BSE-PROPERTY FOR SOME CERTAIN SEGAL AND BANACH ALGEBRAS

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ABSTRACT. For a commutative semi-simple Banach algebra $A$ which is an ideal in its second dual we give a necessary and sufficient condition for an essential abstract Segal algebra in $A$ to be a BSE-algebra. We show that a large class of abstract Segal algebras in the Fourier algebra $A(G)$ of a locally compact group $G$ are BSE-algebra if and only if they have bounded weak approximate identities. Also, in the case that $G$ is discrete we show that $A_{cb}(G)$ is a BSE-algebra if and only if $G$ is weakly amenable. We study the BSE-property of some certain Segal algebras which introduced recently by J. Inoue and S.-E. Takahasi which implemented by local functions. Finally we give a similar construction for the group algebra which implemented by a measurable and sub-multiplicative function.

1. Introduction and Preliminaries

Let $G$ be a locally compact abelian group. A subspace $S$ of $L^1(G)$ is called a (Reiter) Segal algebra if it satisfies the following conditions:

1. $S$ is dense in $L^1(G)$.
2. $S$ is a Banach space under some norm $\| \cdot \|_S$ such that $\|f\|_1 \leq \|f\|_S$ for each $f \in S$.
3. $L_y f$ is in $S$ and $\|f\|_S = \|L_y f\|_S$ for all $f \in S$ and $y \in G$ where $L_y f(x) = f(y^{-1}x)$.
4. For all $f \in S$, the mapping $y \mapsto L_y f$ is continuous.

In [1], J. T. Burnham with changing $L^1(G)$ by an arbitrary Banach algebra $A$, gave a generalization of Segal algebras and introduced the notion of an abstract Segal algebra.

It is well-known that $L^1(G)$ is a commutative semi-simple regular Banach algebra with a bounded approximate identity with compact support; see [14].

Recently, J. Inoue and S.-E. Takahasi in [10] investigated abstract Segal algebras in a non-unital commutative semi-simple regular Banach algebra $A$ such that $A$ has a bounded approximate identity in $A_c$ where

$$A_c = \{a \in A : \hat{a} \text{ has compact support}\},$$

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and \( \hat{a} \) denotes the Gel’fand transform of \( a \). Indeed, they gave the following definition of a Segal algebra in \( A \).

**Definition 1.1.** An ideal \( S \) in \( A \) is called a Segal algebra in \( A \) if it satisfies the following properties:

(1) \( S \) is dense in \( A \).

(2) \( S \) is a Banach space under some norm \( \| \cdot \|_S \) such that \( \|a\|_A \leq \|a\|_S \) for each \( a \in S \).

(3) \( \|ax\|_S \leq \|a\|_A \|x\|_S \) for each \( a \in A \) and \( x \in S \).

(4) \( S \) has approximate units.

Clearly, an abstract Segal algebra in \( A \) (in the sense of Burnham) is a Segal algebra in \( A \) if and only if it possesses approximate units.

A commutative Banach algebra \( A \) is without order if for \( a \in A \), the condition \( aA = \{0\} \) implies \( a = 0 \) or equivalently \( A \) does not have any non-zero annihilator. For example if \( A \) has an approximate identity, then it is without order. A linear operator \( T \) on \( A \) is called a multiplier if it satisfies \( aT(b) = T(a)b \) for all \( a, b \in A \).

Suppose that \( \mathcal{M}(A) \) denotes the space of all multipliers of the Banach algebra \( A \). If \( \Delta(A) \) denotes the space of all characters of \( A \); that is, non-zero homomorphisms from \( A \) into \( \mathbb{C} \), then for each \( T \in \mathcal{M}(A) \), there exists a unique bounded continuous function \( \widehat{T} \) on \( \Delta(A) \) such that \( \widehat{T}(\phi)(\phi) = \widehat{T}(\phi)(\hat{a})(\phi) \) for all \( a \in A \) and \( \phi \in \Delta(A) \); see [14, Proposition 2.2.16]. Let \( \hat{M}(A) \) denote the space of all \( \hat{T} \) corresponding to \( T \in \mathcal{M}(A) \).

A bounded continuous function \( \sigma \) on \( \Delta(A) \) is called a BSE-function if there exists a constant \( C > 0 \) such that for each \( \phi_1, ..., \phi_n \in \Delta(A) \) and complex numbers \( c_1, ..., c_n \), the inequality

\[
\left| \sum_{i=1}^{n} c_i \sigma(\phi_i) \right| \leq C \left\| \sum_{i=1}^{n} c_i \phi_i \right\|_{A^*}
\]

holds. Let \( C_{\text{BSE}}(\Delta(A)) \) be the set of all BSE-functions.

A without order commutative Banach algebra \( A \) is called a BSE-algebra if \( C_{\text{BSE}}(\Delta(A)) = \hat{M}(A) \).

The theory of BSE-algebras for the first time introduced and investigated by Takahasi and Hatori; see [20] and two other notable works [15, 13]. In [13], the authors answered to a question raised in [20]. Examples of BSE-algebras are the group algebra \( L^1(G) \) of a locally compact abelian group \( G \), the Fourier algebra \( A(G) \) of a locally compact amenable group \( G \), all commutative \( C^* \)-algebras, the disk algebra, and the Hardy algebra on the open unit disk.
A net \( \{a_\alpha\} \) in \( A \) is called a bounded weak approximate identity (BWAI) for \( A \) if \( \{a_\alpha\} \) is bounded in \( A \) and
\[
\lim_{\alpha} \phi(a_\alpha a) = \phi(a) \quad (\phi \in \Delta(A), \ a \in A),
\]
or equivalently, \( \lim_{\alpha} \phi(a_\alpha) = 1 \) for each \( \phi \in \Delta(A) \). Clearly, each BAI of \( A \) is a BWAI and the converse is not valid in general; see [11] and [16]. Note that bounded weak approximate identities are important to decide whether a commutative Banach algebra is a BSE-algebra or not. For example, it was shown in [9] that a Segal algebra \( S(G) \) on a locally compact abelian group \( G \) is a BSE-algebra if and only if it has a bounded weak approximate identity.

For undefined concepts and notations appearing in the sequel, one can consult [2, 14].

The outline of the next sections is as follows:

In §2, for a commutative semi-simple Banach algebra \( A \) which is an ideal in its second dual we give a necessary and sufficient conditions for an essential abstract Segal algebra in \( A \) to be a BSE-algebra. We show that a large class of abstract Segal algebras in the Fourier algebra \( A(G) \) of a locally compact group \( G \) are BSE-algebra if and only if they have bounded weak approximate identities. Also, for discrete groups \( G \), we show that \( A_{cb}(G) \) is a BSE-algebra if and only if \( G \) is weakly amenable.

In §3, we study the BSE-property of the Segal algebra \( A_{\tau(n)} \) in \( A \) which introduced by Inoue and Takahasi and in the case that \( (A, \| \cdot \|_X) \) is a BSE-algebra, we show that \( A_{\tau(n)} \) is a BSE-algebra if and only if \( \tau \) is bounded where \( \tau : X \rightarrow C \) is a certain continuous function. Also, we compare the BSE-property between \( A \) and \( A_{\tau(n)} \). In §4, motivated by the definition of \( A_{\tau(n)} \), for an arbitrary locally compact (abelian) group \( G \), and a measurable sub-multiplicative function \( \tau : G \rightarrow C^\times \), we define the Banach algebra \( L^1(G)_{\tau(n)} \). Then we investigate the BSE-property of this algebra.

## 2. BSE-Abstract Segal Algebras

Recall that a Banach algebra \( B \) is an abstract Segal algebra of a Banach algebra \( A \) if

1. \( B \) is a dense left ideal in \( A \),
2. there exists \( M > 0 \) such that \( \|b\|_A \leq M\|b\|_B \) for each \( b \in B \)
3. there exists \( C > 0 \) such that \( \|ab\|_B \leq C\|a\|_A\|b\|_B \) for each \( a, b \in B \).

Endow \( \Delta(A) \) and \( \Delta(B) \) with the Gel’fand topology, the map \( \varphi \mapsto \varphi|_B \) is a homeomorphism from \( \Delta(A) \) onto \( \Delta(B) \); see [1, Theorem 2.1].
Let $B$ be an abstract Segal algebra with respect to $A$. We say that $B$ is essential if $\langle AB \rangle$ is $\| \cdot \|_B$-dense in $B$, where $\langle AB \rangle$ denotes the linear span of $AB = \{ab : a \in A, b \in B\}$.

**Theorem 2.1.** Let $A$ be a semi-simple commutative Banach algebra which is an ideal in its second dual $A^{**}$. Suppose that $B$ is an essential abstract Segal algebra in $A$. Then the following statements are equivalent.

(i) $B$ is a BSE-algebra.

(ii) $B = A$ and $A$ is a BSE-algebra.

**Proof.** Suppose that $B$ is a BSE-algebra. Then by [20, Corollary 5], $B$ has a BWAI, say $(g_\gamma)_\gamma$. It is clear that $(g_\gamma)_\gamma$ is also a BWAI for $A$. So, by [15, Theorem 3.1] $A$ is a BSE-algebra and has a bounded approximate identity, say $(e_\alpha)_\alpha$. Since $B$ is essential, $(e_\alpha)_\alpha$ is also a bounded approximate identity for $B$ in $A$. In fact, for each $b \in B$ and $\varepsilon > 0$, there is $c = \sum_{i=1}^n a_ib_i$ with $1 \leq i \leq n$, $a_i \in A$ and $b_i \in B$ such that $\|b - c\|_B \leq \varepsilon$. Thus for each $\alpha$ we have

$$\|e_\alpha b - b\|_B \leq (1 + K)\varepsilon + C\sum_{i=1}^n \|e_\alpha a_i - a_i\|_A \|b_i\|_B,$$

where $K = \sup \|e_\alpha\|_A$. This shows that $\|e_\alpha b - b\|_B \to 0$ for all $b \in B$. Thus, $B = BA$ by Cohen’s factorization theorem. Now, let $b \in B$ and $(b_n)$ be a bounded sequence in $B$. Then $b = ca$ for some $c \in B$ and $a \in A$. Since $A$ is an ideal in its second dual, it follows that the operator $\rho_a : A \to A$ defined by $\rho_a(a') = a a'$, $(a' \in A)$ is weakly compact; see [3, Lemma 3]. Therefore, if $(b_n)$ is bounded sequence in $A$, there exists a subsequence $(b_{n_k})$ of $(b_n)$ such that $(\rho_a(b_{n_k}))$ is convergent to some $a'$ in the weak topology of $A$. Now, we observe that $f \cdot c \in A^*$ for all $f \in B^*$, where $(f \cdot c)(a) = f(ca)$ for all $a \in A$. This shows that the sequence $(\rho_{ca}(b_{n_k}))$ is convergent to $ca'$ in the weak topology of $B$. Therefore, the operator $\rho_b : B \to B$ is weakly compact which implies that $B$ is an ideal in its second dual. Since $B$ is semi-simple, [15, Theorem 3.1] implies that $B$ has a bounded approximate identity. Thus $A = B$ by [1, Theorem 1.2]. That (ii) implies (i) is trivial. $\square$

**Example 2.2.** Let $G$ be a locally compact group and let $A(G)$ be the Fourier algebra of $G$. It was shown in [15, Theorem 5.1] that $A(G)$ is a BSE-algebra if and only if $G$ is amenable. Moreover, $A(G)$ is an ideal in its second dual if and only if $G$ is discrete; see [5, Lemma 3.3]. Therefore, by Theorem 2.1 if $G$ is discrete, then each essential abstract Segal algebra $SA(G)$ in $A(G)$ is a BSE-algebra if and only if $SA(G) = A(G)$ and $G$ is amenable.
Let $G$ be a locally compact group and let $L^r(G)$ be the Lebesgue $L^r$-space of $G$, where $1 \leq r < \infty$. Then

$$SA^r(G) := L^r(G) \cap A(G)$$

with the norm $|||f||| = \|f\|_r + \|f\|_{A(G)}$ and the pointwise product is an abstract Segal algebra in $A(G)$.

**Corollary 2.3.** Let $G$ be a discrete group and let $1 \leq r \leq 2$. Then $SA^r(G)$ is a BSE-algebra if and only if $G$ is finite.

**Proof.** If $G$ is finite, then $SA^r(G) = A(G)$. So, the result follows from Theorem 2.1.

For the converse, first note that $SA^r(G) = l^r(G)$ and the norms $\| \cdot \|_r$ and $||| \cdot |||$ on $SA^r(G)$ are equivalent by the open mapping theorem. In fact, $l^2(G) \subseteq \delta_e \ast l^2(G) \subseteq A(G)$, where $\delta_e$ is the point mass at the identity element $e$ of $G$. So, if $1 \leq r \leq 2$, then $l^r(G) \subseteq l^2(G)$ and

$$l^r(G) = l^r(G) \cap l^2(G) \subseteq SA^r(G) \subseteq l^r(G).$$

Moreover, it is clear that $l^r(G)$ has an approximate identity and consequently it is an essential abstract Segal algebra in $A(G)$. Therefore, if $SA^r(G) = l^r(G)$ is a BSE-algebra, then $A(G) = l^r(G)$ by Example 2.2. Thus, $A(G) = l^2(G)$, is a reflexive predual of a $W^*$-algebra. This implies, as is known, that $A(G)$ is finite dimensional; see [18]. Thus $G$ is finite, which completes the proof. \[\square\]

For a locally compact group $G$, we recall that $A(G)$ is always an ideal in the Fourier-Stieltjes algebra $B(G)$ and note that $M(A(G)) = B(G)$ when $G$ is amenable. The spectrum of $A(G)$ can be canonically identified with $G$. More precisely, the map $x \mapsto \varphi_x$ where $\varphi_x(u) = u(x)$ for all $u \in A(G)$ is a homeomorphism from $G$ onto $\Delta(A(G))$.

**Theorem 2.4.** Let $G$ be a locally compact group and let $SA(G)$ be an abstract Segal algebra in $A(G)$ such that $B(G) \subseteq M(SA(G))$, i.e., for each $u \in B(G)$, we have $uSA(G) \subseteq SA(G)$. Then $SA(G)$ is a BSE-algebra if and only if $SA(G)$ has a BWAI.

**Proof.** Clearly if $SA(G)$ is a BSE-algebra, then it has a BWAI.

Conversely, suppose that $SA(G)$ has a BWAI, say $(e_\gamma)_\gamma$. Then $M(SA(G)) \subseteq \Delta(SA(G))$ by [20, Corollary 5]. Moreover, it is clear that $(e_\gamma)_\gamma$ is also a BWAI for $A(G)$. Consequently, we conclude that $G$ is amenable by [15, Theorem 5.1]. Now, we need to show the reverse inclusion. Since $SA(G)$ is an abstract
Segal algebra in $A(G)$, there exists $M > 0$ such that $\|u\|_{A(G)} \leq M\|u\|_{SA(G)}$ for all $u \in SA(G)$. Thus, for any $x_1, ..., x_n \in G$ and $c_1, ..., c_n \in \mathbb{C}$,

$$\left\| \sum_{j=1}^{n} c_j \varphi_{x_j} \right\|_{SA(G)^*} \leq M \left\| \sum_{j=1}^{n} c_j \varphi_{x_j} \right\|_{A(G)^*}.$$  

This implies that

$$C_{BSE}(\Delta(SA(G))) \subseteq C_{BSE}(\Delta(A(G))) = \hat{B}(G) \subseteq \mathcal{M}(SA(G)).$$

Hence, $SA(G)$ is a BSE-algebra. □

**Example 2.5.** (1) Let $G$ be a locally compact group and let $1 \leq r < \infty$. Now, since $\|u\|_{\infty} \leq \|u\|_{B(G)}$ for all $u \in B(G)$, it follows that $uL^r(G) \subseteq L^r(G)$. This implies that $B(G) \subseteq \mathcal{M}(SA^r(G))$. Thus $SA^r(G)$ is a BSE-algebra if and only if it has a BWAI.

(2) Let $S_0(G)$ be the Feichtinger Segal algebra in $A(G)$. Then $B(G) \subseteq \mathcal{M}(S_0(G))$; see [19, Corollary 5.2]. Thus $S_0(G)$ is a BSE-algebra if and only if it has a BWAI.

**Corollary 2.6.** Let $G$ be a locally compact group and let $SA(G)$ be an essential abstract Segal algebra in $A(G)$. Then $SA(G)$ is a BSE-algebra if and only if $SA(G)$ has a BWAI.

**Proof.** Suppose that $SA(G)$ has a BWAI. Then $A(G)$ has a bounded approximate identity. Now, by the same argument as the proof of Theorem 2.1 we can show that $SA(G) = A(G)SA(G)$. Consequently, for each $u \in B(G)$

$$uSA(G) = uA(G)SA(G) \subseteq A(G)SA(G) = SA(G),$$

which implies that $B(G) \subseteq \mathcal{M}(SA(G))$. Hence, $SA(G)$ is a BSE-algebra by Theorem 2.4. □

Suppose that $G$ is a locally compact group and $\mathcal{M}_{cb}A(G)$ denotes the Banach algebra of completely bounded multipliers of $A(G)$, that is, the continuous and bounded functions $\nu$ on $G$ such that $\nu A(G) \subseteq A(G)$ and the map $L_{\nu}$ defined by $L_{\nu}(u) = \nu u$ is completely bounded; see [4] for a complete course on operator space theory. Note that, by $A(G) = VN(G)$, $\mathcal{M}_{cb}A(G)$ is a completely contractive Banach algebra, where $VN(G)$ is the group von Neumann algebra. It is well-known that $A(G) \subseteq B(G) \subseteq \mathcal{M}_{cb}A(G)$. Now, let

$$A_{cb}(G) = \overline{A(G)}_{\|\cdot\|_{\mathcal{M}_{cb}A(G)}}.$$
This algebra for the first time introduced by Forrest in [6]. See also some recent works [7, 8, 17] for more details and properties.

We end this section with the following result regarding the BSE-property of $A_{cb}(G)$ in the case that $G$ is discrete. Recall that a locally compact group $G$ is said to be weakly amenable if $A(G)$ has an approximate identity which is bounded in $\| \cdot \|_{A_{cb}(G)}$ or equivalently $A_{cb}(G)$ has a bounded approximate identity; see [7, Proposition 1].

**Theorem 2.7.** Suppose that $G$ is a discrete locally compact group. Then $A_{cb}(G)$ is a BSE-algebra if and only if $G$ is weakly amenable.

**Proof.** By [17, Lemma 4.1] we know that $G$ is discrete if and only if $A_{cb}(G)$ is a closed ideal in its second dual. Also, it is clear that $A_{cb}(G)$ is commutative and semi-simple. Now the result follows by [15, Theorem 3.1].

### 3. Segal Algebras Implemented by Local Functions

In this section we focus on a certain Segal algebra which recently introduced by Inoue and Takahasi. Let $X$ be a non-empty locally compact Hausdorff space. A subalgebra $A$ of $C_0(X)$ is called a Banach function algebra if $A$ separates strongly the points of $X$ (that is, for each $x, y \in X$ with $x \neq y$, there exists $f \in A$ such that $f(x) \neq f(y)$ and for each $x \in X$, there exists $f \in A$ such that $f(x) \neq 0$) and with a norm $\| \cdot \|$, $(A, \| \cdot \|)$ is a Banach algebra.

Suppose that $(A, \| \cdot \|)$ is a natural regular Banach function algebra on a locally compact, non-compact Hausdorff space $X$ with a bounded approximate identity $\{e_\alpha\}$ in $A_c$. We recalling the following definitions from [10].

**Definition 3.1.** A complex-valued continuous function $\sigma$ on $X$ is called a local $A$-function if for all $f \in A_c$, $f \sigma \in A$. The set of all local $A$-functions is denoted by $A_{loc}$.

**Definition 3.2.** For positive integer $n$ and a continuous complex-valued function $\tau$ on $A$, put

$$A_{\tau(n)} = \{ f \in A : f \tau^k \in A \quad (0 \leq k \leq n) \},$$

$$\|f\|_{\tau(n)} = \sum_{k=0}^{n} \|f \tau^k\|.$$

In the sequel of this section, suppose that $n$ is a constant positive integer and $\tau \in A_{loc}$.

By [10, Theorems 5.4], if $\tau \in A_{loc}$, then $(A_{\tau(n)}, \| \cdot \|_{\tau(n)})$ is a Segal algebra in $A$ such that $\Delta(A_{\tau(n)}) = \Delta(A) = X$, that is, $x \rightarrow \phi_x$ is a homeomorphism from $X$ onto $\Delta(A_{\tau(n)})$. 


Also, one can see that $A_{\tau(n)}$ is a Banach function algebra, because for each $x, y \in X$ with $x \neq y$, there exists $f \in A_{\tau(n)}$ such that $f(x) = \phi_x(f) \neq \phi_y(f) = f(y)$ and by using the Urysohn lemma for each $x \in X$, there exists $f \in A_{\tau(n)}$ with $f(x) \neq 0$. Note that by [10, Theorem 3.5], $A_{\tau(n)}$ is Tauberian. Recall that a Banach algebra $A$ is Tauberian if $A_c$ is dense in $A$.

The following theorem is one of our main results in this section.

**Theorem 3.3.** Suppose that $(A, \| \cdot \|_X)$ is a BSE-algebra where $\| \cdot \|_X$ is the uniform norm. Then the following statements are equivalent.

(i) $A_{\tau(n)}$ is a BSE-algebra.

(ii) $\tau$ is bounded.

**Proof.** (i) $\rightarrow$ (ii). Suppose that $A_{\tau(n)}$ is a BSE-algebra, therefore it has a BWAI. So, there exists a constant $M > 0$ such that

$$\|f_\alpha\|_{\tau(n)} < M, \quad \lim_{\alpha} f_\alpha(x) = 1 \quad (x \in X).$$

On the other hand, $\|f_\alpha\|_{\tau(n)} \leq \|f_\alpha\|_{\tau(n)}$, hence we have

$$|\tau(x)| \leq M \quad (x \in X).$$

Therefore, $\tau$ is bounded.

(ii) $\rightarrow$ (i). Let $\tau$ be bounded by $M$, that is, $\|\tau\|_X < M$. Clearly, $A_c \subseteq A_{\tau(n)}$. For each $f \in A$, there exists a net $\{f_\alpha\}$ in $A_c$ such that $\|f_\alpha - f\|_X \rightarrow 0$. Now, $\{f_\alpha\}$ is in $A$ and $f\tau = \lim_{\alpha} f_\alpha\tau$, since

$$\|f\tau - f_\alpha\tau\|_X = \|(f - f_\alpha)\tau\|_X \leq M\|f - f_\alpha\|_X \rightarrow 0.$$ 

So, $f\tau \in A$. Similarly, one can see that $f\tau^k$ for each $1 < k \leq n$ is in $A$. Therefore, $A = A_{\tau(n)}$. Hence $A = A_{\tau(n)}$. Finally, for each $f \in A$

$$\|f\|_X \leq \|f\|_{\tau(n)} \leq \|f\|_X \left(\sum_{k=0}^{n} M^k\right).$$

Therefore, $A_{\tau(n)}$ is topologically isomorphic to $A$, so $A_{\tau(n)}$ is a BSE-algebra. □

Recall that $\tau \in A_{\text{loc}}$ is called a rank $\infty$ local $A$-function, if for each $k = 0, 1, 2, \cdots$, the inclusion $A_{\tau(k)} \supsetneq A_{\tau(k+1)}$ holds. By [10, Proposition 8.2 (ii)], if $\|\tau\|_X = \infty$, then $\tau$ is a rank $\infty$ local $A$-function.

As an application of the above theorem, we give the following result which provide for us examples of Banach algebras without any BWAI.

**Corollary 3.4.** Let $A = C_0(\mathbb{R})$ and $\tau(x) = x$ for every $x \in \mathbb{R}$. Then

$$A \supsetneq A_{\tau(1)} \supsetneq A_{\tau(2)} \supsetneq \cdots \supsetneq A_{\tau(n)} \supsetneq \cdots.$$ (3.1)

For each $k = 1, 2, 3, \cdots$, $A_{\tau(k)}$ is not a BSE-algebra and has no BWAI.
Proof. By [20, Theorem 3], if $A = C_0(\mathbb{R})$, we know that $A$ is a BSE-algebra. Since for each $x \in \mathbb{R}$, there exists $f \in A$ such that $f = \tau$ on a neighborhood of $x$, hence by [10, Proposition 7.2], $\tau$ is an element of $A_{\text{loc}}$. But $\tau$ is not bounded and this implies that $A_{\tau(k)}$ is not a BSE-algebra by Theorem 3.3. Also, since $\tau$ is not bounded, $\tau$ is a rank $\infty$ local $A$-function, hence we have equation 3.1. Finally, if $A_{\tau(k)}$ has a BWAI, then similar to the proof of Theorem 3.3 one has $\|\tau\|_\mathbb{R} < \infty$ which is impossible.

By the above corollary if $A$ is a BSE-algebra, then $A_{\tau(n)}$ is not necessarily a BSE-algebra. For the converse we have the following proposition.

Proposition 3.5. Suppose that $\tau \in A_{\text{loc}}$ and $A$ is an ideal in its second dual. If $A_{\tau(n)}$ is a BSE-algebra, then $A$ is a BSE-algebra.

Proof. If $\{e_\alpha\}$ is a b.a.i for $A$, then $\{e_\alpha\}$ is an approximate identity for $A_{\tau(n)}$. So, $A_{\tau(n)}$ is an essential abstract Segal algebra with respect to $A$. Now, using Theorem 2.1 we have the result.

We do not know whether Proposition 3.5 fails if the assumption that $A$ is an ideal in its second dual is dropped.

Remark 3.6. If $X$ is a discrete space, then $A$ is an ideal in its second dual. Since $A$ is a semi-simple commutative and Tauberian Banach algebra, using [15, Remark 3.5], we conclude that $A$ is an ideal in its second dual.

4. A Construction on Group Algebras

Let $G$ be a locally compact group and let $L^1(G)$ be the space of all measurable and integrable complex-valued functions (equivalent classes with respect to the almost everywhere equality relation) on $G$ with respect to the left Haar measure of $G$. The convolution product of the functions $f$ and $g$ in $L^1(G)$ is defined by

$$f \ast g(x) = \int_G f(y)g(y^{-1}x)dy.$$ 

For each $f \in L^1(G)$, let $\|f\|_1 = \int_G |f(x)|dx$. It is well-known that $L^1(G)$ endowed with the norm $\| \cdot \|_1$ and the convolution product is a Banach algebra called the group algebra of $G$; see [2, Section 3.3] for more details.

The following lemma proved by Bochner and Schoenberg for $G = \mathbb{R}$ in (1934) and by Eberlein for general locally compact abelian (LCA) groups in (1955). Here, using a result due to E. Kaniuth and A. ¨Ulger, we give another proof.

Recall that for a LCA group $G$, the dual group of $G$, $\hat{G}$ defined as the set of all continuous homomorphisms from $G$ to $\mathbb{T}$ where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. It is well-known that $\hat{G}$ is a LCA group with the pointwise operation.
Lemma 4.1. Suppose that $G$ is LCA group. Then $L^1(G)$ is a BSE-algebra.

Proof. We know that $L^1(G)$ is isometrically isomorphic to $A(\hat{G})$. But $\hat{G}$ is amenable. Therefore, by [15, Theorem 5.1], $A(\hat{G})$ and so $L^1(G)$ is a BSE-algebra. □

In the sequel, motivated by the construction of $A_{\tau(n)}$ in the preceding section, we introduce a subalgebra of the group algebra $A = L^1(G)$ where $G$ is a locally compact group.

Recall that $\varphi : G \rightarrow \mathbb{C}^\times$ is sub-multiplicative if

$$|\varphi(xy)| \leq |\varphi(x)||\varphi(y)| \quad (x, y \in G),$$

where $\mathbb{C}^\times$ denotes the multiplicative group of non-zero complex numbers.

For a measurable sub-multiplicative function $\tau : G \rightarrow \mathbb{C}^\times$ and each $n \in \mathbb{N}$, put

$$L^1(G)_{\tau(n)} = \{ f \in L^1(G) : f\tau, \ldots, f\tau^n \in L^1(G) \}$$

$$\|f\|_{\tau(n)} = \sum_{k=0}^{n} \|f\tau^k\|_1 \quad (f \in L^1(G)_{\tau(n)}).$$

As the first result in this section, we show that $L^1(G)_{\tau(n)}$ is a Banach algebra as follows.

Proposition 4.2. $L^1(G)_{\tau(n)}$ is a Banach algebra with the convolution product and the norm $\| \cdot \|_{\tau(n)}$.

Proof. For each $f, g \in L^1(G)_{\tau(n)}$ and $1 \leq k \leq n$, we have

$$\|(f \ast g)\tau^k\|_1 \leq \int \int |f(y)||g(y^{-1}x)||\tau^k(x)|dydx$$

$$= \int \int |f(y)||g(y^{-1}x)||\tau^k(x)|dxdy$$

$$= \int \int |f(y)||g(x)||\tau^k(yx)|dxdy$$

$$\leq \|f\tau^k\|_1 \|g\tau^k\|_1.$$
have
\[
\|f \ast g\|_{\tau(n)} = \|f \ast g\|_1 + \sum_{k=1}^{n} \|(f \ast g)\tau^k\|_1
\]
\[
\leq \|f\|_1\|g\|_1 + \sum_{k=1}^{n} \|f\tau^k\|_1\|g\tau^k\|_1
\]
\[
\leq \|f\|_{\tau(n)}\|g\|_{\tau(n)}.
\]

To see the completeness of \(\| \cdot \|_{\tau(n)}\), let \(\{f_i\}\) be a Cauchy sequence in \(L^1(G)_{\tau(n)}\). So, there exist \(f \in L^1(G)\) and \(g_k \in L^1(G)\) for each \(1 \leq k \leq n\) such that
\[
\lim_{i \to \infty} \|f_i - f\|_1 = 0, \quad \lim_{i \to \infty} \|f_i \tau^k - g_k\|_1 = 0.
\]
Since \(\lim_{i \to \infty} \|f_i - f\|_1 = 0\), there exists a subsequence \(\{f_{i_m}\}\) such that
\[
\lim_{i_m} f_{i_m}(x) = f(x) \text{ a.e., for } x \in G.
\]
Also, since \(\lim_{i_m} \|f_{i_m} \tau^k - g_k\|_1 = 0\), there exists a subsequence \(\{f_{i_{m,k}}\}\) of \(\{f_{i_m}\}\) such that
\[
\lim_{i_{m,k}} f_{i_{m,k}}(x) \tau^k(x) = g_k(x) \text{ a.e., for } x \in G.
\]
Therefore, \(f \tau^k = g_k\) a.e., so, \(f\) is an element of \(L^1(G)_{\tau(n)}\) such that
\[
\|f_i - f\|_{\tau(n)} = \|f_i - f\|_1 + \|f_i \tau - f \tau\|_1 + \cdots + \|f_i \tau^n - f \tau^n\|_1
\]
\[
= \|f_i - f\|_1 + \|f_i \tau - g_1\|_1 + \cdots + \|f_i \tau^n - g_n\|_1 \to 0.
\]
Hence, \((L^1(G)_{\tau(n)}; \| \cdot \|_{\tau(n)})\) is complete. \(\square\)

In the sequel, we suppose that \(G\) is a LCA group.

**Theorem 4.3.** If \(\tau\) is bounded, then \(L^1(G)_{\tau(n)}\) is a BSE-algebra.

**Proof.** If \(\tau\) is bounded by \(M\), then for each \(f \in L^1(G)\) and \(1 \leq k \leq n\) we have \(f \tau^k \in L^1(G)\) and
\[
\|f\|_1 \leq \|f\|_{\tau(n)} = \sum_{k=0}^{n} \|f \tau^k\|_1 \leq \|f\|_1(\sum_{k=0}^{n} M^k).
\]
So, \(L^1(G)_{\tau(n)}\) is topologically isomorphic to \(L^1(G)\) and hence it is a BSE-algebra by Lemma 4.1. \(\square\)

When \(\tau\) satisfying \(|\tau(x)| \geq 1\) a.e., for \(x \in G\), we show in the sequel that \(L^1(G)_{\tau(n)}\) is indeed a Beurling algebra which we recall its definition as follows:

A weight on \(G\) is a measurable function \(w : G \to (0, \infty)\) such that \(w(xy) \leq w(x)w(y)\) for all \(x, y \in G\). The Beurling algebra \(L^1(G, w)\) defined by the space of all measurable and complex-valued functions \(f\) on \(G\) such that \(\|f\|_{1,w} = \int |f(x)|w(x)\, dx < \infty\). The Beurling algebra with the convolution product and
the norm $\| \cdot \|_{1,w}$ is a Banach algebra with $\Delta(L^1(G, w)) = \hat{G}(w)$, where $\hat{G}(w)$ is the space of all non-zero complex-valued continuous homomorphisms $\varphi$ on $G$ such that $|\varphi(x)| \leq w(x)$ for each $x \in G$; see [14].

The space $M(G, w)$ of all complex regular Borel measures $\mu$ on $G$ such that $\mu w \in M(G)$ with convolution product and norm

$$\|\mu\|_{M(G), w} = \|\mu w\|_{M(G)} = \int w(x) \, d|\mu|(x)$$

is a Banach algebra called the weighted measure algebra, where $\mu w$ defined by

$$\mu w(B) = \int_B w(x) \, d\mu(x)$$

for each Borel subset $B$ of $G$.

### Proposition 4.4

If $|\tau| \geq 1$ a.e., then $L^1(G, \tau(n))$ and $L^1(G, |\tau^n|)$ are topologically isomorphic.

**Proof.** Suppose that for almost every $x \in G$, $|\tau(x)| \geq 1$. Clearly if $f \in L^1(G, \tau(n))$, then we have $f \in L^1(G, |\tau^n|)$. On the other hand, if $f \in L^1(G, |\tau^n|)$ by applying $|\tau| \geq 1$ a.e., we conclude that $f \in L^1(G, \tau(n))$, since for all $0 \leq k \leq n$, $|f(x)||\tau^k(x)| \leq |f(x)||\tau^n(x)|$ a.e., therefore, $L^1(G, |\tau^n|) = L^1(G, \tau(n))$ as two sets.

Also, using $|\tau| \geq 1$ a.e., we have

$$\|f\|_{1,w} \leq \|f\|_{\tau(n)} \leq n\|f\|_{1,w},$$

where $w = |\tau^n|$. So, two norms $\| \cdot \|_{1,w}$ and $\| \cdot \|_{\tau(n)}$ are equivalent, which completes the proof. \qed

### Remark 4.5

Note that $L^1(G, \tau(n))$ and $L^1(G, |\tau^n|)$ are not equal in general. For example, let $G = R^+$ be the multiplicative group of all positive real numbers, $n = 2$ and $\tau(x) = \frac{1}{x}$ for all $x \in R^+$. Clearly, $\tau$ is measurable and sub-multiplicative. Also, it is easily verified that $L^1(G, \tau(2)) \subseteq L^1(G, |\tau^2|)$. Now, take $0 < \alpha < 1$ and put

$$f(x) = \begin{cases} 
0, & 0 < x < 1; \\
\alpha x, & 1 \leq x.
\end{cases}$$

One can easily check that $f$ is in $L^1(G, |\tau^2|)$ but it is not in $L^1(G, \tau(2))$. So,

$$L^1(G, |\tau^2|) \neq L^1(G, \tau(2)).$$

Also, if we put $g(x) = \chi_{(0,1]}$, then $g \in L^1(G)$ but $g \notin L^1(G, \tau(2))$. Hence,

$$L^1(G) \neq L^1(G, \tau(2)).$$

### Remark 4.6

Although in general $L^1(G, |\tau^k|) \neq L^1(G, \tau(n))$ for every integer $k$ with $0 \leq k \leq n$, but we have

$$\overline{L^1(G, \tau(n))}_{\| \cdot \|_{1,|\tau^n|}} = L^1(G, |\tau^k|), \quad (0 \leq k \leq n).$$
Because \( C_c(G) \) is dense in \( L^1(G, |\tau^k|) \) and similar to [14, Lemma 1.3.5 (i)], one can see that \( C_c(G) \subseteq L^1(G)_{\tau(n)} \).

**Remark 4.7.** Let \( K \subseteq G \) be a relatively compact neighborhood of \( e \); the identity of \( G \). Put

\[ U_K = \{ U \subseteq K : U \text{ is a relatively compact neighborhood of } e \} \]

For each \( U \in U_K \), let \( f_U = \frac{\chi_U}{|U|} \), where \( |U| \) denotes the Haar measure of \( U \). On the other hand, since \( K \) is relatively compact by [14, Lemma 1.3.3], there exists a positive real number \( b \) such that \( |\tau(x)| \leq b \) for all \( x \in K \). So

\[ \|f_U\|_{\tau(n)} \leq 1 + b + \ldots + b^n \quad (U \in U_K). \]

Also, similar to the group algebra case, for each \( f \in L^1(G)_{\tau(n)} \) we have \( \|f_U * f - f\|_{\tau(n)} \to 0 \) when \( U \) tends to \{e\}. Therefore, \( \{f_U\}_{U \in U_K} \) is a BAI for \( L^1(G)_{\tau(n)} \).

**Remark 4.8.** Using the above remarks one can see that in general \( L^1(G)_{\tau(n)} \) is not an abstract Segal algebra with respect to \( L^1(G) \), because \( L^1(G)_{\tau(n)} \) has a b.a.i and in general \( L^1(G) \neq L^1(G)_{\tau(n)} \). But it is well-known that if \( S \) is an abstract Segal algebra with respect to \( L^1(G) \) such that has a BAI, then \( S = L^1(G) \).

Suppose that \( G \) is compact and \( w \) is a weight on \( G \). So, by Lemma 1.3.3 and Corollary 1.3.4 of [14], there exists positive real number \( b \) such that \( 1 \leq w(x) \leq b \) for all \( x \in G \). Hence, \( L^1(G, w) \) is topologically isomorphic to \( L^1(G) \). Now, using Proposition 4.4 and the proof of Theorem 4.3, we have the following corollary. Note that " \( \cong \) " means topologically isomorphic.

**Corollary 4.9.** If \( G \) is a compact group, then \( L^1(G)_{\tau(n)} \) is a BSE-algebra and we have the following relations:

\[ L^1(G, |\tau^n|) \cong L^1(G)_{\tau(n)} \cong L^1(G). \]

**Remark 4.10.** For every integer \( k \) with \( 0 \leq k \leq n \), we have,

\[ L^1(G)_{\tau(n)} \subseteq L^1(G, |\tau^k|). \]

Therefore,

\[ \hat{G} \cup \hat{G}(|\tau|) \cup \ldots \cup \hat{G}(|\tau^n|) \subseteq \Delta(L^1(G)_{\tau(n)}). \]

Now, using Remark 4.6, clearly every \( \varphi \in \Delta(L^1(G)_{\tau(n)}) \) can be extended to a character of \( L^1(G, |\tau^k|) \) for each \( 0 \leq k \leq n \). So,

\[ \Delta(L^1(G)_{\tau(n)}) = \hat{G} \cup \hat{G}(|\tau|) \cup \ldots \cup \hat{G}(|\tau^n|). \]

Note that by the above relation, we conclude that \( L^1(G)_{\tau(n)} \) is semi-simple.
Remark 4.11. Put $M(G)_{\tau(n)} = \bigcap_{k=1}^{n} M(G, |\tau|^k)$ and define the following norm:
\[ \|\mu\|_{\tau(n)} = \sum_{k=0}^{n} \|\mu|\tau|^k\|_{M(G)} \quad (\mu \in M(G)_{\tau(n)}). \]

A direct use of the convolution product shows that $(M(G)_{\tau(n)}, \|\cdot\|_{\tau(n)})$ is a normed algebra such that
\[ M(G)_{\tau(n)} \subseteq \mathcal{M}(L^1(G)_{\tau(n)}). \tag{4.1} \]

We do not know whether the converse of the above inequality is hold or not. Clearly, if $\tau$ is bounded, then $M(G) \cong M(G)_{\tau(n)}$ and so the converse of the above inequality holds by Wendel’s Theorem.

As it is shown in [12, Remark 3.2], if $B$ is an abstract Segal algebra with respect to $A$, then $C_{BSE}(\Delta(B)) \subseteq C_{BSE}(\Delta(A))$. If $A = L^1(G)$ and $B = L^1(G)_{\tau(n)}$ although $B$ is not an abstract Segal algebra with respect to $A$ in general, but we have a similar result as follows. We prove it for the sake of completeness and convenience of the reader.

**Proposition 4.12.** Let $A = L^1(G)$ and $B = L^1(G)_{\tau(n)}$. Then we have
\[ \widehat{\mathcal{M}(B)} \subseteq C_{BSE}(\Delta(B)) \subseteq C_{BSE}(\Delta(A)) = \widehat{\mathcal{M}(A)}. \]

**Proof.** Since $B$ has a BAI, $\widehat{\mathcal{M}(B)} \subseteq C_{BSE}(\Delta(B))$.

To see the second inequality, let $f \in A^*$. In view of the following relation
\[ |f(b)| \leq \|f\|_{A^*} \|b\|_A \leq \|f\|_{A^*} \|b\|_{\tau(n)} \quad (b \in B), \]
we have
\[ \|f\|_{B^*} \leq \|f\|_{A^*}. \tag{4.2} \]

Now, let $\sigma \in C_{BSE}(\Delta(B))$, so there exists $C > 0$ such that for each $\varphi_1, \ldots, \varphi_n \in \Delta(B)$ and complex numbers $c_1, \ldots, c_n$
\[ \left| \sum_{i=1}^{n} c_i\sigma(\varphi_i) \right| \leq C \left\| \sum_{i=1}^{n} c_i\varphi_i \right\|_{B^*}. \]

If for each $1 \leq i \leq n$ we take $\varphi_i \in \Delta(A) = \widehat{G} \subseteq \Delta(B)$ and using relation 4.2 we conclude that
\[ \left| \sum_{i=1}^{n} c_i\sigma(\varphi_i) \right| \leq C \left\| \sum_{i=1}^{n} c_i\varphi_i \right\|_{B^*} \leq C \left\| \sum_{i=1}^{n} c_i\varphi_i \right\|_{A^*}. \]

Hence $\sigma$ is an element of $C_{BSE}(\Delta(A))$, which completes the proof. \qed

We end this paper with the following questions and conjecture.

**Conjecture:** We conjecture that Theorem 2.7 is valid for every locally compact group.
Question: Are the converse of Theorem 4.3 and Relation 4.1 hold?

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