On the Existence of Perfect Codes for Asymmetric Limited-Magnitude Errors

Sarit Buzaglo  
Dept. of Computer Science  
Technion-Israel Institute of Technology  
Haifa 32000, Israel  
Email: sarahb@cs.technion.ac.il

Tuvi Etzion  
Dept. of Computer Science  
Technion-Israel Institute of Technology  
Haifa 32000, Israel  
Email: etzion@cs.technion.ac.il

Abstract—Block codes, which correct asymmetric errors with limited-magnitude, are studied. These codes have been applied recently for error correction in flash memories. The codes will be represented by lattices and the constructions will be based on a generalization of Sidon sequences. In particular we will consider perfect codes for these type of errors.

I. INTRODUCTION

Asymmetric error-correcting codes were subject to extensive research due to their application in coding for computer memories [3]. The advance of technology and the appearance of new nonvolatile memories, such as flash memory, led to a new type of asymmetric errors which have limited-magnitude. A multilevel flash cell is electronically programmed into threshold levels which can be viewed as elements of the set \{0, 1, \ldots , q - 1\}. Errors in this model are usually in one direction and are not likely to exceed a certain limit. This means that a cell in level \(i\) can be raised by an error to level \(j\), such that \(i < j\) and \(j - i \leq \ell\), where \(\ell\) is the error limited-magnitude.

Asymmetric error-correcting codes with limited-magnitude were proposed in [1] and were first considered for nonvolatile memories in [2]. Recently, several other papers have considered the problem, e.g. [3], [5], [6], [11]. In this work we mainly consider perfect codes for asymmetric errors with limited-magnitude \(q\). We will consider only linear codes, unless otherwise is stated. Each \(t\)-error-correcting perfect code in the Hamming scheme, over GF\((q)\), is also a perfect code for error-correction of \(t\) asymmetric errors with limited-magnitude \(q - 1\) [3]. Especially, a Hamming code of length \(n = \frac{q^t - 1}{q - 1}\), over GF\((q)\), can correct one asymmetric error with limited-magnitude \(q - 1\). Additional perfect codes for correction of one asymmetric error with limited-magnitude \(\ell\) are obtained from tiling of \(Z^n\) with semi-crosses whose arms have length \(\ell\) [10]. Perfect unbalanced limited-magnitude codes were considered in [9].

In this work we mainly consider perfect codes for asymmetric limited-magnitude errors. We will consider only linear codes, unless otherwise is stated. Each \(t\)-error-correcting perfect code in the Hamming scheme, over GF\((q)\), is also a perfect code for error-correction of \(t\) asymmetric errors with limited-magnitude \(q - 1\) [3]. Especially, a Hamming code of length \(n = \frac{q^t - 1}{q - 1}\), over GF\((q)\), can correct one asymmetric error with limited-magnitude \(q - 1\). Additional perfect codes for correction of one asymmetric error with limited-magnitude \(\ell\) are obtained from tiling of \(Z^n\) with semi-crosses whose arms have length \(\ell\) [10]. Perfect unbalanced limited-magnitude codes were considered in [9].

The rest of this work is organized as follows. In Section II we will define the basic concepts for codes which correct \(t\) asymmetric errors with limited-magnitude \(\ell\). We will show a convenient way to handle such codes and discuss three equivalent representations of such codes. In Section III we will present a new construction for perfect codes of length \(n\) which correct \(n - 1\) asymmetric errors with limited-magnitude \(\ell\), for any given \(\ell\). In Section IV we show that perfect codes of length \(n\) which correct \(n - 2\) asymmetric errors with limited-magnitude one cannot exist. We conclude in Section VI.

II. BASIC CONCEPTS

For a word \(X = (x_1, x_2, \ldots , x_n) \in Q^n\), the Hamming weight of \(X\), \(w_H(X)\), is the number of nonzero entries in \(X\), i.e., \(w_H(X) = |\{i : x_i \neq 0\}|\).

A code \(C\) of length \(n\) over the alphabet \(Q = \{0, 1, \ldots , q - 1\}\) is a subset of \(Q^n\). A vector \(E = (e_1, e_2, \ldots , e_n)\) is a \(t\)-asymmetric-error with limited-magnitude \(\ell\) if \(w_H(E) \leq t\) and \(0 \leq e_i \leq \ell\) for each \(1 \leq i \leq n\). The set of all \(t\)-asymmetric-errors with limited-magnitude \(\ell\). A code \(C \subseteq Q^n\) can correct \(t\) asymmetric errors with limited-magnitude \(\ell\) if for any two codewords \(X_1\), \(X_2\), and any two \(t\)-asymmetric-errors with limited-magnitude \(\ell\), \(E_1\), \(E_2\), such that \(X_1 + E_1 \in Q^n\), we have that \(X_1 + E_1 \neq X_2 + E_2\).

For simplicity it is more convenient to consider the code \(C\) as a subset of \(Z_q^n\), where all the additions are performed modulo \(q\). Such a code \(C\) can be viewed also as a subset of \(Z^n\) formed by the set \(\{X + qY \in C, Y \in Z^n\}\).

We will represent a linear code \(C\), over \(Z_q^n\), which corrects \(t\) asymmetric errors with limited-magnitude \(\ell\), in two more different ways. The first is by an integer lattice and the second is by a generalization of the well-known Sidon sequence, the \(B_h\) sequence. We will show an equivalence between the three representations.

An integer lattice \(\Lambda\) is an additive subgroup of \(Z^n\). We will assume that

\[ \Lambda = \{u_1v_1 + u_2v_2 + \cdots + u_nv_n : u_1, u_2, \ldots , u_n \in Z\} \]

where \(\{v_1, v_2, \ldots , v_n\}\) is a set of linearly independent vectors in \(Z^n\). The set of vectors \(\{v_1, v_2, \ldots , v_n\}\) is called the basis for \(\Lambda\), and the \(n \times n\) matrix

\[ G = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix} \]

having these vectors as its rows is said to be the generator matrix for \(\Lambda\).

The volume of a lattice \(\Lambda\), denoted by \(V(\Lambda)\), is inversely proportional to the number of lattice points per a unit volume. There is a simple expression for the volume of \(\Lambda\), namely,

\[ V(\Lambda) = |\det G| \]

A set \(P \subseteq Z^n\) is a packing of \(Z^n\) with a shape \(S\) if copies of \(S\) placed on the points of \(P\) (in the same relative position) are disjoint. A set \(T\) is a tiling of \(Z^n\) if it is a packing and the
disjoint copies of $S$ cover $\mathbb{Z}^n$. A lattice $\Lambda$ is a lattice packing (tiling) with the shape $S$ if $\Lambda$ forms a packing (tiling) with $S$.

The following lemma is well known.

**Lemma 1:** A necessary condition that the lattice $\Lambda$ defines a lattice packing (tiling) with the shape $S$ is that $V(\Lambda) \geq |S|$ ($V(\Lambda) = |S|$), where $|S|$ denote the volume of $S$.

A linear code $C$, over $\mathbb{Z}_q^n$, which corrects $t$ asymmetric errors with limited-magnitude $\ell$ viewed as a subset of $\mathbb{Z}^n$ is equivalent to an integer lattice packing with the shape $S(n, t, \ell)$. Therefore, we will call this a lattice code.

Let $A(n, t, \ell)$ denote the set of lattice codes in $\mathbb{Z}^n$ which correct $t$ asymmetric errors with limited-magnitude $\ell$. A code $L \in A(n, t, \ell)$ is called perfect if it forms a lattice tiling with the shape $S(n, t, \ell)$.

Let $[\ell]$ be the set $\{0, 1, 2, \ldots, \ell\}$ and let $G$ be an Abelian group. A $B_h([\ell])(G)$ sequence of length $m$ is a sequence (set) of $m$ elements in $(G, b_1, b_2, \ldots, b_m)$ such that all sums

$$\sum_{j=1}^{m} \alpha_j b_j,$$

where $\alpha_j \in [\ell]$ and at most $h$ of the the $\alpha_j$'s are nonzero, are distinct elements of $G$. $B_h([\ell])$ sequences were first mentioned in [6] for correction of asymmetric errors with limited-magnitude.

**Lemma 2:** If $\mathcal{L} \in A(n, t, \ell)$ then there exists an Abelian group $G$ of order $|G| = V(\mathcal{L})$ and a $B_h([\ell])(G)$ sequence of length $n$.

**Lemma 3:** Let $G$ be an Abelian group and let $b_1, \ldots, b_n$ be a $B_h([\ell])(G)$ sequence. Then there exists a lattice code $\mathcal{L} \in A(n, t, \ell)$ with $V(\mathcal{L}) \leq |G|$.

**Corollary 1:** A perfect lattice code $L \in A(n, t, \ell)$ exists if and only if there exists an abelian group $G$ of order $|S(n, t, \ell)|$ and a $B_h([\ell])(G)$ sequence of length $n$.

To form a code $C \subseteq \Sigma^n$, where $\Sigma \equiv \{0, 1, \ldots, \sigma - 1\}$, which corrects $t$ asymmetric errors with limited-magnitude $\ell$, one can take a lattice code $\mathcal{L} \in A(n, t, \ell)$. Then $C \equiv \Sigma \cap \Sigma^n$, where $X$ is any element of $\mathbb{Z}^n$ added to all the elements of the lattice $\mathcal{L}$, is an appropriate code. Note that the code $C$ is usually not linear.

**III. Perfect Codes Which Correct $n - 1$ Errors**

To use Corollary 1 we have to compute $S(n, t, \ell)$.

**Lemma 4:** $|S(n, t, \ell)| = \sum_{i=0}^{t} \binom{n}{i} \ell^i$.

**Corollary 2:** $|S(n, n - 1, \ell)| = (\ell + 1)^n - \ell^n$.

For the ring $G = \mathbb{Z}_q$, the ring of integers modulo $q$, let $G^*$ be the multiplicative group of $G$ formed from all the elements of $G$ which have multiplicative inverses in $G$.

**Lemma 5:** Let $n \geq 2$, $\ell \geq 1$, be two integers and let $G$ be the ring of integers modulo $(\ell + 1)^n - \ell^n$. Then, (P1) $\ell$ is an element of $G^*$.

(P2) The element $x = (\ell + 1) \cdot \ell^{-1}$ of $G$ is an element of $G^*$ of order $n$.

(P3) The expression $1 + x + x^2 + \ldots + x^{n-1}$ equals to zero in $G$.

(P1) and (P2) are important in the construction obtained from the following theorem, while (P3) is important for its proof.

**Theorem 6:** For each $n \geq 2$ and $\ell \geq 1$, let $x = (\ell + 1) \cdot \ell^{-1} \in \mathbb{Z}_{(\ell+1)^n - \ell^n}$. Then the set $\{1, x, x^2, \ldots, x^{n-1}\}$ is a $B_{n-1}([\ell])(\mathbb{Z}_{(\ell+1)^n - \ell^n})$ sequence.

**Corollary 3:** For each $n \geq 2$ and $\ell \geq 1$ there exists a perfect lattice code $L \in A(n, n - 1, \ell)$.

**IV. Nonexistence of Some Perfect Codes**

Recall that there exists a perfect lattice code $L \in A(n, t, \ell)$ for various parameters with $t = 1$. Such codes also exist for $t = n$ and all $\ell \geq 1$ and for the parameters of the Golay codes and the binary repetition code of odd length. In Section III we proved the existence of such codes for $t = n - 1$ and all $\ell \geq 1$. Next, we ask whether such codes exist for $t = n - 2$? Unfortunately, if $t = n - 2$ and $\ell = 1$ such codes cannot exist. The proof is based on the following lemma.

**Lemma 7:** If there exists a perfect lattice code in $A(n, n - 2, \ell)$ then $|S(n, n - 2, \ell)|$ divides $(\ell+1)^{n-2} \cdot (\ell+1+\alpha(n-2-\ell))$ for some integer $\alpha$, $0 \leq \alpha \leq \ell$.

**Theorem 8:** There are no perfect lattice codes in $A(n, n - 2, 1)$ for all $n > 3$.

**V. Conclusion**

We discussed three different equivalent ways to consider linear codes which correct $t$ asymmetric errors with limited-magnitude. One of these ways was to consider $B_h$ sequences. We presented a construction of $B_h$ sequences which result in perfect codes of length $n$ for correction of $n - 1$ asymmetric errors with limited-magnitude $\ell$ for any given $\ell$. A related nonexistence result for $n - 2$ errors and limited-magnitude one was given. It is a major research problem to prove whether more such perfect codes exist.

**VI. Note Added**

After the Arxiv submission we became aware of the work in [7] which contains Theorem 6.

**ACKNOWLEDGMENT**

This work was supported in part by the Israel Science Foundation (ISF), Jerusalem, Israel, under Grant No. 230/08.

**References**

[1] R. Ahlswede, H. Aydinian, and L. Khachatrian, “Unidirectional error control codes and related combinatorial problems”, in Proc. Eighth Int. Workshop Algebra. Combin. Coding Theory (ACCT-8), St. Petersburg, pp. 6–9, 2002.

[2] Y. Cassuto, M. Schwartz, V. Bohossian, and J. Bruck, “Codes multi-level flash memories: correcting asymmetric limited-magnitude limited-magnitude errors”, in Proc. IEEE Int. Symp. on Inform. Theory, Nice, pp. 1176–1180, 2007.

[3] Y. Cassuto, M. Schwartz, V. Bohossian, and J. Bruck, “Codes for asymmetric limited-magnitude limited-magnitude errors with application to multi-level flash memories”, IEEE Trans. Inform. Theory, vol IT-56, pp. 1582–1595, 2010.

[4] L. Dolececk, “Towards longer lifetime of emerging technologies using number theory”, in Proc. Workshop on the Application of Commum. Theory to Emerging Memory Technologies, Miami, pp. 1936–1940, 2010.

[5] N. Elarief and B. Bose, “Optimal, systematic, q-ary codes correcting all asymmetric and symmetric errors of limited magnitude”, IEEE Trans. on Inform. Theory, vol IT-56, pp. 979–980, 2010.

[6] T. Klove, B. Bose, and N. Elarief, “Systematic single limited magnitude asymmetric error correcting codes”, IEEE Trans. Inform. Theory, vol IT-57, pp. 4477–4487, 2011.

[7] T. Klove, L. Luo, I. Naydenova, and S. Yari “Some codes correcting asymmetric errors of limited magnitude”, IEEE Trans. Inform. Theory, vol IT-57, pp. 7459–7472, 2011.

[8] R. N. Rao and E. Fujiwara, Error-Control Coding for Computer Systems, London, U.K.: Prentice-Hall, 1989.

[9] M. Schwartz, “Quasi-cross lattice tilings with applications to flash memory”, in Proc. IEEE Intersymp. on Inform. Theory, Saint Petersburg, pp. 2133–2137, 2011.

[10] S. Stein, “Tiling, packing, and covering by clusters”, Rocky Mountain J. Math., vol. 16, pp. 277–321, 1986.

[11] E. Yaakobi, P. H. Siegel, A. Vardy, and J. K. Wolf, “On codes that correct asymmetric errors with graded magnitude distribution”, in Proc. IEEE Intersymp. on Inform. Theory, Saint Petersburg, pp. 1021–1025, 2011.