Data-Dependence of Plateau Phenomenon in Learning with Neural Network — Statistical Mechanical Analysis

Yuki Yoshida  Masato Okada
Department of Complexity Science and Engineering, Graduate School of Frontier Sciences,
The University of Tokyo
5-1-5 Kashiwanoha, Kashiwa, Chiba 277-8561, Japan
{yoshida@mns, okada@edu}.k.u-tokyo.ac.jp

Abstract

The plateau phenomenon, wherein the loss value stops decreasing during the process of learning, has been reported by various researchers. The phenomenon is actively inspected in the 1990s and found to be due to the fundamental hierarchical structure of neural network models. Then the phenomenon has been thought as inevitable. However, the phenomenon seldom occurs in the context of recent deep learning. There is a gap between theory and reality. In this paper, using statistical mechanical formulation, we clarified the relationship between the plateau phenomenon and the statistical property of the data learned. It is shown that the data whose covariance has small and dispersed eigenvalues tend to make the plateau phenomenon inconspicuous.

1 Introduction

1.1 Plateau Phenomenon

Deep learning, and neural network as its essential component, has come to be applied to various fields. However, these still remain unclear in various points theoretically. The plateau phenomenon is one of them. In the learning process of neural networks, their weight parameters are updated iteratively so that the loss decreases. However, in some settings the loss does not decrease simply, but its decreasing speed slows down significantly partway through learning, and then it speeds up again after a long period of time. This is called as "plateau phenomenon". Since 1990s, this phenomena have been reported to occur in various practical learning situations (see Figure 1(a) and Park et al. [2000], Fukumizu and Amari [2000]). As a fundamental cause of this phenomenon, it has been pointed out by a number of researchers that the intrinsic symmetry of neural network models brings singularity to the metric in the parameter space which then gives rise to special attractors whose regions of attraction have nonzero measure, called as Milnor attractor (defined by Milnor [1985]; see also Figure 5 in Fukumizu and Amari [2000] for a schematic diagram of the attractor).

1.2 Who moved the plateau phenomenon?

However, the plateau phenomenon seldom occurs in recent practical use of neural networks (see Figure 1(b) for example).

In this research, we rethink the plateau phenomenon, and discuss which situations are likely to cause the phenomenon. First we introduce the student-teacher model of two-layered networks as an ideal system. Next, we reduce the learning dynamics of the student-teacher model to a small-dimensional order parameter system by using statistical mechanical formulation, under the assumption that the
input dimension is sufficiently large. Through analyzing the order parameter system, we can discuss how the macroscopic learning dynamics depends on the statistics of input data. Our main contribution is the following:

- Under the statistical mechanical formulation of learning in the two-layered perceptron, we showed that macroscopic equations can be derived even when the statistical properties of the input are generalized. In other words, we extended the result of Saad and Solla [1995] and Riegler and Biehl [1995].

- By analyzing the macroscopic system we derived, we showed that the dynamics of learning depends only on the eigenvalue distribution of the covariance matrix of the input data.

- We clarified the relationship between the input data statistics and plateau phenomenon. In particular, it is shown that the data whose covariance matrix has small and dispersed eigenvalues tend to make the phenomenon inconspicuous, by numerically analyzing the macroscopic system.

1.3 Related works

The statistical mechanical approach used in this research is firstly developed by Saad and Solla [1995]. The method reduces high-dimensional learning dynamics of nonlinear neural networks to low-dimensional system of order parameters. They derived the macroscopic behavior of learning dynamics in two-layered soft-committee machine and by analyzing it they point out the existence of plateau phenomenon. Nowadays the statistical mechanical method is applied to analyze recent techniques (Hara et al. [2016], Yoshida et al. [2017], Takagi et al. [2019] and Straat and Biehl [2019]), and generalization performance in over-parameterized setting (Goldt et al. [2019]) and environment with conceptual drift (Straat et al. [2018]). However, it is unknown that how the property of input dataset itself can affect the learning dynamics, including plateaus.

Plateau phenomenon and singularity in loss landscape as its main cause have been studied by Fukumizu and Amari [2000], Wei et al. [2008], Cousseau et al. [2008] and Guo et al. [2018]. On the other hand, recent several works suggest that plateau and singularity can be mitigated in some settings. Orhan and Pitkow [2017] shows that skip connections eliminate the singularity. Another work by Yoshida et al. [2019] points out that output dimensionality affects the plateau phenomenon, in that multiple output units alleviate the plateau phenomenon. However, the number of output elements does not fully determine the presence or absence of plateaus, nor does the use of skip connections. The statistical property of data just can affect the learning dynamics dramatically; for example, see Figure 2 for learning curves with using different datasets and same network architecture. We focus on what kind of statistical property of the data brings plateau phenomenon.

![Figure 1](image_url)

Figure 1: (a) Training loss curves when two-layer perceptron with 4-4-3 units and ReLU activation learns IRIS dataset. (b) Training loss curve when two-layer perceptron with 784-20-10 units and ReLU activation learns MNIST dataset. For both (a) and (b), results of 100 trials with random initialization are overlaid. Minibatch size of 10 and vanilla SGD (learning rate: 0.01) are used.
We consider the situation that the network learns data generated by another network, called “teacher network”, which has fixed weights. Specifically, we consider two-layer perceptron that outputs $Q R T$ in the on-line manner (see Figure 3). We assume that the input $x$ is drawn from some distribution $p(x)$ every time independently. We adopt vanilla stochastic gradient descent (SGD) algorithm for learning. We assume the squared loss function $\varepsilon = \frac{1}{2}(s - t)^2$, which is most commonly used for regression.

2.2 Statistical Mechanical Formulation

In order to capture the learning dynamics of nonlinear neural networks described in the previous subsection macroscopically, we introduce the statistical mechanical formulation in this subsection.

Let $x_i := J_i \cdot \xi$ (1 $\leq i \leq K$) and $y_n := B_n \cdot \xi$ (1 $\leq n \leq M$). Then

$$(x_1, \ldots, x_K, y_1, \ldots, y_M) \sim N(0, [J_1, \ldots, J_K, B_1, \ldots, B_M]^T \Sigma [J_1, \ldots, J_K, B_1, \ldots, B_M])$$

holds with $N \to \infty$ by generalized central limit theorem, provided that the input distribution $p(x)$ has zero mean and finite covariance matrix $\Sigma$.

Next, let us introduce order parameters as following: $Q_{ij} := J_i^T \Sigma J_j = \langle x_i x_j \rangle$, $R_{i n} := J_i^T \Sigma B_n = \langle x_i y_n \rangle$, $T_{m n} := B_i^T \Sigma B_m = \langle y_n y_m \rangle$ and $D_{ij} := w_i w_j$, $E_{i n} := w_i v_n$, $F_{m n} := v_n v_m$. Then

$$(x_1, \ldots, x_K, y_1, \ldots, y_M) \sim N(0, \begin{pmatrix} Q & R \\ T^T & T \end{pmatrix})$$

The parameters $Q_{ij}, R_{i n}, T_{m n}, D_{ij}, E_{i n},$ and $F_{m n}$ introduced above capture the state of the system macroscopically; therefore they are called as “order parameters.” The first three represent the state of the first layers of the two networks (student and teacher), and the latter three represent their connection weights. In contrast, $T_{m n}$ and $F_{m n}$ are constant during learning.
2.2.1 Higher-order order parameters

The important difference between our situation and that of [Saad and Solla, 1995] is the covariance matrix $\Sigma$ of the input $\xi$ is not necessarily equal to identity. This makes the matter complicated, since higher-order terms $\Sigma^e$ ($e = 1, 2, \ldots$) appear inevitably in the learning dynamics of order parameters. In order to deal with these, here we define some higher-order version of order parameters.

Let us define higher-order order parameters $Q^{(e)}_{ij}$, $R^{(e)}_{in}$ and $T^{(e)}_{nm}$ for $e = 0, 1, 2, \ldots$, as $Q^{(e)}_{ij} := J_i^T \Sigma^e J_j$, $R^{(e)}_{in} := J_i^T \Sigma^e B_n$, and $T^{(e)}_{nm} := B_n^T \Sigma^e B_m$. Note that they are identical to $Q_{ij}$, $R_{in}$ and $T_{nm}$ in the case of $e = 1$. Also we define higher-order version of $x_i$ and $y_n$, namely $x^{(e)}_i$ and $y^{(e)}_n$, as $x^{(e)}_i := \xi^T \Sigma^e J_i$, $y^{(e)}_n := \xi^T \Sigma^e B_n$. Note that $x^{(0)}_i = x_i$ and $y^{(0)}_n = y_n$.

3 Derivation of dynamics of order parameters

At each iteration of on-line learning, weights of the student network $J_i$ and $w_i$ are updated with

$$
\Delta J_i = -\frac{\eta}{N} \frac{d\epsilon}{dJ_i} = \eta \frac{1}{N} \left[ \frac{1}{2} (t-s) \cdot \mathbf{w}_i g'(x_i) \xi - \frac{1}{N} \left( \sum_{n=1}^{M} v_n g(y_n) - \sum_{j=1}^{K} w_j g(x_j) \right) \cdot \mathbf{w}_i \right] g'(x_i) \xi,
$$

$$
\Delta w_i = -\frac{\eta}{N} \frac{d\epsilon}{dw_i} = \frac{\eta}{N} g(x_i) \left( \frac{1}{N} \left( \sum_{n=1}^{M} v_n g(y_n) - \sum_{j=1}^{K} w_j g(x_j) \right) \right),
$$

in which we set the learning rate as $\eta/N$, so that our macroscopic system is $N$-independent.

Then, the order parameters $Q^{(e)}_{ij}$ and $R^{(e)}_{in}$ ($e = 0, 1, 2, \ldots$) are updated with

$$
\Delta Q^{(e)}_{ij} = (J_i + \Delta J_i)^T \Sigma^e (J_j + \Delta J_j) - J_i^T \Sigma^e J_j = J_i^T \Sigma^e \Delta J_j + J_i^T \Sigma^e \Delta J_i + \Delta J_i^T \Sigma^e \Delta J_j
$$

$$
= \frac{\eta}{N} \left[ \sum_{p=1}^{M} E_{ip} g'(x_i) x_p g(y_p) - \sum_{p=1}^{K} E_{ip} g'(x_i) x_p g(x_p) + \sum_{p=1}^{M} E_{jp} g'(x_j) x_p g(y_p) - \sum_{p=1}^{K} E_{jp} g'(x_j) x_p g(x_p) \right]
$$

$$
+ \frac{\eta^2}{N^2} \xi^T \Sigma^e \xi \left[ \sum_{p=1}^{K} \sum_{q=1}^{M} E_{ip} E_{j} g'(x_i) y_p g(x_p) g(y_q) + \sum_{p=1}^{M} \sum_{q=1}^{K} E_{ip} E_{j} g'(x_i) y_p g(x_p) g(y_q) - \sum_{p=1}^{K} \sum_{q=1}^{M} E_{ip} E_{j} g'(x_i) y_p g(x_q) g(y_q) \right],
$$

$$
\Delta R^{(e)}_{in} = (J_i + \Delta J_i)^T \Sigma^e B_n - J_i^T \Sigma^e B_n = \Delta J_i^T \Sigma^e B_n
$$

$$
= \frac{\eta}{N} \left[ \sum_{p=1}^{M} E_{ip} g'(x_i) y_p g(y_p) - \sum_{p=1}^{K} E_{ip} g'(x_i) y_p g(x_p) \right].
$$

Figure 3: Overview of student-teacher model formulation.
Since
\[ \xi^T \Sigma^e \xi \approx N \mu_{e+1} \quad \text{where} \quad \mu_d := \frac{1}{N} \sum_{i=1}^{N} \lambda_i^d, \quad \lambda_1, \ldots, \lambda_N : \text{eigenvalues of} \Sigma \]
and the right hand sides of the difference equations are \( O(N^{-1}) \), we can replace these difference equations with differential ones with \( N \rightarrow \infty \), by taking the expectation over all input vectors \( \xi \):
\[
\frac{dQ_{ij}^{(e)}}{d\bar{\alpha}} = \eta \left[ \sum_{p=1}^{M} E_{ip} I_3(x_i, x_j^{(e)}, y_p) - \sum_{p=1}^{K} D_{ip} I_3(x_i, x_j^{(e)}, x_p) + \sum_{p=1}^{K} D_{jp} I_3(x_j, x_j^{(e)}, x_p) \right]
+ \eta^2 \mu_{e+1} \left[ \sum_{p,q}^{K,K} D_{ip} D_{jq} I_4(x_i, x_j, x_p, x_q) + \sum_{p,q}^{M,M} E_{ip} E_{jq} I_4(x_i, x_j, y_p, y_q) - \sum_{p,q}^{K,K} D_{ip} E_{jq} I_4(x_j, x_j, x_p, x_q) \right],
\]
(3)
\[
\frac{dE_{in}}{d\bar{\alpha}} = \eta \left[ \sum_{p=1}^{M} E_{ip} I_3(x_i, y_n^{(e)}, y_p) - \sum_{p=1}^{K} D_{ip} I_3(x_i, y_n^{(e)}, x_p) \right],
\]
(5)
where \( I_3(z_1, z_2, z_3) := \langle g'(z_1)z_2g(z_3) \rangle \) and \( I_4(z_1, z_2, z_3, z_4) := \langle g'(z_1)g'(z_2)g(z_3)g(z_4) \rangle \).
(4)
In these equations, \( \bar{\alpha} := \alpha/N \) represents time (normalized number of steps), and the brackets \( \langle \cdot \rangle \) represent the expectation when the input \( \xi \) follows the input distribution \( p(\xi) \).

The differential equations for \( D \) and \( E \) are obtained in a similar way:
\[
\frac{dD_{ij}}{d\bar{\alpha}} = \eta \left[ \sum_{p=1}^{M} E_{ip} I_2(x_j, y_p) - \sum_{p=1}^{K} D_{ip} I_2(x_j, x_p) \right] + \sum_{p=1}^{K} E_{jp} I_2(x_i, x_p) - \sum_{p=1}^{K} D_{jp} I_2(x_i, x_p),
\]
(6)
\[
\frac{dE_{in}}{d\bar{\alpha}} = \eta \left[ \sum_{p=1}^{M} F_{pn} I_2(x_i, y_p) - \sum_{p=1}^{K} E_{pn} I_2(x_i, x_p) \right],
\]
(7)
where \( I_2(z_1, z_2) := \langle g(z_1)g(z_2) \rangle \).

These differential equations (3) and (5) govern the macroscopic dynamics of learning. In addition, the generalization loss \( \varepsilon_g \), the expectation of loss value \( \varepsilon(\xi) = \frac{1}{2} \| \mathbf{s} - \mathbf{t} \|^2 \) over all input vectors \( \xi \), is represented as
\[
\varepsilon_g = \langle \frac{1}{2} \| \mathbf{s} - \mathbf{t} \|^2 \rangle = \frac{1}{2} \left[ \sum_{p,q}^{M,M} F_{pq} I_2(y_p, y_q) + \sum_{p,q}^{K,K} D_{pq} I_2(x_p, x_q) - \sum_{p,q}^{K,K} E_{pq} I_2(x_p, y_q) \right]
\]
(7)

3.1 Expectation terms

Above we have determined the dynamics of order parameters as (3), (5) and (7). However they have expectation terms \( I_2(z_1, z_2), I_3(z_1, z_2^{(e)}, z_3) \) and \( I_4(z_1, z_2, z_3, z_4) \), where \( z \) is either \( x_i \) or \( y_n \). By studying what distribution \( z \) follows, we can show that these expectation terms are dependent only on 1-st and \((e+1)\)-th order parameters, namely, \( Q^{(1)}, R^{(1)}, T^{(1)} \) and \( Q^{(e+1)}, R^{(e+1)}, T^{(e+1)} \); for example,
\[
I_3(x_i, x_j^{(e)}, y_p) = \int dz_1 dz_2 dz_3 g'(z_1)z_2g(z_3) \mathcal{N}(z|0, \begin{pmatrix} Q^{(1)}_{ij} & Q^{(e+1)}_{ij} \\ Q^{(e+1)}_{ij} & R^{(e+1)}_{ij} \\ R^{(1)}_{ip} & R^{(1)}_{jp} \\ R^{(e+1)}_{ip} & R^{(e+1)}_{jp} \\ P^{(1)}_{ip} & P^{(1)}_{jp} \\ P^{(e+1)}_{ip} & P^{(e+1)}_{jp} \\ T^{(1)}_{ip} & T^{(1)}_{jp} \end{pmatrix} ).
\]
holds, where \( \ast \) does not influence the value of this expression (see Supplementary Material A.1 for more detailed discussion). Thus, we see the ‘speed’ of \( e \)-th order parameters (i.e. 3 and 5) only depends on 1-st and \((e+1)\)-th order parameters, and the generalization error \( \varepsilon_g \) (equation (7)) only depends on 1-st order parameters. Therefore, with denoting \((Q^{(e)}, R^{(e)}, T^{(e)})\) by \( \Omega^{(e)} \) and \((D, E, F)\) by \( \chi \), we can write

\[
\frac{d}{da^e} \Omega^{(e)} = f^{(e)}(\Omega^{(1)}, \Omega^{(e+1)}, \chi), \quad \frac{d}{da^e} \chi = g(\Omega^{(1)}, \chi), \quad \varepsilon_g = h(\Omega^{(1)}, \chi)
\]

with appropriate functions \( f^{(e)}, g \) and \( h \). Additionally, a polynomial of \( \Sigma \)

\[
P(\Sigma) := \prod_{i=1}^{d} (\Sigma - \lambda_i' I_N) = \sum_{e=0}^{d} c_e \Sigma^e
\]

where \( \lambda_1', \ldots, \lambda_d' \) are distinct eigenvalues of \( \Sigma \)
equals to 0, thus we get

\[
\Omega^{(d)} = -\sum_{e=0}^{d-1} c_e \Omega^{(e)}, \quad (8)
\]

Using this relation, we can reduce \( \Omega^{(d)} \) to expressions which contain only \( \Omega^{(0)}, \ldots, \Omega^{(d-1)} \), therefore we can get a closed differential equation system with \( \Omega^{(0)}, \ldots, \Omega^{(d-1)} \) and \( \chi \).

In summary, our macroscopic system is closed with the following order parameters:

- **Order variables**: \( Q^{(0)}_{ij}, Q^{(1)}_{ij}, \ldots, Q^{(d-1)}_{ij}, R^{(0)}_{in}, R^{(1)}_{in}, \ldots, R^{(d-1)}_{in}, D_{ij}, E_{in} \)
- **Order constants**: \( T^{(0)}_{nm}, T^{(1)}_{nm}, \ldots, T^{(d-1)}_{nm}, F_{nm} \) \((d): \text{number of distinct eigenvalues of } \Sigma\)

The order variables are governed by (3) and (5). For the lengthy full expressions of our macroscopic system for specific cases, see Supplementary Material A.2.

### 3.2 Dependency on input data covariance \( \Sigma \)

The differential equation system we derived depends on \( \Sigma \), through two ways; the coefficient \( \mu_{e+1} \) of \( O(\eta^2) \)-term, and how \((d)\)-th order parameters are expanded with lower order parameters (as (8)). Specifically, the system only depends on the eigenvalue distribution of \( \Sigma \).

### 3.3 Evaluation of expectation terms for specific activation functions

Expectation terms \( I_2, I_3 \) and \( I_4 \) can be analytically determined for some activation functions \( g \), including sigmoid-like \( g(x) = erf(x/\sqrt{2}) \) (see [Saad and Solla 1995]) and \( g(x) = \text{ReLU}(x) \) (see [Yoshida et al. 2017]).

### 4 Analysis of numerical solutions of macroscopic differential equations

In this section, we analyze numerically the order parameter system, derived in the previous section\(^4\). We assume that the second layers’ weights of the student and the teacher, namely \( w_i \) and \( v_n \), are fixed to 1 (i.e. we consider the learning of soft-committee machine), and that \( K \) and \( M \) are equal to 2, for simplicity. Here we think of sigmoid-like activation \( g(x) = erf(x/\sqrt{2}) \).

#### 4.1 Consistency between macroscopic system and microscopic system

First of all, we confirmed the consistency between the macroscopic system we derived and the original microscopic system. That is, we computed the dynamics of the generalization loss \( \varepsilon_g \) in two ways: (i) by updating weights of the network with SGD \(^1\) iteratively, and (ii) by solving numerically the differential equations \(^5\) which govern the order parameters, and we confirmed that they accord with each other very well (Figure 4). Note that we set the initial values of order parameters in (ii) as values corresponding to initial weights used in (i). For dependence of the learning trajectory on the initial condition, see Supplementary Material A.3.

\(^1\) We executed all computations on a standard PC.
4.2 Case of scalar input covariance $\Sigma = \sigma I_N$

As the simplest case, here we consider the case that the covariance matrix $\Sigma$ is proportional to unit matrix. In this case, $\Sigma$ has only one eigenvalue $\lambda = \mu_1$ of multiplicity $N$, then our order parameter system contains only parameters whose order is $0$ ($\varepsilon = 0$). For various values of $\mu_1$, we solved numerically the differential equations of order parameters (5) and plotted the time evolution of generalization loss $\varepsilon_g$ (Figure 5(a)). From these plots, we quantified the lengths and heights of the plateaus as following: we regarded the system is plateauing if the decreasing speed of log-loss is smaller than half of its terminal converging speed, and we defined the height of the plateau as the median of loss values during plateauing. Quantified lengths and heights are plotted in Figure 5(b)(c). It indicates that the plateau length and height heavily depend on $\mu_1$, the input scale. Specifically, as $\mu_1$ decreases, the plateau rapidly becomes longer and lower. Though smaller input data lead to longer plateaus, it also becomes lower and then inconspicuous. This tendency is consistent with Figure 2(a)(b), since IRIS dataset has large $\mu_1$ ($\approx 15.9$) and MNIST has small $\mu_1$ ($\approx 0.112$). Considering this, the claim that the plateau phenomenon does not occur in learning of MNIST is controversial; this suggests the possibility that we are observing quite long and low plateaus.

Note that Figure 5(b) shows that the speed of growing of plateau length is larger than $O(1/\mu_1)$. This is contrast to the case of linear networks which have no activation; in that case, as $\mu_1$ decreases, the speed of learning gets exactly $1/\mu_1$-times larger. In other words, this phenomenon is peculiar to nonlinear networks.
4.3 Case of different input covariance $\Sigma$ with fixed $\mu_1$

In the previous subsection we inspected the dependence of the learning dynamics on the first moment $\mu_1$ of the eigenvalues of the covariance matrix $\Sigma$. In this subsection, we explored the dependence of the dynamics on the higher moments of eigenvalues, under fixed first moment $\mu_1$.

In this subsection, we consider the case in which the input covariance matrix $\Sigma$ has two distinct nonzero eigenvalues, $\lambda_1 = \mu_1 - \Delta \lambda/2$ and $\lambda_2 = \mu_1 + \Delta \lambda/2$, of the same multiplicity $N/2$ (Figure 6). With changing the control parameter $\Delta \lambda$, we can get eigenvalue distributions with various values of second moment $\mu_2 = \langle \lambda_i^2 \rangle$.

\[
\begin{align*}
\Delta \lambda \\
\mu_1 - \frac{\Delta \lambda}{2} & \quad \mu_1 + \frac{\Delta \lambda}{2}
\end{align*}
\]

Figure 6: Eigenvalue distribution with fixed $\mu_1$ parameterized by $\Delta \lambda$, which yields various $\mu_2$.

Figure 7(a) shows learning curves with various $\mu_2$ while fixing $\mu_1$ to 1. From these curves, we quantified the lengths and heights of the plateaus, and plotted them in Figure 7(b)(c). These indicate that the length of the plateau shortens as $\mu_2$ becomes large. That is, the more the distribution of nonzero eigenvalues gets broaden, the more the plateau gets alleviated.

5 Conclusion

Under the statistical mechanical formulation of learning in the two-layered perceptron, we showed that macroscopic equations can be derived even when the statistical properties of the input are generalized. We showed that the dynamics of learning depends only on the eigenvalue distribution of the covariance matrix of the input data. By numerically analyzing the macroscopic system, it is shown that the statistics of input data dramatically affect the plateau phenomenon.

Through this work, we explored the gap between theory and reality; though the plateau phenomenon is theoretically predicted to occur by the general symmetrical structure of neural networks, it is seldom observed in practice. However, more extensive researches are needed to fully understand the theory underlying the plateau phenomenon in practical cases.
Acknowledgement

This work was supported by JSPS KAKENHI Grant-in-Aid for Scientific Research(A) (No. 18H04106).

References

Florent Cousseau, Tomoko Ozeki, and Shun-ichi Amari. Dynamics of learning in multilayer perceptrons near singularities. *IEEE Transactions on Neural Networks*, 19(8):1313–1328, 2008.

Kenji Fukumizu and Shun-ichi Amari. Local minima and plateaus in hierarchical structures of multilayer perceptrons. *Neural Networks*, 13(3):317–327, 2000.

Sebastian Goldt, Madhu S Advani, Andrew M Saxe, Florent Krzakala, and Lenka Zdeborová. Dynamics of stochastic gradient descent for two-layer neural networks in the teacher-student setup. *arXiv preprint arXiv:1906.08632*, 2019.

Weili Guo, Yuan Yang, Yingjiang Zhou, Yushun Tan, Haikun Wei, Aiguo Song, and Guochen Pang. Influence area of overlap singularity in multilayer perceptrons. *IEEE Access*, 6:60214–60223, 2018.

Kazuyuki Hara, Daisuke Saitoh, and Hayaru Shouno. Analysis of dropout learning regarded as ensemble learning. In *International Conference on Artificial Neural Networks*, pages 72–79. Springer, 2016.

John Milnor. On the concept of attractor. In *The Theory of Chaotic Attractors*, pages 243–264. Springer, 1985.

A Emin Orhan and Xaq Pitkow. Skip connections eliminate singularities. *arXiv preprint arXiv:1701.09175*, 2017.

Hyeyoung Park, Shun-ichi Amari, and Kenji Fukumizu. Adaptive natural gradient learning algorithms for various stochastic models. *Neural Networks*, 13(7):755–764, 2000.

Peter Riegler and Michael Biehl. On-line backpropagation in two-layered neural networks. *Journal of Physics A: Mathematical and General*, 28(20):L507, 1995.

David Saad and Sara A Solla. On-line learning in soft committee machines. *Physical Review E*, 52 (4):4225, 1995.

Michiel Straat and Michael Biehl. On-line learning dynamics of relu neural networks using statistical physics techniques. *arXiv preprint arXiv:1903.07378*, 2019.

Michiel Straat, Fhi Abadi, Christina Göpfert, Barbara Hammer, and Michael Biehl. Statistical mechanics of on-line learning under concept drift. *Entropy*, 20(10):775, 2018.

Shiro Takagi, Yuki Yoshida, and Masato Okada. Impact of layer normalization on single-layer perceptron—statistical mechanical analysis. *Journal of the Physical Society of Japan*, 88(7):074003, 2019.

Haikun Wei, Jun Zhang, Florent Cousseau, Tomoko Ozeki, and Shun-ichi Amari. Dynamics of learning near singularities in layered networks. *Neural computation*, 20(3):813–843, 2008.

Yuki Yoshida, Ryo Karakida, Masato Okada, and Shun-ichi Amari. Statistical mechanical analysis of online learning with weight normalization in single layer perceptron. *Journal of the Physical Society of Japan*, 86(4):044002, 2017.

Yuki Yoshida, Ryo Karakida, Masato Okada, and Shun-ichi Amari. Statistical mechanical analysis of learning dynamics of two-layer perceptron with multiple output units. *Journal of Physics A: Mathematical and Theoretical*, 2019.
A.1 Properties of expectation term $I_2$, $I_3$ and $I_4$

The differential equations of learning dynamics (3) and (5) in the main text have expectation terms, $I_2(z_1, z_2)$, $I_3(z_1, z_2, z_3)$ and $I_4(z_1, z_2, z_3, z_4)$. Since their $z$s are either $x_i^{(e)} = \xi^T \Sigma^* J_i$ or $y_n^{(e)} = \xi^T \Sigma^* B_n$, any tuple $(z_1, z_2, \ldots)$ follows multivariate normal distribution $\mathcal{N}(z|0, \langle z \cdot z^T \rangle)$ when $N \to \infty$ by generalized central limit theorem, provided that the input $\xi$ has zero mean and finite covariance. Thus the expectation terms only depend on the covariance matrix $\langle z \cdot z^T \rangle$, and their elements can be calculated as $\langle x_i^{(e)} x_j^{(e)} \rangle = Q_i^{(e+1)} j, \langle x_i^{(e)} y_n^{(e)} \rangle = R_{in}^{(e+1)}$ and $\langle y_n^{(e)} y_n^{(e)} \rangle = T_n^{(e+1)}$. For example,

$$
I_2(x_i, y_p) = \int dz_1 dz_2 g(z_1)g(z_2) \mathcal{N}(z|0, \begin{pmatrix} Q_{ii}^{(1)} & R_{ip}^{(1)} \\ R_{jp}^{(1)} & T_{pp}^{(1)} \end{pmatrix}),
$$

$$
I_3(x_i, x_j^{(e)}, y_p) = \int dz_1 dz_2 dz_3 g'(z_1)z_2g(z_3) \mathcal{N}(z|0, \begin{pmatrix} Q_{ii}^{(1)} & Q_{ij}^{(e+1)} & R_{ip}^{(1)} \\ Q_{ji}^{(e+1)} & Q_{jj}^{(e+1)} & R_{jp}^{(1)} \\ R_{ip}^{(1)} & R_{jp}^{(1)} & T_{pp}^{(1)} \end{pmatrix}),
$$

$$
I_4(x_i, x_j, y_p, y_q) = \int dz_1 dz_2 dz_3 dz_4 g(z_1)g(z_2)g(z_3)g(z_4) \mathcal{N}(z|0, \begin{pmatrix} Q_{ii}^{(1)} & Q_{ij}^{(1)} & Q_{ik}^{(1)} & R_{ip}^{(1)} \\ Q_{ji}^{(1)} & Q_{jj}^{(1)} & Q_{jk}^{(1)} & R_{jp}^{(1)} \\ Q_{ki}^{(1)} & Q_{kj}^{(1)} & Q_{kk}^{(1)} & T_{pp}^{(1)} \\ R_{ip}^{(1)} & R_{jp}^{(1)} & T_{pp}^{(1)} & T_{pq}^{(1)} \end{pmatrix}).
$$

Note that all the covariance matrix is symmetric. Their left-bottom sides are not shown for notational simplicity. Substituting these for $I$s shown in equations (3) and (5) in the main text, we see that the ‘speed’ of $e$-th order parameters can be dependent only on 1-st, $(e+1)$-th, and $(2e+1)$-th order parameters.

Here we prove the following proposition, in order to show that the ‘speed’ of $e$-th order parameters are not dependent on $(2e+1)$-th order parameters.

**Proposition.** The expectation term $I_3(z_1, z_2, z_3) := \int dz_1 dz_2 dz_3 g'(z_1)z_2g(z_3) \mathcal{N}(z|0, C)$ does not depend on $C_{22}$.

**Proof.** Since $C$ is positive-semidefinite, we can write $C = VV^T$ for some squared matrix $V$. Thus, when $\xi \sim \mathcal{N}(0, I_N)$, $A\xi \sim \mathcal{N}(0, C)$ holds. Therefore, we can regard that $z_1(i = 1, 2, 3)$ is generated by $z_i = v^T_i \xi$ where $v_i$ is $i$-th row vector of $V$ and $\xi$ follows the standard normal distribution.

We can write $v_2 = c_1 v_1 + c_3 v_3 + v^\perp$ for some coefficient $c_1, c_3 \in \mathbb{R}$ and some vector $v^\perp$ perpendicular to $v_1$ and $v_3$. Then $I_3$ is written as

$$
I_3(z_1, z_2, z_3) = \langle g'(z_1)z_2g(z_3) \rangle = c_1 \langle g'(z_1)z_1g(z_3) \rangle + c_3 \langle g'(z_1)z_3g(z_3) \rangle + \langle g'(z_1)v^\perp T \xi g(z_3) \rangle.
$$

Since $\xi \sim \mathcal{N}(0, I_N)$ and $v^\perp \perp v_1, v_3$ hold, $(z_1, z_3)$ and $v^\perp T \xi$ is independent. Therefore the third term in the right hand side of the equation above is

$$
\langle g'(z_1)v^\perp T \xi g(z_3) \rangle = \langle g'(z_1)g(z_3) \rangle \langle v^\perp T \xi \rangle = 0.
$$

In addition, we can determine $c_1$ and $c_3$ by solving

$$
C_{12} = v^T_1 v_1 = (c_1 v^T_1 + c_3 v^T_3 + v^\perp^T)v_1 = c_1 C_{11} + c_3 C_{13} \quad \text{and}
$$

$$
C_{23} = v^T_3 v_3 = (c_1 v^T_1 + c_3 v^T_3 + v^\perp^T)v_3 = c_1 C_{13} + c_3 C_{33}.
$$

Together with these, we get

$$
I_3(z_1, z_2, z_3) = (C_{12}C_{33} - C_{13}C_{23}) I_3(z_1, z_2, z_3) + (C_{11}C_{23} - C_{12}C_{13}) I_3(z_1, z_3, z_3),
$$

which shows that $I_3$ is independent to $C_{22}$. ■

A.2 Full expression of order parameter system

Here we describe the whole system of the order parameters, with specific eigenvalue distribution of $\Sigma$. 

10
In this case, the order parameters are

\[
\text{Order variables : } Q_{ij}^{(0)}, R_{in}^{(0)}, D_{ij}, E_{in}
\]

\[
\text{Order constants : } T_{nm}^{(0)}, F_{nm}.
\]

Note that \(Q_{ij}^{(1)}\) is identical to \(Q_{ij}^{(0)}\). This is same for \(R\) and \(T\). The order parameter system is described as following, with omitting \(^{(0)}\)-s for notational simplicity:

\[
\frac{dQ_{ij}}{d\alpha} = \eta \left[ \sum_{p=1}^{M} E_{ip} I_3 \left( \frac{Q_{ij}}{R_{ip}} \right) - \sum_{p=1}^{K} D_{ip} I_3 \left( \frac{Q_{ij}}{Q_{ip}} \right) \right] + \sum_{p=1}^{M} E_{jp} I_3 \left( \frac{Q_{ij}}{R_{ip}} \right) - \sum_{p=1}^{K} D_{jp} I_3 \left( \frac{Q_{ij}}{Q_{ip}} \right)
\]

\[
+ \eta^2 \left[ \sum_{p=1}^{K} D_{ip} I_4 \left( \frac{Q_{ij}}{R_{ip}} \right) - \sum_{p=1}^{K} D_{jp} I_4 \left( \frac{Q_{ij}}{Q_{ip}} \right) \right] + \sum_{p=1}^{K} D_{ip} I_4 \left( \frac{Q_{ij}}{Q_{ip}} \right) - \sum_{p=1}^{K} D_{jp} I_4 \left( \frac{Q_{ij}}{R_{ip}} \right)
\]

\[
\frac{dR_{in}}{d\alpha} = \eta \left[ \sum_{p=1}^{M} E_{ip} I_3 \left( \frac{R_{in}}{R_{ip}} \right) - \sum_{p=1}^{K} D_{ip} I_3 \left( \frac{R_{in}}{Q_{ip}} \right) \right] + \sum_{p=1}^{M} E_{jp} I_3 \left( \frac{R_{in}}{R_{ip}} \right) - \sum_{p=1}^{K} D_{jp} I_3 \left( \frac{R_{in}}{Q_{ip}} \right)
\]

and

\[
\frac{dD_{ij}}{d\alpha} = \eta \left[ \sum_{p=1}^{M} E_{ip} I_2 \left( \frac{Q_{ij}}{R_{ip}} \right) - \sum_{p=1}^{K} D_{ip} I_2 \left( \frac{Q_{ij}}{Q_{ip}} \right) \right] + \sum_{p=1}^{M} E_{jp} I_2 \left( \frac{Q_{ij}}{R_{ip}} \right) - \sum_{p=1}^{K} D_{jp} I_2 \left( \frac{Q_{ij}}{Q_{ip}} \right)
\]

\[
\frac{dE_{in}}{d\alpha} = \eta \left[ \sum_{p=1}^{M} F_{pn} I_2 \left( \frac{Q_{ij}}{R_{ip}} \right) - \sum_{p=1}^{K} D_{ip} I_2 \left( \frac{Q_{ij}}{Q_{ip}} \right) \right]
\]

where

\[
I_2(C) = \frac{2}{\pi} \arcsin \frac{C_{12}}{\sqrt{1 + C_{11} + C_{22}}},
\]

\[
I_3(C) = \frac{2}{\pi} \cdot \frac{1}{\sqrt{(1 + C_{11})(1 + C_{33}) - C_{13}^2}} \frac{C_{23}(1 + C_{11}) - C_{12}C_{13}}{1 + C_{11}},
\]

\[
I_4(C) = \frac{4}{\pi^2} \cdot \frac{1}{\sqrt{1 + 2C_{11}}} \arcsin \frac{1}{\sqrt{(1 + 2C_{11})(1 + C_{22}) - 2C_{12}^2 \sqrt{(1 + 2C_{11})(1 + C_{33}) - 2C_{13}^2}}},
\]

for \(g(x) = \text{erf}(x/\sqrt{2})\) activation, as \cite{SaadSolla1995} showed.

\[\text{A.2.2 Case with } \Sigma \text{ which has two distinct eigenvalues, } \lambda_1 \text{ of multiplicity } r_1 N \text{ and } \lambda_2 \text{ of multiplicity } r_2 N\]

In this case, the order parameters are

\[
\text{Order variables : } Q_{ij}^{(0)}, Q_{ij}^{(1)}, R_{in}^{(0)}, R_{in}^{(1)}, D_{ij}, E_{in}
\]

\[
\text{Order constants : } T_{nm}^{(0)}, T_{nm}^{(1)}, F_{nm}.
\]
Since $\Sigma^2 - (\lambda_1 + \lambda_2)\Sigma + \lambda_1\lambda_2 I_N = 0$, the relation $Q^{(2)}_{ij} = (\lambda_1 + \lambda_2)Q^{(1)}_{ij} - \lambda_1\lambda_2 Q^{(0)}_{ij}$ holds. This is same for $R$ and $T$. Then the order parameter system is described as following:

$$
\frac{dQ^{(0)}_{ij}}{d\alpha} = \eta \left[ \sum_{p=1}^{M} E_{ip}I_3 \left( \begin{array}{c} Q^{(1)}_{ii} \\ Q^{(1)}_{ij} \\ R^{(1)}_{ip} \\ T^{(1)}_{ip} \end{array} \right) - \sum_{p=1}^{K} D_{ip}I_3 \left( \begin{array}{c} Q^{(1)}_{ii} \\ Q^{(1)}_{ij} \\ Q^{(1)}_{ip} \\ Q^{(1)}_{pp} \end{array} \right) \right] + \sum_{p=1}^{M} E_{jp}I_3 \left( \begin{array}{c} Q^{(1)}_{jj} \\ Q^{(1)}_{ji} \\ R^{(1)}_{jp} \\ T^{(1)}_{jp} \end{array} \right) - \sum_{p=1}^{K} D_{jp}I_3 \left( \begin{array}{c} Q^{(1)}_{jj} \\ Q^{(1)}_{ji} \\ Q^{(1)}_{jp} \\ Q^{(1)}_{pp} \end{array} \right)
$$

$$
+ \eta^2 (r_1\lambda_1 + r_2\lambda_2) \left[ \sum_{p-q}^{K,K} D_{ip}D_{jq}T_4 \left( \begin{array}{c} Q^{(1)}_{ii} \\ Q^{(1)}_{ij} \\ Q^{(1)}_{ip} \\ Q^{(1)}_{iq} \end{array} \right) + \sum_{p-q}^{M,M} E_{ip}E_{jq}I_4 \left( \begin{array}{c} Q^{(1)}_{ii} \\ Q^{(1)}_{ij} \\ Q^{(1)}_{ip} \\ Q^{(1)}_{iq} \end{array} \right) \right]
$$

$$
- \sum_{p-q}^{K,M} D_{ip}E_{jq}I_4 \left( \begin{array}{c} Q^{(1)}_{ii} \\ Q^{(1)}_{ij} \\ Q^{(1)}_{ip} \\ Q^{(1)}_{iq} \end{array} \right) - \sum_{p-q}^{M,M} E_{ip}D_{jq}I_4 \left( \begin{array}{c} Q^{(1)}_{ii} \\ Q^{(1)}_{ij} \\ Q^{(1)}_{ip} \\ Q^{(1)}_{iq} \end{array} \right)
$$

$$
\frac{dQ^{(1)}_{ij}}{d\alpha} = \eta \left[ \sum_{p=1}^{M} E_{ip}I_3 \left( \begin{array}{c} (\lambda_1 + \lambda_2)Q^{(1)}_{ij} - \lambda_1\lambda_2 Q^{(0)}_{ij} \\ R^{(1)}_{ip} \\ T^{(1)}_{ip} \end{array} \right) - \sum_{p=1}^{K} D_{ip}I_3 \left( \begin{array}{c} (\lambda_1 + \lambda_2)Q^{(1)}_{ij} - \lambda_1\lambda_2 Q^{(0)}_{ij} \\ Q^{(1)}_{pp} \end{array} \right) \right] + \sum_{p=1}^{M} E_{jp}I_3 \left( \begin{array}{c} (\lambda_1 + \lambda_2)Q^{(1)}_{jj} - \lambda_1\lambda_2 Q^{(0)}_{jj} \\ R^{(1)}_{jp} \\ T^{(1)}_{jp} \end{array} \right) - \sum_{p=1}^{K} D_{jp}I_3 \left( \begin{array}{c} (\lambda_1 + \lambda_2)Q^{(1)}_{jj} - \lambda_1\lambda_2 Q^{(0)}_{jj} \\ Q^{(1)}_{pp} \end{array} \right)
$$

$$
+ \eta^2 (r_1\lambda_1^2 + r_2\lambda_1^2) \left[ \sum_{p-q}^{K,K} D_{ip}D_{jq}T_4 \left( \begin{array}{c} Q^{(1)}_{ii} \\ Q^{(1)}_{ij} \\ Q^{(1)}_{ip} \\ Q^{(1)}_{iq} \end{array} \right) + \sum_{p-q}^{M,M} E_{ip}E_{jq}I_4 \left( \begin{array}{c} Q^{(1)}_{ii} \\ Q^{(1)}_{ij} \\ Q^{(1)}_{ip} \\ Q^{(1)}_{iq} \end{array} \right) \right]
$$

$$
- \sum_{p-q}^{K,M} D_{ip}E_{jq}I_4 \left( \begin{array}{c} Q^{(1)}_{ii} \\ Q^{(1)}_{ij} \\ Q^{(1)}_{ip} \\ Q^{(1)}_{iq} \end{array} \right) - \sum_{p-q}^{M,M} E_{ip}D_{jq}I_4 \left( \begin{array}{c} Q^{(1)}_{ii} \\ Q^{(1)}_{ij} \\ Q^{(1)}_{ip} \\ Q^{(1)}_{iq} \end{array} \right)
$$

$$
\frac{dR^{(0)}_{in}}{d\alpha} = \eta \left[ \sum_{p=1}^{M} E_{ip}I_3 \left( \begin{array}{c} R^{(1)}_{in} \\ R^{(1)}_{ip} \\ T^{(1)}_{ip} \end{array} \right) - \sum_{p=1}^{K} D_{ip}I_3 \left( \begin{array}{c} R^{(1)}_{in} \\ Q^{(1)}_{ip} \\ Q^{(1)}_{pp} \end{array} \right) \right] + \sum_{p=1}^{M} E_{ip}I_3 \left( \begin{array}{c} (\lambda_1 + \lambda_2)R^{(1)}_{in} - \lambda_1\lambda_2 R^{(0)}_{in} \\ R^{(1)}_{ip} \\ T^{(1)}_{ip} \end{array} \right) - \sum_{p=1}^{K} D_{ip}I_3 \left( \begin{array}{c} (\lambda_1 + \lambda_2)R^{(1)}_{in} - \lambda_1\lambda_2 R^{(0)}_{in} \\ Q^{(1)}_{pp} \end{array} \right)
$$

$$
\frac{dR^{(1)}_{in}}{d\alpha} = \eta \left[ \sum_{p=1}^{M} E_{ip}I_3 \left( \begin{array}{c} (\lambda_1 + \lambda_2)R^{(1)}_{in} - \lambda_1\lambda_2 R^{(0)}_{in} \\ R^{(1)}_{ip} \\ T^{(1)}_{ip} \end{array} \right) - \sum_{p=1}^{K} D_{ip}I_3 \left( \begin{array}{c} (\lambda_1 + \lambda_2)R^{(1)}_{in} - \lambda_1\lambda_2 R^{(0)}_{in} \\ Q^{(1)}_{pp} \end{array} \right) \right] + \sum_{p=1}^{M} E_{ip}I_3 \left( \begin{array}{c} (\lambda_1 + \lambda_2)R^{(1)}_{in} - \lambda_1\lambda_2 R^{(0)}_{in} \\ Q^{(1)}_{pp} \end{array} \right) - \sum_{p=1}^{K} D_{ip}I_3 \left( \begin{array}{c} (\lambda_1 + \lambda_2)R^{(1)}_{in} - \lambda_1\lambda_2 R^{(0)}_{in} \\ Q^{(1)}_{pp} \end{array} \right)
$$
With \( N \) with multiplicity 0, \( N \) with macroscopic system, and its variability by random weight initialization. Network size: 2.

That point. How close the initial condition is to that point affects how long it takes to break the weight at all. To argue practical learning trajectory, we have to consider the initial value slightly off from the mean and variance of corresponding initial macroscopic parameters.

However, the solution trajectory starting from just \( \mu_e I_k, 0, \mu_e I_M \) cannot break the weight symmetry at all. To argue practical learning trajectory, we have to consider the initial value slightly off from that point. How close the initial condition is to that point affects how long it takes to break the weight.

\begin{align}
\frac{dD_{ij}}{d\alpha} &= \eta \left[ \sum_{p=1}^{M} E_{ip} I_2 \left( Q_{jj}^{(1)} R_{pp}^{(1)} T_{ij}^{(1)} \right) - \sum_{p=1}^{K} D_{ip} I_2 \left( Q_{jj}^{(1)} R_{pp}^{(1)} T_{ij}^{(1)} \right) \right], \\
\frac{dE_{in}}{d\alpha} &= \eta \left[ \sum_{p=1}^{M} E_{ip} I_2 \left( Q_{ii}^{(1)} R_{pp}^{(1)} T_{ip}^{(1)} \right) - \sum_{p=1}^{K} D_{ip} I_2 \left( Q_{ii}^{(1)} R_{pp}^{(1)} T_{ip}^{(1)} \right) \right],
\end{align}

A.3 Dependence of learning trajectory on initial conditions on macroscopic parameters

![Figure A.8: Dynamics of generalization error \( \varepsilon_g \) and order parameters \( Q_{ij} \) and \( R_{in} \) computed with macroscopic system, and its variability by random weight initialization. Network size: \( N \)-2-1. Learning rate: \( \eta = 0.1 \). Eigenvalues of \( \Sigma \): \( \lambda_1 = 0.3 \) with multiplicity \( 0.5 N \), \( \lambda_2 = 1.7 \) with multiplicity \( 0.5 N \). Black lines: dynamics of \( \varepsilon_g \). Blue lines: \( Q_{11}, Q_{12}, Q_{22} \). Green lines: \( R_{11}, R_{12}, R_{21}, R_{22} \). (a) \( N = 10^5 \), (b) \( N = 10^7 \). In both figures, solid curves and shades represent mean and standard deviation of 100 trials, respectively (note that mean and standard deviation of loss are computed in logarithmic scale).]

In the statistical mechanical formulation, by considering \( N \) as large, the dynamics of the system is reduced to macroscopic differential equations with small (\( N \)-independent) dimensions. The macroscopic system we derived is deterministic in the sense that randomness brought by stochastic gradient descent is vanished. However, note that the trajectory of the macroscopic state can vary in accordance with its initial condition. Figure [A.8] shows this variability with shades.

How does the initial condition affect the learning trajectory? Consider a typical initialization that the microscopic parameters \( J_1, J_2, B_1 \) and \( B_2 \) are initialized as \( (J_i)_k, (B_n)_k \sim N(0, 1/N) \). Then the mean and variance of corresponding initial macroscopic parameters \( Q, R \) and \( T \) are

\[
\begin{align*}
\mathbb{E}[Q_{ii}^{(e)}] &= \mu_e, & \mathbb{V}[Q_{ii}^{(e)}] &= \frac{3\mu_e^2}{N}, & \mathbb{E}[Q_{ij}^{(e)}] &= 0, & \mathbb{V}[Q_{ij}^{(e)}] &= \frac{\mu_e^2}{N}, \\
\mathbb{E}[R_{in}^{(e)}] &= 0, & \mathbb{V}[R_{in}^{(e)}] &= \frac{\mu_e^2}{N}, \\
\mathbb{E}[T_{nn}^{(e)}] &= \mu_e, & \mathbb{V}[T_{nn}^{(e)}] &= \frac{3\mu_e^2}{N}, & \mathbb{E}[T_{nm}^{(e)}] &= 0, & \mathbb{V}[T_{nm}^{(e)}] &= \frac{\mu_e^2}{N}.
\end{align*}
\]

With \( N \rightarrow \infty \), these probabilistic parameters converge to \( (Q^{(e)}, R^{(e)}, T^{(e)}) = (\mu_e I_k, 0, \mu_e I_M) \). However, the solution trajectory starting from just \( (\mu_e I_K, 0, \mu_e I_M) \) cannot break the weight symmetry at all. To argue practical learning trajectory, we have to consider the initial value slightly off from that point. How close the initial condition is to that point affects how long it takes to break the weight.
symmetry, that is, the plateau length. This is why Figure A.8(b) with $N = 10^7$ exhibits plateau slightly longer than that of Figure A.8(a) with $N = 10^5$. 