Electrons turn into anyons under an elastic membrane

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We show that electrons acquire anyonic statistics if a two-dimensional electron gas is placed in proximity to a charged magnetic membrane. In this two-layered system, electrons become anyons through electromagnetic field mediated interactions with a membrane phonon. The statistical parameter is irrational and depends on the membrane tension and on the electric charge and magnetic dipolar densities on the membrane. We propose a measurement that yields the value of the statistical parameter.

I. INTRODUCTION

Anyons are particles whose wavefunctions acquire a nontrivial phase upon exchange of coordinates, i.e., a phase factor other than that for bosons (+1) or fermions (−1) [1, 2]. Anyons are possible in two spatial dimensions (2D), where the braiding of the worldlines of the particles is non-trivial. Quasiparticle excitations that obey Abelian anyonic exchange statistics, as well as non-Abelian extensions, are theoretically predicted to exist in fractional quantum Hall (FQH) liquids [3–6]. Much effort, in the form of theoretical proposals [7, 8] and experimental attempts [9–12], has been devoted to the detection of anyons, specifically in the form of theoretical proposals [7, 8] and experimental attempts [9–12], has been devoted to the detection of anyons in 2D, where the braiding of the worldlines of the quasiparticle excitations (2D), where the braiding of the worldlines of the particles is non-trivial. Quasiparticle excitations that obey Abelian anyonic exchange statistics, as well as non-Abelian extensions, are theoretically predicted to exist in fractional quantum Hall (FQH) states [3–6]. Much effort, in the form of theoretical proposals [7, 8] and experimental attempts [9–12], has been devoted to the detection of both Abelian and non-Abelian anyons in FQH liquids.

In the FQH effect, the elementary quasiparticle excitations carry both rational fractional charges and rational statistics (the phases are rational multiples of $\pi$) [3–6]. This “rationality” is rooted in the connection of the FQH effect and Chern-Simons theory [13–17], from which it also follows that the many-body ground state is degenerate on closed surfaces, with the degeneracy depending on the surface genus [18, 19].

Anyons, however, need not be always rational like in the FQH effect. The statistical angles can in principle take irrational values [2], once the physical origin of the anyonic statistics is divorced from Chern-Simons theory. One example where the statistical angle can vary continuously is when the quasiparticles are associated to vortices in a complex mass order parameter (a Higgs field) for 2D Dirac electrons [20, 21]. In these examples both the quasiparticle charge and statistics can vary continuously, but then the vortices are logarithmically confined. The vortices can be deconfined with the addition of an axial gauge field [22], but when they are, the charge and statistics become rational again [20, 21].

The goal of this work is to point out that, given a recently suggested alternative framework for realizing anyonic statistics [23], it is possible to turn ordinary electrons into anyons. This mutation, we show, occurs when a 2D electron gas (2DEG) is placed under an elastic electric and magnetic membrane. In this two-layered system, the anyons reside on the 2DEG layer. The essence of the mechanism is that height fluctuations of the membrane generate fluctuations in the gauge potential seen by the electrons in the 2DEG below, while the charge density of the electrons can affect the height fluctuations of membrane above. Hence, the membrane fluctuations act as a middle man, allowing the electron charge to directly source the magnetic field. This is sufficient to turn an electron into an anyon.

Under the elastic membrane, the electric charge of the particle is rational (in fact, integer), but the statistical angle is not forced to take rational values. Instead, it depends continuously on the elastic membrane tension, $\tau$, as well as on the electric charge and magnetic dipole densities ($\sigma_e, \sigma_m$) on the membrane. The statistical angle is $\pi + \theta$, where

$$\theta = -2\pi^2e^2\sigma_e\sigma_m/\tau. \tag{1}$$

We present the mechanism behind the statistical transmutation of electrons into anyons under the electric and magnetic membrane, and argue that there is an experimentally accessible regime where the effect could be probed.

The basic observation underlying the change in electron statistics is the fact that in 2+1 dimensions a gauge field has the same degrees of freedom as a scalar. Thus, the anyonic phenomena can be obtained via the replacement of the spatial components of a statistical gauge field, $a_i$, by a scalar field: $a_i = \epsilon_{ij} \nabla_j \phi$. In terms of the scalar, the statistical magnetic field $b = \epsilon_{ij} \nabla_i a_j = -\nabla^2 \phi$. We conclude that anyons will result if we can arrange that the electron field, $\psi(\vec{r})$, serves as a source for $\phi(\vec{r})$,

$$\nabla^2 \phi(\vec{r}) = \frac{g}{\tau} \psi(\vec{r}), \tag{2}$$

while at the same time enjoying a coupling to it through a “covariant derivative” which mimics the effects of a gauge field $a_i$:

$$\mathcal{L} = -\frac{1}{2m} \left( \nabla_i - i\alpha \epsilon_{ij} \nabla_j \phi \right) \psi \bar{\psi}^2. \tag{3}$$

In this framework, the statistical parameter $\theta$ is given in terms of the couplings $g/\tau$ and $\alpha$ as $\theta = \frac{ge}{2\tau}$, and can be any real number.

An action which leads to $\frac{ge}{2\tau}$ (at sufficiently large dis-
The vector potential on the metallic sheet is calculated from the membrane and spins along the dipole direction. The external field that aligns the dipoles is gapped by the term $\phi$ corrections due to the tilting of the dipoles is of higher order than the length of the membrane. The configuration is depicted in Fig. 1.

We assume the following Hamiltonian for electrons on the metallic sheet:

$$H_\psi = \frac{1}{2m} \left( \hat{p}^2 + eA \right)^2 - eA_0.$$  

Where $A_0$ is the electric potential, and $A_i$ is the magnetic vector potential.

Our first task is to calculate the vector potential on the metallic sheet induced by the height fluctuations of the membrane. In this calculation we assume that the surface of the magnetic membrane fluctuates while the dipolar moments are fixed in the $z$ direction. This assumption is reasonable for two reasons. The first is that corrections due to the tilting of the dipoles is of higher order in gradients of $\phi$. The second is that the tilting of the dipoles is gapped by the external field that aligns the spins along the $z$ direction.

We assign the surface $X' = (x',y')$ coordinate to the membrane and $X = (x,y)$ to the metallic sheet. The vector potential on the metallic sheet is calculated from

$$A_i(X) = \epsilon_{ij} \int d^2X' \frac{-\tilde{\sigma}_m(X') (X - X')_j}{\left[ (X - X')^2 + (\delta + \phi(X'))^2 \right]^{3/2}},$$

where $\phi(X')$ is the height fluctuation of the membrane around its average position $\delta$. (Notice that $\delta > 0$ if the membrane is placed above the metallic sheet, and $\delta < 0$ when the membrane is below.) We also introduced

$$\tilde{\sigma}_m(X') = \frac{\sigma_m}{\sqrt{1 + (\nabla\phi(X'))^2}},$$

which is related to $\sigma_m$ by considering the tilting of the normal direction. In the absence of height fluctuations, the magnetic membrane induces a constant magnetic field on the metallic sheet. Therefore we can decompose the vector potential as

$$\vec{A} = \vec{A}^{BG} + \vec{A}^\phi.$$  

The contribution $\vec{A}^{BG}$ leads to a uniform background field, which can be added to the externally applied field. The contribution $\vec{A}^\phi$ is due to the height fluctuations of the membrane. We assume that these fluctuations are smooth on the scale of $\delta$. As we show in the Appendix, the vector potential fluctuation is

$$A^\phi(X) \approx 2\pi \sigma_m \epsilon_{ij} \nabla_j \phi(X) \text{ sgn}(\delta).$$

Notice that to leading order the vector potential is independent of the magnitude of $\delta$, but it depends on its sign, $i.e.$, on whether the membrane placed above or below the electronic plane.

A very similar calculation leads to the dependence of the electric potential on the membrane fluctuation:

$$A^0(X) \approx -2\pi \sigma_e \phi(X) \text{ sgn}(\delta).$$

Here, as above, we have ignored the uniform background electric field in the $z$ direction.
FIG. 2. Geometry to detect the quantal phases acquired by electrons due to the membrane above them. The geometry is to be patterned on the 2DEG layer underneath the elastic membrane. Electrons loaded into the metallic island of radius \( R \) cause the membrane to displace vertically. The membrane can be clamped down at a distance \( R_D \) from the center (but not on the 2DEG plane). The membrane height displacements generate a magnetic flux for the electrons in the metallic ring. The flux gives rise to Aharonov-Bohm oscillations of the conductance, which can be measured experimentally. For the electrons confined to their plane, unaware of the membrane in the third dimension, the accumulated quantum mechanical phase is due to anyonic statistics.

Comparing (4) with (5) we can read off the couplings of the membrane phonons with the electrons of the metallic layer:

\[
\alpha = -2\pi \sigma_m e \text{sgn} (\delta) , \quad g = 2\pi \sigma_e e \text{sgn} (\delta) .
\]  

(11)

Thus, in terms of the membrane parameters, the statistical angle \( \theta = \frac{\alpha g}{2\tau} \) is given by the expression presented in Eq. (1). We note that \( \theta \) can be any real number, and can be varied experimentally. Next, we discuss an experimental setup to measure the statistical parameter \( \theta \).

III. PROPOSED EXPERIMENTAL SETUP

Consider the geometry depicted in Fig. 2, where a metallic ring and a central metallic island are defined on the 2DEG plane. The island could be charged by a back gate. The proposed measurement consists of studying the oscillations of the conductance through the ring as function of the charge in the island. These oscillations arise because the charge in the island vertically displaces the membrane, which in turn generates a magnetic flux for the electrons in the ring. For the electrons confined to the plane of the 2DEG, unaware of the membrane in the third dimension, the accumulated quantum mechanical phase is due to anyonic statistics.

Let us compute the Aharonov-Bohm (AB) phase accumulated by the electrons in the ring when one charges up the island with charge \( Q \). The deformation of the membrane follows from the equations of motion for the displacement \( \phi \):

\[
\nabla^2 \phi (r) = \frac{g}{\tau} \rho (r) - \frac{\alpha}{\tau} \hat{\mathbf{z}} \cdot \nabla \times \mathbf{J} (r) ,
\]

where \( \rho (r) \) and \( \mathbf{J} (r) \) are, respectively, the electron number and current densities on the 2DEG plane. For the electrons confined to the plane of the 2DEG, unaware of the membrane in the third dimension, the accumulated quantum mechanical phase is due to anyonic statistics.

The flux in (13) is in addition to any other flux \( \Phi_{\text{ext}} \) due to externally applied magnetic fields. The conductance through the ring, \( G (\Phi_{\text{total}} / \Phi_0) \), oscillates as function of \( \Phi_{\text{total}} = \Phi + \Phi_{\text{ext}} \) with period \( \Phi_0 = \hbar c/e = (2\pi/e) \) in natural units where \( \hbar = c = 1 \). Now, the charge in the island is quantized, \( Q = n e, n \in \mathbb{Z} \). Thus \( \Phi \) can only change in steps as the charge is varied by changing the back gate (\( \Delta Q = C \Delta V \), where \( C \) is the capacitance of the island). Every time extra \( n = -\Delta Q/e \) electrons are added to the island, the pattern of AB oscillations in the conductance \( G \) is shifted by a phase \( 2\pi \Delta Q/e \). So measurements for different values of back gate voltages should yield a family of phase shifted patterns of AB oscillations in the conductance \( G \) as function of the externally applied flux \( \Phi_{\text{ext}} \). The shift in the patterns is \( \Delta \Phi_{\text{ext}} = -\Phi \).

The statistical parameter \( \theta \) of the anyons can then be determined from measurements of \( \Delta \Phi_{\text{ext}} \) and \( \Delta V \):

\[
\theta = \frac{\alpha g}{2\tau} = \frac{\pi}{\Phi_0 / \Phi} \frac{\Delta \Phi_{\text{ext}} / \Phi_0}{\Delta Q/e} = \frac{1}{2} \frac{e^2}{C} \frac{\Delta \Phi_{\text{ext}}}{\Delta V} .
\]

(14)

A comment is in order. The coupling of the electrons in the 2DEG with the phonon mode in the membrane leads to an effective logarithmic potential between the electrons. A possible worry is then that a dynamical phase accumulates as the electrons go around the ring. Such a phase would depend on the detailed shape of the ring. The reason why this additional dynamical phase is not present is that the ring is metallic, and therefore charges can redistribute so as to bring the total potential (the sum of effective plus electric potentials) to a constant value throughout the ring. As long as the width of the ring is smaller than the distance to the membrane, these redistributed charges will not affect the membrane displacement above the ring. Consequently, the electromagnetic field generated by the membrane will not be affected, which means that the AB-phase will not be disturbed.
A. Possible realizations

Let us discuss possible examples of such two-layered magnetic/electronic system. There are many possibilities for the 2DEG, for example Si or a semiconductor GaAs/AlGaAs heterostructure. The ability to gate the electron gas, so as to define an Aharonov-Bohm ring and an island, is important. Alternatively, one could just deposit thin metallic (say gold) layers on a substrate, and write directly the wire and island with metal. Essential is the ability to back gate the island, to control the electron density in it.

As for the possible realization of the electric and magnetic membrane, perhaps the ideal candidate would be a suspended graphene sheet that is doped with magnetic ions. The polarized membrane would result from an externally applied field. By electrically biasing the membrane, one could charge it too. It would thus be possible to control both the charge and magnetic dipole densities \( \sigma_e \) and \( \sigma_m \).

Let us first estimate the size of the various effects we have discussed in this particular realization. We will only provide an order of magnitude estimate. Magnetic impurities spaced about 1 nm away, with each impurity having a magnetic moment of order \( \mu_B \), yields a magnetic density of order \( \sigma_m \sim 3 \times 10^{-5} \text{eV} \). As for the membrane tension and surface charge density, it is more suitable to work with the membrane displacements directly. We shall work with the constraint that the maximum displacement \( \phi_{\text{max}} \) does not exceed the order of 300 nm. (The distance \( \delta \) can be taken to be of order of \( \phi_{\text{max}} \) safely if the membrane is popped upwards.)

If the membrane is clamped (Dirichlet boundary conditions) at a radius \( R_D \) from the center of the island (see Fig. 2), then the displacement due to a charge \( Q \) in the island of radius \( R \) is

\[
\phi(r) = \begin{cases} 
0, & R_D < r \\
-\frac{Q}{2\pi \sigma_m} \ln R_D/r, & R < r \leq R_D \\
-\frac{Q}{2\pi \sigma_m} \ln R_D/R, & r \leq R.
\end{cases}
\]  

We shall consider \( R \sim 3 \mu m \) and \( \Delta R = R_D - R \sim 0.1 R \sim 300 \text{nm} \). The AB ring width must fit within this gap, so a width of \( \sim 30 \text{nm} \) is reasonable.

One can express the statistical phase \( \theta \) in terms of the maximum displacement of the membrane, at the edge of the island:

\[
\frac{\theta}{\pi} = \frac{\alpha Q}{2\pi e} = -\frac{\phi_{\text{max}}}{Q/e} \frac{1}{\ln R_D/R} = -\frac{2\pi \sigma_m \phi_{\text{max}}}{Q/e} \frac{1}{\ln R_D/R}.
\]  

The statistical phase an electron in the ring feels as a result of all the \(-Q/e\) electrons in the island is

\[
-\frac{Q \theta}{e \pi} = 2\pi \sigma_m \phi_{\text{max}} \frac{1}{\ln R_D/R} \approx 2\pi \sigma_m \phi_{\text{max}} \frac{R}{\Delta R} \sim 3 \times 10^{-2}.
\]  

We remark that using ions with magnetic moment larger that one Bohr magneton and increasing the density of magnetic dopants could raise this number by one or two orders of magnitude. Furthermore, a larger maximum displacement \( \phi_{\text{max}} \) and larger radius \( R \) (keeping constant the difference \( \Delta R \)) can increase the effect by perhaps another order of magnitude.

Finally, the value of \( Q = -eN_e \) will ultimately depend on the tension and electric charge density. Because the bending rigidity in graphene is \( \kappa \sim 1 \text{ eV} \), there is a characteristic scale \( L_e^2 = \kappa/\tau \) which needs to be kept smaller than the dimensions in the proposed experimental set up (say \( L_e \sim 10 \text{nm} \)). We can express the membrane displacement as \( \phi_{\text{max}} \sim \sigma_e eL_e^2N_e/\kappa \). Parameterizing the charge density as \( \sigma_e = \frac{e}{\kappa}, \) with \( \Delta \sim 1 \text{nm} \) one would have a charge of about \( 10^{-2} \text{e} \) per atom in the membrane. We thus find that \( N_e \sim 1 \). Therefore the phase picked by the electrons around the AB ring is sizable even for very few electrons in the island.

IV. CONCLUSIONS

In this paper we have shown that it is possible to turn electrons into anyons if they are placed under an elastic membrane that is both charged and magnetized. We have also proposed an experimental set up to detect the anyonic statistics, and estimated the size of the measurable phases for a possible concrete experimental realization using patterned metallic rings and islands under a suspended graphene sheet, which could be both charged and doped with magnetic impurities.

Finally, let us point out that there is nothing intrinsic to our general “membrane” mechanism which necessitates that the “electronyons” feel an induced logarithmic potential. Indeed, had we not charged the membrane at all, the electrons would still become anyons due to their Zeeman coupling to the magnetic field (albeit for distances less than \( L_e \), which could be made large by not tensioning the membrane). However, there would not be a logarithmic potential, as the induced coupling would be of the form \( \nabla^2 \phi \psi^\dagger \sigma_z \psi \). The resulting statistical parameter would be too small in this case.

One could imagine yet a different setup; instead of one membrane, there are three membranes on top of each other, with charge densities \( \sigma_e, -2\sigma_e, \) and \( \sigma_e \), for the top, middle, and bottom membrane respectively. As long as the membranes are very close to one other, and can be made to move together, there will be an effective quadrupole charge density. This will result in a coupling of the
form $\nabla^2 \phi \psi^\dagger \psi$, which again does not induce an effective force between electrons. In this scenario, ignoring the Coulomb force between electrons, the theory will be precisely that of \cite{23} at distances less than $L_r$.

Appendix A: The electric and vector potentials due to membrane fluctuations

In this Appendix we present the potentials $(A^\phi_0, \vec{A}_0)$ due to the membrane phonon. We will present the vector potential calculation in detail. The electric potential is found in a very similar manner. We assign the $X' = (x', y')$ coordinate to the membrane and $\vec{X} = (x, y)$ to the fixed plane. We are interested in the contribution from $\phi(x, y)$. The vector potential on the fixed plane is calculated from:

$$A_i(X) = -\epsilon_{ij} \nabla_j \int d^2 X' \frac{\sigma_m}{(X' - X)^2 + (\delta + \phi(X'))^2}^{1/2}$$

$$= -\frac{4\pi \sigma_m}{(2\pi)^3} \int d^2 X' \int d^2 K' \int d\kappa \frac{e^{i\vec{K}' \cdot (\vec{X}' - \vec{X})} e^{i\kappa(\delta + \phi(X'))}}{K'^2 + \kappa^2}$$

$$= -\frac{4\pi \sigma_m}{(2\pi)^3} \int d^2 X' \int d^2 K' \frac{e^{i\vec{K}' \cdot (\vec{X}' - \vec{X})} e^{-|K'||\delta + \phi(X')|}}{2|K'|} \ . \quad (A1)$$

The $\phi$-dependence of the vector potential can be extracted by linearizing in $\phi$, in the regime that the membrane does not touch the 2DEG, when $\text{sgn}(\delta + \phi) = \text{sgn}(\delta)$:

$$A_i^\phi(X) = \frac{2\pi}{(2\pi)^3} \sigma_m \int d^2 X' \int d^2 K' e^{i\vec{K}' \cdot (\vec{X}' - \vec{X})} e^{-|K'||\delta + \phi(X')|} \text{sgn}(\delta) \ . \quad (A2)$$

We are interested in small fluctuations of the field and in momenta $|K'| \delta \ll 1$. Consequently,

$$A_i^\phi(X) \approx 2\pi \sigma_m \text{sgn}(\delta) \epsilon_{ij} \nabla_j \left( \phi(X) - \int \frac{d^2 K'}{(2\pi)^2} |K'| |\delta| e^{i\vec{K}' \cdot \vec{X}} \tilde{\phi}(K') \right) \ . \quad (A3)$$

Here $\tilde{\phi}$ is the Fourier transform of $\phi$, and we have included the leading correction in $\delta$. It is interesting to note that corrections are non-local, furthermore we see that $\delta$ appears with factors of $K'$, therefore the corrections decouple in the long wavelength limit. An analogous formula holds for the electric potential, $A_0^\phi(X)$, with $\sigma_m$ replaced by $\sigma_c$ (and with the $-\epsilon_{ij} \nabla_j$ derivative absent).

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