Dirac versus reduced phase space quantization for systems admitting no gauge conditions

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Abstract

The constrained Hamiltonian systems admitting no gauge conditions are considered. The methods to deal with such systems are discussed and developed. As a concrete application, the relationship between the Dirac and reduced phase space quantizations is investigated for spin models belonging to the class of systems under consideration. It is traced out that the two quantization methods may give similar, or essentially different physical results, and, moreover, a class of constrained systems, which can be quantized only by the Dirac method, is discussed. A possible interpretation of the gauge degrees of freedom is given.
1 Introduction

Nowadays there are two main methods to quantize the Hamiltonian systems with first class constraints: the Dirac quantization [1] and the reduced phase space quantization [2], while the path integral method [3, 2] and the BRST quantization, being the most popular method for the covariant quantization of gauge–invariant systems [4], are based on and proceed from them [2, 3, 6]. The basic idea of the Dirac method consists in imposing quantum mechanically the first class constraints as operator conditions on the states for singling out the physical ones [1]. The reduced phase space quantization first identifies the physical degrees of freedom at the classical level by the factorization of the constraint surface with respect to the action of the gauge group, generated by the constraints. Then the resulting Hamiltonian system is quantized as a usual unconstrained system [2]. Naturally, the problem of the relationship of these two methods arises. It was discussed in different contexts in literature (see [7]–[13], and references therein), and it was demonstrated for the concrete examples that these two methods give, in general, different results. The corresponding quantum systems may differ, e.g., in their energy level structure, or in the eigenvectors of their Hamiltonians [9, 10]. Nevertheless, there is an opinion that the differences between the two quantization methods can be traced out to a choice of factor ordering in the construction of various physical operators.

In the present paper we investigate the relationship of the two methods of quantization for the special class of Hamiltonian systems with first class constraints corresponding to different physical models of spinning particles. The specific general property of the considered examples of constrained systems is the following: their constraints generate \( \text{SO}(2) \) transformations and, hence, corresponding gauge orbits topologically are one-spheres \( S^1 \). This fact implies that these systems do not admit gauge conditions. Nevertheless, for the first three systems we can construct the corresponding reduced phase spaces. We shall demonstrate that the two methods of quantization give coinciding physical results only for the first of these systems, but they give, in general, essentially different physical results for two other systems.

Since in the present paper we consider the constrained systems which do not admit gauge conditions, we should use a general geometrical approach to the Dirac–Bergmann theory of the constrained systems for the construction of the reduced phase spaces. This approach is reviewed in Section 2 of our paper. Note that from the point of view of this approach, it is not necessary to use the notion of the weak equality of the functions on the phase space, and this notion is not used in our paper at all.

The first concrete example, we consider, is the plane spin model, which is a subsystem of the (3+1)–dimensional models of massless particles with arbitrary helicity [14], and of the (2+1)–dimensional relativistic models of fractional spin particles [15]. This model is related also to the simplest nonrelativistic model of the anyon [16]. The classical and quantum treatment of the model having a cylinder \( S^1 \times \mathbb{R} \) as the initial phase space, is performed in Section 3. Here we trace out an interesting analogy in interpretation of the situation with nonexistence of a global gauge condition for this simple constrained system with the situation taking place for the non-Abelian gauge theories [17].

The second model, which is considered in Section 4, is the rotator spin model, being a subsystem of the (3+1)–dimensional model of the relativistic massive particle with integer
spin [15]. The nontriviality of the model reveals itself in the topology of the constraint surface diffeomorphic to the group manifold of the Lie group SO(3).

Section 5 is devoted to the consideration of the top spin model, which, in turn, is a subsystem in the non–Grassmannian model of the relativistic massive particle with arbitrary integer or half–integer spin [19]. This system is also topologically nontrivial and has the cotangent bundle $T^*\text{SO}(3)$ as the initial phase space.

In section 6 we summarize the results, present conclusions on the relationship of the two methods of quantization and formulate a possible interpretation of the sense of gauge degrees of freedom for constrained systems. Here we also discuss a broad class of pseudoclassical models [20]–[24], i.e., models containing Grassmann variables, whose general feature consists in the presence of the constraints, nonlinear in Grassmann variables and having no nontrivial projections on the unit of Grassmann algebra. For such systems it is not possible to construct the reduced phase space in principle, and they can be quantized only by the Dirac method.

Appendixes A and B are devoted to the presentation of the method of dependent coordinates used in the paper. Actually this method is, in a sense, a reformulation of the Dirac bracket method, but from our point of view it has, in comparison with the former, greater mathematical definiteness, and turns out to be more convenient for our purposes.

Appendix C collects the facts about three dimensional rotation group, which are needed in the main text.

Everywhere in the text repeated indices imply the corresponding summation.

2 Geometrical Background of the Dirac–Bergmann Theory

As it has been noted in the Introduction, the systems we consider in the present paper do not admit gauge conditions. To construct the reduced phase space for them we should use a general geometrical approach (see [25], and references therein). This approach together with its relation to the usual Dirac–Bergmann formalism is briefly described below.

Let $(M, \omega^M)$ be an $m$–dimensional symplectic manifold, and $N$ be an $n$–dimensional submanifold of $M$. Let $\iota : N \to M$ be the inclusion mapping. The two–form $\omega^N = \iota^* \omega^M$ is obviously closed. Suppose that $\omega^N$ has a constant rank, then the set

$$E_{\omega^N} = \{ x \in TN \mid i(x)\omega^N = 0 \}$$

is a distribution on $N$, called the characteristic distribution of $\omega^N$. As $\omega^N$ is closed, $E_{\omega^N}$ is integrable and by the Frobenius theorem defines a foliation $\mathcal{F}_{\omega^N}$ on $N$. Identifying all points of a leaf we get the quotient space $P = N/\mathcal{F}_{\omega^N}$. Assume that $P$ is a manifold with the canonical projection $\pi : N \to P$ being a submersion. In other words, we suppose that $\pi : N \to P$ is a fibered manifold. In this case it can be shown that there exists a closed nondegenerate two–form $\omega^P$, such that

$$\pi^* \omega^P = \omega^N.$$  \hspace{1cm} (2.1)

Thus, the manifold $P$ has a natural structure of a symplectic manifold.
In the Dirac–Bergmann theory the submanifold \( N \) is a manifold defined by equations

\[
\psi_a = 0, \quad a = 1, \ldots, m - n.
\]

It means that

\[
N = \{ p \in M \mid \psi_a(p) = 0, \ a = 1, \ldots, m - n \},
\]

where \( \psi_a \) are differentiable functions on \( M \), such that the mapping \( \psi : M \to \mathbb{R}^{m-n} \), defined by \( p \to (\psi_1(p), \ldots, \psi_{m-n}(p)) \) is of rank \( m - n \) for every \( p \in N \).

Let \( x \in T_pN \), then

\[
d\psi_a(\iota_*(x)) = 0, \quad a = 1, \ldots, m - n.
\]

On the other hand, suppose that \( y \in T_{\iota(p)}M, \ p \in N \), and

\[
d\psi_a(y) = 0, \quad a = 1, \ldots, m - n,
\]

then there exists a unique vector \( x \in T_pN \), such that \( y = \iota_*(x) \).

Recall that the Hamiltonian vector field \( X_f \) on \( M \), corresponding to the function \( f \) on \( M \), is defined by

\[
i(X_f)\omega^M = df,
\]

and for the Poisson bracket of two functions, \( f \) and \( g \), on \( M \) we have the expression

\[
\{ f, g \} = -\omega^M(X_f, X_g) = X_f(g) = -X_g(f).
\]

Introduce the matrix valued function \( \Delta = \|\Delta_{ab}\| \), where

\[
\Delta_{ab} = \{\psi_a, \psi_b\},
\]

and define

\[
\tilde{\Delta}_{ab} = \iota^*\Delta_{ab}.
\]

Suppose, that the matrix \( \tilde{\Delta} \) is degenerate at some point \( p \in N \). Hence we can find a nontrivial set of real numbers \( c^a, \ a = 1, \ldots, m - n, \) such that

\[
c^a\tilde{\Delta}_{ab}(p) = 0.
\]

From Eq. (2.4) we get

\[
c^aX_{\iota(p)}(\psi_b) = d\psi_{\iota(p)}(c^aX_a) = 0,
\]

where we have denoted \( X_{\psi_a} \) through \( X_a \). Thus, there exists a vector \( x \in T_pN \), such that

\[
\iota_*(x) = c^aX_{\iota(p)}.
\]

Let us show that the vector \( x \) belongs to \( \mathcal{E}_{\omega N} \). Indeed, for any \( x' \in T_{x'}N \) we have

\[
[i(x)\omega^N](x') = \omega^N(x, x') = \omega^M(\iota_*(x), \iota_*(x')).
\]

Using Eq. (2.5), we get

\[
[i(x)\omega^N](x') = c^a\omega^M(X_a, \iota_*(x')) = c^ad\psi_a(\iota_*(x')) = 0.
\]
That was to be demonstrated.

On the other hand, suppose that \( x \in E_\omega N \). If \( x \in T_p N \), then for any \( x' \in T_p N \) we have

\[
0 = [i(x)\omega^N](x') = \omega^M(\iota_*(x), \iota_*(x')) = [i(\iota_*(x))\omega^M](\iota_*(x')).
\]

Note that if a one–form \( \mu \in T^*_p(p)M \) is such that

\[
\mu(\iota_*(x)) = 0
\]

for all \( x \in T_p N \), then

\[
\mu = c^a d\psi_{\alpha(p)}
\]

for some constants \( c^a, a = 1, \ldots, m - n \). Hence, we have

\[
\iota_*(x) = c^a X_{\alpha(p)},
\]

and

\[
0 = d\psi_b(\iota_*(x)) = c^a X_{\alpha(p)}(\psi_b) = c^a \Delta_{ab}(p).
\]

Thus, we see that the elements of \( E_\omega N \) and the null vectors of the matrices \( \Delta(p), p \in N, \) are in one–to–one correspondence, and the two–form \( \omega^N \) is of constant rank if and only if the rank of \( \Delta(p) \) is independent of \( p \).

Recall that in the Dirac–Bergmann theory the functions \( \psi_\alpha \) are constraints, the submanifold \( N \) is called the constraint surface, and the symplectic manifold \( P \) is called the reduced phase space.

A function \( f \) on \( M \) is said to be of first class, if

\[
\{ f, \psi_\alpha \} = f^b_\alpha \psi_b
\]

for some functions \( f^b_\alpha \). A constraint being a function of first class, is called the first class constraint. Suppose that the matrix \( \Delta(p) \) is of rank \( k \) at any point \( p \in M \). Let the indices \( \alpha, \beta, \gamma \) take values from 1 to \( k \), while the index \( \mu \) takes values from \( k+1 \) to \( m-n \). Without a loss of generality, we can consider, in fact locally, that the submatrix \( \|\Delta_{\alpha\beta}(p)\| \) is nondegenerate at any point \( p \in M \). In this case we can introduce the functions \( \Delta^{\alpha\beta} \) satisfying the relation

\[
\Delta^{\alpha\gamma}(p)\Delta_{\gamma\beta}(p) = \delta^\alpha_\beta
\]

for all \( p \in N \). It is clear that the functions

\[
\phi_\alpha = \psi_\alpha,
\]

\[
\phi_\mu = \psi_\mu - \{\psi_\mu, \psi_\alpha\}\Delta^{\alpha\beta}\psi_\beta
\]

form an equivalent set of constraints defining the constraint surface \( N \).

The constraints \( \phi_\mu \) are first class constraints. It can be easily shown that the Hamiltonian vector fields \( X_{\phi_\mu} \) are tangent to \( N \), and these fields restricted to \( N \) form a basis of the characteristic distribution \( E_\omega N \). The main property of the constraints \( \phi_\alpha \), which certainly follows from their definition, is that the matrix formed by the Poisson brackets of \( \phi_\alpha \) is nondegenerate at any point of the constraint surface. The constraints \( \phi_\alpha \) are called second class constraints.
If the matrix $\tilde{\Delta}(p)$ is of zero rank at any point $p \in N$, we have only first class constraints. In this case
\[
\{\psi_a, \psi_b\} = \psi_c f^c_{ab}
\]
for some functions $f^c_{ab}$ on $M$. If the matrix $\tilde{\Delta}(p)$ is nondegenerate at any point of $N$ we deal with second class constraints. In this case the corresponding characteristic distribution is trivial, and the reduced phase space coincides with the manifold $N$. A convenient method to work with second class constraints is the Dirac bracket method, or the method of dependent coordinates (see Appendix B). Note that in this case it is convenient to use upper indices for the constraints.

Let us discuss now the procedure of constructing the reduced phase space for the case of first class constraints, according to the Dirac–Bergmann theory.

Below we identify the tangent space $T_pN$ with the set of vectors from $T_{i(p)}M$, satisfying Eq. (2.2), and consider the submanifold $N$ simply as a subset of $M$. If the matrix $\tilde{\Delta}(p)$ is of rank zero at any point $p \in N$, then the Hamiltonian vector fields $X_a$ belong to the characteristic distribution $E_\omega N$, and generate it. In the case when the functions $f^c_{ab}$ from Eq. (2.6) are constant on $M$, we can try to find a group of canonical transformations $G$ such that the functions $\psi_a$ be its generators. This group is called the gauge group, generated by the constraints $\psi_a$. If such a group exists, the leaves of the foliation, defined by the characteristic distribution, are its orbits.

Suppose that the fibered manifold $\pi : N \to P$ has a global section $s : P \to N$. The differentiable mapping $s$ by definition satisfies the relation
\[
\pi \circ s = \text{id}_P.
\]
It is clear that the image $s(P)$ of the section $s$ can be considered as a submanifold of $N$ that is diffeomorphic to $P$. Furthermore, from Eqs. (2.1) and (2.7) it follows that
\[
s^*\omega^N = \omega^P.
\]
Hence,
\[
s^*\omega^{s(P)} = \omega^P,
\]
where $\omega^{s(P)}$ is the restriction of the two–form $\omega^N$ to the submanifold $s(P)$, and $s$ is considered as a mapping from $P$ to $s(P)$. Thus the submanifold $s(P)$ is a symplectic manifold that is symplectomorphic to $P$, and we can identify $P$ with $s(P)$. Choosing different global sections of the fibered manifold $\pi : N \to P$, we come to different realizations of the reduced phase space as a submanifold of $N$. Note that $N$ is a submanifold of the symplectic manifold $M$, hence we actually have realizations of the reduced phase space as a submanifold of $M$.

Assume now that the reduced phase space, realized as a submanifold of $M$, is defined by the equations
\[
\psi_a = 0, \quad \chi^a = 0, \quad a = 1, \ldots, m - n.
\]
The functions $\chi^a$ are called gauges, or gauge conditions. They can be considered as additional constraints. Denote the full set of the constraints by $\psi^\alpha$, $\alpha = 1, \ldots, 2(m - n)$:
\[
\psi^\alpha = \psi_a, \quad \psi^{\alpha+(m-n)} = \chi^a.
\]
As \( P \) is a symplectic submanifold of \( M \), the matrix \( \tilde{\Xi}(p) \) with the matrix elements \( \tilde{\Xi}^{\alpha\beta} = \iota^*\{\psi^\alpha, \psi^\beta\}(p) \) is nondegenerate at any point \( p \in P \). From this it follows that the matrix

\[
\tilde{\Lambda}^b_a(p) = \iota^*\{\psi_a, \chi^b\}(p)
\]  

must be of rank \( m - n \) at any point \( p \in P \).

Note that a section \( s \) allowing to identify the reduced phase space \( P \) with a symplectic submanifold \( s(P) \) of the initial symplectic manifold \( M \) may not exist. In this case we say that the system under consideration does not admit gauge conditions. In such a situation we can always consider a set of local sections \( s_i : U_i \to N \), covering \( P \). For any \( i \) the set \( s_i(U_i) \) is symplectomorphic to the open symplectic submanifold \( U_i \). The total reduced phase space can be considered as the result of natural patching of submanifolds \( s_i(U_i) \). Actually it is more convenient to construct the reduced phase space by the direct geometrical construction described above. It is the method we use in the present paper.

Note also that even if a global section of the fibered manifold \( \pi : N \to P \) exists we cannot, in general, consider \( s(P) \) as a submanifold of \( M \), defined by equations. We shall encounter such a situation in the next section.

### 3 Plane Spin Model

The first model we are going to consider is the plane spin model [14, 15]. This is a very simple model, but nevertheless here we come across peculiarities relevant for a more general case. Therefore, we give for this model a more detailed consideration than it deserves from a first sight. The initial phase space of the model is a cotangent bundle \( T^*S^1 \) of the one-dimensional sphere \( S^1 \), that is a cylinder \( S^1 \times \mathbb{R} \). It can be described locally by an angular variable \( 0 \leq \varphi < 2\pi \) and the conjugate momentum \( S \in \mathbb{R} \). Here the angle variable cannot be considered as a global coordinate. The symplectic two–form \( \omega \) in terms of the local variables \( \varphi, S \) has the form

\[
\omega = dS \wedge d\varphi.
\]

Thus, we have locally

\[
\{\varphi, S\} = 1.
\]

Actually, any \( 2\pi \)–periodical function of the variable \( \varphi \) that is considered as a variable, taking values in \( \mathbb{R} \), can be considered as a function on the phase space, i.e., as an observable, and any observable is connected with the corresponding \( 2\pi \)–periodical function. Therefore, we can introduce the functions

\[
q_1 = \cos \varphi, \quad q_2 = \sin \varphi
\]

as the functions on the phase space of the system. For these functions we have

\[
\{q_1, q_2\} = 0, \quad \{q_1, S\} = -q_2, \quad \{q_2, S\} = q_1. \tag{3.2}
\]

The functions \( q_1 \) and \( q_2 \) are not independent, and satisfy the relation

\[
q_1^2 + q_2^2 = 1.
\]
Note, that any function on the phase space can be considered as a function of dependent coordinates \( q_1, q_2 \) and \( S \). These coordinates will be taken below as the quantities, forming a restricted set of observables whose quantum analogs have the commutators which are in the direct correspondence with their Poisson brackets.

We come to the plane spin model by introducing the ‘spin’ constraint

\[
\psi = S - \theta = 0,
\]

where \( \theta \) is an arbitrary real constant.

Consider now the Dirac quantization of the system. To this end let us take as the Hilbert space the space of complex \( 2\pi \)-periodical functions of the variable \( \varphi \) with the scalar product

\[
(\Phi_1, \Phi_2) = \frac{1}{2\pi} \int_0^{2\pi} \Phi_1(\varphi)\Phi_2(\varphi) \, d\varphi.
\]

The operators \( \hat{q}_1 \) and \( \hat{q}_2 \), corresponding to the functions \( q_1 \) and \( q_2 \), are the operators of multiplication by the functions \( \cos \varphi \) and \( \sin \varphi \), respectively, i.e.,

\[
\hat{q}_1 \Phi = q_1 \Phi, \quad \hat{q}_2 \Phi = q_2 \Phi.
\]

The operator \( \hat{S} \) is defined by

\[
\hat{S} \Phi = \left( -i \frac{d}{d\varphi} + c \right) \Phi,
\]

where \( c \) is an arbitrary real constant. It is clear that the operators \( \hat{q}_1, \hat{q}_2 \) and \( \hat{S} \) are Hermitian operators with respect to the scalar product, defined by Eq. (3.4).

One can easily show that the relations

\[
[\hat{q}_1, \hat{q}_2] = 0, \quad [\hat{q}_1, \hat{S}] = -i \hat{q}_2, \quad [\hat{q}_2, \hat{S}] = i \hat{q}_1
\]

are valid for any value of \( c \) in agreement with Eq. (3.2).

The quantum analog of the constraint (3.3) gives the equation for the physical state wave functions:

\[
(\hat{S} - \theta) \Phi_{phys} = 0.
\]

Decomposing the function \( \Phi_{phys}(\varphi) \) over the orthonormal basis, formed by the functions \( e^{ik\varphi} \):

\[
\Phi_{phys}(\varphi) = \sum_{k \in \mathbb{Z}} \psi_k e^{ik\varphi},
\]

we find that equation (3.5) has a nontrivial solution only when

\[
c = \theta + n,
\]

where \( n \) is some fixed integer, \( n \in \mathbb{Z} \). In this case the corresponding physical normalized wave function is

\[
\Phi_{phys}(\varphi) = e^{in\varphi}.
\]

Here the only physical operator \( \hat{S} \), i.e. an operator commuting with the quantum constraint \( \hat{\psi} \), is \( \hat{S} \), and it is reduced to the constant \( \theta \) on the physical subspace (3.6).
Now we come back to the classical theory in order to construct the reduced phase space of the model. Let us show that for the surface, defined by Eq. (3.3), there is no ‘good’ gauge condition, but, nevertheless, the reduced phase space of the system can be constructed. Indeed, it is clear that the one–parameter group of transformations, generated by the constraint \( \psi \), consists of the rotations of the phase space. This group acts transitively on the constraint surface, and we have only one gauge orbit, which is the constraint surface itself. The gauge conditions must single out one point of an orbit. In our case we have to define only one gauge condition, let us denote it by \( \chi \). The function \( \chi \) must be such that the equations
\[
\psi = 0, \quad \chi = 0
\]
determine a set, consisting of only one point, and in this point we should have
\[
\{ \psi, \chi \} \neq 0
\]
(see Section 2). Recall that any function on the phase space of the system can be considered as a function of the variables \( \varphi \) and \( S \), which is \( 2\pi \)-periodical with respect to \( \varphi \). Thus, we require the \( 2\pi \)-periodical function \( \chi(\varphi, S)|_{S=\theta} \) turns into zero at only one point \( \varphi = \varphi_0 \) from the interval \( 0 \leq \varphi < 2\pi \). Moreover, we should have
\[
\{ \psi, \chi \}(\varphi_0, \theta) = -\left. \frac{\partial \chi(\varphi, \theta)}{\partial \varphi} \right|_{\varphi=\varphi_0} \neq 0.
\]
It is clear that such a function does not exist. Nevertheless, we have here the reduced phase space that consists of only one point. Therefore, the reduced space quantization is trivial: physical operator \( \hat{S} \) takes here constant value \( \theta \) in correspondence with the results obtained by the Dirac quantization method. When the described planar spin model is a subsystem of some other system, the reduction means simply that the cylinder \( T^*S^1 \) is factorized into a point, where \( S = \theta \), and that wave functions do not depend on the variable \( \varphi \).

Let us point out one interesting analogy in interpretation of the situation with nonexistence of a global gauge condition. Here the condition of \( 2\pi \)-periodicity can be considered as a ‘boundary’ condition. If for a moment we forget about it, we can take as a gauge function any monotonic function \( \chi(\varphi, S), \chi \in \mathbb{R} \), such that \( \chi(\varphi_0, \theta) = 0 \) at some point \( \varphi = \varphi_0 \), and, in particular, we can choose the function \( \chi(\varphi, S) = \varphi \). The ‘boundary’ condition excludes all such global gauge conditions. In this sense the situation is similar to the situation in the non–Abelian gauge theories where without taking into account the boundary conditions for the fields it is also possible to find global gauge conditions, whereas the account of those leads, in the end, to the nonexistence of global gauge conditions (see, e.g., [17]).

Now we are going to describe the system using dependent coordinates (see Appendix B). To this end let us take two–dimensional vectors \( \mathbf{q} \) and \( \mathbf{p} \), where \( \mathbf{p} \) is defined as
\[
p_1 = -S \sin \varphi, \quad p_2 = S \cos \varphi.
\]
Let \( \epsilon_{ij} \) be a skew symmetric tensor, normalized by the condition \( \epsilon_{12} = 1 \). For any two dimensional vectors \( \mathbf{a} \) and \( \mathbf{b} \) we use the notation
\[
\mathbf{a} \times \mathbf{b} = \epsilon_{ij} a_i b_j.
\]
It is clear that

\[ S = \epsilon_{ij} q_i p_j = q \times p. \]

The variables \( q \) and \( p \) satisfy the relations

\[ q^2 = 1, \quad qp = 0, \]

and can be considered as dependent coordinates in the phase space. For nonzero Poisson brackets we have the following expressions:

\[ \{p_i, p_j\} = p_i q_j - q_i p_j, \quad \{q_i, p_j\} = \delta_{ij} - q_i q_j. \] (3.7)

Hence, the Hamiltonian vector field, corresponding to constraint (3.3) has, according to Eq. (B.9), the form

\[ X_\psi = -q \times \frac{\partial}{\partial q} - p \times \frac{\partial}{\partial p}. \]

This vector field generates the characteristic distribution of the reduction of the symplectic two–form to the constraint surface, given by Eq. (3.3). The corresponding leaves are the orbits of the one–parameter group of canonical transformations, generated by the function \( \psi \). To find them we can use Eq. (B.10), that leads to

\[ q(\tau) = q \cos \tau + S^{-1} p \sin \tau, \quad p(\tau) = p \cos \tau - S q \sin \tau. \]

It is clear that in accordance with the above consideration the constraint surface consists of only one leaf, and the reduced phase space is a point.

Concluding this section, let us write the corresponding Lagrangian for the described Hamiltonian system:

\[ L = -\theta q \times q - \lambda (q^2 - 1), \] (3.8)

where \( \lambda \) is a Lagrange multiplier. It is not difficult to show that this Lagrangian leads to the Hamiltonian system with two second class constraints

\[ \psi_1 = qp = 0, \quad \psi_2 = \frac{1}{2} (q^2 - 1) = 0, \]

and one first class constraint

\[ \psi = q \times p - \theta = 0. \]

Using the notion of the Dirac bracket (see Appendix B), it is easy to get convinced that the Poisson brackets for the variables \( q_i \) and \( p_i \), restricted to the submanifold, defined by the constraints \( \psi_1 \) and \( \psi_2 \), coincide with those, given by Eq. (3.7). It is convenient further to use the same letters for coordinates and their restrictions to the submanifold under consideration. The exact meaning of the coordinates is always clear from the context.

Using parameterization (3.1), we come from Lagrangian (3.8) to the Lagrangian

\[ L = \theta \dot{\varphi}. \] (3.9)

Note that Lagrangian (3.9) is present as a part of a complete Lagrangian in the nonrelativistic model of the anyon [16].

Concluding this section, stress once more that for the considered simplest system the both quantization methods give coinciding results.
4 Rotator Spin Model

In this section we consider the rotator spin model [18]. The initial phase space of the system is described by a spin three–vector $S$ and a unit vector $q$,

$$q^2 = 1,$$

being orthogonal one to the other,

$$qS = 0.$$

The variables $q_i$ and $S_i$, $i = 1, 2, 3$, can be considered as dependent coordinates in the phase space of the system. The Poisson brackets for these coordinates are

$$\{q_i, q_j\} = 0, \quad \{S_i, S_j\} = \epsilon_{ijk}S_k, \quad \{S_i, q_j\} = \epsilon_{ijk}q_k.$$

Introducing the vector

$$p = S \times q,$$

we get for its components the following Poisson brackets:

$$\{p_i, p_j\} = p_iq_j - q_ip_j, \quad \{q_i, p_j\} = \delta_{ij} - q_ip_j,$$

that is of the same form as Eq. (3.7), and

$$\{S_i, p_j\} = \epsilon_{ijk}p_k.$$

Note that the vector $p$ satisfies the relation

$$qp = 0.$$

Actually, the variables $q_i$ and $p_i$ can be considered as another set of dependent coordinates in the phase space.

Using the expressions for the Poisson brackets of the dependent variables, we find the following expression for the symplectic two–form:

$$\omega = dp_i \wedge dq_i = d(\epsilon_{ijk}S_jq_k) \wedge dq_i. \quad (4.1)$$

Note that the phase space under consideration is the internal phase space of the one–mode relativistic string model [26].

Introducing the spherical angles $\varphi$, $\vartheta$ ($0 \leq \varphi < 2\pi$, $0 \leq \vartheta \leq \pi$) and the corresponding momenta $p_\varphi, p_\vartheta \in \mathbb{R}$, we can write the following parameterization for the vectors $q$ and $p$:

$$q = \begin{pmatrix} \cos \varphi \sin \vartheta \\ \sin \varphi \sin \vartheta \\ \cos \vartheta \end{pmatrix}, \quad p = \begin{pmatrix} \cos \varphi \cos \vartheta p_\varphi - \frac{\sin \varphi}{\sin \vartheta} p_\varphi \\ \sin \varphi \cos \vartheta p_\varphi + \frac{\cos \varphi}{\sin \vartheta} p_\varphi \\ -\sin \vartheta p_\varphi \end{pmatrix}, \quad (4.2)$$

and for the symplectic two–form we get the expression

$$\omega = dp_\vartheta \wedge d\vartheta + dp_\varphi \wedge d\varphi.$$
From this relation we conclude that the initial phase space of the system is symplectomorphic to the cotangent bundle $T^*S^2$ of the two–dimensional sphere $S^2$, furnished with the canonical symplectic structure.

The rotator spin model is obtained from the initial phase space by imposing the constraint

$$\psi = \frac{1}{2}(S^2 - \rho^2) = 0, \quad \rho > 0,$$

fixing the spin of the system. This can be obtained also starting from the Lagrangian

$$L = -\rho \sqrt{\dot{q}^2} - \lambda (q^2 - 1),$$

where $\lambda$ is a Lagrange multiplier. Indeed, the Hamiltonian description of the system with this Lagrangian results in two second class constraints

$$\psi_1 = qp = 0, \quad \psi_2 = \frac{1}{2}(q^2 - 1) = 0,$$

and a first class constraint (4.3) with $S = q \times p$. The reduction to the surface, defined by the second class constraints $\psi_1$ and $\psi_2$, leads to the Poisson brackets which coincide with those given above.

Using the Dirac method, we quantize the model in the following way. The state space is a space of the square integrable functions on the two–dimensional sphere. The scalar product is

$$(\Phi_1, \Phi_2) = \int_{S^2} \Phi_1(\varphi, \theta) \Phi_2(\varphi, \theta) \sin \theta d\theta d\varphi.$$ 

Parameterization (4.2) allows us to use as the operator $\hat{S}$ the usual orbital angular momentum operator expressed via spherical angles [27]. The wave functions as the functions on a sphere are decomposable over the complete set of the spherical harmonics:

$$\Phi(\varphi, \theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Phi_{lm} Y_{lm}(\varphi, \theta),$$

and, therefore, the quantum analog of the first class constraint (4.3),

$$(\hat{S}^2 - \rho^2)\Phi_{phys} = 0,$$

leads to the quantization condition for the constant $\rho$:

$$\rho^2 = n(n + 1),$$

where $n > 0$ is an integer. Only in this case equation (4.4) has a nontrivial solution of the form

$$\Phi^n_{phys}(\varphi, \theta) = \sum_{m=-n}^{n} \Phi_{nm} Y^n_m(\varphi, \theta),$$

i.e., with the choice of (4.3) we get the states with spin equal to $n$:

$$\hat{S}^2 \Phi^n_{phys} = n(n + 1)\Phi^n_{phys}.$$
Thus, we conclude that the Dirac quantization leads to the quantization (4.3) of the parameter $\rho$ and, as a result, the quantum system describes the states with integer spin $n$.

Let us turn now to the construction of the reduced phase space of the system. Then we perform the reduced phase space quantization and reveal that this method of quantization gives results physically different from those obtained by the Dirac quantization method.

The constraint surface of the model can be considered as a set composed of the points specified by two orthogonal normed three–vectors. Each pair of such vectors can be supplemented by a unique third three–vector, defined in such a way that we get an oriented orthonormal basis in three dimensional vector space. It is well known that the set of all oriented orthonormal bases in three dimensional space can be smoothly parameterized by the elements of the Lie group SO(3) (see Appendix C). Thus, the constraint surface in our case is diffeomorphic to the group manifold of the Lie group SO(3).

As in the preceding Section, to find the one parameter group of canonical transformations, generated by the constraint $\psi$, we can use Eq. (3.10). That gives the result

$$q(\tau) = q \cos(S\tau) + \frac{1}{S} S \times q \sin(S\tau), \quad S(\tau) = S, \quad S = \sqrt{S^2}.$$ 

Hence, we see that the transformations we are looking for, are the rotations about the direction, given by the spin vector. Thus, in the case of a general position the orbits of the one–parameter group of transformations under consideration are one dimensional spheres. Note, that only the orbits, belonging to the constraint surface where $S = \rho \neq 0$, are interesting to us. It is clear that an orbit is uniquely specified by the direction of the spin three–vector $S$ whose length is fixed by the constraint $\psi$. As a result of our consideration, we conclude that the reduced phase space of the rotator spin model is the coset space $SO(3)/SO(2)$, which is diffeomorphic to the two–dimensional sphere $S^2$ [28]. Due to the reasons discussed in the preceding section there is no gauge condition in this case either. In fact, since $SO(3)$ is a nontrivial fiber bundle over $S^2$, we can neither find a mapping from $S^2$ to $SO(3)$ whose image would be diffeomorphic to the reduced phase space. In other words, in this case the reduced phase space cannot be considered as a submanifold of the constraint surface.

Our next goal is to write an expression for the symplectic two–form on the reduced phase space. We shall give here two considerations of this question, using independent and dependent coordinates.

Convenient independent coordinates in the initial phase space can be introduced here with the help of the relations

$$S' = S \begin{pmatrix} \sin \vartheta \cos \varphi \\ \sin \vartheta \sin \varphi \\ \cos \vartheta \end{pmatrix}, \quad q = \begin{pmatrix} \cos \gamma \cos \vartheta \cos \varphi - \sin \gamma \sin \varphi \\ \cos \gamma \cos \vartheta \sin \varphi + \sin \gamma \cos \varphi \\ - \cos \gamma \sin \vartheta \end{pmatrix},$$

where $0 \leq \vartheta \leq \pi$, $0 \leq \varphi, \gamma < 2\pi$, and $0 < S < \infty$. In terms of these coordinates the symplectic two–form $\omega$ takes the form

$$\omega = d(Sd\gamma + S \cos \vartheta d\varphi). \quad (4.6)$$

From this relation we get the following expressions for the nonzero Poisson brackets

$$\{\gamma, S\} = 1, \quad \{\varphi, \cos \vartheta\} = S^{-1}, \quad \{\cos \vartheta, \gamma\} = S^{-1} \cos \vartheta.$$
The restriction of the symplectic two–form $\omega$ to the constraint surface, which we denote by the same letter, has the form
\[ \omega = \rho d(\cos \vartheta d\varphi). \] (4.7)

It is clear that the transformations of the group, generated by $\psi$, act on the independent coordinates as follows:
\[ S(\tau) = S, \quad \vartheta(\tau) = \vartheta, \quad \varphi(\tau) = \varphi, \quad \gamma(\tau) = \gamma + S\tau. \]

Thus, we see that the variables $\vartheta, \varphi$ can be considered as coordinates in the reduced phase space, and the symplectic two–form on the reduced phase space is given by (4.7).

The same results, but in a more transparent form, are obtained using dependent coordinates. It is clear that we can consider the variables $S_i$ as dependent coordinates in the reduced phase space, thus the symplectic two–form on it may be expressed in terms of them. Note, to this end, that the vectors $q, s = S/S$ and $q \times s$ form an orthonormal basis, and the differentials of the vectors $q$ and $s$ satisfy the relations
\[ \begin{align*}
qdq &= 0, \quad sds = 0, \quad \text{(4.8)} \\
spdq + qds &= 0. \quad \text{(4.9)}
\end{align*} \]

Write the restriction of the symplectic two–form (4.1) to the constraint surface as
\[ \omega = \rho(ds \times q) \land dq + \rho(s \times dq) \land dq. \]

From Eq. (4.8) it follows that we can write
\[ \begin{align*}
dq &= a q + b(q \times s), \quad \text{(4.10)} \\
ds &= f q + g(q \times s), \quad \text{(4.11)}
\end{align*} \]

where $a, b, f$ and $g$ are some one–forms. From Eq. (4.10) we get
\[ s \times dq = bq, \]
and, therefore,
\[ (s \times dq) \land dq = 0. \]

Further, from Eq. (4.11) we have
\[ ds \times q = gs. \] (4.12)

This equality implies
\[ (ds \times q) \land dq = f \land g, \]
where we have used Eq. (4.9). From Eq. (4.11) we have also
\[ (s \times ds) \land ds = -2f \land g, \]
that gives finally
\[ \omega = \frac{1}{2 \rho^2} (S \times dS) \land dS. \] (4.13)
Thus, we see that the dependent coordinates $S^i$,
\begin{equation}
S^2 = \rho^2,
\end{equation}
in the reduced phase space of the system provide a realization of the basis of the Lie algebra so(3):
\begin{equation}
\{S_i, S_j\} = \epsilon_{ijk} S_k.
\end{equation}

The quantization on the reduced phase space can be performed with the help of the geometric quantization method proceeding from the classical relations (4.13)–(4.15). This was done in detail earlier (see, e.g., Ref. [29]), and we write here only the final results of this procedure. The constant $\rho$ is quantized:
\begin{equation}
\rho = j, \quad 0 < 2j \in \mathbb{Z},
\end{equation}
i.e., it can take only integer or half-integer value, and the Hermitian operators, corresponding to the components of the spin vector, are realized in the form:
\begin{equation}
\hat{S}_1 = \frac{1 - z^2}{2} \frac{d}{dz} + jz, \quad \hat{S}_2 = i \left(\frac{1 + z^2}{2} \frac{d}{dz} - jz\right), \quad \hat{S}_3 = z \frac{d}{dz} - j,
\end{equation}
where
\[ z = e^{-i\phi} \tan \vartheta/2, \]
or, in terms of the dependent coordinates,
\[ z = \frac{S_1 - iS_2}{\rho + S_3}. \]
Operators (4.17) act in the space of holomorphic functions $f(z)$ with the scalar product
\[ (f_1, f_2) = \frac{2j + 1}{\pi} \int \int \frac{f_1(z)f_2(z)}{(1 + |z|^2)^{2j+2}} d^2z, \]
in which the functions
\[ \psi_j^m \propto z^{j+m}, \quad m = -j, -j + 1, ..., j, \]
form the set of eigenfunctions of the operator $\hat{S}_3$ with the eigenvalues $s_3 = m$. These operators satisfy the relation
\[ \hat{S}^2 = j(j + 1) \]
instead of classical relation (4.14), and, therefore, we have the $(2j+1)$–dimensional irreducible representation $D_j$ of the Lie group SU(2).

Thus, we see that for the rotator spin model the reduced phase space quantization method leads to the states with integer or half–integer spin, depending on the choice of the quantized parameter $\rho$, and gives in general the results physically different from the results obtained with the help of the Dirac quantization method. Let us stress once again here that within the Dirac quantization method in this model the spin operator $\hat{S}$ has a nature of the orbital angular momentum operator, and it is this nature that does not allow spin to take half-integer values [30]. In other words, in the Dirac quantization we quantize the system with the phase space being the cotangent bundle to the two–dimensional sphere, defined by the relation $q^2 = 1$, while in the reduced phase space quantization we start with the phase space being a two–dimensional sphere of the radius coinciding with the length of the vector $S$. It is this difference between the phase spaces that leads to the different results of quantizations.
5 Top Spin Model

In this section we consider the top spin model \[19\]. The initial phase space of the model is described by the spin three–vector \( S \), and by three vectors \( e_i \), forming a right orthonormal basis in \( \mathbb{R}^3 \):

\[ e_i e_j = \delta_{ij}, \quad e_i \times e_j = \epsilon_{ijk} e_k \]  

(5.1)

(see Appendix C). Denote the components of the vectors \( e_i \) with respect to the canonical basis of \( \mathbb{R}^3 \) by \( E_{ij} \). The components \( S_i \) of the vector \( S \) and the quantities \( E_{ij} \) form a set of dependent coordinates in the phase space of the system. The corresponding Poisson brackets are

\[
\{ E_{ij}, E_{kl} \} = 0, \\
\{ S_i, E_{jk} \} = \epsilon_{ikl} E_{jl}, \\
\{ S_i, S_j \} = \epsilon_{ijk} S_k. 
\]

(5.2)

(5.3)

The vectors \( e_i \), satisfying Eq. (5.1) form a right orthonormal basis in \( \mathbb{R}^3 \). The set of all such bases can be identified with the three–dimensional rotation group (see Appendix C). Taking into account Eqs. (5.2) and (5.3) we conclude that the initial phase space is actually the cotangent bundle \( T^*\text{SO}(3) \), represented as the manifold \( \mathbb{R}^3 \times \text{SO}(3) \). Using Eqs. (5.2) and (5.3), one can get the following expression for the symplectic two–form \( \omega \) on the initial phase space:

\[
\omega = \frac{1}{2} d((S \times e_l)de_l) = \frac{1}{2} d(\epsilon_{ijk} S_j E_{lk} dE_{li}).
\]

(5.4)

It is useful to introduce the variables \( J_i = e_i S = E_{ij} S_j \). For these variables we have the following Poisson brackets:

\[
\{ J_i, E_{jk} \} = -\epsilon_{ijl} E_{lk}, \quad \{ J_i, J_j \} = -\epsilon_{ijk} J_k.
\]

(5.5)

(5.6)

Note, that we have the equality

\[ S_i S_i = J_i J_i. \]

The phase space of the top spin model is obtained from the phase space, described above, by introducing two first class constraints

\[
\psi = \frac{1}{2} (S^2 - \rho^2) = 0, \\
\chi = Se_3 - \kappa = 0,
\]

(5.5)

(5.6)

where \( \rho > 0, \quad |\kappa| < \rho. \)

The described Hamiltonian system can be also obtained starting from the Lagrangian

\[
L = -\theta \sqrt{\dot{q}_1^2 + \dot{q}_2^2 - (\dot{q}_1 q_2)^2} - \frac{\kappa}{2} (q_1 \dot{q}_2 - q_2 \dot{q}_1) - \lambda_1 (q_1^2 - 1) - \lambda_2 (q_2^2 - 1) - \lambda_3 q_1 q_2,
\]

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where $\lambda_1, \lambda_2$ and $\lambda_3$ are Lagrange multipliers and $\theta \neq 0$ is a real constant. The Hamiltonian description of the system given by this Lagrangian results in the set of six second class constraints

$$
\psi_1 = q_1^2 - 1 = 0, \quad \psi_2 = q_2^2 - 1 = 0, \quad \psi_3 = q_1 q_2 = 0,
$$
$$
\psi_4 = q_1 p_1 = 0, \quad \psi_5 = q_2 p_2 = 0, \quad \psi_6 = q_1 p_2 + q_2 p_1 = 0,
$$

and two first class constraints (5.5) and (5.6) with

$$
\rho_2 = \theta^2 + \kappa^2, \quad S = q_1 \times p_1 + q_2 \times p_2, \quad e_3 = e_1 \times e_2, \quad e_1 = q_1, \quad e_2 = q_2.
$$

The reduction to the surface of second class constraints $\psi_1, \ldots, \psi_6$ leads, in the end, to the description of the system by dependent variables $E_{ij}$ and $S_i$ having the Poisson brackets of the form (5.2) and (5.3).

Consider now the Dirac quantization of the model. Recall that the matrix $E$ can be identified with the corresponding rotation matrix (see Appendix C). Let us parameterize the matrix $E$ by the Euler angles, $E = E(\alpha, \beta, \gamma)$, and use the representation where the operators, corresponding to these angles are diagonal. In this representation state vectors are functions of the Euler angles, and the operators $\hat{S}_i$ and $\hat{J}_i$ are realized as linear differential operators, acting on such functions [31]. The quantum analogs of the constraints $\psi$ and $\chi$ turn into the equations for the physical states of the system:

$$
(\hat{S}^2 - \rho^2)\Phi_{\text{phys}} = 0, \quad (\hat{J}_3 - \kappa)\Phi_{\text{phys}} = 0.
$$

(5.7) (5.8)

An arbitrary state vector can be decomposed over the set of the Wigner functions, corresponding to either integer or half–integer spins [31]:

$$
\Phi(\alpha, \beta, \gamma) = \phi_{jmk} D^j_{mk}(\alpha, \beta, \gamma),
$$

(5.9)

where $j = 0, 1, \ldots$, or $j = 1/2, 3/2, \ldots$, and $k, m = -j, -j + 1, \ldots, j$. The Wigner functions $D^j_{mk}$ have the properties

$$
\hat{S}^2 D^j_{mk} = j(j + 1) D^j_{mk}, \quad \hat{S}_3 D^j_{mk} = m D^j_{mk}, \quad \hat{J}_3 D^j_{mk} = k D^j_{mk}.
$$

From the representation of the state vector (5.9) we see that Eqs. (5.7) and (5.8) have nontrivial solutions only when $\rho^2 = j(j + 1)$, and $\kappa = k$, for some integer or half–integer numbers $j$ and $k$, such that $-j \leq k \leq j$. In other words we get the following quantization condition for the parameters of the model:

$$
\rho^2 = j(j + 1), \quad \kappa = k, \quad -j \leq k \leq j, \quad 0 < 2j \in \mathbb{Z}.
$$

The corresponding physical state vectors have the form

$$
\Phi_{\text{phys}}(\alpha, \beta, \gamma) = \sum_{m=-j}^{j} \varphi_m D^j_{mk}(\alpha, \beta, \gamma).
$$

Thus, we see that the Dirac quantization of the top spin model leads to a pure integer or half–integer spin system.
Proceed now to the construction of the reduced phase space of the system. As the constraints $\psi$ and $\chi$ have zero Poisson bracket, we can consider them consecutively. Let us start with the constraint $\psi$.

From the expressions for the Poisson brackets (5.2) and (5.3) it follows that the group of gauge transformations, generated by the constraint $\psi$, acts in the initial phase space variables as follows:

$$
e_i(\tau) = e_i \cos(S\tau) + \frac{1}{S}(S \times e_i) \sin(S\tau) + \frac{1}{S^2}S(Se_i)(1 - \cos(S\tau)), \quad (5.10)$$

$$S(\tau) = S, \quad S = \sqrt{S^2}. \quad (5.11)$$

Comparing these relations with Eq. (C.5), we see that the transformations under consideration have the sense of the rotation by the angle $S\tau$ about the direction of the spin vector. In matrix notations we can write

$$E(\tau) = ER^T(s, S\tau), \quad s = S/S. \quad (5.12)$$

Recall that the initial phase space of the system is diffeomorphic to $\mathbb{R}^3 \times \text{SO}(3)$. Let us consider this space as a trivial fibre bundle over $\mathbb{R}^3$ with the fibre $\text{SO}(3)$. Gauge transformations (5.10), (5.11) act in fibres of this bundle. It is clear that the constraint surface, defined by the constraint $\psi$, is a trivial fibre subbundle $S^2 \times \text{SO}(3)$. As $\text{SO}(3)/\text{SO}(2) = S^2$, then after the reduction over the action of the gauge group we come to the fibre bundle over $S^2$ with the fibre $S^2$. As it follows from general theory of fibre bundles [32], this fibre bundle is again trivial. Thus the reduced phase space, obtained using only the constraint $\psi$, is the direct product $S^2 \times S^2$. To find the expression for the symplectic two–form on this reduced space we first introduce convenient coordinates in the initial phase space.

Let $R(s, n_3)$ be a rotation that transforms the vector $n_3$ to the vector directed along the vector $s$. This rotation is undefined for $S = 0$, but it is irrelevant for us, because we are finally interested only in the constraint surface where $S = \rho \neq 0$. We can choose $R(s, n_3)$ in the form

$$R(s, n_3) = R(n_3, \varphi)R(n_2, \vartheta),$$

where $\varphi$ and $\vartheta$ are the polar angles of the vector $S$. We shall also use for the $R(s, n_3)$ the notation $R(\varphi, \vartheta)$. From the definition of $R(s, n_3)$ we have

$$S = S R(s, n_3)n_3. \quad (5.13)$$

Using now Eq. (C.5), we can present the matrix $R(s, S\tau)$ as

$$R(s, S\tau) = R(s, n_3)R(n_3, S\tau)R^T(s, n_3).$$

Using this representation in Eq. (5.12), we get the following transformation law for the matrix $F = ER(s, n_3)$:

$$F(\tau) = FR^T(n_3, S\tau). \quad (5.14)$$

The matrix $F$ defines a new right basis $\{f_i\}$, with

$$f_i = F_{ij}n_j = R^{-1}(s, n_3)e_i. \quad (5.15)$$
Let \( \lambda, \mu \) and \( \nu \) be the Euler angles for the matrix \( F \), i.e.,

\[
F = R(n_3, \lambda)R(n_2, \mu)R(n_3, \nu).
\]

From Eq. (5.14) it follows that these angles are transformed under the action of the gauge transformations as follows:

\[
\lambda(\tau) = \lambda, \quad \mu(\tau) = \mu, \quad \nu(\tau) = \nu - S\tau.
\]

It is clear that we can use the quantities \( S, \varphi, \vartheta \) and \( \lambda, \mu, \nu \) as independent coordinates in the initial phase space of the system. The symplectic two–form \( \omega \), written in terms of these coordinates has the form

\[
\omega = d(S \cos \vartheta d\varphi - Sd\nu - S \cos \mu d\lambda).
\]  

(5.16)

To show that Eq. (5.16) is valid we start with expression (5.4). From Eqs. (5.13), (C.12) and (5.15) it follows that

\[
\omega = \frac{1}{2} d[SR(\mathbf{s}, n_3)(\mathbf{n}_3 \times \mathbf{f}_i)d(R(\mathbf{s}, n_3)\mathbf{f}_i)].
\]

Using matrix notation we can rewrite this relation in the form

\[
\omega = -\frac{1}{2} d[S \text{tr} (M_3R^T(\varphi, \vartheta)dR(\varphi, \vartheta))] + \frac{1}{2} d[S \text{tr} (M_3F^T(\lambda, \mu, \nu)dF(\lambda, \mu, \nu))].
\]  

(5.17)

Further, Eqs. (C.9) and (C.11) lead to the equalities

\[
R^T(\varphi, \vartheta)dR(\varphi, \vartheta) = (-\sin \vartheta M_1 + \cos \vartheta M_3)d\varphi + M_2d\vartheta,
\]

\[
F^T(\lambda, \mu, \nu)dF(\lambda, \mu, \nu) = [-\sin \mu(\cos \nu M_1 - \sin \nu M_2) + \cos \mu M_3]d\lambda
\]

\[
+(-\sin \nu M_1 + \cos \nu M_2)d\mu + M_3d\nu.
\]

Now using these expressions in Eq. (5.17), and taking into account Eq. (C.8), we come to Eq. (5.16).

It is instructive to compare Eq. (5.16) with Eq. (4.6). From Eq. (5.16) we see that the restriction of the symplectic two–form to the constraint surface, defined by the constraint \( \psi \), has the form

\[
\omega = \rho d(\cos \vartheta d\varphi - \cos \mu d\lambda).
\]

Using the definition of the variables \( J_i \) and quantities \( F_{ij} \), we see that

\[
J_i = SF_{i3}.
\]

From this relation it is easy to conclude that \( \lambda \) and \( \mu \) can be considered as the spherical angles, parameterizing \( J_i \), that actually are the polar angles of the vector \( \mathbf{S} \) with respect to the basis \( \{ \mathbf{e}_i \} \):

\[
\mathbf{S} = S(\cos \lambda \sin \mu \mathbf{e}_1 + \sin \lambda \sin \mu \mathbf{e}_2 + \cos \mu \mathbf{e}_3).
\]

It is quite clear that the quantities \( S_i \) and \( J_i \) form a set of dependent coordinates in the reduced phase space under consideration. Here we have \( S_i S_i = J_i J_i = \rho^2 \), that confirms
the conclusion made above that the reduced phase space in this case is $S^2 \times S^2$. Using now the experience gained in the preceding Section, we see that the symplectic two–form can be written in terms of these coordinates as

$$\omega = -\frac{1}{2\rho^2}(\epsilon_{ijk}S_idS_j \wedge dS_k - \epsilon_{ijk}J_idJ_j \wedge dJ_k).$$

Let us turn our attention to the constraint $\chi$. It is easy to get convinced that the transformations of the gauge group, generated by this constraint act in the initial phase space in the following way:

$$e_i(\tau) = R(e_3, \tau)e_i = e_jR_{ji}(n_3, \tau),$$

$$S(\tau) = S.$$

We see that the gauge group, generated by the constraint $\chi$, acts only in one factor of the product $S^2 \times S^2$, which is a reduced phase space obtained by us after reduction with the help of the constraint $\psi$. Thus we can consider only that factor, which is evidently described by the quantities $J_i$. From such point of view, the constraint surface, defined by the constraint $\chi$, is a one dimensional sphere $S^1$, where the group of gauge transformations acts transitively. Hence, after reduction we get only one point. Thus, the final reduced phase space is a two–dimensional sphere $S^2$.

In matrix notation we have

$$E(\tau) = R^T(n_3, \tau)E,$$

that implies

$$F(\tau) = R^T(n_3, \tau)F.$$ 

Hence, for the Euler angles $\lambda$, $\mu$, and $\nu$ we get

$$\lambda(\tau) = \lambda - \tau, \quad \mu(\tau) = \mu, \quad \nu(\tau) = \nu.$$ 

It is clear from this consideration, that the reduced phase space is described by the independent coordinates $\varphi$, $\vartheta$. The symplectic two–form on the reduced phase space is

$$\omega = \rho d(\cos \vartheta d\varphi),$$

that can be written in terms of the dependent coordinates $S_i$ as

$$\omega = -\frac{1}{2\rho^2}\epsilon_{ijk}S_idS_j \wedge dS_k.$$ 

The reduced phase space we have obtained, coincides with the reduced phase space from the preceding Section. Hence the geometric quantization method gives again the quantization condition (4.16) for the parameter $\rho$, while the parameter $\kappa$ remains unquantized here. Therefore, while for this model unlike the previous one, two methods of quantization lead to the quantum system, describing either integer or half-integer spin states, nevertheless, the corresponding quantum systems are different: the Dirac method gives discrete values for the observable $\hat{J}_3$, whereas the reduced phase space quantization allows it to take any value $\kappa$, such that $\kappa^2 < j^2$ for a system with spin $j$. 

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Let us note here one interesting property of the system. We can use a combination of the Dirac and reduced phase space quantization methods. After the first reduction with the help of the constraint \( \psi \), the system, described by the spin vector and the ‘isospin’ vector with the components \( I_i = -J_i \), \( S_i S_i = I_i I_i \), can be quantized according to Dirac by imposing the quantum analog of the constraint \( \chi \) on the state vectors for singling out the physical states. In this case we have again the quantization of the parameter \( \kappa \) as in the pure Dirac quantization method, and, therefore, here the observable \( \hat{J}_3 \) can take only integer or half–integer value. Hence, in this sense, such a combined method gives the results coinciding with the results of the Dirac quantization method.

6 Discussion and Conclusions

The plane spin model, considered in Section 3, gives an example of the classical constrained system with finite number of the degrees of freedom for which there is no gauge condition, but nevertheless, the reduced phase space can be represented as a submanifold of the constraint surface. As we have seen, Dirac and reduced phase space quantization methods lead to the coinciding physical results for this plane spin model. Moreover, we have revealed an interesting analogy in interpretation of the situation with nonexistence of a global gauge condition for this simple constrained system with the situation taking place for the non-Abelian gauge theories [17].

The rotator and top spin models give examples of the classical systems, in which there is no global section of the space of gauge orbits. In spite of impossibility to impose gauge conditions such systems admit the construction of the reduced phase space. These two models demonstrate that the reduced phase space and the Dirac quantization methods (or some their combinations), can give essentially different physical results.

Thus, for Hamiltonian systems with first class constraints we encounter two related problems.

The first problem consists in the choice of a ‘correct’ quantization method for such systems. From the mathematical point of view any quantization leading to a quantum system, which has the initial system as its classical limit, should be considered as a correct one, but physical reasonings may distinguish different quantization methods. Consider, for example, the above mentioned systems. The rotator spin model, quantized according to the Dirac method, represents by itself the orbital angular momentum system with additional condition (4.4) singling out the states with a definite eigenvalue of angular momentum operator \( \hat{S}^2 \). This eigenvalue, in turn, is defined by the concrete value of the quantized parameter of the model: \( \rho^2 = n(n + 1) > 0 \). On the other hand, the reduced phase space quantization of the model gives either integer or half–integer values for the spin of the system. If we suppose that the system under consideration is to describe orbital angular momentum, we must take only integer values for the parameter \( \rho \) in the reduced phase space quantization method. But in this case we must, nevertheless, conclude, that the reduced phase space quantization method of the rotator spin model describes a more general system than the quantum system obtained as a result of the Dirac quantization of that classical system.

The Dirac quantization of the top spin model, or its described combination with the reduced phase space quantization gives us a possibility to interpret this system as a system
having spin and isospin degrees of freedom (with equal spin and isospin: \( \hat{S}^2 = \hat{I}_i \hat{I}_i = j(j+1) \)), but in which the isospin degrees of freedom are ‘frozen’ by means of the condition \( \hat{I}_3 \Phi_{\text{phys}} = -k \Phi_{\text{phys}} \). On the other hand, as we have seen, the reduced space quantization method does not allow one to have such interpretation of the system since it allows the variable \( I_3 \) to take any (continuous) value \(-\kappa\) restricted only by the condition \( \kappa^2 < j^2 \), i.e., the operator \( \hat{I}_3 \) (taking here only one value) cannot be interpreted as a component of the isospin vector operator. From this point of view a ‘more correct’ method of quantization is the Dirac quantization method.

In this respect it is worth to point out that there is a class of physical models, for which it is impossible to get the reduced phase space description, and which, therefore, can be quantized only by the Dirac method.

Indeed, there are various pseudoclassical models containing first class nilpotent constraints of the form [20]–[24]:

\[
\psi = \xi_{i_1}...\xi_{i_n} G^{i_1...i_n} = 0,
\]

where \( \xi_{i_k} \), are real Grassmann variables with the Poisson brackets

\[
\{\xi_k, \xi_l\} = -i g_{kl},
\]

\( g_{kl} \) being a real nondegenerate symmetric constant matrix. Here it is supposed that \( G^{i_1...i_n} \), \( n \geq 2 \), are some functions of other variables, antisymmetric in their indices, and all the terms in a sum have simultaneously either even or odd Grassmann parity. For our considerations it is important that constraints (6.1) are the constraints, nonlinear in Grassmann variables, and that they have zero projection on the unit of Grassmann algebra. In the simplest example of relativistic massless vector particle in (3+1)–dimensional space–time [20]–[22] the odd part of the phase space is described by two Grassmann vectors \( \xi^a_{\mu}, a = 1, 2 \), with brackets

\[
\{\xi^a, \xi^b\} = -i \delta^{ab} g_{\mu\nu},
\]

and the corresponding nilpotent first class constraint has the form:

\[
\psi = i \xi^1_{\mu} \xi^2_{\nu} g^{\mu\nu} = 0,
\]

where \( g_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \). This constraint is the generator of the SO(2)–rotations in the ‘internal isospin’ space:

\[
\xi^1_{\mu}(\tau) = \xi^1_{\mu} \cos \tau + \xi^2_{\mu} \sin \tau, \quad \xi^2_{\mu}(\tau) = \xi^2_{\mu} \cos \tau - \xi^1_{\mu} \sin \tau.
\]

The specific property of this transformation is that having \( \xi^a_{\mu}(\tau) \) and \( \xi^a_{\mu} \), we cannot determine the rotation angle \( \tau \) because there is no notion of the inverse element for an odd Grassmann variable. Another specific feature of the nilpotent constraint (6.2) is the impossibility to introduce any, even local, gauge constraint for it. In fact, we cannot find a gauge constraint \( \chi \) such that the Poisson bracket \( \{\psi, \chi\} \) would be invertible. Actually, it is impossible in principle to construct the corresponding reduced phase space for such a system. Obviously, the same situation arises for the constraint of general form (6.1). It is necessary to note here that in the case when the constraint \( \psi \) depends on even variables of the total phase space (see, e.g., ref. [23]), and, therefore, generates also transformations of some of them, we
cannot fix the transformation parameter (choose a point in the orbit) from the transformation law of those even variables, because the corresponding parameter is present in them with a noninvertible factor, nonlinear in Grassmann variables. Therefore, the pseudoclassical systems containing the constraints of form (6.1) can be quantized only by the Dirac method, that was done in original papers \[21, 23, 24\] (see also Ref. \[22\] where the BRST quantization of such systems was considered).

Let us come back to the discussion of the revealed difference between two methods of quantization, and point out that the second related problem is clearing up the sense of gauge degrees of freedom. The difference appearing under the Dirac and reduced phase space quantization methods can be understood as the one proceeding from the quantum ‘vacuum’ fluctuations corresponding to the ‘frozen’ (gauge) degrees of freedom. Though these degrees of freedom are ‘frozen’ by the first class constraints, they reveal themselves through quantum fluctuations, and in the Dirac quantization method they cannot be completely ‘turned off’ due to the quantum uncertainty principle. Thus, we can suppose that the gauge degrees of freedom serve not simply for ‘covariant’ description of the system but have ‘hidden’ physical meaning, in some sense similar to the compactified degrees of freedom in the Kaluza–Klein theories. If we adopt such a physical point of view, we have to use only the Dirac quantization method. Further, the gauge principle cannot be considered then as a pure technical principle. From here we arrive also at the conclusion that the Dirac separation of the constraints into first and second class constraints is not technical, and nature ‘distinguish’ these two cases as essentially different, since gauge degrees of freedom, corresponding to the first class constraints, may reveal themselves at the quantum level (compare with the point of view advocated in Ref. \[34\]).

Finally, let us mention also that since the path integral quantization method of constrained systems with first class constraints is based on the introduction of the gauge conditions for the first class constraints \[3, 4\], it appears that the usage of this method for the systems considered in the present paper is rather problematic.

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### A Pseudoinverse Mappings

In this appendix and in the next one, we describe the method of dependent coordinates, that is a very convenient method to deal with second class constraints (see also \[33\]).

Let \(A\) be a linear mapping from an \(n\)-dimensional linear space \(V\) to an \(m\)-dimensional linear space \(W\). It is known that for any decompositions

\[
V = V_0 \oplus V_1, \quad W = W_0 \oplus W_1,
\]

such that \(V_0 = \text{Ker} A, W_1 = \text{Im} A\), the mapping \(A\) induces an isomorphism \(A'\) of \(V_1\) and \(W_1\). Here we have

\[
A = i^{W_1} A' \pi^{V_1},
\]
where $\pi^{V_i}$ is the projection of $V$ onto $V_1$, and $\iota^{W_i}$ is the inclusion mapping of $W_1$ into $W$. Note that decompositions (A.1) generate the dual decompositions

$$V^* = V^*_0 \oplus V^*_1, \quad W^* = W^*_0 \oplus W^*_1,$$

such that

$$W^*_0 = \ker A^*, \quad V^*_1 = \im A^*,$$

where $A^*: W^* \to V^*$ is the adjoint mapping of the mapping $A$.

The pseudoinverse mapping of the mapping $A$ is defined as

$$A^{-1} = \iota^{V_1} A'^{-1} \pi^{W_1}, \tag{A.3}$$

where $\pi^{W_1}$ is the projection of $W$ onto $W_1$, and $\iota^{V_1}$ is the inclusion mapping of $V_1$ into $V$. From Eqs. (A.2) and (A.3) we get

$$A^{-1} A = \Pi, \quad A A^{-1} = \Sigma,$$

where

$$\Pi = \iota^{V_1} \pi^{V_1}, \quad \Sigma = \iota^{W_1} \pi^{W_1}.$$

For the pseudoinverse mapping $A^{-1}$ we have also

$$\ker A^{-1} = W_0, \quad \im A^{-1} = V_1,$$

furthermore,

$$\ker A^{-1*} = V^*_0, \quad \im A^{-1*} = W^*_1.$$

Choosing different decompositions of the linear spaces $V$ and $W$ we get different pseudoinverse mappings. Actually, it can be shown that if for a given mapping $A$ we fix decompositions (A.1), then there exists a unique mapping $A^{-1}$, defined by the conditions

$$A A^{-1} = \Sigma, \quad \ker A^{-1*} = V^*_0,$$

or by the conditions

$$A^{-1} A = \Pi, \quad \ker A^{-1*} = W_0.$$

Note that decompositions (A.1) can be fixed in the following way. Denote by $p$ the rank of the mapping $A$. Recall that $\text{rank } A = \dim \im A$. Let $\{e_\alpha\}_{\alpha=1,\ldots,n-p}$ be a basis in $V_0 = \ker A$, and $\{\mu^\alpha\}_{\alpha=1,\ldots,n-p}$ be a set of elements of $V^*$ such that

$$\mu^\alpha(e_\beta) = \delta^\alpha_\beta.$$

In this case we can define $V_1$ as

$$V_1 = \{v \in V \mid \mu^\alpha(v) = 0, \alpha = 1, \ldots, n-p\},$$

and we have

$$\Pi = 1 - e_\alpha \otimes \mu^\alpha.$$
As rank $A^* = \text{rank } A$, then \( \dim \ker A^* = m - p \). Let \( \{ \nu^\sigma \}_{\sigma = 1, \ldots, m - p} \) be a basis in \( \ker A^* = W_0^* \), and \( \{ f_\sigma \}_{\sigma = 1, \ldots, m - p} \) be a set of elements of \( W \) such that

\[
\nu^\tau (f_\sigma) = \delta^\tau_\sigma.
\]

The space \( W_0 \) can be defined as the space generated by the set \( \{ f_\sigma \}_{\sigma = 1, \ldots, m - p} \). Indeed, it is not difficult to show that in this case \( W = W_0 \oplus W_1 \). The mapping \( \Sigma \) in this case can be presented in the form

\[
\Sigma = 1 - f_\sigma \otimes \nu^\sigma.
\]

Treating an arbitrary \( n \times m \) matrix as the matrix of a linear mapping from an \( n \)–dimensional linear space to an \( m \)–dimensional linear space, we associate a linear mapping with any matrix. Hence, we can define the pseudoinverse matrix of a matrix as the matrix of the corresponding pseudoinverse operator. We can reformulate the results of the above consideration for the case of matrices as follows.

Let \( a = \| a^r_i \| \) be \( n \times m \) matrix of rank \( p \). Hence, there is a complete set of \( n - p \) independent relations of the form

\[
a^r_i e^i_\alpha = 0, \quad \alpha = 1, \ldots, n - p.
\]

Since the rank of the transposed matrix equals the rank of the initial one, there is a complete set of \( m - p \) independent relations

\[
\nu^\tau a^r_i = 0, \quad \tau = 1, \ldots, m - p.
\]

It is clear that there are quantities \( \mu^\alpha_i \) and \( f^\tau_\tau \) such that

\[
\mu^\alpha_i e^i_\beta = \delta^\alpha_\beta, \quad \nu^\tau f^r_\sigma = \delta^\tau_\sigma.
\]

Define the matrices \( \| \pi^i_j \| \) and \( \| \sigma^r_s \| \) by

\[
\pi^i_j = \delta^i_j - e^i_\alpha \mu^\alpha_j, \quad \sigma^r_s = \delta^r_s - f^r_\tau \nu^\tau.
\]

From the discussion above we see that there exists a unique matrix \( a^{-1} = \| (a^{-1})^r_i \| \), defined by the conditions

\[
a^r_i (a^{-1})^i_s = \sigma^r_s, \quad \mu^\alpha_i (a^{-1})^i_r = 0,
\]

or by the conditions

\[
(a^{-1})^i_r a^r_j = \pi^i_j, \quad (a^{-1})^i_r f^r_\tau = 0.
\]

Consider a special case of skew symmetric \( n \times n \) matrices. Usually, with such matrices we associate linear operators from an \( n \)–dimensional \( V \) space to the dual space \( V^* \). If a matrix \( \| a_{ij} \| \) is skew symmetric then the corresponding mapping \( A \) is skew adjoint, \( A^* = -A \). If we choose in \( (A.1) \)

\[
W_0 = V_0^*, \quad W_1 = V_1^*,
\]

then the pseudoinverse mapping is skew adjoint and the corresponding pseudoinverse matrix is skew symmetric.
B  Dependent Coordinates

Let $N$ be a smooth manifold of the dimension $n$. Suppose that in some open neighborhood $U \subset N$ there is given a set of functions $z^i$, $i = 1, \ldots, m > n$ such that for any point $p \in U$ the rank of the set $dz^i_p$, $i = 1, \ldots, m$, is equal to $n$. In such a case we call the functions $z^i$ local dependent coordinates in $M$. It is clear that if we consider the mapping $\iota : U \to \mathbb{R}^m$, defined by

$$\iota(p) = (z^1(p), \ldots, z^m(p)), \quad p \in U,$$

then we can consider $U$ as a smooth $n$–dimensional submanifold of $\mathbb{R}^m$. In fact the case when $U = M$, i.e., the case of globally defined dependent coordinates, is the most interesting one.

Since for any $p \in U$ the rank of the set $dz^i_p$ is equal to $n$, then there exists a set of smooth functions $\mu^\alpha_i$, $\alpha = 1, \ldots, m - n$, such that

$$\mu^\alpha_i dz^i = 0, \quad \alpha = 1, \ldots, m - n.$$

It is clear that the quantities $\mu^\alpha_i$ can be chosen in such a way that the set of $m - n$ $m$–dimensional vectors $\mu^\alpha_i(p)$ be linearly independent at any point $p \in U$.

Let $u^a$, $a = 1, \ldots, n$, be a set of independent coordinates defined in some open neighborhood $V \subset U$. For the dependent coordinates $z^i$ we get

$$dz^i = z^i_a du^a,$$

where

$$z^i_a = \frac{\partial z^i}{\partial u^a}.$$

The $m \times n$ matrix $\|z^i_a\|$ is of rank $n$, and satisfies the relations

$$\mu^\alpha_i z^i_a = 0.$$

Let $e^i_\alpha$ be a set of smooth functions, such that

$$\mu^\alpha_i e^i_\beta = \delta^\alpha_\beta.$$

Define the matrix $\|\pi^i_j\|$ by

$$\pi^i_j = \delta^i_j - e^i_\alpha \mu^\alpha_j.$$

Denote by $z^a_i$ the matrix elements of the pseudoinverse matrix of the matrix $\|z^i_a\|$, defined by the relations

$$z^a_i z^a_j = \pi^i_j.$$

As it follows from Appendix A, we have the relations

$$z^a_i z^i_b = \delta^a_b, \quad z^a_i e^i_\alpha = 0. \quad (B.1)$$

Now we can introduce a set of the vector fields $\partial/\partial z^i$, defined by

$$\frac{\partial}{\partial z^i} = z^a_i \frac{\partial}{\partial u^a}.$$
These vector fields satisfy the relations
\[ e^i_\alpha \frac{\partial}{\partial z^i} = 0, \]
and they are, in a sense, dual to the differentials \( dz^i \):
\[ dz^i \left( \frac{\partial}{\partial z^j} \right) = \pi^i_j. \] (B.2)

It is quite clear that the vector fields \( \partial/\partial z^i \) are independent of the choice of local coordinates \( u^a \). An arbitrary vector field \( X \) can be represented as
\[ X = X^i \frac{\partial}{\partial z^i}. \]

Note that the functions \( X^i \) in this relation are defined ambiguously. To make them unique we subject them to the relations
\[ \mu^\alpha_i X^i = 0. \] (B.3)

Analogously, for any one–form \( \varphi = \varphi_i dz^i \) the functions \( \varphi_i \) are fixed by the relations
\[ \varphi_i e^i_\alpha = 0. \]

Such a fixation allows one to write
\[ df = \frac{\partial f}{\partial z^i} dz^i \] (B.4)
for any function \( f \), and
\[ \varphi(X) = \varphi_i X^i \]
for any one–form \( \varphi \) and vector field \( X \).

Suppose now that the manifold \( N \) is a symplectic manifold with the symplectic two–form \( \omega \), having in the local independent coordinates \( u^a \) the representation
\[ \omega = \frac{1}{2} \omega_{ab} du^a \wedge du^b. \]

Let \( \| \omega^{ij} \| \) be the matrix composed by the Poisson brackets of the functions \( z^i \):
\[ \{ z^i, z^j \} = \omega^{ij}. \]

Using the coordinates \( u^a \), we can write
\[ \omega^{ij} = \omega^{ab} z^i_\alpha z^j_\beta, \] (B.5)
where \( \omega^{ab} = \{ u^a, u^b \} \). Hence, we have
\[ \mu^\alpha_i \omega^{ij} = 0. \] (B.6)

It is easy to understand that the matrix \( \| \omega^{ij} \| \) is of rank \( n \), therefore, the \( m \)–dimensional vectors \( \mu^\alpha_i \) form a complete system of the vectors, satisfying Eq. (B.6).
It is clear that the symplectic two–form \( \omega \) can be represented as
\[
\omega = \frac{1}{2} \omega_{ij} dz^i \wedge dz^j. \tag{B.7}
\]
The matrix \( \| \omega_{ij} \| \) is supposed to be skew symmetric, and is defined ambiguously. We fix it with the help of the conditions
\[
\omega_{ij} c^j_a = 0. \tag{B.8}
\]
Writing the obvious relation
\[
\omega_{ij} = \omega_{ab} z^a_i z^b_j,
\]
and comparing it with Eq. (B.5) we conclude that
\[
\omega^{ik} \omega_{kj} = \pi^{ij}.
\]
Thus, the matrix \( \| \omega_{ij} \| \) is the pseudoinverse matrix of the matrix \( \| \omega^{ij} \| \). Actually, the inverse statement is also true. Namely, if we have the representation of the symplectic two–form \( \omega \) of type (B.7), where the skew symmetric matrix \( \| \omega_{ij} \| \) satisfies relations (B.8), then the matrix \( \| \omega^{ij} \| \), giving the Poisson brackets of the functions \( z^i \), is the pseudoinverse matrix of the matrix \( \| \omega_{ij} \| \).

The Hamiltonian vector field \( X_f \), corresponding to the function \( f \) on \( N \), can be written as
\[
X_f = X_f^i \frac{\partial}{\partial z^i}.
\]
Using Eqs. (2.3), (B.2), (B.3) and (B.4) we get
\[
X_f^i = \frac{\partial f}{\partial z^j} \omega^{ji}.
\]
Hence, we have
\[
\{ f, g \} = \frac{\partial f}{\partial z^i} \omega^{ij} \frac{\partial g}{\partial z^j},
\]
and, in particular,
\[
X_f^i = \{ f, z^i \}. \tag{B.9}
\]
Thus, our conventions lead to the same formulae that we have in the case of independent coordinates.

In the present paper we suppose that the one parameter group of canonical transformations \( \Phi_\tau \), generated by a function \( \psi \), is defined by the equation
\[
\dot{\Phi}_\tau(p) = -X_\psi(\Phi_\tau(p)), \quad p \in N.
\]
From this equation for any function \( f \) we have
\[
\Phi_\tau^* f = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} \{ \ldots \{ f, \psi \}, \ldots, \psi \}. \tag{B.10}
\]
Consider now the case when the symplectic manifold \( N \) is a symplectic submanifold of an \( m \)-dimensional symplectic manifold \( M \). Let \( \Omega \) be the symplectic two–form on \( M \), and \( \iota : N \to M \) be the inclusion mapping. In this case we have

\[
\omega = \iota^* \Omega.
\]

We assume further that the submanifold \( N \) is a submanifold, defined by the equations

\[
\psi^\alpha = 0, \quad \alpha = 1, \ldots, m - n.
\]

For the functions \( \psi^\alpha \) we have

\[
\iota^* \psi^\alpha = 0,
\]

then

\[
\iota^* d\psi^\alpha = 0. \quad (B.11)
\]

Let \( Z^i, i = 1, \ldots, m \) be a set of independent local coordinates in \( M \). It is clear that the functions \( z^i = \iota^* Z^i \) form a set of dependent coordinates in \( N \). Using the representation

\[
d\psi^\alpha = \frac{\partial \psi^\alpha}{\partial Z^i} dZ^i,
\]

from Eq. (B.11) we get

\[
\iota^* \frac{\partial \psi^\alpha}{\partial Z^i} dz^i = 0.
\]

It is clear that we can put

\[
\mu^\alpha_i = \iota^* \frac{\partial \psi^\alpha}{\partial Z^i}.
\]

As it was proved in Section 2, at any point \( p \in N \) the matrix

\[
\tilde{\Xi}^{\alpha\beta}(p) = \iota^* \{ \psi^\alpha, \psi^\beta \}(p)
\]

is nondegenerate. Using the coordinates \( Z^i \), we can write

\[
\Omega = \frac{1}{2} \Omega_{ij} dZ^i \wedge dZ^j,
\]

hence,

\[
\{ Z^i, Z^j \} = \Omega^{ij},
\]

where \( \| \Omega^{ij} \| \) is the inverse matrix of the matrix \( \| \Omega_{ij} \| \). Thus, we have

\[
\tilde{\Xi}^{\alpha\beta} = \tilde{\Omega}^{ij} \mu^\alpha_i \mu^\beta_j, \quad \tilde{\Omega}^{ij} = \iota^* \Omega^{ij}.
\]

This equality allows us to put

\[
e^i_\alpha = \tilde{\Omega}^{ij} \mu^\beta_j \tilde{\Xi}_{\beta\alpha}, \quad (B.12)
\]

where \( \| \tilde{\Xi}_{\alpha\beta} \| \) is the inverse matrix of the matrix \( \| \tilde{\Xi}^{\alpha\beta} \| \).

For the symplectic two–form \( \omega \) we have

\[
\omega = \tilde{\Omega}_{ij} dz^i \wedge dz^j,
\]
where $\tilde{\Omega}_{ij} = \iota^*\Omega_{ij}$. Hence, the functions $\omega_{ij}$, satisfying Eq. (B.8), are given by

$$\omega_{ij} = \pi^k_i \tilde{\Omega}_{kl}^{\mu l} = \tilde{\Omega}_{ij} - \mu^\alpha_i \xi_{\alpha \beta}^j.$$

For the pseudoinverse matrix we get

$$\omega_{ij} = \tilde{\Omega}_{ij} - \epsilon^i_{\alpha} \xi_{\alpha \beta}^j ,$$

that with the help of Eq. (B.12) can be written as

$$\omega_{ij} = \iota^* \left( \Omega_{ij} - \Omega_{ik} \frac{\partial \psi^\alpha}{\partial Z^k} \frac{\partial \psi^\beta}{\partial Z^l} \Omega_{lj} \right) ,$$

or, equivalently,

$$\{ z^i, z^j \} = \iota^* \left( \{ Z^i, Z^j \} - \{ Z^i, \psi^\alpha \} \xi_{\alpha \beta}^j \{ \psi^\beta, Z^j \} \right) .$$

From this relation for any two functions, $F$ and $G$, on $M$ we get

$$\{ \iota^* F, \iota^* G \} = \iota^* \left( \{ F, G \} - \{ F, \psi^\alpha \} \xi_{\alpha \beta}^j \{ \psi^\beta, G \} \right) .$$

Thus if we define the Dirac bracket of the functions $F$ and $G$ by the usual relation

$$\{ F, G \}^* = \{ F, G \} - \{ F, \psi^\alpha \} \xi_{\alpha \beta}^j \{ \psi^\beta, G \} ,$$

then we have

$$\{ \iota^* F, \iota^* G \} = \iota^* \{ F, G \}^* .$$

C Three–Dimensional Rotation Group

Consider the space $\mathbb{R}^3$ as a three–dimensional linear space with the canonical basis

$$n_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad n_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \quad n_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .$$

The scalar product of two vectors $u = u_i n_i$ and $v = v_i n_i$ is defined by

$$uv = u_1 v_1 + u_2 v_2 + u_3 v_3 ,$$

whereas their vector product is

$$u \times v = \epsilon_{ijk} n_i u_j v_k ,$$

where $\epsilon_{ijk}$ are the components of the totally antisymmetric tensor with $\epsilon_{123} = +1$.

The three–dimensional rotation group consists of all nondegenerate linear transformations of $\mathbb{R}^3$ preserving its orientation and scalar product of the vectors. This group is denoted $SO(3)$, and its elements are called rotations. We specify an orientation in $\mathbb{R}^3$ considering the canonical basis as a right one. Note that the canonical basis is orthonormal. By definition, a rotation $R$ transforms the basis $\{ n_i \}$ to the right orthonormal basis $\{ e_i \}$:

$$e_i = R n_i .$$
We associate the matrix $R$ with the rotation $\mathcal{R}$ through the relation
\[ \mathcal{R} n_i = n_j R_{ji}. \]  
(C.1)

As the rotation $\mathcal{R}$ preserves the scalar product, the matrix $R$ satisfies the relation
\[ R_{ij} R_{kj} = \delta_{ik}, \]  
(C.2)

that can be written in matrix notations as
\[ RR^T = 1. \]  
(C.3)

From this relation it follows that $\det R = \pm 1$. Remembering that $\mathcal{R}$ preserves orientation, we see that actually
\[ \det R = 1. \]  
(C.4)

It is clear that any matrix $R$, satisfying relations (C.3) and (C.4), generates a rotation $\mathcal{R}$, defined by Eq. (C.1).

A simple geometric consideration shows that the rotation $\mathcal{R}(n_1, \varphi)$ by the angle $\varphi$ about the direction of $n_1$ is described by the formulas
\[ \mathcal{R}(n_1, \varphi) n_1 = n_1, \]
\[ \mathcal{R}(n_1, \varphi) n_2 = n_2 \cos \varphi + n_3 \sin \varphi, \]
\[ \mathcal{R}(n_1, \varphi) n_3 = -n_2 \sin \varphi + n_3 \cos \varphi. \]

Hence the matrix $R(n_1, \varphi)$ has the form
\[ R(n_1, \varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{pmatrix}. \]

Analogously, for $R(n_2, \varphi)$ and $R(n_3, \varphi)$ we get
\[ R(n_2, \varphi) = \begin{pmatrix} \cos \varphi & 0 & \sin \varphi \\ 0 & 1 & 0 \\ -\sin \varphi & 0 & \cos \varphi \end{pmatrix}, \quad R(n_3, \varphi) = \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

The rotation by the angle $\varphi$ about the direction of a unit vector $u$ is described by
\[ r' = \mathcal{R}(u, \varphi) r = r \cos \varphi + (u \times r) \sin \varphi + u(u r)(1 - \cos \varphi). \]  
(C.5)

Hence, we have
\[ R_{ij}(u, \varphi) = \delta_{ij} \cos \varphi + \epsilon_{ijk} u_k \sin \varphi + u_i u_j (1 - \cos \varphi). \]  
(C.6)

Let $Q$ be the matrix, corresponding to an arbitrary rotation $\mathcal{Q}$. From (C.6) we can easily get
\[ Q R(u, \varphi) Q^T = R(Q u, \varphi). \]  
(C.7)

It is quite clear that the Lie algebra of the group $SO(3)$ is formed by the matrices $A$, satisfying the relation
\[ A + A^T = 0. \]
A convenient basis in this Lie algebra is formed by the matrices

\[ M_i = \left. \frac{dR(n_i, \varphi)}{d\varphi} \right|_{\varphi=0}. \]

From Eq. (C.6) we get

\[ (M_i)_{jk} = \epsilon_{ijk} = -\epsilon_{ijk}, \]

and, therefore,

\[ [M_i, M_j] = \epsilon_{ijk} M_k. \]

A simple calculation gives

\[ \text{tr} (M_i M_j) = -2\delta_{ij}. \] (C.8)

It is clear also that the following equations are valid:

\[ \frac{dR(n_i, \varphi)}{d\varphi} = M_i R(n_i, \varphi) = R(n_i, \varphi) M_i. \] (C.9)

Writing relation (C.4) in the form

\[ \epsilon_{ijk} R_{il} R_{jm} R_{kn} = \epsilon_{lmn}, \] (C.10)

and using Eq. (C.2), we get

\[ R_{il} \epsilon_{ijk} R_{kn} = R_{jm} \epsilon_{lmn}. \]

This relation in matrix notation has the form

\[ R^T M_i R = R_{ij} M_j. \] (C.11)

Analogously, from Eq. (C.10) for an arbitrary rotation \( \mathcal{R} \) and arbitrary vectors \( \mathbf{u}, \mathbf{v} \) we get the relation

\[ (\mathcal{R} \mathbf{u}) \times (\mathcal{R} \mathbf{v}) = \mathcal{R} (\mathbf{u} \times \mathbf{v}). \] (C.12)

An arbitrary rotation \( \mathcal{R} \) can be parameterized by the Euler angles with the help of the relation

\[ \mathcal{R}(\alpha, \beta, \gamma) = \mathcal{R}(n_3, \alpha) \mathcal{R}(n_2, \beta) \mathcal{R}(n_3, \gamma), \]

where \( 0 \leq \alpha < 2\pi, 0 \leq \beta \leq \pi, \) and \( 0 \leq \gamma < 2\pi. \)

Let \( \{e_i\} \) be an arbitrary orthonormal basis in \( \mathbb{R}^3 \). In this case we have

\[ e_i e_j = \delta_{ij}. \] (C.13)

Expanding the vectors \( e_i \) over the canonical basis \( \{n_i\} \):

\[ e_i = E_{ij} n_j, \] (C.14)

we get the matrix \( E = (E_{ij}) \). From Eq. (C.13) it follows that

\[ EE^T = 1. \] (C.15)
The basis \( \{e_i\} \) is a right one if and only if

\[ \det E = 1. \quad (C.16) \]

Hence, any right orthonormal basis in \( \mathbb{R}^3 \) is connected with an element of the group SO(3). On the other hand, any matrix \( E \), satisfying relations (C.15) and (C.16), specifies a right orthonormal basis in \( \mathbb{R}^3 \) by Eq. (C.14). Thus, we see that there is a one-to-one correspondence between the set of all right orthonormal bases in \( \mathbb{R}^3 \) and the group SO(3).

Note that condition (C.16) can be written as

\[ e_i(e_j \times e_k) = \epsilon_{ijk}, \]

or as

\[ e_i \times e_j = \epsilon_{ijk}e_k. \]
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