0.1 Introduction: the need to go beyond Riemannian manifolds

A number of developments in physics in recent years have evoked the possibility that the treatment of spacetime might involve more than just the Riemannian spacetime of Einstein’s general relativity:

1. The vain effort so far to quantize gravity is, perhaps, the strongest piece of evidence for going beyond a geometry which is dominated by the classical distance concept.

2. Intuitively, the generalization of the three–dimensional theory of elastic continua with microstructure to the four–dimensional spacetime of gravity suggests, in a rather convincing manner, physical interpretations for the newly emerging structures in post–Riemannian spacetime geometry.

3. The description of hadron (or nuclear) matter in terms of extended structures. In particular, the quadrupole pulsation rates of that matter and, in a rest frame, their relation to representations of the volume–preserving three–dimensional linear group $SL(3, R)$ — with the rotation group $SO(3)$ as subgroup — have been established experimentally.

4. The study of the early universe — in the light of the various theorems about a singular origin, the ideas about unification of the fundamental interactions (mostly involving additional dimensions, later compactified) and inflationary models with dilaton–induced Weyl covector —

...and each of these developments necessitates the study of dynamical theories involving post–Riemannian geometries, whether in the context of local field theories or within the framework of string theories. We explain the interest in continua with microstructure, in extended structures, and the problematics of the early universe – as far as these are relevant as motivations for a relaxation of the Riemannian constraint in gravity – in the rest of this chapter, leaving the rather involved issue of quantum gravity to Chap.2.

The smallest departure from a $V_4$ would consist in admitting torsion, the field strength of local translations, arriving thereby at a Riemann-Cartan spacetime $U_4$ and, furthermore, nonmetricity, resulting in a metric-affine $(L_4, g)$ spacetime [He6]. In what follows, starting with Chapter 3, we deal with the geometry of spacetime, with the Euler-Lagrange field equations, with the Noether conservation identities, with the conformal properties, and with a specific model of spontaneous symmetry breakdown, in the framework of such metric-affine spacetimes. For reasons that will become clear in the sequel, we study in particular spacetime models arising from a Weyl/Yang-Mills-like gauge theory approach to gravity.
0.2 Spacetime as a continuum with microstructure

In Einstein’s general relativity theory (GR), the linear connection of its Riemannian spacetime is (i) metric(–compatible), that is, the length and angle measurements are integrable, and (ii) symmetric. The symmetry of the Riemannian (or Levi-Civita) connection translates into the closure of infinitesimally small parallelograms. Already the transition from the flat gravity–free Minkowski spacetime to the Riemannian spacetime in Einstein’s theory can locally be understood as a deformation process. A strain tensor $\varepsilon_{AB}$ in continuum mechanics [La3] measures by its very definition $\varepsilon_{AB} := (g^{\text{defor.}}_{AB} - g^{\text{undef.}}_{AB})/2$ the change of the metric between the undeformed and the deformed state. Thus, because of the pairing of stress and strain, it does not come as a surprise that in GR, according to Hilbert’s definition, the stress-energy-momentum tensor couples the Lagrangian to the metric.

The lifting of the constraints of metric–compatibility and symmetry yields nonmetricity and torsion, respectively. The continuum under consideration, here classical spacetime, is thereby assumed to have a non-trivial microstructure, similar to that of a liquid crystal or a dislocated metal or the like. In particular, to drop the metricity condition, i.e. to allow for nonmetricity $Q_{\alpha\beta} := -Dg_{\alpha\beta} \neq 0$, and to “touch” thereby the lightcone, if parallelly displaced, is classically a step of unusual boldness, but may be unavoidable in quantum gravity. It is gratifying, though, to have the geometrical concepts of nonmetricity and torsion already arising in the (three–dimensional) continuum theory of lattice defects – and there they have concrete interpretations as densities of point defects and line defects (dislocations), respectively, cf. [Kr2, Kr4]. But even more, certain types of “hyperstress” are induced by these post–Riemannian structures: Double–stress without moment relates to nonmetricity, spin moment stress to torsion.

Just as ordinary stress is the analogue of the (Hilbert) energy–momentum density, hyperstress finds its field–theoretical image in the densities of hypermomentum, i.e. in the

$$\text{spin current} \oplus \text{dilation current} \oplus \text{shear current}.$$  \hspace{1cm} (0.2.1)

And these currents ought to couple to the corresponding post–Riemannian structures, a hypothesis which brought the metric–affine gravity theory under way in the first place [He9, He10].

According to Sakharov [Sa2, Sa3], gravitation represents a “metrical elasticity” of space which is brought about by quantum fluctuations of the vacuum. Here we pursue this analogy with continuum mechanics much further and introduce additional nonmetric and torsional degrees of freedom into spacetime, but, we believe, it is done in the same spirit.

\[ ^1 \text{There exists an extended literature on continua with microstructure, see, for example Jaunzemis [Ja2], Mindlin [Mi15], and Nye [Ny1]. Kröner’s articles [Kr1, Kr2, Kr3, Kr4, Kr5, He12] on lattice defects are particularly illuminating, since they relate differential geometric notions to distributions of lattice defects. His article on the lattice interpretation of nonmetricity [Kr5] seems remarkable; however, no use of it has been made so far. The gauge–theoretical point of view is stressed by Kleinert [Kl1]. The analogies between three– and four–dimensional continua with microstructure have been particularly worked out by [Gr8, He1, He22, Mc2].} \]
0.3 Hadrons as extended structures — effective ‘strong gravity’

With the discovery of a spatial spread for the hadrons — first in experiments measuring the electromagnetic form factors, then in the identification of the baryons with an SU(3) octet (rather than with the fundamental representation of the group, as in the Sakata model) and the conception of quarks as constituents — it became important to describe the dynamics and kinematics of quantum extended structures (extendons). The 1965 work dealt with three-dimensional vibrating and rotating "lumps" [Do5]. Then came dual models [Ve1] and their reinterpretation as a quantum string [Na1, Su3], a one-dimensional extendon. It was later understood as an “effective” description of QCD flux tubes, extending between point-quarks [Ni4].

Extendons can be deformed, and thus represent affine geometries in themselves. Hadron excitations show up as Regge trajectories, and the massive states fit $SL(3, R)$ representations — as would indeed be expected from the pulsations of a (consider it as an approximation) fixed-volume 3-extendon [Do5]. The quantum $d$-extendon involves covariance of a $d+1$ manifold (e.g. the world-sheet for the string), the extendon’s time evolution. This resembles gravity, involves gauging geometrical groups and often reproduces the same equations that were derived in the pursuit of quantum gravity.

It is thus not surprising that “effective strong-gravity” theories, in which the Planck length $\ell$ is replaced by the Compton wavelength of the proton, were derived in the same context. One such example [He23] with a confining Pöschl-Teller type potential [Mi6] in the effective radial Schrödinger equation arose in the Poincaré gauge theory and its generalization to $SL(6, C)$ flavor models of Salam et al. [Sa5, Sa5a, Mi3a]. By including the $SU(3)$ color group of QCD one ends up with the $SL(6, C)^f \otimes SL(6, C)^c$ model of color geometrodynamics [Mi4, Mi6]. Another such treatment has used affine manifolds [Ne14, Ne22]. In these models, “low energy” means ”hadron energies”, i.e. 1 – 100 GeV. The slope of the trajectories is of the order of 1 GeV, as against the $10^{19}$ GeV of the theories we mentioned in Sec.1.1 above.

In a recent version of this approach, “chromogravity” [Ne22a, Ne22b, Si8, Si9] is derived from QCD itself, as an “effective” theory. A gravitation-like component is identified in the infrared limit of QCD, its contribution providing for color confinement, for the systematics of the excitations in the hadron spectrum (Regge sequences) and for the forces of longer-range responsible for the nuclear excitation spectrum. This QCD-generated graviton-like component is the analog of van der Waals forces in molecules, where a $J = 2$ combination of two photons is exchanged between atoms; in QCD, a $J = 2$-mediated zero-color component, plus all higher spin zero-color combinations of QCD gluons, make up this pseudo-gravitational component. The emergence of a $J = 2$ contribution from a $J = 1$ force (QCD) in higher orders is similar to the generation of the $J = 2$ gravitational contribution in string theory, from closed strings — i.e. from the contraction of two open strings — an open string corresponding in the massless sector to a $J = 1$-mediated force.
0.4 The early universe (cosmogony)

Already in the seventies, various theorems implied that, with a cosmology based on Riemannian geometry, the universe was forced either to have come out of a singularity — or, inevitably, to fall into one in the future. The simplest way of avoiding such a result is to assume that in the distant past — or the distant future — the geometry is not Riemannian.

In the late seventies and in the eighties, the same conclusion emerged from the new studies of the early universe connected with gauge unification theories\(^\text{2}\) (GUT) [Or0a] and their supersymmetric extensions — later replaced by unification and superunification as derived from the quantum superstring. In these theories the early universe has additional dimensions (and superdimensions). It is assumed that these extra dimensions spontaneously compactify, leaving internal symmetries as residual effects in the final four-dimensional spacetime, cf. [We2b]. The symmetries that we have identified phenomenologically include those of the $SU(3) \times SU(2) \times U(1)$ group of the standard model embedded within higher rank groups such as $E(6)$ or $SU(5)$. All of this implies geometries ranging from Kähler and Calabi-Yau to affine manifolds.

The eighties also ushered in inflationary cosmology [Gu1a, Li0, Al2], a new conception of the very early universe, now from the point of view of cosmology itself, rather than particle physics (though it does affect it too). In the more advanced “extended” models [La1a] one finds it necessary to abandon the Riemannian constraints [St2a], at the very least replacing Einstein’s geometry by Weyl’s. We deal with this situation in an example in Chap. 6.

0.5 Organization of the paper and notation

In Chap.2 we take a tour d’horizon around quantum gravity. We mention the main open questions and unsolved problems.

In Chap.3 we show how, by starting with the affine group $A(n, R)$ and its Yang–Mills type gauging, we eventually arrive at a metric–affine geometry of spacetime, the structures and properties of which we explicate in the rest of this chapter. In particular, the potentials emerging from the affine connection are the coframe and the linear connection. The latter is decomposed into Riemannian and post–Riemannian pieces, and the interrelations of the Chern–Simons terms to the Bianchi identities are exhibited. The rules of exterior calculus we defer to Appendix A and the irreducible decompositions of nonmetricity, torsion, curvature, and of the Bianchi identities to Appendix B. All this is more or less traditional wisdom. However, we stress the post–Riemannian structures in a coherent geometrical framework, such as nonmetricity, the Weyl one–form, and the volume–preserving piece of the connection.

In Chap.4 the question is answered of how one can present especially fermionic mat-

\(^{2}\)Although the initials GUT were originally taken to mean Grand Unified Theories, it was later agreed (1979 HEPAP Conference) to read them as Gauge Unification Theories, in order to leave room, as might be needed, for ‘grander’ theories some day.
ter in such a metric–affine spacetime. The results of this chapter are fairly new and have been found during the last 15 years or so by one of us (YN) and his collaborators. World spinors are defined and their conformal properties studied. Technical details of the unitary irreducible representations of the \( SA(4, R) \) and the corresponding subgroups are collected in Appendix C.

Having now a spacetime arena available and matter fields ‘moving’ therein, we can build up a Lagrangian of this gravitationally interacting matter system and an action function as well. This is done in Chap. 5 in the conventional way. We postulate affine gauge invariance and switch on the Lagrange–Noether machinery. Besides the conventional canonical energy–momentum current, we define, generalizing the spin current, a hypermomentum current that is coupled to the linear connection, i.e. to the new gravitational potential of spacetime.

The Noether identities (5.2.10), (5.2.16) and the general form of the gravitational field equations (5.5.2), (5.5.3), (5.5.4) are derived. We discuss the Belinfante–Rosenfeld symmetrization of the energy–momentum current and study different limiting cases of the gravitational field equations by means of the Lagrange multiplier technique. Finally Astekhar type complex variables are generated by means of a metric–affine Chern–Simons term in the gauge Lagrangian. Whereas most of the material of this chapter appeared before, we claim some originality as to the completeness and the rigor of our presentation.

Up to including Chap. 5, no gravitational gauge Lagrangian is specified explicitly. Thus we provided a ‘kinematical’ framework for metric–affine gauge gravity which has to be filled with physical life. This is done in Chap. 6. Conformally invariant gravitational gauge Lagrangians, including dilaton fields, are studied and compared to alternative approaches in the literature. Various schemes of symmetry reduction from the linear to the Lorentz group are given explicitly. The exact procedure is one of the main open problems. We believe, however, that the solution of exactly this problem is indispensable for future progress in gravity. Moreover, we discuss generalizations of recent inflationary models in our post–Riemannian framework.

The list of literature is organized so as to be of maximal usefulness to the reader. We tried to refer to all material relevant to our task. Should we have overlooked some articles, we would like to ask the authors to let us know, possibly by email to hehl@thp.uni-koeln.de. We may want to supply this additional information in an erratum.

In the body of the paper, the gravitational models used will be abbreviated as follows:

- **\( \text{GR} \)** = general relativity theory, also called Einstein gravity (Riemannian spacetime \( V_4 \)) [Ei0].
- **\( \text{GR}_\parallel \)** = teleparallel (version of general relativity) theory (Weitzenböck spacetime \( W_4 \); Riemann–Cartan spacetime with vanishing (Cartan–)curvature and non–vanishing torsion), see [Ni6b, Sc11].
- **\( \text{EC theory} \)** = Einstein–Cartan (–Sciama–Kibble theory of) gravity (Riemann–Cartan spacetime \( U_4 \); Metric and metric–compatible connection), see [Tr2].
- **\( \text{PG} \)** (theory) = Poincaré (gauge theory of) gravity (Riemann–Cartan spacetime \( U_4 \)),
see [He4].

**MAG** = Metric-affine (gauge theory of) gravity (metric–affine spacetime \((L_4, g)\): Independent \(GL(4, R)\)-connection and independent metric), see [He19].

We denote the covering of a certain group by an overline. We have, for instance, \(SL(2, C) = \overline{SO}(1, 3)\). Sometimes we dispense with the overline for convenience provided it is clear from the context anyways.

### 0.6 Acknowledgments

This paper was only made possible through substantial support of the German-Israeli Foundation for Scientific Research & Development (GIF), Jerusalem & Munich.

Different people helped us at various stages of the writing up of the paper. We are most grateful to all of them. Yuri Obukhov (Moscow/Cologne) read very carefully a preliminary version of our article and came up with numerous suggestions. Djordje Šijački (Belgrade) as well as Tom Laffey (Dublin) and Jürgen Lemke (Cologne/Austin) were of great help in group–theoretical questions. Jörg Hennig (Clausthal) advised us on bundles, Norbert Straumann (Zürich) on densities, Ralf Hecht (Cologne/Chung-Li) on energy complexes, Horst Konzen (Cologne) checked some of the algebra, Romulado Tresguerres (Madrid/Cologne) contributed to our understanding of conformal transformations, and Franz Schunck (Cologne) developed some cosmological models. C.Y. Lee (Seoul) shared with us his quantization experience and J. Godfrey (then Tel-Aviv) his knowledge of projective geometry.

And last but not least, Dietrich Stauffer (Cologne/Antigonish) promoted this project generously by his leave of absence from Cologne, paid by the Canada Council. One of us (Y.N.) was Fall 1993 Joint Royal Society/Israel Academy of Sciences and Humanities Research Professor and would like to thank Prof. D. Lynden-Bell and the University of Cambridge Institute of Astronomy for hospitality during the final stages of this work.