Exact solutions of an elliptic Calogero–Sutherland model

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A model describing $N$ particles on a line interacting pairwise via an elliptic function potential in the presence of an external field is partially solved in the quantum case in a totally algebraic way. As an example, the ground state and the lowest excitations are calculated explicitly for $N = 2$.

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It is well known that the class of exactly solvable problems does not include most physical problems. The development of computer science in the last decades has made possible the use of numerical methods to approximate exact solutions in a wide variety of situations. Yet, the study of exactly solvable models still deserves attention, not only because the knowledge of exact solutions can be used to test approximate methods, but also in its own right, due to the simplicity and mathematical beauty of the models, and the wide range of connections with other fields of physical and mathematical research.

This is illustrated by the renewed interest in the Calogero–Sutherland (CS) models of interacting particles in one dimension, which have been recently applied to many different fields such as quantum spin chains with long range interaction [1], random matrix theory [2], fractional statistics and anyons [3], Yang–Mills theories [4], quantum Hall liquids [5], soliton theory [6], vicinal surfaces in crystals [7], and black holes [8].

The first example of a non-trivial integrable quantum many-body problem was found by Calogero [9], and consists of a system of identical nonrelativistic particles interacting pairwise through an inverse-square potential $v(r) = r^{-2}$, so that

$$H_N = - \sum_{k=1}^{N} \partial_{x_k}^2 + g \sum_{j,k=1}^{N} \delta(x_j - x_k).$$

By integrable we mean here that a complete commuting set of constants of motion can be explicitly constructed. Soon afterwards, Sutherland [10] established the integrability of the model [9] with an inverse sine-square interaction $v(r) = \sin^2(r)$.

The most general interaction potential $v$ for which the Hamiltonian [9] is known to be integrable is the Weierstrass $P$ function, which includes the rational and trigonometric cases as special limits. The integrability of this potential was proved in the classical case by Calogero and Perelomov by means of a Lax pair representation [11], and its explicit integration was performed by Krichever [12], Olshanetsky and Perelomov [13] later showed that all these models have an underlying algebraic structure based on root systems of $A_n$ algebras, and that integrable models associated to other root systems also exist. In the models treated in Ref. [13] the integrals of motion are related to the radial parts of the Laplace–Beltrami operator on a symmetric space associated to the given root system. These integrable models are obtained from the projection of free motion on a higher-dimensional manifold.

However, integrable Hamiltonians are not necessarily solvable, i.e., we might not be able to find explicitly their spectrum and eigenfunctions. The models with inverse-square and inverse sine-square interaction are known to be solvable, and much literature has been devoted to the study of their eigenfunctions [14], but the more general model with the Weierstrass $P$ potential is considerably more difficult. In fact, very few explicit solutions are known in this case [13], and only for a low number of particles.

The purpose of this Letter is to present a model of $N$ particles on a line with elliptic pairwise interaction in an external field for which a finite number of eigenvalues and eigenfunctions can be computed algebraically. We shall only sketch here the main ideas behind the proof of this result, referring the reader to our previous work [16] for a more complete description of the method used.

Consider the $N$-body quantum Hamiltonian

$$H_N = - \sum_{k=1}^{N} \partial_{x_k}^2 + V_N(x)$$

with potential

$$V_N(x) = c_m \sum_{k=1}^{N} P(x_k + i\beta) + 4b(b-1) \sum_{k=1}^{N} P(2x_k) + a(a-1) \sum_{j,k=1}^{N} P(x_j + x_k) + P(x_j - x_k),$$

where $a$ and $b$ are positive real parameters, $m$ is a non-negative integer,

$$c_m = 2[2b + m + a(N-1)] \times [2b + 2m + 2a(N-1) + 1] > 0,$$
and \( \mathcal{P}(z) \equiv \mathcal{P}(z|g_2, g_3) \) denotes the Weierstrass \( \mathcal{P} \) function with invariants \( g_2, g_3 \in \mathbb{R} \). If \( g_2 \) and \( g_3 \) satisfy the inequality \( g_2^2 > 27g_3^2 \), then \( \mathcal{P}(z) \) has two fundamental periods \( 2\alpha \) and \( 2i\beta \) which are real and purely imaginary, respectively [17]. In this case \( \mathcal{P}(x+i\beta) \) is real and regular (analytic) for all real values of \( x \), with real period \( 2\alpha \). On the other hand, \( \mathcal{P}(x) \) is real for real \( x \) and diverges as \( (x-2n\alpha)^{-2} \) when \( x \) tends to an integer multiple \( 2n\alpha \) of the real period \( 2\alpha \). Thus, the configuration space for the Hamiltonian \( \mathcal{P} \)–(3) can be taken as the bounded region of \( \mathbb{R}^N \)

\[
0 < x_N < x_{N-1} < \cdots < x_1 < \alpha .
\]

Since the potential \( \mathcal{P} \) is confining in this region, the spectrum of \( \mathcal{H}_N \) is purely discrete, and the boundary condition satisfied by its eigenfunctions \( \psi_k(\mathbf{x}) \) is their vanishing on the boundary of \( \mathcal{P} \).

The potential \( \mathcal{P} \) with \( c_m = 0 \) is of \( C_N \) type \( \mathcal{P} \)

\[
4b(b-1) \sum_{k=1}^{N} v(2x_k) + \alpha(a-1) \sum_{j<k}^{N} \left[ v(x_j-x_k) + v(x_j+x_k) \right],
\]

with interaction potential \( v(r) = \mathcal{P}(r) \). The term proportional to \( c_m \) in \( \mathcal{P} \) can be viewed as the contribution of an external field with potential \( \mathcal{P}(r+i\beta) \).

We shall now show that, when the parameter \( m \) is a non-negative integer, one can algebraically compute a finite number (depending on \( m \), see [14] below) of eigenvalues and eigenfunctions of the Hamiltonian \( \mathcal{P} \)–(3). These algebraic eigenfunctions have the form

\[
\psi_k(\mathbf{x}) = \mu(\mathbf{x}) \chi_k(\mathbf{z}),
\]

where

\[
\mu(\mathbf{x}) = \prod_{j<k} [\mathcal{P}(x_j+i\beta) - \mathcal{P}(x_k+i\beta)]^a \prod_{k} [\mathcal{P}'(x_k+i\beta)]^{b},
\]

and \( \chi_k(\mathbf{z}) \) is a suitable completely symmetric polynomial of degree at most \( Nm \) in the variables

\[
z_j = \mathcal{P}(x_j+i\beta), \quad j = 1, \ldots, N.
\]

The exact solutions of the trigonometric and rational \( C_N \) (in fact, \( BC_N \)) models also assume the form \( \mathcal{P} \), but in this case \( \mu \) can be factorized over the system of positive roots, i.e., it has the form

\[
\mu = \prod_{j<k} [f(x_j-x_k)f(x_j+x_k)]^a \prod_{k} [f(2x_k)]^b,
\]

where \( f(x) = x \) in the rational case and \( f(x) = \sin x \) in the trigonometric case. Moreover, in both cases \( \mu \) coincides with the ground-state wave function of the system. By contrast, the function \( \mu \) in \( \mathcal{P} \) cannot be factorized over the system of positive roots for any function \( f(x) \), and is not the ground-state wave function of the Hamiltonian \( \mathcal{P} \)–(3). As a matter of fact, it was shown in [13] that the most general potential allowing for the factorization \( \mathcal{P} \)–(3) does not include the elliptic case. This is one of the reasons why it has been so difficult to obtain explicit solutions of the elliptic CS models.

When the parameters \( a \) and \( b \) are positive, the functions \( \mathcal{P} \) are regular in the region \( \mathcal{P} \), and they automatically vanish on its boundary on account of the identities \( \mathcal{P}'(i\beta) = \mathcal{P}'(\alpha+i\beta) = 0 \) (see Ref. [14]). Thus, to show that \( \psi_k \) in Eq. \( \mathcal{P} \) is an eigenfunction of \( \mathcal{H}_N \) we only have to check that it satisfies the Schrödinger equation \( (\mathcal{H}_N - E_k) \psi_k = 0 \) in the open region \( \mathcal{P} \). Equivalently, \( \chi_k \) must be a solution of the equation \( (\mathcal{H}_N - E_k) \chi_k = 0 \), where the gauge Hamiltonian \( \mathcal{H}_N \) is defined by

\[
\mathcal{H}_N = \mu^{-1} \mathcal{H}_N \mu .
\]

Note that, by the standard properties of the Weierstrass function \( \mathcal{P} \), \( \mu \) does not vanish in the region \( \mathcal{P} \).

It can be shown that, provided that \( m \) is a non-negative integer, \( \mathcal{H}_N \) preserves the finite-dimensional polynomial space

\[
\mathcal{M}_m = \text{span} \left\{ \tau_1^{i_1} \tau_2^{i_2} \cdots \tau_N^{i_N} : \sum_{i=1}^{N} i_i \leq m \right\} ,
\]

where

\[
\tau_k = \sum_{i_1 < i_2 < \cdots < i_k} z_{i_1} z_{i_2} \cdots z_{i_k}, \quad 1 \leq k \leq N,
\]

are the elementary symmetric functions of the variables \( z_k \). This is due to the fact that, when \( \mathcal{H}_N \) is written in terms of the symmetric variables \( \tau_1, \ldots, \tau_N \), it can be expressed as a quadratic combination of the generators of \( sl(N+1) \) in the representation

\[
\mathcal{D}_k = \partial_{\tau_k}, \quad \mathcal{N}_{jk} = \tau_j \partial_{\tau_k}, \quad \mathcal{U}_k = \tau_k \left( m - \sum_{i=1}^{N} \tau_i \partial_{\tau_i} \right) ;
\]

\[
j, k = 1, 2, \ldots, N .
\]

Since these generators obviously preserve the subspace \( \mathcal{M}_m \), so does the gauge Hamiltonian \( \mathcal{H}_N \). It follows that \( \mathcal{H}_N \) has (at most)

\[
\dim \mathcal{M}_m = \binom{m + N}{m}
\]

eigenfunctions \( \chi_k \) lying in \( \mathcal{M}_m \), which can be algebraically computed, along with their corresponding eigenvalues, simply by diagonalizing the finite-dimensional matrix of \( \mathcal{H}_N |_{\mathcal{M}_m} \). The elements of \( \mathcal{M}_m \), being polynomials in the symmetric variables \( \tau_k \) of degree at most \( m \), are symmetric polynomials in \( z \) of degree not greater than \( Nm \). Thus the original Hamiltonian
The eigenvalues of this matrix are the algebraic energies of the physical Hamiltonian $H_2$, and its eigenvectors give the components of the corresponding functions $\chi_k$ in Eq. (3) with respect to the canonical basis of $M_m$.

The matrix of the restriction $H_2|_{M_2}$ of the Hamiltonian (2)–(3), is sometimes called a hidden symmetry algebra $\mathfrak{h}$, since in this case the Hamiltonian need not be a Casimir element.

For simplicity (indeed, no conceptual difficulties arise for higher values of $N$ and $m$), let us consider the problem for $N = 2$ and $m = 2$, for which the potential reads

$$V_2(x_1, x_2) = c_2 \sum_{k=1}^{2} |P(x_k + i\beta) + 4b(b - 1) \sum_{k=1}^{2} P(2x_k) + 2a(a - 1)|P(x_1 + x_2) + P(x_1 - x_2)|.$$  (15)

Note that this is intrinsically a two-body problem, since the potential (15) is not translation invariant. The number of algebraic eigenstates is at most $\dim M_2 = 6$, and the matrix of the restriction $H_2|_{M_2}$ with respect to the canonical basis $\{1, \tau_1, \tau_2, \tau_1^2, \tau_1 \tau_2, \tau_2^2\}$ of $M_2$ is given by

$$\begin{pmatrix}
0 & 16a + 4b + 20 & 0 & -2a g_3 & 4g_3 & 0 \\
 16a + 4b + 20 & 0 & 8a + 24b + 12 & 0 & 0 & 0 \\
 0 & 8a + 24b + 12 & 0 & 0 & 0 & 0 \\
 0 & 8a + 12b + 14 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 8a + 12b + 14 & 0 & 0 \\
 0 & 0 & 0 & 0 & 8a + 24b + 28 & 0
\end{pmatrix}.  \tag{16}
$$

The lowest energy states of the system, although there is no guarantee that this should still be true in the general case. The polynomials $\chi_k$ corresponding to the six algebraic eigenfunctions of the potential (15)–(17) are given in Table III. In Figs. 4–8 we present a plot of the ground state and the first two excited wave functions of the system.

| $\chi_k$ | $\tau_1$ | $\tau_2$ | $\tau_1^2$ | $\tau_1 \tau_2$ | $\tau_2^2$ |
|----------|----------|----------|-----------|-------------|-----------|
| $\chi_0$ | -3.0585  | 7.1643   | 2.2349    | -9.778      | 9.0382    |
| $\chi_1$ | -3.3008  | -10.917  | 2.5612    | 13.462      | -24.885   |
| $\chi_2$ | 0.40412  | 5.1067   | -2.2912   | -0.7721     | 6.1599    |
| $\chi_3$ | -3.9273  | -37.989  | 2.0457    | 14.734      | 94.264    |
| $\chi_4$ | 1.6422   | -2.7456  | -0.1761   | -10.472     | -18.108   |
| $\chi_5$ | 3.8942   | 8.3079   | 3.6675    | 15.1        | 14.898    |

TABLE II: Polynomials $\chi_k$ corresponding to the algebraic eigenfunctions $\psi_k = \mu \chi_k$ of the Hamiltonian (2)–(16). In all cases, the coefficient of 1 has been normalized to unity.

The fact that $\psi_0 = \mu \chi_0$ is the ground state of the system is immediately apparent if we note that $\chi_0$ can be expressed in terms of the variables $z_k$ as

$$\chi_0 = 2.2349(z_2 - 0.82835)(z_2 - 0.54017) - 9.778 z_1(z_2 - 0.79769)(z_2 - 0.39212) + 9.0382 z_1^2(z_2 - 0.75385)(z_2 - 0.32801).  \tag{18}$$

For the values of the invariants $g_2$ and $g_3$ given in (17), we have

$$e_3 = -0.72011 \leq z_k = \mathcal{P}(x_k + i\beta) \leq e_2 = -0.240851.$$
We thus see from the previous expression that $\chi_0$ is positive everywhere. Since, by (4), $\mu$ has no zeros in the open triangle $0 < x_2 < x_1 < \alpha$, it follows that $\psi_0$ does not vanish in this triangle.

In conclusion, a finite number of eigenvalues and eigenfunctions of a quantum Hamiltonian describing $N$ particles on a line with elliptic interaction in the presence of an external field have been explicitly calculated by an algebraic method independent of the usual approach based on root systems.

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