Detecting the skewness of data from the sample size and the five-number summary

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Abstract

For clinical studies with continuous outcomes, when the data are potentially skewed, researchers may choose to report the whole or part of the five-number summary (the sample median, the first and third quartiles, and the minimum and maximum values), rather than the sample mean and standard deviation. For the studies with skewed data, if we include them in the classical meta-analysis for normal data, it may yield misleading or even wrong conclusions. In this paper, we develop a flow chart and three new tests for detecting the skewness of data from the sample size and the five-number summary. Simulation studies demonstrate that our new tests are able to control the type I error rates, and meanwhile provide good statistical power. A real data example is also analyzed to demonstrate the usefulness of the skewness tests in meta-analysis and evidence-based practice.

\textit{Key words}: Five-number summary, Flow chart, Meta-analysis, Sample size, Skewness test

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1 Introduction

Meta-analysis is an important tool to synthesize the research findings from multiple studies for decision making. To conduct a meta-analysis, the summary statistics are routinely collected from each individual study, and in particular for continuous outcomes, they consist of the sample mean and standard deviation (SD). In some studies, researchers may instead report the whole or part of the five-number summary \( \{a, q_1, m, q_3, b\} \), where \( a \) is the minimum value, \( q_1 \) is the first quartile, \( m \) is the sample median, \( q_3 \) is the third quartile, and \( b \) is the maximum value. More specifically, by letting \( n \) be the size of the data, the three common scenarios for reporting the five-number summary include

\[
S_1 = \{a, m, b; n\}, \\
S_2 = \{q_1, m, q_3; n\}, \\
S_3 = \{a, q_1, m, q_3, b; n\}.
\]

In practice, however, few existing methods in meta-analysis are able to pool together the studies with the sample mean and SD and the studies with the five-number summary.

To overcome this problem, there are two common approaches in the literature. The first approach is to exclude the studies with the five-number summary from meta-analysis by labeling them as “studies with insufficient data”. This approach was, in fact, quite popular in the early years. Nevertheless, by doing so, valuable information may be excluded so that the final meta-analytical result is less reliable or even misleading, especially when a large proportion of studies are reported with the five-number summary. In contrast, the second approach is to apply the recently developed methods (Hozo et al., 2005; Walter and Yao, 2007; Wan et al., 2014; Luo et al., 2018; Shi et al., 2020) that convert the five-number summary back to the sample mean and SD, and then include them in the subsequent meta-analysis. It is noteworthy that these transformation methods have been attracting more attention and been increasingly applied in meta-analysis and evidence-based practice. For example, the sample SD estimation in Wan et al. (2014) and the sample mean estimation in Luo et al. (2018) have received a total of 1607 and 259 citations, respectively, in Google Scholar as of 10 October 2020.
Despite the popularity of the second approach, we note that most of the aforementioned methods for data transformation are built on the basis of the normality assumption for the underlying data. In Section 2, we will show that once the data are skewed, these normal-based methods will not be able to provide reliable estimates for the transformed sample mean and SD. As a consequence, if we still include the skewed studies in meta-analysis, it may result in misleading or unreliable conclusions. This motivates us to develop new methods for detecting the skewness of data in this paper. Specifically, as presented in the flow chart in Figure 1, only if the skewness test is not rejected, the data can be further transformed back to the sample mean and SD and the associated study can remain to be included in the meta-analysis.

There is a large body of literature on the test of normality, mainly two different types of tests. The first type is the graphical methods that apply, for example, the histogram, Q-Q plot and box plot (Thode, 2002). Yet due to the absence of an objective criterion, the graphical methods are often subjective and their reliability will be heavily dependent on the experience of the evaluators. For more objective methods for detecting the normality, it yields the second type of tests including the chi-square goodness-of-fit test, the empirical distribution function test, the Shapiro-Wilk test, and the moment test (D’Agostino et al., 1990; Cramér, 1928; Kolmogorov, 1933; Anderson and Darling, 1954; Shapiro and Wilk, 1965; Jarque and Bera, 1980). Nevertheless, we note that most existing normality tests require the complete data set so that they are not applicable when the data include only the whole or part of the five-number summary. For this issue, Altman and Bland (1996) also discussed in their short note as follows: “When authors present data in the form of a histogram or scatter diagram then readers can see at a glance whether the distributional assumption is met. If, however, only summary statistics are presented—as is often the case—this is much more difficult.”.

To summarize, when only the five-number summary is available, it is nearly impossible to test whether the underlying data follow a normal distribution. But as an alternative, we note that the information in the five-number summary, together with the sample size, is surprisingly sufficient enough to conduct a skewness test. Further by the symmetry of the normal distribution, if the skewness test shows that the data are significantly skewed, then equivalently we can also conclude that the data are not normally distributed. While
Conduct a systematic review and find a study reported with the whole or part of the five-number summary

Detect the skewness of data under scenario $S_1$, $S_2$ or $S_3$

- Not skewed
  - Scenarios $S_1$ and $S_2$: Estimate the sample mean and SD by Luo et al. (2018) and Wan et al. (2014), respectively
- Skewed
  - Scenario $S_3$: Estimate the sample mean and SD by Luo et al. (2018) and Shi et al. (2020), respectively
  - Exclude the study from meta-analysis

Apply the estimated mean and SD and include the study in the meta-analysis

Figure 1: A flow chart for performing meta-analysis when some studies are reported with the whole or part of the five-number summary.
for the other direction, if the skewness test is not rejected, then we follow the common practice that assumes the reported data to be normal. Following the above procedure, we will have the capacity to rule out the very skewed studies so that the final meta-analysis can be conducted more reliably than the existing methods in the literature. Finally, due to the limited data available, we believe that our proposed flow chart in Figure 1 is the best we can do for combining the studies with two different types of summary data, and we also expect that it may have potential to be widely adopted in meta-analysis and evidence-based practice.

2 Motivating Examples

We first present a simulation study to evaluate the performance of the existing transformation methods when the underlying distribution is skewed away from normality. Specifically, we consider four normal-related distributions (Forbes et al., 2011) as follows: (i) the skew-normal distribution with parameters $\xi = 0$, $\omega = 1$ and $\alpha = -10$, (ii) the half-normal distribution with parameters $\mu = 0$ and $\sigma^2 = 1$, (iii) the log-normal distribution with parameters $\mu = 0$ and $\sigma^2 = 1$, and (iv) the mixture-normal distribution that takes the values from $N(-2, 1)$ with probability 0.3 and from $N(2, 1)$ with probability 0.7. Also to visualize the skewness of the distributions, the probability density functions of the four distributions are plotted in Figure 2.

Next, for each distribution, a sample of size 200 is randomly generated. With the complete sample, we can readily compute the sample mean and SD, and also collect the sample median, the minimum and maximum values. Now to evaluate the normal-based methods under scenario $S_1$, we apply Luo et al. (2018) and Wan et al. (2014) to estimate the sample mean and SD, respectively. With 100,000 simulations, we report the averages (standard errors) of the estimated sample mean and SD, together with the averages (standard errors) of the true sample mean and SD, in Table 1. From the simulated results, it is evident that the converted sample mean and SD using the normal-based methods are less accurate for all four skewed distributions. In particular, we note that the sample SD is significantly overestimated for Log-normal(0, 1), and the sample mean is significantly overestimated for the mixture-normal distribution.
Our second motivating example is from the meta-analysis conducted by Nnoaham and Clarke (2008), which was to study the relationship between the low serum vitamin D levels and tuberculosis. Specifically, as shown in Table 2, a total of six studies with continuous outcomes were included in their conducted meta-analysis. Among them and for both cases and controls, three studies reported the sample median and range (Davies et al., 1985; Grange et al., 1985; Davies et al., 1987), two studies reported the sample mean and SD (Davies et al., 1988; Chan et al., 1994), and one study reported the sample mean and range (Sasidharan et al., 2002). Moreover, for the three studies reported with the sample median and range, we note that the sample median is always closer, by a large margin, to the minimum value than to the maximum value. This indicates that the underlying data for these studies are likely to be positively skewed, as also observed in the literature.
(Sheikh et al., 2012; Fang et al., 2016). More specifically, in Section 5, we will show that the data from Davies et al. (1985) and Grange et al. (1985) are significantly skewed from normality. Such skewed data, if not excluded from meta-analysis, may result in misleading or unreliable conclusions.

Table 1: The true and estimated averages (standard errors) of the sample mean and SD for the four normal-related distributions.

| Distribution       | Sample mean True value | Estimated value | Sample SD True value | Estimated value |
|--------------------|------------------------|----------------|----------------------|----------------|
| Skew-normal        | -0.79 (0.04)           | -0.73 (0.05)   | 0.61 (0.04)          | 0.57 (0.07)    |
| Half-normal        | 0.80 (0.04)            | 0.73 (0.05)    | 0.60 (0.04)          | 0.54 (0.07)    |
| Log-normal         | 1.65 (0.16)            | 1.53 (0.32)    | 2.09 (0.56)          | 3.10 (1.57)    |
| Mixture-normal     | 0.80 (0.15)            | 1.34 (0.14)    | 2.09 (0.08)          | 1.63 (0.11)    |

Table 2: The summary statistics of the six studies included in the meta-analysis of Nnoaham and Clarke (2008), where the studies are reported with sample median (range) [sample size], sample mean±SD [sample size], or sample mean* (range) [sample size].

| Study               | Cases         | Controls       |
|---------------------|---------------|----------------|
| Davies et al. (1985)| 16 (2.25-74.25) [40] | 27.25 (9-132.5) [40] |
| Grange et al. (1985) | 65.75 (43.75-130.5) [40] | 69.5 (48.5-125) [38] |
| Davies et al. (1987) | 39.75 (16.75-89.25) [15] | 65.5 (26.25-114.75) [15] |
| Davies et al. (1988) | 69.5±24.5 [51] | 95.5±29.25 [51] |
| Chan et al. (1994)   | 46.5±18.5 [24] | 52.25±15.75 [24] |
| Sasidharan et al. (2002) | 26.75* (2.5-75) [35] | 48.5* (22.5-145) [16] |

3 New Methods for Detecting the Skewness

As mentioned earlier, for the clinical data reported with the whole or part of the five-number summary, we are interested in testing whether or not the underlying distribution follows a normal distribution. When the normality assumption does not hold, the data will often be skewed, which is, in fact, one main reason why researchers may choose to report the five-number summary. In this section, we will formulate the null and alternative hypotheses for detecting the skewness of data under the three common scenarios, and then construct their test statistics, as well as derive their null distributions and the critical regions.
Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from the normal distribution with mean $\mu$ and variance $\sigma^2$, and $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be the corresponding order statistics. Then, for simplicity, by letting $n = 4Q + 1$ with $Q$ being a positive integer, the five-number summary can be represented as $a = X_{(1)}$, $q_1 = X_{(Q+1)}$, $m = X_{(2Q+1)}$, $q_3 = X_{(3Q+1)}$ and $b = X_{(n)}$. Let also $X_i = \mu + \sigma Z_i$ for $i = 1, \ldots, n$, or equivalently,

$$X_{(i)} = \mu + \sigma Z_{(i)},$$

where $Z_1, Z_2, \ldots, Z_n$ are independent random variables from the standard normal distribution, and $Z_{(1)} \leq Z_{(2)} \leq \cdots \leq Z_{(n)}$ are the order statistics.

### 3.1 Detecting the Skewness under Scenario $S_1$

We first consider scenario $S_1$ where the minimum, median, and maximum values are available together with the sample size. When the data are normally distributed, we expect that the distance between $a$ and $m$ should be not far away from the distance between $m$ and $b$. More specifically, by Lemma [1] in the Appendix B and the facts that $E(m - a) = \sigma E(Z_{(2Q+1)} - Z_{(1)})$ and $E(b - m) = \sigma E(Z_{(n)} - Z_{(2Q+1)})$, we have $E(m-a) = E(b-m)$. In view of this, we define $\theta_1 = E(b-m) - E(m-a) = E(a+b-2m)$ as the level of skewness for the underlying distribution of the data. Then to detect the skewness of data, we propose to consider the following hypotheses:

$$H_0 : \theta_1 = 0 \quad \text{versus} \quad H_1 : \theta_1 \neq 0.$$

When the null hypothesis is rejected, we conclude $\theta_1 \neq 0$ so that the data are significantly skewed. Further by the flow chart in Figure 1 we recommend not to include the corresponding study in the meta-analysis for normal data, as otherwise it may yield misleading results for decision making.

Under scenario $S_1$, the level of skewness $\theta_1$ can be naturally estimated by $a + b - 2m$, which is probably also the only available estimator due to the limited data sources. An illustration example is also given in the Appendix A for demonstrating the suitability of
applying $a + b - 2m$ to detect the skewness of data. More specifically, by the Wald test (Casella and Berger, 2002), we consider the test statistic as

$$W_1 = \frac{a + b - 2m}{SE(a + b - 2m)},$$

where $SE(a + b - 2m)$ denotes the standard error of $a + b - 2m$ under the null hypothesis. By formula (1), we can rewrite $SE(a + b - 2m) = \sigma \delta_1(n)$, where $\sigma$ is the standard deviation of the normal distribution and $\delta_1(n) = SE(Z_{(1)} + \hat{Z}_{(n)} - 2Z_{(2Q+1)})$. Next, for the unknown $\sigma$, we consider to estimate it by the method in Wan et al. (2014). Specifically, we have $\hat{\sigma}_1 = (b - a)/\xi(n)$, where $\xi(n) = 2\Phi^{-1}[(n - 0.375)/(n + 0.25)]$ and $\Phi^{-1}$ is the quantile function of the standard normal distribution. Finally, by noting that $\delta_1(n)$ and $\xi(n)$ are fixed values for any given $n$, we remove them from the Wald statistic and that yields our final test statistic as

$$T_1 = \frac{a + b - 2m}{b - a}.$$  

(2)

3.1.1 The Null Distribution of $T_1$

In this section, we consider two different approaches to derive the null distribution of the test statistic $T_1$. The first one is to derive the asymptotic null distribution when $n$ tends to infinity, and the second one is to derive the exact null distribution for any fixed $n$.

Noting that $T_1$ involves the extreme order statistics, by Ferguson (1996) the asymptotic null distribution will not follow a normal distribution as that for a classical Wald statistic. To further clarify it, when $n$ is large, the extreme order statistics will tend to be less stable than the intermediate order statistics and hence provide a slower convergence rate toward the asymptotic distribution for the given test statistic. Specifically by Theorem [1] in the Appendix B, under the null hypothesis, we show that

$$\sqrt{2\ln(n)}\xi(n)T_1 \xrightarrow{D} \text{Logistic}(0, 1), \text{ as } n \to \infty,$$

(3)

where $\xrightarrow{D}$ denotes the convergence in distribution, and Logistic(0, 1) represents the logistic distribution with location parameter $\mu = 0$ and scale parameter $s = 1$. Noting also that the asymptotic null distribution is symmetric about zero, we can specify the critical
region of size $\alpha$ as \{\(t_{1,\text{obs}} : |t_{1,\text{obs}}| > l_{\alpha/2}/(\sqrt{2\ln(n)}\xi(n))\}\}, where \(t_{1,\text{obs}}\) is the observed value of \(T_1\) and \(l_{\alpha/2}\) is the upper \(\alpha/2\) quantile of Logistic(0, 1). Nevertheless, despite the elegant analytical results, the asymptotic test by (3) will have a serious limitation that the convergence rate is relatively slow at the order of \(\sqrt{2\ln(n)}\xi(n)\). Moreover, the simulation results in Section 4 will show that the asymptotic null distribution fails to control the type I error rates for some small sample sizes.

To improve the detection accuracy, our second approach is to derive the exact null distribution of \(T_1\) for any fixed \(n\). By (1) and (2), we can represent the test statistic as

\[
T_1 = \frac{Z_{(1)} + Z_{(n)} - 2Z_{(2Q+1)}}{Z_{(n)} - Z_{(1)}}. \tag{4}
\]

Since the right-hand side of (1) is purely a function of the order statistics of the standard normal distribution, the null distribution of \(T_1\) will be free of the parameters \(\mu\) and \(\sigma^2\). Moreover, we have derived the sampling distribution of \(T_1\) under the null hypothesis in Theorem 2 in the Appendix B. Further by the symmetry of the null distribution, the critical region of size \(\alpha\) can be specified as

\[
\{t_{1,\text{obs}} : |t_{1,\text{obs}}| > c_{1,\alpha/2}\},
\]

where \(c_{1,\alpha/2}\) is the upper \(\alpha/2\) quantile of the null distribution of \(T_1\). More specifically, if the absolute value of \(t_{1,\text{obs}}\) is larger than the critical value \(c_{1,\alpha/2}\), then we reject the null hypothesis, indicating that the underlying distribution of the data tends to be skewed so that the transformation methods for normal data may not be appropriate to apply. Otherwise if the test is not rejected, we suggest not to exclude the study from meta-analysis, but instead follow the flow chart in Figure 1 to estimate the sample mean and SD.

From the practical point of view, however, the null distribution of \(T_1\) in Theorem 2 in the Appendix B is somewhat complicated and the true values of \(c_{1,\alpha/2}\) may not be easy to obtain. To help practitioners and promote the proposed test, by the R software we have also computed the numerical values of \(c_{1,\alpha/2}\) for \(\alpha = 0.05\) with \(n\) up to 401 in Table 5 in the Appendix C. Specifically in each sampling, we generate a random sample of size \(n\)
Figure 3: The green points represent the exact critical values under scenarios $S_1$, $S_2$ and $S_3$ respectively, and the red lines represent the approximate function of the critical values for $n$ up to 401.
from the standard normal distribution. By collecting \( \{a, m, b\} \) from the whole sample, we further compute the value of \( T_1 \) by formula (2). With 1,000,000 repetitions, we achieve the sampling distribution of \( T_1 \) under the null hypothesis and then compute the numerical values of \( c_{1.0.025} \) without calculating the complicated integral in Theorem 2 in the Appendix B. Moreover, for ease of implementation, we have also provided an approximate formula for the numerical critical values. Specifically, noting that the critical values depend only on the sample size, we consider the formula \( c_{1.0.025} \approx 1.01/\ln(n + 9) + 2.43/(n + 1) \), which, as shown in the top panel of Figure 3, approximates very well the critical values of \( c_{1.0.025} \). Finally, with the approximate formula, it also provides an alternative way to detect the skewness without the help of a computer-aided program. To do so, one can simply compute by hand the absolute value of the observed test statistic and then compare it with \( 1.01/\ln(n + 9) + 2.43/(n + 1) \) for making a final decision.

3.2 Detecting the Skewness under Scenario \( S_2 \)

Under scenario \( S_2 \), the reported summary data include the first quartile \( q_1 \), the sample median \( m \), the third quartile \( q_3 \), and the sample size \( n \). When the data are normally distributed, we expect that the distance between \( q_1 \) and \( m \) should be close to the distance between \( m \) and \( q_3 \). Specifically, by Lemma 1 in the Appendix B and the facts that \( E(m - q_1) = \sigma E(Z_{2Q+1} - Z_{Q+1}) \) and \( E(q_3 - m) = \sigma E(Z_{3Q+1} - Z_{2Q+1}) \), we have \( E(m - q_1) = E(q_3 - m) \). We then define \( \theta_2 = E(q_3 - m) - E(m - q_1) = E(q_1 + q_3 - 2m) \) as the level of skewness for the underlying distribution of the data. Finally for detecting the skewness of data, we consider the following hypotheses:

\[
H_0 : \theta_2 = 0 \quad \text{versus} \quad H_1 : \theta_2 \neq 0.
\]

When the null hypothesis is rejected, we conclude that the underlying distribution of the data is significantly skewed.

Following the same spirit as under scenario \( S_1 \), for the above hypothesis we consider the following test statistic:

\[
T_2 = \frac{q_1 + q_3 - 2m}{q_3 - q_1}.
\] (5)
It is noteworthy that $T_2$ has also been adopted by Groeneveld and Meeden (1984) as a measure of skewness. Moreover, unlike the test statistic $T_1$ that involves the extreme order statistics, the asymptotic normality of $T_2$ can be readily established. Specifically in Theorem 3 in the Appendix B, we have shown that

$$0.74\sqrt{n}T_2 \xrightarrow{D} N(0,1), \text{ as } n \to \infty.$$

(6)

Further by (6), the critical region of size $\alpha$ can be approximately as $\{t_{2,\text{obs}} : |t_{2,\text{obs}}| > z_{\alpha/2}/(0.74\sqrt{n})\}$, where $t_{2,\text{obs}}$ is the observed value of $T_2$ and $z_{\alpha/2}$ is the upper $\alpha/2$ quantile of the standard normal distribution.

Nevertheless, given that the asymptotic critical values can be quite large especially for small sample sizes, the above asymptotic test may not provide an adequate power for detecting the skewness. To further improve the detection accuracy, we have also derived the exact null distribution of $T_2$ in Theorem 4 in the Appendix B for any fixed $n$. Noting also that the null distribution of $T_2$ is symmetric about zero, we can specify the exact critical region of size $\alpha$ as follows:

$$\{t_{2,\text{obs}} : |t_{2,\text{obs}}| > c_{2,\alpha/2}\},$$

where $c_{2,\alpha/2}$ is the upper $\alpha/2$ quantile of the null distribution of $T_2$. When the null hypothesis is rejected so that the underlying distribution of the data is significantly skewed, we recommend to exclude the study from meta-analysis. Otherwise, we follow the flow chart in Figure 1 and estimate the sample mean and SD using the transformation methods for normal data.

Finally, as that for scenario $S_1$, we note that the computation formula for the critical values by Theorem 4 in the Appendix B is rather complicated and not readily accessible. Thus to help practitioners, we have also computed the numerical values of $c_{2,\alpha/2}$ for $\alpha = 0.05$ with $n$ up to 401 in Table 6 in the Appendix C. For ease of implementation, an approximate formula for the critical values is also provided as $c_{2,0.025} \approx 2.66/\sqrt{n} - 5.92/n^2$. From the middle panel of Figure 3, the approximate formula fits the critical values of $c_{2,0.025}$ well and, consequently, it also provides a convenient way for practitioners to detect
the skewness of data by hand.

### 3.3 Detecting the Skewness under Scenario $S_3$

In this section, we consider to detect the skewness of data under scenario $S_3$ where the sample size $n$ and the five-number summary $\{a, q_1, m, q_3, b\}$ are fully available. For normal data, we have $E(m - a) = E(b - m)$ and $E(m - q_1) = E(q_3 - m)$, or equivalently, $\theta_1 = E(a + b - 2m) = 0$ and $\theta_2 = E(q_1 + q_3 - 2m) = 0$. Noting also that the summary data under scenario $S_3$ is the union of those under scenarios $S_1$ and $S_2$, we consider the following joint hypothesis for detecting the skewness of data:

$$H_0 : \theta_1 = 0 \text{ and } \theta_2 = 0 \quad \text{versus} \quad H_1 : \theta_1 \neq 0 \text{ or } \theta_2 \neq 0.$$

If the joint null hypothesis is rejected, then the data will be claimed as significantly skewed, either in the intermediate region or in the tail region of the underlying distribution.

Following the similar arguments as under scenarios $S_1$ and $S_2$, $\hat{\theta}_1 = a + b - 2m$ is the sample estimate of the skewness $\theta_1$, and $\hat{\theta}_2 = q_1 + q_3 - 2m$ is the sample estimate of the skewness $\theta_2$. Thus to test the joint null that $\theta_1 = 0$ and $\theta_2 = 0$, we can take the maximum of their absolute sample estimates as the test statistic. Specifically, if the maximum value is larger than a given threshold, then the test will be rejected so that either $\theta_1$ or $\theta_2$ will be concluded as nonzero. Meanwhile, to make the two test components comparable, we also standardize them and yield the test statistic as $W_3 = \max\{|W_1|, |W_2|\}$, where $W_1 = (a + b - 2m)/\text{SE}(a + b - 2m)$ as already defined and $W_2 = (q_1 + q_3 - 2m)/\text{SE}(q_1 + q_3 - 2m)$.

Further by Theorem 5 in the Appendix B, we replace $\text{SE}(a + b - 2m)$ and $\text{SE}(q_1 + q_3 - 2m)$ by their respective estimates, and formulate the final test statistic as

$$T_3 = \max \left\{ \frac{2.65 \ln(0.6n)}{\sqrt{n}} \left| \frac{a + b - 2m}{b - a} \right|, \left| \frac{q_1 + q_3 - 2m}{q_3 - q_1} \right| \right\}.$$  

(7)

In addition, by (2) and (5) and the fact that the weight to the first component is purely a function of $n$, it follows that $T_3$ will be independent of $\mu$ and $\sigma^2$ under the joint null.
hypothesis. Then accordingly, we can propose the critical region of size $\alpha$ as

$$\{t_{3,\text{obs}} : t_{3,\text{obs}} > c_{3,\alpha}\},$$

where $t_{3,\text{obs}}$ is the observed value of $T_3$, and $c_{3,\alpha}$ is the upper $\alpha$ quantile of its null distribution. Finally, in case the null hypothesis is rejected, we recommend to exclude the study from meta-analysis. Otherwise, we follow the flow chart in Figure 1 and apply the transformation methods for normal data to estimate the sample mean and SD.

The same as before, we also apply the sampling method to numerically compute the critical values of $c_{3,\alpha}$ for $\alpha = 0.05$ with $n$ up to 401, and present them in Table 7 in the Appendix C. Moreover, we provide an approximate formula for the critical values as $c_{3,0.05} \approx 2.97/\sqrt{n} - 39.1/n^3$. The bottom panel of Figure 3 shows that the approximate formula fits the critical values of $c_{3,0.05}$ quite well, and consequently, it also readily provides an alternative way to detect the skewness of the reported data.

4 Simulation Studies

In this section, we conduct simulation studies to evaluate the performance of the three skewness tests. We first assess the type I error rates of the new tests with the asymptotic, exact, and approximated critical values at the significance level of 0.05, and then compute and compare their statistical power under four skewed alternative distributions.

4.1 Type I Error Rates

To examine whether the type I error rates are well controlled, we first generate a sample of size $n$ from the null distribution, and without loss of generality, we consider the standard normal distribution. Then for the proposed test under scenario $S_1$, we record the summary statistics $\{a, m, b\}$ from the simulated sample, and compute the observed value of the test statistic $t_{1,\text{obs}}$ by (2). Further by comparing $t_{1,\text{obs}}$ with the asymptotic, exact, and approximated critical values respectively, we can make a decision whether to reject the null hypothesis, or equivalently, whether a type I error will be made. Finally, we repeat
the above procedure for 1,000,000 times with \( n \) ranging from 5 to 401, compute the type I error rates for the three different tests, and report them in the top panel of Figure 4.

It is evident from the simulated results that, under scenario \( S_1 \), the proposed tests with the exact and approximated critical values perform nearly the same, and they both control the type I error rates at the significance level of 0.05 regardless of the sample size. This, from another perspective, demonstrates that our approximate formula of the critical values is rather accurate and can be recommended for practical use. In contrast, for the asymptotic test with the null distribution specified in (3), the type I error rates are less well controlled, either inflated or too conservative. For example, the type I error rate is as high as 0.057 when \( n = 17 \), and it is always less than 0.05 when \( n \) is large, even though the simulated type I error rate does converge to the significance level as \( n \) goes to infinity which coincides with the theoretical result in Theorem 1 in the Appendix B.

To assess the type I error rates under the last two scenarios, we record instead the summary statistics \( \{q_1, m, q_3\} \) or \( \{a, q_1, m, q_3, b\} \) from the simulated sample, compute the observed value of the test statistic by (5) or (7), and then compare it with the different critical values to determine whether the null hypothesis will be rejected. Furthermore, with 1,000,000 simulations, we report their type I error rates in the middle and bottom panels of Figure 4, respectively. Under scenario \( S_2 \), the tests with the exact and approximated critical values both perform well and control the type I error rates. While for the asymptotic test with the normal approximation in (6), it does not provide a good performance when \( n \) is small. In particular when \( n = 5 \), the type I error rate will be nearly zero so that the test will be extremely conservative. Under scenario \( S_3 \), given that the asymptotic test is not available, we thus report the type I error rates for the tests with the exact and approximated critical values only. From the bottom panel of Figure 4 we note that both tests control the type I error rates and they perform equally well for any fixed sample size.
Figure 4: The type I error rates for the proposed test statistics under three scenarios for $n$ up to 401. The solid orange triangles represent the test with the asymptotic critical values, the empty green points represent the test with the exact critical values, and the solid red points represent the test with the approximated critical values.
4.2 Statistical Power

In this section, we assess the ability of the three tests for detecting the skewness when the alternative distribution is skewed. For this purpose, we reconsider the four normal-related distributions in Section 2, Skew-normal$(0, 1, -10)$, Half-normal$(0, 1)$, Log-normal$(0, 1)$ and $0.3* N(-2, 1) + 0.7* N(2, 1)$, as the alternative distributions for all the tests under different scenarios. Then to numerically compute the statistical power, we follow the same procedure as in Section 4.1 except that the sample data are now generated from the alternative distributions rather than the standard normal distribution. In addition, since the asymptotic test is suboptimal and the other two tests perform nearly the same, we report the simulated power only for the tests with the approximated critical values in Figure 5 based on 1,000,000 simulations.

For the test under scenario $S_1$, we note that it is always very powerful, in particular for the three unimodal alternative distributions. This is mainly because the extreme order statistics, including the minimum and maximum values, are very sensitive to the tail behavior of the underlying distribution. In contrast, the intermediate order statistics, including the first and third quartiles, behave more stably and are less affected by the tail distributions (Gather and Tomkins, 1995). As a consequence, the test under scenario $S_2$ is often less powerful in detecting the skewness of data, as those reflected in the power curves for the three unimodal alternative distributions. But as shown in the bottom right panel of Figure 5, there are also exceptional cases. Specifically for the mixture-normal distribution, since the two tails are both normally shaped, the minimum and maximum values behave similarly so that the mid-range, $(a + b)/2$, is quite stable along with the sample size, and consequently, it diminishes the ability of detecting the skewness. On the other side, we note that the median is more close to the third quartile rather than to the first quartile, and so the test under scenario $S_2$ turns out to be more powerful than the test under scenario $S_1$ in the mixture-normal case. Finally for the test under scenario $S_3$, since it takes into account both the extreme and intermediate order statistics, it is not surprising that it always performs better than, or at least as well as, the other two tests in most settings.

To sum up, by virtue of the well-controlled type I error rates and the reasonable statistical power, we believe that our easy-to-implement tests with the approximated
critical values will have potential to be widely adopted for detecting the skewness away from normality with application to meta-analysis.

Figure 5: The statistical power of the proposed tests under three scenarios for $n$ up to 401. “1” represents the statistical power under scenarios $S_1$, “2” represents the statistical power under scenarios $S_2$, and “3” represents the statistical power under scenarios $S_3$. 
5 Real Data Analysis

To further illustrate the usefulness of our new tests, we revisit the motivating example in Section 2, where Nnoaham and Clarke (2008) studied the relationship between the low serum vitamin D levels and tuberculosis. Recall that there are six studies in their meta-analysis with continuous outcomes, where the summary statistics are reported in Table 2 together with the sample sizes. Specifically, for any study reported with the whole or part of the five-number summary, we first detect whether the data are skewed away from the normal distribution, and then follow the flow chart in Figure 1 as a guideline for the subsequent meta-analysis.

Table 3: The observed values of the test statistic $T_1$, the critical values for the corresponding sample sizes and the decisions of the test.

| Study            | Cases |                  |                  | Controls |                  |                  |
|------------------|-------|------------------|------------------|----------|------------------|------------------|
|                  | $T_1$ | Critical value   | Decision         | $T_1$    | Critical value   | Decision         |
| Davie's et al. (1985) | 0.618 | 0.319            | Reject           | 0.704    | 0.319            | Reject           |
| Grange et al. (1985) | 0.493 | 0.319            | Reject           | 0.451    | 0.325            | Reject           |
| Davie's et al. (1987) | 0.366 | 0.470            | Not reject       | 0.113    | 0.470            | Not reject       |

As pointed out in Section 2, the three studies reported with the sample median and the extreme values are likely be skewed. To detect the potential skewness, we hereby apply our new test under scenario $S_1$ to each study. By comparing the observed values of the test statistic $T_1$ with the approximated critical values at the significance level of 0.05, we report the hypothesis test results in Table 3 and note that the data from Davies et al. (1985) and Grange et al. (1985) are significantly skewed away from normality. Thus by the flow chart in Figure 1, we suggest to exclude these two studies from the subsequent meta-analysis for normal data. While for Davies et al. (1987), since it passes the skewness test, we apply Luo et al. (2018) to estimate the sample mean and Wan et al. (2014) to estimate the sample SD. In addition, for Sasidharan et al. (2002), we apply Wan et al. (2014) to estimate the missing SD so that this study can also be included in the meta-analysis. Finally, together with the remaining two studies, Davies et al. (1988) and Chan et al. (1994), that are reported with the sample mean and SD, we conduct the final meta-analysis with a total of four studies and report the forest plot in Figure 6. By taking the mean difference (MD) as the effect size, the fixed-effect model yields the overall effect size.
-15.9 with 95% confidence interval (CI) being $[-22.4, -9.4]$, and the random-effects model yields the overall effect size -18.1 with the 95% CI being $[-29.1, -7.1]$. Since neither CI includes zero, both models indicate that the low serum vitamin D levels are significantly associated with a higher risk of active tuberculosis.

| Study                  | Experimental | Control | Fixed effect model | Random effects model |
|------------------------|--------------|---------|--------------------|---------------------|
|                        | Total Mean   | SD      | Total Mean         | SD                  |
|                         |              |         |                    |                     |
| Davies et al. (1987)   | 15           | 44.3    | 15                 | 67.2                |
|                        | 20.4         | 24.9    |                    |                     |
| Davies et al. (1988)   | 51           | 69.5    | 51                 | 95.5                |
|                        | 24.5         | 39.2    |                    |                     |
| Chan et al. (1994)     | 24           | 46.5    | 24                 | 52.2                |
|                        | 18.5         | 15.8    |                    |                     |
| Sasidharan et al. (2002)| 35          | 26.8    | 16                 | 48.5                |
|                        | 17.0         | 33.9    |                    |                     |
| Weight (fixed)         | -22.9        | [-39.2; -6.6] | 15.8% | 21.7% |
| Weight (random)        | -26.0        | [-38.7; -13.3] | 26.1% | 26.7% |
| Heterogeneity: $I^2$ = 61%, $\tau^2$ = 75.3232, $p = 0.05$ | | | | |

Figure 6: The forest plot of the meta-analysis that excludes the two studies with significantly skewed data.

| Study                  | Experimental | Control | Fixed effect model | Random effects model |
|------------------------|--------------|---------|--------------------|---------------------|
|                        | Total Mean   | SD      | Total Mean         | SD                  |
|                         |              |         |                    |                     |
| Davies et al. (1985)   | 40           | 20.5    | 40                 | 36.0                |
|                        | 16.5         | 28.2    |                    |                     |
| Grange et al. (1985)   | 40           | 70.0    | 38                 | 73.1                |
|                        | 19.8         | 17.7    |                    |                     |
| Davies et al. (1987)   | 15           | 44.3    | 15                 | 67.2                |
|                        | 20.4         | 24.9    |                    |                     |
| Davies et al. (1988)   | 51           | 69.5    | 51                 | 95.5                |
|                        | 24.5         | 39.2    |                    |                     |
| Chan et al. (1994)     | 24           | 46.5    | 24                 | 52.2                |
|                        | 18.5         | 15.8    |                    |                     |
| Sasidharan et al. (2002)| 35          | 26.8    | 16                 | 48.5                |
|                        | 17.0         | 33.9    |                    |                     |
| Weight (fixed)         | -15.5        | [-25.6; -5.4] | 20.3% | 18.9% |
| Weight (random)        | -22.9        | [-39.2; -6.6] | 7.9%  | 12.8% |
| Heterogeneity: $I^2$ = 65%, $\tau^2$ = 62.2778, $p = 0.02$ | | | | |

Figure 7: The forest plot of the meta-analysis that includes the two studies with significantly skewed data.

Finally, to check what the skewness test brings in, we presume that the two excluded studies are also normally distributed and include them in the meta-analysis. Then by the transformation methods in Luo et al. (2018) and Wan et al. (2014), we have a total of six studies in Figure 7. By comparing the two forest plots, we note that, even though the meta-analytical results are not altered, the two skewed studies do yield a less significant result for both the fixed-effect and random-effects models. To conclude the study, if a
large proportion of studies in the meta-analysis are skewed, then whether to include them or not may result in very different conclusions so that it calls for more attention.

6 Conclusion

For clinical studies with continuous outcomes, the sample mean and standard deviation (SD) are routinely reported as the summary statistics when the data are normally distributed. While in some studies, however, researchers may report the whole or part of the five-number summary, mainly because the data generated from the specific studies are potentially skewed away from normality. For the studies with skewed data, if we include them in the classical meta-analysis for normal data, it may yield misleading or even wrong conclusions. In this paper, we proposed three new tests for detecting the skewness of data from the sample size and the five-number summary. For ease of implementation, we also provided the approximate formulas for the critical values at the significance level of 0.05. Simulation studies demonstrated that our new tests control the type I error rates well and also provide good statistical power for detecting the skewness of data away from normality. The new tests were further applied in a real data example to demonstrate their usefulness in meta-analysis and evidence-based practice. To conclude the paper, we summarized the three skewness tests together with the critical regions of size 0.05 in Table 4. We also provided an online calculator at http://www.math.hkbu.edu.hk/~tongt/papers/median2mean.html to help practitioners perform the skewness tests and follow the flow chart in Figure 1 to conduct the meta-analysis.

Table 4: The summary table of the skewness tests under three scenarios.

| Scenario | Test statistic | Critical region of size 0.05 |
|----------|---------------|--------------------------------|
| $S_1$    | $T_1 = \frac{a + b - 2m}{b - a}$ | $|T_1| > \frac{1.01 \ln(n+9) + 2.43}{n+1}$ |
| $S_2$    | $T_2 = \frac{q_1 + q_3 - 2m}{q_3 - q_1}$ | $|T_2| > \frac{2.66}{\sqrt{n}} - \frac{5.92}{n^2}$ |
| $S_3$    | $T_3 = \max \left\{ \frac{2.65 \ln(0.6n)}{\sqrt{n}} |T_1|, |T_2| \right\}$ | $T_3 > \frac{2.97}{\sqrt{n}} - \frac{39.1}{n^3}$ |
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Appendix A

Under scenario $S_1$, the level of skewness $\theta_1$ can be naturally estimated by $a + b - 2m$, which is probably also the only available estimator due to the limited data sources. Now to explore the sampling distributions of $a + b - 2m$ under both the null and alternative hypotheses, we consider $N(10, \sigma^2)$ as the null distribution and Skew-normal$(0, 1, -10)$ as the alternative distribution, where $\sigma^2 = 1 - 200/(101\pi) = 0.37$ is taken so that the two distributions have the same variance. Then for each setting, we draw a random sample of size 100 and compute the sample value of $a + b - 2m$. We repeat the procedure for 1,000,000 times and report the histograms of $a + b - 2m$ in Figure 8 for both distributions, where the vertical dashed lines represent the critical values at the significance level of 0.05.

![Histograms](image)

Figure 8: The histograms of $a + b - 2m$ for $N(10, 0.37)$ and for Skew-normal$(0, 1, -10)$, where $n = 100$. The vertical dashed lines represent the two critical values such that the null hypothesis is rejected if the observed $a + b - 2m$ is larger than 0.76 or less than -0.76.

For the normal distribution on the left panel, the sampling distribution of $a + b - 2m$ is symmetric about zero, which coincides with the analytical result that the level of skewness is zero when the underlying distribution is normal. While for the skew-normal distribution on the right panel, it is evident that most values of $a + b - 2m$ are smaller than zero with the average about -1.3. Further by the two sampling distributions, if we reject the null hypothesis when the observed value of $a + b - 2m$ is larger than 0.76 or less than -0.76,
then the type I error rate is well controlled at the significance level of 0.05, and meanwhile
the power of the test will be greater than 0.9. To conclude, we have shown by a simulation
study that a powerful test statistic can be built based on $a + b - 2m$ to test whether or
not the underlying distribution of the data is skewed away from the normal distribution.

**Appendix B**

**Lemma 1.** Let $Z_1, Z_2, \ldots, Z_n$ be a random sample from the standard normal distribution,
and $Z(1) \leq Z(2) \leq \cdots \leq Z(n)$ be the corresponding order statistics. We have

$$(Z(1), Z(2), \ldots, Z(n)) \overset{d}{=} (-Z(n), -Z(n-1), \ldots, -Z(1)),$$

where $\overset{d}{=}$ represents that two random vectors follow the same distribution. Thus it is
evident that

$$E(Z_{(i)}) = -E(Z_{(n-i+1)}), \quad 1 \leq i \leq n.$$ 

Specifically, by letting $n = 4Q + 1$ with $Q$ being a positive integer, it directly follows that

$E(Z_{(2Q+1)}) = 0$, $E(Z_{(Q+1)}) = -E(Z_{(3Q+1)})$ and $E(Z_{(1)}) = -E(Z_{(n)})$. 

**Theorem 1.** Under scenario $S_1 = \{a, m, b; n\}$, as $n \to \infty$ under the null hypothesis of
normality, we have

$$\sqrt{2 \ln(n) \xi(n)} \left(\frac{a + b - 2m}{b - a}\right) \overset{D}{\to} \text{Logistic}(0, 1),$$

where $\xi(n) = 2\Phi^{-1}[(n - 0.375)/(n + 0.25)]$. 

**Proof.** Under the null hypothesis of normality, it is evident that

$$\frac{a + b - 2m}{2\sigma} = \frac{Z_{(1)} + Z_{(n)}}{2} - Z_{(2Q+1)}.$$
According to Ferguson (1996), as \( n \to \infty \), we have

\[
\sqrt{2 \ln(n)} \left( \frac{Z(1) + Z(n)}{2} \right) \overset{D}{\to} \text{Logistic}(0, 0.5)
\]

and

\[
\sqrt{n} Z_{(2Q+1)} \overset{D}{\to} N \left( 0, \frac{\pi}{2} \right),
\]

where \( \overset{D}{\to} \) denotes the convergence in distribution. Thus under the null hypothesis, it is evident that

\[
\sqrt{2 \ln(n)} \left( \frac{a + b - 2m}{2\sigma} \right) = I_1 - I_2,
\]

where

\[
I_1 = \sqrt{2 \ln(n)} \left( \frac{Z(1) + Z(n)}{2} \right),
\]

\[
I_2 = \sqrt{2 \ln(n)} Z_{(2Q+1)}.
\]

As \( n \to \infty \), \( I_1 \overset{D}{\to} \text{Logistic}(0, 0.5) \) and \( I_2 \overset{P}{\to} 0 \), where \( \overset{P}{\to} \) denotes the convergence in probability. Then by Slutsky’s Theorem, as \( n \to \infty \), we conclude that

\[
\sqrt{2 \ln(n)} \left( \frac{a + b - 2m}{\sigma} \right) \overset{D}{\to} \text{Logistic}(0, 1).
\]  \( (8) \)

According to Wan et al. (2014), it is evident that \( \hat{\sigma}_1 = (b - a)/\xi(n) \) is a consistent estimator of the standard deviation \( \sigma \) for a normal distribution. With \( \sigma \) estimated by \( \hat{\sigma}_1 \) in \( (8) \) and noting that \( \xi(n) \) is the fixed value for any given \( n \), the final test statistic is derived as

\[
T_1 = \frac{a + b - 2m}{b - a}
\]

and again by Slutsky’s Theorem as \( n \to \infty \), \( \sqrt{2 \ln(n)} \xi(n)T_1 \overset{D}{\to} \text{Logistic}(0, 1) \).

**Theorem 2.** Under scenario \( S_1 = \{a, m, b; n\} \), the null distribution of the test statistic
\[ T_1 = \frac{(a + b - 2m)/(b - a)}{f_1(t_1) = \int \int_{D} \frac{n!}{(2Q-1)!!} \cdot \frac{v-u}{2} \cdot \phi(u)\phi(v)\phi \left( \frac{u+v}{2} - \frac{t_1(v-u)}{2} \right) \times \left[ \Phi \left( \frac{u+v}{2} - \frac{t_1(v-u)}{2} \right) - \Phi(u) \right]^{2Q-1} \times \left[ \Phi(v) - \Phi \left( \frac{u+v}{2} - \frac{t_1(v-u)}{2} \right) \right]^{2Q-1} \, du dv, \]

(9)

where \( D \) is the integral area that satisfies \( u \leq v \). Furthermore, the null distribution of \( T_1 \) is symmetric about zero.

**Proof.** Denote \( z_1 = Z_{(1)}, z_m = Z_{(2Q+1)} \) and \( z_n = Z_{(n)} \). Recall that for \((z_1, z_m, z_n)\), we have the joint distribution as

\[
g_1(z_1, z_m, z_n) = \frac{n!}{(2Q-1)!!} \phi(z_1)\phi(z_m)\phi(z_n) [\Phi(z_m) - \Phi(z_1)]^{2Q-1} [\Phi(z_n) - \Phi(z_m)]^{2Q-1}.
\]

It is evident that \((z_1, z_m, z_n) \rightarrow (u, t_1, v)\) is a one-to-one mapping, where \( u = z_1, t_1 = (z_1 + z_n - 2z_m)/(z_n - z_1) \) and \( v = z_n \), or equivalently, \( z_1 = u, z_m = (u + v)/2 - t_1(v-u)/2 \) and \( z_n = v \). Thus we have the determinant of the Jacobian matrix as

\[
\begin{vmatrix}
1 & 0 & 0 \\
1 + t_1 & u - v & 0 \\
\frac{2}{2} & \frac{2}{2} & \frac{1}{2} \\
\end{vmatrix} = \frac{v - u}{2}.
\]

With the Jacobian transformation, we derive the joint distribution of \((u, t_1, v)\) as

\[
g^*_1(u, t_1, v) = \frac{n!}{(2Q-1)!!} \cdot \frac{v-u}{2} \cdot \phi(u)\phi(v)\phi \left( \frac{u+v}{2} - \frac{t_1(v-u)}{2} \right) \times \left[ \Phi \left( \frac{u+v}{2} - \frac{t_1(v-u)}{2} \right) - \Phi(u) \right]^{2Q-1} \times \left[ \Phi(v) - \Phi \left( \frac{u+v}{2} - \frac{t_1(v-u)}{2} \right) \right]^{2Q-1}.
\]

Further by taking the integrals with respect to \( u \) and \( v \), we achieve the sampling distribution of \( T_1 \) in (9) under the null hypothesis.
In addition, by Lemma 1, we have
\[(Z_1, Z_{2Q+1}, Z_n) \overset{d}{=} (-Z_n, -Z_{2Q+1}, -Z_1).\]

Thus under the null hypothesis, it is evident that
\[T_1 = \frac{Z_1 + Z_n - 2Z_{2Q+1}}{Z_n - Z_1} \quad \text{and} \quad -T_1 = \frac{(-Z_n) + (-Z_1) - 2(-Z_{2Q+1})}{(-Z_1) - (-Z_n)},\]
and therefore the null distribution of \(T_1\) is symmetric about zero.

**Theorem 3.** Under scenario \(S_2 = \{q_1, m, q_3; n\}\), as \(n \to \infty\) under the null hypothesis of normality, we have
\[0.74 \sqrt{n} \left( \frac{q_1 + q_3 - 2m}{q_3 - q_1} \right) \overset{d}{\to} N(0, 1).\]

**Proof.** Under the null hypothesis of normality, it is evident that
\[
\frac{q_1 + q_3 - 2m}{\sigma} = Z_{Q+1} + Z_{3Q+1} - 2Z_{2Q+1}.
\]
According to Ferguson (1996), as \(n \to \infty\), we have that \((Z_{Q+1}, Z_{2Q+1}, Z_{3Q+1})\) follows asymptotically a tri-variate normal distribution with mean vector \((\Phi^{-1}(0.25), 0, \Phi^{-1}(0.75))\) and covariance matrix \(\Sigma\), where
\[
\Sigma = \frac{1}{n} \begin{pmatrix}
1.86 & 0.99 & 0.62 \\
0.99 & \pi/2 & 0.99 \\
0.62 & 0.99 & 1.86
\end{pmatrix}.
\]
Then by the Delta method, as \(n \to \infty\), we have
\[Z_{Q+1} + Z_{3Q+1} - 2Z_{2Q+1} = (1, -2, 1) \begin{pmatrix} Z_{Q+1} \\ Z_{2Q+1} \\ Z_{3Q+1} \end{pmatrix} \overset{d}{\to} N(0, 3.32/n).\]
Therefore, we achieve the asymptotic normality under the null hypothesis that

\[ 0.55\sqrt{n} \left( \frac{q_1 + q_3 - 2m}{\sigma} \right) \xrightarrow{D} N(0, 1). \]  

(10)

According to Wan et al. (2014), it is evident that \( \hat{\sigma}^2 = \frac{q_3 - q_1}{\eta(n)^2} \) is a consistent estimator of the standard deviation \( \sigma \) for a normal distribution, where \( \eta(n) = 2\Phi^{-1}((0.75n - 0.125)/(n + 0.25)) \). Further by Theorem 1 in Shi et al. (2020), as \( n \to \infty \), \( \eta(n) = 2\Phi^{-1}(0.75) = 1.35 \). With \( \sigma \) estimated by \( \hat{\sigma} \) in (10) and noting \( \eta(n) \) is a constant, we propose the test statistic as

\[ T_2 = \frac{q_1 + q_3 - 2m}{q_3 - q_1} \]

and by Slutsky’s Theorem as \( n \to \infty \), \( 0.74\sqrt{n}T_2 \xrightarrow{D} N(0, 1) \).

**Theorem 4.** Under scenario \( S_2 = \{q_1, m, q_3; n\} \), the null distribution of the test statistic \( T_2 = (q_1 + q_3 - 2m)/(q_3 - q_1) \) is

\[ f_2(t_2) = \int_{\mathcal{D}} \frac{n!}{[Q!(Q-1)!]^2} \phi(x) \phi \left( \frac{x + y - \frac{t_2(y - x)}{2}}{2} \right) \phi(y) \Phi(x)^Q [1 - \Phi(y)]^Q \]

\[ \cdot \left[ \Phi \left( \frac{x + y - \frac{t_2(y - x)}{2}}{2} \right) - \Phi(x) \right]^{Q-1} \left[ \Phi(y) - \Phi \left( \frac{x + y - \frac{t_2(y - x)}{2}}{2} \right) \right]^{Q-1} \, dx \, dy, \]

(11)

where \( \mathcal{D} \) is the integral area that satisfies \( x \leq y \). Furthermore, the null distribution of \( T_2 \) is symmetric about zero.

**Proof.** Denote \( z_{q_1} = Z_{(Q+1)} \), \( z_m = Z_{(2Q+1)} \) and \( z_{q_3} = Z_{(3Q+1)} \). Recall that for \( (z_{q_1}, z_m, z_{q_3}) \),
we have the joint distribution as

\[
g_2(z_{q_1}, z_m, z_{q_3}) = \frac{n!}{Q!(Q - 1)!(Q - 1)!} \phi(z_{q_1}) \phi(z_m) \phi(z_{q_3}) [\Phi(z_{q_1})]^Q [\Phi(z_m) - \Phi(z_{q_1})]^{Q-1} \\
\times [\Phi(z_{q_3}) - \Phi(z_m)]^{Q-1} [1 - \Phi(z_{q_3})]^Q.
\]

It is evident that \((z_{q_1}, z_m, z_{q_3}) \rightarrow (x, t_2, y)\) is a one-to-one mapping, where \(x = z_{q_1}, t_2 = (z_{q_1} + q_3 - 2z_m)/(z_{q_3} - z_{q_1})\) and \(y = z_{q_3}\), or equivalently, \(z_{q_1} = x, z_m = (x + y)/2 - t_2(y - x)/2\) and \(z_{q_3} = y\). Thus we have the determinant of the Jacobian matrix as

\[
\begin{vmatrix}
1 & 0 & 0 \\
1 + t_2 & x - y & 0 \\
2 & 0 & 1 + t_2
\end{vmatrix} = \frac{y - x}{2}.
\]

With the Jacobian transformation, we derive the joint distribution of \((x, t_2, y)\) as

\[
g_2^*(x, t_2, y) = \frac{n!}{[Q!(Q - 1)!]^2} \phi(x) \phi \left( \frac{x + y}{2} - \frac{t_2(y - x)}{2} \right) \phi(y) [\Phi(x)]^Q [1 - \Phi(y)]^Q \\
\times \left[ \Phi \left( \frac{x + y}{2} - \frac{t_2(y - x)}{2} \right) - \Phi(x) \right]^{Q-1} \left[ \Phi(y) - \Phi \left( \frac{x + y}{2} - \frac{t_2(y - x)}{2} \right) \right]^{Q-1}.
\]

Further by taking the integrals with respect to \(x\) and \(y\), we achieve the sampling distribution of \(T_2\) in (11) under the null hypothesis. Similar to the proof in Theorem 2, we can prove that the null distribution of \(T_2\) is symmetric about zero.

**Theorem 5.** Under scenario \(S_3 = \{a, q_1, m, q_3, b, n\}\), the test statistic is derived as

\[
T_3 = \max \left\{ \frac{2.65 \ln(0.6n)}{\sqrt{n}} \left| \frac{a + b - 2m}{b - a} \right|, \left| \frac{q_1 + q_3 - 2m}{q_3 - q_1} \right| \right\}.
\]

**Proof.** Under scenario \(S_3\), by taking advantages of both extreme and intermediate order
statistics, we consider to detect the skewness of data with

\[ W_3 = \max \left\{ \left| \frac{a + b - 2m}{\text{SE}(a + b - 2m)} \right|, \left| \frac{q_1 + q_3 - 2m}{\text{SE}(q_1 + q_3 - 2m)} \right| \right\}. \]

Recall that in Section 3.1.1, we have derived \( \text{SE}(a + b - 2m) = \sigma \delta_1(n) \), where \( \delta_1(n) = \text{SE}(Z(1) + Z(n) - 2Z(2Q+1)) \) and \( \sigma \) is estimated with \( (b - a)/\xi(n) \). Similarly, we have \( \text{SE}(q_1 + q_3 - 2m) = \sigma \delta_2(n) \), where \( \delta_2(n) = \text{SE}(Z(Q+1) + Z(3Q+1) - 2Z(2Q+1)) \), and \( \sigma \) can be estimated with \( (q_3 - q_1)/\eta(n) \) in Wan et al. (2014), where \( \eta(n) = 2\Phi^{-1}[(0.75n - 0.125)/(n + 0.25)] \).

Then, the test statistic is specified as

\[ \max \left\{ \frac{\xi(n)}{\delta_1(n)} |T_1|, \frac{\eta(n)}{\delta_2(n)} |T_2| \right\}, \]

where \( T_1 = (a + b - 2m)/(b - a) \) and \( T_2 = (q_1 + q_3 - 2m)/(q_3 - q_1) \) are the test statistics under scenarios \( S_1 \) and \( S_2 \).

To simplify the presentation, we combine \( \eta(n)/\delta_2(n) \) with \( \xi(n)/\delta_1(n) \) in the first term so that only one coefficient \( k(n) = [\xi(n)/\delta_1(n)]/[\eta(n)/\delta_2(n)] \) is needed, and meanwhile \( k(n)|T_1| \) and \( |T_2| \) are still comparable in scale. Further noting that \( k(n) \) is difficult to compute since \( \delta_1(n) \) and \( \delta_2(n) \) involved do not have the explicit forms, the test statistic may not be readily accessible to practitioners. By following the asymptotic form of \( k(n) \), we have provided the approximate formula as \( k(n) \approx 2.65 \ln(0.6n)/\sqrt{n} \) for practical use.

Finally, test statistic is yielded as

\[ T_3 = \max \left\{ \frac{2.65 \ln(0.6n)}{\sqrt{n}} \left| \frac{a + b - 2m}{b - a} \right|, \left| \frac{q_1 + q_3 - 2m}{q_3 - q_1} \right| \right\}. \]
Appendix C

Table 5: The numerical values of $c_{1.025}$ for $1 \leq Q \leq 100$, where $n = 4Q + 1$.

| $Q$ | $c_{1.025}$ | $Q$ | $c_{1.025}$ | $Q$ | $c_{1.025}$ | $Q$ | $c_{1.025}$ |
|-----|-------------|-----|-------------|-----|-------------|-----|-------------|
| 1   | 0.7792      | 21  | 0.2505      | 41  | 0.2094      | 61  | 0.1920      |
| 2   | 0.5706      | 22  | 0.2464      | 42  | 0.2087      | 62  | 0.1905      |
| 3   | 0.4964      | 23  | 0.2433      | 43  | 0.2072      | 63  | 0.1903      |
| 4   | 0.4413      | 24  | 0.2402      | 44  | 0.2067      | 64  | 0.1898      |
| 5   | 0.4032      | 25  | 0.2375      | 45  | 0.2051      | 65  | 0.1892      |
| 6   | 0.3763      | 26  | 0.2352      | 46  | 0.2042      | 66  | 0.1886      |
| 7   | 0.3554      | 27  | 0.2332      | 47  | 0.2031      | 67  | 0.1878      |
| 8   | 0.3395      | 28  | 0.2315      | 48  | 0.2024      | 68  | 0.1877      |
| 9   | 0.3253      | 29  | 0.2286      | 49  | 0.2013      | 69  | 0.1867      |
| 10  | 0.3132      | 30  | 0.2277      | 50  | 0.2000      | 70  | 0.1864      |
| 11  | 0.3045      | 31  | 0.2243      | 51  | 0.1990      | 71  | 0.1858      |
| 12  | 0.2956      | 32  | 0.2238      | 52  | 0.1989      | 72  | 0.1850      |
| 13  | 0.2884      | 33  | 0.2219      | 53  | 0.1979      | 73  | 0.1848      |
| 14  | 0.2812      | 34  | 0.2203      | 54  | 0.1974      | 74  | 0.1840      |
| 15  | 0.2755      | 35  | 0.2183      | 55  | 0.1964      | 75  | 0.1837      |
| 16  | 0.2708      | 36  | 0.2172      | 56  | 0.1949      | 76  | 0.1836      |
| 17  | 0.2660      | 37  | 0.2151      | 57  | 0.1946      | 77  | 0.1823      |
| 18  | 0.2613      | 38  | 0.2135      | 58  | 0.1938      | 78  | 0.1819      |
| 19  | 0.2564      | 39  | 0.2128      | 59  | 0.1928      | 79  | 0.1818      |
| 20  | 0.2535      | 40  | 0.2111      | 60  | 0.1922      | 80  | 0.1811      |

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Table 6: The numerical values of $c_{2,0.025}$ for $1 \leq Q \leq 100$, where $n = 4Q + 1$.

| $Q$ | $c_{2,0.025}$ | $Q$ | $c_{2,0.025}$ | $Q$ | $c_{2,0.025}$ | $Q$ | $c_{2,0.025}$ |
|-----|--------------|-----|--------------|-----|--------------|-----|--------------|
| 1   | 0.9463       | 21  | 0.2861      | 41  | 0.2067       | 61  | 0.1692       |
| 2   | 0.8000       | 22  | 0.2809      | 42  | 0.2034       | 62  | 0.1681       |
| 3   | 0.6913       | 23  | 0.2748      | 43  | 0.2019       | 63  | 0.1667       |
| 4   | 0.6163       | 24  | 0.2685      | 44  | 0.1993       | 64  | 0.1653       |
| 5   | 0.5594       | 25  | 0.2633      | 45  | 0.1975       | 65  | 0.1641       |
| 6   | 0.5177       | 26  | 0.2588      | 46  | 0.1954       | 66  | 0.1627       |
| 7   | 0.4819       | 27  | 0.2538      | 47  | 0.1936       | 67  | 0.1614       |
| 8   | 0.4534       | 28  | 0.2494      | 48  | 0.1914       | 68  | 0.1602       |
| 9   | 0.4297       | 29  | 0.2447      | 49  | 0.1897       | 69  | 0.1593       |
| 10  | 0.4084       | 30  | 0.2403      | 50  | 0.1879       | 70  | 0.1583       |
| 11  | 0.3903       | 31  | 0.2361      | 51  | 0.1854       | 71  | 0.1570       |
| 12  | 0.3744       | 32  | 0.2339      | 52  | 0.1831       | 72  | 0.1561       |
| 13  | 0.3608       | 33  | 0.2298      | 53  | 0.1823       | 73  | 0.1551       |
| 14  | 0.3486       | 34  | 0.2267      | 54  | 0.1804       | 74  | 0.1538       |
| 15  | 0.3372       | 35  | 0.2233      | 55  | 0.1785       | 75  | 0.1527       |
| 16  | 0.3266       | 36  | 0.2204      | 56  | 0.1776       | 76  | 0.1518       |
| 17  | 0.3179       | 37  | 0.2176      | 57  | 0.1757       | 77  | 0.1506       |
| 18  | 0.3085       | 38  | 0.2148      | 58  | 0.1749       | 78  | 0.1496       |
| 19  | 0.2999       | 39  | 0.2112      | 59  | 0.1721       | 79  | 0.1486       |
| 20  | 0.2931       | 40  | 0.2080      | 60  | 0.1718       | 80  | 0.1479       |
Table 7: The numerical values of $c_{3,0.05}$ for $1 \leq Q \leq 100$, where $n = 4Q + 1$.

| $Q$ | $c_{3,0.05}$ | $Q$ | $c_{3,0.05}$ | $Q$ | $c_{3,0.05}$ | $Q$ | $c_{3,0.05}$ |
|-----|--------------|-----|--------------|-----|--------------|-----|--------------|
| 1   | 1.0129       | 21  | 0.3214       | 41  | 0.2305       | 61  | 0.1885       | 81  | 0.1635       |
| 2   | 0.9062       | 22  | 0.3139       | 42  | 0.2271       | 62  | 0.1871       | 82  | 0.1626       |
| 3   | 0.7929       | 23  | 0.3067       | 43  | 0.2247       | 63  | 0.1856       | 83  | 0.1617       |
| 4   | 0.7060       | 24  | 0.3004       | 44  | 0.2223       | 64  | 0.1840       | 84  | 0.1607       |
| 5   | 0.6416       | 25  | 0.2948       | 45  | 0.2193       | 65  | 0.1827       | 85  | 0.1600       |
| 6   | 0.5898       | 26  | 0.2885       | 46  | 0.2173       | 66  | 0.1813       | 86  | 0.1587       |
| 7   | 0.5490       | 27  | 0.2831       | 47  | 0.2149       | 67  | 0.1802       | 87  | 0.1579       |
| 8   | 0.5151       | 28  | 0.2781       | 48  | 0.2129       | 68  | 0.1786       | 88  | 0.1570       |
| 9   | 0.4870       | 29  | 0.2738       | 49  | 0.2104       | 69  | 0.1775       | 89  | 0.1561       |
| 10  | 0.4630       | 30  | 0.2687       | 50  | 0.2082       | 70  | 0.1762       | 90  | 0.1556       |
| 11  | 0.4419       | 31  | 0.2645       | 51  | 0.2065       | 71  | 0.1747       | 91  | 0.1546       |
| 12  | 0.4229       | 32  | 0.2604       | 52  | 0.2043       | 72  | 0.1734       | 92  | 0.1537       |
| 13  | 0.4071       | 33  | 0.2564       | 53  | 0.2024       | 73  | 0.1724       | 93  | 0.1528       |
| 14  | 0.3929       | 34  | 0.2523       | 54  | 0.2004       | 74  | 0.1713       | 94  | 0.1522       |
| 15  | 0.3797       | 35  | 0.2489       | 55  | 0.1986       | 75  | 0.1700       | 95  | 0.1512       |
| 16  | 0.3675       | 36  | 0.2456       | 56  | 0.1971       | 76  | 0.1689       | 96  | 0.1505       |
| 17  | 0.3569       | 37  | 0.2419       | 57  | 0.1953       | 77  | 0.1679       | 97  | 0.1497       |
| 18  | 0.3473       | 38  | 0.2393       | 58  | 0.1933       | 78  | 0.1669       | 98  | 0.1489       |
| 19  | 0.3380       | 39  | 0.2359       | 59  | 0.1920       | 79  | 0.1657       | 99  | 0.1479       |
| 20  | 0.3290       | 40  | 0.2330       | 60  | 0.1902       | 80  | 0.1646       | 100 | 0.1472       |