New characterizations of the region of complete localization for random Schrödinger operators

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Abstract

We study the region of complete localization in a class of random operators which includes random Schrödinger operators with Anderson-type potentials and classical wave operators in random media, as well as the Anderson tight-binding model. We establish new characterizations or criteria for this region of complete localization, given either by the decay of eigenfunction correlations or by the decay of Fermi projections. (These are necessary and sufficient conditions for the random operator to exhibit complete localization in this energy region.) Using the first type of characterization we prove that in the region of complete localization the random operator has eigenvalues with finite multiplicity.

1 Introduction

We study localization in a class of random operators which includes random Schrödinger operators with Anderson-type potentials and classical wave operators in random media, as well as the Anderson tight-binding model. For these operators localization is obtained either by a multiscale analysis [FrS, FrMSS, CKM, Dr, Sp, DrK, KLS, Klo1, FK1, FK2, CoH1, CoH2, FK3, FK4, W1, BCH, KSS, CoHT, GFK1, S1, Klo3, DSS, GK3, GK4, K1K, K], or, in certain cases, by the fractional moment method [AM, A, ASFH, W2, Klo2, AENSS]. In addition to pure point spectrum with exponentially localized eigenfunctions, localization proved by a either a multiscale analysis

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or the fractional moment method always include other properties such as dynamical localization \cite{AG, GD, ANFH, DS, GK1, AENSS}.

In \cite{GK5} we proved a converse to the multiscale analysis: the region of dynamical localization coincides with the region where the multiscale analysis (and the fractional moment method, when applicable) can be performed. We also gave a large list of characterizations of this region of localization, that is, necessary and sufficient conditions to be satisfied by the random operator in this energy region for a multiscale analysis to be performed at these energies \cite[Theorem 4.2]{GK5}. This region of localization is the analogue for random operators of the region of complete analyticity for classical spin systems \cite{DoS1, DoS2}. For this reason we call it the region of complete localization. (Note that the spectral region of complete localization is called the strong insulator region in \cite{GK5}, and the region of complete localization is called the region of dynamical localization in \cite{GKS}.)

In this article we establish two new consequences of the multiscale analysis that are also characterizations of the region of complete localization, given either by the decay of eigenfunction correlations or by the decay of Fermi projections. Using the characterization by the decay of eigenfunction correlations we prove that in the region of complete localization the random operator has eigenvalues with finite multiplicity.

In the one-dimensional case the multiplicity of eigenvalues is easily seen to be always less than or equal to 2. But for \( d > 1 \) this had only been previously known for in two cases. The first is the Anderson tight-binding model with bounded density for the probability distribution of the single site potential, which has simple eigenvalues in the region of localization \cite{S, KlM}. The second is its continuum analogue, Anderson-type Hamiltonians in the continuum with bounded density for the probability distribution of the strength of single site potential, for which the finite multiplicity of eigenvalues in the region of localization is known \cite{CoH1}. (Although Simon’s original proof for the Anderson model \cite{S} does not shed light on the continuum, the recent proof by Klein and Molchanov \cite{KlM} indicates that these Anderson-type Hamiltonians in the continuum should have simple eigenvalues in the region of localization. The missing ingredient is a continuous analogue of Minami’s estimate \cite{M}.)

Our proof of finite multiplicity of eigenvalues only requires the conditions for the multiscale analysis, so it applies in great generality. It neither requires probability distributions with bounded densities, nor the unique continuation property for eigenfunctions, both requirements for the Combes and Hislop result \cite{CoH1}. In particular, our result applies to random Landau Hamiltonians \cite{CoH2, W1, GK3, GKS} and to classical wave operators (e.g., acoustic and Maxwell operators) in random media \cite{GK3, FK1, KKM}.

We first characterize the region of complete localization by the decay
of the expectation of eigenfunction correlations (Theorem 1). We call this characterization the strong form of “Summable Uniform Decay of Eigenfunction Correlations” (SUDEC). SUDEC has also an almost-sure version which is essentially equivalent to the SULE (“Semi Uniformly Localized Eigenfunctions”) property introduced in [DeRJLS1, DeRJLS2]. This almost-sure SUDEC is a modification of the WULE (“Weakly Uniformly Localized Eigenfunctions”) property in [K]. (See also [T] for related properties.) But although SUDEC has a strong form (i.e., in expectation), SULE does not by its very definition.

Recently detailed almost-sure properties of localization like SULE or SUDEC, which go beyond exponential localization or almost-sure dynamical localization, turned out to be crucial in the analysis of the quantum Hall effect. In [EGS], SULE is used to prove the equivalence between edge and bulk conductance in quantum Hall systems whenever the Fermi energy falls into a region of localized states. In [CoG, CoGH], SUDEC is used to regularize the edge conductance in the region of localized states and get its quantization to the desired value. In [GKS], SUDEC is the main ingredient for a new and quite transparent proof of the constancy of the bulk conductance if the Fermi energy lies in a region of localized states.

It is well known that in the region of complete localization the random operator has pure point spectrum with exponentially decaying eigenfunctions [FrMSS, DrK, Kl]. The SULE property is also known with exponentially decaying eigenfunctions [GD, GK1]. Theorem 1 yields easily an almost-sure SUDEC (and SULE) with sub-exponentially decaying eigenfunctions. Combining the proof of [G, Theorem 1.5] with the argument in [DrK, Kl], we obtain a form of SUDEC with exponentially decaying eigenfunctions (Theorem 2). (See [GK6] for more on SUDEC and SULE.)

We conclude with a characterization of the region of complete localization by the decay of the expectation of the operator kernel of Fermi projections (Theorem 3), a crucial ingredient in linear response theory and in explanations of the quantum Hall effect [BES, AG, BoGKS, GKS].

The derivation of SUDEC and of the decay of Fermi projections from the multiscale analysis is based on the methods developed in [GK1] and, in the case of the Fermi projections, the sub-exponential kernel decay for Gevrey-like functions of generalized Schrödinger operators given in [BoGK]. That they characterize the region of complete localization relies on the converse to the multiscale analysis, the fact that slow transport implies that a multiscale analysis can be performed [GK5].

This article is organized as follows: We introduce random operators, state our assumptions, and define the region of complete localization in Section 2. We state our results in Section 3. Theorem 1 and its corollaries are proved in Section 4. Theorem 2 is proved in Section 5. The proof of Theorem 3 is given in Section 6.
Notation: We set \( \langle x \rangle := \sqrt{1 + |x|^2} \) for \( x \in \mathbb{R}^d \). By \( \Lambda_L(x) \) we denote the open cube (or box) \( \Lambda_L(x) \) in \( \mathbb{R}^d \) (or \( \mathbb{Z}^d \)), centered at \( x \in \mathbb{Z}^d \) with side of length \( L > 0 \); we write \( \chi_{x,L} \) for its characteristic function, and set \( \chi_x := \chi_{x,1} \). Given an open interval \( I \subset \mathbb{R} \), we denote by \( C_c^\infty(I) \) the class of real valued infinitely differentiable functions on \( \mathbb{R} \) with compact support contained in \( I \), with \( C_c^\infty(I)^+ \) being the subclass of nonnegative functions. The Hilbert-Schmidt norm of an operator \( A \) is written as \( \| A \|_2 \), i.e., \( \| A \|_2^2 = \text{tr} A^*A \). \( C_{a,b,...}, K_{a,b,...}, \) etc., will always denote some finite constant depending only on \( a, b,... \). (We omit the dependence on the dimension \( d \) in final results.)

2 Random operators and the region of complete localization

In this article a random operator is a \( \mathbb{Z}^d \)-ergodic measurable map \( H_\omega \) from a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) (with expectation \( \mathbb{E} \)) to generalized Schrödinger operators on the Hilbert space \( \mathcal{H} \), where either \( \mathcal{H} = L^2(\mathbb{R}^d, dx; \mathbb{C}^n) \) or \( \mathcal{H} = \ell^2(\mathbb{Z}^d, \mathbb{C}^n) \). Generalized Schrödinger operators are a class of semibounded second order partial differential operators of Mathematical Physics, which includes the Schrödinger operator, the magnetic Schrödinger operator, and the classical wave operators, eg., the acoustic operator and the Maxwell operator. (See \cite{GK2} for a precise definition and \cite{Kl} for examples.) We assume that \( H_\omega \) satisfies the standard conditions for a generalized Schrödinger operator with constants uniform in \( \omega \).

Measurability of \( H_\omega \) means that the mappings \( \omega \to f(H_\omega) \) are weakly (and hence strongly) measurable for all bounded Borel measurable functions \( f \) on \( \mathbb{R} \). \( H_\omega \) is \( \mathbb{Z}^d \)-ergodic if there exists a group representation of \( \mathbb{Z}^d \) by an ergodic family \( \{ \tau_y; y \in \mathbb{Z}^d \} \) of measure preserving transformations on \( (\Omega, \mathcal{F}, \mathbb{P}) \) such that we have the covariance given by

\[
U(y)H_\omega U(y)^* = H_{\tau_y(\omega)} \quad \text{for all } y \in \mathbb{Z}^d,
\]

where \( U(y) \) is the unitary operator given by translation: \( (U(y)f)(x) = f(x - y) \). (Note that for Landau Hamiltonians translations are replaced by magnetic translations.) It follows that there exists a nonrandom set \( \Sigma \) such that \( \sigma(H_\omega) = \Sigma \) with probability one, where \( \sigma(A) \) denotes the spectrum of the operator \( A \). In addition, the decomposition of \( \sigma(H_\omega) \) into pure point spectrum, absolutely continuous spectrum, and singular continuous spectrum is also the same with probability one. (E.g., \cite{PF, St}.)

We assume that the random operator \( H_\omega \) satisfies the hypotheses of \cite{GK1, GK5} in an open energy interval \( \mathcal{I} \). These were called assumptions or properties SGEE, SLI, EDI, IAD, NE, and W in \cite{GK1, GK3, GK5, Kl}.
(Although the results in [GK5] are written for random Schrödinger operators, they hold without change for generalized Schrödinger operators as long as these hypotheses are satisfied.) Although we assume a polynomial Wegner estimate as in [GK5], our results are still valid if we only have a sub-exponential Wegner estimate, with the caveat that one must substitute sub-exponential moments for polynomial moments (see [GK3, Remark 2.3]). In particular, our results apply to Anderson or Anderson-type Hamiltonians without the requirement of a bounded density for the probability distribution of the single site potential.

Property SGEE guarantees the existence of a generalized eigenfunction expansion in the strong sense. We assume that $H_\omega$ satisfies the stronger trace estimate [GK1, Eq. (2.36)], as in [GK5]. (Note that for classical wave operators we always project to the orthogonal complement of the kernel of $H_\omega$, see [GK1, KlKS, KlK].) For some fixed $\kappa > \frac{d}{2}$ (which will be generally omitted from the notation) we let $T_a$ denote the operator on $H$ given by multiplication by the function $\langle x - a \rangle^\kappa$, $a \in \mathbb{Z}^d$, with $T := T_0$. Since $\langle a + b \rangle \leq \sqrt{2} \langle a \rangle \langle b \rangle$, we have

$$\|T_b T_a^{-1}\| \leq 2 \bar{\mu} (b - a)^\kappa.$$  \hfill (2.2)

The domain of $T$, $\mathcal{D}(T)$, equipped with the norm $\|\phi\|_+ = \|T \phi\|$, is a Hilbert space, denoted by $H_+$. The Hilbert space $H_-$ is defined as the completion of $H$ in the norm $\|\phi\|_- = \|T^{-1} \phi\|$. By construction, $H_+ \subset H \subset H_-$, and the natural injections $\iota_+: H_+ \to H$ and $\iota_-: H \to H_-$ are continuous with dense range. The operators $T_+: H_+ \to H$ and $T_-: H \to H_-$, defined by $T_+ = T\iota_+$, and $T_- = \iota_- T$ on $\mathcal{D}(T)$, are unitary. We define the random spectral measure

$$\mu_\omega(B) := \langle T^{-1} P_{B,\omega} T^{-1} \rangle = \|T^{-1} P_{B,\omega}\|_2^2,$$  \hfill (2.3)

where $B \subset \mathbb{R}$ is a Borel set and $P_{B,\omega} = \chi_B(H_\omega)$. It follows from [GK1, Eq. (2.36)] that for $\mathbb{P}$-a.e. $\omega$ we have

$$\mu_\omega(B) = \mu_\omega(B \cap \Sigma) \leq K_{B \cap \Sigma},$$  \hfill (2.4)

where $K_B := K_{B \cap \Sigma}$ is independent of $\omega$, increasing in $B \cap \Sigma$, and $K_B < \infty$ if $B \cap \Sigma$ is bounded. Using the covariance (2.1), for $\mathbb{P}$-a.e. $\omega$ and all $a \in \mathbb{Z}^d$ we have

$$\mu_{a,\omega}(B) := \|T_a^{-1} P_{B,\omega}\|_2^2 = \|T^{-1} P_{B,\omega}\tau(-a)^{-1}\|_2^2 = \mu_{\tau(-a)\omega}(B) \leq K_B.$$  \hfill (2.5)

We have a generalized eigenfunction expansion for $H_\omega$: For $\mathbb{P}$-a.e. $\omega$ there exists a $\mu_\omega$-locally integrable function $P_\omega(\lambda): \mathbb{R} \to T_1(H_+,H_-)$, the Banach space of bounded operators $A: H_+ \to H_-$ with $T^{-1} A T_+$ trace class, such that

$$\tr \{ T^{-1} P_\omega(\lambda) T_+^{-1} \} = 1 \text{ for } \mu_\omega\text{-a.e. } \lambda,$$  \hfill (2.6)
and, for all Borel sets $B$ with $B \cap \Sigma$ bounded,

$$t_-P_\omega(B)t_+ = \int_B P_\omega(\lambda) d\mu_\omega(\lambda),$$

(2.7)

where the integral is the Bochner integral of $T_1(\mathcal{H}_+, \mathcal{H}_-)$-valued functions. Moreover, if $\phi \in \mathcal{H}_+$, then $P_\omega(\lambda)\phi \in \mathcal{H}_-$ is a generalized eigenfunction of $H_\omega$ with generalized eigenvalue $\lambda$ (i.e., an eigenfunction of the closure of $H_\omega$ in $\mathcal{H}_-$ with eigenvalue $\lambda$) for $\mu_\omega$-a.e $\lambda$. (See [GKS, Section 3] for details.)

The multiscale analysis requires the notion of a finite volume operator, a “restriction” $H_{\omega,x,L}$ of $H_\omega$ to the cube (or box) $\Lambda_L(x)$, centered at $x \in \mathbb{Z}^d$ with side of length $L \in 2\mathbb{N}$ (assumed here for convenience; we may take $L \in L_0\mathbb{N}$ for a suitable $L_0 \geq 1$ as in [GKS]), where the “randomness based outside the cube $\Lambda_L(x)$” is not taken into account. We assume the existence of appropriate finite volume operators $H_{\omega,x,L}$ for $x \in \mathbb{Z}^d$ with $L \in 2\mathbb{N}$ satisfying properties SLI, EDI, IAD, NE, and W in the open interval $I$. (See the discussion in [GKS, Section 4].)

The region of complete localization $\Xi^G$ for the random operator $H_\omega$ in the open interval $I$ is defined as the set of energies $E \in \mathcal{I}$ where we have the conclusions of the bootstrap multiscale analysis, i.e., as the set of $E \in \mathcal{I}$ for which there exists some open interval $I \subset \mathcal{I}$, with $E \in I$, such that given any $\zeta$, $0 < \zeta < 1$, and $\alpha$, $1 < \alpha < \zeta^{-1}$, there is a length scale $L_0 \in 6\mathbb{N}$ and a mass $m > 0$, so if we set $L_k+1 = [L_k\zeta]_{6\mathbb{N}}$, $k = 0, 1, \ldots$, we have

$$P \{ R(m, L_k, I, x, y) \geq 1 - e^{-L_k^\zeta} \} \geq 1 - e^{-L_k^\zeta},$$

(2.8)

for all $k = 0, 1, \ldots$, and $x, y \in \mathbb{Z}^d$ with $|x - y| > L_k + \varrho$, where

$$R(m, L, I, x, y) = \{ \omega; \text{for every } E' \in I \text{ either } \Lambda_L(x) \text{ or } \Lambda_L(y) \text{ is } (\omega, m, E')\text{-regular} \}.$$

Here $|K|_{6\mathbb{N}} = \max\{L \in 6\mathbb{N}; L \leq K\}$ (we work with scales in $6\mathbb{N}$ for convenience); $\varrho > 0$ is given in Assumption IAD, if $\text{dist}(\Lambda_L(x), \Lambda_L(x')) > \varrho$, then events based in $\Lambda_L(x)$ and $\Lambda_L(x')$ are independent. Given $E \in \mathbb{R}$, $x \in \mathbb{Z}^d$ and $L \in 6\mathbb{N}$, we say that the box $\Lambda_L(x)$ is $(\omega, m, E)$-regular for a given $m > 0$ if $E \notin \sigma(H_{\omega,x,L})$ and

$$||\Gamma_{x,L}R_{\omega,x,L}(E)\chi_{x,\frac{L}{2}}|| \leq e^{-m\varrho},$$

(2.10)

where $R_{\omega,x,L}(E) = (H_{\omega,x,L} - E)^{-1}$ and $\Gamma_{x,L}$ denotes the characteristic function of the “belt” $\Lambda_{L-1}(x) \setminus \Lambda_{L-3}(x)$. (See [GKS, Section 3].) We will take $\mathcal{H} = L^2(\mathbb{R}^d; dx; C^n)$, but the arguments can be easily modified for $\mathcal{H} = \ell^2(\mathbb{Z}^d; C^n).$
By construction \( \Xi_{CL}^I \) is an open set. It can be defined in many ways, we gave the most convenient definition for our purposes. (We refer to [GK5, Theorem 4.2] for the equivalent properties that characterize \( \Xi_{CL}^I \). The spectral region of complete localization in \( I, \Xi_{CL}^I \cap \Sigma \), is called the “strong insulator region” in [GK5]). Note that \( \Xi_{CL}^I \) is the set of energies in \( I \) where we can perform the bootstrap multiscale analysis. (If the conditions for the fractional moment method are satisfied in \( I, \Xi_{CL}^I \) coincides with the set of energies in \( I \) where the fractional moment method can be performed.) By our definition spectral gaps are (trivially) intervals of complete localization.

3 Theorems and corollaries

In this article we provide two new characterizations of the region of complete localization. The first characterizes the region of complete localization by the decay of the expectation of generalized eigenfunction correlations, the second by the expectation of decay of Fermi projections.

We start with generalized eigenfunctions. Given \( \lambda \in \mathbb{R} \) and \( a \in \mathbb{Z}^d \) we set

\[
W_{\lambda,\omega}(a) := \begin{cases} 
\sup_{\phi \in H_{\omega}^+} \frac{\| \chi_a \mathcal{P}_\omega(\lambda) \phi \|}{\| T_a^{-1} \mathcal{P}_\omega(\lambda) \phi \|} & \text{if } \mathcal{P}_\omega(\lambda) \neq 0, \\
0 & \text{otherwise,}
\end{cases}
\]

(3.1)

\( W_{\lambda,\omega}(a) \) is a measurable function of \((\lambda, \omega)\) for each \( a \in \mathbb{Z}^d \) with

\[
W_{\lambda,\omega}(a) \leq \left( \frac{\sqrt{d}}{2} \right)^\kappa = (1 + \frac{d}{4})^{\frac{\kappa}{2}}.
\]

(3.2)

Our first characterization is given in the following theorem.

**Theorem 1** Let \( I \) be a bounded open interval with \( \bar{I} \subset I \). If \( \bar{I} \subset \Xi_{CL}^I \), then for all \( \zeta \in ]0,1[ \) we have

\[
\mathbb{E} \left\{ \| W_{\lambda,\omega}(x) W_{\lambda,\omega}(y) \|_{L^\infty(I, d\mu_\omega(\lambda))} \right\} \leq C_I, \zeta e^{-|x-y| \zeta} \quad \text{for all } x, y \in \mathbb{Z}^d.
\]

(3.3)

Conversely, if (3.3) holds for some \( \zeta \in ]0,1[ \), then \( I \subset \Xi_{CL}^I \).

Note that the converse will still hold if we only have fast enough polynomial decay in (3.3).

**Remark 1** We may replace the denominator \( \| T_a^{-1} \mathcal{P}_\omega(\lambda) \phi \| \) in (3.1) by

\[
\Theta_a(\phi) := \inf_{b \in \mathbb{Z}^d} \{ (b - a)^\kappa \| T_b^{-1} \mathcal{P}_\omega(\lambda) \phi \| \}.
\]

Since \( \Theta_a(\phi) \leq \| T_a^{-1} \mathcal{P}_\omega(\lambda) \phi \| \), this slightly improves (3.3).
Corollary 1 \( H_\omega \) has pure point spectrum in the open set \( \Xi^{C_L}_I \) for \( \mathbb{P} \)-a.e. \( \omega \), with the corresponding eigenfunctions decaying faster than any sub-exponential. Moreover, we have (with \( P_{\lambda,\omega} := P_{\{\lambda\},\omega} \))

\[
\mathbb{E} \left\{ \| \mu_\omega(\{\lambda\}) \left( \text{tr} \ P_{\lambda,\omega} \right) \|_{L^\infty(I, d\mu_\omega(\lambda))} \right\} \leq C I < \infty, \tag{3.4}
\]

and hence for \( \mathbb{P} \)-a.e. \( \omega \) the eigenvalues of \( H_\omega \) in \( \Xi^{C_L}_I \) are of finite multiplicity.

It is well known that \( H_\omega \) has pure point spectrum in \( \Xi^{C_L}_I \) with exponentially decaying eigenfunctions. Our point is that pure point spectrum follows directly from (3.3), also yielding sub-exponentially decaying eigenfunctions. The estimate (3.4) is new, and it immediately implies that for \( \mathbb{P} \)-a.e. \( \omega \) the random operator \( H_\omega \) has only eigenvalues with finite multiplicity in \( \Xi^{C_L}_I \).

If \( H_\omega \) has pure point spectrum we might as well work with eigenfunctions, not generalized eigenfunctions. Given \( \lambda \in \mathbb{R} \) and \( a \in \mathbb{Z}^d \) we set

\[
W_{\lambda,\omega}(a) := \begin{cases} 
\sup_{\phi \in \mathcal{H}, \phi \neq 0} \frac{\| \chi_a P_{\lambda,\omega} \phi \|}{\| T_a^{-1} P_{\lambda,\omega} \phi \|} & \text{if } P_{\lambda,\omega} \neq 0, \\
0 & \text{otherwise,} 
\end{cases} \tag{3.5}
\]

and

\[
Z_{\lambda,\omega}(a) := \begin{cases} 
\frac{\| \chi_a P_{\lambda,\omega} \|_2}{\| T_a^{-1} P_{\lambda,\omega} \|_2} & \text{if } P_{\lambda,\omega} \neq 0, \\
0 & \text{otherwise.} 
\end{cases} \tag{3.6}
\]

\( W_{\lambda,\omega}(a) \) and \( Z_{\lambda,\omega}(a) \) are measurable functions of \((\lambda, \omega)\) for each \( a \in \mathbb{Z}^d \). They are covariant, that is,

\[
Y_{\lambda,\omega}(a) = Y_{\lambda,\tau(b)\omega}(a+b) \quad \text{for all } b \in \mathbb{Z}^d, \text{ with } Y = W \text{ or } Y = Z. \tag{3.7}
\]

It follows from (2.7) that \( \iota_- P_{\lambda,\omega} \iota_+ = P_\omega(\lambda) \mu_\omega(\{\lambda\}) \). Since \( P_{\lambda,\omega} \neq 0 \) if and only if \( \mu_\omega(\{\lambda\}) \neq 0 \), we have \( W_{\lambda,\omega}(a) = W_{\lambda,\omega}(a) \) if \( \mu_\omega(\{\lambda\}) \neq 0 \) and \( W_{\lambda,\omega}(a) = 0 \) otherwise. Combining this fact with the definition of the Hilbert-Schmidt norm and (3.4) we get

\[
Z_{\lambda,\omega}(a) \leq W_{\lambda,\omega}(a) \leq W_{\lambda,\omega}(a) \leq \left( 1 + \frac{d}{4} \right)^{\frac{1}{2}}. \tag{3.8}
\]

Remark 2 \( H_\omega \) has pure point spectrum in an open interval \( I \) if and only if for \( \mathbb{P} \)-a.e. \( \omega \) we have \( W_{\lambda,\omega}(a) = W_{\lambda,\omega}(a) \) for all \( a \in \mathbb{Z}^d \) and \( \mu_\omega \)-a.e. \( \lambda \in I \).

Thus we have the following corollary to Theorem 1.
Corollary 2 Let $I$ be a bounded open interval with $\bar{I} \subset \mathbb{I}$, $H_\omega$ has pure point spectrum in $I$ for $\mathbb{P}$-a.e. $\omega$ and for all $\zeta \in [0,1]$ and $x, y \in \mathbb{Z}^d$ we have
\[
E \left\{ \| Z_{\lambda,\omega}(x) Z_{\lambda,\omega}(y) \|_{L^\infty(I,d\mu_\omega(\lambda))} \right\} \leq C_{I,\zeta} e^{-|x-y|^\epsilon},
\]
(3.9)
\[
E \left\{ \| W_{\lambda,\omega}(x) W_{\lambda,\omega}(y) \|_{L^\infty(I,d\mu_\omega(\lambda))} \right\} \leq C_{I,\zeta} e^{-|x-y|^\epsilon}.
\]
(3.10)
Conversely, if $H_\omega$ has pure point spectrum in $I$ for $\mathbb{P}$-a.e. $\omega$, and either (3.9) or (3.10) holds for some $\zeta \in [0,1]$, we have $I \subset \Xi^\text{CL}_\omega$.

We now turn to almost sure consequences of Theorem 1.

Corollary 3 Let $I$ be a bounded open interval with $\bar{I} \subset \mathbb{I}$, the following holds for $\mathbb{P}$-a.e. $\omega$: $H_\omega$ has pure point spectrum in $I$ with finite multiplicity, so let $\{E_{n,\omega}\}_{n \in \mathbb{N}}$ be an enumeration of the (distinct) eigenvalues of $H_\omega$ in $I$, with $\nu_{n,\omega}$ being the (finite) multiplicity of the eigenvalue $E_{n,\omega}$.

(i) Summable Uniform Decay of Eigenfunction Correlations (SUDEC): For each $\zeta \in [0,1]$ and $\epsilon > 0$ we have
\[
\| \chi_x \phi \| \| \chi_y \psi \| \leq C_{I,\zeta,\epsilon,\omega} \| T^{-1} \phi \| \| T^{-1} \psi \| \langle y \rangle^{d+\epsilon} e^{-|x-y|^\epsilon},
\]
(3.11)
\[
\| \chi_x \phi \| \| \chi_y \psi \| \leq C_{I,\zeta,\epsilon,\omega} \| T^{-1} \phi \| \| T^{-1} \psi \| \langle x \rangle^{d+\epsilon} \langle y \rangle^{d+\epsilon} e^{-|x-y|^\epsilon},
\]
(3.12)
for all $\phi, \psi \in \text{Ran} \ P_{E_{n,\omega}}$, $n \in \mathbb{N}$, and $x, y \in \mathbb{Z}^d$.

(ii) Semi Uniformly Localized Eigenfunctions (SULE): There exist centers of localization $\{y_{n,\omega}\}_{n \in \mathbb{N}}$ for the eigenfunctions such that for each $\zeta \in [0,1]$ and $\epsilon > 0$ we have
\[
\| \chi_x \phi \| \leq C_{I,\zeta,\epsilon,\omega} \| T^{-1} \phi \| \langle y_{n,\omega} \rangle^{2(d+\epsilon)} e^{-|x-y_{n,\omega}|^\epsilon},
\]
(3.13)
for all $\phi \in \text{Ran} \ P_{E_{n,\omega}}$, $n \in \mathbb{N}$, and $x \in \mathbb{Z}^d$. Moreover, we have
\[
N_{L,\omega} := \sum_{n \in \mathbb{N} : |y_{n,\omega}| \leq L} \nu_{n,\omega} \leq C_{I,\omega} L^d \quad \text{for all} \ L \geq 1.
\]
(3.14)

(iii) SUDEC and SULE for complete orthonormal sets: For each $n \in \mathbb{N}$ let $\{\phi_{n,j,\omega}\}_{j \in \{1,2,\ldots,n_{\omega}\}}$ be an orthonormal basis for the eigenspace $\text{Ran} \ P_{E_{n,\omega}}$, so $\{\phi_{n,j,\omega}\}_{n,j \in \{1,2,\ldots,n_{\omega}\}}$ is a complete orthonormal set of eigenfunctions of $H_\omega$ with energy in $I$. Then for each $\zeta \in [0,1]$ and $\epsilon > 0$ we have
\[
\| \chi_x \phi_{n,i,\omega} \| \| \chi_y \phi_{n,j,\omega} \| \leq C_{I,\zeta,\epsilon,\omega} \sqrt{\alpha_{n,i,\omega}} \sqrt{\alpha_{n,j,\omega}} \langle y \rangle^{d+\epsilon} e^{-|x-y|^\epsilon},
\]
(3.15)
\[
\| \chi_x \phi_{n,i,\omega} \| \| \chi_y \phi_{n,j,\omega} \| \leq C_{I,\zeta,\epsilon,\omega} \sqrt{\alpha_{n,i,\omega}} \sqrt{\alpha_{n,j,\omega}} \langle x \rangle^{d+\epsilon} \langle y \rangle^{d+\epsilon} e^{-|x-y|^\epsilon},
\]
(3.16)
\[
\| \chi_x \phi_{n,j,\omega} \| \leq C_{I,\zeta,\epsilon,\omega} \sqrt{\alpha_{n,j,\omega}} \langle y_{n,\omega} \rangle^{2(d+\epsilon)} e^{-|x-y_{n,\omega}|^\epsilon},
\]
(3.17)
for all $n \in \mathbb{N}$, $i, j \in \{1, 2, \ldots, \nu_{n, \omega}\}$, and $x, y \in \mathbb{Z}^d$, where
\[
\alpha_{n,j,\omega} := \|T^{-1} \phi_{n,j,\omega}\|^2, \quad n \in \mathbb{N}, j \in \{1, 2, \ldots, \nu_{n,\omega}\},
\]
\[
\sum_{j \in \{1, 2, \ldots, \nu_{n,\omega}\}} \alpha_{n,j,\omega} = \mu_{\omega}(\{E_{n,\omega}\}) \quad \text{for all } n \in \mathbb{N},
\]
\[
\sum_{n \in \mathbb{N}, j \in \{1, 2, \ldots, \nu_{n,\omega}\}} \alpha_{n,j,\omega} = \sum_{n \in \mathbb{N}} \mu_{\omega}(\{E_{n,\omega}\}) = \mu_{\omega}(I).
\]

**Remark 3** The statements (i) and (ii) are essentially equivalent, and imply finite multiplicity for eigenvalues, while (iii) does not, see [GK6]. Note that in (ii) eigenfunctions associated to the same eigenvalue have the same center of localization. It is easy to see that (3.11) implies (3.12), the reverse implication also being true up to a change in the constant—both forms of SUDEC are useful.

If $I$ is a bounded open interval with $\bar{I} \subset \Xi_{CL}^I$, it is known that that for $\mathbb{P}$-a.e. $\omega$ the operator $H_{\omega}$ has pure point spectrum in $I$ with exponentially decaying eigenfunctions [FrMSS, DrK, Kl]. The SULE property is also known with exponential decay [GD, GK1]. Combining the proof of [G, Theorem 1.5] with the argument in [DrK, Kl] we also obtain SUDEC with exponential decay for $\mathbb{P}$-a.e. $\omega$.

**Theorem 2** Let $I$ be be a bounded open interval with $\bar{I} \subset \Xi_{CL}^I$. For all $\phi \in \mathcal{H}_+$ and $\lambda \in I$ set $\alpha_{\lambda,\phi} := \|T^{-1} P_{\omega}(\lambda) \phi\|^2$. The following holds for $\mathbb{P}$-a.e. $\omega$ and $\mu_{\omega}$-a.e. $\lambda \in I$: For all $\varepsilon > 0$ there exists $m_{\varepsilon} = m_{I,\varepsilon} > 0$ such that for all $\phi, \psi \in \mathcal{H}_+$ we have
\[
\|\chi_x P_{\omega}(\lambda) \phi \| \| \chi_y P_{\omega}(\lambda) \psi \| \leq C_{I,\varepsilon,\omega} \sqrt{\alpha_{\lambda,\phi} \alpha_{\lambda,\psi}} e^{(\log \langle x \rangle)^{1+\varepsilon}} e^{(\log \langle y \rangle)^{1+\varepsilon}} e^{-m_{\varepsilon}|x-y|}
\]
for all $x, y \in \mathbb{Z}^d$. In particular, it follows that $H_{\omega}$ has pure point spectrum in $I$ with exponentially decaying eigenfunctions.

Unlike Theorem 1, Theorem 2 does not give a characterization of the region of complete localization. But it still implies that $H_{\omega}$ has only eigenvalues with finite multiplicity in $I$ [GK6].

Compared to the rather short and transparent proof of (3.12), the proof of (3.21) is quite technical and involved—an extra motivation for deriving (3.12).

We now turn to the characterization in terms of the decay of Fermi projections. We set $P_{\omega}^{(E)} := P_{[-\infty, E]_{\omega}}$, the Fermi projection corresponding to the Fermi energy $E$.
Theorem 3
Let \( I \) and \( I_1 \) be bounded open intervals with \( \bar{I} \subset I \subset \bar{I}_1 \subset \Xi^\text{CL} \). If \( \bar{I} \subset \Xi^\text{CL} \), then for all \( \zeta \in [0, 1] \) we have
\[
E \left\{ \sup_{E \in I} \| \chi_{x} P_{\omega}^{(E)} \chi_{y} \|_{2}^{2} \right\} \leq C_{I, \zeta} e^{-|x-y|^\zeta} \quad \text{for all } x, y \in \mathbb{Z}^d. \tag{3.22}
\]
Conversely, if (3.22) holds for some \( \zeta \in (0, 1) \), then \( I \subset \Xi^\text{CL} \).

4 Summable Uniform Decay of Eigenfunction Correlations

In this section we prove Theorem 1 and its corollaries.

Proof of Theorem 1
Since \( \bar{I} \subset \Xi^\text{CL} \), given any \( \zeta \), \( 0 < \zeta < 1 \), and \( \alpha \), \( 1 < \alpha < \zeta^{-1} \), there is a length scale \( L_0 \in \mathbb{N} \) and a mass \( m > 0 \), so if we set \( L_{k+1} = \lfloor L_k \alpha \rfloor \), \( k = 0, 1, \ldots, \) we have (2.8) for all \( k = 0, 1, \ldots, \) and \( x, y \in \mathbb{Z}^d \) with \( |x - y| > L_k + \delta \).

Let \( I \subset \Xi^\text{CL} \) be a bounded interval with \( \bar{I} \subset \mathcal{I} \). Note that the quantity \( \| W_{\lambda, \omega}(x) W_{\lambda, \omega}(y) \|_{L^\infty(I, d\mu_{\omega}(\lambda))} \) is measurable in \( \omega \) since the \( L^\infty \) norm on sets of finite measure is the limit of the \( L^p \) norms as \( p \to \infty \). (It is actually covariant in view of the way \( P_{\omega}(\lambda) \) is constructed (see [KKS, Eq. (46)]), and the fact that the measures \( \mu_{\omega} \) and \( \mu_{\tau_a}(\omega) \) are equivalent.)

Lemma 1
Let \( \omega \in R(m, L, I, x, y) \) (defined in (2.9)). Then
\[
\| W_{\lambda, \omega}(x) W_{\lambda, \omega}(y) \|_{L^\infty(I, d\mu_{\omega}(\lambda))} \leq C_{I, m} e^{-m^4}. \tag{4.1}
\]

Proof. Let \( \omega \in R(m, L, I, x, y) \). Then for any \( \lambda \in I \), either \( \Lambda_{x}(x) \) or \( \Lambda_{L}(y) \) is \((m, \lambda)\)-regular for \( H_{\omega} \), say \( \Lambda_{L}(x) \). Given \( \phi \in \mathcal{H}_\omega \), \( P_{\omega}(\lambda) \phi \) is a generalized eigenfunction of \( H_{\omega} \) with eigenvalue \( \lambda \) (perhaps the trivial eigenfunction 0), so it follows from the EDI [GK1 (2.15)], using \( \chi_{x} = \chi_{x, \frac{1}{2}} \chi_{x} \), that
\[
\| \chi_{x} P_{\omega}(\lambda) \phi \| \leq \tilde{c}_{I} \| \Gamma_{x, L} R_{x, L}(\lambda) \chi_{x, L/3} \|_{x, L} \| \Gamma_{x, L} P_{\omega}(\lambda) \phi \|. \tag{4.2}
\]
Thus, using the bound (3.2) for the term in $y$ we obtain since

$$\Lambda_{\mathcal{L}} \leq \bar{\zeta}$$

we have (2.8) for all $k$, since $\Lambda_{\mathcal{L}}$ holds with any $\zeta$.

Lemma 2

For $\omega \in \mathbb{R}$ we have

$$\|\chi_x P_{\omega}(\lambda)\chi_y\|_2 \leq C_d \langle x \rangle^{2\zeta \langle y \rangle} \|W_{\lambda, \omega}(x)\| \|W_{\lambda, \omega}(y)\|$$

for all $x, y \in \mathbb{Z}^d$, $\lambda \in \mathbb{R}$.

Proof. Let $\{\psi_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. We have

$$\|\chi_x P_{\omega}(\lambda)\chi_y\|_2 = \sum_{n \in \mathbb{N}} \|\chi_x P_{\omega}(\lambda)\chi_y \psi_n\|^2$$

$$\leq \|W_{\lambda, \omega}(x)\|^2 \sum_{n \in \mathbb{N}} \|T^{-1}_x P_{\omega}(\lambda)\chi_y \psi_n\|^2$$

$$= \|W_{\lambda, \omega}(x)\|^2 \|T^{-1}_x P_{\omega}(\lambda)\chi_y\|_2^2 \leq C_d \langle x \rangle^{2\zeta \langle y \rangle} \|W_{\lambda, \omega}(x)\|^2,$$

where we used (4.3) and (2.2). Since $\|\chi_x P_{\omega}(\lambda)\chi_y\|_2 = \|\chi_y P_{\omega}(\lambda)\chi_x\|_2$, the lemma follows. \[\square\]
Thus it follows from (3.3) that
\[
\left\| \chi_x f(H_\omega) P_\omega(I) \right\|_2 \leq \sup_{f \in B_1} \int_I |f(\lambda)| \left\| \chi_x P_\omega(\lambda) \chi_0 \right\|_2 \, d\mu_\omega(\lambda) \leq \int_I \left\| \chi_x P_\omega(\lambda) \chi_0 \right\|_2 \, d\mu_\omega(\lambda) \leq C_d^x (\kappa) K_I \left\| W_{\lambda,\omega}(x) W_{\lambda,\omega}(0) \right\|_{L^2(I, d\mu_\omega(\lambda))}.
\]

So now assume (3.3) holds for some \( \zeta \in ]0,1[. \) By \( B_1 = B_1(\mathbb{R}) \) we denote the collection of real-valued Borel functions \( f \) of a real variable with \( \sup_{t \in \mathbb{R}} |f(t)| \leq 1. \) Using the generalized eigenfunction expansion \( [2,4] \), Lemma 2, and (2.4), we get
\[
\sup_{f \in B_1} \left\| \chi_x f(H_\omega) P_\omega(I) \chi_0 \right\|_2 \leq \sup_{f \in B_1} \int_I |f(\lambda)| \left\| \chi_x P_\omega(\lambda) \chi_0 \right\|_2 \, d\mu_\omega(\lambda) \leq \int_I \left\| \chi_x P_\omega(\lambda) \chi_0 \right\|_2 \, d\mu_\omega(\lambda) \leq C_d^x (\kappa) K_I \left\| W_{\lambda,\omega}(x) W_{\lambda,\omega}(0) \right\|_{L^2(I, d\mu_\omega(\lambda))}.
\]

Therefore, it follows from (4.16) that for any \( x, y \in \mathbb{Z}^d \) we have
\[
E \left\{ \sup_{f \in B_1} \left\| \chi_x f(H_\omega) P_\omega(I) \chi_y \right\|_2 \right\} = E \left\{ \sup_{f \in B_1} \left\| \chi_{x-y} f(H_\omega) P_\omega(I) \chi_0 \right\|_2 \right\} \leq C_d^x (\kappa) e^{-\frac{1}{2} |x-y|^2}.
\]

It now follows from \( [GK5] \) Theorem 4.2] that \( I \subset \Xi^C_2 \).
In fact, these eigenvalues have finite multiplicity, a consequence of the estimate (3.4), which is proved as follows: Using (2.5) and (3.8), we have

\[
\mu_\omega(\{\lambda\}) (\text{tr } P_{\lambda,\omega}) = \| T^{-1} P_{\lambda,\omega} \|_2^2 (\text{tr } P_{\lambda,\omega}) \\
\leq C_d \sum_{x,y \in \mathbb{Z}^d} \langle x \rangle^{-2\kappa} \| \chi_x P_{\lambda,\omega} \|_2^2 \| \chi_y P_{\lambda,\omega} \|_2^2 \\
\leq C_d K_i^2 \sum_{x,y \in \mathbb{Z}^d} \langle x \rangle^{-2\kappa} (Z_{\lambda,\omega}(x)Z_{\lambda,\omega}(y))^2 \\
\leq C_d K_i^2 \sum_{x,y \in \mathbb{Z}^d} \langle x \rangle^{-2\kappa} Z_{\lambda,\omega}(x)Z_{\lambda,\omega}(y),
\]

and hence (3.4) follows from Remark 2 and (3.8) (or from (3.9)).

**Lemma 3** Let \( I \) be a bounded interval with \( \bar{I} \subset \Xi_{CL} \). Then for all \( \xi \in [0,1], p \geq 1, \) and \( \mathbb{P}\text{-a.e. } \omega \) we have

\[
\left\| \sum_{x,y \in \mathbb{Z}^d} e^{i\xi x - y} \langle x \rangle^{-2\kappa} [W_{\lambda,\omega}(x)W_{\lambda,\omega}(y)]^p \right\|_{L^\infty(I,d\mu_\omega(\lambda))} \leq C_{I,\xi,p,\omega} < \infty. \quad (4.16)
\]

**Proof.** It follows from (3.3) and (3.2) that for any \( \xi \in [0,1] \) and \( p \geq 1 \) we have

\[
E \left\{ \sum_{x,y \in \mathbb{Z}^d} e^{i\xi x - y} \langle x \rangle^{-2\kappa} \| W_{\lambda,\omega}(x)W_{\lambda,\omega}(y) \|_{L^\infty(I,d\mu_\omega(\lambda))}^p \right\} \leq C_{I,\xi,p} < \infty,
\]

and hence (4.16) follows.

In fact Lemma 3 holds for any \( p > 0 \) by modifying the proof of Theorem 1.

**Proof of Corollary 2** Since when \( H_\omega \) has pure point spectrum in \( I \) for \( \mathbb{P}\text{-a.e. } \omega \) the estimate (3.10) is the same as (3.3), the corollary with (3.10) follows immediately from Theorem 1. The estimate (3.8) follows immediately from from (3.10) in view of (3.8). To prove the converse from (3.9), note that if \( \mu_\omega(\{\lambda\}) \neq 0 \), we have, using (2.2) and (2.0),

\[
\| \chi_x P_{\lambda}(\omega) \chi_y \|_1 = \mu_\omega(\{\lambda\})^{-1} \| \chi_x P_{\lambda,\omega} \chi_y \|_1 \\
\leq \mu_\omega(\{\lambda\})^{-1} \| \chi_x P_{\lambda,\omega} \|_2 \| \chi_y P_{\lambda,\omega} \|_2 \\
= \mu_\omega(\{\lambda\})^{-1} \| T^{-1}_x P_{\lambda,\omega} \|_2 \| T^{-1}_y P_{\lambda,\omega} \|_2 Z_{\lambda,\omega}(x)Z_{\lambda,\omega}(y) \\
\leq C_d \langle x \rangle^\kappa \langle y \rangle^\kappa Z_{\lambda,\omega}(x)Z_{\lambda,\omega}(y). \quad (4.18)
\]
Thus, if $H_\omega$ has pure point spectrum in $I$, (4.11) follows from (3.19), and hence $I \subset \mathcal{L}_I^0$, by [GK5] Theorem 4.2.

**Proof of Corollary 3**

Pure point spectrum almost surely in $I$ with eigenvalues of finite multiplicity follows from Corollary 1. It follows from Lemma 4 that for all $\xi \in [0, 1]$, $p \geq 1$, $x, y \in \mathbb{Z}^d$, $\phi, \psi \in \text{Ran} P_{E_n, \omega}$, $n \in N$, and $i, j \in \{1, 2, \ldots, \nu_n, \omega\}$ we have

$$
\|\chi_x \phi\| \|\chi_y \psi\| \leq \left[W_{E_n, \omega}(x)W_{E_n, \omega}(y)\right] \left[\|T_{-1}^{-1} \phi\| \|T_y^{-1} \psi\|\right] \\
\leq 2^\kappa \langle x \rangle^\kappa \langle y \rangle^\kappa \|T_{-1}^{-1} \phi\| \|T_y^{-1} \psi\| \left[C_{I, \xi, \rho, \omega}(y) 2^{2\kappa} - |x-y|\xi\right] \equiv \left(4.19\right)
$$

where we used (2.26).

The SUDEN estimate (3.11) for given $\varepsilon > 0$ and $\zeta \in [0, 1]$ follows from (4.19) by working with $\frac{d}{2} < \kappa < \frac{d+\varepsilon}{2}$, choosing $p \geq 1$ such that $d + \varepsilon = \frac{2(p+1)\kappa}{p}$, and taking $\xi = \frac{1}{1+\kappa}$.

To prove the SULE-like estimate (3.13), for each $n \in \mathbb{N}$ we take a nonzero eigenfunction $\psi \in \text{Ran} P_{E_n, \omega}$, and pick $y_{n, \omega} \in \mathbb{Z}^d$ (not unique) such that

$$
\|\chi_{y_{n, \omega}} \psi\| = \max_{y \in \mathbb{Z}^d} \|\chi_y \psi\|. \quad \left(4.20\right)
$$

Since for all $a \in \mathbb{Z}^d$ and $\phi \in \mathcal{H}$ we have

$$
\|T^{-1}_a \phi\|^2 = \sum_{y \in \mathbb{Z}^d} \|\chi_y T^{-1}_a \phi\|^2 \leq \max_{y \in \mathbb{Z}^d} \|\chi_y \phi\|^2 \sum_{y \in \mathbb{Z}^d} \|\chi_y T^{-1}_a \phi\|^2 \\
= \max_{y \in \mathbb{Z}^d} \|\chi_y \phi\|^2 \sum_{y \in \mathbb{Z}^d} \|\chi_y T^{-1}_a \phi\|^2 \leq C_d^2 \max_{y \in \mathbb{Z}^d} \|\chi_y \phi\|^2, \quad \left(4.21\right)
$$

we get

$$
\|T^{-1}_a \phi\| \leq C_d \|\chi_{y_{n, \omega}} \psi\| \quad \text{for all } a \in \mathbb{Z}^d. \quad \left(4.22\right)
$$

It now follows from (4.19), taking $\psi$ as in (4.20), $y = y_{n, \omega}$, using (4.22), and choosing $p$ and $\xi$ as above, that for all $x \in \mathbb{Z}^d$, $\psi \in \text{Ran} P_{E_n, \omega}$, and $i \in \{1, 2, \ldots, \nu_n, \omega\}$ we have

$$
\|\chi_x \phi\| \leq C_d^{-1} C_{I, \xi, \rho, \omega} \|T^{-1}_a \phi\| \langle y_{n, \omega}\rangle^{d+\varepsilon} e^{-|x-y_{n, \omega}|\xi}, \quad \left(4.23\right)
$$

which is just (3.13).

SUDEC and SULE for the complete orthonormal set $\{\phi_{n, j, \omega}\}_{n \in \mathbb{N}, j \in \{1, 2, \ldots, \nu_n, \omega\}}$ of eigenfunctions of $H_\omega$ with energy in $I$ follows. Note that the equalities (3.19) and (3.20) follow immediately from (2.6).
To prove (3.14), note that it follows from (3.17) that
\[
\left\| \chi(\{ |x - y_n,\omega| \geq R \}) \phi_{n,j,\omega} \right\|^2 \\
\leq C_{I,\zeta,\epsilon,\omega}^2 \langle y_n,\omega \rangle^{2(d+\epsilon)} e^{-|x - y_n,\omega|^{\zeta}} \\
\leq C_{I,\zeta,\epsilon,\omega}'' \langle y_n,\omega \rangle^{2(d+\epsilon)} e^{-\frac{1}{2} R^{\zeta}} \leq \frac{1}{2},
\]
if we take
\[
R = R_{n,j,\omega} \geq 2 \left\{ \log \left( 2 C_{I,\zeta,\epsilon,\omega}'' \langle y_n,\omega \rangle^{2(d+\epsilon)} \right) \right\}^{\frac{1}{\zeta}}.
\]
Given \( L \geq 1 \), we set
\[
R_{L,\omega} = 2 \left\{ \log \left( 2 C_{I,\zeta,\epsilon,\omega}'' \langle L \rangle^{2(d+\epsilon)} \right) \right\}^{\frac{1}{\zeta}} \leq C_{I,\zeta,\epsilon,\omega}''' (\log L)^{\frac{1}{\zeta}},
\]
\[
S_{L,\omega} = L + 2 R_{L,\omega} \leq C_{I,\zeta,\epsilon,\omega}''' L.
\]
Note that if \( |y_n,\omega| \leq L \) we have \( \| \chi_0, S_{L,\omega} \phi_{n,j,\omega} \|^2 \geq \frac{1}{2} \) for all \( j \in \{1,2,\ldots,\nu_n \} \). Thus, using (4.24), we get
\[
\frac{1}{2} N_L \leq \sum_{n \in \mathbb{N}, j \in \{1,2,\ldots,\nu_n \}} \| \chi_0, S_{L,\omega} \phi_{n,j,\omega} \|^2 = \| \chi_0, S_{L,\omega} P_I,\omega \|^2 \\
\leq \sum_{a \in \mathbb{Z}^d \cap \Lambda_{S_{L,\omega}(0)}} \| \chi_0 P_I,\omega \|^2 = \sum_{a \in \mathbb{Z}^d \cap \Lambda_{S_{L,\omega}(0)}} \| \chi_0 P_I,(-a),\omega \|^2 \\
\leq C_d \sum_{a \in \mathbb{Z}^d \cap \Lambda_{S_{L,\omega}(0)}} \mu_{(-a),\omega}(I) \leq C_d S_{L,\omega}^d K_I \leq \tilde{C}_{I,\zeta,\epsilon,\omega} K_I L^d,
\]
which yields (3.14).}

5 SUDEC with exponential decay

In this section we prove Theorem 2.

Proof of Theorem 2 Let us fix \( \epsilon > 0 \). Since \( \tilde{I} \subset \Xi_{\tilde{I}}^L \), we can pick \( \zeta \in \mathbb{C}^1(0,1] \) and \( \alpha \in \mathbb{C}(1,\zeta^{-1}] \) such that \( \alpha < (1+\epsilon) \zeta \) and there is a length scale \( L_0 \in 6\mathbb{N} \) and a mass \( m = m_\zeta > 0 \), so if we set \( L_{k+1} = [L_k^2]_{6\mathbb{N}}, \) \( k = 0,1,\ldots \), we have \( 2L_k \) for all \( k = 0,1,\ldots \), and \( x,y \in \mathbb{Z}^d \) with \( |x - y| > L_k + \rho \). We fix \( \rho \in \mathbb{R}^1 \) and \( \rho > \frac{1+2\alpha}{1-2\rho} > 1 \). As in [K] Proof of Theorem 6.4], we pick \( \rho \in \mathbb{R}^1, \frac{1}{2}, \) and \( \rho > \frac{1+2\alpha}{1-2\rho} > 1 \), and for each \( x_0 \in \mathbb{Z}^d \) and \( k = 0,1,\ldots \) define
the discrete annuli
\[ A_{k+1}(x_0) = \{ \Lambda_{2L_{k+1}}(x_0) \setminus \Lambda_{2L_k}(x_0) \} \cap \mathbb{Z}^d, \]  
\[ \tilde{A}_{k+1}(x_0) = \{ \Lambda_{\frac{\omega}{1+\varepsilon} L_{k+1}}(x_0) \setminus \Lambda_{\frac{\omega}{1+\varepsilon} L_k}(x_0) \} \cap \mathbb{Z}^d. \] (5.1) (5.2)

We consider the event
\[ F_k = \bigcap_{y \in \mathbb{Z}^d} \bigcap_{\log \langle y \rangle \leq (mL_{k+1})^{(1+)^{-1}}} R(m, L_k, I, x, y), \] (5.3)

with \( R(m, L, I, x, y) \) given in (4.4). It follows from (4.4) that \( \sum_{k=1}^{\infty} \mathbb{P}(F_k) < \infty \), so that the Borel-Cantelli Lemma applies and yields an almost-surely finite \( k_1(\omega) \), such that for all \( k \geq k_1(\omega) \), if \( E \in I \) and \( \log \langle y \rangle \leq (mL_{k+1})^{(1+)^{-1}} \), either \( \Lambda_{k_1}(y) \) is \((\omega, m, E)\)-regular or \( \Lambda_{k_1}(x) \) is \((\omega, m, E)\)-regular for all \( x \in A_k(y) \). For convenience we require \( k_1(\omega) \geq 1 \).

Using [K] Lemma 6.2 we conclude that for all \( y \in \mathbb{Z}^d \), \( \mathbb{P}\text{-a.e.} \), and \( \mu_\omega \text{-a.e.} \lambda \in \mathcal{I} \), there exists a finite \( k_2 = k_2(y, \omega, \lambda) \) such that for all \( k > k_2 \) we have that \( \Lambda_{k_1}(y) \) is \((\omega, m, \lambda)\)-singular, and moreover \( \Lambda_{k_2}(y) \) is \((\omega, m, \lambda)\)-regular unless \( k_2(\omega, y, \lambda) = 0 \).

For each \( y \in \mathbb{Z}^d \) we define \( k_3 := k_3(y) \) by
\[ (mL_{k_3})^{(1+)^{-1}} < \log \langle y \rangle \leq (mL_{k_3+1})^{(1+)^{-1}}, \] (5.4)

when possible, with \( k_3(y) = -1 \) otherwise.

We now set
\[ k_* := k_* (\omega, y, \lambda) = \max\{ k_1(\omega), k_3(y), k_2(\omega, y, \lambda) + 1 \}; \] (5.5)

note that \( 1 \leq k_* (\omega, y, \lambda) < \infty \) for \( \mathbb{P}\text{-a.e.} \), and \( \mu_\omega \text{-a.e.} \lambda \in \mathcal{I} \).

Let \( \phi, \psi \in \mathcal{H}_+ \) be given. Then for \( \mathbb{P}\text{-a.e.} \omega \), and \( \mu_\omega \text{-a.e.} \lambda \in \mathcal{I} \), if \( k \geq k_* \), the box \( \Lambda_{k_1}(y) \) is \((\omega, m, \lambda)\)-singular and thus \( \Lambda_{k_1}(x) \) is \((\omega, m, \lambda)\)-regular for all \( x \in A_{k_1}(y) \). It follows, as in [K] Proof of Theorem 6.4, that for all \( x \in A_{k+1}(y) \) we have
\[ \| \chi_x \mathbf{P}_\omega (\lambda) \psi \| \leq C_{d, m, \langle y \rangle^c} \| T_{x}^{-1} \mathbf{P}_\omega (\lambda) \psi \| e^{-m_\psi |x-y|}, \] (5.6)

where \( m_\psi = \frac{\nu(3\rho+1 )}{2} m \in ]0, m[ \). It remains to consider the case when \( x \in \Lambda_{\frac{\omega}{1+\varepsilon} L_{k_2}}(y) \cap \mathbb{Z}^d \). If \( k_* = \max\{ k_1(\omega), k_3(y) \} \) \( > k_2(\omega, y, \lambda) \), we use (4.2) and, if \( k_* = k_3(y) \), (4.3), getting
\[ \| \chi_x \mathbf{P}_\omega (\lambda) \psi \| \leq C_{d, \langle y \rangle^c} \| T_{x}^{-1} \mathbf{P}_\omega (\lambda) \psi \| e^{m_{L_{k_*}}} e^{-m_{L_{k_*}}} \] (5.7)

\[ \leq \begin{cases} C_{d, \langle y \rangle^c} \| T_{x}^{-1} \mathbf{P}_\omega (\lambda) \psi \| e^{(\log (\langle y \rangle)^{1+}) e^{-m_\psi |x-y|}} & \text{if } k_* = k_3(y) \leq \text{if } k_* = k_1(\omega). \end{cases} \]
Estimating $\|\chi_x P_{\omega}(\lambda)\|\phi$ by (5.2), we get the bound
\[
\|\chi_x P_{\omega}(\lambda)\|\phi \leq C_{d,\omega} |x|^{\kappa} \langle y \rangle^{2\kappa} \sqrt{\alpha_{\omega,\phi}\alpha_{\lambda,\psi}} e^{(\log |y|)^{1+\epsilon}} e^{-m'|x-y|},
\]
with $m' = m_\rho$. If $k_x = k_2(\omega, y)$, we have $k_2 \geq 1$ and hence $\Lambda_{L_{k_2}}(y)$ is $(\omega, m, \lambda)$-regular. Using (5.6) and (2.2), we get
\[
\|\chi_x P_{\omega}(\lambda)\| \leq C_{d,1,m} (\langle y \rangle |T^{-1} P_{\omega}(\lambda)\|e^{-m_{L_{k_2}}}).
\]
If $x \in \Lambda_{\frac{1}{2\rho} L_{k_2}}(y) \cap \mathbb{Z}^d$, we may bound the term in $x$ by (5.2) and get (5.8) with $m' = (1-2\rho) m$ and another constant $C_{d,\omega}$. Since $x \in \Lambda_{\frac{1}{2\rho} L_{k_2+1}}(y) \cap \mathbb{Z}^d$, we cannot have $x \notin \Lambda_{\frac{1}{2\rho} L_{k_2+1}}(y) \cap \mathbb{Z}^d$ by our choice of $b$ and $\rho$. Thus the only remaining case is when $x \in \tilde{A}_{k_2+1}(y)$, where $\tilde{A}_{k_2+1}(y)$ is defined as in (2.2) but with $2\rho$ substituted for $\rho$. If all boxes $\Lambda_{L_{k_2}}(x')$ with $|x' - x| \leq \rho|x-y|$ are $(\omega, m, \lambda)$-regular, the argument in [Kl, Proof of Theorem 6.4] still applies, and hence we also get (5.6) and (5.8) with $m' = m_\rho$. If not, there exists $x' \in \tilde{A}_{k_2+1}(y)$ with $|x' - x| \leq \rho|x-y|$ such that $\Lambda_{L_{k_2}}(x')$ is $(\omega, m, \lambda)$-singular. Clearly, $x' \in \tilde{A}_{k_2+1}(y)$ if and only if $y \in \tilde{A}_{k_2+1}(x')$. In addition, since $k_3(y) \leq k_2(\omega, y)$, we have $k_3(x') \leq k_2(\omega, y, \lambda) + 1$, as
\[
\log \langle x' \rangle \leq \frac{1}{2} \log 2 + \log \langle y \rangle + \log \langle b L_{k_2+1} \rangle \leq (m L_{k_2+1})^{(1+\epsilon)-1}.
\]
Thus, as $k_2 \geq k_1(\omega)$, we can apply the argument to (5.10) in the annulus $A_{k_2+1}(x')$, obtaining
\[
\|\chi_{x'} P_{\omega}(\lambda)\| \leq C_{d,m} (\langle x' \rangle^{\kappa} |T^{-1} P_{\omega}(\lambda)\|e^{-m_{L_{k_2}}} (x')^{(1+\epsilon)-1}).
\]
where we used $|x'-x| \leq \rho|x-y|$ and $|x'-y| \geq |x-y| - |x'-x| \geq (1-\rho)|x-y|$. Estimating $\|\chi_{x'} P_{\omega}(\lambda)\|\phi$ by (5.2), we get the bound
\[
\|\chi_{x'} P_{\omega}(\lambda)\|\|\chi_{x'} P_{\omega}(\lambda)\| \leq C_{d,\omega} (\langle x' \rangle^{\kappa} \langle y \rangle^{2\kappa} \sqrt{\alpha_{\omega,\phi}\alpha_{\lambda,\psi}} e^{-m'|x-y|} (5.13)
\]
with $m' = \rho(1-\rho)m_\rho$. The theorem is proved.

6 Decay of the Fermi projection

In this section we prove Theorem 4.
Proof of Theorem 3 Let $I$ and $I_1$ be bounded open intervals with $\bar{I} \subset I_1 \subset I \subset \mathbb{Z}^d_*$. It follows from [GK1] Theorem 3.8 that for all $\zeta \in [0,1]$ we have

$$\mathbb{E} \left\{ \sup_{f \in B_1} \| \chi_x f(H_\omega) P_{\omega}(I_1) \chi_y \|^2 \right\} \leq C_{1,\zeta} e^{-|x-y|^s} \quad \text{for all } x, y \in \mathbb{Z}^d. \quad (6.1)$$

We write $I = (\alpha, \beta)$, and fix $\delta = \frac{1}{2} \text{dist}(I, \partial I) > 0$. Given $\zeta \in [0,1]$, we choose $\zeta' \in \zeta, [1]$. Since $H_\omega$ is semibounded, we can choose $\gamma > -\infty$ such that $\Sigma \subset [\gamma, \infty]$. We pick a $L^1$-Gevrey function $g$ of class $\frac{1}{\zeta'}$ on $[\gamma, \infty]$, such that $0 \leq g \leq 1$, $g \equiv 1$ on $(-\infty, \alpha + \delta]$, and $g \equiv 0$ on $[\beta + \delta, \infty]$. (See [BoGK] Definition 1.1; such a function always exists.) For all $E \in I$ we have $P_{\omega}^{(E)} = g(H_\omega) + f_E(H_\omega)$, where $f_E(t) = \chi_{[-\infty, E]}(t) - g(t) \in B_1$, with $f_E(H_\omega) = f_E(H_\omega) P_{\omega}(I_1)$. Using [BoGK] Theorem 1.4, for $\mathbb{P}$-a.e. $\omega$ we have

$$\| \chi_x g(H_\omega) \chi_y \| \leq C_{g, \zeta', \zeta''} e^{-C_{g, \zeta', \zeta''}|x-y|^s} \quad \text{for all } x, y \in \mathbb{Z}^d. \quad (6.2)$$

On the other hand, it follows from [GK1] Eq. (2.36) and the covariance that for $\mathbb{P}$-a.e. $\omega$

$$\| \chi_x g(H_\omega) \chi_y \|_1 \leq \| \chi_x g(H_\omega) \chi_y \|_1 \| \chi_y g(H_\omega) \chi_x \|_1 \leq C_g \quad \text{for all } x, y \in \mathbb{Z}^d. \quad (6.3)$$

Since $\| A \|_2 \leq \| A \|_1$ for any operator $A$, we get

$$\| \chi_x g(H_\omega) \chi_y \|_2 \leq C_{g, \zeta', \zeta''} e^{-C_{g, \zeta', \zeta''}|x-y|^s} \quad \text{for all } x, y \in \mathbb{Z}^d. \quad (6.4)$$

The estimate (6.2) for all $\zeta \in [0,1]$ now follows from (6.1) and (6.4).

To prove the converse, let us suppose (6.2) holds for some $\zeta \in [0,1]$. Let $X \in C_{\zeta, \zeta'}^\infty(I)$. By the spectral theorem,

$$e^{-itH_\omega} X(H_\omega) = \int e^{-itE} X(E) P_{\omega}(dE) = -\int (e^{-itE} X(E)') P_{\omega}^{(E)} dE$$

$$= - \int_I (e^{-itE} X(E)') P_{\omega}^{(E)} dE. \quad (6.5)$$

Thus for all $n > 0$ we have

$$\left\| \langle x \rangle^{\frac{n}{2}} e^{-itH_\omega} X(H_\omega) \chi_0 \right\|_2 \leq C_X(1 + t) \int_I \left\| \langle x \rangle^{\frac{n}{2}} P_{\omega}^{(E)} \chi_0 \right\|_2 dE, \quad (6.6)$$
and hence

\[
\begin{align*}
\mathbb{E} \left\{ \left\| \langle x \rangle \mathbb{P} e^{-itH_x} \chi_0 \right\|_2^2 \right\} \\
\leq C_X^2 (1 + t)^2 \mathbb{E} \left\{ \left\| \int_I \langle x \rangle \mathbb{P}^{(E)} \chi_0 \right\|_2^2 \right\} \\
\leq C_X^2 (1 + t)^2 |I| \int_I \mathbb{E} \left\{ \left\| \langle x \rangle \mathbb{P}^{(E)} \chi_0 \right\|_2^2 \right\} dE \leq C_{X,I,n,\zeta} (1 + t)^2,
\end{align*}
\]

(6.7)

where we used (3.22) to get the last inequality. It follows that

\[
\mathcal{M}(n, X, T) := \frac{2}{T} \int_0^\infty e^{-\frac{t}{T}} \mathbb{E} \left\{ \left\| \langle x \rangle \mathbb{P} e^{-itH_x} \chi_0 \right\|_2^2 \right\} dt
\]

\[
\leq C'_{X,I,n,\zeta} (1 + T^2),
\]

(6.8)

hence

\[
\liminf_{T \to \infty} \frac{1}{T^\alpha} \mathcal{M}(n, X, T) < \infty \quad \text{for all } \alpha \geq 2 \text{ and } n > 0.
\]

(6.9)

It now follows from [GK5, Theorem 2.11] that \( I \subset \Xi^{CL} \).

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**References**

[A] Aizenman, M.: Localization at weak disorder: some elementary bounds. Rev. Math. Phys. 6, 1163-1182 (1994)

[AENSS] Aizenman, M., Elgart, A., Naboko, S., Schenker, J.H., Stolz, G.: Moment Analysis for Localization in Random Schrödinger Operators. Preprint

[AG] Aizenman, M., Graf, G.M.: Localization bounds for an electron gas. J. Phys. A: Math. Gen. 31, 6783-6806, (1998)

[AM] Aizenman, M., Molchanov, S.: Localization at large disorder and extreme energies: an elementary derivation. Commun. Math. Phys. 157, 245-278 (1993)

[ASFH] Aizenman, M., Schenker, J., Friedrich, R., Hundertmark, D.: Finite volume fractional-moment criteria for Anderson localization. Commun. Math. Phys. 224, 219-253 (2001)
[BES] Bellissard, J., van Elst, A., Schulz-Baldes, H.: The noncommutative geometry of the quantum Hall effect. J. Math. Phys. 35, 5373-5451 (1994).

[BCH] Barbaroux, J.M., Combes, J.M., Hislop, P.D.: Localization near band edges for random Schrödinger operators. Helv. Phys. Acta 70, 16-43 (1997)

[BoGK] Bouclet, J.M., Germinet, F., Klein, A.: Sub-exponential decay of operator kernels for functions of generalized Schrödinger operators. Proc. Amer. Math. Soc. 132, 2703-2712 (2004)

[BoGKS] Bouclet, J.M., Germinet, F., Klein, A., Schenker, J.H.: Linear response theory for magnetic Schrödinger operators in disordered media. J. Funct. Anal. In press

[CKM] Carmona, R., Klein, A., Martinelli, F.: Anderson localization for Bernoulli and other singular potentials. Commun. Math. Phys. 108, 41-66 (1987)

[CoG] Combes, J.M., Germinet, F.: Edge and Impurity Effects on Quantization of Hall Currents. Commun. Math. Phys. 256, 159-180 (2005)

[CoGH] Combes, J.M., Germinet, F., Hislop, P.: On the quantization of Hall currents in presence of disorder, to appear in the Proceedings of the Conference Q-Math9 (Giens, 2004)

[CoH1] Combes, J.M., Hislop, P.D.: Localization for some continuous, random Hamiltonian in d-dimension. J. Funct. Anal. 124, 149-180 (1994)

[CoH2] Combes, J.M., Hislop, P.D.: Landau Hamiltonians with random potentials: localization and the density of states. Commun. Math. Phys. 177, 603-629 (1996)

[CoHT] Combes, J.M., Hislop, P.D., Tip, A.: Band edge localization and the density of states for acoustic and electromagnetic waves in random media. Ann. Inst. H. Poincare Phys. Theor. 70, 381-428 (1999)

[DSS] Damanik, D., Sims, R., Stolz, G.: Localization for one dimensional, continuum, Bernoulli-Anderson models. Duke Math. J. 114, 59-100 (2002)

[DS] Damanik, D., Stollmann, P.: Multi-scale analysis implies strong dynamical localization. Geom. Funct. Anal. 11, 11-29 (2001)
Del Rio, R., Jitomirskaya, S., Last, Y., Simon, B.: What is Localization? Phys. Rev. Lett. 75, 117-119 (1995)

Del Rio, R., Jitomirskaya, S., Last, Y., Simon, B.: Operators with singular continuous spectrum IV: Hausdorff dimensions, rank one perturbations and localization. J. d’Analyse Math. 69, 153-200 (1996)

Dobrushin, R., Shlosman, S.: Completely analytical Gibbs fields. Prog in Phys. 10, 347-370 (1985)

Dobrushin, R., Shlosman, S.: Completely analytical interactions. J. Stat. Phys. 46, 983-1014 (1987)

von Dreifus, H.: On the effects of randomness in ferromagnetic models and Schrödinger operators. Ph.D. thesis, New York University (1987)

von Dreifus, H., Klein, A.: A new proof of localization in the Anderson tight binding model. Commun. Math. Phys. 124, 285-299 (1989)

Elgart, A., Graf, G.M., Schenker, J.H.: Equality of the bulk and edge Hall conductances in a mobility gap. Preprint (2004)

Figotin, A., Klein, A.: Localization phenomenon in gaps of the spectrum of random lattice operators. J. Stat. Phys. 75, 997-1021 (1994)

Figotin, A., Klein, A.: Localization of electromagnetic and acoustic waves in random media. Lattice model. J. Stat. Phys. 76, 985-1003 (1994)

Figotin, A., Klein, A.: Localization of classical waves I: Acoustic waves. Commun. Math. Phys. 180, 439-482 (1996)

Figotin, A., Klein, A.: Localization of classical waves II: Electromagnetic waves. Commun. Math. Phys. 184, 411-441 (1997)

Fröhlich, J., Martinelli, F., Scoppola, E., Spencer, T.: Constructive proof of localization in the Anderson tight binding model. Commun. Math. Phys. 101, 21-46 (1985)

Fröhlich, J., Spencer, T.: Absence of diffusion with Anderson tight binding model for large disorder or low energy. Commun. Math. Phys. 88, 151-184 (1983)
Germinet, F.: Dynamical localization II with an application to the almost Mathieu operator. J. Stat Phys. 95, 273-286 (1999)

Germinet, F., De Bièvre, S.: Dynamical localization for discrete and continuous random Schrödinger operators. Commun. Math. Phys. 194, 323-341 (1998)

Germinet, F., Klein, A.: Bootstrap Multiscale Analysis and Localization in Random Media. Commun. Math. Phys. 222, 415-448 (2001).

Germinet, F., Klein, A.: Operator kernel estimates for functions of generalized Schrödinger operators. Proc. Amer. Math. Soc. 131, 911-920 (2003).

Germinet, F, Klein, A.: Explicit finite volume criteria for localization in continuous random media and applications. Geom. Funct. Anal. 13, 1201-1238 (2003)

Germinet, F, Klein, A.: High disorder localization for random Schrödinger operators through explicit finite volume criteria. Markov Process. Related Fields. 9, 633-650 (2003)

Germinet, F., Klein, A.: A characterization of the Anderson metal-insulator transport transition. Duke Math. J. 124, 309-351 (2004).

Germinet, F., Klein, A.: Localization zoology for Schrödinger operators. In preparation.

Germinet, F, Klein, A., Schenker, J.: Dynamical delocalization in random Landau Hamiltonians. Preprint (2005)

Kirsch, W., Stollman, P., Stolz, G.: Localization for random perturbations of periodic Schrödinger operators. Random Oper. Stochastic Equations 6, 241-268 (1998)

Klein, A.: Multiscale analysis and localization of random operators. In Random Schrodinger operators: methods, results, and perspectives. Panorama & Synthèse, Société Mathématique de France. To appear.

Klein, A., Koines, A., Seifert, M.: Generalized eigenfunctions for waves in inhomogeneous media. J. Funct. Anal. 190, 255-291 (2002)
[KIK] Klein, A., Koines, A.: A general framework for localization of classical waves: II. Random media. Math. Phys. Anal. Geom. 7, 151-185 (2004)

[KILS] Klein, A., Lacroix, J., Speis, A.: Localization for the Anderson model on a strip with singular potentials. J. Funct. Anal. 94, 135-155 (1990)

[KIM] Klein, A., Molchanov, S.: Simplicity of eigenvalues in the Anderson model. Preprint.

[Klo1] Klopp, F.: Localization for continuous random Schrödinger operators. Commun. Math. Phys. 167, 553-569 (1995)

[Klo2] Klopp, F.: Weak disorder localization and Lifshitz tails. Commun. Math. Phys. 232,125-155 (2002)

[Klo3] Klopp, F.: Weak disorder localization and Lifshitz tails: continuous Hamiltonians. Ann. I.H.P. 3, 711-737 (2002)

[M] Minami, N.: Local fluctuation of the spectrum of a multidimensional Anderson tight binding model. Comm. Math. Phys. 177, 709–725 (1996)

[PF] Pastur, L., Figotin, A.: Spectra of Random and Almost-Periodic Operators. Heidelberg: Springer-Verlag, 1992

[RS1] Reed, M., Simon, B.: Methods of Modern Mathematical Physics I: Functional Analysis, revised and enlarged edition. Academic Press, 1980

[S] Simon, B.: Cyclic vectors in the Anderson model. Special issue dedicated to Elliott H. Lieb. Rev. Math. Phys. 6, 1183-1185 (1994)

[Sp] Spencer, T.: Localization for random and quasiperiodic potentials. J. Stat. Phys. 51, 1009-1019 (1988)

[St] Stollmann, P.: Caught by disorder. Bound States in Random Media. Birkhäuser 2001.

[T] Tcheremchantsev, S.: How to prove dynamical localization, Commun. Math. Phys. 221,27-56 (2001)

[W1] Wang, W.-M.: Microlocalization, percolation, and Anderson localization for the magnetic Schrödinger operator with a random potential. J. Funct. Anal. 146, 1-26 (1997)

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[W2] Wang, W.-M.: Localization and universality of Poisson statistics for the multidimensional Anderson model at weak disorder. Invent. Math. 146, 365-398 (2001)