STABILITY THRESHOLD OF THE COUETTE FLOW FOR NAVIER-STOKES BOUSSINESQ SYSTEM WITH LARGE RICHARDSON NUMBER $\gamma^2 > \frac{1}{4}$

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Abstract. In this paper, we study the nonlinear asymptotic stability of the Couette flow in the stably stratified regime, namely the Richardson number $\gamma^2 > \frac{1}{4}$. Precisely, we prove that if the initial perturbation $(u_{in}, \vartheta_{in})$ of the Couette flow $v_s = (y, 0)$ and the linear temperature $\rho_s = -\gamma^2 y + 1$ satisfies $\|u_{in}\|_{H^{r+1}} + \|\vartheta_{in}\|_{H^{r+2}} \leq \epsilon_0 \nu^{\frac{1}{2}}$, then the asymptotic stability holds.

1. Introduction

The stability of shear flow in a stratified medium is of interest in many fields of research such as fluid dynamics, geophysics, astrophysics, mathematics, etc. Density stratification can strongly affect the dynamic of fluids like air in the atmosphere or water in the ocean and the stability of the question of stratified flows dates back to Taylor [35] and Goldstein [20], and since then there has been an active search towards the understanding of the stability of density-stratified flows.

1.1. Two dimensional Navier-Stokes Boussinesq equations. In this paper, we consider the two-dimensional Navier-Stokes Boussinesq system with full dissipation in $\Omega = \mathbb{T} \times \mathbb{R}$:

$$\begin{cases}
\partial_t v + v \cdot \nabla v - \nu \Delta v + \nabla P = -\rho g e_2 \\
\partial_t \rho + v \cdot \nabla \rho - \mu \Delta \rho = 0 \\
\nabla \cdot v = 0 \\
v|_{t=0} = v_{in}(x, y), \quad \rho|_{t=0} = \rho_{in}(x, y),
\end{cases}$$

(1.1)

where $(x, y) \in \Omega$, $v = (v^1, v^2)$ is the velocity field, $P$ is the pressure, $\rho$ is the temperature (density), $g$ is the gravitational constant, $e_2 = (0, 1)$ is the unit vector in the vertical direction, $\nu$ is the viscosity coefficient and $\mu$ is the thermal diffusivity. In this paper, we focus on the case when $C^{-1} \leq \frac{\nu}{\mu} \leq C$ with some constant $C > 1$ independent of $\nu$. Thus, the three parameters $\mu, \nu$ and $g$ can be normalized to $\mu = \nu$ and $g = 1$.

The system admits a class of steady states, where the velocity field is the Couette flow and the temperature is a linear function of the vertical component, namely,

$$v_s = (y, 0), \quad \rho_s = -\gamma^2 y + 1, \quad \rho_s = \frac{1}{2} \gamma^2 y^2 - y + C_0,$$

(1.2)

where the constant $\gamma^2$ is the Richardson number $[4]$. The Richardson number is one of the control parameters of the stability of stratified shear flows. The Miles-Howard theorem [21, 33] guarantees that any flow in the inviscid non-diffusive limit is linearly stable if the local Richardson number everywhere exceeds the value $\frac{1}{4}$, however, unstable modes can arise when the Richardson number is smaller than $\frac{1}{4}$ [18].

In general, the Richardson number $\gamma^2(y) = -\frac{g \partial_y \vartheta_{in}(y)}{1 + g U(y)}$ is not always a constant, where $g$ is the gravitational constant, $U(y)$ is the horizontal velocity of the shear flow, and $\vartheta_{in}$ is the temperature.
In this paper, we study the asymptotic stability of the Couette flow when the Richardson number is greater than $\frac{1}{4}$, namely,
\[ \gamma^2 > \frac{1}{4}. \]

It is natural to introduce the perturbation. Let $v = u + v_s$, $P = p + p_s$ and $\rho = \gamma^2 \vartheta + \rho_s$, then $(u, p, \vartheta)$ satisfies
\[ \begin{cases} 
\partial_t u + y\partial_x u + \left( \begin{array}{c} u^2 \\ 0 \end{array} \right) + u \cdot \nabla u + \nabla p - \left( \begin{array}{c} 0 \\ \gamma^2 \vartheta \end{array} \right) = 0, \\
\partial_t \vartheta + y\partial_x \vartheta + u \cdot \nabla \vartheta - \nu \Delta \vartheta - u^2 = 0, \\
u |_{t=0} = u_{in}(x, y), \quad \vartheta |_{t=0} = \vartheta_{in}(x, y).
\end{cases} \]

Let $\omega = \nabla \times u = \partial_x u^2 - \partial_y u^1$ be the vorticity, then $(\omega, \vartheta)$ satisfies
\[ \begin{cases} 
\partial_t \omega + y\partial_x \omega + u \cdot \nabla \omega - \nu \Delta \omega = -\gamma^2 \partial_x \vartheta, \\
\partial_t \vartheta + y\partial_x \vartheta + u \cdot \nabla \vartheta - \nu \Delta \vartheta - u^2 = 0, \\
u = \nabla^\perp \psi = (\partial_y \psi, \partial_x \psi), \\
\Delta \psi = \omega.
\end{cases} \]

1.2. Historical comments. In the physics literature, there has been a lot of work devoted to the stability of the Couette flow in the linearized stratified inviscid flow. See [12, 19] and the references therein. There are only few mathematically rigorous results.

Lin and Yang [38] studied the linear asymptotic stability of the steady-state (1.2) for the 2D Euler Boussinesq system. More precisely, they proved that if $\gamma^2 > \frac{1}{4}$, then the solution to the system (2.8) with $\nu = 0$ satisfies
\[ (u_\#, \omega_\#, \vartheta_\#) \lesssim \left( (t)^{-\frac{1}{2}}, (t)^{-\frac{3}{2}}, (t)^{-\frac{1}{2}}, (t)^{-\frac{1}{2}}, (t)^{-\frac{11}{2}} \right), \]
where $P_\# = f_\# = f - P_0 f$ and $P_0 f = f_0 = \frac{1}{2\pi} \int_{\mathbb{T}} f(t, x, y)dx$ denote the non-zero mode and the zero mode. See also [9] for other linear results of general shear flows and [2] for nonlinear results. For the Navier-Stokes Boussinesq system without the heat diffusion ($\nu = 1, \mu = 0$), Masmoudi, Said-Houari and Zhao [28] proved that if the initial perturbations are in Gevrey-$m$ with $1 \leq m < 3$, then the steady state (1.2) is asymptotically stable. For the case $\nu = \mu > 0$, in [16, 29], the authors proved the asymptotic stability of the steady-state (1.2) with $\gamma^2 = 0$, if the initial perturbations $(u_{in}, \vartheta_{in})$ satisfy
\[ \|u_{in}\|_{H^2} \leq \varepsilon_0 \nu^{\frac{1}{2}}, \quad \|\vartheta_{in}\|_{H^1} + \|D_x^\perp \vartheta_{in}\|_{H^1} \leq \varepsilon_1 \nu^{\frac{3}{2}}, \]
for both the finite channel ($\Omega = \mathbb{T} \times [-1, 1]$) and the infinite channel ($\Omega = \mathbb{T} \times \mathbb{R}$) cases. We also refer to [40] for the asymptotic stability result of the Couette flow with a linear temperature.

The mechanism leading to stability is the so-called inviscid damping and enhanced dissipation. Similar phenomena also happen in other fluids systems. One may refer to [6, 23, 24, 31] for the inviscid damping results of Euler equations, and to [8, 10, 25, 30, 32] for the enhanced dissipation results of Navier-Stokes equations around 2D Couette flow, and to [8, 11, 15, 36] for the enhanced dissipation results of Navier-Stokes equations around 3D Couette flow. For other shear flows, one may refer to [27, 22, 37, 26] for Kolmogorov flow and to [14, 15, 17] for Poiseuille flow.

In this paper, we focus on the small viscosity case and study the stability threshold problem, namely,

Given norms $\| \cdot \|_{Y_1}$ and $\| \cdot \|_{Y_2}$, find $\alpha = \alpha(Y_1, Y_2)$ and $\beta = \beta(Y_1, Y_2)$ such that
\[ \|u_{in}\|_{Y_1} \leq \nu^\alpha \text{ and } \|\vartheta_{in}\|_{Y_2} \leq \nu^\beta \Rightarrow \text{stability}; \]
\[ \| u_{in} \|_{Y_1} \gg \nu^{\alpha} \text{ or } \| \vartheta_{in} \|_{Y_2} \gg \nu^{\beta} \Rightarrow \text{instability.} \]

1.3. Main result. Our main result states as follows:

**Theorem 1.1.** Let \( s \geq 6 \) and \( \gamma^2 > \frac{1}{4} \). There exist \( 0 < \epsilon_0 = \epsilon_0(s, \gamma^2) < 1, 0 < \nu_0 = \nu_0(s, \gamma^2) < 1 \), such that for all \( 0 < \nu \leq \nu_0 \) and \( 0 < \epsilon \leq \epsilon_0 \), if the initial data \((u_{in}, \vartheta_{in})\) satisfies

\[ \| u_{in} \|_{H^{s+1}} + \| \vartheta_{in} \|_{H^{s+2}} \leq \epsilon \nu^{\frac{1}{2}}, \]

then the solution \((u, \omega, \vartheta)\) of (1.4) satisfies

\[ \| P_{\neq} u^{1} \|_{L^2} + \langle t \rangle \| P_{\neq} u^{2} \|_{L^2} + \langle t \rangle^{-1} \| P_{\neq} \omega \|_{L^2} + \| P_{\neq} \vartheta \|_{L^2} \leq C \epsilon \nu^{\frac{1}{2}} \langle t \rangle^{-\frac{1}{2}} e^{-c_0 \nu^{\frac{1}{2}} t}, \]

\[ \nu^{-\frac{1}{4}} \| P_{\neq} u \|_{H^s} + \| P_{\neq} \omega \|_{H^s} + \nu^{-\frac{1}{4}} \| P_{\neq} \vartheta \|_{H^s} \leq C \epsilon \nu^{\frac{1}{4}}, \]

where \( C \) and \( c \) are the constants independent of \( t, \epsilon \) and \( \nu \).

**Remark 1.2.** We consider the case \( \gamma^2 - \frac{1}{4} \gg \nu \) in this paper. It is interesting to study the case when \( \gamma^2 - \frac{1}{4} \) is \( \nu \)-related.

**Remark 1.3.** For the Navier-Stokes equation, \( \rho = 0 \), in [32], the authors proved that the asymptotic stability of the Couette flow holds if the initial perturbation satisfies \( \| \omega_{in} \|_{H^s} \leq \epsilon_0 \nu^{\frac{1}{4}} \). Moreover, the enhanced dissipation rate is

\[ \| P_{\neq} \omega \|_{L^2} \leq Ce^{-c_0 \nu^{\frac{1}{4}} t}. \]

For the Navier-Stokes Boussinesq system, when \( \gamma^2 > \frac{1}{4} \), the asymptotic behavior of \( P_{\neq} \omega \) changes. The vorticity has an additional transient growth \( \sqrt{t} \), namely,

\[ \| P_{\neq} \omega \|_{L^2} \leq C \sqrt{\langle t \rangle} e^{-c_0 \nu^{\frac{1}{4}} t}. \]

To stabilize this growth, we need an additional \( \nu^{\frac{1}{4}} \) smallness, due to the fact that \( \sqrt{\langle t \rangle} e^{-c_0 \nu^{\frac{1}{4}} t} \lessapprox \nu^{-\frac{1}{4}} \). Thus it is somehow necessary to prove the asymptotic stability under the assumption that the initial perturbation \( \omega_{in} \) is of size \( \nu^{\frac{1}{4}} \nu^{\frac{1}{4}} = \nu^{\frac{3}{4}} \).

The size of the non-zero mode of the initial perturbations \((P_{\neq} u_{in}, P_{\neq} \theta_{in})\) seems optimal. See Section 2.6 for a formal discussion of the sharpness of the size. We also note that the size of the zero mode of the initial perturbations \((P_\| u_{in}, P_\| \theta_{in})\) can be slightly larger.

1.4. Notations and conventions. By convention, we always use Greek letters such as \( \eta \) and \( \xi \) to denote frequencies in the \( y \) direction and lowercase Latin characters commonly used as indices such as \( k \) and \( l \) to denote frequencies in the \( x \) or \( z \) direction (which are discrete).

For a Schwartz function \( f = f(z, y) : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R} \), we define the Fourier transform as

\[ \hat{f}(k, \eta) = \frac{1}{2\pi} \int_{\mathbb{T} \times \mathbb{R}} f(z, y) e^{-ikz - i\eta y} dz dy, \]
and the Fourier inversion of $\hat{f}(k, \eta)$ is
\[
\mathcal{F}^{-1} f(z, y) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \hat{f}(k, \eta) e^{i k z + i \gamma \eta} d\eta.
\]

With these definitions, we have $\hat{R}$ and the Fourier inversion of $\hat{4}$

Change of coordinates.

2.1. Main idea and the proof of Theorem 1.1

In this section, we give the main idea and sketch of the proof of Theorem 1.1.

For any function $f$ defined on $\mathbb{R}$, we denote its Sobolev norm by
\[
\|f\|_{H^s} = \|\langle D_y \rangle^s f\|_{L^2} = \|\langle \eta \rangle^s \hat{f}\|_{L^2}.
\]

For any function $f$ defined on $\mathbb{T} \times \mathbb{R}$, we denote its Sobolev norm by
\[
\|f\|_{H^s} = \|\langle \nabla \rangle^s f\|_{L^2} = \left( \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \langle k, \eta \rangle^{2s} |\hat{f}(k, \eta)|^2 d\eta \right)^{\frac{1}{2}}.
\]

We denote the projection to the zero mode by
\[
P_0 f(y) = f_0(y) = \frac{1}{2\pi} \int_{\mathbb{T}} f(z, y) dz,
\]
and denote the projection to the non-zero mode by
\[
P_{\neq} f(z, y) = f_{\neq}(z, y) = f(z, y) - P_0 f(y).
\]

For a statement $Q$, $\mathbbm{1}_Q$ will denote the function that equals 1 if $Q$ is true and 0 otherwise.

2. Main idea and the proof of Theorem 1.1

In this section, we give the main idea and sketch of the proof of Theorem 1.1.

2.1. Change of coordinates. First, we introduce the change of coordinates:
\[
\begin{align*}
z &= x - ty, & f(t, z, y) &= w(t, x, y), & \theta(t, z, y) &= \vartheta(t, x, y), \\
\bar{u}(t, z, y) &= u(t, x, y), & \phi(t, z, y) &= \psi(t, x, y).
\end{align*}
\]

Under the change of coordinates, we deduce the following nonlinear equations from (1.4),
\[
\begin{align*}
\partial_t f + \bar{u} \cdot \nabla_L f - \nu \Delta_L f &= -\gamma^2 \partial_x \theta, \\
\partial_t \theta + \bar{u} \cdot \nabla_L \theta - \nu \Delta_L \theta &= \bar{u}^2 = -\partial_x \Delta_L^{-1} f, \\
\bar{u} &= \nabla_L^2 \phi, & \Delta_L \phi &= f.
\end{align*}
\]

We divide the solution $(f, \theta)$ into the zero mode and non-zero modes. Let $f = f_0 + f_{\neq}$, whereas $f_0$ satisfies
\[
\begin{align*}
\begin{cases}
\partial_t f_0 - \nu \partial_{yy} f_0 = -\partial_y (\bar{u}^2_{\neq} f_0) a, \\
f_0|_{t=0} = P_0 f_{in},
\end{cases}
\end{align*}
\]
Let $\theta = \theta_0 + \theta_2 + \theta_\neq$, whereas $\theta_0$ satisfies
\begin{equation}
\begin{cases}
\partial_t \theta_0 - \nu \partial_{yy} \theta_0 = 0, \\
\theta_0|_{t=0} = P_0 \theta_{in},
\end{cases}
\end{equation}
and $\theta_2$ satisfies
\begin{equation}
\begin{cases}
\partial_t \theta_2 - \nu \partial_{yy} \theta_2 + (\tilde{u}_\neq \cdot \nabla_L \theta_\neq) = 0, \\
\theta_2|_{t=0} = 0.
\end{cases}
\end{equation}

Remark 2.1. The main idea of the decomposition of the zero mode of $\theta$, namely $\theta_0 = \theta_0 + \theta_2$, is to divide $\theta_0$ into two parts, the first part $\theta_0$ has higher regularity with larger size, the second part $\theta_2$ has lower regularity with smaller size.

We also obtain that for $(f_\neq, \theta_\neq)$,
\begin{equation}
\begin{cases}
\partial_t f_\neq + (\tilde{u} \cdot \nabla_L f_\neq) - \nu \Delta_L f_\neq = -\gamma^2 \partial_{zz} f_\neq, \\
\partial_t \theta_\neq + (\tilde{u} \cdot \nabla_L \theta_\neq) - \nu \Delta_L \theta_\neq = \tilde{u}_\neq^2, \\
(f_\neq, \theta_\neq)|_{t=0} = (P_\neq f_{in}, P_\neq \theta_{in}).
\end{cases}
\end{equation}

We also need to estimate the zero mode $\tilde{u}_1^0$ when estimating the nonlinear terms of the non-zero modes system. From (1.3), it is easy to get that $\tilde{u}_1^0$ satisfies
\begin{equation}
\begin{cases}
\partial_t \tilde{u}_1^0 - \nu \partial_{yy} \tilde{u}_1^0 = -\partial_y (\tilde{u}_\neq^2 \tilde{u}_\neq^1), \\
\tilde{u}_1^0|_{t=0} = P_0 \tilde{u}_{in}.
\end{cases}
\end{equation}

By using the standard energy estimate, it is easy to get that from (2.4),

Lemma 2.2. It holds that
\[ \frac{d}{dt} \| \theta_0 \|^2_{H^{s+2}} + 2\nu \| \partial_y \theta_0 \|^2_{H^{s+2}} = 0. \]

2.2. Linearized system and good unknowns. Before beginning the proof of Theorem 1.1, we first discuss the corresponding linearized system:
\begin{equation}
\begin{cases}
\partial_t \omega + y \partial_x \omega - \nu \Delta \omega = -\gamma^2 \partial_x \vartheta, \\
\partial_t \vartheta + y \partial_x \vartheta - \nu \Delta \vartheta = \vartheta^2, \\
u = (\vartheta^1, \vartheta^2) = (-\partial_y \psi, \partial_x \psi), \quad \Delta \psi = \omega.
\end{cases}
\end{equation}

Under the new coordinates (2.1), we get that $(f, \theta)$ satisfies
\begin{equation}
\begin{cases}
\partial_t f - \nu \Delta_L f = -\gamma^2 \partial_{zz} \theta, \\
\partial_t \theta - \nu \Delta_L \theta = \tilde{u}^2, \\
\tilde{u} = \nabla_L \phi, \quad \Delta_L \phi = f,
\end{cases}
\end{equation}

where $\nabla_L = \left( \frac{\partial_z}{\partial_y - t \partial_z} \right)$ and $\Delta_L = \partial_z^2 + (\partial_y - t \partial_z)^2$.

For (2.9), by taking Fourier transform in $z$ and $y$, we obtain
\begin{equation}
\begin{cases}
\partial_t \tilde{f} + \nu (k^2 + (\eta - k t)^2) \tilde{f} = -\gamma^2 ik \tilde{\theta}, \\
\partial_t \tilde{\theta} + \nu (k^2 + (\eta - k t)^2) \tilde{\theta} = -\frac{ik}{k^2 + (\eta - k t)^2} \tilde{f}, \\
\tilde{u} = \left( \begin{array}{c}
-i(\eta - k t) \tilde{\phi} \\
\frac{ik}{\tilde{\phi}}
\end{array} \right), \quad -(k^2 + (\eta - k t)^2) \tilde{\phi} = \tilde{f}.
\end{cases}
\end{equation}

For the sake of presentation, we obtain that by denoting $\tilde{\Theta} = ik \tilde{\phi}$,
\begin{equation}
\begin{cases}
\partial_t \tilde{f} + \nu (k^2 + (\eta - k t)^2) \tilde{f} = -\gamma^2 \tilde{\Theta}, \\
\partial_t \tilde{\Theta} + \nu (k^2 + (\eta - k t)^2) \tilde{\Theta} = \frac{k^2}{(k^2 + (\eta - k t)^2)} \tilde{f}.
\end{cases}
\end{equation}
For our purposes, it is more convenient to recall the energy method used in [9], originally introduced to deal with the linear stability of the Couette flow in a compressible fluid [11, 39]. Indeed, we have that for $k \neq 0$ and $t \geq \frac{n}{k}$,

$$
\frac{d}{dt} \left( (1 + (t - \frac{n}{k})^2)^{-\frac{1}{2}} \hat{f}^2 + (1 + (t - \frac{n}{k})^2)^{\frac{1}{2}} \gamma^2 |\Theta|^2 + \hat{f} \Theta \right) \\
+ 2\nu (k^2 + (\eta - kt)^2) \left( (1 + (t - \frac{n}{k})^2)^{-\frac{1}{2}} \hat{f}^2 + (1 + (t - \frac{n}{k})^2)^{\frac{1}{2}} \gamma^2 |\Theta|^2 + \hat{f} \Theta \right)
\lesssim (1 + (t - \frac{n}{k})^2)^{-\frac{3}{2}} \left( (1 + (t - \frac{n}{k})^2)^{-\frac{1}{2}} \hat{f}^2 + (1 + (t - \frac{n}{k})^2)^{\frac{1}{2}} \gamma^2 |\Theta|^2 \right),
$$

and for $t \leq \frac{n}{k}$,

$$
\frac{d}{dt} \left( (1 + (t - \frac{n}{k})^2)^{-\frac{1}{2}} \hat{f}^2 + (1 + (t - \frac{n}{k})^2)^{\frac{1}{2}} \gamma^2 |\Theta|^2 - \hat{f} \Theta \right) \\
+ 2\nu (k^2 + (\eta - kt)^2) \left( (1 + (t - \frac{n}{k})^2)^{-\frac{1}{2}} \hat{f}^2 + (1 + (t - \frac{n}{k})^2)^{\frac{1}{2}} \gamma^2 |\Theta|^2 - \hat{f} \Theta \right)
\lesssim (1 + (t - \frac{n}{k})^2)^{-\frac{3}{2}} \left( (1 + (t - \frac{n}{k})^2)^{-\frac{1}{2}} \hat{f}^2 + (1 + (t - \frac{n}{k})^2)^{\frac{1}{2}} \gamma^2 |\Theta|^2 \right),
$$

which implies that

$$
|\hat{f}(t, k, \eta)| \lesssim \frac{|k^2 + (kt - \eta)^2|^\frac{1}{2}}{|k^2 + \eta^2|^\frac{1}{4}} |\hat{f}_{in}(k, \eta)| e^{-c\nu k^2 t^3},
$$

(2.12)

$$
|\hat{\theta}(t, k, \eta)| \lesssim \frac{|k^2 + \eta^2|^\frac{1}{2}}{|k^2 + (kt - \eta)^2|^\frac{1}{4}} |\hat{\theta}_{in}(k, \eta)| e^{-c\nu k^2 t^3}.
$$

To attack the nonlinear problem, the linear estimate (2.12) is not enough, thus we introduce the good unknowns $(X_1, X_2)$ which enjoy a better system. The idea of finding the good unknowns is to symmetrize the system (2.8) via time-dependent Fourier multipliers. First, let us introduce two time-dependent Fourier multipliers $N = \hat{N}(\nabla L)$ and $\hat{N} = \hat{N}(\nabla L)$, which are defined by

$$
N(\nabla L) f_{\neq} \overset{\text{def}}{=} |\partial_z|^{-\frac{1}{2}} (-\Delta_L)^{\frac{1}{2}} f_{\neq} = \mathcal{F}^{-1} \left( |k|^{-\frac{1}{2}} (k^2 + (\eta - kt)^2)^{\frac{1}{4}} \hat{f}_{\neq}(k, \eta) \right),
$$

and

$$
\hat{N}(\nabla L) f_{\neq} \overset{\text{def}}{=} \frac{1}{2} (\partial_y - t \partial_x) \partial_z |\partial_z|^{-\frac{1}{2}} (-\Delta_L)^{-\frac{3}{2}} f_{\neq} \\
= \mathcal{F}^{-1} \left( \frac{1}{2} (\eta - kt) k |k|^{-\frac{1}{2}} (k^2 + (\eta - kt)^2)^{-\frac{1}{4}} \hat{f}_{\neq}(k, \eta) \right).
$$

Next, let us introduce the good unknowns

$$
X_1 \overset{\text{def}}{=} N^{-1} f_{\neq}, \quad X_2 \overset{\text{def}}{=} (1 - \frac{1}{4\gamma^2})^{-\frac{1}{2}} \left( \gamma^{-1} \hat{N} f_{\neq} + \gamma \hat{N} \partial_x \Theta_{\neq} \right),
$$

and it is easy to obtain that

$$
(2.13) \quad f_{\neq} = N X_1, \quad \Theta_{\neq} = \gamma^{-1} (\partial_x)^{-1} \left( 1 - \frac{1}{4\gamma^2} \right)^{\frac{1}{2}} N^{-1} X_2 - \frac{1}{2} \gamma^{-1} \hat{N} X_1.
$$

Thus, the nonlinear system (2.6) can be rewritten as follows:

$$
\begin{align*}
\partial_t X_1 - \nu \Delta_L X_1 &= -\frac{\sigma}{2} (\hat{N}^{-1})^2 X_2 - N^{-1} (\hat{u} \cdot \nabla L f)_{\neq}, \\
\partial_t X_2 - \nu \Delta_L X_2 &= \frac{\sigma}{2} (\hat{N}^{-1})^2 X_1 + \frac{3}{2} \gamma^{-1} \partial_x (\hat{N} \partial_x \Theta)_{\neq} - \frac{1}{2} \gamma^{-1} \hat{N} (\hat{u} \cdot \nabla L f)_{\neq}, \\
-2\gamma^{-1} \sigma^{-1} \partial_z \hat{N} (\hat{u} \cdot \nabla L \Theta)_{\neq} &= -\sigma^{-1} \hat{N} (\hat{u} \cdot \nabla L f)_{\neq}.
\end{align*}
\quad (2.14)
$$
Lemma 2.3. Here and as follows, we will denote $\sigma = \sqrt{4\gamma^2 - 1}$ for brevity.

In the following, we focus on the equations (2.3), (2.5), (2.7) and (2.14).

2.3. Main multipliers. We use some key time-dependent Fourier multipliers $\mathcal{N}$, $\hat{\mathcal{N}}$, $\mathcal{A}$ and $m$, whereas $\mathcal{N}$ and $\hat{\mathcal{N}}$ are used to capture the linear growth/decay of the nonzero modes of $(f, \theta)$. To control the growth from the nonlinear interactions, we introduce $\mathcal{A}$ and $m$.

2.3.1. Multipliers $\mathcal{A}$. We use the Fourier multiplier operator $\mathcal{A}$ to control the growth of nonzero modes, which is defined as:

$$\mathcal{A}f(t,z,y) = \mathcal{F}^{-1}\left( \mathcal{A}^\ast_k(t,\eta) \hat{f}(t,k,\eta) \right),$$

with

$$\mathcal{A}^\ast_k(t,\eta) = e^{\nu \varphi \frac{t}{4}} \mathcal{M}(t,k,\eta)(k,\eta)^s.$$ Here, the main multiplier $\mathcal{M}$ is defined by

$$\mathcal{M} \overset{\text{def}}{=} \exp\{K(\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3)\},$$

where $K$ is a large constant, and $\mathcal{M}_i$, $i = 1, 2, 3$ are defined by

$$\partial_t \mathcal{M}_1 = -\nu \frac{3}{2} \frac{\varphi'(\nu \frac{1}{2} \frac{1}{2} \text{sgn}(k)(\eta - kt))}{|k|^\frac{3}{2}},$$

$$\partial_t \mathcal{M}_2 = -\frac{k^2}{k^2 + (\eta - kt)^2},$$

$$\partial_t \mathcal{M}_3 = -\frac{1}{(k^2 + (\eta - kt)^2)^\frac{1}{4}}.$$ with the initial data $\mathcal{M}_i|_{t=0} = 0$ for $i = 1, 2, 3$, and here $\varphi \in C^\infty(\mathbb{R})$ is a real-valued, nondecreasing function such that $0 \leq \varphi \leq 1$, $\varphi' \geq 0$ and $\varphi' = \frac{1}{4}$ on $[-1, 1]$. Thus, we can easily get

$$0 < c_0(K) \leq \mathcal{M} \leq 1.$$ As a consequence,

$$\mathcal{A}^\ast_k(t,\eta) \approx e^{\nu \varphi \frac{t}{4}} (k,\eta)^s.$$ The operator $\mathcal{M}_1$ is used to obtain the enhanced dissipation. Indeed, we have the following lemma:

Lemma 2.3. It holds for all $t, \eta$ and $k \neq 0$ that

$$\frac{1}{4} \nu (k^2 + (\eta - kt)^2) + \nu \frac{1}{2} |k|^\frac{3}{2} \varphi' \left( \nu \frac{1}{2} |k|^{-\frac{3}{2}} \text{sgn}(k)(\eta - kt) \right) \geq \frac{1}{4} \nu \frac{1}{2} |k|^\frac{3}{2}.$$ Proof. For the case $|\eta - kt| \leq \nu^{-\frac{1}{2}} |k|^\frac{3}{4}$, by the definition of $\varphi$, we have

$$\nu \frac{1}{2} |k|^\frac{3}{2} \varphi' \left( \nu \frac{1}{2} |k|^{-\frac{3}{2}} \text{sgn}(k)(\eta - kt) \right) \geq \frac{1}{4} \nu \frac{1}{2} |k|^\frac{3}{2}.$$ For the case $|\eta - kt| \geq \nu^{-\frac{1}{2}} |k|^\frac{3}{4}$, it holds that $\frac{1}{4} \nu (k^2 + (\eta - kt)^2) \geq \frac{1}{4} \nu \frac{1}{2} |k|^\frac{3}{2}$. Thus we proved the lemma.

The operator $\mathcal{M}_2$ is to obtain the inviscid damping. For $|k|$ small, the operator $\mathcal{M}_3$ is slightly stronger than $\mathcal{M}_2$ in terms of the time decay. We shall also use the following commutator estimates.
Lemma 2.4. There holds that for $k \neq 0$,
\[
\mathcal{M}(t, k, \eta)\langle k, \eta \rangle^s - \mathcal{M}(t, k, \xi)\langle k, \xi \rangle^s \lesssim |\eta - \xi|(\nu^\frac{1}{s}|k|^{-\frac{1}{s}} + |k|^{-1})\langle k, \xi \rangle^s + |\eta - \xi|\langle k, \eta - \xi \rangle^{s-1}.
\]
Proof. By the definition of $\mathcal{M}$, we have
\[
\mathcal{M}(t, k, \eta) - \mathcal{M}(t, k, \xi)
= \left[ (e^{K\mathcal{M}_1})(t, k, \eta) - (e^{K\mathcal{M}_1})(t, k, \xi) \right] (e^{K\mathcal{M}_2+K\mathcal{M}_3})(t, k, \eta)
+ (e^{K\mathcal{M}_1})(t, k, \xi)(e^{K\mathcal{M}_1})(t, k, \eta) \left[ (e^{K\mathcal{M}_2})(t, k, \eta) - (e^{K\mathcal{M}_2})(t, k, \xi) \right]
+ (e^{K\mathcal{M}_1})(t, k, \xi)(e^{K\mathcal{M}_2})(t, k, \xi) \left[ (e^{K\mathcal{M}_3})(t, k, \eta) - (e^{K\mathcal{M}_3})(t, k, \xi) \right].
\]
(2.17)
According to the definition of $\mathcal{M}_1$, we have
\[
\partial_\xi \mathcal{M}_1(t, k, \xi) = \nu^{\frac{1}{s}}|k|^{-\frac{1}{s}}\text{sgn}(k)\varphi'(\nu^{\frac{1}{s}}|k|^{-\frac{1}{s}}\text{sgn}(k)(\xi - kt)),
\]
and then by (2.15), we obtain
\[
|\langle e^{K\mathcal{M}_1}\rangle(t, k, \eta) - (e^{K\mathcal{M}_1})(t, k, \xi)|
= |(\eta - \xi) \int_0^1 (e^{K\mathcal{M}_1})(t, k, \xi)K\partial_\xi \mathcal{M}_1(t, k, \xi)d\xi|
= |K(\eta - \xi) \int_0^1 (e^{K\mathcal{M}_1})(t, k, \xi)\nu^{\frac{1}{s}}|k|^{-\frac{1}{s}}\text{sgn}(k)\varphi'(\nu^{\frac{1}{s}}|k|^{-\frac{1}{s}}\text{sgn}(k)(\xi - kt))d\xi|
\lesssim |\eta - \xi|\nu^{\frac{1}{s}}|k|^{-\frac{1}{s}}.
\]
(2.18)
By the definition of $\mathcal{M}_2$, we have
\[
\mathcal{M}_2(t, k, \eta) - \mathcal{M}_2(t, k, \xi) = - \int_0^t \frac{k^2}{k^2 + (\eta - k\tau)^2}d\tau + \int_0^t \frac{k^2}{k^2 + (\xi - k\tau)^2}d\tau
- \int_{\frac{2}{\tau}}^{\frac{\xi}{2} - t} \frac{1}{1 + z^2}dz - \int_{\frac{2}{\tau}}^{\frac{\xi}{2} - t} \frac{1}{1 + z^2}dz
= \int_{\frac{2}{\tau}}^{\frac{\xi}{2} - t} \frac{1}{1 + z^2}dz - \int_{\frac{2}{\tau}}^{\frac{\xi}{2} - t} \frac{1}{1 + z^2}dz,
\]
and then we get
\[
|\mathcal{M}_2(t, k, \eta) - \mathcal{M}_2(t, k, \xi)| \lesssim \frac{|\eta - \xi|}{|k|}.
\]
Thus, we obtain that by (2.15),
(2.19)
\[
|\langle e^{K\mathcal{M}_2}\rangle(t, k, \eta) - (e^{K\mathcal{M}_2})(t, k, \xi)| \lesssim |\mathcal{M}_2(t, k, \eta) - \mathcal{M}_2(t, k, \xi)| \lesssim \frac{|\eta - \xi|}{|k|}.
\]
By the similar argument, we get that
\[
|\mathcal{M}_3(t, k, \eta) - \mathcal{M}_3(t, k, \xi)| \lesssim \frac{|\eta - \xi|}{|k|^\frac{2}{s}},
\]
and
(2.20)
\[
|\langle e^{K\mathcal{M}_3}\rangle(t, k, \eta) - (e^{K\mathcal{M}_3})(t, k, \xi)| \lesssim \frac{|\eta - \xi|}{|k|^\frac{2}{s}}.
\]
Then, from (2.17), we obtain that by combining (2.15), (2.18), (2.19) and (2.20),
\[ |\mathcal{M}(t, k, \eta) - \mathcal{M}(t, k, \xi)| \lesssim |\eta - \xi| \left( \nu^{\frac{1}{3}} |k|^{-\frac{1}{3}} + |k|^{-1} \right). \]

Thus, for \( k \neq 0 \), we obtain
\[
\mathcal{M}(t, k, \eta)^s - \mathcal{M}(t, k, \xi)^s
= \left[ \mathcal{M}(t, k, \eta) - \mathcal{M}(t, k, \xi) \right] \langle k, \xi \rangle^s + \mathcal{M}(t, k, \eta) \left[ \langle k, \eta \rangle^s - \langle k, \xi \rangle^s \right]
\lesssim |\eta - \xi| \left( \nu^{\frac{1}{3}} |k|^{-\frac{1}{3}} + |k|^{-1} \right) \langle k, \xi \rangle^s + |\eta - \xi| \left[ \langle k, \eta \rangle^{s-1} + \langle k, \xi \rangle^{s-1} \right]
\lesssim |\eta - \xi| \left( \nu^{\frac{1}{3}} |k|^{-\frac{1}{3}} + |k|^{-1} \right) \langle k, \xi \rangle^s + |\eta - \xi| \langle k, \eta - \xi \rangle^{s-1}.
\]

We complete the proof of the lemma. \( \square \)

2.3.2. Multipliers \( m \). We use the multiplier operator \( m \) to control the nonlinear growth of zero-mode \( \theta_{02} \), see Section 5.3 for more details.

For \( |\eta| \geq 3 \), let
\[
t(\eta) \overset{\text{def}}{=} \frac{2|\eta|}{2E(\sqrt{|\eta|}) + 1},
\]
where \( E(\sqrt{|\eta|}) \) is the integer part of \( \sqrt{|\eta|} \). For \( |k| = 1, 2, 3, \ldots, E(\sqrt{|\eta|}) \) and \( k\eta > 0 \), we denote the resonance interval by
\[
I_{k, \eta} = \left[ \frac{2\eta}{2k + 1}, \frac{2\eta}{2k - 1} \right].
\]
Then \( t(\eta) \approx \sqrt{|\eta|} \) and
\[
[t(\eta), 2|\eta|] = \bigcup_{k=1}^{E(\sqrt{|\eta|})} I_{k, \eta}.
\]

For \( |\eta| < 3 \), we define \( m(t, \eta) \equiv 1 \).

For \( |\eta| \geq 3 \), if \( t \geq 2|\eta| \), we define \( m(t, \eta) \equiv 1 \);

if \( t \in \left[ \frac{2\eta}{3|\eta|}, 2|\eta| \right] = I_{1, \eta} \), we define \( m(t, \eta) \) by solving
\[
\begin{cases}
\partial_t m = -\frac{m}{(1+(t-\eta)^2)^{\frac{5}{4}}}, \\
m(2\eta, \eta) = 1;
\end{cases}
\]
if \( t \in I_{k, \eta} \) with \( k = 2, 3, \ldots, E(\sqrt{|\eta|}) \), we define \( m(t, \eta) \) by solving
\[
\begin{cases}
\partial_t m = -\frac{m}{k^2(1+(t-\eta)^2)^{\frac{5}{4}}}, \\
m\big|_{t=2\eta/k} = m\left( \frac{2\eta}{2k-1}, \eta \right);
\end{cases}
\]
if \( t \leq t(\eta) \), let \( m(t, \eta) = m\left( \frac{2\eta}{2E(\sqrt{|\eta|})+1}, \eta \right) \).

**Lemma 2.5.** It holds that
\[
m(t, \eta) \approx 1.
\]
**Proof.** For any \( t, \eta \), it is easy to see that \( m(t, \eta) \leq 1 \) and
\[
\frac{1}{m\left( \frac{2\eta}{2E(\sqrt{|\eta|})+1}, \eta \right)} = \prod_{k=1}^{E(\sqrt{|\eta|})} \exp \left\{ \int_{2\eta/k}^{2\eta} \frac{1}{k^2} \left( 1 + \frac{\eta}{k} \right)^2 \right\} \frac{3}{4} dt \}.
\]
Thus we have proved the lemma. □

For any function \( f(t,y) \) defined on \( \mathbb{R}^+ \times \mathbb{R} \), we define

\[
\frac{1}{m} f = m(t, \partial_y)^{-1} f = \mathcal{F}^{-1}\left( \hat{f}(t, \eta) \frac{m}{m(\hat{t}, \eta)} \right),
\]

and

\[
\sqrt{\frac{\partial_t m}{m}} f = \sqrt{\frac{\partial_t m(t, \eta)}{m(t, \eta)}} f = \mathcal{F}^{-1}\left( \sqrt{\frac{\hat{\partial_t m}(\eta)}{m(\hat{t}, \eta)}} \hat{f}(t, \eta) \right).
\]

### 2.4. Main energy estimates.

We define

\[
E_{\#} \overset{\text{def}}{=} \frac{1}{2} \left( \|A X_1\|_{L^2}^2 + \|A X_2\|_{L^2}^2 \right),
\]

\[
F_0 \overset{\text{def}}{=} \|f_0\|_{H^s}, \quad H_{02} \overset{\text{def}}{=} \|m^{-1} \theta_{02}\|_{H^s}, \quad V_0 \overset{\text{def}}{=} \|\tilde{u}_0\|_{H^s}.
\]

From the time evolution of \( E_{\#} \), we get

\[
\frac{1}{2} \frac{d}{dt} \left( \|A X_1\|_{L^2}^2 + \|A X_2\|_{L^2}^2 \right) + \nu \left( \|\nabla L A X_1\|_{L^2}^2 + \|\nabla L A X_2\|_{L^2}^2 \right) + \mathcal{C}\mathcal{R}(t)
\]

\[
= c v^2 \left( \|A X_1\|_{L^2}^2 + \|A X_2\|_{L^2}^2 \right) + \frac{3}{2\sigma} \int A \partial_t^2 (-\Delta L)^{-1} N^{-2} X_1 A X_2 dz dy
\]

\[
- \int A N^{-1}(\tilde{u} \cdot \nabla L f)_{\#} A X_1 dz dy
\]

\[
+ \int A \left( -2\gamma^2 \sigma^{-1} \partial_{\#} N(\tilde{u} \cdot \nabla L \theta)_{\#} - \sigma^{-1} N(\tilde{\theta} \cdot \nabla L f)_{\#} \right) A X_2 dz dy
\]

\[
\overset{\text{def}}{=} c v^2 \left( \|A X_1\|_{L^2}^2 + \|A X_2\|_{L^2}^2 \right) + \frac{1}{2}\int A N^{-1} X_2 A X_1 dz dy + \frac{\sigma}{2} \int A N^{-1} X_1 A X_2 dz dy = 0.
\]

Here in (2.23), \( L \) is the linear term, \( NL_1, NL_2 \) are the nonlinear terms and \( \mathcal{C}\mathcal{R} \) stands for "Cauchy-Kovalevskaya",

\[
\mathcal{C}\mathcal{R}(t) \overset{\text{def}}{=} K \sum_{k \neq 0} \int -\partial_t M_1(t, k, \eta) \left( |A_k^1(t, \eta) \tilde{X}_1(t, k, \eta)|^2 + |A_k^2(t, \eta) \tilde{X}_2(t, k, \eta)|^2 \right) d\eta
\]

\[
+ K \sum_{k \neq 0} \int -\partial_t M_2(t, k, \eta) \left( |A_k^1(t, \eta) \tilde{X}_1(t, k, \eta)|^2 + |A_k^2(t, \eta) \tilde{X}_2(t, k, \eta)|^2 \right) d\eta
\]

\[
+ K \sum_{k \neq 0} \int -\partial_t M_3(t, k, \eta) \left( |A_k^2(t, \eta) \tilde{X}_1(t, k, \eta)|^2 + |A_k^2(t, \eta) \tilde{X}_2(t, k, \eta)|^2 \right) d\eta
\]

\[
\overset{\text{def}}{=} \sum_{j=1}^3 \mathcal{C}\mathcal{R}_j(t).
\]
By Lemma 2.3, we get
\[
\frac{1}{4} \nu \left( \| \nabla L A X_1 \|_{L^2}^2 + \| \nabla L A X_2 \|_{L^2}^2 \right) + \mathcal{C} \mathcal{R}_1 \geq \frac{1}{4} \nu \frac{1}{3} \left( \| D_z \|_{L^2}^2 \right)
\]
where \( \mathcal{E} \mathcal{D} \) stands for the enhanced dissipation.

On the other hand, by the definition of the multipliers \( M_2 \) and \( M_3 \), we have
\[
\mathcal{C} \mathcal{R}_2 = K \left( \| \partial_x (\Delta_L)^{-\frac{1}{2}} A X_1 \|_{L^2}^2 + \| \partial_x (\Delta_L)^{-\frac{1}{2}} A X_2 \|_{L^2}^2 \right),
\]
\[
\mathcal{C} \mathcal{R}_3 = K \left( \| (\Delta_L)^{-\frac{1}{2}} A X_1 \|_{L^2}^2 + \| (\Delta_L)^{-\frac{1}{2}} A X_2 \|_{L^2}^2 \right).
\]

For the linear term \( L \), by taking \( K \geq 3 \sigma^{-1} \), we get that
\[
\frac{3}{2 \sigma} \int A (\partial_x^2 (\Delta_L)^{-\frac{1}{2}} N^{-2} X_1) A X_2 dx dy
\]
\[
\leq \frac{3}{2 \sigma} \| \partial_x (\Delta_L)^{-\frac{1}{2}} A X_1 \|_{L^2} \| \partial_x (\Delta_L)^{-\frac{1}{2}} A X_2 \|_{L^2}
\]
\[
\leq \frac{1}{4} K \| \partial_x (\Delta_L)^{-\frac{1}{2}} A X_1 \|_{L^2}^2 + \frac{1}{4} K \| \partial_x (\Delta_L)^{-\frac{1}{2}} A X_2 \|_{L^2}^2.
\]

Thus, by choosing \( 0 < c \leq \frac{1}{8} \) and denoting \( \mathcal{D} \) be the diffusion:
\[
\mathcal{D} \equiv \frac{3}{4} \nu \left( \| \nabla L A X_1 \|_{L^2}^2 + \| \nabla L A X_2 \|_{L^2}^2 \right),
\]
we obtain that from (2.23),
\[
\frac{d}{dt} E_{\neq} + \mathcal{D} + \mathcal{E} \mathcal{D} + \mathcal{C} \mathcal{R}_2 + \mathcal{C} \mathcal{R}_3
\]
\[
\leq \left| \int A N^{-1} (\hat{u} \cdot \nabla f)_{\neq} A X_1 dx dy \right|
\]
\[
+ \left| \int A (2 \gamma^2 \sigma^{-1} \partial_x N (\hat{u} \cdot \nabla f)_{\neq} - \sigma^{-1} \hat{N} (\hat{u} \cdot \nabla f)_{\neq}) A X_2 dx dy \right|
\]
\[
= |NL_1| + |NL_2|.
\]

We define the following control as the bootstrap hypotheses for \( t \geq 0 \).
\[
E_{\neq}(t) + \frac{1}{2} \int_0^t \left( \mathcal{D} + \mathcal{E} \mathcal{D} + \mathcal{C} \mathcal{R}_2 + \mathcal{C} \mathcal{R}_3 \right) (t') dt' \leq (4 \nu \frac{1}{2})^2,
\]
\[
\| f_0 \|_{H^s}^2 + \nu \int_0^t \| \partial_y f_0(t') \|_{H^s}^2 dt' \leq (4 \nu \frac{1}{2})^2,
\]
\[
\| \hat{u}_0 \|_{H^s}^2 + \nu \int_0^t \| \partial_y \hat{u}_0(t') \|_{H^s}^2 dt' \leq (4 \nu \frac{1}{2})^2,
\]
\[
\| m^{-1} \theta_0 \|_{H^s}^2 + \nu \int_0^t \| \partial_y \left( m^{-1} \theta_0(t) \right) \|_{H^s}^2 dt' + \int_0^t \left\| \sqrt{\frac{\partial m}{m}} m^{-1} \theta_0(t') \right\|_{H^s}^2 dt' \leq (4 \nu \frac{3}{2})^2.
\]

Let \( I^* \) be the connected set of time \( t \geq 0 \) such that the bootstrap hypotheses (2.26) are all satisfied. We will work on the regularized solutions for which we know \( E_{\neq}, F_0, H_0, V_0 \) take values continuously in time, and hence \( I^* \) is a closed interval \( [0, T^*] \) with \( T^* \geq 0 \). The
bootstrapping is complete if we show that $I^*$ is also open, which is the purpose of the following proposition, the proof of which constitutes the majority of this work.

**Proposition 2.6.** Let $s \geq 6$ and $\gamma^2 > \frac{1}{4}$. There exist $0 < \epsilon_0 = \epsilon_0(s, \gamma^2 - \frac{1}{4}), \nu_0 = \nu_0(s, \gamma^2 - \frac{1}{4}) < 1$, such that for all $0 < \nu \leq \nu_0$ and $0 < \epsilon \leq \epsilon_0$, such that if the bootstrapping hypotheses \((2.26) - (2.29)\) hold, then for any $t \in [0, T^*)$, we have the following properties $t \in [0, T^*)$:

\[
E_\epsilon(t) + \frac{1}{2} \int_0^t (\mathcal{D} + \mathcal{C} \mathcal{D} + \mathcal{C} \mathcal{R}_2 + \mathcal{C} \mathcal{R}_3)(t') dt' \leq (2\epsilon \nu^2)^2,
\]

\[
\|f_0\|^2_{H^s} + \nu \int_0^t \|\partial_y f_0(t')\|^2_{H^s} dt' \leq (2\epsilon \nu^2)^2,
\]

\[
\|\tilde{u}_0\|^2_{H^s} + \nu \int_0^t \|\partial_y \tilde{u}_0(t')\|^2_{H^s} dt' \leq (2\epsilon \nu^2)^2,
\]

\[
\left\| m^{-1} \theta_0 \right\|^2_{H^s} + \nu \int_0^t \left\| \partial_y \left( m^{-1} \theta_0 \right)(t') \right\|^2_{H^s} dt' + \int_0^t \left\| \sqrt{\frac{m}{m}} m^{-1} \theta_0(t') \right\|^2_{H^s} dt' \leq (2\epsilon \nu^2)^2,
\]

from which it follows that $T^* = +\infty$.

The remainder of the paper is devoted to the proof of Proposition 2.6. One of the key estimates is to control the nonlinear terms $NL_1$ and $NL_2$. By using the fact that

\[
\tilde{u} \cdot \nabla f = \tilde{u} \partial_z f + \tilde{u} \partial_y f + \tilde{u}_0 \partial_z f + \tilde{u}_0 \partial_y f,
\]

and

\[
\tilde{u} \cdot \nabla \theta = \tilde{u} \partial_z \theta + \tilde{u} \partial_y (\theta_0 + \theta_0) + \tilde{u} \partial_z \theta + \tilde{u} \partial_y (\theta_0 + \theta_0)
\]

we have

\[
NL_1 = - \int AN^{-1} (\tilde{u} \partial_z f) AX_1 dz dy - \int AN^{-1} (\tilde{u} \partial_y f) AX_1 dz dy
\]

\[
- \int \left( AN^{-1} (\tilde{u} \partial_z f) AX_1 dz dy - \int AN^{-1} (\tilde{u} \partial_y f) AX_1 dz dy \right)
\]

\[
(2.34) \quad \overset{\text{def}}{=} I_1 + I_2 + I_3 + I_4,
\]

and

\[
NL_2 = \int \left( - 2\gamma^2 \sigma^2 \partial_z N(\tilde{u} \partial_z \theta) - \sigma^2 \tilde{N}(\tilde{u} \partial_z f) \right) AX_2 dz dy
\]

\[
- \frac{2\gamma^2}{\sigma} \int A \partial_z N(\tilde{u} \partial_y (\theta_0 + \theta_0)) AX_2 dz dy - \sigma^2 \int \tilde{N}(\tilde{u} \partial_y f) AX_2 dz dy
\]

\[
- \frac{2\gamma^2}{\sigma} \int A \partial_z N(\tilde{u} \partial_z f) AX_2 dz dy - \frac{2\gamma^2}{\sigma} \int A \partial_z N(\tilde{u} \partial_y f) AX_2 dz dy
\]

\[
- \sigma^2 \left( \int \tilde{N}(\tilde{u} \partial_z f) AX_2 dz dy - \sigma^2 \int \tilde{N}(\tilde{u} \partial_y f) AX_2 dz dy \right)
\]

\[
(2.35) \quad \overset{\text{def}}{=} \sum_{i=1}^7 J_i.
\]
In Section 3, we will give the estimates of the interactions between the zero mode and the nonzero modes: $I_1, I_2, J_1, J_2, J_3$. In Section 4, we will give the estimates of the interactions between non-zero modes: $I_3, I_4, J_4, J_5, J_6, J_7$. In precise, we mainly prove the following lemmas.

**Lemma 2.7.** Under the bootstrap hypotheses, for $t \in [0, T^*]$, it holds that
\begin{align}
|I_1| & \leq C\epsilon \hat{v} \frac{1}{m^2} (D \Phi) \frac{2}{m^2} (C \Phi_2) \frac{1}{m^2} + C\epsilon \nu \Phi,

|I_2| & \leq C\epsilon \hat{v} \frac{1}{m^2} (C \Phi_2) \frac{1}{m^2} + C\epsilon \nu \frac{1}{m^2} (C \Phi_2) \frac{1}{m^2} \|\partial_y (D_g)^s f_0\|_{L^2}^\frac{1}{m^2},

|I_3| + |I_4| & \leq C\epsilon \Phi + C\epsilon \Phi_2.
\end{align}

**Lemma 2.8.** Under the bootstrap hypotheses, for $t \in [0, T^*]$, it holds that
\begin{align}
|J_1| & \leq C\epsilon \hat{v} \frac{1}{m^2} (C \Phi) \frac{2}{m^2} (C \Phi_2) \frac{1}{m^2} + C\epsilon \nu \frac{1}{m^2} (C \Phi_2) \frac{1}{m^2} + C\epsilon \nu \frac{1}{m^2} (C \Phi),

|J_2| & \leq C\epsilon \hat{v} \frac{1}{m^2} (C \Phi_2) \frac{1}{m^2} + C\epsilon \nu \frac{1}{m^2} (C \Phi_2) \frac{1}{m^2} \|\partial_y \theta_{02}\|_{H^s},

|J_3| & \leq C\epsilon \hat{v} \frac{1}{m^2} (C \Phi_2) + C\epsilon \nu \frac{1}{m^2} (C \Phi_2) \frac{1}{m^2} \|\partial_y (D_g)^s f_0\|_{L^2}^\frac{1}{m^2},

|J_4| + |J_5| + |J_6| + |J_7| & \leq C\epsilon \Phi + C\epsilon \Phi_2.
\end{align}

In Section 5, we will prove the following proposition.

**Proposition 2.9.** Under the bootstrap hypotheses, for $t \in [0, T^*]$, there holds that
\begin{align}
\|f_0\|^2_{H^s} + \nu \int_0^t \|\partial_y f_0(t')\|^2_{H^s} dt' & \leq \|P_0 f_{in}\|^2_{H^s} + C\epsilon \nu \frac{1}{m^2},

\|\tilde{u}_0\|^2_{H^s} + \nu \int_0^t \|\partial_y \tilde{u}_0(t')\|^2_{H^s} dt' & \leq \|P_0 \tilde{u}_{in}\|^2_{H^s} + C\epsilon \nu \frac{1}{m^2},

\|m^{-1} \theta_{02}\|^2_{H^s} + \nu \int_0^t \|\partial_y (m^{-1} \theta_{02})(t')\|^2_{H^s} dt' + \int_0^t \left\|\sqrt{\frac{\partial_m}{m}} m^{-1} \theta_{02}(t')\right\|^2_{H^s} dt' & \leq C\epsilon \nu \frac{1}{m^2}.
\end{align}

2.5. **Proof of Theorem 1.1.** In this subsection, we first admit Lemma 2.7 Lemma 2.8 and Proposition 2.9 and prove Proposition 2.6.

**Proof.** From Proposition 2.9, the estimates (2.31), (2.32) and (2.33) can be obtained directly by choosing the initial data satisfying
\[\|P_0 f_{in}\|_{H^s} \leq \epsilon \nu \frac{1}{m^2}, \quad \|P_0 \tilde{u}_{in}\|_{H^s} \leq \epsilon \nu \frac{1}{m^2},\]
and taking $\epsilon$ small enough.

By combining Lemma 2.7 Lemma 2.8 (2.31), (2.33) with (2.25), we have
\[\frac{d}{dt} E_x + D + C \Phi + C \Phi_2 + C \Phi_3 \leq C \epsilon \Phi + C \epsilon \Phi_2 + C \epsilon \Phi_3 \]
\[+ C \epsilon \nu \frac{3}{m^2} (C \Phi_2) \frac{3}{m^2} \|\partial_y (D_g)^s f_0\|_{L^2}^\frac{1}{m^2}, \]
\[+ C \epsilon \nu \frac{3}{m^2} (C \Phi_2) \frac{3}{m^2} \|\partial_y (D_g)^s f_0\|_{L^2}^\frac{1}{m^2}.\]
Then by taking $\epsilon$ small enough, we can directly obtain (2.30).
Thus, we deduced the proof of Proposition 2.6.

Finally, we conclude the proof of Theorem 1.1.

**Proof.** The estimate (1.7) can be directly obtained by Proposition 2.6. By using (2.30) and the definition (2.13), we easily have (1.8) and (1.9).

By using the fact that

\[ \tilde{u}^1_x = (\partial_y - i\partial_z)(-\Delta_L)^{-1} f_x, \quad \tilde{u}^2_x = \partial_x(-\Delta_L)^{-1} f_x, \quad \text{and} \quad f_x = |\partial_x|^{-1/2}(-\Delta_L)^{1/2} X_1, \]

we get that from (2.30),

\[
\| P_x u^1 \|_{L^2} \lesssim \| \partial_x|^{-1/2}(-\Delta_L)^{-1/2} X_1 \|_{L^2} \leq C\nu^{1/2}(t)^{-1/2} e^{-c\nu^{1/2} t},
\]

\[
\| P_x u^2 \|_{L^2} = \| \partial_x|^{-1/2}(-\Delta_L)^{-1/2} X_1 \|_{L^2} \leq C\nu^{1/2}(t)^{-1/2} e^{-c\nu^{1/2} t},
\]

\[
\| P_x f \|_{L^2} \lesssim \| \partial_x|^{-1/2}(-\Delta_L)^{1/2} X_1 \|_{L^2} \leq C\nu^{1/2}(t)^{1/2} e^{-c\nu^{1/2} t},
\]

\[
\| P_x \theta \|_{L^2} \lesssim \| \partial_x|^{-1}(-\Delta_L)^{1/2} X_1 \|_{L^2} + \| \partial_x|^{-1}(-\Delta_L)^{1/2} X_1 \|_{L^2} \leq C\nu^{3/2}(t)^{-1/2} e^{-c\nu^{1/2} t}.
\]

Thus, we proved (1.6).

Thus, we complete the proof of Theorem 1.1.

2.6. Discussion of the optimality of the size. In this section, we will show some evidence of the sharpness of the size $\nu^2$. For clarity, we assume that the size of $X_i$, $i = 1, 2$ is $\nu^2$ and the size of $\theta_{02}$ is $\nu^3$. Let us rewrite the nonlinear system as follows:

\[
\begin{cases}
\partial_t X_2 - \nu \Delta L X_2 = -2\gamma^2 \nu^{-1} \partial_x N(\tilde{u}_x^2 \partial_y \theta_{02}) + \text{good terms}, \\
\partial_t \theta_{02} - \nu \partial_{yy} \theta_{02} = -(\tilde{u}_x^2 (\partial_y - i\partial_z) \theta_x^2) + \text{good terms}.
\end{cases}
\]

For the term $\partial_x N(\tilde{u}_x^2 \partial_y \theta_{02})$, let us focus on the low-high interaction, namely,

\[
\mathcal{F}(\partial_x N(\tilde{u}_x^2 \partial_y \theta_{02})) = \mathcal{F}(\partial_x N((\tilde{u}_x^2)_{\text{low}}(\partial_y \theta_{02}))_{\text{high}}) + \text{good terms}
\]

\[
= \sum_{k \neq 0} \int_{|\eta| > 2|kt|, |\xi - \eta| \leq 1/|\xi|} -k|k|^{-1/2} (k^2 + (\eta - kt)^2)^{1/2} \tilde{u}_k(t, \eta - \xi) \hat{\theta}_{02}(t, \xi) d\xi + \text{good terms}.
\]

Since $\tilde{u}_x^2$ is the lower frequency, we regard it as $\nu^2(t)^{-1/2}$. Thus formally we have

\[
\left| \sum_{k \neq 0} \int_{|\eta| > 2|kt|, |\xi - \eta| \leq 1/|\xi|} -k|k|^{-1/2} (k^2 + (\eta - kt)^2)^{1/2} \tilde{u}_k(t, \eta - \xi) \hat{\theta}_{02}(t, \xi) d\xi \right|
\]

\[
\lesssim \sum_{k \neq 0} \int_{|\eta| > 2|kt|, |\xi - \eta| \leq 1/|\xi|} \nu^2(t)^{-1/2} |\xi|^{1/2} |\hat{\theta}_{02}(t, \xi)| d\xi
\]

and we write the toy model for $X_2$ as follows:

\[
(2.46) \quad \partial_t X_2 - \nu \partial_{yy} X_2 = C\nu^2(t)^{-1/2} |\partial_y|^{1/2} \theta_{02},
\]
where we also formally regard $\Delta_L$ as $\partial_{yy}$.

For the term $(\tilde{u}^2_y(\partial_y - t\partial_z)\theta_{\neq})_0$, we also focus on the low-high interaction, namely,

$$F(\tilde{u}^2_y(\partial_y - t\partial_z)\theta_{\neq})_0 = F\left(\left(\tilde{u}^2_y(\partial_y - t\partial_z)\theta_{\neq}\right)_{\text{low}}\right) + \text{good terms}$$

$$= \sum_{k\neq 0} \int_{|x| > 2|k|, |x-y| \leq \frac{1}{2}|x|} \tilde{u}^2(t, -k, \eta - \xi) i(\xi - kt) \hat{\theta}(t, k, \xi)d\xi$$

$$+ \text{good terms}.$$  

By (2.13), we formally regard $X_1 \sim X_2$, then

$$|F(\tilde{u}^2_y(\partial_y - t\partial_z)\theta_{\neq})_0| \lesssim \sum_{k\neq 0} \int_{|x| > 2|k|, |x-y| \leq \frac{1}{2}|x|} \nu^{\alpha}(t)^{-\frac{2}{3}}|\xi|^{rac{1}{2}}|\hat{X}_2(t, k, \xi)|d\xi,$$

and we have the toy model for $\theta_{0,2}$:

$$\partial_t \theta_{02} - \nu \partial_{yy} \theta_{02} = C\nu^{\alpha}(t)^{-\frac{1}{3}}|\partial_y|^\frac{1}{3}X_2.$$  

To close the energy estimate of the toy model (2.46) and (2.47) in Sobolev space, we have to use the diffusion term and pay the smallness to control the derivative loss. Due to the diffusion effect, formally we regard $\partial_y \sim \nu^{-\frac{1}{4}}$. Then we have the following model:

$$\begin{aligned}
\partial_t X_2 &- \nu \partial_{yy} X_2 = \epsilon \nu^{\alpha-\frac{1}{3}}(t)^{-\frac{3}{2}} \theta_{02}, \\
\partial_t \theta_{02} &- \nu \partial_{yy} \theta_{02} = \epsilon \nu^{\alpha-\frac{1}{4}}(t)^{-\frac{3}{2}} X_2.
\end{aligned}$$

Thus, we have

$$\left\|X_2(t)\right\|^2_{H^s} + \int_0^t \left\|\partial_y X_2(\tau)\right\|^2_{H^s} d\tau \leq \left\|X_2(0)\right\|^2_{H^s} + \int_0^t \epsilon \nu^{\alpha-\frac{1}{3}}(\tau)^{-\frac{3}{2}} d\tau \sup_{\tau \in [0,t]} \left\|X_2(\tau)\right\|_{H^s} \left\|\theta_{02}(\tau)\right\|_{H^s}$$

$$\leq \left\|X_2(0)\right\|^2_{H^s} + \epsilon \nu^{\alpha-\frac{3}{2}} \sup_{\tau \in [0,t]} \left\|X_2(\tau)\right\|_{H^s} \left\|\theta_{02}(\tau)\right\|_{H^s},$$

and

$$\left\|\theta_{02}(t)\right\|^2_{H^s} + \int_0^t \left\|\partial_y \theta_{02}(\tau)\right\|^2_{H^s} d\tau \leq \int_0^t \epsilon \nu^{\alpha-\frac{1}{4}}(\tau)^{-\frac{3}{2}} d\tau \sup_{\tau \in [0,t]} \left\|X_2(\tau)\right\|_{H^s} \left\|\theta_{02}(\tau)\right\|_{H^s}$$

$$\leq \epsilon \nu^{\alpha-\frac{1}{4}} \sup_{\tau \in [0,t]} \left\|X_2(\tau)\right\|_{H^s} \left\|\theta_{02}(\tau)\right\|_{H^s}.$$  

Thus it requires

$$\epsilon \nu^{\alpha-\frac{3}{4}} \nu^\alpha \nu^\beta \leq \nu^{2\alpha}, \quad \epsilon \nu^{\alpha-\frac{1}{4}} \nu^\alpha \nu^\beta \leq \nu^{2\beta},$$

which is $\alpha \geq \frac{1}{4}$ and $\beta \geq \frac{3}{4}$. See the estimate of $J_{22}$ in Section 3.4 and the estimate of $K_{121}$ in Section 5.3 for the rigorous proof.

3. THE INTERACTIONS BETWEEN THE ZERO MODE AND THE NON-ZERO MODE

In this section, we study the nonlinear interaction between the zero mode and the non-zero mode. We will prove (2.36) and (2.37) in Lemma 2.7 and (2.39), (2.40) and (2.41) in Lemma 2.8.
3.1. **Treatment of $I_1$.** For $I_1$, by Young’s inequality and the bootstrap hypotheses, we have

$$I_1 = - \int [A(N^{-1}(\tilde{w}_0^1 \partial_z f_\neq) - A(\tilde{w}_0^1 \partial_z X_1))]AX_1 dz dy$$

$$- \int [A(\tilde{w}_0^1 \partial_z X_1) - \tilde{w}_0^1 \partial_z AX_1]AX_1 dz dy$$

$$\overset{\text{def}}{=} I_1^{\text{com1}} + I_1^{\text{com2}},$$

and here, we used that by integration parts,

$$\int (\tilde{w}_0^1 \partial_z AX_1)AX_1 dz dy = 0.$$

For $I_1^{\text{com1}}$, by using $f_\neq = N X_1$ and the fact that

$$\frac{1}{(k^2 + (\eta - kt)^2)^\frac{3}{2}} - \frac{1}{(k^2 + (\xi - kt)^2)^\frac{3}{2}} \lesssim (k^2 + (\eta - kt)^2)^{-\frac{1}{2}} (k^2 + (\xi - kt)^2)^{-\frac{1}{2}} \frac{|\eta - \xi|}{(k^2 + (\xi - kt)^2)^{\frac{1}{2}} + (k^2 + (\eta - kt)^2)^{\frac{1}{2}}}$$

we obtain

$$|I_1^{\text{com1}}| \lesssim \sum_{k \neq 0} \int \mathcal{A}^*_{k}(t, \eta) \left| \frac{1}{(k^2 + (\eta - kt)^2)^\frac{3}{2}} - \frac{1}{(k^2 + (\xi - kt)^2)^\frac{3}{2}} \right| |\tilde{u}_0^1(t, \eta - \xi)|$$

$$\times |k| (k^2 + (\xi - kt)^2)^{-\frac{1}{2}} |\tilde{X}_1(t, k, \xi)| \mathcal{A}^*_{k}(t, \eta)|X_1(t, k, \eta)| d\xi d\eta$$

$$\lesssim \sum_{k \neq 0} \int \mathcal{A}^*_{k}(t, \eta) (k^2 + (\eta - kt)^2)^{-\frac{1}{2}} \frac{|\eta - \xi| |\tilde{u}_0^1(t, \eta - \xi)|}{(k^2 + (\xi - kt)^2)^{\frac{1}{2}} + (k^2 + (\eta - kt)^2)^{\frac{1}{2}}}$$

$$\times |k| |\tilde{X}_1(t, k, \xi)| \mathcal{A}^*_{k}(t, \eta)|X_1(t, k, \eta)| d\xi d\eta.$$

By (2.16), Young’s inequality and the bootstrap hypotheses, we have

$$|I_1^{\text{com1}}| \lesssim \sum_{k \neq 0} \int \mathcal{A}^*_{k}(t, \xi)|\eta - \xi||\tilde{u}_0^1(t, \eta - \xi)||\tilde{X}_1(t, k, \xi)||k|(k^2 + (\eta - kt)^2)^{-\frac{1}{2}}$$

$$\times \mathcal{A}^*_{k}(t, \eta)|X_1(t, k, \eta)| d\xi d\eta$$

$$\lesssim \|f_0\|_{H^s} \|AX_1\|_{L^2} \|\partial_2 (-\Delta L)^{-\frac{1}{2}} AX_1\|_{L^2} \leq C \epsilon \nu^\frac{1}{2} (\mathcal{E} \mathcal{D})^{\frac{3}{2}} (\mathcal{E} \mathcal{R}_2)^{\frac{1}{2}}.$$

For $I_1^{\text{com2}}$, we get that by Lemma (2.4)

$$|I_1^{\text{com2}}| \lesssim \sum_{k \neq 0} \int \mathcal{A}^*_{k}(t, \eta) - \mathcal{A}^*_{k}(t, \xi) |\tilde{u}_0^1(t, \eta - \xi)||\tilde{X}_1(t, k, \xi)||k|(k^2 + (\eta - kt)^2)^{-\frac{1}{2}}$$

$$\times |\tilde{X}_1(t, k, \eta)| d\xi d\eta$$

$$\lesssim \sum_{k \neq 0} \int |\eta - \xi||k|^\frac{3}{2} |\tilde{u}_0^1(t, \eta - \xi)||\mathcal{A}^*_{k}(t, \xi)||\tilde{X}_1(t, k, \xi)||\mathcal{A}^*_{k}(t, \eta)||\tilde{X}_1(t, k, \eta)| d\xi d\eta$$

$$+ \sum_{k \neq 0} \int |\eta - \xi||\tilde{u}_0^1(t, \eta - \xi)||\mathcal{A}^*_{k}(t, \xi)||\tilde{X}_1(t, k, \xi)||\mathcal{A}^*_{k}(t, \eta)||\tilde{X}_1(t, k, \eta)| d\xi d\eta$$

$$+ \sum_{k \neq 0, |k| \leq |\eta - \xi|} \int |\eta - \xi||\tilde{u}_0^1(t, \eta - \xi)| \epsilon \nu^\frac{1}{2} |k||\tilde{X}_1(t, k, \xi)||\mathcal{A}^*_{k}(t, \eta)||\tilde{X}_1(t, k, \eta)| d\xi d\eta.$$
\[ + \sum_{k \neq 0, |k| > |\eta - \xi|} \int_{\xi, \eta} |\eta - \xi|^{|k| \frac{1}{2}} \langle x \rangle |\hat{u}_0(t, \eta - \xi)| \langle k, \xi \rangle \langle x \rangle \langle x \rangle |A_k(t, \eta)| |\hat{X}_1(t, k, \eta)| |\hat{X}_1(t, k, \eta)| d\xi d\eta \]
\[ \lesssim \nu^{\frac{1}{2}} \| \hat{u}_0 \|_{H^3}' \| \Delta x \|_{L^2} + \| \hat{u}_0 \|_{H^3} ' \| A X \|_{L^2} + \| \hat{u}_0 \|_{H^3} e^{c_\nu^\frac{1}{2} t} X \|_{H^3} \| A X \|_{L^2} \]
\[ \leq C e^{c_\nu^\frac{1}{2} t} \mathcal{D} + C e^{c_\nu^\frac{1}{2} t} \mathcal{D} \leq C e^{c_\nu^\frac{1}{2} t} \mathcal{D}. \]

3.2. Treatment of $I_2$. By using the definition (2.13), we have
\[ |I_2| \lesssim \sum_{k \neq 0} \int_{\xi, \eta} A_k(t, \eta) |k| |k^2 + (\eta - kt)^2|^{-\frac{1}{2}} |\hat{X}_1(t, k, \eta)| d\xi d\eta \]
\[ \lesssim \sum_{k \neq 0} \int_{\xi, \eta, |\xi| \leq |\eta - \xi|} A_k(t, \eta) |k| |k^2 + (\eta - kt)^2|^{-\frac{1}{2}} |\hat{X}_1(t, k, \eta)| d\xi d\eta \]
\[ + \sum_{k \neq 0} \int_{\xi, \eta, |\xi| > |\eta - \xi|} A_k(t, \eta) |k| |k^2 + (\eta - kt)^2|^{-\frac{1}{2}} |\hat{X}_1(t, k, \eta)| d\xi d\eta \]
\[ \overset{\text{def}}{=} I_{21} + I_{22}. \]

For $I_{21}$, by using the fact that $\langle k, \eta \rangle \lesssim \langle k, \eta - \xi \rangle$ and for $k \neq 0$, \[ \frac{k^2 + (\eta - kt)^2}{k^2 + (\eta - \xi - kt)^2} \lesssim 1 + \xi^2, \]
we obtain that by Young's inequality and the bootstrap hypotheses
\[ |I_{21}| \lesssim \sum_{k \neq 0} \int_{|\xi| \leq |\eta - \xi|} \left( \frac{k^2 + (\eta - kt)^2}{k^2 + (\eta - \xi - kt)^2} \right)^{\frac{1}{2}} A_k(t, \eta) |k| |k^2 + (\eta - \xi - kt)^2|^{-\frac{1}{2}} |\hat{X}_1(t, k, \eta)| |\hat{X}_1(t, k, \eta)| d\xi d\eta \]
\[ \lesssim \sum_{k \neq 0} \int_{|\xi| \leq |\eta - \xi|} A_k(t, \eta) |k| |k^2 + (\eta - \xi - kt)^2|^{-\frac{1}{2}} |\hat{X}_1(t, k, \eta)| \| f_0 \|_{H^3} \]
\[ \lesssim \| \partial_x (-\Delta_L)^{\frac{1}{2}} A X \|_{L^2} \| f_0 \|_{H^3} \leq C e^{c_\nu^\frac{1}{2} t} \mathcal{R}_2. \]

For $I_{22}$, we have
\[ |I_{22}| \lesssim \sum_{k \neq 0, |k| > |\eta - \xi|} \int_{|\eta - \xi| < |\xi|} A_k(t, \eta) |k| |k^2 + (\eta - \xi - kt)^2|^{-\frac{1}{2}} |\hat{X}_1(t, k, \eta)| \]
\[ \times |\langle \xi \rangle^2 \hat{f}_0(t, \xi)| |k| |k^2 + (\eta - kt)^2|^{-\frac{1}{2}} A_k(t, \eta) |\hat{X}_1(t, k, \eta)| d\xi d\eta \]
\[ + \sum_{k \neq 0, |k| \leq |\xi|} \int_{|\eta - \xi| < |\xi|} A_k(t, \eta) |k| |k^2 + (\eta - \xi - kt)^2|^{-\frac{1}{2}} |\hat{X}_1(t, k, \eta)| \]
\[ \times |\langle \xi \rangle^2 \hat{f}_0(t, \xi)| |k| |k^2 + (\eta - kt)^2|^{-\frac{1}{2}} A_k(t, \eta) |\hat{X}_1(t, k, \eta)| d\xi d\eta \]
\[ + \sum_{k \neq 0, |k| \leq |\xi|} \int_{|\eta - \xi| < |\xi|} A_k(t, \eta) |k| |k^2 + (\eta - \xi - kt)^2|^{-\frac{1}{2}} |\hat{X}_1(t, k, \eta)| \]
\[ \times |\langle \xi \rangle^2 \hat{f}_0(t, \xi)| |k| |k^2 + (\eta - kt)^2|^{-\frac{1}{2}} A_k(t, \eta) |\hat{X}_1(t, k, \eta)| d\xi d\eta \]
\[ \lesssim \| \partial_x (-\Delta_L)^{\frac{1}{2}} A X \|_{L^2} \| f_0 \|_{H^3} \leq C e^{c_\nu^\frac{1}{2} t} \mathcal{R}_2. \]
\[
\times |(i\xi)\hat{f}_0(t, \xi)|(k^2 + (\eta - kt)^2)^{-\frac{1}{2}}A_k^s(t, \eta)|\hat{X}_1(t, k, \eta)|d\xi d\eta \\
= I_{221} + I_{222}.
\]

For \(I_{221}\), by \(|k| > |\eta - \xi|\), we have \(\langle k, \eta \rangle^s \lesssim \langle k, \eta - \xi \rangle^s + \langle k, \xi \rangle^s \lesssim \langle k \rangle^s\), and
\[
(1 + (t - \frac{\eta - \xi}{k})^2)^{-\frac{3}{4}} \lesssim (t)^{-\frac{3}{2}}.
\]

Then, we get that the bootstrap hypotheses,
\[
|I_{221}| \lesssim \sum_{k \neq 0, |k| > |\xi|} \int_{|\eta - \xi| < |\xi|} e^{cv^\frac{1}{2}t}\langle k \rangle^s|k|(k^2 + (\eta - \xi - kt)^2)^{-\frac{3}{4}}|\hat{X}_1(t, k, \eta - \xi)|
\times |(i\xi)\hat{f}_0(t, \xi)|(k^2 + (\eta - kt)^2)^{-\frac{1}{2}}A_k^s(t, \eta)|\hat{X}_1(t, k, \eta)|d\xi d\eta
\lesssim \sum_{k \neq 0, |k| > |\xi|} \int_{|\eta - \xi| < |\xi|} e^{cv^\frac{1}{2}t}\langle k \rangle^{s-1}(1 + (t - \frac{\eta - \xi}{k})^2)^{-\frac{3}{4}}|\hat{X}_1(t, k, \eta - \xi)|
\times |(i\xi)\hat{f}_0(t, \xi)|A_k^s(t, \eta)|\hat{X}_1(t, k, \eta)|d\xi d\eta
\lesssim \langle t \rangle^{-\frac{3}{2}}\|f_0\|_{H^s}\|AX_1\|_{L^2} \leq Ce^{v^\frac{3}{2}t}(t)^{-\frac{3}{2}}.
\]

For \(I_{222}\), by using \(|\xi|^s \lesssim |\eta - \xi - k\xi|^{\frac{s}{2}} + |\eta - k\xi|^{\frac{s}{2}}, \langle k, \eta \rangle^s \lesssim \langle k, \xi \rangle^s \lesssim \langle \xi \rangle^s\), and
\[
\|(-\Delta_L)^{-\frac{3}{4}}X_1\|_{H^s} \lesssim \langle t \rangle^{-\frac{3}{2}}\|X_1\|_{H^s}.
\]

we obtain
\[
|I_{222}| \lesssim \sum_{k \neq 0, |k| \leq |\xi|} \int_{|\eta - \xi| < |\xi|} A_k^s(t, \eta)|k|(k^2 + (\eta - \xi - kt)^2)^{-\frac{3}{4}}|\hat{X}_1(t, k, \eta - \xi)|
\times |\xi||\hat{f}_0(t, \xi)|(k^2 + (\eta - kt)^2)^{-\frac{1}{2}}A_k^s(t, \eta)|\hat{X}_1(t, k, \eta)|d\xi d\eta
\leq \sum_{k \neq 0, |k| \leq |\xi|} \int_{|\eta - \xi| < |\xi|} |k|(k^2 + (\eta - \xi - kt)^2)^{-\frac{3}{4}e^{cv^\frac{1}{2}t}}|\hat{X}_1(t, k, \eta - \xi)|
\times \langle \xi \rangle^s|\xi|^\frac{1}{2}|\hat{f}_0(t, \xi)|(k^2 + (\eta - kt)^2)^{-\frac{1}{2}}A_k^s(t, \eta)|\hat{X}_1(t, k, \eta)|d\xi d\eta
+ \sum_{k \neq 0, |k| \leq |\xi|} \int_{|\eta - \xi| < |\xi|} |k|(k^2 + (\eta - \xi - kt)^2)^{-\frac{3}{4}e^{cv^\frac{1}{2}t}}|\hat{X}_1(t, k, \eta - \xi)|
\times \langle \xi \rangle^s|\xi|^\frac{1}{2}|\hat{f}_0(t, \xi)|A_k^s(t, \eta)|\hat{X}_1(t, k, \eta)|d\xi d\eta.
\]

Then, by Young’s inequality, the interpolation inequality and the bootstrap hypotheses, we get
\[
|I_{222}| \lesssim \|\partial_x(-\Delta_L)^{-\frac{3}{4}}e^{cv^\frac{1}{2}t}X_1\|_{H^2}\|\partial_y^2(D_y)^s f_0\|_{L^2}\|\langle -\Delta_L \rangle^{-\frac{3}{4}}AX_1\|_{L^2}
+ \|\langle -\Delta_L \rangle^{-\frac{3}{4}}e^{cv^\frac{1}{2}t}X_1\|_{H^3}\|\partial_y^2(D_y)^s f_0\|_{L^2}\|AX_1\|_{L^2}
\lesssim \|\partial_x(-\Delta_L)^{-\frac{3}{4}}AX_1\|_{L^2}\|\partial_y(D_y)^s f_0\|_{L^2}\|\langle -\Delta_L \rangle^{-\frac{3}{4}}AX_1\|_{L^2}\|\partial_y(D_y)^s f_0\|_{L^2}
\lesssim \|\partial_x(-\Delta_L)^{-\frac{3}{4}}AX_1\|_{L^2}\|\partial_y(D_y)^s f_0\|_{L^2}\|\langle -\Delta_L \rangle^{-\frac{3}{4}}AX_1\|_{L^2}\|\partial_y(D_y)^s f_0\|_{L^2}
\lesssim \|\langle -\Delta_L \rangle^{-\frac{3}{4}}AX_1\|_{L^2}\|\partial_y(D_y)^s f_0\|_{L^2}\|\langle -\Delta_L \rangle^{-\frac{3}{4}}AX_1\|_{L^2}\|\partial_y(D_y)^s f_0\|_{L^2}
\lesssim \|\partial_x(-\Delta_L)^{-\frac{3}{4}}AX_1\|_{L^2}\|\partial_y(D_y)^s f_0\|_{L^2}\|\langle -\Delta_L \rangle^{-\frac{3}{4}}AX_1\|_{L^2}\|\partial_y(D_y)^s f_0\|_{L^2}
\lesssim C e^{v^\frac{3}{2}t}(C R_2)^{\frac{3}{2}}\|\partial_y(D_y)^s f_0\|_{L^2}^\frac{3}{2} + C e^{v^\frac{3}{2}t}(t)^{-\frac{3}{2}}\|\partial_y(D_y)^s f_0\|_{L^2}^\frac{3}{2}.
\]
Thus, combining (3.22), (3.33) with (3.3), we obtain that
\[
|J_2| \leq C e^{\frac{k}{2} t} (t)^{-2} + C e^{\frac{k}{2} (t)^{-2} \| \psi(D_g)^s f_0 \|_{L^2}^2} + C e^{\frac{2}{2} (t)^{-2} \| \psi(D_g)^s f_0 \|_{L^2}^2}.
\]

3.3. Treatment of J1. For J1, we can rewrite it as:
\[
J_1 = -2\gamma^2 \sigma^{-1} \left[ AN(\tilde{u}_0^1 \partial_\phi \theta - A(\tilde{u}_0^1 N \partial_\phi \theta \phi)) \right] AX_2 dxdy
- \sigma^{-1} \left[ AN(\tilde{u}_0^1 \partial_z f_\phi - A(\tilde{u}_0^1 N \partial_z f_\phi)) \right] AX_2 dxdy
+ \left[ A(\tilde{u}_0^1 \partial_\phi X_2 - \tilde{u}_0^1 \partial_z AX_2) \right] AX_2 dxdy + \left[ \tilde{u}_0^1 \partial_z AX_2 AX_2 dxdy
= -2\gamma^2 \sigma^{-1} \left[ AN(\tilde{u}_0^1 \partial_\phi \theta - A(\tilde{u}_0^1 N \partial_\phi \theta \phi)) \right] AX_2 dxdy
- \sigma^{-1} \left[ AN(\tilde{u}_0^1 \partial_z f_\phi - A(\tilde{u}_0^1 N \partial_z f_\phi)) \right] AX_2 dxdy
+ \left[ A(\tilde{u}_0^1 \partial_\phi X_2 - \tilde{u}_0^1 \partial_z AX_2) \right] AX_2 dxdy
\]
def \[ J_1^{com1} + J_1^{com2} + J_1^{com3}, \]
whereas in the last second equality, we get that by integration by parts,
\[
\int \tilde{u}_0^1 \partial_z AX_2 AX_2 dxdy = 0.
\]

By using the definition (2.13), we have
\[
|\tilde{\partial}_z \phi(t, k, \eta)| \lesssim |k|^{\frac{1}{2}} (k^2 + (\eta - kt)^2)^{-\frac{1}{4}} |X_2(t, k, \eta)|
+ |\eta - kt||k|^{\frac{1}{2}} (k^2 + (\eta - kt)^2)^{-\frac{1}{4}} |X_2(t, k, \eta)|
= \frac{|\eta - \xi|}{(k^2 + (\eta - kt)^2)^{\frac{1}{4}} + (k^2 + (\xi - kt)^2)^{\frac{1}{4}}},
\]
and \(\langle k, \eta \rangle \lesssim \langle k, \eta - \xi \rangle + \langle k, \xi \rangle\), we get that
\[
|J_1^{com1}| \lesssim \sum_{k \neq 0} \int_{\xi, \eta} A_1^k(t, \eta)|k|^{-\frac{1}{4}} (k^2 + (\eta - kt)^2)^{\frac{1}{4}} - (k^2 + (\xi - kt)^2)^{\frac{1}{4}} |\tilde{u}_0^1(t, \eta - \xi)|
\times |k|^{\frac{3}{2}} (k^2 + (\xi - kt)^2)^{-\frac{1}{4}} (|X_1| + |X_2|)(t, k, \xi) A_3^k(t, \eta)|X_2(t, k, \eta)|d\xi d\eta
\lesssim \sum_{k \neq 0} \int_{\xi, \eta} A_1^k(t, \eta)|\eta - \xi||\tilde{u}_0^1(t, \eta - \xi)||k|(k^2 + (\xi - kt)^2)^{-\frac{1}{4}} (|X_1| + |X_2|)(t, k, \xi)
\times A_3^k(t, \eta)|X_2(t, k, \eta)|d\xi d\eta
\lesssim \sum_{k \neq 0} \int_{\xi, \eta} A_1^k(t, \eta - \xi)|\eta - \xi||\tilde{u}_0^1(t, \eta - \xi)||k|(k^2 + (\xi - kt)^2)^{-\frac{1}{4}} (|X_1| + |X_2|)(t, k, \xi)
\times A_3^k(t, \eta)|X_2(t, k, \eta)|d\xi d\eta
Thus, we obtain that
\[ |J_{11}^{\text{com1}}| \lesssim \sum_{k \neq 0, |k| \leq |\eta - \xi|} \int_{\xi, \eta} \langle \eta - \xi \rangle^{s} |\eta - \xi| |\hat{u}_{0}(t, \eta - \xi)| e^{\nu_{1}t} |k| (k^{2} + (\xi - kt)^{2})^{-\frac{1}{4}} \left( |\hat{X}_{1}| + |\hat{X}_{2}| \right) (t, k, \xi) \]
\[ \times A_{k}^{\epsilon} (t, \eta) |\hat{X}_{2} (t, k, \eta)| d\xi d\eta \]
\[ \overset{\text{def}}{=} J_{11}^{\text{com1}} + J_{12}^{\text{com1}}. \]

For \( J_{11}^{\text{com1}} \), we get that by Young’s inequality and the bootstrap hypotheses,
\[
|J_{11}^{\text{com1}}| \lesssim \sum_{k \neq 0, |k| \leq |\eta - \xi|} \int_{\xi, \eta} \langle \eta - \xi \rangle^{s} |\eta - \xi| |\hat{u}_{0}(t, \eta - \xi)| e^{\nu_{1}t} |k| (k^{2} + (\xi - kt)^{2})^{-\frac{1}{4}} \left( |\hat{X}_{1}| + |\hat{X}_{2}| \right) (t, k, \xi) \\
\times (|\hat{X}_{1}| + |\hat{X}_{2}|) (t, k, \xi) A_{k}^{\epsilon} (t, \eta) |\hat{X}_{2} (t, k, \eta)| d\xi d\eta \\
+ \sum_{k \neq 0, |k| > |\eta - \xi|} \int_{\xi, \eta} \langle |\eta - \xi| \hat{u}_{0}(t, \eta - \xi) |A_{k}^{\epsilon} (t, \xi) |k| (k^{2} + (\xi - kt)^{2})^{-\frac{1}{4}} \right) \\
\times (|\hat{X}_{1}| + |\hat{X}_{2}|) (t, k, \xi) A_{k}^{\epsilon} (t, \eta) |\hat{X}_{2} (t, k, \eta)| d\xi d\eta \\
\leq \| f_{0} \|_{H^{s}} \| A X_{2} \|_{L^{2}} \left( \| e^{\nu_{1}t} \partial_{\xi}(-\Delta L)^{-\frac{1}{2}} X_{1} \|_{H^{2}} + \| e^{\nu_{1}t} \partial_{x}(-\Delta L)^{-\frac{1}{2}} X_{2} \|_{H^{2}} \right) \\
+ \| \hat{u}_{0} \|_{H^{s}} \| A X_{2} \|_{L^{2}} \left( \| \partial_{x}(-\Delta L)^{-\frac{1}{2}} AX_{1} \|_{L^{2}} + \| \partial_{x}(-\Delta L)^{-\frac{1}{2}} AX_{2} \|_{L^{2}} \right) \\
\lesssim \left( \| f_{0} \|_{H^{s}} + \| \hat{u}_{0} \|_{H^{s}} \right) \| A X_{2} \|_{L^{2}} \left( \| \partial_{x}(-\Delta L)^{-\frac{1}{2}} AX_{1} \|_{L^{2}} + \| \partial_{x}(-\Delta L)^{-\frac{1}{2}} AX_{2} \|_{L^{2}} \right) \\
\leq C \nu^{\frac{1}{2}} \| (\mathcal{D}) \frac{1}{2} (C \mathcal{R}_{2}) \frac{1}{2} + C \nu^{\frac{1}{2}} \| (\mathcal{D}) \frac{1}{2} (C \mathcal{R}_{2}) \frac{1}{2} .
\]

And for \( J_{12}^{\text{com1}} \), by Young’s inequality and the bootstrap hypotheses, we have
\[
|J_{12}^{\text{com1}}| \lesssim \sum_{k \neq 0} \int_{\xi, \eta} \langle |\eta - \xi| \hat{u}_{0}(t, \eta - \xi) |A_{k}^{\epsilon} (t, \xi) |k| (k^{2} + (\xi - kt)^{2})^{-\frac{1}{4}} \right) \\
\times (|\hat{X}_{1}| + |\hat{X}_{2}|) (t, k, \xi) A_{k}^{\epsilon} (t, \eta) |\hat{X}_{2} (t, k, \eta)| d\xi d\eta \\
\leq \| \hat{u}_{0} \|_{H^{s}} \| A X_{2} \|_{L^{2}} \left( \| \partial_{x}(-\Delta L)^{-\frac{1}{2}} AX_{1} \|_{L^{2}} + \| \partial_{x}(-\Delta L)^{-\frac{1}{2}} AX_{2} \|_{L^{2}} \right) \\
\leq \| \hat{u}_{0} \|_{H^{s}} \| A X_{2} \|_{L^{2}} \left( \| \partial_{x}(-\Delta L)^{-\frac{1}{2}} AX_{1} \|_{L^{2}} + \| \partial_{x}(-\Delta L)^{-\frac{1}{2}} AX_{2} \|_{L^{2}} \right) \\
\leq C \nu^{\frac{1}{2}} \| (\mathcal{D}) \frac{1}{2} (C \mathcal{R}_{2}) \frac{1}{2} .
\]

Thus, we obtain that
\[
(3.6) \quad |J_{1}^{\text{com2}}| \leq C \nu^{\frac{1}{2}} \| (\mathcal{D}) \frac{1}{2} (C \mathcal{R}_{2}) \frac{1}{2} + C \nu^{\frac{1}{2}} \| (\mathcal{D}) \frac{1}{2} (C \mathcal{R}_{2}) \frac{1}{2} \leq C \nu^{\frac{1}{2}} \| (\mathcal{D}) \frac{1}{2} (C \mathcal{R}_{2}) \frac{1}{2} .
\]

For \( J_{1}^{\text{com2}} \), by the definition (2.13) and using the fact that
\[
(\eta - \xi)(k^{2} + (\eta - \xi)(\xi - \eta)(k^{2} + (\xi - kt)^{2})^{-\frac{1}{4}} \\
\lesssim |\eta - \xi| |(k^{2} + (\xi - kt)^{2})^{-\frac{1}{4}} + |\eta - \xi| |(k^{2} + (\eta - \xi)(k^{2} + (\xi - kt)^{2})^{-\frac{1}{4}} |,
\]
we get that by Young’s inequality and the bootstrap hypotheses,
\[
|J_{1}^{\text{com2}}| \lesssim \sum_{k \neq 0} \int_{\xi, \eta} A_{k}^{\epsilon} (t, \eta) |k| \left( \eta - \xi \right) (k^{2} + (\eta - \xi)(\xi - \eta)(k^{2} + (\xi - kt)^{2})^{-\frac{1}{4}} \\
\times |\hat{u}_{0}(t, \eta - \xi)| |k| \left( \eta - \xi \right) (k^{2} + (\xi - kt)^{2})^{-\frac{1}{4}} |\hat{X}_{1} (t, \xi) A_{k}^{\epsilon} (t, \eta) |\hat{X}_{2} (t, k, \eta)| d\xi d\eta.
\]
From (3.6), (3.7) and (3.8), we get
\[
\lesssim \sum_{k \neq 0} \int A_k^0(t, \eta) |\eta - \xi||\nabla_0(t, \eta - \xi)||k|(k^2 + (\xi - kt)^2)^{-\frac{1}{2}}|\widetilde{X}_1(t, k, \xi)|
\times A_k^0(t, \eta)|\widetilde{X}_2(t, k, \eta)|d\xi d\eta
\]
\[
+ \sum_{k \neq 0} \int A_k^0(t, \eta) |\eta - \xi||\nabla_0(t, \eta - \xi)||k||\widetilde{X}_1(t, k, \xi)|(k^2 + (\eta - kt)^2)^{-\frac{1}{2}}
\times A_k^0(t, \eta)|\widetilde{X}_2(t, k, \eta)|d\xi d\eta
\]
\[
\lesssim \|f_0\|_{H^s} \|AX_1\|_{L^2} \|\partial_z (-\Delta_L)^{-\frac{1}{2}} AX_2\|_{L^2} + \|f_0\|_{H^s} \|AX_2\|_{L^2} \|\partial_z (-\Delta_L)^{-\frac{1}{2}} AX_1\|_{L^2}
\]
\[
\lesssim C\nu^\frac{1}{2} (\mathcal{E} \mathcal{D})^\frac{1}{2} (\mathcal{E} \mathcal{R}_2)^\frac{1}{2}.
\]
For $J_1^{\text{com3}}$, by changing $X_1$ to be $X_2$ in $I_2^{\text{com2}}$, we get
\[
(3.8) \quad |J_1^{\text{com3}}| \leq C\nu^\frac{1}{2} (\mathcal{E} \mathcal{D})^\frac{1}{2}.
\]
From (3.6), (3.7) and (3.8), we get
\[
|J_1| \leq C\nu^\frac{1}{2} (\mathcal{E} \mathcal{D})^\frac{1}{2} (\mathcal{E} \mathcal{R}_2)^\frac{1}{2} + C\nu^\frac{1}{2} (\mathcal{E} \mathcal{D})^\frac{1}{2} (\mathcal{E} \mathcal{R}_2)^\frac{1}{2}.
\]

3.4. Treatment of $J_2$. For $J_2$, we have
\[
J_2 = -2\gamma^2 \sigma^{-1} \int A \partial_z N(u_\nu^2 \partial_y \theta_{01}) AX_2 dxdy - 2\gamma^2 \sigma^{-1} \int A \partial_z N(u_\nu^2 \partial_y \theta_{02}) AX_2 dxdy
\]
\[
\overset{\text{def}}{=} J_{21} + J_{22}.
\]
For $J_{21}$, by (3.1) and using the fact that
\[
(k^2 + (\eta - kt)^2)^\frac{1}{2} \lesssim (k^2 + (\eta - \xi - kt)^2)^\frac{1}{2} + |\xi|^\frac{1}{2},
\]
we get that by Young’s inequality and the bootstrap hypotheses,
\[
|J_{21}| \lesssim \sum_{k \neq 0} \int \xi, \eta A_k^0(t, \eta) |k|(k^2 + (\eta - kt)^2)^\frac{1}{2} (k^2 + (\eta - \xi - kt)^2)^{-\frac{3}{2}}|\widetilde{X}_1(t, k, \eta - \xi)|
\times |\xi \partial_{01}(t, \xi)| A_k^0(t, \eta)|\widetilde{X}_2(t, k, \eta)|d\xi d\eta
\]
\[
\lesssim \sum_{k \neq 0} \int \xi, \eta A_k^0(t, \eta) |k|(k^2 + (\eta - \xi - kt)^2)^{-\frac{1}{2}}|\widetilde{X}_1(t, k, \eta - \xi)||\xi \partial_{01}(t, \xi)|
\times A_k^0(t, \eta)|\widetilde{X}_2(t, k, \eta)|d\xi d\eta + \sum_{k \neq 0} \int \xi, \eta A_k^0(t, \eta) |k|(k^2 + (\eta - \xi - kt)^2)^{-\frac{3}{2}}
\times |\widetilde{X}_1(t, k, \eta - \xi)||\xi|^\frac{1}{2} |\xi \partial_{01}(t, \xi)| A_k^0(t, \eta)|\widetilde{X}_2(t, k, \eta)|d\xi d\eta
\]
\[
\lesssim \|\partial_z (-\Delta_L)^{-\frac{1}{2}} AX_1\|_{L^2} \|\partial_{01}\|_{H^{s+1}} \|AX_2\|_{L^2} + \|\partial_z (-\Delta_L)^{-\frac{1}{2}} AX_1\|_{L^2} \|\partial_{01}\|_{H^{s+2}} \|AX_2\|_{L^2}
\]
\[
\leq C\nu^\frac{1}{2} (\mathcal{E} \mathcal{D})^\frac{1}{2} (\mathcal{E} \mathcal{R}_2)^\frac{1}{2}.
\]
For $J_{22}$, by Young’s inequality, the interpolation inequality and the bootstrap hypotheses, we have
\[
|J_{22}| \lesssim \sum_{k \neq 0} \int \xi, \eta A_k^0(t, \eta) |k|(k^2 + (\eta - kt)^2)^\frac{1}{2} (k^2 + (\eta - \xi - kt)^2)^{-\frac{3}{2}}|\widetilde{X}_1(t, k, \eta - \xi)|
\times |\xi \partial_{02}(\xi)| A_k^0(t, \eta)|\widetilde{X}_2(t, k, \eta)|d\xi d\eta
\]
\[
\lesssim \|(-\Delta L)^{\frac{1}{4}}AX_2\|_{L^2}\|(-\Delta L)^{-\frac{3}{4}}AX_1\|_{L^2}\|\partial_y\theta_{02}\|_{H^s}
\]
\[
\lesssim \|\nabla LAX_2\|_{L^2}^\frac{1}{2}\|AX_2\|_{L^2}^\frac{1}{2}\|(-\Delta L)^{-\frac{1}{2}}AX_1\|_{L^2}\|AX_1\|_{L^2}\|\partial_y\theta_{02}\|_{H^s}
\]
\[
\leq C\epsilon\nu_2\|D^\frac{1}{2}(\mathcal{C}\mathcal{R}_2)\|_{L^2}\|\partial_y\theta_{02}\|_{H^s}.
\]

Thus, we obtain
\[
|J_2| \leq C\epsilon\nu_2\|D^\frac{1}{2}(\mathcal{C}\mathcal{R}_2)\|_{L^2} + C\epsilon\nu_2\|D^\frac{1}{2}(\mathcal{C}\mathcal{R}_2)\|_{L^2}\|\partial_y\theta_{02}\|_{H^s}.
\]

3.5. Treatment of \( J_3 \). For \( J_3 \), similarly as the estimate of \( I_2 \), we get
\[
|J_3| \leq C\epsilon\nu_2\|D\|^{\frac{1}{2}}(\mathcal{C}\mathcal{R}_2)^{\frac{1}{2}} + C\epsilon\nu_2\|D\|^{\frac{1}{2}}(\mathcal{C}\mathcal{R}_2)^{\frac{1}{2}}\|\partial_y\theta_{02}\|_{H^s}.
\]

4. THE INTERACTIONS BETWEEN NON-ZERO MODES

In this section, we study the nonlinear interactions between the non-zero modes. We mainly prove (2.38) in Lemma 2.7 and (2.32) in Lemma 2.8

4.1. Treatment of \( I_3 \). By using the fact that
\[
|\hat{u}(t, k, \eta)| \lesssim \|\eta - kt\| |k|^{-\frac{1}{2}}(k^2 + (\eta - kt)^2)^{-\frac{1}{2}}|\vec{X}_1(t, k, \eta)|
\]
(4.1)
and \( |l|^{\frac{1}{2}} \lesssim |k - l|^{\frac{1}{2}} + |k|^{\frac{1}{2}} \), we get that for \( I_3 \),
\[
|I_3| \lesssim \sum_{k \neq 0, l, \eta, \xi} A_k^* (t, \eta)|k - l|^{-\frac{1}{2}}((k - l)^2 + (\eta - \xi - (k - l)t)^2)^{-\frac{1}{2}}|\vec{X}_1(t, k - l, \eta - \xi)|
\]
\[
\times |l|^{\frac{1}{2}}(l^2 + (\xi - lt)^2)^{\frac{1}{2}}|\vec{X}_1(t, l, \xi)|A_l^* (t, \eta)|k|^{\frac{1}{2}}(k^2 + (\eta - kt)^2)^{-\frac{1}{2}}|\vec{X}_1(t, k, \eta)|d\xi d\eta
\]
\[
\lesssim \sum_{k \neq 0, l, \eta, \xi} A_k^* (t, \eta)((k - l)^2 + (\eta - \xi - (k - l)t)^2)^{-\frac{1}{2}}|\vec{X}_1(t, k - l, \eta - \xi)|
\]
\[
\times (l^2 + (\xi - lt)^2)^{\frac{1}{2}}|\vec{X}_1(t, l, \xi)|A_l^* (t, \eta)|k|^{\frac{1}{2}}(k^2 + (\eta - kt)^2)^{-\frac{1}{2}}|\vec{X}_1(t, k, \eta)|d\xi d\eta
\]
\[
+ \sum_{k \neq 0, l, \eta, \xi} A_k^* (t, \eta)|k - l|^{-\frac{1}{2}}((k - l)^2 + (\eta - \xi - (k - l)t)^2)^{-\frac{1}{2}}|\vec{X}_1(t, k - l, \eta - \xi)|
\]
\[
\times (l^2 + (\xi - lt)^2)^{\frac{1}{2}}|\vec{X}_1(t, l, \xi)|A_l^* (t, \eta)|k|^{\frac{1}{2}}(k^2 + (\eta - kt)^2)^{-\frac{1}{2}}|\vec{X}_1(t, k, \eta)|d\xi d\eta.
\]

Then by Young’s inequality, the interpolation inequality and the bootstrap hypotheses, we get that
\[
|I_3| \lesssim \|(-\Delta L)^{-\frac{1}{2}}AX_1\|_{L^2}\|(-\Delta L)^{\frac{1}{2}}AX_1\|_{L^2}\|\partial_x\frac{1}{2}(-\Delta L)^{-\frac{1}{2}}AX_1\|_{L^2}
\]
\[
+ \|\partial_x\frac{1}{2}(-\Delta L)^{-\frac{1}{2}}AX_1\|_{L^2}\|(-\Delta L)^{\frac{1}{2}}AX_1\|_{L^2}\|\partial_x(-\Delta L)^{-\frac{1}{2}}AX_1\|_{L^2}
\]
\[
\lesssim \|(-\Delta L)^{-\frac{1}{2}}AX_1\|_{L^2}\|\nabla LAX_1\|_{L^2}\|AX_1\|_{L^2}\|\partial_x(-\Delta L)^{-\frac{1}{2}}AX_1\|_{L^2}
\]
\[
+ \|\partial_x(-\Delta L)^{-\frac{1}{2}}AX_1\|_{L^2}\|AX_1\|_{L^2}\|\partial_x(-\Delta L)^{-\frac{1}{2}}AX_1\|_{L^2}
\]
\[
\lesssim \|\partial_x(-\Delta L)^{-\frac{1}{2}}AX_1\|_{L^2}\|\nabla LAX_1\|_{L^2}\|AX_1\|_{L^2} + \|\partial_x(-\Delta L)^{-\frac{1}{2}}AX_1\|_{L^2}\|AX_1\|_{L^2}\|\nabla LAX_1\|_{L^2}
\]
\[
\leq C\epsilon\mathcal{D} + C\epsilon\mathcal{C}\mathcal{R}_2.
\]
4.2. Treatment of $I_4$. For $I_4$, by the definition (2.13), (3.1) and using the fact that

$$(l^2 + (\xi - lt)^2)^{\frac{1}{4}} \lesssim (l^2 + (\eta - \xi - lt)^2)^{\frac{1}{4}} + (l^2 + (\eta - lt)^2)^{\frac{1}{4}},$$

we have

$$|I_4| \lesssim \sum_{k \neq 0, l} \int_{\eta, \xi} A_k^*(t, \eta) |k|^{\frac{1}{2}} (k^2 + (\eta - kt)^2)^{-\frac{1}{4}} |k - l|^{\frac{3}{4}} ((k - l)^2 + (\eta - \xi - (k - l)t)^2)^{-\frac{3}{4}} |\tilde{X}_1(t, k - l, \eta - \xi)|$$

$$\times |\tilde{X}_1(t, k - l, \eta - \xi)| |\xi - kt||l|^{-\frac{1}{4}} |\tilde{X}_1(t, l, \xi)||A_k^*(t, \eta)| |k|^{\frac{1}{2}} (k^2 + (\eta - kt)^2)^{-\frac{1}{4}} |\tilde{X}_1(t, k, \eta)| |\xi - kt||l|^{-\frac{1}{4}} |\tilde{X}_1(t, l, \xi)||A_k^*(t, \eta)| |k|^{\frac{1}{2}} |\tilde{X}_1(t, k, \eta)|$$

$$\lesssim \sum_{k \neq 0, l} \int_{\eta, \xi} A_k^*(t, \eta) |k - l|^{\frac{1}{2}} ((k - l)^2 + (\eta - \xi - (k - l)t)^2)^{-\frac{3}{4}} |\tilde{X}_1(t, k - l, \eta - \xi)|$$

$$\times |\xi - kt||l|^{-\frac{1}{4}} |\tilde{X}_1(t, l, \xi)||A_k^*(t, \eta)| |k|^{\frac{1}{2}} |\tilde{X}_1(t, k, \eta)|$$

$$\overset{\text{def}}{=} I_{41} + I_{42}.$$

For $I_{41}$, we obtain that by Young’s inequality and the bootstrap hypotheses,

$$|I_{41}| \lesssim \|\partial_\xi (-\Delta L)^{-\frac{1}{2}} AX_1\|_{L^2} \|\nabla_L AX_1\|_{L^2} \|AX_1\|_{L^2}$$

$$\leq C \epsilon \mathcal{D} \frac{1}{2} (\mathcal{E}_2)^{\frac{1}{2}}.$$

For $I_{42}$, by using $|k|^{\frac{1}{2}} \lesssim |k - l|^{\frac{1}{2}} + |l|^{\frac{1}{2}}$, by Young’s inequality and the bootstrap hypotheses, we have

$$|I_{42}| \lesssim \sum_{k \neq 0, l} \int_{\eta, \xi} A_k^*(t, \eta) |k - l| ((k - l)^2 + (\eta - \xi - (k - l)t)^2)^{-\frac{3}{4}} |\tilde{X}_1(t, k - l, \eta - \xi)|$$

$$\times |\xi - kt||l|^{-\frac{1}{4}} |\tilde{X}_1(t, l, \xi)||A_k^*(t, \eta)| |\tilde{X}_1(t, k, \eta)|$$

$$\lesssim \|\partial_\xi (-\Delta L)^{-\frac{1}{2}} AX_1\|_{L^2} \|\nabla_L AX_1\|_{L^2} \|AX_1\|_{L^2} \leq C \epsilon \mathcal{D} \frac{1}{2} (\mathcal{E}_2)^{\frac{1}{2}}.$$

Thus, we deduce

$$|I_4| \leq C \epsilon \mathcal{D} \frac{1}{2} (\mathcal{E}_2)^{\frac{1}{2}} \leq C \epsilon \mathcal{D} + C \epsilon \mathcal{E}_2.$$

4.3. Treatment of $J_4$. For $J_4$, we get that by the definition (2.13),

$$J_4 = -2\gamma^2 \sigma^{-1} \int A(\partial_\xi N(\tilde{u}_\rho, \partial_\xi \theta_\rho)) AX_2 dz dy$$

$$= - \int A(\partial_\xi N(\tilde{u}_\rho, N^{-1} X_2)) AX_2 dz dy - \sigma^{-1} \int A(\partial_\xi N(\tilde{u}_\rho, N X_1)) AX_2 dz dy$$

$$= J_{41} + J_{42}.$$
For $J_{41}$, by (4.1) and using the fact that $|k|^\frac{1}{2} \lesssim |k-l|^\frac{1}{2} + |l|^\frac{1}{2}$, we get that by the interpolation inequality and the bootstrap hypotheses,

$$|J_{41}| \lesssim \sum_{k \neq 0, l} \int A_k(t, \eta)|k-l|^\frac{1}{2}(k^2 + (\eta - \xi - (k-l)t)^2)^\frac{1}{2} \left| \bar{X}_1(t, k-l, \eta - \xi) \right|$$

$$\times |l|^\frac{1}{2}(l^2 + (\xi - lt)^2)^{-\frac{1}{4}} \left| \bar{X}_2(t, l, \xi) \right||k|^\frac{1}{2}(k^2 + (\eta - kt)^2)^\frac{1}{2} A_k(t, \eta) \left| \bar{X}_2(t, k, \eta) \right| d\xi d\eta$$

$$\lesssim \sum_{k \neq 0, l} \int A_k(t, \eta)|(k-l)^2 + (\eta - \xi - (k-l)t)^2)^{-\frac{1}{4}} \left| \bar{X}_1(t, k-l, \eta - \xi) \right|$$

$$\times |l|^\frac{1}{2}(l^2 + (\xi - lt)^2)^{-\frac{1}{4}} \left| \bar{X}_2(t, l, \xi) \right||k|^\frac{1}{2}(k^2 + (\eta - kt)^2)^\frac{1}{2} A_k(t, \eta) \left| \bar{X}_2(t, k, \eta) \right| d\xi d\eta$$

$$+ \sum_{k \neq 0, l} \int A_k(t, \eta)|k-l|^\frac{1}{2}(k^2 + (\eta - \xi - (k-l)t)^2)^{-\frac{1}{4}} \left| \bar{X}_1(t, k-l, \eta - \xi) \right|$$

$$\times |l|^\frac{1}{2}(l^2 + (\xi - lt)^2)^{-\frac{1}{4}} \left| \bar{X}_2(t, l, \xi) \right||k|^\frac{1}{2}(k^2 + (\eta - kt)^2)^\frac{1}{2} A_k(t, \eta) \left| \bar{X}_2(t, k, \eta) \right| d\xi d\eta$$

$$\lesssim \left\| (\Delta_L)^{-\frac{1}{4}} A_1 \right\|_{L^2} \left\| (\Delta_L)^{-\frac{1}{4}} A_2 \right\|_{L^2} \left\| \partial_x^2 (\Delta_L)^{-\frac{1}{4}} A_1 \right\|_{L^2}$$

$$+ \left\| \partial_x^2 (\Delta_L)^{-\frac{1}{4}} A_1 \right\|_{L^2} \left\| (\Delta_L)^{\frac{1}{4}} A_2 \right\|_{L^2} \left\| \partial_x (\Delta_L)^{-\frac{1}{4}} A_2 \right\|_{L^2}$$

$$\leq C e^{C D^2 (C R)} + C e^{D^2 (C R)}.$$
5.1. Under the bootstrap hypotheses, it holds that 
\[ \langle \xi - \eta l \rangle \langle l^2 + (\xi - \eta l)^2 \rangle \left( |\tilde{X}_1(t, l, \xi)| + |\tilde{X}_2(t, l, \xi)| \right) \]
\[ \times (k^2 + (\eta - kt)^2) \frac{1}{2} A_k^L(t, \eta) \langle |\tilde{X}_2(t, k, \eta)| \rangle d\xi d\eta \]
\[ \sum_{k \neq 0, l, \xi, \eta} A_k^L(t, \eta) \langle k - l \rangle \langle l^2 + (\eta - (k - l) t)^2 \rangle \left( |\tilde{X}_1(t, k - l, \eta - \xi)| \right) \]
\[ \times \frac{1}{2} A_k^L(t, \eta) \langle |\tilde{X}_2(t, k, \eta)| \rangle d\xi d\eta \]
\[ + \sum_{k \neq 0, l, \xi, \eta} A_k^L(t, \eta) \langle k - l \rangle \langle l^2 + (\eta - (k - l) t)^2 \rangle \left( |\tilde{X}_1(t, k - l, \eta - \xi)| \right) \]
\[ \times \frac{1}{2} A_k^L(t, \eta) \langle |\tilde{X}_2(t, k, \eta)| \rangle d\xi d\eta \]
\[ \times (k^2 + (\eta - kt)^2) \frac{1}{2} A_k^L(t, \eta) \langle |\tilde{X}_2(t, k, \eta)| \rangle d\xi d\eta. \]

And then, by interpolation inequality and the bootstrap hypotheses, we obtain that
\[ |J_6| \lesssim \| \partial_2 (-\Delta_L)^{-\frac{1}{2}} A X_1 \|_{L^2} \left( \| (-\Delta_L)^{\frac{1}{2}} A X_1 \|_{L^2} + \| (-\Delta_L)^{\frac{1}{2}} A X_2 \|_{L^2} \right) \]
\[ \lesssim \| \partial_2 (-\Delta_L)^{-\frac{1}{2}} A X_1 \|_{L^2} \| \nabla_L A X_2 \|_{L^2} \| A X_2 \|_{L^2} \]
\[ + \| \partial_2 (-\Delta_L)^{-\frac{1}{2}} A X_1 \|_{L^2} \| \nabla_L A X_1 \|_{L^2} \| A X_1 \|_{L^2} \| \nabla_L A X_2 \|_{L^2} \| A X_2 \|_{L^2} \]
\[ \leq C \mathcal{D} \frac{1}{2} (\mathcal{C} R_2)^{\frac{1}{2}} \leq C \mathcal{D} + C \mathcal{C} R_2. \]

4.5. Treatment of $J_6$. For $J_6$ and $J_7$, we can get the same estimates as $I_3$ and $I_4$:
\[ |J_6| + |J_7| \lesssim \| \partial_2 (-\Delta_L)^{-\frac{1}{2}} A X_1 \|_{L^2} \| \nabla_L A X_1 \|_{L^2} \| A X_1 \|_{L^2} \| A X_2 \|_{L^2} \| \partial_2 (-\Delta_L)^{-\frac{1}{2}} A X_2 \|_{L^2} \]
\[ + \| \partial_2 (-\Delta_L)^{-\frac{1}{2}} A X_1 \|_{L^2} \| \nabla_L A X_2 \|_{L^2} \| A X_1 \|_{L^2} \| \nabla_L A X_1 \|_{L^2} \| \partial_2 (-\Delta_L)^{-\frac{1}{2}} A X_2 \|_{L^2} \]
\[ \leq C \mathcal{D} + C \mathcal{C} R_2. \]

5. Energy estimates on zero modes

In this section, we mainly prove Proposition 2.3, namely, we study the energy estimates of $f_0$, $\tilde{u}_0$, and $\theta_{02}$.

The most difficult part is the energy estimate on $\theta_{02}$. There are two-type nonlinear interactions $\Gamma_1$ and $\Gamma_2$ (see (5.21)). The estimate of $\Gamma_1$ is easy to obtain as (5.2). The low-high interaction $(\tilde{u}_+^2)_{\text{low}}((\partial_{\gamma} - t \partial_{\eta}) \theta_{\phi})_{\text{high}}$ in $\Gamma_2$ is discussed in Section 2.6. The estimate of the high-low interaction $(\tilde{u}_+^2)_{\text{high}}((\partial_{\gamma} - t \partial_{\eta}) \theta_{\phi})_{\text{low}}$ in $\Gamma_2$ is the most technical part. Indeed, for any fixed frequency $\eta$, when $t \in [t(\eta), 2\eta]$, there is $k$ such that $k \eta > 0$ and $t \in I_{k, \eta}$. At this critical time region $I_{k, \eta}$, the $k$th-mode of $\tilde{u}_+^2$, namely $\tilde{u}_+(t, k, \xi)$ with $\xi \sim \eta$, causes a rapid growth of $\theta_{02}$. Such growth of $\theta_{02}$ varies in time and frequency. Thus we construct the time-dependent multiplier $m$, which allows us to control the growth of $\theta_{02}$.

5.1. Energy estimates on $f_0$. For $f_0$, we have

Lemma 5.1. Under the bootstrap hypotheses, it holds that
\[ \frac{d}{dt} \| f_0 \|_{H^s}^2 + 2\nu \| \partial_\gamma f_0 \|_{H^s}^2 \leq C \nu \mathcal{D} \frac{1}{2} (\mathcal{C} R_2)\| \partial_\gamma f_0 \|_{H^s}. \]
Proof. Due to the fact that the zero mode $f_0$ satisfies
\[ \partial_t f_0 - \nu \partial_{yy} f_0 = -\partial_y (\bar{u}_y^2 f_\neq) , \]
we obtain that by using the energy method,
\[ \frac{d}{dt} \| (D_y)^s f_0 \|_{L^2}^2 + 2\nu \| \partial_y (D_y)^s f_0 \|_{L^2}^2 = -2 \langle (D_y)^s \partial_y (\bar{u}_y^2 f_\neq) , (D_y)^s f_0 \rangle_{L^2} . \]
By (2.7), (5.1), using the interpolation inequality and the bootstrap hypotheses, we get
\[ -2 \langle (D_y)^s \partial_y (\bar{u}_y^2 f_\neq) , (D_y)^s f_0 \rangle_{L^2} \lesssim \| (\Delta L)^{-\frac{s}{2}} AX_1 \|_{L^2} \| (\Delta L)^{\frac{s}{2}} AX_1 \|_{L^2} \| \partial_y f_0 \|_{H^s} \]
\[ \lesssim \| (\Delta L)^{-\frac{s}{2}} AX_1 \|_{L^2} \| \nabla LAX_1 \|_{L^2} \| AX_1 \|_{L^2} \| \partial_y f_0 \|_{H^s} \]
\[ \lesssim C \nu^{\frac{1}{2}} \mathcal{E} \mathcal{D}^{\frac{1}{2}} (\mathcal{E} \mathcal{R}_2)^{\frac{1}{2}} \| \partial_y f_0 \|_{H^s} . \]
Thus, we complete the proof of the lemma. \( \square \)

5.2. Energy estimates on $\bar{u}_0^1$. For $\bar{u}_0^1$, we mainly have the following lemma:

**Lemma 5.2.** Under the bootstrap hypotheses, there holds that
\[ \frac{d}{dt} \| \bar{u}_0^1 \|_{H^s}^2 + 2\nu \| \partial_y \bar{u}_0^1 \|_{H^s}^2 \leq C \nu^{\frac{1}{2}} (\mathcal{E} \mathcal{R}_2)^{\frac{1}{2}} \| \partial_y \bar{u}_0^1 \|_{H^s}^{\frac{1}{2}} + C \nu^{\frac{1}{2}} (\mathcal{E} \mathcal{R}_2)^{\frac{1}{2}} (\mathcal{E} \mathcal{D})^{\frac{1}{2}} \| \partial_y \bar{u}_0^1 \|_{H^s}^{\frac{1}{2}} . \]

**Proof.** From (2.7), we obtain that by using the energy method,
\[ \frac{d}{dt} \| (D_y)^s \bar{u}_0^1 \|_{L^2}^2 + 2\nu \| \partial_y (D_y)^s \bar{u}_0^1 \|_{L^2}^2 = -2 \langle (D_y)^s \partial_y (\bar{u}_y^2 \bar{u}_\neq) , (D_y)^s \bar{u}_0^1 \rangle_{L^2} \overset{\text{def}}{=} \Xi . \]
By using (4.1), (3.1) and the fact that $|\eta|^{\frac{1}{2}} \leq |\xi - kt|^{\frac{1}{2}} + |\eta - \xi + kt|^{\frac{1}{2}}$, we have
\[ |\Xi| \leq \sum_{k \neq 0} \int_{\xi, \eta} \langle \eta \rangle^s (k^2 + (\xi - kt)^2)^{-\frac{s}{2}} |\tilde{X}_1(t,k,\xi)| (k^2 + (\eta - \xi + kt)^2)^{-\frac{s}{2}} \]
\[ \times |\tilde{X}_1(t,-k,\eta)| |\eta|^s |\eta|^{\frac{1}{2}} |\bar{u}_0^1(t,\eta)| d\xi d\eta \]
\[ \leq \sum_{k \neq 0} \int_{\xi, \eta} \langle \eta \rangle^s (k^2 + (\xi - kt)^2)^{-\frac{s}{2}} |\tilde{X}_1(t,k,\xi)| (k^2 + (\eta - \xi + kt)^2)^{-\frac{s}{2}} \]
\[ \times |\tilde{X}_1(t,-k,\eta)| |\eta|^s |\eta|^{\frac{1}{2}} |\bar{u}_0^1(t,\eta)| d\xi d\eta \]
\[ + \sum_{k \neq 0} \int_{\xi, \eta} \langle \eta \rangle^s (k^2 + (\xi - kt)^2)^{-\frac{s}{2}} |\tilde{X}_1(t,k,\xi)| |\tilde{X}_1(t,-k,\eta)| |\eta|^s |\eta|^{\frac{1}{2}} |\bar{u}_0^1(t,\eta)| d\xi d\eta \overset{\text{def}}{=} \Xi_1 + \Xi_2 . \]

Then, by the interpolation inequality and the bootstrap hypotheses, we get
\[ |\Xi_1| \lesssim \| (\Delta L)^{-\frac{s}{2}} AX_1 \|_{L^2} \| (\Delta L)^{-\frac{s}{2}} AX_1 \|_{L^2} \| \partial_y (D_y)^s \bar{u}_0^1 \|_{L^2} \]
\[ \lesssim \| (\Delta L)^{-\frac{s}{2}} AX_1 \|_{L^2} \| (\Delta L)^{-\frac{s}{2}} AX_1 \|_{L^2} \| \partial_y (D_y)^s \bar{u}_0^1 \|_{L^2} \]
\[ \lesssim C \nu^{\frac{1}{2}} (\mathcal{E} \mathcal{R}_2)^{\frac{1}{2}} \| \partial_y (D_y)^s \bar{u}_0^1 \|_{L^2}^{\frac{1}{2}} , \]
and
\[ |\Xi_2| \lesssim \| (\Delta L)^{-\frac{s}{2}} AX_1 \|_{L^2} \| AX_1 \|_{L^2} \| \partial_y (D_y)^s \bar{u}_0^1 \|_{L^2} \]
\[ \lesssim \| (\Delta L)^{-\frac{s}{2}} AX_1 \|_{L^2} \| AX_1 \|_{L^2} \| \partial_y (D_y)^s \bar{u}_0^1 \|_{L^2}^{\frac{1}{2}} \| (D_y)^s \bar{u}_0^1 \|_{L^2}^{\frac{1}{2}} . \]
\[ \leq C\nu \frac{7}{12}(\mathfrak{C}\mathfrak{R}_2)^{\frac{7}{12}}(\mathfrak{E}\mathfrak{D})^{\frac{7}{12}}\|\partial_y (D_y)^s \bar{v}_0\|_{L^2}. \]

Thus, we complete the proof of the lemma. \( \square \)

5.3. Energy estimates on \( \theta_{02} \). To obtain the estimate (2.45), we mainly need to prove the following lemma.

**Lemma 5.3.** Under the bootstrap hypotheses, there holds that

\[
\frac{d}{dt}\left\| (D_y)^s m^{-1} \theta_{02} \right\|_{L^2}^2 + 2\nu \left\| \partial_y (D_y)^s m^{-1} \theta_{02} \right\|_{L^2}^2 + 2\left\| \frac{\partial m}{m} (D_y)^s m^{-1} \theta_{02} \right\|_{L^2}^2 \\
\leq C\nu \frac{7}{12}(\mathfrak{C}\mathfrak{D})^{\frac{7}{12}}(\mathfrak{C}\mathfrak{R}_2)^{\frac{7}{12}} + C\nu \| (D_y)^s m^{-1} \theta_{02} \|_{H^s}^2
\]

**Proof.** From (2.5), by applying \( (D_y)^s \) to (2.5), and multiplying the equation by \( \frac{1}{m^2} (D_y)^s \theta_{02} \), and then integrating over \( \mathbb{R} \), we have that,

\[
\frac{d}{dt}\left\| (D_y)^s m^{-1} \theta_{02} \right\|_{L^2}^2 + 2\nu \left\| \partial_y (D_y)^s m^{-1} \theta_{02} \right\|_{L^2}^2 + 2\left\| \frac{\partial m}{m} (D_y)^s m^{-1} \theta_{02} \right\|_{L^2}^2 \\
= -2\left\langle (D_y)^s \frac{1}{m} (\bar{v}_{02} \partial_z \theta_{02} \bar{v}_{02}), (D_y)^s m^{-1} \theta_{02} \right\rangle_{L^2}
\]

\[
- 2\left\langle (D_y)^s \frac{1}{m} (\bar{v}_{02} (\partial_y - t \partial_z) \theta_{02}), (D_y)^s m^{-1} \theta_{02} \right\rangle_{L^2}
\]

(5.1)

On one hand, by (4.1) and (3.5), we obtain that by the interpolation inequality and the bootstrap hypotheses,

\[
\Gamma_1 \lesssim \sum_k \int_{\xi, \eta} \frac{\langle \eta \rangle s}{m(t, \eta)} (k^2 + (\xi - kt)^2)^{-\frac{3}{4}} |\hat{X}_1(t, k, \xi)| (k^2 + (\eta - \xi + kt)^2)^{-\frac{1}{4}}
\]

\[
\times \left( |\hat{X}_1(t, -k, \eta - \xi)| + |\hat{X}_2(t, -k, \eta - \xi)| \right) \langle \eta \rangle s \frac{\bar{\theta}(t, \eta)}{m(t, \eta)} d\xi d\eta
\]

\[
\lesssim \| (\Delta_L)^{-\frac{1}{2}} AX_1 \|_{L^2} \| (D_y)^s m^{-1} \theta_{02} \|_{L^2} \left( \| (\Delta_L)^{-\frac{1}{2}} AX_2 \|_{L^2} + \| (\Delta_L)^{-\frac{1}{2}} AX_1 \|_{L^2} \right)
\]

\[
\lesssim \| (\Delta_L)^{-\frac{1}{2}} AX_1 \|_{L^2} \| AX_1 \|_{L^2} \| (\Delta_L)^{-\frac{1}{2}} AX_2 \|_{L^2} \| AX_2 \|_{L^2} \| (D_y)^s m^{-1} \theta_{02} \|_{L^2}
\]

\[
+ \| (\Delta_L)^{-\frac{1}{2}} AX_1 \|_{L^2} \| AX_1 \|_{L^2} \| (\Delta_L)^{-\frac{1}{2}} AX_2 \|_{L^2} \| AX_2 \|_{L^2} \| (D_y)^s m^{-1} \theta_{02} \|_{L^2}
\]

(5.2)

On the other hand, by (3.1) and (3.5), we have

\[
\Gamma_2 \lesssim \sum_k \int_{\xi, \eta} \frac{\langle \eta \rangle s}{m(t, \eta)} (k^2 + (\xi - kt)^2)^{-\frac{3}{4}} |\hat{X}_1(t, k, \xi)| (k^2 + (\eta - \xi + kt)^2)^{-\frac{1}{4}}
\]

\[
\times |\hat{X}_2(t, -k, \eta - \xi)| \langle \eta \rangle s \frac{\bar{\theta}(t, \eta)}{m(t, \eta)} d\xi d\eta
\]

\[
+ \sum_k \int_{\xi, \eta} \frac{\langle \eta \rangle s}{m(t, \eta)} (k^2 + (\xi - kt)^2)^{-\frac{3}{4}} |\hat{X}_1(t, k, \xi)| (k^2 + (\eta - \xi + kt)^2)^{-\frac{1}{4}}
\]
\[ \times |\hat{X}_1(t, -k, \eta - \xi)| \langle \eta \rangle^s \frac{\hat{\theta}_{02}(t, \eta)}{m(t, \eta)} d\xi d\eta \]
\[ \overset{\text{def}}{=} K_1 + K_2. \]

In the following, we only need to estimate \( K_1 \), and \( K_2 \) can be similarly estimated. Due to we have
\[ |K_1| \lesssim \int_{\xi, \eta} \sum_k \mathbb{1}_{|k|>|\eta|} \frac{\langle \eta \rangle^s}{m(t, \eta)} (k^2 + (\xi - kt)^2)^{-\frac{3}{4}} |\hat{X}_1(t, k, \xi)| \]
\[ \times (k^2 + (\eta - \xi + kt)^2)^{\frac{1}{4}} |\hat{X}_2(t, -k, \eta - \xi)| \langle \eta \rangle^s \frac{\hat{\theta}_{02}(t, \eta)}{m(t, \eta)} d\xi d\eta \]
\[ + \int_{\xi, \eta} \sum_k \mathbb{1}_{|k|\leq|\eta|} \frac{\langle \eta \rangle^s}{m(t, \eta)} (k^2 + (\xi - kt)^2)^{-\frac{3}{4}} |\hat{X}_1(t, k, \xi)| \]
\[ \times (k^2 + (\eta - \xi + kt)^2)^{\frac{1}{4}} |\hat{X}_2(t, -k, \eta - \xi)| \langle \eta \rangle^s \frac{\hat{\theta}_{02}(t, \eta)}{m(t, \eta)} d\xi d\eta \]
\[ \overset{\text{def}}{=} K_{11} + K_{12}. \]

For \( K_{11} \), by using the fact that
\[ (k^2 + (\eta - \xi + kt)^2)^{\frac{1}{4}} \lesssim \langle t \rangle^{\frac{1}{2}} \langle k, \eta - \xi \rangle^{\frac{1}{2}}, \]
and for \(|k| > |\eta|\),
\[ (k^2 + (\xi - kt)^2)^{-\frac{3}{4}} \lesssim \langle k^2 + (\eta - kt)^2 \rangle^{-\frac{3}{4}} \langle k, \eta - \xi \rangle^{\frac{3}{2}} \lesssim \langle t \rangle^{-\frac{3}{4}} \langle k, \eta - \xi \rangle^{\frac{3}{2}}, \]
and by Young’s inequality and the bootstrap hypotheses, we have that for \( s \geq 6, \)
\[ |K_{11}| \lesssim \int_{\xi, \eta} \sum_k \mathbb{1}_{|k|>|\eta|} \frac{\langle k \rangle^s}{m(t, \eta)} |\hat{X}_1(t, k, \xi)| \frac{\langle k, \eta - \xi \rangle^2}{|k|^3} \langle t \rangle^{-1} \]
\[ \times |\hat{X}_2(t, -k, \eta - \xi)| \langle \eta \rangle^s \frac{\hat{\theta}_{02}(t, \eta)}{m(t, \eta)} d\xi d\eta \]
\[ \lesssim \langle t \rangle^{-1} \|X_1\|_{H^{2+\frac{3}{4}}} \|X_2\|_{H^{2+\frac{3}{4}}} \|D_y\|^s m^{-1} \theta_{02} \|L^2 \]
\[ \lesssim \langle t \rangle^{-1} e^{-c \nu^2 t} \|A\|_{L^2} \|AX_1\|_{L^2} \|AX_2\|_{L^2} \|m^{-1} \theta_{02}\|_{H^s} \]
\[ \leq C e^{3\nu^2 t} \langle t \rangle^{-1} e^{-c \nu^2 t}. \]

For \( K_{12} \), we need to estimate it more carefully. By the high and low frequency decomposition, we have
\[ |K_{12}| \lesssim \int_{\xi, \eta} \sum_k \mathbb{1}_{|\xi|\leq|\eta|} \frac{\langle \eta \rangle^s}{m(t, \eta)} (k^2 + (\xi - kt)^2)^{-\frac{3}{4}} |\hat{X}_1(t, k, \xi)| \]
\[ \times (k^2 + (\eta - \xi + kt)^2)^{\frac{1}{4}} |\hat{X}_2(t, -k, \eta - \xi)| \langle \eta \rangle^s \frac{\hat{\theta}_{02}(t, \eta)}{m(t, \eta)} d\xi d\eta \]
\[ + \int_{\xi, \eta} \sum_k \mathbb{1}_{|\xi|>|\eta|} \frac{\langle \eta \rangle^s}{m(t, \eta)} (k^2 + (\xi - kt)^2)^{-\frac{3}{4}} |\hat{X}_1(t, k, \xi)| \]
\[ \times (k^2 + (\eta - \xi + kt)^2)^{\frac{1}{4}} |\hat{X}_2(t, -k, \eta - \xi)| \langle \eta \rangle^s \frac{\hat{\theta}_{02}(t, \eta)}{m(t, \eta)} d\xi d\eta \]
\( \times (k^2 + (\eta - \xi + kt)^2)^{\frac{1}{2}} |\hat{X}_2(t, -k, \eta - \xi)| \langle \eta \rangle^s \frac{\hat{\theta}_{t,2}(t, \eta)}{m(t, \eta)} d\xi d\eta \)

\[ \text{def} = K_{121} + K_{122}. \]

For \( K_{121} \), by using the fact that \( \langle \eta \rangle^s \leq \langle k, \eta - \xi \rangle^s + \langle k, \xi \rangle^s \leq \langle k, \eta - \xi \rangle^s \), Young's inequality and

\[ \|(-\Delta_L)^{-\frac{s}{2}}X_1\|_{H^2} \lesssim \langle t \rangle^{-\frac{3}{2}} \|X_1\|_{H^2}, \]

we obtain that by the interpolation inequality and the bootstrap hypotheses,

\[ |K_{121}| \lesssim \|(-\Delta_L)^{-\frac{s}{2}}X_1\|_{H^2}\|(-\Delta_L)^{\frac{s}{2}}AX_2\|_{L^2}\|m^{-1}\theta_{t,2}\|_{H^2} \]
\[ \lesssim \langle t \rangle^{-\frac{3}{2}} \|X_1\|_{H^2} \|
abla_L AX_2\|_{L^2} \|AX_2\|_{L^2} \|m^{-1}\theta_{t,2}\|_{H^2} \]
\[ \lesssim \langle t \rangle^{-\frac{3}{2}} \|AX_1\|_{L^2} \|\nabla_L AX_2\|_{L^2} \|m^{-1}\theta_{t,2}\|_{H^2} \]
\[ \lesssim C_\epsilon \frac{\eta^2}{\xi^2} \langle t \rangle^{-\frac{3}{2}} \Omega^\frac{3}{2}. \]

(5.5)

For \( K_{122} \), by using the fact that

\[ (k^2 + (\xi - kt)^2)^{-\frac{3}{4}} \lesssim (k^2 + (\eta - kt)^2)^{-\frac{3}{4}} \frac{\langle k, \eta - \xi \rangle^2}{|k|^2}, \]

we get that

\[ |K_{122}| \lesssim \int_{\xi, \eta} \sum_k \mathbb{1}_{t \geq 2|\eta|} \mathbb{1}_{|x| > |\eta - \xi|} \mathbb{1}_{|k| \leq |\eta|} \langle \eta \rangle^s \frac{\hat{\theta}_{t,2}(t, \eta)}{m(t, \eta)} (k^2 + (\xi - kt)^2)^{-\frac{3}{4}} |\hat{X}_2(t, -k, \eta - \xi)| \]
\[ \times (k^2 + (\eta - \xi + kt)^2)^{\frac{1}{2}} |\hat{X}_2(t, -k, \eta - \xi)| \langle \eta \rangle^s \frac{\hat{\theta}_{t,2}(t, \eta)}{m(t, \eta)} d\xi d\eta \]
\[ + \int_{\xi, \eta} \sum_k \mathbb{1}_{t \geq \frac{3}{2}|\eta|} \mathbb{1}_{|x| > |\eta - \xi|} \mathbb{1}_{|k| \leq |\eta|} \langle \eta \rangle^s \frac{\hat{\theta}_{t,2}(t, \eta)}{m(t, \eta)} (k^2 + (\xi - kt)^2)^{-\frac{3}{4}} |\hat{X}_2(t, -k, \eta - \xi)| \]
\[ \times (k^2 + (\eta - \xi + kt)^2)^{\frac{1}{2}} |\hat{X}_2(t, -k, \eta - \xi)| \langle \eta \rangle^s \frac{\hat{\theta}_{t,2}(t, \eta)}{m(t, \eta)} d\xi d\eta \]
\[ + \int_{\xi, \eta} \sum_k \mathbb{1}_{t < \frac{3}{2}|\eta|} \mathbb{1}_{|x| > |\eta - \xi|} \mathbb{1}_{|k| \leq |\eta|} \langle \eta \rangle^s \frac{\hat{\theta}_{t,2}(t, \eta)}{m(t, \eta)} (k^2 + (\xi - kt)^2)^{-\frac{3}{4}} |\hat{X}_2(t, -k, \eta - \xi)| \]
\[ \times (k^2 + (\eta - \xi + kt)^2)^{\frac{1}{2}} |\hat{X}_2(t, -k, \eta - \xi)| \langle \eta \rangle^s \frac{\hat{\theta}_{t,2}(t, \eta)}{m(t, \eta)} d\xi d\eta \]
\[ \text{def} = K_1 + K_2 + K_3. \]

For \( K_1 \), by using \( \langle \eta \rangle^s \leq \langle k, \eta \rangle^s \leq \langle k, \eta - \xi \rangle^s + \langle k, \xi \rangle^s \leq \langle k, \eta - \xi \rangle^s \), and

\[ (k^2 + (\xi - kt)^2)^{-\frac{3}{4}} \lesssim (k^2 + (\eta - kt)^2)^{-\frac{3}{4}} \frac{(k^2 + (\eta - kt)^2)^{\frac{3}{4}}}{(k^2 + (\xi - kt)^2)^{\frac{3}{4}}} \]
\[ \lesssim (k^2 + (\eta - kt)^2)^{-\frac{3}{4}} \frac{\langle k, \eta - \xi \rangle^2}{|k|^2}, \]
we get that by the interpolation inequality and the bootstrap hypotheses,

\[ |K_1| \lesssim \int_{\xi, \eta} \sum_{k} \mathbb{1}_{t \geq |\eta|} \mathbb{1}_{|\xi| > |\eta| - \varepsilon} \mathbb{1}_{|k| \leq |\eta|} \frac{\langle k, \xi \rangle^s}{m(t, \eta)} (k^2 + (\eta - kt)^2)^{-\alpha} (k, \eta - \xi)^{\frac{\alpha}{2}} |\hat{X}_1(t, k, \xi)| \]

\[ \times (k^2 + (\eta - \xi + kt)^2)^{\frac{\alpha}{2}} |\hat{X}_2(t, -k, \eta - \xi)| \langle \eta \rangle^s \frac{\hat{\theta}_{02}(t, \eta)}{m(t, \eta)} d\xi d\eta \]

\[ \lesssim \int_{\xi, \eta} \sum_{k} \mathbb{1}_{t \geq |\eta|} \mathbb{1}_{|\xi| > |\eta| - \varepsilon} \mathbb{1}_{|k| \leq |\eta|} (t)^{-\frac{5}{2}} (k, \xi)^s |\hat{X}_1(t, k, \xi)| \langle \eta \rangle^s \frac{\hat{\theta}_{02}(t, \eta)}{m(t, \eta)} d\xi d\eta \]

\[ \times |\hat{X}_2(t, -k, \eta - \xi)| \langle \eta \rangle^s \frac{\hat{\theta}_{02}(t, \eta)}{m(t, \eta)} d\xi d\eta \]

\[ \lesssim (t)^{-1} \|A_1\|_H^s \|X_2\|_H^s \|\langle D_y \rangle^s m^{-1} \theta_{02}\|_{L^2} \]

\[ \lesssim (t)^{-1} e^{-ct^\frac{3}{4}} \|A_1\|_{L^2} \|A_2\|_{L^2} \|\langle D_y \rangle^s m^{-1} \theta_{02}\|_{L^2} \]

(5.6) \[ \leq C e^{3\sqrt{\frac{s}{t}}} (t)^{-1} e^{-ct^\frac{3}{4}}. \]

For \( K_3 \), we have

\[ |K_3| \lesssim \int_{\xi, \eta} \sum_{k} \mathbb{1}_{t < |\eta|} \mathbb{1}_{|\xi| > |\eta| - \varepsilon} \mathbb{1}_{|k| \leq \frac{1}{10} \sqrt{|\eta|} m(t, \eta)} (k^2 + (\xi - kt)^2)^{-\frac{\alpha}{2}} \]

\[ \times |\hat{X}_1(t, k, \xi)| (k^2 + (\eta - \xi + kt)^2)^{\frac{\alpha}{2}} |\hat{X}_2(t, -k, \eta - \xi)| \langle \eta \rangle^s \frac{\hat{\theta}_{02}(t, \eta)}{m(t, \eta)} d\xi d\eta \]

\[ + \int_{\xi, \eta} \sum_{k} \mathbb{1}_{t < |\eta|} \mathbb{1}_{|\xi| > |\eta| - \varepsilon} \mathbb{1}_{|k| \leq |\eta| \frac{1}{10} \sqrt{|\eta|}} m(t, \eta) (k^2 + (\xi - kt)^2)^{-\frac{\alpha}{2}} \]

\[ \times |\hat{X}_1(t, k, \xi)| (k^2 + (\eta - \xi + kt)^2)^{\frac{\alpha}{2}} |\hat{X}_2(t, -k, \eta - \xi)| \langle \eta \rangle^s \frac{\hat{\theta}_{02}(t, \eta)}{m(t, \eta)} d\xi d\eta \]

\[ \overset{\text{def}}{=} K_{31} + K_{32}. \]

For \( 1 \leq |k| \leq \frac{1}{10} \sqrt{|\eta|} \), and then by \(|\eta - kt| \geq \frac{1}{2} |\eta| \geq (t)^2 \), we have

\[ (k^2 + (\xi - kt)^2)^{-\frac{\alpha}{2}} \lesssim (k^2 + (\eta - kt)^2)^{-\frac{\alpha}{2}} \frac{\langle k, \eta - \xi \rangle^{\frac{\alpha}{2}}}{|k|^{\frac{\alpha}{2}}} \]

\[ \lesssim \frac{\langle k, \eta - \xi \rangle^{\frac{\alpha}{2}}}{|\eta - kt|^2} \lesssim \frac{\langle k, \eta - \xi \rangle^{\frac{\alpha}{2}}}{\langle t \rangle^s}. \]

And then, by using \( \langle \eta \rangle^s \lesssim \langle k, \xi \rangle^s \) (5.3) and the bootstrap hypotheses, we obtain

\[ |K_{31}| \lesssim \int_{\xi, \eta} \sum_{k} \mathbb{1}_{t < |\eta|} \mathbb{1}_{|\xi| > |\eta| - \varepsilon} \mathbb{1}_{|k| \leq \frac{1}{10} \sqrt{|\eta|} m(t, \eta)} \hat{X}_1(t, k, \xi) \]

\[ \times \langle t \rangle^{-\frac{\alpha}{2}} \langle k, \eta - \xi \rangle^2 |\hat{X}_2(t, -k, \eta - \xi)| \langle \eta \rangle^s \frac{\hat{\theta}_{02}(t, \eta)}{m(t, \eta)} d\xi d\eta \]

\[ \lesssim (t)^{-\frac{5}{2}} e^{-ct^\frac{3}{4}} \|A_1\|_{L^2} \|A_2\|_{L^2} \|\langle D_y \rangle^s m^{-1} \theta_{02}\|_{L^2} \]

\[ \lesssim (t)^{-\frac{5}{2}} e^{-ct^\frac{3}{4}} \|A_1\|_{L^2} \|A_2\|_{L^2} \|\langle D_y \rangle^s m^{-1} \theta_{02}\|_{L^2} \]
\[ \leq C \epsilon^3 \nu^2 \gamma \langle t \rangle^{-3 \over 4} e^{-c \nu \gamma t}. \]

For \( \sqrt{\eta} < |k| \leq |\eta| \), by using the fact that \( \langle \eta \rangle^s \lesssim \langle k, \xi \rangle^s \), and \( |\eta| \)

\[ (k^2 + (\xi - kt)^2)^{-{3 \over 4}} \lesssim k^{-{3 \over 4}} \lesssim \langle t \rangle^{-{3 \over 4}}, \]

we obtain that by Young's inequality and the bootstrap hypotheses,

\[ |K_3| \lesssim \int_{(\xi,\eta)} \sum_k \mathbb{1}_{t < \min(\eta, |\xi|)} \mathbb{1}_{|\xi - k| < |\eta|} \mathbb{1}_{|k| < \sqrt{\eta}} \frac{\langle k, \xi \rangle^s}{m(t, \eta)} |\hat{X}_1(t, k, \xi)| \langle t \rangle^{-1} \times \langle k, \eta - \xi \rangle^3 |\hat{X}_2(t, -k, \eta - \xi)| \langle \eta \rangle^s \frac{\hat{\theta}_0(t, \eta)}{m(t, \eta)} d\xi d\eta \]

\[ \lesssim \langle t \rangle^{-1} e^{-c \nu \gamma t} \|AX_1\|_{L^2} \|X_2\|_{H^{1 \over 2}} \langle D_y \rangle^s \|m^{-1} \hat{\theta}_0 \|_{L^2} \]

\[ \lesssim \langle t \rangle^{-1} e^{-c \nu \gamma t} \|AX_1\|_{L^2} \|AX_2\|_{L^2} \langle D_y \rangle^s \|m^{-1} \hat{\theta}_0 \|_{L^2} \]

\[ \leq C \epsilon^3 \nu^2 \gamma \langle t \rangle^{-3 \over 4} e^{-c \nu \gamma t}. \]

Thus, we get that

\[ |K_3| \leq C \epsilon^3 \nu^2 \gamma \langle t \rangle^{-3 \over 4} e^{-c \nu \gamma t}. \]

Finally, we estimate the term \( K_2 \). For \( t(\eta) \leq t \leq 2|\eta| \), we have \( t \in \left[ \frac{2|\eta|}{2j + 1}, \frac{2|\eta|}{2j - 1} \right] \) for some \( j \in [1, E(\sqrt{\eta})] \). And then, for \( K_2 \), we have

\[ |K_2| \lesssim \int_{(\xi,\eta)} \sum_k \mathbb{1}_{t(\eta) \leq t \leq 2|\eta|} \mathbb{1}_{t \in I_j, \eta} \mathbb{1}_{|\xi| > |\eta|} \mathbb{1}_{|k| < |\eta|} \frac{\langle k, \xi \rangle^s}{m(t, \eta)} \frac{\hat{\theta}_0(t, \eta)}{m(t, \eta)} d\xi d\eta \]

\[ + \int_{(\xi,\eta)} \sum_{j=1}^{E(\sqrt{\eta})} \mathbb{1}_{t \in I_j, \eta} \mathbb{1}_{|\xi| > |\eta|} \mathbb{1}_{|k| < |\eta|} \frac{\langle k, \xi \rangle^s}{m(t, \eta)} \frac{\hat{\theta}_0(t, \eta)}{m(t, \eta)} d\xi d\eta \]

\[ \equiv K_{21} + K_{22}. \]

When \( j \neq k \), we have \( |t - k| \gtrsim \frac{|\eta|}{2j} \), and for \( 1 \leq |j| \leq 4|k| \),

\( (k^2 + (\eta - kt)^2)^{-{3 \over 4}} \lesssim k^{-{3 \over 4}} \langle t \rangle^{-{3 \over 4}} \lesssim k^{-{3 \over 4}} (1 + \frac{|\eta|^2}{k^2})^{-{3 \over 4}} \lesssim (t)^{-{3 \over 4}}, \)

and for \( |j| > 4|k| \), by using \( \frac{2|\eta|}{2j - 1} \geq t \), we have

\( (k^2 + (\eta - kt)^2)^{-{3 \over 4}} = k^{-{3 \over 4}} (1 + \frac{-\eta^2}{k^2})^{-{3 \over 4}} \)

\[ \lesssim k^{-{3 \over 4}} (1 + t^2)^{-{3 \over 4}} \lesssim k^{-{3 \over 4}} (t)^{-{3 \over 4}}. \)
Thus, for $K_{21}$, we get that
\[
|K_{21}| \lesssim \int_{\xi, \eta} \sum_{k \neq j} 1_{t(\eta) \leq t_{\eta} \leq 2t_{\eta}} 1_{t \leq t_{\eta}} \langle k - \xi, \eta \rangle \langle \xi - k, \eta \rangle \left| \hat{X}_1(t, k, \xi) \right| \left| \frac{\partial_\Theta(t, \eta)}{m(t, \eta)} \right| d\xi d\eta
\]
\[
\times |\hat{X}_2(t, -k, \eta - \xi)\langle \eta \rangle \frac{\partial_\Theta(t, \eta)}{m(t, \eta)}| d\xi d\eta
\]
\[
\lesssim \langle t \rangle^{-1} e^{-c_2^{1/4}t} \|A X_1\|_{L^2} \|X_2\|_{H^4} \|D_x\| m^{-1} \|\Theta_2\|_{L^2}
\]
\[
\lesssim \langle t \rangle^{-1} e^{-c_2^{1/4}t} \|A X_1\|_{L^2} \|A X_2\|_{L^2} \|D_x\| m^{-1} \|\Theta_2\|_{L^2}
\]
\[
\leq C e^{3\nu^{1/2} \langle t \rangle^{-1} e^{-c_2^{1/4}t}}.
\]

When $j = k$, we have
\[
|K_{22}| \lesssim \int_{\xi, \eta} \sum_{j=1}^{E(\sqrt{\eta})} 1_{t \leq t_{\eta}} \langle k - \xi, \eta \rangle \langle \xi - k, \eta \rangle \left| \hat{X}_1(t, k, \xi) \right| \left| \frac{\partial_\Theta(t, \eta)}{m(t, \eta)} \right| d\xi d\eta
\]
\[
\times (j^2 + (\eta - \xi + j t)^2)^{1/2} \left| \hat{X}_2(t, -j, \eta - \xi) \right| (1 + (t - \eta j)^2)^{-3/2} \langle \eta \rangle \frac{\partial_\Theta(t, \eta)}{m(t, \eta)} d\xi d\eta
\]
and then, we get that by using the definition of the multiplier $m$ and the bootstrap hypotheses,
\[
|K_{22}| \leq \int_{\xi, \eta} \sum_{j=1}^{E(\sqrt{\eta})} 1_{t \leq t_{\eta}} \langle k - \xi, \eta \rangle \langle \xi - k, \eta \rangle \left| \hat{X}_1(t, k, \xi) \right| \left| \frac{\partial_\Theta(t, \eta)}{m(t, \eta)} \right| d\xi d\eta
\]
\[
\times (j^2 + (\eta - \xi + j t)^2)^{1/2} \left| \hat{X}_2(t, -j, \eta - \xi) \right| (1 + (t - \eta j)^2)^{-3/2} \langle \eta \rangle \frac{\partial_\Theta(t, \eta)}{m(t, \eta)} d\xi d\eta
\]
\[
\lesssim \int_{\xi, \eta} \sum_{j=1}^{E(\sqrt{\eta})} 1_{t \leq t_{\eta}} \langle k - \xi, \eta \rangle \langle \xi - k, \eta \rangle \left| \hat{X}_1(t, k, \xi) \right| \left| \frac{\partial_\Theta(t, \eta)}{m(t, \eta)} \right| d\xi d\eta
\]
\[
\times \frac{1}{t} \left( \langle j, \eta - \xi \rangle \frac{\partial_\Theta(t, \eta)}{m(t, \eta)} \right) \left| \hat{X}_2(t, -j, \eta - \xi) \right| \left( \frac{\partial_\Theta(t, \eta)}{m(t, \eta)} \right) d\xi d\eta
\]
\[
\leq t^{1/2} e^{-c_3^{1/2}t} \left\| \frac{\partial_\Theta(t, \eta)}{m(t, \eta)} \right\|_{H^2} \left\| (-\Delta)^{-3/2} AX_2 \right\|_{L^2} \left\| X_1 \right\|_{H^4}
\]
\[
\lesssim t^{1/2} e^{-c_3^{1/2}t} \left\| \frac{\partial_\Theta(t, \eta)}{m(t, \eta)} \right\|_{H^2} \left\| (-\Delta)^{-3/2} AX_2 \right\|_{L^2} \left\| X_1 \right\|_{L^2}
\]
\[
\leq C e^{3\nu^{1/2} \langle t \rangle^{-1} e^{-c_3^{1/2}t}} \left\| \frac{\partial_\Theta(t, \eta)}{m(t, \eta)} \right\|_{H^2}.
\]
Thus, we get
\[
|K_2| \leq C e^{3\nu^{1/2} \langle t \rangle^{-1} e^{-c_3^{1/2}t}} + C e^{3\nu^{1/2} \langle \xi R_3 \rangle^{1/2} \left\| \frac{\partial_\Theta(t, \eta)}{m(t, \eta)} \right\|_{H^2}}.
\]
From (5.2), (5.4) - (5.8), we complete the proof of Lemma 5.3.

Finally, we conclude that Proposition 2.4 follows directly from Lemma 5.1, Lemma 5.2 and Lemma 5.3 with $\epsilon$ small enough.
Acknowledgements

C. Zhai’s work is supported by a grant from the China Scholarship Council and this work was done when C. Zhai was visiting the center SITE, NYU Abu Dhabi. She appreciates the hospitality of NYU.

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