A REMARK ON COMPACT KÄHLER MANIFOLDS WITH NEF ANTICANONICAL BUNDLES AND ITS APPLICATIONS

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Abstract. Let $(X, \omega_X)$ be a compact Kähler manifold such that the anticanonical bundle $-K_X$ is nef. We prove that the slopes of the Harder-Narasimhan filtration of the tangent bundle with respect to a polarization of the form $\omega_X^{n-1}$ are semi-positive. As an application, we give a characterization of rationally connected compact Kähler manifolds with nef anticanonical bundles. As another application, we give a simple proof of the surjectivity of the Albanese map.

0. Introduction

Compact Kähler manifolds with semipositive anticanonical bundles have been studied in depth in [CDP12], where a rather general structure theorem for this type of manifolds has been obtained. It is a natural question to find some similar structure theorems for compact Kähler manifolds with nef anticanonical bundles. Obviously, we cannot hope the same structure theorem for this type of manifolds (cf. [CDP12, Remark 1.7]). It is conjectured that the Albanese map is a submersion and that the fibers exhibit no variation of their complex structure (cf. [CH13] for some special cases).

In relation with the structure of compact Kähler manifolds with nef anticanonical bundles, it is conjectured in [Pet12, Conj. 1.3] that the tangent bundles of projective manifolds with nef anticanonical bundles are generically nef. We first recall the notion of generically semipositive (resp. strictly positive) (cf. [Miy87, Section 6])

Definition 0.1. Let $X$ be a compact Kähler manifold and let $E$ be a vector bundle on $X$. Let $\omega_1, \cdots, \omega_{n-1}$ be Kähler classes. Let

$$0 \subset E_0 \subset E_1 \subset \cdots \subset E_s = E \quad \text{(resp. } \Omega_X^1)$$

be the Harder-Narasimhan semistable filtration with respect to $(\omega_1, \cdots, \omega_{n-1})$. We say that $E$ is generically $(\omega_1, \cdots, \omega_{n-1})$-semipositive (resp. strictly positive), if

$$\int_X c_1(E_{i+1}/E_i) \wedge \omega_1 \wedge \cdots \wedge \omega_{n-1} \geq 0 \quad \text{(resp. } > 0) \quad \text{for all } i.$$

If $\omega_1 = \cdots = \omega_{n-1}$, we write the polarization as $\omega_1^{n-1}$ for simplicity.

We rephrase [Pet12, Conj. 1.3] as follows

Conjecture 0.1. Let $X$ be a projective manifold with nef anticanonical bundle. Then $T_X$ is generically $(H_1, \cdots, H_{n-1})$-semipositive for any $(n-1)$-tuple of ample divisors $H_1, \cdots, H_{n-1}$.

In this article, we first give a partial positive answer to this conjecture. More precisely, we prove
Theorem 0.2. Let $X$ be a compact Kähler manifold with nef anticanonical bundle (resp. nef canonical bundle). Then $T_X$ (resp. $\Omega^1_X$) is generically $\omega_X^{n-1}$-semipositive for any Kähler class $\omega_X$.

Remark 0.3. If $X$ is projective and $K_X$ is nef, Theorem 0.2 is a special case of [Miy87, Cor. 6.4]. Here we prove it for arbitrary compact Kähler manifolds with nef canonical bundles. If $-K_X$ is nef, Theorem 0.2 is a new result even for algebraic manifolds.

As an application, we give a characterization of rationally connected compact Kähler manifolds with nef anticanonical bundles.

Proposition 0.4. Let $X$ be a compact Kähler manifold with nef anticanonical bundle. Then the following four conditions are equivalent

(i): $H^0(X, (T^*_X)^\otimes m) = 0$ for all $m \geq 1$.

(ii): $X$ is rationally connected.

(iii): $T_X$ is generically $\omega_X^{n-1}$-strictly positive for some Kähler class $\omega_X$.

(iv): $T_X$ is generically $\omega_X^{n-1}$-strictly positive for any Kähler class $\omega_X$.

Remark 0.5. Mumford has in fact stated the following conjecture which would generalize the first part of Proposition 0.4: for any compact Kähler manifold $X$, the variety $X$ is rationally connected if and only if

$$H^0(X, (T^*_X)^\otimes m) = 0 \quad \text{for all } m \geq 1.$$

We thus prove the conjecture of Mumford under the assumption that $-K_X$ is nef.

As another application, we study the pseudo-effectiveness of $c_2(T_X)$. It is conjectured by Kawamata that

Conjecture 0.6. If $X$ is a compact Kähler manifold with nef anticanonical bundle. Then

$$\int_X (c_2(T_X)) \wedge \omega_1 \wedge \cdots \wedge \omega_{n-2} \geq 0,$$

for all nef classes $\omega_1, \cdots, \omega_{n-2}$.

When $\dim X = 3$, this conjecture was partially solved by [Xie05]. Using Theorem 0.2 and an idea of A. Höring, we prove

Proposition 0.7. Let $(X, \omega_X)$ be a compact Kähler manifold with nef anticanonical bundle. Then

$$\int_X c_2(T_X) \wedge (c_1(-K_X) + \epsilon \omega_X)^{n-2} \geq 0$$

for $\epsilon > 0$ small enough. Moreover, if $X$ is projective and the equality holds for some $\epsilon > 0$ small enough, then after a finite étale cover, $X$ is either a torus or a smooth $\mathbb{P}^1$-fibration over a torus.

As the last application, we study the Albanese map of compact Kähler manifolds with nef anticanonical bundles. It should be first mentioned that the surjectivity of the Albanese map has been studied in depth by several authors. If $X$ is assumed to be projective, the surjectivity of the Albanese map was proved by Q. Zhang in [Zha96]. Still under the assumption that $X$ is projective, [LTZZ10] proved that the Albanese map is equidimensional and all the fibres are reduced. Recently, M. Păun [Pău12] proved the surjectivity for arbitrary compact Kähler manifolds with nef
anticanonical bundles, as a corollary of a powerful method based on a direct image argument. Unfortunately, it is hard to get information for the singular fibers from his proof. Using Theorem 1.2 we give a new proof of the surjectivity for the Kähler case, and prove that the map is smooth outside a subvariety of codimension at least 2.

**Proposition 0.8.** Let $X$ be a compact Kähler manifold with nef anticanonical bundle. Then the Albanese map is surjective, and smooth outside a subvariety of codimension at least 2. In particular, the fibers of the Albanese map are connected and reduced in codimension 1.

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1. **Preparatory lemmas**

The results in this section should be well known to experts. For the convenience of readers, we give an account of the proofs here.

**Lemma 1.1.** Let $(X, \omega)$ be a compact Kähler manifold of dimension $n$ and let $E$ be a torsion free coherent sheaf. Let $D_1, \cdots, D_{n-1}$ be nef classes in $H^{1,1}(X, \mathbb{Q})$ and let $A$ be a Kähler class. Let $a$ be a sufficiently small positive number. Then the Harder-Narasimhan semistable filtration of $E$ with respect to $(D_1 + aA, \cdots, D_{n-1} + aA)$ is independent of $a$.

**Remark 1.2.** If $A$ has rational coefficients, Lemma 1.1 is proved in [KMM04]. When $A$ is not necessarily rational, the proof turns out to be a little bit more complicated. We begin with the following easy observation.

**Lemma 1.3.** In the situation of Lemma 1.1, let we take $k \in \{0, 1, 2, \cdots, n\}$ arbitrary. Then we can find a basis $\{e_1, \cdots, e_s\}$ of $H^2_k(X, \mathbb{Q})$ depending only on $A^k$, such that

(i): $A^k = \sum_{i=1}^s \lambda_i \cdot e_i$ for some $\lambda_i > 0$.

(ii): Let $\mathcal{F}$ be a torsion free coherent sheaf. Set

$$ D^t := \sum_{i_1 < i_2 < \cdots < i_t} D_{i_1} \cdots D_{i_t} \quad \text{for any } t \in \mathbb{N}, $$

and

$$ a_i(\mathcal{F}) := c_1(\mathcal{F}) \cdot D^{n-k-1} \cdot e_i. $$

Then the subset $S$ of the set of rational numbers such that

$$ S := \{a_i(\mathcal{F})| \mathcal{F} \subset E, i \in \{1, \cdots, s\}\} $$

is bounded from above, and the denominator (assumed positive) of all elements of $S$ is uniformly bounded from above. Moreover, if $\{\mathcal{F}_t\}_t$ is a sequence of coherent subsheaves of $E$ such that the set $\{c_1(\mathcal{F}_t) \cdot D^{n-k-1} \cdot A^k\}_t$ is bounded from below, then $\{c_1(\mathcal{F}_t) \cdot D^{n-k-1} \cdot A^k\}_t$ is a finite subset of $\mathbb{Q}$. 

Proof. We can take a basis \( \{ e_i \}_{i=1}^s \) of \( H^{2k}(X, \mathbb{Q}) \) in a neighborhood of \( A^k \), such that
\[
A^k = \sum_{i=1}^s \lambda_i \cdot e_i \quad \text{for some } \lambda_i > 0
\]
and \( (e_i)^{k,k} \) can be represented by a smooth \( (k,k) \)-positive form on \( X \) (cf. [127, Chapter 3, Def 1.1] for the definition of \( (k,k) \)-positivity), where \( (e)^{k,k} \) is the projection of \( e \) in \( H^{k,k}(X, \mathbb{R}) \). We now check that \( \{ e_i \}_{i=1}^s \) satisfies the lemma. By construction, \((i)\) is satisfied. As for \((ii)\), since \( e_i \) and \( D_i \) are fixed and \( c_1(\mathcal{F}) \in H^{1,1}(X, \mathbb{Z}) \), the denominator of any elements in \( S \) is uniformly bounded from above.

Thanks to (2), we know that \( S \) is bounded from above by using the same argument as in [Kob87] Lemma 7.16, Chapter 5. For the last part of \((ii)\), since
\[
c_1(\mathcal{F}_i) \cdot D^{n-k-1} \cdot A^k = \sum_i a_i(\mathcal{F}_i) \cdot \lambda_i,
\]
we obtain that \( \{ \sum a_i(\mathcal{F}_i) \cdot \lambda_i \}_i \) is uniformly bounded. Since \( \lambda_i > 0 \) and \( a_i(\mathcal{F}_i) \) is uniformly upper bounded, we obtain that \( a_i(\mathcal{F}_i) \) is uniformly bounded. Combining this with the fact already proved that the denominator of any elements in \( S \) is uniformly bounded, \( \{ c_1(\mathcal{F}_i) \cdot D^{n-k-1} \cdot A^k \}_i \) is thus finite. The lemma is proved. \( \square \)

Proof of Lemma (17). Let \( \{ a_p \}_{p=1}^{\infty} \) be a decreasing positive sequence converging to 0. Let \( \mathcal{F}_p \subset E \) be the first piece of the Harder-Narasimhan filtration of \( E \) with respect to \( (D_1 + a_p \cdot A, \cdots, D_{n-1} + a_p \cdot A) \). Set \( D_k := \sum_{i_1 < i_2 < \cdots < i_k} D_{i_1} \cdot D_{i_2} \cdots D_{i_k} \).

Then
\[
c_1(\mathcal{F}_p) \wedge (D_1 + a_p \cdot A) \wedge \cdots \wedge (D_{n-1} + a_p \cdot A) = \sum_{k=0}^n (a_p)^k \cdot c_1(\mathcal{F}_p) \wedge D^{n-k-1} \wedge A^k.
\]

By passing to a subsequence, we can suppose that \( \text{rank} \mathcal{F}_p \) is constant. To prove Lemma (14) it is sufficient to prove that for any \( k \), after passing to a subsequence, the intersection number \( \{ c_1(\mathcal{F}_p) \wedge D^{n-k-1} \wedge A^k \}_p \) is stationary when \( p \) is large enough (14).

We prove it by induction on \( k \). Note first that, by [Kob87] Lemma 7.16, Chapter 5, the set \( \{ c_1(\mathcal{F}_p) \wedge D^{n-k-1} \wedge A^k \}_p \) is upper bounded. If \( k = 0 \), since
\[
c_1(\mathcal{F}_p) \wedge (D_1 + a_p \cdot A) \wedge \cdots \wedge (D_{n-1} + a_p \cdot A) \geq \frac{\text{rank}(\mathcal{F}_p)}{\text{rank} E} \cdot c_1(E) \wedge (D_1 + a_p \cdot A) \wedge \cdots \wedge (D_{n-1} + a_p \cdot A),
\]
and \( \lim_{p \to \infty} a_p = 0 \), the upper boundedness of \( \{ c_1(\mathcal{F}_p) \wedge D^{n-k-1} \wedge A^k \}_p \) implies that the set \( \{ c_1(\mathcal{F}_p) \wedge D^{n-1} \}_p \) is bounded from below. Then \((ii)\) of Lemma (13) implies that \( \{ c_1(\mathcal{F}_p) \wedge D^{n-1} \}_p \) is a finite set. By the pigeon hole principle, after passing to a subsequence, the set \( \{ c_1(\mathcal{F}_p) \wedge D^{n-1} \}_p \) is stationary.

1: In fact, if \( \mathcal{F} \subset E \) is always the first piece of semistable filtration with respect to the polarization \( (D_1 + a_p \cdot A, \cdots, D_{n-1} + a_p \cdot A) \) for a positive sequence \( \{ a_p \}_{p=0}^{+\infty} \) converging to 0, and \( G \subset E \) is always the first piece of semistable filtration for another sequence \( \{ b_p \}_{p=0}^{+\infty} \) converging to 0, the stability condition implies that
\[
\text{rank}(G) \cdot c_1(\mathcal{F}) \cdot D^k \cdot A^{n-k-1} = \text{rank}(\mathcal{F}) \cdot c_1(G) \cdot D^k \cdot A^{n-k-1}
\]
for any \( k \). Therefore \( G \) has the same slope as \( \mathcal{F} \) with respect to \( (D_1 + a \cdot A, \cdots, D_{n-1} + a \cdot A) \) for any \( a > 0 \). Then \( \mathcal{F} = G \).
\( \{c_1(F_p) \wedge D^{n-t-1} \wedge A^t \}_{p=1} \) is constant for \( p \geq p_0 \), where \( t \in \{0, \ldots, k-1\} \). Our aim is to prove that after passing to a subsequence,

\[ \{c_1(F_p) \wedge D^{n-k-1} \wedge A^k \}_{p=1} \]

is stationary. By definition, we have

\[
c_1(F_p) \wedge (D_1 + a_p \cdot A) \wedge \cdots \wedge (D_{n-1} + a_p \cdot A) \geq c_1(F_{p_0}) \wedge (D_1 + a_p \cdot A) \wedge \cdots \wedge (D_{n-1} + a_p \cdot A)
\]

for any \( p \geq p_0 \). Since \( \{c_1(F_p) \wedge D^{n-t-1} \wedge A^t \}_{p=1} \) is constant for \( p \geq p_0 \) when \( t \in \{0, \ldots, k-1\} \), we obtain

\[
\begin{align*}
&c_1(F_p) \wedge D^{n-k-1} \wedge A^k + \sum_{i \geq 1} (a_p)^i \cdot c_1(F_p) \wedge D^{n-k-1-i} \wedge A^{k+i} \\
&\geq c_1(F_{p_0}) \wedge D^{n-k-1} \wedge A^k + \sum_{i \geq 1} (a_p)^i \cdot c_1(F_{p_0}) \wedge D^{n-k-1-i} \wedge A^{k+i}
\end{align*}
\]

for any \( p \geq p_0 \). Therefore the upper boundedness of \( \{c_1(F_p) \wedge D^{n-k-1-i} \wedge A^{k+i} \}_{p=1} \) implies that \( \{c_1(F_p) \wedge D^{n-k-1} \wedge A^k \}_{p=1} \) is lower bounded. Therefore

\[ \{c_1(F_p) \wedge D^{n-k-1} \wedge A^k \}_{p=1} \]

is uniformly bounded. Using (ii) of Lemma [1.5] \( \{c_1(F_p) \wedge D^{n-k-1} \wedge A^k \}_{p=1} \) is a finite set. By the pigeon hole principle, after passing to a subsequence,

\[ \{c_1(F_p) \wedge D^{n-k-1} \wedge A^k \}_{p=1} \]

is stationary. The lemma is proved. \( \square \)

By the same argument as above, we can easily prove that

**Lemma 1.4.** Let \((X, \omega)\) be a compact Kähler manifold and let \( E \) be a torsion free \( \omega \)-stable coherent sheaf. Then \( E \) is also stable with respect to a small perturbation of \( \omega \).

**Proof.** Using (ii) of Lemma 1.3 (by taking \( D_i = 0, A = \omega \)), we obtain that

\[
\sup \left\{ \frac{1}{\text{rank}(F)} \int_X c_1(F) \wedge \omega^{n-1} |F \text{ a coherent subsheaf of } E \text{ with strictly smaller rank} \right\}
\]

is strictly smaller than the slope of \( E \). Therefore \( E \) is also stable with respect to a small perturbation of \( \omega \). \( \square \)

**Remark 1.5.** If the Kähler metric \( \omega \in H^2(X, \mathbb{Z}) \), the lemma comes directly from the fact that the set

\[
\begin{align*}
\left\{ \int_X c_1(F) \wedge \omega^{n-1} |F \text{ a coherent subsheaf of } E \text{ with strictly smaller rank} \right\}
\end{align*}
\]

takes value in \( \mathbb{Z} \).

We recall a regularization lemma proved in [Jac0] Prop. 3.

**Lemma 1.6.** Let \( E \) be a vector bundle on a compact complex manifold \( X \) and \( F \) be a subsheaf of \( E \) with torsion free quotient. Then after a finite number of blowups \( \pi : \tilde{X} \to X \), there exists a holomorphic subbundle \( F \) of \( \pi^*(E) \) containing \( \pi^*(F) \) with a holomorphic quotient bundle, such that \( \pi_*(F) = F \) in codimension 1.
We need another lemma which is proved in full generality in [DPS94, Prop. 1.15]. For completeness, we give the proof here in an over simplified case, but the idea is the same.

**Lemma 1.7.** Let \((X, \omega)\) be a compact Kähler manifold. Let \(E\) be an extension of two vector bundles \(E_1, E_2\)

\[0 \to E_1 \to E \to E_2 \to 0.\]

We suppose that there exist two smooth metrics \(h_1, h_2\) on \(E_1\) and \(E_2\), such that

\[
i \Theta_{h_1}(E) \wedge \omega^{n-1} \geq c_1 \cdot \text{Id}_{E_1}\]

and

\[
i \Theta_{h_2}(E) \wedge \omega^{n-1} \geq c_2 \cdot \text{Id}_{E_2}\]

pointwise. Then for any \(\epsilon > 0\), there exists a smooth metric \(h_\epsilon\) on \(E\) such that

\[
i \Theta_{h_\epsilon}(E) \wedge \omega^{n-1} \geq (\min(c_1, c_2) - \epsilon) \cdot \text{Id}_E,
\]

and

\[
\|i \Theta_{h_\epsilon}(E)\|_{L^\infty} \leq C \cdot (\|i \Theta_{h_1}(E_1)\|_{L^\infty} + \|i \Theta_{h_2}(E_2)\|_{L^\infty})
\]

for some uniform constant \(C\) independent of \(\epsilon\).

**Proof.** Let \([E] \in H^1(X, \text{Hom}(E_2, E_1))\) be the element representing \(E\) in the extension group. Let \(E_s\) be another extension of \(E_1\) and \(E_2\), such that \([E_s] = s \cdot [E]\), where \(s \in \mathbb{C}^\ast\). Then there exists an isomorphism between these two vector bundles (cf. [adg, Remark 14.10, Chapter V]). We denote the isomorphism by \(\varphi_s : E \to E_s\).

Thanks to (5), if \(|s|\) is small enough with respect to \(\epsilon\), we can find a smooth metric \(h_s\) on \(E_s\) satisfying

\[
i \Theta_{h_s}(E_s) \wedge \omega^{n-1} \geq (\min(c_1, c_2) - \epsilon) \cdot \text{Id}_{E_s},
\]

and

\[
\|i \Theta_{h_s}(E_s)\|_{L^\infty} \leq C \cdot (\|i \Theta_{h_1}(E_1)\|_{L^\infty} + \|i \Theta_{h_2}(E_2)\|_{L^\infty})
\]

for some uniform constant \(C\) (cf. [adg, Prop 14.9, Chapter V]). Let \(h = \varphi_s^\ast(h_s)\) be the induced metric on \(E\). Then for any \(\alpha \in E\),

\[
\langle i \Theta_h(E)\alpha, \alpha \rangle_h = \langle i \Theta_{h_s}(E_s)\varphi_s^\ast(\alpha), \varphi_s^\ast(\alpha) \rangle_{h_s} = \langle i \Theta_{h_s}(E_s)\varphi_s(\alpha), \varphi_s(\alpha) \rangle_{h_s}
\]

Combining this with (7), we get

\[
\langle i \Theta_h(E)\alpha, \alpha \rangle_h \wedge \omega^{n-1} \geq (\min(c_1, c_2) - \epsilon) \cdot \text{Id}_E.
\]

Moreover, (8) implies also (6). The lemma is proved. \(\Box\)

We recall the following well-known equality in Kähler geometry.
Proposition 1.8. Let \((X, \omega_X)\) be a Kähler manifold of dimension \(n\), \(R\) be the curvature tensor and \(\text{Ric}\) be the Ricci tensor (cf. the definition of [Zhe00, Section 7.5]). Let \(i \Theta_{\omega_X} (T_X)\) be the curvature of \(T_X\) induced by \(\omega_X\). We have

\[
\langle i \Theta_{\omega_X} (T_X) \wedge \omega_X^{n-1} u, v \rangle_{\omega_X} = \text{Ric} (u, \overline{v}).
\]

Proof. Let \(\{e_i\}_{i=1}^n\) be an orthonormal basis of \(T_X\) with respect to \(\omega_X\). By definition, we have

\[
\langle i \Theta_{\omega_X} (T_X) \wedge \omega_X^{n-1} u, v \rangle_{\omega_X} = \sum_{1 \leq i \leq n} \langle i \Theta_{\omega_X} (T_X) u, v \rangle (e_i, \overline{e_i}) = \sum_{1 \leq i \leq n} R(e_i, \overline{e_i}, u, \overline{v}).
\]

By definition of the Ricci curvature (cf. [Zhe00, Page 180]), we have

\[
\text{Ric}(u, \overline{v}) = \sum_{1 \leq i \leq n} R(u, \overline{e_i}, e_i, \overline{e_i}).
\]

Combining this with the First Bianchi equality

\[
\sum_{1 \leq i \leq n} R(e_i, \overline{e_i}, u, \overline{v}) = \sum_{1 \leq i \leq n} R(u, \overline{e_i}, e_i, \overline{e_i}),
\]

the proposition is proved. \(\square\)

2. Main theorem

We first prove Theorem 0.2 in the case when \(-K_X\) is nef.

Theorem 2.1. Let \((X, \omega)\) be a compact \(n\)-dimensional Kähler manifold with nef anticanonical bundle. Let

\[
0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_s = T_X
\]

be a filtration of torsion-free subsheaves such that \(\mathcal{E}_{i+1}/\mathcal{E}_i\) is an \(\omega\)-stable torsion-free subsheaf of \(T_X/\mathcal{E}_i\) of maximal slope. Let

\[
\mu(\mathcal{E}_{i+1}/\mathcal{E}_i) = \frac{1}{\text{rank}(\mathcal{E}_{i+1}/\mathcal{E}_i)} \int_X c_1(\mathcal{E}_{i+1}/\mathcal{E}_i) \wedge \omega_X^{n-1}
\]

be the slope of \(\mathcal{E}_{i+1}/\mathcal{E}_i\) with respect to \(\omega_X^{n-1}\). Then

\[
\mu(\mathcal{E}_{i+1}/\mathcal{E}_i) \geq 0 \quad \text{for all } i.
\]

Proof. We first consider a simplified case.

Case 1: \([10]\) is regular, i.e., all \(\mathcal{E}_i, \mathcal{E}_{i+1}/\mathcal{E}_i\) are vector bundles.

By the stability condition, to prove the theorem, it is sufficient to prove that

\[
\int_X c_1(T_X/\mathcal{E}_i) \wedge \omega_X^{n-1} \geq 0 \quad \text{for any } i.
\]

Thanks to the nefness of \(-K_X\), for any \(\epsilon > 0\), there exists a Kähler metric \(\omega_{\epsilon}\) in the same class of \(\omega\) such that (cf. the proof of [DPS93, Thm. 1.1])

\[
\text{Ric}_{\omega_{\epsilon}} \geq -\epsilon \omega_{\epsilon},
\]

\[\text{Using (ii) of Lemma 1.3 (by taking } D_\ast = 0 \text{ and } A = \omega\text{), one can prove the existence of such a filtration by a standard argument [HN95].}\]
where $\text{Ric}_{\omega}$ is the Ricci curvature with respect to the metric $\omega$. Thanks to Proposition 1.8, we have

$$\frac{i \Theta_{\omega_i}(T_X) \wedge \omega_{\epsilon}^{-1}}{\omega_{\epsilon}^n} \alpha, \alpha \omega_i = \text{Ric}_{\omega_i}(\alpha, \pi).$$

Then (12) implies a pointwise estimate

$$\frac{i \Theta_{\omega_i}(T_X) \wedge \omega_{\epsilon}^{-1}}{\omega_{\epsilon}^n} \geq -\epsilon \cdot \text{Id}_{T_X}.$$  \hfill (13)

Taking the induced metric on $T_X/E_i$ (we also denote it by $\omega_{\epsilon}$), we get (cf. [adg, Chapter V])

$$\frac{i \Theta_{\omega_i}(T_X/E_i) \wedge \omega_{\epsilon}^{-1}}{\omega_{\epsilon}^n} \geq -\epsilon \cdot \text{Id}_{T_X/E_i}.$$  \hfill (14)

Therefore

$$\int_X c_1(T_X/E_i) \wedge \omega_{\epsilon}^{-1} \geq -\text{rank}(T_X/E_i) \cdot \epsilon \int_X \omega_{\epsilon}^n.$$  

Combining this with the fact that $[\omega_i] = [\omega]$, we get

$$\int_X c_1(T_X/E_i) \wedge \omega_{\epsilon}^{-1} = \int_X c_1(T_X/E_i) \wedge \omega_{\epsilon}^{-1} \geq -C \epsilon,$$  \hfill (15)

for some constant $C$. Letting $\epsilon \to 0$, (11) is proved.

Case 2: The general case

By Lemma 1.6, there exists a desingularization $\pi : \tilde{X} \to X$, such that $\pi^*(T_X)$ admits a filtration:

$$0 \subset E_1 \subset E_2 \subset \cdots \subset \pi^*(T_X),$$  \hfill (16)

where $E_i, E_i/E_{i-1}$ are vector bundles and $\pi_*(E_i) = E_i$ outside an analytic subset of codimension at least 2. Let $\tilde{\mu}$ be the slope with respect to $\pi^*(\omega)$. Then

$$\tilde{\mu}(E_i/E_{i-1}) = \mu(E_i/E_{i-1})$$  \hfill (17)

(cf. [Jac10, Lemma 2]), and $E_i/E_{i-1}$ is a $\pi^*(\omega)$-stable subsheaf of $\pi^*(T_X)/E_{i-1}$ of maximal slope (cf. Remark 2.2 after the proof).

We now prove that $\tilde{\mu}(E_i/E_{i-1}) \geq 0$. Thanks to (13), for any $\epsilon > 0$ small enough, we have

$$\frac{i \Theta_{\pi^* \omega_i}(\pi^*(T_X)) \wedge (\pi^* \omega_{\epsilon})^{-1}}{(\pi^* \omega_{\epsilon})^n} \geq -\epsilon \cdot \text{Id}_{\pi^*(T_X)},$$

which implies that

$$\frac{i \Theta_{\pi^* \omega_i}(\pi^*(T_X)/E_i) \wedge (\pi^* \omega_{\epsilon})^{-1}}{(\pi^* \omega_{\epsilon})^n} \geq -\epsilon \cdot \text{Id}_{\pi^*(T_X)/E_i}.$$  \hfill (18)

By the same argument as in Case 1, (18) and the maximal slope condition of $E_{i+1}/E_i$ in $\pi^*(T_X)/E_i$ implies that

$$\tilde{\mu}(E_{i+1}/E_i) = \frac{1}{\text{rank}(E_{i+1}/E_i)} \int_{\tilde{X}} c_1(E_{i+1}/E_i) \wedge \pi^* \omega_{\epsilon}^{-1} \geq -C \epsilon$$

for some constant $C$ independent of $\epsilon$. Letting $\epsilon \to 0$, we get $\tilde{\mu}(E_{i+1}/E_i) \geq 0$. Combining this with (17), the theorem is proved.

\begin{remark}
In the situation of (16) in Theorem 2.1, we would like to prove that $E_i/E_{i-1}$ is also stable for $\pi^* \omega + \epsilon \omega_{\tilde{X}}$ for any $\epsilon > 0$ small enough.
\end{remark}
Proof of Remark 2.2. Let $F$ be any coherent sheaf satisfying
\[(19) \quad E_{i-1} \subset F \subset E_i \quad \text{and} \quad \text{rank } F < \text{rank } E_i.\]
It is sufficient to prove that
\[(20) \quad \frac{1}{\text{rank } F / E_i} \int_X c_1(F/E_{i-1}) \wedge (\pi^* \omega + \epsilon \omega)_{X}^{n-1} < \tilde{\mu}(E_i/E_{i-1})\]
for a uniform $\epsilon > 0$, where $\tilde{\mu}$ is the slope with respect to $\pi^* (\omega)$ as defined in Theorem 2.1.

Thanks to the formula
\[\int_X c_1(F) \wedge \pi^*(\omega)_{X}^{n-1} = \int_X c_1(\pi^*(F)) \wedge \omega_{X}^{n-1},\]
Lemma 1.4 and the stability condition of $E_i / E_{i-1}$ imply that the upper bound of the set
\[\left\{ \frac{1}{\text{rank } F / E_i} \int_X c_1(F/E_{i-1}) \wedge \pi^* \omega_{X}^{n-1} | F \text{ satisfies } (19) \right\}\]
is strictly smaller than $\tilde{\mu}(E_i/E_{i-1})$. Combining this with the fact that
\[\int_X c_1(F) \wedge \omega_X^s \wedge \pi^* (\omega)^{n-s-1}\]
is uniformly bounded from above for any $s$, (20) is proved. \(\Box\)

We now prove Theorem 0.2 in the case when $K_X$ is nef.

**Theorem 2.3.** Let $(X, \omega)$ be a compact Kähler manifold with nef canonical bundle. Let
\[(21) \quad 0 \subset E_0 \subset E_1 \subset \cdots \subset E_s = \Omega^1_X\]
be a filtration of torsion-free subsheaves such that $E_{i+1} / E_i$ is an $\omega$-stable torsion-free subsheaf of $T_X / E_i$ of maximal slope. Then
\[\int_X c_1(E_{i+1} / E_i) \wedge \omega^{n-1} \geq 0 \quad \text{for all } i.\]

**Proof.** The proof is almost the same as Theorem 2.1. First of all, since $K_X$ is nef, for any $\epsilon > 0$, there exists a smooth function $\psi$ on $X$, such that
\[\text{Ric}_{\omega} + i \partial \bar{\partial} \psi \leq \epsilon \omega.\]
By solving the Monge-Ampère equation
\[(22) \quad (\omega + i \partial \bar{\partial} \psi)^n = \omega^n \cdot e^{-\psi - \epsilon \varphi},\]
we can construct a new Kähler metric $\omega_\epsilon$ in the cohomology class of $\omega$:
\[\omega_\epsilon := \omega + i \partial \bar{\partial} \varphi.\]
Thanks to (22), we have
\[\text{Ric}_{\omega_\epsilon} = \text{Ric}_{\omega} + i \partial \bar{\partial} \psi + e i \partial \bar{\partial} \varphi \leq \epsilon \omega + e i \partial \bar{\partial} \varphi = \epsilon \omega_\epsilon.\]
We first suppose that (21) is regular, i.e., $E_i$ and $E_{i+1} / E_i$ are free for all $i$. Let $\alpha \in \Omega^1_{X,\omega}$ for some point $x \in X$ with norm $\|\alpha\| = 1$ and let $\alpha^*$ be the dual of $\alpha$ with respect to $\omega$. Then we have also a pointwise estimate at $x$:
\[\left( \frac{i \Theta_{\omega} (\Omega_X)^{\wedge} \omega_\epsilon^{n-1}}{\omega_\epsilon^n} \alpha, \alpha \right) = \left( \frac{i \Theta_{\omega} (T_X)^{\wedge} \omega_\epsilon^{n-1}}{\omega_\epsilon^n} \alpha^*, \alpha^* \right)\]
− \text{Ric}_{\omega}(\alpha^*, \alpha^*) \geq -\epsilon.

By the same proof as in Theorem 2.1, \( \int_X c_1(\mathcal{E}_{i+1}/\mathcal{E}_i) \wedge \omega^{n-1} \) is semi-positive for any \( i \). For the general case, the proof follows exactly the same line as in Theorem 2.1. \( \Box \)

3. Applications

As an application, we give a characterization of rationally connected compact Kähler manifolds with nef anticanonical bundles.

**Proposition 3.1.** Let \( X \) be a compact Kähler manifold with nef anticanonical bundle. Then the following four conditions are equivalent

(i): \( H^0(X, (T^*_X)^{\otimes m}) = 0 \) for all \( m \geq 1 \).

(ii): \( X \) is rationally connected.

(iii): \( T_X \) is generically \( \omega^{n-1}_X \) strictly positive for some Kähler class \( \omega_X \).

(iv): \( T_X \) is generically \( \omega^{n-1}_X \) strictly positive for any Kähler class \( \omega_X \).

**Proof.** The implications (iv) \( \Rightarrow \) (iii), (ii) \( \Rightarrow \) (i) are obvious. For the implication (iii) \( \Rightarrow \) (ii), we first note that (iii) implies (i) by Bochner technique. Therefore \( X \) is projective and any Kähler class can be approximated by rational Kähler classes. Using [BM01, Theorem 0.1], (iii) implies (ii).

We now prove that (i) \( \Rightarrow \) (iv). Let \( \omega \) be any Kähler class. Let

\[
0 \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_s = T_X
\]

be the Harder-Narasimhan semistable filtration with respect to \( \omega^{n-1} \). To prove (iv), it is sufficient to prove

\[
\int_X c_1(T_X/\mathcal{E}_{s-1}) \wedge \omega^{n-1} > 0.
\]

Recall that Theorem 2.1 implies already that

\[
\int_X c_1(T_X/\mathcal{E}_{s-1}) \wedge \omega^{n-1} \geq 0.
\]

We suppose by contradiction that

\[
\int_X c_1(T_X/\mathcal{E}_{s-1}) \wedge \omega^{n-1} = 0.
\]

Let \( \alpha \in H^{1,1}(X, \mathbb{R}) \). We define new Kähler metrics \( \omega_\epsilon = \omega + \epsilon \alpha \) for \( |\epsilon| \) small enough. Thanks to [Miy87, Cor. 2.3], the \( \omega^{n-1}_\epsilon \)-semistable filtration of \( T_X \) is a refinement of (23). Therefore, Theorem 2.1 implies that

\[
\int_X c_1(T_X/\mathcal{E}_{s-1}) \wedge (\omega + \epsilon \alpha)^{n-1} \geq 0
\]

for \( |\epsilon| \) small enough. Then (24) implies that

\[
\int_X c_1(T_X/\mathcal{E}_{s-1}) \wedge \omega^{n-2} \wedge \alpha = 0 \quad \text{for any } \alpha \in H^{1,1}(X, \mathbb{R}).
\]

By the Hodge index theorem, we obtain that \( c_1(T_X/\mathcal{E}_{s-1}) = 0 \). By duality, there exists a subsheaf \( \mathcal{F} \subset \Omega^1_X \), such that

\[
c_1(\mathcal{F}) = 0 \quad \text{and} \quad \det \mathcal{F} \subset (T^*_X)^{\otimes \text{rank}\mathcal{F}}.
\]
Observing that $H^1(X, \mathcal{O}_X) = 0$ by assumption, i.e., the group Pic$^0(X)$ is trivial, hence (25) implies the existence of an integer $m$ such that $(\det \mathcal{F})^\otimes m$ is a trivial line bundle. Observing moreover that $(\det \mathcal{F})^\otimes m \subset (T_X^{\ast})^\otimes m \cdot \text{rank} \mathcal{F}$, then

$$H^0(X, (T_X^{\ast})^\otimes m \cdot \text{rank} \mathcal{F}) \neq 0,$$

which contradicts with (i). The implication $(i) \Rightarrow (iv)$ is proved. □

**Remark 3.2.** One can also prove the implication $(iii) \Rightarrow (ii)$ without using the profound theorem of [BM01]. We give the proof in Appendix A.

The above results lead to the following question about rationally connected manifolds with nef anticanonical bundles.

**Question 3.3.** Let $X$ be a smooth compact manifold. Then $T_X$ is generically $\omega^{n-1}$-strictly positive for any Kähler metric $\omega$ if and only if $X$ is rationally connected manifold with nef anticanonical bundle.

As a second application, we study a Conjecture of Y. Kawamata (cf. [Miy87], Thm. 1.1) for the dual case and [Xie05] for dimension 3.

**Conjecture 3.4.** If $X$ is a compact Kähler manifold with nef anticanonical bundle. Then

$$\int_X c_2(T_X) \wedge \omega_1 \wedge \cdots \wedge \omega_{n-2} \geq 0$$

for all nef classes $\omega_1, \cdots, \omega_{n-1}$.

Using Theorem 2.1 and a more refined argument as in [Miy87], Thm.6.1, we can prove

**Proposition 3.5.** Let $(X, \omega_X)$ be a compact Kähler manifold with nef anticanonical bundle and let $\omega_X$ be a Kähler metric. Then

$$\int_X c_2(T_X) \wedge (c_1(-K_X) + \epsilon \omega_X)^{n-2} \geq 0$$

for any $\epsilon > 0$ small enough.

**Proof.** Let $\text{nd}$ be the numerical dimension of $-K_X$. If $\text{nd} = 0$, then $c_1(K_X) = 0$. It is well know that (29) holds in this case (cf. [Miy87] or [Kob87]). From now on, we suppose that $\text{nd} \geq 1$. Let

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_l = T_X$$

be a stable filtration of the Harder-Narasimhan semistable filtration of $T_X$ with respect to the polarization $(c_1(-K_X) + \epsilon \omega_X)^{n-1}$ for some small $\epsilon > 0$. By Lemma 1.1, the filtration (27) is independent of $\epsilon$ when $\epsilon$ is sufficiently small. By Theorem 2.1, we have

$$c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge (c_1(-K_X) + \epsilon \omega_X)^{n-1} \geq 0$$

for any $i$ and $\epsilon > 0$ sufficiently small. Then

$$c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge (-K_X)^{\text{nd}} \wedge (\omega_X)^{n-1-\text{nd}} \geq 0$$

for any $i$.

Since

$$\sum_i c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge (-K_X)^{\text{nd}} \wedge (\omega_X)^{n-1-\text{nd}} = (-K_X)^{\text{nd}+1} \wedge (\omega_X)^{n-1-\text{nd}} = 0,$$
we obtain
\[ c_1(F_i/F_{i-1}) \wedge (-K_X)^{nd} \wedge (\omega_X)^{n-nd} = 0 \quad \text{for any } i. \]

Combining (29) with (28), we obtain
\[ c_1(F_i/F_{i-1}) \wedge (-K_X)^{nd-1} \wedge (\omega_X)^{n-nd} \geq 0 \quad \text{for any } i. \]

Combining this with the stability condition of the filtration, we can find an integer \( k \geq 1 \) such that
\[ c_1(F_i/F_{i-1}) \wedge (-K_X)^{nd-1} \wedge (\omega_X)^{n-nd} = a_i > 0 \quad \text{for } i \leq k, \]

and
\[ c_1(F_i/F_{i-1}) \wedge (-K_X)^{nd-1} \wedge (\omega_X)^{n-nd} = 0 \quad \text{for } i > k. \]

Case (1): \[ \sum_{i \leq k} a_i \geq 1 \] and \( nd \geq 2 \). Using the Hodge index theorem, we have
\[ c_1(F_i/F_{i-1}) \wedge (-K_X + \omega_X)^{n-2} \geq (\alpha^2 \wedge (-K_X + \omega_X)^{n-2} - (\alpha \wedge (-K_X) \wedge (-K_X + \omega_X)^{n-2})^2, \]
for any \( \alpha \in H^{1,1}(X, \mathbb{R}) \). If we take \( \alpha = c_1(F_i/F_{i-1}) \) in (33) and use (32), we obtain
\[ c_2(T_X) \wedge (-K_X + \omega_X)^{n-2} \geq c_1(-K_X)^2 \wedge (-K_X + \omega_X)^{n-2} - \sum_{i \leq k} \frac{1}{r_i} c_1(F_i/F_{i-1})^2 \frac{(-K_X)^2 \wedge (-K_X + \omega_X)^{n-2}}{(-K_X)^2 \wedge (-K_X + \omega_X)^{n-2}}. \]

Now we estimate the two terms in the right hand side of (34). Using (31), we have
\[ c_1(-K_X)^2 \wedge (-K_X + \omega_X)^{n-2} = \left( \sum_{i \leq k} a_i \right) \epsilon^{n-nd} + O(\epsilon^n). \]

and
\[ \sum_{i \leq k} \frac{1}{r_i} \left( c_1(F_i/F_{i-1}) \wedge (-K_X) \wedge (-K_X + \omega_X)^{n-2} \right)^2 \frac{(-K_X)^2 \wedge (-K_X + \omega_X)^{n-2}}{(-K_X)^2 \wedge (-K_X + \omega_X)^{n-2}} \]
\[ = \frac{1}{\sum_{i \leq k} a_i} \left( \sum_{i \leq k} \frac{a_i^2}{r_i} \right) \epsilon^{n-nd} + O(\epsilon^n). \]

Since \( \sum_{i \leq k} r_i \geq 2 \), we have
\[ \sum_{i \leq k} a_i > \frac{1}{\sum_{i \leq k} a_i} \left( \sum_{i \leq k} \frac{a_i^2}{r_i} \right). \]

Therefore \( c_2(T_X) \wedge (-K_X + \omega_X)^{n-2} \) is strictly positive when \( \epsilon > 0 \) is small enough.

---

3It is important that \( \alpha^2 \wedge (-K_X + \omega_X)^{n-2} \) maybe negative.


**Case (2):** \( \sum_{i \leq k} r_i = 1 \) and \( nd \geq 2 \). In this case, we obtain immediately that \( r_1 = 1 \) and \( k = 1 \). Moreover, (31) in this case means that
\[
c_1(F_1) \cap (-K_X)^{nd-1} \cap (\omega_X)^{n-nd} > 0,
\]
and
\[
c_1(F_i/F_{i-1}) \cap (-K_X)^{nd-s} \cap (\omega_X)^{n-nd+s-1} > 0.
\]

Set \( s \) be the smallest integer such that
\[
c_1(F_2/F_1) \cap (-K_X)^{nd-s} \cap (\omega_X)^{n-nd+s-1} > 0.
\]

Taking \( \alpha = c_1(F_i/F_{i-1}) \) in (33) for any \( i \geq 2 \), we get
\[
(36) \quad c_1(F_i/F_{i-1})^2 \cap (c_1(-K_X) + \epsilon \omega_X)^{n-2}
\]

\[
\leq \frac{(c_1(F_i/F_{i-1}) \cap (-K_X) \cap (-K_X + \epsilon \omega_X)^{n-2})^2}{(-K_X)^2 \cap (-K_X + \epsilon \omega_X)^{n-2}}
\]

\[
\leq \frac{(\epsilon^{n+s-nd-1})^2}{\epsilon^{n-nd}}(1 + O(1)) = \epsilon^{2s+n-nd-2} + O(\epsilon^{2s+n-nd-2}) \text{ for } i \geq 2.
\]

Similarly, if we take \( \alpha = \sum_{i \geq 2} c_1(F_i/F_{i-1}) \) in (33), we obtain
\[
(37) \quad \left( \sum_{i \geq 2} c_1(F_i/F_{i-1}) \right)^2 \cap (c_1(-K_X) + \epsilon \omega_X)^{n-2} \leq \epsilon^{2s+n-nd-2}.
\]

Combining (36), (37) with (22), we obtain
\[
c_2(T_X) \cap (-K_X + \epsilon \omega_X)^{n-2}
\]

\[
\geq \left( c_1(-K_X)^2 - \sum_{i \geq 2} \frac{1}{r_i} c_1(F_i/F_{i-1})^2 \cap (c_1(-K_X) - \sum_{i \geq 2} c_1(F_i/F_{i-1}))^2 \right)(-K_X + \epsilon \omega_X)^{n-2}
\]

\[
= 2c_1(-K_X) \cap \left( \sum_{i \geq 2} c_1(F_i/F_{i-1}) \right)(-K_X + \epsilon \omega_X)^{n-2}
\]

\[
- \left( \sum_{i \geq 2} \frac{1}{r_i} c_1(F_i/F_{i-1})^2 \cap \left( \sum_{i \geq 2} c_1(F_i/F_{i-1}) \right)^2 \right) \cap (-K_X + \epsilon \omega_X)^{n-2}
\]

\[
\geq \epsilon^{n-nd+s-1} - \epsilon^{n-nd+2s-2}.
\]

Let us observe that by (35) we have \( s \geq 2 \). Therefore \( c_2(T_X) \cap (-K_X + \epsilon \omega_X)^{n-2} \) is strictly positive for \( \epsilon > 0 \) small enough.

**Case (3):** \( nd = 1 \). Using (22), we have
\[
c_2(T_X) \cap (-K_X + \epsilon \omega_X)^{n-2} \geq - \sum_{i} \frac{1}{r_i} c_1(F_i/F_{i-1})^2 (-K_X + \epsilon \omega_X)^{n-2}.
\]

By the Hodge index theorem, we obtain
\[
c_2(T_X) \cap (-K_X + \epsilon \omega_X)^{n-2}
\]

\[
\geq \lim_{t \to 0^+} - \sum_{i} \frac{1}{r_i} c_1(F_i/F_{i-1}) \cap (-K_X + t \omega_X) \cap (-K_X + \epsilon \omega_X)^{n-2})^2
\]

\[
\frac{(-K_X + t \omega_X)^2 \cap (-K_X + \epsilon \omega_X)^{n-2}}{(-K_X + t \omega_X)^2 \cap (-K_X + \epsilon \omega_X)^{n-2}}.
\]

Let us observe that by (20) we have
\[
c_1(F_i/F_{i-1}) \cap (-K_X) \cap (\omega_X)^{n-2} = 0 \quad \text{for any } i.
\]

Then
\[
c_2(T_X) \cap (-K_X + \epsilon \omega_X)^{n-2}
\]
$$\geq \lim_{t \to 0^+} - \sum_i \frac{1}{r_i} \frac{(te^{n-2}c_1(F_i/F_{i-1}) \wedge \omega_{X}^{n-1})^2}{t^2e^{n-2}\omega_X^n + te^{n-2}(-K_X)\omega_X^n} = 0.$$ 

\[ \square \]

It is interesting to study the case where the equality holds in (26) of Proposition 3.5. We will prove that in this case, $X$ is either a torus or a smooth $\mathbb{P}^1$-fibration over a torus. Before proving this result, we first prove an auxiliary lemma.

Lemma 3.6. Let $(X, \omega_X)$ be a compact Kähler manifold with nef anticanonical bundle. Let

$$0 = F_0 \subset F_1 \subset \ldots \subset F_i = T_X$$

be a stable filtration of the Harder-Narasimhan semistable filtration of $T_X$ with respect to $(c_1(-K_X) + \epsilon \omega_X)^{n-2}$ as in (27). If $X$ is not a torus and

$$\int_X c_2(T_X) \wedge (c_1(-K_X) + \epsilon \omega_X)^{n-2} = 0$$

for some $\epsilon > 0$ small enough, we have

(i) $\text{nd}(-K_X) = 1$.

(ii) $(F_i/F_{i-1})^{**}$ is projectively flat for all $i$, i.e., $(F_i/F_{i-1})^{**}$ is locally free and there exists a smooth metric $h$ on it such that $i \Theta_h(F_i/F_{i-1})^{**} = \alpha_i \cdot \text{Id}$ for some $\alpha_i \in H^{1,1}(X, \mathbb{Z})$. Moreover, $\alpha_i$ is nef and proportional to $c_1(-K_X)$.

(iii) $c_2(F_i/F_{i-1}) = 0$ for all $i$, and (38) is regular outside a subvariety of codimension at least 3.

Remark 3.7. We first remark that for a vector bundle $V$ of rank $k$ supported on a subvariety $j : Z \subset X$ of codimension $r$, by the Grothendieck-Riemann-Roch theorem, we have

$$c_r(j_*(V)) = (-1)^{r-1}(r-1)!k[Z].$$

Therefore for any torsion free sheaf $E$, we have $c_2(E) \geq c_2(E^{**})$ and the equality holds if and only if $E = E^{**}$ outside a subvariety of codimension at least 3.

Proof. If $\text{nd}(-K_X) = 0$, then $K_X$ is numerically trivial. In this case, we know that (39) implies that $X$ is a torus (cf. for instance, [Tia00, Thm 2.13]). From now on, we suppose that $\text{nd}(-K_X) \geq 1$. By the proof of Proposition 3.5 the equality (39) implies that the filtration (38) is in the case (3), i.e., $\text{nd}(-K_X) = 1$. Then

$$\int_X c_2(T_X) \wedge (c_1(-K_X) + \epsilon \omega_X)^{n-2} = 0.$$

By the proof of Proposition 3.5 the equality (39) implies also that the filtration (38) also satisfies:

$$\int_X c_1(F_i/F_{i-1})^2 \wedge (c_1(-K_X) + \epsilon \omega_X)^{n-2} = 0.$$

Moreover, (39) implies that the equality holds in (42), which implies that, (cf. the proof of [Miy87] Thm 6.1, Cor 4.7)

$$\int_X c_2((F_i/F_{i-1})^{**}) \wedge (c_1(-K_X) + \epsilon \omega_X)^{n-2} = \int_X c_2(F_i/F_{i-1}) \wedge (c_1(-K_X) + \epsilon \omega_X)^{n-2} = 0.$$

We now check (ii) and (iii). By [BS94] Cor 3, (42) and (41) imply that $(F_i/F_{i-1})^{**}$ is locally free and projectively flat. The first part of (ii) is proved.
For the second part of $(ii)$, since $(\mathcal{F}_i/\mathcal{F}_{i-1})^{**}$ is projectively flat, there exists an $\alpha_i \in H^{1,1}(X, \mathbb{Z})$, such that $c_i(\mathcal{F}_i/\mathcal{F}_{i-1}) = \text{rank}(\mathcal{F}_i/\mathcal{F}_{i-1}) \cdot \alpha_i$. By (20), we have

$$\int_X c_i(-K_X) \wedge c_i(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge (c_i(-K_X) + e\omega_X)^{n-2} = 0.$$  

Combining this with (40) and (41), by the Hodge index theorem we obtain that $c_i(\mathcal{F}_i/\mathcal{F}_{i-1}) = a_i \cdot c_i(-K_X)$ for some $a_i \in \mathbb{Q}$. By Theorem 2.1, we have

$$c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge (c_1(-K_X) + e\omega_X)^{n-1} \geq 0.$$ 

Therefore $a_i \geq 0$ and $\alpha_i$ is nef. $(ii)$ is proved. As for $(iii)$, by (i) and $(ii)$, we obtain that $c_2((\mathcal{F}_i/\mathcal{F}_{i-1})^{**}) = 0$. Combining this with (42) and the Remark 3.7, we get $(iii)$. 

Using an idea of A. Höring, we finally prove that

**Proposition 3.8.** Let $(X, \omega_X)$ be a projective manifold with nef anticanonical bundle. We suppose that $\int_X c_2(X) \wedge (c_1(-K_X) + e\omega_X)^{n-2} = 0$ for some $e > 0$ small enough. Then after a finite étale cover, $X$ is either a torus or a smooth $\mathbb{P}^1$-fibration over a torus.

**Proof.** If $X$ is a torus, the proposition is proved. From now on, we suppose that $X$ is not a torus. Let

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_i = T_X$$

a stable filtration of the Harder-Narasimhan semistable filtration of $T_X$ with respect to $(c_1(-K_X) + e\omega_X)^{n-1}$ as in (27). Let $Z$ be the locus where (13) is not regular. By Lemma 3.6, we have $\text{codim}_X Z \geq 3$ and $\text{nd}(-K_X) = 1$. Moreover, $(ii)$ of Lemma 3.6 implies that $T_X$ is nef on any smooth projective curve in $X \setminus Z$.

By the condition $\text{nd}(-K_X) = 1$, we know that $K_X$ is not nef. Then there exists a Mori contraction $\varphi: X \to Y$. Since $\text{nd}(-K_X) = 1$ and $-K_X$ is ample on all the $\varphi$-fibres, we see that all the $\varphi$-fibres have dimension at most one. By Ando’s theorem [And85], we know that $\varphi$ is either a blow-up along a smooth subvariety of codimension two or a conic bundle. There are three cases.

1st case: $\varphi: X \to Y$ is a blow-up along a smooth subvariety $S$ of $Y$ and $\text{codim}_Y S = 2$. Let $E$ be the exceptional divisor. Then a general fiber $F$ of $\varphi: E \to S$ does not meet $Z$, since $\text{codim}_X Z \geq 3$. Therefore $T_X|_F$ is nef. On the other hand, since $F = \mathbb{P}^1$, we have a direct decomposition

$$T_X|_F = T_E|_F \oplus N_{E/X}|_F = T_E|_F \oplus |E||_F.$$

Since $|E||_F$ is strictly positive, $T_X|_F$ must contain a strictly negative part. We get a contradiction.

2nd case: $\varphi: X \to Y$ is a conic bundle with a non-empty discriminant $\Delta \subset Y$. Let $C \subset X$ be an irreducible component of a general fiber of $\varphi: \varphi^{-1}(\Delta) \to \Delta$. Then $T_X|_C$ is not nef (cf. [And85] Thm 3.1, Lemma 1.5). On the other hand, it is known that $\text{codim}_Y \Delta = 1$ (cf. for instance, [Sat82]). Since $C$ is a general fiber over $\Delta$, we obtain that $C \cap Z = \emptyset$. Therefore $T_X|_C$ is nef. We get a contradiction.

\footnote{In fact, let $Q(\alpha, \beta) = \int_X \alpha \wedge \beta \wedge (c_1(-K_X) + e\omega_X)^{n-2}$. Then $Q$ is of index $(1, m)$. Let $V$ be the subspace of $H^{1,1}(X, \mathbb{R})$ where $Q$ is negative definite. If $Q(\alpha_1, \alpha_1) = Q(\alpha_2, \alpha_2) = Q(\alpha_1, \alpha_2) = 0$ for some non trivial $\alpha_1, \alpha_2$, then both $\alpha_1$ and $\alpha_2$ are not contained in $V$. Therefore we can find a $t \in \mathbb{R}$, such that $(\alpha_1 - t\alpha_2) \in V$. Since $Q(\alpha_1 - t\alpha_2, \alpha_1 - t\alpha_2) = 0$, we get $\alpha_1 - t\alpha_2 = 0$. Therefore $\alpha_1$ is proportional to $\alpha_2$.}
3rd case: $\varphi: X \to Y$ is a smooth $\mathbb{P}^1$-bundle. By [Miy83, 4.11] we have
$$\varphi_*(K_X^2) = -4K_Y,$$
so $K_X^2 = 0$ implies that $K_Y \equiv 0$. By the condition
$$\int_X c_2(X) \wedge (c_1(-K_X) + \epsilon \omega_X)^{n-2} = 0,$$
we obtain that $c_2(Y) = 0$. Therefore, after a finite étale cover, $Y$ is a torus. The proposition is proved.

\[ \Box \]

Remark 3.9. In general, if $\int_X c_2(X) \wedge (c_1(-K_X) + \epsilon \omega_X)^{n-2} = 0$, We cannot hope that $X$ can be covered by a torus. In fact, the example [DPS94, Example 3.3] satisfies the equality $c_2(X) = 0$ and $X$ can not be decomposed as direct product of torus with $\mathbb{P}^1$. Using [DHP08], we know that $X$ cannot be covered by torus. Therefore we propose the following conjecture, which is a mild modification of the question of Yau:

Conjecture 3.10. Let $(X, \omega_X)$ be a compact Kähler manifold with nef anticanonical bundle. Then $\int_X c_2(T_X) \wedge \omega_X^{n-2} \geq 0$. If the equality holds for some Kähler metric, then $X$ is either a torus or a smooth $\mathbb{P}^1$-fibration over a torus.

Remark 3.11. If one could prove that $T_X$ is generically nef with respect to the polarizations $(c_1(-K_X) + \epsilon \omega, \omega, \cdots, \omega)$ for any $\epsilon > 0$ small enough, using the same argument as in this section, one could prove this conjecture.

As the last application of Theorem 2.1 we give a new proof of the surjectivity of Albanese map when $X$ is a compact Kähler manifold with nef anticanonical bundle.

Proposition 3.12. Let $(X, \omega)$ be a compact Kähler manifold with nef anticanonical bundle. Then the Albanese map is surjective, and smooth outside a subvariety of codimension at least 2. In particular, the fibers of the Albanese map are connected and reduced in codimension 1.

Proof. Let

\begin{equation}
0 \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_s = T_X
\end{equation}

be a filtration of torsion-free subsheaves such that $\mathcal{E}_{i+1}/\mathcal{E}_i$ is an $\omega$-stable torsion-free subsheaf of $T_X/\mathcal{E}_i$ of maximal slope.

Case 1: $\mathcal{E}_i$ is regular, i.e., all $\mathcal{E}_i$ and $\mathcal{E}_i/\mathcal{E}_{i-1}$ are locally free.

In this case, we can prove that the Albanese map is submersive. Let $\tau \in H^0(X, T_X^*)$ be a nontrivial element. To prove that the Albanese map is submersive, it is sufficient to prove that $\tau$ is non vanishing everywhere. Thanks to Theorem 2.1 and the stability condition of $\mathcal{E}_i/\mathcal{E}_{i-1}$, we can find a smooth metric $h_i$ on $\mathcal{E}_i/\mathcal{E}_{i-1}$ such that
$$\frac{i\Theta_{h_i}(\mathcal{E}_i/\mathcal{E}_{i-1}) \wedge \omega^{n-1}}{\omega^n} = \lambda_i \cdot \text{Id}_{\mathcal{E}_i/\mathcal{E}_{i-1}},$$
for some constant $\lambda_i \geq 0$. Thanks to the construction of $\{h_i\}$ and Lemma 1.7 for any $\epsilon > 0$, there exists a smooth metric $h_\epsilon$ on $T_X$, such that
\begin{equation}
\frac{i\Theta_{h_\epsilon}(T_X) \wedge \omega^{n-1}}{\omega^n} \geq -\epsilon \cdot \text{Id}_{T_X},
\end{equation}
for some constant $\epsilon > 0$.
and the matrix valued (1,1)-form \(i\Theta_{h_\epsilon}(T_X)\) is uniformly bounded. Let \(h_\epsilon^*\) be the dual metric on \(T_X^*\). Then the closed (1,1)-current
\[
T_\epsilon = \frac{i}{2\pi} \partial\overline{\partial} \ln \|\tau\|_{h_\epsilon^*}^2
\]
satisfies
\[
(46) \quad T_\epsilon \geq -\frac{i \Theta_{h_\epsilon^*}(T_X^*) \tau, \tau}{\|\tau\|_{h_\epsilon^*}^2}.
\]
Since \(-\Theta_{h_\epsilon^*}(T_X^*) = i \Theta_{h_\epsilon}(T_X)\), (45) and (46) imply a pointwise estimate
\[
(47) \quad T_\epsilon \wedge \omega^{n-1} \geq -\epsilon \omega^n.
\]

We suppose by contradiction that \(\tau(x) = 0\) for some point \(x \in X\). By Lemma 1.7, \(i\Theta_{h_\epsilon}(T_X)\) is uniformly lower bounded. Therefore, there exists a constant \(C\) such that \(T_\epsilon + C \omega\) is a positive current for any \(\epsilon\). After replacing by a subsequence, we can thus suppose that \(T_\epsilon\) converge weakly to a current \(T\), and \(T + C \omega\) is a positive current. Since \(\tau(x) = 0\), we have
\[
\nu(T_\epsilon + C \omega, x) \geq 1 \quad \text{for any} \quad \epsilon,
\]
where \(\nu(T_\epsilon + C \omega, x)\) is the Lelong number of the current \(T_\epsilon + C \omega\) at \(x\). Using the main theorem in [Siu74], we obtain that
\[
\nu(T + C \omega, x) \geq 1.
\]
Therefore there exists a constant \(C_1 > 0\) such that
\[
\int_{B_x(r)} (T + C \omega) \wedge \omega^{n-1} \geq C_1 \cdot r^{2n-2} \quad \text{for} \quad r \text{ small enough},
\]
where \(B_x(r)\) is the ball of radius \(r\) centered at \(x\). Then
\[
\int_{U_x} T \wedge \omega^{n-1} > 0
\]
for some neighborhood \(U_x\) of \(x\). Therefore
\[
(48) \quad \lim_{\epsilon \to 0} \int_{U_x} T_\epsilon \wedge \omega^{n-1} > 0.
\]
Combining (47) with (48), we obtain
\[
\lim_{\epsilon \to 0} \int_X T_\epsilon \wedge \omega^{n-1} > 0.
\]
We get a contradiction by observing that all \(T_\epsilon\) are exact forms.

**Case 2: General case**

By Lemma 1.6 there exists a desingularization \(\pi : \tilde{X} \to X\), such that \(\pi^*(T_X)\) admits a filtration:
\[
0 \subset E_1 \subset E_2 \subset \cdots \subset \pi^*(T_X)
\]
satisfying that \(E_i, E_i/E_{i-1}\) are vector bundles and \(\pi^*(E_i) = E_i\) on \(X \setminus Z\), where \(Z\) is an analytic subset of codimension at least 2. Let \(\tau \in H^0(X, T^*_X)\) be a nontrivial element. Our aim is to prove that \(\tau\) is non vanishing outside \(Z\).

Let \(x \in \tilde{X} \setminus \pi^*(Z)\). Let \(U_x\) be a small neighborhood of \(x\) such that \(U_x \subset \tilde{X} \setminus \pi^*(Z)\). We suppose by contradiction that \(\pi^*(\tau)(x) = 0\). By [BS94], there exists Hermitian-Einstein metrics \(h_{\epsilon,i}\) on \(E_i/E_{i-1}\) with respect to \(\pi^*\omega + \epsilon \omega_{\tilde{X}}^\epsilon\), and
\{i\Theta_{h_{\epsilon_i}}(E_i/E_{i-1})\}_\epsilon is uniformly bounded on \(U_x\). Combining this with Lemma 1.7, we can construct a smooth metric \(h_\epsilon\) on \(\pi^*(T_X)\) such that
\[
i\Theta_{h_\epsilon}(\pi^*(T_X)) \wedge (\pi^*\omega + \epsilon \omega_X)^n \geq -2C\epsilon \cdot \text{Id}_{\pi^*(T_X)},
\]
and \(i\Theta_{h_\epsilon}(\pi^*(T_X))\) is uniformly bounded on \(U_x\). Let \(T_\epsilon = \frac{i}{2\pi} \partial \overline{\partial} \ln \|\pi^*(\tau)\|^2_{\omega_\epsilon}\). By the same argument as in Case 1, the uniform boundedness of \(i\Theta_{h_\epsilon}(\pi^*(T_X))\) in a neighborhood of \(x\) implies the existence of a neighborhood \(U'_x\) of \(x\) and a constant \(c > 0\), such that
\[
\lim_{\epsilon \to 0} \int_{U'_x} T_\epsilon \wedge (\pi^*(\omega) + \epsilon \omega_X)^{n-1} \geq c.
\]
Combining this with (49), we get
\[
\lim_{\epsilon \to 0} \int_X T_\epsilon \wedge (\pi^*(\omega) + \epsilon \omega_X)^{n-1} \geq c,
\]
which contradicts with the fact that all \(T_\epsilon\) are exact. Therefore \(\tau\) is non vanishing outside \(Z\). Proposition 3.12 is proved. \(\Box\)

**Appendix A. A Bochner technique proof**

We would like to give a proof of the implication \((iii) \Rightarrow (ii)\) in Proposition 3.1 without using [BM01 Thm 0.1].

**Proof.** By [CDP12 Criterion 1.1], to prove the implication, it is sufficient to prove that for some ample line bundle \(F\) on \(X\), there exists a constant \(C_F > 0\), such that
\[
H^0(X, (T_X)^\otimes m \otimes F^\otimes k) = 0 \quad \text{for all } m, k \text{ satisfying } m \geq C_F \cdot k.
\]

Thanks to the condition \((iii)\), there exists a Kähler class \(A\), such that for the Harder-Narasimhan semistable filtration with respect to \(A\)
\[
0 = F_0 \subset F_1 \subset \cdots \subset F_k = T_X,
\]
we have
\[
\mu_A(F_i/F_{i-1}) \geq c \quad \text{for all } i,
\]
for some constant \(c > 0\). Moreover, for the Harder-Narasimhan filtration of \((T_X)^\otimes m\) with respect to \(A\), \(m \cdot c\) is also a lower bound of the minimal slope with respect to the filtration.

We now prove (50) by a basic Bochner technique. After replacing by a more refined filtration, we can suppose that
\[
0 \subset E_0 \subset E_1 \subset \cdots \subset E_s = (T_X)^\otimes m
\]
is a filtration of torsion-free subsheaves such that \(E_{i+1}/E_i\) is an \(\omega\)-stable torsion-free subsheaf of \(T_X/E_i\) of maximal slope for simplicity. Let \(\omega\) be a positive \((1,1)\)-form representing \(c_1(A)\).

---

5In fact, [HS94] proved that \(h_{\epsilon_i} \) and \(h_{\epsilon_i}^{-1}\) are \(C^{1,\alpha}\)-uniform bounded in \(U_x\). Since \(U_x\) is in \(X \setminus Z\), \(\omega := \pi^*\omega + \epsilon \omega_X\) is uniformly strict positive on \(U_x\). By [Ko87 Chapter I, (14.16)] and Hermitian-Einstein condition, we obtain that \(\Delta_{\omega}(h_{\epsilon_i})_{j,k}\) is uniformly \(C^\alpha\) bounded on \(U_x\), where \(\Delta_{\omega}\) is the Laplacian with respect to \(\omega\) and \((h_{\epsilon_i})_{j,k} := h_{\epsilon_i}(e_j, e_k)\) for a fixed base \(\{e_k\}\) of \(E_i/E_{i-1}\). The standard elliptic estimates gives the uniform boundedness of \(i\Theta_{h_{\epsilon_i}}(E_i/E_{i-1})\) on \(U_x\).
If (51) is regular, then there exists a Hermitian-Einstein metric on every quotient. Since $\mu_A(E_i/E_{i-1}) \geq c \cdot m$, thanks to Lemma 1.7 we can construct a smooth metric $h$ on $(T_X)^\otimes m$, such that
\[ i\Theta_h(T_X^\otimes m) \wedge \omega^{n-1} \geq \frac{m \cdot c}{2} \text{Id.} \]
Let $\tau \in H^0(X, (T_X)^\otimes m \otimes F^\otimes k)$. We have
\[ \Delta_\omega(\|\tau\|^2) = \|D'_h\tau\|^2 - \frac{i\Theta_h^\ast((T_X^\otimes m \otimes F^\otimes k)\tau, \tau) \wedge \omega^{n-1}}{\omega^n}. \]
Let $C_F$ be a constant large enough with respect to $c$. If $m \geq C_F \cdot k$, (52) implies that
\[ \int_X \|D'_h\tau\|^2 \omega^n - \langle i\Theta_h^\ast((T_X^\otimes m \otimes F^\otimes k)\tau, \tau) \wedge \omega^{n-1} \rangle \geq c_1 \|\tau\|^2, \]
for some constant $c_1 > 0$. Observing moreover
\[ \int_X \Delta_\omega(\|\tau\|^2)\omega^n = 0, \]
we get $\tau = 0$.

If (51) is not necessary regular, by Lemma 1.6 we can find a resolution $\pi : \tilde{X} \rightarrow X$ such that there exists a regular filtration
\[ 0 \subset E_1 \subset E_2 \subset \cdots \subset \pi^\ast(T_X) \]
and
\[ \mu_{\pi^\ast(A)}(E_i/E_{i-1}) = \mu_A(E_i/E_{i-1}) \geq c \cdot m. \]
Thanks to the strict positivity of $c$, if $\epsilon$ is small enough,
\[ \mu_{\epsilon}(E_i/E_{i-1}) \geq \frac{c \cdot m}{2} \quad \text{for any } i, \]
where $\mu_{\epsilon}$ is the slope with respect to $\pi^\ast(A) + \epsilon \omega_{\tilde{X}}$. Thanks to Remark 2.2, $E_i/E_{i-1}$ are also stable for $\pi^\ast(A) + \epsilon \omega_{\tilde{X}}$ when $\epsilon$ small enough. Therefore there exists a smooth Hermitian-Einstein metric on every quotient $E_i/E_{i-1}$. Using Lemma 1.7, (54) implies the existence of a smooth metric $h_{\epsilon}$ on $\pi^\ast(T_X)^\otimes m$, such that
\[ i\Theta_{h_{\epsilon}}(\pi^\ast(T_X)^\otimes m) \wedge (\pi^\ast(\omega) + \epsilon \omega_{\tilde{X}})^{n-1} \geq \frac{m \cdot c}{4} \text{Id} \]
for $\epsilon$ small enough. Using the same Bochner technique as in (52) and (53), applied to $\pi^\ast(T_X)$ with respect to $\pi^\ast(A) + \epsilon \omega_{\tilde{X}}$, we get
\[ H^0(\tilde{X}, \pi^\ast((T_X)^\otimes m \otimes F^\otimes k)) = 0 \quad \text{for all } m, k \text{ satisfying } m \geq C_F \cdot k. \]
(50) is thus proved. \qed

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