Finger Paint and Physics:  
A simple demonstration of Circular Motion and Conservation of Energy  
Sarah Phan-Budd  
Department of Physics, Winona State University, Winona, MN, 55987

It can be a challenge to come up with simple demonstrations of circular motion and conservation of energy. One such demonstration consists of a large exercise ball, off of which a small solid or hollow ball is rolled. The small balls are coated in finger paint so, after an initial push, they roll nearly without slipping and create a visible tracks that can be measured and compared.

Figure 1 shows the demonstration setup. A hollow plastic ping-pong ball and a small solid metal ball are used for this experiment. The larger exercise ball is set in a container to keep it stable. The floor is covered with a tarp or other materials to protect from paint drips. Students are asked to model the length of the paint track and determine which type of ball (solid or hollow) will create a longer track.

A similar problem is found in many intermediate level mechanics textbooks.\(^1\) A small ball of mass \(m\) that has been released from rest at the top of a large ball (radius \(R\)) and has rolled through a vertical distance \(H\) through an angle \(\theta\), as shown in Figure 2.

Figure 1: Demonstration Setup. The small solid or hollow ball is released from the top of the exercise ball, leaving a visible paint track which can be measured.

Figure 2: A small ball (mass \(m\)) slips a distance \(H\) down a large ball (radius \(R\)) through an angle \(\theta\).
We will first consider only the case where the ball rolls entirely without slipping. This assumption gives an over-estimate of the actual distance the ball will travel. With any finite amount of friction, the small ball will start slipping and slip some distance before falling off the larger ball. Details of the case with a finite amount of friction can be found in Appendix A.

The following conservation of energy formula applies,

\[ mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2. \]  

(1)

Here, \( v = \omega r \) and \( I = cmr^2 \), where \( r \) is the radius of the small ball and the constant \( c \) depends on the shape of the small ball. Solving for the speed,

\[ v^2 = \frac{2gR(1 - \cos(\theta))}{1 + c}. \]  

(2)

In the case that the ball rolls entirely without slipping, the radial equation of motion can also be applied, giving

\[ mg \cos(\theta) - N = \frac{mv^2}{R}. \]  

(3)

Here \( N \) is the normal force. The puck leaves the sphere when the normal force is zero, or

\[ v^2 = gR \cos(\theta). \]  

(4)

Setting the speed obtained from the energy equation equal to the speed obtained from the equation of motion

\[ \cos(\theta) = \frac{2}{3 + c}. \]  

(5)

In the case of a small solid ball, \( c=2/5 \) so \( \cos(\theta) = \frac{10}{17} \). In the case of a small hollow ball, \( c=2/3 \) so \( \cos(\theta) = \frac{6}{11} \). This implies that the solid ball leaves the sphere at about 0.94 radians and the hollow ball leaves the sphere at about 0.99 radians. Both angles are larger than the case of pure slipping, where frictionless object leaves the large ball at about 0.84 radians (or when \( \cos(\theta) = \frac{6}{11} \)).

It is easiest to measure the circumference, \( C \), of the exercise ball and the arclength, \( s \), that each small ball traverses before leaving the exercise ball.

\( C = 2\pi R \) and \( s = R\theta \) so distance the small ball travels down the side of the exercise ball is given by

\[ s = \frac{C\theta}{2\pi}. \]  

(6)

Since the angle at which the solid ball leaves the sphere is smaller than the angle at which the solid ball leaves the sphere, the solid ball leaves a shorter paint track than the hollow ball. Additionally, the lengths of the tracks can be measured and compared with our no-slip model. It is worthwhile to have a classroom discussion of sources of deviation from our simple model, such as those presented by J. Flores, A.G. del Rio, A. Calles and H. Riveros.

A final word to the wise. As your local pre-school teacher will tell you, putting just a little bit of soap in the finger paint helps it wash off more easily, although in this case too much soap will lower the coefficient of friction and cause the balls to start slipping more quickly.
Appendix A: Solution with a finite amount of friction

To find the angle where the small ball will fall off the larger ball with a finite amount of friction, we first find the angle where the small ball begins to slip. We combine the equation for speed from the conservation of energy equation
\[ v^2 = \frac{2gR(1 - \cos(\theta_s))}{1 + c} \] (7)
with the radial equation of motion
\[ m r \alpha = mg \sin(\theta) - f \] (8)
and the torque equation
\[ R f = cmR^2 \alpha. \] (9)
Finally, as long as the ball remains in circular motion
\[ mg \cos(\theta) - N = \frac{mv^2}{R} \] (10)
The ball will start to slip when \( f = \mu_s N \). Combining equations 7-10, iv we can solve for the angle that the ball starts slipping \( \theta_s \), in terms of only the constant \( c \) that is determined by the shape of the small ball. We find
\[ \mu_s = \frac{c \sin(\theta)}{(3 + c) \cos(\theta) - 2}. \] (11)
Solving equation 11 we can find the angle where the ball begins to slip in terms of the coefficient of static friction and \( c \),
\[ \cos(\theta_s) = \frac{2(3 + c)\mu_s^2 + c\sqrt{\mu_s^2(3 + c)^2 - 4\mu_s^2 + c^2}}{\mu_s^2(3 + c)^2 + c^2}. \] (12)
Figure 3 plots \( \theta_s \) as a function of the coefficient of static friction, for a solid and hollow ball.

Figure 3: The angle where the small ball starts to slip, \( \theta_s \) as a function of the coefficient of static friction (\( \mu_s \)), for a solid and hollow ball. The hollow ball starts to slip before the solid ball.

We can also find the speed that the ball begins to slip (\( v_s \)) from \( \theta_s \),
\[ v_s^2 = \frac{2gR(1 - \cos(\theta))}{1 + c}. \]  

(13)

After starting to slip, the small ball will slide some distance before falling off the large ball. The angle of release can be determined from the radial equation of motion

\[ mg \cos(\theta) - N = \frac{mv^2}{R}, \]  

(14)

combined with the tangential equation of motion

\[ mgs \sin(\theta) - f = m \frac{\partial v}{\partial t}. \]  

(15)

Here, \( f = \mu_k N \) once the ball has started slipping. Combining equations 14 and 15 leads to the differential equation for the speed which the ball falls off the big ball, as a function of the angle of release.

\[ mg \sin(\theta) - \mu_k m \left( g \cos(\theta) - \frac{v^2}{R} \right) = \frac{v}{r} \frac{\partial v}{\partial \theta}. \]

Given our initial condition that the speed of the small ball is \( v_s \) at the angle where the small ball starts to slip, \( \theta_s \), we can find the speed of the small ball as a function of angle

\[ v = \sqrt{v_s^2 e^{2\mu_k(\theta - \theta_s)} - \frac{2rg}{1 + 4\mu_k^2} \left[ (1 - 2\mu_k^2) \cos(\theta) + 3\mu_k \sin(\theta) + e^{2\mu_k(\theta - \theta_s)} \left( (2\mu_k^2 - 1) \cos(\theta_s) - 3\mu_k \sin(\theta_s) \right) \right]}. \]  

(16)

The small ball falls off the big ball when the normal force is zero, or

\[ v^2 = gR \cos(\theta) \]  

(17)
as before in the simple case where the ball rolls without slipping.

We can understand the motion of the small ball on the surface of the larger ball by plotting the trajectory in \( v \) vs \( \theta \) space. We plot the simplified case where \( \mu_k \approx \mu_s = \mu \). These plots include both the rolling without slipping and slipping portion of the ball’s motion.

Figures 4 and 5 shows the trajectory for a solid ball and hollow plotted for several values of \( \mu \). As expected, the release angle increases as the coefficient of friction gets larger. In both cases, the trajectory approaches case in which the ball does not slip before falling off the larger ball, labeled no slip, for large values of \( \mu \).
Figure 4: The trajectory of a small solid ball rolling off a large ball, as a function of the coefficient of friction ($\mu$). The small ball falls off the larger ball when the trajectory crosses the line labeled release condition. The trajectories for various values of $\mu$ lie in between the case with no rolling (where the ball falls off at about 0.84 radians) and the case with no slipping (about 0.94 radians).

Figure 4: The trajectory of a small solid ball rolling off a large ball, as a function of the coefficient of friction ($\mu$). The small ball falls off the larger ball when the trajectory crosses the line labeled release condition. The trajectories for various values of $\mu$ lie in between the case with no rolling (where the ball falls off at about 0.84 radians) and the case with no slipping (about 0.99 radians).

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