Abstract: Free vibration analysis of the porous functionally graded circular plates has been presented on the basis of classical plate theory. The three defined coupled equations of motion of the porous functionally graded circular/annular plate were decoupled to one differential equation of free transverse vibrations of plate. The one universal general solution was obtained as a linear combination of the multiparametric special functions for the functionally graded circular and annular plates with even and uneven porosity distributions. The multiparametric frequency equations of functionally graded porous circular plate with diverse boundary conditions were obtained in the exact closed-form. The influences of the even and uneven distributions of porosity, power-law index, diverse boundary conditions and the neglected effect of the coupling in-plane and transverse displacements on the dimensionless frequencies of the circular plate were comprehensively studied for the first time. The formulated boundary value problem, the exact method of solution and the numerical results for the perfect and imperfect functionally graded circular plates have not yet been reported.

Keywords: eigenvalue problem; axisymmetric and non-axisymmetric vibrations; multiparametric special functions; circular plate; functionally graded porous material

1. Introduction

Functionally graded materials (FGMs) are a class of composite materials, which are made of the ceramic and metal mixture such that the material properties vary continuously in appropriate directions of structural components. In the processes of preparing functionally graded material, micro-voids and porosities may appear inside material in view of the technical issues. Zhu et al. [1] reported that many porosities appear in material during the functionally graded material preparation process by the non-pressure sintering technique. Wattanasakulpong et al. [2] reported that many porosities exist in the intermediate area of the functionally graded material fabricated by utilizing a multi-step sequential infiltration technique because of the problem with infiltration of the secondary material into the middle area. In that case, less porosities appear in the top and bottom area of material because infiltration of the material is easier in these zones.

In recent years, a significant number of articles about the free vibrations of porous functionally graded (FGM) plates have appeared in the literature due to their wide applications in many fields of engineering such as aeronautical, civil, mechanical, automotive, and ocean engineering. The gradation of properties in functionally graded materials and the diverse distributions of porosity have a significant effect on distributions of the mass and the stiffness of plates and therefore their natural frequencies. The knowledge about influence of distribution of the material properties on dynamics of plates is very important because it allows us to predict the frequency of plates and find their optimal parameters. Additionally, the comprehensive investigation of the effect of functionally...
graded material with porosities and diverse boundary conditions on the natural frequencies of plates is the first important step to designing their safe and rational active vibration control system.

We note that, in most engineering applications, the classical plate theory is often used to analyze the dynamic behavior of thin lightweight plates. It is impossible to review all works focused on mechanical behavior of porous FGM structures; then, we limit ourselves to chronological review of some of the works focused on mechanical behavior of porous and porous FGM plates that are closely related to our work.

Jabbari et al. [3] studied the buckling of thin saturated porous circular plate with the layers of piezoelectric actuators. Buckling load was obtained for clamped circular plate under uniform radial compressive loading. The same authors presented the buckling analysis of clamped thin saturated porous circular plate with sensor–actuator layers under uniform radial compression [4,5] investigated thermal and mechanical stability of clamped thin saturated and unsaturated porous circular plates with piezoelectric actuators. Rad and Shariyat [6] solved the three-dimensional magneto-elastic problem for asymmetric variable thickness porous FGM circular supported on the Kerr elastic foundation using the differential quadrature method and the state space vector technique. Barati et al. [7] studied buckling of functionally graded piezoelectric rectangular plates with porosities based on the four-variable plate theory. Mechab et al. [8] studied free vibration of the FGM nanoplate with porosities resting on Winkler and Pasternak elastic foundation based on the two-variable plate theory. Mojahedin et al. [9] analyzed buckling of radially loaded clamped saturated porous circular plates based on higher order shear deformation theory. Wang and Zu [10] analyzed vibration behaviors of thin FGM rectangular plates with porosities and moving in the thermal environment using the method of harmonic balance and the Runge–Kutta technique. Gupta and Talha [11] analyzed flexural and vibration response of porous FGM rectangular plates using nonpolynomial higher-order shear and the normal deformation theory. Wang and Zu [12] analyzed vibration characteristics of longitudinally moving sigmoid porous FGM plates based on the von Kármán nonlinear plate theory. Ebrahimi et al. [13] studied free vibration of smart shear deformable rectangular plates made of porous magneto-electro-elastic functionally graded materials. Feyzi and Khorshidvand [14] studied axisymmetric post-buckling behavior of a saturated porous circular plate with simply supported and clamped boundary conditions. Wang and Zu [15] studied large-amplitude vibration of thin sigmoid functionally graded plates with porosities. Wang et al. [16] studied vibrations of longitudinally travelling FGM porous thin rectangular plates using the Galerkin method and the four-order Runge–Kutta method. Ebrahimi et al. [17] used a four-variable shear deformation refined plate theory for free vibration analysis of embedded smart rectangular plates made of magneto-electro-elastic porous functionally graded materials. Shahverdi and Barati [18] developed nonlocal strain-gradient elasticity model for vibration analysis of porous FGM nano-scale rectangular plates. Shojaaee et al. [19] studied free vibration and thermal buckling of micro temperature-dependent FGM porous circular plate using the generalized differential quadrature method. Barati and Shahverdi [20] presented a new solution to examine large amplitude vibration of a porous nanoplate resting on a nonlinear elastic foundation modeled based on the four-variable plate theory. Kiran et al. [21] studied free vibration of porous FGM magneto-electro-elastic skew plates using the finite element formulation. Cong et al. [22] presented an analytical approach to buckling and post-buckling behavior analysis of FGM rectangular plates with porosities under thermal and thermomechanical loads based on the Reddy’s higher-order shear deformation theory. Kiran and Kattimani [23] studied free vibration and static behavior of porous FGM magneto-electro-elastic rectangular plates using the finite element method. Arshid and Khorshidvand [24] analyzed free vibration of saturated porous FGM circular plates integrated with piezoelectric actuators using the differential quadrature method. Shahsavari et al. [25] used the quasi-3D hyperbolic theory for free vibration of porous FGM rectangular plates resting on Winkler, Pasternak and Kerr foundations.
2. Contribution of Current Study

The aim of the paper is to formulate and solve the boundary value problem for the free axisymmetric and non-axisymmetric vibrations of FGM circular plate with even and uneven porosity distributions and diverse boundary conditions. The defined coupled equations of motion for the porous FGM circular plate were decoupled based on the properties of physical neutral surface. The general solution of the decoupled equation of motion of a porous FGM circular plate was defined as the linear combination of the Bessel functions functionally dependent on the material parameters. The obtained characteristic equations allow us to comprehensively study the effect of the distribution of material parameters and the formulated boundary conditions on the natural frequencies of axisymmetric and non-axisymmetric vibrations of the circular plates without the necessity to solve a new eigenvalue problem for plates with a steady distribution of parameters.

Authors of many previous papers (e.g., [26–30]) presented the free transverse vibration analysis of the perfect (without porosity) FGM circular plates using the equation of motion including only the coefficient of the pure bending stiffness varying in the thickness direction of the plate. The coefficients of the extensional stiffness and the bending-extensional coupling stiffness were neglected because the effect of the coupled in-plane and transverse displacements was omitted for obtaining simplified solution to the eigenvalue problem.

In the present paper, the obtained equation of motion of the perfect and imperfect FGM circular plates includes the coefficients of extensional stiffness, bending-extensional coupling stiffness and bending stiffness, which appeared by decoupling the in-plane and transverse displacements using the properties of the physical neutral surface. The differences between the values of numerical results for the eigenfrequencies of the perfect FGM circular plate with and without the coupling effect are shown for diverse boundary conditions.

To the best knowledge of authors, there are no studies which focus on the free axisymmetric and non-axisymmetric vibrations of FGM and porous FGM circular plates. In particular, the obtained exact solution, the multiparametric frequency equations and the calculated eigenfrequencies for the free vibrations of perfect and imperfect FGM circular plates with clamped, simply supported, sliding and free edges have not yet been reported. The present paper fills this void in the literature.

3. FGM Circular Plate with Porosities

Consider a porous FGM thin circular plate with radius R and thickness h presented in the cylindrical coordinate (r, θ, z) with the z-axis along the longitudinal direction. The geometry and the coordinate system of the considered circular plate are shown in Figure 1. The FGM plate contains evenly (e) and unevenly (u) distributed porosities along the plate’s thickness direction. The cross-sections of the FGM circular plates with the two various types of distribution of porosities are shown in Figure 2.

\[\begin{align*}
\nu(z, g, \phi) &= (\nu_0 - \nu_1) \left(1 + \frac{\nu_0 + \nu_1}{2} \phi \right) + \nu_1 - \nu_0 (\nu_0 + \nu_1) \left(1 - \frac{\nu_0 + \nu_1}{2} \phi \right), \\
\rho(z, g, \phi) &= (\rho_0 - \rho_1) \left(1 + \frac{\rho_0 + \rho_1}{2} \phi \right) + \rho_1 - \rho_0 (\rho_0 + \rho_1) \left(1 - \frac{\rho_0 + \rho_1}{2} \phi \right), \\
\varepsilon(z, g) &= \varepsilon_0 - \varepsilon_1, \\
\varepsilon(z, g) &= \varepsilon_0 - \varepsilon_1,
\end{align*}\]

Figure 1. The geometry and the coordinate system of the porous FGM circular plate.
The functionally graded material is a mixture of a ceramic (c) and a metal (m). If the volume fraction of the ceramic part is \( V_c \) and the metallic part is \( V_m \), we have the well-known dependence:

\[
V_c(z) + V_m(z) = 1.
\]  

(1)

Based on the modified rule of mixtures [16] with the porosity volume fraction \( \psi \) \( (\psi \ll 1) \), the Young’s modulus, the density and the Poisson’s ratio for evenly \( (\psi \ll 1) \) distributed porosities over the cross-section of the plate have the general forms:

\[
E^f(z, \psi) = E_c \left[ V_c(z) - \frac{\psi}{2} \right] + E_m \left[ V_m(z) - \frac{\psi}{2} \right],
\]  

(2a)

\[
\rho^f(z, \psi) = \rho_c \left[ V_c(z) - \frac{\psi}{2} \right] + \rho_m \left[ V_m(z) - \frac{\psi}{2} \right],
\]  

(2b)

\[
\nu^f(z, \psi) = \nu_c \left[ V_c(z) - \frac{\psi}{2} \right] + \nu_m \left[ V_m(z) - \frac{\psi}{2} \right].
\]  

(2c)

The volume fraction of the ceramic part changes continually along the thickness and can be defined as [31]

\[
V_c(z, g) = \left( \frac{z}{h} + \frac{1}{2} \right)^g, \quad g \geq 0,
\]  

(3)

where \( g \) is the power-law index of the material. A change in the power \( g \) of functionally graded material results in a change in the portion of the ceramic and metal components in the circular plate. We assume that the composition is varied from the bottom surface \( (z = -h/2) \) to the top surface \( (z = h/2) \) of the circular plate. After substituting the variation of the ceramic part \( V_c(z, g) \) from Equation (3) into Equation (2), the material properties of the functionally graded circular plate with evenly distributed porosities are defined in the final form:

\[
E^f(z, g, \psi) = (E_c - E_m) \left( \frac{z}{h} + \frac{1}{2} \right)^g + E_m - \frac{\psi}{2}(E_c + E_m),
\]  

(4a)

\[
\rho^f(z, g, \psi) = (\rho_c - \rho_m) \left( \frac{z}{h} + \frac{1}{2} \right)^g + \rho_m - \frac{\psi}{2}(\rho_c + \rho_m),
\]  

(4b)

\[
\nu^f(z, g, \psi) = (\nu_c - \nu_m) \left( \frac{z}{h} + \frac{1}{2} \right)^g + \nu_m - \frac{\psi}{2}(\nu_c + \nu_m).
\]  

(4c)

Figure 2. The cross-sections of the porous FGM circular plate: (a) even distribution; (b) uneven distribution.

The volume fraction of the ceramic part changes continually along the thickness and can be defined as [31]

\[
V_c(z, g) = \left( \frac{z}{h} + \frac{1}{2} \right)^g, \quad g \geq 0,
\]  

(3)
For the functionally graded circular plate with unevenly (\(i\)) distributed porosities \([16]\), the material properties in Equations (4) can be replaced by the following forms:

\[
E^u(z, \theta, \psi) = (E_c - E_m) \left( \frac{z}{h} + \frac{1}{2} \right)^3 + E_m - \frac{\psi}{2} (E_c + E_m) \left( 1 - \frac{2|z|}{h} \right),
\]  
(5a)

\[
\rho^u(z, \theta, \psi) = (\rho_c - \rho_m) \left( \frac{z}{h} + \frac{1}{2} \right)^3 + \rho_m - \frac{\psi}{2} (\rho_c + \rho_m) \left( 1 - \frac{2|z|}{h} \right),
\]  
(5b)

\[
v^u(z, \theta, \psi) = (v_c - v_m) \left( \frac{z}{h} + \frac{1}{2} \right)^3 + v_m - \frac{\psi}{2} (v_c + v_m) \left( 1 - \frac{2|z|}{h} \right).
\]  
(5c)

In this case, the porosity linearly decreases to zero at the top and the bottom of the cross-section of the plate. The effect of Poisson’s ratio is much less on the mechanical behavior of FGM plates than the Young’s modulus \([32,33]\), thus the Poisson’s ratio will assume to be constant \(\nu' = \nu^u = \nu\) in the whole volume of the porous FGM circular plate.

4. Constitutive Relations and Governing Equations

In most practical applications, the ratio of the radius \(R\) to the thickness \(h\) of the plate is more than 10; then, the assumptions of classical plate theory (CPT) are applicable and rotary inertia and shear deformation can be successfully omitted.

For a thin circular plate, the displacement field has the form:

\[
u_r\left(r, \theta, z, t\right) = u\left(r, \theta, t\right) - z \frac{\partial w\left(r, \theta, t\right)}{\partial r},
\]  
(6a)

\[
u_\theta\left(r, \theta, z, t\right) = v\left(r, \theta, t\right) - z \frac{\partial w\left(r, \theta, t\right)}{\partial \theta},
\]  
(6b)

\[w\left(r, \theta, z, t\right) = w\left(r, \theta, t\right),
\]  
(6c)

where \(u, v\) and \(w\) are the radial, circumferential and transverse displacements of the midplane \((z = 0)\) of the plate at time \(t\). Based on the linear strain–displacement relations and Hook’s law, the resultant forces and the moments for porous FGM circular plate \((i = \{e, u\})\) can be expressed in the following form \([34]\):

\[
\begin{bmatrix}
N^i_{rr} \\
N^i_{\theta\theta} \\
N^i_{r\theta}
\end{bmatrix} = \begin{bmatrix}
A^{i_{11}} & A^{i_{12}} & 0 \\
A^{i_{12}} & A^{i_{11}} & 0 \\
0 & 0 & A^{i}_{33}
\end{bmatrix} \begin{bmatrix}
\varepsilon^0_{rr} \\
\varepsilon^0_{\theta\theta} \\
\gamma^0_{r\theta}
\end{bmatrix} + \begin{bmatrix}
B^{i}_{11} & B^{i}_{12} & 0 \\
B^{i}_{12} & B^{i}_{11} & 0 \\
0 & 0 & B^{i}_{33}
\end{bmatrix} \begin{bmatrix}
\kappa_{rr} \\
\kappa_{\theta\theta} \\
\kappa_{r\theta}
\end{bmatrix},
\]  
(7a)

\[
\begin{bmatrix}
M^i_{rr} \\
M^i_{\theta\theta} \\
M^i_{r\theta}
\end{bmatrix} = \begin{bmatrix}
B^{i}_{11} & B^{i}_{12} & 0 \\
B^{i}_{12} & B^{i}_{11} & 0 \\
0 & 0 & B^{i}_{33}
\end{bmatrix} \begin{bmatrix}
\varepsilon^0_{rr} \\
\varepsilon^0_{\theta\theta} \\
\gamma^0_{r\theta}
\end{bmatrix} + \begin{bmatrix}
D^{i}_{11} & D^{i}_{12} & 0 \\
D^{i}_{12} & D^{i}_{11} & 0 \\
0 & 0 & D^{i}_{33}
\end{bmatrix} \begin{bmatrix}
\kappa_{rr} \\
\kappa_{\theta\theta} \\
\kappa_{r\theta}
\end{bmatrix},
\]  
(7b)

where

\[
\left(\varepsilon^0_{rr}, \varepsilon^0_{\theta\theta}, \gamma^0_{r\theta}\right) = \left(\frac{\partial u}{\partial r}, \frac{1}{r^2} \frac{\partial v}{\partial \theta}, \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right),
\]  
(8a)

\[
\left(\kappa_{rr}, \kappa_{\theta\theta}, \kappa_{r\theta}\right) = \left(-\frac{\partial^2 w}{\partial r^2}, -\frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial w}{\partial r}, -\frac{2}{r^2} \frac{\partial^2 w}{\partial r \partial \theta} + \frac{2}{r^2} \frac{\partial w}{\partial \theta}\right)
\]  
(8b)

are the in-plane strains and curvatures of midplane, respectively.

We assume that the material properties are varied from the bottom surface \((z = -h/2)\) to the top surface \((z = h/2)\) of the plate; then, the coefficients of extensional stiffness \(A^{i}_{kl}\), bending-extensional
coupling stiffness \( B_{kl}^i \) and bending stiffness \( D_{kl}^i \) can be defined for FGM circular plate with \( i\)-th distribution of porosities in the general forms:

\[
\begin{align*}
(A_{11}^i, B_{11}^i, D_{11}^i) &= \int_{-h/2}^{h/2} \frac{E(z, \psi, \theta)}{1 - v^2} (1, z, z^2) dz, \\
(A_{12}^i, B_{12}^i, D_{12}^i) &= \int_{-h/2}^{h/2} \frac{vE(z, \psi, \theta)}{1 - v^2} (1, z, z^2) dz, \\
(A_{33}^i, B_{33}^i, D_{33}^i) &= \int_{-h/2}^{h/2} \frac{E(z, \psi, \theta)}{2(1 + v)} (1, z, z^2) dz.
\end{align*}
\]  

(9a, 9b, 9c)

Additionally, the stiffness coefficients from Equation (9) satisfy the equations

\[
A_{12}^i + 2A_{33}^i = A_{11}^i, \quad B_{12}^i + 2B_{33}^i = B_{11}^i, \quad D_{12}^i + 2D_{33}^i = D_{11}^i.
\]  

(10)

The resultant forces and the moments can be also defined by

\[
\begin{align*}
(N_{rr}^i, N_{\theta\theta}^i, N_{r\theta}^i) &= \int_{-h/2}^{h/2} \left( \sigma_{rr}^i, \sigma_{\theta\theta}^i, \tau_{r\theta}^i \right) dz, \\
(M_{rr}^i, M_{\theta\theta}^i, M_{r\theta}^i) &= \int_{-h/2}^{h/2} \left( \sigma_{rr}^i z, \sigma_{\theta\theta}^i z, \tau_{r\theta}^i z \right) dz,
\end{align*}
\]  

(11a, 11b)

where the stress components and the strain components have the form:

\[
\begin{pmatrix}
\sigma_{rr}^i \\
\sigma_{\theta\theta}^i \\
\tau_{r\theta}^i
\end{pmatrix} =
\begin{pmatrix}
E(z, \psi, \theta) (\varepsilon_{rr} + v \varepsilon_{\theta\theta}) \\
E(z, \psi, \theta) (\varepsilon_{\theta\theta} + v \varepsilon_{rr}) \\
\frac{E(z, \psi, \theta)}{2(1 + v)} (2 \gamma_{r\theta})
\end{pmatrix},
\]  

(12)

\[
\begin{pmatrix}
\varepsilon_{rr} \\
\varepsilon_{\theta\theta} \\
2\gamma_{r\theta}
\end{pmatrix} =
\begin{pmatrix}
\varepsilon_{rr}^0 + z \kappa_{rr} \\
\varepsilon_{\theta\theta}^0 + z \kappa_{\theta\theta} \\
\gamma_{r\theta}^0 + z \kappa_{r\theta}
\end{pmatrix}.
\]  

(13)

4.1. Coupled Equations of Motion

Using the Hamilton’s principle [34] and ignoring in-plane inertia forces, the equilibrium equations of motion of the porous FGM thin circular plate have the forms:

\[
\frac{\partial N_{rr}^i}{\partial r} + \frac{1}{r} \left( \frac{\partial N_{r\theta}^i}{\partial \theta} + N_{rr}^i - N_{\theta\theta}^i \right) = 0,
\]  

(14a)

\[
\frac{\partial^2 M_{rr}^i}{\partial r^2} + \frac{2}{r} \frac{\partial M_{rr}^i}{\partial r} + \frac{1}{r^2} \frac{\partial^2 M_{r\theta}^i}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial^2 M_{\theta\theta}^i}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial \varepsilon_{rr}^0}{\partial \theta} + \frac{2}{r^2} \frac{\partial \gamma_{r\theta}^0}{\partial \theta} = \rho^i h \frac{\partial^2 w}{\partial r^2},
\]  

(14c)

where the resultant forces and the moments can be obtained using Equations (7) and (8), and can be presented in the following form:

\[
N_{rr}^i = A_{11}^i \frac{\partial u}{\partial r} + A_{12}^i \left( \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} \right) - B_{11}^i \frac{\partial^2 w}{\partial r^2} - B_{12}^i \left( \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right),
\]  

(15a)

\[
N_{r\theta}^i = A_{12}^i \frac{\partial u}{\partial \theta} + A_{11}^i \left( \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} \right) - B_{12}^i \frac{\partial^2 w}{\partial r^2} - B_{11}^i \left( \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right),
\]  

(15b)
By substituting Equation (20) into Equation (15), the in-plane forces \( N_{z} \) on the surface, where the in-plane displacements will be omitted. The in-plane displacements of the midplane surface is a geometrical midplane. We can eliminate this coupling by introducing the physical neutral surface. By substituting \( \rho^i = 1 \int_{-h/2}^{h/2} \rho^i(z, g, \psi) dz, \ i = \{ e, u \}. \)

Substituting Equations (15) and (16) into Equation (14), and using relations given in Equation (10), we get the coupled equilibrium equations of motion of the porous FGM circular plate presented in terms of displacement components:

\[
A_{i1} \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} - \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial^2 v}{\partial \theta^2} \right) + A_{i3} \left( \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial^2 v}{\partial r \partial \theta} + \frac{1}{r^2} \frac{\partial v}{\partial \theta} \right) - B_{i11} \frac{\partial^2 w}{\partial r^2} = 0, \tag{18a}
\]

\[
A_{i1} \left( \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial^2 v}{\partial r \partial \theta} \right) + A_{i3} \left( \frac{1}{r^2} \frac{\partial^2 u}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial v}{\partial r} \right) - B_{i11} \frac{\partial^2 w}{\partial \theta^2} = 0, \tag{18b}
\]

\[
\frac{1}{r^2} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r} \frac{\partial w}{\partial \theta} = -\rho^i \frac{\partial^2 w}{\partial \theta^2}, \tag{18c}
\]

where \( \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \) is the Laplace operator presented in polar coordinates and

\[
\varepsilon = \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r}. \tag{19}
\]

4.2. Decoupled Equation of Motion

Equation (18) show that the in-plane stretching and bending are coupled because the reference surface is a geometrical midplane. We can eliminate this coupling by introducing the physical neutral surface, where the in-plane displacements will be omitted. The in-plane displacements of the midplane can be expressed in terms of the slopes of deflection in the following form:

\[
u(r, \theta, t) = z_0 \frac{\partial w(r, \theta, t)}{\partial \theta}, \tag{20a}
\]

\[
v(r, \theta, t) = z_0 \frac{\partial w(r, \theta, t)}{\partial \theta}, \tag{20b}
\]

where \( z_0 \) is the distance between the midplane and the physical neutral surface. By substituting Equation (20) into Equations (6) and (15) and introducing \( z = z_0 \), the in-plane displacements \( u, v \) and the in-plane forces \( N_{zr}, N_{z\theta}, N_{z\phi} \) must equal zero based on properties of the physical neutral surface. By substituting Equation (20) into Equation (15)

\[
N_{zr} = \left( z_0 A_{11} - B_{11} \right) \frac{\partial^2 w}{\partial r^2} + \left( z_0 A_{12} - B_{12} \right) \left( \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right) = 0, \tag{21a}
\]

\[
N_{z\theta} = \left( z_0 A_{12} - B_{12} \right) \frac{\partial^2 w}{\partial r^2} + \left( z_0 A_{11} - B_{11} \right) \left( \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right) = 0, \tag{21b}
\]
where 

\[ L = \text{constant}, \quad \text{distance } z_0 = \text{constant} \]

and assuming that the Poisson’s ratio is constant, distance \( z_0 \) can be obtained from relations:

\[ z_0 A_{11}^i - B_{11}^i = z_0 A_{12}^i - B_{12}^i = z_0 A_{33}^i - B_{33}^i = 0, \]

where

\[ z_0 = \frac{B_{11}^i}{A_{11}^i} = \frac{B_{12}^i}{A_{12}^i} = \frac{B_{33}^i}{A_{33}^i} = \frac{\int_{R_{1/2}}^{R} E(z, \theta, \psi)dz}{\int_{R_{1/2}}^{R} E(z, \theta, \psi)dz}. \]

By substituting Equations (20) and (23) into Equations (18c) and (19), we obtain the decoupled equation of transverse vibration of the porous thin circular plate in the form:

\[ D^i \nabla^2 w = -\rho^i \frac{\partial^2 w}{\partial t^2}, \]

where

\[ D^i = D_{11}^i - \frac{(B_{11}^i)^2}{A_{11}^i}. \]

5. Solution of the Problem

Taking into account a harmonic solution, the small vibration of the porous FGM circular plate may be expressed as follows:

\[ w(r, \theta, t) = W(r) \cos(n\theta) \cos(\omega t), \]

where \( W(r) \) is the radial mode function as the small deflection compared with the thickness \( h \) of the plate, \( n \) is the integer number of diagonal nodal lines, \( \theta \) is the angular coordinate, and \( \omega \) is the natural frequency. By substituting Equation (26) into Equation (24) using the dimensionless coordinate \( \xi = r/R \) \( 0 < \xi < 1 \), the general governing differential equation assumes the following form:

\[ L_n(W) = \rho^i h \omega^2 W, \]

where \( L_n(\cdot) \) is the differential operator defined by

\[ L_n(\cdot) \equiv \frac{\partial^4}{\partial \xi^4} + \frac{2 \partial^2}{\partial \xi^2} \frac{\partial^2}{\partial \xi^2} + \frac{(1 + 2n^2)}{\xi^2} \frac{\partial^2}{\partial \xi^2} + \frac{(1 + 2n^2)}{\xi^2} \frac{\partial^2}{\partial \xi^2} + \frac{(n^4 - 4n^2)}{\xi^4}. \]

The calculated general forms of material density \( \rho^i \) and the coefficients of extensional stiffness \( A_{11}^i \), extensional-bending coupling stiffness \( B_{11}^i \) and bending stiffness \( D_{11}^i \) for the porous FGM circular plate are presented in the following general forms:

\[ \rho^i = \rho_c(2x - \psi - \psi \gamma) + \rho_m(2x \gamma - \psi - \psi \gamma), \]

\[ A_{11}^i = \frac{E_i h}{1 - \nu^2} \left[ \frac{(2x - \psi - \psi \gamma) + \frac{E_i}{E_m} (2x \gamma - \psi - \psi \gamma)}{2x(1 + \gamma)} \right], \]

\[ B_{11}^i = B_{11}^m \left[ \frac{\gamma(1 - \frac{E_m}{E_i})}{2(1 + \gamma)(2 + \gamma)} \right], \]

\[ D^i = D^m \left[ \frac{\psi(6x^2 + 6x + 12) - \psi(1 + \gamma)(2 + \gamma)(3 + \gamma) + \frac{E_m}{E_i} \psi(2x^2 + 6x + 16x^3 - \psi(1 + \gamma)(2 + \gamma)(3 + \gamma))}{2(1 + \gamma)(2 + \gamma)(3 + \gamma)} \right]. \]
where \( x = y = 1 \) for the even distribution (\( i = e \)) of porosities and \( x = 2, y = 4 \) for the uneven (\( i = u \)) distribution of porosities. The extensional-bending coupling stiffness \( B'_{11} \) has the same form for both types of porosities.

By substituting the obtained forms from Equation (29) into Equation (27), the generalized ordinary differential equation with variable coefficients is obtained as:

\[
\mathcal{L}_n(W) \chi = \lambda^2 \mu^j W,
\]

where

\[
\mathcal{L}_n(W) \chi = (\chi^i_1 + \chi^i_2) \frac{d^4}{d\xi^4} + \frac{2(\chi^i_1 + \chi^i_2)}{\xi} \frac{d^3}{d\xi^3} - \frac{(1+2\eta^2)(\chi^i_1 + \chi^i_2)}{\xi^2} \frac{d^2}{d\xi^2} + \frac{(1+2\eta^2)(\chi^i_1 + \chi^i_2)}{\xi^3} \frac{d}{d\xi} + \left( n^3 - 4n^2 \right) \left( \chi^i_1 + \chi^i_2 \right),
\]

\[
\chi^i_1 = \frac{6\eta^2(E_c - E_m)^2}{E_c(1 + \eta)(2 + \eta)^2[E_c(\psi + g\psi - 2\chi) + E_m(\psi + g\psi - 2\chi\eta)]},
\]

\[
\chi^i_2 = \frac{E_c \left[ \psi(12 + 6\eta + 6\eta^2) - \psi(1 + \eta)(2 + \eta)(3 + \eta) \right] + E_m \left[ \psi(16\eta + 6\eta^2 + 2\eta^3) - \psi(1 + \eta)(2 + \eta)(3 + \eta) \right]}{2\eta^2(3 + \eta)(2 + \eta)(3 + \eta)},
\]

\[
\mu^j = \frac{(-g\psi - \psi + 2\chi) - \frac{\nu \psi}{E_c}(g\psi + \psi - 2\chi\eta)}{2\chi(1 + \eta)},
\]

\[
\lambda = \omega R^2 \sqrt{\rho_c h / D_c},
\]

\[
D_c = \frac{E_c h^3}{12(1 - \nu^2)}.
\]

The boundary conditions on the outer edge (\( \xi = 1 \)) of the porous FGM circular plate may be one of the following: clamped, simply supported, sliding supported and free. These conditions may be written in terms of the radial mode function \( W(\xi) \) in the following form:

- Clamped:

\[
W(\xi) \big|_{\xi = 1} = 0,
\]

\[
\frac{dW}{d\xi} \bigg|_{\xi = 1} = 0.
\]

- Simply supported:

\[
W(\xi) \big|_{\xi = 1} = 0,
\]

\[
M(W) \bigg|_{\xi = 1} = \left[ \frac{d^2 W}{d\xi^2} + \frac{v}{\xi} \frac{dW}{d\xi} - \frac{\nu n^2}{\xi^2} W \right] \bigg|_{\xi = 1} = 0.
\]

- Sliding supported:

\[
\frac{dW}{d\xi} \bigg|_{\xi = 1} = 0,
\]

\[
V(W) \bigg|_{\xi = 1} = \left[ \frac{d^3 W}{d\xi^3} + \frac{1}{\xi} \frac{d^2 W}{d\xi^2} - \left( \frac{1 + 2\nu^2 - \nu n^2}{\xi^2} \right) \frac{dW}{d\xi} + \left( \frac{3\nu^2 - \nu n^2}{\xi^3} \right) W \right] \bigg|_{\xi = 1} = 0.
\]

- Free:

\[
M(W) \big|_{\xi = 1} = 0,
\]

\[
V(W) \big|_{\xi = 1} = 0.
\]

The static forces \( M(W) \) and \( V(W) \) are the normalized radial bending moment and the normalized effective shear force, respectively.
The one multiparametric general solution of the defined differential Equation (30) for FGM circular/annular plates with the two various types of distribution of porosities \( i = \{ e, u \} \) is obtained in the following form:

\[
W_n(\xi, \lambda, g, \psi) = C_1 I_n \left[ (\lambda \sqrt{\Omega})^{1/2} \xi \right] + C_2 I_n \left[ (\frac{\lambda \sqrt{\Omega}}{g})^{1/2} \xi \right] + C_3 Y_n \left[ (\lambda \sqrt{\Omega})^{1/2} \xi \right] + C_4 K_n \left[ (\lambda \sqrt{\Omega})^{1/2} \xi \right],
\]

where \( n (n \in \mathbb{N}^+) \) is the number of nodal lines, \( C_1, C_2, C_3, C_4 \) are the constants of integration, \( I_n \left[ (\lambda \sqrt{\Omega})^{1/2} \xi \right], I_n \left[ (\frac{\lambda \sqrt{\Omega}}{g})^{1/2} \xi \right], Y_n \left[ (\lambda \sqrt{\Omega})^{1/2} \xi \right], K_n \left[ (\lambda \sqrt{\Omega})^{1/2} \xi \right] \) are the Bessel functions as particular solutions of Equation (30), and \( \mathcal{M}_i \) is the generalized multiparametric function defined as:

\[
\mathcal{M}_i \equiv \mathcal{M}_i (x, y, g, \psi, E_m, E_c, \rho_m, \rho_c) = \frac{\Omega_1^i}{\Omega_2^i + \Omega_3^i}, \quad \mathcal{M}_i \geq 1 \forall g \in [0, \infty] \cap \forall \psi \in [0, 1),
\]

where

\[
\Omega_1^i = -E_c x (2 + g)^2 [\rho_c (g \psi + \psi - 2x) + \rho_m (g \psi + \psi - 2x g)],
\]

\[
\Omega_2^i = \frac{12xyg^2 (E_c - E_m) \rho_c}{E_c (g \psi + \psi - 2x) + E_m (g \psi + \psi - 2x g)},
\]

\[
\Omega_3^i = \frac{(2 + g)\rho_c \left[ E_c \left( y^2 + g^2 \right)^2 -\psi (1 + \psi) (2 + g) (3 + g) \right] + E_m \left[ (16g^2 + 6g^2 + 2g^2 - \psi (1 + \psi) (2 + g) (3 + g) \right]}{3g^2}.
\]

The functions \( I_n \left[ (\lambda \sqrt{\Omega})^{1/2} \xi \right] \) and \( I_n \left[ (\frac{\lambda \sqrt{\Omega}}{g})^{1/2} \xi \right] \) are the limited linear independent solutions \( \left( \lim_{\xi \to 0} \left[ (\lambda \sqrt{\Omega})^{1/2} \xi \right] < \infty, \lim_{\xi \to 0} \left[ (\frac{\lambda \sqrt{\Omega}}{g})^{1/2} \xi \right] < \infty \) of Equation (30) for the axisymmetric and non-axisymmetric deflections at center \( (\xi = 0) \) of the porous FGM circular plate and diverse values of the physically justified parameters \( \lambda, g, \) and \( \psi \). The particular solutions \( Y_n \left[ (\lambda \sqrt{\Omega})^{1/2} \xi \right] \) and \( K_n \left[ (\lambda \sqrt{\Omega})^{1/2} \xi \right] \) are the unlimited \( \left( \lim_{\xi \to 0} \left[ (\lambda \sqrt{\Omega})^{1/2} \xi \right] = -\infty, \lim_{\xi \to 0} \left[ (\frac{\lambda \sqrt{\Omega}}{g})^{1/2} \xi \right] = \infty \) for the deflection at the center of the plate, then, the general solution (41) for the porous FGM circular plate can be presented in the new form:

\[
W_n^i(\xi, \lambda, g, \psi) = C_1 \Psi_1 + C_2 \Psi_2,
\]

where

\[
\Psi_1 \equiv I_n \left[ (\lambda \sqrt{\Omega})^{1/2} \xi \right], \quad \Psi_2 \equiv I_n \left[ (\lambda \sqrt{\Omega})^{1/2} \xi \right].
\]

By applying the general solution (44) and the boundary conditions (37–40) as well as assuming the existence of the non-trivial constants \( C_1 \) and \( C_2 \), the general nonlinear multiparametric characteristic equations of the FGM circular plate with the two various types of distribution of porosities were obtained in the form:

- **Clamped (C):**

\[
\Delta_C^i(\lambda, g, \psi, n, x, y) \equiv \left| \begin{array}{cc} \frac{\partial \Psi_1}{\partial \xi} & \frac{\partial \Psi_2}{\partial \xi} \\ \frac{\partial \Psi_1}{\partial \xi} & \frac{\partial \Psi_2}{\partial \xi} \end{array} \right|_{\xi = 1} = 0;
\]

- **Simply supported (SS):**

\[
\Delta_{SS}^i(\lambda, g, \psi, n, x, y) \equiv \left| \begin{array}{cc} \Psi_1 & \Psi_2 \\ M[\Psi_1] & M[\Psi_2] \end{array} \right|_{\xi = 1} = 0;
\]
where will be valid for the FGM circular plates with even (i = e) distribution of porosities. If \( x = 2, y = 4 \) is introduced to Equations (42) and (45), then the obtained characteristic equation (46) will be valid for the FGM circular plates with uneven (i = u) distribution of porosities.

The general solution for the perfect (without porosity) FGM circular plate can be obtained from Equation (44) and presented in the following form:

\[
W_n(\xi, \lambda, g) = C_1 I_n \left[ (\lambda \sqrt{\frac{\psi g}{\rho}}) \frac{1}{2} \zeta \right] + C_2 I_n \left[ (\lambda \sqrt{\frac{\psi g}{\rho}}) \frac{1}{2} \zeta \right].
\]  (47)

After calculations, the final form of general solution for the perfect FGM circular plate is expressed as

\[
W_n(\xi, \lambda, g) = I_n \left[ (\lambda \sqrt{\frac{\psi g}{\rho}}) \frac{1}{2} \zeta \right] + I_n \left[ (\lambda \sqrt{\frac{\psi g}{\rho}}) \frac{1}{2} \zeta \right],
\]  (48)

where

\[\frac{\psi}{\rho} = \frac{E_c(2 + g)^3(3 + g)}{(3 + g)(E_c + gE_m)(\rho_c + g\rho_m)}\frac{1}{12E_c^2 + (28g + 16g^2 + 4g^3)E_cE_m + (7g^2 + 4g^3 + g^4)E_m^2}.\]  (49)

The general solution for the perfect FGM circular plate with negligible effect of the coupling in-plane and transverse displacements \((A_{11} \rightarrow 0, B_{11} \rightarrow 0)\) has the form:

\[
W_n(\xi, \lambda, g) = C_1 I_n \left[ (\lambda \sqrt{\frac{\psi g}{\rho}}) \frac{1}{2} \zeta \right] + C_2 I_n \left[ (\lambda \sqrt{\frac{\psi g}{\rho}}) \frac{1}{2} \zeta \right],
\]  (50)

where

\[\frac{\psi}{\rho} = \frac{E_c(2 + g)(3 + g)(\rho_c + g\rho_m)}{\rho_c[3E_c(2 + g + g^2) + E_m(8g + 3g^2 + g^3)]}.\]  (51)

6. Parametric Study

The every single fundamental and lower dimensionless frequencies of the free axisymmetric and non-axisymmetric vibrations of porous FGM circular plate were calculated for diverse values of the power-law index \( g \), the porosity volume fraction \( \psi \) and different boundary conditions using the Newton method aided by a calculation software.

The Poisson’s ratio is taken as \( v = 0.3 \) and its variation is assumed to be negligible. In the present study, aluminum is taken as the metal and alumina is taken as the ceramic material. The values of Young’s modulus and densities are taken as follows: \( E_m = 70 \text{ GPa}, E_c = 380 \text{ GPa}, \rho_m = 2702 \text{ kg/m}^3, \rho_c = 3800 \text{ kg/m}^3 \).

6.1. Imperfect FGM Circular Plate

The obtained numerical results for the first three dimensionless frequencies \( \lambda = \omega R^2 \sqrt{\rho_c h/D_c} \) of the axisymmetric \((n = 0)\) and non-axisymmetric \((n = 1)\) vibrations of the perfect \((\psi \rightarrow 0)\) homogeneous \((g \rightarrow 0)\) circular plate with various boundary conditions are presented in Table 1 and compared with the results obtained by Wu and Liu [35], Yalcin et al. [36], Zhou et al. [37] and
Duan et al. [38]. The obtained numerical results for the perfect homogeneous circular plate are in excellent agreement with those available in the literature.

Table 1. The dimensionless frequencies of the perfect homogeneous circular plate.

| n | Clamped | Simply Supported | Sliding Supported | Free |
|---|---------|------------------|------------------|------|
| 0  | 10.215  | 21.260           | 4.935            | 13.898|
| 1  | 14.682  | 3.082            | 9.003            | 20.474|

Table 2. The dimensionless fundamental frequencies of the clamped porous FGM circular plate.

| i  | n   | ψ   | g    | λ0   |
|----|-----|-----|------|------|
| 0  | 10.215 | 9.481 | 0.2  | 8.896 |
| 0.05 | 10.286 | 9.522 | 0.4  | 8.436 |
| 0.1  | 10.362 | 9.566 | 0.6  | 7.797 |
| 0.2  | 10.535 | 9.668 | 1    | 7.090 |
| 0.3  | 10.745 | 9.792 | 2    | 6.677 |
| 0.4  | 10.956 | 9.912 | 3    | 6.377 |
| 0.5  | 11.167 | 10.026 | 4    | 6.077 |
| 0.6  | 11.378 | 10.138 | 5    | 5.777 |

The calculated fundamental dimensionless frequencies λ0 of the axisymmetric (n = 0) and non-axisymmetric (n = 1) vibrations of the FGM circular plate with evenly (i = e) and unevenly (i = u) distributed porosity are presented in Tables 2–5. In the parametric study, values of the power-law index of FGMs is taken as $g = \{ 0, 0.2, 0.4, 0.6, 1, 2, 3, 4, 5 \}$ and values of the porosity volume fraction is taken as $\psi = \{ 0, 0.05, 0.1, 0.2, 0.3 \}$. 

Table 3. The dimensionless fundamental frequencies of the clamped FGM circular plate with evenly distributed porosity.

| i  | n   | ψ   | g    | λ0   |
|----|-----|-----|------|------|
| 0  | 10.215 | 9.481 | 0.2  | 8.896 |
| 0.05 | 10.286 | 9.522 | 0.4  | 8.436 |
| 0.1  | 10.362 | 9.566 | 0.6  | 7.797 |
| 0.2  | 10.535 | 9.668 | 1    | 7.090 |
| 0.3  | 10.745 | 9.792 | 2    | 6.677 |
| 0.4  | 10.956 | 9.912 | 3    | 6.377 |
| 0.5  | 11.167 | 10.026 | 4    | 6.077 |
| 0.6  | 11.378 | 10.138 | 5    | 5.777 |

The calculated fundamental dimensionless frequencies λ0 of the axisymmetric (n = 0) and non-axisymmetric (n = 1) vibrations of the FGM circular plate with evenly (i = e) and unevenly (i = u) distributed porosity are presented in Tables 2–5. In the parametric study, values of the power-law index of FGMs is taken as $g = \{ 0, 0.2, 0.4, 0.6, 1, 2, 3, 4, 5 \}$ and values of the porosity volume fraction is taken as $\psi = \{ 0, 0.05, 0.1, 0.2, 0.3 \}$. 

Table 4. The dimensionless fundamental frequencies of the clamped FGM circular plate with unevenly distributed porosity.
Table 3. The dimensionless fundamental frequencies of the simply supported porous FGM circular plate.

| $i$ | $n$ | $\psi$ | 0   | 0.2 | 0.4 | 0.6 | 1   | 2   | 3   | 4   | 5   |
|-----|-----|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|     |     |        | $\lambda_0$ |     |     |     |     |     |     |     |     |
| $e$ | 0   | 4.935  | 4.580 | 4.297 | 4.075 | 3.767 | 3.425 | 3.317 | 3.274 | 3.248 |
|     | 0.05| 4.969  | 4.600 | 4.302 | 4.064 | 3.728 | 3.343 | 3.218 | 3.168 | 3.141 |
|     | 0   | 5.005  | 4.621 | 4.306 | 4.051 | 3.683 | 3.242 | 3.092 | 3.033 | 3.004 |
|     | 0.2 | 5.089  | 4.670 | 4.315 | 4.017 | 3.562 | 2.953 | 2.711 | 2.609 | 2.563 |
|     | 0.3 | 5.190  | 4.730 | 4.322 | 3.965 | 3.378 | 2.432 | 1.908 | 1.600 | 1.412 |
| $u$ | 0   | 13.898 | 12.898 | 12.103 | 11.477 | 10.608 | 9.646 | 9.343 | 9.220 | 9.147 |
|     | 0.05| 13.993 | 12.954 | 12.115 | 11.446 | 10.500 | 9.415 | 9.062 | 8.923 | 8.847 |
|     | 0   | 14.097 | 13.015 | 12.127 | 11.410 | 10.372 | 9.131 | 8.708 | 8.543 | 8.460 |
|     | 1   | 14.333 | 13.153 | 12.152 | 11.312 | 10.032 | 8.218 | 7.635 | 7.349 | 7.218 |
|     | 0.2 | 14.618 | 13.322 | 12.173 | 11.166 | 9.514 | 6.849 | 5.373 | 4.506 | 3.977 |

Table 4. The dimensionless fundamental frequencies of the porous FGM circular plate with sliding support.

| $i$ | $n$ | $\psi$ | 0   | 0.2 | 0.4 | 0.6 | 1   | 2   | 3   | 4   | 5   |
|-----|-----|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|     |     |        | $\lambda_0$ |     |     |     |     |     |     |     |     |
| $e$ | 0   | 14.682 | 13.626 | 12.785 | 12.124 | 11.206 | 10.190 | 9.870 | 9.740 | 9.663 |
|     | 0.05| 14.782 | 13.685 | 12.798 | 12.092 | 11.092 | 9.946 | 9.573 | 9.426 | 9.346 |
|     | 0   | 14.892 | 13.749 | 12.811 | 12.053 | 10.956 | 9.646 | 9.199 | 9.025 | 8.938 |
|     | 0.2 | 15.141 | 13.895 | 12.837 | 11.950 | 10.597 | 8.786 | 8.066 | 7.764 | 7.625 |
|     | 0.3 | 15.442 | 14.074 | 12.860 | 11.796 | 10.051 | 7.236 | 5.676 | 4.761 | 4.201 |
| $u$ | 0   | 3.082  | 2.860  | 2.684  | 2.545  | 2.352  | 2.139  | 2.072  | 2.045  | 2.029 |
|     | 0.05| 3.103  | 2.873  | 2.687  | 2.538  | 2.328  | 2.088  | 2.010  | 1.980  | 1.962 |
|     | 0   | 3.126  | 2.886  | 2.690  | 2.530  | 2.300  | 2.025  | 1.931  | 1.894  | 1.876 |
|     | 0.2 | 3.178  | 2.917  | 2.695  | 2.509  | 2.225  | 1.844  | 1.693  | 1.630  | 1.600 |
|     | 0.3 | 3.242  | 2.954  | 2.700  | 2.476  | 2.110  | 1.519  | 1.191  | 0.999  | 0.882 |
|     | 0   | 14.682 | 13.626 | 12.785 | 12.124 | 11.206 | 10.190 | 9.870 | 9.740 | 9.663 |
|     | 0.05| 14.786 | 13.717 | 12.861 | 12.184 | 11.237 | 10.174 | 9.836 | 9.703 | 9.627 |
|     | 0   | 14.895 | 13.812 | 12.940 | 12.246 | 11.268 | 10.154 | 9.795 | 9.656 | 9.581 |
|     | 0.2 | 15.124 | 14.014 | 13.107 | 12.377 | 11.328 | 10.094 | 9.684 | 9.529 | 9.453 |
|     | 0.3 | 15.372 | 14.233 | 13.289 | 12.518 | 11.388 | 10.002 | 9.518 | 9.334 | 9.253 |
|     | 0   | 3.082  | 2.860  | 2.684  | 2.545  | 2.352  | 2.139  | 2.072  | 2.045  | 2.029 |
|     | 0.05| 3.104  | 2.880  | 2.700  | 2.558  | 2.359  | 2.136  | 2.065  | 2.037  | 2.021 |
|     | 0.1 | 3.127  | 2.890  | 2.716  | 2.571  | 2.365  | 2.131  | 2.056  | 2.027  | 2.011 |
|     | 0.2 | 3.175  | 2.942  | 2.752  | 2.598  | 2.378  | 2.119  | 2.033  | 2.000  | 1.984 |
|     | 0.3 | 3.227  | 2.988  | 2.790  | 2.628  | 2.391  | 2.100  | 1.998  | 1.960  | 1.942 |
The dependences of the fundamental dimensionless frequencies $\lambda_0$ of the free axisymmetric ($n = 0$) and non-axisymmetric ($n = 1$) vibrations of the perfect ($\psi = 0$) FGM circular plate with various boundary conditions.

6.2. Perfect FGM Circular Plate

The obtained general solution (48) and the defined boundary conditions (37 ÷ 40) were used to calculate the first three dimensionless frequencies $\lambda$ of the axisymmetric ($n = 0$) and non-axisymmetric ($n = 1$) vibrations of the perfect ($\psi = 0$) FGM circular plate with various boundary conditions.

The obtained numerical results are presented in Tables 6–9 for selected values of the power-law index $g$. Numerical results obtained for the clamped and simply supported plates (Tables 6 and 7) were compared with the results presented in the paper [27], where the effect of the coupling in-plane and transverse displacements was omitted.
Figure 3. The dependence of the fundamental dimensionless frequencies $\lambda_0$ of the free axisymmetric ($n = 0$) and non-axisymmetric ($n = 1$) vibrations on selected values of the power-law index and the porosity volume fraction of the clamped circular plate with evenly and unevenly distributed porosities.
Figure 4. The dependence of the fundamental dimensionless frequencies $\lambda_0$ of the free axisymmetric ($n = 0$) and non-axisymmetric ($n = 1$) vibrations on selected values of the power-law index and the porosity volume fraction of the simply supported circular plate with evenly and unevenly distributed porosities.
Figure 5. The dependence of the fundamental dimensionless frequencies $\lambda_0$ of the free axisymmetric ($n = 0$) and non-axisymmetric ($n = 1$) vibrations on selected values of the power-law index and the porosity volume fraction of the sliding supported circular plate with evenly and unevenly distributed porosities.
Figure 6. The dependence of the fundamental dimensionless frequencies $\lambda_0$ of the free axisymmetric ($n = 0$) and non-axisymmetric ($n = 1$) vibrations on selected values of the power-law index and the porosity volume fraction of the free circular plate with evenly and unevenly distributed porosities.
Table 6. The dimensionless frequencies of the clamped perfect FGM circular plate.

| $n$ | $\lambda$ | \begin{tabular}{c} \hspace{1cm} \\ \hspace{1cm} \\ \end{tabular} | 1 | 2 | 3 | 4 | 5 | $\infty$ |
|-----|------------|----------------------------------------|---|---|---|---|---|----------|
| 0   | $\lambda_0$ Present \[27\] | 7.797 | 7.090 | 6.867 | 6.777 | 6.724 | 5.199 |
|     | $\lambda_1$ Present \[27\] | 30.357 | 27.604 | 26.737 | 26.386 | 26.177 | 20.243 |
|     | $\lambda_2$ Present \[27\] | 68.012 | 61.845 | 59.902 | 59.116 | 58.649 | 45.352 |
| 1   | $\lambda_0$ Present | 16.228 | 14.756 | 14.292 | 14.105 | 13.993 | 10.821 |
|     | $\lambda_1$ Present | 46.430 | 42.219 | 40.893 | 40.357 | 40.038 | 30.961 |
|     | $\lambda_2$ Present | 91.655 | 83.344 | 80.725 | 79.667 | 79.037 | 61.118 |

Table 7. The dimensionless frequencies of the simply supported perfect FGM circular plate.

| $n$ | $\lambda$ | \begin{tabular}{c} \hspace{1cm} \\ \hspace{1cm} \\ \end{tabular} | 1 | 2 | 3 | 4 | 5 | $\infty$ |
|-----|------------|----------------------------------------|---|---|---|---|---|----------|
| 0   | $\lambda_0$ Present \[27\] | 3.767 | 3.425 | 3.317 | 3.274 | 3.248 | 2.512 |
|     | $\lambda_1$ Present \[27\] | 22.685 | 20.628 | 19.980 | 19.717 | 19.562 | 15.127 |
|     | $\lambda_2$ Present \[27\] | 56.602 | 51.470 | 49.853 | 49.199 | 48.810 | 37.744 |
| 1   | $\lambda_0$ Present | 10.608 | 9.646 | 9.343 | 9.220 | 9.147 | 7.074 |
|     | $\lambda_1$ Present | 37.003 | 33.648 | 32.591 | 32.163 | 31.909 | 24.675 |
|     | $\lambda_2$ Present | 78.446 | 71.332 | 69.091 | 68.185 | 67.646 | 52.310 |

Table 8. The dimensionless frequencies of the free perfect FGM circular plate.

| $n$ | $\lambda$ | \begin{tabular}{c} \hspace{1cm} \\ \hspace{1cm} \\ \end{tabular} | 1 | 2 | 3 | 4 | 5 | $\infty$ |
|-----|------------|----------------------------------------|---|---|---|---|---|----------|
| 0   | $\lambda_0$ Present | 6.872 | 6.248 | 6.052 | 5.973 | 5.926 | 4.582 |
|     | $\lambda_1$ Present | 29.343 | 26.682 | 25.844 | 25.505 | 25.303 | 19.567 |
|     | $\lambda_2$ Present | 66.979 | 60.905 | 58.992 | 58.218 | 57.757 | 44.663 |
| 1   | $\lambda_0$ Present | 15.628 | 14.211 | 13.764 | 13.584 | 13.476 | 10.421 |
|     | $\lambda_1$ Present | 45.653 | 41.513 | 40.209 | 39.682 | 39.368 | 30.443 |
|     | $\lambda_2$ Present | 90.799 | 82.565 | 79.971 | 78.922 | 78.298 | 60.547 |

Table 9. The dimensionless frequencies of the perfect FGM circular plate with sliding support.

| $n$ | $\lambda$ | \begin{tabular}{c} \hspace{1cm} \\ \hspace{1cm} \\ \end{tabular} | 1 | 2 | 3 | 4 | 5 | $\infty$ |
|-----|------------|----------------------------------------|---|---|---|---|---|----------|
| 0   | $\lambda_0$ Present | 11.206 | 10.190 | 9.870 | 9.740 | 9.663 | 7.473 |
|     | $\lambda_1$ Present | 37.568 | 34.161 | 33.088 | 32.654 | 32.396 | 25.051 |
|     | $\lambda_2$ Present | 79.000 | 71.836 | 69.579 | 68.667 | 68.124 | 52.680 |
| 1   | $\lambda_0$ Present | 2.352 | 2.139 | 2.072 | 2.045 | 2.029 | 1.568 |
|     | $\lambda_1$ Present | 21.676 | 19.711 | 19.091 | 18.841 | 18.692 | 14.454 |
|     | $\lambda_2$ Present | 55.612 | 50.569 | 48.981 | 48.338 | 47.956 | 37.084 |

The fundamental dimensionless frequencies of the perfect FGM circular plates with and without the effect of the coupling in-plane and transverse displacements obtained for selected values of the power-law index and diverse boundary conditions are presented in Table 10. Additionally, the differences (errors) between obtained results were calculated according to the equation:
where $\lambda_0$ and $\lambda_0'$ are the fundamental dimensionless frequencies of the perfect FGM circular plate without and with effect of the coupling in-plane and transverse displacements, respectively. Figure 7 presents the dependence of the differences (errors) between obtained results for the power-law index $g \geq 0$.

Table 10. The differences between the fundamental dimensionless frequencies of the perfect FGM circular plates with and without effect of the coupling in-plane and transverse displacements.

| BCs            | $n$ | $\lambda_0$ | $\lambda_0'$ | $\delta$ (%) | $\delta$ (%) | $\delta$ (%) | $\delta$ (%) | $\delta$ (%) | $\delta$ (%) | $\delta$ (%) | $\delta$ (%) |
|----------------|-----|--------------|---------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| Clamped        | 0   | 8.498        | 7.979         | 8.2          | 17.687       | 16.228       | 4.105        | 11.562       | 10.608       | 8.2          | 15.628       |
|                | 1   | 8.123        | 7.090         | 12.7         | 16.406       | 14.756       | 3.924        | 11.051       | 9.646        | 12.7         | 14.211       |
|                |     | 7.911        | 6.867         | 13.2         | 16.464       | 14.292       | 3.821        | 10.762       | 9.343        | 13.2         | 13.764       |
|                |     | 7.733        | 6.777         | 12.3         | 16.094       | 14.105       | 3.736        | 10.521       | 9.220        | 12.3         | 13.584       |
|                |     | 7.573        | 6.724         | 11.2         | 15.762       | 13.993       | 3.658        | 10.303       | 9.147        | 11.2         | 13.051       |
|                |     | 6.977        | 6.512         | 6.6          | 15.420       | 13.552       | 3.370        | 9.492        | 8.859        | 6.6          | 11.945       |
|                |     | 6.064        | 5.960         | 1.7          | 12.621       | 12.404       | 2.929        | 8.250        | 8.108        | 1.7          | 11.332       |
|                |     | 5.687        | 5.654         | 0.5          | 11.835       | 11.767       | 2.747        | 7.737        | 7.692        | 0.5          | 10.421       |
|                |     | 5.199        | 5.199         | 0            | 10.821       | 10.821       | 2.512        | 7.074        | 7.074        | 0            | 7.074        |
| Simply supported | 0   | 3.797        | 3.767         | 8.2          | 11.562       | 10.608       | 3.924        | 8.2          | 10.608       | 8.2          | 11.562       |
|                | 1   | 3.707        | 3.707         | 12.7         | 11.051       | 9.646        | 3.821        | 11.051       | 9.343        | 12.7         | 13.764       |
| Sliding supported | 0   | 8.2          | 8.2           | 13.2         | 16.406       | 14.756       | 3.924        | 11.051       | 9.646        | 13.2         | 13.764       |
|                | 1   | 8.2          | 8.2           | 13.2         | 16.464       | 14.292       | 3.821        | 10.762       | 9.343        | 13.2         | 13.764       |
| Free           | 0   | 8.2          | 8.2           | 13.2         | 15.762       | 13.993       | 3.736        | 10.521       | 9.220        | 13.2         | 13.051       |
|                | 1   | 8.2          | 8.2           | 13.2         | 15.420       | 13.552       | 3.658        | 10.303       | 9.147        | 13.2         | 13.051       |

Figure 7. The dependence of the differences (errors) between the fundamental dimensionless frequencies of the perfect FGM circular plate without $(\lambda_0^P)$ and with $(\lambda_0'^P)$ effect of the coupling in-plane and transverse displacements for diverse values of the power-law index $g \geq 0$. 

\[
\delta(\%) = \left| \frac{\lambda_0 - \lambda_0'}{\lambda_0^P} \right| \times 100%.
\]
7. Discussion

7.1. Imperfect FGM Circular Plate

The numerical results for the fundamental dimensionless frequencies of the porous FGM circular plates presented in Tables 2–5 and Figures 3–6 show the following dependences:

- the fundamental eigenfrequency $\lambda_0$ of the axisymmetric and non-axisymmetric vibrations of the circular plate decreases with the increasing value of the power-law index $g$ for the two considered distributions of porosities and all considered values of the porosity volume fraction $\psi$;
- for the evenly distributed porosities, the fundamental eigenfrequency $\lambda_0$ of the axisymmetric and non-axisymmetric vibrations of the plate increases with the increasing value of the porosity volume fraction $\psi$ for $g \in [0, 0.4]$ and decreases for $g \in [0.6, 5]$;
- for the unevenly distributed porosities, the fundamental eigenfrequency $\lambda_0$ of the axisymmetric and non-axisymmetric vibrations of the plate increases with the increasing value of the porosity volume fraction $\psi$ for $g \in [0, 1]$ and decreases for $g \in [2, 5]$;
- the influence of values of the porosity volume fraction $\psi$ on the values of the fundamental eigenfrequency $\lambda_0$ of the axisymmetric and non-axisymmetric vibrations of the plate is smaller for the unevenly distributed porosities than for the evenly distributed porosities;
- for the evenly distributed porosities, the fundamental eigenfrequency $\lambda_0$ of the axisymmetric and non-axisymmetric vibrations of plate decreases faster for $\psi = 0.3$ with the increasing values of the power-law index $g$ than for $\psi = \{0, 0.1, 0.2\}$;
- for the unevenly distributed porosities, the fundamental eigenfrequency $\lambda_0$ of the axisymmetric and non-axisymmetric vibrations of the plate decreases slowly with the increasing values of the power-law index $g$ for all considered values of the porosity volume fraction $\psi$.

The observed dependences exist because of the diverse influence of porosity distributions, values of the power-law index and the porosity volume fraction on decreasing (increasing) the ratios of mass to stiffness of the considered circular plates. The all observed dependences are independent of the considered boundary conditions which influence only the values of the dimensionless frequencies of the plate.

7.2. Perfect FGM Circular Plate

It can be observed that the values of dimensionless frequencies of the perfect FGM circular plates obtained by omitting the effect of coupling in-plane and transverse displacements are higher than the values of the dimensionless frequencies of the considered plate with the coupling effect. The differences (errors) between the calculated dimensionless frequencies of free axisymmetric and non-axisymmetric vibration of the perfect FGM circular plate with and without the coupling effect are significant for the power-law index $g \in [0, 20]$, but, for $g \in [20, \infty]$, these differences decrease from 2% to 0%. It can be observed from Table 10 that the differences between the calculated dimensionless frequencies are independent of the modes of vibrations and the boundary conditions of the considered circular plate.

8. Conclusions

This paper presents the influence of two different types of distribution of porosities on the free vibrations of the thin functionally graded circular plate with clamped, simply supported, sliding supported, and free edges. To this aim, the boundary value problem was formulated and a solution was obtained in the exact form. The universal multiparametric characteristic equations were defined using the properties of the multiparametric general solution obtained for the plate with even and uneven distribution of porosities. The effects of the power-law index, the volume fraction index and diverse boundary conditions on the values of the dimensionless frequencies of the free axisymmetric and non-axisymmetric vibrations of the circular plate were comprehensively studied. Additionally, the
influences of the power-law index and different boundary conditions on the values of dimensionless frequencies of the FGM circular plate without porosities were also presented.

The presented multiparametric analytical approach can be effectively applying for free vibration of circular and annular plates with other diverse models of an FGM and FGM porous material. The material parameters can be modeled via the exponential or sigmoid functions, as well as Mori–Tanaka functions or other homogenization techniques [39–44]. Diverse applied homogenization techniques only have an influence on the forms of the final replaced plate’s stiffnesses and directly on the function $\varphi_i$ presented in the obtained general solution in the present paper. It will be the goal of future papers.

The obtained multiparametric general solution will allow for studying the influences of diverse additional complicating effects such as stepped thickness, cracks, additional mounted elements expressed by only additional boundary conditions on the dynamic behavior of the porous functionally graded circular and annular plates. The exact frequencies of vibration presented in non-dimensional form can serve as benchmark values for researchers and engineers to validate their analytical and numerical methods applied in design and analysis of porous functionally graded structural elements.

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