COMPARISON OF POISSON STRUCTURES AND POISSON-LIE DYNAMICAL $r$-MATRICES

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Abstract. We construct a Poisson isomorphism between the formal Poisson manifolds $g^*$ and $G^*$, where $g$ is a finite dimensional quasitriangular Lie bialgebra. Here $g^*$ is equipped with its Lie-Poisson (or Kostant-Kirillov-Souriau) structure, and $G^*$ with its Poisson-Lie structure. We also quantize the Poisson-Lie dynamical $r$-matrices of Balog-Fehér-Palla.

INTRODUCTION AND MAIN RESULTS

We construct Poisson isomorphisms between the formal Poisson manifolds $g^*$ and $G^*$, where $g$ is a finite dimensional quasitriangular Lie bialgebra (Thm. 0.1). Here $g^*$ is equipped with its Lie-Poisson (or Kostant-Kirillov-Souriau) structure, and $G^*$ with its Poisson-Lie structure.

Thm. 0.1 may be viewed as a generalization of the formal version of [GW] (later reproved in [Al], Thm. 1, and [Bo]), where Ginzburg and Weinstein construct a Poisson diffeomorphism between the real Poisson manifolds $k^*$ and $K^*$, where $K$ is a compact Lie group and $k$ is its Lie algebra. It can also be viewed as a new result in the subject of linearization of Poisson structures; e.g., in contrast with our result, it has been shown in [Ch] that not all Poisson structures on Poisson-Lie groups are linearizable.

We give two proofs of Thm. 0.1. The first proof relies on a nondegeneracy assumption and is geometric. Namely, it relies on the construction of a map $g(\lambda) : g^* \to G$ satisfying a differential equation, which is achieved by using the theory of the classical Yang-Baxter equation and gauge transformations. Note that a geometric proof of the same result, not relying on the nondegeneracy assumption, is given in [AM2]. The second proof relies of the theory of quantization of Lie bialgebras.

We then apply Thm. 0.1 to the quantization of Poisson-Lie dynamical $r$-matrices introduced in [FM1] (see Thm. 0.4).

We now describe our results in more detail.

0.1. Comparison of Poisson structures: statement and proofs of Thm. 0.1

0.1.1. Formulation of Thm. 0.1 Let $(g, r)$ be a finite dimensional quasitriangular Lie bialgebra over a field $k$ of characteristic 0. Recall that this means that $g$ is a Lie algebra, $r \in g^\otimes 2$, $t := r + r^{2,1}$ is invariant, and CYB($r$) := $[r^{1,2}, r^{1,3}] + [r^{1,2}, r^{2,3}] + [r^{1,3}, r^{2,3}] = 0$. The Lie cobracket on $g$ is defined by $\delta(x) := [x \otimes 1 + 1 \otimes x, r]$ for any $x \in g$.

The Lie bracket on $g$ defines a linear Poisson structure on $g^*$; we also denote by $\mathfrak{g}^*$ the formal neighborhood of the origin in this vector space, which is then a formal Poisson manifold. On the other hand, we denote by $G^*$ the formal Poisson-Lie group with Lie bialgebra $g^*$. Our main result is:

Theorem 0.1. The formal Poisson manifolds $g^*$ and $G^*$ are isomorphic.

0.1.2. Structure of the proof of Thm. 0.1 Thm. 0.1 will be proved in two ways: (a) under the assumption that $t \in S^2(g)$ is nondegenerate as a consequence of Props. 0.2 and 0.3 (proved
0.1.3. The statements leading to the proof of Thm. 0.1.4 We now formulate Props. 0.2 and 0.3. Denote by $G$ the formal group with Lie algebra $\mathfrak{g}$ and by $\text{Map}_0(\mathfrak{g}^*, G)$ the space of formal maps $g : \mathfrak{g}^* \to G$, such that $g(0) = 1$; this is the space of maps of the form $e^{x(\lambda)}$, where $x(\lambda) \in \mathfrak{g} \otimes \hat{S}(\mathfrak{g})_{>0}$. The set $\text{Map}_0(\mathfrak{g}^*, G)$ has a group structure, defined by $(g_1 * g_2)(\lambda) := g_2(\text{Ad}(\mathfrak{g}^*(\lambda))(\lambda))g_1(\lambda)$. Its subspace of all maps $g(\lambda)$ such that

\begin{equation}
1. g_1^{-1}d_2(g_1)(\lambda) - g_2^{-1}d_1(g_2)(\lambda) + [\text{id} \otimes \text{id} \otimes \lambda, [g_1^{-1}d_3(g_1)(\lambda), g_2^{-1}d_3(g_2)(\lambda)]] = 0
\end{equation}

is a subgroup $\text{Map}_0^{\text{ham}}(\mathfrak{g}^*, G)$. Here $g_1^{-1}d_2(g_1)(\lambda) = \sum_{i} g_1^{-1}d_2g(\lambda) \otimes e_i$ is viewed as a formal function $\mathfrak{g}^* \to \hat{\mathfrak{g}}^{\otimes 2}$; $(e_1^i, e_2^i)$ are dual bases of $\mathfrak{g}^*$ and $\mathfrak{g}$, $g_1^{-1}d_2g(\lambda) = (g_1^{-1}d_2(g_1))^{1-i}$ and $\partial x(g(\lambda)) := (d/dx)|_{x=0} g(\lambda + \varepsilon x)$. We will also denote by $g_1^{-1}d_2(g_13)$ the same quantity, viewed as an element of $\hat{\mathfrak{g}}^{\otimes 2} \hat{\otimes} \hat{S}(\mathfrak{g})$.

We have a group morphism $\theta : \text{Map}_0^{\text{ham}}(\mathfrak{g}^*, G) \to \text{Aut}_1(\mathfrak{g})^\text{op}$ to the group (with opposite structure) of Poisson automorphisms of $\mathfrak{g}$* with differential at 0 equal to the identity, taking $g(\lambda)$ to the automorphism $\lambda \to \text{Ad}(\mathfrak{g}^*(\lambda))(\lambda)$.

**Proposition 0.2.** There exists a formal map $g(\lambda) \in \text{Map}_0(\mathfrak{g}^*, G)$, such that

\begin{equation}
2. (g_1)^{-1}d_2(g_1) - (g_2)^{-1}d_1(g_2) + \text{Ad}(\mathfrak{g}^* \otimes \lambda, [(g_1)^{-1}d_3(g_1), (g_2)^{-1}d_3(g_2)]) = \rho_{\text{AM}}.
\end{equation}

(Identity of formal maps $\mathfrak{g}^* \to \Lambda^2(\mathfrak{g})$.) Here $r_0 = (r - r^2)/2$, and $\rho_{\text{AM}}$ is the Alekseev- Meinrenken $r$-matrix \cite{AM1, BDR} given by

$$
\rho_{\text{AM}}(\lambda) = (\text{id} \otimes \varphi(\text{ad} \lambda^\vee))(t),
$$

where $\lambda^\vee = (\lambda \otimes \text{id})(t)$ and $\varphi(z) := -\frac{1}{2} + \frac{1}{2} \cotanh \frac{z}{2}$.

The group $\text{Map}_0^{\text{ham}}(\mathfrak{g}^*, G)$ acts simply and transitively on the space of solutions $g(\lambda)$ of (2), as follows: $(\alpha * g)(\lambda) = g(\text{Ad}(\alpha)(\lambda))(\lambda)\alpha(\lambda)$.

**Proposition 0.3.** Assume that $t \in S^2(\mathfrak{g})$ is nondegenerate. Let $g(\lambda) \in \text{Map}_0(\mathfrak{g}^*, G)$ be as in Prop. 0.2. There exists a unique isomorphism $g^*(\lambda) : \mathfrak{g}^* \to G^*$, defined by the identity

$$
g(\lambda)e^{\lambda^\vee} g(\lambda)^{-1} = L(\mathfrak{g}^*(\lambda))R(\mathfrak{g}^*(\lambda))^{-1}.
$$

Here $L, R : G^* \to G$ are the formal group morphisms corresponding to the Lie algebra morphisms $L, R : \mathfrak{g}^* \to \mathfrak{g}$ given by $L(\lambda) := (\lambda \otimes \text{id})(r)$, $R(\lambda) := -(\lambda \otimes \text{id})(r^2)$. 1

In other words, we have (non-Poisson) formal manifold isomorphisms $\mathfrak{g}^* \xrightarrow{\theta} G \xleftarrow{b} G^*$, $a(\lambda) = g(\lambda)e^{\lambda^\vee} g(\lambda)^{-1}$, $b(\mathfrak{g}^*) = L(\mathfrak{g}^*)R(\mathfrak{g}^*)^{-1}$, and $g^*(\lambda) = b^{-1} \circ a(\lambda)$.

The isomorphism $(\alpha * g)^*(\lambda) : \mathfrak{g}^* \to G^*$ corresponding to $(\alpha * g)(\lambda)$ is such that $(\alpha * g)^*(\lambda) = g^*(\theta(\alpha)(\lambda))$.

Props. 0.2 and 0.3 immediately imply Thm. 0.1 under the assumption that $t \in S^2(\mathfrak{g})$ is nondegenerate.

These propositions also imply that the set of all isomorphisms $\mathfrak{g}^* \to G^*$ that they allow to construct is a principal homogeneous space under the image of $\theta : \text{Map}_0^{\text{ham}}(\mathfrak{g}^*, G) \to \text{Aut}_1(\mathfrak{g}^*)$. When $\mathfrak{g}$ is semisimple, any derivation $\mathfrak{g} \to S^k(\mathfrak{g})$ is inner, so $\theta$ is surjective. So in that case, our construction yields all the Poisson isomorphisms $\mathfrak{g}^* \to G^*$ taking 0 to 1 and with differential at 0 equal to the identity.

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1We denote by $\hat{S}(\mathfrak{g})$ the degree completion $\hat{\mathfrak{g}}_{k>0} S^k(\mathfrak{g})$ of the symmetric algebra $S(\mathfrak{g})$, and set $\hat{S}(\mathfrak{g})_{>1} = \hat{\mathfrak{g}}_{k>0} S^k(\mathfrak{g})$.

2We view $\Lambda^n(\mathfrak{g})$ as a subspace of $\mathfrak{g}^{\otimes n}$. 

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geometrically in Sect. 1.1.1 and using quantization in Sects. 1.2.2 and 1.2.3; (b) unconditionally in Sect. 1.2.1.
0.2. Quantization of Poisson-Lie dynamical $r$-matrices. In [BFP] [FMI], there was introduced the Poisson-Lie dynamical $r$-matrix
\[
\rho_{\text{PL}}(g^*) = \left( \text{id} \otimes (\nu \frac{\text{id} + a(g^*)^{2\nu}}{\text{id} - a(g^*)^{2\nu}} - \frac{1 \text{id} + a(g^*)}{2 \text{id} - a(g^*)}) \right)(t).
\]
Here $a(g^*) : G^* \to GL(g)$ is defined by $a(g^*) = \text{Ad}(L(g^*)R(g^*)^{-1})$ and $\nu$ is a fixed element in $k$. Then $\rho_{\text{PL}}$ is a formal map $G^* \to \wedge^2(g)$, i.e., an element of $\wedge^2(g) \otimes \mathcal{O}_{G^*}$. It is a Poisson-Lie classical dynamical $r$-matrix for $(G^*, g, Z_\nu)$, where $Z_\nu = (\nu^2 - \frac{1}{4})(t^{1,2}, t^{2,3})$ (see also [FEM]).

A quantization of this dynamical $r$-matrix is the data of a quantization $U_h(g)$ of the Lie bialgebra $g$, and a pair $(\hat{J}, \hat{\Phi})$, where:

- $\hat{\Phi}$ is an associator for $U_h(g)$, i.e., $\hat{\Phi} \in U_h(g)^{\otimes 3}$ satisfies the pentagon equation $\hat{\Phi}^{2,3,4} \hat{\Phi}^{1,23,4} \hat{\Phi}^{1,2,3} = \hat{\Phi}^{1,2,3} \hat{\Phi}^{3,2,4} \hat{\Phi}^{1,2,3,4}$, the invariance condition $[\hat{\Phi}, (\Delta \otimes \text{id}) \circ \Delta(a)] = 0$ for any $a \in U_h(g)$, and the expansion $\hat{\Phi} = 1 + h^2 \hat{\phi}_2$, where $\hat{\phi}_2 \in U_h(g)^{\otimes 3}$ satisfies $\sum_{\sigma \in S_3} \text{sign}(\sigma) \sigma(\hat{\phi}_2)_{h=0} = Z_\nu$;
- $\hat{J} \in U_h(g)^{\otimes 2} \otimes U_h(g)^\prime$ satisfies the dynamical twist equation $\hat{J}^{2,3,4} \hat{J}^{2,3,4} \hat{J}^{1,23,4} \hat{J}^{1,2,3,4} = \hat{J}^{1,2,3} \hat{J}^{2,3,4} \hat{J}^{1,2,3,4}$, the invariance condition $[\hat{J}, (\Delta \otimes \text{id}) \circ \Delta(a)] = 0$ for any $a \in U_h(g)$, and the expansion $\hat{J} = 1 + h^2 \hat{J}_1$, where $\hat{J}_1 \in U_h(g)^{\otimes 2} \otimes U_h(g)^\prime$ satisfies $(\hat{J}_1 - \hat{J}_1^{2,1,3})_{h=0} = \rho_{\text{PL}}$.

**Theorem 0.4.** Such a quantization exists (see Thm. [ZJ]).

1. CONSTRUCTION OF ISOMORPHISMS $g^* \to G^*$

We give two families of proofs of the statements announced in Sect. [0.1,1]. The first family is geometric and is contained in Sect. [1.1]. The second family relies on quantization of Lie bialgebras and is contained in Sect. [1.2].

Recall that the statements from Sect. [0.1,1] comprise Thm. [0.1], Prop. [0.2], and Prop. [0.3]. It is noted in Sect. [0.1,3] that Thm. [0.1] follows from Props. [0.2] and [0.3] if $t$ is nondegenerate. We give two proofs of these propositions: (a) a Poisson geometric proof (Sect. [1.1]), and (b) a proof based on the theory of quantization of Lie bialgebras (Sects. [1.2,2] and [1.2,3]). In Sect. [1.2,4] we give an unconditional proof of Thm. [0.1].

1.1. Geometric construction.

1.1.1. Construction of $g(\lambda)$ (proof of Prop. [1.2]). One checks that $\text{Map}_{0}^\text{ham}(g^*, G)$ is a pronilpotent Lie group with Lie algebra $\{ \alpha \in g \otimes \hat{S}(g)_{\geq 1} | \text{Alt} \circ d(\alpha) = 0 \}$. This Lie algebra is isomorphic to $(\hat{S}(g)_{\geq 1}, \{-, -\})$ under $d : f \mapsto d(f)$.

Let us denote by $\mathcal{G}$ the set of all $g \in \text{Map}_{0}^\text{ham}(g^*, G)$ satisfying [2]. This is a subvariety of the proalgebraic variety $\text{Map}_{0}(g^*, G)$. One checks that $(\hat{S}(g)_{\geq 1}, \{-, -\})$ acts by vector fields on $\mathcal{G}$, by
\[
g^{-1} \delta_f(g) = (\text{id} \otimes \text{id} \otimes \lambda, [d_3(f_2), g_1^{-1}d_3(g_12))] - d_1(f_2) \in g \otimes \hat{S}(g)_{>0},
\]
and that the right infinitesimal action of $\text{Map}_{0}^\text{ham}(g^*, G)$ on itself is given by the same formula. It follows that the right action of $\text{Map}_{0}(g^*, G)$ on itself restricts to an action of $\text{Map}_{0}^\text{ham}(g^*, G)$ on $\mathcal{G}$.

We now prove that if $\mathcal{G}$ is nonempty, then $\text{Map}_{0}^\text{ham}(g^*, G)$ acts simply and transitively on $\mathcal{G}$.

Let us show that the action is simple. If $g, g' \in \mathcal{G}$ and $\alpha \in \text{Map}_{0}^\text{ham}(g^*, G)$ are such that $g + \alpha = g'$, then let $a := \text{log}(\alpha)$. Assume that $a \neq 0$ and let $n$ be the smallest integer such that

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3We denote by $\otimes$ the topological tensor product, defined as follows: if $V, W$ are topological vector spaces of the form $V = V_0[[x_1, \ldots, x_n]]$, then $V \otimes W = W_0[[y_1, \ldots, y_m]]$, and $V \otimes W := V_0 \otimes W_0[[x_1, \ldots, x_n, y_1, \ldots, y_m]]$.

4We set $\hat{\Phi}^{1,2,3,4} = (\Delta \otimes \text{id} \otimes \text{id})(\hat{\Phi})$, etc.

5For $\rho = \alpha \otimes f \in \wedge^n(g) \otimes \hat{S}(g)$, we set $d(\rho) := \sum_i \alpha \otimes e_i \otimes (d/d\xi)(\xi = 0) f(\lambda + \epsilon^2)$ and if $\xi \in \wedge^{n-1}(g) \otimes g$, we define $\text{Alt}(\xi \otimes f) := (\xi + \xi^{2,\ldots,n-1} + \cdots + \xi^{n,\ldots,n-1}) \otimes f$.
we determined \( \rho \) in \( U \) satisfies tribute (as the classical limit of \( \Phi \) is proportional to \( g \)).

Let us now prove that the action is transitive. Let \( g, g' \in G \). Set \( A := \log(g) \), \( A' := \log(g') \). Then \( A, A' \in g \otimes \hat{S}(g)_{>0} \). Assume that \( A \neq A' \) and let \( n \) be the smallest integer such that the component of \( A' - A \) in \( g \otimes S^n(g) \) is nonzero; we denote by \( (A' - A)_n \) this component. Comparing equations (2) for \( g \) and \( g' \), we get \( \text{Alt} \, \text{d}(A' - A)_n = 0 \). It follows that there exists \( a \in S^{n+1}(g) \), such that \((A' - A)_n = \text{Alt} \, \text{d}(a)\), i.e., \((A' - A)_n \in \text{Lie Map}_0^{\text{ham}}(g^*, G)\). Let \( \exp_s \) be the exponential map of \( \text{Map}_0^{\text{ham}}(g^*, G) \) and \( \alpha := \exp_s((A' - A)_n) \in \text{Map}_0^{\text{ham}}(g^*, G) \). Then \( \log(\alpha \ast g) = A(\text{Ad}^*(\alpha(\lambda)))(\lambda) + (A' - A)_n \mod g \otimes \hat{S}(g)_{\geq n+1} \); therefore the difference of logarithms of \( \alpha \ast g \) and \( g' \) coincide modulo \( g \otimes \hat{S}(g)_{\geq n+1} \). Working by successive approximations, we construct \( \beta \in \text{Map}_0^{\text{ham}}(g^*, G) \) such that \( \beta \ast g = g' \). This proves that the action is transitive.

Let us now prove that \( G \) is nonempty. Recall that \( r_0 = (r - r^{-2})/2 \). If \( g \in \text{Map}_0^{\text{ham}}(g^*, G) \), set \((r_0)^\theta := 1 \).h.s. of (2).

Lemma 1.1. Assume that \( \log(g) = -r/2 \) modulo \( g \otimes \hat{S}(g)_{>1} \). Set \( \rho := (r_0)^\theta \) and assume that \( \rho = \rho_{\text{inv}} + \alpha \), where \( \rho_{\text{inv}}, \alpha \in \wedge^2(g) \otimes \hat{S}(g) \), \( \rho_{\text{inv}} \) is \( g \)-invariant and \( \alpha \in g \otimes \hat{S}(g)_{\geq n} \). Then

\[
\text{CYB}(\rho) - \text{Alt}(\text{d}(\rho)) = Z_{1,2,3}^{1,2,3} - \frac{1}{2}[r^{1,234}, \alpha^{2,3,4}] \]

modulo \( g^{\otimes 3} \otimes \hat{S}(g)_{\geq n+1} \).

Proof of Lemma. This statement can be proved directly. It can also be viewed as the classical limit of the following statement. Let \( U = U(g)[[h]] \), let \( J = (U^2)^\times \), and let \( \Phi := (J^{1,2,3,4})^J \). If \( K \in (U^2)^\times \) and we set \( \Phi := (K^{1,2,3,4})^J \), then \( \Phi \) satisfies

\[
\text{d}(\Phi) = (\text{d}(\Phi))^{1,2,3,4} \Phi^{1,2,3} = (K^{1,2,3,4})^{1,2,3,4} \Phi^{1,2,3,4} \Phi^{1,2,3} \]

modulo \( g^{\otimes 3} \otimes \hat{S}(g)_{\geq n+1} \).

Here \((a, b) = aba^{-1}b^{-1} \). The statement of the lemma is recovered when \( J, K \in (U^2)^\times \) have the form \( 1 - hr/2 + o(h) \), and \( K \) is admissible with classical limit \( \gamma(\lambda) \); the contribution of \( (K^{1,2,3,4})^{1,2,3,4} \) is \(-\frac{1}{2}[r^{1,234}, \alpha^{2,3,4}] \), while the commutator \((K^{1,2,3,4})^{1,2,3} \) does not contribute (as the classical limit of \( \Phi \) is proportional to \( Z \) and hence invariant).

Let us now prove that \( G \) is nonempty. We will construct a sequence \( g_n \in \text{Map}_0(g^*, G) \), such that \( g_n := (r_0)^\theta \) satisfies \( g_n = \rho_{\text{AM}} \) modulo \( \wedge^3(g) \otimes \hat{S}(g)_{\geq n+1} \).

If \( n = 0 \), we set \( g_0 := \exp(-r/2) \), then \( \rho_0 = 0 = \rho_{\text{AM}} \) modulo \( \wedge^3(g) \otimes \hat{S}(g)_{\geq 1} \). Assume that we determined \( g_n \) and let us construct \( g_{n+1} \). Set \( \alpha := g_n - \rho_{\text{AM}} \), and let \( \alpha_{n+1} \) be the component of \( \alpha \) in \( \wedge^2(g) \otimes S^{n+1}(g) \). Then Lemma 1.1 implies that

\[
\text{CYB}(\rho_{\text{AM}} + \alpha_{n+1}) - \text{Alt}(\text{d}(\rho_{\text{AM}} + \alpha_{n+1})) = Z_{1,2,3}^{1,2,3} - \frac{1}{2}[r^{1,234}, \alpha_{n+1}^{2,3,4}] \mod \wedge^3(g) \otimes S^{n+1}(g).
\]

Since \( \rho_{\text{AM}} \) satisfies the modified classical dynamical Yang-Baxter equation, the component in \( \wedge^3(g) \otimes S^{n}(g) \) of this identity yields \( \text{Alt}(\text{d}(\alpha_{n+1})) = 0 \), so that we have \( \alpha_{n+1} = \text{Alt}(\text{d}(\beta)) \) for some \( \beta \in g \otimes S^{n+2}(g) \). Then we set \( g_{n+1} := \exp_s(-\beta) \ast g_n, \rho_{n+1} := (r_0)^{\theta}_n \). Then \( \rho_{n+1} - \rho_{\text{AM}} = \rho_n - \rho_{\text{AM}} - \text{Alt}(\text{d}(\beta)) = \alpha_{n+1} - \text{Alt}(\text{d}(\beta)) = 0 \mod \wedge^3(g) \otimes \hat{S}(g)_{\geq n+1} \). By successive approximations, we then construct \( g \) such that \((r_0)^{\theta}_0 = \rho_{\text{AM}} \). So \( G \) is nonempty.

1.1.2. Poisson isomorphism \( g^* \rightarrow G^* \) (proof of Prop. [1,3]). In this section, we show that the fact that \( \gamma(\lambda) \) satisfies \[2\] implies that \( \lambda \mapsto \gamma^*(\lambda) \) is a Poisson isomorphism. We will freely use the formalism of differential geometry, even though we work in the formal setup; the computations below make sense (and prove the desired result) when working over an Artinian \( k \)-ring.

\[6\]

Here the logarithm is the inverse of the ordinary exponential map \( g \otimes \hat{S}(g) \rightarrow \text{Map}_0^{\text{ham}}(g^*, G) \).
According to [STS], \( b : G^* \to G \) formal isomorphism if \( t \) is nondegenerate and the image of the Poisson bracket \( \{ -, - \}_{G^*} \) on \( G^* \) under \( b \) is the Poisson bracket on \( G \)

\[
\{F, H\}_{G}(g) = \langle (d_R - d_L)F(g) \otimes d_L H(g), r \rangle + \langle (d_R - d_L)F(g) \otimes d_R H(g), r^{2,1} \rangle,
\]
where \( q \in G, F, H \) are functions on \( G, d_L, d_R F(g), d_R F(g) \in \mathfrak{g}^* \) and right differentials defined by \( \langle d_L F(g), a \rangle = \langle (d/d\varepsilon)\big|_{\varepsilon=0} F(e^{\varepsilon a}), \rangle \langle d_R F(g), a \rangle = \langle (d/d\varepsilon)\big|_{\varepsilon=0} F(ge^{\varepsilon a}) \rangle \) for any \( a \in \mathfrak{g} \).

For \( \xi \in \mathfrak{g}^* \), define \( F_\xi \in \mathcal{O}_G \) by \( F_\xi(\xi) = \langle \xi, \log(g) \rangle \).

**Lemma 1.2.** \((d_L F_\xi)(g) = f(\frac{1}{2} \text{ad}^* (\log g))(\xi) \) and \((d_R F_\xi)(g) = f(-\frac{1}{2} \text{ad}^* (\log g))(\xi) \), where \( f(z) = ze^z/(\sinh z) \), and \( \text{ad}^* \) denotes the coadjoint action of \( \mathfrak{g} \) on \( \mathfrak{g}^* \).

**Proof.** Set \( x := \log(g), \) let \( a \in \mathfrak{g} \), and set \( \tilde{a} := (d/d\varepsilon)_{\varepsilon=0} \log(e^{\varepsilon a}) \). The coefficient in \( e^{\varepsilon a} e^x = e^{x+\tilde{a} + O(\varepsilon)} \) yields \( e^x = \sum_{n \geq 1} (n!)^{-1} \sum_{k=0}^{n-1} \tilde{a}^k x^{n-1-k} \). Applying \( \text{ad}(x) \) to this relation, we get \( \tilde{a} = f(-\frac{1}{2} \text{ad} x)(a) \). Now \( \langle d_L F_\xi(g), a \rangle = \langle (d/d\varepsilon)\big|_{\varepsilon=0} F_\xi(e^{\varepsilon a}), \rangle \), which yields the first formula. On the other hand, we have for any function \( F \) on \( G, d_R F(g) = \text{Ad}^* (g^{-1})(d_L F(g)) \), where \( \text{Ad}^* \) is the coadjoint action of \( G \) on \( \mathfrak{g}^* \), hence \( d_R F_\xi(g) = \text{Ad}^* (e^{-x})(f(\frac{1}{2} \text{ad}^* x))(\xi) = f(-\frac{1}{2} \text{ad}^* x)(\xi) \). \( \square \)

Then we get

\[
\{ F_\xi, F_\eta \}_{G}(g) = \langle \text{ad}^* (x)(\xi) \otimes \text{ad}^* (x)(\eta), (\text{id} \otimes \text{ad}(x))(t) \rangle - \langle \text{ad}^* (x)(\xi) \otimes \eta, t \rangle,
\]
where \( x = \log(g) \) and \( \varphi \) and \( r_0 \) are as in Prop. 1.2.

On the other hand, let \( \{- , - \}_{G^*} \) be the image of the Poisson bracket \( \{- , - \}_G \) by \( a : \mathfrak{g}^* \to G \). Let \( f_\xi(\lambda) := F_\xi \circ a(\lambda), \) then \( f_\xi(\lambda) = \langle \xi \otimes \lambda, (\text{Ad}(g(\lambda)) \otimes \text{id})(t) \rangle \). If \( f \in \mathcal{O}_{G^*}, \) \( \lambda \in \mathfrak{g}^* \), define \( df(\lambda) \in \mathfrak{g} \) by \( (\alpha, df(\lambda)) = \langle (d/d\varepsilon)\big|_{\varepsilon=0} f(\lambda + \alpha), \rangle \). We have \( \{ f, g \}_{G^*}(\lambda) = \langle \lambda, [df(\lambda), dg(\lambda)] \rangle \).

Then \( df(\lambda) = \langle \xi \otimes \text{id}, A(\lambda) \rangle \), where

\[
A(\lambda) = \langle \text{Ad}(g(\lambda)) \otimes \text{id}(t) + [(d_2 g_1) g_t^{-1}, \text{Ad}(g(\lambda))(\lambda^\gamma) \otimes 1] \rangle
\]
So

\[
\{ f_\xi(\lambda), f_\eta(\lambda) \} = \langle \xi \otimes \eta \otimes \lambda, [A^{1,3}(\lambda), A^{2,3}(\lambda)] \rangle
\]

This decomposes as the sum of four terms (we set \( \tilde{x} = \text{Ad}(g(\lambda))(\lambda^\gamma) \)):

\[
\begin{align*}
\langle & \xi \otimes \tilde{x} \otimes \lambda, (\text{Ad}(g(\lambda)) \otimes \text{id})(t^{13}) \rangle = \langle \xi \otimes \eta, (ad(\tilde{x}) \otimes \text{id})(t) \rangle; \\
\langle & \xi \otimes \eta \otimes \lambda, [(d_3 g_1) g_t^{-1}, \text{Ad}(g_1(\lambda))(\lambda^\gamma)], (\text{Ad}(g_2(\lambda))(\lambda^\gamma)^2)] \rangle = \langle \text{ad}^* (\tilde{x})(\xi) \otimes \eta, -\text{Ad}(g(\lambda))^{2,1}(d_2 g_1) \rangle; \\
\langle & \xi \otimes \eta \otimes \lambda, [(d_3 g_2) g_t^{-1}, \text{Ad}(g_2(\lambda))(\lambda^\gamma)] \rangle \rangle = \langle \text{ad}^* (\tilde{x})(\xi) \otimes \eta, \text{Ad}(g(\lambda))^{2,1}(d_1 g_2) \rangle; \\
\langle & \xi \otimes \eta \otimes \lambda, [(d_3 g_1) g_t^{-1}, \text{Ad}(g_1(\lambda))(\lambda^\gamma)], (\text{Ad}(g_2(\lambda))(\lambda^\gamma)^2)] \rangle \rangle = \langle \text{ad}^* (\tilde{x})(\xi) \otimes \eta, \text{Ad}(g(\lambda))^{2,2}(d_2 g_1) \rangle.
\end{align*}
\]

We have \( \{ F_\xi, F_\eta \}_{G^*}(g) = \{ f_\xi, f_\eta \}_{G^*}(a^{-1}(g)) \); then \( \tilde{x} = \tilde{x} \), so \( \{ F_\xi, F_\eta \}_{G^*} = \{ F_\xi, F_\eta \}_{G} \) iff

\[
\langle \text{ad}^* (x)(\xi) \otimes \text{ad}^* (x)(\eta), \text{Ad}(g(\lambda))^{2,1}(d_1 g_2 - g_1^{-1} d_3 g_1) \rangle + \langle (\text{id} \otimes \text{id} \otimes \lambda, [d_3 g_1 g_t^{-1}, d_2 g_2 g_t^{-1}]) \rangle - \langle \text{id} \otimes \text{ad}(x)(t) \rangle = 0,
\]
for which a sufficient condition is that \( g(\lambda) \) satisfies 2.

**Remark 1.3.** Formula 4 implies that the image of \( \{- , - \}_G \) under the map \( \log : G \to \mathfrak{g} \) is given by

\[
\{ f, g \}(x) = (df(x) \otimes dg(x), \langle ad(x) \otimes \frac{1}{2} \text{ad}(x) \coth(\frac{1}{2} \text{ad}(x)) \rangle)(t) + \text{ad}(x)^{\otimes 2}(r_0);
\]
this is a result of [EM2].
1.2. Constructions based on quantization of Lie bialgebras. We now give a proof of Thm. 1.1 based on the theory of quantization of Lie bialgebras (Sect. 1.2.1), and then give proofs of Props. 1.2 and 1.3 based on the same theory (Sects. 1.2.2 and 1.2.3).

1.2.1. Unconditional proof of Thm. 1.1. It follows from 

[3] that the subalgebras \( U(U')' \) and \( U' \) of \( U \) are equal. According to [3], \( (U')' \) is a flat deformation of \( O_{G^*} := O(g)^* \). On the other hand, \( U' \) is a flat deformation of \( O_{G^*} := \widetilde{S}(g) \). So the equality \( i_\hbar : (U')' \to U' \) induces an isomorphism of Poisson algebras \( i : O_{G^*} \to O_{g^*} \), and therefore a Poisson isomorphism \( g^* \to G^* \).

1.2.2. Construction of \( g(\lambda) \) (proof of Prop. 1.3). Set \( U := U(g)[[\hbar]] \), \( U' := \{ x \in U | \forall n \geq 0, \delta_n(x) \in h^n U(\tilde{\otimes}) \} = U(hq[[\hbar]]) \subset U \) (here \( \delta_n = (id - \eta \circ \varepsilon)^n \circ \Delta^{(n)} \), \( \eta \) is the unit map \( k[[\hbar]] \to U \). As a \( k[[\hbar]] \)-algebra, \( U' \) is a flat deformation of \( \widetilde{S}(g) = k[[g^*]] \).

Let \( \Phi \in U^{\hat{\otimes}3} \) be an admissible associator. This means that

\[
\Phi \in 1 + \frac{\hbar^2}{24}[\mu^{1,2}, t^{2,3}] + \hbar^2 U^{\hat{\otimes}3}, \quad \hbar \log(\Phi) \in (U')^{\hat{\otimes}3},
\]

\((U, m, \Delta_0, \mathcal{R}_0 = 1, \Phi)\) is a quasitriangular quasi-Hopf algebra. We also require \( \varepsilon^{(i)}(\Phi) = 1 \), \( i = 1, 2, 3 \) (here \( \varepsilon^{(1)} = \varepsilon \otimes id \otimes id \), etc., \( \varepsilon \) is the counit, \( m \) and \( \Delta_0 \) are the undeformed product and coproduct of \( U \)). According to [3], any universal Lie associator gives rise to an admissible associator.

According to [3], there exists a twist killing \( \Phi \), and according to [3], this twist can then be made admissible by a suitable gauge transformation. The resulting twist \( J \) satisfies the following conditions:

\[
J \in U^{\hat{\otimes}2}, \quad J = 1 - \hbar r/2 + o(h), \quad h \log(J) \in (U')^{\hat{\otimes}2}, \quad (\varepsilon \otimes id)(J) = (id \otimes \varepsilon)(J) = 1,
\]

(5)

Then \( U^J := (U, m, \Delta^J, \mathcal{R}) \) is a quasitriangular Hopf algebra quantizing \( (g, r) \). Here \( \Delta^J(x) = J\Delta_0(x)J^{-1} \) and \( \mathcal{R} = J^{1,2}r^{1,3}/J^{2,3} \).

We have \( \text{Ker}(\varepsilon) \cap U' \subset hU, \) therefore \( log(J) \in U^{\hat{\otimes}U} \). Then its reduction mod \( h \), denoted \( g(\lambda) = g^{1,2} = \log(J)_{|_{h=0}} \), belongs to \( U(g) \otimes \widetilde{S}(g) = U(g)[[g^*]] \) (formal series on \( g^* \) with coefficients in \( U(g) \)). The reduction mod \( h \) of \( \Phi \) is \( g^{1,2} = g^{1,3}g^{2,3} \). Since we also have \( (\varepsilon \otimes id)(\Phi) = 1 \), we get \( g = \exp(A) \), with \( A \in g \otimes \widetilde{S}(g) \geq 0 \).

**Lemma 4.3.** \( g(\lambda) \) satisfies (4).

**Proof.** According to [3], \( \Phi \in U^{\hat{\otimes}2}U \) has the expansion \( 1 + \hbar \phi_1 + o(h^2) \), where \( \phi_1 \in U^{\otimes 2}U \) is such that \( (\phi_1 - \phi^{2,1,3})_{|_{h=0}} = -\rho_{AM} \).

If \( x \in U(g) \otimes \widetilde{S}(g) \), we denote by \( \bar{x} \) a lift of \( x \) in \( U^{\hat{\otimes}U} \).

Let us expand \( \log(J) \) as \( \bar{A} + hA_1 + o(h) \), with \( A_1 \in U^{\hat{\otimes}U} \).

Then \( J^{1,23} = \exp(\bar{A}^{1,3} + \hbar A_1^{1,3} + d_2 A^{1,3} + o(h)) = J^{1,3}(1 + \hbar g_{-1}^{-1}d_2(g_{1}) + o(h)) \).

We have \( [J^{1,3}, J^{2,3}] = \hbar g_{1}^{-1}g_{2}^{2,3} + o(h) \), so \( (J^{1,23})^{-1}[J^{1,3}, J^{2,3}] = \hbar g_{1}^{-1}g_{2}^{2,3} + o(h) \). Therefore we get

\[
J^{1,23} = J^{2,3}J^{1,3}(1 + \hbar \psi_{1} + o(h)),
\]

where \( \psi_{1} \in U^{\otimes 2}U \) is such that \( (\psi_{1} - \psi^{2,1,3})_{|_{h=0}} = (id \otimes id \otimes \lambda, [g^{1,3}d_{3}(g_{1}), g_{2}^{-1}d_{3}(g_{2})]) + o(h) \). Then \( J^{2,3} \) gives

\[
1 + h\phi_{1} + o(h) = (1 - \hbar g_{-1}^{-1}d_2(g_{1}) + o(h))(J^{1,3})^{-1}J^{2,3}^{-1}(1 - \hbar r/2 + o(h))J^{2,3}J^{1,3}(1 + \hbar \psi_{1} + o(h)).
\]

The reduction modulo \( h \) of \( (J^{1,3})^{-1}J^{2,3}^{-1}rJ^{2,3}J^{1,3} \) is \( \text{Ad}(g \otimes g)^{-1}(r) \). Then substracting 1, dividing by \( h \), reducing modulo \( h \) and antisymmetrizing the two first tensor factors, we get the lemma.

\[\square\]
More generally, assume that $J' \in U \otimes^2 \mathbb{C}$ satisfies $J' = 1 - h \varpi/2 + o(h)$, $h \log(J') \in (U') \otimes^2 \mathbb{C}$, $(\varepsilon \otimes \text{id})(J') = 1$ and
\[
\Phi = (J^{2,3} J'^{1,23})^{-1} J'^{1,2} J^{1,3}.
\]
Then $J' \in U \otimes U'$, and its reduction $g'$ modulo $h$ satisfies: $g' \in \exp(\mathfrak{g} \otimes \widehat{S}(\mathfrak{g})_{>0})$, and equation (2) with $\mathfrak{g}$ replaced by $g'$. Moreover, there exists $u \in U^*$ such that $J' = u^2 J(u^{-2})^{-1}$.

**Remark 1.5.** Equation (6) can be interpreted as saying that $J'$ is a vertex-IRF transformation relating $J^{1,2}$ and $\Phi$, and equation (2) for $g'$ is the classical limit of this statement (see [EN]). Vertex-IRF transformations are a special kind of non-invariant dynamical gauge transformations, which map a constant, but non-invariant twist to an invariant, but non-constant (i.e., dynamical) one.

Let $U_0' := \text{Ker}(\varepsilon) \cap U'$. Then $h^{-1} U_0' \subset U$ is a Lie subalgebra for the commutator. This Lie algebra acts on the set of solutions of (6) by $\delta_a(J) = u^2 J - J u^2$ (since it is equal to $\{u^2, J\} - J(u^2 - u^2)$); this means that if $\varepsilon$ is a formal parameter with $\varepsilon^2 = 0$, then $(\text{id} + \varepsilon \delta_a)(J')$ is a solution of (6) if $J'$ is.

The reduction modulo $h$ of this action may be described as follows. The Lie algebra $(\widehat{S}(\mathfrak{g})_{>0}, \{\cdot, \cdot\})$ acts on the set of solutions of (6) by $\delta_f(g) = \{1 \otimes f, g\} - g \cdot df$, i.e., action (6). When restricted to the Lie subalgebra $\widehat{S}(\mathfrak{g})_{>1}$, this action is the infinitesimal of the right action of $\text{Map}^h_{\text{ham}}(\mathfrak{g}^*, G)$ on the set of solutions of (2), given by $(g * \alpha)(\lambda) = g(\text{Ad}^*(\alpha(\lambda)))(\lambda)\alpha(\lambda)$.

1.2.3. **Isomorphism $\mathfrak{g}^* \to G^*$ (proof of Prop. [U,3]).** Let us construct the maps $a : \mathcal{O}_G \to \mathcal{O}_{\mathfrak{g}^*}$ and $b : \mathcal{O}_{\mathfrak{g}^*} \to \mathcal{O}_G$, out of the quantization of Lie bialgebras.

We first construct quantized versions of $a$ and $b$. We have $h \log(J) \in (U') \otimes^2 \mathbb{C}$, therefore $J, J' \in U \otimes U'$ and $J e^{h t} J = e^{h t} J$. We then define $a_h : U^* \to U'$ by $a_h(f) := (f \otimes \text{id})(Je^{h t} J'^{-1})$. Now we also have $J^2 \in U \otimes U'$, hence $R_1, R_1^2 \in U \otimes U' = U \otimes (U^*)'$, we then define the linear map $b_h : U^* \to (U^*)'$ by $b_h(f) := (f \otimes \text{id})(R_1^2)$. Then $a_h b_h^{-1} : U^* \to (U^*)'$ coincides with $b_h$. We define $a, b$ as the reductions modulo $h$ of $a_h, b_h$.

Let us now compute the classical limit of $a_h$. Define maps $j_h, \iota_h, j_h^* : U^* \to U'$ by $j_h(f) = (f \otimes \text{id})(J), j_h^*(f) = (f \otimes \text{id})(J'^{-1})$. Recall that the reductions modulo $h$ of all three elements $J, J', e^{h t}$ in $U(\mathfrak{g}) \otimes \widehat{S}(\mathfrak{g})$ are of the form $K = \exp(k)$, where $k \in \mathfrak{g} \otimes \widehat{S}(\mathfrak{g})_{>0}$.

**Lemma 1.6.** If $K \in U(\mathfrak{g}) \otimes \widehat{S}(\mathfrak{g})$ is of the form $\exp(k)$, where $k \in \mathfrak{g} \otimes \widehat{S}(\mathfrak{g})_{>0}$, then the morphism $\mathcal{O}_G = U(\mathfrak{g})^* \to \widehat{S}(\mathfrak{g}) = \mathcal{O}_{\mathfrak{g}^*}$ given by $f \mapsto (f \otimes \text{id})(K)$ is dual to the morphism $\mathfrak{g}^* \to G$, $\lambda \mapsto e^{k(\lambda)}$.

**Proof.** We compose this morphism with the transpose of the inverse of the symmetrization map $\text{Sym}^{-1} : S(\mathfrak{g})^* \to U(\mathfrak{g})^*$. The morphism $\text{Sym}^{-1}$ corresponds to the logarithm map $G \to \mathfrak{g}$. Now the composed morphism $\mathcal{O}_G \to \widehat{S}(\mathfrak{g})$ is given by $f \mapsto (f \otimes \text{id})(\text{Sym}^{-1} \otimes \text{id})(K)$. Now $\text{Sym}^{-1} \otimes \text{id} : (K) = \exp(k)$, where the exponential is now taken in $S(\mathfrak{g}) \otimes \widehat{S}(\mathfrak{g})$. The morphism $S(\mathfrak{g})^* \to \widehat{S}(\mathfrak{g})$, $f \mapsto (f \otimes \text{id})(\exp(k))$, is an algebra morphism, taking the function $X \mapsto \alpha(X)$ on $\mathfrak{g}$ ($\alpha \in \mathfrak{g}^*$) to the function $\lambda \mapsto \alpha(k(\lambda))$ on $\mathfrak{g}^*$, and therefore corresponds to the morphism $\mathfrak{g}^* \to \mathfrak{g}$, $\lambda \mapsto (k(\lambda))$. Composing with the exponential, we get the announced morphism.

It follows that the reductions modulo $h$ of the morphisms $j_h, \iota_h, j_h^*$ are morphisms $\mathcal{O}_G \to \mathcal{O}_{\mathfrak{g}^*}$, corresponding to morphisms $j, \iota, j' : \mathfrak{g}^* \to G$ such that $j(\lambda) = g(\lambda), \iota(\lambda) = e^{k(\lambda)}$, $j'(\lambda) = g(\lambda)^{-1}$.

Then $a_h = m(2) \circ (j_h \iota_h \iota_h^* \iota_h^*) \circ \Delta(2)$, so $a : \mathcal{O}_{\mathfrak{g}^*} \to \mathcal{O}_G$ corresponds to the composed map $G^* \to (G^*)^2 \to (G^*)^4 \to G$. Therefore we get $a(\lambda) = g(\lambda)e^{k(\lambda)}g(\lambda)^{-1}$.

We now compute the classical limit of $b_h$. Define maps $L_h, R_h^\alpha : (U^*)^2 \to (U^*)'$ by $L_h(f) := (f \otimes \text{id})(R_1^2), R_h^\alpha(f) := (f \otimes \text{id})(R_1)$. Then $L_h$ is an antimorphism of algebras and a morphism
of coalgebras, while \( R'_{th} \) is a morphism of algebras and antimorphism of coalgebras. Their reductions \( L, R' \) modulo \( h \) are morphisms \( \mathcal{O}_G \to \mathcal{O}_G \) (anti-Poisson coalgebra morphism in the case of \( L \), Poisson anti-coalgebra in the case of \( R' \)). Using [EGH], appendix, one shows that these morphisms correspond to the morphisms of formal groups \( L, R' : G^* \to G \) (antimorphism in the case of \( R' \)), corresponding to the morphisms \( L, R' : g^* \to g \), given by \( L(\lambda) := (\lambda \otimes \text{id})(r) \) and \( R'(\lambda) := (\lambda \otimes \text{id})(r^{2,1}) \). Here \( L \) is a Lie algebra, anti-Lie coalgebra morphism and \( R' \) is an anti-Lie algebra, Lie coalgebra morphism. Now \( b_h = m \circ (L_h \otimes R_h) \circ \Delta \), so \( b : \mathcal{O}_G \to \mathcal{O}_G \) corresponds to the composed map \( G^* \xrightarrow{\text{diag}} (G^*)^2 \xrightarrow{(L,R')} G^2 \xrightarrow{\text{product}} G \), i.e., \( g^* \mapsto L(g^*)R'(g^*) = L(g^*)R(g^*)^{-1} \).

Finally, the isomorphism \( i : g^* \to G^* \) is equal to \( b^{-1} \circ a \), so it takes \( \lambda \in g^* \) to \( g^* \in G^* \) such that
\[
L(g^*)R(g^*)^{-1} = g(\lambda) e^{\lambda^v} g(\lambda)^{-1}.
\]

So it coincides with the isomorphism obtained in Prop. 1.3

1.3. On the group \( \text{Map}_0^{\text{ham}}(g^*, G) \). In this section, we give a "quantum" proof of the following statement, which was used in Sect. 1.1 (and can also be proved in the setup of this section).

**Proposition 1.7.** \( \text{Map}_0^{\text{ham}}(g^*, G) \) is a subgroup of \( \text{Map}_0(g^*, G) \).

**Proof.** Set \( U'_0 = \text{Ker}(\varepsilon) \cap U' \). The map of \( U' \to S(g) \) of reduction by \( h \) restricts to \( U'_0 \to S(g)_{>0} \), so we get a map \( U'_0 \to g \).

Consider the set of all \( J' \), such that \( h \log(J') \in (U'_0)^{\otimes 2} \), the image of \( h \log(J') \) in \( g^{\otimes 2} \) is zero, and \( J'^{2,1} \cdot J'^{1,23} = J'^{1,23} \).

This is the set of elements of the form \( J'(u) = u^2(1 - 1)^{-1} \), where \( u \in \exp(U'_0 / \hbar) \) is such that the image of \( \log(u) \) under \( S(g)_{>0} \to g \) is zero. Indeed, one recovers \( u \) from \( J' \) by \( u = (\text{id} \otimes \varepsilon)(J')^{-1} \).

We denote by \( A(\lambda) \) the image of \( h \log(J') \) by \( S(g)_{>0} \otimes S(g)_{>0} \to g \otimes S(g)_{>0,2} \), and set \( g(\lambda) := \exp(A(\lambda)) \). Then \( g(\lambda) \in \text{Map}_0(g^*, G) \).

As in Lemma 1.4, one proves that \( g(\lambda) \) satisfies equation (1). Hence we get a map of sets \( \{ J' \text{ as above} \} \to \text{Map}_0^{\text{ham}}(g^*, G) \), which is surjective.

On the other hand, the set of all \( J'(u) \) is equipped with a product, such that \( J'(u) \cdot J'(v) = J'(uv) \). This product expresses as follows: \( J'(u) \cdot J'(v) = (u^2 J'(v)(u^{-2})^{-1} J'(u)) \). The classical limit of this expression is the product formula for \( \text{Map}_0(g^*, G) \). So \( \text{Map}_0^{\text{ham}}(g^*, G) \) is a subgroup of \( \text{Map}_0(g^*, G) \).

2. Quantization of \( \rho_{FM} \)

In this section, we prove Thm. 0.3.

Let \( \Phi_{\text{univ}} \) be a universal Lie associator defined over \( k \) with parameter \( \mu = 1 \). So \( \Phi_{\text{univ}} = 1 + \frac{1}{\hbar^2}t_{12,23}+ \text{terms of degree } > 2 \), where \( t_{ij} \) is the universal version of \( t^i \cdot t^j \) (see [Dr3]). Set \( \Phi_v := \Phi_{\text{univ}}(2huv^{1,2}, 2huv^{2,3}) \) and \( \Phi := \Phi_{1/2} \). Then
\[
(U(g)[[\hbar]], m, \Delta_0, \Phi_v)
\]
is a quasi-Hopf algebra; its classical limit is the quasi-Lie bialgebra \( (g, \delta = 0, \nu^2 t^{1,2}, t^{2,3}) \). Let \( J \) be an admissible twist killing \( \Phi \), and let us twist this quasi-Hopf algebra by \( J \). We obtain the quasi-Hopf algebra
\[
(U(g)[[\hbar]], m, \Delta'_v, \Phi'_v),
\]
where \( \Delta'_v(x) = J\Delta_0(x)J^{-1} \) and \( \Phi'_v = J^{2,3}J^{1,23}\Phi_v(J^{-1})^{1,2}(J^{-1})^{1,2} \). Its classical limit is the quasi-Lie bialgebra \( (g, \delta(x) = \{x^1 + x^2, r\}, Z_v) \).

Now \( (U(g)[[\hbar]], m, \Delta'_v) \) is a Hopf algebra quantizing \( (g, \delta) \), which we denote by \( U_h(g) \), and \( \Phi'_v \) is an associator for this quantized universal enveloping algebra, with classical limit \( Z_v \).
\[ \Phi_{\nu}^J \text{ clearly satisfies the invariance and pentagon equations. We have } J = 1 + \hbar j_1, \text{ where } j_1 \in U \otimes U', \text{ so } J_{1,3}, J_{1,23}, (J^{-1})_{1,23} \text{ and } (J^{-1})_{1,2} \text{ are all of the form } 1 + \hbar k, k \in U \otimes U'. \text{ Hence } \Phi_{\nu}^J = 1 + \hbar \psi_{\nu}, \text{ where } \psi_{\nu} \in U \otimes U’. \text{ Set } U_h := U_h(g), \text{ then } U_h' = U', \text{ so } \psi_{\nu} \in U_h \otimes U_h'. \text{ We now compute } (\psi_{\nu} - \psi_{\nu}^{2,1,3})_{h=0}; \text{ this is an element of } U(g) \otimes U(g) \cong U(\hat{g}) \otimes U(\hat{g}). \]

\[ \text{We have } \Phi_{\nu}^J = \Phi^J + (\Phi_{\nu} - \Phi)^J = 1 + \hbar (\Phi_{\nu} - \Phi)^J. \text{ Let us define } \phi, \phi_{\nu} \text{ by } \Phi = 1 + \hbar \phi, \Phi_{\nu} = 1 + \hbar \phi_{\nu}, \text{ then } (\psi_{\nu})_{h=0} = ((\phi_{\nu} - \phi)^J)_{h=0}, \text{ therefore}
\]

\[ (\psi_{\nu} - \psi_{\nu}^{2,1,3})_{h=0} = ((\phi_{\nu} - \phi_{\nu}^{2,1,3})^J)_{h=0} - ((\phi - \phi_{\nu}^{2,1,3})^J)_{h=0} = \text{Ad}(g(\lambda))^\otimes 2 (\rho_{AM}(\lambda) - \rho_{AM}^\nu(\lambda)) \]

as a formal function \( g^* \rightarrow \Lambda^2(g) \). Here \( \rho_{AM}^\nu(\lambda) = 2\nu \rho_{AM}(2\nu \lambda) \). Since \( \rho_{AM} \) and \( \rho_{AM}^\nu \) are \( G \)-equivariant, this is

\[ (\rho_{AM} - \rho_{AM}^\nu)((\text{Ad}^*(g(\lambda)))(\lambda)) = \left( \text{id} \otimes \left( \frac{1}{2} e^{2\nu \text{ad}(\lambda^\vee)} + \text{id} \right) - \nu e^{2\nu \text{ad}(\lambda^\vee)} - \text{id} \right)(t) = \rho_{FM}(g^*) \]

Here we set \( \lambda := \text{Ad}^*(g(\lambda)))(\lambda) \) and we use the relation \( L(g^*)G(g)^{-1} = e^{\lambda^\vee} \).

We have proved:

**Theorem 2.1.** The quantized universal enveloping algebra \( U_h(g) = U(g)[[\hbar]]^J \), together with the pair \( (J, \hat{\Phi}) \) defined by \( J = \hat{\Phi} = \Phi_{\nu}^J \), is a quantization of the Poisson-Lie dynamical \( r \)-matrix \( \rho_{AM}(g^*) \) over \( (G^*, \hat{g}, \hat{Z}_\nu) \).

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