Psybrackets, Pseudoknots and Singular Knots

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Abstract

We introduce algebraic structures known as psybrackets and use them to define invariants of pseudoknots and singular knots and links. Psybrackets are Niebrzydowski tribrackets with additional structure inspired by the Reidemeister moves for pseudoknots and singular knots. Examples and computations are provided.

Keywords: Pseudoknots, Singular knots, Psybrackets, Niebrzydowski tribrackets, ternary quasigroups

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1 Introduction

In [19,20] algebraic structures known as knot-theoretic ternary quasigroups were introduced and investigated. With a notational change, these have been studied by the third listed author and collaborators as Niebrzydowski tribrackets in papers such as [6,15,17,18] and used to define invariants of classical knots and links, virtual links and handlebody-links. Related objects known as biquasiles have been investigated by the first and third authors in [10] and by the third author and collaborators in [3,12].

Pseudoknots arose in biology as a way of dealing with knotted objects with only partial information about the crossings; see e.g. [4,5,11,22] etc. The mathematical formulation in [7,8,9] defines pseudoknots and pseudolinks combinatorially as equivalence classes of pseudoknot diagrams, i.e., knot diagrams with ordinary classical crossings together with precrossings in which it is unknown which strand is on top, under the equivalence relation determined by the pseudoknot Reidemeister moves.

Singular knots are rigid vertex isotopy classes of 4-valent spatial graphs. We can think of singular knots and links as knots and links in which some strands are fused together at vertices known as singular crossings. In particular, the cyclic ordering of the edges around each singular crossing is fixed.

Identifying singular crossings with precrossings, the singular Reidemeister moves form a subset of the pseudoknot Reidemeister moves; combinatorially, the two classes of objects differ only by a single move.

In [14], together with two collaborators the third listed author exploited the similarity of the Reidemeister moves for pseudoknots and singular knots to introduce psyquandles, algebraic coloring structures for pseudoknots and singular knots extending the notion of biquandle colorings from the world of classical knots and links. Finite biquandles give rise to integer-valued counting invariants, which can be enhanced in various ways to define new stronger invariants.

In this paper we apply the idea of Niebrzydowski tribrackets to the case of pseudoknots and singular knots, defining psybrackets analogously to the way psyquandles extend biquandles. The paper is organized as follows. In Section 2 we review the basics of pseudoknots and singular knots. In Section 3 we define psybrackets and provide some examples. In Section 4 we define the psybracket counting invariant and provide some computational examples to explore the power of the new invariants. We conclude in Section 5 with some questions for future research.

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2 Pseudoknots and Singular Knots

In this section we review the basics of pseudoknots and singular knots; the remainder of the paper will concern algebraic structures from which we will derive invariants of pseudoknots and singular knots.

Definition 1. An oriented pseudoknot diagram has positive and negative classical crossings but also precrossings.

Replacing a precrossing with a classical crossing is known as resolution.

Precrossings represent classical crossings for which it is unknown which strand passes over and which passes under. We may regard a precrossing as a linear combination of both crossings with a scalar coefficient of $\frac{1}{2}$ for each; extending linearly, we may regard a pseudoknot diagram as a linear combination of its resolutions. Interpreting the scalar weights as probabilities, we obtain from a pseudoknot its wereset or weighted resolution set, a discrete probability distribution whose events are the classical knots obtained by resolving all precrossings, with probabilities given by the scalar coefficients.

Example 1. The pseudolink \[ \text{has wereset} \]

\[ \left\{ \begin{array}{c}
\begin{array}{c}
\frac{1}{4}
\end{array}
\end{array}\right\}, \begin{array}{c}
\frac{1}{4}
\end{array}, \begin{array}{c}
\frac{1}{2}
\end{array}, \begin{array}{c}
\end{array} \right\} \]

Singular knots and links are 4-valent spatial graphs considered up to rigid vertex isotopy, where the cyclic ordering of the edges entering a vertex is fixed. Such a rigid vertex is called a singular crossing; we can imagine singular crossings as points where the knot becomes stuck to itself (transversely, not tangentially).

Identifying precrossings with singular crossings, the Reidemeister moves for pseudolinks and singular links are the same except for one move: precrossings can be introduced or removed via a Reidemeister I type move, while singular crossings cannot. More precisely, a pseudolink is an equivalence class of pseudolink diagrams under the equivalence relation generated by planar isotopy moves, the classical Reidemeister moves.
I, II and III,

and the moves PI, PI', PII, PIII and PIII'.

In particular, in [16] it is shown that these moves form a generating set of oriented pseudoknot Reidemeister moves. A singular link is an equivalence class of singular link diagrams under the equivalence relation generated by planar isotopy, the usual Reidemeister moves I, II and III, and the moves PII, PIII and PIII'. See [2, 7, 8] for more about pseudoknot and singular knot Reidemeister moves.

Remark 1. Singular knots and links may be regarded as the “pseudo-framed case” of pseudoknots and pseudolinks, where the “pseudo-writhe” or number of precrossings is preserved.
As we will see, the algebraic conditions on our psybracket structure coming from moves PI and PI’ are already implied by move PII so the invariants we define will be valid for both pseudolinks and singular links; on the other hand, they will not be able to distinguish singular links which differ only by PI and PI’ moves.

3 Psybrackets

We begin with a definition.

**Definition 2.** Let $X$ be a set. A *psybracket* structure on $X$ consists of two maps $(\langle,\rangle)_c, (\langle,\rangle)_p : X \times X \times X \to X$ such that

(i) For all $a, b, c \in X$ there exist unique $x, y, z, u, v \in X$ such that

\[
\langle a, b, x \rangle_c = c \quad (i.i) \\
\langle a, y, b \rangle_c = c \quad (i.ii) \\
\langle z, a, b \rangle_c = c \quad (i.iii) \\
\langle u, b, c \rangle_p = b \quad (i.iv) \\
\langle a, b, v \rangle_p = b \quad (i.v),
\]

(ii) For all $a, b, c \in X$ we have

\[
\langle a, \langle a, b, c \rangle_c, c \rangle_p = \langle a, \langle a, b, c \rangle_p, c \rangle_c
\]

and

(iii) For all $a, b, c, d \in X$ we have

\[
\langle \langle a, b, c \rangle_c, c, d \rangle_c = \langle \langle a, b, (b, c, d)_c \rangle_c, b, c, d \rangle_c, c \rangle_c \\ = \langle \langle a, b, (b, c, d)_p \rangle_c, b, c, d \rangle_c, c \rangle_c \\ = \langle a, \langle a, b, c \rangle_p, \langle a, b, c \rangle_p, c, d \rangle_c, c \rangle_p \quad (iii.i) \\
\langle (a, b, c)_c, c, d \rangle_c = \langle a, \langle a, b, c \rangle_p, \langle a, b, c \rangle_p, c, d \rangle_c, c \rangle_c \\ = \langle a, \langle a, b, c \rangle_p, \langle a, b, c \rangle_c, c, d \rangle_c, d \rangle_c \quad (iii.ii) \\
\langle a, b, (b, c, d)_c \rangle_c = \langle a, \langle a, b, c \rangle_p, \langle a, b, c \rangle_p, c, d \rangle_c, c \rangle_c \\ = \langle a, \langle a, b, c \rangle_p, \langle a, b, c \rangle_c, c, d \rangle_c, c \rangle_p \quad (iii.iii) \\
\langle a, b, (b, c, d)_p \rangle_c = \langle a, \langle a, b, c \rangle_p, \langle a, b, c \rangle_p, c, d \rangle_c, c \rangle_p \\ = \langle a, \langle a, b, c \rangle_p, \langle a, b, c \rangle_p, c, d \rangle_c, d \rangle_c \quad (iii.iv) \\
\langle a, b, (b, c, d)_p \rangle_c = \langle a, \langle a, b, c \rangle_p, \langle a, b, c \rangle_p, c, d \rangle_c, c \rangle_p \quad (iii.v) \\
\langle a, b, (b, c, d)_p \rangle_c = \langle a, \langle a, b, c \rangle_p, \langle a, b, c \rangle_p, c, d \rangle_c, d \rangle_c \quad (iii.vi)
\]

**Remark 2.** The set $X$ with the map $(\langle,\rangle)_c$ forms a *vertical tribracket* as described in [15], also known as a *Niebrzydowski tribracket* or a *knot-theoretic ternary quasigroup*. In particular, the binary operations

\[
\cdot, \cdot' : X \times X \to X
\]

are quasigroup structures on $X$.

The psybracket axioms are motivated by the following region coloring rules for pseudoknot diagrams:
An assignment of elements of a psybracket $X$ to each region in the complement of a pseudoknot diagram or singular knot diagram $D$ such that the above conditions are satisfied at every crossing is called a *psybracket coloring* or an $X$-coloring of $D$. The psybracket axioms are the conditions needed to ensure that for every psybracket coloring of a diagram before a move, there is a unique coloring of the diagram after the move that agrees with the pre-move coloring outside the neighborhood of the move.

Axioms (i.i)–(i.iv) are the conditions required by the Reidemeister II moves:

The conditions imposed by the classical Reidemeister I moves are special cases of the Reidemeister II conditions. For example, in

we have the requirement that for every $b, c \in X$ there exists a unique $a \in X$ such that $\langle a, b, c \rangle_c = b$, but this is already implied by left-invertibility (i.iii). The other classical Reidemeister I moves are similar.
Axioms (i.iv) and (i.v) are motivated by moves PI and PI':

Axiom (ii) is motivated by move PII:

Axioms (iii.i)–(iii.vi) are motivated by Reidemeister III, PIII and PIII':

\[
\begin{align*}
\langle a, b, c \rangle_p &\quad \xrightarrow{\text{PI}} \quad c &\quad \xrightarrow{\text{PI'}} \quad b &\quad \xrightarrow{\text{PI'}} \quad a &\quad \xrightarrow{\text{PI}} \quad c &\quad \langle a, b, c \rangle_p \\
\langle a, \langle a, b, c \rangle_c, c \rangle_p &\quad \xrightarrow{\text{PII}} \quad \langle a, b, c \rangle_c &\quad \xrightarrow{\text{PII}} \quad \langle a, b, c \rangle_p &\quad \xrightarrow{\text{PII}} \quad \langle a, \langle a, b, c \rangle_p, c \rangle_c \\
\langle a, \langle a, b, c \rangle_c, c \rangle_p &\quad \xrightarrow{\text{III}} \quad \langle a, \langle a, b, c \rangle_p, c \rangle_c &\quad \langle a, \langle b, c, d \rangle_c \rangle &\quad \xrightarrow{\text{III}} \quad \langle a, b, \langle b, c, d \rangle_c \rangle &\quad \langle a, b, \langle b, c, d \rangle_c \rangle &\quad \langle a, \langle b, c, d \rangle_c, d \rangle_c \\
\langle a, \langle a, b, c \rangle_c, \langle a, b, c \rangle_c, c \rangle \rangle &\quad \xrightarrow{\text{III}} \quad \langle a, \langle a, b, c \rangle_p, c \rangle &\quad \langle a, b, \langle b, c, d \rangle_c \rangle &\quad \langle a, b, \langle b, c, d \rangle_c \rangle &\quad \langle a, b, \langle b, c, d \rangle_c, d \rangle_c \\
\end{align*}
\]
Since this set of moves is a generating set of oriented pseudoknot Reidemeister moves, we have:

**Proposition 1.** Let $D$ be an oriented pseudoknot diagram or singular knot diagram and $X$ a psybracket. Then for any $X$-coloring of $D$ before a pseudoknot or singular knot Reidemeister move, there is a unique $X$-coloring of the diagram after the move that agrees with the coloring before the move outside the neighborhood of the move.

**Definition 3.** Let $X$ and $Y$ be psybrackets. A map $f : X \to Y$ is a psybracket homomorphism if for all $a, b, c \in X$ we have

$$\langle f(a), f(b), f(c) \rangle_c = f(\langle a, b, c \rangle_c) \quad \text{and} \quad \langle f(a), f(b), f(c) \rangle_p = f(\langle a, b, c \rangle_p).$$

A bijective psybracket homomorphism is an isomorphism.
Example 2. A Niebrzydowski tribracket can be given the structure of a psybracket by setting
\[ \langle a, b, c \rangle_p = \langle a, b, c \rangle_c. \]
To see that this definition satisfies the axioms, we need only note that replacing the precrossings in the moves PI, PIII and PIII' with positive crossings results in valid classical Reidemeister moves and in the same diagram on both sides of move PII. Similarly, setting
\[ \langle a, b, c \rangle_p = d \]
where
\[ \langle a, d, c \rangle_c = b \]
yields a psybracket, as we can see by resolving the precrossings as negative classical crossings.

Example 3. Let \( G \) be a group. Then \( G \) is a Niebrzydowski tribracket under the operation
\[ \langle a, b, c \rangle = ab^{-1}c \]
known as a Dehn tribracket. Then the two psybracket structures in Example 2 are
\[ \langle a, b, c \rangle_c = \langle a, b, c \rangle_p = ab^{-1}c \]
and
\[ \langle a, b, c \rangle_c = ab^{-1}c, \quad \langle a, b, c \rangle_p = cb^{-1}a \]
respectively; we call these the positive and negative Dehn psybracket structures on \( G \).

We can specify a psybracket structure on a finite set \( X = \{1, 2, \ldots, n\} \) with a pair of operation 3-tensors, i.e. lists of \( n \) square matrices of size \( n \times n \) such that the entry in matrix \( a \) row \( b \) column \( c \) is \( \langle a, b, c \rangle_c \) or \( \langle a, b, c \rangle_p \).

Example 4. The Dehn tribracket structure on \( \mathbb{Z}_4 = \{1, 2, 3, 4\} \) (where we use 4 for the class of zero) is
\[ \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix} \]
The positive and negative Dehn psybracket structures on \( \mathbb{Z}_4 \) both have this same operation 3-tensor for \( \langle ., . \rangle_p \) and for \( \langle ., . \rangle_c \).

Example 5. Let \( X = \{1, 2, 3\} \). Using python code, we compute that there are six isomorphism classes of psybracket structures on \( X \). Representatives of each class are specified by the operation 3-tensors below:
\[ \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \]
\[ \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \]
\[ \begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 2 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 2 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \]
Psybracket Counting Invariants

The motivation for the psybracket definition is to define counting invariants of pseudoknots and singular knots. As with previous knot coloring structures, by Proposition 1 we have the following result:

**Theorem 2.** Let $X$ be a finite psybracket, $K$ an oriented pseudoknot or singular knot diagram, and $C(K, X)$ the set of $X$-colorings of $K$. Then the number of $X$-colorings of $K$,

$$\Phi_{X}^Z(K) = |C(K, X)|$$

is an integer-valued invariant of pseudoknots and singular knots we will call the psybracket counting invariant.

**Example 6.** Let $X$ be the psybracket specified by

$$\begin{bmatrix}
\begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \\
\end{bmatrix},
\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \\
\end{bmatrix},
\begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \\
\end{bmatrix}
\end{bmatrix}_c,$$

and consider the pseudolink

Then for example the assignment of elements in $\{1, 2, 3\}$ to the regions given by
is an $X$-coloring, since we have
\[
\langle 1, 1, 2 \rangle_c = 3 \quad \text{and} \quad \langle 1, 3, 2 \rangle_p = 1.
\]

**Example 7.** Consider the pseudoknot $3_{1.3}$ below.

![Pseudoknot 3_{1.3}](image)

It has two positive crossings and one precrossing; resolving the precrossing one way yields a trefoil and the other way yields an unknot. We can distinguish this pseudoknot from both the trefoil and the unknot using psybracket counting invariants. Specifically, the psybracket
\[
X_1 = \left[\left[\left[ 1 \ 2 \ 3 \right], \left[ 3 \ 1 \ 2 \right], \left[ 2 \ 3 \ 1 \right]\right], \left[\left[ 1 \ 2 \ 3 \right], \left[ 3 \ 1 \ 2 \right], \left[ 2 \ 3 \ 1 \right]\right]\right]
\]
gives us counting invariant values $\Phi_{X_1}(3_{1.3}) = 27$ and $\Phi_{X_1}(\text{Unknot}) = 9$ while the psybracket
\[
X_2 = \left[\left[\left[ 1 \ 2 \ 3 \right], \left[ 3 \ 1 \ 2 \right], \left[ 2 \ 3 \ 1 \right]\right], \left[\left[ 1 \ 2 \ 3 \right], \left[ 3 \ 1 \ 2 \right], \left[ 2 \ 3 \ 1 \right]\right]\right]
\]
gives us counting invariant values $\Phi_{X_2}(3_{1.3}) = 9$ and $\Phi_{X_2}(3_1) = 27$.

**Example 8.** We computed the coloring invariant $\Phi^Z_X$ of a choice of orientation for the 2-bouquet graphs in [21] using the psybracket $X$ with the operation matrix
\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2
\end{bmatrix}, \begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2
\end{bmatrix}, \begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2
\end{bmatrix}, \begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2
\end{bmatrix}
\]
The results are collected in the table.

| $L$ | $\Phi^Z_X(L)$ |
|-----|---------------|
| $0^k_1, 5^k_6, 6^k_3, 11^k_6, 10^k_6$ | 9 |
| $6^k_6, 6^k_2, 5^k_1, 6^k_4, 6^k_12$ | 27 |
| $2^k_{11}, 3^k_{14}, 5^k_9, 6^k_{16}, 6^k_{17}, 6^k_{19}, 4^k_1, 5^k_3, 6^k_1, 6^k_3$ | 81 |
| $5^k_2, 5^k_4, 6^k_{15}, 6^k_{19}, 4^k_1, 6^k_2, 6^k_4, 6^k_5, 6^k_8$ | 243 |
| $4^k_1, 4^k_2, 4^k_3, 5^k_1, 6^k_{11}, 6^k_{13}, 6^k_{14}, 6^k_{16}, 6^k_3, 6^k_9$ | 729 |
| $6^k_{12}, 5^k_2, 6^k_{11}$ | 2187 |
| $6^k_1, 6^k_2, 6^k_5, 6^k_8, 6^k_{10}$ | 6561 |

**Example 9.** We computed the coloring invariant $\Phi^Z_X$ of a choice of orientation for the pseudoknots up to 5 crossings in the pseudoknot tables in [8] using the psybracket $X$ with the operation matrix
\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2
\end{bmatrix}, \begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2
\end{bmatrix}, \begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2
\end{bmatrix}, \begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2
\end{bmatrix}
\]
The results are collected in the table.

| $L$                                      | $\Phi^Z_X(L)$ |
|------------------------------------------|---------------|
| $3_1.1, 3_1.2, 3_1.3, 4_1.1, 4_1.2, 4_1.3, 4_1.4, 4_1.5, 5_2.1$ | 9             |
| $5_2.3, 5_2.4, 5_2.6$                     | 27            |
| $5_2.5, 5_2.7, 5_2.9$                     | 81            |
| $5_2.8, 5_2.10$                          | 729           |

Example 10. We computed the coloring invariant $\Phi^Z_X$ of a choice of orientation for the pseudoknots up to 5 crossings in the pseudoknot tables using the psybracket $X$ with the operation matrix

$$\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2
\end{bmatrix}, \begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2
\end{bmatrix}, \begin{bmatrix}
2 & 3 & 1 \\
3 & 1 & 2 \\
1 & 2 & 3
\end{bmatrix}_c \begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2
\end{bmatrix}, \begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2
\end{bmatrix}_p.$$

The results are collected in the table.

| $L$                                      | $\Phi^Z_X(L)$ |
|------------------------------------------|---------------|
| $3_1.2, 3_1.3, 4_1.2, 4_1.3, 5_1.1, 5_1.3, 5_1.4, 5_2.1$ | 9             |
| $3_1.1, 4_1.1, 4_1.4, 4_1.5, 5_2.1, 2_5.5, 5_2.2$ | 27            |
| $5_2.3, 5_2.4, 5_2.6$                     | 81            |
| $5_2.5, 5_2.7, 5_2.9$                     | 243           |
| $5_2.8, 5_2.10$                          | 729           |

5 Questions

In this paper we have only initiated the study of the new topic of psybrackets and their pseudo/singular knot and link invariants. There are many interesting questions to be explored in this area; we suggest a few of them here.

- As with counting invariants arising from other structures, many types of enhancements are possible. Applying a historically successful strategy, we ask what invariants of psybracket-colored pseudoknots are possible. Ideas might include cocycle enhancements analogous to those in [15], skein enhancements like those in [1], module enhancements like those in [13] and many more.

- What is the structure of psybrackets? What kinds of products, decompositions, functors to and from other algebraic categories are possible?

- What generalizations are possible to the cases of pseudo/singular trivalent graphs and handlebody knots or to the virtual and twisted virtual cases?

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