Borel Resummation of the Perturbative Free Energy
of Hot Yang-Mills Theory

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Abstract

The divergent perturbative expansion of the free-energy density of thermal $SU(3)$ gauge theory is resummed into a rapidly convergent series using a variational implementation of the method of conformal mapping of the corresponding Borel series. The resummed result differs significantly from non-perturbative lattice simulations and the discrepancy is attributed to the presence of a pole on the positive axis of the Borel plane. The position of that pole is determined numerically and the difference between the lattice data and the resummed series is related to a phenomenological bag ‘constant’.

It is generally believed that the collision of heavy-ions at sufficiently high energies will lead to the formation of a new phase of matter, the quark-gluon plasma, and experiments to produce such a plasma are underway at Brookhaven and CERN. As the effective coupling, $\alpha$, of quantum chromodynamics (QCD) decreases with an increase in energy, theorists have used perturbative methods to study properties of the plasma at high temperature.
For example, a completely analytical calculation of the free-energy density of QCD to order $\alpha^{5/2}$ was performed a few years ago [1, 2]. The purely gluonic contribution is given by

$$\frac{F}{F_0} = 1 - \frac{15}{4} \left( \frac{\alpha}{\pi} \right) + 30 \left( \frac{\alpha}{\pi} \right)^{3/2} + \left( 67.5 \ln \left( \frac{\alpha}{\pi} \right) + 237.2 - 20.63 \ln \left( \frac{\bar{\mu}}{2\pi T} \right) \right) \left( \frac{\alpha}{\pi} \right)^2$$

$$- \left( 799.2 - 247.5 \ln \left( \frac{\bar{\mu}}{2\pi T} \right) \right) \left( \frac{\alpha}{\pi} \right)^{5/2},$$

where $F_0 = \frac{-8\pi^2 T^4}{45}$ is the contribution of non-interacting gluons and $\bar{\mu}$ is the renormalization scale in the $\overline{MS}$ scheme. Unfortunately (1) is an oscillatory, non-convergent, series even for $\alpha$ as small as 0.2, which is close to the value of physical interest.

Pade’ Approximants (P.A.’s) were used in Ref.[3] to resum the series (1). It was found that the dependence on the arbitrary scale $\bar{\mu}$ was reduced and the convergence of the series somewhat improved. However P.A.’s have a number of well-known drawbacks. For those and other reasons, in Refs.[4] the authors abandoned the expansion of the free-energy density with respect to the coupling constant and considered instead selective resummations of gauge-invariant diagrams. Though the results of [4] compare favorably with lattice simulations [5], calculations beyond leading order are complicated and thus it seems that the issue of convergence is left open. In Ref.[4], yet another procedure was used to study the free-energy of hot QCD: short distance perturbative effects were handled analytically while long-distance effects were described [4] by an effective three-dimensional theory and studied numerically.

It appears that a number of separate issues concerning Eq.(1) have become confused in the literature. The first issue is whether the given divergent series can be resummed into a convergent series, preferably in a systematic and well-motivated way. The second issue is whether such a resummed series accurately represents the physical quantity. The third issue is whether one obtains any new physical insight in the process. It will be the attempt of this paper to shed some light on these and related questions.

Recall that the divergence of perturbative expansions in quantum field theory is a generic phenomena [6]. Given a series

$$\hat{S}_N(\lambda) = 1 + \sum_{n=1}^{N} f_n \lambda^n,$$  

(2)
where $\lambda$ is the coupling constant, one expects the coefficients $f_n$ to grow as $n!$ for large $n$. It is natural then to introduce the Borel transform

$$B(z) = 1 + \sum_{n=1}^{N} \frac{f_n z^n}{n!} \quad (3)$$

which has better convergence properties than (2). The series (2) may be recovered from (3) through a Laplace transform

$$\hat{S}_N(\lambda) = \frac{1}{\lambda} \int_0^\infty dz \ e^{-z/\lambda} B(z). \quad (4)$$

In order to proceed non-trivially, one first performs an approximate summation of the series (3) so that Eq.(4) then gives the resummed version of (2). Now, suppose that the only singularity of $B(z)$ is at $z = -1/p$, with $p$ real and positive. Then the radius of convergence of the Borel series is $1/p$. In order to perform the integral in (4), one needs to extend the domain of convergence of the Borel series. One way to do this is by the method of conformal mapping\[7, 8\]. Define

$$w(z) = \frac{\sqrt{1 + pz} - 1}{\sqrt{1 + pz} + 1} \quad (5)$$

which maps the Borel plane to a unit circle. The inverse of (5) is given by

$$z = \frac{4w}{p} \frac{1}{(1 - w)^2} \quad (6)$$

The idea is to rewrite (3) in terms of the variable $w$. Therefore, using (5), $z^n$ is expanded to order $N$ in $w$ and substituted into (4). The result is

$$S_N(\lambda) = 1 + \frac{1}{\lambda} \sum_{n=1}^{N} \frac{f_n n!}{n!} \left(\frac{4}{p}\right)^n \sum_{k=0}^{N-n} \frac{(2n + k - 1)!}{k! (2n - 1)!} \int_0^\infty e^{-z/\lambda} w(z)^{(k+n)} \ dz, \quad (7)$$

where $w(z)$ is given by (5). Equation (7) represents a resummation of the original series (2). This technique has been used in determining critical exponents in statistical systems where the singularity at $z = -1/p$ is due to instantons \[4, 5\], and in QCD at zero temperature where the singularity is due to renormalons \[9\].

Currently no information is available about the exact location of singularities in the Borel plane of thermal QCD though undoubtedly there is at least
one on the negative semi-axis. Therefore in order to apply the resummation (7) to Eq.(11), a new idea is introduced in this paper: It is first assumed that the only singularity is at $z = -1/p$, $p > 0$, with the value of $p$ determined by the condition that it be the position of an extremum of (7). That is, $p$ is chosen to be a solution of

$$\left( \frac{\partial S_N(\lambda, p)}{\partial p} \right)_{\lambda=\lambda_0} = 0.$$  \hspace{1cm} (8)

Since $S_N(\lambda, p)$ depends on the coupling $\lambda$, one first fixes $\lambda$ at some reference value $\lambda = \lambda_0$ (say, at the mid-point of the range of interest) in order to solve Eq.(8). Fortunately, it turns out that the solution of (8), and hence the convergence of $S_N$, is not very sensitive to the exact value of $\lambda_0$.

Let me illustrate the technique by applying it to two cases where exact results are known. Consider first the integral

$$I(\lambda) = \int_0^{\infty} dz \, e^{-z/(1+z\lambda)}.$$ \hspace{1cm} (9)

If the right-hand-side of (9) is expanded as a power series in $\lambda$ one obtains at $N$-th order

$$I_N(\lambda) = 1 + \sum_{n=1}^{N} \lambda^n (-1)^n n!.$$ \hspace{1cm} (10)

Clearly, from (9), the exact Borel transform of this series is $B(z) = 1/(1+z)$ with a singularity at $z = -1$. Ignoring this information, let us resum the divergent series (10) using (7) with $f_n = (-1)^n n!$ and values of $p$ determined for each $N$ through equation (8) at the reference value $\lambda_0 = 0.5$. The results for (8) are as follows: There is no extremum for $N = 1$. For $N = 2$, there is a minimum at $p = 2.65$. For $N = 3$ there is a local maximum at $p = 1.5$ and a minimum at $p = 5.1$. For $N = 4$ there is a local minimum at $p = 1.3$, a local maximum at $p = 2.3$ and a global minimum at $p = 8.4$. Jumping ahead to $N = 8$, there is a local minimum at $p = 1.075$, a local maximum at $p = 1.5$ and a global minimum at $p = 3.6$.

Thus in general (8) has more than one solution for a given $N$ and $\lambda_0$. In Fig.(1), Eq.(8) is plotted for $N = 2, 3, 4$ and $8$, at the respective minimum. Notice how the curves converge rapidly to the exact value given by (9). Alternatively, one might choose for each $N$, the value of $p$ (from the multiple solutions of Eq.(8)) which seems to be part of a converging sequence. In this
case the values, $p = 2.65(N = 2), p = 1.5(N = 3), p = 1.3(N = 4),$ and $p = 1.075(N = 8)$, appear to converge to the exact value $p = 1$. The curves are shown in Fig.(2). Clearly the curves in Fig.(2) converge faster to the exact value than those of Fig.(1), but unless one is interested in very high numerical accuracy, the difference is not significant. For example, at $\lambda = 0.5$, the exact value of $\mathfrak{F}$ is 0.722657, while the resummed value for $N = 8$ is given at the global minimum $p = 1.075$ by $S_8(\lambda = 0.5, p = 1.075) = 0.722652$, and at the local minimum $p = 3.6$ by $S_8(\lambda = 0.5, p = 3.6) = 0.722524$.

The main points illustrated by this example, which seem to be common to the other cases studied, are, (i) the rapid convergence of the resummed series represented by (7) compared to the original wildly oscillating series (10), (ii) the relative insensitivity of the convergence and numerical value of the resummed series to the particular extremum chosen among the possible multiple solutions of (8) (for a given $N$ and $\lambda_0$), even if the chosen value of $p$ is quite different from the exact value, (iii) the relative insensitivity of Eq.(8), and hence the convergence of (7) to the precise value of $\lambda_0$.

For another example, consider thermal $O(M) \lambda^2 \phi^4$ field theory in the limit $M \to \infty$. The exact free-energy density in this case has been determined in Ref.[10]. In Eq.(5.8) of that paper the perturbative expansion, in $\lambda$, of the free-energy density is also given up to $\lambda^6$. Defining $S = (F(T) - F(0))/F_{\text{ideal}}$ and choosing $\bar{\mu} = T$ for simplicity, the values of $f_n$ for $2 \leq n \leq 6$ can be read off from Eq.(5.8) of Ref.[10] and (8) solved at some reference value, say $\lambda_0 = 4$. The solutions of (8) in this case are: $N = 3, p = 0.1$ (min); $N = 4, p = 0.05$ (local max), $p = 0.2$ (min); $N = 5, p = 0.025$ (local min), $p = 0.1$ (local max), $p = 0.3$ (global min); $N = 6, p = 0.1$ (local max), $p = 0.45$ (global min). As in the first example, the convergence of the resummed series is found to be rapid even for large coupling, in contrast to the oscillatory behaviour of the ordinary perturbation expansion observed in Ref.[10]. Fig.(3) shows the curves for $S_N(\lambda), 3 \leq N \leq 6$, for the value $p = 0.1$, which seems to be the value $p$ converges to as $N$ increases. The exact value of the free-energy density at $\lambda = 8$ taken from Fig.(6) of Ref.[10] is about 0.875. By comparison the resummed value predicted here is given at sixth order by $S_6(\lambda = 8, p = 0.1) = 0.889$. On the other hand, if one evaluates $S_6$ not at $p = 0.1$ but rather at its global minimum $p = 0.45$, one gets $S_6(\lambda = 8, p = 0.45) = 0.849$, a difference of less than 5%.

\footnote{Note that the definition of the coupling constant used here is different from that in Ref.[10].}
Actually, no information is available about the singularities in the Borel plane for the $O(M \to \infty)$ scalar field studied in [10]. The good agreement of the results obtained here with the exact results of Ref.[10] leads one to conjecture that for the free-energy of this model, the singularity closest to the origin in the Borel plane might be near $p = 0.1$, that is, $z = -10$.

Note that although the coupling constant in the scalar field theory model is $\lambda^2$, the perturbative expansion of the free-energy density contains the odd powers $\lambda^3$ and $\lambda^5$ which is typical of thermal field theories with massless particles (or at very high temperatures) [1, 11]. Physically these are due to collective effects such as Debye screening and it is sometimes suggested in the literature that these terms should be treated on a different footing. However, as the analysis above shows, from a mathematical point of view these odd powers are no different from the other terms in the expansion of the free-energy density and can be resummed as part of a single series.

Finally, the resummation technique of Eqs.(7,8) is applied to the free-energy density of $SU(3)$ gauge theory given in Eq.(1). As in Refs.[4], I replace $(\frac{\alpha}{\pi})^\frac{1}{2}$ by the approximate two-loop running coupling constant defined by

$$
\lambda(c, x) = \frac{2}{\sqrt{11L(c, x)}} \left( 1 - \frac{51}{121} \ln(L(c, x)) \right) \quad (11)
$$

where $L(c, x) = \ln((2.28\pi cx)^2)$, $c = \bar{\mu}/2\pi T$ and $x = T/T_c$, with $T_c \sim 270MeV$ the critical temperature which separates the low and high temperature phases [3]. Furthermore, as in Ref.[3], I have absorbed the $\ln(\alpha)$ term which appears at three-loop order into the coefficient of the $\alpha^2$ term in (1).

Fixing first the reference values $c_0 = 1$, $x_0 = 3$ (which fixes the reference value of $\lambda_0$), the results of (3) are: $N = 2$, no extremum; $N = 3$, $p = 3.2(\text{min})$; $N = 4$, $p = 7.6(\text{min})$; $N = 5$, $p = 13.1(\text{min})$. Since $c$ and $x$ appear in (3) and (4) only logarithmically, changing these values in the range of say, $0.5 < c < 2$, $2 < x < 5$, has almost no impact on the solution of Eq.(3) and hence on the optimal values of $p$.

The curves for the resummed series are plotted in Fig.(4) for $c = 1$, that is at the renormalization scale $\bar{\mu} = 2\pi T$. Again the rapid and monotonic convergence is manifest, the result approaching the ideal gas value even at moderate temperatures $\sim 2T_c$. One can estimate the effect of the unknown higher order, $\lambda^6$, contribution. It turns out to be negligible [12]. Therefore one feels confident that the $N = 5$ curve in Fig.(4) is numerically close to
the (unknown) exact sum of the perturbation series. In Fig.(5) the curve for $S_5(x,c,p = 13.1)$ is plotted for three values of $c$ to indicate its mild dependence (less than 1%) to the arbitrary renormalization scale $\bar{\mu}$.

Lattice results for the free-energy density of pure $SU(3)$ theory are shown in Fig(6). The lattice community has indicated that their errors are under control (less than 5%). In that case, I am left with the task of explaining the significant difference (e.g. $\sim 15\%$ at $T = 3T_c$) between the best analytically resummed result represented by the $N = 5$ curve in Fig.(4) and the lattice data. At zero temperature, it is known that non-abelian gauge theories are not Borel summable [9]. That is, $B(z)$ contains singularities for positive $z$, rendering the integral in (4) ambiguous. The situation is not expected to be different at non-zero temperature. Usually [9], the presence of such singularities is taken to indicate the existence of non-perturbative corrections. One can estimate the ambiguity, $\delta S$, and hence the non-perturbative correction, as the residue of the integrand in (4) at the location of the singularity [9]. If the singularity of $B(z)$, on the positive semi-axis, closest to the origin is a pole at $z = q$, then from (4)

$$\delta S = \frac{A}{\lambda} e^{-q/\lambda}$$

where $A$ is a constant. Assuming that the difference between the lattice data and Fig.(5), is due to (12), the constants $A$ and $q$ can be determined by rewriting (12) as

$$\ln(\lambda \delta S) = \ln(A) - q/\lambda$$

and using for $\delta S$ the difference between Fig.(6) and the median curve in Fig.(5) (i.e. $c = 1$). Fig.(7) shows the left-hand-side of (13) plotted against $1/\lambda$. This gives $A = e^{8.7}$ and $q = 2.62$, and so, with $\lambda(x) \equiv \lambda(c = 1,x)$,

$$S_{latt} = S_{pert} - \frac{1}{\lambda(x)} e^{8.7 - 2.62/\lambda(x)} ,$$

where $S_{latt}$ represents the lattice data for the free-energy, and $S_{pert}$ the Borel resummed perturbative result, both normalized with respect to the ideal gas value.

It is extremely reassuring that both the sign and magnitude of $q$ determined in this way are self-consistent with the assumptions made. In particular, the singularity at $z = q = 2.62$ is more than 30 times away from the origin than the singularity at $z = -1/p = -1/13.1$ and justifies a posteriori
the resummation procedure (7) which considered only the nearest singular-
ity. Furthermore since $S_{pert}$ is extremely close to the ideal gas value, Eq. (14) may be interpreted as a generalisation of phenomenological equations of state [11] for the free-energy where the second term on the right-hand-side of (14) is called a 'bag constant'. In our case the ‘constant’ is really temperature dependent and represents a non-perturbative contribution to the free-energy that vanishes at infinite temperature. Note that this interpretation of the second term on the right-hand-side of (14) is consistent with the usual one only because the resummed perturbative result lies above the lattice data.

Let me now summarize the main results of this paper. A new procedure, a variational version of the well-known conformal-mapping of Borel series, was introduced and its practicability illustrated. It was shown that the badly divergent series for the free-energy density of thermal $SU(3)$ gauge theory (1) could be resummed in a systematic and relatively well-motivated way into a rapidly convergent series. However the final result differed significantly from non-perturbative lattice data, suggesting that the discrepancy is due to the non-Borel-summability of the theory. Numerically, the difference was parametrised in terms of two constants (see (14)), and it is suggested that the ambiguity in the Borel integral is the bag ‘constant’ used in phenomenological models for the free-energy density.

In physical terms, the situation may be described as follows. If the Borel resummed perturbative expansion had agreed with the lattice data, then one would have argued that the high-temperature phase of $SU(3)$ theory is appropriately described by weakly coupled gluons. However, if the deviation of lattice data from the resummed and convergent perturbation expansion found here is taken at face value, then one is led to conclude that even at temperatures as high as three times $T_c$ ($\sim 700 MeV$), the phase of thermal $SU(3)$ is not accurately described by weakly coupled gluons but rather by more complicated structures which are responsible for the non-perturbative bag contribution. If one accepts this conclusion, then one must also be open to the possibility that what will be produced at CERN and Brookhaven might be better characterised as something other than a quark-gluon plasma. For some alternative descriptions of the high-temperature phase of QCD see, for example, [13].

An extension of the analysis presented here to QCD and other thermal gauge theories, a further development of the methodology itself and its applications to other physical problems, will be presented in an accompanying series of papers [12].
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Figure Captions

Figure 1: Plot of Eq.(7) for the model in Eq.(10), for $N = 2, 3, 4$ and 8 at the respective global minimum of Eq.(8). The curves move upwards with increasing $N$. The curves for $N = 3, 4$ practically coincide, while the curve for the exact expression (9) is indistinguishable from that for $N = 8$.

Figure 2: Plot of Eq.(7) for the model in Eq.(10), for $N = 2, 3, 4$ and 8 at the respective values of $p$ from Eq.(8) which converge to 1. The values of the curves at $\lambda = 0.5$ are, $N = 2(0.704)$, $N = 3(0.726)$, $N = 4(0.7219)$, $N = 8(0.7227)$. The curve for the exact expression (9) is hardly distinguishable from that for $N = 3, 4, 8$.

Figure 3: Plot of Eq.(7) for the model in Ref.[10] for $N = 3, 4, 5$ and $N = 6$ at the value $p = 0.1$. The curves move upwards as $N$ increases. For $N = 5, 6$ they are practically identical. The exact curve in Ref.[10] lies slightly below that for $N = 6$: At $\lambda = 8$ the exact curve has the value 0.875.

Figure 4: The resummed (7) free-energy density of hot $SU(3)$ gauge theory for $N = 3, 4$ and 5, at the renormalization point $\bar{\mu} = 2\pi T$. The curves move upwards as $N$ increases.

Figure 5: The fifth order resummed free-energy density (7) of hot $SU(3)$ gauge theory at $p = 13.1$, for the renormalization scale values $\bar{\mu} = 0.5, 1$ and 2. The free-energy density increases with increasing $\bar{\mu}$.

Figure 6: Mean lattice results for the free-energy density of hot $SU(3)$ gauge theory from Ref.[5]. Here $S_{lat}$ refers to the free-energy divided by the free-energy of an ideal gas of gluons.

Figure 7: Plot of the left-hand-side of Eq.(13) against $1/\lambda$ at several points (temperatures).
\[ \ln(\lambda S) \quad \{\text{Fig. 7}\} \]

\[ 1/\lambda \]

\[ 4.4 \quad 4.5 \quad 4.6 \quad 4.7 \quad 4.8 \quad 4.9 \]