Broadly-Pluriminimal Submanifolds of Kähler-Einstein Manifolds

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Abstract: We define broadly-pluriminimal immersed 2n-submanifold $F : M \to N$ into a Kähler-Einstein manifold of complex dimension 2n and scalar curvature $R$. We prove that, if $M$ is compact, $n \geq 2$, and $R < 0$, then: (i) Either $F$ has complex or Lagrangian directions; (ii) If $n = 2$, $M$ is oriented, and $F$ has no complex directions, then it is a Lagrangian submanifold, generalizing the well-known case $n = 1$ for minimal surfaces due to Wolfson. We also prove that, if $F$ has constant Kähler angles with no complex directions, and is not Lagrangian, then $R = 0$ must hold. Our main tool is a formula on the Laplacian of a symmetric function on the Kähler angles.

Key Words: Minimal, pluriharmonic, Lagrangian submanifold, Kähler-Einstein manifold.

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1 Introduction

Let $(N, J, g)$ be a Kähler manifold of complex dimension $2n$, with complex structure $J$, Riemannian metric $g$, and Kähler form $\omega(U, V) = g(JU, V)$. Let $F : M \to N$ be an immersed submanifold of real dimension $2n$. We take on $M$ its induced metric $g_M = F^*g$.

The 2-form $F^*\omega$ on $M$, defines for each $0 \leq k \leq n$, the sets

$$\Omega_{2k} = \{ p \in M : F^*\omega \text{ has rank } 2k \text{ at the point } p \}.$$

We will denote by $\Omega_{2k}^0$ the set of interior points of $\Omega_{2k}$ in $M$. At each point $p \in M$, we identify $F^*\omega$ with a skew-symmetric operator of $T_pM$ by using the musical isomorphism with respect to $g_M$, namely,

$$g_M(F^*\omega(X), Y) = F^*\omega(X, Y).$$

†Deceased on October 2nd, 1999
We take its polar decomposition

\[ F^*\omega = \tilde{g} J_\omega \]  

(1.1)

where \( J_\omega : T_p M \to T_p M \) is a (in fact unique) partial isometry with the same kernel \( \mathcal{K}_\omega \) as of \( F^*w \), and where \( \tilde{g} \) is the positive semidefinite operator \( \tilde{g} = |F^*\omega| = \sqrt{(F^*\omega)^2} \), with kernel \( \mathcal{K}_\omega \) as well. It turns out that \( J_\omega : \mathcal{K}_\omega^\perp \to \mathcal{K}_\omega^\perp \) defines a complex structure on \( \mathcal{K}_\omega^\perp \), the orthogonal complement of \( \mathcal{K}_\omega \) in \( T_p M \). Moreover, \( J_\omega \) is \( g_M \)-orthogonal. Since \( F^*\omega \) is a normal operator, then \( J_\omega \) commutes with \( \tilde{g} \). The map \( P \to |P| \) is Lipschitz in the space of normal operators of a finite dimensional Hilbert space, for the Hilbert Schmidt norm ([Bh] p.215). Thus, the tensor \( \tilde{g} \) is continuous on all \( M \) and locally Lipschitz. On each \( \Omega^0_{2k}, \mathcal{K}_\omega \) and \( \mathcal{K}_\omega^\perp \) are smooth sub-vector bundles of \( TM \) and, from the smoothness of the polar decomposition on invertible operators, \( \tilde{g} \) and \( J_\omega \) are smooth morphisms on these open sets. Let \( \{X_\alpha, Y_\alpha\}_{1 \leq \alpha \leq n} \) be a \( g_M \)-orthonormal basis of \( T_p M \), that diagonalizes \( F^*\omega \) at \( p \), that is

\[ F^*\omega = \bigoplus_{1 \leq \alpha \leq n} \begin{bmatrix} 0 & -a_\alpha \\ a_\alpha & 0 \end{bmatrix}. \]  

(1.2)

Since \( |a_\alpha| \leq 1 \), then \( a_\alpha = \cos \theta_\alpha \) for some angle \( \theta_\alpha \). We can reorder the diagonalizing basis such that \( \cos \theta_1 \geq \cos \theta_2 \geq \ldots \geq \cos \theta_n \geq 0 \). The angles \( \{\theta_\alpha\}_{1 \leq \alpha \leq n} \) are the Kähler angles of \( F \) at \( p \). Thus, \( \forall \alpha, F^*\omega(X_\alpha) = \cos \theta_\alpha Y_\alpha, F^*\omega(Y_\alpha) = -\cos \theta_\alpha X_\alpha \) and if \( k \geq 1 \), where \( 2k \) is the rank of \( F^*\omega \) at \( p \), \( J_\omega X_\alpha = Y_\alpha \) \( \forall \alpha \leq k \). The Weyl’s perturbation theorem (cf. [Bh]), applied to the eigenvalues of the symmetric operator \(|F^*\omega|\), shows that, ordering the \( \cos \theta_\alpha \) in the above way, the map \( p \to \cos \theta_\alpha(p) \) is locally Lipschitz on \( M \), for each \( \alpha \). Considering \( \tilde{g} \) and \( F^*\omega \) 2-tensors, they are related by,

\[ \tilde{g}(X, Y) = F^*\omega(X, J_\omega Y), \quad \forall X, Y \in T_p M. \]

Both \( \tilde{g} \) and \( F^*\omega \) are \( J_\omega \)-invariant, that is \( \tilde{g}(J_\omega X, J_\omega Y) = \tilde{g}(X, Y) \), \( F^*\omega(J_\omega X, J_\omega Y) = F^*\omega(X, Y) \). In particular, on \( \Omega_{2n}, -F^*\omega \) is the Kähler form of \( M \) with respect the almost complex structure \( J_\omega \) and the Riemannian metric \( \tilde{g} \).

A complex direction of \( F \) is a real two plane \( P \) of \( T_p M \) such that \( dF(P) \) is a complex line of \( T_{F(p)}N \), that is, \( JdF(P) \subset dF(P) \). Similarly, \( P \) is said to be a Lagrangian direction of \( F \) if \( \omega \) vanishes on \( dF(P) \), that is, \( JdF(P) \perp dF(P) \). The immersion \( F \) has no complex directions iff \( \cos \theta_\alpha < 1 \) \( \forall \alpha \). The 2-plane \( \{X_\alpha, Y_\alpha\} \) is a complex direction of \( M \) iff \( J \circ dF = \pm dF \circ J_\omega \) on that plane. \( M \) is a complex submanifold iff \( \cos \theta_\alpha = 1 \) \( \forall \alpha \), and is a Lagrangian submanifold iff \( \cos \theta_\alpha = 0 \) \( \forall \alpha \). We say that \( F \) has equal Kähler angles if \( \theta_\alpha = \theta \) \( \forall \alpha \). Complex and Lagrangian submanifolds are examples of such case.

On each \( \Omega^0_{2k}, \mathcal{K}_\omega^\perp \) is a smooth \( J_\omega \)-Hermitian sub-vector bundle of \( TM \). Thus, for each \( p_0 \in \Omega^0_{2k} \) there exists a smooth local \( g_M \)-orthonormal frame of \( \mathcal{K}_\omega^\perp \) defined on a
neighbourhood of \( p_0 \), of the form \( X_1, J_\omega X_1, \ldots, X_k, J_\omega X_k \). We may enlarge it to a smooth \( g_M \)-orthonormal frame on \( M \), on a neighbourhood of \( p_0 \)

\[
X_1, Y_1 = J_\omega X_1, \ldots, X_k, Y_k = J_\omega X_k, X_{k+1}, Y_{k+1}, \ldots, X_n, Y_n
\]

(1.3)

where \( X_{k+1}, Y_{k+1}, \ldots, X_n, Y_n \) is any \( g_M \)-orthonormal frame of \( \mathcal{K}_\omega \). We may require it to be a diagonalizing basis of \( F^*\omega \) at \( p_0 \). Note that in general it is not possible to get smooth diagonalizing \( g_M \)-orthonormal frames in a whole neighbourhood of a point \( p_0 \), unless \( F^*\omega \) has distinct non-zero eigenvalues \( \cos \theta_1, \ldots, \cos \theta_n \) at \( p_0 \), or \( F^*\omega \) has constant rank two, or \( F \) has equal Kähler angles.

Let us denote by \( \nabla_X dF(Y) = \nabla dF(X, Y) \) the second fundamental form of \( F \). It is a symmetric 2-tensor on \( M \) that takes values on the normal bundle \( NM = (dF(TM))^\perp \). Let us denote by \((\ )^\perp \) the orthogonal projection of \( F^{-1}TN \) onto the normal bundle. We denote by \( \nabla \) both Levi-Civita connections of \( M \) and \( N \). We also denote by \( \nabla \) the induced connection on \( F^{-1}TN \). We take on \( NM \) the usual connection \( \nabla^\perp \), given by \( \nabla^\perp X = (\nabla_X U)^\perp \), for \( X \in T_pN \) and \( U \) a smooth section of \( NM \subset F^{-1}TN \). \( F \) is said to be totally geodesic if \( \nabla dF = 0 \), and \( F \) is minimal if \( \text{trace}_{g_M} \nabla dF = 0 \), that is \( F \) is an harmonic immersion.

For a local frame as in (1.3), and that diagonalizes \( F^*\omega \) at \( p_0 \), we take the complex frame of \( T^cM \)

\[
Z_\alpha = \frac{X_\alpha - iY_\alpha}{2} = \alpha^\prime, \quad Z_\bar{\alpha} = \frac{X_\alpha + iY_\alpha}{2} = \bar{\alpha}^\prime, \quad \alpha \in \{1, \ldots, n\}.
\]

We extend by \( \mathcal{C} \)-multilinearity \( g, g_M, \tilde{g}, R^N, F^*\omega \), and any other tensors that may occur. Sometimes we denote by \( \langle \cdot, \cdot \rangle \) the \( \mathcal{C} \)-bilinear extension of \( g_M \).

If \( p_0 \) is a point without complex directions, that is, \( \sin \theta_\alpha \neq 0 \) \( \forall \alpha \), then, \( \{dF(\alpha), dF(\bar{\alpha}), U_\alpha, U_\bar{\alpha}\}_{1 \leq \alpha \leq n} \), where

\[
U_\alpha = \frac{(JdF(\alpha))^\perp}{\sin \theta_\alpha} = \frac{JdF(\alpha) - i \cos \theta_\alpha dF(\alpha)}{\sin \theta_\alpha},
\]

and \( U_\bar{\alpha} = \overline{U_\alpha} \), is (up to a constant factor) an unitary basis of \( T^c_{(p_0)}N \), with \( U_\alpha, U_\bar{\alpha} \) complex basis of the complexified normal bundle. We denote by \( R^N(U, V)W = -\nabla_U \nabla_V W + \nabla_V \nabla_U W + \nabla_{[U, V]} W \) and by \( R^N(U, V, W, Z) = g(R^N(U, V)W, Z) \) the curvature tensor and Riemannian curvature tensor of \( N \) respectively.

**Lemma 1.1** At a point \( p_0 \) without complex directions, the Ricci tensor of \( N \) is given by, for \( U, V \in T^c_{(p_0)}N \)

\[
Ricci^N(U, V) = \sum_{1 \leq \mu \leq n} \frac{4}{\sin^2 \theta_\mu} R^N(U, JV, dF(\mu), JdF(\bar{\mu}) + i \cos \theta_\mu dF(\bar{\mu}))
\]
Proof. Recall that $R^N(JZ, JW, Z', W') = R^N(Z, W, JZ', JW') = R^N(\alpha, \beta) = -\frac{1}{2} \text{trace} \left( Z \rightarrow R^N(U, JV)JZ \right)$. Thus we have,

$$Ricci^N(U, V) = \sum_{\alpha} -\frac{1}{2} \left( 4R^N(U, JV, JdF(\alpha), dF(\alpha)) + 4R^N(U, JV, JU, U\alpha) \right)$$

$$= \sum_{\alpha} \frac{2}{\sin^2 \theta_{\alpha}} \left( R^N(U, JV, dF(\alpha), \sin^2 \theta_{\alpha} JdF(\alpha)) + R^N(U, JV, JdF(\alpha), i \cos \theta_{\alpha} dF(\alpha)) \right)$$

leading the expression in the lemma. \qed

A first conclusion can be obtained from the above lemma:

**Proposition 1.1** Let $F : M \rightarrow N$ be a totally geodesic immersion, or more generally, let $F$ be an immersion such that the normal component of $F^*R^N(X, Y)Z$ vanishes, $\forall X, Y, Z \in TM$. If $F$ has no complex directions, then $F^*\Psi = 0$, where $\Psi(U, V) = Ricci^N(JU, V)$, $\forall U, V \in TN$, is the Ricci form of $N$. In particular, if $N$ is Kähler-Einstein of non-zero Ricci tensor, $F$ is a Lagrangian submanifold.

Proof. We denote by $\nabla_X \nabla dF$ the covariant derivative of $\nabla dF$ as a section of $\otimes^2 T^* M \otimes NM$. Since $U_\mu$ lies in the complexified normal bundle, using Codazzi’s equation, $\forall X, Y \in T_{\mu}M$

$$R^N(dF(X), dF(Y), dF(\mu), JdF(\bar{\mu}) + i \cos \theta_{\mu} dF(\bar{\mu}))$$

$$= g \left( -\nabla_X \nabla dF(Y, \mu) + \nabla_Y \nabla dF(X, \mu), JdF(\bar{\mu}) + i \cos \theta_{\mu} dF(\bar{\mu}) \right)$$

that is zero for $F$ totally geodesic or for $F$ with vanishing normal component of $F^*R^N$. Therefore, $F^*\Psi(X, Y) = Ricci^N(JdF(X), dF(Y)) = -Ricci^N(dF(X), JdF(Y)) = 0$, by Lemma 1.1. \qed

Henceforth, we assume $N$ is Kähler-Einstein with $Ricci^N = Rg$. Our aim is to find conditions for a minimal immersion $F$ to be Lagrangian. Wolfson [W] proves that, for $n = 1$, if $F$ is a minimal real surface without complex directions, immersed into a Kähler-Einstein surface of negative scalar curvature $R$, then $F$ is Lagrangian. His main tool is a formula on the Laplacian of a smooth map $\kappa$ on the Kähler angle of $F$, where the scalar curvature $R$ of $N$ appears. Here and in [S-V] we generalize this map $\kappa$ to any dimension $n$ (see (2.2) below) and compute its Laplacian. The main problem with this generalization for $n \geq 2$, is that $\kappa$, although it is Lipschitz on $M$, under the assumption of no existence of complex directions, it is no longer smooth on sets where $F$ might change the number of Lagrangian directions. Then, for instance, we cannot use Stokes, or apply so easily the maximum principle has Wolfson did. The expression of $\Delta \kappa$ can be simplified in two cases. The first case is when $F$ has equal Kähler angles, that we study in [S-V]. In this
case, we gain more regularity for \( \kappa \), and we could conclude that, if \( R \neq 0 \) and \( n = 2 \), then either \( F \) is a complex or a Lagrangian submanifold, and for \( n \geq 3 \), if \( R < 0 \) and \( F \) has no complex directions, then \( F \) is Lagrangian (see [S-V]). The second case is when \( F \) is a broadly-pluriminimal submanifold, a concept we introduce in the next section. This is a concept that is close to the product of minimal surfaces immersed into Kähler surfaces, possibly with different Kähler angles. In this case the expression for \( \Delta \kappa \) is the simplest, and similar to the one of Wolfson [W]. Using this formula, we obtain a conclusion of Lagrangianity for the case \( n = 2 \) and \( R < 0 \) (Theorem 2.1), generalizing the above result of Wolfson.

2 Broadly-pluriminimal submanifolds

As in the previous section we let \((N, J, g)\) be a Kähler manifold of complex dimension \( 2n \) and \( F : M \to N \) an immersed submanifold of real dimension \( 2n \).

Definition. A map \( F : M \to N \) is said to be broadly-pluriminimal if

(i) \( F \) is minimal,

(ii) On each \( \Omega_{2k} \), for \( 1 \leq k \leq n \), \( F \) is pluriharmonic with respect to any local complex structure \( \tilde{J} = J_\omega \oplus J' \) where \( J' \) is any \( g_M \)-orthogonal complex structure of \( \mathcal{K}_\omega \).

On the open set \( \Omega_{2n} \), (ii) means that \( F \) is pluriharmonic with respect to the complex structure \( J_\omega \). If \( \mathcal{K}_\omega = 0 \), we simply say that \( F \) is pluriminimal. We recall that pluriharmonic maps are harmonic maps (see e.g [O-V]). If \( F \) is broadly-pluriminimal, at each point \( p \in \Omega_{2k} \), \( k \geq 1 \), \( \nabla dF \) is of type \((2,0) + (0,2)\) for any complex structure \( \tilde{J} \) of \( T_p M \) of the above form and properties, or equivalently, the \((1,1)\)-part of it vanishes:

\[
(\nabla dF)^{(1,1)}(X, Y) = \frac{1}{2} \left( \nabla dF(X, Y) + \nabla dF(\tilde{J}X, \tilde{J}Y) \right) = 0.
\]

(2.1)

Examples. (1) Any minimal immersion of an oriented real surface into a Kähler complex surface \( F : M^2 \to N^2 \) is broadly-pluriminimal. In fact, we have \( F^* \omega = fVol \), where \( f : M \to \mathbb{R} \) is a smooth map. Any \( g_M \)-orthonormal basis \( \{X, Y\} \), diagonalizes \( F^* \omega \), with \( F^* \omega(X, Y) = \pm f = \pm \cos \theta \) smooth everywhere. In this case \( J_\omega = \pm J_M \), wherever \( f \) does not vanish, and where \( J_M \) is the natural complex structure defined by a direct orthonormal basis. Since \( F \) is harmonic and \( n = 1 \), it is pluriharmonic with respect to \( J_M \) and with respect to \(-J_M \). Condition (ii) now follows, because, at each point \( p_0 \) either \( \mathcal{K}_\omega = T_{p_0} M \), that is \( k = 0 \), or \( \mathcal{K}_\omega = 0 \), that is \( k = n \).

(2) The product, \( F = F_1 \times \ldots \times F_n : M = M_1 \times \ldots \times M_n \to N_1 \times \ldots \times N_n \), of mini-
normal orientable surfaces $F_i : M_i \to N_i$ immersed into Kähler complex surfaces and their reparametrizations, $\tilde{F} = F \circ \phi : \tilde{M} \to N_1 \times \ldots \times N_n$, where $\phi : \tilde{M}^{2n} \to M = M_1 \times \ldots \times M_n$ is a diffeomorphism between manifolds, are broadly-pluriminimal. More generally, the product of broadly-pluriminimal submanifolds is a broadly-pluriminimal submanifold. We note that, to require the product of two surfaces to have equal Kähler angles, implies each surface to have constant Kähler angle (and equal to each other), for, the two Kähler angles are indexed on independent variables. So, equal Kähler angles and broadly-pluriminimality are independent concepts.

(3) If $F : M \to N$ is a minimal Lagrangian submanifold, then $F$ is broadly-pluriminimal. In this case $\mathcal{K}_\omega = TM$ and so $\Omega^0_{2k} = \emptyset$ for $k \geq 1$. Thus (ii) is satisfied.

(4) If $F : M \to N$ is a complex submanifold, then $F$ is trivially pluriminimal. In this case $J_\omega$ is the induced complex structure from $J$.

(5) If $F : M \to N$ is a minimal immersion with equal Kähler angles and no complex directions, in [S-V] we prove that $F$ is broadly-pluriminimal iff the isomorphism $\Phi : TM \to NM$, given by $\Phi(X) = (JdF(X))^{-1}$, is parallel, considering $TM$ with the conformally equivalent Kähler complex metric $\hat{g}(X,Y) = g_M(X,Y) - g_M(F^*\omega(X), F^*\omega(Y)) = \sin^2 \theta g_M(X,Y)$. Moreover, if this is the case, for $n \geq 2$, it turns out that the Kähler angle is constant, and so, $\Phi : (TM, g_M) \to (NM, g)$ is a parallel homothetic isomorphism. Furthermore, if $F$ is not Lagrangian, $N$ must be Ricci-flat, as a consequence of Corollary 2.1 given in the next section of this paper. This example shows how natural is the definition of broadly-pluriminimal.

(6) Any minimal immersion $F : M^{2n} \to T^{2n}$ into the flat complex torus with no Lagrangian directions and such that $(M, J_\omega, g_M)$ is Kähler, is a pluriminimal submanifold. In fact, from Gauss equation, $\sum_{\alpha,\mu} R^M(\mu, \alpha, \bar{\mu}, \bar{\alpha}) = -\sum_{\alpha,\mu} \|\nabla d\alpha(\mu, \bar{\mu})\|^2$, where $R^M$ is the Riemannian curvature tensor of $M$, that is of type $(1,1)$ with respect to $J_\omega$.

(7) Let $(N, I, J, g)$ be a hyper-Kähler manifold of real dimension 8, and $F : M \to N$ a minimal immersion of a 4-dimensional submanifold with non-negative isotropic sectional curvature, and such that $\forall \nu \phi \in S^2$, $F$ has equal Kähler angles with respect to the complex structure $J_{\nu \phi} = \cos \nu I + \sin \nu \cos \phi J + \sin \nu \sin \phi K$, where $K = IJ$. Set for each unit vector $X \in TM$, $H_X = \text{span}\{X, IX, JX, KX\}$. In [S-V] we prove that, if $\exists p \in M$ and $\exists X \in T_p M$, unit vector, such that $\text{dim}(T_p M \cap H_X) \geq 2$, then there exists $\nu \phi \in S^2$ such that $M$ is a $J_{\nu \phi}$-complex submanifold. Furthermore, if $J_{\nu \phi} = I$ then $F : M \to (N, I, g)$ is obviously pluriminimal. If $J_{\nu \phi} \neq I$ but $T_p M \cap H_X \neq \{0\}$, then $F : M \to (N, I, g)$ is still pluriminimal, with constant Kähler angle $\nu$ or $\nu + \pi$. 

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If \( \{\theta_\alpha\}_{1 \leq n \leq n} \) are the Kähler angles of \( F \), \( g_M \pm \tilde{g} \), where \( \tilde{g} \) is given in (1.1), is represented in the unitary basis \( \{\sqrt{2}\alpha, \sqrt{2}\overline{\alpha}\} \) of \( T_pM \), for \( p \) near \( p_0 \), by a \( 2n \times 2n \) matrix that at \( p_0 \) is the diagonal matrix \( D(1 \pm \cos \theta_1, \ldots, 1 \pm \cos \theta_n, 1 \pm \cos \theta_1, \ldots, 1 \pm \cos \theta_n) \). Thus, 
\[
\det(g_M \pm \tilde{g}) = \prod_{1 \leq \alpha \leq n}(1 \pm \cos \theta_\alpha)^2.
\]
If \( p_0 \) is a point without complex directions, \( \cos \theta_\alpha \neq 1 \), \( \forall \alpha \in \{1, \ldots, n\} \), and so \( \tilde{g} \leq g_M \). Thus, is \( F \) has no complex directions we may consider the map
\[
\kappa = \frac{1}{2} \log \left( \frac{\det(g_M + \tilde{g})}{\det(g_M - \tilde{g})} \right) = \sum_{1 \leq \alpha \leq n} \log \left( \frac{1 + \cos \theta_\alpha}{1 - \cos \theta_\alpha} \right), \tag{2.2}
\]
This map \( \kappa \) is non-negative, continuous on \( M \) and smooth on each \( \Omega^{0}_{2k} \). It is an increasing map on each \( \cos \theta_\alpha \). In [S-V] we compute \( \Delta \kappa \) on \( \Omega^{0}_{2k} \) (see Proposition 2.1 below), for any minimal immersion \( F \). In this paper we will compute \( \Delta \kappa \) for the case of \( F \) broadly-pluriminimal (Proposition 2.2). Then, using such formula we can prove the statements given in the abstract, and some few other ones.

### 2.1 The computation of \( \Delta \kappa \)

Let \( \{X_\alpha, Y_\alpha\} \) be a local \( g_M \)-orthonormal frame satisfying the conditions in (1.3), and that diagonalizes \( F^*\omega \) at \( p_0 \). We define a local \( g_M \)-orthogonal complex structure on a neighbourhood of \( p_0 \in \Omega^{0}_{2k} \) as \( \tilde{J} = J_\omega \oplus J' \), where \( J_\omega \) is defined on \( K^\perp_\omega \) and \( J' \) is the local complex structure on \( K_\omega \), defined on a neighbourhood of \( p_0 \) by \( J'Z_\alpha = iZ_\alpha, J'Z_\overline{\alpha} = -iZ_\overline{\alpha}, \forall \alpha \geq k + 1 \). On a neighbourhood of \( p_0 \), \( Z_\alpha \) is of type (1,0) with respect to \( \tilde{J}, \forall \alpha \), and \( Z_\alpha \) and \( Z_\overline{\alpha} \) are in \( K^c_\omega \), \( \forall \alpha \geq k+1 \). Since \( \tilde{J} \) is \( g_M \)-orthogonal, \( \forall \alpha, \beta, \) on a neighbourhood of \( p_0 \),
\[
\langle \nabla_{\tilde{Z}}(\tilde{J}(\alpha), \beta) \rangle = 2i\langle \nabla_{\tilde{Z}}(\alpha, \beta) \rangle = -\langle \alpha, \nabla_{\tilde{Z}}(\beta) \rangle, \quad \langle \nabla_{\tilde{Z}}(\tilde{J}(\alpha), \beta) \rangle = 0. \tag{2.3}
\]
In particular \( \nabla_{\tilde{Z}}(\tilde{J}(\alpha)) \) is of type (0,1). Note that, considering \( \tilde{g} \) and \( F^*\omega \) 2-tensors, \( \tilde{g}(X,Y) = F^*\omega(X,\tilde{J}Y) \) still holds \( \forall X,Y \in T_pM \). Set \( \tilde{g}_{AB} = \tilde{g}(A,B) \), and define \( B = B, \forall A,B \in \{1, \ldots, n, \overline{1}, \ldots, \overline{n}\} \). Let \( \epsilon_\alpha = 1, \epsilon_\overline{\alpha} = -1, \forall \alpha \in \{1, \ldots, n\} \). Then \( \forall 1 \leq \alpha, \beta \leq n, \forall A,B \in \{1, \ldots, n, \overline{1}, \ldots, \overline{n}\} \) and \( \forall C \in \{1, \ldots, n\} \cup \{k+1, \ldots, \overline{n}\} \),
\[
\begin{align*}
F^*\omega(\alpha, C) &= g(JdF(\alpha), dF(C)) = 0 & \forall p \text{ near } p_0 \\
F^*\omega(\alpha, \overline{\beta}) &= g(JdF(\alpha), dF(\overline{\beta})) = \frac{i}{2}\delta_{\alpha\beta}\cos \theta_\alpha & \text{at } p_0 \\
\tilde{g}_{AB} &= i\epsilon_B F^*\omega(A, B) = i\epsilon_B g(JdF(A), dF(B)) & \forall p \text{ near } p_0 \\
\tilde{g}_{\alpha C} &= \tilde{g}_{\overline{\alpha} C} = 0 & \forall p \text{ near } p_0 \\
\tilde{g}_{\alpha \overline{\beta}} &= \tilde{g}_{\overline{\alpha} \beta} = \frac{1}{2}\delta_{\alpha\beta}\cos \theta_\alpha & \text{at } p_0 
\end{align*}
\]
Now we compute the covariant derivative of \( F^*\omega \). Let \( p \in M, X,Y,Z \in T_pM \). Then
\[
\begin{align*}
d(g(JdF(X), dF(Y)))(Z) &= g(JdF(X), dF(Y)) + g(JdF(Z), dF(Y)) + g(JdF(X), dF(Z)) \\
&= g(JdF(X), \nabla_{\tilde{Z}}dF(Y)) + g(JdF(X), dF(Y)) + g(JdF(X), dF(Z)) \tag{2.5}
\end{align*}
\]
\[ \nabla_Z F^* \omega(X, Y) = -g(\nabla_Z dF(X), JdF(Y)) + g(\nabla_Z dF(Y), JdF(X)). \] (2.6)

For simplicity of notation we denote by
\[ g_{XYZ} = g(\nabla_X dF(Y), JdF(Z)). \]

From (2.5) and (2.4) we have

**Lemma 2.1** \( \forall p \) near \( p_0 \in \Omega^0_{2k} \), \( Z \in T^c_p M \), and \( \mu, \gamma \in \{1, \ldots, n\} \)
\[ d \tilde{g}_{\mu \gamma}(Z) = ig_Z \mu \gamma - ig_Z \gamma \mu + 2 \sum_{\rho} (\langle \nabla_Z \mu, \rho \rangle \tilde{g}_{\rho \gamma} + \langle \nabla_Z \gamma, \rho \rangle \tilde{g}_{\mu \rho}) \]
\[ 0 = d \tilde{g}_{\mu \gamma}(Z) = -ig_Z \mu \gamma + ig_Z \gamma \mu + 2 \sum_{\rho} (\langle \nabla_Z \mu, \rho \rangle \tilde{g}_{\rho \gamma} - \langle \nabla_Z \gamma, \rho \rangle \tilde{g}_{\mu \rho}). \]

**Proposition 2.1** [S-V] If \( F \) is minimal without complex directions, and \( \{X_\alpha, Y_\alpha\} \) is a local orthonormal frame of the form (1.3) that diagonalizes \( F^* \omega \) at \( p_0 \in \Omega^0_{2k} \), \( 0 \leq k \leq n \), then at \( p_0 \)
\[ \Delta \kappa = 4i \sum_{\beta} \text{Ricci}^N(JdF(\beta), dF(\bar{\beta})) \]
\[ + \sum_{\beta, \mu, \rho} \frac{32}{\sin^2 \theta_\mu} \text{Im} \left( R^N(JdF(\beta), dF(\mu), dF(\bar{\beta}), JdF(\bar{\mu}) + i \cos \theta_\mu dF(\bar{\mu})) \right) \]
\[ - \sum_{\beta, \mu, \rho} \frac{64(\cos \theta_\mu + \cos \theta_\rho)}{\sin^2 \theta_\mu \sin^2 \theta_\rho} \text{Re} \left( g_{\beta \mu \rho} g_{\bar{\beta} \rho \bar{\mu}} \right) \]
\[ + \sum_{\beta, \mu, \rho} \frac{32(\cos \theta_\mu - \cos \theta_\rho)}{\sin^2 \theta_\mu \sin^2 \theta_\rho} \left( |g_{\beta \mu \rho}|^2 + |g_{\bar{\beta} \rho \bar{\mu}}|^2 \right) \]
\[ + \sum_{\beta, \mu, \rho} \frac{32(\cos \theta_\rho + \cos \theta_\mu)}{\sin^2 \theta_\mu \sin^2 \theta_\rho} \left( |\langle \nabla_{\beta \mu}, \rho \rangle|^2 + |\langle \nabla_{\beta \rho}, \mu \rangle|^2 \right). \] (2.7)

Now we get

**Proposition 2.2** If \( F \) is broadly-pluriminimal without complex directions, and \( \{X_\alpha, Y_\alpha\} \) is a local orthonormal frame of the form (1.3) that diagonalizes \( F^* \omega \) at \( p_0 \in \Omega^0_{2k} \), \( 0 \leq k \leq n \), then at \( p_0 \)
\[ \Delta \kappa = 4i \sum_{1 \leq \beta \leq n} \text{Ricci}^N(JdF(\beta), dF(\bar{\beta})) \] (2.8)
Proof. First, we rewrite $\Delta \kappa$ of Proposition 2.1 in terms of $(\nabla dF)^{(1,1)}$, the symmetric tensor given by (2.1) on a neighbourhood of $p_0$, and it is the $(1,1)$-part of $\nabla dF$ with respect to the complex structure $\tilde{J}$. From Lemma 2.1, at $p_0$,

$$
|g_Z\mu| = |g_Z\rho\mu|^2 + (\cos \theta_\mu + \cos \theta_\rho)^2(|\nabla_Z\mu, \rho|^2 + 2(\cos \theta_\mu + \cos \theta_\rho)Im(\langle \nabla_Z\mu, \rho \rangle g_Z\tilde{\beta}\tilde{\mu})).
$$

Hence, in the expression of $\Delta \kappa$ of Proposition 2.1,

$$
(2.7) = \sum_{\beta,\mu,\rho} \frac{16(\cos \theta_\rho - \cos \theta_\mu)}{\sin^2 \theta_\mu \sin^2 \theta_\rho} (|g_\beta\mu\rho|^2 - |g_\beta\rho\mu|^2 + |g_\beta\mu\rho|^2 - |g_\beta\rho\mu|^2)
$$

$$
= \sum_{\beta,\mu,\rho} \frac{16(\cos \theta_\rho - \cos \theta_\mu)}{\sin^2 \theta_\mu \sin^2 \theta_\rho} (|\langle \nabla_\beta\mu, \rho \rangle|^2 + |\langle \nabla_\beta\mu, \rho \rangle|^2)
$$

$$
+ \sum_{\beta,\mu,\rho} \frac{32(\cos \theta_\rho - \cos \theta_\mu)}{\sin^2 \theta_\mu \sin^2 \theta_\rho} Im(\langle \nabla_\beta\mu, \rho \rangle g_\beta\tilde{\beta}\tilde{\mu} + \langle \nabla_\beta\mu, \rho \rangle g_\beta\tilde{\beta}\tilde{\mu})
$$

$$
= \sum_{\beta,\mu,\rho} \frac{16(\cos \theta_\rho - \cos \theta_\mu)}{\sin^2 \theta_\mu \sin^2 \theta_\rho} (|\langle \nabla_\beta\mu, \rho \rangle|^2 + |\langle \nabla_\beta\mu, \rho \rangle|^2)
$$

$$
+ \sum_{\beta,\mu,\rho} \frac{32(\cos \theta_\rho - \cos \theta_\mu)}{\sin^2 \theta_\mu \sin^2 \theta_\rho} Im(\langle \nabla_\beta\mu, \rho \rangle g_\beta\tilde{\beta}\tilde{\mu} + \langle \nabla_\beta\mu, \rho \rangle g_\beta\tilde{\beta}\tilde{\mu})
$$

Note that (2.9) = 0, because it is the product of skew-symmetric factor $\rho, \mu$ with a symmetric one. Hence, interchanging $\mu$ with $\rho$ when necessary,

$$
(2.7) = \sum_{\beta,\mu,\rho} \frac{32(\cos \theta_\rho - \cos \theta_\mu)}{\sin^2 \theta_\rho \sin^2 \theta_\mu} Im(\langle \nabla_\beta\rho, \mu \rangle g_\beta\tilde{\mu}\tilde{\rho} + \langle \nabla_\beta\rho, \mu \rangle g_\beta\tilde{\mu}\tilde{\rho})
$$

$$
= \sum_{\beta,\mu,\rho} \frac{32}{\sin^2 \theta_\rho} Im(\langle \nabla_\beta\rho, \mu \rangle g_\beta\tilde{\mu}\tilde{\rho} + \langle \nabla_\beta\rho, \mu \rangle g_\beta\tilde{\mu}\tilde{\rho} - \langle \nabla_\beta\mu, \rho \rangle g_\beta\tilde{\mu}\tilde{\rho} - \langle \nabla_\beta\mu, \rho \rangle g_\beta\tilde{\mu}\tilde{\rho}).
$$

Therefore, and since $\langle \nabla_Z\rho, \mu \rangle = -\langle \nabla_Z\mu, \rho \rangle$,

$$
\Delta \kappa = 4i \sum_{\beta} Ricci^N (JdF(\beta), dF(\tilde{\beta}))
$$

$$
+ \sum_{\beta,\mu,\rho} \frac{32}{\sin^2 \theta_\mu} Im(\langle \nabla_\beta\mu, \rho \rangle g_\beta\tilde{\mu}\tilde{\rho} + \langle \nabla_\beta\mu, \rho \rangle g_\beta\tilde{\mu}\tilde{\rho})
$$

$$
- \sum_{\beta,\mu,\rho} \frac{64(\cos \theta_\mu + \cos \theta_\rho)}{\sin^2 \theta_\mu \sin^2 \theta_\rho} Re(g_\beta\mu\tilde{\beta}g_\beta\tilde{\mu})
$$

$$
- \sum_{\beta,\mu,\rho} \frac{32}{\sin^2 \theta_\mu} Im(\langle \nabla_\beta\mu, \rho \rangle g_\beta\tilde{\mu}\tilde{\rho} + \langle \nabla_\beta\mu, \rho \rangle g_\beta\tilde{\mu}\tilde{\rho} + \langle \nabla_\beta\mu, \rho \rangle g_\beta\tilde{\mu}\tilde{\rho} + \langle \nabla_\beta\mu, \rho \rangle g_\beta\tilde{\mu}\tilde{\rho})
$$

$$
+ \sum_{\beta,\mu,\rho} \frac{32(\cos \theta_\mu + \cos \theta_\rho)}{\sin^2 \theta_\mu} \left( |\langle \nabla_\beta\mu, \rho \rangle|^2 + |\langle \nabla_\beta\mu, \rho \rangle|^2 \right).
Derivating (2.1), considering $\nabla dF$ and $(\nabla dF)^{(1,1)}$ both with values in the normal bundle, we get

$$\nabla_Z(\nabla dF)^{(1,1)}(X, Y) = \frac{1}{2} \left( \nabla_Z \nabla dF(X, Y) + \nabla_Z \nabla dF(\bar{J}X, \bar{J}Y) + \nabla dF(\nabla_Z \bar{J}(X), \bar{J}Y) + \nabla dF(\bar{J}X, \nabla_Z \bar{J}(Y)) \right).$$

Since $\bar{J}Z_\mu = iZ_\mu$ and $\bar{J}Z_{\bar{\mu}} = -iZ_{\bar{\mu}} \forall \mu$, on a neighbourhood of $p_0$, we have, $\forall \alpha, \beta$

$$\nabla_Z(\nabla dF)^{(1,1)}(\alpha, \beta) = \frac{i}{2} \left( \nabla dF(\nabla_Z \bar{J}(\alpha), \bar{J}\beta) + \nabla dF(\alpha, \nabla_Z \bar{J}(\beta)) \right), \quad (2.11)$$

$$\nabla_Z(\nabla dF)^{(1,1)}(\bar{\alpha}, \bar{\beta}) = \frac{i}{2} \left( -\nabla dF(\nabla_Z \bar{J}(\bar{\alpha}), \bar{\beta}) + \nabla dF(\alpha, \nabla_Z \bar{J}(\bar{\beta})) \right). \quad (2.12)$$

Since $F$ is minimal,

$$\sum_{\beta} \nabla_Z \nabla dF(\beta, \bar{\beta}) = \frac{1}{4} \text{trace}_{g_M} \nabla_Z \nabla dF = \frac{1}{4} \nabla_Z(\text{trace}_{g_M} \nabla dF) = 0.$$

Then applying Codazzi’s equation and noting that $JdF(\bar{\mu}) + i \cos \theta_\mu dF(\bar{\mu})$ is in the complexified normal bundle,

$$\sum_{\beta} R^N(dF(\beta), dF(\bar{\mu}), dF(\bar{\beta}), JdF(\bar{\mu}) + i \cos \theta_\mu dF(\bar{\mu})) = \sum_{\beta} g(-\nabla_\beta \nabla dF(\mu, \bar{\beta}), JdF(\bar{\mu})). \quad (2.13)$$

Using (2.3), (2.12) and (2.13)

$$\sum_{\beta} R^N(dF(\beta), dF(\bar{\mu}), dF(\bar{\beta}), JdF(\bar{\mu}) + i \cos \theta_\mu dF(\bar{\mu})) = \sum_{\beta} g(-\nabla_\beta \nabla dF(\mu, \bar{\beta}), JdF(\bar{\mu})). \quad (2.13)$$

Consequently,

$$\Delta \kappa = 4i \sum_{\beta} \text{Ricci}^N(JdF(\beta), dF(\bar{\beta})) - \sum_{\beta, \mu, \rho} \frac{32}{\sin^2 \theta_\mu} \text{Im}(\langle \nabla_\beta (\nabla dF)^{(1,1)}(\mu, \bar{\beta}), JdF(\bar{\mu}) \rangle)
+ \sum_{\beta, \mu, \rho} \frac{64}{\sin^2 \theta_\mu} \text{Im}(\langle \nabla_\beta \mu, \rho \rangle g_\rho \bar{\beta} \bar{\mu})
+ \sum_{\beta, \mu, \rho} \frac{64}{\sin^2 \theta_\mu} \text{Im}(\langle \nabla_\beta \bar{\beta}, \bar{\rho} \rangle g_\rho \bar{\mu} \bar{\rho})
- \sum_{\beta, \mu, \rho} \frac{32}{\sin^2 \theta_\mu} \text{Im}(\langle \nabla_\beta \mu, \rho \rangle g_{\bar{\beta} \bar{\rho}})
- \sum_{\beta, \mu, \rho} \frac{32}{\sin^2 \theta_\mu} \text{Im}(\langle \nabla_\beta \bar{\beta}, \rho \rangle g_{\mu \bar{\rho}})
+ \sum_{\beta, \mu, \rho} \frac{32(\cos \theta_\mu + \cos \theta_\rho)}{\sin^2 \theta_\mu} \left( |\langle \nabla_\beta \mu, \rho \rangle|^2 + |\langle \nabla_\beta \bar{\beta}, \rho \rangle|^2 \right). \quad (2.17)$$

$\text{trace}_{g_M}$ refers to the trace of the metric $g_M$ on the normal bundle.
By lemma 2.1

\[(2.14) + (2.15) = \sum_{\beta, \mu, \rho} \frac{32}{\sin^2 \theta_{\mu}} \text{Im} \left( \langle \nabla_{\beta \mu} \rho \rangle (g_{\beta \rho} \bar{\mu} - g_{\beta \bar{\mu}} \rho) \right) = - \sum_{\beta, \mu, \rho} \frac{32 (\cos \theta_{\mu} + \cos \theta_{\rho})}{\sin^2 \theta_{\mu}} | \langle \nabla_{\beta \mu} \rho \rangle |^2 \]

defines a matrix \( \nabla \). That cancels with some term of (2.17). Similarly

\[(2.16) + \sum_{\beta, \mu, \rho} \frac{32 (\cos \theta_{\mu} + \cos \theta_{\rho})}{\sin^2 \theta_{\mu}} | \langle \nabla_{\beta \mu} \rho \rangle |^2 =
\]
\[= - \sum_{\beta, \mu, \rho} \frac{32}{\sin^2 \theta_{\mu}} \text{Im} \left( \langle \nabla_{\beta \mu} \rho \rangle (g_{\beta \mu} \bar{\rho} - g_{\beta \bar{\mu}} \rho) \right) - \sum_{\beta, \mu, \rho} \frac{64}{\sin^2 \theta_{\mu}} \text{Im} \left( \langle \nabla_{\beta \mu} \rho \rangle g_{\beta \bar{\rho} \bar{\mu}} \right)
\]
\[+ \sum_{\beta, \mu, \rho} \frac{32 (\cos \theta_{\mu} + \cos \theta_{\rho})}{\sin^2 \theta_{\mu}} | \langle \nabla_{\beta \mu} \rho \rangle |^2
\]
\[= - \sum_{\beta, \mu, \rho} \frac{64}{\sin^2 \theta_{\mu}} \text{Im} \left( \langle \nabla_{\beta \mu} \rho \rangle g_{\beta \bar{\rho} \bar{\mu}} \right)
\]

Then,

\[\Delta \kappa = 4i \sum_{\beta} \text{Ricci}^N (JdF(\beta), dF(\bar{\beta})) - \sum_{\beta, \mu, \rho} \frac{32}{\sin^2 \theta_{\mu}} \text{Im} \left( g \left( \nabla_{\beta} (\nabla dF)^{1,1}(\mu, \bar{\beta}), JdF(\bar{\mu}) \right) \right)
\]
\[- \sum_{\beta, \mu, \rho} \frac{64 (\cos \theta_{\mu} + \cos \theta_{\rho})}{\sin^2 \theta_{\mu} \sin^2 \theta_{\rho}} \text{Re} \left( g_{\beta \mu} \bar{\rho} g_{\beta \bar{\rho} \bar{\mu}} \right)
\]
\[+ \sum_{\beta, \mu, \rho} \frac{64}{\sin^2 \theta_{\mu}} \text{Im} \left( \langle \nabla_{\beta \bar{\rho}} \bar{\rho} \rangle g_{\rho \mu} \bar{\mu} \right)
\]
\[- \sum_{\beta, \mu, \rho} \frac{64}{\sin^2 \theta_{\mu}} \text{Im} \left( \langle \nabla_{\beta \mu} \rho \rangle g_{\beta \bar{\rho} \bar{\mu}} \right)
\]

(2.18)

Now, from (2.3), \( (\nabla_{\beta \bar{\rho}})^{1,0} = \frac{i}{2} \nabla_{\beta \bar{J}}(\bar{\beta}) \), and so

\[(2.18) = \sum_{\beta, \mu, \rho} \frac{32}{\sin^2 \theta_{\mu}} \text{Im} \left( g \left( \nabla dF^{1,1}(\mu, \bar{J}(\beta)), JdF(\bar{\mu}) \right) \right)
\]

and from (2.11), (2.3), and that \( \nabla_{\beta \bar{J}}(\mu) \) is of type \((0,1)\)

\[(2.19) = - \sum_{\beta, \mu, \rho} \frac{32}{\sin^2 \theta_{\rho}} \text{Im} \left( g \left( \nabla dF(\beta, -\frac{i}{2} \nabla_{\beta \bar{J}}(\mu)), JdF(\bar{\mu}) \right) \right)
\]

\[= \sum_{\beta, \mu, \rho} \frac{32}{\sin^2 \theta_{\mu}} \text{Im} \left( g \left( \nabla (\nabla dF)^{1,1}(\beta, \mu), JdF(\bar{\mu}) \right) \right)
\]
\[- \sum_{\beta, \mu, \rho} \frac{32}{\sin^2 \theta_{\mu}} \text{Im} \left( g \left( \nabla dF(\frac{i}{2} \nabla_{\beta \bar{J}}(\beta), \mu), JdF(\bar{\mu}) \right) \right)
\]

Using the unitary basis \( \{ \sqrt{2} \alpha, \sqrt{2} \alpha \} \) of \( T_p M \), for \( p \) near \( p_0 \), \( g_M + \tilde{g} \) is represented by the matrix

\[g_M \pm \tilde{g} = \begin{bmatrix}
\delta_{\alpha \gamma} \pm 2 \bar{g}_{\alpha \gamma} & 0 \\
0 & \delta_{\alpha \gamma} \pm 2 \bar{g}_{\alpha \gamma}
\end{bmatrix}
\]
with $\tilde{g}_{\mu\rho} = \tilde{g}_{\rho\mu}$. This matrix is at the point $p_0$ the diagonal matrix $D(1 + \cos \theta_1, \ldots, 1 + \cos \theta_n, 1 + \cos \theta_1, \ldots, 1 + \cos \theta_n)$. Thus (cf. lemma 5.2 of [S-V]), $\forall Z \in T_{p_0}M$,

$$d(det(g_M \pm \tilde{g}))(Z) = \pm 4 \sum_{\mu} \frac{det(g_M \pm \tilde{g})}{(1 + \cos \theta_\mu)} d\tilde{g}_{\mu\bar{\mu}}(Z).$$

Then, using Lemma 2.1

$$2dk_{p_0}(Z) = \frac{d(det(g_M + \tilde{g}))(Z)}{det(g_M + \tilde{g})} - \frac{d(det(g_M - \tilde{g}))(Z)}{det(g_M - \tilde{g})} = 4 \sum_{\mu} \frac{1}{(1 + \cos \theta_\mu)} d\tilde{g}_{\mu\bar{\mu}}(Z) + 4 \sum_{\mu} \frac{1}{(1 - \cos \theta_\mu)} d\tilde{g}_{\mu\bar{\mu}}(Z) \quad (2.20)$$

$$= 8 \sum_{\mu} \frac{i}{\sin^2 \theta_\mu} \left( \langle \nabla dF(Z, \mu), JdF(\bar{\mu}) \rangle - \langle \nabla dF(Z, \bar{\mu}), JdF(\mu) \rangle \right).$$

Thus, since $Im(iA) = Im(i\bar{A})$ for any complex number $A$,

$$-4 \sum \text{Im} \left( dk_{p_0} \left( \nabla_\beta \bar{J}(\beta) \right) \right) =$$

$$= \sum_{\beta,\mu} \frac{32}{\sin^2 \theta_\mu} \left( \text{Im} \left( g(\nabla dF(\frac{i}{2} \nabla_\beta \bar{J}(\beta), \bar{\mu}), JdF(\mu)) \right) - \text{Im} \left( g(\nabla dF(\frac{i}{2} \nabla_\beta \bar{J}(\beta), \mu), JdF(\bar{\mu})) \right) \right) \quad (2.21)$$

Consequently, for $F$ minimal, at $p_0$,

$$\triangle \kappa = 4i \sum_{\beta} \text{Ricci}^{N}(JdF(\beta), dF(\bar{\beta}))$$

$$- \sum_{\beta,\mu} \frac{32}{\sin^2 \theta_\mu} \text{Im} \left( g(\nabla_\beta (\nabla dF)^{(1,1)}(\beta, \mu), JdF(\bar{\mu})) \right) \quad (2.22)$$

$$+ \sum_{\beta,\mu} \frac{32}{\sin^2 \theta_\mu} \text{Im} \left( g(\nabla_\beta (\nabla dF)^{(1,1)}(\beta, \mu), JdF(\bar{\mu})) \right) \quad (2.23)$$

$$- 4 \text{Im} \left( dk_{p_0} \left( \sum_{\beta} \nabla_\beta \bar{J}(\beta) \right) \right) \quad (2.24)$$

$$- \sum_{\beta,\mu,\rho} \frac{64(\cos \theta_\mu + \cos \theta_\rho)}{\sin^2 \theta_\mu \sin^2 \theta_\rho} \text{Re} \left( g_{\beta\mu\rho} \tilde{g}_{\beta\rho\mu} \right) \quad (2.25)$$

Of course we may assume $k \geq 1$, since, on $\Omega^0_{2k}$, the summation on $\mu$ (and $\beta$) of the left-hand side of (2.13) vanish, (because of a symmetry and skew-symmetry argument) and Proposition 2.1 gives Proposition 2.2 on that open set. If $F$ is broadly-pluriminimal then $F$ is pluriharmonic with respect to any local complex structure $\bar{J}$ we described above, on each $\Omega^0_{2k}$. Then $(\nabla dF)^{(1,1)} = 0$ and (2.22), (2.23), and (2.25) vanish. Now we prove that (2.24) also vanish. On a neighbourhood of $p_0$, since $\tilde{g}_{\mu\bar{\mu}} = 0$, for all $\mu \geq k$, then the $\sum_{\mu}$ in (2.20) and in (2.21) can be replaced by $\sum_{1 \leq \mu \leq k}$. On the other hand, from (2.21), (2.3)
and broadly-pluriminimality

\[-4 \sum_{\beta} \text{Im}\left(\frac{dK_{\rho}}{d\rho} \left(\nabla_{\beta} J(\beta)\right)\right) = \sum_{1 \leq \beta \leq n} \sum_{1 \leq \mu \leq k} \frac{64}{\sin^{2} \theta_{\mu}} \text{Im}\left(\sum_{1 \leq \rho \leq n} \langle \nabla_{\beta} \bar{\beta}, \rho \rangle g_{\rho} \mu \mu \right)\]  
(2.26)

From broadly-pluriminimality, for each \(\mu \leq k\), \(g_{\rho} \mu \mu = 0\), \(\forall \rho \geq k + 1\). But on \(K_{\omega}\), the \(g_{M}\)-orthogonal complex structure \(J'\) is arbitrary. In particular we may replace \(J'\) by \(-J'\). This means that we may replace \(\bar{\rho} \geq k + 1\) by \(\rho \geq k + 1\). Therefore, \(g_{\rho} \mu \mu = 0\), \(\forall \rho \geq k + 1\) and \(\forall \mu \leq k\). From Lemma 2.1 and and pluriharmonicity, for \(\rho \leq k\) and \(\forall \beta\),

\[(\cos \theta_{\beta} + \cos \theta_{\rho}) \langle \nabla_{\beta} \bar{\beta}, \rho \rangle = i g_{\beta} \beta \rho - i g_{\bar{\beta}} \rho \bar{\beta} = 0\]

with \((\cos \theta_{\beta} + \cos \theta_{\rho}) > 0\). Then \(\langle \nabla_{\beta} \bar{\beta}, \rho \rangle = 0\), \(\forall \rho \leq k\). Thus, for each \(\mu \leq k\),

\[\langle \nabla_{\beta} \bar{\beta}, \mu \rangle = 0\]

for any \(\rho \leq k\) and also for any \(\rho \geq k + 1\). That is (2.26) = 0, and we have proved Proposition 2.2. \(\Box\)

**Corollary 2.1** In the conditions of Proposition 2.2, if \(N\) is Kähler-Einstein with \(\text{Ricci}^{N} = Rg\), on each \(\Omega_{2k}^{0}\),

\[\Delta \kappa = -2R \left(\sum_{\beta} \cos \theta_{\beta}\right)\]

**Remark.** One may search when \((\Omega_{2n}, J_{\omega}, g_{M})\) is a Kähler manifold, for, it is more usual to define pluriharmonicity on Kähler manifolds then in almost complex manifolds. More generally, we may ask when or where \(\nabla J_{\omega} = 0\) holds. Set, on each \(\Omega_{2k}^{0}\), \(\epsilon'_{\alpha} = +1\), \(\epsilon'_{\alpha} = -1\), \(\forall 1 \leq \alpha \leq k\), and \(\epsilon'_{\alpha} = \epsilon'_{\alpha} = 0\), \(\forall \alpha \geq k + 1\). Then, \(\forall A, B \in \{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}, \forall Z \in T_{p}M\), and \(\forall p\) near \(p_{0} \in \Omega_{2k}^{0}\)

\[\langle \nabla_{Z} J_{\omega}(A), B \rangle = i (\epsilon'_{A} + \epsilon'_{B}) \langle \nabla_{Z} A, B \rangle = -\langle A, \nabla_{Z} J_{\omega}(B) \rangle\]

Then, on \(\Omega_{2k}^{0}\)

\[\nabla_{Z} J_{\omega}(K_{\omega}) \subseteq K_{\omega}^{\perp}\]
\[\nabla_{Z} J_{\omega}(K_{\omega}^{\perp}) \subseteq (K_{\omega}^{\perp})^{0.1} \cup K_{\omega}\]
\[\nabla_{Z} J_{\omega}(K_{\omega}^{\perp}) \subseteq (K_{\omega}^{\perp})^{0.1} \cup K_{\omega}\]

(2.28)

For simplicity, we denote by \(\cos \theta_{\beta} = -\cos \theta_{\bar{\beta}}\). Hence at \(p_{0}\) and \(\forall A, B \in \{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}\), \(\text{F}^{*} \omega(A, B) = \delta_{AB} \frac{1}{2} \cos \theta_{A}\). Derivating at \(p_{0}\)

\[d(\text{F}^{*} \omega(A, B))(Z) = \nabla_{Z} \text{F}^{*} \omega(A, B) - i(\cos \theta_{A} + \cos \theta_{B}) \langle \nabla_{Z} A, B \rangle\]

(2.29)

Consequently, from (2.27) and (2.29), we obtain, at \(p_{0}\)

\[\nabla_{Z} \text{F}^{*} \omega(A, B) = 0\]
\[\forall A, B \in \{k + 1, k + 1, \ldots, n, \bar{n}\}\]
\[\nabla_{Z} \text{F}^{*} \omega(A, B) = \frac{\cos \theta_{A} + \cos \theta_{B}}{\epsilon'_{A} + \epsilon'_{B}} \langle \nabla_{Z} J_{\omega}(A), B \rangle\]
\[\text{whenever} \ \epsilon'_{A} + \epsilon'_{B} \neq 0\]

(2.30)
In particular, (2.28), (2.30) and (2.6) let us to conclude that, \((\Omega_{2n}, J_\omega, g_M)\) is a Kähler manifold iff \(\nabla Z F^*\omega\) is of type \((1,1)\), \(\forall Z \in T^cM\), on \(\Omega_{2n}\) iff \((V, W) \rightarrow g(\nabla Z dF(V), JdF(W))\) is symmetric on \(T^{1,0}M\), \(\forall Z \in T^cM\) on \(\Omega_{2n}\) iff \(\langle \nabla Z J_\omega(\alpha), \beta \rangle = 0\) \(\forall 1 \leq \alpha, \beta \leq n\) and \(\forall Z \in T^cM\), on \(\Omega_{2n}\). In [S-V] we prove that, if \(F^*\omega\) is parallel, that is, \(\nabla Z F^*\omega = 0\), then the Kähler angles are constant, \(K_\omega\), \(K_\omega\) are parallel sub-bundles of \(TM\), and \(\nabla J_\omega = 0\).

### 2.2 Some conclusions

Let \(p_0\) be an absolute maximum of \(\kappa\). Then \(\kappa(p_0) \geq 0\) with equality to zero iff \(F^*\omega = 0\) everywhere, that is, \(F\) is a Lagrangian submanifold. We only know that \(\kappa\) is locally Lipschitz on \(M\) and smooth on each \(\Omega_{2k}^0\). Nevertheless we have the following proposition:

**Proposition 2.3** The map \(\kappa\) is differentiable at \(p_0\) with \(d\kappa(p_0) = 0\).

**Proof.** Let \(2k\) be the rank of \(F^*\omega\) at \(p_0\), and \(\pm i \cos \theta_\alpha(p)\) the eigenvalues of \((F^*\omega)^c\), the \(C\)-linear extension of \(F^*\omega\) to \(T^cM\), for \(p\) near \(p_0\). Of course we may assume \(k \geq 1\). Set

\[
\kappa_1(p) = \sum_{1 \leq \alpha \leq k} \log \left( \frac{1 + \cos \theta_\alpha(p)}{1 - \cos \theta_\alpha(p)} \right), \quad \kappa_2(p) = \sum_{k+1 \leq \alpha \leq n} \log \left( \frac{1 + \cos \theta_\alpha(p)}{1 - \cos \theta_\alpha(p)} \right).
\]

\(\kappa_1\) is the piece of \(\kappa\) defined by angles which cosine is not zero near \(p_0\). The remaining angles, forming the \(\kappa_2\), are zero at \(p_0\), therefore they remain well distinct from the ones which form the \(\kappa_1\), for \(p\) near \(p_0\). Then we may conclude that \(\kappa_1\) is smooth. In fact, if we take \(C\) a contour in the 1-complex space around the eigenvalues \(\pm i \cos \theta_1, \ldots, \pm i \cos \theta_k\) for \(p\) near \(p_0\), not meeting any of all other eigenvalues, then \(P = \frac{1}{2\pi i} \int_C (\lambda I - (F^*\omega)^c)^{-1}d\lambda\), where \(I : T^cM \rightarrow T^cM\) is the identity morphism, is, for each \(p\) on a neighbourhood of \(p_0\), an orthogonal projection (since \(F^*\omega\) is normal) onto the subspace that is the direct sum of the eigenspaces corresponding to the eigenvalues inside \(C\), and \(T = \frac{1}{2\pi i} \int_C \lambda (\lambda I - (F^*\omega)^c)^{-1}d\lambda\) is the restriction of \((F^*\omega)^c\) to the sub-bundle \(E = P(T^cM)\). Clearly \(T\) and \(P\) are smooth sections of \(T^*M \otimes T^cM\), and so \(E\) is smooth. Then \(\kappa_1\) is just defined as \(\kappa\), but relative to \(T\) on \(E\) where it is invertible everywhere. Now we have, for \(p\) near \(p_0\), \(\kappa = \kappa_1 + \kappa_2\), with \(\kappa_1\) smooth. The point \(p_0\) is a maximum of \(\kappa\) and of \(\kappa_1\), and is a minimum of \(\kappa_2\). Then \(d\kappa_1(p_0) = 0\). We may assume \(\kappa\) and \(\kappa_i\) are defined on a open set of \(\mathbb{R}^m\). We will prove that \(\kappa_2\) is stationary at \(p_0\), that is, \(\kappa_2\) is differentiable at \(p_0\) and its differential at \(p_0\) is zero. Thus we want to prove that \(r(p) = \kappa_2(p) - \kappa_2(p_0)\) satisfies

\[
\forall t > 0 \exists s > 0 : ||p - p_0|| \leq s \implies ||r(p)|| \leq t||p - p_0|| \quad (2.31)
\]

Suppose this is not true. Then

\[
\exists t' > 0 \text{ and } p_n \rightarrow p_0 \text{ such that } ||r(p_n)|| > t'||p_n - p_0|| \quad (2.32)
\]
From \( \kappa(p_n) \leq \kappa(p_0) \), \( \kappa_1(p_n) \leq \kappa_1(p_0) \), and \( \kappa_2(p_n) \geq \kappa_2(p_0) \), we have

\[
0 \leq \frac{r(p_n)}{||p_n - p_0||} = \frac{\kappa_2(p_n) - \kappa_2(p_0)}{||p_n - p_0||} = \frac{\kappa(p_n) - \kappa(p_0)}{||p_n - p_0||} \leq -\frac{\kappa_1(p_n) - \kappa_1(p_0)}{||p_n - p_0||} = -d\kappa_1(p_0)\left(\frac{p_n - p_0}{||p_n - p_0||}\right) + \frac{r'(p_n)}{||p_n - p_0||}
\]

where \( r'(p) \) satisfies (3.31). Since \( d\kappa_1(p_0) = 0 \), the last term in the above inequality converges to zero, contradicting (3.32). \( \square \)

In the conditions of Proposition 2.2 and Corollary 2.1 assume that \( M \) is compact and \( N \) is a Kähler-Einstein manifold. If \( R < 0 \), by applying maximum principle to \( \kappa \) at a maximum point \( p_0 \), we immediately conclude from Corollary 2.1:

**Lemma 2.2** If \( R < 0 \), and \( F \) is broadly-pluriminimal, not Lagrangian and with no complex directions, then \( p_0 \) is not in \( \Omega^0_{2k} \forall 0 \leq k \leq n \). In other words, if the rank of \( F^*\omega \) is \( 2k \) at \( p_0 \), then there exists a sequence \( p_m \to p_0 \) such that the rank of \( F^*\omega \) at \( p_m \) is \( > 2k \). In particular \( F^*\omega \) cannot have constant rank and \( F^*\omega \) is degenerated at \( p_0 \).

**Proposition 2.4** If \( M \) is compact and \( R < 0 \), every broadly-pluriminimal immersion either has Lagrangian or complex directions.

**Proof.** If \( F \) has no Lagrangian directions \( M = \Omega_{2n} = \Omega^0_{2n} \). Then, non existence of complex directions contradicts the Lemma. \( \square \)

Of course if \( n = 1 \) the assumption \((ii)\) for broadly-pluriminimal is automatic, and minimal surfaces are broadly-pluriminimal, and we recover some result of Wolfson [W].

**Proposition 2.5** If \( F : M \to N \) is broadly-pluriminimal without complex directions, \( M \) is compact, \( R < 0 \) and if rank \( F^*\omega \leq 2 \), then \( F \) is Lagrangian.

**Proof.** If \( \kappa(p_0) \neq 0 \), then \( F^*\omega \) would have rank \( 2 \) at \( p_0 \), contradicting Lemma 2.2. \( \square \)

An immediate consequence of Corollary 2.1 by making \( \kappa \) constant, is the following proposition:

**Proposition 2.6** If \( F \) is broadly-pluriminimal, not Lagrangian, with constant Kähler angles and no complex directions, then \( R = 0 \) must hold.

The following proposition is announced in [S-V], but only proved in this paper, as a consequence of Corollary 2.1 as well:
**Proposition 2.7** If $M$ is compact, $N$ is Kähler-Einstein with $R < 0$, and if $F$ is broadly-pluriminimal without complex directions and with equal Kähler angles, then $F$ is Lagrangian.

**Proof.** If the Kähler angles are all equal, $\cos \theta_\alpha = \cos \theta, \forall \alpha$, and if this is not zero everywhere, that is, $F$ is not Lagrangian, then $p_0 \in \Omega_{2n}$, what is not possible. □

**Theorem 2.1** If $n = 2$, and if $F : M \rightarrow N$ is broadly-pluriminimal without complex directions, $M$ is compact and orientable, and $N$ is Kähler-Einstein with $R < 0$, then $F$ is Lagrangian.

**Proof.** We have two eigenvalues $\cos \theta_1 \geq \cos \theta_2 \geq 0$, and we suppose $F$ not Lagrangian. Then from Lemma 2.2 we must have $\cos \theta_1(p_0) > 0$, $\cos \theta_2(p_0) = 0$ and there exist $p_\mathbf{m} \rightarrow p_0$ such that $\cos \theta_2(p_\mathbf{m}) > 0$, $\forall \mathbf{m}$. Since $\cos \theta_1 \neq \cos \theta_2$ at $p_0$, by continuity they remain different for $p$ near $p_0$, and $\pm i \cos \theta_1$ do not vanish and have multiplicity one for $p$ near $p_0$. Therefore, an application of the implicit map theorem to the characteristic equation of the complex extension of $F^* \omega$, shows that $\cos \theta_1$ defines a smooth map defined on a neighbourhood of $p_0$. From $F^* \omega(p) = \cos \theta_1(p)X_1^1 \wedge Y_1^1 + \cos \theta_2(p)X_2^2 \wedge Y_2^2$, where $X_1, Y_1, X_2, Y_2$ is a diagonalizing orthonormal basis of $F^* \omega(p)$, we have

\[
(F^* \omega)^2(p) = \epsilon(p)2 \cos \theta_1(p) \cos \theta_2(p)Vol
\]

where $Vol$ is the volume element of $M$, and where $\epsilon(p)$ is +1 or -1 according $X_1, Y_1, X_2, Y_2$ is an direct or inverse basis, respectively. Since $(F^* \omega)^2$ and $\cos \theta_1$ are smooth, then

\[
s_2(p) = \epsilon(p) \cos \theta_2(p) \tag{2.33}
\]

is a smooth map for $p$ near $p_0$. Without loss of generalization we may suppose that $s_2(p_\mathbf{m}) > 0 \forall \mathbf{m}$, by taking a subsequence, and changing the orientation of $M$, if necessary. Then $\epsilon(p_\mathbf{m}) = 1 \forall \mathbf{m}$. Let us take the smooth map, defined on a neighbourhood of $p_0$

\[
\tilde{\kappa} = \log \left( \frac{1 + \cos \theta_1}{1 - \cos \theta_1} \right) + \log \left( \frac{1 + s_2}{1 - s_2} \right)
\]

Since $s_2 = \cos \theta_2$ in the open set of the points where $s_2 > 0$ or equivalently where $\epsilon = +1$ and $\cos \theta_2$ does not vanish, then $\kappa = \tilde{\kappa}$ on that set, and in particular in a neighbourhood of each $p_\mathbf{m}$, for each $\mathbf{m}$. Moreover $\kappa(p_0) = \tilde{\kappa}(p_0)$. We now prove that $p_0$ is also a maximum of $\tilde{\kappa}$. Let $p$ on a neighbourhoud of $p_0$. If $s_2(p) = \cos \theta_2(p)$, then $\tilde{\kappa}(p) = \kappa(p) \leq \kappa(p_0) = \tilde{\kappa}(p_0)$. If $s_2(p) = -\cos \theta_2(p)$ then

\[
\tilde{\kappa}(p) = \kappa(p) - 2 \log \left( \frac{1 + \cos \theta_2(p)}{1 - \cos \theta_2(p)} \right)
\]
with \( \log \left( \frac{1 + \cos \theta_2(p)}{1 - \cos \theta_2(p)} \right) \geq 0 \). From \( \kappa(p) \leq \kappa(p_0) = \tilde{\kappa}(p_0) \) we obtain

\[
\tilde{\kappa}(p) = \kappa(p) - 2 \log \left( \frac{1 + \cos \theta_2(p)}{1 - \cos \theta_2(p)} \right) \leq \tilde{\kappa}(p_0) - 2 \log \left( \frac{1 + \cos \theta_2(p)}{1 - \cos \theta_2(p)} \right) \leq \tilde{\kappa}(p_0).
\]

Therefore, by maximum principle applied to \( \tilde{\kappa} \) at \( p_0 \), by Corollary 2.1 applied to \( p_m \in \Omega_{2n}^0 \), and by continuity of the maps \( \cos \theta_\alpha \),

\[
0 \geq \bigtriangleup \tilde{\kappa}(p_0) = \lim_m \bigtriangleup \tilde{\kappa}(p_m) = \lim_m \bigtriangleup \kappa(p_m) = \lim_m -2R(\cos \theta_1(p_m) + \cos \theta_2(p_m))
\]

\[
= -2R(\cos \theta_1(p_0) + \cos \theta_2(p_0)) = -2R \cos \theta_1(p_0) \geq 0
\]

which implies \( \cos \theta_1(p_0) = 0 \), that is a contradiction. \( \square \)

Unfortunately a similar argument does not work for higher dimensions, since we do not have smooth functions as good as (2.33). We only have smoothness of some symmetric functions of the Kähler angles.

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