Onset of fluidization in vertically shaken granular material

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When granular material is shaken vertically one observes convection, surface fluidization, spontaneous heap formation and other effects. There is a controversial discussion in literature whether there exists a threshold for the Froude number \( \Gamma = A_0 \omega_0^3 / g \) below which these effects cannot be observed anymore. By means of theoretical analysis and computer simulation we find that there is no such single threshold. Instead we propose a modified criterion which coincides with critical Froude number \( \Gamma_c = 1 \) for small driving frequency \( \omega_0 \).

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I. INTRODUCTION

When granular material in a rectangular container is imposed to vertical oscillation under certain conditions one observes a variety of effects, such as convection \( \| \| \), surface fluidization \( \| \| \), spontaneous heap formation \( \| \| \), surface patterns \( \| \| \| \), oscillons \( \| \| \| \) and others. The common feature of all these effects is that particles change their position with respect to each other. Provided the particles do not change their mechanical properties during observation time (by polishing, comminution etc.) the condition for this motion is that neighboring particles separate from each other at least for a small part of the oscillation cycle \( T = 2 \pi / \omega_0 \).

There is a controversial discussion in the literature whether there is a critical value of Froude number

\[
\Gamma_c = A_0 \omega_0^3 / g, \tag{1}
\]

below which the above mentioned effects vanish, with \( A_0 \) and \( \omega_0 \) being the parameters of the sinusoidal motion of the container. In many experimental observations (e.g. \( \| \| \) and computer simulations (e.g. \( \| \| \| \| \) ) such a critical number \( \Gamma_c \) was found. Several authors believe that the value is \( \Gamma_c = 1 \). In numerical simulations, however, surface fluidization and convection has been found for \( \Gamma \lesssim 1 \). Therefore, some authors believe that \( \Gamma \) is not the proper criterion to determine the degree of fluidization of a granular system \( \| \| \| \| \).

In this article we discuss the response of granular material with respect to vertical oscillation in the limit of an one dimensional approach: the lowest bead of a vertical column of \( N \) identical spherical beads is shaken with \( z_0 = A_0 \cos \omega_0 t \) and the other beads move due to their interaction force and gravity \( g \). We study the motion of the entire column and can show that particles can lose contact to their neighbors even for the case that \( \Gamma = A_0 \omega_0^3 / g \) is significantly less than 1.

Adjacent spheres \( k \) and \( k + 1 \) of radius \( r \) and mass \( m \) at vertical positions \( z_k \) and \( z_{k+1} \) interact with their next neighbors by

\[
F_{k,k+1} = -\sqrt{r} \left( \mu \xi_{k,k+1}^3 + \alpha \xi_{k,k+1} \right), \tag{2}
\]

with \( \mu \) and \( \alpha \) being elastic and dissipative material constants, i.e. functions of Young modulus, Poisson ratio and dissipation rate (for details of the derivation of Eq. (2) see [19]). \( \xi \) is the compression \( 2r - |z_k - z_{k+1}| \) of the spheres. The height of the column is \( L = 2Nr \). Expression (2) is valid if the typical relative velocities of adjacent spheres are far below the speed of sound in the material of the spheres. Certainly this condition holds for typical vibration experiments.

Introducing new coordinates \( u_k = z_k - 2rk \) \( (k = 0 \ldots N) \) the compression of two adjacent spheres is

\[
\xi_{k,k+1} = u_k - u_{k+1}. \tag{3}
\]

Applying these definitions in Eq. (3) and adding gravity \( g \) we get

\[
\dot{z}_k = \frac{1}{m} (F_{k,k+1} - F_{k-1,k}) - g \tag{4}
\]

\[
F_{k,k+1} = -\mu \sqrt{r} (u_k - u_{k+1})^{3/2} - \alpha \xi_{k+1} (u_k - u_{k+1}) \sqrt{u_k - u_{k+1}}.
\]

The 0-th sphere is fixed at the oscillating table, hence its position is \( z_0(t) = u_0(t) = A_0 \cos \omega_0 t \).

We are interested in the critical parameters of driving \( (A_0, \omega_0) \) when the \( N \)-th particle loses contact, i.e. when \( u_N > u_{N-1} \). We define the “response” \( R(\omega_0) \) as the ratio \( A_N / A_0 \) where \( A_N \) is the amplitude of the \( N \)-th particle at frequency \( \omega_0 \) and \( A_0 \) is the amplitude of the driving vibration. \( R(\omega_0) \) can be calculated by convoluting the motion \( z_N(t) \) with \( \exp(i \omega_0 t) \). Suppose \( A_N \omega_0^3 / g \geq 1 \) the \( N \)-th particle separates from the \( N - 1 \)-st. If we would find \( A_0 < A_N \) the critical Froude number \( \Gamma_c = A_0 \omega_0^3 / g \) would be less than 1. We will show that there is a range for \( \omega_0 \) where this is the case.

\[
\omega_0 = \frac{1}{\sqrt{2}} \left( \frac{\xi_{k,k+1}}{u_k - u_{k+1}} \right)^{2/3} \sqrt{\frac{g}{\mu \sqrt{r}}},
\]

for \( \mu = \mu(\xi_{k,k+1}) \) in the limit of small amplitudes where the above assumptions are valid. The second assumption is justified by the fact that the threshold \( \Gamma_c \) is significantly less than 1.

\[
\Gamma_c = A_0 \omega_0^3 / g.
\]

The critical Froude number \( \Gamma_c \) is given by the smallest \( \Gamma \) such that \( \dot{z}(\Gamma) = 0 \).

\[
\dot{z}(\Gamma) = A_0 \Gamma \omega_0^3 / g.
\]

The above equation can be solved for \( \Gamma \) and \( \omega_0 \) by numerical methods.

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\omega_0 = \frac{1}{\sqrt{2}} \left( \frac{\xi_{k,k+1}}{u_k - u_{k+1}} \right)^{2/3} \sqrt{\frac{g}{\mu \sqrt{r}}},
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The above equation can be solved for \( \Gamma \) and \( \omega_0 \) by numerical methods.
In the next section we will formulate the problem in a continuum approach and derive a nonlinear partial differential equation for the motion of the column of particles. This equation is solved in section III in the limit of elastic material properties, i.e. by dropping the dissipative terms. Once the solution for the elastic case has been discussed in detail, it is easier to study the influence of the dissipative term and to derive the solution of the full equation of motion which is done in section IV. Section V compares the analytic results with a molecular dynamics simulation of the original (discrete) problem stated in Eq. (4). Finally, we discuss the results.

II. CONTINUUM APPROACH

To study the system analytically we use an one dimensional continuum approach. To this end we perform a Taylor expansion of the force with respect to the radius \( r \) and consequently consider the limit \( r \to 0 \), \( N \to \infty \) with \( 2rN = L = \text{const.} \) First we have to replace the displacements \( u_k \) by \( u(2kr) \) introducing the displacement field \( u(z) \) which is a continuous function of \( z \). With Eq. (3) we find from Taylor expansion

\[
\xi_{k,k+1} = u(2kr) - u(2kr + 2r) = -2r (\partial u / \partial z) |_{z=2kr}
\]

The net force experienced by the \( k \)-th particle is

\[
F_k = F_{k,k+1} - F_{k-1,k} = -\mu \sqrt{\kappa} \left( \xi_{k,k+1}^{3/2} - \xi_{k-1,k}^{3/2} \right)
\]

\[
= -\frac{2}{\sqrt{2}} \mu \left[ \left( -\frac{\partial u_k}{\partial z} \right)^{3/2} - \left( -\frac{\partial u_{k-1}}{\partial z} \right)^{3/2} \right] + 2 \sqrt{2} \mu \left[ \frac{\partial^2 u_k}{\partial t \partial z} \sqrt{-\frac{\partial u_k}{\partial z}} - \frac{\partial^2 u_{k-1}}{\partial t \partial z} \sqrt{-\frac{\partial u_{k-1}}{\partial z}} \right],
\]

with the abbreviations

\[
u_k = u(2kr), \quad \frac{\partial u_k}{\partial z} = \left. \frac{\partial u}{\partial z} \right|_{z=2kr}. \tag{5}
\]

Both expressions in square brackets are expanded again and Eq. (5) turns into

\[
F_k = \frac{3}{\pi \rho} \left[ -\mu \left( -\frac{\partial u}{\partial z} \right)^{3/2} + \alpha \frac{\partial^2 u}{\partial t \partial z} \sqrt{-\frac{\partial u}{\partial z}} \right].
\]

With

\[
\kappa = \frac{3 \sqrt{2} \mu}{\pi \rho}, \quad \beta = \frac{3 \sqrt{2} \alpha}{\pi \rho}
\]

the continuum formulation of (4) is

\[
\frac{\partial^2 u}{\partial t^2} = -\frac{g}{4} - \frac{\partial}{\partial z} \left[ \kappa \left( -\frac{\partial u}{\partial z} \right)^{3/2} - \beta \frac{\partial^2 u}{\partial t \partial z} \sqrt{-\frac{\partial u}{\partial z}} \right], \tag{7}
\]

\[
\left. \frac{\partial u}{\partial z} \right|_{z=L} = 0
\]

where \( g \) accounts for the gravitational force.

III. LIMIT OF ELASTIC PARTICLES

In the following we consider Eq. (5) the limit of no damping (\( \beta = 0 \)). Using new variables

\[
x = 1 - \frac{z}{L}, \quad \tau = \left( \frac{g \kappa^2}{L^5} \right)^{1/6} t
\]

\[
\Omega = \left( \frac{L^5}{g \kappa^2} \right)^{1/6}, \quad \gamma = \left( \frac{g^2 L^5}{\kappa^2} \right)^{1/6}
\]

Eq. (5) turns into

\[
\frac{\partial^2 u}{\partial \tau^2} = -\gamma^2 + \frac{1}{\gamma} \frac{\partial}{\partial \tau} \left[ \left( \frac{\partial u}{\partial \tau} \right)^{3/2} \right] \tag{8}
\]

\[
\left. \frac{\partial u}{\partial \tau} \right|_{\tau=0} = 0. \tag{9}
\]

Equation (8) is defined in the range \( x \in [0,1] \). The time independent solution \( U(x) \) of (8) is

\[
U(x) = \frac{3}{5} \gamma^2 \left( x^{5/3} - 1 \right). \tag{10}
\]

The solution of (8) can be considered as a superposition of the static solution (11) and a perturbation \( w(x, \tau) \). Inserting \( u = U + w \) in (8) we find:

\[
\frac{\partial^2 w}{\partial \tau^2} = -\gamma^2 + \frac{1}{\gamma} \frac{\partial}{\partial \tau} \left[ \left( \frac{\partial U}{\partial \tau} + \frac{\partial w}{\partial \tau} \right)^{3/2} + \frac{3}{2} \sqrt{\frac{\partial U}{\partial \tau} \frac{\partial w}{\partial \tau}} \right]
\]

\[
\approx -\gamma^2 + \frac{1}{\gamma} \frac{\partial}{\partial \tau} \left[ \left( \frac{\partial U}{\partial \tau} \right)^{3/2} + \frac{3}{2} \sqrt{\frac{\partial U}{\partial \tau} \frac{\partial w}{\partial \tau}} \right]
\]

\[
= 3 \frac{\partial}{\partial \tau} \left[ x^{5/3} \frac{\partial w}{\partial \tau} \right] \tag{11}
\]

By separation of variables \( w = T(\tau, \Omega)X(x, \Omega) \), i.e. a standing wave Ansatz we obtain two ordinary differential equations for \( T \) and \( x \):

\[
\ddot{T} = -\Omega^2 T \tag{12}
\]

\[
\frac{3}{2} \frac{d}{dx} \left( x^{1/3} \frac{dX}{dx} \right) = -\Omega^2 X, \tag{13}
\]

with \( \Omega \) being a real number. For \( T(\tau, \Omega) \) one gets

\[
T \sim \exp(i \Omega \tau).
\]
The solution of the spatial equation can be found using the Ansatz
\[ X(x, \Omega) = x^{1/3} f(y), \quad y = \frac{2}{5} \sqrt{\Omega} x^{5/6}, \]
which yields
\[ y^2 \frac{d^2 f}{dy^2} + y \frac{df}{dy} + \left( y^2 - \frac{4}{25} \right) f = 0. \quad (14) \]

Eq. (14) is the Bessel equation of order 2/5. Hence the solution of (13) is
\[ X(x, \Omega) = \left( \frac{6}{25} \right)^{1/5} \Gamma \left( \frac{3}{5} \right) \Omega^{2/5} x^{1/3} J_{2/5} \left( \frac{2}{5} \sqrt{\Omega} x^{5/6} \right). \quad (15) \]

An expression containing \( J_{2/5} \) would be a solution too, however, it does not satisfy the condition (1). The prefactor in (13) has been chosen to assure \( X(0, \Omega) = 1 \).

Hence the solution for a single vibrational mode \( u_\Omega \) is
\[ u_\Omega = \exp (i \Omega \tau) X(x, \Omega). \quad (16) \]

Without prior knowledge the full solution of Eq. (1) has to be assumed to be a superposition of vibrational modes for all real (rescaled) frequencies \( \Omega \):
\[ u = \int_{-\infty}^{\infty} d\Omega A(\Omega) \exp (i \Omega \tau) X(x, \Omega). \quad (17) \]

In the steady state of pure sinusoidal excitation of the base, i.e., when all non-oscillatory perturbations which originate from the initialization have been damped out, Eq. (17) is the full (steady state-) solution of Eq. (11).

The function \( A(\Omega) \) represents the excitation of the mode at frequency \( \Omega \). The boundary condition at the top of the chain is automatically satisfied, whereas the boundary condition at the bottom reads
\[ u(1, \tau) = \int_{-\infty}^{\infty} d\Omega A(\Omega) \exp (i \Omega \tau) X(1, \Omega) \quad (18) \]
\[ = A_0 \cos \Omega_0 \tau. \quad (19) \]

One can see that the integrand of Eq. (15) can be nonzero only for \( \Omega \neq \Omega_0 \). This means that for \( \Omega \neq \Omega_0 \) either \( A(\Omega) \) or \( X(1, \Omega) \) have to be zero, i.e. for all frequencies for which \( X(1, \Omega) \) is nonzero the respective amplitude must be zero, whereas for all frequencies which are a root of \( X(1, \Omega) = 0 \) the amplitude can be nonzero. Therefore, we find that the full solution of Eq. (11) is a superposition of the vibrational mode of the frequency of shaking \( \Omega_0 \) and of a discrete set of frequencies \( \Omega_k \) \( (k = 1 \ldots \infty) \).

Note that \( \Omega_k \) are no rational multiples of each others since the roots of Bessel-functions are incommensurable (see Eq. (13)). Therefore, to determine the maximum acceleration of the topmost particle it is sufficient to consider only the mode of the external excitation. All other vibrational modes can only further increase the maximal acceleration.

The above defined response \( R \) is the ratio \( A_N/A_0 \). Since the zeroth particle corresponds to \( x = 1 \) and the N-th to \( x = 0 \) we can write
\[ R^{-1}(\Omega_0) = \left| \frac{X(1, \Omega_0)}{X(0, \Omega_0)} \right| = |X(1, \Omega_0)| \]
\[ = \left( \frac{6}{25} \right)^{1/5} \Gamma \left( \frac{3}{5} \right) \Omega_0^{2/5} J_{2/5} \left( \frac{2}{5} \sqrt{\Omega_0} \right). \quad (20) \]

The response \( R \) is an amplification factor, hence, the value \( g/R(\Omega_0) \) is the critical acceleration of the driving vibration \( 2\Omega_0 \). \( R \) is larger than 1 for all driving frequencies \( \omega_0 \). This means that for any driving frequency \( \omega_0 \) and driving amplitude \( A_0 \) the amplitude of the top particle of the column \( A_N \) at frequency \( \omega_0 \) will be larger than \( A_0 \). Therefore, for \( A_N\omega_0^2/g = 1 \), i.e. when the N-th particle separates from the \( N-1 \)-st, we find \( A_N\omega_0^2/g = \Gamma_c < 1 \).

According to the above arguments we have to replace the condition \( \Gamma \geq 1 \) which was supposed to be the pre-condition for surface fluidization, convection etc., by
\[ A_0\omega_0^2/g = \Gamma \geq R^{-1}(\omega_0). \quad (21) \]

The function \( R^{-1}(\omega_0) \) over \( \omega_0 \) is drawn in Fig. 4 (dash-dotted line, \( R_\infty^{-1} \)). For the system parameters we used \( A_0 = 0.01 \text{ mm}, \) elastic constant \( \kappa = 2.8 \cdot 10^4 \text{ m}^2/\text{sec}^2 \) (rubber with Young modulus \( Y = 4 \cdot 10^7 \text{ Pa} \)) and \( L = 0.6 \text{ m} \).
The curve reveals pronounced resonances at Eigenfrequencies \( \omega_k \) where \( R^{-1} \) becomes minimal (only the first resonance is shown in Fig. 4).

All experiments on surface fluidization and convection which can be found in literature have been performed far below the first resonance. Therefore, of particular interest to practical purposes is the limit of small frequency \( \omega_0 \), i.e. below the first Eigenfrequency. The Taylor expansion of Eq. (20) yields \( R^{-1}(\Omega_0) \) for small \( \Omega_0 \)
\[ R^{-1} = 1 - \frac{2}{5} \Omega_0^2 + O(\Omega_0^4) \]
\[ = 1 - \frac{2}{5} \left( \frac{L^5}{g\kappa^2} \right)^{1/3} \omega_0^2 + O(\omega_0^4). \quad (22) \]

Given the container vibrates with frequency \( \omega_0 \). Then for the critical amplitude \( A_0 \) of the vibration when the top particle separates, i.e. when the material starts to fluidize, one finds
\[ A_0 = \frac{g}{\omega_0^2} - \frac{2}{5} \left( \frac{L^5}{g\kappa^2} \right)^{1/3}. \quad (23) \]
being the natural length scale coming out from the analysis. By means of the Fourier transform,
\[ W(x, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} w(x, t) \, dt, \]
equation (24) becomes
\[ -\omega^2 W = \frac{3}{2} - i\omega \beta \frac{\partial}{\partial x} \left( x^{1/3} \frac{\partial W}{\partial x} \right), \]
which has the same structure as Eq. (13). Hence the same transformations apply in this case and the general solution reads finally
\[ W(x, \omega) = x^{1/3} \left[ C_1 J_{-\frac{2}{5}} \left( \frac{6 \omega \ell}{\sqrt{2} \kappa - i\omega \beta} \right)^{5/6} \right. \]
\[ + \left. C_2 J_{-\frac{6}{5}} \left( \frac{6 \omega \ell}{\sqrt{2} \kappa - i\omega \beta} \right)^{5/6} \right]. \quad (28) \]

The part of the solution depending on \( J_{2/5} \) carries a divergence at \( x = 0 \) (\( z = L \)), and therefore \( C_1 = 0 \) is required for the solution to be physical. The condition of the free end at \( x = 0 \) (\( z = L \)) is satisfied automatically, as in the case \( \beta = 0 \). The solution has exactly the same structure as the solution of the elastic problem, Eq. (13), and the only change is that the argument of the Bessel function has an imaginary part. If one considers only the mode \( \omega_0 \), which propagates from the bottom (\( x = 1 \)) with amplitude \( A_0 \), the solution reads
\[ w(x, t) = \text{Re} \left\{ A_0 e^{i\omega_0 t} x^{\frac{1}{3}} \left( J_{-\frac{2}{5}} \left( \frac{6 \omega_0 \ell}{\sqrt{2} \kappa - i\omega_0 \beta} \right)^{5/6} \right. \right. \]
\[ \left. \left. - J_{-\frac{6}{5}} \left( \frac{6 \omega_0 \ell}{\sqrt{2} \kappa - i\omega_0 \beta} \right)^{5/6} \right) \right\}. \quad (29) \]

The fluidization condition at the top of the chain \(|\partial^2 w/\partial t^2(x = 0, t)| > g\) can be written in general
\[ R^{-1}(\omega_0) \]
\[ \equiv \Gamma \left( \frac{3}{5} \right) \left| \left( \frac{6 \omega_0 \ell}{\sqrt{2} \kappa - i\omega_0 \beta} \right)^{2/5} J_{-\frac{2}{5}} \left( \frac{6 \omega_0 \ell}{\sqrt{2} \kappa - i\omega_0 \beta} \right) - \frac{A_0 \omega_0^2}{g} \right| \]
\[ < A_0 \omega_0^2 g. \quad (30) \]

**IV. DISSIPATIVE PARTICLE INTERACTION**

We will consider, as before, small perturbations \( w \) about the static deformation of the chain under gravity which propagate from the bottom. The dissipative term is characterized by the parameter \( \beta \) in Eq. (1). From this equation, again introducing the static solution given by Eq. (10) and by using the same transformation for the spatial coordinate \( x = 1 - z/L \), one obtains the corresponding linearized wave equation for dissipative materials,
\[ \frac{\partial^2 w}{\partial t^2} = \left( \frac{g}{\kappa L^5} \right) \frac{3}{2} \kappa \frac{\partial}{\partial x} \left( x^{1/3} \frac{\partial w}{\partial x} \right) + \beta \frac{\partial}{\partial x} \left( x^{1/3} \frac{\partial^2 w}{\partial t \partial x} \right). \quad (24) \]

In this case it is less useful to introduce the rescaled time variable \( \tau \), while it proves convenient to define
\[ \left( \frac{\kappa L^5}{g} \right)^{1/6} \equiv \ell, \quad (25) \]
Since the Bessel functions of the first class only have zeros on the real axis, $R^{-1}(\omega_0)$ can no longer be zero for any frequency if $\beta \neq 0$. This means that the sharp resonances displayed by $R(\omega_0)$ when $\beta = 0$ disappear and are replaced by more or less pronounced minima in dependence on the damping constant $\beta$. This can also be observed in Figure 1 (full line). Increasing values of $\beta$ make the response smoother, deviating from that of the elastic case earlier, and the local minima translate on the frequency axis appreciably, see Figure 2.

Analogue to the elastic case (Eq. (22)), for small frequencies $R^{-1}(\omega_0)$ can be expanded into a Taylor series:

$$R^{-1} \approx 1 - \frac{\omega_0^2}{2} \left( \frac{L^5}{g\kappa^2} \right)^{1/3} + \omega_0^4 \left( \frac{L^5}{g\kappa^2} \right)^{2/3} \left( \frac{3}{100} + \frac{8}{45} \beta^2 \left( \frac{g}{\kappa L^2} \right)^{1/3} \right).$$

The contribution due to the dissipative parameter enters the Taylor expansion at the fourth power of the frequency. Therefore, the analysis of the elastic case given by Eq. (22) remains valid for small frequencies.

It is interesting to note that due to Eq. (31) there always exists a global minimum below the value $R^{-1}(\omega_0) = 1$, regardless of the value of the dissipative parameter. We want to study for which range of frequencies the inverse response $R^{-1}$ is smaller than one for varying damping constant $\beta$: The lower boundary of this interval obviously is $\omega = 0$. To determine the upper boundary $\omega_{\text{max}}$ we solved numerically the equation $R^{-1}(\omega_{\text{max}}) = 1$ for different values of $\beta$. The result of this calculation is shown in Fig. 3.

One can see that for high enough damping this frequency $\omega_{\text{max}}$ varies only slowly with $\beta$. The curve almost saturates at $\omega_{\text{max}} \approx 200$ sec$^{-1}$ which is close to the first maximum of the undamped inverse response (see Fig. 4, dash-dotted line). Although the valley of the inverse response function becomes smaller with increasing damping (Fig. 3), even for larger damping $R^{-1}$ is smaller than one in a finite frequency interval, i.e. the effect of amplitude amplification exists for almost the entire range of frequencies between zero and the first maximum of the undamped inverse response.
For small enough damping the frequency range of amplitude amplification $R^{-1}(\omega_0) < 1$ extends beyond the position of the first maximum of the undamped inverse response. Figure 3 shows the response function for different dissipative parameter $\beta$ together with the elastic case ($\beta = 0$, dash-dotted). The frequency $\omega_0$ at which $R^{-1}(\beta) = 1$ increases with decreasing damping. In this Fig. one can clearly see the extension of the range of amplitude amplification. If we decrease the damping parameter, starting at high values where the amplifying range is limited to the first “well” we cannot expect a large change since, as long as we remain limited to the first well, there is a upper bound (the frequency of the first maximum of $R^{-1}$) to this range. Reducing $\beta$ further we will eventually reach values for which the amplifying range extends to the second well. Even for values of $\beta$ which are only slightly below this threshold the range will now span again almost the entire range of the second well up to the frequency of the second maximum. So there will be rather sharp steps in the dependence of the upper range limit of the damping instead of only gradual changes. This behaviour explains the steps in the closeup of Fig. 3.

V. NUMERICAL RESULTS

To check the analytical results and in particular the validity of the continuum approach we calculated $R^{-1}$ for a finite value of damping $\beta$ from the numerical simulation of Eq. (1). The circles in Fig. 3 display the reciprocal response $R^{-1}$ over $\omega_0$ with fixed amplitude $A_0 = 0.01$ mm, elastic constant $\kappa = 2.8 \times 10^4$ m$^2$/sec$^2$ (rubber with Young modulus $Y = 4 \cdot 10^7$ Pa) and $L = 0.6$ m. Fig. 3 shows that for small frequency $\omega_0$ and small damping $R$ the undamped theoretical curve (dash-dotted) agrees well with numerical data. If one compares the numerical result with the damped solution according to Eq. (2) (full line in Fig. 3) the agreement with theory is very well.

To check the validity of our linear theory we also determined directly by integrating Eq. (4) at which Froude number $\Gamma_c \equiv A_0\omega_0^2/g > 1$ one has to satisfy $A_0\omega_0^2/g > R^{-1}$ where $R^{-1}$ is a function of $\omega_0$. We have shown that independent on the material properties there exists always a range $\omega_0 \in [0..\omega_{\text{max}}]$ for which the amplitude of vibration $A_0$ is amplified, i.e. for which the top particle can separate (the material fluidizes) even if $A_0\omega_0^2/g < 1$. Numerical calculations agree well with the analytic results.

Whereas the critical Froude number $\Gamma_c \geq 1$ is certainly the proper criterion to predict whether a single rigid particle will jump on a vibrating table, we suspect that this number is not suited to describe the behavior of a vibrated column of spheres, and the more it is not a criterion for surface fluidization of a three dimensional granular material.

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VI. DISCUSSION

For the case of a vertical column of viscoelastic spheres we derived a linear wave equation in one dimensional approximation. We have shown that the sphere on top of the column $N$ can separate from the $N - 1$-st even if the container is oscillated with $A_0\omega_0^2/g < 1$. As main result we derived a modified condition for the topmost particle to separate from its neighbor. We could show that instead of the widely accepted condition $\Gamma_c \equiv A_0\omega_0^2/g > 1$ one has to satisfy $A_0\omega_0^2/g > R^{-1}$ where $R^{-1}$ is a function of $\omega_0$. We have shown that independent on the material properties there exists always a range $\omega_0 \in [0..\omega_{\text{max}}]$ for which the amplitude of vibration $A_0$ is amplified, i.e. for which the top particle can separate (the material fluidizes) even if $A_0\omega_0^2/g < 1$. Numerical calculations agree well with the analytic results.

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[20] Here we consider only the mode of external excitation $\omega_0$. Since the higher frequencies $\omega_k$ are noncommensurable with $\omega_0$ and with each others the maximum acceleration of the superposition must be higher than $R(\omega_0) A_0 \omega_0^2$. Hence consideration of higher modes will further increase the effect discussed here.