Further hardness results on the
generalized connectivity of graphs∗

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Abstract

The generalized $k$-connectivity $\kappa_k(G)$ of a graph $G$ was introduced by Chartrand et al. in 1984, which is a nice generalization of the classical connectivity. Recently, as a natural counterpart, Li et al. proposed the concept of generalized edge-connectivity for a graph. In this paper, we determine the computational complexity of the generalized connectivity and generalized edge-connectivity of a graph. Two conjectures are also proved to be true.

Keywords: connectivity, edge-connectivity, Steiner tree, internally disjoint trees, edge-disjoint trees, generalized connectivity, generalized edge-connectivity, complexity, polynomial-time algorithm, $\mathcal{NP}$-complete.

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1 Introduction

All graphs considered in this paper are undirected, finite and simple. We refer to the book [1] for graph theoretical notation and terminology not described here. The generalized connectivity of a graph $G$, introduced by Chartrand et al. in [2], is a natural and nice generalization of the concept of the standard (vertex-)connectivity. For a graph $G(V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, an $S$-Steiner tree or a Steiner tree connecting $S$ (or simply, an $S$-tree) is such a subgraph $T(V', E')$ of $G$ that is a tree with

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for any graph \( G \), we have polynomial-time algorithms to get the connectivity \( \kappa(G) \) and the edge-connectivity \( \lambda(G) \). A natural question is whether there is a
polynomial-time algorithm to get the \( \kappa_3(G) \) or \( \lambda_3(G) \), or more generally \( \kappa_k(G) \) or \( \lambda_k(G) \).

In [12], the authors described a polynomial-time algorithm to decide whether \( \kappa_3(G) \geq \ell \).

**Theorem 1.** [12] Given a fixed positive integer \( \ell \), for any graph \( G \) the problem of deciding whether \( \kappa_3(G) \geq \ell \) can be solved by a polynomial-time algorithm.

As a continuation of their investigation, S. Li and X. Li later turned their attention to the general \( \kappa_k \) and obtained the following results in [11].

**Theorem 2.** [11] For two fixed positive integers \( k \) and \( \ell \), given a graph \( G \), a \( k \)-subset \( S \) of \( V(G) \), the problem of deciding whether there are \( \ell \) internally disjoint trees connecting \( S \) can be solved by a polynomial-time algorithm.

**Theorem 3.** [11] For any fixed integer \( k \geq 4 \), given a graph \( G \), a \( k \)-subset \( S \) of \( V(G) \) and an integer \( \ell \) (\( 2 \leq \ell \leq n - 2 \)), deciding whether there are \( \ell \) internally disjoint trees connecting \( S \), namely deciding whether \( \kappa(S) \geq \ell \), is \( \mathcal{NP} \)-complete.

**Theorem 4.** [11] For any fixed integer \( \ell \geq 2 \), given a graph \( G \) and a subset \( S \) of \( V(G) \), deciding whether there are \( \ell \) internally disjoint trees connecting \( S \), namely deciding whether \( \kappa(S) \geq \ell \), is \( \mathcal{NP} \)-complete.

In Theorem 3, for \( k = 3 \) the complexity problem was not solved in [11], and the problem is still open. So, S. Li in her Ph.D. thesis [8] conjectured that it is \( \mathcal{NP} \)-complete.

**Conjecture 1.** [8] Given a graph \( G \) and a \( 3 \)-subset \( S \) of \( V(G) \) and an integer \( \ell \) (\( 2 \leq \ell \leq n - 2 \)), deciding whether there are \( \ell \) internally disjoint trees connecting \( S \), namely deciding whether \( \kappa(S) \geq \ell \), is \( \mathcal{NP} \)-complete.

Since \( \kappa_k(G) = \min \{ \kappa(S) \} \), where the minimum is taken over all \( k \)-subsets \( S \) of \( V(G) \), S. Li also considered the complexity of the problem of deciding whether \( \kappa_k(G) \geq \ell \), and conjectured that it is \( \mathcal{NP} \)-complete.

**Conjecture 2.** [8] For a fixed integer \( k \geq 3 \), given a graph \( G \) and an integer \( \ell \) (\( 2 \leq \ell \leq n - 2 \)), the problem of deciding whether \( \kappa_k(G) \geq \ell \), is \( \mathcal{NP} \)-complete.

In this paper, we will confirm that these two conjectures are true.

For the generalized \( k \)-edge-connectivity \( \lambda_k(G) \), it is also natural to consider its computational complexity problem: for any two positive integers \( k \) and \( \ell \), given a \( k \)-subset \( S \) of \( V(G) \), is there a polynomial-time algorithm to determine whether \( \lambda(S) \geq \ell \) ?
If both \( k \) and \( \ell \) are fixed integers, we will reduce it to the problem in Theorem 2 and prove that there is a polynomial-time algorithm to determine whether \( \lambda_k(G) \geq \ell \). If one of \( k \) and \( \ell \) is not fixed, then the problem turns out to be \( \mathcal{NP} \)-complete.

The rest of this paper is organized as follows. In next section we give the proofs of Conjectures 1 and 2. Section 3 contains the hardness results of the generalized edge-connectivity.

## 2 Proofs of the two conjectures

In this section, we focus on solving Conjectures 1 and 2. In order to show that these conjectures are correct, we first introduce a basic \( \mathcal{NP} \)-complete problem and a new problem.

**3-DIMENSIONAL MATCHING (3-DM):** Given three sets \( U, V \) and \( W \) with \( |U| = |V| = |W| \), and a subset \( T \) of \( U \times V \times W \), decide whether there is a subset \( M \) of \( T \) with \( |M| = |U| \) such that whenever \((u, v, w)\) and \((u', v', w')\) are distinct triples in \( M \), we have \( u \neq u' \), \( v \neq v' \), and \( w \neq w' \)?

**Problem 1:** Given a tripartite graph \( G = (V, E) \) with three partitions \((\overline{U}, \overline{V}, \overline{W})\), and \(|\overline{U}| = |\overline{V}| = |\overline{W}| = q\), decide whether there is a partition of \( V \) into \( q \) disjoint 3-sets \( V_1, V_2, \ldots, V_q \) such that every \( V_i = \{v_{i1}, v_{i2}, v_{i3}\} \) satisfies that \( v_{i1} \in \overline{U}, v_{i2} \in \overline{V}, v_{i3} \in \overline{W} \), and \( G[V_i] \) is connected?

By reducing 3-DM to Problem 1, we can get the following result.

**Lemma 1.** Problem 1 is \( \mathcal{NP} \)-complete.

**Proof.** It is easy to see that Problem 1 is in \( \mathcal{NP} \) since given a partition of \( V(G) \) into \( q \) disjoint 3-sets \( V_i (i = 1, 2, \ldots, q) \), one can check in polynomial time that every \( V_i = \{v_{i1}, v_{i2}, v_{i3}\} \) satisfies that \( v_{i1} \in \overline{U}, v_{i2} \in \overline{V}, v_{i3} \in \overline{W} \), and \( G[V_i] \) is connected.

We now prove that 3-DM is polynomially reducible to this problem.

Given three sets \( U, V \) and \( W \) with \( |U| = |V| = |W| = n \), and a subset \( T \) of \( U \times V \times W \). Let \( T = \{T_1, T_2, \ldots, T_m\} \). We will construct a tripartite graph \( G[\overline{U}, \overline{V}, \overline{W}] \) with \(|\overline{U}| = |\overline{V}| = |\overline{W}| = q\) such that the desired partition exists for \( G \) if and only if there is a subset \( M \) of \( T \) with \( |M| = n \) and whenever \((u, v, w)\) and \((u', v', w')\) are distinct triples in \( M \), we have \( u \neq u' \), \( v \neq v' \) and \( w \neq w' \).

For each \( T_i = (u_i, v_i, w_i) \), we add 18 new vertices \( V_i^0 = \{t_{i1}, t_{i2}, \ldots, t_{i18}\} \) and 26 edges \( E_i^0 \) which are shown in Figure 1. Thus \( G[\overline{U}, \overline{V}, \overline{W}] \) is defined by

\[
\overline{U} = U \cup \{t_{i3}, t_{i6}, t_{i7}, t_{i10}, t_{i13}, t_{i16} : 1 \leq i \leq m\}
\]
\[ \overline{V} = V \cup \{t_{1i}, t_{i4}, t_{i9}, t_{i12}, t_{i14}, t_{i17} : 1 \leq i \leq m\} \]
\[ \overline{W} = W \cup \{t_{2i}, t_{i5}, t_{i8}, t_{i11}, t_{i15}, t_{i18} : 1 \leq i \leq m\} \]
\[ V = \overline{U} \cup \overline{V} \cup \overline{W}, \quad E = \bigcup_{i=1}^{m} E_i^0. \]

Figure 1. Graphs for Lemma 1.

Note that \(|V| = 3n + 18m\), \(|E| = 26m\). Thus this instance can be constructed in polynomial time from a 3-DM instance. Now that \(q = n + 6m\).

If there is a subset \(M\) of \(T\) with \(|M| = n\), and whenever \((u, v, w)\) and \((u', v', w')\) are distinct triples in \(M\) we have \(u \neq u', v \neq v',\) and \(w \neq w'\), then the corresponding partition \(V = V_1 \cup V_2 \cup \ldots \cup V_q\) is given by taking \(\{u_i, t_{i1}, t_{i2}\}, \{v_i, t_{i7}, t_{i8}\}, \{w_i, t_{i13}, t_{i14}\}, \{t_{i3}, t_{i4}, t_{i5}\}, \{t_{i9}, t_{i10}, t_{i11}\}, \{t_{i15}, t_{i16}, t_{i17}\}, \{t_{i6}, t_{i12}, t_{i18}\}\) from the vertices of \(V_i^0 \cup T_i\) whenever \(T_i = (u_i, v_i, w_i)\) is in \(M\), and by taking \(\{t_{i1}, t_{i2}, t_{i3}\}, \{t_{i4}, t_{i5}, t_{i6}\}, \{t_{i7}, t_{i8}, t_{i9}\}, \{t_{i10}, t_{i11}, t_{i12}\}, \{t_{i13}, t_{i14}, t_{i15}\}, \{t_{i16}, t_{i17}, t_{i18}\}\) from the vertices of \(V_i^0\) whenever \(T_i = (u_i, v_i, w_i)\) is not in \(M\).

Since \(|M| = n\), \(|T \setminus M| = m - n\), we can find \(7n + 6(m - n) = n + 6m = q\) partition sets, each set consists of three vertices which belong to \(\overline{U}, \overline{V}, \overline{W}\), respectively, and they induce a connected subgraph.

Conversely, let \(V_1, V_2, \ldots, V_q\) be the desired partition of \(V(G)\). In the following, we call a 3-set \(\{u, v, w\}\) a partition set, if there is some \(j\) such that \(\{u, v, w\} = V_j\). Then we choose \(T_i \in M\) if \(\{t_{i6}, t_{i12}, t_{i18}\}\) is a partition set.

Now we claim that \(|M| = n\), and whenever \((u, v, w)\) and \((u', v', w')\) are distinct triples in \(M\) we have \(u \neq u', v \neq v',\) and \(w \neq w'\). Indeed, let \(T_i = (u_i, v_i, w_i)\). If \(\{t_{i6}, t_{i12}, t_{i18}\}\) is a partition set, then either \(\{t_{i3}, t_{i4}, t_{i5}\}\) and \(\{u_i, t_{i1}, t_{i2}\}\) are the partition sets, or \(\{t_{i1}, t_{i3}, t_{i5}\}\) and \(\{u_i, t_{i2}, t_{i4}\}\) are the partition sets. In either cases, \(u_i\) must belong to a partition set with the other two elements belong to \(V_i^0\). Similar thing is true for \(v_i\) and \(w_i\). If \(\{t_{i6}, t_{i12}, t_{i18}\}\) is not a partition set, then \(\{t_{i6}, t_{i11}, t_{i12}\}\) or \(\{t_{i4}, t_{i5}, t_{i6}\}\) is a partition set. But \(\{t_{i6}, t_{i11}, t_{i12}\}\) can not be a partition set. If so, then \(\{t_{i8}, t_{i10}, v_i\}, \{t_{i7}, t_{i8}, t_{i9}\}\) or \(\{t_{i7}, t_{i8}, t_{i1}\}\) must be a partition set, and no matter in which cases, \(t_{i6}\) or \(t_{i10}\) can not be in a
partition set. Thus only \( \{t_{i_4}, t_{i_5}, t_{i_6}\} \) is a partition set. Similarly, \( \{t_{i_1}, t_{i_2}, t_{i_3}\}, \{t_{i_7}, t_{i_8}, t_{i_9}\}, \{t_{i_{10}}, t_{i_{11}}, t_{i_{12}}\}, \{t_{i_{13}}, t_{i_{14}}, t_{i_{15}}\}, \{t_{i_{16}}, t_{i_{17}}, t_{i_{18}}\} \) must be partition sets. Then \( u_i, v_i, w_i \) can not be in any partition sets with some vertices in \( V^0_i \).

If \( T_i = (u_i, v_i, w_i) \) and \( T_j = (u_j, v_j, w_j) \) are distinct triples in \( M \), then \( u_i \) is in a partition set with the other two elements in \( V^0_i \), and \( u_j \) is in a partition set with the other two elements in \( V^0_j \), and thus \( u_i \neq u_j \), and so do \( v_i \neq v_j \) and \( v_i \neq v_j \). Assume that \( |M| = \ell \). If \( T_i \in M \), then there are 7 partition sets in \( V^0_i \cup T_i \). If \( T_i \notin M \), then there are 6 partition sets in \( V^0_i \). Since \( 7\ell + 6(m - \ell) = q = n + 6m, \ell = n \), that is \( |M| = n \). The proof is now complete.

Now we prove that Conjecture 1 is true by reducing Problem 1 to it.

**Theorem 5.** Given a graph \( G \), a 3-subset \( S \) of \( V(G) \) and an integer \( \ell \) \((2 \leq \ell \leq n - 2)\), the problem of deciding whether \( G \) contains \( \ell \) internally disjoint trees connecting \( S \) is \( NP \)-complete.

**Proof.** It is easy to see that this problem is in \( NP \).

Let \( G \) be a tripartite graph with partition \((U, V, W)\) and \(|U| = |V| = |W| = q\). We will construct a graph \( G' \), and a 3-subset \( S \) and an integer \( \ell \) such that there are \( \ell \) internally disjoint trees connecting \( S \) in \( G' \) if and only if \( G \) contains a partition of \( V(G) \) into \( q \) disjoint sets \( V_1, V_2, \ldots, V_q \) each having three vertices, such that every \( V_i = \{v_{i_1}, v_{i_2}, v_{i_3}\} \) satisfies that \( v_{i_1} \in U, v_{i_2} \in V, v_{i_3} \in W \), and \( G[V_i] \) is connected.

We define \( G' \) as follows:

- \( V(G') = V(G) \cup \{a, b, c\} \);
- \( E(G') = E(G) \cup \{au : u \in U\} \cup \{bv : v \in V\} \cup \{cw : w \in W\} \).

Then we define \( S = \{a, b, c\} \), and \( \ell = q \).

If there are \( q \) internally disjoint trees connecting \( S \) in \( G' \), then, since \( a, b \) and \( c \) all have degree \( q \), each tree contains a vertex from \( U \), a vertex from \( V \) and a vertex from \( W \), and they induce a connected subgraph. Since these \( q \) trees are internally disjoint, they form a partition of \( V(G) \).

Conversely, if \( V_1, V_2, \ldots, V_q \) is a partition of \( V(G) \) each having three vertices, such that every \( V_i = \{v_{i_1}, v_{i_2}, v_{i_3}\} \) satisfies that \( v_{i_1} \in U, v_{i_2} \in V, v_{i_3} \in W \), and \( G[V_i] \) is connected, then let \( T_i \) be the spanning tree of \( G[V_i] \) together with the edges \( av_{i_1}, bv_{i_2}, cv_{i_3} \), where \( V_i = \{v_{i_1}, v_{i_2}, v_{i_3}\} \). It is easy to check that \( T_1, T_2, \ldots, T_q \) are the desired internally disjoint trees connecting \( S \).

Now from Theorem 3 and Theorem 5 if \( k \geq 3 \) is a fixed integer and \( \ell \) is not a fixed
integer, the problem of deciding whether \( \kappa(S) \geq \ell \) is \( \mathcal{NP} \)-complete. Since \( \kappa_k(G) = \min \{ \kappa(S) : S \subseteq V(G), |S| = k \} \), we have that the problem of deciding whether \( \kappa_k(G) \geq \ell \) is as hard as the problem of deciding whether \( \kappa(S) \geq \ell \). Moreover, the problem of deciding whether \( \kappa_k(G) \geq \ell \) is in \( \mathcal{NP} \), and so it is \( \mathcal{NP} \)-complete. This shows that Conjecture 2 is true.

**Theorem 6.** For a fixed integer \( k \geq 3 \), given a graph \( G \) and an integer \( 2 \leq \ell \leq n - 2 \), the problem of deciding whether \( \kappa_k(G) \geq \ell \) is \( \mathcal{NP} \)-complete.

### 3 Hardness results on generalized edge-connectivity

In this section we consider the computational complexity of the generalized edge-connectivity \( \lambda_k(G) \). Since \( \lambda_k(G) = \min \{ \lambda(S) : S \subseteq V(G), |S| = k \} \), we first consider \( \lambda(S) \), and get the following result.

**Theorem 7.** Given two fixed positive integers \( k \) and \( \ell \), for any graph \( G \) the problem of deciding whether \( \lambda_k(G) \geq \ell \) can be solved by a polynomial-time algorithm.

**Proof.** Given a connected graph \( G \) of order \( n \) and a \( k \)-subset \( S \) of \( V(G) \). Let \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and \( E(G) = \{e_1, e_2, \ldots, e_m\} \). We will construct a graph \( G' \) and a \( k \)-subset \( S' \) of \( V(G') \) such that there are \( \ell \) edge-disjoint trees connecting \( S \) in \( G \) if and only if there are \( \ell \) internally disjoint trees connecting \( S' \) in \( G' \).

We define \( G' \) as follows:

- \( V(G') = V(G) \cup V(L(G)) = \{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_m\} \);
- \( E(G') = \{e_ie_j : e_i, e_j \in E(L(G))\} \cup \{v_ie_j : e_j \text{ is incident to } v_i \text{ in } G\} \);

where \( L(G) \) is the line graph of \( G \). We define \( S' = S \).

If there are \( \ell \) edge-disjoint trees connecting \( S \) in \( G \), say \( T_1, T_2, \ldots, T_\ell \). First for each tree \( T_i \) \((1 \leq i \leq \ell)\), we replace every edge \( e_j = v_jv_{j'} \) by a path \( v_{j_1}e_{j_1}v_{j_2} \). The obtained graph \( T'_i \) now is a tree in \( G' \). Clearly, the trees \( T'_1, T'_2, \ldots, T'_\ell \) are \( \ell \) edge-disjoint trees connecting \( S' = S \) in \( G' \). Consider the tree \( T'_i \). If there is a vertex \( v \in V(T'_i) \setminus S \) such that its neighbors in \( T'_i \) are \( e_{i_1}, e_{i_2}, \ldots, e_{i_p} \), then we delete the vertex \( v \) and its incident edges \( v_{i_1}, v_{i_2}, \ldots, v_{i_p} \), add a path \( e_{i_1}e_{i_2} \ldots e_{i_p} \). We do this operation for all the vertex \( v \in V(T'_i) \setminus S \) for \( 1 \leq i \leq \ell \). The resulting trees are denoted by \( T''_1, T''_2, \ldots, T''_\ell \). It is easy to check that they are internally disjoint trees connecting \( S' \) in \( G' \).

Conversely, if there are \( \ell \) internally disjoint trees \( T_1, T_2, \ldots, T_\ell \) connecting \( S' \) in \( G' \). Consider any tree \( T_i \) \((1 \leq i \leq \ell)\). If there is an edge \( e_{j_1}e_{j_2} \) in \( E(T_i) \), by the definition of \( E(G') \), \( e_{j_1} \) and \( e_{j_2} \) are adjacent in \( G \) and hence they have a common vertex \( v_j \) in \( G \). Note
that \( v_j e_{j_1}, v_j e_{j_2} \) are also edges of \( G' \). Thus we replace the edge \( e_{j_1} e_{j_2} \) by a path \( e_{j_1} v_j e_{j_2} \). We do this for all the edges of this type in \( T_i \). The resulting connected graph is denoted by \( G_i \). Now there is no such edge \( e_i e_j \) in \( G_i \) and \( d_{G_i}(e) \leq 2 \). If \( d_{G_i}(e) = 1 \), we just delete it. If \( d_{G_i}(e) = 2 \), there are two vertices \( v_{j_1} \) and \( v_{j_2} \) adjacent with \( e \), where \( v_{j_1} \) and \( v_{j_2} \) are the endpoints of \( e \) in \( G \), so we delete the vertex \( e \) and add an edge \( v_{j_1} v_{j_2} \); then the obtained graph \( G'_i \) is a connected graph with \( S \subseteq V(G'_i) \). Let \( T'_i \) be the spanning tree of \( G'_i \). It is easy to check that \( T'_1, T'_2, \ldots, T'_\ell \) are \( \ell \) edge-disjoint trees connecting \( S \) in \( G \).

From the above reduction, if we want to know whether there are \( \ell \) edge-disjoint trees connecting \( S \) in \( G \), we can construct a graph \( G' \), and decide whether there are \( \ell \) internally disjoint trees connecting \( S' \) in \( G' \). By Theorem 2, since \( k \) and \( \ell \) are fixed, the problem of deciding whether there are \( \ell \) internally disjoint trees connecting \( S \) can be solved by a polynomial-time algorithm. Therefore, the problem of deciding whether there are \( \ell \) edge-disjoint trees connecting \( S \) can be solved by a polynomial-time algorithm. The proof is complete.

Now we consider the problem of deciding whether \( \lambda_k(G) \geq \ell \), for \( k \geq 3 \) a fixed integer but \( \ell \) a not fixed integer. At first, we denote the case when \( k = 3 \) by Problem 2.

**Problem 2:** Given a graph \( G \), a 3-subset \( S \) of \( V(G) \), and an integer \( \ell \) (\( 2 \leq \ell \leq n - 2 \)), decide whether there are \( \ell \) edge-disjoint trees connecting \( S \), that is, \( \lambda(S) \geq \ell \)?

Notice that the reduction from Problem 1 to the problem in Theorem 2 can also be used to be the reduction from Problem 1 to Problem 2 since the \( q \) internally disjoint trees connecting \( S \) in \( G' \) are also \( q \) edge-disjoint trees connecting \( S \) in \( G' \). On the other hand, if there are \( q \) edge-disjoint trees connecting \( S \) in \( G' \), since the degrees of \( a, b \) and \( c \) are \( q \) we have that each tree \( T_i \) contains one vertex \( u_i \) in \( \overline{U} \), one vertex \( v_i \) in \( \overline{V} \), and one vertex \( w_i \) in \( \overline{W} \), then \( \{u_i, v_i, w_i\} \) (\( 1 \leq i \leq q \)) constitute a partition of \( V(G) \), and each induces a connected subgraph. Thus the following lemma holds.

**Lemma 2.** Problem 2 is \( \mathcal{NP} \)-complete.

Now we show that for a fixed integer \( k \geq 4 \), replacing the 3-subset of \( V(G) \) with a \( k \)-subset of \( V(G) \) in Problem 2, the problem is still \( \mathcal{NP} \)-complete, which can be proved by reducing Problem 2 to it.

**Lemma 3.** For any fixed integer \( k \geq 4 \), given a graph \( G \), a \( k \)-subset \( S \) of \( V(G) \), and an integer \( \ell \) (\( 2 \leq \ell \leq n - 2 \)), deciding whether there are \( \ell \) edge-disjoint trees connecting \( S \), namely deciding whether \( \lambda(S) \geq \ell \), is \( \mathcal{NP} \)-complete.

**Proof.** Clearly, the problem is in \( \mathcal{NP} \).
Given a graph $G$, a 3-subset $S = \{v_1, v_2, v_3\}$ of $V(G)$ and an integer $\ell$ ($2 \leq \ell \leq n - 2$), we construct a graph $G' = (V', E')$ and a $k$-subset $S'$ of $V(G')$ and let $\ell' = \ell$ such that there are $\ell$ edge-disjoint trees connecting $S$ in $G$ if and only if there are $\ell$ edge-disjoint trees connecting $S'$ in $G'$.

We define $G'$ as follows:

- $V'(G') = V(G) \cup \{a^i : 1 \leq i \leq k - 3\} \cup \{a^i_j : 1 \leq i \leq k - 3, 1 \leq j \leq \ell\}$.
- $E'(G') = E(G) \cup \{v_1 a^i_j : 1 \leq i \leq k - 3, 1 \leq j \leq \ell\} \cup \{a^i a^i_j : 1 \leq i \leq k - 3, 1 \leq j \leq \ell\}$.

Let $S' = S \cup \{a^1, a^2, \ldots, a^{k-3}\}$.

It is easy to check that there are $\ell$ edge-disjoint trees connecting $S$ in $G$ if and only if there are $\ell$ edge-disjoint trees connecting $S'$ in $G'$. The proof is complete.

From Lemma 2 and Lemma 3, we obtain the following result.

**Theorem 8.** For any fixed integer $k \geq 3$, given a graph $G$, a $k$-subset $S$ of $V(G)$, and an integer $\ell$ ($2 \leq \ell \leq n - 2$), deciding whether there are $\ell$ edge-disjoint trees connecting $S$, namely deciding whether $\lambda(S) \geq \ell$, is $\mathcal{NP}$-complete.

Similar to the argument in the proof of Conjecture 2, we conclude that if $k \geq 3$ is a fixed integer but $\ell$ is not a fixed integer, the problem of deciding whether $\lambda_k(G) \geq \ell$ is $\mathcal{NP}$-complete. This proves the next result.

**Theorem 9.** For a fixed integer $k \geq 3$, given a graph $G$ and an integer $\ell$ ($2 \leq \ell \leq n - 2$), the problem of deciding whether $\lambda_k(G) \geq \ell$ is $\mathcal{NP}$-complete.

Now we turn to the case that $\ell$ is a fixed integer but $k$ is not a fixed integer. At first, we consider the case $\ell = 2$, and denote it by Problem 3.

**Problem 3:** Given a graph $G$, a subset $S$ of $V(G)$, decide whether there are two edge-disjoint trees connecting $S$, that is $\lambda(S) \geq 2$?

We show that Problem 3 is $\mathcal{NP}$-complete by reducing 3-SAT to it.

**BOOLEAN 3-SATISFIABILITY (3-SAT):** Given a boolean formula $\phi$ in conjunctive normal form with three literals per clause, decide whether $\phi$ is satisfiable?

**Lemma 4.** Problem 3 is $\mathcal{NP}$-complete.

**Proof.** Clearly, Problem 3 is in $\mathcal{NP}$.

Let $\phi$ be an instance of 3-SAT with clauses $c_1, c_2, \ldots, c_m$ and variables $x_1, x_2, \ldots, x_n$. We construct a graph $G_\phi = (V_\phi, E_\phi)$ and define a subset $S$ of $V(G_\phi)$ such that there are two edge-disjoint trees connecting $S$ if and only if $\phi$ is satisfiable.
We define $G_\phi$ as follows:

- $V(G_\phi) = \{\hat{x}_i, x_i, x_i^c : 1 \leq i \leq n\} \cup \{c_i, c_i^c : 1 \leq i \leq m\} \cup \{a, b\}$;
- $E(G_\phi) = \{\hat{x}_i x_i, \hat{x}_i x_i^c : 1 \leq i \leq n\} \cup \{c_i x_i : x_i \in c_j\} \cup \{x_i x_i^c : x_i \in c_j\}$
  $\cup \{x_1 x_1 x_i^c : 2 \leq i \leq n\} \cup \{x_1 x_1^c x_i^c : 2 \leq i \leq n\} \cup \{ab\}$
  $\cup \{ac_i^c, c_i c_i^c : 1 \leq i \leq m\} \cup \{bx_i, bx_i^c : 1 \leq i \leq n\}$.

Now we define $S = \{c_i^c : 1 \leq i \leq m\} \cup \{\hat{x}_i : 1 \leq i \leq n\}$.

Suppose that there are two edge-disjoint trees $T_1$ and $T_2$ connecting $S$. We know that the edge $ab$ cannot be in both $E(T_1)$ and $E(T_2)$, and so assume that $T_1$ does not contain $ab$. Next we claim that $ac_i^c \notin E(T_1)$ for $1 \leq i \leq m$. For otherwise, without loss of generality, let $ac_i^c \in E(T_1)$. Since the degree of $c_i^c$ is 2, $T_1$ contains only one of the edges $ac_i^c$ and $c_i c_i^c$ for $1 \leq i \leq m$. So $c_i c_i^c$ is not in $T_1$. Because $T_1$ is connected, the path from $c_i^c$ to $\hat{x}_j$ in $T_1$ must contain the edges $ac_i^c, a c_i, c_i c_i^c$ for some $1 \leq k \leq m$ and a path from $c_k$ to $\hat{x}_j$. But in this case, the degree of $\hat{x}_j$ in $T_1$ is 2, a contradiction. Therefore, $T_1$ contains all the edges $c_i c_i^c$ ($1 \leq i \leq m$), and for each $c_i$ ($1 \leq i \leq m$), there must exist some $x_j$ in $V(T_1)$ such that $c_i x_j \in E(T_1)$ or $x_j \in V(T_1)$ such that $c_i x_j \in E(T_1)$. As $\hat{x}_i \in S$ ($1 \leq i \leq n$) and the degree of $\hat{x}_i$ is 2, $V(T_1)$ contains only one of the neighbors of $\hat{x}_i$. If $x_i$ is contained in $V(T_1)$, then set $x_i = 1$. Otherwise, set $x_i = 0$. Clearly, we conclude that $\phi$ is satisfiable in this assignment.

On the other hand, suppose that $\phi$ is satisfiable with the assignment $t$. We will find two edge-disjoint trees connecting $S$ as follows.

For each $1 \leq i \leq m$, there must exist a $j$ ($1 \leq j \leq n$) such that $x_j \in c_i$ and $t(x_j) = 1$, or $\overline{x}_j \in c_i$ and $t(x_j) = 0$. We then construct $T_1$ with edge set $\{c_i x_j (or c_i \overline{x}_j), c_i c_i^c : 1 \leq i \leq m\}$. Obviously, $V(T_1)$ can not contain both $x_i$ and $\overline{x}_i$. If none of $x_i$ and $\overline{x}_i$ is in $V(T_1)$, we choose any one of them belonging to $V(T_1)$. Now if $x_i \in V(T_1)$, we add $x_i x_i$ (if $x_i \in V(T_1)$) or $x_i \overline{x}_i$ (if $\overline{x}_i \in V(T_1)$) to $E(T_1)$, for $2 \leq i \leq n$. Otherwise, if $\overline{x}_i \in V(T_1)$, we add $x_i x_i$ (if $x_i \in V(T_1)$) or $\overline{x}_i \overline{x}_i$ (if $\overline{x}_i \in V(T_1)$) to $E(T_1)$, for $2 \leq i \leq n$. Finally, if $\overline{x}_i \in V(T_1)$, we add $\hat{x}_i x_i$ to $E(T_1)$, if $\overline{x}_i \in V(T_1)$, we add $\hat{x}_i \overline{x}_i$ to $E(T_1)$. It is easy to check that the graph $T_1$ is indeed a tree containing $S$. Now let $T_2$ be a tree containing $ab$, $ac_i^c$ for $1 \leq i \leq m$, $bx_j$ and $\hat{x}_j x_j$ (if $\overline{x}_j \in V(T_1)$), $bx_j$ and $\hat{x}_j \overline{x}_j$ (if $x_j \in V(T_1)$). Then we conclude that $T_1$ and $T_2$ are two edge-disjoint trees connecting $S$. The proof is complete.

Now we show that for a fixed integer $\ell \geq 3$, the problem is still $\mathcal{NP}$-complete, which can be proved by reducing Problem 3 to it.

**Lemma 5.** For any fixed integer $\ell \geq 3$, given a graph $G$, a subset $S$ of $V(G)$, deciding whether there are $\ell$ edge-disjoint trees connecting $S$, namely deciding whether $\lambda(S) \geq \ell$, is $\mathcal{NP}$-complete.
Theorem 10. For any fixed integer $\ell \geq 2$, given a graph $G$, a subset $S$ of $V(G)$, deciding whether there are $\ell$ edge-disjoint trees connecting $S$, namely deciding whether $\lambda(S) \geq \ell$, is $\mathcal{NP}$-complete.

Proof. Clearly, the problem is in $\mathcal{NP}$. Let $G$ and a subset $S$ of $V(G)$ be an instance of Problem 3. We will construct a graph $G' = (V', E')$ and a subset $S'$ of $V(G')$ such that there are two edge-disjoint trees connecting $S$ if and only if there are $\ell$ edge-disjoint trees connecting $S'$.

Assume that $S = \{v_1, v_2, \ldots, v_k\}$. To define $G'$, we add $k$ new vertices $\{v'_1, v'_2, \ldots, v'_k\}$. Then for every $v'_i$, we add two paths $v_i v'_1 v'_i$ and $v_i v'_2 v'_i$ for $1 \leq i \leq k$. Finally we add $\ell - 2$ new vertices $\{a_1, a_2, \ldots, a_{\ell-2}\}$, and join each $a_i$ to $v'_1, v'_2, \ldots, v'_k$ for $1 \leq i \leq \ell - 2$. That is,

- $V(G') = V(G) \cup \{v'_1 : 1 \leq i \leq k\} \cup \{v'_1, v'_2 : 1 \leq i \leq k\} \cup \{a_j : 1 \leq j \leq \ell - 2\};$
- $E(G') = E(G) \cup \{v_i v'_1, v_i v'_2 : 1 \leq i \leq k\} \cup \{v_i v'_1, v'_2 v'_i : 1 \leq i \leq k\} \cup \{a_i v'_j : 1 \leq i \leq \ell - 2, 1 \leq j \leq k\}.$

We define $S' = \{v'_1, v'_2, \ldots, v'_k\}$.

Suppose that there are two edge-disjoint trees $T_1$ and $T_2$ connecting $S$ in $G$, let $T'_1$ be the tree containing $T_1$ and the paths $v_i v'_1 v'_i (1 \leq i \leq k)$, $T'_2$ be the tree containing $T_2$ and the paths $v_i v'_2 v'_i (1 \leq i \leq k)$. Then $T'_1$ and $T'_2$ are two edge-disjoint trees connecting $S'$ in $G'$. For each $a_i (1 \leq i \leq \ell - 2)$, let $T_{i+2}'$ be the tree with the edges $a_i v'_j (1 \leq j \leq k)$. Then we find $\ell$ edge-disjoint trees connecting $S'$ in $G'$.

Conversely, suppose that there are $\ell$ edge-disjoint trees $T'_1, T'_2, \ldots, T'_{\ell}$ connecting $S'$ in $G'$. Since the degree of $v'_i$ in $G'$ is $\ell$, $v'_1, v'_2, \ldots, v'_k$ are all leaves in each $T'_j$. If the tree $T'_j$ contains the edge $a_j v'_i$, then $T'_j$ is the star with center vertex $a_j$ and leaves $v'_i (1 \leq i \leq k)$. Otherwise, there is some vertex in $S'$ with its degree at least two, a contradiction. Thus, among $\{T'_1, T'_1, \ldots, T'_{\ell}\}$, there are $\ell - 2$ trees connecting $S'$, each is a star with center vertices $a_j (1 \leq j \leq \ell - 2)$. The remaining two trees must contain the paths $v'_1 v'_1 v_i$ or $v'_1 v'_2 v_i$ for $1 \leq i \leq k$ and two $S$-trees in $G$. Therefore, we find two edge-disjoint trees connecting $S$ in $G$. The proof is complete.

Combining Lemma 4 with Lemma 5 we obtain the following result.

Theorem 10. For any fixed integer $\ell \geq 2$, given a graph $G$, a subset $S$ of $V(G)$, deciding whether there are $\ell$ edge-disjoint trees connecting $S$, namely deciding whether $\lambda(S) \geq \ell$, is $\mathcal{NP}$-complete.

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