THEORETICAL MODELS OF YOUNG OPEN STAR CLUSTERS: EFFECTS OF A GASEOUS COMPONENT AND GAS REMOVAL

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ABSTRACT

We construct a family of semianalytic models for young open clusters, including a gaseous component and varying assumptions about the distribution function for the stellar component. The parameters of these models are informed by observed open clusters and general theoretical considerations regarding cluster formation. We use this framework to estimate the fraction \( f_\text{ex} \) of the stellar component that remains gravitationally bound after the gaseous component disperses. The remaining stellar fraction \( f_\text{ex} \) depends on the distribution function of the stars. We calculate the function \( f_\text{ex}(\epsilon_\text{s}) \) for this class of open cluster models and provide fitting formulae for representative cases.

Subject headings: celestial mechanics, stellar dynamics — open clusters and associations: general — stars: formation

1. INTRODUCTION

In this paper, we study the equilibrium structure and early evolution of young open star clusters. Our long-term goal is to develop a unified treatment that includes cluster formation, removal of the gaseous component, and the longer term evolution of the system. A unified picture of young cluster evolution can be useful in several different contexts. On one hand, we can consider an open cluster as an astrophysical entity and study its birth, evolution, and ultimate demise. On the other hand, we can study the effects of the cluster setting on the star formation process. In this present work, we construct models for young clusters in which a gaseous component is still present. We then study how the clusters react to gas removal. Our results are applicable to robust clusters that live for relatively long times (~100 Myr); perhaps 10% of all star formation takes place in such robust cluster environments (see our companion paper: Adams & Myers 2000).

A collection of previous papers have addressed the question of whether or not a cluster can remain gravitationally bound after the removal of its gaseous component (e.g., Hills 1980; Mathieu 1983; Elmegreen 1983; Lada, Mar- gulis, & Dearborn 1983). Most previous work uses an approach based on the virial theorem, however, and does not explicitly include the distribution function for the stars. In the case of rapid gas removal and a cluster that begins in a state of exact virial equilibrium, the cluster expands by a factor \( f_\text{ex} \) after the gas is removed. In the simplest models, this factor is given by

\[
 f_\text{ex} = \frac{\epsilon_\text{s}}{2\epsilon_\text{s}} - 1 ,
\]

where \( \epsilon_\text{s} \) is the mass fraction of the stellar component in the original cluster. In this approximation, the cluster remains bound for \( \epsilon_\text{s} > \frac{1}{2} \); as \( \epsilon_\text{s} \to \frac{1}{2} \), \( f_\text{ex} \to \infty \) and the cluster becomes unbound. This approach has been generalized to include varying assumptions about the speed of gas removal and departures from exact virial equilibrium, e.g., stellar speeds that are less than the virial speeds in the cluster potential (see, e.g., Verschueren 1990; Verschueren & David 1989; Elmegreen & Clemens 1985; Lada et al. 1984). In this present work, we take into account the distribution function for the stars and, separately, the density distribution of the gaseous component. These complications lead to a much wider range of possible behavior. Because the cluster stars have a velocity distribution, the low-speed stars in the tail of the distribution survive as a gravitationally bound entity even if \( \epsilon_\text{s} < \frac{1}{2} \); the high-speed stars in the opposite tail escape even if \( \epsilon_\text{s} > \frac{1}{2} \). For a given distribution function and a given star formation efficiency \( \epsilon_\text{s} \), a fraction \( f_\text{ex}(\epsilon_\text{s}) \) of the stars will thus remain bound after gas removal. The function \( f_\text{ex}(\epsilon_\text{s}) \) varies smoothly with star formation efficiency rather than exhibiting singular behavior.

This paper is organized as follows. In § 2, we construct equilibrium models of young open clusters including standard forms for the stellar distribution function, a gaseous component, and anisotropy parameters. We study the effects of gas removal in § 3; in particular, we calculate the fraction \( f_\text{ex} \) of stars remaining after gas leaves the system. We compare our results with observed clusters in § 4 and then conclude, in § 5, with a summary and brief discussion of our results. In the Appendix, we also present a crude cluster formation theory, which informs the theoretical models in the main text. Starting with the existing theory for the collapse of molecular cloud cores, we scale up the solutions to describe the collapse flow that forms a star cluster. Using this formalism, we estimate the anisotropy of the velocity distribution for forming clusters.

2. EQUILIBRIA OF STAR CLUSTERS CONTAINING GAS

In this section, we construct a class of cluster models that incorporate both stars and gas. In this standard approach, the structure of clusters is determined by two differential equations. The first is the Poisson equation for the gravitational potential \( \Phi \),

\[
 \nabla^2 \Phi = 4\pi G \rho_\text{tot} ,
\]

where the total density \( \rho_\text{tot} \) includes both stars and gas. The second is the collisionless Boltzmann equation, which takes the form

\[
 \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \nabla \Phi \frac{\partial f}{\partial v} = 0 ,
\]

where \( f = f(x, v, t) \) is the distribution function for the stars.
2.1. Models with Isotropic Velocity Dispersion

As a starting point, we construct equilibrium models with an isotropic velocity dispersion. Making straightforward extensions of the general treatment outlined in Binney & Tremaine (1987), we include a gaseous halo in the formalism. In particular, we use lowered isothermal cluster models by assuming a stellar distribution function \( f(\varepsilon) \) of the form

\[
 f(\varepsilon) = \rho_1 (2\pi\sigma^2)^{-3/2} e^{\varepsilon^2} \big[ e^{\varepsilon^2} - 1 \big],
\]

where the relative energy \( \varepsilon \) is defined by \( \varepsilon = \Psi - \mathcal{V}/2 \), and where the relative potential \( \Psi \) is true to the gravitational potential \( \Phi \) through \( \Psi = -\mathcal{V} + \Phi_0 \). The constant \( \rho_1 \) sets the density scale and the parameter \( \sigma \) sets the scale for the velocity dispersion (notice that \( \sigma \neq \langle v^2 \rangle^{1/2} \)). By assuming a distribution function which is a function of the energy only, we automatically obtain a solution to the collisionless Boltzmann equation (3) for equilibrium configurations with \( \partial f/\partial t = 0 \). We have also implicitly assumed an isotropic distribution in velocity space (we consider the more general case of anisotropic models in a subsequent section).

From the distribution function \( f(\varepsilon) \), we can determine the stellar density \( \rho_\ast \) in terms of the potential. Using the resulting density in the Poisson equation, in conjunction with the additional terms resulting from gas, we can find the potential as a function of radius in the cluster and then find the stellar density as a function of radius. For convenience, we define new variables,

\[
 \psi = \frac{\Psi}{\sigma^2}, \quad \xi = \frac{r}{r_0} \quad r_0 = \left[ \frac{9g_o^2}{4\pi G \rho_0} \right]^{1/2},
\]

where \( r_0 \) is the central density of the stellar component of the cluster. The variable \( r_0 \) is thus the King radius and plays the role of an effective core radius for the stellar component (see Binney & Tremaine 1987).

Using the newly defined variables, we can evaluate the density of stars \( \rho_\ast(r) \) in terms of the reduced potential \( \psi(r) \), i.e.,

\[
 \rho_\ast = \int f d^3v = \rho_1 \left[ e^{\psi} \operatorname{erf} \left( \sqrt{\psi} \right) - \sqrt{\frac{\psi}{\pi}} \left( 1 + \frac{2}{3} \sqrt{\psi} \right) \right],
\]

where \( \operatorname{erf}(z) \) is the error function (e.g., Abramowitz & Stegun 1972). The central density can thus be written

\[
 \rho_0 = \rho_1 \left[ e^{\psi_0} \operatorname{erf} \left( \sqrt{\psi_0} \right) - \sqrt{\frac{\psi_0}{\pi}} \left( 1 + \frac{2}{3} \sqrt{\psi_0} \right) \right],
\]

where \( \psi_0 = \psi(0) = \Psi(0)/\sigma^2 \) determines the depth of the gravitational potential well at the cluster center. We find it convenient to define a function

\[
 g_\ast(x) \equiv e^x \operatorname{erf} \left( \sqrt{x} \right) - \sqrt{\frac{x}{\pi}} \left( 1 + \frac{2}{3} \sqrt{x} \right),
\]

so that \( \rho_\ast = \rho_1 g_\ast(\psi) = \rho_0 g_\ast(\psi_0)/g_\ast(\psi_0) \).

To specify the gas contribution, we use a density distribution much like an isothermal sphere; in particular, we consider a gaseous halo around the cluster with a density profile of the form

\[
 \rho_{gas} = \frac{a^2}{2\pi G (r^2 + r_o^2)}
\]

where \( a \) is the effective sound speed of the gas and where \( r_{0g} \) is the core radius for the gas. For simplicity, we assume that the gas and the stars have the same core radius \( r_0 \); in principle, we could include an additional parameter \( a_{gas} = r_{0g}/r_0 \) in the analysis.

The new version of the Poisson equation (which determines the cluster structure) takes the form

\[
 \xi^2 \frac{d}{d\xi} \left( \xi^2 \frac{d\psi}{d\xi} \right) = -\frac{9}{g_\ast(\psi_0)} \left[ e^\psi \operatorname{erf} \left( \sqrt{\psi} - 2 \sqrt{\frac{\psi}{\pi}} \left( 1 + \frac{2}{3} \sqrt{\psi} \right) \right) - \frac{\Lambda}{1 + \xi^2} \right],
\]

where the constant \( \Lambda \) determines the contribution of the gas relative to that of the stars and is defined by

\[
 \Lambda = \frac{2a^2}{\sigma^2}.
\]

We thus have a two-parameter family of cluster models. To specify each model, we need to choose the value of \( \psi_0 \) to set the depth of the gravitational potential well and the value of \( \Lambda \) to set the gas contribution.

As we integrate equation (10) outward, both the potential \( \psi \) and the stellar density \( \rho_\ast \) decrease and eventually become zero at some outer truncation radius \( r_T \). Notice that the stellar density approaches the form \( \rho_\ast(\psi) \sim (8/15 \sqrt{\pi}) \psi^{3/2} \) in the limit \( \psi \to 0 \). At this outer boundary, the true gravitational potential \( \Phi \) is given by \( \Phi(r_T) = -GM(r_T)/r_T \), where \( M(r_T) \) is the total mass enclosed, including both stars and gas. The value of the true potential at the origin is then \( \Phi(0) = \Phi(r_T) - \Psi(0) \), where \( \Psi(0) = \psi_0 \sigma^2 \). As the input parameter \( \psi_0 \) grows larger, the outer truncation radius \( r_T \) grows accordingly; the mass \( M(r_T) \) also increases, and so does the overall depth of the potential well as determined by \( \lbrack \Phi(0) \rbrack \).

A typical density profile is shown in Figure 1a (for a model using \( \psi_0 = 10 \) and \( \Lambda = 1/2 \)). For this case, the mass in stars and the mass in gas are roughly comparable (specifically, 44% stars and 56% gas). The dashed curve shows the stellar component (eq. [6]) and the dotted curve shows the gaseous component (eq. [9]). The total density is given by the solid curve. After the gas is removed, the remaining stellar component contains \( \sim 73\% \) of the starting inventory, with its resulting density distribution depicted by the dot-dashed curve (the formulation of this gas removal calculation is presented in § 3). For this class of models, the density in the cluster center is dominated by stars and the outside is dominated by gas; the star formation efficiency thus increases toward the inside. This structure is roughly consistent with that expected from a collapse model of cluster formation as outlined in the Appendix. Even though the density is strongly peaked toward the center, most of the mass in the cluster—in both stars and gas—resides in the outer portion of the cluster. Figure 1b shows a collection of density profiles with varying depths of the gravitational potential, determined by \( \psi_0 \) which lies in the range \( 6 \leq \psi_0 \leq 14 \). Similarly, Figure 1c shows a collection of density profiles for \( \psi_0 \) = 10 and varying amounts of gas, determined by \( \Lambda \) which lies in the range \( 0 \leq \Lambda \leq 2 \).

In this present construction, the stars live in a potential well determined by both their own self-gravity and by the gravity of the accompanying gas. The stars have a distribution function which was chosen to be a function of an integral of motion (the energy). Through the Jeans theorem, the distribution function is thus a valid solution to the collisionless Boltzmann equation which determines equilibrium con-
Equilibrium density profiles for clusters with isotropic velocity distributions. The density is scaled to the central density $\rho_0$ of the stellar component and the radius is given in units of the corresponding King radius, i.e., $\xi = r/r_0$. (a) A typical density profile with $\psi_0 = 10$ and $\Lambda = 0$ (44% stars and 56% gas). The dashed curve shows the stellar component, the dotted curve shows the gaseous component, and the solid curve shows the total density. The dot-dashed curve shows the stellar component remaining after gas is removed from the system. (b) Collection of density profiles with varying depths of the gravitational potential and fixed gaseous halo with $\Lambda = 0$. The various curves use $\psi_0 = 6, 8, 10, 12,$ and 14. As the depth $\psi_0$ increases, the outer radius of the stellar component of the cluster grows accordingly (see also Table 1). (c) Collection of density profiles for $\psi_0 = 10$ and varying amounts of gas, determined by $\Lambda = 0, 0.5, 1.0, 1.5,$ and 2.0. As $\Lambda$ increases, the outer radius of the cluster decreases.

2.2. Models with Anisotropic Velocity Dispersion

In this section, we consider the more realistic case of anisotropic models. For an anisotropic velocity dispersion tensor, the distribution function $f$ is a function of both the...
energy $\varepsilon$ and the angular momentum $L$. To obtain a natural
extension of the formulation presented in the previous section, we use a distribution function of the form

$$f(\varepsilon, L) = \rho_1(2\pi\sigma^2)^{-3/2}g\left(\frac{L^2}{r_A^2\sigma^2}\right)e^{\varepsilon/\sigma^2} - 1,$$  \hspace{1cm} (12)

where the function $g$ and the anisotropy radius $r_A$ determine the degree of departure from isotropy. In the limit $r_A \to \infty$, $g \to 1$ and we recover the isotropic models of the previous section. In our cluster formation scenario (see the Appendix), the anisotropy radius $r_A$ is determined by the centrifugal radius $R_C$ of the collapse flow so that $r_A \approx R_C$ (where $R_C$ is evaluated at the end of the cluster formation epoch).

With a distribution function of the form (eq. [12]), the stellar density can be written

$$\rho_\ast = \rho_1\sqrt{\frac{2}{\pi}}\int_0^{\infty} \sin \eta \, d\eta \int_0^{\sqrt{2}\eta} v^2 \, dv \times g\left(\frac{r^2 v^2 \sin^2 \eta}{r_A^2}\right)e^{\eta/\sigma^2} - 1,$$  \hspace{1cm} (13)

where we have scaled out the $\sigma$ dependence. Motivated by our model of cluster formation from a collapse flow, we use an anisotropy function $g(L^2)$ that decreases as one power of $r$ (and, hence, $L$) for large radii. We thus choose the particular form

$$g(L^2) = \frac{1}{(1 + L^2/r_A^2)^{1/2}},$$

$$= \frac{1}{(1 + r^2 v^2 \sin^2 \eta/r_A^2)} \quad \text{and} \quad \frac{1}{(1 + \alpha^2 v^2 \sin^2 \eta)^{1/2}}, \hspace{1cm} (14)$$

where we have defined an anisotropy parameter $\alpha \equiv r/r_A$.

With this choice for $g(L^2)$, the angular integral becomes

$$\int_0^{\infty} \sin \eta \, d\eta \, \left(\frac{1}{1 + \alpha^2 v^2 \sin^2 \eta}\right)^{1/2} \equiv \frac{1}{\alpha v} \tan(\alpha v). \hspace{1cm} (15)$$

The stellar density takes the form

$$\rho_\ast = \rho_1\sqrt{\frac{2}{\pi}} \int_0^{\infty} v \, dv \tan(\alpha v)(\alpha^2 - 1) \equiv \rho_1\sqrt{\frac{2}{\pi}} \int_0^{\infty} I_\rho,$$  \hspace{1cm} (16)

where we have defined the integral $I_\rho$ in the final equality. The integral $I_\rho$ can be partially evaluated to obtain

$$I_\rho = \frac{\sqrt{2\psi}}{2\pi} - \left(\frac{\psi}{2\pi}\right) \left(1 + \psi + \frac{1}{2\pi^2}\right) + \alpha^2K_\rho,$$  \hspace{1cm} (17)

where the remaining integral $K_\rho$ is given by

$$K_\rho(\psi, \alpha) \equiv \int_0^{\sqrt{2}\psi} e^{-\psi/2} \frac{d\psi}{1 + \alpha^2 v^2}.$$  \hspace{1cm} (18)

Although the integral $K_\rho$ cannot be evaluated in terms of elementary functions, we can obtain a good approximation using asymptotic analysis (Bleistein & Handelsman 1986).

In the limit of small $\alpha$, we invoke Laplace’s method and keep only the leading order term to find the following analytic expression:

$$K_\rho \approx \frac{\sqrt{\pi}}{2(1 + 2\pi^2)^{-1/2}} \text{erf}\left[\sqrt{\psi}(1 + 2\pi^2)\right]. \hspace{1cm} (19)$$

This approximation becomes exact in the limit of small $\alpha$ (small departures from isotropy). In the opposite limit of large $\alpha$,

$$K_\rho \approx \frac{\alpha}{\pi} \sqrt{\frac{2\psi}{\alpha}} \tan\left(\frac{\sqrt{2}\psi}{\alpha}\right) \tan\left(\frac{\sqrt{2}\psi}{\alpha}\right). \hspace{1cm} (20)$$

The stellar density contribution is thus given by the combination of equations (16)–(20). Since we have made an approximation in evaluating the density (eqs. [19] and [20]), the distribution function corresponding to this density profile is not exactly of the form given by equation (12), although it is close and it has the correct behavior in the limiting regimes.

The form of the Poisson equation, which determines the potential, now takes the form

$$\frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{d\xi}{dr}\right) = -\frac{9h_\ast(\psi)}{h_\ast(\psi)} - \frac{\Lambda}{1 + \xi^2}, \hspace{1cm} (21)$$

where the function $h_\ast(x)$ is defined by

$$h_\ast(x) = e^{x}K_\rho(x, \alpha) + \sqrt{\frac{2\psi}{2\pi^2}} \frac{\psi}{2\pi^2} - \frac{1}{\alpha} \text{atan}(\alpha\sqrt{2x}) \times \left(1 + \frac{1}{\alpha}\right). \hspace{1cm} (22)$$

The parameter $\alpha = r/r_A = \xi r_0/r_A$, so we define an overall anisotropy parameter $\beta \equiv r_0/r_A$. Our scenario for cluster formation indicates that both the anisotropy radius $r_A$ and the cluster core radius $r_0$ are approximately given by the...
centrifugal radius $R_c$ from the collapse; we thus expect $r_d \approx R_c \approx r_o$ and, hence, $\beta \approx 1$. Notice that in the limit $r \to 0$, $x \to 0$, and the function $h_k$ takes on a limiting form $h_k(x; x \to 0) = \sqrt{\pi}/2 e^x \text{erf}(\sqrt{x}) - \sqrt{2x}(1 + x)$.

Figure 2 shows the density profiles for a representative cluster with an anisotropic velocity distribution, where the parameters are taken to be $\psi_0 = 10$, $\Lambda = \frac{1}{2}$, and $\beta = 1$. The dashed curve shows the stellar component, the dotted curve shows the gaseous component, and the solid curve shows the total density; the dot-dashed curve shows the reduced stellar density profile after gas removal (see the following section). For this class of models, the velocity anisotropy leads to a more extended stellar component compared to the isotropic models of the previous section. For the same values of $\psi_0$ and $\Lambda$, the stellar component extends farther into the gaseous halo and the mass fraction of stars is correspondingly smaller. For this typical model using $\psi_0 = 10$ and $\Lambda = \frac{1}{2}$, for example, the stellar fraction is $27\%$ (compared with $44\%$ for the isotropic case $\beta = 0$). The variation of the density profiles with varying depths of the gravitational potential ($\psi_0$) and varying amounts of gas ($\Lambda$) are similar to those shown in Figure 1 for isotropic models.

3. REMOVAL OF THE GASEOUS COMPONENT

For the next stage of this calculation, we let all of the gas be removed from the cluster on a short timescale. We thus assume that the distribution function of the stars does not have time to adjust and, hence, all of the stars suddenly find themselves in a new environment with a less deep gravitational potential well, but they initially retain the same distribution function, as given by equation (4) or (12) evaluated using the old values of the potential. In nature, the removal of gas does not take place instantaneously, however, so this calculation represents a limiting case of the physical problem.

The energy required to remove gas from a cluster can be easily obtained. The binding energy of a cluster is given by $E = kGM^2/R$, where $k$ is a constant of order unity that depends on the cluster shape. In terms of representative values, we can write

$$E \approx 4 \times 10^{46} \text{ergs} \left(\frac{M}{1000 \ M_\odot}\right)^2 \left(\frac{R}{1 \ \text{pc}}\right)^{-1}. \quad (23)$$

Since a supernova explosion has a typical energy rating of $E_{\text{SN}} \approx 10^{51} \text{ergs}$, one supernova provides enough energy to have a devastating impact on the gaseous component of a cluster. However, radiative cooling is very efficient in molecular clouds and can dispose of much of the injected energy on short timescales (Wheeler et al. 1980; Franco et al. 1994). In addition, gas dispersal takes place efficiently even in the absence of supernovae (Palla & Stahler 2000). As another mechanism, winds from young stellar objects inject a large amount of mechanical luminosity into the surrounding medium, with a typical scaling law of $L_{\text{mech}} \approx 0.01L_*$ (e.g., Lada 1985). According to this relation, a young stellar object (YSO) with $L_* = 3L_\odot$ imparts an energy $\Delta E \approx 10^{45}$ ergs during a time interval $\sim 0.3$ Myr. The outflows from YSOs thus impart enough energy to facilitate gas removal. At the high densities of molecular clouds, however, gas removal by photoionization and photodissociation can be a more effective mechanism (Diaz-Miller et al. 1998). In any case, the energy to remove gas from clusters is easily obtained, but the requisite momentum transfer is more difficult to achieve. Although the actual gas dispersal mechanism remains unclear, gas is observed to dissipate in less than $10^7$ yr, even for aggregates containing no massive stars (Palla & Stahler 2000).

In the models of this paper, we calculate the fraction of stars remaining in a cluster (after gas removal) as a function of star formation efficiency. We note, however, that the star formation efficiency is coupled to the gas removal process. If gas is removed slowly, then the transformation of gas into stars operates over a longer period of time and the star formation efficiency can be larger (although the opposite is not necessarily true).

Against this background, we now proceed with the calculation. The immediate change $\Delta \psi_0$ in the gravitational potential—that just due to gas removal—is given by

$$\Delta \psi_0(\xi) = (\Delta \psi_0)_{\text{r}} + \Lambda \times \left[\frac{1}{\xi_T} \tan \xi - \frac{1}{\xi} \tan \xi + \frac{1}{2} \ln \left(1 + \frac{\xi^2}{\xi_T^2} \right)\right], \quad (24)$$

where the change in potential $(\Delta \psi_0)_{\text{r}}$ at the outer cluster boundary can also be evaluated to obtain

$$\Delta \psi_{\text{o}} = \Lambda \left(1 - \frac{1}{\xi} \tan \xi \right). \quad (25)$$

The physical change in the potential is thus given by $\Delta \psi_0 = \sigma^2 \Delta \psi_0(\xi)$.

With the gas removed from the cluster, the stars find themselves in a new (shallower) potential and, hence, some stars must leave the system. The gravitational potential changes because of both gas removal and the removal of the high-velocity stars. At a given location within the cluster, the original distribution of stars had a range of possible velocities given by

$$0 \leq v^2 \leq 2\Psi. \quad (27)$$

In the new configuration, without the gaseous component, the stars that remain bound to the cluster must have velocities in the range

$$0 \leq v^2 \leq 2(\Psi - \Delta \Psi) \equiv v_m^2, \quad (28)$$

where $\Delta \Psi$ includes the change in potential due to stars leaving the system, and where we have defined a maximum velocity $v_m$ in the final equality. We thus need to solve for the new potential, which we can write as $\phi = v_m^2/2\sigma^2$.

3.1. Reduced Density Profiles for Isotropic Models

For our model with isotropic velocity dispersion, the density of stars remaining is given in terms of $\phi$ (or, equivalently, $v_m$) according to the relation

$$\rho_{\ast}(\text{bound}) = \int_{0}^{\phi} 4\pi v^2 \text{d}v f(v) = \rho_1 \left[ e^\phi \text{erf}(\sqrt{\phi}) - 2\sqrt{\frac{\phi}{\pi}} \left(e^{\phi} + \frac{2}{3\phi}\right) \right]. \quad (29)$$
In the above expression, the old potential $\psi$ no longer has the physical meaning of the gravitational potential. Instead, $\psi(\xi)$ is simply a known function which determines the distribution function of stars, only a fraction of which will remain in the cluster. The fraction that remains depends on the new potential $\phi(\xi)$ which must be determined by solving the following new version of the Poisson equation,

$$\xi^{-2} \frac{d}{d\xi} \left( \xi^2 \frac{d\phi}{d\xi} \right) = -\frac{9}{\rho(\psi_0)} \times \left[ e^{\psi} \text{erf} \left( \frac{\sqrt{\rho}}{\sqrt{2}} \right) - 2 \sqrt{\phi \pi} \left( e^{\psi - \phi} + \frac{2}{\phi} \right) \right],$$

(30)

with $\psi(\xi)$ a known function. To find the new potential and, hence, the new density, we must also specify the boundary condition at the cluster center $\xi = 0$. Here we invoke the condition

$$\phi(0) = \psi(0) - \Delta \psi(0) = \psi(0) - \frac{\Lambda}{2} \ln(1 + \xi_0^2),$$

(31)

which accounts for the change in potential due to gas loss.

With the problem now completely specified, we can solve for the original equilibrium structure of the cluster as a function of the potential well depth $\psi_0$ and the gas parameter $\Lambda$. We can then integrate equation (30) to find the density profile for the stars remaining after the gas is removed. The mass fractions are listed in Table 1 for the representative case of $\psi_0 = 10$. Notice how the results differ from those predicted by equation (1). For $\Lambda = \frac{1}{2}$, for example, the total gas mass is somewhat greater than the original mass in stars. After the gas is removed, nearly 73% of the stellar component remains, whereas the virial argument predicts that none of the stars remain bound.

The difference between these results lies in two key features of this present treatment: (a) By formulating the problem in terms of the distribution function $f(e)$ for the stars, we take into account the low-velocity tail of the distribution (which always tends to remain bound). (b) The gaseous halo in our models is tied to the background molecular cloud and not directly tied to the stellar component of the cluster (although the gravitational interaction is included). This complication allows the gas density distribution to have a markedly different form from that of the stars. In particular, the stellar component exhibits a relatively steep fall off near the edge of the cluster, whereas the gas density continues to follow its usual power-law distribution (see Fig. 1). As a result, the mass of the gaseous component is concentrated more toward the outer portion of the cluster (compared to the stellar component) and gas removal has a less destructive effect on stellar population of the cluster.

Figure 3 shows the fraction $\mathcal{F}_*$ of stars remaining after gas removal as a function of the size of the star formation efficiency, defined as $\epsilon_* \equiv \frac{\rho_* dV}{M}$. Notice that the shape of these curves is markedly different from the singular behavior predicted by equation (1). As a very rough approximation, the entire collection of curves can be described by the simple function $\mathcal{F}_* = 2\epsilon_* - \epsilon_*^2$, as shown by the dashed curve. Much more accurate fits to individual models can be obtained. For the $\psi_0 = 10$ model, for example, the dotted curve shows a fit using a function $\mathcal{F}_* = (2\epsilon - \tilde{\epsilon}^2)^{3/2}$, where $\tilde{\epsilon} \equiv (10\epsilon_\ast - 1)/9$ is a stretched variable.

The clusters produced by this rapid gas removal have a truncated distribution function (by construction). From this state, the cluster tends to evolve toward a new configuration with a smooth distribution function. Because this subsequent evolution takes place through dynamical interactions of the constituent stars, the timescale for readjustment is longer than the gas removal time (but is nonetheless shorter than the total dynamical evolution timescale). This longer term evolution poses an interesting problem for future work.

### 3.2. Reduced Density Profiles for Anisotropic Models

For the anisotropic models developed in the previous section, the new density profile, after the gas is removed

| $\Lambda$ | $\xi_T$ | $m_*$ | $m_{\text{gas}}$ | $m_{\text{new}}$ | $\mathcal{F}_*$ |
|----------|---------|-------|----------------|----------------|-------------|
| 0.00      | 224     | 126   | 0.00           | 126            | 1.000       |
| 0.25      | 150     | 84.9  | 51.7           | 73.4           | 0.865       |
| 0.50      | 106     | 58.9  | 73.0           | 42.8           | 0.727       |
| 0.75      | 78.1    | 42.7  | 80.1           | 26.9           | 0.631       |
| 1.00      | 59.6    | 32.5  | 81.0           | 18.5           | 0.569       |
| 1.25      | 47.1    | 25.8  | 79.4           | 13.6           | 0.526       |
| 1.50      | 38.3    | 21.2  | 77.0           | 10.4           | 0.489       |
| 1.75      | 32.0    | 17.9  | 74.3           | 8.09           | 0.451       |
| 2.00      | 27.2    | 15.5  | 71.8           | 6.36           | 0.412       |
| 2.25      | 23.6    | 13.6  | 69.9           | 4.99           | 0.369       |
| 2.50      | 20.8    | 12.0  | 67.2           | 3.88           | 0.323       |
| 2.75      | 18.5    | 10.8  | 65.3           | 2.96           | 0.274       |
| 3.00      | 16.7    | 9.76  | 63.5           | 2.19           | 0.225       |
| 3.25      | 15.1    | 8.90  | 61.9           | 1.55           | 0.174       |
| 3.50      | 13.9    | 8.16  | 60.4           | 1.00           | 0.123       |
| 3.75      | 12.8    | 7.53  | 59.1           | 0.52           | 0.069       |
| 4.00      | 11.9    | 6.97  | 57.9           | 0.09           | 0.013       |

![Fig. 3.—Fraction $\mathcal{F}_*$ of stars remaining bound to a cluster after gas removal for systems with isotropic velocity distributions. The solid curves show the fraction $\mathcal{F}_*$ as a function of star formation efficiency $\epsilon_*$ for varying $\psi_0$ (here $\psi_0 = 6, 8, 10, 12, \text{and} 14$). The dashed curve shows the function $\mathcal{F}_* = 2\epsilon_* - \epsilon_*^2$, which provides a rough approximation to the entire family of curves. The dotted curve shows a more accurate fit (to the $\psi_0 = 10$ model) using the function $\mathcal{F}_* = (2\epsilon - \tilde{\epsilon}^2)^{3/2}$, where $\tilde{\epsilon} \equiv (10\epsilon_\ast - 1)/9$ is a stretched variable.](image)
pared with the isotropic models of the previous section, the and then use the above formulation to Þnd the density of the cluster in the presence of a gaseous component of the solution to the Poisson equation.

As in the previous case, we can Þnd the equilibrium structure of the cluster in the presence of a gaseous component and then use the above formulation to Þnd the density profile of the stars remaining after gas is removed. Compared with the isotropic models of the previous section, the anisotropy of the velocity distribution allows a greater fraction of the stars to remain after gas removal. For a collection of models with varying ψo, Figure 4 shows the fraction Fs of stars remaining after gas removal as a function of the star formation efficiency ϵs (solid curves). The dashed curve shows the function Fs = (2ϵs − ϵs)1/2, which provides a rough Þt to the family of curves as shown.

Keep in mind that ψ is the original potential, before gas removal, and φ is the physical potential as determined by the solution to the Poisson equation.

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4. COMPARISON WITH OBSERVATIONS

The Trapezium cluster is a nearby young cluster in Orion and is often used an example of a forming bound cluster (e.g., Hillenbrand & Hartmann 1998; McCaughrean & Stauffer 1994; Hillenbrand 1997). Since this cluster is relatively young (the stars have a quoted mean age of ~0.8 Myr), it provides a good point of comparison for the models of newly formed clusters. The total mass within the central 2.06 pc region is estimated to be in the range 4500–4800 M⊙ (Hillenbrand & Hartmann 1998); these same authors estimate that the core radius r0 ≈ 0.2 pc and find a total number of stars N0 ≈ 2300 within the central r = 2.06 pc region. Within this same volume, the total mass of the cluster is larger than that of the stellar component by a factor of about 2; i.e., stars constitute 40%–50% of the cluster mass within the inner 2 pc.

The characteristics of this cluster are roughly consistent with the cluster formation scenario outlined in the Appendix. If we use the isothermal version of the model, for example, we would need an effective sound speed of a ≈ 2.3 km s⁻¹ to form a 4800 M⊙ cluster in 1.7 Myr; these same values imply r0 ≈ 2 pc. In order to obtain a centrifugal radius (and a core radius) of Rc = r0 = 0.2 pc, the initial rotation rate of the core would have to be Ωc ≈ 0.4, which is a reasonable value for molecular clouds. Similarly, for the logatropic model, a pressure scale of P0 ≈ 6.6 × 10⁻¹⁰ dyn cm⁻² would imply r0 = 2 pc and a formation time of 2.9 Myr. To obtain the same core radius of r0 = Rc = 0.2 pc, the required rotation rate is again Ωc ≈ 0.4. We thus argue that inside-out collapse models can be made consistent with this particular observed young cluster.

Hillenbrand & Hartmann (1998) have already used lowered isothermal models to Þt to the stellar component of the Trapezium cluster data and Þnd good agreement using a central potential ψ0 = 9. This model does not include the gaseous component or the possibility of anisotropy in the velocity distribution; however, including these additional parameters can only make the Þt better. We can obtain a reasonable model Ðt to this same cluster using ψ0 ≈ 10. For an isotropic model with Λ = 2, the total mass in mass is comparable to the mass in stars. If the gas is removed from such a cluster, nearly 73% of the stellar mass remains bound, with the remainder escaping to Þll the Þeld with stars. These models extend out to a truncation radius rt which is about 100 core radii and thus extends many times farther out than the observed portion of the cluster. In order to have the gas mass and the stellar mass nearly equal in the (observed) central 2 pc region, we need a larger gas parameter of Λ ≈ 1.5; for this case, only about 48% of the stars remain bound after the gas leaves the system. After gas removal, this simple model thus predicts that the central region of the Trapezium cluster will remain a bound entity with approximately N ~ 1000 stars. If the velocity distribution is not isotropic, but instead becomes more radial in the outer cluster, then an even larger fraction of stars could be retained after gas removal.

In the existing literature, there are few predictions regarding the fate of the Trapezium for comparison. An older study using only 16 stars indicates a half-life of only 10⁶ yr (Allen & Poveda 1975). On the other hand, a recent numerical study (P. Kroupa 2000, private communication) obtains results in good agreement with the semianalytic models of this paper.

5. SUMMARY AND DISCUSSION

In this paper, we have used results from star formation theory and stellar dynamics to develop a working model for the early evolution of open star clusters. More speciÞcally, we have constructed a sequence of equilibrium cluster models, including both isotropic and anisotropic velocity dispersions. These models represent a straightforward generalization of previous work to include a gaseous com-
isotropic models, the fraction can be characterized by a star formation efficiency (see Figs. 3 and 4). As a crude behind after gas removal is a smoothly varying function of the centrifugal radius \( R_c \sim 0.1 \) pc, has time to dynamically relax and tends to exhibit an isotropic velocity dispersion; the centrifugal radius also sets the core radius of the cluster. In the outer portion of the forming cluster, \( r > R_c \), the velocity dispersion becomes anisotropic and nearly radial.

This paper represents a modest step toward a unified picture of cluster formation, early evolution, and dispersal. We have included a gaseous component in the construction of semianalytical equilibrium models of clusters and have considered the initial effects of gas removal. The subsequent development of the resulting stellar systems should be studied next. More detailed models of cluster formation constitute another fruitful area for further work. With a more definitive picture of cluster formation in hand, the distribution function of the stars in the earliest phases can be more precisely specified and the fraction of stars that remain after gas removal can be more accurately determined. Another important complication is to include a different distribution function for stars of different masses. Once we understand these issues of cluster formation and early evolution, we can then determine their effects on the star formation process.

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**APPENDIX**

**A CLUSTER FORMATION MODEL**

In this Appendix, we present a simple model for cluster formation. For robust bound clusters forming within molecular clouds, we expect the original protocluster structure to collapse as a whole. To obtain a mathematical description of this collapse, we consider the flow that eventually produces a cluster to be a scaled up version of the collapse flows that have been studied previously for single star formation (e.g., Shu 1977; Terebey, Shu, & Cassen 1984; Jijina & Adams 1996). This approach should capture the basic essence of the collapse problem. In this case, the collapse of a molecular cloud region proceeds from inside-out. The central portion of the flow approaches a ballistic (pressure-free) form and helps determine the velocity distribution of the forming cluster. Even for infalling gas, the inner limit of the collapse flow always approaches pressure-free conditions; infalling stars are manifestly ballistic. The timescale for individual star formation events is \( \sim 10^5 \) yr (Myers & Fuller 1993; Adams & Fatuzzo 1996), whereas the timescale for the entire cluster to form is somewhat longer (1–3 Myr). We thus expect most of the stars to form while the overall collapse of the cluster is still taking place. In apparent support of this picture, observations suggest that cluster formation takes place within only about one sound crossing time of the system (Elmegreen 2000).

For a given gravitational potential, we find the orbital solutions for stars (or gas parcels) falling toward the cluster center. In the standard infall calculation, the inner solution is derived using the gravitational potential of a point source. Since this potential is spherically symmetric, angular momentum is conserved and the motion is confined to a plane described by the coordinates \((r, \phi)\); the radius \(r\) is the same in both the plane and the original spherical coordinates. The angular coordinate \(\phi\) in the plane is related to the angle in spherical coordinates by the relation \(\cos \phi = \cos \theta \cos \theta_0\), where \(\theta_0\) is the angle of the asymptotically radial streamline (see below). For zero energy orbits, the equations of motion imply a cubic orbit solution,

\[
1 - \frac{\mu}{\mu_0} = \zeta (1 - \mu_0^2) .
\]

(A1)

Here the trajectory that is currently passing through the position given by \(\zeta\) and \(\mu \equiv \cos \theta\) initially made the angle \(\theta_0\) \((\mu_0 = \cos \theta_0)\) with respect to the rotation axis. The quantity \(\zeta\) is defined by

\[
\zeta \equiv \frac{\dot{J}_z}{GMr} = \frac{R_c}{r} ,
\]

(A2)

\[
\frac{1}{2} \frac{d}{dt} R^2 \frac{\dot{\theta}_0}{r} = -GMr \frac{(1 - \cos \theta_0)}{r^2} ,
\]

(A3)

\[
d \theta_0 = \frac{1}{r^2} \left( \frac{1}{2} \frac{\dot{J}_z}{GMr} - \frac{1}{2} \frac{\dot{\theta}_0}{r} \right) dr.
\]

(A4)
where $j_\infty$ is the specific angular momentum of parcels of gas currently arriving at the cluster center along the equatorial plane. The second equality defines a centrifugal radius $R_C$. We assume that the initial cloud is uniformly rotating at a constant rotation rate $\Omega$, so that $j_\infty = \Omega r_\infty^2$, where $r_\infty$ is the starting radius of the material that is arriving at the center at a given time.

To evaluate the radii $R_C$ and $r_\infty$, we invert the mass distribution of the initial state. For an isothermal cloud, the mass profile and the centrifugal radius are given by

$$M(r) = \frac{2a^2r}{G} \quad \text{and} \quad r_\infty = \frac{GM}{2a^2}, \quad R_C = \frac{\Omega^2 G^2 M^3}{16a^8}, \quad (A3)$$

where $a$ is the characteristic scale speed. For this case, the infall collapse solution (Shu 1977) indicates that the cloud exhibits a well defined mass infall rate $\dot{M} = m_0 a^3/G$, where $m_0 \approx 0.975$. To form a cluster with mass $M = 1000 M_\odot$ in 1 Myr, for example, the required effective sound speed is $a \approx 1.63$ km s$^{-1}$. With this central mass, the region that originally filled a volume of radius $r_\infty \approx 0.81$ pc has fallen to the center (within $R_C$). The size of the collapsing region is twice as large, $r_H = at \approx 1.66$ pc, and contains a total mass $M \approx 2000 M_\odot$. The centrifugal radius $R_C \approx 0.1$ pc for $\Omega = 1$ km s$^{-1}$ pc$^{-1}$ (a typical observed value; Goodman et al. 1993). This centrifugal radius is comparable to the expected core radius $r_{core}$ of a newly formed cluster, and we make the rough identification $r_{core} \sim R_C$.

We also consider initial states for nonisothermal conditions. Molecular line widths often show a substantial nonthermal component with a density dependence of the form $\Delta v \propto \rho^{-1/2}$ (e.g., Larson 1985; Myers & Fuller 1992). If we use a "logatropic" equation of state $P = P_0 \log \rho/\rho_0$ to describe such a fluid (Lizano & Shu 1989; Jijina & Adams 1996; McLaughlin & Pudritz 1997; Galli et al. 1999), the equilibrium mass distribution and the corresponding radii $r_\infty$ and $R_C$ are given by

$$M(r) = \left[\frac{2\pi P_0}{G}\right]^{1/2} r^{1/2}, \quad r_\infty = M^{1/2} \left[\frac{2\pi P_0}{G}\right]^{-1/4}, \quad R_C = \frac{\Omega^2 M}{2\pi P_0}, \quad (A4)$$

where $P_0$ is the pressure scale that determines the amount of nonthermal support in the cloud. In this case, the mass infall rate is time dependent. The total mass $M(t)$ that falls to the center of the flow during a time $t$ is given by $M = m_0 t^2(2\pi G P_0)^{3/2} / 16G$, where $m_0 \approx 0.0302$. During logatropic collapse, most of the mass in the original cloud region is still on the way down, rather than at the center. If the cluster encompasses the entire collapsing region, the total mass is about 30 times that of the central core. A typical scenario would thus have $100 M_\odot$ in the central core and $3000 M_\odot$ still falling inward. For this case, the required pressure scale $P_0 \approx 8.9 \times 10^{-10}$ dyn cm$^{-2}$. To obtain a centrifugal radius $R_C \approx 0.1$ pc $\sim r_{core}$ as before, we need $\Omega \approx 3$ km s$^{-1}$ pc$^{-1}$.

Both the isothermal and the logatropic models can produce a cluster in $\sim 1$ Myr using reasonable values for the input parameters ($\Omega$ and $a$ or $P_0$). In both cases, the effective transport speed required to support the initial cloud is comparable to the velocity dispersion of the resulting cluster. For the range of parameters discussed above, this velocity scale is $1-2$ km s$^{-1}$ and is roughly consistent with the velocity dispersion of observed open clusters.

For this cluster formation scenario to be consistent with the current paradigm of star formation (e.g., Shu, Adams, & Lizano 1987), the cluster environment cannot greatly disrupt the collapse of smaller regions that produce individual stars. To fix ideas, we assume that the large-scale collapse of the cluster is determined by logatropic conditions (eq. [A4]) and the collapse of individual star forming regions is given by isothermal conditions (eq. [A3]). To produce a $1 M_\odot$ star, for example, the radial extent of the initial precollapse region is $r_\infty = GM/2a^2 \approx 0.02$ pc. In the central region ($r < 1$ pc) that eventually becomes the cluster, the individual star forming sites do not greatly interfere with each other as long as the number of stars does not exceed $N \approx (1/0.02)^3 \approx 10^3$. Similarly, the collapse of an individual star proceeds largely independent of the tidal forces. For a cluster of mass $M_{clust} \approx 1000 M_\odot$ and size $R_{clust} \approx 1$ pc, the tidal radius $r_T \sim 0.1$ pc, which is much larger than the size of an individual infall region ($\sim 0.02$ pc).

We expect that the stars in the newly formed cluster retain some dynamical memory of the velocity distribution of the collapse flow. In this flow, streamlines entering the central region do not cross each other. As long as the infall time is longer than the timescale for individual star formation events, the core regions that produce stars will not have a chance to interact. (As an aside, note that this cluster formation scenario has an initial transient phase [the first $10^4$ yr] in which the infall of the cluster takes place faster than individual stars can form. Protostars entering the central region during this initial time period thus have an opportunity to interact and merge; this activity may contribute to the production of more massive stars in the cluster center.)

Given the orbital solution, we can find the velocity fields for the collapse flow:

$$v_r = -\frac{GM}{r} \left[2 - \zeta(1 - \mu_0^2)\right]^{1/2}, \quad (A5)$$

$$v_\theta = \left(\frac{GM}{r}\right)^{1/2} \left[1 - \mu_0^2 \left(\frac{\mu_0^2 - \mu^2}{1 - \mu^2}\right)^{1/2}\right], \quad (A6)$$

$$v_\phi = \left(\frac{GM}{r}\right)^{1/2} \left(1 - \mu_0^2\right)^{-1/2} \left(r^{1/2}\right). \quad (A7)$$

Since $\zeta$, $\mu$, and $\mu_0$ are related through the orbit equation (A1), the velocity field is completely determined for any position $(r, \theta)$. With the velocity field specified, we can find the anisotropy in the flow as a function of radius. We define the angular
average for both the perpendicular component of the velocity field and the radial component

$$\langle v_r^2 \rangle = \frac{GM}{r} \zeta I_v, \quad \langle v_\theta^2 \rangle = \frac{GM}{r} (2 - \zeta I_v),$$  \hspace{1cm} (A8)

where the integral $I_v$ is given by

$$I_v \equiv \int_0^1 d\mu (1 - \mu^2).$$  \hspace{1cm} (A9)

To evaluate the integral $I_v$, we change the integration variable from $\mu$ to $\mu_0$ and change the lower end of the range of integration from 0 to a critical value $\mu_C$. This difference arises because streamlines from all initial angles cannot fall to arbitrarily small radii. For large radii, streamlines originating preferentially from the poles reach these smaller radii. The last streamline that reaches a given radius is determined by $\mu_C$. Evaluating the integral $I_v$, we find

$$I_v = \left( \frac{2}{3} - \frac{4}{15} \zeta \right) (1 - \mu_C) + \mu_C \left( \frac{2}{15} + \frac{4}{15} \times \frac{1}{\zeta} \right).$$  \hspace{1cm} (A10)

This expression simplifies in the inner and outer regimes. For large radii, $r \gg R_c$, $\mu_C \to 0$, and $I_v \to \left( \frac{1}{3} \right) (1 - 2\zeta/5)$. In the opposite limit of small radii, $r \ll R_c$, $\mu_C \to 1 - \frac{1}{3} \zeta$, and $I_v \to (8/15)\zeta$.

In the context of cluster formation, we evaluate the anisotropy of the flow in the outer regime and assume that the cluster retains some memory of its initial velocity distribution. Combining equations [A8]–[A10], we find

$$\zeta \equiv \frac{\langle v_\theta^2 \rangle}{\langle v_r^2 \rangle} = \frac{\zeta I_v}{2 - \zeta I_v} = \frac{\zeta (1 - 2\zeta/5)}{3 (1 - \zeta/3)}.$$  \hspace{1cm} (A11)

To leading order, we thus obtain $\zeta \sim \zeta = R_C/r$. For large radii the velocities become nearly radial, whereas for small radii the velocities become more isotropic. Scattering of the newly formed stars will be relatively efficient at small radii and can drive the velocity dispersion even further toward isotropy. More specifically, the relaxation time can be written in the form $t_{\text{relax}} \approx 14 \text{ Myr} \left( r/1 \text{ pc} \right)^2 \left( r/1 \text{ pc} \right)^{-1}$, where we have assumed a typical robust cluster with $N = 1000$ stars. For small radii $r < 0.1 \text{ pc} \approx R_C$, the relaxation time is less than about 0.2 Myr. As a result, the region within the centrifugal radius will experience several relaxation times during the expected timescale for the cluster to form.

In this picture, clusters form with a nearly isotropic velocity dispersion on the inside and a highly radial velocity dispersion on the outside, with the centrifugal radius $R_C$ providing the boundary between the two regimes. Furthermore, the core radius of the cluster is determined by $r_0 \approx R_C \approx 0.1 \text{ pc}$ (for typical initial conditions).

REFERENCES

Abramowitz, M., & Stegun, I. A. 1972, Handbook of Mathematical Functions (New York: Dover)
Adams, F. C., & Fatuzzo, M. 1996, ApJ, 464, 256
Adams, F. C., & Myers, P. C. 2000, ApJ, submitted
Allen, C., & Poveda, A. 1975, PASP, 87, 499
Binney, J., & Tremaine, S. 1987, Galactic Dynamics (Princeton: Princeton Univ. Press)
Bleistein, N., & Handelsman, R. A. 1986, Asymptotic Expansions of Integrals (New York: Dover)
Díaz-Miller, R. I., Franco, J., & Shore, S. N. 1998, ApJ, 501, 192
Elmegreen, B. G. 1983, MNRAS, 203, 1011
———. 2000, ApJ, 530, 277
Elmegreen, B. G., & Clemens, C. 1985, ApJ, 294, 523
Franco, J., Miller, W. W., Arthur, S. J., Tenorio-Tagle, G., & Terlevich, R. 1994, ApJ, 435, 805
Galli, D., Lizano, S., Li, Z. Y., Adams, F. C., & Shu, F. H. 1999, ApJ, 521, 630
Goodman, A. A., Benson, P. J., Fuller, G. A., & Myers, P. C. 1993, ApJ, 406, 528
Hillenbrand, L. A. 1997, AJ, 113, 1733
Hillenbrand, L. A., & Hartmann, L. W. 1998, ApJ, 492, 540
Hills, J. G. 1980, ApJ, 235, 986
Jijina, J., & Adams, F. C. 1996, ApJ, 462, 874
Lada, C. J. 1985, ARA&A, 23, 267
Lada, C. J., Margulis, M., & Dearborn, D. 1984, ApJ, 285, 141
Larson, R. B. 1985, MNRAS, 214, 379
Lizano, S., & Shu, F. H. 1989, ApJ, 342, 834
Mathieu, R. D. 1983, ApJ, 267, L97
McCaughrean, M. J., & Stauffer, J. 1994, AJ, 108, 1382
McLaughlin, D. E., & Pudritz, R. E. 1997, ApJ, 476, 750
McLaughlin, D. E., & Pudritz, R. E. 1997, ApJ, 476, 750
Myers, P. C., & Fuller, G. A. 1992, ApJ, 396, 631
———. 1993, ApJ, 402, 635
Palla, F., & Stahler, S. W. 2000, ApJ, 540, 255
Palla, F., & Stahler, S. W. 2000, ApJ, 540, 255
Shu, F. H. 1977, ApJ, 214, 488
Shu, F. H., Adams, F. C., & Lizano, S. 1987, ARA&A, 25, 23
Terebey, S., Shu, F. H., & Cassen, P. 1984, ApJ, 286, 529
Verschueren, W. 1990, A&A, 234, 156
Verschueren, W., & David, M. 1989, A&A, 219, 105
Wheeler, J. C., Mazurek, T. J., & Sivaramakrishnan, A. 1980, ApJ, 237, 781