Existence of strongly symmetrical weakly pandiagonal graeco-latin squares

Abstract: A graeco-latin square is a pair of orthogonal latin squares. It is a design of experiment in which the experimental units are grouped in three different ways. In this paper, constructions of a pair of orthogonal latin squares which are both strongly symmetrical and weakly pandiagonal are investigated. As a result, it is proved that there exists a pair of strongly symmetrical weakly pandiagonal orthogonal latin squares of order $n$ if and only if $n > 4$ and $n \equiv 0, 1, 3 \pmod{4}$ with only one possible exception for $n = 12$.

Keywords: Latin square, graeco, orthogonal, strongly symmetrical, weakly pandiagonal

1 Introduction

A Latin square of order $n$, denoted by $LS(n)$, is an $n \times n$ array such that every row and every column is a permutation of an $n$-set $S$. A transversal in a Latin square is a set of positions, one per row and one per column, among which the symbols occur precisely once each. A diagonal Latin square is a Latin square with the additional property that the main diagonal and back diagonal are both transversals. Two $LS(n)$s are called orthogonal, denoted by $OLS(n)$, if each symbol in the first square meets each symbol in the second square exactly once when they are superposed. A pair of orthogonal latin squares is also called a graeco-latin square because Euler ([7]) used Greek letters for one square of the pair and latin letters for the other. See also [5] for overview of the Latin squares.

In 1776, Euler presented a paper in which he constructed magic squares of orders 3, 4, and 5 from orthogonal latin squares. He posed the question for order 6, now known as Eulers 36 Officers Problem. He conjectured that no solution exists for order 6. Indeed he conjectured further that there exist orthogonal latin squares of all orders $n$ except when $n \equiv 2 \pmod{4}$ ([7, 9]). Much later, Bose, Shrikhande and Parker [1] proved that the Euler conjecture was false for all orders $n$ of the form $4k + 2$ except $n = 2$ or $6$. Much shorter disproofs of Euler's conjecture have been obtained. See, for example, [4, 15].

Wallis and Zhu [11], Heinrich and Hilton [10], Brown, Cherry, Most, Most, Parker and Wallis [2] investigated the existence of orthogonal diagonal latin squares. They proved that there exists a pair of diagonal $OLS(n)$s if and only if $n \notin \{2, 3, 6\}$. An $LS(n)$ is called self-orthogonal if it is orthogonal to its transpose. Let $n$ be a positive integer. Denote $I_n = \{0, 1, \cdots , n - 1\}$. An $LS(n)$ over $I_n$ $A = (a_{i,j})(i, j \in I_n)$ is called strongly symmetrical if $a_{i,j} + a_{n-1-i,n-1-j} = n - 1$, $i, j \in I_n$. Danhof, Phillips and Wallis [6] investigated the existence of strongly symmetrical $OLS(n)$s. Du and Cao [8] investigated the existence of strongly symmetrical self-orthogonal $LS(n)$. Cao and Li [3] proved that there is a strongly symmetrical self-orthogonal $LS(n)$ if and only if $n \equiv 0, 1, 3 \pmod{4}$ and $n \neq 3$. It results in the following.
Lemma 1.1. ([3]) A pair of strongly symmetrical OLS(n) exists if and only if n ≡ 0, 1, 3 (mod 4) and n ≠ 3.

Given positive integers a, n, we denote a (mod n) by ⟨a⟩n for convenience. For a matrix A of order n and k ∈ In, we call the elements ai, (k+i), 1 ≤ i ≤ n the k-th right pandiagonal and ai, (k−i), 1 ≤ i ≤ n the k-th left pandiagonal. Clearly, 0-th right pandiagonal is the main diagonal and the (n−1)-th left pandiagonal is the back diagonal. Let A be an LS(n) over In. A is weakly pandiagonal if the sum of the n numbers in each pandiagonal is the same. For example, the following is a pair of weakly pandiagonal OLS(n).

$$D = \begin{pmatrix} 0 & 3 & 6 & 5 & 4 & 7 & 2 & 1 \\ 1 & 2 & 7 & 4 & 5 & 6 & 3 & 0 \\ 5 & 6 & 3 & 0 & 1 & 2 & 7 & 4 \\ 4 & 7 & 2 & 1 & 0 & 3 & 6 & 5 \\ 2 & 1 & 4 & 7 & 6 & 5 & 0 & 3 \\ 3 & 0 & 5 & 6 & 7 & 4 & 1 & 2 \\ 7 & 4 & 1 & 2 & 3 & 0 & 5 & 6 \\ 6 & 5 & 0 & 3 & 2 & 1 & 4 & 7 \end{pmatrix}, \quad D' = \begin{pmatrix} 0 & 1 & 5 & 4 & 2 & 3 & 7 & 6 \\ 3 & 2 & 6 & 7 & 1 & 0 & 4 & 5 \\ 6 & 7 & 3 & 2 & 4 & 5 & 1 & 0 \\ 5 & 4 & 0 & 1 & 7 & 6 & 2 & 3 \\ 4 & 5 & 1 & 0 & 6 & 7 & 3 & 2 \\ 7 & 6 & 2 & 3 & 5 & 4 & 0 & 1 \\ 2 & 3 & 7 & 6 & 0 & 1 & 5 & 4 \\ 1 & 0 & 4 & 5 & 3 & 2 & 6 & 7 \end{pmatrix}.$$

Xu and Lu [12] introduced weakly pandiagonal OLS(n) to construct pandiagonal magic squares. They proved that a weakly pandiagonal self-orthogonal LS(n) exists if and only if n ≡ 0, 1, 3 (mod 4) and n ≠ 3, 6 (mod 9).

Zhang, Li and Lei [13] investigated a pair of weakly pandiagonal OLS(n). They proved the following.

Lemma 1.2. ([13]) A pair of weakly pandiagonal OLS(n) exists if and only if n ≡ 0, 1, 3 (mod 4) and n ≠ 3.

We use SPOLS(n) to denote strongly symmetrical weakly pandiagonal OLS(n). Clearly, the above (D, D') is a pair of SPOLS(8). In this paper, the existence of a pair of SPOLS(n) is investigated. The construction of SPOLS(n) is provided in Section 2 and in section 3 the following is proved.

Theorem 1.3. A pair of SPOLS(n) exists if and only if n > 4 and n ≡ 0, 1, 3 (mod 4) with only one possible exception for n = 12.

2 Constructions for SPOLS(n)

Given an m × n matrix A and an r × s matrix B, the Kronecker Product A ⊗ B is an mr × ns matrix given as follows.

$$A ⊗ B = \begin{pmatrix} a_{0,0}B & a_{0,1}B & \cdots & a_{0,n-1}B \\ a_{1,0}B & a_{1,1}B & \cdots & a_{1,n-1}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m-1,0}B & a_{m-1,1}B & \cdots & a_{m-1,n-1}B \end{pmatrix}.$$

Denote Jm×n an m × n matrix with all elements 1s. A Jm×n is denoted by Jn. The Kronecker Product is used to get the following construction.

Construction 2.1. If there exists a pair of SPOLS(m) and there exists a pair of SPOLS(n), then there exists a pair of SPOLS(mn).

Proof Suppose that (A, B) is a pair of SPOLS(m) over Im and (C, D) is a pair of SPOLS(n) over In. Let

$$E = (e_{i,j}) = nA ⊗ J_n + J_m ⊗ C, \quad F = (f_{i,j}) = nB ⊗ J_n + J_m ⊗ D,$$

where

$$e_{i,j} = na_{u,s} + c_{v,t}, \quad f_{i,j} = nb_{u,s} + d_{v,t},$$

$$i = nu + v, \quad j = ns + t, \quad u, s ∈ I_m, \quad v, t ∈ I_n.$$
In [13] Construction 2.1 proved that $(E, F)$ is a pair of weakly pandiagonal OLS$(mn)$. We shall prove that $E, F$ are also strongly symmetrical.

Let $i' = mn - 1 - i$, $j' = mn - 1 - j$, $u' = m - 1 - u$, $v' = n - 1 - u$, $s' = m - 1 - s$, $t' = n - 1 - t$. Clearly, $i' = nu' + v'$, $j' = ns' + t'$. So we have

$$
e_{ij} + e_{i'j'} = na_{u,s} + c_{v,t} + na_{u',s'} + c_{v',t'} = n(a_{u,s} + a_{u',s'}) + (c_{v,t} + c_{v',t'}) = n(m - 1) + n - 1 = mn - 1.$$

Thus $E$ is strongly symmetrical. Similarly, $F$ is also strongly symmetrical. So $(E, F)$ is a pair of SPOLS$(mn)$. □

Let $m$ be an even integer and $n$ be an integer. Let $T = \{T_0, T_1, \ldots, T_{m-1}\}$ be a partition of $I_m$ such that $|T_i| = n, i \in I_m$. Denote $S = \frac{mn(mn-1)}{2}$. Given a rational number $x$, we use $\lfloor x \rfloor$ to denote the largest integer $a$ such that $a \leq x$ as usual.

**Construction 2.2.** Let $(D, D')$ be a pair of strongly symmetrical OLS$(m)$ over $I_m$, and $(M, M')$ be a pair of strongly symmetrical OLS$(n)$ over $I_n$, where $D = (d_{u,v})$, $D' = (d'_{u,v})$, $M = (m_{i,j})$, $M' = (m'_{i,j})$. Let $\varphi_{u,v}, \varphi'_{u,v}$ be a bijection from $I_n$ to $T_v, u, v \in I_m$. Let

$$A = (A_{u,v}), \quad A_{u,v} = (a_{u,v}(i,j))_{n \times n}, \quad a_{u,v}(i,j) = \varphi_{u,d_{u,v}}(m_{i,j}),$$

$$B = (B_{u,v}), \quad B_{u,v} = (b_{u,v}(i,j))_{n \times n}, \quad b_{u,v}(i,j) = \varphi'_{u,d'_{u,v}}(m'_{i,j}),$$

where $u, v \in I_m, i, j \in I_n$. If $\varphi_{u,v}, \varphi'_{u,v}$ satisfy

$$\begin{align*}
\varphi_{u,v}(t) + \varphi_{m-1-u,m-1-v}(n - 1 - t) &= mn - 1, \\
\varphi'_{u,v}(t) + \varphi'_{m-1-u,m-1-v}(n - 1 - t) &= mn - 1,
\end{align*}$$

Then $(A, B)$ is a pair of strongly symmetrical OLS$(mn)$ over $I_{mn}$. Further, if $\varphi_{u,v}$ satisfies

$$\begin{align*}
\sum_{u \in I_m} \sum_{i \in I_n} \varphi_{u,d_{u,v}(i,j)} (m_{i,j})_a &= S, \\
\sum_{u \in I_m} \sum_{i \in I_n} \varphi_{u,d'_{u,v}(i,j)} (m'_{i,j})_a &= S,
\end{align*}$$

and $\varphi'_{u,v}$ satisfies

$$\begin{align*}
\sum_{u \in I_m} \sum_{i \in I_n} \varphi'_{u,d'_{u,v}(i,j)} (m'_{i,j})_a &= S, \\
\sum_{u \in I_m} \sum_{i \in I_n} \varphi'_{u,d'_{u,v}(i,j)} (m'_{i,j})_a &= S,
\end{align*}$$

then $(A, B)$ is a pair of SPOLS$(mn)$ over $I_{mn}$.

**Proof** First we prove that $A, B$ are LS$(mn)$. Let $u \in I_m, i \in I_n$. We consider the entries $a_{u,v}(i,j)$ and $a_{u,v'}(i,j')$ of the $(nu + i)$-th row of $A$, where $v, v' \in I_m, j, j' \in I_n$, and $(v, j) \neq (v', j')$. Since $D$ is a latin square, if $v \neq v'$ then $d_{u,v} \neq d_{u,v'}$, hence $T_{d_{u,v}} \cap T_{d_{u,v'}} = \emptyset$. So $\varphi_{u,d_{u,v}}(m_{i,j}) \neq \varphi_{u,d_{u,v'}}(m_{i,j'})$. Thus $a_{u,v}(i,j) \neq a_{u,v}(i,j')$. If $v = v'$, $j \neq j'$, then the image of $\varphi_{u,d_{u,v}}$ and $\varphi_{u,d_{u,v'}}$ fall into the same set $T_{d_{u,v}}$. Since $M$ is latin square, $m_{i,j} \neq m_{i,j'}$, thus $\varphi_{u,d_{u,v}}(m_{i,j}) \neq \varphi_{u,d_{u,v'}}(m_{i,j'})$. So, $a_{u,v}(i,j) \neq a_{u,v'}(i,j')$. Therefore each row of $A$ is a permutation of $I_{mn}$. Similarly, each column of $A$ is also a permutation of $I_{mn}$. Thus $A$ is an LS$(mn)$ over $I_{mn}$. Similarly, $B$ is also an LS$(mn)$ over $I_{mn}$.

Now we shall prove that $A, B$ are orthogonal. Let

$$a_{u_1,v_1}(i_1,j_1)b_{u_2,v_2}(i_2,j_2) = a_{u_2,v_2}(i_2,j_2)a_{u_1,v_1}(i_1,j_1),$$

where $u_1, u_2, v_1, v_2 \in I_m, i_1, i_2, j_1, j_2 \in I_n$. Then
Let \( \varphi_{u_1, d_{u_1}, \nu_1}(m_{i_1, j_1}) = \varphi_{u_2, d_{u_2}, \nu_2}(m_{j_2, i_2}) \),
\( \varphi'_{u_1, d_{u_1}, \nu_1}(m_{i_1, j_1}) = \varphi'_{u_2, d_{u_2}, \nu_2}(m_{j_2, i_2}) \).

Note that \( \varphi_{u_1, d_{u_1}, \nu_1} \) and \( \varphi'_{u_1, d_{u_1}, \nu_1} \) are bijection from \( I_n \) to \( T_{d_{u_1}, \nu_1} \), respectively, \( i = 1, 2 \). So, \( T_{d_{u_1}, \nu_1} = T_{d_{u_2}, \nu_2} \), \( T_{d_{u_1}, \nu_1} = T_{d_{u_2}, \nu_2} \). Thus \( d_{u_1, \nu_1} = d_{u_2, \nu_2}, d_{u_1, \nu_1} = d_{u_2, \nu_2} \). Since \( D, D' \) are orthogonal, \( u_1 = u_2, v_1 = v_2 \). Since \( \varphi, \varphi' \) are bijection, we have

\[
m_{i_1, j_1} = m_{i_2, j_2}, m_{i_1, j_1} = m_{i_2, j_2}.
\]

Since \( M, M' \) are orthogonal, we have \( i_1 = i_2, j_1 = j_2 \). Thus \( A, B \) are orthogonal.

For any \( u, v \in I_m, i, j \in I_m \), let \( u' = m - 1 - u, v' = m - 1 - v, i' = n - 1 - i, j' = n - 1 - j \). Since \( D, M \) are strongly symmetrical, we get \( d_{u, v} + d_{u', v'} = m - 1, m_{i, j} + m_{i', j'} = n - 1 \). By the condition (L1) we have

\[
a_{u, v}(m_{i, j}) + a_{u', v'}(m_{i', j'}) = \varphi_{u, d_{u, \nu}}(m_{i, j}) + \varphi_{u', d_{u', \nu}}(m_{i', j'})
= \varphi_{u, d_{u, \nu}}(m_{i, j}) + \varphi_{u', m - 1 - d_{u, \nu}}(n - 1 - m_{i, j})
= mn - 1.
\]

Thus \( A \) is strongly symmetrical. Similar, \( B \) is also strongly symmetrical by the condition (L2).

Now we consider the pandiagonals of \( A, B \). For any \( s \in I_{mn} \), the \( s \)-th right pandiagonal of \( A \) consists of the \( (k, (k + s)_{mn}) \)-entry of \( A \), \( k \in I_m \). Let \( k = un + i, s = vn + j \), where \( u, v \in I_m, i, j \in I_n \). Note that \( i + j = n \left[ \frac{i}{n} \right] + (i + j)_n \), we have

\[
\langle k + s \rangle_{mn} = \langle n(u + v) + i + j \rangle_{mn}
= \langle (n + i + j)_{mn} = \langle n(u + v) + n \left[ \frac{i}{n} \right] + (i + j)_n \rangle_{mn}
= n(u + v + \left[ \frac{i}{n} \right]) + (i + j)_n.
\]

So, the \( (k, (k + s)_{mn}) \)-entry of \( A \) is

\[
a_{u, v}(m_{i, \nu_i + \nu_{\left[ \frac{i}{n} \right]}})(i, (j + i)_n),
\]

which is just

\[
\varphi_{u, d_{u, \nu_i + \nu_{\left[ \frac{i}{n} \right]}}}(m_{i, (j + i)_n}).
\]

The sum of each right pandiagonal of \( A \) is the same number \( S \) by (L3). Similarly, the sum of each left pandiagonal of \( A \) is also \( S \) by (L4). So \( A \) is weakly pandiagonal. By (L5) and (L6) \( B \) is also weakly pandiagonal. Thus \( (A, B) \) is a pair of SPOLS(mn) over \( I_{mn} \).

\[
\square
\]

\textbf{Lemma 2.3.} \textit{There exists a pair of SPOLS(24).}

\textbf{Proof} \ Let \( (D, D') \) be the pair of strongly symmetrical OLS(8) given in the introduction. Let

\[
M = \begin{pmatrix}
0 & 2 & 1 \\
2 & 1 & 0 \\
1 & 0 & 2
\end{pmatrix}, \quad
M' = \begin{pmatrix}
1 & 2 & 0 \\
0 & 1 & 2 \\
2 & 0 & 1
\end{pmatrix}.
\]

Clearly, \( (M, M') \) is a pair of strongly symmetrical OLS(3). Let

\[
L = \begin{pmatrix}
0 & 2 & 6 & 3 & 4 & 1 & 8 & 7 & 10 & 11 & 9 & 5 & 21 & 17 & 23 & 20 & 19 & 22 & 15 & 16 & 13 & 14 & 18 & 12
\end{pmatrix},
\]

\[
L' = \begin{pmatrix}
0 & 2 & 6 & 1 & 3 & 4 & 10 & 8 & 7 & 11 & 9 & 5 & 21 & 17 & 23 & 22 & 19 & 20 & 13 & 16 & 15 & 14 & 18 & 12
\end{pmatrix}.
\]

Denote \( L = (L_0, L_1, \cdots, L_7) \), \( L_v = (l_v(0), l_v(1), l_v(2)) \), \( v \in I_8 \). Similarly, we denote \( L' \).

For any \( t \in I_3, v \in I_8 \), let

\[
\varphi_{u, v}(t) = \begin{cases}
L_v(t), & u = 0, 1, 2, 3;
8n - 1 - l_{7-v}(n - 1 - t), & u = 4, 5, 6, 7.
\end{cases}
\]
We introduce the following definition.

**Definition 1.** Let $L$ be a permutation of $0, 1, \cdots, 4n - 1$. Denote $L = (L_0|L_1|L_2|L_3)$, where

\[ L_i = (l_i(0) \ l_i(1) \ \cdots \ l_i(n - 1)), \ i = 0, 1, 2, 3. \]
For any satisfied.

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Proof We use Construction 2.2 by taking \( m = 4 \). Let

\[
D = \begin{pmatrix}
0 & 3 & 1 & 2 \\
2 & 1 & 3 & 0 \\
3 & 0 & 2 & 1 \\
1 & 2 & 0 & 3
\end{pmatrix}, \quad D' = \begin{pmatrix}
2 & 0 & 1 & 3 \\
1 & 3 & 2 & 0 \\
3 & 1 & 0 & 2 \\
0 & 2 & 3 & 1
\end{pmatrix}.
\]

Construction 2.4. If there is an S(4n) and there is a pair of strongly symmetrical OLS(n), then there is a pair of SPOLS(4n).

Proof We use Construction 2.2 by taking \( m = 4 \). Let

Let \( L = (L_0|L_1|L_2|L_3) \) be an S(4n). Denote

\[
L_v = (l_v(0) \ l_v(1) \cdots l_v(n-1)), v \in I_n,
\]

For any \( t \in I_n \), let

\[
\varphi_{u,v}(t) = \begin{cases}
l_v(t), & u = 0, 2; \\
4n - 1 - l_{3-v}(n - 1 - t), & u = 1, 3.
\end{cases}
\]

and

\[
\varphi'_{u,v}(t) = \varphi_{u,v}(t), \quad u, v \in I_4.
\]

Let

\[ T_v = \{l_v(j)j \in I_n\}, v \in I_4, \quad T = \{T_0, T_1, T_2, T_3\}. \]

Since \( L \) is a permutation, \( T \) is a partition of \( I_n \). By (R1), \( T_v = \{4n - 1 - x| x \in T_{3-v}\}, v \in I_4. \)

For fixed \( v \in I_4 \), if \( t \) runs over \( I_n \) then \( \varphi_{0,v}(t), \varphi_{2,v}(t) \) run over \( T_v \), and \( \varphi_{1,v}(t), \varphi_{3,v}(t) \) runs over \( \{4n - 1 - x| x \in T_{3-v}\} \), which is exactly \( T_v \). So, for any \( u, v \in I_4 \), \( \varphi_{u,v} \) is a bijection from \( I_n \) to \( T_v \). By (2) \( \varphi'_{u,v} \) is also a bijection from \( I_n \) to \( T_v \).

Let \( A, B \) be the matrices given by the Construction 2.2. We shall prove that \( \varphi \) and \( \varphi' \) satisfy (L1)-(L6) in Construction 2.2. So, \( (A, B) \) is a pair of SPOLS(4n) by Construction 2.2.

In fact, by (1), for any \( t \in I_n \),

\[
\varphi_{0,v}(t) + \varphi_{3,3-v}(n - 1 - t) = l_v(t) + 4n - 1 - l_v(t) = 4n - 1.
\]

Similarly, one can prove that \( \varphi_{1,v}(t) + \varphi_{2,3-v}(n - 1 - t) = 4n - 1 \). Thus (L1) and (L2) in Construction 2.2 are satisfied.

For any \( t \in I_n \), by (1) and (R2) and (R3) it is readily verified the following.

\[
\varphi_{0,0}(t) + \varphi_{0,2}(t) = \varphi_{0,0}(n - 1 - t) + \varphi_{0,2}(n - 1 - t), \quad (3)
\]

\[
\varphi_{0,1}(t) + \varphi_{0,3}(t) = \varphi_{0,0}(n - 1 - t) + \varphi_{0,2}(n - 1 - t), \quad (4)
\]

\[
\varphi_{0,1}(t) + \varphi_{0,3}(t) = \varphi_{0,1}(n - 1 - t) + \varphi_{0,3}(n - 1 - t). \quad (5)
\]

By (1) we have

\[
\varphi_{0,v}(t) = 4n - 1 - \varphi_{1,3-v}(n - 1 - t) = \varphi_{2,v}(t) = 4n - 1 - \varphi_{3,3-v}(n - 1 - t).
\]
By (3) we have
\[\varphi_{0,0}(t) + \varphi_{1,1}(t) + \varphi_{2,2}(t) + \varphi_{3,3}(t) = 2(4n - 1), \quad t \in I_n.\]
That is \(\sum_{u \in I_n} \varphi_{u,d_u}(t) = 2(4n - 1), \quad t \in I_n.\) Similarly, by (4) and (5) we have
\[\sum_{u \in I_n} \varphi_{u,d_{u(r\cdot n)+\delta}}(t) = 2(4n - 1), \quad \nu = 1, 2, 3, \quad t \in I_n.\]
Thus for any \(\nu \in I_n, \varphi\) satisfies
\[\sum_{u \in I_n} \varphi_{u,d_{u(r\cdot n)+\delta}}(t) = 2(4n - 1).\]
So,
\[\sum_{u \in I_n} \sum_{t \in I_n} \varphi_{u,d_{u(r\cdot n)+\delta}}(m_{i,j+n}) = \sum_{i,j} \left( \sum_{u \in I_n} \varphi_{u,d_{u(r\cdot n)+\delta}}(m_{i,j+n}) \right) + \sum_{i,j} \left( \sum_{u \in I_n} \varphi_{u,d_{u(r\cdot n)+\delta}}(m_{i,j+n}) \right) = 2(4n - 1) + 2(4n - 1) = 2n(4n - 1).\]
It shows that the condition (L3) in Construction 2.2 is satisfied.
The condition (L4) in Construction 2.2 is satisfied by (3), (4) and (5). So (L5) and (L6) are satisfied by (2).

The following example is provided to explain Construction 2.4 and its proof.

**Example 1.** Let \((D, D')\) be the same as in Construction 2.4. Let \(M = D, M' = D'\). Then \(D, D', M, M'\) are strongly symmetrical. Let
\[L = (0 3 5 6 | 2 1 7 4 | 14 13 11 8 | 12 15 9 10).\]
Clearly \(L\) is an S(16). By Construction 2.4 we have
\[
\begin{pmatrix}
0 & 6 & 3 & 5 & 11 & 10 & 15 & 9 & 2 & 4 & 1 & 7 & 14 & 8 & 13 & 11
\end{pmatrix},
\begin{pmatrix}
11 & 14 & 13 & 8 & 5 & 0 & 3 & 6 & 7 & 2 & 1 & 4 & 9 & 12 & 15 & 10
\end{pmatrix}
\]
\[
\begin{pmatrix}
5 & 3 & 6 & 0 & 9 & 15 & 10 & 12 & 7 & 1 & 6 & 2 & 11 & 13 & 8 & 14
\end{pmatrix},
\begin{pmatrix}
13 & 8 & 11 & 4 & 2 & 6 & 5 & 0 & 1 & 4 & 7 & 2 & 15 & 10 & 9 & 12
\end{pmatrix}
\]
\[
\begin{pmatrix}
6 & 0 & 5 & 3 & 10 & 12 & 9 & 15 & 4 & 2 & 7 & 1 & 8 & 14 & 11 & 13
\end{pmatrix},
\begin{pmatrix}
8 & 13 & 14 & 11 & 6 & 3 & 0 & 5 & 4 & 1 & 2 & 7 & 10 & 15 & 12 & 9
\end{pmatrix}
\]
\[
\begin{pmatrix}
3 & 5 & 0 & 6 & 15 & 9 & 12 & 10 & 1 & 7 & 2 & 4 & 13 & 11 & 14 & 8
\end{pmatrix},
\begin{pmatrix}
14 & 11 & 8 & 13 & 0 & 5 & 6 & 3 & 2 & 7 & 4 & 1 & 12 & 9 & 10 & 15
\end{pmatrix}
\]
\[
\begin{pmatrix}
11 & 13 & 8 & 14 & 7 & 1 & 4 & 2 & 9 & 15 & 10 & 12 & 5 & 3 & 6 & 0
\end{pmatrix},
\begin{pmatrix}
2 & 7 & 4 & 1 & 12 & 9 & 10 & 15 & 14 & 11 & 8 & 13 & 0 & 5 & 6 & 3
\end{pmatrix}
\]
\[
\begin{pmatrix}
14 & 8 & 13 & 11 & 2 & 4 & 1 & 7 & 12 & 10 & 15 & 9 & 0 & 6 & 3 & 5
\end{pmatrix},
\begin{pmatrix}
4 & 1 & 2 & 7 & 10 & 15 & 9 & 12 & 8 & 13 & 11 & 6 & 3 & 0 & 5
\end{pmatrix}
\]
\[
\begin{pmatrix}
13 & 11 & 14 & 8 & 1 & 7 & 2 & 4 & 15 & 9 & 12 & 10 & 1 & 6 & 0 & 5 & 3
\end{pmatrix},
\begin{pmatrix}
1 & 4 & 7 & 2 & 15 & 10 & 9 & 12 & 13 & 8 & 11 & 16 & 3 & 5 & 0 & 6
\end{pmatrix}
\]
\[
\begin{pmatrix}
13 & 11 & 14 & 8 & 4 & 2 & 1 & 7 & 10 & 12 & 9 & 15 & 6 & 0 & 5 & 3
\end{pmatrix},
\begin{pmatrix}
7 & 2 & 1 & 4 & 9 & 12 & 15 & 10 & 11 & 14 & 8 & 5 & 0 & 3 & 6
\end{pmatrix}
\]

It is readily verified that \((A, B)\) is a pair of SPOLS(16).
Definition 2. Let $H$ be an $m \times n$ matrix over $I_{mn}$ consisting of distinct numbers. $H$ is called a row magic rectangle if

$$\sum_{i=0}^{n-1} h_{i,j} = \frac{n(mn - 1)}{2}, \quad i \in I_m.$$  

(6)

A row magic rectangle $H$ is called centre-complementary, denoted by $(m, n)$-CCRMR, if $H$ satisfies

$$h_{i,j} + h_{m-1-i,n-1-j} = mn - 1, \quad i \in I_m, j \in I_n.$$  

(7)

See similar definition in [14].

Construction 2.5. Let $n \equiv 3 \pmod{6}$, $n \neq 3$, if there is a $(3, \frac{n}{2})$-CCRMR, then there is a pair of SPOLS(n).

Proof Let $H$ be a $(3, \frac{n}{2})$-CCRMR over $I_n$. Define a penmutation $\sigma$ of $I_n$:

$$\sigma(i) = h_{\langle i \rangle, \lfloor \frac{i}{3} \rfloor}, \quad i \in I_n.$$  

(8)

Let $U$, $V$ be matrices of order $n$ over $I_n$, where

$$u_{i,j} = \sigma(2i + j + 1)n, \quad v_{i,j} = \sigma(2i - j)n, \quad i, j \in I_n.$$  

(9)

It is readily verified that $(U, V)$ is a pair of OLS(n). We shall prove that $U$ and $V$ is a pair of SPOLS(n).

First we shall prove that $U$, $V$ are weakly pandiagonal. Clearly, the left pandiagonals of $U$ and the right pandiagonals of $V$ are transversals. For the left pandiagonal of $U$, fixed $w \in I_n$, let $w = 3s + t$, $s \in I_{\frac{n}{3}}$, $t \in I_3$. By (6), (8) and (9) we have

$$\sum_{i=0}^{n-1} u_{i,(i+w)_n} = \sum_{i=0}^{n-1} \sigma((2i + (i + w)n + 1)n) = \sum_{i=0}^{n-1} \sigma((3(i + s) + t + 1)n),$$  

(10)

We have \{$(3(i + s) + t + 1)n| i \in I_n$\} = $3\{3i + (t + 1)3| i \in I_{\frac{n}{3}}\}$. In fact, since $t + 1 = 3\lfloor \frac{t+1}{3} \rfloor + (t + 1)3$ and there is an integer $q$ such that $i + s + \lfloor \frac{i+1}{3} \rfloor = q \cdot \frac{n}{3} + \lfloor \frac{i+1}{3} \rfloor$, we have

$$\{3(i + s + \lfloor \frac{i+1}{3} \rfloor)n| i \in I_n\} = \{(3(i + s + \lfloor \frac{i+1}{3} \rfloor + (t + 1)3)n| i \in I_n\} = 3\{3i + (t + 1)3| i \in I_{\frac{n}{3}}\}.$$  

Thus the equality (10) is $3\sum_{i=0}^{\frac{n}{3}-1} \sigma(3i + (t + 1)3)$. It also equals to $3\sum_{i=0}^{\frac{n}{3}-1} h_{\langle t+1 \rangle, i}$ by the equality (8). Further, it is just $\frac{n(n-1)}{2}$ by (6). So the sum of the $n$ numbers of each right pandiagonal of $U$ is the same. Similarly, the sum of the $n$ numbers of each left pandiagonal of $V$ is the same. Thus $U$, $V$ are weakly pandiagonal.

Now we prove that $U$, $V$ are strongly symmetrical. For any $i \in I_n$, let $q = \lfloor \frac{i}{3} \rfloor$, $r = \langle i \rangle_3$, then $i = 3q + r$, $q \in I_{\frac{n}{3}}$, $r \in I_3$. So we have $n - 1 - i = 3(\frac{n}{3} - 1 - q) + 2 - r$. Thus by (7) and (8) we have

$$\sigma(i) + \sigma(n-1-i) = h_{r,q} + h_{2-r,\frac{n}{3}-1-q} = n - 1.$$  

for $i, j \in I_n$, we have

$$u_{n-1-i,n-1-j} = \sigma((2(n - 1 - i) + (n - 1 - j) + 1)n) = \sigma((n - 1 - (2i + j + 1)n)n),$$  

which is just $\sigma(n - 1 - (2i + j + 1)n)$. So,

$$u_{i,j} + u_{n-1-i,n-1-j} = \sigma((2i + j+1)n) + \sigma(n - 1 - (2i + j + 1)n) = n - 1.$$  

Thus $U$ is strongly symmetrical. Similarly, one can prove that $V$ is strongly symmetrical. Therefore, $(U, V)$ is a pair of SPOLS(n).
An example is provided to explain the construction 2.5.

**Example 2.** Let $n = 9$, and

$$H = \begin{pmatrix} 0 & 7 & 5 \\ 6 & 4 & 2 \\ 3 & 1 & 8 \end{pmatrix}.$$ 

Clearly, $H$ is a $(3, 3)$-CCRMR. By Construction 2.5, define a permutation of $\sigma$ over $I_9$:

$$\sigma(i) = h_{(i+1/4)}, \ i \in I_9.$$ 

That is

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}.$$ 

Let $(U, V)$ be a pair of OLS(9) given by (9), where

$$u_{i,j} = \sigma((2i + j + 1)/9), \ v_{i,j} = \sigma((2i-j)/9), \ i, j \in I_9.$$ 

So,

$$U = \begin{pmatrix} 6 & 3 & 7 & 4 & 1 & 5 & 2 & 8 & 0 \\ 7 & 4 & 1 & 5 & 2 & 8 & 0 & 6 & 3 \\ 1 & 5 & 2 & 8 & 0 & 6 & 3 & 7 & 4 \\ 2 & 8 & 0 & 6 & 3 & 7 & 4 & 1 & 5 \\ 0 & 6 & 3 & 7 & 4 & 1 & 5 & 2 & 8 \\ 3 & 7 & 4 & 1 & 5 & 2 & 8 & 0 & 6 \\ 4 & 1 & 5 & 2 & 8 & 0 & 6 & 3 & 7 \\ 5 & 2 & 8 & 0 & 6 & 3 & 7 & 4 & 1 \\ 8 & 0 & 6 & 3 & 7 & 4 & 1 & 5 & 2 \end{pmatrix}, \ V = \begin{pmatrix} 0 & 8 & 2 & 5 & 1 & 4 & 7 & 3 & 6 \\ 3 & 6 & 0 & 8 & 2 & 5 & 1 & 4 & 7 \\ 4 & 7 & 3 & 6 & 0 & 8 & 2 & 5 & 1 \\ 5 & 1 & 4 & 7 & 3 & 6 & 0 & 8 & 2 \\ 8 & 2 & 5 & 1 & 4 & 7 & 3 & 6 & 0 \\ 6 & 0 & 8 & 2 & 5 & 1 & 4 & 7 & 3 \\ 7 & 3 & 6 & 0 & 8 & 2 & 5 & 1 & 4 \\ 1 & 4 & 7 & 3 & 6 & 0 & 8 & 2 & 5 \\ 2 & 5 & 1 & 4 & 7 & 3 & 6 & 0 & 8 \end{pmatrix}.$$ 

It is readily verified that $(U, V)$ is a pair of SPOLS(9).

### 3 Proof of Theorem 1.3

In this section we consider the existence of a pair of SPOLS($n$). There doesn’t exist a pair of SPOLS($n$) if $n \equiv 2 \pmod{4}$ by Lemma 1.1. To prove Theorem 1.3 it remains to check the following three cases: (1) $n \equiv 1, 5 \pmod{6}$; (2) $n \equiv 3 \pmod{6}$; (3) $n \equiv 0 \pmod{4}$.

**Case 1.** $n \equiv 1, 5 \pmod{6}$

**Lemma 3.1.** There exists a pair of SPOLS($n$) if $n \equiv 1, 5 \pmod{6}$.

**Proof** Let $U = (u_{i,j})_{n \times n}$, $V = (v_{i,j})_{n \times n}$, where

$$u_{i,j} = (2i + j + 1)_n, \ v_{i,j} = (2i - j)_n, \ i, j \in I_n.$$ 

$(U, V)$ is a pair of OLS($n$) over $I_n$. Now we prove that $U$, $V$ are strongly symmetrical and weakly pandiagonal. It is enough to prove $U$ is strongly symmetrical and weakly pandiagonal. Similarly we have $V$.

For any $i, j \in I_n$, we have

$$u_{i,j} + u_{n-1-i,n-1-j} = (2i + j + 1)_n + (2(n - 1 - i) + (n - 1 - j) + 1)_n$$

$$= (2i + j + 1)_n + (3n - 1 - (2i + j + 1))_n$$

$$= (2i + j + 1)_n + (n - 1 - (2i + j + 1))_n$$

$$= n - 1.$$
The sum of each pair of symmetrical elements is the elements of the
It is readily verified that the elements of
If \( s \) is strongly symmetrical.
Lemma 3.5. There is a pair of SPOLS(\( n \)) if \( n \equiv 3 \pmod{6} \) and \( n \neq 3 \).

Lemma 3.2. There is a \((3, n)\)-CCMR for odd \( n \geq 3 \).

Proof Let \( n \) be odd, \( n \equiv 3 \), denote \( n = 4s + w, w \in \{3, 5\}, s \geq 0 \). We perform induction on \( s \).
If \( s = 0 \), a \((3, 3)\)-CCMR is given by Example 2, and a \((3, 5)\)-CCMR is given below.

For \( s \geq 0 \), suppose that \( A(s, w) \) is a \((3, 4s + w)\)-CCMR. Then the row sum of \( A(s, w) \) is

\[ (4s + w)(12s + 3w - 1)/2. \]

The sum of each pair of symmetrical elements is \( 12s + 3w - 1 \). Let

\[ L = \begin{pmatrix} 0 & 5 & 1 & 4 \\ 2 & 3 \end{pmatrix}, \quad R(s, w) = \begin{pmatrix} 12s + 3w + 8 & 12s + 3w + 9 \\ 12s + 3w + 7 & 12s + 3w + 10 \\ 12s + 3w + 6 & 12s + 3w + 11 \end{pmatrix}. \]

Denote

\[ A(s + 1, w) = (L \mid A(s, w) + 6I_{3, (4s+w)} \mid R(s, w)). \]

It is readily verified that the elements of \( A(s + 1, w) \) run over \( I_{n+2} \). By calculation, for any \( i \in I_3 \), the sum of the elements of the \( i \)-th row of \( A(s + 1, w) \) is

\[ \frac{n(3n-1)}{2} + 6n + 2(3n) + 22 = \frac{(4(s+1)+w)(12(s+1)+3w-1)}{2}, \]

which is independent on \( i \). The sum of each pair of symmetrical elements of \( A(s + 1, w) \) is \( 12s + 3w + 11 = 12(s + 1) + 3w - 1 \), which is also independent on \( i \). Thus \( A(s + 1, w) \) is a \((3, 4(s+1)+w)\)-CCMR. There is a \((3, n)\)-CCMR for any odd \( n \geq 3 \) by induction.

Lemma 3.3. There exists a pair of SPOLS(\( n \)) if \( n \equiv 3 \pmod{6} \) and \( n \neq 3 \).

Proof It follows by Construction 2.5 and Lemma 3.2.

Theorem 3.4. There is a pair of SPOLS(\( n \)) for any odd \( n \geq 3 \).

Proof It follows by Lemma 3.1 and Lemma 3.3.

Case 3. \( n \equiv 0 \pmod{4} \)
Let \( n = 4k, k \geq 1 \). We should deal with the case \( k \equiv 2 \pmod{4} \) and the case \( k \equiv 0, 1, 3 \pmod{4} \).

Lemma 3.5. There is a pair of SPOLS(\( 4k \)) for any positive integer \( k \equiv 2 \pmod{4} \).
Proof The \((D, D')\) given in Section 1 is a pair of SPOLS(8). Lemma 2.3 gives a pair of SPOLS(24). Thus there is a pair of SPOLS(4k), \(k = 2, 6\). For \(k \equiv 2 \pmod{4}\), \(k \geq 10\), there is a pair of SPOLS(\(k\)) by Theorem 3.4. Since there is a pair of SPOLS(8), there is a pair of SPOLS(4k) by Construction 2.1.

Now we consider the case \(k \equiv 0, 1, 3\pmod{4}\).

**Lemma 3.6.** There is an \(S(4k)\) if \(k \equiv 0, 1, 3\pmod{4}\) and \(k > 4\).

**Proof** There is an \(S(4k)\) for \(k = 4, 5, 7\) as follows.

\[
\begin{align*}
k = 4: L &= (0 3 5 6 | 2 1 7 4 | 14 13 11 8 | 12 15 9 10), \\
k = 5: L &= (3 7 10 11 14 | 0 17 6 18 4 | 13 19 1 15 2 | 16 9 5 8 12), \\
k = 7: L &= (1 21 4 12 17 3 | 9 0 8 22 11 2 | 13 | 18 5 27 25 14 19 16 | 10 26 23 15 20 24 6).
\end{align*}
\]

Suppose that there is an \(S(4k)\) for any \(k \geq 4\), \(k \equiv 0, 1, 3\pmod{4}\).

\[
L = (L_0 L_1 L_2 L_3),
\]

where the \(j\)-position of \(L_i\) is \(l_i(j), l \in I_4, j \in I_k\). Let

\[
H = (H_0 H_1 H_2 H_3)
\]

be the \(S(16)\) given above. We shall construct an \(S(4k)\). Let

\[
\tilde{L} = (\tilde{L}_0 \tilde{L}_1 \tilde{L}_2 \tilde{L}_3),
\]

where

\[
\begin{align*}
\tilde{L}_0 &= (h_0(0), h_0(1), l_0(0) + 8, \ldots, l_0(k - 1) + 8, h_0(2), h_0(3)), \\
\tilde{L}_1 &= (h_1(0), h_1(1), l_1(0) + 8, \ldots, l_1(k - 1) + 8, h_1(2), h_1(3)), \\
\tilde{L}_2 &= (h_2(0) + 16, h_2(1) + 16, l_2(0) + 8, \ldots, l_2(k - 1) + 8, h_2(2) + 16, h_2(3) + 16), \\
\tilde{L}_3 &= (h_3(0) + 16, h_3(1) + 16, l_3(0) + 8, \ldots, l_3(k - 1) + 8, h_3(2) + 16, h_3(3) + 16).
\end{align*}
\]

Clearly, the elements of \(\tilde{L}\) run over \(I_{6(k+4)}\), and it satisfy the conditions (R1)-(R3). Thus \(\tilde{L}\) is an \(S(4k)\). The proof is completed by induction.

**Example 3.** By using the \(S(16)\) given in Lemma 3.6 we can get an \(S(32)\) as follows:

\[
L = (0 3 8 11 13 14 5 6 | 2 1 10 9 15 12 7 4 | 30 29 22 21 19 16 27 24 | 28 31 20 23 17 18 25 26).
\]

**Lemma 3.7.** There is a pair of SPOLS(\(4k\)) for \(k \equiv 0, 1, 3\pmod{4}\) and \(k > 4\).

**Proof** Let \(k \equiv 0, 1, 3\pmod{4}, k > 4\). There is a pair of strongly symmetrical OLS(\(k\)) by Lemma 1.1, and there is an \(S(4k)\) by Lemma 3.6. Thus there is a pair of SPOLS(\(4k\)) by Construction 2.4.

**Theorem 3.8.** There is a pair of SPOLS(\(4k\)) for \(k \geq 2\) and \(k \neq 3\).

**Proof** The result follows from Lemma 3.5 and Lemma 3.7.

**Lemma 3.9.** There is no SPOLS(\(4\)).

**Proof** Suppose that there is a pair of SPOLS(\(4\)) over \(I_4, (A, B)\). Let

\[
U = 4A + B = \begin{pmatrix}
a_1 & b_1 & a_2 & b_2 \\
c_1 & d_1 & c_2 & d_2 \\
a_3 & b_3 & a_4 & b_4 \\
c_3 & d_3 & c_4 & d_4
\end{pmatrix}.
\]
Let
\[ e = a_1 + a_2 + a_3 + a_4, \quad f = b_1 + b_2 + b_3 + b_4, \]
\[ g = c_1 + c_2 + c_3 + c_4, \quad h = d_1 + d_2 + d_3 + d_4. \]

Then \( e + f = e + g = 60 \), hence \( f = g \). Since \( A \) and \( B \) are strongly symmetrical, we have \( b_1 + c_4 = c_1 + b_4 = b_2 + c_3 = c_2 + b_3 = 15 \). Thus \( f + g = 60 \). So, \( f = g = 30 \). On the other hand, since \( A \) is weakly pandiagonal, \( b_2 + c_2 + b_3 + c_3 = b_2 + c_1 + b_3 + c_4 = 30 \). So \( c_2 + c_3 = c_1 + c_4 \), \( c_2 + c_4 = 2(c_1 + c_4) = 30 \), it follows that \( c_1 + c_4 = 15 \). Previously we have proved that \( c_1 + b_4 = 15 \), so \( c_4 = b_4 \), a contradiction. Thus there is no \( \text{SPOLS}(4) \).

**Proof of Theorem 1.3** The proof follows by combining Theorem 3.4, Theorem 3.8 and Lemma 3.9.

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