Novel Practicable GLHUA-i,i=1,2,3, Invisible Cloak Absorb the Incident Wave and Create Outgoing Wave Without Exceeding Light Speed Propagation and New N Dimensional Maxwell Electromagnetic Equations

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In this version of our paper, we make two great breakthrough progress contribution:
I. We proposed New N dimensional Maxwell Equation that is different from Gauge potential Maxwell field equation. We propose new N dimensional Curl operator. The 3 dimensional curl operator is well defined, however, for N \( N \neq 3 \) dimensional space, the curl operator is never be defined in other published papers before. As breakthrough, we propose new curl operator in N dimensional space, 
\[
\text{Curl}_F = (\nabla \times)_N = (\nabla \nabla \cdot - \nabla \cdot \nabla) \frac{1}{2}, (1)
\]
and New N Dimensional Maxwell Equations:
\[
\begin{align*}
\text{Curl}_F E &= (\nabla \times)_N E = -\frac{\partial B}{\partial t}, \\
\nabla \cdot B &= 0, \\
\text{Curl}_F H &= (\nabla \times)_N H = \frac{\partial D}{\partial t} + \vec{J}, \\
\nabla \cdot D &= \rho. (2)
\end{align*}
\]
For electric current or magnetic source, the analytic electromagnetic wave field solution of the new N dimensional Maxwell equation (2) are obtained.
For 3 dimension, our curl operator \( \text{Curl}_F \) in (1) become standard 3D curl
\[
\text{Curl}_F = (\nabla \times)_3 E = \begin{vmatrix}
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\
E_{x_1} & E_{x_2} & E_{x_3}
\end{vmatrix}, (3)
\]
our N dimension Maxwell equation in (2) becomes standard 3D Maxwell equations.
\[
\begin{align*}
\nabla \times E &= \begin{vmatrix}
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3}
\end{vmatrix} E_{x_1} E_{x_2} E_{x_3} = -\frac{\partial B}{\partial t}, \\
\nabla \times H &= \begin{vmatrix}
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3}
\end{vmatrix} H_{x_1} H_{x_2} H_{x_3} = \frac{\partial D}{\partial t} + \vec{J}, (4)
\end{align*}
\]
our N dimensional Maxwell equations have great role in the theoretical and practicable applications, in particular, in electromagnetic invisible cloaks and in super sciences. This breakthrough discovery has been announced in March 15, 2020 in Hunan Super Computational Society (in Figure 51). The second author Dr. Feng Xie in GL Geophysical Laboratory and Stanford University made main and great contribution in this breakthrough discovery and creation.

II. We propose three novel practicable GLHUA-1, GLHUA-2 and GLHUA-3 invisible cloaks, some part in our cloak annular layer absorbs the incident wave and, in the meantime, other part of our cloak annular layer creates outgoing wave that make it to be invisible cloak and analytical electromagnetic wave without exceeding light speed propagation through it. Ten more discoveries and creations are reported in this paper. The copyright and patent for this paper are belong to the authors of this paper.

1. New practicable GLHUA-1, GLHUA-2 and GLHUA-3 annular layer electromagnetic (EM) invisible cloak with relative refractive index parameters large or equal to 1 are created. In the GLHUA-1 cloak, all relative EM parameters (25) are large or equal to 1, \( R_1 \leq r \leq R_2 \), \( R_1 > 0, R_2 > R_1, \varepsilon_r = \mu_r = \frac{R_2^2}{R_1^2}, \varepsilon_\theta = \varepsilon_\phi = \mu_\theta = \mu_\phi = \frac{R_2 - R_1}{R_2 + R_1} \). In the GLHUA-2 invisible cloak, relative refractive index parameters (11) are large or equal to 1, the angular parameter (8) are large or equal to...
1. the radial parameters (7) are large than positive constant. The EM wave in GLHUA-1 and GLHUA-2 invisible cloak are physical bounded. In GLHUA-3 invisible cloak, relative refractive index parameters (18) are large or equal to 1, the angular parameter (14) are large or equal to 1, the radial parameters (15) are large than 0.

2. Analytical EM wave solutions of EM full wave equation with the GLHUA-1, GLHUA-2 and GLHUA-3 cloak materials are found in annular layer of the three cloaks.

3. A GLHUA-1 expansion method are propsoed to find an exact analytical EM wave propagation in the GLHUA-1,GLHUA-2 and GLHUA-3 double layer cloaks.

4. We proposed new negative free space concept in arXiv:1612.02857. We define the negative free space in that the sphere radial is negative or zero. Also we define the positive free space in that sphere radial is positive or zero, we live in the positive space. We proved that the electromagnetic equations in the free positive space can analytical be continue to the negative space.

5. let R be sphere radial variable in the basic positive and negative space, r be sphere radial variable in physical plsitive space. We propose a new GLHUANP-2 transformation from the negative space $[-\infty, -R_2]$ to spherical annual layer $[R_1, R_2]$ in positive space, the positive space $[R_2, \infty]$ is invariant. The transformation GLHUANP-2 are total different from Pendry and ULF transformation. The new GLHUANP-2 transformation in this paper is,

$$r = R_1 - \frac{(R_2 - R_1)^2}{R + R_1}, -\infty < R \leq -R_2, R_1 \leq r \leq R_2, \ (5)$$

The transformation maps negative infinity, $R = -\infty$, in negative space to $r = R_1$ in the physical positive space; and maps $R = -R_2$ in negative space to $r = R_2$ in the physical positive space; The inverse transformation of (5) is

$$R = -R_1 - \frac{(R_2 - R_1)^2}{R - R_1}, R_1 \leq r \leq R_2, -\infty < R \leq -R_2, \ (6)$$

6. Using the transformation GLHUANP-2 (5) and (6), a novel GLHUA-2 invisible cloak is created, the relative radial electric permittivity $\varepsilon_r$ and magnetic permeability $\mu_r$ are derived to be,

$$\varepsilon_r = \mu_r = \frac{1}{r^2} \left( \frac{R_2 (r - R_1) + (R_2 - R_1)^2}{(R_2 - R_1)^2} \right), R_1 \leq r \leq R_2, \ (7)$$

The angular realtive electric permittivity and magnetic permeability are

$$\varepsilon_\theta = \mu_\theta = \mu_\phi = \frac{(R_2 - R_1)^2}{(r - R_1)^2}, R_1 \leq r \leq R_2, \ (8)$$

The properties of GLHUA-2 invisible cloak are as following:

1. The EM parameters are continuous on the $r = R_2$

$$\varepsilon_r (R_2) = \mu_r (R_2)$$

$$\varepsilon_r (R_2) = \frac{1}{r^2} \left( \frac{R_1 (R_2 - R_1) + (R_2 - R_1)^2}{(R_2 - R_1)^2} \right)$$

$$= \frac{1}{r^2} \frac{R_2}{R_2 - R_1} = 1, \ (9)$$

$$\varepsilon_r (R_2) = \mu_r (R_2) = 1, \ (10)$$

2. The refractive index

$$1 \leq n \leq +\infty, \ (11)$$

3. The EM wave propagation in the GLHUA-2 cloak without exceeding light speed difficulties.

4. The relative radial electric permittivity $\varepsilon_r$ and magnetic permeability $\mu_r$ in (7) are large than zero. The property (7) is important that makes the maganetic tense and electric tense in GLHUA-2 cloak are physical bounded.

5. By GLHUA-1 analytical expansion method in this paper, we fund an exact analytical EM wave propagation in the GLHUA-2 cloak without exceeding light speed propagation. The analytical electromaganetical wave propagation in the GLHUA-2 cloak prove that the GLHUA-2 cloak is completed practicable invisible cloak. The radial electric wave propagation through the GLHUA-2 cloak is display in the figure 29-36. The videos "Analytical maganetic wave propagation the practicable GLHUA-2 invisible cloak" are showing in

https://video.weibo.com/show?fid=1034:4412551845597236
https://video.weibo.com/show?fid=1034:4413016675135960
https://video.weibo.com/show?fid=1034:4413022027073708

7. GLHUANP-3 radial transformation,

$$r = R_1 + (R_2 - R_1)e^{\frac{R_2 - R_1}{R_2 - R_1}}, \ (12)$$
which also maps negative infinity, $-\infty$ to $R_1$, and maps $-R_2$ to $R_2$.

GLHUANP-3 inverse transformation is

$$R = -R_2 + (R_2 - R_1) \log \left( \frac{r - R_1}{R_2 - R_1} \right).$$  \hspace{1cm} (13)

From the GLHUANP-3 transformation (12), third novel GLHUA-3 invisible cloak with following relative electric permittivity and magnetic permeability are created as

$$\varepsilon_\theta = \mu_\theta = \varepsilon_\phi = \mu_\phi = \frac{R_2 - R_1}{r - R_1},$$  \hspace{1cm} (14)

$$\varepsilon_r = \mu_r = \left( -R_2 + (R_2 - R_1) \log \left( \frac{r - R_1}{R_2 - R_1} \right) \right)^2 \frac{1}{r^2} \frac{r - R_1}{R_2 - R_1}. $$  \hspace{1cm} (15)

The properties of GLHUA-3 invisible cloak are as following:

1. The EM parameters are continuous on the $r = R_2$

$$\varepsilon_\theta(R_2) = \mu_\theta(R_2) = \varepsilon_\phi(R_2) = \mu_\phi(R_2) = 1,$$  \hspace{1cm} (16)

$$\varepsilon_r(R_2) = \mu_r(R_2) = 1. $$  \hspace{1cm} (17)

2. The refractive index

$$1 \leq n \leq +\infty, $$  \hspace{1cm} (18)

(3). The EM wave propagation in the GLHUA-3 cloak without exceeding light speed difficulties.

The radial electric wave propagation through the GLHUA-3 cloak is displayed in the figure 43-50.

(4). We find analytical EM wave without exceeding light speed propagation through (26) cloak.

8. In GLHUA-1, GLHUA-2 and GLHUA-3 cloaks, some part of the cloak layer absorb the incident wave and, in the meantime, other part of cloak layer create outgoing wave that make they to be invisible cloak and analytical electromagnetic wave without exceeding light speed propagation through them.

9. GLHUAF transformation is proposed.

$$r = R_1 + (R_2 - R_1)e^{R_2 - R_1}$$  \hspace{1cm} (19)

Which maps positive infinity $+\infty$ to $R_1$, and $R_2$ is invariant.

10. By GLHUAF transformation, a novel GLHUAF invisible cloak with double negative materials is created.

Use the GLHUAF transformation to transform the radial magnetic equation in free space from positive infinity $+\infty$ to $R_1$, and $R_2$ is invariant.

$$\frac{\partial}{\partial r} \left( \frac{1}{\mu_\phi} \frac{\partial H}{\partial r} \right) + \frac{1}{r^2 \mu_r} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial H}{\partial \phi} \right) + k^2 \varepsilon_\theta H = M_s$$  \hspace{1cm} (20)

novel double negative GLHUAF relative EM parameters are created,

$$\varepsilon_\theta = \mu_\theta = \varepsilon_\phi = \mu_\phi = -\frac{R_2 - R_1}{r - R_1},$$  \hspace{1cm} (21)

$$\varepsilon_r = \mu_r = -\frac{R_2^2}{r^2} \frac{r - R_1}{R_2 - R_1}.$$  \hspace{1cm} (22)

By the inverse transformation of (19), in the (21), the

$$R = R_2 - (R_2 - R_1) \log \left( \frac{r - R_1}{R_2 - R_1} \right),$$  \hspace{1cm} (23)

$$\varepsilon_r = \mu_r = -\frac{1}{r^2} \frac{r - R_1}{R_2 - R_1} \left( R_2 - (R_2 - R_1) \log \left( \frac{r - R_1}{R_2 - R_1} \right) \right)^2.$$  \hspace{1cm} (24)

By installing the GLHUAF relative double negative electric permittivity and magnetic permeability (21) and (24) in the sphere annular layer $R_1 < r \leq R_2$ and installing the free space in the inner sphere $r \leq R_1$, the GLHUAF invisible cloak is created. Novel EM wave propagation through the GLHUAF invisible cloak are presented in Figure 37 to the Figure 42.

We discover a new GLHUA-1 invisible cloak and new EM parameters as follows:

$$\varepsilon_r = \mu_r = R_2^2/r^2,$$

$$\varepsilon_\theta = \varepsilon_\phi = \mu_\theta = \mu_\phi = (R_2 - R_1)/(r - R_1).$$  \hspace{1cm} (25)
We discover an analytical exact radial electric and magnetic wave propagation to satisfy the anisotropic radial electric and magnetic wave equation in the outer annular layer \( R_1 \leq r \leq R_2 \) with the above GLHUA-1 relative EM parameters and in the inner sphere \( 0 \leq r \leq R_1 \) with free space relative EM parameters 1, where \( 0 < R_1 < R_2 \). Moreover, we discover the analytical exact wave propagation to satisfy 3D Maxwell EM equation with GLHUA-1 anisotropic invisible cloak material in the annular layer \( R_1 \leq r \leq R_2 \) and free space inner sphere \( 0 \leq r \leq R_1 \).

It is totally different from the Pendry transform, by using GL no scattering modeling and inversion in [1-3], in this paper, we discover a new GLHUA-1 outer annular layer electromagnetic (EM) invisible cloak with relative parameter not less than 1. The outer layer cloak can be any positive thickness. In GLHUA-1 outer annular layer cloak, \( R_1 \leq r \leq R_2, R_1 > 0, R_2 > R_1, \varepsilon_r = \mu_r = \frac{R_3 - R_1}{R_2 - R_1} \). We discovered an exact analytical EM wave propagation solution of the full wave Maxwell EM equation in the GLHUA-1 outer cloak. In GLHUA-1 outer annular layer cloak, \( R_1 \leq r \leq R_2 \), for the electric source located outside of GLHUA-1 outer layer cloak, \( r_s > R_2 \), the exact analytic radial EM wave field solution of Maxwell radial EM differential equation is as follows

**VERSION 1**

\[
\begin{bmatrix}
E_r(r, \theta, \phi) \\
H_r(r, \theta, \phi)
\end{bmatrix} = 
\sum_{l=1}^{\infty} \sum_{m=-l}^{l} 
\begin{bmatrix}
E_{r,l,1}(r, \theta, \phi) \\
H_{r,l,1}(r, \theta, \phi)
\end{bmatrix} 
\cos \left( k(R_2 - R_1) \log \left( \frac{r - R_1}{R_2 - R_1} \right) \right) 

+ \sum_{l=1}^{\infty} \sum_{m=-l}^{l} 
\begin{bmatrix}
D_e(\theta, \phi) \\
D_h(\theta, \phi)
\end{bmatrix} 
Y_{l}^m(\theta, \phi) Y_{l}^{m*}(\theta_s, \phi_s),
\]

**VERSION 2**

\[
\begin{bmatrix}
E_r(r) \\
H_r(r)
\end{bmatrix} = 
\sum_{l=1}^{\infty} \sum_{m=-l}^{l} 
\begin{bmatrix}
E_{r,l,1}(r) \\
H_{r,l,1}(r)
\end{bmatrix} 
\cos \left( k(R_2 - R_1) \log \left( \frac{r - R_1}{R_2 - R_1} \right) + kR_2 \right) 

+ \sum_{l=1}^{\infty} \sum_{m=-l}^{l} 
\begin{bmatrix}
D_e(\theta, \phi) \\
D_h(\theta, \phi)
\end{bmatrix} 
Y_{l}^m(\theta, \phi) Y_{l}^{m*}(\theta_s, \phi_s).
\]

In the free space outside of the cloak, \( r > R_2, E_r(r) = E_{b,r}(r) \quad H_r(r) = H_{b,r}(r) \), on the boundary \( r = R_2, E_r(r) |_{r = R_2} = E_{b,r}(r) |_{r = R_2}, H_r(r) |_{r = R_2} = H_{b,r}(r) |_{r = R_2} \), and the derivative \( \partial E_r(r) / \partial r |_{r = R_2} = \partial H_r(r) / \partial r |_{r = R_2} \). The exact analytical radial EM wave solution show that there is no scattering from the GLHUA-1 outer layer cloak to disturb the incident EM wave outside of the cloak in free space. The exact analytical radial EM wave solution show that the EM wave can not propagation penetrate into the sphere \( r < R_1 \). In the free space inside of the concealment, \( r < R_1, E_r(r) = 0 \) and \( H_r(r) = 0 \). Therefore, GLHUA-1 outer layer is practicable invisible cloak with cloak concealment sphere \( r < R_1 \). We discover two novel wave in the GLHUA-1 outer layer cloak, \( R_1 \leq r \leq R_2 \), one is enter wave from incident which is absorbed, other one is created wave by GLHUA-1 cloak materials, the absorption wave and created wave make that the EM wave propagation through the GLHUA-1 cloak without exceeding light speed propagation. We propose a GLHUA-1 expansion method to find an exact analytical EM wave propagation in the GLHUA-1 double cloak and other spherical annular layer cloak, and completely solved invisible cloak problem in the spherical annular devices.

We discover negative space first in the world. The space in which the radial variable in the sphere coordinate is negative is called negative space. The space in which the radial variable in the sphere coordinate is positive or zero is called positive space, that is real three dimensional space we are living. The positive space is visible, but the negative space is invisible. The door between the positive space and negative space is 0. In normal general sciences, the positive space and negative space are not connected, the door 0 between the positive space and negative space is closed. Up to now, all scientists are working in the positive space, all sciences principle, for example, conservation of energy law, are hold and satisfied in the positive space. We proved that the radial EM equation and acoustic equation and their solutions in the sphere coordinate in the positive space can be
We create GLHUANP-2 radial transformation and GLHUA-2 invisible cloak. We create GLHUANP-3 radial transformation and GLHUA-3 invisible cloak.

The idea, discoveries, creations in this paper are breakthrough progress and different from all other cloak in other research publications. Our analytical EM wave in GLHUA-1, GLHUA-2 and GLHUA-3 cloak is undisputed evidence and rigorous proof to prove that the GLHUA-1, GLHUA-2 and GLHUA-3 double layer cloak are practicable invisible cloak without exceed light speed propagation.

Copyright and patent of the GLHUA-1, GLHUA-2 and GLHUA-3 EM cloaks and GL modeling and inversion methods and all rights are reserved by authors in GL Geophysical Laboratory.

PACS numbers: 13.40.-f, 41.20.-q, 41.20.jb, 42.25.Bs

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I. SCIENCES DISCOVER REPORT

A. New N dimensional Maxwell Equation

we propsed New N dimensional Maxwell Equation that is different from Gauge potential Maxwell field equation. We propose new N dimensional Curl operator. The 3 dimensional curl operator is well defined, However, for N $N \neq 3$ dimensional space, the curl operator is never be defined in other published papers before. As breakthough, we propose new curl operator in N dimensional space, $\text{Curl}_F = (\nabla \times)_N$:

$$\text{Curl}_F = (\nabla \times)_N = (\nabla \nabla \cdot - \nabla \cdot \nabla)^{\frac{1}{2}},$$

and New N Dimensional Maxwell Equations:

$$\nabla \times E = \frac{\partial}{\partial x_1} \hat{x}_1 \hat{E}_{x_1} + \frac{\partial}{\partial x_2} \hat{x}_2 \hat{E}_{x_2} + \frac{\partial}{\partial x_3} \hat{x}_3 \hat{E}_{x_3} = -\frac{\partial B}{\partial t},$$

$$\nabla \times H = \frac{\partial}{\partial x_1} \hat{x}_1 \hat{H}_{x_1} + \frac{\partial}{\partial x_2} \hat{x}_2 \hat{H}_{x_2} + \frac{\partial}{\partial x_3} \hat{x}_3 \hat{H}_{x_3} = \frac{\partial D}{\partial t} + \vec{J},$$

$$\nabla \cdot B = 0,$$

$$\nabla \cdot D = \rho,$$
Our N dimensional Maxwell equations have great role in the theoretical and practicable applications, in particular, in electromagnetic invisible cloaks and in super sciences. This breakthrough discovery has been announced in March 15, 2020 in Human Super Computational Society (in Figure 51). The second author Dr. Feng Xie in GL Geophysical Laboratory and Stanford University made main and great contribution in this breakthrough discovery and creation.

### B. Super Sciences and General Sciences

To study visible natural sciences and world and invisible social science, thinking sciences an human society are main objects of the general sciences. Inversely, To study the invisible natural sciences and invisible natural world and visible social and thinking sciences are main objects of the novel Super Sciences. N (N > 3) dimensional Maxwell equations is usefull to study combination sciences of natural science and social science. Our GLHUA-1 practicable double cloak and Easton LaChappelles mind control robot hands show that the new novel super science is being born.

### C. Negative Space and Positive Space

We discover negative space first in the world. We proposed new negative free space concept in [arXiv:1612.02583](https://arxiv.org/abs/1612.02583). We define the negative free space in which the sphere radial is negative or zero. Also we define the positive free space in which the sphere radial is positive or zero. that positive space is real three dimensional space we are living. The positive space is visible, but the negative space is invisible. The door between the positive space and negative space is 0. In normal general sciences, the positive space and negative space are not connected, the door 0 between the positive space and negative space is closed. Up to now, all scientists are working in the positive space, all sciences principle, for example, conservation of energy law, are hold and satisfied in the positive space. We proved that the radial EM equation and acoustic equation and their solutions in the sphere coordinate in the positive space can be analytic continuation into negative space with negative radial R < 0.

We proved that the homogeneous Maxwell electromagnetic equations and their solutions in the free positive space r ≥ 0 can analytically be continue to the negative free space r ≤ 0. The detailed proof will given in the following statement 1.

The second order anisotropic homogeneous Maxwell electromagnetic equation system in the sphere coordinate positive space

In paper [2], the anisotropic homogeneous Maxwell electromagnetic equation system in the sphere coordinate positive space, r ≥ 0, can be translated to separate second order radial electric and magnetic tense wave field equations, the homogeneous radial electric and magnetic tense wave field equation (5), (6) in paper [2] is

\[
\frac{\partial}{\partial r} \left[ \frac{1}{\varepsilon_r} \frac{\partial^2 E_r}{\partial r^2} \right] + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} \left[ \frac{E_r}{H_r} \right] + \frac{1}{\sin \theta \partial^2 \theta} \left[ \frac{E_r}{H_r} \right] + k^2 r^2 \left[ \mu_0 \frac{\partial \varepsilon_r E_r}{\varepsilon_0 \partial r} \right] = 0, \tag{28}
\]

Where \( E_r \) is the radial electric tense wave field, \( H_r \) is the radial magnetic tense wave field, other angular electric tense wave field \( E_\theta \), \( E_\phi \) and angular magnetic tense wave field \( H_\theta \), \( H_\phi \), will be found from (7)-(10) in paper [2], the electric permittivity are \( (\varepsilon_\theta, \varepsilon_\phi, \varepsilon_0, \varepsilon_r, \varepsilon_\theta) = (\varepsilon_\phi \varepsilon_\theta) \) are relative electric permittivity, the magnetic permeability \( (\mu_r, \mu_0, \mu_0, \mu_0) = (\mu_0) \) are relative magnetic permeability. \( \varepsilon_0 = 0.8854d - 11 \) is the basic electric permittivity in the free space, \( \mu_0 = 4\pi \times 10^{-7} \) is the basic magnetic permeability in the free space, \( c = 1/\sqrt{\varepsilon_0 \mu_0} \) is the light speed in the free space, \( k^2 = \omega^2 \varepsilon_0 \mu_0 \) is wave number, \( \omega^2 = 2\pi f \) is angular frequency, \( f \) is frequency. We define a new radial electric and magnetic wave field,

\[
\begin{aligned}
E(r, \theta, \phi) &= r^2 \varepsilon_r E_r(r, \theta, \phi) \quad H(r, \theta, \phi) = r^2 \mu_r H_r(r, \theta, \phi). \quad \tag{29}
\end{aligned}
\]

Substitute (29) into equation (28), the equation (28) becomes a new electromagnetic wave field equation

\[
\frac{\partial}{\partial r} \left[ \frac{1}{\varepsilon_r} \frac{\partial E_r}{\partial r} \right] + \frac{1}{r^{-1} \mu_0 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} \left[ \frac{E_r}{H_r} \right] + \frac{1}{r^{-1} \mu_0 \sin \theta \partial^2 \theta} \left[ \frac{E_r}{H_r} \right] + k^2 \left[ \mu_0 \frac{\partial \varepsilon_r E_r}{\varepsilon_0 \partial r} \right] = 0 \tag{30}
\]

In the next, we study the electric and magnetic tense wave field equation (28) and new electric and magnetic tense wave field equation (30). In the free space the relative electric permittivity \( \varepsilon_r = \varepsilon_\theta = \varepsilon_\phi = 1 \), the relative magnetic permeability \( \mu_r = \mu_0 = 1 \), the homogeneous radial electric and magnetic tense wave field equation (28) in the sphere coordinate with radial variable, \( r \geq 0 \), is reduced to

\[
\frac{\partial}{\partial r} \left[ \frac{1}{\varepsilon_r} \frac{\partial E_r}{\partial r} \right] + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} \left[ \frac{E_r}{H_r} \right] + \frac{1}{\sin \theta \partial^2 \theta} \left[ \frac{E_r}{H_r} \right] + k^2 r^2 \left[ \mu_0 \frac{\partial \varepsilon_r E_r}{\varepsilon_0 \partial r} \right] = 0 \tag{31}
\]

The new magnetic wave field equation (32) in free space becomes to,

\[
\begin{aligned}
E(r, \theta, \phi) &= r^2 \varepsilon_r E_r(r, \theta, \phi) \quad H(r, \theta, \phi) = r^2 \mu_r H_r(r, \theta, \phi). \quad \tag{32}
\end{aligned}
\]

\( E_r(r, \theta, \phi) \) is radial electric tense, \( \varepsilon_0 E_r(r, \theta, \phi) \) is radial electric displacement, \( H_r(r, \theta, \phi) \) is radial magnetic tense,
$\mu_0 H_r(r, \theta, \phi)$ is radial magnetic flux in the free space.

$$\begin{bmatrix} D_r \\ B_r \end{bmatrix} = \begin{bmatrix} \varepsilon_0 E_r(r, \theta, \phi) \\ \mu_0 H_r(r, \theta, \phi) \end{bmatrix}$$ (33) (33)

Substitute (32) into (31), the equation (31) becomes to electric and magnetic wave field equation in the free space,

$$\begin{bmatrix} \frac{\partial}{\partial R} \frac{\partial}{\partial R} \\ \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} \end{bmatrix} \begin{bmatrix} E(-R, \theta, \phi) \\ H(-R, \theta, \phi) \end{bmatrix}$$ (34) (34)

Statement 1: The homogeneous electromagnetic wave field equations and their solutions in the free positive space $r \geq 0$ can analytically be continue to the negative free space $R \leq 0$.

Proof: Suppose that the radial electric and magnetic wave field $E(r, \theta, \phi)$ and $H(r, \theta, \phi)$, satisfy the radial electromagnetic wave field equation (34). By direct calculation $E(r, \theta, \phi)$ and $H(r, \theta, \phi)$, also satisfy the equation (34).

For $R < 0$, Substitute $r = -R > 0$, into the (34), because

$$\begin{bmatrix} \frac{\partial}{\partial R} \frac{\partial}{\partial R} \\ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} \end{bmatrix} \begin{bmatrix} E(-R, \theta, \phi) \\ H(-R, \theta, \phi) \end{bmatrix}$$ (35) (35)

We have

$$\begin{bmatrix} \frac{\partial}{\partial R} \frac{\partial}{\partial R} \\ \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} \end{bmatrix} \begin{bmatrix} E(R, \theta, \phi) \\ H(R, \theta, \phi) \end{bmatrix}$$ (36) (36)

From equation (36), the homogeneous radial electric and magnetic equations (34) and their solutions $E(r, \theta, \phi)$ and $H(r, \theta, \phi)$ in the free positive space $r \geq 0$ has analytically been continue to the negative free space $R \leq 0$. Substitute $E(R, \theta, \phi) = R^2 E_r(R, \theta, \phi)$ and $H(R, \theta, \phi) = R^2 H_r(R, \theta, \phi)$ in (32) into (34), the homogeneous radial electric and magnetic field equations (31) and their solutions $E_r(r, \theta, \phi)$ and $H_r(r, \theta, \phi)$ in the free positive space $r \geq 0$ can analytically be continue to the negative free space $R \leq 0$. From (7), (8), (9) and (10) in paper [2], the homogeneous angular electromagnetic field equations and their solutions $E_\theta(r, \theta, \phi)$, $E_\phi(r, \theta, \phi)$, $H_\theta(r, \theta, \phi)$, $H_\phi(r, \theta, \phi)$, in the free positive space $r \geq 0$ can analytically be continue to the negative free space $R \leq 0$. The statement is proved.

D. A novel GLHUA-2 electromagnetic invisible cloak With Refractive Index $n \geq 1$ that absorbs incoming wave and creates outgoing wave

A novel GLHUA-2 electromagnetic invisible cloak

In this paper, we create a novel GLHUA-2 electromagnetic invisible cloak with electromagnetic parameters depended on radial $r$ in sphere annular layer $R_1 \leq r \leq R_2$. The relative refractive index of the invisible cloak is large or equal to one,

$$n(r) = (R_1 + (R_2 - R_1)^2/(r - R_1))/r \geq 1$$

and the relative angular electromagnetic parameters of the cloak are large or equal to one,

$$\varepsilon_\theta(r) = \varepsilon_\phi(r) = (R_2 - R_1)^2/(r - R_1)^2 \geq 1$$

$$\mu_\theta(r) = \mu_\phi(r) = (R_2 - R_1)^2/(r - R_1)^2 \geq 1$$

$$\mu_\tau(r) = 1$$

$$\varepsilon_\tau(r) = 1$$

if $R_2 \geq 2R_1, \varepsilon_\tau(r) \geq 1$, $\mu_\tau(r) \geq 1$, if $R_2 < 2R_1, \varepsilon_\tau(r) < (R_2 - R_1)^2/R_1^2 > 0$, $\mu_\tau(r) \geq (R_2 - R_1)^2/R_1^2 > 0$ these electromagnetic parameters and their properties make that analytical exact electromagnetic tense and magnetic flux and electric displacement wave propagation in the cloak are physical bounded and without exceeding light propagation[1-3]. The invisible cloak is created by our sphere radial transformation, $r(R) = R_3 - (R_2 - R_3)^2/(R + R_1)$, which maps negative infinity, $-\infty$, to $R_1$, and maps $-R_2$ to $R_2$ and derivative of the transformation at $-R_2$ equal to 1. i.e. the transformation maps the sphere annular domain $[-\infty, -R_2]$ in negative free space to sphere annular domain $[R_1, R_2]$ in the positive space; New negative free space we proposed
is in which the sphere radial is negative, \( r \leq 0 \). The conventional free space with sphere radial \( r \geq 0 \) is defined to be positive free space. Maxwell electromagnetic homogeneous equation and its solution in the positive free space can analytically be continue to the negative space. The cloak material is not the conventional light transmitting material. The wave rays entering the cloak does not continuous redirection curvly transmitting the cloak. The incoming electromagnetic wave which entering the cloak are all absorbed by the cloak material. The cloak material creates outgoing wave to propagate to outside of the cloak. In simplicity, the novel GLHUA-2 cloak material absorbs incident electromagnetic wave which entering cloak, and creates outgoing wave which propagate to outside of the cloak that make the device to be invisible cloak. The absorption and creating effects can be happen in same time but different location. The absorption and creating effects can be happen in same location but different time. Absorption and creating time-space action of the electromagnetic wave is light propagation in free time-space or material time-space. Because on the boundary sphere surface \( r = R_2 \), the refractive index and electromagnetic parameters are equal to one and continuous with that in free space \( r \geq R_2 \), by Sell Law and Fresnel equations, there is no any scattering wave ray from the cloak to reflect to the free space, moreover, on the boundary spherical surface \( r = R_2 \) between annular layer cloak \( R_1 \leq r \leq R_2 \) and outside of cloak \( r \geq R_2 \), the cloaking electric and magnetic wave field (49) and the free space incident electric and magnetic wave (62) and their derivative are continuous. This continuity does ensure that there is no additional scattering from the cloak to disturb the incident wave in free space outside of cloak \( r \geq R_2 \), thus the GLHUA-2 cloak is invisible, where the cloaking electric and magnetic wave field is defined to be the electric and magnetic wave field is annular layer cloak \( R_1 \leq r \leq R_2 \), the free space incident electric and magnetic wave field is defined to be the incident electric and magnetic wave in free space \( r \geq R_2 \). Because there is no wave propagation in outside of the full negative and positive free space, i.e. outside of \(( -\infty < r < +\infty )\), the GLHUANP-2 transformation maps the negative spherical annular manifold domain, \(-\infty < R \leq -R_2 \) in the negative space to the spherical annular domain, \( R_1 \leq r \leq R_2 \) in the positive space, thus there is no any electromagnetic wave propagation in the inner sphere \( r \leq R_2 \), moreover, in the analytical electric and magnetic wave field (49) \( \lim_{r \to R_1} R(r) = \lim_{r \to R_1} ( -R_1 - (R_2 - R_1)^2 / (r - R_1) ) = -\infty \), and when \( r \to R_1 \), the phase velocity trets to zero, thus in any finite time the incident wave propagation can not arrive the inner boundary \( r = R_1 \), thus there is no any wave propagation can be penetrated into the inner sphere \( r < R_1 \). The novel GLHUANP-2 transformation map negative space to positive space.

GLHUA-2 invisible cloak is created by a novel GLHUANP-2 transformation. The GLHUANP-2 transformation, \( r(R) = R_1 - \frac{(R_2 - R_1)^2}{R_1 + R_2} \), \( -\infty < R \leq -R_2 \), \( \lim_{R \to -R_2} aR^r(R) = 1 \), maps the negative spherical annular manifold domain, \(-\infty < R \leq -R_2 \), in the negative space to the spherical annular domain, \( R_1 \leq r \leq R_2 \) in the positive space. The detailed transformation will mathematical formulated in the following section. We proposed new negative free space concept [1]. we define the negative free space in which the sphere radial is negative or zero. Also we define the positive free space in which the sphere radial is positive or zero. We proved that the homogeneous Maxwell electromagnetic equations and their solutions in the free positive space \( r \geq 0 \) can analytically be continue to the negative free space \( r \leq 0 \). The detailed proof will given in the above statement 1.

The analytical electromagnetic wave propagation in the GLHUA-2 invisible cloak

By the GLHUANP-2 transformation, we find the analytical electromagnetic wave propagation in the GLHUA-2 cloak, by which we analytically proved that the GLHUA-2 cloak is invisible cloak.

The following figures and their explantions show that the GLHUA-2 invisible cloak material absorb the incident wave and create outgoing wave to propagate to outside of the cloak, that make the device which including sphere annular layer, \( R_1 \leq r \leq R_2 \), with the material and free space inner sphere \( r \leq R_1 \) to be invisible cloak. The electromagnetic wave created by the cloak materials is called the created electromagnetic wave. In the following figures, there are two magnetic wave propagation in the annular layer \( R_1 \leq r \leq R_2 \), one is incident incoming wave (red line (1)), other wave is created wave (blue line (2)), which is created by the cloak materials. The both magnetic wave front propagation red line (1) and blue line (2) in the annular cloak are separated and disconnected, they are only connected in a subsurface. In following figures, \( R_2 = 2R_1 \epsilon_\theta = \mu_\theta = \mu_\phi = (R_2 - R_1)^2 / (r - R_1)^2 \geq 1 \), with Guass source. We define first relative time step to be in which the incident wave arrive and tangent to the sphere surface \( r = R_2 \) in the right side of the cloak.

Figure 29, At first relative time step, the incident magneic wave front (red line (1)) coming and tangent to the sphere surface, the novel curve wave front (blue line (2)) is created by material.

Figure 30, At relative 5th time step, the magneic wave front (red line (1)) incoming to right side of cloak, the material created curve wave front (blue line (2)) expand up and down.
Figure 31. At relative 14 time step, the magnetic wave front (red line (1)) entering to right side of cloak, the material created curve wave front (blue line (2)) form wave front on the left side of cloak. Both wave front near each other but no connected.

Figure 32. At relative 15 time step, the magnetic wave front (red line (1)) entering to right side of cloak, the material created curve wave front (blue line (2)) form wave front in the left side of the cloak, the red incoming wave (1) and blue created wave front (2) are connected.

Figure 33. At relative 16 time step, the curve wave front created by cloak material and incident wave in outside of cloak form wave front in the left side of the cloak (blue line (2)), the incoming magnetic wave front red line (1) right side of cloak is disconnected with created wave (blue line (2)).

Figure 34. At relative 20 time steps, the curve wave front created by the cloak material is connected to the incident wavefront outside the cloak to form a wavefront and propagate to the left side of the cloak (blue line (2)). The incoming magnetic wave (red line (1)) is separated from the created wave (blue line (2)), and the incoming wavefront (red line (1)) is shrink, and wavefront (red line (1)) is behind from the created wavefront (blue line (2)).

Figure 35. At relative 25 time steps, the curve wave front created by cloak material and incident wave front in outside of cloak are connected to form wave front that propagate to boundary, the left side of the cloak (blue line (2)); The incoming magnetic wave (red line (1)) is separated from the created wave (blue line (2)), and wavefront (red line (1)) is behind from the created wavefront (blue line (2)), and the incoming wavefront (red line (1)) is absorbed and shrink.

Figure 36. At relative 25 time steps, the material created curve wave front propagation (blue line (2)) has already been out of cloak and did not disturb incident wave in outside of the cloak and make the cloak is invisible; the incoming magnetic wave front (red line (1)) is shrink to a circle and absorbed and can not be penetrated to the concealment. The inner sphere is really cloaked concealment.

The GLHUANP-2 transformation formulation let $R$ be sphere radial variable in the negative free space, $r$ be sphere radial variable in physical positive space. We propose a new Negative space to Positive space GLHUANP-2 transformation formulation,

$$r(R) = R_1 - \frac{(R_2-R_1)^2}{R}, \quad -R_2 \leq -R_2, R_1 \leq r(R) \leq R_2, \quad R_1 \leq r \leq R_2,$$  \quad (37)

$$\lim_{R \to -\infty} r(R) = R_1, \quad (38)$$

$$\frac{d r}{d R}(-R_2) = 1, \quad (39)$$

The function $r(R)$ in (37) is monotone increase and continuous differentiable function of radial variable $R$ in the negative space, $\infty < R \leq -R_2$, the angular variable $\theta$ and $\phi$ of in the sphere coordinate are not transformed under the GLHUANP-2 transformation. The inverse function of the function (37)

$$R(r) = -R_1 - \frac{(R_2-R_1)^2}{r}, \quad R_1 \leq r \leq R_2, -\infty < R(r) \leq -R_2, \quad (40)$$

Thus, the transformation (37)-(39) one to one and onto maps the domain $[-\infty, -R_2]$ in negative free space to spherical annular layer $[R_1, R_2]$ in positive free space. The negative free space domain $r \geq R_2$ is not transformed under the GLHUANP-2 transformation,

$$r = R, r \geq R_2, R \geq R_2 \quad (41)$$

The GLHUANP-2 invisible cloak with refractive index $n(r) \geq 1$ and without exceeding light speed.

GLHUANP-2 invisible cloak relative anisotropic materials are created by GLHUANP-2 transformation.

To substitute the GLHUANP-2 transformation (37) into (34) in this paper, the radial isotropic homogeneous electric and magnetic field equation (34) in negative free space domain $-\infty \leq R \leq -R_2$ is translated to the radial anisotropic material electric and magnetic equation in annular layer in the positive space $R_1 \leq r \leq R_2$,

$$\frac{\partial}{\partial r} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} E(R, \theta, \phi) \right] + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left[ \frac{E(R, \theta, \phi)}{r \sin \theta} \right] + \frac{k^2}{r^2} \left[ \frac{E(R, \theta, \phi)}{r^2} \right] = 0,$$

$$-\infty < R \leq -R_2, R_1 \leq r(R) \leq R_2,$$  \quad (42)

To compare (42) with (30), we create a novel GLHUANP-2 electromagnetic anisotropic invisible cloak material. The relative radial electric permittivity $\varepsilon_r$ and magnetic permeability $\mu_r$ are,

$$\varepsilon_r(r) = \frac{\varepsilon_r(r)}{\mu_r(r)} = \frac{1}{r^2} \frac{d}{dr} \frac{E(R, \theta, \phi)}{r^2} = \frac{1}{R_1} \left( \frac{R_2-R_1}{R_2-R_1} \right), \quad (43)$$

The angular relative electric permittivity and magnetic permeability are

$$\varepsilon_{\theta}(r) = \frac{\varepsilon_{\theta}(r)}{\mu_{\theta}(r)} = \frac{1}{(r-R_1)^2}, \quad R_1 \leq r \leq R_2,$$  \quad (44)

$$\varepsilon_r = \varepsilon_r = \varepsilon_{\phi} = \varepsilon_\phi = 1, \quad r > R_2, \quad (45)$$
The electromagnetic refractive index
\[ n^2(r) = \varepsilon_\varphi(r) \mu_r(r) \]
\[ = \frac{1}{r^2} \left( R_1 + \frac{(R_2 - R_1)^2}{(r - R_1)^2} \right)^2 , \tag{46} \]
\[ R_1 \leq r \leq R_2, \]
\[ n = 1, r \geq R_2 \tag{47} \]

The properties of GLHU-2 invisible cloak materials (1). The electromagnetic parameters are continuous on the sphere surface, \( r = R_2 \) By equation (43)
\[ \varepsilon_r(R_2) = \mu_r(R_2) \]
\[ = \frac{1}{R_1^2} \left( R_1 + \frac{(R_2 - R_1)^2}{(r - R_1)^2} \right)^2 , \tag{48} \]
\[ R_1 \leq r \leq R_2, \]
\[ n = 1, r \geq R_2 \tag{47} \]

(2) The refractive index are continuous on the sphere surface, \( r = R_2 \) by equation (46), (48) and (49)
\[ n^2(R_2) = n^2(R_2) = \varepsilon_\varphi(R_2) \mu_r(R_2) \]
\[ = \frac{1}{R_1^2} \left( R_1 + \frac{(R_2 - R_1)^2}{(r - R_1)^2} \right)^2 , \tag{50} \]
\[ R_1 \leq r \leq R_2, \]
\[ n^2(R_2) = 1 \tag{50} \]

(3) The refractive index large than or equal to one
\[ \frac{d}{dr} n(r) = \frac{d}{dr} \varepsilon_\varphi(r) \mu_r(r) \]
\[ = \frac{d}{dr} \left( R_1 + \frac{(R_2 - R_1)^2}{(r - R_1)^2} \right)^2 , \tag{51} \]
\[ R_1 \leq r \leq R_2, \]
\[ \frac{d}{dr} n(r) \leq 0, \tag{51} \]

(4) The relative radial electric permittivity \( \varepsilon_r \) and magnetic permeability \( \mu_r \) in (43) are large than positive constant which large than zero, the electric field and magnetic field are bounded in the GLHU-2 invisible cloak.
\[ \frac{d}{dr} \varepsilon_r(r) = \frac{d}{dr} \mu_r(r) \]
\[ = \frac{2}{r^2} \left( R_1 + \frac{(R_2 - R_1)^2}{(r - R_1)^2} \right)^2 \]
\[ + 2 \left( R_1 + \frac{(R_2 - R_1)^2}{(r - R_1)^2} \right)^2 \]
\[ = \frac{2}{r^2} \left( R_1 + \frac{(R_2 - R_1)^2}{(r - R_1)^2} \right)^2 \]
\[ \frac{d}{dr} \varepsilon_r(r) = \frac{2}{r^2} \left( R_1 + \frac{(R_2 - R_1)^2}{(r - R_1)^2} \right)^2 \]
\[ = \frac{2}{r^2} \left( R_1 + \frac{(R_2 - R_1)^2}{(r - R_1)^2} \right)^2 \]
\[ R_1 \leq r \leq R_2, \tag{53} \]

\[ 2R_1 > R_2, \tag{54} \]
then \( R_1^2 - (R_2 - R_2)^2 > 0 \), in (53), \( \frac{d}{dr} \varepsilon_r(r) = \frac{d}{dr} \mu_r(r) > 0, \)
\[ \varepsilon_r(r) = \mu_r(r) \]
\[ \geq \varepsilon_r(R_1) = \mu_r(R_1) \]
\[ = 1 \]

\[ 2R_1 = R_2, \tag{56} \]

(5) The angular relative electric permittivity and magnetic permeability in (44) are large than or equal to one
\[ \varepsilon_\varphi(r) = \mu_\varphi(r) \]
\[ = \mu_\varphi(r) = \mu_\varphi(r) \]
\[ = 1 \]

(5) The angular relative electric permittivity and magnetic permeability in (44) are large than or equal to one
\[ \frac{\partial^2 \varphi}{\partial \varphi^2} \frac{\partial \varphi}{\partial \varphi} \left[ \begin{array}{c} E \\ H \end{array} \right] + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} \left[ \begin{array}{c} E \\ H \end{array} \right] = \frac{J_\varphi}{M_\varphi}, \tag{61} \]

Suppose that the \( J_\varphi, M_\varphi \) electromagnetic source is located at \( \vec{r}_s = (r_s, \theta_s, \phi_s) \) outside of the cloak in the free space, \( r_s > R_2 \) and in \( \vec{r} \) direction, by the sphere harmonic expansion of the (41)-(43) in [2], the incident electromagnetic wave is
\[ \sum_{m=-l}^{l} \left[ \frac{E_l(r, \vec{r}_s)}{H_l(r, \vec{r}_s)} \right] Y_l^m(\theta, \phi) Y_l^m(\theta_s, \phi_s), \tag{62} \]
the incident electric and magnetic wave field \( E_i(r, r_s) \), \( H_i(r, r_s) \), in (62) satisfy equation (61), \( E_{i,l}(r, r_s) \), \( H_{i,l}(r, r_s) \) is the \( l \) harmonic sub radial wave field in the sphere harmonic expansion (62), the subscript \( i \) denotes incident wave, subscript \( l \) denotes \( l \) harmonic component.

**Statement 2**: Suppose that the electric and magnetic wave field

\[
\begin{bmatrix}
E_{i,1} \\
H_{i,1}
\end{bmatrix} = \begin{bmatrix}
R_{ji}(kR) \\
R_{ji}(kR)
\end{bmatrix},
\]

\[
\begin{bmatrix}
E_{i,2} \\
H_{i,2}
\end{bmatrix} = \begin{bmatrix}
R_{ni}(kR) \\
R_{ni}(kR)
\end{bmatrix},
\]

are the two linear independent analytical solution of the following \( l \) harmonic electric and magnetic wave field equations by using the sphere harmonic decomposition of the equation (34)

\[
\frac{\partial}{\partial r} \frac{\partial}{\partial r} \begin{bmatrix}
E_i(r) \\
H_i(r)
\end{bmatrix} - \frac{l(l+1)}{r^2} \begin{bmatrix}
E_i(r) \\
H_i(r)
\end{bmatrix} + k^2 \begin{bmatrix}
\frac{\partial}{\partial r} E_i(r) \\
\frac{\partial}{\partial r} H_i(r)
\end{bmatrix} = 0,
\]

in the free negative space domain \(-\infty < R \leq -R_2\), then

\[
\begin{bmatrix}
E_i(r) \\
H_i(r)
\end{bmatrix} = \begin{bmatrix} -R_1 - \frac{(R_2 - R_1)^2}{(r - R_1)} \\
\frac{A_{E,i} B_{E,i}}{A_{H,i} B_{H,i}}
\end{bmatrix},
\]

\[
j_i \left( k \frac{R_1 - R_1 - \frac{(R_2 - R_1)^2}{(r - R_1)}}{R_1 - \frac{(R_2 - R_1)^2}{(r - R_1)}} \right),
\]

\[
n_i \left( k \frac{R_1 - R_1 - \frac{(R_2 - R_1)^2}{(r - R_1)}}{R_1 - \frac{(R_2 - R_1)^2}{(r - R_1)}} \right),
\]

is the analytical solution of the following \( l \) harmonic magnetic wave field equation (65) in the annular layer \( R_1 \leq r \leq R_2 \), with anisotropic materials (43) and (44),

\[
\frac{\partial}{\partial r} \begin{bmatrix}
\frac{1}{\varepsilon_r} \frac{\partial E_i(r)}{\partial r} \\
\varepsilon_r \frac{\partial H_i(r)}{\partial r}
\end{bmatrix} - \frac{l(l+1)}{r^2} \begin{bmatrix}
E_i(r) \\
H_i(r)
\end{bmatrix} + k^2 \begin{bmatrix}
\frac{\partial}{\partial r} E_i(r) \\
\frac{\partial}{\partial r} H_i(r)
\end{bmatrix} = 0,
\]

where constants \( A_{E,i}, B_{E,i} \) and \( A_{H,i}, B_{H,i} \), are uniquely determined by the continuous of the radial electric and magnetic wave field and their derivatives on the boundary spherical surface \( r = R_2 \).

**Statement 3**: The analytical electric and magnetic wave field propagation through GLHUA-2 cloak in the annular layer \( R_1 \leq r \leq R_2 \) with anisotropic materials in (43) and (44) is

\[
\begin{bmatrix}
E(r, r_s) \\
H(r, r_s)
\end{bmatrix} = \begin{bmatrix} -R_1 - \frac{(R_2 - R_1)^2}{r - R_1} \\
\sum_{l=1}^{\infty} \left( \begin{bmatrix} A_{E,i} & B_{E,i} \\
A_{H,i} & B_{H,i}
\end{bmatrix} \right) \begin{bmatrix} j_l(k(r - R_1 - \frac{(R_2 - R_1)^2}{r - R_1}) \\
n_l(k(r - R_1 - \frac{(R_2 - R_1)^2}{r - R_1}))
\end{bmatrix}
\end{bmatrix}^{\frac{l}{m=-l}} \begin{bmatrix} D_E(\theta, \phi) \\
D_H(\theta, \phi)
\end{bmatrix} Y_{lm}^m(\theta, \phi) Y_{lm}^m(\theta_s, \phi_s),
\]

the analytical electric and magnetic wave field (68) shows that the GLHUA-2 cloak is complete invisible cloak with anisotropic materials in (43) and (44) in the annular layer \( R_1 \leq r \leq R_2 \).

**E. An Active GLHUA-3 Invisible Cloak With Refractive Index \( n \geq 1 \) that absorbs incoming wave and creates outgoing wave**

We propose other GLHUANP-3 transformation which maps \( R \rightarrow -\infty \) in basic negative space to \( r = R_1 \) in physical positive space and maps \( R \rightarrow -R_2 \) in basic negative space to \( r = R_2 \) in physical positive space.

\[
r = R_1 + (R_2 - R_1)e^{\frac{R+R_2}{R_2-R_1}},
\]

(69)

And inverse transformation

\[
R = -R_2 + (R_2 - R_1) \log((r - R_1)/(R_2 - R_1)),
\]

(70)

The invisible cloak GLHUA-3 relative electric permittivity and magnetic permeability (71) and (72) are created by the above radial transformation

\[
\varepsilon_r(r) = \mu_r(r) = \frac{(-R_2 + (R_2 - R_1) \log((r - R_1)/(R_2 - R_1)))^2}{R_2 - R_1},
\]

(71)

\[
R_1 \leq r \leq R_2,
\]

\[
\varepsilon_\theta(r) = \varepsilon_\phi(r) = \mu_\theta(r) = \mu_\phi(r)
\]

(72)

and the relative electromagnetic parameters in \( r \leq R_1 \) or \( r \geq R_2 \) are

\[
\varepsilon_r = \varepsilon_\phi = \mu_\phi = \mu_r = \mu_\theta = 1,
\]

(73)

The parameters have the following properties,

\[
\varepsilon_r(R_2) = \mu_r(R_2) = 1,
\]

\[
\varepsilon_\theta(R_2) = \varepsilon_\phi(R_2) = \mu_\theta(R_2) = \mu_\phi(R_2) = 1,
\]

(74)

\[
n^2(r) = \varepsilon_\theta \mu_\theta = \varepsilon_\phi \mu_\phi = \frac{(-R_2 + (R_2 - R_1) \log((r - R_1)/(R_2 - R_1)))^2}{R_2 - R_1} \geq 1,
\]

(75)

\[
R_1 \leq r \leq R_2,
\]
From property (77) and Sell Law and Fresnel equations, on the boundary, \( r = R_2 \), between the sphere and the free space, there is no any reflection from the sphere \( r \leq R_2 \), thus, the cloak is invisible. From property (75) and (76), the refractive index large or equal 1, thus the electromagnetic or light wave propagation in this invisible cloak is no exceeding light speed. To substitute relative parameters (71) and (72) into the annular cloak [1], for each sphere Harmonic number \( l \), the radial electric and magnetic wave equation, reduced to the sub EM equation. Here the radial magnetic equation becomes the radial \( l \) sub magnetic equation,

\[
\frac{\partial}{\partial r} \left( \frac{r - R_1}{(R_2 - R_1)} \right) \frac{\partial H_l}{\partial r} + \frac{1}{(l+1)(R_2 - R_1)} \frac{H_l}{r} - \frac{(-R_2 + (R_2 - R_1) \log \left( \frac{r - R_1}{R_2 - R_1} \right) \left( r - R_1 \right)}{r - R_1} H_l = 0,
\]

where \( H_l = r^2 B_{r,l} \), \( B_{r,l} \) is \( l \) component of the radial magnetic flux, the radial sub electric equation is same type equation as (78), using our method in [1], to solve equation (78), we obtain analytical \( l \) component radial magnetic wave solution

\[
H_l(r, r_\ast) = \left(-R_2 + (R_2 - R_1) \log((r - R_1)/(R_2 - R_1)) \right) (a_l(r_\ast)j_l(k(-R_2 + (R_2 - R_1) \log((r - R_1)/(R_2 - R_1)))) + b_l(r_\ast)n_l(k(-R_2 + (R_2 - R_1) \log((r - R_1)/(R_2 - R_1))))),
\]

Where \( l = 1, 2, 3 \ldots \) \( j_l \) is the Spherical Bessel Function of the First Kind, \( n_l \) is the Spherical Bessel Function of the Second Kind, or spherical Neumann functions, The two constants \( a_l \) and \( b_l \) are uniquely determined by continuous conditions of the incident wave and its derivative with respect to the radii \( r \) on the boundary \( r = R_2 \).

Because the variable

\[
-\infty < R = -R_2 + (R_2 - R_1) \log((r - R_1)/(R_2 - R_1)) \leq -R_2,
\]

So, \( H_l(r, r_\ast) \) in (79) is bounded continuous differentiable physical wave. Because the \( n(R_2) = 1 \) in (77), there is no any reflection from the sphere surface, \( r = R_2 \), and thus the continuous boundary conditions uniquely determine that all incident light wave to be transmitted into the sphere \( r \leq R_2 \) in (79), in any finite time, the wave (79) can not be arrived to the inner spherical surface \( r = R_1 \), and no wave to be propagation penetrated into the inner sphere, thus the inner sphere \( r \leq R_1 \) is cloaked concealment.

We discover that the electromagnetic spherical coordinate equation and its solution can analytically be continuation to a negative space with negative radii \( r \leq 0 \). The space with sphere radii \( r \geq 0 \) is called the positive space. The space with sphere radii \( r \leq 0 \) is called the negative space. In the following paragraph, we describe the analytical radial magnetic wave propagation through the invisible cloak with the relative materials (71) and (72) in the annular cloak \( R_1 \leq r \leq R_2 \), \( R_1 = 0.5m \) and \( R_2 = 1.0m \). The analytical radial magnetic wave is

\[
H(r, \theta, \phi) = \sum_{l=1}^{LM} H_l(r, r_\ast) D_h(\theta, \phi) \sum_{m=-l}^{l} Y_l^m(\theta, \phi) Y_l^{m*}(\theta_\ast, \phi_\ast),
\]

Where \( H(r) = r^2 B_l(r) \), \( B_l(r) \) is the radial magnetic flux component. In the free space, \( r \geq R_2 \) \( H(r, \theta, \phi) = H_l(r, \theta, \phi) = r^2 B_{l,r}(r, \theta, \phi) \), in inner sphere \( r \leq R_1 \) \( H(r, \theta, \phi) = 0 \), thus the inner sphere is cloaked concealment, the \( B_{l,r}(r, \theta, \phi) \) is incident radial magnetic flux wave in free space, \( r \geq R_2 \) the incident magnetic wave is excited by the electric current time pulse source located in the \((r_\ast, \theta_\ast, \phi_\ast) = (18.3m, r/2, 0)\) with direction \( \vec{y}, \vec{y} = (\sin \theta \sin \phi, \cos \theta \sin \phi, \cos \phi) \). The following figures show the above radial magnetic wave propagation though the invisible cloak without exceeding light speed. There are two waves propagates in the annular layer \( R_1 \leq r \leq R_2 \), one is incident incoming wave, other wave is created wave which is created by the cloak materials, the wave fronts are colored by bold red. When the incident radial magnetic wave excited by right impulse source is propagation in \( r > R_2 \), right and outside of the cloak sphere, the invisible cloak material creates the created wave colored by red in the right part of the inner boundary \( r = R_1 \) in the annular layer. The radial electric wave propagation through the GLHUA-3 cloak is display in the figure 43-50.

II. INTRODUCTION

In 2000, in our Global Integral and Local Differential GILD EM modeling and inversion simulation [4], we observed a strange double layered cloak phenomena in Fig.2 in [5] and in Fig.11 of [5]. The double layered cloak phenomena was called GILD phenomena has been published in SEG Expanded Abstracts, Vol. 21, No. 1, 692-695, 2002, by Jianhua Li at al. [5]. The double layer cloak GILD phenomena is nonzero variation solution imaging of the zero scattering inversion which cloaked the object of the scattering inversion imager. Since then, we created the Global and Local modeling and inversion [6] in 2006,
and used it to create our novel GLLH [11] and GLHUA [1-3] invisible EM cloak. A New computational mirage has been published in PIERs abstract 2005 in Hang Zhon of China, 296[13]. In 2006, using transform optics, Pendry et al proposed first invisible cloak [7] with infinite speed and exceeding light speed propagation. Ulf Leonhardt et al. [27] proposed a cloak with finite speed based on a Euclid and non Euclid joint transform optics. However, ULF cloak [27] still has exceeding light speed difficulty and without theoretical full wave proof and without full wave simulation verification. In 2009, we published paper in arXiv[10], A Double Layer Electromagnetic Cloak And GL EM Modeling, in this paper, the inner layer cloak with relative refractive index not less than 1 was proposed first in the world[10]. In 2010, first in the world, we published GLLH EM Invisible Cloak with Novel Front Branching and without Exceed Light Speed Violation in ArXiv1005.3999 in the May of 2010 [11] and in PIERs 2010[12]. Next year, in 2011, Ulf Leonhardt et al. published paper Invisibility cloaking without superluminal propagation in Arxiv: 1105.0164v3[14]. Ulf did write a complete review and references of cloak history in his paper[14], he did criticism for many papers about Pendrys cloak and all published cloak papers and wrote that The fundamental problem is that perfect invisibility (Pendry cloak[7]) requires that light should propagate in certain cloaking regions with a superluminal phase velocity that tends to infinity. In his paper [14] in 2011, Ulf cited our paper arXiv:1005.3999 in 2010 as his reference [35] and wrote that The preprint [35], proposes a different method for cloaking without superluminal propagation. However, it is needed more communication for understand between our GLHUA cloak and ULF cloak. Up to now, ULFs cloak is still needed to verify by full wave theoretical analysis and is still needed to verify by full wave computational simulation. Pendry cloak [7] is created by a linear transformation which maps zero to \( R_1 \) and \( R_2 \) to \( R_2 \) in positive space. Therefore, Pendry cloak can not be realized because Pendry cloak has infinite speed and exceeding light speed propagation that is fundamental and can not be solved difficults.

We created GILD modeling and inversion method [4][5][6][8] and used it to discover double cloak phenomena in 2000. We did create GL double layer cloak and inner layer cloak without exceeding light speed in 2009[10]. We used our Global and Local GL no scattering and inversion to create GLLH double layer cloak with relative refractive index not less than 1 and without exceeding light speed propagation that published in 2010 [11] first in the world. Practicable GLHUA double layer invisible cloak with relative EM parameter not less than 1 in 2016 arXiv:1612.02857[11] and its theoretical proof in arXiv:1701.00534[2], we create an GLHUA analytical modeling method and a GLHUA-1 invisible cloak and an exact analytical EM wave propagation in the GLHUA-1 double layer cloak in this paper version 1 in 2017. We propose a negative space concept in version 13 of our paper arXiv:1612.02857. We proved statement 1 that "The homogeneous electromagnetic wave field equations and their solutions in the free positive space \( r \geq 0 \) can analytically be continue to the negative free space \( r \leq 0 \) in this paper. We propose Novel thrsformation GLHUANP-1 which maps negative infinity, \(-\infty\) in negative space, to \( R_1 \) in positive space, and maps \( R_2 \) to \( R_2 \). We propose Novel transformation GLHUANP-2 and GLHUANP-3 which map negative infinity, \(-\infty\) in negative space, to \( R_1 \) in positive space, and maps \(-R_2\) to \( R_2 \) in negative space to \( R_2 \) in positive space. We create practicable GLHUA-2 and GLHUA-3 invisible cloak with the relative refractive index large or equal to one and without exceeding light speed propagation difficult. When \( 2R_1 = R_2 \), the GLHUA-2 cloak materials are GLHUA cloak materials with \( \alpha = 2 \), GLHUA-2 cloak materials is created by GLHUANP-2 transformation in this paper, GLHUA cloak materials is created by GL no scattering modeling inversion in [1]and [2]. The relative angular EM parameters

\[
\varepsilon_\theta = \varepsilon_\phi = \mu_0 = \mu_\phi = \frac{(R_2 - R_1)}{(r - R_1)}
\]

of GLHUA cloak with \( \alpha = 1 \) and GLHUA-1 cloaks can be created by GLHUANP-3 transformation. The relative angular EM parameters of GLLH invisible cloak in (5) of arXiv:1005.3999[11]

\[
\varepsilon_\theta = \varepsilon_\phi = \mu_0 = \mu_\phi = \frac{2 \log(e^{1/R_{1}}(R_0 - R_1)/(r - R_1))}{2 \log(e^{1/R_{1}}(R_0 - R_1)/(r - R_1))}
\]

\[
\text{GLLH (5)}
\]

can be created from other nove GLLHNP transformation which map negative infinity, \(-\infty\) in negative space, to \( R_1 \) in positive space, and maps \(-R_2\) in negative space to \( R_2 \) in positive space. There is typo in (5) of the our paper arXiv:1005.3999[11]. The corrected (5) of [11] is the above formula GLLH(5).

### III. Practicable GLHUA-1 OUTER LAYER CLOAK WITH RELATIVE PARAMETER NOT LESS THAN 1

In this section, we created a novel practicable GLHUA-1 outer annular layer EM invisible cloak with relative parameter not less than 1.

In the outer annular layer \( R_1 \leq r \leq R_2 \) the radial relative electric permittivity and magnetic permeability are

\[
\varepsilon_r(r) = \mu_r(r) = \frac{R_2^2}{R_1^2}, \quad (82)
\]

They are monotone decrease function and

\[
\frac{R_2^2}{R_1^2} \geq \varepsilon_r(r) = \mu_r(r) = \frac{R_2^2}{R_1^2} \geq 1, \quad (83)
\]
the angular relative electric permittivity and magnetic permeability are
\[ \varepsilon_\phi(r) = \mu_\theta(r) = \mu_\phi(r) = \frac{R_2 - R_1}{r - R_1}, \] (84)

They are monotone decrease function and
\[ \infty \geq \varepsilon_\theta(r) = \varepsilon_\phi(r) = \mu_\theta(r) = \mu_\phi(r) = 1. \] (85)

From (82) and (84), on the boundary \( r = R_2 \), the relative parameters in GLHUA-1 outer layer cloak are equal to 1 and are continuous across the outer boundary \( r = R_2 \). Summary, we have the following theorem 1.

**Theorem 1**: In GLHUA-1 outer layer cloak, \( R_1 \leq r \leq R_2 \) the radial relative electric permittivity and magnetic permeability in (82) and the angular relative electric permittivity and magnetic permeability in (84) are not less than 1 and are continuous across the outer boundary \( r = R_2 \).

IV. EXACT ANALYTICAL EM WAVE PROPAGATION IN THE GLHUA-1 OUTER LAYER CLOAK

In this section, we create an analytical EM wave field propagation in the GLHUA-1 outer cloak with relative parameters in (82) and (84). For simplicity, we create a new analytical expansion of the solution of the magnetic wave equation in the GLHUA-1 outer layer cloak domain \( R_1 \leq r \leq R_2 \).

A. The spherical harmonic expansion of incident radial GL magnetic wave field

Suppose that the incident radial magnetic wave field \( H^b_r \),
\[ H^b_r = \cos \phi \frac{1}{r} \frac{\partial g}{\partial \theta} - \frac{\cos \theta \sin \phi}{r \sin \theta} \frac{\partial g}{\partial \phi} \] (86)
is incident magnetic wave for GLHUA-1 outer layer cloak, where
\[ g(\vec{r}, \vec{r}_s) = -\frac{1}{4\pi} \frac{e^{ik|\vec{r} - \vec{r}_s|}}{|\vec{r} - \vec{r}_s|}. \] (87)

\( H^b_r \) is excited by the electric point source
\[ \vec{J}_s = \frac{1}{r^2 \sin \theta} \delta(r - r_s) \delta(\theta - \theta_s) \delta(\phi - \phi_s) \vec{e}_y. \] (88)

located in the point \( \vec{r}_s = (r_s, \theta_s, \phi_s), r_s > R_{out} > R_2 \). \( H^b_r \) is the solution of the magnetic wave equation
\[ \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial H^b_r}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial H^b_r}{\partial \theta} + k^2 r^2 \varepsilon_\theta \mu_r H^b_r = 0, \] (89)
in the annular domain \( R_2 < r < R_{out} \). In free space, with relative parameters are equal to 1, \( \mu_\theta = \varepsilon_\theta = 1, \mu_\phi = \varepsilon_\phi = 1, \mu_r = \varepsilon_r = 1 \).

Define Global and Local - GL magnetic wave
\[ H(\vec{r}) = r^2 \mu_r H_r(\vec{r}). \] (90)
The magnetic wave equation (89) is translated to the following GL magnetic wave equation
\[ \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial H}{\partial r} \right) + \frac{1}{\mu_r} \frac{\partial H}{\partial r} + \mu_r \varepsilon_r H = 0, \] (91)
Using the GL electric wave field \( E(\vec{r}^o) = r^2 \varepsilon_r E_r(\vec{r}^o) \), to replace GL magnetic wave field \( H(\vec{r}) = r^2 \mu_r H_r(\vec{r}) \) in (91), the equation (91) become the GL electric wave equation. The GL EM wave propagation in free space satisfies the equation (91) with relative parameter equal to 1; GL EM wave propagation in GLHUA-1 outer layer cloak satisfies the equation (91) with GLHUA-1 relative parameters not less than 1 in (82) and (84).

Define the operator
\[ \Re(\theta, \phi) = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \phi^2} \] (92)
\[ D_h(\theta, \phi) = \cos \phi \frac{\partial}{\partial \theta} - \cos \theta \sin \phi \frac{\partial}{\partial \phi} \] (93)

By direct calculation, we can prove
\[ \Re(\theta, \phi) D_h(\theta, \phi) D_h(\theta, \phi) = D_h(\theta, \phi) \Re(\theta, \phi), \] (94)
The spherical harmonic expansion of the incident GL magnetic wave \( H^{b^{(i)}}(\vec{r}) = r^2 H^{b^{(i)}}(\vec{r}) \) is
\[ H^{b^{(i)}}(r, \theta, \phi) = -ik \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \frac{j_l(kr) h^{(l)}(kr)}{l} \sum_{m=-l}^{l} D_h(\theta, \phi) Y_l^m(\theta, \phi) Y_l^{m*}(\theta, \phi_s), \] (95)

Because the electric point source \( (r_s, \theta_s, \phi_s) \) in outside of the GLHUA-1 cloak, \( r_s > R_{out} > R_2 \) and operator \( D_h(\theta, \phi) \) and \( \Re(\theta, \phi) \) can be commutative (94), substitute (95) into the GL magnetic equation (91) in the sphere annular domain \( \{r, \theta, \phi\} \), without source, the equation (91) is split into the second ordinary differential equation for every \( l, l = 1, 2, \ldots \), as follows,
\[ \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial H^{b^{(i)}}}{\partial r} \right) = \frac{l(l+1)}{r^2} H^{b^{(i)}}(r) + k^2 H^{b^{(i)}}(r) = 0, \] (96)
B. Analytical EM Wave propagation in the GLHUA-1 outer layer cloak

We propose GLHUA-1 expansion method to create analytic expansion of the magnetic wave field in GLHUA-1 outer layer cloak which satisfies the magnetic equation (91) in the outer layer domain with relative parameters (82) and (84). The GLHUA-1 analytic magnetic wave field and incident magnetic wave and their derivative satisfy the boundary continuous condition across the boundary \( r = R_2 \).

Substitute the spherical harmonic expansion of the magnetic wave field

\[
H(r, \theta, \phi) = \sum_{l=1}^{\infty} H_l(r) \sum_{m=-l}^{l} D_{lm}(\theta, \phi) Y_l^m(\theta, \phi) Y_l^{m*}(\theta, \phi),
\]

(97)

into the GL magnetic equation (91) in the GLHUA-1 outer annular layer domain \( R_1 \leq r \leq R_2 \), the equation (91) is split into the second ordinary differential equation for every \( l, j = 1, 2, \cdots \),

\[
\frac{\partial}{\partial r} \left( \frac{1}{\varepsilon_0} \frac{\partial H_l(r)}{\partial r} \right) - \frac{(l+1)}{\mu_r r^2} H_l(r) + k^2 \varepsilon_{0l} H_l(r) = 0,
\]

(98)

By comparison between magnetic equation (98) in GLHUA-1 outer layer cloak and incident magnetic wave equation (96), the \( l \) component magnetic field \( H_l(r) \) and \( l \) component incident magnetic field \( H_l^{(b)}(r) \) are continuous across boundary \( r = R_2 \),

\[
H_l(R_2) = H_l^{(b)}(R_2),
\]

(99)

and their radial derivative are continuous across boundary \( r = R_2 \),

\[
\frac{\partial}{\partial r} H_l(R_2) = \frac{\partial}{\partial r} H_l^{(b)}(R_2),
\]

(100)

Substitute GLHUA-1 cloak relative materials \( \varepsilon_r(r) = \mu_r(r) = \frac{\mu_0}{R_2^2} \) in (82) and relative angular parameter in (84) into the equation (91), we have

\[
\frac{\partial}{\partial r} \left( \frac{r-R_1}{r-R_1} \frac{\partial g_{l,1}(r)}{\partial r} \right) - \frac{(l+1)}{R_1^2} g_{l,1}(r) + k^2 \frac{R_2-R_1}{r-R_1} g_{l,1}(r) = 0,
\]

(101)

We create GLHUA-1 expansion of the \( l \) component GL magnetic field \( H_l(r) \)

\[
H_l(r) = g_{l,1}(r) \cos(k(R_2 - R_1) \log(\frac{r-R_1}{r-R_1})) + g_{l,2}(r) \sin(k(R_2 - R_1) \log(\frac{r-R_1}{r-R_1})),
\]

(102)

By calculation,

\[
\frac{\partial}{\partial r} \left( \frac{r-R_1}{r-R_1} \frac{\partial g_{l,1}(r)}{\partial r} \right) + 2k \frac{\partial}{\partial r} g_{l,2}(r) \cos(k(R_2 - R_1) \log(\frac{r-R_1}{r-R_1})) + 2k \frac{\partial}{\partial r} g_{l,1}(r) \sin(k(R_2 - R_1) \log(\frac{r-R_1}{r-R_1}))
\]

(103)

Substitute \( (102) \) and \( (103) \) into the equation \( (101) \), we obtain a new GLHUA-1 two equation system

\[
\frac{\partial}{\partial r} \left( \frac{r-R_1}{r-R_1} \frac{\partial g_{l,1}(r)}{\partial r} \right) + 2k \frac{\partial}{\partial r} g_{l,2}(r) + \frac{(l+1)}{R_1^2} g_{l,1}(r) = 0,
\]

(104)

\[
-2k \frac{\partial}{\partial r} g_{l,1}(r) + \frac{(l+1)}{R_1^2} g_{l,2}(r) = 0,
\]

(105)

We expand solution \( (g_{l,1}(r), g_{l,2}(r)) \) of the GLHUA-1 two equation system \( (104) \) and \( (105) \) as analytical power series,

\[
g_{l,1}(r) = \sum_{j=0}^{\infty} a_{l,j} (r - R_1)^j,
\]

(106)

\[
g_{l,2}(r) = \sum_{j=0}^{\infty} b_{l,j} (r - R_1)^j,
\]

(107)

By calculation,

\[
(r - R_1) \frac{\partial}{\partial r} g_{l,1}(r) = \sum_{j=0}^{\infty} a_{l,j} (j+1) (r - R_1)^j - \sum_{j=0}^{\infty} a_{l,j+1} (j+1) (r - R_1)^j,
\]

(108)

\[
(r - R_1) \frac{\partial}{\partial r} g_{l,2}(r) = \sum_{j=0}^{\infty} b_{l,j} (j+1) (r - R_1)^j - \sum_{j=0}^{\infty} b_{l,j+1} (j+1) (r - R_1)^j,
\]

(109)

\[
\frac{\partial}{\partial r} g_{l,1}(r) = \sum_{j=0}^{\infty} a_{l,j} (r - R_1)^{j+1} - \sum_{j=0}^{\infty} a_{l,j+1} (r - R_1)^j,
\]

(110)

\[
\frac{\partial}{\partial r} g_{l,2}(r) = \sum_{j=0}^{\infty} b_{l,j} (r - R_1)^{j+1} - \sum_{j=0}^{\infty} b_{l,j+1} (r - R_1)^j,
\]

(111)
Substitute (106-111) into GLHUA-1 equation (104), we have
\[
\begin{align*}
\frac{1}{R_2 - R_1} \sum_{j=1}^{\infty} a_{l,j+1}(j+1)(r-R_1)^j + \frac{1}{R_2 - R_1} \sum_{j=0}^{\infty} a_{l,j+1}(j+1)(r-R_1)^j \\
+ 2k \sum_{j=0}^{\infty} b_{l,j+1}(j+1)(r-R_1)^j - \frac{l(l+1)}{R_2^2} \sum_{j=0}^{\infty} a_{l,j}(r-R_1)^j = 0
\end{align*}
\] (112)

Substitute (106-111) into GLHUA-1 equation (105), we have
\[
\begin{align*}
\frac{1}{R_2 - R_1} \sum_{j=1}^{\infty} b_{l,j+1}(j+1)(r-R_1)^j + \frac{1}{R_2 - R_1} \sum_{j=0}^{\infty} b_{l,j+1}(j+1)(r-R_1)^j \\
- 2k \sum_{j=0}^{\infty} a_{l,j+1}(j+1)(r-R_1)^j - \frac{l(l+1)}{R_2^2} \sum_{j=0}^{\infty} b_{l,j}(r-R_1)^j = 0
\end{align*}
\] (113)

When \( j = 0 \) in (112), we have
\[
\frac{1}{R_2 - R_1} a_{l,1} + 2kb_{l,1} = \frac{l(l+1)}{R_2^2} a_{l,0},
\] (114)

When \( j = 0 \) in (113), we have
\[
\frac{1}{R_2 - R_1} b_{l,1} - 2ka_{l,1} = \frac{l(l+1)}{R_2^2} b_{l,0},
\] (115)

Solve the linear equation (114) and (115), we obtain
\[
\begin{bmatrix}
a_{l,1} \\
b_{l,1}
\end{bmatrix} = \frac{l(l+1)}{1 + 4k(R_2 - R_1)^2 R_2^2}
\begin{bmatrix}
(R_2 - R_1) & -2k(R_2 - R_1)^2 \\
2k(R_2 - R_1)^2 & (R_2 - R_1)
\end{bmatrix}
\begin{bmatrix}
a_{l,0} \\
b_{l,0}
\end{bmatrix}
\] (116)

For every \( j \geq 1 \), in (112) and (113), we have
\[
\frac{1}{R_2 - R_1} (j+1)^2 a_{l,j+1} + 2k(j+1)b_{l,j+1} = \frac{l(l+1)}{R_2^2} a_{l,j},
\] (117)

\[
\frac{1}{R_2 - R_1} (j+1)^2 b_{l,j+1} - 2k(j+1)a_{l,j+1} = \frac{l(l+1)}{R_2^2} b_{l,j},
\] (118)

Solve the linear equation (117) and (118), we obtain
\[
\begin{bmatrix}
a_{l,j+1} \\
b_{l,j+1}
\end{bmatrix} = \frac{l(l+1)}{(j+1)^3 + 4k(R_2 - R_1)^2(j+1) R_2^2}
\begin{bmatrix}
(j+1)(R_2 - R_1) & -2k(R_2 - R_1)^2 \\
2k(R_2 - R_1)^2 & (j+1)(R_2 - R_1)
\end{bmatrix}
\begin{bmatrix}
a_{l,j} \\
b_{l,j}
\end{bmatrix},
\] (119)

There are two linearly independent vector function solution of the GLHUA-1 two equations system (104) and (105), \((g_{l,1,p}(r), g_{l,2,p}(r)), p = 1, 2\). When \( a_{l,0}^{(1)} = 1, b_{l,0}^{(1)} = 0 \) from (116) and (119), we obtain \((a_{l,j}^{(1)}, b_{l,j}^{(1)}), j = 1, 2, \ldots \) and from analytical power series (106) and (107), we obtain the first solution of GLHUA-1 quations (110) and (111),\((g_{l,1,1}(r), g_{l,2,1}(r))\). When \( a_{l,0}^{(2)} = 0, b_{l,0}^{(2)} = 1 \), from (116) and (119), we obtain \((a_{l,j}^{(2)}, b_{l,j}^{(2)}), j = 1, 2, \ldots \), and from analytical power series (106) and (107), we obtain the second solution of GLHUA-1 equations (104) and (105),\((g_{l,1,2}(r), g_{l,2,2}(r))\) , for \( l = 1, 2, \ldots \).

For arbitrary two constants \( A_1 \) and \( A_2 \), the general solution of the GLHUA-1 equation (104) and (105) is
\[
H_{l,1}(r) = A_1 g_{l,1,1}(r) + A_2 g_{l,1,2}(r),
\] (120)
\[
H_{l,2}(r) = A_1 g_{l,2,1}(r) + A_2 g_{l,2,2}(r),
\] (121)

Substitute (120) and (121) into (102), we obtain
\[
\begin{align*}
\frac{d}{dr} \left[ H_{l,1}(r) \right] &= k(R_2 - R_1) \log \left( \frac{r - R_1}{R_2 - R_1} \right) + H_{l,2}(r) \\
&= \left( A_1 g_{l,1,1}(r) + A_2 g_{l,1,2}(r) \right) \\
&+ \left( A_1 g_{l,2,1}(r) + A_2 g_{l,2,2}(r) \right)
\end{align*}
\] (122)

From the continuous condition of \( H_{l}(r) \) and \( H_{l}^{(b)}(r) \) on \( r = R_2 \) (99) and their derivative continuous condition (100), we determine the constants \( A_1 \) and \( A_2 \) in (122). Substitute (122) into (97), we obtained the GL analytic magnetic wave \( H(r) \), in the GLHUA-1 outer layer with relative parameter (82) and (84) not less than 1.

Summary, we have proved the following theorem.

**Theorem 2:** In the outer annular layer cloak, \( R_1 \leq r \leq R_2 \), the radial relative electric permittivity and magnetic permeability, \( \varepsilon_r = \mu_r = \frac{R_1}{R_2} \) in (82), and the angular relative electric permittivity and magnetic permeability, \( \varepsilon_\theta = \varepsilon_\phi = \mu_\theta = \mu_\phi = \frac{R_2 - R_1}{R_2 - R_1} \) in (84).

The incident magnetic wave is excited by the electric point source in the point \( \mathbf{r}_0 = (r_0, \theta_0, \phi_0) \), \( r_0 > R_{\text{out}} > R_2 \).

The analytic GL radial magnetic wave field solution of 3D GL radial magnetic equation (91) and on the boundary \( r = R_2 \) with field and derivative continuous to the incident magnetic wave in (95) is
inner spherical surface, upward crescent type GL wave in the lower part of the same upper part of the inner spherical surface, is very strange, there is a down crescent type GL wave in FIG. 1: (color online) the magnetic wave is being tangent

\[
H(\vec{r}) = \sum_{l=1}^{\infty} H_{l,1}(r) \cos \left( k(R_2 - R_1) \log \left( \frac{r-R_1}{R_2-R_1} \right) \right)
\]

\[
= \sum_{m=-l}^{l} D_h(\theta, \phi) Y_l^m(\theta, \phi) \tilde{Y}_l^{m*}(\theta, \phi)
\]

\[
+ \sum_{l=1}^{\infty} H_{l,2}(r) \sin \left( k(R_2 - R_1) \log \left( \frac{r-R_1}{R_2-R_1} \right) \right)
\]

\[
\sum_{m=-l}^{l} D_h(\theta, \phi) Y_l^m(\theta, \phi) \tilde{Y}_l^{m*}(\theta, \phi),
\]

Where operator

\[
D_h(\theta, \phi) = \cos \phi \frac{\partial}{\partial \theta} - \cos \theta \sin \phi \frac{\partial}{\partial \phi},
\]

\[
\left[ H_{l,1}(r) \right]_{l,j} = \left[ \begin{array}{c} g_{l,1,1}(r) \\ g_{l,1,2}(r) \end{array} \right],
\]

\[
\left[ H_{l,2}(r) \right]_{l,j} = \left[ g_{l,2,1}(r) \\ g_{l,2,2}(r) \right] = \sum_{j=0}^{l} \frac{a_{l,j}^{(1)}}{a_{l,j}^{(2)}} \left( r - R_1 \right)^j,
\]

\[
\left[ \begin{array}{c} a_{l,j+1}^{(1)} \\ a_{l,j+1}^{(2)} \end{array} \right] = \frac{(l+1)!}{(l+j+1)!} \left[ \begin{array}{c} 2k(R_2 - R_1) \\ -2k(R_2 - R_1) \end{array} \right] \left[ \begin{array}{c} a_{l,j}^{(1)} \\ a_{l,j}^{(2)} \end{array} \right],
\]

\[
\left[ g_{l,1,1}(r) g_{l,1,2}(r) \\ g_{l,2,1}(r) g_{l,2,2}(r) \right]
\]

\[
= \left[ \begin{array}{c} A_{l,1} \\ A_{l,2} \end{array} \right],
\]

\[
\left[ \begin{array}{c} H_{l,1}^b(R_2) \\ H_{l,2}^b(R_2) \end{array} \right] = \left[ \begin{array}{c} H_{l,1}^b(R_2) \\ H_{l,2}^b(R_2) \end{array} \right] = \left[ \begin{array}{c} -ik_{l+1}j_l(kR_2)h_l^{(1)}(kR_2) \\ -ik(l+1)j_l(kR_2) - kR_2j_{l+1}(kR_2)h_l^{(1)}(kR_2) \end{array} \right],
\]

The \( j_l(\cdot) \) is spherical Bessel function, \( h_l^{(1)}(\cdot) = j_l(\cdot) + ik_l(\cdot) \) is first type spherical Hankel function. \( y_l(\cdot) \) is spherical Neumann function. Now, we prove that GLHUA-1 exact analytical radial magnetic wave (123) does satisfy the GL radial magnetic equation (91).

Substitute the new GLHUA-1 outer layer invisible cloak (82) and (84) into the GL radial magnetic differential equation (91), we obtain GL radial magnetic equation in GLHUA-1 cloak domain

\[
\frac{\partial}{\partial r} \frac{r-R_1}{R_2-R_1} \frac{\partial}{\partial r} H(\vec{r}) + \frac{1}{R_2^2} \left( \sin \theta \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \phi} \right) H(\vec{r}) + \frac{k^2 R_2^2 - R_1^2}{r-R_1} H(\vec{r}) = 0,
\]

Substitute the GLHUA-1 analytical EM wave (123) into the equation (131), and By definition of operator \( R(\theta, \phi) \) in (92) and definition of operator \( D_h(\theta, \phi) \) in (93), and

\[
\left[ \begin{array}{c} a_{l,0}^{(1)} \\ a_{l,0}^{(2)} \end{array} \right] = \left[ \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right],
\]

FIG. 1: (color online) the magnetic wave is being tangent to the outer boundary \( r = R_2 \) without any scattering. It is very strange, there is a down crescent type GL wave in the upper part of the inner spherical surface, \( r = R_1 \), and upward crescent type GL wave in the lower part of the same inner spherical surface, \( r = R_1 \).

FIG. 2: (color online) Magnetic wave propagation through GLHUA-1 outer layer cloak, wave front at 31 step.
\[ \Re (\vartheta, \phi) D_h(\theta, \phi) = D_h(\theta, \phi) \Re (\vartheta, \phi) \text{ in (94)}, \]

we find that

\[ \frac{\partial}{\partial r} \left( \frac{r}{R_2 - R_1} \right) H(\vec{r}) + \frac{1}{R_2^3} \left( \sin \theta \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{\partial^2}{\partial \phi^2} \right) H(\vec{r}) \]

\[ + \frac{k^2}{R_2} \frac{\partial}{\partial r} H(\vec{r}) = \sum_{l=1}^{\infty} \left( \frac{\partial}{\partial r} \frac{r - R_1}{r - R_2} \frac{\partial}{\partial r} \right) H_{l,1}(r) + 2 \frac{\partial}{\partial r} \frac{H_{l,1}(r)}{R_2} \frac{\partial}{\partial r} \frac{H_{l,1}(r)}{R_2} \]

\[ \cos \left( k(R_2 - R_1) \log \left( \frac{r - R_1}{r - R_2} \right) \right) \]

\[ \left( \sum_{m=-l}^{l} D_h(\theta, \phi) Y_{l,m}^{s}(\theta, \phi) Y_{l,m}^{s}(\theta, \phi) \right) \]

(132)

By (126)-(128), \( g_{l,1} \) and \( g_{l,2} \) satisfy the GLHUA-1 two equation system (104) and (105). With initial (1,0), \( g_{l,1} \) and \( g_{l,2} \) satisfy the GLHUA-1 two equation system (104) and (105). With initial (0,1), By (125), we know that \( H_{l,1} \) and \( H_{l,2} \) satisfy the GLHUA-1 two equation system (104) and (105). Substitute \( H_{l,1} \) and \( H_{l,2} \) in (125) into (132). We have proved that exact analytical GL radial magnetic \( H(\vec{r}) \) in (125) is solution of the GL radial magnetic equation (82). Based on the Condition (129) and (130), on the outer spherical boundary \( r = R_2 \), the magnetic \( H(\vec{r}) \) and its derivative satisfy the continuous boundary conditions (99) and (100). Therefore, the theorem 9 is proved.

Suppose that the incident electric wave is excited by the electric point source (92) located in the point \( \vec{r}_s = (r_s, \theta, \phi_s), r_s > R_{\text{out}} > R_2 \). By same analytic process, from the continuous of the incident GL electric wave and its derivative on the boundary, \( r = R_2 \), we can create the exact GL analytic electric wave in the GLHUA-1 outer layer. \( E(\vec{r}) \).

\[ \text{FIG. 3: (color online) Magnetic wave propagation through GLHUA-1 outer layer cloak, wave front at 37 step.} \]

\[ \text{FIG. 4: (color online) Magnetic wave propagation through GLHUA-1 outer layer cloak, wave front at 43 step.} \]
FIG. 5: (color online) Magnetic wave propagation through GLHUA-1 outer layer cloak, wave front at 49 step.

FIG. 6: (color online) Magnetic wave propagation through GLHUA-1 outer layer cloak, wave front at 55 step.

FIG. 7: (color online) Magnetic wave propagation through GLHUA-1 outer layer cloak, wave front at 61 step.

FIG. 8: (color online) Magnetic wave propagation through GLHUA-1 outer layer cloak, wave front at 67 step.

\[
\begin{bmatrix}
E_r(\vec{r}) \\
H_r(\vec{r})
\end{bmatrix}
= \sum_{l=1}^{\infty} \begin{bmatrix}
\hat{E}_{r,l,1}(\vec{r}) \\
\hat{H}_{r,l,1}(\vec{r})
\end{bmatrix} \cos \left( k(R_2 - R_1) \log \left( \frac{r-R_2}{R_2-R_1} \right) \right) \\
\sum_{m=-l}^{l} \begin{bmatrix}
\frac{E_r(\vec{r})}{H_r(\vec{r})} \\
\frac{H_r(\vec{r})}{E_r(\vec{r})}
\end{bmatrix} \frac{D_c(\theta, \phi)}{D_h(\theta, \phi)} Y_{l,m}^s(\theta, \phi, \phi_s) + \sum_{m=-l}^{l} \begin{bmatrix}
E_r(\vec{r}) \\
H_r(\vec{r})
\end{bmatrix} \frac{D_c(\theta, \phi)}{D_h(\theta, \phi)} Y_{l,m}^s(\theta, \phi, \phi_s).
\]

(135)

In the outside of the cloak, \( r > R_2 \), \( E_r(\vec{r}) = E_{b,r} (\vec{r}) \), \( H_r(\vec{r}) = H_{b,r} (\vec{r}) \), on the boundary \( r = R_2 \), \( E_r(\vec{r})|_{r=R_2} = E_{b,r} (\vec{r})|_{r=R_2} \), \( H_r(\vec{r})|_{r=R_2} = H_{b,r} (\vec{r})|_{r=R_2} \),

\[
\frac{\partial E_r(\vec{r})}{\partial r}|_{r=R_2} = \frac{\partial E_{b,r}(\vec{r})}{\partial r}|_{r=R_2}, \quad \frac{\partial H_r(\vec{r})}{\partial r}|_{r=R_2} = \frac{\partial H_{b,r}(\vec{r})}{\partial r}|_{r=R_2}.
\]

(136)

The exact analytical radial EM wave solution show that there is no scattering from the GLHUA-1 outer layer cloak to disturb the incident EM wave in free space outside of the cloak. Proof: Our theorem 3 is proved by using similar proof in theorem 2.
FIG. 9: (color online) Magnetic wave propagation through GLHUA-1 outer layer cloak, wave front at 73 step.

FIG. 10: (color online) Magnetic wave propagation through GLHUA-1 outer layer cloak, wave front at 79 step.

FIG. 11: (color online) Magnetic wave propagation through GLHUA-1 outer layer cloak, wave front at 85 step.

FIG. 12: (color online) Magnetic wave propagation through GLHUA-1 outer layer cloak, wave front at 91 step.

V. SIMULATION OF THE EXACT ANALYTIC MAGNETIC WAVE PROPAGATION THROUGH THE GLHUA-1 OUTER LAYER CLOAK

In GLHUA-1 outer cloak, $R_1 \leq r \leq R_2$, the incident EM wave field is excited by the electric current point source located in $r_s = (13.0m, 0.5\pi, 0)$, the electric current point source is

$$S(r, r_s) = e^{-i\omega t} \delta(r - r_s) \vec{e}_y,$$  \hspace{1cm} (138)

where

$$\vec{e} = \vec{e}_y = (0, 1, 0)$$
$$= (\sin \theta \sin \phi \cos \theta \sin \phi \cos \phi),$$  \hspace{1cm} (139)

In the free space $r > R_2$, the angular frequency $\omega = 2\pi f$, $f = 0.1669555 \times 10^9 Hz$.

In GLHUA-1 outer cloak, $R_1 \leq r \leq R_2$, $R_1 = 0.66m, R_2 = 1.0m$, the incident radial magnetic wave excited by the above electric current source is

$$H^b = \cos \phi \frac{1}{r} \frac{\partial g}{\partial \theta} - \frac{\cos \theta \sin \phi}{r \sin \theta} \frac{\partial g}{\partial \phi}.$$  \hspace{1cm} (140)

The GLHUA-1 exact analytical EM wave propagation show that on the inner spherical surface boundary $r = R_1$, the GLHUA-1 cloak material does generate a strange wave. We call the strange wave as GLHUA-1 created wave by cloak materials wave, for simply, we call it "GL wave". In Figure 1 (90), the magnetic wave is being tangent to the outer boundary $r = R_2$ without any scattering. It is very strange, there is a down crescent type GL wave in the upper part of the inner spherical surface, $r = R_1$, and upward crescent type GL wave in the lower part of the same inner spherical surface, $r = R_1$. 

In the next step, the magnetic wave smoothly enter the outer annular layer \(R_1 < r < R_2\), is presented in Figure 2. (112) The upper and lower two crescent type GL wave attached on the inner spherical surface, \(r = R_1\), maintain the shape and propagation from left to right; along the same inner spherical surface, a red small part of the arc GL wave generated; The wave propagation phase image in figure 3 (124) is similar with the image in figure 2., the magnetic wave in the GLHUA-1 outer layer, the red arc and crescent GL wave on the inner spherical surface are propagation to left, the arc and crescent GL wave phase shape is growing along on the same inner spherical surface, \(r = R_1\), and the arc GL wave is catching to crescent GL wave. In figure 4 (136), the wave propagation phase image is similar with the image in figure 3, the phase image in figure 4 is propagated to left of image in figure 3, the upper and lower crescent GL wave along on the inner spherical surface, \(r = R_1\) is connected into double crescent GL wave.

The phase image in figure 5 is propagated to left of image in figure 4. In the figure 5, the double crescent GL wave is growing up to a spherical surface GL wave, which is connected with the arc GL wave, the magnetic wave in GLHUA-1 outer layer is propagating to left of it in figure 4, follow and closed to the GL wave along on the inner spherical surface, \(r = R_1\). In the figure 6 (180), in the left side of the inner spherical surface, the spherical surface GL wave is growing up to like a wave front on the left side of the inner spherical surface, in the right side of the inner spherical surface, the arc GL wave is growing as half spherical surface GL wave which phase is right part of the inner spherical surface. The right half spherical surface GL wave is connected with the left side spherical surface GL wave. The magnetic
wave in GLHUA-1 outer layer is propagating to left of it in figure 5, follow and more close to the GL wave along on the inner spherical surface, \( r = R_1 \). The phase image in figure 7 (181) is similar with the phase image in figure 6 and is propagated to left of image in figure 6. The magnetic wave propagation in figure 7 is important step. In the figure 7, the magnetic wave in right side of inner spherical surface meets the spherical surface GL wave in the left side of the inner spherical surface. In figure 8 (19), the magnetic wave use the left spherical GL wave as wave front for new propagation, the original wave front of the magnetic wave is disengaged and as new spherical surface GL wave attached in the right side of the inner spherical surface. The magnetic wave change wave front in the figure 7 shows that the magnetic wave front propagation in the GLHA outer layer cloak is discontinuous and the ray is also discontinuous. The phase image in figure 9 (21) is similar with the phase image in figure 8 and is propagated to left of image in figure 8. In the next magnetic wave propagation step in the figure 9, 10, 11, let P1 be cross point of the wave front and equatorial and P2 be cross point of the outer spherical surface, \( r = R_2 \) and equatorial. Let Q1 be cross point of the wave front and outer spherical surface \( r = R_2 \) and Q2 be the cross point of line \( z = z_{q1} \) of Q1 and line \( x = x_{p2} \) of P2. Let d1 be distance between P1 and P2, \( d_2 \) be the distance between Q1 and Q2. In the figure 9, 10, 11, the magnetic wave is propagating forward to left, on the magnetic wave front, d1 is minimal and phase velocity at point P1 on the wave front is also minimal; \( d_2 \) is maximum and phase velocity at point P1 on the wave front is also minimal; \( d_1 > d_2 \), the difference of \( d_2-d_1 \) in monotone decreasing. In the figure 17, \( d_1=d_2=0 \), the every point on the wave front is also discontinuous.
magnetic wave front are propagating out of the GLHUA-1 cloak simultaneously, the magnetic wave is recovered to the incident wave in free space at same time. In the figure 18, the magnetic wave front is recovered incident wave front in free space that show GLHUA-1 outer layer cloak is invisible. In the figure 9, 10, 11, along inner spherical surface, \( r = R_1 \), the upper GL wave is propagating counterclockwise and the lower GL wave is propagating clockwise with some discontinuous. In figure 12, the right GL wave attached on the right inner spherical surface is deformed to double crescent, and left arc GL wave is disengaged from the right GL wave. In figure 11 (139), the right GL wave attached on the right inner spherical surface is more deformed to double crescent, and left arc GL wave is disengaged from the right GL wave and is more discontinuous. In figure 12 (151), the right GL wave attached on the right inner spherical surface is discontinuously split to the upper crescent GL wave and lower crescent GL wave. In figure 13 (163), along inner spherical surface, \( r = R_1 \), the upper crescent GL wave and arc GL wave is propagating counterclockwise and the lower crescent GL wave and arc GL wave is propagating clockwise with some discontinuous. The phase image of the magnetic wave front and GL wave in figure 14 (175) is similar with the phase image in the figure 13, moreover, the left arc GL wave is shrink to very short. The new magnetic wave front is appear in right side in the new period. The distance between two magnetic wave front is wave length.

The phase Image of the magnetic wave front and GL wave along the inner spherical surface, \( r = R_1 \) in figure 15 (176) and figure 16 (186) is same as phase image
in figure 1 and figure 2, respectively. Summary, the incident magnetic wave in free space never be disturbed by GLHUA-1 outer cloak, GLHUA-1 cloak is invisible; The magnetic wave can not propagating penetrated into inside of the sphere $r < R_1$. The phase velocity of magnetic wave propagation is less than light speed. The exact analytic GLHUA-1 wave propagation simulation shows that the GL wave along the inner spherical surface $r = R_1$ is generated by the GLHUA-1 outer cloak material. The GL wave is a bridge for the EM wave propagation through the GLHUA-1 outer cloak without exceeding light speed. The GL wave is a cloaking making that the EM wave propagation can not penetrate into the concealment.

VI. DISCUSSION AND CONCLUSION ON EXACT ANALYTIC EM WAVE IN GLHUA-2 CLOAK

Substitute the inverse transformation to incident analytical EM wave, we can obtain the analytical EM wave...
propagation in the 0 to \( R_1 \) spherical radial transformation cloak. It is totally different from transformation optic, without using transformation, our GLHUA-1 analytical expansion can also be directly used to find analytical EM wave propagation in the 0 to \( R_1 \) spherical radial transformation cloak. For example, we propose GLHUA-1 analytical expansion method to find analytical EM wave propagation in Pendry cloak.

### A. GLHUA-1 analytical expansion method to find analytical EM wave in Pendry Cloak

Substitute relative EM parameters in Pendry cloak

\[
\varepsilon_r = \mu_r = \frac{R_2}{R_2 - R_1} \frac{(r - R_1)^2}{r^2}, \tag{141}
\]

\[
\varepsilon_\theta = \mu_\theta = \varepsilon_\phi = \mu_\phi = \frac{R_2}{R_2 - R_1}, \tag{142}
\]

into the equation (98), the (98) become to

\[
\frac{\partial}{\partial r} \left( \frac{R_2 - R_1}{R_2^2} \frac{\partial H_i(r)}{\partial r} \right) - \frac{R_2 - R_1}{R_2^2} \frac{1}{(r - R_1)^2} H_i(r) + k^2 \frac{\partial}{\partial r} \left( \frac{R_2 - R_1}{R_2^2} \right) H_i(r) = 0, \tag{143}\]

We create GLHUA-1 analytical expansion of radial magnetic wave \( l \) component \( H_{l,p}, p = 1, 2, \)

\[
H_{l,p} = H_{l,1,p}(r) \cos \left( \frac{k R_2(r - R_1)}{R_2 - R_1} \right) + H_{l,2,p}(r) \sin \left( \frac{k R_2(r - R_1)}{R_2 - R_1} \right), \tag{144}\]

Substitute (144) into (143), we have novel GLHUA-1 dual expansion equation, \( p = 1, 2, \)

\[
\frac{\partial}{\partial r} \left( \frac{R_2 - R_1}{R_2^2} \frac{\partial H_{l,1,p}(r)}{\partial r} \right) + 2k \frac{\partial H_{l,1,p}(r)}{\partial r} = \frac{R_2 - R_1}{R_2^2} \frac{1}{(r - R_1)^2} H_{l,1,p}(r), \tag{145}\]

\[
\frac{\partial}{\partial r} \left( \frac{R_2 - R_1}{R_2^2} \frac{\partial H_{l,2,p}(r)}{\partial r} \right) - 2k \frac{\partial H_{l,2,p}(r)}{\partial r} = \frac{R_2 - R_1}{R_2^2} \frac{1}{(r - R_1)^2} H_{l,2,p}(r), \tag{146}\]

We propose negative power series expansion

\[
H_{l,1,p}(r) = \sum_{j=0}^{\infty} a_{l,j,p} (r - R_1)^{-j}, \tag{147}\]

\[
H_{l,2,p}(r) = \sum_{j=0}^{\infty} b_{l,j,p} (r - R_1)^{-j}, \tag{148}\]

where \( p = 1, 2, \) when \( p = 1, \ a_{l,0,p} = 1, b_{l,0,p} = 0, \) when \( p = 2, \ a_{l,0,p} = 0, b_{l,0,p} = 1. \) Substitute (147) and (148) into GLHUA-1 dual equations (145) and (146), we obtain

\[
H_{l,1} = H_{l,1,1}(r) \cos \left( \frac{k R_2(r - R_1)}{R_2 - R_1} \right) + H_{l,2,1}(r) \sin \left( \frac{k R_2(r - R_1)}{R_2 - R_1} \right), \tag{149}\]

\[
H_{l,2} = H_{l,1,2}(r) \cos \left( \frac{k R_2(r - R_1)}{R_2 - R_1} \right) + H_{l,2,2}(r) \sin \left( \frac{k R_2(r - R_1)}{R_2 - R_1} \right), \tag{150}\]

For arbitrary constants \( A \) and \( B, \)

\[
H_l(r) = AH_{l,1}(r) + BH_{l,2}(r) = A \left( \frac{R_2(r - R_1)}{R_2 - R_1} \right) \cos \left( \frac{k R_2(r - R_1)}{R_2 - R_1} \right) + B \left( \frac{R_2(r - R_1)}{R_2 - R_1} \right) \sin \left( \frac{k R_2(r - R_1)}{R_2 - R_1} \right), \tag{151}\]

From the continuous boundary condition on \( r = R_2, \) (99) and (100) in this paper, we obtained \( B = 0, \)

\[
H_l(r) = -ik \left( \frac{R_2(r - R_1)}{R_2 - R_1} \right) \left( \frac{R_2(r - R_1)}{R_2 - R_1} \right) h_{1l}(kr_s), \tag{152}\]

**Theorem 4:** In the spherical annular layer \( R_1 \leq r \leq R_2, \) the EM relative parameters is presented by (141) and (142), the electric point delta source is denoted by (88) in this paper, then analytical radial GL magnetic wave propagation in Pendry cloak is

\[
H_\rho(r) = -ik \sum_{l=1}^{\infty} \sum_{m=-l}^{l} D_h(\theta, \phi) Y_l^m(\theta, \phi) Y_l^{m*}(\theta, \phi) \tag{153}\]

\[
H_\theta(r) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} D_h(\theta, \phi) Y_l^m(\theta, \phi) Y_l^{m*}(\theta, \phi), \tag{154}\]

\[
H_\phi(r) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} D_h(\theta, \phi) Y_l^m(\theta, \phi) Y_l^{m*}(\theta, \phi). \tag{155}\]
B. Comparison GLHUA-1, GLHUA-2, GLHUA-3 cloak and Pendry Cloak

In GLHUA-1 invisible cloak (82) and (84), GLHUA-2 invisible cloak (7), (8), GLHUA-3 invisible cloak, (14) and (15), the refractive index \( n = \sqrt{\varepsilon_\theta \mu_\theta} \) is large than 1 and going to infinity when \( r \) going to \( R_1 \). The phase velocity of EM wave propagation in GLHUA-1, GLHUA-2, GLHUA-3 cloak are less than light speed and going to zero when \( r \) going to \( R_1 \). GLHUA-1, GLHUA-2, GLHUA-3 cloak are practicable. In Pendry invisible cloak (141) and (142), the refractive index \( n = \sqrt{\varepsilon_\theta \mu_\theta} \) is less than 1 and going to zero when \( r \) going to \( R_1 \). The phase velocity of EM wave propagation in Pendry cloak is exceeding light speed and going to infinity when \( r \) going to \( R_1 \). Pendry invisible cloak is very difficult and can not be practicable. The analytical Magnetic wave in GLHUA-1 cloak (135) in theorem 3 and analytical Magnetic wave in Pendry cloak (154) in theorem 4 are totally different. Any annular layer Pendry cloaking by Pendry transformation will make the phase velocity exceeding light speed and tends to infinity in \( r = R_1 \). Our GLHUA-1, GLHUA-2 and GLHUA-3 invisible cloak without exceeding light speed and without infinity, and overcome the fundamental difficulties of Pendry cloak. Our GLHUA-1 analytical expansion method can be used to find analytical EM wave propagation in the GLHUA-1, GLHUA-2 and GLHUA-3 invisible cloak. Also analytical EM wave propagation in the GLHUA-2 and GLHUA-3 invisible cloak can be found by the GLHUANP-2 and GLHUANP-3 transformation respectively.

C. Analytical EM wave in GLHUA-1, GLHUA-2 and GLHUA-3 cloak are undisputed evidence to prove that GLHUA-1, GLHUA-2 and GLHUA-3 cloak are practicable invisible cloak and super sciences is being born

Our analytical EM wave in GLHUA-1, GLHUA-2 and GLHUA-3 cloak are undisputed evidence and rigorous proof to prove that the GLHUA-1, GLHUA-2 and GLHUA-3 double layer cloak are practicable invisible cloak without exceed light speed propagation. Our analytical EM wave in GLHUA-1, GLHUA-2 and GLHUA-3 cloak is undisputed evidence and rigorous proof to prove GLHUA-1, GLHUA-2 and GLHUA-3 double layer invisible cloak and their theoretical proof in are right.

GL simulation shows that there are novel GLHUA-1 created wave by cloak materials bridge which is generated by the GLHUA-1 outer layer invisible cloak materials in (82) and (84). The GLHUA-1 created wave by cloak materials bridge make the GLHUA-1 analytical EM wave propagation without exceeding light speed. The GLHUA-1 created wave by cloak materials bridge is obvious for \( R_1 = 0.5R_2 \) or \( R_1 = 0.66R_2 \), for \( R_1 > 0.8R_2 \), the GLHUA-1 created wave by cloak materials bridge wave become to the arc GLHUA-1 created wave by cloak materials wave on the inner spherical layer \( r = R_1 \).

\[
\varepsilon_r = 1, \\
\varepsilon_\theta = \varepsilon_\phi = \frac{1}{2} \left( \left( \frac{r-R_2}{R_1-R_2} \right)^\alpha + \left( \frac{R_1-R_2}{R_1-R_2} \right)^\alpha \right),
\]

In our paper [5-6][11-12][15], we proposed GL no scattering modeling and inversion to create practicable GLHUA-1 and GLH two class of the invisible cloak. Our GLHUA-1 cloak by GL no scattering modeling and inversion is different from GLH cloak in 2010. In 1971, we used no scattering inversion idea to construct a novel
Our idea and method are different from Pendry and other scattering modeling and inversion for 6 years since 2000. When Pendry cloak in 2006, we had been working in no very important role for research works in invisible cloak. This scattering modeling and inversion idea and method play in the world [26][27]. GILD and GL scattering and no super convergence of 3D iso parameter finite element first in China and discovered 3D 20 nodes high accurate curve element and developed invisible cloak, therefore the full wave theoretical proof and full wave computational simulation are necessary to verify the invisible cloak. GLHUA-1 invisible cloak and GLLH invisible cloak have been verified by the full wave theoretical proof and full wave simulation using GL full wave no scattering modeling and inversion in [1][2].

FIG. 32: (color online) At relative 15 time step, the magnetic wave front (red line (1)) incoming to right side of cloak, the material created curve wave front (blue line (2)) form wave front on the left side of cloak. Both wave front near each other but no connected.

FIG. 33: (color online) At relative 16 time steps, the curve wave front created by cloak material and incident wave in outside of cloak form wave front in the left side of the cloak (blue line (2)), the incoming magnetic wave front red line (1) right side of cloak is disconnect with created wave (blue line (2)).

FIG. 34: (color online) At relative 20 time steps, the curve wave front created by the cloak material is connected to the incident wavefront outside the cloak to form a wavefront and propagate to the left side of the cloak (blue line (2)). The incoming magnetic wave (red line (1)) is separated from the created wave (blue line (2)), and the incoming wavefront (red line (1)) is shrink, and wavefront (red line (1)) is behind from the created wavefront (blue line (2)).
Magnetic Wave Propagation In Novel GLHUA Outer Layer CLOAK
GLHUA cloak material absorb incident EM wave and create outgoing wave
R1=0.5m, R2=1.0m, dt=3.22e-10, by transformation from infinity to R1, -R2 to R2
Without Exceeding Light Speed Propagation
Jianhua Li, Feng Xie, Lee Xie, Ganquan Xie In GLGEO
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FIG. 35: (color online) At relative 25 time steps, the curve wave front created by cloak material and incident wave front in outside of cloak are connected to form wave front that propagate to boundary, the left side of the cloak (blue line (2)); The incoming magnetic wave (red line (1)) is separated from the created wave (blue line (2)), and wavefront (red line (1)) is behind from the created wavefront (blue line (2)), and the incoming wavefront (red line (1)) is absorbed and shrink.

FIG. 36: (color online) At relative 25 time steps, the material created curve wave front propagation (blue line (2)) has already been out of cloak and did not disturb incident wave in outside of the cloak and make the cloak is invisible; the incoming magnetic wave front (red line (1)) is shrink to a circle and absorbed and can not be penetrated to the concealment. The inner sphere is really cloaked concealment.

FIG. 37: (color online) Radial magnetic wave propagation at 1 time step.

FIG. 38: (color online) Radial magnetic wave propagation at 5 time step.

materials in [13] in ArXiv:1612.02857 show that GL no scattering modeling and inversion is powerful method to make practicable invisible cloak and to investigate new invisible natural science Inventor Easton LaChappelles mind control robot hands is beginning to investigate visible thinking science and visible social science. In current general science, visible natural science and invisible thinking science and social science are main object. Now invisible natural science and invisible thinking science and visible social science will be main object for a novel super science. Our GLHUA-1 practicable double cloak and Easton LaChappelles mind control robot hands show that the new novel super science is being born.
When the radial variable in the sphere space is positive, if the wave solution of the electric wave equation is \( H_r(R, \theta, \phi) \), then \( H_r(-R, \theta, \phi) \) is also solution of homogeneous equation (82) with \( M_s = 0 \).

Proof. For point source at \((r_s, \theta_s, \phi_s)\), if \( H_r(R, \theta, \phi) \) is solution of (82) with point source. By heavy calculus calculation, we prove that the \( H_r(-R, \theta, \phi) \) satisfies homogeneous equation (134) with \( M_s = 0 \).

Theorem 6: The homogeneous radial electric wave equation (134) with \( M_s = 0 \), its solution \( H_r(R, \theta, \phi) \) in positive space can be analytic continuation into negative space with negative radial \( R \).

Proof: Let \( H_r(R, \theta, \phi) \) be the solution of the radial electric equation (134) with point source, by theorem 5, the \( H_r(-R, \theta, \phi) \) is also the solution of the radial electric wave equation.
\[
\frac{\partial}{\partial R} \frac{\partial}{\partial R} R^2 H_r(-R, \theta, \phi) + \frac{1}{\sin \theta} \frac{1}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} H_r(-R, \theta, \phi) + k^2 R^2 H_r(-R, \theta, \phi) = 0, \tag{157}
\]

where 
\[
\frac{\partial}{\partial \xi} \frac{\partial}{\partial \xi} R^2 H_r(-\xi, \theta, \phi) + \frac{1}{\sin \theta} \frac{1}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} H_r(-\xi, \theta, \phi) + k^2 R^2 H_r(-\xi, \theta, \phi) = 0, \tag{158}
\]

and 
\[
\frac{\partial}{\partial \xi} \frac{\partial}{\partial \xi} R^2 H_r(\xi, \theta, \phi) + \frac{1}{\sin \theta} \frac{1}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} H_r(\xi, \theta, \phi) + k^2 R^2 H_r(\xi, \theta, \phi) = 0, \tag{159}
\]

By using \( R = \xi \), we obtain
\[
\frac{\partial}{\partial R} \frac{\partial}{\partial R} R^2 H_r(R, \theta, \phi) + \frac{1}{\sin \theta} \frac{1}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} H_r(R, \theta, \phi) + k^2 R^2 H_r(R, \theta, \phi) = 0, \tag{160}
\]

Therefore, homogeneous radial electric wave equation (134) with \( M_r = 0 \) and its solution \( H_r(R, \theta, \phi) \) in positive space can be analytic continuation into negative space with negative radial \( R \) in equation (137). The theorem 6 is proved.

**B. GLHUANP-4 transformation to map the Negative Infinity \(-\infty\) to \( R_1 \) and \( R_2 \) to \( R_2 \)**

1. **GLHUANP-4 transformation**

We create a novel sphere radial transformation GLHUANP-4 to map the negative infinity, \(-\infty\) in Basic Space (BS) to \( R_1 \) in the Physical space (PS) and keep \( R_2 \) invariant that is as follows

\[
r = R_1 + (R_2 - R_1)e^{(R-R_2)/(R_2-R_1)}, \tag{161}
\]

and its inverse transformation

\[
R = (R_2 - R_1)\log((r-R_1)/(R_2-R_1)) + R_2, \tag{162}
\]

Where \( R \) be radial variable in the Basic Space (BS), \( r \) be radial variable in the Physical Space (PS).

2. **GLHUANP-4 relative electromagnetic permeability and dielectric are created**

By the GLHUANP-4 transformation, new GLHUANP-4 invisible cloak and GLHUANP-4 relative EM parameters are created as follows:

\[
\varepsilon_r = \mu_r = (R_2 - R_1)/(r - R_1)(163) \tag{163}
\]

the angular electric dielectric and magnetic permeability of GLHUANP-4 are same as the relative angular EM parameters of GLHAU-1 EM invisible cloak.

**C. The magnetic wave propagation through the GLHUANP-4 invisible cloak**

The magnetic wave propagation through the GLHUANP-4 invisible cloak with relative GLHUANP-4 electromagnetic parameters (84) in \( R_1 \leq r \leq R_2 \) and free space in \( 0 \leq r \leq R_1 \)

**D. GLHUANP-4 Invisible Cloak Borns GLHAU-3 invisible cloak without exceeding light speed propagation and GLHUANP-4 Invisible Cloak With exceeding light speed propagation**

The GLHUANP-4 radial transformation maps the Negative Infinity \(-\infty\) in BS To \( R_1 \) in PS and \( R_2 \) in BS to \( R_2 \) in PS. By the GLHUANP-4 transformation, the relative electric dielectric and magnetic permeability,

\[
\varepsilon_r = \mu_r = \frac{R_2 - R_1}{\sqrt{r^2 - R_1^2}}, \tag{165}
\]

and

\[
\varepsilon_\theta = \mu_\theta = \frac{R_2 - R_1}{\sqrt{r^2 - R_1^2}}, \tag{166}
\]

in (165)-(166) are installed in the annular layer, \( R_1 \leq r \leq R_2 \). In addition, the inner small sphere \( 0 \leq R \leq R_1 \), and outside of large sphere \( r \geq R_2 \), be free space, a GLHUANPII invisible cloak is created with concealment, \( 0 \leq r \leq R_1 \). Let

\[
H_1 = R_1 + (R_2 - R_1)e^\frac{-R_2}{\sqrt{R_2^2 - R_1^2}}, \tag{167}
\]

and

\[
H_0 = R_1 + (R_2 - R_1)e^\frac{-R_0+R_2}{\sqrt{R_2^2 - R_1^2}}, \tag{168}
\]

In the annular layer \( H_1 \leq r \leq R_2 \), the relative EM GLHUANP-4 refractive index

\[
0 \leq n = \sqrt{\varepsilon_\theta \mu_\theta} \leq 1, \tag{169}
\]

and

\[
n(H_1) = \sqrt{\varepsilon_\theta \mu_\theta}(H_1) = 0, \tag{170}
\]

and

\[
n(R_2) = \sqrt{\varepsilon_\theta \mu_\theta}(R_2) = 1, \tag{171}
\]
and EM parameters to be 1 in the inner sphere free space \(0 \leq r \leq H_1\), GLHUANP-4 cloak become an GLHUANP-4 - 3 invisible cloak with exceeding light speed and infinity velocity at \(r = H_1\), which is similar with, but is different from the Pendry cloak. In the annular layer \(R_1 \leq r \leq H_0\), the relative EM GLHUANP-4 refractive index

\[
1 \leq n = \frac{\varepsilon_\theta \mu_r}{\varepsilon_r \mu_\theta} \leq \infty
\]  

(172)

and,

\[
n(H_0) = \frac{\varepsilon_\theta \mu_r}{\varepsilon_r \mu_\theta} = 1,
\]  

(173)

and EM parameters to be 1 in the inner sphere free space \(0 \leq r \leq R_1\), the GLHUANP-4 cloak becomes GLHUANP-3 invisible cloak which is similar with the GLLHUA-3 invisible cloak with relative EM refractive index parameter not less than one and without exceeding light speed physical difficulties.

\[\frac{\varepsilon_\theta = \varepsilon_\phi = \mu_\theta = \mu_\phi}{\varepsilon_r = \mu_r = 1}, \quad \frac{1}{2} \left(\frac{(r - R_1)}{R_2 - R_1} + \left(\frac{R_2 - R_1}{r - R_1}\right)\right), R_1 < r \leq R_2,\]  

(177)

or GLHUA-1 - III practicable invisible cloak with EM parameter not less one

\[\varepsilon_\theta = \mu_\theta = \frac{(R_2 - R_1)}{(r - R_1)},\]  

(179)

2. By GL super computational sciences no scattering modeling and inversion, we discovered and created GLHUA-1 - I and GLHUA-1 - III practicable invisible cloak with EM parameter not less one, by GL no scattering modeling simulation, we find the numerical electromagnetic wave propagation in the GLHUA-1 - I and GLHUA-1 - III practicable invisible cloak with EM parameter not less one

3. By GL no scattering modeling simulation, the radial magnetic wave propagation in GLHUA-1 practicable invisible cloak

\[\varepsilon_r = \mu_r = 1,\]

(178)

G. A novel and strange GLHUA-1 cloak wave in

\[R_1 \leq r \leq R_2\]  

and around and attached on inner sphere surface the \(r = R_1\), and the GLHUA-1 wave front is discontinuous.

A novel and strange GLHUA-1 cloak wave in \(R_1 \leq r \leq R_2\) and around and attached on inner sphere surface the \(r = R_1\), and the GLHUA-1 wave front is discontinuous. The novel and strange GLHUA-1 cloak wave is created wave by GLHUA-1 cloak materials in (25)

H. Discussions and Conclusions

1. The two novel wave fronts of GLHUA-1 and GLHUA-2 and GLHUA-3 are discontinuous

The GLHUA-1 and GLHUA-2 and GLHUA-3 are practicable invisible cloak with EM parameter not less than one and without exceeding light speed propagation fundamental difficulties. The two novel wave fronts of GLHUA-1 and GLHUA-2 and GLHUA-3 are discontinuous. One wave front is absorbing incoming incident EM wave by the part of GLHUA-1, GLHUA-2 and GLHUA-3 materials. Other wave front is creating outgoing EM wave by the other part of GLHUA-1, GLHUA-2 and GLHUA-3 materials. The novel EM wave front propagation in GLHUA-1, i.e., \(i = 1, 2, 3\) are displaying in Figure 2 to Figure 36 that proved that our GLHUA-1 and GLHUA-2 and GLHUA-3 cloak are invisible cloaks with the refractive index large or equal one.

2. The novel GLHUA-1 practicable invisible cloak with EM parameter not less one

\[\varepsilon_\theta = \mu_\theta = \frac{(R_2 - R_1)}{(r - R_1)},\]

(179)

The novel GLHUA-1 practicable invisible cloak with EM parameter not less one.

\[\varepsilon_r = \mu_r = 1,\]

(176)

In our paper arXiv:1706.10147[3], we proposed the novel GLHUA-1 practicable invisible cloak with EM parameter not less one[3]

\[\varepsilon_r = \mu_r = \frac{R_2^2}{r^2},\]

(174)

\[\varepsilon_\theta = \mu_\theta = \varepsilon_\phi = \mu_\phi = \frac{R_2 - R_1}{r - R_1},\]

(175)

2. In our paper arXiv:1706.10147[3], we create the new GLHUA-1 separate variable method and find analytical radial electric and radial magnetic wave propagation through GLHUA-1 practicable invisible cloak

3. The analytical radial magnetic wave propagation through GLHUA-1 practicable invisible cloak is shown in the Figure 13 and Figure 27

F. The electric and magnetic wave propagation through GLHUA-1 practicable invisible cloak with EM parameter not less one

1. GLHUA-1 practicable invisible cloak with EM parameter not less one

In our papers arXiv:1612.02857[2], arXiv:1701.00534[6], arXiv:1701.00258[7], we created GLHUA-1 practicable invisible cloak with EM parameter not less one.

\[\varepsilon_r = \mu_r = 1,\]

(176)

\[\varepsilon_\theta = \varepsilon_\phi = \mu_\theta = \mu_\phi = \frac{1}{2} \left(\frac{(r - R_1)}{R_2 - R_1} + \left(\frac{R_2 - R_1}{r - R_1}\right)\right), R_1 < r \leq R_2,\]

(177)

In our paper arXiv:1612.02857[2], arXiv:1701.00534[6], arXiv:1701.00258[7], we created GLHUA-1 practicable invisible cloak with EM parameter not less one.

\[\varepsilon_r = \mu_r = 1,\]

(176)
2. The Inner Sphere of Pendry cloak, $0 \leq r \leq R_1$ is not cloaked concealment, if Pendry EM parameters are put in it.

In the Pendry Cloak, the relative EM parameters are from Pendry transformation in the annular layer $R_1 \leq r \leq R_2$, also. Pendry put free space media with relative EM parameter 1 in the inner sphere $0 \leq r \leq R_1$ that create Pendry invisible cloak in his paper in sciences 2006 [4]. In my paper [3], we discovered and proved an important problem that if the Pendry relative EM parameters

$$\varepsilon_r = \mu_r = (R_2/(R_2 - R_1))(r - R_1)^2/r^2$$

(180)

$$\varepsilon_\theta = \mu_\theta = \varepsilon_\phi = \mu_\phi = R_2/(R_2 - R_1)$$

(181)

are sit in the whole sphere $0 < R_1 < R_2$ which include the annular $R_1 \leq r \leq R_2$ and inner sphere $0 \leq r \leq R_1$. What happen is for EM wave propagation? For solving this problem, we discover and propose the novel negative space in which the radial variable of the sphere coordinate is negative. The real 3D space we living with positive or zero radial variable is called positive space. The positive space is visible 3D space we living and From my negative space theory, we prove that if the Pendry EM parameters

$$\varepsilon_r = \mu_r = (R_2/(R_2 - R_1))(r - R_1)^2/r^2$$

are sit in the whole sphere $0 < R_1 < R_2$ which include the annular $R_1 \leq r \leq R_2$ and inner sphere $0 \leq r \leq R_1$, the inner sphere is not cloaked concealment. In Figure in other paper, we present the Pendry EM parameters (90-91) in [4] are sit in the whole sphere $0 < R_1 < R_2$ and inner sphere $0 \leq r \leq R_1$. The radial magnetic wave excited by point source in $r_s >> R_2$ propagate through whole sphere with Pendry EM parameters (180)-(181) in the paper) in [4]. There is radial magnetic wave propagation in the annular layer $R_1 \leq r \leq R_2$ and also in the inner sphere $0 \leq r \leq R_1$, the inner sphere is not cloaked concealment.

In Figure in other paper, we present the Pendry EM parameters (90-91) in [4] are sit in the whole sphere $0 < R_1 < R_2$ and inner sphere $0 \leq r \leq R_1$. The radial magnetic wave excited by point source in $r_s >> R_2$ propagate through whole sphere with Pendry EM parameters (180)-(181) in the paper) in [4]. There is radial magnetic wave propagation in the annular layer $R_1 \leq r \leq R_2$ and also in the inner sphere $0 \leq r \leq R_1$, the inner sphere is not cloaked concealment.

VIII. ANOTHER NOVEL GLHUAF INVISIBLE CLOAK WITH DOUBLE NEGATIVE EM PARAMETER AND SQUARE OF REFRACTIVE INDEX $n^2 \geq 1$

A. Another GLHUAF radial transformation

GLHUANPII radial transformation inspires us to invent another novel radial Transformation GLHUAF radial transformation.

$$r = R_1 + (R_2 - R_1)e^{R_2 - R_1}/(R_2 - R_1)$$

(19)

Which maps positive infinity $+\infty$ to $R_1$, and $R_2$ is invariant. $R$ is radial variant in basic space before transformation, $r$ is radial transformation in the physical space after transformation.

B. GLHUAF Double Negative EM relative parameters

Use the GLHUAF transformation to transform the radial magnetic equation in free space from positive infinity $+\infty$ to $R_1$, and $R_2$ is invariant.

$$\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial H}{\partial r} + \frac{1}{r \mu_\epsilon} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial H}{\partial \theta} \right) + \frac{k^2}{r^2} \varepsilon_\theta H = M_s$$

(20)

novel double negative GLHUAF relative EM parameters are created,

$$\varepsilon_\theta = \mu_\theta = \varepsilon_\phi = \mu_\phi = -\frac{R_2 - R_1}{r - R_1}$$

(21)

$$\varepsilon_r = \mu_r = -\frac{R_2^2}{r^2} \frac{r - R_1}{R_2 - R_1}$$

(22)

By the inverse transformation of (19), in the (21), the

$$R = R_2 - (R_2 - R_1) \log \left( \frac{r - R_1}{R_2 - R_1} \right)$$

(23)

$$\varepsilon_r = \mu_r = -\frac{1}{r^2} \frac{r - R_1}{R_2 - R_1} \left( R_2 - (R_2 - R_1) \log \left( \frac{r - R_1}{R_2 - R_1} \right) \right)^2$$

(24)

C. GLHUAF EM invisible cloak

By installing the GLHUAF relative double negative electric permittivity and magnetic permeability (21) and (24) in the sphere annular layer $R_1 < r \leq R_2$ and installing the free space in the inner sphere $r \leq R_1$, the GLHUAF invisible cloak is created. Novel EM wave propagation through the GLHUAF invisible cloak are presented in Figure 37 to the Figure 42.

IX. GLHUAF-3 INVISIBLE CLOAK BY TRANSFORMATION WHICH MAPS NEGATIVE INFINITY TO $R_1$ AND MAPS $-R_2$ TO $R_2$

We discover additional new invisible cloak GLHUAF-3 in this paper. GLHUANP-3 radial transformation,

$$r = R_1 + (R_2 - R_1)e^{R_2 - R_1}/(R_2 - R_1)$$

(12)

which maps negative infinity, $-\infty$ to $R_1$, and maps $-R_2$ to $R_2$. The GLHUANP-3 inverse transformation is

$$R = -R_2 + (R_2 - R_1) \log \left( \frac{r - R_1}{R_2 - R_1} \right)$$

(13)
From the GLHUANP-3 transformation (12), GLHUA-3 relative electric permittivity and magnetic permeability are created as

\[
\varepsilon_\theta = \mu_\theta = \varepsilon_\phi = \mu_\phi = \frac{R_2 - R_1}{r - R_1}, \quad (14)
\]

\[
\varepsilon_r = \mu_r = \left( -R_2 + (R_2 - R_1) \log \left( \frac{r - R_1}{R_2 - R_1} \right) \right)^2, \quad (15)
\]

The GLHUA-3 radial parameters (15) is as follows, By installing GLHUA-3 EM parameters (14) and (15) in the annular layer \( R_1 \leq r \leq R_2 \) and installing free space in inner sphere \( 0 \leq r \leq R_1 \), the new GLHUA-3 invisible cloak is created. The properties of GLHUA-3 invisible cloak are as following:

1. The EM parameters are continuous on the \( r = R_2 \)

\[
\varepsilon_\theta(R_2) = \mu_\theta(R_2) = \varepsilon_\phi(R_2) = \mu_\phi(R_2) = 1, \quad (16)
\]

\[
\varepsilon_r(R_2) = \mu_r(R_2) = 1, \quad (17)
\]

2. The refractive index

\[
1 \leq n \leq +\infty, \quad (18)
\]

3. The EM wave propagation in the GLHUA-3 cloak without exceeding light speed difficulties.

The radial electric wave propagation through the GLHUA-3 cloak is display in the figure 43-50.

**FIG. 44:** (color online) At relative 5th time step, the magnetic wave front incoming to right side of cloak, the material created curve wave front expand up and down

**FIG. 45:** (color online) At relative 14 time step, the magnetic wave incoming to right side of cloak, the material created curve wave front form wave front on the left side of cloak. Both wave front near each other but no connected.

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Magnetic Wave Propagation In Novel GLHUA Active Invisible CLOAK
The active invisible cloak material absorbs incoming wave and products outcoming wave
GLHUA Cloak is created by -infinity to \( R_1 \) and \( -R_2 \) to \( R_2 - 2 \) radial transformation
\( R_1 = 0.5 \text{ m}, R_2 = 1.0 \text{ m} \), source at \( (18.3, 0, 0) \), \( dt = 2.0 \times 10^{-10} \text{ S} \)

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FIG. 46: (color online) At relative 15 time step, the magnetic wave front incoming to right side of cloak, the material created curve wave front form wave front in the left side of the cloak, the red incoming wave (1) and blue created wave front (2) are connected.

FIG. 47: (color online) At relative 16 time steps, the curve wave front created by cloak material and incident wave in outside of cloak form wave front in the left side of the cloak, the incoming magnetic wave front right side of cloak is disconnected with created wave

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