GENUS ZERO GOPAKUMAR-VAFA TYPE INVARIANTS FOR CALABI-YAU 4-FOLDS

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ABSTRACT: In analogy with the Gopakumar-Vafa conjecture on CY 3-folds, Klemm and Pandharipande defined GV type invariants on Calabi-Yau 4-folds using Gromov-Witten theory and conjectured their integrality. In this paper, we propose a sheaf-theoretic interpretation of their genus zero invariants using Donaldson-Thomas theory on CY 4-folds. More specifically, we conjecture genus zero GV type invariants are DT$_4$ invariants for one-dimensional stable sheaves on CY 4-folds. Some examples are computed for both compact and non-compact CY 4-folds to support our conjectures. We also propose an equivariant version of the conjectures for local curves and verify them in certain cases.

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0. Introduction

0.1. Background. Gromov-Witten invariants \cite{1,24} are rational numbers which virtually count stable maps from complex curves to algebraic varieties (or symplectic manifolds). Because of multiple-cover contributions, they are in general not integers and hence are not honest enumerative invariants. On a Calabi-Yau 3-fold \( Y \), motivated by string duality, Gopakumar-Vafa \cite{10} conjectured the existence of integral invariants \( n_{g,\beta} \) \( (g \geq 0, \beta \in H_2(Y)) \) which determine Gromov-Witten invariants \( GW_{g,\beta} \) by the identity

\[
\sum_{\beta > 0, g \geq 0} GW_{g,\beta} \lambda^{2g-2} t^\beta = \sum_{\beta > 0, g \geq 0, k \geq 1} \frac{n_{g,\beta}}{k} \left( \frac{2 \sin \left( \frac{k \lambda}{2} \right)}{k} \right)^{2g-2} t^k \beta.
\]

In particular, when \( g = 0 \), it recovers the Aspinwall-Morrison multiple cover formula

\[
GW_{0,\beta} = \sum_{k \geq 1, k|\beta} \frac{1}{k^3} n_{0,\beta/k}.
\]

Moreover, the invariants \( n_{g,\beta} \) should be interpreted in a sheaf-theoretic way \cite{10,13,16,18,27}, for example when \( g = 0 \), the invariant \( n_{0,\beta} \) is conjectured to be the Donaldson-Thomas invariant \cite{31} counting one-dimensional stable sheaves \( E \) with \( [E] = \beta, \chi(E) = 1 \) on \( Y \).

As Gromov-Witten invariants can be defined for smooth varieties of any dimension, it is natural to ask for generalizations of GV type invariants in higher dimensions. In \cite{21}, Klemm-Pandharipande gave a definition of GV type invariants on Calabi-Yau 4-folds via Gromov-Witten theory, and conjectured that they are integers.

0.2. GV type invariants on CY 4-folds. Let \( X \) be a smooth projective Calabi-Yau 4-fold \cite{35}. Note that in this case, Gromov-Witten invariants vanish for genus \( g \geq 2 \) by the dimension reason, so one only needs to consider the genus 0 and 1 cases.

The genus 0 GW invariants on \( X \) are defined using insertions: for integral classes \( \gamma_i \in H^{m_i}(X, \mathbb{Z}), 1 \leq i \leq n \), one defines

\[
GW_{0,\beta}(\gamma_1, \ldots, \gamma_n) := \int_{\overline{M}_{0,n}(X,\beta)} \prod_{i=1}^{n} \text{ev}_i^*(\gamma_i),
\]

where \( \text{ev}_i : \overline{M}_{0,n}(X,\beta) \to X \) is the \( i \)-th evaluation map. The invariants \( (0.1) \)

\[
n_{0,\beta}(\gamma_1, \ldots, \gamma_n) \in \mathbb{Q}
\]

are defined in \cite{21} by the identity

\[
\sum_{\beta > 0} GW_{0,\beta}(\gamma_1, \ldots, \gamma_n) q^\beta = \sum_{\beta > 0} n_{0,\beta}(\gamma_1, \ldots, \gamma_n) \sum_{d=1}^{\infty} d^{n-3} q^d \beta.
\]

Conjecture 0.1. (\cite{21}) The invariants \( n_{0,\beta}(\gamma_1, \ldots, \gamma_n) \) are integers.

In \cite{21}, genus zero GW invariants on \( X \) are computed directly in many examples, using either virtual localization or mirror symmetry techniques, in support of their conjectures.

0.3. Our proposal. The aim of this paper is to give a sheaf-theoretic interpretation of the above GV type invariants \( (0.1) \) in terms of Donaldson-Thomas invariants for CY 4-folds (called DT\(_4\)-invariants) introduced by Cao-Leung \cite{7} and Borisov-Joyce \cite{2}. More specifically, we consider the moduli space \( M_\beta \) of 1-dimensional stable sheaves with Chern character \((0, 0, 0, \beta, 1)\). By the results of \cite{7,2}, assuming the existence of a suitable orientation on this space, there exists a DT\(_4\)-virtual class

\[
(M_\beta)^{\text{vir}} \in H_2(M_\beta, \mathbb{Z}).
\]

The virtual class \( (0.2) \) depends on the choice of an orientation on certain line bundle. On each connected component of \( M_\beta \), there are two choices of orientations, which affect the corresponding contribution to the class \( (0.2) \) by a sign (for each connected component).

In order to define the invariants, we require some insertions. Let \( \mathcal{E} \) be a universal sheaf on \( X \times M_\beta \). We define the map \( \tau \) by

\[
\tau : H^m(X) \to H^{m-2}(M_\beta), \quad \tau(\gamma) = \pi_{M_\beta}^*(\pi_X^*\gamma \cup \text{ch}_3(\mathcal{E})).
\]
where \( \pi_X, \pi_M \) are projections from \( X \times M_4 \) to corresponding factors, and \( ch_3(\mathcal{E}) \) is the Poincaré dual to the fundamental cycle of the universal sheaf \( \mathcal{E} \). For \( \gamma_i \in H^m(X, \mathbb{Z}) \), \( 1 \leq i \leq n \), the DT\(_4\) invariant is defined by

\[
\text{DT}_4(\beta \mid \gamma_1, \ldots, \gamma_n) := \int_{[\mathcal{M}_4]^\text{vir}} \prod_{i=1}^{n} \tau(\gamma_i).
\]

**Conjecture 0.2.** (Conjecture 1.3) For a suitable choice of orientation, we have the identity

\[
n_{0,0}(\gamma_1, \ldots, \gamma_n) = \text{DT}_4(\beta \mid \gamma_1, \ldots, \gamma_n).
\]

In particular, we have the multiple cover formula

\[
\text{GW}_{0,0}(\gamma) = \sum_{k|\beta} \frac{1}{k^2} \cdot \text{DT}_4(\beta/k \mid \gamma).
\]

In the current formulation, we do not specify how to choose the orientation in order to match invariants. While a priori there are many choices of orientations, gauge theory arguments (explained later) suggest there are deformation invariant orientations. However, specifying the choice among them requires further investigation.

Our proposal is based on the heuristic argument in Subsection 1.4, where we prove Conjecture 0.2 assuming the CY 4-fold \( X \) to be 'ideal', i.e. curves in \( X \) are smooth of expected dimensions. Apart from that, we verify our conjecture in examples as follows.

**0.4. Verifications of the conjecture I: compact examples.** We first prove Conjecture 0.2 in some examples of compact CY 4-folds.

**Elliptic fibrations.** We consider a projective CY 4-fold \( X \) which admits an elliptic fibration

\[
\pi: X \to \mathbb{P}^3
\]

over \( \mathbb{P}^3 \), given by a Weierstrass model. Let \( f \) be a general fiber of \( \pi \). Then we have

**Proposition 0.3.** (Proposition 2.3) For multiple fiber classes \( \beta = r[f] \), \( r \geq 1 \), Conjecture 0.3 is true.

In this case, we can directly compute the DT\(_4\) invariants and check the compatibility with the computation of GW invariants in [21].

**CY 3-fold fibrations.** We consider a projective CY 4-fold \( X \) which admits a CY 3-fold fibration

\[
\pi: X \to C
\]

over a curve \( C \). In this case, we conjecture a DT\(_4\)/DT\(_3\) correspondence (Conjecture 2.4), which roughly says that DT\(_4\) invariants for one-dimensional stable sheaves supported on general fibers of \( \pi \) equal DT\(_3\) invariants for one-dimensional stable sheaves on those general fibers (CY 3-folds). A special case is when

\[
(0.3) \quad X = Y \times E
\]

where \( Y \) is a CY 3-fold, \( E \) is an elliptic curve and \( \pi \) is the projection to \( E \). In this situation, we verify the conjecture:

**Proposition 0.4.** (Corollary 2.7) Suppose \( X \) is given by (0.3). Then for any \( \beta \in H_2(Y) \subseteq H_2(X) \) and divisor \( H \subseteq X \), we have

\[
(0.4) \quad \text{DT}_4(\beta \mid H \cdot Y) = \text{DT}_3(\beta) \cdot (H \cdot \beta),
\]

for certain choice of orientation in defining the LHS. Here \( \text{DT}_3(\beta) = \deg[M\gamma_3]^{\text{vir}} \) is the DT\(_3\) invariant [31] for one dimensional stable sheaves with Chern character given by \((0, 0, \beta, 1)\).

This would then imply that our Conjecture 0.2 is consistent with the genus zero GV/DT conjecture on CY 3-folds [16]. In particular, by combining the GW/DT/PT correspondence, geometric vanishing and wall-crossing formula on CY 3-folds (see Corollary A.6), we obtain the following result:

**Theorem 0.5.** (Theorem 2.8) Let \( Y \) be a complete intersection CY 3-fold in a product of projective spaces and let \( X = Y \times E \) for an elliptic curve \( E \). Then for any primitive curve class \( \beta \in H_2(Y) \subseteq H_2(X) \) and divisor \( H \subseteq X \), we have

\[
\text{GW}_{0,0}(H \cdot Y) = \text{DT}_4(\beta \mid H \cdot Y),
\]

for certain choice of orientation in defining the RHS, i.e. Conjecture 0.2 holds in this case.
Hyperkähler 4-folds. When the CY 4-fold $X$ has a holomorphic symplectic form, i.e. $X$ is a hyperkähler 4-fold, GW invariants vanish, and so do the GV type invariants. To verify Conjecture [7] we are left to prove the vanishing of DT$_4$ invariants. In [7], such vanishing is shown for torsion-free sheaves by considering the trace map, but this argument does not apply to the case of torsion sheaves. Instead, we construct a cosection map from the (trace-free) obstruction sheaf of moduli spaces of stable sheaves to a trivial bundle which is compatible with Serre duality (Proposition 2.3). We expect the following vanishing result then follows.

Claim 0.6. (Claim 2.11) Let $X$ be a projective hyperkähler 4-fold and $M$ be a proper moduli scheme of simple perfect complexes $F$’s with $\text{ch}_4(F) \neq 0$ or $\text{ch}_3(F) \neq 0$. Then the virtual class $[M]^{\text{vir}} \in H_*(M)$ vanishes.

At the moment, we are lack of Kiem-Li type theory of cosection localization for D-manifolds in the sense of Joyce or Kuranishi space structures in the sense of Fukaya-Oh-Ohta-Ono. We believe that when such a theory is established, our claim should follow automatically. Nevertheless, we have the following evidence for the claim.

1. At least when $M_S$ is smooth, Proposition 2.9 gives the vanishing of virtual class.
2. If there is a complex analytic version of $(−2)$-shifted symplectic geometry [40] and the corresponding construction of virtual classes [2], one could prove the vanishing result as in GW theory, i.e. taking a generic complex structure in the $S^2$-twistor family of the hyperkähler 4-fold which does not support coherent sheaves and then vanishing of virtual classes follows from their deformation invariance.

By taking away trivial obstruction factors, one can define reduced virtual classes and the corresponding invariants. We explicitly do this in an example and prove a version of Conjectures [2.2] for reduced invariants (Proposition 2.11).

0.5. Verifications of the conjecture II: local surfaces. For a smooth projective surface $S$, we consider the non-compact CY 4-fold

$$X = \text{Tot}_S(L_1 \oplus L_2)$$

where $L_1, L_2$ are line bundles on $S$ satisfying $L_1 \otimes L_2 \cong K_S$. We can also investigate an analogue of Conjecture [0.2] previously formulated on projective CY 4-folds. In particular, when $L_1^{-1}, L_2^{-1}$ are ample, the moduli space $M_{X, \beta}$ of one dimensional stable sheaves on $X$ is compact (Proposition 3.1). So DT$_4$ invariants are well-defined and we can study Conjecture [0.2] in this case. In Subsection 3.2 we check this for low degree curves when $S = P^2$ and

$$X = \text{Tot}_{P^2}(\mathcal{O}_{P^2}(-1) \oplus \mathcal{O}_{P^2}(-2)).$$

In general, the moduli space $M_{X, \beta}$ is non-compact. On the other hand, there is a $\mathbb{C}^*$-action on $X$ fiberwise over $S$, which preserves the CY 4-form on $X$, and such that the $\mathbb{C}^*$-fixed locus of $M_{X, \beta}$ is compact. In this case, DT$_4$ invariants may be defined via equivariant residue, yielding rational functions of the equivariant parameters. Namely we define

$$[M_{X, \beta}]^{\text{vir}} := [M_{X, \beta}^{\text{vir}}] \cdot e(\mathcal{R}\text{Hom}_{\pi_M}(\mathcal{E}, \mathcal{E})^{\text{mov}})^{1/2} \in H_*(M_{X, \beta}^{\text{vir}})[t^\pm 1],$$

as in [7] Section 8). Here

$$[M_{X, \beta}^{\text{vir}}] \in H_*(M_{X, \beta})$$

should be the DT$_4$ virtual class of the $\mathbb{C}^*$-fixed locus, $\mathcal{E}$ is the universal sheaf on $X \times M_{X, \beta}$, $\pi_M : X \times M_{X, \beta} \to M_{X, \beta}$ is the projection and $t$ is the equivariant parameter for the $\mathbb{C}^*$-action. Since the localization formula for DT$_4$-virtual class is not yet available, the definition of (0.5) is only heuristic at this moment.

Note that the moduli space $M_{S, \beta}$ of one-dimensional stable sheaves on $S$ is a union of connected components of $M_{X, \beta}$ and we can determine its contribution to (0.5) (Proposition 5.3). When the surface component is the only $\mathbb{C}^*$-fixed locus, i.e. $M_{X, \beta}^{\text{vir}} = M_{S, \beta}$, we can rigorously define (0.5) and the following residue DT$_4$ invariant $\text{DT}_4^{\text{res}}(\beta) \in \mathbb{Z}$ by taking the residue of (0.5) at $t = 0$:

$$\text{DT}_4^{\text{res}}(\beta) := \text{Res}_{t=0} \int_{[M_{X, \beta}^{\text{vir}}]} e(\mathcal{R}\text{Hom}_{\pi_M}(\mathcal{E}, \mathcal{E})^{\text{mov}})^{1/2}. $$
Then we propose an analogue of Conjecture 0.2 (see Conjecture 3.3) for residue invariants as follows

\[(0.6) \quad GW_{0,\beta}^{\text{res}} = \sum_{k|\beta} \frac{1}{k} DT_{4}^{\text{res}}(\beta/k).\]

Here \(GW_{0,\beta}^{\text{res}} \in \mathbb{Q}\) is the corresponding residue GW invariants. Note that the power of \(1/k\) becomes three (instead of two in (0.2)) as there is no insertion here. The above conjecture is verified in the following:

**Theorem 0.7.** (Theorem 3.7, 3.10) Let \(X = \text{Tot}_{S}(O_{S} \oplus K_{S})\). Then \(M_{X,\beta}^{S} = M_{S,\beta}\) and (0.6) is true if

1. \(S\) is a smooth toric del-Pezzo surface and \(\beta \in H_{2}(X)\) is any curve class;
2. \(S\) is a rational elliptic surface, and \(\beta = \beta_{n}\) are primitive classes defined in (0.9).

Similarly to Theorem 0.5, the above result is reduced to the similar result for the non-compact CY 3-fold \(Y = \text{Tot}_{S}(K_{S})\).

### 0.6. Verifications of the conjecture III: local curves

Let \(C\) be a smooth projective curve with genus \(g(C)\). We consider a CY 4-fold \(X\) given by

\[X = \text{Tot}_{C}(L_{1} \oplus L_{2} \oplus L_{3}),\]

where \(L_{1}, L_{2}, L_{3}\) are line bundles on \(C\) satisfying \(L_{1} \otimes L_{2} \otimes L_{3} \cong \omega_{C}\). The three dimensional complex torus \(T = (\mathbb{C}^{*})^{3}\) acts on \(X\) fiberwise over \(C\). The \(T\)-equivariant GW invariants

\[GW_{0,d(C)} \in \mathbb{Q}(\lambda_{1}, \lambda_{2}, \lambda_{3})\]

can be defined via equivariant residue (e.g. [22]). Here \(\lambda_{i}\) are the equivariant parameters with respect to the \(T\)-action.

On the other hand, there is a two dimensional subtorus \(T_{0} \subseteq (\mathbb{C}^{*})^{3}\) which preserves the CY 4-form on \(X\). As in the local surface case (0.5), we may define equivariant \(DT_{4}\) invariants

\[DT_{4}(d(C)) \in \mathbb{Q}(\lambda_{1}, \lambda_{2})\]

as rational functions in terms of equivariant parameters of \(T_{0}\). We explicitly determine \(DT_{4}(d(C))\) for \(d \leq 2\) (Corollary 4.3, 4.9) and propose the following conjecture:

**Conjecture 0.8.** (Conjecture 4.10) For any smooth projective curve \(C\) and line bundles \(L_{i}\) \((i = 1, 2, 3)\) on \(C\) with \(L_{1} \otimes L_{2} \otimes L_{3} \cong \omega_{C}\), we have the identity

\[GW_{0,2(C)} = DT_{4}(2[C]) + \frac{1}{8} DT_{4}([C]) \in \mathbb{Q}(\lambda_{1}, \lambda_{2})\]

after substituting \(\lambda_{3} = -\lambda_{1} - \lambda_{2}\) in the LHS.

By computing \(DT_{4}([2C])\) explicitly and using Mathematica, we prove the following:

**Theorem 0.9.** (Theorem 4.12) Conjecture 0.8 is true if

1. \(g(C) \geq 1\);
2. \(g(C) = 0\) and \(|l_{1}| \leq 10, |l_{2}| \leq 10\) for \(l_{i} = \deg(L_{i})\) with \(l_{1} \geq l_{2} \geq l_{3}\).

Indeed the conjectural formula in Conjecture 0.8 implies some non-trivial identities of rational functions of \(\lambda_{1}, \lambda_{2}\). See Appendix B.

### 0.7. Speculation for genus one GV type invariants

For genus one, virtual dimensions of GW moduli spaces on a CY 4-fold \(X\) are zero, so the GW invariants

\[GW_{1,\beta} = \int_{[M_{1,0}(X,\beta)]^{\text{vir}}} 1 \in \mathbb{Q}\]

can be defined without insertions. The invariants

\[(0.7) \quad n_{1,\beta} \in \mathbb{Q}\]

are defined in [21] by the identity

\[\sum_{\beta > 0} GW_{1,\beta} q^{\beta} = \sum_{\beta > 0} n_{1,\beta} \sum_{d=1}^{\infty} \sigma(d) q^{d\beta} + \frac{1}{24} \sum_{\beta > 0} n_{0,\beta}(c_{2}(X)) \log(1 - q^{\beta}) - \frac{1}{24} \sum_{\lambda_{1}, \lambda_{2}} m_{\lambda_{1}, \lambda_{2}} \log(1 - q^{\lambda_{1} + \lambda_{2}}),\]
where $\sigma(d) = \sum d_i^e$ and $m_{\beta_1, \beta_2} \in \mathbb{Z}$ are called meeting invariants which can be inductively determined by genus zero GW invariants. In [21], the invariants (0.7) are also conjectured to be integers.

It is a natural problem to give a sheaf-theoretic interpretation to the invariants (0.7). In the CY 3-fold case, the current proposals for such invariants involve refined versions of DT invariants (see [13, 19, 27]), which are not available for CY 4-folds. Instead, we speculate that there is an interpretation of the invariants (0.7) using the DT-version of Pandharipande-Thomas’ stable pair invariants [29].

Namely let us consider the stable pair moduli space $P_n(X, \beta)$ which parametrizes stable pairs $(s: \mathcal{O}_X \to F)$, $[F] = \beta$, $\chi(F) = n$ i.e. $F$ is a pure one dimensional sheaf and $s$ is surjective in dimension one. By assuming an orientability result, one can similarly define the virtual class

$$[P_n(X, \beta)]^{vir} \in H_{2n}(P_n(X, \beta), \mathbb{Z}).$$

A difference from the CY 3-fold case is that the virtual dimension of (0.8) depends on the holomorphic Euler characteristic $n$. A special case is when $n = 0$, where the virtual dimension is zero, we can take its degree:

$$P_{0, \beta} := \int [P_0(X, \beta)]^{vir} 1 \in \mathbb{Z}. $$

We expect the above invariants to contain information on the genus one GV type invariants (0.7). Some of our computations suggest the following formula

$$\sum_{\beta \geq 0} P_{0, \beta} q^\beta = \prod_{\beta > 0} M(q)^{m_\beta},$$

where $M(q) = \prod_{k \geq 1} (1 - q^k)^{-k}$ is the MacMahon function. Some more analysis of the above formula may be pursued in a future work.

0.8. Notation and convention. In this paper, all varieties and schemes are defined over $\mathbb{C}$. For a morphism $\pi: X \to Y$ of schemes, and for $\mathcal{F}, \mathcal{G} \in \text{D}^b(\text{Coh}(X))$, we denote by $\mathbf{R}\pi_\ast \mathbf{R}\pi_\ast(\mathcal{F}, \mathcal{G})$ the functor $\mathbf{R}\pi_\ast \mathbf{R}\pi_\ast(\mathcal{F}, \mathcal{G})$. We also denote by $\text{ext}^i(\mathcal{F}, \mathcal{G})$ the dimension of $\text{Ext}^i_X(\mathcal{F}, \mathcal{G})$.

A class $\beta \in H_2(X, \mathbb{Z})$ is called irreducible (resp. primitive) if it is not the sum of two non-zero effective classes (resp. if it is not a positive integer multiple of an effective class).

For a scheme $X$, we denote by $\chi(X)$ its topological Euler characteristic. For a sheaf $\mathcal{F}$ on $X$, we denote by $\chi(\mathcal{F})$ its holomorphic Euler characteristic.

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1. Definitions and conjectures

Throughout this paper, unless stated otherwise, $X$ will always denote a smooth projective Calabi-Yau 4-fold over $\mathbb{C}$, i.e. $K_X \cong \mathcal{O}_X$.

1.1. GW/GV conjecture on Calabi-Yau 4-folds. Let $\overline{M}_{g, n}(X, \beta)$ be the moduli space of genus $g$, $n$-pointed stable maps to $X$ with curve class $\beta$. Its virtual dimension is given by

$$-K_X \cdot \beta + (\dim X - 3)(1 - g) + n = 1 - g + n.$$

For cohomology classes

$$(1.1) \quad \gamma_i \in H^{m_i}(X, \mathbb{Z}), \quad 1 \leq i \leq n,$$

the corresponding Gromov-Witten invariant is defined by

$$(1.2) \quad \text{GW}_{g, \beta}(\gamma_1, \ldots, \gamma_n) = \int_{[\overline{M}_{g, n}(X, \beta)]^{vir}} \prod_{i=1}^n \text{ev}_i^!(\gamma_i),$$

where $\text{ev}_i: \overline{M}_{g, n}(X, \beta) \to X$ is the $i$-th evaluation map.
Conjecture 1.1. and conjecture the following

defined invariants $n_{0,\beta}(\gamma_1, \ldots, \gamma_n)$ on CY 4-folds by the identity

$$\sum_{\beta > 0} \text{GW}_{0,\beta}(\gamma_1, \ldots, \gamma_n)q^\beta = \sum_{\beta > 0} n_{0,\beta}(\gamma_1, \ldots, \gamma_n) \sum_{d=1}^{\infty} d^{n-3} q^{d\beta},$$

and conjecture the following

Conjecture 1.1. (21 Conjecture 0) The invariants $n_{0,\beta}(\gamma_1, \ldots, \gamma_n)$ are integers.

For $g \geq 2$, GW invariants vanish for dimensional reasons, so a GW/GV type integrality conjecture on CY 4-folds makes sense only for genus 0 and 1. We will propose a sheaf-theoretic definition of GV-type integral invariants using DT invariants [7, 2].

1.2. Review of DT invariants. Before stating our proposal, we first review the framework for DT invariants. We fix an ample divisor $\omega$ on $X$ and take a cohomology class $v \in H^*(X, \mathbb{Q})$.

The coarse moduli space $M_\omega(v)$ of $\omega$-Gieseker semistable sheaves $E$ on $X$ with $\text{ch}(E) = v$ exists as a projective scheme. We always assume that $M_\omega(v)$ is a fine moduli space, i.e. any point $[E] \in M_\omega(v)$ is stable and there is a universal family

$$E \in \text{Coh}(X \times M_\omega(v)).$$

In [7, 2], under certain hypotheses, the authors construct a DT virtual class

$$[M_\omega(v)]^{vir} \in H_{2-\chi(v,v)}(M_\omega(v), \mathbb{Z}),$$

where $\chi(-, -)$ is the Euler pairing. Notice that this class will not necessarily be algebraic.

Roughly speaking, in order to construct such a class, one chooses at every point $[E] \in M_\omega(v)$, a half-dimensional real subspace

$$\text{Ext}^2_+ (E, E) \subset \text{Ext}^2(E, E)$$

of the usual obstruction space $\text{Ext}^2(E, E)$, on which the quadratic form $Q$ defined by Serre duality is real and positive definite. Then one glues local Kuranishi-type models of form $\kappa_+ = \pi_+ \circ \kappa : \text{Ext}^1(E, E) \rightarrow \text{Ext}^2_+(E, E),$

where $\kappa$ is a Kuranishi map of $M_\omega(v)$ at $E$ and $\pi_+$ is the projection according to the decomposition $\text{Ext}^2(E, E) = \text{Ext}^2_+ (E, E) \oplus \sqrt{-1} \cdot \text{Ext}^2_-(E, E)$.

In [7], local models are glued in three special cases:

1. when $M_\omega(v)$ consists of locally free sheaves only;
2. when $M_\omega(v)$ is Kuranishi smooth, i.e. local Kuranishi maps $\kappa$’s vanish;
3. when $M_\omega(v)$ is a shifted cotangent bundle of a derived smooth scheme.

And the corresponding virtual classes are constructed using either gauge theory or algebrao-geometric perfect obstruction theory in the sense of Behrend-Fantechi [11] and Li-Tian [24].

The general gluing construction is due to Borisov-Joyce [2, 11], based on Pantev-Tian-Vaquie-Vezzosi’s theory of shifted symplectic geometry [30] and Joyce’s theory of derived $\mathcal{C}^\infty$-geometry. The corresponding virtual class is constructed using Joyce’s D-manifold theory (a machinery similar to Fukaya-Oh-Ohta-Ono’s theory of Kuranishi space structures used in defining Lagrangian Floer theory).

In this paper, all computations and examples will only involve the virtual class constructions in situations (2), (3), mentioned above. We briefly review them as follows:

- When $M_\omega(v)$ is Kuranishi smooth, the obstruction sheaf $\text{Ob} \rightarrow M_\omega(v)$ is a vector bundle endowed with a quadratic form $Q$ via Serre duality. Then the DT virtual class is given by

$$[M_\omega(v)]^{vir} = \text{PD}(e(\text{Ob}, Q)).$$

One needs to assume that $M_\omega(v)$ can be given a $(-2)$-shifted symplectic structure as in Claim 3.29 [2] to apply their constructions.
Here $e(Ob, Q)$ is the half-Euler class of $(Ob, Q)$ (i.e. the Euler class of its real form $Ob_+$), and $PD(-)$ is its Poincaré dual. Note that the half-Euler class satisfies

$$e(Ob, Q)^2 = (-1)^{\frac{rk(Ob)}{2}} e(Ob),$$

if $rk(Ob)$ is even,

$$e(Ob, Q) = 0,$$ if $rk(Ob)$ is odd.

- When $M_d(v)$ is a shifted cotangent bundle of a derived smooth scheme, roughly speaking, this means that at any closed point $[F] \in M_d(v)$, we have Kuranishi map of type

$$\kappa: \text{Ext}^2(F, F) \to \text{Ext}^2(F, F) = V_F \oplus V_F^*,$$

where $\kappa$ factors through a maximal isotropic subspace $V_F$ of $(\text{Ext}^2(F, F), Q)$. Then the DT$_4$ virtual class of $M_d(v)$ is, roughly speaking, the virtual class of the perfect obstruction theory formed by $\{V_F\}_{F \in M_d(v)}$. When $M_d(v)$ is furthermore smooth as a scheme, then it is simply the Euler class of the vector bundle $\{V_F\}_{F \in M_d(v)}$ over $M_d(v)$.

**On orientations.** To construct the above virtual class $\mathbf{1.3}$ with coefficients in $\mathbb{Z}$ (instead of $\mathbb{Z}_2$), we need an orientability result for $M_d(v)$, which is stated as follows. Let

$$\mathcal{L} := \det(\mathbf{RHom}_{M_d}(\mathcal{E}, \mathcal{E})) \in \text{Pic}(M_d(v)), \quad \pi_M: X \times M_d(v) \to M_d(v),$$

be the determinant line bundle of $M_d(v)$, equipped with a symmetric pairing $Q$ induced by Serre duality. An orientation of $(\mathcal{L}, Q)$ is a reduction of its structure group (from $O(1, \mathbb{C})$ to $SO(1, \mathbb{C}) = \{1\}$; in other words, we require a choice of square root of the isomorphism

$$Q: \mathcal{L} \otimes \mathcal{L} \to \mathcal{O}_{M_d(v)}$$

(1.6)

to construct virtual class $\mathbf{1.5}$. An existence result of orientations is proved in [3] Theorem 2.2] for CY 4-folds $X$ such that $\text{Hol}(X) = SU(4)$ and $H^{\text{odd}}(X, \mathbb{Z}) = 0$. Notice that, if orientations exist, their choices form a torsor for $H^0(M_d(v), \mathbb{Z}_2)$.

At the moment, there is no canonical choice of orientation for defining the virtual class $\mathbf{1.5}$. We restrict to the case of CY 4-fold $X$ with $\text{Hol}(X) = SU(4)$ and $H^{\text{odd}}(X, \mathbb{Z}) = 0$ and explain how to obtain a deformation invariant virtual class (hence also the corresponding invariants).

Let $\mathfrak{M}$ be the space of all complex structures on $X$. We take a finite type connected open subset $U$ of $\mathfrak{M}$, and a relative ample line bundle $\mathcal{O}_{X_U}(1)$. Let $\{M_{\mathcal{O}_{X_U}(1)}(v)\}_{U} \in U$ be a family of fine Gieseker moduli space (for some $v \in H^*(X, \mathbb{Q})$). We can do a family of Seidel-Thomas twists which identifies $\{M_{\mathcal{O}_{X_U}(1)}(v)\}_{U} \in U$ with a family of moduli space $\{M_{\mathcal{O}_F}(v)\}_{U} \in U$ of simple holomorphic structures on a complex vector bundle $E$.

Recall that the proof of [3] Theorem 2.2] shows the orientation of $(\mathcal{L}, Q)$ over $M_{\mathcal{O}_{X_U}(1)}(v)$ is given as follows: firstly use S-T twists to identify the moduli space to a moduli of simple holomorphic structures on a complex bundle $E$ and then get an induced orientation from the space $\check{B}_{E, X}$ of gauge equivalence classes of framed connections on $E' = E \oplus (\det E)^{-1} \oplus \mathbb{C}^N$ with $N \gg 0$. Note that $\check{B}_{E, X}$ does not depend on the choice of complex structures on $X$, and we know $\pi_1(\check{B}_{E, X}) = 0$ by [3] Theorem 2.1]. So we can choose orientation of $(\mathcal{L}, Q)$ consistently over the family $\mathfrak{U}$ such that the virtual class is invariant under deformation of complex structures.

**1.3. One-dimensional stable sheaves and the genus zero conjecture.** Let us take $\beta \in H_2(X, \mathbb{Z}) \cong H^0(X, \mathbb{Z})$. By taking the cohomology class

$$v = (0, 0, 0, 0, 1) \in H^0(X) \oplus H^2(X) \oplus H^4(X) \oplus H^6(X) \oplus H^8(X),$$

we set

$$M_\beta = M_d(0, 0, 0, 0, 1).$$

**Remark 1.2.** Note that $M_\beta$ is the moduli space of one-dimensional sheaves $E$’s on $X$ satisfying the following: for any $0 \neq E' \subset E$, we have $\chi(E') \leq 0$. In particular, it is independent of the choice of $\omega$ and a fine moduli space.

Since $\chi(v, v) = 0$ in this case, we have

$$[M_\beta]^{\text{vir}} \in H_2(M_\beta, \mathbb{Z}).$$

We define insertions as follows: for a class $\gamma \in H^m(X, \mathbb{Z})$, set

$$\tau(\gamma) := \pi_{M_\beta}(\pi_{\mathcal{O}_F} \cup \chi_t(E)) \in H^{m-2}(M_\beta, \mathbb{Z}).$$

(1.8)
Here $\pi_X, \pi_M$ are projections from $X \times M_\beta$ onto corresponding factors, and $\chi_3(E)$ is the Poincaré dual to the fundamental class of the universal sheaf \((1.4)\). For integral classes \((1.1)\), we define

\[ DT_4(\beta \mid \gamma_1, \ldots, \gamma_n) = \int_{[M_\beta]^{vir}} \prod_{i=1}^{n} \tau(\gamma_i) \in \mathbb{Z}. \]

Note that \((1.9)\) is zero unless \((1.3)\) holds. We propose the following conjecture, which gives a sheaf theoretic interpretation of Conjecture \((1.3)\).

**Conjecture 1.3.** (Genus 0) We have the identity

\[ n_{0,\beta}(\gamma_1, \ldots, \gamma_n) = DT_4(\beta \mid \gamma_1, \ldots, \gamma_n), \]

for certain choice of orientation in defining the RHS.

In particular, we have the multiple cover formula

\[ GW_{0,\beta}(\gamma) = \sum_{k|\beta} \frac{1}{k^2} \cdot DT_4(\beta/k \mid \gamma). \]

**Remark 1.4.** For the insertions, if $m_n = 2$ in \((1.7)\), we have

\[ n_{0,\beta}(\gamma_1, \ldots, \gamma_n) = (\beta \cdot \gamma_n) \cdot n_{0,\beta}(\gamma_1, \ldots, \gamma_{n-1}), \]

\[ DT_4(\beta \mid \gamma_1, \ldots, \gamma_n) = (\beta \cdot \gamma_n) \cdot DT_4(\beta \mid \gamma_1, \ldots, \gamma_{n-1}). \]

Therefore we may assume that $m_i \geq 3$ for all $i$ in Conjecture \((1.3)\). By \((1.3)\), there are two possibilities

- $n = 1$ and $m_1 = 4$.
- $n = 2$ and $m_1 = m_2 = 3$.

In particular, when $H^{odd}(X, \mathbb{Z}) = 0$, we only need to consider the first case.

1.4. **Heuristic explanation of the conjecture.** In this subsection, we give a heuristic argument to explain why we expect Conjecture \((1.3)\) to be true. In this discussion, we ignore questions of orientation.

Let $X$ be an ‘ideal’ CY$_4$ in the sense that all curves inside are smooth of expected dimensions, i.e.

1. any rational curve in $X$ comes with a compact 1-dimensional smooth family of embedded rational curves, whose general member is smooth with normal bundle $O_{P^1}(-1,-1,0)$.
2. any elliptic curve $E$ in $X$ is smooth, super-rigid, i.e. the normal bundle is $L_1 \oplus L_2 \oplus L_3$ for general degree zero line bundle $L_i$ on $X$ satisfying $L_1 \otimes L_2 \otimes L_3 = O_E$. Furthermore any two elliptic curves are disjoint.
3. there is no curve in $X$ with genus $g \geq 2$.

Let $C$ be a rational curve in $X$ with $[C] = \beta \in H_2(X, \mathbb{Z})$, and $\{C_i\}_{i \in T}$ be the 1-dimensional family $C$ sits in. Any one-dimensional stable sheaf $F \in M_\beta$ supported on $C_i$ is $O_{C_i}$ for some curve $C_i$. For a general $t \in T$ such that $N_{C_i/X} \cong O_{P^1}(-1,-1,0)$, there exists canonical isomorphisms

\[ \text{Ext}^1_X(F, F) \cong H^0(C_i, N_{C_i/X}) \cong \mathbb{C}, \]
\[ \text{Ext}^2_X(F, F) \cong H^1(C_i, N_{C_i/X}) \oplus H^1(C_i, N_{C_i/X})^\vee = 0. \]

Hence the local contribution of $C_i$ to the $DT_4$ virtual class $[M_\beta]^{vir}$ is the fundamental class of the family $[T]$.

Let $E$ be an elliptic curve in $X$ with $[E] = \beta \in H_2(X, \mathbb{Z})$. Then any one dimensional stable sheaf $F \in M_\beta$ supported on $E$ is a line bundle on $E$. A similar calculation shows that we have

\[ \text{Ext}^1_X(F, F) \cong H^0(E, N_{E/X}) \oplus H^0(E, O_E) \cong \mathbb{C}, \]
\[ \text{Ext}^2_X(F, F) \cong H^1(E, N_{E/X}) \oplus H^1(E, N_{E/X})^\vee = 0. \]

Hence the local contribution of $E$ to $DT_4$ virtual class $[M_\beta]^{vir}$ is the fundamental class of the Picard variety $\text{Pic}(E) \cong E$.

Let us take $\gamma \in H^4(X, \mathbb{Z})$ and consider the $DT_4$ invariant

\[ DT_4(\beta \mid \gamma) = \int_{[M_\beta]^{vir}} \tau(\gamma). \]

Since the insertion $\tau(\gamma)$ imposes a codimension one constraint on the deformation space of curves in $X$, an elliptic curve $E \subset X$ does not contribute to \((1.10)\) as it is rigid. By the above argument,
we have
\[ \int_{[\mathcal{M}]_{vir}} \tau(\gamma) = \sum \gamma \cdot [C_T], \]
where \([C_T] \in H_4(X, \mathbb{Z})\) denotes the fundamental class of the \(T_i\) family of rational curves. This heuristic argument confirms Conjecture \(\dagger\) as both \(n_{0,\beta}(\gamma)\) and \(\text{DT}_4(\beta | \gamma)\) are (virtually) enumerating rational curves of class \(\beta\) incident to cycle dual to \(\gamma\).

2. Compact examples

In this section, we verify Conjecture \(\dagger\) for certain compact Calabi-Yau 4-folds.

2.1. Elliptic fibrations. For \(Y = \mathbb{P}^3\), we take general elements
\[ u \in H^0(Y, \mathcal{O}_Y(4K_Y)), \quad v \in H^0(Y, \mathcal{O}_Y(-6K_Y)). \]
Let \(X\) be a CY 4-fold with an elliptic fibration
\[ \pi: X \to Y \]
given by the equation
\[ zy^2 = x^3 + ux^2z + vz^3 \]
in the \(\mathbb{P}^2\)-bundle
\[ \mathbb{P}(\mathcal{O}_Y(-2K_Y) \oplus \mathcal{O}_Y(-3K_Y) \oplus \mathcal{O}_Y) \to Y, \]
where \([x : y : z]\) are homogeneous coordinates for the above projective bundle. A general fiber of \(\pi\) is a smooth elliptic curve, and any singular fiber is either a nodal or cuspidal plane curve. Moreover, \(\pi\) admits a section \(\iota\) whose image correspond to the fiber point \([0 : 1 : 0]\).

Let \(h\) be a hyperplane in \(\mathbb{P}^3\), \(f\) be a general fiber of \(\pi: X \to Y\) and set
\[ B = \pi^* h, \quad E = \iota(\mathbb{P}^3) \in H_6(X, \mathbb{Z}). \]
We consider the moduli space \(\mathcal{M}_r[f]\) of one dimensional stable sheaves on \(X\) in the multiple fiber class \(r[f]\).

Lemma 2.1. For any \(r \in \mathbb{Z}_{\geq 1}\), we have an isomorphism \(\mathcal{M}_r[f] \cong X\), under which the virtual class of \(\mathcal{M}_r[f]\) is given by
\[ [\mathcal{M}_r[f]]_{vir} = \pm \text{PD}(c_3(X)) \in H_2(X, \mathbb{Z}), \]
where the sign corresponds to the choice of an orientation in defining the LHS.

Proof. By stability, any one dimensional stable sheaf \(\mathcal{E}\) with \([\mathcal{E}] = r[f]\) satisfies \(\text{Hom}(\mathcal{E}, \mathcal{E}) = \mathbb{C}\), hence by Lemma \(\dagger\) it is scheme theoretically supported on a fiber of \(\pi\). Therefore \(\mathcal{M}_r[f]\) is identified with the \(\pi\)-relative moduli space of stable sheaves on \(X\). By \([4]\) Theorem 2.1, \(\mathcal{M}_r[f]\) is a smooth CY 4-fold which is derived equivalent to \(X\). There are rational maps
\[ \mathcal{M}_r[f] \xrightarrow{\phi_1} M_r[f] \xrightarrow{\phi_2} X, \]
where \(\phi_1\) sends a stable sheaf \(\mathcal{E}\) on \(\pi^{-1}(p)\) for a general point \(p \in Y\) to \(\det(\mathcal{E})\), and \(\phi_2\) sends \(x \in \pi^{-1}(p)\) to \(I_x\) where \(I_x \subset \mathcal{O}_{\pi^{-1}(p)}\) is the ideal sheaf of \(x\) in \(\pi^{-1}(p)\). It is well-known that \(\phi_1\), \(\phi_2\) are isomorphisms on a general fiber of \(\pi\) by Atiyah’s work, hence they are birational maps. Therefore, \(\mathcal{M}_r[f]\) and \(X\) are connected by a finite number of flops by \([17]\). On the other hand, the Picard number of \(X\) is two by the weak Lefschetz theorem and the extremal rays of its nef cone are given by the divisors \(\mathcal{O}_x\). Therefore \(X\) does not admit any flop, and the birational map \(\phi_2^{-1} \circ \phi_1\) extends to an isomorphism
\[ \phi_2^{-1} \circ \phi_1: \mathcal{M}_r[f] \xrightarrow{\cong} X. \]

Using the derived equivalence between \(\mathcal{M}_r[f]\) and \(X\) proved in \([4]\), the deformation-obstruction spaces (with Serre duality pairing) on \(\mathcal{M}_r[f]\) are identified with those on \(X\) viewed as a moduli space of skyscraper sheaves \(\{\mathcal{O}_x\}_{x \in X}\). Therefore the identity \(\dagger\) holds from \([7]\) Proposition 7.17. \(\square\)

Here we used the following lemma, which is well-known:

Lemma 2.2. Let \(f: X \to Y\) be a morphism of schemes and suppose we have \(E \in \text{Coh}(X)\) which is set theoretically supported on \(f^{-1}(p)\) for some \(p \in Y\) and satisfies \(\text{Hom}(E, E) = \mathbb{C}\). Then \(E\) is scheme-theoretically supported on \(f^{-1}(p)\).
Proof. Let $m_p \subset \mathcal{O}_Y$ be the ideal sheaf of $p$. For $u \in m_p$, the multiplication by $u$ defines the morphism
\begin{equation}
(2.3) \quad u: E \to E.
\end{equation}
Since $\text{Hom}(E, E) = \mathbb{C}$, the morphism (2.3) is either isomorphism or zero map. As $E$ is set theoretically supported on $f^{-1}(p)$, some compositions of (2.3) must be a zero map. Therefore (2.3) is a zero map for any $u \in m_p$, which implies that $E$ is $\mathcal{O}_{f^{-1}(p)}$-module. \qed

We now verify Conjecture [13] for multiple fiber classes.

**Proposition 2.3.** Conjecture [13] is true for $\beta = r[f]$, $\gamma = B^2$ or $B \cdot E$.

Proof. Let $\mathcal{E}$ be the universal sheaf on $X \times M_{r[f]}$. Under the isomorphism $M_{r[f]} \cong X$ in Lemma 2.1 we have
\[ \text{ch}_3(\mathcal{E}) = r[X \times_Y X] \in H^6(X \times X, \mathbb{Z}). \]
Together with the identity (2.2), for $\gamma \in H^4(X, \mathbb{Z})$ we obtain
\[ \text{DT}_4(r[f] \mid \gamma) = \pm r \int_X \pi^* \pi_* \gamma \cup c_3(X). \]
For $\gamma = B^2$, we have $\pi_* \gamma = 0$, hence
\begin{equation}
(2.4) \quad \text{DT}_4(r[f] \mid B^2) = 0, \quad r \in \mathbb{Z}_{\geq 1}.
\end{equation}
For $\gamma = B \cdot E$, we have $\pi^* \pi_* \gamma = B$ and
\begin{equation}
(2.5) \quad \text{DT}_4(r[f] \mid B \cdot E) = \pm r \int_X B \cup c_3(X) = \pm 960r.
\end{equation}
In both cases (2.4), (2.5), the results agree (for a suitable choice of sign) with $n_r[f](\gamma)$ (which is 0, 960r respectively) given in [21, Table 7]. \qed

2.2. Quintic fibrations. We consider a compact CY 4-fold $X$ which admits a quintic 3-fold fibration structure
\[ \pi: X \to \mathbb{P}^1, \]
i.e. $\pi$ is a proper morphism whose general fiber is a smooth quintic 3-fold $Y \subseteq \mathbb{P}^4$. Examples of such CY 4-folds include resolutions of degree 10 orbifold hypersurface in $\mathbb{P}^5(1,1,2,2,2,2)$ and hypersurface of bidegree $(2,5)$ in $\mathbb{P}^1 \times \mathbb{P}^4$ (see [21]). We will explain some part of [21, Table 4, Table 6] for these two examples. Let $b$ be the hyperplane class of $\mathbb{P}^1$ and $B = \pi^* b$ be the fiber class of $\pi$, $F \in H_6(X)$ be a divisor Poincaré dual to degree one curve in the quintic fiber. Notice that the genus zero integral invariants $n^2_{\beta, \beta}$'s (w.r.t. insertion $B \cdot F$) for both tables are the same when $\beta \in H_2(Y) \subseteq H_2(X)$. Moreover, under the identification $H_2(Y) \cong \mathbb{Z}$, $\beta \mapsto d$, these numbers are $d$ times the degree $d$, genus zero Gopakumar-Vafa invariants of quintic 3-fold $Y$ [10] which are conjecturally the same as $\text{DT}_3$ invariants for one dimensional stable sheaves on $Y$ [16]. Hence, Conjecture [13] for this case may be reduced to a relation between $\text{DT}_4$ invariants of $X$ and $\text{DT}_3$ invariants of $Y$.

More generally, we expect the following conjecture:

**Conjecture 2.4.** Let $X$ be a projective CY 4-fold which admits a CY 3-fold fibration
\[ \pi: X \to C \]
over a smooth curve $C$.

Let $X_p = \pi^{-1}(p)$ be a general fiber and $\beta \in \text{Im}(i_* : H_2(X_p) \to H_2(X))$. Then for an ample divisor $H \subseteq X$, we have
\[ \text{DT}_4(\beta \mid H \cdot X_p) = (H \cdot \beta) \cdot \sum_k \text{DT}_3(\beta_k), \]
for certain choice of orientation in defining the LHS. Here $\{\beta_k\} \subseteq H_2(X_p)$ are (finitely many) non-zero effective curve classes of $X_p$ which map to $\beta$. $\text{DT}_3(\beta_k)$ is the $\text{DT}$ invariant [31] for 1-dimensional stable sheaves $\mathcal{F} \in \text{Coh}(X_p)$ with $[\mathcal{F}] = \beta_k$ and $\chi(\mathcal{F}) = 1$.

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\footnote{In [21, Table 7], genus zero GW invariants are computed by Picard-Fuchs equations through mirror symmetry. Fortunately, our CY$_4$ is a hypersurface in a toric variety, mirror principle has been verified in this case by the works of Lian-Liu-Yau and Givental.}
When $\pi: X \to C$ is a trivial CY 3-fold fibration, the above conjecture will be proved in Corollary 2.7. For the general case, we give a heuristic explanation as follows.

Note that the fibration $\pi: X \to C$ induces a fibration on the moduli space $\pi_M: M_\beta \to C$.

As any one dimensional stable sheaf on $X$ is scheme theoretically supported on some fiber (see Lemma 2.2), the fiber $\pi^{-1}_{M}(p)$ is regarded as the moduli space of stable sheaves on $X_p$. For a 'general' $p \in C$, ideally, the moduli space $\pi^{-1}_{M}(p)$ is smooth (Kuranishi maps are zero) of expected dimension 0. We take $t_{*}: E \in \pi^{-1}_{M}(p)$ and have

$$\text{Ext}^1_X(t_{*}E, t_{*}E) \cong \text{Ext}^1_{X_p}(E, E) \oplus \text{Ext}^0_{X_p}(E, E) \cong \mathbb{C},$$

$$\text{Ext}^2_X(t_{*}E, t_{*}E) \cong \text{Ext}^2_{X_p}(E, E) \oplus \text{Ext}^1_{X_p}(E, E)^{\vee} = 0.$$

Then for a small neighborhood $U(p) \subseteq C$, there is an isomorphism

$$\pi^{-1}_M(U(p)) \cong \pi^{-1}_M(p) \times U(p).$$

In [2], to define virtual class of $M_\beta$, local models of type $\kappa_+: \text{Ext}^1(E, E) \to \text{Ext}^2(E, E)$ are glued using partition of unity. Hence, the above local model

$$\kappa_+ = 0: \text{Ext}^1_X(t_{*}E, t_{*}E) \to \text{Ext}^2_X(t_{*}E, t_{*}E)$$

for closed subset $\pi^{-1}_M(p) \subseteq M_\beta$ can be glued over $\beta$. Then the representing manifold of $[M_\beta]^{\text{vir}}$, when viewed as a submanifold of $M_\beta$, can be chosen to be $\pi^{-1}_M(U(p))$ (which is homeomorphic to $\prod_{i=1}^{\text{DT}_3(\beta_k)} U(p)$) near $\pi^{-1}_M(p)$. Note that the insertion $\tau: H^4(X) \to H^2(M_\beta)$ satisfies

$$\tau(H \cdot X_p) = (H \cdot \beta)[\pi^{-1}_M(p)].$$

Hence, we have

$$\text{DT}_4(\beta \mid H \cdot X_p) = \int_{[M_\beta]^{\text{vir}}} \tau(H \cdot X_p) = (H \cdot \beta) \cdot \sum_k \text{DT}_3(\beta_k).$$

**Corollary 2.5.** Assuming Conjecture 2.7 then Conjecture 1.3 is true for $\beta \in H_2(Y) \subseteq H_2(X)$, $\gamma = B \cdot F$ in the quintic fibration examples in Table 4 and Table 6 of [21] if and only if the (CY$_3$) genus zero Gopakumar-Vafa/Donaldson-Thomas conjecture 16 is true for $\beta \in H_2(Y)$. Here $Y$ is the quintic 3-fold realized as a general fiber of $\pi: X \to \mathbb{P}^1$.

**Proof.** A similar proof will be given after Corollary 2.7. \hfill $\square$

### 2.3. Product of elliptic curves and Calabi-Yau 3-folds.

Sitting in between examples given by elliptic and quintic fibrations, in this subsection, we consider a CY 4-fold of type $X = Y \times E$, where $Y$ is a projective CY 3-fold and $E$ is an elliptic curve. We will show our Conjecture 1.3 for CY 4-folds is consistent with the genus zero Gopakumar-Vafa/Donaldson-Thomas conjecture 16 for CY 3-folds.

We pick a reference point $0 \in E$ and embed $i: Y \hookrightarrow X$ via $y \mapsto (y, 0)$. We take a curve class $\beta \in H_2(Y, \mathbb{Z}) \hookrightarrow H_2(X, \mathbb{Z})$ and consider the moduli spaces

$M_{X, \beta} = M(0, 0, 0, \beta, 1), \quad M_{Y, \beta} = M(0, 0, \beta, 1)$

of 1-dimensional stable sheaves on $X$ and $Y$ respectively.

**Lemma 2.6.** There exists an isomorphism $M_{X, \beta} \cong M_{Y, \beta} \times E$ under which the $\text{DT}_4$ virtual class satisfies

$$[M_{X, \beta}]^{\text{vir}} = \text{deg}[M_{Y, \beta}]^{\text{vir}} \cdot [E],$$

for certain choice of orientation in defining the LHS. Here $[M_{Y, \beta}]^{\text{vir}}$ is the $\text{DT}_3$ virtual class.

**Proof.** By Lemma 2.2 any stable sheaf $F \in M_{X, \beta}$ is scheme theoretically supported on $Y \times \{t\}$ for some $t \in E$, i.e. it is written as $F = t_{*}E$ for some $E \in M_{Y, \beta}$ and $t : Y = Y \times \{t\} \hookrightarrow X$. From the spectral sequence

$$\text{Ext}^1_Y(E, \wedge^t \mathcal{N}_{Y/X} \otimes E) \Rightarrow \text{Ext}^1_X(t_{*}E, t_{*}E),$$

and $\mathcal{N}_{Y/X} \cong \mathcal{O}_Y$, we have canonical isomorphisms

$$\text{Ext}^1_Y(F, F) \cong \text{Ext}^1_X(E, E) \oplus \mathbb{C},$$

$$\text{Ext}^2_Y(F, F) \cong \text{Ext}^2_X(E, E) \oplus \text{Ext}^2_Y(E, E)^{\vee},$$

then $\pi^{-1}_M(U(p)) \cong \pi^{-1}_M(p) \times U(p)$.
under which $\operatorname{Ext}^2_Y(\mathcal{E}, \mathcal{E})$ is a maximal isotropic subspace of $(\operatorname{Ext}^2_Y(F, F), Q_{\text{Serre}})$. Moreover there exists a Kuranishi map $\kappa: \operatorname{Ext}^1_Y(F, F) \to \operatorname{Ext}^2_Y(F, F)$ for $M_{X, \beta}$ at $F$ of type

$$\kappa: \operatorname{Ext}^1_Y(\mathcal{E}, \mathcal{E}) \oplus \mathbb{C} \to \operatorname{Ext}^2_Y(\mathcal{E}, \mathcal{E}) \oplus \operatorname{Ext}^2_Y(\mathcal{E}, \mathcal{E})^\vee,$$

where $\kappa_Y$ is a Kuranishi map of $M_{Y, \beta}$ at $Y$. Hence the map

$$\phi: M_{Y, \beta} \times E \to M_{X, \beta}, \quad \phi(\mathcal{E}, t) = (\iota_t)_* \mathcal{E}$$

is an isomorphism.

Similar to [7] Theorem 6.5, [9] Theorem 1.6, under the above isomorphism, we have

$$[M_{X, \beta}]^\text{vir} = \deg[M_{Y, \beta}]^\text{vir} \cdot [E],$$

for certain choice of orientation, where $[M_{Y, \beta}]^\text{vir} \in A_0(M_{Y, \beta})$ is the DT$_3$ virtual class.

**Corollary 2.7.** Let $X = Y \times E$ be a product of a projective CY$_3$ with an elliptic curve. Fix a point $0 \in E$ and denote $Y = Y \times \{0\} \subseteq X$. Then for any $\beta \in H_2(Y) \subseteq H_2(X)$ and divisor $H \subseteq X$, we have

$$\text{DT}_4(\beta \mid H \cdot Y) = \text{DT}_3(\beta) \cdot (H \cdot \beta),$$

for certain choice of orientation in defining the LHS. Here $\text{DT}_3(\beta) = \deg[M_{Y, \beta}]^\text{vir}$ is the DT$_3$ invariant (see Appendix A).

In particular, Conjecture [13] is true for $\beta \in H_2(Y) \subseteq H_2(X)$ and $\gamma = H \cdot Y$ (for any divisor $H \subseteq X$) if and only if the (CY$_3$) genus zero Gopakumar-Vafa/Donaldson-Thomas conjecture [16] (see Conjecture [17]) is true for $\beta \in H_2(Y)$.

**Proof.** Let $\mathcal{F}$ be the universal sheaf for $M_{X, \beta}$. The insertion [18]

$$\tau: H^2(X, \mathbb{Z}) \to H^2(M_{X, \beta}, \mathbb{Z}),$$

$$\tau(\alpha) = (\pi_M)_*(\pi_X^* \alpha \cup [\mathcal{F}])$$

satisfies $\tau(H \cdot Y) = (H \cdot \beta) \cdot [M_{Y, \beta}]$ for any divisor $H \subseteq X$. By Lemma 2.6 then

$$\text{DT}_4(\beta \mid H \cdot Y) = \int_{[M_{X, \beta}]^\text{vir}} \tau(H \cdot Y) = \pm \deg[M_{Y, \beta}]^\text{vir} \cdot (H \cdot \beta).$$

Hence, Conjecture [13] holds for $\beta \in H_2(Y) \subseteq H_2(X)$ and $\gamma = H \cdot Y$ if and only

$$\text{GW}_{0, \beta}(H \cdot Y) = \sum_{d | \beta} \frac{1}{d^2} \cdot \text{DT}_4(\beta/d \mid H \cdot Y) = \sum_{d | \beta} \frac{1}{d^3} \cdot \text{DT}_3(\beta/d) \cdot (H \cdot \beta).$$

Notice that GW invariants satisfy similar formula as (2.6), i.e.

$$\text{GW}_{0, \beta}(H \cdot Y) = \text{GW}_{0, \beta}(Y) \cdot (H \cdot \beta).$$

So (2.7) holds true if and only if

$$\text{GW}_{0, \beta}(Y) = \sum_{d | \beta} \frac{1}{d^3} \cdot \text{DT}_3(\beta/d),$$

i.e. genus zero GV/DT conjecture (Conjecture [17]) for $\beta \in H_2(Y)$ in a CY 3-fold $Y$. \qed

By combining with Corollary [16] we have the following:

**Theorem 2.8.** Let $Y$ be a complete intersection CY 3-fold in the product of projective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, and $X = Y \times E$ for an elliptic curve $E$. Then for any primitive curve class $\beta$ on $Y$ and cycle class $H \cdot Y$ for any divisor $H$ on $X$, Conjecture [15] holds, i.e. the identity

$$\text{GW}_{0, \beta}(H \cdot Y) = \text{DT}_4(\beta \mid H \cdot Y)$$

holds for any primitive curve class $\beta \in H_2(Y) \subseteq H_2(X)$, any divisor $H \subseteq X$ and certain choice of orientation in defining the RHS.
2.4. Hyperkähler 4-folds and cosection localization. In this subsection, we investigate Conjecture [13] for a CY 4-fold X which admits a holomorphic symplectic form, i.e. a hyperkähler 4-fold. GW invariants on hyperkähler manifolds vanish as they are deformation invariants and there are no holomorphic curves for generic complex structures in the $S^2$-twistor family. An alternate way to see this vanishing is through the existence of a nowhere-vanishing cosection (see for example Kiem-Li [19]).

Given a perfect obstruction theory [11] on a Deligne-Mumford stack $M$, the existence of a cosection

$$\varphi : Ob_M \to O_M$$

of the obstruction sheaf $Ob_M$ allows us to localize the virtual class of $M$ to the closed subspace $Z(\varphi) \subseteq M$ where $\varphi$ is not surjective. In particular, if $\varphi$ is surjective everywhere (which is guaranteed by the existence of holomorphic symplectic forms), then the virtual class of $M$ vanishes. Moreover, by truncating the obstruction theory to remove the trivial factor $O_M$, one can define a reduced obstruction theory and reduced virtual class.

To verify Conjecture [13] for hyperkähler 4-folds, we need to prove the vanishing of $DT_4$ invariants for $M_\beta$. Heuristically, in the ideal case, when all curves in $X$ are smooth embedded, one could identify the obstruction theory of $M_\beta$ with obstruction theory of GW theory as in [7] Section 7.2, hence vanishing of invariants follows. We give a cosection argument as follows.

Cosection and vanishing of $DT_4$ virtual classes. Fix a 1-dimensional stable sheaf $F \in M_\beta$. By taking the wedge product with the square $At(F)^2$ of the Atiyah class and contracting with the holomorphic symplectic form $\sigma$, we get a surjective map

$$\phi : \text{Ext}^2(F, F) \xrightarrow{\wedge \chi(F)^2} \text{Ext}^4(F, F \otimes \Omega^2_X) \xrightarrow{\sigma} \text{Ext}^4(F, F) \xrightarrow{\text{tr}} H^4(X, O_X).$$

More generally, we have

**Proposition 2.9.** Let $X$ be a projective hyperkähler 4-fold, $F$ be a perfect complex on $X$ and $Q$ be the Serre duality quadratic form on $\text{Ext}^2(F, F)$. Then the composition map

$$\phi : \text{Ext}^2(F, F) \xrightarrow{\wedge \chi(F)^2} \text{Ext}^4(F, F \otimes \Omega^2_X) \xrightarrow{\sigma} \text{Ext}^4(F, F) \xrightarrow{\text{tr}} H^4(X, O_X)$$

is surjective if either $\chi_4(F) \neq 0$ or $\chi_4(F) \neq 0$. Moreover,

1. if $\chi_4(F) \neq 0$, then we have a $Q$-orthogonal decomposition

$$\text{Ext}^2(F, F) = \text{Ker}(\phi) \oplus \mathbb{C}\langle \text{At}(F)^2 \rangle \sigma,$$

where $Q$ is non-degenerate on each subspace;

2. if $\chi_3(F) = 0$ and $\chi_3(F) \neq 0$, then we have a $Q$-orthogonal decomposition

$$\text{Ext}^2(F, F) = \mathbb{C}\langle \text{At}(F)^2 \rangle \sigma, \chi_X \circ \text{At}(F) \rangle \oplus (\mathbb{C}\langle \text{At}(F)^2 \rangle \sigma, \chi_X \circ \text{At}(F)),$$

where $Q$ is non-degenerate on each subspace. Here $\chi_X$ is the Kodaira-Spencer class which is Serre dual to $\chi_3(F)$.

**Proof.** (1) If $\chi_4(F) \neq 0$, $\text{At}(F)^4 \in \text{Ext}^4(F, F \otimes K_X)$ is a nonzero element since

$$\text{tr}(\text{At}(F)^4) = 4! \cdot \chi_4(F) \in H^{1,4}(X, \mathbb{Q}).$$

We define an inclusion

$$\iota : H^4(X, O_X) \to \text{Ext}^2(F, F),$$

$$1 \mapsto \frac{1}{12 \chi_4(F)} \cdot \langle \text{At}(F)^2 \rangle \sigma.$$

Then $\phi \circ \iota = Id$ and gives a splitting

$$\text{Ext}^2(F, F) = \text{Ker}(\phi) \oplus \mathbb{C}\langle \text{At}(F)^2 \rangle \sigma.$$

Note that $Q(\text{Ker}(\phi), \text{At}(F)^2 \sigma) = 0$ from the definition of $Q$ and $\text{Ker}(\phi)$.

(2) If $\chi_3(F) \neq 0$, we denote $\beta \in H_2(X)$ to be the Poincaré dual of $\chi_3(F)$. By the non-degeneracy of $\sigma$, one can choose a first order deformation $\kappa_X \in H^1(X, TX)$ of $X$ such that

$$\int_{\beta} \kappa_X \cdot \sigma = 1.$$

By [6] Proposition 4.2, the obstruction class $\kappa_X \circ \text{At}(F) \in \text{Ext}^2(F, F)$ satisfies

$$\text{tr}(\kappa_X \circ \text{At}(F) \circ \text{At}(F)^2) = -2 \kappa_X \cdot \chi_3(F).$$
Then
\[ \int_X \phi(\kappa_X \circ \operatorname{At}(F)) \wedge \sigma^2 = - \int_X (\kappa_X \circ \operatorname{ch}_3(F)) \wedge \sigma \]
\[ = \int_X (\kappa_X \wedge \sigma) \wedge \operatorname{ch}_3(F) \]
\[ = \int_\beta \kappa_X \wedge \sigma = 1, \]
where the second equality is because of the homotopy formula [20; Proposition 10]
\[ 0 = \kappa_X \wedge (\operatorname{ch}_3(F) \wedge \sigma) = (\kappa_X \circ \operatorname{ch}_3(F)) \wedge \sigma + (\kappa_X \wedge \sigma) \wedge \operatorname{ch}_3(F). \]
Thus the map
\[ \iota : H^4(X, \mathcal{O}_X) \to \operatorname{Ext}^2(F, F), \]
\[ 1 \mapsto \kappa_X \circ \operatorname{At}(F) \]
satisfies \( \phi \circ \iota = \operatorname{id} \) and hence \( \phi \) is surjective.

Notice that
\[ Q(\kappa_X \circ \operatorname{At}(F), \operatorname{At}(F)^2 \wedge \sigma) = 2 \int_X \phi(\kappa_X \circ \operatorname{At}(F)) \wedge \sigma^2 = 2, \]
\[ \int_X \phi(\operatorname{At}(F)^2 \wedge \sigma) \wedge \sigma^2 = \frac{1}{2} Q(\operatorname{At}(F)^2 \wedge \sigma, \operatorname{At}(F)^2 \wedge \sigma) = 0, \]
since \( \operatorname{ch}_3(F) = 0 \). Thus \( \mathbb{C} \langle \operatorname{At}(F)^2 \wedge \sigma, \kappa_X \circ \operatorname{At}(F) \rangle \) is a two dimensional subspace on which \( Q \) is non-degenerate. The orthogonal complement \( \mathbb{C} \langle \operatorname{At}(F)^2 \wedge \sigma, \kappa_X \circ \operatorname{At}(F) \rangle \perp \) does not contain \( \operatorname{At}(F)^2 \wedge \sigma \) and \( \kappa_X \circ \operatorname{At}(F) \), so we have
\[ \mathbb{C} \langle \operatorname{At}(F)^2 \wedge \sigma, \kappa_X \circ \operatorname{At}(F) \rangle \oplus (\mathbb{C} \langle \operatorname{At}(F)^2 \wedge \sigma, \kappa_X \circ \operatorname{At}(F) \rangle \perp) = \operatorname{Ext}^2(F, F) \]
by dimensions counting. \( \square \)

We claim that the surjectivity of the cosection map leads to the vanishing of virtual class.

**Claim 2.10.** Let \( X \) be a projective hyperkähler 4-fold and \( M \) be a proper moduli scheme of simple perfect complexes \( F \)'s with \( \operatorname{ch}_3(F) \neq 0 \) or \( \operatorname{ch}_3(F) \neq 0 \). Then the virtual class satisfies
\[ [M]^{\text{vir}} = 0. \]

At the moment, we are lack of Kiem-Li type theory of cosection localization for D-manifolds in the sense of Joyce or Kuranishi space structures in the sense of Fukaya-Oh-Ohta-Ono. We believe that when such a theory is established, our claim should follow automatically. Nevertheless, we have the following evidence for the claim.

1. At least when \( M_\beta \) is smooth, Proposition 2.9 gives the vanishing of virtual class.
2. If there is a complex analytic version of \( (-2) \)-shifted symplectic geometry \[30 \] and the corresponding construction of virtual classes \[2 \], one could prove the vanishing result as in GW theory, i.e. taking a generic complex structure in the \( S^2 \)-twistor family of the hyperkähler 4-fold which does not support coherent sheaves and then vanishing of virtual classes follows from their deformation invariance.

**Reduced DT\(_4\) virtual classes, an example.** By taking away the trivial factors in obstruction spaces, one could define reduced invariants, which are computed in the following example.

Let \( p : S \rightarrow \mathbb{P}^1 \) be an elliptic K3 surface with a section \( i \). We assume general fibers of \( p \) are smooth elliptic curves and any singular fiber is either a nodal or cuspidal plane curve.

Fix a CY surface \( T \), we denote
\[ \pi : X = S \times T \rightarrow \mathbb{P}^1 \times T, \]
\[ \pi(s, t) = (p(s), t), \]
which is an elliptic fibration with a section \( s = (i, \text{id}) \). Let \( [f] \) be the fiber class of fibration \( \pi \), and \( \beta = r[f] \in H_2(X) \) with \( r \geq 1 \). As in Lemma 2.11 there exists an isomorphism \( M_\beta \cong X \) such that the DT\(_4\) virtual class satisfies
\[ [M_\beta]^{\text{vir}} = \pm \operatorname{PD}(c_3(X)) = 0. \]

Under the above isomorphism \( M_\beta \cong X \), the obstruction bundle of \( M_\beta \) is
\[ \wedge^2(TX) \cong O_S \oplus O_T \oplus (TS \otimes TT), \]
and the DT$_4$ obstruction bundle can be chosen as 
\[
\lambda^2_+ (TX) = \mathcal{O}_S \oplus (TS \otimes TT)^+,
\]
The trivial factor $\mathcal{O}_S$ in $\wedge^2_+(TX)$ makes the DT$_4$ virtual class vanish. We consider reduced obstruction bundle 
\[
\wedge^2_+(TX)_{\text{red}} := (TS \otimes TT)^+,
\]
and reduced DT$_4$ virtual class 
\[
[M_{\beta}]_{\text{red}} := \text{PD}(\epsilon(\wedge^2_+(TX)_{\text{red}})).
\]
By the property of half Euler class (e.g. Remark 8.3), we have 
\[
e((TS \otimes TT)^+)^2 = e(TS \otimes TT) = -2c_2(S) \cdot c_2(T),
\]
whose square roots are given by 
\[
e(\wedge^2_+(TX)_{\text{red}}) = \pm \sqrt{e(TS \otimes TT)} = \pm (c_2(S) - c_2(T)).
\]
Hence 
\[
[M_{\beta}]_{\text{red}} = \pm (\chi(S) \cdot [T] - \chi(T) \cdot [S]).
\]
As for insertions, we consider 
\[
\tau : H^6(X) \to H^4(M_{\beta}), \quad \tau(\gamma) = (\pi M_{\beta})_* (\pi_X^* \gamma \cup \text{ch}_3(\mathcal{E})),
\]
where $\mathcal{E}$ is the universal sheaf and $\text{ch}_3(\mathcal{E}) = r \lfloor X \times (\mathbb{P}^1 \times T) \rfloor X$.

Then the reduced DT$_4$ invariant satisfies 
\[
\text{DT}^\text{red}_4 (r[f] \mid \gamma) := \int_{[M_{\beta}]_{\text{red}}^{\text{vir}}} \tau(\gamma)
\]
satisfies 
\[
\text{DT}^\text{red}_4 (r[f] \mid E) = r \int_{[M_{\beta}]_{\text{red}}^{\text{vir}}} \pi^* \pi_* (E) = \pm r \cdot \chi(S) \cdot \int_T [t] = \pm 24 r,
\]
where $E = s(\mathbb{P}^1 \times t) \in H_2(X) \cong H^6(X)$ as in Proposition 2.3.

As for the corresponding GW theory, we have 
\[
\overline{M}_{0,1}(X, r[f]) \cong \overline{M}_{0,1}(S, r[f]) \times T
\]
whose virtual class vanishes. By considering the reduced obstruction theory [19] and insertions, the reduced GW invariant satisfies 
\[
\text{GW}^\text{red}_{0,r[f]}(E) = \int_{[\overline{M}_{0,1}(X, r[f])]_{\text{red}}^{\text{vir}}} \text{ev}^*(E)
\]
\[
= \left( \int_{[\overline{M}_{0,1}(S, r[f])]_{\text{red}}^{\text{vir}}} \text{ev}^*(i(\mathbb{P}^1)) \right) \cdot \int_T [pt]
\]
\[
= r \cdot \text{deg} [\overline{M}_{0,0}(S, r[f])]_{\text{red}}^{\text{vir}},
\]
where $E = s(\mathbb{P}^1 \times t) = i(\mathbb{P}^1) \cdot 1 \in H_2(S) \otimes H_0(T) \subset H_2(X)$.

A hyperkähler version of Conjecture 1.3 for reduced invariants is given by 

**Proposition 2.11.** In the above setting, we have a multiple cover formula 
\[
\text{GW}^\text{red}_{0,r[f]}(E) = \sum_{k \mid r} \frac{1}{k^2} \cdot \text{DT}^\text{red}_4 \left( \frac{r}{k} [f] \mid E \right)
\]
for certain choices of orientations in defining the RHS.

**Proof.** We have the following Aspinwall-Morrison formula 
\[
\text{deg} [\overline{M}_{0,0}(S, r[f])]_{\text{red}}^{\text{vir}} = \sum_{k \mid r} \frac{1}{k^3} \cdot n_{0, \frac{r}{k}[f]}(S)
\]
relating reduced GW invariants $\text{deg} [\overline{M}_{0,0}(S, r[f])]_{\text{red}}^{\text{vir}}$ with genus zero BPS numbers $n_{0, \frac{r}{k}[f]}(S)$ for K3 surface $S$ ([36, 20]). Yau-Zaslow formula gives 
\[
n_{0, \frac{r}{k}[f]}(S) = n_{0, [f]}(S) = \chi(S) = 24,
\]
as $r[f] \cdot r[f] = [f] \cdot [f] = 0.$
3. Local surfaces

Let \( (S, O_S(1)) \) be a smooth projective surface and
\[
(3.1) \quad \pi: X = \text{Tot}_S(L_1 \oplus L_2) \to S
\]
be the total space of direct sum of two line bundles \( L_1, L_2 \) on \( S \). Assuming that
\[
(3.2) \quad L_1 \oplus L_2 \cong K_S,
\]
then \( X \) is a non-compact CY 4-fold. In this section, we study Conjecture [15] for \( X \).

3.1. Stable sheaves without thickening. For a non-compact CY 4-fold \( \pi \), we take a curve class
\( \beta \in H_2(X, \mathbb{Z}) \cong H_2(S, \mathbb{Z}) \).

and consider the moduli space \( M_\beta = M_{X, \beta} \) of 1-dimensional stable sheaves \( F \) with \( [F] = \beta \) and \( \chi(F) = 1 \). We also consider \( M_{S, \beta} \), the moduli space of 1-dimensional stable sheaves \( F \) on \( S \) with \( [F] = \beta \) and \( \chi(F) = 1 \). Note that \( M_{S, \beta} \) is compact while \( M_{X, \beta} \) may not be compact in general. On the other hand, for the zero section \( \iota: S \hookrightarrow X \) of the projection \( (3.1) \), we have the push-forward embedding
\[
(3.3) \quad \iota_*: M_{S, \beta} \hookrightarrow M_{X, \beta}.
\]

In the following case, the morphism \( (3.3) \) is an isomorphism and \( M_{X, \beta} \) has well-defined DT\(_4\) virtual class.

**Proposition 3.1.** If \( L_1^{-1} \) and \( L_2^{-1} \) are ample, then \( (3.3) \) is an isomorphism. Under the isomorphism \( (3.3) \), we have
\[
[M_{X, \beta}]^{\text{vir}} = \pm [M_{S, \beta}] \cdot e \left( \mathcal{E}xt^1_{\pi M_\beta}(F, F \otimes L_1) \right),
\]
for certain choices of orientations in defining the LHS. Here \( F \in \text{Coh}(S \times M_{S, \beta}) \) is the universal sheaf and \( \pi M_\beta : S \times M_{S, \beta} \to M_{S, \beta} \) is the projection.

**Proof.** A coherent sheaf \( F \in \text{Coh}(X) \) is determined by \( \pi_* F \in \text{Coh}(S) \) and two morphisms [12, Ex. 5.17 Chapter II]
\[
\phi_i : \pi_* F \to \pi_* F \otimes L_i, \quad i = 1, 2.
\]
We claim \( \phi_i = 0 \) by using the ampleness of \( L_i^{-1} \). Take the Harder-Narasimhan and Jordan-Hölder filtration
\[
o = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = \pi_* F
\]
of \( \pi_* F \), where the quotient \( E_i = F_{i-1}/F_{i-1}'s \) are stable with decreasing reduced Hilbert polynomial.
\[
p(E_1) \geq p(E_2) \geq \cdots \geq p(E_n).
\]

We consider the following diagram
\[
\begin{array}{cccccc}
0 & \to & F_1 & \to & \pi_* F & \to & \pi_* F/F_1 & \to & 0 \\
0 & \to & F_1 \otimes L_i & \to & \pi_* F \otimes L_i & \to & (\pi_* F/F_1) \otimes L_i & \to & 0.
\end{array}
\]
Note that \( p(F_1) \geq p(E_k) > p(E_k \otimes L_i) \) by the ampleness of \( L_i^{-1} \) for any \( k = 1, 2, \cdots, n \), hence \( \text{Hom}(F_1, E_k \otimes L_i) = 0 \) [14, Proposition 1.2.7].

As \( (\pi_* F/F_1) \) fits into extensions of \( \{ E_k \}_{k=2, \cdots, n} \), so
\[
\text{Hom}(F_1, (\pi_* F/F_1) \otimes L_i) = 0.
\]

Hence \( \phi_i \) restricts to \( \phi_i|_{F_1}: F_1 \to F_1 \otimes L_i \). This determines a subsheaf \( \widetilde{F_1} \subseteq F \) on \( X \) such that
\[
p(\pi_* \widetilde{F_1}) = p(F_1) > p(\pi_* F), \quad \text{contradicting with the stability of } F.
\]

Hence, for any \( F \in M_{X, \beta} \), there exists \( E \in M_{S, \beta} \) such that \( F = \iota_* (E) \). To compare the deformation-obstruction theory, we have canonical isomorphisms
\[
\begin{align*}
\text{Ext}_X^1(\iota_* E, \iota_* E) & \cong \text{Ext}^1_S(E, E), \\
\text{Ext}_X^2(\iota_* E, \iota_* E) & \cong \text{Ext}^2_S(E, E \otimes L_1) \oplus \text{Ext}^2_S(E, E \otimes L_1)^\vee,
\end{align*}
\]
where \( \text{Ext}^2_S(E, E \otimes K_S)^\vee = 0 \) and \( \text{Ext}^2_S(E, E \otimes L_i) = 0 \) \( (i = 1, 2) \) by the stability of \( E \). Hence the morphism \( (3.3) \) is an isomorphism. The comparison of virtual classes is similar to [7, Theorem 6.5] (see the last part of Section [12]).
3.2. Computations for $\mathcal{O}_{\mathbb{P}^2}(-1, -2)$. In this subsection, we fix $S = \mathbb{P}^2$ and consider the case $L_1 = \mathcal{O}(-1)$ and $L_2 = \mathcal{O}(-2)$. For $d \in \mathbb{Z}$, we consider the moduli space $M_{X,d}$ of one dimensional stable sheaves $F$’s on $X = \mathcal{O}_{\mathbb{P}^2}(-1, -2)$ with

$$[F] = d \in H_2(X, \mathbb{Z}) \cong H_2(\mathbb{P}^2, \mathbb{Z}) \cong \mathbb{Z}, \quad \chi(F) = 1.$$  

By Lemma 3.2.1, we have the isomorphism $M_{S,d} \cong M_{X,d}$ and a well-defined virtual class on $M_{X,d}$, whose computation is reduced to the one on the moduli space $M_{S,d}$ on $S$. We explain the calculation in degree $d = 3$, since degrees 1 and 2 are easier versions of the same approach.

There is a natural support morphism to the linear system of degree 3 curves

$$M_{S,3} \to |\mathcal{O}(3)| = \mathbb{P}^9.$$  

Moreover, if we denote

$$\mathcal{C} \hookrightarrow \mathbb{P}^9 \times \mathbb{P}^2$$  

the universal curve over this linear system, we have an isomorphism $\mathcal{C} \cong M_{S,3}$ which sends the pair $(\mathcal{C}, p)$ to the dual (on $\mathcal{C}$) of the ideal sheaf $I_{\mathcal{C}, p}$.

Let $V$ denote the vector bundle on $M_{S,3}$ whose fiber at a point $[E]$ is

$$\text{Ext}^1_S(E, E(-1)) = \text{RHom}_S(I_{\mathcal{C}, p}, I_{\mathcal{C}, p}(-1))[1].$$

Its top Chern class can be computed via the diagram:

$$\begin{array}{ccc}
\mathcal{C} \times \mathbb{P}^2 & \overset{j}{\twoheadrightarrow} & \mathbb{P}^9 \times \mathbb{P}^2 \\
\pi_c & \square & \pi_{1,2} \\
& \mathcal{C} & \overset{j}{\twoheadrightarrow} \mathbb{P}^9 \times \mathbb{P}^2.
\end{array}$$

The $K$-theory class of the universal ideal sheaf $I$ on $\mathcal{C} \times \mathbb{P}^2$ is given by the pullback $[I] = j^*[F]$, where $[F]$ is given by

$$[F] = \pi^*_{1,3}[\mathcal{C}] - \pi^*_{2,1}[\mathcal{O}_X] = 3[\mathcal{O}(0, -1, 0)] - [\mathcal{O}(0, -1, -3)] - [\mathcal{O}(0, -1, 1)] - [\mathcal{O}(0, -2, -1)].$$

Therefore

$$[\mathcal{V}] = j^*\pi_{1,2,*}([F]^* \otimes [F] \otimes \mathcal{O}(0, 0, -1))$$

is the pullback of an explicit $K$-theory class $\gamma$ on $\mathbb{P}^9 \times \mathbb{P}^2$. Note that we have

$$j_*[\mathcal{C}] = H_1 + 3H_2 \in H^2(\mathbb{P}^9 \times \mathbb{P}^2, \mathbb{Z}),$$

where $H_1$, $H_2$ are hyperplane classes on $\mathbb{P}^9$, $\mathbb{P}^2$. For the point class $[pt] \in H^4(\mathbb{P}^2, \mathbb{Z})$, we can compute

$$\int_{[M_{X,\beta}]^{\vir}} \tau([pt]) = \pm \int_{[M_{S,3}]} e(\mathcal{V}) \cdot j^*H_1$$

$$= \pm \int_{[\mathbb{P}^9 \times \mathbb{P}^2]} (H_1 + 3H_2) \cdot H_1 \cdot \gamma = \pm 1,$$

which matches the prediction via Gromov-Witten theory in [21] Section 3.2.

The same approach works for curves of degree $(2, 2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and this again matches the answer via Gromov-Witten theory in [21] Section 3.3.

3.3. Localization principle in DT$_4$ theory. In general, $M_{X,\beta}$ may not be compact, so we want to define the integral of $[M_{X,\beta}]^{\vir}$ via virtual localization:

Let $\mathbb{C}^*$ act on the fibers of (3.1) by weight $(1, -1)$, which preserves the CY4-form on $S$. So the action also lifts to the moduli space $M_{X,\beta}$ preserving the Serre duality pairing. Analogous to [7] Section 8, heuristically speaking, one should have virtual localization formula of type

$$[M_{X,\beta}]^{\vir} = [M_{X,\beta}^{\C}]^{\vir} \cdot e(\text{RHom}_{\pi_M, \mathcal{E}, \mathcal{E}}^{\mov})^{1/2} \in H_*(M_{X,\beta}^{\C})[t^{\pm 1}],$$

where $M_{X,\beta}^{\C}$ should have $(-2)$-shifted symplectic structure and $[M_{X,\beta}]^{\vir}$ is its DT$_4$ virtual class, $\mathcal{E} \in \text{Coh}(X \times M_{X,\beta})$ is the universal sheaf and $\pi_M : X \times M_{X,\beta} \to M_{X,\beta}$ is the projection, $t$ is the equivariant parameter for the $\mathbb{C}^*$-action.

Remark 3.2. Suppose that $M_{X,\beta}$ admits a $\mathbb{C}^*$-equivariant virtual class induced by the $\mathbb{C}^*$-equivariant $(-2)$-shifted symplectic structure. Then the RHS of (3.4) may coincide with the integration of the $\mathbb{C}^*$-equivariant virtual class up to sign by a virtual $\mathbb{C}^*$-localization formula.
3.4. Contribution from surface component $M_{S,\beta}$. Notice that the moduli space $M_{S,\beta}$ of one dimensional stable sheaves $F$'s on $S$ with $[F] = \beta$, $\chi(F) = 1$ is a union of connected components of $M^\infty_{\chi=1}$. Below we determine the contribution of the surface component $M_{S,\beta}$ to the equivariant localization formula (3.4) of $M_{X,\beta}$.

Let $\mathbb{F} \in \text{Coh}(S \times M_{S,\beta})$ be the universal sheaf, $\pi_{M_{\beta}} : M_{S,\beta} \times S \to M_{S,\beta}$ be the projection. The standard deformation-obstruction theory

\begin{equation}
(3.5) \quad (\tau_{\beta_1} R \text{Hom}_{\pi_{M_{\beta}}} (\mathbb{F}, \mathbb{F}))^\vee[-1] \to L_{M_{S,\beta}}
\end{equation}

of $M_{S,\beta}$ is perfect [1, 24] and defines a virtual class $[M_{S,\beta}]^{\text{vir}} \in H_{2g+2}(M_{S,\beta}, \mathbb{Z})$.

**Proposition-Definition 3.3.** Suppose that $\text{Hom}(F, F \otimes L_2) = 0$ for any $[F] \in M_{S,\beta}$. The contribution of $M_{S,\beta}$ to the equivariant virtual class (3.4) of $M_{X,\beta}$ is

\begin{equation}
(3.6) \quad \pm [M_{S,\beta}]^{\text{vir}} \cdot c(-R \text{Hom}_{\pi_{M_{\beta}}} (\mathbb{F}, F \otimes L_1) \otimes t).
\end{equation}

**Proof.** Let $j$ be the inclusion $j = (\iota, \text{id}) : S \times M_{S,\beta} \hookrightarrow X \times M_{S,\beta}$, where $\iota$ is the zero section inclusion. We set

$\mathcal{U} := R \text{Hom}_{\pi_{M_{\beta}}} (j_! F, j_! F) \in D^b_c (M_{S,\beta})$.

Then there are isomorphisms

$\mathcal{U} \cong R \text{Hom}_{\pi_{M_{\beta}}} (L j^* j_! F, \mathbb{F})$

$\cong R \text{Hom}_{\pi_{M_{\beta}}} \left( \mathbb{F} \oplus (\mathbb{F} \otimes N_{S/\chi}^\vee)[1] \oplus (\mathbb{F} \otimes \wedge^2 N_{S/\chi}^\vee)[2], \mathbb{F} \right)$

$\cong R \text{Hom}_{\pi_{M_{\beta}}} (\mathbb{F}, \mathbb{F}) \oplus R \text{Hom}_{\pi_{M_{\beta}}} (\mathbb{F}, \mathbb{F} \otimes K_S)[-2]$

$\oplus R \text{Hom}_{\pi_{M_{\beta}}} (\mathbb{F}, \mathbb{F} \otimes L_1) \otimes t[-1] \oplus R \text{Hom}_{\pi_{M_{\beta}}} (\mathbb{F}, \mathbb{F} \otimes L_2) \otimes t^{-1}[-1].$

For the $\mathbb{C}^*$-fixed part, the Grothendieck duality gives

$H^1(\mathcal{U})^{\text{fix}} \cong \text{Ext}_{\pi_{M_{\beta}}}^1 (\mathbb{F}, \mathbb{F}),$

$H^2(\mathcal{U})^{\text{fix}} \cong \text{Ext}_{\pi_{M_{\beta}}}^2 (\mathbb{F}, \mathbb{F}) \oplus \text{Ext}_{\pi_{M_{\beta}}}^2 (\mathbb{F}, \mathbb{F})^\vee.$

By the shifted cotangent bundle argument as in Subsection 1.2, we see that

$[M_{X,\beta}]^{\text{vir}}|_{M_{S,\beta}} = [M_{S,\beta}]^{\text{vir}}.$

For the movable part, using condition (3.2) and Grothendieck duality, we have

$H^1(\mathcal{U})^{\text{mov}} \cong \left( R \text{Hom}_{\pi_{M_{\beta}}} (\mathbb{F}, \mathbb{F} \otimes L_1) \otimes t \right) \oplus \left( R \text{Hom}_{\pi_{M_{\beta}}} (\mathbb{F}, \mathbb{F} \otimes L_2) \otimes t^{-1} \right),$

$H^2(\mathcal{U})^{\text{mov}} \cong \left( \text{Ext}_{\pi_{M_{\beta}}}^1 (\mathbb{F}, \mathbb{F} \otimes L_1) \otimes t \right) \oplus \left( \text{Ext}_{\pi_{M_{\beta}}}^1 (\mathbb{F}, \mathbb{F} \otimes L_1)^\vee \otimes t^{-1} \right).$

By the assumption, we have $R \text{Hom}_{\pi_{M_{\beta}}} (\mathbb{F}, \mathbb{F} \otimes L_2) = 0$ and

$\text{Ext}_{\pi_{M_{\beta}}}^2 (\mathbb{F}, \mathbb{F} \otimes L_1) = R \text{Hom}_{\pi_{M_{\beta}}} (\mathbb{F}, \mathbb{F} \otimes L_2)^\vee = 0.$

Therefore we obtain the desired identity (3.3) following localization formula of type (3.4). \(\square\)

**Definition 3.4.** Suppose that $\text{Hom}(F, F \otimes L_2) = 0$ for any $[F] \in M_{S,\beta}$, and $M^\infty_{\chi=1} = M_{S,\beta}$ holds. We define the residue DT$_4$ invariant by the residue of (3.3) at $t = 0$, i.e.

$\text{DT}_4^{\text{res}}(\beta) := \pm \int_{[M_{S,\beta}]^{\text{vir}}} c_{g+2+1}(-R \text{Hom}_{\pi_{M_{\beta}}} (\mathbb{F}, \mathbb{F} \otimes L_1)).$

In the GW side, let $\pi : C \to \mathcal{M}_{0,0}(S, \beta)$ be the universal curve and $f : C \to S$ be the universal stable map. We define the residue GW invariant by

$\text{GW}^{\text{res}}_{0,\beta} := \text{Res}_{t=0} \int_{[M_{0,0}(S, \beta)]^{\text{vir}}} e(-R \pi_* f^* N_{S/\chi}) \in \mathbb{Q}.$

Here $N_{S/\chi} = (L_1 \otimes t) \oplus (L_2 \otimes t^{-1})$ is the $\mathbb{C}^*$-equivariant normal bundle and $e(-)$ is the $\mathbb{C}^*$-equivariant Euler class. As an analogy of Conjecture 1.3, we propose the following conjecture:

**Conjecture 3.5.** We have the following identity

$$\text{GW}^{\text{res}}_{0,\beta} = \sum_{k|\beta} \frac{1}{k} \text{DT}_4^{\text{res}}(\beta/k),$$

for certain choices of orientations in defining the RHS.
Note that the power of $1/k$ in the coefficients is 3 instead of 2 because we do not put insertions here.

3.5. Computations for $X = \text{Tot}_S(O_S \oplus K_S)$. Let $S$ be a smooth projective surface and consider the non-compact CY 4-fold $X = \text{Tot}_S(O_S \oplus K_S)$. In this case, the residue $DT_4$ invariant in Definition 3.4 is related to the $DT_3$ invariant on the non-compact CY 3-fold $Y = \text{Tot}_S(K_Y)$. We have the following lemma:

**Lemma 3.6.** Suppose that $\text{Hom}(F, F \otimes K_S) = 0$ for any $[F] \in M_{S, \beta}$, and $M_{S, \beta} = M_{Y, \beta}$ holds. Then we have

$$DT_{4, \text{res}}(\beta) = \pm \chi(M_{S, \beta}).$$

In particular, we have $DT_{4, \text{res}}^{\text{res}}(\beta) = \pm DT_3(\beta)$, where the RHS is the $DT_3$ invariant on $Y$.

**Proof.** Since $X = Y \times \mathbb{A}^1$, similarly to Lemma 2.6 we have $M_{X, \beta} = M_{Y, \beta} \times \mathbb{A}^1$. Therefore the assumption $M_{S, \beta} = M_{Y, \beta}$ implies $M_{X, \beta}^c = M_{S, \beta}$. Also the assumption $\text{Hom}(F, F \otimes K_S) = 0$ implies that $\text{Ext}_S^2(F, F) = 0$ for any $[F] \in M_{S, \beta}$. In particular, $M_{S, \beta}$ is non-singular and $[M_{S, \beta}]^{\text{vir}} = [M_{S, \beta}]$. From the definition of $DT_{4, \text{res}}(\beta)$, it follows that

$$DT_{4, \text{res}}(\beta) = \pm \int_{M_{S, \beta}} c(\mathcal{E}xt^1_{\tau_M}(F, F))$$

which is the topological Euler characteristic of $M_{S, \beta}$. Therefore the lemma follows.

In the following examples, conditions in Definition 3.4 are satisfied, so Lemma 3.6 can be used to verify Conjecture 3.5.

**When $S$ is a toric del-Pezzo surface.** In the toric del-Pezzo surface case, we have

**Theorem 3.7.** Let $S$ be a smooth toric del-Pezzo surface. Then Conjecture 2.7 holds for $X = \text{Tot}_S(O_S \oplus K_S)$.

**Proof.** By the virtual localization formula, it is easy to see that the residue GW invariant is the usual GW invariant on $Y = \text{Tot}_S(K_S)$. Then the result follows from Lemma 3.6 and Corollary 3.6.

**When $S$ is a rational elliptic surface.** Let $p : S \to \mathbb{P}^1$ be a rational elliptic surface with a section $s$ and $C \cong s(\mathbb{P}^1) \subseteq S$, $f$ a general fiber of $p$. We consider primitive curve classes

$$\beta_n = [C] + n[f] \in H_2(S, \mathbb{Z}), \quad n \geq 0.$$  

We first show the following:

**Lemma 3.8.** For any $[F] \in M_{S, \beta_n}$, we have $\text{Hom}(F, F \otimes K_S) = 0$ and

$$M_{S, \beta_n} = M_{Y, \beta_n} = M_{X, \beta_n}^c \cong \text{Hilb}^n(S).$$

**Proof.** By [13 Proposition 4.8], we have an isomorphism $M_{S, \beta_n} = M_{Y, \beta_n} \cong \text{Hilb}^n(S)$ given by a Fourier-Mukai transformation. In particular, the equivalence of derived categories implies that for any $F \in M_{S, \beta_n}$, we have

$$\text{Ext}_S^2(F, F) \cong \text{Ext}_S^2(I_Z, I_Z)$$

for some zero dimensional subscheme $Z \subseteq S$ with length $n$. Hence

$$\text{Hom}(F, F \otimes K_S) \cong \text{Ext}^2(F, F)^\vee \cong \text{Ext}^2_S(I_Z, I_Z)^\vee \cong \text{Hom}(I_Z, I_Z \otimes K_S) = 0$$

since $h^{0, i}(S) = 0$ for $i = 1, 2$.

As a corollary, we have the following:

**Corollary 3.9.** In the above situation, we have $DT_{4, \text{res}}(\beta_n) = \pm \chi(\text{Hilb}^n(S))$. In particular if we take the plus sign as the orientation, they fit into the generating series

$$\sum_{n \geq 0} DT_{4, \text{res}}(\beta_n)q^n = \prod_{k \geq 1} \frac{1}{(1 - q^k)^{12}}.$$  

**Proof.** The identity $DT_{4, \text{res}}(\beta_n) = \pm \chi(\text{Hilb}^n(S))$ follows from Lemma 3.6 and Lemma 3.8. By Göttscbe’s formula [11], we obtain the generating series.

Now we have the following:
Theorem 3.10. Let $S$ be a rational elliptic surface and take $\beta_\alpha$ as in \[\ref{(3.2)}\]. Then Conjecture 3.3 is true for $X = \text{Tot}(\mathcal{O}_S \oplus K_S)$ and $\beta = \beta_\alpha$, i.e., $\text{GW}_{0,\beta_\alpha}^{\text{res}} = \text{DT}_{\text{csc}}(\beta_\alpha)$ holds.

Proof. Since the virtual dimension of $\overline{M}_{0,0}(S,\beta_\alpha)$ is zero, we have

$$\text{GW}_{0,\beta_\alpha}^{\text{res}} = \text{deg}(\overline{M}_{0,0}(S,\beta_\alpha))^{\text{vir}}.$$ 

Its generating series fits into Göttsche-Yau-Zaslow formula (see e.g. \[\ref{[5, Theorem 1.2]}\])

$$\sum_{n=0}^{\infty} \text{deg}(\overline{M}_{0,0}(S,\beta_\alpha))^{\text{vir}} q^n = \prod_{k \geq 1} \frac{1}{(1-q^k)^2}.$$ 

Therefore the result follows from Corollary \[\ref{(3.9)}\]. \hfill \Box

4. Local curves

Let $C$ be a smooth projective curve of genus $g(C) = g$, and

$$p: X = \text{Tot}_C(L_1 \oplus L_2 \oplus L_3) \to C$$

be the total space of split rank three vector bundle on it. Assuming that

$$L_1 \oplus L_2 \oplus L_3 \cong \omega_C,$$

then the variety \[\ref{(4.1)}\] is a non-compact CY 4-fold. Below we set $l_i := \text{deg} L_i$, and may assume that $l_1 \geq l_2 \geq l_3$ without loss of generality. In this section, we study an equivariant version of Conjecture \[\ref{(4.3)}\] for $X$. 

4.1. Localization for GW invariants. Let $T = (\mathbb{C}^*)^3$ be the three dimensional complex torus which acts on the fibers of $X$ given by \[\ref{(4.1)}\]. Let $\bullet$ denote $\text{Spec} \mathbb{C}$ with trivial $T$-action. Let $\mathbb{C} \otimes t_i$ be the one dimensional vector space with $T$-action with weight $t_i$, and $\lambda_i \in H_T^*(\bullet)$ its 1st Chern class. We note that

$$H_T^2(\bullet) = \mathbb{C} [\lambda_1, \lambda_2, \lambda_3].$$

Let $j: C \hookrightarrow X$ be the zero section of the projection \[\ref{(4.1)}\]. Note that we have

$$H_2(X,Z) = \mathbb{Z}[C],$$

where $[C]$ is the fundamental class of $j(C)$. For $d \in \mathbb{Z}_{>0}$, we consider the diagram

$$\begin{align*}
\begin{array}{ccc}
C & \xrightarrow{f} & C \\
\downarrow & & \downarrow \\
\overline{M}_h(C,d[C]) & \xrightarrow{h} & C
\end{array}
\end{align*}$$

where $C$ is the universal curve and $f$ is the universal stable map. The $T$-equivariant GW invariant of $X$ is defined by

$$\text{GW}_{n,d}[C] = \text{GW}_{n,d} := \int_{[\overline{M}_h(C,d[C])]^{\text{vir}}} e(-Rf_*f^* N) \in \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3),$$

where $N$ is the $T$-equivariant normal bundle of $j(C) \subset X$:

$$N = (L_1 \otimes t_1) \oplus (L_2 \otimes t_2) \oplus (L_3 \otimes t_3).$$

If $g(C) > 0$, we have the obvious vanishing of genus zero GW invariants

$$\text{GW}_{0,d} = 0, \quad g(C) > 0, \quad d \in \mathbb{Z}_{>0}$$

because $\overline{M}_0(C,d[C]) = \emptyset$.

If $g(C) = 0$, we have

$$\text{GW}_{0,d} = \int_{[\overline{M}_0(\mathbb{P}^1,d)]} e\left(-Rf_*f^* \left(\mathcal{O}_{\mathbb{P}^1}(l_1) t_1 \oplus \mathcal{O}_{\mathbb{P}^1}(l_2) t_2 \oplus \mathcal{O}_{\mathbb{P}^1}(l_3) t_3\right)\right).$$

For example in the $d = 1$ case, $\overline{M}_0(\mathbb{P}^1,1)$ is one point and

$$\text{GW}_{0,1} = \lambda_1^{l_1-1} \lambda_2^{l_2-1} \lambda_3^{l_3-1}.$$
In the $d = 2$ case, a straightforward localization calculation as in \[22\] with respect to the $(\mathbb{C}^*)^2$-action on $\mathbb{P}^1$ gives

\[
\text{GW}_{0,2} = \frac{1}{8} \lambda_1^{-2} \lambda_2^{-1} \lambda_3^{-2} \lambda_4^{-1} \left( (t_1 - t_1 - 1)^2 + \cdots \right) \lambda_1^{-2} + (t_2 - (t_2 - 1)^2 + \cdots) \lambda_2^{-2} + (t_3 - (t_3 - 1)^2 + \cdots) \lambda_3^{-2} + t_1 t_2 \lambda_1^{-1} \lambda_2^{-1} + t_2 t_3 \lambda_2^{-1} \lambda_3^{-1} + t_1 t_2 \lambda_1^{-1} \lambda_3^{-1} \right).
\]

Here we write $T = t$ for $t \geq 0$ and $T = -t - 1$ for $t < 0$.

### 4.2. Localization for stable sheaves on local curves

The $T$-action on $X$ does not preserve its CY 4-form, so we cannot apply the $T$-localization for $\text{DT}_4$-theory. Instead, we consider its restriction to the subtorus

\[
T_0 = \{ t_1 t_2 t_3 = 1 \} \subset T
\]

which preserves the CY 4-form on $X$. Then the Serre duality pairing on moduli space $M_\beta$ is preserved by $T_0$. Similarly, to the case of local surfaces, we will define equivariant virtual classes for $M_\beta$ using localization formula with respect to $T_0$-action, and investigate their relation with equivariant GW invariants.

We consider the $T_0$-action on the moduli space of one dimensional stable sheaves $M_{d[C]}$ on the local curve $X$. Let $E \in \text{Coh}(X \times M_{d[C]}^{T_0})$ be the universal sheaf, and $\pi$ the projection from $X \times M_{d[C]}^{T_0}$ to $M_{d[C]}^{T_0}$. Analogous to \[3.3\], we have the following localization formula for $M_{d[C]}$ should give the following definition:

\[
\text{DT}_4(d[C]) = \int_{[M_{d[C]}]} e(\mathcal{R} \text{Hom}_X(E, E')^\text{mov})^{1/2} \in \mathbb{Q}(\lambda_1, \lambda_2),
\]

where we use the isomorphism

\[
H_{T_0}(\bullet) = \mathbb{C}[\lambda_1, \lambda_2, \lambda_3]/(\lambda_1 + \lambda_2 + \lambda_3) \cong \mathbb{C}[\lambda_1, \lambda_2].
\]

This equivariant invariant \[4.7\] will be rigorously defined for $d = 1$ \[1.3\] and $d = 2$ \[4.7\] respectively below, where the virtual class of the torus fixed locus is its usual fundamental class and there is a preferred choice of square root of the equivariant Euler class of the virtual normal bundle.

### 4.3. Contribution from the component $M_C(d, 1)$

Let $M_C(d, 1)$ be the moduli space of rank $d$ stable vector bundles $F$ on $C$ with $\chi(F) = 1$. By push-forward $j_*$ for the zero section $j: C \to X$, $M_C(d, 1)$ is a connected component of $M_{d[C]}$. Below we use the following fact:

**Lemma 4.1.** For $F, F' \in \text{Coh}(C)$ and push-forwards $j_* F, j_* F' \in \text{Coh}(X)$, we have

\[
\chi(j_* F, j_* F') = \chi_C(F, F') - \chi_C(F, F' \otimes N) + \chi_C(F, F' \otimes \lambda^2 N) - \chi_C(F, F' \otimes \lambda^3 N).
\]

Here $\chi(-, -)$ is the Euler pairing on $X$ and $\chi_C(-, -)$ is the Euler pairing on $C$.

**Proof.** For $F \in \text{Coh}(C)$, we have the isomorphism

\[
\text{L} j^* j_* F \cong \bigoplus_{i \geq 0} F \otimes \wedge^i N^\vee[i].
\]

Therefore the lemma follows by the adjunction. \qed

We now investigate the contribution of $M_C(d, 1)$ to \[4.7\]. For $[F] \in M_C(d, 1)$, by Lemma 4.1 we have

\[
\chi(j_* F, j_* F) = \chi_C(F, F) - \chi_C(F, F \otimes N) + \chi_C(F, F \otimes \lambda^2 N) - \chi_C(F, F \otimes \lambda^3 N).
\]

We set

\[
\chi(j_* F, j_* F)^{1/2} := \chi_C(F, F) - \chi_C(F, F \otimes N).
\]

Then as elements of $T_0$-equivariant $K$-theory of $M_C(d, 1)$, we have

\[
\chi(j_* F, j_* F) = \chi(j_* F, j_* F)^{1/2} + \chi(j_* F, j_* F)^{1/2, \text{mov}}.
\]

The $T_0$-fixed and movable parts of $\chi(j_* F, j_* F)^{1/2}$ are given by

\[
\chi(j_* F, j_* F)^{1/2, \text{fix}} = \chi_C(F, F), \quad \chi(j_* F, j_* F)^{1/2, \text{mov}} = -\chi_C(F, F \otimes N).
\]
The first identity of (4.8) implies that the virtual class $[M^T_{d|C}]^{\text{vir}}$ should be the usual fundamental class $[MC(d,1)]$ on the component $MC(d,1) \subset M^T_{d|C}$. Let $\mathcal{F}$ be a universal vector bundle on $C \times MC(d,1)$ and $\pi$ the projection from $MC(d,1) \times C$ to $MC(d,1)$. By the second identity of (4.8), the contribution of $MC(d,1)$ to (4.7) should be given by

$$\int_{MC(d,1)} e(-\mathbf{R} \mathcal{H}om_{\pi^*}(\mathcal{F}, \mathcal{F} \boxtimes N)) \in \mathbb{Q}(\lambda_1, \lambda_2),$$

where we regard $N$ (4.4) as a $T_0$-equivariant vector bundle on $C$.

**Proposition 4.2.** The integral (4.9) is zero unless $g(C) = 0$ and $d = 1$. If $g(C) = 0$ and $d = 1$, it is equal to

$$\lambda_1^{-l_1-1} \lambda_2^{-l_2-1} (-\lambda_1 - \lambda_2)^{-l_3-1}. \tag{4.10}$$

**Proof.** We first assume that $g(C) = 0$. Then we have $MC(d,1) = \emptyset$ for $d \geq 2$, so (4.9) is zero.

For $g(C) = 0$ and $d = 1$, then $MC(1,1) = \text{Spec } \mathbb{C}$ and the identity (4.10) follows from

$$\mathbf{R} \mathcal{H}om_{\pi^*}(\mathcal{F}, \mathcal{F} \boxtimes N) = \mathbf{R} \Gamma(C, N).$$

Next we assume that $g(C) > 0$ and show the vanishing of (4.9). For $L \in \text{Pic}(C)$, let $MC(d, L) \subset MC(d,1)$ be the closed subscheme given by stable bundles $F$ on $C$ with fixed determinant $det = F = L$. We have an étale surjective map

$$h : \text{Pic}^{0}(C) \times MC(d, L) \to MC(d,1)$$

given by $(L', F) \mapsto L' \otimes F$. By setting $F' = (h \times id)^* F$, it is enough to show the vanishing

$$\int_{\text{Pic}^{0}(C) \times MC(d, L)} e(-\mathbf{R} \mathcal{H}om_{\pi^*}(F', F' \boxtimes N)) = 0,$$

where $\pi'$ is the projection from $\text{Pic}^{0}(C) \times MC(d, L) \times C$ to $\text{Pic}^{0}(C) \times MC(d, L)$.

We set $F_L = F|_{MC(d, L) \times C}$. Since $F'$ is isomorphic to $\mathcal{O}_{\text{Pic}^{0}(C)} \boxtimes F_L$ up to tensoring a line bundle, the above integral coincides with

$$\int_{\text{Pic}^{0}(C) \times MC(d, L)} p_M^e(-\mathbf{R} \mathcal{H}om_{\pi'}(F_L, F_L \boxtimes N)),$$

where $p_M$ is the projection from $\text{Pic}^{0}(C) \times MC(d, L)$ to $MC(d, L)$, and $\pi''$ is the projection from $MC(d, L) \times C$ to $MC(d, L)$. Since $\dim \text{Pic}^{0}(C) = g(C) > 0$, the above integral obviously vanishes. \hfill $\square$

When $d = 1$, $MC(1,1) \cong M^T_{d|C}$ is the only $T_0$-fixed component, we have the following corollary:

**Corollary 4.3.** For $g(C) > 0$, we have $\text{GW}_{0,|C|} = DT_4([C]) = 0$. For $g(C) = 0$, we have $\text{GW}_{0,|C|} = \lambda_3 = -\lambda_1 - \lambda_2 = DT_4([C]) = \lambda_1^{-l_1-1} \lambda_2^{-l_2-1} (-\lambda_1 - \lambda_2)^{-l_3-1}$. \hfill $\square$

**Proof.** For $d = 1$, we have the isomorphism

$$j_* : MC(1,1) = \text{Pic}^{0}(C) \xrightarrow{\cong} M^T_{d|C}.$$

Therefore the result follows from (4.5) and Proposition 4.2. \hfill $\square$

### 4.4. Localization for degree two stable sheaves

We next consider the case of $d = 2$. For $(k, k') \in \mathbb{Z}^2$, we denote by $\text{Pic}^{(k,k')}(C)$ the moduli space of triples

$$(L, L', \iota) \in \text{Pic}^{k}(C) \times \text{Pic}^{k'}(C), \quad \iota : L \hookrightarrow L'$$

Here $\iota$ is an inclusion of line bundles. Then $M^T_{2|C}$ can be described as follows.

**Proposition 4.4.** We have the following isomorphism

$$\int_{MC(2,1) \cup \bigcup_{i=1}^3 \bigcup_{(d_0,d_i) \in \mathbb{Z}^2} \bigcap_{g \leq d_0 \leq g+(l_i-1)/2} \bigcap_{g \leq d_i \leq g+1} \text{Pic}^{(d_0,d_i)}(C) \xrightarrow{\cong} M^T_{2|C}. \tag{4.11}$$

**Proof.** For $[F] \in M^T_{2|C}$, it is either a rank two stable vector bundle on $C$ or thickened into one of the $L_i$-direction. In the latter case, we have

$$p_* F = F_0 \oplus (F_i \otimes t_i^{-1})$$
where $F_0$, $F_i$ are line bundles on $C$. The $\mathcal{O}_X$-module structure of $F$ is given by a non-trivial morphism
\begin{equation}
(4.12) \quad \phi: F_0 \otimes L_i^{-1} \to F_i.
\end{equation}
We write $F_i = F_i' \otimes L_i^{-1}$, and set $d_0 = \deg F_0$, $d_i = \deg F_i'$. By the above argument, the $L_i$-thickened sheaf $[F] \in M_{2|C}^{T_0}$ determines a point
\[(F_0, F_i', \phi: F_0 \hookrightarrow F_i') \in \text{Pic}^{(d_0, d_i)}(C).\]
Conversely by going back the above argument, we easily see that any point in $\text{Pic}^{(d_0, d_i)}(C)$ determines a point in $M_{2|C}^{T_0}$. Therefore it is enough to determine the possible $(d_0, d_i)$.

Since $\chi(F) = 1$, we have the identity
\begin{equation}
(4.13) \quad d_0 + d_i = 2g - 1 + l_i.
\end{equation}
By the exact sequence in $\text{Coh}(X)$
\begin{equation}
(4.14) \quad 0 \to F_i \to F_i' \to F_0 \to 0
\end{equation}
and the stability of $F$, we have $\chi(F_0) \geq 1$, i.e. $d_0 \geq g$. Also since (4.12) is non-zero, we have $d_0 \leq d_i$. By substituting (4.13), we see that
\begin{equation}
(4.15) \quad g \leq d_0 \leq g + \frac{l_i}{2} - \frac{1}{2}.
\end{equation}
Conversely if $(d_0, d_i)$ satisfy the conditions (4.13), (4.15), then the corresponding sheaf on $X$ determines a point in $M_{2|C}^{T_0}$. Therefore we obtain the desired isomorphism (4.11). \hfill \Box

The following lemma is obvious:

**Lemma 4.5.** We have an isomorphism
\[\text{Pic}^{(d, d')} (C) \xrightarrow{\sim} \text{Pic}^{d'} (C) \times S^{d-d'} (C)\]
given by $(L, L', i) \mapsto (L', \text{Supp} (\text{Cok}(i)))$. In particular, $\text{Pic}^{(d, d')}(C)$ is smooth of dimension $g + d' - d$.

**Determine $[M_{2|C}^{T_0}]^\text{vir}$ and square root of Euler class of virtual normal bundle.** To apply localization formula (4.7), we need to determine the virtual class $[M_{2|C}^{T_0}]^\text{vir}$, the equivariant virtual normal bundle of $M_{2|C}^{T_0}$ and square roots of its Euler class. By Proposition 4.2, the component $M_C(2, 1)$ does not contribute to (4.7). We need to consider contributions from thickened sheaves.

For a $L_i$-thickened sheaf $[F] \in M_{2|C}^{T_0}$, the exact sequence (4.14) gives $F = F_0 + F_i \cdot t_i^{-1}$ in the $T_0$-equivariant $K$-theory of $X$. Therefore
\[\chi(F, F) = \chi(j_* F_0, j_* F_0) + \chi(j_* F_0, j_* F_1) t_i^{-1} + \chi(j_* F_1, j_* F_0) t_i + \chi(j_* F_i, j_* F_i).\]
We set
\begin{equation}
(4.16) \quad \chi(F, F)^{1/2} := \chi(j_* F_0, j_* F_0) + \chi(j_* F_0, j_* F_1) t_i^{-1}.
\end{equation}
Then as elements of $T_0$-equivariant $K$-theory of $M_{2|C}^{T_0}$, we obtain
\[\chi(F, F) = \chi(F, F)^{1/2} + \chi(F, F)^{1/2, \text{vir}}.\]
By Lemma 4.4 we have
\[\chi(F, F)^{1/2} = \chi(\mathcal{O}_C) - \chi(N) + \chi(\Lambda^2 N) - \chi(\Lambda^3 N) + (\chi(A) - \chi(A \otimes N) + \chi(A \otimes \Lambda^2 N) - \chi(A \otimes \Lambda^3 N)) t_i^{-1},\]
where we set $A = F_i \otimes F_0$. By the above formula, the $T_0$-fixed part of (4.16) is
\[\chi(F, F)^{1/2, \text{fix}} = \chi(\mathcal{O}_C) - \chi(\Lambda^3 N) - \chi(A \otimes L_i)\]
which is $(1 - g - d_i + d_0)$-dimensional by Riemann-Roch theorem. Therefore
\[\text{Hom}(F, F) - \chi(F, F)^{1/2, \text{fix}}\]
is of dimension $g + d_i - d_0$, which coincides with the dimension of $\text{Pic}^{(d_0, d_i)}(C)$ by Lemma 4.3. It follows that, by Proposition 4.4 the virtual class associated to the $T_0$-fixed obstruction theory on $M_{2|C}^{T_0}$ should be its usual fundamental class.
We now give a definition of $\text{DT}_4(2[C]) \in \mathbb{Q}(\lambda_1, \lambda_2)$. Let 

$$(F_0, F'_0, t), \ i : F_0 \hookrightarrow F'_i$$

be the universal object on $\text{Pic}^{(d_0, d_1)}(C) \times C$, i.e. $F_0, F'_i$ are line bundles on $\text{Pic}^{(d_0, d_1)}(C) \times C$ and $i$ is the universal injection. Let $F'_i : = F'_i \otimes L^{-1}_i$, and consider its push-forward

$$j_* F'_i \in \text{Coh}(\text{Pic}^{(d_0, d_1)}(C) \times X), \ i = 1, 2.$$ Based on the localization formula \eqref{eq:Euler} and the above discussions, we define $\text{DT}_4(2[C])$ to be \eqref{eq:DT4}(17)

$$\sum_{i=1}^{3} \sum_{(d_0, d_1) \in \mathbb{Z}^2} \int_{\text{Pic}^{(d_0, d_1)}(C)} e \left( R \text{Hom}_\pi (j_* F_0, j_* F_0)^{\text{mov}} + R \text{Hom}_\pi (j_* F_0, j_* F'_i \cdot t_i^{-1})^{\text{mov}} \right),$$

as an element in $\mathbb{Q}(\lambda_1, \lambda_2)$. Here $\pi$ is the projection from $M^{d_0}_{2[C]} \times X$ to $M^{d_0}_{2[C]}$, and we have used the isomorphism \eqref{eq:iso}. The second integrations are derived from \eqref{eq:iso}.

The rest of this subsection is devoted to an explicit computation of $\text{DT}_4(2[C])$.

**Lemma 4.6.** Let $Z \subset S^{d_1, d_0}(C) \times C$ be the universal divisor and set

$$A = \mathcal{O}_{S^{d_1, d_0}(C) \times C}(Z) \otimes L_i^{-1}.$$ We set $N_i = N_i - L_i \otimes t_i$ in the $T_0$-equivariant K-theory of $C$, and denote by $p_S, \pi_S$ the projections from $\text{Pic}^{d_1}(C) \times S^{d_1, d_0}(C), S^{d_1, d_0}(C) \times C$ to $S^{d_1, d_0}(C)$ respectively. Then we have the identity

$$\int_{\text{Pic}^{(d_0, d_1)}(C)} e \left( R \text{Hom}_\pi (j_* F_0, j_* F_0)^{\text{mov}} + R \text{Hom}_\pi (j_* F_0, j_* F'_i \cdot t_i^{-1})^{\text{mov}} \right)

= -\lambda_1^{-2l_1-2} \lambda_2^{-2l_2-2} \chi(\lambda_1 + \lambda_2) t_i^{-1}$$

\[ \cdot \left( R \pi_S (A - \pi_S (A \otimes N_i) + R \pi_S (A \otimes \lambda^2 N) - R \pi_S (A \otimes \omega_C)) t_i^{-1} \right). \]

**Proof.** The $T_0$-movable part of \eqref{eq:iso} is

$$\chi(F, F)^{1/2, \text{mov}}

= -\chi(N) + \chi(\lambda^2 N) + (\chi(A) - \chi(A \otimes N_i) + \chi(A \otimes \lambda^2 N) - \chi(A \otimes \lambda N)) t_i^{-1}.$$ Suppose that $(F_0, F'_0)$ corresponds to $(F'_i, Z)$ under the isomorphism in Lemma \ref{lem:iso}. Then

$$A = F_0 \otimes F'_0 = F'_i \otimes F'_i \otimes L_i^{-1} = \mathcal{O}_C(Z) \otimes L_i^{-1}.$$ Therefore the desired identity holds. \hfill \square

Similar to Proposition \ref{prop:vanishing}, we have the following vanishing for higher genus.

**Corollary 4.7.** For $g(C) > 0$, we have $\text{DT}_4(2[C]) = 0$.

For $g(C) = 0$, we compute the integral in Lemma \ref{lem:iso} as follows.

**Lemma 4.8.** Suppose that $g(C) = 0$. By setting $k = d_i - d_0 + 1$, the integral

$$\int_{\text{Pic}^{d_i}(C) \times S^{d_i, d_0}(C)} p_S^\ast e \left( (R \pi_S (A - \pi_S (A \otimes N_i) + R \pi_S (A \otimes \lambda^2 N) - R \pi_S (A \otimes \omega_C)) t_i^{-1} \right)$$

in Lemma \ref{lem:iso} for $i = 1$ is calculated as

$$A(l_1, l_2, l_3, k) := \text{Res}_{h=0} \left\{ h^{-k}(\lambda_1 + h)^{k+l_2} (\lambda_2 + h)^{k+l_1} \right\},$$

$$(-\lambda_1 - \lambda_2 + h)^{k+l_3} (-\lambda_1 + \lambda_2 + h)^{l_1 - l_3 - k} (-2\lambda_1 - \lambda_2 + h)^{l_1 - l_3 - k} (2\lambda_1 + h)^{k-2}.$$

The integral for $i = 2$ is given by

$$B(l_1, l_2, l_3, k) := \text{Res}_{h=0} \left\{ h^{-k}(\lambda_2 + h)^{k+l_1} (-\lambda_2 + \lambda_1 + h)^{k+l_3} (-2\lambda_2 - \lambda_1 + h)^{l_2 - l_3 - k} (2\lambda_2 + h)^{k-2}. \right\}$$

**Proof.** For $g(C) = 0$, we have $\text{Pic}^{d_i}(C) = \text{Spec} \mathbb{C}$ and $S^{d_i, d_0}(C) = \mathbb{P}^{d_i, d_0}$. The universal divisor $Z \subset \mathbb{P}^{d_i, d_0} \times \mathbb{P}^1$ is an $(1, d_i - d_0)$-divisor, so we have

$$A = \mathcal{O}_{\mathbb{P}^{d_i, d_0} \times \mathbb{P}^1}(1, d_i - d_0 - l_i).$$
Therefore we have (here we write $O(1) = O_{x_1-\rho_0}(1)$)
\[
\begin{align*}
R \pi_{S^*} A &= O(1)^{\oplus k-1}, \quad R \pi_{S^*} (A \otimes \omega_C) = O(1)^{\oplus k-l_1-2}, \\
R \pi_{S^*} (A \otimes N_1) &= O(1)^{\oplus k-l_1+1} t_2 \oplus O(1)^{\oplus k-l_1+3} t_3, \\
R \pi_{S^*} (A \otimes \lambda^2 N) &= O(1)^{\oplus k+l_1} t_1 t_2 \oplus O(1)^{\oplus k+l_1} t_1 t_3 \oplus O(1)^{\oplus k-2} t_2 t_3.
\end{align*}
\]

Let $h = c_1(O(1))$. Then the desired integral is the $h^{k-1}$-part of
\[
(-\lambda_1 + h)^2 (-\lambda_1 + \lambda_2 + h)^{-k+l_1-l_2} (-\lambda_1 + \lambda_3 + h)^{-k+l_1-l_3} (\lambda_2 + h)^{k+l_1} (\lambda_3 + h)^{k+l_3} (-\lambda_1 + \lambda_2 + \lambda_3 + h)^{k-2-2l_1}.
\]

Using $\lambda_1 + \lambda_2 + \lambda_3 = 3$, we obtain the desired result. The case of $i = 2$ is similar. \qed

**Corollary 4.9.** For $g(C) = 0$, we have
\[
DT_4(2[C]) = -\lambda_1^{-2l_1-2} \lambda_2^{-2l_2-2} (\lambda_1 + \lambda_2)^{-2l_3-2} \cdot \left( \sum_{1 \leq k \leq l_1, k \equiv l_3 \pmod{2}} A(l_1, l_2, l_3, k) + \sum_{1 \leq k \leq l_2, k \equiv l_3 \pmod{2}} B(l_1, l_2, l_3, k) \right).
\]

**Proof.** Since $l_1 + l_2 + l_3 = -2$ and we have assumed $l_1 \geq l_2 \geq l_3$, so $l_1 \geq 0 > l_3$. In particular, by the inequality
\begin{equation}
0 \leq d_0 \leq \frac{l_i - 1}{2}
\end{equation}
in the definition of $DT_4(2[C])$, there is no contribution from $L_3$-thickened sheaves.

By setting $k = d_i - d_0 + 1$, since $d_0 + d_i = l_i - 1$, we have
\begin{equation}
d_0 = \frac{l_i - k}{2}, \quad d_i = \frac{l_i + k}{2} - 1.
\end{equation}
The inequality (4.19) is then equivalent to $1 \leq k \leq l_i$. Conversely given an integer $k$ satisfying $1 \leq k \leq l_i$, there exists $(d_0, d_i) \in \mathbb{Z}^2$ satisfying (4.19) and (4.20) if and only if $k \equiv l_i (\text{mod } 2)$. Therefore the desired identity holds by Lemma 4.6 and Lemma 4.8. \qed

**4.5. Equivariant version of Conjecture 4.3 for local curves.** We propose an equivariant version of Conjecture 4.3 (without insertions) for degree two case (the degree one case is given by Corollary 4.3).

**Conjecture 4.10.** For any smooth projective curve $C$ and line bundles $L_i$ ($i = 1, 2, 3$) on $C$ with $L_1 \otimes L_2 \otimes L_3 \cong \omega_C$, we have the identity
\[
GW_{0,2[C]} = DT_4(2[C]) + \frac{1}{8} DT_4([C]) \in \mathbb{Q}(\lambda_1, \lambda_2)
\]

after substituting $\lambda_3 = -\lambda_1 - \lambda_2$.

**Remark 4.11.** As there is no insertion here, the coefficient of $DT_4([C])$ is $1/2^3$ instead of $1/2^2$ in Conjecture 4.3.

When $g(C) \geq 1$, the above conjecture is obviously true by Proposition 4.2 and Corollary 4.7. For $g(C) = 0$ case, we fix constants $l_i \in \mathbb{Z}$ ($i = 1, 2, 3$) with $l_1 + l_2 + l_3 = -2$ and may assume $l_1 \geq l_2 \geq l_3$ without loss of generality, the above conjecture is expressed in terms of a polynomial relation in variables $\{\lambda_i^k\}$, we verify the conjecture with the help of software 'Mathematica' in examples as follows.

**Theorem 4.12.** When $g(C) = 0$ and denote $\deg(L_i) = l_i$, then Conjecture 4.10 is true if $|l_1| \leq 10$, $|l_2| \leq 10$.

In the appendix, we will list a computational example (the case of $l_1 = 8$, $l_2 = 6$, $l_3 = -16$) given by 'Mathematica' and one can see there are really non-trivial cancellations involved to make Conjecture 4.10 true in this case.

For curve classes of higher degree, we may define $DT_4(d[C]) \in \mathbb{Q}(\lambda_1, \lambda_2)$ and have
Conjecture 4.13. For any smooth projective curve $C$, line bundles $L_i$ ($i = 1, 2, 3$) on $C$ with $L_1 \otimes L_2 \otimes L_3 \cong \omega_C$ and $d \geq 1$, we have the identity

$$GW_{0,d[C]} = \sum_{k/d} \frac{1}{k^3} \cdot DT_4 \left( \frac{d}{k} [C] \right) \in \mathbb{Q}(\lambda_1, \lambda_2)$$

after substituting $\lambda_3 = -\lambda_1 - \lambda_2$.

APPENDIX A. GENUS ZERO GV/DT CONJECTURE FOR CY 3-FOLDS

Here we recall the genus zero GV/DT conjecture for CY 3-folds and explain how it can be derived from the conjectural GW/PT correspondence, a geometric vanishing conjecture and wall-crossing in the derived category.

Let $Y$ be a smooth projective CY 3-fold. For $\beta \in H_2(Y, \mathbb{Z})$, the virtual dimension of $\overline{M}_g(Y, \beta)$ is zero due to the CY3 condition of $Y$. Hence we have the GW invariant without insertion

$$GW_{g, \beta} = \int_{[\overline{M}_g(Y, \beta)]^{vir}} 1 \in \mathbb{Q}.$$

Its generating series is uniquely written as

$$GW_{0, \beta} = \sum_{k \geq 1, k|\beta} \frac{1}{k^3} n^G_{0, \beta/k}.$$

On the other hand, let $M_3$ be the moduli space of one dimensional stable sheaves $E$ on $Y$ with $[E] = \beta$, $\chi(E) = 1$. Then there is a zero-dimensional virtual fundamental cycle on $M_3$, and its integral yields the DT invariant

$$DT_3(\beta) := \int_{[M_3]^{vir}} 1 \in \mathbb{Z}.$$

The genus zero GV/DT conjecture for CY 3-folds is as follows:

Conjecture A.1. \cite{19} We have the identity $n^G_{0, \beta} = DT_3(\beta)$.

Let $P_n(Y, \beta)$ be the moduli space of stable pairs $(F, s)$ on $Y$ such that $[F] = \beta$, $\chi(F) = n$. Similarly, we have the PT invariant \cite{29}

$$P_{n, \beta} := \int_{[P_n(Y, \beta)]^{vir}} 1 \in \mathbb{Z}.$$

The logarithm of its generating series is uniquely written as

$$\log \left( 1 + \sum_{\beta > 0, n \in \mathbb{Z}} P_{n, \beta} q^n t^\beta \right) = \sum_{\beta > 0, g \in \mathbb{Z}} \sum_{k \geq 1} \frac{n_{g, \beta}}{k} (-1)^{g-1} (-q)^{-1} \chi(q^2 \beta) (-q)^{-2} 2^g 2^{k \beta}.$$

for some $n_{g, \beta} \in \mathbb{Z}$ with $n_{g, \beta} = 0$ for $g \gg 0$. We have the following strong form of the GW/PT correspondence:

Conjecture A.2. We have the identity $n^G_{g, \beta} = n^P_{g, \beta}$ for any $g \in \mathbb{Z}$ and $\beta > 0$.

We have the following:

Lemma A.3. Suppose that $Y$ satisfies Conjecture A.2. Then $Y$ satisfies Conjecture A.1.

Proof. Conjecture A.2 implies that $n^P_{g, \beta} = 0$ for $g < 0$. By \cite{33} Theorem 6.4], the wall-crossing argument in the derived category shows the identity $n^G_{0, \beta} = DT_3(\beta)$. Therefore $n^G_{0, \beta} = DT_3(\beta)$ holds.

Note that the original GW/DT conjecture \cite{25} together with DT/PT correspondence \cite{32} only claims the identity of A.1 and A.2 as rational functions of $q$. In order to further have the identity $n^G_{g, \beta} = n^P_{g, \beta}$, we need to know either one of the following properties:

1. For any fixed $\beta$, we have $n^G_{g, \beta} = 0$ for $g \gg 0$.
2. For any $\beta$, we have $n^P_{g, \beta} = 0$ for $g < 0$.
Indeed if one of the above conditions is satisfied, then the uniqueness of the expressions in the form of \( \text{A.1} \) or \( \text{A.2} \) shows the identity \( n_{g,\beta}^{GW} = n_{g,\beta}^{P} \). So assuming that the GW/PT conjecture holds, the conditions (1), (2), and Conjecture (A.3) are equivalent.

The vanishing \( n_{g,\beta}^{P} = 0 \) for \( g < 0 \) is nothing but Pandharipande-Thomas’ strong rationality conjecture [29]. By the wall-crossing argument in the derived category [33], this vanishing is equivalent to the multiple cover formula of generalized DT invariants [15]. Let \( N_{g,\beta} \in \mathbb{Q} \) be the generalized DT invariant which counts one dimensional semistable sheaves \( E \) on \( Y \) with \( [E] = \beta \), \( \chi(E) = n \). Its multiple cover conjecture is given as follows:

**Conjecture A.4.** [15 33] We have the identity

\[
\text{A.3} \quad N_{n,\beta} = \sum_{k \geq 1, k \mid (n, \beta)} \frac{1}{k!} \text{DT}_{3}(\beta/k).
\]

We have the following lemma:

**Lemma A.5.** For a CY 3-fold \( Y \), suppose that the GW/PT conjecture [25] holds. Moreover suppose that for a fixed \( \beta \in H_{2}(Y, \mathbb{Z}) \), the identity \( \text{A.3} \) holds. Then for any \( k \geq 1 \) with \( k|\beta \), we have the identity \( n_{0,\beta}^{GW} = \text{DT}_{3}(\beta/k) \). In particular, \( n_{0,\beta} = \text{DT}_{3}(\beta) \) holds for any primitive curve class \( \beta \).

**Proof.** By the argument of [33 Theorem 6.4], the identity \( \text{A.3} \) implies that \( n_{g,\beta/k}^{P} = 0 \) for any \( g < 0 \) and \( k \geq 1 \) with \( k|\beta \), and \( n_{0,\beta/k}^{P} = \text{DT}_{3}(\beta/k) \) holds. Then comparing \( \text{A.1} \) with \( \text{A.2} \), we obtain \( n_{0,\beta/k}^{GW} = \text{DT}_{3}(\beta/k) \). The identity \( \text{A.3} \) always holds for primitive curve class \( \beta \) by [31] Lemma 2.12. Therefore \( n_{0,\beta} = \text{DT}_{3}(\beta) \) holds for any primitive \( \beta \).

Then using the result of [28], we have the following:

**Corollary A.6.** Let \( Y \) be a complete intersection CY 3-fold in the product of projective spaces \( \mathbb{P}^{n_{1}} \times \cdots \times \mathbb{P}^{n_{e}} \). Then for any primitive curve class \( \beta \) on \( Y \), we have \( n_{0,\beta}^{GW} = \text{DT}_{3}(\beta) \).

**Proof.** The GW/PT conjecture is proved for such CY threefolds in [28]. Then the result follows from Lemma A.5.

In the case of some toric CY 3-fold, we can derive Conjecture A.4 from the vanishing \( n_{g,\beta}^{GW} = 0 \) for \( g > 0 \). Let \( S \) be a smooth toric del-Pezzo surface and consider the non-compact CY 3-fold \( Y = \text{Tot}_{S}(K_{S}) \). Although \( Y \) is non-compact, all the relevant objects (stable maps, stable pairs, stable sheaves) are supported on the zero section of \( Y \rightarrow S \). Therefore all the above invariants, results are obtained as in the projective CY 3-fold case. We have the following corollary:

**Corollary A.7.** Let \( S \) be a smooth toric del-Pezzo surface. Then Conjecture A.4 holds for \( Y = \text{Tot}_{S}(K_{S}) \).

**Proof.** Since \( S \) is toric, the Conjecture A.2 holds by [25 23]. Therefore the result follows by Lemma A.3.

**Remark A.8.** In particular for \( Y = \text{Tot}_{S}(K_{S}) \) in Corollary A.7, the identity \( \text{A.3} \) holds. We don’t know how to prove this without using the GW/PT correspondence [25].

**Appendix B. An orientability result for smooth moduli spaces of one dimensional stable sheaves**

Let \( X \) be a smooth projective CY 4-fold and \( M_{\beta} \) be the moduli space of one dimensional stable sheaves of Chern character \((0,0,0,\beta,1)\). We denote \( \mathcal{L} \) to be its determinant line bundle (see [10]) and \( Q \) the non-degenerate quadratic form induced by Serre duality. Then we have the following:

**Proposition B.1.** If \( M_{\beta} \) is a normal variety, then \((\mathcal{L}, Q)\) has an orientation.

**Proof.** In fact, we are left to show \( \mathcal{L} \cong \mathcal{O}_{M_{\beta}} \). Since if \( \mathcal{L} \cong \mathcal{O}_{M_{\beta}} \), the square \( \mathcal{L}^{2} \cong \mathcal{O}_{M_{\beta}} \) of this isomorphism, although may be different from the one given by Serre duality (17), its difference with that one gives an isomorphism \( \mathcal{O}_{M_{\beta}} \cong \mathcal{O}_{M_{\beta}} \), which has a square root as \( M_{\beta} \) is compact.

Then the argument follows from [27 Proposition 3.13] which we adapt to our case as follows. Note that for a normal variety, a holomorphic line bundle is determined by its restriction to the smooth locus. Without loss of generality, we may assume \( M_{\beta} \) is a smooth variety. Let \( E \in \text{Coh}(X \times M_{\beta}) \) be the universal family, then \( [E] \in K(X \times M_{\beta}) \) lies in the subgroup \( K^{\geq 3}(X \times M_{\beta}) \) generated by sheaves with codimension \( \geq 3 \). A classical result of Grothendieck gives

\[
[R \text{Hom}(\mathcal{E}, \mathcal{E})] = [\mathcal{E}^{\vee}] \otimes [\mathcal{E}] \in K^{\geq 6}(X \times M_{\beta}).
\]
As $X$ is a 4-fold, we have
\[ [\mathbf{R}\pi_{M_\beta,*}(\mathbf{R}\mathcal{H}\text{om}(\mathcal{E}, \mathcal{E}))] \in K^{2,2}(M_\beta), \]
where $\pi_{M_\beta} : X \times M_\beta \to M_\beta$ is the projection. By taking determinant, we are done.

\section*{Appendix C. Software code and explicit computations of an example}

In the following, we list the computational result of an example (the case of $l_1 = 8$, $l_2 = 6$, $l_3 = -16$) for Conjecture \[ \text{4.10} \] and Theorem \[ \text{4.12} \] with the help of software 'Mathematica'.

\textbf{Software code of Mathematica.}

\begin{align*}
\text{DT}_4([C]) &= \lambda_1^{-1-l_1} \lambda_2^{-1-l_2} (-\lambda_1 - \lambda_2)^{-1-l_3}, \\
A &= \text{Sum}[	ext{Mod}[(k - l_1 + 1), 2] (-\lambda_1^{-2l_1-2})(\lambda_2^{-2l_2-2})(\lambda_1 + \lambda_2)^{-2l_3-2}, \\
&\quad \text{Residue}[(h^k)(-\lambda_1 + h)(\lambda_2 + h)^{k+l_2}(-\lambda_1 - \lambda_2 + h)^{k+l_3}(-\lambda_1 + \lambda_2 + h)^{k-l_2-k}, \\
&\quad (-2\lambda_1 - \lambda_2 + h)^{k-l_2-k}, \{h, 0\}], \{k, l_1\}], \\
B &= \text{Sum}[	ext{Mod}[(k - l_2 + 1), 2] (-\lambda_1^{-2l_1-2})(\lambda_2^{-2l_2-2})(\lambda_1 + \lambda_2)^{-2l_3-2}, \\
&\quad \text{Residue}[(h^k)(-\lambda_2 + h)(\lambda_1 + h)^{k+l_1}(-\lambda_1 - \lambda_2 + h)^{k+l_3}(-\lambda_2 + \lambda_1 + h)^{k-l_1-k}, \\
&\quad (-2\lambda_2 - \lambda_1 + h)^{k-l_1-k}, \{h, 0\}], \{k, l_2\}], \\
\text{DT}_4(2[C]) &= A + B, \\
\text{GW}_{0,2}[C] &= -\frac{1}{8}(\lambda_1^{-2l_1-1} - 1)(\lambda_2^{-2l_2-1})(\lambda_1 + \lambda_2)^{-2l_3-3}, \\
&\quad \left( (\text{Sum}[-1^{i+1}(l_i - (i - 1)^2), \{l_i, 1, 1\}]) \lambda_1^{-2}(\lambda_1 + \lambda_2)^2 + \\
&\quad (\text{Sum}[-1^{i-1}(l_i - (i - 1)^2), \{l_i, 1, 1\}]) \lambda_2^{-2}(\lambda_1 + \lambda_2)^2 + \\
&\quad (\text{Sum}[-1^{i-1}(-l_i - (i - 1)^2), \{l_i, 1, 1\}]) l_1 l_2 \lambda_1^{-1} \lambda_2^{-1}(\lambda_1 + \lambda_2)^2 - l_2 l_3 \lambda_1^{-1}(\lambda_1 + \lambda_2) - l_1 l_3 \lambda^{-1}(\lambda_1 + \lambda_2) \right). \end{align*}

\textbf{An explicit example.} For $l_1 = 8$, $l_2 = 6$, $l_3 = -16$, we have
\begin{align*}
\text{DT}_4([C]) &= \frac{(-\lambda_1 - \lambda_2)^{15}}{\lambda_2^{16}}, \\
\text{DT}_4(2[C]) &= \frac{(-\lambda_1 - \lambda_2)^{15}}{\lambda_2^{16}} (21\lambda_1^{18} + 480\lambda_1^{17}\lambda_2 + 5012\lambda_1^{16}\lambda_2^2 + 31776\lambda_1^{15}\lambda_2^3 + 137460\lambda_1^{14}\lambda_2^4 + 432208\lambda_1^{13}\lambda_2^5 + \\
&\quad 1026480\lambda_1^{12}\lambda_2^6 + 1869676\lambda_1^{11}\lambda_2^7 + 2726437\lambda_1^{10}\lambda_2^8 + 3132120\lambda_1^9\lambda_2^9 + 2845128\lambda_1^8\lambda_2^{10} + 2057120\lambda_1^7\lambda_2^{11} + \\
&\quad 1171716\lambda_1^6\lambda_2^{12} + 518448\lambda_1^5\lambda_2^{13} + 174160\lambda_1^4\lambda_2^{14} + 42816\lambda_1^3\lambda_2^{15} + 7245\lambda_1^2\lambda_2^{16} + 752\lambda_1\lambda_2^{17} + 36\lambda_2^{18}), \\
\text{GW}_{0,2}[C] &= \frac{(-\lambda_1 - \lambda_2)^{23}}{8\lambda_2^{16}} (21\lambda_1^4 + 186\lambda_1^3\lambda_2 + 497\lambda_1^2\lambda_2^2 + 248\lambda_1\lambda_2^3 + 36\lambda_2^4), \\
\frac{1}{\text{DT}_4([C])} + \text{DT}_4(2[C]) - \text{GW}_{0,2}[C] &= 0. \end{align*}

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