Markov chain \((S_i, \theta_i; i \geq 0)\) with transition probabilities [7 – 10]:

\[
P\{S_{i+1}=r, \theta_{i+1} \leq t | S_i = h\} = Q_{hr}(t).
\]

In this case, the components \( S_i \) and \( \theta_i \) conditionally independent, that is, equality is performed:

\[
P\{S_{i+1}=r, \theta_{i+1} \leq t | S_i = h\} = P\{S_{i+1}=r | S_i = h\} \cdot P\{\theta_{i+1} \leq t | S_i = h\}.
\]

Then the components of a semi-Markov matrix for a discrete phase space of States \( E=(l,i) \) equal:

\[
Q_{hr}(t) = P_{hr}G_h(t), \quad h,r \in E,
\]

when \( \sum_{r \in E} Q_{hr}(t) = P\{\theta_{i+1} \leq t | S_i = h\} = G_h(t), \quad h \in E \).

Let us define a stationary distribution \( \pi=(\pi^g_k; g \in E) \) semi-Markov process (SMP) in phase space \( E \) according to the formulas given in [7]:

\[
\pi^g_k = \rho^g_k M^g_k / M; \quad M = \sum_{g \in E} \rho^g_k M^g_k, \quad g \in E,
\]

when \( \rho=(\rho^g_k; g \in E) \) - stationary distribution of nested Markov chains (NMC) \((S_n, n \geq 0)\); \( M^g_k = M(v^g_k) \) - the average residence times in the States SMP; \( M^g_k \) - mathematical expectation of time spent in States \( g \in E \).

Values \( \rho^g_k \) determined from the system of equations [7]:

\[
\rho^g_k = \sum_{r \in E} \rho^r_k P^{rg}_k, \quad (2)
\]

it should be noted that \( \sum_{g} \rho^g_k = 1 \).

The graph states are as follows [6]:

\( S^0_k \) is an instantaneous state corresponding to the end of the measurement process at the previous stage (probability \( P^0_{k+1} = 1 \); \( S^1_k \) is the first measurement with the distribution function \( F^1_k(t) \) and
mathematical expectation $M_1^t$; $S_2^t$ is the second dimension with the distribution function $F_2^t(t)$ and mathematical expectation $M_2^t$; ... $S_m^t$ of the $g$-th measurement with distribution function $F_g^t(t)$ and mathematical expectation $M_m^t$; ... $S_m^t$ of $m$-th measurement with distribution function $F_m^t(t)$ and mathematical expectation $M_m^t$.

Let us make a system of equations based on (2) for determining the stationary distribution of the NMC $\rho_g^k$, $g = (0, m)$, having the form:

$$
\begin{align*}
\rho_k^0 &= \rho_k^1 \cdot f_k^0 + \rho_k^2 \cdot f_k^2 + \rho_k^3 \cdot f_k^3 + \cdots + \rho_k^g \cdot f_k^g + \cdots + \rho_k^m \cdot f_k^m = \sum_{i=0}^m \rho_k^i \cdot f_k^i, \\
\rho_k^1 &= \rho_k^0 \cdot f_k^1, \\
\rho_k^2 &= \rho_k^3 \cdot f_k^2, \\
\rho_k^3 &= \rho_k^2 \cdot f_k^3, \\
&
\vdots \\
\rho_k^g &= \rho_k^{g-1} \cdot f_k^g, \\
\rho_k^m &= \rho_k^{m-1} \cdot f_k^m = \rho_k^0 \cdot f_k^1 \cdot f_k^2 \cdot f_k^3 \cdots f_k^{g-1} \cdot f_k^g \cdots f_k^{m-2} \cdot f_k^{m-1} \cdot f_k^m = \rho_k^0 \cdot \prod_{i=1}^m f_k^{i-1}. 
\end{align*}
$$

The normalization condition has the form

$$
\sum_{g=1}^m \rho_k^g = 1. \tag{4}
$$

In accordance with the graph (Figure 1), we assume,

$$
\begin{align*}
\rho_k^{g=0} &= 1; \\
\rho_k^{g=1} &= 1; \\
\rho_k^{g=2} &= \left(1 - f_k^{g=0}\right); \\
\rho_k^{g=3} &= \left(1 - f_k^{g=2}\right); \\
\rho_k^{g=4} &= \left(1 - f_k^{g=3}\right); \\
\rho_k^{g=5} &= \left(1 - f_k^{g=4}\right); \\
\rho_k^{g=6} &= \left(1 - f_k^{g=5}\right); \\
\rho_k^{g=7} &= \left(1 - f_k^{g=6}\right); \\
\rho_k^{g=8} &= \left(1 - f_k^{g=7}\right); \\
\rho_k^{g=9} &= \left(1 - f_k^{g=8}\right). 
\end{align*}
$$

Substituting all values for $\rho_g^k$, $g = (0, m)$ from (3) to the normalization condition (4) we get

$$
\rho_k^0 (1 + \rho_k^{g=1} + \rho_k^{g=2} + \rho_k^{g=3} + \rho_k^{g=4} + \rho_k^{g=5} + \rho_k^{g=6} + \rho_k^{g=7} + \rho_k^{g=8} + \rho_k^{g=9} + \cdots + \prod_{i=1}^m f_k^{i-1}) = 1
$$

Where from

$$
\rho_k^0 = (1 + \rho_k^{g=1} + \rho_k^{g=2} + \rho_k^{g=3} + \rho_k^{g=4} + \rho_k^{g=5} + \rho_k^{g=6} + \rho_k^{g=7} + \rho_k^{g=8} + \rho_k^{g=9} + \cdots + \prod_{i=1}^m f_k^{i-1})^{-1}. \tag{6}
$$

By substituting expressions (5) in (3), (6) we finally get expressions for defining the NMC:
\[
\begin{align*}
\rho_k^0 &= \left[ 2 + (1 - P_k^{10}) + (1 - P_k^{10}) \cdot (1 - P_k^{20}) + \ldots + \prod_{i=1}^{m} (1 - P_k^{i-10}) \right]^{-1}; \\
\rho_k^1 &= \rho_k^0; \\
\rho_k^2 &= \rho_k^0 \cdot (1 - P_k^{10}); \\
\vdots & \quad \vdots \\
\rho_k^g &= \rho_k^0 \cdot \prod_{i=1}^{g} (1 - P_k^{i-10}); \\
\vdots & \quad \vdots \\
\rho_k^m &= \rho_k^0 \cdot \prod_{i=1}^{m} (1 - P_k^{i-10}).
\end{align*}
\]

On the basis of (1), (7) we determine the stationary distribution of SMP

\[
M = \sum_{i=0}^{m} \rho_k^i \cdot M_k^i = \sum_{i=0}^{m} \left[ \rho_k^0 \prod_{i=1}^{m} (1 - P_k^{i-10}) \right] \cdot M_k^i + \rho_k^0 \cdot M_k^0,
\]

\[
\pi_k^i = \frac{\rho_k^i \cdot M_k^i}{M}.
\]

Using the full probability formula we solve the problem of determining the DR of the measurement time on \( k \) - the first stage.

We accept the following hypotheses: \( H_1 \) - the first measurement is performed; \ldots \( H_{m} \) - the \( m \)th dimension is performed.

The probabilities of these hypotheses with the expression (8) are equal

\[
\begin{align*}
P(H_1) &= \pi_k^1 = \frac{\rho_k^1 \cdot M_k^1}{M}; \\
P(H_2) &= \pi_k^2 = \frac{\rho_k^2 \cdot M_k^2}{M}; \\
\vdots & \quad \vdots \\
P(H_m) &= \pi_k^m = \frac{\rho_k^m \cdot M_k^m}{M}.
\end{align*}
\]

Enter an event \( \mathcal{A} \), by which we will understand the number of measurements at the stage. The conditional probability of event \( \mathcal{A} \) when you perform a hypothesis \( H_1, \ldots, H_m \) equal

\[
\begin{align*}
P(A/H_1) &= p[t_1 < t] = F_{1}(t); \\
P(A/H_2) &= p[t_1 + t_2 < t] = F_{1,2}(t); \\
P(A/H_3) &= p[t_1 + t_2 + t_3 < t] = F_{1,2,3}(t); \\
\vdots & \quad \vdots \\
P(A/H_k) &= p\left[ \sum_{i=1}^{k} t_i < t \right] = F_{1,2,\ldots,k}(t); \\
\vdots & \quad \vdots \\
P(A/H_m) &= p\left[ \sum_{i=1}^{m} t_i < t \right] = F_{1,2,\ldots,m}(t),
\end{align*}
\]

where \( (*)_{m} \) is an operation designation \( m \) is a multiple convolution.
The function of time distribution between requests in a sparse flow of events, determined by the full probability formula, taking into account (9), (10) and has the form:

\[ K_k(t) = P(A) = \sum_{i=1}^{m} P(H_{g_i})P(A/H_{g_i}) = \sum_{i=1}^{m} P_i \cdot F_i^{(s)}(t). \] (11)

In the model, as in the case of [6], it is necessary to take into account that some part of the product \( m \) remains defective after measurements and leaves the system. The probability that the product will be suitable after \( m \) measurements, based on the graph (figure 1), is equal to:

\[ p_{k}^{suit} = 1 - \prod_{i=1}^{m} (1 - P_{i}^{(1,0)}). \] (12)

Then the probability of marriage is determined from the expression:

\[ p_{k}^{det} = 1 - p_{k}^{suit} = \prod_{i=1}^{m} (1 - P_{i}^{(1,0)}). \]

The next step is to dilute the product flow coming out of the \( k \)-stage's. This will allow you to finally get the DR \( K_k^*(t) \) the time between events in the flow of good products coming out from \( k \)-stage's. To do this, it is necessary to substitute a non-simple event stream [6] into the rarefaction formula \( K_k(t) \) and \( p_{k}^{suit} \) from the expression (11), (12):

\[ K_k^*(t) = p_{k}^{suit} K_k(t) + p_{k}^{suit} \sum_{i=1}^{m} \left( \prod_{i=1}^{m} (1 - P_{i}^{(1,0)}) \cdot K_k^{(*)}(t) \right). \] (13)

The expression (13) allows each of the stages of multiple measurements to be replaced by an equivalent stage of single measurements performed by a device that is a simple element with two factor States: a callback and a workable one with a known service time distribution function defined by (13).

The execution time of all stages will be determined as the sum of random variables (the implementation times of each stage), and the DR of this time - as a composition of the required number of distribution laws of the implementation times of each stage, determined by the formula (13).

Mathematical expectation of the execution time \( k \)-ro stages are equal:

\[ m_k = \int_{0}^{1} [1 - K_k^*(t)] dt. \] (14)

Then the mathematical expectation of the task execution time consisting of \( n \) stages \( t_{1,n} \) equally:

\[ t_{1,n} = \sum_{k=1}^{N} m_k, \] (15)

where \( m_k \) is determined by (14).

One of the problems that arise when implementing the resulting model is determining probabilities \( P_{i}^{(1,0)} \) what the product is after \( i \) measurements will be valid, which in turn depend on the number of measurements. The algorithm proposed for their determination is as follows:

We set the confidence probability. We determine the maximum number of measurements that provide the required accuracy in accordance with the methodology given in [4]. We make the first measurement. Further, in accordance with the maximum entropy theorem of a random variable centered on a finite interval [4], which is provided by the composite distribution law, the probability of the measurement result going beyond the tolerance field is determined and compared with the specified value. If it is less than the specified value of the probability of the measurement result going beyond the tolerance field, then the product is considered suitable, otherwise we make the next measurement, reducing the variance accordingly. We perform this sequence until we reach the
specified maximum number of dimensions of the corresponding one $m$. If the $m$ condition is still not met after the measurements, the product is considered defective and leaves the system.

3. Conclusion

Thus, the constructed semi-Markov model allows predicting the performance of a technical system based on formulas (13), (14), (16), which take into account various probabilities of transition from one state to another, which is the main advantage of the model.

In the case where the use of precision measuring equipment is not economically feasible, it makes sense to use the multiple measurement method, which has the drawback mentioned above. In this regard, it is recommended to select the minimum required number of measurements, providing the required accuracy at maximum performance. In this case, an information approach based on the principle of maximum entropy was used to solve the problem. This approach allows you to determine the minimum required number of measurements, while ensuring a given accuracy.

Since the design of any complex system is based on the modeling of indicators of productivity and quality of products, this approach, unlike all others, allows us to determine whether the developed system meets the specified requirements.

In the future, it is planned to extend this approach to the modeling of technical systems with the return of products for re-service, which is widespread when performing finishing operations, such as grinding, honing, polishing.

References

[1] Novitsky P V 1968 Fundamentals of Information Theory of Measuring Devices (Energia)
[2] Novitsky P V and Zograf I A 1991 Estimation of Errors in Measurement Results (Leningrad: Energoatomizdat)
[3] RMG 64-2003 GSI. Providing efficiency in the management of technological processes. Methods and ways to improve the accuracy of measurements
[4] Kopp V, Skidan A, Balakin A and Filipovich O 2007 Proc. of Odessa Polytechnic University: Scientific and industrial-practical collection of technical and natural sciences 1(27) 214-218
[5] Janes E T 1982 Proc. of the Institute of Electrical and Electronics Engineers 70(9) 33-51
[6] Kopp V, Balakin A, Balakina N and Zamoryonov M 2018 Int. Conf. on Modern Trends in Manufacturing Technologies and Equipment (ICMTMTE 2018) 04021
[7] Korolyuk V S and Turbin A F 1989 Stochastic Models of Systems (Kiev: Naukova Dumka)
[8] Korolyuk V S and Turbin A F 1978 Mathematical Foundations of Phase Integration of Complex Systems (Kiev: Naukova Dumka)
[9] Korolyuk V S and Turbin A F 1982 Markov Recovery Processes in Reliability Problems of Systems (Kiev: Naukova Dumka)
[10] Obzherin Y E and Boyko E G 2015 Semi-Markov Models: Control of Restorable Systems with Latent Failures (Elsevier Inc.)