Abstract

In this first part of a larger review undertaking the results of the first author and a part of the second author doctor dissertation are presented. Next we plan to give a survey of a nowadays situation in the area of investigation. Here we report on what follows.

Calculation of the partition function for any vector Potts model is at first reduced to the calculation of traces of products of the generalized Clifford algebra generators. The formula for such traces is derived.

The latter enables one, in principle, to use an explicit calculation algorithm for partition functions also in other models for which the transfer matrix is an element from generalized Clifford algebra.

The method - simple for $Z_2$ case - becomes complicated for $Z_n$, $n > 2$, however everything is controlled due to knowledge of the corresponding algebra properties and those of generalized cosh function.

Hence the work to gain the thermodynamics of the system, though possibly tantalous, looks now a reasonable, tangible task with help of computer symbolic calculations.

The discussion of the content of the in statu nascendi second part is to be found at the end of this Part 1 presentation. This constitutes the section V.

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I. Introduction

The main idea of all calculations to follow (see [8]) is to consider the task of determining the complete partition function for noncritical Potts models - as a problem from the theory of generalized Clifford algebras $C_{2p}^{(n)}$ classified in [15].
As a matter of fact, the transfer matrix approach has led the authors of [1,14 except for 20,21] to these very algebras, although this observation does not seem to be realized by the mentioned authors.

In general - the transfer matrix technique for a statistical system (or - a lattice field theory) with the most general translational invariant and globally symmetric Hamiltonian (Action) on a two-dimensional lattice - does generate appropriate algebras of operators which are of the type of algebra extensions of some groups [8,18,24].

If one considers $Z_n$ cyclic groups as symmetry groups of Hamiltonian (chiral Action) then the algebra generated by transfer matrix approach is the corresponding $C_k^{(n)}$ algebra, and it is due to its properties, that the given model has the duality property [18].

For duality property we refer the reader to the review [19], and for Potts models, in general - to [2,23].

Calculations to be carried out here for the $Z_n$ - vector Potts model simplify tremendously in the case of $n = 2$ i.e. for the Ising model and there lead to the known complete partition function [5] (see also [22] for modern presentation) which after carrying out the thermodynamic limit, goes into the Onsager formula [16,17].

The method we choose to proceed with, is an appropriate generalization of the one used in [22] which consists there in reducing the problem of finding of the partition function for the Ising model to calculation of $\overline{\text{Tr}} (P_1...P_s)$, where $\overline{\text{Tr}}$ is the normalized trace while $P$'s are linear combinations of $\gamma$ matrices - generators of usual Clifford algebra naturally assigned to the lattice.

Then the observation that $\overline{\text{Tr}} (P_1...P_2s)$ is just a Pfaffian [3] of an antisymmetric matrix formed with scalar products of $P$'s leads one to calculation of the determinant from this very matrix.

The method proposed in [5,22] is purely algebraic, and though probably not the shortest one, it lacks ambiguities of other methods and uses well established, simple language of Clifford algebras. For other aspects and connotations of such an approach see [9,10,11,12].

Our paper is organized as follows:

In the second section we write down the $Z_n$-vector Potts model in a form resembling (and generalizing!) the Ising model without external field and then we represent the transfer matrix (an element of $C_2^{(n)}$) as a sum of expressions proportional to $\text{Tr} (\gamma_{i_1}...\gamma_{i_s})$, where this time $\gamma$'s are generalized $\gamma$ matrices.

Equivalently, the Hamiltonian for this Potts model can be looked upon as an Action for the $Z_n$ chiral model on the square lattice.

In the third section we derive the formula for $\text{Tr} (\gamma_{i_1}...\gamma_{i_s})$.

The last section is to supply an inevitable information on $C_k^{(n)}$ algebras for the Reader’s convenience as well as some calculations avoided in the
We would like to end this introduction by quotation from R. J. Baxter’s book (see [2] p. 454): ”The only hope that occurs to me is just as Onsager (1944) and Kaufman (1949) originally solved the zero-field Ising model by using the algebra of spinor operators, so there may be similar algebraic methods for solving the eight-vertex and Potts models”. Our suggestion then is that these very algebras are just generalized Clifford algebras, and the presented paper is aimed to deliver arguments in favor of that point of view.

II. The transfer matrix as a polynomial in $\gamma$’s

In the following the transfer matrix $M$ for the $Z_n$-vector Potts model is represented in a form of “multi-sum” of expressions proportional to $\text{Tr} (\gamma_i \ldots \gamma_i)$. Let us assign to the set of states for this $Z_n$ vector Potts model on a $p \times q$ torus lattice ($p$ rows, $q$ columns), a set

$\text{Def: } S = \{(s_{i,k}) = (p \times q) ; s_{i,k} \in \mathbb{Z}_n\} \diamond

$\text{where we have chosen a multiplicative realization for the cyclic group } \mathbb{Z}_n = (\omega^l)_{l=0}^{n-1} \text{ and } s_{ik} \text{ denotes a matrix element of the } (p \times q) \text{ matrix. Here, naturally, } \omega \text{ denotes the primitive root of unity.}$

The total energy $E$ is then given by:

$$-\frac{E[(s_{i,k})]}{kT} = a \sum_{i,k=1}^{p,q} (s_{i,k}^{-1}s_{i,k+1} + s_{i,k+1}^{-1}s_{i,k}) + b \sum_{i,k=1}^{p,q} (s_{i,k}^{-1}s_{i+1,k} + s_{i+1,k}^{-1}s_{i,k})$$

while the partition function is defined to be:

$$Z = \sum_{(s_{i,k}) \in S} \exp \left\{ -\frac{E[(s_{i,k})]}{kT} \right\}$$

(2.1)

One sees that the total energy of the system as represented by (2.1) is at the same time - a generalization of its own $Z_2$ Ising case and - the Action of the corresponding chiral model (for connection between lattice gauge theory and spin systems see [6] and references therein). The partition function could be written in terms of transfer matrix and for that purpose we introduce the following notation:
Notation:

\[ \vec{s} \cdot \vec{s}' = \sum_{i=1}^{p} s_i s'_i, \quad \vec{s}_k = \begin{pmatrix} s_{1,k} \\ s_{2,k} \\ \vdots \\ s_{p,k} \end{pmatrix}, \quad \vec{s}^*_k = \begin{pmatrix} s_{1,k}^* \\ s_{2,k}^* \\ \vdots \\ s_{p,k}^* \end{pmatrix} \]  \hspace{1cm} (2.3)

\[ (s_{i,k}) = (\vec{s}_1, \vec{s}_2, ..., \vec{s}_q) \]

With the notation (2.3) adopted, the partition function \( Z \) may be now rewritten in a form

\[
Z = \sum_{\vec{s}_1, ..., \vec{s}_q} \exp \left\{ a \sum_{k=1}^{q} (\vec{s}^*_k \cdot \vec{s}_{k+1} + \vec{s}^*_k \cdot \vec{s}_k) + b \sum_{k=1}^{q} \left( \vec{s}^*_k \cdot \sum_{1}^{1} \vec{s}_k + \vec{s}_k \cdot \sum_{1}^{1} \vec{s}^*_k \right) \right\},
\]

after the natural periodicity conditions have been imposed.

**Periodicity conditions:**

\[ \vec{s}_{q+1} = \vec{s}_1, \quad (\vec{s}_k)_1 = (\vec{s}_k)_{p+1}; \quad k = 1, ..., q, \]  \hspace{1cm} (2.5)

where \((\vec{x})_i\) denotes the \(i\)-th component of \(\vec{x}\).

The matrix \(\sum_{1}^{1} \) is a \(p \times p\) generalized Pauli matrix with matrix elements \(\delta_{i+1,j}\), where \(i, j \in \mathbb{Z}'_p = \{0, 1, ..., p-1\}\) and "+" is understood as the \(\mathbb{Z}'_p\) group action.

We introduce also the \(\sigma_1\) generalized Pauli matrix, which is one of the three \(\sigma_1, \sigma_2, \sigma_3\) - playing the same role in representing \(C_{2p}^{(n)}\) generalized Clifford algebras as the usual ones in representing the ordinary \(C_{2p}^{(2)}\) Clifford algebras via well known tensor products of \(\sigma\) matrices [4].

It is now obvious that \(Z\) may be represented as

\[ Z = \text{Tr} \ M^q, \]  \hspace{1cm} (2.6)

as we have

\[ Z = \sum_{\vec{s}_1, ..., \vec{s}_q} M (\vec{s}_1, \vec{s}_2) M (\vec{s}_2, \vec{s}_3) ... M (\vec{s}_q, \vec{s}_1), \]

where matrix elements of the transfer matrix \(M\) are given by:

\[ M (\vec{s}, \vec{s}') = \exp \left\{ 2b \text{Re} \left( \vec{s}' \cdot \sum_{1}^{1} \vec{s} \right) \right\} \exp \left\{ 2a \text{Re} \left( \vec{s}'^* \cdot \vec{s}' \right) \right\}. \]  \hspace{1cm} (2.7)

It is convenient to consider the matrix \(M\) as a product \(M = BA\), where the corresponding matrix elements are identified as

\[ A (\vec{s}', \vec{s}'') = \exp \left\{ 2a \text{Re} \left( \vec{s}'^* \cdot \vec{s}'' \right) \right\}, \]

and

\[ B (\vec{s}, \vec{s}'') = \exp \left\{ 2b \text{Re} \left( \vec{s}' \cdot \sum_{1}^{1} \vec{s} \right) \right\} \delta (\vec{s}, \vec{s}''). \]  \hspace{1cm} (2.8)
As all these $A, B, M$ matrices are multiindexed it is obvious that they might be represented either as tensor products of $(n \times n)$ matrices ($p$ times) or as $(n^p \times n^p)$ matrices. It is not difficult then to see that

$$A = \otimes^p \hat{a} \quad \text{i.e.}$$

(2.9)

$A$ is the $p$-th tensor power of the $(n \times n)$ matrix $\hat{a}$, which has the form of a circulant matrix $W[\sigma_1]$:

$$\hat{a} = (\hat{a}_{I,J}) = (\exp\left\{2a \text{Re} (\omega^{J-I})\right\}) = \sum_{l=0}^{n-1} \lambda_l \sigma_1^l \equiv W[\sigma_1] , \quad (2.10)$$

where $I, J \in Z'_n = \{0, 1, 2, ..., n-1\}$ and

$$\lambda_l = \exp\left\{2a \text{Re} (\omega^J)\right\} . \quad (2.11)$$

In order to see that $A$ and $B$ matrices are just some elements of $C_{2p}^{(n)}$ we shall express them in terms of operators $X_k$ and $Z_k$; $k = 1, 2, ..., p$ i.e. matrices typical for tensor product representation of generalized Clifford algebras via generalized Pauli matrices (see (A.3)).

Def:

$$X_k = I \otimes ... \otimes I \otimes \sigma_1 \otimes I \otimes ... \otimes I \quad (p- \text{terms}),$$

$$Z_k = I \otimes ... \otimes I \otimes \sigma_3 \otimes I \otimes ... \otimes I \quad (p- \text{terms}), \quad \diamond$$

where $\sigma_1$ and $\sigma_3$ are situated on the $k$-th site, counting from the left hand side.

The matrix $A$ may be therefore now rewritten as a product of $(n^p \times n^p)$ matrices

$$A = \prod_{k=1}^{p} W [X_k] , \quad \text{where} \quad W [X_k] = \sum_{l=0}^{n-1} \lambda_l X_k^l . \quad (2.12)$$

Similarly, for the matrix $B$ we derive:

$$B = \exp \left\{b \sum_{k=1}^{p} (Z_{k-1}^{-1} Z_{k+1} + Z_{k+1}^{-1} Z_k) \right\} , \quad (2.13)$$

where $Z_{p+1} = Z_1$.

The formula (2.13) follows from the simple observation that matrix elements of
\[ Z_k^{-1} Z_{k+1} + Z_{k+1}^{-1} Z_k \] (multiindexed by \( \mathcal{S} \) and \( \mathcal{S}' \)) give exactly \( \log \) of the corresponding term of (2.8) expression for \( B \).

The \( \delta \) function arises due to the fact that \( \sigma_3 = (\delta_{IJ} \omega^I) \) (see Appendix) and the exponentiation of matrix elements is possible because \( B \) is simply proportional to unit matrix.

Once \( A \) and \( B \) have been represented as in (2.12) and (2.13) it is easy to express them in terms of generalized \( \gamma \) matrices. Introducing then the tensor product representation (A.3) we get:

\[
X_k = \omega^{n-1} \gamma_k^{n-1} \gamma_k \\
Z_k^{-1} Z_{k+1} = \gamma_k^{n-1} \gamma_k^{n+1} \quad \text{for odd } n, 
\]

and

\[
X_k = \xi \omega^{n-1} \gamma_k^{n-1} \gamma_k \\
Z_k^{-1} Z_{k+1} = \xi \gamma_k^{n-1} \gamma_k^{n+1} \quad \text{for even } n, 
\]

where \( k = 1, 2, \ldots, p-1 \) and \( \xi^2 = \omega \).

The corresponding expression on the boundaries - read:

\[
Z_p^{-1} Z_1 = U \gamma_p^{n-1} \gamma_1 \quad \text{for odd } n, 
\]

and

\[
Z_p^{-1} Z_1 = \xi^{-1} U \gamma_p^{n-1} \gamma_1 \quad \text{for even } n, 
\]

where

\[
\omega \cdot U = \otimes \sigma_1. 
\]

For the proof of (2.14)-(2.17) use (A.6) and (A.7).

From now on we shall proceed with formulas for \( n \) being odd, loosing nothing from generality of considerations while corresponding formulas for the case of \( n \) even are easily derivable from those for the odd case. This in mind we get

\[
A = \prod_{k=1}^{p} W [\omega^{-1} \gamma_k^{n-1} \gamma_k], 
\]

\[
B = \exp \left\{ b \sum_{k=1}^{p-1} (\gamma_k^{n-1} \gamma_{k+1} + \gamma_{k+1}^{n-1} \gamma_k) \right\} \times \exp \left\{ bU \gamma_p^{n-1} \gamma_1 + bU^{-1} \gamma_1^{n-1} \gamma_p \right\}. 
\]

Our first goal is then achieved if one notes that

\[
U = \prod_{k=1}^{p} \gamma_k^{n-1} \gamma_k 
\]

i.e. the transfer matrix \( M \) is now expressed in terms of generalized \( \gamma \) matrices.
Before proceed, it is rather trivial and important to note that $U^n = 1$, $Z^n_k = 1$, $X^n_k = 1$, with obvious implication of the same property for the $n$-th order polynomials in (2.19) and (2.20).

Our second and the main goal of this section is to represent the transfer matrix in a form - reducing the $Tr M^q$ problem to calculation of $Tr (\gamma_1 \gamma_2 \ldots \gamma_s)$ for some collections of $\gamma$’s.

(Note that for $n = 2$ the way to get the complete partition function is shorter as there, it is enough to reduce the $Tr M^q$ problem to calculation of $Tr (P_1 P_2 \ldots P_s)$ where $P$’s are linear combinations of $\gamma$’s. Hence the number of necessary summations is much, much smaller than in the case $n > 2$, where it is rather useless to try to represent $A$ and $B$ matrices in that convenient form).

For that to do we shall deal now with the matrix $B$, to reveal its adequate for the purpose - structure.

Let us start with observations following from $U^n = 1$ property of $U$. Define $V^\pm_k$ matrices (see (A.11)) to be

\[
V^+_k = \frac{1}{n} \sum_{i=0}^{n-1} \omega^{-ki} U^i \\
V^-_k = \frac{1}{n} \sum_{i=0}^{n-1} \omega^{-ki} U^{-i}
\]

and also

\[
\tilde{B}^+_k = \exp\left\{b \omega^k \frac{1}{\gamma_1} \gamma_1 \right\} \\
\tilde{B}^-_k = \exp\left\{b \omega^k \frac{1}{\gamma_1} \gamma_1 \right\}, \quad \text{where} \quad k = 0, 1, \ldots, n - 1.
\]

Then we have

\[
\exp\left\{b U \frac{1}{\gamma_1} \gamma_1 \right\} = \sum_{k=0}^{n-1} \tilde{B}^+_k V^+_k
\]

and

\[
\exp\left\{b U^{-1} \frac{1}{\gamma_1} \gamma_1 \right\} = \sum_{k=0}^{n-1} \tilde{B}^-_k V^-_k.
\]

Equations (2.24) and (2.25) follow from examining $\exp\{U x\}$ via series expansions modulo $n$, as in (A.8)-(A.10).

The matrices $V^\pm_k$ have an important property (see (A.11))

\[
[V^\pm_k]^n = V^\pm_k
\]

hence from (2.20), (2.24), (2.25) and the commutativity of matrix arguments of $B$ ($U, V^\pm_k, \gamma_{k+1} \ldots$ etc.) it follows that:

\[
B = \sum_{l,k=0}^{n-1} (B^+_k V^+_k B^-_l V^-_l)
\]
where
\[ B_k^+ = \exp \left\{ b \sum_{\alpha=1}^{p-1} \gamma_{\alpha-1} \gamma_{\alpha+1} \right\} \tilde{B}_k^+ , \] (2.28)
and
\[ B_k^- = \exp \left\{ b \sum_{\alpha'=1}^{p-1} \gamma_{\alpha'-1} \gamma_{\alpha'+1} \right\} \tilde{B}_k^- . \] (2.29)

Expression (2.27) for \( B \) becomes still simpler due to the remarkable property of \( V_k \)'s:
\[ V_k V_l = 0 ; \quad k \neq l \] (2.30)
where, for the moment, \( V_k = V_k^\pm \), (see Appendix for proof).

Hence
\[ B = \sum_{k=0}^{n-1} B_k^+ B_k^- V_k^+ V_k^- , \] (2.31)
- all terms of the \( k \)-th summand - commuting.

As for the commuting of various matrices involved in representing of the transfer matrix, note that
\[ [A, B] \neq 0 , \quad [U, A] = 0 \quad \text{and} \quad [V_k^+, A] = [V_k^-, A] = 0 . \]

All this is sufficient to write:
\[ M^q = \sum_{l=0}^{n-1} [B_l^+ B_l^- A]^q (V_l^+ V_l^-)^r \] (2.32)
where \( 1 \leq r \leq n - 1 \) and \( V^q = V^r \) as \( V^n = V \).

It is not difficult to see how "\( r \)" arises. Namely "\( r \)" is the residual of the quotient \( \frac{q-n}{n-1} \).

This \( n-1 \) multiplicity in formula (2.32) makes the problem of thermodynamic limit more interesting and involved.

If one assumes however that

**assumption:**
\[ q = n + l (n - 1) , \] (2.33)
with \( l \) an arbitrary integer, then
\[ M^q = \sum_{l=0}^{n-1} [B_l^+ B_l^- A]^q V_l^+ V_l^- . \] (2.34)

Note that there is no multiplicity in (2.32) for the \( Z_2 \) case (Ising model) and also for that case \( V_k^+ = V_k^- \equiv V_k , k = 0, 1 \).

The more: \( V_k V_k = V_k \) and (2.34) reduces its form considerably.
Comparison:

In order to compare (2.34) with the known expression for $M^q$ in the case of Ising model [22] one should note that, for $Z_2$, (2.15) and (2.17) differ only by $i$ and $-i$ from (2.14) and (2.16) correspondingly, hence we have

$$M_q = (B_- A)^q V_+ + (B_+ A)^q V_- ,$$

where

$$B_- = B_0^+ B_0^- = \exp \left\{ 2bi \left( \sum_{\alpha=1}^{n-1} \gamma_\alpha \gamma_{\alpha+1} - \gamma_p \gamma_1 \right) \right\} ,$$

$$B_+ = B_1^+ B_1^- = \exp \left\{ 2bi \left( \sum_{\alpha'=1}^{n-1} \gamma_{\alpha'} \gamma_{\alpha'+1} - \gamma_p \gamma_1 \right) \right\}$$

and

$$V_+ = V_0 = \frac{1}{2} (1 + U) , \quad V_- = V_1 = \frac{1}{2} (1 - U) .$$

The formula (2.35) coincides then with the one known for Ising model [22] apart from the obvious (see (2.1)) and insignificant scaling of constants $a$ and $b$ by factor 2.

Using the formula (2.35), the notion of Pfaffian and its relation to determinant - the author of [22] reobtained the complete partition function leading to the famous Onsager formula.

Having the same goal in mind we are going at first to examine the expression (2.34) in order to see how (the polynomial in $\gamma$'s !) $M^q$ is represented as a multisum of summands proportional to $\text{Tr} (\gamma_{i_1} \ldots \gamma_{i_k})$. For that purpose we write:

$$\tilde{B}_k^+ = \exp (b \rho u_k^+) ,$$

and

$$\tilde{B}_k^- = \exp (b \rho^{-1} u_k^-) ,$$

where

$$! \quad (u_k^+)^n = (u_k^-)^n = 1$$

i.e.

$$u_k^+ \equiv \omega^k \rho^{-1} \gamma_{p-1} \gamma_1 ,$$

$$u_k^- \equiv \omega^k \rho \gamma_{p}^{n-1} \gamma_1 ,$$

while

$$\rho \equiv \rho(n) = \omega^{\frac{n^2-1}{2}} \quad (k = 0, 1, \ldots, n - 1) .$$

Therefore both $\tilde{B}_k^+$ and $\tilde{B}_k^-$ become $n - 1$ order polynomials in $u_k^+$ and $u_k^-$ correspondingly (see (A.10)).

One also shows easily that $v_{\alpha}^+$ and $v_{\alpha}^-$ defined below
\[
\exp \left\{ b \gamma_{\alpha}^{n-1} \gamma_{\alpha+1} \right\} = \exp \left\{ b \rho v^+_{\alpha} \right\}, \quad (2.41)
\]

\[
\exp \left\{ b \gamma_{\alpha+1}^{n-1} \gamma_{\alpha} \right\} = \exp \left\{ b \rho^{-1} v^-_{\alpha} \right\}, \quad (2.42)
\]

do satisfy:

\[
(v^+_{\alpha})^n = (v^-_{\alpha})^n = 1, \quad \alpha = 1, \ldots, p - 1. \quad (2.43)
\]

hence both expressions (2.41) and (2.42) become \( n - 1 \) order polynomials in matrices \( v^+_{\alpha} \) and \( v^-_{\alpha} \) correspondingly.

**n=2:**

For \( n = 2 \) the further job is extremely facilitated due to the fact that \( A \) becomes then of the form

\[
A = \prod_{k=1}^{p} P_k Q_k \quad (2.44)
\]

where \( P \)'s and \( Q \)'s are some known linear combinations of ordinary \( \gamma \) matrices and similar holds for \( B_+ \), \( B_- \) matrices from (2.35).

The matrices \( V_+ \), \( V_- \) from (2.35) have also simple form and thus \( \text{Tr} M^q \) disentangles for \( n = 2 \) to be the sum only four summands of the Pfaffian type i.e. \( \text{Tr} (P_1 P_2 \ldots P_s) \).

**n>2:**

Unfortunately this disentanglement is no more possible for \( n > 2 \) as the polynomial \( W \) (see (2.12)) no more is representable uniquely as the product of linear combinations of \( \gamma \)'s and neither is \( B \).

Hence the multi-sum becomes more complicated.

Nevertheless it is obvious that \( \text{Tr} M^q \) problem reduces to calculation of \( \text{Tr} (\gamma_{i_1} \gamma_{i_2} \ldots \gamma_{i_s}) \) for some collections of \( \gamma \)'s.

**n=2:**

In the case of Ising model, the four arising Pfaffians contribute to the partition function to give [22]:

\[
Z = 2^{p q - 1} \left\{ \prod_{k,l=1}^{p q} \left[ \text{ch} 2a' \text{ch} 2b' - \text{sh} 2a' \cos \frac{2\pi}{q} (2l + 1) - \text{sh} 2b' \cos \frac{2\pi}{p} (2k + 1) \right]^{\frac{1}{2}} + \right.
\]

\[
+ \prod_{k,l=1}^{p q} \left[ \text{ch} 2a' \text{ch} 2b' - \text{sh} 2a' \cos \frac{2\pi}{q} (2l + 1) - \text{sh} 2b' \cos \frac{2\pi}{p} (2k + 1) \right]^{\frac{1}{2}} + \right.
\]

\[
+ \prod_{k,l=1}^{p q} \left[ \text{ch} 2a' \text{ch} 2b' - \text{sh} 2a' \cos \frac{2\pi}{q} (2l + 1) - \text{sh} 2b' \cos \frac{2\pi}{p} (2k + 1) \right]^{\frac{1}{2}} + \right.
\]

\[
- \sigma \prod_{k,l=1}^{p q} \left[ \text{ch} 2a' \text{ch} 2b' - \text{sh} 2a' \cos \frac{2\pi}{q} (2l + 1) - \text{sh} 2b' \cos \frac{2\pi}{p} (2k + 1) \right]^{\frac{1}{2}} \left\} \quad (2.45)
\]
where $\sigma$ denotes the sign of $T - T_c$ and $a' = 2a$, $b' = 2b$. Both the square root and the $\sigma$-sign have appeared here because of the use of $P f^2 = \det$ relation.

$n > 2$:
Again, for $n > 2$, as we shall see in the following section, although the generalization of the Pfaffian is possible to the case of arbitrary $n$, its relation to any valuable generalization of determinant does not to be valid as the arising signum like function no more is an epimorphism of $S_k$ onto $Z_n$, except for $n = 2$ of course. ($S_k$ - the symmetric group of $k$-elemental permutations).

However, one may write $M^q$ as the polynomial in $\gamma$’s and then use the general formula from the following section.

This representation of $M^q$ in terms of $\gamma_i\ldots\gamma_i$ products is given in the Appendix.

As a result we have the following structure of the complete partition function for the $Z_n$ vector Potts models:

$$M^q = \frac{1}{n^2} \sum_{j_1, j_2=0}^{n-1} \sum_{\vec{\Pi} \in \Gamma_m} G(\vec{\Pi}) \hat{\Omega}(\vec{\Pi}; j_1, j_2),$$

where $G(...) \text{ are known functions}$ of parameters $a$ and $b$, (see the Appendix: (A.15)) and

$$\mathbf{Tr} \hat{\Omega} = \omega^i \text{ or } \mathbf{Tr} = 0,$$

where $i = i(\vec{\Pi}, j_1, j_2) \in Z'_n = \{0, 1, \ldots, n - 1\}$.

The dependence of $i$ on its indices is easy to be derived using the most general, appropriate formula for $\mathbf{Tr}(\gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_s})$ supplied by the next section.

III. Trace formula for any element of $C_{2p}^{(n)}$

In this section the explicit formula for trace of any element of $C_{2p}^{(n)}$ algebra is delivered.

The very formula is crucial for getting the complete partition function for Potts models and hence (see [2] p. 454) for solving several major problems of statistical physics being unsolved till now since many years.

The problem of explicit trace formula for $M^q \in C_{2p}^{(n)}$, is decisive in calculation of $Z$ function for those models on the lattice in which the transfer matrix is an element of $C_{2p}^{(n)}$.

We proceed now to derivation of the very formula.

Note! By definition, in this section $\mathbf{Tr}$ map is normalized i.e. $\mathbf{Tr} I = 1$. The derivation has the form of a sequence of lemmas.
Lemma 1.
Let \( k \neq n \mod n, \ k \in N \); then \( \text{Tr} (\gamma_{i_1} \ldots \gamma_{i_k}) = 0 \).

Proof: The same as for usual Clifford algebras. Use the matrix \( U \) defined by (2.18).

Lemma 2.
\( \text{Tr} (\gamma_{i_1} \gamma_{i_2} \ldots \gamma_{i_k}) \neq 0 \) iff there exists permutation \( \delta \in S_{kn} \), such that
\( i_{\sigma(1)} = i_{\sigma(2)} = \ldots = i_{\sigma(n)}, \ i_{\sigma(n+1)} = \ldots = i_{\sigma(2n)}, \ \ldots, \ i_{\sigma(kn-n+1)} = \ldots = i_{\sigma(kn)}. \)

Proof: The proof follows from observation that due to (A.1) if no \( n \)-tuple of the same \( \gamma \)'s exists then \( \text{Tr}(\ldots) = 0 \). Other steps of the proof are reduced to this first one.

It is therefore trivial to note, but important to realize, that:

Lemma 3.
\( \text{Tr} (\gamma_{i_1} \ldots \gamma_{i_k}) = 0 \) or \( \ell \in \mathbb{Z}_n \) - the multiplicative cyclic group of \( n \)-th roots of unity.

In Lemma 3 \( k \) is again an arbitrary integer while in all preceding lemmas, and in the following, \( i_1, i_2, \ldots, i_k \) run from 1 to number of generators of the given algebra. This number was chosen to be even, however note [15] that the "odd case" problem is reduced to this very one due to the properties of generalized Clifford algebra representations.

The major problem now is to determine this value "0 or \( \ell \)" for arbitrary set of indices \( i_1, i_2, \ldots, i_k \).
In order to do that define a signum like function \( K \) (unfortunately it is an epimorphism only for \( n = 2 \)) - as follows:

Def:
\[ K : S_p \rightarrow \mathbb{Z}_n ; \quad \Theta_{\sigma(1)} \Theta_{\sigma(2)} \ldots \Theta_{\sigma(p)} = K(\sigma) \Theta_1 \Theta_2 \ldots \Theta_p \],
where \( \Theta \)'s satisfy (A.1) except for the condition \( \gamma_i^n = 1 \), which is now replaced by \( \Theta_i^2 = 1 \).

This definition being adapted, it is now not very difficult to prove:

Lemma 4.
\[ \text{Tr}(\gamma_{i_1}...\gamma_{i_p}) = K(\Sigma)K(\sigma), \quad \text{for} \]

a) \( i_{\sigma(1)} = ... = i_{\sigma(n)}, \quad ... \), \( i_{\sigma(pn-n+1)} = ... = i_{\sigma(pn)} \) and

b) \( i_{\bar{\sigma}(n)} < i_{\bar{\sigma}(2n)} < ... < i_{\bar{\sigma}(pn)}, \)

where \( \bar{\sigma} = \Sigma \circ \sigma \), while \( \Sigma \) is a permutation of the elements \( \{n, 2n, ..., pn\} \).

(The group of \( \Sigma \)'s is naturally identified with an appropriate subgroup of \( S_{pn} \)).

\[ \diamond \]

Proof: The proof relies on observation that these are only different \( n \)-tuples which are "rigidly" shifted ones through the others, i.e. there is no permutation within any given \( n \)-tuple.

\[ \diamond \]

The generalization of Lemma 4 to the arbitrary case of some of the \( n \)-tuples being equal - is straightforward. (The necessary change of conditions a) and b) is obvious).

This in mind and from other lemmas we finally get:

\[ \text{Theorem:} \quad \text{Tr}(\gamma_{i_1}...\gamma_{i_p}) = \sum'_{\sigma \in S_{pn}} \sum_{\bar{\sigma} \in S_{\vec{p}}} K(\Sigma)K(\sigma) \times \delta(i_{\bar{\sigma}(1)}, ..., i_{\bar{\sigma}(p_1n)}) \times \]

\[ \times \delta(i_{\bar{\sigma}(p_{1}n+1)}, ..., i_{\bar{\sigma}(p_{1}+p_{2}n)}) \times ... \times \delta(i_{\bar{\sigma}(p_{n}n-p_{1}n+1)}, ..., i_{\bar{\sigma}(pn)}) , \]

with the notation to follow.  

\[ \diamond \]

Notation:

\[ \vec{p} = (p_1, p_2, ..., p_l), \quad p_i \geq 1, \sum_{i=1}^{l} p_i = p, \quad \bar{\sigma} = \Sigma \circ \sigma, \quad \text{and} \quad S_{\vec{p}} \text{ is a subgroup of} \ S_{pn} \text{ isomorphic (for example!) to the group of all block matrices obtained via permutations of "block columns" of the matrix} \]

\[ \begin{pmatrix}
I_{p_1n} \\
I_{p_2n} \\
. \\
. \\
I_{p_ln}
\end{pmatrix}, \quad \text{where} \ I_k \text{ is the} \ (k \times k) \text{ unit matrix.} \]

\( \delta \) - here denotes the multi-indexed Kronecker delta i.e. it assigns zero unless all its arguments are equal and in this very case \( \delta(\ldots) = 1 \). The sum \( \Sigma' \) is meant to take into account only those permutations that do satisfy the conditions:

a) \( \sigma(1) < \sigma(2) < ... < \sigma(p_1n), \quad ... \), \( \sigma(pn-pn+1) < ... < \sigma(pn) \), and

b) \( \sigma(1) < \sigma(p_1n+1) < ... < \sigma(pn-pn+1) \).

Comments:

1) For the case of \( n = 2 \) the theorem gives us the Pfaffian of the product \( \gamma_{i_1}, ..., \gamma_{i_2} \), as in the case, (and only! for \( n = 2 \)) \( K(\Sigma) = 1 \) and we are left, as a result with only \( \Sigma' \) sum, while Kronecker deltas become functions of the same number of indices \( i_j \).

2) The theorem solves our problem of \( \text{Tr} M^q \), as any element of generalized Clifford algebra is a polynomial in \( \gamma \)'s satisfying (A.1), including \( M^q \in C_{2p}^{(n)} \).
IV. Final comments for the Part 1 of the presentation

We have carried out our twentieth century investigation for the $Z_n$ vector Potts model known also under the name of planar Potts model. The similar investigation of the other Potts models, i.e. standard Potts models with two-site interaction [23] and multisite interactions as well, is being now carried out.

However, it is to be noted here that the model considered in [14] possesses transfer matrix $M = BA$, where matrix $A$ is a particular case of the one defined by (2.19) while $B$, though also expressed by $Z_k$ matrices defined in section II, has a different polynomial (in these operators) in the exponential.

As for the multisite interactions, the algebras to be used are the universal generalized Clifford algebras, introduced in [7]. Needless to say that these are standard Potts models which are of more interest because of their relation to a number of outstanding problems in lattice statistics [2,23]. To this end let us express our suggestion that the models of lattice statistics could be adapted (thanks to specific interpretation) to the domain of urban economics which, using the notion of entropy and information introduces as a matter of fact a kind of thermodynamics [13].

V. An outline of the second planned part content

The content of the preceding chapters-except for the algorithm for calculation of the complete partition function as presented above - was already published in twenty first century [25],[26]. This especially concerns the Clifford algebra technique omnipresent here and planned to play similar leading role in the next part of our review. The content of [26] from 2001 indicates the idea and a way how to use generalized Clifford algebra for chiral Potts models on the plane. The incessantly growing area of applications of Clifford algebras and naturalness of their use in formulating problems for direct calculation entitles one to call them Clifford numbers. The generalized “universal” Clifford numbers are here introduced via k-ubic form $Q_k$ replacing quadratic one in familiar construction of an appropriate ideal of tensor algebra. One of the epimorphic images of universal algebras $k - C_n \equiv T(V)/I(Q_k)$ is the algebra $Cl_n(k)$ with n generators and these are the algebras to be used here. Because generalized Clifford algebras $Cl_n(k)$ possess inherent $Z_k$ grading - this property makes them an efficient apparatus to deal with spin lattice systems. This efficiency is illustrated in [26] by derivation of two major observations. Firstly, the partition functions for vector and planar Potts models and other model with $Z_n$ invariant Hamiltonian are polynomials in generalized hyperbolic functions of the n-th order. Secondly, the problem of algorithmic calculation of the partition function for any vector Potts model as treated here is reduced to the calculation of traces of products of the generators of the generalized Clifford algebra. Finally the expression for such
traces for arbitrary collection of generator matrices is derived in [26].

Since the same 2001 year, due to the authors of [27] we know the form of the $k$-state Potts model partition function (equivalent to the Tutte polynomial) for a lattice strip of fixed width and arbitrary length.

From 2005 year Alan D. Sokal 54 pages review [28] aimed for mathematicians too one may learn that "the multivariate Tutte polynomial (known to physicists as the Potts-model partition function) can be defined on an arbitrary finite graph $G$, or more generally on an arbitrary matroid $M$, and encodes much important combinatorial information about the graph"."

Alan D. Sokal discusses there "some questions concerning the complex zeros of the multivariate Tutte polynomial, along with their physical interpretations in statistical mechanics (in connection with the Yang–Lee approach to phase transitions) and electrical circuit theory." Quite numerous open problems are also posed in [28]. For many references see both [27] and [28].

Coming back for a while to generalized Clifford algebra we mark their being just only mentioned in the abstract of Baxter paper [29] in which: "The partition function of the N-state superintegrable chiral Potts model is obtained exactly and explicitly (if not completely rigorously) for a finite lattice with particular boundary conditions". "The associated Hamiltonian has a very simple form, suggesting that may be a more direct algebraic method (perhaps a generalized Clifford algebra) for obtaining its eigenvalues." Baxter - including his famous book [1982 Exactly Solved Models in Statistical Mechanics] comes back several time to the idea of Clifford or Clifford-like algebras' potential importance for the still not solved problem of complete partitions function obtained in a way Ising model was spliced with help of Clifford algebras properties being used in a natural and elegant way. Manageable in a understandable way. Let us quote after Baxter from [30]:"

"This is rather intriguing - we are in much the same position with the chiral Potts model as we were in 1951, when Professor Yang entered the field of statistical mechanics by calculating MO for the Ising model. So on the occasion of his 70th birthday we are able to present him not only with this meeting in honor of his great contributions to theoretical physics, but also with an outstanding problem worthy of his mettle. Plus a change, c'est la même chose.""

Here and there

P. P. Martin’s book [31] and his many subsequent papers in twentieth as well in twenty first century [32] are inevitable source of ideas and inspiration. And here now comes the 2005 year. 2005 was declared by some authors to be the major breakthrough for the chiral Potts model. This concerns

an outstanding problem of the order parameters. See Baxter again [33] and [34]. There in [33] and then in [34] Baxter deals again with the problem of the order parameter in the chiral Potts model. He recalls that an elegant conjecture for this was made in 1983 and that it has since been successfully tested against series expansions, but as far as the author of [34] is aware
there is as yet no proof of the conjecture.

2005. Again Professor Baxter. Here is an abstract of his Annual Conference 2005 of the Australian Mathematical Society public talk entitled *Lattice models in statistical mechanics: the chiral Potts model*. There are a few lattice models of interacting systems that can be solved exactly, in the sense that one can calculate the free energy in the thermodynamic limit of a large system. The interesting ones are mostly two-dimensional, such as the Ising model and the six and eight-vertex models. A comparatively recent addition to the list is the chiral Potts model. This is more difficult mathematically than its predecessors. While its free energy was calculated in 1988, until now there has only been a conjecture (a very elegant one) for the order parameters, i.e. the spontaneous magnetizations. This conjecture has now been verified, and in this talk I shall discuss the difficulties encountered and the method used. The solution from 1988 he is referring to apparently refers to papers such as [35], [36], [37], [38] and others later, see [39]. By the way? - the authors of [39] implicitly indicate Generalized Clifford Algebras - in a footnote (3) referring there to Morris papers from 1967 and 1968 [quoted in all Kwasniewski papers on subject]. The 69 pages, 30 figures review [39] was written in honor of Onsager’s ninetieth birthday, also in order to present ‘some exact results in the chiral Potts models and to translate these results into language more transparent to physicists’.

This is more or less what the second part is planned to be about. By no means it sholud include review of numerous contributions of Professor F.Y. Wu including not only Potts models [see: http://www.physics.neu.edu/Department/Vtwo/faculty/wu.../wupubupdated81803.htm but such fascinating papers as [40] of specifically personal interest of the authors [http://ii.uwb.edu.pl/akk/publ1.htm]. Papers published in Advances in Applied Clifford Algebras such as [42] and the papers by the authors are to be included in the second part of this review too.

**Appendix**

1. \( C_{2p}^{(n)} \) generalized Clifford algebra is defined [15] to be generated by \( \gamma_1, \ldots, \gamma_{2p} \) matrices satisfying:

\[
\gamma_i \gamma_j = \omega^{ij} \gamma_i, \quad i < j, \quad \gamma_i^n = 1, \quad i, j = 1, 2, \ldots, 2p.
\]

(A.1)

The very algebra has - up to equivalence - only one irreducible and faithful representation, and its generators can be represented as tensor products of generalized Pauli matrices:

\[
\sigma_1 = (\delta_{i+1,j}), \quad \sigma_2 = (\omega^i \delta_{i+1,j}), \quad \sigma_3 = (\omega^i \delta_{i,j}),
\]

(A.2)

where \( i, j \in Z_n' = \{0, 1, \ldots, n - 1\} \) - the additive cyclic group.

One easily checks, that \( \{\sigma_i\}_{i=1}^3 \) do satisfy (A.1) for \( n \) being odd.
Let $I$ denotes since now the unit $(n \times n)$ matrix and let

$$\gamma_1 = \sigma_3 \otimes I \otimes I \otimes \ldots \otimes I \otimes I,$$
$$\gamma_2 = \sigma_1 \otimes \sigma_3 \otimes I \otimes \ldots \otimes I \otimes I,$$
$$\vdots$$
$$\gamma_p = \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \ldots \otimes \sigma_1 \otimes \sigma_3,$$
$$\bar{\gamma}_1 = \sigma_2 \otimes I \otimes \ldots \otimes I \otimes I,$$
$$\bar{\gamma}_2 = \sigma_1 \otimes \sigma_2 \otimes I \otimes \ldots \otimes I \otimes I,$$
$$\vdots$$
$$\bar{\gamma}_p = \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \ldots \otimes \sigma_1 \otimes \sigma_2,$$

then $\{\gamma_i, \bar{\gamma}_j, i, j = 1, \ldots, p\}$ do satisfy (A.1) with $\omega$ replaced by $\omega^{-1}$, hence (A.3) are generators of the algebra isomorphic to $C_{2p}^{(n)}$ (isomorphism is given by $\sigma_1 \leftrightarrow \sigma_3$ in (A.3)) (This very (A.3) representation was chosen for technical reason - we get, for example, in calculations of section II, the matrix $U$ without coefficients etc.).

It is also to be noted that for $n$ being odd

$$\sigma_3 = \sigma_1^{n-1} \sigma_2 . \quad (A.4)$$

The case of $n$ being even leads to similar representation with $\sigma_1$ un-changed but $\sigma_2$ and $\sigma_3$ now equal to:

$$\sigma_2 = (\xi^i \delta_{i+1,j}), \quad \sigma_3 = \xi \sigma_1^{n-1} \sigma_2 , \quad (A.5)$$

where $\xi$ is a primitive $2n$-th root of unity such that

$$\xi^2 = \omega .$$

(A.3) then with these appropriate for case $n = 2\nu$ generalized Pauli matrices, provides us with the same type representation of $C_{2p}^{(n)}$ as the one for the case $n = 2\nu + 1$.

One then easily proves that

$$\sigma_3^{n-1} \sigma_2 = \omega \sigma_1 ,$$
$$\sigma_2^{n-1} \sigma_1 = \sigma_3^{-1} \quad \text{for} \quad n = 2\nu + 1 \quad (A.6)$$

and

$$\sigma_3^{n-1} \sigma_2 = \xi^{-1} \omega \sigma_1 ,$$
$$\sigma_2^{n-1} \sigma_1 = \xi^{-1} \sigma_3^{-1} \quad \text{for} \quad n = 2\nu . \quad (A.7)$$

2. In this part of the Appendix we derive one useful formula, necessary for section II.

Let $x$ be any element of an associative, finite dimensional algebra with unity $1$. 

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Then
\[ \exp \{ x \} = \sum_{i=0}^{n-1} f_i (x) , \quad \text{where} \]
\[ f_i (x) = \sum_{k=0}^{\infty} \frac{x^{nk+i}}{(nk+i)!} , \quad i = 0, ..., n - 1. \]  \hspace{1cm} (A.8)

We now express these \( f_i \)'s in terms expotentials.
For that to do it is sufficient to note that
\[ f_i (\omega x) = \omega^i f_i (x) , \quad i = 0, 1, ..., n - 1. \]  \hspace{1cm} (A.9)

The (A.9) reveals the \( Z_n \) symmetry properties of these generalized “cosh”
functions and we get from this set of relations
\[ f_i (x) = \frac{1}{n} \sum_{k=0}^{n-1} \omega^{-ki} \exp \left\{ \omega^k x \right\} , \quad i = 0, ..., n - 1. \]  \hspace{1cm} (A.10)

3. For considerations of the section II we need the following

Lemma
Let \( U \) be as \( x \) above and in addition let \( U^n = 1 \). Then \( V \) defined as follows
\[ V = \frac{1}{n} \sum_{i=0}^{n-1} U^i , \quad \text{has the property :} \]
\[ V^n = V. \]  \hspace{1cm} (A.11)

Proof: For the proof, just note that for some \( a_i \)'s
\[ V^n = \sum_{i=0}^{n-1} a_i U^i , \]
and both sides of this identity equation must be symmetric in \( U^i \) monomials.
One concludes therefore that \( a_i = a_j , i, j = 0, 1, ..., n - 1 \), hence - counting
the number of all arising summands - one arrives at the conclusion of the
Lemma.

4. Here, the proof of the (2.30) formula follows.
Let \( k = l + r , 0 < r \leq n - 1 \). Then
\[ V_k^+ V_l^+ = \frac{1}{n^2} \sum_{i_1=0}^{n-1} \sum_{i_2=0}^{n-1} \omega^{-ri_1} (\omega^{-1} U)^{i_1+i_2} . \]

Introduce now new summation indices: \( i_1 \) and \( i = i_1 + i_2 \).
Then we have
\[ V_k^+ V^+_l = \frac{1}{n^2} \sum_{i_1=0}^{n-1} \omega^{-r_{i_1}} \sum_{i=0}^{n-1} (\omega^{-1} U)^i = 0, \]
because the summation over \( i_1 \) gives zero.
The proof for other \((+, -), (-, +), (-, -)\) cases is the same.

5. In this part of the Appendix we derive the multisum structure of the complete partition function \( \sum \), for the \( Z_n \)-vector Potts model with any \( n \geq 2 \). Since now on we shall use the following abbreviations:
\[ \Gamma_k \equiv Z_n' \otimes Z_n' \otimes ... \otimes Z_n' \, \text{, (} k \text{ summands)}, \]
where \( Z_n' = \{0, 1, ..., n-1\} \), and
\[ \omega^{n-2} \equiv \rho (n) \equiv \rho \, . \]
Recall also \( u^+_k, u^-_k ; k \in Z_n' \) and \( v^+_\alpha, v^-_\alpha ; \alpha = 1, ..., p-1 \), defined in section II. Then we have, according to (A.10):
\[ B^+_k = \sum_{l_1=0}^{n-1} f_{l_1} (bp)^{l_1} \prod_{\alpha=1}^{p-1} \sum_{s=0}^{n-1} f_s (bp^\alpha) (v^+_\alpha)^s \]
\[ B^-_k = \sum_{l_2=0}^{n-1} f_{l_2} (bp^{-1})^{l_2} \prod_{\alpha'=1}^{p-1} \sum_{t=0}^{n-1} f_t (bp^{-1}) (v^-_{\alpha'})^t \, . \, \text{ (A.12)} \]

In order to manage with abundance of indices we introduce a further, deliberate notation.

**Notation:**
\[ \vec{L} \in \Gamma_2 \, , \quad \vec{T}, \vec{S} \in \Gamma_{p-1} \quad \text{i.e.} \]
\[ \vec{L} \equiv (l_1, l_2) \text{ where } l_1, l_2 \in Z_n' \text{ etc.} \]

We also define:
\[ F \left( b; k, \vec{L}, \vec{S}, \vec{T} \right) \equiv \omega^{k(l_1+l_2) + p^2 - l_1} f_{l_1} (bp) f_{l_2} (bp^{-1}) \prod_{i=1}^{p-1} f_{s_i} (bp) \rho^{-s_i} \times \prod_{j=1}^{p-1} f_{t_j} (bp^{-1}) \rho^j, \]
\[ \Xi \left( \vec{L}, \vec{S}, \vec{T} \right) \equiv (z_p^{n-1} \gamma_1)^{l_1} (\gamma_1^{n-1} \gamma_p^{l_2}) \prod_{i=1}^{p-1} (\gamma_i^{n-1} \gamma_{i+1})^{s_i} \prod_{j=1}^{p-1} (\gamma_{j+1}^{n-1} \gamma_j)^{l_j} \]
and
\[ B_k \equiv B^+_k B^-_k \, . \, \text{ ♦} \]
Hence we may write
\[ B_k = \sum_{\vec{L}, \vec{S}, \vec{T}} F(\vec{b}; k, \vec{L}, \vec{S}, \vec{T}) \hat{\Xi}(\vec{L}, \vec{S}, \vec{T}) \] (A.13)
where
\[ \vec{L} \in \Gamma_2, \quad \vec{S}, \vec{T} \in \Gamma_{p-1}. \]

It is important to recall now that:
\[ M^q = \sum_{k=0}^{n-1} [B_k, A]^q V_k^+ V_k^- \quad \text{and} \quad [B_k, A] \neq 0. \]

We introduce therefore again an appropriate notation.

Notation:
\[ \Lambda \left( a; \vec{I} \right) \equiv \prod_{r=1}^{p} \lambda_{i_r} \left( a \right) \omega^{-i_r}, \quad \text{and} \]
\[ \hat{\Gamma} \left( \vec{I} \right) = \prod_{r=1}^{p} (\gamma_r^{n-1})^{i_r}, \quad \text{while} \]
(see (2.21))
\[ U = \prod_{r=1}^{p} \gamma_r^{n-1} \]
(recall also (2.22)).}

This being adapted we may write:
\[ A = \sum_{\vec{I} \in \Gamma_p} \Lambda \left( a; \vec{I} \right) \hat{\Gamma} \left( \vec{I} \right) \] (A.14)

Investigation of the multisum structure of \( M^q \) matrix eventuates in rather transparent form of it, if one, (for the last time!) introduces an overall index, or rather "multiindex" for all reappearing summations.

Notation:
\[ \vec{\Pi} \equiv \left( k, \vec{L}_1, ..., \vec{L}_q, \vec{S}_1, ..., \vec{S}_q, \vec{T}_1, ..., \vec{T}_q, \vec{I}_1, ..., \vec{I}_q \right), \]
\[ G \left( \vec{\Pi} \right) = \prod_{r=1}^{q} F \left( b; k, \vec{L}_r, \vec{S}_r, \vec{T}_r \right) \Lambda \left( a; \vec{I}_r \right), \]
\[ \hat{\Omega} \left( \vec{\Pi}, j_1, j_2 \right) = \left( \prod_{r=1}^{q} \hat{\Xi} \left( \vec{L}_r, \vec{S}_r, \vec{T}_r \right) \hat{\Gamma} \left( \vec{I}_r \right) \right) U^{j_1 - j_2} \omega^{-k(j_1 + j_2)}. \]

All this together taken into account leads to
\[ M^q = \frac{1}{n^2} \sum_{j_1, j_2=0}^{n-1} \sum_{\vec{\Pi} \in \Gamma_m} G \left( \vec{\Pi} \right) \hat{\Omega} \left( \vec{\Pi}, j_1, j_2 \right) \] (A.15)
where $\vec{\Pi} \in \Gamma_m$; \quad $m = 3pq + 1$.

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