Abstract

In this short note, we improve and extend Yao’s paper "On the power of quantum fingerprinting" [Yao03] about simulating a classical public coin simultaneous message protocol by a quantum simultaneous message protocol with no shared resource.

1 Introduction

The simultaneous message model of communication complexity can be described as follows. Suppose $f : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}$ is a function. There are three players viz. Alice, Bob and a referee. Alice possesses $x \in \{0,1\}^n$ and Bob possesses $y \in \{0,1\}^n$. Alice and Bob each send a single message to the referee, who then outputs a guess for $f(x, y)$. Alice’s and Bob’s messages can be classical or quantum.

In the classical public coin simultaneous message model, Alice, Bob and the referee know the state of an additional random variable called a public coin, that is chosen independently of the input $(x, y)$ according to some probability distribution. The messages $a$ and $b$ of Alice and Bob respectively are deterministic functions of the state $l$ of the public coin as well as the input $(x, y)$ i.e. $a = a(x, l)$ and $b = b(y, l)$. Suppose $a$ is at most $c_A$ bits long and $b$ is at most $c_B$ bits long, for any input $(x, y)$ and any value $l$ of the public coin. Given the state $l$ of the public coin, the strategy of the referee is deterministic and can be modeled by an $M_A \times M_B$ boolean matrix $D_l$, where $M_A \triangleq 2^{c_A}$, $M_B \triangleq 2^{c_B}$. The rows and columns of $D_l$ are indexed by the possible messages of Alice and Bob respectively. On receiving messages $a$ and $b$ from Alice and Bob respectively, the referee outputs $D_l(a, b)$. We require that the protocol be correct with probability at least $3/4$ for all inputs, that is,

$$\forall x, y \in \{0,1\}^n : \Pr_l[D_l(a(x, l), b(y, l)) = f(x, y)] > 3/4,$$

where the probability is over the choice of the public coin $l$. By a result of Newman [New91], one can assume that the public coin $l$ is chosen uniformly from the set $[L]$, where $L = O(n)$, at the expense of making the correctness probability at least $2/3$. The communication cost of the protocol is defined to be $c_A + c_B$. We let $R^{||,\text{pub}}(f)$ denote the communication complexity of $f$ in this model i.e. the smallest communication cost of a protocol in this model computing $f$.

In the quantum simultaneous message model, Alice, Bob and the referee are quantum computers. There is no prior entanglement amongst them i.e. at the start of the protocol, the states of Alice, Bob and the referee are in tensor with each other. Alice sends a pure state $|u_x\rangle$ on $c_A$ qubits and Bob sends a pure state $|v_y\rangle$ on $c_B$ qubits to the referee. The states $|u_x\rangle$ and $|v_y\rangle$ are called the fingerprints of Alice’s and Bob’s inputs.
x and y respectively. The referee performs a two-outcome POVM on \(|u_x\rangle \otimes |v_y\rangle\) and outputs the result. We require that the protocol be correct with probability at least 3/4 for all inputs. The communication cost of the protocol is defined to be \(c_A + c_B\). We let \(Q(f)\) denote the communication complexity of \(f\) in this model.

In a recent paper, Yao \cite{Yao03} showed how to simulate a classical public coin simultaneous protocol by a quantum simultaneous protocol that has no prior entanglement. The simulation incurs an exponential overhead. More precisely, he showed that \(Q(f) \leq O(2^2R^{\|\text{pub}}(f)(R^{\|\text{pub}}(f) + \log n))\). He also defined a quantity called the convex width \(\text{convw}(D)\) of an \(M \times M\) matrix \(D\), and remarked that \(\text{convw}(D) \leq M\) if all entries of \(D\) are either 0 or 1. If \(D\) is the referee matrix of an optimal classical public coin simultaneous message protocol for \(f\) (note that in Yao’s paper the referee matrix is assumed to be square and independent of the public coin), Yao showed that, in fact, \(Q(f) \leq O(\text{convw}(D)^4(R^{\|\text{pub}}(f) + \log n))\).

In this note, we strengthen and generalise Yao’s results. We start by proving a near quadratic improvement of Yao’s general simulation of a classical public coin simultaneous message protocol by a quantum simultaneous message protocol without prior entanglement. More precisely, we show that \(Q(f) \leq O(2^2R^{\|\text{pub}}(f)(R^{\|\text{pub}}(f) + \log n + 1))\). The same result was independently obtained by Gavinsky, Kempe and de Wolf \cite{GavinskyKempeWolff04}. A similar result for a related setting of communication complexity was recently proved by Gavinsky \cite{Gavinsky05}. We then define a new notion called the row-column width \(\text{rcw}(D)\) of an \(M \times M\) matrix \(D\). For the row-column width, we can assume that our matrix is square without loss of generality, since a non-square matrix can be made square by padding with zeroes without changing its row-column width. For all square matrices \(D\), \(\text{rcw}(D) \leq \text{convw}(D)\). For all square matrices \(D\), \(\text{rcw}(D) \leq \sqrt{M}\). We show that if \(D\) is the referee matrix of an optimal classical public coin simultaneous message protocol for \(f\), \(Q(f) \leq O(\text{convw}(D)^4(R^{\|\text{pub}}(f) + \log n + 1))\).

The notation \(|\cdot|\) below stands for the \(\ell_2\)-norm of a vector.

## 2 An almost quadratic improvement of Yao’s general simulation

Consider an optimal classical public coin simultaneous message protocol for \(f\). Let \(c_A\) and \(c_B\) be upper bounds on the message lengths of Alice and Bob respectively for any input \((x, y)\) and any value \(l\) of the public coin. Define \(M_A \triangleq 2^c_A\), \(M_B \triangleq 2^c_B\). Assume without loss of generality that \(c_A \leq c_B\). Then, \(M_A \leq 2^{R^{\|\text{pub}}(f)/2}\). Define

\[
|u_x\rangle \triangleq \frac{1}{\sqrt{L}} \sum_{l \in L} |l\rangle \otimes |a(x, l)\rangle, \quad |v_y\rangle \triangleq \frac{1}{\sqrt{L}MA} \sum_{l \in L} |l\rangle \otimes D_l|b(y, l)\rangle,
\]

where matrix \(D_l\) is considered as a linear operator from Bob’s message space to Alice’s message space viz. \(D_l|b\rangle \triangleq \sum_a D_l(a, b)|a\rangle\). Now the inner product

\[
\langle u_x|v_y\rangle = \frac{1}{\sqrt{MA}} \sum_{l \in L} \frac{(a(x, l)|D_l|b(y, l))}{L},
\]

that is, \(\langle u_x|v_y\rangle\) is the probability that the classical protocol outputs 1 divided by \(\sqrt{MA}\). \(||u_x||\) = 1. Since each \(D_l\) is an \(M_A \times M_B\) boolean matrix, \(||D_l|b|| \leq \sqrt{M_A}\) for all possible classical messages \(|b\rangle\) of Bob. Hence, \(||v_y|| \leq 1\).

Now define

\[
|\hat{u}_x\rangle \triangleq |0\rangle|u_x\rangle, \quad |\hat{v}_y\rangle \triangleq |0\rangle|v_y\rangle + |1\rangle|\text{junk}_y\rangle,
\]

2
where $|\text{junk}_y\rangle$ is added to ensure that $||\tilde{v}_y|| = 1$. Hence, $\langle\tilde{u}_x|\tilde{v}_y\rangle = \langle u_x|v_y\rangle$. Thus if $f(x,y) = 1$, $\langle\tilde{u}_x|\tilde{v}_y\rangle \geq (3/4) \cdot \sqrt{M_A}$, and if $f(x,y) = 0$, $\langle\tilde{u}_x|\tilde{v}_y\rangle \leq (1/4) \cdot \sqrt{M_A}$. By a Chernoff bound (see e.g. [AS00 Corollary A.1.7]), Alice and Bob can each send $O(M_A^2)$ independent copies of $|\tilde{u}_x\rangle$ and $|\tilde{v}_y\rangle$ respectively to the referee, who can then determine $\langle f(x,y)\rangle$ with error probability at most $1/4$ by inner product estimation via the controlled-swap circuit [BCWdW01, Yao03]. It follows that $Q^\parallel(f) \leq O(2^{R^\parallel \text{pub}(f)}(R^\parallel \text{pub}(f) + \log n + 1))$.

### 3 Fingerprinting and row-column width

We now generalize the construction of the above section, and in the process, also generalize Yao’s convex width [Yao03] to get our new notion of row-column width. Define $M \triangleq \max\{M_A, M_B\}$. For convenience of notation, we assume that the possible classical messages of Alice as well as Bob come from the set $[M]$. Thus, we assume that for any state $l$ of the public coin the referee matrix $D_l$ is an $M \times M$ matrix. This assumption is without loss of generality, as will become clear later.

For an $M \times N$ matrix $Q$, define the column norm of $Q$ as

$$cn(Q) \triangleq \max_{b \in [N]} ||Q_b||,$$

where $Q_b$ denotes the $b$th column of $Q$. The row norm of $Q$, $rn(Q)$, is defined similarly. Fix an integer $K > 0$. For every $l \in [L]$, decompose $D_l$ as a product $D_l = E_lF_l$ for some $M \times K$ matrix $E_l$ and $K \times M$ matrix $F_l$. Define the row width, column width and row-column width of $D \triangleq \{D_l\}_{l \in L}$ according to the above decompositions as follows.

$$\begin{align*}
\text{rw}(D) & \triangleq \sqrt{\frac{1}{L} \sum_{l \in [L]} \text{rn}(E_l)^2}, \\
\text{cw}(D) & \triangleq \sqrt{\frac{1}{L} \sum_{l \in [L]} \text{cn}(F_l)^2}, \\
\text{rcw}(D) & \triangleq \text{rw}(D) \cdot \text{cw}(D).
\end{align*}$$

The row-column width of $D \triangleq \{D_l\}_{l \in L}$ is defined to be the minimum row-column width over all decompositions of $D_l$ into products of $M \times K$ and $K \times M$ matrices, where $K = M^2$.

Fix such optimal decompositions of $D_l$ with $K = M^2$. Define

$$\begin{align*}
|u_x\rangle & \triangleq \frac{1}{\text{rw}(D)\sqrt{L}} \sum_{l \in L} |l\rangle \otimes E^\dagger_l |a(x,l)\rangle, \\
|v_y\rangle & \triangleq \frac{1}{\text{cw}(D)\sqrt{L}} \sum_{l \in L} |l\rangle \otimes F_l |b(y,l)\rangle.
\end{align*}$$

It is easy to check that $||u_x|| \leq 1$ and $||v_y|| \leq 1$. Now the inner product

$$\langle u_x|v_y\rangle = \frac{1}{\text{rcw}(D)} \sum_{l \in [L]} \frac{\langle a(x,l)|D_l||b(y,l)\rangle}{L},$$

that is, $\langle u_x|v_y\rangle$ is the probability that the classical protocol outputs 1 divided by $\text{rcw}(D)$. 

3
Now define
\[ |\tilde{u}_x\rangle \triangleq |00\rangle|u_x\rangle + |01\rangle|\text{junk}_x\rangle, \quad |\tilde{v}_y\rangle \triangleq |00\rangle|v_y\rangle + |10\rangle|\text{junk}^*_y\rangle, \]
where \(|\text{junk}_x\rangle, |\text{junk}^*_y\rangle\) are added to ensure that \(||\tilde{u}_x\rangle|| = 1, ||\tilde{v}_y\rangle|| = 1\) respectively. Hence, \(\langle u_x|\tilde{v}_y\rangle = \langle u_x|v_y\rangle\). Reasoning as in the previous section, we get that \(Q^4(f) \leq O(rcw(D)^4(R^{\text{pub}}(f) + \log n + 1))\).

### 4 Two properties of the row-column width

For \(l \in [L]\), consider the trivial decomposition \(D_l = ID_l\), where \(I\) denotes the \(M \times M\) identity matrix. Since each \(D_l\) is an \(M \times M\) boolean matrix,
\[
cw(D) \leq \sqrt{\frac{1}{L} \sum_{l \in [L]} \text{cn}(D_l)^2} \leq \sqrt{M}
\]
for this decomposition. Also for this decomposition \(rw(D) = 1\). Thus, for \(M \times M\) boolean matrices the row-column width \(rw(D) \leq \sqrt{M}\).

Consider now the case when all the \(D_l\)’s are the same. We shall denote them by \(D\). \(D\) is an \(M \times M\) boolean matrix. We will show below that \(rcw(D) \leq \text{convw}(D)\), where \(\text{convw}(D)\) is the convex width of \(D\) defined by Yao [Yao03]. We first recall the definition of \(\text{convw}(D)\).

**Definition 1 (Yao03)** \(\text{convw}(D)\) is the minimum integer \(W\) for which there exists a decomposition \(D = \sum_{j=1}^{W} G_j P_j\), where each \(P_j\) is an \(M \times M\) permutation matrix and each \(G_j\) is a symmetric positive semidefinite matrix with non-negative real entries.

Yao [Yao03] also remarked that \(\text{convw}(D) \leq M\) for any \(M \times M\) boolean matrix \(D\). Indeed, consider the following cyclic diagonal decomposition \(D = \sum_{j=1}^{M} D_j\), where
\[
D_j(a, b) \triangleq D(a, b) \quad \text{if} \quad b - a \equiv (j - 1) \mod M
\]
\[
\triangleq 0 \quad \text{otherwise}.
\]
Above \(1 \leq a, b \leq M\). Note that \(D_j\) can be obtained by permuting the columns of a diagonal matrix with boolean entries. This decomposition shows that \(\text{convw}(D) \leq M\). In fact, the upper bound can be attained. Consider, for example, the matrix \(Q\) where the first column is filled with all 1’s and all other entries are 0. Any decomposition of \(Q\) as a sum of \(W < M\) matrices with non-negative real entries must contain a matrix \(Q_j\) with at least two non-zero entries in the first column and all zeroes in the remaining columns. No permutation of the columns of \(Q_j\) can make it symmetric. This shows that \(\text{convw}(Q) = M\). Note however that for this example, \(\text{rn}(Q) \leq 1\).

Consider an optimal decomposition \(D = \sum_{j=1}^{W} G_j P_j\), where \(W = \text{convw}(D)\). Write each \(G_j\) as \(G_j = T_j^\text{T}T_j\), where \(T_j\) is a matrix with real entries. Let \(E_j \triangleq T_j^\text{T}\) and \(F_j \triangleq T_jP_j\). Let \(K \triangleq MW\). Since \(W \leq M, K \leq M^2\). Define the \(M \times K\) matrix \(E\) as \(E \triangleq [E_1|\ldots|E_W]\) and the \(K \times M\) matrix \(F\) as \(F^\text{T} \triangleq [F_1|\ldots|F_W]\). Then \(D = EF\). For this decomposition of \(D\) it is easy to see that
\[
\text{rn}(E_j) = \text{cn}(F_j) = \sqrt{\max_{(a,b)\in[M]\times[M]} G_j(a,b)}.
\]
Since for all \((a, b) \in [M] \times [M], 0 \leq G_j(a, b) \leq 1, \) \(\text{rn}(E_j) = \text{cn}(F_j) \leq 1, 1 \leq j \leq W.\) Hence,

\[
\text{rn}(E) \leq \sqrt{\sum_{j=1}^{W} \text{rn}(E_j)^2} \leq \sqrt{W}, \\
\text{cn}(F) \leq \sqrt{\sum_{j=1}^{K} \text{cn}(F_j)^2} \leq \sqrt{W}.
\]

This proves that \(\text{rcw}(D) \leq \text{convw}(D).\)

We make two more easy observations. The first one is that \(\text{rcw}(D) \leq \text{cn}(D) \leq \|D\|,\) where \(\|D\|\) denotes the \(\ell_2\)-operator norm of the matrix \(D.\) The second one is that \(\text{rcw}(D) \leq \text{rank}(D).\)

**Fact 1 ([dW03])** \(O(\sqrt{M})\) is the best possible upper bound for the row-column width of a general \(M \times M\) boolean matrix \(D.\)

**Proof:** For a matrix \(A,\) we consider two norms:

\[\|A\|_{\text{tr}} \overset{\Delta}{=} \text{is the sum of singular values of } A \text{ (trace norm)}\]
\[\|A\|_{F} \overset{\Delta}{=} \sqrt{\sum_{ij} \|A_{ij}\|^2} \text{ is the Frobenius norm}\]

Let \(M = 2n,\) and let \(D\) be the Boolean \(M \times M\) matrix for inner product on \(n\)-bit strings i.e. \(D_{xy} = x \cdot y \mod 2.\) Let \(D_{\pm} = 2D - J,\) where \(J\) is the all-ones matrix. \(J\) has rank 1 and \(\|J\|_{\text{tr}} = M.\) Since \(D_{\pm}\) is the unnormalized \(n\)-qubit Hadamard transform, we have \(D_{\pm}^2 = M \cdot I.\) Hence all singular values of \(D_{\pm}\) are \(\sqrt{M},\) and \(\|D_{\pm}\|_{\text{tr}} = M^{3/2}.\) Therefore, using triangle inequality

\[\|D\|_{\text{tr}} = \|D_{\pm} + J/2\|_{\text{tr}} \geq \frac{1}{2} (\|D\|_{\text{tr}} - \|J\|_{\text{tr}}) = \frac{M^{3/2} - M}{2}.
\]

Let \(D = EF\) be some optimal decomposition of \(D\) for the row-column width. By Holder’s inequality, we have

\[\|D\|_{\text{tr}} = \|EF\|_{\text{tr}} \leq \|E\|_{F} \cdot \|F\|_{F} \leq \sqrt{M \cdot \text{rn}(E)^2} \sqrt{M \cdot \text{cn}(F)^2} = M \cdot \text{rcw}(D)
\]

Combining both inequalities

\[\text{rcw}(D) \geq \frac{\sqrt{M} - 1}{2}.
\]

## 5 Open problem

The main question left open by this work is whether it is possible to overcome the exponential overhead incurred in simulating a classical public coin simultaneous message protocol by a quantum simultaneous message protocol with no shared resource. Interesting progress on this question has been made by the recent paper of Gavinsky, Kempe and de Wolf [GKdW04].

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