Effect of Fear, Treatment, and Hunting Cooperation on an Eco-Epidemiological Model: Memory Effect in Terms of Fractional Derivative

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Abstract
In this paper, we have studied a fractional-order eco-epidemiological model incorporating fear, treatment, and hunting cooperation effects to explore the memory effect in the ecological system through Caputo-type fractional-order derivative. We have studied the behavior of different equilibrium points with the memory effect. The proposed system undergoes through Hopf bifurcation with respect to the memory parameter as the bifurcation parameter. We perform numerical simulations for different values of the memory parameter and some of model parameters. In the numerical results, it appears that the system is exhibiting a stable behavior from a period or chaotic nature with the increase in the memory effect. The system also exhibits two transcritical bifurcations with respect to the growth rate of the prey. At low values of prey’s growth, all species go to extinction, at moderate values of prey’s growth, only preys (susceptible and infected) can survive, and at higher values of prey’s growth, all species survive simultaneously. The paper ended with some recommendations.

Keywords Eco-epidemic model · Fear effect · Hunting cooperation · Caputo fractional-order derivative · Transcritical bifurcation · Hopf bifurcation

1 Introduction
In population dynamics, ecological interactions such as competition, mutualism, and predation, play an essential role. However, parasite infection also affects the size of populations. Thus, prey–predator interactions should not ignore this issue. There have been multiple field studies demonstrating parasitic infections in prey and predators. Parasites can reduce the ability of infected organisms to survive and reproduce by affecting their internal mechanisms. Therefore, we ought to be concerned about predator–prey systems in which both populations are infected. An eco-epidemiological approach focuses on infectious diseases in populations and communities. The step-by-step process of analyzing a problem from a molecular, social, and demographic perspective is considered eco-epidemiology. A system’s dynamics are affected by infection in any part of the population or both populations. In recent years, infectious disease has emerged as a significant factor. Researchers are increasingly studying predator and prey with infectious diseases.

In many studies, predator–prey models have been investigated only with a disease in the prey. Hethcote et al. (2004) presented a predator–prey model in which SIS parasitic infection in prey led to higher rates of predation on infected prey before predator predation. In Sinha et al. (2010), authors explore prey–predator interactions in the context of environmental toxicants and disease. According to their study, the toxicants affect the population, while the infected prey is much more vulnerable to the toxicants as well as predators than sound prey. Shaikh et al. (2021) investigates the dynamics of an eco-epidemic predator–prey system in which disease is spread to prey species and alternative food is provided to predators. Moustafa et al.

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(2020) analyzes a fractional-order eco-epidemiological model with disease in the prey population and showed that the order of fractional derivative stabilizes the coexistence equilibrium. According to Meng et al. (2018), the predator population can survive in a predator–prey system with prey harvesting and disease spreading among prey species. Recently, Sk et al. (2022) have studied a prey–predator model that incorporated infection in prey in both deterministic and stochastic environments, and found that high levels of fear and low levels of refuge can eliminate the disease from the system.

Some infectious diseases can influence the dynamics of predator–prey systems when they enter either the predator body or the prey body (Djilali and Ghanbari 2021) through some pathogens. Pathogens include germs, viruses, fungi, parasites, etc. The objects can spread by direct or indirect contact with animals or some other way such as water and air (Van Seventer and Hochberg 2017). If the infection is not treated, it may be harmful. According to the Food and Agriculture Organization of the United Nations (FAO) (Romain et al. 2020), livestock accounts for (40%) of the total agricultural production in developed countries, and for (20%) in developing countries. The infection agencies may directly infect the prey or the predator, or the predator may become infected after consumption of the infected prey (Andrew et al. 2016).

An important topic in ecological systems with epidemics is the dynamic relationship between predators and their prey. Eco-epidemiological models are investigated the ecological system with infection (Mukherjee 2010; Chakraborty et al. 2011; Chattopadhyay et al. 2002, 1999). The necessity of conserving wild animals has led many ecologists and eco-epidemiologists to become familiar with eco-epidemiology. Eco-epidemiological models discuss the prey–predator relationships when some of the species are infected (Juneja and Agnihotri 2018). One of the main objectives of the investigation of the eco-epidemic model is to control the spreading of diseases when disease and treatment both coexist simultaneously. Many studies have been published on ecological and epidemiological models with the disease either in prey (Meng et al. 2018; Mortoja et al. 2018) or in predator (Rana et al. 2016; Juneja and Agnihotri 2018) or in both prey and predator (Agnihotri and Juneja 2015; Hsieh and Hsiao 2008) and reference therein.

Several field survey data and experimental results on terrestrial vertebrates showed that the fear of predators would cause a substantial variability of prey density (Sarkar and Khajanchi 2020). In Mukherjee (2020); Zhang et al. (2019), the authors have studied some of these types of models in the presence of fear effect and competitor for the prey in the predator–prey model with prey refuge. Treatment of the infected prey populations restores the prey to its previous situation; as a result, availability of susceptible prey becomes plenty to the predator, and dynamics of the system may be more complex compared to other situations (Adnan Thirthar 2020).

There are ample information in the existing literature on predator–prey interactions, which utilize the diversity of functional responses of both prey and predator populations. In the literature, direct assassinations of prey populations by predator populations have been the focus. Several studies have examined predator and prey behavior as well as antipredator activities (Wang et al. 2016; Wang and Zou 2017). Wang et al. (2016) and Zanette et al. (2011) constructed mathematical models for the predator–prey system by including the cost of fear for prey species due to predators, where the cost of fear determines the birth rate of prey species. They showed that the presence of predator-defeating activities or a major cost of fear can eliminate periodic behaviors, excluding the paradox of the enrichment scenario. Moreover, they demonstrated that fear can stabilize the system by eliminating population oscillations. Additionally, oscillations emerge from either supercritical or subcritical Hopf bifurcation under comparatively low cost for fear (Wang et al. 2016). Therefore, the fear effect can produce multi-stability in the predator–prey system. Mondal et al. (2022) showed the existence of saddle-node bifurcation, Hopf bifurcation, and Bogdanov–Takens bifurcation in an imprecise predator–prey system with fear effect and nonlinear harvesting of predators in an uncertain environment.

The authors in Danane et al. (2021) studied a mathematical model that described COVID-19 dynamics with the effects of governmental action and individual risk awareness to reduce the spread of infection effectively. In addition, the authors showed that the infection converges more quickly to its steady state when the fractional derivative is used. Kumar et al. (2022) evaluated the mathematical model of the COVID-19 epidemic using Caputo–Fabrizio fractional derivatives. Boudaoui et al. (2021) employed the Caputo–Fabrizio derivative to model the novel Coronavirus disease COVID-19 and found that fractional-order epidemic models provide more insight into the disease. Using the Caputo operator, Gao and Baskonus (2022) developed a modified epidemiological Susceptible-Infected-Removed model. In their study, Tanriverdi et al. (2021) examined the dynamics and mathematics of the fractional-order atmosphere–soil–land plant carbon cycle system involving the time-dependent variable of carbon flux in the atmosphere. An experimental study on fractional models of complex permittivity of conductor media with relaxation is conducted by Ciancio et al. (2022). Zamir et al. (2021) examined mathematical modeling of the eradication of the COVID-19 infection with the help of almost non-pharmaceutical interventions (NPIs). Danane
et al. (2021) investigated the dynamics of a COVID-19 stochastic model with an isolation strategy using Lévy jump perturbations incorporating toward all compartments of the suggested model. Hama et al. (2022) examined the behavior of a SEIS stochastic model which is performed by a Lévy process. In the research articles (Danane et al. 2021; Kumar et al. 2022; Boudaoui et al. 2021; Gao and Baskonus 2022; Tanriverdi et al. 2021; Ciancio et al. 2022; Zamir et al. 2021; Danane et al. 2021; Hama et al. 2022), the epidemic model with fractional-order derivatives mainly appears. Moustafa et al. (2020) studied a fractional-order eco-epidemiological model with disease in the prey population. But they have not considered any biological effects such as fear and hunting cooperation. In Sk and Pal (2022), authors studied a prey–predator model with infection in a prey population. They have also considered the effects of fear, refuge, and harvesting in their proposed model. But they did not consider any treatment for infected individuals or memory effect. In Moustafa et al. (2022), authors studied a fractional-order prey–predator model with infection in predator and prey harvesting. They did not consider the effects of fear, cooperation, or any treatment on the infected populations. Yousef et al. (2022) studied a fractional-order eco-epidemiological model with fear and hunting cooperation but they did not consider the effect of treatment. Motivating from the above-said articles, we think that there may be some research gap in the memory-dependent eco-epidemiological model with hunting cooperation, fear, and treatment of infected prey simultaneously. To fill the research gap in this paper, we have studied the combined effects of fear and hunting cooperation. Furthermore, we include a treatment term for infected prey and also we have analyzed our model with memory effects.

To formulate the model, we assume that the total prey population is composed of two compartments: One is the class of the susceptible prey, and the other is the class of the infected prey, their population density is denoted by $S(t)$ and $I(t)$, respectively, and the density of predator populations is denoted by $y(t)$. We assume that only susceptible prey is capable of reproducing, the disease is spread among the prey population only and the disease is not genetically inherited. We also assume that infected prey does not compete for the resource of being weak due to disease infection. We consider that the predator eats only the infected prey. Let the rate of incidence of disease transmission be $\beta SI$ and the treatment function $\frac{\rho I}{\sigma + I}$ in which $\rho > 0$ represents the maximum medical resource supplied for treatment, while $\frac{1}{\sigma} > 0$ stands for the saturation factor that measure the effect of the delay in treatment for the infected individuals. Introducing the effect of cooperation of predators are following functional form $(p + by)y$, where $y$ is the predator density, $p > 0$ is the attack rate of the predator on the prey, and $b \geq 0$ describes the predator cooperation in hunting (Mondal et al. 2022, 2022). Here, we consider the fear function in the form $\eta + \frac{v(1 - \eta)}{v + y}$, where $v$ represents the fear level and $\eta \in [0, 1]$ represents the minimum cost of fear, which is extensively described in Sarkar and Khajanchi (2020).

Based on all the above-described situations, the three-dimensional eco-epidemic model can be formulated as follows:

$$\begin{align*}
\frac{dS}{dt} &= \left(\eta + \frac{v(1 - \eta)}{v + y}\right) r S - (d + \beta I) S + \frac{\rho I}{\sigma + I} \\
\frac{dI}{dt} &= \left[\beta S - \delta - (p + by) y - \frac{p}{\sigma + I}\right] I \\
\frac{dy}{dt} &= \left[c(p + by)I - m\right] y
\end{align*}$$

(1)

### Table 1 A description of biological symbols along with their meanings and values

| Parameter          | Environmental Interpretation                                      | Values  |
|--------------------|------------------------------------------------------------------|---------|
| $\eta$             | Minimum cost of fear                                             | 0.0011  |
| $v$                | Level of fear                                                    | 0.12    |
| $r$                | Intrinsic growth rate of prey                                    | 0.112   |
| $d$                | Natural death rate of the susceptible prey                        | 0.12    |
| $\beta$            | Disease transmission rate                                        | 0.32    |
| $\rho$             | Maximum medical resource supplied for treatment                 | 0.9     |
| $\frac{1}{\sigma}$| Stands for the saturation factor that measure the effect of the delay in treatment for the infected individuals | $\frac{1}{0.8}$ |
| $\delta$           | Death rate of the infected prey                                  | 0.12    |
| $p$                | The attack rate of the predator on the infected prey             | 0.29    |
| $b$                | Predator cooperation in hunting                                   | 0.26    |
| $c$                | Efficiency with which predators convert consumed infected prey into new predators | 0.34    |
| $m$                | Natural death rate of the predator                               | 0.39    |
with initial conditions \( S(0) > 0, I(0) \geq 0, y(0) \geq 0 \) and all the model parameters are nonnegative and their biological consequence are described in Table 1.

Fractional-order derivative is the generalization of integer-order derivative to an arbitrary order (Petráš 2011). Ordinary derivatives are memoryless, i.e., they possess local properties. As a fractional-order derivative, it has non-local properties; namely, it carries not only the current circumstance but also its prior historical states (Al-Khaled and Alquran 2014; Das 2008). In the last few years, a lot of research articles have appeared in different scientific journals studying biological, inventory models, or physical problems (Das et al. 2019; Pakhira et al. 2020). Fractal-order differential equations have gained a lot of attention and appreciation recently due to their ability to find accurate descriptions of various nonlinear phenomena. In recent years, dynamical system research has increasingly utilized models based on fractal order (Li et al. 2015; Hegazi et al. 2013) and references therein. Biological systems carry memory effects in searching for food, locating a safe place to live, and finding a mate. The memory effect is included in biological systems by using fractional-order derivatives and in most cases, it has been observed that the memory effect has a stabilization property. The fundamental results of fractional derivative are given in Kilbas et al. (2006); Podlubny (1998); El-Sayed (1996). The authors in Li et al. (2017); Ghosh et al. (2021) used fractional-order derivative to study the physical problems like earthquakes, network problems, Liu system, etc.

The reduction in the fractional-order model from the integer-order model was logically developed through the introduction of the kernel function in Saeedian et al. (2017); Ghosh et al. (2021). Following the methodology developed in Saeedian et al. (2017); Ghosh et al. (2021), the system (1) easily can be reduced into a fractional-order model in the following form:

\[
\begin{align*}
D^\alpha_t S(t) &= \left( \eta + \frac{v(1-\eta)}{v+y} \right) rS - (d + \beta t)S + \frac{\beta I}{\sigma + 1}, \\
D^\alpha_t I(t) &= \left( \beta S - \delta - (p + by)  y - \frac{\rho}{\sigma + 1} \right) I, \\
D^\alpha_t y(t) &= [c(p + by)I - m] y,
\end{align*}
\]

with initial conditions \( S(0) > 0, I(0) \geq 0, y(0) \geq 0 \), \( 0 < \epsilon \leq 1 \) is the order of fractional derivative, \( D^\alpha_t \) is the Caputo type of fractional-order derivative (CFD), and \( t_0 \) is the initial time which is equal or greater than zero. Definition and properties of CFD with reduction of ODE into FDE are described in Sect. 2.

This manuscript is organized as follows: Sect. 2 discusses the definition and some basic properties of CFD with basic mathematical analysis such as the existence of solutions and their positivity and bounds. Section 3 discusses equilibrium points and their stability. Section 4 summarizes numerical results with memory effects. Finally, Sect. 5 concludes with some conclusions.

## 2 Definition and Some Basic Properties of CFD with Basic Mathematical Analysis

Here, we shall describe some basic definition and properties of fractional derivative. The commonly used definitions are the Riemann–Liouville (RL) and Caputo derivatives which are given below:

1. **RL definition of fractional derivative for any \((n - \epsilon)\)**
   time integrable and \( n \) times differentiable function \( g(t) \) on \([t_0, t]\) is defined as
   \[
   R^\eta_t D^\alpha_t g(t) = \frac{1}{\Gamma(n-\epsilon)} \left( \frac{d}{dt} \right)^n \int_{t_0}^t \frac{g(w)}{(t-w)^{n-\epsilon+1}} dw,
   \]
   where \( n-1 < \epsilon \leq n \).

2. **The CFD of order \( \epsilon \)**
   with \( n - 1 < \epsilon \leq n \) for the function \( g \in C^\infty([t_0, +\infty), R] \) is defined as Boukhouima et al. (2017):
   \[
   D^\alpha_t g(t) = \frac{1}{\Gamma(n-\epsilon)} \int_{t_0}^t \frac{g^{(n)}(w)}{(t-w)^{n-\epsilon+1}} dw
   \]
   where \( \Gamma(.) \) is the Gamma function, \( t \geq t_0 \) and \( n \) is a positive integer. Particularly, when \( 0 < \epsilon \leq 1 \), the above definition modified as
   \[
   D^\alpha_t g(t) = \frac{1}{\Gamma(1-\epsilon)} \int_{t_0}^t \frac{g'(w)}{(t-w)^{\epsilon}} dw.
   \]

3. **In the above two cases, the memory is expressed in terms of a singular kernel.**
   Recently, Caputo–Fabrizio fractional derivative is developed to replace the singular kernel with a non-singular kernel, which is given below:
   \[
   \text{CF} D^\alpha_t g(t) = \frac{1}{\Gamma(1-\epsilon)} \int_{t_0}^t g'(w) \exp\left(-\frac{t-w}{1-\epsilon}\right) dw
   \]
   where \( t_0 < 0 \) and \( 0 < \epsilon \leq 1 \).

   Again, any system with the differential equation in the form
   \[
   \frac{dx}{dt} = g(x)
   \]
   can be expressed in terms of the memory kernel function in the following form:
the integration is over suitable domain \([t_0, t]\). The function \(k(\xi - t)\) is the memory-dependent kernel used by different authors in Ghosh et al. (2021). For memoryless system \(k(\xi - t) = \delta(\xi - t)\), where \(\delta(\cdot)\) is the Dirac delta function.

If we consider \(k(\xi - t) = \frac{1}{\Gamma(1 - \epsilon)}(\xi - t)^{\epsilon - 2}\), where \(0 < \epsilon < 1\) the RHS of (3) can be expressed as

\[
\frac{dx(t)}{dt} = -\alpha D_t^{\epsilon - 1}g(x(t)).
\]

Operating Caputo-type functional derivative of order \((\epsilon - 1)\) on both sides, we get

\[
\delta D_t^{\epsilon} (x(t)) = g(x(t))
\]

with initial \(x(0) = X_0\). Using the above methodology system (1) can be converted to the system (2).

In the following, we shall present some preliminary results that are satisfied by Caputo-type differential equations, which will be used to establish different analytical results (Li et al. 2009; Choi et al. 2014).

**Lemma 1** The linear fractional-order differential equation of the form \(D_t^{\epsilon} y(t) = \lambda y(t)\) has solution of the form \(y(t) = y(t_0) E_{\epsilon, \lambda(t - t_0)^{\epsilon}}\) where \(0 < \epsilon < 1\), \(E_t\) is the Mittag-Leffler function with parameter \(\epsilon\) in the domain \([t_0, t]\).

**Lemma 2** The Caputo differential operator is linear, i.e.,

\[
D_t^{\epsilon} (\alpha x(t) + \mu y(t)) = \alpha D_t^{\epsilon} x(t) + \mu D_t^{\epsilon} y(t).
\]

**Lemma 3** For any real-valued continuous function \(x(t)\) with \(t \geq t_0\), the inequality \(\frac{1}{2} D_t^{\epsilon} x(t)^2 \leq D_t^{\epsilon} x(t)\) holds.

**Lemma 4** For any real-valued function \(x(t)\) with \(t \geq t_0\), real constant \(x^*\) and \(0 < \epsilon < 1\) the inequality

\[
D_t^{\epsilon} \left( x(t) - x^* - x^* \ln \left( \frac{x(t)}{x^*} \right) \right) \leq \left( 1 - \frac{x(t)}{x^*} \right) D_t^{\epsilon} x(t)
\]

holds.

**Lemma 5** For any real-valued function \(x(t)\) with \(t \geq t_0\), two arbitrary real constant \(\lambda, \delta\) and \(D_t^{\epsilon} x(t) \leq \lambda x(t) + d \) the following result holds:

\[
x(t) \leq x(t_0) e_\epsilon (\lambda(t-t_0)^\epsilon)^* + d(t-t_0)^\epsilon e_\epsilon (\delta(t-t_0)^\epsilon)^*, t \geq t_0.
\]

**Lemma 6** For the system of \(m\)-dimensional linear fractional-order differential equation \(D_t^{\epsilon} x(t) = B(t)x(t), 0 < \epsilon < 1\) with \(x(t) = (x_1(t), x_2(t), \ldots, x_m(t))\) and \(B = [b_{ij}]_{1 \times 1}\) is the coefficient matrix with eigenvalues \(\lambda_1, \lambda_2, \ldots, \lambda_m\). The stability of the above linear system about the trivial equilibrium point can be verified using the following conditions:

The trivial equilibrium point of the system will be

(a) Locally asymptotically stable if \(\arg(\lambda_j) > \frac{\pi}{2}\) for \(j = 1, 2, 3, \ldots\).

(b) Stable if \(\arg(\lambda_j) \geq \frac{\pi}{2}\) and algebraic multiplicity will be same as the geometric multiplicity for those \(\lambda_j\) for which the sign of equality holds.

(c) Unstable if for at least one \(j\), \(\arg(\lambda_j) < \frac{\pi}{2}\).

Let us denote \(m(\epsilon) = \frac{\epsilon\pi}{2} - \min_{1 \leq j \leq 1} \{\arg(\lambda_j)\}\) then the above system can be easily expressed as: (i) For stable equilibrium point \(m(\epsilon) < 0\) and (ii) For unstable equilibrium point \(m(\epsilon) > 0\). Oscillatory or time steady behavior is highly dependent on the memory parameter when other parameters are fixed.

Here we shall present the existence uniqueness and positivity boundedness conditions of solutions of the proposed fractional-order prey–predator model (2).

### 2.1 Existence and Uniqueness

To prove the existence and uniqueness of the solutions, we have adapted the same methodology which is described in Ghosh et al. (2021); Wang et al. (2019). The existence and uniqueness of the solutions of the fractional-order system (2) are studied in the region \(Y \times (0, T]\) where

\[
Y = \{ (S, I, y) \in R^3; \max(|S|, |I|, |y|) \leq \gamma \}
\]

for some positive constant \(\gamma\).

**Theorem 1** There exists a unique solution \(E(t) \in Y\) of the fractional-order prey–predator model (2) with initial condition \((S_0, I_0, y_0) \in Y\), which is defined for all \(t \geq 0\).

**Proof** To establish the above statement, we shall follow the approach as used by the authors in Ghosh et al. (2021); Boukhouima et al. (2017); Wang et al. (2019). For this purpose, we consider two solutions \(E, \bar{E} \in Y\) and denote the function \(W(E) = (W_1(E), W_2(E), W_3(E))\) such
\[
\begin{align*}
W_1(E) &= \left( \eta + \frac{v(1-\eta)}{v+y} \right) rS - (d + \beta l)S + \frac{\rho l}{\sigma + l} [c(p + by)I - m]y. \\
W_2(E) &= \left[ \beta S - \delta - (p + by)y - \frac{\rho}{\sigma + l} \right] I, \\
W_3(E) &= \left[ (p + by)I - m \right] y.
\end{align*}
\]

Now, \( \| W(E) - W(\bar{E}) \| = | W_1(E) - W_1(\bar{E}) | + | W_2(E) - W_2(\bar{E}) | + | W_3(E) - W_3(\bar{E}) | \leq \eta | S - \bar{S} | + \rho | 1 - \eta | | S - \bar{S} | + \frac{r}{v} (1 - \eta) \chi | S - \bar{S} | \\
&\quad + d | S - \bar{S} | + \beta \chi | S - \bar{S} | + \beta \chi | I - \bar{I} | \\
&\quad + \frac{\rho}{\sigma} | I - \bar{I} | + \beta \chi | S - \bar{S} | \\
&\quad + \beta \chi | I - \bar{I} | + \delta | I - \bar{I} | + p \chi | I - \bar{I} | \\
&\quad + p \chi | y - \bar{y} | + pb \chi | I - \bar{I} | \\
&\quad + p b \chi^2 | y - \bar{y} | + \frac{p}{\sigma} | I - \bar{I} | + cp \chi | I - \bar{I} | \\
&\quad + cp \chi | y - \bar{y} | + cb | I - \bar{I} | \\
&\quad + cb \chi^2 | y - \bar{y} | + m | y - \bar{y} | \\
&\leq \eta r + r | 1 - \eta | + \frac{r}{v} (1 - \eta) \chi + d + 2 \beta \chi | S - \bar{S} | \\
&\quad + \frac{r}{v} (1 - \eta) \chi + p \chi + pb \chi^2 + cp \chi \\
&\quad + cb \chi^2 + \rho | y - \bar{y} | \\
&\quad + (2 \beta \chi + 2 \frac{p}{\sigma} + \delta + p \chi + pb \chi + cp \chi + cb) | I - \bar{I} | \\
&\leq K | E - \bar{E} |,
\end{align*}
\]

where

\[
K = \max \left\{ \eta r + r | 1 - \eta | + \frac{r}{v} (1 - \eta) \chi + d + 2 \beta \chi \frac{r}{v} (1 - \eta) \chi + p \chi + pb \chi^2 + cp \chi + cb \chi^2 + m, \right.
\]

\[
\left. \quad 2 \beta \chi + 2 \frac{p}{\sigma} + \delta + p \chi + pb \chi + cp \chi + cb \chi^2 \right\}.
\]

Thus, \( W(E) \) satisfied the Lipschitz condition, and hence, the proposed fractional-order system has a unique solution in the domain \( \tilde{Y} \).

\[ \square \]

### 2.2 Positivity and Boundedness of the Solutions

In this part, we shall establish the positiveness of the solutions of (2). From (2), we have the following:

\[
\begin{align*}
D_t^\alpha M(t) + \tau M(t) &= \left( \eta + \frac{v(1-\eta)}{v+y} \right) rS - dS - \delta I \\
&\quad - \frac{m}{c} y + \tau S + \frac{\tau}{c} y \\
&= \left( \eta + \frac{v(1-\eta)}{v+y} - d + \tau \right) S + (\tau - \delta) I \\
&\quad + \frac{1}{c} (\tau - m) y
\end{align*}
\]

Now, we choose \( \tau < \min \{ d, \delta, m \} \), then,
where the function $E_c$ is the one parameter Mittag-Leffler function. By Lemma 5 and Corollary 6 in Choi et al. (2014), one gets the following expression:

$$0 \leq M(t) \leq M(0)E_{c,\tau}(-\tau(t)^\gamma) + \frac{r}{d}E_{\alpha,\gamma+1}(-\tau(t)^\gamma),$$

where $\alpha$ is the fractional order of the fractional-order system. This result establishes the boundedness of all solutions of the fractional-order system. Hence, for the fractional-order prey–predator system (4), all its solutions that started in $R^3_+$ are uniformly bounded in the region

$$Y = \{(S, I, y) \in R^3_+, M(t) \leq \frac{r}{d\tau} + \epsilon, \epsilon > 0\}.$$  

\[ \square \]

3 Equilibrium Points and Stabilities

In this section, we shall identify the equilibrium points of the proposed fractional-order system and investigate their nature. The proposed fractional-order system (2) has the following equilibrium points:

1. The species free equilibrium point $E_0(0,0,0)$, which always exists.
2. The predator-free equilibrium point

$$E_1\left(\frac{1}{\beta(l^* + \delta)}, I^*, 0\right),$$

where $l^*$ is a root of the following second degree polynomial:

$$a_0l^2 + a_1l + a_2 = 0$$

where $a_0 = \beta\delta, a_1 = \delta(\beta\sigma + d - r), a_2 = -(\rho + \sigma\delta)(r - d)$. Depending on values of the model parameters the polynomial (5) may have: (i) a unique positive root if $a_1 > 0$; (ii) two positive roots if $a_1 < 0, a_2 > 0$ with $a_1^2 - 4a_2 > 0$; and (iii) no positive root if $a_1 > 0, a_2 < 0$.

3. The coexistence equilibrium point $E^*(S^*, I^*, y^*)$, where

$$S^* = \frac{1}{\beta}\left[\frac{pm(m - cpl^*)}{bc^2l^*^2} + \frac{\rho}{\sigma + l^* + \delta^2}\right], y^* = \frac{m - cpl^*}{cbl^*},$$

and $l^*$ is a root of the following polynomial with fifth degree:

$$a_5l^5 + a_4l^4 + a_3l^3 + a_2l^2 + a_1l + a_0 = 0,$$

where $a_j$ are

$$a_0 = -\beta\delta\rho^2c^3(v - 1),$$
$$a_1 = -\beta\delta\rho c(v - 1)(\beta - r\eta) + \rho\beta c - \beta\rho c$$
$$a_2 = -\beta\delta\sigma c(v - 1)(\beta - r\eta) - \beta\rho c$$
$$a_3 = -\beta\delta\rho c(v - 1)(\beta - r\eta) - \beta\rho c$$
$$a_4 = -\beta\delta\rho c(v - 1)(\beta - r\eta) - \beta\rho c$$
$$a_5 = -\beta\delta\rho c(v - 1)(\beta - r\eta) - \beta\rho c.$$

To find the nature of equilibrium points, we have to find the characteristic roots of the corresponding linear part. For this purpose, we give the transformation about any characteristic roots of the corresponding linear part. For this purpose, we give the transformation about any characteristic roots of the corresponding linear part. For this purpose, we give the transformation about any characteristic roots of the corresponding linear part. For this purpose, we give the transformation about any characteristic roots of the corresponding linear part.

4. The intrinsic growth of the system is a locally asymptotically stable provided that $d > r$.

Proof The characteristic matrix of the fractional-order system is...
The predator-free equilibrium point $E_r$ is stable if $d > r$ and $0 < \epsilon < 1$. Hence, the trivial equilibrium point cannot be changed using the memory effect. If the trivial equilibrium point becomes unstable then the predator-free equilibrium point will generate. In the next theorem, we shall investigate the stability of this equilibrium point.

Another important result is that the system may experience the transcritical bifurcation at $r = d$ for the memoryless system.

**Theorem 4** The predator-free equilibrium point $E_r(S^*, I^*, 0)$ of the fractional-order model (2) is locally asymptotically stable provided $\frac{cp}{I^*} < m$ and $r + \frac{\rho}{\sigma + I^*} > \frac{\epsilon}{2}$.

**Proof** Here the characteristic matrix is

$$L(E_r) = \begin{pmatrix} r - d & \frac{\rho}{\sigma} & 0 \\ 0 & -\delta - \frac{\rho}{\sigma} & 0 \\ 0 & 0 & -m \end{pmatrix}.$$ 

The associated eigenvalues of the $L(E_r)$ are $\lambda_1 = r - d$, $\lambda_2 = -\delta - \frac{\rho}{\sigma}$, and $\lambda_3 = -m$. So, by using Matignon criteria for stability of fractional-order differential equations we have $|\arg(\lambda_2)| = \pi > \frac{\epsilon\pi}{2}$ and $|\arg(\lambda_1)| = \pi > \frac{\epsilon\pi}{2}$, provided that $d > r$ and $0 < \epsilon < 1$. Hence, the trivial equilibrium point $E_0$ is local asymptotically stable if the natural death rate of the susceptible prey is larger than the intrinsic growth rate.

Finally, stability of interior equilibrium point $E^*$ is investigated. The characteristic equation corresponding to the Jacobian matrix $L(E^*)$ is given by:

$$x^2 + R_1 x^2 + R_2 x + R_3 = 0$$

where $R_1 = -(l_{11} + l_{22} + l_{33}), R_2 = l_{11}(l_{22} + l_{33}) + l_{22}l_{33} - l_{23}l_{32} - l_{12}l_{21}, R_3 = l_{11}l_{23}l_{32} + l_{12}l_{21}l_{33} - l_{11}l_{22}l_{33}$.

The local stability of the $E^*$ depends on values of $R_1, R_2$ and $R_3$. Using Routh–Hurwitz criterion, the sign of real part of the equations can be easily determine (Ahmed et al. 2006). The equation (8) has all negative real roots if $R_1 > 0, R_2 > 0, R_3 > 0$ and $R_1 R_2 > R_3$ then $m(\epsilon) < 0$ but if $R_1 R_2 < R_3$ then the stability–instability can be easily controlled using the memory effect. Using the results as stated in Ahmed et al. (2006).

**Theorem 5** The local stability of persistence equilibrium point $E^*$ is determined if one of the following is hold:

1. $Q(R_1, R_2, R_3) > 0, R_1 > 0$, $R_3 > 0$ and $R_1 R_2 > R_3$.
2. $Q(R_1, R_2, R_3) < 0, R_1 > 0$, $R_2 > 0$ and $R_1 R_2 = R_3$.

where $0 < \epsilon < 1$ and $Q(R_1, R_2, R_3)$ is the discriminant of (8) which as follows:

$$Q(R_1, R_2, R_3) = 18 R_1 R_2 R_3 + (R_1 R_2)^2 - 4 R_3 R_1^2 - 4 R_2^3 - 27 R_3^2.$$  

(9)

Next, we shall investigate the existence of Hopf bifurcation of the fractional-order system considering the memory parameter (\epsilon) as the bifurcation parameter. It is obvious that if the conditions stated in Theorem 5 are not satisfied then the system losses stability through the generation of periodic solutions. Since if $R_i > 0$, $i = 1, 2, 3$ and $R_1 R_2 - R_3 < 0$ then roots of the equation (8) have negative real part and two will be complex conjugate with positive real part. In this situation suppose the roots are $\lambda_1 = -\lambda_2$ and $\lambda_{2,3} = \pm \beta_2$ with $\lambda_{1,2} > \beta_2 > 0$ then $m_1(\epsilon) < 0$ and $m_{2,3}(\epsilon) = \frac{\pi\epsilon}{2} - \arg(\lambda_{2,3}) = \frac{\pi\epsilon}{2} - \tan^{-1}(\beta_2/\beta_2)$. From the expression $m_{2,3}$ one can obtain a fixed value $\epsilon_H = \frac{2}{\pi} \tan^{-1}(\beta_2/\beta_2)$ such that $m_{2,3}(\epsilon) < 0$ accordingly as $\epsilon = \epsilon_H$. Thus the system will experiences Hopf bifurcation when it crosses the critical value $\epsilon = \epsilon_H$ as the transversality condition $\frac{d(m_{2,3}(\epsilon))}{d\epsilon} = 0$ is
satisfied. These discussions can be summarized in terms of the following theorem:

**Theorem 6** Interior equilibrium point of the system (2) will experience Hopf bifurcation for \( R_1 > 0, R_2 > 0, R_3 > 0 \) and \( R_1R_2 > R_3 \) when the memory parameter \( (\epsilon) \) will cross the critical value \( \epsilon = e^{H/\delta} \).

### 4 Numerical Simulation

In this section, we shall numerically verify the effect of memory parameter \( (\epsilon) \) and other model parameters on the model dynamics. For this purpose, we have taken the model parameters as shown in Table 1 and different values of \( 0 < \epsilon \leq 1 \). To study the dynamics of the ordinary differential equation model, we have drawn a complete bifurcation diagram considering \( r \) as the bifurcation diagram (see Fig. 1). It is clear from the figure that the system experiences transcritical bifurcation two times, one for generation of planar equilibrium point \( E_1 \) and another for generation of \( E^* \) equilibrium point through stability exchange of \( E_0, E_1 \), respectively. The full phase portrait corresponding to Fig. 1 is presented in Fig. 2. It is clear from Fig. 2a for lower values of \( r(r<0.128) \) all the species go to extinction. Due to the extinction of all species, this situation is harmful to the ecological system. Now, increasing the value of \( r \) (0.128 < \( r < 0.585 \)), we observe that one stable predator-free equilibrium point arises and \( E_0 \) exchange its stability (see Fig. 2b). The situation is also harmful biologically since only prey survives. Again, increasing the value of \( r(r > 0.585) \), the stable interior equilibrium point \( E^* \) generates with \( E_1 \) exchange its stability (see Fig. 2c). All the species in the ecosystem survive simultaneously in this situation, which is biologically significant. For these values of the parameters, the memory effect will not affect the stability of the equilibrium points only the time of reaching time to the equilibrium point will increase. To find the periodic solutions of the system, we enhance the prey birth rate as well as the maximum medical resource to \( r = 0.72 \) and \( \rho = 1.9 \); then, we observed that the interior equilibrium point becomes an unstable spiral (stable limit cycle) from stable spiral (see Fig. 3).

To study the memory effect, we have drawn phase portraits for different values of memory parameter \( \epsilon \) (see Fig. 3e). Figure 3e shows the limit cycle becomes smaller with the decrease of \( \epsilon \). We have verified that the system loses the periodicity and becomes stable for \( \epsilon < 0.88 \) (see Fig. 4). Thus, the memory effect is stabilizing the system when the system is showing unstable behavior. Now, we increased the maximum medical resource \( \rho \) to 2.9 form 1.9 and observed that the system gives two periodic solutions (see Fig. 5). Now, we introduce the memory effect by changing the parameter \( \epsilon \). Figure 5e shows, firstly, the two limit cycle collide on a single limit cycle with the decrease of \( \epsilon \), and the limit cycle becomes smaller, and finally, the system becomes stable for \( \epsilon < 0.7 \) (see Fig. 6). Thus, the memory effect stabilizes the system. Again, we increased the maximum medical resource \( \rho \) to 3.9 from 2.9 and found that the system displayed chaotic behavior (see Fig. 7). Now, we introduce the memory effect changing the parameter \( \epsilon \), and the system goes to stable mode from the earlier process but here system is stable for very low values of \( \epsilon \) (see Fig. 8). Based on the above discussion, a low memory effect will stabilize a system when it shows a periodic solution, but a strong memory effect will stabilize a system when it displays two periods or chaos. Biologically this result is highly sensitive because the system with chaotic behavior (extinction of species) can be stabilized.
Fig. 3 Time series and phase portrait for $r = 0.72$, $\beta = 1.9$ and other parameters are given in Table 1.

Fig. 4 Time series corresponding to the phase portrait Fig. 3(e) for different values of memory parameter ($\epsilon$).
Fig. 5 Time series and phase portrait for $r = 0.72$, $\rho = 3.9$ other parameters are given in Table 1

Fig. 6 Time series corresponding to the phase portrait Fig. 5(e) for different values of memory parameter ($\epsilon$)
Fig. 7 Time series and phase portrait for $r = 0.72, \rho = 3.9$ other parameters are given in Table 1.

Fig. 8 Time series corresponding to the phase portrait Fig. 7(e) for different values of memory parameter ($\epsilon$)
by introducing a high memory effect, i.e., previous memory of the species.

5 Conclusions

In our study, we examined the effect of fear, treatment, and cooperation on assessing an eco-epidemiological model with memory dependence. We have studied the existence, positivity, and boundedness of the solutions of the fractional-order system. The ordinary system experiences transcritical bifurcation considering the prey birth rate as the bifurcation parameter. Transcritical bifurcation is not directly affected by the memory effect. In the presence of the memory effect, the fractional-order system undergoes Hopf bifurcation. The ordinary system exhibits chaotic, period-doubling, or periodic solutions, and memory effects can stabilize these solutions. The numerical results of the simulation show that if the equilibrium point is an unstable node or saddle, the memory effect cannot change its behavior. However, the memory effect can stabilize the system when it is an unstable spiral (including limit cycles or chaotic behavior). The oscillatory behavior (like one or two periodic or chaotic) can be changed to a stable spiral, increasing the memory effect. Fear of predators, hunting cooperation, and treatment of infected prey with memory effects all play a crucial role in preserving biodiversity.

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Data availability All data generated or analyzed during this study are included in this article.

Declarations

Conflicts of interest There is no conflict of interest in this research work as declared by authors.

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