On validity of the original Bell inequality for the Werner nonseparable state

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Abstract

In quant-ph/0406139 we have introduced in a very general setting the new class of quantum states, density source-operator states, satisfying any classical CHSH-form inequality, and shown that any separable state belongs to this class. In the present paper, we prove that, the Werner nonseparable state on $\mathbb{C}^d \otimes \mathbb{C}^d$, $d \geq 2$, also belongs to the class of density source-operator states. Moreover, for any $d \geq 3$, the Werner state is in such a subclass of this class where each density source-operator state satisfies also the perfect correlation form of the original Bell inequality regardless of whether or not the Bell perfect correlation restriction is fulfilled.

1 Introduction

The present paper is a sequel to [1] where, in a very general setting, we have introduced the new class of quantum states satisfying the original CHSH inequality [2] and, more generally, every classical\(^1\) CHSH-form inequality.

Any state in this new class admits a special type dilation to an extended tensor product Hilbert space. We call this dilation a density source-operator and refer to a quantum state with this type of dilation as a density source-operator state. As we specify this in [1], the existence for a quantum state of a density source-operator is a purely geometrical property, and cannot be, in general, linked with the existence for this state of a local hidden-variable (LHV) model.

To analyze the relation between different classes of quantum states satisfying a classical CHSH-form inequality, we show in [1] that all separable states belong to the class of density source-operator states as a particular subclass and that, for the two qubit system, the Werner nonseparable state [3] is a density source-operator state. It is, however, well

\(^1\)The term "classical" specifies the validity in the frame of classical probability.
known [3,4] that the Werner nonseparable state on \( \mathbb{C}^d \otimes \mathbb{C}^d \) satisfies the original CHSH inequality for any dimension \( d \geq 2 \).

In the present paper, we prove:

- for any dimension \( d \geq 2 \), the Werner nonseparable state belongs to the class of density source-operator states on \( \mathbb{C}^d \otimes \mathbb{C}^d \);
- for any dimension \( d \geq 3 \), the Werner nonseparable state satisfies the original Bell inequality [5,6], in its perfect correlation form.

The latter earlier unknown property of the Werner state can be verified experimentally.

2 The Werner state as a density source-operator state

Consider on \( \mathbb{C}^d \otimes \mathbb{C}^d \), \( \forall d \geq 2 \), the Werner nonseparable quantum state [3]:

\[
\rho_d^{(w)} = \frac{1}{d^3} I_{\mathbb{C}^d \otimes \mathbb{C}^d} + \frac{2}{d^2} P_d^{(-)} = \frac{d+1}{d^3} I_{\mathbb{C}^d \otimes \mathbb{C}^d} - \frac{1}{d^2} V_d,
\]

where

\[
P_d^{(-)} = \frac{1}{2}(I_{\mathbb{C}^d \otimes \mathbb{C}^d} - V_d)
\]

is the orthogonal projection onto the antisymmetric subspace of \( \mathbb{C}^d \otimes \mathbb{C}^d \) and \( V_d \) is the flip operator on \( \mathbb{C}^d \otimes \mathbb{C}^d \), defined by the relation:

\[
V_d(\psi_1 \otimes \psi_2) := \psi_2 \otimes \psi_1, \quad \forall \psi_1, \psi_2 \in \mathbb{C}^d.
\]

Recall that \( V_d \) is self-adjoint, with

\[
\text{tr}[V_d] = d, \quad (V_d)^2 = I_{\mathbb{C}^d \otimes \mathbb{C}^d}.
\]

Below, we use the notion of a density source-operator, introduced, in general, in [1], section 3, and the notation for a density source-operator on \( \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d \), specified in [1], section 4.1.

Proposition 1 For any \( d \geq 2 \), the Werner nonseparable state \( \rho_d^{(w)} \) belongs to the class of density source-operator (DSO) states on \( \mathbb{C}^d \otimes \mathbb{C}^d \).

Proof. Consider first the case \( d = 2 \). For the two-qubit Werner state

\[
\rho_2^{(w)} = \frac{3}{8} I_{\mathbb{C}^2 \otimes \mathbb{C}^2} - \frac{1}{4} V_2,
\]
the density source-operator $T^{(2)}$ on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, with the partial traces

$$\operatorname{tr}_{(2)}^{(2)}[T^{(2)}] = \operatorname{tr}_{(3)}^{(2)}[T^{(2)}] = \rho_2^{(w)}$$

(6)

over the elements standing on the $j$-th place, $j = 2, 3$, of tensor products, has been introduced in [1], section 7, example 1, and has the form:

$$T^{(2)} = \frac{1}{4} I_{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2} - \frac{1}{8} V_2 \otimes I_{\mathbb{C}^2} - \frac{1}{8} (I_{\mathbb{C}^2} \otimes V_2)(V_2 \otimes I_{\mathbb{C}^2})(I_{\mathbb{C}^2} \otimes V_2).$$

(7)

To prove the statement for any $d \geq 3$, let us introduce on $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ the orthogonal projection

$$Q_d = \frac{1}{6} \left\{ I_{\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d} - V_d \otimes I_{\mathbb{C}^d} - I_{\mathbb{C}^d} \otimes V_d 
- (I_{\mathbb{C}^d} \otimes V_d)(V_d \otimes I_{\mathbb{C}^d})(I_{\mathbb{C}^d} \otimes V_d) 
+ (I_{\mathbb{C}^d} \otimes V_d)(V_d \otimes I_{\mathbb{C}^d}) + (V_d \otimes I_{\mathbb{C}^d})(I_{\mathbb{C}^d} \otimes V_d) \right\},$$

(8)

with

$$\operatorname{tr}[Q_d] = \frac{d(d-1)(d-2)}{6}. \quad (9)$$

Since in an orthonormal basis $\{e_n; n = 1, 2, ..., d\}$ in $\mathbb{C}^d$ the flip operator $V_d$ admits the representation

$$V_d = \sum_{n,m=1}^d |e_n\rangle\langle e_m| \otimes |e_m\rangle\langle e_n|,$$

(10)

the projection (8) can be written otherwise as:

$$6Q_d = I_{\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d} - \sum_{n,m=1}^d |e_n\rangle\langle e_m| \otimes |e_m\rangle\langle e_n| \otimes I_{\mathbb{C}^d}$$

$$- \sum_{n,m=1}^d I_{\mathbb{C}^d} \otimes |e_n\rangle\langle e_m| \otimes |e_m\rangle\langle e_n| - \sum_{n,m=1}^d |e_n\rangle\langle e_m| \otimes I_{\mathbb{C}^d} \otimes |e_m\rangle\langle e_n|$$

$$+ \sum_{n,m,k=1}^d |e_n\rangle\langle e_m| \otimes |e_m\rangle\langle e_k| \otimes |e_k\rangle\langle e_n| + \sum_{n,m,k=1}^d |e_m\rangle\langle e_n| \otimes |e_k\rangle\langle e_m| \otimes |e_n\rangle\langle e_k|.$$  

(11)

From (11) and (2) it follows that, for any $j = 1, 2, 3$,

$$\operatorname{tr}_{(j)}^{(d)}[Q_d] = \frac{d-2}{6} (I_{\mathbb{C}^d \otimes \mathbb{C}^d} - V_d)$$

$$= \frac{d-2}{3} P_d^{(-)}.$$  

(12)

For any dimension $d \geq 3$, consider on $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ the density operator

$$T^{(d)} = \frac{1}{d^4} I_{\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d} + \frac{6}{d^2(d-2)} Q_d.$$  

(13)
It is easy to verify that each density operator
\[
\text{tr}_{\mathbb{C}^d}^{(j)}[T^{(d)}], \quad \forall j = 1, 2, 3,
\] (14)
on \mathbb{C}^d \otimes \mathbb{C}^d, reduced from \text{T}^{(d)}\text{◮◮}, coincide with the Werner state, that is:
\[
\text{tr}_{\mathbb{C}^d}^{(1)}[T^{(d)}] = \text{tr}_{\mathbb{C}^d}^{(2)}[T^{(d)}] = \text{tr}_{\mathbb{C}^d}^{(3)}[T^{(d)}] = \rho_d^{(w)}.
\] (15)
Hence, by definition of a density source-operator for a quantum state (cf. [1], section 3), the operator \text{T}^{(d)}\text{◮◮} represents a density source-operator for the Werner state \rho_d^{(w)}, \forall d \geq 3.
Thus, for any dimension \(d \geq 2\), the Werner state \rho_d^{(w)} on \mathbb{C}^d \otimes \mathbb{C}^d is a density source-operator state.

As we prove in [1], any density-source operator state satisfies every classical CHSH-form inequality.

### 3 The Werner state and the original Bell inequality

Consider an Alice/Bob joint generalized quantum measurement\(^2\), with outcomes \(|\lambda_1| \leq 1\) on the side of Alice and outcomes \(|\lambda_2| \leq 1\) on the side of Bob, performed on a bipartite quantum system in a state \(\rho\) on \mathbb{C}^d \otimes \mathbb{C}^d and described by the POV measure
\[
M_1^{(a)}(d\lambda_1) \otimes M_2^{(b)}(d\lambda_2),
\] (16)
which is specified by a pair \((a, b)\) of some parameters on the sides of Alice and Bob, respectively. Here, \(M_1^{(a)}\) and \(M_2^{(b)}\) are the POV measures, describing, respectively, marginal experiments of Alice and Bob.

Under the joint quantum measurement (16), the expectation value of the product \(\lambda_1\lambda_2\) of outcomes is given by
\[
\langle \lambda_1\lambda_2 \rangle_{\rho}^{(a,b)} := \int \lambda_1\lambda_2 \text{tr}[\rho(M_1^{(a)}(d\lambda_1) \otimes M_2^{(b)}(d\lambda_2))]
\] (17)
\[
= \text{tr}[\rho(W_1^{(a)} \otimes W_2^{(b)})],
\]
with \(W_1^{(a)}, W_2^{(b)}\) being quantum observables on \mathbb{C}^d, defined by the relations
\[
W_1^{(a)} := \int \lambda_1 M_1^{(a)}(d\lambda_1), \quad \|W_1^{(a)}\| \leq 1,
\] (18)
\[
W_2^{(b)} := \int \lambda_2 M_2^{(b)}(d\lambda_2), \quad \|W_2^{(b)}\| \leq 1,
\]
and representing the marginal experiments on the sides of Alice and Bob, respectively.

\(^2\)See [7,1] for the description of an Alice/Bob joint quantum measurement in a very general setting.
Consider now three Alice/Bob joint quantum measurements (19), specified by pairs of parameters:

\[(a, b_1), \ (a, b_2), \ (b_1, b_2), \quad (20)\]

and satisfying the condition

\[W^{(b_1)}_1 = W^{(b_1)}_2. \quad (20)\]

As we discuss this in [7,1], the latter condition does not imply the validity of the Bell perfect correlation/anticorrelation restrictions \(\langle \lambda_1 \lambda_2 \rangle^{(b_1, b_2)}_\rho = \pm 1\) and is usually fulfilled under Alice/Bob joint quantum measurements. The condition (20) is, for example always true under Alice/Bob projective spin measurements.

From the relation (15) and proposition 2 in [1], it follows that if the above specified Alice/Bob joint measurements are performed on a bipartite quantum system in the Werner nonseparable state \(\rho_d^{(w)}\) on \(\mathbb{C}^d \otimes \mathbb{C}^d\) then, for any dimension \(d \geq 3\), the expectation values

\[\langle \lambda_1 \lambda_2 \rangle_a^{(a,b_1)}_\rho_d^{(w)}, \quad \langle \lambda_1 \lambda_2 \rangle_a^{(a,b_2)}_\rho_d^{(w)}, \quad \langle \lambda_1 \lambda_2 \rangle_{b_1,b_2}^{(b_1,b_2)}_\rho_d^{(w)} \quad (21)\]

satisfy the perfect correlation form of the original Bell inequality, that is:

\[
\begin{align*}
|\langle \lambda_1 \lambda_2 \rangle_a^{(a,b_1)}_\rho_d^{(w)} & - \langle \lambda_1 \lambda_2 \rangle_a^{(a,b_2)}_\rho_d^{(w)} | \quad (22) \\
& = \left| \text{tr}[\rho_d^{(w)}(W_1^{(a)} \otimes W_2^{(b_1)})] - \text{tr}[\rho_d^{(w)}(W_1^{(a)} \otimes W_2^{(b_2)})] \right| \\
& \leq 1 - \text{tr}[\rho_d^{(w)}(W_1^{(b_1)} \otimes W_2^{(b_2)})] \\
& = 1 - \langle \lambda_1 \lambda_2 \rangle_{b_1,b_2}^{(b_1,b_2)}_\rho_d^{(w)}. 
\end{align*}
\]

Written otherwise, the original Bell inequality (22) for the Werner state \(\rho_d^{(w)}, \forall d \geq 3\), reads

\[
\begin{align*}
|\text{tr}[\rho_d^{(w)}(J^{(a)} \otimes J^{(b_1)})] & - \text{tr}[\rho_d^{(w)}(J^{(a)} \otimes J^{(b_2)})] | \quad (23) \\
& \leq 1 - \text{tr}[\rho_d^{(w)}(J^{(b_1)} \otimes J^{(b_2)})] 
\end{align*}
\]

and is valid for any three quantum observables \(J^{(a)}, J^{(b_1)}, J^{(b_2)}\) on \(\mathbb{C}^d\), with the operator norms \(\| \cdot \| \leq 1\).

**Remark 1** Notice that, as is the case for any density source-operator state specified in proposition 2 in [1], for the Werner state \(\rho_d^{(w)}, \forall d \geq 3\), the perfect correlation form of the original Bell inequality is valid regardless of whether or not the Bell perfect correlation restriction \(\langle \lambda_1 \lambda_2 \rangle^{(b_1, b_2)}_\rho_d^{(w)} = 1\) is satisfied.

Thus, for any dimension \(d \geq 3\), the Werner nonseparable state on \(\mathbb{C}^d \otimes \mathbb{C}^d\) satisfies both classical probability constraints - the original CHSH inequality and the original Bell inequality, in its perfect correlation form.

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3See [5,6].
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