LITTLEWOOD–PALEY–RUBIO DE FRANCIA INEQUALITY FOR THE TWO-PARAMETER WALSH SYSTEM

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A version of Littlewood–Paley–Rubio de Francia inequality for the two-parameter Walsh system is proved: for any family of disjoint rectangles $I_k = I_{k_1} \times I_{k_2}$ in $\mathbb{Z}_+ \times \mathbb{Z}_+$ and a family of functions $f_k$ with Walsh spectrum inside $I_k$ the following is true:

$$\left\| \sum_k f_k \right\|_{L^p} \leq C_p \left( \sum_k |f_k|^2 \right)^{1/2} \left\| \sum_k |f_k|^2 \right\|_{L^p}, \quad 1 < p \leq 2,$$

where $C_p$ does not depend on the choice of rectangles $\{I_k\}$ or functions $\{f_k\}$. The arguments are based on the atomic theory of two-parameter martingale Hardy spaces. In the course of the proof, a two-parameter version of Gundy’s theorem on the boundedness of operators taking martingales to measurable functions is formulated, which might be of independent interest. Bibliography: 24 titles.

1. Introduction

Consider a countable index set $\mathbb{Z}$ and an orthonormal basis $\{\phi_n\}_{n \in \mathbb{Z}}$ in the space $L^2$. Define operators $M_I$ for $I \subseteq \mathbb{Z}$ by $M_I f = \sum_{n \in I} \langle f, \phi_n \rangle \phi_n$. Whenever $M_I f = f$, we say that the spectrum of $f$ lies in $I$ and write $\text{spec } f \subseteq I$.

Consider also a partition $\{I_k\}_{k \in \mathbb{N}}$ of the index set $\mathbb{Z}$ and a family of functions $f_k \in L^2$ such that $\text{spec } f_k \subseteq I_k$. Then the following equality holds:

$$\left\| \sum_{k=1}^{\infty} f_k \right\|_{L^2} = \left( \sum_{k=1}^{\infty} |f_k|^2 \right)^{1/2} \left\| \sum_{k=1}^{\infty} |f_k|^2 \right\|_{L^2} .$$

(1)

This follows directly from Parseval’s identity and generalizes, in a sense, this classical result: if $I_k$ are singletons, then we recover Parseval’s identity.

Clearly, if we replace both $L^2$ norms in equation (1) by $L^p$ norms with certain $p \neq 2$, then the identity is not valid. In this case, it is interesting to study a weaker kind of relationships between the left hand side and the right hand side of (1). For instance, for some bases $\{\phi_n\}$ and specific partitions $\mathbb{Z} = \bigcup_{k \in \mathbb{N}} I_k$, this or that one-sided inequality with a multiplicative constant might be true.

The most famous assertion of this kind is the Littlewood–Paley inequality

$$c_p \left( \sum_{k=1}^{\infty} |f_k|^2 \right)^{1/2} \left\| \sum_{k=1}^{\infty} f_k \right\|_{L^p} \leq \sum_{k=1}^{\infty} f_k \left\| \sum_{k=1}^{\infty} |f_k|^2 \right\|_{L^p} \leq C_p \left( \sum_{k=1}^{\infty} |f_k|^2 \right)^{1/2} \left\| \sum_{k=1}^{\infty} f_k \right\|_{L^p}, \quad 1 < p < \infty,$$

where $c_p$ and $C_p$ do not depend on the choice of rectangles $\{I_k\}$ or functions $\{f_k\}$.
where $\phi_n(t) = e^{2\pi int}$, $n \in \mathbb{Z}$, is the standard trigonometric system over the interval $[0, 1]$, $L^2 = L^2([0, 1])$, and $I_k$ is a partition of the set $\mathbb{Z}$ of integers into a Hadamard lacunary sequence of intervals.\footnote{The very same year when Littlewood and Paley [13] introduced the pair of inequalities (2) for the trigonometric system, Paley [20] proved the same pair of inequalities for the Walsh system under consideration in the present paper.}

The corresponding statement for the trigonometric system and partitions of $\mathbb{Z}$ into arbitrary intervals was established by Rubio de Francia [21] in 1985. He showed that in this case the following pair of inequalities holds:

$$\left\| \sum_{k=1}^{\infty} f_k \right\|_{L^p} \leq C_p \left\| \left( \sum_{k=1}^{\infty} |f_k|^2 \right)^{1/2} \right\|_{L^p}, \quad 1 < p \leq 2, \quad (3)$$

$$c_p \left\| \left( \sum_{k=1}^{\infty} |f_k|^2 \right)^{1/2} \right\|_{L^p} \leq \left\| \sum_{k=1}^{\infty} f_k \right\|_{L^p}, \quad p \geq 2. \quad (4)$$

The above estimates are called Littlewood–Paley–Rubio de Francia inequalities or simply Rubio de Francia inequalities. These results have sparked a whole new line of research, yielding a number of extensions published to date.

The majority of the extensions are related to the case of the trigonometric system. Specifically, in [2,11], inequality (3) is generalized to arbitrary exponents $0 < p \leq 2$. In [8,17,19,22], a generalization for the $D$-parameter trigonometric system $\phi_n$, $n \in \mathbb{Z}^D$, and partitions of $\mathbb{Z}^D$ into arbitrary products of intervals is formulated, with inequality (3) similarly extended to arbitrary exponents $0 < p \leq 2$. Rubio de Francia himself in the original paper [21] as well as other authors in [1,9] considered certain weighted generalizations. In [14,18], versions of these inequalities for the Morrey–Companato and Tribel–Lizorkin spaces are proved. Lacey [12] reviews some of the mentioned, as well as certain other extensions of Rubio de Francia inequalities.

Recently, Osipov [16] proved a version of inequality (3), where $\{\phi_n\}$, $n \in \mathbb{Z}_+$, is the Walsh system and the $I_k$ partition the positive integers $\mathbb{Z}_+$ into arbitrary pairwise nonintersecting intervals. In this paper, we take this line of research further by proving (3) for the two-parameter Walsh system $\phi_n$, $n \in \mathbb{Z}_+ \times \mathbb{Z}_+$ and partitions of $\mathbb{Z}_+ \times \mathbb{Z}_+$ into arbitrary pairwise nonintersecting rectangles. Formally, we prove the following statement.

\textbf{Theorem 1.} Consider a family of pairwise nonintersecting rectangles $I_k = I^1_k \times I^2_k$ inside $\mathbb{Z}_+ \times \mathbb{Z}_+$ and a family of functions $f_k$ with Walsh spectrum inside $I_k$, that is,

$$f_k(x_1, x_2) = \sum_{(n_1, n_2) \in I_k} (f_k, w_{n_1} w_{n_2}) w_{n_1}(x_1) w_{n_2}(x_2), \quad (5)$$

where $w_{n_i}$ are the standard Walsh functions in the Paley ordering.

If $1 < p \leq 2$, then

$$\left\| \sum_k f_k \right\|_{L^p} \leq C_p \left\| \left( \sum_k |f_k|^2 \right)^{1/2} \right\|_{L^p}, \quad (6)$$

where $C_p$ does not depend on the choice of rectangles $\{I_k\}$ or functions $\{f_k\}$. The proof is based on a martingale version of the two-parameters singular integral theory of R. Fefferman and Journe [5,8] formulated by Weisz [23].

We use the theory of Weisz to prove Theorem 4, a two-parameter analog of Gundy’s Theorem [7] on the boundedness of operators taking martingales to measurable functions. Theorem 4
helps proving the boundedness of operators that map two-parameter martingales to measurable functions in a rather general setting, hence, it could be of independent interest.\footnote{To the best knowledge of the author, this assertion has not been explicitly formulated in the contemporary literature.}

By applying the combinatorial argument from Osipov’s work on the one-parameter Walsh system \[16\] independently for each variable, we essentially reduce the Rubio de Francia inequality for the two-parameter Walsh system to the question of boundedness for a certain operator, which in its turn resolve by means of Theorem 4.

\section{Preliminaries}

In this section, we present preliminary results used in the proof of Theorem 5. First, we define two-parameter dyadic martingales and introduce the corresponding Hardy spaces. Next, we present notions from the atomic theory of Hardy spaces that are useful for establishing boundedness of operators mapping martingales to measurable functions. Finally, we recall the definition of the classical Walsh basis and define the two-parameter Walsh system.

While we need the theory of \(l^2\)-valued functions and martingales to prove the main theorem, we study in this section only the scalar-valued case to avoid cumbersome notation. In fact, every definition, notion and assertion introduced here is trivially transferable to the \(l^2\)-valued case.

\subsection{Two-parameter dyadic martingales}

The two-parameter dyadic filtration is defined as the family \(\{\mathcal{F}_{n_1,n_2}\}_{n_1 \in \mathbb{Z}_+, n_2 \in \mathbb{Z}_+}\) of \(\sigma\)-algebras generated by the dyadic rectangles of size \(2^{-n_1} \times 2^{-n_2}\), that is,

\[\mathcal{F}_{n_1,n_2} = \sigma\left( \left\{ \left[ \frac{k_1}{2^{n_1}}, \frac{k_1+1}{2^{n_1}} \right] \times \left[ \frac{k_2}{2^{n_2}}, \frac{k_2+1}{2^{n_2}} \right] : 0 \leq k_i < 2^{n_i} \right\} \right), \tag{7}\]

where \(\sigma(\mathcal{H})\) denotes the \(\sigma\)-algebra generated by the elements of \(\mathcal{H}\). Let \(\mathcal{E}_{n_1,n_2}\) denote the operator of conditional expectation with respect to the \(\sigma\)-algebra \(\mathcal{F}_{n_1,n_2}\).

Hereinafter we often denote elements \((n_1,n_2) \in \mathbb{Z}_+^2\) by a single symbol \(n\). For \(n, m \in \mathbb{Z}_+^2\), we write \(n \leq m\) if and only if \(n_1 \leq m_1\) and \(n_2 \leq m_2\). Now, we introduce the following definition.

\textbf{Definition.} A family of integrable functions \(u = \{u_n\}_{n \in \mathbb{Z}_+^2}\) is a two-parameter dyadic martingale (from now on, referred to as a martingale) if the following conditions are fulfilled:

1) for all \(n \in \mathbb{Z}_+^2\), the function \(u_n\) is \(\mathcal{F}_n\)-measurable,

2) we have \(E_n u_m = u_n\) for all \(n, m\) such that \(n \leq m\).

We say that a martingale \(u\) is in \(L^p\) and we write \(u \in L^p\), \(0 < p \leq \infty\), if \(u_n \in L^p\) for all \(n \in \mathbb{Z}_+^2\) and \(\|u\|_{L^p} := \sup_{n \in \mathbb{Z}_+^2} \|u_n\|_{L^p} < \infty\). For two-parameter martingales, as in the classical one-parameter case, the following is true (see \[24\]): if \(u \in L^p\) for \(1 < p < \infty\), then there exists a function \(g \in L^p\) such that \(u_n = E_n g\) and

\[\lim_{\min(n_1,n_2) \to \infty} \|u_n - g\|_{L^p} = 0, \quad \|u\|_{L^p} = \|g\|_{L^p}. \tag{8}\]

Following the common practice, we henceforth identify a martingale \(u\) with the function \(g\) and denote \(g\) by the same symbol \(u\).

The martingale differences \(\Delta_n\) are defined as

\[\Delta_{n_1,n_2} u := u_{n_1,n_2} - u_{n_1-1,n_2} - u_{n_1,n_2-1} + u_{n_1-1,n_2-1}, \tag{9}\]

where the formal symbols \(u_{n_1,-1}\) and \(u_{-1,n_2}\) are assumed to be equal to zero.
2.2. Hardy spaces of two-parameter dyadic martingales. We start with introducing a version of the Littlewood–Paley square function for two-parameter dyadic martingales.

**Definition.** Littlewood–Paley square function is denoted by \( S \) and is given by
\[
S(u) := \left( \sum_{n \in \mathbb{Z}^2_+} |\Delta_n u|^2 \right)^{1/2}.
\]

The expression \( \|S(u)\|_{L^p} \) is a norm that defines the Hardy spaces.

**Definition.** For \( 0 < p < \infty \), the martingale Hardy space \( H^p \) (from now on, referred to as the Hardy space) consists of martingales \( u \) such that
\[
\|u\|_{H^p} := \|S(u)\|_{L^p} < \infty.
\]

It is known (cf. [3, 4, 15]) that \( \|S(u)\|_{L^p} \sim \|u\|_{L^p} \) for \( 1 < p < \infty \), meaning that for such exponents \( p \), the spaces \( L^p \) and \( H^p \) coincide. However, for \( p \leq 1 \), the Hardy spaces constitute an independent and very useful entity.

Finally, we formulate the following interpolation result for the Hardy spaces.

**Theorem 2.** Let \( V \) be a sublinear operator bounded between \( H^{p_0} \) and \( L^{p_0} \) and between \( H^{p_1} \) and \( L^{p_1} \). Then \( V \) is bounded between \( H^p \) and \( L^p \) for \( p_0 < p < p_1 \).

**Proof.** Cf. [23, Theorem A].

2.3. Boundedness of operators in \( H^p \). R. Fefferman’s theorem [5] is an extremely important tool for establishing the boundedness of operators on two-parameter Hardy classes in trigonometric harmonic analysis. It allows one to check rather simple quasi locality conditions, similar to those often used in one-parameter case. As it turns out, the situation in the two-parameter martingale case is similar. The corresponding claim is based, as in the trigonometric case, on the atomic decomposition of the Hardy space. To formulate this claim, we need two definitions.

First, we define the martingale counterpart of R. Fefferman’s rectangle atoms.

**Definition.** A function \( a \in L^2 \) is called a martingale \( H^p \) rectangle atom (from now on, referred to as a rectangle atom) if the following conditions are satisfied:
1) \( \text{supp } a \subseteq F \), where \( F \subseteq [0,1)^2 \) is a dyadic rectangle,
2) \( \|a\|_{L^2} \leq |F|^{1/2 - 1/p} \),
3) for all \( x, y \in [0,1) \), we have \( \int_0^1 a(u,y)du = \int_0^1 a(x,u)du = 0 \).

In accordance with the convention mentioned in Section 2.1, we consider a rectangle atom as a function or a martingale depending on the context.

Second, we introduce a class of operators for which the aforementioned quasi locality condition is satisfied.

**Definition.** An operator \( V \) mapping martingales to measurable functions is said to be \( H^p \) quasi local if there exists \( \delta > 0 \) such that for all \( r \in \mathbb{N} \), for all dyadic rectangles \( R \subseteq [0,1)^2 \), and for all \( H^p \) rectangle atoms \( a \) supported on \( R \), we have
\[
\int_{[0,1)^2 \setminus R_r} |Va|^p \leq C_p 2^{-\delta r},
\]
where \( R_r \) is a dyadic rectangle such that \( R \subseteq R_r \) and \( |R_r| = 2^{2r} |R| \), and \( C_p \) is a constant depending only on \( p \).
Finally, we formulate the claim.

**Theorem 3.** Consider a sublinear operator $V$ that is $H^p$ quasi local for some exponent $0 < p \leq 1$. If $V$ is bounded between $L^2$ and $L^2$, then

$$\|Vu\|_{L^p} \leq C_p \|u\|_{H^p} \text{ for all } u \in H^p.$$  

**Proof.** Cf. [23, Theorem 2]. \hfill \Box

### 2.4. Two-parameter Walsh system

We conclude the preliminaries with defining the two-parameter Walsh system. We start with recalling the definition of the classical one-parameter Walsh system.

**Definition.** The Walsh system $\{w_n\}_{n \in \mathbb{Z}_+}$ is a family of piecewise constant functions of one real variable defined as follows. First, put $w_0 = 0$. Next, if $n > 0$ and $n = 2^{k_1} + \cdots + 2^{k_s}$, $k_1 > k_2 > \cdots > k_s \geq 0$, put

$$w_n(x) := \prod_{i=1}^{s} r_{k_i+1}(x), \quad \text{where} \quad r_k(x) = \text{sgn } \sin 2^k \pi x.$$  

(14)

Here $\{r_k\}_{k \in \mathbb{Z}_+}$ is the Rademacher system. Different orderings of Walsh functions are considered in the literature. The ordering that corresponds to the definition above is called the Paley ordering. Hereinafter we consider precisely this ordering.

The two-parameter Walsh system is defined by the expression

$$w_{n_1,n_2}(x_1, x_2) = w_{n_1}(x_1)w_{n_2}(x_2), \quad n \in \mathbb{Z}_+^2.$$  

(15)

It is an orthonormal basis in $L^2([0,1]^2)$. Moreover, for any function $f$, we have

$$(\mathbb{B}_{k_1,k_2}f)(x_1, x_2) = \sum_{n_1=0}^{2^{k_1}-1} \sum_{n_2=0}^{2^{k_2}-1} \langle f, w_n \rangle \ w_n(x_1, x_2),$$  

(16)

$$(\Delta_{k_1,k_2}f)(x_1, x_2) = \sum_{n \in \delta_{k_1,k_2}} \langle f, w_n \rangle \ w_n(x_1, x_2),$$  

(17)

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2([0,1]^2)$ and

$$\delta_{k_1,k_2} = [2^{k_1-1}, 2^{k_1} - 1] \times [2^{k_2-1}, 2^{k_2} - 1], \quad k_1, k_2 > 0,$$

$$\delta_{0,k_2} = \{0\} \times [2^{k_2-1}, 2^{k_2} - 1], \quad k_2 > 0,$$

$$\delta_{k_1,0} = [2^{k_1-1}, 2^{k_1} - 1] \times \{0\}, \quad k_1 > 0,$$

$$\delta_{0,0} = \{(0,0)\}.$$  

(18)

For a pair $w_n(\cdot), w_m(\cdot)$, $n, m \in \mathbb{Z}_+$ of two one-parameter Walsh functions, we have $w_n(x)w_m(x) = w_{n+m}(x)$, where $\hat{\oplus}$ is the bitwise exclusive disjunction (xor) operation acting on the binary representations of numbers $n$ and $m$: 

$$\left(\sum_{k=0}^{\infty} \alpha_k 2^k\right) \hat{\oplus} \left(\sum_{k=0}^{\infty} \beta_k 2^k\right) := \sum_{k=0}^{\infty} (\alpha_k + \beta_k \mod 2) 2^k.$$  

(19)

If we define the corresponding operation $\hat{\oplus}$ acting on a pair of $n = (n_1, n_2)$ and $m = (m_1, m_2)$ by putting

$$n \hat{\oplus} m = (n_1 \hat{\oplus} m_1, n_2 \hat{\oplus} m_2),$$  

(20)

then obviously

$$w_n(x_1, x_2)w_m(x_1, x_2) = w_{n \hat{\oplus} m}(x_1, x_2).$$  

(21)
Theorem 3 from the previous section enables us to formulate a new version of Gundy’s theorem introduced in [6,7]. Note that our formulation will be closer to the version formulated much later by Kislyakov [10]. This theorem, due to the simplicity of its conditions, can be rather useful in proving the boundedness of operators mapping martingales into measurable functions. This result may be of independent interest and it is the key to proving Theorem 5.

We start with a definition. A martingale $u$ is a simple martingale if there exists an $m \in \mathbb{Z}^+_+$ such that $u_n = E_m u_n$ for all indices $n \in \mathbb{Z}^+_+$. Now, we are ready to formulate the following version of Gundy’s theorem.

**Theorem 4.** Consider a sublinear operator $V$ mapping martingales to measurable functions. Assume the following two conditions:

1) The operator $V$ is bounded between $L^2$ and $L^2$.
2) If $u$ is a simple martingale for which $u_0, 0 = 0$ and

$$\Delta_n u = 1_{e_n} \Delta_n u, \text{ where } e_n \in F_m \text{ for some } m \leq n, m \neq n,$$

then \( \{|Vu| > 0\} \subseteq \bigcup_{n \in \mathbb{Z}^+_+ \setminus \{0\}} e_n. \)

Then $V$ is bounded between $H^p$ and $L^p$ for any $0 < p \leq 1$.

**Proof.** Fix $0 < p \leq 1$. By Theorem 3, it suffices to show that $V$ is $H^p$ quasi local.

Take an $H^p$ rectangle atom $a$ supported on a dyadic rectangle $R \subseteq [0,1)^2$. We have to check that, for all $r \in \mathbb{N}$ and for some $\delta$ not depending on $a$ and $r$,

$$\int_{[0,1)^2 \setminus R^r} |Va|^p (x_1, x_2) \, dx_1 \, dx_2 \leq C_p 2^{-\delta r}. \quad (23)$$

We claim that it suffices to check condition (23) for atoms that are simple martingales. Indeed, assume that (23) holds for all rectangle atoms that are simple martingales. Let us prove that (23) holds for a rectangle atom $a$. It is easy to check that the simple martingale $a_n = E_n a$ remains a rectangle atom. Since $V$ is bounded between $L^2$ and $L^2$,

$$\lim_{\min(n_1, n_2) \to \infty} \|a_n - a\|_{L^2} = 0 \text{ implies } \lim_{\min(n_1, n_2) \to \infty} \|Va_n - Va\|_{L^2} = 0.$$

Hence

$$\lim_{\min(n_1, n_2) \to \infty} \|Va_n - Va\|_{L^p} = 0.$$

This justifies the passage to the limit in inequality (23), which proves the inequality for the initial rectangle atom $a$. Thus, hereinafter in this proof, we assume all rectangle atoms to be simple martingales.

Now, we find an element $N \in \mathbb{Z}^+_+$ such that $R \in F_N$, and for any $n$ such that $R \in F_n$, we have $N \leq n$. Since $\text{supp } a \subseteq R$, item 3 in the definition of a rectangle atom guarantees that

$$\Delta_n a = 1_R \Delta_n a \quad \text{for } n \geq N, n \neq N, \quad (24)$$

$$\Delta_n a = 1_{\emptyset} \Delta_n a \quad \text{otherwise}. \quad (25)$$

Moreover, $a_{0,0} = 0$ due to item 3 in the definition of a rectangle atom. Using assumption 2 of the theorem, we have \( \{(Va) > 0\} \subseteq R \), hence,

$$\int_{[0,1)^2 \setminus R^r} |Va|^p (x_1, x_2) \, dx_1 \, dx_2 \leq \int_{[0,1)^2 \setminus R} |Va|^p (x_1, x_2) \, dx_1 \, dx_2 = 0. \quad (26)$$

The right-hand side is trivially bounded by $C_p 2^{-\delta r}$ for any $\delta > 0$; this proves the claim.

**Corollary.** Under assumptions of Theorem 4, $V$ is bounded between $L^s$ and $L^s$ for $1 < s \leq 2$. 751
Proof. Interpolation between the boundedness of $V : H^p \to L^p$ for some $p \leq 1$ and $V : L^2 \to L^2$ by means of Theorem 2 gives the result. \hfill \Box

4. Auxiliary operator $G$

In this section, we introduce an auxiliary operator $G$, the two-parameter counterpart of the auxiliary operator $G$ introduced by Osipov in [16], and we prove its boundedness using the results of the previous section. This particular operator appears in the proof of the main theorem.

Consider a family of multi-indices $\mathcal{A} \subseteq \mathbb{N} \times \mathbb{Z}_2^+$. Its elements are pairs $(j, k)$, where $j \in \mathbb{N}$, $k \in \mathbb{Z}_2^+$. Let $\delta_k$ be defined by (18). Consider a family $\{a_{j,k}\}_{(j,k) \in \mathcal{A}} \subseteq \mathbb{Z}_2^+$ such that $\{a_{j,k} + \delta_k\}_{(j,k) \in \mathcal{A}}$ consists of pairwise nonintersecting subsets of $\mathbb{Z}_2^+$. The operator $G$ is induced by the family $\{a_{j,k}\}_{(j,k) \in \mathcal{A}}$; we prove its boundedness in the following lemma.

Lemma. The operator $G$ maps any vector-valued function $h = \{h_{j,k}\}_{(j,k) \in \mathbb{N} \times \mathbb{Z}_2^+}$ from the space $L^p(\ell^2_{\mathbb{N} \times \mathbb{Z}_2^+})$, $1 < p \leq 2$, to a measurable function by the following law:

$$(Gh)(x_1, x_2) := \sum_{(j,k) \in \mathcal{A}} w_{a_{j,k}}(x_1, x_2) (\Delta_k h_{j,k})(x_1, x_2).$$

This operator is bounded between $L^p(\ell^2_{\mathbb{N} \times \mathbb{Z}_2^+})$ and $L^p$, that is,

$$\|Gh\|_{L^p} \leq C_p \|h\|_{L^p(\ell^2_{\mathbb{N} \times \mathbb{Z}_2^+})},$$

where the constant $C_p$ depends only on $p$.

Proof. For $1 < p \leq 2$, there exists a one-to-one mapping between the elements of $L^p$ and the martingales from $L^p$, hence, the operator $G$ may be viewed as an operator mapping $\ell^2(\mathbb{N} \times \mathbb{Z}_2^+)$-valued martingales (rather than $\ell^2(\mathbb{N} \times \mathbb{Z}_2^+)$-valued functions) to measurable functions. We will prove that $G$ fulfills the conditions of Theorem 4, or, strictly speaking, its generalization to the case of $\ell^2(\mathbb{N} \times \mathbb{Z}_2^+)$-valued martingales. Here we rely on the fact that the theory is transferable to the $\ell^2$-valued case, as indicated at the beginning of Sec. 2.

Since the operator $G$ is linear, it is sublinear. Next, Plancherel’s identity and the fact that $\{a_{j,k} + \delta_k\}_{(j,k) \in \mathcal{A}}$ is a family of pairwise nonintersecting sets give the boundedness of $G$ between $L^2$ and $L^2$.

Finally, we claim that for a simple $\ell^2(\mathbb{N} \times \mathbb{Z}_2^+)$-valued martingale with $u_{0,0} = 0$ and $\Delta_n u = \mathbbm{1}_{e_n} \Delta_n u$, where $e_n \in \mathcal{F}_m$, $m \leq n$, $m \neq n$, we have $\{|Gu| > 0\} \subseteq \bigcup_{n \in \mathbb{Z}_2^+ \setminus \{0\}} e_n$. Indeed, we have

$$\{|Gu| > 0\} \subseteq \bigcup_{(j,k) \in \mathcal{A}} \{|w_{a_{j,k}} \Delta_k u_{j,k}| > 0\} = \bigcup_{(j,k) \in \mathcal{A}} \{\{|\Delta_k u_{j,k}| > 0\}

= \bigcup_{(j,k) \in \mathcal{A}} \{|\mathbbm{1}_{e_k} \Delta_k u_{j,k}| > 0\} \subseteq \bigcup_{(j,k) \in \mathcal{A}} \{|\mathbbm{1}_{e_k}| > 0\} \subseteq \bigcup_{k \in \mathbb{Z}_2^+ \setminus \{0\}} e_k.
$$

This proves the claim. \hfill \Box

5. Proof of the main theorem

We use the theory developed in the preceding sections to prove the main theorem. We begin with recalling its formulation.

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Consider a family of pairwise no
a,b
= X [2
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= +1
J
As in [16], we partition the rectangles
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×
˙
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a
= (29)
(31)
does not depend on the choice of rectangles
⊆
as above and consider all direct products. Therefore,
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(39)
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by forming the direct product of partitions of intervals
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is a strictly increasing sequence and
let
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de the
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are the standard Walsh functions in the Paley ordering.
proof. As in [16], we partition the rectangles
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k
into fragments that behave well under shifts induced by the operation +. This, together with Lemma and the classical assertion about the boundedness of the Littlewood–Paley square function, will allow us to prove the claim. Let
I
k
= I
k
× I
k
= [a
k
(1)
, b
k
(1) − 1] × [a
k
(2)
, b
k
(2) − 1].
(31)
We build the partition of
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k
by forming the direct product of partitions of intervals
I
k
and
I
k
, while partitioning these individual intervals in the same way as it was done by Osipov [16].
Recall that in [16] an interval
I
= [a, b − 1] ⊆ Z+
+ is partitioned as
I
= \{a\} \cup \left( \bigcup_{j=1}^{r} J_{j} \right) \cup \left( \bigcup_{j=0}^{s} \bar{J}_{j} \right) = \bigcup_{j=0}^{r} J_{j} \cup \bigcup_{j=1}^{s} \bar{J}_{j},
(32)
where
∈ Z+
+ are some numbers,
0
= \{a\}, and
J_{j}, \bar{J}_{j}
are pairwise disjoint sets. Moreover, for
j
> 0, we have \(|J_{j}| = 2^{\gamma_{j}}\), \(|\bar{J}_{j}| = 2^{\gamma_{j}}\), where
j
is a strictly increasing sequence and
j
is a strictly decreasing \(Z_{+}\)-valued sequence.
The most important property of the intervals
J_{j}
and \bar{J}_{j}
is that they can be shifted to become the dyadic intervals
a + J_{0} = \{0\}, \quad a + J_{j} = [2^{\gamma_{j}}, 2^{\gamma_{j}+1} − 1], \quad b + \bar{J}_{j} = [2^{\gamma_{j}}, 2^{\gamma_{j}+1} − 1];
(33)
hence, the following holds:
\Delta_{\kappa_{j+1} + w_{a} f} = w_{a} f \quad \text{if spec } f \subseteq J_{j},
(34)
\Delta_{\gamma_{j+1} + w_{b} f} = w_{b} f \quad \text{if spec } f \subseteq \bar{J}_{j}.
(35)
To partition a rectangle
I
= I
× I
= [a
, b
− 1] × [a
, b
− 1] ⊆ Z
+ , we partition each interval
i
as above and consider all direct products. Therefore,
I
= \bigcup_{j} A_{j} \cup \left( \bigcup_{j} B_{j} \right) \cup \left( \bigcup_{j} C_{j} \right) \cup \left( \bigcup_{j} D_{j} \right),
(36)
where
j
= J_{j}^{(1)}, \quad B_{j} = \bar{J}_{j}^{(1)} × \bar{J}_{j}^{(2)},
(37)
C_{j} = \bar{J}_{j}^{(1)} × J_{j}^{(2)}, \quad D_{j} = J_{j}^{(1)} × \bar{J}_{j}^{(2)},
(38)
where a superscript indicates whether the object belongs to the partition of
i
or
i
.
This partition of
I
has properties similar to those in (34), (35). Define
a, b, c, d
to be the vertices of the rectangle
I
, that is,
\begin{align*}
a := (a^{(1)}, a^{(2)}), & \quad b := (b^{(1)}, b^{(2)}), & \quad c := (b^{(1)}, a^{(2)}), & \quad d := (a^{(1)}, b^{(2)}). \tag{39}
\end{align*}
Then
\[ \Delta_{\kappa_j^{(1)}+1,\kappa_j^{(2)}+1}w_af = w_af \quad \text{if spec } f \subseteq A_j, \quad (40) \]
\[ \Delta_{\gamma_j^{(1)}+1,\gamma_j^{(2)}+1}w_bf = w_bf \quad \text{if spec } f \subseteq B_j, \quad (41) \]
\[ \Delta_{\delta_j^{(1)}+1,\delta_j^{(2)}+1}w_cf = w_cf \quad \text{if spec } f \subseteq C_j, \quad (42) \]
\[ \Delta_{\kappa_j^{(1)}+1,\kappa_j^{(2)}+1}w_df = w_df \quad \text{if spec } f \subseteq D_j. \quad (43) \]

This behavior under shifts will be of utter importance in what follows. Let us similarly partition each \( I_k \), adding yet another index \( k \) to all objects that arise from this construction. Then \( f_k \) can be represented as the sum
\[ f_k = \sum_j f^A_{k,j} + \sum_j f^B_{k,j} + \sum_j f^C_{k,j} + \sum_j f^D_{k,j}, \quad (44) \]
where spec \( f^A_{k,j} \subseteq A_{k,j} \), spec \( f^B_{k,j} \subseteq B_{k,j} \), spec \( f^C_{k,j} \subseteq C_{k,j} \), spec \( f^D_{k,j} \subseteq D_{k,j} \).

Define
\[ g^A_{k,j} = w_{a_k}f^A_{k,j}, \quad g^B_{k,j} = w_{b_k}f^B_{k,j}, \quad g^C_{k,j} = w_{c_k}f^C_{k,j}, \quad g^D_{k,j} = w_{d_k}f^D_{k,j}. \quad (45) \]

Then \( \sum_k f_k \) can be represented as follows:
\[ \sum_k \left( w_{a_k} \sum_j g^A_{k,j} + w_{b_k} \sum_j g^B_{k,j} + w_{c_k} \sum_j g^C_{k,j} + w_{d_k} \sum_j g^D_{k,j} \right). \quad (46) \]

Application of Lemma to this expression (justified by none other than properties (40)–(43)), followed by applying the triangle inequality, gives us\(^3\)
\[ \left\| \sum_k f_k \right\|_{L^p} \lesssim \left( \left( \sum_k \left( \sum_j |g^A_{k,j}|^2 + \sum_j |g^B_{k,j}|^2 + \sum_j |g^C_{k,j}|^2 + \sum_j |g^D_{k,j}|^2 \right) \right)^{1/2} \right. \quad (47) \]
\[ \leq \left( \left( \sum_k \sum_j |g^A_{k,j}|^2 \right)^{1/2} \right) \left( \left( \sum_k \sum_j |g^B_{k,j}|^2 \right)^{1/2} \right) \left( \left( \sum_k \sum_j |g^C_{k,j}|^2 \right)^{1/2} \right) \left( \left( \sum_k \sum_j |g^D_{k,j}|^2 \right)^{1/2} \right) \quad (48) \]
\[ + \left( \left( \sum_k \sum_j |g^A_{k,j}|^2 \right)^{1/2} \right) \left( \left( \sum_k \sum_j |g^B_{k,j}|^2 \right)^{1/2} \right) \left( \left( \sum_k \sum_j |g^C_{k,j}|^2 \right)^{1/2} \right) \left( \left( \sum_k \sum_j |g^D_{k,j}|^2 \right)^{1/2} \right) \quad (49) \]

Hereinafter, the symbol \( \lesssim \) denotes the inequality up to an implicit multiplicative constant. Consider now separately, e.g., the third term. We have
\[ w_{c_k}f_k = w_{c_k+a_k} \sum_j g^A_{k,j} + w_{c_k+b_k} \sum_j g^B_{k,j} + \sum_j g^C_{k,j} + w_{c_k+d_k} \sum_j g^D_{k,j}. \quad (50) \]

Observe that \( \Delta_{\kappa_j^{(1)}+1,\kappa_j^{(2)}+1}w_{c_k}f_k = g^C_{k,j} \). Thus, in the decomposition \( w_{c_k}f_k = \sum_{n \in \mathbb{Z}_+^2} \Delta_n w_{c_k}f_k \), the functions \( g^C_{k,j} \) are among the right-hand side terms. Therefore,
\[ \sum_j |g^C_{k,j}|^2 \leq \sum_{n \in \mathbb{Z}_+^2} |\Delta_n w_{c_k}f_k|^2 = (S(w_{c_k}f_k))^2, \quad (51) \]

\(^3\)Note that \( k \) and \( j \) here correspond to \( j \) and \( k \), respectively, in the formulation of Lemma.
where $S$ is the Littlewood–Paley square function. By leveraging its boundedness (cf. [3, 4, 15], see also book [24], where the scalar-valued version of this statement is proved, from which the vector-valued version follows easily), we have

$$
\left\| \left( \sum_{k} \sum_{j} |g_{k,j}^C|^2 \right)^{1/2} \right\|_{L^p} \leq \left\| \left( \sum_{k} \sum_{n \in 2^k \mathbb{Z}} |\Delta_n w_{ck} f_k|^2 \right)^{1/2} \right\|_{L^p} \tag{52}
$$

$$
\lesssim \left\| \left( \sum_{k} |w_{ck} f_k|^2 \right)^{1/2} \right\|_{L^p} = \left\| \left( \sum_{k} |f_k|^2 \right)^{1/2} \right\|_{L^p}. \tag{53}
$$

Similarly, we estimate each of the four terms in (48) and (49). Combining these inequalities, we finally obtain

$$
\left\| \sum_{k} f_k \right\|_{L^p} \lesssim \left\| \left( \sum_{k} |f_k|^2 \right)^{1/2} \right\|_{L^p}, \tag{54}
$$

which proves the claim.

\[\square\]

**Remark.** In the light of [17] and [19], it is natural to ask whether a similar statement holds for a general multi-parameter Walsh system and a partition of $\mathbb{Z}^D$ into arbitrary products of intervals. The author is going to address this question in the near future. For now, we only mention that there is no direct analog of Theorem 3 in the general multi parameter case and a finer statement would be required.

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