An improved numerical fit to the peak harmonic gravitational wave frequency emitted by an eccentric binary

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ABSTRACT

I present a new numerical fit of $n_{\text{peak}}(e)$ which significantly improves upon that of W03 for low eccentricities ($e \lesssim 0.8$), whereas also accurate for high eccentricities:

$$n_{\text{peak}}(e) \approx 2 \left( 1 + \frac{1}{e^2} \right) \frac{1}{(1-e^2)^{3/2}},$$

Equation (5) is plotted as dashed vertical lines in Fig. 1(a). Although a reasonable fit for high eccentricities, it does not capture the true maximum of $g(n,e)$ very accurately for lower eccentricities. Nevertheless, the fit of W03 is commonly used to estimate the peak GW frequency of eccentric binaries, in particular in the context of population synthesis studies (e.g., Thompson 2011; Antonini & Perets 2012; Samsing et al. 2014; Rodriguez et al. 2018; Hamers et al. 2018; Kremer et al. 2021; Shao & Li 2021; Vynatheya & Hamers 2021).

Here, I present a new numerical fit of $n_{\text{peak}}(e)$ which significantly improves upon that of W03 for low eccentricities ($e \lesssim 0.8$), whereas also accurate for high eccentricities:

$$n_{\text{peak}}(e) \approx 2 \left( 1 + \frac{1}{e^2} \right) \frac{1}{(1-e^2)^{3/2}},$$

where $c_1 = -1.01678$, $c_2 = 5.57372$, $c_3 = -4.9271$, and $c_4 = 1.68506$. The fit retains the factor $(1-e^2)^{-3/2}$ from W03 that dominates at high eccentricities, but the behavior at smaller eccentricities is modified. Equation (6) correctly states that $n_{\text{peak}}(0) = 2$ for circular orbits.

The new fits are indicated in Fig. 1(a) with solid vertical lines, and show significantly better match with the true $n$ corresponding to a maximum in $g(n,e)$ (when $n$ is interpreted as a real number).

Fig. 1(b) plots the peak harmonic $n_{\text{peak}}$ as a function of $e$. Grey open circles show the 'exact' calculation of $n_{\text{peak}}$ by numerically calculating the maximum of $g(n,e)$ for given eccentricity. The latter is carried out in practice for real $n \geq 1$, although it should be understood that $n$ is actually an integer. The exact (real) calculation yields that $n_{\text{peak}}$ decreases below 2 as $e \to 0$. However, it is clear that, in the circular limit, $g(n,0) = 1$ for $n = 2,$
and $g(n,0) = 0$ for all other integer $n$. Therefore, when determining $n_{\text{peak}}$, one should take the integer value and limit to $n_{\text{peak}} \geq 2$ in order to retain the correct behavior in the fitting function as $e \to 0$.

By enforcing that $n_{\text{peak}}(0) = 2$ and given the limitations of the assumed functional form, the fit of W03 significantly overpredicts $n_{\text{peak}}$ for $e \lesssim 0.8$. The new fit better captures the low-eccentricity regime, while also still satisfying $n_{\text{peak}}(0) = 2$ and giving a good description at high eccentricities. This is particularly clear in Fig. 1(c), in which rounded integer numbers are plotted.

Lastly, Fig. 1(d) shows the fractional residuals in the integer $n_{\text{peak}}$ computed as the difference between the ‘exact’ integer-rounded calculation with $n_{\text{peak}} \geq 2$ and the integer-rounded fits, i.e.,

$$\frac{\Delta n_{\text{peak}}}{n_{\text{peak}}} = \frac{\int \lfloor n_{\text{peak}}^{(\text{exact})} \rfloor - \lfloor n_{\text{peak}}^{(\text{fit})} \rfloor}{\int \lfloor n_{\text{peak}}^{(\text{exact})} \rfloor}.$$  

(7)

For $e \lesssim 0.8$, the fit of W03 systematically overpredicts $n_{\text{peak}} (\Delta n_{\text{peak}} < 0)$. This is especially the case for the range $0.17 \lesssim e \lesssim 0.30$, where Equation (5) predicts $n_{\text{peak}} = 3$, whereas it should be $n_{\text{peak}} = 2$. For larger $e$, the discrepancies become less severe, although at $e = 0.8$ the peak harmonic is still overpredicted by $\sim 10\%$. For even higher $e$, $e \gtrsim 0.97$, Equation (5) starts to slightly underpredict $n_{\text{peak}} (\Delta n_{\text{peak}} > 0)$.

In contrast, the new fit Equation (6) has typically zero fractional residuals, with only a few spikes occurring due to rounding effects at transitions where $n_{\text{peak}}$ advances.
by unity. At $e = 0.999$ (the highest eccentricity considered when determining Equation 6), the new fit has zero fractional residuals, i.e., $n_{\text{peak}}(0.999) = 51.755$ according to Equation (6) is consistent with the exact calculation. In contrast, Equation (5) has a fractional error of $\simeq 1\%$ with $n_{\text{peak}}^{(\text{W03})}(0.999) = 51.216$.

When applying the fit presented here, it should be remembered that it relies on the results of Peters & Mathews (1963) based on the lowest-order post-Newtonian terms that describe dissipation due to GW emission (2.5PN). Higher-order post-Newtonian corrections (e.g., Tucker & Will 2021) are not included.

Also to be considered in Equation (6) is that the peak GW frequency does not describe the amplitude, or, more relevantly, the signal-to-noise ratio in the GW detector band. The latter should also be taken into account when making statements about detectability of GW sources (using, e.g., LEGWORK, Wagg et al. 2021).

Lastly, the fits considered by W03 and the new fit here quantify the energy flux, but not the angular momentum flux, which could peak at a different harmonic. The latter should be considered in future work, as it controls orbital circularisation.

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