Comparative analysis of the machine repair Problem with imperfect coverage and service pressure condition

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Abstract. We analyze the warm-standby M/M/R machine repair problem with multiple imperfect coverage which involving the service pressure condition. When an operating machine (or warm standby) fails, it may be immediately detected, located, and replaced with a coverage probability $c$ by a standby if one is available. A recursive method is used to develop the steady-state analytic solutions. The total expected profit function per unit time is derived to determine the joint optimal values at the maximum profit. We utilize the direct search method to measure the various characteristics of the profit function followed by Quasi-Newton method to search the optimal solutions.

1. Introduction

This paper studies the warm-standby M/M/R machine repair problem (MRP) with multiple imperfect coverage which involving the service pressure condition. The faults that switch a failed machine in a spare machine are called to be not covered. The coverage factor $c$ includes the probabilities of successful detection, location, and recovery (see Trivedi [1]). The service pressure coefficient is a positive constant and indicates degree to which the servers increase the service rate in order to reduce the number of failed machines in the system.

The concept of coverage factor in a repairable system has been introduced by several researchers such as Dugan and Trivedi [2], Trivedi [1], Wang and Chiu [3], etc. Dugan and Trivedi [2] considered the coverage modeling for dependability analysis of fault-tolerant systems. Wang and Chiu [3] considered the cost benefit analysis of availability systems with warm standby units and imperfect coverage for three availability models. Profit and cost analysis in the MRP have been investigated by a variety of authors including Sivazlian and Wang [4], Wang, et al. [5], and so on. The Quasi-Newton method (QNM) has been used by many researchers such as Wang, et al. [5], Wang, et al. [6] and so on. Wang, et al. [7] considered the M/M/1 MRP with working vacation. They constructed a cost model to derive the optimal number of operating machines by the direct search method (DSM) and find the optimal values of two different repair rates by the QNM. Wang, et al. [6] considered the profit analysis of the M/M/R MRP with balking, reneging, and standby switching failures. They used the two methods DSM and steepest descent method to find the maximum profit until the availability, balking, and reneging constraints are satisfied.

The purpose of this paper is two-fold. We first use a recursive method for developing steady-state analytic solutions. Next, we develop a profit model to determine the joint optimal values at the maximum profit. We use the DSM to find the optimal number of servers $R$ and the optimal number of warm standby $S$, and then apply the QNM to obtain the optimal service rate $\mu^*$ after $R^*$ and $S^*$ are fixed.
2. Problem statement
We consider a machine repair model with $N=M+S$ identical machines and $R$ servers in the repair facility. As many as $M$ of these can be operating simultaneously in parallel, the rest of the $S$ machines are warm-standby spares. Each of the operating machines fails independently of the state of the others and has an exponential time-to-failure distribution with parameter $\lambda$. Each of the available warm standbys fails independently of the state of all the others and has an exponential time-to-failure distribution with parameter $\alpha$ ($0 < \alpha < \lambda$). When an operating machine (or warm standby) fails, it may be immediately detected, located, and replaced with a coverage probability $c$ by a standby if one is available. The coverage factor is the same for an operating machine failure as that for a warm standby failure and is denoted by $c$. We define the unsafe failure state of the system as any one of the breakdowns is not covered. Operating machine failure (or warm standby failure) in the unsafe failure state is cleared by a reboot. Reboot delay occurs at rate $\beta$ for an operating machine (or warm standby) which is exponentially distributed. Each of the servers has an exponential time-to-repair distribution with parameter $\mu$ until the number of failed machines $n$ is greater than or equal to the number of servers $R$ in the system, and the mean service rate $\mu_n$ is defined as $\mu_n = n\mu$, where $1 \leq n \leq R - 1$. When the number of failed machines $n$ is greater than or equal to the number of servers $R$, the conditions of service pressure coefficient will be considered, and the mean service rate $\mu_n$ is defined as $\mu_n = R\mu \left[ n \left( R + 1 \right) / R \left( n + 1 \right) \right]^{\theta}$, where $R \leq n \leq N$, and $\theta$ is the pressure coefficient. When an operating machine (or warm standby) fails, it is immediately sent to a repair facility where it is repaired in the order of their breakdowns; that is, the first-come, first-served discipline. The server can repair only one operating machine (or warm standby) at a time.

3. Steady-state results
We obtain the steady-state solutions given by

$$P_n = \prod_{j=0}^{n-1} \left[ M \theta_\lambda + (S - j) \theta_\alpha \right] / n! P_0, \quad 1 \leq n \leq R - 1$$ (1)

$$P_n = \prod_{k=R}^{n} \left[ k \left( R + 1 \right) / R \left( k + 1 \right) \right]^{\theta \cdot \prod_{j=0}^{n-1} \left[ M \theta_\lambda + (S - j) \theta_\alpha \right]} / P_0, \quad R \leq n \leq S - 1$$ (2)

$$P_n = \left( N-n \right)! / R! R^{n-R} \prod_{k=R}^{n} \left[ k \left( R + 1 \right) / R \left( k + 1 \right) \right]^{\theta \cdot \prod_{j=0}^{n-1} \left[ M \theta_\lambda + (S - j) \theta_\alpha \right]} / P_0, \quad S \leq n \leq N$$ (3)

$$P_{uf_1} = \frac{\left( M \lambda + S \alpha \right) \left( 1 - c \right)}{\beta} P_0$$ (4)

$$P_{uf_2} = \frac{\left( M \lambda + (S - (n-1)) \alpha \right) \left( 1 - c \right) \prod_{j=0}^{n-2} \left[ M \theta_\lambda + (S - j) \theta_\alpha \right]}{\beta (n-1)!} P_0, \quad 2 \leq n \leq R - 1$$ (5)
\[
P_{uf_s} = \frac{[M \lambda + (S - (n-1)) \alpha] (1 - c) \prod_{k=R}^{a-1} \left(\frac{k(R + 1)}{R(k + 1)}\right)^{\theta} \prod_{j=0}^{a-2} \left[M \theta_{\alpha} + (S - j) \theta_{\beta}\right]}{\beta R! R^{(n-1)-R}} P_0, \quad R \leq n \leq S - 1 \tag{6}
\]

\[
P_{uf_s} = \frac{[M + S - (n-1)] \lambda (1 - c) M! \theta_{\lambda}^{(n-1)-S} \prod_{k=R}^{a-1} \left(\frac{k(R + 1)}{R(k + 1)}\right)^{\theta} \prod_{j=0}^{a-2} \left[M \theta_{\alpha} + (S - j) \theta_{\beta}\right]}{\beta (N - (n-1))! R! R^{(n-1)-R}} P_0, \quad S \leq n \leq N - 1 \tag{7}
\]

where \( \theta_{\lambda} = \frac{\lambda}{\mu} \) and \( \theta_{\alpha} = \frac{\alpha}{\mu} \).

The steady-state solutions \( P_n \) and \( P_{uf_s} \) always exist because the number of states is finite. For \( (1)-(7) \), \( P_0 \) can be solved from the following normalizing equation

\[
\sum_{n=0}^{N} P_n + \sum_{n=1}^{N-1} P_{uf_s} = 1
\]

4. Numerical results

Let \( E[O] \) be the expected number of operating machines in the system, \( E[S] \) be the expected number of spare machines in the system, \( E[I] \) be the expected number of idle servers in the system, \( E[B] \) be the expected number of busy servers in the system, \( MA \) be machine availability, and \( O.U. \) be operative utilization fraction. Let \( p \) represent revenue per unit time when one machine is in an operating state, \( C_1 \) be cost per unit time when one machine is in an operating state, \( C_2 \) be cost per unit time when one machine is functioning as a warm standby, \( C_3 \) be cost per unit time when one server is busy, \( C_4 \) be cost per unit time when one server is idle, and \( C_5 \) denote cost per unit time of providing the service rate \( \mu \). Then the total expected profit function is given by

\[
F(R, S, \mu) = (p - C_1) E[O] - C_2 E[S] - C_3 E[I] - C_4 E[I] - C_5 \mu. \tag{8}
\]

The profit and cost parameters in \( (8) \) are assumed to be linear in the expected number of the indicated quantity. We first use DSM to find the optimal value of the number of servers \( R^* \) and warm standbys \( S^* \), followed by QNM to search the optimal value of \( \mu^* \).

The profit maximization problem can be presented mathematically as

\[
Z^* = F(R^*, S^*, \mu^*) \quad \text{Maximize} \quad F(R, S, \mu).
\]

Since \( R \) and \( S \) are discrete variables, we use direct substitution of successive values of \( R \) and \( S \) into the profit function until the maximum value of \( Z^* \), say \( F(R^*, S^*, \mu^*) \) is obtained. The following numerical results are provided by considering the revenue and cost parameters as follows:

\[
p = $300, \quad C_1 = $175, \quad C_2 = $30, \quad C_3 = $100, \quad C_4 = $60, \quad C_5 = $50.
\]

We fix \( M=15 \), vary \( R \) from 1 to 10 and vary \( S \) from 1 to 15.
Table 1. System performance measures of the warm-standby M/M/R MRP with multiple imperfect coverage and service pressure coefficient ($\alpha = 0.05, \mu = 3.0, c = 0.7, \beta = 2.4$).  

| $(\lambda, \theta)$ | $(1.0, 0.8)$ | $(2.0, 0.8)$ | $(3.0, 0.8)$ | $(1.0, 0.2)$ | $(1.0, 0.4)$ | $(1.0, 0.6)$ |
|---------------------|--------------|--------------|--------------|--------------|--------------|--------------|
| $(R^*, S^*)$        | $(5, 8)$     | $(10, 15)$   | $(5, 8)$     | $(5, 8)$     | $(5, 8)$     |
| $Z^*$               | 1111.12      | 776.82       | 436.52       | 1088.86      | 1096.92      | 1104.32      |

Table 2. QNM and PSO algorithm in searching the global best solution ($\alpha = 0.05, \beta = 2.4$)  

| $(\lambda, \theta, c)$ | $(\mu^*, \mu^*, Z_{QNM}^*)$ | $(\mu^*, \mu^*, Z_{QNM}^*)$ |
|-------------------------|--------------------------------|--------------------------------|
| $(1.0, 0.6, 0.7)$       | $(5, 8, 3.4850, 1123.53)$     | $(6, 12, 4.8788, 936.85)$     |
| $(2.0, 0.6, 0.7)$       | $(6, 12, 4.9430, 932.08)$     | $(7, 15, 5.0484, 760.44)$     |

5. Conclusions  
In this paper, we studied the warm-standby M/M/R MRP with multiple imperfect coverage and service pressure coefficient. We first used state-transition-rate diagram to set up the steady-state equations. Next, we used recursive method to develop the steady-state analytical solutions which are used to calculate various system performance measures, such as the expected number of failed machines, the expected number of idle servers, machine availability and operative utilization, etc. Finally, we derived the expected profit function per unit time to determine the optimal number of servers $R^*$, the optimal number of warm standbys $S^*$, and the optimal service rate $\mu^*$ at the maximum profit by using QNM.

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