On the Factoriality of $q$-Deformed Araki-Woods von Neumann Algebras

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Abstract: The $q$-deformed Araki-Woods von Neumann algebras $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ are factors for all $q \in (-1, 1)$ whenever $\dim(\mathcal{H}_\mathbb{R}) \geq 3$. When $\dim(\mathcal{H}_\mathbb{R}) = 2$ they are factors as well for all $q$ so long as the parameter defining $(U_t)$ is ‘small’ or 1 (trivial) as the case may be.

1. Introduction

To any strongly continuous orthogonal representation $(U_t)$ of $\mathbb{R}$ on a real Hilbert space $\mathcal{H}_\mathbb{R}$ with $\dim(\mathcal{H}_\mathbb{R}) \geq 2$, Hiai in [Hi03] constructed the $q$-deformed Araki-Woods von Neumann algebras (hereafter abbreviated as $q$-Araki-Woods algebras) for $-1 < q < 1$. These are $W^*$-algebras usually arising from non-tracial representations of the $q$-commutation relations (a Yang–Baxter deformation of the canonical commutation relations), thereby yielding an interpolation between the Bosonic and Fermionic statistics. Hiai’s functor is a fusion of Shlyakhtenko’s free CAR functor [S97] (associated $W^*$-algebras are called free Araki-Woods factors) and the $q$-Gaussian functor of Bożejko-Speicher (associated $W^*$-algebras are called Bożejko-Speicher factors) (see [BS91]). All of these constructions are generalizations of Voiculescu’s $C^*$-free Gaussian functor, which is the central object of study in free probability [VDN92]. Note that when $q = 0$, i.e. the Yang–Baxter deformation is trivial (free case), Hiai’s functor reduces to Shlyakhtenko’s functor. However, when $(U_t)$ is trivial, Hiai’s functor reduces to the $q$-Gaussian functor. Further, when $q = 0$ and $(U_t)$ is trivial, one obtains Voiculescu’s functor.

The $q$-Araki-Woods algebras are quite complicated objects. Structural properties of the Bożejko-Speicher factors have been studied in [A11,BS91,BKS97,D14,R05, N04,Sn04,S04,S09], and those of the free Araki-Woods factors have been studied in [H09,HR11,BHV15]; though, this list is by no means complete. On the contrary, very little is known about the $q$-Araki-Woods algebras (see [N04,N06,W21]). In fact, the simplest question regarding its factoriality is not known in full generality. This is because,
unlike the case when $q = 0$, there is very little room to perform meaningful calculations with the standard generators of these algebras owing to the complicated nature of the scalar product (of the GNS space) and the interference of the modular group. The case when $\dim(\mathcal{H}_\mathbb{R}) = 2$ is the hardest and most notorious.

This paper attempts the factoriality problem of the $q$-Araki-Woods algebras. Before describing our results, though, we note the past efforts in this direction.

The factoriality of the Bożejko-Speicher algebras was not a single-handed attempt. In [BKS97], the factoriality was established by Bożejko, Kümmerer and Speicher when $\dim(\mathcal{H}_\mathbb{R})$ is infinite. By making careful estimates of norms of certain operators on the $q$-deformed full Fock space, Śniady established the factoriality when $\dim(\mathcal{H}_\mathbb{R})$ is finite but greater than a constant depending on $q$. It was finally settled in [R05] by showing that any standard generator of the Bożejko-Speicher algebra generates a strongly mixing (see [CFM] for defn.) MASA in the ambient algebra. Therefore, the center of the algebra gets arrested in two orthogonal (with respect to the vacuum state) MASAs and is thus reduced to scalars. By using freeness and modular data, Shlyakhtenko established the factoriality of the free Araki-Woods algebras in [S97].

Hiai established the factoriality of the $q$-Araki-Woods algebras in [Hi03] in the case when the dimension of the almost periodic part of $(U_t)$ is infinite or $(U_t)$ is weakly mixing, by showing that the centralizer of the vacuum state has trivial relative commutant (irreducible inclusion). Unfortunately, there is a gap in the proof of [Hi03, Thm. 3.2]. To be precise, Hiai’s proof holds only in the case when the set of eigenvalues of the analytic generator of $(U_t)$ has a limit point in $\mathbb{R}$ other than 0. Without this assumption, the conclusion ‘$\varphi(y^* x) = 0$’ in the last equation in [Hi03, Thm. 3.2] would fail, and hence, the final statement cannot be concluded. Using the theory of free monotone transport (see [Ne15, Thm. 4.5]), Nelson proved that when $\mathcal{H}_\mathbb{R}$ is finite-dimensional, the $q$-Araki-Woods algebras are isomorphic to the free Araki-Woods factors for sufficiently small values of $|q|$; in particular, they are factors, in this case.

[BM] closely followed [R05]. In [BM], the authors observe that a non-zero vector $\xi \in \mathcal{H}_\mathbb{R}$ fixed by $(U_t)$ enables the construction of an orthonormal basis of analytic vectors (of the GNS space) which behaves well as long as one only considers its interaction with elements of the algebra $v N(s_q(\xi))$, where $s_q(\xi)$ is the standard self-adjoint generator of the $q$-Araki-Woods algebra corresponding to $\xi$. This allows exploiting the ideas in [R05] to show that $v N(s_q(\xi))$ is a strongly mixing MASA living inside the centralizer of the vacuum state. Using this, the factoriality of the $q$-Araki-Woods algebras was shown to be true if $(U_t)$ is not ergodic or has a non-zero weakly mixing component. Irreducibility of the centralizer was also obtained when $(U_t)$ is not ergodic, and the almost periodic component is at least two-dimensional. At the same time, a proof of the same statement on factoriality was obtained independently by Skalski and Wang [SW]; note that the phenomenon of mixing is implicit in this proof too.

When $(U_t)$ is ergodic and almost periodic, the above ideas involving MASAs freeze. This is because the standard self-adjoint generators no longer generate abelian von Neumann subalgebras with appropriate conditional expectations or operator-valued weights. In fact, it is far worse that these algebras are quasi split [BM2] and therefore admit very large relative commutants.

Given the previous attempts, we are left to deal with the case when $\mathcal{H}_\mathbb{R} = \mathbb{R}^2 \oplus K_\mathbb{R}$, where $K_\mathbb{R}$ is a real Hilbert space (could be 0), and $\mathbb{R}^2$ is reducing subspace for $(U_t)$ with associated subrepresentation being ergodic. As discussed before, we cannot rely on MASAs anymore, but our strategy is to leverage with a bit of ‘mixing’. The analysis is split into two cases.
When $K_R = 0$, denote $A$ to be the algebra generated by the centralizer and the commutant of the ambient algebra. The novelty of this approach is to identify a suitable subspace inside the GNS space of the centralizer which possesses ‘sufficient mixing’ (the associated subspace in [BM,R05] was the GNS space of a MASA), and use an appropriate orthonormal basis of that subspace to track a vector $\xi$, so that the cyclic subspace $A\xi$ fully captures the ‘size’ of the relative commutant of the centralizer. However, there is a payoff. Since we are working with a ‘mixing subspace’, we lose algebraic techniques (most importantly cannot locate unitaries) and depend on norm estimates of operators involving creation and annihilation operators. This forces us a bargain with the characteristic parameter that defines the two-dimensional representation, but our results remain valid for all $q \neq 0$.

When $K_R \neq 0$, a similar idea with slight modification works. This time, the increment in dimension allows us to choose unitaries from an orthogonal subalgebra and frees us from any bargain with the characteristic parameter as was in the previous case, and the conclusions are no longer subject to any constraints.

The key to the factoriality is Lemma 3.12 in which ‘some mixing’ is analyzed to control the relative commutant of the centralizer. The main results of this paper are summarized as follows:

**Theorem:** Let $\mathcal{H}_R = \mathbb{R}^2 \oplus K_R$, where $K_R$ is a real Hilbert space (could be 0), and let $\mathbb{R}^2$ be reducing subspace for $(U_t)$ with associated subrepresentation being ergodic. Let $\lambda \in (0, 1)$ be the parameter that defines $(U_t)$ on $\mathbb{R}^2$. Then, the following holds:

1. if $\dim(\mathcal{H}_R) = 2$ and $\lambda$ is small (depending on $q$), then the centralizer of the $q$-quasi free (vacuum) state is irreducible if $q \neq 0$;
2. if $\dim(\mathcal{H}_R) \geq 3$, the $q$-Araki-Woods algebras are factors for all $-1 < q < 1$;
3. if the almost periodic part of $(U_t)$ is sufficiently large, then the centralizer of the $q$-quasi free state is irreducible for all $-1 < q < 1$.

This paper is organized as follows. In Sect. 2, we lay out all the technical prerequisites that are needed to address the problem. The technical lemmas that will be used to deal with factoriality is divided into two groups under Sect. 3; the case when $\dim(\mathcal{H}_R) = 2$ and $(U_t)$ is ergodic appears in Sect. 3.1 and that for all cases ($\dim(\mathcal{H}_R) \geq 2$) appears in Sect. 3.2. The main theorems on factoriality appear in Sect. 4. Irreducibility of the centralizer is discussed in Sect. 5.

### 2. Preliminaries

In this section, we accumulate some well known facts about the $q$-deformed Araki-Woods von Neumann algebras constructed by Hiai in [Hi03] that will be indispensable for our purpose. As a convention, all Hilbert spaces in this paper are separable, all von Neumann algebras have separable preduals, inclusions of von Neumann algebras are unital and inner products are linear in the second variable. This section has overlap with [BM, §2].

#### 2.1. Hiai’s construction.

Let $\mathcal{H}_R$ be a real Hilbert space with $\dim(\mathcal{H}_R) \geq 2$ and let $(U_t)_{t \in \mathbb{R}}$ be a strongly continuous orthogonal representation of $\mathbb{R}$ on $\mathcal{H}_R$. Let $\mathcal{H}_C = \mathcal{H}_R \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of $\mathcal{H}_R$. Denote the inner product and norm on $\mathcal{H}_C$ by $\langle \cdot, \cdot \rangle_{\mathcal{H}_C}$ and $\| \cdot \|_{\mathcal{H}_C}$ respectively. Identify $\mathcal{H}_R$ in $\mathcal{H}_C$ by $\mathcal{H}_R \otimes 1$. Thus, $\mathcal{H}_C = \mathcal{H}_R + i\mathcal{H}_R$, and as a real Hilbert space the inner product of $\mathcal{H}_R$ in $\mathcal{H}_C$ is given by $\Re(\langle \cdot, \cdot \rangle_{\mathcal{H}_C})$. Consider
the bounded anti-linear (complex (left) conjugation) operator $\mathcal{J} : \mathcal{H}_C \to \mathcal{H}_C$ given by $\mathcal{J}(\xi + i\eta) = \xi - i\eta, \xi, \eta \in \mathcal{H}_R$, and note that $\mathcal{J}\xi = \xi$ for $\xi \in \mathcal{H}_R$. Moreover,

$$
(\xi, \eta)_{\mathcal{H}_C} = \overline{(\eta, \xi)_{\mathcal{H}_C}} = (\eta, \mathcal{J}\xi)_{\mathcal{H}_C}, \quad \text{for all } \xi \in \mathcal{H}_C, \eta \in \mathcal{H}_R.
$$

By abuse of notation we denote the linear extension of $(U_t)$ on $\mathcal{H}_C$ by the same notation, which is again a strongly continuous one-parameter group of unitaries in $\mathcal{H}_C$. Let $A$ denote the analytic generator of $(U_t)$. Then $A$ is positive, nonsingular and self-adjoint. Note that $\mathcal{J}A = A^{-1}\mathcal{J}$.

Introduce a new inner product on $\mathcal{H}_C$ by $(\xi, \eta)_U = \left(\frac{2}{1+iA} - 1\right)\xi, \eta \in \mathcal{H}_C$, and let $\|\cdot\|_U$ denote the associated norm on $\mathcal{H}_C$. Let $\mathcal{H}$ denote the complex Hilbert space obtained by completing $(\mathcal{H}_C, \|\cdot\|_U)$. The inner product and norm of $\mathcal{H}$ will respectively be denoted by $(\cdot, \cdot)_U$ and $\|\cdot\|_U$ as well. Then, $(\mathcal{H}_R, \|\cdot\|_{\mathcal{H}_C}) \ni \xi \mapsto \tilde{\xi} \in (\mathcal{H}_C, \|\cdot\|_U) \subseteq (\mathcal{H}, \|\cdot\|_U)$, is an isometric embedding of the real Hilbert space $\mathcal{H}_R$ in $\mathcal{H}$ (in the sense of [S97]). With abuse of notation, we will identify $\mathcal{H}_R$ with its image $\iota(\mathcal{H}_R)$. Then, $\mathcal{H}_R \cap i\mathcal{H}_R = \{0\}$ and $\mathcal{H}_R + i\mathcal{H}_R$ is dense in $\mathcal{H}$ (see pp. 332 [S97]).

Note that $(U_t)$ extends to a strongly continuous unitary representation $(\tilde{U}_t)$ of $\mathbb{R}$ on $\mathcal{H}$. Let $\tilde{A}$ be the analytic generator associated to $(\tilde{U}_t)$, which is obviously an extension of $A$. Any eigenvector of $\tilde{A}$ is an eigenvector of $A$ corresponding to the same eigenvalue [BM, Prop. 2.1]. Since the spectral data of $A$ and $\tilde{A}$ (and hence of $(U_t)$ and $(\tilde{U}_t)$) are essentially the same (see [BM, §2] for details), and $\tilde{U}_t, \tilde{A}$ are respectively extensions of $U_t, A$ for all $t \in \mathbb{R}$, so we would now write $\tilde{A} = A$ and $\tilde{U}_t = U_t$ for all $t \in \mathbb{R}$.

For $q \in (-1, 1)$ and for the Hilbert space $\mathcal{H}$, consider the associated $q$-Fock space $\mathcal{F}_q(\mathcal{H})$ introduced in [BS91]. $\mathcal{F}_q(\mathcal{H})$ is constructed as follows. Let $\Omega$ be a distinguished unit vector in $\mathcal{C}$ usually referred to as the vacuum vector. Denote $\mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$, and, for $n \geq 1$, let $\mathcal{H}^{\otimes n} = \text{span}_C\{\xi_1 \otimes \cdots \otimes \xi_n : \xi_i \in \mathcal{H} \text{ for } 1 \leq i \leq n\}$ denote the algebraic tensor products. Let $\mathcal{F}_\text{fin}(\mathcal{H}) = \text{span}_C(\mathcal{H}^{\otimes n} : n \geq 0)$. For $n, m \geq 0$ and $f = \xi_1 \otimes \cdots \otimes \xi_n \in \mathcal{H}^{\otimes n}$, $g = \xi_1 \otimes \cdots \otimes \xi_m \in \mathcal{H}^{\otimes m}$, the association

$$
(f, g)_q = \delta_{m,n} \sum_{\pi \in S_n} q^{i(\pi)} \langle \xi_1, \xi_{\pi(1)} \rangle_U \cdots \langle \xi_n, \xi_{\pi(n)} \rangle_U,
$$

where $i(\pi)$ denotes the number of inversions of the permutation $\pi \in S_n$, defines a positive definite sesquilinear form on $\mathcal{F}_\text{fin}(\mathcal{H})$ and the $q$-Fock space $\mathcal{F}_q(\mathcal{H})$ is the completion of $\mathcal{F}_\text{fin}(\mathcal{H})$ with respect to the norm $\|\cdot\|_q$ induced by $(\cdot, \cdot)_q$. Denote $\mathcal{H}^{\otimes \alpha n} := \overline{\mathcal{H}^{\otimes n}}^{\|\cdot\|_q}, n \geq 0$. Note that $\|\cdot\|_q = \|\cdot\|_U$ on $\mathcal{H}^{\otimes \alpha - 1} = \mathcal{H}$.

For $\xi \in \mathcal{H}$, the left $q$-creation and $q$-annihilation operators on $\mathcal{F}_q(\mathcal{H})$ are respectively defined by:

$$
c_q(\xi)\Omega = \xi,
$$

$$
c_q(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n,
$$

and,

$$
c_q(\xi)^*\Omega = 0,
$$

$$
c_q(\xi)^*(\xi_1 \otimes \cdots \otimes \xi_n) = \sum_{i=1}^n q^{i-1} \langle \xi, \xi_i \rangle_U \xi_1 \otimes \cdots \otimes \xi_{i-1} \otimes \xi_{i+1} \otimes \cdots \otimes \xi_n,
$$

(2)
where $\xi_1 \otimes \cdots \otimes \xi_n \in \mathcal{H}^{\otimes q^n}$ for $n \geq 1$. The operators $c_q(\xi)$ and $c_q(\xi)^*$ are bounded on $\mathcal{F}_q(\mathcal{H})$ and they are adjoints of each other. Moreover,

$$
\|c_q(\xi)\| = \begin{cases} 
\frac{1}{\sqrt{1-q}} \|\xi\|_U, & \text{if } 0 < q < 1; \\
\|\xi\|_U, & \text{if } -1 < q \leq 0. 
\end{cases}
$$

Moreover, they satisfy the following $q$-commutation relation:

$$
c_q(\xi)^c_q(\zeta) - qc_q(\zeta)c_q(\xi)^* = \langle \xi, \zeta \rangle_U I, \quad \text{for all } \xi, \zeta \in \mathcal{H}. \tag{4}
$$

The following Lemma from [BM] will be crucial for our purpose.

**Lemma 2.1** [BM, Lemma 2.3]. Let $\xi, \xi_i, \eta_j \in \mathcal{H}$, for $1 \leq i \leq n$, $1 \leq j \leq m$. Then,

$$
c_q(\xi)^c(\xi)\left((\xi_1 \otimes \cdots \otimes \xi_n) \otimes (\eta_1 \otimes \cdots \otimes \eta_m)\right) = \left(c_q(\xi)^c(\xi_1 \otimes \cdots \otimes \xi_n)\right) \otimes (\eta_1 \otimes \cdots \otimes \eta_m) + q^n(\xi_1 \otimes \cdots \otimes \xi_n) \otimes \left(c_q(\xi)^c(\eta_1 \otimes \cdots \otimes \eta_m)\right).$$

Following [Hi03, S97], consider the $C^*$-algebra $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t) := C^*[s_q(\xi) : \xi \in \mathcal{H}_\mathbb{R}]$ and the von Neumann algebra $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$, where

$$s_q(\xi) = c_q(\xi) + c_q(\xi)^*, \quad \xi \in \mathcal{H}_\mathbb{R}.$$ 

$\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ is known as the $q$-deformed Araki-Woods von Neumann algebra (see [Hi03, §3]). The vacuum state $\varphi_{q,U} := \langle \Omega, \cdot \Omega \rangle_q$ (also called the $q$-quasi free state), is a faithful normal state of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ and $\mathcal{F}_q(\mathcal{H})$ is the GNS Hilbert space of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ associated to $\varphi_{q,U}$. Thus, $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ acting on $\mathcal{F}_q(\mathcal{H})$ is in standard form [H75].

Making slight violation of the traditional notations, we will use the symbols $\langle \cdot, \cdot \rangle_q$ and $\|\cdot\|_q$ respectively to denote the inner product and two-norm of elements of the GNS Hilbert space.

### 2.2. Modular Theory

Most of what follows in Sects. 2.2 and 2.3 is taken from [S97, Hi03]. We need to have a convenient description of the commutant and centralizer of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ (which has been recorded in the case $q = 0$ in [S97] and a similar collection of operators in the commutant has been identified in [Hi03]). Thus, we need to record some facts related to the modular theory of the $q$-quasi free state $\varphi_{q,U}$. Let $J_{\varphi_{q,U}}$ and $\Delta_{\varphi_{q,U}}$ respectively denote the modular conjugation and modular operator associated to $\varphi_{q,U}$ and let $S_{\varphi_{q,U}} = J_{\varphi_{q,U}}^{\frac{1}{2}} \Delta_{\varphi_{q,U}}^{\frac{1}{2}}$. Then, for $n \in \mathbb{N}$,

$$J_{\varphi_{q,U}}(\xi_1 \otimes \cdots \otimes \xi_n) = A^{-1/2}\xi_n \otimes \cdots \otimes A^{-1/2}\xi_1, \quad \forall \xi_i \in \mathcal{H}_\mathbb{R} \cap \mathcal{D}(A^{-\frac{1}{2}}); \tag{5}$$

$$\Delta_{\varphi_{q,U}}(\xi_1 \otimes \cdots \otimes \xi_n) = A^{-1}\xi_1 \otimes \cdots \otimes A^{-1}\xi_n, \quad \forall \xi_i \in \mathcal{H}_\mathbb{R} \cap \mathcal{D}(A^{-1});$$

$$S_{\varphi_{q,U}}(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_n \otimes \cdots \otimes \xi_1, \quad \forall \xi_i \in \mathcal{H}_\mathbb{R}. $$
The modular automorphism group \((\sigma^{q,U}_t)\) of \(q,U\) is given by \(\sigma^{q,U}_t = \Ad(F(U_t))\), where \(F(U_t) = \text{id} \oplus \oplus_{n \geq 1} U_t^\otimes n\), for all \(t \in \mathbb{R}\). In particular,

\[
\sigma^{q,U}_t(s_q(\xi)) = s_q(U_t \xi), \quad \text{for all } \xi \in \mathcal{H}_\mathbb{R}. \tag{6}
\]

To reduce notation, the complex (left) conjugation \(J\) associated to \(\mathcal{H}_\mathbb{R} \subset \mathcal{H}\) will be denoted by \(\xi + i\eta = \xi - i\eta\) for \(\xi, \eta \in \mathcal{H}_\mathbb{R}\). It corresponds with \(S_{q,U}\).

### 2.3. Commutant

Now we proceed to describe the commutant of \(\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)^\prime\). Consider the set

\[
\mathcal{H}_\mathbb{R}' = \{\xi \in \mathcal{H} : \langle \xi, \eta \rangle_U \in \mathbb{R} \text{ for all } \eta \in \mathcal{H}_\mathbb{R}\}.
\]

Then \(\mathcal{H}_\mathbb{R}'\) is a real subspace. Note that \(\overline{\mathcal{H}_\mathbb{R}'} + i\mathcal{H}_\mathbb{R}' = \mathcal{H}\) and \(\mathcal{H}_\mathbb{R}' \cap i\mathcal{H}_\mathbb{R}' = \{0\}\). It is easy to check that \(A^{-1/2} \xi \in \mathcal{H}_\mathbb{R}'\) for all \(\xi \in \mathcal{D}(A^{-1/2}) \cap \mathcal{H}_\mathbb{R}\).

Now for \(\xi \in \mathcal{H}\), define the right creation operator \(c_{q,r}(\xi)\) on \(F_q(\mathcal{H})\) by

\[
c_{q,r}(\xi)\Omega = \xi, \quad \text{for all } \eta \in \mathcal{H}_\mathbb{R}. \tag{7}
\]

Clearly, \(c_{q,r}(\xi) = j c_{q,\overline{\xi}}(\xi)^\ast\), where \(j : F_q(\mathcal{H}) \to F_q(\mathcal{H})\) is the unitary defined by

\[
j(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_n \otimes \cdots \otimes \xi_1, \quad \text{where } \xi_i \in \mathcal{H} \quad \text{for all } 1 \leq i \leq n, n \geq 1. \tag{8}
\]

\[
j(\Omega) = \Omega.
\]

Therefore, \(c_{q,r}(\xi) \in \mathcal{B}(F_q(\mathcal{H}))\) and its adjoint \(c_{q,r}(\xi)^\ast\) is given by

\[
c_{q,r}(\xi)^\ast \Omega = 0,
\]

\[
c_{q,r}(\xi)^\ast (\xi_1 \otimes \cdots \otimes \xi_n)
\]

\[
= \sum_{i=1}^{n} q^{n-i} (\xi, \xi_i)_U \xi_1 \otimes \cdots \otimes \xi_{i-1} \otimes \xi_{i+1} \otimes \cdots \otimes \xi_n, \quad \xi_i \in \mathcal{H}, n \geq 1. \tag{9}
\]

Write \(s_{q,r}(\xi) = c_{q,r}(\xi) + c_{q,r}(\xi)^\ast, \xi \in \mathcal{H}\). The following result describes the commutant of \(\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)^\prime\).

**Theorem 2.2** [BM, Thm. 2.4]. Suppose \(\xi \in \mathcal{D}(A^{-1}) \cap \mathcal{H}_\mathbb{R}\). Then \(J_{q,U} s_q(\xi) J_{q,U} = s_{q,r}(A^{-\frac{1}{2}} \xi). Moreover, \(\Gamma(\mathcal{H}_\mathbb{R}, U_t)^\prime = \{s_{q,r}(\xi) : \xi \in \mathcal{H}_\mathbb{R}'\}\). The complex (right) conjugation associated to \(\mathcal{H}_\mathbb{R}' \subset \mathcal{H}\) will be denoted by \(\bar{r} = \xi - i\eta\) for \(\xi, \eta \in \mathcal{H}_\mathbb{R}'\). It corresponds with \(J S_{q,U} J^\ast\).
2.4. Notations and some technical facts. In this paper, we are interested in the factoriality of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ and the orthogonal representation remains arbitrary but fixed. Thus, to reduce notation, we will write $M_q = \Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ and $\varphi = \varphi_{q, U}$. We will also denote $J_{\varphi_{q, U}}$ by $J_t$ and $\Delta_{\varphi_{q, U}}$ by $\Delta$. As $\Omega$ is separating for both $M_q$ and $M_q'$, for $\zeta \in M_q \Omega$ and $\eta \in M_q' \Omega$ there exist unique $x_\zeta \in M_q$ and $x_\eta' \in M_q'$ such that $\zeta = x_\zeta \Omega$ and $\eta = x_\eta' \Omega$. In this case, we will write

$$W(\zeta) = x_\zeta \quad \text{and} \quad W_r(\eta) = x_\eta'.$$

Note that $W_r(\eta) = J W(J \eta) J$, as $J \eta \in M_q \Omega$ from Tomita’s fundamental theorem. Thus, for example, as $\xi \in M_q \Omega$ for every $\xi \in \mathcal{H}_\mathbb{R}$, so $W(\xi + i \eta) = s_q(\xi) + i s_q(\eta)$ for all $\xi, \eta \in \mathcal{H}_\mathbb{R}$.

Write $Z(M_q) = M_q \cap M_q'$. Let $M_q^\varphi = \{ x \in M_q : \sigma^\varphi_t(x) = x \ \text{for all} \ t \in \mathbb{R} \}$ denote the centralizer of $M_q$ associated to the state $\varphi$. Recall that $x \in M_q$ is analytic with respect to $(\sigma^\varphi_t)$ if and only if the function $\mathbb{R} \ni t \mapsto \sigma^\varphi_t(x) \in M_q$ extends to a weakly entire function. We say that a vector $\xi \in M_q \Omega$ is analytic, if $W(\xi)$ is analytic for $(\sigma^\varphi_t)$.

In order to control calculations within the page limit, we adopt the following notations for convenience.

1. $\xi_1 \cdots \xi_n := \xi_1 \otimes \cdots \otimes \xi_n \in \mathcal{H}_\otimes q^n$ for $\xi_i \in \mathcal{H}$, $1 \leq i \leq n$;
2. $c_q(\xi) = c_q(\xi_1) \cdots c_q(\xi_n)$ for $\xi = \xi_1 \cdots \xi_n \in \mathcal{H}_\otimes q^n$;
3. $c_{q,r}(\xi) = c_{q,r}(\xi_n) \cdots c_{q,r}(\xi_1)$ for $\xi = \xi_1 \cdots \xi_n \in \mathcal{H}_\otimes q^n$;
4. $C_q = \prod_{i=1}^{\infty} \frac{1}{1 - q^i}$;
5. $d_0 = 1, d_j = \prod_{i=1}^{j} (1 - q^i), j \in \mathbb{N}$, and $d_\infty = \prod_{i=1}^{\infty} (1 - q^i)$;
6. $[n]_q := 1 + q + \cdots + q^{(n-1)}, [n]_q! := \prod_{j=1}^{n} [j]_q$, for $n \geq 1$, and $[0]_q := 0, [0]_q! := 1$ by convention.

Note that $d_k \cdot \frac{1}{d_k} \leq C_q$ for all $k \geq 0$ and $|q| < 1$.

The following norm estimates will be crucial (see [BKS97, BS91, B99, R05]).

- For all $\xi \in \mathcal{H}_\otimes q^n$,
  \[ \|c_q(\xi)\| \leq \sqrt{C_q} \|\xi\|_q. \]  \hspace{1cm} (10)
- If $\xi \in \mathcal{H}$ and $\|\xi\|_U = 1 (= \|\xi\|_q)$, then
  \[ \|\xi^n\|_q = [n]_q! = d_n (1 - q)^{-n}. \]  \hspace{1cm} (11)
- If $\xi_1, \ldots, \xi_n, \xi \in \mathcal{H}$ with $\|\xi_j\|_U = \|\xi\|_U = 1$ for all $1 \leq j \leq n$, then
  \[ \|\xi_1 \cdots \xi_n \xi^{m}\|_q = \|\xi^n \xi_n \cdots \xi_1\|_q \leq C_q^\frac{m}{2} [m]_q!, \ m \geq 0. \]  \hspace{1cm} (12)
- We recall the following $q$-analogue of the Pascal’s identity for $q$-binomial coefficients (cf. [BKS97, Prop. 1.8]):
  \[ q^k \binom{n}{k}_q + \binom{n}{k-1}_q = \binom{n+1}{k}_q, \quad k \leq n. \]  \hspace{1cm} (13)

(We regard $\binom{n}{k}_q = 0$ when $k < 0$.) We also recall the Wick formula from [B99, N04], and its right version which can be obtained using the (right) complex conjugation.
Proposition 2.3. Let $\xi_1, \ldots, \xi_n$ be in $\mathcal{H}_\mathbb{C}$. Then,

$$W(\xi_1 \cdots \xi_n) = \sum_{i=0}^{n} \sum_{\sigma \in S_n} q^{\vert \sigma \vert} c_q(\xi_{\sigma(1)}) \cdots c_q(\xi_{\sigma(i)}) c_q(\xi_{\sigma(i+1)})^* \cdots c_q(\xi_{\sigma(n)})^*,$$

where $S_{n,i}$ is the set of permutations of $\{1, \ldots, n\}$ that are increasing on $\{1, \ldots, i\}$ and $\{i+1, \ldots, n\}$ and $\vert \sigma \vert$ is the number of inversions of $\sigma$. Further, if $\xi_1, \ldots, \xi_n \in \mathcal{H}_{\mathbb{R}}^+ + i\mathcal{H}_{\mathbb{R}}^-$, then

$$W_r(\xi_1 \cdots \xi_n) = \sum_{i=0}^{n} \sum_{\sigma \in S_{n,i}} q^{\text{flip} \circ \sigma} c_{q,r}(\xi_{\sigma(1)}) \cdots c_{q,r}(\xi_{\sigma(i)}) c_{q,r}(\xi_{\sigma(i+1)})^* \cdots c_{q,r}(\xi_{\sigma(n)})^*,$$

where flip is the permutation $\text{flip}(k) = n - k$, $1 \leq k \leq n$.

There is also another convenient way to write the Wick formula using crossings of partitions.

Any $\sigma \in S_{n,i}$ is completely determined by a subset $\mathcal{J} = \{j_1 < \cdots < j_i\}$ with complement $\mathcal{J}^c = \{k_{i+1} < \cdots < k_n\}$, then $\vert \sigma \vert$ is the number of crossings of the partition $\mathcal{J} \cup \mathcal{J}^c$, i.e.,

$$\vert \sigma \vert = c(\mathcal{J}, \mathcal{J}^c) = \#\{(a, b) \mid j_a > k_b\}.$$

Thus,

$$W(\xi_1 \cdots \xi_n) = \sum_{i=0}^{n} \sum_{\mathcal{J} = \{j_1 < \cdots < j_i\}} q^{c(\mathcal{J}, \mathcal{J}^c)} c_q(\xi_{j_1}) \cdots c_q(\xi_{j_i}) c_q(\xi_{k_{i+1}})^* \cdots c_q(\xi_{k_n})^*. \quad (14)$$

2.5. Centralizer. We need a convenient description of the centralizer $M_q^\psi$ which depends on the almost periodic part of the orthogonal representation $(U_t)$. We need some preparation. For details check [BM].

Recall that for a strongly continuous orthogonal representation $t \mapsto U_t$, $t \in \mathbb{R}$, on the real Hilbert space $\mathcal{H}_\mathbb{R}$, there is a unique decomposition (see [S97]),

$$(\mathcal{H}_\mathbb{R}, U_t) = \left( \bigoplus_{j=1}^{N_1} (\mathbb{R}, \text{id}) \right) \oplus \left( \bigoplus_{k=1}^{N_2} (\mathcal{H}_\mathbb{R}(k), U_t(k)) \right) \oplus (\widetilde{\mathcal{H}}_\mathbb{R}, \widetilde{U}_t),$$

where $0 \leq N_1, N_2 \leq \aleph_0$, $\mathcal{H}_\mathbb{R}(k) = \mathbb{R}^2$, $U_t(k) = \begin{pmatrix} \cos(t \log \lambda_k) - \sin(t \log \lambda_k) \\ \sin(t \log \lambda_k) \cos(t \log \lambda_k) \end{pmatrix}$, $0 < \lambda_k < 1$, and $(\widetilde{\mathcal{H}}_\mathbb{R}, \widetilde{U}_t)$ corresponds to the weakly mixing component of the orthogonal representation; thus $\mathcal{H}_\mathbb{R}$ is either 0 or infinite-dimensional.

If $N_1 \neq 0$, let $e_j = 0 \oplus \cdots \oplus 0 \oplus 1 \oplus 0 \oplus \cdots \oplus 0 \in \bigoplus_{j=1}^{N_1} \mathbb{R}$, where 1 appears at the $j$-th place for $1 \leq j \leq N_1$. Similarly, if $N_2 \neq 0$, let $f_k^1 = 0 \oplus \cdots \oplus 0 \oplus \begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus 0 \oplus \cdots \oplus 0 \in \bigoplus_{j=1}^{N_2} \mathbb{R}$. Then...
On the Factoriality of $q$-Deformed Araki-Woods von Neumann... 

On the Factoriality of $q$ of multiplicity more than or equal to 2, then $M$ is not ergodic if and only if it is trivial and in that case $M_q$ is decided in several steps. When $\dim(\mathcal{H}_\mathbb{R}) = 2$, then $(U_i)$ is not ergodic if and only if it is trivial and in that case $M_q$ is a II$_1$ factor. The most important and difficult case is the one when $\dim(\mathcal{H}_\mathbb{R}) = 2$ and $(U_i)$ is ergodic. The difficulty arises in lack of room to perform meaningful calculations.

In this section, we lay out the technical analysis that will lead to the factoriality of $M_q$. This section is divided into two subsections. In Sect. 3.1 we deal with the case when $\dim(\mathcal{H}_\mathbb{R}) = 2$ and $(U_i)$ is ergodic and in Sect. 3.2 we prepare the machinery that will help deal with all the cases.
3.1. $\Gamma(\mathbb{R}^2, U_q)^n$ with small $\lambda$ and $(U_I)$ ergodic. Following the discussion in Sect. 2.5, it follows that $N_1 = 0$ and $N_2 = 1$. (In this case $\hat{\mathcal{H}}_\mathbb{R} = 0$.) Further, there exists a $\lambda \in (0, 1)$ such that

$$U_I = \left( \begin{array}{cc} \cos(t \log \lambda) & -\sin(t \log \lambda) \\ \sin(t \log \lambda) & \cos(t \log \lambda) \end{array} \right).$$

Reducing notation, write $e_1 := e_1^1 = \frac{\sqrt{\lambda + i}}{2} \left( \begin{array}{c} 1 \\ i \end{array} \right)$ and $e_2 := e_1^2 = \frac{\sqrt{\lambda - i}}{2} \left( \begin{array}{c} 1 \\ -i \end{array} \right)$ so that $A e_1 = \frac{1}{\lambda} e_1$ and $A e_2 = \lambda e_2$ and $\{e_1, e_2\}$ forms an orthonormal basis of $(\mathbb{C}^2, \langle \cdot, \cdot \rangle_U)$.

Now we set $e = \lambda^{-\frac{1}{2}} e_1$. Then, $\bar{e} = \lambda^{\frac{1}{2}} e_2$. Note that $\|e\|_U = \lambda^{-\frac{1}{4}}$ and $\|\bar{e}\|_U = \lambda^{\frac{1}{4}}$. Note that in this context $\hat{\mathcal{H}} = \mathbb{C}^2$ is considered with respect to the inner product $\langle \cdot, \cdot \rangle_U$.

We also have the formulas $\bar{e}^r = \lambda^{-1} e$ and $\bar{e}^r = \lambda e$.

We begin with some useful lemmas.

**Lemma 3.1.** For $m \geq 0$ and $\xi \in \mathcal{H}$, we have

$$c_q(\xi)^m(\bar{\xi}^m) = \begin{cases} \frac{[n]_q!}{[m-n]_q!} \|\xi\|^2 \xi^{m-n}, & n \leq m; \\ 0, & n > m. \end{cases}$$

**Proof.** The proof follows directly from Eq. (2). □

**Lemma 3.2.** $W(e^n) = W(e^n) = \sum_{k=0}^{n} \binom{n}{k} c_q(e)^{n-k} c_q(\bar{e})^k$ for $n \geq 1$.

**Proof.** The proof is straightforward and follows by using Eq. (13) and induction. So we omit the proof. □

The next lemma initiates the interplay between the parameters $\lambda$ and $q$ in the case $\dim(\hat{\mathcal{H}}_\mathbb{R}) = 2$, which will force constraints to the factoriality of $M_q$.

**Lemma 3.3.** Let $D(q) = \sup_n d_n$. With the above notations, the following are true:

(i) $\{\lambda^\frac{n}{2} (1-q)^\frac{n}{2} c_q(e^n)\}_{n \geq 1}, \{\lambda^{-\frac{n}{2}} (1-q)^{-\frac{n}{2}} c_q(\bar{e}^n)\}_{n \geq 1}$ are bounded (actually by 1 if $q \geq 0$ and by $\sqrt{C_q} D_q$ if $q < 0$).

(ii) $\{\lambda^\frac{n}{2} (1-q)^{-\frac{n}{2}} W(e^n)\}_{n \geq 1}$ is bounded.

**Proof.** (i). First assume $0 \leq q < 1$. Then, from Eq. (3) it follows that

$$\|c_q(e)^n\| \leq \|c_q(e)\|^n = \|e\|_q^n (1-q)^{-\frac{n}{2}} \leq \lambda^{-\frac{n}{2}} (1-q)^{-\frac{n}{2}}.$$

If $-1 < q < 0$, from Eq. (10) and Eq. (11) it follows that

$$\|c_q(e)^n\| = \|c_q(e)^n\| \leq \sqrt{C_q} \|e^n\|_q = \sqrt{C_q} \|e\|^n_\sqrt{[n]_q!} = \sqrt{C_q} \lambda^{-\frac{n}{2}} \sqrt{[n]_q!} = \sqrt{C_q} d_n \lambda^{-\frac{n}{2}} (1-q)^{-\frac{n}{2}} \leq \sqrt{C_q} D(q) \lambda^{-\frac{n}{2}} (1-q)^{-\frac{n}{2}}.$$

The argument for $\bar{e}$ is similar. Thus (i) follows.
(ii). We just use Lemma 3.2 and the triangle inequality. Note that
\[
\| W(e^n) \| \leq \sum_{k=0}^{n} \frac{d_n}{d_{n-k}d_k} \| c_q(e)^{n-k} \| \cdot \| c_q(\tilde{e})^s \| \quad \text{(use Eq. (11))}
\]
\[
\leq C_q \sum_{k=0}^{n} \frac{d_n}{d_{n-k}d_k} \| e^{n-k} \| q \cdot \| \tilde{e}^k \| q \quad \text{(use Eq. (10))}
\]
\[
= C_q \sum_{k=0}^{n} \frac{d_n}{d_{n-k}d_k} \| e \| _q \cdot \sqrt{[n-k]!} \cdot \| \tilde{e}^k \| \cdot \sqrt{[k]!} \quad \text{(use Eq. (11))}
\]
\[
= C_q \sum_{k=0}^{n} \frac{d_n}{\sqrt{d_{n-k}d_k}} \lambda^{(2k-n)/4} (1-q)^{-n/2} \quad \text{(use Eq. (11)).}
\]

The result follows as \((d_i)\) is bounded from above and below and \(\lambda < 1\). □

Lemma 3.4. Let \(T_n = (1-q)^n \lambda^{n/2} c_q(e)^n c_q(e)^n\) for \(n \geq 0\), then \(T_n\) is norm convergent to some \(T \in C^*(c_q(e))\).

Moreover, if \(q \geq 0\) then \(d_\infty \leq T \leq 1\) and if \(q \leq 0\) then \(d_\infty/(1-q) \leq T \leq 1-q\).

Proof. Let \(\mathcal{K} = \mathbb{C}e\). Then by [W17, Thm. 3.6] (see also [W18, JSW96] Example 2), the \(C^*\)-algebras generated by the creation operators associated to \(e\) in \(\mathcal{F}(\mathcal{H})\) and in \(\mathcal{F}(\mathcal{K})\) are isomorphic. Thus, we only need to prove the result in \(\mathcal{B}(\mathcal{F}(\mathcal{K}))\); but \(\mathcal{F}(\mathcal{K}) = \oplus_{k=0}^{\infty} \mathbb{C}e^k(e^0 := \Omega\) by convention). With respect to this decomposition, \(T_n\) is a diagonal operator with
\[
T_n(e^k) = (1-q)^n \lambda^{n/2} c_q(e)^n c_q(e)^n e^{(n+k)} = (1-q)^n \frac{[n+k]!}{[k]!} e^k
\]
\[
= \left( \prod_{j=1}^{n} (1-q^{k+j}) \right) e^k, \quad \forall k \geq 0, n \geq 1.
\]

Let \(T \in \mathcal{B}(\mathcal{F}(\mathcal{K}))\) be the diagonal operator with eigenvalues \(\frac{d_\infty}{d_k} = \prod_{j=1}^{\infty} (1-q^{k+j})\) and associated eigenvectors \(e^k\) for all \(k \geq 0\). Then,
\[
\|T_n - T\| \leq K \sup_k |1 - \prod_{j=n}^{\infty} (1-q^{k+j})| \to 0 \text{ as } n \to \infty,
\]
where \(K\) is a constant (independent of \(n, k\)). Consequently, \(0 \leq T \in C^*(c_q(e))\).

The estimations for \(T\) are clear from its spectrum if \(q \geq 0\). When \(q < 0\), we have to estimate \(\alpha_q = \inf_k \prod_{j=1}^{\infty} (1-q^{k+j})\) and \(\beta_q = \sup_k \prod_{j=1}^{\infty} (1-q^{k+j})\). Note that for all \(m \geq 0\), we have that \((1-q^{2m})(1-q^{2m+1}) \leq 1\). Thus, it follows that \(\alpha_q = \prod_{j=1}^{\infty} (1-q^{k+j}) = d_\infty/(1-q)\). In the same way since \(\prod_{j=2m}^{\infty} (1-q^{k+j}) \leq 1\), we must have \(\beta_q \leq 1 - q\). □

Remark 3.5. When \(q < 0\), if moreover \(|q|(1+|q|) \leq 1\), we have that for all \(m \geq 0\), \((1-q^{2m+1})(1-q^{2m+2}) \geq 1\). Thus, it follows that \(\beta_q = \sup_k d_\infty/d_k = d_\infty\) in that case.
Now, consider the following operator
\[ S_n = (1 - q)^n \lambda^{\frac{n}{2}} c_q(e)^{*n} W(e^n). \]

Also note that, for each fixed \( k \), as \( n \) goes to infinity, \( \left( \begin{array}{c} n \\ k \end{array} \right)_q = \frac{d_n}{d_{n-k}d_k} \) converges to \( c_k = d_k^{-1} \leq C_q \).

**Lemma 3.6.** The sequence \( S_n \) converges in norm to
\[ S_{\infty} = \sum_{k=0}^{\infty} c_k (1 - q)^k \lambda^{k/2} c_q(e)^{*k} T c_q(\bar{e})^{*k}. \]

Moreover, if \( q > 0 \) and \( \lambda < (1 + \frac{C_q D(q)}{d_{\infty}^2})^{-2} \) or if \( q < 0 \) and \( \lambda < (1 + \frac{C_q D(q)(1-q)^2}{d_{\infty}^2})^{-2} \), the operator \( S_{\infty} \) is invertible.

**Proof.** Let \( B(q) = \sqrt{C_q D(q)} \). First note that the series defining \( S_{\infty} \) converges absolutely. Indeed by Lemma 3.3 (i), we have
\[
\|(1 - q)^{k/2} c_q(e)^k\| \leq \lambda^{-k/4}, \quad \|(1 - q)^{k/2} c_q(\bar{e})^k\| \leq \lambda^{k/4}, \quad \text{when } 0 \leq q < 1;
\]
\[
\|(1 - q)^{k/2} c_q(e)^k\| \leq B(q) \lambda^{-k/4}, \quad \|(1 - q)^{k/2} c_q(\bar{e})^k\| \leq B(q) \lambda^{k/4}, \quad \text{when } -1 < q < 0.
\]

Thus, using Lemma 3.4 we have
\[
\sum_{k=0}^{\infty} \|c_k (1 - q)^k \lambda^{k/2} c_q(e)^{*k} T c_q(\bar{e})^{*k}\| \leq \begin{cases} C_q \sum_{k=0}^{\infty} \lambda^{k/2} < \infty, & \text{if } q \geq 0; \\
C_q^2 D(q)(1-q) \sum_{k=0}^{\infty} \lambda^{k/2} < \infty, & \text{if } q < 0. 
\end{cases}
\]

We note the following:
\[
S_n = (1 - q)^n \lambda^{\frac{n}{2}} c_q(e)^{*n} W(e^n)
\]
\[
= (1 - q)^n \lambda^{\frac{n}{2}} c_q(e)^{*n} \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right)_q c_q(e)^{n-k} c_q(\bar{e})^{*k} \quad \text{(Lemma 3.2)}
\]
\[
= \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right)_q (1 - q)^{k/2} \lambda^{k/2} c_q(e)^{*k} (1 - q)^{n-k} \lambda^{-k/2} c_q(e)^{*(n-k)} c_q(e)^{n-k} c_q(\bar{e})^{*k}
\]
\[
= \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right)_q (1 - q)^{k/2} \lambda^{k/2} c_q(e)^{*k} T_{n-k} c_q(\bar{e})^{*k} \quad \text{(Lemma 3.4).}
\]

We have just seen in Lemma 3.4 that \( T_{n-k} \) is norm convergent to \( T \) (and bounded by 1 if \( q \geq 0 \) or \( 1 - q \) if \( q < 0 \)). Thus, the general term in \( S_n \) converges to the general term of \( S_{\infty} \) in norm and one concludes the convergence in the statement with the dominated convergence theorem as \( \sum_{k=0}^{\infty} \lambda^{k/2} < \infty \).

We also have \( S_{\infty} = T (1 + T^{-1} V) \) (use Lemma 3.4), where
\[
\|V\| \leq \begin{cases} C_q \sum_{k=1}^{\infty} \lambda^{k/2}, & \text{if } q > 0; \\
C_q^2 D(q)(1-q) \sum_{k=1}^{\infty} \lambda^{k/2}, & \text{if } q < 0. 
\end{cases}
\]
The rest follows by ensuring that $\|T^{-1}V\| < 1$ using Lemma 3.4. We know that $\|T^{-1}\| \leq \frac{1}{d_\infty}$ if $q > 0$ and $\|T^{-1}\| \leq \frac{1-q}{d_\infty}$ if $q < 0$ by Lemma 3.4. Thus, $S_\infty$ is invertible if $\frac{C_q}{d_\infty} \sqrt{\frac{\lambda}{\sqrt{\lambda}}} < 1$ when $q > 0$ and $C_q^2 D(q) (1-q)^2 \frac{\sqrt{\lambda}}{1-\sqrt{\lambda}} < 1$ when $q < 0$. □

Remark 3.7. (1) When $q = 0$, $S_\infty = \sum_{k=0}^\infty \lambda^{k/2} c_0(e)^* c_0(\bar{e})^k$. The operators $l_k = c_0(\bar{e})^k c_0(e)^k$ are partial isometries with orthogonal ranges when $k \geq 1$. On the smallest subspace containing $\Omega$ and invariant by $l_k$’s, $S_\infty$ acts like $1 + \sqrt{\frac{\lambda}{1-\lambda}} l^*$ where $l$ is a unilateral shift. Hence, $S_\infty$ has a kernel if $\frac{\lambda}{1-\lambda} > 1$, that is $\lambda > \frac{1}{2}$.

(2) As a result of Lemma 3.6, we have

$$S_\infty^* \Omega = \sum_{k=0}^\infty c_k (1 - q)^k \lambda^{k/2} c_q(\bar{e})^k T c_q(e)^k \Omega = \sum_{k=0}^\infty c_k (1 - q)^k \lambda^{k/2} c_q(\bar{e})^k T (e^k)$$

$$= \sum_{k=0}^\infty c_k (1 - q)^k \lambda^{k/2} c_q(\bar{e})^k \frac{d_\infty}{d_k} e^k \ (\text{Lemma 3.4})$$

$$= d_\infty \sum_{k=0}^\infty c_k^2 (1 - q)^k \lambda^{k/2} \bar{e}^k e^k.$$ 

The vector $S_\infty^* \Omega$ will be useful later in the proof.

As a consequence of the Wick formula, we explicitly get:

**Lemma 3.8.** For every $n \geq 0$, there are reals $q_{k,l}, 0 \leq k, l \leq n$ with $|q_{k,l}| \leq C_q^2 |q|^{(n-k)L}$ such that

$$W(\bar{e}^n e^n) = \sum_{k,l=0}^n q_{k,l} c_q(\bar{e})^k c_q(e)^l c_q(e)^*^{(n-k)} c_q(\bar{e})^{* (n-l)}.$$ 

**Proof.** We use the Wick formula in Eq. (14). Terms of the form $c_q(\bar{e})^k c_q(e)^l c_q(e)^*^{(n-k)} c_q(\bar{e})^{* (n-l)}$ occur when we choose $J = J_1 \cup J_2 \subset \{1, \ldots, 2n\}$ with $J_1 \subset \{1, \ldots, n\}$ of cardinal $k$ and $J_2 \subset \{n + 1, \ldots, 2n\}$ of cardinal $l$. Thus, the formula holds with $q_{k,l} = \sum_{J_1, J_2} q^{n(3,3')}$, where the sum runs over all possibilities as mentioned.

Writing $3^c = K_1 \cup K_2$ with $K_1 = J \cap \{1, \ldots, n\}$ and $K_2 = J \cap \{n + 1, \ldots, 2n\}$, we have $c(3, 3^c) = c(J_1, K_1) + c(J_2, K_2) + (n-k)L$, has all elements in $K_1$ cross those of $J_2$. Hence,
\[ q_{k,l} = q^{(n-k)l} \cdot \sum_{J_1} q^{c(J_1,K_1)} \cdot \sum_{J_2} q^{c(J_2,K_2)}. \]

It follows from [B99] that \(|q_{k,l}| \leq |q|^{(n-k)l} C_q^2\). \(\Box\)

As a result of Lemma 3.8 we have:

**Lemma 3.9.** The sequence \(((1 - q)^n W(\tilde{e}^n e^n))\) is bounded.

**Proof.** We rely on Lemma 3.8 and the triangle inequality with the help of Eq. (10) and Eq. (11). Note that

\[
\|(1 - q)^n W(\tilde{e}^n e^n)\| \leq \sum_{k,l=0}^n C_q^4 |q|^{(n-k)l} \sqrt{d_1d_{n-l}d_kd_{n-k}} \lambda^{(k-l)/2}
\]

\[
\leq \sum_{k,l=0}^n C_q^6 |q|^{(n-k)l} \lambda^{(k-l)/2}
\]

\[
= C_q^6 \sum_{j=-n}^n \lambda^{j/2} \sum_{0 \leq k,l \leq n} |q|^{(n-k)l}. \]

If we denote by \(c_{j,n}\) the last quantity, we have for \(j < 0\)

\[
c_{j,n} = \sum_{s=0}^{n+j} |q|^{(n-s)(s-j)} \leq (n + j + 1)|q|^{-nj}.
\]

If \(j \geq 0\), then

\[
c_{j,n} = \sum_{k=j}^n |q|^{(n-k)(k-j)} \leq \frac{1}{1 - |q|}.
\]

Choose \(n_0\) large so that \(|q|^n \sqrt{\lambda} < 1\) for all \(n \geq n_0\). Combining the estimates

\[
\|(1 - q)^n W(\tilde{e}^n e^n)\| \leq C_q^6 \left( \frac{1}{1 - |q|} \sum_{j=0}^n \lambda^{j/2} + \sum_{j=1}^n \lambda^{-j/2} (n + 1) |q|^{nj} \right)
\]

\[
\leq C_q^6 \left( \frac{1}{1 - |q|} + \frac{(n + 1) |q|^n}{\sqrt{\lambda}} \right), \quad \forall n \geq n_0.
\]

This allows to conclude as \((n + 1) |q|^n \to 0\) as \(n \to \infty\). \(\Box\)

We end this section with the following lemma.

**Lemma 3.10.** Let \(n_1, n_2 \in \mathbb{N} \cup \{0\}\) and \(\Psi \in \mathcal{H}^{\otimes l}\) for some \(l \geq 0\). Then,

\[
(1 - q)^n c_q(e)^*(\tilde{e}^{(n+n_1)} e^{(n+n_2)} \Psi) \to 0,
\]

\[
(1 - q)^{n/2} \lambda^{-n/4} c_q(e)^*(\tilde{e}^{(n+n_1)} \Psi) \to 0, \text{ as } n \to \infty.
\]
Proof. We only prove the first one, the second one is similar. Using Lemma 2.1 and Eq. (2), it follows that

\[
\| (1 - q)^n c_q (e)^{(n+1)} e^{(n+2)} \Psi \|_q = (1 - q)^n \| q \|_{(n+1)} \| e^{(n+1)} (c_q (e)^{(n+2)} \Psi) \|_q = (1 - q)^n \| q \|_{(n+1)} \| c_q (e) \|_{(n+1)} \| c_q (e)^{(n+2)} \Psi \|_q \leq C_q^2 (1 - q)^n \| q \|_{(n+1)} \| e^{(n+1)} \|_q \| \| \Psi \|_q \quad \text{(by (10))}
\]

\[
= C_q^2 (1 - q)^n \| q \|_{(n+1)} \| \Psi \|_q \quad \text{(by Eq. (11))}
\]

\[
\leq C_q^2 (1 - q)^{\frac{n+1+\gamma}{2}} \| q \|_{(n+1)} \| \Psi \|_q \quad \text{as } n \to \infty 0.
\]

\[\square\]

3.2. Technicalities to treat all cases. The results in this section are crucial for tracking the relative commutant of \( M_q^\theta \) and will be used even in the case when \( \text{dim}(\mathcal{H}_\mathbb{R}) \geq 3 \). Thus, we adjust the set up of this section in such a way that it applies to all cases.

For the next three results, we assume \( \mathcal{H}_\mathbb{R} = \mathbb{R}^2 \oplus K_\mathbb{R} \), where \( K_\mathbb{R} \) is a real Hilbert space (could be 0), and \( \mathbb{R}^2 \) is reducing subspace for \( (U_t) \) with associated sub-representation being ergodic i.e., \( (\mathbb{R}^2, U_t) \) is as before. Define \( \mathcal{O} = \{ e, \bar{e}, \Omega \} \cup \{ \xi \in K_\mathcal{C} : W(\xi) \text{ is analytic for } (\sigma_t^\theta) \} \).

The next lemma tracks \( w.o.t. \)-limits of certain sequences of operators, which in turn track the relative commutant of the centralizers.

Lemma 3.11. For any \( \chi \in \mathcal{H}_\mathbb{R}^{\otimes i} \) and \( \eta \in \mathcal{H}_\mathbb{R}^{\otimes j}, a, b, \alpha, \beta \in \mathbb{Z} \), we have

\[
\lim_{n \to \infty} (1 - q)^{2n} \| \eta \|^{a+b} e^{n+\beta} \bar{e}^{n+a} \| \chi \|_q = \delta_{a+b+j} d^{\frac{\alpha}{a+i} = \beta} \| e^{j} \|^{\frac{\chi}{\| e^{j} \|_q}} \chi.
\]

Proof. As one is taking limit as \( n \to \infty \), one can assume that \( n+a, n+\alpha, n+b \) and \( n+\beta \) are all positive and large. First assume that \( \eta \) and \( \chi \) are elementary tensors in the letters from \( \mathcal{O} \). If \( n+a \geq n+\alpha \), then \( a-\alpha \geq 0 \). Consequently, \( (1 - q)^{n} \| e^{n+a} \| \| e^{n+a} \| \Omega = c_q \| \chi \| \chi \Omega \| \Omega \) is a bounded sequence from Eq. (11) and Lemma 3.9. Dealing with the case \( n+a < n+\alpha \) similarly (replacing \( \chi \) by \( e^{\alpha-a} \chi \)), it follows that \( (1 - q)^{n} e^{n+a} \| e^{n+a} \| \chi = 0 \). Similarly, \( (1 - q)^{n} \| e^{n+a} \| \| e^{n+a} \| \Omega \) is bounded.

First, assume that \( \eta \) or \( \chi \) contains a letter different from \( e, \bar{e}, \Omega \). For simplicity assume \( \eta = \eta_1 \cdots \eta_j \) is such that at least one letter in \( \eta \notin \{ e, \bar{e}, \Omega \} \). Let \( T = \prod_{l=1}^{j-1} c_q(\eta_l) \). By Lemma 2.1, it follows that there exists a finite set \( F \) (depending on \( l \)) and scalars \( c_{f,n} \), such that \( c_{f,n} \) is unifomly bounded for all \( f \in F, a, \alpha \in \mathbb{Z} \) and vectors \( \chi \in \mathcal{F}_\mathbb{R}(\mathcal{H}) \) for all \( f \in F \) such that

\[
T \| e^{n+a} \| \| e^{n+a} \| \chi = \sum_{f \in F} c_{f,n} e^{n+a} \| e^{n+a} \| \chi.
\]
Fix $f \in F$. From Lemma 2.1 again, it follows that as $n \to \infty$,
\[(1 - q)^n c_q(\eta) e^{n+a} \chi_f = q^{2n+a} (1 - q)^n e^{n+a} c_q(\eta) \chi_f \to 0.\]
Summing over $f \in F$, it follows that the limit in the statement is 0 and matches with the right hand side of the statement. Arguing similarly with $\chi$ and using right creation operators, it follows that it is sufficient to assume that the letters in $\eta$, $\chi$ are all in $\{e, \bar{e}, \Omega\}$, otherwise both sides in the statement are 0.

We do the proof by induction on $i + j$. Note that $\langle \bar{e}^i, \bar{e}^j \rangle_q = 1$ and similarly for $e$. So, if $\eta$ is an elementary tensor in the letters $e$ and $\bar{e}$, the quantity $\langle \eta, \bar{e}^j \rangle_q = 1$ if $\eta = \bar{e}^j$ and 0 otherwise. Thus, counting the number of letters $e$ and $\bar{e}$ gives the Dirac condition.

We start with $i + j = 0$. The Dirac condition is clear and from Eq. (11) we have
\[\|\bar{e}^{n+a} e^{n+\beta}\|_q^2 = \lambda(\alpha-\beta)/2 [n+a]_q [n+\beta]_q = \lambda(\alpha-\beta)/2 (1 - q)^{-2n-a-\beta} d_{n+a} d_{n+\beta}.\]
(15)
Thus, the limit in the statement is $\lambda(\alpha-\beta)/2 (1 - q)^{-(\alpha+\beta)} d_{2\infty}^2$.

Assume for instance $j > 0$. Let $\eta = e_1 \cdots e_j$ with $e_k \in \{e, \bar{e}\}$. We have
\[(1 - q)^{2n} \langle \eta \bar{e}^{n+b} e^{n+\beta}, e^{n+a} \rangle_q = (1 - q)^{2n} \langle \eta \bar{e}^{n+b} e^{n+\beta}, c_q(e_1) \bar{e}^{n+a} e^{n+a} \rangle_q,\]
(16)
with $\eta' = e_2 \cdots e_j$. By making arguments as in the first paragraph of the proof, it follows that $(1 - q)^n \eta' \bar{e}^{n+b} e^{n+\beta}$ is bounded. Thus, using Lemma 3.10, we get that the limit in Eq. (16) is zero unless $e_1 = \bar{e}$. If this is so, then from Lemma 2.1 we have
\[c_q(e_1) \bar{e}^{n+a} e^{n+\alpha} \chi = \lambda^{1/2} [n+a]_q \bar{e}^{n+a-1} e^{n+a} \chi + q^{2n+a} e^{n+a} \chi (c_q(e_1)^* \chi).\]

Thus, by induction the limit exists and
\[\lim_{n} (1 - q)^{2n} \langle \eta \bar{e}^{n+b} e^{n+\beta}, e^{n+a} \rangle_q = \frac{\lambda^{1/2}}{1 - q} \lim_{n} (1 - q)^{2n} \langle \eta' \bar{e}^{n+b} e^{n+\beta}, e^{n+a-1} e^{n+a} \rangle_q.\]
One can argue in the same way if $i > 0$ using the right creation operators and Eq. (9) when dealing with $\chi$.

By linearity, the statement holds when $\eta$ and $\chi$ are in the span of elementary tensors with letters from $\bar{O}$. Finally use a two-step approximation argument using Eq. (8) and Eq. (10).

The next lemma is the key to factoriality and irreducible centralizers.

**Lemma 3.12.** The w.o.t.-limit of $(1 - q)^n W_r(\bar{e}^n e^n) W(\bar{e}^n e^n)$ exists and is the positive rank-one operator $T_\xi$ where
\[\xi = S_\infty(\Omega) = \| \cdot \|_q - \lim_{n \to \infty} (1 - q)^n \chi^{n/2} W(\bar{e}^n) e^n = d_\infty \sum_{k=0}^\infty \frac{\xi^k}{k!} (1 - q)^k \chi^{k/2} \bar{e}^k e^k.\]

Needless to say, $T_\xi$ is a scalar multiple of the rank-one projection $P_{\xi}$.
Proof. First note that \( e, \tilde{e} \) are analytic vectors for \( (\Delta^l) \), thus words in them belong to the Tomita algebra associated to \( \varphi \). Further, \( \tilde{e}^n e^n \in M_q^\infty \Omega \) and \( J(\tilde{e}^n e^n) = \tilde{e}^n e^n \) for all \( n \) (see Thm. 2.4). Consequently, \( z_n = (1 - q)^2n W_r(e^n e^n) W(\tilde{e}^n e^n) \in B(\mathcal{F}_q(\mathcal{H})) \) is a bounded sequence; also \( \xi_n = (1 - q)^n \tilde{e}^n e^n \) is bounded in \( n \) (by Lemma 3.9).

To show the w.o.t.-convergence, we just need to find the limit of \( \langle \Psi, z_n \Phi \rangle_q \) where \( \Psi \in \mathcal{H}_R^{\otimes q^l} \) and \( \Phi \in \mathcal{H}_R^{\otimes q^k} \), \( k, l \geq 0 \). Since \( \mathcal{H}_R = \tilde{\mathcal{H}} \oplus \mathcal{K}_R \) with \( \tilde{\mathcal{H}} = \mathbb{R}^2 \), we have \( \mathcal{H} = \tilde{\mathcal{H}} \oplus \mathcal{K} \), where \( \mathcal{K} \) is the completion of \( \mathcal{H}_C \) (resp. \( \mathcal{K}_C \)) with respect to the norm induced by \( (U_t|_{\tilde{\mathcal{H}}_C}) \) (resp. \( (U_t|_{\mathcal{K}_C}) \)). Consequently, \( \mathcal{H}_0 := \tilde{\mathcal{H}}_R \oplus \mathcal{K}_R' \subseteq \mathcal{H}_R' \), where the former (real) subspace has the obvious description. Then \( \mathcal{H}_0 + i\mathcal{H}_0 \) contains \( \mathcal{H} \oplus 0 \) and \( 0 \oplus \mathcal{K} \). It follows that \( \mathcal{H}_0 + i\mathcal{H}_0 \) is dense in \( \mathcal{H} \). Let \( \mathcal{O}' = \{ e, \tilde{e}, \Omega \} \cup (K'_R + iK'_R) \). Then the span of elementary tensors with letters from \( \mathcal{O}' \) is dense in \( \mathcal{F}_q(\mathcal{H}) \). Therefore, by density, we can also assume that \( \Phi = e_1 e_2 \cdots e_k \) and \( \Psi = f_1 f_2 \cdots f_l \), where \( f_j \in \mathcal{O} \) and \( e_j \in \mathcal{O}' \). Then, we have

\[
\langle \Psi, z_n \Phi \rangle_q = \langle \Psi, (1 - q)^{2n} W_r(\tilde{e}^n e^n) W(\tilde{e}^n e^n) \Phi \rangle_q = (1 - q)^n \langle W(\Psi) \tilde{e}^n e^n, W_r(\Phi) \tilde{e}^n e^n \rangle_q.
\]

By the Wick formulas (Prop. 2.3),

\[
W(\Phi) = \sum_{0 \leq j \leq l, \sigma \in S_{l,j}} q^{l|\sigma|} u_{\sigma, j} \text{ if } l > 0,
\]

\[
W_r(\Phi) = \sum_{0 \leq i \leq k, \rho \in S_{k,i}} q^{k|\rho|} v_{\rho, i} \text{ if } k > 0,
\]

where

\[
u_{\sigma, j} = c_q(f_{\sigma(1)}) \cdots c_q(f_{\sigma(j)}) c_q(f_{\sigma(j+1)})^* \cdots c_q(f_{\sigma(l)})^*
\]

and

\[
u_{\rho, i} = c_q(r(e_{\rho(1)}) \cdots c_q,r(e_{\rho(i)}) c_q,r(e_{\rho(i+1)})^* \cdots c_q,r(e_{\rho(k)})^*).
\]

Moreover, \( W(\Psi) = 1 \) if \( l = 0 \) and \( W_r(\Phi) = 1 \) if \( k = 0 \).

Let \( \min(k, l) > 0 \). One observes that the left or right annihilation operators in a symbol other than \( e, \tilde{e} \) does not contribute in the inner product \( \langle \Psi, z_n \Phi \rangle_q \). Thus, the contributing factor of \( \langle \Psi, z_n \Phi \rangle_q \) comprises of two scenarios: \( i \) when a generic term in the Wick expansion formula of \( W_r(\Phi) \) and \( W(\Psi) \) both consist of creation operators only, \( ii \) when the annihilations operators in a generic term in the Wick expansion formula of \( W_r(\Phi) \) or \( W(\Psi) \) consists only of the symbols \( e \) or \( \tilde{e} \).

For both cases, if the letters in the creation operators (either left or right or both, as the case may be) consist of a symbol different from \( e \) or \( \tilde{e} \), then by Lemma 3.11, the associated limit contributing to \( \langle \Psi, z_n \Phi \rangle_q \) goes to 0 as \( n \to \infty \). Consequently, we can assume that \( e_i, f_j \in \{ e, \tilde{e} \} \). The same conclusion holds when \( \min(k, l) = 0 \) and \( \max(k, l) > 0 \).

It is clear that this reduction is nothing but compressing \( M_q \) by the Jones’ projection onto its subalgebra \( \Gamma_q(\mathbb{R}^2, U_l)'' \) (which possess \( \varphi \)-preserving conditional expectation) to reduce to the set up when \( \dim(\mathcal{H}_R) = 2 \).

If \( l > 0 \), then using Lemma 3.10 \((l - j)\) times (and Lemma 2.1) assuming \( l - j > 0 \), we get that \( u_{\sigma, j}(\xi_n) \) goes to 0 in \( \| \|_q \) unless \( f_{\sigma(j+1)} = \cdots = f_{\sigma(l)} = e \). If this is so, then using Lemmas 2.1 and 3.1 we have:

\[
u_{\sigma, j}(\xi_n) = \frac{|n_l|_{q^j}!}{|n_i|_{q^j}!} \lambda(l-j)/2(1-q)^n \tilde{e}^{n+l-j}e^n,
\]
where \( \Psi_j = f_{\sigma(1)} f_{\sigma(2)} \cdots f_{\sigma(j)} \). Setting \( \Psi_j^c = f_{\sigma(j+1)} \cdots f_{\sigma(l)} \) (\( \Psi_f^c := \Omega \)), we may rewrite \( u_\sigma(j_\xi) \) in full generality as

\[
u_\sigma(j_\xi) = \frac{[n]_q}{[n - l + j]_q} \lambda^{(l-j)/2}(1 - q)^n \langle \psi_j^c, \bar{e}^{l-j}, q \rangle_{\sigma} \psi_j^c \bar{e}^{n-l+j} e^n.
\]

We can do the same for \( \Phi \) to get \( v_\rho(j_\xi) = \frac{[n]_q}{[n - k + i]_q} \lambda^{(k-i)/2}(1 - q)^n \bar{e}^{n-k+i} e_{\rho(i)} \cdots e_{\rho(1)} \) provided that \( e_{\rho(i)} = \cdots = e_{\rho(j)} = \bar{e} \) if \( k > 0 \) (assuming \( k - i > 0 \) and noting \( \bar{e} = \lambda e \)). Set \( \Phi_i = e_{\rho(i)} \cdots e_{\rho(1)} \) and \( \Phi_i^c = e_{\rho(j)} \cdots e_{\rho(i+1)} \) (\( \Phi_c^f := \Omega \)). Then,

\[
u_\rho(j_\xi) = \frac{[n]_q}{[n - k + i]_q} \lambda^{(k-i)/2}(1 - q)^n \bar{e}^{k-i} e_{\rho(i)} \cdots e_{\rho(1)} \Phi_i^c \bar{e}^n e^n e^{n-k+i} \Phi_i.
\]

We are in position of using (Lemma 3.11) with \( a = \beta = 0 \) to get the existence of

\[
\lim_n \langle u_\sigma(j_\xi), v_\rho(j_\xi) \rangle_q = \delta_{2i-j} \frac{d_\lambda^{(i+j)/2}(1 - q)^j}{2i = k (1 - q)^j} \langle \psi_j^c, \bar{e}^j \rangle_{\sigma} \langle \psi_j^c, \bar{e}^{l-j} \rangle_{\sigma} \langle \psi_j^c, \bar{e}^i \rangle_{\sigma} \langle \psi_j^c, \bar{e}^{k-i} \rangle_{\sigma} \langle \psi_j^c, \bar{e}^{n-k+i} \rangle_{\sigma} \langle \psi_j^c, \bar{e}^n \rangle_{\sigma},
\]

when \( k, l > 0 \). This can be interpreted as decoupled scalar products. When \( k \) or \( l \) is 0, there is no Wick product expansion in terms of creation and annihilation operators, but the obvious modifications justify the existence of the desired limit(s) using Lemma 3.11 and matches Eq. (17).

Thus, summing all the terms we get that for some \( \chi_k \in \mathcal{H}^{\otimes q} \) and \( \eta_l \in \mathcal{H}^{\otimes q} \),

\[
\lim_n \langle \psi, z_n \Phi \rangle_q = \langle \chi_k, \Phi \rangle_q \langle \psi, \eta_l \rangle_q, \quad \forall k, l \geq 0.
\]

Now we will use some arguments to avoid the use of \( q \)-symmetrization operators to identify the vectors. Since \( z_n \) is bounded, this justifies that the \( w.o.t.- \)limit exists and has rank 1. As \( z_n = z_n^* \), we have \( \psi_l = \eta_l \) for all \( l \geq 0 \). Taking \( \Phi = \Omega \), we have that \( z_n \Omega \) is weakly converging to \( \zeta = \langle \chi_0, \Omega \rangle_q \left( \oplus_{m=0}^{\infty} \chi_m \right) \).

We want to identify \( \xi = \oplus_{m=0}^{\infty} \chi_m \) as \( z_n \xrightarrow{w.o.t.} T_\xi \) (the rank-one limit), and to do so we identify \( \zeta \). To find \( \zeta \), we want to consider \( \lambda^{-n/2} (1 - q)^n \langle W(\psi) \bar{e}^n, \bar{e}^n \rangle_q \). This is a bounded sequence (see Lemma 3.3).

Note that if \( l = 0 \) (i.e., \( \psi = \Omega \), then \( \lim_n \lambda^{-n/2} (1 - q)^n \langle W(\psi) \bar{e}^n, \bar{e}^n \rangle_q = d_\infty \). Let \( l > 0 \). As above, we can use the Wick formula for \( \psi \) and the situation as in Lemma 3.11. We get that \( u_\sigma(j \lambda^{-n/4} \bar{e}^n) \) is 0 unless \( s_{\sigma(j+1)} = \cdots = s_{\sigma(l)} = e \). If this is so, \( u_\sigma(j \lambda^{-n/4} \bar{e}^n) = [n]_q^{l-j}/[n-j+l]_q \lambda^{(l-j)/2} \bar{e}^{j} \bar{e}^{n-j-l} \) (Lemma 3.1). Taking scalar product with \( \lambda^{-n/4} \bar{e}^n \), recalling that \( \| \bar{e}^n \|^2_q = [n]_q \lambda^{n/2} \) and computing again as in Lemma 3.11, we have

\[
\lim_n \lambda^{-n/2} (1 - q)^n \langle u_\sigma(j \bar{e}^n), \bar{e}^n \rangle_q = \delta_{2j = d_\infty \lambda^{(n/2)/2}} (1 - q)^{-j} \langle \psi_j^c, \bar{e}^j \rangle_{\sigma} \langle \psi_j^c, \bar{e}^{n-l-j} \rangle_{\sigma} \langle \psi_j^c, \bar{e}^{n-j} \rangle_{\sigma} \langle \psi_j^c, \bar{e}^{n-j} \rangle_{\sigma}.
\]

Thus, comparing with Eq. (17) (setting \( k = 0 \)) we have

\[
\lim_n \langle \psi, z_n \Omega \rangle_q = d_\infty \lim_n \lambda^{-n/2} (1 - q)^n \langle W(\psi) \bar{e}^n, \bar{e}^n \rangle_q.
\]
But using Eq. (5), we have
\[ (W(\Psi)\vec{\varepsilon}^n, \vec{\varepsilon}^n)_q = (\Psi, W(\vec{\varepsilon}^n)W_1(\vec{\varepsilon}^n)^*\Omega)_q = \lambda^n (\Psi, W(\vec{\varepsilon}^n)W(e^n)\Omega)_q. \]

Since \( S^*_n(\Omega) = \lambda^{n/2}(1 - q^n)W(\vec{\varepsilon}^n)W(e^n)\Omega \) (use Lemma 3.2 or Eq. (5)), we get that \( \xi = d_\infty S^n(\Omega) \). To compute its value, one can use Lemmas 3.1 and 3.2 to derive
\[ \lambda^{n/2}(1 - q^n)W(\vec{\varepsilon}^n)e^n = \sum_{k=0}^n (1 - q)^k \frac{d_n^2}{d_{n-k}d_k^2} \lambda^{k/2}\vec{\varepsilon}^k e^k. \]

By taking the limit in \( n \) and using the dominated convergence theorem, we have
\[ \zeta = d_\infty^2 \sum_{k=0}^\infty c_k^2 (1 - q)^k \lambda^{k/2}\vec{\varepsilon}^k e^k, \]
and this expression tallies with the expression in (2) of Rem. 3.7.

Comparing the two expressions for \( \zeta \) we have \( \langle \chi_0, \Omega \rangle q \chi_0 = d_\infty^2 \Omega \). Since \( \chi_0 = \kappa \Omega \) where \( \kappa \in \mathbb{C} \), it follows that \( |\kappa|^2 = d_\infty^2 \) (scalar product is linear on the right), i.e., \( |\kappa| = d_\infty \). Consequently,
\[ \xi = \frac{1}{(\chi_0, \Omega)_q} \zeta = \frac{1}{(\chi_0, \Omega)_q} d_\infty^2 \sum_{k=0}^\infty c_k^2 (1 - q)^k \lambda^{k/2}\vec{\varepsilon}^k e^k = e^{i\arg(\kappa)} d_\infty \sum_{k=0}^\infty c_k^2 (1 - q)^k \lambda^{k/2}\vec{\varepsilon}^k e^k. \]
Finally, note that a rank-one self-adjoint operator is uniquely determined up to a phase factor of the associated vector. Also note that \( \xi \) is not an unit vector, thus \( T_{\xi} \) is a positive scalar multiple of \( P_{\vec{\varepsilon}^k e^k} \).

The next lemma will be used particularly when \( \dim(\mathcal{H}_\mathbb{R}) \geq 3 \). It describes some amount of mixing available when \( \dim(\mathcal{H}_\mathbb{R}) \) is ‘large’, which frees one from any bargain with the parameters \( \lambda \) and \( q \) to decide the factoriality unlike the case when \( \dim(\mathcal{H}_\mathbb{R}) = 2 \).

**Lemma 3.13.** Suppose \( \dim(\mathcal{H}_\mathbb{R}) \geq 3 \) with \( (U_i) \) as in the set up. Let \( 0 \neq \eta \in \mathcal{H}_\mathbb{R} \) be such that \( \eta \perp \vec{\varepsilon}, e \) in \( \langle \cdot, \cdot \rangle_q \). Further, let \( \{u_n\} \) be a sequence of unitaries in \( vN(W(\eta)) \) converging to \( 0 \) in the w.o.t. Then, \( u_n W(\xi)u_n^* \rightarrow c_1 \) in the w.o.t. for some \( c > 0 \), where
\[ \xi = d_\infty \sum_{k=0}^\infty c_k^2 (1 - q)^k \lambda^{k/2}\vec{\varepsilon}^k e^k. \]

**Proof.** Firstly, note that \( vN(W(\eta)) \) is a diffuse abelian von Neumann algebra [BM, §4]. Thus, the desired sequence \( \{u_n\} \) exists as in the statement.

By Lemma 3.12, the series defining \( \xi \) is convergent in \( \mathcal{F}_q(\mathcal{H}) \) in \( \|\cdot\|_q \). Further, \( \{(1 - q)^k W(\vec{\varepsilon}^k e^k)\} \) is a bounded sequence by Lemma 3.9. Since \( c_k \) is a bounded sequence, so \( d_\infty \sum_{k=0}^N c_k^2 (1 - q)^k \lambda^{k/2} W(\vec{\varepsilon}^k e^k) \) is convergent in norm as \( N \rightarrow \infty \) in \( M_\mathbb{R}^\psi \). Using Thm. 2.4, it follows that
\[ W(\xi) = d_\infty \sum_{k=0}^\infty c_k^2 (1 - q)^k \lambda^{k/2} W(\vec{\varepsilon}^k e^k) \in M_\mathbb{R}^\psi. \]

Therefore, it is sufficient to show that \( (1 - q)^k u_n W(\vec{\varepsilon}^k e^k)u_n^* \) converges to \( 0 \) in the w.o.t as \( n \rightarrow \infty \) for each \( k \geq 1 \). Further, the boundedness of the sequence entails that it is sufficient to verify the convergence with vectors of the form \( \Psi = e_1 e_2 \cdots e_l \in \mathcal{H}^\otimes l \)}
and \( \Phi = f_1 f_2 \cdots f_{l'} \in \mathcal{H}^{\otimes q} \) where \( e_i, f_j \in \mathcal{O}' \) and \( l, l' \geq 0 \) (see Lemma 3.12 for defn. of \( \mathcal{O}' \)).

Now, note that
\[
\langle \Phi, u_n W(\tilde{e}^k e^k)u_n^* \Psi \rangle_q = \langle u_n^* \Phi, W(\tilde{e}^k e^k)u_n^* \Psi \rangle_q
= \langle W_r(\Phi)u_n^* \Omega, W(\tilde{e}^k e^k) W_r(\Psi)u_n^* \Omega \rangle_q
= \langle W_r(\Psi)^* W_r(\Phi)u_n^* \Omega, W(\tilde{e}^k e^k)u_n^* \Omega \rangle_q.
\]

When \( k > 0 \), this last quantity goes to 0 as \( n \to \infty \) as \( \tilde{e}^k e^k \perp \text{span}\{\eta^l : l \geq 0\} \) (see [BM, §4]). This is a consequence of the Wick formula and Lemma 1 in [SW] as explained around (4.2) there.

As \( \text{span}_{\mathbb{C}} \mathcal{O}' \) is dense in \( \mathcal{H} \) and \( (1 - q)^k W(\tilde{e}^k e^k) \) is uniformly bounded (Lemma 3.9), it follows that \( (1 - q)^k u_n W(\tilde{e}^k e^k)u_n^* \to 0 \) as \( n \to \infty \) in the w.o.t. for each \( k \geq 1 \). Clearly \( c = d_\infty \) and the proof is complete. \( \Box \)

4. Factoriality

This is the main section of this paper. Here we establish the factoriality of \( M_q \). The set up of Sect. 3.2 will be in force in this section. As informed earlier, we will have to compromise with the parameter \( \lambda \) to assert the factoriality of \( M_q \) when \( \text{dim}(\mathcal{H}_\mathbb{R}) = 2 \).

In this case though, we will directly deduce that \( M_q^\varrho \) is irreducible. When \( \text{dim}(\mathcal{H}_\mathbb{R}) \geq 3 \), \( M_q \) is a factor regardless of the value of the parameter \( \lambda \) defining \( (U_t) \) on \( \mathbb{R}^2 \) and \(-1 < q < 1 \). However, deducing that \( M_q^\varrho \) is irreducible will not be direct and will be constrained. When \( q = 0 \), \( M_0 \) is the free Araki-Woods factor [S97] and there is nothing to prove in this case.

As an outcome of the results in Sect. 3, we have the following:

**Lemma 4.1.** With \( \xi \) as in Lemma 3.12, let \( P \) be the orthogonal projection onto \( [M_q^\varrho M_q' \xi]^{\perp} \). Then, \( (M_q^\varrho)' \cap M_q \) is trivial if and only if \( P = 0 \). In particular, \( M_q^\varrho \) and \( M_q \) are factors if and only if \( P = 0 \).

**Proof.** It is clear that \( P \in (M_q^\varrho)' \cap M_q \). Suppose that \( (M_q^\varrho)' \cap M_q = \mathbb{C}1 \). Then, \( P = 0 \) as \( \xi \neq 0 \) and \( M_q^\varrho \) and \( M_q \) are factors.

Conversely, by Lemma 3.12 it follows that \( T_\xi \in M_q^\varrho \cap M_q' \). Let \( x \in (M_q^\varrho)' \cap M_q \) so that \( xT_\xi = T_\xi x \). In particular, \( x(\xi) = \lambda_0 \xi \) for some \( \lambda_0 \in \mathbb{C} \). Then, for any \( m \in M_q^\varrho \) and \( n \in M_q' \), \( x(mn(\xi)) = \lambda_0 mn(\xi) \). If \( P = 0 \), this means that \( x(\eta) = \lambda_0 \eta \) for all \( \eta \in \mathcal{F}_q(\mathcal{H}) \) and thus \( x = \lambda_0 1 \). \( \Box \)

Now we are in position to state the first theorem on factoriality.

**Theorem 4.2.** Let \( q \neq 0 \). If \( \text{dim}(\mathcal{H}_\mathbb{R}) = 2 \) and \( (U_t) \) is ergodic then \( (M_q^\varrho)' \cap M_q = \mathbb{C}1 \) (and hence \( M_q^\varrho \) and \( M_q \) are factors) when
\[
\lambda < \begin{cases} 
(1 + \frac{C_q}{d_\infty})^{-2}, & \text{if } q > 0; \\
(1 + \frac{(C_q(1-q))^2 D(q)}{d_\infty})^{-2}, & \text{if } q < 0.
\end{cases}
\]
Proof. Consider $P$ from Lemma 4.1. Then, we have $P(\xi) = 0$. Thanks to Lemma 3.12 and Eq. (5), we have that
\[
0 = \langle P(\xi), \Omega \rangle_q = \lim_{n} \lambda^{n/2}(1 - q)^n \langle PW(e^n)^* W(e^n) \Omega, \Omega \rangle_q
= \lim_{n} \lambda^{n/2}(1 - q)^n \langle W(e^n)^* W(e^n) P \Omega, P \Omega \rangle_q
= \lim_{n} \lambda^{n/2}(1 - q)^n \|W(e^n)P(\Omega)\|^2.
\]
Thus, $\eta_n = \lambda^{n/4}(1 - q)^{n/2}W(e^n)P(\Omega)$ goes to 0 in $\|\cdot\|_q$. Thanks to Lemma 3.3 (i), we still get that $\lambda^{n/4}(1 - q)^{n/2}c_q(e)^{\eta_n} = S_n(P(\Omega))$ goes to 0. Consequently, $S(\infty)(P(\Omega)) = 0$. But $S(\infty)$ is invertible by Lemma 3.6 and the hypothesis, and hence $P(\Omega) = 0$ forcing $P = 0$. Now use Lemma 4.1. □

Now, we assume that $\mathcal{H}_R = \mathbb{R}^2 \oplus K_R$ where $K_R$ is a non-zero real Hilbert space, and $\mathbb{R}^2$ is reducing subspace for $(U_l)$ with associated sub representation being ergodic i.e., $(\mathbb{R}^2, U_l)$ is as in Sect. 3.1. It should be noted that unlike Thm. 4.2, the invertibility of the operator $S(\infty)$ has no role to play in deciding the factoriality in this case.

Theorem 4.3. Suppose $\dim(\mathcal{H}_R) \geq 3$. Then $M_q$ is a factor.

Proof. Let $P$ be the orthogonal projection onto $[M_q M_q^\prime \xi]\perp$, where $\xi$ is as in Lemma 3.13. Since $\dim(\mathcal{H}_R) \geq 3$, choose $0 \neq \eta \in \mathcal{H}_R$ such that $s_q(\eta)$ is analytic for $(\sigma^q_t)$ and $\eta \perp \{e, \bar{e}\}$ in $\langle \cdot, \cdot \rangle_q$. Then, $u_n = e^{-inx_q(\eta)}$ is an analytic sequence of unitaries such that $u_n \rightarrow 0$ in the w.o.t.-topology. The w.o.t.-convergence holds as the spectral measure of $s_q(\eta)$ is Lebesgue absolutely continuous (see [BM, §4]). By Lemma 3.13, it follows that
\[
J(\sigma^q_t(u_n))^* J u_n^* \xi = u_n^* W(\xi) u_n \Omega \rightarrow c\Omega \text{ weakly as } n \rightarrow \infty.
\]
It follows that $[M_q M_q^\prime \xi]$ is weakly dense in $\mathcal{F}_q(\mathcal{H})$, and hence norm dense in $\mathcal{F}_q(\mathcal{H})$ by a theorem of Mazur. It follows that $P = 0$.

From Lemma 3.12, we have $T_\xi \in M_q^\prime \vee M_q^\prime$. Let $x \in \mathcal{Z}(M_q)$. Then, $x T_\xi = T_\xi x$.
Replacing the role of $M_q^\prime$ by $M_q$ in the proof of Lemma 4.1 (second paragraph), the argument is verbatim. □

Remark 4.4. The following comments are in order.

1. In view of the main results in [BM, R05] and the results in this section, the factoriality problem for $\Gamma_q(\mathcal{H}_R, U_l)^\prime$ remains open only in the case when $\dim(\mathcal{H}_R) = 2$, $(U_l)$ is ergodic and $\lambda$ is not small in the sense of Thm. 4.2.
2. When $\dim(\mathcal{H}_R) = 2$, our proof is different than the existing proofs of the factoriality of $M_q$ under various assumptions (in [BM, Hi03, R05, S97, Ne15]) because of the role of the operator $S(\infty)$ in the proof, and this process will not work when $q = 0$ and $\lambda > 1/2$ (see Rem. 3.7).
3. Let $\eta \in \mathcal{T}_R$ be such that $\|\eta\|_U = 1 (= \|\eta\|_q)$ and $\eta$ is not fixed by $(U_l)$. Then, either $(U_l)$ has a non-zero weakly mixing component or has a two-dimensional ergodic sub representation characterized by $\lambda \in (0, 1)$. If $\dim(\mathcal{H}_R) \geq 3$ or $\dim(\mathcal{H}_R) = 2$, $q \neq 0$ and $\lambda$ is ‘small’ in the sense of Thm. 4.2 or $\dim(\mathcal{H}_R) = 2$ and $q = 0$, then $\nu (W(\eta)) \subseteq M_q$ is a split inclusion (see [BM2, Thm. 4.6] and also results in Sect. 5).
5. Revisiting Centralizer

The first result on $M^q$ is Thm. 3.2 of [Hi03]. In Sect. 2.5 and in [BM, §7] we have discussed on the structure of $M^q$. In this section, we continue our discussion on $M^q$. Note that $M^q$ depends on the almost periodic component of $(U_i)$ (see Thm. 2.4). Thm. 7.1 of [BM] says that if there is a non-trivial fixed point of $(U_i)$ and the almost periodic component of $(U_i)$ is at least two-dimensional then $(M^q)' \cap M_q = C1$.

Consequently, when $\text{dim}(\mathcal{H}_R) \geq 2$ the last statement together with Thm. 4.2 provide all that is known about $M^q$. When the almost periodic part of $(U_i)$ is three-dimensional or five-dimensional one can apply [BM, Thm. 7.1]. Now we proceed to describe the structure of $M^q$ when the almost periodic part of $(U_i)$ is sufficiently large.

**Theorem 5.1.** Let $\mathcal{H}^ap \subseteq \mathcal{H}_R$ denote the almost periodic part of $(U_i)$. If $\dim(\mathcal{H}^ap) \geq 5$, then $(M^q)' \cap M_q = C1$. In particular, $M^q$ is a factor.

**Proof.** If $(U_i)$ has a non-zero fixed point then there is nothing to prove. So we only have to prove in the case when $\dim(\mathcal{H}^ap) \geq 6$ and $(U_i)$ is ergodic. In this case, $\mathcal{H}_R = \bigoplus_{i=1}^3 \mathbb{R}^2 \oplus K_R$, where $\mathbb{R}_i^2 := \mathbb{R}^2$ for $1 \leq i \leq 3$ and $K_R$ are invariant subspaces of $(U_i)$.

Denote $M^q(1) = \Gamma_q(\mathbb{R}_1^2, U_{1,1})^\prime$, $M^q(2) = \Gamma_q(\mathbb{R}_2^2 + \mathbb{R}_3^2, U_{1,1})^\prime$. Then $M^q(1)$, $M^q(2)$ are unital subalgebras of $M_q$ via the $q$-Gaussian functoriality [Hi03]. Note that $M^q(1)$ and $M^q(2)$ possess $\varphi$-preserving conditional expectations by Takesaki’s theorem [T72].

By Thm. 4.3, it follows that $M_q$ and $M^q(2)$ are both factors. We claim that both $M_q$ and $M^q(2)$ cannot be of type I and hence they are diffuse. Let us first assume this claim and finish the proof. Since $(U_{1,1})$ is almost periodic, $(M^q(2)) \varphi \mid M_q$ is also diffuse (see [DM, Thm. 7.9] and [U11, Lemma 2.1]). Let $u_n \in (M^q(2)) \varphi \mid M_q$ be a sequence of unitaries that go to 0 in the w.o.t. By Thm. 2.4, it follows that $u_n \in M^q$ as well.

Let $\xi$ be as in Lemma 3.13 obtained by regarding $M^q(1)$ as the von Neumann algebra in Sect. 3.1. From Lemma 3.13, it follows that $u_n^* W(\xi) u_n \overset{\text{w.o.t}}{\to} c1$, where $c > 0$ is a scalar. It follows that $\{M^q, M^q(2), M^q(3), \xi\} = F_q(\mathcal{H})$. Thanks to Lemma 4.1, the argument is complete.

Now it remains to establish the claim. The proof of the claim is identical for both $M^q(2)$ and $M_q$. So we prove it for $M_q$ and work with the sub representation $(\mathbb{R}_1^2, U_{1,1})$ as in Sect. 3.2.

Recall that a factor $M$ equipped with a faithful normal state $\psi$ is of type I if and only if, $M \ni x \mapsto \frac{1}{\psi(x)} x \Omega_\psi \in L^2(M, \psi)$ is a compact embedding (see [BDL, Cor. 2.9] or [DM, Thm. 7.13]). Note that $\{1 - q^n W(\bar{e}^n e^n)\}_{n \geq 1} \subseteq M^q$ is bounded from Lemma 3.9. However, $\frac{1}{\psi} (1 - q^n W(\bar{e}^n e^n) \Omega = (1 - q^n \bar{e}^n e^n), and this sequence has no converging subsequence in $\mathcal{F}_q(\mathcal{H})$. Therefore, the symmetric embedding of $M_q$ in $\mathcal{F}_q(\mathcal{H})$ is not compact proving the claim.

**Remark 5.2.** (1) If $\dim(\mathcal{H}^ap) = 4$ and $(U_i)$ is ergodic, then upon assuming that the parameter $\lambda \in (0, 1)$ which defines a two-dimensional sub representation is small in the sense of Thm. 4.2, one can still conclude that $(M_q)' \cap M_q = C1$ (when $q \neq 0$).

(2) In all cases where $(M_q)' \cap M_q = C1$, the $S$-invariant of Connes can be directly deduced from the modular theory of the vacuum state and is exactly as in [BM, Thm. 8.2].
(3) Note that the last part of the proof of Thm. 5.1 along with Thm. 5.2 of [DM2] says that if $\mathcal{H}_{\mathbb{R}}^{op} \neq 0$, then $M_q^+$ cannot be discrete (direct sum of matrix algebras).

(4) Suppose that $\dim(\mathcal{H}_{\mathbb{R}}) = 4$ and $(U_t)$ is ergodic. In this case if $\xi \in \mathcal{H}_{\mathbb{R}}$ be such that $\|\xi\|_q = 1$, then $M_q = v N(\sigma_q(\xi)) \subseteq M_q^+$ is a quasi-split inclusion (see [BM2, Thm. 4.6]). Since $M_q$ is a factor (Thm. 4.3), $M_q \otimes 1 \subseteq M_q \otimes \mathcal{B}(\mathcal{K})$ is a split inclusion (see [BM2, Thm. 3.8]), where $\dim(\mathcal{K}) = \aleph_0$. It follows that there is no normal conditional expectation from $M_q$ onto $M_{\xi}$. This forces that $M_q$ cannot be a $\text{II}_1$ factor, for if it were there would have been a normal conditional expectation from $M_q$ onto $M_{\xi}$ preserving the canonical trace.

In view of Rem. 5.2, we are still not in position to compute the $S$-invariant when $\dim(\mathcal{H}_{\mathbb{R}}) = 4$ and $(U_t)$ is ergodic. Therefore, we conclude this section by proving that $M_q$ is never semifinite.

**Theorem 5.3.** When $(U_t)$ is non-trivial $M_q$ is not semifinite. In particular, if $\dim(\mathcal{H}_{\mathbb{R}}) = 4$ then $M_q$ is a type III factor.

**Proof.** If $(U_t)$ has a non-trivial weakly mixing component, the statement follows from [BM]. Therefore, we assume that $(\mathcal{H}_{\mathbb{R}}, U_t)$ is as in the set up of Sect. 3.2.

We claim that $(\sigma_t^q)$ cannot be inner. Suppose there exists a one-parameter group of unitaries $(u_t) \subseteq M_{\mathbb{C}}^q$ such that $\sigma_t^q(x) = u_t x u_t^*$ for all $x \in M_q$ and $t \in \mathbb{R}$. Then, by Lemma 3.12, we have $u_t T_{\xi} = T_{\xi} u_t$ for all $t \in \mathbb{R}$. Therefore, there exists scalars $z_t$ with $|z_t| = 1$ such that $u_t \xi = z_t \xi$ for all $t$. Replacing $u_t$ by $\overline{z_t} u_t$, we may assume that $z_t = 1$ for all $t$. Then, $u_t W(\xi) = W(\xi)$ for all $t$.

Let $\mathcal{K} = \text{Ker}(W(\xi))$ and $\mathcal{L} = \mathcal{K}^\perp$. Then, $u_t|_{\mathcal{L}} = 1_{\mathcal{L}}$. Put $w_n = |W((\lambda^{\frac{1}{2}}(1-q)^\frac{1}{2}e)^n)|^2$. This is a bounded sequence by Lemma 3.3. By the proof of Thm. 3.12, we have

$$W(\xi) = w.o.t.- \lim_n w_n.$$ 

Hence, $\mathcal{K} = \{\xi \in \mathcal{F}_q(\mathcal{H}) : w_n \xi \xrightarrow{w} 0\}$ and it follows that $W(e)\mathcal{K} \subseteq \mathcal{K}$ (use Lemma 3.2). Thus, decomposing $\mathcal{B}(\mathcal{F}_q(\mathcal{H}))$ with respect to $\mathcal{K} \oplus \mathcal{L}$, one has the form

$$W(e) = \begin{pmatrix} x & y \\ 0 & w \end{pmatrix} \quad \text{and} \quad u_t = \begin{pmatrix} v_t & 0 \\ 0 & 1 \end{pmatrix} \quad \forall t.$$ 

Since $u_t W(e) u_t^* = \sigma_t^q(W(e)) = e^{it \log \lambda} W(e)$ for all $t$ (use Eq. (5)), so choosing $t$ such that $t \log \lambda \notin 2\pi \mathbb{Z}$ it follows that $w = 0$. Thus, $\text{Ran}(W(e)) \subseteq \mathcal{K}$.

This is false as $e \notin \mathcal{K}$. Indeed,

$$W(\xi) = w.o.t.- \lim_n \lambda^n (1-q)^n W(\bar{\xi}) W(e)^n = \lambda^{\frac{1}{2}} (1-q) W(\bar{\xi}) W(\xi) W(e).$$

Consequently, $\xi = \lambda^{\frac{1}{2}} (1-q) W(\bar{\xi}) W(\xi) e \neq 0$. This completes the proof. \(\blacksquare\)

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