Canonical partition functions: ideal quantum gases, interacting classical gases, and interacting quantum gases

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Abstract. In statistical mechanics, for a system with a fixed number of particles, e.g. a finite-size system, strictly speaking, the thermodynamic quantity needs to be calculated in the canonical ensemble. Nevertheless, the calculation of the canonical partition function is difficult. In this paper, based on the mathematical theory of the symmetric function, we suggest a method for the calculation of the canonical partition function of ideal quantum gases, including ideal Bose, Fermi, and Gentile gases. Moreover, we express the canonical partition functions of interacting classical and quantum gases given by the classical and quantum cluster expansion methods in terms of the Bell polynomial in mathematics. The virial coefficients of ideal Bose, Fermi, and Gentile gases are calculated from the exact canonical partition function. The virial coefficients of interacting classical and quantum gases are calculated from the canonical partition function by using the expansion of the Bell polynomial, rather than calculated from the grand canonical potential.

Keywords: anyons and fractional statistical models, quantum gases, rigorous results in statistical mechanics
1. Introduction

It is often difficult to calculate the canonical partition function. In the canonical ensemble, there is a constraint on the total number of particles. Once interactions also exist, whether quantum exchange interactions or classical inter-particle interactions, the calculation in canonical ensembles becomes complicated. In ideal quantum gases, there are quantum exchange interactions among gas molecules; in interacting classical gases, there are classical interactions among gas molecules; in interacting quantum gases, there are both quantum exchange interactions and classical inter-particle interactions among gas molecules. In an interacting system there is no single-particle state, and in a canonical ensemble there is a constraint on the total particle number. Therefore, in the...
calculation of the canonical partition functions for such systems, we have to take these two factors into account simultaneously, and this makes the calculation difficult. It is a common practice to avoid such a difficulty by, instead of canonical ensembles, turning to grand canonical ensembles in which there is no constraint on the total particle number. In grand canonical ensembles one uses the average particle number. In this paper, we suggest a method to calculate canonical partition functions.

The method suggested in the present paper is based on the mathematical theory of the symmetric function and the Bell polynomial.

The symmetric function was first studied by Hall in the 1950s [1, 2]. Now it is studied as the ring of symmetric functions in algebraic combinatorics. The symmetric function is closely related to the integer partition in number theory and plays an important role in the theory of group representations [3].

The Bell polynomial is a special function in combinatorial mathematics [4, 5]. It is useful in the study of set partitions and is related to the Stirling and Bell number. For more details of the Bell polynomial, one can refer to [4–8].

First, using the method, we will calculate the exact canonical partition function for ideal quantum gases, including ideal Bose gases, ideal Fermi gases, and ideal Gentile gases. In ideal quantum gases, quantum exchange interactions exist, as will be shown later, the canonical partition functions are symmetric functions. Based on the theory of the symmetric function, the canonical partition function is represented as a linear combination of $S$-functions. The $S$-function is an important class of symmetric functions, and they are closely related to integer partitions [3] and permutation groups [1, 2].

Second, we show that the canonical partition functions of interacting classical gases and interacting quantum gases given by the classical and quantum cluster expansions are the Bell polynomial. In interacting classical gases, classical inter-particle interactions exist and in interacting quantum gases, both quantum exchange interactions and classical inter-particle interactions exist. After showing that the canonical partition function of interacting gases is the Bell polynomial, we can calculate, e.g. the virial coefficient by the property of the Bell polynomial.

We will calculate the virial coefficients of ideal Bose gases, ideal Fermi gases, and ideal Gentile gases from their exact canonical partition functions. Moreover, based on the property of the Bell polynomial, we directly calculate the virial coefficients of interacting classical and quantum gases from the canonical partition function directly, rather than, as in the cluster expansion method, from the grand canonical potential. We also compare the virial coefficients calculated in the canonical ensemble with those calculated in the grand canonical ensemble. From the results one can see that the virial coefficients are different at small $N$ and they are consistent with each other as $N$ goes to infinity.

There are many studies on canonical partition functions. Some canonical partition functions of certain statistical models are calculated, for example, the canonical partition function for a two-dimensional binary lattice [9], the canonical partition function for quon statistics [10], a general formula for the canonical partition function expressed as sums of the $S$-function for a parastatistical system [11], the canonical partition function of freely jointed chain model [12], an exact canonical partition function for ideal Bose gases calculated by the recursive method [13], and the canonical partition function of fluids calculated by simulations [14]. Some methods for the calculation of the canonical partition function are developed, for example, the numerical method [13, 15], the recursion relation of the canonical partition function [15–17]. The behaviour of the
canonical partition function is also discussed, for example, the zeroes of the canonical partition function [18, 19] and the classical limit of the quantum-mechanical canonical partition function [20].

There are also many studies on the symmetric function and the Bell polynomial. In mathematics, some studies are devoted to the application of the $S$-function, also known as the Schur function, for example, the application of $S$-function in the symmetric function space [21], the application of the $S$-function in probability and statistics [22], and the supersymmetric $S$-function [23]. In physics, there are also applications of the $S$-function, for example, the canonical partition function of a parastatistical system is expressed as a sum of $S$-functions [11], the factorial $S$-function, a generalization of the $S$-function, times a deformation of the Weyl denominator may be expressed as the partition function of a particular statistical-mechanical system [24] and the application to statistical mechanics and representation theory [25]. The Bell polynomial can be used in solving the water wave equation [26], in seventh-order Sawada–Kotera–Ito equation [27], in combinatorial Hopf algebras [28]. Moreover, various other applications can be found in [5, 29–31].

There are studies devoted to the application of the Bell polynomial in statistical mechanics. The canonical partition function of an interacting system is expressed in terms of the Bell polynomial by expanding the grand canonical partition function in power series of the fugacity $z$ [32]. As further applications of the Bell polynomial, a microscopic interpretation of the grand potential is given in [33]. The partition function of the zero-field Ising model on the infinite square lattice is given by resorting to the Bell polynomial [34]. Reference [35] proves an assertion suggested in [33], which provides a formula for the canonical partition function using Bell polynomials and shows that the Bell polynomial is very useful in statistical mechanics. In this paper, we will compare our method in obtaining the canonical partition function of interacting gases with that in the literature.

This paper is organized as follows. In section 2, we give a brief review for the integer partition and the symmetric function. In sections 3 and 4, the canonical partition functions of ideal Bose and Fermi gases are given. In section 5, the canonical partition function of ideal Gentile gases is given. In sections 6 and 7, we show that the canonical partition functions of interacting classical and quantum gases are indeed Bell polynomials. In section 8, we calculate the virial coefficients of ideal Bose, Fermi, and Gentile gases in the canonical ensemble and compare them with those calculated in the grand canonical ensemble. In section 9, we calculate the virial coefficients of interacting classical and quantum gases in the canonical ensemble and compare them with those calculated in the grand canonical ensemble. The conclusions are summarized in section 10. In appendix, the details of the calculation about Gentile gases is given.

2. The integer partition and the symmetric function: a brief review

The main result of the present paper is to represent the canonical partition function as a linear combination of symmetric functions. The symmetric function is closely related to the problem of integer partitions. In this section, we give a brief review of symmetric functions and integer partitions. For more details on this part, one can refer to [1–3].

**Integer partitions.** The integer partitions of $N$ are representations of $N$ in terms of other positive integers which sum up to $N$. The integer partition is denoted by
$(\lambda) = (\lambda_1, \lambda_2, ...)$, where $\lambda_1$, $\lambda_2$, ... are called elements, and they are arranged in descending order. For example, the integer partition of 4 are $(\lambda) = (4)$, $(\lambda)' = (3, 1)$, $(\lambda)'' = (2^2)$, $(\lambda)''' = (2, 2^2)$, and $(\lambda)''' = (1^4)$, where, e.g. the superscript in $1^2$ means 1 is appearing twice, the superscript in $2^2$ means 2 is appearing twice, and so on.

Arranging integer partitions in a prescribed order. An integer $N$ has many integer partitions and the unrestricted partition function $P(N)$ counts the number of integer partitions [3]. For a given $N$, one arranges the integer partition in the following order: $(\lambda)$, $(\lambda)'$, when $\lambda_1 > \lambda'_1$; $(\lambda)'$, when $\lambda_1 = \lambda'_1$ but $\lambda_2 > \lambda'_2$; and so on. One keeps comparing $\lambda_i$ and $\lambda'_i$ until all the integer partitions of $N$ are arranged in a prescribed order. We denote $(\lambda)_j$ the $j$th integer partition of $N$. For example, the integer partitions arranged in a prescribed order of 4 are $(\lambda)_1 = (4)$, $(\lambda)_2 = (3, 1)$, $(\lambda)_3 = (2^2)$, $(\lambda)_4 = (2, 1^2)$, and $(\lambda)_5 = (1^4)$. For any integer $N$, the first integer partition is always $(\lambda)_{j=1} = (N)$ and the last integer partition is always $(\lambda)_{j=|P(N)|} = (1^N)$. We denote $\lambda_{j,i}$ the $i$th element in the integer partition $(\lambda)_j$. For example, for the second integer partition of 4, i.e. $(\lambda)_2 = (3, 1)$, the elements of $(\lambda)_2$ are $\lambda_{2,1} = 3$ and $\lambda_{2,2} = 1$, respectively. For any integer $N$, we always have $\lambda_{1,1} = N$, and $\lambda_{P(N),1} = \lambda_{P(N),2} = ... = \lambda_{P(N),N} = 1$.

The symmetric function. A function $f(x_1, x_2, ..., x_n)$ is called a symmetric function if it is invariant under the action of the permutation group $S_n$; that is, for $\sigma \in S_n$,

$$\sigma f(x_1, x_2, ..., x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)}) = f(x_1, x_2, ..., x_n), \quad (2.1)$$

where $x_1, x_2, ..., x_n$ are $n$ independent variables of $f(x_1, x_2, ..., x_n)$.

The symmetric function $m(\lambda)(x_1, x_2, ..., x_l)$. The symmetric function $m(\lambda)(x_1, x_2, ..., x_l)$ is an important kind of symmetric function. One of its definitions is [2]

$$m(\lambda)_I(x_1, x_2, ..., x_l) = \sum_{\text{perm}}^{\lambda_{1,1}}_{x_i_1}^{\lambda_{1,2}}_{x_i_2} ...^{\lambda_{1,N}}_{x_i_N}, \quad (2.2)$$

where $\sum_{\text{perm}}$ indicates that the summation runs over all possible monotonically increasing permutations of $x_i$, $(\lambda)_I$ is the $l$th integer partition of $N$, and $\lambda_{I,j}$ is the $j$th element in $(\lambda)_I$. Here, the number of variables is $l$, and $l$ should be larger than $N$ as required in the definition. $l$, the number of $x_i$, could be infinite and in the problem considered in this paper, $l$ is always infinite. Each integer partition $(\lambda)_I$ of $N$ corresponds to a symmetric function $m(\lambda)_I(x_1, x_2, ...)$ and vice versa.

The $S$-function $(\lambda)(x_1, x_2, ...)$. The $S$-function $(\lambda)(x_1, x_2, ...)$, another important kind of symmetric functions, among their many definitions, can be defined as [1, 2]

$$(\lambda)_I(x_1, x_2, ...) = \sum_{j=1}^{P(N)} \frac{g_I}{N!} \chi_J^I \prod_{m=1}^{k} \left( \sum_{i}^{x_i^m} a_{j,m} \right), \quad (2.3)$$

where $(\lambda)_I$ is the $l$th integer partition of $N$, $a_{j,m}$ counts the times of the number $m$ appeared in $(\lambda)_I$, and $\chi_J^I$ is the simple characteristic of the permutation group of order $N$. $g_I$ is defined as $g_I = N! \left( \prod_{j=1}^{N} j^{a_{j,I}} \right)^{-1}$. Here we only consider the case that the number of variables is always infinite. Each integer partition $(\lambda)_I$ of $N$ corresponds to an $S$-function $(\lambda)_I(x_1, x_2, ...)$ and vice versa.
The relation between the $S$-function $(\lambda)(x_1, x_2, ...)$ and the symmetric function $m_{(\lambda)}(x_1, x_2, ...)$. There is a relation between the $S$-function $(\lambda)(x_1, x_2, ...)$ and the symmetric function $m_{(\lambda)}(x_1, x_2, ...)$: the $S$-function $(\lambda)(x_1, x_2, ...)$ can be expressed as a linear combination of the symmetric function $m_{(\lambda)}(x_1, x_2, ...)$,

$$(\lambda)_K(x_1, x_2, ...) = \sum_{I=1}^{P(N)} k^I_K m_{(\lambda)}^I(x_1, x_2, ...),$$

(2.4)

where the coefficient $k^I_K$ is the Kostka number [2].

3. The canonical partition function of ideal Bose gases

In this section, we present an exact expression of the canonical partition function of ideal Bose gases.

**Theorem 1.** For ideal Bose gases, the canonical partition function is

$$Z_{BE}(\beta, N) = (N) \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ... \right),$$

(3.1)

where $(N)(x_1, x_2, ...)$ is the $S$-function corresponding to the integer partition $(N)$ defined by equation (2.3) and $\varepsilon_i$ is the single-particle eigenvalue.

**Proof.** By definition, the canonical partition function of an $N$-body system is

$$Z(\beta, N) = \sum_s e^{-\beta E_s},$$

(3.2)

where $E_s$ denotes the eigenvalue of the system with the subscript $s$ labeling the states, $\beta = 1/(kT)$, $k$ is the Boltzmann constant, and $T$ is the temperature.

Rewrite equation (3.2) in terms of the occupation number. For an ideal quantum gas, particles are randomly distributed on the single-particle states. Collect the single-particle states occupied by particles by weakly increasing order of energy, i.e. $\varepsilon_{i1} \leq \varepsilon_{i2} \leq \varepsilon_{i3} \leq \cdots$, and denote the number of particle occupying the $j$th state by $a_j$. For Bose gases, there is no restriction on $a_j$, since the maximum occupation number is infinite. Note that here $a_i \geq 1$ rather than $a_i \geq 0$. This is because here only the occupied states are reckoned in. Equation (3.2) can be re-expressed as [36]

$$Z_{BE}(\beta, N) = \sum_s e^{-\beta E_s} = \sum_{\{a_i\}} \sum_{\text{perm}} e^{-\beta \varepsilon_{i1} a_1 - \beta \varepsilon_{i2} a_2 - \cdots},$$

(3.3)

where the sum $\sum_{\text{perm}}$ runs over all possible monotonically increasing permutations of $\varepsilon_i$ and the sum $\sum_{\{a_i\}}$ runs over all the possible occupation numbers restricted by the constraint

$$\sum a_i = N, \quad a_i \geq 1.$$  

(3.4)
The constraint (3.4) ensures that the total number of particles in the system is a constant.

Equalling the occupation number $a_i$ and the element $\lambda_i$, equalling the variable $x_i$ and $e^{-\beta \varepsilon_i}$, and comparing the canonical partition function, equation (3.3), with the definition of the symmetric function $m(\lambda) (x_1, x_2, ...)$, equation (2.2), give

$$Z_{BE} (\beta, N) = \sum_{I=1}^{P(N)} m(\lambda)_I (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...). \quad (3.5)$$

Substituting equations (2.4) into (3.5) and setting all the coefficients $k_k^I$ to 1 give equation (3.1), where the relation $\sum_{I=1}^{P(N)} m(\lambda)_I (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) = (\lambda)_1 (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...)$ [2] and $(\lambda)_1 = (N)$ are used. ■

In a word, the canonical partition function of ideal Bose gases is a $S$-function.

Moreover, by one kind of expression of the $S$-function given in [1, 2], we can obtain the canonical partition function of ideal Bose gases given in [13]. In [13], the author obtained a recurrence relation of the canonical partition function: $Z_{BE} (\beta, N) = \frac{1}{N} \sum_{k=1}^{N} (\sum_{i} e^{-\beta \varepsilon_i}) Z_{BE} (\beta, N - k)$, based on a result given by Mastubara [37] and Feynman [38]. Starting from this recurrence relation, the author obtains the canonical partition function of ideal Bose gases in matrix form by introducing an $\infty \times \infty$ triangularized matrix and computing the inverse of a matrix.

In [1, 2], the $S$-function is represented as the determinant of a certain matrix:

$$(N) (x_1, x_2, ...) = \frac{1}{N!} \det \left( \begin{array}{cccccc}
\sum_i x_i & -1 & 0 & ... & 0 \\
\sum_i x_i^2 & \sum_i x_i & -2 & ... & 0 \\
\sum_i x_i^3 & \sum_i x_i^2 & \sum_i x_i & ... & ... \\
... & ... & ... & ... & ... \\
\sum_i x_i^N & \sum_i x_i^{N-1} & \sum_i x_i^{N-2} & ... & \sum_i x_i \\
\end{array} \right). \quad (3.6)$$

The canonical partition function (3.1), by equation (3.6), can be equivalently expressed as

$$Z_{BE} (\beta, N) = (N) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...)$$

$$= \frac{1}{N!} \det \left( \begin{array}{cccccc}
Z (\beta) & -1 & 0 & ... & 0 \\
Z (2\beta) & Z (\beta) & -2 & ... & 0 \\
Z (3\beta) & Z (2\beta) & Z (\beta) & ... & ... \\
... & ... & ... & ... & ... \\
Z (N\beta) & Z (N\beta - \beta) & Z (N\beta - 2\beta) & ... & Z (\beta) \\
\end{array} \right), \quad (3.7)$$

where

$$Z (\beta) = \sum_{i} e^{-\beta \varepsilon_i} \quad (3.8)$$

is the single-particle partition function of ideal classical gases. For a free ideal classical gas, the single-particle partition function is
\[ Z(\beta) = \frac{V}{\lambda^3} \quad (3.9) \]

with \( V \) the volume and \( \lambda = \hbar \sqrt{\frac{\beta}{2\pi m}} \) the thermal wave length \([39]\).

It is worthy to note that different expressions of the \( S \)-function will give different expressions of canonical partition functions.

### 4. The canonical partition function of ideal Fermi gases

In this section, we present an exact expression of the canonical partition function of ideal Fermi gases.

**Theorem 2.** For ideal Fermi gases, the canonical partition function is

\[ Z_{FD}(\beta, N) = \left(1^N\right) \left(e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots\right), \quad (4.1) \]

where \( \left(1^N\right) (x_1, x_2, \ldots) \) is the \( S \)-function corresponding to the integer partition \( \left(1^N\right) \) defined by equation (2.3).

**Proof.** Rewrite equation (3.2) in terms of occupation numbers. The difference between Fermi and Bose systems is that the maximum occupation number should be 1, i.e. \( a_i = a_j = \ldots a_k = 1 \). Equation (3.2) then can be re-expressed as \([36]\)

\[ Z_{FD}(\beta, N) = \sum_s e^{-\beta E_s} = \sum_{\text{perm}} e^{-\beta \epsilon_1 - \beta \epsilon_2 - \ldots - \beta \epsilon_N}, \quad (4.2) \]

where the sum \( \sum_{\text{perm}} \) runs over all possible monotonically increasing permutations of \( \epsilon_i \). The constraint on the total number of particles of the system now becomes \( a_1 = a_2 = \ldots = a_N = 1 \).

Equalling the occupation number \( a_i \) and the element \( \lambda_i \), equalling the variable \( x_i \) and \( e^{-\beta \epsilon_i} \), and comparing the canonical partition function, equation (4.2), with the definition of the symmetric function \( m(\lambda) (x_1, x_2, \ldots) \), equation (2.2), give

\[ Z_{FD}(\beta, N) = m(\lambda)_{P(N)} \left(e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots\right) = m_{\left(1^N\right)} \left(e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots\right), \quad (4.3) \]

where \( (\lambda)_{P(N)} = \left(1^N\right) \) is used. Then we have

\[ Z_{FD}(\beta, N) = \left(1^N\right) \left(e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots\right), \quad (4.4) \]

where the relation \( m_{\left(1^N\right)} \left(e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots\right) = (\lambda)_{P(N)} \left(e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots\right) \) is used \([2]\). \hfill \blacksquare

Similarly, the \( S \)-function \( \left(1^N\right) \left(e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots\right) \) can be represented as the determinant of a certain matrix \([1, 2]\):
The canonical partition function (4.1), by equation (4.5), can be equivalently expressed as

\[
Z_{FD}(\beta, N) = (1^N) \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right) = \frac{1}{N!} \det \begin{pmatrix}
Z(\beta) & 1 & 0 & \ldots & 0 \\
Z(2\beta) & Z(\beta) & 2 & \ldots & 0 \\
Z(3\beta) & Z(2\beta) & Z(\beta) & \ldots & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
Z(N\beta) & Z(N\beta - \beta) & Z(N\beta - 2\beta) & \ldots & Z(\beta)
\end{pmatrix}.
\] (4.6)

5. The canonical partition function of ideal Gentile gases

The Gentile statistic is a generalization of Bose and Fermi statistics [40, 41]. The maximum occupation number of a Fermi system is 1 and of a Bose system is \(\infty\). As a generalization, the maximum occupation number of a Gentile system is an arbitrary integer \(q\) [40–42]. Beyond commutative and anticommutative quantization, which corresponds to the Bose case and the Fermi case respectively, there are also some other effective quantization schemes [43, 44]. It is shown that the statistical distribution corresponding to various \(q\)-deformation schemes are in fact various Gentile distributions with different maximum occupation numbers \(q\) [45]. There are many physical systems obey intermediate statistics, for example, spin waves, or, magnons, which is the elementary excitation of the Heisenberg magnetic system [46], deformed fermion gases [47, 48], and deformed boson gases [49]. Moreover, there are also generalizations for Gentile statistics, in which the maximum occupation numbers of different quantum states take on different values [50].

In this section, we present the canonical partition function of ideal Gentile gases.

**Theorem 3.** For ideal Gentile gases with a maximum occupation number \(q\), the canonical partition function is

\[
Z_q(\beta, N) = \sum_{I=1}^{P(N)} Q_I(q) (\lambda)_I \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right),
\] (5.1)

where the coefficient

\[
Q_I(q) = \sum_{K=1}^{P(N)} \left( k^*_K \right)^{-1} \Gamma^K(q)
\] (5.2)

with \(\Gamma^K(q)\) satisfying

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\[ \Gamma^K(q) = 0 \] when \( \lambda_{K,1} > q \),

\[ \Gamma^K(q) = 1 \] when \( \lambda_{K,1} \leq q \),

(5.3)

and \( (k^I_K)^{-1} \) satisfying

\[ \sum_{L} (k^I_K)^{-1} k^L_I = \delta_K^I \] (5.4)

with \( k^L_I \) the Kostka number \([2]\).

**Proof.** Rewrite equation (3.2) in terms of the occupation number. Here, the occupation number \( a_i \) is restricted by \( 0 < a_j \leq q \). Equation (3.2) can be re-expressed as

\[ Z_q(\beta, N) = \sum_s e^{-\beta E_s} = \sum_{\{a_i\}_q} \sum_{\text{perm}} e^{-\beta \varepsilon_1 a_1 - \beta \varepsilon_2 a_2 \ldots}, \] (5.5)

where the sum \( \sum_{\text{perm}} \) runs over all possible monotonically increasing permutation of \( \varepsilon_i \), and the sum \( \sum_{\{a_i\}_q} \) runs over all possible occupation number restricted by the constraints

\[ \sum a_i = N, \quad 1 \leq a_j \leq q. \] (5.6)

Equalling the occupation number \( a_i \) and the element \( \lambda_i \), equalling the variable \( x_i \) and \( e^{-\beta \varepsilon_i} \), and comparing the canonical partition function, equation (5.5), with the definition of the symmetric function \( m(\lambda)(x_1, x_2, \ldots) \), equation (2.2), give

\[ Z_q(\beta, N) = \sum_{K=1}^{P(N)} \Gamma^K(q) m(\lambda)_K \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right). \] (5.7)

By introducing \( \Gamma^K(q) \), the constraint on \( a_j \), i.e. \( 1 \leq a_j \leq q \), is automatically taken into account.

Introducing \( \left( k^I_K \right)^{-1} \), which satisfies equation (5.4), multiplying \( \left( k^I_K \right)^{-1} \) to both sides of equation (2.4), and summing over the indice \( I \), we arrive at

\[ m(\lambda)_K \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right) = \sum_{I=1}^{P(N)} \left( k^I_K \right)^{-1} (\lambda)_I \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right). \] (5.8)

Substituting equations (5.8) into (5.7) gives

\[ Z_q(\beta, N) = \sum_{I=1}^{P(N)} \sum_{K=1}^{P(N)} \Gamma^K(q) \left( k^I_K \right)^{-1} (\lambda)_I \left( e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots \right). \] (5.9)

Substituting \( Q^J(q) \) which is defined by equations (5.2) into (5.9) gives equation (5.1) directly.

\[ \text{https://doi.org/10.1088/1742-5468/aaa37e} \]
In appendix A.2, as examples, we calculate the canonical partition function of a Gentile gas based on equation (5.1) for \( N = 3, 4, 5, \) and 6.

## 6. The canonical partition function of interacting classical gases

In this section, we show that the canonical partition function of an interacting classical gas given by the classical cluster expansion is indeed a Bell polynomial \([1–3]\).

**Theorem 4.** The canonical partition function of an interacting classical gas with \( N \) particles is

\[
Z(\beta, N) = \frac{1}{N!} B_N (\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_N),
\]

where \( B_N (x_1, x_2, ..., x_N) \) is the Bell polynomial \([4, 5]\) and \( \Gamma_i \) is defined as

\[
\Gamma_i = \frac{\lambda^i}{\lambda^3} b_i
\]

with \( b_i \) the expansion coefficient in the classical cluster expansion \([39]\).

**Proof.** The canonical partition function given by the classical cluster expansion is \([39]\)

\[
Z(\beta, N) = \sum_{\{m_l\}} \left[ \prod_{l=1}^{N} \left( \frac{1}{l!} \frac{1}{\lambda^3} \right)^{m_l} \frac{1}{m_l!} \right].
\]

Comparing equation (6.3) with expression of the Bell polynomial \([4, 5]\)

\[
B_N (x_1, x_2, ..., x_N) = N! \sum_{\{m_l\}} \left[ \prod_{l=1}^{N} \left( \frac{1}{l!} x_l \right)^{m_l} \frac{1}{m_l!} \right],
\]

we immediately arrive at equation (6.1).

Here, for completeness, we list some \( \Gamma_i \) in the classical cluster expansion. For an interacting classical gas with the Hamiltonian \( H_N = -\sum_{l=1}^{N} \frac{p_l^2}{2\lambda^3} + \sum_{j<l} U_{jl}, \) \( \Gamma_i \) is given as \([39, 51]\)

\[
\Gamma_1 = \frac{V}{\lambda^3},
\]

\[
\Gamma_2 = \frac{1}{\lambda^6} \int d^3q_1 \int d^3q_2 \left( e^{-\beta U_{12}} - 1 \right), \ldots
\]
7. The canonical partition function of interacting quantum gases

The interacting quantum gas is always an important issue in statistical mechanics. In a quantum hard-sphere gas, there are two interplaying effects: quantum exchange interactions and classical inter-particle interactions [52–54]. In this section, we show that the canonical partition function can be represented by the Bell polynomial.

**Theorem 5.** The canonical partition function of an interacting quantum gas with \( N \) particles is

\[
Z(\beta, N) = \frac{1}{N!} B_N(\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_N),
\]

where \( B_N(x_1, x_2, ..., x_N) \) is the Bell polynomial [4, 5] and \( \Gamma_i \) is defined as

\[
\Gamma_i = \frac{\ell!V}{\lambda^\ell} b_i
\]

with \( b_i \) the expansion coefficient in the quantum cluster expansion [39, 51].

Here, for completeness, we list some first expansion coefficients in the quantum cluster expansion. For an interacting quantum gas with the Hamiltonian \( H = -\frac{\hbar^2}{2m} \sum_{l=1}^i \nabla_l^2 + \sum_{j<l} U_{jl} \), the quantum cluster expansion coefficient is given by [39, 51]

\[
b_i = \frac{1}{\ell!V \lambda^\ell} \int d^3q \langle q_1, q_2, q_3, ..., q_l | U_1 | q_1, q_2, q_3, ..., q_l \rangle,
\]

with \( \langle q'_1 | U_1 | q_1 \rangle = \lambda^2 \langle q'_1 | e^{-\beta H_1} | q_1 \rangle \) and \( \langle q'_1, q'_2 | U_2 | q_1, q_2 \rangle = \lambda^6 \left[ \langle q'_1, q'_2 | e^{-\beta H_2} | q_1, q_2 \rangle - \langle q'_1 | e^{-\beta H_1} | q_1 \rangle \langle q'_2 | e^{-\beta H_1} | q_2 \rangle \right] \). The quantum exchange interaction is reflected in the symmetric property of the state vector \( | q_1, q_2, q_3, ... \rangle \), symmetric for bosons and antisymmetric for fermions.

**Proof.** The canonical partition function given by the quantum partition cluster expansion is [39]

\[
Z(\beta, N) = \sum_{\{m_l\}} \left[ \prod_{l=1}^N \left( \frac{1}{\ell!} \frac{V}{\lambda^\ell} \right)^{m_l} \frac{1}{m_l!} \right].
\]

Comparing equation (7.4) with the expression of the Bell polynomial defined by equation (6.4) gives equation (7.1) directly.

Comparing equations (6.1) and (7.1), we can see that the only difference between the canonical partition functions of an interacting classical gas and an interacting quantum gas is the variation of the expansion coefficients.

The above result agrees with the canonical partition function given in [32]. In [32], the canonical partition function is obtained by starting with an expansion of the grand canonical partition function \( \Xi(z, \beta) = \exp \left( -\beta \Phi(z, \beta) \right) = \exp \left( -\beta \sum_{m=1}^\infty \frac{\phi_m(\beta)}{m!} \right) = 1 + \sum_{N=1}^\infty \frac{z^N}{N!} B_N(w_1, ..., w_N) \) with \( w_l = -\beta \phi_l(\beta) = -\beta \frac{\partial \Phi}{\partial z} \bigg|_{z=0} \).
and $\Phi(z, \beta)$ the grand potential, and then comparing with the relation between the grand canonical partition function and the canonical partition function, 
\[ \Xi(z, \beta) = \sum_{N=0}^{\infty} Z(\beta, N) z^N \] [39]. In [32], the canonical partition function is obtained from a grand partition function. In the present paper, the canonical partition function is calculated in the canonical ensemble. Concretely, in this paper, the canonical partition function for interacting classical and quantum gases is obtained by comparing the canonical partition function of interacting classical and quantum gases given by the cluster expansion method [39] and the definition of the Bell polynomial [4]. This allows us to give the parameter $\Gamma_l$ in the canonical partition function explicitly.

8. The virial coefficients of ideal quantum gases in the canonical ensemble

Based on the canonical partition function of ideal quantum gases given in sections 3–5, we calculate the virial coefficient for ideal quantum gases and then compare them with those calculated in the grand canonical ensemble.

8.1. The virial coefficients of ideal Bose and Fermi gases in the canonical ensemble

In this section we calculate the virial coefficients of ideal Bose and Fermi gases directly from the canonical partition functions.

In terms of the $S$-function, the canonical partition functions of ideal Bose and Fermi gases can be expressed by the partition function of a classical free particle. Substituting the partition function of a classical free particle (3.9) into equations (3.7) and (4.6) and re-expressing them in terms of the $V/\lambda^3$ give the canonical partition function of ideal Bose gases

\[ Z_{BE}(\beta, N) = \frac{1}{N!} \left( \frac{V}{\lambda^3} \right)^N + \frac{1}{2^{3/2}} \frac{1}{2(N-2)!} \left( \frac{V}{\lambda^3} \right)^{N-1} + \cdots \] (8.1)

and the canonical partition function of ideal Fermi gases

\[ Z_{FD}(\beta, N) = \frac{1}{N!} \left( \frac{V}{\lambda^3} \right)^N - \frac{1}{2^{3/2}} \frac{1}{2(N-2)!} \left( \frac{V}{\lambda^3} \right)^{N-1} + \cdots . \] (8.2)

In order to calculate the virial coefficients, we first calculate the equation of state from the canonical partition function. The equation of state is [39]

\[ P = \frac{1}{\beta} \frac{\partial \ln Z(N, \beta)}{\partial V}. \] (8.3)

The equation of state (8.3) can be expressed by the virial expansion [39]:

---

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$PV NkT = \sum_n a_n \left( \frac{N\lambda^3}{V} \right)^{n-1}$

with $a_n$ the virial coefficient [39].

Substituting equations (8.1) and (8.2) into (8.3), the coefficients of $(N\lambda^3/V)^2$ give the second virial coefficients of ideal Bose and Fermi gases, respectively:

$$a_2 = \pm \frac{1}{2^{5/2}} \left( 1 - \frac{1}{N} \right),$$

(8.5)

where ‘+’ stands for Bose gases and ‘−’ stands for Fermi gases.

Comparing the second virial coefficient obtained in the canonical ensemble, equation (8.5) with the second virial coefficient obtained in the grand canonical ensemble [39],

$$a_2^{\text{grand}} = \pm \frac{1}{2^{5/2}},$$

(8.6)

we can see that the second virial coefficient in the canonical partition function depends on the total particle number $N$. As $N$ tends to infinity, $a_2$ recovers the second virial coefficient obtained in the grand canonical ensemble $a_2^{\text{grand}}$.

8.2. The virial coefficients of Gentile gases in the canonical ensemble

For Gentile gases, the maximum occupation number is an integer $q$. In this section, we list some virial coefficients of Gentile gases. The details of the calculation will be given in the appendix A.1.

In the following, we compare the results obtained in canonical ensembles and in grand canonical ensembles in tables 1–5, respectively. The virial coefficient of Gentile gases in the grand canonical ensemble can be found in [55].

From tables 1–5, one can see that the virial coefficients calculated from in the canonical ensemble are different from those calculated in the grand canonical ensemble. However, as $N$ increases, the result obtained in the canonical ensemble tends to the result obtained in the grand canonical ensemble.

9. The virial coefficients of interacting classical and quantum gases in the canonical ensemble

In this section, based on the expansion of the Bell polynomial, we calculate the second virial coefficient of interacting classical and quantum gases in canonical ensemble and then compare the results with those calculated in the grand canonical ensemble.

The Bell polynomial can be expanded as

$$B_N [\Gamma_1, \Gamma_2, ..., \Gamma_N] = \Gamma_1^N + \frac{N!}{2(N-2)!} \Gamma_1^{N-2}\Gamma_2 + ... + \Gamma_N.$$

(9.1)

Then the canonical partition function (6.1) becomes

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The second virial coefficients of interacting classical and quantum gases can be directly obtained from the canonical partition function. By equation (6.2), we have $$\Gamma_1 = \frac{V}{x^2}$$ and $$\Gamma_2 = \frac{2V}{x^2}b_2$$. Then equation (9.2) becomes
\[
Z(\beta, N) = \frac{1}{N!} \left( \frac{V}{x^3} \right)^N + \frac{1}{(N-2)!} \left( \frac{V}{x^3} \right)^{N-1} b_2 + \cdots.
\] (9.3)

Substituting equations (9.3) into (8.3) and selecting the coefficients of $$(N\lambda^3/V)^2$$ give the second virial coefficients:
\[
a_2 = b_2 \left( 1 - \frac{1}{N} \right). \tag{9.4}
\]

One can see that the second virial coefficients are functions of the total particle number $N$.

Comparing the second virial coefficient obtained in the canonical ensemble, equation (9.4), with the second virial coefficient obtained in the grand canonical ensemble \[39\],
\[
a_2^\text{grand} = b_2, \tag{9.5}
\]
we can see that the result obtained in the canonical partition function depends on the total particle number $N$ and recovers the result obtained in the grand canonical ensemble when $N$ tends to infinite.

10. Conclusions and discussions

Solving an $N$-body system is an important problem in physics. There are many mechanical and statistical-mechanical methods developed for this problem.

The mechanical treatment for an $N$-body system is to find the eigenvalues and eigenstates of the $N$-body Hamiltonian of the system. In the mechanical treatment, both information of eigenvalues and eigenfunctions are taken into account. However, as is well known, solving an $N$-body system in mechanics is very difficult. In an interacting quantum system, two factors, classical inter-particle interactions and the quantum exchange interactions, are intertwined together.

The statistical-mechanical treatment for an $N$-body system, essentially, is to use average values instead of exact values. Only the information of eigenvalues is taken into account. All thermodynamic quantities can be obtained from, for example, the canonical partition function $Z(\beta, N) = \sum_s e^{-\beta E_s}$ which is determined only by the eigenvalues $E_s$ of the $N$-body system. At the expense of losing the eigenfunctions information, the statistical-mechanical treatment is much easier compared with mechanical treatment. In other words, the statistical-mechanical treatment can be regarded as an approximate method for $N$-body problems.

A rigorous description of an $N$-body system in statistical mechanics should be made in a canonical ensemble with fixed particle number $N$. The core task in a canonical
ensemble is to seek the canonical partition function \( Z(\beta, N) \). The calculation of the canonical partition function is in fact a sum over all states with a constraint of the total particle number of the system. Nevertheless, it is very difficult to calculate the canonical partition function when one has to deal with the difficulties from the classical inter-particle interactions and quantum exchange interactions, and take the constraints of the total particle number of the system into consideration simultaneously.

In order to avoid the difficulty caused by the constraint of the total particle number, conventionally, one introduces the grand canonical ensemble which has no constraint on the particle number, and, instead of canonical partition function, one turns to calculate the grand canonical partition function. The advantage is that the constraint of particle numbers on the sum is now removed; the price is that the particle number is now not a constant. In the grand canonical ensemble, the total particle number is not rigorously equal to an exact number \( N \), but is equal to the average particle number \( \langle N \rangle \). That is to say, an \( N \)-body problem is approximately converted to an \( \langle N \rangle \)-body problem, with a deviation of \( \frac{1}{\sqrt{N}} \). Unless in the thermodynamic limit, i.e. \( N \to \infty \), the result obtained in the grand canonical ensemble is not the same as that obtained in the canonical ensemble. Thus, for an \( N \)-body problem, the grand canonical ensemble is an approximate method even in statistical-mechanical treatment.

The method suggested in the paper is based on the mathematical theory of the symmetric function and the Bell polynomial. In this paper, for ideal quantum gases, including Bose, Fermi, and Gentile gases, we suggest a method to calculate exact canonical partition functions, and show that the canonical partition functions of ideal Bose, Fermi, and Gentile gases can be represented as linear combinations of the \( S \)-function. For interacting classical and quantum gases, we point out that the canonical partition functions given by the cluster expansion method are indeed the Bell polynomial. Starting from the exact canonical partition functions of ideal quantum gases, we calculate the virial coefficients in canonical ensemble. For interacting gases, we calculate the virial coefficients in canonical ensembles instead of in the grand canonical ensembles of the literature.

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Appendix. Details of the calculation for ideal Gentile gases in the canonical ensemble

In this appendix, we give some details for the calculation of the ideal Gentile gases in the canonical ensemble.

A.1. The coefficient \( \mathcal{Q}^I(q) \)

The canonical partition function of ideal Gentile gases, equation (5.1), can be represented as a linear combination of the \( S \)-function \( (\lambda)_I (e^{-\beta e_1}, e^{-\beta e_2}, \ldots) \) and the corresponding coefficient \( \mathcal{Q}^I(q) \) is defined by equation (5.2). That is to say, the canonical
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partition function of ideal Gentile gases will be obtained once the coefficient $Q^I(q)$ is known. In this appendix, we show how to calculate the coefficient $Q^I(q)$ for some given $N$.

$N=3$. The Kostka number $k^I_K$ for $N=3$ reads $k^{(3)}_1 = k_1^1 = 1$, $k^{(3)}_{(2,1)} = k_2^1 = 0$, $k^{(3)}_{(1^3)} = k_3^1 = 0$, $k^{(2,1)}_{(3)} = k_1^1 = 1$, $k^{(2,1)}_{(2,1)} = k_2^2 = 1$, $k^{(2,1)}_{(1^3)} = k_3^1 = 0$, $k^{(3)}_{(1^3)} = k_1^1 = 1$, $k^{(3)}_{(1^3)} = k_2^2 = 2$, and $k^{(3)}_{(1^3)} = k_3^3 = 1$ [1, 2]. For clarity, we can rewrite the Kostka number in a matrix form:

$$k = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \quad (A.1)$$

where we take the upper index of $k^I_K$ as a row index and the lower index as a column index.

For $q=3$, by the definition of $\Gamma^K(q)$, equation (5.3), we have $\Gamma^{(3)}_3 = \Gamma_1(3) = 1$, $\Gamma^{(2,1)}_{(3)} = \Gamma_2(3) = 1$, and $\Gamma^{(1^3)}_{(3)} = \Gamma_3(3) = 1$. For clarity, we express $\Gamma^K(q)$ as

$$\Gamma(3) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad (A.2)$$

where we take the upper index of $\Gamma^K(q)$ as a row index. Again we express equation (5.2) in the matrix form

$$Q(q) = k^{-1} \Gamma(q), \quad (A.3)$$

where $Q(q)$ denotes the vector form of $Q^I(q)$ by taking the upper index as a row index, and $k^{-1}$ is the matrix form of $(k^{-1})^I_K$.

Then substituting equations (A.1) and (A.2) into (A.3) gives

$$Q(3) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad (A.4)$$

That is to say $Q^{(3)}(3) = Q^I_1(3) = 1$, $Q^{(2,1)}_{(3)} = Q^2(3) = 0$, $Q^{(1^3)}_{(3)} = Q^3(3) = 0$.

Therefore, when $N=3$ and $q=3$, equation (5.1) can be expressed as

$$Z_3(\beta, 3) = 1 \times (3) \left( e^{-\beta_1}, e^{-\beta_2}, ... \right) + 0 \times (2, 1) \left( e^{-\beta_1}, e^{-\beta_2}, ... \right) + 0 \times (1^3) \left( e^{-\beta_1}, e^{-\beta_2}, ... \right)$$
$$= (3) \left( e^{-\beta_1}, e^{-\beta_2}, ... \right), \quad (A.5)$$

where $(\lambda)_1 \left( e^{-\beta_1}, e^{-\beta_2}, ... \right) = (3) \left( e^{-\beta_1}, e^{-\beta_2}, ... \right)$, $(\lambda)_2 \left( e^{-\beta_1}, e^{-\beta_2}, ... \right) = (2, 1) \left( e^{-\beta_1}, e^{-\beta_2}, ... \right)$, and $(\lambda)_3 \left( e^{-\beta_1}, e^{-\beta_2}, ... \right) = (1^3) \left( e^{-\beta_1}, e^{-\beta_2}, ... \right)$.

Equation (A.5) is just the canonical partition function for Gentile gases with the particle number $N=3$ and the maximum occupation number $q=3$.

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For $q = 2$, by equation (5.3), we have
\[
\Gamma(2) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} .
\] (A.6)
Substituting equations (A.6) and (A.1) into (A.3) gives
\[
Q(2) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} .
\] (A.7)
Substituting the coefficient $Q(2)$, equations (A.7), into (5.1) gives the canonical partition function
\[
Z_2(\beta, 3) = 0 \times (3) (e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... ) + 1 \times (2, 1) (e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... ) - 1 \times (1^3) (e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... )
\]
\[
= (2, 1) (e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... ) - (1^3) (e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... ) .
\] (A.8)
For $q = 1$, by equation (5.3), we have
\[
\Gamma(1) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .
\] (A.9)
Substituting equations (A.9) and (A.1) into (A.3) gives
\[
Q(1) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .
\] (A.10)
Substituting the coefficient $Q(1)$ given by equations (A.10) into (5.1) gives the canonical partition function
\[
Z_1(\beta, 3) = 0 \times (3) (e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... ) + 0 \times (2, 1) (e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... ) + 1 \times (1^3) (e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... )
\]
\[
= (1^3) (e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... ) .
\] (A.11)
\[
N = 4. The Kostka number for N = 4 is [1, 2]
\[
k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\
1 & 3 & 2 & 3 & 1 \end{pmatrix} .
\] (A.12)
For $q = 4$, by equation (5.3), we have
\[
\Gamma(4) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} .
\] (A.13)
Substituting equations (A.12) and (A.13) into (A.3) gives

\[ Q(4) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 3 & 2 & 3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \]  

(A.14)

Substituting the coefficient \( Q(4) \) given by equations (A.14) into (5.1) gives the canonical partition function

\[
Z_4(4, \beta) = 1 \times (4) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) + 0 \times (3, 1) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) + 0 \times (2^2) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) \\
+ 0 \times (2, 1^2) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) + 0 \times (1^5) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) \\
= (4) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...). 
\]

For \( q = 3 \), by equation (5.3), we have

\[ \Gamma(3) = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \]  

(A.16)

Substituting equations (A.12) and (A.16) into (A.3) gives

\[ Q(3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 3 & 2 & 3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}. \]  

(A.17)

Substituting the coefficient \( Q(3) \) given by (A.17) into equation (5.1) gives the canonical partition function

\[
Z_3(\beta, 4) = 0 \times (4) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) + 0 \times (3, 1) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) + 0 \times (2^2) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) \\
- 1 \times (2, 1^2) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) + 1 \times (1^5) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) \\
= (3, 1) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) - (2, 1^2) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) + (1^5) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...). 
\]

For \( q = 2 \), by equation (5.3), we have

\[ \Gamma(2) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \]  

(A.19)

Substituting equations (A.12) and (A.19) into (A.3) gives
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\[ Q(2) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 \\
1 & 3 & 2 & 3 & 1
\end{pmatrix}^{-1} \begin{pmatrix}
0 \\
0 \\
1 \\
1 \\
1
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
1 \\
1 \\
-1
\end{pmatrix}. \quad (A.20)

Substituting the coefficient \( Q(2) \) given by equations (A.20) into (5.1) gives the canonical partition function

\[
Z_2(\beta, 4) = 0 \times (4) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) + 0 \times (3, 1) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) + 1 \times (2^2) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) \\
+ 0 \times (2, 1^2) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) - 1 \times (1^5) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) \\
= (2^2) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) - (1^5) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...). \quad (A.21)
\]

For \( q = 1 \), by equation (5.3), we have

\[
\Gamma(1) = \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}. \quad (A.22)
\]

Substituting equations (A.12) and (A.22) into (A.3) gives

\[
Q(1) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 \\
1 & 3 & 2 & 3 & 1
\end{pmatrix}^{-1} \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
1
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
1
\end{pmatrix}. \quad (A.23)
\]

Substituting the coefficient \( Q(2) \) given by equations (A.20) into (5.1) gives the canonical partition function

\[
Z_1(\beta, 4) = 0 \times (4) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) + 0 \times (3, 1) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) + 0 \times (2^2) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) \\
+ 0 \times (2, 1^2) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) + 1 \times (1^5) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...) \\
= (1^5) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, ...). \quad (A.24)
\]

\( N = 5 \). The Kostka number for \( N = 5 \) is \([1, 2]\)

\[
k = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 & 0 \\
1 & 2 & 2 & 1 & 1 & 0 \\
1 & 3 & 3 & 3 & 2 & 1 \\
1 & 4 & 5 & 6 & 5 & 4 \\
1 & 5 & 6 & 5 & 4 & 1
\end{pmatrix}. \quad (A.25)
\]

For \( q = 5 \), by equation (5.3), we have

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\[
\Gamma(5) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \tag{A.26}
\]

Substituting equations (A.25) and (A.26) into (A.3) gives

\[
Q(5) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 & 1 & 0 \\ 1 & 3 & 3 & 3 & 2 & 1 \\ 1 & 4 & 5 & 6 & 5 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{A.27}
\]

Substituting the coefficients \(Q(5)\) given by equation (A.27) into (5.1) gives the canonical partition function

\[
Z_5(\beta, 5) = 1 \times \begin{pmatrix} e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots \end{pmatrix} + 0 \times (4, 1) \begin{pmatrix} e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots \end{pmatrix} + 0 \times (3, 2) \begin{pmatrix} e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots \end{pmatrix} + 0 \times (3, 1^2) \begin{pmatrix} e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots \end{pmatrix} + 0 \times (2^2, 1) \begin{pmatrix} e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots \end{pmatrix} + 0 \times (2, 1^3) \begin{pmatrix} e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots \end{pmatrix} + 0 \times (1^5) \begin{pmatrix} e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, \ldots \end{pmatrix}.
\]

For \(q = 4\), by equation (5.3), we have

\[
\Gamma(4) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \tag{A.29}
\]

Substituting equations (A.25) and (A.29) into (A.3) gives

\[
Q(4) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 & 1 & 0 \\ 1 & 3 & 3 & 3 & 2 & 1 \\ 1 & 4 & 5 & 6 & 5 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{A.30}
\]

https://doi.org/10.1088/1742-5468/aaa37e
Substituting the coefficient $Q(4)$ given by equation (A.30) into (5.1) gives the canonical partition function

$$Z_4(\beta, 5) = 0 \times (5) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right) + 1 \times (4, 1) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right) + 0 \times (3, 2) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right) - 1 \times (3, 1^2) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right) + 0 \times (2^2, 1) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right) + 1 \times (2, 1^3) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right) - 1 \times (1^5) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right) = (4, 1) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right) - (3, 1^2) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right) + (2, 1^3) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right) - (1^5) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right).$$

For $q = 3$, by equation (5.3), we have

$$\Gamma(3) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$  \hspace{1cm} \text{(A.31)}$$

Substituting equations (A.25) and (A.32) into (A.3) gives

$$Q(3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 & 1 & 0 \\ 1 & 3 & 3 & 3 & 2 & 1 \\ 1 & 4 & 5 & 6 & 5 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$  \hspace{1cm} \text{(A.32)}$$

Substituting the coefficients $Q(3)$ given by equation (A.33) into (5.1) gives the canonical partition function

$$Z_3(\beta, 5) = 0 \times (5) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right) + 0 \times (4, 1) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right) + 1 \times (3, 2) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right) + 0 \times (3, 1^2) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right) - 1 \times (2^2, 1) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right) + 0 \times (2, 1^3) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right) + 1 \times (1^5) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right) = (3, 2) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right) - (2^2, 1) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right) + (2, 1^3) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right) + (1^5) \left( e^{-\beta e_1}, e^{-\beta e_2}, \ldots \right).$$

For $q = 2$, by equation (5.3), we have

$$\Gamma(2) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$  \hspace{1cm} \text{(A.34)}$$
Substituting equations (A.25) and (A.35) into (A.3) gives

\[
Q(2) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 & 0 \\
1 & 2 & 2 & 1 & 1 & 0 \\
1 & 3 & 3 & 3 & 2 & 1 \\
1 & 4 & 5 & 6 & 5 & 4 & 1
\end{pmatrix}^{-1}
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
1 \\
1 \\
1
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
1 \\
1 \\
0
\end{pmatrix}.
\]  
(A.36)

Substituting the coefficients \(Q(2)\) given by equations (A.36) into (5.1) gives the canonical partition function

\[
Z_2(\beta, 5) = 0 \times (5) \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right) + 0 \times (4, 1) \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right) + 0 \times (3, 2) \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right) \\
+ 0 \times (3, 1^2) \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right) + 1 \times (2^2, 1) \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right) - 1 \times (2, 1^3) \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right) \\
+ 0 \times (1^5) \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right)
= (2^2, 1) \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right) - 1 \times (2, 1^3) \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right).
\]  
(A.37)

For \(q = 1\), by equation (5.3), we have

\[
\Gamma(1) = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
1
\end{pmatrix}.
\]  
(A.38)

Substituting equations (A.25) and (A.38) into (A.3) gives

\[
Q(1) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 & 0 \\
1 & 2 & 2 & 1 & 1 & 0 \\
1 & 3 & 3 & 3 & 2 & 1 \\
1 & 4 & 5 & 6 & 5 & 4 & 1
\end{pmatrix}^{-1}
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
1 \\
0 \\
1
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
1
\end{pmatrix}.
\]  
(A.39)

Substituting the coefficients \(Q(2)\) given by equation (A.36) into (5.1) gives the canonical partition function

\[
Z_1(\beta, 5) = 0 \times (5) \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right) + 0 \times (4, 1) \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right) + 0 \times (3, 2) \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right) \\
+ 0 \times (3, 1^2) \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right) + 0 \times (2^2, 1) \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right) + 0 \times (2, 1^3) \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right) \\
+ 1 \times (1^5) \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right)
= (1^5) \left( e^{-\beta \epsilon_1}, e^{-\beta \epsilon_2}, ... \right).
\]  
(A.40)
A.2. The canonical partition function

In appendix A.1, we show how to calculate the coefficients $Q'(q)$ from equation (5.2). The canonical partition function of ideal Gentile gases can be represented as a linear combination of the $S$-functions provided the coefficients $Q'(q)$ are given.

In this section, we calculate the canonical partition function and express the canonical partition function in terms of the single-particle partition function defined by equation (3.8), $Z(β)$.

$N = 3$. The simple characteristics $χ^K_I$ of the permutation group $S_3$ for $N = 3$ is [1]

$$χ = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix},$$

(A.41)

where $χ^K_I$ is expressed in the matrix form by taking the upper index as the row index and the lower index as the column index. Substituting the simple characteristics $χ^K_I$, equation (A.41), into the definition of the $S$-function, equation (2.3), gives the expression of $S$-functions, $(λ)_I (e^{-βε_1}, e^{-βε_2}, ...)$, in terms of the single-particle partition functions $Z(β)$. Then substituting the $S$-function $(λ)_I (e^{-βε_1}, e^{-βε_2}, ...)$ into the canonical partition function, equation (A.5), gives the expression of the canonical partition function in terms of the single-particle partition function $Z(β)$:

$$Z_2(β, 3) = (2, 1) (e^{-βε_1}, e^{-βε_2}, ...) - (1^3) (e^{-βε_1}, e^{-βε_2}, ...)$$

$$= \frac{1}{3!} Z(β)^3 + \frac{1}{2} Z(β) Z(2β) - \frac{2}{3} Z(3β).$$

(A.42)

$N = 4$. The simple characteristic $χ^K_I$ of the permutation group $S_4$ for $N = 4$ is [1]

$$χ = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 0 & -1 & 1 \\ 2 & -1 & 2 & 0 \\ 3 & 0 & -1 & -1 \\ 1 & 1 & 1 & -1 \end{pmatrix},$$

(A.43)

Substituting the simple characteristics (A.43) into the $S$-function in equations (A.21) and (A.18), respectively, gives

$$Z_2(β, 4) = (2^2) (e^{-βε_1}, e^{-βε_2}, ...) - (1^4) (e^{-βε_1}, e^{-βε_2}, ...)$$

$$= \frac{1}{4!} Z(β)^4 + \frac{1}{4} Z(β)^2 Z(2β) + \frac{1}{8} Z(2β)^2 - \frac{2}{3} Z(3β) Z(β) + \frac{1}{4} Z(4β)$$

(A.44)

and

$$Z_3(β, 4) = (3, 1) (e^{-βε_1}, e^{-βε_2}, ...) - (2, 1^2) (e^{-βε_1}, e^{-βε_2}, ...) + (1^4) (e^{-βε_1}, e^{-βε_2}, ...)$$

$$= \frac{1}{4!} Z(β)^4 + \frac{1}{4} Z(β)^2 Z(2β) + \frac{1}{8} Z(2β)^2 + \frac{1}{8} Z(3β) Z(β) - \frac{3}{4} Z(4β).$$

(A.45)

$N = 5$. The simple characteristic $χ^K_I$ of the permutation group $S_5$ for $N = 5$ is [1]
\[\chi = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & -1 & 1 & 0 & 1 \\
0 & -1 & 1 & -1 & 1 & -1 \\
1 & 0 & 0 & -2 & 0 & 6 \\
0 & 1 & -1 & -1 & 1 & -1 \\
-1 & 0 & 1 & 1 & 0 & 1 \\
1 & -1 & -1 & 1 & 1 & 1
\end{pmatrix}. \quad (A.46)\]

Substituting the simple characteristics (A.46) into the \(S\)-functions in equations (A.37), (A.34), and (A.34), respectively, gives

\[Z_2(\beta, 5) = \frac{1}{5!}Z(\beta)^5 + \frac{1}{12}Z(\beta)^3Z(2\beta) + \frac{1}{8}Z(\beta)Z(2\beta)^2, \quad (A.47)\]

\[-\frac{1}{3}Z(\beta)^2Z(3\beta) - \frac{1}{3}Z(2\beta)Z(\beta) + \frac{1}{4}Z(\beta)Z(4\beta) + \frac{1}{5}Z(5\beta), \quad (A.48)\]

\[Z_3(\beta, 5) = \frac{1}{5!}Z(\beta)^5 + \frac{1}{12}Z(\beta)^3Z(2\beta) + \frac{1}{8}Z(\beta)Z(2\beta)^2 \]

\[+ \frac{1}{6}Z(\beta)^2Z(3\beta) + \frac{1}{6}Z(2\beta)Z(\beta) - \frac{3}{4}Z(\beta)Z(4\beta) + \frac{1}{5}Z(5\beta), \quad (A.50)\]

and

\[Z_4(\beta, 5) = \frac{1}{5!}Z(\beta)^5 + \frac{1}{12}Z(\beta)^3Z(2\beta) + \frac{1}{8}Z(\beta)Z(2\beta)^2 \]

\[+ \frac{1}{6}Z(\beta)^2Z(3\beta) + \frac{1}{6}Z(2\beta)Z(\beta) + \frac{1}{4}Z(\beta)Z(4\beta) - \frac{4}{5}Z(5\beta). \quad (A.51)\]

\(N = 6\). The simple characteristics \(\chi^K_I\) of the permutation group \(S_6\) for \(N = 6\) can be found in [1]. A similar procedure gives

\[Z_2(\beta, 6) = \frac{1}{6!}Z(\beta)^6 \]

\[+ \frac{1}{48}Z(\beta)^4Z(2\beta) + \frac{1}{16}Z(\beta)^2Z(2\beta)^2 \]

\[+ \frac{1}{48}Z(2\beta)^3 - \frac{1}{9}Z(\beta)^3Z(3\beta) - \frac{1}{3}Z(\beta)Z(2\beta)Z(3\beta) \]

\[+ \frac{2}{9}Z(3\beta)^2 + \frac{1}{8}Z(\beta)^2Z(4\beta) + \frac{1}{8}Z(2\beta)Z(4\beta) + \frac{1}{5}Z(5\beta)Z(\beta) - \frac{1}{3}Z(6\beta), \quad (A.52)\]
Canonical partition functions

\[ Z_3(\beta, 6) = \frac{1}{6!} Z(\beta)^6 \]
\[ + \frac{1}{48} Z(\beta)^4 Z(2\beta) + \frac{1}{16} Z(\beta)^2 Z(2\beta)^2 \]
\[ + \frac{1}{48} Z(2\beta)^3 + \frac{1}{18} Z(\beta)^3 Z(3\beta) + \frac{1}{6} Z(\beta) Z(2\beta) Z(3\beta) \]
\[ + \frac{1}{18} Z(3\beta)^2 - \frac{3}{8} Z(\beta)^2 Z(4\beta) - \frac{3}{8} Z(2\beta) Z(4\beta) + \frac{1}{5} Z(5\beta) Z(\beta) + \frac{1}{6} Z(6\beta), \]  
(A.53)

\[ Z_4(\beta, 6) = \frac{1}{6!} Z(\beta)^6 \]
\[ + \frac{1}{48} Z(\beta)^4 Z(2\beta) + \frac{1}{16} Z(\beta)^2 Z(2\beta)^2 \]
\[ + \frac{1}{48} Z(2\beta)^3 + \frac{1}{18} Z(\beta)^3 z(3\beta) + \frac{1}{6} Z(\beta) z(2\beta) z(3\beta) \]
\[ + \frac{1}{18} z(3\beta)^2 + \frac{1}{8} z(\beta)^2 z(4\beta) + \frac{1}{8} z(2\beta) z(4\beta) - \frac{4}{5} z(5\beta) z(\beta) + \frac{1}{6} z(6\beta), \]  
(A.54)

and

\[ Z_5(\beta, 6) = \frac{1}{6!} Z(\beta)^6 \]
\[ + \frac{1}{48} Z(\beta)^4 Z(2\beta) + \frac{1}{16} Z(\beta)^2 Z(2\beta)^2 \]
\[ + \frac{1}{48} Z(2\beta)^3 + \frac{1}{18} Z(\beta)^3 Z(3\beta) + \frac{1}{6} Z(\beta) Z(2\beta) Z(3\beta) \]
\[ + \frac{1}{18} Z(3\beta)^2 + \frac{1}{8} Z(\beta)^2 Z(4\beta) + \frac{1}{8} Z(2\beta) Z(4\beta) + \frac{1}{5} Z(5\beta) Z(\beta) - \frac{5}{6} Z(6\beta). \]  
(A.55)

The simple characteristics \( \chi^K \) of the permutation group \( S_6 \) is not listed in the paper, but it can be found in [1]. The calculation of the canonical partition function for Gentile gases for \( N = 6 \) are not listed in this paper, but can be calculated by the method provided in this paper.

From the results, equations (A.42), (A.44), (A.45), (A.48), (A.50)–(A.55), one can see that the canonical partition functions \( Z_q(\beta, N) \) of ideal Gentile gases can be written in the form of \( Z_q(\beta, N) = \frac{1}{N!} Z(\beta)^N + \text{corrections} \), where \( Z(\beta)^N \) is the canonical partition function of ideal classical gases with \( N \) particles and the correction depends on both \( N \) and \( q \).

A.3. The virial coefficients

In this section, we calculate the virial coefficients from the canonical partition function.

The exact canonical partition function of ideal Gentile gases can be obtained by substituting the single-particle partition function \( Z(\beta) \), equation (3.9), into the canonical partition functions, equations (A.42), (A.44), (A.45), (A.48), (A.50)–(A.55).

For \( q = 2 \),

https://doi.org/10.1088/1742-5468/aaa37e
Z_2(\beta, 3) = \frac{1}{6} \left( \frac{V}{\lambda^3} \right)^3 + \frac{1}{4\sqrt{2}} \left( \frac{V}{\lambda^3} \right)^2 - \frac{2}{9\sqrt{3}} \frac{V}{\lambda^3}, \quad (A.56)

Z_2(\beta, 4) = \frac{1}{24} \left( \frac{V}{\lambda^3} \right)^4 + \frac{1}{8\sqrt{2}} \left( \frac{V}{\lambda^3} \right)^3 + \frac{1}{64} \left( \frac{V}{\lambda^3} \right)^2 - \frac{2}{9\sqrt{3}} \frac{V}{\lambda^3} + \frac{1}{32} \left( \frac{V}{\lambda^3} \right), \quad (A.57)

Z_2(\beta, 5) = \frac{1}{120} \left( \frac{V}{\lambda^3} \right)^5 + \frac{1}{24\sqrt{2}} \left( \frac{V}{\lambda^3} \right)^4 + \frac{1}{64} \left( \frac{V}{\lambda^3} \right)^3 - \frac{1}{9\sqrt{3}} \frac{V}{\lambda^3}^3 + \frac{1}{32} \left( \frac{V}{\lambda^3} \right)^2 - \frac{1}{18\sqrt{6}} \left( \frac{V}{\lambda^3} \right)^2 + \frac{1}{25\sqrt{5}} \frac{V}{\lambda^3}, \quad (A.58)

Z_2(\beta, 6) = \frac{1}{720} \left( \frac{V}{\lambda^3} \right)^6 + \frac{1}{96\sqrt{2}} \left( \frac{V}{\lambda^3} \right)^5 + \frac{1}{128} \left( \frac{V}{\lambda^3} \right)^4 - \frac{1}{27\sqrt{3}} \left( \frac{V}{\lambda^3} \right)^4 + \frac{1}{64} \left( \frac{V}{\lambda^3} \right)^3 - \frac{1}{768\sqrt{2}} \left( \frac{V}{\lambda^3} \right)^3 + \frac{1}{18\sqrt{6}} \left( \frac{V}{\lambda^3} \right)^3 + \frac{2}{243} \left( \frac{V}{\lambda^3} \right)^2 + \frac{1}{128\sqrt{2}} \left( \frac{V}{\lambda^3} \right)^2 + \frac{1}{25\sqrt{5}} \left( \frac{V}{\lambda^3} \right)^2 - \frac{1}{18\sqrt{6}} \left( \frac{V}{\lambda^3} \right)^2. \quad (A.59)

Substituting equations (A.56)–(A.59) into the equation of state (8.3) respectively gives the virial coefficients a_i listed in Table 1.

For q = 3,

Z_3(\beta, 3) = \frac{1}{6} \left( \frac{V}{\lambda^3} \right)^3 + \frac{1}{4\sqrt{2}} \left( \frac{V}{\lambda^3} \right)^2 + \frac{1}{9\sqrt{3}} \frac{V}{\lambda^3}, \quad (A.60)

Z_3(\beta, 4) = \frac{1}{24} \left( \frac{V}{\lambda^3} \right)^4 + \frac{1}{8\sqrt{2}} \left( \frac{V}{\lambda^3} \right)^3 + \frac{1}{64} \left( \frac{V}{\lambda^3} \right)^2 + \frac{1}{9\sqrt{3}} \frac{V}{\lambda^3}^2, \quad (A.61)

Z_3(\beta, 5) = \frac{1}{120} \left( \frac{V}{\lambda^3} \right)^5 + \frac{1}{24\sqrt{2}} \left( \frac{V}{\lambda^3} \right)^4 + \frac{1}{64} \left( \frac{V}{\lambda^3} \right)^3 + \frac{1}{18\sqrt{3}} \frac{V}{\lambda^3}^3 - \frac{3}{32} \left( \frac{V}{\lambda^3} \right)^2 + \frac{1}{36\sqrt{2}} \left( \frac{V}{\lambda^3} \right)^2 + \frac{1}{25\sqrt{5}} \frac{V}{\lambda^3}, \quad (A.62)
Canonical partition functions

$$Z_3(\beta, 6) = \frac{1}{720} (V_{\lambda^3})^6 + \frac{1}{96\sqrt{2}} (V_{\lambda^3})^5 + \frac{1}{128} (V_{\lambda^3})^4 + \frac{1}{54\sqrt{3}} (V_{\lambda^3})^4$$
$$- \frac{3}{64} (V_{\lambda^3})^3 + \frac{1}{768\sqrt{2}} (V_{\lambda^3})^3 + \frac{1}{36\sqrt{6}} (V_{\lambda^3})^3 + \frac{1}{486} (V_{\lambda^3})^2$$
$$- \frac{3}{128\sqrt{2}} (V_{\lambda^3})^2 + \frac{1}{25\sqrt{5}} (V_{\lambda^3})^2 + \frac{1}{36\sqrt{6}} (V_{\lambda^3})^2. \quad (A.63)$$

Substituting equations (A.60)–(A.63) into the equation of state (8.3) respectively gives the virial coefficients $a_i$ listed in table 2.

For $q = 4$,

$$Z_4(\beta, 4) = \frac{1}{24} (V_{\lambda^3})^4 + \frac{1}{8\sqrt{2}} (V_{\lambda^3})^3 + \frac{1}{64} (V_{\lambda^3})^2$$
$$+ \frac{1}{9\sqrt{3}} (V_{\lambda^3})^2 + \frac{1}{32} (V_{\lambda^3}). \quad (A.64)$$

$$Z_4(\beta, 5) = \frac{1}{120} (V_{\lambda^3})^5 + \frac{1}{24\sqrt{2}} (V_{\lambda^3})^4 + \frac{1}{64} (V_{\lambda^3})^3$$
$$+ \frac{1}{18\sqrt{3}} (V_{\lambda^3})^3 + \frac{1}{32} (V_{\lambda^3})^2 + \frac{1}{36\sqrt{6}} (V_{\lambda^3})^2 - \frac{4}{25\sqrt{5}} V_{\lambda^3}. \quad (A.65)$$

$$Z_4(\beta, 6) = \frac{1}{720} (V_{\lambda^3})^6 + \frac{1}{96\sqrt{2}} (V_{\lambda^3})^5 + \frac{1}{128} (V_{\lambda^3})^4 + \frac{1}{54\sqrt{3}} (V_{\lambda^3})^4$$
$$+ \frac{1}{64} (V_{\lambda^3})^3 + \frac{1}{768\sqrt{2}} (V_{\lambda^3})^3 + \frac{1}{36\sqrt{6}} (V_{\lambda^3})^3 + \frac{1}{486} (V_{\lambda^3})^2$$
$$+ \frac{1}{128\sqrt{2}} (V_{\lambda^3})^2 - \frac{4}{25\sqrt{5}} (V_{\lambda^3})^2 + \frac{1}{36\sqrt{6}} (V_{\lambda^3})^2. \quad (A.66)$$

Substituting equations (A.64)–(A.66) into the equation of state (8.3) respectively gives the virial coefficients $a_i$ listed in table 3.

For $q = 5$,

$$Z_5(\beta, 5) = \frac{1}{120} (V_{\lambda^3})^5 + \frac{1}{24\sqrt{2}} (V_{\lambda^3})^4 + \frac{1}{64} (V_{\lambda^3})^3$$
$$+ \frac{1}{18\sqrt{3}} (V_{\lambda^3})^3 + \frac{1}{32} (V_{\lambda^3})^2 + \frac{1}{36\sqrt{6}} (V_{\lambda^3})^2 + \frac{1}{25\sqrt{5}} V_{\lambda^3}. \quad (A.67)$$

\begin{table}[h]
\centering
\caption{The virial coefficients of ideal Gentile gases: $q = 3$.}
\begin{tabular}{|c|c|c|c|c|}
\hline
 & $N = 3$ & $N = 4$ & $N = 5$ & $N = 6$ \\
\hline
$a_1$ & 1 & 1 & 1 & 1 \\
$a_2$ & -0.11785 & -0.13258 & -0.14142 & -0.14731 \\
$a_3$ & 0.01316 & 0.01048 & 0.00842 & 0.00685 \\
$a_4$ & 0.00039 & 0.00367 & 0.00739 & 0.01061 \\
\hline
\end{tabular}
\end{table}
Substituting equations (A.67) and (A.68) into (8.3) respectively gives the virial coefficients listed in table 4.

For $q = 6$,

$$Z_6 (\beta, 6) = \frac{1}{720} \left( \frac{V}{\lambda^3} \right)^6 + \frac{1}{96\sqrt{2}} \left( \frac{V}{\lambda^3} \right)^5 + \frac{1}{128} \left( \frac{V}{\lambda^3} \right)^4 + \frac{1}{54\sqrt{3}} \left( \frac{V}{\lambda^3} \right)^4 \begin{array}{c} + \frac{1}{64} \left( \frac{V}{\lambda^3} \right)^3 + \frac{1}{768\sqrt{2}} \left( \frac{V}{\lambda^3} \right)^3 + \frac{1}{36\sqrt{6}} \left( \frac{V}{\lambda^3} \right)^3 + \frac{1}{486} \left( \frac{V}{\lambda^3} \right)^2 \end{array} \begin{array}{c} + \frac{1}{128\sqrt{2}} \left( \frac{V}{\lambda^3} \right)^2 + \frac{1}{25\sqrt{5}} \left( \frac{V}{\lambda^3} \right)^2 - \frac{5}{36\sqrt{6}} \left( \frac{V}{\lambda^3} \right)^2. \end{array} \tag{A.69}$$

Substituting equation (A.69) into the equation of state (8.3) gives the virial coefficients listed in table 5.
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