On some comparison of multistep second derivative methods with the multistep hybrid methods and their application to solve integro-differential equations

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Abstract. As is known the necessity to solve the initial-value problem for the Volterra integro-differential equations arises in investigation of the residual knowledge of some objects. Volterra by using the integro-differential equations has studied the memory of land and many other important problems which have been arisen in the study of some phenomena from the different industries of natural sciences. By using a similar form of the initial-value problem for both Volterra integro-differential equation and ODEs have established some relation. Here, by continuing this investigation receives the direct relation between ODE and Volterra integro-differential equation. By using this relation to solve Volterra integro-differential equation have applied the methods which are used in solving of the initial-value problem for ODEs. For this aim have proposed to use the general form of the multistep second derivative hybrid methods. Demonstrated the advantages of this method and constructed one-step methods with order of exactness \( p \leq 10 \). The constructed here methods have compared with the known methods. And also by using the model problem have been illustrated the advantages of the received here results.

1. Introduction

There are wide classes of problems the investigation of which have reduced to solving of the initial-value problem for the ODEs fundamentally investigated by many known authors. It is not difficult to show that the named problem can be received from the initial-value problem for the Volterra integro-differential equation as the partial case. For this, let us consider the following initial-value problem for the Volterra integro-differential equation:
\[ y'(x) = f(x, y) + \lambda \int_{x_0}^{x} K(x, s, y(s))ds, \quad y(x_0) = y_0, \quad x_0 \leq x \leq X. \quad (1) \]

Here \( \lambda \) - some parameter and the continuous on totality of arguments functions of \( f(x, y) \) and \( K(x, s, y) \) are define in some close sets, where they have partial derivatives to up some order including \( p \). To determine the numerical solution of the problem (1) let us divide the segment \([x_0, X]\) to \( N \) equal parts by using the mesh points \( x_i = x_0 + ih \quad (i = 0, 1, \ldots, N) \). Here, \( 0 < h \) is the step size. This problem has been investigated by many known authors, beginning from the Vito-Volterra which opened the new direction in the mathematics named as the Volterra integro-differential equation (see for example [1]-[9]).

If in the equation (1) put \( \lambda = 0 \), then receive:
\[ y'(x) = f(x, y), \quad y(x_0) = y_0, \quad x_0 \leq x \leq X, \quad (2) \]
which is the initial-value problem for the ODE of the first order. And now let us consider the following Volterra integral equation of the second kind:
\[ y(x) = f(x) + \int_{x_0}^{x} K(x, s, y(s))ds, \quad x_0 \leq s \leq x \leq X. \quad (3) \]

Let us suppose that the solution of the equation (3) has been found by some way and after taking, that in equation (3) receive the equality from which one can be found the following:
\[ y'(x) = f'(x) + K(x, x, y) + \int_{x_0}^{x} K'(x, s, y(s))ds, \quad (4) \]
\[ y(x_0) = f(x_0), \quad x_0 \leq x \leq X. \]

This is same with the problem (1). By using the above –mentioned relations the scientists have proposed to use the following way for solving of the problem (1) (see for example [3]-[13]):
\[ y'(x) = f(x, y) + v(x), \quad y(x_0) = y_0, \quad x_0 \leq x \leq X, \quad (5) \]

Here \( v(x) = \lambda \int_{x_0}^{x} K(x, s, y(s))ds \).

Here we want to prove that in this case one can be solved the problems (1)-(5) by the the same method and how can be adapted that to other cases.

2. **Construction of the general method for solving of the named problems**

Let us suppose that the function of \( K(x, s, y) \) is degenerate and can be written as follows:
\[ K(x, s, y) = \sum_{j=0}^{m} a_j(x)b_j(s, y). \quad (6) \]

By taking the equality of (6) into account in the equation of (3), receive:
\[ y(x) = f(x) + \sum_{j=0}^{m} a_j(x)v_j(x), \]
here functions \( v_j(x), \quad (j = 0, 1, \ldots, m) \) can be defined as follows:
\[ v_j'(x) = b_j(x, y), \quad v_j(x_0) = 0, \quad j = 0, 1, \ldots, m. \quad (7) \]

By the simple comparison of the problems (7) and (2) receive that these problems are equivalent. In other words these problems can be solved by using one and the same methods. And now let us consider the comparison of the problems (1) and (2). For this aim take into account the equality of (6) in the equation of (5). In this case receive the following:
\[ y'(x) = f(x, y) + \sum_{j=0}^{m} a_j(x)v_j(x), \quad (8) \]
\[ y(x_0) = y_0, \quad x_0 \leq x \leq X, \]

Here the function \( v_j(x) \quad (j = 0,1,\ldots,m) \) are the solutions of the problems (7). By taking into account that the functions \( v_j(x) \quad (j = 0,1,\ldots,m) \) are known, receive that the problems of (8) and (2) are equivalent. For the shake of objectivity let us note that if the function \( K(x,s,y) \) independent from the variable \( x \), in other words if the function of \( a_j(x) \quad (j = 0,1,\ldots,m) \) are constant (it is to say that \( K(x,s,y) = b(s,y) \) and \( m = 1 \)), then receive that the problems (2) and (3) are equivalent and the problem (5) can be joint to them by using the vector \( z(x) = (y(x),v(x)). \)

And now let us replace this way of the construction of the numerical methods for solving of the problem (1). It is known that the function \( K(x,s,y) \) can be presented in the following form by using Lagrange interpolation polynomials with respect to \( x \) which have the next form:

\[
K(x,s,y) = \sum_{j=0}^{k} l_j(x) \int_{s_0}^{s} K(x_j,s,y(s))ds + R_k(x),
\]

\[
x \in [x_0, x_{n+1}],
\]

Here \( l_j(x) \quad (i = 0,1,\ldots,k) \) are the Lagrange base functions (see, [14,p. 106]). If use this equality in the correlation of (3), then receive:

\[
y(x) = f(x) + \sum_{j=0}^{k} l_j(x) \int_{s_0}^{s} K(x_j,s,y(s))ds, \quad x \in [x_n, x_{n+1}].
\]

If here denote \( b_j(s,y) = K(x_j,s,y(s)) \quad (j = 0,1,\ldots,k) \) and using them in the equality of (9), then receive the similar case to the above-mentioned degenerate case. By using this way the equation (9) can be rewritten as:

\[
y(x) = f(x) + \sum_{j=0}^{k} l_j(x)v_j(x), \quad v_j'(x) = K(x_j,x,y), \quad v_j'(x_0) = 0, \quad (j = 0,1,\ldots,k).
\]

Here for solving the equation of (3), proposed to use the solution of the initial-value problem for the system of ODEs. But in the works ([15]-[21]) have proposed the methods, which have more simple form than solving of the problem (10). Therefore, here we used the similarly ways.

By the above described way prove that one can be applied some methods to solve the above-mentioned problem which have used in solving of the problem (1). Here we will show that in some cases arises the necessity to use some modification of proposed above methods.

As is known one of popular methods, which have applied to solve the problem (2), is the multistep methods with the constant coefficients. Simple or classical multistep method for solving of the problem (2) can be written as:

\[
\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i},
\]

This method was fundamentally investigated by some authors (see for example [15]-[21]).

Dahlquist prove that if the method (11) is stable, then \( p \geq 2[k/2]+2 \) is hold (here \( p \) is the degree for the method (11)). The conceptions of the degree and stability are defined by the way proposed in the [15]. For the construction more exact method the specialists have proposed to use the following multistep second derivative methods:

\[
\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i s_{n+i},
\]
here the function \( g(x, y) = f'_y(x, y) + f'_x(x, y)f(x, y) \).

Let us note that the method (12) has investigated by many scientists (see for example [22]-[25]). Suppose that \( \alpha_k \neq 0 \). And in [26] prove that if the method (12) is stable, then there are stable methods of type (12) with the degree \( p_{\text{max}} = 2k + 2 \). In [25] prove that if \( \alpha_k = 0 \) then there are the stable methods of the type (12) with the degree \( p > 2k + 2 \). Therefore the method (12) in the work [25] has investigated in more general form, which can be written as:

\[
\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i g_{n+i}.
\]

(13)

Note that this method can be applied to solve the initial-value problem for the ODEs of the second order.

For the construction of the stable methods with the higher exactness Butcher and Gear have proposed to use the hybrid methods (see [26], [27]). Here by continuing that way have constructed hybrid method by using the formula (13) which in one variant can be written as the following:

\[
\sum_{i=0}^{k} \alpha_i y'_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i g_{n+i},
\]

(14)

\[
+ h^3 \sum_{i=0}^{k} \gamma_i y''_{n+i} + h^3 \sum_{i=0}^{k} \gamma_i v''_{n+i},
\]

\[
(\max(\|f\|, \|v\|) < 1, \; i = 0, 1, \ldots, k).
\]

In first let us apply the method (11) to solving of the problem (1) and after them adapted method (14) to find of the numerical solution of the problem (1).

In this case, receive:

\[
\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i},
\]

(15)

\[
\sum_{i=0}^{k} \alpha_i v_{n+i} = h \sum_{i=0}^{k} \beta_i v_{n+i}.
\]

(16)

It is not difficult to prove that the method (16) has constructed by using the method (11) and therefore they are same. For this it is enough to take into account the following relationship between coefficients \( \beta_i \) and \( \beta^{(j)}_i \) \((i, j = 0, 1, \ldots, k)\):

\[
\sum_{i=0}^{k} \beta^{(j)}_i = \beta_i \quad (i = 0, 1, \ldots, k).
\]

(17)

It is evident that one of the solutions of this system of linear algebraic equations is \( \beta^{(j)}_i = \beta_i \) \((i = 0, 1, \ldots, k)\). It follows from here that the method (16) has constructed by using the coefficients of the method (11). In this mean they are same.

It is not difficult to understand that the method (16) can be applied to solve the equation (3). Thus prove that the method (11) can be applied to solve all problems which have given above.

And now let us consider the application of the method (14) to solving of above-mentioned problems. For this aim, method (14) is written in the following form:

\[
\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i g_{n+i} + h^3 \sum_{i=0}^{k} \gamma_i y''_{n+i},
\]

(18)

\[
+ h^3 \sum_{i=0}^{k} \gamma_i v''_{n+i} + h^3 \sum_{i=0}^{k} \gamma_i v''_{n+i},
\]

\[
(\max(\|f\|, \|v\|) < 1, \; i = 0, 1, \ldots, k).
\]
\[
\sum_{j=0}^{k-m} \alpha_j v_{x_1} = h \sum_{j=0}^{k-m} \beta_j \psi(x_{x_1}, x_{x_2}, y_{x_1}) + h \sum_{j=0}^{k-m} \beta_j \psi(x_{x_1}, x_{x_2}, y_{x_1}) + h^2 \sum_{j=0}^{k-m} \gamma_j \psi(x_{x_1}, x_{x_2}, y_{x_1}),
\]

(19)

\[
v_{x_1} = \psi(x_{x_1}, x_{x_2}, y_{x_1}).
\]

To construct the concrete methods of the type (19) there is needed to present some ways for finding of the values of the coefficients \(\hat{\beta}_j, \gamma_j, \hat{\gamma}_j\) (\(i, j = 0, 1, 2, \ldots, k\)). For this aim, let us consider the special case and put \(K(x, y) = \phi(s, y)\). In this case receive that:

\[
\sum_{j=0}^{k-m} \beta_j = \hat{\beta}_j, \sum_{j=0}^{k-m} \gamma_j = \gamma_j; \sum_{j=0}^{k-m} \hat{\gamma}_j = \hat{\gamma}_j
\]

(20)

\((i = 0, 1, \ldots, k)\).

If there are known the values of the coefficients of \(\beta_i, \hat{\beta}_j, \gamma_j, \hat{\gamma}_j\) (\(i, j = 0, 1, 2, \ldots, k\)), then by using the above received system of algebraic equations one can find the values of the coefficients \(\beta_j, \hat{\beta}_j, \gamma_j, \hat{\gamma}_j\) (\(i, j = 0, 1, 2, \ldots, k\)). Thus has been demonstrated that by using the values of the coefficients of the method (14) one can determine the values of the coefficients of the method (19). And now let us consider determination of the values of the coefficients for the method (14). By using Taylor series for the functions \(y(x + ih), y'(x + ih), y''(x + ih), y'(x + m, h), y''(x + m, h)\) in the following asymptotic equality:

\[
\sum_{j=0}^{k-m} \alpha_j y(x + ih) - h \sum_{j=0}^{k-m} \left(\beta_j y'(x + ih) - \hat{\beta}_j y'(x + (i+1)h)\right) = 0
\]

(21)

\[
h^2 \sum_{j=0}^{k-m} \left(y_j y''(x + ih) - \hat{\gamma}_j y''(x + (i+1)h)\right) = O(h^{s+1}), h \to 0,
\]

receive the following lemma.

Lemma. For holding of the asymptotic equality (21) the satisfying of the coefficients \(\alpha_i, (i = 0, 1, \ldots, k - m)\), \(\beta_i, \hat{\beta}_i, \gamma_j, \hat{\gamma}_j\) (\(i = 0, 1, 2, \ldots, k\)) of the following system is necessary and sufficient:

\[
\sum_{j=0}^{k-m} \alpha_i \alpha_j = 0; \sum_{j=0}^{k-m} \left(\beta_i + \hat{\beta}_i\right) = \sum_{j=0}^{k-m} i \alpha_j;
\]

\[
\sum_{j=0}^{k-m} \left(\gamma_j + \hat{\gamma}_j\right) + \sum_{j=0}^{k-m} \left(\beta_j + (i+1)\beta_j\right) = \sum_{j=0}^{k-m} i^2 \alpha_j;
\]

\[
(s-1)\sum_{j=0}^{k-m} \left(\gamma_j + (i+1)\gamma_j\right) + \sum_{j=0}^{k-m} \left(\beta_j + (i+1)\beta_j\right) = \sum_{j=0}^{k-m} i^s \alpha_j.
\]

(22)

\((s = 3, 4, \ldots, p)\).

It is evident that the amount of the unknowns in this system equal to \(7(k+1) - m\) but amount of the equations in this system equal to \(p+1\). Thus have received that one can construct method with the degree \(p\) by using of the solution of the system (22).

Let us note that the integer value of \(p\) is called as the degree for the method (14) if the asymptotic equality of (21) is hold. It is known that both theoretical and practical interest to represent the convergent of the numerical methods. It is known that if the method (14) is convergent, then its coefficients satisfy to the following conditions:

A. Coefficients \(\alpha_i, (i = 0, 1, \ldots, k - m)\), \(\beta_j, \hat{\beta}_j, \gamma_j, \hat{\gamma}_j\) (\(j = 0, 1, \ldots, k\)) are real numbers and \(\alpha_{k,m} \neq 0\).

B. The following polynomials

\[
\rho(\lambda) = \sum_{i=0}^{k-m} \lambda_i; \ \sigma(\lambda) = \sum_{i=0}^{k-m} \beta_i \lambda_i; \ \sigma^*(\lambda) = \sum_{i=0}^{k-m} \hat{\beta}_i \lambda_i; \ \gamma(\lambda) = \sum_{j=0}^{k-m} \gamma_j \lambda_j; \ \hat{\gamma}(\lambda) = \sum_{j=0}^{k-m} \hat{\gamma}_j \lambda_j;
\]

have no common factor different from constant.
C. The conditions $\rho'(l) = \sigma(l) + \rho(l) \neq 0$, $\rho'^*(l) \neq 0$ and $p \geq 2$ are holds. However, if $\sigma(l) = \rho(l) = 0$ then the degree of the method (14) determined by using the following asymptotic equalities:

$$
\sum_{i=0}^{k-n} \alpha_i y(x + ih) - h^2 \sum_{i=0}^{k} y_i y^*(x + (i+\alpha) h) = O(h^{n+2}), h \to 0.
$$

These conditions can be transformed to the coefficients of the method (19) by using the system (20). And now let us consider the construction of some concrete methods with different degrees. Let us consider the simple case $\nu_i = l_i = 0$ ($i = 0, 1, \ldots, k$) and put $m = 1, k = 3$. In this case $p_{\text{max}} = 3k + 1 - m$ or $p_{\text{max}} = 9$. This method is unstable therefore let us consider construction of the stable methods with higher degrees. If the method (14) stable in the case $\nu_i = l_i = 0$ ($i = 0, 1, \ldots, k$) then receive that there is a stable method with the degree $p = 2k + 2 + m$. From here receive that in our case there is stable method with the degree $p = 9$, which can be written as:

$$
y_{n+2} = \frac{416}{313} y_{n+1} - \frac{103}{313} y_n + h(157 y_{n+1} + 1123 y_{n+2} + 8451 y_{n+3} - 2830 y''_n) / 25353
$$

$$+ h^2(-11 y''_{n+1} - 630 y''_{n+2} + 1557 y''_{n+3} - 92 y''_{n+4}) / 8451, \quad (24)$$

The remainder term can be written as: $103h^3 y''_{1/2} / 212965200$.

If to put $m = 0$, then for the stable methods receive $p_{\text{max}} = 2k + 2$ and for the value $k = 3$ there are exist the stable methods with the degree $p = 8$ which can be written as the following:

$$
y_{n+3} = \frac{1}{3} (y_{n+2} + y_{n+1} + y_n) + h(10781 y'_n + 22707 y'_{n+1} + 16659 y'_{n+2} + 5285 y'_{n+3}) / 27216
$$

$$- h^2(2099 y''_n - 7227 y''_{n+1} - 2853 y''_{n+2} - 979 y''_{n+3}) / 45360, \quad (25)$$

The remainder term can be written as: $3h^3 y''_{1/2} / 156800$.

And now let us construct hybrid method in simple form. For this aim, let us use the method of (15) and also the solution of the system (22). One of the stable methods constructed by the described way can be written as:

$$
y_{n+1} = y_n + h(29 y'_n + 24 y'_{n+1} - y''_{n+2}) / 180 + h(62 y'_{n+1/2} + 2 y'_{n+3/2}) / 90. \quad (26)$$

The forward-jumping method in very simple form can be written as follows:

$$
y_{n+1} = y_n + h(8 y'_n + 5 y'_{n+1} - y''_{n+2}) / 12, \quad (27)$$

which has the degree $p = 3$, but method (24) has the degree $p = 5$.

Let us note that in the case $m = 0$ and $k = 1$ from the formula (14) one can be received the following method for the values $y_i = \hat{y}_i = 0$ ($i = 0, 1, \ldots, k$):

$$
y_{n+1} = y_n + h(y'_{n+1/2,0} + y'_{n+1/2,\alpha}), \quad \alpha = \sqrt{3} / 6. \quad (28)$$

This method is stable and has the degree $p = 4$.

Let us apply these methods to solving of the following problems:

1. $y' = -\int_0^x y(s) ds, \quad y(0) = -1, \quad 0 \leq x \leq 2$ the exact solution is $y(x) = -\cos x$.

2. $y' = -4\int_0^x x y(s) ds, \quad y(0) = -1/4, \quad 0 \leq x \leq 2$ the exact solution is $y(x) = -\cos x^2 / 4$.

The results of the examples 1 and 2 for $h = 0.1$ tabulated in the Table 1.
Then one can be constructed the results have been shown that this method generalizes all known multistep methods.

And now let us tabulate the results receiving in the solving of the examples 1 and 2 for the step-size $h = 0.05$ in the Table 2.

### Table 1. The results of the examples 1 and 2 for $h = 0.1$.

| $x$     | Example (1) by the method (26) | Example (2) by the method (26) | Example (1) by the method (28) | Example (2) by the method (28) |
|---------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|
| 0.1     | 1.72E-10                       | 2.64E-10                       | 1.12E-9                        | 1.68E-10                       |
| 0.6     | 9.70E-10                       | 2.82E-8                        | 1.71E-6                        | 2.27E-7                        |
| 1.0     | 1.43E-9                        | 6.98E-8                        | 8.12E-6                        | 5.69E-6                        |
| 1.5     | 1.68E-9                        | 2.00E-7                        | 2.51E-5                        | 1.26E-4                        |
| 2.0     | 1.50E-9                        | 1.20E-8                        | 5.05E-5                        | 2.43E-4                        |

Remark. As was noted above for the construction of more exact methods some authors proposed to use the multistep second derivative methods. And by taking into account that the forward-jumping methods are more exact than the implicit methods here considered the construction of the stable methods of the type (13). Therefore let us consider the case $k = 2$, $s = 3$. And for the finding of the coefficients $\alpha_i$, $\beta_i$, $\gamma_i$ ($j = 0, 1, 2, i = 0, 1, 2, 3$) here proposed to use the following systems of algebraic equations which can be received from the system (22):

$$\sum_{i=0}^{k-m} \alpha_i = 0; \quad \sum_{i=0}^{k-m} i\alpha_i = \sum_{i=0}^{k-m} \beta_i, \quad \sum_{i=0}^{k-m} \frac{i^l}{l!} \alpha_i = \sum_{i=0}^{k} \left( \frac{i-1}{(l-1)!} \beta_i + \frac{i-2}{(l-2)!} \gamma_i \right) (l = 2, p).$$

By solving the system (29) have been constructed the methods (24), (25) and the following method:

$$y_{n+2} = \frac{1}{2} y_n + \frac{1}{2} y_{n+1} - \frac{h}{907200} (26912 y'_n - 652725 y'_{n+1} - 501525 y'_{n+2} - 22075 y''_{n+3}) + \frac{h^2}{60480} (10409 y''_{n+1} - 1537 y''_{n+2} - 8523 y''_{n+3} - 349 y'''_{n+4}) + \frac{173 h^3 y^{(10)}(\xi)}{16934400},$$

Thus have been demonstrated that there is stable forward-jumping methods with the degree $p > 2k + 2$. As was noted above the methods of the type (14) are more exact than the stable methods of the type (13). For the illustration of this explanation let us put $m = 0$ and $k = 1$. Then one can be constructed the stable methods of the type (14) will have the degree $p = 10$ (see for example [28]).

### Table 2. The results of the examples 1 and 2 for $h = 0.05$.

| $x$     | Example (1) by the method (26) | Example (2) by the method (26) | Example (1) by the method (28) | Example (2) by the method (28) |
|---------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|
| 0.1     | 5.41E-12                       | 5.79E-12                       | 6.25E-10                       | 9.75E-12                       |
| 0.6     | 3.04E-11                       | 8.26E-10                       | 2.37E-7                        | 1.80E-8                        |
| 1.0     | 4.52E-11                       | 2.18E-9                        | 1.07E-6                        | 7.24E-7                        |
| 1.5     | 5.33E-11                       | 5.76E-9                        | 3.25E-6                        | 1.67E-5                        |
| 2.0     | 4.81E-11                       | 1.88E-9                        | 6.43E-6                        | 2.55E-5                        |

3. Conclusion

Here has considered the construction of the general hybrid multistep second derivative methods of forward-jumping type. And has been shown that this method generalizes all known multistep methods...
with constant coefficients. Therefore, here are determined the advantages and disadvantages of proposed methods, which have been compared with the known multistep methods. Here also constructed some stable method of the hybrid multistep second derivative type. Has been given the new direction for construction of the stable methods with the high exactness, which are applied to solving of the initial-value problem for ODE, Volterra integro-differential equation and also Volterra integral equation. We hope that this direction will find its followers.

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