Explicit Gravitational Radiation in Hyperbolic Systems for Numerical Relativity

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A method for studying the causal structure of space-time evolution systems is presented. This method, based on a generalization of the well-known Riemann problem, provides intrinsic results which can be interpreted from the geometrical point of view. A one-parameter family of hyperbolic evolution systems is presented and the physical relevance of their characteristic speeds and eigenfields is discussed. The two degrees of freedom corresponding to gravitational radiation are identified in an intrinsic way, independent of the space coordinate system. A covariant interpretation of these degrees of freedom is used to solve the gravitational radiation degrees of freedom of the field equations from the second order field equations.

1. INTRODUCTION

The structure of Einstein field equations has deserved great interest since the very beginning of General Relativity. It was early noticed that, by rearranging the order or partial derivatives, the principal part of the four dimensional Ricci tensor could be written as a sort of generalized wave equation:

\[ 2 \, R_{\mu \nu} = -\Box g_{\mu \nu} + \partial_\mu \Gamma_\nu + \partial_\nu \Gamma_\mu + \ldots \] (1)

where the box stands for the d’Alembert operator acting on functions and we have written for short

\[ \Gamma^\mu = g^{\alpha \sigma} \Gamma^\mu_{\rho \sigma} = -\Box x^\mu . \] (2)

This opened the way to the use of harmonic spacetime coordinates (\( \Box x^\mu = 0 \)) in order to obtain an hyperbolic evolution system.(See Ref. [10] for a detailed comparison of the ”old” and ”new” hyperbolic systems).

Suddenly, the problem of having too few hyperbolic formalisms for Numerical Relativity turned into the opposite problem of having too many of them. Some of the works considered even multiparameter families of hyperbolic systems [14,10,26]. Faced with the problem of choosing arbitrary space coordinates on every \( t = constant \) slice. This allows for instance to use normal coordinates ensuring hyperbolicity while keeping the freedom of choosing arbitrary space coordinates with the evolution equation (5), leading in each case to a new brand of hyperbolic systems.

\[ (\partial_t - \mathcal{L}_\beta) \gamma_{ij} = -2 \alpha K_{ij} \] (4)
\[ (\partial_t - \mathcal{L}_\beta)K_{ij} = -\nabla_i \alpha_j + \alpha \left[ R_{ij} - 2K^2_{ij} + \text{tr}K K_{ij} \right] \] (5)

where \( \mathcal{L} \) stands for the Lie derivative (we restrict ourselves to the vacuum case for simplicity). The remaining four equations could instead be expressed as constraints:

\[ (3) \, R - \text{tr}(K^2) + (\text{tr}K)^2 = 0 \] (6)
\[ \nabla_k K^k_i - \partial_i (\text{tr}K) = 0 . \] (7)

This opened the door to a new way of obtaining hyperbolic evolution systems. The key point was to use in one or another way the momentum constraint (7) to ensure hyperbolicity while keeping the freedom of choosing characteristic speeds (\( \beta^i = 0 \)) without affecting the mathematical structure of the evolution system. (See Ref. [11] for a detailed comparison of the ”old” and ”new” hyperbolic systems).

These findings came at the right moment for people working on Numerical Relativity with a view on the gravitational waves detector projects starting by the turn of the century. Following the wake of these first works, many groups found their own way of combining the momentum constraint with the evolution equation (5), leading in each case to a new brand of hyperbolic systems.

There are many answers, of course, but we can get a hint from the physical point of view when we realize that all but one of the resulting characteristic speeds can be altered when playing with these arbitrary parameters (see the work of the Cornell group [26] for a fairly complete analysis). The only one that has an intrinsic value (light speed) is associated with the two (transverse traceless) degrees of freedom of gravitational waves. The evident question now is whether or not we can directly relate the corresponding explicit eigenfields with gravitational radiation propagation. We will see below that a positive answer actually determines one of the parameters, introduced in the Cornell paper [26]. We claim that the requirement of having explicit eigenvectors describing the gravitational waves degrees of freedom in an intrinsic way (for an arbitrary choice of space coordinates) is a strong
benchmark to discriminate between different hyperbolic evolution systems.

In order to obtain our results, it has been crucial the use of an intrinsic method in order to find the characteristic speeds and the corresponding eigenfields. This method, based on a General Relativistic analogue of the well known Riemann problem of Computational Fluid Dynamics (CFD), is completely independent of the space coordinate system used at any given \( t = \text{constant} \) hypersurface. It is presented in Section 3 and then applied in Section 4 to a new simple one-parameter family of hyperbolic systems in order to illustrate our arguments.

II. A FIRST ORDER EVOLUTION SYSTEM

The evolution system (11) is of first order in time but second order in space. To obtain a system which is also of first order in space, we will follow the standard procedure by considering the first space derivatives of the metric coefficients \( \gamma_{ij} \) as new independent variables:

\[
D_{kij} = 1/2 \partial_k \gamma_{ij} .
\]

The evolution equation of these new variables \( D_{kij} \) can then be taken to be:

\[
\partial_t D_{kij} + \partial_k (\alpha \ K_{ij}) = 0
\]

where we are using for simplicity normal coordinates, so that \( \beta^i = 0 \). The relationship (5) can then be understood as a first integral of the extended system (3).

In order to complete this system, however, we need to provide evolution equations for the lapse function \( \alpha \) and its first derivatives. It can be done in a straightforward way as follows:

\[
\partial_t \ln \alpha = -\alpha \ Q
\]

\[
A_i = \partial_i \ln \alpha
\]

\[
\partial_t A_i + \partial_i (\alpha \ Q) = 0
\]

where, again, the equation (8) must be considered as a first integral of (11). Notice that one could instead prescribe a more general evolution equation for \( A_i \):

\[
\partial_t A_i + \partial_i (\alpha \ Q) + \mu \left( \nabla_k K^{ki} - \partial_t tr K \right) = 0
\]

where \( \mu \) is an arbitrary parameter. In that case, (11) would be a first integral of (11) if and only if the momentum constraint (3) is satisfied.

The complete first order evolution system can then be expressed in a balance law form, namely:

\[
\partial_t u + \partial_k F^k = S ,
\]

where \( u \) is the array of independent variables (the metric coefficients and their first derivatives), and both the Fluxes \( F^k \) vector array and the Sources \( S \) scalar array depend of \( u \).

Note however that the actual form of the fluxes associated to \( K_{ij} \) in eq. (3), will depend on the expression of the three-dimensional Ricci tensor (3) \( R_{ij} \) in terms of the metric derivatives (see for instance Ref. 26 for a discussion of this point). We will take in this paper the original expression, namely:

\[
(3) R_{ij} = \partial_k \Gamma^{k}_{ij} - \partial_j \Gamma^{k}_{ki} + \Gamma^{k}_{kr} \Gamma^{r}_{ij} - \Gamma^{k}_{ri} \Gamma^{r}_{kj}
\]

This is in contrast with the actual choice in most of the hyperbolic formalisms published up to now, where a three-dimensional version of the decomposition (3) is used instead of (15) in order to mimic the properties of generalized wave equations.

III. THE GENERAL RELATIVISTIC RIEMANN PROBLEM

The balance law form (14) of the evolution system is familiar from the Computational Fluid Dynamics (CFD) domain, where we can consider (14) in the sense of distributions, so that discontinuous solutions for \( u \) are allowed (weak solutions). The discontinuity surface \( \Sigma \) can be given by:

\[
\phi(x^i, t) = \text{constant}
\]

\[
\partial_t \phi + v^i \partial_i \phi = 0 ,
\]

where \( v^i \) is the velocity of the \( x^i = \text{constant} \) points of the surface \( \Sigma \). The consistency of these weak solutions with (14) requires the exact cancellation of the Dirac \( \delta \) terms arising where the derivatives of discontinuous functions are performed. One gets then the well known Rankine-Hugoniot conditions:

\[
v[u] = n_k [F^k]
\]

where the square bracket stands for the jump of the quantities across \( \Sigma \), \( n_k \) is the unit normal to \( \Sigma \) and \( v \equiv v^i n_i \) is the propagation speed of the discontinuity front. The eigenvalue problem (16) is usually known in CFD as the “Riemann problem”.

In General Relativity, we are familiar with ‘matched’ metrics having discontinuous first order derivatives and other less common weak solutions, like colliding waves. A peculiar feature in General Relativity is however that the non-linear terms are limited to the Sources \( S \) in (14), which can not originate \( \delta \) terms and do not appear then in (16). This means that the General Relativistic Riemann problem is a linear one and can be solved explicitly. This implies also that the eigenvalues \( v \) of (14) will coincide with the characteristic speeds, contrary to what happens in CFD, where one finds a much richer structure so that, in addition to plain discontinuities, one can get also shocks and rarefaction waves propagating at non-characteristic speeds. We will take advantage of these simplifications in what follows.
Let us then write down the Riemann problem corresponding to our evolution system \( \text{(4, 5, 9, 13)} \), taking from granted that the metric coefficients are continuous so that the only jumps can be on the first derivatives quantities, namely
\[
\begin{align*}
  v[K_{ij}] &= \alpha n_k \lambda_{ij}^k \\
  v[D_{ki}^j] &= \alpha n_k [K^j_{ik}] \\
  v[A_i] &= \alpha n_k [Q \delta^i_k + \mu (K^k_i - tr K \delta^i_k)]
\end{align*}
\] (17-19)
where we have written, allowing for \( \text{(15)} \),
\[
\begin{align*}
  \lambda_{ij}^k &= -\Gamma^k_{ij} + 1/2 \delta^k_i (A_j + D_j) + 1/2 \delta^k_j (A_i + D_i) \\
  D_k &= \gamma^{ij} D_{kj}
\end{align*}
\] (20)
and \( \Gamma^k_{ij} \) Figure 1.

Let us remark that the Riemann problem \( \text{(17-19)} \) is stated for a generic orientation of the unit normal \( n_k \), in a way that is independent of the space coordinate system. In this sense, it provides an intrinsic method to obtain the characteristic speeds and the corresponding eigenfields in terms of geometrical objects related with both the \( t = \text{constant} \) hypersurface and the characteristic surface \( \Sigma \).

### IV. GETTING AN STRONGLY HYPERBOLIC SYSTEM

We will now solve the eigenvalue problem \( \text{(17-19)} \) in order to display the causal structure of the evolution system \( \text{(4, 5, 9, 13)} \). We remember that the system will be strongly hyperbolic if and only if all the eigenvalues \( v \) are real and if the set of eigenfields is complete, in the sense that it spans all our variable space. It follows easily from \( \text{(18)} \) that
\[
v[D_{\perp ij}] = 0
\] (21)
where the symbol \( \perp \) replacing one index means that we are taking the corresponding components tangent to \( \Sigma \) (orthogonal to \( n_k \)). The remaining degrees of freedom in \( \text{(18)} \) can then be written as
\[
v[D^a_{ji}] = \alpha [K_{ij}]
\] (22)
where we have noted for short \( D^a_{ji} \equiv n_k D^k_{ij} \).

Let us now analyze the “mixed” components of \( \text{(17)} \)
\[
\begin{align*}
  v[K^a_{\perp}] &= \alpha [-D^a_{\perp} n^a + 1/2 A_{\perp}] \\
  v[A_{\perp}] &= \alpha \mu [K_{\perp}]
\end{align*}
\] (23-24)
so that, allowing for \( \text{(23)} \), one gets the following set of eigenvectors
\[
\sqrt{\mu / 2} K_{\perp} \pm (1/2 A_{\perp} - D_{\perp}^n n^a)
\] (25)
with characteristic speeds \( v = \pm \alpha \sqrt{\mu / 2} \), respectively. This means that our evolution system will be (strongly) hyperbolic only if \( \mu > 0 \). This surprising parameter-dependence of the characteristic speeds can be understood if we recall that physical solutions must verify the momentum constraint \( \text{(5)} \). For a discontinuous solution one gets
\[
[K^a_{\perp}] = n_i [tr K]
\] (26)
so that the first term of \( \text{(23)} \) comes from the transverse part of \( \text{(26)} \). This means that the characteristic cones spanned by \( \text{(23)} \) are just mathematical artifacts in order to get an hyperbolic system: they can not propagate physical information. We will take \( \mu = 2 \) to impose this arbitrary characteristic speed to coincide with light speed.

Let us focus now on the purely transverse components in \( \text{(17)} \). We get
\[
\begin{align*}
  v[K_{\perp}] &= \alpha [-\Gamma_{\perp}^a - \alpha n^a (D^a_{\perp} - 2 D_{\perp}^a n^a)] \\
  v[D^a_{\perp}] &= \alpha [K_{\perp}]
\end{align*}
\] (27-28)
so that, allowing again for \( \text{(21)} \), one gets the following eigenfields
\[
K_{\perp} \pm (D^a_{\perp} - 2 D_{\perp}^a n^a)
\] (29)
with characteristic speed \( v = \pm \alpha \), independently of the choice of \( \mu \). Notice that the longitudinal component of \( \text{(29)} \) can be written as
\[
(\gamma^{ij} - n^i n^j) [K_{ij}] = 0
\] (30)
so that, again, the trace component in \( \text{(29)} \) can not describe the propagation of any physical information. See for instance Ref. \( \text{(24)} \) to verify that the corresponding characteristic speed can be also modified at will by using the energy constraint in a suitable way. We can conclude that only the traceless part of the transverse components \( \text{(29)} \) can actually propagate physical information. These two degrees of freedom are then good candidates to describe intrinsic features, like gravitational waves, as we will discuss further in the next section.

The remaining degrees of freedom in the original system \( \text{(17-19)} \) are given by:
\[
\begin{align*}
  v[tr K] &= \alpha [A^n + 2 (D^n - D_k^k n^a)] \\
  v[A^n] &= \alpha [2 (K^{nn} - tr K) + Q] \\
  v[\gamma^{ij} D^a_{ij}] &= \alpha [tr K]
\end{align*}
\] (31-33)
where now we must specify the gauge-related function \( Q \) in terms of other quantities. We have postponed the gauge specification up to that point to emphasize that our previous analysis is gauge independent. Now we will take the simple prescription
\[
Q = f tr K
\] (34)
where \( f \) can be again an arbitrary parameter. We get then, allowing once more for \( \text{(29)} \):
\[ v[A^n + 2(D^n - D_k^k n) - f D^n] = 0 \] (35)
and also that the following gauge-dependent combinations
\[ \sqrt{f} trK \pm (A^n + 2D^n - 2D_k^k n) \] (36)
are eigenfields with characteristic speeds \( v = \pm \alpha \sqrt{f} \), so that the system can be strongly hyperbolic only if \( f > 0 \). We can take for instance \( f = 1 \) if we insist in having light speed as a characteristic speed also here.

The analysis is complete now, because for \( \mu = 2, f > 0 \), the eigenfields (21), (25), (29), (35) and (36) span the whole space of variables. It follows that the resulting evolution system is strongly hyperbolic, no matter of the particular orientation \( n_k \) of the characteristic surface \( \Sigma \).

\[ v[A^n + 2(D^n - D_k^k n) - f D^n] = 0 \] (35)

V. GEOMETRICAL AND PHYSICAL INTERPRETATION

Let us now decompose the line element of the three-dimensional \( t = \text{constant} \) hypersurfaces in the following way \( (2+1 \) decomposition):
\[ \gamma_{ij} dx^i dx^j = -N^2 dz^2 + \sigma_{ab} (dx^a + \lambda^a dz) (dx^b + \lambda^b dz) \quad a, b = 1, 2 \] (37)
where the coordinates \( x^a \) span the characteristic surface \( \Sigma \), \( \sigma_{ab} = \gamma_{ab} \) is the induced metric on \( \Sigma \) and \( z \) is a transverse coordinate so that the unit normal to \( \Sigma \) can be expressed as
\[ n_i = N \delta_i^z \]. (38)
The eigenfields (29) can then be written as
\[ K_{ab} \pm N (D^z_{ab} - D_{ab} z - D_{ba} z) \] (39)
and a straightforward calculation shows that the extrinsic curvature of the two-dimensional surface \( \Sigma \), namely
\[ \kappa_{ab} \equiv 1/(2N) [\partial_z \sigma_{ab} - \mathcal{L}_\lambda(\sigma_{ab})] \] (40)
is given by
\[ \kappa_{ab} = N (D^z_{ab} - D_{ab} z - D_{ba} z) \] (41)
so that the eigenfields (29) can be expressed as
\[ K_{ab} \pm \kappa_{ab} \] (42)
which shows that they are two-dimensional tensors associated to the characteristic surface \( \Sigma \) in an intrinsic, coordinate-independent way.

Looking for a physical interpretations, let us note that the traceless part of (12) contains the shear degrees of freedom as measured by a two-dimensional array of observers sitting on the surface \( \Sigma \). This is precisely the kind of effect one expects from gravitational waves with wavefront \( \Sigma \). Our results can then be interpreted as an extension of the standard description of the effect of a gravitational wave on an array of freely falling (geodesic, \( \alpha = \text{constant} \)) observers. This is an extension in the sense that our gauge conditions (34) apply to more general kinds of observers (like static ones, where both \( Q \) and \( trK \) vanish separately so that (34) holds for any value of \( f \)). This can be relevant for modelling Earth-based gravitational wave detectors, which are certainly not in free fall.

Notice, however, that the eigenfields expression (29) is sensitive to the ordering ambiguity of space derivatives when passing from (1) to a first order system (see Ref. [24] for details). The tensor character of the eigenfields is completely lost if one starts with a three-dimensional version of (1) instead of (14). The intrinsic information resides of course in the system (14), where the divergence of the fluxes is taken, but the local covariance of the fluxes themselves is lost if we make a choice different of (14). We claim that our results provide a criterion for solving the ordering ambiguity: the eigenfields describing the gravitational radiation degrees must have an intrinsic geometrical meaning independent of the space coordinate system. This opens the way to a coordinate-independent local description of gravitational wave detectors at Earth-based laboratories.

VI. CONCLUSIONS AND OUTLOOK

We have started this paper by proposing a new simple method, based on the General Relativistic Riemann problem, in order to analyze the causal structure of any first order flux-conservative evolution system in an intrinsic way, independent of the space coordinates. To compare with previous results [12], we can look at the characteristic speeds and see how we get now light speed as \( v = \pm \alpha \) (length per unit coordinate time), whereas in coordinate-dependent analysis, the speed along the z axis is given by \( v = \pm \alpha \sqrt{z} \) (coordinate displacement per unit coordinate time).

The use of this intrinsic method has allowed ourselves to explain the appearance of arbitrary parameter-dependent characteristic speeds [28] as a direct consequence of the mixing of evolution and constraint equations. As we emphasized in Section IV, only two degrees of freedom are intrinsically related with causal propagation of physical information. Hyperbolic formalisms, which can only consist of characteristic cones and lines, apply instead to an extended, unconstrained, space of solutions. The subspace of physical solutions is recovered by restricting the initial conditions to those that satisfy the constraints. The extra degrees of freedom are then to be regarded as mathematical artifacts devised to ensure hyperbolicity and their corresponding (arbitrary) characteristic speeds are irrelevant from the physical point of view.
Regarding the two remaining degrees of freedom, that can consistently be interpreted as describing gravitational radiation, we have provided an intrinsic geometrical interpretation of the corresponding eigenfields in terms of the shear of the congruence of eulerian (laboratory) observers combined with the shear of the front-wave surfaces Σ. The requirement that the eigenfields admit this geometrical interpretation allows one to solve the ordering ambiguity for space derivatives when passing from the second order form of Einstein evolution equations to a first order system.

The fulfillment of this requirement must be regarded as an important benchmark to check any hyperbolic formalism. This is actually a strong requirement: none of the hyperbolic formalisms proposed previously by the Palma group verifies it. It is not difficult, however, to modify the existing formalisms to comply with it [27].

This “explicit gravitational radiation” requirement is mandatory if we want to provide a local description of gravitational radiation in order to model instance the process of detection of these waves by a given array of observers. In this sense, our results generalize the well-known textbook results for freely falling observers, based in the geodesic deviation equations. In particular, our results can be useful to model Earth-based laboratories, where the Earth gravitational field can be approximated by the well known Schwarzchild solution and the other effects, including the gravitational wave itself, can be included using perturbation theory.

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