Acoustic Peak Spacing, Cosmological Density, and Equation of State

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ABSTRACT

The spacing of the acoustic peaks in the cosmic microwave background radiation anisotropy multipole spectrum has been claimed to provide the value of the total cosmological density overtly, “written on the sky.” Through a semianalytic analysis of the cosmological evolution of the sound horizon and the physics of decoupling we address the robustness of the relation between the peak spacing and the cosmological density. In fact, the asymptotic distance and horizon scalings often used are not good approximations, and the individual densities and equations of state of different components do enter the problem. An observed spacing could be fit by models with different total densities. We investigate the different regions of density-equation of state parameter space and also provide accurate fitting formulas for the peak spacing as a function of matter density, total density, and additional component equation of state (e.g. cosmological constant or cosmic strings). Limits provided by peak spacing measurements on the number of neutrino species and the baryon-photon ratio are also addressed.

1. Introduction

The cosmic microwave background (CMB) radiation provides a cornucopia of cosmological information about the early universe and the formation of large scale structure. With the increasing sensitivity of medium scale angular anisotropy experiments, both ground based and the near future space missions MAP and Planck, further information on the properties of our universe is imminent. Measurements on scales of 10-60 arcminutes cover the horizon scales around the epochs of recombination and last scattering and promise a wealth of data on the large scale properties of our universe as well as indications of the early universe origin of fluctuations that become cosmological structure.

In particular, they hold the hope of measuring the total cosmological energy density and hence elucidating the ultimate fate of our universe, as well as providing evidence on the validity of the inflationary picture of perturbation generation. Such a lofty goal would be reached through a series of measurements using straightforward, well understood, physical theory, and would avoid more tortuous and uncertain paths such as the distance ladder and the nonlinear evolution of complex galaxies and stars.

With the prospect of the revelatory data looming, it behooves us to analyze carefully the extent to which the translation from peak multipole spacing measurements to the single number of the total energy density of the universe is indeed obvious and clean. At first sight it seems astonishing that despite having different species of cosmological inhabitants – photons, baryons, neutrinos, cold dark matter, cosmological constant, etc. – and an uncertain expansion rate (Hubble constant) that the spacing should depend on this one single parameter. Fortunately, the physics of baryon-photon decoupling and cosmological
distances is sufficiently simple that we can go a long way through analytic investigation. Of course, numerical calculations of not only the peak spacing but their amplitudes and the full multipole spectrum of microwave background anisotropies have been carried out (e.g. Zaldarriaga, Seljak, & Bertschinger 1997; Kamionkowski, Spergel, & Sugiyama 1994), and a rigorous derivation of cosmological parameter values should rely on the full information and accuracy they possess (e.g. Zaldarriaga, Spergel, & Seljak 1997; Bond, Efstathiou, & Tegmark 1997). But for understanding the physical origin of the results in different regions of parameter space, this analytic exploration provides a heuristic and surprisingly accurate portrait.

2. Acoustic Oscillation Anisotropies

On medium angular scales, $\theta \approx 10'-2^\circ$ or multipoles $l \approx 100 - 1000$, the dominant anisotropy in the CMB is expected to be acoustic, or Doppler, oscillations (Hu, Sugiyama, & Silk 1997; Bond et al. 1994; occasionally these are called Sakharov (1966) oscillations). These are due to harmonic density perturbations in the coupled baryon-photon fluid in the prerecombination epoch. While after recombination these will grow to form the large scale structure of the universe, the density perturbations cannot grow as long as the matter is tightly coupled to the radiation, e.g. through Thomson scattering. This situation persists even after the end of the radiation dominated era due to the high entropy, or low baryon to photon ratio. Thus even when the matter energy density governs the expansion rate of the universe, the coupling forces the behavior to be oscillational – a sound or acoustic wave.

As the universe expands, the Thomson scattering rate drops below the cosmological expansion rate and the coupling reactions become inefficient. Matter recombines to neutral atoms and the ionization fraction freezes out. The photons become free streaming and the lack of further interaction preserves the density irregularities imprinted on the photon field by the matter oscillations. These appear as angular anisotropies, with the largest angular scale arising from the largest matter wavelength $\lambda$ – that of the sound horizon at decoupling.

In CMB studies one often works in multipole space $l \sim \theta^{-1} \sim \lambda^{-1}$. A length scale translates to an angle $\theta$, or multipole $l$, through the angular diameter distance $r_a$ by $l = k r_a = 2\pi r_a/\lambda = \pi/\theta$, where $k$ is the wavenumber. Thus one expects a harmonic series of anisotropies corresponding to acoustic normal modes, with peaks at $l_1 : l_2 : l_3 \ldots \approx 1 : 2 : 3 \ldots$. The peak spacing $\Delta l$ is a direct measure of the maximum wavelength, the horizon diameter $\lambda = 2r_h$. Thus,

$$\Delta l = \pi r_a/r_h. \quad (1)$$

In attempting to learn about cosmological parameters by observing the anisotropy acoustic peak spacing or locations, we are simply applying the classical angular diameter distance test to the CMB. This has many virtues over its use with galaxies and other conventional objects, including astrophysical simplicity and physical understanding of the source, lack of evolutionary or selection effects, and the high redshift improving the discrimination between cosmological models, as advocated by Linder (1988a).

In order to compare observations of the multipole spacing to cosmological models we need three quantities: the angular diameter distance $r_a$ out to redshift $z$, the sound
horizon scale $r_h$ at redshift $z$, and the redshift of decoupling $z_{dec}$, at which we evaluate these functions. Each depends on numerous subsidiary cosmological variables. Before looking at these dependences let us note a few issues regarding the applicability of this acoustic oscillation picture.

The ansatz relies on the presence of harmonic perturbations in the coupled baryon-photon fluid prior to decoupling. While this is a generic feature of the adiabatic fluctuations generated in inflationary scenarios, it does not have to exist generally. The presence of isocurvature perturbations, such as those arising from topological defect models, suppresses some of the peaks and changes the peak spacing (see Hu & White 1996). It is also possible to arrange active, causal processes in such a way as to create peaks that mimic the results of inflation (Turok 1996).

Moreover, although in a purely harmonic mode the first peak occurs at multipole $l_1 = \Delta l$ and the spacing $l_{m+1} - l_m = \Delta l$ for all $m$, not just $m \gg 1$, gravitational forcing terms in the oscillations slightly change these predictions (Hu & White 1996). Given a theory for the origin of the perturbations, however, the locations and small $m$ spacings (they reach the asymptotic value by $m \approx 3$) can be related to the pure harmonic value $\Delta l$. Thus, though we phrase all our results in terms of the spacing $\Delta l$, they can be applied as well to the peak locations.

Although adiabatic perturbations arise naturally within inflationary theory, we will not restrict ourselves to flat cosmological models, both because of the possibility of open inflationary universes, and because adiabatic perturbations can be generated via other, causal processes. With these points in mind, this paper operates within the picture of the acoustic peaks directly tracing the sound horizon scale, for however far that is valid.

### 3. Physical Variables

As illustrated in (1), the peak multipole spacing depends on two distance scales and the redshift at which they are evaluated. The distance-redshift relations are straightforward cosmological expressions. The redshift is that of decoupling, for this is when the physical imprinting of the acoustic anisotropies in the CMB temperature pattern occurs, when the photons become unaffected by further interactions with the matter.

#### 3a. Angular Diameter Distance

In a homogeneous and isotropic universe, i.e. a Friedmann-Robertson-Walker (FRW) model, the angular diameter distance is given by

$$r_a(z) = (1 + z)^{-1}(1 - \Omega_T)^{-1/2} \sinh \left( (1 - \Omega_T)^{1/2} \int_1^{1+z} dy/H(y) \right), \quad (2)$$

where $\Omega_T$ is the total cosmological energy density in units of the critical density $\rho_c = 3H^2(0)/8\pi$, $y = 1 + z$, and $H(y)$ is the Hubble parameter,

$$H(y) = H(0) \left[ \sum_{\sigma} \Omega_\sigma y^{3(1+\sigma)} + (1 - \Omega_T)y^2 \right]^{1/2}. \quad (3)$$
(See Linder 1988a,b, 1997 for general distance-redshift and cosmological parameter expressions for multiple components and equations of state.) The equation of state of a cosmological component is given by \( \sigma = p/\rho \), where \( p \) is its pressure, \( \rho \) its energy density, and \( \Omega_\sigma \) its dimensionless density \( \rho/\rho_c \).

Note that \( r_a \) depends not only on \( \Omega_T = \sum \Omega_\sigma \) but also on each component individually, though generally one particular equation of state will dominate at a given redshift. One subtlety involves the use of the FRW model despite the clumpiness of matter – i.e. structure – along the line of sight, at least at low redshift. The question of when to use clumpy model distances instead of FRW distances is not wholly settled, but according to recent criteria of Linder (1998), FRW relations can be applied with confidence when considering angular scales \( \theta \gtrsim 5'' \) (\( l \ll 10^5 \)), as we do.

3b. Sound Horizon Scale

The particle horizon distance is

\[
r_p = a \int \frac{dt}{a} = \int dt/a,
\]

where \( a(t) \) is the expansion parameter, or scale factor, and \( t \) is the time. To obtain the sound horizon we simply correct this for the speed of propagation, or sound speed \( c_s \), of the acoustic waves:

\[
r_h(z) = (1 + z)^{-1} \int_{1+z}^{\infty} dy \frac{c_s(y)}{H(y)},
\]

since \( H(y) = a^{-1} \frac{da}{dt} \) and \( y = a^{-1} \).

The sound speed is closely related to the equation of state parameter \( \sigma \): \( c_s = \sqrt{\frac{dp}{d\rho}} \) while \( \sigma = p/\rho \). At the epoch of decoupling and earlier we expect the only two significant components to be nonrelativistic (pressureless) matter, \( \sigma \approx (1/3)v^2/c^2 \approx 0 \), and relativistic particles – photons and neutrinos – with \( \sigma = 1/3 \). The only acoustic coupling that exists, however, is between the baryons and photons. Thus the total pressure fluctuation in the acoustic medium is \( dp = dp_\gamma = (1/3)d\rho_\gamma \) and the total energy density fluctuation is \( d\rho = d\rho_\gamma + d\rho_b \).

For adiabatic perturbations the total entropy fluctuation must vanish; since the entropy is directly proportional to the baryon-photon number ratio \( \eta = n_b/n_\gamma \), this implies \( (\delta n/n)_b = (\delta n/n)_\gamma \). For matter, \( (\delta n/n)_b = (\delta \rho/\rho)_b \) while for blackbody radiation the number density goes as the temperature \( T^3 \) and the energy density goes as \( T^4 \), so \( (\delta n/n)_\gamma = (3/4)(\delta \rho/\rho)_\gamma \). Thus, \( \delta \rho_b = (3/4)(\rho_b/\rho_\gamma) \delta \rho_\gamma \). Finally, the evolution with expansion of the two components is \( \rho_b \sim y^3 \) and \( \rho_\gamma \sim y^4 \), so we find

\[
c_s(y) = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{1}{3} \frac{1}{1 + (3/4)(\Omega_b/\Omega_\gamma)y^{-1}}}. \tag{5}
\]

Defining a redshift of baryon-photon equality, \( y_{eq(b,\gamma)} = \Omega_b/\Omega_\gamma \), we see that for \( y \gg y_{eq(b,\gamma)} \), when photons dominate, the effective equation of state is \( \sigma_{eff} \equiv c_s^2 = 1/3 \), while as the universe expands and baryons come to dominate for \( y \ll y_{eq(b,\gamma)} \), \( \sigma_{eff} \to 0 \). Thus the sound speed varies between \( 1/\sqrt{3} \) and 0, decreasing as baryons become more important (this can be viewed as increasing the effective mass of the acoustic phonon – see Hu, Sugiyama, & Silk 1997). Note that the redshift of baryon-photon equality is not the same
as the redshift of matter-radiation equality, unless all matter is baryonic (no cold dark matter, for example), and all radiation is photons (e.g. no light or massless neutrinos).

3c. Decoupling Redshift

The last ingredient necessary is the redshift at which to evaluate the two distance scales, that of decoupling when the photons receive their final anisotropy imprint. The coupling process is dominated by Thomson scattering between the photons and the free electrons of the perennially ionized hydrogen and proceeds at a time dependent reaction rate \( \Gamma(z) = \sigma_T n_e(z) \), where \( \sigma_T \) is the Thomson cross section and \( n_e \) the electron number density.

As the universe expands, the photon gas cools, eventually dropping below the ionization energy of hydrogen, leading to recombination of the electrons into atoms. Due to the high entropy, or low baryon-photon number \( \eta \), this actually occurs when the photon temperature is \( 2 \ln \eta \approx 1/40 \) of the hydrogen ground state ionization energy. However, although related, the decoupling of the photons and baryons is a slightly different situation: what is important here is the number of interactions ongoing between the two components. When the Thomson reaction rate drops below the expansion rate \( H \sim t^{-1} \) of the universe, or equivalently the photon mean free path exceeds the light horizon, then interactions effectively cease: \( N_{\text{int}} \approx \int_{t_{\text{dec}}}^{\infty} \Gamma \, dt < 1 \).

The criteria for decoupling, therefore, is \( \Gamma \leq H \) (cf. Kolb & Turner 1990). Both rates are redshift dependent: as the universe expands (redshift decreases) the electron density is diluted and also the Hubble parameter decreases. Setting \( \Gamma = H \) defines the desired redshift of decoupling, \( z_{\text{dec}} \). To find the electron density \( n_e = X_e \eta n_\gamma(T_\gamma) \) in \( \Gamma \) one cannot simply use the Saha expression for the ionization fraction \( X_e = n_e/n_b \) because the atomic levels are not in statistical equilibrium. Jones & Wyse (1985) provide an excellent analysis of the relevant physics and applicable approximations. We follow their work, generalizing it slightly. By comparing various atomic excitation rates, they reduce the expression for the ionization fraction as a function of redshift to a Riccati equation and find that near decoupling a quasiequilibrium WKBJ solution obtains, similar to the analysis for the nucleosynthesis rate equations done by Esmailzadeh, Starkman, & Dimopoulos (1991).

This fixed point solution gives \( X_e \sim (\Omega_T h^2)^{1/2} f^{-1}(z, \Omega_T) (\Omega_b h^2)^{-1} \), where \( h = H(0)/(100 \, \text{km s}^{-1} \text{Mpc}^{-1}) \) and \( f(z, \Omega_T) \) involves the ratio of the decoupling redshift to the redshift of equality of energy density in the matter and radiation, \( z_{\text{eq}} \). Retracing their analysis for a general, not just dust (zero pressure), universe, i.e. allowing arbitrary equation of state components with density \( \Omega_\sigma \), reveals that the factor \( (\Omega_T h^2)^{1/2} f^{-1}(z, \Omega_T) \) arises from \( (dt/dz)^{-1} \sim H(z) \) and the \( \Omega_b h^2 \) factor comes from \( n_b \), since the total recombination rate is proportional to \( n_e n_p \sim n_b^2 \) while the competing total ionization rate is proportional to \( n_b \) (\( n_e, n_p \) are the number densities of electrons and protons, respectively). Therefore the generalized decoupling condition \( \Gamma = H \) is

\[
\Gamma(z_{\text{dec}}) = \sigma_T n_e X_e n_b \sim H(z) n_b^{-1} n_b = H(z_{\text{dec}}) \\
\Rightarrow \quad z_{\text{dec}} = z_{\text{dec}}(T_\gamma),
\]

i.e. \( z_{\text{dec}} \) is practically a constant (except that the atomic transitions also depend on the background temperature \( T_\gamma \)).
One has the remarkable conclusion that cancellations in \( H(z) \) and \( n_b \) lead to the redshift of decoupling being extraordinarily insensitive to the baryon content of the universe, the total density and density of individual components, and the overall expansion rate. The redshift is almost completely determined by the photon temperature, which is of course fixed by CMB measurements, and implies \( z_{dec} = 1100 \). This is the value at which we evaluate the distances \( r_a \) and \( r_h \) for all models.

Jones & Wyse (1985) carefully examine the approximations they use and these appear robust. We will note that the leading correction term (the last term in their equation A13) is proportional to \( \Omega_b/\Omega_T \) and so the approximation should be especially good for universes with \( \Omega_b \ll \Omega_T \) (as they state; also see underived equation 3.101 of Kolb & Turner 1990) – this holds for all models we consider.

### 3d. Summary of Cosmological Dependences

Before calculating \( r_a(z_{dec}) \) and \( r_h(z_{dec}) \) to obtain the multipole peak spacing we pause to review the cosmological ingredients that enter into the problem. In broad terms: \( r_a \) involves \( \Omega_T \) and \( H(z) \); \( r_h \) involves \( c_s(z) \) and \( H(z) \); \( z_{dec} \) involves \( \Gamma(z) \) and \( H(z) \). Each of these major variables, however, has subsidiary elements entering:

- \( H(z) \) – component densities \( \Omega_\sigma \) and equations of state \( \sigma \); current value \( h = H(0)/(100 \, \text{km s}^{-1} \, \text{Mpc}^{-1}) \)
- \( c_s(z) \) – baryon-photon ratio \( \eta \); helium abundance \( Y \) through translation of \( n_b \) to \( \Omega_b \) (i.e. average mass of a baryonic particle: see Section 5b)
- \( \Omega_{1/3} \) – number of neutrino species \( N_\nu \); Hubble constant \( h \) affects translation of \( T_\gamma \) to \( \Omega_{1/3} \sim h^{-2}T_\gamma^4 \); \( \Omega_{1/3} \) enters into \( H(z) \) and definition of \( z_{eq} = \Omega_{1/3}/\Omega_o \)
- \( r_a(z) \) – clumpiness of density distribution, as discussed in Section 3a

We keep these in mind for two reasons: 1) to understand the original claims that the peak spacing \( \Delta l \) was degenerate in these ingredients and depended only on \( \Omega_T \), and 2) to attempt to use the wealth and accuracy of CMB anisotropy data to probe these additional cosmological quantities when we see how the putative degeneracy breaks down. After obtaining results for \( \Delta l \) in the next section we return to this question in Section 5.

### 4. Peak Spacing

Using equations (1)-(5) and \( z_{dec} = 1100 \) we obtain analytic or quadrature expressions for the anisotropy multipole peak spacing \( \Delta l \). We examine open and flat models, \( \Omega_T \leq 1 \), with three cases of components: pure dust (\( \sigma = 0 \)) with density \( \Omega_o \); dust plus a cosmological constant (\( \sigma = -1 \)) with densities \( \Omega_o, \Omega_{-1} \); dust plus a \( \sigma = -1/3 \) component with densities \( \Omega_o, \Omega_{-1/3} \). \( [\Omega_{-1/3} \text{ does not include curvature energy } 1 - \Omega_T; \text{ it is a distinct component with } p = -(1/3)\rho, \text{ corresponding perhaps to a cosmic string network. Its function is simply to illustrate the effect of the equation of state on the anisotropy multipole pattern. It is also interesting to consider because it doesn’t affect the kinematics of the universe; that is, a universe with } \Omega_T = \Omega_o + \Omega_{-1/3} \text{ has the same } H(z) \text{ as one with just } \Omega_T' = \Omega_o \text{ (Linder 1988b).}] \)

We do not restrict to the flat inflationary case of \( \Omega_o + \Omega_{-1} = 1 \). We take \( T_\gamma = 2.73 \, K \), i.e. \( \Omega_\gamma = 2.5 \times 10^{-5}h^{-2} \), and three neutrino species so \( \Omega_{1/3} = 4.2 \times 10^{-5}h^{-2} \). We ignore

\[ \]
Y, take $\Omega_bh^2 = 0.0125$, and consider $h = 0.5$ and $h = 1$ cases. Effects of changing these parameters are discussed in Section 5b.

First, let us explore the early argument that $\Delta l$ is degenerate with respect to all parameters but $\Omega_T$. Initially consider a dust universe. At asymptotically high redshift, the angular diameter distance $r_a \approx 2H^{-1}(0)\Omega_o^{-1/2}z^{-1}$, as can be seen by manipulation of (2) into the Mattig (1958; more accessible in Linder 1988b, eqs. 30, A4) analytic form. Similarly, from (4) the sound horizon $r_h \rightarrow (2/\sqrt{3})H^{-1}(0)\Omega_o^{-1/2}z^{-3/2}$ since the baryon loading is negligible at very high redshift. Thus $\Delta l \rightarrow \pi\sqrt{3}z^{1/2}\Omega_o^{-1/2}$. If one assumes that $z_{dec} = 1100$ is in the asymptotic regime then one would obtain $\Delta l \approx 180\Omega_o^{-1/2}$ (i.e. $\theta \approx 1^\circ \times \Omega_o^{1/2}$). For a flat inflationary universe, $\Omega_o + \Omega_{-1} = 1$, $r_h$ is little affected and one asymptotically has $r_a \sim \Omega_o^{-1/2}$ so $\Delta l \sim \Omega_o^{0}$– independent of the individual components.

This sort of estimation led to the early belief that the peak spacing depended only on $\Omega_T$, not $\Omega_o$ and $\Omega_{-1}$ individually (of course this overlooked the point that by assuming a flat universe one already knew $\Omega_T$ – what we supposedly were trying to find from the CMB acoustic anisotropy location).

However, one is in fact not in the asymptotic regime. In the dust case, the leading correction to $r_a$ is of order $(\Omega_o z)^{-1/2}$, which can be as large as 10%, the influence of the radiation component on $H(z)$ is not negligible for $r_h$ since $z_{eq}/z_{dec} \geq (1/20)$ (it is a fairly small effect for $r_a$, $\leq 3\%$), and the baryon loading effect in $c_s$ is at maximum of order $(3/4)z_{eq}(b, c)/z_{dec} \approx 34\%$ (smaller when integrated over $z > z_{dec}$). One actually finds $r_a \sim \Omega_o^{-0.93}$ and $r_h \sim \Omega_o^{-0.32}$ so $\Delta l \sim \Omega_o^{-0.6}$. Remarkably, for a flat inflationary universe $\Omega_o + \Omega_{-1} = 1$, $r_a \sim \Omega_o^{-0.40}$, $r_h \sim \Omega_o^{-0.32}$ so $\Delta l \sim \Omega_o^{-0.08}$ – indeed rather insensitive to the individual component densities. This breaks down, however, for nonflat universes.

In the past few years numerous numerical computations (e.g. Zaldarriaga, Seljak, & Bertschinger 1997; Bond, Efstathiou, & Tegmark; Kamionkowski, Spergel, & Sugiyama 1994), properly including the effects discussed here, have calculated the multipole spectrum including the peak spacings. However, no systematic analysis (beyond the useful animations of Hu – see http://www.sns.ias.edu/~whu/physics/physics.html) has been made of the physical scalings, especially for open models, providing simple analytical fits to the parameter dependences and demonstrating the role of individual component densities and equations of state. Indeed, the myth persists within large sections of the astrophysical community that one or two imminent measurements of the first few Doppler (acoustic) peak locations will fix the total density and hence fate of the universe. Here we attempt to portray the results almost as accurately as the matter Boltzmann–radiative transfer numerical simulations but with explicit fits and scaling of physical dependences, to illustrate the true blend of simplicity and complexity in the results.

Figures 1 show the multipole peak spacing for the various models. Note that the vertical spread of the curves demonstrates the lack of degeneracy with respect to the individual components – i.e. it is not just $\Omega_T$ that defines $\Delta l$. We do see that flat models have a small dispersion, as previously mentioned. The bottom dashed curve gives the naive, often quoted $\Omega_T^{-1/2}$ dependence; this is never an acceptable fit. The bottom solid curve gives a derived $\Omega_T^{-0.59}$ ($\Omega_T^{-0.69}$ for $h = 0.5$) fit for pure dust ($\Omega_o = \Omega_T$) universes. Clearly the presence of individual components does significantly affect the peak spacing. Conversely, observations of acoustic peak locations does not uniquely determine the total
Figure 1: The peak multipole spacing $\Delta l$ is plotted against the total cosmological density $\Omega_T$ for a variety of models. Each panel is labelled by the component equation of state $\sigma$ (all include dust as well) and Hubble constant $h$. The curves from the top down in each panel are for constant $\Omega_o$, starting from $\Omega_o = 0.1$ (solid), $\Omega_o = 0.2$ (dotted), $\Omega_o = 0.3$ (dashed), etc. They run for values of $\Omega_T \geq \Omega_o$. The bottom two curves are fits for pure dust models (where $\Omega_T = \Omega_o$), serving as the lower envelope. The solid bottom curve shows the fit of this paper (see equation 7 and Table 1) while the bottom dashed curve gives the naive asymptotic fit proportional to $\Omega_T^{-1/2}$.

density of the universe. This is discussed further in Section 5a.

As the equation of state parameter $\sigma$ of the second component approaches zero (that of the first component, dust), the dispersion at fixed $\Omega_T$ lessens, as expected. But there is no
qualitative, smoking gun difference between adding different equation of state components, no obvious observational signature. [The pure dust results ($\Omega_o = \Omega_T$) are of course the same in Figures 1a and 1b, and 1c and 1d.]

Decreasing the Hubble constant has a large effect, increasing both the values and dispersion of the peak spacings. Again this is in contradiction to the accepted asymptotic picture, that held that the Hubble constant only entered into the heights and not locations of the anisotropy peaks. (Rigorously, $\Delta l$ would be independent of $h$ only for flat universes where $z_{eq} \gg z_{dec}$. ) Numerical simulations, however, show the same dependence exhibited here; for example Figure 2 of Hu & White (1997; also see Figure 3) finds $\Delta l(\Omega_o = 1, \Omega_T = 1)_{h=0.5} \approx 285$, in excellent agreement with our value of 283. Note that the dashed asymptotic behavior $\Omega_T^{-1/2}$ is here normalized to $\Delta l(1, 1)$ for each $h$; if we used the true naive approach of $\Delta l(1, 1)$ being independent of $h$, the fit of this curve to the spacings would be even worse.

Analytic fits to $\Delta l$ incorporating scaling with the relevant physical variables: the matter density $\Omega_o$, total density $\Omega_T$, and Hubble constant $h$ are useful in presenting the essential physics. Of course in the old $\Omega_T^{-1/2}$ fit neither $\Omega_o$ nor $h$ would appear. The general form we derive is

$$
\Delta l(\Omega_o, \Omega_T) / \Delta l(1, 1) = \Omega_o^{-x} \left( \Omega_o^{-y} - \Omega_o^{-x} \right) \left( \frac{1 - \Omega_T}{1 - \Omega_o} \right)^z. \tag{7}
$$

Two particular cases of interest are the flat (possibly inflationary) case when $\Omega_T = 1$: here $\Delta l \sim \Omega_o^{-x}$, and the pure dust case, $\Omega_o = \Omega_T$, with $\Delta l \sim \Omega_o^{-y}$. Table 1 gives the values $x, y, z$ for the equations of state and Hubble constants adopted. Equation (7) is a central result of this paper and provides an excellent fit to the results for $\Delta l$ plotted in Figures 1, obtained from solving equations (1)-(5). The approximations are good to about 1% rms with maximum deviation about 2%, and thus serve as useful and accurate guides to the physical dependences, short of carrying out the full numerical generation of the CMB multipole spectrum.

| $h$ | $\sigma$ | $x$ | $y$ | $z$ |
|-----|--------|----|----|----|
| 1   | -1     | 0.06 | 0.59 | 0.91 |
| 1   | -1/3   | -0.05 | 0.59 | 1.21 |
| 0.5 | -1     | 0.17 | 0.69 | 0.92 |
| 0.5 | -1/3   | 0.05 | 0.69 | 1.18 |

We can also fit the change in peak spacing with $h$ as

$$
\Delta l(\Omega_o, \Omega_T)_{h} / \Delta l(\Omega_o, \Omega_T)_{h=1} = h^{-f(\Omega_o)}
$$

$$
f(\Omega_o) = 0.30\Omega_o^{-0.38}. \tag{8}
$$

This is good to better than 2% for $h = 0.5$. Note that $f$ does not depend on $\Omega_T$. Also, the ratio of the peak spacings for the pure dust case $(\Omega_o, \Omega_o)$ to the flat case $(\Omega_o, 1)$ are independent of $h$, depending only on the equation of state, as can be seen from the quantity $y - x$ in Table 1.
5. Sensitivity to Cosmological Parameters

We now address how the information resident in the acoustic peaks traces and constrains the cosmological constituents. See Hu & White (1996), especially the excellent and comprehensive Section 5.6, for an outline of how the underlying physical process generating the matter perturbations influences the anisotropies.

5a. Determining Cosmological Density

As illustrated in the previous section, the dependence of the acoustic peak locations on the cosmological parameters is more complicated than initially believed. In particular, the peak locations and spacing certainly do not completely determine the total density $\Omega_T$; there is no unique value derivable without additional assumptions.

To make this explicit, consider a putative determination of peak spacing at $\Delta l = 350$. Even if we restrict ourselves to only models with dust and cosmological constant, and with $h = 1$ — i.e. we somehow have prior knowledge that Figure 1a is the appropriate one — a horizontal line with $\Delta l = 350$ intersects the curves at the following values:

$$(\Omega_o, \Omega_T) = (0.48, 0.48), (0.4, 0.57), (0.3, 0.69), (0.2, 0.8), (0.1, 0.92).$$

This means that a whole family of models is allowed, and these have total densities ranging from $\Omega_T = 0.48 - 0.92$, when $\Omega_o \geq 0.1$. This is very far from a definitive determination of the cosmological density and the fate of the universe, even with prior perfect knowledge of the types of cosmological constituents and of the Hubble constant. (Even if one restricted to flat inflationary models, note that a measured $\Delta l$ of 290, say, could arise from $\Omega_o = 0.08$, $\Omega_{-1} = 0.92$, $h = 1$ or $\Omega_o = 0.52$, $\Omega_{-1/3} = 0.48$, $h = 0.5$ or $\Omega_o = 0.83$, $\Omega_{-1} = 0.17$, $h = 0.5$ – quite a range of models.) So the next ground based anisotropy detection hinting at a peak should not be expected to resolve or even significantly constrain our estimation of the density of the universe.

Lest we be too pessimistic, we note that much additional information exists in the amplitudes of the multipole peaks and this will help constrain the cosmological parameters. Analysis of these, however, is not so amenable to the quasianalytic arguments employed here and best relies on the excellent full numerical codes available.

5b. Constraining Subsidiary Cosmological Parameters

Turning from pessimism to optimism, let us consider what we can learn in that bright future (when MAP and Planck are aloft?) when superbly detailed CMB and other observations have fixed the major cosmological parameters like $\Omega_o$, $\Omega_T$, $h$, and $\sigma$. As mentioned in Section 3d a number of other cosmological quantities influence the acoustic peak spacing in a subsidiary role. The two most promising for constraint are the number of neutrino species $N_\nu$ and the baryon-photon ratio $\eta$.

Figure 2 illustrates the sensitivity of $\Delta l$ to both of these. As discussed in Section 3d, the number of neutrino species adds to the relativistic energy density, $\Omega_{1/3}$, and hence affects the redshift of matter-radiation equality, $z_{eq}$ (which for three species is $21.6 \Omega_o h^2 \times z_{dec}$). The other key redshift is that of baryon-photon equality, $z_{eq(b,\gamma)}$, a direct measure of $\eta$. The curves are labelled with the baryon loading parameter $\beta = (3/4)z_{eq(b,\gamma)}/z_{eq} = 0.0158 (\Omega_b h^2 / 0.0125) (\Omega_o h^2)^{-1}$.

As one increases the number of neutrino species, $N_\nu$, one decreases $z_{eq}$, and increases
Figure 2: The peak multipole spacing is plotted relative to what it would be neglecting baryon loading and the radiation contribution to the expansion rate. This probes the influence of the baryon content of the universe, extra relativistic degrees of freedom, or anything that affects the ratio of the redshifts of matter-radiation equality and decoupling. The arrows indicate the nominal values (with three light neutrino species) for universes with $\Omega_o h^2 = 0.1, 0.25, 1$, from left to right. The parameter $\beta$ is proportional to the baryon density. Dotted curves neglect the effects on $r_a$ to show that the major influence is on the sound horizon, dashed curves include the effects on $r_a$ for $\Omega_o = 1$, and solid curves for $\Omega_o = 0.2$.

$\Delta l$ if the baryon loading is small. What is plotted is the ratio of $\Delta l$ derived with some value of $z_{eq}$ compared to the asymptotic case where the universe at decoupling is completely matter dominated ($z_{eq} \gg z_{dec}$). So to find the effect of changing $N_\nu$ one simply takes the ratio of the multipole values for the two $N_\nu$'s considered. For example, the value at the middle arrow (corresponding to three neutrino species in a $\Omega_o h^2 = 0.25$ dust universe) is 1.54 for the $\beta = 0.01$ curve. Adding one light neutrino species decreases $\log(z_{eq}/z_{dec})$ by 0.055, bringing the value to 1.58, so $\Delta l$ increases by 2.5%. A reasonable fit is

$$(\Delta l)_{N_\nu}/(\Delta l)_3 = [1 + 0.135 (N_\nu - 3)]^{g(\Omega_o h^2)},$$

$$g(\Omega_o h^2) = 0.07 (\Omega_o h^2)^{-0.54},$$

(9)

taking into account that $\beta$ changes with $\Omega_o h^2$. The effects tend to be in the 1-3.5% range.
One can also fix the number of neutrino species, and hence $\Omega_{1/3}/\Omega_o$ and $z_{eq}/z_{dec}$, and consider changing the baryon-photon ratio $\eta$, which is proportional to $\beta$. One has $\beta = \left[45\zeta(3)/2\pi^4\right] m_b T_\gamma (\Omega_{1/3}/\Omega_o) \eta$, where $\zeta$ is the Riemann zeta function and $m_b = \rho_b/n_b$ the average baryonic particle mass. So if all other parameters were determined, one could in principle measure the baryon-photon ratio, or equivalently the specific entropy of the universe, $s \sim \eta^{-1}$, from the shift of the acoustic peak spacing relative to its expected value for some fiducial $\eta$. Note that even the helium abundance $Y$ enters, through the average baryon mass $m_b$. Ignoring this last, tiny effect, one finds that $\Delta l \sim \eta^{0.11}$ for $\beta$ in the range 0.01-0.1, leading to a 8% shift for a factor of two difference in $\eta$.

Recall that for a dust, $h = 1$ model $\Delta l \sim \Omega_T^{-0.59}$ so a 3% change in $\Delta l$ in this case, say, shifts $\Omega_T$ by 5%. The variation of subsidiary cosmological parameters leads only to small effects, but the Planck Surveyor should be able to pinpoint the multipole spacing to better than half a percent, so they are not wholly beyond detection.

6. Conclusion

The physics behind the generation of the CMB acoustic peak anisotropies is well defined and simple. However, common, asymptotic expressions for the distance behaviors and peak spacings are woefully inadequate. Fortunately, there exist comprehensive numerical solutions, and this paper presents accurate analytic fits to the multipole peak locations taking into account the range of relevant cosmological parameters, elucidating the physical inputs. While naive application of the peak location to determining the total density of the universe is doomed, full use of detailed anisotropy observations does indeed carry the hope of deriving not only the total density, but that of the individual components – baryons, cosmological constant, etc. – and even such variables as the number of light neutrino species and the entropy of the universe.

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