Self-similar continued root approximants

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Abstract

A novel method of summing asymptotic series is advanced. Such series repeatedly arise when employing perturbation theory in powers of a small parameter for complicated problems of condensed matter physics, statistical physics, and various applied problems. The method is based on the self-similar approximation theory involving self-similar root approximants. The constructed self-similar continued roots extrapolate asymptotic series to finite values of the expansion parameter. The self-similar continued roots contain, as a particular case, continued fractions and Padé approximants. A theorem on the convergence of the self-similar continued roots is proved. The method is illustrated by several examples from condensed-matter physics.

Keywords: Perturbation theory, Asymptotic series, Extrapolation problem, Self-similar approximation theory, Strong-coupling limit, Condensed-Matter Physics

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1 Introduction

The standard difficulty, repeatedly arising in various problems of condensed-matter physics, statistical and chemical physics, field theory, and many other problems of theoretical physics and applied mathematics, is the necessity of using perturbation theory in powers of a parameter assumed to be asymptotically small, while in reality this parameter is either finite or even tending to infinity. How then it would be possible to extrapolate the asymptotic series of perturbation theory to the finite, or even to infinite values of the parameter? The most often used methods of extrapolation are based on Padé approximants [1].

But, when extrapolating an asymptotic expansion \( f_k(x) \) to large values of the parameter \( x \), by means of the Padé approximants, one meets the ambiguity, since the straightforward use of these approximants yields

\[
P_{M/N}(x) \sim x^{M-N} \quad (x \to \infty),
\]

which, depending on the relation between \( M \) and \( N \), can tend to either infinity (when \( M > N \)), to zero (when \( M < N \)), or to a constant (if \( M = N \)).

When the character of the large-variable limit is known, one can invoke the two-point Padé approximants [1]. However the accuracy of the latter is not high and their definition contains several drawbacks. (i) First of all, when constructing these approximants, one often obtains spurious poles yielding unphysical singularities, sometimes with a large number of poles [1,2]. (ii) Second, there are the cases when Padé approximants are not able to sum perturbation series even for small values of an expansion parameter [3]. (iii) Third, in the majority of cases, for achieving a reasonable accuracy, one needs to have tens of terms in perturbative expansions [1], while interesting problems provide, as a rule, only a few terms. (iv) Fourth, defining the two-point Padé approximants, one always confronts the ambiguity in distributing the coefficients for deciding which of these must reproduce the left-side expansion and which the right-side series. This ambiguity aggravates with the increase of the approximants orders, making it difficult to compose two-point Padé tables. For the case of a few terms, this ambiguity makes the two-point Padé approximants practically unapplicable. For example, it has been shown [4] that, for the same problem, one may construct different two-point Padé approximants, all having correct left and right-side limits, but differing from each other in the intermediate region by 100% of uncertainty. Hence, in the case of short series the two-point Padé approximants do not allow for getting a reliable description. (v) Fifth, the two-point Padé approximants can be used for interpolating between two different expansions not always, but only when these two expansions enjoy compatible variables [1]. When the expansions have incompatible variables, the two-point Padé approximants cannot be defined in principle. (vi) Sixth, interpolating between two points, one of which is finite and another is at infinity, one is able to characterize the large-variable limit of only rational powers [1]. (vii) Finally, it may happen that in the large-variable limit only the power is known, while the amplitude is not. Then the two-point Padé approximants cannot be defined.

There exists a more general approach for extrapolating asymptotic series in powers of a small parameter, or a variable, to finite and even infinite values of such variables. This approach is based on the self-similar approximation theory [5-17]. In the frame of this theory, we have developed the methods of extrapolating asymptotic series by using several
types of self-similar approximants, such as optimized approximants, nested exponentials, nested roots, iterated roots, and factor approximants [5-17].

In the present paper, we advance a novel type of self-similar approximants that may be called self-similar continued root approximants, or, for short, self-similar continued roots. In a particular case, these continued roots reduce to continued fractions [18] and, respectively, to Padé approximants. But, generally, their form is different and not reducible to continued fractions. The self-similar continued roots could be transformed into expressions of the type of the numerical nested radicals [19-22], which, however, is not convenient for the extrapolation procedure applied to functions.

In Sec. 2, we explain how the self-similar continued roots arise in the process of the self-similar renormalization of asymptotic series and prove the convergence of these root approximants. In Sec. 3, we demonstrate, by several examples from condensed-matter physics, that the continued roots can be employed as approximants extrapolating asymptotic series and providing good accuracy. Possible generalizations for the continued root approximants are also mentioned.

2 Construction and convergence of self-similar continued roots

Assume that we are looking for the solution of a complicated problem characterized by a real function $f(x)$ of a real variable $x \in \mathbb{R}$. And let this problem be not solvable explicitly, but allowing only for an approximate solution at small values of $x$,

$$f(x) \simeq f_k(x) \quad (x \to 0),$$

where it is represented by asymptotic series of orders $k = 1, 2, \ldots$,

$$f_k(x) = \sum_{n=0}^{k} a_n x^n \quad (a_0 = 1).$$

(2)

Here, without the loss of generality, we set $a_0 = 1$. This is because any function

$$g(x) \simeq g_k(x) \quad (x \to 0),$$

that at small $x$ is given by the series

$$g_k(x) = g_0(x) \sum_{n=0}^{k} a_n x^n,$$

can always be written in such a form, where $a_0 = 1$. Here $g_0(x)$ is assumed to be known. Then the considered function is defined as

$$f(x) \equiv \frac{g(x)}{g_0(x)}.$$

Respectively, the series at small $x$ take the form

$$f_k(x) = \frac{g_k(x)}{g_0(x)} = \sum_{n=0}^{k} a_n x^n,$$
corresponding to Eq. (2) with $a_0 = 1$.

To apply the self-similar renormalization procedure [5-17] to series (2), recall that the first-order series, having the linear form $f_1(x) = 1 + a_1 x$, transforms into $f_1^*(x) = (1 + A_1 x)^s$. Notice that series (2) can be rewritten in the form

$$f_k(x) = \left(1 + a_1 x \left(1 + a_2 \frac{x}{a_1} \left(1 + \ldots + a_{k-1} \frac{x}{a_{k-2}} \left(1 + \frac{a_k}{a_{k-1}} x\right)\right)\right)\right) \ldots \right).$$

Then applying the self-similar renormalization sequentially to each of the terms inside the brackets, we come to the self-similar continued root

$$f_k^*(x) = (1 + A_1 x (1 + A_2 x (1 + A_3 x))^s \ldots)^s . \quad (3)$$

For instance, in low orders we have

$$f_1^*(x) = (1 + A_1 x)^s , \quad f_2^*(x) = (1 + A_1 x (1 + A_2 x)^s)^s ,$$
$$f_3^*(x) = (1 + A_1 x (1 + A_2 x (1 + A_3 x)^s))^s .$$

Note that, generally speaking, instead of the same power $s$, we could take different powers $s_k$. This, however, would increase the number of parameters that need to be defined, which would require to have more information on the sought function. The use of the same powers $s$ simplifies the problem.

One can notice that in the particular case of $s = -1$, the self-similar continued roots reduce to the continued fractions [18] and, respectively, to the Padé approximants. Thus,

$$f_{2k}^*(x) = P_{k/k}(x) ,$$
$$f_{2k+1}^*(x) = P_{k/k+1}(x) \quad (s = -1) .$$

In this way, the self-similar continued roots (3) generalize the notion of continued fractions.

It is easy to see that if we require the continued roots to be real-valued quantities for all $x \in [0, \infty)$, the parameters $A_n$ have to be non-negative,

$$A_n \geq 0 \quad (n = 1, 2, \ldots, k) . \quad (4)$$

The parameters $A_n$ are to be defined by the accuracy-through-order procedure, by comparing the like orders in the small-variable expansion for the continued roots (3) with the corresponding series (2),

$$f_k^*(x) \simeq f_k(x) \quad (x \to 0) . \quad (5)$$

For instance, in the first order, the expansion of approximant (3) is

$$f_1^*(x) \simeq 1 + s A_1 x \quad (x \to 0) ,$$

which gives

$$A_1 = \frac{a_1}{s} .$$

In the second order, at small $x$, we have

$$f_2^*(x) \simeq 1 + s A_1 x + \frac{s A_1}{2} (s A_1 - A_1 + 2 s A_2)x^2 ,$$

$$f_3^*(x) \simeq 1 + s A_1 x + \frac{s A_1}{2} (s A_1 - A_1 + 2 s A_2)x^2 ,$$

...
from where we get the same parameter $A_1$ and
\[ A_2 = \frac{(1-s)a_1^2 + 2sa_2}{2s^2a_1}. \]

What is left to be defined is the power $s$. Assume that the large-variable asymptotic behaviour of the sought function is known to be
\[ f(x) \simeq Bx^\beta \quad (x \to \infty). \tag{6} \]
In turn, the large-variable behaviour of approximant (3) is
\[ f^*_k(x) \simeq B_k x^{\beta_k} \quad (x \to \infty), \tag{7} \]
with the amplitude
\[ B_k = \prod_{n=1}^{k} A_n^s \tag{8} \]
and the power
\[ \beta_k = \sum_{n=1}^{k} s^n = \frac{s - s^{k+1}}{1 - s}. \tag{9} \]
Let us assume (which will be proved below) that the approximants converge, under $k \to \infty$, so that there exists the limit
\[ \lim_{k \to \infty} \beta_k = \beta. \tag{10} \]
This requires that the power $s$, by modulus, be smaller than one, as a result of which, Eqs. (9) and (10) yield
\[ \beta = \frac{s}{1 - s} \quad (|s| < 1). \tag{11} \]
Inverting the latter equality gives
\[ s = \frac{\beta}{1 + \beta} \quad \left( \beta > -\frac{1}{2} \right). \tag{12} \]

We have assumed above that the sequence of approximants (3) converges as $k$ increases. Now, we provide the proof of this.

**Theorem.** Suppose that $x$ lays in a finite interval
\[ x \in [0, L] \quad (L < \infty), \tag{13} \]
the power $s$ is such that $|s| < 1$ and the parameters $A_n$ are non-negative and bounded,
\[ 0 \leq A_n \leq M \quad (n = 1, 2, \ldots). \tag{14} \]
Then the sequence of approximants (3) converges,
\[ \lim_{k \to \infty} f^*_k(x) = f^*_\infty(x). \tag{15} \]
Proof. The self-similar continued root (3) can be reduced to the form of the nested radicals

\[ f_k^*(x) = (1 + (x_2 + (x_3 + \ldots x_{k-1})^s)\ldots)^s, \]

in which the notation

\[ x_n \equiv x^{\gamma_n} \prod_{j=1}^{n-1} A_j^{s_j-n} \quad (x_1 \equiv 1) \]

is used, where

\[ \gamma_n \equiv \frac{1 - s^{n-1}}{(1 - s)s^{n-1}} \quad (n = 2, 3, \ldots). \]

By the Herschfeld theorem [19], the sequence of the nested radicals, with non-negative \( x_n \), converges if and only if all \( x_n^{s_n} \) are bounded. Condition (14) shows that

\[ \prod_{j=1}^{n-1} A_j^{s_j} \leq M^{\gamma_n s_n}, \]

with

\[ \gamma_n s_n = \frac{s - s^n}{1 - s}. \]

Taking into account condition (13), we see that the terms \( x_n^{s_n} \) are bounded for all finite \( n \),

\[ x_n^{s_n} \leq (LM)\gamma_n s_n. \]

When \( n \to \infty \), under \(|s| < 1\), then

\[ x_n^{s_n} \leq (LM)^{s/(1-s)} \quad (n \to \infty). \]

Therefore all terms \( x_n^{s_n} \) are bounded for any \( n \), including \( n \to \infty \). Hence the sequence of the nested radicals converges, as well as the sequence of the self-similar continued roots (3).

Remark. When considering the large-variable behaviour, it is always possible to treat this variable as asymptotically large, but finite, so that condition (13) be valid.

3 Extrapolation by means of self-similar continued roots

The self-similar continued roots (3) can be used for approximating functions in the whole region of the variable, between zero and infinity. Below, we illustrate that these approximants extrapolate, with a good accuracy, the small-variable asymptotic expansion (2) to the large-variable region, where the sought function behaves as in Eq. (6). Defining the power \( s \) according to Eq. (12), we shall calculate the amplitude \( B_k \), evaluating the accuracy of the found approximations (8) by comparing it with the known value \( B \).
3.1 Nonlinear Schrödinger equation

Many problems in quantum physics, condensed-mater physics, and optics reduce to the nonlinear Schrödinger equation of the type

$$-\frac{1}{2} \frac{d^2 \psi}{dx^2} + \left( \frac{1}{2} x^2 + g |\psi|^2 \right) \psi = E \psi .$$

(16)

Here we consider the one-dimensional case, where \( x \in (-\infty, \infty) \). The first term corresponds to kinetic-energy operator. The second term represents a harmonic external potential. The third term describes the interactions of atoms, typical, e.g., of trapped atomic gases [23-25], with a coupling parameter \( g \).

The ground-state solution represents equilibrium Bose-Einstein condensate. And the higher-energy solutions characterize coherent modes [25]. The spectrum of coherent modes for Eq. (16) can be written [13] in the form

$$E_n = \left( n + \frac{1}{2} \right) f(\alpha) ,$$

(17)

where \( n = 0, 1, 2, \ldots \) and the function \( f(\alpha) \) can be found by means of perturbation theory as an expansion

$$f_k(\alpha) = 1 + \sum_{m=1}^{k} a_m \alpha^m$$

(18)

in powers of the effective coupling parameter

$$\alpha \equiv \frac{2J_n}{1 + 2n} g ,$$

(19)

in which

$$J_n \equiv \frac{1}{2^n \pi n!} \int_{-\infty}^{\infty} H_n^4(x)e^{-2x^2} dx ,$$

with \( H_n(x) \) being Hermite polynomial. The first five coefficients in expansion (18) are

$$a_1 = 1 , \quad a_2 = -\frac{1}{8} , \quad a_3 = \frac{1}{32} , \quad a_4 = -\frac{1}{128} , \quad a_5 = \frac{3}{2048} .$$

At large \( \alpha \), we have

$$f(\alpha) \simeq \frac{3}{2} \alpha^{2/3} \quad (\alpha \to \infty) .$$

(20)

Hence, \( \beta = 2/3 \), which, in view of Eq. (12), gives \( s = 2/5 \).

The continued roots (3), at large \( \alpha \), lead to the form

$$f_k^*(\alpha) \simeq B_k \alpha^{2/3} .$$

(21)

For the amplitude we find the approximations

$$B_2 = 1.549484 \quad (3.3\%) ,$$

$$B_3 = 1.554034 \quad (3.6\%) ,$$

$$B_4 = 1.539048 \quad (2.6\%) ,$$

$$B_5 = 1.523475 \quad (1.6\%) ,$$

where in brackets the related percentage errors are shown.
3.2 Fröhlich optical polaron

The ground-state energy of the Fröhlich optical polaron can be written as a function

$$ E(\alpha) = -\alpha f(\alpha) \quad (22) $$

of the effective coupling parameter $\alpha$. The function $f(\alpha)$ here can be calculated in the second-order perturbation theory giving [26] the expression

$$ f_2(\alpha) = 1 + a_1 \alpha + a_2 \alpha^2, \quad (23) $$

with the coefficients

$$ a_1 = 1.591962 \times 10^{-2}, \quad a_2 = 0.806070 \times 10^{-3}. $$

The strong-coupling asymptotic behavior was found by Miyake [27,28] as

$$ f(\alpha) \simeq B\alpha \quad (\alpha \to \infty), \quad (24) $$

with the amplitude $B = 0.108513$. This gives for the ground-state energy (22)

$$ E(\alpha) \simeq -B\alpha^2 \quad (\alpha \to \infty). \quad (25) $$

For function (24), we have $\beta = 1$, hence $s = 0.5$. Extrapolating expansion (23) by the second-order continued root, we get $B_2 = 0.1044$, with the error of 3.8%.

3.3 Fluctuating fluid membrane

The pressure of a fluctuating fluid membrane, as a function of stiffness $g$ can be represented in the form

$$ P(g) = \frac{\pi^2}{8g^2} f(g). \quad (26) $$

Perturbation theory for the function $f(g)$ has been done [29] up to the six-th order, resulting in the expansion

$$ f_k(g) = 1 + \sum_{n=1}^{k} a_n g^n, \quad (27) $$

with the coefficients

$$ a_1 = \frac{1}{4}, \quad a_2 = \frac{1}{32}, \quad a_3 = 2.176347 \times 10^{-3}, $$

$$ a_4 = 0.552721 \times 10^{-4}, \quad a_5 = -0.721482 \times 10^{-5}, \quad a_6 = -1.777848 \times 10^{-6}. $$

The pressure of the membrane between hard walls corresponds to the stiffness $g \to \infty$. The hard-wall limit was calculated by Monte Carlo techniques [30], yielding

$$ P(\infty) = 0.0798 \pm 0.0003. \quad (28) $$

This corresponds to the asymptotic behaviour of the function $f(g)$ as

$$ f(g) \simeq 0.064683 g^2 \quad (g \to \infty). \quad (29) $$
Therefore here $\beta = 2$ and $s = 2/3$.

Employing the self-similar continued roots, we find the following approximations for the hard-wall pressure:

\[
\begin{align*}
P_2^*(\infty) &= 0.047705 \quad (-40\%) , \\
P_3^*(\infty) &= 0.061904 \quad (-22\%) , \\
P_4^*(\infty) &= 0.072407 \quad (-9.3\%) , \\
P_5^*(\infty) &= 0.079569 \quad (-0.29\%) , \\
P_6^*(\infty) &= 0.083702 \quad (4.9\%)
\end{align*}
\]

where in brackets the corresponding percentage errors are given.

### 3.4 Fluctuating fluid string

The free energy of a fluctuating fluid string coincides with the ground-state energy of a particle in a box [31,32]. The latter, as a function of the wall stiffness $g$, can be written as

\[
E(g) = \frac{\pi^2}{8g^2} f(g) ,
\]

with the notation

\[
f(g) = 1 + \frac{g^2}{32} + \frac{g^4}{4\sqrt{1 + \frac{g^2}{64}}} .
\]

Perturbative expansion in powers of $g$ yields

\[
f_k(g) = 1 + \sum_{n=1}^{k} a_n g^n ,
\]

with the coefficients

\[
a_1 = \frac{1}{4} , \quad a_2 = \frac{1}{32} , \quad a_3 = \frac{1}{512} , \quad a_4 = 0 ,
\]

\[
a_5 = -\frac{1}{131072} , \quad a_6 = 0 , \quad a_7 = \frac{1}{16777216} ,
\]

and so on.

The case of rigid walls corresponds to $g \to \infty$, which results in

\[
E(\infty) = \frac{\pi^2}{128} = 0.077106 .
\]

This implies the asymptotic behaviour of function (31) as

\[
f(g) \simeq 0.0624998g^2 \quad (g \to \infty) .
\]

From here, we have $\beta = 2$ and $s = 2/3$. 

9
We calculate the rigid-wall limit by extrapolating expansions (32) by means of the self-similar continued roots. The results for different approximations \( \varepsilon_k^*(\infty) \) are presented in the Table, together with the related percentage errors

\[
\varepsilon_k \equiv \frac{E_k^*(\infty) - E(\infty)}{E(\infty)} \times 100\%.
\]

### Table: Self-similar continued root approximants of different orders \( k \) for the hard-wall energy \( E_k^*(\infty) \), with the related percentage errors.

| \( k \) | \( E_k^*(\infty) \) | \( \varepsilon_k \) |
|---|---|---|
| 2 | 0.047705 | -38% |
| 3 | 0.061362 | -20% |
| 4 | 0.070598 | -8.4% |
| 5 | 0.076155 | -1.2% |
| 6 | 0.079097 | 2.6% |
| 7 | 0.080336 | 4.2% |
| 8 | 0.080533 | 4.4% |
| 9 | 0.080141 | 3.9% |
| 10 | 0.079540 | 3.2% |
| 11 | 0.079199 | 2.7% |
| 12 | 0.079123 | 2.6% |
| 13 | 0.078363 | 1.6% |

#### 3.5 Discussion and possible generalization

The problem is considered of extrapolating asymptotic series in powers of a small variable to large values of this variable. A new structure is shown to arise as a result of the self-similar renormalization, giving the self-similar continued root approximants. The use of these approximants for extrapolation procedure requires the knowledge of the power for the large-variable behaviour. The convergence of the sequence of these approximants is proved. Several examples from quantum condensed matter physics illustrate good accuracy of the approximations. These examples show that convergence is not necessarily monotonic, but can be oscillating. Nevertheless, the convergence theorem tells us that the sequence of these approximants does converge.

The method is applicable, when all parameters \( A_n \) in form (3) are non-negative. It may happen that for particular cases some of these parameters occur to become negative, which would yield to complex expressions for the roots. In that situation, the method cannot be used directly, but one can proceed by taking the highest available real-valued root approximation and using the following terms for constructing corrected approximants as is explained in Ref. [17].

We have checked the applicability of the method for a number of other problems. As a rule, the low-order approximants are usually real, giving good accuracy, but may become
complex for higher-order approximants. For instance, if we try to approximate, by continued
roots, the statistical sum of the zero-dimensional oscillator as a function of a coupling pa-
rameter [33], when $\beta = -1/4$ and $s = -1/3$, we get in the second order the strong-coupling
amplitude 0.970, whose error is 5%. The third and fourth approximants are real, though
slightly worse than the second one. However, the fifth approximant is complex.

In another example, we calculate the expansion factor of a three-dimensional polymer
chain as a function of an effective coupling parameter [34,35], when $\beta = 0.3544$ and $s =
0.261666$. The second-order approximant for the amplitude of the strong-coupling behavior
gives 1.554, which is within an error of 1.5%. The third-order approximant is yet real, but
the fourth is complex. Thus, in the above two examples, we need to limit calculations by
the low-order approximants. However, it is worth recalling that the majority of the most
interesting problems in condensed-matter physics rarely enjoy the luxury of having many
expansion terms. The standard situation is when one is able to derive just a few lowest
terms of perturbation theory.

In the main expression (3) that has been used throughout the paper, we have taken the
same powers $s$. Strictly speaking, it has been admissible to write a more general expression

$$f_k(x) = (1 + A_1 x (1 + A_2 x \ldots (1 + A_k x)^s_k \ldots)^{s_{k-1}} \ldots)^{s_1},$$

with different powers $s_n$. But then, it would be necessary to have more information on the
large-variable behaviour in order to define all these different powers.

Concluding, we have suggested a novel method for extrapolating the small-variable
asymptotic series to the large values of the variable. The method produces a convergent
sequence of approximants and can be used for extending the validity of perturbation theory.

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References

[1] G.A. Baker, P. Graves-Moris, Padé Approximants, Cambridge University, Cambridge, 1996.

[2] E.B. Saff, R.S. Varga, Numer. Math. 26 (1976) 245–354.

[3] B. Simon, Bull. Am. Math. Soc. 24 (1991) 303–319.

[4] O.V. Selyugin, M.A. Smodyrev, Phys. Stat. Sol. B 155 (1989) 155–167.

[5] V.I. Yukalov, Phys. Rev. A 42 (1990) 3324–3334.

[6] V.I. Yukalov, Physica A 167 (1990) 833–860.

[7] V.I. Yukalov, J. Math. Phys. 32 (1991) 1235–1239.

[8] V.I. Yukalov, J. Math. Phys. 33 (1992) 3994–4001.

[9] V.I. Yukalov, E.P. Yukalova, Physica A 198 (1993) 573–592.

[10] V.I. Yukalov, E.P. Yukalova, Physica A 206 (1994) 553–580.

[11] V.I. Yukalov, E.P. Yukalova, Physica A 225 (1996) 336–362.

[12] V.I. Yukalov, S. Gluzman, Phys. Rev. E 58 (1998) 1359–1382.

[13] V.I. Yukalov, E.P. Yukalova, S. Gluzman, Phys. Rev. A 58 (1998) 96–115.

[14] V.I. Yukalov, E.P. Yukalova, Chaos Solit. Fract. 14 (2002) 839–861.

[15] V.I. Yukalov, S. Gluzman, D. Sornette, Physica A 328 (2003) 409–438.

[16] V.I. Yukalov, E.P. Yukalova, Phys. Lett. A 368 (2007) 341–347.

[17] S. Gluzman, V.I. Yukalov, J. Math. Chem. 48 (2010) 883–913.

[18] A.Y. Khinchin, Continued Fractions, University of Chicago, Chicago, 1964.

[19] A. Herschfeld, Am. Math. Monthly 42 (1935) 419–429.

[20] W.S. Sizer, Math. Mag. 59 (1986) 23–27.

[21] J.M. Borwein, G. de Barra, Am. Math. Monthly 98 (1991) 735–739.

[22] D.J. Jones, Math. Mag. 68 (1995) 387–392.

[23] C.J. Pethik, H. Smith, BoseEinstein Condensation in Dilute Gases, Cambridge University, Cambridge, 2008.

[24] V.I. Yukalov, Laser Phys. 19 (2009) 1–110.

[25] V.I. Yukalov, Phys. Part. Nucl. 42 (2011) 460–513.
[26] C. Alexandrou, R. Rosenfelder, Phys. Rep. 215 (1992) 1–48.
[27] S.J. Miyake, J. Phys. Soc. Jpn. 38 (1975) 181–182.
[28] S.J. Miyake, J. Phys. Soc. Jpn. 41 (1976) 747–752.
[29] B. Kastening, Phys. Rev. E 73 (2006) 011101.
[30] G. Gompper, D.M. Kroll, Eur. Phys. Lett. 9 (1989) 59–64.
[31] H. Kleinert, Phys. Lett. A 257 (1999) 269–274.
[32] B. Kastening, Phys. Rev. E 66 (2002) 061102.
[33] V.I. Yukalov, Int. J. Mod. Phys. B 7 (1993) 1711–1730.
[34] M. Muthukumar, B.G. Nickel, J. Chem. Phys. 86 (1987) 460–476.
[35] B. Li, N. Madras, A.D. Sokal, J. Stat. Phys. 80 (1995) 661–754.