Choreographies as Functions

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Abstract
We propose a new interpretation of choreographies as functions, whereby coordination protocols for concurrent and distributed systems are expressed in terms of a $\lambda$-calculus. Our language is expressive enough to enable, for the first time, the writing of higher-order protocols that do not require central control. Nevertheless, it retains the simplicity and elegance of the $\lambda$-calculus, and it is possible to translate choreographies into endpoint implementations.

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1 Introduction

Choreographic Languages and Endpoint Projection
Choreographies are coordination plans for concurrent and distributed systems, which define the communications that should be enacted by a system of processes [25, 34, 38]. Implementing choreographies is notoriously hard, because of the usual issues of concurrent programming. Notably, it requires predicting how processes will interact at runtime, for which programmers do not receive adequate help from mainstream programming technology [26, 30, 35]. This challenge has spawned a prolific area of research within the communities of concurrency theory and programming languages, which focuses on the definition of choreographic languages (languages for expressing choreographies) and how terms in such languages can be correctly translated into abstract models of implementations [3, 24].

Current formulations of choreographic languages are based on the theories of communicating automata [4, 10] and process calculi [37, 5, 23], inspired by earlier studies on message sequence charts [2, 25]. The starting point for these works was to use these well-known theories to formulate the communication primitive of choreographic languages. This key primitive, which comes straight from the “Alice and Bob” notation of security protocols [33], allows for moving data from one process to another. In this context, processes are usually called roles. Many choreographic languages come with a translation of choreographies into abstractions of implementations, typically called Endpoint Projection (EPP), and a proof of its correctness, typically defined as an operational correspondence result [6].

The Issue of Compositionality
In practice, choreographies are large—some even over a hundred pages of text [36]. Thus, it is important to understand the principles of how
choreographies can be made modular, enabling the writing (preferably disciplined by types) of large choreographies as compositions of smaller, reusable ones.

Previous work investigated of how choreographic languages can be extended with parametric procedures, by introducing ad-hoc extensions informally inspired by the \(\lambda\)-calculus and its types [7] [12] [15]. However, none of these choreographic languages were designed with the \(\lambda\)-calculus as basis. This led to visible fragmentation due to differences in how procedures are formulated (the proverbial wheel has been reinvented many times), and also to a series of technical shortcomings, for example: partial application is not supported [10]; most works do not support higher-order composition of choreographies [12], and those that do require centralised coordination when entering a procedure (going against distribution, which is instead assumed in all other syntactic constructs) [15, 21]; and abstractions of parameters of different types are distinguished syntactically [7].

To date, whether the elegance, power, and canonicity of the \(\lambda\)-calculus can be adopted for the composition of choreographies remains unclear.

This Article We present the Choreographic \(\lambda\)-calculus, Chor\(\lambda\) for short, the first \(\lambda\)-calculus that supports the writing of choreographies.

In Chor\(\lambda\), the communication primitive of choreographies is formulated as a function: \(\text{com}_{S,R}\), which is a \(\lambda\)-expression that takes a value at a role \(S\) and returns the same value at another role \(R\). In general, for the first time, all choreographies are \(\lambda\)-terms that can be composed following the functional programming style.

Terms in Chor\(\lambda\) are located at roles, to reflect distribution. For example, the value \(5@\text{Alice}\) reads “the integer 5 at Alice”. Terms are typed with novel data types that are annotated with roles. In this case, \(5@\text{Alice}\) has the type \(\text{Int} @ \text{Alice}\), read “an integer at Alice”. We write type assignments in the usual way: \(\text{Int} @ \text{Alice}\). In addition to values such as integers, we can also have functions at roles. The choreography \(\lambda f : \text{Int} @ \text{Alice} \rightarrow \text{Int} @ \text{Alice}. \lambda x : \text{Int} @ \text{Alice}. (f y)\) takes a function located at Alice and applies it to an integer located at Alice. Since Chor\(\lambda\) is a higher-order choreographic language, we can even parametrise our choreographies on functions located at multiple roles. If a function \(f\) for example wants to send \(5@\text{Alice}\) from Alice to Bob, it may decide to simply use \(\text{com}\), or it may instead allow the programmer to specify an alternative function which includes e.g. an encryption. We would express this as \(f = \lambda x : \text{Int} @ \text{Alice} \rightarrow \text{Int} @ \text{Bob}. (x 5 @ \text{Alice})\). Higher-order choreographies can also be used for more complex scenarios where instead of simply communicating a value from one role to another the protocol is parametrised on ways to authenticate or what to do in case the protocol fails.

We use types to define a typing discipline for Chor\(\lambda\) that checks that choreographies make sense. Consider the function \(f\) defined as \(\lambda x : \text{Int} @ \text{Alice}. \text{com}_{\text{Proxy}, \text{Bob}} (\text{com}_{\text{Alice}, \text{Proxy}} x)\), which communicates an integer from Alice to Bob by passing through an intermediary Proxy. For any term \(M\), the composition \(f M\) makes sense if the evaluation of \(M\) returns something of the type expected by \(f\), that is \(\text{Int} @ \text{Alice}\). The composition \(f 5 @ \text{Alice}\) makes sense, but \(f 5 @ \text{Bob}\) does not, because the argument is not at the role expected by \(f\). We define an operational semantics for Chor\(\lambda\) and prove that our type system supports type preservation and progress.

After the presentation of Chor\(\lambda\), we define Endpoint Projection (EPP): a translation from terms in Chor\(\lambda\) to implementations in a concurrent \(\lambda\)-calculus (borrowing techniques from process calculi), where roles are enacted by processes that can communicate by the usual send and receive primitives. Our main result is that EPP is sound and complete, in terms of an operational correspondence: the implementation of a choreography enacts only and all the communications defined in the originating choreography. As a corollary of progress for
well-typed choreographies and the correctness of EPP, we obtain that implementations of choreographies generated by our EPP always progress.

The expressivity of Chorλ is illustrated with a series of representative examples: remote procedure calls, remote transformations of lists, the Diffie-Hellman protocol for secure key exchange, and a distributed authentication protocol. Notably, we leverage compositionality to show how choreographies can be parameterised over different communication semantics, enabling protocol layering. In particular, we combine distributed authentication with a choreography for encryption to secure communications.

We believe that our results are promising not only for the future development of more expressive choreographic languages in practice, but also for bridging the community of functional programming to that of choreographic languages: this is the first time that choreographies are explained in terms of the solid foundations of λ-calculus.

2 Related Work

Choreographic languages and EPP have been successfully employed in the verification, monitoring, and synthesis of concurrent and distributed programs [3, 24]. For example, in multiparty session types, choreographies are translated to types that used to check that processes written in (variations of) the π-calculus communicate as expected [23].

In some settings, choreographies need to define computation at roles. For instance, many security protocols define how data should be encrypted and/or anonymised, and parallel algorithms define how each process implements its part of a computation. Choreographies that include computation can be defined in choreographic programming, which elevates choreographic languages to full-fledged programming languages [31]. Choreographic programming languages showed promise in a number of contexts, including parallel algorithms [11], cyber-physical systems [29, 28, 19], self-adaptive systems [14], system integration [18], information flow [27], and the implementation of security protocols [19].

Technically, Chorλ is a member of choreographic programming, but we believe that our principles could be applied also to other kinds of choreographic languages (e.g., by abstracting from the concrete values that are transmitted). In particular, there are several implementations of choreographic languages that are equipped with ad-hoc, limited variations of choreographic procedures (functions in Chorλ) that could benefit from our results [7, 22, 14, 19].

Our data types are inspired by the Choral programming language, an object-oriented language where object types are annotated with roles to capture choreographies [19]. Choral does not come with a formal model: its semantics and typing are only informally described. In a sense, Chorλ can be seen as the first formal investigation of the principles that underpin Choral. Compared to Choral our formalisation supports partial application (thus, e.g., configurations parameters of choreographies can be set at different stages), the term and type languages are much simpler, and we provide a provably-correct translation of choreographies to concurrent implementations.

Pirouette is a higher-order choreographic language, which has been developed concurrently to this work [21]. Chorλ and Pirouette have been developed independently and have different designs. Pirouette is fully formalised in Coq, but it is more complex: the languages for writing choreographies and the local computation performed by roles are separate, and in particular come with dedicated syntax and semantics for applications (depending on whether the arguments are local expressions or choreographies). By contrast, Chorλ offers a single, general language where local terms are just degenerate instances of choreographic programs, thus retaining the elegance of λ-calculus. Because of the type system that we had
to develop in order to reason about this kind of programs, there is no need to define separate semantics (nor syntax) for local and choreographic applications. Another relevant difference between Chorλ and Pirouette is that Chorλ expresses fully decentralised choreographies: synchronisation in Chorλ takes place only if the choreography specifies it. In Pirouette, instead, the reduction of an application performs a global synchronisation among all roles (even those not involved in the application). Similar limitations are shared by previous works, which require synchronisations whenever a higher-order procedure is entered [15]. Defining a semantics for decentralised execution in Chorλ required adding sophistication: we had to add rules that are not normally required in λ-calculus, in order to capture concurrent execution of terms located at different roles.

Another related line of work is that on multitier programming and its progenitor calculus, Lambda 5 [32]. Similarly to Chorλ, Lambda 5 and multitier languages have data types with locations [39]. However, they are used very differently. In choreographic languages (thus Chorλ), programs have a “global” point of view and express how multiple roles interact with each other. By constrast, in multitier programming programs have the usual “local” point of view of a single role but they can nest (local) code that is supposed to be executed remotely. The reader interested in a detailed comparison of choreographic and multitier programming can consult [20], which presents algorithms for translating choreographies to multitier programs and vice versa. The correctness of these algorithms has never been proven, because they use the informally-specified Choral language as a representative choreographic language. We conjecture that the introduction of Chorλ could be the basis for a future investigation of formal translations between choreographic programs (in terms of Chorλ) and multitier programs (in terms of Lambda 5). In a similar direction, [9] presented a simple first-order multitier language from which it is possible to infer abstract choreographies (computation is not included) that describe the communication flows that multitier programs enact. This language, like all existing multitier languages, does not support higher-order composition of multitier programs. Establishing translations between Chorλ and multitier languages might provide insight on how multitier languages can support higher-order composition (as in our approach).

3 The Choreographic λ-calculus

In this section we introduce the Choreographic λ-calculus, Chorλ, which extends the simply typed λ-calculus [10] with roles and communication.

Syntax

Definition 1. The syntax of Chorλ is given by the following grammar

\[ M ::= V | f(\bar{R}) | M \; M | \text{case } M \text{ of } \text{Inl } x \Rightarrow M; \; \text{Inr } x \Rightarrow M | \text{select } R,R \ell l M \]

\[ V ::= x | \lambda x : T.M | \text{Inl } V | \text{Inr } V | \text{fst} | \text{snd} | \text{Pair } V \; V | () @ R | \text{com } R,R \]

\[ T ::= T \rightarrow \rho T | T + T | T \times T | ()@R | t@\bar{R} \]

where \( M \) is a choreography, \( V \) is a value, \( T \) is a type, \( x \) is a variable, \( \ell \) is a label, \( f \) is a choreography name, \( R \) is a role, \( X \) is a role variable, \( \rho \) is a set of roles, and \( t \) is a type name.

Abstraction \( \lambda x : T.M \), variable \( x \) and application \( MM \) are as in the standard (typed) λ-calculus. Likewise for pairs and sums. For simplicity, constructors for sums (Inl and Inr) and products (Pair) are only allowed to take values as inputs, but this is only an apparent
restriction: we can define, e.g., a function \texttt{inl} as \( \lambda x : T. \texttt{inl} \ x \) and then apply it to any choreography. Similarly, we define the functions \texttt{inr} and \texttt{pair} (the latter is for constructing pairs). We use these utility functions in our examples. Sums and products are deconstructed in the usual way, respectively by the \texttt{case} construct and by the \texttt{fst} and \texttt{snd} primitives.

The primitives \texttt{select}_{S,R} \ l \ M \texttt{ and com}_{S,R} \texttt{ come from choreographies and are the only primitives of Chor}\lambda\texttt{ that introduce interaction between different roles. The term select}_{S,R} \ l \ M \texttt{ is a selection, where } S \texttt{ informs } R \texttt{ that it has selected the label } l. \texttt{ Selections choreographically represent the communication of an internal choice made by } S \texttt{ to } R. \texttt{ (In the implementation of choreographies, a selection corresponds to an internal choice at the sender and an external choice at the receiver.)} \texttt{ The term com}_{S,R} \texttt{, instead, is a communication; a communication is in effect a } \lambda\texttt{-expression that takes a value at role } S \texttt{ and returns the same value at role } R. \texttt{ }

Finally, \( f(\vec{R}) \) is a name for a function \( f \) instantiated with the roles \( \vec{R} \), which evaluates to a corresponding choreography with the same number of roles \( f(\vec{R}) \) as defined by an environment of definitions. Choreography functions are used to model recursion. In the typing and semantics of Chor\lambda\texttt{, we will use } D \texttt{ to range over mappings of choreography names to choreographies. In examples we will sometimes use } M\mathrel{\mathrel{\@}} R, V \mathrel{\mathrel{\@}} R, \texttt{ and } T \mathrel{\mathrel{\@}} R \texttt{ to denote a choreography, value, or type which is located entirely at the role } R. \texttt{ }

In Chor\lambda\texttt{ types record the distribution of values across roles: if role } R \texttt{ occurs in the type given to } V \texttt{ then part of } V \texttt{ will be located at } R. \texttt{ Because a function may involve more roles besides those listed in the types of their input and output, the type of abstractions } T \to \rho T' \texttt{ is annotated with a set of roles } \rho \texttt{ denoting the roles that may participate in the computation of a function with that type besides those occurring in the input } T \texttt{ or the output } T'. \texttt{ In the sequel we will often omit this annotation if the set of additional roles is empty thus writing } T \to T' \texttt{ instead of } T \to \rho T'. \texttt{ The types for sums and products are the usual ones (forming a sum or product of } T \texttt{ and } T' \texttt{ does not introduce new roles besides those already listed in } T \texttt{ and } T'). \texttt{ The type of units is annotated with the role where each unit is located; } (\cdot)\mathrel{\mathrel{\@}} R \texttt{ is the type of the unit value available (only) at role } R. \texttt{ Named types } t \texttt{ are annotated with the roles } \vec{R}, \texttt{ instantiating the roles occurring in their definition (we will discuss type definitions later in this section). The set of roles in a type is formally defined as follows.}

\textbf{Definition 2 (Roles of a type).} The roles of a type \( T, \texttt{ roles}(T) \), are defined as follows.

\[
\begin{align*}
\texttt{roles}(&@\vec{R}) = \vec{R} \\
\texttt{roles}(T \to \rho T') = \texttt{roles}(T) \cup \texttt{roles}(T') \cup \rho \\
\texttt{roles}(()@R) = \{R\} \\
\texttt{roles}(T + T') = \texttt{roles}(T) \cup \texttt{roles}(T') \\
\texttt{roles}(T \times T') = \texttt{roles}(T) \cup \texttt{roles}(T')
\end{align*}
\]

A choreography term \( M \) may involve more roles besides those listed in its type. For instance, the choreography \texttt{com}_{S,R} (\cdot)\@S \texttt{ has type } (\cdot)\@R \texttt{ but involves also role } S. \texttt{ }

A key concern of choreographic languages is knowledge of choice: the property that when a choreography chooses between alternative branches (as with our \texttt{case} primitive), all roles that need to behave differently in the branches are properly informed via appropriate selections \[8\]. We give an example of how selections should be used, and postpone a formal discussion of how knowledge of choice is checked for to our presentation of Endpoint Projection.

\textbf{Example 3 (Remote Map).} The choreography below defines a remote map function, where \( f \) (available at role \( R \)) is applied to all elements of a list (available at role \( S \)). It consists of two functions: \texttt{remoteFunction}, which applies \( f \) to a single element, and \texttt{remoteMap}, which iterates this application over the input list.

\begin{verbatim}
remoteFunction(R,S) := \lambda f : \texttt{Int} \mathrel{\mathrel{\@}} R \to \texttt{Int} \mathrel{\mathrel{\@}} R. \lambda val : \texttt{Int} \mathrel{\mathrel{\@}} S. \texttt{com}_{R,S} (f (\texttt{com}_{S,R} val))
\end{verbatim}
We now show how to type choreographies following the intuitions already given earlier. Typing judgements have the form \( M; \Sigma; \Gamma \vdash M : T \), where: \( \Theta \) the set of roles that can be used for typing \( M \); \( \Sigma \) is collection of type definitions i.e. expressions of the form \( t_\rho = T \); and \( \Gamma \) is a typing environment for variables (and choreography names) that are free in \( M \). We further require that a type name \( t \) is defined at most once in \( \Sigma \), that definitions are contractive, and that \( \text{roles}(T) = \rho \) for any \( t_\rho = T \in \Sigma \). We call \( \Theta; \Sigma; \Gamma \) jointly a typing context. Many of the rules resemble those for simply typed \( \lambda \)-calculus, but with roles added, and the additional requirements that only the roles in the type are used in the term being typed. We include some representative ones in Figure 1 (the complete set of typing rules is given in Appendix A).

Rules \text{TVar}, \text{TDef}, and \text{TAbs} exemplify how role checks are added to the standard typing rules for simply typed \( \lambda \)-calculus. Rule \text{TCom} types communication actions, moving subterms that were placed at role \( S \) to role \( R \) (\( T[S := R] \) is the type expression obtained by replacing \( S \) with \( R \)). Rule \text{TSel} types selections as no-ops, again checking that the sender

\[
\text{remoteMap}(R, S) = \lambda f : \text{Int}@R \to \text{Int}@R. \ \lambda \text{ListInt}@S.
\]

\[
\begin{align*}
\text{case list of} \\
\text{Inl } x & \Rightarrow \text{select}_{S,R} \text{stop } ()@S; \\
\text{Inr } x & \Rightarrow \text{select}_{S,R} \text{again} \\
\text{cons}(S) & \Rightarrow \text{remoteFunction}(R, S) f \ (\text{fst } x) \ (\text{remoteMap}(R, S) f \ (\text{snd } x))
\end{align*}
\]

Here, \text{ListInt}@S is the recursive type satisfying \text{ListInt}@S = ()@S + \text{Int}@S \times \text{ListInt}@S and, for simplicity, a primitive type for integers is assumed. When we introduce typing judgements, we will show how to work with this kind of types.

Notice how the \text{case} is evaluated on data at role \( S \), so that role is the only one initially knowing which branch has been chosen. Each branch, however, starts with a selection from role \( S \) to role \( R \). Since \( R \) receives a different label in the two branches, respectively \text{stop} and \text{again}, it can use this information to figure out whether it should terminate (stop) or the choreography continues (again): from its point of view, \( R \) is reactively handling a stream.

We can encode conditional statements in the standard way: we define a type \text{Bool}@R as ()@R + ()@R, and if \( M \text{ then } M' \text{ else } M'' \) an abbreviation for \text{case } M \text{ of } \text{Inl } x \Rightarrow M'; \text{Inr } x \Rightarrow M''.

Free and bound variables are defined as expected, noting that \( x \) and \( y \) are bound in \text{case } M \text{ of } \text{Inl } x \Rightarrow M'; \text{Inr } y \Rightarrow M''. We write \( \text{fv}(M) \) for the set of free variables in choreography \( M \), and likewise for types. A choreography \( M \) is closed if \( \text{fv}(M) = \emptyset \). The formal definition is given in Appendix A.

\begin{definition}[Well-formed Choreography] We say that \( M \) is a well-formed choreography with regards to \( D \) if:
\begin{itemize}
\item Any \( f(\vec{R}) \) called by \( M \) or \( D \) is defined in \( D \) as \( f(\vec{R'}) \) with \( |\vec{R}| = |\vec{R'}| \).
\item Any \( f(\vec{R}) \) called by \( M \) or \( D \), elements of \( \vec{R} \) are distinct from each other.
\item In \( t@\vec{R} \), elements of \( \vec{R} \) are distinct from each other.
\item Any \( f \) only has one definition in \( D \).
\item If \( D(f(\vec{R})) = M' \) then \( M' \) does not contain roles not in \( \vec{R} \).
\item \( M \) is closed with respect to role and normal variables.
\end{itemize}
\end{definition}

\section*{Typing}

We now show how to type choreographies following the intuitions already given earlier. Typing judgements have the form \( \Theta; \Sigma; \Gamma \vdash M : T \), where: \( \Theta \) the set of roles that can be used for typing \( M \); \( \Sigma \) is collection of type definitions i.e. expressions of the form \( t_\rho = T \); and \( \Gamma \) is a typing environment for variables (and choreography names) that are free in \( M \). We further require that a type name \( t \) is defined at most once in \( \Sigma \), that definitions are contractive, and that \( \text{roles}(T) = \rho \) for any \( t_\rho = T \in \Sigma \). We call \( \Theta; \Sigma; \Gamma \) jointly a typing context. Many of the rules resemble those for simply typed \( \lambda \)-calculus, but with roles added, and the additional requirements that only the roles in the type are used in the term being typed. We include some representative ones in Figure 1 (the complete set of typing rules is given in Appendix A).

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\[
\begin{align*}
\frac{x : T \in \Gamma \quad \text{roles}(T) \subseteq \Theta}{\Theta; \Sigma; \Gamma \vdash x : T} & \quad \text{[TVAR]} \\
\frac{f(\vec{R}')) : T \in \Gamma \quad \text{roles}(T) \subseteq \vec{R} \subseteq \Theta}{\Theta; \Sigma; \Gamma \vdash f(\vec{R}) : T[\vec{R}/\vec{R}']} & \quad \text{[TDEF]} \\
\frac{\rho \cup \text{roles}(T) \cup \text{roles}(T') : \Sigma; \Gamma, x : T \vdash M : T'}{\Theta; \Sigma; \Gamma \vdash \lambda x : T. M : T \rightarrow_{\rho} T'} & \quad \text{[TABS]} \\
\frac{\text{roles}(T) = \{S\} \quad \{S, R\} \subseteq \Theta}{\Theta; \Sigma; \Gamma \vdash \text{com}_{S,R} : T \rightarrow_{\rho} T[S := R]} & \quad \text{[TCom]} \\
\frac{\Theta; \Sigma; \Gamma \vdash M : T \quad \{S, R\} \subseteq \Theta}{\Theta; \Sigma; \Gamma \vdash \text{select}_{S,R} \ M \vdash T} & \quad \text{[TSel]} \\
\frac{\Theta; \Sigma; \Gamma \vdash \text{com}_{S,R} : T \rightarrow_{\rho} T'[\vec{R}/\vec{R}']}{\Theta; \Sigma; \Gamma \vdash \text{com}_{S,R} \ M \vdash T'} & \quad \text{[TEq]} \\
\end{align*}
\]

Figure 1: Typing rules for Chor\(\lambda\) (representative selection).

and receiver of the selection are legal roles. Rule [TEq] allows rewriting a type according to \(\Sigma\) in order to mimic recursive types (see Example 3).

We also write \(\Theta; \Sigma; \Gamma \vdash D\) to denote that a set of definitions \(D\), mapping names to choreographies, is well-typed. Sets of definitions play a key role in the semantics of choreographies, and can be typed by the rule below.

\[
\forall f(\vec{R}) \in \text{domain}(D) \quad f : T \in \Gamma \quad \Sigma; \Gamma \vdash D(f) : T \quad \text{[TDefs]}
\]

Example 5. The set of definitions in Example 3 can be typed in the typing context \(\Theta; \Sigma; \Gamma\) where \(\Theta = \{R, S\}\), \(\Sigma = \{\text{ListInt}\@S = (\@S + (\@S \times \text{ListInt}\@S))\}\) and \(\Gamma = \{\text{remoteFunction} : (\@R \rightarrow \@R) \rightarrow \@S \rightarrow \@S, \text{remoteMap} : (\@R \rightarrow \@R) \rightarrow \text{ListInt}\@S \rightarrow \text{ListInt}\@S\}\).

Semantics

Chor\(\lambda\) comes with a reduction semantics that captures the essential ingredients of the calculi that inspired it: \(\beta\) - and \(\eta\) - reduction, from \(\lambda\)-calculus, and the usual reduction rules for communications and selections. Some representative rules are given in Figure 4.

Proposition 6. Given a choreography \(M\), if \(M \xrightarrow{\ell,R} M'\) and \(M \xrightarrow{\ell,R'} M''\) with \(M' \neq M''\), then \(R \cap R' = \emptyset\).

Proof. Follows from induction on \(M\).

Rules [APPABS], [APP1] and [APP2] implement a call-by-value \(\lambda\)-calculus. Rules [CASE] and [CASER] implement \(\eta\)-reductions for sums, and likewise for rules [PRO1] and [PRO2] wrt pairs. The communication rule [COM] changes the associated role of a value, moving it from \(S\) to \(R\), while the selection rule [SEL] implements selection as a no-op. Rule [DEF] allows reductions to use choreographies defined in \(D\).

In addition to the fairly standard \(\lambda\)-calculus semantics, we have some additional rules for out-of-order execution. These include rewriting terms as described in Figure 2 and being able to propagate some transitions past an abstraction, case, and selection as in rules [INCASE] and [INSEL]. We also have almost full \(\beta\)-reduction by using rule [APP3]
x \notin \text{fv}(M')
\frac{\((\lambda x : T.M) N\) \leadsto (\lambda x : T.(M' M')) N}{[\text{R-AbsR}]}
\frac{x \notin \text{fv}(M') \quad \text{sroles}(M') \varcap \text{roles}(N) = \emptyset}{M' ((\lambda x : T.M) N) \leadsto (\lambda x : T.(M' M)) N}{[\text{R-AbsL}]}
\frac{x, x' \notin \text{fv}(M) \quad \text{sroles}(M) \varcap \text{roles}(N) = \emptyset}{(\text{case } N \text{ of } \text{inl} \ x \Rightarrow M_1; \ \text{inr} \ x' \Rightarrow M_2) \leadsto \text{case } N \text{ of } \text{inl} x \Rightarrow M_1 M; \ \text{inr} x' \Rightarrow M_2 M}{[\text{R-CaseR}]}
\frac{M \ (\text{case } N \text{ of } \text{inl} \ x \Rightarrow M_1; \ \text{inr} \ x' \Rightarrow M_2) \leadsto \text{case } N \text{ of } \text{inl} x \Rightarrow M M_1; \ \text{inr} x' \Rightarrow M M_2}{\text{sroles}(M) \varcap \text{roles}(N) = \emptyset}{[\text{R-CaseL}]}
\frac{(\text{select}_{S,R} \ l \ N) \leadsto \text{select}_{S,R} \ l \ (N \ M)}{[\text{R-SelR}]}
\frac{M \ (\text{select}_{S,R} \ l \ N) \leadsto \text{select}_{S,R} \ l \ (M \ N)}{[\text{R-Sell}]}

\text{Figure 2} \text{ Rewriting of Chor} \lambda.

\frac{\lambda x : T.M \ x \to V \ M \to M[x := V]}{[\text{AppAbs}]}
\frac{M \to D \ M'}{\lambda x : T.M \to D \ \lambda x : T.M'}{[\text{InAbs}]}
\frac{M \to D \ M'}{\ell = \lambda \varcap \text{roles}(N) = \emptyset}{[\text{App1}]}
\frac{M \to D \ M' \ N}{\ell = \lambda \varcap \text{roles}(N) = \emptyset}{[\text{App2}]}
\frac{M \to D \ M' \ N}{M \to D \ M \ N'}{[\text{App3}]}
\frac{N \to D \ N' \ [\text{Case}]}{\text{case } N \text{ of } \text{inl} \ x \Rightarrow M; \ \text{inr} \ x' \Rightarrow M' \to \text{case } N' \text{ of } \text{inl} \ x \Rightarrow M; \ \text{inr} \ x' \Rightarrow M'}
\frac{M_1 \to D \ M_1'; \ M_2 \to D \ M_2'; \ \text{R} \varcap \text{roles}(N) = \emptyset}{\text{case } N \text{ of } \text{inl} \ x \Rightarrow M_1; \ \text{inr} \ x' \Rightarrow M_2 \to \text{case } N' \text{ of } \text{inl} \ x \Rightarrow M_1'; \ \text{inr} \ x' \Rightarrow M_2'}{[\text{InCase}]}
\frac{\text{case } \text{inl} \ V \text{ of } \text{inl} \ x \Rightarrow M; \ \text{inr} \ x' \Rightarrow M' \to \text{case } \text{inl} \ V \text{ of } \text{inl} \ x \Rightarrow M; \ \text{inr} \ x' \Rightarrow M'}{[\text{CaseL}]}
\frac{\text{fst Pair} \ V \ V' \to D \ V \ M[x := V] \ [\text{Def}]}{\frac{\text{f}(\text{f}v(V)) = M}{[\text{Def}]}}
\frac{\text{com}_{S,R} \ V \ \ell \to D \ V[S := \text{R}]}{[\text{Com}]}
\frac{M \to D \ M' \ \text{R} \varcap \{S,R\} = \emptyset}{[\text{InSel}]}\frac{\text{select}_{S,R} \ l \ M \to D \ \text{select}_{S,R} \ l \ M'}{M \to D \ M''}{[\text{Sel}]}
\frac{\text{select}_{S,R} \ l \ M \to D \ \text{select}_{S,R} \ l \ M'}{[\text{Sel}]}\frac{M \to D \ M''}{M \to D \ M''}{[\text{Sel}]}

\text{Figure 3} \text{ Semantics of Chor} \lambda.
Example 7. Consider the choreography $M = f(R') \ ((\lambda x : T@R.V@R') V'@R)$. In $R$, this choreography is the application $((\lambda x : T.(\_)) V')$, but without full $\beta$-reduction, $M$ would be unable to do this application before $f(R')$ finished running, which may be never if $f$ diverges. In order for $M$ to mimic the action available at $R$, we therefore need rule $\text{APP3}$.

In Example 5 we see why we need the rewriting rules in order to do out-of-order executions.

Example 8 (Rewriting). Consider the choreography $((\lambda x : ()@R.\lambda x : T@R'.M) f(R)) V@R'$. At the role $R'$, this corresponds to $(\lambda x : T.M) V@R'$, but if $f$ diverges we need rule $\text{R-ABS}$ to get $\lambda x : T@R'.M$ and $V@R'$ next to each other by rewriting to $((\lambda x : ()@R.(\lambda x : T@R'.M) V@R') f(R))$ and rule $\text{INABS}$ to propagate the application of $(\lambda x : T@R'.M)$ and $V@R'$ past $\lambda x : ()@R$.

This problem can also be seen in e.g. $(\lambda y : T@R'.M@R')$ case $f(R)$ of $\text{ln} \ x \Rightarrow V@R'; \text{lnr} \ x' \Rightarrow V@R'$ where, since the conditional is not located at $R'$ it is projected to the value $V$. This gives us the application $(\lambda y : T.M) V$, which we therefore similarly need to be able to perform at the choreography as well as the process.

These out-of-order-execution rules have restrictions on them because we want to avoid there being more than one communication or synchronisation available at the same time on the same roles.

Example 9 (Communication order). Consider a choreography with two communications between the same roles, $\lambda x : T@R.(\text{com}_{S,R} V@S) (\text{com}_{S,R} V'@S)$. This has a similar structure to $((\lambda x : ()@R.(\lambda x : T@R'.M) V@R') f(R))$ from Example 8 but if we can do either communication in any order we please then it would be reasonable that would mean $S$ is currently able to send either $V$ or $V'$ with no guarantee that if $S$ chooses to send $V$ first $R$ will also choose to use its left receive action or vice versa.

We therefore restrict the out-of-order communications.

We also use the label $\lambda$ on in rule $\text{INABS}$ to restrict these out-of-order communications, since we do not know which roles we need to restrict communication on in Example 9 until we reach the application, at which point the label becomes a $\tau$ again if it is allowed to propagate.

Since we restrict out-of-order communication in the other out-of-order execution rules, we need to be the same in the rewriting rules as shown in Example 11. For this purpose we use the concept of synchronisation roles.

Example 10 (sroles). We define the set of synchronising roles of a choreography, $\text{sroles}$ in the following way:

- $\text{sroles}(\text{com}_{S,R}) = \{S, R\}$
- $\text{sroles}(\text{select}_{S,R} l M) = \{S, R\} \cup \text{sroles}(M)$

With all other constructs being defined homomorphically.

Example 11. Consider the choreography $(\text{com}_{S,R} V@S) ((\lambda x : T@R.M) (\text{com}_{S,R} V'@S))$. Here, thanks to the restriction on synchronisation in rule $\text{APP3}$ only the left $\text{com}_{S,R}$ on $V$ is available. If we were to rewrite the choreography to $((\lambda x : T@R.(\text{com}_{S,R} V@S) M) (\text{com}_{S,R} V'@S))$, we would instead have the rightmost $\text{com}_{S,R}$ on $V'$ available. This means we have both communication available depending on whether we decide to rewrite and we have the same problem as in Example 7. We therefore do not allow such a rewrite and use synchronisation roles to prevent it.

We focus on well-formed choreographies. Our first result shows that well-formed choreographies remain well-formed under reductions.
Proposition 12. Let $M$ be a well-formed choreography. If $M \rightarrow_D M'$ then $M'$ is well-formed.

Proof. Straightforward from the semantics.

One of the hallmark properties of choreographies is that well-typed choreographies should continue to reduce until they reach a value. We split this result in two independent statements.

Theorem 13 (Progress). Let $M$ be a well-formed choreography and $D$ a collection of named choreographies with all the necessary definitions for $M$. If there exists a typing context $\Theta; \Sigma; \Gamma$ such that $\Theta; \Sigma; \Gamma \vdash M : T$ and $\Theta; \Sigma; \Gamma \vdash D$, then either $M$ is a value (and $M \not\rightarrow_D$) or there exists a choreography $M'$ such that $M \rightarrow_D M'$.

Proof. Follows by induction on the typing derivation of $\Theta; \Sigma; \Gamma \vdash M : T$.

Theorem 14 (Type Preservation). Let $M$ be a well-formed choreography. If there exists a typing context $\Theta; \Sigma; \Gamma$ such that $\Theta; \Sigma; \Gamma \vdash M : T$ and $\Theta; \Sigma; \Gamma \vdash D$, then $\Theta; \Sigma; \Gamma \vdash M' : T$ for any $M'$ such that $M \rightarrow_D M'$.

Proof. Follows from the typing and semantic rules.

Combining these results, we conclude that if $M$ is a well-typed, well-formed, choreography, then either $M$ is a value or $M$ reduces to some well-typed, well-formed choreography $M'$. Since $M'$ still satisfies the hypotheses of the above results, either it is a value or it can reduce.

4 Endpoint Projection

In order to implement a choreography, one must determine how each individual role behaves. We introduce a process calculus to specify these behaviours, and show how to generate implementations of choreographies automatically. In this context, roles are implemented by processes and we use the two terms interchangeably.

Process Language

Definition 15. The syntax of behaviours, local values and local types is defined by the following grammar.

$$B ::= L \mid f(\bar{R}) \mid B \mid \text{case } B \text{ of } \text{Inl } x \Rightarrow B; \text{Inr } x \Rightarrow B \mid \oplus_R \ell B \mid \&_R\{\ell_1 : B_1, \ldots, \ell_n : B_n\}$$

$$L ::= x \mid \lambda x : T.B \mid \text{Inl } L \mid \text{Inr } L \mid \text{fst } L \mid \text{snd } L \mid \text{Pair } L L \mid () \mid \text{recv}_R \mid \text{send}_R \mid \bot$$

Behaviours correspond directly to local counterparts of choreographic actions. The terms from the $\lambda$-calculus are unchanged (except that there are no role annotations now); the choreographic actions generate two terms each. Selection yields the offer term $\&_R\{\ell_1 : B_1, \ldots, \ell_n : B_n\}$, which offers a number of different ways it can continue for another process $R$ to choose from, and the choice term $\oplus_R \ell B$, which directs $R$ to continue as the behaviour labelled $\ell$. Likewise, communication has been divided into a send to $R$ action, $\text{send}_R$, and a receive from $R$ action, $\text{recv}_R$. We also add a $\bot$ value and type, which we use when projecting choreographies or types onto roles where they are not present, e.g. $(\bot)@R$ projected to roles $S$ will be $\bot$ and also have the type $\bot$. Types are otherwise defined exactly as for choreographies, but without roles.

Definition 16. A network $N$ is a finite map from a set of processes to behaviours.
This means that we do not allow out-of-order execution of communications or selections. Using choreography semantics, we can perform the corresponding action resulting in because an action in the choreography can correspond to an internal action at multiple roles. Therefore, we often write equivalent to a parallel composition of networks with singleton domain, and therefore we also need to be able to do this application, but as the conditional is located would be to be ready to synchronise before performing the application resulting in. Since we have out-of-order-execution rules in our choreography semantics, we can perform the corresponding action

This means that \( S \) also needs to be able to do this application, but as the conditional is located at \( S \), \( M \) at \( S \) would be \( \text{case } N \text{ of } \text{lnl } x \Rightarrow (\lambda y. y) \Rightarrow \text{send}_R; \text{lnr } x' \Rightarrow (\lambda y. y) \Rightarrow \text{send}_R \). We therefore need to allow out-of-order execution at the processes as well as the choreographies.

Note that since \( R \) must wait for \( S \) to be ready to synchronise before performing the receive, we do not allow out-of-order execution of communications or selections.

\[
((\lambda x. B') B'' \rightarrow (\lambda x. B B'') B') \quad \text{[LR-AbsR]}
\]

\[
\text{roles}(B'') = \emptyset \quad \text{[LR-AbsL]}
\]

\[
(\text{case } B \text{ of } \text{lnl } x \Rightarrow B_1; \text{lnr } x \Rightarrow B_2) \rightarrow \text{case } B \text{ of } \text{lnl } x \Rightarrow B' \quad \text{[LR-CaseR]}
\]

\[
\text{roles}(B') = \emptyset \quad \text{[LR-CaseL]}
\]

\[
B' (\text{case } B \text{ of } \text{lnl } x \Rightarrow B_1; \text{lnr } x \Rightarrow B_2) \rightarrow \text{case } B \text{ of } \text{lnl } x \Rightarrow B' \quad \text{[LR-CaseL]}
\]

\[
\text{roles}(B) = \emptyset \quad \text{[LR-OffL]}
\]

\[
(\&_R\{\ell_1 : B_1, \ldots, \ell_n : B_n\}) B \rightarrow \&_R\{\ell_1 : B_1, \ldots, \ell_n : B_n\} \quad \text{[LR-OffR]}
\]

\[
\text{roles}(B') = \emptyset \quad \text{[LR-Chol]}
\]

\[
(\&_R l B) B' \rightarrow (B B') \quad \text{[LR-Chol]}
\]

\[
\bot \rightarrow \bot \quad \text{[LR-BotM]}
\]
Figure 5  Network semantics (representative rules).
We now have the necessary ingredients to define the endpoint projection (EPP) of a choreography, where a function and a value are both sent from a role. The need for this rule is illustrated by Example 18.

**Endpoint Projection (EPP)**

Example 18. Consider the choreography \((\text{com}_{S,R} (\lambda x : T@S.M@S)) (\text{com}_{S,R} V@S)\), where a function and a value are both sent from \(S\) to \(R\) before being applied at \(R\). At \(R\) this is the process \((\text{recv}_S \perp) (\text{recv}_S \perp)\), which executes as we want. However, at roles \(S\) this choreography becomes the process \((\text{send}_R (\lambda x : T,M)) (\text{send}_R V)\), which after the two communications are executed becomes \(\perp \perp\). After the choreography has executed both communications it is \((\lambda x : T@R,M@R) V@R\), which at \(S\) becomes \(\perp\), since \(S\) is not part of the rest of the choreography. We therefore need a way to make the two correspond. The rewriting rule \([\text{LR-Botm}]\) serves this purpose.

Our calculus includes a typing system for typing behaviours. Typing judgements now have the form \(\Sigma; \Gamma \vdash B : T\). Most of the rules are direct counterparts to those in Figure 1, obtained by removing \(\Theta\) and all side conditions involving roles. We also add the \(\perp\) type, used for typing \(\perp\) values or choreographies consisting entirely of \(\perp\) values, which can always be reduced to \(\perp\) using rule \([\text{LR-Botm}]\). The new rules are given in Figure 6.

**Endpoint Projection (EPP)**

We now have the necessary ingredients to define the endpoint projection (EPP) of a choreography \(M\) for an individual role \(R\) given a typing derivation showing that \(\Theta; \Sigma; \Gamma \vdash M : T\) for some type \(T\). Formally, the definition of EPP depends on this derivation; but to keep notation simple we write \(\llbracket M \rrbracket_R\).

Intuitively, the projection simply translates each choreography action to the corresponding local behaviour. For example, a communication action projects to a send (for the sender), a receive (for the receiver), or a unit (for the remaining processes). In order to define EPP precisely, we need a few additional ingredients, which we briefly describe.

Projecting a term requires knowing the roles involved in its type. This is implicitly given in the derivation provided to EPP. It can easily be shown by structural induction that, if the derivation contains two different typing judgements for the same term, then the roles involved in that term’s type are the same. So we write without ambiguity roles(type(M)) for this set of roles.

The second ingredient concerns knowledge of choice. When projecting case \(M\) of \text{Inl} \(x \Rightarrow M’\); \text{Inr} \(y \Rightarrow M”\), roles not occurring in \(M\) cannot know what branch of the choreography is chosen; therefore, the projections of \(M’\) and \(M”\) must be combined in a uniquely defined behaviour. This is done by means of a standard partial merge operator (\(\sqcup\)), adapted from [5] [12] [24], whose key property is

\[ \&\{\ell_i : B_i\}_{i \in I} \sqcup \&\{\ell_j : B_j’\}_{j \in J} = \&\{\ell_k : B_k \sqcup B_k’\}_{k \in I \cup J} \cup \{\ell_i : B_i\}_{i \in I \setminus J} \cup \{\ell_j : B_j’\}_{j \in J \setminus I} \]
and which is homomorphically defined for the remaining constructs (see the Appendix A for the full definition). Merging of incompatible behaviours is undefined.

**Definition 19.** The EPP of a choreography $M$ for role $R$ is defined by the rules in Figure 7.

To project a network from a choreography, we therefore project the choreography for each role and combine the results in parallel: $[M] = \prod_{R \in \text{roles}(M)} R([M]_R)$.

Intuitively, projecting a choreography to a role that is not involved in it returns a $\perp$. More complex choreographies, though, may involve roles that are not shown in their type. This explains the first clause for projecting an application: even if $R$ does not appear in the type of $M$, it may participate in interactions inside $M$. A similar observation applies to the projection of case, where merging is also used.

Selections and communications follow the intuition given above, with one interesting detail: self-selections are ignored, and self-communications project to the identity function. This is different from many standard choreography calculi, where self-communications are not allowed – we do not want to impose this in Chorλ, since one of the planned future developments for this language is to add polymorphism.

Likewise, projecting a type yields $\perp$ at any role not used in that type. The projection of a set of function definitions maps choreography names to behaviours.

**Proposition 20.** Let $M$ be a well-formed choreography. If $\Theta; \Sigma; \Gamma \vdash M : T$, then for any role $R$ appearing in $M$, we have that $[\Sigma]; [\Gamma] \vdash [M]_R : [T]_R$, where $[\Sigma]$ and $[\Gamma]$ are defined by applying EPP to all the types occurring in those sets.

**Proof.** Straightforward from the typing and projection rules.

**Example 21.** The projections of the choreographies in Example 3 are the following.

\[
[D(\text{remoteFunction}(R, S))]_1(R) = \lambda f : (\text{Int} \to \text{Int}). \lambda \text{val} : \perp. \text{send}_R(f(\text{recv}_S \perp)) \\
[D(\text{remoteFunction}(R, S))]_2(R) = \lambda f : \perp. \lambda \text{val} : \text{Int}. \text{recv}_R(\text{send}_R \text{val}) \\
[D(\text{remoteMap}(R, S))]_1(R) = \lambda f : \text{Int} \to \text{Int}. \lambda \text{list} : \perp. &_S\{\text{stop} : \perp\text{again} : (\text{remoteFunction}_1(S) f \perp) (\text{remoteMap}_1(S) f \perp)} \\
[D(\text{remoteMap}(R, S))]_2(R) = \lambda f : \perp. \lambda \text{list} : \text{List} \text{Int}. &_R\{\text{case list of} \ \text{Inl} \ x \Rightarrow (\oplus_R \text{stop} (\perp)) ; \\
& \text{Inr} \ x \Rightarrow (\oplus_R \text{again} (\text{cons} (\text{remoteFunction}_2(R) \perp (\text{fst} \ x)) \\
& (\text{remoteMap}_2(R) \perp (\text{snd} \ x))))\}
\]

This example illustrates the key features discussed in the text: projection of communications as two dual actions; the use of merge in the projection of case; and the way function applications are projected when the role does not appear in the function’s type.

We now show that there is a close correspondence between the executions of choreographies and of their projections. Intuitively, this correspondence states that a choreography can execute an action iff its projection can execute the same action, and both transition to new terms in the same relation. However, this is not completely true: if a choreography $C$ reduces by rule case, then the result has fewer branches than the network obtained by performing the corresponding reduction in the projection of $C$.

In order to capture this, we revert to the notion of pruning [6, 13], defined by $B \triangledown B'$ iff $B \sqcup B' = B$. Intuitively, if $B \triangledown B'$, then $B$ offers the same and possibly more behaviours than $B'$. This notion extends to networks by defining $\mathcal{N} \triangledown \mathcal{N}'$ to mean that, for any role $R$, $\mathcal{N}(R) \triangledown \mathcal{N}'(R)$. 
Example 22. Consider the choreography

\[ C = \text{case Inl } (\_ \oplus R \text{ of Inl} x \Rightarrow \text{select}_{R,S} \text{ left } 0 \oplus S; \text{ lnr } y \Rightarrow \text{select}_{R,S} \text{ right } 1 \oplus S). \]

Its projection for role \( S \) is \([C]_S = \&_R\{\text{left : 0, right : 1}\}\).

After entering the conditional in the choreography, \( C \) reduces to \( C' = \text{select}_{R,S} \text{ left } 0 \oplus S \), whereas \( S \) does not have a corresponding action and its behaviour remains \( \&_R\{\text{left : 0, right : 1}\} \), which is not the projection of \( C' \). However, \( \&_R\{\text{left : 0, right : 1}\} \sqcup \&_R\{\text{left : 0, right : 1}\} = \&_R\{\text{left : 0, right : 1}\} \), so \([C']_S \) is a pruning of this behaviour.

In addition to pruning we need one more equivalence for our semantics of choreographies and networks to correspond.

Definition 23. \( P \equiv \lambda x : \bot. P \bot \)

\[ \Pi_R P_R \equiv \Pi_R P'_R \text{ if } P_R \equiv P'_R \text{ for all } R \]

\[ N' \subseteq N'' \text{ if } N \sqsubseteq N'' \text{ and } N'' \equiv N' \]

We can finally show that the EPP of a choreography can do all that (completeness) and only what (soundness) the original choreography does.

Theorem 24 (Completeness). Given a well-formed choreography \( M \), if \( M \xrightarrow{\tau_R} D M' \) and \( \Theta; \Sigma; \Gamma \vdash M : T \), then there exist networks \( N \) and \( M'' \) such that: \( [M] \rightarrow^{+}_{[D]} N ; M' \rightarrow D M'' \); and \( N \equiv [M''] \).

Proof. By structural induction on the derivation of \( M \rightarrow D M' \).

Theorem 25 (Soundness). Given a well-formed choreography \( M \), if \( \Theta; \Sigma; \Gamma \vdash M : T \) and \( [M] \rightarrow^{+}_{[D]} N \) for some network \( N \), then there exist a choreography \( M' \), a set mapping \( D \), and a network \( N' \) such that: \( M \rightarrow^{+}_{D} M' \); \([D] = D; N \rightarrow D N' \); and \( N' \equiv [M'] \).

Proof. By structural induction on \( M \).

From Theorems 13, 14, 24, and 25, we get the following corollary, which states that a network derived from a well-typed well-formed choreography can continue to reduce until all roles contain only local values.

Corollary 26. Given a well-formed choreography \( M \) and a function environment \( D \) containing all the functions of \( M \), if \( \Theta; \Sigma; \Gamma \vdash M : T \) and \( \Theta; \Sigma; \Gamma \vdash D \), then: whenever \( [M] \rightarrow^{+}_{[D]} N \) for some network \( N \), either there exists \( R \) such that \( N \rightarrow^{+}_{\{D\}} N' \) or \( N = \prod_{R \in \text{roles}(M)} R[L_R] \).
Choreographies as Functions

\[
[M \cdot N]_R = \begin{cases} 
[M]_R \cdot [N]_R & \text{if } R \in \text{roles}(\text{type}(M)) \text{ or } R \in \text{roles}(M) \cap \text{roles}(N) \\
⊥ & \text{if } [M]_R = [N]_R = ⊥ \\
[M]_R & \text{if } [N]_R = ⊥ \\
[N]_R & \text{otherwise}
\end{cases}
\]

\[
[λx:T.M]_R = \begin{cases} 
λx:[T]_R \cdot [M]_R & \text{if } R \in \text{roles}(x:T.M) \\
⊥ & \text{otherwise}
\end{cases}
\]

\[
\text{case } M \text{ of } \text{lnl } x ⇒ N; \text{lnl } x' ⇒ N' ]_R =
\begin{cases} 
\text{case } [M]_R \text{ of } \text{lnl } x ⇒ [N]_R; \text{lnl } x' ⇒ [N']_R & \text{if } R \in \text{roles}(\text{type}(M)) \\
⊥ & \text{if } [M]_R = [N]_R = [N']_R = ⊥ \\
[M]_R & \text{if } [N]_R = ⊥ \\
[N]_R \cup [N']_R & \text{if } [M]_R = ⊥ \\
(λx'':⊥.[N]_R \cup [N']_R) \cdot [M]_R & \text{for some } x'' \notin \text{fv}(N) \cup \text{fv}(N') \\
\end{cases}
\]

\[
\text{select}_{S,S'} \cdot ℓ \cdot M ]_R = \begin{cases} 
\text{select}_{S,S'} \cdot ℓ \cdot [M]_R & \text{if } R = S \neq S' \\
& \& \text{select}_{S'} \cdot ℓ \cdot [M]_R & \text{if } R = S' \neq S \\
[M]_R & \text{otherwise}
\end{cases}
\]

\[
\text{com}_{S,S'} ]_R = \begin{cases} 
λx:[T]_R \cdot x & \text{if } R = S = S' \text{ and } \text{type(} \text{com}_{S,S'} \text{)} = T \to ℓ T' \text{ and } \text{type}(\text{com}_{S,S'}) = T \to ℓ T' \\
\text{send}_{S'} & \text{if } R = S \neq S' \\
\text{recv}_S & \text{if } R = S' \neq S \\
⊥ & \text{otherwise}
\end{cases}
\]

\[
[()@S]_R = \begin{cases} 
() & \text{if } S = R \\
⊥ & \text{otherwise}
\end{cases}
\]

\[
[x]_R = \begin{cases} 
x & \text{if } R \in \text{roles}(\text{type}(x)) \\
⊥ & \text{otherwise}
\end{cases}
\]

\[
[f(\vec{R})]_R = \begin{cases} 
f_1((R_1, \ldots, R_{i-1}, R_{i+1}, \ldots, R_n)) & \text{if } \vec{R} = (R_1, \ldots, R_{i-1}, R, R_{i+1}, \ldots, R_n) \\
⊥ & \text{otherwise}
\end{cases}
\]

Types:

\[
[()@S]_R = \begin{cases} 
() & \text{if } S = R \\
⊥ & \text{otherwise}
\end{cases}
\]

\[
[T \to ρ \cdot T']_R = \begin{cases} 
[T]_R \to [T']_R & \text{if } R \in ρ \cup \text{roles}(T) \cup \text{roles}(T') \\
⊥ & \text{otherwise}
\end{cases}
\]

\[
[ρ@\vec{R}]_R = \begin{cases} 
τ \cdot ρ & \text{if } \vec{R} = (R_1, \ldots, R_{i-1}, R, R_{i+1}, \ldots, R_n) \\
⊥ & \text{otherwise}
\end{cases}
\]

\[
[T \times T']_R = \begin{cases} 
[T]_R \times [T']_R & \text{if } R \in \text{roles}(T \times T') \\
⊥ & \text{otherwise}
\end{cases}
\]

Definitions:

\[
[D] = \{f_i(R_1, \ldots, R_{i-1}, R_{i+1}, \ldots, R_n) \mapsto [D(f(R_1, \ldots, R_n))]_R | f(R_1, \ldots, R_n) \in \text{domain}(D)\}
\]

- Figure 7: Projecting a choreography in Chorλ onto a role.
5 An Illustrative Example: Secure Authentication

In this section, we illustrate more in depth the expressivity of Chor$\lambda$: we write a distributed authentication protocol inspired by the OpenID specification [36] and modularly combine it with the Diffie–Hellman protocol for key exchange [17] to secure its communications. Since we do not have polymorphism, our implementation is not completely generic; in particular, we can only communicate strings, and we need multiple implementations of the same function for different roles. For simplicity, we assume primitive types $\text{Int}$ and $\text{String}$.

Our protocol involves three roles: a client $C$, a server $S$, and an identity provider $IP$. We assume the existence of a couple of functions whose implementation is immaterial for the presentation.

The essential functions for implementing the cryptographical security of the protocol allow us to compute powers with a given modulo, and encrypt and decrypt messages:

- $\text{modPow}(R) : \text{Int}@R \rightarrow \text{Int}@R \rightarrow \text{Int}@R$
- $\text{encrypt}(R) : \text{Int}@R \rightarrow \text{String}@R \rightarrow \text{String}@R$
- $\text{decrypt}(R) : \text{Int}@R \rightarrow \text{String}@R \rightarrow \text{String}@R$

There are also functions for working with the credentials

- $\text{username}(R) : \text{Credentials}@R \rightarrow \text{String}@R$
- $\text{password}(R) : \text{Credentials}@R \rightarrow \text{String}@R$
- $\text{calcHash}(R) : \text{String}@R \rightarrow \text{String}@R \rightarrow \text{String}@R$

computing, respectively, the username and password from a local type $\text{Credentials}@R$ (which can be implemented as a pair, for example), and the hash of a string with a given salt. These are mainly used by the client.

Finally, there exist functions for retrieving the salt, checking the hash, and creating a token for a given username, which are used by the identity provider.

- $\text{getSalt}(R) : \text{String}@R \rightarrow \text{String}@R$
- $\text{check}(R) : \text{String}@R \rightarrow \text{String}@R \rightarrow \text{Bool}@R$
- $\text{createToken}(R) : \text{String}@R \rightarrow \text{String}@R$

We denote by $\Gamma$ the set of typings of all the above functions.

The first step is implementing the Diffie–Hellman algorithm for each pair of roles. This is done by means of the following choreography, which takes all of the necessary parameters of the algorithm as arguments:

\[
\text{diffieHellman}(P, Q) = \\
\lambda \text{psk} : \text{Int}@P, \lambda \text{qsk} : \text{Int}@Q, \lambda \text{psg} : \text{Int}@P, \lambda \text{qsg} : \text{Int}@Q, \lambda \text{psp} : \text{Int}@P, \lambda \text{qsp} : \text{Int}@Q. \\
(\lambda \text{rk} : \text{Int}@P \times \text{Int}@Q. \text{pair}(\text{modPow}(P) \text{psg} (\text{fst} \text{rk}) \text{psp})(\text{modPow}(Q) \text{qsg} (\text{snd} \text{rk}) \text{qsp})) \\
((\lambda \text{pk} : \text{Int}@P \times \text{Int}@Q. \text{pair}(\text{com}_Q,P (\text{snd} \text{pk}))(\text{com}_P,Q (\text{fst} \text{pk})))) \\
((\lambda \text{sk} : \text{Int}@P \times \text{Int}@Q. \text{pair}(\text{modPow}(P) \text{qsg} (\text{fst} \text{sk}) \text{qsp})(\text{modPow}(Q) \text{psg} (\text{snd} \text{sk}) \text{psp})) \\
(Pair \text{psk qsk})))
\]

This choreography applies the sequence of distributed transformations specified in the Diffie–Hellman key exchange algorithm to a pair of secret keys. We can check that
The second ingredient is a choreography to create pairs of secure channels between two roles, encrypting and decrypting messages with the appropriate key:

\[
\text{makeSecureChannels}(P, Q) = \lambda \text{key} : \text{Int} @ P \times \text{Int} @ Q.
\]

\[
\text{Pair} \ (\lambda \text{val} : \text{String} @ P. \ (\text{decrypt}(P) (\text{snd} \ \text{key}) \ (\text{com} P, Q \ (\text{encrypt}(P) (\text{fst} \ \text{key}) \ \text{val}))))
\]

\[
(\lambda \text{val} : \text{String} @ Q. \ (\text{decrypt}(P) (\text{fst} \ \text{key}) \ (\text{com} Q, P \ (\text{encrypt}(Q) (\text{snd} \ \text{key}) \ \text{val}))))
\]

Again, we can show that

\[
\Theta; \Sigma; \Gamma \vdash \text{makeSecureChannels}(P, Q) : \text{((Int} @ P \times \text{Int} @ Q) \to ((\text{String} @ P \to \text{String} @ Q) \times (\text{String} @ Q \to \text{String} @ P))
\]

The third ingredient is an authentication protocol where the client and the server both get a unique token from the identity provider if authentication succeeds. We use if-then-else as syntactic sugar in the following choreography:

\[
\text{authenticate}(S, C, I) = \lambda \text{credentials} : \text{Credentials} @ C. \ \\
(\lambda \text{comipc} : \text{String} @ C \to \text{String} @ I. \ (\lambda \text{comipc} : \text{String} @ I \to \text{String} @ C. \ \\
\ (\lambda \text{comips} : \text{String} @ I \to \text{String} @ S. \ \\
\ (\lambda \text{user} : \text{String} @ I. \ (\lambda \text{salt} : \text{String} @ C. \ (\lambda \text{hash} : \text{String} @ I. \ \\
\ (\text{if check}(I) \ \text{user} \ \text{hash} \ \text{then} \ \\
\ (\text{select}_I, C \ \text{ok} (\text{select}_I, S \ \text{ok}) \ \\
\ (\text{axtoken} : \text{String} @ I. \ \text{inl} (\text{pair} (\text{comipc} \ \text{token}) (\text{comips} \ \text{token})))) \ (\text{createToken}(I) \ \text{user})) \ \\
\ \text{else} \ \\
\ (\text{select}_I, C \ \text{ko} (\text{select}_I, S \ \text{ko} \ \text{inr} () @ I)) \ \\
\ (\text{comipc} (\text{calcHash}(C) \ \text{salt} (\text{password}(C) \ \text{credentials})))) \ \\
\ (\text{comipc} (\text{getSalt}(I) \ \text{user})) ) \ \\
\ (\text{comipc} (\text{username}(C) \ \text{credentials}))))
\]

This protocol is parameterised over three channels between the participants. The client sends their username to the identity provider, who replies with the appropriate salt; the client then then uses this to hash their password and send it back to the identity provider, who checks the result and either sends a token to both participants or returns a unit. We can type this choreography as follows:

\[
\Theta; \Sigma; \Gamma \vdash \text{authenticate} : \text{Credentials} @ C \to (\text{String} @ C \to \text{String} @ I) \to \\
(\text{String} @ I \to \text{String} @ C) \to (\text{String} @ I \to \text{String} @ S) \to ((\text{String} @ C \times \text{String} @ S) + () @ I)
\]

We now write the choreography main which ties everything together. Using the secure channels created by makeSecureChannels, which are backed by an encryption key provided by diffieHellman, the three roles execute the authentication protocol specified by authenticate:

\[
\text{main}(S, C, I) = \\
(\lambda \text{kl1} : \text{Int} @ C \times \text{Int} @ I. \lambda \text{kl2} : \text{Int} @ I \times \text{Int} @ S. \\
(\lambda \text{cl1} : (\text{String} @ C \to \text{String} @ I) \times (\text{String} @ I \to \text{String} @ C). \\
\lambda \text{cl2} : (\text{String} @ I \to \text{String} @ S) \times (\text{String} @ S \to \text{String} @ I)).
\]
In this example, the protocol simply tells the client whether authentication has succeeded; but this could of course be replaced with more meaningful code.

Let $\Gamma'$ be obtained from $\Gamma$ by adding the types for $\text{diffieHellman}$, $\text{makeSecureChannels}$ and $\text{authenticate}$ given earlier. Then this choreography has type $\text{String}@C$ in the typing environment $\Theta; \Sigma; I'$. To complete this section, we illustrate how implementations of the individual roles can be obtained by projecting each choreography. Projecting our choreographies $\text{diffieHellman}_{P,Q}$ and $\text{makeSecureChannels}_{P,Q}$ for role $P$ yields the following behaviours:

In turn, $\text{authenticate}$ has the following projections for the client and the server.

It is simple to check that the types of these projections are indeed the projections of the types of the original choreographies.
20 Choreographies as Functions

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Definition 27 (Free Variables). Given a choreography $M$, the free variables of $M$, $\text{fv}(M)$, are defined as:

$$\text{fv}(N N') = \text{fv}(N) \cup \text{fv}(N')$$

$$\text{fv}(x) = x$$

$$\text{fv}((\theta R)) = \emptyset$$

$$\text{fv}(f) = \emptyset$$

$$\text{fv}(\text{case } N \text{ of } \text{lnl } x \Rightarrow M; \text{lnr } y \Rightarrow M') = \text{fv}(N) \cup (\text{fv}(M) \setminus \{x\}) \cup (\text{fv}(M') \setminus \{y\})$$

$$\text{fv}(\text{fst}) = \text{fv}(\text{snd}) = \emptyset$$

$$\text{fv}(\text{lnl } V) = \text{fv}(\text{lnr } V) = \text{fv}(V)$$

Definition 28 (Merging). Given two behaviours $B$ and $B'$, $B \sqcup B'$ is defined as follows.

$$B_1 \sqcup B_2 \sqcup B'_1 = (B_1 \sqcup B'_1) \sqcup (B_2 \sqcup B'_2)$$

$$\text{case } B_1 \text{ of } \text{lnl } x \Rightarrow B_2; \text{lnr } y \Rightarrow B_3 \sqcup \text{case } B'_1 \text{ of } \text{lnl } x \Rightarrow B'_2; \text{lnr } y \Rightarrow B'_3 =$$

$$\text{case } (B_1 \sqcup B'_1) \text{ of } \text{lnl } x \Rightarrow (B_2 \sqcup B'_2); \text{lnr } y \Rightarrow (B_3 \sqcup B'_3)$$

$$\sqcup_R \ell B \sqcup \sqcup_R \ell B' = \sqcup_R \ell (B \sqcup B')$$

$$\&\{\ell_i : B_i\}_{i \in I} \sqcup \&\{\ell_j : B'_j\}_{j \in J} = \&\{\ell_k : B_k \sqcup B'_k\}_{k \in I \cup J} \sqcup \{\ell_i : B_i\}_{i \in I \cup J} \sqcup \{\ell_j : B'_j\}_{j \in I \cup J}$$

$$\text{fst} \sqcup \text{fst} = \text{fst} \quad \text{snd} \sqcup \text{snd} = \text{snd}$$

$$\text{lnl } L \sqcup \text{lnl } L' = \text{lnl } (L \sqcup L') \quad \text{lnr } L \sqcup \text{lnr } L' = \text{lnr } (L \sqcup L')$$

$$\text{Pair } L_1 \sqcup \text{Pair } L'_1 = \text{Pair } (L_1 \sqcup L'_1) \quad f \sqcup f = f$$

$$\text{recv}_R \sqcup \text{recv}_R = \text{recv}_R \quad \text{send}_R \sqcup \text{send}_R = \text{send}_R$$

A.1 Proof of Theorem 24

Lemma 29. Given a choreography $M$, if $\Theta; \Sigma; \Gamma \vdash M : T$ then for any role $R$ in roles($M$), $[M]_R = L$ if $M = V$. 
Proof. Straightforward from the projection rules.

**Lemma 30.** Given a type \( T \), for any role \( R \notin \text{roles}(T) \), \( [T]_R = \bot \).

Proof. Straightforward from induction on \( T \).

**Lemma 31.** Given a value \( V \), for any role \( R \notin \text{roles(type}(V)) \), we have \( [V]_R = \bot \).

Proof. Follows from Lemmas 29 and 30 and the projection rules.

**Lemma 32.** Given a well-formed choreography, \( M \), if \( \Theta; \Sigma; \Gamma \vdash M : T \) and \( R \notin \text{roles}(M) \cup \text{roles}(T) \) then \( [M]_R = \bot \).

Proof. Straightforward from Lemmas 29 and 31 and induction on \( \Theta; \Sigma; \Gamma \vdash M : T \).

**Lemma 33.** If \( M \leadsto M' \) and \( M \xrightarrow{\tau_R} D M'' \) then \( M' \xrightarrow{\tau_R} D M''' \) such that \( M'' \leadsto^* M''' \)

Proof. Follows from case analysis on \( M \leadsto M' \).

**Lemma 34.** If \( M \leadsto M' \) then for any role \( R \), \( [M]_R \leadsto \leadsto^* B \) such that \( B \equiv [M']_R \)

Proof. Follows from case analysis on \( M \leadsto M' \).
\[
\lambda x : T. M \xrightarrow{\tau} D M[x := V] \quad \text{APPABS}
\]

\[
M \xrightarrow{\ell \cdot R} D M' \quad \text{INABS}
\]

\[
\lambda x : T. M \xrightarrow{\lambda \cdot R} D \lambda x : T. M' \quad \text{INABS}
\]

\[
M \xrightarrow{\ell \cdot R} D M' \quad \ell = \lambda \Rightarrow R \cap \text{roles}(N) = \emptyset \quad \text{APP1}
\]

\[
M \xrightarrow{\tau \cdot R} D M' N \quad \text{INABS}
\]

\[
N \xrightarrow{\tau \cdot R} D N' \quad \text{APP2}
\]

\[
V N \xrightarrow{\tau \cdot R} D V N' \quad \text{APP3}
\]

\[
\lambda x : T. M \xrightarrow{\tau \cdot D} D M' N \quad \text{APP1}
\]

\[
\text{case } N \text{ of } \text{inl } x \Rightarrow M_1; \text{inr } x' \Rightarrow M_2 \xrightarrow{\tau \cdot D} D M' \quad \text{INCASE}
\]

\[
\text{case } \text{inl } V \text{ of } \text{inl } x \Rightarrow M; \text{inr } x' \Rightarrow M' \xrightarrow{\tau \cdot D} D M' \quad \text{INCASE}
\]

\[
\text{case } \text{inr } V \text{ of } \text{inl } x \Rightarrow M_1; \text{inr } x' \Rightarrow M_2 \xrightarrow{\tau \cdot D} D M' \quad \text{CASEL}
\]

\[
\text{case } \text{inr } V \text{ of } \text{inl } x \Rightarrow M_1; \text{inr } x' \Rightarrow M_2 \xrightarrow{\tau \cdot D} D M' \quad \text{CASER}
\]

\[
\text{fst } V V' \xrightarrow{\tau \cdot D} D V' \quad \text{PROJ} \quad \text{snd } V V' \xrightarrow{\tau \cdot D} D V' \quad \text{PROJ2}
\]

\[
f(V) = \emptyset \quad \text{DEF}
\]

\[
\text{com}_{S,R} V \xrightarrow{\tau \cdot (S,R)} D V[S := R] \quad \text{COM}
\]

\[
\text{select}_{S,R} \ell M \xrightarrow{\tau \cdot (S,R)} D M \quad \text{SEL}
\]

\[
\text{select}_{S,R} \ell M \xrightarrow{\tau \cdot R} D M' \quad \text{INSEL}
\]

\[
M \xrightarrow{\tau \cdot R} D M' \quad \text{STR}
\]

\[\text{Figure 9 Semantics of Chor} \lambda\]
Lemma 35. If $M \xrightarrow{\tau} D M'$ for $R \neq \emptyset$ then $[[M]] \xrightarrow{\tau_R} [[M']]$

Lemma 36. If $M \xrightarrow{\tau} D M'$ then for any role $R \not\in R$, $[[M]]_R = [[M']]_R$

Proof of Theorem 24. We prove this by structural induction on $M \rightarrow \ell D M'$.

- Assume $M = \lambda x : T.N V$ and $M' = N[x := V]$. Then for any role $R \in \text{roles(type}(\lambda x : T.N))$, we have $[[M]]_R = (\lambda x : [[T]]_R \cdot [[N]]_R) \cdot [[V]]_R$ and $[[M']]_R = N[x := [[V]]_R]$. And for any $R' \not\in \text{roles(type}(\lambda x : T.N))$, we have $[[M]]_{R'} = [[M']]_{R'} = \bot$. We therefore get $R[[M]]_R \xrightarrow{\tau_{[D]}} [[M']]_R$ for all $R \in \text{roles(type}(\lambda x : T.N))$ and define $N = \prod_{R \in \text{roles(type}(\lambda x : T.N))} R[[M]]_R \prod_{R' \not\in \text{roles(type}(\lambda x : T.N))} R'[\bot]$ and the result follows.

- Assume $M = N M''$, $M' = N' M''$, and $N \xrightarrow{\tau} D N'$. Then for any role $R \in \text{roles(type}(N))$, $[[N]]_R = [N]_R \cdot [[M'']]_R$ and $[[M']]_R = [N']_R \cdot [[M'']]_R$. For any role $R'$ such that $[[N']]_{R'} = [[M''']]_{R'} = \bot$, by induction we have $[[N']]_{R'} = \bot$, and therefore $[[M]]_{R'} = [[M']]_{R'} = \bot$. For any other role $R'$ such that $[[M'']]_{R''} = \bot$, we get $[[M]]_{R'} = [[N']]_{R'}$ and $[[M']]_{R'} = [[N']]_{R'}$. And by induction $[[N]] \xrightarrow{\tau_{[D]}} N_N$ and $N' \xrightarrow{\tau_{[D]}} N''$ for $N_N \supseteq [N']$. For any role $R$ we therefore get $[[N]]_R \xrightarrow{\mu_0} [D]_R \xrightarrow{\mu_1} \ldots [D]_R$ for $R \supseteq [[N]]_R$ for some sequences of transitions $\mu_0 \xrightarrow{\mu_1} \ldots$, and from the network semantics we get

$$[[M]] \xrightarrow{\ast} \prod_{R \in \text{roles(type}(N)))} R[[B]_R] \prod_{R' \not\in \text{roles(type}(N)))} R'[\bot] = N$$

and $M' \xrightarrow{\ast} N''$. And since $[[N]] \xrightarrow{\tau_{[D]}} N'$ and $[[N']] \xrightarrow{\tau_{[D]}} N''$, we know these sequences of transitions can synchronise when necessary, and if $[[N'']] \neq [[N']]$, then we can do the extra application to get rid of this unit.

- Assume $M = V N$, $M' = V N'$, and $N \xrightarrow{\tau} D N'$. Then for any role $R \in \text{roles(type}(V))$, $[[V]]_R = [V]_R \cdot [[N]]_R$ and $[[M']]_R = [V]_R \cdot [[N']]_R$. Since $V$ is a value, for any role $R' \not\in \text{roles(type}(V))$, we have $[[V]]_{R'} = \bot$ and so for any role $R'$ such that $[[V]]_{R'} = [V]_{R'} = \bot$, by induction we get $[[N']]_{R'} = \bot$ and therefore $[[M]]_{R'} = [[M']]_{R'} = \bot$. For any other role $R'$ such that $[[V]]_{R'} = \bot$, we have $[[M]]_{R'} = [[N]]_{R'}$ and $[[M']]_{R'} = [[N']]_{R'}$. By induction, $[[N]] \xrightarrow{\tau_{[D]}} N_N$ and $N' \xrightarrow{\tau_{[D]}} N''$ for $N_N \supseteq [N']$. For any role $R$ we therefore get $[[N]]_R \xrightarrow{\mu_0} [D]_R \xrightarrow{\mu_1} [D]_R \ldots [D]_R$ for $R \supseteq [[N]]_R$ for some sequences of transitions $\mu_0 \xrightarrow{\mu_1} \ldots$, and from the network semantics we get

$$[[M]] \xrightarrow{\ast} \prod_{R \in \text{roles(type}(V)))} R[[V]]_R B_R \prod_{R' \not\in \text{roles(type}(V)))} R'[B_R] = N$$

and $M' \xrightarrow{\ast} V N''$ and the result follows.

- Assume $M = M'' N$, $M' = M'' N'$, and $N \xrightarrow{\tau} D N'$. Then for any $R \in R$, $\text{roles}(M'') \cap R = \emptyset$ and the result follows from induction and using rule $\text{NA}pp2$. 

$\square$
Assume $M = \text{case } N \text{ of } \text{l}n x \mapsto N''; \text{l}n x' \mapsto N'''$, $M' = \text{case } M'' \text{ of } \text{l}n x \mapsto N'$; $\text{l}n x \mapsto N''$, and $R \xrightarrow{\tau R} D M''$. Then for any role $R'$ such that $R \in \text{roles}(\text{type}(N))$, $[M]_{R'} = \text{case } [N]_{R} \text{ of } \text{l}n x \mapsto [N']_{R} \text{ if } \text{l}n x' \mapsto [N'']_{R}$ and $[M']_{R'} = \text{case } [M'']_{R} \text{ of } \text{l}n x \mapsto [N']_{R}$. For any other role $R''$ such that $[N]_{R''} = [N'']_{R''}$, we get $[M]_{R''} = [M'']_{R''} = [N']_{R''} \cup [N''']_{R''}$. For any other roles $R''' \in \text{roles}(\text{type}(N))$, $[M]_{R'''} = [M'']_{R'''}$. For any other role $R''''$, we have $[M]_{R''''} = \lambda x. [N]_{R''''} \cup [N'']_{R''''}$. This case is similar to case four.

Assume $M = \text{case } N \text{ of } \text{l}n x \mapsto N'$; $\text{l}n x' \mapsto N''$, and $M' = N'[x := x']$. Then for any role $R \in \text{roles}(\text{type}(\text{V}))$, we have $[M]_{R} = \text{case } \text{V} \text{ of } \text{l}n x \mapsto [N]_{R}; \text{l}n x' \mapsto [N'']_{R}$. By Lemma [31], $[N][x := [V]_{R}] = \lambda x. [N]_{R}[x := [V]_{R}]$. For any other role $R' \notin \text{roles}(\text{type}(\text{V}))$, $[\text{V}]_{R} = \bot$, and therefore $[M]_{R'} = [M']_{R'}$. The result follows.

Assume $M = \text{case } \text{inl} \text{ of } \text{l}n x \mapsto N'$; $\text{l}n x' \mapsto N''$, and $M' = N'[x := x']$. This case is similar to the previous.

Assume $M = \text{case } \text{n} \text{l} \text{n} \text{x} \mapsto N'$; $\text{l}n x' \mapsto N''$, and $M' = N'[x := x']$. Then for any role $R \in \text{roles}(\text{type}(\text{n} \text{l} \text{n} \text{x}))$, we have $[M]_{R} = \text{case } \text{n} \text{l} \text{n} \text{x} \text{ of } \text{l}n x \mapsto [N]_{R}; \text{l}n x' \mapsto [N'']_{R}$, and $[M']_{R} = [M']_{R'}$. The result follows.

Assume $M = \text{select}_{S,R} l M'$. Then $[M]_{S} = \bot \text{ if } [M']_{S}, [M]_{R} = \& l : [M']_{R}$, and for any $R' \notin \{S,R\}$, $[M]_{R'} = [M']_{R'}$. We therefore get $[M]_{ \text{select}_{S,R} l M' } = [M]_{ \text{select}_{S,R} l M' } \mid [M]_{R'}$. This result follows.

Assume $M = \text{select}_{S,R} l N, M' = \text{select}_{S,R} l N'$, and $R \cdot N \xrightarrow{\text{R}} D N'$. Then $[M]_{S} = \bot \text{ if } [N]_{S}, [M]_{S} = \bot \text{ if } [N']_{S}, [M]_{R} = \& l : [N]_{R}, [M']_{R} = \& l : [N']_{R},$ and for any $R' \notin \{S,R\}$, $[M]_{R'} = [N]_{R'}$ and $[M']_{R'} = [N']_{R'}$. The result follows from induction and using rules \text{NOFF2} and \text{VCHO2}.

Assume $M = \text{fst Pair } V$ and $M' = V$. Then for any role $R \in \text{roles}(\text{type}(\text{Pair } M' V))$, $[M]_{R} = \text{fst Pair } [M']_{R}$ and for any other role $R' \notin \text{roles}(\text{type}(\text{Pair } M' V))$, we have $[M]_{R'} = \bot$ and $[M']_{R} = \bot$. We define $N' = N'' = [M']_{R'}$. This result follows.

Assume $M = \text{snd Pair } V$ and $M' = V$. Then the case is similar to the previous.
A.2 Proof of Theorem 25

Definition 37. Given a network $N = \prod_{R \in \rho'} R[B_R]$, we have $N' = \prod_{R \in \rho} R[B_R]$.

Lemma 38. For any role $R$ and network $N$, if $N \xrightarrow{\tau} N'$ and $R \notin R'$ then $N'(R) = N^\prime(R)$.

Proof. Straightforward from the network semantics.

Lemma 39. For any set of roles $R$ and network $N$, if $N \xrightarrow{\tau} N'$ and $R \cap R' = \emptyset$ then $N \setminus R \xrightarrow{\tau} N' \setminus R$.

Proof. Straightforward from the network semantics.

Lemma 40. If $[M]_{R} \xrightarrow{\tau} B$ then there exists $M'$ such that $M \xrightarrow{\tau} M'$ and $B \equiv [M']_{R}$.

Proof. Follows from case analysis on $[M]_{R} \xrightarrow{\tau} B$ keeping in mind that $[M]_{R}$ cannot be $\bot$.

Proof. If $[M] \xrightarrow{\tau} [M]_{R}$ uses rule $\text{NSTr}$ then this follows from Lemma 40. Otherwise we prove this by structural induction on $M$.

- Assume $M = V$. Then for any role $R$, $[M]_{R} = L$, and therefore $[M] \xrightarrow{\tau} [M]_{R}$.
- Assume $M = N_1 \cdot N_2$. Then for any role $R \in \text{roles}(\text{type}(N_1)) \cup (\text{roles}(N_1) \cap \text{roles}(N_2))$, $[M]_{R} = [N_1]_{R} \cdot [N_2]_{R}$, for any role $R'$ such that $[N_1]_{R'} = [N_2]_{R'} = \bot$, we have $[M]_{R'} = \bot$. For any other role $R''$ such that $[N_1]_{R''} = \bot$, $[M]_{R''} = [N_2]_{R''}$. For any other role $R'''$ such that $[N_2]_{R'''} = \bot$, we get $[M]_{R'''} = [N_1]_{R'''}$. We then have 2 cases.
  - Assume $N_2 = V$. Then $[N_2]_{R} = L$ by Lemma 29, and for any $R'$ such that $R' \notin \text{roles}(\text{type}(N_2)) \subseteq \text{roles}(\text{type}(N_1))$, by Lemma 31 $[N_2]_{R'} = \bot$ and therefore $[M]_{R'} = [N_1]_{R'}$, and we have 5 cases.
    - Assume $N_1 = \lambda x : T.N_3$. Then for any role $R \in \text{roles}(\text{type}(N_1))$, $[N_1]_{R} = \lambda x : [T]_{R} \cdot [N_3]_{R}$. And for any role $R' \notin \text{roles}(\text{type}(N_1))$, $[N_1]_{R'} = \bot$. We have two cases, using either rule $\text{NApp}$ or rules $\text{NSApp}$ and $\text{NApp}$.
      If we use rule $\text{NApp}$, then there exists $R''$ such that $R = R''$ and $R' \notin \text{roles}(\text{type}(N_1))$. Then we get $N = [M] \setminus \{R''\} \cdot R''[N_3]_{R''}[x := [N_2]_{R''}]$. We say that $M' = N_3[x := N_2]$ and the result follows from using rule $\text{NApp}$ in every role in $\text{roles}(\text{type}(N_1))$.
      If we use rules $\text{NSApp}$ and $\text{NApp}$ then there exists $R''$ such that $R = R''$ and $[N_3]_{R''}$ is $\bot$ and $B$ and $N = [M] \setminus \{R''\} \cdot R''[\lambda x.B]_{[N_2]_{R''}}$. By induction, $N_3 \xrightarrow{\tau} N'_3$ and $([N_3] \setminus \{R''\} \cdot R''[\lambda x.B] \xrightarrow{\tau} N''$ such that $[N_3] \supseteq N''$, and we define $M' \lambda x : T.N_3N_2$ and $\lambda x : T.N_3N_2$. Then if $S \neq R$, $[M]_{S} = \text{send}_{R} [N_2]_{S}$, $[M]_{R} = \text{recv}_{R} \bot$, and for $R' \notin \{S, R\}$, $[N_1]_{R'} = \bot = [M]_{R'}$, and therefore $R = S, R$, and if $S = R$ then $[N_1]_{R} = \lambda x.x$. Then the result follows from using rule $\text{NSApp}$ in every role in $\text{roles}(\text{type}(N_3))$. The result follows by structural induction on $M$. 

If \( R = S, R \) then \( N = [M] \setminus \{ S, R \} | S \setminus R | R[[N_2]_S | S := R] \). Because \([N_2]_R = \perp\) and \([N_2]_S = V, N_2 = V\). Therefore \( M \xrightarrow{R} D \) \( V | S := R \) and the result follows.

If \( R = R \) then \( S = R, N = [M] \setminus \{ R \} | R[[N_2]_R \) and the rest is similar to above.

* Assume \( N_1 = \text{fst} \). Then \( N_2 = \text{Pair} \ V \ V' \) and for any role \( R \in \text{roles}([\text{fst} \ V \ V'])\), \([M]_R = \text{fst} \text{Pair} \ [V]_R \ [V']_R \) and for any other role \( R' \notin \text{roles}([\text{Pair} \ V \ V'])\), by Lemma \([\text{PAIR}]\) we have \([M]_R = [N_2]_R = \perp\) and therefore \([M]_{R'} \not\rightarrow\).

If \( R = R \in \text{roles}([\text{Pair} \ V \ V'])\) then \( N = [M] \setminus \{ R \} | R[[V]_R] \) and \( M \xrightarrow{R} D \). The result follows by use of rule \([\text{PAIR}]\) and Lemma \([\text{PAIR}]\).

* Assume \( N_1 = \text{snd} \). This case is similar to the previous.

* Otherwise, \( N_1 \neq V \) and either \( R = R \) or \( R = R, S \).

If \( R = R \) then either \([N_1]_R \xrightarrow{\text{snd}} B \) and \( R \in \text{roles}([\text{snd}])\), \( N = [M] \setminus \{ R \} | R[B] \) and by induction, \( N_1 \rightarrow D_0 \ N_1' \) such that \([N_1]_R \setminus \{ R \} | R[B] \rightarrow D \ N_1' \supseteq [N'_1]_R \). Since all these transitions can be propagated past \( N_2 \) in the network \([N_2]_R \) in any role \( R' \) involved, we get the result for \( M' = N'_1 \).

If \( R = R, S \) then the case is similar.

- If \( N_2 \neq V \) then we have 2 cases.
  * If \( R = R \) then either \([N_1]_R \xrightarrow{\text{snd}} B \) or \([N_2]_R \xrightarrow{\text{snd}} B \) and the case is similar to the previous.
  * If \( R = R \) then there exists \( L \) such that either \([N_1]_R \xrightarrow{\text{send}_R} B_S \) or \([N_2]_R \xrightarrow{\text{send}_S} B_S \).

If \( [N_1]_R \xrightarrow{\text{send}_R} B_S \) then \([N_1]_R \neq L' \) and therefore \([N_1]_R \rightarrow D \ N_1' \). And for any other role \( \text{roles}([\text{snd}])\), \([M]_{R'} = (\lambda x. [N'_1]_{R'} \cup [N''_1]_{R'}) \ [N']_{R'} \). We have three cases.

* Assume \( N = \text{inl} \ V \). Then \([N]_R = \text{inl} \ [V]_R \) and \( N' = [M] \setminus \{ R \} | R[[N']_R | R[[V]_R] \). We define \( M' = N' \) and since \([N']_R \supseteq [N']_{R'} \cup [N'']_{R'} \) the result follows from using rules \([\text{INL}])\) and \([\text{NCASE})\).

* Assume \( N = \text{inr} \ V \). Then the case is similar to the previous.

* Otherwise, we use either rule \([\text{NCASE})\) or rule \([\text{NCASE2})\). If we use rule \([\text{NCASE})\) we have a transition \([N]_{R'} \rightarrow B \) such that

\[
N = [M] \setminus \{ R \} | R\text{case} \ B \text{ of inl} \ V \Rightarrow [N']_{R'} | \text{inr} \ x' \Rightarrow [N'']_{R'}
\]

and the result follows from induction similar to the last application case.

If we use rule \([\text{NCASE2})\) then \([N']_R \rightarrow_D B \) and \([N''_R] \rightarrow_D B \). If \([N']_R \rightarrow_D B \) then by induction, \( N' \rightarrow_D N'' \) and \([N']_R \setminus \{ R \} | R[B] \rightarrow_D N'' \) such that \( N'' \supseteq [N'']_{R'} \) and \( N'' \rightarrow_D N'' \) and \([N'']_R \setminus \{ R \} | R[B] \rightarrow_D N'' \) such that \( N'' \supseteq [N'']_R \). Since \( N' \) and \( N'' \) are mergeable on other roles, the result follows from using rule \([\text{INCASE})\).

* Assume \( R = R \notin \text{roles}([\text{snd}]) \). Then we have three cases.

  * Assume \( N = \text{inl} \ V \). Then \([N]_R = \perp \) and \( N = [M] \setminus \{ R \} | R[[N']_R \cup [N'']_R]. \) We define \( M' = N' \) and the result follows.

  * Assume \( N = \text{inr} \ V \). Then the case is similar to the previous.
Otherwise, $[N]_R \neq L$ and we therefore have $[N]_R \xrightarrow{\tau} B$ and $\mathcal{N} = [M] \setminus \{R\} \mid R[[\lambda x.[N']_R \sqcup [N'']_R]_R] \backslash \{R\} \mid R[B]$, and by induction, $N \rightarrow_D N''$ such that $[N] \setminus \{R\} \mid R[B] \rightarrow^* N''$ for $N'' \supseteq [N''']$. Since all these transitions can be propagated past $N_2$ in the network and the conditional or $(\lambda x.[N']_R \sqcup [N'']_R)$ in any other role $R'$ involved, we get the result for $M' = \text{case} \ N''$ of $\text{lnl} \ x \Rightarrow N'; \text{lnr} \ x' \Rightarrow N''$.

Assume $R = S, R$. Then the logic is similar to the third subcases of the previous two cases.

Assume $M = \text{select}_{S, R} \ell N$. This is similar to the $N_1 = \text{com}_{S, R}$ case above.

Assume $M = f(R_1, \ldots, R_n)$. Then $[M] = \prod_{i=1}^n R_i[f_i(R_1, \ldots, R_{i-1}, R_{i+1}, \ldots, R_n)] \mid \prod_{R \notin \langle R_1, \ldots, R_n \rangle} R[\perp]$. We therefore have some role $R$ such that $R = R$ and $\mathcal{N} = ([M] \setminus R_i) \mid R_i[[D](f_i(\bar{R}))\{\bar{R}'/(R_1, \ldots, R_{i-1}, R_{i+1}, \ldots, R_n)\}]$. We then define $M' = D(f(R_1, \ldots, R_n))\{\langle R'_{i-1}, R_{i+1}, \ldots, R_n \rangle/(R_1, \ldots, R_n)\}$ and

$$\mathcal{N}' = [M'] = \prod_{i=1}^n R_i[[D](f_i(\bar{R}))\{\bar{R}'/(R_1, \ldots, R_{i-1}, R_{i+1}, \ldots, R_n)\}] \mid \prod_{R \notin \langle R_1, \ldots, R_n \rangle} R[\perp]$$

and the result follows. ▼
Figure 10 Semantics of networks.
\[
\begin{align*}
\Gamma, x : T & \vdash B : T & \text{(NTVar)} \\
\Sigma ; \Gamma & \vdash B : T & \text{(NTAbs)} \\
\Sigma, \Gamma & \vdash \lambda x : T. B : T' & \text{(NTAbs)} \\
\Sigma & \vdash f : T & \text{(NTDef)} \\
\forall f \in \text{domain}(D) & \quad f : T \in \Gamma & \Sigma, \Gamma & \vdash D(f) : T & \text{(NTDefS)}
\end{align*}
\]

\[\Sigma; \Gamma \vdash B : T \text{ for } 1 \leq i \leq n\]

\[\Sigma; \Gamma \vdash \& R \{ \ell_1 : B_1, \ldots, \ell_n : B_n \} : T\]

\[\Sigma; \Gamma \vdash \text{send}_R : T \rightarrow \perp \text{ (NTSend)}\]

\[\Sigma; \Gamma \vdash \text{recv}_R : \perp \rightarrow T \text{ (NTRcv)}\]

\[\Sigma; \Gamma \vdash \text{case } B \text{ of Inl } x \Rightarrow B' ; \text{ Inr } x' \Rightarrow B'' : T \text{ (NTCase)}\]

\[\Sigma; \Gamma \vdash f : T \in \Gamma \text{ (NTDef)}\]

\[\Sigma; \Gamma \vdash (\text{Pair} : T \rightarrow \perp \rightarrow_\partial (T \times T')) \text{ (NTPair)}\]

\[\Sigma; \Gamma \vdash \text{fst} : (T \times T') \rightarrow_\partial T \text{ (NTProj1)}\]

\[\Sigma; \Gamma \vdash \text{snd} : (T \times T') \rightarrow_\partial T' \text{ (NTProj2)}\]

\[\Sigma; \Gamma \vdash B : T' \quad \{T = T', T' = T\} \cap \Sigma \neq \emptyset \quad \Sigma ; \Gamma \vdash B : T \text{ (NTEq)}\]

\[\Sigma; \Gamma \vdash (\bigoplus R \ell B) : T \text{ (NTChor)}\]

\[\Sigma; \Gamma \vdash \&_R \{\ell_1 : B_1, \ldots, \ell_n : B_n\} : T \text{ (NTOff)}\]

\[\Sigma; \Gamma \vdash \text{recv}_R : \perp \rightarrow T \text{ (NTRcv)}\]

\[\Sigma; \Gamma \vdash \text{send}_R : T \rightarrow \perp \text{ (NTSend)}\]

\[\Sigma; \Gamma \vdash \text{recv}_R : \perp \rightarrow T \text{ (NTRcv)}\]

\[\Sigma; \Gamma \vdash \text{case } B \text{ of Inl } x \Rightarrow B' ; \text{ Inr } x' \Rightarrow B'' : T \text{ (NTCase)}\]

\[\Sigma; \Gamma \vdash f : T \in \Gamma \text{ (NTDef)}\]

\[\Sigma; \Gamma \vdash (\text{Pair} : T \rightarrow \perp \rightarrow_\partial (T \times T')) \text{ (NTPair)}\]

\[\Sigma; \Gamma \vdash \text{fst} : (T \times T') \rightarrow_\partial T \text{ (NTProj1)}\]

\[\Sigma; \Gamma \vdash \text{snd} : (T \times T') \rightarrow_\partial T' \text{ (NTProj2)}\]

\[\Sigma; \Gamma \vdash B : T' \quad \{T = T', T' = T\} \cap \Sigma \neq \emptyset \quad \Sigma ; \Gamma \vdash B : T \text{ (NTEq)}\]

\[\forall f \in \text{domain}(D) \quad f : T \in \Gamma \quad \Sigma, \Gamma \vdash D(f) : T \text{ (NTDefS)}\]

\textbf{Figure 11} Typing rules for simple processes.
$\llbracket M N \rrbracket_R = \begin{cases} \llbracket M \rrbracket_R \land \llbracket N \rrbracket_R & \text{if } R \in \text{roles}(\text{type}(M)) \text{ or } R \in \text{roles}(M) \cap \text{roles}(N) \\ \perp & \text{if } \llbracket M \rrbracket_R = \llbracket N \rrbracket_R = \perp \\ \llbracket M \rrbracket_R & \text{if } \llbracket N \rrbracket_R = \perp \\ \llbracket N \rrbracket_R & \text{otherwise} \end{cases}$

$\llbracket \lambda x : T . M \rrbracket_R = \begin{cases} \lambda x . \llbracket M \rrbracket_R & \text{if } R \in \text{roles}(\text{type}(x : T . M)) \\ \perp & \text{otherwise} \end{cases}$

$\llbracket \text{case } M \text{ of } \text{inl } x \Rightarrow N ; \text{inr } x' \Rightarrow N' \rrbracket_R = \begin{cases} \text{case } \llbracket M \rrbracket_R \text{ of } \text{inl } x \Rightarrow \llbracket N \rrbracket_R ; \text{inr } x' \Rightarrow \llbracket N' \rrbracket_R & \text{if } R \in \text{roles}(\text{type}(M)) \\ \perp & \text{if } \llbracket M \rrbracket_R = \llbracket N \rrbracket_R = \llbracket N' \rrbracket_R = \perp \\ \llbracket M \rrbracket_R \cup \llbracket N' \rrbracket_R & \text{if } \llbracket N \rrbracket_R = \perp \\ \llbracket M \rrbracket_R & \text{otherwise} \\ (\lambda x'' : \perp . \llbracket N \rrbracket_R \cup \llbracket N' \rrbracket_R) \Rightarrow \llbracket M \rrbracket_R & \text{for some } x'' \notin \text{fv}(N) \cup \text{fv}(N') \text{ otherwise} \end{cases}$

$\llbracket \text{select}_{S,S'} \ell M \rrbracket_R = \begin{cases} \oplus_{S'} \ell \llbracket M \rrbracket_R & \text{if } R = S \neq S' \\ \&_{S} \ell \llbracket M \rrbracket_R & \text{if } R = S' \neq S \\ \llbracket M \rrbracket_R & \text{otherwise} \end{cases}$

$\llbracket \text{com}_{S,S'} \rrbracket_R = \begin{cases} \lambda x . x & \text{if } R = S = S' \\ \text{send}_{S'} & \text{if } R = S \neq S' \\ \text{recv}_{S} & \text{if } R = S' \neq S \\ \perp & \text{otherwise} \end{cases}$

$\llbracket () @ S \rrbracket_R = (\ ) \text{ if } S = R \\ \perp \text{ otherwise} 
\llbracket x \rrbracket_R = \begin{cases} x & \text{if } R \in \text{roles}(\text{type}(x)) \\ \perp & \text{otherwise} \end{cases}$

$\llbracket f(\vec{R}) \rrbracket_R = \begin{cases} f_i(\langle R_1, \ldots, R_{i-1}, R_{i+1}, \ldots, R_n \rangle) & \text{if } \vec{R} = \langle R_1, \ldots, r_{i-1}, R, R_{i+1}, \ldots, R_n \rangle \\ \perp & \text{otherwise} \end{cases}$

$\llbracket \text{Pair } V V' \rrbracket_R = \begin{cases} \text{Pair } \llbracket V \rrbracket_R \llbracket V' \rrbracket_R & \text{if } R \in \text{roles}(\text{type}(V) \times \text{type}(V')) \\ \perp & \text{otherwise} \end{cases}$

$\llbracket \text{fst} \rrbracket_R = \begin{cases} \text{fst} & \text{if } R \in \text{roles}(\text{type}(\text{fst})) \\ \perp & \text{otherwise} \end{cases}$

$\llbracket \text{snd} \rrbracket_R = \begin{cases} \text{snd} & \text{if } R \in \text{roles}(\text{type}(\text{snd})) \\ \perp & \text{otherwise} \end{cases}$

$\llbracket \text{inl } V \rrbracket_R = \begin{cases} \text{inl } \llbracket V \rrbracket_R & \text{if } R \in \text{roles}(\text{type}(\text{inl } V)) \\ \perp & \text{otherwise} \end{cases}$

$\llbracket \text{inr } V \rrbracket_R = \begin{cases} \text{inr } \llbracket V \rrbracket_R & \text{if } r \in \text{roles}(\text{type}(\text{inr } V)) \\ \perp & \text{otherwise} \end{cases}$

Types:

$\llbracket T \rightarrow_{\rho} T' \rrbracket_R = \begin{cases} \llbracket T \rrbracket_R \rightarrow \llbracket T' \rrbracket_R & \text{if } R \in \rho \cup \text{roles}(T) \cup \text{roles}(T') \\ \perp & \text{otherwise} \end{cases}$

$\llbracket T \times T' \rrbracket_R = \begin{cases} \llbracket T \rrbracket_R \times \llbracket T' \rrbracket_R & \text{if } R \in \text{roles}(T \times T') \\ \perp & \text{otherwise} \end{cases}$

$\llbracket T + T' \rrbracket_R = \begin{cases} \llbracket T \rrbracket_R + \llbracket T' \rrbracket_R & \text{if } R \in \text{roles}(T + T') \\ \perp & \text{otherwise} \end{cases}$

$\llbracket t @ \vec{R} \rrbracket_R = \begin{cases} t_i & \text{if } \vec{R} = \langle R_1, \ldots, R_{i-1}, R, R_{i+1}, \ldots, R_n \rangle \\ \perp & \text{otherwise} \end{cases}$