A NOTE ON REAL KILLING SPINORS IN WEYL GEOMETRY

VOLKER BUCHHOLZ
Humboldt Universität zu Berlin, Institut für Reine Mathematik, Ziegelstraße 13a, D-10099 Berlin.

Abstract: This text is dedicated to the real Killing equation on 3-dimensional Weyl manifolds. Any manifold admitting a real Killing spinor of weight 0 satisfies the conditions of a Gauduchon-Tod geometry. Conversely, any simply connected Gauduchon-Tod geometry has a 2-dimensional space of solutions of the real Killing equation on the spinor bundle of weight 0.

Subj. Class: Differential Geometry
1991 MSC: 53C05;53C10;53A30
Keywords: Weyl geometry; Spin geometry; Killing equation

1 Introduction

In [2] we introduced the spinor geometry on Weyl manifolds and investigated the Dirac-, Twistor- and Killing equation in this context. Concerning the real Killing equation we presented in [2] the following result:

Theorem 1.1 (see [2], Theorem 3.1) Let \( \psi \in \Gamma(S^w) \) be a real Killing spinor on a Weyl manifold \((M^n, c, W)\), i.e. there exists a density \( \beta \in \Gamma(L^{-1}) \) for which

\[
\nabla S^w \psi = \beta \otimes \nu \psi, \quad \beta \in \Gamma(C \otimes L^{-1})
\]

is satisfied. Then the following statements hold:

1. \( R = 4n(n-1)\beta^2 \).
2. \( w \neq 0 \): \( W \) is exact and Einstein-Weyl.
3. \( w = 0, n \geq 4 \): \( W \) is exact and Einstein-Weyl.

The following equations were obtained within the proof and will play an important role in the sequel:

\[
\mu^2 \text{Ric'} \otimes \psi = 2 \left( n - 1 - \frac{n-1}{n-2} \right) \nabla \beta \otimes \psi + \left( 1 - \frac{n-1}{n-2} \right) \mu^{12} \text{Alt} \nabla \beta \otimes c \otimes \psi + \frac{R}{n} \nu \psi - \mu^2 F \otimes \psi \quad (1)
\]
\[
F \cdot \psi = -\frac{4(n-1)}{n-2} \nabla \beta \cdot \psi.
\] (2)

Theorem 1.1 gives no statement for the case \( n = 3 \) and \( w = 0 \). In the next section we prove that in three dimensions the existence of a real Killing spinor of weight 0 is essentially equivalent to the fact that this manifold is a Gauduchon-Tod geometry:

**Definition 1.2** (see [4], Proposition 5) A 3-dimensional Weyl manifold \((M^3, W, c)\) is called Gauduchon-Tod geometry, if there exists a density \( \beta \in \Gamma(L^{-1}) \) such that the following conditions are satisfied:

1. \( W \) is Einstein-Weyl;
2. \( R = 24\beta^2 \);
3. \( 4\nabla \beta = \ast F \).

**Remark:** The \( \kappa \in C^\infty(M, \mathbb{R}) \) in ([4], Proposition 5) is related to the \( \beta \in \Gamma(L^{-1}) \) of Definition 1.2 in the following way: \( \kappa_{\ast \gamma} = -4\beta \). For more information on Gauduchon-Tod geometries, e.g. their classification, see [4] and the references therein.

Hence, the main result of this text is as follows:

**Theorem 1.3** Let \((M^3, c, W)\) be a CSpin-manifold.

1. If \( \psi \in \Gamma(S^0) \) is a real Killing spinor then the space of solutions of the Killing equation is 2-dimensional and \((M^3, c, W)\) is a Gauduchon-Tod geometry.
2. Conversely, any simply connected Gauduchon-Tod geometry has a two dimensional space of Killing spinors of weight 0.

**Acknowledgement:** I would like to thank N. Ginoux for pointing out a gap in a former version of [2]. This concerns the real Killing equation in dimension 3 and for weight 0.

## 2 The proof of Theorem 1.3

Let \((M^3, c, W)\) be a CSpin-manifold. The curvature tensor of the Weyl structure \( W \) is given by

\[
\mathcal{R} = \text{Ric}^N \triangle c + F \otimes c,
\]

where \( \triangle : T^{2,0} \times T^{2,0} \rightarrow T^{4,0} \)

\[
\omega \triangle \eta := [(23) + (12)(24)(34) - (24) - (12)(23)]\omega \otimes \eta, \quad \omega, \eta \in T^{2,0}
\]

is the so called Kulkarni-Nomizu product (see [1]) and

\[
\text{Ric}^N := -\text{sym}_0 \text{Ric} - \frac{1}{12} \text{Rc} + \frac{1}{2} F
\] (3)

is the normalized Ricci tensor of \( W \) (see [3]). \text{sym}_0 denotes the symmetric trace free part of a \((2,0)\)-tensor. The following Lemma is a tool for calculations with Kulkarni-Nomizu products in spin geometry:

**Lemma 2.1** Let \( \omega \) be a \((2,0)\)-tensor. Then the following algebraic identity holds in any dimension:

\[
\mu^{34} \omega \triangle c = 2 \text{Alt} \mu^2 \omega - 2 \text{Alt} \omega.
\]
Proof:

\[ \mu^{34} \omega \bigtriangleup c = [ \mu^{34} + (12)(24)(34) - (24) - (12)(23)] \omega \otimes c = [ \mu^{24} + (12)\mu^{32} - \mu^{32} - (12)\mu^{24}] \omega \otimes c \]

\[ = [\mu^{23} + (12)\mu^{32} - \mu^{32} - (12)\mu^{23}] \omega \otimes c = [-2\mu^{23} + 2tr^{23} + 2(12)\mu^{32} - 2(12)tr^{23}] \omega \otimes c \]

\[ = -2|\mu^{32} - (12)\mu^{12}] \omega \otimes c - 2 \text{Alt} \omega = -2[1 - (12)]|\mu^{2}\omega \otimes c - 2 \text{Alt} \omega \]

\[ = 2 \text{Alt} \mu^{2}\omega - 2 \text{Alt} \omega. \]

\[ \square \]

Lemma 2.2 Let \(4\nabla \beta = *F\) be satisfied on \((M^3, c, W)\). Then the following identities are true for any spinor \(\psi \in \Gamma(S^w)\):

\[ \frac{1}{4} \text{Alt} \nu^2 F \otimes \psi - \text{Alt} \nabla \beta \otimes \nu \psi - \frac{1}{2} F \otimes \psi = 0. \] (4)

and

\[ (\nu \nabla \beta - \nabla \beta \cdot \nu) \cdot \psi - \frac{1}{2} \mu^2 F \otimes \psi = 0. \] (5)

Proof: Denote by \((e_1, e_2, e_3)\) a local weightless conformal frame on \((M^3, c, W)\) as well as \((\sigma_1, \sigma_2, \sigma_3)\) its dual. In dimension 3 we have the important relation

\[ e_i \cdot e_j \cdot \psi = -\sum_{k=1}^{3} \epsilon_{ijk} e_k \cdot \psi. \] (6)

Here \(\epsilon_{ijk}\) denotes the Levi-Civita symbol. Since the \(*\)-operator on 2-forms is defined by the formula

\[ *F = \frac{1}{2} \sum_{i,j=1}^{3} F(e_i, e_j) * (\sigma_i \wedge \sigma_j) = \frac{1}{2} \sum_{i,j,k=1}^{3} F(e_i, e_j) e_{ijk} \sigma_k \]

we can rewrite the assumption as follows:

\[ 8 \nabla_{e_k} \beta = \sum_{i,j=1}^{3} \epsilon_{ijk} F(e_i, e_j). \] (7)

From the algebraic identity

\[ F \cdot \psi = \sum_{i,j=1}^{3} F(e_i, e_j) e_i \cdot e_j \cdot \psi = -\sum_{i,j,k=1}^{3} \epsilon_{ijk} F(e_i, e_j) e_k \cdot \psi = -8 \sum_{k=1}^{3} (\nabla_{e_k} \beta) e_k \cdot \psi \]

and (7) then follows:

\[ F \cdot \psi = -8 \nabla \beta \cdot \psi, \] (8)

Using (7) and (8) we get:

\[ \frac{1}{4} \text{Alt} \nu^2 F \otimes \psi - \text{Alt} \nabla \beta \otimes \nu \psi - \frac{1}{2} F \otimes \psi \]

\[ = \frac{1}{4} \sum_{i,j,k=1}^{3} F(e_j, e_k) \sigma_i \wedge \sigma_j \otimes e_i \cdot e_k \cdot \psi - \sum_{i,j=1}^{3} (\nabla_{e_i} \beta) \sigma_i \wedge \sigma_j \otimes e_j \cdot \psi - \frac{1}{2} \sum_{i,j=1}^{3} F(e_i, e_j) \sigma_i \wedge \sigma_j \otimes \psi \]
\[
0 = \frac{1}{4} \mu^2 \text{Alt} \nu \mu^2 F \otimes \psi - \mu^2 \text{Alt} \nabla \beta \otimes \nu \psi - \frac{1}{2} \mu^2 F \otimes \psi
\]

\[
= \frac{1}{4} \mu^2 \nu \mu^2 F \otimes \psi - \mu^2 \nabla \beta \otimes \nu \psi - \frac{1}{2} \mu^2 F \otimes \psi - \frac{1}{4} \mu^4 \nu \mu^2 F \otimes \psi + \mu^2 \nabla \beta \otimes \nu \psi
\]

\[
= \frac{1}{4} \nu F \cdot \psi - \frac{1}{2} \mu^2 F \otimes \psi + 3 \nabla \beta \otimes \psi - \frac{1}{2} \mu^2 F \otimes \psi + \frac{3}{4} \mu^2 F \otimes \psi + \nabla \beta \cdot \nu \psi
\]

\[
= \frac{1}{4} \nu F \cdot \psi - \frac{1}{2} \mu^2 F \otimes \psi + 3 \nabla \beta \otimes \psi - \frac{1}{2} \mu^2 F \otimes \psi + \frac{3}{4} \mu^2 F \otimes \psi + \nabla \beta \cdot \nu \psi
\]

\[
= \nu \nabla \beta \cdot \psi + \nabla \beta \otimes \psi - \frac{1}{4} \mu^2 F \otimes \psi
\]

\[
0 = \frac{1}{4} (\nu \nabla \beta - \nabla \beta \cdot \nu) \cdot \psi - \frac{1}{4} \mu^2 F \otimes \psi.
\]

Hence (\ref{2}) is true. \hfill \Box

After these preliminary calculations we get to the proof of Theorem 1.3. Let \( \psi \in \Gamma(S^0) \) be a real Killing spinor. The first statement follows immediately from the existence of an equivariant quaternionic structure \( j \) on the spinor module, which commutes with the Clifford multiplication. By Theorem 1.1 and its proof we have

\[
R = 24 \beta^2; \quad F \cdot \psi = -8 \nabla \beta \cdot \psi.
\]

Since \( \psi \) vanishes nowhere the second equation is equivalent to

\[
4 \nabla \beta = \ast F
\]

by (\ref{2}). Therefore we have already verified the conditions 2. and 3. of the definition of a Gauduchon-Tod geometry. It remains to proof that the manifold is Einstein-Weyl. To this end we have to
simplify (1) by means of \( R = 24\beta^2 \) and \( Ric' = \text{sym}_qRic + \frac{1}{2}Rc - \frac{1}{2}F \) to

\[
\mu^2\text{sym}_qRic' \otimes \psi = -((\nabla\beta \cdot \nu - \nu\nabla\beta) \cdot \psi - \frac{1}{2}\mu^2 F \otimes \psi)
\]

But the righthandside vanishes according to (5). Hence \( W \) is Einstein-Weyl.

Conversely, let \( W \) be a simply connected Gauduchon-Tod geometry. Is is sufficient to show that \( S^0 \) is flat with respect to \( \nabla^\beta = \nabla^S,0 - \beta \otimes \nu \). To this end, we have to proof that the curvature of \( \nabla^\beta \) vanishes. We use the properties of Gauduchon-Tod geometries given in Definition 1.2, the result of Lemma 2.1 and the equations (4) and \( R^{S,0} = \frac{1}{4}\mu^{34}Ric^N \triangle c = -\frac{1}{4}\mu^{34} (\frac{1}{12}Rc - \frac{1}{2}F) \triangle c \).

\[
\begin{align*}
\mathcal{R}^\beta &= \text{Alt}\nabla^\beta \circ \nabla^\beta = \text{Alt}\nabla^\beta \circ (\nabla^{S,0} - \beta \otimes \nu) \\
&= \text{Alt} \left( \nabla^{S,0} \circ \nabla^{S,0} - \nabla^\beta \otimes \nu - (12)\beta \otimes \nu \nabla^{S,0} - \beta \nu \nabla^{S,0} + \beta^2 \nu \nu \right) \\
&= \mathcal{R}^{S,0} - \text{Alt}(\nabla^\beta)\nu + \beta^2 \text{Alt} \nu^{12} = -\frac{1}{4}\mu^{34} \left( \frac{1}{12}Rc - \frac{1}{2}F \right) \triangle c - \text{Alt}(\nabla^\beta)\nu + \beta^2 \text{Alt} \nu^{12} \\
&= -\frac{1}{4} \left( \frac{1}{6} R \text{Alt} \nu^2 c - \text{Alt} \nu^2 F + 2F \right) - \text{Alt}(\nabla^\beta)\nu + \beta^2 \text{Alt} \nu^{12} \\
&= -\frac{1}{24} R \text{Alt} \nu^{12} + \frac{1}{4} \text{Alt} \nu^2 F - \frac{1}{2} F - \text{Alt}(\nabla^\beta)\nu + \beta^2 \text{Alt} \nu^{12} \\
&= \frac{1}{4} \text{Alt} \nu^2 F - \frac{1}{2} F - \text{Alt}(\nabla^\beta)\nu \\
&= 0.
\end{align*}
\]

\( \square \)

References

[1] A.L. Besse. *Einstein manifolds*. Springer Verlag, 1987.

[2] V. Buchholz. Spinor equations in Weyl geometry. to appear in *Proc. of the Winter School on Geometry and Physics, Srni*, 1999; math.DG/9901125 v2.

[3] D. Calderbank, H. Pederson. Einstein-Weyl geometry. *Odense Universitet, preprint*, No. 40, 1997.

[4] P. Gauduchon, K.P. Tod. Hyper-hermitian metrics with symmetry. *Journal of Geometry and Physics*, pages 291–304, 1998.