Black Tori Solutions in Einstein and 5D Gravity

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Abstract

The 'anholonomic frame' method [1–3] is applied for constructing new classes of exact solutions of vacuum Einstein equations with off–diagonal metrics in 4D and 5D gravity. We examine several black tori solutions generated by anholonomic transforms with non–trivial topology of the Schwarzshild metric, which have a static toroidal horizon. We define ansatz and parametrizations which contain warping factors, running constants (in time and extra dimension coordinates) and effective nonlinear gravitational polarizations. Such anisotropic vacuum toroidal metrics, the first example was given in [1], differ substantially from the well known toroidal black holes [4] which were constructed as non–vacuum solutions of the Einstein–Maxwell gravity with cosmological constant. Finally, we analyze two anisotropic 5D and 4D black tori solutions with cosmological constant.

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I. INTRODUCTION

Black hole - torus systems [3] and toroidal black holes [1,4] became objects of astrophysical interest since it was shown that they are inevitable outcome of complete gravitational collapse of a massive star, cluster of stars, or can be present in the center of galactic systems.

Black hole and black tori solutions appear naturally as exact solutions in general relativity and extra dimension gravity theories. Such solutions can be constructed in both asymptotically flat spacetimes and in spacetimes with cosmological constant, posses a specific supersymmetry and could be with toroidal, cylindrical or planar topology [4].

String theory suggests that we may live in a fundamentally higher dimensional spacetime [6]. The recent approaches are based on the assumption that our Universe is realized as a three dimensional (in brief, 3D) brane, modeling a 4D pseudo–Riemannian spacetime, embedded in the 5D anti–de Sitter (AdS₅) bulk spacetime. It was proposed in the Rundall

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and Sundrum (RS) papers \[7\] that such models could be with relatively large extra dimension as a way to solve the hierarchy problem in high energy physics.

In the present paper we explore possible black tori solutions in 5D and 4D gravity. We obtain a new class of exact solutions to the 5D vacuum Einstein equations in the bulk, which have toroidal horizons and are related via anholonomic transforms with toroidal deformations of the Schwarzchild solutions. The solutions could be with warped factors, running constants and anisotropic gravitational polarizations. We then consider 4D black tori solutions and generalize both 5D and 4D constructions for spacetimes with cosmological constant.

We also discuss implications of existence of such anisotropic black tori solutions with non-trivial topology to the extra dimension gravity and general relativity theory. We prove that warped metrics can be obtained from vacuum 5D gravity and not only from a brane configurations with specific energy–momentum tensor.

We apply the Salam, Strathee and Peracci \[8\] idea on a gauge field like status of the coefficients of off–diagonal metrics in extra dimension gravity and develop it in a new fashion by applying the method of anholonomic frames with associated nonlinear connections on 5D and 4D (pseudo) Riemannian spaces \[1–3\].

We use the term 'locally anisotropic' spacetime (or 'anisotropic' spacetime) for a 5D (4D) pseudo–Riemannian spacetime provided with an anholonomic frame structure with mixed holonomic and anholonomic variables. The anisotropy of gravitational interactions is modeled by off–diagonal metrics, or, equivalently, by theirs diagonalized analogs given with respect to anholonomic frames.

The paper is organized as follow: In Sec. II we consider two off–diagonal metric ansatz, construct the corresponding exact solutions of 5D vacuum Einstein equations and illustrate the possibility of extension by introducing matter fields and the cosmological constant term. In Sec. III we construct two classes of 5D anisotropic black tori solutions and consider subclasses and reparametizations of such solutions in order to generate new ones. Sec. IV is devoted to 4D black tori solutions. In Sec. V we extend the approach for anisotropic 5D and 4D spacetimes with cosmological constant and give two examples of 5D and 4D anisotropic black tori solution. Finally, in Sec. VI, we conclude and discuss the obtained results.

II. OFF–DIAGONAL METRIC ANSATZ

We introduce the basic denotations and two ansatz for off–diagonal 5D metrics (see details in Refs. \[1–3\]) to be applied in definition of anisotropic black tori solutions.

Let us consider a 5D pseudo–Riemannian spacetime provided with local coordinates \(u^a = (x^i, y^4, y^5)\), for indices like \(i, j, k, .. = 1, 2, 3\) and \(a, b, ... = 4, 5\). The \(x^i\)–coordinates are called holonomic and \(y^a\)–coordinates are called anholonomic (anisotropic); they are given respectively with respect to some holonomic and anholonomic subframes (see the formulae \(4\) and \(8\)). Every coordinate \(x^i\) or \(y^a\) could be a time like, 3D space, or the 5th (extra dimensional) coordinate; we shall fix on necessity different parametrizations.

We investigate two classes of 5D metrics:

The first type of metrics are given by a line element

\[
d s^2 = g_{\alpha\beta} \left( x^i, v \right) du^\alpha du^\beta
\]  

(1)
with the metric coefficients $g_{\alpha\beta}$ parametrized with respect to the coordinate co-frame $du^\alpha$, being dual to $\partial_\alpha = \partial/\partial u^\alpha$, by an off-diagonal matrix (ansatz)

$$
\begin{bmatrix}
  g_1 + w_1^2 h_4 + n_1^2 h_5 & w_1 w_2 h_4 + n_1 n_2 h_5 & w_1 w_3 h_4 + n_1 n_3 h_5 & w_1 h_4 & n_1 h_5 \\
  w_1 w_2 h_4 + n_1 n_2 h_5 & g_2 + w_2^2 h_4 + n_2^2 h_5 & w_2 w_3 h_4 + n_2 n_3 h_5 & w_2 h_4 & n_2 h_5 \\
  w_1 w_3 h_4 + n_1 n_3 h_5 & w_2 w_3 h_4 + n_2 n_3 h_5 & g_3 + w_3^2 h_4 + n_3^2 h_5 & w_3 h_4 & n_3 h_5 \\
  w_1 h_4 & w_2 h_4 & w_3 h_4 & h_4 & 0 \\
  n_1 h_5 & n_2 h_5 & n_3 h_5 & 0 & h_5 
\end{bmatrix},
$$

where the coefficients are some necessary smoothly class functions of type:

$$
g_1 = \pm 1, g_{2,3} = g_{2,3}(x^2, x^3), h_{4,5} = h_{4,5}(x^i, v),$$

$$w_i = w_i(x^i, v), n_i = n_i(x^i, v).$$

The second type of metrics are given by a line element (with a conformal factor $\Omega(x^i, v)$ and additional deformations of the metric via coefficients $\zeta_i(x^i, v)$, indices with 'hat' take values like $i = 1, 2, 3, 5$) written as

$$
ds^2 = \Omega^2(x^i, v)\hat{g}_{\alpha\beta}(x^i, v) du^\alpha du^\beta,
$$

were the coefficients $\hat{g}_{\alpha\beta}$ are parametrized by the ansatz:

$$
\begin{bmatrix}
  g_1 + (w_1^2 + \zeta_1^2) h_4 + n_1^2 h_5 & (w_1 w_2 + \zeta_1 \zeta_2) h_4 + n_1 n_2 h_5 & (w_1 w_3 + \zeta_1 \zeta_3) h_4 + n_1 n_3 h_5 & (w_1 + \zeta_1) h_4 & n_1 h_5 \\
  (w_1 w_2 + \zeta_1 \zeta_2) h_4 + n_1 n_2 h_5 & g_2 + (w_2^2 + \zeta_2^2) h_4 + n_2^2 h_5 & (w_2 w_3 + \zeta_2 \zeta_3) h_4 + n_2 n_3 h_5 & (w_2 + \zeta_2) h_4 & n_2 h_5 \\
  (w_1 w_3 + \zeta_1 \zeta_3) h_4 + n_1 n_3 h_5 & (w_2 w_3 + \zeta_2 \zeta_3) h_4 + n_2 n_3 h_5 & g_3 + (w_3^2 + \zeta_3^2) h_4 + n_3^2 h_5 & (w_3 + \zeta_3) h_4 & n_3 h_5 \\
  (w_1 + \zeta_1) h_4 & (w_2 + \zeta_2) h_4 & (w_3 + \zeta_3) h_4 & h_4 & 0 \\
  n_1 h_5 & n_2 h_5 & n_3 h_5 & 0 & h_5 + \zeta_4 h_4 
\end{bmatrix}.
$$

For trivial values $\Omega = 1$ and $\zeta_i = 0$, the line interval (3) transforms into (7).

The quadratic line element (7) with metric coefficients (2) can be diagonalized,

$$
\delta s^2 = [g_1(dx^1)^2 + g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\delta v)^2 + h_5(\delta y^5)^2],
$$

with respect to the anholonomic co–frame $(dx^i, \delta v, \delta y^5)$, where

$$\delta v = dv + w_i dx^i$$

and

$$\delta y^5 = dy^5 + n_i dx^i$$

which is dual to the frame $(\delta_i, \partial_4, \partial_5)$, where

$$\delta_i = \partial_i + w_i \partial_4 + n_i \partial_5.$$

The bases (3) and (7) are considered to satisfy some anholonomic relations of type

$$\delta_i \delta_j - \delta_j \delta_i = W^k_{ij} \delta_k$$

for some non–trivial values of anholonomic coefficients $W^k_{ij}$. We obtain a holonomic (coordinate) base if the coefficients $W^k_{ij}$ vanish.

The quadratic line element (7) with metric coefficients (2) can be also diagonalized,

$$
\delta s^2 = \Omega^2(x^i, v)[g_1(dx^1)^2 + g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\hat{\delta} v)^2 + h_5(\hat{\delta} y^5)^2],
$$

where

$$
\hat{\delta} = \delta - \Omega^{-1}\partial_\alpha \
$$

is the anholonomic co-frame.
but with respect to another anholonomic co–frame \( \left( dx^i, \delta v, \delta y^5 \right) \), with

\[
\delta v = dv + (w_i + \zeta_i) dx^i + \zeta_5 \delta y^5 \quad \text{and} \quad \delta y^5 = dy^5 + n_i dx^i
\]  

(10)

which is dual to the frame \( \left( \hat{\delta}_i, \partial_4, \hat{\delta}_5 \right) \), where

\[
\hat{\delta}_i = \partial_i - (w_i + \zeta_i) \partial_4 + n_i \partial_5, \quad \hat{\delta}_5 = \partial_5 - \zeta_5 \partial_4.
\]  

(11)

The nontrivial components of the 5D Ricci tensor, \( R^\beta_\alpha \), for the metric (5) given with respect to anholonomic frames (6) and (7) are

\[
R^2_2 = R^3_3 = -\frac{1}{2g_2g_3} \left[ g_{3}^{**} - g_{2}^{*}g_{3}^{*} - \frac{(g_{3}^{*})^2}{2g_2} + g'' - \frac{g_2'g_3}{2g_3} - \frac{(g_{2}^{*})^2}{2g_2} \right],
\]  

(12)

\[
R^4_4 = R^5_5 = -\frac{\beta}{2h_4h_5},
\]  

(13)

\[
R_{4i} = -\frac{w_i}{2h_5} - \frac{\alpha_i}{2h_5},
\]  

(14)

\[
R_{5i} = -\frac{h_5}{2h_4} \left[ n_i^{**} + \gamma n_i^{*} \right]
\]  

(15)

where

\[
\alpha_i = \partial_i h_5^* - h_5^* \partial_i \ln \sqrt{|h_4h_5|}, \quad \beta = h_5^{**} - h_5^{*} \ln \sqrt{|h_4h_5|}^*, \quad \gamma = 3h_5^*/2h_5 - h_4^*/h_4.
\]  

(16)

For simplicity, the partial derivatives are denoted like \( a^* = \partial a/\partial x^1, a^* = \partial a/\partial x^2, a' = \partial a/\partial x^3, a^* = \partial a/\partial v \).

We obtain the same values of the Ricci tensor for the second ansatz (9) if there are satisfied the conditions

\[
\hat{\delta}_ih_4 = 0 \quad \text{and} \quad \hat{\delta}_i\Omega = 0
\]  

(17)

and the values \( \zeta_i = (\zeta_i, \zeta_5 = 0) \) are found as to be a unique solution of (17); for instance, if

\[
\Omega^{q_1/q_2} = h_4 \quad (q_1 \text{ and } q_2 \text{ are integers}),
\]  

(18)

the coefficients \( \zeta_i \) must solve the equations

\[
\partial_4 \Omega - (w_i + \zeta_i)\Omega^* = 0.
\]  

(19)

The system of 5D vacuum Einstein equations, \( R^\beta_\alpha = 0 \), reduces to a system of nonlinear equations with separation of variables,

\[
R^2_2 = 0, \quad R^4_4 = 0, \quad R_{4i} = 0, \quad R_{5i} = 0,
\]  

which together with (19) can be solved in general form [3]: For any given values of \( g_2 \) (or \( g_3 \)), \( h_4 \) (or \( h_5 \)) and \( \Omega \), and stated boundary conditions we can define consequently the set of metric coefficients \( g_3 \) (or \( g_2 \)), \( h_4 \) (or \( h_5 \)), \( w_i \), \( n_i \) and \( \zeta_i \).
The introduced ansatz can be used also for constructing solutions of 5D and 4D Einstein equations with nontrivial energy-momentum tensor

\[ R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \kappa \Upsilon_{\alpha\beta}. \]

The non–trivial diagonal components of the Einstein tensor, \( G^\alpha_\beta = R^\alpha_\beta - \frac{1}{2} R \delta^\alpha_\beta \), for the metric (3), given with respect to anholonomic frames, are

\[ G^1_1 = - \left( R^2_2 + S^4_4 \right), \quad G^2_2 = G^3_3 = - S^4_4, \quad G^4_4 = G^5_5 = - R^2_2. \] (20)

So, we can extend the system of 5D vacuum Einstein equations by introducing matter fields for which the energy–momentum tensor \( \Upsilon_{\alpha\beta} \) given with respect to anholonomic frames satisfy the conditions

\[ \Upsilon^1_1 = \Upsilon^2_2 + \Upsilon^4_4, \quad \Upsilon^2_2 = \Upsilon^3_3, \quad \Upsilon^4_4 = \Upsilon^5_5. \] (21)

We note that, in general, the tensor \( \Upsilon_{\alpha\beta} \) may be not symmetric because with respect to anholonomic frames there are imposed constraints which makes non symmetric the Ricci and Einstein tensors (1)–(3).

In the simplest case we can consider a "vacuum" source induced by a non–vanishing 4D cosmological constant, \( \Lambda \). In order to satisfy the conditions (21) the source induced by \( \Lambda \) should be in the form \( \kappa \Upsilon_{\alpha\beta} = (2\Lambda g_{11}, \Lambda g_{22}) \), where underlined indices \( \alpha, \beta, \ldots \) run 4D values 2, 3, 4, 5. We note that in 4D anholonomic gravity the source \( \kappa \Upsilon_{\alpha\beta} = \Lambda g_{\alpha\beta} \) satisfies the equalities \( \Upsilon^2_2 = \Upsilon^3_3 = \Upsilon^4_4 = \Upsilon^5_5 \).

By straightforward computations we obtain that the nontrivial components of the 5D Einstein equations with anisotropic cosmological constant, \( R_{11} = 2\Lambda g_{11} \) and \( R_{22} = \Lambda g_{22} \), for the ansatz (4) and anholonomic metric (9) given with respect to anholonomic frames (10) and (11), are written in a form with separated variables:

\[ g^\ast_3 = g^\ast_2 g^\ast_3 - \frac{(g^\ast_3)^2}{2g_3} + g^\ast_2 - \frac{g^\ast_2 g^\ast_3}{2g_3} - \frac{(g^\ast_2)^2}{2g_2} = 2\Lambda g_2 g_3, \] (22)

\[ h^\ast_5 - h^\ast_5 \ln \sqrt{|h_4 h_5|} = 2\Lambda h_4 h_5, \] (23)

\[ w_i, + \alpha_i = 0, \] (24)

\[ n^\ast_i + \gamma n^\ast_i = 0, \] (25)

\[ \partial_i \Omega - (w_i + \zeta_i) \Omega = 0. \] (26)

where

\[ \alpha_i = \partial_i h^\ast_5 - h^\ast_5 \partial_i \ln \sqrt{|h_4 h_5|}, \beta = 2\Lambda h_4 h_5, \gamma = 3h^\ast_5/2h_5 - h^\ast_4/h_4. \] (27)

In the vacuum case (with \( \Lambda = 0 \)) these equations are compatible if \( \beta = \alpha_i = 0 \) which results that \( w_i (x^i, v) \) could be arbitrary functions; this reflects a freedom in definition of the holonomic coordinates. For simplicity, for vacuum solutions we shall put \( w_i = 0 \). Finally, we remark that we can "select" 4D Einstein solutions from an ansatz (2) or (4) by considering that the metric coefficients do not depend on variable \( x^1 \), which mean that in the system of equations (22)–(26) we have to deal with 4D values \( w_i (x^\bot, v), n^\ast_i (x^\bot, v), \zeta_i (x^\bot, v), \) and \( h_4 (x^\bot, v), h_5 (x^\bot, v), \Omega (x^\bot, v) \).
III. 5D BLACK TORI

Our goal is to apply the anholonomic frame method as to construct such exact solutions of vacuum (and with cosmological constant) 5D Einstein equations as they have a static toroidal horizon for a metric ansatz (2) or (4) which can be diagonalized with respect to some well defined anholonomic frames. Such solutions are defined as some anholonomic transforms of the Schwarzschild solution to a toroidal configuration with non–trivial topology. In general form, they could be defined with warped factors, running constants (in time and extra dimension coordinate) and nonlinear polarizations.

A. Toroidal deformations of the Schwarzschild metric

Let us consider the system of isotropic spherical coordinates \((\rho, \theta, \varphi)\), where the isotropic radial coordinate \(\rho\) is related with the usual radial coordinate \(r\) via the relation

\[
r = \rho (1 + r_g/4\rho)^2 \quad \text{for} \quad r_g = 2Gm_0/c^2
\]

being the 4D gravitational radius of a point particle of mass \(m_0\), \(G = 1/M^2_P\) is the 4D Newton constant expressed via Plank mass \(M_P\) (following modern string/brane theories, \(M_P\) can be considered as a value induced from extra dimensions). We put the light speed constant \(c = 1\).

This system of coordinates is considered for the so–called isotropic representation of the Schwarzschild solution [10]

\[
ds^2 = \left(\frac{\hat{\rho} - 1}{\hat{\rho} + 1}\right)^2 dt^2 - \rho_g^2 \left(\frac{\hat{\rho} + 1}{\hat{\rho}}\right)^4 \left(d\hat{\rho}^2 + \hat{\rho}^2 d\theta^2 + \hat{\rho}^2 \sin^2\theta d\varphi^2\right),
\]

where, for our further considerations, we re–scaled the isotropic radial coordinate as \(\hat{\rho} = \rho/\rho_g\), with \(\rho_g = r_g/4\). The metric (28) is a vacuum static solution of 4D Einstein equations with spherical symmetry describing the gravitational field of a point particle of mass \(m_0\). It has a singularity for \(r = 0\) and a spherical horizon for \(r = r_g\), or, in re–scaled isotropic coordinates, for \(\hat{\rho} = 1\). We emphasize that this solution is parametrized by a diagonal metric given with respect to holonomic coordinate frames.

We also introduce the toroidal coordinates (in our case considered as alternatives to the isotropic radial coordinates) \((\sigma, \tau, \varphi)\), running values \(-\pi \leq \sigma < \pi, 0 \leq \tau \leq \infty, 0 \leq \varphi < 2\pi\), which are related with the isotropic 3D Cartezian coordinates via transforms

\[
\begin{align*}
\tilde{x} &= \frac{\hat{\rho} \sinh \tau}{\cosh \tau - \cos \sigma} \cos \varphi, \\
\tilde{y} &= \frac{\hat{\rho} \sinh \tau}{\cosh \tau - \cos \sigma} \sin \varphi, \\
\tilde{z} &= \frac{\hat{\rho} \sin \sigma}{\cosh \tau - \cos \sigma}
\end{align*}
\]

and define a toroidal hypersurface

\[
\left(\sqrt{\tilde{x}^2 + \tilde{y}^2} - \rho \cosh \tau \sinh \tau\right)^2 + \tilde{z}^2 = \frac{\rho^2}{\sinh^2 \tau}.
\]

The 3D metric on a such toroidal hypersurface is

\[
ds^2_{(3D)} = g_{\sigma\sigma} d\sigma^2 + g_{\tau\tau} d\tau^2 + g_{\varphi\varphi} d\varphi^2,
\]

where
\[ g_{\sigma\sigma} = g_{\tau\tau} = \frac{\tilde{\rho}^2}{(\cosh \tau - \cos \sigma)^2}, \quad g_{\varphi\varphi} = \frac{\tilde{\rho}^2 \sinh^2 \tau}{(\cosh \tau - \cos \sigma)^2}. \]

We can relate the toroidal coordinates \((\sigma, \tau, \varphi)\) from [29] with the isotropic radial coordinates \((\tilde{\rho}, \theta, \varphi)\), scaled by the constant \(\rho_g\), from [28] as
\[
\tilde{\rho} = 1, \quad \sinh^{-1} \tau = \tilde{\rho}
\]
and transform the Schwarzschild solution into a new metric with toroidal coordinates by changing the 3D radial line element into the toroidal one and stating the \(tt\)-coefficient of the metric to have a toroidal horizon. The resulting metric is
\[
ds^2_{(S)} = \left(\sinh \frac{\tau - 1}{\sinh \tau + 1}\right)^2 dt^2 - \tilde{\rho}^2 \frac{(\sinh \tau + 1)^4}{(\cosh \tau - \cos \sigma)^2} \left( d\sigma^2 + d\tau^2 + \sinh^2 \tau d\varphi^2 \right),
\]
(30)

Such deformed Schwarzchild like toroidal metric is not an exact solution of the vacuum Einstein equations, but at long radial distances it transform into usual Schwarzschild solution with the 3D line element parametrized by toroidal coordinates.

For our further considerations we introduce two Classes (A and B) of 4D auxiliary pseudo-Riemannian metrics, also given in toroidal coordinates, being some conformal transforms of (30), like
\[
ds^2_{(S)} = \Omega_{A,B} (\sigma, \tau) ds^2_{(A,B)},
\]
but which are not supposed to be solutions of the Einstein equations:

- Metric of Class A:
\[
ds^2_{(A)} = -d\sigma^2 - d\tau^2 + a(\tau) d\varphi^2 + b(\sigma, \tau) dt^2, \quad (31)
\]
where
\[
a(\tau) = -\sinh^2 \tau \quad \text{and} \quad b(\sigma, \tau) = -\frac{(\sinh \tau - 1)^2 (\cosh \tau - \cos \sigma)^2}{\rho_g^2 (\sinh \tau + 1)^6},
\]
which results in the metric (30) by multiplication on the conformal factor
\[
\Omega_A (\sigma, \tau) = \rho_g^2 \frac{(\sinh \tau + 1)^4}{(\cosh \tau - \cos \sigma)^2}. \quad (32)
\]

- Metric of Class B:
\[
ds^2_{(B)} = g(\tau) \left( d\sigma^2 + d\tau^2 \right) - d\varphi^2 + f(\sigma, \tau) dt^2, \quad (33)
\]
where
\[
g(\tau) = -\sinh^{-2} \tau \quad \text{and} \quad f(\sigma, \tau) = \rho_g^2 \left( \frac{\sinh^2 \tau - 1}{\cosh \tau - \cos \sigma} \right)^2,
\]
which results in the metric (30) by multiplication on the conformal factor
\[
\Omega_B (\sigma, \tau) = \rho_g^{-2} \frac{(\cosh \tau - \cos \sigma)^2}{(\sinh \tau + 1)^2}.
\]
We shall use the metrics (30), (31) and (33) in order to generate exact solutions of
the Einstein equations with toroidal horizons and anisotropic polarizations and running of
constants by performing corresponding anholonomic transforms as the solutions will have
a horizon parametrized by a torus hypersurface and gravitational (extra dimensional, or
nonlinear 4D) renormalizations of the constant \( \rho_g \) of the Schwarzschild solution, \( \rho_g \to \overline{\rho}_g = \omega \rho_g \), where the dependence of the function \( \omega \) on some holonomic or anholonomic coordinates
will depend on the type of anisotropy. For some solutions we shall treat \( \omega \) as a factor
modeling running of the gravitational constant, induced, induced from extra dimension,
in another cases we will consider \( \omega \) as a nonlinear gravitational polarization which models
some anisotropic distributions of masses and matter fields and/or anholonomic vacuum
gravitational interactions.

B. Toroidal 5D metrics of Class A

In this subsection we consider four classes of 5D vacuum solutions which are related to
the metric of Class A (31) and to the toroidally deformed Schwarzschild metric (30).
Let us parametrize the 5D coordinates as \((x^1 = \chi, x^2 = \sigma, x^3 = \tau, y^4 = v, y^5 = p)\), where
the solutions with the so–called \( \varphi \)–anisotropy will be constructed for \((v = \varphi, p = t)\) and the
solutions with \( t \)–anisotropy will be stated for \((v = t, p = \varphi)\) (in brief, we write respectively,
\( \varphi \)–solutions and \( t \)–solutions).

1. Class A of vacuum solutions with ansatz (2):

We take an off–diagonal metric ansatz of type (2) (equivalently, (1 )) by representing
\[ g_1 = \pm 1, g_2 = -1, g_3 = -1, h_4 = \eta_4(\sigma, \tau, v)h_{4(0)}(\sigma, \tau) \text{ and } h_5 = \eta_5(\sigma, \tau, v)h_{5(0)}(\sigma, \tau), \]
where \( \eta_{4,5}(\sigma, \tau, v) \) are corresponding "gravitational renormalizations" of the metric coefficients
\( h_{4,5(0)}(\sigma, \tau) \). For \( \varphi \)–solutions we state \( h_{4(0)} = a(\tau) \) and \( h_{5(0)} = b(\sigma, \tau) \) (inversely, for
\( t \)–solutions, \( h_{4(0)} = b(\sigma, \tau) \) and \( h_{5(0)} = a(\sigma, \tau) \)).

Next we consider a renormalized gravitational 'constant' \( \overline{\rho}_g = \omega \rho_g \), were for \( \varphi \)–solutions the receptivity \( \omega = \omega(\sigma, \tau, v) \) is included in the gravitational polarization \( \eta_5 \) as
\( \eta_5 = [\omega(\sigma, \tau, \varphi)]^{-\frac{1}{2}} \), or for \( t \)–solutions is included in \( \eta_4 \), when \( \eta_4 = [\omega(\sigma, \tau, t)]^{-\frac{1}{2}} \). We can construct an exact solution of the 5D vacuum Einstein equations if, for explicit dependencies
on anisotropic coordinate, the metric coefficients \( h_4 \) and \( h_5 \) are related by the equation (23),
which in its turn imposes a corresponding relation between \( \eta_4 \) and \( \eta_5 \),
\[ \eta_4 h_{4(0)} = \frac{\sqrt{\eta_5}}{h_{5(0)}} \left( \left[ \sqrt{|\eta_5|} \right]^{\frac{1}{2}} \right)^2, \quad h_{2(0)}^2 = \text{const.} \]
In result, we express the polarizations \( \eta_4 \) and \( \eta_5 \) via the value of receptivity \( \omega \),
\[ \eta_4 (\chi, \sigma, \tau, \varphi) = h_{2(0)}^2 \frac{b(\sigma, \tau)}{a(\tau)} \left\{ \left[ \omega^{-1}(\chi, \sigma, \tau, \varphi) \right]^{\frac{1}{2}} \right\}^2, \eta_5 (\chi, \sigma, \tau, \varphi) = \omega^{-2}(\chi, \sigma, \tau, \varphi), \quad (34) \]
for \( \varphi \)–solutions , and
\[ \eta_4 (\chi, \sigma, \tau, t) = \omega^{-2} (\chi, \sigma, \tau, t), \eta_5 (\chi, \sigma, \tau, t) = h_{00}^{-2} \frac{b(\sigma, \tau)}{a(\tau)} \left[ \int dt \omega^{-1} (\chi, \sigma, \tau, t) \right]^2, \]  

(35)

for \( t \)-solutions, where \( a(\tau) \) and \( b(\sigma, \tau) \) are those from (31).

For vacuum configurations, following (24), we put \( w_i = 0 \). The next step is to find the values of \( n_i \) by introducing \( h_4 = \eta_4 h_{4(0)} \) and \( h_5 = \eta_5 h_{5(0)} \) into the formula (24), which, for convenience, is expressed via general coefficients \( \eta_4 \) and \( \eta_5 \). After two integrations on variable \( v \), we obtain the exact solution

\[ n_k(\sigma, \tau, v) = n_{k[1]}(\sigma, \tau) + n_{k[2]}(\sigma, \tau) \int [\eta_4/(\sqrt{\eta_5})]^3 dv, \eta_5^* \neq 0; \]

(36)

\[ = n_{k[1]}(\sigma, \tau) + n_{k[2]}(\sigma, \tau) \int \eta_4 dv, \eta_5^* = 0; \]

\[ = n_{k[1]}(\sigma, \tau) + n_{k[2]}(\sigma, \tau) \int [1/(\sqrt{\eta_5})]^3 dv, \eta_5^* = 0, \]

with the functions \( n_{k[2]}(\sigma, \tau) \) are defined as to contain the values \( h_{00}^2, a(\tau) \) and \( b(\sigma, \tau) \).

By introducing the formulas (34) for \( \varphi \)-solutions (or (33) for \( t \)-solutions) and fixing some boundary condition, in order to state the values of coefficients \( n_{k[1,2]}(\sigma, \tau) \) we can express the ansatz components \( n_k(\sigma, \tau, \varphi) \) as integrals of some functions of \( \omega(\sigma, \tau, \varphi) \) and \( \partial_\varphi \omega(\sigma, \tau, \varphi) \) (or, we can express the ansatz components \( n_k(\sigma, \tau, t) \) as integrals of some functions of \( \omega(\sigma, \tau, t) \) and \( \partial_t \omega(\sigma, \tau, t) \)). We do not present an explicit form of such formulas because they depend on the type of receptivity \( \omega = \omega(\sigma, \tau, v) \), which must be defined experimentally, or from some quantum models of gravity in the quasi classical limit. We preserved a general dependence on coordinates \( (\sigma, \tau) \) which reflect the fact that there is a freedom in fixing holonomic coordinates (for instance, on a toroidal hypersurface and its extensions to 4D and 5D spacetimes). For simplicity, we write that \( n_i \) are some functionals of \( \{\sigma, \tau, \omega(\sigma, \tau, v), \omega^*(\sigma, \tau, v)\} \)

\[ n_i \{\sigma, \tau, \omega, \omega^*\} = n_i \{\sigma, \tau, \omega(\sigma, \tau, v), \omega^*(\sigma, \tau, v)\}. \]

(37)

In conclusion, we constructed two exact solutions of the 5D vacuum Einstein equations, defined by the ansatz (2) with coordinates and coefficients stated by the data:

\[ \varphi \text{-solutions} : (x^1 = \chi, x^2 = \sigma, x^3 = \tau, y^4 = v = \varphi, y^5 = p = t), g_1 = \pm 1, \]

\[ g_2 = -1, g_3 = -1, h_4(0) = a(\tau), h_5(0) = b(\sigma, \tau), \text{see (31)}; \]

\[ h_4 = \eta_4(\sigma, \tau, \varphi) h_{4(0)}, h_5 = \eta_5(\sigma, \tau, \varphi) h_{5(0)}, \]

\[ \eta_4 = h_{00}^2 \frac{b(\sigma, \tau)}{a(\tau)} \left\{ \left[ \omega^{-1}(\chi, \sigma, \tau, \varphi) \right]^* \right\}^2, \eta_5 = \omega^{-2}(\chi, \sigma, \tau, \varphi), \]

\[ w_i = 0, n_i \{\sigma, \tau, \omega, \omega^*\} = n_i \{\sigma, \tau, \omega(\sigma, \tau, \varphi), \omega^*(\sigma, \tau, \varphi)\}. \]

(37)

and

\[ t \text{-solutions} : (x^1 = \chi, x^2 = \sigma, x^3 = \tau, y^4 = v = t, y^5 = p = \varphi), g_1 = \pm 1, \]

\[ g_2 = -1, g_3 = -1, h_4(0) = b(\sigma, \tau), h_5(0) = a(\tau), \text{see (31)}; \]

\[ h_4 = \eta_4(\sigma, \tau, t) h_{4(0)}, h_5 = \eta_5(\sigma, \tau, t) h_{5(0)}, \]

\[ \eta_4 = \omega^{-2}(\chi, \sigma, \tau, t), \eta_5 = h_{00}^{-2} \frac{b(\sigma, \tau)}{a(\tau)} \left\{ \int dt \omega^{-1}(\chi, \sigma, \tau, t) \right\}^2, \]

\[ w_i = 0, n_i \{\sigma, \tau, \omega, \omega^*\} = n_i \{\sigma, \tau, \omega(\sigma, \tau, t), \omega^*(\sigma, \tau, t)\}. \]

(38)
Both types of solutions have a horizon parametrized by torus hypersurface (as the condition of vanishing of the "time" metric coefficient states, i.e. when the function \( b(\sigma, \tau) = 0 \)). These solutions are generically anholonomic (anisotropic) because in the locally isotropic limit, when \( n_\alpha, n_\beta, \omega \rightarrow 1 \) and \( n_i \rightarrow 0 \), they reduce to the coefficients of the metric \( (31) \). The last one is not an exact solution of 4D vacuum Einstein equations, but it is a conformal transform of the 4D Schwarzschild metric deformed to a torus horizon, with a further trivial extension to 5D. With respect to the anholonomic frames adapted to the coefficients \( n_i \) (see \( (6) \)), the obtained solutions have diagonal metric coefficients being very similar to the metric \( (30) \) written in toroidal coordinates. We can treat such solutions as black tori with the mass distributed linearly on the circle which can not transformed in a point, in the center of torus.

The solutions are constructed as to have singularities on the mentioned circle. The initial data for anholonomic frames and the chosen configuration of gravitational interactions in the bulk lead to deformed toroidal horizons even for static configurations. The solutions admit anisotropic polarizations on toroidal coordinates \((\sigma, \tau)\) and running of constants on time \( t \) and/or on extra dimension coordinate \( \chi \). Such renormalizations of constants are defined by the nonlinear configuration of the 5D vacuum gravitational field and depend on introduced receptivity function \( \omega(\sigma, \tau, v) \) which is to be considered an intrinsic characteristics of the 5D vacuum gravitational 'ether', emphasizing the possibility of nonlinear self-polarization of gravitational fields.

Finally, we point that the data \((37)\) and \((38)\) parametrize two very different classes of solutions. The first one is for static 5D vacuum black tori configurations with explicit dependence on anholonomic coordinate \( \phi \) and possible renormalizations on the rest of 3D space coordinates \( \sigma \) and \( \tau \) and on the 5th coordinate \( \chi \). The second class of solutions is similar to the static ones but with an emphasized anholonomic running on time of constants and with possible anisotropic dependencies on coordinates \((\sigma, \tau, \chi)\).

2. Class A of vacuum solutions with ansatz \((4)\):

We construct here 5D vacuum \( \varphi \)- and \( t \)-solutions parametrized by an ansatz with conformal factor \( \Omega(\sigma, \tau, v) \) (see \((1)\) and \((2)\)). Let us consider conformal factors parametrized as \( \Omega = \Omega_{[0]}(\sigma, \tau)\Omega_{[1]}(\sigma, \tau, v) \). We can generate from the data \((37)\) (or \((38)\)) an exact solution of vacuum Einstein equations if there are satisfied the conditions \((17), (18) \) and \((19)\), i.e.

\[
\Omega_{[0]}^{q_1/q_2} \Omega_{[1]}^{q_1/q_2} = \eta_4 h_{4(0)},
\]

for some integers \( q_1 \) and \( q_2 \), and there are defined the second anisotropy coefficients

\[
\zeta_i = \left( \partial_i \ln |\Omega_{[0]}| \right) \left( \ln |\Omega_{[1]}| \right)^* + \left( \Omega_{[1]}^* \right)^{-1} \partial_i \Omega_{[1]}.
\]

So, taking a \( \varphi \)- or \( t \)-solution with corresponding values of \( h_4 = \eta_4 h_{4(0)} \), for some \( q_1 \) and \( q_2 \), we obtain new exact solutions, called in brief, \( \varphi_c \)- or \( t_c \)-solutions (with the index "c" pointing to an ansatz with conformal factor), of the vacuum 5D Einstein equations given in explicit form by the data:

\( \varphi_c \)-solutions : \((x^1 = \chi, x^2 = \sigma, x^3 = \tau, y^4 = v = \varphi, y^5 = p = t), g_1 = \pm 1, g_2 = -1, g_3 = -1, h_{4(0)} = a(\tau), h_{5(0)} = b(\sigma, \tau), \) see \((31)\);
\[ h_4 = \eta_4(\sigma, \tau, \varphi)h_{4(0)}, h_5 = \eta_5(\sigma, \tau, \varphi)h_{5(0)}, \]
\[ \eta_4 = h_{4(0)}^2 \frac{b(\sigma, \tau)}{a(\tau)} \{ \left[ \omega^{-1}(\chi, \sigma, \tau, \varphi) \right]^* \}^2, \eta_5 = \omega^{-2}(\chi, \sigma, \tau, \varphi), \] (39)
\[ w_i = 0, n_i\{\sigma, \tau, \omega, \omega^*\} = n_i\{\sigma, \tau, \omega(\sigma, \tau, \varphi), \omega^*(\sigma, \tau, \varphi)\}, \Omega = \Omega_{[0]}(\sigma, \tau)\Omega_{[1]}(\sigma, \tau, \varphi) \]
\[ \zeta_i = \left( \partial_i \ln |\Omega_{[0]}| \right) \left( \ln |\Omega_{[1]}| \right)^* + \left( \Omega_{[1]}^* \right)^{-1} \partial_i \Omega_{[1]}, \eta_4 a = \Omega_{[0]}^{q_1/q_2}(\sigma, \tau)\Omega_{[1]}^{q_1/q_2}(\sigma, \tau, \varphi). \]

and
\[ t_c\text{-solutions : } (x^1 = \chi, x^2 = \sigma, x^3 = \tau, y^4 = v = t, y^5 = p = \varphi), g_1 = \pm 1, \]
\[ g_2 = -1, g_3 = -1, h_{4(0)} = b(\sigma, \tau), h_{5(0)} = a(\tau), \text{see (31)}; \]
\[ h_4 = \eta_4(\sigma, \tau, t)h_{4(0)}, h_5 = \eta_5(\sigma, \tau, t)h_{5(0)}, \]
\[ \eta_4 = \omega^{-2}(\chi, \sigma, \tau, t), \eta_5 = h_{4(0)}^{-2}b(\sigma, \tau) \left[ \int dt \omega^{-1}(\chi, \sigma, t) \right]^2, \] (40)
\[ w_i = 0, n_i\{\sigma, \tau, \omega, \omega^*\} = n_i\{\sigma, \tau, \omega(\sigma, \tau, t), \omega^*(\sigma, \tau, t)\}, \Omega = \Omega_{[0]}(\sigma, \tau)\Omega_{[1]}(\sigma, \tau, t) \]
\[ \zeta_i = \left( \partial_i \ln |\Omega_{[0]}| \right) \left( \ln |\Omega_{[1]}| \right)^* + \left( \Omega_{[1]}^* \right)^{-1} \partial_i \Omega_{[1]}, \eta_4 a = \Omega_{[0]}^{q_1/q_2}(\sigma, \tau)\Omega_{[1]}^{q_1/q_2}(\sigma, \tau, t). \]

These solutions have two very interesting properties: 1) they admit a warped factor on the 5th coordinate, like \( \Omega_{[1]}^{q_1/q_2} \sim \exp[-k|\chi|] \), which in our case is constructed for an anisotropic 5D vacuum gravitational configuration and not following a brane configuration like in Refs. [12]; 2) we can impose such conditions on the receptivity \( \omega(\sigma, \tau, v) \) as to obtain in the locally isotropic limit just the toroidally deformed Schwarzschild metric (30) trivially embedded into the 5D spacetime.

We analyze the second property in details. We have to chose the conformal factor as to be satisfied three conditions:
\[ \Omega_{[0]}^{q_1/q_2} = \Omega_A, \Omega_{[1]}^{q_1/q_2} \eta_4 = 1, \Omega_{[1]}^{q_1/q_2} \eta_5 = 1, \] (41)
were \( \Omega_A \) is that from (32). The last two conditions are possible if
\[ \eta_4^{-q_1/q_2} \eta_5 = 1, \] (42)
which selects a specific form of receptivity \( \omega(x^i, v) \). Putting into (42) the values \( \eta_4 \) and \( \eta_5 \) respectively from (39), or (40), we obtain some differential, or integral, relations of the unknown \( \omega(\sigma, \tau, v) \), which results that
\[ \omega(\sigma, \tau, \varphi) = (1 - q_1/q_2)^{-1/q_1/q_2} h_{4(0)} b(\sigma, \tau) \left[ \frac{1}{\omega_{[0]}(\sigma, \tau)} \right]^{q_1/q_2}, \text{ for } \varphi_c\text{-solutions}; \] (43)
\[ \omega(\sigma, \tau, \varphi) = \left( q_1/q_2 - 1 \right) h_{4(0)} \sqrt{|\omega_{[1]} - \omega_{[0]}(\sigma, \tau)|} \left[ \omega_{[0]}(\sigma, \tau) \right]^{1/q_1/q_2}, \text{ for } t_c\text{-solutions,} \]
for some arbitrary functions \( \omega_{[0]}(\sigma, \tau) \) and \( \omega_{[1]}(\sigma, \tau) \). So, receptivities of particular form like (43) allow us to obtain in the locally isotropic limit just the toroidally deformed Schwarzschild metric.

We conclude this subsection: the vacuum 5D metrics solving the Einstein equations describe a nonlinear gravitational dynamics which under some particular boundary conditions and parametrizations of metric’s coefficients can model anisotropic, topologically not trivial, solutions transforming, in a corresponding locally isotropic limit, in some toroidal
or ellipsoidal deformations of the well known exact solutions like Schwarzschild, Reissner-Nördstrom, Taub NUT, various type of wormhole, solitonic and disk solutions (see details in Refs. [1–3]). We emphasize that, in general, an anisotropic solution (parametrized by an off–diagonal ansatz) could not have a locally isotropic limit to a diagonal metric with respect to some holonomic coordinate frames. This was proved in explicit form by choosing a configuration with toroidal symmetry.

C. Toroidal 5D metrics of Class B

In this subsection we construct and analyze another two classes of 5D vacuum solutions which are related to the metric of Class B (33) and which can be reduced to the toroidally deformed Schwarzschild metric (34) by corresponding parametrizations of receptivity \( \varphi (\sigma, \tau, v) \). We emphasize that because the function \( g(\sigma, \tau) \) from (33) is not a solution of equation (22) we introduce an auxiliary factor \( \varpi (\sigma, \tau) \) which are related to the metric of Class B (33) and which can be reduced to the toroidally deformed Schwarzshild metric (30) by corresponding parametrizations of receptivity \( \varphi (\sigma, \tau, v) \).

Because the method of definition of such solutions is similar to that from previous subsection, in our further considerations we shall omit computations and present directly the data which select the respective configurations for \( \varphi_c \)-solutions and \( t_c \)-solutions.

The Class B of 5D solutions with conformal factor are parametrized by the data:

\[ \varphi_c \text{-solutions} : (x^1 = \chi, x^2 = \sigma, x^3 = \tau, y^4 = v = \varphi, y^5 = p = t), \ g_1 = \pm 1, \]

\[ g_2 = g_3 = \varpi (\sigma, \tau)g(\sigma, \tau), h_{4(0)} = -\varpi (\sigma, \tau), h_{5(0)} = \varpi (\sigma, \tau)f(\sigma, \tau), \text{see} (33); \]

\[ \varpi = g^{-1}(\sigma, \tau)\varpi_0 \exp[a_2 \sigma + a_3 \tau], \ \varpi_0, a_2, a_3 = \text{const}; \]

\[ h_4 = \eta_4(\sigma, \tau, \varphi)h_{4(0)}, h_5 = \eta_5(\sigma, \tau, \varphi)h_{5(0)}; \]

\[ \eta_4 = -h^2_{(0)}f(\sigma, \tau) \left\{ [\varomega^{-1}(\chi, \sigma, \tau, \varphi)]^* \right\}^2, \ \eta_5 = \varomega^{-2}(\chi, \sigma, \tau, \varphi), \]

\[ w_i = 0, n_i \{\sigma, \tau, \omega, \omega^*\} = n_i \{\sigma, \tau, \omega(\sigma, \tau, \varphi), \omega^*(\sigma, \tau, \varphi)\}, \ \Omega = \varomega^{-1}(\sigma, \tau)\Omega_{[2]}(\sigma, \tau, \varphi) \]

\[ \zeta_i = \partial_i \ln |\varomega| \left( \ln |\Omega_{[2]}| \right)^* + \left( \Omega^*_{[2]} \right)^{-1} \partial_i \Omega_{[2]}, \eta_4 = -\varomega^{-n_1/q_2}(\sigma, \tau)\Omega^{n_1/q_2}_{[2]}(\sigma, \tau, \varphi). \]

and

\[ t_c \text{-solutions} : (x^1 = \chi, x^2 = \sigma, x^3 = \tau, y^4 = v = t, y^5 = p = \varphi), g_1 = \pm 1, \]

\[ g_2 = g_3 = \varpi (\sigma, \tau)g(\sigma, \tau), h_{4(0)} = \varpi (\sigma, \tau)f(\sigma, \tau), h_{5(0)} = -\varpi (\sigma, \tau), \text{see} (33); \]

\[ \varpi = g^{-1}(\sigma, \tau)\varpi_0 \exp[a_2 \sigma + a_3 \tau], \ \varpi_0, a_2, a_3 = \text{const}; \]

\[ h_4 = \eta_4(\sigma, \tau, t)h_{4(0)}, h_5 = \eta_5(\sigma, \tau, t)h_{5(0)}; \]

\[ \eta_4 = \varomega^{-2}(\chi, \sigma, \tau, t), \ \eta_5 = -h^2_{(0)}f(\sigma, \tau) \left[ \int dt \ \omega^{-1}(\chi, \sigma, \tau, t) \right]^2; \]

\[ w_i = 0, n_i \{\sigma, \tau, \omega, \omega^*\} = n_i \{\sigma, \tau, \omega(\sigma, \tau, \varphi), \omega^*(\sigma, \tau, \varphi)\}, \ \Omega = \varomega^{-1}(\sigma, \tau)\Omega_{[2]}(\sigma, \tau, t) \]

\[ \zeta_i = \partial_i (\ln |\varomega|) \left( \ln |\Omega_{[2]}| \right)^* + \left( \Omega^*_{[2]} \right)^{-1} \partial_i \Omega_{[2]}, \eta_4 = -\varomega^{-n_1/q_2}(\sigma, \tau)\Omega^{n_1/q_2}_{[2]}(\sigma, \tau, t). \]

where the coefficients \( n_i \) can be found explicitly by introducing the corresponding values \( \eta_4 \) and \( \eta_5 \) in formula (30).
By a procedure similar to the solutions of Class A (see previous subsection) we can find the conditions when the solutions (44) and (45) will have in the locally anisotropic limit the toroidally deformed Schwarzschild solutions, which impose corresponding parametrizations and dependencies on \( \Omega_{[2]}(\sigma, \tau, v) \) and \( \omega(\sigma, \tau, v) \) like (11) and (13). We omit these formulas because, in general, for anholonomic configurations and nonlinear solutions there are not hard arguments to prefer any holonomic limits of such off–diagonal metrics.

Finally, in this Section, we remark that for the considered classes of black tori solutions the so–called \( t \)–components of metric contain modifications of the Schwarzschild potential

\[
\Phi = -\frac{M}{P[4]} r \quad \text{into} \quad \Phi = -\frac{M \omega(\sigma, \tau, v)}{P[4]} r,
\]

where \( P[4] \) is the usual 4D Plank constant, and this is given with respect to the corresponding anholonomic frame of reference. The receptivity \( \omega(\sigma, \tau, v) \) could model corrections warped on extra dimension coordinate, \( \chi \), which for our solutions are induced by anholonomic vacuum gravitational interactions in the bulk and not from a brane configuration in \( AdS_5 \) spacetime. In the vacuum case \( k \) is a constant which characterizes the receptivity for bulk vacuum gravitational polarizations.

**IV. 4D BLACK TORI**

For the ansatz (2), with trivial conformal factor, a black torus solution of 4D vacuum Einstein equations was constructed in Ref. [1]. The goal of this Section is to consider some alternative variants, with trivial or nontrivial conformal factors and for different coordinate parametrizations and types of anisotropies. The bulk of 5D solutions from the previous Section are reduced into corresponding 4D ones if we eliminate the 5th coordinate \( \chi \) from the the off–diagonal ansatz (2) and (4) and corresponding formulas and solutions.

**A. Toroidal 4D vacuum metrics of Class A**

Let us parametrize the 4D coordinates as \((x^2, y^a) = (x^2 = \sigma, x^3 = \tau, y^4 = v, y^5 = p)\); for the \( \varphi \)–solutions we shall take \((v = \varphi, p = t)\) and for the solutions \( t \)–solutions will consider \((v = t, p = \varphi)\). For simplicity, we write down the data for solutions without proofs and computations.

1. **Class A of vacuum solutions with ansatz (2):**

The off–diagonal metric ansatz of type (2) (equivalently, (3)) with the data

- \( \varphi \)–solutions : \((x^2 = \sigma, x^3 = \tau, y^4 = v = \varphi, y^5 = p = t)\)
- \( g_2 = -1, g_3 = -1, h_{4(0)} = a(\tau), h_{5(0)} = b(\sigma, \tau), \) see (31);
- \( h_4 = 4_4(\sigma, \tau, \varphi)h_{4(0)}, h_5 = 5_5(\sigma, \tau, \varphi)h_{5(0)}, \)
- \( \eta_4 = h_4^2 \frac{b(\sigma, \tau)}{a(\tau)} \left\{ \left[ \frac{w(\sigma, \tau, \varphi)}{a(\tau)} \right]^* \right\}^2, \eta_5 = \omega^2(\sigma, \tau, \varphi), \)
- \( w_i = 0, n_2^i(\sigma, \tau, \omega, \omega^*) = n_2^i(\sigma, \tau, \omega(\sigma, \tau, \varphi), \omega^*(\sigma, \tau, \varphi)). \)
\begin{align}
t-solutions : \ (x^2 = \sigma, x^3 = \tau, y^4 = v = t, y^5 = p = \varphi) \\
g_2 = -1, g_3 = -1, h_{4(0)} = b(\sigma, \tau), h_{5(0)} = a(\tau), \text{see (31)}; \\
h_4 = \eta_4(\sigma, \tau, t) h_{4(0)}, h_5 = \eta_5(\sigma, \tau, t) h_{5(0)}, \\
\eta_4 = \omega^{-2}(\sigma, \tau, t), \eta_5 = h_{5(0)}^{-1} b(\sigma, \tau) \left[ \int dt \ \omega^{-1}(\sigma, \tau, t) \right]^2, \\
w_i = 0, n_i \{ \sigma, \tau, \omega, \omega^* \} = n_i \{ \sigma, \tau, \omega(\sigma, \tau, t), \omega^*(\sigma, \tau, t) \}. \tag{47}
\end{align}

where the \( n_i \) are computed
\begin{align}
n_k(\sigma, \tau, v) &= n_{k[1]}(\sigma, \tau) + n_{k[2]}(\sigma, \tau) \int [\eta_4/(\sqrt{|\eta_5|})]^{3}] dv, \eta_5^* \neq 0; \tag{48} \\
 &= n_{k[1]}(\sigma, \tau) + n_{k[2]}(\sigma, \tau) \int \eta_4 dv, \eta_5^* = 0; \\
 &= n_{k[1]}(\sigma, \tau) + n_{k[2]}(\sigma, \tau) \int [1/(\sqrt{|\eta_5|})]^{3}] dv, \eta_5^* = 0.
\end{align}

when the integration variable is taken \( v = \varphi \), for \( (13) \), or \( v = t \), for \( (17) \). These solutions have the same toroidal symmetries and properties stated for their 5D analogs \( (37) \) and for \( (38) \) with that difference that there are not any warped factors and extra dimension dependencies. Such solutions defined by the formulas \( (14) \) and \( (17) \) do not result in a locally isotropic limit into an exact solution having diagonal coefficients with respect to some holonomic coordinate frames. The data introduced in this subsection are for generic 4D vacuum solutions of the Einstein equations parametrized by off–diagonal metrics. The renormalization of constants and metric coefficients have a 4D nonlinear vacuum gravitational nature and reflects a corresponding anholonomic dynamics.

2. Class A of vacuum solutions with ansatz \( (4) \):

The 4D vacuum \( \varphi^* \) and \( t \)-solutions parametrized by an ansatz with conformal factor \( \Omega(\sigma, \tau, v) \) (see \( (4) \) and \( (9) \)). Let us consider conformal factors parametrized as \( \Omega = \Omega_{[0]}(\sigma, \tau) \Omega_{[1]}(\sigma, \tau, v) \). The data are
\begin{align}
\varphi^*-solutions : \ (x^2 = \sigma, x^3 = \tau, y^4 = v = \varphi, y^5 = p = t) \\
g_2 = -1, g_3 = -1, h_{4(0)} = a(\tau), h_{5(0)} = b(\sigma, \tau), \text{see (31)}; \\
h_4 = \eta_4(\sigma, \tau, \varphi) h_{4(0)}, h_5 = \eta_5(\sigma, \tau, \varphi) h_{5(0)}, \Omega = \Omega_{[0]}(\sigma, \tau) \Omega_{[1]}(\sigma, \tau, \varphi), \\
\eta_4 = h_{4(0)}^{-1} b(\sigma, \tau) \left[ \omega^{-1}(\sigma, \tau, \varphi) \right]^{*} \eta_5 = \omega^{-2}(\sigma, \tau, \varphi), \tag{49} \\
w_i = 0, n_i \{ \sigma, \tau, \omega, \omega^* \} = n_i \{ \sigma, \tau, \lambda, \omega(\sigma, \tau, \varphi), \omega^*(\sigma, \tau, \varphi) \}, \\
\zeta_i = \left( \partial_i \ln|\Omega_{[0]}| \right) \left( \ln|\Omega_{[1]}| \right)^* + \left( \Omega_{[1]} \right)^{1/2} \partial_i \Omega_{[1]}, \eta_4 a = \Omega_{[0]}^{n_1/q_2}(\sigma, \tau) \Omega_{[1]}^{n_2/q_2}(\sigma, \tau, \varphi).
\end{align}

and
\( t_{c}\)-solutions: \((x^2 = \sigma, x^3 = \tau, y^4 = v = t, y^5 = p = \varphi)\)

\[g_2 = -1, g_3 = -1, h_{4(0)} = b(\sigma, \tau), h_{5(0)} = a(\tau), \text{see (31)};\]
\[h_4 = \eta_4(\sigma, \tau, t)h_{4(0)}, h_5 = \eta_5(\sigma, \tau, t)h_{5(0)}, \Omega = \Omega_{[0]}(\sigma, \tau)\Omega_{[1]}(\sigma, \tau, t),\]
\[\eta_4 = \omega^{-2}(\sigma, \tau, t), \eta_5 = h_{5(0)}^{-2}b(\sigma, \tau)\left(\int dt \; \omega^{-1}(\sigma, \tau, t)\right)^2,\]
\[w_i = 0, n_4\{\sigma, \tau, \omega, \omega^*\} = n_5\{\sigma, \tau, \omega(\sigma, \tau, t), \omega^*(\sigma, \tau, t)\},\]
\[\zeta_i = \left(\partial_i \ln |\Omega_{[0]}|\right) \left(\ln |\Omega_{[1]}|\right)^* + \left(\Omega_{[1]}^{-1}\partial_i \Omega_{[1]}\right), \eta_4 = \Omega_{[0]}^{q_1/q_2}(\sigma, \tau)\Omega_{[1]}^{q_1/q_2}(\sigma, \tau, t),\]

where the coefficients the \( n_i \) are given by the same formulas (43).

Contrary to the solutions (46) and for (47) theirs conformal anholonomic transforms, respectively, (49) and (50), can be subjected to such parametrizations of the conformal factor and conditions on the receptivity \( \omega(\sigma, \tau, v) \) as to obtain in the locally isotropic limit just the toroidally deformed Schwarzschild metric (30). These conditions are stated for \( \eta_4^{-q_1/q_2}\eta_5 = 1 \), which selects a specific form of the receptivity \( \omega \). Putting the values \( \eta_4 \) and \( \eta_5 \), respectively, from (49), or (50), we obtain some differential, or integral, relations of the unknown \( \omega(\sigma, \tau, v) \), which results that

\[\omega(\sigma, \tau, \varphi) = (1 - q_1/q_2)^{-1-q_1/q_2} \left[h_{-1(0)}^{-1}\sqrt{|a/b|\varphi + \omega_{[0]}(\sigma, \tau)}\right], \text{for } \varphi_{c}\]-solutions;
\[\omega(\sigma, \tau, t) = \left[(q_1/q_2 - 1) h_{0(0)} \sqrt{|a/b|t + \omega_{[1]}(\sigma, \tau)}\right]^{1-q_1/q_2}, \text{for } t_{c}\]-solutions,

for some arbitrary functions \( \omega_{[0]}(\sigma, \tau) \) and \( \omega_{[1]}(\sigma, \tau) \). The obtained formulas for \( \omega(\sigma, \tau, \varphi) \) and \( \omega(\sigma, \tau, t) \) are 4D reductions of the formulas (11) and (43).

**B. Toroidal 4D vacuum metrics of Class B**

We construct another two classes of 4D vacuum solutions which are related to the metric of Class B (33) and can be reduced to the toroidally deformed Schwarzschild metric (30) by corresponding parametrizations of receptivity \( \omega(\sigma, \tau, v) \). The solutions contain a 2D conformal factor \( \varpi(\sigma, \tau) \) for which \( \varpi \) becomes a solution of (23) and a 4D conformal factor parametrized as \( \Omega = \varpi^{-1}(\sigma, \tau)\Omega_{[2]}(\sigma, \tau, v) \) in order to set the constructions into the ansatz (4) and anholonomic metric interval (3).

The data selecting the 4D configurations for \( \varphi_{c}\)-solutions and \( t_{c}\)-solutions:

\( \varphi_{c}\)-solutions: \((x^2 = \sigma, x^3 = \tau, y^4 = v = \varphi, y^5 = p = t)\)

\[g_2 = g_3 = \varpi(\sigma, \tau)g(\sigma, \tau), h_{4(0)} = -\varpi(\sigma, \tau), h_{5(0)} = \varpi(\sigma, \tau)f(\sigma, \tau), \text{see (33)};\]
\[\varpi = g^{-1} \varpi \exp[a_2\sigma + a_3\tau], \quad \varpi, a_2, a_3 = \text{const};\]
\[h_4 = \eta_4(\sigma, \tau, \varphi)h_{4(0)}, h_5 = \eta_5(\sigma, \tau, \varphi)h_{5(0)}, \Omega = \varpi^{-1}(\sigma, \tau)\Omega_{[2]}(\sigma, \tau, \varphi)\]
\[\eta_4 = -h_{0(0)}^{-2}f(\sigma, \tau) \left[\left(\omega^{-1}(\sigma, \tau, \varphi)\right)^*\right]^2, \eta_5 = \omega^{-2}(\sigma, \tau, \varphi),\]
\[w_i = 0, n_4\{\sigma, \tau, \omega, \omega^*\} = n_5\{\sigma, \tau, \omega(\sigma, \tau, \varphi), \omega^*(\sigma, \tau, \varphi)\},\]
\[\zeta_i = \partial_i \ln |\varpi| \left(\ln |\Omega_{[2]}|\right)^* + \left(\Omega_{[2]}^{-1}\partial_i \Omega_{[2]}\right), \eta_4 = -\varpi^{-q_1/q_2}(\sigma, \tau)\Omega_{[2]}^{q_1/q_2}(\sigma, \tau, \varphi).\]
and

\( t_c \)-solutions : \((x^2 = \sigma, x^3 = \tau, y^4 = v = t, y^5 = p = \varphi)\)

\begin{align*}
g_2 &= g_3 = \varpi(\sigma, \tau) \varphi(\sigma, \tau), \quad h_{4(0)} = \varpi(\sigma, \tau) f(\sigma, \tau), \quad h_{5(0)} = -\varpi(\sigma, \tau), \text{ see (33)}; \\
\varpi &= g^{-1}_{00} \exp[a_2 \sigma + a_3 \tau], \quad \varpi_0, a_2, a_3 = \text{const}, \\
h_4 &= \eta_4(\sigma, \tau, t) h_{4(0)}, \quad h_5 = \eta_5(\sigma, \tau, t) h_{5(0)}, \quad \Omega = \varpi^{-1}(\sigma, \tau) \Omega[2](\sigma, \tau, t) \\
\eta_4 &= \omega^{-2}(\sigma, \tau, t), \quad \eta_5 = -\varpi^{-2}_{(0)} f(\sigma, \tau) \left[ \int dt \omega^{-1}(\sigma, \tau, t) \right]^{2}, \\
w_i &= 0, n_i{\{\sigma, \tau, \omega, \omega^*\}} = n_i{\{\sigma, \tau, \omega(\sigma, \tau, t), \omega^*(\sigma, \tau, t)\}}, \\
\zeta_i &= \partial_i(\ln |\varpi|) \left( \ln |\Omega[2]| \right)^{*} + \left( \Omega[2]^* \right)^{-1} \partial_i \Omega[2], \quad \eta_4 = -\omega^{-q_1/q_2}(\sigma, \tau) \Omega_{[2]}^{q_1/q_2}(\sigma, \tau, t).
\end{align*}

where the coefficients \( n_i \) can be found explicitly by introducing the corresponding values \( \eta_4 \) and \( \eta_5 \) in formula (36).

For the 4D Class B solutions, some conditions can be imposed (see previous subsection) when the solutions (51) and (52) have in the locally anisotropic limit the toroidally deformed Schwarzschild solution, which imposes some specific parametrizations and dependencies on \( \Omega[2](\sigma, \tau, \omega) \) and \( \omega(\sigma, \tau, \omega) \) like (11) and (13). We omit these considerations because for anholonomic configurations and nonlinear solutions there are not arguments to prefer any holonomic limits of such off–diagonal metrics.

We conclude this Section by noting that for the constructed classes of 4D black tori solutions the so–called \( t \)–component of metric contains modifications of the Schwarzschild potential

\[ \Phi = -\frac{M}{M_{P[4]}^2 r} \]

\[ \text{into } \Phi = -\frac{M \omega(\sigma, \tau, v)}{M^2_{P[4]} r}, \]

where \( M_{P[4]} \) is the usual 4D Plank constant; the metric coefficients are given with respect to the corresponding anholonomic frame of reference. In 4D anholonomic gravity the receptivity \( \omega(\sigma, \tau, v) \) is considered to renormalize the mass constant. Such gravitational self–polarizations are induced by anholonomic vacuum gravitational interactions. They should be defined experimentally or computed following a model of quantum gravity.

V. THE COSMOLOGICAL CONSTANT AND ANISOTROPY

In this Section we analyze the general properties of anholonomic Einstein equations in 5D and 4D gravity with cosmological constant and consider two examples of 5D and 4D exact solutions.

A non–vanishing \( \Lambda \) term in the system of Einstein’s equations instrucutes substantial differences because \( t \beta \neq 0 \) and, in this case, one could be \( w_i \neq 0 \); The equations (22) and (23) are of more general nonlinearity because of presence of the \( 2 \Lambda g_2 g_3 \) and \( 2 \Lambda h_4 h_5 \) terms. In this case, the solutions with \( g_2 = \text{const} \) and \( g_3 = \text{const} \) (and \( h_4 = \text{const} \) and \( h_5 = \text{const} \)) are not admitted. This makes more sophisticate the procedure of definition of \( g_2 \) for a stated \( g_3 \) (or inversely, of definition of \( g_3 \) for a stated \( g_2 \)) from (22) [similarly of constructing \( h_4 \) for a given \( h_5 \) from (23) and inversely], nevertheless, the separation of variables is not affected.
by introduction of cosmological constant and there is a number of possibilities to generate exact solutions.

The general properties of solutions of the system (22)–(26), with cosmological constant \( \Lambda \), are stated in the form:

- The general solution of equation (24) is to be found from the equation
  \[
  \varpi \varpi'' - (\varpi^*)^2 + \varpi \varpi' - (\varpi')^2 = 2\Lambda \varpi^3.
  \]
  for a coordinate transform coordinate transforms \( x^{2,3} \to \tilde{x}^{2,3} (u, \lambda) \) for which
  \[
  g_2(\sigma, \tau)(d\sigma)^2 + g_3(\sigma, \tau)(d\tau)^2 \to \varpi \left[(d\tilde{x}^2)^2 + \epsilon(dx^3)^2\right], \epsilon = \pm 1
  \]
  and \( \varpi^* = \partial \varpi / \partial \tilde{x}^2 \) and \( \varpi' = \partial \varpi / \partial \tilde{x}^3 \).

- The equation (23) relates two functions \( h_4 (\sigma, \tau, v) \) and \( h_5 (\sigma, \tau, v) \) with \( h_5^* \neq 0 \). If the function \( h_5 \) is given we can find \( h_4 \) as a solution of
  \[
  h_4^* + \frac{2\Lambda}{\pi} (h_4)^2 + 2 \left( \frac{\pi^*}{\pi} - \pi \right) h_4 = 0,
  \]
  where \( \pi = h_5^*/2h_5 \).

- The exact solutions of (24) for \( \beta \neq 0 \) is
  \[
  w_k = -\alpha_k / \beta,
  \]
  \[
  = \partial_k \ln \left[ \sqrt{|h_4 h_5|/|h_5^*|} \right] / \partial v \ln \left[ \sqrt{|h_4 h_5|/|h_5^*|} \right],
  \]
  for \( \partial_v = \partial / \partial v \) and \( h_5^* \neq 0 \).

- The exact solution of (25) is
  \[
  n_k = n_{k[1]} (\sigma, \tau) + n_{k[2]} (\sigma, \tau) \int [h_4/(\sqrt{|h_5^*|})^3] dv,
  \]
  \[
  = n_{k[1]} (\sigma, \tau) + n_{k[2]} (\sigma, \tau) \int [1/(\sqrt{|h_5^*|})^3] dv, \ h_4^* = 0,
  \]
  for some functions \( n_{k[1,2]} (\sigma, \tau) \) stated by boundary conditions.

- The exact solution of (26) is given by
  \[
  \zeta_i = -w_i + (\Omega^*)^{-1} \partial_i \Omega, \ \Omega^* \neq 0,
  \]
We note that by a corresponding re-parametrizations of the conformal factor $\Omega (\sigma, \tau, v)$ we can reduce (53) to

$$\varpi \varpi^{**} - (\varpi^*)^2 = 2\Lambda \varpi^3$$

(58)

which gives and exact solution $\varpi = \varpi (\tilde{x}^2)$ found from

$$(\varpi^*)^2 = \varpi^3 \left( C \varpi^{-1} + 4\Lambda \right), C = \text{const},$$

(or, inversely, to reduce to

$$\varpi \varpi'' - (\varpi')^2 = 2\Lambda \varpi^3$$

with exact solution $\varpi = \varpi (\tilde{x}^3)$ found from

$$(\varpi')^2 = \varpi^3 \left( C \varpi^{-1} + 4\Lambda \right), C = \text{const}).$$

The inverse problem of definition of $h_5$ for a given $h_4$ can be solved in explicit form when $h_4^* = 0$, $h_4 = h_{4(0)}(\sigma, \tau)$. In this case we have to solve

$$h_5^{**} + \frac{(h_5^*)^2}{2h_5} - 2\Lambda h_{4(0)}h_5 = 0,$$

(59)

which admits exact solutions by reduction to a Bernulli equation.

The outlined properties of solutions with cosmological constant hold also for 4D anholonomic spacetimes with ”isotropic” cosmological constant $\Lambda$. To transfer general solutions from 5D to 4D we have to eliminate dependencies on the coordinate $x^1$ and to consider the 4D ansatz without $g_{11}$ term.

A. A 5D anisotropic black torus solution with cosmological constant

We give an example of generalization of anisotropic black hole solutions of Class A, constructed in the Section III as they will contain the cosmological constant $\Lambda$; we extend the solutions given by the data (39).

Our new 5D $\varphi-$ solution is parametrized by an ansatz with conformal factor $\Omega(x^i, v)$ (see (4) and (9)) as $\Omega = \varpi^{-1}(\sigma)\Omega_{[0]}(\sigma, \tau)\Omega_{[1]}(\sigma, \tau, v)$. The factor $\varpi(\sigma, \tau)$ is chosen as to be a solution of (58). This conformal data must satisfy the condition (18), i.e.

$$\varpi^{-q_1/q_2}\Omega_{[0]}^{q_1/q_2}\varpi^{q_1/q_2} = \eta_4 \varpi h_{4(0)}$$

for some integers $q_1$ and $q_2$, where $\eta_4$ is found as $h_4 = \eta_4 \varpi h_{4(0)}$ satisfies the equation (54) and $\Omega_{[0]}(\sigma, \tau)$ could be chosen as to obtain in the locally isotropic limit and $\Lambda \to 0$ the toroidally deformed Schwarzschild metric (30). Choosing $h_5 = \eta_5 \varpi h_{5(0)}$, $\eta_5 h_{5(0)}$ is for the ansatz for (39), for which we compute the value $\pi = h_5^*/2h_5$, we obtain from (54) an equation for $\eta_4$,

$$\eta_4^* + \frac{2\Lambda}{\pi} \varpi h_{4(0)}(\eta_4)^2 + 2\left(\frac{\pi^*}{\pi} - \pi\right) \eta_4 = 0$$

18
which is a Bernulli equation \([14]\) and admit an exact solution, in general, in non explicit form, \(\eta_4 = \eta_4^{[\text{bern}]}(\sigma, \tau, v, \Lambda, \varpi, \omega, a, b)\), were we emphasize the functional dependencies on functions \(\varpi, \omega, a, b\) and cosmological constant \(\Lambda\). Having defined \(\eta_4^{[\text{bern}]}\), \(\eta_5\) and \(\varpi\), we can compute the \(\alpha_i\), \(\beta_i\), and \(\gamma_i\)-coefficients, expressed as \(\alpha_i = \alpha_i^{[\text{bern}]}(\sigma, \tau, v, \Lambda, \varpi, \omega, a, b), \beta = \beta^{[\text{bern}]}(\sigma, \tau, v, \Lambda, \varpi, \omega, a, b)\) and \(\gamma = \gamma^{[\text{bern}]}(\sigma, \tau, v, \Lambda, \varpi, \omega, a, b)\), following the formulas \([16]\).

The next step is to find
\[
w_i = w_i^{[\text{bern}]}(\sigma, \tau, v, \Lambda, \varpi, \omega, a, b)\]
and
\[
n_i = n_i^{[\text{bern}]}(\sigma, \tau, v, \Lambda, \varpi, \omega, a, b)
\]
as for the general solutions \([55]\) and \([54]\).

At the final step we are able to compute the the second anisotropy coefficients
\[
\zeta_i = -w_i^{[\text{bern}]} + (\partial_1 \ln |\varpi^{-1}\Omega_0|) \left(\ln |\Omega_0|\right)^* + \left(\Omega_1^*\right)^{-1} \partial_1 \Omega_1,
\]
which depends on an arbitrary function \(\Omega_0(\sigma, \tau)\). If we state \(\Omega_0(\sigma, \tau) = \Omega_A\), as for \(\Omega_A\) from \([33]\), see similar details with respect to formulas \([11]\), \([12]\) and \([13]\).

The data for the exact solutions with cosmological constant for \(v = \varphi\) can be stated in the form

\[
\varphi_c\text{-solutions} : \ (x^1 = \chi, x^2 = \sigma, x^3 = \tau, y^4 = v = \varphi, y^5 = p = t), \ g_1 = \pm 1,
\]
\[
g_2 = \varpi(\sigma), \ g_3 = \varpi(\sigma), \ h_4(0) = a(\tau), \ h_5(0) = b(\sigma, \tau), \ \text{see (31) and (38)}; \]
\[
h_4 = \eta_4(\sigma, \tau, \varpi), h_4(0) = b(\sigma, \tau), \ h_5 = \eta_5(\sigma, \tau, \varpi), \ \text{see (31) and (38)};
\]
\[
\eta_4 = \eta_4^{[\text{bern}]}(\sigma, \tau, v, \Lambda, \varpi, \omega, a, b), \eta_5 = \omega^{-2}(\chi, \sigma, \tau, \varphi),
\]
\[
w_i = w_i^{[\text{bern}]}(\sigma, \tau, v, \Lambda, \varpi, \omega, a, b), \ n_i = \{\sigma, \tau, \omega, \omega^*\} = n_i^{[\text{bern}]}(\sigma, \tau, v, \Lambda, \varpi, \omega, a, b),
\]
\[
\Omega = \varpi^{-1}(\sigma)\Omega_0(\sigma, \tau)\Omega_1(\sigma, \tau, \varphi), \eta_4a = \Omega^{q_1/q_2} \Omega_1^{q_1/q_2}(\sigma, \tau, \varphi).
\]

We note that a solution with \(v = t\) can be constructed as to generalize \([40]\) to the presence of \(\Lambda\). We can not present such data in explicit form because in this case we have to define \(\eta_5\) by solving a solution like \([23]\) for \(h_5\), for a given \(h_4\), which can not be integrated in explicit form.

The solution \([60]\) preserves the two interesting properties of \([39]\): 1) it admits a warped factor on the 5th coordinate, like \(\Omega_1^{q_1/q_2} \sim \exp\{-k|\chi|\}\), which in this case is constructed for an anisotropic 5D vacuum gravitational configuration with anisotropic cosmological constant but not following a brane configuration like in Refs. \([7]\). 2) we can impose such conditions on the receptivity \(\omega(\sigma, \tau, \varphi)\) as to obtain in the locally isotropic limit just the toroidally deformed Schwarzschild metric \([30]\) trivially embedded into the 5D spacetime.

**B. A 4D anisotropic black torus solution with cosmological constant**

The symplest way to construct a such solution is to take the data \([50]\), for \(v = \varphi\), to eliminate the variable \(\chi\) and to reduce the 5D indices to 4D ones. We obtain the 4D data:
\( \varphi_c \)-solutions : \((x^2 = \sigma, x^3 = \tau, y^4 = v = \varphi, y^5 = p = t)\),
\[
\begin{align*}
g_2 &= \varpi(\sigma), g_3 = \varpi(\sigma), h_{4(0)} = a(\tau), h_{5(0)} = b(\sigma, \tau), \text{see (31) and (58)}; \\
h_4 &= \eta_4(\sigma, \tau, \varphi)\varpi(\sigma)h_{4(0)}, h_5 = \eta_5(\sigma, \tau, \varphi)\varpi(\sigma)h_{5(0)}, \\
\eta_4 &= \sqrt[\text{bern}]{\eta_4(\sigma, \tau, \varphi, \Lambda, \varpi, \omega, a, b), \eta_5 = \omega^{-2}(\sigma, \tau, \varphi)}, \\
w_i &= \sqrt[\text{bern}]{w_i(\sigma, \tau, \varpi, \omega, a, b), n_i(\sigma, \tau, \omega, \omega^*) = n_i(\sigma, \tau, \varpi, \omega, a, b),} \\
\Omega &= \varpi^{-1}(\sigma)\Omega_{[0]}(\sigma, \tau)\Omega_{[1]}(\sigma, \tau, \varphi), \eta_4 a = \Omega_{[0]}^{q_1/q_2}(\sigma, \tau)\Omega_{[1]}^{q_1/q_2}(\sigma, \tau, \varphi). \\
\zeta_i &= -w_i^{[\text{bern}]} + \left( \partial_i \ln |\varpi^{-1}\Omega_{[0]}| \right) \left( \ln |\Omega_{[1]}| \right)^* + \left( \Omega_{[1]}^* \right)^{-1} \partial_i \Omega_{[1]}.
\end{align*}
\]

The solution (61) describes a static black torus solution in 4D gravity with cosmological constant, \( \Lambda \). The parameters of solutions depends on the \( \Lambda \) as well are renormalized by nonlinear anholonomic gravitational interactions. We can consider that the mass associated to such toroidal configuration can be anisotropically distributed in the interior of the torus and gravitationally polarized.

Finally, we note that in a similar manner like in the Sections III and IV we can construct another classes of anisotropic black holes solutions in 5D and 4D spacetimes with cosmological constants, being of Class A or Class B, with anisotropic \( \varphi \)-coordinate, or anisotropic \( t \)-coordinate. We omit the explicit data which are some nonlinear anholonomic generalizations of those solutions.

**VI. CONCLUSIONS AND DISCUSSION**

We have shown that static black tori solutions can be constructed both in vacuum Einstein and five dimensional (5D) gravity. The solutions are parametrized by off–diagonal metric ansatz which can diagonalized with respect to corresponding anholonomic frames with mixtures of holonomic and anholonomic variables. Such metrics contain a toroidal horizon being some deformations with non-trivial topology of the Schwarzschild black hole solution.

The solutions were constructed by using the anholonomic frame method \[1, 3\] which results in a very substantial simplification of the Einstein equations which admit general integrals for solutions.

The constructed black tori metrics depend on classes of two dimensional and three dimensional functions which reflect the freedom in definition of toroidal coordinates as well the possibility to state by boundary conditions various configurations with running constants, anisotropic gravitational polarizations and (in presence of extra dimensions) with warping geometries. The new toroidal solutions can be extended for spacetimes with cosmological constant.

In view of existence of such solutions, the old problem of the status of frames in gravity theories rises once again, now in the context of ”effective” diagonalization of off–diagonal metrics by using anholonomic transforms. The bulk of solutions with spherical, cylindrical and plane symmetries were constructed in gravitational theories of diverse dimensions by using diagonal metrics (sometimes with off–diagonal terms) given with respect to ”pure” coordinate frames. Such solutions can be equivalently re-defined with respect to arbitrary frames of reference and usually the problem of fixing some reference bases in order to state
the boundary conditions is an important physical problem but not a dynamical one. This problem becomes more sophisticated when we deal with generic off–diagonal metrics and anholonomic frames. In this case some 'dynamical, components of metrics can be transformed into 'non–dynamical' components of frame bases, which, following a more rigorous mathematical approach, reflects a constrained nonlinear dynamics for gravitational and matter fields with both holonomic (unconstrained) and anholonomic (constrained) variables. In result there are more possibilities in definition of classes of exact solutions with non–trivial topology, anisotropies and nonlinear interactions.

The solutions obtained in this paper contain as particular cases (for corresponding parametrizations of considered ansatz) the 'black ring' metrics with event horizon of topology $S^1 \times S^2$ analyzed in Refs. [12]. In our case we emphasized the presence of off–diagonal terms which results in warping, anisotropy and running of constants. Here it should be noted that the generic nonlinear character of the Einstein equations written with respect to anholonomic frames connected with diagonalization of off–diagonal metrics allow us to construct different classes of exact 5D and 4D solutions with the same or different topology; such solutions can define very different vacuum gravitational and gravitational–matter field configurations.

The method and results presented in this paper provide a prescription on anholonomic transforming of some known locally isotropic solutions from a gravity/string theory into corresponding classes of anisotropic solutions of the same, or of an extended theory:

A vacuum, or non-vacuum, solution, and metrics conformally equivalent to a known solution, parametrized by a diagonal matrix given with respect to a holonomic (coordinate) base, contained as a trivial form of ansatz [3], or [4], can be transformed into a metric with non-trivial topological horizons and then generalized to be an anisotropic solution with similar but anisotropically renormalized physical constants and diagonal metric coefficients, given with respect to adapted anholonomic frames; the new anholonomic metric defines an exact solution of a simplified form of the Einstein equations (22)–(26) and (19); such types of solutions are parametrized by off–diagonal metrics if they are re–defined with respect to usual coordinate frames.

We emphasize that the anholonomic frame method and constructed black tori solutions conclude in a general formalism of generating exact solutions with off–diagonal metrics in gravity theories and may have a number of applications in modern astrophysics and string/M–theory gravity.

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