HIGH ENERGY EVOLUTION BEYOND
THE BALITSKY - KOVCHEGOV EQUATION *

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We present a short review of recent theoretical activity which is an attempt to learn
about low x evolution beyond the Balitsky - Kovchegov equation.

1 Balitsky-Kovchegov equation

The Balitsky - Kovchegov equation (BK) [1, 2, 3, 4, 5] is the best presently available tool
to study saturation phenomena at high energies. Contrary to many models the BK has
solid grounds in perturbative QCD. The equation reads

\[ \frac{d N(x_{01}, y; b)}{d y} = \frac{\alpha_s}{2} \int d^2 x_2 \frac{x_{01}^2}{x_{02}^2 x_{12}^2} \times \]

\[ \left( 2 N(x_{02}, y; b - \frac{1}{2} x_{12}) - N(x_{01}, y; b) - N(x_{02}, y; b - \frac{1}{2} x_{12}) N(x_{12}, y; b - \frac{1}{2} x_{02}) \right) \]

The function \( N(r_\perp, y; b) \) stands for imaginary part of the amplitude for a dipole of size
\( r_\perp = x_{10} \) elastically scattered at the impact parameter \( b \). Eq. (1) has a simple meaning:
the dipole of size \( x_{10} \) decays in two dipoles of sizes \( x_{12} \) and \( x_{02} \) with the decay probability
\( \frac{x_{01}^2}{x_{02}^2 x_{12}^2} \). These two dipoles then interact with the target. The non-linear term takes into
account a simultaneous interaction of two produced dipoles with the target. The linear
part of Eq. (1) is the LO BFKL equation, which describes the evolution of the multiplicity
of the fixed size color dipoles with respect to the rapidity (energy) \( y \). For the discussion
below we introduce a short notation for the BK: \( \frac{d N}{d y} = \alpha_s K \epsilon r \otimes (N - N N) \).

The theoretical success associated with the BK is based on the following facts: • The
BK is based on the correct high energy dynamics which is taken into account via the LO
BFKL evolution kernel. • The BK restores the s-channel unitarity of partial waves (fixed

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impact parameter) which is badly violated by the linear BFKL evolution. • The BK is believed to describe gluon saturation, a phenomenon expected at high energies. • The BK resolves the infrared diffusion problem associated with the linear BFKL evolution. This means the equation is much more stable with respect to possible corrections coming from the nonperturbative domain. • The BK has met with phenomenological successes when confronted against DIS data from HERA [6, 7, 8].

The BK is not exact and has been derived in several approximations. • The LO BFKL kernel is obtained in the leading soft gluon emission approximation and at fixed $\alpha_s$. • The Large $N_c$ limit is used in order to express the nonlinear term as a product of two functions $N$. This limit is in the foundation of the color dipole picture. To a large extent the large $N_c$ limit is equivalent to a mean field theory without dipole correlations. • The BK assumes no target correlations. Contrary to the large $N_c$ limit which is a controllable approximation within perturbative QCD, the absence of target correlations is of pure nonperturbative nature. This assumption has grounds for asymptotically heavy nuclei, but it is likely not be valid for proton or realistic nucleus target.

There are several quite serious theoretical problems which need to be resolved in the future. • The BK is not symmetric with respect to target and projectile. While the latter is assumed to be small and perturbative, the former is treated as a large nonperturbative object. The fan structure of the diagrams summed by the BK violates the t-channel unitarity. A first step towards restoration of the t-channel unitarity would be an inclusion of Pomeron loops.

• Though the BK respects the s-channel unitarity \(^1\) it violates the Froissart bound for energy behavior of the total cross section. One needs gluon saturation and confinement in order to respect the Froissart bound. On one hand, the BK provides the gluon saturation at fixed impact parameter. On the other hand, being purely perturbative it cannot generate a mass gap needed to ensure a fast convergence of the integration over the impact parameter $b$. Because of this problem, up to now all the phenomenological applications of the BK were based on model assumptions regarding the $b$-dependence. It is always assumed that the $b$-dependence factorizes and in practice the BK is usually solved without any trace of $b$. An attempt to go beyond this approximation was reported in [10].

• It is very desirable to go beyond the BK and relax all underlying assumptions outlined above. The high order corrections are most needed. In particular it is important to learn how to include running of $\alpha_s$ though in the phenomenological applications the

\(^{1}\)There was a recent claim of Mueller and Shoshi [9] that the s-channel unitarity is in fact violated during the evolution.
running of $\alpha_s$ has been usually implemented.

- The LO BFKL kernel does not have the correct short distance limit responsible for the Bjorken scaling violation. As a result the BK does not naturally match with the DGLAP equation. Though several approaches for unification of the BK and DGLAP equations were proposed [11, 6, 12], the methods are not fully developed. All approaches deal with low $x$ and gluon sector only. We would like to have a unified evolution scheme both for small and large $x$ with quarks equally treated.

2 Beyond the BK

We now present a short (not complete) review of a recent theoretical activity which is an attempt to go beyond the BK.

- Though the BFKL kernel is known at next-to-leading order, a nonlinear equation at NLO has not been derived yet. The authors of [13] have been able to compute a single NLO contribution which has maximal nonlinearity, namely the $N^3$ term:

$$\frac{dN}{dy} = \alpha_s Ker \otimes (N - NN) - \alpha_s^2 \tilde{Ker} \otimes NN.$$  \hspace{1cm} (2)

In [14] the NLO BFKL at presence of a saturation boundary was considered. The results show a decrease in the saturation scale growth as a function of rapidity towards the value $\lambda \approx 0.3$ observed experimentally.

In [15] a study of the BK at presence of a rapidity veto was made. Rapidity veto means that no parton emissions are allowed which are separated in rapidity by less than the veto $\eta$. At high energies the method of rapidity veto is known to mimic higher order corrections. The application of the method to the BK equation makes it nonlocal in rapidity

$$\frac{dN(y)}{dy} = \alpha_s Ker \otimes (N(y-\eta) - N(y-\eta)) N(y-\eta)).$$

The veto delays saturation in accord with the expectations associated with the next-to-leading order corrections. If the veto is put on top of the BK equation with running $\alpha_s$, then the effect of additional NLO corrections is significantly reduced. This observation gives support to phenomenological studies of Refs. [6, 16].

Another approach to partially include NLO corrections into BK equation is to implement in its linear term a unified BFKL-DGLAP framework developed in [12, 16].

- The $N_c$ corrections can be accounted for through JIMWLK functional equation [17], which is equivalent to the Balitsky’s infinite hierarchy of equations [2]. Introducing
$N$ as a target expectation value of a certain operator (product of two Wilson lines), $N \equiv \langle W \rangle_{\text{target}}$, the first couple of equations are

$$\frac{d \langle W \rangle}{d y} = \alpha_s \text{Ker} \otimes (\langle W \rangle - \langle WW \rangle).$$  \hspace{1cm} (3)$$

$$\frac{d \langle WW \rangle}{d y} = \alpha_s \text{Ker} \otimes (\langle WW \rangle - \langle WWWW \rangle).$$  \hspace{1cm} (4)

The large $N_c$ limit and the absence of the target correlations used by Kovchegov is equivalent to a mean field approximation which allows to express a correlator of a product as a product of correlators: $\langle WW \rangle = \langle W \rangle \langle W \rangle = N N$; $N_c \to \infty$. Thus the first equation of the Balitky’s chain closes to the BK.

A first numerical solution of the JIMWLK equation was reported in [18]. They do not find any qualitative deviation from solutions of the BK. The $N_c$ corrections were found to be at a level of few percents.

The authors of [19] have considered $N_c$ corrections to the triple Pomeron vertex:

$$\frac{d N}{d y} = \alpha_s \text{Ker} \otimes (N - N N - \frac{1}{N_c^2} n)$$

An additional equation for $n$ was proposed in [19].

- For proton and realistic (not very dense) nucleus targets a systematic approach towards inclusion of target correlations has been developed in [20]. Target correlations can be introduced via certain linear functional differential equation. In general, this linear functional equation cannot be reformulated as a nonlinear equation. However, in a particular case when all $n$-dipole correlations can be accounted for by a single correlation parameter, the equation can be brought to a modified version of the BK:

$$\frac{d N}{d y} = \alpha_s \text{Ker} \otimes (N - \kappa NN),$$  \hspace{1cm} (5)

with $\kappa \geq 1$ being the correlation parameter to be found from a model for the target.

- Inclusion of Pomeron loops is the first step towards restoration of the $t$-channel unitarity. Iancu and Mueller [21] has recently considered rare fluctuations which were interpreted in [22] as pomeron loop contributions. Unfortunately, it looks like contributions of the pomeron loops cannot be incorporated in a framework of a single equation. They modify the asymptotic behavior of the amplitude $N$ in the deep saturation limit:

$$N(Y) = 1 - e^{-c(Y - Y_0)^2}; \quad Y \to \infty \quad c = 2\bar{\alpha_s} \quad \text{BK}$$
\[ N(Y) = 1 - e^{-1/2e(Y-Y_0)^2}; \quad Y \to \infty \]

Pom Loops

- It is claimed that the BK sums all possible contributions which are not suppressed either by \( \alpha_s \) or \( N_c \). For example, the cubic term which is in Eq. 2 appears at next-to-leading \( \alpha_s \) order only. In particular it is implied that all multi-pomeron exchanges and multipomeron vertices are either absorbed by the triple pomeron vertex of the BK or suppressed. It was argued in [20] that this might be false. They argue that in addition to a possibility for a pomeron to split into two, there exists a local in rapidity process of multi-pomeron exchange. After these contributions were resummed in the eikonal approximation, a new modification of the BK was proposed:

\[
\frac{dN}{dy} = (1 - N) \alpha_s K_{er} \otimes (N - NN) \quad (6)
\]

- Outlook. It is essential for the future phenomenological studies to eliminate the model dependent treatments of the impact parameter. Though the BK has been solved numerically with the full \( b \)-dependence traced [10], the results are not yet suitable for phenomenological applications.

A further study of the relation between the dipole picture vs. traditional diagramatics based on the \( s \)-channel unitarity is needed. In particular, it is not clear if the dipole picture survies at NLO. In general there is a quest for a simple effective Reggeon field theory in QCD.

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