Scaling of Particle Trajectories on a Lattice II:
The Critical Region

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Abstract

The scaling behavior of the closed trajectories of a moving particle generated by randomly placed rotators or mirrors on a square or triangular lattice in the critical region are investigated. We study numerically two scaling functions: $f(x)$ related to the trajectory length distribution $n_S$ and $h(x)$ related to the trajectory size $R_S$ (gyration radius) as introduced by Stauffer for the percolation problem, where $S$ is the length of a closed trajectory. The scaling function $f(x)$ is in most cases found to be symmetric double Gaussians with the same characteristic size exponent $\sigma = 0.43 \approx 3/7$ as was found at criticality. In contrast to previous assumptions of an exponential dependence of $n_S$ on $S$, the Gaussian functions lead to a stretched exponential dependence of $n_S$ on $S$, $n_S \sim e^{-S^{6/7}}$. However, for the rotator model on the partially occupied square lattice, an alternative scaling function near criticality is found, leading to a new exponent $\sigma' = 1.6 \pm 0.3$ and a super exponential dependence of $n_S$ on $S$. The appearance of the same exponent $\sigma = 3/7$ describing the behavior at and near the critical point is discussed. Our numerical simulations show that $h(x)$ is essentially a constant, which depends on the type of lattice and on the concentration of the scatterers.

Key words: Lattice, particle trajectories, percolation, scaling function, critical exponents.
1 Introduction

In a previous paper\(^{(1)}\) we studied numerically the scaling behavior of extended (possibly infinite) particle trajectories generated by randomly distributed (rotator or mirror) scatterers on a square or a triangular lattice. We restricted ourselves there to the scaling behavior of structural properties of particle trajectories — such as of the asymptotic number of right rotators on a particle trajectory or of the asymptotic number of sites visited by the moving particle on its trajectory with different frequencies — that occurred strictly at criticality. In this paper we will discuss the scaling behavior found in the region near criticality, i.e. we study the critical behavior when one approaches the critical point from the outside. This involves, in addition to the scaling exponents \(\tau, d_f\) and \(\sigma\), which already appeared in previous papers\(^{(1-4)}\), the determination of a scaling function defined by:

\[
n_S = S^{-\tau+1} f[(C_R - C_{R_c})S^\sigma] \tag{1}
\]

where \(n_S\) is the probability to find a closed trajectory of length \(S\), \(C_R\) and \(C_{R_c}\) are the concentration (i.e. the fraction) of right scatterers and the critical concentration of right scatterers on the lattice, respectively. For both the square and the triangular lattices fully occupied by rotators, one can show, by mapping the rotator model onto a percolation problem\(^{(5,6)}\) that has been solved exactly before\(^{(8)}\), that \(C_{R_c} = 1/2, \tau = 15/7, d_f = 7/4\) and \(\sigma = 3/7\). The scaling function \(f(x)\) yields more detailed information about the trajectory size distribution than the exponent \(\sigma\), where \(x = (C_R - C_{R_c})S^\sigma\). Since there are no infinitely extended trajectories away from criticality, \(f(x)\) must vanish when \(x \rightarrow \infty\). The exponent \(-\tau + 1\) instead of the usual \(\tau\) occurs because the trajectories are constructed here from, what corresponds in percolation theory to, a seed and the number of choices to put the seed on a trajectory is proportional to the size of the trajectory itself\(^{(1,2,9,10)}\). When \(x = 0\) and
\( f(x) \) becomes a constant, we have, from Eq. (1), \( n_S \sim S^{-\tau+1} \), recovering the results obtained previously at criticality \(^{(1,2)}\).

One important feature about the scaling function \( f(x) \) is that it is essentially invariant once we get into the critical region, i.e. the scaling function we obtain at one concentration near criticality is essentially the same as that obtained at another concentration. Note that since \( x = (C_R - C_{RC})S^\sigma \), to make \( f(x) \) invariant, i.e. concentration independent, we need to give \( C_{RC} \) and \( \sigma \) each a unique value. This provides an alternative method to compute both the exponent \( \sigma \) and the critical concentration \( C_{RC} \), just from numerical data obtained at a few different concentrations in the critical region.

In addition to \( f(x) \), there is another scaling function \( h(x) \) introduced by Stauffer \(^{(11)}\) some time ago,

\[
R_S = S^{1/d_f} h((C_R - C_{RC})S^\sigma) \tag{2}
\]

Here \( R_S \) is the gyration radius of a closed trajectory of length \( S \), in other words, \( R_S^2 \) is defined as the mean square displacement from the origin of all trajectories of length \( S \). The scaling function \( h(x) \) characterizes corrections to the fractal dimension as one moves away from criticality. From our numerical simulations, we found, however, that \( h(x) \) is essentially a constant for all \( x \). This implies that \( R_S \sim S^{2/d_f} \) holds quite accurately in practice even for small trajectories, i.e. away from criticality.

Previous numerical determinations of \( \sigma \) were based on the calculation of the first moment of the trajectory length distribution \(^{(2-4,12)}\), \( <S> \), which is divergent at criticality,

\[
<S> = \sum_{S}^{\infty} S n_S = \sum_{S}^{\infty} S^{-\tau + 2} f((C_R - C_{RC})S^\sigma) \sim (C_R - C_{RC})^{-(3-\tau)/\sigma} \tag{3}
\]
Since $<S> \sim (C_R - C_{Rc})^{-\gamma}$ by definition, one has $\gamma = (3 - \tau)/\sigma = 2$. In this paper we compute the average of the mean square displacement of all trajectories, $<R^2>$, which also diverges at criticality,

$$
<R^2> \quad = \quad \sum_{S}^{\infty} R^2_{S} n_{S} \\
= \quad \sum_{S}^{\infty} S^{-(\tau + 1 + 2/d_f)} f((C_R - C_{Rc})S^\sigma) h^2((C_R - C_{Rc})S^\sigma) \\
\sim \quad (C_R - C_{Rc})^{-(2 + 2/d_f - \tau)/\sigma}
$$

(4)

If we define an exponent $\rho$ by $<R^2> \sim (C_R - C_{Rc})^{-\rho}$ one has $\rho = (2 + 2/d_f - \tau)/\sigma = 7/3 = 2.333$. The hyperscaling relation in Eq. (4) is independent of the specific form of $f(x)$ and $h(x)$. Note that the upper limit of the summations in Eq. (3) and Eq. (4) is infinity, which poses a difficulty in getting very close to the critical point, since the dominant closed trajectories are then too large to generate on the computer. On the other hand, we can still compute the scaling function $f(x)$ by making a large cutoff ($2^{24}$) in $S$, so that trajectories whose length is bigger than $2^{24}$ are not considered. Although then the tail of $f(x)$ will get truncated by the cutoff, the remaining part still yields sufficient information to determine $\sigma$.

The numerical algorithm used to generate the particle trajectories here has been explained in the previous paper (1). This algorithm is very powerful due to its speed and its ability to generate large particle trajectories by using a virtual lattice scheme and a dynamic memory allocation technique. The calculation of each scaling function $f(x)$ or $h(x)$ and the calculation of $<S>$ or $<R^2>$ at each concentration involve 500,000 and 30,000 particle trajectories, respectively.

The plan of this paper is the following. In section 2 the critical region for trajectories generated by rotators on a square lattice is discussed. For the critical behavior near $C_{Rc} = C_{Lc} = 1/2$, two qualitatively different scaling functions, $f(x)$ a double Gaussian and $f'(x)$ an exponential function, corre-
sponding to two different values of $\sigma$: $\sigma = 0.43 \approx 3/7$ and $\sigma' = 1.6$, respectively, near this critical point have been determined, depending on how one approaches this point in the phase diagram. While $\sigma = 0.43$ leads to a stretched exponential behavior of $n_S$ along the line $C_R + C_L = 1$, $\sigma' = 1.6$ leads to, what one could call, a super-exponential behavior along the line $C_R = C_L$. $h(x)$ is obtained by computing $f(x)h(x)$, which appears to be proportional to $f(x)$, so that $h(x)$ is essentially a constant. In section 3 we discuss the mirror model on the square lattice and a so-called quasi-rotator model deduced from the mirror model. In section 4, the critical region on the triangular lattice is discussed, which obtains both for rotators and mirrors. We found that the exponent $\sigma$ along the critical line is the same as that for $C = 1$, i.e. $\sigma = 0.43$, and that the scaling functions $f(x)$ and $h(x)$ are qualitatively the same and differ only by the values of the constants occurring in them. A discussion of the results obtained in this and the previous paper is given in section 5.

2 Critical region for the rotator model on the square lattice

2.1 The fully occupied lattice

As we have shown in the previous paper for $C = 1$, the trajectories generated by a moving particle through randomly placed rotators can be mapped onto the perimeters of bond percolation clusters. Therefore the theory that has been developed for this percolation problem can be applied directly here. The critical concentration is $C_{Rc} = C_{Lc} = 1/2$ and the exact values for $\sigma$, $d_f$ and $\tau$ are $3/7$, $7/4$ and $15/7$, respectively. From these known exponents and the critical concentration, we can compute the scaling function directly from Eq. (1),

$$f((C_R - \frac{1}{2})S^\sigma) = \frac{n_S}{S^{-\tau+1}}$$
Note that since \( n_S \) is the probability to find a trajectory of length \( S \), the right hand side of Eq. (5) is an average of \( S^{8/7} \), taken over trajectories of length \( S \) and can be easily determined in our numerical simulations. It is not difficult to see that \( f(x) \) must be symmetric with respect to \( x = 0 \), since the probability to generate the same trajectory is invariant under the transformation of interchanging \( C_R \) and \( C_L \), i.e.

\[
f((C_R - \frac{1}{2})S^\sigma) = f((C_L - \frac{1}{2})S^\sigma) = f(-(C_R - \frac{1}{2})S^\sigma)
\]  

(6)

Our numerical calculations of the scaling function \( f(x) \) were carried out at \( C_R = 0.47, 0.48 \) and \( 0.49, \) respectively. We found that the scaling functions obtained at these three concentrations collapse to a single curve, which could be fitted to a double Gaussian, i.e a sum of two overlapping Gaussians, (Fig. 1),

\[
f(x) = 1.03e^{-2.25(x+0.86)^2} + 1.03e^{-2.25(x-0.86)^2}
\]  

(7)

Note that we determined 60 values of \( f(x) \) for each \( C_R \), so that the curve in Fig. 1 contains 180 points. When \( x \gg 1 \), \( f(x) \) can be approximated by

\[
f(x) \sim e^{-2.25x^2}
\]  

(8)

Therefore, since \( n_S \) is proportional to \( f(x) \), \( n_S \) exhibits a stretched exponential decay for large \( S \): \( n_S \sim e^{-S^{6/7}} \), in contrast to the exponential decay reported in the literature\(^{3,4} \) based on the solution of the percolation problem on the Bethe lattice.

In order to calculate \( h(x) \), we first computed the product of \( h(x)f(x) \). From Eq. (1) and Eq. (2), we have

\[
h(C_R - C_{R_c})f(C_R - C_{R_c}) = n_S \frac{R_S}{S^{r-1+d_f}}
\]  

(9)
The right hand side of Eq. (9) is just the average of $R_S/S^{\tau-1+d_f}$ for fixed $S$, which can be easily calculated numerically. Our numerical results show then that $h(x)f(x)$ is proportional to $f(x)$, Eq. (7), (Fig. 2),

$$f(x)h(x) = 0.39e^{-2.25(x+0.86)^2} + 0.39e^{-2.25(x-0.86)^2}$$

so that $h(x)$ is essentially a constant, 0.37.

We have also obtained the first moment of the trajectory length distribution as a function of $C_R-C_{Rc}$. Our numerical calculations show the following power law behavior,

$$< S > = (C_R-C_{Rc})^{-\gamma}$$

where $\gamma = 2.00 \pm 0.01$, (Fig. 3), in good agreement with the exact result $\gamma = (3-\tau)/\sigma = 2$. Our numerical results for the mean square displacement of the trajectories also show a power law behavior,

$$< R^2 > \sim (C_R-C_{Rc})^{-\rho}$$

where $\rho = 2.33 \pm 0.01$, (Fig. 4), in good agreement with the exact result $\rho = (2+2/d_f-\tau)/\sigma = 1/\sigma = 7/3 = 2.333$.

As a check, we also used a different method, the so called histogram method, to calculate the scaling function $f(x)$ for different concentrations from data computed at a given concentration $C_R$. A similar method has been used by Leath\(^{(9,10)}\) to study the scaling behavior of percolation clusters and by Ferrenberg and Swendsen\(^{(13)}\) to study the critical behavior of the Ising model. This calculation is based on the following observation. Each trajectory that is generated at one concentration of right rotators ($C_R$) and left rotators ($C_L$), can also be generated at other concentrations. However, the probability with which the trajectory is generated depends on the concentrations,

$$P(N_R,N_L) = A C_R^{N_R} C_L^{N_L}$$
where $A$ is the normalization factor for $P(N_R, N_L)$ and $N_R$ and $N_L$ are the number of right rotators and left rotators contained in the trajectory, respectively. $A$ can also be defined as the total number of all different closed trajectories generated on the lattice, so that $A$ does not depend on the concentration. The probability to generate the same trajectory at another concentration $C'_R$ and $C'_L$ is

$$P'(N_R, N_L) = A' C'_R^{N_R} C'_L^{N_L}$$

$$= P(N_R, N_L) \frac{A'}{A} \left( \frac{C'_R}{C_R} \right)^{N_R} \left( \frac{C'_L}{C_L} \right)^{N_L}$$

(14)

where $A'$ is the normalization factor for $P'(N_R, N_L)$. Note that if $C'_R/C_R > 1$, then $C'_L/C_L < 1$, or vice versa. To evaluate $(C'_R/C_L)^{N_R}$ and $(C'_L/C_L)^{N_L}$ separately for large $N_R$ and $N_L$ is not feasible on the computer, since one term is too large and the other term is too small. In our simulations, their product is computed from an alternative expression, $e^{N_R \ln(C'_R/C_R) + N_L \ln(C'_L/C_L)}$, which works quite well. Here, we emphasize that, although in theory $A$ and $A'$ should be the same, for numerical simulations, $A$ and $A'$ are not equal due to systematic errors in the Monte Carlo samplings, so that they have to be determined from Eq. (13) and Eq. (14), respectively. $n_S$ at concentration $C'_R$ can be obtained from $P'(N_R, N_L)$ as,

$$n_S = \sum_{N_R} \sum_{N_L} P'(N_R, N_L) \delta(S - S'(N_R, N_L))$$

(15)

where $S'(N_R, N_L)$ is the length of a trajectory containing $N_R$ left rotators and $N_L$ left rotators. Once $n_S$ is obtained, $f(x)$ can be calculated for any other concentration according Eq. (13).

The standard data that we used were computed at the concentration $C_R = 0.51$. From these we generated the scaling functions $f(x)$ for $C'_R = 0.52, 0.53, 0.54, 0.55$, all of which fall on the same curve (Fig. 5). Thus this method is consistent with the first one. These results also show that the critical region in concentration is at least as large as 0.05.
2.2 The partially occupied lattice

For $C < 1$, the particle trajectories cannot be mapped onto a percolation problem anymore, because the trajectories can cross themselves. It was found by Cohen and Wang\textsuperscript{(5)} that there are two nonlinear critical lines symmetric with respect to the line $C_R = C_L$, merging at $C_R = C_L = 1/2$, (Fig. 6). Moreover, we found that they appear to be tangent to the line $C = 1$. Previous numerical results have shown that the trajectory fractal dimension $d_f$ and the exponent $\tau$ along the critical lines are the same as that in the case $C_R = C_L = 1/2$. Here we show numerically, however, that in addition to the exponent $\sigma$ which we have investigated in the previous section, a different exponent $\sigma'$ also appears when $C < 1$, suggesting a new scaling behavior.

For $C < 1$, the simplest way to study the critical behavior near criticality is to approach the critical point $C_{Rc} = C_{Lc} = 1/2$ along the line $C_R = C_L$, which is perpendicular to the critical line at $C_R = C_L = 1/2$. We assume that the cluster size distribution has a form similar to Eq. (1),

$$n_S = S^{-\tau+1} f'( (1 - C) S^{\sigma'} )$$ \hspace{1cm} (16)

where $f'(x)$ is a new scaling function in contrast to $f(x)$, which we have obtained before and $x$ is defined as $(1 - C) S^{\sigma'}$. Since $x$ cannot be negative, $f'(x)$ must be asymmetric with respect to $x = 0$, in contrast to $f(x)$ of Eq. (4). The exponent $\tau$, however, must be the same as before, since for $C = 1$, the previously established relation $n_S \sim S^{-\tau+1}$ must be recovered.

The computation of the new scaling function $f'(x)$ is much more time consuming than that for the rotator model on the fully occupied lattice, although the procedure is similar. The reason is the following. When $C$ is small, the moving particle becomes less frequently scattered, therefore, on average, it takes longer for the particle trajectory to get closed. On the other hand, if $C$ is close to one, i.e. near criticality, the particle trajectories also close very
slowly. On the line $C_R = C_L$, we found that the trajectories close fastest at $C_R = C_L = 0.45$, when essentially, all trajectories are closed after $2^{21}$ time steps (Fig. 7). However, this concentration is not useful to calculate the scaling function, because it is not in the critical region.

As we explained in the introduction, if the critical behavior appears only for very large trajectories, direct calculation of $\gamma'$ and $\rho'$ — the exponents corresponding to $\gamma$ and $\rho$, respectively — from equations Eq. 3 and Eq. 4 becomes very difficult, because the large trajectories are too large to generate on the computer. On the other hand, we can still compute the scaling function $f'(x)$ by making a cutoff, i.e. trajectories whose length is bigger than a fixed number are disregarded. In our simulations, we took this fixed number or cutoff as sixteen million time steps. By choosing $\sigma' = 1.6$, we find that the scaling functions obtained for different concentrations, $C = 0.99, 0.985, 0.98$, all collapse to a single curve, (Fig. 8),

$$f'(x) = 0.475e^{-(1.65\times10^{-8}x)}$$  \hspace{1cm} (17)

where for each $C$, 25 points have been computed, so that the curve in Fig. 8 contains 75 points. Note that since $x \sim S^{1.6}$, $f'(x)$ decays as a super-exponential function of $S$, $n_S \sim e^{-S^{1.6}}$. In fact, eq.(17) obtains for values of $\sigma' = 1.6 \pm 0.3$.

The other scaling function $h'(x)$ is obtained from the calculation of $f'(x)h'(x)$, which we found again to be proportional to $f'(x)$, (Fig. 9),

$$f'(x)h'(x) = 0.115e^{-(1.65\times10^{-8}x)}$$  \hspace{1cm} (18)

so that $h'(x) = 0.115/0.475 = 0.24$ independent of $x$. The exponents $\gamma'$ and $\rho'$ can be obtained from $\gamma' = (3 - \tau)/\sigma' = 0.54$ and $\rho' = (2 + 2/d_f - \tau)/\sigma' = 0.63$, respectively, both much smaller than those for $C = 1$. 


3 Critical region for the mirror model on the square lattice

It has been shown before, by mapping the mirror model to a percolation problem, that $C = 1$ is a critical line\(^{(5,6)}\), i.e. the size distribution and the fractal dimension of the trajectories have the same power law behavior as for the corresponding percolation problem at criticality, $n_S \sim S^{-\tau+1} \sim S^{-8/7}$ or $\tau = 15/7$ and $d_f = 7/4$, respectively. However, the trajectory size average $< S >$ and the mean square displacement $< R^2 >$ are both divergent along the critical line. Therefore, the exponent $\sigma$ does not appear here. Although the mirror model can be mapped onto a quasi-rotator model which exhibits the same scaling behavior as that for the ordinary rotator model at criticality\(^{(1)}\), the quasi-rotator model cannot be used to calculate $\sigma$, $\gamma$ and $\rho$, because at present we do not know how to generate the particle trajectories directly rather than deducing them from the mirror model.

For $C < 1$, the mirror model cannot be mapped onto a percolation problem anymore. It was found numerically that the critical exponents are drastically changed from those at $C = 1$; $n_S \sim (\ln S)^{-1}$ and $d_f = 2$ with logarithmic corrections\(^{(5,6)}\). This new critical behavior exists in the whole $(C_R, C_L)$ plane, where super-diffusion occurs\(^{(5)}\), except at the three boundary lines $C = 1$, $C_R = 0$ and $C_L = 0$, where the moving particle simply zig-zags to infinity. Since $< S >$ and $< R^2 >$ are both divergent everywhere, $\sigma$ cannot be defined either.

4 Critical region for the rotator and the mirror model on the triangular lattice
4.1 The fully occupied lattice

Since the rotator model and mirror model are equivalent on the triangular lattice\(^{(1,5,7)}\), we only study the rotator model here. Similar to the rotator model on the square lattice, the rotator model on the triangular lattice can be mapped onto a percolation problem\(^{(1,5,7)}\), a site percolation problem here, for which \(\tau = 15/7\), \(d_f = 7/4\) and \(\sigma = 3/7\), while the critical concentration is \(C_{Rc} = C_{Lc} = 1/2\).

Our numerical calculations of the scaling function \(f(x)\) were carried out at \(C_R = 0.47, 0.48\) and 0.49, respectively. The scaling functions for these three concentrations collapsed to a single curve, which could be fitted again to a double Gaussian, (Fig. 10),

\[
f(x) = 0.94e^{-1.80(x+0.96)^2} + 0.94e^{-1.80(x-0.96)^2}
\]  

where for each \(C_R\), more than 60 points were computed and the curve comprises therefore more than 180 points. As before, \(f(x)\) must be symmetric with respect to \(x = 0\), the reason being that the probability to generate the same trajectory is invariant under the transformation of interchanging \(C_R\) and \(C_L\).

Eq.(7) and Eq.(19) are very similar except that the constants are different. When \(x \gg 1\), \(f(x) \sim e^{-x^2}\), so \(n_S \sim e^{-S^6/7}\) which is the same for the square lattice, i.e. the number of trajectories decays with a stretched exponential law in \(S\).

The calculation of the scaling function \(h(x)\) is similar to that for the square lattice. We found numerically, \(f(x)h(x)\) is proportional to \(f(x)\), (Fig. 11),

\[
f(x)h(x) = 0.28e^{-1.80(x+0.96)^2} + 0.28e^{-1.80(x-0.96)^2}
\]  

so that \(h(x)\) is a constant, \(0.28/0.94 = 0.30\).

The average trajectory size diverges as one approaches criticality with an exponent \(\gamma\), \(< S >\sim (C_R - C_{Rc})^{-\gamma}\). Our numerical simulations show that
\( \gamma = 2.0 \pm 0.01 \), (Fig. 12), in good agreement with the exact result, \( \gamma = 2 \). The mean square displacement of the particle trajectories diverges with an exponent \( \rho \), \( \langle R^2 \rangle \sim (C_R - C_{Rc})^{-\rho} \), the value of \( \rho \) obtained from our numerical simulations is \( 2.33 \pm 0.01 \) (Fig. 13), also in good agreement with the exact result \( \gamma = 7/3 \).

Using the histogram method, we computed the scaling functions also from the standard data obtained at the concentration \( C_R = 0.51 \), for the other concentrations \( C_R = 0.52, 0.53, 0.54, 0.55 \). All the scaling functions fall on the same curve, Eq. (19), (Fig. 14), showing that this method is consistent with the first one.

### 4.2 The partially occupied lattice

For \( C < 1 \), the rotator model and the mirror model can be still mapped onto each other. However neither model can be mapped onto a percolation problem now. Here we only consider the rotator model. It was found before that there is only one critical line \( C_R = C_L \) rather than two critical lines as in the case of the rotator model on the partially occupied square lattice\(^{(1,5)}\). The critical exponents \( \tau \) and \( d_f \) along the critical line are found numerically to be the same as those at the critical point, \( C_R = C_L = 1/2 \), viz. 15/7 and 7/4, respectively\(^{(1,5,7)}\).

To study the critical behavior near criticality, we first need to choose the direction in which we approach criticality. The most obvious choice is the direction perpendicular to the critical line. Although the rotator model cannot be mapped onto a percolation problem, we found that, if we choose \( \sigma = 3/7 \), the scaling functions computed at different \( C_R \) collapse very well onto a single curve. This suggests that randomly distributed empty sites on the lattice are irrelevant for the critical behavior. The scaling functions were computed in the critical region near \( C_{Rc} = C_{Lc} = 0.425 \) at three different \( C_R, C_R = C_{Rc} + 0.01, \)
\[ C_R = C_{R_c} + 0.015, \quad C_R = C_{R_c} + 0.02 \text{ along the line } C = 0.85. \]  
We found that these scaling functions could be fitted to a double Gaussian, (Fig. 15),

\[ f(x) = 1.05e^{-1.18(x+1.19)^2} + 1.05e^{-1.18(x-1.19)^2} \]  \hspace{1cm} (21)

where we computed 80 points for each \( C_R \), leading to 240 points on the curve. From Eq. (19) and Eq. (21), one can see that all three constants contained in \( f(x) \) are concentration dependent.

The product of \( f(x)h(x) \) is again found to be proportional to \( f(x) \), (Fig. 16),

\[ f(x)h(x) = 0.30e^{-1.18(x+1.19)^2} + 0.30e^{-1.18(x-1.19)^2} \]  \hspace{1cm} (22)

so that \( h(x) = 0.30/1.05 = 0.28. \)

To investigate whether the critical exponent \( \sigma \) has the same value along the critical line, we also computed \( <S> \) and \( <R^2> \) in the critical region around \( C_{R_c} = C_{L_c} = 0.45 \) and \( C_{R_c} = C_{L_c} = 0.4 \), respectively. In both cases, we found, from our numerical simulations, that the exponents \( \gamma = 2.00 \pm 0.02 \) and \( \rho = 2.33 \pm 0.02 \) are very close to their values at \( C_{R_c} = C_{L_c} = 0.5: \) 2 and \( 7/3 \), respectively, (Fig. 17-18), suggesting that the critical behavior along the critical line \( C_R = C_L \) belongs indeed to the same universality class.

## 5 Conclusion

We end with a few remarks.

1. We have studied the critical behavior of particle trajectories on both the square lattice and the triangular lattice in the critical region, for both \( C = 1 \) and \( C < 1 \). Our methods were based on calculations of two scaling functions \( f(x) \) (which yields \( \sigma \)) and \( h(x) \), as well as the two exponents \( \gamma \) and \( \rho \) from \( <S> \) and \( <R^2> \) respectively. From these two scaling functions, one derived and verified numerically the two scaling relations, \( \gamma = (3 - \tau)/\sigma \) and \( \rho = (2 + 2/d_f - \tau)/\sigma \). Our results are summarized in table I.
2. For $C = 1$, the rotator model on the square lattice, and the rotator and the mirror model on the triangular lattice can all be mapped onto a percolation problem. Our numerical results show that near $C_{Rc} = C_{LC} = 1/2$, $f(x)$ can be fitted to a sum of two overlapping Gaussians, i.e. a double Gaussian. Our numerical calculations of $<S>$ and $<R^2>$ show that $\gamma = 2.00 \pm 0.01$ and $\rho = 2.33 \pm 0.01$, respectively, in excellent agreement with the exact results. Weinrib and Trugman\(^{(14)}\) have argued that the scaling function $f(x)$ for the perimeter of percolation clusters should be proportional to the scaling function $g(x)$ for the percolation clusters themselves. It would be interesting to check this for the $f(x)$ we obtained here.

3. For $C < 1$, however, the critical behavior of the rotator model on the square lattice and that of the rotator and mirror model on the triangular lattice are very different. None of these models can be mapped onto a percolation problem. For the square lattice, we obtained numerically a new exponent $\sigma' = 1.6 \pm 0.3$ which is significantly larger than the usual $\sigma = 3/7$. This indicates that the typical size of closed trajectories upon approach to criticality increases much slower to infinity along the line $C_R = C_L$ than along the line $C_R + C_L = 1$. The new scaling function $f'(x)$, corresponding to the exponent $\sigma' = 1.6$, appears to be an exponential function, rather than a double Gaussian as for $C = 1$, but the origin of this exceptional critical behavior is unclear to us. For the triangular lattice, however, we found numerically, that $\sigma$ is very close to $3/7$ which also obtains for $C = 1$, and that the scaling function $f(x)$ can still be described by a double Gaussian. This suggests that the critical behavior in the direction perpendicular to the critical line $C_R = C_L$ is in the same universality class. For dilute thermal systems, it is also found that the critical behavior is independent of the density of impurities if the specific heat exponent $\alpha$ is negative\(^{(15)}\), but to what extent this is similar to the independence of the number of empty sites found here, is unclear to us.
4. The scaling function \( h(x) \) appears to behave like a constant in all cases, indicating that there are no noticeable corrections to the universal fractal dimension \( d_f = 7/4 \). This simply means that the gyration radius of the extended trajectories is independent of the degree of occupation of the lattice by scatterers and only trivially dependent on the nature of the lattice.

5. We have studied in the previous paper the scaling behavior of extended particle trajectories as their sizes approach infinity at criticality. For example, we have obtained for the rotator model on the fully occupied square lattice that \( <S/N - 3/2> \sim N^{-0.57} \), where the exponent is very close to \( 1 - \sigma = 4/7 = 0.571 \), with \( \sigma \) characterizing the trajectory size distribution in the critical region near criticality. We surmise that this may not be a coincidence and conjecture that there is a possible relation between the scaling behavior of extended closed trajectories at criticality and in the critical region near criticality. If so, the question arises whether also other exponents such as \( \gamma \) and \( \rho \) could be determined from the scaling behavior at criticality, rather than as we have done here, in the critical region.

6. Finally, we remark that the diffusive behavior of the moving particle away from criticality, which has been studied extensively before\(^{(5,7)}\), is in fact controlled by the scaling function \( f(x) \). The time dependent diffusion constant \( D(t) \) is defined as \( D(t) = <R^2(t)> / (4t) = \sum R^2_S n_S / 4t \), where the summation is taken over all closed trajectories upto length \( t \). For \( (C_R - C_{R_c}) t^{\sigma} \ll 1 \), when \( f(x) \) is essentially a constant, \( D(t) \) is then virtually the same as at criticality, while for \( (C_R - C_{R_c}) t^{\sigma} \gg 1 \), when \( f(x) \) is very small, \( D(t) \) decays to zero as \( 1/t \).

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Figure captions

Fig. 1. The scaling function \( f(x) \) vs \( x \) for the rotator model on a fully occupied square lattice, computed at \( C = C_{Rc} + 0.01 \) (\( \bigcirc \)), \( C = C_{Rc} + 0.02 \) (+) and \( C = C_{Rc} + 0.03 \) (\( \square \)). The curve is described by a double Gaussian, 
\[
f(x) = 1.03e^{-2.25(x+0.86)^2} + 1.03e^{-2.25(x-0.86)^2}.
\]
The deviations between \( f(x) \) and the numerical data near \( x = 0 \) in this and later similar figures are due to the failure of scaling for small trajectories.

Fig. 2. The scaling function \( f(x)h(x) \) vs \( x \) for the rotator model on a fully occupied square lattice, computed at \( C = C_{Rc} + 0.01 \) (\( \bigcirc \)), \( C = C_{Rc} + 0.02 \) (+) and \( C = C_{Rc} + 0.03 \) (\( \square \)). The curve is described by a double Gaussian,
\[
f(x)h(x) = 0.39e^{-2.25(x+0.86)^2} + 0.39e^{-2.25(x-0.86)^2}.
\]

Fig. 3. \( \ln <S> \) vs \( -\ln(C_{R} - C_{Rc}) \) for the rotator model on the fully occupied square lattice. The slope of the fitting line is 2.00.

Fig. 4. \( \ln <R^2> \) vs \( -\ln(C_{R} - C_{Rc}) \) for the rotator model on the fully occupied square lattice. The slope of the fitting line is 2.33.

Fig. 5. The scaling function \( f(x) \) vs \( x \) for the rotator model on the fully occupied square lattice, obtained by using the histogram method. Computed for \( C = C_{Rc} + 0.02 \) (\( \bigcirc \)), \( C = C_{Rc} + 0.03 \) (+), \( C = C_{Rc} + 0.04 \) (\( \square \)) and \( C = C_{Rc} + 0.05 \) (\( \times \)). The curve is described by a double Gaussian, 
\[
f(x) = 1.03e^{-2.25(x+0.86)^2} + 1.03e^{-2.25(x-0.86)^2}.
\]

Fig. 6. Part of the phase diagram for the rotator model on the square lattice obtained from numerical simulations for \( 0.65 \leq C \leq 1 \). The lines are drawn to guide the eye.

Fig. 7. \( n_S \) vs \( S \), at concentrations \( C_R = C_L = 0.5 \) (\( \bigcirc \)), \( C_R = C_L = 0.485 \) (+), \( C_R = C_L = 0.45 \) (\( \square \)) and \( C_R = C_L = 0.40 \) (\( \times \)).

Fig 8. The scaling function \( f'(x) \) vs \( x \) for the rotator model on the partially occupied square lattice, computed at \( C = C_{Rc} + 0.02 \) (\( \bigcirc \)), \( C = C_{Rc} + 0.025 \) (+)
and \( C = C_{Re} + 0.01 \) (\( \square \)). The curve is described by an exponential function, \( f'(x) = 0.475e^{-0.165 \times 10^{-8}x} \).

Fig 9. The scaling function \( f'(x)h'(x) \) vs \( x \) for the rotator and mirror model on the partially occupied square lattice, computed at \( C = C_{Re} + 0.02 \) (\( \diamond \)), \( C = C_{Re} + 0.025 \) (\( + \)) and \( C = C_{Re} + 0.01 \) (\( \square \)). The curve is described by an exponential function, \( f'(x)h'(x) = 0.115e^{-0.165 \times 10^{-8}x} \).

Fig 10. The scaling function \( f(x) \) vs \( x \) for the rotator and mirror model on the fully occupied triangular lattice, computed at \( C = C_{Re} + 0.01 \) (\( \diamond \)), \( C = C_{Re} + 0.02 \) (\( + \)) and \( C = C_{Re} + 0.03 \) (\( \square \)). The curve is described by a double Gaussian, \( f(x) = 0.94e^{-1.8(x+0.96)^2} + 0.94e^{-1.8(x-0.96)^2} \).

Fig 11. The scaling function \( f(x)h(x) \) vs \( x \) for the rotator and mirror model on the fully occupied triangular lattice, computed at \( C = C_{Re} + 0.01 \) (\( \diamond \)), \( C = C_{Re} + 0.02 \) (\( + \)) and \( C = C_{Re} + 0.03 \) (\( \square \)). The curve is described by a double Gaussian, \( f(x)h(x) = 0.28e^{-1.8(x+0.96)^2} + 0.28e^{-1.8(x-0.96)^2} \).

Fig 12. \( \ln <S> \) vs \( -\ln |C_R - C_{Re}| \) for the rotator and mirror model on the fully occupied triangular lattice. The slope of the fitting line is 2.00.

Fig 13. \( \ln <R^2> \) vs \( -\ln |C_R - C_{Re}| \) for the rotator and mirror model on the fully occupied triangular lattice. The slope of the fitting line is 2.33.

Fig 14. The scaling function \( f(x) \) vs \( x \), for the rotator and mirror model on the fully occupied triangular lattice, computed by using the histogram method, for \( C = C_{Re} + 0.02 \) (\( \diamond \)), \( C = C_{Re} + 0.03 \) (\( + \)), \( C = C_{Re} + 0.04 \) (\( \square \)) and \( C = C_{Re} + 0.05 \) (\( \times \)). The curve is described by a double Gaussian, \( f(x) = 0.94e^{-2.25(x+0.96)^2} + 0.94e^{-2.25(x-0.96)^2} \).

Fig 15. The scaling function \( f(x) \) vs \( x \) for the rotator and mirror model on the partially occupied triangular lattice, computed at \( C = C_{Re} + 0.01 \) (\( \diamond \)), \( C = C_{Re} + 0.015 \) (\( + \)) and \( C = C_{Re} + 0.02 \) (\( \square \)). The curve is described by a double Gaussian \( f(x) = 1.05e^{-1.18(x+1.19)^2} + 1.05e^{-1.18(x-1.19)^2} \).

Fig 16. The scaling function \( f(x)h(x) \) vs \( x \) for the rotator and mirror model
on the partially occupied triangular lattice, computed at $C = C_{Rc} + 0.01$ ($\bigcirc$), $C = C_{Rc} + 0.015$ (+) and $C = C_{Rc} + 0.02$ (□). The curve is described by a double Gaussian $f(x)h(x) = 0.30e^{-1.18(x+1.19)^2} + 0.30e^{-1.18(x-1.19)^2}$.

Fig. 17. $\ln < S >$ vs $-\ln |C - C_{Rc}|$ for the partially occupied triangular lattice, for $C = 0.9$ ($\bigcirc$) and $C = 0.8$ (+). The slope of the fitting lines is 2.00.

Fig. 18. $\ln < R^2 >$ vs $-\ln |C - C_{Rc}|$ for the partially occupied triangular lattice, for $C = 0.9$ ($\bigcirc$) and $C = 0.8$ (+). The slope of the fitting lines is 2.33.
Table I: Scaling functions and critical exponents

| Scaling Functions       | Square (C = 1) | Triangular (C < 1) |
|-------------------------|----------------|--------------------|
| \( f(x) \) (double Gaussian) | \( A_1 \) | 1.03 | 0.94 | 1.05 |
|                         | \( \alpha_1 \) | 2.25 | 1.80 | 1.18 |
|                         | \( a_1 \) | 0.86 | 0.96 | 1.19 |
| \( h(x) \) (constant) | 0.38 | 0.30 | 0.28 |
| \( f'(x) \) (exponential) | \( A'_1 \) | 0.475 | 0.165 \( \times 10^{-8} \) |
|                         | \( \alpha'_1 \) | 1.65 \( \times 10^{-8} \) |
|                         | \( a'_1 \) | 0 |
| \( h'(x) \) (constant) | 0.24 |

| Exponents               | \( \tau \) | 15/7 | 15/7 | 15/7 | 15/7 |
|-------------------------| \( d_f \) | 7/4 | 7/4 | 7/4 | 7/4 |
|                         | \( \sigma \) (\( \sigma' \)) | 3/7 | 1.6 | 3/7 | 3/7 |
|                         | \( \gamma \) (\( \gamma' \)) | 2 | 0.54 | 2 | 2 |
|                         | \( \rho \) (\( \rho' \)) | 7/3 | 0.63 | 7/3 | 7/3 |

Scaling functions and critical exponents obtained for closed trajectories on the square and triangular lattice. \( f(x) = A_1 \exp[-\alpha_1(x - a_1)^2] + A_1 \exp[-\alpha_1(x + a_1)^2] \) and \( f'(x) = A'_1 \exp[-\alpha'_1(x - a'_1)] \). The critical exponents for \( C = 1 \) are known exactly, those for \( C < 1 \) only numerically. The square lattice for \( C < 1 \) behaves exceptional if the critical point \( C_R = C_L = 1/2 \) is approached along the line \( C_R = C_L \), rather than along the line \( C = 1 \); the primed quantities refer to this case.
Fig. 1
Fig. 3
Fig. 4
Fig. 6
Fig. 7
Fig. 8
Fig. 9
Fig. 10
Fig. 12
Fig. 13
Fig. 14
Fig. 15
Fig. 16
Fig. 17
