An isoperimetric inequality for a biharmonic Steklov problem

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Abstract

For the biharmonic Steklov eigenvalue problem considered in this paper, we show that among all bounded Euclidean domains of class $C^1$ with fixed measure, the ball maximizes the first positive eigenvalue.

1 Introduction

The study of isoperimetric problems is a hot topic in Differential Geometry and of course in Spectral Geometry. The famous Faber-Krahn inequality says that the ball minimizes the first Dirichlet eigenvalue of the Laplacian among all domains with fixed measure (see [9, 10]). While Szegő [16, 17] and Weinberg [18] showed that among all domains with fixed measure, the ball maximizes the first nonzero Neumann eigenvalue of the Laplacian. Except Dirichlet and Neumann cases, other boundary conditions for this kind of isoperimetric problems can also be proposed and similar conclusions can be expected (see, e.g., [4]). These conclusions reveal the dependence of eigenvalues of the Laplacian on bounded Euclidean domains with fixed volume. Naturally, one might ask:

Question. Can similar spectral isoperimetric inequalities be obtained for other elliptic operators?

The answer is of course positive and many facts have been known. In fact, even for the biharmonic operator (also called the bi-Laplace operator), although generally the corresponding eigenvalue equations (with different boundary conditions) are fourth-order PDEs, some spectral isoperimetric inequalities can also be achieved. For instance,

- Lord Rayleigh conjectured:

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Among open sets in the Euclidean n-space $\mathbb{R}^n$ with the same measure, the ball mini-
mizes the fundamental tone of the clamped plate problem (i.e., the first eigenvalue of
the biharmonic operator with the Dirichlet and Neumann boundary conditions).

For this conjecture, Nadirashvili [15] solved the case $n = 2$ while Ashbaugh and Benguria
[2] solved the case $n = 3$. However, the case $n \geq 4$ still remains open.

- For a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial \Omega$, $n \geq 2$, Chasman [3] considered
the following eigenvalue problem of free plate

$$
\begin{cases}
\Delta^2 u - \tau \Delta u = \Lambda u & \text{in } \Omega, \\
\frac{\partial^2 u}{\partial v^2} = 0 & \text{on } \partial \Omega, \\
\tau \frac{\partial u}{\partial v} - \text{div}_{\partial \Omega} (\text{Proj}_{\partial \Omega} [(D^2 u)\vec{v}]) - \frac{\partial \Lambda u}{\partial v} = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(1.1)

where $\tau \in \mathbb{R}$, $\Delta$ is the Laplacian, $\vec{v}$ denotes the outward unit normal vector of $\partial \Omega$, $\Delta^2$ is
the biharmonic operator in $\Omega$, div$_{\partial \Omega}$ is the surface divergence on $\partial \Omega$, the operator Proj$_{\partial \Omega}$
projects onto the space tangent to $\partial \Omega$, and $D^2 u$ denotes the Hessian matrix. Physically, when
$n = 2$, $\Omega$ is the shape of a homogeneous, isotropic plate, and the parameter $\tau$ is the ratio of
lateral tension to flexural rigidity of the plate. Positive $\tau$ corresponds to a plate under tension,
while negative $\tau$ gives us a plate under compression. Chasman [3, Section 4] proved that if
$\tau \geq 0$, the operator $\Delta^2 - \tau \Delta$ in the boundary value problem $\text{(1.1)}$ has a discrete spectrum and
all the eigenvalues, with finite multiplicity, in this spectrum can be listed non-decreasingly
as follows

$$
0 = \Lambda_1(\Omega) \leq \Lambda_2(\Omega) \leq \Lambda_3(\Omega) \leq \cdots \uparrow \infty.
$$

She also showed that among all domains with fixed volume, the lowest nonzero eigenvalue
$\Lambda_2(\Omega)$ for a free plate under tension (i.e., $\tau > 0$) is maximized by a ball (see [3, Theorem
1]). Later, Chasman considered the following eigenvalue problem of free plate under tension
and with nonzero Poisson’s ratio

$$
\begin{cases}
\Delta^2 u - \tau \Delta u = \Gamma u & \text{in } \Omega, \\
(1 - \sigma) \frac{\partial^2 u}{\partial v^2} + \sigma \Delta u = 0 & \text{on } \partial \Omega, \\
\tau \frac{\partial u}{\partial v} - (1 - \sigma) \text{div}_{\partial \Omega} (\text{Proj}_{\partial \Omega} [(D^2 u)\vec{v}]) - \frac{\partial \Lambda u}{\partial v} = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(1.2)

where $\sigma$ is the Poisson’s ratio and other same symbols have the same meanings as those in
(1.1). Clearly, if $\sigma = 0$ then the BVP (1.2) degenerates into the BVP (1.1). Naturally, one
might ask “Whether the isoperimetric inequality for the first nonzero eigenvalue $\Lambda_2(\Omega)$ in
the BVP (1.1) can be improved to the case of the BVP (1.2) or not? Is this possible improvement
too direct and without any difficulty?” Chasman [7] gave an answer to these questions in
details. In fact, she (see [7, Section 4]) explained that if $\tau \geq 0$ and $\sigma \in (-1/(n - 1), 1)$, the
operator $\Delta^2 - \tau \Delta$ in the BVP (1.2) has a discrete spectrum and all the eigenvalues, with finite
multiplicity, in this spectrum can be listed non-decreasingly as follows

$$
0 = \Gamma_1(\Omega) \leq \Gamma_2(\Omega) \leq \Gamma_3(\Omega) \leq \cdots \uparrow \infty.
$$

1 BVP for short.
2 For the physical explanation of Poisson’s ratio $\sigma$ in the BVP (1.2), see the footnote on the 2nd page of [14].
She also proved that the ball with the same volume maximizes $\Gamma_2(\Omega)$ if the free plate is under tension and one of the followings holds:

1. $n = 2$ and $\sigma > -51/97$ or $\tau \geq 3(1 - \sigma)/(\sigma + 1)$,
2. $n = 3$,
3. $n \geq 4$ and $\sigma \leq 0$ or $\tau \geq (n + 2)/2$.

However, numerical and analytic evidences suggest that this fact should hold for $\tau > 0$, $\sigma \in (-1/(n - 1), 1)$ – see [7, Section 8] for details. Based on this, Chasman [7] conjectured:

- Among all domains with fixed volume, the lowest nonzero eigenvalue $\Gamma_2(\Omega)$ for a free plate under tension, with Poisson’s ratio $\sigma \in (-1/(n - 1), 1)$, is maximized by a ball.

This conjecture is open, and the best partial answers so far are due to Chasman [3, 7].

- Buoso and Provenzano [5] considered a Steklov-type eigenvalue problem therein and showed that among all bounded Euclidean domains of class $C^1$ with fixed measure, the ball maximizes the first positive eigenvalue.

Except the above isoperimetric results, much less is known for the biharmonic operator. The purpose of this paper is trying to get a new isoperimetric inequality for the biharmonic operator in a BVP having physical background.

Throughout this paper, let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^1$, $n \geq 2$. Inspired by Chasman’s work [3, 7], we consider the following Steklov-type eigenvalue problem

\[
\begin{cases}
\Delta^2 u - \tau \Delta u = 0 & \text{in } \Omega, \\
(1 - \sigma) \frac{\partial u}{\partial \vec{v}} + \sigma \Delta u = 0 & \text{on } \partial \Omega, \\
\tau \frac{\partial u}{\partial \vec{v}} - (1 - \sigma) \text{div}_{\partial \Omega} \left[ \text{Proj}_{\partial \Omega} \left[ (D^2 u) \vec{v} \right] \right] - \frac{\partial \Delta u}{\partial \vec{v}} = \lambda u & \text{on } \partial \Omega,
\end{cases}
\tag{1.3}
\]

where $\sigma \in \mathbb{R}$ and other same symbols have the same meanings as those in (1.1). For this BVP, if $\tau > 0$ and $\sigma \in (-1/(n - 1), 1)$, then it only has discrete spectrum and all the eigenvalues, with finite multiplicity, in this spectrum can be listed non-decreasingly as follows (see Section 2 for details)

\[0 = \lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \cdots \uparrow \infty.\]

Moreover, we can prove:

**Theorem 1.1.** For the Steklov-type problem (1.3), if $\tau > 0$ and $\sigma \in (-1/(n - 1), 1)$, then $\lambda_2(\Omega) \leq \lambda_2(\Omega^*)$, where $\Omega^*$ is the Euclidean ball having the same measure as $\Omega$.

**Remark 1.2.** (1) Chasman’s experience [7] shows that for the free plate problem of the biharmonic operator, there exists the essential difference between the nonzero Poisson’s ratio case and the null case, i.e., the spectral isoperimetric result of the BVP (1.1) cannot be simply extended to the case of BVP (1.2). This is exactly the motivation why we consider the boundary conditions given in (1.3) for the equation $\Delta^2 u - \tau \Delta u = 0$ in $\Omega$.

(2) Clearly, if the parameter $\sigma = 0$, then the BVP (1.3) becomes the one considered in [5], and correspondingly, the isoperimetric inequality shown in Theorem 1.1 here degenerates into the one
obtained in [5, Corollary 5.20]. Besides, one can see [5, Section 2] for the physical background of the BVP (1.3) in the case \( \sigma = 0 \) and \( n = 2 \), where it was used to describe the transverse vibrations of a thin plate in the theory of linear elasticity.

(3) As shown in Subsection 2.3, we would like to show that if additionally \( \Omega \) is of class \( C^2 \), then eigenvalues and eigenfunctions of the Steklov-type problem (1.3) can be converged by eigenvalues and eigenfunctions of the eigenvalue problem of the operator \( \Delta^2 - \tau \Delta \) subject to Neumann boundary conditions. This fact gives a further interpretation of problem (1.3) as the equation of a free vibrating plate (under tension and with nonzero Poisson’s ratio) whose mass is concentrated at the boundary in the case of domains of class \( C^2 \).

(4) The BVP (1.3) has already been considered in [8] by the corresponding author, Prof. J. Mao, with his collaborators, and a lower bound for the sums of the reciprocals of the first \( n \) nonzero eigenvalues \( \lambda_i(\Omega) \) has been obtained (see [8, Theorem 1.5]).

2 Analysis of the spectrum: characterization and asymptotic behavior

2.1 Characterization

First, we would like to show that the boundary conditions in (1.3) are reasonable. In order to explain clearly, we consider a slight more general version of the problem (1.3) as follows

\[
\begin{align*}
\Delta^2 u - \tau \Delta u &= 0 \quad \text{in } \Omega, \\
(1 - \sigma) \frac{\partial^2 u}{\partial x^2} + \sigma \Delta u &= 0 \quad \text{on } \partial \Omega, \\
\tau \frac{\partial u}{\partial \nu} - (1 - \sigma) \text{div}_{\partial \Omega} \left( \text{Proj}_{\partial \Omega} \left[ (D^2 u) \vec{v} \right]\right) - \frac{\partial \Delta u}{\partial \nu} &= \lambda \rho u \quad \text{on } \partial \Omega,
\end{align*}
\]

(2.1)

where the positive weight \( \rho \in L^\infty(\partial \Omega) \) denotes a mass density. Now, consider the weak eigenvalue equation for eigenfunction \( u \) with the eigenvalue \( \lambda \in \mathbb{R} \) and choose some test function \( \phi \in H^2(\Omega) \), one has

\[
\int_\Omega \left[ (1 - \sigma) D^2 u : D^2 \phi + \sigma \Delta u \Delta \phi + \tau \nabla u \cdot \nabla \phi \right] dx = \lambda \int_{\partial \Omega} \rho u \phi dS,
\]

(2.2)

where

\[
D^2 u : D^2 \phi = \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 \phi}{\partial x_i \partial x_j},
\]

with \( \{x_i\}_{1 \leq i \leq n} \) Cartesian coordinates of \( \mathbb{R}^n \), denotes the Frobenius product, \( \nabla \) is the gradient operator, and \( dx, dS \) are volume densities of \( \Omega \) and \( \partial \Omega \) respectively. By the divergence theorem, one has

\[
\int_\Omega \nabla u \cdot \nabla \phi dx = \int_{\partial \Omega} \phi \frac{\partial u}{\partial \nu} dS - \int_\Omega \phi (\Delta u) dx,
\]

(2.3)
which implies the Hessian term becomes
\[
\int_{\Omega} (1 - \sigma) D^2 u : D^2 \phi \, dx = (1 - \sigma) \sum_{i,j=1}^{n} \int_{\Omega} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} \, dx = (1 - \sigma) \sum_{j=1}^{n} \int_{\Omega} D(u_{x_j}) \cdot D(\phi_{x_j}) \, dx
\]
\[
= (1 - \sigma) \left[ \sum_{j=1}^{n} \int_{\partial \Omega} \phi_{x_j} \frac{\partial (u_{x_j})}{\partial n} \, dS - \sum_{j=1}^{n} \int_{\Omega} \Delta (u_{x_j}) \cdot \phi_{x_j} \, dx \right]
\]
\[
= (1 - \sigma) \left\{ \int_{\partial \Omega} \left[ \frac{\partial \phi}{\partial \nu} \frac{\partial^2 u}{\partial \nu^2} - \phi \cdot \text{div}_{\partial \Omega}(D^2 u \cdot \nu) - \phi \frac{\partial (\Delta u)}{\partial \nu} \right] \, dS
\right\}
\]
\[
+ \int_{\Omega} \phi \cdot \Delta^2 u \, dx \right\}.
\]
Define the tangential divergence \(\text{div}_{\partial \Omega}\) of a vector field \(F\) as \(\text{div}_{\partial \Omega} F = \text{div}F|_{\partial \Omega} - (DF \cdot \nu) \cdot \nu\).
Applying the divergence theorem twice, we can get
\[
\int_{\Omega} \sigma \Delta u \cdot \Delta \phi \, dx = \sigma \int_{\partial \Omega} \Delta u \frac{\partial \phi}{\partial \nu} \, dS - \sigma \int_{\Omega} D(\Delta u) \cdot D \phi \, dx
\]
\[
= \sigma \int_{\partial \Omega} \left[ \Delta u \frac{\partial \phi}{\partial \nu} - \phi \frac{\partial (\Delta u)}{\partial \nu} \right] \, dS + \sigma \int_{\Omega} \Delta^2 u \cdot \phi \, dx.
\]
Therefore, the weak eigenvalue equation (2.2) can be written as
\[
\int_{\Omega} \phi (\Delta^2 u - \tau \Delta u) \, dx + \int_{\partial \Omega} \frac{\partial \phi}{\partial \nu} \left[ (1 - \sigma) \frac{\partial^2 u}{\partial \nu^2} + \sigma \Delta u \right] \, dS +
\int_{\partial \Omega} \phi \left[ \frac{\partial u}{\partial \nu} - \frac{\partial \Delta u}{\partial \nu} - (1 - \sigma) \text{div}_{\partial \Omega}(D^2 u \cdot \nu) - \lambda \rho u \right] \, dS = 0,
\]
which holds for all \(\phi \in H^2(\Omega)\) and, of course, implies the BVP (2.10) directly.

### 2.2 Analysis of the spectrum of the Steklov-type eigenvalue problem (2.10)

Let
\[
\rho \in \mathcal{B}^S := \{\rho \in L^\infty(\partial \Omega) | \text{essinf}_{x \in \partial \Omega} \rho(x) > 0\}.
\]
As shown in Subsection 2.1, the weak formulation of BVP (2.10) is given by (2.2). Clearly, \(\lambda = 0\) is an eigenvalue whose eigenfunctions are nonzero constant functions. Hence, we need to consider the problem in the quotient space \(H^2(\Omega) / \mathbb{R}\). Let \(\mathcal{S}^S_{\rho}\) be a continuous embedding of \(L^2(\partial \Omega)\) into \(H^2(\Omega)^{\prime}\) defined by
\[
\mathcal{S}^S_{\rho}[u][\phi] := \int_{\partial \Omega} \rho u \phi \, dS, \quad \forall u \in L^2(\partial \Omega), \phi \in H^2(\Omega).
\]
Set
\[
H^2_{\rho} S(\Omega) := \left\{ u \in H^2(\Omega) \left| \int_{\partial \Omega} \rho u \, dS = 0 \right. \right\}.
\]
In $H^2(\Omega)$, consider the following bilinear form
\[
\langle u, \phi \rangle = \int_{\Omega} [(1 - \sigma)D^2u : D^2\phi + \sigma \Delta u \cdot \Delta \phi + \tau \nabla u \cdot \nabla \phi] \, dx,
\] (2.6)
which, by applying Poincaré-Wirtinger inequality, turns out to be a scalar product on $H^2_S(\Omega)$. Now, we make an agreement as follows:

- In the sequel, we shall treat $H^2_S(\Omega)$ as the functional space defined by (2.3) and endowed with the form (2.6).

Define a set $F(\Omega)$ as $F(\Omega) := \{ G \in H^2(\Omega) | G[1] = 0 \}$. Now, one can define an operator $\mathcal{R}_\rho^S : H^2_S \mapsto F(\Omega)$ given by
\[
\mathcal{R}_\rho^S[u][v] := \int_{\Omega} [(1 - \sigma)D^2u : D^2v + \sigma \Delta u \cdot \Delta v + \tau \nabla u \cdot \nabla v] \, dx,
\] (2.7)
where $u \in H^2_S(\Omega), v \in H^2(\Omega)$. It is not hard to know that $\mathcal{R}_\rho^S$ is a homeomorphism. Define an operator $\pi_\rho^S : H^2(\Omega) \mapsto H^2_S$ as follows
\[
\pi_\rho^S[u] := u - \frac{\int_{\partial \Omega} \rho u dS}{\int_{\partial \Omega} \rho dS}.
\] (2.8)
Consider the space $H^2(\Omega)/\mathbb{R}$ equipped with the bilinear form induced by (2.6), which is exactly a Hilbert space, and then one can define a map $\pi_\rho^{s, S} : H^2(\Omega)/\mathbb{R} \mapsto H^2_S(\Omega)$ determined by the equality $\pi_\rho^S = \pi_\rho^{s, S} \circ \rho$, with $p$ a canonical projection of $H^2(\Omega)$ onto $H^2(\Omega)/\mathbb{R}$. It is not hard to know that $\pi_\rho^{s, S}$ is a homeomorphism. We can define a differential operator $T_\rho^S$ on $H^2(\Omega)/\mathbb{R}$ as follows
\[
T_\rho^S := (\pi_\rho^{s, S})^{-1} \circ (\mathcal{R}_\rho^S)^{-1} \circ \mathcal{J}_\rho^S \circ \text{Tr} \circ \pi_\rho^{s, S},
\] (2.9)
with $\text{Tr}$ the trace operator acting from $H^2(\Omega)$ to $L^2(\partial \Omega)$. The operator $T_\rho^S$ can be shown to be a nonnegative compact self-adjoint in $H^2(\Omega)/\mathbb{R}$ under the suitable assumption. However, before that, we need the following fact:

- For any function $u \in H^2(\Omega)$, we have the sharp bound $(\Delta u)^2 \leq n|D^2u|^2$.

This fact can be directly obtained by the Cauchy-Schwarz inequality. Besides, for the operator $T_\rho^S$, it is easy to find the following fact:

- The pair $(\lambda, u)$ of set $(\mathbb{R} \setminus \{0\}) \times (H^2_S(\Omega) \setminus \{0\})$ satisfies (2.2) if and only if $\lambda \neq 0$ and the pair $(\lambda^{-1}, p[u])$ of set $\mathbb{R} \times ((H^2(\Omega)/\mathbb{R}) \setminus \{0\})$ satisfies the equation $\lambda^{-1} p[u] = T_\rho^S p[u]$.

Now, we have:

**Theorem 2.1.** Assume that $\tau > 0, \sigma \in (-1/(n-1), 1)$. The operator $T_\rho^S$ defined by (2.9) is a nonnegative compact self-adjoint operator in $H^2(\Omega)/\mathbb{R}$, and its eigenvalues are the reciprocals of the positive eigenvalues of problem (2.2). In particular, the set of eigenvalues of problem (2.2) consists the image of a sequence contained in $[0, \infty)$ and increasing to $+\infty$. Besides, the multiplicity of each eigenvalue is finite.
Proof. Let \( u \in H^2(\Omega)/\mathbb{R} \). By the orthogonal decomposition, we have

\[
u = \pi^S_R[u] + (\pi^S_R[u])^\perp,
\]

where \( \pi^S_R[u] \in H^{2,S}_R \), \((\pi^S_R[u])^\perp \in (H^{2,S}_R)^\perp \). So, for \( v \in H^2(\Omega)/\mathbb{R} \), one has

\[
\langle \pi^S_R[u], v \rangle_{H^2(\Omega)/\mathbb{R}} = \langle \pi^S_R[u], \pi^S_R[v] + (\pi^S_R[v])^\perp \rangle_{H^2(\Omega)/\mathbb{R}} = \langle \pi^S_R[u], \pi^S_R[v] \rangle_{H^2(\Omega)/\mathbb{R}} = \langle u, \pi^S_R[v] \rangle_{H^2(\Omega)/\mathbb{R}},
\]

where, of course, \( \langle \cdot, \cdot \rangle_{H^2(\Omega)/\mathbb{R}} \) stands for the bilinear form in the quotient space \( H^2(\Omega)/\mathbb{R} \) induced by (2.6). So, we can deduce that

\[
\langle T^S_R u, v \rangle_{H^2(\Omega)/\mathbb{R}} = \langle (\pi^S_R)^{-1} \circ (\mathcal{P}^S_R)^{-1} \circ \mathcal{J}^S_R \circ \text{Tr} \circ \pi^S_R u, v \rangle_{H^2(\Omega)/\mathbb{R}} = \langle \mathcal{J}^S_R \circ \text{Tr} \circ \pi^S_R u, \pi^S_R v \rangle_{H^2(\Omega)/\mathbb{R}} = \int_{\partial \Omega} \rho \pi^S_R u \cdot \pi^S_R v dS.
\]

Similarly, one can obtain

\[
\langle u, T^S_R v \rangle_{H^2(\Omega)/\mathbb{R}} = \int_{\partial \Omega} \rho \pi^S_R u \cdot \pi^S_R v dS.
\]

So,

\[
\langle u, T^S_R v \rangle_{H^2(\Omega)/\mathbb{R}} = \langle T^S_R u, v \rangle_{H^2(\Omega)/\mathbb{R}},
\]

which implies the self-adjointness of the operator \( T^S_R \) directly.

The compactness of \( T^S_R \) can be obtained from the fact that the trace operator \( \text{Tr} : H^1(\Omega) \to L^2(\partial \Omega) \) is compact. Since \( \sigma \in (-1/(n-1), 1) \) and \((\Delta u)^2 \leq n|D^2 u|^2\), one knows \((1-\sigma)|D^2 u|^2 + \sigma(\Delta u)^2 > 0\), which, together with \( \sigma > 0 \), implies the nonnegativity of the operator \( \mathcal{P}^S_R \) defined by (2.7). Naturally, the nonnegativity of the operator \( T^S_R \) follows directly. This completes the proof of Theorem 2.1.

Applying Theorem 2.1 directly, one knows that if \( \tau > 0 \), \( \sigma \in (-1/(n-1), 1) \), then the problem (2.2) only has the discrete spectrum and all its elements (i.e., eigenvalues) can be listed non-decreasingly as follows

\[
0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \uparrow +\infty.
\]
In fact, it is easy to check that \( \lambda = 0 \) is an eigenvalue of the problem \((2.2)\) with constant functions as its eigenfunctions. In contrast, suppose now \( u \) is an eigenfunction of the eigenvalue \( \lambda = 0 \), and then we have

\[
\int_\Omega [(1 - \sigma)|D^2 u|^2 + \sigma|\Delta u|^2 + \tau|\nabla u|^2] \, dx = 0,
\]

where \( |D^2 u|^2 := \sum_{i,j=1}^n \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 \). So, we have \( \nabla u = 0 \), which implies \( u \) is constant. Moreover, by Courant’s principle, it is not hard to know that the eigenvalue \( \lambda = 0 \) should have multiplicity 1.

As inspired by the free plate problem with nonzero Poisson’s ratio discussed by Chasman [7], we know that the bilinear form defined by \((2.7)\) might be coercive for \( \sigma \) not in \((-1/(n-1), 1)\) if imposing some restrictions on \( \tau \). For instance, if \( \tau > 0 \) and \( \sigma = 1 \), then the bilinear form defined \((2.7)\) becomes

\[
\int_\Omega [\Delta u \cdot \Delta v + \tau \nabla u \cdot \nabla v] \, dx,
\]

which is obviously coercive. Besides, in this setting, the problem \((2.2)\) degenerates into

\[
\int_\Omega [\Delta u \Delta \phi + \tau \nabla u \cdot \nabla \phi] \, dx = \lambda \int_{\partial \Omega} \rho u \phi \, dS,
\]

with its strengthened version

\[
\begin{aligned}
\Delta^2 u - \tau \Delta u &= 0 & \text{in } \Omega, \\
\Delta u &= 0 & \text{on } \partial \Omega, \\
\tau \frac{\partial u}{\partial v} - \frac{\partial \Delta u}{\partial v} &= \lambda \rho u & \text{on } \partial \Omega.
\end{aligned}
\]

Following almost the same argument, it is easy to know that for the above eigenvalue problem, the operator \( \Delta^2 - \tau \Delta \) only has discrete spectrum (with nonnegative eigenvalues inside) provided \( \tau > 0 \). Although, in the situation \( \tau > 0 \) and \( \sigma = 1 \), for the problem \((2.2)\), one might also have similar conclusions to Theorem 2.1, the boundary conditions lose the physical background, which is not the case we really want to discuss. Besides, note that if \( \tau = 0 \) and \( \sigma = 1 \), then all harmonic functions in \( H^2 \) are eigenfunctions with eigenvalue zero to the problem \((2.2)\), and of course, one has an eigenvalue of infinite multiplicity. Based on these reasons, we prefer to study the spectral properties of the problem \((2.2)\) under the constraint that \( \tau > 0 \), \( \sigma \in (-1/(n-1), 1) \). Besides, in this constraint, by means of variational principle, the first nonzero eigenvalue \( \lambda_2 \) of the problem \((2.2)\) can be characterized as follows

\[
\lambda_2 = \min \left\{ \frac{\int_\Omega [(1 - \sigma)|D^2 u|^2 + \sigma|\Delta u|^2 + \tau|\nabla u|^2] \, dx}{\int_{\partial \Omega} \rho u^2 \, dS} \right\} \quad \left( \forall u \in H^2(\Omega), \int_{\partial \Omega} \rho u \, dS = 0 \right).
\]

### 2.3 Asymptotic behavior

Consider the following eigenvalue problem of the biharmonic operator with the Neumann boundary conditions

\[
\begin{aligned}
\Delta^2 u - \tau \Delta u &= \lambda \rho u & \text{in } \Omega, \\
(1 - \sigma) \frac{\partial^2 u}{\partial v^2} + \sigma \Delta u &= 0 & \text{on } \partial \Omega, \\
\tau \frac{\partial u}{\partial v} - (1 - \sigma) \text{div}_{\partial \Omega} \left( \text{Proj}_{\partial \Omega} [(D^2 u) \vec{v}] \right) - \frac{\partial \Delta u}{\partial v} &= 0 & \text{on } \partial \Omega,
\end{aligned}
\] (2.10)
where \( \rho \in \mathcal{B}^N := \{ \rho \in L^\infty(\Omega) \mid \text{essinf}_{x \in \Omega} \rho(x) > 0 \} \) is a positive weight. This problem arises in the study of free vibrating plate under tension and with nonzero Poisson’s ratio. One can see Section 1 for a brief introduction of some interesting conclusions to this eigenvalue problem.

Define \( \Omega_\varepsilon := \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \varepsilon \} \), with \( \text{dist}(\cdot, \cdot) \) the Euclidean distance between two geometric objects. As shown in [5, Subsection 3.2], one can fix a positive number \( M > 0 \) and choose the family of densities \( \rho_\varepsilon \) defined by

\[
\rho_\varepsilon(x) = \begin{cases} 
\varepsilon, & \text{if } x \in \Omega \\
\frac{M - \varepsilon |\Omega_\varepsilon|}{|\Omega \setminus \Omega_\varepsilon|}, & \text{if } x \in \Omega \setminus \overline{\Omega_\varepsilon}
\end{cases}
\] (2.11)

for \( \varepsilon \in (0, \varepsilon_0) \) with \( \varepsilon_0 > 0 \) sufficiently small. Furthermore, assume that:

- \( \Omega \) is of class \( C^2 \), \( \varepsilon_0 \) can be chosen in such a way that the map \( x \mapsto x - \varepsilon \tilde{v} \) is a diffeomorphism between \( \partial \Omega \) and \( \partial \Omega_\varepsilon \) for all \( \varepsilon \in (0, \varepsilon_0) \).

By (2.11), it is not hard to check that \( \int_\Omega \rho_\varepsilon dx = M \) for all \( \varepsilon \in (0, \varepsilon_0) \). The quantity \( M \) is called the total mass of the body (see [5, Subsection 3.2]).

As pointed out in Section 1, it is easy to know that if \( \tau > 0, \sigma \in ( -1/(n-1), 1) \), the eigenvalue problem (2.10) only has the discrete spectrum, its nonnegative eigenvalues (of finite multiplicity) can be listed non-decreasingly to infinity, and moreover, all the eigenfunctions form a Hilbert basis of \( L^2(\Omega) \). Now, we consider the weak formulation of problem (2.10) with density \( \rho_\varepsilon \) as follows

\[
\int_\Omega \left[ (1 - \sigma)D^2 u : D^2 \phi + \sigma \Delta u \cdot \Delta \phi + \tau \nabla u \cdot \nabla \phi \right] dx = \lambda \int_\Omega \rho_\varepsilon u \phi dx, \quad (2.12)
\]

for any \( \phi \in H^2(\Omega) \) with the unknowns \( u \in H^2(\Omega), \lambda \in \mathbb{R} \). Define

\[
\mathcal{J}_{\rho_\varepsilon}^N[u][\phi] := \int_\Omega \rho_\varepsilon u \phi dx, \quad \forall u \in L^2(\Omega), \quad \phi \in H^2(\Omega),
\]

which is a continuous embedding from \( L^2(\Omega) \) into \( H^2(\Omega) \). Set

\[
H^2_{\rho_\varepsilon}^N(\Omega) := \left\{ u \in H^2(\Omega) \mid \int_\Omega u \rho_\varepsilon dx = 0 \right\}.
\]

The space \( H^2_{\rho_\varepsilon}^N(\Omega) \) can be endowed with the form (2.12) and then this form defines on \( H^2_{\rho_\varepsilon}^N(\Omega) \) a scalar product, whose induced norm is equivalent to the standard one. Define a map \( \pi_{\rho_\varepsilon}^N : H^2(\Omega) \mapsto H^2_{\rho_\varepsilon}^N(\Omega) \) given by

\[
\pi_{\rho_\varepsilon}^N[u] := u - \frac{\int_\Omega u \rho_\varepsilon dx}{\int_\Omega \rho_\varepsilon dx}
\]

for all \( u \in H^2(\Omega) \). Then we can define the map \( \pi_{\rho_\varepsilon}^N : H^2 / \mathbb{R} \mapsto H^2_{\rho_\varepsilon}^N(\Omega) \) given by the equality \( \pi_{\rho_\varepsilon}^N = \pi_{\rho_\varepsilon}^N \circ p \), which is a homeomorphism. Similar to (2.7), we can define a map \( \mathcal{P}_{\rho_\varepsilon}^N : H^2_{\rho_\varepsilon}^N(\Omega) \mapsto F(\Omega) \) as follows

\[
\mathcal{P}_{\rho_\varepsilon}^N[u][v] := \int_\Omega \left[ (1 - \sigma)D^2 u : D^2 v + \sigma \Delta u \cdot \Delta v + \tau \nabla u \cdot \nabla v \right] dx,
\]
with \( u \in H^{2,N}_\rho(\Omega), v \in H^2(\Omega) \), which is a linear homeomorphism of \( H^{2,N}_\rho(\Omega) \) onto \( F(\Omega) \). Finally, we can define an operator \( T^N_\rho : H^2(\Omega)/\mathbb{R} \mapsto H^2(\Omega)/\mathbb{R} \) as follows

\[
T^N_\rho := (\pi^N_\rho)^{-1} \circ (\mathcal{P}^N_\rho)^{-1} \circ \mathcal{J}^N_\rho \circ i \circ \pi^N_\rho.
\]

(2.13)

For the operator \( T^N_\rho \), it is easy to find the following fact:

- The pair \((\lambda, u)\) of set \((\mathbb{R} \setminus \{0\}) \times (H^{2,S}_\rho(\Omega) \setminus \{0\})\) satisfies (2.12) if and only if \( \lambda \neq 0 \) and the pair \((\lambda^{-1}, p[u])\) of set \(\mathbb{R} \times ((H^2(\Omega)/\mathbb{R}) \setminus \{0\})\) satisfies the equation \( \lambda^{-1} p[u] = T^N_\rho p[u] \).

Similar to Theorem 2.1, we have:

**Theorem 2.2.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) of class \( C^1 \) and \( \varepsilon \in (0, \varepsilon_0) \), and assume that \( \sigma \in (-1/(n-1), 1), \tau > 0. \) The operator \( T^N_\rho \) defined by (2.13) is a non-negative compact self-adjoint operator in \( H^2(\Omega)/\mathbb{R} \), where the eigenvalues of \( T^N_\rho \) are the reciprocals of the positive eigenvalues \( \lambda_j(\rho_{\varepsilon}) \) of the problem (2.10) for all \( j \in \mathbb{N} \). In particular, the set of eigenvalues of problem (2.10) consists of a sequence contained in \([0, \infty)\) and increasing to \(+\infty. \) Besides, the multiplicity of each eigenvalue is finite.

We have the following spectral conclusion:

**Theorem 2.3.** The first eigenvalue \( \lambda_1 \) of problem (2.10) is equal to zero whose eigenfunctions are the constant functions. Moreover, \( \lambda_2(\rho_{\varepsilon}) > 0. \)

Now, we would like to show the asymptotic property between the problem (2.2) and the problem (2.10) if \( \Omega \) is of class \( C^2. \) However, in order to get that, we need to make some preparations. In fact, if \( \Omega \) is of class \( C^2, \) then the Tubular Neighborhood Theorem can be used to perform computations on the strip \( \Omega \setminus \Sigma_{\varepsilon}. \) Moreover, following a standard argument similar to the one shown in [13], we can get the following conclusion:

**Lemma 2.4.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) of class \( C^2. \) Let \( \rho_{\varepsilon} \in \mathcal{R}^N \) defined by (2.71). Then we have the followings:

- For all \( \varphi \in H^2(\Omega)/\mathbb{R}, \pi^N_{\rho_{\varepsilon}}[\varphi] \rightarrow \pi^N_1[\varphi] \) in \( L^2(\Omega) \) (also in \( H^2(\Omega) \)) as \( \varepsilon \rightarrow 0; \)

- If \( u_{\varepsilon} \rightharpoonup u \) in \( H^2(\Omega)/\mathbb{R}, \) then (possibly passing to a subsequence) \( \pi^N_{\rho_{\varepsilon}}[u_{\varepsilon}] \rightarrow \pi^N_1[u] \) in \( L^2(\Omega) \) as \( \varepsilon \rightarrow 0; \)

- Assume that \( u_{\varepsilon}, u, w_{\varepsilon}, w \in H^2(\Omega) \) are functions such that \( u_{\varepsilon} \rightharpoonup u, w_{\varepsilon} \rightharpoonup w \) in \( L^2(\Omega), \) \( \text{Tr}[u_{\varepsilon}] \rightarrow \text{Tr}[u], \text{Tr}[w_{\varepsilon}] \rightarrow \text{Tr}[w] \) in \( L^2(\partial \Omega) \) as \( \varepsilon \rightarrow 0. \) Moreover, assume that there exists a constant \( C > 0 \) such that \( \|\nabla u_{\varepsilon}\|_{L^2(\Omega)} \leq C, \|\nabla w_{\varepsilon}\|_{L^2(\Omega)} \leq C \) for all \( \varepsilon \in (0, \varepsilon_0). \) Then

\[
\int_{\Omega} \rho_{\varepsilon}(u_{\varepsilon} - u) w_{\varepsilon} dx \rightarrow 0
\]

and

\[
\int_{\Omega} \rho_{\varepsilon}(w_{\varepsilon} - w) u dx \rightarrow 0
\]

as \( \varepsilon \rightarrow 0. \)
By applying Lemma 2.4, we can obtain:

**Theorem 2.5.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ of class $C^2$. Let the operators $T^S_{\partial M}$ and $T^N_{\rho^e}$ from $H^2(\Omega)/\mathbb{R}$ to itself be defined as in (2.9) and (2.13), respectively. Then the sequence $\{T^N_{\rho^e}\}_{\varepsilon \in (0, \varepsilon_0)}$ converges in norm to $T^S_{\partial M}$ as $\varepsilon \to 0$.

**Proof.** By (2.8), one can easily get

$$\pi^\varepsilon_c = \pi^1_c$$

for all $c \in \mathbb{R}$ with $c \neq 0$. We need to show that the family of compact operators $\{T^N_{\rho^e}\}_{\varepsilon \in (0, \varepsilon_0)}$ converges compactly to the compact operator $T^S_{\partial M}$, that is to say,

$$\lim_{\varepsilon \to 0} \left\| \left( T^N_{\rho^e} - T^S_{\partial M} \right)^2 \right\|_{L(H^2(\Omega)/\mathbb{R}, H^2(\Omega)/\mathbb{R})} = 0. \tag{2.14}$$

The operators $\{T^N_{\rho^e}\}_{\varepsilon \in (0, \varepsilon_0)}$ and $T^S_{\partial M}$ are self-adjoint, so we only need to prove that $\{T^N_{\rho^e}\}_{\varepsilon \in (0, \varepsilon_0)}$ converges to $T^S_{\partial M}$ in norm. By (2.13), we know that the operator $T^N_{\rho^e}$ compactly converges to $T^S_{\partial M}$ if the following requirements are satisfied:

- (R1) If $\|u_e\|_{H^2(\Omega)/\mathbb{R}} \leq C$ for all $\varepsilon \in (0, \varepsilon_0)$, then the family $\{T^N_{\rho^e}u_e\}_{\varepsilon \in (0, \varepsilon_0)}$ is relatively compact in $H^2(\Omega)/\mathbb{R}$;
- (R2) if $u_e \to u$ in $H^2(\Omega)/\mathbb{R}$, then $T^N_{\rho^e}u_e \to T^S_{\partial M}u$ in $H^2(\Omega)/\mathbb{R}$.

We show (R1) first. For a fixed $u \in H^2(\Omega)/\mathbb{R}$, by Lemma 2.4, we have

$$\lim_{\varepsilon \to 0} \int_\Omega \rho^e \pi_{\rho^e}^\varepsilon N[u] \, dx = \lim_{\varepsilon \to 0} \int_\Omega \rho^e (\pi_{\rho^e}^\varepsilon N[u] - \pi_1^S[u]) \, dx + \left( \lim_{\varepsilon \to 0} \int_\Omega \rho^e \pi_1^S[u] \, dx - \frac{M}{|\partial \Omega|} \int_{\partial \Omega} \pi_1^S[u] \, dS \right) + \frac{M}{|\partial \Omega|} \int_{\partial \Omega} \pi_1^S[u] \, dS = \frac{M}{|\partial \Omega|} \int_{\partial \Omega} \pi_1^S[u] \, dS.$$

Moreover, the equality $(\pi_{\rho^e}^\varepsilon N)^{-1} \circ (\mathcal{P}_{\rho^e})^{-1} = (\pi_1^S)^{-1} \circ (\mathcal{P}_S)^{-1}$ holds, which implies that $T^N_{\rho^e}u$ is bounded for each $u \in H^2(\Omega)/\mathbb{R}$. By Banach-Steinhaus Theorem, there exists a non-negative constant $C'$ such that $\|T^N_{\rho^e}u\|_{L(H^2(\Omega)/\mathbb{R}, H^2(\Omega)/\mathbb{R})} \leq C'$ for all $\varepsilon \in (0, \varepsilon_0)$. Moreover, since $\|u_e\|_{H^2(\Omega)/\mathbb{R}} \leq C$ for $\varepsilon \in (0, \varepsilon_0)$, possibly passing to a subsequence, we have $u_e \to u$ in $H^2(\Omega)/\mathbb{R}$ for some $u \in H^2(\Omega)/\mathbb{R}$. Hence, possibly passing to a subsequence, $T^N_{\rho^e}u_e \to w$ in $H^2(\Omega)/\mathbb{R}$ as $\varepsilon \to 0$. 

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We can show $w = T^S_M u$. Set $w^e := T^{N_e}_{[\Omega^e]} u^e$. By Lemma 2.4, one has

$$
\lim_{\varepsilon \to 0} \int_{\Omega} \left[ (1 - \sigma) D^2(\pi^N_{\rho\varepsilon^e} [w^e]) : D^2(\pi^N_{\rho\varepsilon^e} [\phi]) + \sigma \Delta(\pi^N_{\rho\varepsilon^e} [w^e]) \cdot \Delta(\pi^N_{\rho\varepsilon^e} [\phi]) \\
+ \tau \nabla(\pi^N_{\rho\varepsilon^e} [w^e]) \cdot \nabla(\pi^N_{\rho\varepsilon^e} [\phi]) \right] dx
$$

for all $\phi \in H^2(\Omega)/\mathbb{R}$. On the other hand, from the equality $(\mathcal{P}^N_{\rho} \circ \pi_{\rho\varepsilon^e}^N)w^e = (\mathcal{P}^N_{\rho} \circ i \circ \pi_{\rho\varepsilon^e}^N)u^e$, it follows that

$$
\int_{\Omega} \left[ (1 - \sigma) D^2(\pi^N_{\rho\varepsilon^e} [w^e]) : D^2(\pi^N_{\rho\varepsilon^e} [\phi]) + \sigma \Delta(\pi^N_{\rho\varepsilon^e} [w^e]) \cdot \Delta(\pi^N_{\rho\varepsilon^e} [\phi]) \\
+ \tau \nabla(\pi^N_{\rho\varepsilon^e} [w^e]) \cdot \nabla(\pi^N_{\rho\varepsilon^e} [\phi]) \right] dx
$$

Then, by the third claim of Lemma 2.4, we have

$$
\langle w, \phi \rangle_{H^2(\Omega)/\mathbb{R}} = \lim_{\varepsilon \to 0} \langle w^e, \phi \rangle_{H^2(\Omega)/\mathbb{R}} = \lim_{\varepsilon \to 0} \int_{\Omega} \rho^e \pi^N_{\rho\varepsilon^e} [u^e] \pi^N_{\rho\varepsilon^e} [\phi] dx
$$

and therefore $w = T^S_M u$. Similarly, one has $\|w^e\|_{H^2(\Omega)/\mathbb{R}} \to \|w\|_{H^2(\Omega)/\mathbb{R}}$. Then

$$
\lim_{\varepsilon \to 0} \|w^e\|_{H^2(\Omega)/\mathbb{R}}^2 = \lim_{\varepsilon \to 0} \int_{\Omega} \rho^e (\pi^N_{\rho\varepsilon^e} [u^e] - \pi^S_{\rho\varepsilon^e} [u]) \pi^N_{\rho\varepsilon^e} [w^e] dx
$$

$$
+ \lim_{\varepsilon \to 0} \int_{\Omega} \rho^e \pi^S_{\rho\varepsilon^e} [u] (\pi^N_{\rho\varepsilon^e} [w^e] - \pi^S_{\rho\varepsilon^e} [w^e]) dx
$$

$$
+ \lim_{\varepsilon \to 0} \int_{\Omega} \rho^e \pi^S_{\rho\varepsilon^e} [u] (\pi^S_{\rho\varepsilon^e} [w] - \pi^S_{\rho\varepsilon^e} [w]) dx
$$

$$
+ \lim_{\varepsilon \to 0} \int_{\Omega} \rho^e \pi^S_{\rho\varepsilon^e} [u] \pi^S_{\rho\varepsilon^e} [w] dS
$$

$$
= \frac{M}{|\partial \Omega|} \int_{\partial \Omega} \pi^S_{\rho\varepsilon^e} [u] \pi^S_{\rho\varepsilon^e} [w] dS
$$

$$
= \|w\|_{H^2(\Omega)/\mathbb{R}}^2,
$$
which finishes the proof of (R1).

Let \( u_\varepsilon \to u \) in \( H^2(\Omega)/\mathbb{R} \). Then there exists some non-negative constant \( C'' \) such that
\[
\|u_\varepsilon\|^2_{H^2(\Omega)/\mathbb{R}} \leq C''
\]
for all \( \varepsilon \). Then, using a similar argument to that of the claim (R1), for each sequence \( \varepsilon_j \to 0 \), possibly passing to a subsequence, we have \( T^{N}_{p_\varepsilon} u_{\varepsilon} \to T^{S}_{M} u \). Since this is true for each \( \{\varepsilon_j\}_{j \in \mathbb{N}} \), we have the convergence for the whole family, i.e., \( T^{N}_{p_\varepsilon} u_{\varepsilon} \to T^{S}_{M} u \), which completes the proof of (R2).

In the sequel, we will explore the maximum value of the fundamental tone of problem (1.3) and prove that when \( \Omega^{*} \) is a ball such that \( |\Omega| = |\Omega^{*}| \), this maximum value can be taken by the corresponding eigenvalue of \( \Omega^{*} \).

### 3 Eigenvalues and eigenfunctions on the ball

When \( B \) is a unit ball in \( \mathbb{R}^n \) centered at the origin, we would like to characterize the eigenvalues and the corresponding eigenfunctions of (1.3). We will use spherical coordinates \((r, \theta)\) to do calculations, with \( \theta = (\theta_1, \cdots, \theta_{n-1}) \in S^{n-1} \) and \( S^{n-1} \) the \((n-1)\)-dimensional unit Euclidean sphere. In fact, the corresponding coordinate transformation should be
\[
\begin{align*}
x_1 &= r \cos(\theta_1), \\
x_2 &= r \sin(\theta_1) \cos(\theta_2), \\
&\quad \vdots \\
x_{n-1} &= r \sin(\theta_1) \sin(\theta_2) \cdots \sin(\theta_{n-2}) \cos(\theta_{n-1}), \\
x_n &= r \sin(\theta_1) \sin(\theta_2) \cdots \sin(\theta_{n-2}) \sin(\theta_{n-1}).
\end{align*}
\]
Here, if \( n > 2 \), then \( \theta_1, \cdots, \theta_{n-2} \in [0, \pi], \theta_{n-1} \in [0, 2\pi) \), while if \( n = 2 \), then \( \theta_1 \in [0, 2\pi) \). In spherical coordinates, the boundary condition of problem (1.3) can be written as
\[
\begin{cases}
(1 - \sigma) \frac{\partial^2 u}{\partial r^2} \bigg|_{r=1} + \sigma \Delta u = 0, \\
\tau \frac{\partial u}{\partial r} \bigg|_{r=1} - \frac{1-\sigma}{r^2} \Delta S (\frac{\partial u}{\partial r} - \frac{u}{r}) \bigg|_{r=1} - \frac{\partial^2 u}{\partial r^2} \bigg|_{r=1} = \lambda u |_{r=1},
\end{cases}
\tag{3.1}
\]
where \( \Delta S \) is the angular part of the Laplacian. Using a similar argument to [6, 7], we know that for any nonnegative eigenvalue \( \lambda \), its eigenfunction can be written as a product of a radial part and an angular part. Besides, the radial part of the fundamental tone of the unit ball should be a linear combination of Bessel functions. As we know, the ultraspherical Bessel functions \( j_\ell(z) \) of the first kind are defined as
\[
j_\ell(z) := z^{1-\ell} J_{\frac{\ell-1}{2}+\ell}(z)
\]
with the modified Bessel functions $J_v$ solving the following Bessel equation

$$z^2y''(z) + zy'(z) + (z^2 - v^2)y(z) = 0.$$  

The first and the second kinds of hypersphere modified Bessel functions $i_l(z), k_l(z)$ are defined separately as follows

$$i_l(z) := z^{1 - \frac{v}{2}}I_{\frac{v}{2}-1+l}(z)$$

and

$$k_l(z) := z^{1 - \frac{v}{2}}K_{\frac{v}{2}-1+l}(z),$$

where the modified Bessel functions $I_v$ and $K_v$ are solutions to the following modified ultraspherical Bessel equation

$$z^2y''(z) + zy'(z) + (z^2 + v^2)y(z) = 0.$$  

By [1, §9.6], one knows that $i_l(z)$ and all its derivatives are positive on $(0, +\infty)$. Based on these facts, we can prove:

**Theorem 3.1.** Let $\Omega$ be the unit ball, centered at the origin, in $\mathbb{R}^n$. Any eigenfunction $u_l$ of the problem (3.3) is of the form $u_l(r, \theta) = R_l(r)Y_l(\theta)$, where $Y_l(\theta)$ is a spherical harmonic function of some order $l \in \mathbb{N}$ and

$$R_l(r) = A_l r^l + B_l i_l(\sqrt{\sigma} r),$$

where $A_l$ and $B_l$ are suitable constants such that

$$B_l = \frac{(1 - \sigma)l(l - 1)A_l}{\tau l''_l(\sqrt{\tau}) + \sqrt{\tau} \sigma(n - 1) i'_l(\sqrt{\tau}) - \sigma l(l + n - 2) i_l(\sqrt{\tau}) + (1 - \sigma)l(l - 1) i_l(\sqrt{\tau})}.$$

Moreover, the eigenvalue $\lambda_{(l)}$ associated with the eigenfunction $u_l$ is delivered by formula

$$\lambda_{(l)} = \left( \frac{\tau l''_l(\sqrt{\tau}) + \sqrt{\tau} \sigma(n - 1) i'_l(\sqrt{\tau}) - \sigma l(l + n - 2) i_l(\sqrt{\tau}) + (1 - \sigma)l(l - 1) i_l(\sqrt{\tau})}{\tau l^2 + \sigma l + l(1 - \sigma)(l - 1)(\sigma l + \sigma n - 3 - \sigma))i_l(\sqrt{\tau}) + \tau \sqrt{\tau} l \cdot \\
\hspace{1cm}\sigma n + l \sigma - 2 \sigma - l + 1 \right) + \sqrt{\tau} (1 - \sigma) l(l - 1)(l + n - 2)(\sigma n - \sigma - 2l + \sigma l) - n + 1) i'_l(\sqrt{\tau}) + \tau (1 - \sigma)(l + 2n - 3)(l - 1) i''_l(\sqrt{\tau})$$

for any $l \in \mathbb{N}$.

**Proof.** It is easy to know that the solution $u$ of the Steklov-type eigenvalue problem (3.3) in the unit ball is smooth (see, e.g., [11]). We divide into the argument into two cases:

**Case 1.** Assume that $\Delta u = 0$. The Laplace operator in spherical coordinates can be written as

$$\Delta = \partial_r + \frac{n - 1}{r} \partial_r + \frac{1}{r^2} \Delta_S.$$
Separating variables so that \( u = R(r)Y(\theta) \), we can obtain
\[
R'' + \frac{n-1}{r} R' - \frac{l(l+n-2)}{r^2} R = 0
\] (3.3)
with
\[
\Delta_s Y = -l(l+n-2)Y.
\] (3.4)
Solving the ODE (3.3) yields \( R(r) = ar^l + br^{2-n-l} \), where \( l > 0, n \geq 2 \). Besides, if \( l = 0, n = 2, R(r) = a + b \log r \). When \( r = 0 \), if \( b \neq 0 \), then \( u \) blows up at \( r = 0 \), and so we have to impose \( b = 0 \). Besides, the solutions of (3.4) are the spherical harmonic functions of order \( l \). So, we have
\[
u(r, \theta) = a_i \nu^l Y_l(\theta).
\]
for some \( l \in \mathbb{N} \).

**Case 2.** Assume that \( \Delta u \neq 0 \). Let \( v = \Delta u \), and then the Equation \( \Delta^2 u - \tau \Delta u = 0 \) can be written as
\[
\Delta v = \tau v.
\]
Similarly, separating variables so that \( v = R(r)Y(\theta) \), we have
\[
R'' + \frac{n-1}{r} R' - \frac{l(l+n-2)}{r^2} R = \tau R,
\] (3.5)
with \( Y \) satisfying (3.4). For the ODE (3.5), using a similar argument to that of ODEs (3.3) and (3.4) yields that \( v \) should be
\[
v(r, \theta) = b_{l_1} i_{l_1}(\sqrt{\tau}r)Y_{l_1}(\theta),
\]
for some \( l_1 \in \mathbb{N} \). Since \( v = \frac{\Delta u}{\tau} = \Delta u \), we have
\[
u(r, \theta) = \frac{b_{l_1}}{\tau} i_{l_1}(\sqrt{\tau}r)Y_{l_1}(\theta) - c_{l_2} r^l Y_{l_2}(\theta),
\] (3.6)
where \( l_2 \in \mathbb{N} \). Rewriting the boundary condition \( (1 - \sigma) \frac{\partial^2 u}{\partial r^2} \bigg|_{r=1} + \sigma \Delta u = 0 \) as follows
\[
0 = b_{l_1} i''_{l_1}(\sqrt{\tau})Y_{l_1}(\theta) + \sigma(n-1) \frac{b_{l_1}}{\sqrt{\tau}} i'_{l_1}(\sqrt{\tau})Y_{l_1}(\theta) - \sigma \frac{b_{l_1}}{\tau} i_{l_1}(\sqrt{\tau})l_1(l_1+n-2)Y_{l_1}(\theta)
- l_2(l_2-1)c_{l_2} Y_{l_2}(\theta) - \sigma(n-1)l_2 c_{l_2} Y_{l_2}(\theta) + \sigma c_{l_2} l_2 (l_2+n-2) Y_{l_2}(\theta)
= \left[ \frac{l_1(l_1-1)(1-\sigma)}{\tau} i_{l_1}(\sqrt{\tau}) + \frac{2l_1 + \sigma(n-1) + 1}{\sqrt{\tau}} i_{l_1+1}(\sqrt{\tau}) + i_{l_1+2}(\sqrt{\tau}) \right] b_{l_1} Y_{l_1}(\theta)
- c_{l_2} l_2(l_2-1)(1-\sigma) Y_{l_2}(\theta).
\] (3.7)
We recall that \( i_l(z) \) and all its derivatives are positive on \((0, +\infty)\). Combining the fact that \( Y_{l_1} \) and \( Y_{l_2} \) are linearly independent, we can easily obtain that \( b_{l_1} = 0 \) and \( l_2 = 0 \) or \( l_2 = 1 \). Then, together with (3.6), it follows that
\[
u_l(r, \theta) = -c_{l_2} r^l + \frac{b_{l_1}}{\tau} i_l(\sqrt{\tau}r)Y_l(\theta).
\]
By (3.7) and a direct calculation, one has

\[ u_l(r, \theta) = [A_l r^l + B_l i_l(\sqrt{\tau}r)]Y_l(\theta). \] (3.8)

By (3.7) and a direct calculation, one has

\[ B_l = -\frac{(1 - \sigma) l(l - 1)}{\tau^l l(\sqrt{\tau}) + \sqrt{\tau} \sigma (n - 1) l' l(\sqrt{\tau}) - \sigma l \tau + (l + 2) i_l(\sqrt{\tau})} A_l. \] (3.9)

We know that \( u_l \) given by (3.8) is an eigenfunction of the eigenvalue problem (I.3) on the unit ball. Combining the boundary condition

\[ \tau \frac{\partial u}{\partial r} - \frac{1 - \sigma}{r^2} \Delta S \left( \frac{\partial u}{\partial r} - \frac{u}{r} \right) - \partial \Delta u \bigg|_{r = 1} = \lambda u \bigg|_{r = 1} \]

(3.8) yields

\[ A_l \left[ (1 - \sigma) l(l + n - 2)(l - 1) + B_l \{ -l(l + n - 2)(3 - \sigma) i_l(\sqrt{\tau}) + \sqrt{\tau} \tau + (2 - \sigma) l(l + n - 2) + n - 1 \} l' l(\sqrt{\tau}) - \tau(n - 1) l'' l(\sqrt{\tau}) - \tau \sqrt{\tau} i'(\sqrt{\tau}) \right] = \lambda \tau [A_l + B_l i_l(\sqrt{\tau})] \]

The conclusion (3.2) follows directly by substituting (3.9) into the above equality. \( \square \)

Now, we can give the characterization to the first nonzero eigenvalue and also its eigenfunctions.

**Theorem 3.2.** Let \( \Omega = B_1 \) be the unit ball in \( \mathbb{R}^n \) centered at the origin. The first positive eigenvalue of the eigenvalue problem (I.3) is \( \lambda_2 = \lambda(1) = \tau. \) The corresponding eigenspace is generated by \( \{x_1, x_2, \ldots, x_n\}. \)

**Proof.** By Theorem 3.1, we can get \( 0 = \lambda(0) < \tau = \lambda(1) \). Recall the recursive relation of some known hypersphere Bessel functions (see, e.g., [1, p. 376])

\[ i_l(z) = \frac{2 + 2l}{z} i_{l+1}(z) + i_{l+2}(z) \]
\[ i_l'(z) = \frac{l}{z} i_l(z) + i_{l+1}(z), \]
\[ i_l''(z) = \frac{l(l - 1)}{z^2} i_l(z) + \frac{2l + 1}{z} i_{l+1}(z) + i_{l+2}(z), \]
\[ i_l'''(z) = \frac{l(l - 1)(-1 - 2l)}{z^3} i_l(z) + \frac{3l^2 + 3l + 3}{z^2} i_{l+1}(z) + \frac{3l^2 + 3l + 3}{z} i_{l+2}(z) + i_{l+3}(z). \]

Let \( D(l) \) be the denominator of the RHS of (3.2) and \( N(l) \) be the corresponding numerator, i.e., \( \lambda(l) = \frac{N(l)}{D(l)}. \) By the recursive formula, we can calculate the denominator \( D(l) \) as follows

\[ D(l) = \tau i_l'(\sqrt{\tau}) + \sqrt{\tau} \sigma (n - 1) i_l(\sqrt{\tau}) - \sigma l \tau + (l + 2) i_l(\sqrt{\tau}) - (1 - \sigma) l(l - 1) i_l(\sqrt{\tau}) \]
\[ = \tau \left( \frac{(l - 1) l}{\tau} i_l(\sqrt{\tau}) + \frac{2l + 1}{\sqrt{\tau}} i_{l+1}(\sqrt{\tau}) + i_{l+2}(\sqrt{\tau}) \right) + \sqrt{\tau} \sigma (n - 1) \left( \frac{l}{\sqrt{\tau}} i_l(\sqrt{\tau}) + i_{l+1}(\sqrt{\tau}) \right) \]
\[ - \sigma l(l + n - 2) i_l(\sqrt{\tau}) - (1 - \sigma) l(l - 1) i_l(\sqrt{\tau}) \]
\[ = \sqrt{\tau}(2l + \sigma n + 1 - \sigma) i_{l+1}(\sqrt{\tau}) + \tau i_{l+2}(\sqrt{\tau}). \]
Moreover, the numerator \( N_{(l)} \) can be computed as follows

\[
N_{(l)} = -l^2(l + n - 2)(\sigma \tau + (1 - \sigma)(l - 1)(\sigma l + \sigma n - 3 - \sigma))i_l(\sqrt{\tau}) + \left[ \tau \sqrt{\tau}l \cdot \right.
\]
\[
(\sigma n + l \sigma - 2 \sigma - l + 1) + \sqrt{\tau}(1 - \sigma)l(l - 1)((l + n - 2)(\sigma n - \sigma - 2l + \sigma l) - n + 1) \cdot \n\]
\[
i_l'(\sqrt{\tau}) + \tau l(\tau + (1 - \sigma)(l + 2n - 3)(l - 1))i''_l(\sqrt{\tau}) + \tau \sqrt{\tau}(1 - \sigma)l(l - 1)i''_l(\sqrt{\tau}) \n\]
\[
= -l^2(l + n - 2)(\sigma \tau + (1 - \sigma)(l - 1)(\sigma l + \sigma n - 3 - \sigma))i_l(\sqrt{\tau}) + \n\]
\[
\left[ \tau \sqrt{\tau}l(\sigma n + l \sigma - 2 \sigma - l + 1) \n\]
\[
+ \sqrt{\tau}(1 - \sigma)l(l - 1)((l + n - 2)(\sigma n - \sigma - 2l + \sigma l) - n + 1) \right] \left( \frac{l}{\sqrt{\tau}}i_l(\sqrt{\tau}) + i_{l+1}(\sqrt{\tau}) \n\right.
\]
\[
+ \tau l(\tau + (1 - \sigma)(l + 2n - 3)(l - 1)) \left( \frac{l(l - 1)}{\tau}i_l(\sqrt{\tau}) + \frac{2l + 1}{\sqrt{\tau}}i_{l+1}(\sqrt{\tau}) + i_{l+2}(\sqrt{\tau}) \n\right.
\]
\[
+ \tau \sqrt{\tau}(1 - \sigma)l(l - 1) \left( \frac{l(l - 1)(l - 2)}{\tau \sqrt{\tau}}i_l(\sqrt{\tau}) \n\right.
\]
\[
\left. + \frac{3l^2}{\tau}i_{l+1}(\sqrt{\tau}) + \frac{3l + 3}{\sqrt{\tau}}i_{l+2}(\sqrt{\tau}) + i_{l+3}(\sqrt{\tau}) \right) \n\]
\[
= [\tau \sqrt{\tau}l(\sigma n + \sigma l - 2 \sigma + l + 2) + \sqrt{\tau}(1 - \sigma)l(l - 1)((l + n - 2)(\sigma n - \sigma + \sigma l + 1) + \n\]
\[
3l^2 + 2nl - 2l)i_{l+1}(\sqrt{\tau}) + [\tau l(\tau + (1 - \sigma)(l + 2n)(4l + 2n))i_{l+2}(\sqrt{\tau}) + \n\]
\[
\tau \sqrt{\tau}(1 - \sigma)l(l - 1)i_{l+3}(\sqrt{\tau})]. \n\]

Then applying the recursion formula again and the conclusion [3,2], we have

\[
\lambda_{(l)} = \frac{N_{(l)}}{D_{(l)}} = (\sqrt{\tau}(2l + \sigma n + 1 - \sigma)i_{l+1}(\sqrt{\tau}) + \sqrt{\tau}i_{l+2}(\sqrt{\tau}))^{-1} \left[ \tau \sqrt{\tau}l(\sigma n + \sigma l - 2 \sigma + l + 2) \n\right.
\]
\[
+ \sqrt{\tau}(1 - \sigma)l(l - 1)((l + n - 2)(\sigma n - \sigma + \sigma l + 1) + 3l^2 + 2nl - 2l)i_{l+1}(\sqrt{\tau}) + [\tau l(\tau + \n\]
\[
(1 - \sigma)(l - 1)(4l + 2n))i_{l+2}(\sqrt{\tau}) + \tau \sqrt{\tau}l(1 - \sigma)l(l - 1)i_{l+3}(\sqrt{\tau}) \right] \n\]
\[
= (2l + \sigma n + 1 - \sigma)i_{l+1}(\sqrt{\tau}) + \sqrt{\tau}i_{l+2}(\sqrt{\tau})^{-1} \left[ \tau l(\sigma n - \sigma + 2l + 1 + \sigma l - \sigma - l + 1) \n\right.
\]
\[
+ (1 - \sigma)l(l - 1)((l + n - 2)(\sigma n - \sigma + \sigma l + 1) + 3l^2 + 2nl - 2l)i_{l+1}(\sqrt{\tau}) \n\]
\[
+ \sqrt{\tau}l(1 - \sigma)(l - 1)(4l + 2n))i_{l+2}(\sqrt{\tau}) + \tau \sqrt{\tau}l(1 - \sigma)l(l - 1)i_{l+3}(\sqrt{\tau}) \right] \n\]
\[
= \tau l + (2l + \sigma n + 1 - \sigma)i_{l+1}(\sqrt{\tau}) + \sqrt{\tau}i_{l+2}(\sqrt{\tau})^{-1} \left[ \tau l(\sigma l - \sigma - l + 1) \n\right.
\]
\[
+ (1 - \sigma)l(l - 1)((l + n - 2)(\sigma n - \sigma + \sigma l + 1) + 3l^2 + 2nl - 2l)i_{l+1}(\sqrt{\tau}) \n\]
\[
+ \sqrt{\tau}l(1 - \sigma)(l - 1)(4l + 2n))i_{l+2}(\sqrt{\tau}) + \tau (1 - \sigma)l(l - 1)i_{l+3}(\sqrt{\tau}) \right] \n\]
\[
= \tau l + (2l + \sigma n + 1 - \sigma)i_{l+1}(\sqrt{\tau}) + \sqrt{\tau}i_{l+2}(\sqrt{\tau})^{-1} \left[ \tau l(\sigma l - \sigma - l + 1) \n\right.
\]
\[
+ (1 - \sigma)l(l - 1)((l + n - 2)(\sigma n - \sigma + \sigma l + 1) + 3l^2 + 2nl - 2l)i_{l+1}(\sqrt{\tau}) + \n\]
\[
\sqrt{\tau}(2l + n - 2)(l - 1)l(1 - \sigma)i_{l+2}(\sqrt{\tau}) \right]. \n\]
Hence, for \( l \geq 2 \), one has

\[
\lambda_{(l)} - \lambda_{(1)} = \tau l + \left( (2l + \sigma n + 1 - \sigma)i_{l+1}(\sqrt{\tau}) + \sqrt{\tau}i_{l+2}(\sqrt{\tau}) \right)^{-1} \left\{ \left[ \tau l(\sigma l - \sigma - l + 1) \right. \\
+ (1 - \sigma)(l-1)((l+n-2)(\sigma n - \sigma + \sigma l + 1) + 3l^2 + 2nl - 2l)i_{l+1}(\sqrt{\tau}) + \\
\sqrt{\tau}(2l+n-2)(l-1)l(1-\sigma)i_{l+2}(\sqrt{\tau}) \right] - \tau \\
= \tau(l-1) + \left( (2l + \sigma n + 1 - \sigma)i_{l+1}(\sqrt{\tau}) + \sqrt{\tau}i_{l+2}(\sqrt{\tau}) \right)^{-1} \left\{ \left[ \tau l(\sigma l - \sigma - l + 1) \right. \\
+ (1 - \sigma)(l-1)((l+n-2)(\sigma n - \sigma + \sigma l + 1) + 3l^2 + 2nl - 2l)i_{l+1}(\sqrt{\tau}) + \\
\sqrt{\tau}(2l+n-2)(l-1)l(1-\sigma)i_{l+2}(\sqrt{\tau}) \right].
\]

Since \( \sigma \in (-\frac{1}{n-1}, 1) \), it is easy to have \( \lambda_{(l)} - \lambda_{(1)} > 0 \), which implies

\[
\lambda_{(l)} > \lambda_{(1)} = \tau > 0.
\]

Therefore, we obtain that the first nonzero eigenvalue \( \lambda_2 \) of the problem (1.3) is \( \lambda_{(1)} \). For each \( l \in \mathbb{N} \), when \( \Omega = B_1 \) is the unit ball in \( \mathbb{R}^n \) centered at the origin, one knows

\[
\lambda_{(l)} = \inf_{\Omega} \frac{\int_{B_1} [(1 - \sigma)|D^2 u|^2 + \sigma|\nabla u|^2 + \tau|\nabla u|^2] \, dx}{\int_{\partial B_1} u^2 \, dS},
\]

where the infimum is taken among all functions \( u \) that are \( L^2(\partial B) \)-orthogonal to the first \( m - 1 \) eigenfunctions \( u_i \), with \( m \in \mathbb{N} \), such that \( \lambda_{(l)} = \lambda_m \) is the \( m \)-th eigenvalue of the eigenvalue problem (1.3). Then, after sorting \( \lambda_{(l)} \), it is not hard to obtain a non-negative spectrum of eigenvalues of the problem (1.3). As we all known, eigenfunctions of the form \( u_i = R_i(r)Y_i(\theta) \) should achieve the infimum in (3.10). This completes the proof of Theorem 3.2. \( \square \)

4 The isoperimetric inequality

The so-called *Fraenkel asymmetry* is the following: for any open set \( \Omega \in \mathbb{R}^n \) with finite measure,

\[
\mathcal{A}(\Omega) := \inf \left\{ \frac{\| \chi_\Omega - \chi_B \|_{L^1(\mathbb{R}^n)}}{|\Omega|} \left| B \text{ is the ball with } |B| = |\Omega| \right. \right\},
\]

where \( \mathcal{A}(\Omega) \) is the distance in the \( L^1(\mathbb{R}^n) \) norm of a set \( \Omega \) from the set of all balls of the same measure as \( \Omega \). This quantity turns out to be a suitable distance between sets for the purposes of stability estimates of eigenvalues.

In order to prove Theorem 1.1, we need the following two facts:

**Lemma 4.1.** ([3]) Let \( \Omega \) be an open set with Lipschitz boundary and \( p > 1 \). Then

\[
\int_{\partial \Omega} |x|^p \, dS \geq \int_{\partial \Omega^*} |x|^p \, dS \left( 1 + c_{n,p} \left( \frac{|\Omega \Delta \Omega^*|^2}{|\Omega|} \right)^{2} \right),
\]

where \( \Omega^* \) is the Fraenkel asymmetry of \( \Omega \).
where $\Omega^*$ is the ball centered at zero with the same measure as $\Omega$, $\Omega \Delta \Omega^*$ is the symmetric difference of $\Omega$ and $\Omega^*$, and $c_{n,p}$ is a constant depending only on $n$ and $p$ given by

$$c_{n,p} := \frac{(n + p - 1)(p - 1) \sqrt{2} - 1}{4} \left( \min_{t \in [1, \sqrt{2}]} t^{p - 1} \right).$$

Using a similar argument to that of [12, Theorem 1], we can get the following result.

**Lemma 4.2.** Let $\Omega$ be a bounded domain of class $C^1$ in $\mathbb{R}^n$. Then the eigenvalues of problem (1.3) on $\Omega$ satisfy

$$\sum_{l=k+1}^{k+n} \frac{1}{\lambda_l(\Omega)} = \max \left\{ \sum_{l=k+1}^{k+n} \int_{\partial \Omega} v_l^2 dS \right\}. \quad (4.1)$$

Moreover, if the families $\{v_l\}_{l=k+1}^{k+n}$ satisfy

$$\int_{\Omega} \left[ (1 - \sigma) D^2 v_i : D^2 v_j + \sigma \Delta v_i \cdot \Delta v_j + \tau \nabla v_i : \nabla v_j \right] dx = \delta_{ij}$$

and $\int_{\partial \Omega} v_i u_j dS = 0$ for all $i = k + 1, \ldots, k + n$ in $H^2(\Omega)$, where $u_1, u_2, \ldots, u_k$ are the first $k$ eigen-functions of the problem (1.3), then the maximum value can be attained.

Now, we can prove:

**Theorem 4.3.** For every domain $\Omega$ in $\mathbb{R}^n$ of class $C^1$, we have

$$\lambda_2(\Omega) \leq \lambda_2(\Omega^*) \left( 1 - \delta_{n,\mathcal{O}}(\Omega)^2 \right), \quad (4.2)$$

where $\delta_n$ is given by

$$\delta_n := \frac{n + 1}{8n} (\sqrt{2} - 1),$$

and $\Omega^*$ is a ball with the same measure as $\Omega$.

**Proof.** We divide our proof into two steps:

**Step 1.** Assume first that $\Omega$ is a bounded domain of class $C^1$ in $\mathbb{R}^n$ with the same measure as the unit Euclidean ball $B$. Let trial functions be defined by $v_l = (\tau |\Omega|)^{-\frac{1}{2}} x_l$, with $l = 2, \ldots, n + 1$, such that they have zero integral mean over $\partial \Omega$. This can be always assured. In fact, by coordinate transformation $x = y - \frac{1}{|\partial \Omega|} \int_{\partial \Omega} y dS$, it is easy to know that the trial functions have zero integral mean over $\partial \Omega$. Moreover, functions $v_l$ also satisfy the normalization condition of Lemma 4.2. Choosing $k = 1, l = 2, p = 2$, and applying Lemma 4.1, we can obtain

$$\sum_{l=2}^{n+1} \frac{1}{\lambda_l(\Omega)} \geq \frac{1}{\tau |\Omega|} \int_{\partial \Omega} |x|^2 dS$$

$$\geq \frac{1}{\tau |\Omega|} \left( 1 + c_{n,2} \left( \frac{|\Omega \Delta B|}{|\Omega|} \right)^2 \right) \int_{\partial B} |x|^2 dS$$

$$\geq \sum_{l=2}^{n+1} \frac{1}{\lambda_l(B)} \left( 1 + c_{n,2} \left( \frac{|\Omega \Delta B|}{|\Omega|} \right)^2 \right).$$
Recall that \( \lambda_2(\Omega) \leq \lambda_l(\Omega) \) for \( l \geq 3 \). Then, combing (4.3) and the definition of \( \mathcal{A}(\Omega) \), one has

\[
\lambda_2(\Omega)(1 + c_{n,2}\mathcal{A}(\Omega)^2) \leq \lambda_2(B_1).
\]

Clearly, the above inequality implies the eigenvalue inequality with \( \Omega \) also holds for general bounded domains of class \( C_\lambda \). Theorem 1.1 tells us that for the functional \( \mathcal{A}(\Omega) \), and the fact that \( A \)

Further study

Step 1. This step deals with the proof for general finite values of \( |\Omega| \), which relies on the scaling properties of the eigenvalues. For all \( s > 0 \), noting

\[ s\Omega := \left\{ x \in \mathbb{R}^n \middle| \frac{x}{s} \in \Omega \right\}. \]

Let \( \lambda(\tau, \sigma, \Omega) \) be an eigenvalue of the eigenvalue problem (1.3). For any \( u \in H^2(\Omega) \) with \( f_\Omega u dx = 0 \), let \( \tilde{u}(x) \) be a valid trial function on \( s\Omega \) that makes \( \tilde{u}(x) = u(\frac{x}{s}) \), and then the Rayleigh quotient \( \mathcal{Q}_{s^{-2}\tau, \sigma, s\Omega}[\tilde{u}] \) can be treated as follows:

\[
\mathcal{Q}_{s^{-2}\tau, \sigma, s\Omega}[\tilde{u}] = \frac{\int_{s\Omega} [(1 - \sigma)|D^2\tilde{u}|^2 + \sigma|\Delta\tilde{u}|^2 + s^{-2}\tau|\nabla\tilde{u}|^2] dx}{\int_{\partial s\Omega} \tilde{u}^2 dS} = \frac{\int_{s\Omega} [(1 - \sigma)s^{-2}D^2u(\frac{x}{s})^2 + \sigma s^{-2}\Delta u(\frac{x}{s})^2 + s^{-2}\tau s^{-1}\nabla u(\frac{x}{s})^2] dx}{\int_{\partial s\Omega} u(\frac{x}{s})^2 dS}
\]

(taking \( y = \frac{x}{s} \))

\[
=s^{-4+n}\int_{\Omega} [(1 - \sigma)|D^2u|^2 + \sigma|\Delta u|^2 + \tau|\nabla u|^2] dy
\]

\[
= s^{-3} \mathcal{Q}_{\tau, \sigma, \Omega}[u],
\]

which implies

\[
\lambda(\tau, \sigma, \Omega) = s^3 \lambda(s^{-2}\tau, \sigma, s\Omega).
\]

Together this scaling property with the conclusion in Step 1, we know that the eigenvalue inequality (4.2) also holds for general bounded domains of class \( C^1 \) in \( \mathbb{R}^n \). This completes the proof of Theorem 4.3.

Proof of Theorem 4.4. The conclusion of Theorem 4.1 follows directly by applying Theorem 4.3 and the fact that \( \mathcal{A}(\Omega^*) = 0 \).

5 Further study

Theorem 1.1 tells us that for the functional \( \lambda_2(\cdot) : \Omega \mapsto \mathbb{R}^+ \) with \( |\Omega| \) fixed, \( \Omega^* \) is a maximum value of \( \lambda_2(\cdot) \). A natural question is:

Open problem. Is \( \Omega^* \) the only maximum value of the functional \( \lambda_2(\cdot) \) defined above? Can we get the rigidity from the isoperimetric inequality \( \lambda_2(\Omega) \leq \lambda_2(\Omega^*) \) with \( |\Omega| \) fixed? Is the claim “if the equality in Theorem 4.1 holds, then \( \Omega \) must be \( \Omega^* \)” true?

We believe that the answer to the above Open problem is positive. However, so far, we do not know how to prove it. We have tried the approach shown in [5, Subsection 4.1] to explain that \( \Omega^* \) would be the unique critical point of the functional \( \lambda_2(\cdot) \), but we fail. Clearly, we definitely hope that this difficulty can be overcome in the future.
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