Phase-noise protection in quantum-enhanced differential interferometry

M Landini\textsuperscript{1,2}, M Fattori\textsuperscript{1,2,3}, L Pezzè\textsuperscript{1,2,4} and A Smerzi\textsuperscript{1,2,4}

\textsuperscript{1} Istituto Nazionale di Ottica-CNR (INO-CNR), Via Nello Carrara 1, I-50019 Sesto Fiorentino, Italy
\textsuperscript{2} European Laboratory for Non-Linear Spectroscopy (LENS) and Dipartimento di Fisica, Università di Firenze, Via Nello Carrara 1, I-50019 Sesto Fiorentino, Italy
\textsuperscript{3} Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Firenze, Via Sansone 1, I-50019 Sesto Fiorentino, Italy
\textsuperscript{4} QSTAR, Largo Enrico Fermi 2, I-50125 Firenze, Italy
E-mail: luca.pezze@ino.it

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Abstract
Differential interferometry (DI) with two coupled sensors is a most powerful approach for precision measurements in the presence of strong phase noise. However, DI has been studied and implemented only with classical resources. Here we generalize the theory of differential interferometry to the case of entangled probe states. We demonstrate that, for perfectly correlated interferometers and in the presence of arbitrary large phase noise, sub-shot noise sensitivities—up to the Heisenberg limit—are still possible with a special class of entangled states in the ideal lossless scenario. These states belong to a decoherence free subspace where entanglement is passively protected. Our work paves the way to the full exploitation of entanglement in precision measurements.

Keywords: nonclassical interferometry, atom interferometry, quantum noise

1. Introduction

Atom interferometers [1] nowadays offer unprecedented precision in the measurement of gravity [2], inertial forces [3], atomic properties [4], and fundamental constants [5]. Their large
sensitivity makes their coupling to the environment almost unavoidable, which mainly results in a random noise which affects the signal phase. In order to overcome this limitation, many experiments aiming at precision measurements adopt a differential scheme: two interferometers operating in parallel are affected by the same phase noise and accumulate a different phase shift induced by the measured field. Estimation of the differential phase allows high resolution, thanks to noise cancellation [6]. Schemes based on this concept have proven crucial for the precision measurement of rotations [7], gradients [8], and fundamental constants [9]. Differential atom interferometers have also been proposed for tests of general relativity [10], equivalence principle [11], atom neutrality [12], and for detection of gravitational waves [13]. So far, differential interferometry (DI) has only been exploited with classical resources. Its sensitivity is thus ultimately bounded by the shot noise (SN) limit, \( \Delta \theta_{\text{SN}} \approx \sqrt{N} \), where \( N \) is the number of particles in the input. For the single interferometer operation, a significant enhancement of phase sensitivity up to the Heisenberg limit (HL) \( \Delta \theta_{\text{HL}} \approx 1/N \) can be obtained by using particle-entangled input states [14–17]. This prediction is under intense experimental investigation with cold [18] and ultracold [19–24] atoms. However, the analysis of a single interferometer has emphasized [25–29] that sub-SN cannot be reached in presence of strong phase noise. Is it possible to exploit DI with highly entangled states to overcome the SN\(^5\) in such a noisy environment?

In this manuscript we study DI with two sensors implementing quantum resources and affected by phase noise of an arbitrarily large amplitude (see figure 1). Our analysis takes into account the correlations of the two interferometer outcomes. It goes beyond the trivial subtraction of the two output phases estimated independently, which does not offer any significant quantum enhancement of phase sensitivity. We provide the necessary and sufficient condition, based on the Fisher information, for an entangled state to allow sub-SN phase sensitivity. We also demonstrate that the HL, which is believed to be achievable only in noiseless quantum interferometers [25–28], is preserved by the lossless differential scheme, as long as relative noise fluctuations are also at the HL. While the HL is saturated by maximally entangled states that are extremely fragile to particle losses, the SN can be overcome by less entangled and more robust states, such as those experimentally created via particle–particle interaction in Bose–Einstein condensates (BECs) [19–24]. These findings open the door to full exploitation of quantum resources in realistic devices, provided that a DI scheme is implemented.

### 2. Parameter estimation in differential interferometry

Figure 1(a) shows the general DI scheme discussed in this manuscript. It consists of two interferometers running in parallel. The input state \( \hat{\rho} \) is transformed by \( \hat{U}(\theta, \epsilon_1, \epsilon_2) = e^{-i(\theta+\epsilon_1)\hat{J}_1} \otimes e^{-i\epsilon_2\hat{J}_2} \), where \( \hat{J}_{1,2} \) are collective spin operators for the first and second interferometer, respectively. The phase shift in the first (second) interferometer is \( \theta + \epsilon_1 (\epsilon_2) \), where \( \theta \) is the ‘signal phase’ to be estimated and \( \epsilon_1, \epsilon_2 \in [-\pi, \pi] \) is the phase noise accumulated during the

\(^5\) Quantum-enhanced noiseless differential interferometry is studied in [30].

\(^6\) An optical analogous to the differential scheme discussed here (a differential Michelson-Morley interferometer with squeezed states of light as input, in particular) has been recently investigated [31]. See also quenching in a correlated emission laser, originally proposed in [32] and experimentally demonstrated in [33].
interferometer operations. The values of \( \epsilon_1 \) and \( \epsilon_2 \) change randomly in repeated shots, with probability distribution \( P(\epsilon_1, \epsilon_2) \). Our general formalism does not assume a specific noise model and encompasses both Markovian and non-Markovian dephasing [we will later discuss specific forms of \( P(\epsilon_1, \epsilon_2) \) and focus on the case of a correlated interferometer, where \( \epsilon_1 = \pm \epsilon_2 \)]. We consider a general positive-operator value measure (POVM) \( \mu_B(\hat{\xi}) \) on the output state and use an unbiased estimator \( \Theta_{\text{est}}(\mu_1, \ldots, \mu_n) \), which is a function of the results obtained in \( m \) repeated independent measurements [36]. The variance of the estimator fulfills \( \Delta \Theta_{\text{est}} \geq \Delta \Theta_{\text{CR}} \) [34], where

\[
\Delta \theta_{\text{CR}} = \frac{1}{\sqrt{mF(\theta)}} \quad (1)
\]

is the the Cramer–Rao (CR) bound,

\[
F(\theta) = \sum_{\mu} \frac{1}{P(\mu|\theta)} \left( \frac{dP(\mu|\theta)}{d\theta} \right)^2 \quad (2)
\]

is the Fisher information (FI),

\[
P(\mu|\theta) = \int_{-\pi}^{\pi} d\epsilon_1 \int_{-\pi}^{\pi} d\epsilon_2 P(\epsilon_1, \epsilon_2) P(\mu|\theta, \epsilon_1, \epsilon_2) \quad (3)
\]

\[\text{Figure 1.}\] Differential scheme discussed in this manuscript. (a) Two Mach–Zehnder interferometers affected by shot-to-shot random phase noise \( \epsilon_1 \) and \( \epsilon_2 \). The signal \( \theta \) can be estimated in the presence of arbitrary noise, provided that relative noise fluctuations are sufficiently small. (b) Application to Bose–Einstein condensates (with spatial density represented by blue filled regions) trapped in a superlattice potential (gray curve). Splitting operations in each double-well are obtained by tuning the inter-well barrier. Short range forces between atoms and a nearby surface induce a phase shift \( \theta \). Trapping-potential fluctuations lead to correlations between \( \epsilon_1 \) and \( \epsilon_2 \).
are conditional probabilities, and \( P(\mu|\theta, e_1, e_2) = \text{Tr} [\hat{E}(\mu) \hat{U}(\theta, e_1, e_2) \hat{\rho} \hat{U}^{\dagger}(\theta, e_1, e_2)] \). In particular, if the state \( \hat{\rho} \) is separable in the two interferometers, \( \hat{\rho} = \hat{\rho}_1 \otimes \hat{\rho}_2 \), and the measurements in each interferometer are independent, \( \hat{E}(\mu) = \hat{E}_1(\mu_1) \otimes \hat{E}_2(\mu_2) \) \( [\mu \equiv (\mu_1, \mu_2)] \), then equation (3) becomes

\[
P(\mu_1, \mu_2|\theta) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \text{d}e_1 \text{d}e_2 P(e_1, e_2) P(\mu_1|\theta + e_1) P(\mu_2|e_2). \tag{4}
\]

Equation (1) takes into account the full quantum correlations of the interferometers outcomes and provides the lowest possible phase uncertainty, given the conditional probability distribution \( P(\mu|\theta) \). It can be saturated for large \( m \) by the maximum-likelihood estimator \([34]\).

3. Phase sensitivity

Here we calculate the highest sensitivity allowed by the above DI scheme. We rewrite equation (3) as

\[
\hat{\rho}_{\text{eff}} = \int_{-\pi}^{\pi} \text{d}e_1 \text{d}e_2 P(e_1, e_2) \hat{U}(0, e_1, e_2) \hat{\rho} \hat{U}^{\dagger}(0, e_1, e_2). \tag{5}
\]

The noisy differential interferometer with input \( \hat{\rho} \) is thus equivalent to a noiseless interferometer with effective input density matrix \( \hat{\rho}_{\text{eff}} \). This equivalence can be used to minimize \( \Delta \theta_{\text{CR}} \) over all possible POVMs \([35, 36]\). We have

\[
\Delta \theta_{\text{CR}} \geq \frac{1}{\sqrt{mF_Q[\hat{\rho}_{\text{eff}}]}}. \tag{6}
\]

where \( F_Q[\hat{\rho}_{\text{eff}}] = 4(\Delta \hat{R})^2 \) is the quantum Fisher information (QFI) and \( \hat{R} \) is obtained by solving \( \{\hat{R}, \hat{\rho}_{\text{eff}}\} = i [\hat{\rho}_{\text{eff}}, \hat{J}_1 \otimes 1_2] \). Taking \( |n_i\rangle \) the eigenbasis of \( \hat{J}_1 \), \( |n_1,n_2\rangle = n_1 |n_1\rangle, -N_i/2 \leq n_i \leq N_i/2 \), where \( i = 1, 2 \), labels the interferometer, we have

\[
|n_1, n_2\rangle |\hat{\rho}_{\text{eff}} \rangle |m_1, m_2\rangle = C_{m_1, m_2}^{n_1, n_2} |n_1, n_2\rangle |\hat{\rho} \rangle |m_1, m_2\rangle, \tag{7}
\]

where

\[
C_{m_1, m_2}^{n_1, n_2} = \int_{-\pi}^{\pi} \text{d}e_1 \text{d}e_2 P(e_1, e_2) e^{-i[e_1(n_1-m_1)+e_2(n_2-m_2)]}. \tag{8}
\]

Depending on \( P(e_1, e_2) \), the DI may admit a decoherence free subspace (DFS) spanned by states of the system that experience no evolution under the noise\(^7\). For uncorrelated noise \( P(e_1, e_2) = P_1(e_1)P_2(e_2) \), we obtain \( C_{m_1, m_2}^{n_1, n_2} = 1 \) if and only if \( n_{1,2} = m_{1,2} \); i.e., the DFS simply reduces to the eigenstates of \( \hat{J}_{1,2} \). These states are insensitive to the phase shift and thus useless for phase estimation. As common in several differential atom interferometers, we assume that \( P(e_1, e_2) = P_{\pm}(e_1)P_{\pm}(e_2) \), where \( e_{\pm} = (e_1 \pm e_2)/2 \) indicates the total (‘+’ sign) and relative (‘-’ sign) noise. Equation (8) becomes

\[
C_{m_1, m_2}^{n_1, n_2} = \tilde{P}_{\pm}(n_1-m_1+n_2-m_2)\tilde{P}_{\pm}(n_1-m_1-n_2+m_2). \tag{9}
\]

\(^7\) The concept of DFS for general quantum information purposes was introduced in \([36]\) and experimentally investigated in \([37]\).
where $\hat{P}_\pm(k) = \int_{-\epsilon}^\epsilon d\epsilon P_\pm(\epsilon) e^{-ik\epsilon}$. A non-trivial DFS, defined by the condition $n_1 + n_2 = m_1 + m_2$ [$n_1 - n_2 = m_1 - m_2$], exists in the limit of vanishing relative $P_-(\epsilon_-) = \delta(\epsilon_-)$ noise fluctuations. Such a DFS can be decomposed in subspaces defined by constant values of $M = \pm (N_1 + N_2)/2$, contain coherence terms and are thus relevant for phase estimation. For vanishing total (relative) noise and in presence of large relative (total) noise, $\rho_{\text{eff}}$ becomes block diagonal (see figure 2(a)), describing a statistical mixture of states with definite $M$ values. We thus write $\rho_{\text{eff}} = \sum Q\hat{\rho}_M$, where $\hat{\rho}_M = \hat{x}_M \hat{\rho} \hat{x}_M$, $\hat{x}_M$ are projectors into the fixed-$M$ subspace, and $Q_M = \text{Tr} [\hat{x}_M \hat{\rho} \hat{x}_M]$ are weights satisfying $\sum Q_M = 1$. We have $F_Q[\hat{\rho}_{\text{eff}}] \leq \sum Q_M F_Q[\hat{\rho}_M] \leq 4 \sum Q_M (\Delta \hat{J}_M)^2$, where $\Delta \hat{J}_M$ is the variance of $\hat{J}_M$ calculated for $\hat{\rho}_M$, and the second bound can be saturated by pure states. For separable states, following [16] and assuming, for simplicity, $N_1 = N_2 = N$, we have

$$F_Q[\rho_{\text{eff}}] \leq N - \sum_{M=M}^N (Q_M + Q_{-M}) [N - (N - M)^2] \leq N,$$

(10)

where $\hat{M}$ is the solution of $(N - M)^2 = N$. In general,

$$F_Q[\rho_{\text{eff}}] \leq N^2 - \sum_{M=1}^N (Q_M + Q_{-M})(2N - M)M \leq N^2.$$

(11)

In equations (10) and (11), the QFI is maximized by populating only the $M = 0$ subspace. We recover the same phase uncertainty bounds as in the ideal noiseless case: the SN limit,
\[ \Delta \theta_{SN} = 1/\sqrt{mN}, \] for separable states, and the HL, \[ \Delta \theta_{HL} = 1/\sqrt{mN}, \] for general quantum states. In other words, the condition \[ F_Q[\hat{\rho}_{eff}] > N \] is necessary and sufficient for reaching sub-SN sensitivities. Moreover, according to equation (10), overcoming the SN necessarily requires particle entanglement in the effective input state. In full analogy to the noiseless case, there exist optimal entangled states providing a quadratic enhancement of phase sensitivity even in the presence of large phase noise. The optimal states for DI are (in the \[ |m_1, m_2 \rangle \] basis, see figure 2(b))

\[
|\psi_{opt}\rangle = \begin{cases} 
\frac{|N/2, -N/2\rangle + | -N/2, N/2\rangle}{\sqrt{2}} & \text{if } P_-(e_-) = \delta(e_-) \\
\frac{|N/2, N/2\rangle + | -N/2, -N/2\rangle}{\sqrt{2}} & \text{if } P_+(e_+) = \delta(e_+) 
\end{cases}. \tag{12}
\]

These have \[ F_Q = N^2 \] and are not affected by phase noise. The states (12) have been experimentally realized with two [39] and up to eight [40] trapped ions and further investigated in [41]. While the saturation of the HL requires entangled interferometers, we can still have an HL scaling, i.e., \[ F_Q \propto N^2, \] if we consider states which are separable in the two interferometers \[ \hat{\rho} = \hat{\rho}_1 \otimes \hat{\rho}_2. \] A prominent example is the product of NOON states,

\[
|\psi_{NOON}\rangle = \left( \frac{|N/2\rangle + | -N/2\rangle}{\sqrt{2}} \right)_1 \otimes \left( \frac{|N/2\rangle + | -N/2\rangle}{\sqrt{2}} \right)_2, \tag{13}
\]

which, as shown in figure 2(c) does not (entirely) belong to the DFS and has \[ F_Q = N^2/4 \] both when \[ P_-(e_-) = \delta(e_-) \] and when \[ P_+(e_+) = \delta(e_+). \]

### 4. Differential interferometry with NOON states

In this section we study the DI scheme \[ \hat{U}(\theta, \epsilon_1, \epsilon_2) = (e^{-i\tilde{J}z_1} e^{-i(\theta+\epsilon_1)\tilde{J}_1})_1 \otimes (e^{-i\tilde{J}z_2} e^{-i\epsilon_2\tilde{J}_2})_2, \] each interferometer being represented by a phase shift rotation around the \[ z \] axis followed by a 50–50 beam splitter. We further assume the phase noise distribution \[ P(\epsilon_1, \epsilon_2) = P_+(\epsilon_+)P_-(\epsilon_-). \] As an input state we take the direct product of NOON states, \[ |\psi_{NOON}\rangle \equiv |\text{NOON}\rangle_z \otimes |\text{NOON}\rangle_z, \] with components along the \[ z \] direction, \[ |\text{NOON}\rangle_z = (|N/2\rangle_z + | -N/2\rangle_z)/\sqrt{2} \] [42], \[ |\mu\rangle_z \] being an eigenstate of \[ \hat{J}_z \] with eigenvalue \[ \mu. \] In the following we provide the conditional probabilities and FI when \[ P_+(\epsilon_+) \] are even functions of \[ e_+ \] and specialize to the case a Gaussian noise distribution. For a discussion on more general noise functions, see appendix C.

The probability of measuring a relative number of particles \[ \mu_1 \] at the output of interferometer 1 and \[ \mu_2 \] at the output of interferometer 2 can be calculated analytically and is given by (see appendix C for details on the derivation of the equations below)

\[
P(\mu_1, \mu_2|\theta) = \left( \frac{N!}{2^N} \right)^2 \frac{1}{(N/2 - \mu_1)! (N/2 + \mu_1)! (N/2 - \mu_2)! (N/2 + \mu_2)!} \times 
\left[ 1 + (-1)^{\mu_2} V_N^+ V_N^+ + (-1)^{\mu_1} V_N^+ V_N^+ + \frac{(-1)^{\mu_2}}{2} \left( V_{2N}^+ V_{2N}^+ + V_{2N}^+ V_{2N}^+ \right) \cos N\theta \right]. \tag{14}
\]
where $V_k^\pm \equiv \int_\pi^\pi d\epsilon \sum P_\pm (\epsilon) \cos (K\epsilon)$. The FI is

$$F(\theta) = \frac{N^2\sin^2N\theta}{2} \left[ \frac{A_+B_-^2}{A_-^2 + B_-^2 \cos^2N\theta} + \frac{A_+B_+^2}{A_+^2 - B_+^2 \cos^2N\theta} \right].$$

(15)

where $A_\pm (N) \equiv 1 \pm V_N^\pm V_N^\mp, B_\pm (N) \equiv V_N^\pm V_N^\mp \pm (V_{2N}^+ + V_{2N}^-)/2$. The optimal value of the FI, $F \equiv \max_\theta F(\theta)$, is reached for $\cos N\theta = 0$,

$$F = \frac{N^2}{2} \left[ \frac{B_-^2 (N)}{A_- (N)} + \frac{B_+^2 (N)}{A_+ (N)} \right].$$

(16)

Let us discuss the different limit values of equation (16), taking into account that $V_k^\pm = 1$ when $P_\pm (\epsilon) = \delta(\epsilon)$ and $V_k^\pm = 0$ when $P_\pm (\epsilon) = 1/2\pi$. If relative noise fluctuations are vanishingly small, $P_\pm (\epsilon) = \delta(\epsilon)$, equation (16) ranges from $F = N^2$ (if also $P_\pm (\epsilon) = \delta(\epsilon)$, corresponding to the ideal noiseless limit) to $F = N^2/2$ (when $P_\pm (\epsilon) = 1/2\pi$). In other words, if the relative noise between the two interferometers is fixed, a phase sensitivity at the HL can be obtained for an arbitrary large total noise (i.e., arbitrary large noise in each interferometer). If total noise fluctuations are large, $P_\pm (\epsilon) = 1/2\pi$, we obtain $F = N^2 (V_{2N}^\pm)^2$, which predicts the HL for $V_{2N}^\pm \approx 1$ and sub-SN for $V_{2N}^\pm > 1/\sqrt{N}$. In figure 3(c) we plot equation (16) as a function of $\sigma_\pm$, taking

$$P(\epsilon) = \frac{e^{(\cos \epsilon_\pm)/\sigma_\pm^2}}{2\pi I_0(1/\sigma_\pm^2)}.$$

(17)

where $I_0(x)$ is the modified Bessel function of the first kind. This noise function continuously interpolates from a Gaussian distribution of width $\sigma_\pm$, when $\sigma_\pm \ll 1$, to a flat distribution, when $\sigma_\pm \gg 1$. The condition $V_{2N}^\pm \approx 1$ is thus equivalent to $\sigma^- \approx 1/N$, while $V_{2N}^\pm \gg 1/\sqrt{N}$ is recovered.
for $\sigma_- \lesssim \sqrt{\log N}/N$. These results show that reaching the HL in the differential interferometer requires relative noise fluctuations at the HL itself.

5. Precision limit for differential interferometry with Bose–Einstein condensates

In this section we discuss a differential Mach–Zehnder (MZ) interferometer $\hat{U}(\theta, \epsilon_1, \epsilon_2) = (e^{-i(\theta+\epsilon_1)\hat{J}_1})_1 \otimes (e^{-i\epsilon_2\hat{J}_2})_2$ with input states that can be created with two-mode BECs. We consider phase estimation from the measurement of the number of particles in output. We further take vanishing relative phase noise fluctuations in the two interferometers, $P_\pm(e_-) = \delta(e_-)$, and $P(e_+)$, given by equation (17).

We first consider adiabatic state preparation [43], focusing on the ground state $|\psi_{gs}(\Lambda)\rangle$ of the Hamiltonian $\hat{H} = \hbar^2\hat{J}_z^2 - \hbar\Omega \hat{J}_z$, with $\Lambda \equiv N\chi/\Omega$. This can be implemented in a double-well trap (see figure 1(b)) with $\chi$ and $\Omega$ interaction and tunneling parameters, respectively [19, 22]. We can distinguish Rabi $0 < \Lambda \leq 1$, Josephson $1 < \Lambda \leq N^2$, and Fock $\Lambda > N^2$ regimes. In the ideal case, these regimes are characterized by different scalings of the FI (optimized in $\theta$): $F \sim N$ ($F = N$ for the spin coherent state, $|N/2\rangle_x$, $\Lambda = 0$), $F \sim N^{3/4}$, and $F \sim N^2$.

![Figure 4](image_url)
(\(F = N^2/2 + N\) [44] for the twin-Fock state, \(|\theta\rangle_z, \Lambda = \infty\)), respectively [45]. In figure 4(a) we report the FI (optimized in \(\theta\)) as a function of \(\Lambda\) for the differential MZ with input \(|\psi_{gs}(\Lambda)\rangle \otimes |\psi_{gs}(\Lambda)\rangle\). Different lines refer to different values of \(N \times \sigma_+\), ranging from the noiseless case (\(\sigma_+ = 0\), thick dashed line) to the uniform phase noise (\(\sigma_+ = \infty\), thick solid line). The twin-Fock is optimal for \(\sigma_+ \lesssim 1/N\), while for \(\sigma_+ \to \infty\), the FI is maximized at \(\Lambda \approx N\).

We further consider states that are created by the nonlinear evolution \(|\psi_{dyn}(\tau)\rangle = e^{i\hat{\alpha}_\tau} e^{-i/2} |\psi_{\Lambda} \rangle \otimes |\psi_{\Lambda} \rangle\), starting from a spin coherent state [14, 20, 46], where \(\tau = \chi t\). Here \(e^{i\hat{\alpha}_\tau}\) rotates the state so to maximize the FI in the noiseless case [16]8. In figure 4(b) we show the FI for the differential MZ with input \(|\psi_{dyn}(\tau)\rangle \otimes |\psi_{dyn}(\tau)\rangle\). Lines are as in figure 4(a).

Interestingly, the short time dynamics (\(\tau \lesssim 1/\sqrt{N}\), where spin squeezing is created [46]) is robust, and sub-SN is found also for large phase noise. The characteristic plateau (\(F = N^2/2\) for the ideal case [16]) is washed out when \(\sigma_+ \gtrsim 1/N\). The FI at large values of \(\sigma_+\) is characterized by several peaks, the most prominent found in correspondence to the creation of macroscopic superposition states (‘phase cats’) with multiple (larger than two) components. The long-time dynamics at \(\tau = \pi/2\) leads to maximally entangled states (a two-components phase cat) having \(F = N^2/4\), as discussed in the previous section.

We have repeated the previous analysis for specific input states and large values of \(N\) (up to \(N \approx 1000\)) and \(\sigma_+ = \infty\) (flat total noise case). The results are shown in figure 4(c). The FI reaches an asymptotic power law scaling \(F = \beta N^\alpha\): \(\beta = 0.5, \alpha = 1\) for coherent spin state (green diamonds); \(\beta = 0.2, \alpha = 1.5\) for the optimal states of the adiabatic preparation at \(\Lambda \approx N\) (red squares); \(\beta = 0.3, \alpha = 1.4\) for the optimal states of the diabatic preparation at time \(\tau \approx 1/N^{3/4}\) (blue circles); \(\beta = 0.39, \alpha = 1.17\) for the twin-Fock state (black dots). The solid black line is the analytical NOON state result (\(\alpha = 2, \beta = 1/4\)) discussed previously. The twin-Fock state is an interesting and experimentally relevant [21] example. It is strongly entangled and reaches an HL scaling in the single noiseless MZ; however, it performs only slightly better than the SN in the differential MZ with large noise and a large number of particles. In figure 4(d) we further investigated the FI for the twin-Fock state as a function of \(N\) for different values of \(\sigma_+\) (dots). For \(N \lesssim 1/\sigma_+\) and \(\sigma_+ \ll 1\), the FI follows the ideal behavior \(F = N^2/2 + N\) (dashed line). For \(N \gg 1/\sigma_+\), we recover roughly the same scaling of FI \((F \propto N^{1.17})\) as in the large phase noise case.

6. Conclusions

In this manuscript we have extended the analysis of DI to the domain of entangled states. It is not obvious, a priori, that DI can suppress spurious phase noise when highly entangled—and thus extremely fragile against phase noise fluctuations—states are used. Our analysis reveals that when the phase noise is perfectly correlated in the two interferometers, and losses can be neglected, there exists a DFS where entanglement is passively protected. We have thus identified a class of entangled input state that can provide a sub-SN sensitivity in a differential interferometer up to the HL, even for large noise. This class is nontrivial, fully characterized by the FI, and includes states that have been recently created experimentally. We expect our results to be a guideline for quantum-enhanced realistic interferometers in the near future.

8 For the effect of phase noise in the state preparation see [47].
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Appendix A. Derivation of equation (3)

We here derive equation (3) from first principles. We start from the joint probability density

\[ P(\mu, \theta, \epsilon_1, \epsilon_2) = \int_{-\pi}^{\pi} d\epsilon_1 d\epsilon_2 P(\mu, \theta, \epsilon_1, \epsilon_2) \]

so as to eliminate nuisance parameters. By using the relation \( P(x, y) = P(x|y)P(y) \) between joint and conditional probabilities, where \( x \) and \( y \) are random variables, we have

\[ P(\mu|\theta) = \frac{P(\mu, \theta)}{P(\theta)} = \int_{-\pi}^{\pi} d\epsilon_1 d\epsilon_2 P(\mu|\theta, \epsilon_1, \epsilon_2)P(\epsilon_1, \epsilon_2|\theta). \]

Since \( \epsilon_{1,2} \) do not depend on \( \theta \), i.e., \( P(\epsilon_1, \epsilon_2|\theta) = P(\epsilon_1, \epsilon_2) \), we recover equation (3).

Appendix B. Derivation of inequalities (10) and (11)

Here we detail the calculation of \( 4 \sum_M Q_M(\Delta \hat{J}_l)_M^2 \), where \( (\Delta \hat{J}_l)_M^2 \) is the variance of the operator \( \hat{J}_l \) calculated on the fixed-\( M \) subspace. We consider the case \( N_1 = N_2 = N/2 \) for simplicity. We have

\[ 4(\Delta \hat{J}_l)_M^2 \leq (\max_M n_1 - \min_M n_1)^2, \]

where equality of \( \rho_M \) is the equal-weighted superposition of states with the maximum and the minimum value of \( n_1 \) in the fixed-\( M \) DFS. In general, \( \max_M n_1 = \min (M + N/2, N/2) \), and \( \min_M n_1 = \max (M - N/2, -N/2) \). We thus have

\[ 4\left(\Delta \hat{J}_l\right)_M^2 \leq (N - |M|)^2, \]

where equality of \( \rho_M \) is the equal-weighted superposition of states with the maximum and the minimum value of \( n_1 \) in the fixed-\( M \) DFS. Taking into account that \( \sum_{M = -N}^{N} Q_M = 1 \)

\[
4 \sum_M Q_M(\Delta \hat{J}_l)_M^2 = Q_0 N^2 + \sum_{M=1}^{N} (Q_M + Q_{-M})(N - M)^2 \\
= N^2 - \sum_{M=1}^{N} (Q_M + Q_{-M})M(2N - M).
\]

We thus recover equation (11). Since \( M \leq 2N \), the second term in the equation above is nonnegative, and we find \( 4 \sum_M Q_M(\Delta \hat{J}_l)_M^2 \leq N^2 \). For separable states, we need to further take into account that \( 4(\Delta \hat{J}_l)_M^2 \leq N \) [16]. We thus have

\[ 4(\Delta \hat{J}_l)_M^2 \leq \min \left[(N - |M|)^2, N\right]. \]
and obtain

\[ 4 \sum_{M} Q_{M}(\Delta \hat{J})_{M}^{2} = Q_{0}N + \sum_{M=1}^{N} (Q_{M} + Q_{-M}) \min[(N - M)^{2}, N] \]

\[ = Q_{0}N - \sum_{M=M}^{N} (Q_{M} + Q_{-M}) \left[ N - (N - M)^{2} \right], \]

which follows, since for \( \tilde{M} \leq M \leq N \) we have \( \min[N, (N - M)^{2}] = (N - M)^{2} \). We recover equation (10). Since for \( \tilde{M} \leq M \leq N \), we have \( N - (N - M)^{2} \geq 0 \). The second term in the above equation is nonnegative, and we have \( 4 \sum_{M} Q_{M}(\Delta \hat{J})_{M}^{2} \leq N \) for separable states.

**Appendix C. Extension of the discussion of section 4 to arbitrary noise distributions**

Here we provide a detailed derivation of the equations presented in section 4 and extend the discussion to arbitrary noise distributions. Let us first calculate the conditional probability distribution of the relative number of particles for the single interferometer with a NOON probe state:

\[ P(\mu|\phi) = \left\{ \begin{array}{l}
\frac{\varepsilon^{2}N!}{2^{N/2}N!} \left| e^{-i\frac{\mu}{2}} N \right|^{2} \\
\frac{\varepsilon^{2}N!}{2^{N/2}N!} \left| e^{-i\frac{\mu}{2}} N \right|^{2}
\end{array} \right. \]

The rotation matrix elements \( \varepsilon\langle \mu | e^{-i\frac{\mu}{2}} | \pm N/2 \rangle \) are given by

\[ \varepsilon\langle \mu | e^{-i\frac{\mu}{2}} + N/2 \rangle = e^{-i\frac{\mu}{2}} \frac{N!}{2^{N/2}N!} \sqrt{(N/2 - \mu)!}(N/2 + \mu)! \]

\[ \varepsilon\langle \mu | e^{-i\frac{\mu}{2}} - N/2 \rangle = e^{-i\frac{\mu}{2}} \frac{N!}{2^{N/2}N!} \sqrt{(N/2 + \mu)!}(N/2 + \mu)! \]

We thus obtain

\[ P(\mu|\phi) = \frac{N!}{2^{N} (N/2 - \mu)!((N/2 + \mu)!} \left| e^{-i\frac{\mu}{2}} + (-1)^{\mu}e^{i\frac{\mu}{2}} \right|^{2}, \]

with

\[ \left| e^{-i\frac{\mu}{2}} + (-1)^{\mu}e^{i\frac{\mu}{2}} \right|^{2} = \left\{ \begin{array}{l}
1 + (-1)^{\mu} \cos N\phi \quad \text{if } N \text{ is even}, \\
1 + (-1)^{\mu+1/2} \sin N\phi \quad \text{if } N \text{ is odd}.
\end{array} \right. \]

We now consider the differential sensor described by the unitary operator \( \hat{U}(\theta, \epsilon_{1}, \epsilon_{2}) = (e^{-i\frac{\theta}{2}}e^{-i(\theta+\epsilon_{1})\frac{\epsilon_{1}}{2}})_{1} \otimes (e^{-i\frac{\theta}{2}}e^{-i(\phi_{2})\frac{\epsilon_{2}}{2}})_{2} \), each interferometer being given by the transformation \( (e^{-i\frac{\theta}{2}}e^{-i(\phi_{i})\frac{\epsilon_{i}}{2}})_{i} \), \( i = 1, 2 \), \( \phi_{1} = \theta + \epsilon_{1} \) and \( \phi_{2} = \epsilon_{2} \). We take a NOON state of \( N \) particles as the input of each interferometer (without loss of generality, we assume \( N \)
to be even) and estimate the phase shift from the measurement of the relative number of particles at the output ports of each interferometer, $\hat{E}(\mu) \equiv \hat{E}(\mu_1, \mu_2) = (|\mu_1\rangle\langle\mu_1|) \otimes (|\mu_2\rangle\langle\mu_2|)$. Taking $P(\epsilon_1, \epsilon_2) = P_+(\epsilon_+)P_-(\epsilon_-)$, where $\epsilon_{\pm} = (\epsilon_1 \pm \epsilon_2)/2$, equation (4) writes

$$P(\mu_1, \mu_2) = \int_{-\pi}^{\pi} d\epsilon_+ P_+(\epsilon_+) \int_{-\pi}^{\pi} d\epsilon_- P_-(\epsilon_-) \times P(\epsilon_1 + \epsilon_+, \epsilon_- + \epsilon_-)P(\mu_1, \mu_2),$$

with $P(\mu_i, \epsilon_i)$ ($i = 1, 2$) given by equation (C.1). After straightforward algebra, we obtain

$$P(\mu_1, \mu_2) = \left( \frac{N!}{2^N} \right)^2 A_N(\mu_2) + C_N(\mu_1, \mu_2) \cos N\theta - S_N(\mu_1, \mu_2) \sin N\theta,$$

where

$$A_N(\mu_2) = 1 + (-1)^{\mu_2} \left[ V_N^+V_N^- + W_N^+W_N^- \right],$$

$$C_N(\mu_1, \mu_2) = (-1)^{\mu_1} \left[ V_N^+V_N^- - W_N^+W_N^- \right] + (-1)^{\mu_1+\mu_2} \left( \frac{V_{2N}^+ + V_{2N}^-}{2} \right),$$

$$S_N(\mu_1, \mu_2) = (-1)^{\mu_1} \left[ V_N^+W_N^- + W_N^+V_N^- \right] + (-1)^{\mu_1+\mu_2} \left( \frac{W_{2N}^+ + W_{2N}^-}{2} \right).$$

and

$$V_K^\pm \equiv \int_{-\pi}^{\pi} d\epsilon_+ P_\pm(\epsilon_\pm) \cos (K \epsilon_\pm), W_K^\pm \equiv \int_{-\pi}^{\pi} d\epsilon_+ P_\pm(\epsilon_\pm) \sin (K \epsilon_\pm),$$

$K$ being an integer number. We are now ready to compute the FI, equation (2),

$$F(\theta) = \sum_{\mu_1, \mu_2 = -N/2}^{N/2} \frac{1}{P(\mu_1, \mu_2) \theta} \left( \frac{dP(\mu_1, \mu_2) \theta}{d\theta} \right)^2.$$

The FI can be written as the sum of three terms:

$$F(\theta) = N^2 \left[ F_C(N, \theta) \cos^2 N\theta + F_{SC}(N, \theta) \sin 2N\theta + F_S(N, \theta) \sin^2 N\theta \right],$$

where the coefficients $F_C(N, \theta)$, $F_S(N, \theta)$, and $F_{SC}(N, \theta)$ are functions of $N$ and $N\theta$ and are given by sums over $\mu_1$ and $\mu_2$. To compute the sums, we separate the sum over odd $\mu_{1,2}$ into the sum over odd $\mu_{1,2}$ and sum over even $\mu_{1,2}$ (since $N$ is assumed to be even, $\mu_1$ and $\mu_2$ are integer numbers) and take into account that

$$\sum_{\mu, \text{odd}} \frac{1}{2^N} \frac{N!}{(N/2 - \mu)! (N/2 + \mu)!} = \sum_{\mu, \text{even}} \frac{1}{2^N} \frac{N!}{(N/2 - \mu)! (N/2 + \mu)!} = \frac{1}{2}.$$
We thus obtain
\[
F_C(N, \theta) = \frac{1}{2} A_N^2(1) - \left[ C_N(0, 1) \cos N\theta - S_N(0, 1) \sin N\theta \right]^2 + \frac{1}{2} A_N^2(0) - \left[ C_N(0, 0) \cos N\theta - S_N(0, 0) \sin N\theta \right]^2.
\]
\[
F_S(N, \theta) = \frac{1}{2} A_N^2(1) - \left[ C_N(0, 1) \cos N\theta - S_N(0, 1) \sin N\theta \right]^2 + \frac{1}{2} A_N^2(0) - \left[ C_N(0, 0) \cos N\theta - S_N(0, 0) \sin N\theta \right]^2,
\]
and
\[
F_{SC}(N, \theta) = \frac{1}{2} A_N^2(1) - \left[ C_N(0, 1) \cos N\theta - S_N(0, 1) \sin N\theta \right]^2 + \frac{1}{2} A_N^2(0) - \left[ C_N(0, 0) \cos N\theta - S_N(0, 0) \sin N\theta \right]^2.
\]

The above equations allow us to calculate the FI, given arbitrary relative and total noise functions. For the case of NOON input states considered here, the FI ultimately depends on the eight Fourier coefficients $\pm V_1^N$, $\pm V_2^N$, $\pm W_1^N$, and $\pm W_2^N$. Below, we first show how the calculation of the FI simplifies when noise distributions are even functions of $\epsilon_\pm$. Furthermore, we study the case of perfectly correlated relative noise and arbitrary total noise distribution.

**Symmetric noise distributions.** If $P_\pm(\epsilon_\pm)$ are even functions of $\epsilon_\pm$, the calculation of the Fisher information simplifies notably. We have $W_K^2 = 0$, which implies $S_N(\mu_1, \mu_2) = 0$ for all $\mu_1$ and $\mu_2$. $F_C(N, \theta) = 0$, and $F_{SC}(N, \theta) = 0$. We also have $A_N^2(0) = 1 + V_N^+ V_N^- = A_+, A_N^2(1) = 1 - V_N^+ V_N^- = A_-, C_N(0, 1) = V_N^+ V_N^- - (V_{2N}^+ + V_{2N}^-)/2 = B_-$, and $C_N(0, 0) = V_N^+$, $V_N^- + (V_{2N}^+ + V_{2N}^-)/2 = B_+$, where $A_\pm$ and $B_\pm$ have been introduced in section 4. The conditional probability and the FI reduce to equations (14) and (15), respectively.

**Perfectly correlated relative noise.** In the following we consider the ideal case of perfectly correlated relative noise, $P_{\pm}(\epsilon_-) = \delta(\epsilon_-)$. This implies $V_N^- = V_{2N}^- = 1$, $W_N^- = W_{2N}^- = 0$, and equations (C.2) simplify to
\[
A_N(\mu_2) = 1 + (-1)^{\mu_2} V_N^+,
\]
\[
C_N(\mu_1, \mu_2) = (-1)^{\mu_1} V_N^+ + (-1)^{\mu_1+\mu_2} \left( \frac{1 + V_{2N}^+}{2} \right),
\]
\[
S_N(\mu_1, \mu_2) = (-1)^{\mu_1} W_N^+ + (-1)^{\mu_1+\mu_2} \frac{W_{2N}^+}{2}.
\]

These equations are the basis for further considerations. For instance, if $P_\epsilon(\epsilon)$ (we indicate $\epsilon \equiv \epsilon_+$ to simplify the notation) is an odd function of $\epsilon$ plus a constant providing normalization in the $2\pi$ interval, then $V_K^+ = 0$, and we can expand it in Fourier series as
The condition \( \varepsilon \geq \frac{P}{0} \) implies \( \pi \varepsilon \sum \leq \frac{\infty}{+} + W_K K \), which, integrating over \( \varepsilon \), gives \( \sum \leq \frac{\infty}{+} + W_K K \). In this case, evaluating \( \theta \equiv \theta \max () \) at phase values \( \theta = N \cos 0 \), we have (we recall that \( \theta \equiv \theta \max () \))

\[
P_\varepsilon (\varepsilon) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{K=1}^{+\infty} W_K^+ \sin Ke.
\]

The condition \( P_\varepsilon (\varepsilon) \geq 0 \) implies \( 4\pi^2 |\sum_{K=1}^{+\infty} W_K^+ \sin Ke|^2 \leq 1 \), which, integrating over \( \varepsilon \), gives \( \sum_{K=1}^{+\infty} (W_K^+)^2 \leq 1/2 \). In this case, evaluating \( F(\theta) \) at phase values \( \theta \) such that \( \cos N\theta = 0 \), we have (we recall that \( F \equiv \max_\theta F(\theta) \))

\[
F \geq \frac{N^2}{8} \left( \frac{1}{1 - (W_N - W_{2N}^+)/2} + \frac{1}{1 - (W_N + W_{2N}^+)/2} \right).
\]

The term between brackets does not diverge because of the condition \( \sum_{K=1}^{+\infty} (W_K^+)^2 \leq 1/2 \), and it is always larger than two. It implies that in this case, \( F \geq N^2/4 \). To treat a more general case, we consider the noise distribution

\[
P_\varepsilon (\varepsilon) \propto \sum_{n=1}^{M} e^{\cos(x_n)/\sigma^2}
\]

which is a normalized sum of \( M \) peaks of width \( \sigma \) (for \( \sigma < 1 e^{\cos(x_n)/\sigma^2} \approx e^{-(x_n)^2/2\sigma^2} \), the cos function being used to take into account the \( 2\pi \)-periodicity) centered at random positions \( x_1, x_2, \ldots, x_M \in [-\pi, \pi] \). For random choices of \( x_1, x_2, \ldots, x_M \), we calculate the FI and maximize over \( \theta \). In figure C1 we plot the statistical distribution of \( F = \max_\theta F(\theta) \) as a function of \( N \) and \( M \).

For sufficiently large values of \( N \) and/or \( M \), the noise distribution (C.5) has vanishing high frequency Fourier components. When increasing \( M \) (at a fixed value of \( N \) and \( \sigma \)), this is due to the fact that the noise distribution tends to become flat in most of the random realizations (i.e.,
for most of the random choices of \(x_1, x_2, ..., x_M\). When increasing \(N\) (at fixed \(\sigma \) and \(\sigma\)), this is due to the vanishing tails in the Fourier spectrum of \(e^{\cos(\xi_n)/\sigma^2}\). In both cases, the coefficients \(V_N^+, W_N^+, V_{2N}^+, \) and \(W_{2N}^-\) are vanishing small, and we have

\[
A_N(\mu_2) = 1, \quad C_N(\mu_1, \mu_2) = \frac{(-1)^{\mu_1+\mu_2}}{2}, \quad S_N(\mu_1, \mu_2) = 0,
\]

and

\[
F(\theta) = \frac{N^2 \sin^2 N\theta}{4 - \cos^2 N\theta},
\]

giving \(F \equiv \max_\theta F(\theta) = N^2/4\). In figure C1 we indeed observe that the distribution of \(F\) peaks around 1/4 for sufficiently large values of \(N\) and \(M\).

For small values of \(M\) and \(N\), we may have a situation where \(F/N^2\) is very small. In general, for a fixed number of particles, it is possible to derive pathologic noise distributions for which the FI vanishes. To see this, it is convenient to rewrite \(F(\theta)\) as

\[
\frac{F(\theta)}{N^2} = \frac{(1 - V_N^+)U_1^2(\theta)}{2D_1(\theta)} + \frac{(1 + V_N^+)U_0^2(\theta)}{2D_0(\theta)},
\]

with

\[
U_j(\theta) = \left( W_N^+ + (-1)^j \frac{W_{2N}^+}{2} \right) \cos N\theta
\]

\[
+ \left( V_N^+ + (-1)^j \frac{1 + V_{2N}^+}{2} \right) \sin N\theta,
\]

and

\[
D_j(\theta) = \left( 1 - (-1)^j V_N^+ \right)^2 - \left[ \left( V_N^+ + (-1)^j \frac{1 + V_{2N}^+}{2} \right) \cos N\theta \right.
\]

\[
- \left. \left( W_N^+ + (-1)^j \frac{W_{2N}^+}{2} \right) \sin N\theta \right]^2,
\]

with \(j = 0,1\). It’s possible to demonstrate that \(D_{1,0} > 0 \ \forall \ \theta\). Therefore the Fisher is zero only if both numerators are zero. It is also possible to see that the cases involving \(V_N^+ = \pm 1\) and \(U_{0,1} = 0\) lead to nonphysical probability distributions. The only remaining option is to have both \(U_{0,1}(\theta) = 0 \ \forall \ \theta\). This in turn corresponds to a probability distribution with \(V_N^+ = 0\), \(W_N^+ = 0\), \(W_{2N}^+ = 0\), and \(V_{2N}^- = -1\). Recalling the definition of \(V_{2N}^+\), we thus have that \(F(\theta) = 0\) if and only if

\[
\int \text{d} e \ P(e) \cos^2 Ne = 0. \quad (C.8)
\]

This integral involves two positive functions. Equation (C.8) is thus fulfilled only if \(P(e)\) has support in correspondence to the zeroes of \(\cos Ne\). A total noise distribution \(P(e)\) for which the FI vanishes is therefore obtained, as a normalized sum of Dirac deltas symmetrically centered at the zeroes of \(\cos Ne\). We argue that this situation is pathological for NOON states, where the FI
is entirely determined by the Fourier components of $P(\varepsilon)$, equation (C.3), at $K = N$ and $K = 2N$. Furthermore, if $P(\varepsilon)$, instead of being a sum of Dirac peaks, is a sum of peaks of finite width, we recover, as noticed above, equation (C.6) for $N$ sufficiently large.

Appendix D. Numerical method to compute the Fisher information

Here we report a method for the numerical calculation of the FI that we used to obtain the results of section 5. We consider a differential interferometer and indicate with $\mu_1$ and $\mu_2$ the results of a measurement at the outputs of the two devices. The differential interferometer transformation is $\hat{U}(\theta, \varepsilon_1, \varepsilon_2) = e^{-i(\theta+\varepsilon_1)\hat{h}} \otimes e^{-i\varepsilon_2\hat{h}}$, and the joint conditional probability reads

$$P(\mu_1, \mu_2|\theta) = \int_{-\pi}^{\pi} de \left(\mu_1|\theta + e\right)P\left(\mu_2|e\right)P(e),$$  \hspace{1cm} (D.1)

where we have assumed $P_-(\varepsilon_-) = \delta(\varepsilon_-)$. Noticing that the functions $P(\mu_i|x)$, $i = 1, 2$ are $2\pi$ periodic in $x$, it is therefore possible to make a Fourier expansion of the functions. This is conveniently done with a fast Fourier transform algorithm. Furthermore, the discretized atom number poses a maximum allowed frequency in the decomposition given by Shannon’s criterion:

$$P(\mu_i|x) = \frac{1}{2N} \sum_{k=-N}^{N} a_k(\mu_i) \cos(kx) + b_k(\mu_i) \sin(kx),$$

where $a_k(\mu_i)$ and $b_k(\mu_i)$ are Fourier coefficients of $P(\mu_i|x)$. We thus find

$$P(\mu_1, \mu_2|\theta) = \frac{1}{4} \sum_{k=-N}^{N} A_k^{(\eta)}(\mu_1, \mu_2) \cos(k\theta) + B_k^{(\eta)}(\mu_1, \mu_2) \sin(k\theta),$$

where the coefficients are given by

$$A_k^{(\eta)}(\mu_1, \mu_2) = a^T(\mu_1)C \cdot a(\mu_2) + b^T(\mu_1)S \cdot b(\mu_2)$$

and

$$B_k^{(\eta)}(\mu_1, \mu_2) = b^T(\mu_1)C \cdot a(\mu_2) - a^T(\mu_1)S \cdot b(\mu_2),$$

$a(\mu_i) \equiv (a_{-N}(\mu_i), ..., a_N(\mu_i))$ (and an analogous definition for $b(\mu_i)$) are vectors of Fourier coefficients, and the matrices $C$ and $S$ have components

$$C_{k,k'} \equiv \int_{0}^{2\pi} de \ P(e) \cos(ke) \cos(k'e), \quad S_{k,k'} \equiv \int_{0}^{2\pi} de \ P(e) \cos(ke) \sin(k'e),$$

respectively. Notice that it is immediate to take the derivative of equation (D.1)) with respect to $\theta$, as required for the calculation of the FI. This is an advantage of this method.

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