Rational Matrix-Valued Pick Functions of Several Variables

M.F. Bessmertnyi

Abstract. For a rational function of several variables with nonnegative imaginary part on the upper poly-half-plane, the matrix representations are obtained.

MSC 2010. 32A08, 32A10, 47A56

Key words. long-resolvent representation, transfer-function realisation, positive-kernel decomposition.

1 Introduction

Let \( \Pi^d = \{ z \in \mathbb{C}^d \mid \text{Im} \, z_1 > 0, \ldots, \text{Im} \, z_d > 0 \} \) be an open upper poly-half-plane. The Pick class \( \mathcal{P}^{m \times m}_d \) consists of \( \mathbb{C}^{m \times m} \)-valued functions \( f(z_1, \ldots, z_d) \) holomorphic in \( \Pi^d \) and satisfying the condition

\[
(f(z) - f(z)^*)/2i \geq 0 \quad \text{for} \quad z \in \Pi^d,
\]

where * means transition to the Hermitian conjugate matrix. We will consider only rational function of the Pick class, as well as a subclass \( \mathcal{IP}^{m \times m}_d \) of rational Pick functions, which are Cayley inner. The latter means that such a function takes Hermitian matrix values almost everywhere on distinguished boundary \( \mathbb{R}^d \) of the upper poly-half-plane \( \Pi^d \), or equivalently, is the double Cayley transform (over the variables and over the matrix values) of an inner function on the unit polydisk \( \mathbb{D}^d = \{ z \in \mathbb{C}^d \mid |z_1| < 1, \ldots, |z_d| < 1 \} \). The structure of scalar rational inner functions in polydisk is considered in [15].

For one-variable functions \( f(\lambda) \) of the Pick class\(^1\), Nevanlinna’s integral representation is well known. Pick’s functions of several variables have been studied by many authors (see, for example, bibliography in

\(^1\)Other popular titles for the same class of one-variable functions are “Nevanlinna”, “Nevanlinna-Pick”, and “R-functions”.
In [1] for the Löwner class a generalisation of the classical Nevanlinna theorem to several variables was considered. The Löwner class $\mathcal{L}_d$ is a set of scalar Pick functions $h$ in $d$ variables such that

$$h(z) - h(w) = \sum_{k=1}^{d} (z_k - w_k)\Phi_k(z, w), \quad (1.2)$$

where $\Phi_k(z, w)$ is positive semidefinite, that is, for all $n \geq 1$, $z_1, \ldots, z_n \in \Pi^1$, $c_1, \ldots, c_n \in \mathbb{C}$,

$$\sum_{i,j=1}^{n} \Phi_k(z_j, z_i) c_i c_j \geq 0.$$

Note the checking of condition (1.2) for a function is a difficult.

The Herglotz-Agler class consists of the functions which are holomorphic on the right poly-half-plane and whose values on any commutative $d$-tuple of strictly accretive operators on Hilbert space have positive semidefinite real part. For rational Cayley inner functions $f(z)$ of the Gerglotz-Agler class, a long-resolvent representation was obtained in [2]:

$$f(z) = A_{11}(z) - A_{12}(z)A_{22}(z)^{-1}A_{21}(z), \quad (1.3)$$

where

$$\begin{pmatrix} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & A_{22}(z) \end{pmatrix} = H + z_1 A_1 + \cdots + z_d A_d,$$

and matrix coefficients satisfy the conditions

$$H^* = -H, \quad A_k \geq 0, \quad k = 1, \ldots, d. \quad (1.4)$$

It is easy to see that if in (1.4) the matrix $H$ satisfies condition

$$(H - H^*)/2i \geq 0, \quad (1.5)$$

then (1.3) is a rational matrix-valued function of the Pick class. Indeed, since $A_{22}(z)$ is an invertible matrix, we see that for $z, \zeta \in \mathbb{C}^d$:

$$f(z) = \begin{pmatrix} I_m \\ -A_{22}(\zeta)^{-1}A_{21}(\zeta) \end{pmatrix}^* \begin{pmatrix} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & A_{22}(z) \end{pmatrix} \begin{pmatrix} I_m \\ -A_{22}(z)^{-1}A_{21}(z) \end{pmatrix}.$$

From here we get

$$\frac{f(z) - f(\zeta)^*}{2i} =$$

$$\begin{pmatrix} I_m \\ -A_{21}(\zeta)^*A_{22}(\zeta)^{-1*} \end{pmatrix} \frac{H - H^*}{2i} \begin{pmatrix} I_m \\ -A_{22}(z)^{-1}A_{21}(z) \end{pmatrix} +$$

$$\sum_{k=1}^{d} \frac{z_k - \zeta_k}{2i} \begin{pmatrix} I_m \\ -A_{21}(\zeta)^*A_{22}(\zeta)^{-1*} \end{pmatrix} A_k \begin{pmatrix} I_m \\ -A_{22}(z)^{-1}A_{21}(z) \end{pmatrix}. \quad (1.6)$$

For $\zeta = z$ from (1.6) we obtain $f(z) \in \mathcal{P}_{d \times m}^m$. 

2
Remark 1.1. Let be \((H - H^*)/2i = B_0^*B_0, \ A_k = B_k^*B_k, \ k = 1, \ldots, d\).
Consider rational functions
\[\theta_k(z) = B_k \left( I_m - A_{22}(z)^{-1}A_{21}(z) \right), \ k = 0, 1, \ldots, d.\]
Then from (1.6) we obtain the positive-kernel decomposition
\[
\frac{f(z) - f(\zeta)^*}{2i} = \theta_0(\zeta)^*\theta_0(z) + \sum_{k=1}^{d} \frac{z_k - \zeta_k}{2i} \theta_k(\zeta)^*\theta_k(z).
\]

Another way to obtain rational functions of the Pick class is as follows. Let
\[
H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \frac{H - H^*}{2i} \geq 0
\]
be a \(z\)-independent block matrix, where block \(D\) has size \(n \times n\). If
\[
Z_n = \text{diag}\{I_{n_0}, z_1I_{n_1}, \ldots, z_dI_{n_d}\}, \quad n_0 + n_1 + \cdots + n_d = n,
\]
then function
\[
f(x) = A + BZ_n(I_n - DZ_n)^{-1}C \tag{1.7}
\]
is the rational matrix-valued function of the Pick class. Indeed, \(f(z) = A + BZ_n(I - DZ_n)^{-1}C = A - B(D - Z_n^{-1})^{-1}C\). Then
\[
f(z) = \left( \begin{pmatrix} I \\ -(D - Z_n^{-1})^{-1}C \end{pmatrix}^* \begin{pmatrix} A & B \\ C & D - Z_n^{-1} \end{pmatrix} \begin{pmatrix} I \\ -(D - Z_n^{-1})^{-1}C \end{pmatrix} \right).
\]
From here
\[
\frac{f(z) - f(z)^*}{2i} = \begin{pmatrix} I \\ -(D - Z_n^{-1})^{-1}C \end{pmatrix}^* \frac{H - H^*}{2i} \begin{pmatrix} I \\ -(D - Z_n^{-1})^{-1}C \end{pmatrix} + C^*(I - DZ_n)^{-1} \begin{pmatrix} Z_n - Z_n^* \\ 2i \end{pmatrix} (I - DZ_n)^{-1}C. \tag{1.8}
\]
It follows from (1.8) that \(f(z) \in \mathcal{P}_{d}^{m \times m}\).

For functions of the Pick class, the representation (1.7) is analog of the transfer-function realisation.

In this article, we will show that functions of the form (1.7) and (1.3) (subject to condition (1.5)) exhaust rational \(\mathbb{C}^{m \times m}\)-valued functions of the Pick class \(\mathcal{P}_{d}^{m \times m}\):
Theorem 1.1. (Main Theorem). Let $f(z)$ be a $C^{m \times m}$ valued function of $d$ complex variables. The following statements are equivalent.

(0) $f(z)$ is a rational Pick function.

(1) There exist $n, n_1, \ldots, n_d \in \mathbb{Z}_+, n_0 \geq 0, n = n_0 + n_1 + \cdots + n_d$ and the $z$-independent block $(m + n) \times (m + n)$ matrix

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \frac{H - H^*}{2i} \geq 0,$$

with block $D$ of size $n \times n$ such that

$$f(z) = A - B(D + Z_n)^{-1}C,$$

where $Z_n = \text{diag}\{0_{n_0}, z_1I_{n_1}, \ldots, z_dI_{n_d}\}$.

(2) There exist $n, n_1, \ldots, n_d \in \mathbb{Z}_+, n_0 \geq 0, n = n_0 + n_1 + \cdots + n_d$ and the $z$-independent block $(m + n) \times (m + n)$ matrix

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \frac{H - H^*}{2i} \geq 0,$$

with block $D$ of size $n \times n$ such that

$$f(z) = A + BZ_n(I_n - DZ_n)^{-1}C,$$

where $Z_n = \text{diag}\{I_{n_0}, z_1I_{n_1}, \ldots, z_dI_{n_d}\}$.

(3) The function $f(z)$ has a long-resolvent representation

$$f(z) = A_{11}(z) - A_{12}(z)A_{22}(z)^{-1}A_{21}(z),$$

where

$$\begin{pmatrix} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & A_{22}(z) \end{pmatrix} = H + z_1A_1 + \cdots + z_dA_d,$$

and the matrix coefficients satisfy the conditions:

$$(H - H^*)/2i \geq 0, \quad A_k \geq 0 \quad \text{for} \quad k = 1, \ldots, d.$$ (1.15)

If $f(z)$ is an inner function, then the matrices $H$ in (1), (2), (3) can be chosen as Hermitian ($H = H^*$).

The article is organized as follows. In Section 2, we reduce the study of rational Pick function to the study of a multi-affine function of the Pick class, the degree of which in each variable does not exceed 1.

Section 3 proved (Theorem 3.1) that every rational Pick function in $d$ variables can be obtained from the Cayley inner function in $d + 1$
variables of the Pick class. In Theorem 3.2 (based on Theorem 3.3), a
criterion is obtained for a multi-affine function to be an inner function
of the Pick class.

The necessary apparatus for obtaining representations is developed
in Sections 4, 5 and 6.

The basis is the Sum-of-Squares Theorem (Theorem 4.1). The proof
of the Sum-of Squares Theorem is given in Appendix (Section 9).

In Section 5, for multi-affine functions of the Pick class in several
variables, a generalization of the Darlington method of realization a
function is obtained (Theorem 5.1).

A superposition of matrix coefficients of fractional linear transfor-
mations of a special form is considered in Proposition 6.1.

The matrix representation of Cayley inner functions of the Pick class
was obtained in Section 7 (Theorem 7.1). The proof of the Main Theo-
rem is given in Section 8.

Preliminary information is provided in each section as needed.

2 Reduction to the Multi-Affine Functions

The rational $\mathbb{C}^{m \times m}$-valued function will be written in the form $f(z) = P(z)/q(z)$, where $P(z)$ is a $\mathbb{C}^{m \times m}$-valued polynomial and $q(z)$ is a scalar $\mathbb{C}$-valued polynomial. In fact, division $P(z)/q(z)$ is the standard operation of multiplying of the matrix $P(z)$ by the number $q(z)^{-1}$.

We say that the polynomial $p(z)$ affine in $z_k$ if $\deg_z p(z) = 1$, and we say $p(z)$ is multi-affine, if it is affine in $z_k$ for all $k = 1, \ldots, d$. A rational function $P(z)/q(z)$ will be called multi-affine if

$$\max\{\deg_z p(z), \deg_z q(z)\} = 1, \quad \forall k = 1, \ldots, d.$$ 

Elementary symmetric polynomial are examples of multi-affine functions:

$$\sigma_k(\zeta_1, \ldots, \zeta_n) = \sum_{i_1 < i_2 < \cdots < i_k} \zeta_{i_1}\zeta_{i_2} \cdots \zeta_{i_k}, \quad 0 < k \leq n, \quad \sigma_0(\zeta_1, \ldots, \zeta_n) \equiv 1.$$ 

Reduction of the problem to the multi-affine case is based on the use
of the degree reduction operator [9, 13].

Definition 2.1. Let $p(z_0, z) = \sum_{k=0}^{n_0} p_k(z) z_0^k$ be a polynomial and $n_0 \leq n \in \mathbb{N}$. A map

$$D^n_{z_0} \colon \sum_{k=0}^{n_0} p_k(z) z_0^k \mapsto \sum_{k=0}^{n_0} p_k(z) \binom{n}{k}^{-1} \sigma_k(\zeta_1, \ldots, \zeta_n) \quad (2.1)$$
is called a degree reduction operator in the variable $z_0$. If $f(z_0, z) = P(z_0, z)/q(z_0, z)$ is rational function and $\deg_{z_0} f(z_0, z) = n_0$, then the degree reduction operator is defined as

$$D^n_{z_0}[P(z_0, z)/q(z_0, z)] := D^n_{z_0}[P(z_0, z)]/D^n_{z_0}[q(z_0, z)].$$

Under the condition $\zeta_1 = \cdots = \zeta_n = z_0$ we get the original function. Thus, the operator $D^n_{z_0}$ is invertible. It turns out that the degree reduction operator (2.2) has the following important property.

**Theorem 2.1.** Let $f(z_0, z) = P(z_0, z)/q(z_0, z) \in \mathcal{P}^{m \times m}_{d+1}$ and $P, q$ are coprime polynomials. If $\deg_{z_0} f(z_0, z) = n_0$ and $n_0 \leq n \in \mathbb{N}$, then $\hat{f}(\zeta_1, \ldots, \zeta_n, z) = D^n_{z_0}[f(z_0, z)]$ is the Pick function of the class $\mathcal{P}^{m \times m}_{d+n}$, affine and symmetric in variables $\zeta_1, \ldots, \zeta_n$. Moreover,

$$\hat{f}(z_0, \ldots, z_0, z) = f(z_0, z).$$

We need the following statement.

**Theorem (Grace-Walch-Szegő).** (see [9], Theorem 2.12). Let $p$ be a symmetric multi-affine polynomial in $n$ complex variables, let $C$ be an open or closed circular region in $\mathbb{C}$, and let $\zeta_1, \ldots, \zeta_n$ be any fixed points in the region $C$. If $\deg p = n$ or $C$ is convex, then there exists at least one point $\xi \in C$ such that $p(\zeta_1, \ldots, \zeta_n) = p(\xi, \ldots, \xi)$.

**Remark 2.1.** Recall that a circular region is a proper subset of the complex plane, which is bounded by circles (straight lines). In particular, the half-plane is a convex circular region.

**Proof of Theorem 2.1.** A matrix-valued function $f(z_0, z)$ is the Pick function if and only if for any row vector $\eta \in \mathbb{C}^m$ scalar function $\eta f(z_0, z)\eta^*$ is Pick’s function. The coprime numerator and denominator of the scalar Pick function $p(z_0, z)/q(z_0, z)$ do not vanish $\Pi^{d+1}$. Since addition does not deduce from the class of Pick function, we see that $z_{d+1} + p(z_0, z)/q(z_0, z)$ is the Pick function in variables $z_0, z, z_{d+1}$. Then its numerator satisfies the condition

$$z_{d+1}q(z_0, z) + p(z_0, z) \neq 0 \quad \text{for} \quad \Im z_0 > 0, \ z \in \Pi^d, \ \Im z_{d+1} > 0. \quad (2.4)$$

Let us prove that the affine and symmetric in the variables $\zeta_1, \ldots, \zeta_n$ polynomial

$$\bar{p}(\zeta_1, \ldots, \zeta_n, z, z_{d+1}) = D^n_{z_0}[z_{d+1}q(z_0, z) + p(z_0, z)] = z_{d+1}D^n_{z_0}[q(z_0, z)] + D^n_{z_0}[p(z_0, z)], \quad (2.5)$$
do not vanish at
\[ \text{Im } \zeta_j > 0, \ j = 1, \ldots, n, \ z \in \Pi^d, \ \text{Im } z_{d+1} > 0. \] (2.6)

Indeed, if the variables satisfy (2.6) and \( z, z_{d+1} \) is fixed, then by the Grace-Walsh-Szegő Theorem there is \( \xi, \text{Im } \xi > 0 \) such that
\[ \tilde{p}(\zeta_1, \ldots, \zeta_n, z, z_{d+1}) = \tilde{p}(\xi, \ldots, \xi, z, z_{d+1}) = z_{d+1}q(\xi, z) + p(\xi, z) \neq 0. \]

Let \( \text{Im } \zeta_j > 0, \ j = 1, \ldots, n, \ z \in \Pi^d \) be fixed. Since the polynomial (2.5) vanishes at the point \( z_{d+1} = -\text{D}_{z_0}^n[p(z_0, z)]/\text{D}_{z_0}^n[q(z_0, z)] \) from the closed lower half-plane, we see that
\[ \text{Im } \tilde{f}(\zeta_1, \ldots, \zeta_n, z) = \text{Im } \left( \text{D}_{z_0}^n[p(z_0, z)]/\text{D}_{z_0}^n[q(z_0, z)] \right) \geq 0. \]

The relation (2.3) is obvious \( \square \)

3 Cayley inner functions of the Pick class

We will reduce the problem of obtaining a representation of an arbitrary rational Pick function to the problem of representing a rational Pick function, which is Cayley inner. Then, by Theorem 2.1, it suffices to restrict ourselves to multi-affine Cayley inner Pick functions.

In addition, we obtain a criterion (Theorem 3.2) for a multi-affine function to belong to the subclass \( \mathcal{IP}_d^m \times m \) of Cayley inner functions of Pick class \( \mathcal{P}_d^m \times m \).

At the points of continuity, rational Cayley inner function \( f(z) = P(z)/q(z) \) of the Pick class satisfies the condition
\[ f(\overline{z}) = f(z). \] (3.1)

(Here, the bar denotes complex conjugation: \( \overline{z} = (\overline{z}_1, \ldots, \overline{z}_d) \)). It follows from (3.1) that as the coefficients of the denominator \( q(z) \), we can always choose real numbers \( \overline{q(\overline{z})} = q(z) \). Then the coefficients of the numerator \( P(z) \) are Hermitian \( m \times m \) matrices.

It turns out that every rational Pick function in \( d \) variables can be obtained from the Cayley inner function in \( d + 1 \) variables:

**Theorem 3.1.** Let \( f(z) = P(z)/q(z) \) be a rational \( \mathbb{C}^{m \times m} \)-valued Pick function in \( d \) variables, where \( P(z), q(z) \) are coprime polynomials. If
\[
\begin{align*}
P_1(z) &= \frac{[P(z) - P(\overline{z})^*]}{2i}, & P_2(z) &= \frac{[P(z) + P(\overline{z})^*]}{2}, \\
qu_1(z) &= \frac{[q(z) - q(\overline{z})]}{2i}, & q_2(z) &= \frac{[q(z) + q(\overline{z})]}{2},
\end{align*}
\] (3.2)
then
\[ g(z_0, z) = \frac{z_0 P_1(z) + P_2(z)}{z_0 q_1(z) + q_2(z)} \quad (3.3) \]
is the rational Cayley inner function in \((d + 1)\) variables of the Pick class. Moreover,
\[ g(i, z) = f(z). \quad (3.4) \]

For proof this, we need several known statements.

We say that a multivariate polynomial \(p(z)\) with complex coefficients is \textit{stable} if it is nonzero whenever \(z \in \Pi^d\). A stable polynomial with real coefficients will be called \textit{a real stable}. The ring of the polynomials with real coefficients is denoted by \(R[z_1, \ldots, z_d]\).

**Lemma 3.1** ([8], Corollary 5.5, [7], Lemma 2.2). Let \(p(z) + iq(z) \neq 0\), where \(p, q \in R[z_1, \ldots, z_d]\) and let \(z_{d+1}\) be a new indeterminate. Then the following are equivalent.
(a) \(p(z) + iq(z)\) is stable,
(b) \(p(z) + z_{d+1}q(z)\) is real stable,
(c) all nonzero polynomials in the pencil
\[ \{ \alpha p(z) + \beta q(z) \mid \alpha, \beta \in \mathbb{R} \} \]
are real stable,
(d) \(\text{Im} p(z)/q(z) \geq 0 \) whenever \(\text{Im} z_i > 0\) for all \(1 \leq i \leq d\).

**Proof of Theorem 3.1.** Since \(f(z) = P(z)/q(z) \in \mathcal{P}_d^{m \times m}\), we see that
\[ f_\eta(z) = \frac{\eta P(z)\eta^*}{q(z)} \]
is the scalar function of the Pick class for every row vector \(\eta \in \mathbb{C}^m\). \(f_\eta(z) + \alpha, \) where \(\alpha \in \mathbb{R}\), is also function of the Pick class. Then its numerator \(\alpha q(z) + \eta P(z)\eta^*\) is a stable polynomial with complex coefficients. Using (3.2), we represent it the form
\[ \alpha q(z) + \eta P(z)\eta^* = i(\alpha q_1(z) + \eta P_1(z)\eta^*) + (\alpha q_2(z) + \eta P_2(z)\eta^*), \]
where \((\alpha q_1(z) + \eta P_1(z)\eta^*), (\alpha q_2(z) + \eta P_2(z)\eta^*) \in \mathbb{R}[z_1, \ldots, z_d].\)

Let \(z_0\) be a new indeterminate. By Lemma 3.1, for every \(\eta \in \mathbb{C}^m\)
\[ \alpha(z_0 q_1(z) + q_2(z)) + \eta(z_0 P_1(z) + P_2(z))\eta^* \]
\[ \tag{2} \]
This terminology may differ from the designations of other authors.
is the real stable polynomial in \((d+1)\) variables. Since \(\eta \in \mathbb{C}^m\) is an arbitrary vector, we see that for every \(\alpha, \beta \in \mathbb{R}\) all nonzero polynomials in the pencil
\[
\alpha(z_0q_1(z) + q_2(z)) + \beta(\eta(z_0P_1(z) + P_2(z))\eta^*)
\]
are real stable. Then
\[
(z_0q_1(z) + q_2(z)) + z_{d+1}(\eta(z_0P_1(z) + P_2(z))\eta^*)
\]
is the real stable polynomial in \((d+2)\) variables. Therefore, for every \(\eta \in \mathbb{C}^m\)
\[
\text{Im} \frac{\eta(z_0P_1(z) + P_2(z))\eta^*}{z_0q_1(z) + q_2(z)} \geq 0, \quad \text{whenever } \text{Im} \ z_i > 0 \text{ for all } 0 \leq i \leq d.
\]
From this
\[
g(z_0, z) = \frac{z_0P_1(z) + P_2(z)}{z_0q_1(z) + q_2(z)} \in \mathcal{P}_{d+1}^{m \times m}.
\]
For real values of variables, the values of \(g(x_0, x)\) are Hermitian matrices. Then \(g(z_0, z) \in \mathcal{I} \mathcal{P}_{d+1}^{m \times m}\). Identity (3.4) is obvious. \(\Box\)

We say that a matrix-valued polynomial \(F(z)\) is positive semidefinite or PSD if \(F(x) \geq 0\) for all \(x \in \mathbb{R}^d\).

For Cayley inner Pick functions, the following statement holds:

**Proposition 3.1.** If \(f(z) = P(z)/q(z) \in \mathcal{I} \mathcal{P}_{d}^{m \times m}\), then partial Wronskians
\[
W_k[q, P] = q(z)\frac{\partial P}{\partial z_k}(z) - P(z)\frac{\partial q}{\partial z_k}(z), \quad k = 1, \ldots, d \quad (3.5)
\]
are PSD polynomials.

**Proof.** Suppose \(k = 1\). Let us \(\varphi(\zeta) = f(\zeta, \hat{x})\). If \(\hat{x} = (x_2, \ldots, x_d) \in \mathbb{R}^{d-1}\), then \(\text{Im} \varphi(\zeta) \geq 0\) for \(\text{Im} \zeta > 0\) and \(\text{Im} \varphi(\zeta) = 0\) for \(\text{Im} \zeta = 0\). Hence inequality \(d\varphi(\zeta)/d\zeta |_{\zeta \in \mathbb{R}} \geq 0\) holds. From this
\[
W_{z_1}[q, P](x) = q(x)^2d\varphi(\zeta)/d\zeta |_{\zeta = x_1} \geq 0, \quad x \in \mathbb{R}^d. \quad \Box
\]

For multi-affine functions, the nonnegativity of the Wronskians is also a sufficient condition.

**Theorem 3.2.** A rational multi-affine matrix-valued function \(f(z) = P(z)/q(z)\) belongs to the class \(\mathcal{I} \mathcal{P}_{d}^{m \times m}\) if and only if all Wronskians
\[
W_k[q, P] = q(z)\frac{\partial P}{\partial z_k}(z) - P(z)\frac{\partial q}{\partial z_k}(z), \quad k = 1, \ldots, d
\]
are matrix-valued PSD polynomials.
To prove this theorem, we need the following statement.

**Theorem 3.3.** Let \( f(z) \) be a rational matrix-valued function. Let us assume that for every \( j = 1, 2, \ldots, d \) and for every real \( x_1, x_2, \ldots, x_d \), the functions

\[
f_j(z) = f(x_1, \ldots, x_{j-1}, z_j, x_{j+1}, \ldots, x_d)
\]

satisfy the conditions

\[
\frac{(f_j - f_j^*)}{2i} \geq 0 \quad \text{for} \quad \text{Im} \, z_j > 0.
\]

Then the inequality

\[
\frac{(f(z) - f(z)^*)}{2i} \geq 0 \quad \text{for} \quad z \in \Pi^d
\]

holds.

**Proof.** For row vector \( \xi \) does not depend on \( z \), let us consider the function \( f_\xi(z) = \xi f(z) \xi^* \). It is clear that

\[
\text{Im} \, f_\xi(z) = \xi \left[ \frac{(f(z) - f(z)^*)}{2i} \right] \xi^*.
\]

Therefore, if the inequality

\[
\text{Im} \, f_\xi(z) \geq 0 \quad \text{for} \quad z \in \Pi^d
\]

holds for every \( \xi \), then the inequality

\[
\frac{(f(z) - f(z)^*)}{2i} \geq 0 \quad \text{for} \quad z \in \Pi^d
\]

holds us well. Therefore, it is enough to consider scalar functions only. Let a scalar function \( f(z) \) satisfy the assumptions of the theorem. We consider rational function

\[
u(\zeta_1, \ldots, \zeta_d) = \frac{f \left( i \frac{1 + \zeta_1}{1 - \zeta_1}, \ldots, i \frac{1 + \zeta_d}{1 - \zeta_d} \right) - i}{f \left( i \frac{1 + \zeta_1}{1 - \zeta_1}, \ldots, i \frac{1 + \zeta_d}{1 - \zeta_d} \right) + i}.
\]

(3.6)

Let \( T \) be the unit circle. it is clear, that for every \( j = 1, 2, \ldots, d \) and for every \( t_1 \in T, \ldots, t_{j-1} \in T, t_{j+1} \in T, \ldots, t_d \in T \), the function \( u(t_1, \ldots, t_{j-1}, \zeta_j, t_{j+1}, \ldots, t_d) \) is holomorphic for \( |\zeta_j| < 1 \) and satisfies the inequality

\[
|u(t_1, \ldots, t_{j-1}, \zeta_j, t_{j+1}, \ldots, t_d)| < 1 \quad \text{for} \quad |\zeta_j| < 1.
\]

(3.7)
Let us show that the function \( u(\zeta_1, \ldots, \zeta_d) \) is holomorphic in polydisk
\[ D^d = \{ \zeta \in \mathbb{C}^d \mid |\zeta_1| < 1, \ldots, |\zeta_d| < 1 \} \]
and satisfies the inequality
\[ |u(\zeta_1, \ldots, \zeta_d)| < 1 \quad \text{for} \quad \zeta \in D^d. \]

To prove this, we consider the Fourier coefficients \( \hat{u}(k_1, \ldots, k_d) \) of the function \( u(t_1, \ldots, t_d) \) considered on the torus
\[ T^d = \{ t \in \mathbb{C}^d \mid |t_1| = 1, \ldots, |t_d| = 1 \}: \]
\[ \hat{u}(k_1, \ldots, k_d) = \int_{T^d} u(t_1, \ldots, t_d) t_1^{-k_1} \cdots t_d^{-k_d} m(dt_1) \cdots m(dt_d). \]

\((m(dt)\) is one-dimensional normalized Lebesgue measure). The function \( u \) is contractive on \( T^d \): \(|u(t)| \leq 1 \) for \( t \in T^d \). Therefore, its Fourier coefficients \( \hat{u}(k_1, \ldots, k_d) \) exist. If \( k_j < 0 \) at least for one \( j = 1, 2, \ldots, d \), then \( \hat{u}(k_1, \ldots, k_d) = 0 \). Indeed, for definiteness, let \( k_1 < 0 \). Then
\[ \hat{u}(k_1, \ldots, k_d) = \int_{T^{d-1}} \int_T u(t_1, \ldots, t_d) t_1^{-k_1} \cdots t_d^{-k_d} m(dt_1) \cdots m(dt_d). \]

By condition (3.7), the inner integral in (3.8) vanishes. Therefore, the Fourier coefficients \( \hat{u}(k_1, \ldots, k_d) \) determine the function
\[ g(\zeta_1, \ldots, \zeta_d) = \sum_{\forall k} \hat{u}(k_1, \ldots, k_d) \zeta_1^{k_1} \cdots \zeta_d^{k_d}, \]
which is holomorphic in \( D^d \). On the other hand, denoting \( \zeta_j = r_j t_j \) \((r_j \geq 0, \ t_j \in T)\), we obtain
\[ g(\zeta_1, \ldots, \zeta_d) = \sum_{\forall k} r_1^{k_1} \cdots r_d^{k_d} \hat{u}(k_1, \ldots, k_d) t_1^{k_1} \cdots t_d^{k_d}. \]

Therefore, \( g(\zeta_1, \ldots, \zeta_d) \) is the convolution of the function \( u(t_1, \ldots, t_d) \) and the Poisson kernel
\[ P(r, t) = \sum_{\forall k} r_1^{k_1} \cdots r_d^{k_d} t_1^{k_1} \cdots t_d^{k_d}. \]

Since \(|u(t_1, \ldots, t_d)| \leq 1 \) on \( T^d \) and
\[ \int_{T^d} P(r_1, \ldots, r_d, t_1, \ldots, t_d) m(dt_1) \cdots m(dt_d) = 1, \]
we have
\[ |g(\zeta_1, \ldots, \zeta_d)| \leq 1 \quad \text{for} \quad \zeta \in D^d. \]
Since the Poisson kernel is approximate identity,
\[ \lim_{r \to 1^{-}} g(r_1 t_1, \ldots, r_d t_d) = u(t_1, \ldots, t_d) \]
in every point \((t_1, \ldots, t_d) \in \mathbb{T}^d\) where the function \(u\) is continuous. By the uniqueness theorem,
\[ u(\zeta) = g(\zeta) \quad \text{for} \quad \zeta \in \mathbb{D}^d. \]
Returning to \(f(z_1, \ldots, z_d)\) by means of the transformation that is inverse to the transformation (3.6) we obtain the statement of theorem.

**Proof of Theorem 3.2.** The necessity is proved in Proposition 3.1. Let us prove the sufficiency. Since \(f(z)\) is multi-affine, we see that
\[ f(z) = z_k P_1(\hat{z}) + P_2(\hat{z}), \quad \hat{z} = (z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_d). \]
From here
\[ \text{Im} \ f(z_k, \hat{x}) = \text{Im} \ z_k \frac{W_k[q,P](\hat{x})}{|z_k q_1(\hat{x}) + q_2(\hat{x})|^2}, \quad \hat{x} \in \mathbb{R}^{d-1}. \] (3.9)
Hence \(\text{Im} \ f(z_k, \hat{x}) \geq 0, \text{Im} \ z_k > 0\) for each \(k = 1, \ldots, d\) (for any other real variables). By Theorem 3.3, \(\text{Im} \ f(z) \geq 0\) for \(z \in \Pi^d\). If \(\text{Im} \ z_k = 0, k = 1, \ldots, d\), then \(\text{Im} \ f(z_k, \hat{z}) = 0\), that is, \(f(z) \in \mathcal{I} \mathcal{P}^m_{d \times m}\).

### 4 PSD Polynomials and Sums of Squares

Recall that a matrix-valued polynomial \(F(z)\) is called a **positive semidefinite** or PSD if its values \(F(x)\) are positive semidefinite Hermitian matrices: \(F(x) \geq 0\) for every \(x \in \mathbb{R}^d\).

We say that a \(\mathbb{C}^{m \times m}\)-valued PSD polynomial \(F(z)\) is a **sum of squares** or **SOS** if
\[ F(z) = H(z) H(\bar{z})^*, \] (4.1)
where \(H(z)\) is some \(\mathbb{C}^{m \times k}\)-valued polynomial.

Not every PSD polynomial is an SOS. An amazing fact is that the PSD polynomials (3.5) associated with rational Cayley inner functions of the Pick class are SOS polynomials:

**Theorem 4.1.** (Sum-of-Squares Theorem). If \(f(z) = P(z)/q(z) \in \mathcal{I} \mathcal{P}^m_{d \times m}\), then
\[ W_k[q,P] = q(z) \frac{\partial P}{\partial z_k}(z) - P(z) \frac{\partial q}{\partial z_k}(z), \quad k = 1, \ldots, d \] (4.2)
are matrix-valued SOS polynomials.

For the convenience of the reader, the proof of this theorem is considered in Appendix.
5 Generalization of Darlington’s Theorem

The application of Darlington’s theorem to obtain representations of matrix functions in one variable of some classes was considered in [11]. In [13], an analogue of Darlington’s method was used for a positive real functions of several variables. This method allows us to reduce the question of representing a function in \(d\) variables to the question of representing a similar function in \(d - 1\) variables.

For multi-affine Cayley inner functions of the Pick class, the following generalization of Darlington theorem holds.

**Theorem 5.1.** Let a multi-affine function \(f(z_0, z) \in \mathcal{IP}^{m \times m}_{d+1}\) be represented as

\[
f(z_0, z) = \frac{P(z_0, z)}{q(z_0, z)} = \frac{z_0 P_1(z) + P_2(z)}{z_0 q_1(z) + q_2(z)}, \quad z \in \mathbb{C}^d,
\]

and let

\[
W_{z_0}[q, P] = P_1(z)q_2(z) - P_2(z)q_1(z) = \Phi_0(z)\Phi_0(\bar{z})^*\]

be an SOS polynomial. Suppose \(\Phi_0(z)\) has the size \(m \times r\). Then

(i) If \(q_1(z) \neq 0\), then

\[
g(z) = \begin{pmatrix} g_{11}(z) & g_{12}(z) \\ g_{21}(z) & g_{22}(z) \end{pmatrix} = \frac{1}{q_1(z)} \begin{pmatrix} P_1(z) & \Phi_0(z) \\ \Phi_0(\bar{z})^* & q_2(z)I_r \end{pmatrix}
\]

is a multi-affine function of class \(\mathcal{IP}^{(m+r) \times (m+r)}_d\) and

\[
f(z_0, z) = g_{11}(z) - g_{12}(z)(g_{22}(z) + z_0 I_r)^{-1}g_{21}(z). \quad (5.2)
\]

(ii) If \(q_2(z) \neq 0\), then

\[
g(z) = \begin{pmatrix} g_{11}(z) & g_{12}(z) \\ g_{21}(z) & g_{22}(z) \end{pmatrix} = \frac{1}{q_2(z)} \begin{pmatrix} P_2(z) & \Phi_0(z) \\ \Phi_0(\bar{z})^* & -q_1(z)I_r \end{pmatrix}
\]

is a multi-affine function of class \(\mathcal{IP}^{(m+r) \times (m+r)}_d\) and

\[
f(z_0, z) = g_{11}(z) - g_{12}(z)(g_{22}(z) - z_0^{-1} I_r)^{-1}g_{21}(z). \quad (5.4)
\]

**Proof.** Both cases are proved in similar way. Let us prove, for example, (ii). Representation (5.4) follows from the obvious identity

\[
f(z_0, z) = \frac{z_0 P_1(z) + P_2(z)}{z_0 q_1(z) + q_2(z)} = \frac{P_2}{q_2} - \frac{\Phi_0(z)\Phi_0(\bar{z})^*}{q_2^2(-q_1/q_2 - z_0^{-1})}. \quad (5.5)
\]
The multi-affinity of \( g(z) \) is obvious. Let us prove \( g(z) \in \mathcal{IP}^{(m+r)\times(m+r)} \). By Theorem 3.2, it suffices to prove \( F_k(z) = q_k^2 \partial g(z)/\partial z_k, \ k = 1, \ldots, d \) are PSD polynomials. \( f(z_0, z) \) is multi-affine. Then

\[
f(z_0, z) = \frac{P(z_0, z)}{q(z_0, z)} = \frac{z_0z_k\hat{P}_1 + z_0\hat{P}_2 + z_k\hat{P}_3 + \hat{P}_4}{z_0z_k\hat{q}_1 + z_0\hat{q}_2 + z_k\hat{q}_3 + \hat{q}_4}. \tag{5.6}
\]

From (5.6) and (5.3) we get

\[
W_{z_0}[q, P] = \Phi_0(z)\Phi_0(\bar{z})^* = z_k^3(\hat{q}_3\hat{P}_1 - \hat{q}_1\hat{P}_3) + \\
z_k(\hat{q}_4\hat{P}_1 - \hat{q}_1\hat{P}_4 + \hat{q}_3\hat{P}_2 - \hat{q}_2\hat{P}_3) + (\hat{q}_4\hat{P}_2 - \hat{q}_2\hat{P}_4), \tag{5.7}
\]

\[
F_k = q_k^2 \frac{\partial g(z)}{\partial z_k} = \left( \frac{\hat{P}_3\hat{q}_4 - \hat{P}_4\hat{q}_3}{\Phi_k(z)} \frac{\Phi_k(z)}{(\hat{q}_2\hat{q}_3 - \hat{q}_1\hat{q}_4)I_r} \right), \tag{5.8}
\]

where \( \Phi_k(z) = (z_k\hat{q}_3 + \hat{q}_4)\partial \Phi_0/\partial z_k - \hat{q}_3\Phi_0, k = 1, \ldots, d \). Note that \( \Phi_k(z) \) is actually independent of \( z_0 \) and \( z_k \).

The diagonal elements \( f_{ii}(z_0, z) \) of a rational matrix-valued function \( f(z_0, z) \in \mathcal{IP}^{m\times m} \) satisfy the condition \( f_{ii}(\bar{z}_0, \bar{z}) = f_{ii}(z_0, z) \). Since \( f_{ii} = p_{ii}/q \), we see that the coefficients of the polynomial \( q(z_0, z) \) can be considered real numbers. Note the identity

\[
(\hat{P}_3\hat{q}_4 - \hat{P}_4\hat{q}_3) = \Phi_k(z)(\hat{q}_2\hat{q}_3 - \hat{q}_1\hat{q}_4)^{-1}\Phi_k(\bar{z})^*, \tag{5.9}
\]

or, equivalently,

\[
(\hat{P}_3\hat{q}_4 - \hat{P}_4\hat{q}_3)(\hat{q}_2\hat{q}_3 - \hat{q}_1\hat{q}_4) = \Phi_k(z)\Phi_k(\bar{z})^*. \tag{5.10}
\]

Indeed, since the coefficients of the polynomials \( \hat{q}_i(z) \) are real, then

\[
\Phi_k(z)\Phi_k(\bar{z})^* = (z_k\hat{q}_3 + \hat{q}_4)^2\frac{\partial \Phi_0(z)}{\partial z_k}\frac{\Phi_0(\bar{z})^*}{\partial z_k} - \\
(z_k\hat{q}_3^2 + \hat{q}_3\hat{q}_4) \left( \frac{\partial \Phi_0(z)}{\partial z_k} \Phi_0(\bar{z})^* + \Phi_0(z)\frac{\partial \Phi_0(\bar{z})^*}{\partial z_k} \right) + \hat{q}_3^2\Phi_0(z)\Phi_0(\bar{z})^*. \tag{5.11}
\]

\( \Phi_0(\bar{z}) \) is a multi-affine polynomial. Differentiating (5.7) in \( z_k \) and substituting the obtained expressions into (5.11), we get (5.10).

Since

\[
h = \frac{q}{\partial q/\partial z_0}\bigg|_{z_0=0} = \frac{z_k\hat{q}_3 + \hat{q}_4}{z_k\hat{q}_1 + \hat{q}_2} \in \mathcal{IP}_d,
\]

we see that \( Q(z) = \hat{q}_3(z)\hat{q}_3(z) - \hat{q}_1(z)\hat{q}_4(z) \) is PSD polynomial. For \( x \in \mathbb{R}^{d-1} \), from (5.8), (5.9) we obtain

\[
F_k(x) = \left( I_m \begin{array}{c} \Phi_k(x)Q^{-1}(x) \\ 0 \end{array} \right) \left( \begin{array}{cc} 0 & 0 \\ 0 & Q(x)I_r \end{array} \right) \left( I_m \begin{array}{c} \Phi_k(x)^T \\ 0 \end{array} \right) \geq 0,
\]

i.e., \( F_k(z), k = 1, \ldots, d \) are the PSD polynomials. By Theorem 3.2, the function \( g(z) \) (5.3) belongs to the class \( \mathcal{IP}_d^{(m+r)\times(m+r)} \). □
6 Superposition of Coefficient Matrix of Fractional Linear Transformations

We will need a somewhat unusual superposition of fractional linear transformations of the form

\[ W = g_{11} - g_{12}(g_{22} + Z)^{-1}g_{21}. \]  

(6.1)

The unusual thing is that the argument of transformation (6.1) is assumed to be the matrix

\[ \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \]

and the matrix \( Z \) plays the role of parameter defining transformation (6.1). When considering the superposition of such transformations, it is convenient to use the following notation for transformation (6.1):

\[ W = g_{11} - g_{12}(g_{22} + Z)^{-1}g_{21} := \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} + Z \end{pmatrix}. \]  

(6.2)

The following statement holds.

**Proposition 6.1.** If

\[ W = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} + Z_1 \end{pmatrix} \]  

(6.3)

and

\[ \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}, \]  

(6.4)

then

\[ W = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} + Z_1 & a_{23} \\ a_{31} & a_{32} & a_{33} + Z_2 \end{pmatrix}. \]  

(6.5)

**Proof.** It follows from (6.4) and (6.3) that the matrices \( S = (a_{33} + Z_2) \) and \( g_{22} + Z_1 = (a_{22} + Z_1) - a_{23}(a_{33} + Z_2)^{-1}a_{32} \) are invertible. By the Frobenius formula [10], there exists

\[ \begin{pmatrix} a_{22} + Z_1 & a_{23} \\ a_{32} & a_{33} + Z_2 \end{pmatrix}^{-1} = \begin{pmatrix} (g_{22} + Z_1)^{-1} & -(g_{22} + Z_1)^{-1}a_{23}S^{-1} \\ -S^{-1}a_{32}(g_{22} + Z_1)^{-1} & S^{-1} + S^{-1}a_{32}(g_{22} + Z_1)^{-1}a_{23}S^{-1} \end{pmatrix}. \]  

(6.6)
For the right-hand side of (6.5) we obtain
\[
\begin{align*}
a_{11} - (a_{12}, a_{13}) \times \\
(g_{22} + Z_1)^{-1} &- (g_{22} + Z_1)^{-1}a_{23}S^{-1} \\
-S^{-1}a_{32}(g_{22} + Z_1)^{-1} &- S^{-1}a_{32}(g_{22} + Z_1)^{-1}a_{23}S^{-1}
\end{align*}
\]
\[
\begin{pmatrix}
(a_{21} \\
(a_{31})
\end{pmatrix}
= a_{11} - a_{12}(g_{22} + Z_1)^{-1}a_{21} + a_{13}S^{-1}a_{32}(g_{22} + Z_1)^{-1}a_{21} + \\
a_{12}(g_{22} + Z_1)^{-1}a_{23}S^{-1}a_{31} - \\
a_{13}S^{-1}a_{31} - a_{13}S^{-1}a_{32}(g_{22} + Z_1)^{-1}a_{23}S^{-1}a_{31} = \\
(a_{11} - a_{13}S^{-1}a_{31}) - (a_{12} - a_{13}S^{-1}a_{32})(g_{22} + Z_1)^{-1}a_{21} + \\
(a_{11} - a_{13}S^{-1}a_{31})(g_{22} + Z_1)^{-1}a_{23}S^{-1}a_{31} = \\
(a_{11} - a_{13}S^{-1}a_{31}) - (a_{12} - a_{13}S^{-1}a_{32})(g_{22} + Z_1)^{-1}(a_{21} - a_{23}S^{-1}a_{31}) = \\
g_{11} = g_{12}(g_{22} + Z_1)^{-1}g_{21} = W,
\end{align*}
\]
which was required.

7 Representation of Cayley inner functions of the Pick class

For Cayley inner rational functions of the Pick class, the following statement holds:

**Theorem 7.1.** Let \( f \) be a \( \mathbb{C}^{m \times m} \)-valued function of \( d \) complex variables. The following statements are equivalent.

(0) \( f(z) \) is a rational Cayley inner function of the Pick class.

(1) There exist \( n, n_1, \ldots, n_d \in \mathbb{Z}_+ \), \( n_0 \geq 0 \), \( n = n_0 + n_1 + \cdots + n_d \) and the \( z \)-independent Hermitian \((m + n) \times (m + n)\) matrix

\[
B = \begin{pmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{pmatrix}, \quad B = B^*
\]

with block \( B_{22} \) of size \( n \times n \) such that

\[
f(z) = B_{11} - B_{12}(B_{22} + Z_n)^{-1}B_{21}
\]  

where \( Z_n = \text{diag}\{0_{n_0}, z_1I_{n_1}, \ldots, z_dI_{n_d}\} \).

**Proof.** (0) \( \Rightarrow \) (1). Let \( P(z), q(z) \) be coprime polynomials and \( f(z) = P(z)/q(z) \in \mathcal{L} \mathbb{P}_d^{m \times m} \). If

\[
\max\{\text{deg}_{z_j} P(z), \text{deg}_{z_j} q(z)\} = k_j, \quad j = 1, \ldots, d,
\]

16
then, by Theorem 2.1, a multi-affine function
\[ h_0(\zeta_1, \ldots, \zeta_k) = D_z^{k_1} \cdots D_z^{k_d}[f(z_1, \ldots, z_d)], \tag{7.2} \]
where \( k = k_1 + \cdots + k_d \), belongs to the class \( \mathcal{IP}_k^{m \times m} \). Note that the set of variables \( \{\zeta_1, \ldots, \zeta_k\} \) of the function (7.2) is the union of \( d \) subsets. By identifying all variables of every \( i \)-th subset with the variable \( z_i \), we obtain the original function \( f(z_1, \ldots, z_d) \).

We will construct the representation of the function (7.2) step by step. At the \( j \)-th step, we will obtain a multi-affine Cayley inner matrix-valued function \( h_j(\zeta) \) of the Pick class, depending on the \((k-j)\) variables.

**Step 1.** Since \( h_0(\zeta) \) is multi-affine, we see that
\[ h_0(\zeta_1, \ldots, \zeta_k) = \frac{\zeta_1 P_1(\zeta) + P_2(\zeta)}{\zeta_1 q_1(\zeta) + q_2(\zeta)}, \tag{7.3} \]
where \( P_1, P_2, q_1, q_2 \) are independent of \( \zeta_1 \). By Proposition 3.1, the Wronskian
\[ W_{\zeta_1}[q, P] = P_1(\zeta)q_2(\zeta) - P_2(\zeta)q_1(\zeta) \tag{7.4} \]
is the PSD polynomial.

By Sum-of-Squares Theorem, there exists \( \mathbb{C}^{m \times r_1} \)-valued polynomial \( \Phi_1(\zeta) \) such that
\[ W_{\zeta_1}[q, P] = \Phi_1(\zeta)\Phi_1(\zeta)^*. \]
The polynomial \( \Phi_1(\zeta) \) does not depend on \( \zeta_1 \) and has degree at most 1 for every variable.

For the denominator \( \zeta_1 q_1(\zeta) + q_2(\zeta) \) in (7.3) there are 2 possibilities:

(i) \( q_1(\zeta) \neq 0 \). By Theorem 5.1,
\[ h_0(\zeta) = g_{11} - g_{12}(g_{22} + \zeta_1 I_{r_1})^{-1} g_{21} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} + \zeta_1 I_{r_1} \end{pmatrix}, \tag{7.5} \]
where multi-affine function
\[ g_1(\zeta) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \frac{1}{q_1(\zeta)} \begin{pmatrix} P_1(\zeta) & \Phi_1(\zeta) \\ \Phi_1(\zeta)^* & q_2(\zeta)I_{r_1} \end{pmatrix} \tag{7.6} \]
belongs to the class \( \mathcal{IP}_{k-1}^{(m+r_1) \times (m+r_1)} \).

(ii) \( q_1(\zeta) \equiv 0 \). Then \( q_2(\zeta) \neq 0 \). By Theorem 5.1,
\[ h_0(\zeta) = s_{11} - s_{12}(s_{22} - \zeta_1^{-1} I_{r_1})^{-1} s_{21} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} - \zeta_1^{-1} I_{r_1} \end{pmatrix}, \tag{7.7} \]
where
\[
\begin{pmatrix}
s_{11} & s_{12} \\
s_{21} & s_{22}
\end{pmatrix} = \frac{1}{q_2(\zeta)} \begin{pmatrix} P_2(\zeta) & \Phi_1(\zeta) \\ \Phi_1(\zeta)^* & 0_r \end{pmatrix} \in \mathcal{IP}^{(m+r_1)\times(m+r_1)}_{k-1}. \tag{7.8}
\]
By the Frobenius formula [10], there exists a matrix
\[
\begin{pmatrix}
s_{22} & I_{r_1} \\
I_{r_1} & \zeta_1 I_{r_1}
\end{pmatrix}^{-1} = \begin{pmatrix} (s_{22} - \zeta_1^{-1})^{-1} & * \\ * & * \end{pmatrix}.
\]
Then
\[
h_0(\zeta) = \begin{pmatrix}
s_{11} & s_{12} \\
s_{21} & s_{22} - \zeta_1^{-1} I_{r_1}
\end{pmatrix} = \begin{pmatrix}
s_{11} & s_{12} & 0 \\
s_{21} & s_{22} & I_{r_1} \\
0 & I_{r_1} & 0 + \zeta_1 I_{r_1}
\end{pmatrix}. \tag{7.9}
\]
It is easy to see that the matrix of coefficients
\[
s_1(\zeta) = \begin{pmatrix}
s_{11}(\zeta) & s_{12}(\zeta) & 0 \\
s_{21}(\zeta) & s_{22}(\zeta) & I_{r_1} \\
0 & I_{r_1} & 0
\end{pmatrix} \tag{7.10}
\]
of the fractional linear transformation (7.9) is a multi-affine Cayley inner function of the class \( \mathcal{IP}^{(m+2r_1)\times(m+2r_1)}_{k-1} \).

In both cases (i), (ii) we have
\[
h_0(\zeta) = \frac{h_1(\zeta)}{h_2(\zeta) + \Lambda_1}, \tag{7.11}
\]
where the multi-affine coefficient matrix
\[
h_1(\zeta) = \begin{pmatrix} h_{11}(\zeta) & h_{12}(\zeta) \\ h_{21}(\zeta) & h_{22}(\zeta) \end{pmatrix}
\]
does not depend on \( \zeta_1 \) and has the form (7.6) or (7.10), and
\[
\Lambda_1 = \zeta_1 I_{r_1} \quad \text{or} \quad \Lambda_1 = \text{diag}\{0_{r_1}, \zeta_1 I_{r_1}\}.
\]

**Step j**, \( 2 \leq j \leq k \). Suppose
\[
h_0(\zeta_1, \ldots, \zeta_k) = \frac{\hat{h}_{11}(\zeta) \quad \hat{h}_{12}(\zeta)}{\hat{h}_{21}(\zeta) \quad \hat{h}_{22}(\zeta) + \Lambda_{j-1}}, \tag{7.12}
\]
where
\[
\begin{pmatrix}
\hat{h}_{11}(\zeta) & \hat{h}_{12}(\zeta) \\
\hat{h}_{21}(\zeta) & \hat{h}_{22}(\zeta)
\end{pmatrix} = h_{j-1}(\zeta). \tag{7.13}
\]
is a multi-affine Cayley inner function of the Pick class independent of the variables $\zeta_1, \ldots, \zeta_{j-1}$ and

$$\Lambda_{j-1} = \text{diag}\{0_{j-1}, \zeta_1 I_{r_1}, \ldots, \zeta_{j-1} I_{r_{j-1}}\}.$$  

We represent (7.13) in the form

$$h_{j-1}(\zeta) = \frac{\zeta_j P_1(\zeta) + P_2(\zeta)}{\zeta_j q_1(\zeta) + q_2(\zeta)}, \quad (7.14)$$

Repeating the reasoning from Step 1 for $h_{j-1}(\zeta)$, we obtain

$$h_{j-1}(\zeta) = \left(\begin{array}{c} \hat{h}_{11} \\ \hat{h}_{21} \end{array} \right) = \left(\begin{array}{cc} a_{11}(\zeta) & a_{12}(\zeta) \\ a_{21}(\zeta) & a_{22}(\zeta) \end{array} \right) + \Lambda_{j-1}, \quad \Lambda_{j-1} = \text{diag}\{0_{j-1}, \zeta_1 I_{r_1}, \ldots, \zeta_{j-1} I_{r_{j-1}}\}, \quad (7.15)$$

where

$$\Lambda = \text{diag}\{0_{r_j}, \zeta_j I_{r_j}\}.$$  

By Proposition 6.1, from (7.12), (7.15) we obtain

$$h_0(\zeta_1, \ldots, \zeta_k) = \left(\begin{array}{c} a_{11}(\zeta) & a_{12}(\zeta) & a_{13}(\zeta) \\ a_{21}(\zeta) & a_{22}(\zeta) + \Lambda_{j-1} & a_{23}(\zeta) \\ a_{31}(\zeta) & a_{32}(\zeta) & a_{33}(\zeta) + \Lambda \end{array} \right), \quad (7.16)$$

If necessary, then by permutation the rows and corresponding columns in the last matrix and introducing a new division into blocks, we get

$$h_0(\zeta_1, \ldots, \zeta_k) = \left(\begin{array}{c} h_{11}(\zeta) \\ h_{21}(\zeta) \end{array} \right) = \left(\begin{array}{c} h_{12}(\zeta) \\ h_{22}(\zeta) + \Lambda_j \end{array} \right), \quad (7.17)$$

where $\{h_{kl}(\zeta)\}_{k,l=1}^{2} = h_j(\zeta)$ is multi-affine Cayley inner function of the Pick class independent of the variables $\zeta_1, \ldots, \zeta_j$, and

$$\Lambda_j = \text{diag}\{0_j, \zeta_1 I_{r_1}, \ldots, \zeta_{j-1} I_{r_{j-1}}, \zeta_j I_{r_j}\}.$$  

At the last step (for $j = k$), we obtain the matrix of coefficients of the fractional linear transformation (7.17), which does not depend on any of the variables. Since such a Cayley inner function of the Pick class is a constant Hermitian matrix, we see that

$$h_k(\zeta) \equiv B = \left(\begin{array}{cc} B_{11} & B_{12} \\ B_{21} & B_{22} \end{array} \right), \quad B = B^*$$

and

$$h_0(\zeta_1, \ldots, \zeta_k) = \left(\begin{array}{c} B_{11} \\ B_{21} \end{array} \right) = \left(\begin{array}{c} B_{12} \\ B_{22} + \Lambda_k \end{array} \right) = B_{11} - B_{12}(B_{22} + \Lambda_k)^{-1}B_{21}, \quad (7.18)$$
where
\[ \Lambda_k = \text{diag}\{0_k, \zeta_1 I_{r_1}, \ldots, \zeta_k I_{r_k}\} \].

Returning to the function \( f(z_1, \ldots, z_d) \) by identifying the variables of each \( i \)-th subset of variables with \( z_i \), we obtain the (7.1).

\[ (1) \Rightarrow (0) \]. The trivial identity
\[
\begin{pmatrix}
  f(z) \\
  0
\end{pmatrix} =
\begin{pmatrix}
  B_{11} & B_{12} \\
  B_{21} & B_{22} + Z_n
\end{pmatrix}
\begin{pmatrix}
  I_m \\
  -(B_{22} + Z_n)^{-1}B_{21}
\end{pmatrix}
\]

implies
\[
f(z) = B_{11} - B_{12}(B_{22} + Z_n)^{-1}B_{21} = (I_m, -B_{21}^*(B_{22} + Z_n)^{-1^*}) \times
\begin{pmatrix}
  B_{11} & B_{12} \\
  B_{21} & B_{22} + Z_n
\end{pmatrix}
\begin{pmatrix}
  I_m \\
  -(B_{22} + Z_n)^{-1}B_{21}
\end{pmatrix},
\] (7.19)

From (7.19) it follows that
\[
\frac{f(z) - f(z)^*}{2i} = B_{21}^*(B_{22} + Z_n)^{-1^*}Z_n - Z_n^* (B_{22} + Z_n)^{-1}B_{21},
\] (7.20)

that is, \( f(z) \) is a Cayley inner function of the Pick class.

\section{8 Proof of the Main Theorem}

Theorem 3.1 and 7.1 allows us to obtain a representation for a rational function \( f(z) \) of the Pick class.

\textit{Proof of Main Theorem.}\n
\( (0) \Rightarrow (1) \). Let be \( f(z) \in P_{d}^{m \times m} \). By Theorem 3.1, there exists \( g(z_0, z) \in T^{m \times m}_{d+1} \) such that
\[
g(i, z) = f(z).
\]

By Theorem 7.1, the function \( g(z_0, z) \) has a representation
\[
g(z_0, z) = B_{11} - B_{12}(B_{22} + Z)^{-1}B_{21} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} + Z \end{pmatrix}.
\] (8.1)

We rewrite the representation (8.1) in a more detailed form
\[
g(z_0, z) = \begin{pmatrix} B_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} + z_0 I_{r_0} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} + \hat{Z}_n \end{pmatrix},
\] (8.2)
where

\[ \hat{Z}_n = \text{diag}\{z_1I_{n_1}, \ldots, z_dI_{n_d}\}. \]

Setting in (8.2) \( z_0 = i \), we obtain

\[
f(z_1, \ldots, z_d) = \left( \begin{array}{ccc}
B_{11} & C_{12} & C_{13} & C_{14} \\
C_{21} & C_{22} & C_{23} & C_{24} \\
C_{31} & C_{32} & C_{33} + iI_{r_n} & C_{34} \\
C_{41} & C_{42} & C_{43} & C_{44} + \hat{Z}_n
\end{array} \right).
\] (8.3)

The matrix \( H \) of the fractional linear transformation (8.3) satisfies the condition \( (H - H^*)/2i \geq 0 \), as required.

(1) \( \Rightarrow \) (2). If \( f(z_1, \ldots, z_d) \in P_m^{d \times m} \), then

\[
g(z_1, \ldots, z_d) = f(-1/z_1, \ldots, -1/z_d) \in P_m^{d \times m}.
\]

By (1), we have

\[
g(z) = \left( \begin{array}{ccc}
H_{11} & H_{12} & H_{13} \\
H_{21} & H_{22} & H_{23} \\
H_{31} & H_{32} & H_{33} + \hat{Z}_n
\end{array} \right),
\]

where

\[ \hat{Z}_n = \text{diag}\{z_1I_{n_1}, \ldots, z_dI_{n_d}\}. \]

Then

\[
f(z_1, \ldots, z_d) = g(-1/z_1, \ldots, -1/z_d) =
\[
\left( \begin{array}{ccc}
H_{11} & H_{12} & H_{13} \\
H_{21} & (H_{22} + I_{r_n}) - I_{r_n}^{-1} & H_{23} \\
H_{31} & H_{32} & H_{33} - \hat{Z}_n^{-1}
\end{array} \right).
\] (8.4)

From (8.4) it follows that

\[
f(z) = A - B(D - Z_n^{-1})^{-1}C = A + BZ_n(I - DZ_n)^{-1}C,
\] (8.5)

where

\[
A = H_{11}, \quad B = (H_{12}, H_{13}), \quad C = \left( \begin{array}{c}
H_{21} \\
H_{31}
\end{array} \right), \quad D = \left( \begin{array}{cc}
H_{22} + I_{r_n} & H_{23} \\
H_{32} & H_{33}
\end{array} \right),
\]

and \( Z_n = \text{diag}\{I_{r_n}, z_1I_{n_1}, \ldots, z_dI_{n_d}\} \). The inequality (1.11) is obvious.

(2) \( \Rightarrow \) (1). If \( f(z) \in P_m^{d \times m} \), then

\[
g(z_1, \ldots, z_d) = f(-1/z_1, \ldots, -1/z_d) \in P_m^{d \times m}.
\]
By (2), we have
\[ g(z) = \hat{A} + \hat{B}Z(I - \hat{D}Z)^{-1}\hat{C} = \hat{A} - \hat{B}(\hat{D} - Z^{-1})^{-1}\hat{C}, \]
where
\[ Z = \begin{pmatrix} I_{n_0} & 0 \\ 0 & \hat{Z}_n \end{pmatrix}, \quad \hat{Z}_n = \text{diag} \{ z_1 I_{n_1}, \ldots, z_d I_{n_d} \}. \]

Then
\[ g(z) = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} - I_{n_0}^{-1} & H_{23} \\ H_{31} & H_{32} & H_{33} - \hat{Z}_n^{-1} \end{pmatrix}. \]

From here
\[ f(z) = g(-1/z_1, \ldots, -1/z_d) = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} + \hat{Z}_n \end{pmatrix}, \]
where \( \hat{H}_{22} = H_{22} - I_{n_0}^{-1} \), that is,
\[ f(z) = A - B(D + Z_n)^{-1}C, \quad Z_n = \text{diag} \{ 0_{n_0}, z_1 I_{n_1}, \ldots, z_d I_{n_d} \}. \]

(1) \( \Rightarrow \) (3). In fact, the representation (1.11)
\[ f(z) = A - B(D + Z_n)^{-1}C = \begin{pmatrix} H_{11} & H_{12} & H_{13} & \cdots & H_{1,d+2} \\ H_{21} & H_{22} & H_{23} & \cdots & H_{2,d+2} \\ H_{31} & H_{32} & H_{33} + z_1 I_{n_1} & \cdots & H_{3,d+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{d+2,1} & H_{d+2,2} & H_{d+2,3} & \cdots & H_{d+2,d+2} + z_d I_{n_d} \end{pmatrix} \tag{8.6} \]
is a representation of \( f(z) \) in the form of a long-resolvent [3, 4]:
\[ f(z) = A_{11}(z) - A_{12}(z)A_{22}(z)^{-1}A_{21}(z), \]
where the positive semidefinite matrix coefficients \( A_k \) for independent variables of the matrix pencil
\[ A(z) = \begin{pmatrix} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & A_{22}(z) \end{pmatrix} = H + z_1 A_1 + \cdots + z_d A_d \]
are pairwise orthogonal projectors: \( A_k^2 = A_k = A_k^*, k = 1, \ldots, d, \ A_jA_k = A_kA_j = 0, j \neq k. \)
Since $A_{22}(z)$ is an invertible matrix, we see that

$$f(z) = (I_m, -A_{21}^*A_{22}^{-1^*}) \left( \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right) \left( \begin{array}{c} I_m \\ -A_{22}^{-1}A_{21} \end{array} \right).$$

From here we get

$$\frac{f(z) - f(z)^*}{2i} =$$

$$(I_m, -A_{21}(z)^*A_{22}(z)^{-1^*}) \frac{H - H^*}{2i} \left( \begin{array}{c} I_m \\ -A_{22}(z)^{-1}A_{21}(z) \end{array} \right) + \sum_{k=1}^{d} \frac{z_k - \bar{z}_k}{2i} (I_m, -A_{21}(z)^*A_{22}(z)^{-1^*})A_k \left( \begin{array}{c} I_m \\ -A_{22}(z)^{-1}A_{21}(z) \end{array} \right), \quad (8.7)$$

that is, $f(z) \in \mathcal{P}_d^{m \times m}$.

The last statement of the theorem is obvious. \hfill \Box

### 9 Appendix. Sum-of-Squares Theorem

On $\mathbb{R}^d$, the values of $\mathbb{C}^{m \times m}$-valued Cayley inner functions $f(z_1, \ldots, z_d)$ of the Pick class are complex Hermitian matrices. For this case, the proof of Sum-of-Squares Theorem differs somewhat from given in [5], where scalar functions were considered.

First of all, consider several auxiliary concepts and theorem.

If $K$ is a field, then $K(x_1, \ldots, x_d)$ denotes the set of rational functions in variables $x_1, \ldots, x_d$ with coefficients from the fields $K$. Artin’s solution to Hilbert’s 17th problem for scalar polynomials is well known (see, for example, [14], Chapter XI, Corollary 3.3).

**Artin’s Theorem.** Let $K$ be a real fields admitting only one ordering. Let $f(x) \in K(x_1, \ldots, x_d)$ be a rational function that does not take negative values: $f(a) \geq 0$ for all $a = (a_1, \ldots, a_d) \in K^d$, in which $f(a)$ is defined. Then $f(x)$ is a sum of squares in $K(x_1, \ldots, x_d)$.

**Remark 9.1.** This means: for any polynomial $g(z) \in \mathbb{R}[z_1, \ldots, z_d]$ that is nonnegative on $\mathbb{R}^d$, there exists a polynomial $s(z) \in \mathbb{R}[z_1, \ldots, z_d]$ such that $s(z)^2g(z)$ is an SOS polynomial.

Polynomial $s(z)$ will be called an *Artin denominator* of the nonnegative polynomial $g(x) \geq 0$, $x \in \mathbb{R}^d$.

The $\mathbb{C}^{m \times m}$-valued PSD polynomial also has Artin’s denominator, which is a scalar polynomial with real coefficients.
Proposition 9.1. Let $F(z)$, $z \in \mathbb{C}^d$ be the $\mathbb{C}^{m \times m}$-valued PSD polynomial. Then there exists a scalar polynomial $s(z) \in \mathbb{R}[z_1, \ldots, z_d]$ such that

$$s(z)^2 F(z) = M(z)M(\overline{z})^*, \quad (9.1)$$

where $M(z)$ is some $\mathbb{C}^{m \times k}$-valued polynomial. That is, $s(z)^2 F(z)$ is an SOS polynomial.

Proof. Let $\Phi(x)$ be $\mathbb{R}^{m \times m}$-valued PSD polynomial. Using the Jacobi method of reducing a quadratic form to canonical form, Sylvester’s criterion and Artin’s Theorem we obtain: there exists a polynomial $s(z) \in \mathbb{R}[z_1, \ldots, z_d]$ such that $s(z)^2 \Phi(z) = R(z)R(z)^T$, where $R(x)$ is some $\mathbb{R}^{m \times k}$-valued polynomial.

Let $F(z)$ be a $\mathbb{C}^{m \times m}$-valued PSD polynomial and

$$A(z) = \frac{F(z) + F(\overline{z})}{2}, \quad B(z) = \frac{F(z) - F(\overline{z})}{2i}. \quad (9.2)$$

Since $F(\overline{z}) = F(z)^*$, we see that $A(x)$, $B(x)$ are $\mathbb{R}^{m \times m}$-valued polynomial satisfying the conditions

$$A(z)^T = A(z), \quad \text{and} \quad B(z)^T = -B(z). \quad (9.3)$$

For every row vector $\eta = \xi_1 + i\xi_2 \in \mathbb{C}^m$, where $\xi_1, \xi_2 \in \mathbb{R}^m$, we have

$$\eta F(x) \eta^* = (\xi_1 + i\xi_2)(A(x) + iB(x))(\xi_1^T - i\xi_2^T) =$$

$$(\xi_1, \xi_2) \begin{pmatrix} A(x) & B(x) \\ B(x)^T & A(x) \end{pmatrix} \begin{pmatrix} \xi_1^T \\ \xi_2^T \end{pmatrix} \geq 0. \quad (9.4)$$

Thus,

$$\Phi(x) = \begin{pmatrix} A(x) & B(x) \\ B(x)^T & A(x) \end{pmatrix}\begin{pmatrix} \xi_1^T \\ \xi_2^T \end{pmatrix}$$

is the $\mathbb{R}^{2m \times 2m}$-valued PSD polynomial. Therefore, there exist $s(z) \in \mathbb{R}[z_1, \ldots, z_d]$ and $\mathbb{R}^{m \times k}$-valued polynomials $R_1(x)$, $R_2(x)$ for which

$$s(z)^2 \begin{pmatrix} A(z) & B(z) \\ B(z)^T & A(z) \end{pmatrix} = \begin{pmatrix} R_1(z) \\ R_2(z) \end{pmatrix} \begin{pmatrix} R_1(z)^T \\ R_2(z)^T \end{pmatrix}. \quad (9.5)$$

From (9.3), (9.5) it follows that

$$R_1(z)R_1(z)^T = R_2(z)R_2(z)^T = s(z)^2 A(z), \quad (9.6)$$

$$R_1(z)R_2(z)^T = s(z)^2 B(z) = -R_2(z)R_1(z)^T. \quad (9.7)$$

24
Let us $H(z) = (R_1(z) - iR_2(z))/\sqrt{2}$. Taking into account (9.6), (9.7), we obtain

$$
H(z)H(\bar{z})^* = (R_1(z) - iR_2(z))(R_1(z)^T + iR_2(z)^T)/2 = (R_1(z)R_1(z)^T + R_2(z)R_2(z)^T)/2 + i(-R_2(z)R_1(z)^T + R_1(z)R_2(z)^T)/2 = s(z)^2(A(z) + iB(z)) = s(z)^2F(z), \quad (9.8)
$$

i.e., $s(z)^2F(z)$ is the $\mathbb{C}^{m \times m}$-valued SOS polynomial. \hfill \Box

**Definition 9.1.** Artin’s denominator $s$ of PSD not SOS polynomial $F$ is called an Artin minimal denominator, if a polynomial $\hat{s} = s/s_j$ is not Artin’s denominator of $F$ for every irreducible factor $s_j$ of $s$.

**Theorem 9.1.** Each $\mathbb{C}^{m \times m}$-valued PSD not SOS polynomial $F(z)$ has a non-constant Artin minimal denominator $s(z)$. The irreducible factors of Artin’s minimum denominator do not change sign on $\mathbb{R}^d$.

We need some additional considerations. If $F(z)$ is an SOS polynomial, then $s(z)^2F(z)$ is also an SOS polynomial for every polynomial $s(z) \in \mathbb{R}[z_1, \ldots, z_d]$. If PSD polynomial $F(z)$ is not representable as sum of squares of polynomials, then the question arises: for which $s(z)$ is the polynomial $s(z)^2F(z)$ also PSD not SOS?

**Proposition 9.2.** ([12], Lemma 2.1). Let $F(x)$ be a scalar PSD not SOS polynomial and $s(x)$ an irreducible indefinite polynomial in $\mathbb{R}[x_1, \ldots, x_d]$. Then $s^2F$ is also a PSD not SOS polynomial.

**Proof.** Clearly $s^2F$ is PSD. If $s^2F = \sum_k h_k^2$, then for every real tuple $a$ with $s(a) = 0$, it follows that $s^2F(a) = 0$. This implies $h_k(a)^2 = 0$ \forall k. So on the real variety $s = 0$, we have $h_k = 0$ as well. So (see [6], Theorem 4.5.1) for each $k$, there exists $g_k$ so that $h_k = sg_k$. This gives $F = \sum_k g_k^2$, which is a contradiction. \hfill \Box

**Corollary 9.1.** Let $F(z)$ be a $\mathbb{C}^{m \times m}$-valued PSD not SOS polynomial, and $s(z)$ an irreducible polynomial in $\mathbb{R}[z_1, \ldots, z_d]$. Then $s^2F$ is also a PSD not SOS polynomial.

**Proposition 9.3.** If $r(z)^2F(z)$ is an SOS polynomial and all irreducible factors of polynomial $r(z) \in \mathbb{R}[z_1, \ldots, z_d]$ are indefinite, then $F(z)$ is also an SOS polynomial.

**Proof.** Suppose $F(z)$ is a PSD not SOS polynomial. Let

$$
r(z) = r_1(z) \cdots r_k(z)
$$

25
be the decomposition of \( r(z) \) into irreducible factors. Successively applying Corollary 9.1 to the polynomials
\[
F_1 = r_1^2 F(z), \ F_2 = r_2^2 F_1, \ldots, F_k = r_k^2 F_{k-1}
\]
we get \( F_k = r^2 F \) is PSD not SOS polynomial. Contradiction. \( \Box \)

**Proof of Theorem 9.1.** by Proposition 9.1, for \( \mathbb{C}^{m \times m} \)-valued PSD polynomial \( F(z) \) there exists Artin’s denominator \( a(z) \in \mathbb{R}[z_1, \ldots, z_d] \) for which \( a(z)^2 F(z) \) is an SOS polynomial. Each irreducible factor of \( a(z) \) is either indefinite or does not change sign on \( \mathbb{R}^d \). Then \( a(z) = r(z)s(z) \), where all irreducible factors of \( s(z) \) do not change sign on \( \mathbb{R}^d \), and the irreducible factors of \( r(z) \) are indefinite. Let us \( F_1(z) = s(z)^2 F(z) \). By condition, \( r(z)^2 F_1(z) = a(z)^2 F(z) \) is the SOS polynomial. Since all irreducible factors of \( r(z) \) are indefinite, we see that \( F_1(z) \) is an SOS polynomial (Proposition 9.3). Then \( s(z) \) is also Artin’s denominator for \( F(z) \). Let \( s_j(z) \) be some irreducible factor of \( s(z) \). If \( s(z)/s_j(z) \) remains the Artin denominator of \( F \), then the factor \( s_j(z) \) is removed from \( s(z) \). Removing all “excess” irreducible factors from the polynomial \( s(z) \), we obtain Artin’s denominator with the required properties. \( \Box \)

To prove Sum-of-Squares Theorem, we need some statements from [5]:

**Proposition 9.4.** ([5], Proposition 3.4). Let \( s(z) \in \mathbb{R}[z_1, \ldots, z_d] \) be a irreducible polynomial that does not change sign on \( \mathbb{R}^d \). If \( \partial s(z)/\partial z_1 \neq 0 \), then there exists a point \( z' = (z'_1, x'_2, \ldots, x'_d) \), \( \text{Im} z'_1 > 0 \), \( x'_2, \ldots, x'_d \in \mathbb{R} \) such that \( s(z') = 0 \). Moreover, there exists a point \( z'' \in \mathbb{R}^d \) for which \( s(z'') = 0 \).

Let \( Z(h) = \{ z \in \mathbb{C} | h(z) = 0 \} \) be a zero set of the polynomial \( h \).

**Proposition 9.5.** ([5], Proposition 7.1). Let \( s(z), h(z) \in \mathbb{R}[z_1, \ldots, z_d] \) be coprime polynomials and \( s(z'_1, z') = h(z'_1, z') = 0 \) for fixed \( z'_1 \in \mathbb{C} \), \( z' \in \mathbb{R}^{d-1} \), where \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \). Put \( \Omega = \Omega_1 \times \Omega_\mathbb{K} \), where \( \Omega_1 \subset \mathbb{C} \) is some neighborhood of \( z'_1 \), and \( \Omega_\mathbb{K} \subset \mathbb{K}^{d-1} \) is a some neighborhood of \( z' \). Then neither of the set \( Z(s) \cap \Omega \) and \( Z(h) \cap \Omega \) is subset of the other.

**Remark 9.2.** This means: if \( s(z'_1, z') = h(z'_1, z') = 0 \), then in any neighborhood \( \Omega = \Omega_1 \times \Omega_\mathbb{K} \) of the point \( (z'_1, z') \) there is a point \( (z''_1, z'') \) for which \( s(z''_1, z'') = 0 \) and \( h(z''_1, z'') \neq 0 \).

**Theorem 9.2.** ([5], Theorem 6.1). Let \( f(z) = p(z)/q(z) \) be rational function with real coefficients and \( s(z) \in \mathbb{R}[z_1, \ldots, z_d] \). If
\[
s(z)^2 W_{z_1}[q, p] = s(z)^2 \left( q(z) \frac{\partial p(z)}{\partial z_1} - p(z) \frac{\partial q(z)}{\partial z_1} \right) = H(z)H(z)^T
\]

26
is the SOS polynomial, then there exist a real symmetric matrices $A_j$, $j = 0, 1 \ldots , d$, where $A_1$ is positive semidefinite such that

$$f(z) = \frac{\Psi(\zeta)}{q(\zeta)s(\zeta)}(A_0 + z_1A_1 + \cdots + z_dA_d)\frac{\Psi(z)^T}{q(z)s(z)}, \quad \zeta, z \in \mathbb{C}^d, \quad (9.9)$$

$$W_{z_1}[q, p] = \frac{\Psi(z)}{s(z)}A_1\frac{\Psi(z)^T}{z(z)} = \frac{H(z)H(z)^T}{s(z)^2}, \quad (9.10)$$

where $\Psi(z) = (z^{\alpha_1}, \ldots , z^{\alpha_N})$ is the row vector of all monomials $z^{\alpha_i} = z_1^{\delta_1} \cdots z_d^{\delta_d}$ satisfies the conditions

$$\deg z^{\alpha_i} \leq \deg f + \deg s(z), \quad \deg z^{\alpha_i} \leq \deg z \cdot \deg s(z).$$

Proof of Sum-of-Squares Theorem.

Let $f(z) = P(z)/g(z)$ be a $\mathbb{C}^{m \times m}$-valued rational Cayley inner function of the Pick class. Without loss of generality, we can assume that $\overline{q(z)} = q(z)$ and $P(\overline{z})^* = P(z)$. By Proposition 3.1, all Wronskians $W_{z_k}[q, p]$, $k = 1, \ldots , d$ are $\mathbb{C}^{m \times m}$-valued PSD polynomials.

Suppose one of the Wronskians, say $W_{z_1}[q, p]$, is a PSD not SOS polynomial. By Proposition 9.1, there exists Artin’s denominator $s(z) \in \mathbb{R}[z_1, \ldots , z_d]$ such that

$$s(z)^2W_{z_1}[q, p] = M(z)M(\overline{z})^*, \quad (9.11)$$

where $M(z)$ is some $\mathbb{C}^{m \times k}$-valued polynomial. By Theorem 9.1, we can assume that $s(z)$ is Artin’s minimal denominator. Let $s_0(z) \in \mathbb{R}[z_1, \ldots , z_d]$ be an irreducible factor of $s(z)$ different from the constant. Among the elements $m_{ki}(z)$ of the matrix $M(z)$ there is a polynomial

$$m_{ij}(z) = a_{ij}(z) + ib_{ij}(z), \quad (9.12)$$

where $a_{ij}(z), b_{ij}(z) \in \mathbb{R}[z_1, \ldots , z_d]$, for which $s_0(z)$ is not a divisor. Indeed, if $s_0(z)$ is a divisor of all elements of $m_{ki}(z) = a_{kl}(z) + ib_{kl}(z)$, then $s_0(z)$ is the divisor of $a_{kl}(z)$ and $b_{kl}(z)$. Therefore, it is the divisor of $m_{kl}(\overline{z}) = a_{kl}(z) - ib_{kl}(z)$. Then $s(z)/s_0(z)$ is also the Artin denominator for $W_{z_1}[q, p]$, which contradicts the minimality of $s(z)$.

The element $m_{ij}(z)$ (9.12) belongs to the $i$-th row $M_i(z)$ of the matrix $M(z)$. We represent it in the form

$$M_i(z) = A(z) + iB(z),$$

where the elements of the row vector $A(z), B(z)$ are polynomials with real coefficients.
Note that the \( i \)-th diagonal element \( f_{ii}(z) = p_{ii}(z)/q(z) \) of the original function \( f(z) \) is a scalar rational Cayley inner function of the Pick class with real coefficients. From (9.11) we get the polynomial

\[
s(z)^2 W_{z_1} [q, p_{ii}] = M_i(z) M_i(\overline{z})^* \]

with real coefficients. From here

\[
s(z)^2 W_{z_1} [q, p_{ii}] = (A(z) + iB(z))(A(z) - iB(z))^T = H(z)H(z)^T, \quad (9.13)
\]

where \( H(z) = (A(z), B(z)) = (h_1(z), \ldots, h_{2k}(z)) \). The irreducible factor \( s_0(z) \) is not a divisor of at least one of the elements \( a_{ij}(z), b_{ij}(z) \) (9.12) of \( H(z) \). Without loss of generality, we can assume that \( a_{ij}(z) = h_1(z) \) and \( s_0(z) \) are coprime polynomial.

By Theorem 9.1, for \( f_{ii}(z) = p_{ii}(z)/q(z) \) there exist a real symmetric matrices \( A_j, j = 0, 1, \ldots, d \), where \( A_1 \) is positive semidefinite, such that

\[
f_{ii}(z) = \Psi(\zeta)/q(\zeta)s(\zeta) (A_0 + z_1 A_1 + \cdots + z_d A_d) \Psi(z)T/q(z)s(z), \quad \zeta, z \in \mathbb{C}^d, \quad (9.14)
\]

\[
W_{z_1}[q, p_{ii}] = \frac{\Psi(z)}{s(z)} A_1^{ij} \frac{\Psi(z)^T}{z(z)} H(z)H(z)^T/s(z) = \sum_{j=1}^{2k} \frac{h_j(z)^2}{s(z)^2}. \quad (9.15)
\]

Setting \( \zeta = \overline{z} \), from (9.14), (9.15) we obtain

\[
\text{Im} f_{ii}(z) = \text{Im} z_1 \sum_{j=1}^{2k} \frac{|h_j(z)|^2}{|q(z)s(z)|^2} + \sum_{k=1}^{d} \text{Im} z_k \frac{\Psi(z)A_k \Psi(z)^*}{|s(z)q(z)|^2}. \quad (9.16)
\]

Note that the expression on the left in (9.15) is a polynomial. Therefore, \( \sum_{j=1}^{2k} h_j(z)^2 / s(z)^2 \) is the polynomial. In contrast to (9.15), the expression \( \sum_{j=1}^{2k} |h_j(z)|^2 / |s(z)|^2 \) in (9.16) may not be a polynomial, which is important in our case.

For the Artin denominator \( s(z) \) there are 2 possibilities: (a) there exists a irreducible factor \( s_0(z) \) of the polynomial \( s(z) \) such that \( \partial s_0/\partial z_1 \neq 0 \), (b) \( \partial s(z)/\partial z_1 = 0 \).

**Case (a).** \( \partial s_0/\partial z_1 \neq 0 \). The polynomial \( s_0(z) \) depends on at least 2 variables. By Proposition 9.3, there exists a point \( z' = (z'_1, x'_2, \ldots, x'_d) \), \( \text{Im} z'_1 > 0, x'_2, \ldots, x'_d \in \mathbb{R} \) such that \( s(z') = s_0(z') = 0 \). Without loss of generality, we can assume that \( h_1(z') \neq 0 \). Indeed, \( h_1(z), s_0(z) \) are coprime polynomials. If \( h_1(z') = s_0(z') = 0 \), then by Proposition 9.5 there is a point \( \tilde{z}' = (\tilde{z}'_1, \tilde{x}') \), \( \text{Im} \tilde{z}'_1 > 0, \tilde{x}' \in \mathbb{R}^{d-1} \) for which \( h_1(\tilde{z}') \neq 0, s_0(\tilde{z}') = 0 \).
Since \( q(z) \) is a real stable polynomial, we see that for fixed \( x' \in \mathbb{R}^{d-1} \) the equation \( q(z_1, x') = 0 \) has only real roots. Therefore, \( q(z_1', x') \neq 0 \) and rational function \( f_{ii}(\zeta, x') = p_{ii}(\zeta, x')/q(\zeta, x') \) is holomorphic at \( \text{Im} \, \zeta > 0 \).

On the other hand, since \( s(z_1', x') = 0 \) and \( h_1(z_1', x') \neq 0 \), then from (9.16) we obtain

\[
\lim_{\zeta \to z_1'} \text{Im} \, f_{ii}(\zeta, x') = \lim_{\zeta \to z_1'} \text{Im} \, \zeta \cdot \sum_{j=1}^{2k} \frac{|h_j(\zeta, x')|^2}{|s(\zeta, x') q(\zeta, x')|^2} = +\infty,
\]

that is, \( f_{ii}(\zeta, x') \) has a pole at \( \zeta = z_1, \text{Im} \, z_1 > 0 \). A contradiction.

**Case (b).**

If \( \partial s/\partial z_1 \equiv 0 \), then for some \( k \neq 1 \) there exists an irreducible factor \( s_0(z) \), \( z \in \Pi^{d-1} \) such that \( \partial s_0/\partial z_k \neq 0 \) holds. In addition, \( s_0(z) \) depends on at least 2 variables. By proposition 9.3, there exists a point \( z'' \in \Pi^{d-1} \) for which \( s_0(z'') = 0 \). Without loss of generality, for fixed \( x_1 \in \mathbb{R} \), we can assume that \( h_1(x_1, z'') \neq 0 \). Indeed, if \( h_1(x_1, z'') = s_0(z'') = 0 \), then by Proposition 9.5 there is a point \( (x''_1, z'') \). \( x''_1 \in \mathbb{R}, z'' \in \Pi^{d-1} \), for which \( h_1(x''_1, z'') \neq 0 \), \( s_0(z'') = 0 \).

Since \( q(z) \) is a real stable polynomial, we see that \( q(x_1, z'') \neq 0 \) for \( z'' \in \Pi^{d-1} \). Then there exists neighborhood \( \Omega'' \subset \Pi^{d-1} \) of point \( z'' \in \Pi^{d-1} \) such that \( f_{ii}(x_1, z) \) is holomorphic for \( z \in \Omega'' \) and hence bounded. Then for some \( 0 < C < +\infty \), using (9.16), we have

\[
|\text{Im} \, f_{ii}(x_1, z)| = \sum_{k=2}^{d} \text{Im} \, z_k \frac{\Psi(x_1, z) A_k \Psi(x_1, z)^*}{|q(x_1, z) s(z)|^2} \leq C, \quad z \in \Omega''. \tag{9.17}
\]

Since \( q(x_1, z) \neq 0 \), we see that \( q(z_1, z) \neq 0 \) holds for \( z \in \Omega'' \subset \Pi^{d-1} \) and all \( z_1 \) from some complex neighborhood \( \Omega_1 \subset \mathbb{C} \) of the point \( x_1 \in \mathbb{R} \).

Then

\[
|q(z'_1, z) s(z)| \geq \varepsilon > 0, \quad \text{for } z \in \Omega'' \quad \text{and fixed } z'_1 \in \Omega_1, \text{Im} \, z'_1 > 0.
\]

Thus, there exists \( 0 < C_1 < +\infty \) such that

\[
\sum_{k=2}^{d} \text{Im} \, z_k \frac{\Psi(z'_1, z) A_k \Psi(z'_1, z)^*}{|q(z'_1, z) s(z)|^2} \leq C_1, \quad z \in \Omega'', \text{Im} \, z'_1 > 0. \tag{9.18}
\]
On the other hand, for a point \((z'_1, z) \in \Omega_1 \times \Omega''\), from (9.16) we obtain

\[
\text{Im} f_{ii}(z'_1, z) = \text{Im} z'_1 \sum_{j=1}^{2k} \frac{|h(z'_1, z)|^2}{|q(z'_1, z)s(z)|^2} + \\
\sum_{k=2}^{d} \text{Im} z_k \frac{\Psi(z'_1, z)A_k \Psi(z'_1, z)^*}{|q(z'_1, z)s(z)|^2}, \text{ Im } z'_1 > 0, z \in \Pi^{d-1}. \tag{9.19}
\]

According to (9.18), the second term in (9.19) is bounded for \(z \in \Omega''\). Since \(h_1(z'_1, z'') \neq 0\) and \(s(z'') = s_0(z'') = 0\), we see that the first term increases indefinitely at \(z \to z'' \in \Pi^{d-1}\), which contradict holomorphy of \(f(z)\) in \(\Pi^d\).

We assumed that the PSD polynomial \(W_{z_1}[q, P]\) is not SOS, and we obtained a contradiction. Then \(W_{z_1}[q, P]\) is a SOS polynomial.

**References**

[1] J. Agler, R. Tully-Doyle and N. J. Young, *Nevanlinna Representations in Several Variables* // Journal of Functional Analysis, Vol. 270, (2016), 3000–3046.

[2] J. A. Ball, D. S. Kaliuzhnyi-Verbovetskiï. *Rational Cayley inner Herglotz-Agler function: Positive kernel decompositions and transfer-function realizations* // Linear Algebra and its Applications, 456, (2014), 138–156.

[3] M. F. Bessmertnyï. *Functions of Several Variables in the Theory of Finite Linear Structures* // Ph.D. Thesis, Kharkov University, Kharkov, (1982), 143pp. (Russian).

[4] M. F. Bessmertnyï, *On realization of rational matrix functions of several variables* // Oper. Theory Adv. Appl., Vol. 134, (2002), 157–185, Birkhauser-Verlag, Basel.

[5] M. F. Bessmertnyï, *Holomorphic Rational Functions of Several Variables and Sum of Squares of Polynomials* // arXiv:2106.15996v1[math.CV] 30 Jun 2021.

[6] J. Bochnak, M. Coste, M.-F. Roy, *Real algebraic geometry* // Vol. 95, Springer Berlin, 1998.

[7] J. Borcea, P. Brändén, *Multivariate Pólya-Schur Classification Problems in the Weyl Algebra* // arXiv:math/0606360v8[math.CA] 6 Nov 2009.
[8] P. Brändén, *Polynomials with the Half-Plain Property and Matroid Theory* // arXiv:math/0605678v4[math.CO] 9 May 2007.

[9] Y.-B. Choe, J. G. Oxley, A. D. Sokal, D. G. Wagner, *Homogeneous Multivariate Polynomials with Half-Plane Property* // arXiv:math/0202034v2 4 Dec 2002.

[10] F. R. Gantmacher, *The Theory of Matrices*, Vol. 1, Chelsea, New York, 1959.

[11] A. V. Efimov, V. P. Potapov. *J-expanding matrix functions and their role in the analytical theory of electrical circuits* // Russ. Math, Surveys, 28:1 (1973), 69–140.

[12] C. Goel, S. Kuhlmann, B. Reznick, *On the Choi-Lam analogue of Hilbert’s 1888 theorem for symmetric forms* // arXiv:1505.08145v2[math. AG] 19 Aug 2015.

[13] T. Koga, *Synthesis of finite passive n-ports with prescribed positive real matrices of several variables* // IEEE Trans. Circuit Theory, CT-15(1), (1968), 2–23.

[14] S. Leng, *Algebra* // 3rd ed., Springer-Verlag, New York, 2002.

[15] W. Rudin, *Function Theory in Polydisks* // Benjamin, New York–Amsterdam, 1969.