An A-Stable Block Integrator Scheme for the Solution of First Order System of IVP of Ordinary Differential Equations

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Authors’ contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this article, we present an A-stable block integrator scheme for the solution of first order system of IVP of ordinary differential equations. The block scheme at a single integration step produces four approximate solution values of \( y_{n+1}, y_{n+2}, y_{n+3} \) and \( y_{n+4} \) at point \( x_{n+1}, x_{n+2}, x_{n+3} \) and \( x_{n+4} \) respectively. The order and stability property of the scheme are checked, the method is zero stable, A-stable and of order 6. Some test problems are solved with the proposed scheme and the result are compared with some existing method. The proposed method found to have advantages in terms of accuracy, minimum errors and less computational time. Hence, the method is recommended for solving first order system of IVP of ordinary differential equations.

Keywords: Zero stable; A-stable; IVPs; order; ordinary differential equation.

1 Introduction

A number of real life issues that we encounter, especially in the field of engineering, sciences both physical, social and life sciences can be modeled in Mathematics as differential equations. Considering the vast application of differential equations, analytical and numerical methods are being developed to find solutions.

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This study considers a method for solving systems of first order initial value problems of ordinary differential equation of the form:

\[ y'(x) = f(x, y), \quad y(a) = y_0, \quad a \leq x \leq b \]  \hspace{1cm} (1)

\[ \tilde{y} = (y_1, y_2, y_3, \ldots, y_n), \quad \tilde{\eta} = (\eta_1, \eta_2, \eta_3, \ldots, \eta_n) \]

Ordinary differential equations can be solved by analytical and numerical methods. The solutions generated by the analytical method are generally exact values, whereas with the numerical method an approximation is given as a solution approaching the real value [1]. Implicit numerical schemes proved to be more efficient in solving problems than explicit ones. Most common implicit algorithms are based on Backward Differentiation Formula (BDF). The BDF first appeared in the work of [2]. Researchers continued to improve on the BDF methods. Such improvements include the Extended Backward Differential Formula by [3], modified extended backward differential formula by [4], 2 point diagonally implicit super class of backward differentiation formula [5], an order five implicit 3-step block method for solving ordinary differential equation [6], Implicit r-point block backward differentiation formula for solving first-order stiff ODEs [7], a new variable step size block backward differentiation formula for solving stiff initial value problems [8], a new fifth order implicit block method for solving first order stiff ordinary differential equations by [9], an accurate computation of block hybrid method for solving stiff ODEs [10], One-leg Multistep Method for first Order Differential Equations [11], Sagir [12], Numerical Treatment of Block Method for the Solution of Ordinary Differential equations. Order and Convergence of Enhanced 3 point fully implicit super class of block backward differentiation formula for solving first order stiff initial value problems [13]. All the schemes mentioned above developed by different scholars possess various sorts of accuracy, minimum error and less computation time at one step or the other. However, there is need of developing a numerical algorithm that will solve system of ODEs with minimal computational time and converge faster, hence the motivation for this research.

2 Preliminaries

The following are definitions of the basic terms used in this research.

2.1 Definition 1 (Ordinary Differential Equation)
A differential equation involving derivatives with respect to a single independent variable is called an ordinary differential equation.

2.2 Definition 2 (Order of the Differential Equation)
The order of a differential equation is the order of the highest differential coefficient present in the equation. A differential equation that has the second derivatives as the highest derivatives is said to be of second order.

2.3 Definition 3 (Solution of ODEs)
An equation containing dependent variable \( y \) and independent variable \( x \) and free from the derivative, which satisfies the differential equation is called the solution (primitive) of the differential equation.

2.4 Definition 4 (Initial Value problems)
A differential equation along with initial conditions on the unknown function and its derivatives, all given at the same value of the independent variable, constitutes an initial-value problem.

2.5 Definition 5 (Linear multi-step method)
A general linear multi-step method (LMM) has the following form:

\[ \sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \]
Where $\sigma, \beta$ are constants and $\alpha_k \neq 0$. $\alpha_0$ and $\beta_0$ cannot both be zero at the same time, for any linear k-step method, $\alpha_k$ is normalized to 1.

2.6 Definition 6 (Explicit and Implicit method)

The general linearmulti-step method is said to be Explicit if $\beta_k = 0$, otherwise it is Implicit (i.e. $\beta_k \neq 0$).

2.7 Definition 7 (Linear Difference Operator L)

The linear difference operator $L$ associated with the linear multi-step method is defined by:

$$L(y(x), h) = \sum_{j=0}^{k} [\sigma_j y(x + jh) - h\beta_j y'(x + jh)].$$

Where $y(x)$ is an arbitrary test function and it is continuously differential on $[a, b]$. Expanding $y(x + jh)$ and $y'(x + jh)$ as a Taylor’s series about $x$, and expanding the common terms yields:

$$L(y(x), h) = c_0 y(x_n) + c_1 y'(x_n) + c_2 y''(x_n) + \cdots + c_q h^q y^{(q)}(x_n) +$$

Where $c_q$ are common constants given by:

$$c_0 = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_k$$

$$c_1 = \alpha_1 + 2\alpha_2 + \cdots + k\alpha_k - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k)$$

$$c_q = \frac{1}{q!} (\alpha_1 + 2^q \alpha_2 + \cdots + k^q \alpha_q) - \frac{1}{(q-1)!} (\beta_1 + 2^{q-1} \beta_2 + \cdots + k^{q-1} \beta_k)$$

$q = 2, 3, \ldots, \ldots$

2.8 Definition 8 (Zero stability)

A linear multi-step method (2) is said to be zero stable if all the roots of first characteristics polynomial have modulus less than or equal to unity and those roots with modulus unity are simple.

2.9 Definition 9 (A-stability)

A linear multi-step method (2) is said to be A-stable if the stability region covers the entire negative half plane.

2.10 Definition 10 (Block method)

A method is called Block if it computes more than one solution values at different points per step concurrently. Let $Y_m$ and $F_m$ be vectors defined by:

$$Y_m = [y_n, y_{n+1}, y_{n+2}, \ldots, y_{n+r-1}]^T.$$  

$$F_m = [f_n, f_{n+1}, f_{n+2}, \ldots, f_{n+r-1}]^T.$$  

Then a general k-block, r-point method is a matrix of finite difference equation of the form:

$$Y_m = \sum_{i=0}^{k} A_i Y_{m-i} + \sum_{i=0}^{k} B_i F_{m-i}$$

$A_i$ and $B_i$ are $r \times r$ coefficient matrices.
3 Analysis of the Proposed Method

3.1 Formulation of the Method

Consider the general k-step linear multistep method in definition (5)

\[ \sum_{j=0}^{k} a_j y_{n+j} = h \beta_k y_{n+k} \]  

(2)

This study consider adding a future point in (2), with three step backward, to came-up with the formula of the form:

\[ \sum_{j=0}^{7} a_j y_{n+j-3} = h \beta_k y_{n+k-3} \quad k = 1, 2, 3, 4 \]  

(3)

The implicit four point method (3) is constructed using a linear operator \( L_i \). To derive the four point, define the linear operator \( L_i \) associated with (3) as:

\[ L_i[y(x_i), h] = \alpha_0 y_{n-1} + \alpha_1 y_{n-2} + \alpha_2 y_{n-3} + \alpha_3 y_{n-4} + \alpha_4 y_{n+1} + \alpha_5 y_{n+2} + \alpha_6 y_{n+3} + \alpha_7 y_{n+4} - \frac{h \beta_k y_{n+k-3}}{k = 1, 2, 3, 4} \]  

(4)

To derive the first, second, third, and fourth points as \( y_{n+1}, y_{n+2}, y_{n+3}, \) and \( y_{n+4} \) respectively Using Taylor series expansion in (4) and normalizing \( \alpha_3 = 1, \alpha_4 = 1, \alpha_5 = 1 \) and \( \beta_6 = 1 \) as coefficient’s of the four points, \( k = 1, k = 2, k = 3 \) and \( k = 4 \) respectively. To obtain

\[ y_{n+1} = -1298881 \times 341643939 y_{n-3} + 3461643939 y_{n-2} - 127003623 y_{n-1} + 426060731 y_{n+1} + 2774637 y_{n+2} - 141999876 y_{n+3} + 141999876 y_{n+4} \]  

(5)

3.2 Order of the Method

In this section, we derive the order of the methods (5). It can be transform to a general matrix form as follows:

\[ \sum_{j=0}^{1} C_j y_{m+j-1} = h \sum_{j=0}^{1} D_j y_{m+j-1} \]

Let \( C_0, C_1, D_0 \) and \( D_1 \) be block matrices defined by:

\[ C_0 = [C_0, C_1, C_2, C_3], C_1 = [C_4, C_5, C_6, C_7], D_0 = [D_0, D_1, D_2, D_3], D_1 = [D_4, D_5, D_6, D_7] \]

Where \( C_0, C_1, D_0 \) and \( D_1 \) are square matrices and \( Y_{m-1}, Y_m, F_{m-1} \) and \( F_m \) are column vectors defined by

\[ Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix}, Y_{m-1} = \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix}, F_{m-1} = \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \end{bmatrix}, F_m = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} \]
Thus, equations (5) can be rewritten as

\[
F_m = \begin{bmatrix}
    f_{n+1} \\
    f_{n+2} \\
    f_{n+3} \\
    f_{n+4}
\end{bmatrix} = \begin{bmatrix}
    f_{3m+1} \\
    f_{3m+2} \\
    f_{3m+3} \\
    f_{3m+4}
\end{bmatrix}
\]

From the (6) we have

\[
\begin{bmatrix}
    y_{n-3} \\
    y_{n-2} \\
    y_{n-1} \\
    y_n
\end{bmatrix} = \begin{bmatrix}
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{bmatrix} + h \begin{bmatrix}
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
    f_{n-3} \\
    f_{n-2} \\
    f_{n-1} \\
    f_n
\end{bmatrix}
\]

From the (6) we have

\[
C_0^* = \begin{bmatrix}
    1298881 & 341643939 & -72003623 & 426060731 \\
    341643939 & 569406565 & -113881313 & 341643939 \\
    -79696 & 41929759 & -68414023 & 189894686 \\
    -845265 & 9861425 & -3944570 & 5916855 \\
    70450 & 1295843 & 5593225 & 676840 \\
    1797393 & -1198262 & 926014617 & 105729 \\
    338687 & 353855969 & -2326014617 & 49388481 \\
    348237 & 77076456 & 19269114 & -115614684 \\
    1 & -6274637 & 143998979 & 9603792 \\
    7210474 & 16268759 & 1708219695 & 113881313 \\
    394457 & -5916855 & 1972285 & 42690 \\
    11495780 & -6496015 & 1 & 599131 \\
    599131 & 1198262 & 495749336 & 1 \\
    1117145237 & 1129625017 & 28903671 & 1
\end{bmatrix}
\]

\[
C_1^* = \begin{bmatrix}
    1 & -6274637 & 143998979 & 9603792 \\
    7210474 & 16268759 & 1708219695 & 113881313 \\
    394457 & -5916855 & 1972285 & 42690 \\
    11495780 & -6496015 & 1 & 599131 \\
    599131 & 1198262 & 495749336 & 1 \\
    1117145237 & 1129625017 & 28903671 & 1
\end{bmatrix}
\]
\[
D_5 = \begin{bmatrix}
0 & \frac{9603792}{113881313} & 0 & 0 & 0 \\
0 & 0 & \frac{19789614}{1972285} & 0 & 0 \\
0 & 0 & 0 & \frac{845710}{46087} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
D_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Where

\[
C_0 = \begin{bmatrix}
1298881 \\
341643939 \\
845265 \\
1797393 \\
348237
\end{bmatrix}, \quad C_1 = \begin{bmatrix}
341643939 \\
569406565 \\
9861425 \\
1198262 \\
348237
\end{bmatrix}, \quad C_2 = \begin{bmatrix}
72003623 \\
113881313 \\
1295843 \\
19269114 \\
348237
\end{bmatrix}, \quad C_3 = \begin{bmatrix}
426060731 \\
341643939 \\
599131 \\
113881313 \\
1972285
\end{bmatrix}
\]

\[
C_4 = \begin{bmatrix}
1 \\
7210474 \\
394457 \\
1198262 \\
19269114
\end{bmatrix}, \quad C_5 = \begin{bmatrix}
6274637 \\
16268759 \\
6496015 \\
11296250177 \\
77076456
\end{bmatrix}, \quad C_6 = \begin{bmatrix}
1439989797 \\
1708219695 \\
59168550 \\
495749336 \\
28903671
\end{bmatrix}, \quad C_7 = \begin{bmatrix}
9603792 \\
113881313 \\
14016 \\
113881313 \\
1972285
\end{bmatrix}
\]

From definition the order of the block method (5) and its associated linear operator are given by:

\[
L[y(x); h] = \sum_{j=0}^{7} [C_jy(x + jh)] - h \sum_{j=0}^{7} [D_jy'(x + jh)]
\]

Where \(p\) is unique integer such that

\[E_q = 0, \quad q = 0, 1, \ldots, p \quad \text{and} \quad E_{p+1} \neq 0, \text{where the } E_q \text{ are constant Matrix.} \]
With
\[
E_0 = \sum_{j=0}^7 C_j = 0, E_1 = \sum_{j=0}^7 [jC_j - 2D_j] = 0, E_2 = \sum_{j=0}^7 \left[ \frac{1}{2!} j^2 C_j - 2 j D_j \right] = 0, \quad E_3 = \sum_{j=0}^7 \left[ \frac{1}{3!} j^3 C_j - 21D_j / 2D = 0. \right.
\]
\[
E_4 = \sum_{j=0}^7 \left[ \frac{1}{4!} j^4 C_j - 2 \frac{1}{3!} j^3 D_j \right] = 0, E_5 = \sum_{j=0}^7 \left[ \frac{1}{5!} j^5 C_j - 2 \frac{1}{4!} j^4 D_j \right] = 0, E_6 = \sum_{j=0}^7 \left[ \frac{1}{6!} j^6 C_j - 215D_j / 5D = 0E7 = j = 0717 / 7Cj - 216 / 6D \neq 0. \right.
\]

Therefore, the method is of order 6, with error constant as:

\[
E_7 = \begin{bmatrix}
210 \\
8293585 \\
324 \\
3184255 \\
981 \\
6926402 \\
563 \\
5947583
\end{bmatrix}
\]

(7)

3.3 Zero Stability of the Method

The method (5) is converted into matrix form as:

\[
\begin{bmatrix}
1 & -6274637 & 143998979 & -9603792 \\
7210474 & -16268759 & 1708219695 & -113881313 \\
394457 & 1 & -59168550 & 1972285 \\
11495780 & -6496015 & 1 & -42690 \\
599131 & 1198262 & 495749336 & -599131 \\
1117145237 & 11296250177 & 1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
y_{n+1} \\
y_{n+2} \\
y_{n+3} \\
y_{n+4}
\end{bmatrix}
= h
\]

\[
\begin{bmatrix}
y_{n-3} \\
y_{n-2} \\
y_{n-1} \\
y_{n}
\end{bmatrix}
+ h
\]

The equation above can be written in matrix form as:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
f_{n-3} \\
f_{n-2} \\
f_{n-1} \\
f_n
\end{bmatrix}
+ h
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

(8)
\[ A_0 Y_m = A_1 Y_{m-1} + h(B_0 F_{m-1} + B_1 F_m) \]  

Where

\[
A_0 = \begin{bmatrix}
1 & -6274637 & 143998979 & -9603792 \\
7210474 & 16268759 & 1708219695 & 21582821 & 113881313 & 14016 \\
394457 & 1 & -59168550 & -1972285 & -42690 & -59131 \\
11495780 & -6496015 & 1 & 77076456 & -599131 \\
599131 & 1198262 & 1129625017 & 19269114 & 1117145237 \\
1117145237 & 1129625017 & 495749336 & 1 & 1 \\
19269114 & 77076456 & 28903671 & 128903671 & 495749336 & 77076456 & 71129625017 & 1129625017 & 19269114 & 1117145237 \\
\end{bmatrix},
\]

\[
A_1 = \begin{bmatrix}
1298881 & -341643939 & 72003623 & -426060731 \\
341643939 & 569406565 & 113881313 & 341643939 \\
79696 & 41929759 & 68414023 & 189894868 \\
845265 & 9861425 & 3944570 & 5916855 \\
70450 & 1295843 & 5593225 & 676840 \\
1797393 & 1198262 & 599131 & 105729 \\
338687 & 353855969 & 2326014617 & 49388481 \\
348237 & 77076456 & 19269114 & 115614684 \\
\end{bmatrix}
\]

\[
B_0 = \begin{bmatrix}
0 & 9603792 & 0 & 0 \\
0 & 113881313 & 0 & 0 \\
0 & 0 & 19789614 & 0 \\
0 & 0 & 1972285 & 46087 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[ Y_{m-1}, Y_m, F_{m-1} \text{ and } F_m \text{ are column vectors defined by}
\]

\[
Y_m = \begin{bmatrix}
y_{n+1} \\
y_{n+2} \\
y_{n+3} \\
y_{n+4} \\
\end{bmatrix}, \quad Y_{m-1} = \begin{bmatrix}
y_{n-1} \\
y_{n-2} \\
y_{n-3} \\
y_{n-4} \\
\end{bmatrix}, \quad Y_{n-3} = \begin{bmatrix}
y_{n-3} \\
y_{n-4} \\
y_{n-5} \\
y_{n-6} \\
\end{bmatrix}, \quad Y_{3(m-1)+1} = \begin{bmatrix}
y_{3(m-1)+1} \\
y_{3(m-1)+2} \\
y_{3(m-1)+3} \\
y_{3(m-1)+4} \\
\end{bmatrix}
\]

\[
F_m = \begin{bmatrix}
f_{n+1} \\
f_{n+2} \\
f_{n+3} \\
f_{n+4} \\
\end{bmatrix}, \quad F_{m-1} = \begin{bmatrix}
f_{n-1} \\
f_{n-2} \\
f_{n-3} \\
f_{n-4} \\
\end{bmatrix}, \quad F_{3(m-1)+1} = \begin{bmatrix}
f_{3(m-1)+1} \\
f_{3(m-1)+2} \\
f_{3(m-1)+3} \\
f_{3(m-1)+4} \\
\end{bmatrix}
\]

Substituting scalar test equation \( y' = \lambda y \) (\( \lambda < 0, \lambda \text{ complex} \)) into (9) and using \( \lambda h = \bar{h} \) gives:

\[ A_0 Y_m = A_1 Y_{m-1} + \bar{h}(B_0 Y_{m-1} + B_1 Y_m) \]  

(10)

The stability polynomial of (5) is obtained by evaluating

\[
\text{det}[A_0 - \bar{h}B_1]t - (A_1 + \bar{h}B_0) = 0
\]
By putting $\bar{h} = 0$ in (11), we obtain the first characteristic polynomial as

$$R(\bar{h}, t) = \frac{996090316807547559435103304^4}{1320753239763668203573654609} \times \frac{19573318495604949376200^4}{2560186942430681261642609} \times \frac{664876629681834101786723}{63817166368971096642244097621} \times \frac{106380267494904356265392369}{567630008250718660657235153941} \times \frac{398925257789010461072096380}{636761357124782350323834144} \times \frac{1}{5114435612937185398836021} \times \frac{3059571243351831086667155247}{53190130374567281429461840} \times \frac{446737796802968675844731429}{106380267494904356265392369} \times \frac{153560628739855687511705383}{8350980539874554834320} \times t^4 - t^3 h - 664876629681834101786723 + \frac{19946298090453023536048190}{1567643838863926834777839112886} \times t^3 h + \frac{33243831484091705089341365}{28312643087512266327961051136} \times t^2 h^2 + \frac{23812643087512266327961051136}{5114435612937185398836021} \times t^3 \quad (11)$$

By putting $\bar{h} = 0$ in (11), we obtain the first characteristic polynomial as

$$R(0, t) = \frac{446737796802968675844731429}{106380267494904356265392369} \times \frac{153560628739855687511705383}{8350980539874554834320} \times t^4 - t^3 h - \frac{60664876629681834101786723}{5114435612937185398836021} \times t^3 \quad (12)$$

Since, the roots of (12) are $t_1 = 1$ and $t_2, t_3, t_4 \leq 0$

Therefore, the method (5) is zero Stable by definition (8).
\[ y'_2 = y_1y_2(0) = 1 \quad 0 \leq x \leq 100 \]

Exact Solution
\[ y_1(x) = e^x \]
\[ y_2(x) = e^x \]
Source: [14]

**Problem 2:** \[ y'_1 = 198y_1 + 199y_1y_1(0) = 1 \]

\[ 0 \leq x \leq 10 \]
\[ y'_2 = -398y_1 - 399y_2 \quad y_2(0) = -1 \]

Exact solution
\[ y_1(x) = e^{-x} \]
\[ y_2(x) = -e^{-x} \]
Eigen values \(-1\) and \(-200\)
Source: [7];

**Problem 3:** \[ y'_1 = y_2, y_1(0) = 0 \]

\[ 0 \leq x \leq 20 \]
\[ y'_2 = -y_1y_2(0) = 1 \]

Exact solution
\[ y_1(x) = \sin x \]
\[ y_2(x) = \cos x \]
Source: [15]

**Problem 4:** \[ y'_1 = y_2y_1(0) = 0 \]

\[ y'_2 = -2y_2y_2(0) = 0 \quad 0 \leq x \leq 4\pi \]
\[ y'_3 = y_2 + 2y_3y_3(0) = 1 \]

Exact Solution
\[ y_1(x) = 2\cos x + 6\sin x - 6x - 2 \]
\[ y_2(x) = -2\sin x + 6\cos x - 6 \]
\[ y_3(x) = 2\sin x - 2\cos x + 3 \]
Source: (Sulaiman, 1989)

### 4.2 Numerical Results

The problems sampled in this research are solved using the developed scheme. The results are tabulated, compared; and the graphs highlighting the performance of these methods are plotted. The acronyms below are used in the tables.

\[ h = \text{step-size}; \]

MAXE = Maximum Error;
T=Time in second;
3ESBBDF = 3 Point enhanced fully implicit Super Class of Block Backward Differentiation
F_{3}3ESBBDF = Family of block 3 Super class of Block Backward Differentiation
ABISBDF = A-stable block integrator scheme of Backward Differentiation Formula for solving Stiff IVPs.

Similarly, to highlight the performance of the proposed methods, ABISBDF in relation to the other methods, 3ESBSBDF and \( F_{3}3SBBDF \). The graphs of \( \log_{10}(\text{MAXE}) \) against the step size, \( h \) for the 4 problems are plotted accordingly as shown below.

### 4.3 Discussion of the results

From the numerical problems solved in the Table 1 (comprising problem 1&2), it has been shown that the proposed scheme, ABISBDF outperformed both the 3ESBSDF and \( F_{3}3SBBDF \) in terms of minimum error and
Table 1. Comparison of errors between proposed method and other methods for problem 1 & 2

| h   | Method | MAXE      | TIME     | h   | Method | MAXE      | TIME     |
|-----|--------|-----------|----------|-----|--------|-----------|----------|
| $10^{-2}$ | $F_1$SBBD | 3.30736e-002 | 4.23434e-1 | $10^{-2}$ | $F_1$SBBD | 3.23032e-002 | 3.77590e-002 |
|     | 3ESBSBD | 3.51456e-002 | 3.52416e-4 |     | 3ESBSBD | 3.98707e-002 | 2.63337e-002 |
|     | ABISBDF | 5.82117e-004 | 4.23441e-4 |     | ABISBDF | 5.83217e-003 | 5.68676e-002 |
| $10^{-3}$ | $F_1$SBBD | 5.41853e-003 | 1.81850e-3 | $10^{-3}$ | $F_1$SBBD | 4.76165e-003 | 5.66636e-001 |
|     | 3ESBSBD | 5.20191e-003 | 2.50367e-3 |     | 3ESBSBD | 4.40956e-003 | 2.60816e-001 |
|     | ABISBDF | 6.95338e-005 | 4.65467e-4 |     | ABISBDF | 6.05338e-005 | 5.64515e-001 |
| $10^{-4}$ | $F_1$SBBD | 5.44701e-005 | 1.71443e-2 | $10^{-4}$ | $F_1$SBBD | 4.66516e-004 | 5.64385e-001 |
|     | 3ESBSBD | 5.20417e-005 | 2.56918e-2 |     | 3ESBSBD | 5.08942e-005 | 2.60725e-001 |
|     | ABISBDF | 6.95692e-007 | 4.84833e-3 |     | 4ESBSBD | 6.26692e-007 | 5.68143e+000 |
| $10^{-5}$ | $F_1$SBBD | 5.44971e-007 | 1.70042e-1 | $10^{-5}$ | $F_1$SBBD | 4.68707e+005 | 5.63788e+000 |
|     | 3ESBSBD | 5.25030e-007 | 2.34880e-1 |     | 3ESBSBD | 5.21534e+007 | 2.60597e+000 |
|     | ABISBDF | 6.95974e-009 | 4.8687e-2  |     | ABISBDF | 6.32740e-009 | 5.59821e+001 |
| $10^{-6}$ | $F_1$SBBD | 5.44998e-009 | 1.70308e-00 | $10^{-6}$ | $F_1$SBBD | 4.69123e-006 | 5.65356e+001 |
|     | 3ESBSBD | 5.25648e-009 | 2.35791e-00 |     | 3ESBSBD | 5.89872e-009 | 2.60700e+001 |
|     | ABISBDF | 7.18636e-011 | 4.23434e-1 |     | ABISBDF | 6.33362e-011 | 5.53567e+002 |

Table 2. Comparison of Errors between Proposed Method and other Methods for Problem 3 & 4

| h   | Method | MAXE      | TIME     | h   | Method | MAXE      | TIME     |
|-----|--------|-----------|----------|-----|--------|-----------|----------|
| $10^{-2}$ | $F_1$SBBD | 2.07208e-002 | 1.37500e-2 | $10^{-2}$ | $F_1$SBBD | 2.83032e-002 | 3.67590e-002 |
|     | 3ESBSBD | 2.53437e-002 | 1.20934e-3 |     | 3ESBSBD | 2.48705e-002 | 2.63337e-002 |
|     | ABISBDF | 2.83117e-004 | 7.36289e-2 |     | ABISBDF | 3.83217e-003 | 5.88676e-002 |
| $10^{-3}$ | $F_1$SBBD | 3.20160e-004 | 2.72200e-2 | $10^{-3}$ | $F_1$SBBD | 3.76163e-003 | 8.56636e-002 |
|     | 3ESBSBD | 3.02893e-004 | 1.25972e-2 |     | 3ESBSBD | 3.40956e-003 | 2.60816e-001 |
|     | ABISBDF | 4.05338e-006 | 5.81512e-2 |     | ABISBDF | 4.05338e-005 | 5.54515e-001 |
| $10^{-4}$ | $F_1$SBBD | 3.20233e-006 | 2.02700e-01 | $10^{-4}$ | $F_1$SBBD | 3.76154e-005 | 8.54385e-001 |
|     | 3ESBSBD | 3.08956e-006 | 1.25148e-1 |     | 3ESBSBD | 3.48942e-005 | 2.60725e+000 |
|     | ABISBDF | 4.26592e-008 | 5.81491e-1 |     | ABISBDF | 4.26690e-007 | 5.58143e-001 |
| $10^{-5}$ | $F_1$SBBD | 3.20261e-008 | 1.92600e-0 | $10^{-5}$ | $F_1$SBBD | 3.70705e-005 | 8.53788e+000 |
|     | 3ESBSBD | 3.10157e-008 | 1.25471e-0 |     | 3ESBSBD | 3.58532e-005 | 2.60597e+001 |
|     | ABISBDF | 4.32640e-010 | 5.81122e-0 |     | ABISBDF | 4.32740e-009 | 5.49821e+000 |
| $10^{-6}$ | $F_1$SBBD | 3.20263e-010 | 1.91700e-0 | $10^{-6}$ | $F_1$SBBD | 3.71121e-007 | 8.53356e+001 |
|     | 3ESBSBD | 3.41129e-010 | 1.24892e-0 |     | 3ESBSBD | 3.69872e-007 | 2.60700e+002 |
|     | ABISBDF | 4.33262e-012 | 5.79887e-1 |     | ABISBDF | 4.3335e-009  | 5.43567e+001 |
Fig. 2. Graph of $\log_{10}(MAXE)$ against $h$ for problem 1
Fig. 3. Graph of $\log_{10}(MAXE)$ against $h$ for problem 2
Fig. 4. Graph of $\log_{10}(MAXE)$ against $h$ for problem 3
Fig. 5. Graph of $\log_{10}(MAXE)$ against $h$ for problem 4
less computational time. Also, from table 2 (comprising problem 2&3) the proposed scheme have good advantage in terms of scale error over the two methods compared. But, F3SBBDF has advantages over the new method ABISBDF in execution time. To visibly highlight the performance of the proposed method, ABISBDF in relation to the other methods, 3ESBSBDF and F3SBBDF. The graphs of Log10(MAXE) against the step size, $h$ for the 1-4 problems are plotted accordingly in figure (2,3,4,5), the method has minimum scaled error in the entire problems considered. The proposed scheme is recommended for solving first order system of initial value problems of ordinary differential equation.

5. Conclusion

An A stable block integrator scheme is proposed. The order and stability properties of the method are investigated, the scheme found to be zero stable, A-stable and of order 6. The developed method is implicit methods, can computes four solution values at a time per step, concurrently. The results from the tested problems shows that the new method has advantages in terms of accuracy of the scaled error and computational time when compared with the 3ESBSBDF and also has advantages in terms of accuracy of the scaled error over F3SBBDF method. The proposed scheme can be used in solving a system of first order initial value problem of ordinary differential equations.

Competing Interests

Authors have declared that no competing interests exist.

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