GLOBAL EXISTENCE AND LARGE TIME BEHAVIOR OF A 2D KELLER-SEGEL SYSTEM IN LOGARITHMIC LEBESGUE SPACES

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ABSTRACT. This paper is devoted to the global analysis for the two-dimensional parabolic-parabolic Keller-Segel system in the whole space. By well balanced arguments of the $L^1$ and $L^\infty$ spaces, we first prove global well-posedness of the system in $L^1 \times L^\infty$ which partially answers the question posted by Kozono et al in [19]. For the case $\mu_0 > 0$, we make full use of the linear parts of the system to get the improved long time decay property. Moreover, by using the new formulation involving all linear parts, introducing the logarithmic-weight in time to modify the other endpoint space $L^\infty \times L^\infty$, and carefully decomposing time into several pieces, we are able to establish the global well-posedness and large time behavior of the system in $L^\infty_{\ln} \times L^\infty$.

1. Introduction. We study the Cauchy problem of the two-dimensional (2D) Keller-Segel system

\[
\begin{align*}
    u_t - \Delta u + \nabla \cdot (u \nabla v) &= 0 \quad \text{in } (0, \infty) \times \mathbb{R}^2, \\
    v_t - \Delta v + \mu_0 v - u &= 0 \quad \text{in } (0, \infty) \times \mathbb{R}^2, \\
    (u, v)|_{t=0} &= (u_0, v_0) \quad \text{in } \mathbb{R}^2,
\end{align*}
\]

where $\mu_0 \geq 0$, $u = u(t, x)$ and $v = v(t, x)$ are density of amoebae and concentration of chemical attractant respectively, $(u_0, v_0)$ is the given initial value. Chemotaxis is a biological phenomenon describing the change of motion of a cell population density in response to an external chemical stimulus spreads in the environment where the cells reside. Chemotaxis plays essential roles in various biological processes such as embryonic development, wound healing, and disease progression.

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derivation of the system of equations (1), we refer the readers to Childress-Percus [3] and Keller-Segel [14].

When $\mu_0=0$, (1) is invariant under the following transformations
\[
(u(t,x),v(t,x)) \to (\lambda^2 u(\lambda^2 t, \lambda x), v(\lambda^2 t, \lambda x)) \quad \text{for any } \lambda > 0.
\]

The study of system (1) in a functional space setting with scaling invariant originates from several papers: global existence of mild solution for $(u_0, v_0) \in H^{\frac{7}{5}-\epsilon, \frac{2}{5}}(\mathbb{R}^n) \times H^{\frac{3}{5}-\epsilon, \frac{1}{5}}(\mathbb{R}^n)$ with $\max\{1, \frac{n}{2}\} < r < \frac{n}{2}$ in [17], for $(u_0, v_0) \in L^{\frac{n}{2}}(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ with $n \geq 3$ in [18], and for $(u_0, v_0) \in \dot{L}^{2}(\mathbb{R}^n) \times H^{2\alpha, \frac{3}{2\alpha}}(\mathbb{R}^n)$ with $n \geq 3$ and $\frac{n}{2(n+2)} < \alpha \leq \frac{1}{2}$ in [19]. It is also known that apart from existence and uniqueness of mild solutions in the invariant space, there are papers on asymptotic behaviors, see e.g. [12, 32] and stationary solutions, see e.g. [10, 26].

Our first goal is to answer Kozono, Sugiyama and Wachi's question proposed in [19]: whether there exists a solution to (1) even locally in time for $(u_0, v_0) \in L^1(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)$? We give an affirmative answer to the existence of small solution in Theorem 1.2 by introducing $\|u(t,\cdot)\|_\infty$ in the $L^p$-framework, see e.g. [13].

When $\mu_0 > 0$, the chemical concentration decays exponentially for large time and the coupled linear operators yields better a-priori estimates, which indicate that $L^\infty(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)$ is another critical Lebesgue space for (1). Indeed, by making full use of the linear part of (1) and by defining
\[
S_\lambda(t) = \exp\{t\Delta - \lambda t\} = \exp\{-\lambda t\}S_0(t)
\]
where $\lambda \geq 0$ and $S_0(t) = \exp\{t\Delta\}$, we rewrite (1) into integral equations via Duhamel’s principle as
\[
\begin{cases}
u = \frac{1 - \exp\{-\mu_0 t\}}{\mu_0} S_0(t)u_0 + \exp\{-\mu_0 t\} S_0(t)v_0 & \\
+ \int_0^t \frac{1 - \exp\{-\mu_0 (t-\tau)\}}{\mu_0} S_0(t-\tau) \nabla \cdot (u \nabla v) d\tau.
\end{cases}
\]

It is known that a solution to system (4) is called a mild solution of system (1). It is worth pointing out that system (4) captures all the linear information of the original system (1) and it does not treat $u$ as an external force, which is one of our new points in the current paper.

It seems to be an interesting problem that whether global existence is true for system (1) when $\|u\|_\infty + \|v\|_\infty$ is small.

Our second goal is to prove global well-posedness of (1) in a subspace $L^\infty_\mu \times L^\infty_\nu$, a natural substitution of $L^\infty \times L^\infty$. The results are summarized in Theorem 1.3.

Now we recall some results concerning the parabolic-elliptic/parabolic-hyperbolic Keller-Segel systems. For the parabolic-elliptic Keller-Segel model
\[
\begin{cases}
u_1 = \nabla \cdot (\mu \nabla u - \chi u \nabla v), \\
u_2 = f u - g v + \nu \nabla^2 v,
\end{cases}
\]
where $u$ and $v$ are concentration of cell and chemical, $\mu, \chi, f, g, \nu > 0$ are constants, Childress and Percus conjectured in [4] that, in a radial symmetric two dimensional domain $\Omega$, there exists a critical number $\epsilon^*$ such that if $\int_\Omega u_0(x) dx < \epsilon^*$, then the radial symmetric solution exists globally in time, and if $\int_\Omega u_0(x) dx > \epsilon^*$, then blowup happens. For different versions of the Keller-Segel models, the conjecture
has been essentially proved. For a complete review of this topic, we refer the readers to [11] and the references therein, also see e.g. Diaz-Nagai-Rakotoson [7], Blanchet-Dolbeault-Perthame [1].

A parabolic-hyperbolic system which was derived from the Keller-Segel model

\[
\begin{aligned}
&\begin{cases}
  u_t - \nabla \cdot (uv) = D \Delta u, \\
v_t - \nabla u = 0
\end{cases}
\tag{6}
\end{aligned}
\]

where \( u \) is the concentration of cell, \( v = \sum c \) and \( c \) is the concentration of chemical, \( D > 0 \), was studied in [31, 24] for one dimensional case, was extended to multidimensional cases in [8, 23, 22], and was studied in [21, 28] with a comprehensive qualitative and numerical analysis. We refer the readers to references [5, 6, 9, 15, 16, 25, 27, 29, 30, 33] for more discussions in this direction.

Throughout this paper, let \( C_{\alpha, \beta} \) and \( c_{\alpha, \beta} \) be positive constants which depend on \( \alpha, \beta, \cdots \) and which may vary from line to line. We denote

\[
(t) = e + t
\tag{7}
\]

for short. For any \( 1 \leq q \leq \infty \), we denote \( L^q(\mathbb{R}^2) \) by \( L^q \). Here and hereafter, we focus on the case \( n = 2 \) and denote \( \| \cdot \|_{L^q(\mathbb{R}^2)} \) by \( \| \cdot \|_q \).

**Definition 1.1.** \( L^q_{\infty} \) is defined to be the set of all distributions \( w \) such that

\[
\| w \|_{L^q_{\infty}} := \sup_{t > 0} \| \ln(t) S_0(t) w \|_{\infty} < \infty.
\tag{8}
\]

**Remark 1.** From the definition of \( L^q_{\infty} \), it is easy to check that

\[
\| w \|_{L^q_{\infty}} = \| \ln(t) S_0(t) w \|_{\infty}
\leq \sup_{0 < t < \infty} \| \ln 2e \, S_0(t) w \|_{\infty} + \sup_{t \geq e} \| \ln(t) S_0(t) w \|_{\infty}
\leq \sup_{0 < t < \infty} \| \ln 2e \, S_0(t) w \|_{\infty} + c_q \sup_{t \geq e} \| t^{\frac{1}{q}} S_0(t) w \|_{\infty}
\leq c_q \left( \| w \|_{\infty} + \| w \|_q \right)
\tag{9}
\]

where \( 1 \leq q < \infty \). Therefore, any bounded \(^1 L^q \) function with \( 1 \leq q < \infty \) belongs to \( L^q_{\infty} \), for instance,

\[
w_1 = \varepsilon^{\frac{q}{2}} \varphi(\varepsilon x)
\]

for small \( \varepsilon > 0 \) and \( \varphi \in L^1 \). In fact, by standard argument of the characterization theory for Besov space, it is easy to know that \( \sup_{t \geq \varepsilon} \| \ln(t) S_0(t) w \|_{\infty} \) is determined only by the low frequency information of \( w \). As a consequence, \( L^q_{\infty} \) can be thought as a set of bounded functions with some \( L^q \)-perturbations, see (9). Moreover, the Fourier transformation of the perturbations are supported in a bounded domain. For instance,

\[
w_2(x) = \sin \ell x + 2^{2sk} \varphi(2^k x),
\]

where \( s \) is a small positive constant, \( \ell \) is a large positive integer, \( k \) is a small negative number, and \( \varphi(\xi) \) is compactly supported near the origin with \( \int \varphi(\xi) d\xi = 1 \).

**Theorem 1.2.** Let \( \mu_0 \geq 0 \). For any \( (u_0, v_0) \in L^1 \times L^\infty \) with \( \| u_0 \|_1 + \| v_0 \|_{\infty} \) being small enough, there exists a unique global mild solution \( (u, v) \) to (1) such that

\[
u \in C([0, \infty); L^1), \quad v \in C_*([0, \infty); L^\infty)
\]

\(^1 w \) is a bounded \( L^q \) function if \( w \in L^q \cap L^\infty \) for any \( 1 \leq q < \infty \).
with \( C_*([0, \infty); L^\infty) \) being the set of weakly-star continuous functions on \([0, \infty)\) valued in Banach space \( L^\infty \), and
\[
\|u\|_\infty = O(t^{-1}), \quad \|\nabla v\|_\infty = O(t^{-\frac{1}{2}})
\] as \( t \to +\infty \). Moreover, if \( \mu_0 > 0 \), then we have
\[
\|u\|_\infty = O\left(t^{-1}\right), \quad \|\nabla v\|_\infty = O\left(t^{-\frac{3}{2}} \ln(t)\right)
\] as \( t \to +\infty \), where \( \langle t \rangle \) is defined in (7).

**Remark 2.** From (10)–(11) we observe that solutions to system (1) with \( \mu_0 > 0 \) decay faster than that of \( \mu_0 = 0 \). Moreover, in the case \( \mu_0 > 0 \), one can also prove the existence of small global mild solution to system (1) of \((u_0, v_0) \in L^q \times L^\infty\) for any \( 1 < q < \infty \), as well as its long time decay estimates. However, it seems quite difficult to establish the desired a priori bilinear estimates with \((u_0, v_0) \in L^\infty \times L^\infty\).

To complete the investigation in Lebesgue space framework, it remains to analyze the other end-point case, i.e., \( L^\infty \times L^\infty \). However, as mentioned above, this case seems quite difficult. As a consequence, we modify the \( L^\infty \) slightly and introduce the logarithmic bounded space \( L^\infty_{ln} \) instead. Luckily, this logarithmic space works and yields the following result.

**Theorem 1.3.** Let \( \mu_0 > 0 \). For any \((u_0, v_0) \in L^\infty_{ln} \times L^\infty\) with \( \|u_0\|_{L^\infty_{ln}} + \|v_0\|_{\infty} \) being sufficiently small, there exists a unique global mild solution \((u, v)\) to (1) such that
\[
u \in C_*([0, \infty); L^\infty_{ln}), \quad v \in C_*([0, \infty); L^\infty)
\] with \( C_*([0, \infty); L^\infty_{ln}) \) being the set of weakly-star continuous functions on \([0, \infty)\) valued in Banach space \( L^\infty_{ln} \), and
\[
\|u\|_{\infty} = O\left(\ln^{-1}(t)\right), \quad \|\nabla v\|_{\infty} = O\left(t^{-\frac{1}{2}} \ln^{-1}(t)\right)
\] as \( t \to +\infty \).

**Remark 3.** In the 2D case, it should be interesting to consider various interpolations of (11)–(12) and figure out the connection of time-logarithmic-weight and space-logarithmic-weight (see [2], Theorem 1.1, for \( L^1(R^2, \ln(1 + |x|^2)dx \) and \( u_0 \ln u_0 \in L^1(R^2) \) as well as the endpoint case of Hardy-Littlewood-Sobolev inequality).

**Plan of the paper:** In Section 2, we give preliminary. We prove Theorem 1.2 in Section 3. In Section 4, we give several important Definitions and Lemmas on the analysis in the logarithmic space \( L^\infty_{ln} \times L^\infty \) which lead to the proof of Theorem 1.3.

2. Preliminaries.

**Lemma 2.1.** For any \( t > 0 \), \( 0 \leq \alpha < 1 \), \( \beta > \frac{1}{2} \). Then there exists a positive constant \( C_{\alpha, \beta} \) depending only on \( \alpha \), \( \beta \) such that
\[
\int_0^t \frac{d\tau}{(1 + t - \tau) \tau^\alpha \ln^{2\beta}(e + \tau)} \leq C_{\alpha, \beta} \frac{1}{t^\alpha \ln^{2\beta - 1}(e + t)}.
\] (13)

Moreover, for any \( \gamma > 0 \), there exists a positive constant \( C_{\gamma} \) depending only on \( \gamma \) such that
\[
\int_0^t \frac{d\tau}{\sqrt{t - \tau} \sqrt{\tau} \ln^{2\gamma}(e + \tau)} \leq C_{\gamma} \frac{1}{\ln^{2\gamma}(e + t)}.
\] (14)
Proof. We prove (13) first. In order to give the detailed proof, we divide the proof into two cases: \(0 < t \leq 9\) and \(t > 9\).

**Case 1.** \(0 < t \leq 9\). It is easy to check that \(1 \leq \ln^{2\beta-1}(e + t) \leq 3^{2\beta-1}\). As a consequence, we get

\[
\int_0^t \frac{d\tau}{(1 + t - \tau)^{\tau^\alpha} \ln^{2\beta}(e + \tau)} \leq \int_0^t \frac{d\tau}{\tau^{\alpha}} \leq C_{\alpha,\beta} \frac{1}{t^{\alpha} \ln^{2\beta-1}(e + t)}. \tag{15}
\]

**Case 2.** \(t > 9\). It is easy to check that \(3 < \sqrt{t} < \frac{t}{2}\) and

\[
\ln(e + t) < 2\ln(e + \sqrt{t}) < 2\ln(e + \frac{t}{2}) < 2\ln(e + t).
\]

Hence we get

\[
\int_0^t \frac{d\tau}{(1 + t - \tau)^{\tau^\alpha} \ln^{2\beta}(e + \tau)} \leq \int_{\sqrt{t}}^{\sqrt{t}} \frac{d\tau}{(1 + t - \tau)^{\tau^\alpha} \ln^{2\beta}(e + \tau)} + \int_{\sqrt{t}}^{t} \frac{d\tau}{(1 + t - \tau)^{\tau^\alpha} \ln^{2\beta}(e + \tau)} + \int_{\sqrt{t}}^{\sqrt{t}} \frac{d\tau}{(1 + t - \tau)^{\tau^\alpha} \ln^{2\beta}(e + \tau)} = A_{11} + A_{12} + A_{13}, \tag{16}
\]

where

\[
A_{11} \leq \int_0^{\sqrt{t}} \frac{d\tau}{\tau^\alpha \sqrt{t}} \leq C_{\alpha} \frac{1}{t^{1+\alpha}}
\]

\[
\leq C_{\alpha} \frac{1}{t^{\alpha} \ln^{2\beta-1}(e + t) t^{\frac{1}{2}}} \leq C_{\alpha,\beta} \frac{1}{t^{\alpha} \ln^{2\beta-1}(e + t)}, \tag{17}
\]

\[
A_{12} \leq \int_{\sqrt{t}}^{\sqrt{t}} \frac{d\tau}{\tau \ln^{2\beta}(e + \tau)} \leq C_{\alpha} \frac{1}{t^{\alpha} \ln^{2\beta}(e + \sqrt{t})}
\]

\[
\leq C_{\alpha} \frac{1}{t^{\alpha} \ln^{2\beta-1}(e + t) \ln^{2\beta}(e + \sqrt{t})} \leq C_{\alpha,\beta} \frac{1}{t^{\alpha} \ln^{2\beta-1}(e + t)}, \tag{18}
\]

\[
A_{13} \leq C_{\alpha} \frac{1}{t^{\alpha} \ln^{2\beta}(e + \frac{t}{2})} \int_{\sqrt{t}}^{t} \frac{1}{1 + t - \tau} d\tau
\]

\[
\leq C_{\alpha} \frac{1}{t^{\alpha} \ln^{2\beta-1}(e + t) \ln^{2\beta}(e + \frac{t}{2})} \leq C_{\alpha,\beta} \frac{1}{t^{\alpha} \ln^{2\beta-1}(e + t)}. \tag{19}
\]

Combining the estimates (15)–(19), we prove (13).

It remains to show (14). Similar to (13), we also consider two cases: \(0 < t \leq 9\) and \(t > 9\). If \(0 < t \leq 9\), then we have
Then from (22)–(23), we have

$$\int_0^t \frac{d\tau}{\sqrt{t - \tau} \sqrt{\ln^2(e + \tau)}} \leq \int_0^t \frac{d\tau}{\sqrt{t - \tau} \sqrt{\ln^2(e + \tau)}} \leq C \frac{\ln^2(e + t)}{\ln^2(e + t)}$$

$$\leq C_1 \frac{1}{\ln^2(e + t)};$$

(20)

Else if \( t > 9 \), then we have

$$\int_0^t \frac{d\tau}{\sqrt{t - \tau} \sqrt{t} \ln^2(e + \tau)}$$

$$\leq \int_0^{\sqrt{t}} \frac{d\tau}{\sqrt{t} \sqrt{\ln^2(e + \tau)}} + \int_{\sqrt{t}}^t \frac{d\tau}{\sqrt{\ln^2(e + \sqrt{t})}}$$

$$+ \int_0^t \frac{d\tau}{\sqrt{t - \tau} \sqrt{t} \ln^2(e + \tau)} \leq \frac{C}{\ln^2(e + t)} \left( \frac{\ln^2(e + t)}{t^4} + \frac{\ln^2(e + \sqrt{t})}{\ln^2(e + \frac{t}{2})} \right)$$

$$\leq C_1 \frac{1}{\ln^2(e + t)}. \quad (21)$$

Combining (20)–(21), we obtain (14), and finish the proof.

The last lemma of this section is a slightly different version of the well-known Picard contraction principle, see [20], Theorem 13.2, p.124.

**Lemma 2.2.** Let \( \mu_0 > 0 \). Let \( (X \times Y, \| \cdot \|_X + \| \cdot \|_Y) \) be an abstract Banach product space, and \( B_0 : X \times Y \to X \) and \( B_1 : X \times Y \to Y \) be two bilinear operators. If for any \((u, v) \in X \times Y\), there exists a positive constant \( c \) such that if

$$\left\| \left( B_0(u, v), B_1(u, v) \right) \right\|_{X \times Y} \leq c \left\| (u, v) \right\|_{X \times Y}^2,$$

(22)

then for any \((u_0, v_0)\) satisfying

$$A \triangleq \left\| S_0(t)u_0 \right\|_X + \left\| \frac{S_0(t) - S_{\mu_0} (t)}{\mu_0} u_0 + S_{\mu_0} (t)v_0 \right\|_Y \leq \frac{2}{9c}, \quad (23)$$

the following system

$$u(t) = \left( S_0(t)u_0, \frac{S_0(t) - S_{\mu_0} (t)}{\mu_0} u_0 + S_{\mu_0} (t)v_0 \right) + \left( B_0(u, v), B_1(u, v) \right) \quad (24)$$

has a solution \((u, v)\) in \( X \times Y \). In particular, the solution is the only one such that

$$\frac{1}{2} A \leq \left\| (u, v) \right\|_{X \times Y} \leq \frac{3}{2} A.$$  \quad (25)

**Proof.** For completeness, we sketch the proof.

As usual, we first define the map \( \mathcal{J} \) as follows

$$\mathcal{J} : X \times Y \to X \times Y, \quad \mathcal{J}(u, v) \triangleq \text{"Right hand side of (24)"}. \quad (26)$$

Then from (22)–(23), we have

$$\left\| \mathcal{J}(u, v) \right\|_{X \times Y} \leq A + c \left\| (u, v) \right\|_{X \times Y}^2.$$

(27)

Applying (23) to (27), we obtain that \( \mathcal{J}(u, v) \) maps a bounded ball centered at origin of radius \( \frac{3}{2} A \) in \( X \times Y \) into itself.
Next, for any \((u^1, v^1), (u^2, v^2)\) satisfy (25), we derive that
\[
\|J(u^1, v^1) - J(u^2, v^2)\|_{X \times Y} \leq \frac{1}{3} \|(u^1 - u^2, v^1 - v^2)\|_{X \times Y}.
\]

In fact,
\[
\|J(u^1, v^1) - J(u^2, v^2)\|_{X \times Y} = \|J(u^1, v^1) - J(u^1, v^2)\|_{X \times Y} + \|J(u^1, v^2) - J(u^2, v^2)\|_{X \times Y} \\
\leq \|J(u^1, v^1 - v^2)\|_{X \times Y} + \|J(u^1 - u^2, v^2)\|_{X \times Y} \\
\leq c\|u^1\|_{X} \|v^1 - v^2\|_{Y} + c\|u^1 - u^2\|_{X} \|v^2\|_{Y} \\
\leq \frac{3A}{2} \|v^1 - v^2\|_{Y} + c\|u^1 - u^2\|_{X} \\
\leq \frac{3A}{2} \|(u^1 - u^2, v^1 - v^2)\|_{X \times Y} \\
\leq \frac{1}{3} \|(u^1 - u^2, v^1 - v^2)\|_{X \times Y}.
\]

From the definition of \(J\), (26), (24), we have that
\[
\|J(u, v)\|_{X \times Y} \geq \| \left( S_0(t)u_0, \frac{S_0(t) - S_\mu(t)}{\mu_0}u_0 + S_\mu(t)v_0 \right) \|_{X \times Y} \\
- \| \left( B_0(u, v), B_1(u, v) \right) \|_{X \times Y} \\
\geq \|S_0(t)u_0\|_X + \| \frac{S_0(t) - S_\mu(t)}{\mu_0}u_0 + S_\mu(t)v_0\|_Y - c\|(u, v)\|_{X \times Y}^2 \\
\geq A - \frac{9A}{4} A \geq A - \frac{9A}{4} A = \frac{1}{2} A.
\]

As a direct consequence of contraction mapping theorem, the existence and uniqueness of solution follow. It remains to prove the inequality on the left of (25). In fact, it follows immediately from
\[
\|J(u, v)\|_{X \times Y} \geq A - c\|(u, v)\|_{X \times Y}^2 \geq \frac{1}{2} A.
\]
Hence we complete the proof.

3. Proof of Theorem 1.2.

3.1. Analysis in \(L^1 \times L^\infty\). In this subsection, we prove global well-posedness of (1) with \((u_0, v_0) \in L^1 \times L^\infty\) in the Kato \(L^p\)-framework. At first, we define the working space as follows:
\[
X = \{ u \in S'(0, \infty) \times \mathbb{R}^2) ; \sup_{t > 0} \|u(\cdot, t)\|_1 + \sup_{t > 0} \|tu(\cdot, t)\|_\infty < \infty \}, \quad (28)
\]
\[
Y = \{ v \in S'(0, \infty) \times \mathbb{R}^2) ; \sup_{t > 0} \|v(\cdot, t)\|_\infty + \sup_{t > 0} \|t^{\frac{1}{2}} \nabla v(\cdot, t)\|_\infty < \infty \}. \quad (29)
\]

The next lemma is about the estimates for the linear parts of (4).
Lemma 3.1. Assume that \( \mu_0 \geq 0 \). For any \((u_0, v_0) \in L^1 \times L^\infty\), there exists a positive constant \(c_0\) independent of \(\mu_0\) such that for any \(t > 0\) there holds
\[
\|S_0(t) u_0\|_X + \left\| \frac{1 - \exp\{-\mu_0 t\}}{\mu_0} S_0(t) u_0 + S_{\mu_0}(t) v_0 \right\|_Y \leq c_0 (\|u_0\|_1 + \|v_0\|_\infty). \tag{30}
\]
In particular, if \(\mu_0 > 0\), then we have
\[
\|S_0(t) u_0\|_\infty \leq c_0 \|u_0\|_1 t^{-1} \tag{31}
\]
and
\[
\left\| \frac{1 - \exp\{-\mu_0 t\}}{\mu_0} \nabla S_0(t) u_0 + \nabla S_{\mu_0}(t) v_0 \right\|_\infty \leq c_0 \left(\|u_0\|_1 + \|v_0\|_\infty\right) t^{-\frac{3}{2}}. \tag{32}
\]

Proof. Noticing that the two-dimensional heat kernel is \(\frac{1}{4\pi t} \exp\{-\frac{|x|^2}{4t}\}\), then for any \(t > 0\), \(1 \leq p \leq \infty\),
\[
\left\| \frac{1}{4\pi t} \exp\{-\frac{|x|^2}{4t}\} \right\|_p \leq c_0 t^{-1 + \frac{1}{p}}, \quad \left\| \frac{1}{4\pi t} \nabla \exp\{-\frac{|x|^2}{4t}\} \right\|_p \leq c_0 t^{-\frac{3}{2} + \frac{1}{p}}. \tag{33}
\]
As a direct consequence of (33), \(\frac{1 - \exp\{-\mu_0 t\}}{\mu_0} \leq 1\) and the Young’s inequality, it follows
\[
\left\| \frac{1 - \exp\{-\mu_0 t\}}{\mu_0} S_0(t) u_0 \right\|_\infty + \left\| \frac{1 - \exp\{-\mu_0 t\}}{\mu_0} S_0(t) \nabla v_0 \right\|_\infty \leq c_0 \|u_0\|_1.
\]
The remained parts follow in the similar way. Hence we finish the proof. \(\square\)

It remains to establish the key bilinear estimates in \(X \times Y\). We denote
\[
\begin{cases}
B_0(u, v) = \int_0^t S_0(t - \tau) \nabla \cdot (u \nabla v) d\tau, \\
B_1(u, v) = \int_0^t \frac{1 - \exp\{-\mu_0 (t - \tau)\}}{\mu_0} S_0(t - \tau) \nabla \cdot (u \nabla v) d\tau.
\end{cases} \tag{34}
\]

Lemma 3.2. Assume that \(\mu_0 \geq 0\). There exists a constant \(c > 0\) independent of \(\mu_0\) such that
\[
\|B_0(u, v)\|_X + \|B_1(u, v)\|_Y \leq c \|u\|_X \|v\|_Y. \tag{35}
\]
Moreover, if \(\mu_0 > 0\), then we have
\[
\|B_0(u, v)\|_\infty \leq c \|u\|_X \|v\|_Y t^{-1}, \quad \|\nabla B_1(u, v)\|_\infty \leq \frac{c}{\mu_0} \|u\|_X \|v\|_Y t^{-\frac{3}{2}} \ln(t). \tag{36}
\]

Proof. In order to estimate \(\|B_0(u, v)\|_X\), we apply (33) to (34) to get
\[
\|B_0(u, v)\|_X = \sup_{t > 0} \left\| \int_0^t S_0(t - \tau) \nabla \cdot (u \nabla v) d\tau \right\|_1 + \sup_{t > 0} \left\| \int_0^t S_0(t - \tau) \nabla \cdot (u \nabla v) d\tau \right\|_\infty
\]
\[
\leq \sup_{t > 0} \int_0^t (t - \tau)^{-\frac{3}{2}} \|u \nabla v\|_1 d\tau + \sup_{t > 0} \int_0^t (t - \tau)^{-\frac{3}{2}} \|u \nabla v\|_1 d\tau
\]
\[
+ \sup_{t > 0} \int_0^t (t - \tau)^{-\frac{3}{2}} \|u \nabla v\|_\infty d\tau.
\]
\[ \leq \sup_{t > 0} \left( \int_0^t (t - \tau)^{-\frac{1}{2}} \tau^{-\frac{1}{2}} \, d\tau + \frac{1}{\sqrt{t}} \int_0^t \tau^{-\frac{1}{2}} \, d\tau \right) \sup_{\tau > 0} \frac{\|u \nabla v(\tau)\|_1}{\tau^{\frac{1}{2}}} \]

\[ + \sup_{t > 0} \int_0^t (t - \tau)^{-\frac{1}{2}} \tau^{-\frac{1}{2}} \, d\tau \sup_{\tau > 0} \frac{\|u \nabla v(\tau)\|_\infty}{\tau^{\frac{1}{2}}} \]

\[ \leq c \left( \sup_{\tau > 0} \|u(\tau)\|_1 + \sup_{\tau > 0} \|\tau u(\tau)\|_\infty \right) \sup_{\tau > 0} \|\tau^{\frac{1}{2}} \nabla v(\tau)\|_\infty \]

\[ \leq c \|u\|_X \|v\|_Y. \]

(37)

Similarly, by applying (33)–(34) and \( \frac{1 - \exp[-\mu_0(t-\tau)]}{\mu_0(t-\tau)} \leq 1 \) to \( B_1(u, v) \), we obtain

\[ \|B_1(u, v)\|_Y = \sup_{t > 0} \|B_{\mu_0}(u, v)\|_\infty + \sup_{t > 0} \|t^{\frac{1}{2}} \nabla B_{\mu_0}(u, v)\|_\infty \]

\[ = \sup_{t > 0} \left( \int_0^t \frac{1 - \exp[-\mu_0(t-\tau)]}{\mu_0(t-\tau)} S_0(t - \tau) \nabla \cdot (u \nabla v) \, d\tau \right) \|_\infty \]

\[ + \sup_{t > 0} \int_0^t \frac{1 - \exp[-\mu_0(t-\tau)]}{\mu_0(t-\tau)} \left( (\sqrt{\tau} u \nabla v)(\tau) \right) \|_1 \, d\tau \]

\[ + \sup_{t > 0} \int_0^t \frac{1 - \exp[-\mu_0(t-\tau)]}{\mu_0(t-\tau)} \left( (\tau^{\frac{1}{2}} u \nabla v)(\tau) \right) \|_\infty \, d\tau \]

\[ \leq c \left( \sup_{\tau > 0} \|u(\tau)\|_1 + \sup_{\tau > 0} \|\tau u(\tau)\|_\infty \right) \sup_{\tau > 0} \|\sqrt{\tau} \nabla v(\tau)\|_\infty \]

\[ \leq c \|u\|_X \|v\|_Y. \]

(38)

Combining (37)–(38), we prove (35).

Next we prove the remained part.

\[ \|B_0(u, v)\|_\infty = \| \int_0^t S_0(t - \tau) \nabla \cdot (u \nabla v) \, d\tau \|_\infty \]

\[ \leq \int_0^t (t - \tau)^{-\frac{1}{2}} \tau^{-\frac{1}{2}} \|\tau^{\frac{1}{2}} u \nabla v\|_1 \, d\tau + \int_0^t (t - \tau)^{-\frac{1}{2}} \tau^{-\frac{1}{2}} \|\tau^{\frac{1}{2}} u \nabla v\|_\infty \, d\tau \]

\[ \leq c \|u\|_X \|v\|_Y t^{-\frac{1}{2}}. \]

(39)

\[ \|\nabla B_1(u, v)\|_\infty = \| \int_0^t \frac{1 - \exp[-\mu_0(t-\tau)]}{\mu_0} S_0(t - \tau) \nabla \nabla \cdot (u \nabla v) \, d\tau \|_\infty \]

\[ \leq \int_0^t \frac{1 - \exp[-\mu_0(t-\tau)]}{\mu_0(t-\tau)^{\frac{1}{2}} \tau^{\frac{1}{2}}} \|\tau^{\frac{1}{2}} u \nabla v\|_1 \, d\tau \]

\[ + \int_0^t \frac{1 - \exp[-\mu_0(t-\tau)]}{\mu_0(t-\tau)^{\frac{1}{2}} \tau^{\frac{1}{2}}} \|\tau^{\frac{1}{2}} u \nabla v\|_\infty \, d\tau \]

\[ \leq \left( \frac{1}{t^2} \int_0^t \frac{1}{\mu_0(t-\tau)} \, d\tau + \frac{1}{t^{\frac{3}{2}}} \int_0^t \frac{1}{1 + \mu_0(t-\tau)} \, d\tau \right) \|u\|_X \|v\|_Y \]

\[ \leq c \|u\|_X \|v\|_Y t^{-\frac{1}{2}} \ln(t). \]

(40)

Therefore, we complete the proof. \( \square \)
3.2. Proof of Theorem 1.2. The proof of Theorem 1.2 is a direct consequence of Lemmas 3.1, 3.2 and 2.2. In fact, we can define a map by using (4) and (34) such that (24) of Lemma 2.2 is applied. Then combining the a priori bounds established by Lemmas 3.1 and 3.2 with Lemma 2.2, we get the desired result. Hence we omit the details.

4. Analysis in the logarithmic space \( L^\infty_t \times L^\infty \). In this section, we apply the logarithmic time weighted norm to the local well-posedness obtained in Section 2 to get global well-posedness.

Define
\[
Z_0 = \left\{ w; \sup_{t > 0} \| \ln(e + t)w(t) \|_\infty < \infty \right\},
\]
\[
Z_1 = \left\{ w; \sup_{t > 0} \| \ln(e + t)w(t) \|_\infty + \sup_{t > 0} \| \sqrt{t} \ln(e + t)\nabla w \|_\infty < \infty \right\}.
\]

Lemma 4.1. Assume that \( u_0 \in L^\infty_t \) and \( v_0 \in L^\infty \). Then there exists a positive constant \( c \) such that
\[
\| S_1(t)v_0 \|_{Z_1} \leq c \| v_0 \|_\infty
\]
and
\[
\| S_0(t)u_0 \|_{Z_0} + \| (S_0(t) - S_1(t))u_0 \|_{Z_1} \leq c \| u_0 \|_{L^\infty_\infty}.
\]

Proof. From (3), (42), and Definition 1.1, we get
\[
\| S_1(t)v_0 \|_{Z_1}
= \sup_{t > 0} \| \ln(e + t)S_1(t)v_0 \|_\infty + \sup_{t > 0} \| \sqrt{t} \ln(e + t)\nabla S_1(t)v_0 \|_\infty
\leq \sup_{t > 0} \left( \| \exp\{-t\} \ln(e + t)S_0(t)v_0 \|_\infty + \| \exp\{-t\} \sqrt{t} \ln(e + t)\nabla S_0(t)v_0 \|_\infty \right)
\leq \sup_{t > 0} \| S_0(t)v_0 \|_\infty + \sup_{t > 0} \| S_0(t)\sqrt{t}\nabla v_0 \|_\infty
\leq c \| v_0 \|_\infty.
\]
From (41) (42) and Definition 1.1, we have
\[
\| S_0(t)u_0 \|_{Z_0} = \sup_{t > 0} \| \ln(e + t)S_0(t)u_0 \|_\infty = \| u_0 \|_{L^\infty_\infty}
\]
and
\[
\| (S_0(t) - S_1(t))u_0 \|_{Z_1} = \sup_{t > 0} \| \ln(e + t)(1 - \exp\{-t\})S_0(t)u_0 \|_\infty + \sup_{t > 0} \| \sqrt{t} \ln(e + t)(1 - \exp\{-t\})\nabla S_0(t)u_0 \|_\infty
\leq c \| u_0 \|_{L^\infty_\infty}.
\]
Indeed, the last inequality follows by using (33) and \( 0 \leq 1 - \exp\{-t\} \leq 1 \), we have
\[
\| \sqrt{t} \ln(e + t)\nabla S_0(t)u_0 \|_\infty \leq c \left\| S_0(t)\sqrt{t}\nabla \left( \ln \left( e + \frac{t}{2} \right) + \ln 2 \right)S_0(t)u_0 \right\|_\infty
\leq c \left( \ln \left( e + \frac{t}{2} \right)S_0(t)u_0 \right)_\infty + c \left( S_0(t)u_0 \right)_\infty
\]
Therefore, combining (45)–(47), we obtain the desired estimates.

\[ \leq c \left\| \ln \left( e + \frac{t}{2} \right) S_0 \frac{t}{2} u_0 \right\|_\infty \]

\[ \leq c \| u_0 \|_{L^\infty_n} . \]

Therefore, combining (45)–(47), we obtain the desired estimates.

\[ \square \]

4.1. Bilinear estimates.

**Lemma 4.2.** There exists a positive constant \( c \) such that

\[ \| B_0 (u, v) \|_{Z_0} + \| B_1 (u, v) \|_{Z_1} \leq c \| u \|_{Z_0} \| v \|_{Z_1} . \]  

(48)

**Proof.** From (41) and the estimate (14) with \( \gamma = 1 \), we have

\[ \| B_0 (u, v) \|_{Z_0} \]

\[ = \sup_{t > 0} \left\| \int_0^t \ln (e + \tau) S_0 (t - \tau) \nabla \cdot (u \nabla v) d\tau \right\|_\infty \]

\[ \leq \sup_{t > 0} \int_0^t \frac{\ln (e + \tau)}{\sqrt{(t - \tau) \ln^2 (e + \tau)}} d\tau \sup_{\tau > 0} \ln (e + \tau) u \| \| \sqrt{\tau} \ln (e + \tau) \nabla v \|_\infty \]

\[ \leq c \| u \|_{Z_0} \| v \|_{Z_1} . \]

(49)

Similarly, we obtain

\[ \| B_1 (u, v) \|_{Z_1} = \sup_{t > 0} \left\| \sqrt{t} \ln (e + t) \int_0^t \left( 1 - \exp \{- t + \tau \} \right) S_0 (t - \tau) \nabla \cdot (u \nabla v) d\tau \right\|_\infty \]

\[ + \sup_{t > 0} \left\| \int_0^t \left( 1 - \exp \{- t + \tau \} \right) S_0 (t - \tau) \nabla \cdot (u \nabla v) d\tau \right\|_\infty \]

\[ \leq \sup_{t > 0} \int_0^t \frac{(1 - \exp \{- t + \tau \}) \sqrt{t} \ln (e + t) d\tau}{(t - \tau) \sqrt{\tau} \ln^2 (e + \tau)} \| u \|_{Z_0} \| v \|_{Z_1} \]

\[ + \sup_{t > 0} \int_0^t \frac{(1 - \exp \{- t + \tau \}) \ln (e + t) d\tau}{\sqrt{t - \tau} \sqrt{\tau} \ln^2 (e + t)} \| u \|_{Z_0} \| v \|_{Z_1} \]

\[ \leq \sup_{t > 0} \int_0^t \frac{\sqrt{t} \ln (e + t) d\tau}{\left( 1 + t - \tau \right) \sqrt{\tau} \ln^2 (e + \tau)} \| u \|_{Z_0} \| v \|_{Z_1} \]

\[ + \sup_{t > 0} \int_0^t \frac{\ln (e + t) d\tau}{\sqrt{t - \tau} \sqrt{\tau} \ln^2 (e + \tau)} \| u \|_{Z_0} \| v \|_{Z_1} \]

\[ \leq c \| u \|_{Z_0} \| v \|_{Z_1} . \]

(50)

where in the last inequality we used Lemma 2.1.

Combining (49) and (50), we prove (48).

\[ \square \]

4.2. Proof of Theorem 1.3. The proof of Theorem 1.3 is a direct consequence of Lemmas 4.1, 4.2 and 2.2. Hence we omit the details.

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