ON LOCALIZATION OF SCHRÖDINGER MEANS

PER SJÖLIN

Abstract. Localization properties for Schrödinger means are studied in dimension higher than one.

1. Introduction

Let \( f \) belong to the Schwartz class \( \mathcal{S}(\mathbb{R}^n) \) where \( n \geq 1 \). We define the Fourier transform \( \hat{f} \) by setting

\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) dx, \quad \xi \in \mathbb{R}^n.
\]

For \( f \in \mathcal{S}(\mathbb{R}^n) \) we also set

\[
S_t f(x) = \int_{\mathbb{R}^n} e^{i\xi \cdot x} e^{it\|\xi\|^2} \hat{f}(\xi) d\xi, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}.
\]

If we set \( u(x,t) = S_t f(x)/(2\pi)^n \), then \( u(x,0) = f(x) \) and \( u \) satisfies the Schrödinger equation \( i\partial u/\partial t = \Delta u \).

It is well-known that \( e^{it\|\xi\|^2} \) has the Fourier transform \( K(x) = ce^{-i|x|^2/4} \), where \( c \) denotes a constant, and \( e^{it\|\xi\|^2} \) has the Fourier transform

\[
K_t(x) = \frac{1}{t^{n/2}} K\left( \frac{x}{t^{1/2}} \right), \quad x \in \mathbb{R}^n, \quad t > 0.
\]

One has \( S_t f(x) = K_t \ast f(x) \) for \( f \in \mathcal{S}(\mathbb{R}^n) \) and \( t > 0 \), and we set

\[
S_t f(x) = K_t \ast f(x), \quad t > 0,
\]

for \( f \in L^1(\mathbb{R}^n) \). For \( f \in L^2(\mathbb{R}^n) \) we define \( S_t f \) by formula (1.1).

We introduce Sobolev spaces \( H_s = H_s(\mathbb{R}^n) \) by setting

\[
H_s = \{ f \in S' : \| f \|_{H_s} < \infty \}, \quad s \in \mathbb{R},
\]

where

\[
\| f \|_{H_s} = \left( \int_{\mathbb{R}^n} (1 + \| \xi \|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2}
\]

In the case \( n = 1 \) it is well-known (see Carleson [3] and Dahlberg and Kenig [4]) that

\[
\lim_{t \to 0} \frac{1}{2\pi} S_t f(x) = f(x)
\]

almost everywhere if \( f \in H_{1/4} \). Also it is known that \( H_{1/4} \) cannot be replaced by \( H_s \) if \( s < 1/4 \). In the case \( n \geq 2 \) Sjölin [6] and Vega [9] proved independently that

\[
\lim_{t \to 0} \frac{1}{(2\pi)^n} S_t f(x) = f(x)
\]

almost everywhere if \( f \in H_s, \) and \( s > 1/2 \). This result was improved by Bourgain [1], who proved that \( f \in H_s(\mathbb{R}^n), \) \( s > 1/2 - 1/4n \), is sufficient for convergence almost everywhere.
In the case $n = 2$, Du, Guth, and Li [5] have recently proved that the condition $s > 1/3$ is sufficient. On the other hand Bourgain [2] has proved that $s \geq n/2(n + 1)$ is necessary for convergence for all $n \geq 2$.

We shall here study localization of Schrödinger means and shall first state a result on localization everywhere (see Sjölin [7]).

**Theorem A.** Assume $n \geq 1$. If $f \in H_{n/2}(\mathbb{R}^n)$ and $f$ has compact support then
\[
\lim_{t \to 0} S_t f(x) = 0
\]
for every $x \in \mathbb{R}^n \setminus (\text{supp} f)$.

It is also proved in [7] that this result is sharp in the sense that $H_{n/2}$ cannot be replaced by $H_s$ with $s < n/2$.

We say that one has localization almost everywhere for functions in $H_s$ if for every $f \in H_s$ one has
\[
\lim_{t \to 0} S_t f(x) = 0
\]
almost everywhere in $\mathbb{R}^n \setminus (\text{supp} f)$.

In the case $n = 1$ Sjölin and Soria proved that there is no localization almost everywhere for functions in $H_s$ if $s < 1/4$ (see Sjölin [8]). In fact they proved that there exist two disjoint compact intervals $I$ and $J$ in $\mathbb{R}$ and a function $f$ which belongs to $H_s$ for every $s < 1/4$, with the properties that $\text{supp} f \subset I$ and for every $x \in J$ one does not have
\[
\lim_{t \to 0} S_t f(x) = 0.
\]

In the case $n \geq 2$ Sjölin and Soria also proved that one does not have localization almost everywhere for functions in $H_s(\mathbb{R}^n)$ if $s < 1/4$.

We shall here improve this result and prove that there is no localization almost everywhere for functions in $H_s(\mathbb{R}^n)$ if $n \geq 2$ and $s < n/2(n + 1)$. In fact we shall prove the following theorem.

**Theorem 1.1.** If $n \geq 2$ and $s < n/2(n + 1)$ there exist a function $f$ in $H_s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and a set $F$ with positive Lebesgue measure such that $F \subset \mathbb{R}^n \setminus (\text{supp} f)$ and for every $x \in F$ one does not have $\lim_{t \to 0} S_t f(x) = 0$.

To prove this result we shall combine the method in [8] with an estimate of Bourgain [2].

If $A$ and $B$ are numbers we write $A \lesssim B$ if there is a positive constant $C$ such that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$ we write $A \sim B$.

We introduce the inverse Fourier transform by setting
\[
\tilde{f}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi \cdot x} f(\xi) d\xi, \quad x \in \mathbb{R}^n,
\]
for $f \in L^1(\mathbb{R}^n)$.

Also $B(x; r)$ denotes a ball with center $x$ and radius $r$.

2. **Proof of the theorem**

We start by taking $v_1$ such that $0 < v_1 < 1$ and set $v_k = \varepsilon_k v_{k-1}^\mu$ for $k = 2, 3, 4, \ldots$, where $\varepsilon_k = 2^{-k}$ and $\mu = \max(n, 2 + n/4)$. Then one has $v_k < 2^{-k}$ for $k \geq 2$ and $(v_k)_1^{\infty}$ is a decreasing sequence tending to zero.

We then choose $g \in S(\mathbb{R})$ such that $\text{supp} \tilde{g} \subset (-1, 1)$, $\tilde{g}(x_1) = 1$ for $|x_1| < 1/2$, and set $f_v(x_1) = e^{-ix_1/v^2} \tilde{g}(x_1/v)$, $0 < v < 1$, $x_1 \in \mathbb{R}$. 
The functions $f_v$ were used in Sjölin [8] to study the localization problem in the case $n = 1$. Also let $\Phi \in S(\mathbb{R}^n)$ have $\text{supp} \, \Phi \subset B(0; 1)$ and $\Phi(0) = 1$. We then take $R = 1/v^2$ and set
\[
G_v(x') = R^{-(n-1)/4} \Phi(x') \prod_{j=2}^n \left( \sum_{R/2D < l_j < R/D} e^{iDl_j x_j} \right), \quad 0 < v < 1,
\]
where $x' = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$, $l = (l_2, \ldots, l_n) \in \mathbb{Z}^{n-1}$, and $D = R^{(n+2)/(n+1)}$. We may also assume that $\Phi(x') = \psi(x_2) \cdots \psi(x_n)$ for some $\psi \in S(\mathbb{R})$.

We then set
\[
h_v(x) = h_v(x_1, x') = f_v(x_1)G_v(x'), \quad 0 < v < 1.
\]
In [2] Bourgain studies functions similar to $h_v$. However, in [2] our function $\tilde{g}$ is replaced by a function $\varphi$ with the property that $\tilde{\varphi}$ has compact support. In our argument it will be important that $\tilde{g}$ has compact support so that
\[
\text{supp} \, f_v \subset (-v, v).
\]
We then observe that
\[
S_th_v(x_1, x') = S_t f_v(x_1)S_t G_v(x').
\]
It is proved in Bourgain [2], p.394, that if one assumes $|x| < c$ and $|t| < c/R$ and sets
\[
t = \frac{x_1}{2R} + \tau
\]
with $|\tau| < R^{-3/2}/10$, then
\[
|S_t f_v(x_1)| \gtrsim |\tilde{g}(R^{1/2}x_1 - 2tR^{3/2})| = |\tilde{g}(2\tau R^{3/2})| \gtrsim c_0.
\]
We then take $v = v_k$ for $k = 1, 2, 3, \ldots$, and apply an estimate in [2], p.395, namely that there exists a set $E_k \subset B(0; 1)$ such that for every $x \in E_k$ there exists $t = t_k(x)$ such that
\[
|S_{t_k(x)} G_{v_k}(x')| \gtrsim R^{-(n-1)/4} \prod_{j=2}^n \left| \sum_{l_j} e^{iDl_j x_j} e^{iD^2l_j t} \right| \gtrsim c_0.
\]
Also one has $mE_k \geq c_1 > 0$, where $m$ denotes Lebesgue measure.

We then choose $\delta > 0$ so small that if $F_k = E_k \cap \{x; \ |x| > \delta/2\}$ then one has $mF_k \geq c_1/2 = c_2$ for $k = 1, 2, 3, \ldots$. We may assume that $\delta < 1$.

We then set $F = \bigcap_{k=1}^{\infty} \left( \bigcup_{j=k}^{\infty} F_j \right)$ so that $F$ is the set of all $x$ which belong to infinitely many $F_k$. Since the sets $\bigcup_{j=k}^{\infty} F_j$ decrease as $k$ increases, one obtains $mF \geq c_0$.

It also follows from (2.2) that $|S_{t_k(x)} f_{v_k}(x)| \geq c_0$ for $x \in F_k$. From (2.1) one also concludes that
\[
|t_k(x)| \sim \frac{1}{R_k} = v_k^2
\]
where $R_k = 1/v_k^2$.

We now choose $K$ so large that $v_K < \delta/4$ and set
\[
h(x) = \sum_{k=K}^{\infty} h_{v_k}(x) = \sum_{k=K}^{\infty} f_{v_k}(x_1)G_{v_k}(x').
\]
One has $F \subset \mathbb{R}^n \setminus (\text{supp} \, h)$ since $\text{supp} \, h \subset \{x; \ |x| < \delta/4\}$.

If $x = (x_1, x') \in F$ there exists a sequence $(k_j)_{j=1}^{\infty}$ such that $x \in F_{k_j}$ and
\[
|S_{t_{k_j}} G_{v_{k_j}}(x')| \geq c_0 \quad \text{and} \quad |S_{t_{k_j}} f_{v_{k_j}}(x_1)| \geq c_0.
\]
We shall prove that
\[ |S_{t,k_j}(x)h(x)| \geq c_0 \]
for \( j \) large, and also that \( h \in H_s \cap L^1 \) for \( s < n/(2(n+1)) \). It follows that one does not have localization almost everywhere in \( H_s \) if \( s < n/(2(n+1)) \).

We shall first estimate \( \|h_v\|_{H_s} \). We begin by studying \( f_v \) and \( G_v \). According to Sjölin [8], p. 143, one has
\[ \hat{f}_v(\xi) = v g(v\xi + 1/v) = R^{-1/2} g(R^{-1/2} \xi_1 + R^{1/2}) \]
and for \( |\xi| \geq AR \) we have
\[ R^{-1/2} |\xi_1| \geq R^{1/2} A, \]
where \( A \) is a large constant. It follows that \( |\xi_1| \geq AR \) implies
\[ |R^{-1/2} \xi_1 + R^{1/2}| \geq BR^{-1/2} |\xi_1| \]
and
\[ |\xi| \geq |\xi_1|^{1/2} A_1 R^{1/2}. \]
Hence
\[ |R^{-1/2} \xi_1 + R^{1/2}| \geq B_1 |\xi_1|^{1/2} \]
and
\[ |\hat{f}_v(\xi)| \leq C |\xi|^{-N} \quad \text{for} \quad |\xi| \geq AR, \]
where \( N \) is large. It follows that
\[ (2.4) \quad \int_{|\xi_1| \geq AR} |\hat{f}_v(\xi_1)|^2 (1 + |\xi_1|^2)^{\ast} d\xi_1 \leq CR^{-N} \]
and it is also easy to see that
\[ (2.5) \quad \|f_v\|_2 \sim v^{1/2}. \]

We have
\[ \hat{G}_v(\xi') = R^{-(n-1)/4} \sum_l \hat{\Phi}(\xi' - D l) \]
and it follows that \( \text{supp} \hat{G}_v \subset \{ \xi' : |\xi'| \sim R \} \) and
\[ |\hat{G}_v(\xi')|^2 = R^{-(n-1)/2} \sum_l |\hat{\Phi}(\xi' - D l)|^2. \]
Integrating we obtain
\[ \|G_v\|_2^2 = R^{-(n-1)/2} \sum_l \|\hat{\Phi}\|_2^2 \sim R^{-(n-1)/2} \left( \frac{R}{D} \right)^{n-1}. \]
We have \( D = R^{(n+2)/2(n+1)} \) and \( R/D = R^{n/2(n+1)} \) and hence
\[ \|G_v\|_2^2 \sim R^{-(n-1)/2} R^{n(n-1)/2(n+1)} = R^{-(n-1)/2(n+1)} \]
and
\[ (2.6) \quad \|G_v\|_2 \sim R^{-(n-1)/4(n+1)}. \]
For \( s > 0 \) one obtains

\[
\|h_v\|^2_{H_s} \sim \int |\hat{h}_v(\xi)|^2 |\xi|^{2s} d\xi = \int_{|\xi| < R} |\hat{h}_v(\xi_1)|^2 |\hat{G}_v(\xi')|^2 |\xi|^{2s} d\xi \lesssim \\
\int_{|\xi'| < R} |\hat{h}_v(\xi_1)|^2 |\hat{G}_v(\xi')|^2 |\xi'|^{2s} d\xi' + \int_{|\xi'| \geq AR} |\hat{h}_v(\xi_1)|^2 |\hat{G}_v(\xi')|^2 |\xi|^{2s} d\xi_1 \xi' = I_1 + I_2.
\]

It follows that

\[
I_1 \lesssim R^{2s} \left( \int_{|\xi'| \leq AR} |\hat{h}_v(\xi_1)|^2 d\xi_1 \right) \left( \int |\hat{G}_v(\xi')| d\xi' \right) \lesssim R^{2s} ||f_v||^2_G \lesssim R^{2s} R^{-1/2} R^{-(n-1)/2(n+1)} = R^{2s-n/(n+1)}.
\]

From (2.4) and (2.6) one also obtains

\[
I_2 \lesssim R^{-N}
\]

and hence

\[
I_1 + I_2 \lesssim R^{2s-n/(n+1)}.
\]

It follows that

\[
\|h_v\|_{H_s} \lesssim R^{s-n/2(n+1)} = v^{2(n/2(n+1)-s)}.
\]

Since \( v_k < \varepsilon_k \) and

\[
\|h_k\|_{H_s} \lesssim \sum K \|h_{v_k}\|_{H_s} \lesssim \sum K v_k^{2(n/2(n+1)-s)} < \infty,
\]

it follows that \( h \in H_s \) if \( s < n/2(n+1) \).

We also need some estimates for \( S_t f_v \) and \( S_t G_v \). In Sjölin [8] (see Lemmas 3 and 4) it is proved that

\[
|S_t f_v(x_1)| \lesssim \frac{v}{|t|^{1/2}}
\]

and

\[
|S_t f_v(x_1)| \lesssim \frac{|t|}{v^2}
\]

for \( 0 < v < \delta/4 \), \( |x_1| > \delta/2 \), and \( 0 < |t| < 1 \). Actually it is assumed in [8] that \( t > 0 \) but the same proofs work for \( t < 0 \).

We also have the following estimates for \( S_t G_v \).

**Lemma 2.1.** For \( 0 < v < \delta/4 \), \( 0 < |t| < 1 \), and \( x' \in \mathbb{R}^{n-1} \) one has

\[
|S_t G_v(x')| \lesssim v^{(n-1)/2} \log 1/v)^{n-1} |t|^{-(n-1)/2}
\]

and

\[
|S_t G_v(x')| \lesssim v^{-(n-1)/2(n+1)}.
\]

**Proof.** Choose the integer \( p \) so that \(|4p - R/D| \leq 4\). Then one has

\[
\sum_{R/2D < t < R/D} e^{id_{t|x}} = \sum_{2p} e^{iD|x} + O(1)
\]
and 
\[
\left| \sum_{2p}^{4p} e^{itDx_j} \right| = \left| \sum_{-p}^{p} e^{iD(t+3p)x_j} \right| = \left| \sum_{-p}^{p} e^{itDx_j} \right| = D_p(Dx_j),
\]
where \(D_p\) denotes the Dirichlet kernel. Setting \(y = Dx_j\) one obtains
\[
\int_{a}^{a+1/D} |D_p(Dx_j)|dx_j = \int_{Da}^{Da+1} |D_p(y)|dy \lesssim \frac{1}{D} \log p \sim \frac{1}{D} \log R
\]
for every \(a \in \mathbb{R}\). It follows that
\[
\int_{a}^{a+1} |D_p(Dx_j)|dx_j \lesssim \log R
\]
for every \(a \in \mathbb{R}\).

Letting \(Q\) denote the cube \([0, 1]^{n-1}\) we obtain
\[
\|G_v\|_1 = \int_{\mathbb{R}^{n-1}} |G_v(x')|dx' = \sum_{m \in \mathbb{Z}^{n-1}} \int_{m+Q} |G_v(x')|dx' \lesssim \frac{1}{n+Q} \left( \prod_{m} \left| \sum_{j=2}^{n} e^{itjDx} \right| \right)dx' \lesssim R^{-\frac{n-1}{4}} \sum_{m} \frac{1}{1 + |m|^N} R^{-\frac{n-1}{4}} \sum_{m} \frac{1}{1 + |m|^N} (\log R)^{n-1}
\]
and hence
\[
\|G_v\|_1 \lesssim R^{-\frac{n-1}{4}} (\log R)^{n-1} \sim v^{(n-1)/2} (\log 1/v)^{n-1}.
\]
We have
\[
S_t G_v(x') = \int K_t(x' - y')G_v(y')dy'
\]
where
\[
|K_t(y')| \lesssim |t|^{-(n-1)/2}
\]
and it follows that
\[
|S_t G_v(x')| \lesssim |t|^{-(n-1)/2} \|G_v\|_1
\]
and hence we obtain (2.9).

We also have
\[
S_t G_v(x') = \int_{\mathbb{R}^{n-1}} e^{itx'} e^{it|\xi'|^2} \hat{G}_v(\xi') \xi' dt = R^{-\frac{n-1}{4}} \sum_{l} \int_{\mathbb{R}^{n-1}} e^{itx'} e^{it|\xi'|^2} \hat{\Phi}(\xi' - DL) d\xi'
\]
and we get
\[
|S_t G_v(x')| \lesssim R^{-\frac{n-1}{4}} \sum_{l} ||\hat{\Phi}||_1 \lesssim R^{-\frac{n-1}{4}} (R/D)^{n-1} = R^{(n-1)^2/4(n+1)} = v^{-(n-1)^2/2(n+1)}
\]
and the proof of Lemma 2.1 is complete. \(\square\)
Multiplying (2.7) and (2.9) one obtains
\[
|S_t h_v(x)| \lesssim v^{(n+1)/2}(\log 1/v)^{n-1}|t|^{-n/2} \lesssim \frac{v}{|t|^{n/2}} = \frac{v}{|t|^\gamma}
\]
and combining (2.8) and (2.10) one gets
\[
|S_t h_v(x)| \lesssim \frac{|t|}{v^{4+(n-1)^2/2(n+1)}} \lesssim \frac{|t|}{v^{4+n/2}} = \frac{|t|}{v^\beta}
\]
where \( \gamma = n/2, \beta = 4 + n/2, 0 < \gamma < \delta/4, 0 < |t| < 1, \) and \( |x_1| > \delta/2. \)

We remark that it also follows from the above estimates that \( h \in L^1(\mathbb{R}^n). \) In fact it is easy to see that \( ||f_v||_1 \sim v \) and we have proved that
\[
||G_v||_1 \leq v^{(n-1)/2}(\log 1/v)^{n-1} \leq v^{1/4}
\]
and hence \( ||h_v||_1 \leq v^{5/4}. \) It follows that
\[
||h||_1 \leq \sum_{K}^\infty ||h_{v_k}||_1 < \infty.
\]

We shall now finish the proof of the following result.

**Theorem 2.2.** For \( x \in F \) one has
\[
|S_{t_{k_j}} h(x)| \geq c_0 > 0
\]
for \( j \) large. It follows that there is no localization almost everywhere of Schrödinger means for functions in \( H_s(\mathbb{R}^n) \) if \( s < n/2(n+1). \)

**Proof.** We have
\[
|S_{t_{k_j}} h(x)| \geq |S_{t_{k_j}} h_{v_{k_j}}(x)| - \left| \sum_{i \neq k_j} S_{t_{k_j}} h_{v_i}(x) \right|.
\]
The first term on the right hand side is larger than a positive number \( c_0 \) and it suffices to prove that the second term is small. For simplicity we write \( k \) instead of \( k_j \) in the following formulas.

We have \( 0 < v_i \leq v_{i-1}/2 \) and \( 0 < v_i \leq 2^{-i} \) and it follows that
\[
\sum_{i=k+1}^\infty v_i \leq 2v_{k+1}
\]
and one also has
\[
\sum_{i=K}^{k-1} \frac{1}{v_i^\beta} \lesssim \frac{1}{v_{k-1}^\beta}
\]
since \( 1/v_{i-1} \leq 1/2v_i \) implies \( 1/v_i^\beta \leq \frac{1}{2^{i-1}} \cdot \frac{1}{v_{i-1}^\beta}. \)

For \( i \geq k + 1 \) the inequality (2.11) and the formula (2.3) give
\[
|S_{t_k} h_{v_i}(x)| \leq \frac{v_i}{(v_k^\gamma)} = \frac{v_i}{v_k^\gamma}
\]
and invoking (2.13) we obtain
\[
\sum_{i=k+1}^\infty |S_{t_k} h_{v_i}(x)| \leq \frac{v_{k+1}}{v_k^\gamma} \leq \varepsilon_{k+1},
\]
where we have used the fact that \( v_{k+1} \leq v_k \) and \( v_k \leq 2^{-i} \).
since \( \mu \geq 2\gamma \) and \( v_{k+1} = \varepsilon_{k+1}v_k^\mu \leq \varepsilon_{k+1}v_k^{2\gamma} \). For \( K \leq i \leq k-1 \) the inequality (2.12) gives
\[
|S_{tk}(x)h_{v_i}(x)| \leq \frac{|t_k(x)|}{v_i^\beta} \leq \frac{v_k^2}{v_i^\beta}
\]
and invoking (2.14) one obtains
\[
\sum_{i=K}^{k-1} |S_{tk}(x)h_{v_i}(x)| \leq v_k^2 \sum_{i=K}^{k-1} 1 \leq \frac{v_k^2}{v_{k-1}^\beta}.
\]
Since \( \mu \geq \beta/2 \) we get \( v_k \leq \varepsilon_kv_{k-1}^{\beta/2} \) and \( v_k^2 \leq \varepsilon_k^2v_{k-1}^\beta \). Hence the sum in (2.15) is majorized by \( C\varepsilon_k^2 \). Since \( \varepsilon_k \to 0 \) as \( k \to \infty \) the proof of the theorem is complete.

\[\Box\]

**References**

[1] J. Bourgain, On the Schrödinger maximal function in higher dimensions, Proc. Steklov Inst. Math. 280 (2013), 46-60.

[2] J. Bourgain, A note on the Schrödinger maximal function, J. Anal. Math. 130 (2016), 393-396.

[3] L. Carleson, Some analytical problems related to statistical mechanics, in Euclidean Harmonic Analysis, Lecture Notes in Math. 779 (1979), 5-45.

[4] B.E.J. Dahlberg, and C.E. Kenig, A note on the almost everywhere behaviour of solutions to the Schrödinger equation, in Harmonic Analysis, Lecture Notes in Math. 908 (1982), 205-209

[5] X. Du, L. Guth, and X. Li, A sharp Schrödinger maximal estimate in \( \mathbb{R}^2 \), arXiv:1612.08946v1

[6] P. Sjölin, Regularity of solutions to the Schrödinger equation, Duke Math. J. 55 (1987), 699-715.

[7] P. Sjölin, Some remarks on localization of Schrödinger means, Bull. Sci. Math. 136 (2012), 638-647

[8] P. Sjölin, Nonlocalization of operators of Schrödinger type, Ann. Acad. Sci. Fenn. Math. 38 (2013), 141-147.

[9] L. Vega, Schrödinger equations: pointwise convergence to the initial data, Proc. Amer. Math. Soc. 102 (1988), 874-878.

**Department of Mathematics, KTH Royal Institute of Technology, 100 44 Stockholm, Sweden**

E-mail address: persj@kth.se