Updating structured matrix pencils with no spillover effect on unmeasured spectral data and deflating pair

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Abstract. This paper is devoted to the study of perturbations of a matrix pencil, structured or unstructured, such that a perturbed pencil will reproduce a given deflating pair while maintaining the invariance of the complementary deflating pair. If the latter is unknown, it is referred to as no spillover updating. The specific structures considered in this paper include symmetric, Hermitian, ⋆-even, ⋆-odd and ⋆-skew-Hamiltonian/Hamiltonian pencils. This study is motivated by the well-known Finite Element Model Updating Problem in structural dynamics, where the given deflating pair represents a set of given eigenpairs and the complementary deflating pair represents the remaining larger set of eigenpairs. Analytical expressions of structure preserving no spillover updating are determined for deflating pairs of structured matrix pencils. Besides, parametric representations of all possible unstructured perturbations are obtained when the complementary deflating pair of a given unstructured pencil is known. In addition, parametric expressions are obtained for structured updating with certain desirable structures which relate to existing results on structure preservation of a symmetric positive definite or semi definite matrix pencil.

Keywords. Model updating, structured matrix pencils, inverse eigenvalue problem, deflating subspace

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1 Introduction

The model updating problem (MUP) with no spillover effect on unmeasured spectral data has found its place in the core research areas of numerical linear algebra due to its importance in real world applications, for example, in vibration industries including automobile, space and aircraft industries [13, 11, 15, 29]. The problem is to update a quadratic matrix polynomial in such a way that a small number of measured eigenvalues and eigenvectors are reproduced by the updated model while maintaining the no spillover of the large number of remaining unmeasured eigenpairs. It is of utmost practical interest that the finite-element inherited structures, such as the symmetry, positive definiteness or semi-definiteness are preserved in the updated model. The quadratic finite element model associated with the MUP is given by

\[ M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = 0 \]

where \( M, D, K \) are square matrices of dimension, say \( n \times n \), \( x(t) \) is a column vector of order \( n \). Usually, \( M \) is called mass matrix which is Hermitian positive definite, \( K \) is Hermitian positive semi-definite and called stiffness matrix, and \( D \) is a Hermitian matrix which is called the damping matrix [11, 21, 13]. The equation (1) represents an undamped model if \( D \) is the zero matrix. Solutions of (1) can be obtained as \( x(t) = x_0 e^{\lambda_0 t} \), where \( (\lambda_0, x_0) \) turns out to be eigenpairs of the quadratic matrix polynomial \( Q(\lambda) = \lambda^2 M + \lambda D + K \in \mathbb{C}^{n \times n}[\lambda] \).

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Let \( \{ (\lambda_i, x_i) : i = 1, \ldots, 2n \} \) be a collection of eigenpairs of \( Q(z) \). Then given a positive integer \( p \leq 2n \) and a set of scalars \( \mu_i, i = 1, \ldots, p \), the model updating problem is concerned with finding structure preserving quadratic matrix polynomials \( \triangle Q(z) = \lambda^2 \triangle M + \lambda \triangle D + \triangle K \in \mathbb{C}^{n \times n}[\lambda] \) such that

\[
(Q(\mu_i) + \triangle Q(\mu_i))y_i = 0, \quad i = 1, \ldots, p
\]

for some \( y_i \neq 0 \). In addition, if \( (\lambda_j, x_j), j = p+1, \ldots, 2n \) are not known then it is a no spillover updating. That is,

\[
(Q(\lambda_j) + \triangle Q(\lambda_j))x_j = 0, \quad j = p+1, \ldots, 2n
\]

for such \( \triangle Q(z) \). In the context of applications, equation (1) represents a theoretical finite-element model of a structure that needs to be updated by a few measured eigenvalues \( (\mu_i, i = 1, \ldots, p) \) obtained from the real structure without disturbing the unmeasured eigenvalues \( (\lambda_j, j = p+1, \ldots, 2n) \) of the model. Several attempts have been made to solve the problem both by finding analytical and algorithmic solutions [1, 5, 7, 10, 35, 43, 8, 3, 30, 9, 11, 12, 14]. However, a complete characterization of solution sets describing \( \triangle Q(z) \) which satisfy (2) and (3) remains an open problem [21].

We emphasize that a solution of the no spillover quadratic model updating does not necessarily yield a solution of the no spillover linear updating, just be setting the damped matrix to be the null matrix. For example:

- Consider the solution sets proposed in [12] and [14] for quadratic models. In [12], \( M \) is symmetric positive definite, \( D \) is symmetric and \( K \) is symmetric positive definite, and in [14], the authors consider a same structure of \( Q(\lambda) \) but \( K \) is semi-definite. Setting \( D = 0 \) in the solutions proposed both in [12] and [14], it can be seen that the perturbation \( \triangle D \) is a nonzero matrix. Hence the proposed solutions do not solve the MUP with no spillover for undamped structural models.

- In [21], the authors consider quadratic models \( Q(\lambda) \), where \( M \) is a real symmetric nonsingular matrix, \( D \) and \( K \) are symmetric matrices. However, it can be easily checked that setting \( D = 0 \), the proposed solution provides \( \triangle D \neq 0 \).

- In [22] and [23], the author considers the MUP problem with/without spillover for quadratic models where \( M \) is symmetric/Hermitian positive definite, \( D \) and \( K \) are symmetric/Hermitian matrices. However the author utilizes the Jordan pair of \( Q(\lambda) \) in order to redefine the problem in terms of self-adjoint triple, and the coefficient matrices \( M, D, K \) are written using the moments of the corresponding system. Due to this formulation, it is not clear how setting \( D \) to be the zero matrix will produce structured perturbations of the linear pencil from the solution of quadratic model, unless the Jordan pair satisfies an orthogonality condition.

Thus it may be concluded that the MUP with/without spillover for quadratic models and undamped models are inherently different if \( M \) is a positive definite matrix. In this paper we consider the MUP with no spillover for undamped models \( M \dot{x}(t) + K x(t) = 0 \) represented by structured matrix pencils described as follows.

For \( A \in \mathbb{C}^{n \times n} \) let \( A^T \) denote its transpose and let \( A^* = A^T \) denote its conjugate transpose. Let \( * \in \{*, T\} \) and \( \epsilon_1, \epsilon_2 \in \{-1, 1\} \). We say that the pencil \( L(\lambda) = \lambda M + K \in \mathbb{C}^{n \times n}[\lambda] \) has \((*, \epsilon_1, \epsilon_2)\)-structure if

\[
M^* = \epsilon_1 M, \quad K^* = \epsilon_2 K.
\]

Pencils of this form are known under the following names.
The set of these pencils is denoted by \( \mathbb{P}_n(\epsilon_1, \epsilon_2) \). We also consider \( *\)-skew-Hamiltonian/Hamiltonian matrix pencils \( L(\lambda) = \lambda M + K \in \mathbb{C}^{2n \times 2n} [\lambda] \) which appear in different applications including gyroscopic systems and linear response theory, where \( (JM)^* = -JM, (JK)^* = JK \) and \( J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \). Thus \( JL(\lambda) \in \mathbb{P}_{2n}(\epsilon, -1, 1) \). These structured matrix pencils arise in a variety of real world problems, see [25, 33].

Now, we define MUP with no spillover effect on unmeasured spectral data for pencils \( L(\lambda) = \lambda M + K \) as follows.

**P1** (model updating problem with no spillover) Let \( (\lambda_i, x_i), i = 1, \ldots, p \) be a collection of given eigenpairs of \( L(\lambda) \). Suppose \( (\lambda_j, x_j), j = p + 1, \ldots, n \) is a collection of complementary eigenpairs of \( L(\lambda) \), that is \( \{x_1, \ldots, x_n\} \) is nonsingular. Let \( \lambda_i^0 \) and \( x_i^0 \) be a collection of given scalars and nonzero vectors respectively, \( i = 1, \ldots, p \). Then determine perturbations \( (\Delta M, \Delta K) \) such that \( (\lambda_i^0, x_i^0) \) become eigenpairs of \( L(\lambda) = \lambda(M + \Delta M) + (K + \Delta K) \), and the corresponding complementary eigenpairs of \( L(\lambda) \) are given by \( (\lambda_j, x_j), j = p + 1, \ldots, n \). (The notations \( ^c, ^f, ^a \) stand for change, fixed and aimed respectively.)

Besides, determine \( \Delta M, \Delta K \) such that \( L(\lambda) \in \mathbb{S} \subseteq \mathbb{P}_n(\epsilon_1, \epsilon_2) \) whenever \( L(\lambda) \in \mathbb{S} \) and \( (\lambda_j, x_j), j = p + 1, \ldots, n \) are not known, where \( \mathbb{S} \) is a set of structured matrix pencils.

Setting \( \Lambda_a = \text{diag}\{\lambda_i^0 : i = 1, \ldots, p\} \), \( X_a = [x_1^0, x_2^0, \ldots, x_p^0] \), \( \Lambda_f = \text{diag}\{\lambda_j^0 : j = p + 1, \ldots, n\} \), and \( X_f = [x_{p+1}^f, x_{p+2}^f, \ldots, x_n^f] \), it follows from Problem (P1) that the desired perturbations \( (\Delta M, \Delta K) \) should satisfy

\[
(M + \Delta M)X_a\Lambda_a + (K + \Delta K)X_a = 0, \quad (M + \Delta M)X_f\Lambda_f + (K + \Delta K)X_f = 0.
\]

The matrix pairs \( (X, \Lambda) \) with \( M\Lambda + KX = 0 \) are called deflating pairs of \( \lambda M + K \) \([20]\). Here it is not required that \( \Lambda \) is to be diagonal. However, to avoid redundancies \( X \) should have full column rank. Two deflating pairs \( (X_1, \Lambda_1), (X_2, \Lambda_2) \) are said to be complementary if \( [X_1 \quad X_2] \) is a nonsingular square matrix. With this terminology the following extended problem can be formulated.

**P2** (change of deflating pairs with no spillover) Let \( (X_c, \Lambda_c) \in \mathbb{C}^{n \times p} \times \mathbb{C}^{p \times p} \) and \( (X_f, \Lambda_f) \in \mathbb{C}^{n \times (n-p)} \times \mathbb{C}^{(n-p) \times (n-p)} \) be complementary deflating pairs of a matrix pencil \( L(\lambda) = \lambda M + K \). Let \( (X_a, \Lambda_a) \) be a matrix pair of the same dimension as \( (X_c, \Lambda_c) \) such that \( [X_a \quad X_f] \) is nonsingular. Find perturbations \( (\Delta M, \Delta K) \) such that \( (X_a, \Lambda_a) \) and \( (X_f, \Lambda_f) \) are complementary deflating pairs of the perturbed pencil \( L(\lambda) = (M + \Delta M)\lambda + (K + \Delta K) \).

Moreover, determine pair of structured perturbations \( (\Delta M, \Delta K) \) such that \( L(\lambda) \in \mathbb{S} \subseteq \mathbb{P}_n(\epsilon_1, \epsilon_2) \) whenever \( L(\lambda) \in \mathbb{S} \) and \( (X_f, \Lambda_f) \) is not known, where \( \mathbb{S} \) is a set of structured matrix pencils (Note that \( \Lambda_c, \Lambda_a, \Lambda_f \) need not be diagonal matrices).

Let us call the complementary deflating pairs \( (X_c, \Lambda_c) \) and \( (X_f, \Lambda_f) \) of a pencil \( L(\lambda) \in \mathbb{C}^{n \times n}[\lambda] \) as change and fixed deflating pairs respectively. Then it follows that the Problem (P1) is a special case of Problem (P2).
Problem (P1) for Hermitian pencils defines the standard MUP with no spillover for an undamped model by setting $\lambda = z^2$. It is extensively studied in literature. See \cite{27, 39, 34, 24, 11, 35} and the references therein. However, explicit parametric expressions of $\Delta M, \Delta K$ are obtained only in a few articles when both the coefficient matrices of $L(\lambda)$ are positive definite or semi-definite. For example:

- In \cite{11}, Carvalho et al. have derived solutions of problem (P1) which are of the form $\Delta M = 0, \Delta K = -MX_c \Psi X_c^T M$ for an undamped model $L(z^2) = z^2 M + K \in \mathbb{R}^{n \times n}$ where both $M$ and $K$ are symmetric positive definite, and $\{\lambda_1^c, \ldots, \lambda_p^c\} \cap \{\lambda_{p+1}^f, \ldots, \lambda_{2n}^f\} = \emptyset$. Here $\Psi$ is a (symmetric) solution of a (matrix) linear system, which has to obtained by solving the system numerically.

- Solvability conditions and explicit expressions for solution pairs $(\Delta M, \Delta K)$ are obtained by Mao et al. in \cite{27} for $L(\lambda) = \lambda M - K \in \mathbb{R}^{n \times n}$, where $M$ positive definite and $K$ is positive semi-definite.

Analytical expressions of the updating matrices are also obtained for undamped models in \cite{11} and \cite{10} by treating the MUP as a residual minimization problem and matrix pencil nearness problem respectively. An optimization approach is also considered in \cite{7} to obtain the updates. Determination of explicit expressions for updating matrices is motivated by the fact that it gives more suitable results than the same obtained by using iterative methods \cite{37}. Particular classes of solutions are also obtained for specific structural undamped models \cite{38, 42}. To the best of the knowledge of the authors, no explicit solution sets are available in literature for the undamped model when the corresponding matrix pencils are not Hermitian.

The contribution of this work are as follows. Let $L(\lambda) = \lambda M + K$.

1. First, a general expression is obtained for all possible unstructured perturbations which solves the Problem (P2) when the fixed (unmeasured) deflating pair of the corresponding pencil is known.

2. Next, parametric expressions are determined for structure preserving perturbations which solve the Problem (P2) when $L(\lambda) \in \mathbb{L}_n(\ast, \epsilon_1, \epsilon_2)$. In this case, the fixed (unmeasured) deflating pair of $L(\lambda)$ is unknown, and $\sigma(L_\lambda) \cap \sigma(\epsilon_1 \epsilon_2 \lambda^*_f) = \emptyset$.

3. Finally, parametric solutions of the Problem (P2) are obtained for especially structured pencils $L(\lambda) \in S \subset \mathbb{L}_n(\ast, \epsilon_1, \epsilon_2)$. The pencils $L(\lambda) \in S$ have the following structures: Hermitian pencils with $M$ positive definite, $\ast$-odd pencils with $M$ positive definite; $\ast$-even pencils with $K$ positive definite; and $\ast$-skew-Hamiltonian/Hamiltonian matrix pencils $L(\lambda)$, that is, $JL(\lambda) \in L_{2n}(\ast, -1, 1)$.

Moreover, parametric solution sets for the Problem (P1) are obtained by utilizing the solutions of the Problem (P2) when $L(\lambda) \in S$. It is also shown that the proposed solution realizes the solution obtained by Carvalho et al. in \cite{11} as a special case (see Remark \ref{remark}). Besides, the proposed solution also identifies the solution proposed by Mao et al. in \cite{27} (see Remark \ref{remark}). It is also to be noted in this context that our results can not be obtained as special cases of the existing structured preserving results of the quadratic FEM updating just by setting the damping matrix to be the null matrix.

The obtained results are supported with numerical examples.

The paper is organized as follows. In the next two sections we present elementary facts on deflating pairs and pencils with $(\ast, \epsilon_1, \epsilon_2)$-structure. Though all these fact are known we give some proofs for the convenience of the reader. In Section \ref{section} we discuss Problem (P2) for unstructured perturbations. We give a general solution formula provided for the case that $(X_f, \Lambda_f)$ is completely known. The latter rarely happens in practical applications. However, for pencils with $(\ast, \epsilon_1, \epsilon_2)$-structure the complete knowledge of $(X_f, \Lambda_f)$ is not required for solving the problem. Instead, only a certain spectral condition is needed. This is the content
of Section 5 in which we present our main result. In the remaining sections we discuss special cases and show numerical examples.

**Notation.** As usual, $\mathbb{R}$ and $\mathbb{C}$ denote the field of real and complex numbers respectively.

A pencil $L(\lambda) = \lambda M + K \in \mathbb{C}^{n \times n}[\lambda]$ is said to be regular if its characteristic polynomial $\chi(\lambda) = \det(\lambda M + K)$ is not zero polynomial. In this paper we consider only regular pencils. The zeros of $\chi$ are called the finite eigenvalues of $L(\lambda)$. The pencil is said to have eigenvalue infinity if $M$ is singular. Let $\lambda_0 \in \mathbb{C}$ be a finite eigenvalue. Then there exists a nonzero eigenvector $x \in \mathbb{C}^n$ such that $\lambda_0 Mx + Kx = 0$. The pair $(\lambda_0, x)$ is called an eigenpair of $L(\lambda)$.

Recall from the introduction that a matrix pair $(X, \Lambda) \in \mathbb{C}^{n \times p} \times \mathbb{C}^{p \times p}$ with rank $X = p \leq n$ is said to be a deflating pair for the pencil $L(\lambda)$ if

$$MX\Lambda + KX = 0.$$  

The latter is equivalent to the equation $L(\lambda)X = M X(\lambda I - \Lambda)$. The range of $X$ is then called a deflating subspace. If $p = 1$ then $(\Lambda, X)$ is an eigenpair of $L(\lambda)$. In general the eigenvalues of the square matrix $\Lambda$ form a subset of the set of eigenvalues of $L(\lambda)$. More precisely, if $\xi$ is an eigenvector of $\Lambda$ to the eigenvalue $\lambda_0 \in \mathbb{C}$ (that is $\Lambda \xi = \lambda_0 \xi$) then $(\lambda_0, X\xi)$ is an eigenpair of $L(\lambda)$. In particular, if $\Lambda$ is diagonal then the columns of $X$ are eigenvectors of $L(\lambda)$. Furthermore, for any $\xi_0 \in \mathbb{C}^p$ the function $x(t) = X e^{t \Lambda} \xi_0$ fulfills the differential equation $M \dot{x}(t) + K x(t) = 0$. We say that two deflating pairs $(X, \Lambda), (\bar{X}, \bar{\Lambda})$ of $L(\lambda)$ are complementary if $[X, \bar{X}]$ is a nonsingular square matrix. In this case $([X, \bar{X}], \text{diag}(\Lambda, \bar{\Lambda}))$ is a deflating pair and

$$L(\lambda) = M \left[ X \quad \bar{X} \right] \left( \lambda I - \text{diag}(\Lambda, \bar{\Lambda}) \right) \left[ X \quad \bar{X} \right]^{-1}.$$  

If $(X, \Lambda)$ is a deflating pair then $(XZ, Z^{-1} \Lambda Z)$ is also a deflating pair for any nonsingular matrix $Z \in \mathbb{C}^{p \times p}$. The associated deflating subspaces coincide. A simple application of this fact is as follows. Suppose $M$ and $K$ are real matrices and $(\lambda, x)$ is an eigenpair with nonreal $\lambda$. Then the conjugate pair $(\lambda, \bar{x})$ is also an eigenpair. Suppose that $M$ is nonsingular. Then $\lambda \neq \bar{\lambda}$ implies that the vectors $x, \bar{x}$ are linearly independent and hence, the matrices $\Lambda = \text{diag}(\lambda, \bar{\lambda}), X = \begin{bmatrix} x & \bar{x} \end{bmatrix}$ form a deflating pair. A real deflating pair $(X_r, \Lambda_r)$ with range $X_r = \text{range } X$ is

$$X_r = XZ = \begin{bmatrix} \text{re}(x) & \text{im}(x) \end{bmatrix}, \quad \Lambda_r = Z^{-1} \Lambda Z = \begin{bmatrix} \text{re}(\lambda) & \text{im}(\lambda) \\ -\text{im}(\lambda) & \text{re}(\lambda) \end{bmatrix}, \quad \text{where } Z = \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}.$$  

### 3 Structured pencils

Let $\star \in \{\ast, T\}$ and $\epsilon_1, \epsilon_2 \in \{-1, 1\}$. Recall from the introduction that $L(\lambda) = \lambda M + K \in \mathbb{C}^{n \times n}[\lambda]$ is said to have $(\ast, \epsilon_1, \epsilon_2)$-structure if

$$M^\star = \epsilon_1 M, \quad K^\star = \epsilon_2 K.$$  

The set of these pencils is denoted by $\mathbb{L}_n(\ast, \epsilon_1, \epsilon_2)$. The number $x_1^\ast M x_2 \in \mathbb{C}$ is called the $M$-scalar product of the vectors $x_1, x_2$. For $z \in \mathbb{C}$ we define $z^\ast = \bar{z}$ (the conjugate of $z$) if
\[ \ast = \ast \text{ and } z^* = z \text{ if } \ast = T. \] Then we have \( x_2^* M x_1 = \epsilon_1 (x_1^* M x_2)^* \). This yields

\[
x^* M x \begin{cases} 
\in \mathbb{R} & \text{if } (\ast, \epsilon_1) = (\ast, 1), \\
\in i\mathbb{R} & \text{if } (\ast, \epsilon_1) = (\ast, -1), \\
= 0 & \text{if } (\ast, \epsilon_1) = (T, -1).
\end{cases}
\]

In the first of these cases (\( M \) Hermitian) the matrix \( M \) is said to be positive definite if \( x^* M x > 0 \) for all \( x \neq 0 \). If \( x_1^* M x_2 = 0 \) then the vectors \( x_1, x_2 \) are said to be \( M \)-orthogonal. For a matrix \( X \in \mathbb{C}^{n \times p} \) with columns \( x_i \) the associated \( M \)-Gramian is \( G = X^* M X = [x_i^* M x_j] \in \mathbb{C}^{n \times p} \). Obviously, \( G^* = \epsilon_1 G \).

The proposition below lists elementary properties of pencils with \((\ast, \epsilon_1, \epsilon_2)\)-structure.

**Proposition 3.1.** Let \( L(\lambda) = \lambda M + K \in \mathbb{L}(\ast, \epsilon_1, \epsilon_2) \). Then

(i) \( \lambda_0 \in \mathbb{C} \) is an eigenvalue of \( L(\lambda) \) if and only if \( \epsilon_1 \epsilon_2 \lambda_0^\ast \) is an eigenvalue of \( L(\lambda) \).

Let \( X_j \in \mathbb{C}^{n \times p_j} \), let \( G_{jk} = X_j^* M X_k \) and \( F_{jk} = X_j^* K X_k \) for \( j, k \in \{1, 2\} \). Then

(ii) the pencil \([X_1, X_2]^* L(\lambda)[X_1, X_2] = \lambda \begin{bmatrix} G_{11} & G_{12} \\
G_{21} & G_{22} \end{bmatrix} + \begin{bmatrix} F_{11} & F_{12} \\
F_{21} & F_{22} \end{bmatrix} \) has \((\ast, \epsilon_1, \epsilon_2)\)-structure.

In particular, \( G_{jk} = \epsilon_1 G_{jj} \), \( F_{jk} = \epsilon_2 F_{kk} \) and \( \lambda G_{jj} + F_{jj} \in \mathbb{L}(\ast, \epsilon_1, \epsilon_2) \).

Suppose \((X_j, \Lambda_j), j = 1, 2\) are deflating pairs of \( L(\lambda) \). Then for \( j, k \in \{1, 2\} \),

(iii) \( G_{jk} \Lambda_k = -F_{jk} = \epsilon_1 \epsilon_2 \Lambda_j^\ast G_{jk} \),

(iv) the spectral property \( \sigma(\Lambda_k) \cap \sigma(\epsilon_1 \epsilon_2 \Lambda_j^\ast) = \emptyset \) implies \( G_{jk} = F_{jk} = 0 \),

(v) if \( \sigma(\Lambda_k) \cap \sigma(\epsilon_1 \epsilon_2 \Lambda_j^\ast) = \emptyset \) then

\[ [X_1, X_2]^* L(\lambda)[X_1, X_2] = \text{diag}(\lambda G_{11} - G_{11} \Lambda_1, \lambda G_{22} - G_{22} \Lambda_2). \]

In particular \( G_{11} \) and \( G_{22} \) are both nonsingular if \((X_1, \Lambda_1)\) and \((X_2, \Lambda_2)\) are complementary and \( M \) or \( K \) is nonsingular.

**Proof.** The matrix \( \lambda_0 M + K \) is singular if an only if the matrix \( \epsilon_1 \epsilon_2 \lambda_0^\ast M^* + K^* = \epsilon_2 (\lambda_0 M + K)^* \) is singular. Thus, (i) holds. (ii) is immediate from (i). Multiplying the relation \( M X_k \Lambda_k + K X_k = 0 \) from the left with \( X_j^\ast \) yields the first identity of (iii). The second identity then follows from (ii). Reordering terms in (iii) we get the Sylvester equation \( G_{jk} \Lambda_k - \epsilon_1 \epsilon_2 \Lambda_j^\ast G_{jk} = 0 \). By an elementary result on Sylvester equations we have \( G_{jk} = 0 \) if the matrices \( \Lambda_k \) and \( \epsilon_1 \epsilon_2 \Lambda_j^\ast \) have disjoint spectra. Hence, (iv). (v) is immediate from (ii) and (iv).

The matrices \( X_1 \) and \( X_2 \) in Proposition 3.1 may be identical. In this case we obtain from statement (iv) the following corollary.

**Corollary 3.2.** Let \((X, \Lambda)\) be a deflating pair of \( \lambda M + K \in \mathbb{L}(\ast, \epsilon_1, \epsilon_2) \) such that \( \sigma(\Lambda) \cap \sigma(\epsilon_1 \epsilon_2 \Lambda^\ast) = \emptyset \). Then \( X^* M X = X^* K X = 0 \).

A further corollary of Proposition 3.1 is obtained if \( X_1, X_2 \) are chosen to be column vectors.

**Corollary 3.3.** Let \((\lambda_1, x_1)\) and \((\lambda_2, x_2)\) be eigenpairs of \( \lambda M + K \in \mathbb{L}(\ast, \epsilon_1, \epsilon_2) \). If \( \lambda_2 \neq \epsilon_1 \epsilon_2 \lambda_1^\ast \) then \( x_1^* M x_2 = x_1^* K x_2 = 0 \).

If \((\lambda_0, x)\) is an eigenvpair of \( \lambda M + K \) then by multiplying the relation \((\lambda_0 M + K)x = 0 \) from the left with \( x^\ast \) we get

\[ \lambda_0 = -x^* K x / x^* M x \] (7)

provided that \( x^* M x \neq 0 \). The latter trivially holds if \( \ast = \ast \) and \( M \) is Hermitian and positive definite. However, by the corollary above we have \( x^* M x = 0 \) whenever \( \lambda_0 \neq \epsilon_1 \epsilon_2 \lambda_0^\ast \). In this case we have the following statement which is immediate from the previous results in this section.
Corollary 3.4. Let \((\lambda_0, x)\) be an eigenpair of \(\lambda M + K \in \mathbb{C}_n(\star, \epsilon_1, \epsilon_2)\) such that \(\lambda_0 \neq \epsilon_1 \epsilon_2 \lambda_0^*\). By part (i) of Proposition 6.4 there exists an eigenpair \((\epsilon_1 \epsilon_2 \lambda_0^*, \hat{x})\). Set \(X := [x, \hat{x}]\), \(g := \hat{x}^* M x\). Then \((X, \text{diag}(\lambda_0, \epsilon_1 \epsilon_2 \lambda_0^*))\) is a deflating pair of \(L(\lambda)\), and

\[
X^* M X = \begin{bmatrix} 0 & \epsilon_1 g^* \\ g & 0 \end{bmatrix}, \quad X^* K X = \begin{bmatrix} 0 & -\epsilon_2 \lambda_0^* g^* \\ -\lambda_0 g & 0 \end{bmatrix}.
\]

By scaling of \(x\) one can achieve that \(g = 1\) or \(g = 0\).

The identity \(\mathbf{4}\) yields the following basic fact.

Proposition 3.5. Let \(M\) be Hermitian and positive definite. Then all eigenvalues of \(\lambda M + K\) are real if \(K\) is Hermitian. They are all negative if \(K\) is Hermitian and positive definite. The eigenvalues are all purely imaginary or 0 if \(K\) is skew-Hermitian.

It is a well known fact that to a Hermitian pencil with positive definite \(M\) there exists a basis \(\{x_i, i = 1, \ldots, n\}\) of eigenvectors such that \(x_i^* M x_j = 0\) for \(i \neq j\). The general eigenstructure of pencils with \((\star, \epsilon_1, \epsilon_2)\)-symmetry is somehow involved and will not be discussed here. We refer to the literature \([1, 19, 32, 26]\). The next proposition shows how to construct a complementary deflating pair to a given one.

Proposition 3.6. Let \((X_1, \Lambda_1) \in \mathbb{C}^{n \times p} \times \mathbb{C}^{p \times p}\) be a deflating pair of \(\lambda M + K \in \mathbb{C}_{n}(\star, \epsilon_1, \epsilon_2)\). Suppose that \(M\) and \(G_1 := X_1^* M X_1\) are both nonsingular. Then \(X \in \mathbb{C}^{n \times (n - p)}\) be such that \([X_1, X]\) is nonsingular. Set \(X_2 := X - X_1 G_1^{-1}(X_1^* M X)\). Then

(i) \(X_1^* M X_2 = X_1^* K X_2 = 0\) and \(G_2 := X_2^* M X_2\) is nonsingular.

(ii) Set \(\Lambda_2 := -G_2^{-1}(X_2^* K X_2)\). Then \((X_2, \Lambda_2)\) is a deflating pair of \(L(\lambda)\) which is complementary to \((X_1, \Lambda_1)\).

Proof. (i) The identity \(X_1^* M X_2 = 0\) is easily verified. The identity \(X_1^* K X_2 = \epsilon_2 (X_1^* K X_1)^* = 0\) follows from \(X_2^* M X_1 = \epsilon_1 (X_1^* M X_2)^* = 0\) by multiplying \(M X_1 \Lambda_1 + K X_1 = 0\) with \(X_2^*\) from the left. The nonsingularity of \(G_2\) follows from \([X_1, X_2]^* M [X_1, X_2] = \text{diag}(G_1, G_2)\) and the nonsingularity of the matrices on the left hand side. (ii) The matrix \([X_1, X_2]^* [I, I]^{-1}(X_1^* M X)\] is nonsingular. Thus, \(X_2\) has full column rank. The results obtained so far imply that \([X_1, X_2]^* (M X_2 \Lambda_2 + K X_2) = 0\). Thus, \(M X_2 \Lambda_2 + K X_2 = 0\). □

4 Unstructured updates

We now discuss the updating problem \((\mathbf{P2})\) for pencils without any prescribed structure. By assumption \((X_f, \Lambda_f)\) and \((X_c, \Lambda_c)\) are complementary deflating pairs of \(L(\lambda) = \lambda M + K\). Thus,

\[
M X_f \Lambda_f + K X_f = 0, \quad M X_c \Lambda_c + K X_c = 0.
\]

Since \((X_f, \Lambda_f)\) and \((X_a, \Lambda_a)\) should be complementary deflating pairs of the updated pencil \(L_\Delta(\lambda) = (\lambda M + \Delta M) + (K + \Delta K)\) the matrices \(\Delta M, \Delta K\) we seek for should satisfy

\[
(M + \Delta M) X_f \Lambda_f + (K + \Delta K) X_f = 0,
\]

\[
(M + \Delta M) X_a \Lambda_a + (K + \Delta K) X_a = 0.
\]

Because of \((\mathbf{5})\) an equivalent system of equations is

\[
\Delta M X_f \Lambda_f + \Delta K X_f = 0, \quad \Delta M X_a \Lambda_a + \Delta K X_a = R_a,
\]

where

\[
R_a := -(M X_a \Lambda_a + K X_a) = M(X_c \Lambda_c - X_a \Lambda_a) + K(X_c - X_a).
\]
Notice that
\[ R_a = MX_c(A_c - A_a) \quad \text{if} \quad X_a = X_c. \] (12)
Equations (10) can be written as
\[
\begin{bmatrix} \triangle M & \triangle K \\ \Delta & \mathbf{Y} \end{bmatrix} \begin{bmatrix} X_fA_f \\ X_f \\ X_aA_a \\ X_a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ B \end{bmatrix}.
\] (13)
According to a basic result on linear matrix equations the general solution of (13) is
\[ Y = BA^\dagger + Z(I - AA^\dagger), \quad Z \in \mathbb{C}^{n \times 2n} \text{ arbitrary}, \]
where \( A^\dagger = (A^*A)^{-1}A^* \) is the Moore-Penrose generalized inverse of \( A \). Observe that in the present case \( A^*A \) is indeed nonsingular since \( A \) has full column rank. The latter holds because \( [X_f \quad X_a] \) is nonsingular by assumption. Hence we have obtained a parametrization of all possible updates \( Y = [\Delta M \quad \Delta K] \) that solve problem (P2). However, that the solution requires the knowledge of the matrix \( A \) and hence the knowledge of \( (X_f, A_f) \). This information is often not available in the applications. In the next section on structured pencils we will derive updates whose construction only requires the knowledge of \( (X_c, A_c) \) and a property of the spectrum \( \sigma(A_f) \) which is generically satisfied.

The theorem below provides a convenient subset of the general solution set to Problem (P2). This theorem prepares the result on structured pencils in the next section.

**Theorem 4.1.** Suppose that the assumptions of Problem (P2) hold. Let \( U \in \mathbb{C}^{n \times p} \) be the unique matrix satisfying \( U^*X_f = 0 \), and \( U^*X_a = I_p \) \( \text{ (i.e. U = ([I, 0] [X_a, X_f]^{-1})^*)} \), where \( \ast \in \{*, T\} \). Let \( \tilde{M}, \tilde{K} \in \mathbb{C}^{n \times p} \) be such that
\[ \tilde{M} A_a + \tilde{K} = R_a. \] (14)
Then the matrices \( \Delta M = \tilde{M}U^* \) and \( \Delta K = \tilde{K} U^* \) satisfy the requirements of problem (P2).

**Proof.** The proof is a straightforward verification using (10). \( \square \)

Notice that to any \( \tilde{M} \) there is a unique \( \tilde{K} \) that solves (14), namely \( \tilde{K} = R_a - \tilde{M}A_a \). This yields a parametrization of all solutions. Another parametrization is obtained as follows. Equation (14) can be written in the form
\[
\begin{bmatrix} \tilde{M} & \tilde{K} \end{bmatrix} \begin{bmatrix} A_a \\ I_p \end{bmatrix} = R_a.
\]
Thus, all its solutions are given (see 2) via the Penrose inverse as
\[
\begin{bmatrix} \tilde{M} & \tilde{K} \end{bmatrix} = R_a \left[ A_a \right]^\dagger + [Z_1, Z_2] \left[ \begin{array}{cc} I_p & 0 \\ 0 & I_p \end{array} \right] - [A_a \left[ A_a \right]^\dagger, \quad Z_1, Z_2 \in \mathbb{C}^{n \times p} \text{ arbitrary}.
\]
More explicitly, with the notation \( H_a := (A_a^*A_a + I_p)^{-1} \),
\[
\tilde{M} = R_a H_a A_a^* + Z_1(I_p - A_a H_a A_a^*) - Z_2 H_a A_a^*, \quad \tilde{K} = R_a H_a - Z_1 A_a H_a + Z_2(I_p - H_a).
\] (15)

5 A general update result for pencils with symmetry

We now discuss the updating problem (P2) for pencils with \( \ast, \epsilon_1, \epsilon_2 \)-symmetry. The update method below only changes \( A_c \) and fixes \( X_c \) as well as \( X_f \), that is \( X_a = X_c \). For changing \( X_a \) see the Remark (2). The main requirement that makes our method work is the spectral assumption (a) in the theorem below.
Theorem 5.1. Let \((X_c, \Lambda_c)\) and \((X_f, \Lambda_f)\) be complementary deflating pairs of the pencil
\[ L(\lambda) = \lambda M + K \in \mathbb{L}_n(\epsilon_1, \epsilon_2), \] where \(\Lambda_c \in \mathbb{C}^{p \times p}\). Suppose that

(a) \(\sigma(\Lambda_c) \cap \sigma(\epsilon_1 \epsilon_2 \Lambda_f^*) = \emptyset\) and (b) \(G := X_c^* M X_c\) is nonsingular.

Let \(\Lambda_a, \hat{M}, \hat{K} \in \mathbb{C}^{p \times p}\) be such that

\[ \hat{M} \Lambda_a + \hat{K} = G(\Lambda_c - \Lambda_a). \] (16)

Set

\[ \Delta M := U \hat{M} U^*, \quad \Delta K := U \hat{K} U^*, \quad \text{where} \quad U := MX_c G^{-1}. \]

Then \((X_c, \Lambda_a)\) and \((X_f, \Lambda_f)\) are complementary deflating pairs of the pencil \(L_\Delta(\lambda) = (M + \Delta M) + (K + \Delta K)\). Furthermore, \(L_\Delta(\lambda) \in \mathbb{L}_n(\star, \epsilon_1, \epsilon_2)\) whenever \(\lambda M + \hat{K} \in \mathbb{L}_p(\star, \epsilon_1, \epsilon_2)\).

The latter holds if and only if \(\lambda M + (M + G) \Lambda_a \in \mathbb{L}_p(\star, \epsilon_1, \epsilon_2)\).

Proof. Obviously, \(X_c^* U = I\). By the by part (iv) of Proposition 3.1 and the spectral condition (a) we have \(X_c^* U = 0\). For \(X_a = X_c\), the matrix \(R_a\) from (11) satisfies \(R_a = MX_c(\Lambda_c - \Lambda_a) = U G(\Lambda_c - \Lambda_a)\). Hence (16) implies

\[ (U \hat{M}) \Lambda_a + (U \hat{K}) = R_a. \]

Thus, the first statement of the theorem follows from Theorem 5.1. The other statements are obvious. \(\blacksquare\)

Remark 5.2. (i) If \(X_c^* K X_c\) is nonsingular then \(G = X_c^* M X_c\) is also nonsingular, and the matrix \(U\) in Theorem 5.1 may be written in terms of \(K\) as \(U = K X_c(X_c^* K X_c)^{-1}\).

To see this, multiply \(M X_c \Lambda_a + K X_c = 0\) from the left with \(X_c^*\) and reorder terms so that \(G \Lambda_a = -X_c^* K X_c\). Thus \(G\) and \(\Lambda_c\) are nonsingular and \(U = MX_c G^{-1} = -K X_c \Lambda_c^{-1} G^{-1} = K X_c(X_c^* K X_c)^{-1}\).

(ii) For a given \(\hat{M}\) there is a unique \(\hat{K}\) that solves (16), namely \(\hat{K} = G(\Lambda_c - \Lambda_a) - \hat{M} \Lambda_a\). This yields a parameterization of all solution pairs \((\hat{M}, \hat{K})\). Analogously to the formula (14) an alternative parameterization of all solutions of (16) is given by

\[ \hat{M} = G(\Lambda_c - \Lambda_a) H_a \Lambda_a^* + Z_1(I_p - \Lambda_a H_a \Lambda_a^*) - Z_2 H_a \Lambda_a^*, \quad \hat{K} = G(\Lambda_c - \Lambda_a) H_a - Z_1 \Lambda_a H_a + Z_2(I_p - H_a). \] (17)

where \(H_a = (\Lambda_a^* \Lambda_a + I_p)^{-1}\) and \(Z_1, Z_2 \in \mathbb{C}^{p \times p}\) are arbitrary. Indeed note that the equation (16) can be written as

\[ \begin{bmatrix} \hat{M} & \hat{K} \end{bmatrix} \begin{bmatrix} \Lambda_a \\ I_p \end{bmatrix} = G(\Lambda_c - \Lambda_a) \]

which is a linear system of the form \(AX = B\), where \(X\) is a full rank matrix and \(A\) is unknown. All such \(A\) can be written as \(A = B X^\dagger + Z(I - X^\dagger X)\) for any arbitrary matrix \(Z\) of compatible dimension, where \(X^\dagger\) denotes the pseudoinverse of \(X\) if the pair \((X, B)\) satisfies \(B X^\dagger X = B\), see [2]. Thus the expression given by (17) can be obtained. Further, it may be noted that structured solution of the equation (16) can be obtained by imposing structural conditions on the parameters \(Z_1, Z_2\).

(iii) Let \(Z \in \mathbb{C}^{p \times p}\) be nonsingular. Let \((\hat{M}, \hat{K})\) be solutions of the modified equation

\[ \hat{M}(Z \Lambda_a Z^{-1}) + \hat{K} = G(\Lambda_c - Z \Lambda_a Z^{-1}).\]

Then by Theorem 5.1, \((X_c, Z \Lambda_a Z^{-1})\) is a deflating pair of the associated pencil \(L_\Delta(\lambda)\). Thus \((X_c Z, \Lambda_a)\) is also a deflating pair of \(L_\Delta(\lambda)\).
and \( \Lambda \) in the situation of Theorem 5.1 let \( \Delta \) given by

\[
\Delta = H(\Lambda) = G(\Lambda - (1 + t)\Lambda_a) \quad \text{for some } t \in \mathbb{R},
\]

Then \( X_c, \Lambda_a \) and \( (X_f, \Lambda_f) \) are complementary deflating pairs of \( L_\Delta(\lambda) \). Suppose that \( \Lambda_a \) satisfies \( G\Lambda_a = \epsilon_2(G\Lambda_a)^* \). Then \( L_\Delta(\lambda) \in \mathbb{R}(\epsilon, \epsilon_1, \epsilon_2) \).

6 Updates for especially structured matrix pencils

In this section we determine parametric updates which solve the problem (P1) for specific structured matrix pencils which are subsets of Hermitian, \( * \)-odd, \( * \)-even matrix pencils.

6.1 The Hermitian case with positive definite \( M \)

Suppose that \( L(\lambda) = \lambda M + K \in \mathbb{L}_{\text{Herm}} \) with positive definite \( M \). Then all eigenvalues of \( L(\lambda) \) are real and there exists a basis \( x_1^*, \ldots, x_p^*, x_{p+1}^*, \ldots, x_n^* \) of eigenvectors such that

\[
L(\lambda_i^*)x_i^* = L(\lambda_j^*)x_j^* = 0. \quad \text{By normalizing the eigenvectors (apply for Gram-Schmidt if some } \lambda_i \text{ coincide) we may assume that } (x_i^*)^*Mx_j^* = 0 \text{ for } i \neq j \text{ and } (x_i^*)^*Mx_i^* = 1. \quad \text{Thus, the } M \text{-Gramian of the matrix } X_c = [x_1^* \ldots x_p^*] \text{ satisfies } G = X_c^*MX_c = I_p. \quad \text{Let } \Lambda_c = \text{diag}(\lambda_i^*), \quad \Lambda_f = \text{diag}(\lambda_j^*). \quad \text{The spectral condition } (a) \text{ in Theorem 5 reads}
\]

\[
\{\lambda_1^*, \ldots, \lambda_p^*\} \cap \{\lambda_{p+1}^*, \ldots, \lambda_n^*\} = \emptyset.
\]

If this condition is fulfilled the update matrices in Theorem 5.1 are

\[
\Delta M = MX_c\hat{M}X_c^*M, \quad \Delta K = MX_c(\Lambda_c - \Lambda_a - \hat{M}\Lambda_a)X_c^*M.
\]

Both matrices are Hermitian if \( \hat{M} \) and \( \Lambda_a \) are diagonal and real. If \( M \) and \( K \) are real matrices then \( X_c \) can also be chosen to be real, and consequently the update matrices are real, too.

Remark 6.1. (Recovery of results in Carvalho et al. [10]) If \( \Lambda_a \) is a real diagonal matrix then choosing \( M = 0 \), we obtain \( \Delta M = 0 \) and \( \Delta K = MX_c(\Lambda_c - \Lambda_a)X_c^*M \) from (18). On the other hand, putting \( \hat{K} = 0 \) and assuming \( \Lambda_a \) to be nonsingular, we achieve \( \Delta K = 0 \) and \( \Delta M = MX_c(\Lambda_c\Lambda_a^{-1} - I_p)X_c^*M \).

Here we mention that when \( M \) and \( K \) are real symmetric positive definite matrices then the solution \( \Delta M = 0 \) and \( \Delta K = MX_c(\Lambda_c - \Lambda_a)X_c^*M \) realizes the solution obtained by Carvalho et al. in [11] for undamped models of the form \( L(\lambda) = \lambda^2M + K \). In addition, in their paper, the authors provide the solution where \( \Delta K = MX_c\Psi X_c^*M \) and \( \Psi \) has to be obtained by solving a matrix equation numerically. In contrast, the proposed solution here can be obtained directly by setting \( \Psi = (\Lambda_c^2 - \Lambda_a^2) \).

Remark 6.2. (Recovery of results in Mao et al. [27]) If \( \{\lambda_1^*, \ldots, \lambda_p^*\} \cap \{\lambda_{p+1}^*, \ldots, \lambda_n^*\} = \emptyset \) and \( \Lambda_a \) is a real diagonal matrix then the Hermitian update matrices in Theorem 5.1 are given by \( \Delta M = MX_c\hat{M}X_c^*M \) and \( \Delta K = MX_c\hat{K}X_c^*M \) with

\[
\hat{M} = H_a[(\Lambda_c - \Lambda_a)\Lambda_a + Z_1 - Z_2\Lambda_a], \quad \hat{K} = H_a[(\Lambda_c - \Lambda_a) - Z_1\Lambda_a + Z_2\Lambda_a^2],
\]

where \( H_a = (\Lambda_a^2 + I_p)^{-1} \) and \( Z_1, Z_2 \) are arbitrary real diagonal matrices of compatible sizes.
It is be noted that these solution sets identify the solutions given by Mao et al. in [27]. The perturbations obtained in their paper are given by
\[
\Delta M = M X_c (\Phi - \delta_{p+1} I_p) X_c^T M + (\delta_{p+1} - 1) M
\]
\[
\Delta K = M X_c (\Phi \Lambda_a - \delta_{p+1} \Lambda_c) X_c^T M + (\delta_{p+1} - 1) K
\]
where \( \Phi \) is a symmetric positive definite matrix which satisfies \( \Phi \Lambda_a = \Lambda_a \Phi \), and \( \delta_{p+1} > 0 \) is a real number. Setting \( Z_1 = H_a^{-1} (\Phi - I_p) \), \( Z_2 = \Lambda_c - \Lambda_a \), the perturbations derived in this paper become
\[
\Delta M = M X_c \hat{M} X_c^T M, \quad \Delta K = M X_c \hat{K} X_c^T M
\]
which realizes Mao et al.'s solution when \( \delta_{p+1} = 1 \), where \( \hat{M}, \hat{K} \) are given by equation (12).

Moreover, if the diagonal matrices \( Z_1 \) and \( Z_2 \) are chosen such that \( (\Lambda_c - \Lambda_a) X_a + Z_1 - Z_2 \Lambda_a \) is a diagonal matrix with non-negative diagonal entries then \( \Delta M \) is a positive semi-definite matrix, that is \( M + \Delta M > 0 \).

**Corollary 6.3.** Let the conditions of Remark 6.2 be satisfied. Besides, assume that \( K > 0 \) and \( \lambda_i^2 < 0, i = 1, \ldots, p \). Then if \( Z_1 = \text{diag} (z_i^{(1)}, \ldots, z_p^{(1)}) \) and \( Z_2 = \text{diag} (z_i^{(2)}, \ldots, z_p^{(2)}) \) are chosen such that
\[
z_i^{(1)} - z_i^{(2)} \lambda_i^2 \geq \max \{ (\lambda_i^a - \lambda_i^a) \lambda_i^a, (\lambda_i^c/\lambda_i^a - 1) \}, i = 1, \ldots, p
\]
then the perturbations \( \Delta M, \Delta K \) in Remark 6.2 are positive semi-definite matrices.

**Proof.** Note that \( \lambda_i^c < 0 \) since \( \lambda_i^c = -\frac{(x_i^c)^T K x_i^c}{(x_i^c)^T M x_i^c} \). Then the proof is straightforward and easy to check. \( \square \)

In the following we explain how the above results can be used to solve the standard model updating problem with no spillover effect for undamped models. Suppose that \( M > 0 \) and \( K^* = K \) are complex matrices of order \( n \). Then the eigenvalues of the matrix pencil \( L(\lambda) = \lambda^2 M + K \) occur in pair \( (\lambda, -\lambda) \) corresponding to an eigenvector \( x \in \mathbb{C}^n \). Besides, \( \lambda \) is either a real number or a purely imaginary number.

Let \( (\pm \lambda_i^c, x_i^c), i = 1, \ldots, p \) denote the eigenpairs of \( L(\lambda) \) that are to be changed to the aimed eigenvalues \( \pm \lambda_i^a, i = 1, \ldots, p \) of \( L(\lambda) = \lambda^2 (M + \Delta M) + (K + \Delta K) \), for some positive semi-definite Hermitian matrix \( \Delta M \) and \( \Delta K \in \mathbb{H}_n \). Setting \( \Lambda_c = \text{diag} (\lambda_1^c, \lambda_2^c, \ldots, \lambda_p^c)^2, \Lambda_a = \text{diag} (\lambda_1^a, \lambda_2^a, \ldots, \lambda_p^a)^2, \Lambda_f = \text{diag} (\lambda_{p+1}^f, \lambda_{p+2}^f, \ldots, \lambda_n^f)^2, \) and \( X_c = [x_1^c \ x_2^c \ \ldots \ x_p^c] \), the MUP with no spillover effect for \( L(\lambda) \) translates to the problem (P1).

We depict the same in the following example which is taken from [11].

**Example 6.4.** This example has been taken from [11]. Suppose \( L(\lambda) = \lambda^2 M + K \) with \( M = \text{diag} (1.294, 1.294, 1.294, 1.294, 1.294) > 0 \) and
\[
K = \begin{bmatrix}
1188.5000 & 196.6000 & 0 & 0 & -642.4000 \\
196.6000 & 626.3000 & 0 & -555.6000 & 0 \\
0 & 0 & 1188.5000 & -196.6000 & -546.1000 \\
0 & -555.6000 & -196.6000 & 626.3000 & 196.6000 \\
-642.4000 & 0 & -546.1000 & 196.6000 & 4019.1000
\end{bmatrix} > 0.
\]

Let \( \lambda_1^c = 57.4206i, \lambda_2^c = 4.8629i \) and \( \lambda_3^c = 57.4274i, \lambda_4^c = 4.8112i \). Suppose that we want to replace the set of eigenvalues \( \{\lambda_1^c, -\lambda_1^c, \lambda_2^c, -\lambda_2^c\} \) of \( L(\lambda) \) by the desired set of eigenvalues \( \{\lambda_1^a, -\lambda_1^a, \lambda_2^a, -\lambda_2^a\} \) respectively. Thus \( \Lambda_c = \text{diag} (-3297.13, -23.648), \Lambda_a = \text{diag} (-3297.6, -23.148) \) and
Then the update matrices in Theorem 5.1 are
\[
X_c = \begin{bmatrix}
-0.177539 & 0.125286 \\
-0.018246 & -0.611759 \\
-0.153557 & -0.085635 \\
0.056719 & -0.611759 \\
0.845073 & 0.038600
\end{bmatrix}.
\]

Then by Corollary 6.3 choosing \(Z_1 = \text{diag}(0, 0.021592), Z_2 = \text{diag}(0.47136, 0)\) we obtain
\[
\Delta M = 10^{-3} \begin{bmatrix}
0.5674 & -2.7703 & -0.3878 & -2.7695 & 0.1747 \\
-2.7703 & 13.5270 & 1.8935 & 13.5231 & -0.8535 \\
-0.3878 & 1.8935 & 0.2651 & 1.8930 & -0.1196 \\
-2.7695 & 13.5231 & 1.8930 & 13.5191 & -0.8532 \\
0.1747 & -0.8535 & -0.1196 & -0.8532 & 0.0543
\end{bmatrix} \geq 0 \text{ and }
\]
\[
\Delta K = 10^{-2} \begin{bmatrix}
2.4878 & 0.2557 & 2.1517 & -0.7948 & -11.8415 \\
0.2557 & 0.0263 & 0.2211 & -0.0817 & -1.2170 \\
2.1517 & 0.2211 & 1.8611 & -0.6874 & -10.2400 \\
-0.7948 & -0.0817 & -0.6874 & 0.2539 & 3.7831 \\
-11.8415 & -1.2170 & -10.2400 & 3.7831 & 56.3650
\end{bmatrix} \geq 0.
\]

On taking \(\Lambda_f = \text{diag}(-679.39, -942.69, -968.03)\) and \(X_f = \begin{bmatrix}
0.547227 & 0.642402 & -0.115946 \\
-0.262485 & 0.244128 & -0.519345 \\
0.522356 & -0.545451 & -0.414139 \\
0.313086 & -0.033487 & 0.544433 \\
0.183201 & 0.043366 & -0.147365
\end{bmatrix}\)
we obtain \(\|(M + \Delta M)X_f\Lambda_f + (K + \Delta K)X_f\|_F = 7.7524 \times 10^{-13}\) which shows that the unmeasured spectral data remain undisturbed.

Hence we conclude that eigenvalues of \(L_\Delta(\lambda) = \lambda^2 (M + \Delta M) + (K + \Delta K)\) are \(\{\lambda_1^a, -\lambda_1^a, \lambda_2^a, -\lambda_2^a\}\). Therefore eigenvalues of \(L(\lambda)\) are replaced by the desired eigenvalues with maintaining no spillover condition.

### 6.2 The \(s\)-odd matrix pencils with positive definite \(M\)

Suppose that \(L(\lambda) = \lambda M + K\) is a \(s\)-odd matrix pencil with positive definite \(M\). Then all eigenvalues of \(L(\lambda)\) are either zero or purely imaginary number and there exists a basis \(x_1^c, \ldots, x_p^c, x_{p+1}^c, \ldots, x_n^c\) of eigenvectors such that \(L(\lambda_i^c)x_i^c = L(\lambda_i^f)x_i^f = 0\). By normalizing the eigenvectors, we may assume that \((x_i^c)^*Mx_i^c = 0\) for \(i \neq j\) and \((x_i^c)^*Mx_i^c = 1\). Then the \(M\)-Gramian is given by \(G = X_c^*MX_c = I_p\) where \(X_c = [x_1^c \ldots x_p^c]\). Let \(\Lambda_c = \text{diag}(\lambda_i^c)\), \(\Lambda_f = \text{diag}(\lambda_i^f)\). Assuming the spectral condition (a) in Theorem 5.1, let
\[
\{\lambda_1^c, \ldots, \lambda_p^c\} \cap \{\lambda_{p+1}^f, \ldots, \lambda_n^f\} = \emptyset.
\]

Then the update matrices in Theorem 5.1 are
\[
\Delta M = MX_c \dot{M} X_c^* M, \quad \Delta K = MX_c (\Lambda_c - \Lambda_a - \dot{M}\Lambda_a) X_c^* M. \quad (20)
\]

Thus \(\Delta M = (\Delta M)^*, \Delta K = -(\Delta K)^*\) if \(\dot{M}\) is a real diagonal matrix and \(\Lambda_a\) is an imaginary diagonal matrix.

**Remark 6.5.** If \(\dot{M}\) in (20) is chosen to be a diagonal matrix with non-negative entries then \(\dot{M} + \Delta M > 0\).

**Remark 6.6.** If \(\{\lambda_1^c, \ldots, \lambda_p^c\} \cap \{\lambda_{p+1}^f, \ldots, \lambda_n^f\} = \emptyset\) and \(\Lambda_a\) is a diagonal matrix with purely imaginary complex numbers, the structured update matrices in Theorem 5.1 are given by
\[
\Delta M = MX_c \dot{M} X_c^* M \quad \text{and} \quad \Delta K = MX_c \dot{K} X_c^* M
\]
with
\[
\dot{M} = H_a [(\Lambda_a - \Lambda_c)\Lambda_a + Z_1 + Z_2\Lambda_a], \quad \dot{K} = H_a [(\Lambda_c - \Lambda_a) - Z_1\Lambda_a - Z_2\Lambda_a^2]. \quad (21)
\]
where \( H_a = (I_p - \Lambda_a^2)^{-1} \), \( Z_1 \) is an arbitrary real diagonal matrix and \( Z_2 \) is an arbitrary diagonal matrix with purely imaginary diagonal entries.

Moreover, if the diagonal matrices \( Z_1 \) and \( Z_2 \) are chosen such that \((\Lambda_a - \Lambda_c)\Lambda_a + Z_1 + Z_2\Lambda_a\) is a diagonal matrix with non-negative diagonal entries then \( \Delta M \) is a positive semi-definite matrix, that is \( M + \Delta M > 0 \).

Now let us consider \( T \)-odd matrix pencils \( L(\lambda) = \lambda M + K \in \mathbb{R}^{n \times n}[\lambda] \). Then obviously the complex eigenvalues of \( L(\lambda) \) are purely imaginary which exist in conjugate pairs, whereas zero can be the only real eigenvalue of \( L(\lambda) \). Moreover if \( x \) is an eigenvector corresponding to the complex eigenvalue \( \lambda \) then \( \overline{x} \) is an eigenvector corresponding to the eigenvalue \( \overline{\lambda} = -\lambda \) of \( L(\lambda) \). As usual, let the nonzero eigenvalues \( \lambda_1^c, \lambda_2^c, \ldots, \lambda_p^c \) with no spillover effect in the (structured) perturbed pencil \( L_\Delta(\lambda) \). If \( x_i^c \) denotes the normalized (complex) eigenvector corresponding to the eigenvalue \( \lambda_i^c, 1 \leq i \leq p \) then we may assume that \( X_c^a M X_c = 2I_{2p} \) where \( X_c = [x_1^c x_2^c \ldots x_p^c x_p^c] \). This implies \( X_c^a M X_c = 2I_{2p} \) where \( X_c = [x_1^c x_1^c \ldots x_p^c x_p^c] \). Indeed, note that \( X_c = X_c Z \) where

\[
Z = \text{diag} \left( \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}, \ldots, \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \right) \in \mathbb{C}^{2p \times 2p}.
\]

Let \( \Lambda_c = \text{diag}(\Lambda_1^c, \ldots, \Lambda_p^c) \) and \( \Lambda_a = \text{diag}(\Lambda_1^a, \ldots, \Lambda_p^a) \) where \( \Lambda_j^c = \begin{bmatrix} 0 & \text{im}(\lambda_j^c) \\ -\text{im}(\lambda_j^c) & 0 \end{bmatrix} \) and \( \Lambda_j^a = \begin{bmatrix} 0 & \text{im}(\lambda_j^a) \\ -\text{im}(\lambda_j^a) & 0 \end{bmatrix} \). If the condition (a) of Theorem 5.4 is met, then the update matrices are

\[
\Delta M = \tilde{X}_c M \tilde{X}_c^T M, \quad \Delta K = \tilde{X}_c K \tilde{X}_c^T M
\]

where \( \tilde{M} \) and \( \tilde{K} \) are solutions of equation (10) in which \( X_c \) is replaced by \( \tilde{X}_c \).

Moreover setting \( M = \text{diag}(\alpha_1 I_2, \ldots, \alpha_p I_2) \) \( \alpha_1, \ldots, \alpha_p \in \mathbb{R} \) and \( K = \Lambda_c - \Lambda_a - \tilde{M} \Lambda_a \) we obtain \( \Delta M = \Delta M^T \) and \( \Delta K = -\Delta K^T \).

**Remark 6.7.** If the spectral condition (a) in Theorem 5.4 is met, then the structured update matrices are given by \( \Delta M = \tilde{X}_c M \tilde{X}_c^T M \) and \( \Delta K = \tilde{X}_c K \tilde{X}_c^T M \) with

\[
\tilde{M} = H_a \left[ (\Lambda_a - \Lambda_c) \Lambda_a + Z_1 + Z_2 \Lambda_a \right], \quad \tilde{K} = H_a \left[ (\Lambda_c - \Lambda_a) - Z_1 \Lambda_a - Z_2 \Lambda_a^2 \right]
\]

where \( H_a = (I_p - \Lambda_a^2)^{-1} \) and \( Z_k = \text{diag}(Z_1^{(k)}, \ldots, Z_p^{(k)}), k = 1, 2 \) with \( Z_1^{(1)} = \alpha_1 I_2 \) and \( Z_2^{(1)} = \alpha_j I_2 \). Thus \( L_\Delta(\lambda) = \lambda (M + \Delta M) + (K + \Delta K) \) is a real \( T \)-odd pencil.

Moreover, if the matrices \( Z_1 \) and \( Z_2 \) are chosen such that \((\Lambda_a - \Lambda_c) \Lambda_a + Z_1 + Z_2 \Lambda_a \) is a diagonal matrix with non-negative diagonal entries then \( \Delta M \) is a positive semi-definite matrix, that is \( M + \Delta M > 0 \).

Now we consider an example to obtain solution of (P1) for undamped models \( L(\lambda) = \lambda^2 M + K \) with \( M > 0 \) and \( K^* = -K \), by utilizing Remark 6.6. The values of \( \lambda^2 \) to satisfy \( \det(\lambda^2 M + K) = 0 \) are either zero or purely imaginary numbers (not necessary to have self conjugate pair), that is, either \( \lambda = \pm \sqrt{a/2}(1 + i) \) or \( \lambda = \pm \sqrt{a/2}(1 - i) \) for some \( a \geq 0 \). So, here we define the set \( \mathcal{E} = \left\{ \pm \sqrt{a/2}(1 + i), \pm \sqrt{a/2}(1 - i) : a \geq 0 \right\} \). Then the eigenvalues of the matrix pencil \( L(\lambda) = \lambda^2 M + K \) occur in pair \((\lambda, -\lambda)\) corresponding to an eigenvector \( x \in \mathbb{C}^n \) for some \( \lambda \in \mathcal{E} \).

Let \((\pm \lambda_i^c, x_i^c), i = 1, \ldots, p\) denote the eigenpairs of \( L(\lambda) = \lambda^2 M + K \) and \( \pm \lambda_i^c \) are to be changed to the aimed eigenvalues \( \pm \lambda_i^a, i = 1, \ldots, p \) of \( L_\Delta(\lambda) = \lambda^2 (M + \Delta M) + (K + \Delta K) \), for some positive semi-definite matrix \( \Delta M \) and \( \Delta K = - (\Delta K)^* \) without spillover effect. Setting \( \Lambda_c = \text{diag}(\lambda_1^c, \lambda_2^c, \ldots, \lambda_p^c) \), \( \Lambda_a = \text{diag}(\lambda_1^a, \lambda_2^a, \ldots, \lambda_p^a) \), \( \Lambda_f = \text{diag}(\lambda_{p+1}^f, \lambda_{p+2}^f, \ldots, \lambda_n^f) \)
and $X_c = [x_1^c, x_2^c, \ldots, x_n^c]$, the MUP with no spillover effect for $L(\lambda)$ translates to the problem (P1).

We consider the following example.

**Example 6.8.** Suppose $L(\lambda) = \lambda^2 M + K$ with

\[
M = \begin{bmatrix}
7.37863 + 0.00000i & -1.98637 - 4.01069i & 4.09960 - 3.39198i & -0.13418 + 2.89422i \\
-1.98637 + 4.01069i & 6.55893 + 0.00000i & 1.90812 + 3.90598i & -2.03549 + 1.81182i \\
4.09960 + 3.39198i & 1.90812 - 3.90598i & 6.65654 + 0.00000i & 1.02186 + 1.42954i \\
-0.13418 - 2.89422i & -2.03549 - 1.81182i & 1.02186 - 1.42954i & 6.46256 + 0.00000i \\
\end{bmatrix} > 0,
\]

\[
K = \begin{bmatrix}
0.00000 + 3.90601i & 2.0140 - 0.30415i & 1.34863 + 1.79442i & 0.05369 - 1.38714i \\
-2.0140 - 0.30415i & 0.00000 + 2.49371i & 0.30279 + 1.11588i & -0.05369 + 1.38714i \\
-1.34863 + 1.79442i & -0.30279 + 1.11588i & 0.00000 - 0.49211i & -0.05369 - 1.38714i \\
-0.05369 - 1.38714i & 0.30279 - 1.11588i & 0.00000 + 0.49211i & 0.00000 - 1.85364i \\
\end{bmatrix}.
\]

Let $\lambda_1^c = 1.30078(1 + i)$, $\lambda_2^c = 0.80933(1 - i)$ and $\lambda_3^c = 0.82134(1 - i)$, $\lambda_4^c = 0.56214(1 + i)$. Thus we want to replace the eigenvalues $\lambda_1^c$, $-\lambda_1^c$, $\lambda_2^c$, $-\lambda_2^c$ of $L(\lambda)$ by the desired eigenvalues $\lambda_1^c$, $-\lambda_1^c$, $\lambda_2^c$, $-\lambda_2^c$ respectively. So we form $A_c = \text{diag}(3.38411i, -1.31000i)$, $A_n = \text{diag}(-1.34921i, 0.63200i)$ and

\[
X_c = \begin{bmatrix}
0.776569 - 0.000000i & 0.617954 - 0.000000i \\
0.747129 - 0.098152i & 0.153552 + 0.005888i \\
-0.714782 - 0.126987i & -0.229136 - 0.266691i \\
0.444742 + 0.301815i & 0.039872 + 0.500083i \\
\end{bmatrix}.
\]

Therefore by setting $Z_1 = \text{diag}(8.9752, 2.5715)$ and $Z_2 = \text{diag}(-0.00717i, -0.60271i)$ we obtain

\[
\Delta M = \begin{bmatrix}
2.91691 - 0.000000i & -1.34898 + 0.69543i & 0.58908 - 1.38017i & -2.65147 - 0.99875i \\
1.34898 - 0.69543i & 1.59117 + 0.000000i & -0.77640 + 0.88115i & 1.14417 + 1.21855i \\
0.58908 + 1.38017i & -0.77640 - 0.88115i & 0.99350 + 0.000000i & -0.03741 - 1.55808i \\
-2.65147 - 0.99875i & 1.14417 - 1.21855i & -0.03741 + 1.55808i & 2.80818 + 0.000000i \\
\end{bmatrix} \geq 0,
\]

\[
\Delta K = \begin{bmatrix}
0.00000 - 5.25520i & -0.87564 + 0.59438i & -1.14421 - 1.67866i & -1.92869 + 4.08990i \\
0.87564 + 0.59438i & 0.00000 + 6.19427i & -2.65507 - 1.39903i & -1.45840 + 0.46343i \\
1.14421 - 1.67866i & 2.65507 + 1.39903i & 0.00000 - 0.98530i & -0.92466 - 1.08993i \\
1.92869 - 4.08990i & 1.45840 - 0.46343i & 0.09246 + 1.08993i & 0.00000 + 3.49178i \\
\end{bmatrix}.
\]

Taking $\Lambda_f = \text{diag}(-0.28296i, 0.42255i)$ and $X_f = \begin{bmatrix}
0.196502 + 0.024767i & -0.048688 + 0.190081i \\
-0.036828 + 0.054982i & 0.095288 - 0.254723i \\
-0.150920 + 0.086267i & 0.466261 + 0.000000i \\
0.231864 + 0.000000i & 0.099775 + 0.083410i \\
\end{bmatrix}$ we obtain $\| (M + \Delta M) X_c \Lambda_f + (K + \Delta K) X_f \|_F = 1.2209 \times 10^{-14}$ which shows that the no spillover for the unmeasured spectral data is guaranteed.

Thus we conclude that eigenvalues of $L_\Delta(\lambda) = \lambda^2 (M + \Delta M) + (K + \Delta K)$ are $\lambda_1^c$, $-\lambda_1^c$, $\lambda_2^c$, $-\lambda_2^c$. Hence eigenvalues of $L(\lambda)$ are replaced by the desired eigenvalues with maintaining no spillover effect.

### 6.3 The $*$-even matrix pencils with positive definite $K$

Let $L(\lambda) = \lambda M + K$ be a $*$-even matrix pencil with $K > 0$. Then all eigenvalues of $L(\lambda)$ are purely imaginary and there exists a basis $x_1^c, \ldots, x_p^c, x_{p+1}^f, \ldots, x_n^f$ of eigenvectors such that $L(\lambda_1^c)x_i^c = L(\lambda_i^c)x_i^c = 0$. By normalizing the eigenvectors, we may assume that $(x_i^c)^* K x_i^c = 0$ for $i \neq j$ and $(x_i^c)^* K x_j^f = 1$. Thus, the $M$-Gramian of the matrix $X_c = [x_1^c \ldots x_p^c]$ satisfies

$G = X_c^* M X_c = -\Lambda_c^{-1}$ as $X_c^* K X_c = I_p$, where $\Lambda_c = \text{diag}(\lambda_c^c)$, $\Lambda_f = \text{diag}(\lambda_f^c)$, and $\lambda_c^c \neq 0$. Assuming the spectral condition (a) as given in Theorem 6.1 we have

\[
\{\lambda_1^c, \ldots, \lambda_p^c\} \cap \{\lambda_{p+1}^f, \ldots, \lambda_n^f\} = \emptyset
\]
fulfilling which the update matrices in Theorem 6.9 are

\[ \Delta M = K X_c \hat{M} X_c^* K, \quad \Delta K = K X_c (\Lambda_c^{-1}(\Lambda_a - \Lambda_c) - \hat{M} \Lambda_a) X_c^* K. \]  

(23)

Thus \( \Delta M = - (\Delta M)^* \), \( \Delta K = (\Delta K)^* \) when \( \hat{M} \) and \( \Lambda_a \) are diagonal matrices with purely imaginary diagonal entries.

Remark 6.9. If \( \hat{M} = 0 \) in (23), then we obtain \( \Delta M = 0 \) and \( \Delta K = K X_c \Lambda_c^{-1}(\Lambda_a - \Lambda_c) X_c^* K \) is a Hermitian matrix. On the other hand, assuming \( \Lambda_a \) as nonsingular and setting \( \hat{M} = \Lambda_c^{-1} - \Lambda_a^{-1} \), we obtain \( \Delta K = 0 \) and \( \Delta M = K X_c (\Lambda_c^{-1} - \Lambda_a^{-1}) X_c^* K \) is skew-Hermitian.

Remark 6.10. If \( \{ \lambda_1, \ldots, \lambda_p \} \cap \{ \lambda_{p+1}, \ldots, \lambda_n \} = \emptyset \) and \( \Lambda_a \) is a diagonal matrix with purely imaginary diagonal entries then the structured update matrices in Theorem 6.1 are given by \( \Delta M = K X_c M X_c^* K \) and \( \Delta K = K X_c K X_c^* K \) with

\[ \hat{M} = H_a \left[ \Lambda_c^{-1}(\lambda_a - \Lambda) \Lambda_a + Z_1 + Z_2 \Lambda_a \right], \quad \hat{K} = H_a \left[ \Lambda_c^{-1}(\lambda_a - \Lambda) - Z_1 \Lambda_a - Z_2 \Lambda_a^2 \right] \]  

(24)

where \( H_a = (I_p - \Lambda_a^2)^{-1} \) and \( Z_1 \) is an arbitrary imaginary diagonal matrix, while \( Z_2 \) is an arbitrary real diagonal matrix.

Moreover, if the diagonal matrices \( Z_1 \) and \( Z_2 \) are chosen such that \( \Lambda_c^{-1}(\lambda_a - \Lambda) - Z_1 \Lambda_a - Z_2 \Lambda_a^2 \) is a diagonal matrix with non-negative diagonal entries then \( \Delta K \) is a positive semi-definite matrix, that is \( K + \Delta K > 0 \).

Now we consider \( T \)-even pencils \( L(\lambda) = \lambda M + K \in \mathbb{R}^{n \times n}[\lambda] \) where \( K > 0 \). The structured updates for \( L(\lambda) \) can be obtained following a similar procedure as described for the case of \( T \)-odd matrix pencils. Indeed, observe that nonzero complex eigenvalues of \( L(\lambda) \) are purely imaginary. Let \( (\lambda_i, x_i), (\lambda_i^*, x_i^*) , 1 \leq i \leq p \) be eigenpairs of \( L(\lambda) \) where the eigenvectors are normalized and \( \lambda_i^* \neq 0 \). Then it may be assumed that \( \hat{X}_c^T K \hat{X}_c = I_{2p} \) where \( \hat{X}_c = [\text{re}(x_1^*) \text{im}(x_1^*) \ldots \text{re}(x_p^*) \text{im}(x_p^*)] \).

Then the \( M \)-Gramian of the matrix \( \hat{X}_c \) satisfies \( G = \hat{X}_c^T M \hat{X}_c = - \Lambda_c^{-1} \) where \( \Lambda_c = \text{diag}(\Lambda_1^c, \ldots, \Lambda_p^c) \) and \( \Lambda_a = \text{diag}(\Lambda_1^a, \ldots, \Lambda_p^a) \) with \( \Lambda_j^c = \left[ \begin{array}{cc} 0 & \text{im}(\lambda_j^c) \\ -\text{im}(\lambda_j^c) & 0 \end{array} \right] \) and \( \Lambda_j^a = \left[ \begin{array}{cc} 0 & \text{im}(\lambda_j^a) \\ -\text{im}(\lambda_j^a) & 0 \end{array} \right] \).

If the condition (a) of Theorem 6.1 is met, then the update matrices are

\[ \Delta M = K \hat{X}_c \hat{M} \hat{X}_c^T K, \quad \Delta K = K \hat{X}_c \hat{K} \hat{X}_c^T K \]

where \( \hat{M} \) and \( \hat{K} \) are solutions of equation (16) in which \( X_c \) is replaced by \( \hat{X}_c \).

It may also be noted that choosing \( \hat{M} = \text{diag}(\hat{M}_{11}, \ldots, \hat{M}_{pp}) \) and \( \hat{K} = \Lambda_c^{-1}(\lambda_a - \Lambda_c) - \hat{M} \Lambda_a \) where \( \hat{M}_{jj} = \left[ \begin{array}{cc} 0 & \alpha_j \\ -\alpha_j & 0 \end{array} \right] \) for some \( \alpha_1, \ldots, \alpha_p \in \mathbb{R} \), we obtain \( \Delta M = - \Delta M^T \) and \( \Delta K = \Delta K^T \).

Remark 6.11. If the spectral condition (a) in Theorem 6.1 is met, then the structured updates matrices are given by \( \Delta M = K \hat{X}_c \hat{M} \hat{X}_c^T K \) and \( \Delta K = K \hat{X}_c \hat{K} \hat{X}_c^T K \) with

\[ \hat{M} = H_a \left[ \Lambda_c^{-1}(\lambda_a - \Lambda) \Lambda_a + Z_1 + Z_2 \Lambda_a \right], \quad \hat{K} = H_a \left[ \Lambda_c^{-1}(\lambda_a - \Lambda) - Z_1 \Lambda_a - Z_2 \Lambda_a^2 \right] \]  

(25)

where \( H_a = (I_{2p} - \Lambda_a^2)^{-1} \) and \( Z_k = \text{diag}(Z_1^{(k)}, \ldots, Z_p^{(k)}) \), \( k = 1, 2 \) having \( Z_1^{(1)} = \left[ \begin{array}{cc} 0 & \alpha_j \\ -\alpha_j & 0 \end{array} \right] \) and \( Z_2^{(2)} = \beta_j I_2, \alpha_j, \beta_j \in \mathbb{R} \). Obviously, \( L_\Delta(\lambda) = \lambda (M + \Delta M) + (K + \Delta K) \) is a \( T \)-even real matrix pencil with \( K + \Delta K > 0 \), if \( Z_1, Z_2 \) are chosen such that \( \Lambda_c^{-1}(\lambda_a - \Lambda_c) - Z_1 \Lambda_a - Z_2 \Lambda_a^2 \) is a diagonal matrix with non-negative diagonal entries.

Now we consider an example to obtain solution of (P1) for undamped models \( L(\lambda) = \lambda^2 M + K \) with \( K > 0 \) by utilizing Remark 6.10. The values of \( \lambda^2 \) to satisfy \( \text{det}(\lambda^2 M + K) = 0 \) are purely imaginary numbers (not necessary to have self conjugate pair), that is, \( \lambda \in \mathbb{E} \setminus \{0\} \).
Then the eigenvalues of the matrix pencil $L(\lambda) = \lambda^2 M + K$ occur in pair $(\lambda, -\lambda)$ corresponding to an eigenvector $x \in \mathbb{C}^n$ for some $\lambda \in \mathbb{C} \setminus \{0\}$.

Let $(\pm \xi_i^f, x_i^f)$, $i = 1, \ldots, p$ denote the eigenpairs of $L(\lambda) = \lambda^2 M + K$ and $\pm \xi_i^c$ are to be changed to the aimed eigenvalues $\pm \lambda_i^c$, $i = 1, \ldots, p$ of $L_\Delta(\lambda) = \lambda^2 (M + \Delta M) + (K + \Delta K)$, for some positive semi-definite matrix $\Delta K$ and skew-Hermitian $\Delta M$ with no spillover effect. Setting $\Lambda_c = \text{diag}(\lambda_1^c, \lambda_2^c, \ldots, \lambda_p^c)$, $\Lambda_a = \text{diag}(\lambda_1^a, \lambda_2^a, \ldots, \lambda_p^a)$, $\Lambda_f = \text{diag}(\lambda_f^{p+1}, \lambda_f^{p+2}, \ldots, \lambda_f^p)$ and $X_c = [x_1^f x_2^f \ldots x_p^f]$, the MUP with no spillover effect for $L(\lambda)$ translates to the problem (P1). We consider the following example.

**Example 6.12.** Suppose $L(\lambda) = \lambda^2 M + K$ with

$$
M = \begin{bmatrix}
0.00000 + 0.20972i & -0.10697 + 0.96717i & 0.04080 - 0.91135i & -3.59068 + 1.77061i \\
0.10697 + 0.96717i & 0.00000 - 0.94422i & -0.98779 + 1.35265i & 3.55621 - 0.03449i \\
-0.04080 - 0.91135i & 0.98779 + 1.35265i & 0.00000 - 0.79806i & -0.50440 - 0.71953i \\
3.59068 - 1.77061i & -3.55621 - 0.03449i & 0.50440 - 0.71953i & 0.00000 - 1.82468i \\
\end{bmatrix},
$$

$$
K = \begin{bmatrix}
5.25927 + 0.00000i & -1.36185 - 0.39225i & -1.02993 + 3.85132i & 3.10502 - 0.94912i \\
-1.36185 + 0.39225i & 5.18833 + 0.00000i & 0.25646 + 2.08573i & -0.35504 + 4.89141i \\
-1.02993 - 3.85132i & 0.25646 - 2.08573i & 12.57576 + 0.00000i & 9.24337 + 0.00000i \\
3.10502 + 0.94912i & 2.82543 + 1.42028i & -0.35504 + 4.89141i & 9.24337 + 0.00000i \\
\end{bmatrix} > 0.
$$

Let $\lambda_1^c = 1.8663(1 + i)$, $\lambda_2^c = 0.96032(1 + i)$ and $\lambda_1^a = 1.9538(1 + i)$, $\lambda_2^a = 1.1696(1 + i)$. Thus we want to replace the eigenvalues $\lambda_1^c$, $-\lambda_1^c$, $\lambda_2^c$, $-\lambda_2^c$ of $L(\lambda)$ by the desired eigenvalues $\lambda_1^c$, $-\lambda_1^c$, $\lambda_2^c$, $-\lambda_2^c$ respectively. So we form $\Lambda_c = \text{diag}(6.96617i, 1.84442i)$, $\Lambda_a = \text{diag}(7.63484i, 2.73573i)$ and

$$
X_c = \begin{bmatrix}
0.269248 - 0.049496i & 0.365254 + 0.00000i \\
0.360869 + 0.00000i & 0.021572 + 0.085644i \\
0.105515 - 0.042953i & 0.074614 + 0.141519i \\
-0.030283 + 0.036643i & 0.024397 - 0.220546i \\
\end{bmatrix},
$$

Therefore by setting $Z_1 = \text{diag}(0.10025i, 0.47934i)$ and $Z_2 = \text{diag}(0.26054, 0.84128)$ we obtain

$$
\Delta M = \begin{bmatrix}
0.00000 + 0.52241i & -0.06183 - 0.22791i & 0.00122 - 0.11173i & -0.41289 + 0.39407i \\
0.06183 - 0.22791i & 0.00000 + 0.20921i & -0.13366 + 0.07568i & 0.28312 - 0.05364i \\
-0.00122 + 0.11173i & 0.13366 - 0.07568i & 0.00000 + 0.17138i & 0.18351 - 0.13284i \\
0.41289 + 0.39407i & -0.28312 - 0.05364i & -0.13284 + 0.09000 + 0.70160i \\
\end{bmatrix},
$$

$$
\Delta K = \begin{bmatrix}
3.00449 + 0.00000i & -1.05675 + 0.30905i & -0.52100 + 0.27793i & 2.41288 + 2.20342i \\
-1.05675 - 0.30905i & 1.57244 + 0.00000i & 0.52079 + 1.32381i & 0.10989 - 1.66602i \\
-0.52100 - 0.27793i & 0.52079 - 1.32381i & 1.79857 + 0.00000i & -0.75754 - 1.70191i \\
2.41288 - 2.20342i & 0.16989 + 1.66602i & -0.75754 - 1.70191i & 4.44366 + 0.00000i \\
\end{bmatrix} \geq 0.
$$

Taking $\Lambda_f = \text{diag}(-5.38777i, -0.38831i)$ and $X_f = \begin{bmatrix}
0 & I_n & 0 \\
-I_n & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}$, we obtain $\| (M + \Delta M)X_f \Lambda_f + (K + \Delta K)X_f \|_F = 1.8766 \times 10^{-14}$ which shows that the no spillover for the unmeasured spectral data is guaranteed.

Thus we conclude that eigenvalues of $L_\Delta(\lambda) = \lambda^2 (M + \Delta M) + (K + \Delta K)$ are $\lambda_1^c$, $-\lambda_1^c$, $\lambda_2^c$, $-\lambda_2^c$.

Hence eigenvalues of $L(\lambda)$ are replaced by the desired eigenvalues with maintaining no spillover effect.

### 7 Updates for **-skew-Hamiltonian/Hamiltonian pencils**

Recall that a matrix pencil $L(\lambda) = \lambda M + K \in \mathbb{C}^{2n \times 2n}[\lambda]$ is said to be **-skew-Hamiltonian/Hamiltonian (SHH) pencil** if $M$ is a **-skew-Hamiltonian matrix and $K$ is a **-Hamiltonian matrix, that is $JM = -(JM)^*$ and $JK = (JK)^*$ where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ and $\in \{*, T\}$. It is
also clear that if $L(\lambda)$ is $\ast$-skew-Hamiltonian/Hamiltonian then $JL(\lambda)$ is $\ast$-even. It is also well-known that if $\lambda$ is a simple eigenvalue of $L(\lambda)$ with $\text{re}(\lambda) \neq 0$ then so is $-\lambda$, however a purely imaginary eigenvalue need not occur in pairs [6]. Besides, $\lambda$ and $-\lambda$ have the same partial multiplicities [28]. Our next proposition is about the solution of the problem (P2) for $\ast$-SHH pencil $L(\lambda)$.

**Proposition 7.1.** Let $(X_c, \Lambda_c)$ and $(X_f, \Lambda_f)$ be complementary deflating pairs of the pencil $L(\lambda) = \lambda M + K$, where $\Lambda_c \in \mathbb{C}^{p \times p}$. Suppose that

(a) $\sigma(\Lambda_c) \cap \sigma(-\Lambda_c^\ast) = \emptyset$ and (b) $G := X_c^* JMX_c$ is nonsingular.

Let $\Lambda_a, M, K \in \mathbb{C}^{p \times p}$ be such that

$$\hat{M} \Lambda_a + \hat{K} = G(\Lambda_c - \Lambda_a).$$

(26)

Set

$$\Delta M := J^* U \hat{M} U^*,$$

$$\Delta K := J^* U \hat{K} U^*,$$

where $U := JMX_cG^{-1}$.

Then $(X_c, \Lambda_a)$ and $(X_f, \Lambda_f)$ are complementary deflating pairs of the pencil $L_{\Delta}(\lambda) = (M + \Delta M) \lambda + (K + \Delta K)$. Furthermore, $L_{\Delta}(\lambda)$ is a SHH pencil whenever $\hat{M} M + \hat{K} K \in \mathbb{L}_p(\ast, -1, 1)$. The latter holds if and only if $\lambda \hat{M} M + (\lambda + G) \Lambda_a \in \mathbb{L}_p(\ast, -1, 1)$.

**Proof.** The proof follows easily from Theorem 5.1. \hfill \blacksquare

The next result is about the solution of the problem (P1) for $\ast$-SHH matrix pencil.

**Corollary 7.2.** Suppose $L(\lambda) = \lambda M + K$ is a $\ast$-SHH matrix pencil. Let $(\Lambda_c, X_c)$ be a deflating pair of $L(\lambda)$ where $\Lambda_c = \text{diag}(\lambda_1, -\lambda_1^\ast, \ldots, \lambda_m, -\lambda_m^\ast, \ldots, \lambda_p, -\lambda_p^\ast)$, $\text{re}(\lambda_j^\ast) \neq 0$, $j = 1, \ldots, m$ and $\lambda_j^\ast, k = m+1, \ldots, p$ are purely imaginary numbers. Let $\Lambda_a = \text{diag}(\lambda_1^a, -\lambda_1^a, \ldots, \lambda_m^a, -\lambda_m^a, \lambda_{m+1}^a, \ldots, \lambda_p^a)$ where $\text{re}(\lambda_j^a) \neq 0$ and $\lambda_j^a$ are purely imaginary, $j = 1, \ldots, m$, $k = m+1, \ldots, p$.

Then by Proposition 7.1, if it satisfies the conditions (a), (b) and $\lambda_j^a$'s are simple eigenvalues then the update matrices are $\Delta M = J^* U \hat{M} U^*$, $\Delta K = J^* U (\text{GA}_c - (G + \hat{M}) \Lambda_a) U^*$ for which $(X_c, \Lambda_a), (X_f, \Lambda_f)$ are complementary deflating pairs of $L_{\Delta}(\lambda)$, where $M$ is an arbitrary matrix of compatible size. This solves problem (P1) by unstructured updates.

Further, on choosing $\hat{M} = \text{diag}(\hat{M}_1, \ldots, \hat{M}_m, \hat{M}_{m+1}, \ldots, \hat{M}_p)$ where $\hat{M}_j = \begin{bmatrix} 0 & \alpha_j \\ -\frac{\alpha_j}{\beta_j} & 0 \end{bmatrix}$, $\text{re}(\alpha_j) \neq 0$, and $\hat{M}_k$ are purely imaginary numbers, $L_{\Delta}(\lambda)$ becomes $\ast$-SHH pencil which solves the problem (P1) using structured updates.

**Proof.** Since $\lambda_j^a$s are simple eigenvalues of $L(\lambda)$, the matrix $G$ has the form $G = \text{diag}(G_1, \ldots, G_m, G_{m+1}, \ldots, G_p)$ where $G_j = \begin{bmatrix} 0 & g_j \\ -g_j & 0 \end{bmatrix}$ with $g_j \in \mathbb{C}$, $1 \leq j \leq m$ and $g_k, m+1 \leq k \leq p$ are imaginary numbers.

The rest follows from proposition 7.1. \hfill \blacksquare

Another parametric structured updates are given as follows.

**Remark 7.3.** If the assumptions of Corollary 7.2 holds then $\Delta M = J^T U \hat{M} U^*$ and $\Delta K = J^T U \hat{K} U^*$ solves the problem (P1), where

$$\hat{M} = G(\Lambda_c - \Lambda_a) H_a \Lambda_a^\ast + Z_1(I_p - \Lambda_a H_a \Lambda_a^\ast) - Z_2 H_a \Lambda_a^\ast,$$

$$\hat{K} = G(\Lambda_c - \Lambda_a) H_a - Z_1 \Lambda_a H_a + Z_2 (I_p - H_a),$$

with $H_a = (\Lambda_a^\ast \Lambda_a + I_p)^{-1}$, $G = X_c^* JMX_c$, $U = JMX_cG^{-1}$ and $Z_i = \text{diag}(Z_{i1}, \ldots, Z_{im}, Z_{m+1}, \ldots, Z_{ip})$, $i = 1, 2$,

$$Z_{j1}^{(1)} = \begin{bmatrix} 0 & \alpha_j \\ -\alpha_j & 0 \end{bmatrix}, Z_{j2}^{(2)} = \begin{bmatrix} 0 & \beta_j \\ -\beta_j & 0 \end{bmatrix}, \alpha_j, \beta_j \in \mathbb{C}, 1 \leq j \leq m$$

and $z_{k1}^{(1)}, m+1 \leq k \leq p$ are imaginary numbers and $z_{k2}^{(2)}$ are reals. Besides $\Delta M, \Delta K$ are $\ast$-skew-Hamiltonian and $\ast$-Hamiltonian matrix respectively.
Now we consider $T$-SHH matrix pencils $L(\lambda) = \lambda M + K \in \mathbb{R}^{2n \times 2n}[\lambda]$. Note that for an eigenvalue $\lambda$ of $L(\lambda)$ with $\text{re}(\lambda) \neq 0 \neq \text{im}(\lambda)$, $\lambda$, $-\lambda$, $-\lambda$ are also eigenvalues of $L(\lambda)$. Moreover, if $x$ and $\hat{x}$ are eigenvectors corresponding to $\lambda$, $-\lambda$, respectively, then $\overline{x}$ and $\overline{\hat{x}}$ are eigenvectors corresponding to $\overline{\lambda}$ and $-\lambda$ respectively. If $\text{re}(\lambda) = 0$ then $\lambda$, $-\lambda$ form a pair of eigenvalues of $L(\lambda)$, whereas if $\text{re}(\lambda) = 0$ then $\lambda$, $-\lambda$ are eigenvalues in pairs. Thus for real structured updates of $L(\lambda)$ the eigenvalues are to be replaced as tuples depending on the real and imaginary parts of the eigenvalues. Thus we assume that the quadruple of eigenvalues $(\lambda_j^c, -\lambda_j^c, -\lambda_j^a, \lambda_j^a)$ of $L(\lambda)$ is to be changed by a quadruple $(\lambda'_j^c, -\lambda'_j^c, -\lambda'_j^a, \lambda'_j^a)$ when both the real and imaginary parts of $\lambda_j^c$ and $\lambda_j^a$ are non zero, where $1 \leq j \leq m_1$. The pair of eigenvalues $(\lambda_k^c, -\lambda_k^c)$ is to be changed by a pair $(\lambda_k^a, -\lambda_k^a)$ when the real parts of $\lambda_k^c, \lambda_k^a$ are zero, $m_1 + 1 \leq k \leq m_2$. Finally a pair of eigenvalues $(\lambda_l^c, -\lambda_l^c)$ of $L(\lambda)$ is to be changed by a pair $(\lambda_l^a, -\lambda_l^a)$ when the imaginary parts of $\lambda_l^c, \lambda_l^a$ are zero, $m_2 + 1 \leq k \leq p$. Obviously, $2m_1 + 2p < n$.

Let

$$X_c = [X_c^c \ldots X_{m_1}^c X_{m_1+1}^c \ldots X_{m_2}^c X_{m_2+1}^c \ldots X_p^c]$$

where

$$X_j^c = \begin{bmatrix} \text{re}(x_j^c) & \text{im}(x_j^c) \\ \text{re}(\hat{x}_j^c) & \text{im}(\hat{x}_j^c) \end{bmatrix}, \quad X_k^c = \begin{bmatrix} \text{re}(x_k^c) \\ \text{im}(x_k^c) \end{bmatrix}, \quad X_l^c = \begin{bmatrix} x_l^c & \hat{x}_l^c \end{bmatrix}$$

$x_j^c$ and $\hat{x}_j^c$ denote the eigenvectors corresponding to $\lambda_j^c$ and $-\lambda_j^c$ respectively, and $x_k^c$, $x_l^c$ and $\hat{x}_l^c$ denote the eigenvectors corresponding to the eigenvalues $\lambda_k^c$, $\lambda_l^c$ and $-\lambda_l^c$ respectively.

Further, suppose

$$\Lambda_c = \text{diag}(\Lambda_1^c, \ldots, \Lambda_{m_1}^c, \Lambda_{m_1+1}^c, \ldots, \Lambda_{m_2}^c, \Lambda_{m_2+1}^c, \ldots, \Lambda_p^c)$$

$$\Lambda_a = \text{diag}(\Lambda_1^a, \ldots, \Lambda_{m_1}^a, \Lambda_{m_1+1}^a, \ldots, \Lambda_{m_2}^a, \Lambda_{m_2+1}^a, \ldots, \Lambda_p^a)$$

where

$$\Lambda_j^c = \text{diag}(\hat{\Lambda}_j^c, -\hat{\Lambda}_j^c)^T, \quad \Lambda_k^c = \begin{bmatrix} 0 & \text{im}(\lambda_k^c) \\ -\text{im}(\lambda_k^c) & 0 \end{bmatrix}, \quad \Lambda_j^a = \text{diag}(\hat{\Lambda}_j^a, -\hat{\Lambda}_j^a)^T, \quad \Lambda_k^a = \begin{bmatrix} 0 & \text{im}(\lambda_k^a) \\ -\text{im}(\lambda_k^a) & 0 \end{bmatrix}$$

and

$$\hat{\Lambda}_j^c = \begin{bmatrix} \text{re}(\lambda_j^c) & \text{im}(\lambda_j^c) \\ -\text{im}(\lambda_j^c) & \text{re}(\lambda_j^c) \end{bmatrix}, \quad \hat{\Lambda}_j^a = \begin{bmatrix} \text{re}(\lambda_j^a) & \text{im}(\lambda_j^a) \\ -\text{im}(\lambda_j^a) & \text{re}(\lambda_j^a) \end{bmatrix}, \quad j = 1, \ldots, m_1, k = m_1+1, \ldots, m_2, l = m_2 + 1, \ldots, p.$$
Proof. As the eigenvalues of $\Lambda_c$ are distinct so the matrix $G = X^T J M X_c$ is of the form $G = \text{diag}(G_1, \ldots, G_m, G_{m+1}, \ldots, G_{m+n}, G_{m+n+1}, \ldots, G_p)$ where $G_j = \begin{bmatrix} 0 & u_j I_2 + v_j J_2 \\ -u_j I_2 + v_j J_2 & 0 \end{bmatrix}$, $G_k = v_k I_2$, $G_l = v_l J_2$ for some real numbers $u_j, v_j, v_k, v_l$, $j = 1, \ldots, m_1$, $k = m_1 + 1, \ldots, m_2$, $l = m_2 + 1, \ldots, p$. Rest of the proof follows from Proposition 7.1. 

Another parametric updates $\Delta M, \Delta K$ which solves the problem (P1) for $T$-SHH pencils can be represented as follows.

Remark 7.5. If the assumptions of Theorem 7.4 hold then $T$-skew-Hamiltonian update matrix $\Delta M = J^T U M U T$ and $T$-Hamiltonian matrix is given by $\Delta K = J^T U K U T$ which solves the problem (P1), where

$$M = G(\Lambda_c - \Lambda_a) H_a \Lambda_a^T + Z_1(I_{2m_1+2p} - \Lambda_a H_a \Lambda_a^T) - Z_2 H_a \Lambda_a^T,$$

$$K = G(\Lambda_c - \Lambda_a) H_a - Z_1 \Lambda_a H_a + Z_2(I_{2m_1+2p} - H_a)$$

with $H_a = (\Lambda_a^T H_a + I_{2m_1+2p})^{-1}, G = X_c^T J M X_c$, $U = J M X_c G^{-1}$.

$$Z_i = \text{diag}(Z_{i1}^{(1)}, \ldots, Z_{i1}^{(p)}, \ldots, Z_{i2}^{(1)}, \ldots, Z_{i2}^{(p)}), i = 1, 2,$$

$$Z_{j1}^{(1)} = \begin{bmatrix} 0 & \alpha_j I_2 + \beta_j J_2 \\ -\alpha_j I_2 + \beta_j J_2 & 0 \end{bmatrix}, Z_{j2}^{(2)} = \begin{bmatrix} 0 & u_j I_2 + v_j J_2 \\ -u_j I_2 + v_j J_2 & 0 \end{bmatrix}, Z_{k1}^{(1)} = \beta_k J_2, Z_{k2}^{(2)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ and } \alpha_j, \beta_j, u_j, v_j, \beta_k, u_k, \beta_l, u_l \text{ are arbitrary real numbers, } j = 1, \ldots, m_1, k = m_1 + 1, \ldots, m_2, l = m_2 + 1, \ldots, p.$$

Now we apply the above results on a numerical example to examine the validity of the results.

Example 7.6. Consider a $*$-SHH pencil $L(\lambda) = \lambda M + K$ with

$$M = \begin{bmatrix}
-0.25455 + 0.95256i & 0.02934 + 0.05513i & 0.00000 - 1.83635i & 0.08681 - 1.45077i \\
2.5023 - 0.01156i & 1.14852 - 1.53017i & -0.08681 - 1.45077i & 0.00000 + 1.40120i \\
0.00000 - 0.96582i & -0.22366 - 0.46730i & -0.25455 - 0.95256i & 2.25023 + 0.01156i \\
0.22366 - 0.46730i & 0.00000 - 1.00248i & 0.02934 - 0.05513i & 1.14852 + 1.53017i \\
\end{bmatrix},$$

$$K = \begin{bmatrix}
3.02148 + 1.90489i & 1.10499 + 1.16245i & -1.26366 + 0.00000i & 1.65942 + 0.71011i \\
0.44232 - 1.07299i & 0.29350 + 0.24688i & 1.65942 - 0.71011i & -0.19304 + 0.00000i \\
1.30628 + 0.00000i & -0.42739 + 0.75761i & -3.02148 + 1.90489i & -0.44232 - 1.07299i \\
-0.42739 - 0.75761i & 0.52491 + 0.00000i & -1.10499 + 1.16245i & -0.29350 - 0.24688i \\
\end{bmatrix}.$$

Let $\lambda_1 = -0.92332 - 0.75639i$, $\lambda_2 = -0.12114i$ and $\lambda_3 = -0.76954 + 0.53243i$, $\lambda_4 = -3.22147i$. Suppose that we want to replace the set of eigenvalues $\{\lambda_1, -\lambda_2, \lambda_3\}$ of $L(\lambda)$ by the desired set of eigenvalues $\{\lambda_1^*, -\lambda_2^*, \lambda_3^*\}$ respectively. Thus $\Lambda_c = \text{diag}(\lambda_1^*, -\lambda_2^*, \lambda_3^*)$, $\Lambda_a = \text{diag}(\lambda_4^*, -\lambda_1^*, \lambda_2^*)$ and

$$X_c = \begin{bmatrix}
1.00000 + 0.00000i & -0.43182 + 0.23755i & -0.20930 + 0.22721i \\
-0.32603 - 0.60175i & 1.00000 + 0.00000i & -0.67852 - 0.58802i \\
0.72475 + 0.50622i & -0.01383 + 0.37218i & 0.21160 - 0.29125i \\
-0.20761 + 0.69892i & 0.09784 + 0.45636i & 1.00000 + 0.00000i \\
\end{bmatrix}.$$

Then by remark 7.5 choosing $Z_1 = \begin{bmatrix}
0 & 0.06022 + 0.19082i \\
-0.06022 + 0.19082i & 0 \\
0 & 0 \\
1.19827i \\
\end{bmatrix}$. 

19
complementary if $X$ deflating pairs under some generic assumptions. When the matrices $\Theta$ maintaining no spillover condition. We examine the validity of the theoretical results by considering several numerical dating problem with no spillover.

On taking $f = 4.51104 i$ and $X_f = \begin{bmatrix} 0.20548 + 0.72300 i \\ -0.52204 + 0.39798 i \\ 1.00000 - 0.00000 i \\ -0.61073 + 0.21633 i \end{bmatrix}$ we obtain $\|(M + \Delta M)X_f\|_F = 1.5519 \times 10^{-14}$, which shows that the unmeasured spectral data remain undisturbed.

Hence we conclude that eigenvalues of the SHH pencil $\lambda = \lambda(M + \Delta M) + (K + \Delta K)$ are $\{X_1^a, -F_1, \lambda_2^a\}$. Therefore eigenvalues of $L(\lambda)$ are replaced by the desired eigenvalues with maintaining no spillover condition.

**Conclusion** Given a matrix pencil $L(\lambda) = \lambda M + K \in \mathbb{C}^{n \times n}[\lambda]$, a matrix pair $(X, \Lambda) \in \mathbb{C}^{n \times p} \times \mathbb{C}^{p \times p}$ is said to be a deflating pair of $L(\lambda)$ if $MX + \Lambda X = 0$. Two such deflating pairs $(X_1, \Lambda_1) \in \mathbb{C}^{n \times p} \times \mathbb{C}^{(n-p) \times (n-p)}$ and $(X_2, \Lambda_2) \in \mathbb{C}^{n \times (n-p)} \times \mathbb{C}^{(n-p) \times p}$ are called complementary if $X_1 X_2$ is invertible. Given the complementary deflating pairs $(X_a, \Lambda_a)$ and $(X_f, \Lambda_f)$ of a structured matrix pencil $L(\lambda)$, and another matrix pair $(X_a, \Lambda_a)$ we determine computable expressions of structured and unstructured updates $\Delta M, \Delta K$ such that the updated matrix pencil $L(\lambda) = \lambda(M + \Delta M) + (K + \Delta K)$ inherit $(X_a, \Lambda_a), (X_f, \Lambda_f)$ as complementary deflating pairs under some generic assumptions. When the matrices $\Lambda_a, \Lambda_f$ and $\Lambda_a$ are diagonal matrices then the above problem is called the model updating problem with no spillover, in which the diagonal entries of $\Lambda_a$ and $\Lambda_f$ are the measured and unmeasured eigenvalues of an undamped finite element model associated with the pencil $L(\lambda)$. However, in general $(X_f, \Lambda_f)$ is not known and with this assumption we derive explicit parametric expression of unstructured and structured updates for a variety of structured matrix pencils which include symmetric, Hermitian, even, odd and skew-Hamiltonian/Hamiltonian matrix pencils. We examine the validity of the theoretical results by considering several numerical examples. We plan to extend the proposed framework to finite element quadratic model updating problem with no spillover.

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**References**

[1] B. Adhikari and R. Alam, *Structured backward errors and pseudospectra of structured matrix pencils*, SIAM Journal on Matrix Analysis and Applications. 31.2 (2009), pp. 331 - 359.
2. B. Adhikari and R. Alam, Structured mapping problems for linearly structured matrices, Linear Algebra Appl., 444(2014), pp. 132-145.

3. Z.-J. Bai, B.N. Datta and J. Wang, Robust and minimum norm partial quadratic eigenvalue assignment for vibrating systems: a new optimization approach, Mech. Syst. Signal Process. 24 (2010), pp. 766-783.

4. M. Baruch, Optimization procedure to correct stiffness and flexibility matrices using vibration tests, AIAA J. 16 (1978), pp. 1208-1210.

5. M. Baruch, Optimal correction of mass and stiffness matrices using measured modes, AIAA J. 20 (1982), pp. 1623-1626.

6. P. Benner, R. Byers, V. Mehrmann, and H. Xu, "Numerical computation of deflating subspaces of skew-Hamiltonian/Hamiltonian pencils. SIAM Journal on Matrix Analysis and Applications 24(1) (2002), pp. 165-190.

7. A. Berman and E.J. Nagy, Improvement of a large analytical model using test data, AIAA journal, 21(8) (1983), pp. 1168-1173.

8. S. Brahma and B.N. Datta, An optimization approach for minimum norm and robust partial quadratic eigenvalue assignment problems for vibrating structures, J. Sound Vib. 324 (2009), pp. 471-489.

9. B. Caesar and J. Peter, Direct update of dynamic mathematical models from modal test data, AIAA J. 25 (1987), pp. 1494-1499.

10. J.B. Carvalho, B.N. Datta, W.W. Lin and C.S. Wang, Symmetry preserving eigenvalue embedding in finite-element model updating of vibrating structures, Journal of Sound and Vibration, 290.3(2006), pp. 839-864.

11. J.B. Carvalho, B.N. Datta, A. Gupta and M. Lagadapati, A direct method for model updating with incomplete measured data and without spurious modes, Mechanical Systems and Signal Processing, 21.7(2007), pp. 2715-2731.

12. D. Chu, M.T. Chu and W.-W. Lin, Quadratic model updating with symmetry, positive definiteness, and no spill-over, SIAM J. Matrix Anal. Appl. 31 (2009). pp. 546-564.

13. M.T. Chu, B.N. Datta, W.W. Lin and S. Xu, Spillover phenomenon in quadratic model updating, AIAA journal, 46.2(2008), pp. 420-428.

14. M.T. Chu, W.-W. Lin and S.-F. Xu, Updating quadratic models with no spillover effect on unmeasured spectral data, Inverse Prob. 23 (2007), pp. 243.

15. B.N. Datta, Numerical linear algebra and applications, SIAM, 2010.

16. B.N. Datta and D. Sarkissian, Theory and computations of some inverse eigenvalue problems for the quadratic pencil, Contemporary Mathematics, 280(2001), pp. 221-240.

17. S. Elhay, Some inverse eigenvalue and pole placement problems for linear and quadratic pencils, Numerical Linear Algebra in Signals, Systems and Control. Springer Netherlands, (2011), pp. 217-249.

18. M.I. Friswell and J.E. Mottershead, Finite element model updating in structural dynamics, Springer Science and Business Media, Vol. 38,(2013).

19. I. Gohberg, P. Lancaster and L. Rodman, Indefinite Linear Algebra and Applications, Birkhäuser, (2005).
[20] V. IONESCU, C. OAR, AND M. WEISS, Generalized Riccati theory and robust control: a Popov function approach, John Wiley (1999).

[21] Y.C. Kuo and B.N. Datta, Quadratic model updating with no spill-over and incomplete measured data: Existence and computation of solution, Linear Algebra Appl., 436.7(2012), pp. 2480-2493.

[22] P. Lancaster, Model-updating for self-adjoint quadratic eigenvalue problems, Linear Algebra Appl., 428.11(2008), pp. 2778-2790.

[23] P. Lancaster, Model-updating for symmetric quadratic eigenvalue problems, (2006).

[24] L.W. Li, A new method for structural model updating and stiffness identification, Mech. Syst. Signal Process. 16 (2002), pp. 155167.

[25] S. D. Mackey, N. Mackey, C., and V. Mehrmann, Structured polynomial eigenvalue problems: Good vibrations from good linearizations. SIAM Journal on Matrix Analysis and Applications 28, no. 4 (2006) pp. 1029-1051.

[26] W.-W. Lin, V. Mehrmann, and H. Xu, Canonical forms for Hamiltonian and symplectic matrices and pencils, Linear Algebra and its Applications 302 (1999), pp. 469-533.

[27] X. Mao and H. Dai, Finite element model updating with positive definiteness and no spill-over, Mechanical Systems and Signal Processing 28 (2012), pp. 387-398.

[28] C. Mehl, Condensed forms for skew-Hamiltonian/Hamiltonian pencils. SIAM Journal on Matrix Analysis and Applications. 21.2 (2000), pp. 454-76.

[29] J.E. Mottershead and M.I. Friswell, Model updating in structural dynamics: a survey, Journal of sound and vibration, 167.2(1993), pp. 347-375.

[30] J. Qian, S.-F. Xu, F.-S. Bai, Symmetric low-rank corrections to quadratic models, Numer. Linear Algebra Appl. 16 (2009), pp. 397413.

[31] H. Sarmadi, A. Karamodin and A. Entezami, A new iterative model updating technique based on least squares minimal residual method using measured modal data Applied Mathematical Modelling, 40(2016), pp. 10323-10341.

[32] R.C. Thompson, Pencils of Complex and Real Symmetric and Skew Matrices, Linear Algebra and its Applications 147(1991), pp. 323-371.

[33] F. Tisseur and K. Meerbergen, The quadratic eigenvalue problem, SIAM review, 43(2001), pp.235-286.

[34] F.-S. Wei, Analytical dynamic model improvement using vibration test data, AIAA journal 28.1 (1990), pp. 175-177.

[35] F.-S. Wei, Mass and stiffness interaction effects in analytical model modification, AIAA J. 28 (1990), pp. 1686-1688.

[36] D. Xie, A numerical method of structure-preserving model updating problem and its perturbation theory, Applied Mathematics and Computation, 217 (2011), pp. 6364-6371.

[37] Y.B. Yang and Y.J. Chen, Direct versus iterative model updating methods for mass and stiffness matrices International Journal of Structural Stability and Dynamics, 10 (2010), pp. 165-186.
[38] Y. Yuan, *Structural dynamics model updating with positive definiteness and no spillover*, Mathematical Problems in Engineering 2014 (2014).

[39] Q. Yuan, *Dual approaches to finite element model updating*, Journal of Computational and Applied Mathematics 236.7 (2012) pp. 1851-1861.

[40] Q. Yuan and H. Dai, *The matrix pencil nearness problem in structural dynamic model updating*, Journal of Engineering Mathematics, 93(2015), pp. 131-143.

[41] Y. Yuan Y, K. Zuo and J. Chen, *Updating undamped structural models using displacement output feedback* Applied Mathematical Modelling. 46 (2017), pp. 218-226.

[42] Y.X. Yuan and H. Dai, *A generalized inverse eigenvalue problem in structural dynamic model updating*, Journal of Computational and Applied Mathematics 226.1 (2009), pp.42-49.

[43] D.C. Zimmerman and M. Windengren, *Correcting finite element models using a symmetric eigenstructure assignment technique*, AIAA J. 28 (1990), pp. 16701676.