Extensions of Instantons

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Abstract

In this paper one studies a correspondence between extensions $E$ of instanton bundles $F$ on $\mathbb{P}^3$ (by line bundles or by twisted bundles of forms) and potentials (connections) on certain bundles on the grassmanian $\text{Gr}(2, 4)$, or the sphere $S^4$. Of special interest is the case of extensions by $\mathcal{O}(-1)$, when the $E$’s are stable. One shows also how to obtain the monads of the $E$’s, knowing those of the $F$’s. The main ingredients are the Atiyah-Ward correspondence (cf. [AW]) and a theorem of Hitchin (cf. [H]).

1 Introduction.

In the fundamental papers [W], [AW] of Ward and Atiyah and Ward a 1-1 correspondence between $\{\text{vector bundles on the grassmanian } \text{Gr}(2, 4) \text{ (respectively } S^4) \text{ with an antiself-dual connection}\}$ and $\{\text{holomorphic vector bundles on } \mathbb{P}^3 \text{ (respectively vector bundles on } \mathbb{P}^3 \text{ with a positive real form)}\}$ is obtained. In particular, the bundles on $\mathbb{P}^3$ in this correspondence have necessarily $c_1 = 0$.

A natural question, which in fact the second named author learned from a discussion with M. F. Atiyah in 1981 in Bucharest, is to find a similar correspondence between vector bundles on $\mathbb{P}^3$ with arbitrary $c_1$ and some objects of differential geometry on $\text{Gr}(2, 4)$ or $S^4$. In this direction, an answer was given by Leiterer in [L], who established a correspondence between arbitrary holomorphic vector bundles on $\mathbb{P}^3$ and holomorphic bundles over $\text{Gr}(2, 4)$ with a certain differential operator which he named partial connection.

The aim of this paper is to describe a construction by which to certain holomorphic vector bundles $E$ on $\mathbb{P}^3$ with any $c_1$, namely extensions of an instanton bundle, one makes to correspond bundles on $\text{Gr}(2, 4)$ equipped with certain connections constructed from data associated with the instanton. The ingredients of the construction are the Atiyah-Ward correspondence and the choice of a spinor on $\text{Gr}(2, 4)$ coupled with the Yang-Mills potential obtained by the twistor.
transform. Besides these, we use an elementary construction of a connection on a direct sum of vector bundles when there are known two connections on the two bundles and a section on their tensor product.

When \( c_1(E) = -1 \) one uses a theorem of Hitchin (cf. [H]) to obtain a bijection between certain vector bundles on \( \mathbb{P}^3 \) and a special class of connections in an associated vector bundle over \( S^4 \).

At the end of Section 2 we extend our method for obtaining families of connections on bundles over \( \text{Gr}(2, 4) \) associated to families of pairs of a cohomology class of an instanton bundle on \( \mathbb{P}^3 \) and a spinor field on \( \text{Gr}(2, 4) \) coupled with the Yang-Mills potential which corresponds to the instanton.

The Atiyah-Ward correspondence makes the construction of instantons on \( \mathbb{P}^3 \) of a special relevance. A basic tool is the \textit{monads method} of G. Horrocks (cf. [OSS], [A]). In Section 3 it is shown how to obtain monads of our vector bundles \( E \) from the monads of the instantons \( F \).

It should be pointed that, in "physicists" terms, our construction can be summarised by saying that, from a Yang-Mills potential and a spinor field with arbitrary helicity, one constructs a potential (i.e. connection) on a vector bundle of higher rank.

\textbf{Notation} We explain here some of the notation we use:

\begin{itemize}
  \item \( \mathbb{P} := \mathbb{P}(U^*) \cong \mathbb{P}^3 \) = projective space of lines in \( U \), where \( U \) is a \( \mathbb{C} \)-vector space of dimension 4
  \item \( \mathcal{G} := \text{Gr}(2, 4) \) = the Grassmannian of 2-dimensional subspaces in a vector space of dimension 4
  \item \( \mathcal{S}_+ := \text{positive spinor bundle on} \ \mathcal{G}, \ \text{i.e. universal subbundle on} \ \mathcal{G} \)
  \item \( \mathcal{I} := \text{incidence variety} \{(x, l) : x \in \mathbb{P}^3, l \in \mathcal{G} \ \text{with the property} \ x \in L_l, L_l \ \text{being the line in} \ \mathbb{P} \ \text{represented by} \ l \}
  \item \( S^4 := \text{the 4-sphere} \)
  \item \( F \) a mathematical instanton bundle on \( \mathbb{P}^3 \), i.e a self dual vector bundle on \( \mathbb{P} \) of arbitrary rank, such that \( H^1(F(-2)) = 0 \)
  \item \( V \mathcal{E} := V \otimes E \), where \( V, E \) are vector bundles, even when \( V \) is a vector space interpreted as a constant vector bundle
  \item \( V\mathcal{O} \) for \( V \) a vector space considered as constant vector bundle
\end{itemize}
2 Construction of the connection

2.1 General construction

Let $G$ be a differentiable manifold and $F_1, F_2$ vector bundles over $G$ endowed with connections $\nabla_1, \nabla_2$. Let $\varphi_1 \in \Gamma(F_1 \otimes F_2^*)$ and $\varphi_2 \in \Gamma(F_2 \otimes F_1^*)$ be two differentiable sections, where by $*$ we denote the dual of the corresponding vector bundle. Let $F = F_1 \oplus F_2$. To the quadruple $(\varphi_1, \varphi_2, \nabla_1, \nabla_2)$ we associate the following connection $\nabla$ on $F$: relative to the splitting $F = F_1 \oplus F_2$, $\nabla$ has the block decomposition

$$
\begin{pmatrix}
\nabla_1 & \nabla_{12} \varphi_1 \\
\nabla_{21} \varphi_2 & \nabla_2
\end{pmatrix}
$$

where $\nabla_{12}$ and $\nabla_{21}$ are the canonical induced connections on $F_1 \otimes F_2^*$ and $F_2 \otimes F_1^*$. If we change the splitting it is obvious that the connection is changed by a gauge transformation and so we obtain the following simple lemma:

**Lemma 2.1.** In the situation from above to any quadruple $(\varphi_1, \varphi_2, \nabla_1, \nabla_2)$ it is associated a unique gauge equivalence class of connections on $F$.

2.2 Modification of the Atiyah-Ward transform

Recall the notation summarized in Introduction: $G$ will denote the grassmanian of 2-planes in $\mathbb{C}^4$, $\mathbb{P}$ the projective 3-space and $F$ the (flag) incidence variety. We shall use the following twistor diagram (cf., for instance, [A], [WW]):

Recall that $G$ is realized as a quadric $Q_4$ in $\mathbb{P}^5$ via the Plücker embedding (famous Klein representation of all lines in $\mathbb{P}^3$). A real structure on $\mathbb{P}^3$ induces a real structure on $Q_4$. Taking the real structure defined by

$$
\sigma: (z_1, z_2, z_3, z_4) \mapsto (-\bar{z}_2, \bar{z}_1, -\bar{z}_4, \bar{z}_3)
$$
one shows that although the above map has no fixed points in $\mathbb{P}$, it has fixed lines, which are called \textit{real lines}. The real lines are parametrized by $S^4 \hookrightarrow G$, and $G$ is the complexification of $S^4$. The map $p$ associates to any point $x$ in $\mathbb{P}$ the point in $S^4 \subset G$ representing the real line determined by $x$ and $\sigma(x)$.

Let $F$ be a rank-2 instanton bundle on $\mathbb{P}$ i.e. a mathematical instanton trivial on all the real lines. Let $\mathcal{U} \subset G$ be the open subset which corresponds to all lines in $\mathbb{P}$ on which $F$ is trivial. $\mathcal{U}$ will therefore necessarily contain $S^4$. Let $\mathcal{U}' = \mu \circ \nu^{-1}(\mathcal{U})$.

It is well known that $H^1(F(k)) = 0$ for $k \leq -2$ and thus we start the construction with an element $e \in H^1(F(k)) \cong \text{Ext}(F, \mathcal{O}(k))$ where $k \geq -1$. To $e$ it corresponds an extension of the form:

$$0 \to \mathcal{O}(k) \to E \to F \to 0$$

and we want to apply the twistor transform to the preceding sequence. First of all, the condition $k \geq -1$ implies that the restriction of $E$ to all real lines is of the form $\mathcal{O}(k) \oplus \mathcal{O} \oplus \mathcal{O}$, because on $\mathbb{P}^1$ we have $\text{Ext}^1(\mathcal{O} \oplus \mathcal{O}, \mathcal{O}(k)) = 0$ if $k \geq -1$. So, applying $\mu^*$ and $\nu^*$ one obtains the following sequence on an open set $\mathcal{U}$ containing $S^4$ which is exact also on the right because on $\mathbb{P}^1$, $H^1(\mathcal{O}(k)) = 0$ if $k \geq -1$:

$$0 \to S_k \to \tilde{E} \to \tilde{F} \to 0.$$

Now, $S_k$ is the $k$-symmetric power of the positive spinor bundle over $G$ and it has therefore the canonical Levi-Civita connection denoted by $\nabla_k$. $\tilde{F}$ is the Atiyah-Ward transform of $F$ and it has the Yang-Mills connection $\tilde{\nabla}$. According to 2.1, for constructing a connection on $\tilde{E}$, we need also two sections

$$\varphi_1 \in \Gamma(\tilde{F}^* \otimes S_k) \text{ and } \varphi_2 \in \Gamma(\tilde{F} \otimes S_k^*).$$

But, using the canonical metric on $S_k$ and the fact that $\tilde{F} \cong \tilde{F}^*$, any section $\varphi_1$ will automatically determine a unique section $\varphi_2 = \varphi_1^*$. The preceddings remarks and the \textbf{Lemma 2.1} imply therefore the following result:

\textbf{Theorem 2.2.} Consider the following set of data: $F$ a rank 2 instanton bundle on $\mathbb{P}^3$, $e \in H^1(F(k))$ where $k \geq -1$, $\varphi \in \Gamma(\tilde{F}^* \otimes S_k)$. Then on $\tilde{E}$, the twistor transform of the extension of $F$ corresponding to $e$, there exist a canonical connection associated to the triple $(F, e, \varphi)$. 

\hfill $\blacksquare$
2.3 The case $k = -1$

The value $k = -1$ is special for at least two reasons:
First of all, in this case the $E$’s are stable, because $H^0(E) = H^0(E^*(-1)) = 0$
(cf. [OSS] Remark 1.2.6 Chap. II). This observation suggests that an interesting
problem would be the study of the geometry of these families of extensions of
instantons in the ambient moduly spaces.
Secondly, as it was proved by Hitchin in [H], $H^1(F(-1))$ can be identified with
the subspace of sections in $\Gamma(S_+ \otimes \tilde{F})$ which satisfy the Dirac equation coupled
with the Yang-Mills potential on $\tilde{F}$, where now, $\tilde{F}$ denotes the restriction of
the twistor transform of $F$ to $S^4$, and $S_+$ the positive spinor bundle over $S^4$.
But now one can apply directly to $\tilde{F} \oplus S_+$ the construction from Section 2.1,
without using $E$ directly, but using only its extension class $e \in H^1(F(-1))$ for
obtaining the section $\psi \in \Gamma(S_+ \otimes \tilde{F})$ by [H].
Conversely, let $\mathcal{A}$ be the space of connections on $\tilde{F} \oplus S_+$ such that in the block
decomposition
\[
\begin{pmatrix}
\tilde{\nabla} & \Theta_1 \\
\Theta_2 & \nabla_1
\end{pmatrix}
\]
$\Theta_1$ and $\Theta_2$ have the followings properties:
\[
\begin{align*}
(i) & \quad \Theta_1 = \nabla(\psi_1) \\
(ii) & \quad \Theta_2 = \nabla(\psi_2) \\
(iii) & \quad \psi_2^* = \psi_1^* \\
(iv) & \quad \text{Cliff}(\Theta_1) = 0
\end{align*}
\]
where Cliff denotes the Clifford multiplication.
For any connection in $\mathcal{A}$ one obtains a stable bundle on $\mathbb{P}^3$ and so we have the
following:

**Corollary 2.3.** There exists a canonical bijection between the class of stable
vector bundles on $\mathbb{P}^3$ which are extensions of the instanton bundle $F$ by $O(-1)$
and connections in the class $\mathcal{A}$.

2.4 Extensions by twisted holomorphic forms

The aim of this section is to find an analogue of the above Corollary when
one starts with an extension of the following form:
\[
0 \to \Omega^1 \to E \to F \to 0.
\]
Let $e \in H^1(\Omega^1 \otimes F)$ be the class of this extension. Using the standard sequence on $\mathbb{P}$,

$$0 \to \Omega^1 \to U\mathcal{O}(-1) \to \mathcal{O} \to 0 \quad (\ast)$$

(we recall that $U = H^0(\mathcal{O}(1))$, and tensoring it by $F$, one obtains the following part of the cohomology sequence:

$$H^0(F) \to H^1(\Omega^1 F) \to UH^1(F(-1)) \to H^1(F).$$

The first term on the left is 0 by the stability of the instanton bundle $F$. Denote by $m$ the last arrow on the right, which in fact is a component of the structural multiplication of the $\oplus H^0(\mathcal{O}(n))$-module $\oplus H^1(F(n))$. The cohomology class $e$ will determine a quadruple of elements $(e_i)$ so that $m((e_i)) = 0$ and conversely any such quadruple will determine a unique extension of $F$ by $\Omega^1$. Moreover, using $[H]$ as in the preceding section, any such quadruple corresponds to four Dirac spinor on $S^4$ coupled with the Yang-Mills potential $\tilde{F}$. Therefore we obtain a canonical connection on $\tilde{F} \oplus US_+$ in which the off diagonal terms $\Theta_1$ and $\Theta_2$ have the followings properties:

1. $\Theta_1 = \nabla(\psi_1)$
2. $\Theta_2 = \nabla(\psi_2)$
3. $\psi_2 = \psi_1^*$(i)
4. $\text{Cliff}(\Theta_1) = 0$
5. $m(\psi_1) = 0$

where Cliff denotes the Clifford multiplication, $\psi_1$ is a quadruple of Dirac spinors and we identify according to $[H]$ the space of Dirac spinors with the corresponding cohomology group on $\mathbb{P}$. Conversely, for any connection with these properties we obtain a unique vector bundle on $\mathbb{P}$ which is an extension of the instanton $F$ by $\Omega^1$ and so we have the following:

**Corollary 2.4.** There exists a canonical bijection between the classes of extensions on $\mathbb{P}^3$ of instantons by $\Omega^1$ and connections on $S^4$ which satisfy the five properties above.

We consider now extensions of the form:

$$0 \to \Omega^2(1) \to E \to F \to 0.$$
We shall use the following standard exact sequence:

\[ 0 \to \Omega^2 \to \wedge^2 U\mathcal{O}(-2) \to \Omega^1 \to 0. \]

By tensoring the last sequence with \( F(1) \) one obtains the following exact cohomology sequence:

\[ H^0(F\Omega^1(1)) \to H^1(\Omega^2 F(1)) \to \wedge^2 U H^1(F(-1)) \to H^1(\Omega^1 F(1)). \]

The first term is 0, as one sees multiplying the exact sequence (*) with \( F(1) \), taking the cohomology and using the stability of \( F \). The last arrow on the right is the multiplication \( m : H^0(\Omega^1(2)) \otimes H^1(F(-1)) \to H^1(\Omega^1 F(1)) \).

Therefore one obtains the following analogue of the preceding Corollary:

**Corollary 2.5.** There exist a canonical bijection between the class of extensions on \( \mathbb{P}^3 \) of instantons by \( \Omega^2(1) \) and connections of \( \tilde{F} \oplus \wedge^2 U \otimes S_+ \) over \( S^4 \) which satisfy the analogues of the five properties above.

\[ \square \]

**2.6. Remark.** Let now \( F \) be a rank two holomorphic bundle over \( \mathbb{P}^3 \) which is not an instanton, namely, it satisfies the following:

\( F \) is trivial on a line and \( H^1(F(k)) \neq 0 \) for some \( k \leq -2 \).

For such a \( F \) one can still consider extensions of the form:

\[ 0 \to \mathcal{O}(k) \to E \to F \to 0, \]

One restricts \( E \) to the set \( \mathcal{U}' = \mu(\nu^{-1}(\mathcal{U})) \), where \( \mathcal{U} \) is the open set in \( \mathbb{G} \) which parametrise the lines on which \( F \) is trivial, and one takes the cohomology class \( e \in H^1(\mathcal{U}, F(k)) \). This cohomology group can be identified by [WW] with the space of massless fields on \( \mathcal{U} \) of helicity \( \frac{-k-2}{2} \), coupled with the Yang-Mills potential obtained on \( \tilde{F} \) by the Penrose transform. As in the preceding sections it will be a canonical connection on \( S_{-k-2} \oplus \tilde{F} \).

**2.5 Towers of connections**

The aim of this section is to generalize the construction in 2.2 in order to obtain towers of connections on the grassmanian.
We start with the following set of data, for $k \geq -1$: $F$ a rank-2 instanton bundle on $\mathbb{P}^3$, $e_{k-i} \in H^1(F(i))$ for all $i$, $-1 \leq i \leq k$ and a set of sections $\varphi_{k-i} \in \Gamma(\tilde{F}^* \otimes S_i)$. We describe only the first steps of the construction and skip the inductive argument which is obvious.

For $i = k$ the construction is the same as in 2.2 with $e$ replaced by $e_0$, $\varphi$ by $\varphi_0$ and $E$ by $E_0$. We denote by $\nabla_0$ the corresponding connection on $\tilde{E}_0$.

Let be $i = k - 1$. Dualizing and twisting the exact sequence

$$0 \to O(k) \to E_0 \to F \to 0,$$

one gets the exact sequence:

$$0 \to F^*(k - 1) \to E_0^*(k - 1) \to O(-1) \to 0.$$

By taking the cohomology one obtains:

$$H^1(F^*(k - 1)) \cong H^1(E_0^*(k - 1))$$

and, as $F \cong F^*$, we find that $e_{k-1}$ determines an extension of the following type:

$$0 \to O(k - 1) \to E_1 \to E_0 \to 0.$$

On the grassmanian we will have therefore the following sequence:

$$0 \to S_{k-1} \to \tilde{E}_1 \to \tilde{E}_0 \to 0.$$

To construct a connection on $\tilde{E}_1$ one needs a section in $\Gamma(\tilde{E}_0^* \otimes S_{k-1})$. To obtain it, we use the element $\varphi_{k-1} \in \Gamma(\tilde{F}^* \otimes S_{k-1})$ and the canonical homomorphism

$$\Gamma(\tilde{F}^* \otimes S_{k-1}) \to \Gamma(\tilde{E}_0^* \otimes S_{k-1}).$$

We denote by $\tilde{\varphi}_{k-1}$ the image of $\varphi_{k-1}$ by the preceding homomorphism. Following the construction in Section 2 we obtain a canonical connection on $\tilde{E}_1$.

This construction can be performed inductively and gives:

**Corollary 2.7.** For $k \geq -1$, to the data: $F$ a rank-2 instanton bundle on $\mathbb{P}^3$, $e_{k-i} \in H^1(F(i))$ for $-1 \leq i \leq k$ and sections $\varphi_{k-i} \in \Gamma(\tilde{F}^* \otimes S_i)$ there corresponds a canonical tower of connections on the Penrose transformed of the successive extensions of $F$.

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**2.8. Remark.** Two interesting special cases arise:

1) the preceding construction can be modified to extend $F$ successively by $O$

2) extend $F$ successively by $O(k_i)$, with increasing $k_i$'s.
3 Monads for Extensions of Instanton Bundles on $\mathbb{P}^3$

3.1. Vector bundles can be constructed using the Horrock’s monads, i.e. complexes of vector bundles of the type

$$0 \to \mathcal{W}_1 \xrightarrow{A} \mathcal{V} \xrightarrow{B} \mathcal{W}_2 \to 0,$$

where $A$ and $B$ are injective, respectively surjective, homomorphisms of vector bundles. The object of homology $\text{Ker } B/\text{Im } A$ is a vector bundle. According to a theorem of Barth (cf. [BH], or [OSS]), a symplectic or orthogonal vector bundle $F$ on $\mathbb{P}^3$, of rank $r$ which satisfies the conditions:

- are trivial on some line $\ell$ in $\mathbb{P}^3$,
- $H^1(F(-2)) = 0$,

conditions fulfilled by the the instantons we considered so far, is given by a monad of the special type:

$$0 \to \mathcal{W}(-1) \xrightarrow{A} \mathcal{V} \xrightarrow{A^*} W^*\mathcal{O}(1) \to 0$$

where $W, V$ are vector spaces of dimensions $n$, respectively $2n + r$, $A$ is a $(2n + r) \times n$-matrix of linear forms of constant rank $n$ on $\mathbb{P}^3$, $W^*$ is the dual of $W$, $A^*$ is the dual of $A$.

To the monad one associates the so-called display, i.e. the commutative diagram with exact rows and columns:

$$
\begin{array}{ccc}
0 & \to & W\mathcal{O}(-1) \\
\downarrow & & \downarrow \\
0 & \to & W^*\mathcal{O}(1)
\end{array}
\quad
\begin{array}{ccc}
\mathcal{V} & \xrightarrow{A^*} & W^*\mathcal{O}(1) \\
\downarrow & & \downarrow \\
\mathcal{Q} & \xrightarrow{\mathcal{A}^*} & \mathcal{Q} \\
\downarrow & & \downarrow \\
\mathcal{F} & \to & 0
\end{array}
\quad
\begin{array}{ccc}
0 & \to & \mathcal{W}(1) \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
$$
It is interesting to interpret the data in the above monad in terms of the cohomology of $F$, (cf. [DM] or [A]):

\[
W^* \cong H^1(F(-1)) \\
V \cong H^1(F\Omega^1)
\]

and the linear map $A^*$ can be identified with the canonical map

\[
H^1(F\Omega^1) \to H^1(F(-1)) \otimes H^0(\mathcal{O}(1)),
\]

which comes from the cohomology applied to the Euler exact sequence

\[
0 \to \Omega^1 \to H^0(\mathcal{O}(1)) \otimes \mathcal{O}(-1) \to \mathcal{O} \to 0
\]

multiplied (tensorially) by $F$.

**Lemma 3.2.** Let $F$ be an instanton bundle on $\mathbb{P}^3$ and

\[
\begin{array}{c}
0 \\
\end{array} \xrightarrow{A} \begin{array}{c}
W \mathcal{O}(-1) \\
\end{array} \xrightarrow{A^*} \begin{array}{c}
V \mathcal{O} \\
\end{array} \xrightarrow{} \begin{array}{c}
W^* \mathcal{O}(1) \\
\end{array} \xrightarrow{} 0
\]

its monad. If $K$ is a vector bundle on $\mathbb{P}^3$ and $f \in \text{Hom}(W \mathcal{O}(-1), K)$ is mapped to $e \in H^1(F \otimes K) \cong \text{Ext}^1(F, K)$ by the canonical homomorphism

\[
\text{Hom}(W \mathcal{O}(-1), K) \cong H^1(F(-1)) \otimes H^0(K) \to H^1(F \otimes K),
\]

then the monad

\[
\begin{array}{c}
0 \\
\end{array} \xrightarrow{A} \begin{array}{c}
W \mathcal{O}(-1) \\
\end{array} \xrightarrow{f} \begin{array}{c}
V \mathcal{O} \oplus K \\
\end{array} \xrightarrow{(A^*, 0)} \begin{array}{c}
W^* \mathcal{O}(1) \\
\end{array} \xrightarrow{} 0 \quad (**)
\]

defines a vector bundle $E$ which is the extension of $F$ by $K$:

\[
0 \to K \to E \to F \to 0,
\]

corresponding to the element $e \in \text{Ext}^1(F, K)$.

**Proof.** Exercise in homological algebra. ■
Proposition 3.3. (i) All vector bundles which are extensions of an instanton by an \( \mathcal{O}(k) \) are given by a monad of the form:

\[
\begin{array}{c}
0 \rightarrow W\mathcal{O}(-1) \rightarrow V\mathcal{O} \oplus \mathcal{O}(k) \rightarrow W^*\mathcal{O}(1) \rightarrow 0
\end{array}
\]

where \( f \) is a row vector of \( \dim W \) forms of degree \( k + 1 \).

(ii) Any extension of an instanton \( F \) by \( \Omega^1(k) \) with \( k \geq 1 \) or by \( \Omega^2(k) \) with \( k \geq 2 \) is given by a monad of the shape (**) .

Proof. (i) The map

\[
\text{Hom}(W\mathcal{O}(-1), \mathcal{O}(k)) \cong \text{Hom}(\mathcal{O}(k + 1)) \otimes H^1(F(-1)) \rightarrow H^1(F(k))
\]

is surjective, the \( k[X_0, \ldots, X_3]-\text{module} \ \bigoplus_{i \geq -1} H^1(F(i)) \) being generated by \( H^1(F(-1)) \) (cf. [A] or [OSS]).

(ii) Multiplying the monad of \( F \) by \( \Omega = \Omega^1(k) \), respectively by \( \Omega = \Omega^2(k) \) and taking the cohomology, one obtains the following commutative and exact diagram:

\[
\begin{array}{c}
H^0(W^*\Omega(k + 1)) \rightarrow H^0(W^*\Omega(k + 1)) \\
\downarrow \quad \downarrow \\
H^1(Q^*\Omega(k)) \rightarrow H^1(F\Omega(k)) \rightarrow 0 \\
\downarrow \\
H^1(V\Omega(k)) = 0
\end{array}
\]

which shows that the map

\[
\text{Hom}(W\mathcal{O}(-1), \Omega(k)) = H^0(\Omega(k + 1)) \otimes H^1(F(-1)) \rightarrow H^1(F\Omega(k))
\]

is surjective.

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