DIFFERENTIAL EQUATIONS FOR JACOBI-PIÑEIRO POLYNOMIALS

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Abstract. For \( r \in \mathbb{Z}_{\geq 0} \), we present a Fuchsian linear differential operator of order \( r + 1 \) with three singular points at 0, 1, \( \infty \). This operator annihilates the \( r \)-multiple Jacobi-Piñeiro polynomial.

1. Introduction

Let \( r \) be a natural number. Consider a Fuchsian differential operator

\[
D = \sum_{i=0}^{r+1} c_i(x) \frac{d^i}{dx^i}
\]

with singular points at \( z_1, \ldots, z_n, \infty \) and with kernel consisting of polynomials only. An interest to such operators had arisen recently in relation with the Bethe ansatz method in the Gaudin model, where such operators were used to construct eigenvectors of the Gaudin Hamiltonians, see [SeV], [MV1]-[MV3], [MTV1], [MTV2].

In the Gaudin model, one considers the tensor product \( M = M_1 \otimes \cdots \otimes M_n \) of finite dimensional irreducible \( \mathfrak{gl}_{r+1} \)-modules, located respectively at \( z_1, \ldots, z_n \). The module \( M_s \), sitting at \( z_s \), is determined by the exponents of \( D \) at \( z_s \). One constructs \( r + 1 \) one-parameter families of commuting linear operators \( H_i(x) : M \to M, \ i = 1, \ldots, r+1 \), acting on \( M \) and called the Gaudin Hamiltonians. The problem is to construct eigenvectors and eigenvalues of the Gaudin Hamiltonians.

It turns out, that having the kernel of the differential operator \( D \), i.e. the \( r + 1 \)-dimensional vector space of polynomials, one constructs (under certain conditions) an eigenvector \( v_D \in M \) of the Gaudin Hamiltonians with corresponding eigenvalues being the coefficients of \( D \),

\[
H_i(x) v_D = c_i(x) v_D \quad i = 1, \ldots, r + 1.
\]

The Bethe ansatz idea is to construct all eigenvectors of the Gaudin Hamiltonians by choosing different operators \( D \) with the same singular points and exponents.

This philosophy motivates the detailed study of Fuchsian operators with prescribed singular points, exponents, and polynomial kernels.

The important model case is the study of operators with three singular points 0, 1, \( \infty \). The operators with special exponents 0, \( k+1, k+2, \ldots, k+r \) at \( x = 1 \) and arbitrary exponents at \( x = 0, \infty \) were studied in [MV2]. It was discovered in [MV2] that the kernel

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of such a differential operator consists of Jacobi-Piñeiro polynomials, a special type of multiple orthogonal polynomials, see Lemma 4.4 in \cite{MV2}.

This appearance of orthogonal polynomials in the Bethe ansatz constructions helped us in \cite{MV2} study eigenvectors of the Gaudin Hamiltonians.

In this short paper, we give an example of a reverse implication, namely, that the Bethe ansatz considerations may be useful in studying orthogonal polynomials. We construct a Fuchsian differential operator with singular points at 0, 1, \infty annihilating the Jacobi-Piñeiro polynomial, see the precise statement and the discussion of the result in Section 5. Such an operator can be used in studying the Jacobi-Piñeiro polynomials.

We thank referees for helping to improve the exposition.

2. JACOBI-PiÑEIRO POLYNOMIALS

Let \( l_1, \ldots, l_r \) be integers such that \( l_1 \geq \cdots \geq l_r \geq 0 \). Let \( m_1, \ldots, m_r \) and \( k \) be negative real numbers. We use the notation \( \mathbf{m} = (m_1, \ldots, m_r), \mathbf{l} = (l_1, \ldots, l_r) \).

The Jacobi-Piñeiro polynomial \( P \) is the unique monic polynomial of degree \( l_1 \) whose coefficients are rational functions of \( \mathbf{m}, \mathbf{l}, k \) and which is orthogonal to functions

\[
\frac{1}{(x-1)^{k-1}} x^{m_1-1}, x^{-m_2-1}, \ldots, x^{m_r-1}
\]

with respect to the scalar product given by

\[
(f(x), g(x)) = \int_0^1 f(x)g(x)(x-1)^{-k-1}x^{m_1-1}dx.
\]

We denote the Jacobi-Piñeiro polynomial by \( P_{\mathbf{m}, \mathbf{l}, k}(x) \).

If \( l_2 = l_3 = \cdots = l_r = 0 \), then the Jacobi-Piñeiro polynomial is the classical Jacobi polynomial \( P_{l, \alpha, \beta}(x) \) on interval \([0, 1]\) with \( l = l_1, \alpha = -k - 1, \beta = -m_1 - 1 \).

The Jacobi-Piñeiro polynomial may be given by the Rodrigues-type formula, see \cite{ABV}:

\[
P_{\mathbf{m}, \mathbf{l}, k}(x) = c (x-1)^{k+1} x^{\sum_{i=1}^r m_i - r} \times
\]

\[
\times \frac{d^{l_1-l_r+1}}{dx^{l_r-l_r+1}} x^{l_1-l_r+1-m_r-1} \frac{d^{l_{r-1}-l_r}}{dx^{l_{r-1}-l_r}} \cdots x^{l_2-l_3-m_2-1} \frac{d^{l_1-l_2}}{dx^{l_1-l_2}} (x^{l_1-l_2-m_1-1}(x-1)^{l_1-k-1}),
\]

where \( c \) is a nonzero constant.

The coefficients of the Jacobi-Piñeiro polynomial \( P_{\mathbf{m}, \mathbf{l}, k}(x) \) are rational functions of \( \mathbf{m}, \mathbf{l}, k \) and therefore the polynomial \( P_{\mathbf{m}, \mathbf{l}, k}(x) \) is well defined for almost all complex \( m_1, \ldots, m_r, k \).

3. SPACES OF POLYNOMIALS THE FIRST AND SECOND TYPE

We describe remarkable spaces of polynomials which contain Jacobi-Piñeiro polynomials. See \cite{MV2} for the relation of these spaces to the Bethe Ansatz method.
Parameters \((m, l, k)\) are called consistent if all \(m_i, l_i\) and \(k\) are nonnegative integers satisfying
\[
k \geq l_1 \geq l_2 \geq \cdots \geq l_r \geq 0, \quad l_s - l_{s+1} \leq m_s \quad (s = 1, \ldots, r).
\]
(3.1)

Let \((m, l, k)\) be consistent. We use the convention:
\[
l_0 = k, \quad l_1 = 0.
\]

We call a complex \(r + 1\)-dimensional vector space of polynomials \(V(m, l, k) \subset \mathbb{C}[x]\) the space of polynomials of the first type associated to \((m, l, k)\) if the space satisfies the following two conditions:

- The space \(V(m, l, k)\) has a basis of the form
  \[
  \{v_0(m, l, k), v_1(m, l, k)x^{m_1+1}, v_2(m, l, k)x^{m_1+m_2+2}, \ldots, v_r(m, l, k)x^{\sum_{i=1}^r m_i + r}\}, \quad (3.2)
  \]
  where for \(i = 0, \ldots, r\), the polynomial \(v_i(m, l, k) \in \mathbb{C}[x]\) is a monic polynomial of degree \(k - l_i + l_{i+1}\).

- If a polynomial \(p \in V(m, l, k)\) vanishes at \(x = 1\), then the multiplicity of zero at \(x = 1\) is at least \(k + 1\).

It is easy to see that the basis polynomials in (3.2) have increasing degrees.

Below we will show that for any consistent parameters \((m, l, k)\), there exists a unique space of polynomials of the first type associated to \((m, l, k)\), see Theorem 5.2. Moreover, we will show that this space contains the Jacobi-Piñeiro polynomial \(P(m, l, k)\), see Lemma 4.1.

We call a complex \(r + 1\)-dimensional vector space of polynomials \(U(m, l, k) \subset \mathbb{C}[x]\) the space of polynomials of the second type associated to \((m, l, k)\) if the space satisfies the following two conditions:

- The space \(U(m, l, k)\) has a basis of the form
  \[
  \{u_0(m, l, k), u_1(m, l, k)x^{m_r+1}, u_2(m, l, k)x^{m_r+m_{r-1}+2}, \ldots, u_r(m, l, k)x^{\sum_{i=1}^r m_i + r}\}, \quad (3.3)
  \]
  where for \(i = 0, \ldots, r\), the polynomial \(u_i(m, l, k) \in \mathbb{C}[x]\) is a monic polynomial of degree \(l_{r-i} - l_{r-i+1}\).

- There exists a nonzero polynomial \(p \in U(m, l, k)\) which has zero at \(x = 1\) of order \(k + r\).

It is easy to see that the basis polynomials in (3.3) have increasing degrees.

Below we will show that for any consistent parameters \((m, l, k)\), there exists a unique space of polynomials of the second type associated to \((m, l, k)\), see Theorem 5.2.

The spaces of the first type and of the second type are dual in the sense of [MV1] which we now describe.

Define an \(r\)-tuple \(T = (T_1, \ldots, T_r)\) of polynomials in \(x\) by
\[
T_1 = (x - 1)^k x^{m_1}, \quad T_i = x^{m_i} \quad (i = 2, \ldots, r).
\]
(3.4)
For functions $f_1, \ldots, f_s$ of $x$, the Wronskian $W(f_1, \ldots, f_s)$ is defined by

$$W(f_1, \ldots, f_s) = \det \left( \frac{d^i}{dx^i} f_j \right)_{i,j=1,\ldots,s}.$$ 

For functions $f_1, \ldots, f_s$ of $x$, define the divided Wronskians $W^V(f_1, \ldots, f_s)$ and $W^U(f_1, \ldots, f_s)$ by

$$W^V(f_1, \ldots, f_s) = W(f_1, \ldots, f_s) T_1^{1-s} T_2^{2-s} \cdots T_{s-1}^{-1},$$
$$W^U(f_1, \ldots, f_s) = W(f_1, \ldots, f_s) T_r^{1-s} T_{r-1}^{2-s} \cdots T_{r-s+2}^{-1}.$$ 

**Lemma 3.1.** Let $(m, l, k)$ be consistent parameters.

Let $V$ be a space of the first type associated to $(m, l, k)$. Then the space

$$U = \{ W^V(f_1, \ldots, f_r), \ f_1, \ldots, f_r \in V \}$$

is a space of polynomials of the second type associated to $(m, l, k)$.

Let $U$ be a space of the second type associated to $(m, l, k)$. Then the space

$$V = \{ W^U(f_1, \ldots, f_r), \ f_1, \ldots, f_r \in U \}$$

is a space of polynomials of the first type associated to $(m, l, k)$.

**Proof.** The lemma follows from the definitions. \hfill \Box

4. **Recursion for spaces $V(m, l, k)$**

We show the existence of spaces $V(m, l, k)$ of the first type by constructing them recursively as follows.

Let $m_1, \ldots, m_r$ be nonnegative numbers. Let $0 = (0, \ldots, 0)$. Then clearly the parameters $(m_0, 0, k = 0)$ are consistent.

Introduce the numbers

$$e_i = i + \sum_{j=1}^{i} m_j, \quad (i = 0, \ldots, r).$$

In particular, $e_0 = 0$.

**Lemma 4.1.** The space

$$V(m_0, 0) = \text{span} \{ 1 = x^{e_0}, \ x^{e_1}, \ldots, \ x^{e_r} \}$$

is a space of the first type associated to $(m_0, 0)$.

**Proof.** The lemma is proved by direct verification. \hfill \Box

For $i = 0, 1, \ldots, r$, introduce the first order linear differential operators

$$D_i(m, l, k) = x(x - 1) \frac{d}{dx} - (k + \sum_{s=1}^{i} m_s - l_i + l_{i+1} + i)(x - 1) - k - 1.$$ (4.1)
For $i = 1, \ldots, r$, let $1_i = (1, \ldots, 1, 0, \ldots, 0)$ be the $r$-tuple where we have $i$ ones and $r - i$ zeros. Let $1_0 = 0 = (0, \ldots, 0)$.

For all $i, j \in \{0, 1, \ldots, r\}$, we have

$$D_j(m, l + 1_i, k + 1) D_i(m, l, k) = D_i(m, l, k) D_j(m, l + 1_j, k + 1).$$

**Lemma 4.2.** Suppose $(m, l, k)$ and $(m, l + 1, k + 1)$ are consistent parameters. Let $V(m, l, k)$ be a space of the first type associated to $(m, l, k)$. Then the space

$$V(m, l + 1, k + 1) = \{ D_i(m, l, k) v \mid v \in V(m, l, k) \}$$

is a space of the first type associated to $(m, l + 1, k + 1)$.

**Proof.** The proof is straightforward. \qed

**Theorem 4.3.** Let $(m, l, k)$ be consistent parameters. Then there exist a space $V(m, l, k)$ of the first type and a space $U(m, l, k)$ of the second type associated to $(m, l, k)$.

**Proof.** Let $i(1), \ldots, i(k) \in \{0, \ldots, r\}$ be any sequence of indices such that $0$ occurs in the sequence exactly $k - l_1$ times and every number $i = 1, \ldots, r$ occurs in the sequence exactly $l_i - l_{i+1}$ times. Then $l = \sum_{s=1}^{k} 1_{i(s)}$ and for every for $j = 0, \ldots, k$, the tuple $(m, \sum_{s=1}^{j} 1_{i(s)}, j)$ forms a consistent set of parameters.

Introduce the linear differential operator

$$D_{m,l,k} = D_{j(k)}(m, \sum_{s=1}^{k-1} 1_{j(s)}, k - 1) \ldots D_{j(2)}(m, 1_{j(1)}, 1) D_{j(1)}(m, 0, 0) \quad (4.2)$$

of order $k$.

By Lemma 4.1 a space $V(m, l, k)$ of the first type associated to $(m, l, k)$ can be constructed by application of the operator $D_{m,l,k}$ to the space $V(m, 0, 0)$ of Lemma 4.1.

A space $U(m, l, k)$ of the second type associated to $(m, l, k)$ can be constructed from the space of the first type by the construction of Lemma 3.1. \qed

Let $(m, l, k)$ be consistent parameters. Let $V(m, l, k)$ be the space of the first type associated to $(m, l, k)$. Let $v_0(m, l, k) \in V(m, l, k)$ be the monic polynomial of degree $l_1$. Such a polynomial in $V(m, l, k)$ is unique according to the definition of the space of the first type associated to $(m, l, k)$.

**Lemma 4.4.** The polynomial $v_0(m, l, k)$ is the Jacobi-Piñeiro polynomial $P(m, l, k)$.

**Proof.** The polynomial $v_0(m, l, k)$ is obtained by application of the operator $D_{m,l,k}$ to the function $1$. It is a straightforward calculation to check that this formula for $v_0(m, l, k)$ coincides with the Rodrigues-type formula for the Jacobi-Piñeiro polynomial $P(m, l, k)$ in formula (2.1). \qed
5. The differential operator $D^\gamma_{m,l,k}$.

Let $(m,l,k)$ be consistent parameters. We recall our convention $l_0 = k$, and $l_{r+1} = 0$. Consider two sets of numbers,

$$d_i(m,l,k) = \sum_{s=r+1-i}^{r} m_s - l_{r-i+1} + l_{r-i} + i,$$

$$a_i(m,l,k) = \sum_{s=r+1-i}^{r} m_s + i,$$

where $i = 0, \ldots, r$, and two polynomials in $\alpha$ whose roots are those numbers,

$$d(\alpha; m,l,k) = \prod_{i=0}^{r} (\alpha - d_i(m,l,k)), \quad a(\alpha; m,l,k) = \prod_{i=0}^{r} (\alpha - a_i(m,l,k)).$$

Define the numbers $A_i$ and $B_i$ for $i = 0, \ldots, r + 1$, as the coefficients of the following decompositions

$$d(\alpha; m,l,k) = A_0(m,l,k) + \sum_{i=1}^{r+1} A_i(m,l,k) \alpha(\alpha - 1) \ldots (\alpha - i + 1),$$  

$$(5.1)$$

$$a(\alpha; m,l,k) = B_0(m,l,k) + \sum_{i=1}^{r+1} B_i(m,l,k) \alpha(\alpha - 1) \ldots (\alpha - i + 1).$$  

$$(5.2)$$

Clearly we have $A_{r+1}(m,l,k) = B_{r+1}(m,l,k) = 1$, $A_0(m,l,k) = d(0; m,l,k)$, $B_0(m,l,k) = a(0; m,l,k) = 0$.

**Remark.** If $f$ is a polynomial in $\alpha$ and $\Delta f(\alpha) = f(\alpha + 1) - f(\alpha)$, then

$$f(\alpha) = \sum_{i} \frac{\Delta^i f(0)}{i!} \alpha(\alpha - 1) \ldots (\alpha - i + 1).$$

Introduce the monic linear differential operator

$$D^\gamma_{m,l,k} = \sum_{i=0}^{r+1} A_i(m,l,k) \frac{x - B_i(m,l,k)}{x^{r+1-i}(x - 1)} \frac{d^i}{dx^i}$$

of order $r + 1$.

**Lemma 5.1.** Let $U(m,l,k)$ be a space of the second type associated to $(m,l,k)$. Then $U(m,l,k)$ is the kernel of $D^\gamma_{m,l,k}$.

**Proof.** Let

$$\tilde{D} = \sum_{i=0}^{r+1} c_i(x) \frac{d^i}{dx^i}$$

be the monic differential operator of order $r + 1$ with kernel $U(m,l,k)$. The operator $\tilde{D}$ is a Fuchsian differential operator with singular points at $0,1,\infty$. The coefficients $c_i$ are rational functions in $x$. A coefficient $c_i$ may have poles only at $x = 0$ and $x = 1$ of orders at most $i - r - 1$ and the degree of $c_i$ at infinity is at most $i - r - 1$. It is easy to see that
the poles of coefficients $c_i$ at $x = 1$ are at most simple, cf. for example, formula (5.1) in \cite{MV1}. Therefore, the coefficients $c_i$ can be written in the form

$$c_i = \frac{\tilde{A}_i x - \tilde{B}_i}{x^{r+1-i}(x-1)}.$$  

From the characteristic equation for exponents of $\tilde{D}$ at $x = \infty$ and formula (5.1), we conclude that $\tilde{A}_i = A_i(m, l, k)$. From the characteristic equation for exponents of $\tilde{D}$ at $x = 0$ and formula (5.2), we conclude that $\tilde{B}_i = B_i(m, l, k)$. \hfill $\square$

**Theorem 5.2.** The space $V(m, l, k)$ of the first type and the space $U(m, l, k)$ of the second type associated to $(m, l, k)$ are unique.

**Proof.** The space of the second type is unique by Lemma 5.1. The space of the first type is unique by Lemma 3.1, since formulas of Lemma 3.1 allow us recover uniquely the space of the first type from the unique space of the second type. \hfill $\square$

**Remark.** Set

$$T(x) = (x-1)^k x^{\sum_{i=1}^{r} i m_i}. \quad (5.3)$$

It is easy to see that if $f_1, \ldots, f_{r+1}$ is a basis of $U(m, l, k)$, then the Wronskian of $f_1, \ldots, f_{r+1}$ is equal to $T$ up to multiplication by a number.

6. **DIFFERENTIAL EQUATION FOR THE JACOBI-PiñeIRO POLYNOMIAL**

In this section we present a linear differential operator of order $r + 1$ annihilating the Jacobi-Piñeiro polynomial $P_{m,l,k}(x)$.

Set

$$\tau(x) = (x-1)^k x^{\sum_{i=1}^{r} i m_i}.$$  

Define the linear differential operator,

$$D_{m,l,k} = \tau(x) \sum_{i=0}^{r+1} (-1)^{r+1+i} \frac{d^{i}}{dx^{i}} \frac{A_i(m, l, k) x - B_i(m, l, k)}{x^{r+1-i}(x-1)} \frac{1}{\tau(x)}.$$  

The operator has order $r + 1$ and rational coefficients. Being written in the form

$$D_{m,l,k} = \sum_{i=0}^{r+1} c_i(x) \frac{d^{i}}{dx^{i}},$$

the operator has the leading coefficient $c_{r+1}$ equal to one.

For example, for $r = 1$, the operator is the classical hypergeometric differential operator

$$\frac{d^{2}}{dx^{2}} - \frac{k x + m_1 (x-1)}{x(x-1)} \frac{d}{dx} + \frac{l_1 (k + m_1 + 1 - l_1)}{x(x-1)}.$$
Theorem 6.1. Let \((m, l, k)\) be consistent parameters. Let \(V(m, l, k)\) be the space of the first type associated to \((m, l, k)\). Then the kernel of \(D_{m,l,k}\) is \(V(m, l, k)\).

Proof. The operator \(D_{m,l,k}\) is obtained from the operator \(D_{m,l,k}^{\tau}\) by form conjugation followed by the conjugation with the operator of multiplication by the function \(\tau(x)\).

The kernel of the operator \(D_{m,l,k}^{\tau}\) is the space \(U(m, l, k)\). Then, by standard arguments, the kernel of the operator formally conjugated to \(D_{m,l,k}^{\tau}\) consists of the functions

\[
\left\{ \frac{W(g_1, \ldots, g_r)}{T(x)}, \ g_1, \ldots, g_r \in U(m, l, k) \right\}
\]

where \(T(x)\) is defined in formula (3.3).

Then the kernel of the operator \(D_{m,l,k}\) consists of the functions

\[
\left\{ \frac{\tau(x)W(g_1, \ldots, g_r)}{T(x)}, \ g_1, \ldots, g_r \in U(m, l, k) \right\} = \left\{ W_{U}^{\tau}(g_1, \ldots, g_r), \ g_1, \ldots, g_r \in U(m, l, k) \right\}.
\]

This space coincides with \(V(m, l, k)\) by Lemma 3.1. \(\square\)

Corollary 6.2. For any integers \(l_1, \ldots, l_r, \ l_1 \geq \cdots \geq l_r \geq 0\), and any parameters \(m, k\), the Jacobi-Piñeiro polynomial \(P_{m,l,k}(x)\) is annihilated by the operator \(D_{m,l,k}\).

Proof. If the parameters \((m, l, k)\) are consistent, then the Jacobi-Piñeiro polynomial \(P_{m,l,k}(x)\) belongs to the space \(V(m, l, k)\) by Lemma 4.4. Therefore, the Jacobi-Piñeiro polynomial is annihilated by the operator \(D_{m,l,k}\) for consistent parameters \((m, l, k)\). This implies Corollary 6.2 since the polynomial \(P_{m,l,k}(x)\) and the operator \(D_{m,l,k}\) depend on the parameters \((m, l, k)\) as rational functions. \(\square\)

For \(r = 1\), the statement of the corollary is classical. It says that the Jacobi polynomial is a solution of the hypergeometric differential equation. For \(r = 2\), see both the theorem and the corollary in [MV2]. In [ABV], a recurrent procedure in \(r\) is given to construct a differential operator annihilating the Jacobi-Piñeiro polynomial \(P_{m,l,k}(x)\) and for \(r = 2\) the operator is given explicitly. One of the referees of this paper pointed to us the paper [CV], in which another (or maybe the same) differential operator annihilating the Jacobi-Piñeiro polynomial is constructed. It would be interesting to check if the differential operators of [ABV] and [CV] coincide with \(D_{m,l,k}\).

Remark. The differential operator \(D_{m,l,k}\), annihilating the Jacobi-Piñeiro polynomial \(P(m, l, k)\), is uniquely determined by the following properties:

- Coefficients of \(D_{m,l,k}\) are rational functions of \(x, m, l, k\).
- If \((m, l, k)\) are consistent, then the kernel of \(D_{m,l,k}\) consists of polynomials only.
- The singular points of \(D_{m,l,k}\) are at \(x = 0, 1, \infty\). All singular points are regular. The exponents of \(D_{m,l,k}\) at \(x = 0\) are 0, \(m_1 + 1, m_1 + m_2 + 2, \ldots, m_1 + \cdots + m_r + r\). The exponents at \(x = 1\) are 0, \(k + 1, k + 2, \ldots, k + r\). The exponents at \(x = \infty\) are the numbers \(k + \sum_{i=1}^{r} m_s - l_i + l_{i+1} + i\) for \(i = 0, \ldots, r\).
However, the characteristic equations for exponents of $D_{m,l,k}$ at $x = 0, 1, \infty$ do not determine the coefficients of $D_{m,l,k}$. The triviality of the monodromy of $D_{m,l,k}$ is essential for the uniqueness of the operator $D_{m,l,k}$, in contrast with the situation for the operator $D'_{m,l,k}$, where the uniqueness of the operator $D'_{m,l,k}$ is determined by the characteristic equations for exponents of $D'_{m,l,k}$ only.

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