New John–Nirenberg–Campanato-Type Spaces Related to Both Maximal Functions and Their Commutators

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Abstract  Let \( p, q \in [1, \infty], \alpha \in \mathbb{R}, \) and \( s \) be a non-negative integer. In this article, the authors introduce a new function space \( \tilde{JN}_{(p,q,s)}(X) \) of John–Nirenberg–Campanato type, where \( X \) denotes \( \mathbb{R}^n \) or any cube \( Q_0 \) of \( \mathbb{R}^n \) with finite edge length. The authors give an equivalent characterization of \( \tilde{JN}_{(p,q,s)}(X) \) via both the John–Nirenberg–Campanato space and the Riesz–Morrey space. Moreover, for the particular case \( s = 0 \), this new space can be equivalently characterized by both maximal functions and their commutators. Additionally, the authors give some basic properties, a good-\( \lambda \) inequality, and a John–Nirenberg type inequality for \( \tilde{JN}_{(p,q,s)}(X) \).

1 Introduction

Throughout this article, a cube \( Q_0 \) always means that \( Q_0 \) has finite edge length, all its edges parallel to the coordinate axes, but \( Q_0 \) is not necessary to be open or closed. We always let \( X \) represent either \( \mathbb{R}^n \) or a cube \( Q_0 \) of \( \mathbb{R}^n \) with finite edge length. For any integrable function \( f \) and any cube \( Q \), let

\[
f_Q := \frac{1}{|Q|} \int_Q f.
\]

Here and thereafter, in all integral representation, in order to simplify the presentation, we always omit the differential \( dx \) if there exists no confusion.

The John–Nirenberg space \( JN_p \) has attracted a lot of attention in recent years. It is a byproduct appearing in the study of John and Nirenberg [19] on functions with bounded mean oscillation (the celebrated space BMO), and was further used in the interpolation theory by Stampacchia [24]. Both \( JN_p \) and BMO are function spaces based on mean oscillations of functions, and we refer the reader to [3, 6, 7, 8, 12, 13, 14, 18, 21, 22] for more related researches. Obviously, we have \( JN_1(Q_0) = L^1(Q_0) \) and \( JN_{\infty}(Q_0) = \text{BMO}(Q_0) \); see, for instance, [27]. Moreover, an interesting result of Dafni et al. [9] shows the non-triviality of \( JN_p(Q_0) \) with \( p \in (1, \infty) \) via constructing a surprising function belonging to \( JN_p(Q_0) \) but not an element of \( L^p(Q_0) \). This means that \( JN_p(Q_0) \) is strictly larger than \( L^p(Q_0) \). Later, a \( JN_p \)-type space mixed with Campanato
structure, called the John–Nirenberg–Campanato space $JN_{(p,q,s)}(\mathcal{X})$, was introduced and studied in [29]; see Definition 2.3 below. Indeed, for any $p, q \in [1, \infty]$, $s \in \mathbb{Z}_+$, and $\alpha \in \mathbb{R}$,

$$
JN_{(p,q,s)}(\mathcal{X}) = \begin{cases} 
\text{the John–Nirenberg space } JN_p(\mathcal{X}), & q = 1, \ s = 0 = \alpha, \\
\text{the Campanato space } C_{p,q,s}(\mathcal{X}), & p = \infty, \\
\text{the space } \text{BMO}(\mathcal{X}), & p = \infty, \ \alpha = 0,
\end{cases}
$$

and hence $JN_{(p,q,s)}(\mathcal{X})$ combines some futures of both the John–Nirenberg space and the Campanato space. However, as is mentioned in [9], the structure of $JN_p$ is largely a mystery, and so does $JN_{(p,q,s)}(\mathcal{X})$. For instance, even for some classical operators (such as the Hardy–Littlewood maximal operator, the Calderón-Zygmund operator, and the fractional integral), it is still unclear whether or not it is bounded on $JN_p(\mathcal{X})$ or $JN_{(p,q,s)}(\mathcal{X})$; we refer the reader to [15][16][17][20][27] for some related studies. The main purpose of this article is to investigate the John–Nirenberg–Campanato space from the point of view of the negative part $f^- := -\min(f, 0)$, and to shed some light on the structure of John–Nirenberg-type spaces associated with both maximal functions and their commutators.

The negative part reveals some significant properties of $\text{BMO}(\mathbb{R}^n)$ related to maximal functions. Precisely, let $M$ denote the Hardy–Littlewood maximal operator defined by setting, for any $f \in L^{1}_{\text{loc}}(\mathbb{R}^n)$ (the set of all locally integrable functions) and any $x \in \mathbb{R}^n$,

$$
M(f)(x) := \sup_{\text{cube } Q \ni x} \frac{1}{\mu(Q)} \int_{Q} |f(y)| \, dy.
$$

Moreover, for any given cube $Q$ of $\mathbb{R}^n$ and for any $f \in L^{1}_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$
M_Q(f)(x) := \sup_{\text{cube } Q : x \in Q \subset Q} \frac{1}{\mu(Q)} \int_{Q} |f(y)| \, dy.
$$

For any $b \in L^{1}_{\text{loc}}(\mathbb{R}^n)$, define the commutator $[b, M]$ by setting, for any $f \in L^{1}_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$
[b, M](f)(x) := b(x)M(f)(x) - M(bf)(x).
$$

Then Bastero et al. [2] proved that the following three statements are mutually equivalent:

(i) for any (or some) $p \in (1, \infty)$, the commutator $[b, M]$ is bounded on $L^p(\mathbb{R}^n)$;

(ii) $b \in \text{BMO}(\mathbb{R}^n)$ with $b^- \in L^\infty(\mathbb{R}^n)$;

(iii) for any (or some) $q \in [1, \infty)$,

$$
\sup_{Q} \frac{1}{\mu(Q)} \int_{Q} |b - M_Q(b)|^q < \infty,
$$

where the supremum is taken over all cubes $Q$ of $\mathbb{R}^n$.

Very recently, Wang and Shu [30] further showed that, if we replace $M_Q(b)$ in (iii) by $|b|_{L^Q}$, then the above equivalence still holds true. Indeed, they obtain the equivalence between (ii) and
(iv) \( \sup_{Q} |f|_{b|Q|} < \infty \), where the supremum is taken over all cubes \( Q \) of \( \mathbb{R}^n \).

This implies that (i) \( \iff \) (ii) \( \iff \) (iii) \( \iff \) (iv). Moreover, Wang and Shu [30] also studied this phenomenon in the context of Campanato spaces, and introduced a new function spaces of Morrey–Campanato type in [30]. As applications, they obtained an integral characterization of the non-negative Hölder continuous functions. Note that \( f \) is non-negative if and only if \( f^- = 0 \). Thus, the structure of this new space of Morrey–Campanato type is also completely determined by the classical Campanato space and the boundedness of the negative part \( f^- \).

In this article, we introduce a new John–Nirenberg–Campanato-type space \( \overline{JN}_{(p,q,s)u}(X) \) with \( p, q \in [1, \infty] \), \( \alpha \in \mathbb{R} \), and \( s \) being a non-negative integer; see Definition 2.3 below. We give an equivalent characterization of \( \overline{JN}_{(p,q,s)u}(X) \) via the John–Nirenberg–Campanato space and the Riesz–Morrey space. Moreover, for the particular case \( s = 0 \), this new space can be equivalently characterized by maximal functions and their commutators. Additionally, we give some basic properties, a good-\( \lambda \) inequality, and a John–Nirenberg type inequality for \( \overline{JN}_{(p,q,s)u}(Q_0) \).

The only difference between the “norms” of \( JN_{(p,q,s)u}(X) \) and \( \overline{JN}_{(p,q,s)u}(X) \) is that we change the mean oscillation

\[
\int_{Q} |f - P_{Q}^{(s)}(f)|^{q} \int_{Q} |f - P_{Q}^{(s)}(|f|)|^{q},
\]

where \( P_{Q}^{(s)}(\cdot) \) denotes the unique polynomial satisfying (2.1). Since \( P_{Q}^{(0)}(f) \) coincides with \( f_{Q} \), this is a natural generalization of the above (iv). Thus, a natural question appears:

Whether or not \( f^- \) is still a bridge connecting \( JN_{(p,q,s)u}(X) \) and \( \overline{JN}_{(p,q,s)u}(X) \)?

Indeed, we give a positive answer to this question in Theorem 2.9 below, and the key ingredient is the Riesz–Morrey space \( RM_{p,q,a}(X) \) studied in [28, 31]. Correspondingly, the only difference between the norms of \( JN_{(p,q,s)u}(X) \) and \( RM_{p,q,a}(X) \) is that we change the mean oscillation

\[
\int_{Q} |f - P_{Q}^{(s)}(f)|^{q} \int_{Q} |f|^{q};
\]

see Definition 2.5 below. Moreover, for the case \( s = 0 \), we further obtain a corresponding equivalence of above (i) \( \iff \) (ii) \( \iff \) (iii) \( \iff \) (iv) on \( JN_{(p,q,s)u}(X) \) via maximal functions and their commutators; see Theorem 3.10 and Corollary 3.12 below.

The remainder of this article is organized as follows.

In Section 2, we introduce a “new” John–Nirenberg–Campanato type space \( \overline{JN}_{(p,q,s)u}(X) \), and prove that

\[
RM_{p,q,a}(X) \subset \overline{JN}_{(p,q,s)u}(X) \subset JN_{(p,q,s)u}(X)
\]

in Proposition 2.7 below. Indeed, Remark 2.8 shows that this is truly a new space between the Riesz–Morrey space and the John–Nirenberg space, namely, the inclusions in (1.3) are proper. Moreover, we give an equivalent characterization of \( \overline{JN}_{(p,q,s)u}(X) \) in Theorem 2.9 below via the John–Nirenberg–Campanato space \( JN_{(p,q,s)u}(X) \) and the Riesz–Morrey space \( RM_{p,q,a}(X) \). The proofs of these results rely heavily on [28] below and the linearity of \( P_{Q}^{(s)}(\cdot) \).
In Section 3, we first give some basic properties of \( \tilde{JN}_{(p,q,s)}(\mathcal{X}) \), including the monotonicity on the subindices \( p, q, \) and \( s \) (see Proposition 3.1 below), the limit behavior as \( p \to \infty \) (see Proposition 3.2 and Corollary 3.3 below), and the invariance on the second subindex \( q \) (see Proposition 3.5 below). Moreover, for the case \( s = 0 \), \( P^q_Q(|f|) = |f|_Q \) is closely connected with the maximal function \( M_Q(f) \), and hence we further study the relation between \( \tilde{JN}_{(p,q,s)}(\mathcal{X}) \) and maximal functions. On one hand, the maximal operator is bounded on \( L^q(\mathcal{X}) \) with \( q \in (1,\infty) \) but not bounded on \( L^1(\mathcal{X}) \). On the other hand, when dominating mean oscillations by maximal functions, the linearity of \( \| \cdot \|_{L^1(\mathcal{X})} \) plays a key role, which is no longer feasible for \( L^q(\mathcal{X}) \) with \( q \in (1,\infty) \); see Remark 3.11 below. To coordinate these two conflicts (namely, \( q = 1 \) and \( q > 1 \)), we give another characterization of \( \tilde{JN}_{(p,q,s)}(\mathcal{X}) \) in Proposition 3.8 below. With the aid of Propositions 3.5 and 3.8, we flexibly change the index \( q \) and hence characterize \( \tilde{JN}_{(p,q,s)}(\mathcal{X}) \) via maximal functions in Theorem 3.10 below. Furthermore, a related characterization via their commutators is also established in Corollary 3.12 below.

In Section 4, we prove a John–Nirenberg type inequality and a good-\( \lambda \) inequality for the space \( \tilde{JN}_{(p,q,s)}(Q_0) \), respectively, in Theorem 4.1 and Lemma 4.3 below. As an application, we use this John–Nirenberg type inequality to obtain another proof of Proposition 3.5 at the end of this section.

Finally, we make some conventions on notation. Throughout this article, \( \mathcal{X} \) always represents either \( \mathbb{R}^n \) or a cube \( Q_0 \) of \( \mathbb{R}^n \) with finite edge length. Let \( \mathbb{N} := \{1,2,\ldots\} \) and \( \mathbb{Z}_+ := \mathbb{N} \cup \{0\} \). For any real-valued function \( f \), we use \( f^- := -\min\{f,0\} \) to denote its negative part. Let \( I_E \) denote the characteristic function of any set \( E \subset \mathbb{R}^n \), and \( \mathcal{P}_\lambda(Q) \) the set of all polynomials of degree not greater than \( s \in \mathbb{Z}_+ \) on \( Q \). We denote by both \( C \) and \( \tilde{C} \) positive constants which are independent of the main parameters, but may vary from line to line. Moreover, we use \( C_{(\gamma, \beta, \ldots)} \) to denote a positive constant depending on the indicated parameters \( \gamma, \beta, \ldots \). Constants with subscripts, such as \( C_0 \) and \( A_1 \), do not change in different occurrences. Moreover, the symbol \( f \leq g \) represents that \( f \leq Cg \) for some positive constant \( C \). If \( f \leq g \) and \( g \leq f \), we then write \( f \sim g \). If \( f \leq Cg \) and \( g = h \) or \( g \leq h \), we then write \( f \approx h \) or \( f \approx g \), rather than \( f \leq g = h \) or \( f \leq g \approx h \). For any \( p \in [1,\infty] \), let \( p' \) be its conjugate index, that is, \( p' \) satisfies \( 1/p + 1/p' = 1 \).

## 2 New Function Spaces of John–Nirenberg–Campanato Type

To introduce the space \( \tilde{JN}_{(p,q,s)}(\mathcal{X}) \), we first recall the following basic concepts.

- For any \( s \in \mathbb{Z}_+ \) (the set of all non-negative integers), \( \mathcal{P}^s_Q(f) \) denotes the unique polynomial of degree not greater than \( s \) such that, for any \( |\gamma| \leq s \),

\[
\int_Q [f(x) - P^s_Q(f)(x)] x^\gamma \, dx = 0,
\]

where \( \gamma := (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}_+^n := (\mathbb{Z}_+)^n \), \( |\gamma| := \gamma_1 + \cdots + \gamma_n \), and \( x^\gamma := x_1^{\gamma_1} \cdots x_n^{\gamma_n} \) for any \( x := (x_1, \ldots, x_n) \in \mathbb{R}^n \). It is well known that \( P^q_Q(0) = f_Q \) and, for any \( s \in \mathbb{Z}_+ \), there exists a constant \( C_{(s)} \in [1,\infty) \), independent of both \( f \) and \( Q \), such that, for any \( x \in Q \),

\[
|P^s_Q(f)(x)| \leq C_{(s)} \int_Q |f|;
\]

\[
\|f\|_{\tilde{JN}_{(p,q,s)}(\mathcal{X})} := \sup\{|\langle f, \varphi \rangle| : \varphi \approx h, \|\varphi\|_{L^p(\mathcal{X})} \leq 1\}.
\]
Let Definition 2.1. 

(i) The Campanato space \( \mathcal{C}_{\alpha,q,s}(\mathcal{X}) \), introduced by Campanato [5], is defined by setting

\[
\mathcal{C}_{\alpha,q,s}(\mathcal{X}) := \left\{ f \in L^q_{\text{loc}}(\mathcal{X}) : \| f \|_{\mathcal{C}_{\alpha,q,s}(\mathcal{X})} < \infty \right\}
\]

with

\[
\| f \|_{\mathcal{C}_{\alpha,q,s}(\mathcal{X})} := \sup_{Q \subseteq \mathcal{X}} \left( |Q|^{-\alpha} \left( \int_{Q} |f - P_{Q}^{(s)}(f)|^{q} \right)^{\frac{1}{q}} \right),
\]

where \( P_{Q}^{(s)}(f) \) is the same as in (2.1) and the supremum is taken over all cubes \( Q \) of \( \mathcal{X} \).

(ii) The \( \text{BMO}(\mathcal{X}) \), introduced by John and Nirenberg [19] in 1961 to study the functions of bounded mean oscillation, is defined by setting

\[
\text{BMO}(\mathcal{X}) := \left\{ f \in L_{\text{loc}}^{1}(\mathcal{X}) : \| f \|_{\text{BMO}(\mathcal{X})} = \sup_{\text{cube } Q \subseteq \mathcal{X}} \int_{Q} |f - f_{Q}| < \infty \right\}
\]

with the supremum taken over all cubes \( Q \) of \( \mathcal{X} \).

**Definition 2.1.** Let \( \alpha \in \mathbb{R}, q \in [1, \infty), \) and \( s \in \mathbb{Z}_{+} \).

(i) The Campanato space \( \mathcal{C}_{\alpha,q,s}(\mathcal{X}) \), introduced by Campanato [5], is defined by setting

\[
\mathcal{C}_{\alpha,q,s}(\mathcal{X}) := \left\{ f \in L^q_{\text{loc}}(\mathcal{X}) : \| f \|_{\mathcal{C}_{\alpha,q,s}(\mathcal{X})} < \infty \right\}
\]

with

\[
\| f \|_{\mathcal{C}_{\alpha,q,s}(\mathcal{X})} := \sup_{Q \subseteq \mathcal{X}} \left( |Q|^{-\alpha} \left( \int_{Q} |f - P_{Q}^{(s)}(f)|^{q} \right)^{\frac{1}{q}} \right),
\]

where \( P_{Q}^{(s)}(f) \) is the same as in (2.1) and the supremum is taken over all cubes \( Q \) of \( \mathcal{X} \).
(ii) The space $\tilde{C}_{a,q,s}(\mathcal{X})$ is defined by setting

$$\tilde{C}_{a,q,s}(\mathcal{X}) := \left\{ f \in L^q_{\text{loc}}(\mathcal{X}) : \|f\|_{\tilde{C}_{a,q,s}(\mathcal{X})} < \infty \right\}$$

with

$$\|f\|_{\tilde{C}_{a,q,s}(\mathcal{X})} := \sup \left\{ \|Q\|^{-\alpha} \left( \int_{\tilde{Q}} \left| f - P_{\tilde{Q}}(f) \right|^q \right)^{\frac{1}{q}} \right\},$$

where $P_{\tilde{Q}}(f)$ is the same as in (2.1) and the supremum is taken over all cubes $Q$ of $\mathcal{X}$.

**Remark 2.2.** (i) The “norm” $\| \cdot \|_{\tilde{C}_{a,q,s}(\mathcal{X})}$ is defined modulo polynomials. For simplicity, we regard $C_{a,q,s}(\mathcal{X})$ as the quotient space $C_{a,q,s}(\mathcal{X})/\mathcal{P}_s(\mathcal{X})$. However, $\| \cdot \|_{\tilde{C}_{a,q,s}(\mathcal{X})}$ is not a norm of $L^q_{\text{loc}}(\mathcal{X})$ because the triangular inequality does not hold true.

(ii) Comparing $\tilde{C}_{a,q,s}(\mathcal{X})$ in Definition 2.1(ii) with $\mathcal{H}^{p, q, d}(\mathcal{X})$ in [30], it is easy to find that

$$\tilde{C}_{a,q,s}(\mathcal{X}) = \mathcal{H}^{p, q, n(aq+1)}(\mathcal{X}),$$

where $n$ denotes the dimension of $\mathcal{X}$.

**Definition 2.3.** Let $q \in [1, \infty]$, $s \in \mathbb{Z}_+$, and $\alpha \in \mathbb{R}$.

(i) If $p \in [1, \infty)$, then the **John–Nirenberg–Campanato space** $JN_{(p,q,s), a}(\mathcal{X})$, introduced in [29], is defined by setting

$$JN_{(p,q,s), a}(\mathcal{X}) := \left\{ f \in L^q_{\text{loc}}(\mathcal{X}) : \|f\|_{JN_{(p,q,s), a}(\mathcal{X})} < \infty \right\}$$

with

$$\|f\|_{JN_{(p,q,s), a}(\mathcal{X})} := \sup \left\{ \sum_{Q_i} |Q_i|^{-\alpha} \left( \int_{Q_i} \left| f - P_{Q_i}(f) \right|^q \right)^{\frac{1}{q}} \right\}^\frac{1}{p},$$

where $P_{Q_i}(f)$ for any $i$ is the same as in (2.1) with $Q$ replaced by $Q_i$ and the supremum is taken over all the collections of interior pairwise disjoint cubes $\{Q_i\}$ of $\mathcal{X}$.

(ii) If $p \in [1, \infty)$, then the **John–Nirenberg–Campanato-type space** $\tilde{JN}_{(p,q,s), a}(\mathcal{X})$ is defined by setting

$$\tilde{JN}_{(p,q,s), a}(\mathcal{X}) := \left\{ f \in L^q_{\text{loc}}(\mathcal{X}) : \|f\|_{\tilde{JN}_{(p,q,s), a}(\mathcal{X})} < \infty \right\}$$

with

$$\|f\|_{\tilde{JN}_{(p,q,s), a}(\mathcal{X})} := \sup \left\{ \sum_{Q_i} |Q_i|^{-\alpha} \left( \int_{Q_i} \left| f - P_{Q_i}(f) \right|^q \right)^{\frac{1}{q}} \right\}^\frac{1}{p},$$

where $P_{Q_i}(f)$ for any $i$ is the same as in (2.1) with both $Q$ replaced by $Q_i$ and $f$ replaced by $|f|$, and where the supremum is taken over all the collections of interior pairwise disjoint cubes $\{Q_i\}$ of $\mathcal{X}$. 
(iii) Let \( JN_{(\alpha,q,s),\alpha}(X) := C_{\alpha,q,s}(X) \) and \( \overline{JN}_{(\alpha,q,s),\alpha}(X) := \overline{C}_{\alpha,q,s}(X) \).

**Remark 2.4.** Let \( p, q, s, \) and \( \alpha \) be the same as in Definition 2.3.

(i) Let \( f \) be any non-negative function on \( X \). Then \( f \in JN_{(p,q,s),\alpha}(X) \) if and only if \( f \in \overline{JN}_{(p,q,s),\alpha}(X) \).

(ii) \( \| \cdot \|_{\overline{JN}_{(p,q,s),\alpha}(X)} \) is not a norm of \( L^q_{\text{loc}}(X) \) because the triangular inequality does not hold true.

The following Riesz–Morrey space was introduced in [28] as a bridge connecting Lebesgue spaces and Morrey spaces.

**Definition 2.5.** Let \( p, q \in [1, \infty] \) and \( \alpha \in \mathbb{R} \). Then the **Riesz–Morrey space** \( RM_{p,q,\alpha}(X) \) is defined by setting

\[
RM_{p,q,\alpha}(X) := \{ f \in L^q_{\text{loc}}(X) : \| f \|_{RM_{p,q,\alpha}(X)} < \infty \}
\]

with

\[
\| f \|_{RM_{p,q,\alpha}(X)} := \begin{cases}
\sup \left\{ \sum_{i} |Q_i|^{-\alpha} \left( \int_{Q_i} |f|^q \right)^{\frac{1}{q}} \right\}^{\frac{1}{p}} & \text{if } p \in [1, \infty), \ q \in [1, \infty], \\
\sup \limits_{\text{cube } Q \subset X} |Q|^{-\alpha} \left( \int_{Q} |f|^q \right)^{\frac{1}{q}} & \text{if } p = \infty, \ q \in [1, \infty],
\end{cases}
\]

where the first supremum is taken over all the collections of interior pairwise disjoint cubes \( \{Q_i\}_i \) of \( X \) and the second supremum is taken over all cubes \( Q \) of \( X \).

**Remark 2.6.** The relation between Riesz–Morrey spaces and Lebesgue spaces is completely clarified over all indices in [31 Corollary 3.7]. For the convenience of the reader, we list all cases of \( RM_{p,1,\alpha}(X) \) as follows.

(i) Let \( p \in (1, \infty] \). Then

\[
RM_{p,1,\alpha}(\mathbb{R}^n) = \begin{cases}
L^1(\mathbb{R}^n) & \text{if } \alpha = \frac{1}{p} - 1, \\
L^{1/p}(\mathbb{R}^n) & \text{if } \alpha \in \left( \frac{1}{p} - 1, 0 \right), \\
L^p(\mathbb{R}^n) & \text{if } \alpha = 0, \\
\{0\} & \text{if } \alpha \in (-\infty, \frac{1}{p} - 1) \cup (0, \infty).
\end{cases}
\]

In particular, for any \( \alpha \in (-1, 0) \), \( RM_{\infty,1,\alpha}(\mathbb{R}^n) = M_{1,\alpha}(\mathbb{R}^n) \) which is the Morrey space.

(ii)

\[
RM_{1,1,\alpha}(\mathbb{R}^n) = \begin{cases}
L^1(\mathbb{R}^n) & \text{if } \alpha = 0, \\
\{0\} & \text{if } \alpha \in \mathbb{R} \setminus \{0\}.
\end{cases}
\]

(iii) Let \( p \in (1, \infty] \) and \( Q_0 \) be any cube of \( \mathbb{R}^n \). Then

\[
RM_{p,1,\alpha}(Q_0) = \begin{cases}
L^1(Q_0) & \text{if } \alpha \in (-\infty, \frac{1}{p} - 1], \\
L^{1/p}(Q_0) & \text{if } \alpha \in \left( \frac{1}{p} - 1, 0 \right), \\
L^p(Q_0) & \text{if } \alpha = 0, \\
\{0\} & \text{if } \alpha \in (0, \infty).
\end{cases}
\]
In particular, $RM_{\infty,1,\alpha}(Q_0) = M_{1,\alpha}(Q_0)$ if $\alpha \in (-1, 0)$.

(iv) Let $Q_0$ be any cube of $\mathbb{R}^n$. Then

$$RM_{1,1,\alpha}(Q_0) = \begin{cases} L^1(Q_0) & \text{if } \alpha \in (-\infty, 0], \\ \{0\} & \text{if } \alpha \in (0, \infty). \end{cases}$$

**Proposition 2.7.** Let $p, q \in [1, \infty]$, $s \in \mathbb{Z}_+$, and $\alpha \in \mathbb{R}$. Then

$$RM_{p,q,\alpha}(\mathcal{X}) \subset \bar{JN}_{(p,q,s)_a}(\mathcal{X}) \subset JN_{(p,q,s)_a}(\mathcal{X})$$

and

$$[1 + C(s)]^{-1} \|JN_{(p,q,s)_a}(\mathcal{X})\| \leq \|\cdot\|_{\bar{JN}_{(p,q,s)_a}(\mathcal{X})} \leq [1 + C(s)] \|\cdot\|_{RM_{p,q,\alpha}(\mathcal{X})},$$

where $C(s) \in [1, \infty)$ is the same as in (2.2).

**Proof.** To show Proposition 2.7, it is enough to prove that (2.3) holds true.

First, we show the first inequality of (2.3). To this end, for any $f \in L^q_{\text{loc}}(\mathcal{X})$ and any interior pairwise disjoint cubes $\{Q_i\}_i$ of $\mathcal{X}$, from the Minkowski inequality, the linearity of $\{p^{(s)}_{Q_i}\}_i$, (2.2), and the Hölder inequality, it follows that

$$\left\{ \sum_i |Q_i| |Q_i|^{-\alpha} \left( \int_{Q_i} |f - p^{(s)}_{Q_i}(f)|^q \right)^{\frac{p}{q}} \right\}^{\frac{1}{p}}$$

$$\leq \left\{ \sum_i |Q_i|^{1-p\alpha} \left[ \int_{Q_i} |f - p^{(s)}_{Q_i}(|f|)|^q \right]^{\frac{p}{q}} + \left[ \int_{Q_i} |p^{(s)}_{Q_i}(|f|) - p^{(s)}_{Q_i}(f)|^q \right]^{\frac{p}{q}} \right\}^{\frac{1}{p}}$$

$$\leq \|f\|_{\bar{JN}_{(p,q,s)_a}(\mathcal{X})} + \left\{ \sum_i |Q_i|^{1-p\alpha} \left[ \int_{Q_i} |p^{(s)}_{Q_i}(|f|) - f|^q \right]^{\frac{p}{q}} \right\}^{\frac{1}{p}}$$

$$\leq \|f\|_{\bar{JN}_{(p,q,s)_a}(\mathcal{X})} + C(s) \left\{ \sum_i |Q_i|^{1-p\alpha} \left[ \int_{Q_i} |p^{(s)}_{Q_i}(|f|) - f|^p \right]^{\frac{1}{p}} \right\}^{\frac{1}{q}}$$

$$\leq \|f\|_{\bar{JN}_{(p,q,s)_a}(\mathcal{X})} + C(s) \left\{ \sum_i |Q_i|^{1-p\alpha} \left[ \int_{Q_i} |p^{(s)}_{Q_i}(|f|) - f|^q \right]^{\frac{p}{q}} \right\}^{\frac{1}{q}}$$

which implies that the first inequality of (2.3) holds true.

Next, we show the second inequality of (2.3). To this end, for any $f \in L^q_{\text{loc}}(\mathcal{X})$ and any interior pairwise disjoint cubes $\{Q_i\}_i$ of $\mathcal{X}$, by the Minkowski inequality, (2.2), and the Hölder inequality,
we conclude that

$$\left\{ \sum_i |Q_i^{\alpha_1} \left( \frac{\| f - P_{Q_i}(|f|) \|^q}{|Q_i^{\alpha_1}} \right) \right\}^{\frac{1}{q}} \leq \left\{ \sum_i |Q_i^{\alpha_2} \left( \frac{\| f \|^q}{|Q_i^{\alpha_2}} \right) \right\}^{\frac{1}{q}} + \left\{ \sum_i |Q_i^{\alpha_3} \left( \frac{\| P_{Q_i}(|f|) \|^q}{|Q_i^{\alpha_3}} \right) \right\}^{\frac{1}{q}}$$

$$\leq \| f \|_{RM_{p,q,s}(\mathcal{X})} + C(s) \left\{ \sum_i |Q_i^{\alpha_1} \left( \frac{\| f \|^q}{|Q_i^{\alpha_1}} \right) \right\}^{\frac{1}{q}}$$

$$\leq \| f \|_{RM_{p,q,s}(\mathcal{X})} + C(s) \left\{ \sum_i |Q_i^{\alpha_2} \left( \frac{\| f \|^q}{|Q_i^{\alpha_2}} \right) \right\}^{\frac{1}{q}}$$

$$\leq [1 + C(s)] \| f \|_{RM_{p,q,s}(\mathcal{X})},$$

which implies that the second inequality of (2.3) holds true. Therefore, (2.3) holds true, which completes the proof of Proposition 2.7.

\[\square\]

**Remark 2.8.** Let $p \in (1, \infty)$ and $Q$ denote any given cube of $\mathbb{R}^n$. Then we claim that

$$RM_{p,1,0}(Q) \subseteq \widetilde{JN}_{p,1,0}(Q) \subseteq JN_{p,1,0}(Q).$$

Indeed, by [28] Proposition 1, we have $RM_{p,1,0}(Q) = L^p(Q)$. Moreover, it was proved in [9] Corollary 4.2 that there exists a non-negative function $f \in JN_{p,1,0}(Q) \setminus L^p(Q)$. From Remark 2.4(i), we infer that this non-negative function $f \in JN_{p,1,0}(Q)$. Therefore, we have

$$f \in \widetilde{JN}_{p,1,0}(Q) \setminus L^p(Q) = \widetilde{JN}_{p,1,0}(Q) \setminus RM_{p,1,0}(Q).$$

Moreover, let $g := -f \leq 0$. Then we have $g \in JN_{p,1,0}(Q)$ and $g^- = f \notin L^p(Q) = RM_{p,1,0}(Q)$. Thus, from Theorem 2.9 below, we deduce that

$$g \in JN_{p,1,0}(Q) \setminus \widetilde{JN}_{p,1,0}(Q).$$

Therefore, the above claim holds true.

Now, we state the first main result of this article.

**Theorem 2.9.** Let $p, q \in [1, \infty]$, $s \in \mathbb{Z}_+$, and $\alpha \in \mathbb{R}$. Then $f \in \widetilde{JN}_{p,q,s,\alpha}(\mathcal{X})$ if and only if $f \in JN_{p,q,s,\alpha}(\mathcal{X})$ and $f^- \in RM_{p,1,\alpha}(\mathcal{X})$. Moreover, for any $f \in L^q_{\text{loc}}(\mathcal{X})$,

$$\| f \|_{\widetilde{JN}_{p,q,s,\alpha}(\mathcal{X})} \sim \left[ \| f \|_{JN_{p,q,s,\alpha}(\mathcal{X})} + \| f^- \|_{RM_{p,1,\alpha}(\mathcal{X})} \right]$$

with the positive equivalence constants depending only on both $s$ and $n$. 
Proof. Let \( p, q, s, \) and \( \alpha \) be the same as in the present theorem. We only consider the case \( p < \infty \) because the case \( p = \infty \) can be similarly proved.

We first claim that \( \|\cdot\|_{RM_{p,1,q,\alpha}(X)} \lesssim \|\cdot\|_{\overline{JN}_{(p,q,s,\alpha)}(X)} \). Indeed, for any cube \( Q \subset X \) and any \( f \in L^q_{\text{loc}}(X) \), from the linearity of \( P^{(s)}_Q \), \( P^{(s)}_Q((p^{(s)}_Q(f)) = P^{(s)}_Q(|f|) \), and (2.2), we deduce that

\[
2P^{(s)}_Q(f^-) = |P^{(s)}_Q(|f| - f)| = |P^{(s)}_Q(|f|) - P^{(s)}_Q(f)| = |P^{(s)}_Q(P^{(s)}_Q(|f|)) - P^{(s)}_Q(f)|
\]

Moreover, by (2.1), we obtain

\[
\int_Q |f^-| = \int_Q f^- = \int_Q P^{(s)}_Q(f^-) \leq \int_Q |P^{(s)}_Q(f^-)|
\]

Therefore, for any \( f \in \overline{JN}_{(p,q,s,\alpha)}(X) \) and any interior pairwise disjoint cubes \( \{Q_i\}_i \) of \( X \), from this, (2.4), and the Hölder inequality, it follows that

\[
\left\{ \sum_i |Q_i|^{1-p_{\alpha}} \left[ \int_{Q_i} |f^-| \right]^p \right\}^{\frac{1}{p}} \\
\leq \left\{ \sum_i |Q_i|^{1-p_{\alpha}} \left[ \int_{Q_i} |P^{(s)}_Q(f^-)| \right]^p \right\}^{\frac{1}{p}} \\
\leq \frac{C(s)}{2} \left\{ \sum_i |Q_i|^{1-p_{\alpha}} \left[ \int_{Q_i} |f - P^{(s)}_Q(|f|)| \right]^p \right\}^{\frac{1}{p}} \\
\leq \frac{C(s)}{2} \left\{ \sum_i |Q_i|^{1-p_{\alpha}} \left[ \int_{Q_i} |f - P^{(s)}_Q(|f|)|^q \right]^p \right\}^{\frac{1}{p}} \\
\leq \frac{C(s)}{2} \int |f|_{\overline{JN}_{(p,q,s,\alpha)}(X)} < \infty,
\]

which implies that

\[
\|f^-\|_{RM_{p,1,q,\alpha}(X)} \leq \frac{C(s)}{2} \|f\|_{\overline{JN}_{(p,q,s,\alpha)}(X)} < \infty
\]

and hence \( f^- \in RM_{p,1,q,\alpha}(X) \). This shows that the above claim holds true.

Moreover, using Proposition 2.7, we find that \( f \in JN_{(p,q,s,\alpha)}(X) \) and

\[
\|\cdot\|_{JN_{(p,q,s,\alpha)}(X)} \leq \|\cdot\|_{\overline{JN}_{(p,q,s,\alpha)}(X)}.
\]

To sum up, \( \|\cdot\|_{JN_{(p,q,s,\alpha)}(X)} + \|\cdot\|_{RM_{p,1,q,\alpha}(X)} \leq \|\cdot\|_{\overline{JN}_{(p,q,s,\alpha)}(X)} \) holds true.

It remains to prove \( \|\cdot\|_{\overline{JN}_{(p,q,s,\alpha)}(X)} \leq \left[ \|\cdot\|_{JN_{(p,q,s,\alpha)}(X)} + \|\cdot\|_{RM_{p,1,q,\alpha}(X)} \right] \). For any \( f \in JN_{(p,q,s,\alpha)}(X) \) with \( f^- \in RM_{p,1,q,\alpha}(X) \), and any interior pairwise disjoint cubes \( \{Q_i\}_i \) of \( X \), from the Minkowski...
inequality, the linearity of \( P^{(s)}_{\tilde{Q}} \), and (2.2), we deduce that

\[
\left\{ \sum_{i} |Q_i|^{-\alpha} \left| \int_{Q_i} \left| f - P^{(s)}_{\tilde{Q}}(f) \right|^q \right|^p \right\}^{1/p}
\leq \left\{ \sum_{i} |Q_i|^{1-p\alpha} \left[ \left\{ \int_{Q_i} \left| f - P^{(s)}_{\tilde{Q}}(f) \right|^q \right\}^{\frac{p}{q}} \right] \right\}^{\frac{1}{p}}
\leq \left\{ \sum_{i} |Q_i|^{1-p\alpha} \left[ \left\{ \int_{Q_i} \left| f - P^{(s)}_{\tilde{Q}}(f) \right|^q \right\}^{\frac{p}{q}} \right] \right\}^{\frac{1}{p}}
\leq \left\{ \sum_{i} |Q_i|^{1-p\alpha} \left[ \left\{ \int_{Q_i} \left| f - P^{(s)}_{\tilde{Q}}(f) \right|^q \right\}^{\frac{p}{q}} \right] \right\}^{\frac{1}{p}}
\leq \|f\|_{JN_{(p,q,s,\alpha)}(X)} + 2C(s) \left\{ \sum_{i} |Q_i|^{1-p\alpha} \left[ \left\{ \int_{Q_i} |f| \right\}^{\frac{p}{q}} \right] \right\}^{\frac{1}{p}}
\leq \|f\|_{JN_{(p,q,s,\alpha)}(X)} + 2C(s)\|f\|_{\text{RM}_{p,1,q}(X)} < \infty,
\]

which further implies that

(2.6) \[\|f\|_{\tilde{JN}_{(p,q,s,\alpha)}(X)} \leq \|f\|_{JN_{(p,q,s,\alpha)}(X)} + 2C(s)\|f\|_{\text{RM}_{p,1,q}(X)} < \infty\]

and hence \( f \in \tilde{JN}_{(p,q,s,\alpha)}(X) \). This shows that

\[\|\cdot\|_{\tilde{JN}_{(p,q,s,\alpha)}(X)} \leq \left[ \|\cdot\|_{JN_{(p,q,s,\alpha)}(X)} + \|\cdot\|_{\text{RM}_{p,1,q}(X)} \right],\]

which completes the proof of Theorem 2.9. \( \Box \)

**Remark 2.10.** We have the following observations on Theorem 2.9

(i) This theorem gives an equivalent characterization of \( \tilde{JN}_{(p,q,s,\alpha)}(X) \).

(ii) Let \( p \in (1, \infty) \), \( q \in [p, \infty) \), \( \alpha = 0 \), \( s \in \mathbb{Z}_+ \), and \( Q_0 \) be a cube of \( \mathbb{R}^n \). By Remark 2.6 (iii), we have

\[\text{RM}_{p,1,q}(Q_0) = L^q(Q_0) \supset L^{q/2}(Q_0) = L^{q/2}_{\text{loc}}(Q_0) \supset JN_{(p,q,s,\alpha)}(Q_0).\]

For any \( f \in JN_{(p,q,s,\alpha)}(Q_0) \), from this, we deduce that \( f^{-} \in \text{RM}_{p,1,0}(Q_0) \). By this and Theorem 2.9, we have

\[\tilde{JN}_{(p,q,s,\alpha)}(Q_0) = JN_{(p,q,s,\alpha)}(Q_0).\]

From this and [29, Proposition 2.5], it follows that

\[|Q_0|^{-\frac{1}{p}}\tilde{JN}_{(p,q,s,\alpha)}(Q_0) = |Q_0|^{-\frac{1}{p}}JN_{(p,q,s,\alpha)}(Q_0) = L^q(Q_0, |Q_0|^{-1}dx)/\mathcal{P}_s(Q_0).\]
Let $p, q \in [1, \infty)$, $s \in \mathbb{Z}_+$, $\alpha \in \left(\frac{1}{n}, \infty\right)$, and $Q_0$ be a cube of $\mathbb{R}^n$. Using both [29] Proposition 2.9 and [4] Lemma 3.1, we have $JN_{(p,q,s)}(Q_0) = P_s(Q_0)$. Moreover, by Remark 2.6, we find that $RM_{p,1,\sigma}(Q_0) = \{0\}$. Therefore, combining these and Theorem 2.9, we obtain
\[
\tilde{JN}_{(p,q,s)}(Q_0) = \{f \in P_s(Q_0) : f \geq 0\}.
\]

(iv) Observe that $C_{0,1,0}(X) = \text{BMO}(X)$ and also that $RM_{\infty,1,0} = L^\infty(X)$ due to Remark 2.6. From these and Theorem 2.9, we deduce that $f \in C_{0,1,0}(X)$ if and only if $f \in \text{BMO}(X)$ and $f^- \in L^\infty(X)$. This coincides with [30] Theorem 3.5.

3 Basic Properties and Characterizations via Maximal Functions

In this section, we first give some basic properties of the spaces $\tilde{JN}_{(p,q,s)}(X)$ and then characterize them when $s = 0$ via both maximal functions and their commutators.

**Proposition 3.1.** Let $p, q \in [1, \infty)$, $s \in \mathbb{Z}_+$, and $\alpha \in \mathbb{R}$. Then

(i) for any $p \in [p, \infty]$, $\tilde{JN}_{(p,q,s)}(Q_0) \subset \tilde{JN}_{(p,q,s)}(Q_0)$ and
\[
|Q_0|^{-1/p} \|JN_{(p,q,s)}(Q_0) \leq |Q_0|^{-1/p} \|JN_{(p,q,s)}(Q_0)\;
\]
(ii) for any $q \in [q, \infty]$, $\tilde{JN}_{(p,q,s)}(X) \subset \tilde{JN}_{(p,q,s)}(X)$ and
\[
\| \cdot \|JN_{(p,q,s)}(X) \leq \| \cdot \|JN_{(p,q,s)}(X)\;
\]
(iii) $\tilde{JN}_{(p,q,s+1)}(X) \subset \tilde{JN}_{(p,q,s)}(X)$ and
\[
\| \cdot \|JN_{(p,q,s+1)}(X) \leq \left[1 + C_{1,s+1}\right]\left[1 + C_{s+1}\right] + C_{s+1}C_{1,s+1} \| \cdot \|JN_{(p,q,s)}(X),
\]
where $C_{s+1} \in [1, \infty)$ is the same as in (2.2).

**Proof.** First, (i) follows from the Jensen inequality and (ii) follows from the Hölder inequality; we omit the details here.

It remains to prove (iii) of the present proposition. To this end, it suffices to show that (3.1) holds true. We only consider the case $p < \infty$ because the case $p = \infty$ can be similarly proved.

Let $p \in [1, \infty)$, $q \in [1, \infty]$, and $\alpha \in \mathbb{R}$. For any $s \in \mathbb{Z}_+$, we claim that
\[
\| \cdot \|JN_{(p,q,s+1)}(X) \leq \left[1 + C_{1,s+1}\right]\| \cdot \|JN_{(p,q,s)}(X),
\]
Indeed, for any $f \in L^q_{\text{loc}}(X)$ and any interior pairwise disjoint cubes $\{Q_i\}_i$ of $X$, by the Minkowski inequality, $P_{Q_i}^s(f) = P_{Q_i}^{s+1} P_{Q_i}^s(f)$, the linearity of $P_{Q_i}^{s+1}$, and (2.2), we have
\[
\left\{ \sum_i |Q_i| \left| \int_{Q_i} \left| \int_{Q_i} \left| f - P_{Q_i}^{s+1}(f) \right|^q \right|^\frac{1}{q} \right|^p \right\}^\frac{1}{p}.
Let \( \text{Proposition 3.2.} \) which implies that (3.1) holds true. This finishes the proof of Proposition 3.1.

Now, for any \( s \in \mathbb{Z}_+ \) and \( f \in L^q_{\text{loc}}(X) \), from (2.6), the above claim, Proposition 2.7, and (2.5), we deduce that

\[
\|f\|_{\tilde{J}N_{(p,q,s),h}(X)} \leq \|f\|_{JN_{(p,q,s),h}(X)} + \sum_j |Q_j|^{-\alpha} \left\{ \left( \int_{Q_j} \left| f - P_{Q_j}^{(s)}(f) \right|^q \right)^{\frac{1}{q}} \right\}^\frac{1}{p}.
\]

which implies that (3.1) holds true. This finishes the proof of Proposition 3.1.

**Proposition 3.2.** Let \( \alpha \in \mathbb{R}, q \in [1, \infty), \) and \( s \in \mathbb{Z}_+. \) Then, for any

\[ f \in \bigcup_{r \in [1, \infty)} \bigcap_{p \in [r, \infty)} \tilde{J}N_{(p,q,s),h}(X), \]

it holds true that

\[
\lim_{p \to \infty} \|f\|_{\tilde{J}N_{(p,q,s),h}(X)} = \|f\|_{\tilde{C}_{\alpha,q,s}(X)}.
\]

**Proof.** Let \( \alpha, q, \) and \( s \) be the same as in the present proposition. Let \( \overline{Q} \) be a cube of \( X \) and \( \{Q_i\}_i \) a collection of interior pairwise disjoint cubes of \( X \) which contains \( \overline{Q} \) as its element. Then, for any \( p \in [1, \infty) \) and \( f \in L^q_{\text{loc}}(X), \)

\[
\|f\|_{\tilde{J}N_{(p,q,s),h}(X)} \geq \sum_i |Q_i|^{-\alpha} \left( \int_{Q_i} \left| f - P_{Q_i}^{(s)}(f) \right|^q \right)^{\frac{1}{q}} \geq |\overline{Q}|^{-\alpha} \left( \int_{\overline{Q}} \left| f - P_{\overline{Q}}^{(s)}(f) \right|^q \right)^{\frac{1}{q}}.
\]
Thus,

$$\liminf_{p \to \infty} \|f\|_{\widetilde{JN}_{(p,q,s)}(X)} \geq |\Phi|^{-\alpha} \left( \int_{\Phi} |f - P_{\Phi}^{(s)}(f)|^q \right)^{\frac{1}{q}}.$$ 

By the arbitrariness of $\Phi$, we obtain \( \liminf_{p \to \infty} \|f\|_{\widetilde{JN}_{(p,q,s)}(X)} \geq \|f\|_{\tilde{C}_{a,q,s}(X)} \) for any \( f \in L_{loc}^q(X) \).

Now, let \( f \in \bigcup_{r \in [1,\infty)} \bigcap_{p \in [r,\infty)} \widetilde{JN}_{(p,q,s)}(X) \). Then, for any \( f \in \widetilde{JN}_{(p,q,s)}(X) \) for any \( p \in [r_0, \infty) \). We now show that

\[
(3.3) \quad \limsup_{p \to \infty} \|f\|_{\widetilde{JN}_{(p,q,s)}(X)} \leq \|f\|_{\tilde{C}_{a,q,s}(X)}.
\]

Indeed, if \( \|f\|_{\tilde{C}_{a,q,s}(X)} = \infty \), then (3.3) holds true trivially. If \( \|f\|_{\tilde{C}_{a,q,s}(X)} \) is finite, then, without loss of generality, we may assume \( \|f\|_{\tilde{C}_{a,q,s}(X)} = 1 \) because \( \| \cdot \|_{\widetilde{JN}_{(p,q,s)}(X)} \) and \( \| \cdot \|_{\tilde{C}_{a,q,s}(X)} \) are positively homogeneous. Then, for any \( p \in [r_0, \infty) \), we have

\[
\|f\|_{\widetilde{JN}_{(p,q,s)}(X)}^p = \sup_{i} \left( |Q_i|^{-\alpha} \left( \int_{Q_i} |f - P_{Q_i}^{(s)}(f)|^q \right)^{\frac{1}{q}} \right)^p \leq \sup_{i} \left( |Q_i|^{-\alpha} \left( \int_{Q_i} |f - P_{Q_i}^{(s)}(f)|^q \right)^{\frac{1}{q}} \right)^{r_0} \leq \|f\|_{\widetilde{JN}_{(p,q,s)}(X)}^r,
\]

where the suprema are taken over all the collections of interior pairwise disjoint cubes \( \{Q_i\} \) of \( X \) and the first inequality holds true because

\[
|Q_i|^{-\alpha} \left( \int_{Q_i} |f - P_{Q_i}^{(s)}(f)|^q \right)^{\frac{1}{q}} \leq \|f\|_{\tilde{C}_{a,q,s}(X)} = 1.
\]

Letting \( p \to \infty \), we then obtain

\[
\limsup_{p \to \infty} \|f\|_{\widetilde{JN}_{(p,q,s)}(X)} \leq 1 = \|f\|_{\tilde{C}_{a,q,s}(X)},
\]

and hence (3.3) holds true. This finishes the proof of Proposition 3.2 \( \square \)

**Remark 3.3.** Let \( \alpha, q, \) and \( s \) be the same as in Proposition 3.2. From Proposition 3.2, it follows that, if \( f \in \bigcup_{r \in [1,\infty)} \bigcap_{p \in [r,\infty)} \widetilde{JN}_{(p,q,s)}(X) \) and \( \liminf_{p \to \infty} \|f\|_{\widetilde{JN}_{(p,q,s)}(X)} < \infty \), then

\[
f \in \tilde{C}_{a,q,s}(X) \quad \text{and} \quad \|f\|_{\tilde{C}_{a,q,s}(X)} = \lim_{p \to \infty} \|f\|_{\widetilde{JN}_{(p,q,s)}(X)}.
\]

**Corollary 3.4.** Let \( q \in [1,\infty), \alpha \in \mathbb{R}, s \in \mathbb{Z}_+, \) and \( Q_0 \) be a cube of \( \mathbb{R}^n \). Then

\[
\tilde{C}_{a,q,s}(Q_0) = \left\{ f \in \bigcap_{p \in [1,\infty)} \widetilde{JN}_{(p,q,s)}(Q_0) : \lim_{p \to \infty} \|f\|_{\widetilde{JN}_{(p,q,s)}(Q_0)} < \infty \right\}
\]

and, for any \( f \in \tilde{C}_{a,q,s}(Q_0) \),

\[
\|f\|_{\tilde{C}_{a,q,s}(Q_0)} = \lim_{p \to \infty} \|f\|_{\widetilde{JN}_{(p,q,s)}(Q_0)}.
\]
Proof. Let $q$, $\alpha$, and $s$ be the same as in the present corollary. For any given
\[ f \in \bigcap_{p \in [1, \infty)} JN_{(p,q)}(Q_0) \subset \bigcup_{r \in [1, \infty)} \bigcap_{p \in [r, \infty)} \tilde{JN}_{(p,q)}(Q_0) \]
with $\lim_{p \to \infty} \|f\|_{JN_{(p,q)}(Q_0)} < \infty$, by Remark 3.3 we find that $f \in \tilde{C}_{\alpha,q}(Q_0)$ and $\|f\|_{\tilde{C}_{\alpha,q}(Q_0)} = \lim_{p \to \infty} \|f\|_{JN_{(p,q)}(Q_0)}$.

Conversely, for any given $f \in \tilde{C}_{\alpha,q}(Q_0) = \tilde{JN}_{(\infty,q)}(Q_0)$, from Proposition 3.1(i), it follows that $f \in \tilde{JN}_{(p,q)}(Q_0)$ for any $p \in [1, \infty)$, which implies that
\[ f \in \bigcap_{p \in [1, \infty)} JN_{(p,q)}(Q_0) \subset \bigcup_{r \in [1, \infty)} \bigcap_{p \in [r, \infty)} \tilde{JN}_{(p,q)}(Q_0). \]
Therefore, by Proposition 3.2, we obtain
\[ \lim_{p \to \infty} \|f\|_{JN_{(p,q)}(Q_0)} = \|f\|_{\tilde{C}_{\alpha,q}(Q_0)} < \infty. \]
This finishes the proof of Corollary 3.4. □

Proposition 3.5. Let $1 \leq q < p \leq \infty$, $s \in \mathbb{Z}_+$, and $\alpha \in [0, \infty)$. Then
\[ \tilde{JN}_{(p,q)}(X) = JN_{(p,1)}(X) \quad \text{and} \quad \|f\|_{\tilde{JN}_{(p,q)}(X)} \sim \|f\|_{JN_{(p,1)}(X)}. \]

Proof. Let $p$, $q$, $s$, and $\alpha$ be the same as the present proposition. We only consider the case $p < \infty$ because the case $p = \infty$ can be similarly proved.

By [29] Proposition 4.1, we have $JN_{(p,q)}(X) = JN_{(p,1)}(X)$ with equivalent norms. From this and Theorem 2.9 we deduce that, for any $f \in JN_{(p,1)}(X),$
\[ \|f\|_{\tilde{JN}_{(p,q)}(X)} \sim \|f\|_{JN_{(p,q)}(X)} + \|f\|_{RM_{p,1,q}}(X) \sim \|f\|_{JN_{(p,1)}(X)} + \|f\|_{RM_{p,1,q}}(X) < \infty. \]
This shows that $\tilde{JN}_{(p,1)}(X) \subset \tilde{JN}_{(p,q)}(X)$ and $\|\cdot\|_{\tilde{JN}_{(p,q)}(X)} \leq \|\cdot\|_{\tilde{JN}_{(p,1)}(X)}$.

On the other hand, for any $f \in \tilde{JN}_{(p,q)}(X)$, by the Hölder inequality, we have
\[ \|f\|_{\tilde{JN}_{(p,1)}(X)} \leq \|f\|_{\tilde{JN}_{(p,q)}(X)}. \]
This shows that $\tilde{JN}_{(p,q)}(X) \subset \tilde{JN}_{(p,1)}(X)$ and $\|\cdot\|_{\tilde{JN}_{(p,1)}(X)} \leq \|\cdot\|_{\tilde{JN}_{(p,q)}(X)}$, which completes the proof of Proposition 3.5. □

Remark 3.6. Let $\alpha = 0 = s$ and $q \in [1, \infty)$. By Proposition 3.5 we have
\[ \tilde{C}_{0,q,0}(Q_0) = \tilde{C}_{0,1,0}(Q_0) \quad \text{and} \quad \|\cdot\|_{\tilde{C}_{0,q,0}(Q_0)} \sim \|\cdot\|_{\tilde{C}_{0,1,0}(Q_0)}. \]
This coincides with [30] Theorem 3.3.
Remark 3.7. From Remark 2.10(iii), we deduce that $\widetilde{JN}_{(p,q,s)}(Q_0) = \widetilde{JN}_{(q,q,s)}(Q_0)$ for any given $p \in (1, \infty), q \in [p, \infty)$, and $s \in \mathbb{Z}_+$. By this and Proposition 3.3, we conclude that, for any $p \in (1, \infty)$ and $s \in \mathbb{Z}_+$,

$$\widetilde{JN}_{(p,q,s)}(Q_0) = \begin{cases} \widetilde{JN}_{(p,1,s)}(Q_0), & q \in [1, p), \\ \widetilde{JN}_{(q,q,s)}(Q_0), & q \in [p, \infty). \end{cases}$$

Moreover, from Proposition 3.11(i), it follows that, for any given $p \in [1, \infty), q \in [p, \infty), s \in \mathbb{Z}_+$, $\alpha \in \mathbb{R}$, and any given cube $Q_0$ of $\mathbb{R}^n$, we have

$$\widetilde{JN}_{(q,q,s)}(Q_0) \subset \widetilde{JN}_{(p,q,s)}(Q_0) \quad \text{and} \quad |Q_0|^{-\frac{1}{q}} \cdot \|\tilde{JN}_{(q,q,s)}(Q_0) \leq |Q_0|^{-\frac{1}{q}} \cdot \|\tilde{JN}_{(p,q,s)}(Q_0)\|.$$

Now, we further investigate the case $s = 0$ because, in this case, $P_Q^0(|f|) = |f|_Q$ is closely connected with the maximal function $M_Q(f)$. To this end, we first establish the following equivalent characterization.

Proposition 3.8. Let $p, q \in [1, \infty]$ and $\alpha \in \mathbb{R}$. Then $f \in \widetilde{JN}_{(p,q,0)}(X)$ if and only if $f \in L^q_{\text{loc}}(X)$ and

$$\sup \left\{ \sum_i |Q_i|^{1-pq} \inf_{c \in [0, \infty)} \|f - c\|_{L^q(Q_i \cap Q_{\alpha}\{c\})}^p \right\}^{\frac{1}{p}} < \infty,$$

where the infimum is taken over all non-negative real number and the supremum is taken over all the collections of interior pairwise disjoint cubes $\{Q_i\}_i$ of $X$. Moreover, for any $f \in L^q_{\text{loc}}(X)$, $\|f\|_{\widetilde{JN}_{(p,q,0)}(X)} \sim$ the left-hand side of (3.4) with the positive equivalence constants independent of $f$.

Proof. Since the case $p = \infty$ was obtained in [30, Theorem 3.6], it suffices to consider the case $p \in [1, \infty)$.

We first show that “only if” part. For any cube $Q \subset X$, by $P_Q^0(|f|) = |f|_Q \geq 0$, we conclude that

$$\inf_{c \in [0, \infty)} \|f - c\|_{L^q(Q \cap Q_{\alpha}\{c\})} = \inf_{c \in [0, \infty)} \left\{ \int_Q |f - c|^{\frac{q}{p}} \right\}^{\frac{1}{q}} \leq \left\{ \int_Q |f - |f|_Q|^{\frac{q}{p}} \right\}^{\frac{1}{q}}.$$

Therefore, for any $f \in \widetilde{JN}_{(p,q,0)}(X) \subset L^q_{\text{loc}}(X)$, we have

$$\sup \left\{ \sum_i |Q_i|^{1-pq} \inf_{c \in [0, \infty)} \|f - c\|_{L^q(Q_i \cap Q_{\alpha}\{c\})}^p \right\}^{\frac{1}{p}} < \infty,$$

Next, we show that “if” part. Let $Q$ denote any cube of $X$. Then, for any given $f \in L^q(Q)$ and for any $c \in [0, \infty)$, we have

$$|c - |f|_Q| = |f|_Q - c \leq |f_Q - c| \leq \int_Q |f - c|.$$
From this, the Minkowski inequality, and the Hölder inequality, it follows that
\[
\left[ \int_Q |f - |f||^q \right]^\frac{1}{q} \\
\leq \left[ \int_Q |f - c|^q \right]^\frac{1}{q} + \left[ \int_Q |c - |f||^q \right]^\frac{1}{q} \\
\leq \left[ \int_Q |f - c|^q \right]^\frac{1}{q} + \left[ \int_Q |f - c|^q \right]^\frac{1}{q} = 2 \left[ \int_Q |f - c|^q \right]^\frac{1}{q},
\]
which further implies that
\[
\left[ \int_Q |f - |f||^q \right]^\frac{1}{q} \leq 2 \inf_{c \in [0, \infty)} \|f - c\|_{L^q(Q; |Q|^{-1} dx)}.
\]
By this, we conclude that, for any \( f \in L^q_{\text{loc}}(\mathcal{X}) \),
\[
\|f\|_{\widetilde{JN}_{(p,q,s_0)}(\mathcal{X})} = \sup \left\{ \sum_i |Q_i| \left[ |Q_i|^{-\alpha} \left( \int_{Q_i} |f - |f||^q \right)^\frac{1}{q} \right]^p \right\}^\frac{1}{p} \\
\leq \sup \left\{ \sum_i |Q_i|^{1-p\alpha} \left[ 2 \inf_{c \in [0, \infty)} \|f - c\|_{L^q(Q; |Q|^{-1} dx)} \right]^p \right\}^\frac{1}{p} \\
= 2 \sup \left\{ \sum_i |Q_i|^{1-p\alpha} \inf_{c \in [0, \infty)} \|f - c\|_{L^q(Q; |Q|^{-1} dx)}^p \right\}^\frac{1}{p} < \infty,
\]
which implies \( f \in \widetilde{JN}_{(p,q,s_0)}(\mathcal{X}) \). This finishes the proof of Proposition 3.8. \( \square \)

**Remark 3.9.** (i) Proposition 3.8 implies that, for any \( c_0 \in [0, \infty) \),
\[
\| \cdot \|_{\widetilde{JN}_{(p,q,s_0)}(\mathcal{X})} \sim \| \cdot + c_0 \|_{\widetilde{JN}_{(p,q,s_0)}(\mathcal{X})}.
\]
(ii) It is still unclear whether or not Proposition 3.8 holds true for \( s \in \mathbb{N} \), namely, whether or not \( f \in \widetilde{JN}_{(p,q,s_0)}(\mathcal{X}) \) if and only if
\[
\sup \left\{ \sum_i |Q_i|^{1-p\alpha} \inf_{m \in P_i(Q); m \geq 0} \|f - m\|_{L^q(Q; |Q|^{-1} dx)}^p \right\}^\frac{1}{p} < \infty.
\]
Indeed, the “if” part can be similarly proved as in Proposition 3.8. But the “only if” part is no longer obvious because \( P^{(s)}_Q(\{f\}) \geq 0 \) may be false when \( s > 0 \). For instance, let \( Q := [-1, 1] \), \( a \in (0, 1] \), and \( f(x) := aI_{[0,1/a]}(x) \) for any \( x \in Q \). Then it is easy to find that
\[
P^{(1)}_Q(|f|)(x) = P^{(1)}_Q(f)(x) = \frac{3}{4a} x + \frac{1}{2}
\]
and hence \( P^{(1)}_Q(|f|)(x) < 0 \) when \( x \in [-1, -\frac{2a}{3}] \).
In [2] Proposition 4, Bastero et al. showed that \( f \in \text{BMO}(\mathbb{R}^n) \) and \( f^− \in L^\infty(\mathbb{R}^n) \) if and only if
\[
\sup_Q \int_Q |f - \mathcal{M}_Q(f)|^q < \infty,
\]
where \( q \in [1, \infty) \), \( \mathcal{M}_Q \) is the same as in [11], and the supremum is taken over all cubes \( Q \) of \( X \). Correspondingly, we have the following equivalence, which is the second main result of this article.

**Theorem 3.10.** Let \( 1 \leq q < p \leq \infty \) and \( \alpha \in [0, \infty) \). Then \( f \in \widetilde{J}\mathcal{N}_{(p,q,0)_\alpha}(X) \) if and only if \( f \in L^q_{\text{loc}}(X) \) and
\[
\text{(3.6)} \quad \sup \left\{ \sum_i |Q_i| \left( \int_{Q_i} |f - \mathcal{M}_{Q_i}(f)|^q \right)^{1/q} \right\} < \infty,
\]
where the supremum is taken over all the collections of interior pairwise disjoint cubes \( \{Q_i\}_i \) of \( X \). Moreover, for any \( f \in L^q_{\text{loc}}(X) \), \( \|f\|_{\widetilde{J}\mathcal{N}_{(p,q,0)_\alpha}(X)} \) is the left-hand side of (3.6) with the positive equivalence constants independent of \( f \).

**Proof.** We first show that “only if” part by considering the following two cases.

Case (i) \( 1 < q < p \leq \infty \). In the case, for any cube \( Q \) of \( X \) and for any non-negative constant \( c \), we have
\[
\text{(3.7)} \quad \left| \int_Q f - \mathcal{M}_Q(f) \right|^q \leq \left| \int_Q f - c \right|^q + \left| \int_Q c - \mathcal{M}_Q(f) \right|^q.
\]
In addition, from \( c \geq 0 \), it follows that, for any \( x \in Q \),
\[
| \mathcal{M}_Q(f)(x) - c | = \sup_{\text{cube } Q : x \in Q \subset Q} \int_Q |f(y)| dy - c = \sup_{\text{cube } Q : x \in Q \subset Q} \int_Q |f(y)| dy \leq \sup_{\text{cube } Q : x \in Q \subset Q} \int_Q |f(y) - c| dy = \mathcal{M}_Q(f - c)(x),
\]
which, together with the boundedness of \( \mathcal{M}_Q \) on \( L^q(Q) \) (see, for instance, [11] Theorem 2.2)), further implies that
\[
\text{(3.8)} \quad \left| \int_Q |\mathcal{M}_Q(f) - c|^q \right|^{1/q} \leq \left| \int_Q |\mathcal{M}_Q(f)|^q \right|^{1/q} \leq C_{(q,n)} \left| \int_Q |f - c|^q \right|^{1/q},
\]
where the positive constant \( C_{(q,n)} \) depends only on both \( q \) and \( n \). Thus, by (3.7) and (3.8), we obtain
\[
\left| \int_Q |f - \mathcal{M}_Q(f)|^q \right|^{1/q} \leq \left[ 1 + C_{(q,n)} \right] \left| \int_Q |f - c|^q \right|^{1/q},
\]
which, combined with the arbitrariness of $c \geq 0$, further implies that

$$
\left[ \frac{1}{f_Q} \left| f - M_Q(f) \right|^q \right]^\frac{1}{q} \leq \left[ 1 + C_{(q,n)} \right] \inf_{c \in [0,\infty)} \left[ \frac{1}{f_Q} \left| f - c \right|^q \right]^\frac{1}{q} = \left[ 1 + C_{(q,n)} \right] \inf_{c \in [0,\infty)} \| f - c \|_{L^q(Q)}^{1/2}.
$$

From this, it follows that, for any $f \in \overline{JN}_{(p,q,0)_0}(X)$ and any interior pairwise disjoint cubes $\{Q_i\}_i$ of $X$,

$$
\left\{ \sum_i |Q_i| \left| |Q_i|^{-\alpha} \left( \int_{Q_i} \left| f - M_{Q_i}(f) \right|^q \right)^{\frac{1}{q}} \right| \right\}^{\frac{1}{p}} \leq \left[ 1 + C_{(q,n)} \right] \left[ \sum_i |Q_i|^{1-p\alpha} \inf_{c \in [0,\infty)} \| f - c \|_{L^p(Q_i)}^{1} \right]^{\frac{1}{p}}.
$$

By this and (3.5), we conclude that

$$
\sup \left\{ \sum_i |Q_i| \left| |Q_i|^{-\alpha} \left( \int_{Q_i} \left| f - M_{Q_i}(f) \right|^q \right)^{\frac{1}{q}} \right| \right\}^{\frac{1}{p}} \leq \left[ 1 + C_{(q,n)} \right] \sup \left[ \sum_i |Q_i|^{1-p\alpha} \inf_{c \in [0,\infty)} \| f - c \|_{L^p(Q_i)}^{1} \right]^{\frac{1}{p}} \leq \left[ 1 + C_{(q,n)} \right] \| f \|_{\overline{JN}_{(p,q,0)_0}(X)} < \infty.
$$

Case (ii) $1 = q < p \leq \infty$. In the case, let $r := \frac{1+\min[p,3]}{2} \in (1, p)$. Then, by the Hölder inequality, Case (i) with $q$ replaced by $r$, and Proposition (3.5) we have, for any $f \in \overline{JN}_{(p,1,0)_0}(X)$

$$
\sup \left\{ \sum_i |Q_i| \left| |Q_i|^{-\alpha} \left( \int_{Q_i} \left| f - M_{Q_i}(f) \right|^r \right)^{\frac{1}{r}} \right| \right\}^{\frac{1}{p}} \leq \sup \left\{ \sum_i |Q_i| \left| |Q_i|^{-\alpha} \left( \int_{Q_i} \left| f - M_{Q_i}(f) \right|^q \right)^{\frac{1}{q}} \right| \right\}^{\frac{1}{p}} \leq \left[ 1 + C_{(q,n)} \right] \| f \|_{\overline{JN}_{(p,1,0)_0}(X)} \leq \| f \|_{\overline{JN}_{(p,1,0)_0}(X)} < \infty.
$$

Combining both Cases (i) and (ii), we find that the “only if” part holds true.

Next, we show that “if” part by considering the following two cases.

Case (i) $1 = q < p \leq \infty$. In this case, let $f \in L^p_{\infty}(X)$, $Q$ be a given cube of $X$, $E := \{ x \in Q : f(x) \leq f_Q \}$, and $F := \{ x \in Q : f(x) > f_Q \}$. The following equality is trivially true:

(3.9) \[
\int_E |f - f_Q| = \int_F |f - f_Q|.
\]
Combining this and the observation \( f(x) \leq f_Q \leq M_Q(f)(x) \) for any \( x \in E \), we obtain
\[
\int_Q |f - f_Q| = \frac{2}{|Q|} \int_E |f - f_Q| \leq \frac{2}{|Q|} \int_E |f - M_Q(f)| \leq 2 \int_Q |f - M_Q(f)|. \tag{3.10}
\]
Moreover, notice that \( M_Q(f)(x) \geq |f(x)| \) for almost every \( x \in Q \). Thus, we have, for almost every \( x \in Q \),
\[
0 \leq f^-(x) \leq M_Q(f)(x) - f^+(x) + f^-(x) = M_Q(f)(x) - f(x),
\]
which implies that
\[
f^-_Q = \int_Q |f^-| \leq \int_Q |f - M_Q(f)|. \tag{3.11}
\]
From (3.10) and (3.11), it follows that
\[
\int_Q |f - |f_Q|| = \int_Q |f - f_Q - 2f^-| \leq \int_Q |f - f_Q| + 2f^- \leq 4 \int_Q |f - M_Q(f)|.
\]
By this, we conclude that, for any interior pairwise disjoint cubes \( \{Q_i\} \) of \( X \),
\[
\left\{ \sum_i |Q_i| \left[ |Q_i|^{-\alpha} \left\{ \int_{Q_i} |f - |f_Q|| \right\}^p \right]^{\frac{1}{p}} \right\} \leq 4 \left\{ \sum_i |Q_i| \left[ |Q_i|^{-\alpha} \left\{ \int_{Q_i} |f - M_Q(f)| \right\}^p \right]^{\frac{1}{p}} \right\},
\]
which implies that
\[
\|f\|_{\widetilde{J}_N(p,1,0,\alpha)(X)} \leq 4 \sup \left\{ \sum_i |Q_i| \left[ |Q_i|^{-\alpha} \left\{ \int_{Q_i} |f - M_Q(f)| \right\}^p \right]^{\frac{1}{p}} \right\} < \infty
\]
and hence \( f \in \widetilde{J}_N(p,1,0,\alpha)(X) \).

Case (ii) \( 1 < q < p \leq \infty \). In this case, for any \( f \in L^q_{\text{loc}}(X) \), from Proposition 3.5 Case (i), and the Hölder inequality, we deduce that
\[
\|f\|_{\widetilde{J}_N(p,q,0,\alpha)(X)} \leq \|f\|_{\widetilde{J}_N(p,1,0,\alpha)(X)} \leq \sup \left\{ \sum_i |Q_i| \left[ |Q_i|^{-\alpha} \left\{ \int_{Q_i} |f - M_Q(f)| \right\}^p \right]^{\frac{1}{p}} \right\} \leq \sup \left\{ \sum_i |Q_i| \left[ |Q_i|^{-\alpha} \left\{ \int_{Q_i} |f - M_Q(f)|^q \right\}^{\frac{q}{p}} \right]^{\frac{1}{q}} \right\} < \infty
\]
and hence \( f \in \widetilde{J}_N(p,q,0,\alpha)(X) \). Combining Cases (i) and (ii), we find that the “if” part also holds true. This finishes the proof of Theorem 3.10. \( \square \)
Remark 3.11. We prove Theorem [3.10] under the assumptions on $p$, $q$, and $\alpha$ same as in Proposition [3.5]. This is reasonable because, based on the following two observations, it is necessary to apply the equivalence of $\tilde{JN}_{(p,q,0)}(X)$ between $q = 1$ and $q \in (1, \infty)$ in the proof of Theorem [3.10]:

(i) The maximal operator is bounded when $q \in (1, \infty)$, but not bounded when $q = 1$;

(ii) The domination of the mean oscillation of functions by their maximal functions, namely, (3.10), follows essentially from the linearity of the integral, namely, (3.9). This linearity holds true only when $q = 1$.

Indeed, when proving Theorem [3.10] we flexibly change the index $q$ such that all these $q$ correspond to the same space, which is just the conclusion of Proposition [3.5].

In what follows, we still use $M$ to denote the Hardy–Littlewood maximal operator on $X$ even when $X = Q_0$. Namely, for any $f \in L^1(Q_0)$ and $x \in Q_0$,

$$M(f)(x) := \sup_{\text{cube } Q.: x \in Q, cQ_0} \int_Q |f(y)| \, dy.$$ 

Based on this, even when $X = Q_0$, we still use $[b, M]$ as in (1.2) to denote the commutator generated by $M$ defined on $Q_0$.

Corollary 3.12. Let $p, q \in [1, \infty]$ and $\alpha \in \mathbb{R}$.

(I) The following three statements are mutually equivalent:

(i) $f \in \tilde{JN}_{(p,q,0)}(X)$;

(ii) $f \in JN_{(p,q,0)}(X)$ and $f^- \in RM_{p,1,\alpha}(X)$;

(iii) $f \in L^q_{\text{loc}}(X)$ and

$$\sup \left\{ \sum_i |Q_i|^{1-p\alpha} \inf_{c \in [0,\infty)} \|f - c\|_{L^1(Q_i;|Q_i|^{-1}dx)}^p \right\}^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all the collections of interior pairwise disjoint cubes $\{Q_i\}$ of $X$.

Moreover, for any $f \in L^q_{\text{loc}}(X)$, $\|f\|_{\tilde{JN}_{(p,q,0)}(X)} \sim \|f\|_{JN_{(p,q,0)}(X)} + \|f^-\|_{RM_{p,1,\alpha}(X)} \sim$ the left-hand side of (3.12) with the positive equivalence constants independent of $f$.

(II) The following two statements are mutually equivalent:

(iv) $f \in L^q_{\text{loc}}(X)$ and

$$\sup \left\{ \sum_i |Q_i|^{1-a} \left\{ \int_{Q_i} \left| f - M_{Q_i}(f) \right|^q \right\}^{\frac{1}{q}} \right\}^{\frac{1}{p}} < \infty,$$

where the supremum is taken the same as in (iii);
(v) \( f \in L^q_{\text{loc}}(X) \) and

\[
\left(3.14\right) \quad \sup \left\{ \sum_{i} |Q_i|^{-\alpha} \left( \frac{\int_{Q_i} [f - M_Q(f)]^q}{\|f \cdot M_Q(1_{Q_i})\|^q} \right)^{\frac{1}{q}} \right\}^p < \infty,
\]

where the supremum is taken the same as in (iii).

Moreover, for any \( f \in L^q_{\text{loc}}(X) \), the left-hand sides of both \( 3.13 \) and \( 3.14 \) are equivalent with the positive equivalence constants independent of \( f \).

(III) If \( q < p \) and \( \alpha \geq 0 \), then all these five statements in both (I) and (II) are mutually equivalent and so do the corresponding quantities with the positive equivalence constants independent of \( f \).

**Proof.** Let \( p, q, \) and \( \alpha \) be the same as in the present corollary. By Theorem \( 2.9 \) we find that (i) \( \iff \) (ii). From Proposition \( 3.8 \) it follows that (i) \( \iff \) (iii). Moreover, Theorem \( 3.10 \) shows that (i) \( \iff \) (iv) when \( q < p \) and \( \alpha \geq 0 \). Therefore, it suffices to prove (iv) \( \iff \) (v). Indeed, for any given cube \( Q \) of \( X \) and any \( x \in Q \), we have

\[ M_Q(1_{Q})(x) = 1_{Q}(x) \quad \text{and} \quad M(f 1_{Q})(x) = M_{Q}(f)(x), \]

and hence, for any \( q \in [1, \infty] \),

\[
\left[ \frac{\int_{Q} f - M_{Q}(f)}{\|f \cdot M_{Q}(1_{Q})\|^q} \right]^{\frac{1}{q}} = \left[ \frac{\int_{Q} [f - M_{Q}(1_{Q}) - M_{Q}(f 1_{Q})]^q}{\|f \cdot M_{Q}(1_{Q}) - M_{Q}(f 1_{Q})\|^q} \right]^{\frac{1}{q}}
\]

\[ = \left[ \frac{\int_{Q} [f - M(1_{Q})]^q}{\|f \cdot M(1_{Q})\|^q} \right]^{\frac{1}{q}} \]

By this, we conclude that

\[
\sup \left\{ \sum_{i} |Q_i|^{-\alpha} \left( \frac{\int_{Q_i} [f - M_Q(f)]^q}{\|f \cdot M_Q(1_{Q_i})\|^q} \right)^{\frac{1}{q}} \right\}^p
\]

\[ = \sup \left\{ \sum_{i} |Q_i|^{-\alpha} \left( \frac{\int_{Q_i} [f - M(1_{Q_i})]^q}{\|f \cdot M(1_{Q_i})\|^q} \right)^{\frac{1}{q}} \right\}^p,
\]

which shows the equivalence (iv) \( \iff \) (v). This finishes the proof of Corollary \( 3.12 \). \( \square \)

**Remark 3.13.** Let \( p = \infty \) and \( \alpha = 0 \). In this case, if \( q \in (1, \infty) \) and \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) is real-valued, then the boundedness of \([f, M]\) on \( L^q(\mathbb{R}^n) \) is equivalent to that \( 3.14 \) holds true, which was proved by Bastero et al. \( 2 \) via the real interpolation techniques.
4 John–Nirenberg-Type Inequality

In this section, we prove a good-λ inequality (namely, Lemma 4.3 below) and apply this good-λ inequality to prove a John–Nirenberg type inequality on $\tilde{J}N_{(p,q,s),0}(Q_0)$ (namely, Theorem 4.1 below) via borrowing some ideas from the proof of [29, Theorem 4.3]. Our main tool is the Calderón–Zygmund decomposition. As an application, we use this John–Nirenberg type inequality to obtain another proof of Proposition 3.5 at the end of this section.

In what follows, for any given $p \in [1, \infty)$, the weak Lebesgue space $L^{p,\infty}(X)$ is defined to be the set of all the measurable functions $f$ on $X$ such that

$$\|f\|_{L^{p,\infty}(X)} := \sup_{\lambda \in (0,\infty)} \lambda \|\{x \in X : |f(x)| > \lambda\}\|^\frac{1}{p} < \infty.$$ 

Now, we state the third main result of this article.

**Theorem 4.1.** Let $p \in (1, \infty)$, $s \in \mathbb{Z}_+$, $\alpha \in \mathbb{R}$, and $Q_0$ be a cube of $\mathbb{R}^n$. If $f \in \tilde{J}N_{(p,1,s),0}(Q_0)$, then $f - P_{Q_0}^{(s)}(|f|) \in L^{p,\infty}(Q_0)$ and there exists a positive constant $C_{(n,p,s)}$, depending only on $n$, $p$, and $s$, but independent of $f$, such that

$$\|f - P_{Q_0}^{(s)}(|f|)\|_{L^{p,\infty}(Q_0)} \leq C_{(n,p,s)}\|f\|_{\tilde{J}N_{(p,1,s),0}(Q_0)}.$$ 

Recall that, for any cube $Q$ and any $\ell \in \mathbb{Z}_+$,

$$D_Q^{(\ell)} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \text{for any } i \in \{1, \ldots, n\}, x_i \in \left[a_i + k_i 2^{-\ell} l(Q), a_i + (k_i + 1) 2^{-\ell} l(Q)\right] \text{ with } k_i \in \{0, \ldots, 2^\ell - 2\} \text{ or } x_i \in \left[a_i + (1 - 2^{-\ell}) l(Q), a_i + l(Q)\right],$$

where $l(Q)$ denotes the edge length of $Q$ and $(a_1, \ldots, a_n)$ is a left and lower vertex of $Q$, which means that, for any $(x_1, \ldots, x_n) \in Q$, $x_i \geq a_i$ for any $i \in \{1, \ldots, n\}$. Moreover, the dyadic family $D_Q$ on $Q$ is defined by setting

$$D_Q := \bigcup_{\ell \in \mathbb{Z}_+} D_Q^{(\ell)}.$$ 

In what follows, for any cube $Q$ of $\mathbb{R}^n$, $\mathcal{M}_{Q}^{(d)}$ denotes the dyadic maximal operator related to the dyadic family $D_Q$ on $Q$, namely, for any $f \in L^1(Q)$ and $x \in Q$,

$$\mathcal{M}_{Q}^{(d)}(f)(x) := \sup_{Q_{(x)}} \int_{Q_{(x)}} |f(y)| \, dy,$$

where the supremum is taken over all the dyadic cubes $Q_{(x)} \in D_Q$ containing $x$. The following Calderón–Zygmund decomposition, which is just [25, p.150, Lemma 1], is needed in the proof of Theorem 4.1.

**Lemma 4.2.** Let $Q_0$ be a cube of $\mathbb{R}^n$, $f \in L^1(Q_0)$, and $\lambda \geq \int_{Q_0} |f|$. Then there exist disjoint dyadic cubes $\{Q_k\}_k \subset D_{Q_0}$ such that

(i) $\{x \in Q_0 : \mathcal{M}_{Q_0}^{(d)}(f)(x) > \lambda\} = \bigcup_k Q_k.$
Indeed, for any \( x \in Q_0 \):

(4.5) \( \lambda < \int_{Q_0} |f| \leq 2^n \lambda \) for any \( k \in \mathbb{N} \);

(3) \( \| x \in Q_0 : \mathcal{M}^{(d)}(f)(x) > \lambda \| \leq \frac{1}{\lambda} \int_{x \in Q_0, \mathcal{M}^{(d)}(f)(x) > \lambda} |f|; \)

(4) \( f(x) \leq \lambda \) for almost every \( x \in Q_0 \setminus \bigcup_k Q_k \).

Now, we prove Theorem 4.1 via beginning with the following good-\lambda inequality. We borrow some ideas from the proof of [11 Lemma 4.5] (see also [29, Lemma 4.6]) with suitable modifications.

**Lemma 4.3.** Let \( p \in (1, \infty) \), \( s \in \mathbb{Z}_+ \), \( C(s) \in [1, \infty) \) be the same as in (4.2), \( \theta \in (0, 2^{-n}C(s)^{-1}) \), \( Q_0 \) be a cube of \( \mathbb{R}^n \), and \( f \in \mathcal{J}_N(p, 1, s)(Q_0) \). Then, for any \( \lambda \geq \frac{1}{\theta} \int_{Q_0} |f - P^{(s)}_{Q_0}(|f|)| \),

\[
\| x \in Q_0 : \mathcal{M}^{(d)}_{Q_0} \left( f - P^{(s)}_{Q_0}(|f|) \right)(x) > \lambda \| \leq \frac{1 + C(s)}{1 - 2^n \theta C(s)} \left( \frac{\| f \|_{\mathcal{J}_N(p, 1, s)(Q_0)}}{\lambda} \right)^{\frac{1}{p'}},
\]

where \( p' \) denotes the conjugate index of \( p \), that is, \( p' \) satisfies \( 1/p + 1/p' = 1 \).

**Proof.** Let all symbols be the same as in the present lemma. Applying Lemma 4.2 to \( f - P^{(s)}_{Q_0}(|f|) \) on \( Q_0 \) at height \( \theta \lambda \), we find interior pairwise disjoint dyadic cubes \( \{Q_k\}_k \subset \mathcal{D}_{\mathcal{Q}_0} \) such that

\[
\left\{ x \in Q_0 : \mathcal{M}^{(d)}_{Q_0} \left( f - P^{(s)}_{Q_0}(|f|) \right)(x) > \theta \lambda \right\} = \bigcup_k Q_k.
\]

Since \( \theta < 2^{-n}C(s)^{-1} < 1 \), it follows that

\[
\left\{ x \in Q_0 : \mathcal{M}^{(d)}_{Q_0} \left( f - P^{(s)}_{Q_0}(|f|) \right)(x) > \lambda \right\} \subset \left\{ x \in Q_0 : \mathcal{M}^{(d)}_{Q_0} \left( f - P^{(s)}_{Q_0}(|f|) \right)(x) > \theta \lambda \right\}
\]

and hence

\[
(4.3) \quad \left\{ x \in Q_0 : \mathcal{M}^{(d)}_{Q_0} \left( f - P^{(s)}_{Q_0}(|f|) \right)(x) > \lambda \right\} = \bigcup_k \left\{ x \in Q_k : \mathcal{M}^{(d)}_{Q_k} \left( f - P^{(s)}_{Q_0}(|f|) \right)(x) > \lambda \right\}.
\]

We now claim that, for any \( k \in \mathbb{N} \),

\[
(4.4) \quad \left\{ x \in Q_k : \mathcal{M}^{(d)}_{Q_0} \left( f - P^{(s)}_{Q_0}(|f|) \right)(x) > \lambda \right\} \subset \left\{ x \in Q_k : \mathcal{M}^{(d)}_{Q_k} \left( \left| f - P^{(s)}_{Q_0}(|f|) \right| \mathbf{1}_{Q_k} \right)(x) > \frac{[1 - 2^n \theta C(s)] \lambda}{1 + C(s)} \right\}.
\]

Indeed, for any \( x \in Q_k \) with \( \mathcal{M}^{(d)}_{Q_0} \left( f - P^{(s)}_{Q_0}(|f|) \right)(x) > \lambda \), by Lemma 4.2 we know that there exists a dyadic cube \( Q(s) \ni x \in \mathcal{D}_{Q_0} \) such that

\[
(4.5) \quad \int_{Q(s)} \left| f - P^{(s)}_{Q_0}(|f|) \right| > \lambda.
\]
Since $Q_k$ is the maximal dyadic cube satisfying $\int_Q |f - P_{Q_k}^d(|f|)| > \theta \lambda$, then we have $Q_{(x)} \subset Q_k$. From this and (4.3), it follows that

$$M_{Q_0}^{(d)} \left( \left[ f - P_{Q_0}^d(|f|) \right] 1_{Q_k} \right)(x) \geq \int_{Q_{(x)}} \left| f - P_{Q_0}^d(|f|) \right| > \lambda.$$

By this, $P_{Q_k}^d(P_{Q_0}^d(|f|)) = P_{Q_k}^d(|f|)$, $P_{Q_k}^d(P_{Q_0}^d(|f|)) = P_{Q_k}^d(|f|)$, the linearity of $P_{Q_k}^d$, (2.2), and Lemma 4.2(ii), we conclude that, for any $x \in Q_k$,

$$\lambda < M_{Q_0}^{(d)} \left( \left[ f - P_{Q_0}^d(|f|) \right] 1_{Q_k} \right)(x) = \sup_{Q_{(x)}} \int_Q \left[ \left| f - P_{Q_0}^d(|f|) \right| 1_{Q_k} \right](y) \, dy$$

$$\leq \sup_{Q_{(x)}} \int_Q \left[ f(y) - P_{Q_0}^d(|f|)(y) \right] 1_{Q_k}(y) \, dy + \int_Q \left| P_{Q_0}^d(|f|)(y) - P_{Q_0}^d(f)(y) \right| 1_{Q_k}(y) \, dy$$

$$+ \int_Q \left| P_{Q_k}^d(f - P_{Q_0}^d(|f|))(y) \right| 1_{Q_k}(y) \, dy$$

$$\leq M_{Q_0}^{(d)} \left( \left[ f - P_{Q_0}^d(|f|) \right] 1_{Q_k} \right)(x) + C_{(s)} \int_{Q_k} |f - P_{Q_0}^d(|f|) - C_{(s)} \int_{Q_k} |f - P_{Q_0}^d(|f|)|$$

$$\leq [1 + C_{(s)}] M_{Q_0}^{(d)} \left( \left[ f - P_{Q_0}^d(|f|) \right] 1_{Q_k} \right)(x) + 2^n \theta C_{(s)} \lambda.$$

This shows that the above claim (4.3) holds true.

Next, we prove that, for any $k$,

$$\left| \left\{ x \in Q_k : M_{Q_0}^{(d)} \left( f - P_{Q_0}^d(|f|) \right) (x) > \lambda \right\} \right| \leq \frac{1 + C_{(s)}}{1 - 2^n \theta C_{(s)} \lambda} \int_{Q_k} \left| f - P_{Q_0}^d(|f|) \right|.$$

Indeed, if $\int_{Q_k} \left| f - P_{Q_0}^d(|f|) \right| > \frac{1 - 2^n \theta C_{(s)} \lambda}{1 + C_{(s)}}$, then we have $|Q_k| < \frac{1 + C_{(s)}}{1 - 2^n \theta C_{(s)} \lambda} \int_{Q_k} \left| f - P_{Q_0}^d(|f|) \right|$ due to $\theta < 2^n C_{(s)}$. Thus,

$$\left| \left\{ x \in Q_k : M_{Q_0}^{(d)} \left( f - P_{Q_0}^d(|f|) \right) (x) > \lambda \right\} \right| \leq |Q_k| < \frac{1 + C_{(s)}}{1 - 2^n \theta C_{(s)} \lambda} \int_{Q_k} \left| f - P_{Q_0}^d(|f|) \right|,$$

and hence (4.6) holds true in this case. If $\int_{Q_k} \left| f - P_{Q_0}^d(|f|) \right| \leq \frac{1 - 2^n \theta C_{(s)} \lambda}{1 + C_{(s)}}$, then, applying Lemma 4.2 to $f - P_{Q_0}^d(|f|)$ on $Q_k$ at height $\frac{1 - 2^n \theta C_{(s)} \lambda}{1 + C_{(s)}}$, we obtain

$$\left| \left\{ x \in Q_k : M_{Q_0}^{(d)} \left( \left[ f - P_{Q_k}^d(|f|) \right] 1_{Q_k} \right) (x) > \lambda \right\} \right| \leq \frac{1 + C_{(s)}}{1 - 2^n \theta C_{(s)} \lambda} \int_{Q_k} \left| f - P_{Q_0}^d(|f|) \right|.$$
From this, \( \mathcal{M}_{Q_0}(\cdots 1_{Q_0}) = \mathcal{M}_{Q_0}(\cdots 1_{Q_0}) \), and (4.4), we deduce that
\[
\left| \left\{ x \in Q_k : \mathcal{M}_{Q_0}^{(d)} \left( f - P_{Q_0}^{(s)}(l_f) \right)(x) > \lambda \right\} \right| \\
\leq \left| \left\{ x \in Q_k : \mathcal{M}_{Q_0}^{(d)} \left( f - P_{Q_0}^{(s)}(l_f) \right) 1_{Q_k}(x) > \frac{1 - 2^n \theta C_{(s)} \lambda}{1 + C_{(s)}} \right\} \right| \\
= \left| \left\{ x \in Q_k : \mathcal{M}_{Q_0}^{(d)} \left( f - P_{Q_0}^{(s)}(l_f) \right) 1_{Q_k}(x) > \frac{1 - 2^n \theta C_{(s)} \lambda}{1 + C_{(s)}} \right\} \right| \\
\leq \frac{1 + C_{(s)}}{1 - 2^n \theta C_{(s)} \lambda} \int_{Q_k} \left| f - P_{Q_0}^{(s)}(l_f) \right|. 
\]
Thus, (4.6) holds true. From (4.3), (4.6), the Hölder inequality, and the construction of \( \{Q_k\}_k \), we deduce that
\[
\left| \left\{ x \in Q_0 : \mathcal{M}_{Q_0}^{(d)} \left( f - P_{Q_0}^{(s)}(l_f) \right)(x) > \lambda \right\} \right| \\
\leq \sum_k \left| \left\{ x \in Q_k : \mathcal{M}_{Q_0}^{(d)} \left( f - P_{Q_0}^{(s)}(l_f) \right)(x) > \lambda \right\} \right| \\
\leq \sum_k \frac{[1 + C_{(s)}]}{[1 - 2^n \theta C_{(s)} \lambda]} |Q_k|^\frac{1}{p} \left[ \sum_k |Q_k|^{-p} \left[ \int_{Q_k} \left| f - P_{Q_0}^{(s)}(l_f) \right| \right]^p \right]^{\frac{1}{p}} \left| \sum_k \right| \left| \int_{Q_k} \left| f - P_{Q_0}^{(s)}(l_f) \right| \right|^p \lesssim \lambda |f||\tilde{J}_N(p, \lambda, \eta_0)(Q_0)|, 
\]
which shows that (4.2) holds true. This finishes the proof of Lemma 4.3. 

Proof of Theorem 4.7 We first prove (4.1) for \( \alpha = 0 \). Let \( \theta := 2^{-i(n+1)}C_{(s)}^{-1} \), where \( C_{(s)} \) is the same as (2.2), and let \( \eta := \frac{|f||\tilde{J}_N(p, \lambda, \eta_0)(Q_0)|}{\theta |Q_0|^{1/p}} \). We show that
\[
\| f - P_{Q_0}^{(s)}(l_f) \|_{L^p(Q_0)} = \sup_{\lambda \in (0, \infty)} \lambda \left| \left\{ x \in Q_0 : \left| f - P_{Q_0}^{(s)}(l_f)(x) \right| > \lambda \right\} \right|^\frac{1}{p} \lesssim \| f \|_{\tilde{J}_N(p, \lambda, \eta_0)(Q_0)} 
\]
by considering the following two cases.

Case (i) \( \lambda \leq \eta \), namely, \( \lambda \leq \frac{|f||\tilde{J}_N(p, \lambda, \eta_0)(Q_0)|}{\theta |Q_0|^{1/p}} \). In this case, \( \lambda |Q_0|^\frac{1}{p} \leq 2^{n+1}C_{(s)}|f|\|f\|_{\tilde{J}_N(p, \lambda, \eta_0)(Q_0)} \) and hence
\[
\sup_{\lambda \in (0, \eta]} \lambda \left| \left\{ x \in Q_0 : \left| f - P_{Q_0}^{(s)}(l_f)(x) \right| > \lambda \right\} \right|^\frac{1}{p} \lesssim \sup_{\lambda \in (0, \eta]} \left| \lambda |Q_0|^\frac{1}{p} \right| \leq 2^{n+1}C_{(s)}|f|\|f\|_{\tilde{J}_N(p, \lambda, \eta_0)(Q_0)},
\]
which is the desired estimate in this case.

Case (ii) $\lambda > \eta$. In this case, by the definition of $\|f\|_{\tilde{N}_{p,1,\lambda}^{(d)}(Q_0)}$ and $\theta < 1$, we have

$$\lambda > \eta = \frac{\|f\|_{\tilde{N}_{p,1,\lambda}^{(d)}(Q_0)}}{\theta |Q_0|^{1/p}} \geq \frac{|Q_0|^{1/p} \int_{Q_0} |f - P_{Q_0}^{(s)}(|f|)|}{\theta |Q_0|^{1/p}} > \int_{Q_0} |f - P_{Q_0}^{(s)}(|f|)|.$$

Next, we show that

$$\sup_{\lambda \in (\eta, \infty)} \lambda \left| \int_{\mathcal{D}_0} M_{Q_0}^{(d)} \left( f - P_{Q_0}^{(s)}(|f|) \right) (x) > \lambda \right|^{1/p} \leq \|f\|_{\tilde{N}_{p,1,\lambda}^{(d)}(Q_0)}.$$

Let $j_0$ be the smallest non-negative integer such that $\theta^{-j_0} \eta < \lambda$. By (4.2), $\theta \lambda \leq \theta^{-j_0} \eta < \lambda$, and Lemma 4.2(iii), we have

$$\left| \int_{\mathcal{D}_0} M_{Q_0}^{(d)} \left( f - P_{Q_0}^{(s)}(|f|) \right) (x) > \lambda \right|^{1/p} \leq \frac{D_0}{\theta^{-j_0} \eta} \left| \int_{\mathcal{D}_0} M_{Q_0}^{(d)} \left( f - P_{Q_0}^{(s)}(|f|) \right) (x) > \theta^{-j_0} \eta \right|^{(p')-1} \times \cdots \times \left| \int_{\mathcal{D}_0} M_{Q_0}^{(d)} \left( f - P_{Q_0}^{(s)}(|f|) \right) (x) > \eta \right|^{(p')^{-j_0}} \leq \frac{D_0}{\theta^j \lambda} \left| \int_{\mathcal{D}_0} M_{Q_0}^{(d)} \left( f - P_{Q_0}^{(s)}(|f|) \right) (x) > \eta \right|^{(p')^{-j_0}}.$$

where $D_0 := 2 \|f\|_{\tilde{N}_{p,1,\lambda}^{(d)}(Q_0)}$. Observe that $|Q_0|^{1/p} \int_{Q_0} |f - P_{Q_0}^{(s)}(|f|)| \leq \|f\|_{\tilde{N}_{p,1,\lambda}^{(d)}(Q_0)}$. We obtain

$$\frac{1}{\eta} \int_{Q_0} |f - P_{Q_0}^{(s)}(|f|)| = \frac{\theta |Q_0|^{p'}}{\|f\|_{\tilde{N}_{p,1,\lambda}^{(d)}(Q_0)}} \int_{Q_0} |f - P_{Q_0}^{(s)}(|f|)| \leq \theta |Q_0|,$$
From this, \(1 + 2(p')^{-1} + \cdots + j_0(p')^{-j_0+1} = p^2[1 - (p')^{-j_0}] - p_j(p')^{-j_0} \leq p^2\), and

\[1 + (p')^{-1} + (p')^{-2} + \cdots + (p')^{-j_0+1} = p[1 - (p')^{-j_0}],\]

we deduce that

\[(4.8) \quad \left| \left\{ x \in Q_0 : \mathcal{M}_{Q_0}^{(d)} (f - P_{Q_0}^{(d)}(|f|)) (x) > \lambda \right\} \right| \]

\[\leq \left( \frac{1}{\theta} \right)^p \left\{ 2[1 + C(s)] \frac{\|f\|_{\mathcal{N}_{(p,1,j_0)}(Q_0)}}{\lambda} \right\}^{p[1-(p')^{-j_0}]}

\[= [2[1 + C(s)] \right\} \left( \frac{1}{\theta} \right)^p \left[ \frac{\|f\|_{\mathcal{N}_{(p,1,j_0)}(Q_0)}}{\lambda} \right]^{p[1-(p')^{-j_0}]}

\[\leq \left\{ 2[1 + C(s)] \right\} \left( \frac{1}{\theta} \right)^p \left[ \frac{\|f\|_{\mathcal{N}_{(p,1,j_0)}(Q_0)}}{\lambda} \right]^{p[1-(p')^{-j_0}]}

By the definitions of both \(\eta\) and \(j_0\), we have

\[(4.9) \quad \frac{\|f\|_{\mathcal{N}_{(p,1,j_0)}(Q_0)}}{\lambda} \leq \frac{\eta^{-j_0-1}\eta}{\theta} = \theta^{-j_0-2}.

We now claim that, for any \(j \in \mathbb{N}\),

\[(4.10) \quad (j + 2)(p')^{-j} \leq \max(p,2).

Indeed, for any \(j \in \mathbb{Z}_+\), let \(F(j) := (j + 2)(p')^{-j}\). Then \(F\) attains its maximal value at some \(j_1 \in \mathbb{Z}_+\). If \(j_1 = 0\), then \((j_1 + 2)(p')^{-j_1} = 2 \leq \max(p,2)\). If \(j_1 \in \mathbb{N}\), then

\[\frac{F(j_1 + 1)}{F(j_1)} = \frac{(j_1 + 1)(p')^{-j_1+1}}{(j_1 + 2)(p')^{-j_1}} \leq 1,

which implies that \(j_1 + 2 \leq \frac{1}{p'} + 1 = p\) and hence \((j_1 + 2)(p')^{-j_1} \leq p \leq \max(p,2)\). This proves (4.10). Therefore, by (4.8), (4.9), (4.10), and \(\theta = 2^{-(n+1)C_{(s)}}\), we conclude that

\[\left| \left\{ x \in Q_0 : \mathcal{M}_{Q_0}^{(d)} (f - P_{Q_0}^{(d)}(|f|)) (x) > \lambda \right\} \right|

\[\leq 2^p \left\{ 1 + C(s) \right\} \left[ \frac{1}{\theta} \right]^p \left[ \frac{\|f\|_{\mathcal{N}_{(p,1,j_0)}(Q_0)}}{\lambda} \right]^{p[p^2-p]} \]

\[= 2^p \left\{ 1 + C(s) \right\} \left[ \frac{1}{\theta} \right]^p \left[ \frac{\|f\|_{\mathcal{N}_{(p,1,j_0)}(Q_0)}}{\lambda} \right]^{p[p^2-p \max\{p,2\}]} \]

\[= 2^{p^2(n+1)p^2-p \max\{p,2\}} C_{(s)}^p \left[ 1 + C(s) \right]^p \left[ \frac{\|f\|_{\mathcal{N}_{(p,1,j_0)}(Q_0)}}{\lambda} \right]^{p[p^2-p \max\{p,2\}]}.\]
This implies that (4.7) holds true. Moreover, using Lemma 4.2 iv, we find that
\[ \left\{ x \in Q_0 : \left| f - P_Q^{(s)}(f) \right| > \alpha \right\} \subset \left\{ x \in Q_0 : M_{Q_0}^{(d)} \left( f - P_Q^{(s)}(f) \right)(x) > \alpha \right\}. \]
From this and (4.7), it follows that
\[ \sup_{\lambda \in (0, \infty)} \left| \left\{ x \in Q_0 : \left| f - P_Q^{(s)}(f) \right| > \lambda \right\} \right|^\frac{1}{p} \leq \| f \|_{\tilde{J}N_{(p,1,\alpha)}(Q_0)}. \]
Therefore, (4.11) for \( \alpha = 0 \) holds true by combining Case (i) and Case (ii) and letting
\[ C_{(n,p,s)} := \max \left\{ 2^{n+1} C_{(s)}, 2^{p+1(n+1)(p^2 - p \max|p,2|)} C_{(s)}^{p^2 - p \max|p,2|} \left( 1 + C_{(s)} \right)^p \right\}. \]
Finally, for any \( \alpha \in [0, \infty) \), by (4.11) with \( \alpha = 0 \), we find that
\[ \left\| f - P_Q^{(s)}(f) \right\|_{L^p(Q_0)} \]
\[ \leq C_{(n,p,s)} \| f \|_{\tilde{J}N_{(p,1,\alpha)}(Q_0)} = C_{(n,p,s)} \sup \left\{ \sum_i |Q_i| \left[ \left( \int_{Q_i} \left| f - P_Q^{(s)}(f) \right| \right)^p \right]^{\frac{1}{p}} \right\} \]
\[ \leq C_{(n,p,s)} Q_0^p \sup \left\{ \sum_i |Q_i| \left[ |Q_i|^{-\alpha} \left( \int_{Q_i} \left| f - P_Q^{(s)}(f) \right| \right)^p \right] \right\} \]
\[ \leq C_{(n,p,s)} Q_0^p \| f \|_{\tilde{J}N_{(p,1,\alpha)}(Q_0)}. \]
This finishes the proof of Theorem 4.1.

As an application of Theorem 4.1, we give another proof of Proposition 3.5 as follows.

Another Proof of Proposition 3.5. We only consider the case \( p < \infty \) because the case \( p = \infty \) can be similarly proved. For any \( f \in JN_{(p,q,s)}(X) \) with \( 1 \leq q < p < \infty \) and for any \( \alpha \in \mathbb{R} \), by the Hölder inequality, we obtain
\[ \| f \|_{\tilde{J}N_{(p,q,s)}(X)} = \sup \left\{ \sum_i |Q_i| \left[ |Q_i|^{-\alpha} \left( \int_{Q_i} \left| f - P_Q^{(s)}(f) \right| \right)^p \right]^{\frac{1}{p}} \right\} \]
\[ \leq \sup \left\{ \sum_i |Q_i| \left[ |Q_i|^{-\alpha} \left( \int_{Q_i} \left| f - P_Q^{(s)}(f) \right| \right)^q \right]^{\frac{1}{q}} \right\} \]
\[ = \| f \|_{\tilde{J}N_{(p,q,s)}(X)}. \]
This shows that \( \tilde{J}N_{(p,q,s)}(X) \subset \tilde{J}N_{(p,1,\alpha)}(X) \).

On the other hand, for any \( 1 \leq q < p < \infty \) and any cube \( Q \) of \( \mathbb{R}^n \), by the embedding \( L^{p,\infty}(Q) \subset L^q(Q) \) (see, for instance, [10, p. 14, Exercises 1.1.11]) and Theorem 4.1 we have
\[ (4.11) \left[ \int_Q \left| f - P_Q^{(s)}(f) \right| \right]^q \leq |Q|^{-\frac{1}{p}} \left\| f - P_Q^{(s)}(f) \right\|_{L^{p,\infty}(Q)} \leq |Q|^{-\frac{1}{p}} \| f \|_{\tilde{J}N_{(p,1,\alpha)}(Q)}. \]
Now, for any given interior pairwise disjoint cubes \( \{Q_i\}_i \) of \( X \), from the definition of \( \tilde{JN}(p,1,\alpha)_{b}(Q_i) \), we deduce that, for any \( i \), there exist interior pairwise disjoint cubes \( \{Q_{i,j}\}_j \) of \( Q_i \) such that

\[
(4.12) \quad \tilde{JN}(p,1,\alpha)_{b}(Q_i) \leq \left\{ \sum_j |Q_{i,j}| \left| \int_{Q_{i,j}} |f - P_{Q_{i,j}}^{(s)}(f)| \right|^p \right\}^{\frac{1}{p}}.
\]

By (4.11), (4.12), and \( \alpha \in [0, \infty) \), we obtain

\[
\sum_i |Q_i|^{\frac{1}{p}} \left( \int_{Q_i} f - P_{Q_{i,j}}^{(s)}(f) \right)^p \leq \sum_i |Q_i|^{\frac{1}{p}} \left( \int_{Q_i} f - P_{Q_{i,j}}^{(s)}(f) \right)^p \leq \sum_i |Q_i|^{\frac{1}{p}} \left( \int_{Q_i} f - P_{Q_{i,j}}^{(s)}(f) \right)^p \leq \sum_i \sum_j |Q_{i,j}| \left| \int_{Q_{i,j}} f - P_{Q_{i,j}}^{(s)}(f) \right|^p \leq \left\| f \right\|_{\tilde{JN}(p,1,\alpha)_{b}(Q_i)}^p.
\]

This implies that \( \left\| f \right\|_{\tilde{JN}(p,1,\alpha)_{b}(X)} \leq \left\| f \right\|_{JN(p,1,\alpha)_{b}(X)}^p \) and hence \( \tilde{JN}(p,1,\alpha)_{b}(X) \subset JN(p,1,\alpha)_{b}(X) \).

To sum up, \( \tilde{JN}(p,q,s)_{b}(X) = \tilde{JN}(p,1,\alpha)_{b}(X) \) and \( \left\| f \right\|_{\tilde{JN}(p,q,s)_{b}(X)} \sim \left\| f \right\|_{JN(p,1,\alpha)_{b}(X)} \). This finishes the proof of Proposition \( 3.3 \). \( \square \)

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