Experimental investigation of convex mixtures of Markovian and Non-Markovian single qubit channels on NISQ devices

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Noisy Intermediate Scale Quantum (NISQ) devices have been proposed as a versatile tool for simulating open quantum systems. Recently, the use of NISQ devices as simulators for non-Markovian open quantum systems has helped verify the current descriptions of non-Markovianity in quantum physics. In this work, convex mixtures of channels are simulated using NISQ devices and classified as either Markovian or non-Markovian using the CP-divisibility criteria. Two cases are considered: two Markovian channels being convexly mixed to form a non-Markovian channel and vice versa. This work replicates the experiments performed in a linear optical setup, using NISQ devices, with the addition of a convex mixture of non-Markovian channels that was designed to address some of the problems faced in the experiments performed in the linear optical setup. The results obtained show that, using NISQ devices and within some error, convex mixtures of Markovian channels lead to a non-Markovian channel and vice versa.

I. INTRODUCTION

The theory of open quantum systems describes the dynamics of a quantum system that interacts with its surrounding environment [1,2]. It provides the tools to understand processes such as dissipation and decoherence [1,2] that are responsible for noise in quantum technologies. By studying open quantum systems, insights into minimizing the effects of noise in quantum technologies are gained. These insights, when applied, improve the performance of the quantum technologies in their various applications.

The dynamics of an open quantum system is described by a master equation. The solution of this master equation is a dynamical map, also known as a quantum channel, which describes the evolution of an open quantum system. Under certain assumptions, such as the Born-Markov approximation, the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) form of the master equation can be derived [3,4]. The GKSL form of the master equation describes Markovian dynamics, where all memory effects are neglected. Non-Markovian dynamics differ from Markovian dynamics in that it allows for information to flow back into the system from the environment and it does not neglect memory effects [5]. While everyone agrees that the time-independent GKSL generator with non-negative damping constants describes the quantum Markov process, there are several approaches to defining the non-Markovianity in quantum physics [6,9]. This makes the study of non-Markovianity in quantum physics a highly non-trivial problem.

There has been much interest in the study of the non-Markovian dynamics of an open quantum system [6,8,11,15]. Recently, simulations of open quantum systems have been used to further understand non-Markovian dynamics [16,17].

Due to the growing accessibility of NISQ computers through the cloud from platforms such as the IBM Quantum Experience (IBM QE), there has also been much interest in the use of NISQ devices in simulating quantum systems [18-20]. The IBM QE has been proposed as a versatile tool for the simulation of open quantum systems [17].

Using a linear optical setup, Uriri et al. [16] simulated convex mixtures of two Markovian channels leading to a non-Markovian channel (M + M = nM) and convex mixtures of two non-Markovian channels leading to a Markovian channel (nM + nM = M). However, in [10], the intended Markovian channels could not be definitively verified as Markovian due to experimental error and channel design challenges. Utilizing a general approach to the simulation of the open quantum systems on NISQ devices implemented in [17], this work will focus on the quantum simulation of the convex mixtures of quantum channels. Throughout this work, the CP-divisibility criteria [7] is used to classify a channel as either Markovian and non-Markovian.

The result of this work is to show that, through simulations using NISQ devices, convex mixtures of Markovian channels can lead to a non-Markovian channel (M + M = nM) and vice versa (nM + nM = M). Furthermore, for the convex mixtures from [16], our simulations yielded better approximations of the intended channels than the simulations in [16]. For our designed convex mixture of two non-Markovian channels, the resulting channel obtained could be more definitively verified as Markovian.

This paper is outlined as follows: in section II we provide some background and theory, in section III the method for simulating and characterizing the channels as Markovian or non-Markovian is discussed, section IV outlines how the simulated channels were theoretically constructed, section V contains the results and discussion and in section VI we summarize our findings and make concluding remarks.
II. BACKGROUND AND THEORY

The dynamics of open quantum systems are usually described by a dynamical map $\Lambda_t$ where $t \geq 0$ and $\Lambda_0 = \mathbb{1}$ i.e. a family of single parameter completely positive (CP) maps. If $\rho(0)$ is the initial state of the system then $\rho(t) = \Lambda_t \rho(0)$ represents the density operator at some time $t$. A dynamical map is also referred to as a quantum channel, these shall be used interchangeably throughout this work. One can assume that the map $\Lambda_t$, in most practical cases, satisfies the time-local master equation:

$$\frac{d}{dt} \rho(t) = \mathcal{L}(t) \rho(t) \Leftrightarrow \frac{d}{dt} \Lambda_t = \mathcal{L}(t) \Lambda_t,$$  

(1)

where $\mathcal{L}(t)$ is the time-local generator and has the known form:

$$\mathcal{L}(t) \rho = -i[\hat{\mathcal{H}}(t), \rho] + \sum_k \gamma_k(t) \left( \hat{V}_k(t) \rho \hat{V}_k^\dagger(t) - \frac{1}{2} \{\hat{V}_k^\dagger(t) \hat{V}_k(t), \rho\} \right),$$  

(2)

where $\hat{\mathcal{H}}(t)$ is the time dependent Hamiltonian, $\gamma_k(t)$ and $\hat{V}_k(t)$ are the time dependent decay rates and noise operators respectively. The form of the generator in (2) is very general and gives both Markovian and non-Markovian dynamics. The approach which shall be outlined in more detail in the next section, in studying the Markovianity of a dynamical map as Markovian or non-Markovian [7], which shall be used will be the quantum maps and mas- ters. We are interested in classifying dynamical maps as either Markovian or non-Markovian. The approach we are interested in classifying dynamical maps as Markovian or non-Markovian. We shall use the CP divisibility criteria to characterize the channel $\Lambda_t$ as Markovian or non-Markovian. We start by writing the dynamical map $\Lambda_t$ in the following way:

$$\Lambda_t = V_{t,s} \mathbb{1},$$  

(5)

where $V_{t,s}$ is called the intermediate map (IM) from time $s$ to $t$. The maps $V_{t,s}$ form a family of two parameter propagators. We say that $\Lambda_t$ is CP divisible if, $V_{t,s}$ is completely positive (CP) for all $t \geq s \geq 0$. Now the goal is to check that the map $V_{t,s}$ is CP, we will do this by using the techniques in [16]. It is known that a quantum channel $\Lambda_t$ has a Kraus representation [33]:

$$\Lambda_t \rho = \sum_K \tilde{K}_\alpha \rho \tilde{K}_\alpha^\dagger,$$  

(6)

where $\tilde{K}_\alpha$ are the Kraus operators that satisfy $\sum_K \tilde{K}_\alpha^\dagger \tilde{K}_\alpha = 1$. In this work we will consider the case of a single qubit channel then the Kraus operators are $2 \times 2$ matrices. If we choose a complete basis for the Kraus operators of a single qubit channel as $\{\sigma_0 = \mathbb{1}, \sigma_1, \sigma_2, \sigma_3\}$,
where \( \sigma_i \) are the usual Pauli matrices. Then we can expand the Kraus operators in terms of this basis to get the \( \chi \) matrix representation of the quantum channel for a single qubit:

\[
\Lambda_t \rho = \sum_{m,n=0}^{3} \chi_{mn} \sigma_m \rho \sigma_n. \tag{7}
\]

Here \( \chi_{mn} \) is a positive and Hermitian \( 4 \times 4 \) matrix called the \( \chi \) matrix and shall be determined using a quantum process tomography \[30\] \[34\]. Now if we know the \( \chi \) Matrix then we have a complete description of the channel \( \Lambda_t \).

To determine the elements of the \( \chi \) matrix we need to choose a complete set of input states, we choose the states \( \{|0\rangle, |1\rangle, |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |+_y\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)\} \). These states form a complete set as the projectors constructed from each ket vector in this set can be used to construct the density operator of any physical single qubit state. Now we send these input states through the channel \( \Lambda_t \). We can prepare the initial state of the qubit as each of the input states. Using quantum state tomography we reconstruct the state after each input state is passed through the channel \[35\]. Then the formulas from \[30\] \[34\] are used to construct the \( \chi \) matrix which allows us to reconstruct the channel \( \Lambda_t \).

To simulate the quantum channels using a quantum computer we need to perform a process tomography on a quantum computer. We start by constructing the circuits to simulate the quantum channel for a single qubit \[17\]. The system qubit is prepared in one of the input states above and is then passed through the channel. We then perform the state tomography on the system qubit for each input state and get the corresponding counts. Using the counts obtained from the tomographic circuits we can construct the \( \chi \) matrix \[34\]. The \( \chi \) matrix obtained, using the results from the quantum circuits that were run on the quantum computer, will not be positive and Hermitian. This is due to the fact that we can only make a finite number of measurements on the system qubit.

To correct this we shall use Maximum Likelihood Estimation (MLE) to find the closest possible \( \chi \) matrix, denoted \( \chi_c \), to the ideal \( \chi \) matrix which we denote as \( \chi_{\text{id}} \) \[34\]. Refer to Appendix A and B for more information about MLE technique used to obtain the positive and Hermitian \( \chi \) matrices for the channels and the quantum circuits used to simulate the channels. Now \( \chi_c \) can be used to reconstruct \( \Lambda_t \).

With a knowledge of the \( \chi \) matrices for two different time duration’s \( s \) and \( t \) we can check whether the map \( V_{t,s} \), that evolves the system from time \( s \) to time \( t \), is completely positive. To check this, we make use of the transfer matrices \( F(s) \) and \( F(t) \) for the maps \( \Lambda_s \) and \( \Lambda_t \) respectively. The transfer matrix \( F(t) \) of a map \( \Lambda_t \) is a concrete matrix representation of the map in a given orthonormal basis \[31\]. The transfer-matrix approach is useful as it allows us to represent the density matrix \( \rho \) as a stacked vector \( |\rho\rangle \), now the evolution of the vector \( |\rho(0)\rangle \) can be written as \( |\rho(t)\rangle = F(t)|\rho(0)\rangle \) which is nothing more than the action of the transfer matrix on the stacked vector \[10\]. The elements of a transfer matrix \( F(t) \) are given explicitly as:

\[
F_{\alpha,\beta}(t) = \text{Tr}[G^\alpha_{\beta} \Lambda_t G^\beta]\tag{8}
\]

where \{\( G^\alpha \)\} are a set of orthonormal operators with respect to the Hilbert-Schmidt inner product \[31\]. We choose the set \{\( G^\alpha \)\} to be the standard matrix basis of \( \mathcal{M}_2(\mathbb{C}) \) i.e. \{\( G_1 = |0\rangle \langle 0|, G_2 = |0\rangle \langle 1|, G_3 = |1\rangle \langle 0|, G_4 = |1\rangle \langle 1|\}\), where \{\( |0\rangle, |1\rangle\}\} are the standard computational basis vectors of the single qubit. If we know the \( \chi \) matrix for some time \( t \) we can use this to calculate \( \Lambda_t G_\beta \) and hence calculate \( F(t) \). Now using equation (5) and writing it in terms of transfer matrices we arrive at, \( F(t) = F(t,s)F(s) \). This tells us that if we have the transfer matrix for two times \( t \) and \( s \) we can get the transfer matrix of the intermediate map i.e. \( F(t,s) = F(t)F^{-1}(s) \). For a given transfer matrix \( F(t) \) we can obtain the Choi matrix \( W(t) \) for a single qubit this can be written as:

\[
W(t) = \frac{1}{2} \sum_{\alpha,\beta=-1}^{4} F_{\alpha,\beta}(t)(G^\beta \otimes G^\alpha)\tag{9}
\]

This is derived by applying \( \Lambda_t \) to a single qubit of the maximally entangled state \( |\beta_00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \), hence \( W(t) = (1 \otimes \Lambda_t)|\beta_00\rangle/|\beta_00\rangle \). By the Choi-Jamiolkowski isomorphism, a dynamical map \( \Lambda_t \) is completely positive if and only if the corresponding Choi matrix of the map is positive \[30\] \[37\]. This tells us that the map \( \Lambda_t \) is CP if all the eigenvalues of the Choi matrix \( W(t) \) are non-negative. So for two times \( s \) and \( t \) such that \( t \geq s \geq 0 \), if the eigenvalues of the Choi matrix \( W(t,s) \), of the intermediate map \( V_{t,s} \), are non-negative and \( ||W(t,s)|| = 1 \) (since \( \Lambda_t \) is TP) then the intermediate map is completely positive and by our definition of CP divisibility this tells us that the dynamical map \( \Lambda_t = V_{t,s} \Lambda_s \) is Markovian. Any deviation from this leads to non-Markovian dynamics.

**IV. THEORETICAL DESIGN OF CHANNELS**

In the previous section we outlined a method to characterize a quantum channel as either Markovian or Non-Markovian. In this section we shall theoretically analyse the channels we aim to simulate. We first need to consider the type of channels we will simulate, we shall consider single qubit pauli channels. These are very simple channels and will make the simulation and analysis very easy. The single qubit Pauli channel is of the form:

\[
\Lambda_t \rho = \sum_{\alpha=0}^{3} p_\alpha(t) \sigma_\alpha \rho \sigma_\alpha, \tag{10}
\]

where \( \sigma_0 = 1 \) and \( p_\alpha(t) \) is a time dependent probability distribution such that \( p_0(0) = 1 \) and \( p_i(0) = 0 \) for
\( i \in \{1, 2, 3\} \) and \( \sum_{i=0}^{3} p_i(t) = 1 \) for all \( t \geq 0 \). This channel was studied extensively in [14], it should also be noted that this channel also falls into a larger class of channels called random unitary channels [13]. To make a statement about whether the Pauli channel is Markovian or not we need to analyse the decay rates of the generator of the Pauli channel. The generator of the Pauli channel in equation (10) is,

\[
\mathcal{L}(t)\rho = \sum_{k=1}^{3} \gamma_k(t)(\sigma_k \rho \sigma_k - \rho),
\]

where \( \gamma_k(t) \), are the decay rates and have the form,

\[
\gamma_k(t) = \frac{1}{4} \sum_{\beta=0}^{3} H_{k\beta} \left\{ \frac{\sum_{\nu=0}^{3} H_{\beta\nu} \rho(t)}{\sum_{\sigma=0}^{3} H_{\beta\sigma} \rho(t)} \right\},
\]

for \( k = 1, 2, 3 \). For more information on derivation of the generator of the pauli channel as well as its decay rates refer to appendix C and [11]. Now from [11, 13, 14] we use the following result. The pauli channel (10) is Markovian (i.e \( \Lambda_t \) is divisible) if and only if \( \gamma_k(t) \geq 0 \) for all \( t \geq 0 \) and for \( k = 1, 2, 3 \). This tells us that if the decay rates of the generator of a channel are non-negative then that channel is Markovian and any deviation from this leads to a the channel becoming non-Markovian. We know that from the time local generator \( \mathcal{L}(t) \) we can get the channel \( \Lambda_t \) from the relation,

\[
\Lambda_t = \mathcal{T} e^{\int dt \mathcal{L}(t)}
\]

where \( \mathcal{T} \) is the chronological time ordering operator. Now we note that a linear combinations of Markovian generators \( \mathcal{L}_1(t) \) and \( \mathcal{L}_2(t) \) i.e. \( \alpha_1 \mathcal{L}_1(t) + \alpha_2 \mathcal{L}_2(t) \) with \( \alpha_1, \alpha_2 \geq 0 \), is also a Markovian generator. Hence Markovian generators form a convex set in the space of admissible generators. The same is not true on the channel level, since for two Markovian channels \( \Lambda^{(1)}_t \) and \( \Lambda^{(2)}_t \) it is not always the case that there convex linear combination will be Markovian i.e. \( \eta \Lambda^{(1)}_t + (1-\eta) \Lambda^{(2)}_t \), \( \eta \in [0, 1] \), is not necessarily Markovian [12]. We can now condiser the two cases of convex mixing of channels outlined in section II i.e. \( \text{M+M=nM and nM+nM=M} \). We shall find pauli channels that satisfy these two cases.

\subsection*{A. Markovian channel addition (M+M=nM)}

The goal is to find two Markovian channels that when convexly combined yield a non-Markovian channel, we shall use the channels from [16] that demonstrate this. We start by defining the following two channels:

\[
\Lambda^{(1)}_t \rho = p(t)\rho + (1-p(t))\sigma_1 \rho \sigma_1
\]

\[
\Lambda^{(2)}_t \rho = p(t)\rho + (1-p(t))\sigma_2 \rho \sigma_2
\]

where the probability \( p(t) = \frac{1+e^{-t}}{2} \), refer to Fig. 1 to see a plot of this function. Using equation (11) and (12) above we can find the generators of these channels. We find that for channels \( \Lambda^{(1)}_t = e^{t\mathcal{L}_1} \) and \( \Lambda^{(2)}_t = e^{t\mathcal{L}_2} \) the generators are given by:

\[
\mathcal{L}_1 \rho = \frac{1}{2}(\sigma_1 \rho \sigma_1 - \rho), \quad \mathcal{L}_2 \rho = \frac{1}{2}(\sigma_2 \rho \sigma_2 - \rho).
\]

Since the decay rates in both these generators are non-negative for all \( t \geq 0 \) then the channels in equation (14) and (15) are Markovian. We can now consider the convex combination of these channels in the following way:

\[
\Lambda^{(T)}_t \rho = \frac{1}{2} \Lambda^{(1)}_t \rho + \frac{1}{2} \Lambda^{(2)}_t \rho = \frac{1+e^{-t}}{2} \rho + \frac{1-e^{-t}}{4}(\sigma_1 \rho \sigma_1 + \sigma_2 \rho \sigma_2)
\]

where \( \Lambda^{(T)}_t \) is the total channel. Now for this total channel the generator is:

\[
\mathcal{L}_T(t)\rho = \frac{1}{4}(\sigma_1 \rho \sigma_1 - \rho) + \frac{1}{4}(\sigma_2 \rho \sigma_2 - \rho) - \frac{\tanh(t/2)}{4}(\sigma_3 \rho \sigma_3 - \rho)
\]

Since the decay rate \( \gamma_3(t) < 0 \) in equation (18), the total channel is non-Markovian more so this channel is eternally non-Markovian [11, 12]. Now that we have an example of the addition of two Markovian channels using the techniques in [17] we can construct quantum circuits to simulate these channels. Refer to Appendix B for the quantum circuit diagrams that simulate these channels.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Plot showing the probability \( p(t) = \frac{1+e^{-t}}{2} \) for the channels used in Markovian channel addition (M+M=nM).}
\end{figure}

\subsection*{B. Non-Markovian Channel Addition (nM+nM=M)}

The goal now is to find an example of non-Markovian channels that when convexly mixed yield a Markovian
channel. We shall first make use of the channels from [10] that demonstrate this in the linear optical setup. It will then become clear from the results in the next section that the channels from [10], have the problem that the minimum eigenvalue of the Choi matrix of the intermediate map for the total Markovian channel is zero, making it hard to definitively show Markovianity. We shall address this issue by designing our own channels that attempt to solve the problem of the zero eigenvalue.

Firstly let us examine the channels in [10]. We consider the following two channels:

\[ \Lambda_t^{(1)} \rho = p_1(t) \rho + (1-p_1(t)) \sigma_1 \rho \sigma_1 \]  
\[ \Lambda_t^{(2)} \rho = p_2(t) \rho + (1-p_2(t)) \sigma_1 \rho \sigma_1 \]

where the probabilities \( p_1(t) \) and \( p_2(t) \) are given respectively by:

\[ p_1(t) = \frac{3}{2} \left( 1 + e^{-t} - \frac{\cos^2(t)}{3} \right), \quad p_2(t) = \cos^2(t). \]

Refer to Fig. 2 to see these functions plotted. Now we can construct the generators for the channels above,

\[ \mathcal{L}_1(t) \rho = \frac{8}{1} \sin(t) \cos(t) - \frac{6}{4} (\sigma_1 \rho \sigma_1 - \rho), \]
\[ \mathcal{L}_2(t) \rho = \tan(2t) (\sigma_1 \rho \sigma_1 - \rho). \]

For the decay rate in \( \mathcal{L}_1(t) \) it is negative for certain values of \( t \) and hence \( \Lambda_t^{(1)} \) is globally non-Markovian. Similarly the decay rate in \( \mathcal{L}_2(t) \) is periodic and negative in some time interval making \( \Lambda_t^{(2)} \) globally non-Markovian. Now we can look at a convex combination of \( \Lambda_t^{(1)} \) and \( \Lambda_t^{(2)} \) to get the total channel:

\[ \Lambda_t^{(T)} \rho = \frac{2}{3} \Lambda_t^{(1)} \rho + \frac{1}{3} \Lambda_t^{(2)} \rho = \frac{1 + e^{-t}}{2} \rho + \frac{1 - e^{-t}}{2} (\sigma_1 \rho \sigma_1). \]

We know from the Markovian channel addition that the generator for this channel is,

\[ \mathcal{L}_T \rho = \frac{1}{2} (\sigma_1 \rho \sigma_1 - \rho). \]

We know that this channel is Markovian, so we now have an example of convex mixing of non-Markovian channels leading to a Markovian channel [12][16]. Refer to Appendix B for the circuits that simulate these channels. Now to solve the problem of the zero eigenvalue we need to consider a special case of the Pauli channel in equation (10) i.e. the depolarizing channel [17]. We can obtain the depolarizing channel from equation (10) by parameterizing the probabilities \( p_\alpha(t) \) as follows:

\[ \Lambda_t \rho = (1 - \frac{3p(t)}{4}) \rho + \frac{p(t)}{4} \sum_{\alpha=1}^{3} \sigma_\alpha \rho \sigma_\alpha \]

We can obtain the total Markovian channel addition (nM+nM=M) experiment in [16].

where \( 0 \leq p(t) \leq 1 \) for all times \( t \geq 0 \). Now from the fact that \( \Lambda_0 = 1 \) we see that \( p(0) = 0 \). The decay rates for the depolarizing channel can now be calculated using equation (12) as:

\[ \gamma_k(t) = \frac{\dot{p}(t)}{4(1-p(t))} \]

for \( k=1,2,3 \). From equation (26) we can tighten the bounds on \( p(t) \) i.e. \( 0 \leq p(t) < 1 \). Now the form of the decay rate in equation (26) tells us that for the channel to be Markovian the function \( p(t) \) should satisfy \( \dot{p}(t) \geq 0 \) for all \( t \geq 0 \) and for the channel to be non-Markovian there should exist some time \( t' \geq 0 \) such that \( \dot{p}(t') \leq 0 \).

Now let us consider the following two individual channels:

\[ \Lambda_t^{(1)} \rho = (1 - \frac{3q(t)}{4}) \rho + \frac{q(t)}{4} \sum_{\alpha=1}^{3} \sigma_\alpha \rho \sigma_\alpha \]
\[ \Lambda_t^{(2)} \rho = (1 - \frac{3r(t)}{4}) \rho + \frac{r(t)}{4} \sum_{\alpha=1}^{3} \sigma_\alpha \rho \sigma_\alpha. \]

The decay rates for both the individual channels in equation (27) are:

\[ \gamma_k^{(1)}(t) = \frac{\dot{q}(t)}{4(1-q(t))} \], \[ \gamma_k^{(2)}(t) = \frac{\dot{r}(t)}{4(1-r(t))} \]

for \( k=1,2,3 \). Taking a convex combination of the individual channels in equation (27) we obtain the total channel:

\[ \Lambda_t^{(T)} \rho = \eta \Lambda_t^{(1)} \rho + (1-\eta) \Lambda_t^{(2)} \rho \]
\[ = (1 - \frac{3}{4} w(t)) \rho + \frac{w(t)}{4} \sum_{\alpha=1}^{3} \sigma_\alpha \rho \sigma_\alpha \]

where \( \eta \in [0,1] \) and \( w(t) = \eta q(t) + (1-\eta) r(t) \). Now the decay rates for the total channel are:
It is clear from equation (30) that it is possible for there to exist times $t'$ and $t''$ such that $\dot{q}(t') < 0$ and $\dot{r}(t'') < 0$ and $w(t) \geq 0$ for all times $t$. This tells us that it is possible using the depolarizing channel in equation (25) to show that we can convexly mix two non-Markovian channels to get a total channel that is Markovian. The goal now would be to pick functions $q(t)$ and $r(t)$ such that $\Lambda_{t}^{(1)}$ and $\Lambda_{t}^{(2)}$ are non-Markovian and the total channel $\Lambda_{t}^{(T)}$ is Markovian. We observe that if we parameterize $q(t)$ and $r(t)$ as:

$$q(t) = a(t) + b(t)$$

$$r(t) = a(t) - b(t)$$

and setting $\eta = \frac{1}{2}$ in equation (29), then using the bounds on $q(t)$ and $r(t)$ we get $b(t) \leq a(t) < 1 - b(t)$. Taking into consideration all the constraints outlined above we choose the functions $a(t)$ and $b(t)$ as follows:

$$a(t) = \frac{0.5}{1 + \exp(-4(t - 2))} + \frac{0.48}{1 + \exp(-4.5(t - 6))}$$

$$b(t) = 0.49\exp(-(t - 4)^6).$$

Refer to Fig. 3 for the plots of the functions given in equation (31) above. From equation (28) we see that the decay rates for the individual channels $\Lambda_{t}^{(1)}$ and $\Lambda_{t}^{(2)}$ are non-Markovian. This is because for some time interval $\dot{q} < 0$ and for some other interval of time $\dot{r} < 0$ making the decay rates negative leading to the channels being non-Markovian. Fig. 4 shows the plots of $\dot{q}(t)$ and $\dot{r}(t)$ which shows that both are negative for some times. The total channel $\Lambda_{t}^{(T)}$ is parameterized by the function $w(t)$ and it is clear from Fig. 5 that $\dot{w}(t) \geq 0$ for all times $t \geq 0$, so the total channel $\Lambda_{t}^{(T)}$ is Markovian. Refer to Appendix D for more intuition about how the functions $q(t)$ and $r(t)$ were chosen. Hence we have found an example of the convex sum of two non-Markovian channels $\Lambda_{t}^{(1)}$ and $\Lambda_{t}^{(2)}$ yielding a Markovian total channel $\Lambda_{t}^{(T)}$.

V. RESULTS AND DISCUSSION

In both Markovian channel addition and non-Markovian channel addition, we use the circuits shown in Appendix B to implement the channels. The tomographic circuits needed to do a process tomography are created, using the circuits from Appendix B, and a process tomography is performed on the channels for each time step in the interval $t \in [0, 3.8]$ for the Markovian channel addition ($M+M=M$), the time interval $t \in [0, 3.7]$ for the replicated non-Markovian channel addition ($nM+nM=M$) from [16] and the time interval $t \in [0, 8.8]$ for the non-Markovian channel addition ($nM+nM=M$) that we have designed to address the problem of the zero eigenvalue. We use a time step of 0.1 seconds. The counts obtained from the results of these circuits are used to generate the $\chi$ matrices for each time step. By adding Gaussian fluctuations we generate 100 $\chi$ matrices for each time step. Now using MLE we obtain the closest possible $\chi$ matrices to the ideal chi matrices. We then pick a time $s$ such that $t \geq s \geq 0$ and for each pair $(t, s)$ we obtain 10000 $\chi$ matrices. We use these in our analysis. In both cases we will compute the process fidelity of the $\chi$ matrix for each time $t$ after the MLE,
channels Λ matrix channel addition and non-Markovian channel addition. forming the process tomography the experiments. After simulating the channels and per-

We used the 5 qubit ibmqx2 quantum computer to run channels on the IBM quantum computer in the cloud.

s not affect our results and calculations as we shall always t = 0 where the fidelity is approximately 0.5, this does χ tells us that after MLE our (1) the fidelities for the channel Λ

of 0.1 from 0 to 3.8 seconds. We see that in Fig. 6. (a) Fig. 6. we plot the process fidelities for each time step

This is done to measure the quality of the χ matrices obtained after MLE to see how close it is to the ideal matrix χ_{id}. We note that F_p ∈ [0, 1], when F_p = 1 this tells us that the χ matrix is the same as the ideal i.e. χ = χ_{id} and when F_p = 0 the χ matrix is far from the ideal chi matrix χ_{id}. We shall compute the process fidelities for the channels Λ^{(1)}, Λ^{(2)}, Λ^{(T)} for both the cases of Markovian channel addition and non-Markovian channel addition.

A. Markovian Channel Addition (M+M=nM)

In the Markovian Channel addition we simulate the channels on the IBM quantum computer in the cloud. We used the 5 qubit ibmqx2 quantum computer to run the experiments. After simulating the channels and performing the process tomography the χ matrices are constructed and MLE is performed on these χ matrices. In Fig. 6 we plot the process fidelities for each time step of 0.1 from 0 to 3.8 seconds. We see that in Fig. 6 (a) the fidelities for the channel Λ^{(1)} are very close to 1 this tells us that after MLE our χ matrices are very close to the ideal case. The fidelities for the channel Λ^{(2)} (see Fig. 6 (b)) are also very close to ideal except for time t = 0 where the fidelity is approximately 0.5, this does not affect our results and calculations as we shall always consider times s and t that are bigger than 0. In Fig. 6 (c) we see that the fidelities for the total channel Λ^{(T)} while not as high as the other two channels are still relatively good enough for the purposes of our experiment.

It should be noted that the Fidelity is lower in this case due to decoherence and dissipation in the quantum computer. Now using the characterization method outlined in Section III to classify the channels. We calculate the Eigenvalues of the Choi matrix of the IM for each of the time pairs (t, s). We then plot the minimum eigenvalues for each Choi Matrix W(t, s) of the IM, the reason we look at the minimum eigenvalue is because if the minimum eigenvalue is non-negative then the other eigenvalues will also be non-negative and hence the dynamical map is Markovian with respect to the CP divisibility criteria [7]. We shall also compare the eigenvalues to the theoretical eigenvalues.

In Fig. 7 (a) we see the minimum eigenvalues of the Choi matrix of the IM for the Markovian channel Λ^{(1)} for s = 0.5, here we observe that the eigenvalues for this channel are negative and this is the same problem the authors experienced in [10]. This is due to the fact that the minimum eigenvalue of the Choi matrix of the IM for this channel is zero, as we can not definitively show that something in an experiment is zero as there is always some standard deviation.

A way to overcome this would be to design two Markovian channels and convexly mix them such that the total channel is non-Markovian and none of these channels have minimum eigenvalues of the Choi matrices of their intermediate maps being zero, this will allow us to definitively make a statement about the Markovianity while still leaving room for error. We see that there is a single point that deviates from the rest of the data and has a very large standard deviation, this is due to noise in the quantum computer. Another possible cause is that the NISQ computers can sometimes struggle in the implementation of rotation gates for certain angles.

Although even with these draw backs these results are still a good enough approximation of zero as the theoretical curve is within the standard deviation of the points. In Fig. 7 (b) we show the minimum eigenvalues of the Choi matrix of the IM for the Markovian channel Λ^{(2)} when s = 0.5, here the results are much better as we do not have points that deviate far from the rest of the data but the problem of the zero eigenvalue is still present here as well. Overall the results are a good enough approximation of zero.

In Fig. 7 (c) we see the minimum eigenvalues of the Choi matrix of the IM for the total channel Λ^{(T)} for s = 0.5. The results show definitively that this channel is non-Markovian as the eigenvalues within their standard deviation are all negative and fully cover the theoretical curve. The results do fluctuate due to noise, but they are still good and follow the behaviour of the theoretical curve. It is interesting to note that the results obtained from the quantum computer are actually better than the results obtained in the linear optical setup in [10]. Hence the results shown above tell us that it is possible to simulate convex mixtures of Markovian channels leading to a non-Markovian channel on a quantum computer.
Fig. 6. The process fidelities for the $\chi$ matrices of the implemented channels after MLE for times from 0 to 3.8 seconds with time step 0.1 seconds. (a) and (b) show the fidelities for the Markovian channels $\Lambda_t^{(1)}$ and $\Lambda_t^{(2)}$ respectively. (c) shows the fidelity for the total non-Markovian channel $\Lambda_t^{(T)}$.

Fig. 7. The plot (a) shows the minimum eigenvalues of the IM for the Markovian channel $\Lambda_t^{(1)}$, it is clear that the experimental results are a good enough approximation of the zero eigenvalue. (b) Shows the minimum eigenvalues of the IM for the channel $\Lambda_t^{(2)}$, the experimental results in this case are better than for the first channel as the standard deviation is small for all the points. (c) shows the minimum eigenvalues of the IM for the total non-Markovian channel $\Lambda_t^{(T)}$, the points cover the theoretical curve and hence this channel is definitively non-Markovian by the CP divisibility criteria [7].

B. non-Markovian Channel Addition
(nM+nM=M)

For the non-Markovian channel addition we first replicated the experiments done in [16], on a NISQ device and then we performed the experiments using the channels that we have designed from the previous section that solves the problem of the minimum eigenvalue being zero. First let us consider the channels from [16].

For the non-Markovian channel addition that replicated the experiments in [16] we used the 5 qubit ibmq_santiago quantum computer. After simulating the channels and performing the process tomography the $\chi$ matrices are constructed and MLE is performed on these $\chi$ matrices for each time step in the interval $t \in [0, 3.7]$. We plot the process fidelities for the channels in Fig. 8.

We see that for Fig. 8 (a) and (c) the fidelities for the channels are relatively high and the quality of the $\chi$ matrices is good. In Fig. 8 (b) the results are not as good and they fluctuate due to decoherence and dissipation, but these fidelities are still high enough for the $\chi$ matrices to be used in our analysis. In Fig. 9 (a) we plot the minimum eigenvalues for the non-Markovian channel $\Lambda_t^{(1)}$ for $s = 0.5$ and we observe a good correspondence with the theoretical eigenvalues. There is some deviation due to noise but overall the points are close enough to the theoretical curve to be a good approximation. Since the eigenvalues are negative for some time interval the channel is therefore globally non-Markovian by the CP divisibility criteria.

The results in Fig. 9 (b) also show that the channel $\Lambda_t^{(2)}$ is non-Markovian, as the eigenvalues are negative for some time interval. Although here the data deviates from the theoretical curve, the channel can still be classified as non-Markovian. This deviation could be due to the fact that the $\chi$ matrices for this channel in Fig. 8 (b) have a low fidelity. Lastly in Fig. 9 (c), for the channel $\Lambda_t^{(T)}$, we see the data is a good enough approximation for the zero eigenvalue. The simulation of this channel experienced the same issues as the channels in Section V.A. The results of this experiment are also better than the results obtained in the linear optical setup.
Fig. 8. The process fidelities for the $\chi$ matrices of the implemented channels after MLE for times from 0 to 3.7 seconds with time step 0.1 seconds. (a) shows the fidelities of the non-Markovian channel $\Lambda_1^{(1)}$ which is good indicating chi matrices that are close to ideal. (b) show the fidelities for the non-Markovian channel $\Lambda_2^{(2)}$, the fidelities here fluctuate and aren’t as good as in (a), this is due to noise from the quantum computer. (c) shows the fidelity for the total non-Markovian channel $\Lambda^{(T)}$, the fidelity here is very high and close to one, indicating that the chi matrices obtained are close to ideal.

Fig. 9. The plot (a) shows the minimum eigenvalues of the IM for the non-Markovian channel $\Lambda_1^{(1)}$, it is clear that the experimental results cover the theoretical curve and this indicates this channel is non-Markovian by CP divisibility. (b) Shows the minimum eigenvalues of the IM for the channel $\Lambda_2^{(2)}$, the experimental results in this case show that the channel is non-Markovian as they are negative for some time interval, but they deviate from the theoretical curve, this could be due to the low fidelities of the chi matrices obtained for this channel. (c) shows the minimum eigenvalues of the IM for the total Markovian channel $\Lambda^{(T)}$. The experimental points, although below zero, are still a good enough approximation of zero within some error. So that we can say by the CP divisibility the total channel is Markovian.

Fig. 10. as our data points fully cover the curve in Fig. 9 (a), for the results in Fig. 9 (b) they are similar to 10. Overall the results obtained from the simulation performed on a quantum computer are better. Now we simulate the channels we have designed, to demonstrate the non-Markovian channel addition, on 16 qubit IBM quantum computer ibmq_guadalupe in the cloud. After simulating the channels and performing the process tomography the $\chi$ matrices are constructed and MLE is performed on these $\chi$ matrices for each time step in the interval $t \in [0, 8.8]$. We plot the process fidelities for the channels in Fig. 10. We see that for Fig. 10 (a) and 10 (b) the fidelities for the individual channels are high and also have a value of one for large parts of the time interval indicating that the quality of the $\chi$ matrices is good. In Fig. 10 (c) the fidelity of the total channel is very good although at time $t = 1.8$ s the fidelity is low this tells us that the $\chi$ matrices for the total channel will be close to ideal for most of the interval but at $t = 1.8$ s the $\chi$ matrix will be far from the ideal matrix. These fidelities are high enough such that the accuracy for the $\chi$ matrices, to be used in our analysis, is high. In Fig. 11 (a) we plot the minimum eigenvalues of the Choi matrix of the IM for the non-Markovian channel $\Lambda_1^{(1)}$ for $s = 3$ and we observe a good correspondence with the theoretical eigenvalues. Since the eigenvalues are negative for some time interval the channel is therefore globally non-Markovian by the CP divisibility criteria.

Fig. 11 (b) shows the minimum eigenvalues of the Choi matrix of the IM for the channel $\Lambda_2^{(2)}$, the points do approximate the shape of the theoretical curve but there
Fig. 10. The process fidelities for the $\chi$ matrices of the implemented channels after MLE for times from 0 to 8.8 seconds with time step 0.1 seconds. (a) and (b) show the fidelities for the non-Markovian channels $\Lambda_{t}^{(1)}$ and $\Lambda_{t}^{(2)}$ respectively, the fidelities for these channels are good as for $\Lambda_{t}^{(1)}$ the fidelity is 1 for a large part of the time interval, for $\Lambda_{t}^{(2)}$ the fidelities do fluctuate but are also very good. (c) shows the fidelity for the total Markovian channel $\Lambda_{t}^{(T)}$. There is just an outlier (at $t = 1.8$ s) where the fidelity is low, this could be due to noise and decoherence in the quantum computer.

Fig. 11. The plot (a) shows the minimum eigenvalues of the IM for the non-Markovian channel $\Lambda_{t}^{(1)}$, it is clear that the experimental results are in agreement with the theoretical minimum eigenvalues of the IM. (b) Shows the minimum eigenvalues of the IM for the channel $\Lambda_{t}^{(2)}$, the experimental results in this case deviate from the theoretical curve but this channel is still non-Markovian by CP divisibility. (c) shows the minimum eigenvalues of the IM for the total Markovian channel $\Lambda_{t}^{(T)}$, the points cover the theoretical curve and the initial points although negative are still a good enough approximation of zero, this is a problem face when the minimum eigenvalue is zero. Hence this channel is Markovian by the CP divisibility criteria [7].

is deviation from the theoretical eigenvalues, this could be due to the noise in the quantum computer as seen in the fluctuating fidelity in Fig. 10 (b), but the results obtained are still a good enough approximation of the theoretical curve to make a statement about the Markovianity of the channel. Now even though there is a deviation from the theoretical curve the minimum eigenvalue of the Choi matrix of the IM for the channel $\Lambda_{t}^{(2)}$ is negative for some time which by CP divisibility tells us that this channel is globally non-Markovian. In Fig. 11 (c) the minimum eigenvalues of the Choi matrix of the IM for the Markovian channel $\Lambda_{t}^{(T)}$ are shown. The plot shows that the minimum eigenvalues agree with the the shape of the theoretical curve, but there is deviation from the theoretical curve. We notice the the experimental points deviate from the theoretical curve when the theoretical minimum eigenvalue is zero, but this is the problem with the eigenvalue being zero. This means that the experimental results are negative for the first few seconds which would indicate a non-Markovian channel, but here again these points are close enough to zero to be a good approximation. After a few seconds the minimum eigenvalues become larger than zero and the experimental results are positive telling us that the channel is Markovian. If we consider the experimental minimum eigenvalues during the initial few seconds after $s = 3$ to be a good enough approximation of zero then the channel $\Lambda_{t}^{(T)}$ is Markovian by the CP divisibility criteria. Therefore we have demonstrated above that it is possible to simulate the convex sum of two non-Markovian channels leading to a Markovian channel on a quantum computer. It should be noted that the problem of the minimum eigenvalue of the Choi matrix of the IM being zero was also faced in [16]. We attempted to address the problem of the zero eigenvalue by using two depolarizing channels, as in section IV B, but the functions that were choose still lead to the minimum eigenvalue, of the Choi matrix of the IM.
of the total channel, being zero initially. It is possible to correct this by choosing the functions $q(t)$ and $r(t)$ such that this minimum eigenvalue is bigger than zero but at present the functions $q(t)$ and $r(t)$ have not been found. It is also interesting to note that the quality of results obtained in our experiment on the quantum computer are better than that of the linear optical setup in [16], as our experimental results cover the theoretical curves.

VI. CONCLUSION

We have demonstrated that convex mixtures of Markovian channels leading to a non-Markovian channel as well as convex mixtures of non-Markovian channels leading to a Markovian channel can be simulated on a quantum computer. We also observed that the results of the Markovian channel addition $(M+M=nM)$ were better than the results obtained from the implementation of these channels in a linear optical setup [16]. It should be noted it is still an open problem to find an example of a convex mixture of Markovian channels leading to a non-Markovian channel in which all channels do not have zero as the minimum eigenvalue of the Choi matrices of their respective intermediate maps.

We have demonstrated by replicating the experiments in [16], that we can convexly mix two non-Markovian channels to yield a Markovian channel $(nM+nM=M)$ and the results obtained were a better approximation of the theoretical curves than in [16]. We have also shown in the non-Markovian channel addition $(nM+nM=M)$ that it is possible to design an experiment that addresses the problems faced in [16], namely the problem of the zero eigenvalue, while still demonstrating that we can convexly mix two non-Markovian channels to yield a Markovian channel.

Another problem one may attempt to solve is to find the functions $q(t)$ and $r(t)$ that give rise to individual non-Markovian channels and a total Markovian channel whose minimum eigenvalue is not zero for all times. We also have shown that it is possible to implement a process tomography on the IBM QE quantum computers in the cloud which, at the time of running these simulations, qiskit is unable to do as the built in process tomography tool gives a map that is not CPTP. This work has successfully shown that a NISQ computer can be used to simulate the evolution of an open quantum system and be used as a tool to simulate convex mixtures of quantum channels.

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Appendix A: MLE for the $\chi$ matrix

To perform the MLE on the chi matrices we need the counts obtained from the results of the tomographic circuits used to perform a process tomography on the channel. A vector of 16 counts will be obtained, we denote these counts by $n_{\nu}$ where $\nu = 1, 2, ..., 16$. Each $n_{\nu}$ corresponds to the counts obtained from the tomographic circuit based on the input state used and the number of times the output state was measured. The complete set of states used as the input states are $\{|0\rangle, |1\rangle, |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |+y\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle\}$. TABLE 1 below shows which $n_{\nu}$ corresponds to the input used and output states measured in the tomographic circuits. Now that we have the counts from the tomographic circuits, we can construct the initial $\chi$ matrix which we shall denote as $\chi_p$. This chi matrix will not be positive and Hermitian so we can now use MLE to get the closest possible chi matrix which we shall call $\chi_c$. Start by parameterizing $\chi_c$ as follows:

$$\chi_c(x_1, x_2, ..., x_{16}) = \frac{T^\dagger T}{\text{Tr}[T^\dagger T]} \quad (A1)$$

where $T = T(x_1, x_2, ..., x_{16})$ is a $4 \times 4$ triangular matrix.
that is a function of 16 real variables \(x_1, x_2, ..., x_{16}\) and is shown below in matrix form:

\[
T = \begin{pmatrix}
  x_1 & 0 & 0 & 0 \\
  x_5 + ix_6 & x_2 & 0 & 0 \\
  x_{11} + ix_{12} & x_7 + ix_8 & x_3 & 0 \\
  x_{15} + ix_{16} & x_{13} + ix_{14} & x_9 + ix_{10} & x_4
\end{pmatrix}
\]  

(A2)

A likelihood function can now be constructed using this parameterized form of \(\chi\). First we define the set of operators:

\[
\begin{align*}
H &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\
D &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, R = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}
\end{align*}
\]

These operators will be used to obtain the average counts from the \(\chi\) matrix. Now we define the following array of these operators:

\[
P = \begin{pmatrix}
  (H, H) \\
  (H, V) \\
  (H, D) \\
  (H, R) \\
  (V, V) \\
  (V, H) \\
  (V, D) \\
  (V, R) \\
  (D, H) \\
  (D, V) \\
  (D, D) \\
  (D, R) \\
  (R, H) \\
  (R, V) \\
  (R, D) \\
  (R, R)
\end{pmatrix}
\]

(A3)

this allows us to use index notation to define the likelihood function. The array \(P\) is indexed as follows \(P_{i,j}\) where \(i\) is the \(i\)th tuple in the array and \(j\) is either element 1 or 2 of the \(i\)th tuple. We can now construct the likelihood function \(L = L(x_1, x_2, ..., x_{16})\). This is done by assuming that counts obtained from the results of the quantum circuits have a Gaussian probability distribution. Then then likelihood function is constructed as follows [35]. The probability \(P\) of obtaining a set of 16 counts \(\{x_1, x_2, ..., x_{16}\}\) is:

\[
P(x_1, x_2, ..., x_{16}) = \frac{1}{N_{\text{norm}}} \prod_{\nu=1}^{16} \exp \left[ -\frac{(n_\nu - \bar{n}_\nu)^2}{2\sigma_\nu^2} \right] (A4)
\]

where \(\sigma_\nu\) is the standard deviation the \(\nu\)th count and \(\bar{n}_\nu\) is the average number of counts for the \(\nu\)th count. Now we can approximate \(\sigma_\nu\) by \(\sqrt{\bar{n}_\nu}\). To calculate \(\bar{n}_\nu\) we make use of equation (7) by substituting \(\chi\) with \(\chi_c\), now we can calculate \(\bar{n}_\nu\):

\[
\bar{n}_\nu = \langle n_\nu \rangle = \text{Tr}[P_{\nu,2}A_t(P_{\nu,1})]
\]

\[
= \text{Tr}[P_{\nu,2} \left( \sum_{m,n=0}^{3} (\chi_c)_{mn}\sigma_mP_{\nu,1}\sigma_n \right)] (A5)
\]

Substituting this into \(P\) we obtain:

\[
P(x_1, x_2, ..., x_{16}) = \frac{1}{N_{\text{norm}}} \prod_{\nu=1}^{16} \exp \left[ -\frac{(\text{Tr}[P_{\nu,2}A_t(P_{\nu,1})] - n_\nu)^2}{2\text{Tr}[P_{\nu,2}A_t(P_{\nu,1})]} \right] (A6)
\]

Now to obtain the parameters \(x_1, ..., x_{16}\) that gives the closest possible \(\chi\) matrix to the ideal, we need to maximize the function \(P(x_1, ..., x_{16})\) with respect to \(x_1, ..., x_{16}\). We note that maximizing \(P\) is mathematically equivalent to minimizing the natural log of \(P\). So rather than maximizing \(P\) we minimize the function \(\mathbb{L}(x_1, ..., x_{16}) = \ln[P(x_1, ..., x_{16})]\), with respect to \(x_1, ..., x_{16}\). We use the function \(\mathbb{L}(x_1, ..., x_{16})\) as our likelihood function and it is given as:

\[
\mathbb{L}(x_1, x_2, ..., x_{16}) = \frac{1}{N_{\text{norm}}} \prod_{\nu=1}^{16} \exp \left[ -\frac{(\text{Tr}[P_{\nu,2}A_t(P_{\nu,1})] - n_\nu)^2}{2\text{Tr}[P_{\nu,2}A_t(P_{\nu,1})]} \right] (A7)
\]

Using equation (A5) we can rewrite the likelihood function as:

\[
\mathbb{L}(x_1, ..., x_{16}) = \sum_{\nu=1}^{16} \frac{\text{Tr}[P_{\nu,2} \left( \sum_{m,n=0}^{3} (\chi_c(x_1, ..., x_{16}))_{mn}\sigma_mP_{\nu,1}\sigma_n \right)] - n_\nu)^2}{2\text{Tr}[P_{\nu,2} \left( \sum_{m,n=0}^{3} (\chi_c(x_1, ..., x_{16}))_{mn}\sigma_mP_{\nu,1}\sigma_n \right)]} (A8)
\]

The final step of MLE is to minimize \(\mathbb{L}\) with respect to \(x_1, x_2, ..., x_{16}\). This is done using the minimization methods in the scipy python package. To obtain initial values for \(x_1, x_2, ..., x_{16}\), for the methods in scipy, we use the matrix \(\chi_c\) and solve for the initial conditions [35]. The values \(x_1', x_2', ..., x_6'\) for which \(\mathbb{L}\) is a minimum are then substituted into \(\chi_c\), this gives a positive and Hermitian \(\chi\) matrix that is as close as possible to the ideal \(\chi\) matrix. This \(\chi_c\) is the matrix that will be used in the rest of the calculations in the characterization method.

Appendix B: Circuits for Simulation of Quantum Channels

To simulate the quantum channels in section IV, we need to construct a quantum circuit that implements the channel. This is done using the Stinespring representation of the channel [39]. Since the channels apply noise operators with some probability, by making use of ancilla qubits we can easily design circuits that can implement the channels.
1. Circuits for Markovian Channel addition

To implement the Markovian channel $\Lambda^{(1)}_t$ with probability $p(t)$ we use the circuit shown in Fig. 12 below. The circuit is understood as follows it will apply the gate $\sigma_1 = X$ to the input state $\rho_{in}$ with probability $|\sin(\theta/2)|^2$ and it will leave the state unchanged with probability $|\cos(\theta/2)|^2$ (i.e. when the ancilla is in the state $|0\rangle$ do not change the state $\rho_{in}$ and when the ancilla is in state $|1\rangle$ apply the $X$ gate to the input state $\rho_{in}$). From this we can get the value of the angles in terms of the probability $p(t)$ as:

$$\theta = 2\arccos(\sqrt{p(t)}) = 2\arccos\left(\frac{1 + e^{-t}}{2}\right) \quad (B1)$$

Since the angle $\theta$ is written in terms of $p(t)$ the circuit in Fig. 12 will leave $\rho_{in}$ unchanged with probability $p(t)$ and it will apply $\sigma_1$ to $\rho_{in}$ with probability $1 - p(t)$. Hence the circuit implements the channel $\Lambda^{(1)}_t$.

Similarly we can design a circuit that implements $\Lambda^{(2)}_t$, by using the same ideas used to design the previous circuit. Fig. 13 shows the quantum circuit that implements the channel $\Lambda^{(2)}_t$ where the $Y$ gate is applied to the input state with probability $p(t)$. The angle $\theta$ is obtained using equation (B1). Now that we have designed circuits that implement the individual Markovian channels we design a circuit that implements the total non-Markovian channel $\Lambda^{(2)}_t$. In designing this circuit we keep in mind that we want to leave the input state $\rho_{in}$ unchanged with probability $p(t)$ and apply the noise operators $\sigma_1$ and $\sigma_2$ to the input state with probability $1 - p(t)$. Refer to Fig. 14 for the quantum circuit that implements the total channel. This circuit implements the channel by first preparing the two ancilla qubits in the state: $\cos(\theta_1/2)|00\rangle - \sin(\theta_1/2)\sin(\theta_2/2)|01\rangle - \sin(\theta_1/2)\cos(\theta_2/2)|11\rangle$ now using the ancilla qubits the circuit will apply $X$ to the input state if the ancillas are in the state $|01\rangle$, it will apply $Y$ to the input state when the ancillas are in the state $|11\rangle$ (NB: the circuit applies $XZ$ to the input state which is the same as applying $Y$ up to a phase) and the circuit leaves the input state unchanged when the ancillas are in the state $|00\rangle$. From the state of the ancillas we can determine the angles $\theta_1$ and $\theta_2$:

$$\theta_1 = 2\arccos(\sqrt{p(t)}), \quad \theta_2 = \frac{\pi}{2} \quad (B2)$$

Now that we have the angles in terms of $p(t)$ we have found the circuit that implements the channel $\Lambda^{(T)}_t$.

Fig. 13. Quantum circuit implementing the Markovian channel $\Lambda^{(2)}_t$ for probability $p(t)$.

Fig. 14. Quantum circuit implementing the total non-Markovian channel $\Lambda^{(T)}_t$ for probability $p(t)$.

2. Circuits for non-Markovian Channel addition

We observe that the individual non-Markovian channels used in [16] have the same form as the channel in the previous section but with a different probability. The circuit that implements the non-Markovian channel $\Lambda^{(1)}_t$ for probability $p_1(t)$, given in equation (25), is shown in Fig. 15. Where the angle is $\theta = 2\arccos(\sqrt{p_1(t)})$. Using the same circuit we can implement the channel $\Lambda^{(2)}_t$ by changing the angle so that it is in terms of $p_2(t)$. The total channel is implemented in a similar way.

Fig. 15. Quantum circuit implementing the non-Markovian individual channels $\Lambda^{(1)}_t$ and $\Lambda^{(2)}_t$ for probabilities $p_1(t)$ and $p_2(t)$ respectively. This circuit also implements the total Markovian channel $\Lambda^{(T)}_t$ with a probability of $p(t)$.

To implement the channels that we have designed for the non-Markovian channel addition we need a circuit that can implement the depolarizing channel that is parameterized by some function $p(t)$ (Refer to equation (25)). To do this we use the circuit that simulates the depolarizing channel in [17].
Fig. 16. Circuit implementing the depolarizing channel for a single system qubit. The angle $\theta$ is determined by the formula $\theta(t) = \frac{1}{2} \arccos(1 - 2p(t))$.

**Appendix C: Derivation of decay rates of Pauli channels**

Consider the following pauli channel for a single qubit:

$$\Lambda(t) \rho = \sum_{\alpha=0}^{3} p_\alpha(t) \sigma_\alpha \rho \sigma_\alpha$$  \hspace{1cm} (C1)

where $\sigma_\alpha = 1$ and $p_\alpha(t)$ is a time dependent probability distribution such that $p_0(0) = 1$ and $p_i(0) = 0$ for $i \in \{1, 2, 3\}$. This channel was studied extensively in [13], it should also be noted that this channel also falls into a larger class of channels called random unitary channels [13]. To make a statement about whether this channel is Markovian, which is necessary in designing the channels to simulate, we need to look at its time-local generator $\mathcal{L}(t)$. From the time-local master equation (1) we see that $\mathcal{L}(t) = \tilde{\Lambda}_1 \Lambda_1^{-1}$, this tells us that we need to compute $\Lambda_1^{-1}$ to calculate the generator $\tilde{\Lambda}_1$. Let us note that:

$$\Lambda(t) (\sigma_\alpha) = \lambda_\alpha(t) \sigma_\alpha$$  \hspace{1cm} (C2)

where the time dependent eigenvalues are,

$$\lambda_\alpha(t) = \sum_{\beta=0}^{3} H_{\alpha\beta} p_\beta(t)$$  \hspace{1cm} (C3)

with $\lambda_0(t) = 1$ and $H_{\alpha\beta}$ being the Hadamard matrix defined as:

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$  \hspace{1cm} (C4)

We should note that $\lambda_0(t) = 1$ and $|\lambda_k(t)| \leq 1$ for $k = 1, 2, 3$. Now it is clear that:

$$\mathcal{L}(t) \sigma_\alpha = \mu_\alpha(t) \sigma_\alpha$$  \hspace{1cm} (C5)

where $\mu_\alpha(t) = \frac{\lambda_\alpha(t)}{\lambda_0(t)}$ and in particular $\mu_0(t) = 0$ since $\lambda_0(t) = 1$. Now by introducing the decay rate $\gamma_\alpha(t)$ we get the local generator as:

$$\mathcal{L}(t) \rho = \sum_{\alpha=0}^{3} \gamma_\alpha(t) \sigma_\alpha \rho \sigma_\alpha$$  \hspace{1cm} (C6)

where the decay rates $\gamma_\alpha(t)$ are defined as:

$$\gamma_\alpha(t) = \frac{1}{4} \sum_{\beta=0}^{3} H_{\alpha\beta} \mu_\beta(t).$$  \hspace{1cm} (C7)

By observing that $\sum_{\alpha=0}^{3} \gamma_\alpha = 0$, we arrive at the standard form for the generator of the pauli channel [11] as:

$$\mathcal{L}(t) \rho = \sum_{k=1}^{3} \gamma_k(t) (\sigma_k \rho \sigma_k - \rho).$$  \hspace{1cm} (C8)

Hence we get the expression for $\gamma_k(t)$ as:

$$\gamma_k(t) = \frac{1}{4} \sum_{\beta=0}^{3} H_{\alpha\beta} \left\{ \sum_{\nu=0}^{3} H_{\beta\nu} \dot{\rho}_{\nu}(t) \right\}.$$  \hspace{1cm} (C9)

**Appendix D: Intuition for the choice of functions in the non-Markovian channel addition**

Choosing the functions $q(t)$ and $r(t)$ for the non-Markovian channel addition was a non-trivial task. We shall provide some intuition on how these functions were chosen and the logic behind these choices. The calculations in section III B. give us the following conditions on the functions $q(t)$ and $r(t)$:

$$0 \leq q(t) < 1 \quad \text{and} \quad q(0) = 0,$$

$$0 \leq r(t) < 1 \quad \text{and} \quad r(0) = 0.$$  \hspace{1cm} (D1)

Now from section III B. we have that for the channels $\Lambda_1^{(1)}$ and $\Lambda_1^{(2)}$ to be non-Markovian their respective decay rates should be negative for some time interval. From equation (28) we see that:

$$\dot{q}(t') < 0 \quad \text{for some} \quad t' \geq 0$$

$$\dot{r}(t'') < 0 \quad \text{for some} \quad t'' \geq 0.$$  \hspace{1cm} (D2)

Equation (30) tells us that for the total channel $\Lambda_1^{(T)}$ to be Markovian we must have:

$$h \dot{q}(t) + (1-h) \dot{r}(t) \geq 0 \quad \forall t \geq 0,$$  \hspace{1cm} (D3)

where $h \in [0, 1]$. Now setting $h = \frac{1}{2}$ as in section III we get the decay rate for the total channel as $\frac{1}{2} (\dot{q}(t) + \dot{r}(t))$. The intuition behind how to choose $q, r$ as is follows. We need to choose the functions $q(t), r(t)$ such that when $\dot{q}(t') < 0$ on some interval $t' \in [a, b]$, then $\dot{r}(t') > 0$ for
and taking their convex mixture yields,
\[
\frac{1}{2} \dot{q}(t) + \dot{r}(t) = \dot{a}(t). \quad (D7)
\]
From equations (C5)-(C7) we have the conditions on the functions \(a(t)\) and \(b(t)\). We note that since \(a(t)\) is bounded between the functions \(b(t)\) and \(1 - b(t)\), if we choose \(b(t)\) to have the shape of a Plank distribution where \(b(0) \approx 0\) then \(a(t)\) just needs to satisfy \(a(0) \approx 0\) and equation (C5). We choose \(a(t)\) to be the sum of two sigmoid functions, so that we can satisfy the conditions on \(a(t)\). Now that we have the general shape of both \(a(t)\) and \(b(t)\) by using translation, and scaling factors we can transform the general shapes of these functions to satisfy all the bounds in equations (C5)-(C7). Hence the choice of the functions \(a(t)\) and \(b(t)\) in equation (32). This is the intuition behind the design of the non-Markovian channels in the \((nM+nM=M)\) experiment, a similar approach can be followed for designing other experiments.

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