Symmetric groups and checker triangulated surfaces

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We consider triangulations of surfaces with edges painted three colors so that edges of each triangle have different colors. Such structures arise as Belyi data (or Grothendieck dessins d’enfant), on the other hand they enumerate pairs of permutations determined up to a common conjugation. The topic of these notes is links of such combinatorial structures with infinite symmetric groups and their representations.

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1 Checker triangulations and Belyi data

1.1. Checker triangulations. Consider an oriented compact closed two-dimensional surface $P$ (we do not require that a surface be connected). Consider a finite graph $\gamma$ on $P$ separating $P$ into triangles. Let edges of the graph be colored blue, red, or yellow and let each triangle have edges of all 3 types. We say that a triangle is white if the order of blue, red, yellow edges is clockwise and black in the opposite case, see Fig. 1. We call such structures $(P, \gamma)$ checker surfaces.

Two checker surfaces $(P, \gamma)$ and $(P, \gamma')$ are equivalent if there is an orientation preserving homeomorphism $P \to P'$ identifying colored graphs $\gamma$ and $\gamma'$. Therefore checker surfaces are purely combinatorial objects, we also can regard them as simplicial cell complexes with colorings of triangles and edges.

We denote the set of all checker surfaces having $2n$ triangles by $\Xi_n$.

Example. Consider a sphere with a cycle composed of blue, red, and yellow edges, see Fig. 2. We call such checker surface a double triangle.

Remarks. a) Under our conditions orders of all vertices are even. Edges adjacent to a given vertex have two colors, and these colors are interlacing, see Fig. 1. Therefore we can split vertices into 3 classes, blue, red, yellow: we say that a vertex is blue, if opposite edges are blue, etc. (colors of vertices are not shown on Fig. 1).

b) An intersection of two white (or two black) triangles is the empty set or a vertex. An intersection of a black and a white triangles can be the empty

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set, a vertex, an edge, two edges, three edges, see Fig. 2a, 2b. We avoid a term triangulation since the most common definition of triangulations forbids the last two possibilities. Formally, our objects satisfy a definition of a simplicial cell complex, see, e.g., [12].

"c) We admit empty surfaces.

Remark. Apparently, such colored structures firstly appeared in combinatorial topology in [34], see also [5], [6]."

1.2. Dessins d’enfant. There is the following equivalent language for description of the same objects. We can remove red and yellow edges and blue vertices. After this we get a graph with blue edges on the surface, vertices are painted red and yellow, and colors of the ends of each edge are different. Therefore we get a bipartite graph on the surface, a coloring of vertices is determined by the graph modulo a reversion of colors of all vertices. Such a graph separates the surface into polygonal domains.

To reconstruct a checker structure, we put a point to interior of each domain, connect it by non-intersecting arcs with vertices of the polygon, and paint these arcs in a unique admissible way, see Fig. 3.

1.3. Belyi data. Denote by Ĉ the Riemann sphere, by ℚ the algebraic closure of ℚ, by \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) the Galois group of \( \overline{\mathbb{Q}} \) over \( \mathbb{Q} \). Recall the famous Belyi theorem [1], [2], see expositions in [17], [8].

**Theorem 1.1** Let \( P \) be a nonsingular complex algebraic (projective) curve. A covering holomorphic map \( \pi : P \to \overline{\mathbb{C}} \) whose critical values are contained in the
set \{0, 1, \infty\} exists if and only if \(P\) is defined over \(\overline{\mathbb{Q}}\). For a curve defined over \(\mathbb{Q}\) a map \(\pi\) can be chosen defined over \(\overline{\mathbb{Q}}\).

Words a curve \(P\) is defined over \(\mathbb{Q}\) mean that \(P\) can be determined by a system of algebraic equations in a complex projective space \(\mathbb{CP}^N\) with coefficients in \(\mathbb{Q}\).

Such functions are called Belyi functions, a pair \((P, \pi)\) is called Belyi data. The Galois group \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\) acts on Belyi data changing coefficients of equations determining curves.

1.4. Correspondence between Belyi data and checker surfaces. The real line \(\mathbb{R}\) splits \(\hat{\mathbb{C}}\) into two half-planes. Let us paint the segment \([\infty, 0]\) blue, the segment \([0, 1]\) red, and the segment \([1, \infty]\) yellow. The upper half-plane of \(\hat{\mathbb{C}}\) becomes white and the low half-plane becomes black. Thus we get a structure of a double triangle on \(\hat{\mathbb{C}}\), see Fig. 2b.

For a Belyi map \(P \to \hat{\mathbb{C}}\) we take the preimage of \(\mathbb{R}\). It is a colored graph on \(\mathbb{R}\) splitting \(P\) into white and black triangles. Thus we get a structure of a checker surface on \(P\).

Conversely, let us for a checker surface \((P, \gamma)\) define a complex structure on \(P\) and a holomorphic function \(P \to \hat{\mathbb{C}}\). Consider an equilateral triangle \(T^{\text{white}}\) whose sides are colored blue, red, yellow clockwise and an equilateral triangle \(T^{\text{black}}\) whose sides are colored blue, red, yellow anti-clockwise. We identify \(T^{\text{white}}\) conformally with the white half-plane of \(\hat{\mathbb{C}}\) sending vertices to 0, 1, \(\infty\) and edges to segments \([-\infty, 0], [0, 1], [1, \infty]\) according colors (such a map exists and is unique). We also identify \(T^{\text{black}}\) with the black half-plane in a similar way. We can think that a surface \((P, \gamma)\) is glued from copies of \(T^{\text{white}}\) and \(T^{\text{black}}\). This defines a complex structure on \(P\) outside vertices. If we have \(2k\) triangles adjacent to a vertex \(v\), then a punctured neighborhood of \(v\) has a structure of \(k\)-covering of a punctured circle \(D : 0 < |z| < \varepsilon\). This covering is itself holomorphically equivalent to \(D\), and this gives us a chart on \(P\) in a neighborhood of \(v\). Thus we get a complex structure on \(P\).

Next, we take a map from \(P\) to \(\hat{\mathbb{C}}\) sending white triangles to the white half-plane and black triangles to the black half-plane. By the Riemann–Schwarz
reflection principle the map is holomorphic at interior points of edges. By the theorem on removable singularities the map is also holomorphic at vertices.

Thus we get Belyi data. It is known, that combinatorial data determine a Belyi map up to a natural equivalence (upto an automorphism of $P$). Notice, that in our case $P$ can be disconnected, so we have Belyi functions on all components of the curve.

In particular the group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the space of checker surfaces. In 1984 Grothendieck [10] initiated a program of investigation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ based on dessins d’enfant, see e.g., [35], [36], [39], [17], [8], [37], [38].

2 Correspondence with symmetric groups

2.1. The basic construction. Consider the product $G_n = S_n \times S_n \times S_n$ of 3 copies of the symmetric group $S_n$. An element of this group is a triple of permutations, we denote such triples as

$$p = (p_{\text{blue}}, p_{\text{red}}, p_{\text{yellow}}).$$ (2.1)

Denote by $K_n$ the diagonal subgroup in $G_n$, it consists of triples of the form

$$h = (h, h, h).$$ (2.2)

Take a triangle $T_{\text{white}}^0$, paint its sides blue, red, yellow clockwise. Consider $n$ copies of the triangle $T_{\text{white}}^0$, paint them white and attribute to the triangles labels 1, 2, $\ldots$, $n$. Thus we get triangles $T_{\text{white}}^1, \ldots, T_{\text{white}}^n$.

Consider another triangle $T_{\text{black}}^0$ obtained from $T_{\text{white}}^0$ by a reflection. Its sides are colored blue, red, yellow anti-clockwise. Take $n$ copies of this triangle $T_{\text{black}}^1, \ldots, T_{\text{black}}^n$, see Fig. 4.

Fix an element $p \in G_n$, For each $j = 1, \ldots, n$ we glue the triangle $T_{\text{white}}^j$ with the triangle $T_{\text{black}}^{p_{\text{red}}j}$ along blue sides according orientations. Repeat the same operation for $p_{\text{red}}$ and $p_{\text{yellow}}$. We get a two-dimensional surface equipped with a checker structure. We also have an additional structure, namely we have a bijective map (labeling) from the set $\{1, \ldots, n\}$ to the set of white triangles and the bijective map from the same set to the set of black triangles.
We call the object $P$ obtained in this way a \textit{completely labeled checker surface}. Denote by $\tilde{\Xi}_n$ the set of all such surfaces.

2.2. The inverse construction. Let us describe the inverse map $\tilde{\Xi}_n \to G_n$. Consider a completely labeled surface $P$. Take a blue edge $v$. Let $i_v$ be the label on the white side of $v$, and $j_v$ the label on the black side. Then we set
\[
p^{\text{blue}} : \ i_v \mapsto j_v.
\]
and get $p^{\text{blue}}$. In the same way we reconstruct $p^{\text{red}}$ and $p^{\text{yellow}}$.

2.3. Remarks. a) Thus for any element of $G_n$ we get a surface with $2n$ triangles and $3n$ edges.

b) Blue vertices of $P$ are in one-to-one correspondence with independent cycles of the permutation $(p^{\text{yellow}})^{-1}p^{\text{red}}$. The order of a vertex is the duplicated length of a cycle. A similar statement holds for red and yellow vertices.

c) Take the subgroup $U_p \subset S_n$ generated by $(p^{\text{yellow}})^{-1}p^{\text{blue}}$ and $(p^{\text{yellow}})^{-1}p^{\text{red}}$. Components $P_j$ of $P$ are in one-to-one correspondence with orbits $O_j$ of $U_p$ on $\{1, \ldots, n\}$. A given orbit splits into independent cycles of $(p^{\text{yellow}})^{-1}p^{\text{red}}$, this gives us an enumeration of blue vertices of the component. So we have an expression for the Euler characteristic of a component in group-theoretical terms:
\[
\chi(P_j) = -\#(O_j) + \left\{ \text{number of cycles of } (p^{\text{yellow}})^{-1}p^{\text{blue}} \text{ on } O_j \right\} + \left\{ \text{number of cycles of } (p^{\text{yellow}})^{-1}p^{\text{red}} \text{ on } O_j \right\} + \left\{ \text{number of cycles of } (g^{\text{red}})^{-1}g^{\text{blue}} \text{ on } O_j \right\}.
\]
d) The pass $p \mapsto p^{-1}$ is a reversion of colors of triangles black↔white and a reversion of orientations of all components.

e) For any element $h \in K_n$ the corresponding surface is a disjoint union of double triangles.

2.4. Concatenations. Now let us describe the product in the group $G_n$ on the geometric language.

Let $p, q \in G_n$. Consider the corresponding elements $P, \Omega \in \tilde{\Xi}_n$. For each $j = 1, 2, \ldots, n$ we identify $j$-th black triangles of $\Omega$ and $j$-th white triangle of $P$ according colors of edges. In this way we get a 2-dimensional simplicial cell complex $\mathcal{R}^\circ$ having $3n$ two-dimensional cells, namely:

- $n$ labeled images of white triangles of $\Omega$;
- $n$ labeled images of black triangles of $P$;
- $n$ (labeled) triangles that are results of gluing (let us call such cells \textit{grey}).

Each edge has a color (blue, red, or yellow) and is contained in 3 triangles (white, black, and grey). Next, we remove interiors of all grey triangles from the simplicial cell complex $\mathcal{R}^\circ$ (and forget their labels). We get a complex $\mathcal{R}^e$ having $n$ white triangles and $n$ black triangles. These cells inherit labelings. Moreover, each edge is contained in precisely two triangles.
Figure 5: A normalization.

In fact, we get a surface, but some vertices of the surface are glued one with another, see Fig. 5a. Cutting such gluings we get a surface $\mathcal{R}$ corresponding to the product $r = pq \in G_n$.

Remark. Of course we can transpose an order of the gluing and the making holes. Namely, let $\mathcal{P}$ and $\mathcal{Q}$ be completely labeled checker surfaces. We remove interiors of black triangles from $\mathcal{P}$ and interiors of white triangles from $\mathcal{Q}$ and get new cell complexes $\mathcal{P}^\square$ and $\mathcal{Q}^\square$. For each $j \leq n$ we identify the boundary of former $j$-th black triangle of $\mathcal{Q}$ and the boundary of former $j$-th white triangle of $\mathcal{P}$ according colors of edges. We get a two-dimensional cell complex and normalize it as above.

2.5. Notation. Conjugacy classes and double cosets. Let $H$ be a group, $L, M \subset H$ subgroups. We denote by $L \setminus H / M$ the space of double cosets of $H$ with respect to $L, M$, i.e., the quotient of $H$ with respect to the equivalence

$$h \sim lhm,$$

where $l \in L, m \in M$.

An element of this space is a subset in $H$ of the form $L \cdot h \cdot M$.

Denote by $H / L$ the space of conjugacy classes of a group $H$ with respect to a subgroup $L$, i.e., the quotient of $H$ with respect to the equivalence

$$h \sim l^{-1}hl,$$

where $l \in L$.

Remark. Conjugacy classes can be regarded as a special case of double cosets for the following reason. Consider the group $H \times L$ and the embedding $\iota : L \to H \times L$ given by $\iota(l) = (l, l)$. Obviously,

$$H / L \simeq \iota(L) \setminus (H \times L) / \iota(L).$$

2.6. Relabeling. Let us multiply an element $p \in G_n$ by an element $h \in K_n$, see (2.1) and (2.2). Clearly the operation $p \mapsto hp$ is equivalent to a permutation

\[2\] Let us describe this cutting (a normalization) formally (this is a special case of procedure of normalization of pseudomanifolds from [9]). Let us equip the space $\mathbb{R}^3$ by some natural metric (for instance we assume that all triangles are equilateral triangles with side 1, this determines a 'geodesic' distance on the whole complex. Let us remove all vertices, consider the geodesic distance on this space, and complete our space with respect to the new distance.
of labels on black triangles of the corresponding surface. A right multiplication $p \mapsto ph^{-1}$ is equivalent to a permutation of labels on white triangles.

Consider the space of double cosets of $K_n \backslash G_n/K_n$ of $G_n$ with respect to $K_n$. This means that we consider elements of $\tilde{\Xi}_n$ up to a permutation of labels. In other words, we forget labels.

Thus we get a bijection

$$K_n \backslash G_n/K_n \longleftrightarrow \Xi_n.$$ 

This also gives a geometrical description of pairs of permutations determined up to a common conjugation, i.e., up to the equivalence

$$(g_1, g_2) \sim (h^{-1}g_1h, h^{-1}g_2h).$$

Indeed, we have the space of conjugacy classes of $S_n \times S_n$ with respect to the diagonal subgroup $\text{diag}(S_n)$. According to the remark in previous subsection,

$$(S_n \times S_n)\backslash\text{diag}(S_n) \simeq \text{diag}(S_n) \backslash (S_n \times S_n \times S_n) / \text{diag}(S_n) = K_n \backslash G_n/K_n.$$ 

3 Category of checker cobordisms

3.1. Infinite symmetric group. Denote by $S_\infty$ the group of finitely supported permutations of $\mathbb{N}$, by $\mathcal{S}_\infty$ the group of all permutations. The group $S_\infty$ is a countable discrete group, the group $\mathcal{S}_\infty$ has cardinality continuum and is equipped with a natural topology discussed in the next section. As usual, we can represent elements of these groups by 0-1-matrices having precisely one 1 in each column and each row.

Consider the group $G := S_\infty \times S_\infty \times S_\infty$ and its diagonal $K \simeq S_\infty$. We repeat the construction of the previous section for $n = \infty$. For any $g \in G$ we get a countable disjoint union of compact checker surfaces, all but a finite number of components are double triangles, moreover for all but a finite number of components labels on the black and white sides of a double triangles coincide. As above, we get a correspondence between the group and the set of completely labeled checker surfaces.

3.2. Multiplication of double cosets. Denote by $\mathbb{Z}_+$ the set of non-negative integers. For $\alpha \in \mathbb{Z}_+$ denote by $K[\alpha] \subset K$ the subgroup in $K = S_\infty$ consisting of permutations fixing points $1, \ldots, \alpha \in \mathbb{N}$; for $\alpha = 0$ we set $K[0] := K$. The subgroups $K[\alpha]$ are isomorphic to $S_\infty$.

Consider double cosets spaces

$$\mathcal{M}(\alpha, \beta) := K[\alpha] \backslash G/K[\beta].$$

It turns out to be that for any $\alpha, \beta, \gamma \in \mathbb{Z}_+$ there is a natural multiplication

$$\mathcal{M}(\alpha, \beta) \times \mathcal{M}(\beta, \gamma) \rightarrow \mathcal{M}(\alpha, \gamma)$$
defined in the following way. Consider a sequence \( \theta_j[\beta] \) in \( S_\infty \) given by

\[
\theta_j[\beta] := \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

By \( \Theta_j[\beta] \) denote the corresponding element of \( K \):

\[
\Theta_j[\beta] = (\theta_j[\beta], \theta_j[\beta], \theta_j[\beta]) \in K[\beta].
\]

Let \( p \in M(\alpha, \beta), \ q \in M(\beta, \gamma) \). Choose their representatives \( p \in p, \ q \in q \). The following statements are semi-obvious if to look to them for a sufficiently long time (see formal proofs in [7]).

**Lemma 3.1** The following sequence of double cosets

\[
K[\alpha] \cdot p \cdot \Theta_j[\beta] \cdot q \cdot K[\gamma] \in M(\alpha, \gamma)
\]  

is eventually constant. Its limit\(^3\) does not depend on a choice of representatives \( p \in p, \ q \in q \).

Denote by \( p \oplus q \) the limit of the sequence (3.1). The multiplication \( \oplus \) is associative in the following sense:

**Lemma 3.2** For any \( \alpha, \beta, \gamma, \delta \in \mathbb{Z}_+ \) and any

\( p \in M(\alpha, \beta), \ q \in M(\beta, \gamma), \ r \in M(\gamma, \delta) \),

we have

\[
(p \oplus q) \oplus r = p \oplus (q \oplus r).
\]

Thus we can define the following category \( \mathcal{K} \). The set of its objects is \( \mathbb{Z}_+ \) and the set of morphisms from \( \beta \) to \( \alpha \) is \( M(\alpha, \beta) \).

This operation is a representative of huge zoo of train constructions for infinite dimensional groups, see end of the present section. The main property of \( \oplus \)-multiplication is Theorem 3.3 below.

**3.3. The involution on** \( \mathcal{K} \). The map \( p \mapsto p^{-1} \) induces maps

\[
M(\alpha, \beta) \to M(\beta, \alpha),
\]

we denote these maps by \( p \mapsto p^* \). Obviously, * satisfies the usual properties of involutions

\[
(p \oplus q)^* = q^* \oplus p^*, \quad p^{**} = p.
\]

**3.4. Explicit description of the product of double cosets.** We say that a labeled checker surface is the following collection of combinatorial data:

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\(^3\)I.e., a stable value. However, \( M(\alpha, \gamma) \) is a quotient of a discrete topological space, so it has a discrete topology, and we have a limit in the sense of the formal definition of a limit.
— a finite disjoint union $P$ of checker surfaces;
— an injective map from the set $\{1, \ldots, \alpha\}$ to the set of black triangles of $P$ and an injective map from the set $\{1, \ldots, \beta\}$ to the set of white triangles (we call these maps by *labelings*).
— additionally, we require that any double-triangle component of $P$ keeps at least one label.

Denote by $\tilde{\Xi}[\alpha, \beta]$ the set of such objects.

Clearly, there is a canonical one-to-one correspondence

$$\mathcal{M}(\alpha, \beta) \longleftrightarrow \tilde{\Xi}[\alpha, \beta].$$

Namely, we take a representative of a double coset $p$ and draw the corresponding completely labeled checker surface. A pass to the double coset means forgetting white labels with numbers $> \beta$ and black labels $> \alpha$. Finally, we remove all double triangles without labels and get a finite object.

The product of double cosets corresponds to the concatenation (as in Subsect. 2.4). Namely, let $P$, $Q$ be labeled checker surfaces corresponding to $p$ and $q$. For each $j = 1, \ldots, \beta$ we remove interiors of labeled black triangles of $Q$ and labeled white triangles of $P$ (so we get $\beta$ 'holes' on each surface). For each $j \leq \beta$ we identify boundary of the former black triangle of $Q$ and the former white triangle of $P$ according colors of edges. We get a new 2-dimensional simplicial cell complex and normalize it as above. This surface inherits $\alpha$ labels on white triangles and $\gamma$ labels on black triangles. Finally, we remove label-less double triangles.

Thus we get an operation similar to a concatenation of cobordisms, we glue cell complexes with additional structure instead of manifolds with boundaries, also our operation includes a normalization.

**Remark.** The operation of normalization does not arise if labeled black triangles of $Q$ (or labeled white triangles of $P$) have no common vertices.

**Remark.** The $\ast$-product on $\tilde{\Xi}[0, 0]$ is a disjoint union of label-less checker surfaces,

$$P \ast Q = P \coprod Q.$$  

In particular, this operation is commutative.

**3.5. Multiplicativity theorem.** Let $\rho$ be a unitary representation of the group $G = S_\infty \times S_\infty \times S_\infty$ in a Hilbert space $V$. For each $\alpha \in \mathbb{Z}_+$ denote by

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4 After the previous operation we have a countable collection of label-less double triangles, which do not keep any information. It is more convenient to remove them.

5 For many classes of groups (as finite groups, compact groups, semisimple Lie groups, nilpotent Lie groups, semisimple $p$-adic groups) any irreducible representation of $G_1 \times G_2$ is a tensor product of an irreducible representation $\rho_1$ of $G_1$ and an irreducible representation $\rho_2$ of $G_2$. In fact, this holds if at least one of groups has type I, see, e.g., [18], §5, §13.1. For the group $S_\infty$ this implication does not valid. Also, in the next section we observe that actually we work with a certain completion of the group $G$ and this completion is not a product of groups.
the space of all $K[\alpha]$-fixed vectors $v \in V$,

$$\rho(h)v = v \quad \text{for all } h \in K[\alpha].$$

Remark. Obviously,

$$V[0] \subset V[1] \subset V[2] \subset \cdots \subset V.$$

In this moment we do not require $\cup_\alpha V[\alpha]$ be dense in $V$. However, below we will observe that this requirement is natural.

Denote by $P[\alpha]$ the orthogonal projection operator $V \to V[\alpha]$.

For $\alpha, \beta \in \mathbb{Z}_+$, and $p \in G$ consider the operator

$$\overline{\rho}(p) : V[\beta] \to V[\alpha]$$

given by

$$\overline{\rho}_{\alpha,\beta}(p) := P[\alpha] \rho(p) \bigg|_{V[\beta]}.$$

In other words, we write $\rho(p)$ as a block operator

$$\rho_{\alpha,\beta}(p) = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} : \quad V[\beta] \oplus V[\beta]^\perp \to V[\alpha] \oplus V[\alpha]^\perp$$

and set

$$\overline{\rho}(p) := R_{11}.$$

Then $\overline{\rho}(p)$ depends only on a double coset containing $p$ (this is a standard fact), so actually we get a function $\overline{\rho}_{\alpha,\beta}(p)$ on $M(\alpha,\beta)$.

**Theorem 3.3** The operators $\overline{\rho}_{\alpha,\beta}(-)$ determine a representation of the category $\mathcal{K}$ of double cosets, i.e., for any $\alpha, \beta, \gamma \in \mathbb{Z}_+$ for any $p \in M(\alpha,\beta), q \in M(\beta,\gamma)$,

$$\overline{\rho}_{\alpha,\beta}(p) \overline{\rho}_{\beta,\gamma}(q) = \overline{\rho}_{\alpha,\gamma}(p \oplus q).$$

Moreover, this is a $\ast$-representation of the category $\mathcal{K}$, i.e.,

$$\overline{\rho}_{\alpha,\beta}(p)^\ast = \overline{\rho}_{\beta,\alpha}(p^\ast).$$

See [23], [27].

Thus any unitary representation $\rho$ of $G$ generates a $\ast$-representation $\overline{\rho}$ of the category $\mathcal{K}$. However, generally speaking $\rho$ can not be reconstructed from $\overline{\rho}$. In the next section we describe a more perfect version of this correspondence.

**3.6. Zoo.** Apparently, a first example of a multiplication on double cosets spaces $K \setminus G / K$ and a multiplicativity theorem was discovered by R. S. Ismagilov [13] for a pair $G \supset K$, where $G = \text{SL}_2$ over a non-locally compact non-Archimedean field and $K$ is $\text{SL}_2$ over integer elements of the field.

Later many cases of multiplication were examined by G. I. Olshanski, see [33]. In particular, he considered the case $G = S_\infty \times S_\infty$ and $K = K[\alpha]$, see
there is a well-developed theory of representations of this group, see [14], [3] and further references in this book.

In [19], Sect. VIII.5, it was observed that multiplicativity theorems are a quite general phenomenon, the main obstacle for a further progress was a problem of explicit descriptions of double coset spaces. The construction of this section was obtained in [23]. It has numerous variations with different pairs of groups \(G \supset K\) related to infinite symmetric groups, see [27], [7] as

\[
G = (S_\infty)^n, \quad K = \text{diag}(S_\infty);
G = S_{n\infty}, \quad K = S_\infty \ltimes (\mathbb{Z}_n)^\infty;
G = S_{n\infty}, \quad K = S_\infty \ltimes (S_n)^\infty;
G = S_{n\infty}, \quad K = (S_\infty)^n.
\]

For the last case Nessonov (see [29], [27]) obtained a classification of all \(K\)-spherical representations of \(G\).

On the other hand, there is parallel picture for infinite dimensional real classical groups, [32], [19] (Sect. IX.3), [21], [22]. There is a poorly studied mass construction of the same type for infinite-dimensional groups over \(p\)-adic field [26] and an example over finite fields [25]. On the other hand, this story has a counterpart for groups of transformations of measure spaces [19], Chapter X, [24].

4 The infinite tri-symmetric group

Here we describe a natural completion \(G\) of \(G = S_\infty \times S_\infty \times S_\infty\),

\[S_\infty \times S_\infty \times S_\infty \subset G \subset S_\infty \times S_\infty \times S_\infty\]

and discuss representations of this completion.

4.1. Representations of a full symmetric group \(S_\infty\). We say that a sequence \(g_j \in S_\infty\) converges to \(g\) if for each \(k \in \mathbb{N}\) we have \(g_j k = g k\) for sufficiently large \(j\). This determines a structure of a complete separable topological group on \(S_\infty\) (and this topology is unique in any reasonable sense, see [15]).

Denote by \(S_\infty[\alpha] \subset S_\infty\) the subgroup consisting of elements fixing points 1, 2, \ldots, \(\alpha \in \mathbb{N}\). These subgroups form a base of neighborhoods of unit in \(S_\infty\) (and this can be regarded as another definition of topology in \(S_\infty\)).

A classification of unitary representations of \(S_\infty\) was obtained by A. Lieberman [18] (see also expositions in [30], [19], Sect. VIII.1-2), the classification is relatively simple but it is not necessary for us in the sequel.\(^6\) Representations of \(S_\infty\) that can be extended to \(S_\infty\) can be described in the following form without explicit reference to the topology.

\(^6\)Any irreducible unitary representation of \(S_\infty\) has the following form. Fix \(\alpha \in \mathbb{Z}_+\) and an irreducible representation \(\rho\) of \(S_\alpha\). Consider a subgroup \(S_\alpha \times S_\infty[\alpha]\) and its representation that is trivial on \(S_\infty[\alpha]\) and equal \(\rho\) on \(S_\alpha\). The induced representation of \(S_\infty\) is irreducible, and all irreducible representations of \(S_\infty\) have this form. Any unitary representation is a direct sum of irreducible representations. A decomposition of a unitary representation into a direct sum of irreducible representations is unique.
Proposition 4.1 Let \( \rho \) be a unitary representation of \( S_\infty \) in a Hilbert space \( V \). As above, denote by \( V[\alpha] \) the space of \( S_\infty[\alpha] \)-fixed vectors.

a) A representation \( \rho \) admits a continuous extension to \( S_\infty \) if and only if \( \cup_\alpha V[\alpha] \) is dense in \( V \).

b) An irreducible representation \( \rho \) admits a continuous extension to \( S_\infty \) if and only if \( \cup_\alpha V[\alpha] \) is non-zero.

4.2. Tri-symmetric group. Consider the group \( S_\infty \times S_\infty \times S_\infty \) and its diagonal subgroup \( K \cong S_\infty \). We define the tri-symmetric group \( G \) as the subgroup in \( S_\infty \times S_\infty \times S_\infty \) generated by \( K \) and \( G = S_\infty \times S_\infty \times S_\infty \). In other words

\[
(p^{\text{blue}}, p^{\text{red}}, p^{\text{yellow}}) \in G \iff p^{\text{blue}}(p^{\text{red}})^{-1}, p^{\text{red}}(p^{\text{yellow}})^{-1} \in S_\infty.
\]

The coset space \( G/K \) is countable. We define a topology on \( G \) assuming that \( K \) has the natural topology and the space \( G/K \) is discrete.

For the group \( G \) we repeat the considerations of Subsections 3.2–3.4. In fact, we have

\[
K[\alpha] \setminus G/K[\beta] = K[\alpha] \setminus G/K[\beta],
\]

therefore we get the same category \( K \) of double cosets. For any unitary representation \( \rho \) of \( G \) we have a \( \ast \)-representation of the category \( K \) (since a representation of \( G \) is also a representation of \( G \)).

4.3. The equivalence theorem. See [27], Sect.3.

Theorem 4.2 The map \( \rho \mapsto \overline{\rho} \) establishes a bijection between the set of unitary representations of \( G \) and the set of \( \ast \)-representations of the category \( K \).

5 Spherical functions

5.1. Sphericity.

Theorem 5.1 The pair \( G \supseteq K \) is spherical, i.e., for any irreducible unitary representation of \( G \) the space of \( K \)-fixed vectors has dimension \( \leq 1 \).

Remark. This is a trivial corollary of multiplicativity Theorem 3.3 and the commutativity of the semigroup \( K \setminus G/K \). It is worth noting that for the pair

\[
G_n = S_n \times S_n \times S_n, \quad K_n = \text{diag}(S_n)
\]

the sphericity does not hold.

\footnote{It is worth noting that applying the same construction to an element of \( S_\infty \times S_\infty \times S_\infty \) we generally get a non-locally finite cell complex, which seems to be pathological. In our case we get a disjoint union of finite complexes and all but a finite number of components are double triangles with coinciding labels on white and black sides.}
Consider a $K$-fixed vector $v$ (a spherical vector) for an irreducible representation $\rho$. Consider the following matrix element (a spherical function)

$$\Phi(p) := \langle \rho(p)v, v \rangle.$$  

It is easy to see that for any $r_1, r_2 \in K$ we have $\Phi(r_1pr_2) = \Phi(p)$, i.e., $\Phi$ actually is a function on the double coset space $K \backslash G/K$, i.e., on the space $\Xi[0, 0]$ of checker surfaces.

5.2. Example. Here we present a construction of a family of representations of $G$, for more constructions and a discussion of a classification problem, see [27], Sect.3. Consider three Hilbert spaces $H^{blue}$, $H^{red}$, $H^{yellow}$ (they can be finite or infinite-dimensional). Consider the tensor product

$$X := H^{blue} \otimes H^{red} \otimes H^{yellow},$$  

and fix a unit vector $\xi \in X$. Consider an infinite tensor product

$$V := (X, \xi) \otimes (X, \xi) \otimes (X, \xi) \otimes \ldots$$  

(5.1)

Now let the first copy of $S_\infty$ acts by permutations of factors $\rho^{blue}$, the second copy by permutations of $H^{red}$, and the third by permutations of $Y^{yellow}$. The subgroup $K$ acts by permutations of factors $X$.

5.3. An expression for spherical function. Consider a representation $\rho$ of $G$ constructed in the previous subsection. Consider the vector

$$v := \xi \otimes \xi \otimes \xi \otimes \ldots \in V,$$

it is a unique $K$-invariant vector in $V$. We present a formula for the corresponding spherical function

$$\Phi(p) := \langle \rho(p)v, v \rangle.$$  

---

8Let $(Y_j, v_j)$ be a sequence of Hilbert spaces with distinguished unit vectors $v_j \in Y_j$. To define their tensor product $W$, we consider a system of formal vectors $y_1 \otimes y_2 \otimes \ldots$ (decomposable vectors) such that $y_1 = v_1$ starting some place. We assign for such vectors the following inner products

$$(y_1 \otimes y_2 \otimes \ldots y_1' \otimes y'_2 \otimes \ldots) = \prod_j \langle y_j, y'_j \rangle v_j,$$

and assume that linear combinations of such vectors are dense in a Hilbert space $W$. This uniquely determines a Hilbert space $W$. See [40], Section 4.1, or Addendum to [11].

9We emphasize that a finitely supported permutation of factors $H^{blue}$ send decomposable vectors to decomposable vectors, for a permutation that is not finitely supported an image of a decomposable vector generally makes no sense.

10Generally speaking, $\rho(p)$ is irreducible. However, the cyclic span $V' \subset V$ of $v$ is an irreducible representation of $G$. Otherwise, we decompose $V' = W_1 \oplus W_2$. Indeed, projections of $v$ to the both summands must be $K$-invariant, but a $K$-invariant vector is unique. Therefore $\Phi(p)$ is the spherical function of the restriction of the representation $\rho(p)$ to $V'$. Assume that there are no proper subspaces $Z^{blue} \subset H^{blue}$, $Z^{red} \subset H^{red}$, $Z^{yellow} \subset H^{yellow}$ such that $\xi \in Z^{blue} \otimes Z^{red} \otimes Z^{yellow}$. Denote by $U$ the group consisting of all triples of unitary operators $(A^{blue}, A^{red}, A^{yellow})$ acting in the corresponding spaces such that $A^{blue} \otimes A^{red} \otimes A^{yellow} \xi = \xi$. It is easy to show, that $U$ is a product of compact Lie groups and all but a finite number of factors are Abelian. For $v$ being in a general position (and dim $H^{blue}$, $H^{red}$, $H^{yellow} \geqslant 2$) the group $U$ is trivial. The group $U$ acts in each factor of the tensor product (5.1) preserving distinguished vector and therefore acts in the whole space $V$. There arises a natural conjecture: $U$ and $G$ are dual in the Schur–Weyl sense (for the spherical pair $S_\infty \times S_\infty \supset S_\infty$ this property was obtained in [31], Subsect 5.5). If this is correct, then $\rho$ is irreducible for $\xi$ being in a general position.
Let us choose orthonormal bases $e_i^{\text{blue}} \in H^{\text{blue}}$, $e_j^{\text{red}} \in H^{\text{red}}$, $e_k^{\text{yellow}} \in H^{\text{yellow}}$, this defines a basis in the tensor product $X$. We decompose $\xi \in X$ in this basis

$$\xi = \sum_{i,j,k} \xi_{i,j,k}^{\text{blue}} \otimes e_j^{\text{red}} \otimes e_k^{\text{yellow}}.$$ 

Fix a checker surface. Assign to each blue edge a vector $e_i^{\text{blue}}$, to each red edge a vector $e_j^{\text{red}}$, to each yellow edge a vector $e_k^{\text{yellow}}$. Let us call such data assignments. For a triangle $A$ denote by $\xi_{\text{blue}}(A)$, $\xi_{\text{red}}(A)$, $\xi_{\text{yellow}}(A)$ the numbers of basis vectors on its sides.

**Proposition 5.2** (see [23]) The spherical function $\Phi(\Xi)$ is given by

$$\Phi(\Xi) = \sum_{\text{assignments}} \prod_{\text{white } A} \xi_{\text{blue}}(A), \xi_{\text{red}}(A), \xi_{\text{yellow}}(A) \times \prod_{\text{white } B} \xi_{\text{blue}}(B), \xi_{\text{red}}(B), \xi_{\text{yellow}}(B).$$

### 6 Convolution algebras and their limit

Here we explain how the multiplication of double cosets arises as a limit of convolution algebras on finite groups $G_n$.

**6.1. Convolution algebras.** Let $H$ be a finite group, $S$ its subgroup. Denote by $C[H]$ the group algebra of $H$ equipped with convolution $\ast$. Set

$$\delta_S := \frac{1}{\#(S)} \sum_{h \in S} \delta_h,$$

where $\#(\cdot)$ denotes the number of elements in a finite set and $\delta_h$ is the delta-function on $H$ supported by $h$.

Denote by $C[S \setminus H/S] \subset C[H]$ the space of $S$-biinvariant functions, i.e., functions satisfying the condition

$$f(s_1 hs_2) = f(h), \text{ where } s_1, s_2 \in S.$$ 

Equivalently,

$$f \ast \delta_S = \delta_S \ast f = f. \quad (6.1)$$

Clearly, $C[S \setminus H/S]$ is a subalgebra in $C[H]$.

Let $\rho$ be a unitary finite-dimensional representation of $H$ in the space $V$, denote by the same letter $\rho(\cdot)$ the corresponding representation of the group algebra. Denote by $V^S$ the space of $S$-fixed vectors in $V$. Obviously, $\rho(\delta_S)$ is the operator of orthogonal projection to $V^S$. For $f \in C[S \setminus H/S]$ let us represent $\rho(f)$ as a block operator

$$\rho(f) : V^S \oplus (V^S)^\perp \to V^S \oplus (V^S)^\perp.$$
By (6.1), it has the form
\[ \rho(f) = \begin{pmatrix} \mathfrak{p}(f) & 0 \\ 0 & 0 \end{pmatrix}. \]
Moreover operators \( \mathfrak{p}(f) \) determine a representation of the algebra \( \mathbb{C}[S \setminus H/S] \) in \( V^S \). Standard arguments show that if \( \rho \) is irreducible and \( V^S \neq 0 \), then an initial representation \( \rho \) is uniquely determined by representation \( \mathfrak{p} \).

There is a natural basis in \( \mathbb{C}[S \setminus H/S] \). Namely, for each double coset \( p \), we consider the element of \( \mathbb{C}[H] \) given by
\[ \delta_p := \frac{1}{\#p} \sum_{g \in p} \delta_g. \]
We have
\[ \delta_p \ast \delta_q = \sum c_{pq}^{r} \delta_r, \]
where \( c_{pq}^{r} \in \mathbb{Q} \) are certain structure constants,
\[ \sum_r c_{pq}^{r} = 1, \quad c_{pq}^{r} \geq 0. \]

**Remark.** These considerations with minor variations can be repeated for a locally compact group \( H \) and a compact subgroup \( S \).

**Remark.** An algebra of biinvariant functions (if we are able to understand it) is an important tool of an investigation of representations of \( H \). However, such algebras are difficult objects and there few families of such objects that actually are explored. The most important examples are:

— the Hecke–Iwahori algebras (as \( H = \text{GL} \) over a finite field and \( S \) is the group of all upper triangular matrices);

— the affine Hecke algebras (as \( H = \text{GL} \) over a \( p \)-adic field and \( S \) is the Iwahori subgroup);

— algebras of functions on a semisimple Lie groups biinvariant with respect to its maximal compact subgroup (as \( H = \text{GL}(n, \mathbb{R}) \) and \( S \) is the orthogonal group \( O(n) \); or (a dual version) \( H \) is a unitary group \( U(n) \) and \( S = O(n) \)).

Also recall that the Hecke and the affine Hecke algebras live by their own life independently on the initial groups. There is a huge literature on this topics.

6.2. **Concentration of convolutions.** Now consider a group
\[ H = G_n = S_n \times S_n \times S_n \]
and the subgroup \( K_n[\alpha] \) in the diagonal fixing \( 1, 2, \ldots, \alpha \). Consider algebras \( \mathbb{C}[K_n[\alpha] \setminus G_n/K_n[\alpha]] \) for fixed \( \alpha \) and \( n \to \infty \).

We assign a checker surface to an element of \( K_n[\alpha] \setminus G_n/K_n[\alpha] \) as above. Thus we get an embedding
\[ \theta_n : K_n[\alpha] \setminus G_n/K_n[\alpha] \to \mathcal{M}[\alpha, \alpha] \simeq K[\alpha] \setminus G/K[\alpha]. \]

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The image consists of all surfaces having \( \leq 2n \) triangles. Let \( p \in K[\alpha] \setminus G/K[\alpha] \) contain \( 2l \) triangles. For each \( n \in \mathbb{N} \) we define \( \delta_p(n) \in \mathbb{C}[K_n[\alpha] \setminus G_n/K_n[\alpha]] \) by

\[
\delta_p(n) = \begin{cases} 
0, & \text{if } n < l; \\
q^{-1}p, & \text{if } n \geq l.
\end{cases}
\]

**Theorem 6.1** Let \( p, q \) be contained in the image of \( \theta_m \), i.e., they define elements of \( K_m[\alpha] \setminus G_m/K_m[\alpha] \). Let \( n > m \). Decompose the following convolution as

\[
\delta_p(n) * \delta_q(n) = \sigma_n \delta(p \circ q)(n) + \sum_{r \neq p \circ q} b^{r(n)}_{p(n),q(n)} \delta_r(n).
\]

Then the coefficient \( \sigma_n \) tends to 1 as \( n \to \infty \).

Thus the sum of remaining coefficients tends to 0.

See a formal proof in [20], [7], it is based on arguments from [33].

Multiplicativity Theorem 3.3 can be derived from the concentration of convolutions in a straightforward way, see [7].

### 7 A pre-limit convolution algebra

**7.1. Algebras of conjugacy classes.** Consider the group \( Q_n := S_n \times S_n \), its diagonal \( L_n \simeq S_n \) and the space of conjugacy classes \( Q_n//S_n \). As we noticed above,

\[
(S_n \times S_n)//L_n \simeq K_n \setminus (S_n \times S_n \times S_n)/K_n \simeq \Xi_n.
\]

Consider the space \( \mathbb{C}[Q_n//L_n] \) of all \( f \in \mathbb{C}[Q_n] \) satisfying

\[
f(hgh^{-1}) = f(g), \quad \text{where } g \in Q_n, \; h \in L_n.
\]

Obviously, this space is closed with respect to convolutions. We intend to present a quasi-description of this algebra.

**7.2. The Ivanov-Kerov algebra.** Consider a disjoint union of all sets \( Q_m//L_m \)

\[
\hat{\Xi} := \coprod_{m=0}^{\infty} \Xi_n \simeq \coprod_{m=0}^{\infty} Q_m//L_m,
\]

*(below use the same notation for a conjugacy class and the corresponding checker surface)*.

**Remark.** Emphasize that \( \hat{\Xi} \neq \Xi[0,0] \). Indeed, an element of \( \hat{\Xi} \) is an arbitrary finite checker surface, and an element of \( \Xi[0,0] \) is a checker surface without double triangles. A removing double triangles determines a map \( \hat{\Xi} \to \Xi[0,0] \). A preimage of any point is countable.

For any \( n \) and \( p \in Q_n//L_n \) we define an element \( \Delta_p^{n,n} \) of the group algebra \( \mathbb{C}[Q_n] \) by

\[
\Delta_p^{n,n} := \sum_{g \in p} \delta_g.
\]
Let $m \geq n$ and $p \in Q_m//L_n$. Consider the corresponding element $p(m) \in Q_m//L_m$, i.e., we take the corresponding checker surface and add $m - n$ double triangles. We set

$$\Delta_{p}^{m,n} := \frac{(N-n)!}{(N-m)!} \Delta_{p}^{n,n}.$$ 

**Theorem 7.1** There exists an associative algebra $A$ with a basis $u_p$, where $p$ ranges in $\hat{\Xi}$, and a product

$$u_p \circ u_q = \sum_{s \in \hat{\Xi}} a_{pq}^{r} u_r$$

such that

- $a_{pq}^{r} \in \mathbb{Z}_+$.  
- Consider the linear map $\Pi_n : A \rightarrow \mathbb{C}[Q_n//L_n]$ defined on $u_p$, $u \in Q_k//L_k$ by

$$\Pi_n u_p := \begin{cases} \Delta_{p}^{n,k}, & \text{if } k \leq n \\ 0, & \text{if } k > n. \end{cases}$$

Then $\Pi_n$ is a homomorphism from $A$ to $\mathbb{C}[Q_n//L_n]$.

- Let $p \in Q_n//L_n$, $q \in Q_m//L_m$, $r \in Q_l//L_l$. Then $a_{pq}^{r}$ are zero if $l > m + n$ or if $l < \min(m,n)$.

A similar statement for $\mathbb{C}[S_n//S_n]$ was obtained by Ivanov and Kerov in [14], it was extended to more general objects including $\mathbb{C}[Q_n//L_n]$ in [28].

**7.3. A description of the algebra $A$.** See [28]. Let $X$, $Y$ be finite sets. We say that a partial bijection $s : X \rightarrow Y$ is a bijection of a subset $B \subset X$ to a subset $B \subset Y$, we say $\text{dom}(s) := A$, $\text{im}(s) := B$.

Let $p$, $q \in \hat{\Xi}$. Consider a partial bijection $s$ from the set of black triangles of $p$ to the set of white triangles of $q$. For each $A \in \text{dom} s$ we remove the interior of the black triangle $A \subset p$ and the interior of the white triangle $s(A) \subset q$ and identify their boundaries according colors of edges. We normalize the resulting cell complex and come to a new checker surface $p \circ_s q$. The formula for the product in $A$ is

$$u_p \circ u_q = \sum_{s} u_{p \circ_s q},$$

where the summation is taken over all partial bijection.

**7.4. The Poisson algebra.** Denote by $\mathfrak{A}_n$ the subspace in $A$ generated by all $u_p$, where $p$ ranges in $\prod_{i=0}^{n} Q_n//K_n$. The get a filtration,

$$v \in \mathfrak{A}_n, \quad w \in \mathfrak{A}_m \Rightarrow v \circ w \in \mathfrak{A}_{m+n}.$$ 

The corresponding graded associative algebra $\oplus_{j=0}^{\infty} \mathfrak{A}_n/\mathfrak{A}_{n-1}$ is commutative, a product of basis elements is given by

$$u_p \cdot u_q = u_p \Pi_q.$$
where \( p \coprod q \) is a disjoint union of checker surfaces.

Also, we have a structure of a graded Poisson Lie algebra on \( \bigoplus_{n=0}^{\infty} \mathfrak{A}_n/\mathfrak{A}_{n-1} \), namely, for any \( x \in \mathfrak{A}_m, \ y \in \mathfrak{A}_n \) we define their bracket \( \{x, y\} \) as the image of \( x \ast y - y \ast x \) in \( \mathfrak{A}_{m+n-1}/\mathfrak{A}_{m+n-2} \). Let us describe the bracket. Consider set \( M(p, q) \) of all pairs \( \varphi = (B, C)\), where \( B \) is a black triangle of \( p \) and \( C \) is a white triangle of \( q \). For any \( \psi \) we define a surface \( p \circlearrowleft \varphi q \), it is obtained by removing the interior of \( B \) from \( p \) and the interior of \( C \) from \( q \) and identification of boundaries of these triangles. In this notation,

\[
\{u_p, u_q\} = \sum_{\varphi \in M(p, q)} u_{p \circlearrowleft \varphi q} - \sum_{\psi \in M(q, p)} u_{q \circlearrowleft \psi p}.
\]

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