Gravitational Field Equations in a Braneworld
With Euler-Poincaré Term

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Abstract

We present the effective gravitational field equations in a 3-brane world with Euler-Poincaré term and a cosmological constant in the bulk spacetime. The similar equations on a 3-brane with $Z_2$ symmetry embedded in a five dimensional bulk spacetime were obtained earlier by Maeda and Torii using the Gauss-Codazzi projective approach in the framework of the Gaussian normal coordinates. We recover these equations on the brane in terms of differential forms and using a more general coordinate setting in the spirit of Arnowitt, Deser and Misner (ADM). The latter allows for acceleration of the normals to the brane surface through the lapse function and the shift vector. We show that the gravitational effects of the bulk space are transmitted to the brane through the projected “electric” 1-form field constructed from the conformal Weyl curvature 2-form of the bulk space. We also derive the evolution equations into the bulk space for the electric 1-form field, as well as for the “magnetic” 2-form field parts of the bulk Riemann curvature 2-form. As expected, unlike on-brane equations, the evolution equations involve terms determined by the nonvanishing acceleration of the normals in the ADM-type slicing of spacetime.
1 Introduction

Brane world theories are strictly motivated by the string models. They were mainly proposed to provide new solutions to hierarchy problem and compactification of extra dimensions. The main content of the braneworld idea is that we live in a four dimensional world embedded in a higher dimensional bulk spacetime. In braneworld models, it is shown that gauge fields, fermions and scalar fields of the Standard Model are localised on the brane, while gravity can freely propagate into the higher dimensional bulk space. Such a localization can be seen in Horava-Witten model [1]. It is based on the idea that strongly coupled 10-dimensional $E_8 \times E_8$ heterotic string theory can be related to 11-dimensional theory (M-theory) compactified on an $S^1/Z_2$ orbifold with gauge fields propagating on two 10-dimensional branes located on the boundary hyperplanes with a $Z_2$ symmetry. It has been shown that the subsequent compactification of this model on a deformed Calabi-Yau space leads to a five dimensional spacetime with boundary hyperplanes becoming two 3-branes describing our 4-dimensional world [2, 3].

In a model proposed by Arkani-Hamed-Dimopoulos-Dvali (ADD) [4, 5], the idea of a 3-brane universe is combined with the idea of Kaluza-Klein compactification to solve the hierarchy problem of high-energy physics. This model considers our observable four-dimensional spacetime as a 3-brane with all matter fields localised on it, while gravity can "leak" into all extra spatial dimensions. The size of the extra dimensions may be much greater than the conventional Planckian length. Therefore, unlike the original Kaluza-Klein scenario, the ADD model implies that the extra dimensions can manifest themselves as physical ones.

An alternative braneworld model has been proposed by Randall and Sundrum [6, 7]. Indeed, they proposed two different models consisting of a single extra spatial dimension. In their first model, the two 3-branes were located at the boundaries of an $S^1/Z_2$ orbifold in a 5-dimensional Anti-de Sitter ($AdS_5$) spacetime. It has been shown that under a certain fine balance between the self-gravity of the branes and the bulk cosmological constant the ultraviolet scale is generated from the large Planckian scale through an exponential warp function of a small compactification radius. In the second model, the authors suggested that our observable universe is of a single self-gravitating 3-brane with positive tension and the extra dimension may have even infinite size. In this case, the fine tuning between the gravitational effects of the brane and the bulk results in the localization of 5-dimensional graviton zero mode on the 3-brane, while massive Kaluza-Klein modes die off rapidly at large distances. In other words, 4-dimensional Newtonian gravity
is recovered with high enough accuracy at low energy scales.

In a covariant approach the localization of gravity on a 3-brane was studied by Shiromizu, Maeda and Sasaki [8, 9]. These authors derived covariant gravitational field equations on a 3-brane embedded in a five-dimensional bulk spacetime with $\mathbb{Z}_2$ symmetry. Subsequent generalizations of this study can be made by including dilaton fields as well as the combinations of higher order ($N \geq 2$) Euler-Poincaré densities [10, 11, 12, 13]. (See also [14, 15]). In [16, 17] the covariant gravitational field equations on a singular $\mathbb{Z}_2$ symmetric 3-brane were generalized by including a combination of second order ($N = 2$) Euler-Poincaré gravity which is also known as the Gauss-Bonnet combination. The case of thick branes when Gauss-Bonnet self-interactions are included was studied in [18]. We recall that these type of combinations naturally arise in the low energy action of superstring theories. The higher order curvature combinations can help in the resolution of initial singularities, inflation and fine tuning of cosmological constant [15, 17].

In papers [9, 17] and other related works, the authors project the field equations in the bulk spacetime on a 3-brane using the Gauss-Codazzi projective approach with subsequent specialization to the Gaussian normal coordinates. However, the Gaussian normal coordinates imply a very special slicing of spacetime in the sense that the geodesics orthogonal to a given hypersurface remain orthogonal to all successive hypersurfaces in the slicing. This may not affect the field equations on the brane, since the use of the Gaussian normal coordinates is at least well justified in a close neighborhood of the brane. Indeed, in [21] using a more general ADM-type coordinate setting it has been shown that the effective gravitational field equations on the 3-brane obtained earlier in [9] remain unchanged, however, the evolution equations off the brane are significantly modified due to the acceleration of normals to the brane surface in the nongeodesic, ADM slicing of spacetime.

The use of ADM type coordinates implies the slicing of spacetime by timelike hypersurfaces pierced by a congruence of spacelike curves that are not geodesics and do not intersect the hypersurfaces orthogonally. In other words, in ADM type setting one can allow for acceleration of the normals to the brane surface through the lapse function and the shift vector. Therefore, the consistent analysis of the brane dynamics would benefit from complete freedom in the slicing of five dimensional spacetime in the spirit of Arnowitt, Deser and Misner [22]. For instance, a black hole on a 3-brane world would have a horizon that extends into the bulk space (see Ref. [23] for a comprehensive description of the situation). Clearly, the use of Gaussian normal coordinates may not be appropriate for this kind of situations and the ADM type approach may become very important.
In this paper, using the language of differential forms we generalize the results of [21] by including \((N = 2)\) Euler-Poincaré term in a five-dimensional bulk spacetime. We show that the gravitational influence of the bulk space is felt on the brane through the projected “electric” 1-form field constructed from the bulk Weyl curvature 2-form. We also derive the evolution equations into the bulk space for the electric 1-form field, as well as for the “magnetic” 2-form field parts of the bulk Riemann curvature 2-form. We show that, unlike the on-brane equations, the evolution equations are drastically changed due to the nonvanishing acceleration of the normals in the ADM-type slicing of spacetime.

2  Spacetime geometry and gravitational field equations

We consider a five dimensional bulk spacetime manifold \(M\) equipped with a metric \(G\). We suppose that bulk spacetime includes a 3-brane, 4-dimensional hypersurface endowed with a metric \(g\). Our 5-dimensional action with a cosmological constant and Euler-Poincaré term can be written as

\[
S_5 = \int_M \left\{ \frac{1}{2} R^{AB} \wedge # (E_A \wedge E_B) + \frac{\eta}{4} L_{EP} - \Lambda #1 \right\} + \int_{\partial M} \mathcal{L}(Brane) + \int_M \mathcal{L}(Bulk). \tag{2.1}
\]

Here, the basic gravitational field variables are the coframe 1-forms \(E^A\), \(\{A = 0, 1, 2, 3, 5\}\), in terms of which the spacetime metric \(G = \eta_{AB} E^A \otimes E^B\) where \(\eta_{AB} = diag(- + + + +)\). 5-dimensional Hodge map \(#\) is defined so that oriented volume form \(#1 = E^0 \wedge E^1 \wedge E^2 \wedge E^3 \wedge E^5\), while \(\Lambda\) is the cosmological constant in the bulk spacetime. The torsion-free connection 1-forms \(\Omega^A{}_B\) are obtained from the first Cartan structure equations

\[
dE^A + \Omega^A{}_B \wedge E^B = 0 \tag{2.2}
\]

where the metric compatibility requires \(\Omega_{AB} = - \Omega_{BA}\). Corresponding curvature 2-forms are obtained from the second Cartan structure equations

\[
R^{AB} = d\Omega^{AB} + \Omega^A{}_C \wedge \Omega^C{}_B. \tag{2.3}
\]

\(N = 2\) Euler-Poincaré form action density [14, 15] in 5-dimensional spacetime is

\[
\mathcal{L}_{EP} = R^{AB} \wedge R^{CD} \wedge #(E_A \wedge E_B \wedge E_C \wedge E_D) \tag{2.4}
\]
which can also be written as

\[ L_{EP} = 2R_{AB} \wedge \#R^{AB} - 4P_{A} \wedge \#P^{A} + R_{(5)}^{2} \#1 \]  

(2.5)

in terms of the 5-dimensional Ricci 1-form \( P^{A} = \iota_{B} R^{BA} \) and curvature scalar \( R_{(5)} = \iota_{A} \iota_{B} R^{BA} \). Then the field equation obtained from the variation of action (2.1) with respect to co-frame \( E^{C} \), is

\[
\frac{1}{2} R^{AB} \wedge \#(E_{A} \wedge E_{B} \wedge E_{C}) = -\frac{\eta}{4} R^{AB} \wedge R^{DG} \wedge \#(E_{A} \wedge E_{B} \wedge E_{D} \wedge E_{G} \wedge E_{C}) + \Lambda \#E_{C} - \#\tau_{C}(\text{Bulk}) - \#\tau_{C}(\text{Brane}) \]  

(2.6)

where \( \#T_{C} \) is the stress-energy tensor in the bulk space and \( \#\tau_{C} \) is the stress-energy tensor on 3-brane obtained from the variation of \( \int_{\partial M} L(\text{brane}) \). We take our 5-dimensional local coordinate chart as \( X^{M} = \{ x^{\mu}, y \} \) where \( x^{\mu} \) denotes the local coordinates on the brane. We assume that normal to our brane surface accelerates through the lapse function \( \phi(x, y) \) and the shift vector \( N^{a}(x, y) \). Then in terms of the lapse function and the shift vector the 5-dimensional spacetime metric \( G \) can be written as

\[ G = g + N_{\mu} dx^{\mu} \otimes dy + N_{\mu} dy \otimes dx^{\mu} + (N_{\mu} N^{\mu} + \phi^{2}) dy \otimes dy \]  

(2.7)

where \( g = g_{\mu\nu} dx^{\mu} \otimes dx^{\nu} \) is the 3-brane spacetime metric. We choose the orthonormal co-frame 1-forms as

\[ E^{a}(x, y) = e^{a}(x, y) + N^{a}(x, y) dy \quad a = 0, 1, 2, 3 \quad E^{5} = \phi(x, y) dy. \]  

(2.8)

If we define shift 1-form \( N = N_{a} e^{a} \), then \( N^{a} \) are the orthonormal components of the shift vector \( \vec{N} \). By using inner product identity \( \iota_{B} E^{A} = \delta_{B}^{A} \), we can obtain the components of corresponding inner product operators \( \iota_{A} \). They are given by

\[ \iota_{a} = \iota_{a}, \quad \iota_{5} = \frac{1}{\phi} \iota_{y} - \frac{1}{\phi} N^{a} \iota_{a} \]  

(2.9)

where \( \iota_{a} \) are the inner product operators in 4-dimensional brane spacetime satisfying \( \iota_{b} e^{a} = \delta_{b}^{a} \). We define \( \iota_{y} = \iota_{\frac{\partial}{\partial y}} \) shortly. From (2.2), we can determine the torsion-free connection 1-forms \( \Omega^{A}_{\ B} \) as

\[ 2\Omega^{AB} = \iota^{B} dE^{A} - \iota^{A} dE^{B} + (\iota^{A} \iota^{B} dE_{C}) E^{C}. \]  

(2.10)

On the other hand, we can write \( e^{a} = e^{a}_{\mu}(x, y) dx^{\mu} \) in terms of local coordinate basis one-forms \( dx^{\mu} \). Then since \( e^{a} = e^{a}(x, y) \), we have the following
equation satisfied by torsion-free connection 1-forms $\omega^a{}_b$ and orthonormal co-frames $e^a$ of 3-brane spacetime:

$$de^a = H^a{}_b dy \wedge e^b - \omega^a{}_b \wedge e^b,$$

where

$$H^a{}_b = \partial_y e^a{}_{\mu} \tilde{e}^\mu{}_b.$$  \hfill (2.12)

We note that $e^a{}_{\mu}$ and $\tilde{e}^\mu{}_b$ satisfy the orthogonality relation $e^a{}_{\mu} \tilde{e}^\mu{}_b = \delta^a{}_b$. Then using (2.10), we calculate the components of connection 1-forms $\Omega^A{}_B$ as

$$\Omega^{ab} = \omega^{ab} + \frac{1}{\phi} \lambda^{ab} E^5$$  \hfill (2.13)

where

$$\lambda_{ab} = \frac{1}{2} (H_{ba} - H_{ab} + \iota_b DN_a - \iota_a DN_b)$$  \hfill (2.14)

and

$$\Omega^{5a} = -K^a{}_b e^b - \frac{1}{\phi} \left( K^a{}_b N^b - \partial^a \phi \right) E^5$$  \hfill (2.15)

where we have introduced the quantity

$$K_{ab} = \frac{1}{2\phi} \{(H_{ab} + H_{ba}) - (\iota_a DN_b + \iota_b DN_a)\}$$  \hfill (2.16)

as an extrinsic curvature term and $K = \eta_{ab} K^{ab}$. We also note that we can write connection 1-forms $\omega^{ab}$ of brane spacetime as $\omega^{ab} = \omega^{ab} dx^\mu$. Then since $\omega^{ab} = \omega^{ab}(x, y)$, we have the following equation satisfied by the curvature 2-forms $R^{ab}$ and connection 1-forms $\omega^{ab}$ on the brane

$$d\omega^{ab} = H^{ab}{}_c dy \wedge e^c - \omega^a{}_c \wedge \omega^{cb} + R^{ab}$$  \hfill (2.17)

where

$$H^{ab}{}_c = \partial_y \omega^{ab} e^c.$$  \hfill (2.18)

Next, using (2.3), we calculate the orthonormal components of curvature 2-forms

$$\mathcal{R}^{ab} = \pi^{ab} + \tau^{ab} \wedge E^5.$$  \hfill (2.19)

Here

$$\pi^{ab} = R^{ab} - K^a{}_c K^b{}_d e^c \wedge e^d,$$  \hfill (2.20)

and

$$\tau^{ab} = \frac{1}{\phi} \left\{ D \lambda^{ab} + K^a{}_c K^b{}_d N^c e^d - K^a{}_c K^b{}_d N^d e^c \\
+ K^a{}_c \partial^b \phi e^c - \partial^a \phi K^b{}_d e^d - H^{ab} e^c \right\}$$  \hfill (2.21)
where $R^{ab}$ are the curvature 2-forms on the brane with connection $\omega^a{}_b$. We also calculate the remaining curvature components as

$$R^{5a} = \rho^a + \sigma^a \wedge E^5$$  \hspace{1cm} (2.22)

where

$$\rho^a = -D(K^a{}_b) \wedge e^b$$  \hspace{1cm} (2.23)

and

$$\sigma^a = \frac{1}{\phi} \left\{ \partial_y K^a{}_b e^b - D(K^a{}_b N^b) + D(\partial^a \phi) - \lambda^a c e^c + K^a c H^c b e^b \right\}.$$  \hspace{1cm} (2.24)

By using equations (2.19) and (2.22), we can calculate the components of Ricci 1-form $P^a$ as

$$P^a = \iota^b \pi^{ba} - \frac{1}{\phi} N^b l^b \rho^a - \sigma^a + \left( \iota^b \tau^{ba} - \frac{1}{\phi} N^b l^b \sigma^a \right) E^5$$  \hspace{1cm} (2.25)

and

$$P^5 = -\iota^b \rho^b - \iota^b \sigma^b E^5.$$  \hspace{1cm} (2.26)

Using $\mathcal{R}^{(5)} = t_A t_B \mathbb{R}^{BA}$, we can calculate the curvature scalar $\mathcal{R}^{(5)}$ of bulk spacetime in terms of curvature scalar $\mathcal{R}^{(4)}$ of brane spacetime defined by $\mathcal{R}^{(4)} = t_a t_b R^{ba}$:

$$\mathcal{R}^{(5)} = \mathcal{R}^{(4)} - K^2 + K_{ab} K^{ab} + \frac{2}{\phi} \left\{ N^b l^b DK - t_a D(\partial^a \phi) + t_a D N^b K^{ba} - \partial_y K - H^b a K^a b \right\}.$$  \hspace{1cm} (2.27)

Next, using the relation

$$t_a D N_b - H_{ba} = \lambda_{ba} - \phi K_{ba}$$  \hspace{1cm} (2.28)

and the symmetry of $K_{ba}$ and the anti-symmetry of $\lambda_{ba}$, we can simplify equation (2.27) as

$$\mathcal{R}^{(5)} = \mathcal{R}^{(4)} - K^2 - K_{ab} K^{ab} + \frac{2}{\phi} \left\{ \mathcal{L}_N K - \partial_y K - t_a D(\partial^a \phi) \right\}$$  \hspace{1cm} (2.29)

where

$$\mathcal{L}_N K = N^b t_b DK.$$  \hspace{1cm} (2.30)
is the Lie derivative of $K$ along the vector $\vec{N}$. In order to obtain the gravitational field equations on the brane, we note the following identities between 5-dimensional Hodge map $\#$ and 4-dimensional Hodge map $\ast$:

\begin{align}
\#1 &= \ast 1 \wedge E^5, \\
\#E^a &= \ast e^a \wedge E^5, \\
\#E^5 &= \ast 1 - \frac{N^a}{\phi} \ast e_a \wedge E^5, \\
\#(E_a \wedge E_b \wedge E_c) &= \ast (e_a \wedge e_b \wedge e_c) \wedge E^5, \\
\#(E_b \wedge E_c \wedge E_5) &= \ast (e_b \wedge e_c) - \frac{N^a}{\phi} \ast (e_b \wedge e_c \wedge e_a) \wedge E^5.
\end{align}

We substitute (2.19) and (2.22) into field equation (2.6) and obtain reduced gravitational field equations on the 3-brane. For $C = c$, we obtain the field equations,

\begin{align}
\frac{1}{2} \pi^{ab} \wedge \ast (e_a \wedge e_b \wedge e_c) - \frac{1}{\phi} N^d \rho^b \wedge \ast (e_b \wedge e_c \wedge e_d) + \sigma^b \wedge \ast (e_b \wedge e_c) &= \pi_a \{ \pi^{ab} \wedge \sigma^d + \tau^{ab} \wedge \rho^d \} e_{abcd} \\
+ \Lambda \ast e_c - T_{cd} \ast e^d + \frac{1}{\phi} T_{c5} N_a \ast e^a - \tau_{cd} \ast e^d
\end{align}

and

\begin{align}
DK^b_d \wedge e^d \wedge \ast (e_b \wedge e_c) &= \eta \left\{ R^{ab} - K^a_s K^b_u e_s \wedge e^u \right\} \wedge DK^d_l \wedge e^l e_{abcd} \\
+ T_{c5} \ast 1.
\end{align}

We see that (2.33) is the momentum constraint equation with Euler-Poincaré term, $T_{cd}$ is the energy-momentum tensor of bulk spacetime projected on the brane surface, while $T_{c5} = J_c$ describes the momentum flux from the brane or onto the brane. We note that,

\begin{align}
\# \tau_c &= \tau_{cd} \ast e^d \wedge E^5
\end{align}

where the stress-energy tensor on the brane is of the form

\begin{align}
\tau_{cd} &= \tilde{\tau}_{cd} \frac{\delta(y)}{\phi}
\end{align}

locating the brane hypersurface at $y = 0$. We relate $\delta$-function behaviour in brane energy-momentum tensor on the right hand side of equation (2.32) to the jump in the extrinsic curvature terms of brane \[19, 20\]. At this stage, we impose $Z_2$ mirror symmetry and integrate equation (2.32) across
the brane surface along the orbits of evolution vector $Z^A$. We assume that quantities $K_{ab}^+$ and $K_{ab}^-$ evaluated on both sides of the brane, respectively, remain bounded. Then we obtain the following junction conditions

$$e_a^+ = e_a^-$$

and

$$[K_{ac}]^- - [K]^+ \eta_{ac} - \eta \left\{ 2R_{abcd}[K^{db}]^+ + 2P_{ab}[K^b_c]^+ - 2P_{ac}[K]^+ + 2P_{cb}[K_a b]^+ - \mathcal{R}(4)[K_{ca}]^+ + (\mathcal{R}(4)[K]^+ - 2P_{bd}[K^{db}]^+)\eta_{ca} \right\}$$

$$+ \eta \left\{ 2[KK^b eK_{ba}]^+ - 2[K_{aa}K^{sb}K_{bc}]^+ + [K_{ba}K^{sb}K_{ca}]^+ - [K^2K_{ca}]^+_+ \left( \frac{2}{3}[K^s bK^b \eta] + [K^{bd}K_{db}]^+ + \frac{1}{3}[K^3]^+ \right) \eta_{ca} \right\} = -\tilde{\tau}_{\text{ca}} (2.37)$$

where we have defined the components of the Riemann tensor $R_{abcd}$ and the Ricci tensor $P_{ab}$ from the relations $R_{ab} = \frac{1}{2}R_{abcd}e^c \wedge e^d$ and $P_a = P_{abc} e^b$. Due to $Z_2$-symmetry, $K_{ab}^+ = -K_{ab}^-$. Using this fact and dropping the ± indices, we obtain the equation that relates stress-energy tensor of the brane to the extrinsic curvature terms $K_{ab}$ and the intrinsic curvature terms on either side of the brane surface:

$$K_{ac} - K \eta_{ac} - \eta \left\{ 2R_{abcd}K^{db} + 2P_{ab}K^b c - 2P_{ac}K + 2P_{cb}K_a b - \mathcal{R}(4)K_{ca} \right\} + (\mathcal{R}(4)K - 2P_{ab}K^{ba})\eta_{ca}$$

$$+ \eta \left\{ 2[KK^b eK_{ba}] - 2[K_{aa}K^{sb}K_{bc}] + [K_{ba}K^{sb}K_{ca}] - [K^2K_{ca}] \right\} = -\frac{1}{2}\tilde{\tau}_{ab} (2.38)$$

Next, we obtain the reduced field equations for $C = 5$:

$$\frac{1}{2} \tau^{ab} \wedge *(e_a \wedge e_b) - \frac{1}{2\phi} N^{d \pi} \pi^{ab} \wedge *(e_a \wedge e_b \wedge e_d) = -\frac{\eta}{2\phi} \pi^{ab} \wedge \tau^{cd} \epsilon_{abcd} - \frac{1}{\phi} \Lambda N_a \epsilon^a - T_{bd} \epsilon^d + \frac{1}{\phi} T^5_b N_a \epsilon^a, \quad (2.39)$$

$$\frac{1}{2} \left\{ R^{ab} - K^a cK^b d e^c \wedge e^d \right\} \wedge *(e_a \wedge e_b) = -\frac{\eta}{4} \left\{ R^{ab} \wedge R^{cd} \epsilon_{abcd} - 2R^{ab} \wedge K^c K^d u e^a \wedge e^d \epsilon_{abcd} + K^a \phi K^b n K^c K^d \eta \wedge e^a \wedge e^b \wedge e^c \wedge e^d \epsilon_{abcd} \right\} + \Lambda \ast 1 - T^5_b \ast 1. \quad (2.40)$$
The expression (2.40) is the Hamiltonian constraint equation with Euler-Poincaré term. The component $T_5 \equiv P$ can be interpreted as some kind of pressure term. Next, we decompose the bulk curvature 2-form $\mathbb{R}^{5a}$ into its “electric” and “magnetic” parts. Writing $\mathbb{R}^{5a}$ in the form

$$\mathbb{R}^{5a} = \frac{1}{2} B^{a}{}_{bc} E^{b} \wedge E^{c} + \tilde{\sigma}^{a}{}_{b} E^{5} \wedge E^{b},$$

(2.41)

we can obtain the magnetic tensor component

$$B^{a}{}_{bc} = \rho^{a}{}_{bc} = \iota_{c} D K^{a}{}_{b} - \iota_{b} D K^{a}{}_{c},$$

(2.42)

and the electric tensor component

$$\tilde{\sigma}^{a}{}_{b} = \frac{1}{\phi} \left\{ L_{\vec{N}} K^{a}{}_{b} - \partial_{y} K^{a}{}_{b} + K_{sb} H^{as} - H_{sb} K^{sa} - \iota_{b} D(\partial^{a} \phi) \right\} - \phi K_{sb} K^{sa},$$

(2.43)

where

$$L_{\vec{N}} K^{a}{}_{b} = N^{s} \iota_{s} D K^{a}{}_{b} + K^{a}{}_{s} \iota_{b} D N^{s} - K^{s}{}_{b} \iota_{s} D N^{a}$$

(2.44)

is the Lie derivative of $K^{a}{}_{b}$ along the vector $\vec{N}$. From (2.42), we let the magnetic 2-forms to be given by

$$B^{a} = \frac{1}{2} B^{a}{}_{bc} e^{b} \wedge e^{c} = \rho^{a}.$$ 

(2.45)

We note that the trace of the magnetic tensor is not zero, i.e $\iota_{a} \rho^{a} \neq 0$. On the other hand from (2.43), we let the electric 1-form to be given by

$$\tilde{\sigma}^{a} = \tilde{\sigma}^{a}{}_{b} e^{b}.$$ 

(2.46)

We note that the following relation between $\tilde{\sigma}^{a}$ and $\sigma^{a}$ holds:

$$\tilde{\sigma}^{a} = \frac{1}{\phi} \left\{ N^{s} \iota_{s} D K^{a}{}_{b} e^{b} - N^{b} D K^{a}{}_{b} \right\} - \sigma^{a}$$

$$= - \frac{N^{b}}{\phi} \iota_{b} \rho^{a} - \sigma^{a}.$$ 

(2.47)

From equation (2.43) or (2.47), we can calculate the trace $\tilde{\sigma}$ of the electric tensor as

$$\tilde{\sigma} = \iota_{a} \tilde{\sigma}^{a} = \left\{ L_{\vec{N}} K - \partial_{y} K - \phi K_{ab} K^{ab} - \iota_{a} D(\partial^{a} \phi) \right\} \frac{1}{\phi}.$$ 

(2.48)
By comparing equations (2.29) and (2.48), we obtain the following equation between the trace $\tilde{\sigma}$ and the curvature scalars $R(4)$ and $R(5)$:

$$\tilde{\sigma} = \frac{1}{2} \left\{ R(5) - (R(4) - K^2 + K_{ab} K^{ab}) \right\}. \quad (2.49)$$

On the other hand, we take the trace of field equation (2.6) by considering its exterior product with $E^C$ and obtain the curvature scalar $R(5)$ as

$$\frac{3}{2} R(5) \ast 1 = -\frac{\eta}{4} \pi^{ab} \wedge \pi^{cd} \epsilon_{abcd} - \eta \left\{ \pi^{ab} \wedge \sigma^d \wedge e^c + \tau^{ab} \wedge \rho^d \wedge e^c \right\} \epsilon_{abcd} - T \ast 1 + 5 \Lambda \ast 1 \quad (2.50)$$

where

$$T = T_{cd} \eta^{cd} + P. \quad (2.51)$$

We can express equation (2.50) in terms of the electric 1-form $\tilde{\sigma}^a$ and the magnetic 2-form $\rho^a$. Before that, we simplify $\tau^{ab}$ as

$$\tau^{ab} = -\rho_s^{ab} e^s - \frac{N_s}{\phi} \epsilon_s^{ab} \quad (2.52)$$

where

$$\rho_s^{ab} = \iota^b DK_s^a - \iota^a DK_s^b. \quad (2.53)$$

Then using (2.52), equation (2.50) becomes

$$\frac{3}{2} R(5) \ast 1 = -\frac{\eta}{4} \pi^{ab} \wedge \pi^{cd} \epsilon_{abcd} + \eta \pi^{ab} \wedge \tilde{\sigma}^d \wedge e^c \epsilon_{abcd} + \eta \rho_s^{ab} e^s \wedge \rho^d \wedge e^c \epsilon_{abcd} - T \ast 1 + 5 \Lambda \ast 1. \quad (2.54)$$

Furthermore, using momentum constraint equation (2.33) and equation (2.52), we can also express our reduced gravitational field equation (2.32) in terms of $\tilde{\sigma}^a$ and $\rho^a$ approaching the brane from either positive (+) or (−) side:

$$\frac{1}{2} \pi^{ab} \wedge * (e_a \wedge e_b \wedge e_c) - \tilde{\sigma}^b \wedge * (e_b \wedge e_c) = -\eta \tilde{\sigma}^{ab} \wedge \tilde{\sigma}^d \epsilon_{abcd} - \eta \rho_s^{ab} e^s \wedge \rho^d \epsilon_{abcd} - T_{cd} \ast e^d + \Lambda \ast e_c. \quad (2.55)$$

Now, we can define the trace-free electric 1-form $\tilde{\sigma}^a$ (i.e. $\iota_a \tilde{\sigma}^a = 0$) in terms of $\tilde{\sigma}^a$ and $\tilde{\sigma}$ as

$$\tilde{\sigma}^a = \tilde{\sigma}^a - \frac{1}{4} \tilde{\sigma} e^a. \quad (2.56)$$
Then in terms of $\bar{\sigma}^a$, the effective gravitational field equation (2.55) can be written as

$$\frac{1}{2} \pi^{ab} \wedge *(e_a \wedge e_b \wedge e_c) - \sigma^b \wedge *(e_b \wedge e_c) + \frac{3}{4} \bar{\sigma} \wedge \bar{\sigma}^d \epsilon_{abcd}$$

$$- \frac{\eta}{4} \bar{\pi}^{ab} \wedge *(e_a \wedge e_b \wedge e_c) - \eta \rho_s^{ab} \epsilon^s \wedge \rho^d \epsilon_{abcd} - T_{cd} \wedge e^d + \Lambda \wedge e_c \quad (2.57)$$

where

$$\bar{\pi} = \frac{1}{3 + \frac{\eta}{2 \pi}} \left\{ \frac{\eta}{4} \pi^{ab} \wedge \pi^{cd} \epsilon_{abcd} - \eta \pi^{ab} \wedge \bar{\sigma}^d \epsilon_{abcd}$$

$$- \eta \rho_s^{ab} \wedge (\epsilon^s \wedge \rho^d \wedge \epsilon^e) \epsilon_{abcd} - T - \frac{3}{2} \bar{\pi} + 5 \Lambda \right\} \quad (2.58)$$

in terms of $\bar{\sigma}^a$ and we define $\bar{\pi} = \iota_a \iota_b \pi^{ba}$. Explicitly, $\bar{\pi}$ can be written as

$$\bar{\pi} = K^2 + \bar{K}_{ab} \epsilon^{ba} \quad (2.59)$$

Next, we consider the Weyl (conformal) curvature 2-form

$$C_{AB} = R_{AB} - \frac{1}{3} (E^A \wedge \mathbb{P}^B - E^B \wedge \mathbb{P}^A) + \frac{1}{12} E^A \wedge E^B R_{(5)}. \quad (2.60)$$

Writing the bulk conformal 2-form $C^{5a}$ as

$$C^{5a} = \frac{1}{2} M^{bc} e^b \wedge e^c + \mathcal{E}^a \wedge E^5 \wedge E^b, \quad (2.61)$$

we can obtain the conformal magnetic tensor component

$$M^{ab} = \rho^{ab} - (\rho^s \rho^t \eta^a \wedge \eta^b - \rho^s \eta^a \wedge \rho^t \eta^b) \frac{1}{3}, \quad (2.62)$$

from which the conformal magnetic 2-form $M^a$ follows as

$$M^a = \frac{1}{2} M^{bc} e^b \wedge e^c = \rho^a + \frac{1}{3} \iota_b \rho^b \wedge e^a. \quad (2.63)$$

We note that $M^{bc}$ is traceless, i.e $\iota_a M^a = 0$. On the other hand, we obtain the conformal electric tensor component in the form

$$\mathcal{E}^a = \frac{2}{3} \bar{\pi}^a b - \frac{1}{3} \bar{\pi}^a \rho^b \wedge \epsilon^c - \frac{1}{3} \iota_b \bar{\pi}^a \wedge e^c + \frac{1}{12} \pi^a \wedge \bar{\pi}^b \wedge \epsilon(f). \quad (2.64)$$

If one defines the conformal electric 1-form $\mathcal{E}^a = \mathcal{E}^a \wedge \epsilon^b$, then

$$\mathcal{E}^a = \frac{2}{3} \bar{\pi}^a + \frac{1}{12} \bar{\pi}^a \wedge e^a - \frac{1}{3} \iota_b \bar{\pi}^a \wedge e^c \quad (2.65)$$

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being expressed in terms of the traceless electric 1-forms \( \bar{\sigma}^a \). Therefore, in terms of the conformal electric 1-form \( E^a \), the effective field equation (2.57) becomes

\[
\frac{1}{2} \pi^{ab} \wedge *(e_a \wedge e_b \wedge e_c) - \frac{3}{2} \mathcal{E}^b \wedge *(e_b \wedge e_c) - \left( \frac{9\bar{\pi}}{8 (3 + \frac{3}{2}\bar{\pi})} - \frac{3}{4} \hat{c} \right) * e_c
\]

\[
- \frac{1}{2} \tau_s \pi^{sb} \wedge *(e_b \wedge e_c) - \frac{3\eta}{8 (3 + \frac{3}{2}\bar{\pi})} \kappa * e_c = - \frac{3}{2} \eta \pi^{ab} \wedge \mathcal{E}^d \epsilon_{abcd}
\]

\[
- \left( \frac{\eta}{4} \hat{c} - \frac{3\eta\bar{\pi}}{8 (3 + \frac{3}{2}\bar{\pi})} \right) \pi^{ab} \wedge *(e_a \wedge e_b \wedge e_c) - \frac{1}{2} \eta \pi^{ab} \wedge \tau_n \pi^{nd} \epsilon_{abcd}
\]

\[
+ \frac{\eta^2}{8 (3 + \frac{3}{2}\bar{\pi})} \kappa \pi^{ab} \wedge *(e_a \wedge e_b \wedge e_c) - \eta \rho_s^{ab} \epsilon^s \wedge \rho^d \epsilon_{abcd} + \Lambda * e_c
\]

\[
-T_{cd} * e^d (2.66)
\]

where

\[
\kappa = *(\pi^{sn} \wedge \tau_k \pi^{km} \wedge \epsilon^l) \epsilon_{smdl} \quad (2.67)
\]

and

\[
\hat{c} = \frac{1}{3 + \frac{3}{2}\bar{\pi}} \left\{ \frac{\eta}{4} * (\pi^{ab} \wedge \pi^{cd}) \epsilon_{abcd} - \frac{3\eta}{2} * (\pi^{ab} \wedge \mathcal{E}^d \wedge e^c) \epsilon_{abcd}
\]

\[
- \eta \rho_s^{ab} * (\epsilon^s \wedge \rho^d \wedge e^c) \epsilon_{abcd} - T - \frac{3}{2} \bar{\pi} + 5\Lambda \right\}. \quad (2.68)
\]

Finally, we note that the effective gravitational field equations do not explicitly involve acceleration terms determined through the derivative of the lapse function \( \phi \) and the shift vector \( N^a \). The effective gravitational field equations on the brane can be described in terms of either electric 1-form coming from bulk Riemann curvature 2-form \( \mathbb{R}^{5a} \) or conformal electric 1-form coming from bulk conformal curvature 2-form \( \mathbb{C}^{5a} \).

## 3 Evolution equations

We note that the effective gravitational field equations on the brane are not closed. To obtain a closed system, we should derive the off-brane evolution equations. These equations are found by considering the 5-dimensional Bianchi identity

\[
\mathcal{D} \mathcal{R}^{AB} = 0. \quad (3.1)
\]
From
\[ DR^{ab} = 0, \] (3.2)
we obtain
\[ D\pi^{ab} + K^a e^c \wedge \rho^b - K^b e^c \wedge \rho^a = 0 \] (3.3)
and
\[ \frac{1}{2} \partial y \pi^{ab} cd e^c \wedge e^d + \frac{1}{2} \pi^{ab} cd H^c \wedge e^d + \frac{1}{2} \pi^{ab} cd H^d e^c \wedge e^n + d\phi \wedge \sigma^{ab} + \phi D\tau^{ab} + \lambda^a c e^c \wedge \rho^b + K^a c e^c \wedge \sigma^b + K^a c N^c \rho^b - \partial^a \phi \rho^b + \lambda^b d \pi^{ad} - \phi K^b e^c \wedge \sigma^a - K^b c N^c \rho^a + \partial^b \phi \rho^a = 0. \] (3.4)

From equation (3.3), we obtain the 4-dimensional Bianchi identity
\[ DR^{ab} = 0. \] (3.5)
Then using equations (2.47) and (2.52) we can simplify (3.4) as
\[ \frac{1}{2} \partial y \pi^{ab} cd e^c \wedge e^d - L_{N^a} \pi^{ab} + \frac{1}{2} \pi^{ab} cd H^c \wedge e^d + \frac{1}{2} \pi^{ab} cd H^d e^c \wedge e^n - \rho_s^{ab} d\phi \wedge e^s - \phi D(\rho_s^{ab}) \wedge e^s + \lambda^a c e^c \wedge \sigma^b + K^a c e^c \wedge \sigma^b + K^a c N^c \rho^b - \partial^a \phi \rho^b + \lambda^b d \pi^{ad} + \partial^b \phi \rho^a = 0. \] (3.6)
where
\[ L_{N^a} \pi^{ab} = D(N^a t_s \pi^{ab}) + N^a t_s D\pi^{ab} \] (3.7)
is Lie derivative of \( \pi^{ab} \) along \( N^a \). We note that the components of the acceleration of normal to the brane surface are \( a^b = -\frac{1}{\phi} \partial^b \phi \) and \( a^5 = -\frac{1}{\phi} \partial_4 \phi N^a \) where \( \partial_4 \phi = t_b d\phi \). On the other hand, from
\[ DR^{5a} = 0, \] (3.8)
we obtain
\[ D\rho^a - K_{cs} e^s \wedge \pi^{ca} = 0 \] (3.9)
and
\[ \frac{1}{2} \partial y \rho^a bc e^b \wedge e^c + \frac{1}{2} \rho^a bc H^b \wedge e^c + \frac{1}{2} \rho^a bc H^c s e^b \wedge e^s + d\phi \wedge \sigma^a + \phi D\sigma^a - \phi K_{cs} e^s \wedge \tau^{ca} - K_{cs} N^s \pi^{ca} + \partial_c \phi \pi^{ca} + \lambda^a b \rho^b = 0. \] (3.10)
Using equations (2.47), (2.52) and (3.9), we can write equation (3.10) as

\[
\frac{1}{2} \partial_y \rho^a b c e^b \wedge e^c - \mathcal{L}_{\vec{N}} \rho^a + \frac{1}{2} \rho^a b c H^b l e^l \wedge e^c + \frac{1}{2} \rho^a b c H^c s e^b \wedge e^s - d\phi \wedge \tilde{\sigma}^a - \phi D\tilde{\sigma}^a + \phi K e s \rho^a e^s \wedge e^n + \partial_c \phi c a + \lambda^a b \rho^b = 0 \quad (3.11)
\]

where

\[
\mathcal{L}_{\vec{N}} \rho^a = D(N^s t_s \rho^a) + N^s t_s D\rho^a. \quad (3.12)
\]

Equation (3.11) is the evolution equation for magnetic 2-form \( \rho^a \). In order to obtain the evolution equation for electric 1-form \( \tilde{\sigma}^a \), we should act by \( (\partial_y - \mathcal{L}_{\vec{N}}) \) to both sides of equation (2.55) and compare it with equation (3.6). These equations are complicated and will not be given here.

4 Conclusion

Using the language of differential forms we have presented the gravitational field equations of 3-brane dynamics with the Euler-Poincaré Lagrangian and a cosmological constant in five-dimensions with \( \mathbb{Z}_2 \) symmetry. We have used a more general ADM-type approach in which the normal to the brane surface possesses an acceleration through the lapse function \( \phi \) and the shift vector \( N^a \). In this approach effective gravitational field equations on the brane obtained earlier in [17] still remain to be the same, however in our case they are given in terms of differential forms namely, either the electric 1-form coming from the bulk Riemann curvature 2-form \( R^{5a} \), or the conformal electric 1-form coming from the bulk conformal curvature 2-form \( C^{5a} \). The field equations involve the terms due to the cosmological constant as well. We have also derived the evolution equations into the bulk space which, unlike the on-brane field equations involve the terms determined by the acceleration of the normals to the brane surface in the ADM-type non-geodesic slicing of the bulk spacetime. In this sense, the evolution off-brane equations can be thought of as more general ones and the consistent analysis of the non-linear brane dynamics would certainly benefit from it. Therefore, the developed formalism will be useful in studying the exact gravitational solutions on the brane, like for instance black holes and cosmological perturbations.

5 Acknowledgement

HC is supported through a Post-Doctoral Research Fellowship by the Scientific and Technical Research Council of Turkey (TÜBİTAK). TD acknowledges partial support by the Turkish Academy of Sciences (TÜBA).
References

[1] Horava P and Witten E 1996 Nucl. Phys. B 460 506
[2] Witten E 1996 Nucl. Phys. B 471 135
[3] Lukas A, Ovrut B A, Stelle K S and Waldram D 1999 Phys. Rev. D 59 086001
[4] Arkani-Hamed N, Dimopoulos S and Dvali G 1998 Phys. Lett. B 429 263
[5] Antoniadis I, Arkani-Hamed N, Dimopoulos S and Dvali G 1998 Phys. Lett. B 436 257
[6] Randall L and Sundrum R 1999 Phys. Rev. Lett. 83 3370
[7] Randall L and Sundrum R 1999 Phys. Rev. Lett. 83 4960
[8] Shiromizu T, Maeda K and Sasaki M 2000 Phys. Rev. D 62 024012
[9] Sasaki M, Shiromizu T and Maeda K 2000 Phys. Rev. D 62 024008
[10] Lovelock D 1971 J. Math. Phys. 12 498
[11] Boulware D G and Deser S 1985 Phys. Rev. Lett. 55 2656
[12] Zwiebach B 1985 Phys. Lett. B 156 315
[13] Zumino B 1986 Phys. Rep. 137 109
[14] Arık M and Dereli T 1987 Phys. Lett. B 189 96
[15] Arık M and Dereli T 1989 Phys. Rev. Lett. 62 5
[16] Maeda K and Wands D 2000 Phys. Rev. D 62 124009
[17] Maeda K and Torii T 2004 Phys. Rev. D 69 024002
[18] Giovannini M 2001 Phys. Rev. D 64 124004
[19] Davis S C 2003 Phys. Rev D 67 024030
[20] Gravanis E and Willison S 2003 Phys. Lett. B 562 118
[21] Aliev A N and Gümrükçüoğlu A E 2004 Class. Quant. Grav. 21 5081
[22] Arnowitt R, Deser S and Misner C W In *Gravitation: an introduction to current research*, L. Witten, ed. Wiley, New York, p.227 (1962); gr-qc/0405109

[23] Aliev A N and Gümrukçüoğlu A E 2005 Phys. Rev. D 71 104027