Synchronization and secure communication using some chaotic systems of fractional differential equations

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January 19, 2009

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Abstract: Using Caputo fractional derivative of order \( \alpha, \alpha \in (0, 1) \), we consider some chaotic systems of fractional differential equation. We will prove that they can be synchronized and anti-synchronized using suitable nonlinear control function. The synchronized or anti-synchronized error system of fractional differential equations is used in secure communication.

MSC2000: 65P20, 94A05, 11T71.

Keywords: chaotic system of fractional differential equations, synchronization, anti-synchronization, cryptography, encryption, decryption.

1 Introduction

Synchronization phenomenon has been studied intensively because of its application in many fields, and one of it is secure communication. There are many ways for synchronization, such as feedback method, adaptive techniques, time delay feedback approach, backstepping method, with nonlinear control [10]. We will prove here synchronization and anti-synchronization between two chaotic systems of differential equations, by considering a suitable nonlinear control function.

In the first section we will show synchronization between some representative chaotic systems of fractional differential equations, coupled fractional systems T and between system T and Rössler system. In Section 2 we will present anti-synchronization of the same chaotic systems and we will compare the two methods. In Section 3 we will apply synchronization in secure communication. Numerical simulations are done using Adams-Bashforth-Moulton algorithm [4]. In last section some conclusions are presented.

We will briefly give the definition of fractional derivative, of the following form

\[
D^\alpha_t x(t) := I^{m-\alpha} \left( \frac{d}{dt} \right)^m x(t), \quad \alpha > 0,
\]
where $m = |\alpha|$, \( \left( \frac{d}{dt} \right)^m = \frac{d}{dt} \circ \ldots \circ \frac{d}{dt} \), $I^\beta$ is the $\beta-$order Riemann-Liouville integral operator and it is expressed as

$$I^\beta_t x(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} x(s) ds, \quad \beta > 0,$$

where $\Gamma$ is the Gamma function \[4\].

Along this paper we will work with the following chaotic systems of fractional differential equations:

1. T system of fractional differential equations \[9\]
   \[\begin{align*}
   D_t^{\alpha_1} x_1(t) &= a_1(y_1(t) - x_1(t)), \\
   D_t^{\alpha_2} y_1(t) &= (c_1 - a_1)x_1(t) - a_1 x_3(t) z_1(t), \\
   D_t^{\alpha_3} z_1(t) &= x_1(t)y_1(t) - b_1 z_1(t),
   \end{align*}\]
   (2)
   where $\alpha_1, \alpha_2, \alpha_3 \in (0, 1)$ and $a_1, b_1, c_1$ are the parameters of the system. It is known that the system has a chaotic behaviour for $a_1 := 2.1, b_1 := 0.6$ and $c_1 := 30$;

2. Fractional version of Rössler system with $\alpha_1, \alpha_2, \alpha_3 \in (0, 1)$ is
   \[\begin{align*}
   D_t^{\alpha_1} x_1(t) &= -y_1(t) - z_1(t), \\
   D_t^{\alpha_2} y_1(t) &= x_1(t) + a_3 y_1(t), \\
   D_t^{\alpha_3} z_1(t) &= b_2 + z_1(t)(x_1(t) - c_2)
   \end{align*}\]
   If $a_2 := 0.2, b_2 := 0.2$ and $c_2 := 5.7$, then the system is chaotic.

2 Synchronization between two chaotic fractional systems via nonlinear control

The systems implied in synchronization are called drive (master) and response (slave) systems. Let us consider the drive system given in the form $D_t^{\alpha} x(t) = f(x(t))$ and the response system $D_t^{\alpha} y(t) = g(x(t), y(t))$, where $x(t) = (x_1(t), x_2(t), x_3(t)) \in \mathbb{R}^3$ and $y(t) = (y_1(t), y_2(t), y_3(t)) \in \mathbb{R}^3$ are the phase space variables and $f, g$ the corresponding nonlinear functions. The two systems will be synchronous if the trajectory of drive system follows the same path as the response system, that means

$$|x(t) - y(t)| \to c, \quad t \to \infty. \quad (4)$$

If $c = 0$, then the synchronization is called complete synchronization. Synchronization can be bone using drive and response systems of the same type, or different types of chaotic systems of fractional differential equations. We will consider both these types of synchronization using the systems \[2\] and \[9\] \[3\] \[8\].

We will begin with the synchronization between two identical fractional T systems (coupled) in the following manner. We consider the drive system \[2\] and the response system with a control $u(t) = [u_1(t), u_2(t), u_3(t)]^T$

\[\begin{align*}
   D_t^{\alpha_1} x_2(t) &= a_1(y_2(t) - x_2(t)) + u_1(t), \\
   D_t^{\alpha_2} y_2(t) &= (c_1 - a_1)x_2(t) - a_1 x_2(t) z_2(t) + u_2(t), \\
   D_t^{\alpha_3} z_2(t) &= x_2(t)y_2(t) - b_1 z_2(t) + u_3(t),
   \end{align*}\]
   (5)
Proposition 1

The systems of fractional differential equations and we get

\[
\begin{align*}
D_t^{\alpha} e_1(t) &= a_1(e_2(t) - e_1(t)) + u_1(t), \\
D_t^{\alpha} e_2(t) &= (c_1 - a_1)e_1(t) - a_1(x_2(t)z_2(t) - x_1(t)z_1(t)) + u_2(t), \\
D_t^{\alpha} e_3(t) &= -b_1e_3(t) + x_2(t)y_2(t) - x_1(t)y_1(t) + u_3(t),
\end{align*}
\]

where \( e_1(t) := x_2(t) - x_1(t), \ e_2(t) := y_2(t) - y_1(t), \ e_3(t) := z_2(t) - z_1(t). \) We will re-write the control \( u \) by considering another control \( v \) suitable chosen that is a function of error states \( e_1, e_2, e_3. \) We re-refine \( u \) as

\[
\begin{align*}
u_1(t) &= v_1(t), \\
u_2(t) &= a_1(x_2(t)z_2(t) - x_1(t)z_1(t)) + v_2(t), \\
u_3(t) &= x_1(t)y_1(t) - x_2(t)y_2(t) + v_3(t).
\end{align*}
\]

Substituting (7) in (6) we get

\[
\begin{align*}
D_t^{\alpha} e_1(t) &= a_1(e_2(t) - e_1(t)) + v_1(t), \\
D_t^{\alpha} e_2(t) &= (c_1 - a_1)e_1(t) + v_2(t), \\
D_t^{\alpha} e_3(t) &= -b_1e_3(t) + v_3(t).
\end{align*}
\]

We want to prove that the two considered systems are globally synchronized, that means we have to choose the feedback control \( v \) such that the error to converge to 0, when \( t \to \infty. \) We choose \( v \) such that

\[
[v_1(t), v_2(t), v_3(t)]^T = A \cdot [e_1(t), e_2(t), e_3(t)]^T,
\]

where \( A \in M_{3 \times 3}. \) One choice of \( A \) is

\[
A := \begin{bmatrix}
0 & a_1 & 0 \\
-(c_1 - a_1) & c_1 & 0 \\
0 & 0 & 2b_1
\end{bmatrix}
\]

and the feedback functions are

\[
\begin{align*}
v_1(t) &= a_1e_2(t), \\
v_2(t) &= -(c_1 - a_1)e_1(t) + c_1e_2(t), \\
v_3(t) &= 2b_1e_3(t).
\end{align*}
\]

The error system (8) becomes

\[
\begin{align*}
D_t^{\alpha} e_1(t) &= a_1e_1(t), \\
D_t^{\alpha} e_2(t) &= c_1e_2(t), \\
D_t^{\alpha} e_3(t) &= b_1e_3(t).
\end{align*}
\]

**Proposition 1** The systems of fractional differential equations (2) and (5) will approach global asymptotical synchronization for any initial conditions and with the feedback control (10).

**Proof:** We will prove asymptotic stability of the system (11) by using the Laplace transform (7). We take Laplace transform in both sides of (11), with \( E_i(s) = \mathcal{L}(e_i(t)), i = 1, 2, 3, \) where \( \mathcal{L}(D^{\alpha} e_i(t)) = s^{\alpha_i} E_i(s) - s^{\alpha_i - 1} e_i(0), i = 1, 2, 3 \) and we get

\[
\begin{align*}
E_1(s) &= \frac{s^{\alpha_1 - 1} e_1(0)}{s^{\alpha_1} - a_1}, \ E_2(s) = \frac{s^{\alpha_2 - 1} e_2(0)}{s^{\alpha_2} - c_1}, \ E_3(s) = \frac{s^{\alpha_3 - 1} e_3(0)}{s^{\alpha_3} - b_1}.
\end{align*}
\]
By final-value theorem of the Laplace transformation \[7\], we have
\[
\lim_{t \to \infty} e_1(t) = \lim_{s \to 0} sE_1(s) = 0, \quad \lim_{t \to \infty} e_2(t) = \lim_{s \to 0} sE_2(s) = 0, \quad \lim_{t \to \infty} e_3(t) = \lim_{s \to 0} sE_3(s) = 0.
\]
Therefore, systems (2) and (5) can achieve asymptotic synchronization for any initial conditions and with the control (10).

For numerical simulation we use Adams-Bashforth-Moulton algorithm. For \(\alpha_1 = 0.9\), \(\alpha_2 = 0.5\), \(\alpha_3 = 0.6\) and initial conditions \(x_1(0) = 0.01\), \(x_2(0) = 0.01\), \(x_3(0) = 0.01\). Orbits of the error system (11) are represented in the above figures.

We will consider synchronization between two different chaotic systems of fractional differential equations, Rössler system as drive system (3) and T system as response system
\[
\begin{aligned}
D_\alpha^x e_1(t) = & -a_1 e_1(t) + (a_1 - 1) e_2(t) - e_3(t) + a_1 y_1(t) - x_1(t) + y_2(t) + z_2(t) + u_1(t), \\
D_\alpha^x e_2(t) = & (c_1 - a_1) x_2(t) - a_2 x_2(t) z_2(t) + u_2(t), \\
D_\alpha^x e_3(t) = & x_2(t) y_2(t) - b_1 z_2(t) + u_3(t).
\end{aligned}
\]

The error system is obtained by subtracting (12) and (3) and it has the form
\[
\begin{aligned}
D_\alpha^x e_1(t) = & -a_1 e_1(t) + (a_1 - 1) e_2(t) - e_3(t) + a_1 y_1(t) - x_1(t) + y_2(t) + z_2(t) + u_1(t), \\
D_\alpha^x e_2(t) = & (c_1 - a_1) e_1(t) + a_2 e_2(t) + (c_1 - a_1) x_1(t) - x_2(t) - a_2 x_2(t) z_2(t) -
-a_2 y_2(t) + u_2(t), \\
D_\alpha^x e_3(t) = & -(b_1 + c_2) e_3(t) - b_2 + x_2(t) y_2(t) - b_1 z_1(t) - z_1(t) x_1(t) + c_2 z_2(t) + u_3(t),
\end{aligned}
\]
where \(e_1(t) := x_2(t) - x_1(t)\), \(e_2(t) := y_2(t) - y_1(t)\), \(e_3(t) := z_2(t) - z_1(t)\). We re-write control \(u\) by considering the control \(v\) such that
\[
\begin{align*}
u_1(t) := & -a_1 y_1(t) - x_1(t) - y_2(t) - z_2(t) + v_1(t), \\
u_2(t) := & -(c_1 - a_1) x_1(t) + x_2(t) + a_2 x_2(t) z_2(t) + a_2 y_2(t) + v_2(t), \\
u_3(t) := & b_2 - x_2(t) y_2(t) + b_1 z_1(t) + z_1(t) x_1(t) - c_2 z_2(t) + v_3(t),
\end{align*}
\]
and the error system (13) becomes
\[
\begin{aligned}
D_\alpha^x e_1(t) = & -a_1 e_1(t) + (a_1 - 1) e_2(t) - e_3(t) + v_1(t), \\
D_\alpha^x e_2(t) = & (c_1 - a_1) e_1(t) + a_2 e_2(t) + v_2(t), \\
D_\alpha^x e_3(t) = & -(b_1 + c_2) e_3(t) + v_3(t).
\end{aligned}
\]
As in the case above, we have to choose the feedback control $v$ such that the relation (9) to be fulfilled. One choice of matrix $A$ will be

\[
A := \begin{bmatrix}
2a_1 & -(a_1 - 1) & 1 \\
-(c_1 - a_1 + 1) & 0 & 0 \\
0 & 2b_1 + c_2 & 0
\end{bmatrix}
\]

and the feedback functions are

\[
\begin{align*}
v_1(t) &= 2a_1 e_1(t) - (a_1 - 1)e_2 + e_3(t), \\
v_2(t) &= -(c_1 - a_1 + 1)e_1(t), \\
v_3(t) &= (2b_1 + c_2)e_3(t).
\end{align*}
\]

(16)

The error system (15) becomes

\[
\begin{align*}
D^\alpha_1 e_1(t) &= a_1 e_1(t), \\
D^\alpha_2 e_2(t) &= a_2 e_2(t), \\
D^\alpha_3 e_3(t) &= b_1 e_3(t).
\end{align*}
\]

(17)

Proposition 2: The systems of fractional differential equations (3) and (12) will approach global asymptotical synchronization for any initial conditions and with the feedback control (16).

Proof: The proof is similar to that of Proposition 1. □

The error system (17) is represented in the above figures, for $\alpha_1 = 0.9$, $\alpha_2 = 0.5$, $\alpha_3 = 0.6$ and initial conditions $x_1(0) = 0.01$, $x_2(0) = 0.01$, $x_3(0) = 0.01$.

3 Anti-synchronization between two chaotic fractional systems via nonlinear control

The drive and response systems will be anti-synchronous if

\[
|x(t) + y(t)| \to c, \ t \to \infty.
\]

If $c = 0$, then the anti-synchronization is called complete anti-synchronization. Anti-synchronization can also be done using drive and response systems of the same type, or
different types of chaotic systems of fractional differential equations, as above. We will consider both these types of anti-synchronization using the systems (2) and (3). 

The anti-synchronization between two identical fractional T systems is done in a similar manner like in the synchronization case. We consider the drive system (2) and the response system with a control $u(t) = [u_1(t), u_2(t), u_3(t)]^T$.

In this case the error system is given by adding (10) and (2), and we get

$$\begin{align*}
D_t^{\alpha_1}e_1(t) &= a_1(e_2(t) - e_1(t)) + u_1(t), \\
D_t^{\alpha_2}e_2(t) &= (c_1 - a_1)e_1(t) - a_1(x_2(t)z_2(t) + x_1(t)z_1(t)) + u_2(t), \\
D_t^{\alpha_3}e_3(t) &= -b_1e_3(t) + x_2(t)y_2(t) + x_1(t)y_1(t) + u_3(t),
\end{align*}$$

(18)

where $e_1(t) := x_2(t) + x_1(t)$, $e_2(t) := y_2(t) + y_1(t)$, $e_3(t) := z_2(t) + z_1(t)$. Now we will re-write the control $u$ by choosing a suitable control $v$ as a function depending on the error states $e_1, e_2, e_3$. We re-refine $u$ as

$$\begin{align*}
u_1(t) &= v_1(t), \\
u_2(t) &= a_1(x_2(t)z_2(t) + x_1(t)z_1(t)) + v_2(t), \\
u_3(t) &= -x_1(t)y_1(t) - x_2(t)y_2(t) + v_3(t).
\end{align*}$$

(19)

Substituting (19) in (18) we obtain

$$\begin{align*}
D_t^{\alpha_1}e_1(t) &= a_1(e_2(t) - e_1(t)) + v_1(t), \\
D_t^{\alpha_2}e_2(t) &= (c_1 - a_1)e_1(t) + v_2(t), \\
D_t^{\alpha_3}e_3(t) &= -b_1e_3(t) + v_3(t).
\end{align*}$$

(20)

Two considered systems are globally synchronized if relation the error to converge to 0, when $t \rightarrow \infty$. We have to choose the feedback control $v$ such that relation (19) is fulfilled. One choice of $A$ is

$$A := \begin{bmatrix} 0 & a_1 & 0 \\ -(c_1 - a_1) & c_1 & 0 \\ 0 & 0 & 2b_1 \end{bmatrix}$$

and the feedback functions are

$$\begin{align*}
v_1(t) &= a_1e_2(t), \\
v_2(t) &= -(c_1 - a_1)e_1(t) + c_1e_2(t), \\
v_3(t) &= 2b_1e_3(t).
\end{align*}$$

(21)

The error system (20) becomes

$$\begin{align*}
D_t^{\alpha_1}e_1(t) &= a_1e_1(t), \\
D_t^{\alpha_2}e_2(t) &= c_1e_2(t), \\
D_t^{\alpha_3}e_3(t) &= b_1e_3(t).
\end{align*}$$

(22)

**Proposition 3** The systems of fractional differential equations (2) and (3) will approach global asymptotical anti-synchronization for any initial conditions and with the feedback control (21).

**Proof:** The proof is similar to that of Proposition 1. \(\square\)
In case of Rössler system as drive system (3) and T system (2) as response system, the error system is obtained by adding (12) and (3) and it has the form

\[
\begin{align*}
D_t^{\alpha_1} e_1(t) &= -a_1 e_1(t) + (a_1 - 1) e_2(t) - e_3(t) + a_1(x_1(t) - y_1(t)) + y_2(t) + z_2(t) + u_1(t), \\
D_t^{\alpha_2} e_2(t) &= (c_1 - a_1 + 1) e_1(t) + a_2 e_2(t) - (c_1 - a_1)x_1(t) - x_2(t) - a_2 x_3(t) z_2(t) - a_2 y_2(t) + u_2(t), \\
D_t^{\alpha_3} e_3(t) &= -(b_1 + c_2) e_3(t) + b_2 + x_2(t) y_2(t) + b_1 z_1(t) + z_1(t) x_1(t) + c_2 z_2(t) + u_3(t),
\end{align*}
\]

where \( e_1(t) := x_2(t) + x_1(t), e_2(t) := y_2(t) + y_1(t), e_3(t) := z_2(t) + z_1(t). \) We re-write control \( u \) by considering the control \( \nu \) such that

\[
\begin{align*}
u_1(t) &= -a_1 (x_1(t) - y_1(t)) - y_2(t) - z_2(t) + v_1(t), \\
u_2(t) &= (c_1 - a_1)x_1(t) + x_2(t) + a_2 x_3(t) z_2(t) + a_2 y_2(t) + v_2(t), \\
u_3(t) &= -b_2 - x_2(t) y_2(t) - b_1 z_1(t) - z_1(t)x_1(t) - c_2 z_2(t) + v_3(t),
\end{align*}
\]

and the error system (23) becomes

\[
\begin{align*}
D_t^{\alpha_1} e_1(t) &= -a_1 e_1(t) + (a_1 - 1) e_2(t) - e_3(t) + v_1(t), \\
D_t^{\alpha_2} e_2(t) &= (c_1 - a_1 + 1) e_1(t) + a_2 e_2(t) + v_2(t), \\
D_t^{\alpha_3} e_3(t) &= -(b_1 + c_2) e_3(t) + v_3(t).
\end{align*}
\]

As in the case above, we have to chose the feedback control \( \nu \) such that the relation (9) to be fulfilled. One choice of matrix \( A \) will be

\[
A := \begin{bmatrix}
2a_1 & -(a_1 - 1) & 1 \\
-(c_1 - a_1 + 1) & 0 & 0 \\
0 & 0 & 2b_1 + c_2
\end{bmatrix}
\]

and the feedback functions are

\[
\begin{align*}
u_1(t) &= 2a_1 e_1(t) - (a_1 - 1) e_2 + e_3(t), \\
\nu_2(t) &= -(c_1 - a_1 + 1) e_1(t), \\
\nu_3(t) &= (2b_1 + c_2) e_3(t).
\end{align*}
\]

The error system (25) becomes

\[
\begin{align*}
D_t^{\alpha_1} e_1(t) &= a_1 e_1(t), \\
D_t^{\alpha_2} e_2(t) &= a_2 e_2(t), \\
D_t^{\alpha_3} e_3(t) &= b_1 e_3(t).
\end{align*}
\]

**Proposition 4** The systems of fractional differential equations (3) and (2) will approach global asymptotical synchronization for any initial conditions and with the feedback control (26).

**Proof:** The proof is similar to that of Proposition 1.

\[ \square \]

### 4 Secure information using synchronized chaotic systems of fractional differential equations

In the following we will do the encryption and decryption of a message using coupled chaotic systems of fractional differential equations and their synchronization. For this we use two
coupled systems, for example \([2]\) as drive system and \([5]\) as response. They have a chaotic behaviour and are also synchronized and anti-synchronized. The sender uses the fractional system \([2]\) and the receiver uses \([5]\). They both choose values for variables \(z_1(t)\) and \(z_2(t)\) as public keys, after a period of time, after the synchronization between the considered fractional systems took place \([2]\), \([5]\). The fractional derivative is also sent as a public key to the receiver.

The message that one part wants to send is called plaintext and its correspondent by decryption is called ciphertext. The plaintext and the ciphertext are represented by numbers, each letter from the alphabet is replaced by a corresponding number. So instead of letters and numbers we will use numbers from 0 to 35, in this case. The most general case is to consider small letters, capital letters and special characters. To each of it, a corresponding number is associated, in ASCII representation. In this case the formula corresponding for encryption, (respectively for decryption) is given by:

\[
c_i := p_i + k_i \mod (36),
\]

\[
p_i := c_i - k_i \mod (36),
\]

where \(k_i\) are public keys that mask the message. For each letter or number we use a randomly generated key \(\{k_1, k_2, \ldots, k_n\}\). Actually, each key \(k_j\) hides the piece of message \(p_j\).

We consider the example message "Hello Oscar". This is represented in the following table, each piece of message with the corresponding randomly generated key, and in the next table the plaintext, the ciphertext and the message received:

| Key | \(k_1\) | \(k_2\) | \(k_3\) | \(k_4\) | \(k_5\) | \(k_6\) | \(k_7\) | \(k_8\) | \(k_9\) | \(k_{10}\) |
|-----|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| Message | h | e | l | l | o | o | s | c | a | r |

Beginning with \(t_0\), in our example equal with 1300, both dynamical systems are synchronized, so \(t\) takes values greater than \(t_0 = 1300\), in the synchronized state. The first system sends the encoded message, letter by letter, and for each letter a key is randomly generated. The second system receives the encrypted message and also the key with which is was encrypted. In this example we work with the third component of the considered fractional systems, \(z_1(t)\) and with the keys \(k_i, i = 1, 10\). Decryption is done using the third component of the second system \(z_2(t)\), with \(t \geq t_0\), as it is illustrated in Table 2.

5 Conclusions

In this paper we have presented chaotic behavior of some fractional systems and we have presented a way of encryption and decryption a message. These techniques can be done using other chaotic systems of fractional differential equations, like Lorenz, Chua, hyperchaotic Rössler systems and pair of these fractional systems, but also on some systems of fractional differential equations endowed with another structure, such as metriplectic structure, (almost) Leibniz structure \([3]\). A suitable control can be chosen to achieve synchronization or anti-synchronization.

For a suitable control \(u\), respectively \(v\), synchronization and anti-synchronization behave in the same manner, as the error system of fractional differential equations coincide.
| Time $t$ | $z_1(t)$ | Key $k$ | Plaintext $p$ | Ciphertext $c = p + k \mod (36)$ |
|---------|-----------|---------|---------------|---------------------------------|
| 1301    | 4416      | 4227111... | h(18)         | 0                              |
| 1302    | 4433      | 312595...  | e(15)         | 33                             |
| 1303    | 4449      | 155104...  | l(22)         | 15                             |
| 1304    | 4466      | 974929...  | l(22)         | 15                             |
| 1305    | 4483      | 271362...  | o(25)         | 9                              |
| 1306    | 4500      | 333989...  | o(25)         | 9                              |
| 1307    | 4518      | 649588...  | s(29)         | 14                             |
| 1308    | 4535      | 799296...  | c(13)         | 19                             |
| 1309    | 4552      | 379435...  | a(11)         | 17                             |
| 1310    | 4569      | 963282...  | r(28)         | 22                             |

| Time $t$ | $z_2(t)$ | Key $k$ | Ciphertext $p$ | Plaintext $p = c - k \mod (36)$ |
|---------|-----------|---------|---------------|---------------------------------|
| 1301    | 4416      | 4227111... | 0             | 18(h)                           |
| 1302    | 4433      | 312595...  | 33            | 15(e)                           |
| 1303    | 4449      | 155104...  | 15            | 22(l)                           |
| 1304    | 4466      | 974929...  | 15            | 22(l)                           |
| 1305    | 4483      | 271362...  | 9             | 25(o)                           |
| 1306    | 4500      | 333989...  | 9             | 25(o)                           |
| 1307    | 4518      | 649588...  | 14            | 29(s)                           |
| 1308    | 4535      | 799296...  | 19            | 13(c)                           |
| 1309    | 4552      | 379435...  | 17            | 11(a)                           |
| 1310    | 4569      | 963282...  | 22            | 28(r)                           |
Encryption and decryption was done here letter by letter, using a Maple 11 program. In a discrete case the procedure is more "commercial" because encoding and decoding the message is done for the entire message at the same time.

Another approach for synchronization has been studied and presented in my PhD thesis [1].

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