Coupled coincidence point theorems for mixed 
\((G, S)\)-monotone operators on partially ordered metric 
spaces and applications

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Abstract

In this paper, we introduce the concept of mixed \((G, S)\)-monotone mappings and 
prove coupled coincidence and coupled common fixed point theorems for such mappings 
satisfying a nonlinear contraction involving altering distance functions. Presented 
theorems extend, improve and generalize the very recent results of Harjani, López 
and Sadarangani [J. Harjani, B. López and K. Sadarangani, Fixed point theorems for 
mixed monotone operators and applications to integral equations, Nonlinear Analysis 
(2010), doi:10.1016/j.na.2010.10.047] and other existing results in the literature. Some 
applications to periodic boundary value problems are also considered.

Key words: Coincidence point, coupled common fixed point, \((G, S)\)-monotone 
mapping, ordered set.

1 Introduction and preliminaries

Fixed point problems of contractive mappings in metric spaces endowed with a partially 
order have been studied by many authors (see [1]-[17]). Bhaskar and Lakshmikantham [3] 
introduced the concept of a coupled fixed point and studied the problems of a uniqueness 
of a coupled fixed point in partially ordered metric spaces and applied their theorems 
to problems of the existence of solution for a periodic boundary value problem. In [8], 
Lakshmikantham and Ćirić established some coincidence and common coupled fixed point 
theorems under nonlinear contractions in partially ordered metric spaces. Very recently, 
Harjani, López and Sadarangani [7] obtained some coupled fixed point theorems for a 
mixed monotone operator in a complete metric space endowed with a partial order by 
using altering distance functions. They applied their results to the study of the existence 
and uniqueness of a nonlinear integral equation.

Now, we briefly recall various basic definitions and facts.

**Definition 1.1** (see Bhaskar and Lakshmikantham [3]). Let \((X, \preceq)\) be a partially ordered 
set and \(F : X \times X \to X\). Then the map \(F\) is said to have mixed monotone property if 
\(F(x, y)\) is monotone non-decreasing in \(x\) and is monotone non-increasing in \(y\), that is, for 
any \(x, y \in X\),

\[ x_1 \preceq x_2 \text{ implies } F(x_1, y) \preceq F(x_2, y) \text{ for all } y \in X \]
and

\[ y_1 \preceq y_2 \implies F(x, y_2) \preceq F(x, y_1) \text{ for all } x \in X. \]

The main result obtained by Bhaskar and Lakshmikantham [3] is the following.

**Theorem 1.1 (see Bhaskar and Lakshmikantham [3]).** Let \((X, \preceq)\) be a partially ordered set and suppose there is a metric \(d\) on \(X\) such that \((X, d)\) is a complete metric space. Let \(F : X \times X \to X\) be a mapping having the mixed monotone property on \(X\). Assume that there exists \(k \in [0, 1)\) such that

\[ d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)] \text{ for each } u \preceq x \text{ and } y \preceq v. \]

Suppose either \(F\) is continuous or \(X\) has the following properties:

(i) if a non-decreasing sequence \(x_n \to x\), then \(x_n \preceq x\) for all \(n\),
(ii) if a non-increasing sequence \(x_n \to x\), then \(x \preceq x_n\) for all \(n\).

If there exist \(x_0, y_0 \in X\) such that \(x_0 \preceq F(x_0, y_0)\) and \(F(y_0, x_0) \preceq y_0\), then \(F\) has a coupled fixed point.

Inspired by Definition 1.1, Lakshmikantham and Ćirić in [8] introduced the concept of a \(g\)-mixed monotone mapping.

**Definition 1.2 (see Lakshmikantham and Ćirić [8]).** Let \((X, \preceq)\) be a partially ordered set, \(F : X \times X \to X\) and \(g : X \to X\). Then the map \(F\) is said to have mixed \(g\)-monotone property if \(F(x, y)\) is monotone \(g\)-non-decreasing in \(x\) and is monotone \(g\)-non-increasing in \(y\), that is, for any \(x, y \in X\),

\[ gx_1 \preceq gx_2 \implies F(x_1, y) \preceq F(x_2, y) \text{ for all } y \in X \]

and

\[ gy_1 \preceq gy_2 \implies F(x, y_2) \preceq F(x, y_1) \text{ for all } x \in X. \]

**Definition 1.3 (see Lakshmikantham and Ćirić [8]).** Let \(X\) be a non-empty set, and let \(F : X \times X \to X\), \(g : X \to X\) be given mappings. An element \((x, y) \in X \times X\) is called a coupled coincidence point of the mappings \(F\) and \(g\) if \(F(x, y) = gx\) and \(F(y, x) = gy\).

**Definition 1.4 (see Lakshmikantham and Ćirić [8]).** Let \(X\) be a non-empty set. Then we say that the mappings \(F : X \times X \to X\) and \(g : X \to X\) are commutative if

\[ g(F(x, y)) = F(gx, gy). \]

The main result of Lakshmikantham and Ćirić [8] is the following.
Theorem 1.2 (see Lakshmikantham and Ćirić [8]). Let \((X, \preceq)\) be a partially ordered set and suppose there is a metric \(d\) on \(X\) such that \((X, d)\) is a complete metric space. Assume there is a function \(\phi : [0, +\infty) \to [0, +\infty)\) with \(\phi(t) < t\) and \(\lim_{r \to t^+} \phi(r) < t\) for each \(t > 0\) and also suppose \(F : X \times X \to X\) and \(g : X \to X\) are such that \(F\) has the mixed \(g\)-monotone property on \(X\) and satisfying
\[
d(F(x, y), F(u, v)) \leq \phi \left( \frac{d gx, gu + d gy, gv}{2} \right)
\]
for all \(x, y, u, v \in X\) with \(gx \leq gu\) and \(gv \leq gy\). Assume that \(F(X \times X) \subseteq g(X)\), \(g\) is continuous and commutes with \(F\) and also suppose either \(F\) is continuous or \(X\) has the following properties:

(i) if a non-decreasing sequence \(x_n \to x\), then \(x_n \preceq x\) for all \(n\),
(ii) if a non-increasing sequence \(x_n \to x\), then \(x \preceq x_n\) for all \(n\).

If there exist \(x_0, y_0 \in X\) such that \(gx_0 \preceq F(x_0, y_0)\) and \(F(y_0, x_0) \preceq gy_0\) then there exist \(x, y \in X\) such that \(gx = F(x, y)\) and \(gy = F(y, x)\), that is, \(F\) and \(g\) have a coupled coincidence point.

Very recently, Harjani, López and Sadarangani [7] established coupled fixed point theorems for a mixed monotone operator satisfying contraction involving altering distance functions in a complete partially ordered metric space. Denote by \(F\) the set of functions \(\varphi : [0, +\infty) \to [0, +\infty)\) satisfying the following properties:
(a) \(\varphi\) is continuous and non-decreasing,
(b) \(\varphi(t) = 0\) if and only if \(t = 0\).

Theorem 1.3 (Harjani, López and Sadarangani [7]). Let \((X, \preceq)\) be a partially ordered set and \(d\) be a metric on \(X\) such that \((X, d)\) is a complete metric space. Let \(F : X \times X \to X\) be a mapping having the mixed monotone property on \(X\) and satisfying
\[
\varphi(d(F(x, y), F(u, v)) \leq \varphi(\max\{d(x, u), d(y, v)\}) - \Phi(\max\{d(x, u), d(y, v)\})
\]
for all \(x, y, u, v \in X\) with \(u \preceq x\) and \(y \preceq v\), where \(\varphi, \psi \in F\). Suppose either \(F\) is continuous or \(X\) has the following properties:

(i) if a non-decreasing sequence \(x_n \to x\), then \(x_n \preceq x\) for all \(n\),
(ii) if a non-increasing sequence \(x_n \to x\), then \(x \preceq x_n\) for all \(n\).

If there exist \(x_0, y_0 \in X\) such that \(x_0 \preceq F(x_0, y_0)\) and \(F(y_0, x_0) \preceq y_0\) then \(F\) has a coupled fixed point.

In this paper, we introduce the concept of mixed \((G, S)\)-monotone mappings and prove coupled coincidence and coupled common fixed point theorems for such mappings satisfying a nonlinear contraction involving altering distance functions. Presented theorems extend, improve and generalize the results of Harjani, López and Sadarangani [7]. As applications of our obtained results, we study the existence and uniqueness of solution to periodic boundary value problem.
2 Main Results

Now, we introduce the concept of mixed \((G, S)\)-monotone property.

**Definition 2.1** Let \(X\) be a non-empty set endowed with a partial order \(\preceq\). Consider the mappings \(F : X \times X \to X\) and \(G, S : X \to X\). We say that \(F\) has the mixed \((G, S)\)-monotone property on \(X\) if for all \(x, y \in X\),

\[
\begin{align*}
    x_1, x_2 \in X, & \quad G(x_1) \leq S(x_2) \Rightarrow F(x_1, y) \leq F(x_2, y), \\
    x_1, x_2 \in X, & \quad G(x_1) \geq S(x_2) \Rightarrow F(x_1, y) \geq F(x_2, y), \\
    y_1, y_2 \in X, & \quad G(y_1) \leq S(y_2) \Rightarrow F(x, y_1) \geq F(x, y_2), \\
    y_1, y_2 \in X, & \quad G(y_1) \geq S(y_2) \Rightarrow F(x, y_1) \leq F(x, y_2).
\end{align*}
\]

**Remark 1** If we take \(G = S\), then \(F\) has the mixed \((G, S)\)-monotone property implies that \(F\) has the mixed \(G\)-monotone property.

Now, we state and prove our first result.

**Theorem 2.1** Let \((X, \preceq)\) be a partially ordered set and suppose that there exists a metric \(d\) on \(X\) such that \((X, d)\) is a complete metric space. Let \(G, S : X \to X\) and \(F : X \times X \to X\) be a mapping having the mixed \((G, S)\)-monotone property on \(X\). Suppose that

\[
\varphi(d(F(x, y), F(u, v))) \leq \varphi(\max\{d(Gx, Su), d(Sy, Gv)\}) - \phi(\max\{d(Gx, Su), d(Sy, Gv)\}),
\]

for all \(x, y, u, v \in X\) with \(G(x) \preceq S(u)\) or \(G(x) \succeq S(u)\) and \(S(y) \preceq G(v)\) or \(S(y) \succeq G(v)\), where \(\varphi, \phi \in \mathcal{F}\). Assume that \(F(X \times X) \subseteq G(X) \cap S(X)\) and assume also that \(G, S\) and \(F\) satisfy the following hypotheses:

(I) \(F, G\) and \(S\) are continuous,

(II) \(F\) commutes respectively with \(G\) and \(S\).

If there exist \(x_0, y_0, x_1\) and \(y_1\) such that

\[
\begin{align*}
    \{ & \quad G(x_0) \preceq S(x_1) \preceq F(x_0, y_0), \\
    \{ & \quad G(y_0) \succeq S(y_1) \succeq F(y_0, x_0),
\end{align*}
\]

then there exist \(x, y \in X\) such that

\[
G(x) = S(x) = F(x, y) \quad \text{and} \quad G(y) = S(y) = F(y, x),
\]

that is, \(G, S\) and \(F\) have a coupled coincidence point \((x, y) \in X \times X\).

**Proof.** Let \(x_0, y_0, x_1, y_1 \in X\) such that

\[
G(x_0) \preceq S(x_1) \preceq F(x_0, y_0) \quad \text{and} \quad G(y_0) \succeq S(y_1) \succeq F(y_0, x_0).
\]

Since \(F(X \times X) \subseteq G(X) \cap S(X)\), we can choose \(x_2, y_2, x_3, y_3 \in X\) such that

\[
\begin{align*}
    \{ & \quad G(x_2) = F(x_0, y_0) \quad \text{and} \quad S(x_3) = F(x_1, y_1), \\
    \{ & \quad G(y_2) = F(y_0, x_0) \quad \text{and} \quad S(y_3) = F(y_1, x_1).
\end{align*}
\]

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Continuing this process we can construct sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that

\[
\begin{align*}
G(x_{2n+2}) &= F(x_{2n}, y_{2n}), \\
G(y_{2n+2}) &= F(y_{2n}, x_{2n}), \\
S(x_{2n+3}) &= F(x_{2n+1}, y_{2n+1}), \\
S(y_{2n+3}) &= F(y_{2n+1}, x_{2n+1})
\end{align*}
\]  
for all \( n \geq 0 \). \hspace{1cm} (2)

We shall show that for all \( n \geq 0 \),

\[
G(x_{2n}) \leq S(x_{2n+1}) \leq G(x_{2n+2})
\]  
and

\[
G(y_{2n}) \geq S(y_{2n+1}) \geq G(y_{2n+2}).
\]  
(3)

As \( G(x_0) \leq S(x_1) \leq F(x_0, y_0) = G(x_2) \) and \( G(y_0) \geq S(y_1) \geq F(y_0, x_0) = G(y_2) \), our claim is satisfied for \( n = 0 \).

Suppose that \((3)\) and \((4)\) hold for some fixed \( n \geq 0 \). Since \( G(x_{2n}) \leq S(x_{2n+1}) \leq G(x_{2n+2}) \) and \( G(y_{2n}) \geq S(y_{2n+1}) \geq G(y_{2n+2}) \), and as \( F \) has the mixed \((G,S)\)-monotone property, we have

\[
G(x_{2n+2}) = F(x_{2n}, y_{2n}) \leq F(x_{2n+1}, y_{2n}) \leq F(x_{2n+1}, y_{2n+1}) \leq F(x_{2n+2}, y_{2n+1}) \leq F(x_{2n+2}, y_{2n+2}),
\]
then

\[
G(x_{2n+2}) \leq S(x_{2n+3}) \leq G(x_{2n+4}).
\]

On the other hand,

\[
G(y_{2n+2}) = F(y_{2n}, x_{2n}) \geq F(y_{2n+1}, x_{2n}) \geq F(y_{2n+1}, x_{2n+1}) \geq F(y_{2n+2}, x_{2n+1}) \geq F(y_{2n+2}, x_{2n+2}),
\]
then

\[
G(y_{2n+2}) \geq S(y_{2n+3}) \geq G(y_{2n+4}).
\]

Thus by induction, we proved that \((3)\) and \((4)\) hold for all \( n \geq 0 \).

We complete the proof in the following steps

**Step 1:** We will prove that

\[
\lim_{n \to +\infty} d(F(x_n, y_n), F(x_{n+1}, y_{n+1})) = \lim_{n \to +\infty} d(F(y_n, x_n), F(y_{n+1}, x_{n+1})) = 0.
\]  
(5)

From \((3)\), \((4)\) and \((5)\), we have

\[
\phi\left(d(F(x_{2n}, y_{2n}), F(x_{2n+1}, y_{2n+1}))\right) \leq \phi\left(\max\{d(Gx_{2n}, Sx_{2n+1}), d(Gy_{2n}, Sy_{2n+1})\}\right) - \phi\left(\max\{d(Gx_{2n}, Sx_{2n+1}), d(Gy_{2n}, Sy_{2n+1})\}\right)
\]

\[
\leq \phi\left(\max\{d(Gx_{2n}, Sx_{2n+1}), d(Gy_{2n}, Sy_{2n+1})\}\right).
\]  
(6)

Since \( \phi \) is a non-decreasing function, we get that

\[
d(F(x_{2n}, y_{2n}), F(x_{2n+1}, y_{2n+1})) \leq \max\{d(Gx_{2n}, Sx_{2n+1}), d(Gy_{2n}, Sy_{2n+1})\}.
\]

Therefore

\[
d(Gx_{2n+2}, Sx_{2n+3}) \leq \max\{d(Gx_{2n}, Sx_{2n+1}), d(Gy_{2n}, Sy_{2n+1})\}.
\]  
(8)
Again, using (3), (4) and (1), we have

\[ \phi(d(F(y_{2n}, x_{2n}), F(y_{2n+1}, x_{2n+1}))) \]
\[ \leq \phi(\max\{d(G_{2n}, S_{2n+1}), d(G_{2n}, S_{2n+1})\}) - \phi(\max\{d(G_{2n}, S_{2n+1}), d(G_{2n}, S_{2n+1})\}) \]
\[ \leq \phi(\max\{d(G_{2n}, S_{2n+1}), d(G_{2n}, S_{2n+1})\}). \quad (9) \]

Since \( \phi \) is non-decreasing, we have

\[ d(F(y_{2n}, x_{2n}), F(y_{2n+1}, x_{2n+1})) \leq \max\{d(G_{2n}, S_{2n+1}), d(G_{2n}, S_{2n+1})\}. \]

Therefore

\[ d(G_{2n+2}, S_{2n+3}) \leq \max\{d(G_{2n}, S_{2n+1}), d(G_{2n}, S_{2n+1})\}. \quad (10) \]

Combining (5) and (10), we obtain

\[ \max\{d(G_{2n+2}, S_{2n+3}), d(G_{2n+2}, S_{2n+3})\} \leq \max\{d(G_{2n}, S_{2n+1}), d(G_{2n}, S_{2n+1})\}. \]

Then \( \{\max\{d(G_{2n}, S_{2n+1}), d(G_{2n}, S_{2n+1})\}\} \) is a positive decreasing sequence. Hence there exists \( r \geq 0 \) such that

\[ \lim_{n \to +\infty} \max\{d(G_{2n}, S_{2n+1}), d(G_{2n}, S_{2n+1})\} = r. \]

Combining (7) and (9), we obtain

\[ \max\{\phi(d(G_{2n+2}, S_{2n+3})), \phi(d(G_{2n+2}, S_{2n+3}))\} \]
\[ \leq \phi(\max\{d(G_{2n}, S_{2n+1}), d(G_{2n}, S_{2n+1})\}) - \phi(\max\{d(G_{2n}, S_{2n+1}), d(G_{2n}, S_{2n+1})\}). \]

Since \( \phi \) is non-decreasing, we get

\[ \phi(\max\{d(G_{2n+2}, S_{2n+3}), d(G_{2n+2}, S_{2n+3})\}) \]
\[ \leq \phi(\max\{d(G_{2n}, S_{2n+1}), d(G_{2n}, S_{2n+1})\}) - \phi(\max\{d(G_{2n}, S_{2n+1}), d(G_{2n}, S_{2n+1})\}). \]

Letting \( n \to +\infty \) in the above inequality, we get

\[ \phi(r) \leq \phi(r) - \phi(r), \]

which implies that \( \phi(r) = 0 \) and then, since \( \phi \) is an altering distance function, \( r = 0 \). Consequently

\[ \lim_{n \to +\infty} \max\{d(F(x_{2n}, y_{2n}), F(x_{2n+1}, y_{2n+1})), d(F(y_{2n}, x_{2n}), F(y_{2n+1}, x_{2n+1}))\} = 0. \quad (11) \]

By the same way, we obtain

\[ \lim_{n \to +\infty} \max\{d(F(x_{2n+1}, y_{2n+1}), F(x_{2n+2}, y_{2n+2})), d(F(y_{2n+1}, x_{2n+1}), F(y_{2n+2}, x_{2n+2}))\} = 0. \quad (12) \]

Finally, (11) and (12) give the desired result, that is, (5) holds.
Step 2: We will prove that $F(x_n, y_n)$ and $F(y_n, x_n)$ are Cauchy sequences. From (5), it is sufficient to show that $F(x_{2n}, y_{2n})$ and $F(y_{2n}, x_{2n})$ are Cauchy sequences.

We proceed by negation and suppose that at least one of the sequences $F(x_{2n}, y_{2n})$ or $F(y_{2n}, x_{2n})$ is not a Cauchy sequence.

This implies that $d(F(x_{2n}, y_{2n}), F(x_{2m}, y_{2m})) \to 0$ or $d(F(y_{2n}, x_{2n}), F(y_{2m}, x_{2m})) \to 0$ as $n, m \to +\infty$.

Consequently

$$\max\{d(F(x_{2n}, y_{2n}), F(x_{2m}, y_{2m})), d(F(y_{2n}, x_{2n}), F(y_{2m}, x_{2m}))\} \to 0, \quad \text{as } n, m \to +\infty.$$  

Then there exists $\varepsilon > 0$ for which we can find two subsequences of positive integers $\{m(i)\}$ and $\{n(i)\}$ such that $n(i)$ is the smallest index for which $n(i) > m(i) > i$,

$$\max\{d(F(x_{2m(i)}, y_{2m(i)}), F(x_{2n(i)}, y_{2n(i)})), d(F(y_{2m(i)}, x_{2m(i)}), F(y_{2n(i)}, x_{2n(i)}))\} \geq \varepsilon. \quad (13)$$

This means that

$$\max\{d(F(x_{2m(i)}, y_{2m(i)}), F(x_{2n(i)}-2, y_{2n(i)}-2)), d(F(y_{2m(i)}, x_{2m(i)}), F(y_{2n(i)}-2, x_{2n(i)}-2))\} < \varepsilon. \quad (14)$$

From (2.1), (13) and using the triangular inequality, we get

$$\varepsilon \leq \max\{d(F(x_{2m(i)}, y_{2m(i)}), F(x_{2n(i)}-2, y_{2n(i)}-2)), d(F(y_{2m(i)}, x_{2m(i)}), F(y_{2n(i)}-2, x_{2n(i)}-2))\}$$
$$\leq \max\{d(F(x_{2m(i)}, y_{2m(i)}), F(x_{2n(i)}-2, y_{2n(i)}-2)), d(F(y_{2m(i)}, x_{2m(i)}), F(y_{2n(i)}-2, x_{2n(i)}-2))\}$$
$$+ \max\{d(F(x_{2m(i)}-2, y_{2n(i)}-2), F(x_{2n(i)}-1, y_{2n(i)}-1)), d(F(y_{2m(i)}-2, y_{2n(i)}-2), F(y_{2n(i)}-1, x_{2n(i)}-1))\}$$
$$+ \max\{d(F(x_{2n(i)}-1, y_{2n(i)}-1), F(x_{2n(i)}), y_{2n(i)})), (F(y_{2n(i)}-1, x_{2n(i)}-1), F(y_{2n(i)}, x_{2n(i)}))\}$$
$$< \varepsilon + \max\{d(F(x_{2n(i)}-2, y_{2n(i)}-2), F(y_{2n(i)}-1, x_{2n(i)}-1)), d(F(y_{2n(i)}-2, y_{2n(i)}-2), F(y_{2n(i)}-1, x_{2n(i)}-1))\}$$
$$+ \max\{d(F(x_{2n(i)}-1, y_{2n(i)}-1), F(x_{2n(i)}), y_{2n(i)})), d(F(y_{2n(i)}-1, x_{2n(i)}-1), F(y_{2n(i)}, x_{2n(i)}))\}.$$  

Letting $i \to +\infty$ in above inequality and using (5), we obtain that

$$\lim_{i \to +\infty} \max\{d(F(x_{2m(i)}, y_{2m(i)}), F(x_{2n(i)}), y_{2n(i)})), d(F(y_{2m(i)}, x_{2m(i)}), F(y_{2n(i)}, x_{2n(i)}))\} = \varepsilon. \quad (15)$$

Also, we have

$$\varepsilon \leq \max\{d(F(x_{2m(i)}, y_{2m(i)}), F(x_{2n(i)}-1, y_{2n(i)}-1)), d(F(y_{2m(i)}, x_{2m(i)}), F(y_{2n(i)}-1, x_{2n(i)}-1))\}$$
$$\leq \max\{d(F(x_{2m(i)}, y_{2m(i)}), F(y_{2m(i)}-1, x_{2m(i)}-1)), d(F(y_{2m(i)}, x_{2m(i)}), F(y_{2n(i)}-1, x_{2n(i)}-1))\}$$
$$+ \max\{d(F(x_{2n(i)}-1, y_{2n(i)}-1), F(x_{2n(i)}), y_{2n(i)})), d(F(y_{2n(i)}-1, x_{2n(i)}-1), F(y_{2n(i)}, x_{2n(i)}))\}$$
$$\leq 2 \max\{d(F(x_{2m(i)}, y_{2m(i)}), F(x_{2n(i)}-1, y_{2n(i)}-1)), d(F(y_{2m(i)}, x_{2m(i)}), F(y_{2n(i)}-1, x_{2n(i)}-1))\}$$
$$+ \max\{d(F(x_{2n(i)}-1, y_{2n(i)}-1), F(x_{2n(i)}), y_{2n(i)})), d(F(y_{2n(i)}-1, x_{2n(i)}-1), F(y_{2n(i)}, x_{2n(i)}))\}.$$  

Using (5), (15) and letting $i \to +\infty$ in the above inequality, we obtain

$$\lim_{i \to +\infty} \max\{d(F(x_{2m(i)}-1, y_{2m(i)}-1), F(x_{2n(i)}-1, y_{2n(i)}-1)), d(F(y_{2m(i)}-1, x_{2m(i)}-1), F(y_{2n(i)}, x_{2n(i)}))\} = \varepsilon. \quad (16)$$
On other hand, we have
\[
\max\{d(F(x_{2m(i)}, y_{2m(i)}), F(x_{2n(i)}, y_{2n(i)})), d(F(y_{2m(i)}, x_{2m(i)}), F(y_{2n(i)}, x_{2n(i)})) \}
\]
\[
\leq \max\{d(F(x_{2m(i)}, y_{2m(i)}), F(x_{2n(i)+1}, y_{2n(i)+1})), d(F(y_{2m(i)}, x_{2m(i)}), F(y_{2n(i)+1}, x_{2n(i)+1})) \}
\]
\[
+ \max\{d(F(x_{2n(i)+1}, y_{2n(i)+1}), F(x_{2n(i)}, y_{2n(i)})), d(F(y_{2n(i)+1}, x_{2n(i)+1}), F(y_{2n(i)}, x_{2n(i)})) \}.
\]

Since \( \varphi \) is a continuous non-decreasing function, it follows from the above inequality that
\[
\varphi(\varepsilon) \leq \limsup_{i \to +\infty} \varphi(\max\{d(F(x_{2m(i)}, y_{2m(i)}), F(x_{2n(i)+1}, y_{2n(i)+1})), d(F(y_{2m(i)}, x_{2m(i)}), F(y_{2n(i)+1}, x_{2n(i)+1})) \}).
\]

Using the contractive condition, on one hand we have
\[
\varphi(\max\{d(F(x_{2m(i)}, y_{2m(i)}), F(x_{2n(i)+1}, y_{2n(i)+1})), d(Gx_{2m(i)}), Gy_{2n(i)+1}) \}
\]
\[
- \varphi(\max\{d(Gx_{2m(i)}, Sx_{2n(i)+1}), d(Gy_{2n(i)+1}, Sy_{2n(i)+1}) \}) \leq \varphi(\max\{d(F(x_{2n(i)+2}, y_{2n(i)+2}), F(x_{2n(i)+1}, y_{2n(i)+1})), d(F(y_{2n(i)+2}, y_{2n(i)+2}), F(y_{2n(i)+1}, y_{2n(i)+1})) \})
\]
\[
- \varphi(\max\{d(F(x_{2n(i)+2}, y_{2n(i)+2}), F(x_{2n(i)+1}, y_{2n(i)+1})), d(F(x_{2n(i)+2}, y_{2n(i)+2}), F(y_{2n(i)+1}, x_{2n(i)+1})) \}).
\]

On the other hand we have
\[
\varphi(\max\{d(F(y_{2n(i)}, x_{2n(i)}) \}) \leq \varphi(\max\{d(Gx_{2m(i)}, Sx_{2n(i)+1}), d(Gy_{2n(i)+1}, Sy_{2n(i)+1}) \})
\]
\[
- \varphi(\max\{d(Gx_{2m(i)}, Sx_{2n(i)+1}), d(Gy_{2n(i)+1}, Sy_{2n(i)+1}) \}) \leq \varphi(\max\{d(F(x_{2n(i)+2}, y_{2n(i)+2}), F(y_{2n(i)+1}, x_{2n(i)+1})), d(F(x_{2n(i)+2}, y_{2n(i)+2}), F(x_{2n(i)+1}, y_{2n(i)+1})) \})
\]
\[
- \varphi(\max\{d(F(x_{2n(i)+2}, y_{2n(i)+2}), F(y_{2n(i)+1}, x_{2n(i)+1})), d(F(x_{2n(i)+2}, y_{2n(i)+2}), F(x_{2n(i)+1}, y_{2n(i)+1})) \}).
\]

Therefore
\[
\max\{\varphi(\max\{d(F(x_{2m(i)}, y_{2m(i)}), F(x_{2n(i)+1}, y_{2n(i)+1}), d(F(y_{2m(i)}, x_{2m(i)}) \}) \}
\]
\[
- \varphi(\max\{d(Gx_{2m(i)}, Sx_{2n(i)+1}), d(Gy_{2n(i)+1}, Sy_{2n(i)+1}) \}) \leq \varphi(\max\{d(F(x_{2n(i)+2}, y_{2n(i)+2}), F(x_{2n(i)+1}, y_{2n(i)+1})), d(F(x_{2n(i)+2}, y_{2n(i)+2}), F(x_{2n(i)+1}, x_{2n(i)+1})) \})
\]
\[
- \varphi(\max\{d(F(x_{2n(i)+2}, y_{2n(i)+2}), F(x_{2n(i)+1}, y_{2n(i)+1})), d(F(x_{2n(i)+2}, y_{2n(i)+2}), F(x_{2n(i)+1}, y_{2n(i)+1})) \}).
\]

We claim that
\[
\max\{d(F(x_{2i+2}, y_{2i+2}), F(x_{2n(i)+1}, y_{2n(i)+1}), d(F(y_{2i+2}, y_{2n(i)+2}), F(x_{2n(i)+1}, x_{2n(i)+1})) \}
\]
\[
- \varepsilon \text{ as } i \to +\infty.
\]

In fact, using the triangular inequality, we have
\[
d(F(x_{2m(i)-2}, y_{2m(i)-2}), F(x_{2n(i)-1}, y_{2n(i)-1}))
\]
\[
\leq d(F(x_{2m(i)-2}, y_{2m(i)-2}), F(x_{2m(i)-1}, y_{2m(i)-1})) + d(F(x_{2m(i)-1}, y_{2m(i)-1}), F(x_{2n(i)}, y_{2n(i)}))
\]
\[
+ d(F(x_{2n(i)}, y_{2n(i)}), F(x_{2n(i)-1}, y_{2n(i)-1})).
\]

Letting \( i \to +\infty \) in the above inequality and using (5) and (16), we obtain
\[
\lim_{i \to +\infty} d(F(x_{2m(i)-2}, y_{2m(i)-2}), F(x_{2n(i)-1}, y_{2n(i)-1})) \leq \varepsilon.
\]

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On the other hand, we have
\[
d(F(x_{2m(i)} - 1, y_{2m(i)} - 1), F(x_{2n(i)}, y_{2n(i)})) \\
\leq d(F(x_{2m(i)} - 1, y_{2m(i)} - 1), (F(x_{2m(i)} - 1, 2m(i)) + d(F(x_{2m(i)} - 2, y_{2m(i)} - 2), F(x_{2n(i)} - 1, y_{2n(i)} - 1))) \\
+ d(F(x_{2n(i)} - 1, y_{2n(i)} - 1), F(x_{2n(i)} - 1, y_{2n(i)}))
\]

Letting \( i \to +\infty \) in the above inequality and using (5) and (10), we obtain
\[
\varepsilon \leq \lim_{i \to +\infty} d(F(x_{2m(i)} - 2, y_{2m(i)} - 2), F(x_{2n(i)} - 1, y_{2n(i)} - 1)).
\]

Combining (20) and (21), we get
\[
\lim_{i \to +\infty} d(F(x_{2m(i)} - 2, y_{2m(i)} - 2), F(x_{2n(i)} - 1, y_{2n(i)} - 1)) = \varepsilon.
\]

By the same way, we obtain
\[
\lim_{i \to +\infty} d(F(y_{2m(i)} - 2, x_{2m(i)} - 2), F(y_{2n(i)} - 1, x_{2n(i)} - 1)) = \varepsilon.
\]

Thus we proved (19). Finally, letting \( i \to +\infty \) in (13), using (17), (19) and the continuity of \( \varphi \) and \( \phi \), we get \( \varphi(\varepsilon) \leq \varphi(\varepsilon) - \phi(\varepsilon) \), which implies that \( \phi(\varepsilon) = 0 \), that is, \( \varepsilon = 0 \), a contradiction. Thus \( (F(x_{2n}, y_{2n})) \) and \( (F(y_{2n}, x_{2n})) \) are Cauchy sequences in \( X \), which gives us that \( (F(x_n, y_n)) \) and \( (F(y_n, x_n)) \) are also Cauchy sequences.

**Step 3:** Existence of a coupled coincidence point.
Since \((F(x_n, y_n))\) and \((F(y_n, x_n))\) are Cauchy sequences in the complete metric space \((X, d)\), there exist \( \alpha, \alpha' \in X \) such that:
\[
\lim_{n \to +\infty} F(x_n, y_n) = \alpha \quad \text{and} \quad \lim_{n \to +\infty} F(y_n, x_n) = \alpha'.
\]

Therefore, \( \lim_{n \to +\infty} G(x_{2n+2}) = \alpha, \quad \lim_{n \to +\infty} G(y_{2n+2}) = \alpha', \quad \lim_{n \to +\infty} S(x_{2n+3}) = \alpha \) and \( \lim_{n \to +\infty} S(y_{2n+3}) = \alpha' \).

Using the continuity and the commutativity of \( F \) and \( G \), we have
\[
G(G(x_{2n+2})) = G(F(x_{2n}, y_{2n})) = F(Gx_{2n}, Gy_{2n}) \quad \text{and} \quad G(G(y_{2n+2})) = G(F(y_{2n}, x_{2n})) = F(Gy_{2n}, Gx_{2n}).
\]

Letting \( n \to +\infty \), we get \( G(\alpha) = F(\alpha, \alpha') \) and \( G(\alpha') = F(\alpha', \alpha) \).

Using also the continuity and the commutativity of \( F \) and \( S \), by the same way, we obtain \( S(\alpha) = F(\alpha, \alpha') \) and \( S(\alpha') = F(\alpha', \alpha) \).

Therefore
\[
G(\alpha) = F(\alpha, \alpha') = S(\alpha) \quad \text{and} \quad G(\alpha') = F(\alpha', \alpha) = S(\alpha').
\]

Thus we proved that \( (\alpha, \alpha') \) is a coupled coincidence point of \( G, S \) and \( F \).

In the next result, we prove that the previous theorem is still valid if we replace the continuity of \( F \) by some conditions.

**Theorem 2.2** If we replace the continuity hypothesis of \( F \) in Theorem 2.1 by the following conditions:
(i) if \((x_n)\) is a non-decreasing sequences with \(x_n \to x\) then \(x_n \leq x\) for each \(n \in \mathbb{N}\),

(ii) if \((y_n)\) is a non-increasing sequences with \(y_n \to y\) then \(y \leq y_n\) for each \(n \in \mathbb{N}\),

(iii) \(x, y \in X, \quad x \leq y \Rightarrow Gx \leq Sy,\)

(iv) \(x, y \in X, \quad x \geq y \Rightarrow Gx \geq Sy.\)

Then \(G, S\) and \(F\) have a coupled coincidence point.

**Proof.** Following the proof of Theorem 2.1 we have that \(F(x_n, y_n)\) and \(F(y_n, x_n)\) are Cauchy sequences in the complete metric space \((X, d)\), there exist \(\alpha, \alpha' \in X\) such that

\[
\lim_{n \to +\infty} F(x_n, y_n) = \alpha \quad \text{and} \quad \lim_{n \to +\infty} F(y_n, x_n) = \alpha'.
\]

Therefore \(\lim_{n \to +\infty} F(x_{2n}, y_{2n}) = \alpha\) and \(\lim_{n \to +\infty} F(y_{2n}, x_{2n}) = \alpha'.\) Hence \(\lim_{n \to +\infty} G(x_{2n+2}) = \alpha,\)

\[
\lim_{n \to +\infty} G(y_{2n+2}) = \alpha', \quad \lim_{n \to +\infty} S(x_{2n+3}) = \alpha \quad \text{and} \quad \lim_{n \to +\infty} S(y_{2n+3}) = \alpha'.
\]

Using the commutativity of \(F\) and \(G\) and of \(F\) and \(S\) and the contractive condition, it follows from conditions (iii)-(iv) that

\[
\varphi(d(G(F(x_{2n}, y_{2n})), S(F(x_{2n+1}, y_{2n+1})))) \\
= \varphi(d(F(Gx_{2n}, Gy_{2n}), F(Sx_{2n+1}, Sy_{2n+1}))) \\
\leq \varphi(\max\{d(G(Gx_{2n}), S(Sx_{2n+1})), d(G(Gy_{2n}), S(Sy_{2n+1}))\}) \quad (22) \\
- \phi(\max\{d(G(Gx_{2n}), S(Sx_{2n+1})), d(G(Gy_{2n}), S(Sy_{2n+1}))\}). \quad (23)
\]

Similarly, we have

\[
\varphi(d(G(F(y_{2n}, x_{2n})), S(F(y_{2n+1}, x_{2n+1})))) \\
= \varphi(d(F(Gy_{2n}, Gx_{2n}), F(Sy_{2n+1}, Sx_{2n+1}))) \\
\leq \varphi(\max\{d(G(Gy_{2n}), S(Sy_{2n+1})), d(G(Gx_{2n}), S(Sx_{2n+1}))\}) \quad (24) \\
- \phi(\max\{d(G(Gy_{2n}), S(Sy_{2n+1})), d(G(Gx_{2n}), S(Sx_{2n+1}))\}). \quad (25)
\]

Combining (22), (24) and the fact that \(\max\{\varphi(a), \varphi(b)\} = \varphi(\max\{a, b\})\) for \(a, b \in [0, +\infty)\), from (iii)-(iv), we obtain

\[
\varphi(\max\{d(G(F(x_{2n}, y_{2n})), S(F(x_{2n+1}, y_{2n+1}))), d(G(F(y_{2n}, x_{2n})), S(F(y_{2n+1}, x_{2n+1})))\}) \\
\leq \varphi(\max\{d(G(Gx_{2n}), S(Sx_{2n+1})), d(G(Gy_{2n}), S(Sy_{2n+1}))\}) \\
- \phi(\max\{d(G(Gx_{2n}), S(Sx_{2n+1})), d(G(Gy_{2n}), S(Sy_{2n+1}))\}).
\]

Letting \(n \to +\infty\) in the last expression, using the continuity of \(G\) and \(S\), we get

\[
\varphi(\max\{d(G(\alpha), S(\alpha)), d(G(\alpha'), S(\alpha'))\}) \\
\leq \varphi(\max\{d(G(\alpha), S(\alpha)), d(G(\alpha'), S(\alpha'))\}) - \phi(\max\{d(G(\alpha), S(\alpha)), d(G(\alpha'), S(\alpha'))\}).
\]

This implies that \(\phi(\max\{d(G(\alpha), S(\alpha)), d(G(\alpha'), S(\alpha'))\}) = 0\) and, since \(\phi\) is an altering distance function, then

\[
\max\{d(G(\alpha), S(\alpha)), d(G(\alpha'), S(\alpha'))\} = 0.
\]
Consequently
\[ G(\alpha) = S(\alpha) \quad \text{and} \quad G(\alpha') = S(\alpha'). \] (26)

To finish the proof, we claim that \( F(\alpha, \alpha') = G(\alpha) = S(\alpha) \) and \( F(\alpha', \alpha) = G(\alpha') = S(\alpha') \).

Indeed, using the contractive condition, it follows from (i)-(iv) that
\[
\varphi(d(F(Gx_{2n}, Gy_{2n}), F(\alpha, \alpha')))
\leq \varphi(\max\{d(G(Gx_{2n}), S(\alpha)), d(G(Gy_{2n}), S(\alpha'))\}) - \phi(\max\{d(G(Gx_{2n}), S(\alpha)), d(G(Gy_{2n}), S(\alpha'))\})
\leq \varphi(\max\{d(G(Gx_{2n}), S(\alpha)), d(G(Gy_{2n}), S(\alpha'))\}).
\]

Using the fact that \( \varphi \) is non-decreasing, we get
\[
d(F(Gx_{2n}, Gy_{2n}), F(\alpha, \alpha')) \leq \max\{d(G(Gx_{2n}), S(\alpha)), d(G(Gy_{2n}), S(\alpha'))\}. \] (27)

Similarly, we have
\[
\varphi(d(F(Gy_{2n}, Gx_{2n}), F(\alpha', \alpha)))
\leq \varphi(\max\{d(G(Gy_{2n}), S(\alpha')), d(G(Gx_{2n}), S(\alpha))\})
- \phi(\max\{d(G(Gy_{2n}), S(\alpha')), d(G(Gx_{2n}), S(\alpha))\})
\leq \varphi(\max\{d(G(Gy_{2n}), S(\alpha')), d(G(Gx_{2n}), S(\alpha))\}).
\]

Using the fact that \( \varphi \) is non-decreasing, we see that
\[
d(F(Gy_{2n}, Gx_{2n}), F(\alpha', \alpha)) \leq \max\{d(G(Gy_{2n}), S(\alpha')), d(G(Gx_{2n}), S(\alpha))\}. \] (28)

Combining (27) and (28), we get
\[
\max\{d(F(Gx_{2n}, Gy_{2n}), F(\alpha, \alpha')), d(F(Gy_{2n}, Gx_{2n}), F(\alpha', \alpha))\)
\leq \max\{d(G(Gy_{2n}), S(\alpha')), d(G(Gx_{2n}), S(\alpha))\}.
\]

Using the commutativity of \( F \) and \( G \), we write
\[
\max\{d(G(F'x_{2n}, y_{2n})), F(\alpha, \alpha'), d(G(F'y_{2n}, x_{2n})), F(\alpha', \alpha))\)
\leq \max\{d(G(Gy_{2n}), S(\alpha')), d(G(Gx_{2n}), S(\alpha))\}.
\]

Letting \( n \to +\infty \), using the continuity of \( G \), we obtain
\[
\max\{d(G(\alpha), F(\alpha, \alpha')), d(G(\alpha'), F(\alpha', \alpha))\} \leq \max\{d(G(\alpha), S(\alpha)), d(G(\alpha'), S(\alpha'))\}.
\]

Looking at (26), we deduce that
\[
\max\{d(G(\alpha), F(\alpha, \alpha')), d(G(\alpha'), F(\alpha', \alpha))\} = 0.
\]

Therefore, \( d(G(\alpha), F(\alpha, \alpha')) = 0 \) and \( d(G(\alpha'), F(\alpha', \alpha)) = 0 \).

Consequently
\[ G(\alpha) = F(\alpha, \alpha') \quad \text{and} \quad G(\alpha') = F(\alpha', \alpha). \] (29)

By the same way, we get
\[ S(\alpha) = F(\alpha, \alpha') \quad \text{and} \quad S(\alpha') = F(\alpha', \alpha). \] (30)
Finally, combining (26), (29) and (30), we deduce that \((\alpha, \alpha')\) is a coupled coincidence point of \(F, G\) and \(S\).

Now, we give a sufficient condition for the existence and the uniqueness of the coupled common fixed point. Notice that if \((X, \preceq)\) is a partially ordered set, we endow \(X \times X\) with the following partial order relation:

\[
\text{for } (x, y), (u, v) \in X \times X, \quad (x, y) \preceq (u, v) \iff x \preceq u \text{ and } y \succeq v.
\]

**Theorem 2.3** In addition to the hypotheses of Theorem 2.1 (resp. Theorem 2.2), suppose that for every \((x, y), (x^*, y^*) \in X \times X\) there exists a \((u, v) \in X \times X\) such that \((F(u, v), F(v, u))\) is comparable to \((F(x, y), F(y, x))\) and \((F(x^*, y^*), F(y^*, x^*))\). Then \(F, G\) and \(S\) have a unique coupled common fixed point, that is, there exist a unique \((x, y) \in X \times X\) such that

\[
x = G(x) = F(x, y) = S(x) \quad \text{and} \quad y = G(y) = F(y, x) = S(y).
\]

**Proof.** We know, from Theorem 2.1 (resp. Theorem 2.2), that exists a coupled coincidence point. We suppose that exist \((x, y)\) and \((x^*, y^*)\) two coupled coincidence points, that is, \(G(x) = F(x, y) = S(x), G(y) = F(y, x) = S(y), G(x^*) = F(x^*, y^*) = S(x^*)\) and \(G(y^*) = F(y^*, x^*) = S(y^*)\). We claim that

\[
G(x) = G(x^*) = S(x^*) = S(x) \quad \text{and} \quad G(y) = G(y^*) = S(y^*) = S(y). \tag{31}
\]

If \((F(x, y), F(y, x))\) is comparable to \((F(x^*, y^*), F(y^*, x^*))\), it is easy to reach the result, then we suppose the general case.

By assumption there is \((u, v) \in X \times X\) such that \((F(u, v), F(v, u))\) is comparable to \((F(x, y) F(y, x))\) and \((F(x^*, y^*) F(y^*, x^*))\). We distinguish two cases:

**First case:** We assume that

\[
(F(x, y), F(y, x)) \preceq (F(u, v), F(v, u)) \quad \text{and} \quad (F(x^*, y^*), F(y^*, x^*)) \preceq (F(u, v), F(v, u)).
\]

Put \(u_0 = u\) and \(v_0 = v\) and we choose \(u_1\) and \(v_1\) such that \(G(u_0) \preceq S(u_1) \preceq F(u_0, v_0), G(v_0) \succeq S(v_1) \succeq F(v_0, u_0)\).

Similarly as in the proof of Theorem 2.1 we can construct sequences \(\{u_n\}\) and \(\{v_n\}\) in \(X\) such that

\[
\begin{align*}
G(u_{2n+2}) &= F(u_{2n}, v_{2n}) & \quad \text{and} \quad S(u_{2n+3}) &= F(u_{2n+1}, v_{2n+1}) \\
G(v_{2n+2}) &= F(v_{2n}, u_{2n}) & \quad \text{and} \quad S(v_{2n+3}) &= F(v_{2n+1}, u_{2n+1})
\end{align*}
\]

for all \(n \geq 0\).

Looking at the proof of Theorem 2.1 precisely at (30), we see that \(\{G(u_{2n})\}\) is a non-decreasing sequence, \(G(u_{2n}) \preceq S(u_{2n+1})\), and \(\{G(v_{2n})\}\) is a non-increasing sequence, \(G(v_{2n}) \succeq S(v_{2n+1})\).

Therefore, we have

\[
G(x) = F(x, y) \preceq F(u_0, v_0) = G(u_2) \preceq G(u_{2n}) \preceq S(u_{2n+1})
\]

and

\[
G(y) = F(y, x) \succeq F(v_0, u_0) = G(v_2) \succeq G(v_{2n}) \succeq S(v_{2n+1}). \tag{32}
\]

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Similarly, we have

\[ G(x^*) = F(x^*, y^*) \preceq F(u_0, v_0) = G(u_2) \preceq G(u_{2n}) \preceq S(u_{2n+1}) \]

and

\[ G(y^*) = F(y^*, x^*) \succeq F(v_0, u_0) = G(v_2) \succeq G(v_{2n}) \succeq S(v_{2n+1}). \]

Using (32) and the contractive condition, we write

\[ \varphi(d(F(x, y), F(u_{2n+1}, v_{2n+1}))) \leq \varphi(\max\{d(Gx, Su_{2n+1}), d(Gy, Sv_{2n+1})\}) \]

\[ -\phi(\max\{d(Gx, Su_{2n+1}), d(Gy, Sv_{2n+1})\}) \]

and

\[ \varphi(d(F(y, x), F(v_{2n+1}, u_{2n+1}))) \leq \varphi(\max\{d(Gy, Sv_{2n+1}), d(Gx, Su_{2n+1})\}) \]

\[ -\phi(\max\{d(Gy, Sv_{2n+1}), d(Gx, Su_{2n+1})\}). \]

Therefore

\[ \varphi(\max\{d(F(x, y), F(u_{2n+1}, v_{2n+1})), d(F(y, x), F(v_{2n+1}, u_{2n+1})\}) \]

\[ \leq \varphi(\max\{d(Gx, Su_{2n+1}), d(Gy, Sv_{2n+1})\}) \]

\[ -\phi(\max\{d(Gx, Su_{2n+1}), d(Gy, Sv_{2n+1})\}). \]

Therefore

\[ \varphi(\max(d(G(x), Su_{2n+3}), d(Gy, Sv_{2n+3}))) \leq \varphi(\max(d(G(x), Su_{2n+1}), d(Gy, Sv_{2n+1}))) \]

\[ -\phi(\max(d(G(x), Su_{2n+1}), d(Gy, Sv_{2n+1}))). \]

We see that

\[ \varphi(\max\{d(G(x), Su_{2n+3}), d(Gy, Sv_{2n+3})\}) \leq \varphi(\max\{d(Gx, Su_{2n+1}), d(Gy, Sv_{2n+1})\}). \]

Using the non-decreasing property of \( \varphi \), we get

\[ \max\{d(Gx, Su_{2n+3}), d(Gy, Sv_{2n+3})\} \leq \max\{d(Gx, Su_{2n+1}), d(Gy, Sv_{2n+1})\}. \]

This implies that \( \max\{d(Gx, Su_{2n+1}), d(Gy, Sv_{2n+1})\} \) is a non-increasing sequence. Hence, there exists \( r \geq 0 \) such that

\[ \lim_{n \to +\infty} \max\{d(Gx, Su_{2n+1}), d(Gy, Sv_{2n+1})\} = r. \]

Passing to limit in (33) as \( n \to +\infty \), we obtain

\[ \varphi(r) \leq \varphi(r) - \phi(r), \]

which implies that \( \phi(r) = 0 \) and then, since \( \phi \) is an altering distance function, \( r = 0 \). We deduce that

\[ \lim_{n \to +\infty} \max\{d(Gx, Su_{2n+1}), d(Gy, Sv_{2n+1})\} = 0. \] (35)
Similarly, one can prove that

\[
\lim_{n \to +\infty} \max\{d(Gx^*, Su_{2n+1}), d(Gy^*, Sv_{2n+1})\} = 0. \tag{36}
\]

By the triangle inequality, (35) and (36),

\[
d(Gx, Gx^*) \leq d(Gx, Su_{2n+1}) + d(G(x^*), Su_{2n+1}) \to 0 \text{ as } n \to +\infty, \tag{37}
\]

\[
d(Gy, Gy^*) \leq d(Gy, Sv_{2n+1}) + d(G(y^*), Sv_{2n+1}) \to 0 \text{ as } n \to +\infty. \tag{38}
\]

Hence

\[
G(x) = G(x^*) \text{ and } G(y) = G(y^*). \tag{39}
\]

This prove the claim (31) in this case.  

**Second case:** We assume that \((F(x, y), F(y, x)) \preceq (F(u, v), F(v, u))\) and \((F(x^*, y^*), F(y^*, x^*)) \preceq (F(u, v), F(v, u))\).

Put \(u_0 = u\) and \(v_0 = v\) and we choose \(u_1\) and \(v_1\) such that \(G(u_0) \succeq S(u_1) \succeq F(u_0, v_0)\), \(G(v_0) \preceq S(v_1) \preceq F(v_0, u_0)\).

Similarly as in the proof of Theorem 2.1 we can construct sequences \(\{u_n\}\) and \(\{v_n\}\) in \(X\) such that

\[
\begin{align*}
\{ G(u_{2n+2}) = F(u_{2n}, v_{2n}) \} \quad \text{and} \\
\{ G(v_{2n+2}) = F(v_{2n}, u_{2n}) \}
\end{align*}
\]

for all \(n \geq 0\).

Looking at the proof of Theorem 2.1 precisely at (39), we see that \(\{G(u_{2n})\}\) is a non-increasing sequence, \(G(u_{2n}) \succeq S(u_{2n+1})\), and \(\{G(v_{2n})\}\) is a non-decreasing sequence, \(G(v_{2n}) \preceq S(v_{2n+1})\).

Therefore, we have

\[
G(x) = F(x, y) \succeq F(u_0, v_0) = G(u_2) \succeq G(u_{2n}) \succeq S(u_{2n+1})
\]

and

\[
G(y) = F(y, x) \preceq F(v_0, u_0) = G(v_2) \preceq G(v_{2n}) \preceq S(v_{2n+1}).
\]

Similarly, we have

\[
G(x^*) = F(x^*, y^*) \preceq F(u_0, v_0) = G(u_2) \succeq G(u_{2n}) \succeq S(u_{2n+1})
\]

and

\[
G(y^*) = F(y^*, x^*) \preceq F(v_0, u_0) = G(v_2) \preceq G(v_{2n}) \preceq S(v_{2n+1}).
\]

From this, we complete the proof identically as in the first case and we obtain the claim (31) in this case. Since \(G(x) = F(x, y) = S(x)\) and \(G(y) = F(y, x) = S(y)\), by the commutativity of \(F, G\) and \(F, S\), we have

\[
\begin{align*}
\{ G(G(x)) = G(F(x, y)) = F(Gx, Gy) \} \quad \text{and} \\
\{ G(G(y)) = G(F(y, x)) = F(Gy, Gx) \}
\end{align*}
\]

and

\[
\begin{align*}
\{ S(S(x)) = S(F(x, y)) = F(S(x), S(y)) \}
\quad \text{and} \quad \\
\{ S(S(y)) = S(F(y, x)) = F(S(y), S(x)) \}
\end{align*}
\]

(40)

Set \(G(x) = a = S(x)\), \(G(y) = b = S(y)\). Then from (40),

\[
G(a) = F(a, b) = S(a) \quad \text{and} \quad G(b) = F(b, a) = S(b). \tag{41}
\]
Thus \((a, b)\) is a coupled coincidence point. Then from (31) with \(x^* = a\) and \(y^* = b\) it follows that \(G(a) = G(x) = S(a)\) and \(G(b) = G(y) = S(b)\). Therefore

\[
G(a) = a = S(a) \quad \text{and} \quad G(b) = b = S(b).
\] (42)

We deduce that \((a, b)\) is a coupled common fixed point. To prove the uniqueness, assume that \((c, d)\) is another coupled common fixed point. Then by (31) and (42) we have \(c = G(c) = G(a) = a\) and \(d = G(d) = G(b) = b\).

\[\blacksquare\]

**Remark 2**

Taking \(G = S = I_X\) (the identity mapping of \(X\)) in Theorem 2.1, we obtain [7, Theorem 2].

Taking \(G = S = I_X\) in Theorem 2.2, we obtain [7, Theorem 3].

Taking \(S = G\) in Theorem 2.3, we obtain the following result.

**Corollary 2.1** Let \((X, \preceq)\) be a partially ordered set and suppose that there exists a metric \(d\) on \(X\) such that \((X, d)\) is a complete metric space. Let \(G : X \to X\) be two mappings and \(F : X \times X \to X\) be a mapping with the mixed \(G\)-monotone property and satisfying

\[
\varphi(d(F(x, y), F(u, v))) \leq \varphi(\max\{d(Gx, Gu), d(Gy, Gv)\}) - \phi(\max\{Gx, Gu\), d(Gy, Gv)\})
\]

for all \(x, y, u, v \in X\) with \(G(x) \preceq G(u)\) or \(G(x) \succeq G(u)\) and \(G(y) \succeq G(v)\) or \(G(y) \preceq G(v)\), where \(\varphi\) and \(\phi\) are altering distance functions. Assume that \(F(X \times X) \subseteq G(X)\) and assume also the following hypotheses:

1. \(G\) is continuous,
2. \(F\) is continuous or \(G\) is non-decreasing mapping and \(X\) satisfies the following properties:
   - if \((x_n)\) is a non-decreasing sequences with \(x_n \to x\) then \(x_n \preceq x\) for each \(n \in \mathbb{N}\),
   - if \((y_n)\) is a non-increasing sequences with \(y_n \to y\) then \(y \preceq y_n\) for each \(n \in \mathbb{N}\);
3. for every \((x, y), (x^*, y^*) \in X \times X\) there exists a \((u, v) \in X \times X\) such that \((F(u, v), F(v, u))\) is comparable to \((F(x, y), F(y, x))\) and \((F(x^*, y^*), F(y^*, x^*))\),
4. \(F\) commutes with \(G\).

If there exist \(x_0, y_0 \in X\) such that

\[
\begin{cases}
G(x_0) \preceq F(x_0, y_0) \\
G(y_0) \succeq F(y_0, x_0)
\end{cases}
\]

then there exists a unique \((x, y) \in X \times X\) such that

\[x = G(x) = F(x, y) \quad \text{and} \quad y = G(y) = F(y, x),\]

that is, \(G\) and \(F\) have a unique coupled common fixed point.
3 Applications to periodic boundary value problems

In this section, we study the existence and uniqueness of solution to a periodic boundary value problem, as an application to the fixed point theorem given by Corollary 2.1.

Let $C([0, T], \mathbb{R})$ be the set of all continuous functions $u : [0, T] \to \mathbb{R}$ and consider a mapping $G : C([0, T], \mathbb{R}) \to C([0, T], \mathbb{R})$.

Consider the periodic boundary value problem

\begin{align*}
u' &= f(t, u) + h(t, u), \quad t \in (0, T) \quad (43) \\
u(0) &= u(T), \quad (44)
\end{align*}

where $f$, $h$ are two continuous functions satisfying the following conditions:

There exist positive constants $\lambda_1$, $\lambda_2$, $\mu_1$ and $\mu_2$, such that for all $u, v \in (C([0, T], \mathbb{R})$, $Gv(t) \leq Gu(t)$,

\begin{align*}
0 &\leq (f(t, u(t)) + \lambda_1 u(t)) - (f(t, v(t)) + \lambda_1 v(t)) \leq \mu_1 \ln[(Gu(t) - Gv(t))^2 + 1] \quad (45) \\
- \mu_2 \ln[(Gu(t) - Gv(t))^2 + 1] &\leq (h(t, u(t)) + \lambda_2 u(t)) - (h(t, v(t)) + \lambda_2 v(t)) \leq 0 \quad (46)
\end{align*}

with

\begin{equation}
\frac{2\max\{\mu_1, \mu_2\}}{\lambda_1 + \lambda_2} < 1. \quad (47)
\end{equation}

We firstly study the existence of a solution of the following periodic system:

\begin{align*}
u' + \lambda_1 u - \lambda_2 v &= f(t, u) + h(t, v) + \lambda_1 u - \lambda_2 v \\
v' + \lambda_1 v - \lambda_2 u &= f(t, v) + h(t, u) + \lambda_1 v - \lambda_2 u, \quad (48)
\end{align*}

with the periodicity condition

\begin{equation}
u(0) = u(T) \quad \text{and} \quad v(0) = v(T). \quad (49)
\end{equation}

This problem is equivalent to the integral equations:

\begin{align*}
u(t) &= \int_0^T k_1(t, s)[f(s, u) + h(s, v) + \lambda_1 u - \lambda_2 v] + \int_0^T k_2(t, s)[f(s, v) + h(s, u) + \lambda_1 v - \lambda_2 u]ds \\
v(t) &= \int_0^T k_2(t, s)[f(s, v) + h(s, u) + \lambda_1 v - \lambda_2 u] + \int_0^T k_1(t, s)[f(s, u) + h(s, v) + \lambda_1 u - \lambda_2 v]ds
\end{align*}

where

\begin{align*}
k_1(t, s) &= \begin{cases}
\frac{1}{2} \left[ \frac{e^{\sigma_1(t-s)}}{1 - e^{\sigma_1 T}} + \frac{e^{\sigma_2(t-s)}}{1 - e^{\sigma_2 T}} \right] & 0 \leq s < t \leq T \\
\frac{1}{2} \left[ \frac{e^{\sigma_1(t+s)}}{1 - e^{\sigma_1 T}} + \frac{e^{\sigma_2(t+s)}}{1 - e^{\sigma_2 T}} \right] & 0 \leq t < s \leq T
\end{cases} \\
k_2(t, s) &= \begin{cases}
\frac{1}{2} \left[ \frac{e^{\sigma_2(t-s)}}{1 - e^{\sigma_2 T}} + \frac{e^{\sigma_1(t-s)}}{1 - e^{\sigma_1 T}} \right] & 0 \leq s < t \leq T \\
\frac{1}{2} \left[ \frac{e^{\sigma_2(t+s)}}{1 - e^{\sigma_2 T}} + \frac{e^{\sigma_1(t+s)}}{1 - e^{\sigma_1 T}} \right] & 0 \leq t < s \leq T
\end{cases}
\end{align*}
Here, $\sigma_1 = -(\lambda_1 + \lambda_2)$ and $\sigma_2 = (\lambda_2 - \lambda_1)$.

From [3, Lemma 3.2], we have

$$k_1(t, s) \geq 0, \quad 0 \leq t, s \leq T \quad \text{and} \quad k_2(t, s) \leq 0, \quad 0 \leq t, s \leq T. \quad (50)$$

We assume that there exist $\alpha, \beta \in C([0, T])$ such that

$$G(\alpha(t)) \leq \int_0^1 k_1(t, s)(f(s, \alpha(s)) + h(s, \beta(s)) + \lambda_1 \alpha(s) - \lambda_2 \beta(s))ds$$

$$+ \int_0^1 k_2(t, s)(f(s, \beta(s)) + h(s, \alpha(s)) + \lambda_1 \beta(s) - \lambda_2 \alpha(s))ds$$

$$\quad (51)$$

and

$$G(\beta(t)) \geq \int_0^1 k_1(t, s)(f(s, \beta(s)) + h(s, \alpha(s)) + \lambda_1 \beta(s) - \lambda_2 \alpha(s))ds$$

$$+ \int_0^1 k_2(t, s)(f(s, \alpha(s)) + h(s, \beta(s)) + \lambda_1 \alpha(s) - \lambda_2 \beta(s))ds. \quad (52)$$

We endow $X = C([0, T], \mathbb{R})$ with the metric $d(u, v) = \max_{t \in [0, T]} |u(t) - v(t)|$ for $u, v \in X$.

This space can be equipped with a partial order given by

$$x, y \in C([0, T]), \quad x \preceq y \iff x(t) \leq y(t), \quad \text{for any } t \in [0, T].$$

In $X \times X$ we define the following partial order

$$(x, y), \ (u, v) \in X \times X, \quad (x, y) \preceq (u, v) \iff x \preceq u \quad \text{and} \quad y \succeq v.$$ 

Since for any $x, y \in X$ we have that $\max(x, y)$ and $\min(x, y) \in X$, assumption 3 of Corollary 2.1 is satisfied for $(X, \preceq)$. Moreover in [10] it is proved that $(X, \preceq)$ satisfies assumption 2 of Corollary 2.1.

Now, we shall prove the following result.

**Theorem 3.1** Suppose that $G : X \to X$ is a non-decreasing continuous mapping. Suppose also that (45)-(47) and (51)-(52) hold. Then (48)-(49) has a unique solution. Therefore (43)-(44) has also a unique solution.

**Proof.** We introduce the operator $F : X \times X \to X$ defined by

$$F(u, v)(t) = \int_0^T k_1(t, s)[f(s, u) + h(s, v) + \lambda_1 u - \lambda_2 v] \, ds$$

$$+ \int_0^T k_2(t, s)[f(s, v) + h(s, u) + \lambda_1 v - \lambda_2 u] \, ds$$

for all $u, v \in X$ and $t \in [0, T]$. 

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We claim that $F$ has the mixed $G$-monotone property.
In fact, for $Gx_1 \leq Gx_2$ and $t \in [0, T]$, we have

$$F(x_1, y)(t) - F(x_2, y)(t) = \int_0^T k_1(t, s)(f(s, x_1(s)) - f(s, x_2) + \lambda_1(x_1(s) - x_2(s))ds$$

$$+ \int_0^T k_2(t, s)(h(s, x_1(s)) - h(s, x_2) - \lambda_2(x_1 - x_2))ds.$$  

From (45), (46) and (50), for all $t \in [0, T]$, we have

$$F(x_1, y)(t) - F(x_2, y)(t) \leq 0.$$  

This implies that

$$F(x_1, y) \leq F(x_2, y).$$  

Also, for $Gy_1 \leq Gy_2$ and $t \in [0, T]$, we have

$$F(x, y_1)(t) - F(x, y_2)(t) = \int_0^T k_1(t, s)(h(s, y_1(s)) - h(s, y_2) - \lambda_2(y_1(s) - y_2(s))ds$$

$$+ \int_0^T k_2(t, s)(f(s, y_1(s)) - f(s, y_2) + \lambda_1(y_1 - y_2))ds.$$  

Looking at (45), (46) and (50), for all $t \in [0, T]$, we have

$$F(x_1, y)(t) - F(x_2, y)(t) \geq 0,$$

that is,

$$F(x, y_1) \geq F(x, y_2).$$  

Thus, we proved that $F$ has the mixed $G$-monotone property.

For $G(x) \leq G(u)$ and $G(y) \geq G(v)$, we have $F(x, y) \geq F(u, v)$ and

$$d(F(x, y), F(u, v)) = \max_{t \in [0, T]} |F(x, y)(t) - F(u, v)(t)|$$

$$= \max_{t \in [0, T]} (F(x, y)(t) - F(u, v)(t))$$

$$= \max_{t \in [0, T]} \int_0^T k_1(t, s)((f(s, x(s)) - f(s, u(s)) + \lambda_1(x - u)) - (h(s, v(s)) - h(s, y(s)) - \lambda_2(y - v)))ds$$

$$- \int_0^T k_2(t, s)((f(s, v(s)) - f(s, y(s)) + \lambda_1(v - y)) - (h(s, u(s)) - h(s, x(s)) - \lambda_2(u - x)))ds.$$  

Using (45) and (46) we get

$$d(F(x, y), F(u, v))$$

$$\leq \max_{t \in [0, T]} \int_0^T k_1(t, s)\left(\mu_1 \ln[(Gx(s) - Gu(s))^2 + 1] + \mu_2 \ln[(Gy(s) - Gv(s))^2 + 1]\right)ds$$

$$+ \int_0^T (-k_2(t, s))\left(\mu_1 \ln[(Gv(s) - Gy(s))^2 + 1] + \mu_2 \ln[(Gx(s) - Gu(s))^2 + 1]\right)ds$$

$$\leq \max(\mu_1, \mu_2) \max_{t \in [0, T]} \int_0^T (k_1(t, s) - k_2(t, s)) \ln[(Gx(s) - Gu(s))^2 + 1]ds$$

$$+ \int_0^T (k_1(t, s) - k_2(t, s)) \ln[(Gy(s) - Gv(s))^2 + 1]ds.$$  

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An easy computation yields
\[
d(F(x, y), F(u, v)) \leq \left( \max_{t \in [0, T]} \int_0^T (k_1(t, s) - k_2(t, s))ds \right) \max(\mu_1, \mu_2) \left( \ln[(d(Gx, Gu))^2 + 1] + \ln[(d(Gy, Gv))^2 + 1] \right)
\]
\[
\leq 2 \left( \max_{t \in [0, T]} \int_0^T (k_1(t, s) - k_2(t, s))ds \right) \max(\mu_1, \mu_2) \ln[(\max(d(Gx, Gu), d(Gy, Gv)))^2 + 1]
\]
\[
\leq 2 \max(\mu_1, \mu_2) \max_{t \in [0, T]} \left| \int_0^t \frac{e^{\sigma_1(t-s)}}{1 - e^{\sigma_1 T}}ds + \int_t^T \frac{e^{\sigma_1(T+s)}}{1 - e^{\sigma_1 T}}ds \right| \ln[(\max(d(Gx, Gu), d(Gy, Gv)))^2 + 1].
\]

After integrating, we get
\[
d(F(x, y), F(u, v)) \leq 2 \max(\mu_1, \mu_2) \max_{t \in [0, T]} \left| \int_0^t \frac{e^{\sigma_1(t-s)}}{1 - e^{\sigma_1 T}}ds + \int_t^T \frac{e^{\sigma_1(T+s)}}{1 - e^{\sigma_1 T}}ds \right| \ln[(\max(d(Gx, Gu), d(Gy, Gv)))^2 + 1].
\]

From (47), we obtain
\[
d(F(x, y), F(u, v)) \leq \ln[(\max(d(Gx, Gu), d(Gy, Gv)))^2 + 1]
\]
which implies that
\[
(d(F(x, y), F(u, v)))^2 \leq (\ln[(\max(d(Gx, Gu), d(Gy, Gv)))^2 + 1])^2.
\]

Then,
\[
(d(F(x, y), F(u, v)))^2 \leq (\max(d(Gx, Gu), d(Gy, Gv)))^2
-[\ln[(\max(d(Gx, Gu), d(Gy, Gv)))^2 + 1]]^2.
\]

Set \(\varphi(t) = t^2\) and \(\phi(t) = t^2 - \ln(t^2 + 1).\) Clearly \(\varphi\) and \(\phi\) are altering distance functions and from the above inequality, we obtain
\[
\varphi(d(F(x, y), F(u, v))) \leq \varphi(\max\{d(Gx, Gu), d(Gy, Gv)\}) - \phi((\max\{d(Gx, Gu), d(Gy, Gv)\}))
\]
for all \(x, y, u, v \in X\) such that \(G(x) \preceq G(u)\) and \(G(y) \succeq G(v)\).

Now, let \(\alpha, \beta \in X\) be the functions given by (51) and (52). Then, we have
\[
G(\alpha) \preceq F(\alpha, \beta) \quad \text{and} \quad F(\beta, \alpha) \preceq G(\beta).
\]

Thus, we proved that all the required hypotheses of Corollary 2.1 are satisfied. Hence, \(G\) and \(F\) have a unique coupled fixed point \((u, v) \in X \times X\), that is, \((u, v)\) is the unique solution of (48)-(49). ■

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