Variational Principle for Nonhyperbolic Ergodic Measures: Skew Products and Elliptic Cocycles

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Abstract: For a large class of transitive non-hyperbolic systems, we construct nonhyperbolic ergodic measures with entropy arbitrarily close to its maximal possible value. The systems we consider are partially hyperbolic with one-dimensional central direction for which there are positive entropy ergodic measures whose central Lyapunov exponent is negative, zero, or positive. We construct ergodic measures with zero central Lyapunov exponent whose entropy is positive and arbitrarily close to the topological entropy of the set of points with central Lyapunov exponent zero. This provides a restricted variational principle for nonhyperbolic (zero exponent) ergodic measures. The result is applied to the setting of \(\text{SL}(2, \mathbb{R})\) matrix cocycles and provides a counterpart to Furstenberg’s classical result: for an open and dense subset of elliptic \(\text{SL}(2, \mathbb{R})\) cocycles we construct ergodic measures with upper Lyapunov exponent zero and with metric entropy arbitrarily close to the topological entropy of the set of infinite matrix products with subexponential growth of the norm.

1. Introduction

Topological entropy, metric entropy, and Lyapunov exponents are key concepts in ergodic theory and thermodynamical formalism to quantify the complexity of dynamical systems. Several classical results such as the variational principle for entropy [30] and Ruelle’s inequality [27] provide relations between them. On the other hand, Oseledets’ theorem establishes the framework to study the Lyapunov exponents of invariant measures, see [28]. An ergodic measure is nonhyperbolic if its Oseledets splitting has some bundle whose Lyapunov exponent is zero. Otherwise the measure is called hyperbolic. In what follows, we use the terms nonhyperbolic and hyperbolic only for ergodic measures.

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A particularly interesting setting are partially hyperbolic systems which, by assumption, have several globally defined, continuous, and invariant subbundles which carry implicitly information about Lyapunov exponents. When investigating nonhyperbolic measures of those systems, it suffices to focus on the “central bundle” $E^c$ which neither displays uniform contraction nor expansion and hence detects nonhyperbolicity. We study settings where $E^c$ is one-dimensional and nonhyperbolic measures are robustly present and essential: they exist, some of them have positive entropy, and these two properties hold also for small perturbations of the dynamics. Moreover, the systems are genuinely nonhyperbolic as they display simultaneously nonhyperbolicity and hyperbolicity of different types in the central bundle: there exist ergodic measures $\mu$ for which the central-Oseledets exponent $\chi^c(\mu)$ (relative to $E^c$) is negative, zero or positive. The nonhyperbolic nature of these systems is also reflected by the fact that specification-like properties are not satisfied and that inside this class robust heterodimensional cycles occur densely.

We aim to understand the “total amount” of nonhyperbolicity that can be detected on the ergodic level. Here our focus is on entropy. To put this discussion into a broader context, recall the concept of topological entropy $h_{\text{top}}$ of continuous maps on general sets (not necessarily compact or invariant) introduced by Bowen [11]. One of the key results in [11] is that the entropy of an ergodic measure bounds from below the entropy of the set of its generic points. This result has an immediate consequence in the study of the “set of nonhyperbolicity” when the Lyapunov exponent $\chi^c(x)$ of a point $x$ is the Birkhoff average of a continuous potential at that point. Bowen’s result then implies that the entropy of a nonhyperbolic ergodic measure bounds from below the entropy of the set of nonhyperbolic points

$$\sup\{h(\mu): \mu \text{ ergodic, } \chi^c(\mu) = 0\} \leq h_{\text{top}}(\mathcal{L}^c(0)),$$

where

$$\mathcal{L}^c(0) \overset{\text{def}}{=} \{x: \chi^c(x) = 0\}.$$ 

In general, it is unknown if, in terms of entropy, the set $\mathcal{L}^c(0)$ can be larger. One of the goals of this paper is to explore this relation. We exhibit chaotic settings where the above is an equality. Let us now be more precise.

1.1. Skew products and cocycles. Given $N \geq 2$, consider a finite family $f_i: \mathbb{S}^1 \to \mathbb{S}^1$, $i = 1, \ldots, N$, of $C^1$ diffeomorphisms and the associated step skew product

$$F: \Sigma_N \times \mathbb{S}^1 \to \Sigma_N \times \mathbb{S}^1, \quad F(\xi, x) = (\sigma(\xi), f_{\xi_0}(x)),$$

(1.1)

where $\Sigma_N = \{1, \ldots, N\}^{\mathbb{Z}}$ and $\sigma$ is the usual left shift in this space. Given $X = (\xi, x) \in \Sigma_N \times \mathbb{S}^1$, consider its (fiber) Lyapunov exponent

$$\chi(X) \overset{\text{def}}{=} \lim_{n \to \pm \infty} \frac{1}{n} \log |(f^{
}_{\xi})'(x)|,$$

where for $\xi = (\ldots, \xi_{-2}\xi_{-1}\xi_0, \xi_1, \ldots)$ we write

$$f_{\xi}^{-n} \overset{\text{def}}{=} f_{\xi_{n-1}}^{-1} \circ \cdots \circ f_{\xi_{-1}}^{-1} \quad \text{and} \quad f_{\xi}^{n} \overset{\text{def}}{=} f_{\xi_{n-1}} \circ \cdots \circ f_{\xi_0},$$

(1.2)
and we assume that both limits $n \to \pm \infty$ exist and coincide. Otherwise we say that the Lyapunov exponent $\chi(X)$ does not exist. We will analyze the topological entropy of the level sets of Lyapunov exponents: given $\alpha \in \mathbb{R}$ let

$$\mathcal{L}(\alpha) \overset{\text{def}}{=} \{ X \in \Sigma_N \times S^1 : \chi(X) = \alpha \}. \tag{1.3}$$

Denote by $\mathcal{M}_{\text{erg}}(F)$ the space of $F$-ergodic measures. Given $\mu \in \mathcal{M}_{\text{erg}}(F)$, denote by $h(F, \mu)$ its metric entropy and by $\chi(\mu)$ its Lyapunov exponent

$$\chi(\mu) \overset{\text{def}}{=} \int \log |(f_{i0})'(x)| \, d\mu(\xi, x).$$

In continuation to our introduction above, maps $F$ can be viewed as a special case of partially hyperbolic diffeomorphisms whose one-dimensional central bundle $E^c$ is integrable. Moreover, $E^c$ is tangent to the circle fibers and $\chi^c = \chi$. The class $\text{SP}^1_{\text{shyp}}(\Sigma_N \times S^1)$ of maps $F$ studied here was introduced in [15], see Sect. 2.1 for its definition. They capture the relevant properties of the so-called robustly nonhyperbolic transitive sets, reformulating the main properties of those systems in the setting of skew products over the shift of $N$ symbols whose fiber maps are $C^1$ diffeomorphisms of the circle $S^1$. Every $F \in \text{SP}^1_{\text{shyp}}(\Sigma_N \times S^1)$ is transitive and has so-called contracting and expanding blenders which, by transitivity, are connected; see also the contraction-expansion-rotation examples as introduced in [20]. For details see the discussion in [15, Section 8]. A paradigmatic setting arises from the projective action of $2 \times 2$ matrix-cocycles. In particular, our approach applies to an open and dense subset of the so-called elliptic $\text{SL}(2, \mathbb{R})$ cocycles.

In our setting, by [20], besides hyperbolic measures with negative or positive exponent, there are nonhyperbolic measures. As a consequence of [5], they can be chosen with positive entropy and hence $h_{\text{top}}(F, \mathcal{L}(0)) > 0$. The arguments in [5] are based on the construction of a compact invariant set with positive topological entropy consisting only of points with zero Lyapunov exponent. Hence the existence of nonhyperbolic measures with positive entropy is a consequence of the classical variational principle for entropy [30]. Though, the construction in [5] studies a very specific region of the space (the dynamics associated to some robust cycle involving a blender) and presumably the captured entropy is much smaller than $h_{\text{top}}(F, \mathcal{L}(0))$. A natural question is if there exist nonhyperbolic measures whose entropy is equal or arbitrarily close to $h_{\text{top}}(F, \mathcal{L}(0))$. We answer positively the second question. The notoriously much harder question about the existence of measures maximizing entropy remains open.

On the other hand, by [16], for $F \in \text{SP}^1_{\text{shyp}}(\Sigma_N \times S^1)$ the closure of the ergodic measures is the union of two Poulsen simplices (corresponding to negative and positive Lyapunov exponent, respectively) which “glue along” nonhyperbolic measures. In particular, any nonhyperbolic ergodic measure is a weak$^\ast$ and entropy-limit of hyperbolic ones. Moreover the spectrum of the exponent $\chi$ is a closed interval containing negative and positive numbers and for every $\alpha \neq 0$ it holds

$$\sup\{h(F, \mu) : \mu \in \mathcal{M}_{\text{erg}}(F), \chi(\mu) = \alpha\} = h_{\text{top}}(F, \mathcal{L}(\alpha))$$

and

$$\sup\{h(F, \mu) : \mu \in \mathcal{M}_{\text{erg}}(F), \chi(\mu) = 0\} \leq \lim_{\varepsilon \to 0} \sup\{h(F, \mu) : \mu \in \mathcal{M}_{\text{erg}}(F), \chi(\mu) \in (-\varepsilon, 0) \cup (0, \varepsilon)\} = h_{\text{top}}(F, \mathcal{L}(0)). \tag{1.4}$$
This shows that measures with “weak hyperbolicity” are well inserted in this space and are key ingredients to describe nonhyperbolic ones.

The previous analysis is however insufficient to state in “what amount” weakly hyperbolic (and nonhyperbolic) measures contribute to the complexity of the dynamics. For example, in general it is unknown if any term in (1.4) attains the maximal entropy perbolic (and nonhyperbolic) measures contribute to the complexity of the dynamics. This shows that measures with “weak hyperbolicity” are well inserted in this space and project to the entropy-maximizing Bernoulli measure in $\Sigma_N$. In particular, proximality implies that all terms in (1.4) are strictly less than $\log N$.

The main result claims that in our setting there are nonhyperbolic ergodic measures whose entropy is as large as possible. As a consequence, a restricted variational principle holds for those nonhyperbolic measures.

**Theorem A.** For every $F \in \text{SP}_{\text{shyp}}^1(\Sigma_N \times S^1)$, $N \geq 2$, it holds

$$h_{\text{top}}(F, \mathcal{L}(0)) = \sup\{h(F, \mu) : \mu \in \mathcal{M}_{\text{erg}}(F), \chi(\mu) = 0\}$$

$$= \lim_{\varepsilon \to 0} \sup\{h(F, \mu) : \mu \in \mathcal{M}_{\text{erg}}(F), \chi(\mu) \in (-\varepsilon, 0)\}$$

$$= \lim_{\varepsilon \to 0} \sup\{h(F, \mu) : \mu \in \mathcal{M}_{\text{erg}}(F), \chi(\mu) \in (0, \varepsilon)\}.$$

Theorem A follows from a more quantitative result, see Theorem C stated below.

Let us now draw a consequence for 2 × 2-matrix cocycles generated by a finite collection of matrices $A = \{A_1, \ldots, A_N\}$ in $\text{SL}(2, \mathbb{R})_N$, $N \geq 2$. The action of any matrix on the projective line $\mathbb{P}^1$ (which is topologically the circle $S^1$) is a very special diffeomorphism. Given $A_i$, we define

$$f_i = f_{A_i} : \mathbb{P}^1 \to \mathbb{P}^1, \quad f_i(v) \overset{\text{def}}{=} \frac{A_iv}{\|A_iv\|} \quad (1.5)$$

and denote by $F_A$ the associated skew product generated by the maps $f_1, \ldots, f_N$ as in (1.1). Note that the spectrum of Lyapunov exponents of the cocycle $A$ and the fiber Lyapunov spectrum of $F_A$ are related (see [16, Section 11] for details) and hence our results can be translated to the elliptic cocycles setting. We postpone the details to Sect. 2.3.

Consider the Lyapunov exponent of $\xi^+ = (\xi_0, \xi_1, \ldots) \in \Sigma_N^+ \overset{\text{def}}{=} \{1, \ldots, N\}^\mathbb{N}_0$,

$$\lambda_1(A, \xi^+) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n} \log \|A_{\xi^{n-1}} \circ \cdots \circ A_{\xi_0}\| \quad (1.6)$$

whenever this limit exists. Given an ergodic measure $\nu^+$ with respect to the left shift $\sigma^+$ on $\Sigma_N^+$, by the subadditive ergodic theorem, almost surely it holds

$$\lambda_1(A, \nu^+) = \lambda_1(A, \xi^+) = \lim_{n \to \infty} \frac{1}{n} \log \|A_{\eta^{n-1}} \circ \cdots \circ A_{\eta_0}\| d\nu^+(\eta^+).$$

Note that $\lambda_1(A, \nu^+) \geq 0$. Moreover, given any (nondegenerate) Bernoulli measure $\nu^+$, Furstenberg’s theorem [19] states that, assuming that the semi-group generated by $A$ is

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1 Proximity holds if for every pair of points $x, y \in S^1$ there is $\xi \in \Sigma_N$ so that $|f^N_\xi(x) - f^N_\xi(y)| \to 0$ and $|f^{-N}_\xi(x) - f^{-N}_\xi(y)| \to 0$ as $n \to \infty$. 

not relatively compact and there is no finite set $\emptyset \neq L \subset \mathbb{P}^1$ such that $A(L) = L$ for every $A \in A$, then $\lambda_1(A, b^+) > 0$.²

Bernoulli measures are rather specific ergodic measures and, besides Lyapunov-maximizing ones, very little is known about the ergodic theory of measures in this context. The following result complements this line of research (see also [6]) and can be read as a study of Lyapunov-minimizing measures of matrix cocycles in our non-rigid context. Recall that $A$ is elliptic if its associated multiplicative semigroup contains some elliptic element $R$ (i.e., the absolute value of the trace of $R$ is less than 2). The set $\mathcal{E}_N$ of elliptic cocycles is open. In [16] it is introduced an open and dense subset $\mathcal{E}_{N,\text{shyp}}$ of $\mathcal{E}_N$, the so-called elliptic cocycles having some hyperbolicity. The key property is that if $A \in \mathcal{E}_{N,\text{shyp}}$ then $F_A \in \text{SP}^1_{\text{shyp}}(\Sigma_N \times \mathbb{P}^1)$, see Sect. 2.3 for details. Also observe that we are in the case of proximality, which implies that entropy is positive and less than $\log N$.

Analogously to (1.3) define the set of nonhyperbolic matrix concatenations by

$$\mathcal{L}_A^+(0) \overset{\text{def}}{=} \{ \xi^+ \in \Sigma_N^+ : \lambda_1(A, \xi^+) = 0 \}. \quad (1.7)$$

Theorem B. For every $N \geq 2$ and every $A$ in the open and dense subset $\mathcal{E}_{N,\text{shyp}}$ of $\mathcal{E}_N$ it holds

$$0 < h_{\text{top}}(\sigma^+, \mathcal{L}_A^+(0)) = \sup\{ h(\sigma^+, v^+) : v^+ \in \mathcal{M}_{\text{erg}}(\sigma^+), \lambda_1(A, v^+) = 0 \} < \log N.$$

Let us now discuss the tools to prove the above results and describe their context.

### 1.2. Nonhyperbolic measures: constructions and tools.

When dealing with nonhyperbolic ergodic measures, one major problem is that, at the current state of the art, there are not as many tools available as there are for hyperbolic ones (for example Pesin theory [3]). Among the few tools available to deal with nonhyperbolic measures are the so-called invariance principles in the spirit of Furstenberg’s result [19] and also [2,13,24]. This principle is very well adapted to dynamics arising from cocycles and, in very rough terms, states that if the fiber Lyapunov exponent is zero then the fiber dynamics carries some transversally invariant structure. Though, these tools apply only to base measures which have a local product structure. In general, it is unknown if measures maximizing (1.4) are of this type.

An alternative approach is the explicit construction of nonhyperbolic measures. Naively, one can think of taking a weak$^*$ limit of hyperbolic ergodic measures with central exponents approaching zero. Though, it is in general not guaranteed that the limit measure is ergodic and nontrivial (i.e., with uncountable support). The control of entropy of the limit measure is another issue.³ This approach was implemented and improved in several steps. It was initiated by the so-called GIKN construction in [20] for

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² Furstenberg’s result states the dichotomy “positive Lyapunov exponent versus rigid dynamics”. As we are, by hypotheses, in a non-rigid context, this implies always positive exponent.

³ By [14, Corollary 1.2] (see also [12]), in our setting, the entropy map is upper semi-continuous.
circle fiber-skew products. It was generalized first in [22] for certain partially hyperbolic
diffeomorphisms and thereafter in [8,18,31] for nonhyperbolic homoclinic classes.\(^4\)\(^5\)

Our construction is naively inspired by the GIKN method in [20] that we proceed to
sketch. This construction starts from an appropriate sequence of periodic orbits \(O_n\), say,
expanding in the fiber direction. In very rough terms, each periodic orbit has two parts:
one “repeats and shadows” the previous orbit and the second part is a “tail”. There is
a balance between both parts. The tail is used to “spread” the support of the measures
and to decrease the Lyapunov exponent by visiting a fiber-contracting region. A general
criterium in [20] (see also Proposition 11.1) guarantees ergodicity of any limit measure.

As stated in [23], any limit measure of a GIKN construction involving a limit of a
sequence of periodic measures has zero entropy. Thus, this method is not useful for our
purpose since we aim for positive (maximal) entropy. We extend the GIKN approach of
repeating and tailing to a much broader context to enable to capture positive entropy.
In our construction we replace periodic measures by specifically chosen “Bernoulli-
suspended measures on horseshoes” carrying enough entropy. This choice is based on
a “skeleton property” of hyperbolic measures with negative Lyapunov exponent. More
precisely, we choose sufficiently many orbit pieces capturing the ergodic properties
of the measure in finite time (compare also [15, Section 4]) to “transfer a substantial
amount” of its entropy to the constructed nonhyperbolic measure (see Theorem C). The
skeletons provide the “repeat”-part. We combine this approach with ideas in the proof
of [15, Theorem 5 item 2] which states how much entropy from measures with negative
exponent can “carry over” to those with positive exponent. This provides the “tail”-part.
To prove Theorem A, we will consider measures with exponent close to zero and entropy
close to the target entropy, though our construction is general.

The verification of ergodicity of the limit measure (see [20] and Proposition 11.1)
relies on the control of Birkhoff averages simultaneously on all scales on large measure
sets. It is relatively easy to check if the measures of the sequence are periodic (that is,
supported on a periodic orbit). Indeed, on a periodic orbit Birkhoff averages converge
uniformly, which makes this verification relatively simple. In our study, checking ergod-
icity of limit measures is much more intricate and requires a new approach. Here, we will
deal with ergodic measures on horseshoes and have to rely on large deviation arguments
from probability theory to identify appropriate sets with control of Birkhoff averages.

Our approach has roughly two, somewhat independent, parts: an abstract model for
the repeat-and-tail scheme and its implementation. As abstract model we consider a
cascade\(^6\) of abstract suspension spaces of Bernoulli shifts where we perform the large
deviation control. This model is chosen such that it extends the corresponding measure
preserving systems on a cascade of horseshoes in the product space \(\Sigma_N \times \mathbb{S}^1\). Each
horseshoe has a coded system inside the base shift space \(\Sigma_N\) which is obtained by a
“repeat-and-tail process”. Each coded system is uniquely left decipherable, which is the
key ingredient to the fact that entropy is not lost when considering this factor.

\(^4\) Our focus here is on as-large-as-possible entropy. The GIKN construction can be adapted and extended
to produce nonhyperbolic measures with zero entropy and full support (see [7,8,10]). The method in [5] was
modified in [9] to get nonhyperbolic measures with positive entropy and also full support. It was adapted also in
[6] to deal with matrix cocycles. The constructions in this paper lay the foundations to construct nonhyperbolic
measures with entropy as large as possible and also full support, following the ideas in [8,9].

\(^5\) Concerning nonhyperbolic measures with several zero Lyapunov exponents (that is, a higher-dimensional
central bundle), the state of the art is very incipient, see results in [4] for iterated function systems and in [31]
for some nonhyperbolic homoclinic classes.

\(^6\) As in this paper we consider plenty of sequence spaces, we prefer this terminology.
We perform this analysis in our context of skew products. However, the general idea of an entropy preserving cascade of suspensions of coded systems is fairly general and can be applied to partially hyperbolic diffeomorphisms following the scheme sketched in [15, Section 8.3] and implemented with all details in [17], see also [33]. But this goes beyond the goal of this paper.

1.3. Organization. In Sect. 2, we state the remaining main results. In particular, we provide Theorem C which is the key result towards Theorem A. Section 2.1 describes our axiomatic setting and defines the class \( \text{SP}_{\text{shyp}}^{1} (\Sigma_{N} \times S^{1}) \). We also detail the consequences for elliptic cocycles in Sect. 2.3.

The abstract model is developed in Sects. 3, 4 and 5. Section 3 collects some basic properties of coded systems. In Sect. 4, we define the suspension of a Bernoulli shift and recall some fundamental properties. In particular, we state a key result on large deviations. In Sect. 5, we consider a cascade of those abstract suspension spaces assuming some growth condition of the associated roof functions.

The implementation of this abstract model is done in Sects. 6, 7, 8 and 9. In Sect. 6, we return to consider our skew product setting and study horseshoes which are defined by attractors of contracting iterated function systems (CIFS) induced by the family \( \{ f_{i} \}_{i=1}^{N} \) on some interval \( J \subset S^{1} \). Here the idea of skeleton plays an important role. Section 7 introduces a repeat-and-tail scheme. It induces a cascade of CIFSs and hence a cascade of horseshoes whose properties are studied in Sect. 8. In particular, in Sect. 9 we describe the inherited internal self-similar structures across this cascade.

The proof of Theorem C is split into Sects. 10 and 11.

2. Statement of Results

Before stating the main results in Sect. 2.2, let us first give the complete description of our setting. In Sect. 2.3 we discuss matrix cocycles in detail.

2.1. Axiomatic setup. For \( n \in \mathbb{N} \) let \( \Sigma_{N}^{n} \gdef [1, \ldots, N]^{n} \) and define \( \Sigma_{N}^{*} \gdef \bigcup_{n=1}^{\infty} \Sigma_{N}^{n} \). Given a finite sequence \( \xi = (\xi_{0}, \ldots, \xi_{n-1}) \in \Sigma_{N}^{*} \), we denote by \( |\xi| = n \) its length. Given \( x \in S^{1} \), using the notation in (1.2), consider its forward and backward orbits defined by

\[
\mathcal{O}^{+}(x) \defeq \bigcup_{n \geq 0} \bigcup_{\xi \in \Sigma_{N}} f_{\xi}^{n}(x) \quad \text{and} \quad \mathcal{O}^{-}(x) \defeq \bigcup_{m \geq 1} \bigcup_{\xi \in \Sigma_{N}} f_{\xi}^{-m}(x),
\]

respectively. Let \( \mathcal{O}^{\pm}(H) \defeq \bigcup_{x \in H} \mathcal{O}^{\pm}(x) \) for any subset \( H \subset S^{1} \). Given an interval \( H \), we denote by \( |H| \) its length. We assume that \( S^{1} \) has length one.

We require the following properties to be satisfied.

T (Transitivity) There is \( x \in S^{1} \) such that \( \mathcal{O}^{+}(x) \) and \( \mathcal{O}^{-}(x) \) are both dense in \( S^{1} \).

CEC+(J+) (Controlled Expanding forward Covering). The set \( J^{+} \subset S^{1} \) is a non-trivial closed interval so that there exist positive constants \( K_{1}, \ldots, K_{5} \) so that for every interval \( H \subset S^{1} \) intersecting \( J^{+} \) with \( |H| < K_{1} \) it holds

- (controlled covering) there exists a finite sequence \( (\eta_{0}, \ldots, \eta_{\ell-1}) \) for some positive integer \( \ell \leq K_{2} \log |H| + K_{3} \) such that
\[
(f_{\eta_{\ell-1}} \circ \cdots \circ f_{\eta_{0}})(H) \supset B(J^{+}, K_{4}),
\]
where \( B(J^+, \delta) \) is the \( \delta \)-neighborhood of the set \( J^+ \),

- (controlled expansion) for every \( x \in H \) we have

\[
\log |(f_{\eta_{l-1}} \circ \cdots \circ f_{\eta_0})'(x)| \geq \ell K_5.
\]

**CEC \(^{-1} \)(\( J^- \)) (Controlled Expanding backward Covering).** The step skew product \( F^{-1} \) satisfies the Axiom CEC\(^+ \)(\( J^- \)).

**Acc\(^+ \)(\( J^+ \)) (forward Accessibility relative to \( J^+ \)).** \( 0^+(\text{int} \ J^+) = S^1 \).

\( \text{Acc}^- (J^-) \) (backward Accessibility relative to \( J^- \)). \( 0^- (\text{int} \ J^-) = S^1 \).

**Definition 2.1 (The set SP\(^1 \)shyp \((\Sigma_N \times S^1)\)).** A skew product \( F \) as in (1.1) belongs to \( \text{SP}\(^1 \)shyp \((\Sigma_N \times S^1)\) if it satisfies Axioms T (transitivity), CEC\( \pm (J^\pm) \), and \( \text{Acc} \pm (J^\pm) \) for some closed intervals \( J^- \), \( J^+ \subset S^1 \), which are called backward and forward blending intervals, respectively.

We state some consequences of our axioms.

**Remark 2.2 (Common blending interval).** Let \( F \in \text{SP}\(^1 \)shyp \((\Sigma_N \times S^1)\). \) By [15, Lemma 2.3], there are positive constants \( K_1, \ldots, K_5 \), and \( K_6 \) such that for every \( \delta \in (0, K_6/2) \) and \( x \in S^1 \) the interval \( J = [x - 2\delta, x + 2\delta] \) satisfies Axioms CEC\( \pm (J) \) and \( \text{Acc} \pm (J) \) with these constants. We call such \( J \) a blending interval.

The next observation is an immediate consequence of the compactness of \( S^1 \).

**Claim 2.3** [15, Remark 2.1 and Lemma 2.2]. Assume Axioms T, CEC\( \pm (J) \) and \( \text{Acc} \pm (J) \) are satisfied for some closed interval \( J \). Then for every closed subinterval \( I \) of \( J \) there exists \( m_c = m_c(I) \in \mathbb{N} \) such that for every \( x \in S^1 \) there are finite sequences \( (\theta_1, \ldots, \theta_r) \) and \( (\beta_1, \ldots, \beta_s) \) with \( r, s \leq m_c \) such that

\[
(f_{\beta_r} \circ \cdots \circ f_{\beta_1})(x) \in I \quad \text{and} \quad (f_{\theta_r}^{-1} \circ \cdots \circ f_{\theta_1}^{-1})(x) \in I.
\]

**Definition 2.4 (The constant \( L_1 \)).** Given \( F \in \text{SP}\(^1 \)shyp \((\Sigma_N \times S^1)\) and a blending interval \( J = [x - 2\delta, x + 2\delta] \) with associated constants \( K_1, \ldots, K_5 \), define

\[
L_1 = L_1(F, J) \overset{\text{def}}{=} K_2(2 + |\log(4\delta)| + K_3) + m_c([x - \delta, x + \delta]).
\]

### 2.2. Key results

We are now ready to state the key result towards the proof of Theorem A. First note that by [16, Theorem A and Lemma 5.2] we have \( \mathcal{L}(0) \neq \emptyset \) and there holds the inequality

\[
\sup\{h(F, \mu) : \mu \in \mathcal{M}_{\text{erg}}(F), \chi(\mu) = 0\} \leq h_{\text{top}}(F, \mathcal{L}(0)). \tag{2.1}
\]

By [16, Theorem A], there are numbers \( \alpha_{\min} < 0 < \alpha_{\max} \) such that \( \mathcal{L}(\alpha) \neq \emptyset \) if and only if \( \alpha \in [\alpha_{\min}, \alpha_{\max}] \). Moreover, the map \( \alpha \mapsto h_{\text{top}}(F, \mathcal{L}(\alpha)) \) is continuous on the interval \([\alpha_{\min}, \alpha_{\max}]\). Finally, for every \( \epsilon > 0 \) there exists some ergodic measure \( \mu \) with negative Lyapunov exponent satisfying \( \alpha = \chi(\mu) \in (-\epsilon, 0) \) and \( h(F, \mu) > h_{\text{top}}(F, \mathcal{L}(0)) - \epsilon \); analogously for \( \alpha \in (0, \epsilon) \). With these results at hand, Theorem A is now an immediate consequence of the following.

**Theorem C** (Transfer of entropy to nonhyperbolic measures). For every $F \in \text{SP}_{\text{shyp}}^1(\Sigma_N \times S^1)$, $N \geq 2$, there is some constant $L_1 = L_1(F) > 0$ such that for every $F$-invariant ergodic measure $\mu$ with negative Lyapunov exponent $\alpha \defeq \chi(\mu) < 0$ and positive entropy $h(F, \mu)$ and every $\varepsilon_H \in (0, h(F, \mu))$ there is a sequence of ergodic measures $(\mu_n)_n$ with negative Lyapunov exponents which converges weak* to an ergodic measure $\mu_\infty$ satisfying

$$\chi(\mu_\infty) = 0 \quad \text{and} \quad h(F, \mu_\infty) \geq e^{-L_1|\alpha|}(h(F, \mu) - \varepsilon_H).$$

The proof of Theorem C is given in Sects. 10 and 11. A byproduct of our construction is the following fact, which we prove at the end of Sect. 8.

**Proposition D.** Let $F \in \text{SP}_{\text{shyp}}^1(\Sigma_N \times S^1)$, $N \geq 2$, and $\mu$ be an $F$-invariant ergodic measure with negative Lyapunov exponent and positive entropy $h(F, \mu)$. Then for every $\varepsilon_H \in (0, h(F, \mu))$ there are sequences of compact $F$-invariant sets $\Gamma_n \subset \Sigma_N \times S^1$ and numbers $\alpha_n \leq \beta_n < 0$ with

$$\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = 0$$

having the following properties: for every $n \in \mathbb{N}$

1. the set $\Gamma_n$ has uniform fiber contraction in the sense that $\alpha_n \leq \chi(\bar{\mu}) \leq \beta_n < 0$ for every $\bar{\mu} \in \mathcal{M}_{\text{erg}}(F|\Gamma_n)$,
2. natural projection of $\Gamma_n$ to $\Sigma_N$ is a coded subshift and for every $\xi$ in this projection the fiber $\{(\xi) \times S^1\} \cap \Gamma_n$ is a finite set,
3. $\limsup_{n \to \infty} h_{\text{top}}(F, \Gamma_n) \geq e^{-L_1|\alpha|}(h(F, \mu) - \varepsilon_H)$.

### 2.3. Consequences for elliptic cocycles.

The space of cocycles $\text{SL}(2, \mathbb{R})^N$ roughly splits into the disjoint union of the sets of hyperbolic and elliptic cocycles: these sets are open and their union is dense in $\text{SL}(2, \mathbb{R})^N$, see [34, Proposition 6]. The set of hyperbolic cocycles, including the description of its boundary, is quite well understood, see [1]. However, much less is known about the elliptic cocycles $\mathcal{E}_N$. In [16, Section 11] it is introduced an open and dense subset $\mathcal{E}_{N, \text{shyp}}$ of $\mathcal{E}_N$, the set of elliptic cocycles having some hyperbolicity. For our purposes, the key property of the set $\mathcal{E}_{N, \text{shyp}}$ is that it consists of cocycles $A$ whose associated skew products $F_A$ with fiber maps defined as in (1.5) are contained in $\text{SP}_{\text{shyp}}^1(\Sigma_N \times \mathbb{P}^1)$.

Instead of giving the precise definition of $\mathcal{E}_{N, \text{shyp}}$, let us describe its essential properties. First, recall that an element $A \in \text{SL}(2, \mathbb{R})$ is hyperbolic if the absolute value of its trace is larger than 2, which means that the matrix $A$ has one eigenvalue with absolute value bigger than one and one smaller than one. In particular, the union of the disjoint open sets

$$B^-_A \defeq \{v \in \mathbb{P}^1 : |f'_A(v)| < 1\} \quad \text{and} \quad B^+_A \defeq \{v \in \mathbb{P}^1 : |f'_A(v)| > 1\}$$

is dense in $\mathbb{P}^1$. The set $\mathcal{E}_{N, \text{shyp}}$ consists of cocycles $A \in \mathcal{E}_N$ such that:

1. Some hyperbolicity: The semi-group generated by $A$ contains a hyperbolic element.
2. Transitions in finite time: There is $M \geq 1$ such that for every $v \in \mathbb{P}^1$ there are sequences $\beta^+, \beta^+ \in \Sigma_N^\pm$ such that $f_{\theta_{s-1}} \circ \cdots \circ f_{\theta_0}(v) \in B^+_A$ and $f_{\theta_{r-1}} \circ \cdots \circ f_{\theta_0}(v) \in B^-_A$ for some $s, r \leq M$. 
These properties are just the translation of the properties of maps in $\text{SP}^1_{\text{shyp}}(\Sigma_N \times \mathbb{P}^1)$ to skew products arising from cocycles. They also immediately refer precisely to the context considered in [20]. The set $\mathcal{E}_{N,\text{shyp}}$ is open and dense in $\mathcal{E}_N$, see [16, Proposition 11.23].

Let us now recall some results relating the (upper) Lyapunov exponent $A$ with the (fiber) Lyapunov exponent of its associated skew product $F_A$ on $\Sigma_N \times \mathbb{P}^1$. Let us consider the (forward) Lyapunov exponent

$$\chi^+(\xi^+, v) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n} \log |(f_{\xi^+}^n)'(v)|,$$

using the notation analogous to (1.2), provided this limit exists. For the next results recall the definitions of $\lambda_1(A, \xi^+)$ in (1.6) and $L^+_A(0)$ in (1.7).

**Lemma 2.5** [16, Theorem 11.1, Claim 11.21 for $\alpha = 0$]. For all $A \in \text{SL}(2, \mathbb{R})^N$,

$$h_{\text{top}}(\sigma^+, L^+_A(0)) = h_{\text{top}}(F_A, L(0)).$$

Given $\xi^+ \in \Sigma^+_N$ and $\ell \in \mathbb{N}$, denote by $v_+(\xi^+, \ell) \in \mathbb{P}^1$ a vector at which $|(f_{\xi^+}^\ell)'|$ attains its maximum; note that this vector is unique unless $f_{\xi^+}^\ell$ is an isometry.

**Lemma 2.6** [16, Proposition 11.5]. Assume $\xi^+ \in \Sigma^+_N$ satisfies $\lambda_1(A, \xi^+) = \alpha$.

1. If $\alpha = 0$, then $\chi^+(\xi^+, v) = 0$ for all $v \in \mathbb{P}^1$.
2. If $\alpha > 0$, then the limit $v_0(\xi^+) = \lim_{\ell \to \infty} v_+(\xi^+, \ell)$ exists and it holds

$$\chi^+(\xi^+, v) = \begin{cases} 2\alpha & \text{for } v = v_0(\xi^+), \\ -2\alpha & \text{otherwise.} \end{cases}$$

**Proof of Theorem B.** By Lemma 2.5, it suffices to study the metric entropy of measures $\nu^+ \in \mathcal{M}_{\text{erg}}(\sigma^+)$ satisfying $\lambda_1(A, \nu^+) = 0$. Consider the projections $\pi^+: \Sigma_N \to \Sigma^+_N$, $\pi^+(\xi^+, \ell) = \xi^+$, and $\pi_1: \Sigma_N \times \mathbb{P}^1 \to \Sigma_N$, $\pi_1(\xi, x) = \xi$.

**Claim 2.7.** Given $\mu \in \mathcal{M}_{\text{erg}}(F_A)$, let $v^+ \overset{\text{def}}{=} (\pi^+ \circ \pi_1)_\ast \mu$. If $\chi(\mu) = 0$ then $\lambda_1(A, \nu^+) = 0$.

**Proof.** By ergodicity, $\chi(\xi, v) = 0$ for $\mu$-almost every $(\xi, v)$. Denote by $\mu^+$ the ergodic measure obtained as the push-forward of $\mu$ by the map $(\xi, v) \mapsto (\xi^+, v)$. Hence for $\mu^+$-almost every $(\xi^+, v)$ it holds $\chi^+(\xi^+, v) = 0$. It follows from Lemma 2.6 (1) that $\lambda_1(A, \nu^+) = 0$ for $\nu^+$-almost every $\xi^+$. Note that $\nu^+$ is ergodic. Hence, by the subadditive ergodic theorem, the claim follows.

**Claim 2.8.** For every $\nu^+ \in \mathcal{M}_{\text{erg}}(\sigma^+)$ with $\lambda_1(A, \nu^+) = 0$ there exists $\mu \in \mathcal{M}_{\text{erg}}(F_A)$ satisfying $\chi(\mu) = 0$.

**Proof.** Given $\nu^+ \in \mathcal{M}_{\text{erg}}(\sigma^+)$ there is $\mu \in \mathcal{M}_{\text{erg}}(F_A)$ such that $\nu^+ = (\pi^+ \circ \pi_1)_\ast \mu$. It follows from Lemma 2.6 (1) that $\chi^+(\xi^+, v) = 0$ for $\nu^+$-almost every $\xi^+$ and any $v$. Hence, $\chi(\xi, v) = 0$ for $\mu$-almost every $(\xi, v)$, which gives $\chi(\mu) = 0$.

It follows\(^7\) from [25] that $h(\sigma^+, \nu^+) = h(F_A, \mu)$. Hence

$$\sup_{\mu \in \mathcal{M}_{\text{erg}}(F_A), \chi(\mu) = 0} h(F_A, \mu) = \sup_{\nu^+ \in \mathcal{M}_{\text{erg}}(\sigma^+), \lambda_1(A, \nu^+) = 0} h(\sigma^+, \nu^+).$$

\(^7\) It holds

$$\sup_{\mu: \mu \circ (\pi^+ \circ \pi_1)^{-1} = \nu^+} h(F_A, \mu) = h(\sigma^+, \nu^+) + \int_{\Sigma^+_N} h_{\text{top}}(F_A, (\pi^+ \circ \pi_1)^{-1})(\xi^+) \, d\nu^+(\xi^+).$$

It is straightforward to check that $h_{\text{top}}(F_A, (\pi^+ \circ \pi_1)^{-1})(\xi^+) = 0$ for every $\xi^+$. 

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Hence, the equality follows from Theorem A. The fact that the entropy of this level set is less than \( \log N \) and positive was shown in [16, Theorem B]. \( \square \)

3. Coded Systems

3.1. Preliminaries and decipherability. Throughout this section, let \( \mathcal{A} \) be a finite collection of symbols, also called an alphabet. The full shift over \( \mathcal{A} \), denoted by \( \mathcal{A}^\mathbb{Z} \), is the collection of all bi-infinite sequences of symbols from \( \mathcal{A} \). We write an element of this space as \( a = (a_k)_{k \in \mathbb{Z}} = (\ldots, a_{-1}|a_0, a_1, \ldots) \), where \( a_k \in \mathcal{A} \) for all \( k \in \mathbb{Z} \). Equipped with the metric \( d_1(a, b) \stackrel{\text{def}}{=} e^{-\inf\{|k|: a_k \neq b_k\}} \) this space is compact. Given \( a \in \mathcal{A}^\mathbb{Z} \) and \( n \in \mathbb{N} \), we denote by

\[
[a_0, \ldots, a_n] = \{ b \in \mathcal{A}^\mathbb{Z} : b_k = a_k \text{ for all } k = 0, \ldots, n-1 \}
\]

its \( n \)-cylinder or simply cylinder associated to \( (a_0, \ldots, a_{n-1}) \).

A word over \( \mathcal{A} \) is a finite sequence of symbols in \( \mathcal{A} \). The length of a word \( a \) is the number of symbols it contains and is denoted by \( |a|_A \). A \( k \)-word is a word of length \( k \) and the set of all \( k \)-words over \( \mathcal{A} \) is denoted by \( \mathcal{A}^k \). The empty word \( e \) is the unique word with no symbols and of length zero. The set of all words (including the empty word) over the alphabet \( \mathcal{A} \) is denoted by \( \mathcal{A}^* \). A concatenation of a pair of words \( a = (a_1, \ldots, a_k) \) and \( b = (b_1, \ldots, b_{\ell}) \) is the word \( (a, b) \stackrel{\text{def}}{=} (a_1, \ldots, a_k, b_1, \ldots, b_{\ell}) \). The concatenation of any number of words is analogous. A prefix of a word \( b \in \mathcal{A}^* \) is a word \( a \in \mathcal{A}^* \) such that \( b = (a, c) \) for some \( c \in \mathcal{A}^* \). A suffix of a word \( b \in \mathcal{A}^* \) is a word \( a \in \mathcal{A}^* \) such that \( b = (c, a) \) for some \( c \in \mathcal{A}^* \).

The shift \( \sigma_A : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z} \), defined by \( (\sigma_A(a))_k = a_{k+1} \) for every \( k \in \mathbb{Z} \), is a continuous map. A subshift is a closed subset of \( \mathcal{A}^\mathbb{Z} \) which is \( \sigma_A \)-invariant. Replacing \( \mathbb{Z} \) by \( \mathbb{N}_0 \) and by \( -\mathbb{N} \) one obtains one-sided shift spaces.

A coded system (CS) over the alphabet \( \mathcal{A} \) is a compact subshift \( S \subset \mathcal{A}^\mathbb{Z} \) such that there is a collection \( \mathcal{W} \subset \mathcal{A}^* \) of words over \( \mathcal{A} \) such that \( S \) is the shift invariant closure of all bi-infinite concatenations of words in \( \mathcal{W} \). Here we always assume that \( \mathcal{W} \) is finite.\(^8\) Any such collection \( \mathcal{W} \) is called a code for \( S \). Let us be a bit more precise in our notation. Each code \( \mathcal{W} \subset \mathcal{A}^* \) by itself can be considered as an alphabet giving rise to the space \( \mathcal{W}^\mathbb{Z} \). Consider the map \( \iota_\mathcal{W} : \mathcal{W} \to \mathcal{A}^* \) which sends each element \( w \in \mathcal{W} \) to the corresponding word in \( \mathcal{A}^* \) and let

\[
\iota_\mathcal{W} : \mathcal{W}^* \to \mathcal{A}^*, \quad \iota_\mathcal{W}(w_1, \ldots, w_k) \stackrel{\text{def}}{=} (\iota_\mathcal{W}(w_1), \ldots, \iota_\mathcal{W}(w_k)).
\]

Consider the extension of this map to the space of bi-infinite concatenations

\[
\iota_\mathcal{W} : \mathcal{W}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z},
\]

\[
\iota_\mathcal{W}(w, w_1, w_2, \ldots) \stackrel{\text{def}}{=} (\iota_\mathcal{W}(w_1) \iota_\mathcal{W}(w_0), \iota_\mathcal{W}(w_1), \ldots),
\]

which identifies the bi-infinite concatenation \( (\ldots, w_{-1}|w_0, w_1, \ldots) \in \mathcal{W}^\mathbb{Z} \) of elements in \( \mathcal{W} \) (that is, words over \( \mathcal{A} \)) with the corresponding sequence in \( \mathcal{A}^\mathbb{Z} \). Let

\[
\text{PCS}(\mathcal{W}) \stackrel{\text{def}}{=} \iota_\mathcal{W}(\mathcal{W}^\mathbb{Z}) \subset \mathcal{A}^\mathbb{Z}
\]

\(^8\) The general definition of a coded system allows \( \mathcal{W} \) to be infinite. However, this will not be needed in this paper.
denote the *pre-coded system* defined by \( \mathcal{W} \).

As we assume \( \mathcal{W} \) to be finite,

\[
\text{CS}(\mathcal{W}) \overset{\text{def}}{=} \bigcup_{k \in \mathbb{Z}} \sigma_k^\mathcal{W}(\text{PCS}(\mathcal{W})) = \left\{ (\sigma_k^\mathcal{W} \circ \xi_\mathcal{W})(w) : w \in \mathbb{W}^k, k \in \mathbb{Z} \right\}
\]

(3.4)

is a coded system (that is, this set is, in particular, compact and shift-invariant). Note that every coded system is transitive. We refer to [26, Chapter 13.5] for more details and references on coded systems.

Let \( \mathcal{W} \) be a finite collection of (nonempty) words over the alphabet \( \mathcal{A} \). We say that \( \mathcal{W} \) is *disjoint* if no element in \( \mathcal{W} \) is a prefix of another element in \( \mathcal{W} \). This term is justified by the fact that for any disjoint collection of (nonempty) words their associated cylinders are pairwise disjoint. We say that \( \mathcal{W} \) is *uniquely left decidable* if whenever \( \nu_\mathcal{W}(w_1, \ldots, w_m) \) is a prefix of \( \nu_\mathcal{W}(u_1, \ldots, u_n) \) where \( w_i, u_j \in \mathcal{W} \), then \( m \leq n \) and \( w_i = u_i \) for every \( i = 1, \ldots, m \).

**Lemma 3.1.** Every disjoint finite collection of words is uniquely left decidable.

**Proof.** Let \( \mathcal{W} \) be a disjoint collection of words over the alphabet \( \mathcal{A} \). We proceed by induction over the length of a prefix of a word over the alphabet \( \mathcal{W} \). As \( \mathcal{W} \) by hypothesis is disjoint, the statement is true for \( k = 1 \). Assuming the statement is true for \( k \in \mathbb{N} \), suppose that \( \nu_\mathcal{W}(w_1, w_2, \ldots, w_k, w_{k+1}) \) is a prefix of \( \nu_\mathcal{W}(u_1, u_2, \ldots, u_{m}) \) for some \( m \in \mathbb{N} \). Again invoking our hypothesis that \( \mathcal{W} \) is disjoint, it follows \( w_{i_1} = u_{j_1} \). Hence \( \nu_\mathcal{W}(w_1, \ldots, w_k, w_{k+1}) \) is a prefix of \( \nu_\mathcal{W}(u_1, \ldots, u_{m}) \). Then the induction hypothesis implies \( w_{i_n} = u_{j_n} \) for every \( n = 2, \ldots, k + 1 \), proving the lemma. \( \square \)

In what follows we assume that \( \mathcal{W} = \{w_1, \ldots, w_M\} \). Given \( a \in \text{PCS}(\mathcal{W}) \) a bi-infinite concatenation of words in \( \mathcal{W} \), a *decoding* of \( a \) in \( \mathcal{W} \) is a sequence \((\ldots, i_{-1}|i_0, i_1, \ldots) \in \{1, \ldots, M\}^\mathbb{Z}\) such that

\[
a = \nu_\mathcal{W}(\ldots, w_{i_{-1}}|w_{i_0}, w_{i_1}, \ldots), \quad \text{that is} \quad a \in \nu_\mathcal{W}^{-1}(w).
\]

We call then the one-sided sequences \((\ldots, i_{-1}) \in \{1, \ldots, M\}^{-\mathbb{N}}\) and \((i_0, i_1, \ldots) \in \{1, \ldots, M\}^{\mathbb{N}}\) a *backward* and *forward* decoding of \( a \), respectively.

Let us denote by

\[
\xi^+_{\mathcal{W}} : \mathcal{W}^\mathbb{N} \rightarrow \mathcal{A}^\mathbb{N} \quad \text{and} \quad \xi^-_{\mathcal{W}} : \mathcal{W}^{-\mathbb{N}} \rightarrow \mathcal{A}^{-\mathbb{N}}
\]

the maps analogously defined to the one in (3.2), then a code \( \mathcal{W} \) is uniquely left decipherable if, and only if, \( \xi^+_{\mathcal{W}} \) is invertible. So Lemma 3.1 implies the following.

**Corollary 3.2.** Let \( \mathcal{W} \) be a disjoint finite collection of words over \( \mathcal{A} \). Then every element in \( \text{PCS}(\mathcal{W}) \) has a unique forward decoding in \( \mathcal{W} \) and hence \( \xi^+_{\mathcal{W}} \) is invertible.

**Remark 3.3.** In general, a (even disjoint) collection of words defines sequences which cannot be uniquely decoded. For example, for the alphabet \( \mathcal{A} = \{a, b\} \) the collection \( \mathcal{W} = \{(a, b, a), (a, b, b), (b, a), (b, b, a)\} \) is disjoint, but

\[
\{(a, b, a)^{-\mathbb{N}}, (a, b, a|\ldots) = ((b, b, a)^{-\mathbb{N}}, (b, a|\ldots) \in \text{PCS}(\mathcal{W}) \subset \mathcal{A}^{\mathbb{Z}}
\]

can be written in two different ways as concatenation of words in \( \mathcal{W} \). Nevertheless, the following fundamental fact holds. For completeness, we provide its proof.

\[9\] Note that this is slightly weaker than being uniquely decipherable as in [26, Definition 8.1.21].
Lemma 3.4. Let \( \mathcal{W} \) be a disjoint finite collection of words over \( \mathcal{A} \). Then every element in \( \text{PCS}(\mathcal{W}) \) has at most \( R \) decodings in \( \mathcal{W} \), where \( R \) is the largest length of a word in \( \mathcal{W} \).

Proof. By Corollary 3.2 it suffices to study backward decodings of a sequence. By contradiction, suppose that there exists \( a \in \text{PCS}(\mathcal{W}) \) which has \( R + 1 \) backward decodings. We consider a family \( \mathcal{I} \subset \{1, \ldots, M\}^{-N} \) of decodings of \( a \) having \( R + 1 \) elements. Given two such backward decodings \( i^- = (\ldots, i_{-1}) \) and \( j^- = (\ldots, j_{-1}) \) in \( \mathcal{I} \), for \( k \in \mathbb{N} \) let

\[
I_k \overset{\text{def}}{=} |\text{t}_{\mathcal{W}}(w_{i_{-k}}, \ldots, w_{i_{-1}})|_{\mathcal{A}} \quad \text{and} \quad J_k \overset{\text{def}}{=} |\text{t}_{\mathcal{W}}(w_{j_{-k}}, \ldots, w_{j_{-1}})|_{\mathcal{A}}.
\]

If there is \( k \in \mathbb{N} \) such that there exists an index \( \ell \in \mathbb{N} \) with \( I_k = J_\ell \), then corresponding parts of the decodings coincide as concatenated words over the alphabet \( \mathcal{A} \), that is,

\[
\text{t}_{\mathcal{W}}(w_{i_{-k}}, \ldots, w_{i_{-1}}) = \text{t}_{\mathcal{W}}(w_{j_{-k}}, \ldots, w_{j_{-1}}).
\]

As we assume that \( \mathcal{W} \) is disjoint, by Lemma 3.1 it is uniquely left decipherable. Hence, it follows \( k = \ell \) and \( i_{-n} = j_{-n} \) for every \( n = k, \ldots, 1 \). If there are infinitely many such indices \( k \in \mathbb{N} \), then it follows \( i^- = j^- \), contradicting the fact that we consider two distinct backward decodings. Hence, there exists one largest such index \( k_0 = k_0(i^-, j^-) \). Let

\[
N \overset{\text{def}}{=} \max_{i^-, j^- \in \mathcal{I}} k_0(i^-, j^-)
\]

and note that \( N \) is finite as \( \mathcal{I} \) has \( R + 1 \) elements. Hence, by uniquely left decipherability, this implies that

\[
\{ \text{t}_{\mathcal{W}}(w_{i_{-n}}, \ldots, w_{i_{-1}}) : i^- \in \mathcal{I}, w_{-\ell} \in \mathcal{W}, \ell = N, \ldots, 1 \}
\]

is a collection of distinct words. Because every word in \( \mathcal{W} \) has a length bounded by \( R \), for every \( i^- \in \mathcal{I} \) there exists some index \( n \geq N \) such that

\[
-N R - R \leq - \sum_{\ell=1}^{n} |\text{t}_{\mathcal{W}}(w_{i_{-\ell}})|_{\mathcal{A}} \leq -N R - 1.
\]

In other terms, the marker of the starting position of the \( n \)th word in this decoding is between position \( -N R - R \) and position \( -N R - 1 \) of the “spelling” of \( a \) in the alphabet \( \mathcal{A} \). As by assumption we have \( R + 1 \) such decodings, by the pigeonhole principle (at least) two of them must end at the same position. Say, there are \( i^-, j^- \in \mathcal{I} \) and indices \( k \) and \( \ell \), respectively, such that

\[
-|\text{t}_{\mathcal{W}}(w_{i_{-k}})|_{\mathcal{A}} - \ldots - |\text{t}_{\mathcal{W}}(w_{i_{-1}})|_{\mathcal{A}} = -|\text{t}_{\mathcal{W}}(w_{j_{-\ell}})|_{\mathcal{A}} - \ldots - |\text{t}_{\mathcal{W}}(w_{j_{-1}})|_{\mathcal{A}}.
\]

As in the argument before, uniquely left decipherability implies

\[
\text{t}_{\mathcal{W}}(w_{i_{-k}}, \ldots, w_{i_{-1}}) = \text{t}_{\mathcal{W}}(w_{j_{-\ell}}, \ldots, w_{j_{-1}})
\]

and hence \( k = \ell \) and \( i_n = j_n \) for every \( n = k, \ldots, 1 \). In particular, the latter implies \( i_{-n} = j_{-n} \) for every \( n = N, \ldots, 1 \). This contradicts the fact that all words in (3.5) are distinct. This proves the lemma. \( \square \)

The next facts are immediate consequences of the definition of disjointness.

Corollary 3.5. Let \( \mathcal{W} \) be a disjoint collection of words over the alphabet \( \mathcal{A} \) and \( m \in \mathbb{N} \). Then \( \mathcal{W}^m \) is a disjoint collection of words over \( \mathcal{A} \).

Corollary 3.6. Let \( \mathcal{W}' = \{w'_1, \ldots, w'_M\} \) and \( \mathcal{W} = \{w_1, \ldots, w_M\} \) be two collections of words over the alphabet \( \mathcal{A} \) such that \( w_i \) is a prefix of \( w'_i \) for every \( i = 1, \ldots, M \). If \( \mathcal{W} \) is disjoint then \( \mathcal{W}' \) is disjoint.
3.2. Coded subsystems of the sequence space $\Sigma_N$. The base space $\Sigma_N = \mathcal{A}^\mathbb{Z}$ of the skew product (1.1) with $\mathcal{A} = \{1, \ldots, N\}$ is a special case of the above.

**Notation 3.7.** In Sects. 6, 7, 8 and 10, our base alphabet will always be $\mathcal{A} = \{1, \ldots, N\}$. In Sects. 4 and 5, the alphabet $\mathcal{A}$ is unspecified. We are also going to use two families of other alphabets, $(\mathcal{A}_n)_n$ and $(\mathcal{W}_n)_n$, defined by some finite families of words from $\mathcal{A}^\ast$. All of those alphabets are going to be disjoint. For better readability, we will identify the finite words in alphabets $\mathcal{A}_n$ and $\mathcal{W}_n$ with the corresponding finite words in $\mathcal{A}$. Note that, because of disjointness, we have decipherability of any finite word, hence this convention is not going to lead to any ambiguity. For infinite or bi-infinite words in those alphabets we will use the precise notation, to always keep track whether we are talking about an element of $\mathcal{W}_n^\mathbb{Z}$ or an element of $\mathcal{A}^\mathbb{Z}$: Note that $\text{PCS}(\mathcal{W})$ is a subset of $\mathcal{A}^\mathbb{Z}$.

Given $w_1, \ldots, w_m \in \mathcal{W}$, we denote by

$$|(w_1, \ldots, w_m)| \overset{\text{def}}{=} |(\iota_{\mathcal{W}}(w_1), \ldots, \iota_{\mathcal{W}}(w_m))|_{\mathcal{A}}$$

the length of the concatenated word (spelled in the alphabet $\{1, \ldots, N\}$). We let

$$[w]^{+} \overset{\text{def}}{=} [\iota_{\mathcal{W}}(w)]^{+}$$

(3.6)

denote the cylinder in $\Sigma_N^\mathbb{Z}$. Given $u \in \Sigma_N^\ast$ we denote by

$$(w_1, \ldots, w_m, u) \overset{\text{def}}{=} (\iota_{\mathcal{W}}(w_1, \ldots, w_m), u) \in \Sigma_N^\ast$$

the corresponding concatenated word (in the alphabet $\{1, \ldots, N\}$). Finally, let

$$\mathcal{W}^m \overset{\text{def}}{=} \{(w_1, \ldots, w_m) : w_k \in \mathcal{W}, k = 1, \ldots, m \} \subset \mathcal{A}^\ast.$$  

Analogously to notation (1.2), for $k = 1, \ldots, |(w_1, \ldots, w_m)| \overset{\text{def}}{=} L$, we denote by

$$f^k_{[w_1, \ldots, w_m]} \overset{\text{def}}{=} f_{\xi_{k-1}} \circ \cdots \circ f_{\xi_0}, \text{ where } (\xi_0, \xi_1, \ldots, \xi_{L-1}) \overset{\text{def}}{=} (\iota_{\mathcal{W}}(w_1, \ldots, w_m)) \in \Sigma_N^\ast.$$  

the map obtained by concatenating the maps from the family $\{f_i\}_i$ which are indexed by the first $k$ elements of the concatenated words $w_1, \ldots, w_m$ (spelled in $\{1, \ldots, N\}$). Moreover, for $w \in \mathcal{W}$ let

$$f^{|w|}_w \overset{\text{def}}{=} f^{|w|}_{[w]}.$$  

(3.7)

4. Suspensions of Shift Spaces

We consider measure preserving systems obtained from suspensions of Bernoulli shifts, collect some standard facts (Sect. 4.1), and discuss large deviation results (Sect. 4.2). Throughout this section, we fix a finite initial alphabet $\mathcal{A}$. Later (starting in Sect. 6) we assume that this alphabet is $\{1, \ldots, N\}$ and also invoke the axioms in Sect. 2.1, however these ingredients are irrelevant in this section.
4.1. Suspension model in the full shift over $\mathcal{A}$. Given a function $R : \mathcal{A} \to \mathbb{N}$, we extend it to a step function on the sequence space $\mathcal{A}^\mathbb{Z}$ by

$$R : \mathcal{A}^\mathbb{Z} \to \mathbb{N}, \quad a = (\ldots, a_{-1}|a_0, a_1, \ldots) \in \mathcal{A}^\mathbb{Z} \mapsto R(a) = R(a_0).$$

We define the discrete-time suspension space

$$S_{\mathcal{A}, R} \overset{\text{def}}{=} (\mathcal{A}^\mathbb{Z} \times \mathbb{N}_0) / \sim,$$

defined as the quotient space of $\mathcal{A}^\mathbb{Z} \times \mathbb{N}_0$ modulo the equivalence relation $\sim$ that identifies $(a, s)$ with $(\sigma_\mathcal{A}(a), s - R(a))$ for every $s \geq R(a)$ and $a \in \mathcal{A}^\mathbb{Z}$. For convenience, we represent each class by its element $(a, s)$ with $s \in \{0, \ldots, R(a) - 1\}$, called its canonical representation.

The suspension of $\sigma_\mathcal{A}$ by $R$ is the map

$$\Phi_{\mathcal{A}, R} : S_{\mathcal{A}, R} \to S_{\mathcal{A}, R}, \quad \Phi_{\mathcal{A}, R}(a, s) \overset{\text{def}}{=} \begin{cases} (a, s + 1) & \text{if } 0 \leq s + 1 < R(a), \\ (\sigma_\mathcal{A}(a), 0) & \text{if } s = R(a) - 1. \end{cases}$$

We also consider the ground floor

$$G_{\mathcal{A}, R} \overset{\text{def}}{=} \mathcal{A}^\mathbb{Z} \times \{0\} \subset S_{\mathcal{A}, R}.$$

Remark 4.1 ((Piecewise constant) roof function). We can view the map $R$ as a roof function over $\mathcal{A}^\mathbb{Z}$. By our choice, this function is piecewise constant on each cylinder determined by a symbol of $\mathcal{A}$, that is, for every $a_0 \in \mathcal{A}$ it holds

$$R(a) = R(a_0)$$

for every $a \in [a_0]$. Let

$$M \overset{\text{def}}{=} \text{card } \mathcal{A}, \quad b_\mathcal{A} \text{ be the } (\frac{1}{M}, \ldots, \frac{1}{M})\text{-Bernoulli measure on } \mathcal{A}^\mathbb{Z}, \text{ and } m \text{ be the counting measure on } \mathbb{Z}. \text{ Define the measure } \lambda_{\mathcal{A}} \text{ on } S_{\mathcal{A}, R} \text{ by}

$$\lambda_{\mathcal{A}} \overset{\text{def}}{=} \frac{1}{(b_\mathcal{A} \times m)(S_{\mathcal{A}, R})} (b_\mathcal{A} \times m)|_{S_{\mathcal{A}, R}} = \frac{(b_\mathcal{A} \times m)|_{S_{\mathcal{A}, R}}}{\int R \, db_\mathcal{A}}. \quad (4.1)$$

Given a continuous function $\psi : S_{\mathcal{A}, R} \to \mathbb{R}$, define $\Delta \psi : \mathcal{A}^\mathbb{Z} \to \mathbb{R}$ by

$$\Delta \psi(a) \overset{\text{def}}{=} \sum_{k=0}^{R(a)-1} \psi(a, k). \quad (4.2)$$

Lemma 4.2 (Abramov’s formula). The measure $\lambda_{\mathcal{A}, R}$ is a $\Phi_{\mathcal{A}, R}$-invariant and ergodic Borel probability measure satisfying

$$h_{\text{top}}(\Phi_{\mathcal{A}, R}, S_{\mathcal{A}, R}) \geq h(\Phi_{\mathcal{A}, R}, \lambda_{\mathcal{A}, R}) = \frac{\log \text{card } \mathcal{A}}{\int R \, db_\mathcal{A}} = \frac{\log M}{\frac{1}{M} \sum_{a \in \mathcal{A}} R(a)}.$$

Moreover, for any continuous function $\psi : S_{\mathcal{A}, R} \to \mathbb{R}$ it holds

$$\int \psi \, d\lambda_{\mathcal{A}, R} = \frac{\int \Delta \psi \, db_\mathcal{A}}{\int R \, db_\mathcal{A}}.$$
4.2. Controlled large deviations. We now study the fluctuation of Birkhoff sums of repeated returns to the ground floor of the suspension space. For that, given a potential $\psi : S_{A,R} \to \mathbb{R}$ and $\Delta \psi : \mathcal{A}^\mathbb{Z} \to \mathbb{R}$ defined as above, let

$$\var_{\mathcal{A}}(\Delta \psi) \overset{\text{def}}{=} \max_{a \in \mathcal{A}} \max_{b,c \in [a]} \left\{ (\Delta \psi(b) - \Delta \psi(c)) \right\}.$$  \hfill (4.3)

**Proposition 4.3.** Let $\psi : S_{A,R} \to \mathbb{R}$ be a continuous potential. For every $\varepsilon > 0$ there exists $N_0 = N_0(\psi, \varepsilon) \in \mathbb{N}$ such that if $m \geq N_0$ then there exists a set $A \in \mathcal{A}^\mathbb{Z}$ such that

$$\var_{\mathcal{A}}(\Delta \psi) \geq 1 - \varepsilon$$

and for every $a \in A$, $i = 0, \ldots, m - 1$, and $k \in \{1, \ldots, m\}$ it holds

$$\left| \sum_{j=1}^{i+k-1} \left( R(\sigma^j_A(a)) - \int \mathbb{R} d\mathbb{b}_A \right) \right| < m\varepsilon,$$

$$\left| \sum_{j=1}^{i+k-1} \left( \Delta \psi(\sigma^j_A(a)) - \int \Delta \psi d\mathbb{b}_A \right) \right| < m(2 \var_{\mathcal{A}}(\Delta \psi) + \varepsilon).$$

**Proof.** We use the following probability result based on the Bernstein inequality.

**Lemma 4.4.** Let $X$ be a bounded random variable with expected value $\mathbb{E}(X)$. Then for every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for every $m \geq N_0$ the following holds. Let $X_1, \ldots, X_{2m}$ be independent and identically distributed copies of $X$. Then for every $k \in \{1, \ldots, m\}$ it holds

$$\mathbb{P}\left( \left| \sum_{j=1}^{i+k-1} (X_j - \mathbb{E}(X)) \right| < m\varepsilon \quad \text{for every } i \in \{1, \ldots, m\} \right) > 1 - \varepsilon.$$

**Proof.** Take $C > 0$ such that $|X - \mathbb{E}(X)| \leq C$. By the Bernstein inequality, for every $m \in \mathbb{N}$, $i \in \{0, \ldots, m - 1\}$, and $k \in \{1, \ldots, m\}$ it holds

$$\mathbb{P}\left( \left| \sum_{j=1}^{i+k-1} (X_j - \mathbb{E}(X)) \right| < m\varepsilon \right) > 1 - 2e^{-3m\varepsilon/2C}.$$

Hence, it follows

$$\mathbb{P}\left( \left| \sum_{j=1}^{i+k-1} (X_j - \mathbb{E}(X)) \right| < m\varepsilon \quad \text{for every } i \in \{1, \ldots, m\} \right) > 1 - 2me^{-3m\varepsilon/2C}.$$  \hfill \square

The assertion follows taking $m$ large enough.

To continue with the proof of the proposition, given $\mathcal{A} = \{a_1, \ldots, a_M\}$, consider the partition $\{[a_k] : k = 1, \ldots, M\}$ of $\mathcal{A}^\mathbb{Z}$ into cylinders and let

$$B(a_k) \overset{\text{def}}{=} \max_{[a_k]} \Delta \psi.$$  \hfill \hfill (4.4)

Let

$$\mathfrak{B} \overset{\text{def}}{=} \frac{1}{M}(B(a_1) + \cdots + B(a_M)), \quad \mathfrak{R} \overset{\text{def}}{=} \frac{1}{M}(R(a_1) + \cdots + R(a_M)).$$
Take the Bernoulli measure $b_A$ on $A^{\mathbb{Z}}$ and the random variables on this space

$$X_j(a) = B(a_j) \quad \text{and} \quad Y_j(a) = R(a_j),$$

which are bounded from above by $\max_k B(a_k)$ and $\max_k R(a_k)$, respectively. They are independent and identically distributed and have expected values $\mathcal{B}$ and $\mathcal{R}$, respectively. Given $\varepsilon > 0$, applying Lemma 4.4 to each of those variables, there exists $N_0 \in \mathbb{N}$ such that for every $m \geq N_0$ there is a set $A$ satisfying $b_A(A) > 1 - \varepsilon$ such that for every $a \in A, i = 0, \ldots, m - 1$, and $k \in \{1, \ldots, m\}$ it holds

$$\left| \sum_{j=i}^{i+k-1} B(a_j) - k\mathcal{B} \right| \leq m\varepsilon, \quad \left| \sum_{j=i}^{i+k-1} R(a_j) - k\mathcal{R} \right| \leq m\varepsilon. \quad (4.4)$$

The second inequality in (4.4) and the fact that $\mathcal{R}$ is piecewise constant proves the first claim of the proposition as $\mathcal{R} = \int R \, db_A$.

To prove the second claim, recall that by the definition of $\var_A$ in (4.3) it holds

$$\left| \Delta \psi(\sigma^j_A(a)) - B(a_j) \right| \leq \var_A(\Delta \psi) \quad (4.5)$$

for every $j \in \mathbb{Z}$. For every $i = 0, \ldots, m - 1$, by summing over $j = i, \ldots, i + k - 1$, it follows

$$\left| \sum_{j=i}^{i+k-1} \Delta \psi(\sigma^j_A(a)) - k\mathcal{B} \right| \leq \sum_{j=i}^{i+k-1} \left( \Delta \psi(\sigma^j_A(a)) - B(a_j) \right) + \sum_{j=i}^{i+k-1} B(a_j) - k\mathcal{B} \leq k \var_A(\Delta \psi) + m\varepsilon \leq m \var_A(\Delta \psi) + m\varepsilon.$$ 

Also note that

$$|\mathcal{B} - \int \Delta \psi \, db_A| \leq \var_A(\Delta \psi),$$

which together with $k \leq m$ and the above implies

$$\left| \sum_{j=i}^{i+k-1} \Delta \psi(\sigma^j_A(a)) - k \int \Delta \psi \, db_A \right| \leq \left| \sum_{j=i}^{i+k-1} \Delta \psi(\sigma^j_A(a)) - k\mathcal{B} \right| + m \var_A(\Delta \psi) \leq m \left( \var_A(\Delta \psi) + \varepsilon + \var_A(\Delta \psi) \right).$$

This implies the second claim of the proposition. $\square$

5. Cascade of Alphabets: Repetition and Tailing

Fix a finite alphabet $A$ and any increasing sequence of positive integers $(m_n)_{n \geq 1}$. In Sect. 5.1, we present an inductive construction of a cascade of alphabets $A_n$, each obtained by concatenating $m_n$ words of the former. Each such alphabet comes with a sequence space and corresponding suspension space as in Sect. 4. The inductive definition gives some “self-similar” structure in the sense that each space “contains copies” of the precedents. Section 5.2 studies this structure and develops some terminology. As our measure preserving suspensions are defined by means of Bernoulli measures, all their “copy measures” coincide, see Sect. 5.3. In Sect. 5.4 we collect some relations between the roof functions across the cascade.
5.1. Inductive definition of alphabets. Let $\mathcal{A}_0 \overset{\text{def}}{=} \mathcal{A}$, $M_0 \overset{\text{def}}{=} \text{card } \mathcal{A}_0 = M$. For $n \geq 1$, assume that we have given the collection $\mathcal{A}_{n-1}$ of $M_{n-1}$ finite words over $\mathcal{A}$. Let

$$\mathcal{A}_n \overset{\text{def}}{=} (\mathcal{A}_{n-1})^{m_n} \subset \mathcal{A}^n.$$  

Note that

$$\text{card } \mathcal{A}_n = M_n \overset{\text{def}}{=} (M_{n-1})^{m_n}.$$  

This concludes the inductive definition of the cascade of alphabets $\mathcal{A}_n$.

Note that $\mathcal{A}_n$ is a finite collection of words over the alphabet $\mathcal{A}$: the collection of all $(m_1 \cdot m_2 \cdots m_n)$-words over $\mathcal{A}$. Indeed, each word is obtained by a concatenation of $m_n$-words in the alphabet $\mathcal{A}_{n-1}$. On one hand, by convention, each concatenation of words over $\mathcal{A}$ again is a word over $\mathcal{A}$. On the other hand, the collection $\mathcal{A}_n$ by itself can serve as an alphabet, each of its elements being a symbol in $\mathcal{A}_n$. In each step, we have the corresponding canonical bijection between the symbols in $\mathcal{A}_n$ and the $m_n$-words over the alphabet $\mathcal{A}_{n-1}$. This defines a substitution map from $\mathcal{A}_n$ to $\mathcal{A}_{n-1}$:

$$\mathcal{S}_{n,n-1} : \mathcal{A}_n \to (\mathcal{A}_{n-1})^{m_n}.$$  

We extend this map to the bijection between the corresponding spaces of sequences

$$\overline{\mathcal{S}}_{n,n-1} : (\mathcal{A}_n)^Z \to ((\mathcal{A}_{n-1})^{m_n})^Z = (\mathcal{A}_{n-1})^Z,$$

$$\overline{\mathcal{S}}_{n,n-1}(\ldots |a_0^{(n)}, a_1^{(n)}, \ldots) \overset{\text{def}}{=} (\ldots |\mathcal{S}_{n,n-1}(a_0^{(n)}), \mathcal{S}_{n,n-1}(a_1^{(n)}), \ldots).$$  

Denote by $\sigma_n = \sigma_{\mathcal{A}_n} : (\mathcal{A}_n)^Z \to (\mathcal{A}_n)^Z$ the left shift over $\mathcal{A}_n$. Let $b_n = b_{\mathcal{A}_n}$ be the $(1/m_n, \ldots, 1/m_n)$-Bernoulli measure on $(\mathcal{A}_n)^{\mathbb{N}}$.

Remark 5.1. Note that $\overline{\mathcal{S}}_{n,n-1}$ is a bijection. Moreover $\sigma_n : (\mathcal{A}_n)^Z \to (\mathcal{A}_n)^Z$ is topologically conjugate with $\sigma_{\mathcal{A}_{n-1}}^{m_n} : (\mathcal{A}_{n-1})^Z \to (\mathcal{A}_{n-1})^Z$ by $\mathcal{S}_{n,n-1}$. Note that

$$(\overline{\mathcal{S}}_{n,n-1})_* b_n = b_{n-1}. $$  

Thus,

$$\overline{\mathcal{S}}_{n,n-1} : ((\mathcal{A}_n)^Z, \sigma_n, b_n) \to ((\mathcal{A}_{n-1})^Z, \sigma_{\mathcal{A}_{n-1}}^{m_n}, b_{n-1})$$  

is a metric isomorphism. Hence, applying the previous argument inductively, we obtain that for every $n \in \mathbb{N}$ and $\ell \in \{n-1, \ldots, 0\}$ it holds that

$$\overline{\mathcal{S}}_{n,\ell} \overset{\text{def}}{=} \overline{\mathcal{S}}_{\ell+1,\ell} \circ \cdots \circ \overline{\mathcal{S}}_{n,n-1} : ((\mathcal{A}_n)^Z, \sigma_n, b_n) \to ((\mathcal{A}_\ell)^Z, \sigma_{\mathcal{A}_\ell}^{m_{\ell+1} \cdots m_n}, b_\ell)$$  

(5.3)

is a metric isomorphism. In particular, taking $\ell = 0$, the map

$$\overline{\mathcal{S}}_{n,0} : ((\mathcal{A}_n)^Z, \sigma_n, b_n) \to (\mathcal{A}^Z, \sigma_{\mathcal{A}}^{m_1 \cdots m_n}, b),$$  

(5.4)

where $\sigma_\mathcal{A}$ is the shift in the original alphabet $\mathcal{A}$, is a metric isomorphism.
5.2. Suspension spaces and their internal structure. We now invoke the suspension model in Sect. 4.1 and apply it to each alphabet $A_n$, $n \in \mathbb{N}$. Let $R_n : A_n \to \mathbb{N}$ be some function and consider the corresponding function $R_n : (A_n)^\mathbb{Z} \to \mathbb{N}$. Consider the suspension space $S_{n} = S_{A_n}.R_n$ and the suspension of $\sigma_n = \sigma_{A_n}$ by $R_n$ and denote it by $\Phi_n = \Phi_{A_n}.R_n$. Denote by $\sim$ the corresponding equivalence relation in the definition of the suspension space. Recalling (4.1), consider the $\Phi_n$-ergodic Borel probability measure
\[
\lambda_n = \lambda_{A_n}.R_n \overset{\text{def}}{=} \frac{1}{(b_n \times m)(S_n)}(b_n \times m)|_{S_n}.
\]
(5.5)

We will always use these short notations, unless there is risk of confusion.

The suspension spaces have a “self-similar” internal structure that we will study.

5.2.1. Roofs and tailing functions We impose the following assumption about the roof functions across the cascade. Most of this section only requires the lower bound in Assumption 5.2. We will invoke the upper bound only in Proposition 5.12 to estimate the “expected roof heights” and “expected length of tails”.

Assumption 5.2 (Roof functions). There exists $K > 0$ such that for every $b \in A_n$ with $S_{n,n-1}(b) = (a_0, \ldots, a_{m_n-1}) \in (A_{n-1})^{m_n}$ it holds
\[
\sum_{k=0}^{m_n-1} R_{n-1}(a_k) < R_n(b) \leq \left(1 + K 2^{-(n-1)}\right) \sum_{k=0}^{m_n-1} R_{n-1}(a_k).
\]

Assumption 5.2 allows to define the “tailing length” function
\[
t_n : A_n \to \mathbb{N}, \quad t_n(b) \overset{\text{def}}{=} R_n(b) - \sum_{k=0}^{m_n-1} R_{n-1}(S_{n,n-1}(b))_k.
\]
(5.6)

As before, extend $t_n$ to $A_n^\mathbb{Z}$ by $t_n(b) \overset{\text{def}}{=} t_n(b_0), b = (... , b_{-1}|b_0, b_1, \ldots) \in (A_{n})^\mathbb{Z}$.

5.2.2. Ground floors Consider the $n$th level ground floor
\[
G_n = G_{A_n}.R_n \overset{\text{def}}{=} (A_n)^\mathbb{Z} \times \{0\} \subset S_n.
\]
(5.7)

Given $(a, s) \in S_n$ in its canonical form, define the natural projection
\[
p_n(a, s) \overset{\text{def}}{=} a
\]
from the suspension space “to its ground floor”. By definition,
\[
\Phi_n^R \circ p_n(b_0, 0) = \Phi_n^R(b, 0) = (\sigma_n(b), 0) \in S_n \text{ for every } (b_0, 0) \in G_n.
\]
(5.8)

Note that (5.8) is a return map on the ground floor.
5.2.3. **Intermediate floors of first order** We now extend the concept of *ground floor* $\mathcal{G}_n$ to so-called *intermediate floors*. For that we first divide the suspension space into its *principal part* and its *tail*,
\[ S_n = \mathcal{P}_n \cup \mathcal{T}_n , \]
as follows. A point $(b, s) \in S_n$ (canonically represented) is in $\mathcal{P}_n$ if and only if
\[ 0 \leq s < \sum_{k=0}^{m_n-1} R_{n-1}(\mathcal{S}_{n,n-1}(b))_k = R_n(b) - t_n(b) . \]
Otherwise it belongs to $\mathcal{T}_n$. We define the map
\[ P_{n,n-1}: \mathcal{P}_n \to S_{n-1}, \quad P_{n,n-1}(b, s) \overset{\text{def}}{=} (\mathcal{S}_{n,n-1}(b), s) . \]
Note that for $s > R_{n-1}(\mathcal{S}_{n,n-1}(b))$, the latter is not in its canonical representation. To obtain this representation, one has to take into account (possibly several times) the identification
\[ (\mathcal{S}_{n,n-1}(b), s) \sim_{n-1} (\sigma_{n-1}(\mathcal{S}_{n,n-1}(b)), s - R_{n-1}(\mathcal{S}_{n,n-1}(b))) . \]
By construction, the following holds.

**Lemma 5.3.** The suspension map $\Phi_{n-1}$ on $S_{n-1}$ is a topological factor of the map $\Phi_n$ restricted to $\mathcal{P}_n$ by the factor map $P_{n,n-1}$.

We now extend the term *ground floor*. The quotient map $P_{n,n-1}$ is $m_n$-to-1. Indeed, this follows since $P_{n,n-1}$ precisely restricts to the principal part $\mathcal{P}_n$ of the suspension space. Moreover, there is a natural order of the preimages given by the number of times we take into account the identification $\sim_{n-1}$ in order to obtain the canonical representation. Denote by $\mathcal{G}_n^{(i)}$ the $i$th preimage of the $(n-1)$st level ground floor $\mathcal{G}_{n-1}$ under $P_{n,n-1}$, for $i = 1, \ldots, m_n - 1$, and call them *intermediate floors*. More precisely, let $\mathcal{G}_n^{(0)} \overset{\text{def}}{=} \mathcal{G}_n$ and for $i = 1, \ldots, m_n - 1$ let
\[ \mathcal{G}_n^{(i)} \overset{\text{def}}{=} \left\{(b, s) \in S_n : s = \sum_{\ell=0}^{i-1} R_{n-1}(\mathcal{S}_{n,n-1}(b))_\ell \right\} \subset S_n . \]
They are “lifted copies” of the ground floor $\mathcal{G}_{n-1}$ in the suspension space $S_n$, using the fact that any symbol in the alphabet $\mathcal{A}_n$ is obtained as a concatenation of words in the lower-level alphabet $\mathcal{A}_{n-1}$.

Note that the intermediate floors separate the suspension space $S_n$ into the *strips*
\[ \mathcal{L}_n^{(i)} \overset{\text{def}}{=} \left\{(b, s) \in S_n : \sum_{\ell=0}^{i-1} R_{n-1}(\mathcal{S}_{n,n-1}(b))_\ell \leq s < \sum_{\ell=0}^{i} R_{n-1}(\mathcal{S}_{n,n-1}(b))_\ell \right\} , \]
for $i = 0, \ldots, m_n - 1$ (where for $i = 0$ the first sum is understood to be 0).

**Remark 5.4.** A point $(b, s) \in S_n$ (in its canonical representation) belongs to the strip $\mathcal{L}_n^{(i)}$ with address $(i)$ if the first (symbolic) coordinate of the canonical representation of $P_{n,n-1}(b, s)$ is $\sigma_{n-1}^i(\mathcal{S}_{n,n-1}(b))$.

By construction, the following holds.
Lemma 5.5. Each strip \( \mathcal{L}_n^{(i)} \) is mapped by \( P_{n,n-1} \) onto \( \mathcal{S}_{n-1} \) in a bijective way.

\[
P_{n,n-1}(\mathcal{L}_n^{(i)}) = \mathcal{S}_{n-1}.
\]

Moreover, the (disjoint) union of all strips is the principal part, that is,

\[
P_n = \bigcup_{i=0}^{m_n-1} \mathcal{L}_n^{(i)}.
\]

5.2.4. Intermediate floors of higher order: inductive definition. Above we defined intermediate floors \( \mathcal{G}_n^{(i)} \) by means of the roof function defined on the alphabet \( \mathcal{A}_{n-1} \). Recalling that any word in \( \mathcal{A}_{n-1} \) is in turn spelled in the symbols of the alphabet \( \mathcal{A}_{n-2} \), inside any strip \( \mathcal{L}_n^{(i)} \) that is bounded by the intermediate floors \( \mathcal{G}_n^{(i)} \) and \( \mathcal{G}_n^{(i+1)} \) (for some index \( i = 0, \ldots, m_n-1 \)) we will introduce further, deeper-level, intermediate floors and strips, and we will continue from level \( n - 2 \) down to any level \( \ell \in \{1, \ldots, n-1\} \).

For reasons which will be apparent in what follows, let us first proceed with one further step. As in (5.9), the \( (n-1) \)st level suspension space splits into its principal part and its tail, \( \mathcal{S}_{n-1} = \mathcal{P}_{n-1} \cup \mathcal{T}_{n-1} = \mathcal{P}_{n-1}^{(n-2)} \cup \mathcal{T}_{n-1}^{(n-2)} \). Let

\[
\mathcal{P}_n^{(n-1)} \overset{\text{def}}{=} \mathcal{P}_n \quad \text{and} \quad \mathcal{T}_n^{(n-1)} \overset{\text{def}}{=} \mathcal{T}_n
\]

emphasizing the level \( n - 1 \) that was taken into account in the definition. Analogously, let

\[
\mathcal{G}_n^{(n-1, (i))} \overset{\text{def}}{=} \mathcal{G}_n^{(i)} \quad \text{and} \quad \mathcal{L}_n^{(n-1, (i))} \overset{\text{def}}{=} \mathcal{L}_n^{(i)}.
\]

Before giving the full, inductive, definition, let us first proceed with one further step. As in (5.9), the \( (n-1) \)st level suspension space splits into its principal part and its tail, \( \mathcal{S}_{n-1} = \mathcal{P}_{n-1} \cup \mathcal{T}_{n-1} = \mathcal{P}_{n-1}^{(n-2)} \cup \mathcal{T}_{n-1}^{(n-2)} \). Let

\[
\mathcal{P}_n^{(n-2)} \overset{\text{def}}{=} P_{n,n-1}(\mathcal{P}_n^{(n-1)}) \subset \mathcal{P}_n^{(n-1)} \quad \text{and} \quad \mathcal{T}_n^{(n-2)} \overset{\text{def}}{=} P_{n,n-1}(\mathcal{T}_n^{(n-1)}) \cup \mathcal{T}_n^{(n-1)}.
\]

Hence, the \( n \)th level suspension space splits as

\[
\mathcal{S}_n = \mathcal{P}_n^{(n-2)} \cup \mathcal{T}_n^{(n-2)}.
\]

We subdivide the principal part \( \mathcal{P}_n^{(n-2)} \) into strips

\[
\mathcal{L}_n^{(n-2, (j, i))} \overset{\text{def}}{=} \left( P_{n,n-1} \big|_{\mathcal{L}_n^{(n-1, (i))}} \right)^{-1} \left( \mathcal{L}_n^{(n-2, (j))} \right)
\]

using Lemma 5.5 = \( \left( P_{n,n-1} \big|_{\mathcal{L}_n^{(n-1, (i))}} \right)^{-1} \circ \left( P_{n-1,n-2} \big|_{\mathcal{L}_n^{(n-2, (j))}} \right)^{-1} \left( \mathcal{S}_{n-2} \right) \),

where \( i = 0, \ldots, m_n-1 \) and \( j = 0, \ldots, m_{n-1}-1 \). Each such strip is separated by the corresponding intermediate floors, defined by

\[
\mathcal{G}_n^{(n-2, (j, i))} \overset{\text{def}}{=} \left( P_{n,n-1} \big|_{\mathcal{G}_n^{(n-1, (i))}} \right)^{-1} \left( \mathcal{G}_n^{(n-2, (j))} \right).
\]

The pair of indices \( \mathbf{a} = (j, i) \) above labels what we call the \( (n - 2, n) \)-address of the strip and the intermediate floor.

Below we will consider further levels of our construction. Let us describe our general terminology.
Fig. 1. Ground floor $\mathcal{G}_n$ and addresses of intermediate floors at levels $\ell = n - 3, n - 2,$ and $n - 1$ (from top to bottom), respectively. All are subsets of the suspension space $S_n$. The shaded regions indicate the principle part $\mathcal{P}_n^{(\ell)}$ (blue) and the tail part $\mathcal{T}_n^{(\ell)}$ (green) for $\ell = n - 3, n - 2,$ and $n - 1$ (from top to bottom), respectively.

Notation 5.6 (Addresses). Each intermediate floor and strip in $S_n$ will be indexed by $n$ and $(\ell, a)$, where $\ell \in \{0, \ldots, n - 1\}$ and $a = (a_\ell, \ldots, a_{n-1})$ is a tuple with $a_k \in \{0, \ldots, m_{k+1} - 1\}$ for $k = \ell, \ldots, n - 1$. Here the lower index $n$ indicates to which suspension space the defined set belongs and the upper index $(\ell, a)$ indicates which previous levels are taken into account. The length of the tuple $a$ indicates the difference of levels, it also implicitly determines $\ell$ which we kept in the notation for better readability. This notation will be also used for other objects of our construction. Compare Fig. 1.

We now give the full inductive definition. For $n \in \mathbb{N}$ and $\ell = n - 2, \ldots, 0$, assuming that the principal part $\mathcal{P}_{n-1}^{(\ell)}$ and the tail part $\mathcal{T}_{n-1}^{(\ell)}$ are defined, let

$$\mathcal{P}_n^{(\ell)} \overset{\text{def}}{=} P_{n,n-1}^{-1}(\mathcal{P}_{n-1}^{(\ell)}), \quad \mathcal{T}_n^{(\ell)} \overset{\text{def}}{=} \mathcal{T}_{n-1}^{(\ell)} \cup P_{n,n-1}^{-1}(\mathcal{T}_{n-1}^{(\ell)}).$$

Remark 5.7. Note that $\mathcal{T}_n^{(\ell)}$ gathers all tails added at levels $\ell + 1, \ldots, n$, that is, in each fiber of the suspension space there are

- one tail added at level $n$ of length $t_n(\cdot)$,
- $m_n$ tails added at level $n - 1$ of length $t_{n-1}(\cdot)$, \ldots,
- $m_n \cdots m_{\ell+1}$ tails added at level $\ell$ of length $t_\ell(\cdot)$.

We subdivide the principal part $\mathcal{P}_n^{(\ell)}$ into strips. For that, call $a = (a_\ell, \ldots, a_{n-1})$, where $a_k \in \{0, \ldots, m_k - 1\}$ for every $k = \ell, \ldots, n$, an $(\ell, n)$-address. We define the map that “lifts” the $\ell$th level suspension space to the $n$th level suspension space,

$$L_n^{(\ell,a)} : S_\ell \rightarrow S_n,$$

$$L_n^{(\ell,a)} \overset{\text{def}}{=} (P_{n,n-1}|_{L_n^{(n-1,a_{n-1})}})^{-1} \circ \cdots \circ (P_{\ell+1,\ell}|_{L_n^{(\ell,a\ell)}})^{-1}, \quad (5.12)$$
by concatenating the corresponding inverse branches. Define by

\[ L_n^{(\ell, a)} \overset{\text{def}}{=} L_n^{(\ell, a)}(S_\ell) \quad \text{and} \quad G_n^{(\ell, a)} \overset{\text{def}}{=} L_n^{(\ell, a)}(G_\ell) \quad (5.13) \]

the strip and the intermediate floor with \((\ell, n)\)-address \(a\), respectively. By construction, the following holds.

**Lemma 5.8.** Every strip \(L_n^{(\ell, a)}\) is the bijective image of \(S_\ell\) under \(L_n^{(\ell, a)}\). Every intermediate floor \(G_n^{(\ell, a)}\) is the bijective image of the ground floor \(G_\ell\) under \(L_n^{(\ell, a)}\).

Finally note that

\[ S_n = \mathcal{Y}_n^{(\ell)} \cup \mathcal{Y}_n^{(\ell)} \quad \text{and} \quad \mathcal{P}_n^{(\ell)} = \bigcup_a L_n^{(\ell, a)}, \quad (5.14) \]

where in the latter the union is taken over all \((\ell, n)\)-addresses \(a\).

**Remark 5.9.** (Factors between the principal part and lower-level suspension spaces) For every \(n \in \mathbb{N}\) and \(\ell \in \{n-1, \ldots, 0\}\) the map

\[ P_{n,\ell} \overset{\text{def}}{=} P_{\ell+1, \ell} \circ \cdots \circ P_{n,n-1} : \mathcal{P}_n^{(\ell)} \to S_\ell, \]

defines a “factor map”, though \(\mathcal{P}_n^{(\ell)}\) is not a \(\Phi_n\)-invariant set. Given a canonically represented point \((a, s)\),

\[(a, s) \overset{\text{def}}{=} P_{n,n-1}(b, t) \in S_{n-1}, \quad \text{where} \quad (b, t) \in L_n^{(n-1, (a_{n-1}))} \subset S_n,\]

then

\[ a = (\sigma_{n-1} \circ S_{n,n-1})(b) \]

and by (5.4) it follows that

\[ S_{n-1,0}(a) = (S_{n-1,0} \circ \sigma_{n-1}^{a_{n-1}})(S_{n,n-1}(b)) = ((\sigma_A^{m_1 \cdots m_{n-1}})^{a_{n-1}} \circ S_{n-1,0})(S_{n,n-1}(b)) \]

where \(\sigma_A = \sigma_0\) is the shift in the original alphabet \(A\). In other words, using the notation above,

\[ S_{n-1,0} \circ p_{n-1} \circ P_{n,n-1} \big|_{L_n^{(n-1, (a_{n-1}))}} = \sigma_A^{a_{n-1} - m_1 \cdots m_{n-1}} \circ S_{n,0} \circ p_n. \]

Analogously, for \(\ell \in \{0, \ldots, n-1\}\) and \((\ell, n)\)-address \(a\) it holds

\[ S_{\ell,0} \circ p_\ell \circ P_{n,\ell} \big|_{L_n^{(\ell,a)}} = \sigma_A^{a_{\ell} - m_1 \cdots m_{n-1}} \circ S_{n,0} \circ p_n. \quad (5.15) \]
5.2.5. Localization of intermediate floors in the suspension space

Given \( n \in \mathbb{N} \), for every \((a, 0) \in S_n\), \( \ell \in \{0, \ldots, n - 1\} \), and \((\ell, n)\)-address \( a = (a_\ell, \ldots, a_{n-1}) \) there is a unique \( s_{n, a}(a) \in \mathbb{N}_0 \) such that \((a, s_{n, a}(a)) \in S_n^{(\ell, a)}\) (in its canonical form). Lemma 5.10, that we postpone to the end of the section, precisely describes this number. To state and prove this lemma, we start by introducing the necessary notation that will be used thereafter.

Given numbers \( n \in \mathbb{N}, \ell \in \{1, \ldots, n-1\} \), and some \((\ell, n)\)-address \( a = (a_\ell, \ldots, a_{n-1}) \), for \( j \in \{0, \ldots, m_\ell - 1\} \) denote by \( j a \overset{\text{def}}{=} (j, a_\ell, \ldots, a_{n-1}) \) the corresponding \((\ell - 1, n)\)-address. Given \( \ell \in \{0, \ldots, n-1\} \), define the collection of all \((\ell, n)\)-addresses

\[
W_{n, a}^{(\ell)} \overset{\text{def}}{=} \{ a_k \in \{0, \ldots, m_{k+1} - 1\}, k = \ell, \ldots, n-1 \}
\]

and let

\[
W_n \overset{\text{def}}{=} \bigcup_{\ell=0}^{n-1} W_{n, a}^{(\ell)}.
\]

Consider on \( W_n \) the equivalence relation obtained by identifying \( a \in W_{n, a}^{(\ell)} \) with \( 0a \in W_{n, a}^{(\ell-1)} \) and denote by \( \tilde{W}_n \) the corresponding collection of equivalence classes. For simplicity, by a slight abuse of notation, we denote by \( a \) the equivalence class it represents.

An element in \( \tilde{W}_n \) can have several representations. Given \( a = (a_\ell, \ldots, a_{n-1}) \in \tilde{W}_n \), its simplified representation

\[ w(a) \overset{\text{def}}{=} k. \]

The simplified representation of a tuple \( \overline{0} \) consisting only of 0s is 0, and in this case we let \( w(\overline{0}) = 0 \). Given \( a = (a_\ell, \ldots, a_{n-1}) \in \tilde{W}_n \), let

\[
||a|| \overset{\text{def}}{=} \sum_{k=w(a)}^{n-1} a_k.
\]

Note that this value does not depend on the representation of \( a \in \tilde{W}_n \).

Given \( a = (a_\ell, \ldots, a_{n-1}) \in \tilde{W}_n \), to “move between intermediate floors”, assuming \( a_\ell < m_\ell - 2 \), let us introduce the notation

\[
a + 1_\ell \overset{\text{def}}{=} (a_\ell, \ldots, a_{n-1}) + 1_\ell \overset{\text{def}}{=} (a_\ell + 1, \ldots, a_{n-1}).
\]

Analogously, if \( \ell \in \{1, \ldots, n - 1\} \), let

\[
a + 1_{\ell - 1} \overset{\text{def}}{=} (a_\ell, \ldots, a_{n-1}) + 1_{\ell - 1} \overset{\text{def}}{=} (1, a_\ell, \ldots, a_{n-1}).
\]

\[10\] We use the term simplified to avoid confusion with the term canonical defined above.
For every $\mathbf{a} \in \tilde{W}_n$ there exists a unique sequence $\mathbf{a}^{(0)} = 0, \mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(\|\mathbf{a}\|)} = \mathbf{a}$ of elements of $\tilde{W}_n$ such that

$$\mathbf{a}^{(k+1)} = \mathbf{a}^{(k)} + 1_{\mathbf{w}(\mathbf{a}^{(k+1)})}.$$  

Indeed, if $\mathbf{a} = (a_\ell, \ldots, a_{n-1})$ is in its simplified representation, then:

$$
\begin{align*}
\mathbf{a}^{(0)} &= (0) \\
\mathbf{a}^{(1)} &= (1), \mathbf{a}^{(2)} = (2), \ldots, \mathbf{a}^{(a_{n-1})} = (a_{n-1}), \\
\mathbf{a}^{(a_{n-1}+1)} &= (1, a_{n-1}), \mathbf{a}^{(a_{n-1}+2)} = (2, a_{n-1}), \ldots, \\
\mathbf{a}^{(a_{n-1}+a_{n-2})} &= (a_{n-2}, a_{n-1}), \\
\ldots, \\
\mathbf{a}^{(\sum_{k=\ell+1}^{a_{n-1}} a_k+1)} &= (1, a_{\ell+1}, \ldots, a_{n-1}), \ldots, \mathbf{a}^{(\|\mathbf{a}\|)} = (a_\ell, \ldots, a_{n-1}).
\end{align*}
$$

(5.16)

Given $\mathbf{a} \in (A_n)^Z$, recall that by (5.3) the sequence $b = S_{n, \ell}(\mathbf{a}) \in (A_\ell)^Z$ is obtained by “reading $\mathbf{a}$ in its spelling in the alphabet $A_\ell$.” Recall also that if $b = (\ldots, b_{-1}|b_0, b_1, \ldots)$ then $b_k$ denotes the $k$th element in this bi-infinite sequence (in the alphabet $(A_\ell)^Z$). The above discussion proves the lemma that we finally state. Compare also Fig. 2.

**Lemma 5.10.** For every $n \in \mathbb{N}$, $\ell \in \{0, \ldots, n-1\}$, and $(\ell, n)$-address $\mathbf{a} = (a_\ell, \ldots, a_{n-1})$ the following is true. Let

$$k = k(n, \ell, \mathbf{a}) = \sum_{i=\ell}^{n-1} a_i \cdot m_{\ell+1} \cdots m_{i+1}.$$  

and consider the map

$$\varsigma_n^{(\ell, \mathbf{a})} : (A_n)^Z \to A_\ell, \quad \varsigma_n^{(\ell, \mathbf{a})}(\mathbf{a}) \overset{\text{def}}{=} (S_{n, \ell}(\mathbf{a}))_k = \left( (\sigma^k_\ell \circ S_{n, \ell})(\mathbf{a}) \right)_0.$$  

(5.17)
Then the number

\[ s = s_n^{(\ell, a)}(a) \overset{\text{def}}{=} \sum_{i=1}^{\|a\|} R_{\ell_i}(s_n^{(\ell, a^{(i-1)})}(a)), \quad \text{where} \quad \ell_i = w(a^{(i)}) \]

is the unique number such that \((a, s) \in S_n^{(\ell, a)}\) (in its canonical form).

Note that, by definition, it holds \(b_n \circ P_n^{(\ell, a)} = b_n \circ P_n^{(\ell-1, a)}\). Hence

\[ b_n^{(\ell, a)} = b_n^{(\ell-1, 0a)}. \]

**Lemma 5.11.** The measure \(b_n^{(\ell, a)}\) is a Borel probability measure on \(S_n^{(\ell, a)}\) satisfying

\[ b_n^{(\ell, a)} = b_\ell \circ p_\ell \circ P_n^{(\ell, a)}. \]

**Proof.** Note that, by definition, it holds

\[ p_n(S_n^{(\ell, a)}) = (A_n)^Z \]

and that \(p_n\) bijectively maps \(S_n^{(\ell, a)}\) onto \((A_n)^Z\). Recall that \(\Sigma_n, 0\) is a bijection and that, by definition, it holds

\[ b_n = b_0 \circ \Sigma_n, 0. \]  \hspace{1cm} (5.19)

As \((\Sigma_n, 0 \circ p_n^{(\ell, a)})\) is bijective, to see that both measures coincide, it suffices to check

\[ (\Sigma_n, 0 \circ p_n^{(\ell, a)})_* b_n^{(\ell, a)} = (\Sigma_n, 0 \circ p_n^{(\ell, a)})_* (b_\ell \circ p_\ell \circ P_n^{(\ell, a)}). \]

On one hand (5.19) immediately implies

\[ (\Sigma_n, 0 \circ p_n^{(\ell, a)})_* b_n^{(\ell, a)} = (\Sigma_n, 0 \circ p_n^{(\ell, a)})_* (b_n \circ p_n^{(\ell, a)}) = b_0. \]

On the other hand, by definition and using (5.15), it holds

\[
\begin{align*}
(\Sigma_n, 0 \circ p_n^{(\ell, a)})_* (b_\ell \circ p_\ell \circ P_n^{(\ell, a)}) &= (\Sigma_n, 0 \circ p_n^{(\ell, a)})_* (b_\ell \circ p_\ell \circ P_n^{(\ell, a)}), \\
&= (\Sigma_n, 0 \circ p_n^{(\ell, a)})_* (b_\ell \circ p_\ell \circ P_n^{(\ell, a)}), \\
&= (\sigma_A^{-\sum_{i=1}^{n-1} a_i} \cdot m_{1 \cdots m_j} \circ \Sigma_{\ell, 0} \circ p_\ell \circ P_n^{(\ell, a)})_* b_\ell = (\sigma_A^{-\sum_{i=1}^{n-1} a_i} \cdot m_{1 \cdots m_j})_* b_0 = b_0,
\end{align*}
\]

where the latter follows from the fact that \(b_0\) is Bernoulli and so \(\sigma_A\)-invariant. \(\square\)
5.4. Roof functions: upper bounds and estimates. Recall the constant $K$ in Assumption 5.2. Let

$$L_2 \overset{\text{def}}{=} 2 \max \left\{ K e^K \frac{\max R_0}{\min R_0}, e^K \frac{\min R_0}{\max R_0} \right\}.$$  

**Proposition 5.12** (Estimates on roof functions). Let $\mathcal{R}_n \overset{\text{def}}{=} \int R_n \, db_n$. Under Assumption 5.2 it holds

1. $m_n \max R_n - 1 < \max R_n$,
2. $\frac{\max R_n}{\mathcal{R}_n} \leq \frac{\max R_n}{\min R_n} \leq \prod_{k=\ell}^n \left( 1 + \frac{1}{2^{k-1} K} \right) \frac{\max R_{\ell-1}}{\min R_{\ell-1}} < L_2$,
3. $1 < \frac{\mathcal{R}_n}{m_n \mathcal{R}_{n-1}} < 1 + L_2 \frac{1}{2^n}$,
4. $\max |t_n| \leq L_2 \frac{1}{2^n} \mathcal{R}_n$, where $t_n$ is as in (5.6).

**Proof.** By Assumption 5.2, it holds

$$m_n \max R_{n-1} - 1 < \max R_n.$$

Observe that this implies property (1). Integrating the above, using that $b_{n-1}$ is $\sigma_{n-1}$-invariant and (5.2), we obtain the following estimate which we use below

$$m_n \mathcal{R}_{n-1} < \int R_n \circ S_{n,n-1}^{-1} \, db_{n-1} = \int R_n \, db_n = \mathcal{R}_n. \quad (5.20)$$

To prove (2), observe that the second inequality in Assumption 5.2 implies that

$$R_n \circ S_{n,n-1}^{-1} \leq \left( 1 + K \frac{1}{2^{n-1}} \right) \sum_{j=0}^{m_{n-1}} R_{n-1} \circ \sigma_{n-1}^j.$$

Hence, together with $\min R_n > m_n \min R_{n-1}$, it follows

$$\frac{\max R_n}{\min R_n} \leq \left( 1 + K \frac{1}{2^{n-1}} \right) \frac{\max R_{n-1}}{\min R_{n-1}} \leq \prod_{k=\ell}^n \left( 1 + K \frac{1}{2^{k-1}} \right) \frac{\max R_{\ell-1}}{\min R_{\ell-1}} < e^K \frac{\max R_0}{\min R_0}, \quad (5.22)$$

proving (2).

Recalling again that $b_{n-1}$ is $\sigma_{n-1}$-invariant, we note that by (5.21)

$$\mathcal{R}_n = \int R_n \, db_n = \int R_n \circ S_{n,n-1}^{-1} \, db_{n-1}$$

$$\leq \left( 1 + K \frac{1}{2^{n-1}} \right) \int \sum_{j=0}^{m_{n-1}} R_{n-1} \circ \sigma_{n-1}^j \, db_{n-1} = \left( 1 + L_2 \frac{1}{2^n} \right) m_n \mathcal{R}_{n-1},$$

where we also used $2K \leq L_2$. This together with (5.20) implies (3).
To get (4), by the second inequality in Assumption 5.2 and item (2) it holds
\[ |t_n| \leq K \frac{1}{2^{n-1}} \cdot m_n \max R_{n-1} < K \frac{1}{2^{n-1}} \cdot \max R_n \leq K \frac{1}{2^{n-1}} \cdot \min R_n \cdot R_n. \]
The estimate in (4) now follows from (5.22).

6. Contracting IFSs and Horseshoes

In what follows, we consider \( C^1 \) diffeomorphisms \( f_1, \ldots, f_N : \mathbb{S}^1 \to \mathbb{S}^1 \) and its associated skew product \( F \) as in (1.1). In Sect. 6.1, given an appropriate finite collection of words \( \mathcal{W} \subset \Sigma_N^* \), following [21], we consider the attractor of an associated contracting IFS. In Sect. 6.2, we introduce a contracting IFS with further quantifiers. In Sect. 6.3, we explain how such collection is derived from an \( F \)-ergodic hyperbolic measure with negative Lyapunov exponent. Here we invoke the idea of skeletons associated to an ergodic measure, relying on the axioms stated in Sect. 2.1. The main result of this section is Theorem 6.5. In Sect. 6.4 we derive an auxiliary distortion result. Finally, in Sect. 6.5 we explain how these attractors lead to horseshoes invariant under the skew product \( F \).

In this section, all words are over the alphabet \( \{ 1, \ldots, N \} \). We will invoke the concepts and objects in Sects. 3 and 4 in this particular case. Given a finite collection of words \( \mathcal{W} \subset \Sigma_N^* \), let
\[
\text{CS}(\mathcal{W}) = \bigcup_{k \in \mathbb{Z}} \sigma^k(\text{PCS}(\mathcal{W})) = \bigcup_k (\sigma^k \circ \iota_{\mathcal{W}})(\mathcal{W}\Sigma)
\]
as in (3.4), where \( \sigma \) is the usual shift in \( \Sigma_N \) and \( \iota_{\mathcal{W}} : \mathcal{W}\Sigma \to \Sigma_N \) as in (3.2). We will drop the corresponding index \( \mathcal{W} \) unless there is risk of confusion. Recall our simplifying Notation 3.7. In particular, for \( w \in \mathcal{W} \) we denote by \(|w|\) the length of this word “spelled in \( \{ 1, \ldots, N \} \)”.

**Notation 6.1.** Denote by \( \mathcal{W}^{-N} \) and \( \mathcal{W}^{N_0} \) the respective one-sided shift spaces, similarly \( \Sigma_N^- = \{ 1, \ldots, N \}^{-N} \) and \( \Sigma_N^+ = \{ 1, \ldots, N \}^{N_0} \). Given a sequence \( \xi = (\ldots, \xi_{-1}|\xi_0, \xi_1, \ldots) \in \Sigma_N \) we write \( \xi = \xi^-|\xi^+ \), where \( \xi^+ \in \Sigma_N^+ \) and \( \xi^- \in \Sigma_N^- \). Denote by \( \pi^\pm : \Sigma_N \to \Sigma_N^\pm \) the projections
\[
\pi^- (\xi^-|\xi^+) \overset{\text{def}}{=} \xi^-, \quad \pi^+ (\xi^-|\xi^+) \overset{\text{def}}{=} \xi^+.
\]
For \( n \in \mathbb{N}_0 \), let \( [\xi_0, \ldots, \xi_n]^+ = \pi^+([\xi_0, \ldots, \xi_n]) \). We consider the distance
\[
d^+_1 (\xi^+, \eta^+) \overset{\text{def}}{=} e^{-n(\xi^+, \eta^+)}, \quad \text{where} \quad n(\xi^+, \eta^+) \overset{\text{def}}{=} \inf \{ \ell : \xi^+_{\ell} \neq \eta^+_{\ell} \},
\]
on \( \Sigma_N^+ \). Analogously, we define \( d^-_1 \) on \( \Sigma_N^- \).

6.1. The attractor of a contracting IFS. Recall our notation \( f_{[w]} \) for \( w \in \mathcal{W} \subset \Sigma_N^* \) in (3.7).

**Definition 6.2 [CIFS].** A finite collection of words \( \mathcal{W} \subset \Sigma_N^* \) defines a **contracting iterated function system** (CIFS) on a closed interval \( J \subset \mathbb{S}^1 \) if for every \( w \in \mathcal{W} \) it holds
(a) \( f_{[w]}(J) \subset J \),
Proposition 6.3 (Attractor of a CIFS). Let $\mathcal{W} \subset \Sigma_N^+$ be a finite collection of words over the alphabet $\{1, \ldots, N\}$ defining a CIFS on a closed interval $J \subset \mathbb{S}^1$. The map

$$x : \mathcal{W}^\mathbb{Z} \to \mathbb{S}^1, \quad x(w) \overset{\text{def}}{=} \lim_{n \to \infty} \left( f_{|w_{-1}|} \circ \cdots \circ f_{|w_{-n}|} \right)(x_0),$$

is well defined for every $w = (\ldots, w_{-1}|w_0, w_1, \ldots) \in \mathcal{W}^\mathbb{Z}$ and independent of the point $x_0 \in J$. Moreover, the map

$$\Pi_{\mathcal{W}} : \mathcal{W}^\mathbb{Z} \to \Sigma_N \times \mathbb{S}^1, \quad \Pi_{\mathcal{W}}(w) \overset{\text{def}}{=} (\iota_{\mathcal{W}}(w), x(w))$$

is continuous and satisfies

$$\left( \Pi_{\mathcal{W}} \circ \sigma_{\mathcal{W}} \right)(\ldots, w_{-1}|w_0, w_1, \ldots) = \left( F_{|w_0|} \circ \Pi_{\mathcal{W}} \right)(\ldots, w_{-1}|w_0, w_1, \ldots).$$

Consider the attractor associated to $\mathcal{W},$

$$\Lambda(\mathcal{W}) \overset{\text{def}}{=} \Pi_{\mathcal{W}}(\mathcal{W}^\mathbb{Z}).$$

The map

$$(w, x) \mapsto F_{|w_0|}(\iota_{\mathcal{W}}(w), x)$$

is a return map on $\Lambda(\mathcal{W}).$

Moreover, if $\mathcal{W}$ is disjoint then $\Pi_{\mathcal{W}}$ is uniformly finite-to-one so that

$$\card \Pi_{\mathcal{W}}^{-1}([X]) \leq \max_{w \in \mathcal{W}} |w|, \quad \text{for every} \quad X \in \Lambda(\mathcal{W}).$$

Proof. Let $\mathcal{W} = \{w_1, \ldots, w_M\}$ be some enumeration. Recall that the (bi-)infinite concatenation of words in $\mathcal{W}$ gives a (bi-)infinite sequence in $\Sigma_N$. We will first consider one-sided sequences in $\mathcal{W}^- \subset \Sigma_N^-$. For every $k = 1, \ldots, M$ define

$$\hat{\sigma}_k^{-1} : \mathcal{W}^- \to \mathcal{W}^-, \quad \hat{\sigma}_k^{-1}(\ldots, w_{i-2}, w_i, w_{i-1}) \overset{\text{def}}{=} (\ldots, w_{i-2}, w_{i-1}, w_k)$$

and consider the map

$$\hat{f}_k \overset{\text{def}}{=} \hat{\sigma}_k^{-1} \times f_{|w_k|} : D \to \mathcal{W}^- \times \mathbb{S}^1, \quad \text{where} \quad D \overset{\text{def}}{=} \mathcal{W}^- \times J.$$

By construction, $\hat{f}_k(D) \subset D$ and $\hat{f}_k$ is uniformly contracting for every $k$. Therefore, $\{\hat{f}_1, \ldots, \hat{f}_M\}$ is a finite family of contractions on $D$. By [21], we can consider its associated attractor $\text{Att}^-(\mathcal{W}) \subset D$. Every point $(\underline{w}, x) \in \text{Att}^-(\mathcal{W})$, with $\underline{w} = (\ldots, w_{i-2}, w_{i-1}) \in \mathcal{W}^-$, is uniquely defined by its first coordinate. Indeed, the map

$$\underline{w} \in \mathcal{W}^- \mapsto (\underline{w}, \hat{x}(\underline{w})), \quad \hat{x}(\underline{w}) \overset{\text{def}}{=} \lim_{n \to \infty} \left( f_{|w_{-1}|} \circ \cdots \circ f_{|w_{-n}|} \right)(x_0), \quad x_0 \in J,$$

is continuous and onto $\text{Att}^-(\mathcal{W})$ (and in particular, it does not depend on $x_0$). Let

$$\text{Att}(\mathcal{W}) \overset{\text{def}}{=} \text{Att}^-(\mathcal{W}) \times \mathcal{W}_N^0.$$
For convenience, write a point \((w^-, \hat{x}(w^-), w^+) \in \text{Att}(\mathcal{W})\) as \((w^-, \hat{x}(w^-), w^+) = (w^-|w^+, \hat{x}(w^-)) = (w, \hat{x}(w^-))\). Letting \(x(w) \equiv \hat{x}(w^-)\), this ends the definition of \(\Pi_{\mathcal{W}}\).

By construction, for every \(w = (\ldots, w_{i-1}|w_{i_0}, w_{i_1}, \ldots)\) it holds
\[
(\Pi_{\mathcal{W}} \circ \sigma_{\mathcal{W}})(w) = ((\pi_{\mathcal{W}} \circ \sigma_{\mathcal{W}})(w), x(\sigma_{\mathcal{W}}(w)))
\]
\[
= ((\pi_{\mathcal{W}} \circ \sigma_{\mathcal{W}})(w), f_{|w_{i_0}}(x(w))) = F^{w_{i_0}}(\pi_{\mathcal{W}}(w), x(w))
\]
\[
= (F^{w_{i_0}} \circ \Pi_{\mathcal{W}})(w).
\]

Finally, we check the cardinality of the set of preimages \(\Pi_{\mathcal{W}}^{-1}(X)\) for any point \(X = (\tau_{\mathcal{W}}(w), x(w))\). Note that if \(\mathcal{W}\) disjoint then together with Lemma 3.4 every element in \(\text{PCS}(\mathcal{W}) = \tau_{\mathcal{W}}(\mathcal{W}^2)\) has at most \(\max_{w \in \mathcal{W}}|w|\) decodings in \(\mathcal{W}\). \(\square\)

6.2. Contracting IFS with quantifiers. The following extends Definition 6.2, adding some contraction quantifiers.

**Definition 6.4 (CIFS with quantifiers).** A finite collection of words \(\mathcal{W} \subset \Sigma_N^*\) defines a **contracting iterated function system (CIFS)** on an interval \(J \subset \mathbb{S}^1\) relative to \(K \geq 1\), \(\alpha_0 < 0\), \(\alpha < 0\), and \(\varepsilon \in (0, |\alpha|)\) if
(a) for every \(w \in \mathcal{W}\) it holds \(f_{|w|}(J) \subset J\),
(b) for every \(m \in \mathbb{N}, w_1, \ldots, w_m \in \mathcal{W}, y \in J\), and \(k = 1, \ldots, |(w_1, \ldots, w_m)|\) it holds
\[
|(f_{|w_1,\ldots,w_m|}^k)'(y)| \leq K e^{k\alpha_0},
\]
(c) the spectrum of finite-time Lyapunov exponents satisfies
\[
\left\{ \frac{1}{|w|} \log |(f_{|w|})'(x)| : x \in J, w \in \mathcal{W} \right\} \subset (\alpha - \varepsilon, \alpha + \varepsilon).
\]

6.3. Existence of contracting IFS with quantifiers. Given any \(F\)-ergodic measure with negative Lyapunov exponent, the following theorem provides a collection of words which defines a CIFS with quantifiers. It builds on the existence of “skeletons”, that is, orbit pieces which “ergodically mimic” the measure, see Claim 6.6 and [15, Section 4] for further discussion.

**Theorem 6.5 (Existence of a CIFS with quantifiers).** Let \(F \in \text{SP}^1_{\text{shyp}}(\Sigma_N \times \mathbb{S}^1)\) and \(\mu\) an \(F\)-ergodic hyperbolic measure with Lyapunov exponent \(\alpha = \chi(\mu) < 0\) and entropy \(h = h(F, \mu) > 0\). Then for every \(\varepsilon_E \in (0, |\alpha|/4)\) and \(\varepsilon_H \in (0, h)\) there exist a closed interval \(J \subset \mathbb{S}^1\) and a finite disjoint collection of words \(\mathcal{W} \subset \Sigma_N^*\) defining a CIFS on \(J\) relative to some constant \(K > 1\) and \(\alpha + \varepsilon_E, \alpha, \text{and } \varepsilon_E\) such that
\[
\min_{w \in \mathcal{W}} |w|(h - \varepsilon_H) \leq \log \text{card } \mathcal{W} \leq \max_{w \in \mathcal{W}} |w|(h + \varepsilon_H).
\]

**Proof.** Let \(F \in \text{SP}^1_{\text{shyp}}(\Sigma_N \times \mathbb{S}^1)\) with associated constants \(K_1, \ldots, K_5, K_6\) as in Remark 2.2. Fix \(\varepsilon_E \in (0, |\alpha|/4)\) and \(\varepsilon_H \in (0, h)\). We use the following result. \(\square\)
Claim 6.6 (Existence of skeletons, [15, Proposition 4.11]). There exist \( K_0, L_0 \geq 1 \), and \( n_0 \in \mathbb{N} \) such that for every \( n \geq n_0 \) there exists a finite set \( \mathcal{X} = \mathcal{X}(n) = \{(\xi^i, x_i)\} \subset \Sigma_N \times \mathbb{S}^1 \), where \( \xi^i = (\ldots, \xi^i_{-1}, \xi^i_{1}, \ldots) \), satisfying:

(i) the set \( \mathcal{X} \) has cardinality
\[
L_0^{-1} e^{n(h - \varepsilon_H/2)} \leq \text{card} \mathcal{X} \leq L_0 e^{n(h+\varepsilon_H/2)},
\]
(ii) the words \((\xi^i_0, \ldots, \xi^i_{n-1})\) are all different, and
(iii) for every \( k = 1, \ldots, n \) it holds
\[
K_0^{-1} e^{k(\alpha - \varepsilon_E/4)} \leq |(f^k_{\xi^i})'(x_i)| = |(f^{[i]}_{x^i_0, \ldots, x^i_{i-1}})'(x_i)| \leq K_0 e^{k(\alpha+\varepsilon_E/4)}.
\]

Control of distortion. Let \( K_0, L_0 \), and \( n_0 \) be as in Claim 6.6. We need some auxiliary distortion results. Let
\[
\|F\| \overset{\text{def}}{=} \max \left\{ |f^i_1(x)|, |(f^i_1)'(x)| : i = 1, \ldots, N, x \in \mathbb{S}^1 \right\}.
\]

Let
\[
\text{Mod}_F(\varepsilon) \overset{\text{def}}{=} \left\{ \left| \log |f^i_j(y)| - \log |f^i_j(x)| \right| : i = 1, \ldots, N, x, y \in \mathbb{S}^1, |y-x| \leq \varepsilon \right\}.
\]

Clearly, \( \text{Mod}_F(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \). Let \( r \in (0, 1) \) so that
\[
\text{Mod}_F(K_0r) \leq \varepsilon_E/4 \quad \text{and} \quad 2r < K_1.
\]

Lemma 6.7. Let \((\xi, x) \in \Sigma_N \times \mathbb{S}^1 \) and \( n \in \mathbb{N} \) such that for every \( k = 0, \ldots, n \) it holds
\[
K_0^{-1} e^{k(\alpha - \varepsilon_E/4)} \leq |(f^k_\xi)'(x)| \leq K_0 e^{k(\alpha+\varepsilon_E/4)}.
\]

Then, with \( r \) satisfying (6.3), for every \( y \in B(x, r) \) and \( k = 0, \ldots, n \) it holds
\[
K_0^{-1} e^{k(\alpha - \varepsilon_E/2)} \leq |(f^k_\xi)'(y)| \leq K_0 e^{k(\alpha+\varepsilon_E/2)}.
\]

Proof. The claim is true for \( k = 0 \). By induction, suppose that the claim is true for \( k \). Then \( y \in B(x, r) \) satisfies
\[
|f^k_\xi(y) - f^k_\xi(x)| \leq K_0 e^{k(\alpha+\varepsilon_E/2)r} < K_0r.
\]

The above choice of \( r \) implies
\[
|\log |f^i_{\xi^i}(f^k_\xi(y))|\) - \log |f^i_{\xi^i}(f^k_\xi(x))|| < \frac{\varepsilon_E}{4},
\]
and hence the claim for \( k + 1 \). \( \square \)

Fixing a covering by blending intervals. Recalling Remark 2.2, fix
\[
\delta \in \left(0, \min\left\{ \frac{r}{4}, K_6/2 \right\} \right)
\]
and take a cover of \( \mathbb{S}^1 \) by finitely many intervals \( J_j = [y_j - 2\delta, y_j + 2\delta] \). Let
\[
m_c \overset{\text{def}}{=} \max_j m_c(I_j), \quad \text{where} \quad I_j \overset{\text{def}}{=} [y_j - \delta, y_j + \delta]
\]
and $m_c(I_j)$ is as in Claim 2.3, and let

$$L_1 \overset{\text{def}}{=} \max_j L_1(F, J_j),$$

where $L_1(F, J_j)$ is as in Definition 2.4.

**Choice of further constants.** Let

$$K \overset{\text{def}}{=} K_0 \|F\|^{m_c e^{-m_c(\alpha + \varepsilon_E/2)}}.$$

Consider now $n_1 \in \mathbb{N}$ large enough such that

$$2r K_0 \|F\|^{m_c e^{n_1(\alpha + \varepsilon_E/2)}} < \delta,$$

$$\frac{1}{n_1} \log K < \frac{\varepsilon_E}{4}, \quad \frac{1}{n_1} \log L_0 < \frac{\varepsilon_H}{2}.$$ (6.5)

**Choice of the IFS.** Fix any integer $n \geq \max\{n_0, n_1\}$ let $X = X(n) = \{(\xi^i, x_i)\}_i$ be the set provided by Claim 6.6 so that for every $i$ and $k = 1, \ldots, n$,

$$K_0^{-1} e^{k(\alpha - \varepsilon_E/4)} \leq |(f_{\xi^i}^k)'(x_i)| \leq K_0 e^{k(\alpha + \varepsilon_E/4)}.$$ (6.6)

By Lemma 6.7, for every $i, y \in B(x_i, r)$, and $k = 1, \ldots, n$ it holds

$$K_0^{-1} e^{k(\alpha - \varepsilon_E/2)} \leq |(f_{\xi^i}^k)'(y)| \leq K_0 e^{k(\alpha + \varepsilon_E/2)}.$$ (6.7)

Hence, in particular,

$$|f_{\xi^i}^n(B(x_i, r))| \leq 2r K_0 e^{n(\alpha + \varepsilon_E/2)}.$$ (6.8)

**Choice of a common blending interval.** Choose now an index $j$ for which $N_j \overset{\text{def}}{=} \text{card}(J_j \cap \{x_i\}_i)$ is maximal and let $J = J_j$, $I = I_j$, and $N = N_j$. Observe that by Claim 6.6 (i) and the choice of $N$

$$L_0 e^{n(h + \varepsilon_H/2)} \geq \text{card} X \geq N \geq \frac{1}{2\delta} \cdot \text{card} X \geq \frac{1}{2\delta} \cdot \frac{1}{L_0} e^{n(h - \varepsilon_H/2)}.$$ (6.9)

We can, renumbering this set of points, assume that $x_1, \ldots, x_N \in J$. By Claim 2.3, there are words $(\beta^1_i, \ldots, \beta^s_i)$, $s_i \leq m_c$, such that $f_{\xi^1_i, \ldots, \xi^n_i, \beta^1_i, \ldots, \beta^s_i}(x_i) \in I$. Let now

$$\mathcal{W} \overset{\text{def}}{=} \{w_i\}_{i=1}^N, \quad \text{where} \quad w_i \overset{\text{def}}{=} (\xi^1_i, \ldots, \xi^n_i, \beta^1_i, \ldots, \beta^s_i).$$

**Lemma 6.8.** The collection of words $\mathcal{W}$ is disjoint.

**Proof.** This is an immediate consequence of Claim 6.6 (ii). \quad \Box

**Checking properties of a CIFS with quantifiers.**

**Lemma 6.9.** The collection of words $\mathcal{W}$ satisfies properties (a), (b), and (c) of a CIFS on $J$ relative to $K$, $\alpha + \varepsilon_E$, $\alpha$, and $\varepsilon_E$.

The following two claims prove the above lemma.

**Claim 6.10.** Property (a) holds.
Proof. Observe that (6.8), (6.5), and \( n \geq n_1 \) together imply
\[
|f_{[\xi_1^i, \ldots, \xi_n^i, \beta_1^i, \ldots, \beta_n^i]}(B(x_i, r))| \leq \|F\|^m_c \cdot 2r K_0 e^{n(\alpha + \varepsilon_E/2)} < \delta.
\]
Recall again that \( f_{[\xi_1^i, \ldots, \xi_n^i, \beta_1^i, \ldots, \beta_n^i]}(x_i) \in I \) and \( I \subset J \) is a concentric subinterval of length \( 2\delta \), and \( J \) contains a \( \delta \)-neighborhood of \( I \). Hence, together with (6.4), it follows
\[
f_{[w_1]}(J) = f_{[\xi_1^i, \ldots, \xi_n^i, \beta_1^i, \ldots, \beta_n^i]}(J) \subset f_{[\xi_1^i, \ldots, \xi_n^i, \beta_1^i, \ldots, \beta_n^i]}(B(x_i, r)) \subset J,
\]
which gives property (a). \( \Box \)

Notice that for every \( i \) it holds
\[
|w_i| = n + s_i > n_1.
\]
Using (6.7) together with the estimates \( s_i \leq m_c \) and (6.5), for every \( y \in J \) and \( k = 1, \ldots, |w_i| \) it holds
\[
|(f_{[w_i]}^k)'(y)| \leq K_0\|F\|^m_c e^{-m_c(\alpha + \varepsilon_E/2)} \cdot e^{k(\alpha + \varepsilon_E/2)} \leq K e^{k(\alpha + \varepsilon_E/2)},
\]
which is a first step towards proving (b) and also (c).

Claim 6.11. Properties (b) and (c) hold.

Proof. We first prove a slightly stronger version of property (c). Observe that (6.11) and (6.5) together with (6.10) imply
\[
\frac{1}{|w_i|} \log |(f_{[w_i]}^k)'(y)| \leq \frac{1}{n + s_i} \log K + \frac{1}{2} \varepsilon_E < \alpha + \frac{3}{4} \varepsilon_E,
\]
which together with the analogous lower bound.

By (6.11) property (b) holds for \( m = 1 \). For \( m \geq 1 \), let \( w_1, \ldots, w_m, w_{m+1} \in \mathcal{W} \). For every \( k \in \{|(w_1, \ldots, w_m)| + 1, \ldots, |(w_1, \ldots, w_{m+1})|\} \), by (6.12) and (6.11) it follows
\[
|(f_{[w_1, \ldots, w_m, w_{m+1}]}^k)'(y)| \leq e^{\varepsilon_E k^{\varepsilon_E/4}} \ldots e^{\varepsilon_E k^{\varepsilon_E/4}} K e^{k(\alpha + \varepsilon_E/2)} < K e^{k(\alpha + \varepsilon_E)}.
\]
This proves property (b). \( \Box \)

Cardinality of \( \mathcal{W} \). Recall that \( \max_{w \in \mathcal{W}} |w| > n \). By (6.9) and using (6.5) we get
\[
\log \text{card } \mathcal{W} = \log N \leq \log L_0 + n(h + \frac{\varepsilon_H}{2})
\leq \max_{w \in \mathcal{W}} |w|(h + \frac{\varepsilon_H}{2} + \frac{1}{n} \log L_0) < \max_{w \in \mathcal{W}} |w|(h + \varepsilon_H),
\]
the lower estimate is analogous, adapting the choice of \( n \), proving the theorem. \( \Box \)
6.4. Distortion. In what follows, given a function \( \phi: \Sigma_N \times \mathbb{S}_1 \to \mathbb{R} \), for each \( n \in \mathbb{N} \) we denote by

\[
S_n \phi \overset{\text{def}}{=} \phi + \phi(F) + \cdots + \phi \circ F^{n-1}.
\]

the corresponding Birkhoff sum of \( \phi \) (relative to \( F \)). Denote \( \| \phi \| = \sup |\phi| \).

**Proposition 6.12.** Let \( \mathcal{W} \subset \Sigma_N^* \) be a finite collection of words defining a CIFS on a compact interval \( J \subset \mathbb{S}_1 \) relative to \( K, \alpha_0, \alpha, \) and \( \varepsilon \). Then for every \( \phi: \Sigma_N \times \mathbb{S}_1 \to \mathbb{R} \) continuous and \( \tau > 0 \) there exists \( N_1 = N_1(\phi, \tau) \in \mathbb{N} \) such that for every \( m \geq N_1 \) and finite sequence of concatenated words \( (w_1, \ldots, w_m) \in \mathcal{W}^m \) it holds

\[
\max_{x, y \in \Sigma_N^* \times \mathbb{S}_1} |S_n \phi(x) - S_n \phi(y)| < \tau n, \quad \text{where } n = \sum_{j=1}^{m} |w_j|.
\]

**Proof.** The function \( \phi \) is uniformly continuous and hence there is \( \delta > 0 \) so that at any pair of points in distance at most \( \delta \) the values of \( \phi \) differ at most by \( \tau/2 \). Fix \( \ell, N_1 \in \mathbb{N} \) so that

\[
|J| \cdot K e^{\ell \alpha_0} \leq \ell, \quad e^{-\ell} \leq \delta, \quad \text{and } N_1 > \max \left\{ 2\ell, \frac{2}{\tau} \cdot 4\ell \| \phi \| \right\}. \tag{6.13}
\]

Fix \( m \geq N_1 \) and let \((w_1, \ldots, w_m) \in \mathcal{W}^m \) and \( n = \sum_{j=1}^{m} |w_j| \geq N_1 \). Consider

\[
H = \Sigma_N^* \times [w_1, \ldots, w_m]^+ = \Pi_k \mathbb{S}_1^+ \times \mathbb{S}_1^- \times \mathbb{S}_1^+ = \Pi_k \mathbb{S}_1.
\]

As by property (b) of a CIFS, every map \( f_{[w_j]} \) is a step skew product, for every \( j = 0, \ldots, n \), the image \( F_j(H) \) is a cartesian product of a cylinder of level \( j \), a cylinder of level \( n - j \), and an interval.

For the course of this proof, denote by \( \pi_k, k = 1, 2, 3 \), the projection to the \( k \)th component of the product space \( \mathbb{S}_1^- \times \mathbb{S}_1^+ \times \mathbb{S}_1^+ \). As by property (b) of a CIFS, every map \( f_{[w_j]} \) is a contraction, together with (6.13) it follows

\[
|\pi_3(F_j(H))| \leq \delta \quad \text{for all } j = 0, \ldots, n.
\]

Recall the metrics \( d^\pm \) on \( \Sigma_N^+ \) defined in Sect. 3.2. Note that for every \( j = 0, \ldots, n \)

\[
diam_{d^-}(\pi_1(F_j(H))) \leq e^{-j} \quad \text{and} \quad diam_{d^+}(\pi_2(F_j(H))) \leq e^{-n+j}.
\]

Together with (6.13) it then follows that for every \( j = 0, \ldots, n - \ell \) it holds

\[
diam_{d^-}(\pi_1(F_j(H))) \leq \delta, \quad diam_{d^+}(\pi_2(F_j(H))) \leq \delta, \quad |\pi_3(F_j(H))| \leq \delta.
\]

Thus, for every \( X, Y \in H \) we obtain

\[
|S_n \phi(X) - S_n \phi(Y)| \leq \ell 2\| \phi \| + (n - 2\ell) \frac{\tau}{2} + \ell 2\| \phi \| < \frac{n}{2} + 4\ell \| \phi \|
\]

using (6.13) \( < \frac{n}{2} + \frac{\tau}{2} N_1 \leq \frac{n}{2} + \frac{\tau}{2} n = \tau n \).

This finishes the proof. \( \Box \)
6.5. Horseshoes associated to CIFSs. For every CIFS $W \subset \Sigma_N^+$, Proposition 6.3 asserts the existence of its associated attractor $\Lambda(W) \subset \Sigma_N \times S_1$. Moreover, if $W$ is a CIFS on an interval $J$ then

$$\Lambda(W) = \Pi_W(W^\mathbb{Z}) \subset PCS(W) \times J \subset \Sigma_N \times J.$$  \hspace{1cm} (6.14)  

**Proposition 6.13** (Horseshoe induced by a CIFS). Let $W = \{w_1, \ldots, w_M\} \subset \Sigma_N^+$ be a finite disjoint collection of words defining a CIFS on an interval $J$ relative to $K$, $\alpha_0$, $\alpha$, and $\epsilon$, and $\Lambda(W)$ its associated attractor. Let

$$\Gamma(W) \overset{\text{def}}{=} \bigcup_{k=0}^{R-1} F^k(\Lambda(W)),$$

where $R \overset{\text{def}}{=} \max_{w \in W} |w|$. Then $\Gamma(W)$ is a compact $F$-invariant set such that every ergodic Borel probability measure $\mu' \in M(F|_{\Gamma(W)})$ satisfies

$$\chi(\mu') \in (\alpha - \epsilon, \alpha + \epsilon).$$

**Proof.** By Proposition 6.3, $\Lambda(W)$ is the image of a compact set under a continuous map, and hence compact. The semi-conjugation in Proposition 6.3 implies that $\Gamma(W)$ is $F$-invariant.

Hence, the property of the range of Lyapunov exponents is an immediate consequence of property (c) of a CIFS and the fact that the orbit of every point generic for an $F$-ergodic measure is described by an infinite concatenation of fiber maps $f_{[w]}$ with $w \in W$. \hfill $\Box$

**Remark 6.14** (Horseshoes). The set $\Gamma(W)$ can be seen as a $F$-invariant multi-variable-time horseshoe as in [15, Section 5]. For simplicity, we will refer to such sets simply as horseshoes.

### 7. Repetition and Tailing Scheme

In this section, we introduce the repeat-and-tail scheme which will provide us a cascade of collections of words $W$ over the alphabet $\{1, \ldots, N\}$. By writing $|w|$ for some $w \in W$ we always mean its length as spelled in $\{1, \ldots, N\}$.

**Definition 7.1.** Let $W \subset \Sigma_N^+$ be a collection of (nonempty) words. Given $m \in \mathbb{N}$, consider a tailing map $t = t_{W,m} \colon W^m \to \Sigma_N^*$ and define

$$(W^m)_t \overset{\text{def}}{=} \{(w_1, \ldots, w_m, t(w_1, \ldots, w_m)) : w_k \in W \text{ for } k = 1, \ldots, m\} \subset \Sigma_N^*.$$  

We say that $(W^m)_t$ $m$-times repeats and $t$-tails $W$. We define the tail-adding map

$$T_{(W^m)_t} : W^m \to (W^m)_t, \ T_{(W^m)_t}(w_1, \ldots, w_m) \overset{\text{def}}{=} (w_1, \ldots, w_m, t(w_1, \ldots, w_m)).$$

In the above definition we use our simplifying Notation 3.7. We point out that the words in $W$ may have different length. The same applies to words in $(W^m)_t$. The following is an immediate consequence of Corollary 3.6 and Lemma 3.1.

**Corollary 7.2.** Let $W \subset \Sigma_N^*$ be a finite collection of words which is disjoint. Let $m \in \mathbb{N}$ and consider a tailing map $t = t_{W,m} : W^m \to \Sigma_N^*$. Then $T_{(W^m)_t}$ is bijective. Moreover, $(W^m)_t$ is disjoint and hence uniquely left decipherable.
The next theorem is a key ingredient. It provides a choice of CIFS’s (and hence of the associated attractors and the horseshoes they generate) whose Lyapunov exponent drops by a controlled amount. The estimate on the length of the tails also allows to control the drop of entropy of the horseshoes.

**Theorem 7.3** (Choice of a tailing map). Consider \( F \in \text{SP}_{\text{shyp}}^1(\Sigma_N \times \mathbb{S}^1), \ N \geq 2 \). Let \( J \subset \mathbb{S}^1 \) be a blending interval. Let \( \mathcal{W} \) be a finite disjoint collection of words defining a CIFS on \( J \) relative to \( K \geq 1 \), \( a_0 = \alpha + \varepsilon < 0 \), \( \alpha < 0 \), and \( \varepsilon \), for some \( \varepsilon \in (0, |\alpha|/2) \).

There is \( N_2 = N_2(\mathcal{W}) \in \mathbb{N} \) such that for every \( m \geq N_2 \) there exists a tailing map \( t = t_{\mathcal{W},m} : \mathcal{W}^m \to \Sigma_N^* \) such that the \( m \)-times repeated and \( t \)-tailed collection of words \( (\mathcal{W}^m)_t \) defines a CIFS on \( J \) relative to \( K, \alpha_0', \alpha', \) and \( \varepsilon' \), where

\[
\alpha_0' = \frac{1}{2}(\alpha + \varepsilon), \quad \alpha' = \frac{1}{2}\alpha, \quad \varepsilon' = \frac{\varepsilon}{2}.
\]

Moreover, the tailing map satisfies for every \( w_1, \ldots, w_m \in \mathcal{W} \)

\[
|t(w_1, \ldots, w_m)| \leq L_1 \sum_{j=1}^m |w_j||\alpha|, \tag{7.1}
\]

where \( L_1 = L_1(F, J) > 0 \) is as in Definition 2.4.

**Proof.** Similarly to the proof of Theorem 6.5, we consider the constants \( K_1, \ldots, K_5 \) associated to the blending interval \([x - 2\delta, x + 2\delta] \subset J\). Let \( I = [x - \delta, x + \delta] \) and \( m_c = m_c(I) \) as in Claim 2.3. Recall that

\[
L_1 = L_1(F, J) = K_2(2 + |\log(4\delta)| + K_3) + m_c. \tag{7.2}
\]

**Choice of quantifiers.** Choose \( r > 0 \) such that

\[
r < \min\{K_1, K_4, \|F\|^{-m_c}\delta\}, \quad \text{and} \quad \text{Mod}_F(r) < \frac{\varepsilon}{8}, \tag{7.3}
\]

where \( \|F\| \) is defined in (6.1) and \( \text{Mod}_F(r) \) in (6.2).

**Claim 7.4.** For \( N_2 \in \mathbb{N} \) large enough, \( m \geq N_2 \), and \( w_1, \ldots, w_m \in \mathcal{W} \) it holds

\[
\log \sup_{x, y \in I} \frac{|(f_{[w_1, \ldots, w_m]})(x)|}{|(f_{[w_1, \ldots, w_m]})(y)|} \leq \sum_{j=1}^m |w_j| \frac{\varepsilon}{8}.
\]

**Proof.** Since \( \mathcal{W} \) is a CIFS on \( J \), applying repeatedly its maps to \( J \) shrinks this interval exponentially fast, which implies that the modulus of continuity of \( f_{[w_j]} \) on the corresponding image interval also decreases. Then

\[
\max_{x, y \in I} \log \frac{|(f_{[w_1, \ldots, w_m]})(x)|}{|(f_{[w_1, \ldots, w_m]})(y)|} \leq \sum_{j=1}^{m-1} \max_{x, y \in f_{[w_1, \ldots, w_j]}(I)} \log \frac{|(f_{[w_{j+1}]})(x)|}{|(f_{[w_{j+1}]})(y)|},
\]

where the latter is a finite sum of terms tending to zero. This implies the claim. \( \Box \)
We also assume the following properties to be satisfied for $N_2$:

$$
|J| \max \left\{ e^{N_2 \min_{w \in \mathcal{W}} |w| (\alpha + \varepsilon)}, K e^{N_2 \frac{1}{2} \min_{w \in \mathcal{W}} |w| \alpha} \right\} < r, \quad (7.4)
$$

$$
1 \leq N_2 |\alpha|, \quad \frac{1}{N_2} \log \|F\|^{1+m_c} < \frac{\varepsilon}{8}.
$$

In the following, let

$$
m \geq N_2.
$$

**Length of iterates of $J$.** Fix some enumeration $\mathcal{W} = \{w_1, \ldots, w_M\}$. Given $(i_1, \ldots, i_m) \in \{1, \ldots, M\}^m$, by property (a) of the CIFS, it holds

$$
H_{i_1, \ldots, i_m} \overset{\text{def}}{=} f_{[w_{i_1}, \ldots, w_{i_m}]}(J) \subset J.
$$

Property (c) of the CIFS first implies that for every $y \in J$ it holds

$$
e^{\sum_{j=1}^m |w_{i_j}| (\alpha - \varepsilon)} \leq |(f_{[w_{i_1}, \ldots, w_{i_m}]} - f_{[w_{i_1}, \ldots, w_{i_m}]})(y)| \leq e^{\sum_{j=1}^m |w_{i_j}| (\alpha + \varepsilon)}
$$

and hence

$$
|J| e^{\sum_{j=1}^m |w_{i_j}| (\alpha - \varepsilon)} \leq |H_{i_1, \ldots, i_m}| = |f_{[w_{i_1}, \ldots, w_{i_m}]}(J)| \leq |J| e^{\sum_{j=1}^m |w_{i_j}| (\alpha + \varepsilon)}
$$

by (7.4) and $m \geq N_2 \leq |J| e^{m \min_{w \in \mathcal{W}} |w| (\alpha + \varepsilon)} < r. \quad (7.5)$

**Definition of the tailing map.** Given $(i_1, \ldots, i_m) \in \{1, \ldots, M\}^m$, it holds $H_{i_1, \ldots, i_m} \subset J$. By Axiom CEC+($J$), there exists a finite expanding and covering sequence $(\eta_1, \ldots, \eta_L)$ such that

$$
f_{[\eta_1, \ldots, \eta_L]}(H_{i_1, \ldots, i_m}) \supset B(J, K_4),
$$

where $L \in \mathbb{N}$ satisfies

$$
L \leq K_2 \log |H_{i_1, \ldots, i_m}| + K_3
$$

using (7.5) \leq K_2 \sum_{j=1}^m |w_{i_j}| |\alpha - \varepsilon| + K_2 \log |J| | + K_3 \quad (7.6)

with $|\alpha - \varepsilon| = |\alpha| + \varepsilon < 2|\alpha| \leq \sum_{j=1}^m |w_{i_j}| |\alpha| \left( 2K_2 + \frac{1}{m|\alpha|} (K_2 \log |J| | + K_3) \right).

In the following, instead of “going all the way” to cover the blending interval $J$, we will only consider a certain truncated sequence $(\eta_1, \ldots, \eta_\ell)$ for some $\ell \leq L$. Indeed, choose $\ell \in \{1, \ldots, L\}$ to be the smallest number satisfying

$$
|f_{[\eta_1, \ldots, \eta_\ell]}|(H_{i_1, \ldots, i_m}) \geq |J| e^{\frac{1}{2} \sum_{j=1}^\ell |w_{i_j}| \alpha}. \quad (7.7)
$$

Note that the estimate in (7.5) together with $\alpha + \varepsilon < \frac{1}{2} |\alpha| < 0$ implies $\ell \geq 1$.

By Claim 2.3, there exists a finite sequence $(\beta_1, \ldots, \beta_\ell)$, $s \leq m_c$, such that

$$
(f_{[\beta_1, \ldots, \beta_\ell]} \circ f_{[\eta_1, \ldots, \eta_\ell]})(H_{i_1, \ldots, i_m}) \cap I \neq \emptyset. \quad (7.8)
$$
Define now the tailing map

\[ t: \mathcal{W}^m \to \sum_N^*, \quad t(w_{i_1}, \ldots, w_{i_m}) \overset{\text{def}}{=} (\eta_1, \ldots, \eta_\ell, \beta_1, \ldots, \beta_s). \]

Using (7.4) we have \( m|\alpha| \geq N_2|\alpha| \geq 1. \) Observing that \( \ell \leq L, \) together with (7.6) it hence follows

\[ \ell + s \leq \sum_{j=1}^{m} |w_{i_j}| |\alpha| \left( 2K_2 + \frac{1}{m|\alpha|}(K_2|J|| + K_3) \right) + m_c \leq L_1 \sum_{j=1}^{m} |w_{i_j}| |\alpha|, \]

where \( L_1 \) as in (7.2). This implies property (7.1).

What remains to prove is that \( (\mathcal{W}^m)_t \overset{\text{def}}{=} \{(w_{i_1}, \ldots, w_{i_m}, t(w_{i_1} \ldots w_{i_m})): (w_{i_1}, \ldots, w_{i_m}) \in \mathcal{W}^m\} \)
defines a CIFS on \( J \) with the claimed quantifiers.

**Checking properties of a CIFS with quantifiers.**

**Lemma 7.5.** The collection of words \( (\mathcal{W}^m)_t \) satisfies properties (a), (b), and (c) of a CIFS on \( J \) relative to \( K, \frac{1}{2}(\alpha + \epsilon), \frac{1}{2}\alpha, \) and \( \frac{1}{2}\epsilon. \)

We split the proof of this lemma into claims.

**Claim 7.6.** Property (a) holds.

**Proof.** By the choice of \( \ell \) in (7.7), for every \( k = 1, \ldots, \ell - 1 \) it holds

\[ |(\eta_{[\eta_1, \ldots, \eta_k]}(H_{i_1, \ldots, i_m})| < |J| e^{\frac{1}{2} \sum_{j=1}^{m} |w_{i_j}| |\alpha| < r, \quad (7.9) \]

where for the latter inequality we used (7.4) and \( m \geq N_2. \) This together with \( s \leq m_c \) and (7.3) implies

\[ |(\beta_{[\beta_1, \ldots, \beta_s]} \circ (\eta_{[\eta_1, \ldots, \eta_{\ell - 1}]}) (H_{i_1, \ldots, i_m})| \leq r \|F\| e^{\frac{1}{2} \sum_{j=1}^{m} |w_{i_j}| |\alpha| < \delta. \quad (7.10) \]

Hence, by the intersection property in (7.8) and the fact that \( I = [x - \delta, x + \delta] \subset J = [x - 2\delta, x + 2\delta], \) it follows

\[ (\beta_{[\beta_1, \ldots, \beta_s]} \circ (\eta_{[\eta_1, \ldots, \eta_{\ell - 1}]}) (H_{i_1, \ldots, i_m}) \subset J. \]

This implies the claim. \( \square \)

**Claim 7.7.** Property (b) holds.

**Proof.** By our choice of \( \ell \) in (7.7), there exists \( y \in J \) such that

\[ |(\eta_{[\eta_1, \ldots, \eta_{\ell}]})' (y)| \geq e^{\frac{1}{2} \sum_{j=1}^{m} |w_{i_j}| |\alpha|. \quad (7.10) \]

As \( \ell \) is minimal satisfying (7.7), for every \( k = 1, \ldots, \ell \) there exists \( z_k \in J \) so that

\[ |(\eta_{[\eta_1, \ldots, \eta_k]} \circ (\eta_{[\eta_1, \ldots, \eta_{\ell - 1}]})' (z_k)| < \|F\| e^{\frac{1}{2} \sum_{j=1}^{m} |w_{i_j}| |\alpha|. \quad (7.11) \]

Using (7.9), by the choice of \( r \) in (7.3), for every \( k = 1, \ldots, \ell \)

\[ \log \sup_{x, y \in H_{i_1, \ldots, i_m}} \frac{|(\eta_{[\eta_1, \ldots, \eta_k]}') (x)|}{|(\eta_{[\eta_1, \ldots, \eta_k]}') (y)|} \leq \frac{k \epsilon}{8}. \quad (7.12) \]
Hence, together with (7.11), for every \( z \in J \) and \( k = 1, \ldots, \ell \) it holds
\[
| (f_{[\eta_1, \ldots, \eta_k]} \circ f_{[w_1, \ldots, w_{m_1}]}) (z) | \leq \| F \| e^{\frac{1}{2} \sum_{j=1}^{m} | w_j | \alpha} \cdot e^{k \varepsilon / 8}. \tag{7.13}
\]
On the other hand, by (7.12) and (7.10) and also distortion Claim 7.4, for every \( z \in J \) and \( k = \ell \)
\[
e^{- \sum_{j=1}^{m} | w_j | \varepsilon / 8} e^{- \varepsilon \ell / 8} \leq \frac{| (f_{[\eta_1, \ldots, \eta_\ell]} \circ f_{[w_1, \ldots, w_{m_1}]}) (z) |}{e^{\frac{1}{2} \sum_{j=1}^{m} | w_j | \alpha}} \leq \| F \| \cdot e^{\ell \varepsilon / 8}. \tag{7.14}
\]
Further for every \( x \in J \) and \( k = 1, \ldots, \ell \), using (7.13), it holds
\[
\frac{1}{\sum_{j=1}^{m} | w_j | + k} \log | (f_{[\eta_1, \ldots, \eta_k]} \circ f_{[w_1, \ldots, w_{m_1}]}) (x) |
\leq \frac{1}{\sum_{j=1}^{m} | w_j | + k} \left( \log \| F \| + \sum_{j=1}^{m} | w_j | \alpha + k \varepsilon \right)
\]
by (7.4) and \( m \geq N_2 \) \( < \frac{\varepsilon}{8} + \frac{1}{2} \alpha + \frac{\varepsilon}{8} < \frac{1}{2} \alpha + \frac{\varepsilon}{2} \). \tag{7.15}
Moreover, for every \( x \in J \) and \( k = 1, \ldots, s \) it holds
\[
\frac{1}{\sum_{j=1}^{m} | w_j | + \ell + k} \log | (f_{[\beta_1, \ldots, \beta_k]} \circ f_{[\eta_1, \ldots, \eta_\ell]} \circ f_{[w_1, \ldots, w_{m_1}]}) (x) |
\]
using (7.13) for \( \ell \) \( \leq \frac{1}{\sum_{j=1}^{m} | w_j | + \ell + k} \log \| F \|^{1+k} + \frac{1}{2} \alpha + \frac{\varepsilon}{8} \)
using (7.4) and \( m \geq N_2 \) \( < \frac{\varepsilon}{8} + \frac{1}{2} \alpha + \frac{\varepsilon}{8} < \frac{1}{2} \alpha + \frac{\varepsilon}{2} \). \tag{7.16}
Hence, the hypothesis on \( W \) defining a CIFS with quantifiers, (7.15), and (7.16) together imply property (b). \( \square \)

**Claim 7.8.** Property (c) holds.

**Proof.** The upper bound for the spectrum follows from (7.16). It remains to prove the lower one. Note that (7.10) together with (7.14) implies
\[
\frac{1}{\sum_{j=1}^{m} | w_j | + \ell + s} \log | (f_{[\beta_1, \ldots, \beta_s]} \circ f_{[\eta_1, \ldots, \eta_\ell]} \circ f_{[w_1, \ldots, w_{m_1}]}) (x) |
\geq \frac{1}{\sum_{j=1}^{m} | w_j | + \ell + s} \left( \log \| F \|^{-m_1} + \frac{1}{2} \sum_{j=1}^{m} | w_j | \alpha - \ell \varepsilon / 8 - \sum_{j=1}^{m} | w_j | \varepsilon / 8 \right)
\]
\[
> \frac{1}{\sum_{j=1}^{m} | w_j | + \ell + s} \log \| F \|^{-m_1} + \frac{\sum_{j=1}^{m} | w_j |}{\sum_{j=1}^{m} | w_j | + \ell + s} \left( \frac{1}{2} \alpha - \frac{\varepsilon}{8} \right) - \frac{\varepsilon}{8}
\]
by (7.4) \( > - \frac{\varepsilon}{8} + \frac{1}{2} \alpha - \frac{\varepsilon}{4} > \frac{1}{2} \alpha - \frac{\varepsilon}{2} \),
proving the claim. \( \square \)

This finishes the proof of the theorem. \( \square \)
8. Cascades of Horseshoes

Throughout this section, consider \( F \in \text{SP}^1_{\text{shyp}}(\Sigma_N \times \mathbb{S}^1) \), \( N \geq 2 \). Let \( \mu \) be an ergodic measure with Lyapunov exponent \( \alpha = \chi(\mu) < 0 \) and entropy \( h = h(F, \mu) > 0 \). Fix \( \epsilon \in (0, |\alpha|/4) \) and \( \epsilon_H \in (0, h) \). Let \( J \subset \mathbb{S}^1 \) be a blending interval and \( \mathcal{W} \subset \Sigma^*_N \) be a finite disjoint collection of words as provided by Theorem 6.5, defining a CIFS on \( J \) relative to some constant \( K > 1 \) and \( \alpha_0 = \alpha + \epsilon, \alpha, \) and \( \epsilon \). In particular, it holds
\[
(h(F, \mu) - \epsilon_H) \min_{w \in \mathcal{W}} |w| \leq \log \text{card } \mathcal{W} \leq (h(F, \mu) + \epsilon_H) \max_{w \in \mathcal{W}} |w|.
\] (8.1)

Let \( L_1 = L_1(F, J) \) as in Definition 2.4.

In Sect. 8.1, we construct two cascades of alphabets \((A_n)_n\) and \((\mathcal{W}_n)_n\). Every alphabet \( \mathcal{W}_n \) is formed by words in \( \{1, \ldots, N\} \) and obtained from the previous one \( \mathcal{W}_{n-1} \) by the repeat-and-tail scheme with tailing functions \( t_n \) as in Theorem 7.3. Moreover, every \( \mathcal{W}_n \) defines a CIFS with associated attractor \( \Lambda_n = \Lambda(\mathcal{W}_n) \) which in turn generates a horseshoe \( \Gamma_n \) (as in Propositions 6.3 and 6.13). Each alphabet \( A_n \) is the abstract companion of \( \mathcal{W}_n \) and gives rise to a suspension space for an appropriate roof function \( R_n : A_n \to \mathbb{N} \), see (8.4). Note that all these objects depend on \( \mu \).

In Sect. 8.2, we see that those horseshoes are factors of the suspension spaces. Using the latter, in Sect. 8.3 we obtain estimates of entropy and exponents of the horseshoes. We conclude this section by proving Proposition D.

8.1. Construction of a cascade of horseshoes. In the following we introduce the two cascades \((A_n)_n\) and \((\mathcal{W}_n)_n\) of alphabets. Our scheme is fairly general and only requires \( \mathcal{W}_0 \) and an initially fixed sequence \((m_n)_n\). We always denote by \(|\cdot|\) the length of the corresponding word spelled in \( \{1, \ldots, N\} \).

8.1.1. Inductive definition of alphabets We proceed inductively. Let \( \mathcal{W}_0 \overset{\text{def}}{=} \mathcal{W}, M_0 \overset{\text{def}}{=} \text{card } \mathcal{W}_0 \), and put \( \mathcal{W}_0 = \{w_1^{(0)}, \ldots, w_{M_0}^{(0)}\} \). By hypothesis in the beginning of this section, \( \mathcal{W}_0 \) defines a CIFS on \( J \) relative to \( K, \alpha_0 = \alpha + \epsilon, \alpha, \) and \( \epsilon \). Let
\[
A_0 \overset{\text{def}}{=} \mathcal{W}_0
\]
For \( n \geq 1 \), assume that there is a finite disjoint collection of words
\[
\mathcal{W}_{n-1} = \{w_1^{(n-1)}, \ldots, w_{M_{n-1}}^{(n-1)}\} \subset \Sigma^*_N,
\]
which defines a CIFS on \( J \) relative to \( K, 2^{-(n-1)}\alpha_0, 2^{-(n-1)}\alpha, \) and \( 2^{-(n-1)}\epsilon \), a collection \( A_{n-1} \), and \( m_{n-1} \in \mathbb{N} \). Let \( N_2 = N_2(\mathcal{W}_{n-1}) \in \mathbb{N} \) as in Theorem 7.3 and choose \( m_n \geq N_2 \) with \( m_n \geq m_{n-1} \). Consider the tailing map \( t_n = t_{\mathcal{W}_{n-1}, m_n} : (\mathcal{W}_{n-1})^{m_n} \to \Sigma^*_N \) as provided in that theorem and, recalling Definition 7.1, denote by
\[
\mathcal{W}_n \overset{\text{def}}{=} (\mathcal{W}_{n-1})^{m_n}
\]
the collection which \( m_n \)-times repeats and \( t_n \)-tails \( \mathcal{W}_{n-1} \). Let
\[
A_n \overset{\text{def}}{=} (A_{n-1})^{m_n}.
\]
Note that for every \( n \in \mathbb{N} \)
\[
\text{card } A_n = M_n = M_0^{m_1 \cdots m_{n-1} m_n}.
\] (8.2)
This concludes the inductive description of the alphabets.
8.1.2. Dictionaries  Let us point out natural “dictionaries” between \((W_n)_n\) and \((A_n)_n\) and the corresponding cascade of sequence spaces. For every \(n \in \mathbb{N}\), \(W_n\) and \(A_n\) have the same cardinality, the former is a collection of words that almost coincide with the words in \(A_n\) up to a cascade of tails which were introduced at every intermediate level. Recursively, we define a bijection between each such pair of collections:

- \(C_0\) is the identity on \(A_0 = W_0 = W\),
- for every \(w = (w_{i_1}^{(n-1)}, \ldots, w_{i_m}^{(n-1)}, t_n(w_{i_1}^{(n-1)}, \ldots, w_{i_m}^{(n-1)})) \in W_n\), let \(C_n(w) \equiv (C_{n-1}(w_{i_1}^{(n-1)}), \ldots, C_{n-1}(w_{i_m}^{(n-1)}))\).

The map \(C_n\) “cuts out any tail” added in the definitions of \(W_1, \ldots, W_n\). We let

\[
C_n : (W_n)^Z \to (A_n)^Z,
\]

\[
C_n(\ldots, w_{-1}|w_0, w_1, \ldots) \defeq (\ldots, C_n(w_{-1})|C_n(w_0), C_n(w_1), \ldots).
\]

To prove the next lemma just note that both alphabets have the same cardinality.

**Lemma 8.1.** For every \(n \in \mathbb{N}\), the maps \(\sigma_{W_n}\) on \((W_n)^Z\) and \(\sigma_{A_n}\) on \((A_n)^Z\) are topologically conjugate by \(C_n\).

8.1.3. Definition and control of roof functions  For every \(n \in \mathbb{N}_0\) define the roof function

\[
R_n : A_n \to \mathbb{N}, \quad R_n(a) \defeq \left| C_n^{-1}(a) \right|,
\]

where the length is considered in \(\Sigma_N^*\) identifying each concatenation of words with the corresponding word spelled in \(\{1, \ldots, N\}\). As before, we also consider the associated map \(R_n : (A_n)^Z \to \mathbb{N}\).

The next corollary estimates the lengths of the (inductively defined) tails added in each step. It is an immediate consequence of Theorem 7.3 and Corollary 7.2.

**Corollary 8.2.** (Control of tail-lengths) For every \(n \in \mathbb{N}\), \(W_n \subset \Sigma_N^*\) is a finite disjoint collection of words which defines a CIFS on \(J\) relative to \(K\), \(2^{-n} \alpha_0\), \(2^{-n} \alpha\), and \(2^{-n} \epsilon\). Moreover,

\[
|t_n(w_{i_1}^{(n-1)}, \ldots, w_{i_m}^{(n-1)})| \leq L_1 \frac{1}{2^{n-1}} \sum_{j=1}^{m_n} |w_{ij}^{(n-1)}| |\alpha|.
\]

The following corollary puts the above bounds on the tailing map into the context of roof functions in our abstract model suspension spaces.

**Corollary 8.3.** (Estimates on roof functions) The associated family of roof functions \((R_n)_n\) satisfies Assumption 5.2 with \(K = L_1 |\alpha|\).

**Proof.** The first inequality in Assumption 5.2 holds true by construction. Recalling that \(S_{n,n-1}\) denoted the substitution map from \((A_n)^Z\) to \((A_{n-1})^Z\) as defined in (5.1), from Corollary 8.2 it follows that

\[
R_n \circ S_{n,n-1}^{-1} \leq (1 + L_1 \frac{1}{2^{n-1}} |\alpha|) \sum_{j=0}^{m_n-1} R_{n-1} \circ \sigma_{A_{n-1}}^j,
\]

which implies the second inequality taking \(K = L_1 |\alpha|\). \(\square\)
8.1.4. Inductive definition of horseshoes  For every $n \in \mathbb{N}$, let
\[ \Lambda_n \overset{\text{def}}{=} \Lambda(W_n) \subset \Sigma_N \times J \quad \text{and} \quad \Gamma_n \overset{\text{def}}{=} \Gamma(W_n) \supset \Lambda_n \] (8.5)
be as in Propositions 6.3 and 6.13, respectively. We can view each set $\Lambda_n$ as the “ground floor” of the horseshoe $\Gamma_n$ (the reason for this notation will become clear thereafter). Recall that $(w, x) \mapsto F^{[u_0]}(w, x)$ is a return map on $\Lambda_n$. Moreover, the map
\[ \Pi_n \overset{\text{def}}{=} \Pi_{\Lambda_n} : (W_n)^Z \to \Lambda_n, \] (8.6)
provided by Proposition 6.3 satisfies
\[ (\Pi_n \circ \sigma_{W_n})(w, x) = (F^{[u_0]} \circ \Pi_n)(w, x). \] (8.7)

As $W_n$ is disjoint, by Proposition 6.3, the map $\Pi_n$ is $(\max R_n)$-to-one. Recall that, in general, $\Pi_n$ is uniformly finite-to-one and hence $\Pi_n^{-1}$ is multivalued. However, as $W_n$ is disjoint and hence, by Lemma 3.1, is uniquely left decipherable, the value
\[ R_n \circ \mathcal{C}_n \circ \Pi_n^{-1} \] (8.8)
is well-defined.

8.2. Horseshoes are factors of suspension spaces.  We now invoke the construction in Sect. 4.1 to obtain the cascade of suspension spaces $S_n = S_{A_n, R_n}$ associated to the cascade of words $A_n$ and roof functions $R_n$, $n \in \mathbb{N}$. We will also consider the suspension of $\sigma_n = \sigma_{A_n}$ by $R_n$ and denote it by $\Phi_n = \Phi_{A_n, R_n}$. Recall the definition of the ground floor $G_n = (A_n)^Z \times \{0\}$ in (5.7). By (5.8) and using Notation 8.5, it holds
\[ \Phi_n \circ \mathcal{C}_n \mid G_n = \sigma_n \times \text{id}. \]

By construction of the space $S_n$, the map $\Phi_n \circ \mathcal{C}_n$ is the first return-map on $G_n$.

Recall the homeomorphism $\mathcal{C}_n : (W_n)^Z \to (A_n)^Z$ in (8.3) and the continuous surjective finite-to-one map $\Pi_n : (W_n)^Z \to \Lambda_n$ in (8.6). The following (commuting) diagrams put into relation the shift map on our abstract shift space $(A_n)^Z$, the shift map on the word space $(W_n)^Z \subset \Sigma_N$, and the induced return map on the part of the horseshoe obtained as the attractor of the CIFS at level $n$. The map $h_n$ is defined in (8.11) below.

\[
\begin{align*}
(A_n)^Z & \xrightarrow{\sigma_n = \sigma_{A_n}} (A_n)^Z \\
(W_n)^Z & \xrightarrow{\sigma_{W_n}} (W_n)^Z \\
\Lambda_n & \xrightarrow{F_{R_n} \circ \mathcal{C}_n \circ \Pi_n^{-1}} \Lambda_n
\end{align*}
\]

\[
\begin{align*}
(A_n)^Z \times \{0\} & \xrightarrow{\Phi_n \circ \mathcal{C}_n} (A_n)^Z \times \{0\} \\
(W_n)^Z & \xrightarrow{\mathcal{C}_n^{-1} \circ \Pi_n} (W_n)^Z \\
\Lambda_n & \xrightarrow{F_{R_n} \circ \mathcal{C}_n \circ \Pi_n^{-1}} \Lambda_n
\end{align*}
\]

Hence, the map
\[ \varrho \in (A_n)^Z \mapsto (\Pi_n \circ \mathcal{C}_n^{-1})(\varrho) \in \Lambda_n \] (8.9)
is continuous, onto $\Lambda_n$, at most (max $R_n$)-to-one, and the above diagram comutes. The key result in this section is to extend the map (8.9) to a factor map between the suspension space and the full horseshoe, considering corresponding invariant measures. For that consider the $\Phi_n$-ergodic Borel probability measure

$$\lambda_n \equiv \lambda_{A_n, R_n}$$

on the suspension space $S_n$ as defined in (5.5).

**Proposition 8.4.** There is a continuous surjective map $H_n : S_n = S_{A_n, R_n} \to \Gamma_n$ which is uniformly finite-to-one such that

$$\text{card } H_n^{-1}(\{X\}) \leq (\max R_n)^2, \text{ for every } X \in \Gamma_n,$$

satisfying

$$H_n \circ \Phi_n = F \circ H_n.$$ 

Letting

$$\mu_n \equiv (H_n)_* \lambda_n,$$

the measure preserving system $(\Gamma_n, F, \mu_n)$ is a factor of the measure preserving system $(S_n, \Phi_n, \lambda_n)$ by $H_n$. Moreover, $(\Gamma_n, F, \mu_n)$ is ergodic and it holds

$$h(F, \mu_n) = h(\Phi_n, \lambda_n) = \frac{\log M_n}{M_n \sum_{a \in A_n} R_n(a)}, \text{ where } M_n = \text{card } A_n.$$ 

The following commuting diagram illustrates the above proposition.

$$\begin{array}{c}
(A_n) \times \{0\} \subset \Lambda_n \subset \Gamma_n \\
h_n \downarrow \downarrow \downarrow \downarrow \\
S_n \xrightarrow{\Phi_n} S_n \xrightarrow{H_n} \Gamma_n \xrightarrow{(H_n)_*} \mu_n
\end{array}$$

To prove the proposition, we need a preliminary result and some notation.

**Notation 8.5.** (Return maps) Below we consider several types of return maps. Given a map $S : X \to X$, a set $A \subset X$, and a function $R : A \to \mathbb{N}$, we let

$$S^R : A \to X, \quad S^R(x) \equiv S^{R(x)}(x) = (S \circ \cdots \circ S)(x).$$

**Lemma 8.6.** Let $S : X \to X$ and $T : Y \to Y$ be two homeomorphisms on compact metric spaces. Assume that there are sets $A \subset X$ and $B \subset Y$ and continuous function $R_S : A \to \mathbb{N}$ and $R_T : B \to \mathbb{N}$ such that $S^{R_S} : A \to A$ is the first return-map on $A$ and $T^{R_T} : B \to B$ is a (not necessarily first) return map on $B$. Suppose that there is a continuous surjective map $h : A \to B$ satisfying

$$R_T \circ h = R_S \quad \text{and} \quad h \circ S^{R_S} = T^{R_T} \circ h.$$ 

Let $A' \equiv \bigcup_{k \geq 0} S^k(A)$ and $B' \equiv \bigcup_{k \geq 0} T^k(B)$. Then there exists a continuous surjective map $H : A' \to B'$ which extends $h$ to $A'$ such that

$$H \circ S|_{A'} = T \circ h.$$
Moreover, if there is \( K \in \mathbb{N} \) satisfying
\[
\text{card } h^{-1}(\{b\}) \leq K \quad \text{for every } b \in B
\]
then \( H \) is finite-to-one with
\[
\text{card } H^{-1}(\{y\}) \leq K \sup R_T \quad \text{for every } y \in B'.
\]

**Proof.** Let us first define \( H : A' \to B' \). Given \( x' \in A' \), as \( S^{R_S} \) is a first return to \( A \), there are uniquely determined \( x \in A \) and \( k \in \{0, \ldots, R_S(x) - 1\} \), so that \( x' = S^k(x) \). Let
\[
H(x') = T^k(h(x)).
\]

Noting that \( y' = S(x') = S^{k+1}(x) \) satisfies \( H(y') = T^{k+1}(h(x)) \), it holds
\[
H \circ S(x') = H \circ S^{k+1}(x) = T^{k+1} \circ h(x) = T(T^k \circ h(x)) = T(H \circ S^k(x))
\]
proving that the maps \( S \) and with the notation of this lemma,

**Proof of Proposition 8.4.** Note that
\[
(\mathbb{C}^{-1}_n \circ p_n)((A_n)^\mathbb{Z} \times \{0\}) = (W_n)^\mathbb{Z} \quad \text{and} \quad \Pi_n((W_n)^\mathbb{Z}) = \Lambda_n.
\]

Let
\[
h_n : (A)_{\mathbb{Z}}^X \times \{0\} \to \Lambda_n, \quad h_n(\mathbb{a}) \overset{\text{def}}{=} \left( \Pi_n \circ \mathbb{C}^{-1}_n \circ p_n \right)(\mathbb{a}, 0) \quad (8.11)
\]

We apply Lemma 8.6 letting
\[
X = S_n, \quad S = F_n, \quad A = (A_n)^\mathbb{Z} \times \{0\}, \quad R_S = R_n \circ p_n,
\]
\[
Y = \Gamma_n, \quad T = F, \quad B = \Lambda_n, \quad R_T = R_n \circ \mathbb{C}_n \circ \Pi_n^{-1}, \quad h = h_n.
\]

Recall that by (8.8), the function \( R_T \) is well-defined. Recall that, by the definition of the suspension space and with the notation of this lemma,
\[
A' = S_n = \bigcup_k \Phi_k^X((A_n)^\mathbb{Z} \times \{0\}) \quad \text{and} \quad B' = \Gamma_n.
\]

In the next two claims we check the hypotheses of Lemma 8.6.

**Claim 8.7.** \( (R_n \circ \mathbb{C}_n \circ \Pi_n^{-1}) \circ h_n = R_n \circ p_n \) on \( A \).

**Proof.** Using the definition of \( h_n \) in (8.11) and (8.8), we get
\[
(R_n \circ \mathbb{C}_n \circ \Pi_n^{-1}) \circ h_n = (R_n \circ \mathbb{C}_n \circ \Pi_n^{-1}) \circ \left( \Pi_n \circ \mathbb{C}^{-1}_n \circ p_n \right) = R_n \circ p_n,
\]
proving the claim. \( \square \)
Claim 8.8. \( h_n \circ \Phi_{n,n}^n = F R_n \circ C_n \circ \Pi_n^{-1} \circ h_n \) on \( A \).

Proof. Note that on \((A_n)^\infty \times \{0\}\) it holds
\[
\Phi_{n,n}^n = \sigma_n \times \text{id}. \tag{8.12}
\]

Using the definition of \( h_n \) in (8.11) and (8.12), it follows
\[
(h_n \circ \Phi_{n,n}^n)(a, 0) = \left( (\Pi_n \circ C_n^{-1} \circ p_n) \circ (\sigma_n \times \text{id}) \right)(a, 0)
\]
by Lemma 8.1
\[
= (\Pi_n \circ (C_n^{-1} \circ \sigma_n))(a) = (\Pi_n \circ (\sigma_n \circ C_n^{-1}))(a)
\]
by (8.7)
\[
= (F R_n \circ C_n \circ \Pi_n^{-1} \circ (\Pi_n \circ C_n^{-1}))(a)
\]
\[
= (F R_n \circ C_n \circ \Pi_n^{-1} \circ (\Pi_n \circ C_n^{-1} \circ p_n))(a, 0)
\]
\[
= (F R_n \circ C_n \circ \Pi_n^{-1} \circ h_n)(a, 0),
\]
proving the claim. \( \square \)

Claims 8.7 and 8.8 allow us to apply Lemma 8.6 as explained above. Hence, there is a continuous surjective map \( H_n : S_n \to \Gamma_n \) which extends \( h_n \) and satisfies
\[
F \circ H_n = H_n \circ \Phi_n.
\]

The property about cardinality of preimages also follows from Lemma 8.6 together with the fact that the map in (8.9) is at most \( R_n \)–to–one.

The factor property and ergodicity are consequences of this semiconjugation and the definition of \( \mu_n \). Moreover, as the map \( H_n \) is finite–to–one, by [25], it holds
\[
\sup_{\lambda : (H_n)_\lambda \equiv \mu_n} h(\Phi_n, \lambda) = h(F, \mu_n) + \int_{CS_n} h_{\text{top}}(\Phi_n, H_n^{-1}(X)) \, d\mu_n(X).
\]

In the integral, \( h_{\text{top}}(\cdot) \) denotes the topological entropy, which is zero for every \( X \) because \( H_n^{-1}(X) \) is finite for every \( X \). As the entropy of a factor system is always less than or equal to the entropy of its extension, this implies
\[
h(\Phi_n, \lambda_n) = h(F, \mu_n).
\]
The assertion about entropy now is a consequence of Lemma 4.2. \( \square \)

8.3. Entropy and Lyapunov exponents of horseshoes. We now put the previous results into the context of the cascade of horseshoes \((\Gamma_n)_n\) in (8.5). Recall that our construction, in particular those horseshoes, depend on the initially fixed ergodic measure \( \mu \) as stated in the beginning of Sect. 8. Recall that \( L_1 = L_1(F, J) \) as in Definition 2.4.

Corollary 8.9. For every \( \tilde{\mu} \in M_{\text{erg}}(F|\Gamma_n) \) it holds
\[
\chi(\tilde{\mu}) \leq \left( \frac{1}{2^n}(\alpha - \varepsilon), \frac{1}{2^n}(\alpha + \varepsilon) \right).
\]
Moreover, the measure \( \mu_n \in M_{\text{erg}}(F|\Gamma_n) \) defined in (8.10) satisfies
\[
h(F, \mu_n) \geq e^{-L_1(1-2^{-n})|\alpha|} (h(F, \mu) - \varepsilon_H).
\]
In particular, any measure \( \mu_\infty \) which is weak*-accumulated\(^{11}\) by \((\mu_n)_n\) satisfies
\[
\chi(\mu_\infty) = 0 \quad \text{and} \quad h(F, \mu_\infty) \geq e^{-L_1|\alpha|} \cdot (h(F, \mu) - \varepsilon_H).
\]

**Proof.** By Corollary 8.2, every collection of words \( \mathcal{W}_n \) defines a CIFS on \( J \) relative to \( K, 2^{-n}\alpha_0, 2^{-n}\alpha, \) and \( 2^{-n}\varepsilon \). Hence, Proposition 6.13 (1) implies the statement about the exponents for any measure in \( \mathcal{M}_{\text{erg}}(F|\Gamma_n) \).

Recall that, by Corollary 8.3, Assumption 5.2 is satisfied with \( K = L_1|\alpha| \). By Proposition 8.4, if follows
\[
h(F, \mu_n) = h(\Phi_n, \lambda_n) = \frac{\log M_n}{\sum_{a \in A_n} R_n(a)}
\]
using \( M_n = M_{n-1}^{m_n} \)
\[
\geq \frac{m_n \log M_{n-1}}{\max_{a \in A_n} R_n(a)}
\]
by Corollary 8.2
\[
\geq \frac{m_n \log M_{n-1}}{m_n(1 + L_1 2^{-(n-1)|\alpha|}) \max_{a \in A_n} R_n(a)}
\]
\[
\geq \cdots \geq \prod_{k=0}^{n-1} (1 + L_1 2^{-k}|\alpha|) \max_{a \in A_n} R_n(a)
\]
\[
\geq e^{-L_1(1-2^{-n})|\alpha|} \cdot \frac{\log M_0}{\max_{a \in A_n} R_n(a)}
\]
by (8.1), also using \( R_0(a) = |a| \)
\[
\geq e^{-L_1(1-2^{-n})|\alpha|} \cdot (h(F, \mu) - \varepsilon_H),
\]
proving the lower bound for entropy.

Finally, recalling that \( \chi(\mu_n) \) is the integral of a continuous function, we get \( \chi(\mu_\infty) = 0 \) for any limit measure \( \mu_\infty \). Further, as the entropy map is upper semi-continuous, the result about entropy follows taking limits as \( n \to \infty \). \( \square \)

8.4. Proof of Proposition D. For every \( n \in \mathbb{N} \) consider the numbers
\[
\alpha_n \overset{\text{def}}{=} \inf \{ \chi(\tilde{\mu}) \} \quad \text{and} \quad \beta_n \overset{\text{def}}{=} \sup \{ \chi(\tilde{\mu}) \},
\]
where inf and sup are taken over all \( \tilde{\mu} \in \mathcal{M}_{\text{erg}}(F|\Gamma_n) \). By Corollary 8.9
\[
\frac{1}{2^n}(\alpha - \varepsilon) < \alpha_n < \beta_n < \frac{1}{2^n}(\alpha + \varepsilon),
\]
and
\[
h_{\text{top}}(F, \Gamma_n) \geq h(F, \mu_n) \geq e^{-L_1(1-2^{-n})|\alpha|} (h(F, \mu) - \varepsilon_H).
\]
The natural projection of \( \Gamma_n \) to its first coordinate is the set \( \text{CS}(\mathcal{W}_n) \) which, by definition, is a coded shift. The fact that \( \Lambda_n \) is the attractor of a CIFS implies that every fiber intersecting \( \Lambda_n \) contains only one point. Hence the claimed property of the cardinality a fiber intersecting \( \Gamma_n \) follows. This proves the proposition. \( \square \)

\(^{11}\) Indeed, our particular choice of \((m_n)_n\) in Sect. 10.1 implies convergence, see Lemma 10.2.
9. Inherited Internal Structure of Horseshoes

In this section, we see how the structure of ground and intermediate floors in the suspension space $S_n$ described in Sect. 5.2 passes on to a corresponding internal structure of the horseshoe $\Gamma_n$ via the factor map $H_n$ in Proposition 8.4.

Recall the definition of intermediate floors $G^{(j)}_{n} = G^{(n-1,.j)}_{n}$, $j \in \{0, \ldots, \mu_n - 1\}$ in (5.10) and the notation in (5.11). Let

$$\Lambda^{n-1,.j}_{n} \overset{\text{def}}{=} H_n(G^{(n-1,.j)}_{n}).$$

Analogously, taking into account inductively the spelling of a word in $W_n$ in the alphabet $W_\ell$, for some $\ell \in \{0, \ldots, n - 1\}$, we consider the intermediate floor with $(\ell, n)$-address $a = (a_\ell, \ldots, a_{n-1})$. Recalling the definition of the intermediate floor $G^{(\ell,a)}_{n}$ in (5.13), let

$$\Lambda^{(\ell,a)}_{n} \overset{\text{def}}{=} H_n(G^{(\ell,a)}_{n}).$$

**Remark 9.1 (Almost-return maps).** To motivate the above definitions, consider some point $X = (\xi, x(\xi)) \in \Lambda_n$ and its forward orbit. As $\xi = \pi_1(X) \in \text{PCS}(W_n)$, it has a “spelling in the alphabet of words $W_n$” as

$$\xi = \iota_{W_n}(\ldots, w^{(n)}_i, w^{(n)}_j, \ldots).$$

Recall that by (8.7) it holds

$$F^{[w^{(n)}_i]}(X) \in \Lambda_n,$$

that is, this map is a return map on $\Lambda_n$. Recall that, by (6.14), it holds

$$\Lambda_n \subset \text{PCS}(W_n) \times J \subset \Sigma_1 - N \times \Sigma_1 + N \times J.$$

We now refine this relation to any address.

For our main argument in Sect. 10 the following observation will be essential. Recall that $w^{(n)}_{i_0}$ resulted from our repeat-and-tail scheme:

$$w^{(n)}_{i_0} = (w^{(n-1)}_{j_1}, \ldots, w^{(n-1)}_{j_{\mu_n}}, t_n(w^{(n-1)}_{j_1}, \ldots, w^{(n-1)}_{j_{\mu_n}}),$$

analogously for the other elements $w^{(n)}_{i_k}$. By definition (9.1), the set $\Lambda^{n-1,.j}_{n}$ contains the point on the forward orbit of $X$ whose position on the trajectory is determined by the position of the word $w^{(n-1)}_j$ in the sequence $\xi$. By Lemma 9.2 below it holds

$$\Lambda^{n-1,.j}_{n} \subset \Sigma_1 - N \times [w^{(n-1)}_j]^+ \times J$$

and hence the corresponding part of the orbit of $X$ after few iterations is very close to the corresponding orbit starting in $\Lambda_{n-1}$. Indeed, Lemma 9.2 takes any address $a$ and provides an even finer description. In very rough terms, this lemma states

$$\Lambda^{(\ell,a)}_{n} \subset \left(\Sigma_1 - N \times (W_\ell \times \Sigma_1 + N)\right) \times J,$$

though this formula is not precise for two reasons: first, it does not mark the 0th position of the two-sided sequence and second, $W_\ell$ is the union of words which possibly do not have equal length. Let us hence state the precise statement.
Recall notations in Lemma 5.10 and that \( C_{\ell}^{-1} \) defined in (8.3) “adds tails” up to level \( \ell \). Recall also Notation 3.7.

**Lemma 9.2.** For every \( n \in \mathbb{N}, \ell \in \{0, \ldots, n - 1\} \), and \((\ell, n)\)-address \( a \) and \( \bar{a} \in (\mathcal{A}_n)^{\mathbb{Z}} \), it holds

\[
H_n(a, s_n^{(\ell,\bar{a})}) \in \Sigma_N^{-} \times \left[ (C_{\ell}^{-1} \circ s_n^{(\ell,\bar{a})})(a) \right]^+ \times J.
\]

**Proof.** To prepare the proof, recall the terminology in Sect. 5.2.5. Without loss of generality, we assume that the address \( a = (a_0, \ldots, a_{n-1}) \) has simplified representation, that is, \( a_{\ell} \neq 0 \). Consider the sequence of addresses (each with simplified representation) \( a^{(0)} = 0, \ldots, a^{(k)}, \ldots, a^{(||a||)} = a \) as in (5.16) and denote by \( \ell_k = w(a^{(k)}) \) the corresponding levels which go from \( \ell_0 = n - 1 \) down to \( \ell_{||a||} = \ell \). Let \( \alpha_k \overset{\text{def}}{=} s_n^{(\ell_k, a^{(k)})}(a) = \left( (\sigma_{\ell_k} \circ S_{n,\ell_k})(a) \right)_0 \in \mathcal{A}_{\ell_k} \), where

\[
\tau_k = \sum_{i=0}^{n-1} a_i \cdot m_{\ell_k+1} \cdots m_i+1.
\]

Also consider the corresponding times in the suspension space for \( k = 0, \ldots, ||a|| \)

\[
t_0 = 0, \ldots, \ t_k = \sum_{i=1}^{||a||} R_{\ell_k} \left( s_n^{(\ell_k, a^{(i-1)})}(a) \right) = s_n^{(\ell_k, a^{(k)})}(a), \ldots, \quad (9.2)
\]

\[
t_{||a||} = s_n^{(\ell, a)}(a).
\]

For later reference, also recalling (8.4), note that for every \( k \) it holds

\[
t_k - t_{k-1} = R_{\ell_k} \left( s_n^{(\ell_k, a^{(k-1)})}(a) \right) = R_{\ell_k}(\alpha_k) = |C_{\ell_k}^{-1}(\alpha_k)|, \quad (9.3)
\]

where \(|\cdot|\) is the length of the corresponding word in the alphabet \( \{1, \ldots, N\} \).

We are now prepared to prove the lemma. Let

\[
H_n(a, t_k) = \left( \eta^{(k)}(a), x_k \right) \in \Sigma_N \times \mathcal{S}^{1}, \quad \text{where} \quad \eta^{(k)} \overset{\text{def}}{=} \sigma_{\ell_k}(\eta^{(0)}).
\]

The proof will be by induction on \( k = 0, \ldots, ||a|| \). We have that

\[
(\eta^{(||a||)}, x_{||a||}) = H_n(a, t_{||a||})
\]

(by definition of the suspension space) \( = (H_n \circ \Phi^{||a||}_n)(a, 0) \)

(by semiconjuation in Proposition 8.4) \( = (F^{||a||} \circ H_n)(a, 0) \).

Thus, by definition of the skew product \( F \) together with (9.2), it follows

\[
H_n(a, 0) = F^{-t_{||a||}}(\eta^{(||a||)}, x_{||a||}) = (\sigma^{-t_{||a||}}(\eta^{(||a||)}), x_0) = (\eta^{(0)}, x_0).
\]

One the one hand, \( H_n \) extends \( h_n \) and maps \( (\mathcal{A}_n)^{\mathbb{Z}} \times \{0\} \) into \( \text{PCS}(\mathcal{W}_n) \times J \) and hence \( \eta^{(0)} \in \text{PCS}(\mathcal{W}_n) = \tau_{\mathcal{W}_n}(\mathcal{W}_n^{\mathbb{Z}}) \). On the other hand, \( \eta^{(0)} = (\tau_{\mathcal{W}_n} \circ C_{n}^{-1})(a) \). As done before, in the following we will use that \( \mathcal{W}_n \) is disjoint and hence uniquely left decipherable. Hence, using the notation above, for \( a = (\ldots, a_{-1}|a_0, a_1, \ldots) \) the associated
By definition of the skew product $F$.

Case 1: $\eta(k)$, and we obtain

$$[C_n^{-1}(a_0)]^+ = [(\epsilon_\mathcal{W}_n \circ C_n^{-1})(a_0)]^+$$

and we obtain

$$H_n(a, 0) \in \Sigma_N^- \times [C_n^{-1}(a_0)]^+ \times J.$$ This proves the assertion for $k = 0$.

Assume that the assertion was shown for $k - 1$ for some $k \geq 1$. It holds

$$(\eta(k), x_k) = H_n(a, t_k) = H_n(\Phi_n^{t_k}(a, 0)) = H_n(\Phi_n^{t_k-1} \circ \Phi_n^{t_k}(a, 0))$$

(by the factor property)

$$= F^{t_k-1}(H_n(\Phi_n^{t_k-1}(a, 0)))$$

$$= F^{t_k-1}(H_n(a, t_k-1))$$

$$= F^{t_k-1}(\eta(k-1), x_{k-1}).$$

By induction hypothesis, it holds

$$(\eta(k-1), x_{k-1}) \in \Sigma_N^- \times [C_{\ell_{k-1}}(\alpha_{k-1})]^+ \times J.$$ By definition of the skew product $F$ and (9.4), it follows

$$(\eta(k), x_k) = (\sigma^{t_k-1}(\eta(k-1)), f_{[\eta(k-1)]}^{R_k}(x_{k-1}))$$

using (9.3) $= (\sigma^{R_k}(\eta(k-1)), f_{[\eta(k-1)]}^{R_k}(x_{k-1}))$, where $R_k \overset{\text{def}}{=} R_{\ell_k}(\alpha_k)$.

To finish the proof, it is enough to check the following.

**Claim 9.3.** The first $R_k$ symbols of $\eta^{(k-1)}$ (in the alphabet $\{1, \ldots, N\}$) form a word in $\mathcal{W}_{\ell_k}$. In particular, $f_{[\eta(k-1)]}^{R_k}$ is a map of the CIFS on $J$ defined by $\mathcal{W}_{\ell_k}$.

**Proof.** By definition of $C_{\ell_k}$, it holds $C_{\ell_k}^{-1}(\alpha_k) \in \mathcal{W}_{\ell_k}$. Note also that it is a subword of $\eta^{(k-1)}$. There are two cases to check.

**Case 1:** $\ell_k = \ell_{k-1}$. Then, $C_{\ell_k}^{-1}(\alpha_k)$ is the second element in the bi-infinite concatenation of words (in the alphabet $\mathcal{W}_{\ell_k} = \mathcal{W}_{\ell_{k-1}}$) forming $\eta^{(k-1)}$, that is, with

$$\eta^{(k-1)} = (\ldots, \eta_{i_0}^{(k-1)}, \eta_{i_1}^{(k-1)}, \ldots)$$

it holds $C_{\ell_k}^{-1}(\alpha_k) = \eta_{i_1}^{(k-1)} \in \mathcal{W}_{\ell_k}$ and $\eta^{(k)} = \sigma^{R_k}(\eta^{(k-1)}) = (\ldots, \eta_{i_0}^{(k-1)}, \eta_{i_1}^{(k-1)}, \ldots)$, and we are done.

**Case 2:** $\ell_k < \ell_{k-1}$. In this case, we recall that $\eta^{(k-1)}$ is a bi-infinite concatenation of words in $\mathcal{W}_{\ell_{k-1}},$

$$\eta^{(k-1)} = (\ldots, |\eta_{i_0}^{(k-1)}, \eta_{i_1}^{(k-1)}, \ldots)$$

where each subword is a $m_{\ell_k}$-times repeated and tailed version of words in $\mathcal{W}_{\ell_k},$

$$\eta_{i_0}^{(k-1)} = (w_{j_1}^{(\ell_k)}, w_{j_2}^{(\ell_k)}, \ldots, w_{j_{m_{\ell_k-1}}}^{(\ell_k)}, \ell_{k+1}(w_{j_1}^{(\ell_k)}, w_{j_2}^{(\ell_k)}, \ldots, w_{j_{m_{\ell_k-1}}}^{(\ell_k)})).$$

In particular, $C_{\ell_k}^{-1}(\alpha_k) = w_{j_2}^{(\ell_k)} \in \mathcal{W}_{\ell_k}$ and $\eta^{(k)} = (\ldots, w_{j_2}^{(\ell_k)}, \ldots)$.

The proof of the lemma is now complete. □

The proof of the lemma is now complete. □


10. Core of the Proof of Theorem C

This section puts together all ingredients developed throughout this paper. Let us collect the main ones to state the key result towards the proof of Theorem C.

Given $F \in \mathcal{S}^1_{\text{shyp}}(\Sigma_N \times S^1)$ and some $F$-ergodic measure $\mu$ with negative Lyapunov exponent, by Theorem 6.5 we obtain an initial collection of words $\mathcal{W}_0$ which defines a CIFS with quantifiers on some interval $J$. Given a sequence $(m_n)_n$, we define a cascade of collections of words $(\mathcal{W}_n)_n$ where each word in $\mathcal{W}_n$ is the $m_n$-repeated and tailed version of words in $\mathcal{W}_{n-1}$, the tailing map $\tau_n$ as in Theorem 7.3. By Proposition 6.3, each collection $\mathcal{W}_n$ has an associated horseshoe $\Gamma_n$ of $F$. The horseshoe length defines a corresponding roof function $R_n$.

Our construction is accompanied by a cascade of abstract alphabets $(A_n)_n$. Each $A_n$ defines a shift space, endowed with the shift map $\sigma_n = \sigma_{A_n}$ and the Bernoulli measure $\nu_n$, which forms the ground floor $\mathfrak{g}_n$ of the suspension space $S_n = S_{A_n, R_n}$, with associated suspension map $\Phi_n$ and measure $\lambda_n$, see Sect. 4.1. The measure preserving system $(S_n, \Phi_n, \lambda_n)$ is an extension of $(\Gamma_n, F, \mu_n)$ by the factor map $H_n$, see Proposition 8.4.

Proposition 10.1. Let $F \in \mathcal{S}^1_{\text{shyp}}(\Sigma_N \times S^1)$, $N \geq 2$, and consider an $F$-invariant ergodic measure $\mu$ with Lyapunov exponent $\alpha \overset{\text{def}}{=} \chi(\mu) < 0$ and positive entropy $h = h(F, \mu)$. For every $\varepsilon > 0$, and $\varepsilon_E \in (0, |\alpha|/4)$, there are a closed interval $J \subset S^1$ and an initial finite disjoint collection of words $\mathcal{W}_0 \subset \Sigma_N^+ \overset{\text{def}}{=} \Sigma_N^*$ defining a CIFS on $J$ relative to some $K > 1$, $\alpha + \varepsilon_E$, $\alpha$, and $\varepsilon_E$ satisfying

$$\min_{w \in \mathcal{W}_0} |w|(h - \varepsilon_E) \leq \log \text{card} \mathcal{W}_0 \leq \max_{w \in \mathcal{W}_0} |w|(h + \varepsilon_E).$$

Moreover, there is a sufficiently fast growing sequence of natural numbers $(m_n)_n$ such that the measure preserving systems $(S_n = S_{A_n, R_n}, \Phi_n, \lambda_n)$ satisfy the following. For every continuous function $\phi : \Sigma_N \times S^1 \to \mathbb{R}$ and $\varepsilon > 0$, there exists $L_0 = L_0(\phi, \varepsilon) \in \mathbb{N}$ such that for every $\ell \geq L_0$ and $n \geq \ell + 1$, there exists a subset $S_n, \phi, \varepsilon$ of $S_n$ such that $\lambda_n(S_n, \phi, \varepsilon) > 1 - \varepsilon$ and for every $(a, s) \in S_n, \phi, \varepsilon$ it holds

$$\left| \frac{1}{\mathcal{R}_\ell} \sum_{k=0}^{\mathcal{R}_\ell-1} \psi_n(\Phi_n^s(a, s)) - \int \phi d\mu \right| < \varepsilon,$$

where

$$\mathcal{R}_\ell \overset{\text{def}}{=} \int R_n \, db_n \quad \text{and} \quad \psi_n : S_n \to \mathbb{R}, \quad \psi_n(a, s) \overset{\text{def}}{=} (\phi \circ H_n)(a, s).$$

The proof of the above proposition will be split into subsections.

First, note that by Theorem 6.5 there exist a closed interval $J \subset S^1$ and a finite disjoint collection of words $\mathcal{W}_0 \subset \Sigma_N^*$ defining a CIFS on $J$ relative to some constant $K > 1$ and $\alpha + \varepsilon_E$, $\alpha$, and $\varepsilon_E$, and also satisfying (10.1). Let $A_0 = \mathcal{W}_0$.

10.1. Choice of the fast growing sequence $(m_n)_n$. We start by fixing a dense sequence of continuous functions $\phi_k : \Sigma_N \times S^1 \to \mathbb{R}$, $k \in \mathbb{N}$.

The sequence $(m_n)_n$ is defined inductively over $n \in \mathbb{N}_0$. Let $m_0 = 1$. Suppose that for $n \in \mathbb{N}$ all numbers $\{m_0, m_1, \ldots, m_{n-1}\}$ are chosen and hence $\mathcal{W}_{n-1}$ and $A_{n-1}$ are defined and verify:
• the collection $\mathcal{W}_{n-1}$ defines a CIFS on $J$ relative to $K \geq 1$, $2^{-n}(\alpha + \varepsilon_E)$, $2^{-n}\alpha$, and $2^{-n}\varepsilon_E$,
• there are the associated attractor $\Lambda_{n-1}$ for the CIFS and the horseshoe $\Gamma_{n-1} \supset \Lambda_{n-1}$ generated by it, see Proposition 6.3,
• the word length on $\mathcal{W}_{n-1}$ defines the roof function $R_{n-1}$,
• the abstract Bernoulli shift $((\mathcal{A}_{n-1})^\mathbb{Z}, \sigma_{n-1}, b_{n-1})$ forms the ground floor $S_{n-1}$ for the measure preserving suspension system $(S_{n-1}, \Phi_{n-1}, \lambda_{n-1})$ (Sect. 4.1). This is an entropy-preserving extension of $(\Gamma_{n-1}, F, \mu_{n-1})$ by the factor map $H_{n-1}$ (Proposition 8.4).

To define $m_n$, for $k \in \{1, \ldots, n\}$ consider the auxiliary lifted potentials

$$\psi_{n-1,k} : S_{n-1} \to \mathbb{R}, \quad \text{where} \quad \psi_{n-1,k}(a, s) \overset{\text{def}}{=} (\phi_k \circ H_{n-1})(a, s),$$

and let

(I) (controlled large deviation) $N_0(\psi_{n-1,k}, 2^{-n}) \in \mathbb{N}$ be as in Proposition 4.3 applied to $A_{n-1}$ and $R_{n-1}$,

(II) (controlled distortion) $N_1(\phi_k, 2^{-n}) \in \mathbb{N}$ be as in Proposition 6.12 applied to $\mathcal{W}_{n-1}$,

(III) (tailing map) $N_2 = N_2(\mathcal{W}_{n-1}) \in \mathbb{N}$ be as in Theorem 7.3 applied to $\mathcal{W}_{n-1}$.

We now define $m_n$ by

$$m_n \overset{\text{def}}{=} \max \left\{ \max_{k=1, \ldots, n} N_0(\psi_{n-1,k}, 2^{-n}), \max_{k=1, \ldots, n} N_1(\phi_k, 2^{-n}), N_2, m_{n-1} \right\} + 1.
$$

We define $\mathcal{W}_n$ as the $m_n$-times repeated and tailed version of words in $\mathcal{W}_{n-1}$,

$$\mathcal{W}_n = (\mathcal{W}_{n-1}^{m_n})_{t_n},$$

with the tailing map $t_n$ as in Theorem 7.3. We also let $A_n = (A_{n-1})^{m_n}$. This finishes the inductive definition.

Note that by Corollary 8.3 we have that Assumption 5.2 about the roof functions $R_n$ is satisfied taking $K = L_1|\alpha|$.

10.2. General scheme of the proof of Proposition 10.1. Let us sketch the steps of the proof and recall the main ingredients which will be implemented, compare also Fig. 3.

We first show that the sequence $(\mu_n)_n$ converges in the weak* topology, see Sect. 10.3.

To prove the proposition, we need to show the approximation property (10.2) for a sufficiently large set of points. To do so, given $\phi$ first find $k_0 \in \mathbb{N}$ such that $\phi_{k_0}$ from our dense family is close to it. Choose large $\ell \geq k_0$ and let $n \geq \ell + 1$, see Sect. 10.4.

By implementing item (I) above, controlled large deviation on level $\ell - 1$ provides us a large set of good orbit pieces on $S_{\ell-1}$ which (up to $m_\ell$ consecutive times) run from the $(\ell - 1)$st level ground floor to its roof. Here each piece has a close-to-expected length and a close-to-expected finite Birkhoff sum of the lift of the potential $\phi_{k_0}$ to $S_{\ell-1}$. As $\phi_{k_0}$ and $\phi$ are close, these properties extend to the lift of $\phi$. See Sect. 10.5.

Consider the principal part $\mathcal{P}_{n}(\ell) \subset S_{n}$ defined in (5.14) which decomposes into strips $L_{n}(\ell, a)$ indexed by $(\ell, a)$-addresses

$$\mathcal{P}_{n}(\ell) = \bigcup_{a=(a_{\ell}, \ldots, a_{n-1})} L_{n}(\ell, a).$$
Every strip with \((\ell, n)\)-address \(a\) decomposes into \((\ell - 1, n)\)-substrips

\[
\mathcal{L}_n^{(\ell, a)} = \bigcup_{j=0,\ldots,m_{\ell-1}} \mathcal{L}_n^{(\ell-1, ja)}
\]

which start at the corresponding intermediate floors \(S_n^{(\ell-1, ja)}\). Recall the definition of the map \(L_n^{(\ell-1, ja)}\) in (5.12) mapping the “model suspension space” \(S_{\ell-1}\) bijectively onto the substrip \(\mathcal{L}_n^{(\ell-1, ja)}\). In this way, each good orbit piece obtained by controlled large deviation is sent to its counterpart on the level \(n\)-suspension space \(S_n\). This is more precisely stated in Main Lemma 10.6 in Sect. 10.6 whose proof is postponed to Sect. 10.8.

Assuming Main Lemma 10.6, in Sect. 10.7 we conclude the proof of the proposition. The following are the main ingredients. The Bernoulli measure \(b_{\ell-1}\) on the \((\ell - 1)\)st level ground floor lifts isomorphically to the Bernoulli measure \(b_n\) on the \(n\)th level ground floor which, in turn, lifts naturally to its copy \(b_n^{(\ell, a)}\) on the \((\ell, n)\)-intermediate floor. This allows us to conclude that the large measure set on the model space has its large measure counterpart on each strip. Stitching together all strips provides a large measure subset of the principal part \(P_n^{(\ell)}\). Finally we will see that, by construction, the tail part \(T_n^{(\ell)}\) has comparably small measure.
10.3. Weak∗ convergence of the factor measures μₙ. Our construction provides sequences of probability measures (μₙ)ₙ = (Hₙ)ₙλₙ ∈ Mₜₜ(F), see (8.10). We first show that this sequence converges in the weak∗ topology.

**Lemma 10.2.** The sequence (μₙ)ₙ converges in the weak∗ topology.

Let us first state a preliminary result which is a direct consequence of item (II) above about distortion control, together with Proposition 5.12 (1)–(2). Recall the definition of the sum ∆ψₙ,k in (4.2) and in its variation varₜₜ in (4.3). We consider the abstract alphabets Aₙ and, for simplicity, write [·]ₙ for the cylinder in the sequence space (Aₙ)Z.

**Claim 10.3.** For every n ∈ ℕ and k ∈ {1, . . . , n} it holds

\[
\text{var}_{Aₙ}(\Delta \psiₙ,k) ≤ \frac{1}{2^n} mₙ \max Rₙ-1 ≤ L² \frac{1}{2^n} \mathcal{R}_n.
\]

Moreover, reformulating the above taking into account the bijection Sₙ,n₋₁ between Aₙ and (Aₙ₋₁)ₘₙ, for every a ∈ Aₙ it holds

\[
\max_{a, a' \in [a]ₙ} \left| \sum_{i=0}^{mₙ-1} \Delta \psiₙ₋₁,k(\sigma₋₁ₙ₋₁(S₋₁ₙ₋₁(a))) - \Delta \psiₙ,k(a') \right| ≤ L² \frac{1}{2^n} \mathcal{R}_n.
\]

**Proof of Lemma 10.2.** It suffices to show that for the dense sequence of continuous functions (φₖ)ₖ, it holds

\[
\left| \int φₖ dμₙ - \int φₖ dμₙ₋₁ \right| < C(k, n), \quad (10.3)
\]

for some summable sequence (C(k, n))ₙ. Recall that bₙ([a]ₙ) = M₋₁ₙ₋₁ for every a ∈ Aₙ. Covering the sequence space (Aₙ)Z by the cylinders {[a]ₙ : a ∈ Aₙ}, it holds

\[
\frac{1}{\mathcal{R}_n} \sum_{a ∈ Aₙ} \min_{a ∈ [a]ₙ} \Delta \psiₙ,k(a) \frac{1}{Mₙ} \leq \int φₖ dμₙ \quad (10.4)
\]

together with the analogous upper bound. Analogously, covering the sequence space (Aₙ₋₁)Z by cylinders of length mₙ, it holds

\[
\int φₖ dμₙ₋₁ ≤ \frac{1}{\mathcal{R}_n₋₁} \sum_{(b₁, ..., bₘₙ)} \max \frac{1}{mₙ} \sum_{i=0}^{mₙ-1} \Delta \psi₋₁ₙ₋₁,k(\sigma₋₁ₙ₋₁(b)) \frac{1}{M₋₁ₙ₋₁} , \quad (10.5)
\]

where the maximum is taken over all sequences b in the cylinder [b₁, ..., bₘₙ]ₙ₋₁ ⊂ (Aₙ₋₁)Z and the sum is taken over all (b₁, ..., bₘₙ) ∈ (Aₙ₋₁)ₘₙ. The lower bound is analogous.

By Proposition 5.12 (3), it holds

\[
\frac{1}{\mathcal{R}_n} < \frac{1}{mₙ \mathcal{R}_n₋₁} < \frac{1}{\mathcal{R}_n} + \frac{1}{\mathcal{R}_n} L² \frac{1}{2^n}. \quad (10.6)
\]
We now estimate (10.3), let us compare (10.5) with (10.4). Applying (10.6) and taking into consideration the bijective map $S_{n,n-1}$ between $A_n$ and $(A_{n-1})^{m_n}$, together with $m_{n-1} = M_n, m_n$ max $R_{n-1} < \max R_n$, and Claim 10.3, it follows

\[
\left| \frac{1}{m_n R_{n-1}} \sum_{b \in (A_{n-1})^{m_n}} \max_{b \in [b]_{n-1}} \sum_{i=0}^{m_n-1} \Delta \psi_{n-1,k}(\sigma_{n-1}^i(b)) - \frac{1}{R_n} \sum_{a \in A_n} \min_{a \in [a]_n} \Delta \psi_{n,k}(a) \right| \\
\leq \frac{1}{R_n} \left| \sum_{b \in (A_{n-1})^{m_n}} \max_{b \in [b]_{n-1}} \sum_{i=0}^{m_n-1} \Delta \psi_{n-1,k}(\sigma_{n-1}^i(b)) - \sum_{a \in A_n} \min_{a \in [a]_n} \Delta \psi_{n,k}(a) \right| \\
+ \frac{L_2}{2^n} \cdot (\card A_{n-1})^{m_n} \cdot m_n \max R_{n-1} \cdot \| \phi_k \| \\
\leq \frac{1}{R_n} \cdot \card A_n \cdot L_2 \cdot \frac{1}{2^n} \cdot m_n \cdot \max R_n \cdot \| \phi_k \| \\
\leq \frac{1}{R_n} \cdot \card A_n \cdot L_2 \cdot \frac{1}{2^n} \cdot m_n \cdot \max R_n \cdot \| \phi_k \| \\
\leq M_n \cdot C(k, n), \quad \text{where } C(k, n) \equiv \left\{ L_2 \frac{1}{2^n} + L_2 \frac{1}{2^n} \left\| \phi_k \right\| \right\}
\]

where for the last estimate we used (8.2) $\card A_n = M_n$ and Proposition 5.12 (2). The analogous estimate holds exchanging max and min.

Clearly, $(C(k, n))_n$ is summable. Combining the estimates (10.4) and (10.5) of the integrals, it follows (10.3). This finishes the proof. □

10.4. Choice of quantifiers. Fix a continuous map $\phi: \Sigma_N \times S^1 \to \mathbb{R}$ and $\epsilon \in (0, 1/3)$. Choose $k_0 \in \mathbb{N}$ and $L_0 = L_0(\phi, \epsilon) \geq k_0 + 2$ so that for all $\ell \geq L_0$

\[
\| \phi_{k_0} - \phi \| < \epsilon, \quad \frac{L_2}{2^{\ell-2}} \max \{ 1, \| \phi \| \} < \epsilon, \quad 2\| \phi \| \frac{1}{\ell} L_2 < \epsilon.
\] (10.7)

Hence, for every $n \geq L_0$ and $k \in \{1, \ldots, n\}$ the assertions in (I)–(III) apply to the function $\phi_k$ and its lift $\psi_{n-1,k} = \phi_k \circ H_{n-1}$ and $m_n$. By Lemma 10.2, we can assume that $L_0 \in \mathbb{N}$ is large enough that for every $\ell \geq L_0$ it holds

\[
\left| \int \phi d\mu_{\ell-1} - \int \phi d\mu \right| < \epsilon.
\] (10.8)

10.5. Invoking assertions (I)–(II) to restate large deviation control. Let us restate the estimate in item (I) in a more convenient way.

**Lemma 10.4** (Controlled large deviation). For every $\ell \geq L_0 + 1$ there exists a set

\[
A \subset (A_{\ell-1})^2 \quad \text{satisfying} \quad b_{\ell-1}(A) > 1 - \epsilon
\]
so that for every \( b \in A, i = 0, \ldots, m \ell - 1, \) and \( k \in \{1, \ldots, m \ell \} \) we have

\[
\left| \sum_{j=i}^{i+k-1} \left( R_{\ell-1}(\sigma_{\ell-1}^j(b)) - \mathcal{R}_{\ell-1} \right) \right| < m \ell \frac{1}{2^\ell} < m \ell \varepsilon \tag{10.9}
\]

and for \( \psi_{\ell-1} \overset{\text{def}}{=} \phi \circ H_{\ell-1} \) it holds

\[
\left| \sum_{j=i}^{i+k-1} \left( \Delta \psi_{\ell-1}(\sigma_{\ell-1}^j(b)) - \int \Delta \psi_{\ell-1} \, db_{\ell-1} \right) \right| < 2 \varepsilon L_2 \mathcal{R}_\ell + 2 \varepsilon k. \tag{10.10}
\]

**Proof.** As \( \ell - 1 \geq L_0, \) by assertion (I), Proposition 4.3 applied to \( \psi_{\ell-1,k_0}, 2^{-\ell} < \varepsilon, A_{\ell-1}, \) and \( R_{\ell-1} \) let us control large deviations on certain orbits and “transport” them to level \( n. \) If \( \ell \) so that for every \( b \in A, i = 0, \ldots, m \ell - 1, \) and \( k \in \{1, \ldots, m \ell - 1\} \) we have

\[
\left| \sum_{j=i}^{i+k-1} \left( \Delta \psi_{\ell-1,k_0}(\sigma_{\ell-1}^j(b)) - \int \Delta \psi_{\ell-1,k_0} \, db_{\ell-1} \right) \right| < m \ell (2 \var_{A_{\ell-1}}(\Delta \psi_{\ell-1,k_0}) + \varepsilon).
\]

To estimate the right hand side, apply Claim 10.3 and Proposition 5.12 (3) to get

\[
2m \ell \var_{A_{\ell-1}}(\Delta \psi_{\ell-1,k_0}) \leq m \ell L_2 \frac{1}{2^{\ell-2}} \mathcal{R}_{\ell-1} < \mathcal{R}_\ell \frac{L_2}{2^{\ell-2}}.
\]

Analogously, using also \( m \ell < m \ell \max R_{\ell-1} < L_2 \mathcal{R}_\ell \) and then (10.7), it follows

\[
\left| \sum_{j=i}^{i+k-1} \left( \Delta \psi_{\ell-1,k_0}(\sigma_{\ell-1}^j(b)) - \int \Delta \psi_{\ell-1,k_0} \, db_{\ell-1} \right) \right| < 2 \varepsilon L_2 \mathcal{R}_\ell.
\]

Finally, to substitute the approximating function \( \psi_{\ell-1,k_0} \) by \( \psi_{\ell-1} \), using the first estimate in (10.7) we get \( \| \psi_{\ell-1,k_0} - \psi_{\ell-1} \| < \varepsilon \) and hence we obtain (10.10). \( \square \)

### 10.6. Transporting good orbits from \( S_{\ell-1} \) to \( S_n \)

We now study appropriate subsets of the principal part \( \mathcal{P}_n \) of the suspension space \( S_n. \) We invoke Lemma 10.4 on level \( \ell - 1 \) to control large deviations on certain orbits and “transport” them to level \( n. \) Note that if \( a \) is an \((\ell, n)\)-address, then \( 0 a \) is an \((\ell - 1, n)\)-address. Given \( A \subset (A_{\ell-1})^Z \) as in Lemma 10.4, recalling the definition of the map \( L_n^{(\ell,0)a} \) in (5.12), let

\[
A_n^{(\ell,0)a} \overset{\text{def}}{=} L_n^{(\ell,0)a}(A \times \{0\}) = P_{n,\ell-1}^{-1}(A \times \{0\}) \subset \mathcal{G}_n^{(\ell,0)a} = \mathcal{G}_n^{(\ell,a)}.
\]

Recall the definition of the measure \( b_n^{(\ell,a)} \) in (5.18). By Lemma 5.11, the Bernoulli measure \( b_{\ell-1} \) lifts naturally to its copy \( b_n^{(\ell,0)a} \) and it holds

\[
b_n^{(\ell,0)a}(A_n^{(\ell,0)a}) = (b_{\ell-1} \circ p_{\ell-1} \circ P_{n,\ell-1}^{-1})(P_{n,\ell-1}^{-1}(A \times \{0\})) = b_{\ell-1}(A) > 1 - \varepsilon. \tag{10.11}
\]
Let us consider the following set of addresses

$$
\mathcal{A}_n^\ell \overset{\text{def}}{=} \left\{ a = (a_\ell, \ldots, a_{n-1}) : a_\ell \in \{0, \ldots, m_{\ell+1} - 2\}, a_k \in \{0, \ldots, m_{k+1} - 1\} \text{ for } k \neq \ell \right\}
$$

(10.12)

to which our following arguments can be applied. This restriction on \(a_\ell\) will be explained in the beginning of the proof of Main Lemma 10.6 in Sect. 10.8. Given \(a \in \mathcal{A}_n^\ell\), consider all points whose orbits start in \(A_n^{(\ell-1,0)a}\) and pass through the corresponding “good set” \(A_n^{(\ell-1,0a+1,\ell)}\) in the adjacent intermediate floor:

$$
B_n^{(\ell-1,0a)} \overset{\text{def}}{=} \left\{ \zeta \in A_n^{(\ell-1,0a)} : \Phi_n^{R_\ell \circ P_n,\ell} (\zeta) \in A_n^{(\ell-1,0a+1,\ell)} \right\} \subset \mathcal{L}_n^{(\ell,a)}.
$$

(10.13)

Indeed, by our restriction on the index \(a_\ell\) in address \(a\) this adjacent address \(0a + 1_\ell\) is admissible. Analogously to (10.11), replacing \(0a\) by \(0a + 1_\ell\), it holds

$$
b_n^{(\ell-1,0a+1,\ell)} (A_n^{(\ell-1,0a+1,\ell)}) > 1 - \varepsilon.
$$

Applying again Lemma 5.11, the following holds.

**Claim 10.5.** For every \(a \in \mathcal{A}_n^\ell\), it holds

$$
b_n^{(\ell-1,0a)} (B_n^{(\ell-1,0a)}) > 1 - 2\varepsilon.
$$

Consider the set of points in the substrip \(C_n^{(\ell,a)}\) whose orbit starts in \(B_n^{(\ell-1,0a)}\),

$$
C_n^{(\ell,a)} \overset{\text{def}}{=} \left\{ \Phi_n^k (\zeta) : \zeta \in B_n^{(\ell-1,0a)}, k \in \mathbb{N}_0 \right\} \cap \mathcal{L}_n^{(\ell,a)}.
$$

As the strips \(\mathcal{L}_n^{(\ell,a)}\) are pairwise disjoint for different \((\ell, n)\)-addresses, these sets are also pairwise disjoint.

**Main Lemma 10.6.** There are constants \(C = C(\phi) > 1\) and \(L'_0 \geq L_0(\phi, \varepsilon)\) such that for every \(\ell \geq L'_0, n \geq \ell + 1\), and \((\ell, n)\)-address \(a \in \mathcal{A}_n^\ell\) it holds

(i) (Birkhoff averages) for every \(\zeta \in C_n^{(\ell,a)}\)

$$
\left| \frac{1}{\mathcal{R}_\ell} \sum_{s=0}^{\mathcal{R}_\ell-1} \psi_n (\Phi_n^s (\zeta)) - \int \phi \, d\mu \right| < C\varepsilon.
$$

(ii) (Expected roof functions) for every \(\zeta \in B_n^{(\ell-1,0a)}\)

$$
\left| \sum_{i=0}^{m_{\ell-1}-1} \left( R_{\ell-1} \circ \sigma_{\ell-1}^i \circ p_{\ell-1} \circ P_{n,\ell-1} \right) (\zeta) - m_{\ell} \mathcal{R}_{\ell-1} \right| < m_{\ell}\varepsilon.
$$

We postpone the proof of Main Lemma 10.6 to Sect. 10.8.
10.7. End of the proof of Proposition 10.1. Let \( \ell \geq L'_0, n \geq \ell + 1 \), and \( a \in \varOmega_n^\ell \). By Main Lemma 10.6, every point in the set \( C_n^{(\ell, a)} \) satisfies the claimed approximation property of its \( \mathcal{R}_\ell \)-Birkhoff sum. To finish the proof of the proposition, we need to show that this set has large \( \lambda_n \)-measure.

**Lemma 10.7.** For every \( \ell \geq L'_0 \) sufficiently large and \( n \geq \ell + 1 \) it holds

\[
\lambda_n \left( \bigcup_{a \in \varOmega_n^\ell} C_n^{(\ell, a)} \right) > 1 - 3\varepsilon.
\]

**Proof.** To estimate the \( \lambda_n \)-measure of the union of all such points, first note that

\[
(b_n \times m) \left( \bigcup_{a \in \varOmega_n^\ell} C_n^{(\ell, a)} \right) = \sum_{a \in \varOmega_n^\ell} (b_n \times m) (C_n^{(\ell, a)})
\]

using (5.18) \[\int_{B_n^{(\ell-1,0)a}} m \left( C_n^{(\ell, a)} \cap ([\xi] \times \mathbb{N}) \right) d b_n^{(\ell-1,0a)}(\xi)\] by Lemma 5.11

\[
= \sum_{a \in \varOmega_n^\ell} \int_{B_n^{(\ell-1,0)a}} \sum_{i=0}^{m_{\ell-1}} \left( R_{\ell-1} \circ \sigma_{\ell-1}^i \circ P_{\ell-1} \circ P_n, \ell-1 \right) d \left( b_{\ell-1} \circ (p_{\ell-1} \circ P_n, \ell-1) \right)
\]

by Main Lemma 10.6 \[\geq \sum_{a \in \varOmega_n^\ell} b_n^{(\ell-1,0a)} (B_n^{(\ell-1,0a)}) \cdot m_{\ell} (\mathcal{R}_{\ell-1} - \varepsilon)\]

by Claim 10.5 \[= \text{card } \varOmega_n^\ell \cdot (1 - 2\varepsilon) \cdot m_{\ell} (\mathcal{R}_{\ell-1} - \varepsilon)\]

by (10.12) \[> m_n \cdot \ldots \cdot m_{\ell+2} (m_{\ell+1} - 1) \cdot (1 - 2\varepsilon) \cdot m_{\ell} (\mathcal{R}_{\ell-1} - \varepsilon)\]

\[> m_n \cdot \ldots \cdot m_{\ell} \mathcal{R}_{\ell-1} \left( 1 - \frac{1}{m_{\ell+1}} \right) \cdot (1 - 2\varepsilon) (1 - \varepsilon).\]

On the other hand, by (5.14),

\[
(b_n \times m) (\mathcal{S}_n) = (b_n \times m) (\mathcal{P}_n^{(\ell-1)} \cup \mathcal{T}_n^{(\ell-1)})
\]

\[= (b_n \times m) (\mathcal{P}_n^{(\ell-1)}) + (b_n \times m) (\mathcal{T}_n^{(\ell-1)})\]

\[= \sum_a \sum_{j=0}^{m_{\ell-1}} (b_n \times m) (\mathcal{C}_{n}^{(\ell-1),ja}) + (b_n \times m) (\mathcal{T}_n^{(\ell-1)}),\]

where the sum is taken over all \((\ell, n)\)-addresses \(a\). To estimate \((b_n \times m) (\mathcal{T}_n^{(\ell-1)})\), recall Remark 5.7 about the length of tails added at each step. To simplify the estimate, as the formal localization of the intermediate floors where tails are added is rather involved, we use again Lemma 5.11 to “move between the measures” on intermediate floors. Together with Proposition 5.12 (4) we get

\[
(b_n \times m) (\mathcal{T}_n^{(\ell-1)}) = \sum_{k=\ell}^{n} m_n \cdot \ldots \cdot m_k \int |t_{k-1}| \, db_{k-1} \leq L_2 \sum_{k=\ell}^{n} m_n \cdot \ldots \cdot m_k \mathcal{R}_{k-1} \cdot \frac{1}{2^{k-1}}.
\]

Since by Proposition 5.12 (3) it holds

\[
\mathcal{R}_{k-1} < (1 + L_2 \cdot \frac{1}{2^{k-1}}) m_{k-1} \mathcal{R}_{k-2},
\]

we can bound the sum by a geometric series.

\[
\sum_{k=\ell}^{n} \left( 1 + L_2 \cdot \frac{1}{2^{k-1}} \right) m_{k-1} \mathcal{R}_{k-2} \leq \frac{1}{1 - \frac{1}{2}} \mathcal{R}_{\ell-2}.
\]

Combining this with the previous estimate, we obtain

\[
\lambda_n \left( \bigcup_{a \in \varOmega_n^\ell} C_n^{(\ell, a)} \right) > 1 - 3\varepsilon.
\]
it follows
\[(b_n \times m)(\g_T^{(\ell-1)}) < L_2 \sum_{k=\ell}^{n} m_n \cdots m_{\ell} \mathcal{R}_{\ell-1} \cdot \prod_{j=\ell+1}^{k} \left(1 + L_2 \frac{1}{2^{j-1}} \right) \leq \frac{4L_2^2 m_n \cdots m_{\ell} \mathcal{R}_{\ell-1}}{2^\ell} \cdot e^{2L_2/2^\ell}.
\]

On the other hand, using analogous estimates for the principal (that is, non-tail) part
\[(b_n \times m)(\g_T^{(\ell-1)}) = \sum_{a} \sum_{j=0}^{m_{\ell-1}} \int (\mathcal{R}_{\ell-1} \circ p_{\ell-1} \circ P_{n,\ell-1}) d\nu_n^{(\ell-1),ja} \leq \sum_{a} \sum_{j=0}^{m_{\ell-1}} \mathcal{R}_{\ell-1} = (m_n \cdots m_{\ell+1}) \cdot m_{\ell} \cdot \mathcal{R}_{\ell-1}.
\]

Putting together the previous estimates, we obtain
\[
\lambda_n \left( \bigcup_{a \in \mathcal{A}^\ell_n} C_n^{(\ell,a)} \right) = \frac{(b_n \times m) \left( \bigcup_{a \in \mathcal{A}^\ell_n} C_n^{(\ell,a)} \right)}{(b_n \times m)(\g_T^{(\ell-1)})} \geq \frac{(1 - \frac{1}{m_{\ell+1}})(1 - 2\varepsilon)(1 - \varepsilon)}{1 + 4L_2^2 2^{-\ell} e^{2L_2/2^\ell}}.
\]

To conclude the proof, it suffices to take $L_0' \geq L_0$ sufficiently large. \(\square\)

To conclude the proof of Proposition 10.1 it remains to prove Main Lemma 10.6.

### 10.8. Proof of main Lemma 10.6
Before starting the proof, let us sketch its main steps. The elements of the alphabet $\mathcal{A}_n$ are obtained by concatenating elements on the lower level $\ell - 1$. The substitution map $S_{n,\ell-1}$ translates between $\mathcal{A}_n$ and $\mathcal{A}_{\ell-1}$. Accordingly, words in the collection $\mathcal{W}_n$ are obtained by our repeat-and-tail procedure applied to words on each lower level, in particular on level $\ell - 1$. By construction, the images of good orbit pieces under the factor $H_{\ell-1}$ are sufficiently close to the images of their counterparts under the factor $H_n$. By implementing controlled distortion, this allows us to compare the finite Birkhoff sums of $\phi$ of their lifts on $S_{\ell-1}$ with their counterparts on $S_n$.

The large deviation result Proposition 4.3 was obtained for (at most $m_{\ell}$ consecutive Birkhoff sums on level $\ell - 1$. This corresponds to taking concatenated orbit pieces on consecutive substrips $\mathcal{L}_n^{(\ell-1),ja}$ that stretch over two adjacent strips $\mathcal{L}_n^{(\ell,a)}$ and $\mathcal{L}_n^{(\ell,a+1)}$. Together they will form an orbit piece of close-to-expected length $\mathcal{R}_\ell$. This now explains our choice of addresses $\mathcal{A}^\ell_n$ in (10.12): if we started from inside the last strip, $a_{\ell} = m_{\ell+1} - 1$, then what follows after it is not the next strip but the tail.

Fix some $(n, \ell)$-address $a \in \mathcal{A}^\ell_n$. Consider a point
\[
\gamma \in B_n^{(\ell-1,0a)} \subset \mathcal{G}_n^{(\ell,a)}.
\]

We will show that item (ii) in Main Lemma is true for every such $\gamma$ (Lemma 10.11). We also show that item (i) in Main Lemma holds for every point in the slice which is “in the same fiber” of the suspension space as $\gamma$, that is, for every
\[
\zeta \in \mathcal{L}_n^{(\ell,a)} \quad \text{so that} \quad p_n(\zeta) = p_n(\gamma).
\]

To prove the lemma, in Step 2 we first consider $\zeta$ in some intermediate floor, see Lemma 10.9. The general case is concluded in Step 3, see Lemma 10.10. But first in Step 0 we fix some notation and in Step 1 we implement Sect. 9.
Fig. 4. Points “in the same fiber” as \( \gamma = \zeta_0 \) in the suspension space \( S_n \) project to their counterparts in the model space \( S_{\ell-1} \).

**Step 0: Auxiliary codification of orbits.** For the following see Fig. 4. For \( \gamma \) as in (10.14), let

\[
\mathfrak{a} \overset{\text{def}}{=} \mathsf{p}_n(\gamma) \in (\mathcal{A}_n)^\mathbb{Z}.
\]

Recalling Lemma 5.10 which expresses the unique point of the intersection of \( \mathfrak{a} \)-fiber with the intermediate floor \( S_n^{(\ell \cdot \mathfrak{a})} \), for \( j = 0, \ldots, m_\ell - 1 \) write

\[
\zeta_j \overset{\text{def}}{=} (\mathfrak{a}, s_n^{(\ell-1,j\mathfrak{a})}(\mathfrak{a})) \in S_n^{(\ell-1,j\mathfrak{a})} \subset \mathcal{L}_n^{(\ell,\mathfrak{a})} \subset S_n.
\] (10.15)

Note that

\[
\zeta_0 = (\mathfrak{a}, s_n^{(\ell-1,0\mathfrak{a})}(\mathfrak{a})) = (\mathfrak{a}, s_n^{(\ell,\mathfrak{a})}(\mathfrak{a})) = \gamma.
\]

Note that every \( \zeta_j \) is in the orbit of \( \zeta_0 = \gamma \) (with respect to the suspension map \( \Phi_n \)). Analogously, choose points on intermediate floors within the adjacent slice addressed by \( \mathfrak{a} + 1_\ell \) by letting

\[
w_j \overset{\text{def}}{=} (\mathfrak{a}, s_n^{(\ell-1,j(\mathfrak{a}+1_\ell))}(\mathfrak{a})) \in S_n^{(\ell-1,j(\mathfrak{a}+1_\ell))} \subset \mathcal{L}_n^{(\ell,\mathfrak{a}+1_\ell)}
\]

and note that \( w_j \) is also on the orbit of \( \gamma \).

To define the counterparts of \( \zeta_0, w_0 \) on the model space, let

\[
b \overset{\text{def}}{=} S_{n,\ell-1}(\mathfrak{a}) = (\mathsf{p}_{\ell-1} \circ P_{n,\ell-1})(\zeta_0) \in (\mathcal{A}_{\ell-1})^\mathbb{Z}
\]

\[
c \overset{\text{def}}{=} (\mathsf{p}_{\ell-1} \circ P_{n,\ell-1})(w_0) \in (\mathcal{A}_{\ell-1})^\mathbb{Z}
\]
(recall the definition of $S_{n,t-1}$ in (5.3) and compare Fig. 4). Recalling the choice of $A \subset (A_{t-1})^Z$ in Lemma 10.4 and the definition of $B_n^{(t-1,0a)}$ in (10.13), it holds $b, c \in A$. For $j = 0, \ldots, m_t - 1$ let
\begin{equation}
\begin{aligned}
\xi_j &\overset{\text{def}}{=} (\sigma_{t-1}^j(b), 0) \in S_{t-1}, \\
n_j &\overset{\text{def}}{=} (\sigma_{t-1}^j(c), 0) \in S_{t-1}.
\end{aligned}
\end{equation}

One checks that, using the notation (5.17), it holds
\begin{equation}
\begin{aligned}
S_{n_j}^{(j+1)a+1}(a) &= b_n^{(j+1)}, \quad \text{where } b = (\ldots, b_{j-1}^{(j)} | b_0^{(j)}, \ldots), \\
n_{n_j}^{(j+1)a+1}(a) &= c_n^{(j+1)}, \quad \text{where } c = (\ldots, c_{j-1}^{(j)} | c_0^{(j)}, \ldots).
\end{aligned}
\end{equation}

Also let
\begin{equation}
\begin{aligned}
r_j &\overset{\text{def}}{=} R_{t-1}(b_n^{(j)}), \\
r_{n_j} &\overset{\text{def}}{=} R_{t-1}(c_n^{(j)}).
\end{aligned}
\end{equation}

Finally, by the estimate of the maximal length of a tail added at level $\ell$ in Proposition 5.12 (4) and using (10.7), it holds
\begin{equation}
\begin{aligned}
S_{n_j}^{(j+1)a+1}(a) - (S_{n_j}^{(j+1)a+1}(a) + r_{n_j}^{(j+1)a+1}) \leq \max(|\xi_j| \\
&\leq L_2 \frac{1}{2^\ell} \mathcal{R}_{n_j} < \varepsilon \cdot \mathcal{R}_{n_j}.
\end{aligned}
\end{equation}

**Step 1: Implementing the internal structure of horseshoes.** Let us now use Sect. 9. Recall the factor map $H_{t-1}: S_{t-1} \to \Gamma_{t-1}$. By definition of $\xi_j, n_j$ together with Lemma 9.2, we have
\begin{equation}
\begin{aligned}
H_{t-1}(\xi_j) &= \Sigma_N - \times [C_{t-1}^\ell(b_n^{(j)})]^+ \times J, \\
H_{t-1}(n_j) &= \Sigma_N - \times [C_{t-1}^\ell(c_n^{(j)})]^+ \times J.
\end{aligned}
\end{equation}

By definition, the above points are in $\Lambda_{t-1}$. By definition of $\xi_j$ in (10.15) together with Lemma 9.2 it follows, it holds
\begin{equation}
\begin{aligned}
H_n(\xi_j) &= H_n(a, s_{n_j}(a)) \in \Sigma_N - \times [(C_{t-1}^\ell \circ S_{n_j}^{(j+1)a})(\tilde{a})]^+ \times J, \\
&\quad \text{using (10.17)} \quad \Sigma_N - \times [C_{t-1}^\ell(b_n^{(j)})]^+ \times J.
\end{aligned}
\end{equation}

Analogously,
\begin{equation}
\begin{aligned}
H_n(n_j) &= \Sigma_N - \times [C_{t-1}^\ell(c_n^{(j)})]^+ \times J.
\end{aligned}
\end{equation}

Note that $H_n(\xi_j), H_n(n_j)$ are points in $\Lambda_n$.

Proposition 6.12 implies the following key distortion estimate. For its statement and proof we use the usual notation for a Birkhoff sum $S_n \varphi = \varphi + \varphi \circ G + \cdots + \varphi \circ G^{n-1}$; the map $G$ is given by the context. Recall that $\psi_n$ is the lift of $\phi$ to $\tilde{S}_n$. 

Claim 10.8. With the notation above, for every \( j = 0,\ldots, m_\ell - 1 \) it holds
\[
\left| S_{r_j} \psi_n(\zeta_j) - S_{r_\ell-1}(\zeta_j) \right| = \left| S_{r_j} \phi(H_n(\zeta_j)) - S_{r_\ell-1}(\zeta_j) \right| < \varepsilon r_j,
\]
\[
\left| S_{s_j} \psi_n(w_j) - S_{s_\ell-1}(w_j) \right| = \left| S_{s_j} \phi(H_n(w_j)) - S_{s_\ell-1}(w_j) \right| < \varepsilon s_j.
\]

Step 2: Birkhoff sums for \( \zeta_j \).

Lemma 10.9. Item (i) in Main Lemma 10.6 holds for every \( \zeta_j, j \in \{0,\ldots, m_\ell - 1\} \), taking \( C = C_0 \overset{\text{def}}{=} 5\|\phi\| + 8 + 4L_2 \).

Proof. Given \( j \in \{0,\ldots, m_\ell - 1\} \), to estimate the Birkhoff sum
\[
\sum_{s=0}^{m_\ell-1} \psi_n(\Phi_n^s(\zeta_j)), \tag{10.21}
\]
we separate \( m_\ell \) (disjoint) Birkhoff sums at level \( \ell - 1 \) which start at intermediate floors with \((\ell - 1, n)\)-addresses
\[
j a, (j + 1)a, \ldots, (m_\ell - 1)a \quad \text{and} \quad 0(a + 1), 1(a + 1), \ldots, (j - 1)(a + 1),
\]
respectively. Compare Fig. 5. Let
\[
S_1 = S_1(j) \overset{\text{def}}{=} \sum_{i=j}^{m_\ell-1} S_{r_i} \psi_n(\zeta_i), \quad S_2 = S_2(j) \overset{\text{def}}{=} \sum_{i=0}^{j-1} S_{s_i} \psi_n(w_i),
\]
where \( S_1 \) corresponds to the first collection of addresses and \( S_2 \) to the second one. Note that \( S_1 + S_2 \) is almost equal to the Birkhoff sum (10.21) except for the following two facts:
(a) the sum \( S_1 + S_2 \) takes into account (disjoint) orbit pieces whose total length is in general close to but not equal to the “expected” value \( \mathcal{N}_\ell \),
(b) the sum \( S_1 + S_2 \) ignores all values of \( \psi_n \) at points of the tail added at the \( \ell \)th level.
To address (a), note that calculating $S_1 + S_2$ we sum over $m_\ell$ orbit pieces each having a length very close to the expected one $\mathcal{R}_{\ell-1}$. Let us estimate this deviation:

$$D_1 \overset{\text{def}}{=} \left| m_\ell \mathcal{R}_{\ell-1} - \sum_{i=j}^{m_\ell-1} r_i - \sum_{i=0}^{j-1} s_i \right|$$

$$\left(10.18), (10.19)\right) = \left| m_\ell \mathcal{R}_{\ell-1} - \sum_{i=j}^{m_\ell-1} R_{\ell-1}(\sigma_{\ell-1}(b)) - \sum_{i=0}^{j-1} R_{\ell-1}(\sigma_{\ell-1}(c)) \right|$$

$$\leq \left| (m_\ell - j)\mathcal{R}_{\ell-1} - \sum_{i=j}^{m_\ell-1} R_{\ell-1}(\sigma_{\ell-1}(b)) \right| + \left| j\mathcal{R}_{\ell-1} - \sum_{i=0}^{j-1} R_{\ell-1}(\sigma_{\ell-1}(c)) \right|$$

(10.22)

To address now item (b), first recall that by (10.20) the length of the “tail between the orbit pieces” where the Birkhoff sums $S_1$ and $S_2$ are taken is at most $\varepsilon \mathcal{R}_\ell$. Further, by Proposition 5.12 (3), the estimate (10.22) of $D_1$, and (10.7) and the choice of $\ell$, it holds

$$\left| \mathcal{R}_\ell - \sum_{i=j}^{m_\ell-1} r_i - \sum_{i=0}^{j-1} s_i \right| \leq |\mathcal{R}_\ell - m_\ell \mathcal{R}_{\ell-1}| + D_1 \leq L_2 \frac{1}{\ell^2} \mathcal{R}_\ell + 2\varepsilon \mathcal{R}_\ell$$

(10.23)

$$\leq 3\varepsilon \mathcal{R}_\ell.$$

The Birkhoff sum in (10.21) takes values over the same collection of points as in the sum $S_1 + S_2$, except for two blocks of points. The first block consists of points on the tail (at most $\varepsilon \mathcal{R}_\ell$ points). The second block consists of points at the end of the orbit piece in (10.21) (at most the difference between $\mathcal{R}_\ell$ and the sum of terms in the tail, $S_1$, and $S_2$; that is, at most $\varepsilon \mathcal{R}_\ell + 3\varepsilon \mathcal{R}_\ell$ terms). Hence, it follows

$$\left| \mathcal{R}_\ell - \sum_{s=0}^{\mathcal{R}_{\ell-1}} \psi_n(\Phi^s_n(\xi_j)) - (S_1 + S_2) \right| \leq (\varepsilon \mathcal{R}_\ell + (\varepsilon \mathcal{R}_\ell + 3\varepsilon \mathcal{R}_\ell))\|\psi_n\|$$

$$\leq 5\varepsilon \|\phi\| \mathcal{R}_\ell,$$  

(10.24)

where for the second inequality we also used $\|\psi_n\| \leq \|\phi\|$. This concludes the discussion of the obstructions (a) and (b).

As next step let us estimate $S_1$ and $S_2$. First note that

$$\left| S_1 - (m_\ell - j)\mathcal{R}_{\ell-1} \int \psi_{\ell-1} d\lambda_{\ell-1} \right|$$

$$= \left| \sum_{i=j}^{m_\ell-1} S_{ri} \psi_n(\xi_i) - (m_\ell - j)\mathcal{R}_{\ell-1} \int \psi_{\ell-1} d\lambda_{\ell-1} \right|$$

$$\leq \left| \sum_{i=j}^{m_\ell-1} \left( S_{ri} \psi_n(\xi_i) - S_{ri} \psi_{\ell-1}(\xi_i') \right) \right| + \left| \sum_{i=j}^{m_\ell-1} \left( S_{ri} \psi_{\ell-1}(\xi_i') - \mathcal{R}_{\ell-1} \int \psi_{\ell-1} d\lambda_{\ell-1} \right) \right|.$$
To estimate the first term, by Claim 10.8, we obtain

$$\left| \sum_{i=j}^{m_\ell-1} \left( S_{r_i} \psi_{r_i}(\zeta_i) - S_{r_i} \psi_{r_i-1}(\zeta_i') \right) \right| \leq \varepsilon \sum_{i=j}^{m_\ell-1} r_i.$$  

To estimate the second term, note that

$$\left| \sum_{i=j}^{m_\ell-1} \left( S_{r_i} \psi_{r_i-1}(\zeta_i') - \mathcal{R}_{r_i-1} \int \psi_{r_i-1} d\lambda_{r_i-1} \right) \right|$$

by definition of $\zeta_i'$ in (10.16) = $$\left| \sum_{i=j}^{m_\ell-1} \left( \Delta \psi_{r_i-1}(\sigma_{r_i-1}(\mathbf{b})) - \mathcal{R}_{r_i-1} \int \psi_{r_i-1} d\lambda_{r_i-1} \right) \right|$$

by Abramov’s formula in Lemma 4.2 = $$\left| \sum_{i=j}^{m_\ell-1} \left( \Delta \psi_{r_i-1}(\sigma_{r_i-1}(\mathbf{b})) - \int \Delta \psi_{r_i-1} d\mathbf{b}_{r_i-1} \right) \right|$$

by (10.10) < $2\varepsilon L_2 R_{r_i} + 2\varepsilon (m_\ell - i)$.

Putting the previous estimates together, we get

$$\left| S_1 - (m_\ell - j)\mathcal{R}_{r_i-1} \int \psi_{r_i-1} d\lambda_{r_i-1} \right| \leq \varepsilon \sum_{i=j}^{m_\ell-1} r_i + 2\varepsilon L_2 R_{r_i} + 2\varepsilon (m_\ell - i).$$

We get the analogous estimate for $S_2$,

$$\left| S_2 - j\mathcal{R}_{r_i-1} \int \psi_{r_i-1} d\lambda_{r_i-1} \right| \leq \varepsilon \sum_{i=0}^{j-1} s_i + 2\varepsilon L_2 R_{r_i} + 2\varepsilon j.$$

This implies

$$\left| (S_1 + S_2) - m_\ell \mathcal{R}_{r_i-1} \int \psi_{r_i-1} d\lambda_{r_i-1} \right|$$

$$\leq \varepsilon \left( \sum_{i=j}^{m_\ell-1} r_i + \sum_{i=0}^{j-1} s_i \right) + 4\varepsilon L_2 R_{r_i} + 2\varepsilon m_\ell$$

(10.25)

by (10.23) and $m_\ell < \mathcal{R}_{r_i} < \varepsilon \mathcal{R}_{r_i} + \varepsilon 3\varepsilon \mathcal{R}_{r_i} + 4\varepsilon L_2 R_{r_i} + 2\varepsilon \mathcal{R}_{r_i}$

$$= \varepsilon \left( 3 + 3\varepsilon + 4L_2 \right) \mathcal{R}_{r_i}.$$  

This finishes the estimate of $S_1$ and $S_2$.

Note that by assumption $\ell \geq L_0$ with (10.8) and using $\psi_{r_i-1} = \phi \circ H_{r_i-1}$ and $\mu_{r_i-1} = (H_{r_i-1})_*\lambda_{r_i-1}$ we have

$$\left| \int \psi_{r_i-1} d\lambda_{r_i-1} - \int \phi d\mu \right| = \left| \int \phi d\mu_{r_i-1} - \int \phi d\mu \right| < \varepsilon.$$
Finally, with the above, we conclude

\[
\left| m_\ell R_\ell - \int \psi_\ell d\lambda_\ell_1 - R_\ell \int \phi d\mu \right| 
\leq \left| m_\ell R_\ell - R_\ell \right| \cdot \|\phi\| + R_\ell \cdot \left| \int \psi_\ell d\lambda_\ell_1 - \int \phi d\mu \right|
\]

by Proposition 5.12(3)

\[
\leq L_2 \frac{1}{2\ell} R_\ell \cdot \|\phi\| + R_\ell \cdot \varepsilon = \left( L_2 \frac{1}{2\ell} \|\phi\| + \varepsilon \right) R_\ell
\]

by (10.7)

\[
\leq 2\varepsilon \cdot R_\ell. \tag{10.26}
\]

Hence, (10.24), (10.25), and (10.26) together imply

\[
\left| \frac{1}{R_\ell} \sum_{s=0}^{\mathcal{R}_\ell-1} \psi_n(\Phi_n^s(\xi_j)) - \int \phi d\mu \right| \leq (5\|\phi\| + 8 + 4L_2)\varepsilon.
\]

This proves the lemma. \(\square\)

**Step 3: Birkhoff sums for any other** \(\xi \in C_n^{(\ell,a)}\).

**Lemma 10.10.** Item (i) in Main Lemma 10.6 holds for every \(\xi \in C_n^{(\ell,a)}\) taking \(C = C_0 + 1\).

**Proof.** Choose \(j \in \{0, \ldots, m_\ell - 1\}\) which addresses the previous intermediate floor of level \((\ell - 1)\), that is, using notation (10.15), choose the minimal index \(j\) for which there is \(r \geq 0\) such that \(\xi = \Phi_n^r(\xi_j)\) (compare Fig. 6).

To estimate the differences between the Birkhoff sums along the orbit segments of length \(R_\ell\) starting at \(\xi\) and \(\xi_j\), respectively, just note that both are on the same orbit and both have the same length and hence share most of its terms \(r + 1, \ldots, R_\ell - r\). Hence,

\[
\left| \sum_{s=0}^{\mathcal{R}_\ell-1} \psi_n(\Phi_n^s(\xi)) - \sum_{s=0}^{\mathcal{R}_\ell-1} \psi_n(\Phi_n^s(\xi_j)) \right| \leq 2r \|\psi_n\| \leq 2r \|\phi\|. \tag{10.27}
\]

Note that together with Proposition 5.12 (1)–(2) it holds

\[
r \leq \max R_{\ell-1} \leq \frac{1}{m_\ell} \max R_\ell \leq \frac{1}{m_\ell} L_2 R_\ell. \tag{10.28}
\]
Hence
\[
\left| \frac{1}{\mathcal{R}_\ell} \sum_{s=0}^{\mathcal{R}_\ell-1} \psi_n(\Phi^s_n(\xi)) - \int \phi \, d\mu \right| \leq \frac{1}{\mathcal{R}_\ell} \left| \sum_{s=0}^{\mathcal{R}_\ell-1} \psi_n(\Phi^s_n(\xi)) - \sum_{s=0}^{\mathcal{R}_\ell-1} \psi_n(\Phi^s_n(\xi_j)) \right| 
\]
\[
+ \left| \frac{1}{\mathcal{R}_\ell} \sum_{s=0}^{\mathcal{R}_\ell-1} \psi_n(\Phi^s_n(\xi_j)) - \int \phi \, d\mu \right|
\]
(by (10.27), (10.28), (10.7), Lemma 10.9)
\[
\leq 2\|\phi\| \frac{1}{m_\ell} L_2 + (5\|\phi\| + 8 + 4L_2)\varepsilon
\]
\[
\leq (1 + C_0)\varepsilon,
\]
proving the lemma.  \[\square\]

**Lemma 10.11.** Item (ii) in Main Lemma 10.6 is true for every \(\gamma \in B^{(\ell-1,0a)}_n\).

**Proof.** It suffices to take \(j = 0\) in (10.22) to recall that \(\zeta_0 = \gamma\).  \[\square\]

As \(\gamma \in B^{(\ell,a)}_n\) was arbitrary, Main Lemma 10.6 (i) now is a consequence of Lemmas 10.9 and 10.10. Main Lemma 10.6 (ii) follows from Lemma 10.11. This finishes the proof.  \[\square\]

11. Proof of Theorem C

Our construction provides the sequences of horseshoes \((\Gamma_n)\), as in (8.5), and Borel probability measures \((\mu_n) \subset \mathcal{M}_{\text{erg}}(F)\), as in (8.10). By Lemma 10.2, the sequence \((\mu_n)\) weak* converges to some probability measure \(\mu_\infty\) as \(n \to \infty\). By Corollary 8.9, it holds
\[
\chi(\mu_\infty) = 0 \quad \text{and} \quad h(F, \mu_\infty) \geq e^{-L_1|\alpha|}(h(F, \mu) - \varepsilon_H).
\]
It remains to show that \(\mu_\infty\) is ergodic. For that we will use Proposition 11.1 below that is a minor extension of [20, Lemma 2] and also essentially stated in [32, Chapter 4] \[12\].

For completeness, we prove it in the Appendix.

For every continuous \(\phi: \Sigma_N \times \mathbb{S}^1 \to \mathbb{R}\) and \(\varepsilon > 0\) let \(L_0 = L_0(\phi, \varepsilon) \in \mathbb{N}\) as in Proposition 10.1. Hence, for every \(\ell \geq L_0\) and \(n \geq \ell + 1\) the subset
\[
\Gamma_{n,\phi,\varepsilon} \overset{\text{def}}{=} H_n(S_{n,\phi,\varepsilon}) \subset \Gamma_n
\]
satisfies
\[
\mu_n(\Gamma_{n,\phi,\varepsilon}) = (H_n)_*\lambda_n(\Gamma_{n,\phi,\varepsilon}) = \lambda_n(S_{n,\phi,\varepsilon}) > 1 - \varepsilon.
\]
It also follows that for every \(X = H_n(a, s) \in H_n(S_{n,\phi,\varepsilon}) = \Gamma_{n,\phi,\varepsilon}\) it holds
\[
\left| \frac{1}{\mathcal{R}_\ell} \sum_{k=0}^{\mathcal{R}_\ell-1} \phi(F^k(X)) - \int \phi \, d\mu \right| < \varepsilon,
\]
where we also used the fact that by Proposition 8.4 the maps \(\Phi_n\) and \(F|\Gamma_n\) are semi-conjugate by \(H_n\). We now use the following result.

\[12\] In [20], it is assumed that every measure in the sequence is uniformly distributed on a periodic orbit. The results in [32, Chapter 4] imply this criterion, and are in fact considerably more general, but this exact formulation does not appear there. For this reason we state this proposition and, for completeness, also its proof.
Proposition 11.1. Let $G: X \to X$ be a homeomorphism of a compact metric space. Consider sequences of Borel measurable subsets $(\Upsilon_n)_n$ of $X$, Borel measures $(\varrho_n)_n$ on $X$ weak$^*$ converging to some Borel measure $\varrho$, and positive integers $(T(n))_n$ tending to $\infty$. Assume that for every $\phi: X \to \mathbb{R}$ continuous and $\varepsilon > 0$, there exists $L = L(\phi, \varepsilon) \in \mathbb{N}$ such that for every $\ell \geq L$ there exists $N = N(\ell) \geq \ell$ such that for every $n \geq N$ there exists a measurable subset $\Upsilon_{n, \phi, \varepsilon} \subset \Upsilon_n$ with $\varrho_n(\Upsilon_{n, \phi, \varepsilon}) > 1 - \varepsilon$ such that

$$\left| \frac{1}{T(\ell)} \sum_{k=0}^{T(\ell)-1} \phi(G^k(x)) - \int \phi d\varrho \right| < \varepsilon \quad \text{for every} \ x \in \Upsilon_{n, \phi, \varepsilon}.$$

Then $\varrho$ is $G$-ergodic.

The comments above imply that we can apply Proposition 11.1 with $G = F$, $X = \Sigma_N \times S^1$, $\Upsilon_n = \Gamma_n$, $\varrho_n = \mu_n$, $T(n) = R_n$, $L = L_0$, $N = L_0$, $\Upsilon_{n, \phi, \varepsilon} = \Gamma_{n, \phi, \varepsilon}$. Therefore, the limit measure $\mu_\infty$ is ergodic. \qed

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Appendix: Proof of Proposition 11.1

Given $\phi: X \to \mathbb{R}$, denote

$$\underline{\phi}(x) \overset{\text{def}}{=} \liminf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(G^k(x)) \quad \text{and} \quad \overline{\phi}(x) \overset{\text{def}}{=} \limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(G^k(x)).$$

Given $\varepsilon > 0$, denote the upper topological limit of $(\Upsilon_{n, \phi, \varepsilon})_n$ by $\Upsilon_{\phi, \varepsilon}$, that is,

$$\Upsilon_{\phi, \varepsilon} \overset{\text{def}}{=} \cap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \Upsilon_{n, \phi, \varepsilon} = \{ y \in X : \exists n_k \to \infty, y_k \in \Upsilon_{n_k, \phi, \varepsilon}, y = \lim_{k \to \infty} y_k \}.$$

We use the following fact that is straightforward to check.

Claim 11.2. For every continuous $\phi: X \to \mathbb{R}$ and $\varepsilon > 0$ it holds

$$\varrho(\Upsilon_{\phi, \varepsilon}) \geq \limsup_{n \to \infty} \varrho_n(\Upsilon_{n, \phi, \varepsilon}).$$

Lemma 11.3. For every continuous $\phi: X \to \mathbb{R}$ and $\varepsilon > 0$ there exists a set $\Xi_{\phi, \varepsilon} \subset \Upsilon_{\phi, \varepsilon}$ such that $\varrho(\Xi_{\phi, \varepsilon}) > 1 - \varepsilon$ and

$$\int \phi d\varrho - \varepsilon < \overline{\phi}(x) = \underline{\phi}(x) < \int \phi d\varrho + \varepsilon$$

for every $x \in \Xi_{\phi, \varepsilon}$. 
Proof. Given $\phi$ and $\varepsilon$, let $L = L(\phi, \varepsilon)$ and for $\ell \geq L$ let $N = N(\ell)$ be as in the hypothesis of the proposition. By Claim 11.2 and our hypothesis,

$$\rho(\Upsilon_{\phi, \varepsilon}) \geq \limsup_n \rho_n(\Upsilon_{n, \phi, \varepsilon}) > 1 - \varepsilon.$$ 

Every $x \in \Upsilon_{\phi, \varepsilon}$ is the limit of some sequence of points $x_i$ in $\Upsilon_{n_i, \phi, \varepsilon}$, $n_i \geq \ell$. Hence, for $n_i \geq \ell$ sufficiently large, it holds

$$\left| \frac{1}{T(\ell)} \sum_{k=0}^{T(\ell) - 1} \phi(G^k(x)) - \frac{1}{T(\ell)} \sum_{k=0}^{T(\ell) - 1} \phi(G^k(x_i)) \right| < \varepsilon.$$ 

By our hypothesis on $x_i \in \Upsilon_{n_i, \phi, \varepsilon}$, it holds

$$\left| \frac{1}{T(\ell)} \sum_{k=0}^{T(\ell) - 1} \phi(G^k(x_i)) - \int \phi \, d\rho \right| < \varepsilon.$$ 

Hence, for every $x \in \Upsilon_{\phi, \varepsilon}$ and $\ell \geq 1$ sufficiently large it holds

$$\left| \frac{1}{T(\ell)} \sum_{k=0}^{T(\ell) - 1} \phi(G^k(x)) - \int \phi \, d\rho \right| < 2\varepsilon.$$ 

Therefore, with the notation above, for every $x \in \Upsilon_{\phi, \varepsilon}$

$$\overline{\phi}(x) > \int \phi \, d\rho - 2\varepsilon \quad \text{and} \quad \underline{\phi}(x) < \int \phi \, d\rho + 2\varepsilon.$$ 

Applying the Birkhoff theorem to the invariant measure $\rho$, we get a set $Z$ with $\rho(Z) = 1$ so that at every $z \in Z$ it holds $\overline{\phi}(z) = \phi(z)$. By the above, for every $z \in Z_{\phi, \varepsilon} \defeq Z \cap \Upsilon_{\phi, \varepsilon}$ it holds $|\overline{\phi}(z) - \phi(\rho)| < 3\varepsilon$ and $\rho(Z_{\phi, \varepsilon}) = \rho(\Upsilon_{\phi, \varepsilon}) > 1 - \varepsilon$. This proves the lemma.

Let us now prove that $\rho$ is ergodic. Take a dense set of continuous functions $\{\phi_k\}_k$ and a summable sequence of positive numbers $(\varepsilon_k)_k$. As

$$\sum_k \rho(\Upsilon_{\phi_k, \varepsilon_k}) \leq \sum_k \varepsilon_k < \infty,$$ 

by the Borel–Cantelli lemma, there is a set $\Upsilon$ satisfying $\rho(\Upsilon) = 1$ such that every $x \in \Upsilon$ is contained in only finitely many sets $\Upsilon_{\phi_k, \varepsilon_k}^c$. It follows that for every continuous $\phi$ and $x \in \Upsilon$ Birkhoff averages of $\phi$ converge to $\int \phi \, d\rho$. This implies that $\rho$ is $G$-ergodic.
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