A Polynomial Sieve and Sums of Deligne Type

Dante Bonolis

Abstract

Let \( f \in \mathbb{Z}[T] \) be any polynomial of degree \( d > 1 \) and \( F \in \mathbb{Z}[X_0, \ldots, X_n] \) an irreducible homogeneous polynomial of degree \( e > 1 \) such that the projective hypersurface \( V(F) \) is smooth. In this paper we give a bound for

\[
N(f, F, B) := | \{ x \in \mathbb{Z}^{n+1} : \max_{0 \leq i \leq n} |x_i| \leq B, \exists t \in \mathbb{Z} \text{ such that } f(t) = F(x) \} |.
\]

To do this, we introduce a generalization of the power sieve ([HB84], [Mun09]) and we extend two results by Deligne and Katz on estimates for additive and multiplicative characters in many variables.

A fundamental role in Analytic Number Theory is played by the combination of sieve methods and bounds for sums involving algebraic functions as the additive and multiplicative characters. In this paper we present a blend of this type, introducing a generalization of the following results

(i) The square sieve, or the more general power sieve of Heath-Brown and Munshi ([HB84], [Mun09]),

(ii) Deligne’s and Katz’s estimates for additive and multiplicative characters in many variables ([Del74], [Kat99] and [Kat02]),

and combining them to establish the following:

**Theorem 0.1.** Let \( f \in \mathbb{Z}[T] \) be any polynomial of degree \( d > 1 \) and \( F \in \mathbb{Z}[X_0, \ldots, X_n] \) an irreducible homogeneous polynomial of degree \( e > 1 \) such that the projective hypersurface \( V(F) \) is smooth. We denote

\[
N(f, F, B) := | \{ x \in \mathbb{Z}^{n+1} : \max_{0 \leq i \leq n} |x_i| \leq B, \exists t \in \mathbb{Z} \text{ such that } f(t) = F(x) \} |.
\]

Then we have

\[
N(f, F, B) \ll_{d,e,n,\|f\|,\|F\|} B^{n+\frac{1}{d+1}} \log B.
\]

Where \( \|F\| \) and \( \|f\| \) are the heights of \( F \) and \( f \) respectively.

**Remark 1.** Using the large sieve and a result by Cohen ([Coh81]), Serre has shown that ([Ser97, Theorem 2, Chapter 13]):

\[
N(f, F, B) \ll_{d,e,n,\|f\|,\|F\|} B^{n+\frac{1}{2}}.
\]

in particular, we improve Serre’s bound as soon as \( n > 2 \).

**Remark 2.** Theorem 0.1 generalizes [Mun09, Theorem 1.1], there \( f = T^d \) with \( d \geq 2 \).

As we already mentioned, Theorem 0.1 is a combination of the polynomial sieve (which we do not state here since it is a bit technical) with bounds of the following type:

*correspondence address: dante.bonolis@math.ethz.ch.*
Theorem 0.2. Let $p$ be a prime number and $F \in \mathbb{F}_p[X_0, ..., X_n]$ an irreducible homogeneous polynomial of degree $d \geq 1$ such that the projective hypersurface $V(F) \subset \mathbb{P}^{n-1}_F$ is smooth. Then

$$\sum_{x \in \mathbb{F}_p^n} \text{Kl}_m(F(x); p) \ll_{d,e,n} p^\frac{d}{2},$$

where $\text{Kl}_m(a; p)$ denotes the $m$-th Hyper-Kloosterman sum of parameter $(a, p)$.

Remark 3. Actually, we show a much more general version of Theorem 0.2: first of all, we prove this result for general trace functions, not only for $\text{Kl}_n$. Moreover $F$ will be a polynomial of Deligne type (see Definition 2.3) of which the homogeneous polynomials as in Theorem 0.2 are particular cases.

Theorem 0.3. Let $p$ be a prime number and $m \geq 2$, and $F \in \mathbb{F}_p[X_0, ..., X_n]$ an irreducible homogeneous polynomial of degree $d>1$ with $d \neq m$ such that the projective hypersurface $V(F) \subset \mathbb{P}^{n-1}_F$ is smooth. For any $u \in \mathbb{F}_p^n$ such that $V(\langle x, u \rangle)$ is not tangent to $V(F)$ (i.e. $V(F) \cap V(\langle x, u \rangle)$ is smooth of codimension 2 in $\mathbb{P}^{n-1}_F$) one has

$$\sum_{x \in \mathbb{F}_p^n} \text{Kl}_m(F(u); p)e\left(\frac{\langle x, u \rangle}{p}\right) \ll_{d,n} p^\frac{d}{2},$$

where $e(z) := e^{2\pi i z}$.

Remark 4. Also in this case, we prove Theorem 0.2 not only for $\text{Kl}_n$ but for a more general class of trace functions. Moreover we consider two irreducible polynomials $F,G \in \mathbb{F}_p[X_0, ..., X_n]$ of degree $d,e>1$ such that $V(F), V(G), V(F) \cap V(G) \subset \mathbb{P}^{n-1}_F$ are smooth. On the other hand, the assumption of $F$ and $G$ homogeneous will be crucial for our proof.

0.0.1 Organization of the paper

In the next section we introduce the Polynomial sieve. In section 2 we prove the bounds for sum of trace functions over polynomials of Deligne type. Finally in the last part of this paper we prove Theorem 0.1.

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1 The polynomial sieve

1.0.1 The power sieve

Let $A := (a(n))_n$ be a sequence of non-negative numbers such that

$$\sum_n a(n) < \infty,$$

and assume one knows the distribution of the sequence $A$ for certain moduli. Then, one can ask what is the contribution of the $d$-power in the above sum, i.e. what is the size of

$$\mathcal{V}_{d^a}(A) := \sum_n a(n^d).$$

This question was investigated by Heath-Brown ([HBS4]), in the case of squares, and by Munshi ([Mun99]) for general powers. Their arguments rely on two observations:

i) if a natural number $n$ is a $d$-th power then $n$ is a $d$-th power for any prime $p \mid n$,
ii) if \( p \equiv 1 \mod d \), one decomposes the characteristic function of \( d \)-th powers modulo \( p \) using multiplicative characters of order \( d \)

\[
1_{T^d,p} = \frac{1}{d} + \frac{1}{d} \sum_{\chi_d p = 1, \chi_p \neq 1} \chi_p.
\]

These two remarks lead to the following

**Lemma 1.1** ([HB84], [Mun09]). Let \( \mathcal{P} \) be a finite set of prime numbers \( p \equiv 1 \mod d \) and \( \mathcal{P} := |\mathcal{P}| \). Assume that \( (a(n))_n \) is a sequence of non-negative numbers such that \( a(n) = 0 \) if \( n = 0 \) or \( n \geq e^{\mathcal{P}} \). Then

\[
V_{T^d}(A) \ll_d P^{-1} \sum_n a(n) + P^{-2} \sum_{p \neq q \in \mathcal{P}} \sum_{\chi_p \neq 1, \chi_p = 1} \left| \sum_n a(n) \chi_p(n) \chi_q(n) \right|.
\]

1.0.2 A polynomial sieve

Our goal is to generalize the power sieve to any polynomial with coefficients over \( \mathbb{Z} \): let \( h \in \mathbb{Z}[T] \) with \( \deg h > 1 \), we denote

\[
\mathcal{P}_h := \{ p \text{ prime: } h(F_p) \neq F_p \}.
\]

We want to provide an upper bound for

\[
V_h(A) := \sum_{n \in h(\mathbb{Z})} a(n),
\]

where \( (a(n))_{n \in \mathbb{N}} \) is as before. Also in this case, one has that if \( n \in h(\mathbb{Z}) \) then \( n \in h(F_p) \) for all prime number \( p \). Then, as in the case of the \( d \)-powers, to generalize the power sieve we need a decomposition of the characteristic function \( 1_h(F_p) \) for any \( p \in \mathcal{P}_h \). This is done in [FKM14a Proposition 6.7], indeed Fouvry, Kowalski and Michel showed the following:

**Proposition 1.2.** Let \( p, \ell \) be two distinct primes and let \( h \in \mathbb{F}_p[T] \) a non trivial polynomial of degree \( \deg h < p \). There exists an integer \( k_p \geq 1 \) and a finite number of trace function \( t_{i,p} \) associated to middle-extension \( \ell \)-adic sheaves \( F_{i,p} \), \( 1 \leq i \leq k_p \) which are pointwise pure of weight 0 and algebraic numbers \( c_{i,p} \in \mathbb{Q} \) such that

\[
1_h(F_p)(x) = \sum_{i=1}^{k_p} c_{i,p} t_{i,p}(x)
\]

for all \( x \in F_p \setminus S_{h,p} \), where \( S_{h,p} := \{ h(x) : x \in F_p, h'(x) = 0 \} \cap F_p \) and with the following properties:

i) the constants \( k_p, |c_{i,p}| \) and the conductor of \( F_{i,p} \) (see Definition [2.1]), \( c(F_{i,p}) \), are bounded only in terms of \( \deg g \),

ii) the sheaf \( F_{i,p} \) is trivial and none of \( F_{i,p} \) for \( i \neq 1 \) is geometrically trivial, and moreover

\[
c_{1,p} = \frac{|h(F_p)|}{p} + O_{\deg h}(p^{-\frac{1}{2}})
\]

iii) all \( F_{i,p} \) are tame.

Using this we can prove
Lemma 1.3. Let $h$ be a polynomial in $\mathbb{Z}[T]$ with $\deg h = d \geq 2$. Let $\mathcal{P} \subset \mathcal{P}_h$ be a finite subset of $\mathcal{P}_h$ and denote $P := |\mathcal{P}|$. Then for any sequence $(a(n))_n$ of non-negative numbers such that $a(n) = 0$ if $|\{p \in \mathcal{P}: n \in S_{h,p} \mod p\}| \geq \frac{P}{2d}$ we have

$$V_h(A) \ll_d P^{-1} \sum_n a(n) + P^{-2} \sum_{p \# q \in \mathcal{P}} \sum_{i,j} \left| \sum_n a(n) t_{i,p}(n) T_{j,q}(n) \right|,$$

where the functions $t_{i,p}()$ and the sets $S_{h,p}$ are the ones in the decomposition of $1_{k(h,p)}$ in Proposition 1.2.

Proof. For any $p \in \mathcal{P}$ with $p > d$, an application of Proposition 1.2 leads to the decomposition

$$1_{k(h,p)} = \frac{|h(F_p)|}{p} + \sum_{i=2}^{k_p} c_{i,p} t_{i,p} + O_d(p^{-\frac{1}{2}})$$

over $F_p \setminus S_{h,p}$, where $S_{h,p} := \{ x \in F_p : h'(x) = 0 \}$ and with $k_p, c_{i,p} \ll_d 1$. Then we consider the weighted sum

$$\Sigma := \sum_n a(n) \left( \sum_{p \in \mathcal{P}} \sum_{i=2}^{k_p} c_{i,p} t_{i,p}(n) \right)^2. \tag{5}$$

Thanks to the fact that $h(F_p) \neq F_p$ (since $\mathcal{P} \subset \mathcal{P}_h$), one has that (Tur95 Proposition 2.11)

$$|h(F_p)| \leq p - \frac{p - 1}{d}.$$

Thus we have that for any $p \in \mathcal{P}$ and for any $z \in h(F_p) \setminus S_{h,p}$ the following inequality

$$\sum_{i=2}^{k_p} c_{i,p} t_{i,p}(z) = 1 - \frac{|h(F_p)|}{p} + O_d(p^{-\frac{1}{2}}) \geq \frac{1}{d} + O_d(p^{-\frac{1}{2}}),$$

holds. Hence if $n \in h(\mathbb{Z})$ one has

$$\sum_{p \in \mathcal{P}} \sum_{i=2}^{k_p} c_{i,p} t_{i,p}(n) \geq \frac{P}{d} + O_d(p^{\frac{1}{2}}) - \sum_{p \in \mathcal{P}, n \in S_{h,p}} 1 \geq \frac{P}{2d} + O_d(p^{\frac{1}{2}}),$$

where the last step uses the vanishing assumption on the sequence $(a(n))_n$. Hence $P^2 V_h(A) \ll_d \Sigma$ by positivity. Opening the square in $\Sigma$ we get the result.

Remark 5. Taking the polynomial $h = T^d$ we recover Lemma 1.3. See [Bro15], for another way to generalize the Power Sieve.

Remark 6. The vanishing condition on the sequence $(a(n))_n$ is necessary: let $h = T^d$ and $m$ be a number such that $p|m$ for any $p \in \mathcal{P}$, then $m = 0$ or $m \geq e^P$. Consider the sequence

$$a(n) = \begin{cases} 1 & \text{if } n = m^d \\ 0 & \text{if } n \neq m^d. \end{cases}$$

For this sequence $V_{T^d}(A) = 1$ while the right hand side in (4) is $O(P^{-1})$.

2 Sums of Trace functions over polynomials of Deligne type

Notation end statements of the general version of Theorem 1.2 and Theorem 1.3

In this section we recall some notion of the formalism of trace functions and state the general version of Theorem 1.2 and Theorem 1.3. For a general introduction on this subject we refer to [FKM15]. Basic statements and references can also be founded in [FKM15]. The main examples of trace functions we should have in mind are
i) For any \( f \in \mathbb{F}_p[T] \), the function \( x \mapsto c(f(x)/p) \): this is the trace function attached to the Artin-Schreier sheaf \( L_{c(f/p)} \).

ii) For any \( h \in \mathbb{F}_p[T] \), the trace functions \( t_{i,p} \) appearing in Proposition \( \text{(2.3)} \). Notice that for \( h = T^d \), the \( t_{i,p} \)s are just the multiplicative characters of order \( d \).

iii) The \( n \)-th Hyper-Kloosterman sums: the map

\[
x \mapsto \text{Klo}_n(x; q) := \frac{(-1)^{n-1}}{q^{(n-1)/2}} \sum_{y_1, \ldots, y_n \in \mathbb{F}_p^* \atop y_1 \cdots y_n = x} \psi(y_1 + \cdots + y_n).
\]

can be seen as the trace function attached to the Kloosterman sheaf \( \mathcal{K}_n \) (see \cite{Kat88} for the definition of such sheaf and for its basic properties).

**Definition 2.1.** Let \( \mathcal{F} \) be a constructible \( \ell \)-adic sheaf on \( \mathbb{F}_q^1 \) and \( j : U \to \mathbb{F}_q^1 \) the largest dense open subset of \( \mathbb{F}_q^1 \) where \( \mathcal{F} \) is lisse. The **conductor of** \( \mathcal{F} \) is defined as

\[
c(\mathcal{F}) := \text{Rank}(j_*j^*\mathcal{F}) + |\text{Sing}(\mathcal{F})| + \sum_x \text{Swan}_x(j_*j^*\mathcal{F}) + \dim H^0_j(\mathbb{F}_q^1, \mathcal{F}).
\]

**Remark 7.** We recall that if \( \mathcal{F} \) is middle-extension sheaf then

\[
c(\mathcal{F}) := \text{Rank}(\mathcal{F}) + |\text{Sing}(\mathcal{F})| + \sum_x \text{Swan}_x(\mathcal{F}),
\]

since in this case \( \mathcal{F} \cong j_*j^*\mathcal{F} \) and \( \dim H^0_j(\mathbb{F}_q^1, \mathcal{F}) = 0 \).

**Definition 2.2.** Let \( \mathcal{F} \) be a Fourier sheaf and \( \psi \) a non trivial character over \( \mathbb{F}_q \). Fix \( e \in \mathbb{N}_{\geq 1} \) and consider the morphism

\[
\mathbb{A}^1_{\mathbb{F}_q} \to \mathbb{A}^1_{\mathbb{F}_q}, \quad x \mapsto x^e
\]

Then we can define the \( \ell \)-adic sheaf \( T_\ell(\mathcal{F}) \) as

\[
T_\ell(\mathcal{F}) := FT_{\psi} \left([x \mapsto x^e], \mathcal{F}\right).
\]

Notice that

\[
t_{T_\ell(\mathcal{F})}(y) = -\frac{1}{q^{1/2}} \sum_{x \in \mathbb{F}_q} \psi(xy)t_{[x \mapsto x^e], \mathcal{F}}(x)
\]

\[
= -\frac{1}{q^{1/2}} \sum_{x \in \mathbb{F}_q} \psi(xy) \sum_{z^e = x} t_{\mathcal{F}}(z)
\]

\[
= -\frac{1}{q^{1/2}} \sum_{z \in \mathbb{F}_q} \psi(z^e) t_{\mathcal{F}}(z).
\]

Observe that combining \cite{Kat88} Paragraph 1.3 with \cite{FKM14} Paragraph 3.4 one obtains

\[
c(T_\ell(\mathcal{F})) \ll e c(\mathcal{F})^4.
\]

**Definition 2.3.** Let \( f \in \mathbb{F}_q[X_1, \ldots, X_n] \) be a polynomial in \( n \) variables of degree \( d \geq 1 \), say:

\[
f = f_d + \cdots + f_0,
\]

where \( f_i \) are homogeneous polynomials of degree \( i \). We say that \( f \) is of Deligne type if \( p \nmid d \) and the zero set, \( V(f_d) \), of \( f_d \) defines a smooth hypersurface in \( \mathbb{F}_q^{n-1} \).
Notation 2.1. We use the following conventions:

i) if $Y$ is a scheme over a field $k$ and $\mathcal{F}$ is a constructible $\ell$-adic sheaf on $Y$ we will denote

$$H^i_c(Y, \mathcal{F}) := H^i_c(Y \times \overline{k}, \mathcal{F}).$$

Moreover, we will denote by $t_{\mathcal{F}}$ the trace function attached to $\mathcal{F}$. ii) Any scheme is a scheme over $\mathbb{F}_q$ and any morphism is an $\mathbb{F}_q$-morphism.

iii) For any $g \in \mathbb{P}_q[X_1, \ldots, X_n]$ we denote by $G \subset \mathbb{P}_q[X_0, X_1, \ldots, X_n]$ its homogenization. Moreover, we denote the affine variety associated to $g$ by $V(g)$ and the projective one associated to $G$ by $V(G)$.

Now we can state our main results:

Theorem 2.1. Fix $d, e \geq 1$, $p$ a prime number such that $p \nmid d,e$ and $q = p^\alpha$ for some $\alpha \geq 1$. Let $\ell \neq p$ be a prime number and let $\mathcal{F}$ be a middle-extension $\ell$-adic sheaf on $\mathbb{P}_q^2$ of weight 0. We assume that the geometrically irreducible components of $\mathcal{F}$ are no geometrically trivial, i.e.

$$H^2_c(\overline{\mathbb{F}_q}, \mathcal{F}) = 0.$$

Let $t_{\mathcal{F}}$ be the trace function of $\mathcal{F}$. Let $U$ be the maximal dense open subset of $\overline{\mathbb{F}_q}$ where $\mathcal{F}$ is lisse. Let $f, g \in \mathbb{P}_q[X_1, \ldots, X_{n+1}]$ be polynomials of Deligne type in $n+1$ variables respectively of degree $d$ and $e$. Assume that $V(G)$ is smooth projective variety, $V(G) \cap V(F)$ is smooth of codimension 1 in $V(G)$, and that $V(G) \cap V(F) \cap V(X_0)$ is a smooth projective variety of codimension 2 in $V(G)$. Let us consider the morphism $f : V(g) \to \overline{\mathbb{F}_q}$. If any geometrically irreducible component of $\mathcal{F}$, $\mathcal{F}_i$, satisfies one of the following conditions:

i) there exists $s_i \in \text{Sing}(\mathcal{F}_i)$ such that $f^{-1}(s_i)$ is smooth;

ii) the sheaf $\mathcal{F}_i$ is wildly ramified at $\infty$,

then we have

$$\sum_{x \in \mathbb{F}_q^{n+1}} t_{\mathcal{F}}(f(x)) = q^{n-1} \sum_{x \in \mathbb{F}_q} t_{\mathcal{F}}(x) + O_{d,e,n,c(\mathcal{F})}(q^{2}).$$

(6)

Theorem 2.2. Fix $d, e \geq 1$, $p$ a prime number such that $p \nmid d,e$ and $q = p^\alpha$ for some $\alpha \geq 1$. Let $F, G \subset \mathbb{P}_q[X_1, \ldots, X_n]$ be homogeneous polynomials in $n$ variables of degree $d$ and $e$ respectively. Suppose $V(G), V(F) \subset \mathbb{P}_q^{n-1}$ are smooth and that $V(F) \cap V(G)$ is smooth and of codimension 2 in $\mathbb{P}_q^{n-1}$. Moreover assume that $T_c([x \mapsto x^d]^* \mathcal{F})$ is geometrically irreducible, not geometrically trivial and is either

i) ramified at 0,

ii) wild ramified at $\infty$ and some slope at $\infty$ of $\mathcal{F}_i$ is $\neq 1$.

Then one has

$$\left| \sum_{x \in \mathbb{F}_q^n} t_{\mathcal{F}}(F(x)) \psi(G(x)) \right| \ll_{d,e,n,c(\mathcal{F})} q^{2},$$

(7)

where $\psi$ is a non-trivial additive character of $\mathbb{F}_q$. 

6
Remarks and related works

i) Using [Del74, Theorem 8.4] one can prove that
\[ \sum_{x \in \mathbb{F}_q^n} t(f(x)) = q^{n-1} \sum_{a \in \mathbb{F}_q} t(a) + O_{d,n,c}(\mathbb{F}_q^{n+1}), \] (8)
and
\[ \left| \sum_{x \in \mathbb{F}_q^n} t_{\mathcal{F}}(F(x)) \psi(G(x)) \right| \ll d,e,n,c(\mathbb{F}_q^{n+1}). \] (9)

Hence, we improve the error terms in (8) and (9) of a factor \( q^{1/2} \) and \( q \) respectively.

ii) Theorem 2.1 was already established in the case of \( t_{\mathcal{F}} = \psi \) a non-trivial additive character ([Del74]) and in the case of \( t_{\mathcal{F}} = \chi \) a non-trivial multiplicative character ([Kat99]).

iii) A more general version of Theorem 2.2 was known already in the case \( t_{\mathcal{F}} = \chi \) a non-trivial multiplicative character ([Kat02]).

Proofs of Theorem 2.1 and Theorem 2.2

The Incidence variety

Let \( g(X_1, \ldots, X_n) \) be as in Theorem 2.1 and \( G(X_0, \ldots, X_n) \) its homogenization and consider \( V(G) \subset \mathbb{P}_q^{n+1} \). Let \( F(X_0, \ldots, X_n) \) be the homogenization of \( f(X_1, \ldots, X_n) \), we denote \( U_0 := V(G) \cap V(X_0) \). Let us consider
\[ f = F/X^d_0 : U_0 \longrightarrow \mathbb{P}_q^1, \]
and the constructible \( \mathbb{Q}_l \)-sheaf \( f^* \mathcal{F} \) on \( U_0 \). By the Lefschetz trace formula, we get
\[ \sum_{x \in U_0(\mathbb{F}_q)} t_{\mathcal{F}}(F(x)) = \sum_{x \in U_0(\mathbb{F}_q)} t_{f^* \mathcal{F}}(x) = \sum_{i} (-1)^i \text{Tr}(\text{Fr}_{\mathbb{F}_q}|H^i_c(U_0, f^* \mathcal{F})). \] (10)

To compute the right hand side of the equation (10) we are going to use the same strategy of [Del74, Theorem 8.4], [Kat99] and [Kat02]. One introduces the incidence variety
\[ \tilde{X} = \{ (x, \lambda) \in V(G) \times \mathbb{P}_q^1 : F(x) - \lambda x_0^d = 0 \} \]
which is a smooth variety for any polynomial of Deligne type \( f \) ([Kat80, pages 173–174]). The second projection of \( V(G) \times \mathbb{P}_q^1 \) induces a proper flat morphism
\[ \tilde{f} : \tilde{X} \longrightarrow \mathbb{P}_q^1 \]
with
\[ \tilde{f}^{-1}(\lambda) = V(G) \cap V(F - \lambda X_0^d), \]
for any \( \lambda \in \mathbb{P}_q^1 \). Moreover notice that \( U_0 \) can be viewed as an open subset of \( \tilde{X} \) thanks to the map
\[ x \mapsto (x, \frac{F(x)}{x_0^d}), \]
and by definition of \( \tilde{f} \) one has:
\[ \tilde{f}|_{U_0} = f. \]
The closed complement of \( U_0 \) is given by:
\[ Z := (V(F) \cap V(G) \cap V(X_0)) \times \mathbb{A}_q^1. \]
Theorem 2.5. One has that:

\[ H^i_c(\overline{\mathcal{X}}, R^f \tilde{f}_* \mathcal{F}) = \begin{cases} H^0_c(\overline{\mathcal{X}}, \mathcal{P}), & \text{if } i = 0, \\ 0, & \text{if } i = 1, \\ H^2_c(\overline{\mathcal{X}}, j_* j^* R^f \tilde{f}_* \mathcal{F}_{\ell}) & \text{if } i = 2. \end{cases} \]

Hence in order to compute the right hand side of the above equality we need to understand the size of the cohomology groups \( H^i_c(\overline{\mathcal{X}}, \mathcal{F} \otimes R^f \tilde{f}_* \mathcal{F}_{\ell}) \) and the action of the Frobenius automorphism on these groups.

**Properties of** \( R^f \tilde{f}_* \mathcal{F}_{\ell} \)

We can state the following proposition concerning the property of the higher direct image sheaves \( R^f \tilde{f}_* \mathcal{F}_{\ell} \) on \( \overline{\mathcal{X}} \) (recall that \( \tilde{f} \) is proper).

**Proposition 2.3.** Consider the morphism

\[ \tilde{f} : X \longrightarrow \overline{\mathcal{X}} \]

then the following properties hold:

i) if \( \lambda \in \mathcal{K}_{\mathcal{F}_q} \) is such that \( \tilde{f}^{-1}(\lambda) \) is smooth then \( \tilde{f} \) is proper smooth of relative dimension \( n - 1 \) in a Zariski open neighborhood of \( \lambda \). Morover, the sheaf \( R^f \tilde{f}_* \mathcal{F}_{\ell} \) is lisse in a Zariski open neighborhood of \( \lambda \).

ii) If \( \lambda \in \mathcal{K}_{\mathcal{F}_q} \) is such that \( \tilde{f}^{-1}(\lambda) \) is singular, then \( \tilde{f}^{-1}(\lambda) \) has only isolated singularities.

iii) for any \( i \geq n + 1 \) the sheaf \( R^i \tilde{f}_* \mathcal{F}_{\ell} \) is lisse on \( \overline{\mathcal{X}} \) and pure of weight \( i \),

iv) the sheaves \( R^i \tilde{f}_* \mathcal{F}_{\ell} \) are tamely ramified at \( \infty \) for all \( i \).

v) for \( i = n \), denote \( j : U \longrightarrow \overline{\mathcal{X}} \) the inclusion of an open dense subset \( U \) of \( \overline{\mathcal{X}} \) where \( R^n \tilde{f}_* \mathcal{F}_{\ell} \) is lisse, then one has the exact sequence:

\[ 0 \longrightarrow \mathcal{P} \longrightarrow R^n \tilde{f}_* \mathcal{F}_{\ell} \longrightarrow j_* j^* R^n \tilde{f}_* \mathcal{F}_{\ell} \longrightarrow 0, \tag{11} \]

where \( j_* j^* R^n \tilde{f}_* \mathcal{F}_{\ell} \) is geometrically constant and \( \mathcal{P} \) punctual.

**Proof.** See [Kat02] Proposition 7.1 and [SGA4] Exposé XV, Theorem 2.1 and Exposé XVI, Theorem 2.1 for these arguments.

**Corollary 2.4.** For any \( i \geq n + 1 \), the sheaf \( R^i \tilde{f}_* \mathcal{F}_{\ell} \) is geometrically constant on \( \overline{\mathcal{X}} \).

**Proof.** Thank to points (ii) and (iii) of the previous proposition we have that for any \( i \geq n + 1 \) the sheaves \( R^i \tilde{f}_* \mathcal{F}_{\ell} \) are lisse on \( \overline{\mathcal{X}} \) and tamely ramified at \( \infty \), so they are geometrically constant.

We conclude this section by calculating the cohomology of the sheaf \( R^n \tilde{f}_* \mathcal{F}_{\ell} \):

**Lemma 2.5.** One has that:

\[ H^i_c(\overline{\mathcal{X}}, R^n \tilde{f}_* \mathcal{F}_{\ell}) = \begin{cases} H^0_c(\overline{\mathcal{X}}, \mathcal{P}), & \text{if } i = 0, \\ 0, & \text{if } i = 1, \\ H^2_c(\overline{\mathcal{X}}, j_* j^* R^n \tilde{f}_* \mathcal{F}_{\ell}) & \text{if } i = 2. \end{cases} \]
Proof. The short exact sequence in \((\text{II})\) leads to the long exact sequence for cohomology groups

\[
0 \rightarrow H^0_c(\mathcal{K}_{\mathcal{F}_q}, \mathcal{P}) \rightarrow H^0_c(\mathcal{K}_{\mathcal{F}_q}, R^i j_* \mathcal{F}_\mathcal{E}) \rightarrow H^0_c(\mathcal{K}_{\mathcal{F}_q}, j_* j^* R^i \mathcal{F}_\mathcal{E}) \rightarrow \]
\[
\rightarrow H^1_c(\mathcal{K}_{\mathcal{F}_q}, \mathcal{P}) \rightarrow H^1_c(\mathcal{K}_{\mathcal{F}_q}, R^i j_* \mathcal{F}_\mathcal{E}) \rightarrow H^1_c(\mathcal{K}_{\mathcal{F}_q}, j_* j^* R^i \mathcal{F}_\mathcal{E}) \rightarrow \]
\[
\rightarrow H^2_c(\mathcal{K}_{\mathcal{F}_q}, \mathcal{P}) \rightarrow H^2_c(\mathcal{K}_{\mathcal{F}_q}, R^i j_* \mathcal{F}_\mathcal{E}) \rightarrow H^2_c(\mathcal{K}_{\mathcal{F}_q}, j_* j^* R^i \mathcal{F}_\mathcal{E}) \rightarrow 0.
\]

Then the result follows thanks to the fact that \(H^1_c(\mathcal{K}_{\mathcal{F}_q}, \mathcal{P}) = 0\) for \(i \geq 1\) (\(\mathcal{P}\) is punctual) and that \(H^1_c(\mathcal{K}_{\mathcal{F}_q}, j_* j^* R^i \mathcal{F}_\mathcal{E}) = 0\) for any \(i \neq 2\) (\(j_* j^* R^i \mathcal{F}_\mathcal{E}\) is geometrically constant). \(\square\)

Bounds for the conductor of the cohomology groups

Lemma 2.6. We have

\[H^1_c(\mathcal{K}_{\mathcal{F}_q}, \mathcal{F} \otimes R^i j_* \mathcal{F}_\mathcal{E}) = 0,\]

if \(j = 2\) or \(i \geq 2n - 1\).

Proof. We have two cases

\(i\) for \(j = 2\) and any \(i\) we can argue as follow: we can consider an open Zariski dense set \(\mathcal{U}_i\) where \(\mathcal{F}\) and \(R^i j_* \mathcal{F}_\mathcal{E}\) are both lisse and pointwise pure. Observe that \(H^2_c(\mathcal{U}_i, \mathcal{F} \otimes R^i j_* \mathcal{F}_\mathcal{E})\) does not vanish if and only if there exists a geometrically irreducible component in the Jordan-Holder decomposition of \(\mathcal{F}\) and \(\mathcal{F}_\mathcal{E}\). We can argue as follow: we can consider an open Zariski dense set \(\mathcal{U}_i\) where \(\mathcal{F}\) and \(R^i j_* \mathcal{F}_\mathcal{E}\) are both lisse and pointwise pure. Observe that \(H^2_c(\mathcal{U}_i, \mathcal{F} \otimes R^i j_* \mathcal{F}_\mathcal{E})\) does not vanish if and only if there exists a geometrically irreducible component in the Jordan-Holder decomposition of \(\mathcal{F}\) and \(\mathcal{F}_\mathcal{E}\). Now this is not the case: indeed if \(\mathcal{F}_i\) is wildly ramified at \(\infty\) then a contradiction arises since \(\mathcal{G}\) is tame at \(\infty\) because the sheaves \(R^i j_! \mathcal{F}_\mathcal{E}\) is tame at \(\infty\) (Proposition 2.3 (iii)). If there exists a \(s_i \in \text{Sing}(\mathcal{F}_i)\) such that \(\tilde{f}^{-1}(s)\) is smooth, then by part (ii) of Proposition 2.3 \(R^i j_! \mathcal{F}_\mathcal{E}\) is lisse at \(s_i\) and thus \(\mathcal{G}\) is lisse at \(s_i\). Then \(\mathcal{F}_i\) and \(\mathcal{G}\) are not geometrically isomorphic because \(s_i \in \text{Sing}(\mathcal{F}_i)\) and \(s_i \notin \text{Sing}(\mathcal{G})\).

\(ii\) if \(i \geq 2n - 1\) then a consequence of the proper base change Theorem tells us that for any \(\lambda \in \mathcal{K}_{\mathcal{F}_q}\)

\[(R^i j_! \mathcal{F}_\mathcal{E})_{\lambda} \cong H^1_c(j^{-1}(\lambda), \mathcal{F}_\mathcal{E}).\]

On the other hand, the right hand side of the equation above vanishes since \(\tilde{f}\) has dimension \(n - 1\). \(\square\)

Lemma 2.7. For any \(i\) one has that

\[c(\mathcal{F} \otimes R^i j_* \mathcal{F}_\mathcal{E}) \ll_{d,e,n,c(\mathcal{F})} 1.\]  

(12)

Proof. Thanks to the inequality \([\text{PKM15}, \text{Proposition 8.2, part (3)}]\) it is enough to prove that

\[c(R^i j_* \mathcal{F}_\mathcal{E}) \ll_{d,e,n} 1\]

First of all we will bound the size of \(\text{Sing}(R^i j_* \mathcal{F}_\mathcal{E})\). Thanks to the Proposition 2.3 we know that if the morphism \(\tilde{f}\) is smooth at a closed point \(\lambda \in \mathcal{K}_{\mathcal{F}_q}\) then the sheaf \(R^i j_* \mathcal{F}_\mathcal{E}\) is lisse at that point. So we have that

\[|\text{Sing}(R^i j_* \mathcal{F}_\mathcal{E})| \leq |\{\lambda \in \mathcal{K}_{\mathcal{F}_q} : \tilde{f} \text{ is not smooth in } \lambda\}|.\]
Recall that \( \tilde{f}^{-1}(\lambda) = V(G) \cap V(F - \lambda X_0^d) \). Since by hypothesis \( V(G) \cap V(F - \lambda X_0^d) \cap V(X_0) \) is smooth, it is enough to bound the number of \( \lambda \)'s such that the affine variety \( V(g) \cap V(f - \lambda) \) is singular. Then \( V(g) \cap V(f - \lambda) \) is singular if the Jacobian matrix

\[
\begin{pmatrix}
\frac{\partial \lambda}{\partial X_1} & \frac{\partial g}{\partial X_1} \\
\vdots & \vdots \\
\frac{\partial \lambda}{\partial X_{n+1}} & \frac{\partial g}{\partial X_{n+1}}
\end{pmatrix}
\]

has rank \( \leq 1 \) for some \( \mathbf{v} := [v_1 : \ldots : v_n] \in \mathbb{A}^{n+1}_F \). Let us consider the polynomials

\[
H_{\lambda,0} := \det \left( \frac{\partial f}{\partial X_1} - \lambda \frac{\partial g}{\partial X_1} \right), \quad H_{\lambda,n+1} := \det \left( \frac{\partial f}{\partial X_{n+1}} - \lambda \frac{\partial g}{\partial X_{n+1}} \right),
\]

and for any \( i = 1, \ldots, n \)

\[
H_{\lambda,i} := \det \left( \frac{\partial f}{\partial X_1} \ldots \frac{\partial f}{\partial X_{i-1}} \frac{\partial g}{\partial X_{i+1}} \ldots \frac{\partial g}{\partial X_{n+1}} \right).
\]

If \( V(g) \cap V(f - \lambda) \) is singular then there exists \( \mathbf{v} \in \mathbb{P}^{n+1}_F \) such that

\[
H_{\lambda,i}(\mathbf{v}) = 0, \quad \text{for } i = 0, \ldots, n + 1.
\]

Notice that \( H_{\lambda,i} \) are polynomial of degree \( (e-1)(d-1) \) for \( i = 1, \ldots, n \) and of degree \( \max(d(e-1), (d-1)e) \) for \( i = 0, n + 1 \) and that \( H_{\lambda,i} \) does not depend on \( \lambda \) for \( i = 1, \ldots, n \). Recall that the resultant of \( k \) polynomials in \( k + 1 \) variables \( f_1, \ldots, f_k \) of degree respectively of degree \( d_1, \ldots, d_k \) is an irreducible polynomial in the coefficients of \( f_1, \ldots, f_k \) which vanishes if \( f_1, \ldots, f_k \) have a common root. Using this we can conclude that if \( V(g) \cap V(f - \lambda) \) is singular then

\[
\text{Res}(H_{\lambda,0}, \ldots H_{\lambda,n+1}) = 0.
\]

On the other hand for any \( i = 1, \ldots, n \) the coefficients of \( H_{\lambda,i} \) are independent of \( \lambda \) and the ones of \( H_{\lambda,0}, H_{\lambda,n+1} \) can be viewed as linear polynomial in \( \lambda \), then

\[
r(\lambda) := \text{Res}(H_{\lambda,0}, \ldots H_{\lambda,n+1})
\]

is a polynomial. Moreover \( r(\lambda) \) is not the zero polynomial because \( r(0) \neq 0 \) by hypothesis \( (V(G) \cap V(F)) \) is smooth. Then

\[
|\{ \lambda \in \mathbb{A}_F^{n+1} : \tilde{f} \text{ is not smooth in } \lambda \}| \leq \deg(r(\lambda)) \ll_{d,e,n} 1,
\]

thanks to [GKZ08 Chapter 13, Proposition 1.1]. Let us now calculate Rank\((R^i \tilde{f}_* \overline{\mathcal{M}}_\ell)\). To do this it’s enough to calculate the rank of the geometric fibers \((R^i \tilde{f}_* \overline{\mathcal{M}}_\ell)_\lambda\) where \( \lambda \) is a lisse point of \( R^i \tilde{f}_* \overline{\mathcal{M}}_\ell \). We have already observed that

\[
(R^i \tilde{f}_* \overline{\mathcal{M}}_\ell)_\lambda \cong H^i_c(\tilde{f}^{-1}(\lambda), \overline{\mathcal{M}}_\ell), \quad \text{(13)}
\]

so in order to find the Rank\((R^i \tilde{f}_* \overline{\mathcal{M}}_\ell)_\lambda\) it is enough to calculate the dimension of the cohomology groups in the right-hand side of (13). If \( \tilde{f} \) is smooth at \( \lambda \), using [De74 Theorem 8.1] we get

\[
\text{Rank}(R^i \tilde{f}_* \overline{\mathcal{M}}_\ell)_\lambda = \begin{cases} 0 & \text{if } 0 \leq i < 2n - 2, \ 2 \nmid i \text{ and } i \neq n - 1, \\ 1 & \text{if } 0 \leq i < 2n - 2, \ 2 \mid i \text{ and } i \neq n - 1, \\ b_{n-1}(\tilde{f}^{-1}(\lambda)) - \frac{1 + (-1)^{n-1}}{2} & \text{if } i = n - 1, \end{cases} \quad \text{(14)}
\]
where \( b_{n-1} (\tilde{f}^{-1}(\lambda)) \) is the \((n-1)\)-th Betti number which can be bounded in terms of \( d, e, n \) only. Then \( \text{Rank}(R^i \tilde{f}_* \mathbb{Q}_\ell) \ll_{d,e,n} 1 \). Now we turn our attention to the size of \( \dim H^0_c(\overline{k}_{\mathbb{F}_q}, R^i \tilde{f}_* \mathbb{Q}_\ell) \).

For any \( \ell \)-adic sheaf \( \mathcal{G} \) one has that (see [Kat80, 4.4,4.5])

\[
\dim H^0_c(\overline{k}_{\mathbb{F}_q}, \mathcal{G}) \leq \sum_{s \in S} \text{Rank}(\mathcal{G}_s),
\]

where \( S \) is the set of point where \( \mathcal{G} \) is not lisse. Let \( \lambda \in \text{Sing}(R^i \tilde{f}_* \mathbb{Q}_\ell) \). Then, since \( (R^i \tilde{f}_* \mathbb{Q}_\ell)_\lambda \cong H^0_c(\tilde{f}^{-1}(\lambda), \mathbb{Q}_\ell) \), it is enough to show that that \( \dim H^0_c(\tilde{f}^{-1}(\lambda), \mathbb{Q}_\ell) \ll_{d,e,n} 1 \) in the case where \( f^{-1}(\lambda) \) is a variety with at most isolated singularities, this is done for example in [Kat91, Appendix, Theorem 1]. Hence

\[
\dim H^0_c(\overline{k}_{\mathbb{F}_q}, R^i \tilde{f}_* \mathbb{Q}_\ell) \leq | \text{Sing}(R^i \tilde{f}_* \mathbb{Q}_\ell) | \cdot \max_{\lambda \in \text{Sing}(R^i \tilde{f}_* \mathbb{Q}_\ell)} (\text{Rank}(R^i \tilde{f}_* \mathbb{Q}_\ell)_\lambda) \ll_{d,e,n} 1.
\]

To conclude the proof we need to bound the Swan conductors at singular points. We denote by

\[
U := \overline{k}_{\mathbb{F}_q} \setminus \{ \lambda \in \overline{k}_{\mathbb{F}_q} : \tilde{f} \text{ is not smooth in } \lambda \},
\]

and by \( j : U \hookrightarrow \overline{k}_{\mathbb{F}_q} \) the associated open immersion. Recall that by part (i) of Proposition 2.3 the sheaves \( R^i \tilde{f}_* \mathbb{Q}_\ell \) are lisse on \( U \) for any \( i = 1, \ldots, 2n \). We first compute \( \dim H^i_c(\overline{k}_{\mathbb{F}_q}, R^i \tilde{f}_* \mathbb{Q}_\ell) \).

These cohomology groups are the starting objects for the Leray Spectral sequence arising from the map

\[
\tilde{f} : \tilde{X} \longrightarrow \overline{k}_{\mathbb{F}_q}
\]

and the \( \ell \)-adic sheaf \( \mathbb{Q}_\ell \) on \( \overline{k}_{\mathbb{F}_q} \), i.e. \( H^i_c(\overline{k}_{\mathbb{F}_q}, R^i \tilde{f}_* \mathbb{Q}_\ell) = E^{i,0}_2 \Rightarrow E^{i+j} = H^{i+j}(\tilde{X}, \mathbb{Q}_\ell) \). On the other hand using the decomposition of \( \tilde{X} = U_0 \sqcup ((V(F) \cap V(G) \cap V(X_0)) \times \overline{k}_{\mathbb{F}_q}) \), one proves

**Lemma 2.8.** We have

\[
\dim H^k_c(\tilde{X}, \mathbb{Q}_\ell) = \begin{cases} 
1, & \text{if } 2 \leq k \leq 2n, 2|k \text{ and } k \neq n, \\
b_n(U_0) + b_{n-2}((V(F) \cap V(G) \cap V(X_0)) & \text{if } k = n, \\
0 & \text{otherwise}.
\end{cases}
\]

Let us assume the Lemma above. Observing that \( E^{1,0}_2 = H^1_c(\overline{k}_{\mathbb{F}_q}, R^i \tilde{f}_* \mathbb{Q}_\ell) \), one has

\[
\dim H^1_c(\overline{k}_{\mathbb{F}_q}, R^i \tilde{f}_* \mathbb{Q}_\ell) \leq \dim H^{i+1}_c(\tilde{X}, \mathbb{Q}_\ell) \ll_{d,e,n} 1, \tag{15}
\]

since \( b_n(U_0), b_{n-2}((V(F) \cap V(G) \cap V(X_0)) \ll_{d,e,n} 1 \). At this point, one can apply the Grothendieck–Ogg–Shafarevich Formula getting

\[
\sum_{i=0}^2 (-1)^i \dim (H^i_c(U, j_* j^* R^i \tilde{f}_* \mathbb{Q}_\ell)) = \text{Rank}(j_* j^* R^i \tilde{f}_* \mathbb{Q}_\ell) \cdot |2 - (\overline{k}_{\mathbb{F}_q} \setminus U)(\mathbb{F}_q)|
\]

\[
- \sum_x \text{Swan}_x(j_* j^* R^i \tilde{f}_* \mathbb{Q}_\ell).
\]

On the other hand we have \( H^0_c(U, j_* j^* R^i \tilde{f}_* \mathbb{Q}_\ell) = 0 \) and \( \dim(H^2_c(U, j_* j^* R^i \tilde{f}_* \mathbb{Q}_\ell)) \leq \text{Rank}(R^i \tilde{f}_* \mathbb{Q}_\ell) \).

Moreover, using the fact that \( H^1_c(U, j_* j^* R^i \tilde{f}_* \mathbb{Q}_\ell) \cong H^1_c(\overline{k}_{\mathbb{F}_q}, R^i \tilde{f}_* \mathbb{Q}_\ell) \), we obtain

\[
\dim H^1_c(U, j_* j^* R^i \tilde{f}_* \mathbb{Q}_\ell) \ll_{d,e,n} 1,
\]

thanks to (15). Hence

\[
\sum_x \text{Swan}_x(R^i \tilde{f}_* \mathbb{Q}_\ell) \ll_{e,d,n} 1.
\]

\[\square\]
To complete the proof of Lemma 2.7 we need to prove Lemma 2.8.

proof of Lemma 2.8 We know that $\tilde{X} = U_0 \sqcup Z$ with $Z = (V(F) \cap V(G) \cap V(X_0)) \times K_{\ell}^1$. Let us denote by $j : U_0 \hookrightarrow \tilde{X}$ the open embedding of $U_0$ in $\tilde{X}$ and by $i : Z \hookrightarrow \tilde{X}$ the closed embedding of $Z$ in $\tilde{X}$. Then we have the exact sequence

$$0 \to j^*\mathbb{Q}_\ell \to \mathbb{Q}_\ell \to i_*i^*\mathbb{Q}_\ell \to 0,$$

where $\mathbb{Q}_\ell$ denotes the trivial sheaf. This short exact sequence leads to the long exact sequence

$$\cdots \to H^i_c(\tilde{X}, \mathbb{Q}_\ell) \to H^i_c(Z, \mathbb{Q}_\ell) \to H^{i+1}_c(U_0, \mathbb{Q}_\ell) \to \cdots,$$

combining this with [Del74, Theorem 8.1], the Lemma follows.

Frobenius action

Lemma 2.9. For any even $i \geq n + 1$ one has that:

$$t_{R^n j^*}{\mathfrak{F}}(\lambda) = q^\frac{i}{2},$$

for any $\lambda \in \mathbb{R}_{\ell}^1(\mathbb{F}_q)$. Moreover

$$t_{j^*R^n}{\mathfrak{F}}(\lambda) = q^\frac{i}{2}(1 + (-1)^n),$$

for any $\lambda \in \mathbb{R}_{\ell}^1(\mathbb{F}_q)$.

Proof. By Proposition 2.3 we know that for $i \geq n + 1$ even the sheaf $R^n j^* F$ is geometrically irreducible and geometrically constant i.e.:

$$R^n j^* F \cong \chi_i \otimes \mathbb{Q}_\ell,$$

where $\chi_i : \text{Gal}(\mathbb{F}_q/\mathbb{F}_q) \to \mathbb{Q}_\ell^\times$ is a character, hence

$$t_{R^n j^*}{\mathfrak{F}}(\lambda) = \alpha_i,$$

where $\alpha_i = \chi_i(\text{Fr}_{\text{geom}})$ and $\alpha_i$ is a $q$-Weil number of weight $\leq i$ ([Del80, Theorem 1]). Applying the Grothendieck-Lefschetz trace formula ([SGA4.5, Exposé VI]) one has that

$$\alpha_i q = \sum_{x \in \mathbb{F}_q} t_{\chi_i \otimes \mathbb{Q}_\ell}(x) = \text{Tr}(\text{Fr} | H^i_c(\mathbb{F}_q, \chi_i \otimes \mathbb{Q}_\ell)).$$

For the sheaf $j_* j^* R^n j^* F$ we have to distinguish two cases:

i) $n$ odd. In this case one has $\text{Rank}(R^n j_* j^* F) = 0$, then $\text{Rank}(j_* j^* R^n j^* F) = 0$ and this implies $t_{j_* j^* R^n j^*}{\mathfrak{F}}(\lambda) = 0$ for any $\lambda \in \mathbb{R}_{\ell}^1$.

ii) $n$ even. By Proposition 2.3 the sheaf $j_* j^* R^n j^* F$ is geometrically constant of rank 1, then

$$j_* j^* R^n j^* F \cong \chi_n \otimes \mathbb{Q}_\ell,$$

for some character $\chi_n : \text{Gal}(\mathbb{F}_q/\mathbb{F}_q) \to \mathbb{Q}_\ell^\times$, hence

$$t_{j_* j^* R^n j^*}{\mathfrak{F}}(\lambda) = \alpha_n,$$

with $\alpha_n = \chi_n(\text{Fr}_{\text{geom}})$, and $\alpha_n$ is a $q$-Weil number of weight $\leq n$ ([Del80, Theorem 1]).
We can state both cases by writing
\[ t_{j,j^* R^n F,\tilde{\mathcal{L}}}(\lambda) = \frac{\alpha_n (1 + (-1)^n)}{2}. \]

By the Grothendieck-Lefschetz trace formula we get
\[ \frac{q \alpha_n (1 + (-1)^n)}{2} = \sum_{x \in F_q} t_{\chi_n \otimes \tilde{\mathcal{L}}}(x) = \text{Tr}(\text{Fr} | H^2_c(\mathbb{A}_{F_q}, \chi_n \otimes \tilde{\mathcal{L}})). \]

This shows that in order to compute \( \alpha_i \), it is enough compute \( \text{Tr}(\text{Fr} | H^2_c(\mathbb{A}_{F_q}, R^n F, \tilde{\mathcal{L}})) \) and \( \text{Tr}(\text{Fr} | H^2_c(\mathbb{A}_{F_q}, j_* j^* R^n F, \tilde{\mathcal{L}})). \) Using \([\text{Kat}02, \text{Lemma 11}]\), we can write
\[ |\tilde{X}(F_q)| = \sum_{i \text{ even}, i \geq n+1} \text{Tr}(\text{Fr} | H^2_c(\mathbb{A}_{F_q}, R^n F, \tilde{\mathcal{L}})) \]
\[ + (-1)^n \text{Tr}(\text{Fr} | H^2_c(\mathbb{A}_{F_q}, \mathcal{P})) + \text{Tr}(\text{Fr} | H^2_c(\mathbb{A}_{F_q}, j_* j^* R^n F, \tilde{\mathcal{L}})) \]
\[ + \sum_{i < n} (-1)^i \sum_{j=0}^{2} (-1)^j \text{Tr}(\text{Fr} | H^2_c(\mathbb{A}_{F_q}, R^n F, \tilde{\mathcal{L}})). \]

Applying Lemma \([2.7]\) and \([\text{Del}80, \text{Theorem 1, Theorem 2}]\) one gets
\[ |\tilde{X}(F_q)| = \frac{q \alpha_n (1 + (-1)^n)}{2} + \sum_{i=n+1, i \text{ even}}^{2n-2} q \alpha_i + O_{d,n,c(F)}(q^{\frac{n+1}{2}}), \tag{16} \]

where for any \( i \geq n \) in the above sum, \( |q \alpha_i| = q^{\frac{i}{2} + 1} \). On the other hand one easily compute \( |\tilde{X}(F_q)| \) just using the decomposition \( \tilde{X} = U_0 \sqcup Z \), obtaining
\[ |\tilde{X}(F_q)| = \frac{q^{\frac{n+1}{2}} (1 + (-1)^n)}{2} + \sum_{b=0}^{b=2n} q^{\frac{b}{2}} + O_{d,n,c(F)}(q^{\frac{n+1}{2}}). \tag{17} \]

Comparing the right hand side of \((16)\) with the one of \((17)\) (replacing \( F_q \) by a suitable extension \( \mathbb{F}_q \), if necessary) we obtain that \( q \alpha_i = q^{\frac{i}{2} + 1} \), thus
\[ \alpha_i = q^{\frac{i}{2}} \]
for any \( i \geq n \) as we want. \( \square \)

**Corollary 2.10.** For any \( i \geq n + 1 \), one has
\[ \sum_{j=0}^{1} \text{Tr}(\text{Fr} | H^2_c(\mathbb{A}_{F_q}, \mathcal{F} \otimes R^j F, \tilde{\mathcal{L}})) = q^{\frac{i}{2}} \sum_{x \in F_q} t_{\mathcal{F}}(x), \quad \text{if } i \text{ is even} \]
\[ 0 \quad \text{if } i \text{ is odd}. \]

Moreover
\[ \sum_{j=0}^{1} \text{Tr}(\text{Fr} | H^2_c(\mathbb{A}_{F_q}, \mathcal{F} \otimes j_* j^* R^j F, \tilde{\mathcal{L}})) = \frac{q^{\frac{n+1}{2}} (1 + (-1)^n)}{2} \sum_{x \in F_q} t_{\mathcal{F}}(x). \]
Proof. For $i$ odd $H^j_i(\mathfrak{X}_{\mathbb{F}_q}, \mathcal{F} \otimes R^i \tilde{f}_* \mathbb{Q}_j) = 0$ for $j = 0, 1$ and there is nothing to prove. Otherwise $\mathcal{F} := \mathcal{F} \otimes R^i \tilde{f}_* \mathbb{Q}_j \simeq_{\text{geom}} \mathcal{F}$ and this implies that:

$$t_{\mathcal{F}}(x) = \alpha_i t_{\mathcal{F}}(x)$$

for any $x \in \mathbb{F}_q$, thanks to the previous lemma we know that in this case $\alpha_i = q^{2i}$ and the result follows. The same argument can be used to prove the second part of the statement. 

The action of the Frobenius on the cohomology groups $H^j_i(\mathbb{Z}, \tilde{f}^* \mathcal{F}|_Z)$ instead can be calculated by observing that $Z = (V(F) \cap V(G) \cap V(X_0)) \times \mathbb{A}^k_{\mathbb{F}_q}$ and using the Künneth formula (see [Mil80 VI.8])

$$H^j_i(\mathbb{Z}, \tilde{f}^* \mathcal{F}|_Z) = \bigoplus_{b=0}^{1} H^j_i(V(F) \cap V(G) \cap V(X_0), \mathbb{Q}_\ell) \otimes H^b_i(\mathbb{A}^k_{\mathbb{F}_q}, \mathcal{F}).$$

Combining this with the functoriality of the Frobenius one gets

$$\text{Tr}(\text{Fr} | H^j_i(\mathbb{Z}, \tilde{f}^* \mathcal{F}|_Z)) = \sum_{b=0}^{1} \text{Tr}(\text{Fr} | H^j_i(V(F) \cap V(G) \cap V(X_0), \mathbb{Q}_\ell)) \text{Tr}(\text{Fr} | H^b_i(\mathbb{A}^k_{\mathbb{F}_q}, \mathcal{F}))$$

and this leads to:

**Lemma 2.11.** With the same notation as above one has

$$\sum_{i=0}^{2(n-1)} (-1)^i \text{Tr}(\text{Fr} | H^j_i(\mathbb{Z}, \tilde{f}^* \mathcal{F}|_Z)) = \left( \sum_{x \in \mathbb{F}_q} t_{\mathcal{F}}(x) \right) \left( \sum_{i=n, i \text{ even}} q^{2i} \right) + O_{d,n}(q^{2n}).$$

**Proof.** Combine the above discussion with [Del74 Theorem 8.1].

**End of the proof of Theorem 2.1**

We are finally ready to prove Theorem 2.1. First of all observe that

$$\sum_{x \in X(\mathbb{F}_q)} t_{\tilde{f}^* \mathcal{F}}(x) = \sum_{x \in U_{\alpha}^{\prime}(\mathbb{F}_q)} t_{\tilde{f}^* \mathcal{F}}(x) + \sum_{x \in Z(\mathbb{F}_q)} t_{\tilde{f}^* \mathcal{F}}(x).$$

Arguing as [Kat99 Lemma 11], one has

$$\sum_{x \in X(\mathbb{F}_q)} t_{\tilde{f}^* \mathcal{F}}(x) = \sum_{i,j} (-1)^{i+j} \text{Tr}(\text{Fr} | H^j_i(\mathbb{Z}, \tilde{f}^* \mathcal{F}|_Z))$$

thanks to Lemma 2.10 and Corollary 2.11 one gets that

$$\sum_{x \in \tilde{X}(\mathbb{F}_q)} t_{\tilde{f}^* \mathcal{F}}(x) = \left( \sum_{x \in \mathbb{F}_q} t_{\mathcal{F}}(x) \right) \left( \sum_{i=n, i \text{ even}} q^{2i} \right) + O_{d,n,c(\mathcal{F})}(q^{2n}).$$  \hspace{1cm} (18)

On the other hand combining the Lefschetz trace formula for $\tilde{f}^* \mathcal{F}|_Z$ on $Z$ with Lemma 2.11 we get

$$\sum_{x \in Z(\mathbb{F}_q)} t_{\tilde{f}^* \mathcal{F}|_Z}(x) = \left( \sum_{x \in \mathbb{F}_q} t_{\mathcal{F}}(x) \right) \left( \sum_{i=n, i \text{ even}} q^{2i} \right) + O_{d,n,c(\mathcal{F})}(q^{2n}),$$  \hspace{1cm} (19)

subtracting (19) to (18) we get:

$$\sum_{x \in U_{\alpha}^{\prime}(\mathbb{F}_q)} t_{\tilde{f}^* \mathcal{F}}(x) = q^{n-1} \sum_{x \in \mathbb{F}_q} t_{\mathcal{F}}(x) + O_{d,n,c(\mathcal{F})}(q^{2n}),$$

as we want.
Proof of Theorem 2.2

We start writing
\[
\sum_{x \in \mathbb{F}_q^n} t_F(x) \psi(G(x)) = \sum_{(a,b) \in \mathbb{F}_q^2} t_F(a) \psi(b) N(a,b,F,G),
\]
where for any \(a,b \in \mathbb{F}_q\)
\[
N(a,b,F,G) := |\{x \in \mathbb{F}_q^n : F(x) = a \text{ and } G(x) = b\}|.
\]

At this point it is useful to

i) recall that for any \(a,b \in \mathbb{F}_q\) one has that
\[
N(a,b,F,G) = q^{n-2} + O_d,c,a(q^{\frac{1}{2}})
\]
if \(V(F - a) \cap V(G - b)\) is a smooth variety, and
\[
N(a,b,F,G) = q^{n-2} + O_d,c,a(q^{\frac{1}{2}})
\]
if \(V(F - a) \cap V(G - b)\) is singular,

ii) observe that if \(a \neq 0\)
\[
\sum_{b \in \mathbb{F}_q} N(a,b,F,G) = N(a,F) = q^{n-1} + O_d,c,a(q^{\frac{1}{2}})
\]
where \(N(a,F) := |\{x \in \mathbb{F}_q^n : F(x) = a\}|\).

Since \(F,G\) are homogeneous, for \(a,b \in \mathbb{F}_q\) and \(\eta \in \mathbb{F}_q^d\) we have
\[
N(a,b,F,G) = N(\eta^d a, \eta^d b, F, G),
\]
this can be proven by using the transformation \((x_i) \mapsto (\eta x_i)\). On the other hand the morphism
\[
\varphi : \mathbb{F}_q^\times \times \mathbb{F}_q^\times \times \mathbb{F}_q^\times \times \mathbb{F}_q^\times
\]
\[
\rightarrow \mathbb{F}_q^\times \times \mathbb{F}_q^\times
\]
\[
(\alpha, \eta, b) \mapsto (\eta^d \alpha, \eta^d b)
\]
is a surjection onto \(\mathbb{F}_q^\times \times \mathbb{F}_q^\times\) with \(\ker(\varphi) = (d, q - 1)\). Thus we can rewrite (20) as
\[
\sum_{x \in \mathbb{F}_q^n} t_F(x) \psi(G(x)) = \sum_{(a,b) \in \mathbb{F}_q^2} t_F(a) \psi(b) N(a,b,F,G)
\]
\[
= \sum_{a \in \mathbb{F}_q} t_F(a) N(a,0,F,G) + \sum_{b \in \mathbb{F}_q} t_F(0) \psi(b) N(0,b,F,G)
\]
\[
+ \frac{1}{(d,q-1)} \sum_{\alpha \in \mathbb{F}_q^d/\mathbb{F}_q^{d-1}} \sum_{b \in \mathbb{F}_q} N(\alpha,b,F,G) \sum_{\eta \in \mathbb{F}_q^d} t_F(\alpha \eta^d) \psi(b \eta^e)
\]
\[
- t_F(0) N(0,0,F,G).
\]

Now we remove the condition \(\eta \in \mathbb{F}_q^d\) in the last sum. To do so, we observe that
\[
\frac{1}{(d,q-1)} \sum_{\alpha \in \mathbb{F}_q^d/\mathbb{F}_q^{d-1}} \sum_{b \in \mathbb{F}_q^d} N(\alpha,b,F,G) t_F(0) = \frac{1}{(d,q-1)} \sum_{\alpha \in \mathbb{F}_q^d/\mathbb{F}_q^{d-1}} \sum_{b \in \mathbb{F}_q^d} N(\alpha,b,F,G) t_F(0)
\]
\[
= \frac{1}{(d,q-1)} \sum_{\alpha \in \mathbb{F}_q^d/\mathbb{F}_q^{d-1}} N(\alpha,0,F,G) t_F(0)
\]
\[
= \frac{1}{(d,q-1)} \sum_{\alpha \in \mathbb{F}_q^d/\mathbb{F}_q^{d-1}} N(\alpha,F) t_F(0)
\]
\[
= \frac{1}{(d,q-1)} \sum_{\alpha \in \mathbb{F}_q^d/\mathbb{F}_q^{d-1}} N(0,0,F,G) t_F(0).
\]
where in the last step we used (21). On the other hand we have

\[ N(\alpha, F) = q^{n-1} + O_{d,e,n}(q^{\frac{1}{d}}) \]

\[ N(\alpha, 0, F, G) = q^{n-2} + O_{d,e,n}(q^{\frac{1}{d}}) \]

for any \( \alpha \in \mathbb{F}_q^{\times} / \mathbb{F}_q^{\times d} \). Thus

\[
\frac{1}{(d, q - 1)} \sum_{\alpha \in \mathbb{F}_q^{\times} / \mathbb{F}_q^{\times d}} \sum_{b \in \mathbb{F}_q^*} N(\alpha, b, F, G) t_F(0) = q^{n-2}(q - 1) t_F(0) + O_{d,e,n,\epsilon(F)}(q^{\frac{1}{d}}). 
\]

Moreover \( N(0, 0, F, G) = q^{n-2} + O_{d,e,n}(q^{\frac{1}{d}}) \) because by hypothesis the affine variety \( \{ x \in \mathbb{F}_q^n : F(x) = 0 \text{ and } G(x) = 0 \} \) is singular only at the origin. So we can rewrite (22) as

\[
\sum_{x \in \mathbb{F}_q^n} t_F(F(x)) \psi(G(x)) = \sum_{(a, b) \in \mathbb{F}_q^n} t_F(a) \psi(b) N(a, b, F, G)
\]

\[
= \sum_{a \in \mathbb{F}_q^n} t_F(a) N(a, 0, F, G) + \sum_{b \in \mathbb{F}_q^n} t_F(0) \psi(b) N(0, b, F, G)
\]

\[
+ \frac{1}{(d, q - 1)} \sum_{\alpha \in \mathbb{F}_q^{\times} / \mathbb{F}_q^{\times d}} \sum_{b \in \mathbb{F}_q^*} N(\alpha, b, F, G) \sum_{\eta \in \mathbb{F}_q^*} t_F(\alpha \eta^d) \psi(b \eta^e) + E(q).
\]

where

\[ E(q) = -q^{n-1} t_F(0) + O_{d,e,n,\epsilon(F)}(q^{\frac{1}{d}}). \]

Let us discuss first

\[
M := \frac{1}{(d, q - 1)} \sum_{(a, b) \in \mathbb{F}_q^n / \mathbb{F}_q^{\times d}} \sum_{b \in \mathbb{F}_q^*} N(\alpha, b, F, G) \sum_{\eta \in \mathbb{F}_q^*} t_F(\alpha \eta^d) \psi(b \eta^e)
\]

To simplify the notation we will denote \( G_{\alpha,e} := [x(-1)^r T_e([x] \alpha)] [x \mapsto x^d] F \). Observe that

\[
t_G_{\alpha,e}(b) = \frac{1}{\sqrt{q}} \sum_{\eta \in \mathbb{F}_q^*} t_F(\alpha \eta^d) \psi(b \eta^e)
\]

for any \( b \in \mathbb{F}_q^* \). Then \( M \) become

\[
M = \frac{\sqrt{q}}{(d, q - 1)} \sum_{(a, b) \in \mathbb{F}_q^n / \mathbb{F}_q^{\times d}} \sum_{b \in \mathbb{F}_q^*} N(\alpha, b, F, G) t_G_{\alpha,e}(b)
\]

\[
= \frac{\sqrt{q}}{(d, q - 1)} \sum_{(a, b) \in \mathbb{F}_q^n / \mathbb{F}_q^{\times d}} \sum_{\eta \in \mathbb{F}_q^*} t_G_{\alpha,e}(G(x))
\]

\[
= \frac{\sqrt{q}}{(d, q - 1)} \sum_{(a, b) \in \mathbb{F}_q^n / \mathbb{F}_q^{\times d}} \sum_{\eta \in \mathbb{F}_q^*} t_G_{\alpha,e}(G(x))
\]

\[
= \frac{\sqrt{q}}{(d, q - 1)} \sum_{(a, b) \in \mathbb{F}_q^n / \mathbb{F}_q^{\times d}} N(\alpha, 0, F, G) t_G_{\alpha,e}(0).
\]

On the other hand one has that \( G_{\alpha,e} \) is irreducible and not trivial since \( G_{1,\epsilon} = [x(-1)^r T_e([x] \epsilon)] [x \mapsto x^d] F \) is so (see Lemma 24, Chapter 7). Moreover \( F - \alpha \) and \( G \) are polynomials of Deligne type and

i) \( V(F - \alpha X_0^d) \) is a smooth projective variety for \( \alpha \neq 0 \).

ii) \( V(F - \alpha X_0^d) \cap V(G) \cap V(X_0) \) is smooth of codimension 2 in \( V(F - \alpha X_0^d) \) by hypothesis.

Combining this with Har77, Proposition 7.2, Chapter 1 one obtains that \( V(F - \alpha X_0^d) \cap V(G) \) is of codimension 1 in \( V(F - \alpha X_0^d) \).
$$iii) \quad V(F - \alpha X^d) \cap V(G) \text{ is smooth. Indeed looking at the Jacobian matrix}$$

$$\begin{pmatrix}
  \frac{\partial F}{\partial x_1} & \frac{\partial G}{\partial x_1} \\
  \vdots & \vdots \\
  \frac{\partial F}{\partial x_{n+1}} & \frac{\partial G}{\partial x_{n+1}}
\end{pmatrix}$$

we conclude that $P$ is a singular point if $P = [1 : 0 : \ldots : 0]$ or $P \in V(X_0)$. Now $[1 : 0 : \ldots : 0] \notin V(F - \alpha X^d)$ because $\alpha \neq 0$. Also the other case is impossible because $V(F - \alpha X^d) \cap V(G) \cap V(X_0)$ is smooth by $(ii)$.

$$iv) \quad \text{We distinguish two cases: if } T_\epsilon([x \mapsto x^d]*F) \text{ is singular at 0 then some slope at } \infty \text{ of } [x \mapsto x^d]*F \text{ is } < 1 \text{ ([Kat88 Paragraph 1.13 and Proposition 8.5.8.])}. \text{ On the other hand the slope at } \infty \text{ of } [x \mapsto x^d]*F \text{ are the same of } [x \mapsto x^d]*F. \text{ Hence } G_{\alpha,e} \text{ is singular at } 0. \text{ If } T_\epsilon([x \mapsto x^d]*F) \text{ is wild ramified at } \infty \text{ with some } T_\epsilon([x \mapsto x^d]*F)\text{-break } \neq 1, \text{ we argue as before using [Kat12 Proposition 7.5.4].}$$

Hence the sheaves $G_{\alpha,e}$ are geometrically irreducible, not geometrically trivial, and they are either ramified in 0 or wild ramified at $\infty$. Moreover $V(F - \alpha X^d)$ is smooth for any $\alpha \neq 0$. Then we can apply Theorem [2.1] getting

$$\sum_{x : F(x) = \alpha} t_{G_{\alpha,e}}(G(x)) = q^n - 2 \sum_{b \in F_q^*} t_{G_{\alpha,e}}(b) + O_{d,e,n,c(F)}(q^{n - 2})$$

$$= q^n - 2 \sum_{b \in F_q^*} t_{G_{\alpha,e}}(b) + q^{n - 2} t_{G_{\alpha,e}}(0) + O_{d,e,n,c(F)}(q^{n - 2}).$$

Hence we get

$$M = \frac{\sqrt{n} \cdot q^{n - 2}}{d, 0} \sum_{b \in F_q^*} t_{G_{\alpha,e}}(b) + O_{d,e,n,c(F)}(q^{\frac{n}{2}})$$

$$= \frac{q^{n - 2}}{d, 0} \sum_{b \in F_q^*} t_{F(\alpha^d)\psi(bq)} + O_{d,e,n,c(F)}(q^{\frac{n}{2}})$$

$$= \frac{q^{n - 2}}{d, 0} \sum_{b \in F_q^*} t_{F(0)} + O_{d,e,n,c(F)}(q^{\frac{n}{2}})$$

For the first term of [22] we can argue as follows

$$\sum_{a \in F_q} t_{F(a)|N(a,0,F,G)} = \sum_{x : G(x) = 0} t_{F(x)}$$

$$= q^{n - 2} \sum_{a \in F_q} t_{F(a)} + O_{d,e,n,c(F)}(q^{n - 2}).$$
again using Theorem 2.11 and similarly for the second term \(\sum_{b \in F_q} t_F(0) \psi(b) N(0,b)\). So (22) becomes

\[
\sum_{x \in F_q^*} t_F(F(x)) \psi(G(x)) = q^{n-2} \sum_{a \in F_q} t_F(a) + q^{n-2} \sum_{b \in F_q} t_F(0) \psi(b) \\
+ q^{n-2} \sum_{a \in F_q} \sum_{b \in F_q} t_F(a) \psi(b) \\
+ (q-1)q^{n-2} t_F(0) + E(q) + O_{d,e,n,c(F)}(q^{2})
\]

Using the fact that the term \(q^{n-2} t_F(0)\) is counted twice in the left hand side and recalling the definition of \(E(q) = -q^{n-1} t_F(0) + O_{d,e,n,c(F)}(q^{2})\) we get

\[
\sum_{x \in F_q^*} t_F(F(x)) \psi(G(x)) = q^{n-2} \sum_{(a,b) \in F_q^2} t_F(a) \psi(b) \\
= O_{d,e,n,c(F)}(q^{2})
\]

since \(\sum_{b \in F_q} \psi(b) = 0\).

2.0.1 Some examples.

In the following we will denote \(\mathfrak{F}\) the family of geometrically irreducible middle-extension \(\ell\)-adic sheaf on \(\mathbb{K}_q\) pure of weight 0.

**Lemma 2.12.** Let \(F \in \mathfrak{F}\) then:

i) If \(e = 1\) and \(F \in \mathfrak{F}\) is a Fourier shear, then \(T_1(F) = FT\psi(F) \in \mathfrak{F}\).

ii) If \(e > 1\) and \([x\eta]^* F \neq _{geom} F\) for every non trivial \(\ell\)-root of unity \(\eta\), then \(T_e(F) \in \mathfrak{F}\).

iii) if \(T_e(F) \in \mathfrak{F}\), for any \(\alpha \neq 0\), \(T_e([\alpha]^* F) \in \mathfrak{F}\).

**Proof of Lemma 2.12** One argues as follows:

i) Is just an application of [Kat88, Theorem 8.4.1].

ii) First observe that we may assume that \(F\) is not geometrically trivial otherwise the result is straightforward. Let us start writing

\[
\frac{1}{q} \sum_{x \in F_q} |t_F(x)|^2 = \frac{1}{q^2} \sum_{x \in F_q} \left| \sum_{y \in F_q} t_F(y) \psi(xy) \right|^2 \\
= \frac{1}{q^2} \sum_{x \in F_q} \sum_{(y_1, y_2) \in F_q^2} t_F(y_1) t_F(y_2) \psi(xy_1^e - y_2^e) \\
= \frac{1}{q^2} \sum_{(y_1, y_2) \in F_q^2} t_F(y_1) t_F(y_2) \sum_{x \in F_q} \psi(xy_1^e - y_2^e)
\]

If \(e > 1\) and \([x\eta]^* F \neq _{geom} F\) for every non trivial \(\ell\)-root of unity \(\eta\), we obtain

\[
\frac{1}{q} \sum_{x \in F_q} |t_F(x)|^2 = \frac{1}{q} \sum_{\eta^e = 1} \sum_{y_1 \in F_q} t_F(y_1) \psi(\eta y_1) = 1 + O(q^{1/2}).
\]

Applying [Kat96, Lemma 7.0.3] we get the result.
iii) Observing that
\[
\begin{align*}
t_{T_{c}((x^{d})^{*}, F)}(x) &= -\frac{1}{\sqrt{q}} \sum_{z \in F} \psi(z^{d} x) t_{F}(\alpha z) \\
&= -\frac{1}{\sqrt{q}} \sum_{w \in F} \psi((\pi w)^{d} x) t_{F}(w) \\
&= t_{T_{c}(F)}(\pi x)
\end{align*}
\]
for any \( x \in F_{q} \), we get
\[
\frac{1}{q} \sum_{x \in F_{q}} \left| t_{T_{c}((x^{d})^{*}, F)}(x) \right|^{2} = \frac{1}{q} \sum_{x \in F_{q}} \left| t_{T_{c}(F)}(x) \right|^{2} = 1 + O(q^{1/2}),
\]
by the hypothesis on \( T_{c}(F) \). Thus applying [Kat96, Lemma 7.0.3] the result follows. 

Then we can prove

**Corollary 2.13.** Let \( h \in F[T] \) be a polynomial and \( t \) a trace function appearing in the decomposition of \( 1_{h}(F_{p}) \) in Proposition 2.3 Then
\[
\sum_{x \in \overline{F}_{p}} t(F(x); p) e\left( \frac{\langle x, u \rangle}{p} \right) \ll_{d,n} p^{\frac{d}{2}},
\]
for any \( F \in \overline{F}_{p}[X_{0}, ..., X_{n}] \) irreducible homogeneous polynomial of degree \( d \geq 1 \) such that \( V(F) \subset \overline{F}_{p}^{n-1} \) is smooth and for any \( u \in \overline{F}_{p}^{n} \) such that \( V((x, u)) \) is not tangent to \( V(F) \).

**Proof.** Let us denote \( F \) the \( \ell \)-adic sheaf attached to \( t \). If \( F =_{\text{geom}} K_{\chi(T)} \) a Kummer sheaf attached to a character \( \chi \) of order dividing \( d \) then the result is a special case of [Kat07, Theorem 1]. So we may assume \( F \neq_{\text{geom}} K_{\chi(T)} \) for any Kummer sheaf attached to a character \( \chi \) of order dividing \( d \). Let us start proving that \( [x \mapsto x^{d}]^{*} F \) is not geometrically trivial. We start writing
\[
\sum_{x \in \overline{F}_{p}} t_{[x \mapsto x^{d}]^{*} F}(x) = \sum_{x \in \overline{F}_{p}} t(x^{d})
\]
\[
= \sum_{x \in \overline{F}_{p}} t(z) \sum_{\chi|z^{d}=1} \chi(z)
\]
\[
\ll_{s_{c}(F),d} \sqrt{p},
\]
because \( F \neq_{\text{geom}} K_{\chi} \) for any character \( \chi \) of order dividing \( d \). Thus \( [x \mapsto x^{d}]^{*} F \) is not geometrically trivial. Moreover \( [x \mapsto x^{d}]^{*} F \) is tame because \( F \) is tame ([Kat88, Paragraph 13]). Hence, \( [x \mapsto x^{d}]^{*} F \) is a Fourier sheaf and any irreducible component of \( T_{1}(F) = FT(F) \) is singular at 0 ([Kat88, Theorem 8.4.1 and Proposition 8.5.8]). So we can apply Theorem 2.2.

**Corollary 2.14.** Let \( p \) be a prime number and \( m \geq 2 \), and \( F \in \overline{F}_{p}[X_{0}, ..., X_{n}] \) an irreducible homogeneous polynomial of degree \( d \geq 1 \) with \( d \neq m \) such that the projective hypersurface \( V(F) \subset \overline{F}_{p}^{n-1} \) is smooth. For any \( u \in \overline{F}_{p}^{n} \) such that \( V((x, u)) \) is not tangent to \( V(F) \) (i.e. with \( V(F) \cap V((x, u)) \) is smooth of codimension 2 in \( \overline{F}_{p}^{n-1} \) ) one has
\[
\sum_{x \in \overline{F}_{p}} Kl_{m}(F(u); p) e\left( \frac{\langle x, u \rangle}{p} \right) \ll_{d,n} p^{\frac{d}{2}},
\]
Proof. Let start proving that \([x \mapsto x^d]^* \mathcal{K}_m\) is irreducible. Thanks to [Kat96, Lemma 7.0.3] it is enough to show that
\[
\frac{1}{q} \sum_{x \in \mathbb{F}_q} |K_m(x^d)|^2 = 1 + O(p^{-1/2})
\]
Using the same argument as in Corollary 2.13 one gets
\[
\frac{1}{q} \sum_{x \in \mathbb{F}_q} |K_m(x^d)|^2 = \sum_{\chi: \chi^d = 1} \sum_{x \in \mathbb{F}_q} K_m(z)^2 \chi(z)
= 1 + \sum_{\chi \neq 1: \chi^d = 1} \sum_{x \in \mathbb{F}_q} K_m(z)^2 \chi(z) + O(p^{-1/2}).
\]
On the other hand, one has that
\[
\frac{1}{q} \sum_{x \in \mathbb{F}_q} K_m(z)^2 \chi(z) = 1 + O(p^{-1/2})
\]
if and only if \(\mathcal{K}_m \otimes \mathcal{K}_m = \text{geom} \mathcal{K}_{\chi(T)}\) but this is not the case since \(\mathcal{K}_m \otimes \mathcal{K}_m\) is wild ramified at \(\infty\) (Kat88 Proposition 10.4.1) while \(\mathcal{K}_{\chi(T)}\) is tame everywhere. Thus \([x \mapsto x^d]^* \mathcal{K}_m\) is irreducible. Moreover \(\mathcal{K}_m\) is a Fourier sheaf, thus \(T_1([x \mapsto x^d]^* \mathcal{K}_m)\) is irreducible (Lemma 2.12 part (i)). Now we have to distinguish two cases:

i) \(d < m\), in this case \([x \mapsto x^d]^* \mathcal{K}_m(\infty)\) has only one break at \(\frac{d}{m} \leq 1\), thus \(T_\epsilon([x \mapsto x^d]^* \mathcal{K}_m)\) is ramified at \(0\).

ii) \(d > m\), in this case \([x \mapsto x^d]^* \mathcal{K}_m(\infty)\) has only one break at \(\frac{d}{m} > 1\), thus \(T_\epsilon([x \mapsto x^d]^* \mathcal{K}_m)\) is wild at \(\infty\).

In both cases we can apply Theorem 2.2 and we get the result. \(\square\)

Proof of Theorem 0.1

The strategy of the proof of Theorem 0.1 is similar to the one presented by Munshi in [Mun09] with some modification. Let \(W : \mathbb{R}^{n+1} \rightarrow \mathbb{R}\) be a non-negative smooth function with support in the box \([-B, B]^{n+1}\) satisfying

\[
\frac{\partial^{i_0+\ldots+i_n} W(X_0, \ldots, X_n)}{\partial X_{i_0} \ldots \partial X_{i_n}} \ll B^{-(i_0+\ldots+i_n)},
\]

these property leads to the following bound for the Fourier transform

\[
\hat{W}(\mathbf{u}) \ll B^{n+1} \prod_{i=0}^n (1 + |u_i|B)^{-2},
\]

where \(\mathbf{u} = (u_0, \ldots, u_n)\). Let \(\mathcal{P} \subset \mathcal{P}_f\) be a finite subset of \(\mathcal{P}_f\) to be chosen later and define

\[
a(k) := \begin{cases} 0 & \text{if } k \in S, \\ \sum_{x \in \mathbb{Z}^{n+1}} W(x) & \text{otherwise,}
\end{cases}
\]

where \(S := \{k : |\{p \in \mathcal{P} : k \in S_{p, p} \mod p}\} | \geq \frac{p}{2^j}\}\). By definition \((a(k))_k\) satisfies the hypothesis of Lemma 1.3 then the polynomial sieve gives

\[
\nu_f(A) \ll_d P^{-1} \sum_k a(k) + P^{-2} \sum_{p \neq q \in \mathcal{P}} \sum_{i,j} \sum_k a(k) t_{i,p}(k) t_{j,q}(k),
\]

20
We estimate the first term in the right hand side trivially as $\sum a(k) \ll B^{n+1}$. Thus to prove Theorems \[\text{[1,1]}\] we have to bound

$$S(p, q, i, j) := \sum_k a(k)t_{i, p}(k)\tilde{t}_{j, q}(k)$$

$$= \sum_{x \in \mathbb{Z}^{n+1}} W(x)t_{i, p}(F(x))\tilde{t}_{j, q}(F(x)) - E(p, q, i, j)$$

(24)

for any $p \neq q$, where

$$E(p, q, i, j) := \sum_{k \in S} W(x)t_{i, p}(k)\tilde{t}_{j, q}(k).$$

Using the Poisson summation formula we can rewrite (24) as

$$S(p, q, i, j) = (pq)^{-(n+1)} \sum_{u \in \mathbb{Z}^{n+1}} g(u, p, q, i, j)\hat{W}\left(\frac{u}{pq}\right) - E(p, q, i, j),$$

where

$$g(u, p, q, i, j) := \sum_{a \mod pq} t_{i, p}(F(a))\tilde{t}_{j, q}(F(a))e\left(\frac{\langle a, u \rangle}{pq}\right).$$

Using the Chinese Remainder Theorem we can split the above sum as

$$g(u, p, q, i, j) = g(\mathbb{Z}u, t_{i, p})g(\mathbb{Z}u, \tilde{t}_{j, q}),$$

with

$$g(u, t_{i, p}) := \sum_{a \in \mathbb{Z}^{n+1}} t_{i, p}(F(a))e\left(\frac{\langle a, u \rangle}{p}\right).$$

(25)

The problem now is to give a good bound for $g(u, t_{i, p})$. Assume that the smooth variety $V(F)$ is still smooth modulo $p$ and denote it by $V_p(F)$. Moreover we denote by $V(F)^* \subset \mathbb{P}^n$ its dual variety. Recall that the dual variety of an hypersurface is still an hypersurface; we denote by $G$ the homogeneous polynomial of degree $e$ such that $V(G) = V(F)^* \subset \mathbb{P}^n$. We can distinguish three situations:

i) $u \equiv 0 \mod p$. In this case we say that $u$ is of 0-type,

ii) $u$ is non-zero modulo $p$ and the associate hyperplane $\langle a, u \rangle = 0$ is not tangent to $V_p(F)$.

In this case we say that $u$ is good,

iii) $u$ is non-zero modulo $p$ and the associate hyperplane $\langle a, u \rangle = 0$ is tangent to $V_p(F)$. In this case we say that $u$ is bad.

Let us discuss cases (i) and (ii). Recall that the $t_{i, p}$s are trace function attached to middle-extension sheaves of weight $0$ which are tame and geometrically irreducible. Moreover $F$ is a polynomial of Deligne type and if $u$ is good we have that $V_p(F) \cap V_p(\langle a, u \rangle)$ is smooth of codimension $2$ in $\mathbb{P}^n$. Then we can apply Theorem \[\text{[2.1]}\] and Theorem \[\text{[2.13]}\] getting

$$\sum_{a \in \mathbb{Z}^{n+1}} t_{i, p}(F(a))e\left(\frac{\langle a, u \rangle}{p}\right) = \delta_{u = 0, p}(u)p^n \sum_{a \in \mathbb{Z}^{n+1}} t_{i, p}(a) + O_{d, e, n, c(F)}(p^{\frac{n+1}{2}}),$$

where $\delta_{u = 0, p}(u) = 1$ if $u \equiv 0 \mod p$ and $0$ otherwise. Observing that $c(F) \ll_1 1$ (part (i) of Proposition \[\text{[1.2]}\] we get

$$g(0, t_{i, p}) \ll_{d, e, n} p^{n+\frac{1}{2}},$$

and for $u$ good

$$g(u, t_{i, p}) \ll_{d, e, n} p^{\frac{n+1}{2}}.$$
Lemma 2.15. If $u$ is bad

$$g(u, t_{i,p}) \ll_{d,e,n} \sqrt{pq}^{n+1}.$$  

Proof. We denote be $K_{i,p}$ the normalize Fourier transform of $t_{i,p}$ (which is as well a trace function because $\mathcal{F}_i$ is tame and then a Fourier sheaf)

$$K_{i,p}(y) = -\frac{1}{\sqrt{p}} \sum_{b \in \mathbb{F}_p} t_{i,p}(x) e\left(\frac{xy}{p}\right).$$

Starting from the Fourier inversion formula of $t_{i,p}$ one has

$$t_{i,p}(F(a)) = -\frac{1}{\sqrt{p}} \sum_{b \in \mathbb{F}_p} K_{i,p}(b) e\left(\frac{bF(a)}{p}\right),$$

and this leads to

$$g(u, t_{i,p}) = \sum_{a \in \mathbb{F}_p^{n+1}} \left(-\frac{1}{\sqrt{p}} \sum_{b \in \mathbb{F}_p} K_{i,p}(b) e\left(\frac{bF(a)}{p}\right)\right) e\left(\frac{\langle a, u \rangle}{p}\right) \ll_{d,e,n} \sqrt{pq}^{n+1},$$

where in the final bound we used the fact that $\|K_{i,p}\|_{\infty} \ll c(F_i) \ll_{d,1} 1$ (FKM14b Page 7 combined with Paragraph 3.4) and the Deligne’s bound for additive character sums ([Del74, Theorem 8.4]).

To conclude now it is enough to analyze the contribution of any $u$ in $S(p, q, i, j)$. We prove the following Lemma:

Lemma 2.16. Let $p, q \in \mathcal{P}$ and assume that $p, q \leq B \leq pq$. Then

$$S(p,q,i,j) \ll_{d,e,n} (pq)^{n+1} + B(pq)^n + B^{n+1}_{pq}.$$  

Proof. Let us denote $S_{g,g}$ (resp. $S_{0,0}, S_{0,q}, S_{0,i}, S_{ib}, S_{bi}, S_{b,0}, S_{0,b}, S_{b,b}$), the contribution to $S(p,q,i,j)$ of the $u$’s which are good for both $p$ and $q$ (resp. the contribution to $S(p,q,i,j)$ of the $u$’s which are of type 0 for both $p$ and $q$, and so on). Let us start with the contribution of the $u \in \mathbb{Z}^{n+1}$ which are good for both $p$ and $q$:

$$S_{g,g} \ll B^{n+1}_{pq} (pq) \sum_{u \in \mathbb{Z}^{n+1}} \prod_{u \neq (0)} \left(1 + \frac{|u|}{B}\right)^2 \ll (pq)^{n+1}$$

where in the last step we are assuming that $pq \geq B$. Let us now discuss the contribution of $S_{0,0}$

$$S_{0,0} = (pq)^{-n} \sum_{u \in \mathbb{Z}^{n+1}, u \equiv 0} g(u, p, q, i, j)\hat{W}(\frac{u}{pq})$$

$$\ll B^{n+1}_{pq} (pq)^n \sum_{u \in \mathbb{Z}^{n+1}, u \equiv 0} \prod_{i=0}^{n} \left(1 + \frac{|u|}{B}\right)^{-2} \ll \frac{B^{n+1}}{pq}. $$

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let us continue with the contribution of $S_{b,b}$

$$S_{b,b} = \sum_{u \in \mathbb{Z}^{n+1}}' \tilde{W}(\frac{u}{pq}) g(u, p, q),$$

where the prime indicates that the sum is restricted over those vectors $u \in \mathbb{Z}^{n+1}$ which are bad for both for $p$ and $q$. First observe that $u = (u_0, ..., u_n)$ is bad for $p > e$ if and only if $([u_0 : ... : u_n] \mod p) \in V_p(F) \subset \mathbb{P}^n$ the reduction modulo $p$ of the dual variety. Then we conclude that $u = (u_0, ..., u_n)$ is bad for $p > e$ if and only if $(u \mod p) \in \{ \mathbb{Z}^{n+1} : G(u) = 0 \mod p \}$. Thus

$$S_{b,b} \ll B^{n+1} \left( \frac{pq}{(pq)^{n+1}} \right) \sum_{u \in \mathbb{Z}^{n+1}} \prod_{i=0}^{n} \left( 1 + \frac{|u_i|B}{pq} \right)^{-2}$$

(26)

and we can rewrite the inner sum as

$$\Sigma := \sum_{u \in \mathbb{Z}^{n+1}} \prod_{i=0}^{n} \left( 1 + \frac{|u_i|B}{pq} \right)^{-2}$$

For the following $v := (u_1, ..., u_n)$. Observe that we can write $G(X)$ as

$$G(X) = a_0 X_0^n + X_0^{e-1} G_1(V) + ... + X_0 G_{e-1}(V) + G_e(V),$$

where $G_i$ are homogeneous polynomials of degree $i$ in the variables $X_1, ..., X_n$. Without loss of generality using a different variables than $X_0$, we may assume that there exists $0 \leq i \leq e-1$ such that $G_i(V)$ is not identically zero (for $i = 0$ we mean $G_0 = a_0$). Then

$$\Sigma \leq \sum_{v \in \mathbb{Z}^n \mod pq} \prod_{i=0}^{n} \left( 1 + \frac{|u_i|B}{pq} \right)^{-2} \sum_{G(u_0,v) = 0 \mod pq} \left( 1 + \frac{|u_0|B}{pq} \right)^{-2}$$

(27)

$$+ \sum_{G(u_0,v) \in \mathbb{Z}^{n+1} \mod pq} \prod_{i=0}^{n} \left( 1 + \frac{|u_i|B}{pq} \right)^{-2}.$$  

If $G(X_0, v)$ is not the zero polynomial it has at most $e^2$ roots modulo $pq$ then

$$\sum_{G(u_0,v) = 0 \mod pq} \left( 1 + \frac{|u_0|B}{pq} \right)^{-2} \ll 1$$

then the first term in the right hand side of (27) is bounded by

$$\sum_{v \in \mathbb{Z}^n \mod pq} \prod_{i=1}^{n} \left( 1 + \frac{|u_i|B}{pq} \right)^{-2} \ll \sum_{v \in \mathbb{Z}^n} \prod_{i=1}^{n} \left( 1 + \frac{|u_i|B}{pq} \right)^{-2} \ll \left( \frac{pq}{B} \right)^n.$$

Let us bound the second sum in the right hand side of (27). First observe that

$$\sum_{G(u_0,v) \in \mathbb{Z}^{n+1} \mod pq} \prod_{i=0}^{n} \left( 1 + \frac{|u_i|B}{pq} \right)^{-2} \leq \sum_{G(u_0,v) = 0 \mod pq} \prod_{i=0}^{n} \left( 1 + \frac{|u_i|B}{pq} \right)^{-2}$$

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If the polynomial \( G_i \mod pq \) is a constant (i.e. \( i = 0 \)) we are done since for \( B \) large enough the sum is then empty. Otherwise we can rewrite it as

\[
\left( \sum_{u \in \mathbb{Z}} \left( 1 + \left| \frac{|u|B}{pq} \right| \right)^{-2} \right) \left( \sum_{v \in \mathbb{Z} \mod pq} \prod_{i=1}^{n} \left( 1 + \left| \frac{|u_i|B}{pq} \right| \right)^{-2} \right) \leq \frac{pq}{B} \sum_{v \in \mathbb{Z} \mod pq} \prod_{i=1}^{n} \left( 1 + \left| \frac{|u_i|B}{pq} \right| \right)^{-2}.
\]

Thus we can repeat the argument above applied to \( G_i \) getting

\[
\Sigma \ll \left( \frac{pq}{B} \right)^n.
\]

Then

\[
S_{b,b} \ll B^{n+1} \frac{(pq)^{n+2}}{(pq)^{n+1}} \Sigma \ll B^{n+1} \frac{(pq)^{n+2}}{(pq)^{n+1}} \left( \frac{pq}{B} \right)^n = B(pq)^{2n}.
\]

The bounds for \( S_{q,q}, S_{g,b} \) and \( S_{0,b} \) are similar.

Now we choose the set \( \mathcal{P} \) as

\[
\mathcal{P} := \{ p : f(F_p) \neq \mathbb{F}_p, \ p \text{ is of good reduction for } V(F) \text{ and } Q \leq p \leq 2Q \},
\]

where \( Q \) is such that \( Q \leq B \leq Q^2 \) to be chosen later. Then \( |\mathcal{P}| \gg \frac{Q}{\log Q} \) (the set of prime \( p \) such that \( f(F_p) \neq \mathbb{F}_p \) has positive density in the set of the prime numbers) and any \( p \in \mathcal{P} \) has size \( Q \) so

\[
\mathcal{V}_f(A) \ll_{d,e,n} \frac{B^{n+1} \log Q}{Q} + Q^{n+1} + BQ^n + \frac{(\log Q)^2}{Q^2} \sum_{p \neq q} \sum_{i,j} E(p,q,i,j).
\]

Lemma 2.17. For any \( p, q \in \mathcal{P}, \) one has

\[
E(p,q,i,j) \ll_{d,e,n,F,F} B^n,
\]

provided \( Q > B^\perp. \)

Proof. Thanks to [HB02, Theorem 1] one has

\[
\sum_{k \in S} W(k)_{i,p}(k) \ll_{d,e,n,F,F} B^n \sum_{k \in S, k \leq M, F(k) = k} 1,
\]

where \( M := \max_{k \in [-B,B]} |F(x)| \ll_{\| \cdot \| F} B^\perp. \) Let \( L \) be the splitting field of \( f'(x) = 0 \) and let us denote \( V := \{ h(\frac{\alpha}{\beta}) : \alpha, \beta \text{ integral, } h(\frac{\alpha}{\beta}) = 0 \}. \) If \( k \in S_{h,p} \) and \( k \notin V, \) then there exists \( \frac{\alpha}{\beta} \in V \) such that

\[
p|\beta^e(k - h(\frac{\alpha}{\beta})).
\]

Thus

\[
p \prod_{\frac{\alpha}{\beta} \in V} (\beta^e(k - h(\frac{\alpha}{\beta}))) \ll_{\| \cdot \| F} k^e.
\]

Now suppose that \( k \in S \) and \( k \notin V. \) Since \( S = \{ k : \{ p \in \mathcal{P} : (k \mod p \in S_{h,p}) \} \geq \frac{Q}{\log Q} \}, \) we have that

\[
Q \ll_{\| \cdot \| F} k^e,
\]

and this implies that \( k \gg M^2 \) if \( Q > B^\perp. \) Using this we can conclude

\[
E(p,q,i,j) \ll_{d,e,n,F,F} B^n \cdot |\{ k \in \mathbb{Z} : k \in V \}| \ll_{d,e,n,F,F} B^n,
\]

as we wanted.

Choosing now

\[
Q := \left( \frac{B \log B}{Q} \right)^{\perp} > B^\perp
\]

one gets

\[
\mathcal{V}_f(A) \ll_{d,e,n,F,F} B^{n+\perp} \log B,
\]

as we wanted.
References

[Bro15] T. D. Browning. The polynomial sieve and equal sums of like polynomials. *Int. Math. Res. Not. IMRN*, (7):1987–2019, 2015.

[Coh81] S. D. Cohen. The distribution of Galois groups and Hilbert’s irreducibility theorem. *Proc. London Math. Soc. (3)*, 43(2):227–250, 1981.

[Del74] P. Deligne. La conjecture de Weil. I. *Inst. Hautes Études Sci. Publ. Math.*, (43):273–307, 1974.

[Del80] P. Deligne. La conjecture de Weil. II. *Inst. Hautes Études Sci. Publ. Math.*, (52):137–252, 1980.

[FKM14a] É. Fouvry, E. Kowalski, and P. Michel. Algebraic trace functions over the primes. *Duke Math. J.*, 163(9):1683–1736, 2014.

[FKM14b] Étienne Fouvry, Emmanuel Kowalski, and Philippe Michel. Trace functions over finite fields and their applications. In *Colloquium De Giorgi 2013 and 2014*, volume 5 of *Colloquia*, pages 7–35. Ed. Norm., Pisa, 2014.

[FKM15] É. Fouvry, E. Kowalski, and P. Michel. Algebraic twists of modular forms and Hecke orbits. *Geom. Funct. Anal.*, 25(2):580–657, 2015.

[GKZ08] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants and multidimensional determinants*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008. Reprint of the 1994 edition.

[HB02] D. R. Heath-Brown. The density of rational points on curves and surfaces. *Ann. of Math. (2)*, 155(2):553–595, 2002.

[HB84] D. R. Heath-Brown. The square sieve and consecutive square-free numbers. *Math. Ann.*, 266(3):251–259, 1984.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Kat02] N. M. Katz. Estimates for nonsingular multiplicative character sums. *Int. Math. Res. Not.*, (7):333–349, 2002.

[Kat07] N. M. Katz. Estimates for nonsingular mixed character sums. *Int. Math. Res. Not. IMRN*, (19), 2007.

[Kat12] Nicholas M. Katz. *Convolution and equidistribution*, volume 180 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2012. Sato-Tate theorems for finite-field Mellin transforms.

[Kat80] N. M. Katz. *Sommes exponentielles*, volume 79 of *Astérisque*. Société Mathématique de France, Paris, 1980. Course taught at the University of Paris, Orsay, Fall 1979, With a preface by Luc Illusie, Notes written by Gérard Laumon, With an English summary.

[Kat88] Nicholas M. Katz. *Gauss sums, Kloosterman sums, and monodromy groups*, volume 116 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1988.

[Kat91] C. Hooley. On the number of points on a complete intersection over a finite field. *J. Number Theory*, 38(3):338–358, 1991. With an appendix by Nicholas M. Katz.
[Kat96] N. M. Katz.  *Rigid local systems*, volume 139 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1996.

[Kat99] N. M. Katz.  Estimates for “singular” exponential sums.  *Internat. Math. Res. Notices*, (16):875–899, 1999.

[Mil80] J. S. Milne.  *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.

[Mun09] R. Munshi.  Density of rational points on cyclic covers of $\mathbb{P}^n$.  *J. Théor. Nombres Bordeaux*, 21(2):335–341, 2009.

[SGA4] M. Artin, A. Grothendieck, and J.-L. Verdier.  *Théorie des topos et cohomologie étale des schémas (SGA 4)*, volume 269,270,305 of *Lecture Notes in Mathematics*. Springer-Verlag, 1972.  Séminaire de Géométrie Algébrique du Bois Marie, 1962, Augmenté d’un exposé de Michèle Raynaud. [With an exposé by Michèle Raynaud], With a preface and edited by Yves Laszlo, Revised reprint of the 1968 French original.

[SGA4.5] P. Deligne.  *Cohomologie étale (SGA 4½)*, volume 569 of *Lecture Notes in Mathematics*. Springer-Verlag, 1977.  Séminaire de Géométrie Algébrique du Bois Marie, 1962, Augmenté d’un exposé de Michèle Raynaud. [With an exposé by Michèle Raynaud], With a preface and edited by Yves Laszlo, Revised reprint of the 1968 French original.

[Ser97] J.-P. Serre.  *Lectures on the Mordell-Weil theorem*.  Aspects of Mathematics. Friedr. Vieweg & Sohn, Braunschweig, third edition, 1997.  Translated from the French and edited by Martin Brown from notes by Michel Waldschmidt, With a foreword by Brown and Serre.

[Tur95] G. Turnwald.  A new criterion for permutation polynomials.  *Finite Fields Appl.*, 1(1):64–82, 1995.