Pseudoholomorphic curves in four-orbifolds and some applications

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The main purpose of this paper is to summarize the basic ingredients, illustrated with examples, of a pseudoholomorphic curve theory for symplectic 4-orbifolds. These are extensions of relevant work of Gromov, McDuff and Taubes on symplectic 4-manifolds concerning pseudoholomorphic curves and Seiberg-Witten theory (cf. [19, 34, 35, 44, 46]). They form the technical backbone of [9, 10] where it was shown that a symplectic s-cobordism of elliptic 3-manifolds (with a canonical contact structure on the boundary) is smoothly a product. We believe that this theory has a broader interest and may find applications in other problems. One interesting feature is that existence of pseudoholomorphic curves gives certain restrictions on the singular points of the 4-orbifold contained by the pseudoholomorphic curves.

There are four sections. The first one is concerned with the Fredholm theory for pseudoholomorphic curves in symplectic orbifolds, which is based on the theory of maps of orbifolds developed in [8], particularly the topological structure of the corresponding mapping spaces. In the second section, we discuss the orbifold version of adjunction formula and a formula expressing the intersection number of two distinct pseudoholomorphic curves in terms of contributions from each point in the intersection. Unlike the manifold case, the contribution from a singular point may not be integral, but it can be determined from the local structure of the pseudoholomorphic curves near the singular point, which is important in applications. The third section is occupied by a brief summary of the Seiberg-Witten theory for smooth 4-orbifolds and a discussion on the orbifold analog of the theorems of Taubes concerning existence of pseudoholomorphic curves. We also include here an example to illustrate how a combination of the various ingredients is put into use. In the last section, we discuss some applications and work in progress.

1 Moduli space of pseudoholomorphic maps

We begin with a brief review on the theory of maps of orbifolds developed in [8]. (For an earlier version of this work, see [7].)

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First of all, we recall the definition of (smooth) orbifolds from Satake \[42\] (see also \[25\]). Let \( X \) be a paracompact, Hausdorff space. An \( n \)-dimensional orbifold structure on \( X \) is given by an atlas of local charts \( U \), where \( U = \{ U_i \mid i \in I \} \) is an open cover of \( X \) satisfying the following condition:

\((*)\) For any \( p \in U_i \cap U_j, U_i, U_j \in U \), there is a \( U_k \in U \) such that \( p \in U_k \subset U_i \cap U_j \).

Moreover, (1) for each \( U_i \in U \), there exists a triple \( (V_i, G_i, \pi_i) \), called a local uniformizing system, where \( V_i \) is an \( n \)-dimensional smooth manifold, \( G_i \) is a finite group acting smoothly and effectively on \( V_i \), and \( \pi_i : V_i \rightarrow U_i \) is a continuous map inducing a homeomorphism \( U_i \cong V_i/G_i \). (2) for each pair \( U_i, U_j \in U \) with \( U_i \subset U_j \), there is a set \( Innj(U_i, U_j) = \{ (\phi, \lambda) \} \), whose elements are called injections, where \( \phi : V_i \rightarrow V_j \) is a smooth open embedding and \( \lambda : G_i \rightarrow G_j \) is an injective homomorphism, such that \( \phi \) is \( \lambda \)-equivariant and satisfies \( \pi_i = \pi_j \circ \phi \), and \( G_i \times G_j \) acts transitively on \( Innj(U_i, U_j) \) by

\[(g, g') \cdot (\phi, \lambda) = (g' \circ \phi \circ g^{-1}, Ad(g') \circ \lambda \circ Ad(g^{-1})), \forall g \in G_i, g' \in G_j, (\phi, \lambda) \in Innj(U_i, U_j),\]

(3) the injections are closed under composition for any \( U_i, U_j, U_k \in U \) with \( U_i \subset U_j \subset U_k \).

Notice that for any refinement of \( U \) satisfying \((*)\), there is an induced orbifold structure on \( X \). Two orbifold structures on \( X \) are called equivalent if they induce isomorphic orbifold structures. With the preceding understood, an orbifold is a paracompact, Hausdorff space with an equivalence class of orbifold structures. One can similarly define orbifolds with boundary, fiber bundles over orbifolds, etc. See \[42\] for more details.

Let \( X \) be an orbifold, and \( U \) be an orbifold structure on \( X \). One may assume without loss of generality that for any \( p \in X \), there is a \( U_p \in U \) centered at \( p \) in the sense that in the corresponding local uniformizing system \( (V_p, G_p, \pi_p) \), \( V_p \subset \mathbb{R}^n \) is an open ball centered at 0 with \( \pi_p(0) = p \) and \( G_p \) acts linearly on \( V_p \). The group \( G_p \) is called the isotropy group at \( p \), and \( p \in X \) is called a singular (resp. smooth) point if \( G_p \neq \{e\} \) (resp. \( G_p = \{e\} \)).

**Example 1.1** A natural class of orbifolds is provided by the global quotients \( X = Y/G \), where \( Y \) is a smooth manifold and \( G \) is a discrete group acting smoothly and effectively on \( Y \) with only finite isotropy. Such orbifolds are called “good” orbifolds in Thurston \[54\].

For example, consider the orbifold \( X = \mathbb{S}^2/\mathbb{Z}_n \), where \( \mathbb{S}^2 \subset \mathbb{R}^3 \) is the unit sphere, and the \( \mathbb{Z}_n \)-action is generated by a rotation of angle \( 2\pi/n \) about the \( z \)-axis. There are two singular points in \( X \), the “north pole” and the “south pole”, which are the orbits of \((0,0,1)\) and \((0,0,-1)\) under the \( \mathbb{Z}_n \)-action respectively, both having isotropy group \( \mathbb{Z}_n \).

Not every orbifold is good. For example, consider the “tear-drop” orbifold in Thurston \[54\], where the space \( X = \mathbb{S}^2 \), and the orbifold has only one singular point \( p \) such that a local uniformizing system centered at \( p \) is given by \( (D, \mathbb{Z}_n, \pi) \), where \( D \subset \mathbb{C} \) is a disc centered at 0, the action of \( \mathbb{Z}_n \) is generated by a rotation of angle \( 2\pi/n \), and \( \pi(z) = z^n, \forall z \in D \).

Both orbifolds, \( \mathbb{S}^2/\mathbb{Z}_n \) and the “tear-drop” orbifold, are examples of orbifold Riemann surfaces. A general orbifold Riemann surface is an orbifold where the underlying space is a Riemann surface and the orbifold has finitely many singular points, each having a local uniformizing system defined as in the case of the “tear-drop” orbifold above.
The concept of orbifold was introduced by Satake [42] under the name “V-manifold”, where orbifolds were perceived as a class of singular spaces to which the usual differential geometry on smooth manifolds can be suitably extended by working locally and equivariantly on each local uniformizing system. For example, a differential form on an orbifold is a family of equivariant differential forms on the local uniformizing systems which are compatible under the injections. The concept was later rediscovered by Thurston [54], who also invented the more popular name “orbifold”. Thurston was more concerned with the topological aspects of orbifolds. For instance, Thurston introduced through a novel covering theory the important concept of orbifold fundamental group, which is different from the usual fundamental group. For example, the orbifold fundamental groups of $S^2/\mathbb{Z}_n$ and the “tear-drop” orbifold in Example 1.1 are $\mathbb{Z}_n$ and the trivial group respectively, even though both orbifolds have the same underlying space $S^2$. Orbifolds can be defined equivalently using the categorical language of étale topological groupoids, where the orbifold fundamental group may be interpreted as the fundamental group of an étale topological groupoid (cf. eg. [21]).

Despite the fact that the concept of orbifold was extensively used in the literature, there has not been a well-developed, satisfactory theory of maps for orbifolds. The notion of maps that has been used mostly in the literature, which is usually called a $C^\infty$ map (sometime it is called an orbifold map), is essentially the notion of V-manifold map introduced by Satake in [42], which is roughly speaking a continuous map that can be locally lifted to a smooth map between local uniformizing systems. Although such a notion of maps is sufficient for the purpose of defining diffeomorphisms of orbifolds, or defining sections of an orbifold vector bundle, etc., cf. Remark 1.7 (3) below, it has serious drawbacks in the fact that the space of maps does not have a well-understood topological structure in general, and that it is not compatible with the distinct topological structure of orbifolds discovered by Thurston which we alluded to earlier. On the other hand, the topological aspect of orbifolds has been dealt with mainly through the classifying spaces of topological groupoids (cf. eg. [21]), rather than building upon a satisfactory theory of maps as we do in the manifold case.

With the preceding understood, we now turn to the theory of maps of orbifolds developed in [8]. The main advantage of this theory lies in the fact that it accommodates at the same time both the differential geometric and the topological aspects of orbifolds. More concretely, in Part I of [8], it was shown that the space of maps in this theory has a natural infinite dimensional orbifold structure, hence it is amenable to techniques of differential geometry and global analysis on orbifolds. On the other hand, in Part II of [8], a basic homotopy theory and an analog of CW-complex theory were developed (based upon the use of maps rather than classifying spaces), which provide a machinery for studying homotopy class of maps between orbifolds, and substantially extend the existing homotopy theory defined via classifying spaces of topological groupoids (cf. [21], compare also [37]).

The theory of maps in [8] was developed in the more general context of orbispaces which are generalizations of orbifolds. For technical reason we adopted a modified formalism, cf. Remark 1.7 (4) below.

**Definition 1.2** (See [8], also [7]) Let $X$ be a locally connected topological space. An ‘orbispace structure’ on $X$ is given by an atlas of local charts $U$, where $U = \{U_i | i \in I\}$ is an open cover of $X$ which satisfies the following condition.
Each $U_i \in \mathcal{U}$ is connected, and for any $U_i \in \mathcal{U}$, a connected open subset of $U_i$ is also an element of $\mathcal{U}$.

Moreover,

- for each $U_i \in \mathcal{U}$, there is a triple $(\hat{U}_i, G_{U_i}, \pi_{U_i})$, where $\hat{U}_i$ is a locally connected and connected topological space with a continuous left action of a discrete group $G_{U_i}$, and $\pi_{U_i} : \hat{U}_i \to U_i$ is a continuous map inducing a homeomorphism $\hat{U}_i/G_{U_i} \cong U_i$.
- there exists a family of discrete sets

$$T = \{ T(U_i, U_j) | U_i, U_j \in \mathcal{U} \text{ such that } U_i \cap U_j \neq \emptyset \}$$

which obeys

- $T(U_i, U_i) = G_{U_i}$, and for any $i \neq j$, if we let $\{ W_u | u \in I_{ij} \}$ be the set of connected components of $U_i \cap U_j$, then there are discrete sets $T_{W_u}(U_i, U_j)$ such that $T(U_i, U_j) = \sqcup_{u \in I_{ij}} T_{W_u}(U_i, U_j)$.
- Each $\xi \in T_{W_u}(U_i, U_j)$ is assigned with a homeomorphism $\phi_\xi$, whose domain and range are connected components of the inverse image of $W_u$ in $\hat{U}_i$ and $\hat{U}_j$ respectively, such that $\pi_{U_i} | \text{Domain} (\phi_\xi) = \pi_{U_j} \circ \phi_\xi$.

Moreover, $\xi \neq I_{ij}$, $\eta \in T(U_j, U_k)$, and any $x \in \text{Domain} (\phi_\xi)$ such that $\phi_\xi(x) \in \text{Domain} (\phi_\eta)$, there exists an $\eta \circ \xi(x) \in T(U_i, U_k)$ such that $\phi_\eta(\eta \circ \xi(x)) = \phi_\eta(\phi_\xi(x))$. Moreover, $x \mapsto \eta \circ \xi(x)$ is locally constant, and the composition $(\xi, \eta) \mapsto \eta \circ \xi(x)$ is associative with the group multiplication in $G_{U_i}$ when restricted to $T(U_i, U_i) = G_{U_i}$.

Each $\xi \in T(U_i, U_j)$ has an inverse $\xi^{-1} \in T(U_j, U_i)$, with $\text{Domain} (\phi_\xi) = \text{Range} (\phi_{\xi^{-1}})$, $\text{Domain} (\phi_{\xi^{-1}}) = \text{Range} (\phi_\xi)$, and $\xi^{-1} \circ \xi(x) = e \in G_{U_i} \forall x \in \text{Domain} (\phi_\xi), \xi \circ \xi^{-1}(x) = e \in G_{U_j} \forall x \in \text{Domain} (\phi_{\xi^{-1}})$.

For any refinement of $\mathcal{U}$ satisfying (#), there is an induced orbispace structure on $X$. Two orbispace structures on $X$ are called ‘equivalent’ if they induce isomorphic orbispace structures. Finally, an ‘orbispace’ is a locally connected topological space with an orbispace structure, which is uniquely determined up to equivalence.

Remark 1.3 (1) The orbifolds in $[8]$ are defined in the formalism of Definition 1.2. They differ from those defined in Satake [42] in two aspects. The first one which is more essential is that the group action in a local uniformizing system is no longer required to be effective in [8]. An orbifold in the classical sense (ie. the local group actions are effective) is called ‘reduced’ (cf. [13]). The second aspect is that the technical requirement that for any $U_i, U_j \in \mathcal{U}$ in the atlas of local charts, a connected component of $U_i \cap U_j$ is also an element of $\mathcal{U}$ (note that $T(U_i, U_j)$, which corresponds to $I_{U_j}(U_i, U_j)$ in Satake’s formalism when $U_i \subset U_j$, is defined as long as $U_i \cap U_j \neq \emptyset$). This technical assumption can always be met by replacing $\mathcal{U}$ with a refinement of $\mathcal{U}$ which defines an equivalent orbifold structure (cf. Proposition 2.1.3, Part I of [8]).

(2) The orbispace in the sense of Definition 1.2 belong to a restricted class of orbispace in Haefliger [22], which were defined using the language of étale topological groupoids. To be more precise, if we interpret the orbispace structures in Definition 1.2 in terms of étale topological groupoids, the formalism of Definition
1.2 (which is a modified version of Satake’s) requires the corresponding étale topological
groupoids to satisfy some additional conditions (i.e. the conditions (C1),
(C2) in Part I of [8]). Moreover, simple examples show that these conditions are
not preserved under the Morita equivalence of topological groupoids. Besides orbifolds, it is known that global quotient orbispaces \(X = Y/G\) (where \(Y\) is a locally
connected topological space and \(G\) is a discrete group) and the orbihedra defined
in [22] satisfy the conditions of Definition 1.2. See [8] for more details.

Let \(X, X'\) be any orbispaces in the sense of Definition 1.2, with orbispace structures \(U, U'\) respectively. In order to define maps between \(X\) and \(X'\), we consider
sets of data \([[\hat{f}_\alpha], \{\rho_{\beta\alpha}\}]\) (which will be called a ‘compatible system’ below), where

- there is an open cover \(\{U_\alpha\}\) of \(X\) with \(U_\alpha \subset U\), and a correspondence \(U_\alpha \rightarrow U'_\alpha \subset U'\),
- each \(\hat{f}_\alpha : \hat{U}_\alpha \rightarrow \hat{U}'_\alpha\) is a continuous map, and each \(\rho_{\beta\alpha} : T(U_\alpha, U_\beta) \rightarrow T(U'_\alpha, U'_\beta)\) is a mapping of discrete sets, such that
  (a) \(\phi_{\rho_\alpha(\xi)} \circ \hat{f}_\alpha(x) = \hat{f}_\beta \circ \phi_\xi(x)\) for any \(\alpha, \beta\), where \(\xi \in T(U_\alpha, U_\beta), \ x \in \text{Domain} (\phi_\xi)\).
  (b) \(\rho_\alpha(\eta \circ \xi(x)) = \rho_{\gamma\beta}(\eta) \circ \rho_{\beta\alpha}(\xi)(\hat{f}_\alpha(x))\) for any \(\alpha, \beta, \gamma\), where \(\xi \in T(U_\alpha, U_\beta), \ \eta \in T(U_\beta, U_\gamma), \ x \in \phi_{\xi}^{-1}(\text{Domain} (\phi_\eta)).\) (Note: \(\hat{f}_\alpha(x) \in \phi_{\rho_\alpha(\xi)^{-1}}^{-1}(\text{Domain} (\phi_\eta))\), which follows from (a) and the assumption
  that \(x \in \phi_{\xi}^{-1}(\text{Domain} (\phi_\eta))\).

(Observe that if set \(\rho_\alpha \equiv \rho_{\alpha\alpha} : G_{U_\alpha} \rightarrow G_{U'_\alpha}\), then condition (b) above
implies that \(\rho_\alpha\) is a homomorphism and condition (a) above implies that \(\hat{f}_\alpha\)
is \(\rho_{\alpha\alpha}\)-equivariant.

One can introduce an equivalence relation between compatible systems as follows. Suppose \([[\hat{f}_\alpha], \{\rho_{\beta\alpha}\}]\) is another compatible system. We say that \([[\hat{f}_\alpha], \{\rho_{\beta\alpha}\}]\)
is ‘induced’ from \([[f_\alpha], \{\rho_{\beta\alpha}\}]\), if there exists a set of data \(\{(\theta, \xi_\alpha, \xi'_\alpha)\}\),
where \(\theta : a \mapsto \alpha\) is a mapping of indices satisfying \(U_\alpha \subset U_{\theta(a)}, \ \xi_\alpha \in T(U_a, U_{\theta(a)}),
\xi'_\alpha \in T(U'_a, U'_{\theta(a)}),\) such that

- \(\hat{f}_\alpha = (\phi_{\xi'_\alpha})^{-1} \circ \hat{f}_{\theta(a)} \circ \phi_{\xi_\alpha}\), and
- \(\rho_{\beta\alpha}(\eta) = (\xi'_\alpha)^{-1} \circ \rho_{\beta\theta(\alpha)}(\eta) \circ \xi^\theta_\alpha(x), \ \forall x \in \hat{f}_\alpha(\text{Domain}(\phi_\eta)), \ \theta(\eta) \equiv \xi_b \circ \eta \circ (\xi'_\alpha)^{-1}(z) \in T(U_{\theta(b)}, U_{\theta(a)}), \ \forall z \in \phi_{\xi_b}(\text{Domain}(\phi_\eta)).\)

**Definition 1.4** (See [8], also [7]) Two compatible systems are said to be ‘equivalent’ if they induce a common compatible system.

A map between two orbispaces is defined to be an equivalence class of compatible
systems in the sense of Definition 1.4. (It is shown in [8] that the relation in
Definition 1.4 is indeed an equivalence relation.) With such a notion of maps it is
shown in [8] that the set of orbispaces in the sense of Definition 1.2 form a category.
Observe that a map \(f : X \rightarrow X'\) between orbispaces induces a continuous map
between the underlying topological spaces, which, when \(X, X'\) are reduced orbifolds
and \(f\) is smooth, is a V-manifold map of Satake [12]. We denote by \([X; X']\) the
space of maps between \(X\) and \(X'\). Notice that \([X; X']\) is uniquely determined up
to a bijection by the equivalence class of orbispace structures on \(X, X'\).

The central result in Part I of [8], which is also the technical foundation for the
subsequent development in Part II of [8], is the following structural theorem.
Theorem 1.5 Let $X,X'$ be any orbispaces with orbispace structures $U,U'$ respectively. Then $[X;X']$ is naturally an orbispace in the sense of Haefliger \cite{22}, provided that the following conditions are satisfied by $X$:

1. the underlying topological space is paracompact, locally compact and Hausdorff,
2. the orbispace structure $U$ obeys: for each $U \subseteq U$, the space $\hat{U}$ is locally compact and Hausdorff, and $\pi_U : \hat{U} \to U$ is proper.

Specializing in the case of orbifolds, we have the following theorem. Here $[X;X']^r$ denotes the space of $C^r$ maps between smooth orbifolds $X$ and $X'$, where a $C^r$ map is the equivalence class of a compatible system $((\hat{f}_\alpha),\{\rho_{\beta\alpha}\})$ such that each $\hat{f}_\alpha$ is a $C^r$ map between smooth manifolds.

Theorem 1.6 Let $X,X'$ be smooth orbifolds where $X$ is compact and connected. Then $[X;X']^r$ is naturally a Banach orbifold (Hausdorff and second countable), such that at any $f \in [X;X']^r$, there is a uniformizing system $(V_f,G_f)$, where $V_f$ is an open ball centered at 0 in the space of $C^r$ sections of $f^*(TX')$, and $G_f$ acts on $V_f$ linearly. Moreover, for any $f \in [X;X']^r$ and any point $p$ in the image of $f$, $G_f$ is naturally a subgroup of the isotropy group $G_p$ at $p$. In particular, a $C^r$ map is a smooth point in $[X;X']^r$ if its image contains a smooth point of $X'$.

Remark 1.7 (1) In Theorem 1.6, the orbifolds $X,X'$ are not necessarily reduced, and $X$ can be more generally an orbifold with boundary. For any $C^r$ map $f : X \to X'$ and orbifold vector bundle $E$ over $X'$, the pull-back bundle of $E$ via $f$, which is denoted by $f^*E$, is well-defined as a $C^r$ orbifold vector bundle up to isomorphisms. Finally, by the image of a map between orbifolds, we always mean the image of the induced continuous map between the underlying spaces.

(2) A Banach orbifold is the straightforward generalization (ie., not in the modified formalism of Definition 1.2) of Satake’s orbifolds to the infinite dimensional setting, except that the group action on each local uniformizing system is not required to be effective. In particular, the action of $G_f$ in Theorem 1.6 may not be effective in general.

(3) Suppose both $X,X'$ are reduced. Let $f^\circ$ be any V-manifold map from $X$ to $X'$ such that the inverse image of the smooth locus of $X'$ is dense in $X$. Then $f^\circ$ uniquely determines a map $f : X \to X'$ of orbifolds as follows. Pick any set of local liftings $\{\hat{f}_\alpha\}$ of $f^\circ$, there is a uniquely determined set of mappings $\{\rho_{\beta\alpha}\}$ such that $((\hat{f}_\alpha),\{\rho_{\beta\alpha}\})$ form a compatible system, whose equivalence class is easily seen to depend on $f^\circ$ only (cf. \cite{13}). This perhaps explains why the notion of V-manifold map is sufficient as far as only diffeomorphisms of orbifolds or sections of an orbifold vector bundle are concerned.

(4) The notion of maps in \cite{8} has its root in the construction of Gromov-Witten invariants of symplectic orbifolds in \cite{14}, where the main technical obstacle was the fact that the notion of V-manifold map is not sufficient to define pull-back bundles. The notion of compatible systems$^1$ was introduced in \cite{13} as a device to define pull-back bundles, and a V-manifold map was called a ‘good map’ in \cite{13} if it supports such a compatible system. Moreover, two compatible systems in \cite{13} were defined to be ‘isomorphic’ if they define isomorphic pull-back bundles. For reduced orbifolds,

$^1$We would like to point out that the present definition of compatible systems, which relies on the modified definition of orbifolds in Definition 1.2, is different from that in \cite{13}, and this modified formalism (along with Definition 1.4) is crucial in proving Theorems 1.5 and 1.6.
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one can show that this somewhat indirect, implicit definition is actually equivalent to the direct, more explicitly defined one in Definition 1.4, due to the fact explained in Remark 1.7 (3) above. The bottom line is: The Gromov-Witten invariants based on [8] are the same as those defined in [13]. However, the treatment in [13] was ad hoc in nature and can be substantially simplified in the framework of [8].

(5) In terms of étale topological groupoids, a compatible system may be interpreted as a groupoid homomorphism, and a map between orbispaces in [8] may be regarded as a groupoid homomorphism modulo certain Morita equivalence of topological groupoids. There are several related notions of maps in the literature, see Haefliger [20], Hilsum-Skandalis [23], Pronk [41], and Abramovich-Vistoli [1] (in the context of Deligne-Mumford stacks). Compare also [37]. □

We illustrate Theorem 1.6 with the following example.

Example 1.8 Let \( X = Y/G \) and \( X' = Y'/G' \), where \( Y \) is compact and connected, and \( G, G' \) are finite. (Here the actions of \( G, G' \) are not assumed to be effective). Consider

\[
[(Y, G); (Y', G')]^r \equiv \{ (\hat{f}, \rho) | \hat{f} : Y \to Y' a \ \rho\text{-equiv. } C^r \text{ map, } \rho : G \to G' \text{ a hom.} \}
\]

which is a Banach manifold with a smooth action of \( G' \) defined by

\[
g \cdot (\hat{f}, \rho) = (g \circ \hat{f}, Ad(g) \cdot \rho), \ \forall g \in G', (\hat{f}, \rho) \in [(Y, G); (Y', G')]^r.
\]

Note that each \((\hat{f}, \rho) \in [(Y, G); (Y', G')]^r\) is naturally a compatible system. With this understood, the correspondence

\[
(\hat{f}, \rho) \mapsto \text{the equivalence class of } (\hat{f}, \rho) \text{ in } [X; X']^r
\]

identifies the Banach orbifold \([(Y, G); (Y', G')]^r / G' \) as an open and closed subspace of \([X; X']^r \), which is a proper subspace in general when \( \pi_1(Y) \) is nontrivial. (In fact, each \( \hat{f} \in [X; X']^r \) induces a homomorphism \( f_* : \pi_1^{arb}(X) \to \pi_1^{arb}(X') \) between orbifold fundamental groups, and \( \hat{f} \in [(Y, G); (Y', G')]^r \) iff \( f_*(\pi_1(Y)) \subset \pi_1(Y') \).)

A regular neighborhood of \((\hat{f}, \rho)\) in \([(Y, G); (Y', G')]^r \) can be described as follows. Note that there is a \( G \)-bundle \( E_{(\hat{f}, \rho)} \to Y \) which is the pull-back of \( TY' \) by \((\hat{f}, \rho)\). Let \( V_{(\hat{f}, \rho)} \) be a sufficiently small open ball centered at the origin in the space of \( G \)-sections of \( E_{(\hat{f}, \rho)} \) of \( C^r \) class. The isotropy subgroup \( G_{(\hat{f}, \rho)} \) at \((\hat{f}, \rho)\), which consists of \( g \in G' \) such that \( g \cdot \hat{f}(p) = \hat{f}(p), g\rho(h) = \rho(h)g \) for all \( p \in Y \) and \( h \in G \), acts linearly on \( V_{(\hat{f}, \rho)} \) by

\[
g \cdot s = g \circ s, \ \forall g \in G_{(\hat{f}, \rho)}, s \in V_{(\hat{f}, \rho)}.
\]

With this understood, a regular neighborhood of \((\hat{f}, \rho)\) can be identified with \( V_{(\hat{f}, \rho)} \) via the \( G_{(\hat{f}, \rho)} \)-equivariant map defined by

\[
s(p) \mapsto (exp_{\hat{f}(p)} s(p), \rho), \ \text{where } p \in Y, s \in V_{(\hat{f}, \rho)}.
\]

In other words, \((V_{(\hat{f}, \rho)}, G_{(\hat{f}, \rho)})\) is a uniformizing system at the image of \((\hat{f}, \rho)\) in \([X; X']^r \).

Finally, observe that \( G_{(\hat{f}, \rho)} \) is a subgroup of the isotropy subgroup at \( \hat{f}(p) \) for any \( p \in Y \). □

With these preparations, we now discuss the Fredholm theory for pseudoholomorphic curves in symplectic orbifolds.
Definition 1.9  (1) A symplectic structure on an orbifold $X$ is a closed, non-degenerate 2-form $\omega$, which is given by a family of closed, non-degenerate 2-forms $\{\omega_i\}$ on $V_i$ for each local uniformizing system $(V_i, G_i, \pi_i)$, which are equivariant under the local group actions and compatible with respect to the injections.

(2) An almost complex structure on an orbifold $X$ is an endomorphism $J : TX \to TX$ with $J^2 = -1$, which is given by a family of endomorphisms $\{J_i : TV_i \to TV_i\}$ with $J_i^2 = -1$ for each local uniformizing system $(V_i, G_i, \pi_i)$, which are equivariant under the local group actions and compatible with respect to the injections. Let $\omega$ be any symplectic structure on $X$. An almost complex structure $J$ is called $\omega$-compatible if $\omega(\cdot, J\cdot)$ defines a Riemannian metric on $X$.

(We remark that when $X = Y/G$ is a global quotient by a finite group, a symplectic (resp. almost complex) structure on the orbifold $X$ is given by a $G$-equivariant symplectic (resp. almost complex) structure on $Y$.)

As for the general setup, let $(X, \omega)$ be a closed, reduced symplectic orbifold and $J$ be a fixed $\omega$-compatible almost complex structure on $X$. We will be considering $J$-holomorphic maps (always assuming nonconstant, i.e. the image does not consist of a single point) from an orbifold Riemann surface $(\Sigma, j)$ into $(X, J)$. (A map $f : \Sigma \to X$ is called $J$-holomorphic if each $\tilde{f}_\alpha$ in a corresponding compatible system is $J$-holomorphic, i.e., $df_{\tilde{f}} + J_{\alpha} \circ df_{\tilde{f}} \circ J_{\alpha} = 0$. ) Here an additional assumption is made that each homomorphism $\rho_\alpha \equiv \rho_\alpha \alpha$ in the compatible systems of a $J$-holomorphic map is injective. However, we do not assume that the orbifold Riemann surface $\Sigma$ is reduced. Furthermore, we will adopt the following convention for $\Sigma$: a point $z \in \Sigma$ where the isotropy group $G_z$ acts trivially is called a ‘regular point’; otherwise, it is called an ‘orbifold point’.

The most relevant information concerning a $J$-holomorphic map is its local structure at each point. More precisely, let $f : \Sigma \to X$ be a $J$-holomorphic map, let $z \in \Sigma$, $p \in X$ such that $f(z) = p$, and let $(D_{z}, G_z)$, $(V_p, G_p)$ be local uniformizing systems at $z$, $p$ respectively. (Note that when $\Sigma$ is not reduced, $G_z$ can be any finite group in general.) Then $f$ determines a germ of pairs $(\tilde{f}_z, \rho_z)$, where $\rho_z : G_z \to G_p$ is an injective homomorphism, and $\tilde{f}_z : D_z \to V_p$ is $\rho_z$-equivariant and $J$-holomorphic. Moreover, $(\tilde{f}_z, \rho_z)$ is unique up to a change $(\tilde{f}_z, \rho_z) \mapsto (g \circ \tilde{f}_z, Ad(g) \circ \rho_z)$ for some $g \in G_p$. We will call $(\tilde{f}_z, \rho_z)$ a ‘local representative’ of $f$ at $z$.

With Theorem 1.6, the Fredholm theory in the orbifold setting is in complete analogy with that in the smooth setting. We shall illustrate it by considering the case where $X$ has at most isolated singularities, which suffices for most of the applications in dimension 4. Notice that in this case the orbifold Riemann surface $\Sigma$ is necessarily reduced by our assumptions.

Fixing a sufficiently large $r > 0$, we consider the space $[\Sigma; X]^r$, which, under the assumption on $X$, may be regarded as a Banach manifold because each nonconstant $J$-holomorphic map from $\Sigma$ to $X$ contains a smooth point of $X$ in its image, hence is a smooth point in $[\Sigma; X]^r$ by Theorem 1.6. On the other hand, it is easily seen that the space $J^r$ of $\omega$-compatible almost complex structures of $C^r$ class is always a Banach manifold, with the tangent space at $J \in J^r$ being the space of $C^r$ sections $A$ of the orbifold vector bundle $End \ TX$, which obeys (1) $AJ + JA = 0$, (2) $A^t = A$. (Here the transpose $A^t$ is taken with respect to $\omega(\cdot, J\cdot)$.)

We also need to consider the moduli space of complex structures on the orbifold Riemann surface $\Sigma$, which can be identified with the moduli space of complex
structures on the corresponding marked Riemann surface (with the orbifold points being the marked points), cf. [13]. For simplicity, we assume $\Sigma$ is an orbifold Riemann sphere of $k \leq 3$ orbifold points. Then $\Sigma$ has a unique complex structure $j$ with a $(6 - 2k)$-dimensional automorphism group $G$.

Now for any $(f,J) \in [\Sigma; X]^r \times J^r$, let $E_{(f,J)}$ be the space of $C^{r-1}$ sections $s$ of the orbifold bundle $\text{Hom}(T\Sigma, f^*(TX))$ such that $s \circ j = -J \circ s$. Then there is a Banach bundle $E \to [\Sigma; X]^r \times J^r$ whose fiber at $(f,J)$ is $E_{(f,J)}$. The zero set of $L : [\Sigma; X]^r \times J^r \to E$, where

$$L(f,J) = df + J \circ df \circ j,$$

consists of pairs $(f,J)$ such that $f$ is $J$-holomorphic. We set

$$\dot M \equiv \{ (f,J) \vert (f,J) \in L^{-1}(0), f \text{ is not multiply covered} \},$$

and for any $J \in J^r$, set $\dot M_J = \{ f \vert (f,J) \in \dot M \}$. (Here we say $f$ is not multiply covered if the induced map is not multiply covered.) The following lemma follows by the same analysis as in the manifold case (cf. eg. [36]) with an application of the Sard-Smale theorem [43].

**Lemma 1.10** The subspace $\dot M \subset [\Sigma; X]^r \times J^r$ is a Banach submanifold, and for a generic $J \in J^r$ which may be chosen smooth, $\dot M_J$ is a smooth, finite dimensional manifold.

The dimension of $\dot M_J$ can be calculated using Kawasaki’s orbifold index theorem [26], see also Lemma 3.2.4 in [13]. We state the formula here for the case where $\dim X = 4$ (and $\Sigma$ is a reduced orbifold Riemann sphere with $k \leq 3$ orbifold points). Let $f \in \dot M_J$ and let $z_i \in \Sigma$ be the orbifold points with orders $m_i$. Suppose in a local representative $(f_{z_i}, \rho_{z_i}) : (D_1, \mathbb{Z}_{m_i}) \to (V_i, G_i)$ of $f$ at $z_i$, the action of $\rho_{z_i}(\mu_{m_i})$ on $V_i$ (here $\mu_m \equiv \exp(\sqrt{-1} \omega_m)$) is given by

$$\rho_{z_i}(\mu_{m_i}) \cdot (w_1, w_2) = (\mu_{m_i}^{m_{i,1}} w_1, \mu_{m_i}^{m_{i,2}} w_2), \ 0 < m_{i,1}, m_{i,2} < m_i.$$

Then

$$\dim \dot M_J = 2\hat c + \hat c \cdot \dim \Sigma$$

The automorphism group $G$ of the complex structure $j$ on $\Sigma$ acts smoothly on $\dot M_J$ (see the end of §3.3 in Part I of [8] for a general discussion), which is free because the maps in $\dot M_J$ are not multiply covered. Consequently, $M_J \equiv \dot M_J / G$ is also a smooth manifold with $\dim M_J = \dim \dot M_J - (6 - 2k)$.

It is well-known that in the smooth case, Lemma 1.10 implies nonexistence of certain $J$-holomorphic curves in a symplectic 4-manifold when $J$ is chosen generic. For instance, there exists no embedded $J$-holomorphic 2-sphere $C$ with $C^2 = -2$ and $J$ generic. We will see that this is no longer true if there is a finite group action and $J$ is chosen equivariant. However, existence of $J$-holomorphic curves for a generic equivariant $J$ will put certain restrictions on the fixed-point set of the action, as shown in the following example.

**Example 1.11** Let $\mathbb{Z}_n$ act on a symplectic 4-manifold $Y$ preserving the symplectic structure, and let $p_1, p_2 \in Y$ be two isolated fixed points. Suppose for any generic equivariant $J$, there exists a $\mathbb{Z}_n$-invariant, embedded $J$-holomorphic 2-sphere $C$ with $C^2 = -2$ such that $p_1, p_2 \in C$. We apply Lemma 1.10 to the
orbifold $X \equiv Y/\mathbb{Z}_n$ and $J$-holomorphic maps $\Sigma \equiv S^2/\mathbb{Z}_n \to X$. Then $M_J$, which is a smooth manifold by Lemma 1.10, must have $\dim M_J = 2d \geq 0$ at $C/\mathbb{Z}_n$. Here by the aforementioned formula for $\dim \tilde{M}_J$,
\[
d = \frac{1}{n} c_1(TY) \cdot C + 2 - \frac{1 + m_1}{n} - \frac{1 + m_2}{n} - (3 - 2) = 1 - \frac{1 + m_1}{n} - \frac{1 + m_2}{n},
\]
where $0 < m_1, m_2 < n$ are the weights of the complex representations of $\mathbb{Z}_n$ at $p_1, p_2$ in the direction normal to $C$. Note that $d \in \mathbb{Z}$, which implies that either $m_1 + m_2 + 2 = n$ or $m_1 + m_2 + 2 = 2n$. But the latter case, which implies $m_1 = m_2 = n - 1$, can be ruled out because $d \geq 0$.

Note that a discussion concerning Lemma 1.10 and Example 1.11 where $\Sigma$ is a general orbifold Riemann surface may be found in [11].

Besides Lemma 1.10, there is another type of regularity criterion which is the orbifold version of Lemma 3.3.3 in [36]. See Lemma 3.2 of [10]. Here the almost complex structure $J$ is not necessarily generic.

**Lemma 1.12** Let $f : \Sigma \to X$ be a $J$-holomorphic map from a reduced orbifold Riemann surface into a symplectic 4-orbifold with only isolated singularities. Suppose for any $z \in \Sigma$, the map $\tilde{f}_z$ in a local representative $(\tilde{f}_z, \rho_z)$ of $f$ at $z$ is embedded. Then $f$ is a smooth point in the space of $J$-holomorphic maps if $c_1(T\Sigma) \cdot [\Sigma] > 0$ and $c_1(TX) \cdot f_*([\Sigma]) > 0$.

Finally, we state an orbifold version of the Gromov compactness theorem. For simplicity, we assume that the orbifold Riemann surface $\Sigma$ is reduced.

**Theorem 1.13** Let $(X, \omega)$ be any compact closed symplectic orbifold, $J$ be any $\omega$-compatible almost complex structure, and $0 \neq A \in H_2(X; \mathbb{R})$. For any sequence $f_n : \Sigma \to X$ of $J$-holomorphic maps with $(f_n)_*([\Sigma]) = A$, there is a subsequence, reparametrized if necessary and still denoted by $f_n$, for simplicity, having the following significance: There are at most finitely many simple closed loops $\gamma_1, \ldots, \gamma_l \subset \Sigma$ containing no orbifold points, and a nodal orbifold Riemann surface $\Sigma' = \cup_{\nu} \Sigma_{\nu}$ obtained by collapsing $\gamma_1, \ldots, \gamma_l$, and a $J$-holomorphic map $f : \Sigma' \to X$ with $f_*([\Sigma]) = A \in H_2(X; \mathbb{R})$, such that

1. $f_n$ converges to $f$ in $C^\infty$ on any given compact subset in the complement of $\gamma_1, \ldots, \gamma_l$,
2. if $z_\nu \in \Sigma_\nu, z_\omega \in \Sigma_\omega$ are two distinct points with orders $m_\nu, m_\omega$ respectively, such that $z_\nu, z_\omega$ are the image of the same simple closed loop collapsed under $\Sigma \to \Sigma'$, then $m_\nu = m_\omega = m$, and there exist local representatives $(\tilde{f}_\nu, \rho_\nu), (\tilde{f}_\omega, \rho_\omega)$ of $f$ at $z_\nu, z_\omega$, which obey $\rho_\nu(\mu_m) = \rho_\omega(\mu_m)^{-1}$ (here $\mu_m = \exp(\sqrt{-1}\pi m)$),
3. if $f$ is constant over $\Sigma_\nu$, then either the underlying Riemann surface $|\Sigma_\nu|$ has nonzero genus, or $\Sigma_\nu$ contains at least 3 special points where a special point is either an orbifold point inherited from $\Sigma$ or the image of some $\gamma_i$, $i = 1, \ldots, l$, under $\Sigma \to \Sigma'$.

Theorem 1.13 can be proved in the framework of [8] along the lines of Parker-Wolfson [39] or Ye [57], particularly, with help of an orbifold version of Arzela-Ascoli Theorem. A version of Theorem 1.13 was proved in [13] in an *ad hoc* manner without using the theory in [8].

**Theorem 1.14** (Orbifold Arzela-Ascoli Theorem, cf. [8]) Let $X, X'$ be any complete Riemannian orbifolds where $X$ is compact (with or without boundary).
For any sequence of $C^r$ maps from $X$ to $X'$ which have bounded $C^r$ norms, there is a subsequence which converges to a $C^{r-1}$ map in the $C^{r-1}$ topology.

2 Adjunction and intersection formulae

In this section, we explain the adjunction and intersection formulae for pseudoholomorphic curves in an almost complex 4-orbifold, which were proved in [9] and are the orbifold versions of the relevant results of Gromov [19] and McDuff [34, 35].

First of all, to set the stage we let $(X, J)$ be a closed, reduced almost complex 4-orbifold. A $J$-holomorphic curve $C$ in $X$ is a closed subset which is the image of a $J$-holomorphic map $f : \Sigma \to X$. One can always arrange that the following conditions hold.

- The map $f$ is not multiply covered, i.e., the induced map is not multiply covered.
- For all but at most finitely many regular points $z \in \Sigma$, the homomorphism $\rho_z$ in a local representative $(\tilde{f}_z, \rho_z)$ of $f$ at $z$ is an isomorphism. It is easily seen that with this assumption, the order $m_\Sigma$ of the isotropy groups at regular points of $\Sigma$ depends on $C$ only and equals the order of the isotropy group $G_p$ at a generic point $p \in C$. We will denote this number by $m_C$ and call it the \textquoteleft multiplicity\textquoteright of $C$.

Such a $J$-holomorphic map $f$ is called a \textquoteleft parametrization\textquoteright of $C$. Note that $\Sigma$ may not be reduced, in which case $m_\Sigma = m_C > 1$. See [9] for more details.

Next we define an intersection product for any two $J$-holomorphic curves and a pairing between a class in $H^2(X; \mathbb{Q})$ and a $J$-holomorphic curve, cf. [9].

**Definition 2.1** (1) For any $J$-holomorphic curve $C$ in $X$, the \textquoteleft Poincaré dual\textquoteright of $C$ is defined to be the class $PD(C) \in H^2(X; \mathbb{Q})$ which is uniquely determined by

$$PD(C) \cup \alpha[X] = m_C^{-1} \alpha[C], \ \forall \alpha \in H^2(X; \mathbb{Q}),$$

where $[C]$ is the class of $C$ in $H_2(X; \mathbb{Z})$. (Note: $X$ and $C$ are canonically oriented by $J$, and the cup product on $H^2(X; \mathbb{Q})$ is non-degenerate because $X$ is a $\mathbb{Q}$-homology manifold.)

(2) The intersection number of two $J$-holomorphic curves $C, C'$ (not necessarily distinct) is defined to be

$$C \cdot C' = PD(C) \cup PD(C')[X].$$

(3) The pairing of a class $\alpha \in H^2(X; \mathbb{Q})$ with a $J$-holomorphic curve $C$ is defined to be

$$\alpha \cdot C = \alpha \cup PD(C)[X], \ \forall \alpha \in H^2(X; \mathbb{Q}).$$

The point here is that the fundamental class of an orbifold should be counted only as a fraction of the fundamental class of the underlying space when the orbifold is not reduced (cf. [13, 9]). Note that Definition 2.1 coincides with the usual definition when $m_C = m_{C'} = 1$, which is always the case when $X$ has only isolated singularities.

Now we digress on the local intersection number and local self-intersection number of pseudoholomorphic discs in $\mathbb{C}^2$. To this end, we recall the following local analytic property of pseudoholomorphic curves. Let $D = \{z \mid |z| < 1\}$ be the unit disc in $\mathbb{C}$, and let $f : (D, 0) \to (\mathbb{C}^2, 0)$ be a $J$-holomorphic map which is embedded on $D \setminus \{0\}$. Then for any sufficiently small $\epsilon > 0$, there is an almost complex structure $J_\epsilon$ and a $J_\epsilon$-holomorphic immersion $f_\epsilon$ such that as $\epsilon \to 0$, $J_\epsilon \to J$ in $C^1$.
topology and \( f_e \to f \) in \( C^2 \) topology. Moreover, given any annuli \( \{ \lambda \leq |z| < 1 \} \) and \( \{ \lambda' \leq |z| \leq \lambda \} \) in \( D \), one can arrange to have \( f = f_e \) in \( \{ \lambda \leq |z| < 1 \} \) and to have \( J_e = J \) except in a chosen neighborhood of the image of \( \{ \lambda' \leq |z| \leq \lambda \} \) under \( f \) by letting \( \epsilon > 0 \) sufficiently small. See [34, 35] for more details.

**Definition 2.2** (See [34, 35], and also [9])

1. Let \( C, C' \) be any \( J \)-holomorphic discs in \( \mathbb{C}^2 \) parametrized by \( f : (D, 0) \to (\mathbb{C}^2, 0), f' : (D, 0) \to (\mathbb{C}^2, 0) \), such that both \( f, f' \) are embedded on \( D \setminus \{ 0 \} \), and \( C, C' \) intersect at \( 0 \in \mathbb{C}^2 \) only. Then the local intersection number \( C \cdot C' \) is defined using the following recipe: perturb \( f, f' \) into \( J_e \)-holomorphic immersions \( f_e, f'_e \), then

\[
C \cdot C' = \sum_{\{(z,z') : f_e(z) = f'_e(z') \}} t_{(z,z')} \tag{1}
\]

where \( t_{(z,z')} = 1 \) when \( f_e(z) = f'_e(z') \) is a transverse intersection, and \( t_{(z,z')} = n \geq 2 \) when \( f_e(z) = f'_e(z') \) has tangency of order \( n \).

2. Let \( C \) be any \( J \)-holomorphic disc in \( \mathbb{C}^2 \) parametrized by \( f : (D, 0) \to (\mathbb{C}^2, 0) \) where \( f \) is embedded on \( D \setminus \{ 0 \} \). Then the local self-intersection number \( C \cdot C \) is defined using the following recipe: perturb \( f \) into a \( J_e \)-holomorphic immersion \( f_e \), then

\[
C \cdot C = \sum_{\{(z,z') : z \neq z', f_e(z) = f_e(z') \}} t_{[z,z']} \tag{2}
\]

where \([z,z']\) denotes the unordered pair of \( z, z' \), and where \( t_{[z,z']} = 1 \) when \( f_e(z) = f_e(z') \) is a transverse intersection, and \( t_{[z,z']} = n \geq 2 \) when \( f_e(z) = f_e(z') \) has tangency of order \( n \).

We remark that: (1) the local intersection number \( C \cdot C' \) depends only on the germs of \( C, C' \) at \( 0 \in \mathbb{C}^2 \), it is always positive, and \( C \cdot C' = 1 \) iff \( C, C' \) are both embedded and intersect at \( 0 \in \mathbb{C}^2 \) transversely, and (2) the local self-intersection number \( C \cdot C \) depends only on the germ of \( C \) at \( 0 \in \mathbb{C}^2 \), and it is non-negative with \( C \cdot C = 0 \) iff \( C \) is embedded.

Before stating the orbifold versions of the adjunction and intersection formulae, we first introduce the relevant notation involved in the statements.

(1) Let \( C \) be a \( J \)-holomorphic curve parametrized by \( f : \Sigma \to X \). For any \( z \in \Sigma \) with \( p = f(z) \), let \( (\hat{f}_z, \rho_z) \) be a local representative of \( f \) at \( z \) and \((V_p, G_p)\) be the local uniformizing system at \( p \). Recall that pseudoholomorphic curves (in manifolds) have only isolated singular points (cf. [34, 35]). Since \( f \) is not multiply covered, it is easily seen that we may assume without loss of generality that \( \hat{f}_z \) is an embedding in \( V_p \setminus \{ 0 \} \). (See [9] for more details.) With this understood, we introduce

\[
\Lambda(C)_z \equiv \{ \text{Im} \ (g \circ \hat{f}_z) | g \in G_p \},
\]

which is a set of \( J \)-holomorphic discs in \( V_p \subset \mathbb{C}^2 \) whose elements are naturally parametrized by the coset \( G_p/\text{Im} \rho_z \). We call an element of \( \Lambda(C)_z \) a ‘local representative’ of \( C \) at \( z \). (In connection with \( \Lambda(C)_z \), observe that when \( p \) is an isolated singular point, \( (\hat{f}_z, \rho_z) \) can be interpreted geometrically as follows: set \( U_p \equiv V_p/G_p \), then \( \text{Im} \ \hat{f}_z \setminus \{ 0 \} \) is a connected component of the inverse image of \( C \cap U_p \setminus \{ p \} \) under the finite covering \( V_p \setminus \{ 0 \} \to U_p \setminus \{ p \} \), and \( \text{Im} \ \rho_z \) is the subgroup of deck transformations which leaves \( \text{Im} \ \hat{f}_z \) invariant.)
(2) For any J-holomorphic curve C in X, the ‘virtual genus’ of C is defined to be
\[ g(C) = \frac{1}{2}(C \cdot C - c_1(TX) \cdot C) + \frac{1}{m_C}. \]
Here \( m_C \) is the multiplicity of C, i.e., the order of \( G_p \) at a generic point \( p \in C \).

(3) Let \( \Sigma \) be an orbifold Riemann surface whose orbifold points have orders \( m_i \), \( i = 1, \ldots, k \), and whose regular points have order \( m_{\Sigma} \), and let \( g[\Sigma] \) be the genus of the underlying Riemann surface. Then the ‘orbifold genus’ of \( \Sigma \) is defined to be
\[ g_\Sigma = \frac{g[\Sigma]}{m_{\Sigma}} + \sum_{i=1}^{k} \left( \frac{1}{2m_i} - \frac{1}{2m_{\Sigma}} \right). \]

Note that with this definition, \( c_1(T\Sigma) \cdot [\Sigma] = 2m_{\Sigma}^{-1} - 2g_\Sigma \). (Here \( [\Sigma] \) is \( m_{\Sigma}^{-1} \) times the fundamental class of the underlying Riemann surface, see the remarks below Definition 2.1.)

Now we state the adjunction and intersection formulae in [9].

**Theorem 2.3 (Adjunction Formula)** Let C be a J-holomorphic curve which is parametrized by \( f: \Sigma \rightarrow X \). Then
\[ g(C) = g_\Sigma + \sum_{\{z,z'\}|z \neq z', f(z) = f(z')} k_{[z,z']} + \sum_{z \in \Sigma} k_z, \]
where \([z,z']\) denotes the unordered pair of \( z, z' \), and \( k_{[z,z']}, k_z \) are defined as follows.

- Let \( G_{[z,z']} \) be the isotropy group at \( f(z) = f(z') \) and \( \Lambda(C)_z = \{C_{z,\alpha}\}, \Lambda(C)_{z'} = \{C_{z',\alpha'}\} \), then
  \[ k_{[z,z']} = \frac{1}{|G_{[z,z']}|} \sum_{\alpha,\alpha'} C_{z,\alpha} \cdot C_{z',\alpha'}. \]

- Let \( G_z \) be the isotropy group at \( f(z) \) and \( \Lambda(C)_z = \{C_{z,\alpha}\} \), then
  \[ k_z = \frac{1}{2|G_z|} \left( \sum_{\alpha} C_{z,\alpha} \cdot C_{z,\alpha} + \sum_{\alpha,\beta} C_{z,\alpha} \cdot C_{z,\beta} \right). \]

(Note: the second sum \( \sum_{\alpha,\beta} C_{z,\alpha} \cdot C_{z,\beta} \) is over all \( \alpha, \beta \) which are not necessarily distinct, so it is the same as the self-intersection \( (\sum_{\alpha} C_{z,\alpha}) \cdot (\sum_{\alpha} C_{z,\alpha}) \).)

**Theorem 2.4 (Intersection Formula)** Let \( C, C' \) be distinct J-holomorphic curves parametrized by \( f: \Sigma \rightarrow X, f': \Sigma' \rightarrow X \) respectively. Then the intersection number
\[ C \cdot C' = \sum_{\{z,z'\}|f(z) = f'(z')} k_{(z,z')} \]
where \( k_{(z,z')} \) is defined as follows. Let \( G_{(z,z')} \) be the isotropy group at \( f(z) = f'(z') \) and \( \Lambda(C)_z = \{C_{z,\alpha}\}, \Lambda(C')_{z'} = \{C'_{z',\alpha'}\} \), then
\[ k_{(z,z')} = \frac{1}{|G_{(z,z')}|} \sum_{\alpha,\alpha'} C_{z,\alpha} \cdot C'_{z',\alpha'}. \]

(Note: the intersection formula is topological in nature in the sense that it can be established for more general oriented surfaces, singular or not, as long as the local intersection numbers \( C_{z,\alpha} \cdot C'_{z',\alpha'} \) can be properly defined.)

Before we illustrate these formulae with examples, let’s first make some observations.
Remark 2.5 (1) The adjunction formula implies \( g(C) \geq g_Z \), with \( g(C) = g_Z \) iff \( k_{\{z,z'\}} = 0 \) and \( k_z = 0 \). The geometric meaning of these conditions is that \( C \) is topologically embedded and at each singular point \( p = f(z) \), \( C \) has only one local representative which is embedded (particularly, \( \text{Im } \rho_z = G_z \equiv G_p \)). This is equivalent to saying that \( C \) is a suborbifold.

(2) In many cases \( \text{Im } \rho_z \neq G_p \), for instance, when \( G_p \) is non-abelian and \( p \) is isolated (hence \( \text{Im } \rho_z \) is abelian). It is useful to observe that in this case,

\[
k_z \geq \frac{1}{2m_z} \left( \frac{|G_p|}{m_z} - 1 \right) \geq \frac{1}{2m_z}, \quad \text{where } m_z \text{ is the order of the isotropy group at } z.
\]

Moreover, \( k_z = \frac{1}{2m_z} \left( \frac{|G_p|}{m_z} - 1 \right) \) iff each local representative of \( C \) at \( z \) is embedded and any two different ones intersect transversely.

(3) Assume \( X \) has only isolated singularities. Then a combination of the adjunction and intersection formulae and the index formula may be used to extract information about the local structure of a \( J \)-holomorphic curve near a singular point. To be more precise, let \( \tilde{f}_z : (D_z, \mathbb{Z}_{m_z}) \to (\mathbb{V}_p, G_p) \) be a local representative of \( f \) at \( z \) where \( p = f(z) \in X \) is a singular point, and let \( 0 < m_1, m_2 < m_z \) such that the action of \( \text{Im } \rho_z \) on \( \mathbb{V}_p \) is given by

\[
\rho_z(\mu_{m_z}) \cdot (z_1, z_2) = (\mu_{m_z}^m z_1, \mu_{m_z}^m z_2), \quad \text{where } \mu_m = \exp(\sqrt{-1} \frac{2\pi}{m}).
\]

Then on the one hand, \( m_1, m_2 \) are constrained by the appearance in the index formula associated to the dimension of the moduli space of \( J \)-holomorphic curves (cf. the paragraph below Lemma 1.10), and on the other hand, \( m_1, m_2 \) are related to \( k_{\{z,z'\}}, k_z, \) and \( k_{\{z,z'\}} \) as follows: By assuming \( J \) integrable near the singular points (note: this does not effect Lemma 1.10), we may write \( \tilde{f}_z(w) = (u(w), v(w)) \) for some holomorphic functions \( u(w) = aw^{l_1} + \cdots \) and \( v(w) = bw^{l_2} + \cdots \), where \( l_i \equiv m_i \mod m_z \), \( i = 1, 2 \). As shown in Lemma 2.6 below, \( k_{\{z,z'\}}, k_z, k_{\{z,z'\}} \) can be estimated in terms of \( l_1, l_2 \), hence in terms of \( m_1, m_2 \).

\[\square\]

Lemma 2.6 (1) Let \( C \) be a holomorphic disc in \( \mathbb{C}^2 \) parametrized by \( f(z) = (a(z^{l_1} + \cdots), z^{l_2}) \), which is embedded in \( \mathbb{C}^2 \setminus \{0\} \). Then the local self-intersection number satisfies \( C \cdot C \geq \frac{1}{2}(l_1 - 1)(l_2 - 1) \).

(2) Let \( C, C' \) be distinct holomorphic discs in \( \mathbb{C}^2 \) which are parametrized by \( f(z) = (a(z^{l_1} + \cdots), z^{l_2}) \) and \( f'(z) = (a'(z^{l_1'} + \cdots), z^{l_2'}) \) respectively, such that \( f, f' \) are embedded in \( \mathbb{C}^2 \setminus \{0\} \). Then the local intersection number \( C \cdot C' \geq \min(l_1l_2', l_2l_1') \). Here \( l_1 = \infty \) (resp. \( l_1' = \infty \)) if \( a = 0 \) (resp. \( a' = 0 \)).

Lemma 2.6 is straightforward from the definition of local intersection and self-intersection numbers in Definition 2.2, cf. [34, 35].

We end this section with the following example.

Example 2.7 Consider the weighted projective space \( \mathbb{P}(d_1, d_2, d_3) \), which is the quotient of \( \mathbb{C}^3 \setminus \{0\} \) under the \( \mathbb{C}^* \)-action

\[
z \cdot (z_1, z_2, z_3) = (z_1^{d_1}, z_2^{d_2}, z_3^{d_3}), \quad \forall z \in \mathbb{C}^* \equiv \mathbb{C} \setminus \{0\}.
\]

Here we assume that \( d_1, d_2, d_3 \) are pairwise relatively prime and satisfy \( 1 < d_1 < d_2 < d_3 \). Then \( \mathbb{P}(d_1, d_2, d_3) \) is canonically a complex 4-orbifold, with three isolated singular points \( p_1 = [1 : 0 : 0], p_2 = [0 : 1 : 0] \) and \( p_3 = [0 : 0 : 1] \) of isotropy group \( \mathbb{Z}_{d_1}, \mathbb{Z}_{d_2}, \mathbb{Z}_{d_3} \), respectively. Moreover, the set of orbifold complex line bundles over \( \mathbb{P}(d_1, d_2, d_3) \) is generated by the holomorphic orbifold line bundle \( E_0 \equiv \mathbb{P}(d_1, d_2, d_3) \).
We will say that an orbifold complex line bundle $E$ (or the Poincaré dual of $c_1(E)$) has degree $d$ if $E = E^{td}_d$. It is easy to check that the canonical bundle $K$ has degree $-(d_1 + d_2 + d_3)$. Finally, we observe that $c_1(E_0) \cdot c_1(E_0) = (d_1d_2d_3)^{-1}$. 

With the preceding understood, let’s apply the adjunction formula to the holomorphic curve $C \equiv \{z_1 = 0\}$, which has degree $d_1$ and passes through $p_2, p_3$. The virtual genus

$$g(C) = \frac{1}{2}(C \cdot C + c_1(K) \cdot C) + 1 = \frac{1}{2}(\frac{d_1^2d_2^2 + (d_1+d_2+d_3)d_1d_2d_3}{d_1d_2d_3}) + 1 = 1 - \frac{1}{2d_2} \cdot \frac{1}{2d_3}.$$ 

On the other hand, each singular point $p_2$ or $p_3$ will contribute at least $\frac{1}{2} - \frac{1}{2d_2}$ or $\frac{1}{2} - \frac{1}{2d_3}$ to the right hand side of the adjunction formula. It follows easily that $C$ is a suborbifold.

Next we look at the family of holomorphic curves

$$C_\lambda \equiv \{az^d + bz^d = 0\}, \text{ where } \lambda = [a:b] \in \mathbb{CP}^1 \text{ with } ab \neq 0.$$ 

Each $C_\lambda$ has degree $d_2d_3$ and passes through the singular point $p_1$. To apply the adjunction formula to $C_\lambda$, we first calculate

$$g(C_\lambda) = \frac{1}{2}(\frac{(d_2d_3)^2}{d_1d_2d_3} - \frac{(d_1+d_2+d_3)d_2d_3}{d_1d_2d_3}) + 1 = \frac{1}{2} - \frac{1}{2d_1} + \frac{(d_2 - 1)(d_3 - 1)}{2d_1}.$$ 

On the other hand, by Lemma 2.6 (1), the contribution $k_2$ at $p_1$ is at least $\frac{(d_2 - 1)(d_3 - 1)}{2d_1}$. It follows easily that $C_\lambda$ is a 2-sphere with only one singularity $p_1$.

Finally, we consider the intersection of $C_\lambda, C_{\lambda'}$ where $\lambda \neq \lambda'$. First, observe that $C_\lambda, C_{\lambda'}$ intersect at $p_1$ with local contribution $k_{(z,z')}$ of at least $\frac{d_2d_3}{d_1}$ by Lemma 2.6 (2). On the other hand, $C_\lambda \cdot C_{\lambda'} = \frac{(d_2d_3)^2}{d_1d_2d_3} = \frac{d_2d_3}{d_1}$. The intersection formula implies that $C_\lambda \cap C_{\lambda'} = \{p_1\}$.

All of the claims above can be easily verified directly. 

3 Seiberg-Witten theory and Taubes’ theorems

In this section, we first give a brief summary of the Seiberg-Witten theory for smooth 4-orbifolds, which is completely parallel to that for smooth 4-manifolds (see e.g. [24] for a nice introduction to Seiberg-Witten theory of smooth 4-manifolds). We then discuss the symplectic 4-orbifold analog of the theorems of Taubes in [24] concerning nonvanishing of the Seiberg-Witten invariant and the existence of certain pseudoholomorphic curves. At the end of this section, we present an example which combines various ingredients of the pseudoholomorphic curve theory and show the existence of a symplectic 2-sphere of degree $d_1$ or $-d_1$ in the weighted projective space $\mathbb{P}(d_1, d_2, d_3)$ in Example 2.7 for any given symplectic structure.

All 4-orbifolds in this section are assumed to be reduced, closed and oriented. Let $X$ be a 4-orbifold. We denote by $b_i(X)$ the dimension of $H^i(X; \mathbb{Q})$, and by $b^+_2(X)$ the dimension of the maximal positive subspace for the cup product on $H^2(X; \mathbb{Q})$.

Now let $X$ be a smooth 4-orbifold. Given any Riemannian metric on $X$, a $Spin^c$ structure is an orbifold principal $Spin^c(4)$ bundle over $X$ which descends to the orbifold principal $SO(4)$ bundle of oriented orthonormal frames under the canonical homomorphism $Spin^c(4) \to SO(4)$. There are two associated orbifold
$U(2)$ vector bundles (of rank 2) $S_+, S_-$ with $\det(S_+) = \det(S_-)$, and a Clifford multiplication which maps $T^*X$ into the skew adjoint endomorphisms of $S_+ \oplus S_-$.  

The Seiberg-Witten equations associated to the $Spin^c$ structure (if there is one) are equations for a pair $(A, \psi)$, where $A$ is a connection on $\det(S_+)$ and $\psi$ is a section of $S_+$. Recall that the Levi-Civita connection together with $A$ defines a covariant derivative $\nabla_A$ on $S_+$. On the other hand, there are two maps $\sigma : S_+ \otimes T^*X \to S_-$ and $\tau : \text{End}(S_+) \to \Lambda_+ \otimes \mathbb{C}$ induced by the Clifford multiplication, with the latter being the adjoint of $c_+ : \Lambda_+ \to \text{End}(S_+)$, where $\Lambda_+$ is the orbifold bundle of self-dual 2-forms. With these understood, the Seiberg-Witten equations read

$$D_A \psi = 0 \text{ and } P_+ F_A = \frac{1}{4} \tau(\psi \otimes \psi^*) + \mu,$$

where $D_A \equiv \sigma \circ \nabla_A$ is the Dirac operator, $P_+ : \Lambda^2 T^*X \to \Lambda_+$ is the orthogonal projection, and $\mu$ is a fixed, imaginary valued, self-dual 2-form which is added as a perturbation term.

The Seiberg-Witten equations are invariant under the gauge transformations $(A, \psi) \mapsto (A - 2\phi^{-1} d\phi, \phi \psi)$, where $\phi \in C^\infty(X; S^1)$ are circle-valued smooth functions on $X$. The space of solutions modulo gauge equivalence, denoted by $M$, is compact, and when $b_2^+(X) \geq 1$ and when it is nonempty, $M$ is a smooth orientable manifold for a generic choice of $(g, \mu)$, where $g$ is the Riemannian metric and $\mu$ is the self-dual 2-form of perturbations. Furthermore, $M$ contains no classes of reducible solutions (i.e., those with $\psi \equiv 0$), and if we let $M^0$ be the space of solutions modulo the based gauge group, i.e., those $\phi \in C^\infty(X; S^1)$ such that $\phi(p_0) = 1$ for a fixed base point $p_0 \in X$, then $M^0 \to M$ defines a principal $S^1$-bundle. Let $\mathcal{C}$ be the first Chern class of $M^0 \to M$, $d = \dim M$, and fix an orientation of $M$. Then the Seiberg-Witten invariant associated to the $Spin^c$ structure is defined as follows.

- When $d < 0$ or $d = 2n + 1$, the Seiberg-Witten invariant is zero.
- When $d = 0$, the Seiberg-Witten invariant is a signed count of points in $M$.
- When $d = 2n > 0$, the Seiberg-Witten invariant equals $\mathcal{C}[M]$.

As in the case of smooth 4-manifolds, the Seiberg-Witten invariant of $X$ is well-defined when $b_2^+(X) \geq 2$, depending only on the diffeomorphism class of $X$ (as orbifolds). Moreover, there is an involution on the set of $Spin^c$ structures which preserves the Seiberg-Witten invariant up to a change of sign. When $b_2^+(X) = 1$, there is a chamber structure and the Seiberg-Witten invariant also depends on the chamber which the pair $(g, \mu)$ is in. Moreover, the change of the Seiberg-Witten invariant when crossing a wall of the chambers can be similarly analyzed as in the smooth 4-manifold case.

Above is a brief summary of the Seiberg-Witten theory for smooth 4-orbifolds. In the case of global quotient $X = Y / G$, it corresponds to the $G$-equivariant theory on $Y$.

Now we focus on the case where $X$ is a symplectic 4-orbifold. Let $\omega$ be a symplectic form on $X$. We orient $X$ by $\omega \wedge \omega$, and fix an $\omega$-compatible almost complex structure $J$. Then with respect to the associated Riemannian metric $g = \omega(\cdot, J \cdot)$, $\omega$ is self-dual with $|\omega| = \sqrt{2}$. The set of $Spin^c$ structures on $X$ is nonempty. In fact, the almost complex structure $J$ gives rise to a canonical $Spin^c$ structure where the associated orbifold $U(2)$ bundles are $S_+^0 = I \oplus K_X^{-1}$, $S_-^0 = T^{0,1}X$. Here $I$ is the trivial orbifold complex line bundle and $K_X$ is the canonical bundle $\det(T^{1,0}X)$. Moreover, the set of $Spin^c$ structures is canonically identified with the set of orbifold complex line bundles where each orbifold complex line bundle $E$ corresponds to a
Section \( \psi \) A

Furthermore, by fixing such an equation, by a real number \( X \) to a distinguished family of the Seiberg-Witten equations on \( X \), which is considered as the section of \( \psi \), which is parametrized by a connection \( A \) on \( X \). This particular section \( \psi \) of \( E \) is determined by a connection \( A \) on \( X \). With these understood, there is a distinguished family of the Seiberg-Witten equations on \( X \), which is parametrized by a real number \( r > 0 \) and is for a triple \( (a, \alpha, \beta) \), where in the equations, the section \( \psi \) of \( E \) is written as \( \psi = \sqrt{r} \) and the perturbation term \( \mu \) is taken to be \(- \sqrt{-1} (4^{-1} r \omega) + P_+ F_A \). (Here \( \alpha \) is a section of \( E \) and \( \beta \) a section of \( K_X^{-1} \).)

Note that when \( b_2^+ (X) = 1 \), this distinguished family of Seiberg-Witten equations belongs to a specific chamber for the Seiberg-Witten invariant. This particular chamber will be referred to as the Taubes' chamber. Now consider

**Theorem 3.1** Let \((X, \omega)\) be a symplectic 4-orbifold. Then the following are true.

1. The Seiberg-Witten invariant (in the Taubes' chamber when \( b_2^+ (X) = 1 \)) associated to the canonical \( \text{Spin}^c \) structure equals \( \pm 1 \). In particular, the Seiberg-Witten invariant corresponding to the canonical bundle \( K_X \) equals \( \pm 1 \) when \( b_2^+ (X) \geq 2 \).

2. Let \( E \) be an orbifold complex line bundle. Suppose there is an unbounded sequence of values for the parameter \( r \) such that the corresponding Seiberg-Witten equations have a solution \((a, \alpha, \beta)\). Then for any \( \omega \)-compatible almost complex structure \( J \), there are \( J \)-holomorphic curves \( C_1, C_2, \ldots, C_k \) in \( X \) and positive integers \( n_1, n_2, \ldots, n_k \) such that \( c_1 (E) = \sum_{i=1}^k n_i PD (C_i) \). Moreover, if a closed subset \( \Omega \subset X \) is contained in \( a^{-1} (0) \) throughout, then \( \Omega \subset \cup_{i=1}^k C_i \) also.

(Here a \( J \)-holomorphic curve in \( X \) is as defined in the beginning of \S 2. For the definition of \( PD (C) \), the Poincaré dual of \( C \), see Definition 2.1.)

Theorem 3.1 is the orbifold analog of the relevant theorems of Taubes in [14, 40]. A proof of Theorem 3.1 can be found in [10].

**Remark 3.2** (1) Here is a typical source for the closed subset \( \Omega \) in Theorem 3.1. Suppose \( p \in X \) is a singular point such that \( G_p \) acts nontrivially on the fiber of \( E \) at \( p \). Then \( p \in a^{-1} (0) \) for any solution \((a, \alpha, \beta)\), and consequently \( p \in \cup_{i=1}^k C_i \).

Now let's try this for the case where \( X = Y / \mathbb{Z}_n \) and \( E = K_X \). It follows that if the space of \( \mathbb{Z}_n \)-equivariant self-dual harmonic 2-forms on \( Y \) has dimension \( \geq 2 \), then for any given \( \mathbb{Z}_n \)-equivariant \( J \) on \( Y \), \( c_1 (K_Y) = \sum_{s} n_s PD (C_s) \) for a finite set of \( J \)-holomorphic curves \( \{ C_s \} \) in \( Y \), such that \( \cup_s C_s \) is \( \mathbb{Z}_n \)-invariant, and contains any fixed point of \( \mathbb{Z}_n \), at which the complex representation of \( \mathbb{Z}_n \) on the tangent space has weights other than \((1, n-1)\).

(2) Even though, as we remarked in (1) above, that the \( J \)-holomorphic curves \( \{ C_i \} \) contain certain singular points of \( X \), nothing can be said directly from the theorem about the local structure of these curves near the singular points. To obtain such information, one may use a combination of the adjunction and intersection formulae plus the index formula, cf. Remark 2.5 (3). Here we point out another tactic, which was used in [10]. Suppose the Seiberg-Witten invariant corresponding
to $E$ is nonzero and the dimension of the corresponding Seiberg-Witten moduli space $M$ is $d = 2n > 0$. Then the $J$-holomorphic curves $\{C_i\}$ may be required to contain any given subset $\Omega$ of less than or equal to $n$ points in $X$. One may further combine this with the Gromov compactness theorem (cf. Theorem 1.13) by varying $\Omega$ to produce more desirable $J$-holomorphic curves.

In light of Remark 3.2 (2), we give a formula for the dimension of the Seiberg-Witten moduli space. For simplicity, we assume $X$ has only isolated singular points. For the general case, a formula can be found in Appendix A of [10].

**Lemma 3.3** Suppose the symplectic 4-orbifold $X$ has only isolated singular points $p_1, \cdots, p_l$. Then for any orbifold complex line bundle $E$, the dimension $d(E)$ of the Seiberg-Witten moduli space corresponding to $E$ is given by

$$d(E) = c_1(E)^2 - c_1(E) \cdot c_1(K_X) + \sum_{i=1}^{l} I_i,$$

where $K_X$ is the canonical bundle, and $I_i$ can be determined from the singular point $p_i$ as follows. Let $\rho_{p_i, E}: G_{p_i} \to \mathbb{C}^*$ be the complex representation of $G_{p_i}$ on the fiber of $E$ at $p_i$, and $\rho_{p_i, j}(g) = \exp(\sqrt{-1} \delta_j g)$, $j = 1, 2$, be the eigenvalues of $g \in G_{p_i}$ associated to the complex representation of $G_{p_i}$ on the tangent space at $p_i$. Then

$$I_i = \frac{1}{|G_{p_i}|} \sum_{g \in G_{p_i}\{e\}} \frac{2(\rho_{p_i, E}(g) - 1)}{\prod_{j=1}^{2}(1 - \rho_{p_i, j}(g^{-1}))}, \forall i = 1, \cdots, l.$$

We illustrate the formula with the following example.

**Example 3.4** Let $E$ be the orbifold complex line bundle of degree $d_1$ over the weighted projective space $\mathbb{P}(d_1, d_2, d_3)$ in Example 2.7. We will show that $d(E) = 0$.

Let $K$ be the canonical bundle. By Lemma 3.3,

$$d(E) = c_1(E)^2 - c_1(E) \cdot c_1(K) + I_1 + I_2 + I_3,$$

where $I_1 = 0$, and with $\mu_m = \exp(\sqrt{-1} \frac{2\pi}{m})$,$$

I_2 = \frac{1}{d_2} \sum_{x=1}^{d_2-1} \frac{2(\mu_{d_2}^{d_1}x - 1)}{(1 - \mu_{d_2}^{d_1}x)(1 - \mu_{d_2}^{-d_1}x)}, I_3 = \frac{1}{d_3} \sum_{x=1}^{d_3-1} \frac{2(\mu_{d_3}^{d_1}x - 1)}{(1 - \mu_{d_3}^{d_1}x)(1 - \mu_{d_3}^{-d_1}x)}.$$

We claim that

$$I_2 = \frac{d_2 - 1 - 2\delta_2}{d_2}, I_3 = \frac{d_3 - 1 - 2\delta_3}{d_3},$$

where $\delta_2, \delta_3$ are uniquely determined by the conditions $0 \leq \delta_2 \leq d_2 - 1$, $0 \leq \delta_3 \leq d_3 - 1$, and $d_1 - d_3 \delta_2 = 0$ (mod $d_2$), $d_1 - d_2 \delta_3 = 0$ (mod $d_3$). Assuming validity of the claim, we have

$$d(E) = \frac{d_1^2}{d_1 d_2 d_3} - \frac{d_2}{d_1 d_2 d_3} + \frac{d_2}{d_2 d_3} + \frac{d_3}{d_2 d_3} = \frac{2(d_1 + d_2 d_3 - \delta_3 d_2 - \delta_2 d_3)}{d_2 d_3}.$$

Observe that (1) $\delta_3 d_2 + \delta_2 d_3 = d_1$ (mod $d_2 d_3$) and (2) $d_1 < \delta_3 d_2 + \delta_2 d_3 < 2d_2 d_3$, which implies that $\delta_3 d_2 + \delta_2 d_3 = d_1 + d_2 d_3$, and hence $d(E) = 0$. 


It remains to verify the claim. We shall only check for $I_2$, the case for $I_3$ is completely parallel. Introduce

$$\varphi(t) = \sum_{x=1}^{d_2-1} \frac{\mu_{d_2}^{d_1 x}}{1 - \mu_{d_2}^{-d_3 x} t}$$

Then by putting $\varphi(t)$ into power series of $t$, we have

$$\varphi(t) = \sum_{x=1}^{d_2-1} \mu_{d_2}^{d_1 x} \sum_{l=0}^{\infty} \mu_{d_2}^{-(d_1 - d_2)x} t^l$$

$$= \sum_{l=0}^{\infty} \sum_{x=1}^{d_2-1} \mu_{d_2}^{(d_1 - d_2)x} t^l$$

$$= \sum_{l=0}^{\infty} (-1)^l t^l + d_2 t^{d_2} \sum_{l=0}^{\infty} t^{d_2 l}$$

$$= (t - 1)^{-1} + d_2 t^{d_2} (1 - t^{d_2})^{-1},$$

which gives $\varphi(1) = \frac{1}{2}(d_2 - 1) - \delta_2$. Now it follows easily that $I_2 = \frac{2}{d_2} \varphi(1) = \frac{d_2 - 1 - 2\delta_2}{d_2}$.

We end this section by illustrating how the various ingredients are put into use with the following example.

**Example 3.5** We consider the 4-orbifold $X = \mathbb{P}(d_1, d_2, d_3)$, the weighted projective space in Example 2.7. Let $\omega$ be any symplectic structure which evaluates positively on a homology class in $H_2(X; \mathbb{Q})$ of positive degree.

**Claim 1:** For any $\omega$-compatible $J$, there exists a unique $J$-holomorphic 2-sphere of degree $d_1$, which is an embedded suborbifold containing the singular points $p_2, p_3$ but not $p_1$.

The basic idea of the proof goes as follows. Let $E_1$ be the orbifold complex line bundle of degree $d_1$. As in the smooth case, one can show that the Seiberg-Witten invariant corresponding to $E_1$ is nonzero in the Taubes’ chamber, hence by Theorem 3.1, there exists a finite set of $J$-holomorphic curves $\{C_\alpha\}$ such that

$$c_1(E_1) = \sum_\alpha n_\alpha PD(C_\alpha)$$

for some integers $n_\alpha > 0$.

We need to show that $\{C_\alpha\}$ consists of only one element of degree $d_1$, from which Claim 1 follows easily with an application of the adjunction formula. It is fairly easy to show that $\{C_\alpha\}$ contains no more than two elements, but it is harder to show that the single element has degree $d_1$. The point is that the smaller the degree is, the larger the virtual genus is, and the worse the situation is. The key observation here is that, on the other hand, the more singular points the $J$-holomorphic curve contains, the larger the right hand side of the adjunction formula is, and the better the situation is. For this reason, we need to consider simultaneously the orbifold complex line bundle $E_2$ of degree $d_2$.

Now the details of the proof. Let $K_\omega$ be the canonical bundle of $(X, \omega)$. We will first show that $K_\omega$ has degree $-(d_1 + d_2 + d_3)$. To see this, note that if $e(X), p_1(X)$ denote the Euler class and the first Pontryagin class of $X$, then

$$c_1(K_\omega)^2 = 2e(X) + p_1(X) = c_1(K)^2,$$
where \( K \) is the canonical bundle for the standard complex orbifold structure on \( X \), which has degree \(-(d_1 + d_2 + d_3)\). It is clear that \( \deg K_\omega = -(d_1 + d_2 + d_3) \) if we show that \( \deg K_\omega < 0 \), which follows from \( c_1(K_\omega) \cdot [\omega] < 0 \) because \( [\omega] \cdot c_1(E_1) > 0 \) by the assumption on \( \omega \). To see \( c_1(K_\omega) \cdot [\omega] < 0 \), note that \( b_1(X) = 0 \) and \( b_3^+(X) = b_2(X) = 1 \), so that in this case the chamber structure for the Seiberg-Witten invariants is very simple: there are only two chambers, the 0-chamber and the Taubes’ chamber. The Seiberg-Witten invariant of the canonical \( \text{Spin}^C \) structure in the Taubes’ chamber is nonzero by Theorem 3.1, while the invariant is zero in the 0-chamber because \( X \) has a metric of positive scalar curvature (cf. [4].) Thus the wall-crossing number is nonzero, which implies that \( c_1(K_\omega) \cdot [\omega] < 0 \).

Next we claim that the Seiberg-Witten invariants corresponding to \( E_1, E_2 \) are nonzero in the Taubes’ chamber. To see this, note that by the calculation in Example 3.4, the dimension \( d(E_1) \) of the Seiberg-Witten moduli space is zero. A similar calculation shows that \( d(E_2) \geq 0 \). It is easily seen that the claim follows from a wall-crossing argument by observing that \( c_1(E_i) \cdot [\omega] > 0, i = 1, 2 \), and that \( X \) has a metric of positive scalar curvature.

Now by Theorem 3.1, there exist finite sets of \( J \)-holomorphic curves \( \{C_\alpha\}, \{C_\beta\} \) such that 
\[
c_1(E_1) = \sum_\alpha n_\alpha PD(C_\alpha), \quad c_1(E_2) = \sum_\beta n_\beta PD(C_\beta)
\]
for some integers \( n_\alpha, n_\beta > 0 \).

Moreover, it is easily seen that at \( p_2, p_3 \), the isotropy groups \( G_{p_2}, G_{p_3} \) act non-trivially on the fiber of \( E_1 \), hence \( p_2, p_3 \in \cup_\alpha C_\alpha \) (cf. Remark 3.2 (1)). Similarly, \( p_1, p_3 \in \cup_\beta C_\beta \).

Before we proceed further, let’s make the following observation.

**Claim 2:** The degree of a \( J \)-holomorphic curve in \( X = \mathbb{P}(d_1, d_2, d_3) \) is always integral.

Assuming the validity of Claim 2, we first consider the case where there are \( C_1 \in \{C_\alpha\}, C_2 \in \{C_\beta\} \) such that \( C_1 \neq C_2 \). Observe that \( C_1 \cap C_2 \neq \emptyset \), because otherwise, \( C_1 \cdot C_2 = 0 \) which contradicts \( b_2(X) = 1 \). Pick any point \( p \in C_1 \cap C_2 \), and set \( r_i = \deg C_i, i = 1, 2 \). Then by the intersection formula,
\[
\frac{r_1 r_2}{d_1 d_2 d_3} = C_1 \cdot C_2 \geq \frac{1}{|G_p|} \geq \frac{1}{d_3},
\]
which implies that \( r_1 r_2 \geq d_3 d_2 \). But \( r_1 \leq d_1, i = 1, 2 \), so that \( r_1 = d_1, r_2 = d_2 \). Now it is clear that in this case, \( \{C_\alpha\} \) consists of only \( C_1 \) with \( \deg C_1 = d_1 \), which contains both \( p_2, p_3 \). By the adjunction formula, \( C_1 \) is a 2-square embedded as a suborbifold in \( X \) (see Example 2.7). Moreover, \( C_1 \) does not contain the singular point \( p_1 \). Hence Claim 1 is proved in this case except for the uniqueness. To see the uniqueness, suppose there is a \( C_1' \) such that \( \deg C_1' = d_1 \) and \( C_1' \neq C_1 \). Then similarly we have \( C_1 \cap C_1' \neq \emptyset \), and
\[
\frac{d_1^2}{d_1 d_2 d_3} = C_1 \cdot C_1' \geq \frac{1}{d_3},
\]
which contradicts \( d_1 < d_2 \). Hence the uniqueness.

There is still another possibility that both \( \{C_\alpha\}, \{C_\beta\} \) consist of the same, single \( J \)-holomorphic curve \( C_0 \). But this possibility can be ruled out as follows.

Suppose both \( \{C_\alpha\}, \{C_\beta\} \) consist of \( C_0 \). Then \( C_0 \) contains all three singular points \( p_1, p_2 \) and \( p_3 \). Moreover, \( \deg C_0 \in \mathbb{Z} \) by Claim 2, which implies that \( \deg C_0 =
1 because \( \text{deg } C_0 \) is a common divisor of \( d_1, d_2 \). Let \( C_0 \) be parametrized by a \( J \)-holomorphic map \( f : \Sigma \to X \). Since \( p_1, p_2, p_3 \in C_0 \), there are \( z_1, z_2, z_3 \in \Sigma \) such that \( f(z_1) = p_1, f(z_2) = p_2 \) and \( f(z_3) = p_3 \). Let \( m_1, m_2, m_3 \) be the order of \( z_1, z_2, z_3 \) respectively.

First, we observe that if \( m_i < d_i \) where \( i = 1, 2 \) or 3, then the following inequality holds for the contribution \( k_{z_i} \) on the right hand side of the adjunction formula:

\[
k_{z_i} \geq \frac{1}{2d_i} \left( d_i - 1 \right) \frac{d_i}{m_i} \geq \frac{1}{2m_i}
\]

Second, we introduce the expression

\[
I \equiv 2(g(C_0) - 1) + \frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3}.
\]

Then \( \text{deg } C_0 = 1 \) implies that

\[
I = \frac{1}{d_1} + \frac{(d_2 + d_3 - 1)(d_1 - 1)}{d_1 d_2 d_3}.
\]

It follows easily from the adjunction formula that \( I \geq 1 \), which implies that

\[
\frac{1}{d_1} + \frac{d_2 + d_3 - 1}{d_2 d_3} > I \geq 1.
\]

But this contradicts the assumption that \( d_1 > 1 \) and \( d_1, d_2, d_3 \) are pairwise relatively prime. Hence the second possibility is ruled out.

It remains to verify Claim 2, which is a consequence of the adjunction formula. More precisely, let \( C \) be a \( J \)-holomorphic curve and let \( r = \text{deg } C \in \mathbb{Q} \). Then the virtual genus

\[
g(C) = \frac{r^2 - (d_1 + d_2 + d_3)r}{2d_1 d_2 d_3} + 1.
\]

On the other hand, suppose \( C \) is parametrized by a \( J \)-holomorphic map \( f : \Sigma \to X \) with orbifold genus \( g_C = g_{[\Sigma]} + \sum_{i=1}^{3} (\frac{1}{2} - \frac{1}{2m_i}) \), where \( m_i \) are the orders of the orbifold points on \( \Sigma \). Then each \( m_i \) is a divisor of either \( d_1, d_2 \) or \( d_3 \). It is easily seen from the adjunction formula that \( r^2 - (d_1 + d_2 + d_3)r \in \mathbb{Z} \), which implies that \( r = \text{deg } C \in \mathbb{Z} \).

This completes the proof of Claim 1.

Let \( C \) be the \( J \)-holomorphic 2-sphere in Claim 1. It is interesting to compare \( C \) with the standard 2-sphere \( C_0 \equiv \{ z_1 = 0 \} \). In this regard, we have

**Claim 3:** \( (X,C) \) and \( (X,C_{0}) \) are diffeomorphic as orbifold pairs.

To see this, we first note that \( C \) and \( C_0 \) have isomorphic normal bundles hence have diffeomorphic regular neighborhoods in \( X \). This is because the bundles have the same Euler number (which is \( C \cdot C = C_0 \cdot C_0 = \frac{d_1 d_2 d_3}{2d_1 d_2 d_3} \)) hence the same Seifert invariant, due to the fact that \( d_2, d_3 \) are relatively prime. Secondly, the complements of the regular neighborhoods are also diffeomorphic, i.e., \( C \) is not knotted in \( X \). This follows by observing that the complement \( W \) of a regular neighborhood of \( C \cup \{ p_1 \} \) is a symplectic \( \mathbb{Z} \)-homology cobordism between lens spaces (with a canonical contact structure on the boundary, see §4.1 below for more details). By the main result in [7], \( W \) is smoothly a product, which shows that \( C \) is unknotted. \( \square \)
4 Some applications

This section is devoted to applications (or potential ones) of the pseudoholomorphic curve theory for symplectic 4-orbifolds. More concretely, we will discuss some questions concerning the following subjects where symplectic 4-orbifolds are naturally involved: (1) smooth s-cobordisms of elliptic 3-manifolds, (2) symplectic finite group actions on 4-manifolds, (3) symplectic circle actions on 6-manifolds, (4) algebraic surfaces with isolated quotient singularities.

4.1 Smooth s-cobordisms of elliptic 3-manifolds. An elliptic 3-manifold is the quotient of $S^3 \subset \mathbb{R}^4$ under a free action of some finite subgroup of $SO(4)$. An s-cobordism of elliptic 3-manifolds is a 4-manifold with boundary the disjoint union of two elliptic 3-manifolds such that the inclusion of each boundary component is a homotopy equivalence. (Note that in the present case the Whitehead torsion vanishes, and the two boundary components are homeomorphic, cf. [29].) An s-cobordism is called trivial if it is the product of an elliptic 3-manifold with the closed interval $[0, 1]$.

It has been a longstanding open question as whether there exist exotic smooth structures on a trivial s-cobordism of elliptic 3-manifolds (cf. [2]), or whether any of the nontrivial topological s-cobordisms in [6, 28] is smoothable. In [10], a “geometrization” program was proposed to tackle these questions. At the center of the program is the following conjecture.

Conjecture 4.1 A smooth s-cobordism of elliptic 3-manifolds is smoothly trivial iff its universal cover is smoothly trivial.

The first step in the geometrization program, which was undertaken in [10], is to prove the triviality of a smooth s-cobordism of elliptic 3-manifolds under the assumption of existence of a certain symplectic structure on the s-cobordism. More precisely, given any s-cobordism of elliptic 3-manifolds, which is clearly orientable, one can fix an orientation and identify each of the two boundary components with $S^3/G \subset \mathbb{C}^2/G$ for some finite subgroup $G \subset U(2)$ by an orientation-preserving diffeomorphism (cf. [10]). Let $\omega_0$ be the canonical symplectic structure on $\mathbb{C}^2/G$, which is the descendant of $\sqrt{-1}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ on $\mathbb{C}^2$.

Theorem 4.2 (See [10]) An s-cobordism of elliptic 3-manifolds is smoothly trivial if it has a symplectic structure which equals a positive multiple of $\omega_0$ in a neighborhood of each boundary component.

(The case of lens space was treated in [4], where it was actually shown that a $\mathbb{Z}$-homology cobordism having such a symplectic structure is smoothly trivial.)

The second step in the geometrization program is to show that a smooth s-cobordism of elliptic 3-manifolds must be symplectic if its universal cover is smoothly trivial. It was explained in [10] that this can be reduced to a special case of the problem of constructing symplectic forms from generic self-dual harmonic ones discussed in Taubes [47]. To be more precise, it was shown in [10] that given any s-cobordism with each boundary component identified with $S^3/G \subset \mathbb{C}^2/G$ for some $G \subset U(2)$, there exists a self-dual harmonic 2-form which has regular zeroes and equals a positive multiple of $\omega_0$ in a neighborhood of each boundary component. It is easily seen that when the universal cover is smoothly trivial, one can

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2I am grateful to Dusa McDuff for communicating this problem to me.
compactify the universal cover into $\mathbb{CP}^2$, and reduce the existence of a symplectic structure on the $s$-cobordism to the following problem, cf. [10].

**Problem 4.3** Let $G$ be a finite group acting smoothly on $\mathbb{CP}^2$ which has an isolated fixed point $p$ and an invariant embedded 2-sphere $S$ disjoint from $p$, such that $S$ is symplectic with respect to the Kähler structure $\omega_0$ and generates the second homology. Suppose $\omega$ is a $G$-equivariant, self-dual harmonic form which vanishes transversely in the complement of $S$ and $p$. Modify $\omega$ in the sense of Taubes [47], away from $S$ and $p$, to construct a $G$-equivariant symplectic form on $\mathbb{CP}^2$.

Recall that Taubes’ program in [47] is based on the following two facts: (1) on an oriented 4-manifold with $b_2^+ \geq 1$, a self-dual harmonic 2-form has only regular zeroes for a generic metric, which is a union of embedded circles, (2) if the Seiberg-Witten invariant is nonzero, there are pseudoholomorphic subvarieties in the complement of the zero set, which homologically bound the zero set. Taubes in [47] asked under what conditions one can modify the self-dual 2-form in a neighborhood of the pseudoholomorphic subvarieties to construct a symplectic form. As explained in [47], when the 4-manifold is $S^1 \times M^3$ and the self-dual 2-form is $S^1$-equivariant, the problem is equivalent to cancellation of all critical points of a circle-valued Morse function on $M^3$ to make a fibration over $S^1$.

Although Taubes has since written a number of foundational papers on this subject (cf. [48, 49, 50, 51, 52, 53], see also Gay-Kirby [16] and Auroux-Donaldson-Katzarkov [3]), the problem still remains largely open. However, if successful, it would provide a very interesting alternative way of constructing symplectic structures on 4-manifolds, which is somewhat more abstract compared with the existing methods of symplectic sum and Stein surface construction (cf. [17, 18, 31]). We hope that Conjecture 4.1 and Problem 4.3 may serve as a testing ground for Taubes’ problem as well.

In connection with the last remark above, we mention the Cappell-Shaneson’s $s$-cobordism $W_{CS}$. Akbulut in [2] showed that the universal cover of $W_{CS}$ is smoothly trivial. Thus it would be very interesting to determine by direct means (such as in [2]) as whether $W_{CS}$ or any of its two finite covers (other than the universal cover) is smoothly exotic.

A few words about the proof of Theorem 4.2. The case where the elliptic 3-manifold is $S^3/G$ with $G$ nonabelian is more involved, and its proof requires the full power of the pseudoholomorphic curve theory. However, the basic idea of the proof can be simply described as follows. Let $W$ be a symplectic $s$-cobordism of $S^3/G$ where $G$ is nonabelian. We compactify $W$ into a symplectic 4-orbifold $X$ such that the concave end of $W$ is coned off and the convex end is collapsed into a 2-orbifold along the fibers of the Seifert fibration. Then $X$ is homotopy equivalent to $X_0$, where $X_0$ stands for $\mathbb{B}^4/G$ with boundary collapsed along the Seifert fibration. It is shown that $X$ and $X_0$ are actually diffeomorphic by recovering $X$ from the moduli space of pseudoholomorphic curves in $X$ which corresponds to the family of complex lines in $\mathbb{B}^4/G$. It follows easily from $X \cong X_0$ that $W$ is smoothly trivial.

### 4.2 Symplectic finite group actions on 4-manifolds

Let $Y$ be a symplectic 4-manifold and $G$ be a finite group acting on $Y$ symplectically. Then the quotient space $Y/G$ is a symplectic 4-orbifold, and the pseudoholomorphic curve theory, when applied to $Y/G$, may help extract information about the action of
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The basic idea is that Taubes’ existence theorem will give a set of $J$-holomorphic curves in $Y$ which is invariant under the group action, so that one may read off information about the group action on the 4-manifold $Y$ (particularly the fixed-point data) from the induced action on the 2-dimensional subset.

One of the themes of symplectic topology is the comparison between symplectic manifolds and Kähler manifolds. Extending to group actions, it is natural to compare symplectic group actions on symplectic manifolds with holomorphic actions on Kähler manifolds.

It is known that holomorphic actions on compact Kähler surfaces are rigid in some aspects. For example, it was shown in [40] (compare also [38]) that for a large class of compact Kähler surfaces, a holomorphic action must be trivial if it induces identity on the cohomology ring. (To the contrary, it was demonstrated in [14] that topological actions are extremely flexible in this regard.) A study of homological rigidity of symplectic finite group actions on 4-manifolds was initiated in [12].

For another rigidity phenomenon of holomorphic actions, we mention a theorem of Gang Xiao [56] which generalizes a classical result of Hurwitz on Riemann surfaces. Xiao’s theorem says that the order of automorphism group of a minimal algebraic surface of general type is bounded by a multiple of the Chern number $c_2^1$. It would be an interesting question as whether Xiao’s theorem has any symplectic analog.

Finally, we remark that a particular advantage of pseudoholomorphic curves over the holomorphic ones is that the almost complex structure $J$ may be chosen generic. For an application of this aspect to automorphisms of Kähler surfaces (in connection with Lemma 1.10 and Example 1.11), see the recent preprint [11].

### 4.3 Symplectic circle actions on 6-manifolds

It was shown in [33] that to each symplectic $S^1$-action on a closed symplectic manifold $(X, \omega)$, one can associate a generalized moment map $\mu : X \to S^1$, which is defined (up to a constant) by the equation $\omega(\xi, \cdot) = d\mu$, where $\xi$ = the vector field generating the $S^1$-action. Moreover, one can analyze the $S^1$-action by looking at the reduced spaces $\mu^{-1}(\lambda)/S^1$, $\lambda \in S^1$, which are symplectic orbifolds in general for regular values of $\mu$. Particularly, the $S^1$-action is Hamiltonian if $\mu$ is not onto. When $X$ is 6-dimensional, the reduced spaces $\mu^{-1}(\lambda)/S^1$ are symplectic 4-orbifolds for regular values $\lambda$, so the pseudoholomorphic curve theory may be applied in this situation.

An Hamiltonian $S^1$-action must have fixed points, which correspond to the critical points of the moment map. A natural question is whether the converse is true. It is known to be true for Kähler manifolds (cf. [13]).

The first counterexample was given by McDuff in [33], where the author constructed a 6-dimensional closed symplectic manifold with a non-Hamiltonian $S^1$-action which is not fixed-point free. In the same paper, it was also proved that an $S^1$-action on a symplectic 4-manifold is Hamiltonian iff it has fixed points. The method of the proof is to show that the generalized moment map is not onto, which can be verified by analyzing the change of the reduced spaces when crossing a critical value of the generalized moment map. In general, the problem boils down to finding an effective invariant which can tell the change of the reduced spaces. In dimension 6, it would be interesting to see if there is any version of Gromov invariants of symplectic 4-orbifolds that can be used to detect the change.
In the aforementioned counterexample of McDuff, the fixed-point set is a union of tori. It is a conceivable conjecture that an $S^1$-action is Hamiltonian if its fixed-point set is nonempty and isolated. So far the conjecture has only been verified for semi-free actions (cf. §3).

4.4 Algebraic surfaces with isolated quotient singularities. Let $(X, \omega)$ be a symplectic 4-orbifold with $b^+_2(X) \geq 2$. By the orbifold version of Taubes’ theorem (cf. Theorem 3.1 in §3), the canonical class of $X$ is represented by a set of $J$-holomorphic curves for any given $\omega$-compatible $J$. Moreover, a singular point $p \in X$ is contained in these $J$-holomorphic curves as long as the complex representation of the isotropy group $G_p$ on the tangent space is not contained in $SL_2(\mathbb{C})$ (cf. Remark 3.2 (1)). Thus the pseudoholomorphic curve theory may be used to extract global restrictions on these singular points of $X$.

An interesting class of symplectic 4-orbifolds is provided by algebraic surfaces with isolated quotient singularities. Indeed, one can adopt the method in §2 of symplectic gluing along contact hypersurfaces to show that an algebraic surface $X$ with isolated quotient singularities has a natural orbifold symplectic structure such that the corresponding canonical class is the same as that of $X$ as a complex 4-orbifold. Note that in this case, singularities $p \in X$ with $G_p$ not contained in $SL_2(\mathbb{C})$ are precisely the non-Du Val singularities.

Algebraic surfaces with isolated quotient singularities form a very natural class of singular spaces. (For a recent survey on this subject, see [27]) In fact, quotient singularities in dimension 2 are nothing but the log terminal singularities, and the set of algebraic surfaces with isolated quotient singularities form the smallest category containing nonsingular surfaces, which is closed under the log minimal model program (cf. §8).

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