Travelling wave solution of Dodd-Bullough-Mikhailov equation: a comparative study between Generalized Kudryashov and improved F-expansion methods

Naima Islam, Kamruzzaman Khan and Md Hamidul Islam
1 Department of Mathematics, Pabna University of Science and Technology, Pabna-6600, Bangladesh
2 Department of Mathematics and Physics, North South University, Dhaka 1229, Bangladesh
E-mail: hamidul.islam@northsouth.edu

Abstract
We investigate the efficiency of Generalized Kudryashov and improved F-expansion methods in solving nonlinear partial differential equations. The Dodd-Bullough-Mikhailov equation is considered to implement these methods. Both methods allow us to construct a number of travelling wave solutions of the governing equation. However, the Generalized Kudryashov method is found more direct, effective and requires less tedious symbolic computations compared to the improved F-expansion method. Our analysis also reveal that the basic version of either of the methods could be effective enough to acquire the fundamental wave solutions of the governing equation.

1. Introduction

The partial differential equation (PDE) is a useful tool for describing the phenomena that arise in mathematical physics and engineering. For example, heat flow, wave propagation, dispersion of chemically reactive material, fluid dynamics, quantum mechanics, electricity, plasma physics, propagation of shallow water waves, to name a few. Although linear partial differentials equations are sometimes used to model these problems, nonlinear equations are the best fit for these problems. Numerical solution techniques are often used to solve these nonlinear equations. However, there has been an ever-increasing interest from scientists and engineers in the analytical techniques for addressing these nonlinear problems as that is perhaps the most challenging, and promising area of modern mathematics.

In the past two decades, the nonlinear partial differential equations (NLPDEs) have undergone remarkable developments. A large number of methods have been developed to examine NLPDEs. For example, F-expansion method [1, 2], tanh–function method [3, 4], the Kudrayshov method [5, 6], the modified simple equation method [7–9], the Jacobi elliptic–function method [10], the Bernoulli sub-ODE method [11, 12], the Exp–function method [13, 14], the multiple Exp–function method [15, 16], the (G'/G)–expansion method [17, 18], the variational iteration method [19], the homotopy perturbation method [20], the exp (−φ(ξ))–expansion method [21], the homotopy analysis method [22, 23], the extended tanh–method [24–26], the enhanced (G'/G)–expansion method [27–29], the Generalized Kudrayshov method [30–32], and the improved F–expansion method [33–36], to name a few. Few other methods have been proposed in recent years, for example, Shehu transform [37], Caputo fractional partial derivatives [38] and local fractional homotopy analysis method [39]. These methods allow to construct the travelling wave solutions of NLPDEs, which happen to be the remarkable achievements in the field of applied sciences and hence has drawn attraction of many researchers around the world.

However, because of a large number of methods, there is always a daunting problem of finding a suitable method for an equation under consideration. The purpose of this study is to find a suitable method for solving the Dodd-Bullough-Mikhailov equation (DBM) equation, which may serve as a guideline to look for a suitable method for other nonlinear equations. The DBM equation is a well-known nonlinear partial differential
equation introduced by Roger Dodd, Robin Bullough and Alexander Mikhailov [40], which is a modified version of the Zhiber–Shabat equation

\[ u_{xt} + pe^{u} + re^{-u} + qe^{-2u} = 0, \]

obtained by setting \( r = 0 \). The DBM equation is closely connected with SL(3, \( R \)) sigma-model describing the field triplet in the three-dimensional unimodular affine space [40, 41]. Further, it appears in several other scientific applications, such as fluid dynamics, nonlinear optics, quantum field theory, electromagnetic waves and solid state physics. We solve the governing equation by using the Generalized Kudryashov and improved F-expansion methods.

The remaining part of the paper is organized as follows. Section 2 introduces the working principle of the Generalized Kudryashov and improved F-expansion methods. Section 3 represents the solutions of the DBM equation. Section 4 summarizes the obtained results and compares the solution methods. Finally, section 5 concludes this work with the possible future research direction.

2. Methodology

A general nonlinear partial differential equation (NLEE) is written as

\[ P(u, u_t, u_{xx}, u_{xxx}, \ldots) = 0, \quad x \in \mathbb{R}, \quad t \geq 0, \quad (2.1) \]

where \( u = u(x, t) \) is an unknown function and \( P \) is a polynomial of \( u \) and its various partial derivatives, in which the highest-order derivatives and nonlinear terms are involved. In order to construct the travelling wave solutions, we introduce the transformation \( u(x, t) = u(\xi), \) where \( \xi = x - \omega t \). This transformation converts equation (2.1) into an ordinary differential equation of the form

\[ E(u, u', u''', \ldots) = 0. \quad (2.2) \]

Here \( \omega \) represents the speed of the travelling wave. The sign of \( \omega \) determines the direction of the wave, that is, whether the wave travels in a positive or negative direction.

In the following, we briefly describe the main structures of the solution methods. We begin by introducing the well-known Riccati differential equation, which is often written as

\[ \frac{dQ(\xi)}{d\xi} = k + Q^2(\xi). \quad (2.3) \]

Depending on the value of the parameter \( k \), the equation (2.3) has three different types of solutions, which are as follows:

- When \( k < 0 \), the governing equation possesses two equilibria, a stable node and unstable saddle point, which are born through saddle-node bifurcation. In this case, the solutions are

\[ Q_1 = -\sqrt{-k} \tanh(\sqrt{-k} \xi), \quad (2.4) \]
\[ Q_2 = -\sqrt{-k} \coth(\sqrt{-k} \xi). \quad (2.5) \]

- When \( k \) changes from negative to zero, the equilibria that are born through saddle-node bifurcation get closer and finally collides when \( k = 0 \). The origin, in this case, is the only equilibrium which is semi-stable, and the solution has the following form

\[ Q_3 = -\frac{1}{\xi}. \quad (2.6) \]

- Finally, when \( k > 0 \), the governing equation (2.3) has no equilibrium and hence, has the following two periodic solutions

\[ Q_4 = \sqrt{k} \tan(\sqrt{k} \xi), \quad (2.7) \]
\[ Q_5 = -\sqrt{k} \cot(\sqrt{k} \xi). \quad (2.8) \]

The integrating constant is assumed zero in all three cases above. The working steps of the Generalized Kudryashov and improved F-expansion methods are described in the following two subsections.

2.1. The generalized Kudryashov method [30–32]

The main steps of Generalized Kudryashov method are as follows:
Step-1 Let us consider that the trial function of analytical solution for equation (2.2) as follows

\[ u(\xi) = \sum_{i=0}^{N} a_i Q^i(\xi) + \sum_{j=0}^{M} b_j Q^j(\xi), \]  

(2.1.1)

where \( a_i (i = 0, 1, 2, ..., N) \) and \( b_j (j = 0, 1, 2, ..., M) \) are arbitrary constants to be determined, and \( Q = Q(\xi) \) satisfies the Riccati equation (2.3).

Step-2 We determine the positive integers \( N \) and \( M \) appearing in equation (2.1.1) by considering the homogeneous balance between the highest order derivatives and nonlinear terms in equation (2.2).

Step-3 Substituting equation (2.1.1) in equation (2.2) together with the value of \( N \) and \( M \) obtained in step 2, we get polynomials in \( Q^{-1} \), \( (i, j = 0, 1, 2, ...) \). We then set each coefficient of the resulting polynomial to zero to obtain a system of algebraic equations. The system of algebraic equations is then solved by using Maple to obtain the unknown parameters \( a_i (i = 0, 1, 2, ..., N), b_j (j = 0, 1, 2, ..., M) \), and \( \omega \). We finally substitute the obtained value in equation (2.1.1) to construct the exact travelling wave solutions of equation (2.1).

2.2. The improved \( F \)-expansion method [33–36]

The main steps involved in \( F \)-expansion method are as follows:

Step-1 Let us assume the traveling wave solution of equation (2.2) can be written as follows

\[ u(\xi) = \alpha(\xi) + F(\xi), \]  

(2.2.1)

where \( \alpha = \alpha(m + F(\xi))^{-1} \), \( \beta(i = 1, 2, 3, ..., N) \), \( \omega \) and \( m \) are arbitrary constants to be determined, and \( F = F(\xi) \) satisfies Riccati equation

\[ F'(\xi) = k + F^2(\xi), \]  

(2.2.2)

Step-2 The positive integer \( N \) appearing in equation (2.2.1) can be obtained by taking the homogeneous balance between the highest order derivatives and nonlinear terms in equation (2.2).

Step-3 Substituting equations (2.2.1) and (2.2.2) in equation (2.2) along with the value of \( N \) obtained in step 2, we get a polynomial in \( F(\xi) \). We then set the coefficients of the resulted polynomial to zero to obtain a system of algebraic equations. The system of algebraic equations is then solved by using Maple to determine the unknown parameters \( \alpha_N, \beta_N, M \) and \( \omega \). Finally, setting the obtained values in equation (2.2.1), we obtain the exact travelling wave solutions to equation (2.1).

3. Application of the methods

In this section, we apply the idea of the considered methods to solve the DBM equation. As mentioned in section 1, the DBM equation is given by

\[ u_{tt} + p e^u + q e^{-2u} = 0. \]

Using the transformation \( v = e^u \), where \( v(x, t) = V(\xi) \), and \( \xi = x - \omega t \), we can recast the above equation as follows

\[ \omega(V')^2 - \omega VV'' + p V^3 + q = 0. \]

(3.1.1)

3.1. The implementation of the generalized Kudryashov method

We apply the homogeneous balance between \( V^3 \) and \( VV'' \) as described in Step-2 in section 2.1, which gives \( N = M + 2 \). Setting \( M = 1 \), we obtain \( N = 3 \).

Hence, equation (2.1.1) becomes,

\[ V(\xi) = \frac{a_0 + a_i Q + a_2 Q^2 + a_3 Q^3}{b_0 + b_1 Q}. \]

(3.1.2)

Now substituting equation (3.1.2) into equation (3.1.1), we obtain a polynomial in \( Q(\xi) \). Equating the coefficients of the powers of \( Q(\xi) \) to zero, we obtain a system of algebraic equations. We use Maple to solve this system for \( p, q, a_0, a_1, a_2, a_3, b_0, b_1 \), and \( \omega \), which gives the following set of solutions:
Set-1

\[ p = \frac{2\omega b_1}{a_3}, \quad q = \frac{16\omega^2 k^3}{27b_1^2}, \quad a_0 = \frac{k a_3 b_0}{a_1}, \quad a_1 = \frac{k a_3}{b_1}, \quad a_2 \]

and

Set-2

\[ p = \frac{2\omega b_1}{a_3}, \quad q = 0, \quad a_0 = \frac{k a_3 b_0}{b_1}, \quad a_1 = k a_3, \quad a_2 = \frac{a_3 b_0}{b_1}, \]

\[ \times a_3 = a_3, \quad b_0 = b_0, \quad b_1 = b_1. \]

Therefore, according to \( k < 0, k > 0, \) and \( k = 0, \) as described in section 2, we obtain the following families of solutions.

When \( k < 0, \) we get the following hyperbolic trigonometric solutions:

Family 1

\[ u_1(x, t) = \ln \left( \frac{k a_3}{b_1} \left( \frac{1}{3} - \tanh (\sqrt{k} (x - \omega t)) \right) \right), \]

\[ u_2(x, t) = \ln \left( \frac{k a_3}{b_1} \left( \frac{1}{3} - \coth (\sqrt{k} (x - \omega t)) \right) \right). \]

Family 2

\[ u_3(x, t) = \ln \left( \frac{k a_3}{b_1} (1 - \tanh (\sqrt{k} (x - \omega t))) \right), \]

\[ u_4(x, t) = \ln \left( \frac{k a_3}{b_1} (1 - \coth (\sqrt{k} (x - \omega t))) \right). \]

When \( k > 0, \) we get the following trigonometric solutions:

Family 3

\[ u_5(x, t) = \ln \left( \frac{k a_3}{b_1} \left( \frac{1}{3} + \tan (\sqrt{k} (x - \omega t)) \right) \right), \]

\[ u_6(x, t) = \ln \left( \frac{m a_3}{b_1} \left( \frac{1}{3} + \cot (\sqrt{k} (x - \omega t)) \right) \right). \]

Family 4

\[ u_7(x, t) = \ln \left( \frac{k a_3}{b_1} (1 + \tan (\sqrt{k} (x - \omega t))) \right), \]

\[ u_8(x, t) = \ln \left( \frac{k a_3}{b_1} (1 + \cot (\sqrt{k} (x - \omega t))) \right). \]

Finally, when \( k = 0, \) we get the following solution,

Family 5

\[ u_9(x, t) = \ln \left( \frac{a_3}{b_1 (x - \omega t)^2} \right). \]

3.2. The implementation of the improved \( F \)-expansion method

To apply the \( F \)-expansion method on equation (3.1.1), we first apply the homogeneous balance between \( V^3 \) and \( VV'' \), as described in Step-2 in section 2.2, which gives \( N = 2. \)

Hence, for \( N = 2 \) the equation (2.2.1) reduces to

\[ V(\xi) = a_0 + a_1 (m + F(\xi)) + a_2 (m + F(\xi))^2 + \beta_1 (m + F(\xi))^{-1} + \beta_2 (m + F(\xi))^{-2} \quad (3.2.1) \]

Now substituting equation (3.2.1) into equation (3.1.1), we get a polynomial in \( F(\xi) \). Equating the coefficient of the same power of \( F(\xi) \) to zero, we obtain a system of algebraic equations. With the help of Maple,
we have solved this system of algebraic equations and found the following set of values for
\( p, q, \alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2, \omega, \) and \( m \):

**Set-1**

\[
p = \frac{2\omega(k + m^2)}{\alpha_0}, \quad q = 0, \quad \alpha_0 = \alpha_0, \quad \alpha_1 = 0, \quad \alpha_2 = 0, \\
\beta_1 = -2m\alpha_0, \quad \beta_2 = \alpha_0(k + m^2),
\]

**Set-2**

\[
p = -\frac{4\omega m(k + m^2)}{\beta_1}, \quad q = \frac{4\omega \beta_1^2 k^3}{27 m^2(k + m^2)^2}, \\
\alpha_0 = -\frac{1}{6} \beta_1(k + 3m^2), \quad \alpha_1 = 0, \quad \alpha_2 = 0, \quad \beta_1 = \beta_2, \\
\beta_2 = -\frac{1}{2} \frac{\beta_1(k + m^2)}{m},
\]

**Set-3**

\[
p = \frac{2\omega}{\alpha_2}, \quad q = 0, \quad \alpha_0 = \alpha_2k + \alpha_2m^2, \quad \alpha_1 = -2\alpha_2m, \\
\alpha_2 = \alpha_2, \quad \beta_1 = 0, \quad \beta_2 = 0,
\]

and

**Set-4**

\[
p = \frac{2\omega}{\alpha_2}, \quad q = \frac{16\omega \alpha_2^2 k^3}{27}, \quad \alpha_0 = \frac{1}{3} \alpha_2k + \alpha_2m^2, \\
\alpha_1 = -2\alpha_2m, \quad \alpha_2 = \alpha_2, \quad \beta_1 = 0, \quad \beta_2 = 0.
\]

According to \( k < 0 \), \( k > 0 \), and \( k = 0 \), as described in section 2, we obtain the following families of solutions.

When \( k < 0 \), we get the following hyperbolic trigonometric solutions:

**Family 1**

\[
u_1(x, t) = \ln \left( \frac{\alpha_0 k((1 - \tanh(k(x - \omega t)))}{(m + \sqrt{-k} \tanh(\sqrt{-k}(x - \omega t)))^2} \right), \\
u_2(x, t) = \ln \left( \frac{\alpha_0 k((1 - \coth(k(x - \omega t)))}{(m + \sqrt{-k} \coth(\sqrt{-k}(x - \omega t)))^2} \right).
\]

**Family 2**

\[
u_3(x, t) = \ln \left( \frac{\beta_1 k((k + 3m^2) \tanh(\sqrt{-k}(x - \omega t)) + 4m\sqrt{-k} \tanh(\sqrt{-k}(x - \omega t)) - (m^2 + 3k))}{6m(k + m^2)(m + \sqrt{-k} \tanh(\sqrt{-k}(x - \omega t)))^2} \right), \\
u_4(x, t) = \ln \left( \frac{\beta_1 k((k + 3m^2) \coth(\sqrt{-k}(x - \omega t)) + 4m\sqrt{-k} \coth(\sqrt{-k}(x - \omega t)) - (m^2 + 3k))}{6m(k + m^2)(m + \sqrt{-k} \coth(\sqrt{-k}(x - \omega t)))^2} \right).
\]

**Family 3**

\[
u_5(x, t) = \ln(\alpha_2 k(1 - \tanh(\sqrt{-k}(x - \omega t)))), \\
u_6(x, t) = \ln(\alpha_2 k(1 - \coth(\sqrt{-k}(x - \omega t))).
\]

**Family 4**

\[
u_7(x, t) = \ln \left( \frac{\alpha_2 k(\frac{1}{3} - \tanh(\sqrt{-k}(x - \omega t)))}{3} \right), \\
u_8(x, t) = \ln \left( \frac{\alpha_2 k(\frac{1}{3} - \coth(\sqrt{-k}(x - \omega t)))}{3} \right).
\]

When \( k > 0 \), we get the following trigonometric solutions:
Family 5
\[ u_9(x, t) = \ln \left( \frac{\alpha_9 k (1 + \tan^2 (\sqrt{k} (x - \omega t)))}{(-m + \sqrt{k} \tan (\sqrt{k} (x - \omega t)))^2} \right), \]
\[ u_{10}(x, t) = \ln \left( \frac{\alpha_9 k (1 + \cot^2 (\sqrt{k} (x - \omega t)))}{(-m + \sqrt{k} \cot (\sqrt{k} (x - \omega t)))^2} \right). \]

Family 6
\[ u_{11}(x, t) = \ln \left( \frac{\beta_1 k ((k + 3m^2) \tan^2 (\sqrt{k} (x - \omega t)) + 4m \sqrt{k} \tan (\sqrt{k} (x - \omega t)) + m^2 + 3k)}{6m(k + m^2)(-m + \sqrt{k} \tan (\sqrt{k} (x - \omega t)))^2} \right), \]
\[ u_{12}(x, t) = \ln \left( \frac{\beta_1 k ((k + 3m^2) \cot^2 (\sqrt{k} (x - \omega t)) - 4m \sqrt{k} \cot (\sqrt{k} (x - \omega t)) + m^2 + 3k)}{6m(k + m^2)(m + \sqrt{k} \cot (\sqrt{k} (x - \omega t)))^2} \right). \]

Family 7
\[ u_{13}(x, t) = \ln (\alpha_2 k (1 + \tan^2 (\sqrt{k} (x - \omega t)))) \]
\[ u_{14}(x, t) = \ln (\alpha_2 k (1 + \cot^2 (\sqrt{k} (x - \omega t)))), \]

Family 8
\[ u_{15}(x, t) = \ln \left( \frac{\alpha_2 k \left( \frac{1}{3} + \tan^2 (\sqrt{k} (x - \omega t)) \right)}{} \right), \]
\[ u_{16}(x, t) = \ln \left( \frac{\alpha_2 k \left( \frac{1}{3} + \cot^2 (\sqrt{k} (x - \omega t)) \right)}{} \right). \]

Finally, when \( k = 0 \), we get the following solutions:

Family 9
\[ u_{17}(x, t) = \ln \left( \frac{\alpha_0}{(-m x + m \omega t + 1)^2} \right). \]

Family 10
\[ u_{18}(x, t) = \ln \left( \frac{-\beta_1}{2m(-m x + m \omega t + 1)^2} \right). \]

Family 11
\[ u_{19}(x, t) = \ln \left( \frac{\alpha_2}{(x - \omega t)^2} \right). \]

4. Comparison between the methods

In this section, we summarize the obtained results and make a comparison between the two methods. Besides, we present the graphs of several solutions demonstrating well-known wave shapes. MATLAB surface plot with convergent mesh size for individual solution was used to produce these graphs.

As seen in section 3, both methods provide a large number of the family of solutions of the governing equation, including some repeating and singular solutions. For example, the Generalized Kudryashov method provides a total of five families of solutions, where Family 1 and Family 2 are qualitatively identical, so are Family 3 and Family 4. Therefore, the Generalized Kudryashov method provides only three families of distinct solutions. On the other hand, the improved F-expansion method provides a total of eleven families of solutions, where Family 3 and Family 4 are qualitatively identical, so are Family 7 and Family 8. Families 9, 10 and 11 are also qualitatively identical.

Furthermore, if we compare the solutions obtained by two methods, some solutions can also be seen qualitatively identical. For example, the solution Families 1, 2 produced by the Generalized Kudryashov method are identical to the solution Families 3, 4 produced by the F-expansion method. The solution Families 3, 4 produced by the Generalized Kudryashov method are also identical to the solution Families 7, 8 produced by the F-expansion method. These repeated solutions can be ignored. Besides, the other families of non-soliton...
solutions, obtained by both methods, can also be ignored, as they are singular and do not produce any useful wave solutions. After ignoring the repeating and singular solutions, we get only six families of distinct solutions as given below.

Hyperbolic family 1

\[ u_{h1}(x, t) = \ln \left( \frac{k a t}{b_1} \left( \frac{1}{3} - \tanh^2(\sqrt{-k}(x - \omega t)) \right) \right) , \]

\[ u_{h2}(x, t) = \ln \left( \frac{k a t}{b_1} \left( \frac{1}{3} - \coth^2(\sqrt{-k}(x - \omega t)) \right) \right) . \]

Hyperbolic family 2

\[ u_{h3}(x, t) = \ln \left( \frac{\alpha_0 k (1 - \tanh^2(\sqrt{-k}(x - \omega t)))}{(m + \sqrt{-k} \tanh(\sqrt{-k}(x - \omega t)))^2} \right) , \]

\[ u_{h4}(x, t) = \ln \left( \frac{\alpha_0 k (1 - \coth^2(\sqrt{-k}(x - \omega t)))}{(m + \sqrt{-k} \coth(\sqrt{-k}(x - \omega t)))^2} \right) . \]

Hyperbolic family 3

\[ u_{h5}(x, t) = \ln \left( \frac{\beta_1 k ((k + 3m^2) \tanh^2(\sqrt{-k}(x - \omega t)) + 4m^2 \sqrt{-k} \tanh(\sqrt{-k}(x - \omega t)) - (m^2 + 3k))}{6m(k + m^2)(m + \sqrt{-k} \tanh(\sqrt{-k}(x - \omega t)))^2} \right) , \]

\[ u_{h6}(x, t) = \ln \left( \frac{\beta_1 k ((k + 3m^2) \coth^2(\sqrt{-k}(x - \omega t)) + 4m^2 \sqrt{-k} \coth(\sqrt{-k}(x - \omega t)) - (m^2 + 3k))}{6m(k + m^2)(m + \sqrt{-k} \coth(\sqrt{-k}(x - \omega t)))^2} \right) . \]

Trigonometric family 1

\[ u_{t1}(x, t) = \ln \left( \frac{k a t}{b_1} \left( \frac{1}{3} + \tan^2(\sqrt{k}(x - \omega t)) \right) \right) , \]

\[ u_{t2}(x, t) = \ln \left( \frac{m a t}{b_1} \left( \frac{1}{3} + \cot^2(\sqrt{k}(x - \omega t)) \right) \right) . \]

Trigonometric family 2

\[ u_{t3}(x, t) = \ln \left( \frac{\alpha_0 k (1 + \tan^2(\sqrt{k}(x - \omega t)))}{(m + \sqrt{k} \tan(\sqrt{k}(x - \omega t)))^2} \right) , \]

\[ u_{t4}(x, t) = \ln \left( \frac{\alpha_0 k (1 + \cot^2(\sqrt{k}(x - \omega t)))}{(m + \sqrt{k} \cot(\sqrt{k}(x - \omega t)))^2} \right) . \]

Trigonometric family 3

\[ u_{t5}(x, t) = \ln \left( \frac{\beta_1 k ((k + 3m^2) \tan^2(\sqrt{k}(x - \omega t)) + 4m \sqrt{k} \tan(\sqrt{k}(x - \omega t)) + m^2 + 3k))}{6m(k + m^2)(m + \sqrt{k} \tan(\sqrt{k}(x - \omega t)))^2} \right) , \]

\[ u_{t6}(x, t) = \ln \left( \frac{\beta_1 k ((k + 3m^2) \cot^2(\sqrt{k}(x - \omega t)) - 4m \sqrt{k} \cot(\sqrt{k}(x - \omega t)) + m^2 + 3k))}{6m(k + m^2)(m + \sqrt{k} \cot(\sqrt{k}(x - \omega t)))^2} \right) . \]

Hyperbolic solutions 1 and 2 represent soliton waves of blow-up type, as shown in figures 1 and 2. The hyperbolic Family 2 looks little different from the Family 1, which, however, produces a similar wave shape as Family 1, see figure 3. In fact, Family 2 is completely identical to Family 1 for \( m = 0 \). The solutions in hyperbolic Family 3 are undefined for \( m = 0 \), or \( \pm \sqrt{-k} \). For other combination of the parameters, for example, \( m = -0.2 \), \( k = -1 \), \( \omega = 1 \), and \( \beta_1 = -1 \), the Family 3 produces similar wave shapes as Family 1, suggesting that Family 3 is also identical to Family 1. In the same way, it can also be conjectured that the three trigonometric families are also identical and produce qualitatively similar periodic wave shape as shown in figure 4.

The above discussion suggests that the DBM equation possesses only three types of travelling wave solutions: the singular solitons of bright and dark types, and the periodic solution. Other studies, for example [40–43], also...
Figure 1. Singular soliton of bright type given by $u_{\text{hyp}}(x, t)$ with parameters $a_1 = b_1 = \omega = 1$ and $k = -1$.

Figure 2. Singular soliton of dark type given by $u_{\text{hyp}}(x, t)$ with parameters $a_1 = b_1 = \omega = 1$ and $k = -3$.

Figure 3. Singular soliton of bright type given by $u_{\text{hyp}}(x, t)$ with parameters $a_0 = k = 1$, $\omega = 1$ and $m = 0.1$. 
suggest these three basic types of travelling wave solutions of the DBM equation, which justify the efficiency of both the methods that are considered in this study.

Further, the above discussion reveals that the Generalized Kudryashov method produces less number of repeated solution. As a result, it was found more direct compared to the F-expansion method and required less tedious symbolic computations. However, if we could omit the repeated solutions produced by this method, the symbolic calculation is expected to be more straightforward. Similar to the basic Kudryashov method, a basic F-expansion method that can be obtained by setting \( m = 0 \) in equation (3.2.1), could also help us to omit the unnecessary solutions and reduce the volume of symbolic calculations. A further study showing a comparison between the Generalized Kudryashov and improved F-expansion methods with their simplified versions will be published elsewhere.

5. Conclusion

We investigated the travelling wave solutions of the DBM equation using Generalized Kudryashov and the improved F-expansion methods. Three basic types of wave solutions, such as bright singular solitons, dark singular solitons and periodic solutions were found. Several other repeated and singular solutions were also found. The existing studies, however, suggest that the basic solutions are the only fundamental wave solutions of governing equation. Therefore, the repeated and additional singular solutions are physically insignificant, thus ignored.

Among the two methods, the Generalized Kudryashov method was found more direct and produced less number of insignificant solutions, and the symbolic calculations were straightforward than the other. We, therefore, presume that if the methods are modified in such a way so that the extra solutions are omitted, the volume of symbolic calculations will be reduced significantly. The basic version of either of the methods could be a better alternative to get rid of these insignificant solutions. However, a further study showing a comparison between these methods with their simplified versions may help to justify this conclusion. Similar comparative studies on other nonlinear equations using the methods used in this study, or other methods if needed, may help to find the best effective method for the respective equation.

ORCID iDs

Naima Islam  @ https://orcid.org/0000-0001-8395-9652
Kamruzzaman Khan  @ https://orcid.org/0000-0002-4531-288x
Md Hamidul Islam  @ https://orcid.org/0000-0003-3258-0700

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Figure 4. Periodic solution given by \( u_{\text{Trig}}(x, t) \) with parameters \( a_3 = b_1 = \omega = 1 \) and \( k = 1 \).
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