Geodesic Webs and PDE Systems of Euler Equations

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Abstract

We find necessary and sufficient conditions for the foliation defined by level sets of a function \( f(x_1, \ldots, x_n) \) to be totally geodesic in a torsion-free connection and apply them to find the conditions for \( d \)-webs of hypersurfaces to be geodesic, and in the case of flat connections, for \( d \)-webs \((d \geq n + 1)\) of hypersurfaces to be hyperplanar webs. These conditions are systems of generalized Euler equations, and for flat connections we give an explicit construction of their solutions.

1 Introduction

In this paper we study necessary and sufficient conditions for the foliation defined by level sets of a function to be totally geodesic in a torsion-free connection on a manifold and find necessary and sufficient conditions for webs of hypersurfaces to be geodesic. These conditions has the form of a second-order PDE system for web functions. The system has an infinite pseudogroup of symmetries and the factorization of the system with respect to the pseudogroup leads us to a first-order PDE system. In the planar case (cf. [1]), the system coincides with the classical Euler equation and therefore can be solved in a constructive way. We provide a method to solve the system in arbitrary dimension and flat connection.

2 Geodesic Foliations and Flex Equations

Let \( M^n \) be a smooth manifold of dimension \( n \). Let vector fields \( \partial_1, \ldots, \partial_n \) form a basis in the tangent bundle, and let \( \omega^1, \ldots, \omega^n \) be the dual basis. Then

\[
[\partial_i, \partial_j] = \sum_k c_{ij}^k \partial_k
\]
for some functions $c^k_{ij} \in C^\infty(M)$, and
\[ d\omega^k + \sum_{i<j} c^k_{ij} \omega^i \wedge \omega^j = 0. \]

Let $\nabla$ be a linear connection in the tangent bundle, and let $\Gamma^k_{ij}$ be the Christoffel symbols of second type. Then
\[ \nabla_i (\partial_j) = \sum_k \Gamma^k_{ij} \partial_k, \]
where $\nabla_i \overset{\text{def}}{=} \nabla_{\partial_i}$, and
\[ \nabla_i (\omega^k) = -\sum_j \Gamma^k_{ij} \omega^j. \]

In [1] we proved the following result.

**Theorem 1** The foliation defined by the level sets of a function $f(x_1, \ldots, x_n)$ is totally geodesic in a torsion-free connection $\nabla$ if and only if the function $f$ satisfies the following system of PDEs:
\[ \frac{\partial_i (f_j)}{f_i f_j} - \frac{\partial_j (f_i)}{f_i f_j} + \frac{\partial_j (f_j)}{f_j f_j} = \sum_k \left( \frac{\Gamma^k_{ii}}{f_i f_i} \Gamma^k_{jj} - \frac{\Gamma^k_{ij} f_k}{f_i f_j} \right) \]
for all $i < j, i, j = 1, \ldots, n$; here $f_i = \frac{\partial f}{\partial x_i}$.

We call such a system a **flex system**.

Note that conditions [1] can be used to obtain necessary and sufficient conditions for a $d$-web formed by the level sets of the functions $f_\alpha(x_1, \ldots, x_n), \alpha = 1, \ldots, d$, to be a geodesic $d$-web, i.e., to have the leaves of all its foliations to be totally geodesic: one should apply conditions [1] to the all web functions $f_\alpha, \alpha = 1, \ldots, d$.

### 2.1 Geodesic Webs on Manifolds of Constant Curvature

In what follows, we shall use the following definition.

**Definition 2** We call by $(\text{Flex } f)_{ij}$ the following function:
\[ (\text{Flex } f)_{ij} = f^2_j f_{ii} - 2f_i f_j f_{ij} + f^2_i f_{jj}, \]
where $i, j = 1, \ldots, n$, $f_i = \frac{\partial f}{\partial x_i}$ and $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$.

It is easy to see that $(\text{Flex } f)_{ij} = (\text{Flex } f)_{ji}$, and $(\text{Flex } f)_{ii} = 0$. 

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Proposition 3 Let \((\mathbb{R}^n, g)\) be a manifold of constant curvature with the metric tensor
\[
g = \frac{dx_1^2 + ... + dx_n^2}{(1 + \kappa (x_1^2 + ... + x_n^2))^2},
\]
where \(\kappa\) is a constant. Then the level sets of a function \(f(x_1, ..., x_n)\) are geodesics of the metric \(g\) if and only if the function \(f\) satisfies the following PDE system:
\[
(Flex f)_{ij} = \frac{2\kappa (f_i^2 + f_j^2)}{1 + \kappa (x_1^2 + ... + x_n^2)} \sum_k x_k f_k
\]
for all \(i, j\).

Proof. To prove formula (2), first note that the components of the metric tensor \(g\) are
\[
g_{ii} = b^2, \quad g_{ij} = 0, \quad i \neq j,
\]
where
\[
b = \frac{1}{1 + \kappa (x_1^2 + ... + x_n^2)}.
\]
It follows that
\[
g^{ii} = g_{ii}^{-1}, \quad g^{ij} = 0, \quad i \neq j.
\]
We compute \(\Gamma^i_{jk}\) using the classical formula
\[
\Gamma^i_{jk} = \frac{1}{2} g^{ki} \left( \frac{\partial g_{ii}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right)
\]
and get
\[
\Gamma^k_{ii} = 2\kappa x_k b, \quad k \neq i; \quad \Gamma^i_{ii} = -2\kappa x_i b; \quad \Gamma^k_{ij} = 0, \quad i, j \neq k; \quad i \neq j;
\]
\[
\Gamma^i_{ij} = -2\kappa x_j b, \quad i \neq j; \quad \Gamma^j_{ij} = -2\kappa x_i b, \quad i \neq j.
\]
Substituting these values of \(\Gamma^i_{jk}\) into the right-hand side of formula (1), we get formula (2).

Note that if \(n = 2\), then PDE system (2) reduces to the single equation
\[
(Flex f)_{ij} = \frac{2\kappa (f_i^2 + f_j^2)}{1 + \kappa (x_1^2 + x_2^2)} \sum_k x_k f_k.
\]
The left-hand side of equation (4) does not depend on \(i\) and \(j\). Thus we have
\[
\frac{(\text{Flex } f)_{ij}}{f_i^2 + f_j^2} = \frac{(\text{Flex } f)_{kl}}{f_k^2 + f_l^2}
\]
for any \(i, j, k,\) and \(l.\)

It follows that \( (\text{Flex } f)_{ij} = 0 \) \((5)\) for some fixed \(i\) and \(j,\)

In other words, one has the following result.

**Theorem 4** Let \(W\) be a geodesic \(d\)-web on the manifold \((\mathbb{R}^n, g)\) given by web-functions \(\{f^1, \ldots, f^d\}\) such that \((f^a_k)^2 + (f^a_l)^2 \neq 0\) for all \(a = 1, \ldots, d\) and \(k, l = 1, 2, \ldots, n.\) Assume that the intersections of \(W\) with the planes \((x_{i_0}, x_{j_0})\), for given \(i_0\) and \(j_0,\) are linear planar \(d\)-webs. Then the intersection of \(W\) with arbitrary planes \((x_i, x_j)\) are linear webs too.

### 2.2 Geodesic Webs on Hypersurfaces in \(\mathbb{R}^n\)

**Proposition 5** Let \((M, g) \subset \mathbb{R}^n\) be a hypersurface defined by an equation \(x_n = u(x_1, \ldots, x_{n-1})\) with the induced metric \(g\) and the Levi-Civita connection \(\nabla.\) Then the foliation defined by the level sets of a function \(f(x_1, \ldots, x_{n-1})\) is totally geodesic in the connection \(\nabla\) if and only if the function \(f\) satisfies the following system of PDEs:

\[
(F \text{lex } f)_{ij} = \frac{u_1 f_1 + \cdots + u_{n-1} f_{n-1}}{1 + u_1^2 + \cdots + u_{n-1}^2}((f_j^2 u_{ii} - 2 f_j f_i u_{ij} + f_i^2 u_{jj}). \quad (6)
\]

**Proof.** To prove formula \((6),\) note that the metric induced by a surface \(x_n = u(x_1, \ldots, x_{n-1})\) is

\[
g = ds^2 = \sum_{k=1}^{n-1} (1 + u_k^2) dx_k^2 + 2 \sum_{j=1}^{n-1} u_i u_j dx_i dx_j.
\]

Thus the metric tensor \(g\) has the following matrix:

\[
(g_{ij}) = \\
\begin{pmatrix}
1 + u_1^2 & u_1 u_2 & \cdots & u_1 u_{n-1} \\
u_2 u_1 & 1 + u_2^2 & \cdots & u_2 u_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
u_1 u_{n-1} & u_{n-1} u_2 & \cdots & 1 + u_{n-1}^2
\end{pmatrix}
\]
and the inverse tensor $g^{-1}$ has the matrix

$$(g^{ij}) = \frac{1}{1 + \sum_{k=1}^{n-1} (1 + u_k^2)} \begin{pmatrix}
\sum_{k=2}^{n-1} (1 + u_k^2) & -u_1 u_2 & \ldots & -u_1 u_{n-1} \\
-u_2 u_1 & \sum_{k=1, k \neq 2}^{n-1} (1 + u_k^2) & \ldots & -u_2 u_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
-u_{n-1} u_1 & -u_{n-1} u_2 & \ldots & \sum_{k=1}^{n-2} (1 + u_k^2)
\end{pmatrix}.$$  

Computing $\Gamma^i_{jk}$ by formula (3), we find that

$$\Gamma^i_{jk} = \frac{u_k u_{ij}}{1 + \sum_{k=1}^{n-1} (1 + u_k^2)}.$$  

Applying these formulas to the right-hand side of (1), we get formula (6).

We rewrite equation (6) in the form

$$(\text{Flex } f)_{ij} f^2_j u_{ii} - 2 f_i f_j u_{ij} + f^2_i u_{jj} = (\text{Flex } f)_{kl} f^2_k u_{kk} - 2 f_k f_l u_{kl} + f^2_k u_{ll}$$  

It follows that the left-hand side of (7) does not depend on $i$ and $j$, i.e., we have

$$(\text{Flex } f)_{ij} = 0$$

for some fixed $i$ and $j$, then

$$(\text{Flex } f)_{kl} = 0$$

for any $k$ and $l$.

In other words, we have a result similar to the result in Theorem 4.

**Theorem 6** Let $W$ be a geodesic $d$-web on the hypersurface $(M, g)$ given by web functions $\{f^1, \ldots, f^d\}$ such that $(f^a_i) u_{ii} - 2 f^a_i f^a_j u_{ij} + (f^a_i)^2 u_{jj} \neq 0$, for all $a = 1, \ldots, d$ and $k, l = 1, 2, \ldots, n$. Assume that the intersections of $W$ with the planes $(x_i, x_j)$, for given $i_0$ and $j_0$, are linear planar $d$-webs. Then the intersection of $W$ with arbitrary planes $(x_i, x_j)$ are linear webs too.
3 Hyperplanar Webs

In this section we consider hyperplanar geodesic webs in $\mathbb{R}^n$ endowed with a flat linear connection $\nabla$.

In what follows, we shall use coordinates $x_1, \ldots, x_n$ in which the Christoffel symbols $\Gamma^i_{jk}$ of $\nabla$ vanish.

The following theorem gives us a criterion for a web of hypersurfaces to be hyperplanar.

**Theorem 7** Suppose that a $d$-web of hypersurfaces, $d \geq n + 1$, is given locally by web functions $f_\alpha(x_1, \ldots, x_n), \alpha = 1, \ldots, d$. Then the web is hyperplanar if and only if the web functions satisfy the following PDE system:

$$(\text{Flex } f)_{st} = 0, \quad (8)$$

for all $s < t = 1, \ldots, n$.

**Proof.** For the proof, one should apply Theorem 1 to all foliations of the web.

In order to integrate the above PDEs system, we introduce the functions

$$A_s = \frac{f_s}{f_{s+1}}, \quad s = 1, \ldots, n - 1,$$

and the vector fields

$$X_s = \frac{\partial}{\partial x_s} - A_s \frac{\partial}{\partial x_{s+1}}, \quad s = 1, \ldots, n - 1.$$

Then the system can be written as

$$X_s (A_t) = 0,$$

where $s, t = 1, \ldots, n - 1$.

Note that

$$[X_s, X_t] = 0$$

if the function $f$ is a solution of $(8)$.

Hence, the vector fields $X_1, \ldots, X_{n-1}$ generate a completely integrable $(n - 1)$-dimensional distribution, and the functions $A_1, \ldots, A_{n-1}$ are the first integrals of this distribution.

Moreover, the definition of the functions $A_s$ shows that

$$X_s(f) = 0, \quad s = 1, \ldots, n - 1,$$

also.

As a result, we get that

$$A_s = \Phi_s (f), \quad s = 1, \ldots, n - 1,$$

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for some functions $\Phi_s$.

In these terms, we get the following system of equations for $f$:

\[
\frac{\partial f}{\partial x_s} = \Phi_s(f) \frac{\partial f}{\partial x_{s+1}}, \quad s = 1, ..., n-1,
\]
or

\[
\frac{\partial f}{\partial x_s} = \Psi_s(f) \frac{\partial f}{\partial x_n}, \quad s = 1, ..., n-1,
\]

where $\Psi_{n-1} = \Phi_{n-1}$, and

\[
\Psi_s = \Phi_{n-1} \cdots \Phi_s
\]

for $s = 1, ..., n-2$.

This system is a sequence of the Euler-type equations and therefore can be integrated. Keeping in mind that a solution of the single Euler-type equation

\[
\frac{\partial f}{\partial x_s} = \Psi_s(f) \frac{\partial f}{\partial x_n}
\]

is given by the implicit equation

\[
f = u_0(x_n + \Psi_s(f) x_s),
\]

where $u_0(x_n)$ is an initial condition, when $x_s = 0$, and $\Psi_s$ is an arbitrary nonvanishing function, we get solutions $f$ of system (8) in the form:

\[
f = u_0(x_n + \Psi_{n-1}(f) x_{n-1} + \cdots + \Psi_1(f) x_1),
\]

where $u_0(x_n)$ is an initial condition, when $x_1 = \cdots = x_{n-1} = 0$, and $\Psi_s$ are arbitrary nonvanishing functions.

Thus, we have proved the following result.

**Theorem 8** Web functions of hyperplanar webs have the form

\[
f = u_0(x_n + \Psi_{n-1}(f) x_{n-1} + \cdots + \Psi_1(f) x_1),
\]

where $u_0(x_n)$ are initial conditions, when $x_1 = \cdots = x_{n-1} = 0$, and $\Psi_s$ are arbitrary nonvanishing functions.

**Example 9** Assume that $n = 3$, $f_1(x_1, x_2, x_3) = x_1$, $f_2(x_1, x_2, x_3) = x_2$, $f_3(x_1, x_2, x_3) = x_3$, and take $u_0 = x_3$, $\Psi_1(f_1) = f_1^2$, $\Psi_2(f_1) = f_1$ in (10). Then we get the hyperplanar 4-web with the remaining web function

\[
f_4 = \frac{x_2 - 1 \pm \sqrt{(x_2 - 1)^2 - 4x_1x_3}}{2x_1}.
\]

It follows that the level surfaces $f_4 = C$ of this function are defined by the equation

\[x_1(C^2x_1 - Cx_2 + x_3 + C) = 0,\]
i.e., they form a one-parameter family of 2-planes

\[ C^2x_1 - Cx_2 + x_3 + C = 0. \]

Differentiating the last equation with respect to \( C \) and excluding \( C \), we find that the envelope of this family is defined by the equation

\[ (x_2)^2 - 4x_1x_3 - 2x_2 + 1 = 0. \]

Therefore, the envelope is the second-degree cone.

**Example 10** Assume that \( n = 3 \), \( f_1(x_1, x_2, x_3) = x_1, \ f_2(x_1, x_2, x_3) = x_2, \ f_3(x_1, x_2, x_3) = x_3, \) and take \( u_0 = x_3, \Psi_1(f_4) = 1, \Psi_2(f_4) = f_4^2 \) in (10). Then we get the linear 4-web with the remaining web function

\[ f_4 = \left( \frac{1 \pm \sqrt{1 - 4x_2(x_1 + x_3)}}{2x_2} \right)^2. \]

The level surfaces \( f_4 = C^2 \) of this function are defined by the equation

\[ x_2(x_1 + C^2x_2 + x_3 - C) = 0, \]

i.e., they form a one-parameter family of 2-planes

\[ x_1 + C^2x_2 + x_3 - C = 0. \]

Differentiating the last equation with respect to \( C \) and excluding \( C \), we find that the envelope of this family is defined by the equation

\[ 4x_1x_2 + 4x_2x_3 - 1 = 0. \]

Therefore, the envelope is the hyperbolic cylinder.

In the next example no one foliation of a web \( W_3 \) coincides with a foliation of coordinate lines, i.e., all three web functions are unknown.

**Example 11** Assume that \( n = 3 \) and take

(i) \( u_{01} = x_3, \ \Psi_1(f_1) = f_1^2, \ \Psi_2(f_1) = f_1; \)

(ii) \( u_{02} = x_3, \ \Psi_1(f_2) = 1, \ \Psi_2(f_2) = f_2^2; \)

(iii) \( u_{03} = x_3^3, \ \Psi_1(f_3) = f_3, \ \Psi_2(f_3) = 1; \)

(iv) \( u_{04} = x_3, \ \Psi_1(f_4) = \Psi_2(f_4) = f_4 \)

in (10). Then we get the linear 4-web with the web functions

\[ f_1 = \frac{x_2 - 1 \pm \sqrt{(x_2 - 1)^2 - 4x_1x_3}}{2x_1}, \]

\[ f_2 = \left( \frac{1 \pm \sqrt{1 - 4x_2(x_1 + x_3)}}{2x_2} \right)^2. \]
(see Examples 9 and 10) and

\[ f_3 = \left( \frac{1 \pm \sqrt{1 - 4x_1(x_2 + x_3)}}{2x_1} \right)^2, \]
\[ f_4 = \frac{x_3}{1 - x_1 - x_2}. \]

It follows that the leaves of the foliation \( X_1 \) are tangent 2-planes to the second-degree cone

\[ (x_2)^2 - 4x_1x_3 - 2x_2 + 1 = 0 \]
(cf. Example 9 and 10), the leaves of the foliation \( X_2 \) and \( X_3 \) are tangent 2-planes to the hyperbolic cylinders

\[ 4x_1x_2 + 4x_2x_3 - 1 = 0 \]
\[ 4x_1x_2 + 4x_1x_3 - 1 = 0 \]
(cf. Example 10), and the leaves of the foliation \( X_4 \) are 2-planes of the one-parameter family of parallel 2-planes

\[ Cx_1 + C'x_2 + x_3 = 1, \]
where \( C \) is an arbitrary constant.

References

[1] Goldberg, V. V. and V. V. Lychagin, *Geodesic webs on a two-dimensional manifold and Euler equations*, Acta Math. Appl., 2009 (to appear).

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