Self-Gravitating Non-Abelian Cosmic String Solution

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Abstract

The coupled Einstein-Yang-Mills equations on a time dependent axially symmetric spacetime are investigated, without imposing a priori any conditions on the gauge field. There is numerical evidence for the existence of a regular solution with the desired asymptotic features. Just as in the supermassive abelian counterpart model, the formation of a singularity at finite distance of the core of the string depends critically on a parameter of the model, i.e., the constant value of one of the magnetic components of the YM potentials. The multiple-scale method could supply decisive answers concerning the stability of the solution.

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I. INTRODUCTION

After the astonishing numerical solution of Bartnik and McKinnon [1] of the static spherically symmetric Einstein-Yang-Mills equations with a SU(2) gauge group, a fundamental question arose: do there exist essentially non-abelian static globally regular solutions or blackholes, which carry non-zero electric and/or magnetic charges [2,3]? In first instance, one could prove in the spherically symmetric case [4] the existence of static solutions which are asymptotically flat and with finite mass. The members of this family of solutions could be characterized by the number of zeros of one of the gauge potentials. In fact the repulsive Yang-Mills force can balance the gravitational attractive force and prevent the formation of singularities in spacetime. Later [5] the class of solutions were extended to blackhole solutions with a horizon at some $r_h$ and oscillating solutions which are not asymptotically flat. These non-abelian blackholes could represent counter examples to the ‘no-hair’ conjecture. Further, it was claimed [6] that both regular and blackhole solution in the spherically symmetric case will have gravitational-like and sphaleron-like instabilities. When it was found that the BM solution is unstable, a lot of researchers (see for example [7,8]) tackled the general stability problem of the coupled EYM system and investigated the possible critical behavior of the solutions. It was realized that they had much in common with the electroweak sphalerons. In the spherical symmetric case the critical behavior was formulated by Choptuik [9,10]. It occurred at the boundary in phase space between initial data which eventually form a blackhole and data which do not. Recently it was found, in a two dimensional Weinberg-Salam model with an axially symmetric ansatz, that there is evidence for the existence of an electrically charged sphaleron state [11]. Here we consider the EYM system on an axially symmetric time dependent spacetime and will compare our investigations with the spherical symmetric case and the well known abelian cosmic string solution [12,13], particularly the supermassive case. In these U(1)-gauge cosmic string models it was found that as the energy scale of symmetry breaking increases, the geometry around the string changes from conical to an analog of a Kasner spacetime. But supermassive cosmic strings
may also arise at GUT scales if the coupling between scalar and gauge fields is weak. Further, these low-energy supermassive strings are closely related to global strings: they both show singular behavior at finite distance from the core of the string. We conjecture that the string-like solution in the Eintein-Yang-Mills model shows some similar behavior. The plan of this paper is as follows. In section II we derive the time dependent equations for the Einstein-Yang-Mills system on an axially symmetric spacetime using the algebraic manipulation program MAPLE. In section III we consider the static situation in order to gain insight into the asymptotic and singular behavior. In section IV we solve the equations numerically and in section V we present a conclusion and prospect concerning the stability.

We conjecture that conventional (linear) stability analysis is inadequate to be applied to the situation where a singularity is formed. The reason is that the oscillatory behavior of some of the gauge field components are of high-frequency. We suggest to apply the multiple-scale method [16–18], a method suitable for handling high-frequency perturbations. In a future work, we will extend the investigation to the spinning case, as initiated before [19,20].

II. THE FIELD EQUATIONS

Consider the Lagrangian of the SU(2) EYM system

$$S = \int d^4x \sqrt{-\det g} \left[ \frac{\mathcal{R}}{16\pi G} - \frac{1}{4} \mathcal{F}^a_{\mu\nu} \mathcal{F}^{\mu\nu a} \right], \quad (2.1)$$

with the YM field strength

$$\mathcal{F}^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g \epsilon^{abc} A^b_\mu A^c_\nu, \quad (2.2)$$

$g$ the gauge coupling constant, $G$ Newton’s constant, $A^a_\mu$ the gauge potential and $\mathcal{R}$ the curvature scalar. The field equations then become

$$G_{\mu\nu} = -8\pi G \mathcal{T}_{\mu\nu}, \quad (2.3)$$

$$\mathcal{D}_\mu \mathcal{F}^{\mu\nu a} = 0, \quad (2.4)$$
with \( \mathcal{T} \) the energy momentum tensor

\[
\mathcal{T}_{\mu\nu} = \mathcal{F}_\mu^a \mathcal{F}_\nu^{\alpha a} - \frac{1}{4} g_{\mu\nu} \mathcal{F}_\alpha^{\alpha \beta} \mathcal{F}_\beta^{\alpha \beta a}.
\]  

(2.5)

We are interested in solutions on the cylindrical symmetric spacetime

\[
ds^2 = -e^{2(K-U)}(dt^2 - dr^2) + e^{-2U}W^2d\varphi^2 + e^{2U}dz^2,
\]  

(2.6)

where \( U, K \) and \( W \) are functions of \( t \) and \( r \). This cylindrical symmetric line element can formally be obtained from the stationary axisymmetric line element used by Kleihaus and Kunz [21]

\[
ds^2 = -fdt^2 + m f (dr^2 + dz^2) + lr^2 f d\varphi^2,
\]  

(2.7)

with \( f, m \) and \( l \) functions of \( r \) and \( z \), by the complex substitution \( t \rightarrow iz \) and \( z \rightarrow it \) [22]. Following Bais and Sasaki [23] we then can specify the gauge field potentials \( A_\mu = A_\mu^a t_a \), where the \( t_a \) are the anti-hermitian generators of the gauge group, by

\[
A_\mu dx^\mu = A_0 \tilde{\tau}_\varphi dt + A_1 \tilde{\tau}_r dr + [\eta_2 \tilde{\tau}_r + \eta_1 \tilde{\tau}_z] dz + [\chi_2 \tilde{\tau}_r + \chi_1 \tilde{\tau}_z] d\varphi,
\]  

(2.8)

with \( A_i, \eta_i \) and \( \chi_i \) functions of \( t \) and \( r \) and where \( \tilde{\tau}_\varphi, \tilde{\tau}_r \) and \( \tilde{\tau}_z \) are the axial generators of the SU(2) normalized such that \([\tilde{\tau}_i, \tilde{\tau}_j] = \epsilon_{ijk} \tilde{\tau}_k\). One can reduce the number of gauge field potentials. On the Euclidean spacetime, the self-duality condition eliminates two of the six functions [24]. One also can use the additional U(1) gauge freedom on the \( A_\mu^a \), \( A \rightarrow h^{-1} Ah + h^{-1} dh \), with \( h = \exp[\psi(r,t)\tilde{\tau}_\varphi] \) [2]. Here we use the condition that on the spacetime (2.6), the energy-momentum component \( \mathcal{T}_{z\varphi} \) must vanish. The most simple way to fulfil this condition is \( \eta_1 = \chi_2 \) and \( \eta_2 = \chi_1 \). There are some other possibilities, comparable with those found by Manton [24]. However, these are quite complicated as already noticed by Manton. So in our case, \( \mathcal{T}_{z\varphi} = 0 \) becomes

\[
\mathcal{T}_{z\varphi} = g\eta_1(A_1^2 - A_0^2)(1 + 2g\eta_2) + 2(\partial_t \eta_1 \partial_r \eta_2 - \partial_t \eta_1 \partial_t \eta_2) + A_0 \partial_r \eta_2 + A_1 \partial_r \eta_2 = 0.
\]  

(2.9)

We then have two possibilities: i. \( \eta_2 = \text{constant} = \eta_0 \) and \( A_0 = \pm A_1 \), or, ii. \( \eta_2 = \frac{-1}{2g} \). We consider here case i. The condition of case ii means that the YM mass scale \( M_{YM} \equiv g\eta_0 = -\frac{1}{2} \) In section 3 we shall see that it corresponds in the static case with a singular solution.
The differential equations for the several field variables become (from now on we set 
\( \eta_1 \equiv \eta \))

\[
\partial_t^2 \eta - \partial_r^2 \eta = - \frac{1}{W} \partial_t \eta (\partial_t W - 2W \partial_t U) + \frac{1}{W} \partial_r \eta (2W \partial_r U - \partial_r W) - g_0 (\partial_r A_0 + \partial_t A_0) + \frac{g_\eta A_0}{W} (2W \partial_t U + 2W \partial_r U - \partial_r W - \partial_t W) + \frac{g_\eta e^{2K}}{W^2} [\eta_0 + g (\eta_0^2 - \eta^2)],
\] (2.10)

\[
\partial_t^2 A_0 - \partial_r^2 A_0 = \frac{e^{2K}}{W^2} [g_\eta_0 (1 - W^2 e^{-4U}) + 1] (\partial_t \eta - \partial_r \eta) \\
+ (\partial_r A_0 + \partial_t A_0) (\frac{\partial_r W}{W} - \frac{\partial_t W}{W} + 2 \partial_r U - 2 \partial_t U + 2 \partial_r K - 2 \partial_t K)
\]
\[-2 \frac{e^{2K} A_0}{W^2} [g^2 \eta^2 + (1 + 2g \eta)] - 2e^{2K-4U} A_0 g^2 (\eta^2 + \eta_0^2),
\] (2.11)

\[
\partial_t^2 W - \partial_r^2 W = - \frac{8\pi G}{W} [e^{2K-2U} [g (\eta_0^2 \eta^2) + \eta_0]^2 + e^{-2K+2U} W^2 (\partial_t A_0 + \partial_r A_0)^2],
\] (2.12)

\[
\partial_t^2 K - \partial_r^2 K = (\partial_t U)^2 - (\partial_r U)^2 + \frac{4\pi G}{W} [e^{-2U+2K} [g (\eta_0^2 \eta^2) + \eta_0]^2 \\
+ W^2 e^{2U-2K} (\partial_t A_0 + \partial_r A_0)^2 - (e^{2U} - W^2 e^{-2U}) (\partial_t \eta)^2 - (\partial_r \eta)^2)
\]
\[+ 2A_0 [ g_\eta_0 W^2 \partial_t \eta + (g_\eta_0 + 1) e^{2U} \partial_r \eta + \partial_t \eta] \]
(2.13)

and

\[
\partial_t^2 U - \partial_r^2 U = \partial_t U \partial_t \ln W - \partial_r U \partial_r \ln W - \frac{4\pi G}{W^2} [e^{-2U+2K} [g (\eta_0^2 \eta^2) + \eta_0]^2 \\
+ W^2 e^{2U-2K} (\partial_t A_0 + \partial_r A_0)^2 + (e^{2U} - W^2 e^{-2U}) (\partial_t \eta)^2 - (\partial_r \eta)^2) \\
- 2A_0 [g_\eta_0 W^2 e^{-2U} + (g_\eta_0 + 1) e^{2U}] (\partial_r \eta + \partial_t \eta)].
\] (2.14)

From two combinations of the Yang-Mills equations (2.4), i.e.,

\[
[YM]_{\nu=\varphi,a=3} \pm \left( \cos \varphi \cdot [YM]_{\nu=\varphi,a=1} + \sin \varphi \cdot [YM]_{\nu=\varphi,a=2} \right),
\]

we obtain an expression for \( A_0 \),

\[
A_0 = \frac{e^{2K-4U} W^2 [\eta_0^2 + g_\eta_0 (\eta_0^2 - \eta^2)] + e^{2K} [\eta_0 (1 + g \eta_0^2) - \eta^2 (1 + g \eta_0)]}{2\eta W [\partial_t W + \partial_r W - 2W (\partial_t U + \partial_r U)]}
\] (2.15)

and a first order equation for \( A_0 \).
$$\partial_t A_0 + \partial_r A_0 = -\frac{2A_0}{\eta}(\partial_t \eta + \partial_r \eta) - \frac{e^{2K-4U}}{2\eta}[\eta_0^2 + g\eta_0(\eta_0^2 - \eta^2)]$$

$$-\frac{e^{2K}}{2g\eta W^2}[g\eta^2(1 + g\eta_0) - \eta_0(1 + g\eta_0)^2].$$

(2.16)

In the abelian situation, the equations could be simplified, due to the condition $T_{tt} = -T_{zz}$, and one obtains [15] $K = 2U$. This is the familiar self-gravitating Nielsen-Olesen vortex model studied first by Garfinkle [12] in the static case. In our case we obtain the condition

$$e^{-4U}W^2T_{zz} + T_{\varphi\varphi} - (T_{tt} - T_{rr})W^2e^{-2K} = 0,$$

(2.17)

which yields

$$\partial_r^2 K - \partial_t^2 K = \partial_r^2 U - \partial_t^2 U + \frac{1}{W}(\partial_r U \partial_r W - \partial_t U \partial_t W) - (\partial_r U)^2 + (\partial_t U)^2 - \frac{1}{W}(\partial_t^2 W - \partial_r^2 W).$$

(2.18)

This relation can be used as constraint in the numerical code.

### III. THE STATIC CASE

In order to obtain boundary conditions for the numerical integration of the differential equations, we first consider the static situation. The field equations reduce to

$$\partial_r^2 W = -\frac{8\pi G}{W}[e^{-2K+2U}W^2(\partial_r A_0)^2 + e^{2K-2U}[\eta_0 + g(\eta_0^2 - \eta^2)]^2],$$

(3.1)

$$\partial_r[W \partial_r U] = -\frac{4\pi G}{W}[(W^2e^{-2U} - e^{2U})(\partial_r \eta)^2 + W^2e^{2U-2K}(\partial_r A_0)^2$$

$$-2[(g\eta_0 + 1)e^{2U} + g\eta_0 W^2e^{-2U}]A_0 \partial_r \eta + e^{2K-2U}[\eta_0 + g(\eta_0^2 - \eta^2)]^2].$$

(3.2)

$$\partial_r[W \partial_r K] = \frac{8\pi G}{W}[(g^2(\eta_0^2 + \eta^2)A_0^2W^2e^{-2U} + e^{2U}[\partial_r \eta + (1 + g\eta_0)A_0]^2 + g^2e^{2U}\eta^2 A_0^2].$$

(3.3)

$$\partial_r^2 \eta = -\frac{1}{2}\partial_r A_0 + \frac{(1 + 2g\eta_0)A_0}{2W}(\partial_r W - 2W \partial_r U)$$

$$-\frac{g\eta}{2W^2}e^{2K}(W^2e^{-4U} + 1)[\eta_0 + g(\eta_0^2 - \eta^2)].$$

(3.4)

and
\[ \partial_r^2 A_0 = \partial_r A_0 (2 \partial_r K - \partial_r W - 2 \partial_r U) + \frac{A_0}{W^2} e^{2K} [(1 + g \eta_0)^2 + g^2 \eta^2] + g^2 e^{2K-4U} A_0 (\eta_0^2 + \eta^2). \tag{3.5} \]

Further, (2.15) now reads
\[ A_0 = \frac{e^{2K-4U} W^2 [\eta_0^2 + g \eta_0 (\eta_0^2 - \eta^2)] + e^{2K} \left[ \frac{\eta_0}{g} (1 + g \eta_0)^2 - \eta^2 (1 + g \eta_0) \right]}{2 \eta W (\partial_r W - 2 \partial_r U)}. \tag{3.6} \]

From the condition that in the static situation now \( T_{tr} = 0 \), we obtain
\[
\partial_r \eta = \frac{g^2 A_0 W^2 (\eta_0^2 + \eta^2) + A_0 e^{4U} [(1 + g \eta_0)^2 + g^2 \eta^2]}{g \eta_0 (W^2 - e^{4U}) - e^{4U}}, \tag{3.7}
\]
which is consistent with one of the YM equations. For the special case \( g \eta_0 = -\frac{1}{2} \) we obtain from (3.7)
\[
\frac{\partial_r \eta}{(\frac{1}{g \eta_0})^2 + 1} = -\frac{1}{2} A_0, \tag{3.8}
\]
which can be integrated together with (3.6) for some given spacetime. However, we observe from (3.6) that \( A_0 \) becomes singular for \( W \to e^{2U} \). We then obtain from the YM equations just the condition \( g \eta_0 = -\frac{1}{2} \).

The asymptotic features of the system can be analyzed by following Garfinkle [14]. We write the stress tensor on an orthonormal basis as
\[
T_{\mu \nu} = \sigma \hat{t}_{\mu} \hat{t}_{\nu} + \mathcal{P}_{r} \hat{r}_{\mu} \hat{r}_{\nu} + \mathcal{P}_{z} \hat{z}_{\mu} \hat{z}_{\nu} + \mathcal{P}_{\phi} \hat{\phi}_{\mu} \hat{\phi}_{\nu}, \tag{3.9}
\]
where \( \hat{t}_{\mu} = e^{-K+U} \left( \frac{\partial}{\partial t} \right)^\mu, \hat{r}_{\mu} = e^{-K+U} \left( \frac{\partial}{\partial r} \right)^\mu, \hat{z}_{\mu} = e^{-U} \left( \frac{\partial}{\partial z} \right)^\mu, \hat{\phi}_{\mu} = \frac{e^{U}}{W} \left( \frac{\partial}{\partial \phi} \right)^\mu \). Using the conservation of stress energy
\[
\nabla_\mu T^{\mu \nu} = 0, \tag{3.10}
\]
one obtains
\[
\frac{d}{dr} [W \mathcal{P}_{r}] = (\sigma + \mathcal{P}_{r} + \mathcal{P}_{z} - \mathcal{P}_{\phi}) \Theta_1 - (\sigma + \mathcal{P}_{r}) \Theta_3 + \mathcal{P}_{\phi} \Theta_2, \tag{3.11}
\]
where
\[ \Theta_1 \equiv W \partial_r U, \quad \Theta_2 \equiv \partial_r W, \quad \Theta_3 \equiv W \partial_r K. \]  \hfill (3.12)

Let us assume that

\[
\lim_{r \to \infty} 8\pi GW^2 e^{2K-2U} \mathcal{P}_r \to 0, \hfill (3.13)
\]

and that \( \Theta_i \) approach constant values as \( r \to \infty \) \[12\], which are fairly weak assumptions. Then one can write with the help of (3.1)-(3.3)

\[
\frac{d}{dr} \left[ 8\pi GW^2 e^{2K-2U} \mathcal{P}_r \right] = \frac{d}{dr} (\Theta_2 \Theta_3 - \Theta_1^2). \hfill (3.14)
\]

One then obtains the asymptotic condition

\[
\lim_{r \to \infty} (\Theta_2 \Theta_3 - \Theta_1^2) = 0. \hfill (3.15)
\]

So one has the two possibilities

\[
i : \quad W \partial_r U|_{\infty} = 0, \quad W \partial_r W \partial_r K|_{\infty} = 0
\]

\[
ii : \quad (W \partial_r U|_{\infty})^2 = W \partial_r W \partial_r K|_{\infty}. \hfill (3.16)
\]

If we denote by \( W_{\infty}, U_{\infty} \) and \( K_{\infty} \) the values of the metric fields far from the string, one obtains in the first case: \( U_{\infty} = a_1, W_{\infty} = a_2 r + a_3 \) and \( K_{\infty} = a_4 \), where the \( a_i \) are constants. The metric approaches in this case a conical spacetime \[13\]

\[
ds^2 = e^{2(a_4-a_1)}[-dt^2 + dz^2] + e^{2a_1}dz^2 + e^{-2a_1}(a_2 r + a_3)^2 d\phi^2. \hfill (3.17)
\]

In the second case one obtains in the special case \( K = 2U \) the Kasner-like spacetime

\[
ds^2 = -(b_1 r + b_2)^4 e^{2b_3}[dt^2 - dr^2 - dz^2] + \frac{e^{-2b_4}}{(b_1 r + b_2)^2} d\phi^2, \hfill (3.18)
\]

with \( b_i \) constants. Now we can investigate the asymptotic behavior of \( \eta \) and \( A_0 \) on the conical spacetime (3.17) and compare the asymptotic features with the abelian counterpart model. Substituting the conical spacetime (3.17) into (3.6) and (3.7) one obtains solutions for \( \eta \) and \( A_0 \). The solution is not sensitive for \( a_1, a_3 \) and \( a_4 \). For positive \( a_2 \) the solution will possess singular behavior at finite distance of the core, as expected from the abelian investigations.
of Laguna and Garfinkle [13]. In figure 1-3 we plotted \( \eta \) and \( A_0 \) for \( a_1 = a_2 = a_4 = 1 \) and \( a_3 = 2 \) and for different values of \( \eta_0 \). It is observed that the singularity is pushed to infinity for smaller values of \( \eta_0 \) and \( a_2 > 1 \). This kind of singular behavior is also encountered in the abelian Higgs model [13,14] when the scale of the symmetry breaking is far beyond the GUT scale, i.e., \( 8\pi G \nu^2 >> 10^{-6} \), where \( \nu \) is the vacuum expectation value of the Higgs field. In fact, the conical picture of the string then fades away. It was found that the metric of the transition between conical and Kasner-like is cylindrical, i.e., \( R^3 \times S^1 \). Further it was found that supermassive strings can be formed at GUT scales if \( \alpha \equiv \frac{4\pi Ge^2}{\lambda} \), where \( e \) is the gauge coupling constant in the abelian model and \( \lambda \) the Higgs self-coupling constant, is very small. The singularity must then occur at a distance from the core of the string which is many orders of magnitude greater than the present Hubble scale. In our model we find a comparable dependency on \( \eta_0 \) of the behavior of the singularity. In general, however, one has to solve simultaneously the coupled system. Moreover, one should make a distinction between the interior and exterior field equations with proper matching conditions, in order to get insight in the behavior close to the z-axis [20].

IV. NUMERICAL SOLUTION

The set of equation (2.10)-(2.14) can be solved numerically. We used the ISML software package for numerically solving coupled systems of nonlinear partial differential equations. The package implements finite element collocation methods based on piecewise polynomials for the spatial discretization techniques. The time integration process is then accomplished for a set of ordinary differential equations using banded Jacobians. For the order of the piecewise polynomial space we took 5 and for the number of subintervals into which the spatial domain is to be divided we took 11. The relative error bound was \( 10^{-11} \). In order to obtain from (2.15) a regular and asymptotically correct initial value for \( A_0 \), we choose as initial values \( \eta(r,0) = e^{-r^2} + 1 \), \( U(r,0) = K(r,0) = 0 \) and \( W(r,0) = r + 1 \). Figures 4, 5 and 6 show a typical regular solution of \( A_0, \eta \) and the metric component \( e^{2K-2U} \) for some values.
of $\eta_0$ and $4\pi G$ and suitable boundary conditions. The behavior of $\eta$ remains regular everywhere. In figure 7 we plotted $A_0$ for a smaller value of $\eta_0$. We observe that $A_0$ approaches asymptotically a constant value. For runs where $A_0$ starts to oscillate, $e^{2K-2U}$ decreases strongly, signalling the formation of a singularity. However, a thoroughly investigation of the dependency on $g, \eta_0$ and $G$ will be necessary. This is currently under study.

V. CONCLUSIONS AND OUTLOOK

We investigated the existence of a possible solution of the coupled EYM system on a time-dependent axially symmetric spacetime. At least on the conical spacetime we find evidence for regular behavior of the electric and magnetic components of the YM field for suitable values of the parameters $\eta_0$, $g$ and $4\pi G$. The formation of a singularity at finite distance of the core of the string depends critically on $\eta_0$, the constant value of one of the gauge field potentials, and $4\pi G$. Just as in the static spherically symmetric EYMH case, where the ratio $\frac{M_{YM}}{M_{Pl}} = \frac{g\nu}{\sqrt{4\pi G}}$ ($\nu$ is the vacuum expectation value of the Higgs field) plays a crucial role in the behavior of the solution, we find a crucial dependency on the ratio $\frac{M_{YM}}{M_{Pl}} = \frac{\eta_0}{\sqrt{4\pi G}}$. In the abelian supermassive model [14] a similar behavior is encountered where the singularity arises at finite distance from the core of the string not only for large symmetry breaking scales as found by [13], but also for GUT scale. There is for fixed symmetry breaking scale of order of the GUT scale a critical value for the coupling of the scalar to the gauge field (i.e., $\frac{4\pi Ge^2}{\lambda}$ with $e$ the coupling constant and $\lambda$ the Higgs self coupling) for which the singularity occurs at finite distance of the core. These low energy solutions are more realistic because the larger the energy scale the larger the angle deficit and it no longer makes sense to talk about string type solutions. In our model, there will be a critical value $\frac{\eta_0}{\sqrt{4\pi G}}$ for which the singularity is pushed to infinity.

In order to analyze the stability of the solutions, one usually linearizes the field equations [6], or one expands the field variables in a physically unclear small parameter [25]. One better can apply the so called multiple-scale (or two-timing) method, developed decades ago
by Taub [16] and Choquet-Bruhat [17]. This method is particularly useful for constructing uniformly valid approximations to solutions of perturbation problems. The idea is to expand the several field variables in power series of the ratio of the characteristic wavelength of the perturbations and the characteristic dimension of the background. One writes [18]

$$
g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{1}{\omega} h_{\mu\nu}(x^\sigma; \xi) + \frac{1}{\omega^2} k_{\mu\nu}(x^\sigma; \xi) + ...$$

$$A_\mu^a = \bar{A}_\mu^a + \frac{1}{\omega} B_\mu^a(x^\sigma; \xi) + \frac{1}{\omega^2} C_\mu^a(x^\sigma; \xi) + ...,$$  \hspace{1cm} (5.1)

where \( \xi \equiv \omega \Pi(x^\sigma) \) and \( \Pi \) a phase function. The parameter \( \omega \) measures the ratio of the fast scale to the slow one. The rapid variation only occur in the direction of the vector \( l_\sigma \equiv \frac{\partial \Pi}{\partial x^\sigma} \).

For a function \( \Psi(x^\sigma; \xi) \) one has

$$\frac{\partial \Psi}{\partial x^\sigma} = \partial_\sigma \Psi + \omega l_\sigma \dot{\Psi},$$  \hspace{1cm} (5.2)

where \( \partial_\sigma \Psi \equiv \frac{\partial \Psi}{\partial x^\sigma} |_{\xi fixed} \) and \( \dot{\Psi} \equiv \frac{\partial \Psi}{\partial \xi} |_{x^\sigma fixed} \). Substituting the expansions of the field variables into the equations and collecting terms of equal orders of \( \omega \), one obtains propagation equations for \( \dot{B}_\mu^a \) and \( \dot{h}_{\mu\nu} \) and 'back-reaction' equations for \( \bar{h}_{\mu\nu} \) and \( \bar{A}_\mu^a \). It is clear from the propagation equation that there will be a coupling between the high-frequency gravitational field and the high-frequency behavior of \( A_0 \) when the singularity will be approached. On his turn, \( A_0 \) will create a high-frequency perturbation in \( \eta \).

In a subsequent paper we will present this investigation.
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FIG. 1. Plot of the gauge components $A_0$ and $\eta$ for $\eta_0 = 0.5$. The singularity appears when $\eta$ approaches zero.
FIG. 2. As figure 1, with $\eta_0 = 0.2$. The singularity is encountered at larger r value.
FIG. 3. As figure 1, with \( \eta_0 = 0.01 \). The singularity is pushed to infinity.
FIG. 4. *Plot of a long time-run of the gauge component* $A_0$ *for* $g = -1$, $4\pi G = 0.2$ *and* $\eta_0 = 0.2$.

FIG. 5. *Plot of* $\eta$ *for the situation of figure 4*.
FIG. 6. Plot of the metric component $e^{2K - 2U}$ for the situation of figure 4.

FIG. 7. As figure 4, but now for $\eta_0 = 0.01$. 

