Analytical Fresnel laws at generic curved interfaces

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Fresnel laws, the quantitative information of the amount of light that is reflected from a planar interface in dependence on its angle of incidence, are at the core of ray optics. However, these formulae do not hold at curved interfaces and deviations are appreciable when wavelength and radius of curvature are comparable. This is of particular interest for optical microcavities that play an important role in many modern research fields and applications such as microlasers. Their convexly curved interfaces modify Fresnel’s law in a characteristic manner: the onset of total internal reflection is shifted to angles larger than the critical angle. Here, we derive the missing Fresnel laws for concavely curved refractive index boundaries, enabling the analytical description of light in complex mesoscopic optical structures that will be important in future nano- and microphotonic applications.

I. INTRODUCTION

The Fresnel equations derived by Augustine-Jean Fresnel in 1823 quantify the amount of the reflected, $R$, and transmitted, $T = 1 - R$, intensity of a plane wave incident under a certain angle of incidence $\chi$ at a planar interface between two isotropic optical media of refractive indices $n_1$ and $n_2$. In their original form, they apply to flat interfaces with a relative index of refraction $n = n_1/n_2$. For internal reflection configurations ($n>1$, reflection at the optically thinner medium) total internal reflection occurs above the critical angle of incidence given by $\chi_{cr} = \arcsin 1/n$.

Whereas the effect of the curvature is negligible in many cases when a ray optics description is adequate, it has to be taken into account when the radii $a$ of curvature of the interfaces become comparable to the wavelength $\lambda$ of the incident light. This applies in particular to optical microcavities with typical sizes of a few dozens micrometers across operated with infrared light. A reliable description of the reflection process that captures the deviations from the planar case is indispensable to ensure the correct prediction of the far-field emission of microcavities as well as semiclassical corrections to the ray picture. This implies in particular to deviations from Snell’s law as a result of the so-called Fresnel-filtering effect. These deviations have been studied in detail for convexly shaped interfaces based on the analytical Fresnel formulæ available in this case to capture the reflection and transmission of light beams at a curved interface with radius of curvature $a$ as function of the (relative) refractive index $n$, the wave number of light $k = 2\pi/\lambda$, and the angle of incidence $\chi$.

Here, we derive the missing analytical formulæ for concavely curved boundaries as illustrated in Fig. 1. They will provide the basis for reliable ray-based simulations of photonic devices with convex or concave interfaces or complex boundaries that combine curved segments of both types. We point out that the analytical Fresnel laws that we present here apply to this general situation. To this end, the local radius of curvature $a$ has to be used in the equations, i.e. the interface at the point of incidence of the incident ray is approximated by a cylinder of radius $a$.

The paper is organized as follows: We first state the analytical results for the Fresnel equations at curved interfaces (generalized Fresnel laws) both at convex and concave interfaces and discuss the deviations from the planar case. We then outline their derivation based on the transfer matrix approach that nicely illustrates the dual character of the convex-concave reflection situation.

II. FRESNEL REFLECTION COEFFICIENTS AT CURVED INTERFACES

The Fresnel reflection coefficient for the reflected amplitude ratio $r$ of light propagating in a medium with relative refractive index $n > 1$ (internal reflection configuration) with an angle of incidence $\chi$ are known to read for transverse magnetic (TM) and transverse electric (TE), note that we use the convention where TE features the Brewster angle $\chi_{Br} = \arctan(1/n)$ polarization, respectively,

$$r_p^{TM} = \frac{n \cos \chi - \cos \eta}{n \cos \chi + \cos \eta} \quad (1)$$
$$r_p^{TE} = \frac{-n \cos \eta - \cos \chi}{n \cos \eta + \cos \chi} \quad (2)$$

Here, $\eta$ is the angle of the transmitted (refracted) light.
and given by Snell’s law via \(n \sin \chi = \sin \eta\).

\[ \begin{align*}
\text{Fig. 2.} & \quad \text{Fresnel reflection coefficients } R = |r|^2, |r_{\text{cx}}|^2, |r_{\text{cv}}|^2 \text{ for TM-polarized light and } n = 1.5 \\
& \quad \text{at planar interface (black) and at curved interfaces with } ka = 50 \text{ (green), and } ka = 15 \text{ (red). Ray path reversal amounts to switching convex/concave and requires to renormalise the wavenumber } ka \text{ by } n \text{ (orange and blue curves). Note the delayed onset of total internal reflection as } ka \text{ is reduced.}
\end{align*} \]

\[ \text{Fig. 3.} & \quad \text{Same as Fig. 2 above, but for TE-polarized light. Note, however, that the Brewster angle reflectivity remains small but finite at curved interfaces. As for the TM-case, deviations from the planar limit are important for reflection above the critical angle } \chi_{\text{cr}} = \arcsin 1/n \text{ where curved interfaces are more leaky than planar boundaries.}
\]

### A. Convex case

We shall now see that the backbone of these equations is transferred to the curved interfaces. For the convex case, see\(^1\) and the alternative derivation below in Sec. III, we find the convex Fresnel reflection amplitude \(r_{\text{cv}}\) (assuming \(n > 1\))

\[ r_{\text{cv}} = \frac{\cos \chi + i F_m(ka)}{\cos \chi - i F_m(ka)}. \tag{3} \]

Here, \(ka\) is the dimensionless wave number given as product of the wave number \(k\) in free space and the local radius of curvature \(a\). The real number \(m\) is related to the angle of incidence \(\chi\) via \(m = nka \sin \chi\) and determines the order of the Hankel function \(H_{l(2)}\) of the first (second) kind (cf.\(^12\) for details), and

\[ \begin{align*}
F_{\text{TM}}^m(z) &= \frac{H_{l-1}^1(z)}{n H_l^1(z)} \sin \chi \quad \text{(4)} \\
F_{\text{TE}}^m(z) &= n^2 F_{\text{TM}}^m(z). \quad \text{(5)}
\end{align*} \]

### B. Concave case

The result for the concave reflection amplitude \(r_{\text{cv}}\) the central result of this Letter, has the very similar, noteworthy structure,

\[ r_{\text{cv}} = \frac{\cos \chi - i F_m^*(ka)}{\cos \chi + i F_m^*(ka)} \tag{6} \]

with the complex conjugation \((\cdot)^*\) and \(F_m\) as given in Eqs. (4,5). It implies that the reflected intensity \(R\) is the same at a convex and concave interface, respectively. Note however, that reversal of the light path and the accompanying change from a concave to a convex interface boundary (cf. Fig. 1) will change \(R\) as we discuss below in the context of the convex-concave-duality.

The results are illustrated in Figs. 2 and 3 for TM and TE polarized light, respectively. The deviation from the planar (flat) curve (black curve) is clearly visible and characterized by a much later onset of the regime of total internal reflection. This effect is more pronounced for smaller wavenumbers \(ka\) (higher curvature). The planar case result is approached in the limit \(ka \to \infty\). The reduced total internal reflection at curved boundaries implies a deterioration of the cavity quality \((Q)\) factor and is thus important for many applications. We also point out that the drop of the reflectivity in TE polarization at the Brewster angle is less pronounced at all curved refractive index boundaries and does not (quite) reach zero\(^4,10\).

### C. External reflection configuration, \(n < 1\)

Having considered the important case of optical microcavities where \(n > 1\), we now generalize the Fresnel equations for curved boundaries to relative refractive indices \(n < 1\) and find.

\[ \begin{align*}
\tilde{r}_{\text{cx}} &= -\frac{\cos \eta + i G_m^*(ka)}{\cos \eta + i G_m(ka)} \quad \text{(7)} \\
\tilde{r}_{\text{cv}} &= -\frac{\cos \eta - i G_m^*(ka)}{\cos \eta - i G_m(ka)} \quad \text{(8)}
\end{align*} \]

with

\[ \begin{align*}
G_{\text{TM}}^m(z) &= \frac{n H_{l-1}^1(z)}{H_l^1(z)} - \sin \eta, \quad G_{\text{TE}}^m(z) = G_{\text{TM}}^m(z)/n^2. \quad \text{(9)}
\end{align*} \]

The results are shown in Figs. 4 and 5. As before, the reflectivity remains finite around the Brewster angle.
The most striking difference to the planar case is the rather low reflectivity near grazing incidence at curved interfaces, confirming their larger leakage that we already observed for $n > 1$.

D. Convex-concave duality

The principle of ray path reversal as well as the transfer matrix approach outlined below suggest to consider a convex interface together with its concave counterpart as two possible deviations from the planar case for a given $n$. This implies, however, a renormalisation of the reference wavenumber $ka$ in the reversed situation by a factor $1/n$. These results are included in Figs. 2 – 5. In Figs. 2 and 3 the effect of curvature increases thereby (the factor $n$ can be captured in a decrease from $a$ to $a/n$, or alternatively, in an increase from $\lambda$ to $n\lambda$) and the curves are further away from the ray limit. For $n < 1$, the same reasoning yields the opposite behavior, cf. Figs. 4 and 5.

To this end we point out a symmetry relation between the convex and concave reflectance for a given order of the Hankel function $n$: $|r_{cx}(n \rightarrow n_2)|^2 = |r_{cv}(n \rightarrow n_2)|^2$, i.e., both coincide and deviate in the same manner from the planar case result. Note however that $r_{cx}$ and $r_{cv}$ differ in a phase such that $r_{cx}(n \rightarrow n_2) = r_{cv}(n \rightarrow n_2)$. Deviations from this symmetry may occur when light beams are considered.

III. TRANSFER MATRICES, RESONANCES, AND FRESNEL COEFFICIENTS

A. The transfer matrix

The transfer matrix relates incoming and outgoing wave amplitudes at (dielectric) interfaces, cf. Fig. 6. Here $A_0$, $B_0$ and $A_1$, $B_1$ are the amplitudes of the incoming and outgoing waves, respectively, being related

\[ M A = (A_1 B_1) = M (A_0 B_0) \]

where

\[ M = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} \]

Due to the presence of time-reversal symmetry, the $2 \times 2$ matrix $M$ takes the preceding form, where $a$ and $b$ are complex numbers. The relation to Fresnel coefficients is achieved when writing each outgoing amplitude in terms of a reflected and a transmitted contribution, see Fig. 6, that are related by

\[ A_1 = r'B_1 + tA_0 \]
\[ B_0 = t'B_1 + rA_0 \]

Here $r$ and $t$ ($r'$ and $t'$) are the inner (outer) Fresnel reflection and transmission coefficients, respectively. Using the symmetry relations yields the following representation of $M$ that holds independent of the curvature of the interface,

\[ M = \begin{pmatrix} 1/t' & r'/t' \\ r'/t' & 1/t' \end{pmatrix} \]

Fig. 5. Same as Fig. 4, but for TE-polarized light.

Fig. 4. Same as Fig. 2 above, but now for the external reflection configuration, $n < 1$. Note that the reflectivity at curved boundaries remains moderate even at grazing incidence. As before, the planar limit is reached as $ka$ is increased.

Fig. 6. Incident, transmitted, and reflected ray amplitudes are related via the transfer matrix at a curved interface ($\alpha_p$) or a plane. Here, $n$ is the refractive index of the cavity that we assume to be embedded in air. See text for details.
B. From the cavity transfer matrix to Fresnel coefficients

In the following we will use the electromagnetic wave functions $\psi_0, \psi_1$ on either side of the interface (cf. Fig. 6) which are nothing else but the $z, \psi$ numbers obtained by solving the equation (19).

Whereas in the planar case the $\psi$’s are plane waves and a plane divides space into two half planes, a circle/cylinder takes this role in the presence of curvature. Consequently, we use Hankel functions as incoming and outgoing waves $\psi$ because they accommodate the cylindrical symmetry that we assume (locally) for a curved interface.

The derivation of Fresnel coefficients at convex interfaces is based on finding the resonances in a disk cavity, and relating their real and imaginary part to the Fresnel reflection coefficient.

Here, we use an approach that is applicable to concave interfaces and a plane divides space into two half planes, a circle/cylinder takes this role in the presence of curvature.

Consequently, we use Hankel functions as incoming and outgoing waves $\psi$ because they accommodate the cylindrical symmetry that we assume (locally) for a curved interface.

The analogy to the single-transmission case is established by introducing resonance-dressed Fresnel coefficients, characterized by the presence of an additional phase, namely

$$r_r = r e^{2i n_x k_x a_p}, \quad r_r' = r e^{-2i k_x a_p}$$

$$t_r = t e^{i(n_x-1)k_x a_p}, \quad t_r' = t e^{-i(n_x-1)k_x a_p}$$

The straightforward explanation of the resonance formation is gained when expressing the amplitude $I$ in terms of a geometric series of resonance-dressed Fresnel coefficients, nicely illustrating the successive reflections of light rays,

$$I = \frac{t_r'}{1 - r_r} = t_r' + t_r r_r + t_r' r_r^2 + t_r' r_r^3 + \cdots.$$  

Note that the wave numbers $k$ at resonance can be obtained by solving the equation $1 - r_r = 0$ and that $|I/2|^2$ will oscillate with its maxima reached at resonant wave numbers $k$.

C. Fresnel coefficients at curved interfaces

We adapt the ansatz for the planar cavity (Eqs. (14a,b)) to the rotationally invariant case relevant for curved interfaces:

$$\psi_0(r, \varphi) = \frac{I}{2} (H^1_m(nkr) + H^2_m(nkr)) = I J_m(nkr) e^{im\varphi}$$

$$\psi_1(r, \varphi) = (H^2_m(kr) + S H^1_m(kr)) e^{im\varphi}.$$

Note that we switch from this convex case to the concave situation by exchanging incoming and outgoing Hankel functions (i.e., indices 1 and 2) while properly accounting for the scaling factor $n$, whereas the distinction between TM and TE polarized light originates in the well-known difference in the transition conditions:

$$\psi(a_p - 0) = \psi(a_p + 0)$$

$$\psi'(a_p - 0) = \begin{cases} \psi'(a_p + 0) & \text{TM} \\ n^2 \psi'(a_p + 0) & \text{TE}. \end{cases}$$

The transfer matrix takes then the following form:

$$M^{(TM)} = \frac{1}{D} \begin{pmatrix} D_{12} & D_{22} \\ -D_{11} & -D_{21} \end{pmatrix}$$

$$M^{(TE)} = \frac{\sin \chi}{D} \left( n - \frac{1}{n} \right) \begin{pmatrix} -Q_{12} & -Q_{22} \\ Q_{11} & Q_{21} \end{pmatrix}$$

with (realizing that $a_p$ takes the role of $a$, and $\{\alpha, \beta\} \in \{1, 2\}$)

$$D_{\alpha\beta} = H^\alpha_m(nka) H^{\beta}_{m-1}(ka) - n H^\alpha_{m-1}(nka) H^\beta_m(ka)$$

$$Q_{\alpha\beta} = H^\alpha_m(nka) H^{\beta}_m(ka).$$

By analyzing the resonance-dressed (or multiple) reflection coefficients obtained from Eqs. (21, 22), we find the following additional phases in the Fresnel reflection coefficients (cf. Eqs. (16,17) for the planar case):

convex: $H^1_m(nka), \quad$ concave: $H^2_m(ka)$

The resulting reflection coefficient at a convex interface reads thus

$$r^{(TM)} = \frac{H^1_{m-1}(ka)}{H^1_m(ka)} - \frac{n H^2_{m-1}(nka)}{H^2_m(nka)}$$

$$\frac{H^1_{m-1}(nka)}{H^2_m(nka)} \approx e^{-i(x - \frac{\pi}{2})} = \sin \chi + i \cos \chi$$

This coefficient will not oscillate when changing the argument $nka$. It is therefore suitable to describe open segment boundaries (which also releases the integer-$m$ constraint required for rotational symmetry). We proceed by simplifying the expressions using the large-argument approximations, e.g.
for \( nka \gg 1 \), and analogous relations for the \( n < 1 \), \( k\alpha \gg 1 \) situation (that even hold for moderate values \( k\alpha \)) to arrive at (7) and (8) for the concave case (exchanging 1,2), respectively. For derivation of the TE-case results in Chapter II, we proceed as before but use the appropriate transfer matrix (22).

To summarize, we have completed the picture of Fresnel coefficients at generic curved interfaces by deriving a formula for the concave case in addition to the previously known convex case, and illustrated the capability of transfer matrix approaches.

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