SWAP ACTION ON MODULI SPACES OF POLYGONAL LINKAGES

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Abstract. The basic object of the paper are the moduli spaces $M_2(L)$ and $M_3(L)$ of a closed polygonal linkage either in $\mathbb{R}^2$ or in $\mathbb{R}^3$. As was originally suggested by G. Khimshiashvili, the space $M_2(L)$ is equipped with the oriented area function $A$. In turn, we equip the space $M_3(L)$ with the vector area function $S$. The latter are generically Morse functions, whose critical points have a nice description. In the paper, we define a swap action (that is, the action of some group generated by edge transpositions) on the spaces $M_2(L)$ and $M_3(L)$ which preserves the functions $A$ and $S$ and the Morse points. We prove that the commutant of the group acts trivially, present some computer experiments and formulate a conjecture.

1. Introduction

We study the moduli spaces $M_2(L)$ and $M_3(L)$ of a closed polygonal linkage either in $\mathbb{R}^2$ or in $\mathbb{R}^3$. These spaces attract special attention firstly because of practical applications, and secondly because they can be equipped by additional structures. In this respect we briefly mention the papers by A. Klyachko [4], and by M. Kapovich, J. Millson [2].

In the paper, we consider the oriented area function $A$ defined on the space $M_2(L)$, and the vector area function $S$ defined on the space $M_3(L)$. Generically, $A$ is a Morse functions, and $S$ is a Morse-Bott function, whose critical points have a nice description. In the paper, we enrich this structure by defining a swap action (that is, the action of some group generated by edge transpositions) on the spaces $M_2(L)$ and $M_3(L)$ which preserves the functions $A$ and $S$ and the Morse points. We show that this action factors through a factor group of the group of pure balanced annular braids. Besides, we prove that commutant of the group acts trivially, present some computer experiments and formulate a natural conjecture.

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2. Moduli space and oriented area

A polygonal $n$-linkage is a sequence of positive numbers $l_1, \ldots, l_n$. It should be interpreted as a collection of rigid bars of lengths $l_i$ joined consecutively by revolving joints in a closed chain. We study its flexes with allowed self-intersections. This is formalized in the following definition:
Figure 1. Basic notation for a pentagonal cyclic configuration with $E = (-1, -1, -1, 1, -1)$

Definition 2.1. For a linkage $L$, a configuration in the Euclidean space $\mathbb{R}^d$ is a sequence of points $R = (p_1, \ldots, p_n)$, $p_i \in \mathbb{R}^d$ with $l_i = |p_i, p_{i+1}|$, $l_n = |p_n, p_1|$. The moduli space of $L$ is the set $M_d(L)$ of all such configurations modulo the action of orientation preserving isometries.

In the paper we make use of the signed area function as the Morse function on $M_2(L)$ and of the vector area function on $M_3(L)$.

We start with the 2D-case.

Definition 2.2. The signed area of a polygon $P \subset \mathbb{R}^2$ with the vertices $p_i = (x_i, y_i)$ is defined by

$$2A(P) = (x_1y_2 - x_2y_1) + \ldots + (x_ny_1 - x_1y_n).$$

Definition 2.3. A polygon $P$ is called cyclic if all its vertices $p_i$ lie on a circle.

Cyclic polygons arise in the framework of the paper as critical points of the signed area:

Theorem 2.4. [5] Generically, a polygon $P$ is a critical point of the signed area function $A$ iff $P$ is a cyclic configuration.

The following notation (see Fig.1) is used throughout the paper for closed cyclic configurations:

- $r = r(P)$ is the radius of the circumscribed circle.
- A cyclic configuration is called central if one of its edges contains $O$.
- For a non-central configuration, $\varepsilon_i$ is the orientation of the edge $p_ip_{i+1}$:

$$\varepsilon_i = \begin{cases} 
1, & \text{if the center } O \text{ lies to the left of } p_ip_{i+1}; \\
-1, & \text{if the center } O \text{ lies to the right of } p_ip_{i+1}.
\end{cases}$$

- $E = E(P) = (\varepsilon_1, \ldots, \varepsilon_n)$ is the string of orientations of all the edges.
Now we pass to the 3D-case.

**Definition 2.5.** The vector area of a polygon \( P \subset \mathbb{R}^3 \) with the vertices \( p_i = (x_i, y_i) \) is defined by

\[
\begin{aligned}
2S(P) &= p_1 \times p_2 + p_2 \times p_3 + \cdots + p_n \times p_1, \\
2S(P) &= |p_1 \times p_2 + p_2 \times p_3 + \cdots + p_n \times p_1|.
\end{aligned}
\]

**Theorem 2.6.** Assume that \( S(P) \neq 0 \) for a configuration \( P \in M_3(L) \). Generically, \( P \) is a critical point of the vector area function \( S \) if and only if the two following conditions hold:

1. The orthogonal projection of \( P \) onto the plane \( S(P) \perp \) is a cyclic polygon.
2. For every \( i \), the vectors \( \overrightarrow{T_i}, \overrightarrow{S}, \) and \( \overrightarrow{d_i} \) are coplanar.

Here \( \overrightarrow{d_i} \) is the \( i \)-th short diagonal, \( \overrightarrow{T_i} \) is the vector area of the triangle \( p_{i-1}p_ip_{i+1} \), see Fig. 2, right.

Proof. We list \((2n - 6)\) flexes that generate \((2n - 6)\) elements of the tangent space \( T_P(M_3(L)) \). Generically, these vectors are linearly independent. Therefore, the point \( P \in M_3(L) \) is critical if and only if the function \( S \) has a zero derivative in all these directions.

1. Denote by \( pr \ P \) the orthogonal projection of \( P \) onto \( S(P) \perp \). Each flex of \( pr \ P \) in the plane \( S(P) \perp \) generates a flex of \( P \) in the space \( \mathbb{R}^3 \). During the flex, we maintain the slopes of the edges with respect to the plane \( S(P) \perp \). Since \( \text{dim } M_2(pr \ P) = n - 3 \), we can choose \((n - 3)\) linearly independent tangent vectors of this type.
2. Let us bend the triangle \( T_i \) around the diagonal \( d_i \) keeping the rest of configuration \( P \) frozen. We choose \((n - 3)\) linearly independent tangent vectors of this type.

The flexes of the first (respectively, second) type provide the statement 1 (respectively, statement 2) of the theorem.

\[ \square \]

3. Swap action

We assume that a polygonal linkage \( L \) with \( n \) edges and with all \( l_i \) different is fixed. We make a convention that the numbering is modulo \( n \), that is, for instance, \( n + 1 = 1 \).

**Definition 3.1.** Let \( P \in M_{2,3}(L) \) be a polygon. For \( i = 1, \ldots, n \), denote by \( s_i(P) \) the polygon obtained from \( P \) by transposing of the two edges adjacent to the vertex \( p_i \) (see Fig. 3). For the dimension three, we assume that the new pair of edges lies in the plane spanned by the old one.

We get a homeomorphism

\[
s_i : M_{2,3}(L) \to M_{2,3}(\sigma_i L),
\]
where the element of the symmetric group $\sigma_i \in S_n$ is a transposition induced by $s_i$. Define by $F_n$ the free group whose generators are the abstract symbols $s_i$.

**Lemma 3.2.**

1. The action of $F_n$ respects the functions $A$ and $\vec{S}$.
2. For $n = 4$, the action of $F_4$ respects the volume of the convex hull $V(\text{Conv}(P))$. \hfill $\Box$

However, $F_n$ acts on the disjoint union of moduli spaces $\bigsqcup M_{2,3}(\sigma_iL)$. We wish to restrict ourselves by just one moduli space. This means that we take only those elements that take a configuration to the same moduli space. We formalize this as follows: There is a natural mapping to the symmetric group

$$\pi : F_n \to S_n,$$

which maps $s_i$ to $\sigma_i$. Clearly its kernel $F_0^n$ acts on the moduli space $M_{2,3}(L)$.

**Lemma 3.3.** For a 4-linkage $L$, the group $F_4^0$ acts trivially on $M_{2,3}(L)$.

Proof. (2D). For a 4-gon $P = (p_1, p_2, p_3, p_4)$ denote by $O = O(P)$ the intersection point of perpendicular bisectors to the segments $p_1p_3$ and $p_2p_4$. Denote also

$$r_i(P) = |Op_i|, \quad \beta_i(P) = \angle p_iOp_{i+1}.$$

The lemma follows from the three geometrical observations:

1. A 4-gon is completely defined by $r(P) = ((r_1(P), r_2(P), r_3(P), r_4(P)))$, and $\beta(P) = (\beta_1(P), \beta_2(P), \beta_3(P), \beta_4(P))$.
2. The action of $F_n$ preserves the point $O(P)$ and the vector $r(P)$.
3. The group $F_n$ acts on $\beta(P)$ by permutations: $\beta(s(P)) = \pi(s)\beta(P)$. 

![Diagram](image-url)
(3D). By analyticity reasons it is enough to prove that \( s = (s_1s_2)^3 \) acts trivially on some open subset \( U \) of the space of all 4-gons.

Take an equilateral 4-gon \( P_0 \) (that is, a rhombus but not a square). The swap \( s \) obviously takes \( P_0 \) to itself. Now, let \( P \) be a quadrilateral close to \( P_0 \). Its image \( sP \) is close to \( P \) and has the same values of \( S(P) \) and \( V(Conv(P)) \). By continuity reasons, \( sP = P \). In other words, \( s = (s_1s_2)^3 \) acts trivially on a neighborhood of \( P \) which is an open set. \( \square \)

**Definition 3.4.** Denote by \( Stab = \text{Stab}(M_{2,3}(L)) \subset F_n^0 \) the pointwise stabilizer of the space \( M_{2,3}(L) \), that is, the the group of all elements with the trivial action. Denote also the factor \( F_n^0/\text{Stab} \) by \( SW_n = SW_n(L) \).

**Proposition 3.5.** Generically, the group \( \text{Stab} \) does not depend on \( L \). \( \square \)

**Definition 3.6.** Define \( R \subset F_n^0 \) as the subgroup generated by the elements of the following three types:

1. \( s_i^2 \),
2. \( s_is_j s_is_j^{-1} \), whenever \( |i-j| > 1 \), and
3. \( s_is_{i+1}s_is_{i+1}^{-1}s_is_{i+1}^{-1} \).

**Proposition 3.7.** The group \( R \) is a subgroup of the stabilizer \( \text{Stab} \).

Proof. The first two items are obvious. The third one follows from Lemma 3.3. \( \square \)
Figure 4. The $i$-th generator of the group $F^0_n/R$ represented by a balanced annular braid ($i = 2, \ldots, n$).

Theorem 3.8.  
- The group $F^0_n/R$ acts on the moduli spaces $M_{2,3}(L)$.
- The group $F^0_n/R$ is isomorphic to the described below factor group of the group of pure balanced annular braids. Thus the elements of the group $F^0_n/R$ can be represented by balanced annular braids. For instance, Fig. 4 depicts the generators of the group.
- The group $F^0_n/R$ is isomorphic to $\mathbb{Z}^{n-1}$, and is therefore commutative.

Proof.  
The first statement follows from the above discussion. To prove the third statement, we construct an explicit homomorphism

$$\phi: F^0_n/R \rightarrow \mathbb{Z}^{n-1} \cong \{(w_1, w_2, \ldots, w_n) \in \mathbb{Z}^n : \sum_{i=1}^{n} w_i = 0\}$$

We start with the balanced annular braid group which is defined as follows:

$$B_n = \langle \Sigma_1, \Sigma_2, \ldots, \Sigma_n \mid \Sigma_i \Sigma_j = \Sigma_j \Sigma_i, \Sigma_i \Sigma_{i'} \Sigma_i = \Sigma_{i'} \Sigma_i \Sigma_{i'} \text{ whereas } i-j \neq \pm 1, i-i' = \pm 1 \rangle.$$  

Next, we take the group $B^0_n$ of pure braids, that is, the kernel of the natural map $B_n \rightarrow S_n$ which maps $\Sigma_i$ to $\sigma_i$.

As usual, we visualize a braid as $n$ non-intersecting strands living in a ”thick” cylinder and going from the top to the bottom, see Fig. 4.

Finally, we introduce the group $\overline{B^0_n}$, that is, the group $B^0_n$ factorized by all relations of type $(\Sigma_i)^2 = 1$. The factorization means that the strands can pass freely through each other, but not through the central part of the cylinder.

There is a natural isomorphism

$$\psi: F^0_n/R \rightarrow \overline{B^0_n}$$

which maps $s_i$ to $\Sigma_i$.

Besides, there is a homomorphism

$$w: \overline{B^0_n} \rightarrow \mathbb{Z}^n, b \mapsto w(b) = (w_1(b), w_2(b), \ldots, w_n(b)).$$
where $w_i(b)$ is a winding number of the $i$-th strut of the braid $b$ around the central part of the cylinder. It is easy to check that for any pure braid $b$, we have

$$\sum_{i=1}^{n} w_i(b) = 0.$$  

Taken together, the two maps give the homomorphism

$$w \circ \psi : F_0^n / R \to \{(w_1, w_2, \ldots, w_n) \in \mathbb{Z}^n : \sum_{i=1}^{n} w_i = 0 \},$$

which is obviously bijective.

Figure 4 depicts the preimage of the vector $(1, 0, 0, \ldots, 0, -1, 0, 0, \ldots, 0)$ with just two non-zero entries. The preimage of the vector in the group $F_0^n / R$ is represented by

$$s_{i+1}s_{i+2} \ldots s_{i-1}s_{n-1}s_{n-2} \ldots s_2s_1.$$  

**Proposition 3.9.** The critical points of the function $A$ and $S$ (such that $S \neq 0$) are stable under the action of $F_0^n$.

**Proof.**

(2D). Critical points of the function $A$ are known to be cyclic polygons (see Theorem 2.4). A cyclic polygon $P$ is completely determined by $r(P)$, $L$ and $E(P)$. The action of $F_0^n$ preserves them all.

(3D). Assume that $P$ is a critical point such that $S(P) \neq 0$. Fix a polygon $P$ and a plane $\overrightarrow{S}$. First observe that a critical point is uniquely determined by radius $r(prP)$ of the circumscribing circle, the edge orientations $E(prP)$, and the heights $h_i = \text{dist}(p_i, \overrightarrow{S})$, $i = 1, \ldots, n$.

Let $g$ be an element of $F_0^n / R$. Theorem 2.6 implies that the swap $s_i$ permutes the height differences $h_{i+1} - h_i$ and $h_i - h_{i-1}$. Therefore, $g$ maintains the height differences $h_{i+1} - h_i$. Besides, $g$ maintains both $E(prP)$ and $r(prP)$. By the above observation, $g$ maps $P$ to itself. □

Computer experiments show the following:

**Example 3.10.** Let $g = s_4s_3s_2s_1$. There exists a pentagon $P$ such that $g^kP$ are all different for $k = 1, 2, \ldots, 8!$ (see Fig. 5).

**Example 3.11.** For the pentagon $P$ depicted in Fig. 6, the pentagons $s_4s_3s_2s_1(P)$ and $s_5s_4s_3s_2(P)$ are different. This means that they are different for a generic pentagon.

These two examples motivate the following conjecture:

**Conjecture 1.** For a generic polygonal linkage, the groups $Stab$ and $R$ coincide, i.e., $SW_n = F_0^n / R = \mathbb{Z}^{n-1}$. 

Figure 5. We depict here the polygons (0) $P$ and the iterated actions $g^k(P)$.

Figure 6. The action of the first generator. We depict here a) $P$, b) $P \rightarrow s_1(P)$, c) $s_1(P) \rightarrow s_2s_1(P)$, ..., e) $s_3s_2s_1(P) \rightarrow s_4s_3s_2s_1(P)$, f) $s_4s_3s_2s_1(P)$.

References

[1] Farber M., Schütz D., Homology of planar polygon spaces. Geom. Dedicata, 2007, 125, 18, 75-92.

[2] Kapovich M., Millson J., On the moduli spaces of polygons in the Euclidean plane. Journal of Diff. Geometry, Vol. 42 (1995) N 1, p. 133-164.
Figure 7. The action of the second generator. We depict a) $P$, b’) $P \rightarrow s_2(P)$, c’) $s_1(P) \rightarrow s_3s_2(P)$, ..., e’) $s_4s_3s_2(P) \rightarrow s_5s_4s_3s_2(P)$, f’) $s_5s_4s_3s_2(P)$.

[3] Kent R.P. IV, Peifer D., *A geometric and algebraic description of annular braid groups*. Int. J. Alg. Comp. 12 (2002), 85-97.

[4] Klyachko A., *Spatial polygons and stable configurations of points in the projective line*. Tikhomirov, Alexander (ed.) et al., Algebraic geometry and its applications. Proceedings of the 8th algebraic geometry conference, Yaroslavl’, Russia, August 10-14, 1992. Braunschweig: Vieweg. Aspects Math. E 25, 67-84 (1994).

[5] Khimshiashvili G., Panina G., *Cyclic polygons are critical points of area*. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 2008, 360, 8, 238–245.

[6] Khimshiashvili G., Siersma D., Preprint ICTP, IC/2009/047. 11 p.

[7] Khimshiashvili G., Panina G., Siersma D., Zhukova A., *Extremal configurations of polygonal linkages*. An Oberwolfach preprint, to appear.

[8] Panina G., Zhukova A., *Morse index of a cyclic polygon*. Cent. Eur. J. Math., 9(2) (2011), 364-377.