On the removable singularities
of complex analytic sets

E.M. Chirka *
Steklov Math. Institute RAS
chirka@mi.ras.ru

There is proved the sufficiency of several conditions for the removability of
singularities of complex-analytic sets in domains of $\mathbb{C}^n$.

1. Introduction. A closed subset $\Sigma$ of a complex manifold $M$ is
called below $p$-removable if for every respectively closed purely $p$-dimensional
complex analytic subset $A \subset M \setminus \Sigma$ its closure in $M$ is an analytic set.
We omit "complex" in what follows. Thus "analytic set" below means a
set $A \subset M$ such that for every point $a \in A$ there exists a neighborhood
$U \subset M$ of $a$ such that $A \cap U$ is the set of common zeros of a family of
holomorphic functions in $U$. Such a set $A$ has in general singular points
but the set $sng A$ of them is removable in sense of given definition and in
essence we study below the boundary sets $\Sigma$ the adding of which to $A$ does
not spoil the analyticity.

By $\mathcal{H}^m$ we denote the Hausdorff measure of dimension $m \geq 0$ (see e.g. [5]
Ch.III). There is well-known sufficient metrical condition (Shiffman theorem):
$\Sigma$ is $p$-removable if $\mathcal{H}^{2p-1}(\Sigma) = 0$; see [8] or [2] § 4.4. We assume some
smooth metric on $M$ being fixed and Hausdorff measures are taken with
respect to this metric. The vanishing of $\mathcal{H}^m$-measure of a set in $M$ does not
depend on the choice of smooth metric.

The condition of Shiffman theorem is non-improvable in the scale of Haus-
dorff measures and for its weakening one needs in additional assumptions.
For $p = 1$ the following remarkable theorem is obtained by Lee Stout ([10]
Theorem 3.8.18):

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Let $D$ be an arbitrary domain in $\mathbb{C}^n$ and $A$ is relatively closed purely one-dimensional analytic subset of $D \setminus E$ where $E$ is a compact in $\bar{D}$ such that $H^2(E) = 0$, $H^1(E, \mathbb{Z}) = 0$ and the set $E \cap \partial D$ is either empty or a point. Then $\bar{A} \cap D$ is one-dimensional analytic set.

This is the first statement on removable singularities which I know with conditions on the boundaries of tested sets.

Here $H^1$ denotes Čech cohomology and the condition $H^1(E, \mathbb{Z}) = 0$ is purely topological. By Bruschlinsky theorem it is equivalent to the condition that any continuous function without zeros on $E$ has continuous logarithm (see [11] or [10] p.19). Such sets are called also simply co-connected. There are for instance all totally disconnected compact sets (the connected components are points), simple Jordan arcs, plane compact sets with connected complements e.t.c.

Some conditions on $\bar{\Sigma} \cap \partial D$ are necessary in general for the removability of $\Sigma \subset D$. If for example $D$ is the unit ball in $\mathbb{C}^n$ and $\Sigma$ is its diameter on the axis $x_1 = \text{Re} z_1$ then $\Sigma \cap \partial D$ consists of two points only but for the semi-disk $A = D \cap \{\text{Im} z_1 > 0, z_2 = \ldots = z_n = 0\}$ the set $\bar{A} \cap D$ is not analytic. Nevertheless, the condition $\# (E \cap \partial D) \leq 1$ in Stout theorem can be weakened in the following way.

**Theorem 1.** Let $D$ be an arbitrary domain in $\mathbb{C}^n$ and $\Sigma$ be its bounded relatively closed subset. Assume that $H^2(\bar{\Sigma}) = 0$ and the one-point compactification $\Sigma \sqcup \circ$ is simply co-connected. Then $\Sigma$ is 1-removable.

One-point compactification of a topological space $X$ is the topological space $X \sqcup \circ$ with the same topology on $X$ and additional point $\circ$ which punctured neighborhoods are the complements to compact subsets of the space $X$. (Here and everywhere below $\sqcup$ means the union of disjoint sets.) If $\bar{\Sigma} \setminus \Sigma$ is a point (as in Stout theorem) then $\Sigma \sqcup \circ = \bar{\Sigma}$ as topological spaces, and in the example with diameter of a ball the set $\Sigma \sqcup \circ$ is homeomorphic to a circle and thus is not simply co-connected. The condition $H^1(X \sqcup \circ, \mathbb{Z}) = 0$ for relatively closed $X \subset D \Subset \mathbb{R}^N$ is evidently equivalent to the condition that every continuous function without zeros on $\bar{X}$ and constant on $\bar{X} \setminus X$ has on $\bar{X}$ a continuous logarithm which is also constant on $\bar{X} \setminus X$.

The problems with singularities for analytic sets of dimension $p > 1$ are in some sense simpler due to Harvey – Lawson theorem (see for instance theorem 20.2 on compact singularities in [2]), but the proofs need in additional pseudo-convexity type assumptions. The following theorem is valid for arbitrary
\[ p \geq 1. \]

**Theorem 2.** Let \( D \) be a domain in \( \mathbb{C}^n \), \( \Sigma \) be its relatively closed bounded subset, \( \Sigma_b := \overline{\Sigma} \cap \partial D \) and \( p \in \mathbb{N}, p \leq n \). Assume that \( H^{2p}(\Sigma) = 0, H^{2p-1}(\Sigma \cup \circ, \mathbb{Z}) = 0 \) and \( \hat{\Sigma}_b \cap \Sigma = \emptyset \). Then \( \Sigma \) is \( p \)-removable.

Here \( \hat{X} \) means the polynomially convex hull of a compact set \( X \subset \mathbb{C}^n \) that is the set \( \{ z : |P(z)| \leq \max_X |P| \} \) for any polynomial \( P \).

Remark that the Theorems, Propositions and Corollaries below are extending in obvious way onto domains in Stein manifolds (instead of \( D \subset \mathbb{C}^n \) as in the text) in view of the proper imbeddability of such manifolds into suitable \( \mathbb{C}^N \). One should only to substitute the polynomials and polynomially (or rationally) convex hulls by global holomorphic (meromorphic) functions and corresponding hulls. If \( M \) is a complex submanifold in \( \mathbb{C}^N \), \( G \) is a domain on \( M \) and \( \Sigma, A \subset G \) then \( D = \mathbb{C}^N \setminus (M \setminus G) \) is a domain in \( \mathbb{C}^N \), the sets \( \Sigma, A \) are contained in \( D \) and one can apply the results in \( \mathbb{C}^N \) restricting to \( M \).

The paper is organized as follows. In sect.2 we prove mainly topological preliminaries in spirit of argument principle and degrees of continuous mappings. Sect.3 contains the proof of Theorem 1 in more general situation of Proposition 1. Examples in sect.4 stress that simple sufficient conditions of Theorem 1 are not necessary at all. The proof of Theorem 2 is placed in sect.5 and the corollaries of both theorems are collected in sect.6. At the end we discuss natural relations with removable singularities of holomorphic (and meromorphic) functions and state some open questions.

There are many references in the text on the book of E.L.Stout [10] but the given proofs are complete; the references indicate simply the corresponding statements and arguments in [10].

**2. Preliminaries.** First of all we endow \( \Sigma \cup \circ \) by the structure of metrical space inducing the same topology as described above. Let \( \Sigma \) be relatively closed and bounded subset of a domain \( D \subset \mathbb{C}^n \). The point \( \circ \) is connected and we have to transform \( \Sigma_b \) into a connected set. Define \( \Sigma_0 := \{(tz, t) : z \in \Sigma_b, 0 \leq t \leq 1\} \). Let \( \varphi \) be a continuous nonnegative function in \( \mathbb{C}^n \times \mathbb{R}_t \) with zero set \( \Sigma_0 \) and \( \tilde{\rho}(x, x') := \inf_{\gamma} \int_{\gamma} \varphi \, ds \) where \( ds \) is the euclidean metric in \( \mathbb{C}^n \times \mathbb{R} \) and the infimum is taken by all smooth curves \( \gamma \) containing the points \( x, x' \). Then \( \tilde{\rho} \) is symmetric and satisfies the triangle inequality. It is degenerated on \( \Sigma_0 \times \Sigma_0 \) but the corresponding distance
function \( \rho(z, z') := \tilde{\rho}((z, 1), (z', 1)) \) for \( z, z' \in \Sigma \), \( \rho(\cdot, \cdot) := \tilde{\rho}((\cdot, 1), \Sigma_0) \) for \( z \in \Sigma \) and \( \tilde{\rho}(\cdot, \cdot) := 0 \) defines a metric on \( \Sigma \cup \circ \) as we need.

The following reduction is used in both proofs.

**Lemma 1.** Let \( \Sigma \) be relatively closed subset of a domain \( D \subset \mathbb{C}^n \) such that \( \mathcal{H}^{2p}(\Sigma) = 0 \) and \( A \subset D \setminus \Sigma \) is relatively closed purely \( p \)-dimensional analytic subset such that \( \overline{A_0} \cap D \) is analytic for any irreducible component \( A_0 \) of \( A \). Then \( \overline{A} \cap D \) is analytic.

Let \( A = \cup A_j \) be the decomposition onto irreducible components and \( a \in \overline{A} \cap \Sigma \). As \( \mathcal{H}^{2p}(\Sigma) = 0 \) there is a complex plane \( L \ni a \) of complex dimension \( n - p \) such that the set \( \Sigma \cap L \) is locally finite. Without loss of generality we can assume that \( a = 0 \) and \( L \) is the coordinate plane \( z' := (z_1, ..., z_p) = 0 \). Then there is \( r > 0 \) such that \( \Sigma \cap L \cap \{|z| \leq r\} = \emptyset \) and \( L \cap \{|z| \leq r\} \subset D \). Let us show that there exists a neighborhood \( U \ni a \) intersecting only finite number of \( A_j \).

Assume not. Then there is a sequence of points \( a_k \to a \) such that \( a_k \in A_k \) and \( A_k \neq A_l \) for \( k \neq l \). As the decomposition \( A = \cup A_j \) is locally finite in \( D \setminus \Sigma \) we can assume (passing to a subsequence) that there is \( r' > 0 \) such that \( A_k \cap \{|z'| \leq r', |z| = r\} = \emptyset \) for all \( A_k \ni a_k \). Then the restrictions of the projection \( z \mapsto z' \) onto \( A_k \cap \{|z'| < r', |z| < r\} \) are proper, in particular, their images contain the ball \( \{|z'| < r'\} \subset \mathbb{C}^p \). The projection of \( \Sigma \) into \( \mathbb{C}^p \) has zero volume and thus there is a point \( b' \notin z'(\Sigma) \) with \( |b'| < r' \). By the construction there are points \( b_k \in A_k \) such that \( z'(b_k) = b' \). Passing to a subsequence we can assume that there is \( b \in D \) such that \( b_k \to b \) as \( k \to \infty \). But then \( b \in D \setminus \Sigma \) and we obtain the contradiction with local finiteness of the decomposition into irreducible components (in a neighborhood of \( b \)).

Thus there is a neighborhood \( U \ni a \in D \) and a finite number of indexes \( j_1, ..., j_N \) such that \( U \cap A_j = \emptyset \) if \( j \notin \{j_1, ..., j_N\} \). By the condition the sets \( A_{j_v} \cap D \) are analytic and thus \( \overline{A} \cap U = \cup_{v=1}^N (A_{j_v} \cap U) \) is also analytic. As \( a \in \overline{A} \cap \Sigma \) is arbitrary the set \( \overline{A} \cap D \) is analytic. \( \square \)

For the proofs of main results we need in the following lemmas in a spirit of argument principle.

**Lemma 2.** Let \( E \) be a compact subset of zero \( \mathcal{H}^m \)-measure in the closure of a domain \( D \subset \mathbb{R}^N \), \( N > m \in \mathbb{N} \), and \( K \subset E \) is compact. Then every continuous map \( f : K \to \mathbb{R}^m \setminus 0 \) which is constant on \( K \cap \partial D \) extends to a continuous map of \( E \to \mathbb{R}^m \setminus 0 \) which is constant on \( E \cap \partial D \).
\[\text{Lemma 3.} \quad \text{Let } E' \subset E \text{ be compact sets in } \mathbb{C}^n \text{ and } A \subset \mathbb{C}^n \setminus E \text{ is a bounded purely } p \text{-dimensional analytic set, } 1 \leq p \leq n, \text{ with } \partial A \subset E. \]

\[\text{Assume that} \]

1) \[\mathcal{H}^{2p}(E' \setminus E') = 0 \text{ and} \]

2) \[\mathcal{H}^{2p-1}(E') = 0 \text{ or } A \not\subset \widehat{E}'. \]

\[\text{Then } \mathcal{H}^{2p-1}((E \setminus E') \cup \partial \circ, \mathbb{Z}) \neq 0. \]

\[\text{\langle \text{See Theorem 3.8.15 in [10].} \rangle} \]

Consider first the case when \[\mathcal{H}^{2p-1}(E') = 0. \]

Then \[\mathcal{H}^{2p}(E) = 0 \text{ and there is an affine map } f : \mathbb{C}^n_z \rightarrow \mathbb{C}^n_w \text{ such that} \]

\[0 \in f(A) \text{ but } |f| > r > 0 \text{ on } f(E). \]

As \[\mathcal{H}^{2p-1}(E') = 0 \text{ then one can assume also that the ray } \text{Im } w_1 = 0, \text{ Re } w_1 < 0, w_j = 0, 1 < j \leq p, \text{ does not intersect } f(E'). \]

\[\text{Corollary 1.} \quad \text{If } H^{m-1}((E \cap D) \cup \partial \circ, \mathbb{Z}) = 0 \text{ then } H^{m-1}(X \cup \partial \circ, \mathbb{Z}) = 0 \]

for any relatively closed subset \[X \subset E \cap D. \]

\[\text{\langle \text{The notion of Hausdorff measures is well defined for an arbitrary metric space. With the metric on } (E \cap D) \cup \circ \text{ defined above we have evidently} \]

\[\mathcal{H}^m((E \cap D) \cup \circ) = \mathcal{H}^m(E) = 0, \text{ hence } \mathcal{H}^m(X \cup \circ) = 0. \text{ With this property, the condition} \]

\[H^{m-1}(X \cup \circ, \mathbb{Z}) = 0 \text{ is equivalent to that every continuous map} \]

\[f \text{ of } X \cup \circ \text{ into the unit sphere } S^{m-1} \text{ in } \mathbb{R}^m \text{ is homotopic to a constant one in the class of continuous mappings into } S^{m-1} \text{ (see Theorems VII.3 and VIII.2 in [6]). The "projection" } \]

\[\pi : \hat{X} \rightarrow X \cup \circ \text{ such that } \pi(x) = x \text{ for } x \in X \]

\[\text{and } \pi(x) = \circ \text{ for } x \in \hat{X} \setminus X \text{ is continuous. The map} \]

\[f \circ \pi : \hat{X} \rightarrow S^{m-1} \text{ is constant on } \hat{X} \setminus X. \]

By Lemma 2 there is a continuous map \[\hat{f} : E \rightarrow \mathbb{R}^m \setminus \circ \]

\[\text{equal to } f \circ \pi \text{ on } \hat{X} \text{ and to a constant } (\equiv a) \text{ on } E \cap \partial D. \]

Set \[F := \hat{f}/|\hat{f}| \text{ on } E \cap D \text{ and } F(\circ) := a/|a|. \text{ As } H^1((E \cap D) \cup \circ, \mathbb{Z}) = 0 \text{ the map} \]

\[F \text{ is homotopic to a constant one in the class of continuous mappings into } S^{m-1}. \]

The same is true for \[F|_{X \cup \circ} \text{ and thus } H^{m-1}(X \cup \circ, \mathbb{Z}) = 0. \]

\[\Box \]
Then there exists a homotopy $\varphi_t : \mathbb{C}^p \to \mathbb{C}^p$, $0 \leq t \leq 1$, such that $\varphi_0(w) \equiv w$, $\varphi_t(w) = w$ if $|w| < r$, $|\varphi_t(w)| \geq r$ if $|w| \geq r$ and $\varphi_1 \equiv 1$ on $f(E')$. Set $F := \varphi_1 \circ f$. Analytic set $A \cap \{F = 0\}$ is compact and so is finite. Hence there exists $\varepsilon \in (0, r)$ and $k \in \mathbb{N}$ such that $F : A \cap \{|F| < \varepsilon\} \to \{|w| < \varepsilon\}$ is $k$-sheeted analytic covering (see [2]).

Let $\rho$ be a smooth real function in $\mathbb{C}^p$ equal to $|F|$ when $|F| < \varepsilon$, $= 1$ on $E$ and $0 \leq \rho < 1$ on $A$. By Sard theorem for analytic sets (Proposition 14.3.1 in [2]) $A_t := A \cap \{\rho < t\}$ is analytic set with a border for almost all $t \in (0, 1)$. Let $t_j, j = 0, 1, \ldots$, be increasing sequence of such values with $t_j \to 1$ as $j \to \infty$, $t_0 < \varepsilon$ and $\Gamma_j := A \cap \{\rho = t_j\}$.

Let $\theta := d^c \log |F|^2 \wedge (d\bar{\theta})^{2p-1}$ where $d^c := i(\bar{\partial} - \partial)$. As $(d\bar{\theta})(w^2)^p = 0$ in $\mathbb{C}^p \setminus 0$ we have $d\theta = 0$ on $A \setminus \{F = 0\}$. By Stokes theorem for analytic sets (Theorem 14.3 in [2])

$$\int_{\Gamma_j} \theta = \int_{\Gamma_0} \theta = t_0^{-2p} \int_{\Gamma_0} d^c |F|^2 \wedge (d\bar{\theta})^{2p-1} =$$

$$t^{-2p} \int_{A_t} (d\bar{\theta})(w^2)^p = k t^{-2p} \int_{|w| < t_0} (d\bar{\theta})(w^2)^p > 0$$

because the function $|w|^2$ is strictly plurisubharmonic. It follows that the map $F : \Gamma_j \to \mathbb{C}^p \setminus 0$ is not homotopic to a constant. Then the same is true for the map $F/|F| : \Gamma_j \to S^{2p-1}$ to the unit sphere in $\mathbb{C}^p$. If $F/|F|$ would be homotopic to a constant on $E$ then it would be homotopic to a constant map in a neighborhood of $E$ and so on $\Gamma_j$ for $j$ large enough. But it is not the case by the proving above, hence $F/|F| : E \to S^{2p-1}$ is not homotopic to a constant. The map $F$ is constant on $E'$ and thus it induces the continuous map $h : (E \setminus E') \sqcup 0 \to S^{2p-1}$, $h(z) := (F/|F|)(z)$ for $z \in E \setminus E'$ and $h(0) := (F/|F|)(E')$ which is also not homotopic to a constant map (into $S^{2p-1}$). And it follows that $H^{2p-1}(E \setminus E', \mathbb{Z}) \neq 0$ (see the same theorems in [3]).

If $A \not\subset \widehat{E'}$ we can argue as follows. By the definition of polynomially convex hull for given $a \in A \setminus \widehat{E'}$ there is a polynomial $f_1$ such that $f_1(a) = 0$ and $Re f_1 > r > 0$ on $E'$. As $\mathcal{H}^{2p}(E \setminus E') = 0$ there is a polynomial map $f : \mathbb{C}^n_z \to \mathbb{C}^n_w$, $f = (f_1, \ldots, f_p)$, such that $0 \in f(A)$ but $Re f_1 > 0$ on $E$. The rest is the same as in the case $\mathcal{H}^{2p-1}(E') = 0$ because the ray $\{Im w_1 = 0, Re w_1 < 0, w_j = 0, j = 2, \ldots, n\}$ does not intersect $f(E')$. \hfill \Box
Lemma 4. Let $E$ be a closed subset of a Riemann surface $S$ such that $H^1(E \cup \circ, \mathbb{Z}) = 0$. Then the complement $S \setminus E$ is connected.

Let $a \neq b \in S \setminus E$. Then there is a meromorphic function $f$ on $S$ with only simple zero at $a$ and only simple pole at $b$. Let $\gamma$ be a smooth Jordan arc in $S$ with endpoints $a, b$. Then the multivalued function $\text{Log} \ f$ has a singlevalued holomorphic brunch $\log f = \log |f| + i \arg f$ in $U \setminus \gamma$ for some neighborhood $U \supset \gamma$ homeomorphic to a disk.

Let $\rho$ be a smooth function on $S$ with zero-set $\gamma$ such that $0 \leq \rho < 1$ and $\rho(z_n) \to 1$ for any sequence $\{z_n\} \subset S$ without cluster points. Then there is $r > 0$ such that $\{\rho \leq r\} \subset U$. Let $\lambda(t)$ be a smooth function on $\mathbb{R}_+$ such that $\lambda(t) = 1$ for $0 \leq t \leq r/2$ and $\lambda(t) = 0$ for $t \geq r$. Define $f_1 := \exp((\lambda \circ \rho) \log f)$ on $S \setminus \gamma$, $f_1 = f$ on $\gamma$. Then $f_1 = f$ in $\{\rho \leq r/2\}$ and $f_1 = 1$ in $\{\rho \geq r\}$. Extend $f_1$ onto $E \cup \circ$ setting $f_1(\circ) := 1$. Then $f_1$ is continuous and zero-free on $E \cup \circ$. By Bruschlinsky theorem it has continuous logarithm $\log f_1$ on $E \cup \circ$ such that $(\log f)(\circ) = 0$. By continuity $\log f = 0$ in a neighborhood $V \ni \circ$ in $E \cup \circ$. Let $V \subset \{\rho > r\}$ be an open subset of $S$ such that $V \cap E \subset V_0$. Then $\log f_1$ extended by zero on $V \setminus E$ is continuous logarithm of $f_1$ on $V \cup E$.

Fix a continuous complete distance $\text{dist}$ on $S$ and denote by $\omega$ the modulus of continuity of $f_1$ on $S \setminus V$. If $z \in S$ is such that $\omega(\text{dist}(z, E)) < (1/4) \min_{E \setminus V} |f_1|$ then we define

$$(\log f_1)(z) := (\log f_1)(z_0) + \log \left(1 + \frac{f_1(z) - f_1(z_0)}{f_1(z_0)}\right)$$

where $z_0$ is a nearest point to $z$ on $E$ and $\log(1 + \eta)$ in $\{\eta \in \mathbb{C} : \text{Re} \eta > 0\}$ is the continuous brunch of $\text{Log} \eta$ defined by the condition $\log 1 = 0$. (It follows from the definition that $\log f_1(z)$ does not depend on the choice of nearest point in $E$.) Thus we have defined a continuous logarithm of $f_1$ in a neighborhood $V_1 \supset V \cup E$.

Now assume that $E$ divides $a$ and $b$ and denote by $W$ the connected component of $S \setminus E$ containing $a$. Let $\rho_1$ be a smooth function on $W$ with zero set $\{a\}$ such that $0 \leq \rho_1 < 1$ and $\rho_1(z_n) \to 1$ for any sequence $\{z_n\} \subset W$ without cluster points in $W$. Choose $r_1 > 0$ such that $\{\rho_1 \leq r_1\} \subset \{\rho < r/2\}$, then $r_2 \in (r_1, 1)$ such that $\{\rho_1 \geq r_2\} \subset V_1$ and the levels $\{\gamma_j : \rho_1 = r_j\}$ are smooth. The form $f_1^{-1}df_1$ is closed in $W \setminus \circ$ hence

$$2\pi i = \int_{\gamma_1} f_1^{-1}df = \int_{\gamma_1} f_1^{-1}df_1 = \int_{\gamma_2} f_1^{-1}df_1 = \int_{\gamma_2} d \log f_1 = 0$$
and this contradiction proves that $S \setminus E$ indeed is connected. \hfill $\blacksquare$

**Corollary 2.** Let $A$ be irreducible one-dimensional analytic set in $\mathbb{C}^n$ and $\Sigma$ is relatively closed subset of $A$ such that $\Sigma \sqcup \circ$ is simply co-connected. Then the complement $A \setminus \Sigma$ is connected.

Let $\pi : S \to A$ be the normalization of $A$ i.e. $S$ is Riemann surface and $\pi$ is proper holomorphic map one-to-one over $\text{reg} \, A$ and such that $\# \pi^{-1}(a)$ for any $a \in A$ equals to the number of irreducible germs of $A$ at the point $a$. Then $\pi$ extends to continuous map of compactifications $S \sqcup \circ' \to A \sqcup \circ \supset \Sigma \sqcup \circ$. Set $E := \pi^{-1}(\Sigma)$ and show that $E \sqcup \circ'$ is simply co-connected. If $\Sigma \cap \text{sng} \, A = \emptyset$ then there is nothing to prove. In general $\Sigma \cap \text{sng} \, A = \{a_1, a_2, \ldots\}$ is discrete set and the only possible cluster point of this set in $\Sigma \sqcup \circ$ is $\circ$. Similarly, $E \cap \pi^{-1}(\text{sng} \, A)$ is discrete with only possible cluster point $\circ'$ in the compact set $E \sqcup \circ'$.

Let $f$ be a continuous function without zeros on $E \sqcup \circ'$, $f(\circ') = 1$, extended by continuity onto $S \sqcup \circ'$. Then $f$ does not vanish in a neighborhood $V \supset E \sqcup \circ'$ in $S \sqcup \circ'$. Let $U_j \ni a_j$ be mutually disjoint neighborhoods of $a_j$ in $\mathbb{C}^n$ such that $\pi^{-1}(U_j \cap A)$ are disjoint unions of holomorphic disks $V_{jk} \Subset V$ and the continuous variation of argument of $f$ on each disk $V_{jk}$ is less than $\pi$. Let $\lambda \in C_0^\infty(\cup U_j)$, $0 \leq \lambda \leq 1$, $\lambda(a_j) = 1$ and $(\log f)_{jk}$ are continuous branches of logarithm in $V_{jk}$ such that $|\text{Im} \, (\log f)_{jk}| < 2\pi$. Then $f_0 := f \circ \pi^{-1}$ on $(A \sqcup \circ) \setminus (\cup U_j)$, $f_0 := \exp((1 - \lambda)(\log f)_{jk} \circ \pi^{-1})$ in $\pi(V_{jk})$ is continuous function without zeros on $\Sigma \sqcup \circ$, $f_0(a_j) = 1$. As $\Sigma \sqcup \circ$ is simply co-connected there is continuous logarithm $\log f_0$ in a neighborhood of $\Sigma \sqcup \circ$ in $A \sqcup \circ$. Set $h := f/(f_0 \circ \pi)$. Then $h$ is continuous on $S \sqcup \circ'$, equals to 1 on $(S \sqcup \circ') \setminus (\cup V_{jk})$ and the variation of argument of $h$ on each $V_{jk}$ is less than $\pi$. It follows that $h$ has continuous logarithm $\log h$ on $E \sqcup \circ'$ vanishing on $(E \sqcup \circ') \setminus (\cup V_{jk})$. Thus $f$ has on $E \sqcup \circ'$ continuous logarithm equal to $(\log f_0) \circ \pi + \log h$. By Bruschlinsky theorem $E \sqcup \circ'$ is simply co-connected, by Lemma 4 $S \setminus E$ is connected hence $A \setminus \Sigma = \pi(S \setminus E)$ is connected too.\hfill $\blacksquare$

3. The proof of Theorem 1. We prove more general statement.

**Proposition 1.** Let $\Sigma$ be a bounded relatively closed subset of a domain $D \subset \mathbb{C}^n$ and $\Sigma_b := \Sigma \cap \partial D$. Assume that
1) $\mathcal{H}^2(\Sigma_b) = 0$,
2) $H^1(\Sigma \sqcup \circ, \mathbb{Z}) = 0$ and
3) $\Sigma_b \cap \Sigma = \emptyset$.
Then \( \Sigma \) is 1-removable.

Here \( \hat{X}^r \) means the rationally convex hull of a compact set \( X \subset \mathbb{C}^n \) that is the set \( \{ z : P(z) \in P(X) \text{ for any polynomial } P \} \). Equivalent definition: \( \hat{X}^r \) is the set of points \( z \in \mathbb{C}^n \) such that \( |r(z)| \leq \max_X |r| \) for every rational function \( r \) with poles outside of \( X \cup \{ z \} \). Compact sets of zero area (\( \mathcal{H}^2 \)) in \( \mathbb{C}^n \) are rationally convex (coincide with hulls) and thus Theorem 1 follows from Proposition 1.

We can assume that \( D \) is bounded. Let \( A \subset D \setminus \Sigma \) be purely 1-dimensional relatively closed analytic subset. By Corollary 1 we can assume that \( \Sigma = (\bar{A} \setminus A) \cap D \).

Let \( \rho, \ 0 \leq \rho < 1 \), be a smooth function in \( D \) equal to 0 on \( \Sigma \) and tending to 1 as \( z \to \partial D \setminus \Sigma \). By Sard theorem almost all levels \( \{ \rho = t \} \) are smooth hypersurfaces. The set of singular points of \( A \) is discrete, hence almost all levels \( A \cap \{ \rho = t \} \) are smooth one-dimensional manifolds. (They can not be empty because \( \Sigma \subset \bar{A} \) and otherwise the boundary of \( A \) would be contained in \( \Sigma \) in contradiction with Lemma 3 and the condition 2.) Fix such a \( t \in (0, 1) \) with these two properties and set \( \Omega := \{ z \in D \setminus \Sigma^r : \rho < t \}, \gamma := \partial \Omega \cap A \).

Now we construct one-dimensional relatively closed analytic subset \( A^0 \subset \Omega \) containing \( A \cap \Omega \) (the main part of the proof).

Let \( U \) be a neighborhood of \( \hat{\Sigma}^r \) such that \( \Sigma \not\subset U \) (see condition 3). Then there is a compact rational polyhedron \( V = \{ z \in \mathbb{C}^n : |p_j(z)| \leq 1, |q_j(z)| \geq 1, j = 1, ..., N \} \) with polynomials \( p_j, q_j \) such that \( \hat{\Sigma}^r \subset V \subset U \). Let \( q(z) \) be a polynomial dividing by \( q_1, ..., q_n \) and such that \( \{ q = 0 \} \cap \Sigma = \emptyset \) (it exists due to condition 1). As \( q(\Sigma \cup \gamma) \subset \mathbb{C} \) has zero area and \( \{ q = 0 \} \cap \Sigma = \emptyset \) we can assume (slightly varying \( q_j, q \) and \( V \)) that \( \{ q = 0 \} \cap (\Sigma \cup \gamma) = \emptyset \). Choose \( \varepsilon > 0 \) so small that \( \{|q| \leq \varepsilon \} \cap (\Sigma \cup \gamma) = \emptyset \) and set \( \Omega' := \Omega \cap \{|q| > \varepsilon \}, A' = A \cap \Omega' \) and \( V' = V \cap \{|q| > \varepsilon \} \). For \( \varepsilon \) small \( A' \) is obtained from \( A \cap \Omega \) by removing of finite number of closed holomorphic disks.

Let \( M \) be the hypersurface \( w \cdot q(z) = 1 \) in \( \mathbb{C}^{n+1} = \mathbb{C}^n \times \mathbb{C}_w \). Denote by \( \pi \) the projection \( (z, w) \mapsto z \) and lift the picture in \( \mathbb{C}^n \setminus \{ q = 0 \} \) onto \( M \) setting \( \hat{X} \) be the subset of \( M \) with given projection \( X \). Then \( \hat{V}' = M \cap \{|p_j(z)| < 1, |w \cdot (q_j/q)| \leq 1, j = 1, ..., N \} \) is compact polynomially convex subset of \( \mathbb{C}^{n+1} \) and \( \hat{\gamma} \) is smooth one-dimensional manifold closed in \( \mathbb{C}^{n+1} \setminus \hat{\Sigma}_b \). Show that \( A' \) is contained in polynomially convex hull of the
compact set \( Y := \Sigma_b \cup \tilde{\gamma}' \).

Assume it is not so. Then there is a point \( a \in \tilde{A}' \) and a polynomial \( P \) in \( (z, w) \) such that \( P(a) = 1, |P| < 1 \) on \( Y \) and the set \( P(\tilde{A}') \) contains a neighborhood of 1 in \( \mathbb{C} \). As \( P(\Sigma) \) has zero area there is \( a' \in \tilde{A}' \) such that \( |P(a')| > 1 \), in particular, \( a' \not\in \tilde{Y} \). The boundary of \( \tilde{A}' \) is contained in \( Y \cup \tilde{\Sigma} \). As \( \tilde{A}' \ni a' \not\in \tilde{Y} \) we obtain by Lemma 3 (with \( E' = Y, E = Y \cup \tilde{\Sigma} \)) that \( H^1(\Sigma \cup \circ, \mathbb{Z}) \neq 0 \) in contradiction with the condition 2 of Proposition 1. Thus \( \tilde{A}' \subset \tilde{Y} \).

The set \( Y \) is contained in \( \tilde{V}' \cup \tilde{\gamma}' =: X \) and \( \tilde{V}' \) is polynomially convex. By Stolzenberg theorem \([9]\) \( \tilde{X} \setminus X =: \tilde{A}' \) being non-empty (it contains \( \tilde{A}' \)) is bounded purely one-dimensional analytic set with boundary in \( \tilde{X} \). As \( \tilde{Y} \subset \tilde{X} \) this analytic set contains \( \tilde{A}' \setminus X \). Denote by \( \tilde{A}'' \) the projection of \( \tilde{A}' \) into \( \mathbb{C}^2_1 \). Then \( \tilde{A}'' \cup (A \cap \Omega \cap \{|q| \leq \varepsilon\}) \) is relatively closed analytic subset of \( \mathbb{C}^n \setminus (V \cup \gamma) \) containing \( (A \cap \Omega) \setminus V \). (It is obtained from \( \tilde{A}'' \) by adding of finite number of holomorphic disks which were removed from \( A \cap \Omega \) before the lifting onto \( M \).) Denote by \( A_V \) the union of all irreducible components of this set having relatively open parts of intersections with \( A \cap \Omega \setminus V \). By the uniqueness theorem for analytic sets (Proposition 5.6.1 in \([2]\) the set \( A_V \) does not depend on the choice of \( q \) and \( \varepsilon \) with properties pointed above.

Now we can represent \( \Sigma_k' \) as the intersection of decreasing sequence of rational polyhedrons \( V_k = \{ z \in \mathbb{C}^n : |p_{kj}(z)| \leq 1, |q_{kj}(z)| \geq 1, k = 1, ..., N_k \} \) such that \( q_k := \prod_j q_{kj} \) divides \( q_m \) if \( m > k \). By the construction above we have purely one-dimensional analytic sets in \( \Omega \setminus (V_k \cup \gamma) \) containing \( (A \cap \Omega) \setminus V_k \). Let now \( \tilde{V}_k \) be the lifting of \( V_k \) onto \( M_m : w \cdot q_m(z) = 1, m > k \), and \( \varepsilon_m > 0 \) is chosen as above. As \( q_k \) divides \( q_m \) the set \( \tilde{V}_k \cap \{|w| \leq \varepsilon_m \} \) is polynomially convex and contains \( \tilde{V}_m \), the lifting of \( V_m \). Thus the polynomially convex hull of \( \tilde{V}_k \cap \{|w| \geq 1/\varepsilon_m \} \cup \tilde{\gamma}_m' \) contains the hull of \( \tilde{V}_m \cup \tilde{\gamma}_m' \) and it follows that \( A_k := A_{\tilde{V}_k} \) contains \( A_m \setminus \tilde{V}_k \). Both these sets contain \( (A \cap \Omega) \setminus V_k \) and thus they coincide by the uniqueness theorem. It follows that the union \( \cup_k A_k =: A^0 \) is purely one-dimensional analytic set relatively closed in \( \Omega \) and containing \( A \cap \Omega \). \( \bullet \)

By Lemma 1 we can assume that \( A \cap \Omega \) is irreducible. Let \( A^1 \) be the irreducible component of \( A^0 \cap \Omega \) containing \( A \cap \Omega \). Then \( \Sigma \subset A^1 \) because \( \Sigma \subset \tilde{A} \). By Lemma 4 the set \( (reg A^1) \setminus \Sigma \) is connected. But then it coincides with its open subset \( A \cap \Omega \setminus sng A^1 \) because \( A \cap \Omega \) is closed in \( \Omega \setminus \Sigma \). It follows that \( \tilde{A} \cap \Omega = A^1 \). \( \square \)
Theorem 2 for \( p = 1 \) follows from Proposition 1 because \( \hat{\Sigma}' \subset \hat{\Sigma}_b \). \( D \) is arbitrary and we can substitute \( D \) onto (connected components of) \( D \setminus \hat{\Sigma}_b \).

Remark. If one knows that the hulls of \( V_k \cup \gamma \) are contained in \( \Omega \) then the proof of the last part does not need in Corollary 2. Indeed, if then \( A' \) is an irreducible component of \( A^0 \setminus \Sigma \) then by Lemma 3 it has non-empty open part of its boundary placed on \( \gamma' \). By boundary uniqueness theorem as in the proof above \( A' \) has non-empty open intersection with \( A \cap \Omega \). As \( A' \) and \( A \cap \Omega \) are closed in \( \Omega \setminus \Sigma \) and \( A' \) is irreducible it follows that \( A' \subset A \). The property that the hull is contained in \( \Omega \) (what is used implicitly in the proof of Theorem 3.8.18 in [10]) is fulfilled, say, for polynomially convex (Runge) domains \( D \) but it is not valid in general, \( A'' \not\subset D \) and \( A'' \cap \partial \Omega \not\subset \gamma \) for common domains \( D \) because \( \gamma \) can be not connected even if \( A \cap \Omega \) is irreducible (purely one-dimensional specificity). In the proof above this difficulty is overcome due to Lemma 4 and Corollary 2.

4. Examples. No one of essential conditions of Theorem 1 is necessary for the removability of \( \Sigma \).

1. The circle \( \gamma = \{ z_2 = 0, |z_1| = 1, y_1 \leq 0 \} \cup \{ z_2 = z_1^2 - 1, |z_1| = 1, y_1 \geq 0 \} \) in \( \mathbb{C}^2 \) is not simply co-connected but it is removable for purely one-dimensional analytic sets in \( D : |z| < 3 \). Indeed, let \( A \) be such a set, irreducible and closed in \( D \setminus \gamma \). As the singularities of zero length are removable then \( \partial A \) contains a part of \( \gamma \) of positive length and by the boundary uniqueness theorem \( A \) coincides either with \( (D \setminus \gamma) \cap \{ z_2 = 0 \} \) and then \( \bar{A} \cap D = \{ z_2 = 0 \} \cap D \) or with \( (D \setminus \gamma) \cap \{ z_2 = z_1^2 - 1 \} \) and then \( \bar{A} \cap D = \{ z_2 = z_1^2 - 1 \} \cap D \).

2. Let \( E \) be a closed totally disconnected set of finite length in the unit disk \( \mathbb{D} \) and \( \Sigma := E^n \subset \mathbb{D}^n \). Let \( A \) be an irreducible one-dimensional relatively closed analytic subset of \( \mathbb{D}^n \setminus \Sigma \) and \( a \in \Sigma \) is its cluster point. One can assume that \( A \) is not contained in a plane \( z_1 = c_1 \) for \( c_1 \in \mathbb{D} \). Then \( A \cap \{ z_1 = c_1 \} \) is a discrete set and its union with \( \Sigma \cap \{ z_1 = c_1 \} \) is closed and totally disconnected. Thus there exists a neighborhood \( V \) of the point \( (a_2, ..., a_n) \) in \( \mathbb{D}^{n-1} \) such that \( a_1 \times \partial V \) does not intersect \( (A \cup \Sigma) \cap \{ z_1 = c_1 \} \). Let \( r > 0 \) be so small that \( U := \{|z_1 - a_1| < r\} \subset \mathbb{D} \) and \( U \times \partial V \) does not intersect \( A \cup \Sigma \) also. Then the projection \( z \mapsto z_1 \) of the set \( (A \cup \Sigma) \cap (U \times V) \) is proper, hence there exists \( k \in \mathbb{N} \) such that the number of points in \( A \cap (c_1 \times V) \) counting with multiplicities is equal to \( k \) for all \( c_1 \in U \setminus E \). As \( (A \cup \Sigma) \cap (c_1 \times V) \) is totally disconnected for each \( c_1 \in U \)
then $\# \tilde{A} \cap (c_1 \times V) \leq k$, $c_1 \in U$, and at each point $c \in \tilde{A} \cap (U \times V)$ the multiplicity of $z_1$ is well defined so that the number of points in the intersection of $\tilde{A} \cap (U \times V)$ and $\{z_1 = c_1\}$ counting the multiplicity is equal to $k$. Thus the projections of the set $\tilde{A} \cap (U \times V)$ into $\mathbb{C}^2_{z_1z_j}$, $j = 2, \ldots, n$, are given by corresponding equations $z_j^k + s_{j1}(z_1)z_j^{k-1} + \cdots + s_{jk}(z_1) = 0$, where the functions $s_{ji}$ are continuous in $U$ and holomorphic in $U \setminus E$. As the length of $E \cap U$ is finite the functions $s_{ji}$ are holomorphic in $U$ (see e.g. Theorem A 1.5 in [2]) hence, the projections $\tilde{A} \cap (U \times V)$ to $\mathbb{C}^2_{z_1z_j}$ are analytic. It follows evidently the analyticity of $\tilde{A} \cap (U \times V)$ and $A \cap \mathbb{D}^n$. In this example $n > 1$ is arbitrary and $\Sigma$ is removable in spite of $\mathcal{H}^n(\Sigma) > 0$.

5. The proof of Theorem 2. We need in some properties of solutions of the Plateau problem for analytic sets (see §19.3 in [2]).

A real $C^1$-manifold $\Lambda$ of dimension $2p - 1$, $p > 1$, in a complex manifold $M$ is called maximally complex if the dimension of its complex tangent space $T_a\Lambda \cap iT_a\Lambda \subset T_aM$ has complex dimension $p - 1$ at every point $a \in \Lambda$.

A closed subset $\Gamma$ of a complex manifold $M$ is called maximally complex cycle (of dimension $2p - 1$, $p > 1$) if the measure $\mathcal{H}^{2p-1}\mid \Gamma$ is locally finite and there is a closed (maybe empty) subset $\sigma \subset \Gamma$ of zero $\mathcal{H}^{2p-1}$-measure such that $\Gamma \setminus \sigma$ is a smooth ($C^1$) oriented maximally complex manifold of dimension $2p - 1$ and the current of integration on $\Gamma$ (of smooth differential forms of degree $2p - 1$ with compact supports in $M$) is closed. Such a cycle is called irreducible if it contains no proper maximally complex cycle of the same dimension.

If $A$ is a closed purely $p$-dimensional analytic subset in $M$ and $\rho$ is a real smooth function on $M$ then for almost every $t \in \rho(A)$ the set $\Gamma_t := A \cap \{\rho = t\}$ with smooth part oriented as the boundary of $A_t := A \cap \{\rho < t\}$ (with canonical orientation corresponding to the complex structure on $M$) is a maximally complex cycle of dimension $2p - 1$ (Proposition 14.3.1 in [2]). In this case Stokes formula is valid: $\int_{A_t} d\phi = \int_{\Gamma_t} \phi$ for every smooth form of degree $2p - 1$ on $M$ (Theorem 14.3 in [2]). As the complex dimension of the set of singular points in $A$ is not more than $p - 1$ then for almost every $t \in \rho(A)$ the singular set $\sigma_t$ from the definition has locally finite $\mathcal{H}^{2p-2}$-measure and the irreducible components of $\Gamma_t$ are precisely the closures of connected components of $\Gamma_t \setminus \sigma_t$.

If $Y$ is polynomally convex compact subset in $\mathbb{C}^n$ and $\Gamma$ is a bounded maximally complex cycle in $\mathbb{C}^n \setminus Y$ of dimension $2p - 1$, $p > 1$, then by generalized Harvey – Lawson theorem (Theorem 19.6.2 in [2]) there exists a
bounded closed in $\mathbb{C}^n \setminus (Y \cup \Gamma)$ purely $p$-dimensional analytic set $A'$ such that $\Gamma \subset \overline{A'}$. By the theorem on boundary regularity (Theorem 19.1 in [2]) and boundary uniqueness theorem (Proposition 19.2.1 in [2]) the set $A'$ is irreducible if such is the cycle $\Gamma$.

Reformulate Theorem 2 for $p > 1$ taking in mind Lemma 1 (the case $p = 1$ is already considered in Proposition 1).

Proposition 2. Let $D$ be a domain in $\mathbb{C}^n$, $\Sigma \subset D$ is bounded relatively closed subset, $\Sigma_b := \overline{\Sigma} \cap \partial D$ and $A$ is relatively closed irreducible analytic set of dimension $p > 1$ in $D \setminus \Sigma$. Assume that

1) $H^{2p}(\Sigma) = 0$,
2) $H^{2p-1}(\Sigma \cup \sigma, \mathbb{Z}) = 0$ and
3) $\widehat{\Sigma}_b \cap \Sigma = \emptyset$.

Then $\overline{A} \cap D$ is analytic and $\Sigma$ is $p$-removable.

Substituting $\Sigma$ onto $\Sigma \cap \overline{A}$ one can assume (due to Corollary 1) that $\Sigma \subset \overline{A}$ and we suppose this in what follows.

Let $\rho$, $0 \leq \rho < 1$, be a smooth function in $D$ equal to 0 on $\Sigma \cup (\widehat{\Sigma}_b \cap D)$ and tending to 1 as $z \to \partial D \setminus \widehat{\Sigma}_b$. By Sard theorem for analytic sets (Proposition 14.3.1 in [2]) almost all levels $\overline{A} \cap \{\rho = t\}$ are either empty or maximally complex cycles in $D \setminus \widehat{\Sigma}_b$ of dimension $2p - 1$. If $\overline{A} \cap \{\rho = t\}$ is empty then either $\overline{A} \cap \{\rho < t\}$ is empty or it is nonempty closed in $A$. In the first case there is nothing to prove ($\Sigma \cap \overline{A} = \emptyset$) and in the second case $A$ is contained in $\{\rho < t\}$ because $A$ is irreducible. But then $\partial A \subset \Sigma$ in contradiction with the condition 2 and Lemma 3. Thus one can assume that almost all levels $\overline{A} \cap \{\rho = t\}$ are maximally complex cycles in $D$ of dimension $2p - 1$.

From Lemma 3 with $E' = \widehat{\Sigma}_b$ and $E = E' \cup \Sigma$ we obtain that $\partial A \nsubseteq \widehat{\Sigma}_b \cup \Sigma$, in particular, $A \nsubseteq \widehat{\Sigma}_b$. Hence, there is $t_1 > 0$ such that $A' := A \cap \{\rho < t\} \nsubseteq \widehat{\Sigma}_b$ and $\partial A \nsubseteq \widehat{\Sigma}_b \cup \Sigma$ for $t \geq t_1$. Fix $t_0 > t_1$ such that $A'^{t_0}$ is analytic set with maximally complex cycle-border $(\partial A'^{t_0}) \setminus (\widehat{\Sigma}_b \cup \Sigma)$. Denote by $\Gamma_0$ an irreducible component of this cycle and by $A_0$ the irreducible component of $A'^{t_0} \setminus \widehat{\Sigma}_b$ whose boundary contains $\Gamma_0$. Set $X := \widehat{\Sigma}_b \cup \Gamma_0$.

By the generalized Harvey – Lawson theorem (Theorem 19.6.2 in [2]) there exists bounded irreducible $p$-dimensional analytic set $A'$ in $\mathcal{X} \setminus X$ with boundary in $X$ such that $\Gamma_0 \subset \partial A'$. By the boundary uniqueness theorem (Proposition 19.2.1 in [2]) and Shiffman theorem there is a neighborhood $U \supset \Gamma_0$ in $D \setminus \Sigma$ and a relatively closed set $\sigma \subset \Gamma_0$ (maybe empty) of
zero $H^{2p-1}$-measure such that either $(A' \cup \Gamma_0 \cup A_0) \cap (U \setminus \sigma)$ is analytic or $A' \cup \Gamma_0 \setminus \sigma$ is a smooth manifold with boundary $\Gamma_0 \setminus \sigma$ in $U \setminus \sigma$.

In the first case $A' \cup \Gamma_0 \cup A_0$ is $p$-dimensional analytic set with boundary in $\hat{\Sigma}_b \cup \Sigma$ what is impossible by Lemma 3 and the condition 2. In the second case $A_0 \cap A'$ have $p$-dimensional intersection what follows that $A_0 \subset A'$ because $A_0$ is irreducible and $A'$ is closed in $\mathbb{C}^n \setminus X$. By the boundary uniqueness theorem $A'$ and $A_0$ coincide in a neighborhood of $\Gamma_0 \setminus \sigma$. It follows that $A' \setminus (A_0 \cup (\Sigma \cap \overline{A_0}))$ if it is non-empty is $p$-dimensional analytic set with boundary in $\hat{\Sigma}_b \cup \Sigma$ what again is impossible. Hence, $A' = A_0 \cup (\Sigma \cap \overline{A_0})$ and thus $\overline{A_0}$ is analytic in a neighborhood of $\Sigma$ in $D$. \qed

Remark. Proposition 2 can be generalized by weakening the last condition to $\hat{\Sigma}_b \cap \Sigma = \emptyset$ (lifting the picture into $\mathbb{C}^{n+1}$ as in the proof of Proposition 1). But in case $p > 1$ more natural would be not the rational convexity but some convexity with respect to polynomial mappings to $\mathbb{C}^p$. But the method used above does not work for such more weak conditions.

The proof of Proposition 2 is essentially simpler than that one for Proposition 1 because of two crucial advantages of the case $p > 1$. The first one is in that then one does not need in the proof of the existence of analytic set $A'$ with boundary in $\hat{\Sigma}_b \cup \Gamma_0$ because it follows from generalized Harvey – Lawson theorem. The second one is in that the circle $\Gamma_0$ is irreducible if (and only if) $A'$ is irreducible.

6. Corollaries. Several consequences in spirit of § 3.8 in [10]. First one is about the "infection" property of removability (see Theorem 18.2.1 in [2]).

Corollary 3. Let $\gamma$ be an open Jordan arc relatively closed in a domain $D \subset \mathbb{C}^n$ such that $H^2(\gamma) = 0$. Let $A$ be purely one-dimensional analytic set in $D$ such that $(\overline{A} \setminus A) \cap D \subset \gamma$. If $\overline{A}$ is analytic (maybe empty) in a neighborhood of a point $a \in \gamma$ then $\overline{A} \cap D$ is analytic set.

< Let $\gamma : (0, 1) \to D$ be a parametrization, $\gamma(0) = a$. For every $\varepsilon \in (0, 1/2)$ there is a domain $D_\varepsilon \subset D$ such that $\gamma \cap D_\varepsilon = \{\gamma(t) : \varepsilon < t < 1 - \varepsilon\}$ and $\# \gamma \cap D_\varepsilon = 2$. By the condition there exists an arc $\gamma' \subset \gamma \cap D_\varepsilon$ such that $\overline{A}$ is analytic in its neighborhood. The set $(\gamma \setminus \gamma') \cap D_\varepsilon$ if it is non-empty has not more than two connected components and each of them satisfies the conditions of Stout theorem (in domain $D_\varepsilon$). It follows that $\overline{A} \cap D_\varepsilon$ is analytic for all $\varepsilon$. \qed

The corresponding statement for $p > 1$ is true for connected $C^1$-manifolds
of dimension $2p - 1$ (instead of $\gamma$; see Theorem 18.2.1 in [2]). For general topological manifolds of zero $H^{2p}$-measure it is verisimilar that the statement is valued also but the proof like given above does not work for $p > 1$ because of non-checked in this case condition 3 in Proposition 2.

**Corollary 4.** Let $\Sigma$ be a closed subset of zero $H^{2p}$-measure in a domain $D \subset \mathbb{C}^n$ such that $H^{2p-1}(\Sigma, \mathbb{Z}) = 0$ and each connected component of $\Sigma$ is compact (say, $\Sigma$ is totally disconnected). Then $\Sigma$ is $p$-removable.

$\triangleleft$ Let $A \subset D$ be purely $p$-dimensional analytic set such that $(\bar{A} \setminus A) \cap D \subset \Sigma$ and $\Sigma' \subset \Sigma$ be a connected component. Then there exists a neighborhood $U \supset \Sigma'$ with compact closure in $D$ such that $\partial U \cap \Sigma = \emptyset$. Then $\Sigma \cap U$ is a compact and $H^{2p-1}(\Sigma \cap U, \mathbb{Z}) = 0$. simply co-connected set by Lemma 1. By Theorem 2 the set $\bar{A} \cap U$ is analytic. Thus $\bar{A} \cap D$ is analytic in a neighborhood of each point of $D$. $\square$

And finally a consequence of pseudoconvexity condition on $D$.

**Corollary 5.** Let $D$ be a bounded domain in $\mathbb{C}^n$ such that $\bar{D}$ is the intersection of a decreasing sequence of pseudoconvex domains and $\partial D = \partial (\text{int} \bar{D})$. Let $A \subset D$ be purely $p$-dimensional analytic set closed in $D$ and such that $H^{2p}(\partial A) = 0$. Then $H^{2p-1}(E, \mathbb{Z}) \neq 0$ for every simultaneously open and closed subset $E \subset \partial A$.

$\triangleleft$ Assume there is be an open-closed subset $E \subset \partial A$ with $H^{2p-1}(E, \mathbb{Z}) = 0$. Then there exists a neighborhood $U \supset E$ in $\mathbb{C}^n$ such that $\partial U \cap \partial A = \emptyset$, $U \cap \partial A = E$ and thus $A \cap U$ is an analytic set closed in $U \setminus E$. By Theorem 2 $\bar{A} := (A \cap U) \cup E$ is analytic set in $U$. By the construction, its boundary $\Gamma$ is placed in $D \cap \partial U$ on the positive distance $\delta > 0$ from $\partial D$. By the condition there exists a pseudoconvex domain $D' \supset \bar{D}$ such that the distance of $E$ to $\partial D'$ is less then $\delta$. By the maximum principle for $\bar{A}$ the set $E$ is contained in the hull of $\Gamma$ convex with respect to the algebra of functions holomorphic in $D'$. But the distance of this hull to $\partial D'$ can not be less than $\delta$ due to the pseudoconvexity of $D'$ (see [7] Theorem ?). The contradiction shows that $E = \emptyset$. $\square$

In particular, no open subset in $\partial A$ can be totally disconnected (see [10] Corollary 3.8.20). However, there remains open question (even if $D$ is a ball and $p = 1$) if $\partial A$ can have a connected component consisting of one point.

7. Comments and questions. The graphs of holomorphic functions are
special analytic sets and thus Theorem 1, 2 can be applied to removability of singularities of holomorphic functions (see [3, 4]). But in the case of graphs the conditions on singular sets can be essentially weakened. Quote for comparing the main result of [2]:

Let $E$ be closed subset of a Riemann surface $S$ and $f$ is a meromorphic function on $S \setminus E$ such that $\mathcal{H}^2(\mathcal{C}_f(E)) = 0$ and the cluster set $C(f, z) \subset \mathbb{C}$ at any point of $E$ is connected and has connected complement. Then $f$ extends to meromorphic function on $S$.

(Here $C(f, z)$ is the set of cluster values of $f(\zeta)$ as $\zeta \to z$, $\zeta \in S \setminus E$ and $C_f(E) := \bigcup_{z \in E} \{z\} \times C(f, z)$, the "graph" of cluster values of $f$ at the points of $E$.)

Note that the statement does not follow from Theorem 1 because there is no condition on boundary behaviour and global topology of $\Sigma := C_f(E)$. Nevertheless, the proof (based on argument principle too) is simpler due to simplicity of graphs with respect to general analytic sets.

An analogy of theorems on removable singularities for functions continuous on $S$ and holomorphic on $S \setminus E$ is the following (Proposition 19.2.1 in [2]).

Let $M$ be a connected $(2p - 1)$-dimensional $C^1$-submanifold of a complex manifold $\Omega$, $p \geq 1$, and $A_1$, $A_2$ are different irreducible $p$-dimensional analytic sets relatively closed in $\Omega \setminus M$ and such that $M = \bar{A}_1 \cap \bar{A}_2$. Then $A_1 \cup M \cup A_2$ is analytic subset of $\Omega$.

In view of Theorems 1, 2 the natural question here is if the condition $M \in C^1$ can be weakened to, say, $M$ is connected and $\mathcal{H}^{2p-1}(M) < \infty$ (or even to $\mathcal{H}^{2p}(M) = 0$?). Maybe some additional topological conditions? The question is open even if $M$ is topological manifold of finite $\mathcal{H}^{2p-1}$-measure.

As noted after Corollary 3 the third condition in Proposition 2 (which is not necessary at all) is rather restrictive for applications. The natural desire arises to substitute it by something like that in Theorem 1 (say, $\mathcal{H}^{2p}(\bar{\Sigma}) = 0$) or similar one. In any case it would be useful to substitute the condition $\bar{\Sigma}_b \cap \Sigma = \emptyset$ by one simpler for checking.
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