LONG RAINBOW CYCLES AND HAMILTONIAN CYCLES USING MANY COLORS IN PROPERLY EDGE-COLORED COMPLETE GRAPHS

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Abstract. We prove two results regarding cycles in properly edge-colored graphs. First, we make a small improvement to the recent breakthrough work of Alon, Pokrovskiy and Sudakov who showed that every properly edge-colored complete graph $G$ on $n$ vertices has a rainbow cycle on at least $n - O(n^{3/4})$ vertices, by showing that $G$ has a rainbow cycle on at least $n - O(\log n \sqrt{n})$ vertices. Second, by modifying the argument of Hatami and Shor which gives a lower bound for the length of a partial transversal in a Latin Square, we prove that every properly colored complete graph has a Hamilton cycle in which at least $n - O((\log n)^2)$ different colors appear. For large $n$, this is an improvement of the previous best known lower bound of $n - \sqrt{2n}$ of Andersen.

1. Introduction

Let $G$ be a graph. An edge-coloring is a proper edge-coloring if the set of edges incident to a vertex are given distinct colors. If $G$ is edge-colored, we say that $H \subset G$ is rainbow if the colors assigned to the edges of $H$ are distinct.

There has been extensive research on rainbow properties of properly edge-colored graphs. One problem that has seen significant recent interest is to find many edge-disjoint rainbow spanning trees, see Akbari & Alipour [1], Balogh, Liu & Montgomery [5], Pokrovskiy & Sudakov [13], Carraher, Hartke & Horn [6], and Horn & Nelson [12]. A related old conjecture of Andersen [4] is the following, which if true would be best possible.

Conjecture 1. Every properly edge-colored complete graph on $n$ vertices contains a rainbow path on $n - 1$ vertices.

In the same paper, Andersen proved the following result.

Theorem 2. Every properly edge-colored complete graph on $n$ vertices contains a Hamiltonian cycle in which at least $n - \sqrt{2n}$ distinct colors appear.

In this paper, we make the following improvement (for large $n$) to Theorem 2.

Theorem 3. There exists a constant $C$ such that for every $n$ the following holds. Every properly edge-colored complete graph on $n$ vertices contains a Hamiltonian cycle in which at least $n - C(\log n)^2$ distinct colors appear.
Note that our proof of Theorem 3 very closely follows the proof of Hatami & Shor’s [11] result on the length of a partial traversal in a Latin square.

Instead of asking for a Hamiltonian cycle that uses many colors, another way to approach Conjecture 1 is to attempt to find a long rainbow cycle. This problem has received recent interest. Akbari, Etesami, Mahini & Mahmoody [2] proved that a cycle of length \( n/2 - 1 \) exists in every properly colored complete graph \( G \) on \( n \) vertices. Then Gyárfás & Mhalla [9] proved that a rainbow path of length \((2n + 1)/3\) exists in \( G \) provided that, for every color \( \alpha \) used, the set of edges given the color \( \alpha \) forms a perfect matching in \( G \). Not much later Gyárfás, Ruszinkó, Sárközy, & Schelp [10] showed that \( G \) contains a rainbow cycle of length \((4/7 + o(1))n\) for every proper edge-coloring. Then, independently, both Gebauer & Mousset [8] and Chen & Li [7] showed that a path of length \((3/4 - (1))n\) exists when \( G \) is properly colored. This was the best known lower bound until very recently Alon, Pokrovskiy & Sudakov [3] established the following breakthrough result.

**Theorem 4.** For all sufficiently large \( n \), every properly edge-colored complete graph on \( n \) vertices contains a rainbow cycle on at least \( n - 24n^{3/4} \) vertices.

Heavily relying on the methods developed in [3], we make the following improvement to Theorem 4.

**Theorem 5.** There exists a constant \( C \) such that the following holds. If \( G \) is a properly edge-colored complete graph on \( n \) vertices for \( n \) sufficiently large, then there exists a rainbow cycle in \( G \) on at least \( n - C \log n \sqrt{n} \) vertices.

To prove Theorem 5, we use the following theorem which is an extension of Theorem 1.3 in [3]. Our improvement, and one of the main observations that drives our proof, is that essentially the same argument as the one given in [3] works when the sizes of the two sets are unbalanced, i.e., only one of the two sets needs to have order \( \Omega \left( \frac{(\log n/p)^2}{2} \right) \), the other can be as small as \( \Omega \left( \frac{(\log n/p)}{2} \right) \).

**Theorem 6.** For every sufficiently small \( \varepsilon > 0 \) there exists a constant \( C \) such that the following holds. Let \( G \) be a properly edge-colored graph on \( n \) vertices such that \( \delta(G) \geq (1-\delta)n \) for some \( \delta = \delta(n) \). Let \( H \) be the spanning subgraph obtained by choosing every color class independently at random with probability \( p \). Then the following holds with high probability.

(a) If \((1-\delta)n \geq C \log \frac{n}{p}\), then all vertices \( v \) have degree \((1 \pm \varepsilon)p \cdot d_G(v)\) in \( H \).

(b) For every pair \( A \) and \( B \) of disjoint vertex sets, if \(|A| \geq C \log \frac{n}{p}\) and \(|B| \geq \max\{C(\log \frac{n}{p})^2, C\delta n\}\), then \( e_H(A, B) \geq (1 - \varepsilon)p|A||B|\).

1.1. Definitions and notation. Most of our notation is standard except possibly the following. Let \( G \) be a graph and let \( A \) and \( B \) be disjoint vertex subsets. We let \( E_G(A, B) = \{xy \in E(G) : x \in A \text{ and } y \in B\} \), and let \( e_G(A, B) = |E_G(A, B)| \). For a vertex subset \( U \), we let \( N_G(U) = \bigcup_{u \in U} N_G(u) \).

For a path \( P = v_1, \ldots, v_m \) we call \( v_1 \) and \( v_m \) the endpoints of \( P \) and we call a path \( Q \) a subpath of \( P \) if \( Q = v_i, \ldots, v_j \) for some \( 1 \leq i \leq j \leq m \). For a set \( X \) and \( 0 \leq y \leq |X| \), we
let \( \binom{X}{y} = \{Y \subseteq X : |Y| = y\} \) be the set of all subsets of \( X \) that have order exactly \( y \). All logarithms are base 2 unless otherwise specified.

2. Proof of Theorem 6

We use the following form of the well-known Chernoff bound.

**Lemma 7** (Chernoff bound). Let \( X \) be a binomial random variable with parameters \((n, p)\). Then for every \( \varepsilon \in (0, 1) \) we have that

\[
\mathbb{P}(|X - pn| \geq \varepsilon pn) \leq 2e^{-p\varepsilon^2/3}.
\]

For an \( \varepsilon > 0 \), call a pair of disjoint vertex subsets \( A, B \) of a properly edge-colored graph \( G \) \( \varepsilon \)-nearly-rainbow if there are \((1 - \varepsilon)|A||B|\) different colors that appears on the edges \( E_G(A, B) \).

The following is essentially equivalent to Lemma 2.2 in [3] and is a simple application of the Chernoff bound (Lemma 7).

**Lemma 8.** For every \( \varepsilon > 0 \), there exists a constant \( C \) such that the following holds. Let \( G \) be a properly edge-colored graph on \( n \) vertices, and let \( H \) be the spanning subgraph of \( G \) obtained by choosing every color class independently at random with probability \( p \). Then, with high probability, for every \( \varepsilon \)-nearly rainbow pair \( S, T \) such that \(|S| = |T| \geq C\frac{\log n}{p}\) we have that

\[
e_H(S, T) \geq (1 - 2\varepsilon)p|S||T|.
\]

The proof of the following lemma is very similar to Lemma 2.3 in [3]. The main differences are that it can be applied to graphs that are not complete and that the sets \( A \) and \( B \) can be of different sizes.

**Lemma 9.** For every sufficiently small \( \varepsilon > 0 \), there exists \( C \) such that when \( n \) is sufficiently large the following holds. Let \( G \) be a graph on \( n \) vertices and let \( A \) and \( B \) be two disjoint vertex subsets of size \( a \) and \( b \), respectively, with \( a \leq b \). Suppose \( \delta(G) \geq (1 - \delta)n \) for some \( \delta = \delta(n) > 0 \). If \( y \) divides both \( a \) and \( b \), and \( b \geq \max\{Cy^2, C\delta n\} \) and \( y \geq C \), then there exists a partition \( \{A_i\} \) of \( A \) into parts of size \( y \) and a partition \( \{B_j\} \) of \( B \) into parts of size \( y \) such that all but \( \varepsilon \cdot \frac{ab}{y^2} \) of the pairs \( A_i, B_j \) are \( \varepsilon \)-nearly-rainbow.

**Proof.** Let \( \varepsilon > 0 \) be sufficiently small. We choose \( C \geq 2\varepsilon^{-2} \) so we have that \( y^2/b \leq 1/C \leq \varepsilon^2/2 \) and \( \delta n/b \leq 1/C \leq \varepsilon^2/2 \).

Let

\[
\Omega = \{\alpha : \exists e \in E(A, B) \text{ such that } e \text{ is colored with } \alpha\}
\]

be the set of colors used on the edges of \( E(A, B) \). Assume \( S \) and \( T \) are selected uniformly at random from \( \binom{A}{y} \) and \( \binom{B}{y} \), respectively.

For every distinct \( e, e' \in E(S, T) \) that do not share an endpoint

\[
\mathbb{P}(e \in E(S, T)) = \frac{y^2}{ab} \quad \text{and} \quad \mathbb{P}(e, e' \in E(S, T)) = \frac{y^2(y - 1)^2}{ab(a - 1)(b - 1)}.
\]

Let \( \alpha \in \Omega \) and \( E_\alpha = \{e \in E(A, B) : e \text{ is given color } \alpha\} \). Since the edge-coloring is proper, no two edges in \( E_\alpha \) share an endpoint, which implies that \( |E_\alpha| \leq \min\{a, b\} = a \). Therefore,
using inclusion-exclusion and (1), we get the following lower bound on the probability that the color \( \alpha \) is used on an edge in \( E(S, T) \):

\[
\mathbb{P}(|E_\alpha \cap E(S, T)| \geq 1) \geq \sum_{e \in E_\alpha} \mathbb{P}(e \in E(S, T)) - \sum_{\{e, e' \} \in E_\alpha} \mathbb{P}(e, e' \in E(S, T))
\]

\[
= \frac{y^2}{ab}|E_\alpha| - \frac{y^2(y - 1)^2}{ab(a - 1)(b - 1)} \left( \frac{|E_\alpha|}{2} \right) = \left( 1 - \frac{(y - 1)^2}{2(a - 1)(b - 1)} \right) \frac{y^2}{ab}|E_\alpha| \geq \left( 1 - \frac{\varepsilon^2}{2} \right) \frac{y^2}{ab}|E_\alpha|.
\]

Let \( Z \) be the number of different colors used on the edges of \( E(S, T) \). Using linearity of expectation and the fact that

\[
\sum_{\alpha \in \Omega} |E_\alpha| = |\bigcup_{\alpha \in \Omega} E_\alpha| = |E(A, B)| \geq ab - a\delta n
\]

we have that

\[
\mathbb{E}(Z) \geq \sum_{\alpha \in \Omega} \left( 1 - \frac{\varepsilon^2}{2} \right) \frac{y^2}{ab}|E_\alpha| \geq (1 - \frac{\varepsilon^2}{2}) \frac{y^2}{ab}(ab - a\delta n) \geq \left( 1 - \frac{\varepsilon^2}{2} - \frac{\delta n}{b} \right) y^2 \geq (1 - \varepsilon^2) y^2.
\]

Clearly, \( Z \leq |E(S, T)| \leq y^2 \), so \( y^2 - Z \geq 0 \), and \( \mathbb{E}(y^2 - Z) \leq \varepsilon^2 y^2 \). Markov’s inequality then implies that \( \mathbb{P}(y^2 - Z \geq \varepsilon y^2) \leq \varepsilon \), so the probability that \( S, T \) is \( \varepsilon \)-nearly rainbow is at least \( 1 - \varepsilon \).

Select a partition \( \{A_i\} \) of \( A \) into parts of size \( y \) and a partition \( \{B_j\} \) of \( B \) into parts of size \( y \) uniformly at random. The expected fraction of the \( \frac{ab}{y} \) pairs of sets \( A_i \) and \( B_j \) that are not \( \varepsilon \)-nearly rainbow is at most \( \varepsilon \), so there exists a partition such that all but at most \( \varepsilon \frac{ab}{y} \) pairs of sets are \( \varepsilon \)-nearly rainbow.

Proof of Theorem 6 Assume \( \varepsilon > 0 \) is sufficiently small, and that \( C \geq 6\varepsilon^{-2} \) is large enough so that both Lemmas 8 and 9 hold. We will show that the conditions of Theorem 6 hold with \( C' = 4C^3\varepsilon^{-1} \) and \( 5\varepsilon \) playing the roles of \( C \) and \( \varepsilon \), respectively.

Let \( v \in V(G) \). Because \( G \) is properly edge-colored, for every \( v \in V(G) \), the edges incident to \( v \) are rainbow, so the number of edges incident to \( v \) that are in \( H \) is binomial distributed with parameters \( (d_G(v), p) \). Because \( p \cdot d_G(v) \geq p(1 - \delta)n \geq C\log n \), we have that

\[
n \cdot 2e^{-p \cdot d_G(v) \varepsilon^2/3} \leq 2n^{-1},
\]

so the Chernoff bound (Lemma 7) and the union bound imply that

\[
d_H(v) = (1 \pm \varepsilon)p d_G(v)
\]

for every \( v \in V(G) \) with high probability.

Fix \( y = \lceil C\log n/p \rceil \). By Lemma 8 with high probability, for every \( \varepsilon \)-nearly rainbow pair \( S, T \) such that \( |S| = |T| = y \), we have that

\[
\epsilon_H(S, T) \geq (1 - 2\varepsilon)y^2.
\]

Now fix \( a = y \) and let \( b \) be the smallest number larger than \( \max\{Cy^2, C\delta n\} \) that is divisible by \( y \). Let \( A \) and \( B \) be vertex disjoint subsets of orders \( a \) and \( b \), respectively. Lemma 9 implies that there exists a partition \( \{B_j\} \) of \( B \) into parts of size \( y \) such that all but an \( \varepsilon \) fraction
of the pairs $A, B_j$ are $\varepsilon$-nearly-rainbow, i.e., if we let $J = \{j : A, B_j$ is $\varepsilon$-nearly-rainbow$, $|J| \geq (1 - \varepsilon)ab/y^2 = (1 - \varepsilon)b/y$. Therefore, with (2), we have that with high probability
\begin{equation}
(3) \quad e_H(A, B) \geq \sum_{j \in J} e_H(A, B_j) \geq (1 - \varepsilon)ab/y^2 \cdot (1 - 2\varepsilon)y^2 = (1 - 3\varepsilon)ab.
\end{equation}

Finally, to complete the proof, let $A$ and $B$ be disjoint vertex subsets such that $|A| \geq C' \log n/p$, $|B| \geq \max\{C'(\log n/p)^2, C'\delta n\}$. Note that $|A| \geq \varepsilon^{-1}a$ and $|B| \geq \varepsilon^{-1}b$, so there are at least $(1 - \varepsilon)|A|/a$ disjoint subsets of $A$ of size $a$ and at least $(1 - \varepsilon)|B|/b$ disjoint subsets of $B$ of size $b$. Hence, (3) implies that with high probability
\[ e_H(A, B) \geq (1 - \varepsilon)|A|/a \cdot (1 - \varepsilon)|B|/b \cdot (1 - 3\varepsilon)ab = (1 - 5\varepsilon)|A||B|. \]

\section{Long rainbow cycles}

This appears as Lemma 3.1 in [3].

\textbf{Lemma 10.} For all $\gamma, \delta, n$ with $\delta \geq \gamma$ and $3\gamma \delta - \gamma^2/2 > n^{-1}$ the following holds. Let $G$ be a properly edge-colored graph on $n$ vertices such that $\delta(G) \geq (1 - \delta)n$. Then $G$ contains a rainbow path forest $\mathcal{P}$ with at most $\gamma n$ paths and $|E(\mathcal{P})| \geq (1 - 4\delta)n$.

For $0 < a \leq b$, call a graph $H$ an $(a, b)$-expander, if the following holds:
\begin{enumerate}[(E1)]
\item $\delta(H) \geq a$;
\item if $A \subseteq V(H)$ such that $|A| \geq a$, then $|N_H(A)| \geq n - a - b$; and
\item if $A$ and $B$ are disjoint subsets of order $a$ and $b$, respectively, then $E_H(A, B) \neq \emptyset$.
\end{enumerate}

Note that (E3) implies (E2), because (E3) implies $|V(G) \setminus (N_H(A') \cup A')| \leq b$ for every $A' \in \binom{[n]}{a}$, but it is more convenient to state (E2) separately.

\textbf{Lemma 11.} Let $0 < a \leq b \leq n/4$, $r > 0$, and let $G, H_1, H_2$ and $H_3$, be edge-disjoint spanning subgraphs of the complete graph on $n$ vertices whose edges are edge-colored by pairwise disjoint sets of colors, such that $H_1$, $H_2$ and $H_3$ are each $(a, b)$-expanders. Suppose $\mathcal{P} = \{P_1, \ldots, P_r\}$ is a rainbow path forest in $G$ such that for $U = V(G) \setminus V(\mathcal{P})$ we have $|U| \leq b$. If $|P_i| \leq n - a - |U|$, then there exists $e_j \in E(H_j)$ for $j \in [3]$, and $i \in [r]$, such that there are two disjoint paths $P_i'$ and $P_i''$ in the graph induced in $G$ by $V(P_i) \cup V(P_i') \cup U$ with the additional edges $\{e_1, e_2, e_3\}$ where
\begin{enumerate}[(1)]
\item $P_i'$ is a subpath of $P_i$ on less than $|P_i|/2$ vertices (we allow $P_i'$ to be the path without vertices here),
\item $|P_i'| \geq |P_i| + |P_i| - |P_i|$, and
\item $\mathcal{P}' = \mathcal{P} - P_i - P_i \cup \{P_i', P_i''\}$ is a rainbow path forest.
\end{enumerate}

\textbf{Proof.} Assume $|P_i| \leq n - a - |U|$ and that the statement of the lemma does not hold. Let $v_1, \ldots, v_m$ be the vertices of $P_i$ in the order they appear on the path, let $T$ be the set of vertices on the paths $P_2, \ldots, P_r$, and recall that $U = V(G) \setminus V(\mathcal{P})$. We have that
\begin{equation}
(4) \quad |T| = n - |P_i| - |U| \geq a.
\end{equation}

\textbf{Claim 11.1.} Let $\sigma$ be a permutation of $\{1, 2, 3\}$. For every path $P$ such that
\begin{itemize}
\item $|P| \geq |P_i|$, 
\item $V(P) \subseteq V(P_i) \cup U$, and
\item $E(P) \subseteq E(P_i) \cup \{e_{\sigma(1)}, e_{\sigma(2)}\}$ where $e_{\sigma}(j) \in E(H_{\sigma(j)})$ for $j \in [2]$,
\end{itemize}
there are no edges in $E(H_{\sigma(3)})$ incident to an endpoint of $P$ and a vertex in $T$.

**Proof.** Suppose, for a contradiction, that $v$ is an endpoint of such a path $P$ and there exists $x \in N_{H_{\sigma(3)}}(v) \cap T$. Let $P_i \in \{P_2, \ldots, P_r\}$ be the path containing $x$. We can construct $P'_i$ by combining $P$ with the longer of the two subpaths in $P_i$ that have $x$ as an endpoint. By letting $P'_i$ be subpath of $P_i$ with the vertex set $V(P_i) \setminus V(P')$ (recall that the statement of the theorem allows $P'_i$ to be the path without vertices), we have two paths $P'_i$ and $P'_i$ that satisfy the conditions of the lemma. \hfill $\square$

Let $\mathcal{P}$ be the set of all paths that do not contain an edge from $E(H_3)$ and that meet the conditions of Claim III. Let

$$X = \{ x \in V(G) : \text{there exists } P \in \mathcal{P} \text{ such that } x \text{ is an endpoint of } P \}.$$  

Claim III implies that $E(H_3)(X, T) = \emptyset$, so, with (II), we will contradict (E3) and prove the lemma if we can show that $|X| \geq b$.

Claim III implies that, in $H_1$, $v_1$ does not have a neighbor in $T$, so if we let $\nabla = (N_{H_1}((v_1) \cap U) \cup \{v_j : v_{j+1} \in N_{H_1}((v_1) \cap V(P_1))\}$,

(5) $|N_{H_2}(\nabla)| \geq n - a - b \geq 2b$.

We also have that $\nabla \subseteq X$. To see this, observe that if $y \in Y \cap U$, then $y, v_1, \ldots, v_m$ is in $\mathcal{P}$, and if $y \in Y \setminus U$, then it is on $P_1$, so $y = v_j$ for some $j \in [m]$ and the path $v_j, v_{j+1}, \ldots, v_m$ is in $\mathcal{P}$.

We will now describe a mapping from $N_{H_2}(\nabla)$ to $X$ such that, for every $x \in X$, at most two vertices in $N_{H_2}(\nabla)$ are mapped to $x$. By (5), this will imply that $|X| \geq b$, which, as was previously mentioned, will prove the claim. To this end, let $v \in N_{H_2}(\nabla)$, and arbitrarily select some $y \in N_{H_2}(v) \cap \nabla$. Recall that there exist $P_y \in \mathcal{P}$ that has $y$ as an endpoint and that does not contain edges from either $H_2$ or $H_3$. First assume that $v$ is not on $P_1$. Then, by using $vy \in E(H_2)$, we can append $v$ to $P_y$ to create an element of $\mathcal{P}$ with $v$ as an endpoint. Therefore, $v \in X$, so, in this case, we map $v$ to itself. Now assume that, $v = v_k$ for some $k \in [m]$, and recall that $\nabla$ does not intersect $T$, so $y \in V(P_1) \cup U$. If $y \in U$, then $v_k \neq v_1$, since $v_1 \in E(H_1)$, so we can map $v_k$ to $v_{k-1}$, because $v_1 \ldots v_y v_k \ldots v_m$ is in $\mathcal{P}$. If $y \in P_1$, then $v = v_j$ for some $j \in [m]$. Recall that by the definition of $\nabla$, $v_1 v_{j+1}$ is an edge in $H_1$. If $k \geq j + 1$, the path $v_{k-1}, \ldots, v_{j+1}, v_{j}, v_{j+1}, \ldots, v_m$ is in $\mathcal{P}$, so we map $v_k$ to $v_{k-1}$. Similarly, if $k \leq j - 1$, the path $v_{k+1}, \ldots, v_j, v_k, v_1, v_{j+1}, \ldots, v_m$ is in $\mathcal{P}$, so we map $v_k$ to $v_{k+1}$. Note that we have now proved the lemma, because for every $x \in X$, at most two vertices in $N_{H_2}(\nabla)$ are mapped to $x$; if $x \in X \cap U$, then the only vertex that can be mapped to $x$ is $x$ itself, and if $x = v_k$ for some $v_k \in X \cap V(P_1)$ then $v_{k-1}$ and $v_{k+1}$ are the only vertices that can be mapped to $x$. \hfill $\square$

**Proof of Theorem 5.** Assume $\varepsilon > 0$ is sufficiently small, and pick $C$ large enough so that, provided $n$ is sufficiently large, the conditions of Theorem 6 hold with $\varepsilon$ and $C$. Let $p = C \frac{\log n}{\sqrt{n}}$, $a = \sqrt{n}$, $b = n/4$, $\delta = 4p$, and $\gamma = \frac{1}{C \log a \sqrt{n}}$. Let $G_1$ be a properly edge-colored complete graph on $n$ vertices.

**Claim 5.1.** There exist edge-disjoint spanning subgraphs, $G$, $H_1$, $H_2$ and $H_3$, of $G_1$ that are properly edge-colored with pairwise disjoint sets of colors such that
(a) $\delta(G) \geq (1 - \delta)n$, and,
(b) for every $m \leq \sqrt{n}$ and $i \in [3]$, if $m$ color classes are removed from $H_i$, then the resulting graph is an $(a, b)$-expander.

Proof. We form $H_1$ by selecting the color classes of $G_1$ randomly and independently with probability $p$. With high probability, the conditions of Theorem 6 hold. We fix such a subgraph $H_1$.

Note that every vertex has degree $(1 \pm \varepsilon)p(n - 1)$ in $H_1$, so if we let $G_2$ be the graph formed by removing the edges of $H_1$ from $G_1$, we have that every vertex has degree $(1 - p(1 \pm \varepsilon))(n - 1)$ in $G_2$. Therefore, if we form $H_2$, by selecting the color classes of $G_2$ randomly and independently with probability $p$, the conditions of Theorem 6 hold in $H_2$ with high probability. We fix such a graph $H_2$ and note that every vertex has degree $(1 \pm 1.1\varepsilon)p(n - 1)$ in $H_2$. We then let $G_3$ be the graph formed by removing the edges of $H_2$ from $G_2$. Every vertex has degree $(1 - 2(p \pm 1.1\varepsilon))(n - 1)$ in $G_3$, so if we form $H_3$ by selecting the color classes of $G_3$ randomly and independently with probability $p$, $H_3$ satisfies the conditions of Theorem 6 with high probability, so we can fix such an $H_3$. We now have that for every $j \in [3]$, and every vertex $v$,

$$d_{H_j}(v) \geq (1 \pm 1.2\varepsilon)p(n - 1) > 2\sqrt{n}.$$ 

Let $G$ be the graph formed by removing the edges of $H_3$ from $G_3$, and note that, with (6), for every vertex $v$ we have that

$$d_G(v) = (n - 1) - \sum_{j \in [3]} d_{H_j}(v) \geq (n - 1) - 3(1 + 1.2\varepsilon)np \geq (1 - \delta)n.$$

Because $a \geq C\frac{\log n}{p}$, $b \geq C\left(\frac{\log n}{p}\right)^2$ and $b \geq C\delta n$, the conditions of Theorem 6 imply that, for $j \in [3]$ and every pair of disjoint vertex sets $A$ and $B$ with sizes at least $a$ and $b$, respectively, $e_{H_j}(A, B) \geq (1 - \varepsilon)p|A||B|$.

For each $j \in [3]$, form $H_j'$ by removing an arbitrary set of $m \leq \sqrt{n}$ color classes from $H_j$. By (6), and the fact that $H_j$ is properly edge-colored, we have that $d_{H_j'}(v) \geq d_{H_j}(v) - m \geq a$, so (E1) holds. For every pair of disjoint sets $A$ and $B$ with orders at least $a$ and $b$, respectively, since $pb = C\sqrt{n}\log n/3 > 2\sqrt{n} \geq 2m$,

$$e_{H_j'}(A, B) \geq e_{H_j}(A, B) - m|A| \geq (1 - \varepsilon)p|A||B| - m|A| = |A|((1 - \varepsilon)p|B| - m) > 0.$$

Hence, we have established that (E1) and (E3) from the definition of an $(a, b)$-expander hold in $H_j'$. As was mentioned in the definition of an $(a, b)$-expander, (E3) implies (E2) so this completes the proof of the claim.

Because $\gamma\delta = 4/n$ and $\gamma^2 = o(1/n)$, we have that $3\gamma\delta - \gamma^2/2 \geq 1/n$, so we can apply Lemma 11 to form a rainbow path forest $P = \{P_1, \ldots, P_r\}$ such that $|V(P)| \geq (1 - 4\delta)n$ and $r \leq \gamma n$.

We now apply the following algorithm to $P$.

- If $|P_1| \geq n - 4\delta n - a$ or one of $H_1$, $H_2$, or $H_3$ is not an $(a, b)$-expander, then terminate.
- Otherwise, we let $P'$ and $e_1, e_2$ and $e_3$ be as in the statement of Lemma 11.
- For each $j \in [3]$, remove the color class corresponding to $e_j$ from $H_j$ and then repeat with $P = P'$.
Note that at most \((r - 1) \log n < \sqrt{n}\) iterations of the algorithm will execute, since each of the \(r - 1\) paths \(\{P_2, \ldots, P_r\}\) can be used to extend \(P_1\) at most \(\log n\) times. To see this, observe that every time such a path \(P_i\) is used to extend \(P_1\), at least half of the remaining vertices in \(P_1\) are removed. Therefore, by Claim 5.1, the algorithm must terminate with \(|P_1| \geq n - 4\delta n - a\).

After the algorithm terminates, we have \((a, b)\)-expanders \(H_1, H_2\) and \(H_3\) and a rainbow path \(P_1\) on at least \(n - 4\delta n - a\) vertices such that the edges of \(H_1, H_2, H_3\) and \(P_1\) are colored with disjoint sets of colors. We can now use a procedure similar to the one in the proof of Theorem 12 to form a rainbow cycle of length at least \(n - 4\delta n - 3a\) and this will complete the proof. Let \(v_1, \ldots, v_m\) be the vertices of \(P_1\) in the order they appear on the path. Let \(A_1 = \{v_1, \ldots, v_a\}\) be the first \(a\) vertices on \(P_1\) and let \(A_2 = \{v_{m-(a-1)}, \ldots, v_m\}\) be the last \(a\) vertices on \(P_1\). Assume \(E_{H_1}(A_1, A_2) = \emptyset\), since otherwise we have the desired cycle. Let

\[
B = \{v_j : v_{j+1} \in N_{H_1}(A_1) \cap (V(P_1) \setminus A_1)\},
\]

and note that \(A_2\) and \(B\) are disjoint, and, because \(H_1\) is an \((a, b)\)-expander,

\[
|B| \geq |N_{H_1}(A_1)| - |V(G) \setminus V(P_1)| - |A_1| \geq (n - a - b) - (4\delta n + a) - a \geq b.
\]

Therefore, there exists \(v_j \in B\) and \(v_k \in A_2\) such that \(v_kv_j \in E_{H_2}(A_2, B)\). Recall that there exists \(v_i \in A_1\) such that \(v_iv_{j+1} \in E(H_1)\) and note that \(i < j < k\). Because \(v_i, v_{j+1}, \ldots, v_k, v_j, v_{j-1}, \ldots, v_i\) is a cycle that contains all of the vertices \(v_i, \ldots, v_k\), we have the desired cycle.

4. Spanning Rainbow Path Forest with Few Paths

In this section we prove Theorem 3. In fact, we prove the following more general result which implies the theorem.

**Theorem 12.** There exists a constant \(C\) such that for every \(n\) and for all \(\delta = \delta(n) > 0\) the following holds. If \(G\) is a properly edge-colored graph on \(n\) vertices and \(\delta(G) \geq (1 - \delta) n\), then \(G\) contains a spanning rainbow path forest \(\mathcal{P}\) with at most \(C(\log n)^2 + 3\delta n\) paths.

We will need the following technical lemma. We defer its proof until after the proof of Theorem 12.

**Lemma 13.** Suppose that \(0 < c \leq 1\) and that \(n_1, \ldots, n_k\) is a sequence of strictly increasing positive integers such that for all \(m \leq j < \ell \leq k\)

\[
n_j - n_{j-1} \geq \frac{n_\ell - n_j}{n_j} ((1 + c) n_j - (2n_\ell - n_{\ell-1})).
\]

Then \(k \leq (\log_r n_k)^2 + 2 \log_r n_k + m + 1\) where \(r = 1 + c/3\).

**Proof of Theorem 12.** Let \(G\) be a properly edge-colored graph on \(n\) vertices such that \(\delta(G) \geq (1 - \delta)n\). For a rainbow path forest \(F\), let \(p(F)\) be the number of paths in \(F\) and let \(A(F)\) be the set of endpoints of paths in \(F\). For two rainbow path forests, \(F\) and \(F'\), we say that \(F'\) is obtained from \(F\) by a *swap* if there exists an edge \(e\) in \(G\) that is incident to endpoints of distinct paths in \(F\) such that either \(F' = F + e\) and there are no edges in \(E(F)\) given the same color as \(e\), or \(F' = F + e - e'\) where \(e' \in E(F)\) such that \(e'\) and \(e\) are given the same color. Note that when \(F'\) is obtained from \(F\) by a swap, there is a unique color, say \(\alpha\), that is used on the edges in \(E(F') \triangle E(F)\). We call \(\alpha\) the color *associated* with the swap.
Let $\mathcal{S}(F)$ be the set of rainbow path forests $F'$ that can be obtained from $F$ by a sequence of swaps, i.e., $F' \in \mathcal{S}(F)$ if there exists a sequence of rainbow path forests $F = F_1, F_2, \ldots, F_m = F'$ such that, for every $i \in [m-1]$, $F_{i+1}$ is obtained from $F_i$ by a swap. Note that $p(F) \geq p(F')$ for all $F' \in \mathcal{S}(F)$. If, for all $F' \in \mathcal{S}(F)$, we have that $p(F') = p(F)$ then we say that $F$ is swap-optimal. Let $C(\mathcal{S}(F))$ contain the set of colors $\alpha$ such that $\alpha$ is the color associated with a swap between two forests in $\mathcal{S}(F)$. For every collection of path forests $\mathcal{F}$, let $A(\mathcal{F}) = \bigcup_{F \in \mathcal{F}} A(F)$ be the set of vertices $x$ for which there exists at least one path forest in $\mathcal{F}$ in which $x$ is an endpoint of a path.

Let $k$ be the minimum number of paths in a spanning rainbow path forest of $G$ and fix $F_k$ a rainbow spanning path forest with $k$ paths. Note that $F_k$ is swap-optimal. We use the following iterative procedure to select swap-optimal forests $F$ in $\mathcal{S}(F)$ that are assigned a color from $C$. For every $F \in \mathcal{S}(F)$ and each $x \in F$ containing $x$ is as short as possible.

Note that for every $P$ and $F$ selected in this way, $F - P$ is swap-optimal. To see this, first note that $\alpha$ implies that the colors in $C(\mathcal{S}(F - P))$ are not used on the edges of $P$. But then, for every $F' \in \mathcal{S}(F - P)$, the forest $P + F'$ is rainbow, so $P + F' \in \mathcal{S}(F)$. This implies that $p(P + F') = p(F) = j$, so $p(F') = p(F - P) = j - 1$, which further implies that $F - P$ is swap-optimal. Define $P_{j-1} = P$ and $F_{j-1}^x = F - P_{j-1}$. To complete the procedure for constructing $F_{j-1}$, pick $x \in A(S(F_j))$ so that $|A(S(F_{j-1}))|$ is as small as possible, and then let $F_{j-1} = F_{j-1}^x$.

For every $j \in [k]$, define $A_j = A(S(F_j))$, $n_j = |A_j|$, $C_j = C(S(F_j))$ and $G_j$ to be the graph with vertex set $V(G)$ that contains only the edges of $G$ that are assigned a color from $C_j$. Define $d_j(x) = d_{G_j}(x)$ for $x \in V(G)$, and, for $U \subseteq V(G)$, let $d_j(x, U) = |N_{G_j}(x) \cap U|$.

For every $j \in [k]$, define $A_j = A(S(F_j))$, $n_j = |A_j|$, $C_j = C(S(F_j))$ and $G_j$ to be the graph with vertex set $V(G)$ that contains only the edges of $G$ that are assigned a color from $C_j$. Define $d_j(x) = d_{G_j}(x)$ for $x \in V(G)$, and, for $U \subseteq V(G)$, let $d_j(x, U) = |N_{G_j}(x) \cap U|$.

Similarly, for disjoint vertex subsets $A$ and $B$, we let $E_j(A, B) = E_{G_j}(A, B)$ and $e_j(A, B) = e_{G_j}(A, B)$.

The following claim summarizes some of the important facts implied by this construction.

Claim 12.1. For every $1 \leq j \leq k$, we have that $F_j$ is swap-optimal. For every $2 \leq j \leq k$, every $x \in A_j$ and every $F \in S(F_{j-1}^x)$, we have that $P_{j-1}^x + F \in S(F_j)$, so $A(S(F_{j-1}^x)) \subseteq A_j$. We also have that $|A(S(F_{j-1}^x))| \geq n_j - 1$.

Claim 12.2. For every $1 \leq j \leq k$, we have that $\frac{1}{2}n_j - j \leq |C_j| \leq n_j - j$.

Proof. Let $H$ be the subforest of $F_j$ created by first removing from $F_j$ all edges that were not assigned a color from $C_j$ and then removing all isolated vertices that are not an endpoint of a path in $F_j$. Recall that for $v \in V(F_j)$, we have that $v \in A_j$ if and only if there exists a path forest in $S(F_j)$ in which $v$ is an endpoint of a path. Therefore, $A_j$ contains all of the endpoints of paths in $F_j$ and these endpoints are also in $V(H)$. Now consider a vertex $v$ in $V(F_j)$ that is not the endpoint of a path in $F_j$. Then $v \in A_j$ if and only if one of the two edges incident to $v$ in $F_j$ is colored with a color from $C_j$. Note we have established that $A_j = V(H)$.

Since $H$ is rainbow and $C_j$ is exactly the set of colors used on the edges of $H$, $|E(H)| = |C_j|$. Because $H$ is a path forest and $|V(H)| = |A_j| = n_j$, we have that $|C_j| = |E(H)| = n_j - p(H)$. Therefore, to complete the proof, we only need to show that the number of paths in $H$,
For every \( \alpha \in H \), is between \( j \) and \( \frac{1}{2}n_j + j \). The lower bound on \( p(H) \) follows because \( p(F_j) = j \) and \( H \subseteq F_j \). The upper bound on \( p(H) \) comes from the fact that the isolated vertices in \( H \) must be endpoints of paths in \( F_j \). Therefore, there are at most \( 2j \) isolated vertices in \( H \). Since all of the paths in \( H \) that are not isolated vertices must contain at least 2 vertices, \( p(H) \leq \frac{1}{2}(n_j - 2j) + 2j = \frac{1}{2}n_j + j. \)

**Claim 12.3.** For every \( j \in [k] \) and \( x \in A_j \), we have that \( d_j(x) \leq |C_j| \) and \( d_j(x, A_j) \geq n_j - \delta n. \)

**Proof.** The first inequality, \( d_j(x) \leq |C_j| \), follows from the fact that \( G_j \) is properly edge-colored and only uses colors from \( C_j \). To establish the second inequality, \( d_j(x, A_j) \geq n_j - \delta n \), we first note that, by Claim [12.2], \( A(S(F^x_j)) \) is a subset of \( A_j \), and that \( |A(S(F^x_j))| \geq n_j - \delta n \). Therefore, since \( \delta(G) \geq n - \delta n \), we will prove the second inequality by showing that if \( y \in A(S(F^x_j)) \) such that \( xy \in E(G) \), then the edge \( xy \) is assigned a color from \( C_j \).

To this end, let \( y \in A(S(F^x_j)) \cap N_G(x) \) and let \( \alpha \) be the color assigned to \( xy \). Recall that \( x \) is one of the endpoints of \( P^x_{j-1} \) and that, by the definition of \( A(S(F^x_j)) \), there exists \( F \in S(F^x_j) \) in which \( y \) is the endpoint of a path. By Claim [12.1] if \( F' = P^x_{j-1} + F \), then \( F' \in S(F_j) \). Furthermore, because \( F_j \) is swap-maximal, there exists \( e \in E(F') \) such that \( e \) is assigned the color \( \alpha \). Therefore, \( F' + xy - e \in S(F_j) \), which implies \( \alpha \in C_j \). Since \( F' \) is obtained from \( F \) by a swap and \( \alpha \) is the color associated with this swap, we have that \( \alpha \in C_j \). This implies the claim.

**Claim 12.4.** For every \( 2 \leq j < \ell \leq k \),

\[
(n_{\ell} - n_j) \left( \frac{3}{2}n_j - 2n_\ell + n_{\ell-1} - \delta n \right) \leq e_j(A_\ell \setminus A_j, A_j) \leq n_j(n_j - n_{j-1} - j + \delta n).
\]

**Proof.** Let \( x \in A_\ell \setminus A_j \). First note that \( d_j(x, A_j) \geq d_\ell(x, A_\ell) - |C_\ell \setminus C_j| \) because the edges of \( G \) and hence \( G_\ell \) are properly colored. Because Claim [12.2] implies that

\[
|C_\ell \setminus C_j| \leq (n_{\ell} - \ell) - \left( \frac{1}{2}n_j - j \right),
\]

we have that

\[
(8) \quad d_j(x, A_j) \geq d_\ell(x, A_\ell) - |C_\ell \setminus C_j| \geq d_\ell(x, A_\ell) - (n_{\ell} - \ell) + \left( \frac{1}{2}n_j - j \right).
\]

We also have that,

\[d_\ell(x, A_\ell) \geq d_\ell(x, A_\ell) - |A_\ell \setminus A_j| = d_\ell(x, A_\ell) - (n_{\ell} - n_j),\]

and then by Claim [12.3]

\[d_\ell(x, A_\ell) \geq n_{\ell-1} - \delta n,\]

which using (8) gives us that

\[
d_j(x, A_j) \geq \left( (n_{\ell-1} - \delta n) - (n_j - n_j) \right) - (n_{\ell} - \ell) + \left( \frac{1}{2}n_j - j \right)
\]

\[
= \frac{3}{2}n_j + n_{\ell-1} - 2n_\ell + \ell - j - \delta n \geq \frac{3}{2}n_j - 2n_\ell + n_{\ell-1} - \delta n.
\]

Summing up the above relation over all vertices in \( A_\ell \setminus A_j \) gives the lower bound.
For every $x \in A_j$, by Claims \textbf{[12.3]} and \textbf{[12.2]} $d_j(x, A_\ell) \leq d_j(x) \leq |C_j| \leq n_j - j$. By Claim \textbf{[12.3]} we also have that $d_j(x, A_j) \geq n_{j-1} - \delta n$. Therefore,
\[
d_j(x, A_\ell \setminus A_j) = d_j(x, A_\ell) - d_j(x, A_j) \leq (n_j - j) - (n_{j-1} - \delta n).
\]
Summing over all vertices in $A_j$ gives us the upper bound.

Let $m = [3\delta n]$, so, for every $m \leq j \leq k$, because $n_j \geq j \geq 3\delta n$,
\[
\frac{3}{2} n_j - 2n_\ell + n_\ell - j \geq \frac{7}{6} n_j - (2n_\ell - n_\ell - 1)
\]
and
\[
n_j - n_{j-1} - j + \delta n \leq n_j - n_{j-1}.
\]
Therefore, Claim \textbf{[12.4]} implies that, for every $m \leq j < \ell \leq k$,
\[
n_j - n_{j-1} \geq n_j - n_{j-1} - j + \delta n \geq \frac{n_\ell - n_j}{n_j} \left( \frac{3}{2} n_j - 2n_\ell + n_\ell - j - \delta n \right) \geq \frac{n_\ell - n_j}{n_j} \left( \left( \frac{1}{6} \right) n_j - (2n_\ell - n_\ell - 1) \right).
\]
We can then apply Lemma \textbf{[13]} to $n_1, \ldots, n_k$ to deduce that $k \leq (\log_r n_k)^2 + 2 \log_r n_k + m + 1$ where $r = 19/18$. Hence, Theorem \textbf{[12]} holds.

\textbf{Proof of Lemma \textbf{[13]}}

\textbf{Claim 13.1.} For all $m \leq j < \ell \leq k$, if $n_\ell \leq rn_j$, then $n_\ell - n_j \leq r^{-1}(n_\ell - n_{j-1})$.

\textit{Proof.} We have that
\[
2n_\ell - n_\ell - 1 \leq 2n_\ell - n_j \leq (2r - 1)n_j.
\]
With the fact that $c = 3r - 3$, this implies that
\[
(1 + c)n_j - (2n_\ell - n_\ell - 1) \geq (1 + c - (2r - 1))n_j = (r - 1)n_j.
\]
Combining this with (7) gives us that $(r - 1)(n_\ell - n_j) \leq n_j - n_{j-1}$, so
\[
r(n_\ell - n_j) = n_\ell - n_j + (r - 1)(n_\ell - n_j) \leq n_\ell - n_j + n_j - n_{j-1} = n_\ell - n_{j-1},
\]
which proves the claim.

\textbf{Claim 13.2.} For all $m \leq j < \ell \leq k$, if $n_\ell \leq rn_j$ and $j \leq \ell - 1$, then for every $i$ such that $j \leq i \leq \ell - 1$, we have that $n_\ell \leq rn_j \leq rn_i$. Therefore, Claim \textbf{[13.1]} implies $n_\ell - n_i \leq r^{-1}(n_\ell - n_{i-1})$, which further implies that
\[
1 \leq n_\ell - n_{\ell-1} \leq r^{-(\ell-1)(n_\ell - n_j)},
\]
and $\ell - (j + 1) \leq \log_r(n_\ell - n_j)$. Using this and the fact that our assumption $n_\ell \leq rn_j$ implies that $n_\ell - n_j \leq (r - 1)n_j$ we have that
\[
\ell - (j + 1) \leq \log_r(n_\ell - n_j) \leq \log_r((r - 1)n_j).
\]
Therefore, because $1 < r \leq 4/3$ implies that $r - 1 < r^{-1}$, i.e., $\log_r(r - 1) < -1$, we have
\[
\log_r n_j \geq \ell - j - 1 - \log_r(r - 1) > \ell - j,
\]
and the claim holds.
Let \( t = \lceil \log_r n_k \rceil \) and \( s = \lfloor k - m \rfloor / t \). Note that, for every \( 1 \leq i \leq s \),
\[
(it + m) - ((i - 1)t + m) = t \geq \log_r n_k,
\]
and, therefore, Claim 13.2 implies that
\[
n_{it+m} \geq r^{n_{(i-1)t+m}},
\]
so \( n_{it+m} \geq r^i n_m \), and
\[
n_k \geq n_{st+m} \geq r^s n_m \geq r^s.
\]
Therefore,
\[
\log_r n_k \geq s \geq \frac{k - m}{t} - 1 \geq \frac{k - m}{(\log_r n_k) + 1} - 1,
\]
which implies
\[
k \leq (\log_r n_k)^2 + 2 \log_r n_k + m + 1.\]
\[\square\]

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