On a pre-Jacobi-Jordan algebra: relevant properties and double construction

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Abstract. We introduce a pre-Jacobi-Jordan algebra and study some relevant properties such as bimodules, matched pairs. Besides, we established a pre-Jacobi-Jordan algebra built as a direct sum of a given pre-Jacobi-Jordan algebra \((A, \cdot)\) and its dual \((A^*, \circ)\), endowed with a non-degenerate symmetric bilinear form \(B\), where \(\cdot\) and \(\circ\) are the products defined on \(A\) and \(A^*\), respectively. Finally, after pre-Jacobi-Jordan algebras classification in dimension two, we thoroughly give some double constructions of pre-Jacobi-Jordan algebraic structures.

Keywords. (pre)Jacobi-Jordan algebra, bimodule, matched pair, double construction

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1. Introduction

Jacobi-Jordan algebras (JJ algebras for short) were introduced in [5] in 2014 as vector spaces \(A\) over a field \(k\), equipped with a bilinear map \(\cdot : A \times A \to A\) satisfying the Jacobi identity and instead of the skew-symmetry condition valid for Lie algebras the commutativity \(x \cdot y = y \cdot x\), for all \(x, y \in A\) is imposed. Apparently this class of algebras appear under different names in the literature reflecting, perhaps, the fact that it was considered from different viewpoints by different communities, sometimes not aware of each other’s results. In [6–8] and other Jordan literature, these algebras are called Jordan algebras of nil index 3. In [10] they are called Lie-Jordan algebras (superalgebras are also considered there). In [12] and [13] they were called mock-Lie algebras.

In [9] Wörz-Busekros relates these type of algebras with Bernstein algebras. One crucial remark is that JJ algebras are examples of the more popular and well-referenced Jordan algebras [1][11] introduced in order to achieve an axiomatization for the algebra of observables in quantum mechanics. In [5] the authors achieved the classification of these algebras up to dimension 6 over an algebraically closed field of characteristic different from 2 and 3. As it was explained in the paper and in [1], JJ algebras are objects fundamentally different from both associative and Lie algebras though their definition differs from the latter only modulo a sign. In [1] it’s proven that there exists a rich and very interesting theory behind the JJ algebras which deserves to be developed further mainly for three important reasons (see [1] for more details). They mainly discuss the general extension (GE) problem (which is a kind of generalization of the classical Holder extension problem) for JJ algebras. Interestingly, they authors prove that a finite dimensional JJ algebras is Frobenius if and only if there exists an invariant non degenerate bilinear form (Proposition 1.8).

A Frobenius algebra is an associative algebra equipped with a non-degenerate invariant bilinear form. This type of algebras plays an important role in different areas of mathematics and physics, such as statistical models over two-dimensional graphs [3] and topological quantum field theory [4].
On the other hand, an antisymmetric bilinear form on an associative algebra $\mathcal{A}$ is an antisymmetric bilinear form on $\mathcal{A}$ which is a $1$–cocycle, or Connes cocycle, for the Hochschild cohomology.

In [2], C. Bai described associative analogs of Drinfeld’s double constructions for Frobenius algebras and for associative algebras equipped with non-degenerate Connes cocycles. We note that there are two different types of constructions involved:

i) the Drinfeld’s double type constructions, from a Frobenius algebra or from an associative algebra equipped with a Connes cocycle, and

ii) the Frobenius algebra obtained from an anti-symmetric solution of the associative Yang-Baxter equation and the non-degenerate Connes cocycle obtained from a symmetric solution of a $D$-equation.

Using the main fact that JJ algebras are Frobenius under susmentionned condition, our purpose is to proceed a double construction for JJ algebras and for pre-JJ algebras.

We use in this paper the double construction’s technique by Bai in [2]. The aim of this paper is to give some basics of JJ algebras and, study pre-JJ algebras specially their bimodules and matched pairs. The double construction of symmetric (pre)JJ algebras. Furthermore, a two dimensional classification of pre-JJ algebras is given with a special emphasis on the corresponding double construction.

The paper is organized as follow. In Sec.2, after recalling some basics concepts and necessary properties on antiassociative and JJ algebras, we discuss on the matched pairs of JJ algebras. On the other hand we discuss some basics properties of pre-JJ algebras. Sec.3 is devoted for bimodules and matched pairs of pre-JJ algebras. In Sect.4, we give results on double construction symmetric (pre)JJ algebras. A detailed survey is done for a two dimensional pre-JJ algebras in Sec.5. In Sec.6, we end with some concluding remarks.

2. A pre-Jacobi-Jordan algebra: definition and main results

2.1. Representations and Matched pair of Jacobi-Jordan algebras. Throughout this work, we consider $\mathcal{A}$ a finite dimensional vector space over the field $K$ of characteristic different from 2,3 together with a bilinear product "" defined as: $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ such that $(x, y) \mapsto x \cdot y$.

Definition 2.1. [10] Let "" be a bilinear product in a vector space $\mathcal{A}$. Suppose that it satisfies the following law:

$$(x \cdot y) \cdot z = -x \cdot (y \cdot z). \tag{2.1}$$

Then, we call the pair $(\mathcal{A}, \cdot)$ an antiassociative algebra.

Definition 2.2. [13] An algebra $(\mathcal{A}, \circ)$ over $K$ is called JJ if it is commutative:

$$x \circ y = y \circ x, \tag{2.2}$$

and satisfies the Jacobi identity:

$$(x \circ y) \circ z + (z \circ x) \circ y + (y \circ z) \circ x = 0 \tag{2.3}$$

for any $x,y,z \in \mathcal{A}$.

Theorem 2.3. [13] Given an antiassociative algebra $(\mathcal{A}, \cdot)$, the new algebra $\mathcal{A}^\dagger$ with multiplication give by the "anticommutator"

$$a \circ b = \frac{1}{2} (a \cdot b + b \cdot a),$$

is a JJ algebra.

Since JJ algebras are commutative, the left and right actions of an algebra coincide, so we can speak about just modules.

Definition 2.4. [13] A vector space $V$ is a module over a JJ algebra $\mathcal{A}$, if there is a linear map (a representation) $\rho : \mathcal{A} \rightarrow \text{End}(V)$ such that

$$\rho(x \circ y)(v) = -\rho(x)(\rho(y)v) - \rho(y)(\rho(x)v) \tag{2.4}$$

for any $x, y \in \mathcal{A}$ and $v \in V$. 

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[10]

[13]

[2]
Proposition 2.5. Let \((g, \circ)\) be a JJ algebra and \((V, \rho)\) be a representation of \(g\). The direct summand \(g \oplus V\) with a bracket defined by
\[
(x + u) \circ (y + w) := x \circ y + \rho(x)(w) + \rho(y)(u) \quad \forall x, y \in g \ \forall u, w \in V
\]
is a JJ algebra.

Proof. The symmetry of the bracket is obvious. We show that the Jacobi identity is satisfied:
Let \(x, y, z \in g\) and \(\forall u, v, w \in V\).
\[
\begin{align*}
\circ(x, u, (y, v), (z, w)) (x + u) \circ ((y + v) \circ (z + w)) &= \circ(x, u, (y, v), (z, w)) (x + u) \circ (y \circ z + \rho(y)(v) + \rho(z)(u)) \\
&= \circ(x, u, (y, v), (z, w)) x \circ (y \circ z) + \rho(x)(\rho(y)(v) + \rho(z)(u)) + \rho(y \circ z)(u) \\
&= \circ(x, u, (y, v), (z, w)) \rho(x(\rho(y)(v)) + \rho(x(\rho(z)(u)) + \rho(y(\rho(z)(u))) \\
&+ \rho(z(\rho(y)(u))) \\
&\quad + \rho(y(\rho(z)(u)) + \rho(x(\rho(z)(v)) + \rho(x(\rho(z)(v)) \rho(z(\rho(y)(u))) \\
&+ \rho(x(\rho(z)(v)) + \rho(z(\rho(y)(u)) + \rho(x(\rho(y)(w)) + \rho(y(\rho(x)(w)))) \\
&\quad = 0
\end{align*}
\]
where \(\circ(x, u, (y, v), (z, w))\) denotes summation over the cyclic permutation on \((x, u), (y, v), (z, w)\).

Definition 2.6. Let \((g, \circ)\) be a JJ algebra. Two representations \((V, \rho)\) and \((V', \rho')\) of \(g\) are said to be isomorphic if there exists a linear map \(\phi : V \to V'\) such that
\[
\forall x \in g, \quad \rho'(x) \circ \phi = \phi \circ \rho(x).
\]

Proposition 2.7. Let \((g, \circ)\) be a JJ algebra and \(L_\circ : g \to \text{End}(g)\) be an operator defined for \(x \in g\) by \(L_\circ(x)(y) = x \circ y\). Then \((g, L_\circ)\) is a representation of \(g\).

Proof. Since \(g\) is JJ algebra, the Jacobi condition on \(x, y, z \in g\) is
\[
x \circ (y \circ z) + y \circ (z \circ x) + z \circ (x \circ y) = 0
\]
and may be written
\[
L_\circ(x \circ y)(z) = -L_\circ(x)(L_\circ(y)(z)) - L_\circ(y)(L_\circ(x)(z))
\]
Then the operator \(ad\) satisfies
\[
L_\circ[x, y] = -L_\circ(x) \circ L_\circ(y) - L_\circ(y) \circ L_\circ(x).
\]
Therefore, it determines a representation of the JJ algebra \(g\). 

Let \((g, \circ)\) be a JJ algebra and \((V, \rho)\) be a representation of \(g\). Let \(V^*\) be the dual vector space of \(V\). We define a linear map
\[
\rho^* : g \to \text{gl}(V^*)
\]
\[
V^* \to V^*
\]
x \quad \mapsto \rho^*_x : u^* \mapsto \rho^*_x u^* : V \to \mathbb{K} \quad \langle \rho^*_x u^*, v \rangle := \langle u^*, \rho_x v \rangle
\]
Let \(f \in V^*, \ x, y \in g\) and \(u \in V\). We have
\[
\langle \rho^*(x \circ y)f, u \rangle = \langle f, \rho(x \circ y)u \rangle \\
= \langle f, -\rho(x)\rho(y)u - \rho(y)\rho(x)u \rangle \\
= \langle -\rho^*(x)\rho^*(y)f - \rho^*(y)\rho^*(x)f, u \rangle.
\]
Therefore, We have the following proposition:

Proposition 2.8. Let \((g, \circ)\) be a JJ algebra and \((V, \rho)\) be a representation of \(g\), where \(V\) is a finite dimensional vector space. The following conditions are equivalent:

1. \((V, \rho)\) is a bimodule of \(g\).
(2) \((V^*, \rho^*)\) is a bimodule of \(g\).

**Corollary 2.9.** Let \((g, \circ)\) be a JJ algebra and \((g, L_0)\) be the adjoint representation of \(g\), where \(L_0 : g \rightarrow \text{End}(g)\). We set \(L_\circ : g \rightarrow \text{End}(g^*)\) and
\[
L_\circ^* : g \rightarrow g(V^*) \\
x \rightarrow L_\circ^*(x) : V^* \rightarrow V^* \\
\mathbf{u} \rightarrow L_\circ^*(x)\mathbf{u}^* : V \rightarrow \mathbb{K} \\
f \mapsto \langle L_\circ^*(x)\mathbf{u}^*, \mathbf{v} \rangle := \langle \mathbf{u}^*, L_\circ(x)\mathbf{v} \rangle.
\]

Then \((g^*, L_\circ^*)\) is a representation of \(g\).

**Theorem 2.10.** Let \((G, \circ)\) and \((H, \bullet)\) be two JJ algebras and let \(\mu : H \rightarrow \text{gl}(G)\) and \(\rho : G \rightarrow \text{gl}(H)\) be two JJ algebra representations. Then, \((G, H, \mu, \rho)\) is called a matched pair of the JJ algebras \(G\) and \(H\), denoted by \(G \bowtie_{\mu^{-1}\rho} H\) if and only if \(\mu\) and \(\rho\) satisfy: for all \(x, y, z \in G, a, b \in H\),
\[
\rho(x)(a \bullet b) + \rho(x)a \bullet b + a \bullet \rho(x)b + \rho(\mu(a)x)b + \rho(\mu(b)x)a = 0, \tag{2.8}
\]
\[
\mu(a)(x \circ y) + \mu(a)x \circ y + x \circ \mu(a)y + \mu(\rho(x)a)y + \mu(\rho(y)a)x = 0. \tag{2.9}
\]

In this case, \((G \oplus H, \ast)\) defines a JJ algebra with respect to the product \(\ast\) satisfying:
\[
(x + a) \ast (y + b) = x \circ y + \mu(a)y + \mu(b)x + a \bullet b + \rho(x)b + \rho(y)a. \tag{2.10}
\]

**Proof:** We have:
\[
\begin{align*}
\{(x + a) \ast (y + b)\} \ast (z + c) &+ \{(y + b) \ast (z + c)\} \ast (x + a) + \{(z + c) \ast (x + a)\} \ast (y + b) \\
&= \{(x \circ y) \circ z + (y \circ z) \circ x + (z \circ x) \circ y\} + \{(a \bullet b) \circ c + (b \bullet c) \circ a + (c \bullet a) \circ b\} \\
&+ \{(\mu(c)(x \circ y) + (\mu(c)x \circ y) + x \circ \mu(c)y + \mu(\rho(c)x)y + \mu(\rho(y)c)x)\} \\
&+ \{(\rho(\mu(z)a \bullet b) + \rho(z)a \bullet b + a \bullet \rho(z)b + \rho(\mu(a)z)b + \rho(\mu(b)z)a)\} \\
&+ \{(\mu(a)(y \circ z) + \mu(y \circ z) + y \circ \mu(a)z + \mu(\rho(y)a)z + \mu(\rho(z)a)y)\} \\
&+ \{(\rho(\mu(c)b \bullet c) + \rho(b \bullet c) + b \bullet \rho(c)x + \rho(\mu(c)b)c + \rho(\mu(b)c)b\} \\
&+ \{(\mu(c) \circ a) + \rho(y)(c \circ a) + c \circ \rho(y)a + \rho(\mu(c)y)a + \rho(\mu(a)y)c\} \\
&\begin{cases}
\rho(x)(a \bullet b) + \rho(x)a \bullet b + a \bullet \rho(x)b + \rho(\mu(a)x)b + \rho(\mu(b)x)a = 0, \\
\mu(a)(x \circ y) + \mu(a)x \circ y + x \circ \mu(a)y + \mu(\rho(x)a)y + \mu(\rho(y)a)x = 0,
\end{cases}
\end{align*}
\]

and using the linearity of the representations \(\mu\) and \(\rho\), the Jacobi identity
\[
((x + a) \ast (y + b)) \ast (z + c) + ((y + b) \ast (z + c)) \ast (x + a) + ((z + c) \ast (x + a)) \ast (y + b) = 0
\]
is satisfied. This end the proof.

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**2.2. Definition and main properties of pre-Jacobi-Jordan algebras.**

**Definition 2.11.** \((A, \cdot)\), (or simply \(A\)), is said to be a left pre-JJ-algebra if \(\forall x, y, z \in A\), the antiaassociator of the bilinear product \(\cdot\) defined by \(\langle x, y, z \rangle \rangle := \langle x \cdot y \rangle \cdot z + x \cdot (y \cdot z)\), is symmetric in \(x\) and \(y\), i.e.,
\[
\langle x, y, z \rangle \rangle = -\langle y, x, z \rangle \rangle.
\]

As matter of notation simplification, we will denote \(x \cdot y\) by \(xy\) if not any confusion.

**Proposition 2.12.** Any antiaassociative algebra is a left pre-JJ-algebra, and
\[
\mu \circ (\mu \otimes \text{id}) + \mu \circ (\text{id} \otimes \mu) + \mu \circ (\mu \otimes (\mu \otimes \tau)) + \mu \circ (\text{id} \otimes (\mu \otimes \tau)) = 0, \tag{2.13}
\]
where \(\mu\) is the multiplication operator and \(\tau\) is such that \(\tau(x \circ y) = y \circ x\).
Proof. Let \((\mathcal{A}, \cdot)\) be an antiaassociative algebra. We have \(\forall x, y, z \in \mathcal{A}\)
\[(x \cdot y) \cdot z = -x \cdot (y \cdot z)\]
\[\Leftrightarrow (x, y, z)_{-1} = 0 = -(y, x, z)_{-1}.
\]
Thus \((x, y, z)_{-1} + (y, x, z)_{-1} = 0\), which implies that
\[(xy)z + x(yz) + (yx)z + yxz = 0.
\]
Supposing \(\mu\) be the bilinear product on \(\mathcal{A}\), then the previous relation can be written as
\[
\mu(\mu(x \otimes y) \otimes z) + \mu(x \otimes \mu(y \otimes z)) + \mu((\mu \circ \tau)(x \otimes y) \otimes z) + \mu(y \otimes (\mu \circ \tau)z \otimes x) = 0
\]
\[\Leftrightarrow \mu(\mu \circ id) + \mu \circ (id \otimes \mu) + \mu \circ ((\mu \circ \tau) \otimes id) + \mu(id \otimes (\mu \circ \tau)) = 0 \quad (2.14)
\]

Definition 2.13. An algebra \((\mathcal{A}, \cdot)\) over \(\mathbb{K}\) with the bilinear product given by \((x, y) \mapsto x \cdot y\) is called right pre-JJ algebra if the associator associated to the bilinear product on \(\mathcal{A}\) is symmetric in right, i.e. for all \(x, y, z \in \mathcal{A}\)
\[(x, y, z)_{-1} = -(x, z, y)_{-1} \quad (2.15)
\]

Proposition 2.14. The opposite algebra of a left pre-JJ algebra is a right pre-JJ algebra.

Proof:
Suppose that \((\mathcal{A}, \cdot)\) is a pre-JJ algebra and \((\mathcal{A}, \circ)\) the opposite algebra of the algebra \(\mathcal{A}\). For any \(x, y, z \in \mathcal{A}\) we have:
\[(x, y, z)_{-1, \circ} = (x \circ y) \circ z + x \circ (y \circ z) = z \cdot (y \cdot x) + (z \cdot y) \cdot x = (z, y, x)_{-1} = -(y, z, x)_{-1}
\]
\[= -y \cdot (z \cdot x) - (y \cdot z) \cdot x = -(x \circ z) \circ y - x \circ (z \circ y) = -(x, z, y)_{-1, \circ}.
\]
Therefore, for all \(x, y, z \in \mathcal{A}\), \((x, y, z)_{-1, \circ} = -(x, z, y)_{-1, \circ} \quad \Box
\]

In the following, left pre-JJ algebras and right pre-JJ algebras are equivalent. Any left(right) pre-JJ algebra will be right(left) pre-JJ algebra under new multiplication \((x, y) \rightarrow y \cdot x\). Thus for the rest of this paper, without any further clarification, left pre-JJ algebra are called pre-JJ algebra. In addition, to have an ease in manipulations, we replace \(\circ\) by \([., .]\).

Considering the representations of the left \(L\) and right \(R\) multiplication operations:
\[
L : \mathcal{A} \rightarrow \mathfrak{gl}(\mathcal{A})
\]
\[
x \mapsto L_x : \mathcal{A} \rightarrow \mathcal{A}
\]
\[
y \mapsto x \cdot y,
\]
\[
R : \mathcal{A} \rightarrow \mathfrak{gl}(\mathcal{A})
\]
\[
x \mapsto R_x : \mathcal{A} \rightarrow \mathcal{A}
\]
\[
y \mapsto y \cdot x,
\]
we infer the adjoint representation \(\text{ad} := L + R\) of the sub-adjacent JJ algebra of a pre-JJ algebra \(\mathcal{A}\) as follows:
\[
\text{ad} : \mathcal{A} \rightarrow \mathfrak{gl}(\mathcal{A})
\]
\[
x \mapsto \text{ad}_x : \mathcal{A} \rightarrow \mathcal{A}
\]
\[
y \mapsto \text{ad}_x(y),
\]
such that \(\forall x, y \in \mathcal{A}, \text{ad}_x(y) := (L_x + R_x)(y).

Proposition 2.15. Let \((\mathcal{A}, \cdot)\) be a pre-JJ algebra. For any \(x, y \in \mathcal{A}\), the following relations are satisfied:
- The anticommutator associated to the bilinear product \(\cdot\) given by \([x, y] = x \cdot y + y \cdot x\) defines a JJ algebra structure on \(\mathcal{A}\).
• The left multiplication operator gives a representation of the JJ algebra, that is

\[ L_{[x,y]} = -[L_x, L_y], \]

and the following relation is also satisfied

\[ [L_x, R_y] = -R_{xy} - R_y R_x, \]

where the linear map \( R \) is the right multiplication operator associated to the bilinear product \( \cdot \) on \( A \).

- \( [L_x, R_y] = -[R_x, L_y] \)
- \( L_{xy} + L_x L_y = -L_{yx} - L_y L_x \)
- \( ad = L + R \) a linear representation of the sub-adjacent JJ algebra of \( (A, \cdot) \) and,

\[ [ad_x, ad_y] = ad_{[x,y]}, \]

**Proof:**

Consider the pre-JJ algebra \( (A, \cdot) \). For any \( x, y, z \in A \) we have

\[
[x, [y, z]] + [y, [z, x]] + [z, [y, x]] = [x, yz + zy] + [y, zx +xz] + [z, xy +yx] = x(yz) + x(yz) + (yz)x + (zy)x + y(zx) + y(zx) + (zx)y + (zx)y + z(xy) + z(xy) + (xy)z = \{x(yz) + (yz)x + y(zx) + (zx)y + z(xy) + (xy)z\} = \{(x, y, z)_{-1} + (y, z, x)_{-1} + (z, x, y)_{-1}\} = 0.
\]

On other hand, we have for all \( x, y, z \in A \)

\[
(x, y, z)_{-1} = -(y, x, z)_{-1} \iff (xy)z + x(yz) = -(yx)z = y(xz) = (xy)z + (yx)z = -x(yz) - y(xz) = (L_{xy} + L_{yx})(z) = -(L_x L_y + L_y L_x)(z) = 0.
\]

Therefore, the relation for all \( x, y \in A \)

\[ L_{[x,y]} = -[L_x, L_y] \]

holds.

We have for all \( x, y, z \in A \)

\[
(x, y, z)_{-1} = -(y, x, z)_{-1} \iff (xy)z + x(yz) = -(yx)z = y(xz) \iff (R_z L_x - L_z R_x)(y) = -(R_z R_x + R_x R_z)(y) \iff [R_z, L_x](y) = -(R_z R_x + R_x R_z)(y)
\]

Therefore, the following relation holds

\[ [L_x, R_y] = -(R_{xy} + R_y R_x). \]
We have for all $x, y, z \in \mathcal{A}$

$$[L_x, R_y](z) = L_x (R_y(z)) + R_y (L_x(z))$$

$$= x(zy) + (zx)y$$

$$= (x, z, y) - (z, x, y)$$

$$= - ((zx)y + z(xy)) = - (R_y (R_x(z)) + R_{xy}(z))$$

$$= - (R_y R_x + R_{xy})(z)$$

$$= (z, y, x) = (zy)x + z(yx) = R_x R_y(z) + R_{xy}(z)$$

$$= -(y, z, x) - (yz)x - y(zx)$$

$$= -R_x L_y(z) - L_y R_x(z)$$

$$= -[R_x, L_y](z).$$

Therefore $[L_x, R_y] = -[R_x, L_y]$.

We have for all $x, y, z \in \mathcal{A}$

$$(L_{xy} + L_x L_y)(z) = L_{xy}(z) + L_x (L_y(z)) = (xy)z + x(yz)$$

$$= (x, y, z) - (y, x, z) = -(yx)z + y(xz)$$

$$= - (L_{yx}(z) + L_y(L_x(z)))$$

$$= -(L_{yx} + L_y L_x)(z).$$

Thus $L_{xy} + L_x L_y = -L_{yx} - L_y L_x$.

We have for all $x, y \in \mathcal{A}$,

$$[ad_x, ad_y] = [L_x, L_y] + [L_x, R_y] + [R_x, L_y] + [R_x, R_y]$$

$$= [L_x, L_y] + [R_x, R_y]$$

$$= (L_x L_y + R_x R_y) + (L_y L_x + R_y R_x)$$

$$= (L + R)_{xy} + (L + R)_{yx}$$

$$= (L + R)|_{[x, y]}$$

$$= ad_{[x, y]}$$

Thus, $(\mathcal{A}, \cdot)$ can be called the compatible pre-JJ algebra product of the JJ algebra $\mathcal{G}(\mathcal{A})$.

3. Bimodules and matched pairs of pre-Jacobi-Jordan algebras

**Definition 3.1.** Let $\mathcal{A}$ be a pre-JJ algebra, $V$ be a vector space. Suppose $l, r : \mathcal{A} \to \mathfrak{gl}(V)$ be two linear maps satisfying: for all $x, y \in \mathcal{A}$,

$$[l_x, r_y] = -[l_y, r_x] \quad (3.1)$$

$$l_x y + l_y = -l_{yx} - l_y l_x. \quad (3.2)$$

Then, $(l, r, V)$ (or simply $(l, r)$) is called bimodule of the pre-JJ algebra $\mathcal{A}$.

**Proposition 3.2.** Let $(\mathcal{A}, \cdot)$ be an pre-JJ algebra and $V$ be a vector space over $\mathbb{K}$. Consider two linear maps, $l, r : \mathcal{A} \to \mathfrak{gl}(V)$. Then, $(l, r, V)$ is a bimodule of $\mathcal{A}$ if and only if, the semi-direct sum $\mathcal{A} \oplus V$ of vector spaces is turned into a pre-JJ algebra by defining the multiplication in $\mathcal{A} \oplus V$ by $\forall x_1, x_2 \in \mathcal{A}, v_1, v_2 \in V$,

$$(x_1 + v_1) \ast (x_2 + v_2) = x_1 \cdot x_2 + (l_{x_1} v_2 + r_{x_2} v_1),$$

We denote it by $\mathcal{A} \ltimes_{l, r}^{-1} V$ or simply $\mathcal{A} \ltimes^{-1} V$. 

□
Proof:
It is obvious that the semi-direct sum of two vector spaces is also a vector space. Now suppose that \((l, r, V)\) is a bimodule of \(A\) and show that \((A \oplus V, \ast)\) is a pre-JJ algebra. Since \(\ast\) is a bilinear product,
for all \(x_1, x_2, x_3 \in A\) and for all \(v_1, v_2, v_3 \in V\), we have:

\[
\begin{align*}
((x_1 + v_1), (x_2 + v_2), (x_3 + v_3))_{-1} &= ((x_1 + v_1) \ast (x_2 + v_2)) \ast (x_3 + v_3) \\
&= (x_1 + v_1) \ast ((x_2 + v_2) \ast (x_3 + v_3)) \\
&= (x_1 x_2) x_3 + l x_2 v_3 + r x_3 \left( l x_1 v_2 + r x_2 v_1 \right) \\
&= x_1 (x_2 x_3) + l x_1 (l x_2 v_3) + r x_1 (r x_2 v_1) + r x_2 x_3 v_1 \\
&= (x_1 x_2) x_3 + l x_2 v_3 + r x_3 + r x_2 (r x_1 v_1) \\
&= x_1 (x_2 x_3) + l x_2 (l x_3 v_3) + r x_1 (r x_2 v_1) + r x_2 x_3 v_1,
\end{align*}
\]

\[
\begin{align*}
((x_1 + v_1), (x_2 + v_2), (x_3 + v_3)) &= (x_1, x_2, x_3)_{-1} + (l x_2, l x_1) v_3 \\
&= (r x_2 l x_1 + l x_1 r x_3) v_2 + (r x_3 r x_2 + r x_2 x_3) v_1.
\end{align*}
\]

Therefore,

\[
\begin{align*}
(x_1 + v_1, x_2 + v_2, x_3 + v_3)_{-1} &= -(x_3 + v_3, x_2 + v_2, x_1 + v_1)_{-1} \\
&= \begin{cases} \\
- l x_1 l x_2 = - l x_2 l x_1, \\
- r x_3 l x_1 = - r x_1 r x_3, \\
- r x_2 r x_3 = - r x_3 r x_2 \\
\end{cases} \\
\iff (A \oplus V, \ast) \text{ is a pre-JJ algebra.}
\end{align*}
\]

Furthermore, we derive the next result.

**Proposition 3.3.** Let \(A\) be a pre-JJ algebra and \(V\) be a vector space over \(\mathbb{K}\). Consider two linear maps, \(l, r : A \to \mathfrak{gl}(V)\), such that \((l, r, V)\) is a bimodule of \(A\). Then, the map: \(l + r : A \to \mathfrak{gl}(V)\) \(x \mapsto l + r,\) is a linear representation of the sub-adjacent JJ algebra of \(A\).

**Proof:**
Let \((l, r, V)\) be a bimodule of the pre-JJ algebra \(A\). Then, \(\forall x, y \in A\) \(\left[ l_x, r_y \right] = - [l_y, r_x]; l_{xy} + l_x l_y = - l_y l_x - l_{yx}.\) Besides, it is a matter of straightforward computation to show that \(l + r\) is a linear map on \(A\). Then, we have:

\[
\begin{align*}
[(l + r)(x), (l + r)(y)] &= [l_x + r_x, l_y + r_y] = [l_x, l_y] + [l_x, r_y] + [r_x, l_y] + [r_x, r_y] \\
&= \{ l_x, l_y \} + \{ r_x, r_y \} \\
&= l_x l_y + l_y l_x + r_x r_y + r_x r_y = \{ l_x l_y + r_x r_y \} + \{ l_y l_x + r_y r_x \} \\
&= \{ l_{xy} + r_{yx} \} + \{ r_{yx} + r_{yx} \} = (l + r)(x y) + (l + r)(y x) = (l + r)(x y).
\end{align*}
\]

Therefore, \((l, r, V)\) is a bimodule of \(A\) implies that \(l + r\) is a representation of the linear representation of the sub-adjacent JJ algebra of \(A\). \(\square\)

**Example 3.4.** According to the Proposition [2.13] one can deduce that \((L, R, A)\) is a bimodule of the pre-JJ algebra \(A\), where \(L\) and \(R\) are the left and right multiplication operator representations, respectively.

**Theorem 3.5.** Let \((A, \cdot)\) and \((B, \circ)\) be two pre-JJ algebras. Suppose that \((l_A, r_A, B)\) and \((l_B, r_B, A)\) are bimodules of \(A\) and \(B\), respectively, obeying the relations:

\[
r_A(x)(a, b) = r_A(l_B(b)x)a + r_A(l_B(a)x)b + a \circ (r_A(x)b) + b \circ (r_A(x)a),
\]

\[
- l_A(x)(a \circ b) = l_A(l_B(b)x + r_B(a)x)b + l_A(x)a + r_A(x)a \circ b \\
+ r_A(r_B(b)x)a + a \circ (l_A(x)b),
\]

\text{(3.3)}

\text{(3.4)}
\[
\begin{align*}
    \quad r_A(a)[x,y] &= r_B(l_A(y)a)x + r_B(l_A(x)a)y + x(r_B(a)y + y(r_B(a)x)), \\
    -l_B(a)(xy) &= l_B(l_A(x)a)y + (r_B(a)x)y + x(r_B(a)y) + r_B(r_A(y)a)x \\
    &\quad + (l_B(a)x) + l_B(l_A(x)a)y + l_B(l_B(a)(xy)),
\end{align*}
\]

for all \( x, y \in A \) and \( a, b \in B \). Then, there is a pre-JJ algebra structure on \( A \oplus B \) given by:
\[
(x + a) \cdot (y + b) = (x \cdot y + l_B(a)y + r_B(b)x) + (a \circ b + l_A(x)b + r_A(y)a).
\]

We denote this pre-JJ algebra by \( A \otimes_B l_A, r_A, l_B, r_B \), or simply by \( A \otimes_B B \). Then \( (A, B, l_A, r_A, l_B, r_B) \) satisfying the above conditions is called matched pair of the pre-JJ algebras \( A \) and \( B \).

**Proof:**

Consider \( x, y \in A \) and \( a, b \in B \). We have
\[
\begin{align*}
    (x + a) \cdot (y + b) &= (x + l_B(a)y + r_B(b)x) + (a \circ b + l_A(x)b + r_A(y)a), \\
    x \circ a &= r_B(a)x + l_A(x)a, \quad b = 0, y = 0, \\
    a \circ y &= l_B(a)x + r_A(y)a, \quad x = 0, b = 0.
\end{align*}
\]

Since \( (x, y, z)_{-1} = (xy)z + x(zy) \), we have
\[
\begin{align*}
    (x, b, c)_{-1} &= (x \cdot b) \cdot c + (b \cdot c) = (r_B(b)x + l_A(x)x) \cdot c + x \cdot (b \cdot c) \\
    &= r_B(c)(r_B(b)x + (l_A(x)x)b) \circ c + l_A(r_B(b)x)c + r_B(b \cdot c)x + l_A(x)(b \cdot c), \\
    (a, y, c)_{-1} &= (a \circ y) \cdot c + a \cdot (y \cdot c) \\
    &= (l_B(a)y + r_A(y)a) \cdot c + a \cdot (l_A(y)c + r_B(c)y) \\
    &= l_A(l_B(a)y)c + r_B(c)(l_B(a)y) + (r_A(y)a) \circ c \\
    &\quad + a \circ (l_A(y)c) + l_B(a)(r_B(c)y) + r_A(r_B(c)y)a, \\
    (x, y, c)_{-1} &= (x \cdot y) \cdot c + x \cdot (y \cdot c) \\
    &= (l_A(x)c + r_B(c)(x \cdot y)) + x \cdot (r_B(c)y) + l_A(x)(l_A(y)c) + r_B(l_A(y)c)x, \\
    (x, b, z)_{-1} &= (x \cdot b) \cdot z + x \cdot (b \cdot z) \\
    &= (l_A(x)b + r_B(b)x) \cdot z + x \cdot (r_B(b)b + r_A(z)b) \\
    &= l_B(l_A(x)b)z + r_A(z)(l_A(x)b) + (r_B(b)b) \cdot z \\
    &\quad + x \cdot (l_B(b)b) + l_A(x)(r_A(z)b) + r_B(r_A(z)b)x, \\
    (a, b, c)_{-1} &= (a \circ b) \circ c + a \circ (b \cdot c) \\
    &= (l_B(a \circ b) \cdot z + r_A(z)(a \circ b)) + a \cdot (l_B(b)z + r_A(z)b) \\
    &= l_B(a \circ b)z + r_A(z)(cb) + a \circ (r_A(z)b) + l_B(a)(l_B(b)z) - r_A(l_B(b)z)a.
\end{align*}
\]

The first part of the associator reads:
\[
\begin{align*}
    \{(x + a) \star (y + b)\} \star (z + c) &= \{(xy + l_B(a)y + r_B(b)x) + (a \circ b + l_A(y)a + r_A(y)a)\} \star (z + c) \\
    &= (xy)z + l_B(a)(y)z + l_B(a \circ b + l_A(x)b) + r_A(z)(a \circ b + l_A(y)a)z \\
    &\quad + r_B(c)(xy)z + l_B(a)y + r_B(b)x + (a \circ b + l_A(x)b) + r_A(y)a) \circ c \\
    &\quad + l_A(a)(yz + l_B(b)z + r_B(c)y) + r_A(z)(a \circ b + l_A(x)b) + r_A(y)a) \\
    &= (xy)z + l_B(a)(y)z + r_B(c)(xy) + r_B(c)(l_B(a)y) + r_B(c)(r_B(b)x) \\
    &\quad + (a \circ b) \circ c + (l_A(x)b) \circ c + (r_A(y)a) \circ c + l_A(a)(yz) \\
    &\quad + l_A(a)(l_B(b)z) + l_A(a)(r_B(c)y) + r_A(z)(a \circ b)
\end{align*}
\]
while its second part:

\[
(x + a) \ast ((y + b) \ast (z + c)) = (x + a) \ast \{y + l_y(b)z + r_y(b, c) + l_y(z, a)y\} = x(y + l_y(b)z + r_y(b, c)y + l_y(z, a)y) + r_y(b, c)y + r_x(b, c)
\]

\[
+ l_y(z, a)y \ast r_y(b, c)y + l_y(z, a)y \ast r_x(b, c)
\]

and the associator takes the form:

\[
(x + a, y + b, z + c) = \{((xy)z - x(yz)) + ((a \circ b) \circ c - a \circ (b \circ c)) + \{r_y(c)(r_y(b)x + l_y(x)b) \circ c + l_y(r_y(b)x)c\} - r_x(b, c)\}
\]

Further, we have

\[
(x + a, y + b, z + c) = r_y(b)(x \ast y) + x \ast (r_y(b)c) + r_x(b)(l_y(x)c) + l_y(x)(r_y(b)c)\]

which can also be re-expressed as:

\[
(x + a, y + b, z + c) = (x + a, y + c) + (x + b, z + c) + (x + b, y + z)
\]

Similarly,

\[
(y + b, x + a, z + c) = (y + b, x + c) + (y + b, x + z) + (y + b, x + a)
\]

Since \(A\) and \(B\) are pre-JJ algebras, we have

\[
(x, y, z) = -(y, x, z)
\]

\[
(a, b, c) = -(b, a, c)
\]
Hence,
\[(x, b, z)_{-1} = -(b, x, z)_{-1} \iff (y, a, z)_{-1} = -(a, y, z)_{-1}\{x \to y, b \to a, z \to z\}\]
\[(x, b, c)_{-1} = -(b, x, c)_{-1} \iff (a, y, c)_{-1} = -(y, a, c)_{-1}\{x \to y, b \to a, c \to c\}.
\]
Then, it remains to show that:
\[(x, a, y)_{-1} = -(a, x, y)_{-1}, \quad (3.10)\]
\[(x, a, b)_{-1} = -(a, x, b)_{-1}, \quad (3.11)\]
\[(x, y, a)_{-1} = -(y, x, a)_{-1}, \quad (3.12)\]
\[(a, b, x)_{-1} = -(b, a, x)_{-1}. \quad (3.13)\]

We have
\[(3.10) \iff l_B(l_A(x)a)y + r_A(y)(l_A(x)a) + (r_B(a)x)y + x(l_B(a)y) + l_A(x)(r_A(y)a) + r_B(r_A(y)a)x = -\{(l_B(a)x)y + l_B(r_A(x)a)y + r_A(y)(r_A(x)a) + l_B(a)(xy) + r_A(xy)\} \]
\[\iff -l_B(a)(xy) = l_B(l_A(x)a)y + (r_B(a)x)y + x(l_B(a)y) + r_B(r_A(y)a)x + l_B(a)(xy) \quad \iff (3.6) \]
since \([l_x, r_y] = -r_{xy} - r_y r_x.\]
\[(3.11) \iff r_B(b)(r_B(a)x) + (l_A(x)a) \circ b + l_A(r_B(a)x)b + r_B(a \circ b)x + l_A(x)(a \circ b) = -\{(l_B(a)x)b + r_B(b)(l_B(a)x) + (r_A(x)a) \circ b + a \circ (l_A(x)b) + l_B(a)(r_B(b)x) + r_A(r_B(b)x)\} \]
\[\iff -l_A(x)(a \circ b) = l_A(l_B(a)x) + r_B(a)x + (l_A(x)a) + r_A(x)a \circ b + r_A(r_B(b)x)a + a \circ (l_A(x)b) \quad \iff (3.4) \]
with, \((l_B, r_B)\) is bimodule of \(\mathcal{B}\).
\[(3.12) \iff l_A(xy)a + r_B(a)(xy) + x(r_B(a)y) + l_A(x)(l_A(y)a) + r_B(l_A(y)a)x = -\{(l_A(yx)a + r_B(a)(yx) + y(r_B(a)x) + l_A(y)(l_A(x)a) + r_B(l_A(x)a)y\} \]
\[\iff r_A(a)(x, y) = r_B(l_A(y)a)x + r_B(l_A(x)a)y + x(r_B(a)y) + y(r_B(a)x) \quad \iff (3.5) \]
since \([3.2]\) hold.
\[(3.13) \iff l_B(a \circ b)x + r_A(x)(a \circ b) + a \circ (r_A(x)b) + r_A(l_B(a)x) + l_B(a)(l_B(b)x) = -\{l_B(b \circ a)x + r_A(x)(b \circ a) + b \circ (r_A(x)a) + l_B(b)(r_B(a)x) + r_A(l_B(a)x)b\} \]
\[(3.14) \iff r_A(x)([a, b]) = r_A(l_B(b)x)a + r_A(l_B(a)x)b + a \circ (r_A(x)b) + b \circ (r_A(x)a) \quad \iff (3.2) \]
and \(l_B\) is a linear representation of the sub-adjacent Jacobi-Jordan algebra \(G(B)\).

Hence, \(\mathcal{A} \bowtie^{-1} \mathcal{B}\) is an pre-JJ algebra if and only if \((l_A, r_A)\) is a bimodule of \(\mathcal{A}\) and \((l_B, r_B)\) is a bimodule of \(\mathcal{B}\) and equations \([3.3] - [3.6]\) hold.

On the other hand, if \(\mathcal{A}\) and \(\mathcal{B}\) are pre-JJ sub-algebras of a pre-JJ algebra \(C\) such that \(C = A \oplus B\) which is a direct sum of the underlying vector spaces of \(\mathcal{A}\) and \(\mathcal{B}\), then the linear maps
\[l_A, r_A : A \to gl(B), \quad l_B, r_B : B \to gl(A),\]
defined by
\[(3.14) \quad x \ast a = l_A(x)a + r_B(a)x\]
\[(3.15) \quad a \ast x = l_B(a)x + r_A(x)a\]
satisfy the equations \([3.3] - [3.6]\). In addition, \((l_A, r_A)\) is a bimodule of \(\mathcal{A}\) and \((l_B, r_B)\) is a bimodule of \(\mathcal{B}\). □
COROLLARY 3.6. Let $\langle A, B, l_A, r_A, l_B, r_B \rangle$ be a matched pair of pre-JJ algebras. Then, $(G(A), G(B), l_A + r_A, l_B + r_B)$ is a matched pair of sub-adjacent JJ algebras $G(A)$ and $G(B)$.

Proof:
By using the Proposition 3.3 and the bimodules $(l_A, r_A, B)$ and $(l_B, r_B, A)$, we have: $\text{ad}_A := l_A + r_A$ and $\text{ad}_B := l_B + r_B$ are the linear representations of the sub-adjacent JJ algebras $G(A)$ and $G(B)$ of the pre-JJ algebras $A$ and $B$, respectively. Then, the statement that $G(A) \cong_{\omega_\text{adj}} \text{ad}_A G(B)$ is a matched pair of the JJ algebras $G(A)$ and $G(B)$ follows from Theorem 3.3. By analogous step giving:

$$\text{ad}_A(x) [a, b] - [\text{ad}_A(x)a, b] - [a, \text{ad}_A(x)b] - \text{ad}_A(\text{ad}_B(\text{ad}_B(x)a)b - \text{ad}_A(\text{ad}_B(\text{ad}_B(x)b)a = 0} \quad (3.16)$$

$$\text{ad}_B(a) [x, y] - [\text{ad}_B(a)x, y] - [x, \text{ad}_B(a)y] - \text{ad}_B(\text{ad}_A(\text{ad}_A(x)a)y - \text{ad}_B(\text{ad}_A(\text{ad}_A(y)a)x = 0} \quad (3.17)$$

Thus, we first have:

$$\text{ad}_A(x) [a, b] - [\text{ad}_A(x)a, b] - [a, \text{ad}_A(x)b] - \text{ad}_A(\text{ad}_B(\text{ad}_B(x)a)b - \text{ad}_A(\text{ad}_B(\text{ad}_B(x)b)a = 0$$

Thus:

$$\text{ad}_B(a) [x, y] - [\text{ad}_B(a)x, y] - [x, \text{ad}_B(a)y] - \text{ad}_B(\text{ad}_A(\text{ad}_A(x)a)y - \text{ad}_B(\text{ad}_A(\text{ad}_A(y)a)x = 0.$$
Definition 3.7. Let \( (l, r, V) \) be a bimodule of a pre-JJ algebra \( A \), where \( V \) is a finite dimensional vector space. The dual maps \( l^*, r^* \) of the linear maps \( l, r \), are defined, respectively, as:

\[
l^* : A \rightarrow \mathfrak{gl}(V^*) \quad V^* \rightarrow V^* \\
x \mapsto l^*_x : u^* \mapsto l^*_x u^* : V \rightarrow K, \quad \langle l^*_x u^*, v \rangle := \langle u^*, l_x v \rangle,
\]

(3.18)

\[
r^* : A \rightarrow \mathfrak{gl}(V^*) \quad V^* \rightarrow V^* \\
x \mapsto r^*_x : u^* \mapsto r^*_x u^* : V \rightarrow K, \quad \langle r^*_x u^*, v \rangle := \langle u^*, r_x v \rangle.
\]

(3.19)

Proposition 3.8. Let \((A, \cdot)\) be a pre-JJ algebra and \( l, r : A \rightarrow \mathfrak{gl}(V) \) be two linear maps, where \( V \) is a finite dimensional vector space. The following conditions are equivalent:

1. \((l, r, V)\) is a bimodule of \( A \).
2. \((r^*, l^*, V^*)\) is a bimodule of \( A \).

Proof:

(1)\(\Rightarrow\)(2) Suppose that \((l, r, V)\) is a bimodule of \((A, \cdot)\) and show that \((r^*, l^*, V^*)\) is also a bimodule of \((A, \cdot)\).

We have:

\[
\langle (l^*_x + l^*_y)u^*, v \rangle = \langle l^*_x u^*, v \rangle + \langle l^*_y u^*, v \rangle = \langle l_x v, u^* \rangle + \langle l_y(v), u^* \rangle
\]

\[
= \langle (l_x + l_y)v, u^* \rangle = \langle -l_{y-x} - l_{x-y}(v), u^* \rangle
\]

\[
= -\langle l_{y-x}u^*, v \rangle - \langle l^*_y u^*, v \rangle = -\langle (l^*_x + l^*_y)u^*, v \rangle.
\]

Therefore,

\[
l^*_x + l^*_y = -l^*_y - l^*_x, \quad \forall x, y \in A
\]

(3.20)

\[
\langle [l^*_x, r^*_y]u^*, v \rangle = \langle l^*_x (r^*_y u^*), v \rangle + \langle r^*_x (l^*_y) u^* , v \rangle = \langle l_x v, r^*_y u^* \rangle + \langle r^*_x v, l^*_y u^* \rangle
\]

\[
= \langle r^*_y (l_x v), u^* \rangle + \langle l_x (r_y(v)), u^* \rangle = \langle r^*_y l_x v, u^* \rangle
\]

\[
= \langle -[r^*_y l_x v, u^* \rangle = -\langle (r^*_y l_x) v, u^* \rangle
\]

\[
= -\langle l^*_x (r^*_y + r^*_x) u^*, v \rangle = \langle -[l^*_x, r^*_y] u^*, v \rangle
\]

Therefore

\[
[l^*_x, r^*_y] = -[r^*_y, l^*_x], \quad \forall x, y \in A.
\]

(3.21)

By considering the relations (3.20) and (3.21), we conclude that \((r^*, l^*, V)\) is a bimodule of \((A, \cdot)\).

(2)\(\Rightarrow\)(1) The converse, (i.e., by supposing that \((r^*, l^*, V)\) is a bimodule of \((A, \cdot)\) then \((l, r, V)\) is also a bimodule of \((A, \cdot)\)), can be proved by direct calculations by using similar relations as for the first part of the proof.

Theorem 3.9. Let \((A, \cdot)\) be a pre-JJ algebra. Suppose that there exists a pre-JJ algebra structure "\(\circ\)" on its dual space \(A^*\). Then, \((A, A^*, R^*, L^*, R_x^*, L_y^*)\) is a matched pair of pre-JJ algebras \(A\) and \(A^*\) if and only if

\[
(G(A), G(A^*), -\text{ad}^*, -\text{ad}^*_x) \quad \text{is a matched pair of JJ algebras} \ G(A) \quad \text{and} \quad G(A^*).
\]

Proof:

By considering the Theorem (3.3) setting \( l_A := R^*, r_A := L^*, l_B := R_x^*, r_B := L_y^* \), and exploiting the Definition (2.10) with \( G := G(A) \), \( H := G(A^*), \rho := R^* + L^*, \mu := R_x^* + L_y^* \), and the relations (3.16) and (3.17), we have
The equation \((R^* + L^*)[(x)[a, b] - [(R^* + L^*)(x)a, b] - [a, (R^* + L^*)(x)b]
- (R^* + L^*)((R^* + L^*)((x)a)b - (R^* + L^*)((R^* + L^*)(b)x)a
= R^*(x)[a, b] + L^*(x)[a, b] - [R^*(x)a, b] - [L^*(x)a, b] - [a, R^*(x)b] - [a, L^*(x)b] - R^*(R^*(x)a)b
- R^*(L^*o(x)a)b - R^*(R^*(x)b)a - R^*(L^*o(x)b)a - L^*(L^*o(x)a)b - L^*(R^*(x)a)b
- L^*(R^*(x)b)a = R^*(a o b) + R^*(b o a) + L^*(x)[a, b] - (R^*(x)a) o b o (R^*(x)a) - (L^*(x)a) o b o (L^*(x)a)
- a o (R^*(x)b) - (R^*(x)b)a - a o (L^*(x)b) - (L^*(x)b)a - a o (R^*(x)b)a - R^*(L^*o(a)x)b - R^*(L^*o(a)x)b
- R^*(L^*o(b)x)a - R^*(L^*o(b)x)a - R^*(L^*o(a)x)b - L^*(R^*(x)a)b - L^*(R^*(x)b)a - L^*(R^*(x)b)a
- L^*(R^*(x)b)a = 0.

The two first relations in brace on the last equality give zero (see (3.4)) and the last one brace also yields zero (see \((3.5)\)).

The equation \((R^* + L^*)[(x)[a, b] - [(R^* + L^*)(x)a, b] - [a, (R^* + L^*)(x)b]
- (R^* + L^*)((R^* + L^*)((x)a)b - (R^* + L^*)((R^* + L^*)(b)x)a
= R^*(x)[a, b] + L^*(x)[a, b] - [R^*(x)a, b] - [L^*(x)a, b] - [a, R^*(x)b] - [a, L^*(x)b] - R^*(R^*(x)a)b
- R^*(L^*o(x)a)b - R^*(R^*(x)b)a - R^*(L^*o(x)b)a - L^*(L^*o(x)a)b - L^*(R^*(x)a)b
- L^*(R^*(x)b)a = R^*(a o b) + R^*(b o a) + L^*(x)[a, b] - (R^*(x)a) o b o (R^*(x)a) - (L^*(x)a) o b o (L^*(x)a)
- a o (R^*(x)b) - (R^*(x)b)a - a o (L^*(x)b) - (L^*(x)b)a - a o (R^*(x)b)a - R^*(L^*o(a)x)b - R^*(L^*o(a)x)b
- R^*(L^*o(b)x)a - R^*(L^*o(b)x)a - R^*(L^*o(a)x)b - L^*(R^*(x)a)b - L^*(R^*(x)b)a - L^*(R^*(x)b)a
- L^*(R^*(x)b)a = 0.

The two first relations in brace on the last equality gives zero (see \((3.5)\)) and the last one brace also leads to zero (see \((3.0)\)).

Therefore, hold the equivalences. \(\square\)

4. Double constructions of symmetric (pre)Jacobi-Jordan algebras

In this section, we define and establish the double constructions of the symmetric JJ algebras and symmetric pre-JJ algebras.

**Definition 4.1.** We call \((\mathcal{J}, B)\) a double construction of a symmetric JJ algebra associated to \(\mathcal{J}_1\) and \(\mathcal{J}_1^*\) if it satisfies the conditions

1. \(\mathcal{J} = \mathcal{J}_1 \oplus \mathcal{J}_1^*\) as the direct sum of vector spaces;
2. \(\mathcal{J}_1\) and \(\mathcal{J}_1^*\) are JJ subalgebras of \(\mathcal{J}\);
3. \(B\) is the natural non-degenerate invariant symmetric bilinear form on \(\mathcal{J}_1 \oplus \mathcal{J}_1^*\) given by
   \[
   B(x + a^*, y + b^*) = (x, b^*) + (a^*, y) \quad (4.1)
   \]
   for all \(x, y \in \mathcal{J}_1, a^*, b^* \in \mathcal{J}_1^*\) where \((, )\) is the natural pair between the vector space \(\mathcal{J}_1\) and its dual space \(\mathcal{J}_1^*\).

**Theorem 4.2.** Let \((\mathcal{J}, \circ)\) be a JJ algebra. Suppose that there is a JJ algebra algebra structure "\(\circ\)" on its dual space \(\mathcal{J}^*\). Then, there is a double construction of a symmetric JJ algebra associated to \((\mathcal{J}, \cdot)\) and \((\mathcal{J}, \circ)\) if and only if \((\mathcal{J}, \mathcal{J}_1, L^*, L_1^*)\) is a matched pair of JJ algebras.
PROOF. From Theorem 2.10 we know that \((\mathcal{J} \oplus \mathcal{J}^*, \star)\) is a JJ algebra, where \(*\) is given by
\[
(x + a^\star)(y + b^\star) = x \circ y + L^*_x(b^\star) + L^*_y(a^\star) + b^\star \cdot a^\star \cdot b^\star + L^*_z(a^\star)y + L^*_y(b^\star)x
\] (4.2)
if and only if \((\mathcal{J}, \mathcal{J}^*, L^*_x, L^*_z)\). Furthermore, \(B\) is invariant with the product \(*\), that is \(B[(x + a^\star) \star (y + b^\star), (z + c^\star)] = B[(x + a^\star), (y + b^\star) \star (z + c^\star)]\). Indeed, we have
\[
B[(x + a^\star) \star (y + b^\star), (z + c^\star)] = \langle x \cdot y, c^\star \rangle + \langle c^\star \circ a^\star, y \rangle + \langle b^\star \circ c^\star, x \rangle
+ \langle a^\star \circ b^\star, z \rangle + \langle z \cdot x, b^\star \rangle + \langle y \cdot z, a^\star \rangle
= B[x + a^\star, (y + b^\star) \star (z + c^\star)]
\]
\(\square\)

DEFINITION 4.3. We call \((A, B)\) a double construction of a symmetric pre-JJ algebra associated to \(A_1\) and \(A_1^*\) if it satisfies the conditions
\begin{enumerate}
  \item \(A = A_1 \oplus A_1^*\) as the direct sum of vector spaces;
  \item \(A_1\) and \(A_1^*\) are JJ subalgebras of \(A\);
  \item \(B\) is the natural non-degenerate invariant symmetric bilinear form on \(A_1 \oplus A_1^*\) given by
\[
B(x + a^\star, y + b^\star) = \langle x, b^\star \rangle + \langle a^\star, y \rangle
\] (4.3)
for all \(x, y \in A_1, a^\star, b^\star \in A_1^*\) where \(\langle \cdot, \cdot \rangle\) is the natural pair between the vector space \(A_1\) and its dual space \(A_1^*\).
\end{enumerate}

THEOREM 4.4. Let \((A, \cdot)\) be a pre-JJ algebra. Suppose that there is a pre-JJ algebra structure \(\circ\) on its dual space \(A^*\). Then, there is a double construction of a symmetric pre-JJ algebra associated to \((A, \cdot)\) and \((A^*, \circ)\) if and only if \((A, A^*, R^*, L^*, R^*_0, L^*_0)\) is a matched pair of pre-JJ algebras.

Proof:
By considering that \((A, A^*, R^*, L^*: R^*_0, L^*_0)\) is a matched pair of pre-JJ algebras, it follows that the bilinear product \(*\) defined in the Theorem 4.3 is pre-JJ on the direct sum of underlying vector spaces, \(A \oplus A^*\). We have \(\forall x, y, z \in A; a, b, c \in A^*\).

\[
\mathcal{B}_A((x + a) \star (y + b), z + c) = (xy + R^*_x(a)y + L^*_y(b)x + \langle z, a \circ b + R^*_z(x)b + L^*_y(b)y \rangle a)
= \langle xy, c \rangle + \langle R^*_x(a)y, c \rangle + \langle L^*_y(b)x, c \rangle + \langle z, a \circ b \rangle + \langle z, R^*_z(x)b \rangle + \langle z, L^*_y(y)a \rangle
+ \langle L^*_x(x, c) \rangle + \langle R^*_z(x, a) \rangle + \langle z, a \circ b \rangle + \langle z, x, b \rangle + \langle y, z, a \rangle.
\]

\[
\mathcal{B}_A((x + a), (y + b) \star (z + c)) = \langle x, b \circ c + R^*_y(y)c + L^*_x(z)b \rangle + \langle y, z + R^*_x(b)z \rangle
+ \langle L^*_y(c)y, a \rangle + \langle x, b \circ c \rangle + \langle x, R^*_y(y)c \rangle + \langle x, L^*_z(x)b \rangle
+ \langle y, z, a \rangle + \langle z, R^*_b(a) \rangle + \langle y, L^*_c(a) \rangle
= \langle x, b \circ c \rangle + \langle x, c \rangle + \langle z, x, b \rangle + \langle y, z, a \rangle + \langle z, a \circ b \rangle + \langle y, c \circ a \rangle.
\]

Therefore, the following relation
\[
\mathcal{B}_A((x + a) \star (y + b), (z + c)) = \mathcal{B}_A((x + a), (y + b) \star (z + c))
\] (4.4)
holds, which expresses the invariance of the standard bilinear form on \(A \oplus A^*\). Therefore, \((A \oplus A^*, B)\) is the standard double construction of the pre-JJ algebras \(A\) and \(A^*\).

PROPOSITION 4.5. Let \((A, \cdot)\) be a pre-JJ algebra and \((A^*, \circ)\) be a pre-JJ algebra structure on its dual space \(A^*\). Then the following conditions are equivalent:
\begin{enumerate}
  \item \((A \oplus A^*, B)\) is the standard double construction of considered pre-JJ algebras;
  \item \((\mathcal{G}(A), \mathcal{G}(A^*), -ad^*, -ad^*_0)\) is a matched pair of sub-adjacent JJ algebras;
  \item \((A, A^*, R^*, L^*, R^*_0, L^*_0)\) is a matched pair of pre-JJ algebras.
\end{enumerate}
Proof:
From Theorem [3,9] (2) $\iff$ (3), while from Theorem [4,3] shows that (1) $\iff$ (3). Then (1) $\iff$ (2). $\square$

5. Computations in dimension two

In this section, we investigate the classification of 2-dimensional complex pre-JJ algebras and some double constructions.

Let $\mathcal{A}$ be a pre-JJ algebra such that there is a pre-JJ structure " $\circ$ " on its dual space $\mathcal{A}^*$ spanned by $\{e_1, e_2\}$ and $\{e_1^*, e_2^*\}$ respectively. Formula (2.12) leads to the following relations:

$$(e_i \cdot e_j) \cdot e_k + e_i \cdot (e_j \cdot e_k) = -(e_j \cdot e_i) \cdot e_k - e_j \cdot (e_i \cdot e_k), \quad \text{where} \quad i, j, k = 1, 2. \quad (5.1)$$

Let $e_1 \cdot e_1 = a_1 e_1 + a_2 e_2$, $e_1 \cdot e_2 = b_1 e_1 + b_2 e_2$, $e_2 \cdot e_1 = c_1 e_1 + c_2 e_2$, $e_2 \cdot e_2 = d_1 e_1 + d_2 e_2$ where $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{C}$.

**Proposition 5.1.** There are four non-isomorphic 2-dimensional pre-JJ algebras $\mathcal{A}$ given by the following:

$$e_i \cdot e_j = 0, \quad e_1 \cdot e_1 = e_2, \quad e_2 \cdot e_1 = e_2, \quad e_2 \cdot e_2 = e_1. \quad (5.2)$$

**Proof.** Let $\mathcal{A}$ be a 2-dimensional antiassociative algebra with basis $\{e_1, e_2\}$. Suppose $x, y, z \in \mathcal{A}$ such that $x = x_1 e_1 + x_2 e_2$, $y = y_1 e_1 + y_2 e_2$ and $z = z_1 e_1 + z_2 e_2$ with $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{C}$. By antiassociativity and ignoring the coefficients, we get the relations

$$(e_i \cdot e_j) \cdot e_k = -e_i \cdot (e_j \cdot e_k), \quad i, j, k = 1, 2.$$ 

Setting $e_1 e_1 = a_1 e_1 + a_2 e_2$, $e_1 e_2 = b_1 e_1 + b_2 e_2$, $e_2 e_1 = c_1 e_1 + c_2 e_2$ and $e_2 e_2 = d_1 e_1 + d_2 e_2$, in the previous relations one have the following eight relations equivalent to a system of 32 equations:

$$\begin{aligned}
2a_1^2 + a_2 e_1 + a_2 b_1 &= 0, \\
2a_1 a_2 + a_2 c_1 + a_2 b_2 &= 0, \\
a_1 b_1 + a_2 d_1 + b_1 a_1 + b_2 b_1 &= 0, \\
a_1 a_2 + a_2 d_2 + b_1 a_2 + b_2 &= 0, \\
b_1 a_1 + b_2 c_1 + c_1 a_1 + c_2 b_1 &= -c_1 a_1 - c_2 c_1 - a_1 c_1 - a_2 d_1, \\
b_1 a_2 + b_2 c_2 + c_1 a_2 + c_2 b_2 &= -c_2 c_2 - c_2 c_2 - a_1 c_2 - a_2 d_2, \\
b_2^2 + b_2 d_1 + d_1 a_1 + d_2 b_1 &= -c_1 b_1 - c_2 d_1 - b_1 c_1 - b_2 d_1, \\
b_1 b_2 + b_2 d_2 + d_1 a_2 + d_2 b_2 &= -c_1 b_2 - c_2 d_2 - b_1 c_2 - b_2 d_2, \\
c_1 a_1 + c_2 c_1 + a_1 c_1 + a_2 d_1 &= -a_1 b_1 - b_2 c_1 - a_1 c_1 - c_2 b_1, \\
c_1 a_2 + c_2^2 + a_1 c_2 + a_2 d_2 &= -b_1 a_2 - b_2 c_2 - a_1 c_2 - c_2 b_2, \\
c_1 b_1 + c_2 d_1 + b_1 c_1 + b_2 d_1 &= -b_2^2 - b_2 d_1 - b_1 a_1 - b_2 b_1, \\
c_1 b_2 + c_2 d_2 + b_1 c_2 + b_1 c_2 + b_2 d_2 &= -b_2 b_2 - b_1 b_2 - b_2 d_2 - d_1 a_2 - d_2 b_2, \\
d_1 a_1 + d_2 c_1 + c_2^2 + c_2 d_1 &= -d_1 a_1 - d_2 c_1 - c_2^2 - c_2 d_1, \\
d_1 a_2 + d_2 c_1 + c_1 c_2 + c_2 d_2 &= -d_1 a_2 - d_2 c_2 - c_1 c_2 - c_2 d_2, \\
d_1 b_1 + d_1 a_1 + d_1 c_1 + d_2 d_1 &= -d_1 b_1 - d_2 d_1 - d_1 c_1 - d_2 d_1, \\
d_1 b_2 + d_2^2 + d_1 c_2 + d_2^2 &= -d_1 b_2 - d_2^2 - d_1 c_2 - d_2^2.
\end{aligned} \quad (5.3)$$

For

$$a_1 = 0$$

$$a_2 = 0 \Rightarrow b_2 = 0, c_2 = 0, d_2 = 0, c_1 = 0, b_1 = 0$$

**class:** $e_2 e_2 = d_1 e_1$. 

For
$a_1 = 0$
$a_2 \neq 0$

$b_1 = 0 \Rightarrow c_1 = 0, d_1 = 0, d_2 = 0, c_2 = 0, b_2 = 0$

class : $e_1 e_1 = a_2 e_2$.

For

$a_1 = 0$

$d_1 \neq 0 \Rightarrow c_1 = 0, d_2 = 0, b_1 = 0, b_2 = 0$

class : $e_2 e_1 = c_2 e_2$.

The last class is the trivial one ie. $e_i e_j = 0$. The other cases lead to an absurdity. Therefore, by isomorphism we get the following four classes: $e_i e_j = 0$, $e_1 e_1 = e_2$, $e_2 e_1 = e_2$ and $e_2 e_2 = e_1$. □

Now, we discuss of the double constructions.

Case (I). $e_1 \cdot e_1 = e_2$. The considered product on the dual space is $e_2^* \circ e_2^* = e_1^*$. Using relation (3.7) when $l_A = R^*, r_A = L^*, l_B = l_A^*, r_B = r_A^* = L_B^*$, we obtain the double construction of pre-JJ algebra $(A \oplus A^*, *, B)$ associated to $(A, \cdot)$ and $(A^*, \circ)$ given explicitly
by the following relations:

\((e_1 + e_1^*) (e_1 + e_1^*) = (e_1 \cdot e_1 + R_0^c(e_1^*) e_1 + L_0^c(e_1^*) e_1) + (e_1^* \cdot e_1 + R_0^c(e_1^*) e_1 + L_0^c(e_1^*) e_1), \)
\[ = e_2, \]
\((e_1 + e_1^*) (e_1 + e_2^*) = (e_1 \cdot e_1 + R_0^c(e_2^*) e_1 + L_0^c(e_2^*) e_1) + (e_1^* \cdot e_2 + R_0^c(e_1^*) e_2 + L_0^c(e_1^*) e_1), \)
\[ = e_{2} + e_1^*, \]
\((e_1 + e_1^*) (e_2 + e_1^*) = (e_1 \cdot e_2 + R_0^c(e_1^*) e_2 + L_0^c(e_1^*) e_1) + (e_1^* \cdot e_1 + R_0^c(e_1^*) e_1 + L_0^c(e_1^*) e_1), \)
\[ = 0, \]
\((e_1 + e_1^*) (e_2 + e_2^*) = (e_1 \cdot e_2 + R_0^c(e_2^*) e_2 + L_0^c(e_2^*) e_1) + (e_1^* \cdot e_2 + R_0^c(e_1^*) e_2 + L_0^c(e_1^*) e_1), \)
\[ = 2e_2 + e_1^*, \]
\((e_1 + e_1^*) (e_1 + e_2^*) = (e_1 \cdot e_1 + R_0^c(e_1^*) e_1 + L_0^c(e_1^*) e_1) + (e_1^* \cdot e_2 + R_0^c(e_1^*) e_1 + L_0^c(e_1^*) e_1), \)
\[ = 2e_2, \]
\((e_1 + e_2^*) (e_1 + e_1^*) = (e_1 \cdot e_1 + R_0^c(e_2^*) e_1 + L_0^c(e_2^*) e_1) + (e_1^* \cdot e_1 + R_0^c(e_1^*) e_1 + L_0^c(e_1^*) e_1), \)
\[ = 2e_2, \]
\((e_1 + e_2^*) (e_2 + e_1^*) = (e_1 \cdot e_2 + R_0^c(e_1^*) e_2 + L_0^c(e_1^*) e_1) + (e_2^* \cdot e_1 + R_0^c(e_2^*) e_1 + L_0^c(e_2^*) e_1), \)
\[ = 3e_2 + 3e_1^*, \]
\((e_2 + e_1^*) (e_2 + e_1^*) = (e_2 \cdot e_1 + R_0^c(e_1^*) e_1 + L_0^c(e_1^*) e_1) + (e_1^* \cdot e_1 + R_0^c(e_2^*) e_1 + L_0^c(e_2^*) e_1), \)
\[ = e_2, \]
\((e_2 + e_1^*) (e_1 + e_1^*) = (e_2 \cdot e_1 + R_0^c(e_1^*) e_1 + L_0^c(e_1^*) e_1) + (e_2^* \cdot e_1 + R_0^c(e_2^*) e_1 + L_0^c(e_2^*) e_1), \)
\[ = 0, \]
\((e_2 + e_2^*) (e_1 + e_1^*) = (e_2 \cdot e_1 + R_0^c(e_2^*) e_1 + L_0^c(e_2^*) e_1) + (e_2^* \cdot e_1 + R_0^c(e_2^*) e_1 + L_0^c(e_2^*) e_1), \)
\[ = e_2 + e_1^*, \]
\((e_2 + e_2^*) (e_1 + e_2^*) = (e_2 \cdot e_1 + R_0^c(e_2^*) e_1 + L_0^c(e_2^*) e_1) + (e_2^* \cdot e_2 + R_0^c(e_1^*) e_1 + L_0^c(e_1^*) e_1), \)
\[ = e_2 + e_1^*, \]
\((e_2 + e_2^*) (e_2 + e_1^*) = (e_2 \cdot e_2 + R_0^c(e_2^*) e_2 + L_0^c(e_2^*) e_2) + (e_2^* \cdot e_2 + R_0^c(e_2^*) e_2 + L_0^c(e_2^*) e_2), \)
\[ = 0, \]
\((e_2 + e_2^*) (e_2 + e_2^*) = (e_2 \cdot e_2 + R_0^c(e_2^*) e_2 + L_0^c(e_2^*) e_2) + (e_2^* \cdot e_2 + R_0^c(e_2^*) e_2 + L_0^c(e_2^*) e_2), \)
\[ = e_1^*, \]
\((e_1 + e_2^*) (e_1 + e_2^*) = (e_1 \cdot e_2 + R_0^c(e_1^*) e_2 + L_0^c(e_1^*) e_1) + (e_2^* \cdot e_2 + R_0^c(e_1^*) e_2 + L_0^c(e_1^*) e_2), \)
\[ = 0, \]
\((e_1 + e_2^*) (e_1 + e_2^*) = (e_1 \cdot e_2 + R_0^c(e_2^*) e_2 + L_0^c(e_2^*) e_1) + (e_2^* \cdot e_2 + R_0^c(e_1^*) e_2 + L_0^c(e_1^*) e_2), \)
\[ = e_2 + 2e_1^*, \]
\((e_2^* + e_1^*) (e_2^* + e_2^*) = (e_2 \cdot e_2 + R_0^c(e_1^*) e_2 + L_0^c(e_1^*) e_2) + (e_1^* \cdot e_2 + R_0^c(e_2^*) e_2 + L_0^c(e_2^*) e_2), \)
\[ = 0, \]
\((e_2^* + e_1^*) (e_2^* + e_2^*) = (e_2 \cdot e_2 + R_0^c(e_1^*) e_2 + L_0^c(e_1^*) e_2) + (e_1^* \cdot e_2 + R_0^c(e_2^*) e_2 + L_0^c(e_2^*) e_2), \)
\[ = 0, \]
\((e_2^* + e_1^*) (e_2^* + e_2^*) = (e_2 \cdot e_2 + R_0^c(e_1^*) e_2 + L_0^c(e_1^*) e_2) + (e_1^* \cdot e_2 + R_0^c(e_2^*) e_2 + L_0^c(e_2^*) e_2), \)
\[ = 0. \]

Case(II). \(e_2 \cdot e_1 = e_2\). The product on the dual space is given by: \(e_i^* \cdot e_j^* = 0, \quad i, j = 1, 2\). The double construction of pre-JJ algebra \((A \oplus A^*, *, B)\) associated to \((A, \cdot)\) and \((A^*, \circ)\) given
explicitly by the following relations:

\[(e_1 + e_1^*) \cdot (e_1 + e_1^*) = 0,\]
\[(e_1 + e_1^*) \cdot (e_1 + e_2^*) = e_2^*,\]
\[(e_1 + e_1^*) \cdot (e_2 + e_1^*) = 0,\]
\[(e_1 + e_1^*) \cdot (e_2 + e_2^*) = e_2^*,\]
\[(e_1 + e_1^*) \cdot (e_1 + e_1^*) = e_2^*,\]
\[(e_1 + e_1^*) \cdot (e_1 + e_2^*) = e_2,\]
\[(e_1 + e_1^*) \cdot (e_2 + e_1^*) = e_2^*,\]
\[(e_2 + e_1^*) \cdot (e_1 + e_2^*) = e_2,\]
\[(e_2 + e_1^*) \cdot (e_1 + e_1^*) = e_2,\]
\[(e_2 + e_2^*) \cdot (e_1 + e_2^*) = e_1^*,\]
\[(e_2 + e_2^*) \cdot (e_1 + e_1^*) = e_1^*,\]
\[(e_2 + e_1^*) \cdot (e_2 + e_1^*) = e_2,\]
\[(e_2 + e_1^*) \cdot (e_2 + e_1^*) = e_2^*,\]
\[(e_2 + e_1^*) \cdot (e_1 + e_2^*) = 0,\]
\[(e_2 + e_1^*) \cdot (e_2 + e_1^*) = 0.\]

Case (III). \(e_2 \cdot e_2 = e_1\). The product on the dual space is given by the following relations: \(e_2^* \circ e_1^* = e_2^*\). The double construction of pre-JJ algebra \((A \oplus A^*, \circ, B)\) associated to \((A, \cdot)\) and \((A^*, \circ)\) given explicitly by the following relations:

\[(e_1 + e_1^*) \cdot (e_1 + e_1^*) = 0,\]
\[(e_1 + e_1^*) \cdot (e_1 + e_2^*) = 0,\]
\[(e_1 + e_1^*) \cdot (e_2 + e_1^*) = e_2 + e_2^*,\]
\[(e_1 + e_1^*) \cdot (e_2 + e_2^*) = e_2 + e_2^*,\]
\[(e_1 + e_2^*) \cdot (e_1 + e_1^*) = e_2^*,\]
\[(e_1 + e_2^*) \cdot (e_1 + e_2^*) = e_2^*,\]
\[(e_1 + e_2^*) \cdot (e_2 + e_1^*) = 0,\]
\[(e_2 + e_1^*) \cdot (e_1 + e_1^*) = e_2^*,\]
\[(e_2 + e_1^*) \cdot (e_1 + e_2^*) = e_1 + e_2^*,\]
\[(e_2 + e_1^*) \cdot (e_2 + e_2^*) = 2e_2^*,\]
\[(e_2 + e_1^*) \cdot (e_1 + e_1^*) = 2e_2^*,\]
\[(e_2 + e_2^*) \cdot (e_1 + e_1^*) = e_1,\]
\[(e_2 + e_2^*) \cdot (e_1 + e_2^*) = e_1 + 2e_2^*,\]
\[(e_2 + e_2^*) \cdot (e_2 + e_1^*) = 2e_1,\]
\[(e_1 + e_2^*) \cdot (e_2 + e_2^*) = e_2,\]
\[(e_1 + e_2^*) \cdot (e_1 + e_1^*) = e_2^*,\]
\[(e_1 + e_2^*) \cdot (e_2 + e_1^*) = 0,\]
\[(e_2 + e_1^*) \cdot (e_2 + e_1^*) = e_1 + e_2^*,\]
\[(e_2 + e_1^*) \cdot (e_2 + e_2^*) = 2e_1 + e_2 + 2e_2^*.

Remark 5.2. Let \((A, \cdot)\) be a one dimensional pre-JJ algebra with a basis \(\{e_1\}\). We obtain the trivial class \(e_1 \cdot e_j = 0\).
6. Concluding remarks

In this work, we defined pre-JJ algebras and, discussed their bimodule and matched pair. We established the double construction of JJ algebras and pre-JJ algebras. Finally, we described some double contructions of symmetric pre-JJ algebras in dimension two.

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