Distribution of extremes in the fluctuations of two-dimensional equilibrium interfaces

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We investigate the statistics of the maximal fluctuation of two-dimensional Gaussian interfaces. Its relation to the entropic repulsion between rigid walls and a confined interface is used to derive the average maximal fluctuation $\langle m \rangle \sim \sqrt{2/(\pi K)} \ln N$ and the asymptotic behavior of the whole distribution $P(m) \sim N^2 e^{-2m} \int e^{-\sqrt{2m} r} \, dr$ for $m$ finite with $N^2$ and $K$ the interface size and tension, respectively. The standardized form of $P(m)$ does not depend on $N$ or $K$, but shows a good agreement with Gumbel’s first asymptote distribution with a particular non-integer parameter. The effects of the correlations among individual fluctuations on the extreme value statistics are discussed in our findings.

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A common nature of floods, stock market crashes, and the Internet failures is that they hardly occur but, once they do, have significant consequences for the corresponding systems. These rare but fatal events correspond to the appearances of extreme values in the fluctuating variables such as the daily discharge of a river, the stock index, and the router load, and therefore it has been in great demand to be able to estimate and predict the typical magnitude of the largest values and the probability of a given extreme value. The extreme value theory has been of great importance also in various physics contexts since the physics of complex systems is often governed by the extreme value statistics. Recently, Bramwell and co-workers discovered a universal distribution for the order parameters of a class of model systems for ferromagnet, confined turbulent flow, avalanche, and granular media. The universal distribution turns out to be consistent with Gumbel’s first asymptote distribution that is one of the well-known probability distributions, to be precise, of the $r$th largest value among $N$ independent (uncorrelated) random variables. Interestingly, the parameter $n$ takes a non-integer value and this has been considered as related to strong correlations in those systems. While the reason for this universality is still debated, it renewed the interest in the extreme value statistics of correlated variables.

In this Letter we study the statistics of the maximal fluctuation of two-dimensional (2D) equilibrium interfaces. The Gaussian ensemble is considered, where an interface configuration $\{\phi\} = \{\phi(\vec{r})\} = \{x,y\}, -L/2 \leq x,y < L/2\}$ under the periodic boundary condition has the probability $P(\{\phi\}) \equiv e^{-2H(\{\phi\})}$ with the Gaussian Hamiltonian $H_0(\{\phi\}) = \int d^2r (\nabla \phi)^2$ and $K$ the interface tension. The spatial average $\bar{\phi} = L^{-2} \int d^2r \phi(\vec{r})$ is set to be zero. This Gaussian model has been used to study rough (self-affine) interfaces ubiquitously in nature, the interacting spin system at low temperatures through the 2D XY model with $\phi$ representing spin orientations, and so on. We define the maximal fluctuation for a given interface $\{\phi\}$ as $m = \max_{\vec{r} \in R} \{\phi(\vec{r})\}$, where $R \equiv \{(a_n x, a_n y) | n_1, n_2 = -N/2 + 1, -N/2 + 2, \ldots, N/2\}$ with $a$ the lattice constant and $N = \sqrt{2m}$ the interface size. It is an extreme value among the individual fluctuations, $\phi$’s, that are correlated via $H_0$ as $\langle \phi(\vec{r}) \phi(\vec{r}') \rangle \sim \ln [L/|\vec{r} - \vec{r}'|]$ and we study the statistics of $m$. The effects of such correlations have been studied by a renormalization group approach in the context of glass transition, but the nature of the statistics of $m$ remains to be understood. The maximal fluctuation has its own technological significance as well, for instance, in the onset of the breakdown of corroded surfaces and the occurrence of a short circuit by the metal surface of one electrode reaching the opposite one.

The exact distributions of the maximal height and the width square have been obtained for the 1D Gaussian interface. For two dimensions, however, the distribution of the maximal fluctuation is not known while the width-square distribution is well understood. Of our particular interest are the functional form of the maximal fluctuation distribution, $P(m)$, and its first few moments. We use the relation of $P(m)$ to the entropic repulsion between an interface and rigid walls to find $m \equiv \int dm \, m P(m)$ and the asymptotic behavior of $P(m)$, which is shown to resemble Gumbel’s first asymptotic distribution. The standardized distribution of the maximal fluctuation does not depend on any system parameter, and furthermore, Gumbel’s first asymptote distribution with a specific parameter fits excellently the standardized distribution of $m$. These results are compared with the statistics that would appear without the correlations among $\phi(\vec{r})$’s to illuminate the correlation effect on the extreme value statistics.

When an interface is confined between two rigid walls, the free energy is increased due to the entropic repulsion against the walls. The maximal fluctuation distribution $P(m)$ of a free interface is related to this free energy increase with the walls at $\phi = m$ and $\phi = -m$, $\Delta f(m)$, as

$$e^{-\Delta f(m)} \equiv \frac{\text{Tr} e^{-2H_0(\{\phi\})} \chi_m(\{\phi\})}{\text{Tr} e^{-2H_0(\{\phi\})}} = \int_0^m dm' P(m'),$$

where $\chi_m(\{\phi\})$ is 1 for $m = \max_{\vec{r} \in R} \{\phi(\vec{r})\} \leq m$ and 0 otherwise, and thus $\chi_m(\{\phi\}) = \prod_{\vec{r} \in R} \theta(m - |\phi(\vec{r})|)$ with $\theta(x) = 1$ for $x \geq 0$ and 0 otherwise. We first derive $\Delta f(m)$ and obtain $P(m)$ using Eq. (1).

For the soft walls restricting the width square $w_2 = \phi(\vec{r})^2$ to be less than $x^2$, the free energy increase is given as $\Delta f(x) \sim e^{-(\text{const}) x^2}$.
proportional to the projected area of the interface segment \( \ell^2 \) that is so large that the average fluctuation over the segment, \( \sim \sqrt{\ln \ell} \), is comparable to \( x \). For the rigid-wall problem, one may expect the same result if the average fluctuation and the maximal fluctuation scale as functions of the segment area in the same way, which is true for 1D interface [12, 13], but not for the 2D one. Bricmont et al. have shown, by means of a rigorous proof as well as an heuristic argument, that \( \Delta F(x) \sim e^{-c|x|} \) for 2D confined Gaussian interfaces in the thermodynamic limit \( N \to \infty \) [13].

Reviewing briefly the heuristic argument introduced therein, we compute \( \Delta F(x) \) keeping the lateral size \( N \) large but finite, which leads us to find out \( \langle m \rangle \) and two distinctive behaviors of \( P(m) \) for \( m \ll \langle m \rangle \) and \( m \gg \langle m \rangle \), respectively.

One of the most prominent effects of confinement on an interface is the reduction of the correlation length; the height fluctuations are hardly correlated across the interface-wall collisions. The correlation length can be obtained in a self consistent way by assuming that the effective Hamiltonian \( \mathcal{H}_{\text{eff}}(x) \) satisfying \( \text{Tr} e^{-\Delta \mathcal{H}_{\text{eff}}(x)} / \text{Tr} e^{-\mathcal{H}_0} = e^{-\Delta F(x)} \) with \( \Delta F(x) \) in Eq. (1) takes the form of a massive Gaussian Hamiltonian, \( \mathcal{H}_{\text{eff}}(x) = \int d^2r [(K/2)(\nabla \phi)^2 + (\mu(x)/2)(\phi(x))^2] \). The effective mass \( \mu(x) \) is the inverse of the correlation length and obtained self-consistently. Differentiating Eq. (1) with respect to \( m \) and using \( \mathcal{H}_{\text{eff}}(x) \), one obtains an approximate expression for the derivative of the free energy increase as

\[
-\frac{\partial}{\partial x} \Delta F(x) = \frac{\text{Tr} e^{-\mathcal{H}_0} \int \frac{d^2r}{2\pi} \delta(x - |\phi(\vec{r})|) \prod_{\vec{r} \neq \vec{r}'} \theta(x - |\phi(\vec{r'})|)}{\text{Tr} e^{-\mathcal{H}_0} \chi_x} \approx \int \frac{d^2r}{2\pi} \delta(x - |\phi(\vec{r})|) \langle \mathcal{H}_{\text{eff}}(x) \rangle \approx \frac{2N^2}{\sqrt{2\pi W(x)^2}} e^{-x^2/(2W(x)^2)},
\]

where \( e^{-\mathcal{H}_0} \chi_x \) is approximated by \( e^{-\mathcal{H}_{\text{eff}}(x)} \) and \( \langle A \rangle_{\mathcal{H}_{\text{eff}}(x)} = \text{Tr} A e^{-\mathcal{H}_{\text{eff}}(x)}/\text{Tr} e^{-\mathcal{H}_{\text{eff}}(x)} \). Also we used the relation that \( \langle \delta(\phi - x) \rangle_{\mathcal{H}_{\text{eff}}(x)} = e^{-x^2/(2W(x)^2)} / \sqrt{2\pi W(x)^2} \) with the roughness \( W(x) = \sqrt{\langle \phi(\vec{r})^2 \rangle_{\mathcal{H}_{\text{eff}}(x)}} \) given by

\[
W(x) = \left( \frac{\sum_q K|q|^2 + \mu(x)^2}{2q} \right)^{1/2}.
\]

For Eq. (3), we considered the Fourier components \( \phi_q = L^{-1} \int d^2r e^{-i\vec{q}\cdot\vec{r}} \phi(\vec{r}) \) for \( q = (\pi/L)(2m_n - 1, 2n_y - 1) \), which are decoupled in \( \mathcal{H}_{\text{eff}}(x) \), allowing another expression for \( -(\partial/\partial x) \Delta F(x) \) as

\[
-\frac{\partial}{\partial x} \Delta F(x) \approx \frac{1}{\text{Tr} e^{-\mathcal{H}_0}} \ln \left( \frac{\text{Tr} e^{-\mathcal{H}_{\text{eff}}(x)}}{\text{Tr} e^{-\mathcal{H}_0}} \right) \approx \frac{1}{2} \left( \frac{\partial}{\partial x} \mu(x)^2 \right) \sum_q K|q|^2 + \mu(x)^2.
\]

Equations (2)–(4) give a self-consistent equation for \( \mu(x) \), which can be solved numerically with the condition \( \mu(\infty) = 0 \).

\[\text{FIG. 1: Values of } -\Delta F(x)/\partial x \text{ with } K = 0.92 \text{ calculated by substituting } P(m) \text{ from the Monte Carlo simulation to Eq. (1) } [N = 32 (\square), 64 (\circ), 128 (\triangle)] \text{ and by the self-consistent solution to Eqs. (2)–(4) } [N = 32 \text{ (dotted line), 64 \text{ (dotted-dashed), 128 (dashed))}.}\]

\[\text{FIG. 2: (a) Plots of } \ln \langle m \rangle \text{ and } \ln W \text{ versus } \ln \langle m \rangle \text{ for 2D free Gaussian interfaces with } K = 0.50, 0.92, \text{ and } 1.50. \text{ The upper solid line has the slope 1 while the lower dashed line has 1/2. The dotted lines connecting data points are guide for the eye. (b) Semilog plot of } \langle m \rangle \text{ versus } N \text{ for the same values of } K \text{ as in (a). The lines have the slopes } \sqrt{2/\pi K}, \text{ respectively.}\]

A comparison of the values of \( -\partial \Delta F(x)/\partial x \) from this equation and from a Monte Carlo simulation for free interfaces giving \( P(m) \) and in turn, \( -\partial \Delta F(x)/\partial x \) through Eq. (1), can be a test for the validity of the adopted approximation. As shown in Fig. 1 both are in excellent agreement except for a slight constant deviation that is presumably due to higher-order (in \( \phi^2 \)) contributions to the singular factor \( \theta(x - |\phi|) = \lim_{x \to -\infty} e^{-\langle\phi(x)^2\rangle} \). This justifies the use of \( \mathcal{H}_{\text{eff}}(x) \) for the study of the maximal fluctuation distribution of a free interface as well as the free energy increase of a confined interface.

The large-\( x \) behavior of the effective mass \( \mu(x) \) can be derived analytically. From Eq. (4), \( W(x)^2 \) is represented as \( (2\pi K)^{-1} \ln [\mu_0/\mu(x)] \) for \( \mu_0/\mu(x) \ll N \) and \( (2\pi K)^{-1} \ln N \).
for $\mu_0/\mu(x) \gg N$, where $\mu_0 = \sqrt{K}\pi/a$.
Solving Eqs. 2 and 3 with these values of $W(x)^2$, one finds that, for large $x$, $\mu(x)/\mu_0 \sim e^{-\sqrt{2\pi K}/\sqrt{x}}$ for $1 \ll x \ll x_c$ and $\mu(x)/\mu_0 \sim e^{-\pi k x^2/(2\ln N)}$ for $x \gg x_c$ with $x_c = \sqrt{2/(\pi K) \ln N}$. It is natural to think that the interaction between the walls and the interface is negligible and that the correlation length diverges when the distance to the wall $x$ is much larger than $\langle m \rangle$. Therefore, we expect the characteristic distance $x_c$ to be equal to $\langle m \rangle$,

$$\langle m \rangle \approx \sqrt{\frac{2}{\pi K} \ln N}.$$  

(5)

This relation is in agreement with the result in Ref. [11] and is also verified by our Monte Carlo simulation data for $\langle m \rangle$ as a function of $N$ plotted in Fig. 2. The scaling of $\langle m \rangle$ is distinguishable from that of the roughness $W \sim \sqrt{\ln N}$ [Fig. 4]. This is contrasted to the 1D Gaussian interface, where both the maximal fluctuation and the roughness scale as $\sqrt{N}$ [12-13].

Using the obtained functional form of $\mu(x)$, one can also obtain the free energy increase $\Delta f(x)$ and in turn, the maximal fluctuation distribution $P(m)$ through Eq. 1:

$$P(m) \sim \begin{cases} N^2 e^{-(\text{const}) N^2 e^{-\sqrt{2\pi K} m - \sqrt{2\pi K} m}} (1 \ll m \ll \langle m \rangle), \\ N^2 e^{-(\text{const}) N^2 e^{-\frac{\pi K x^2}{2\ln N}} - \frac{\pi K x^2}{2\ln N}} (m \gg \langle m \rangle). \end{cases}$$

(6)

The asymptotic behavior of $P(m)$ for $m \ll \langle m \rangle$ in Eq. 6 is consistent with Gumbel’s first asymptote distribution that represents the limiting distribution of the $n$ th largest value, $m_n$, among $\mathcal{X} \rightarrow \infty$ independent random variables following an exponential-type distribution [11]. In terms of the standardized variable $z = (m_n - \langle m \rangle)/\sigma_{m_n}$ with $\langle m \rangle$ and $\sigma_{m_n}$ the average and the standard deviation of $m_n$, Gumbel’s first asymptote distribution takes the standardized form $g(z;n)$ given by

$$g(z;n) = \omega e^{-n [e^{b(z-n)} - b(z-n)]},$$

where $b = \sqrt{\psi(n)}$, $s = (\ln n - \psi(n))/b$, and $\omega = n^b/\Gamma(n)$ with $\Gamma(x)$ the Gamma function and $\psi(x) = d\ln \Gamma(x)/dx$. 4.

It is $g(z;n)$ with a non-integer value of $n$ that fits the universal distribution in a class of correlated systems [3-4]. The 2D Gaussian interface belongs to this class since the standardized distribution of the width square $Q_{w_2}(z) = \sigma_{w_2} P(w_2 = \langle w_2 \rangle + z\sigma_{w_2})$ with $\langle w_2 \rangle$ and $\sigma_{w_2}$ average and the standard deviation of $w_2$, respectively, is fitted by $g(z;n = n_{w_2} \approx 1.58)$ without regard to the system size $N$ or tension $K$ [3-4]. To check whether such a unique distribution also exists for the maximal fluctuation, we computed the skewness of $P(m)$, $\gamma_m \equiv \langle (m - \langle m \rangle)^3 \rangle/\sigma_m^3$ with $\sigma_m$ the standard deviation of $m$. According to our data shown in Fig. 3, $\gamma_m$ hardly varies with $N$ or $K$, but stays at 0.68(2), which leads us to expect a similar universality in the maximal fluctuation statistics to in the width-square one. That particular value of the skewness $\gamma_m$ is reproduced in Gumbel’s first asymptote distribution with $n = n_{w_2} \approx 2.6$ as shown in Fig. 3 and we compare the standardized distribution $Q_m(z) = \sigma_m P(m = \langle m \rangle + z\sigma_m)$ for different values of $N$ and $K$ with Gumbel’s first asymptote distribution $g(z;n_{m})$ in Fig. 4. One can identify the agreement of $Q_m(z)$’s and $g(z;n_{m})$ as well as the data collapse of $Q_m(z)$’s for different $N$’s and $K$’s. In addition, this agreement enables one to predict that $\sigma_m$ is given by $\sigma_m = b/\sqrt{2\pi K}$ with $b = \sqrt{\psi(n_{w_2})} \approx 0.69(3)$, which is confirmed numerically in Fig. 4.

Now let us look into the effects of the correlations among $\phi(\vec{r})$’s on $\langle m \rangle$ in Eq. 5 and on $P(m)$ in Eq. 6. For comparison, we consider the statistics of $m$ that would emerge with no correlation for a Gaussian distribution of $\mathcal{X}$ individual fluctuations $p(\phi) = e^{-\phi^2/(2W^2)}/\sqrt{2\pi W^2}$. In this case we could apply the following relation to obtain $\langle m \rangle$: $\int_{-\infty}^{\langle m \rangle} \phi^2 p(\phi) \approx (\mathcal{X} - 1)/\mathcal{X}$ [1]. This relation yields $\langle m \rangle \sim W \sqrt{\ln(\mathcal{X}/W)}$ [12]. The
\sqrt{\ln N}$-scaling of \( \langle m \rangle \) is thus generic when the roughness \( W \) is finite. A different scaling of \( \langle m \rangle \) shows up with a diverging roughness. For instance, \( \langle m \rangle \sim \sqrt{N \ln N} \) in 1D Gaussian interfaces where \( N = N \) and \( W \sim \sqrt{N} \) and \( \langle m \rangle \sim \ln N \) in 2D ones where \( N = N^2 \) and \( W \sim \sqrt{\ln N} \). While the \( \sqrt{N \ln N} \) behavior deviates from the true behavior of \( \langle m \rangle \) in the 1D Gaussian interface \([12, 13]\), the \( \ln N \) behavior in two dimensions is consistent with Eq. (5). Without the correlations, the distribution of the maximal fluctuation could also be obtained by \([1]\) \( P(m) \simeq 2 \mathcal{N} \left( \frac{m}{m_{\text{av}}} d \phi \right)^{N-1} \phi(m) \). This gives \( P(m) \sim \mathcal{N} e^{-(\text{const})N e^{-m^2/2(W^2)-m^2/(2W^2)}} \) as \( P(m) \) for \( m \gg \langle m \rangle \), but is not capable of reproducing the behavior of \( P(m) \) for \( m \) finite in Eq. (6), which demonstrates the essential role of the correlations. The extreme statistics without correlations has been found in small-world networks where the correlation length is finite due to random links \([13]\). The correlation length becomes finite also in the presence of external field \([20]\). \( P(m) \) is more skewed, that is, the skewness is larger, with larger external field \([21]\). On the contrary, the width-square distribution recovers its Gaussian form as it loses the correlations \([20]\).

Our results demonstrate the availability of Gumbel’s first asymptote distribution with a non-integer parameter \( n \) for the maximal fluctuation distribution of the 2D Gaussian interface. The value of \( n \) can be different according to the type of the correlations: It takes a value close to \( \pi/2 \) in the case of the maximal avalanche distribution in the Snepen depinning model \([8]\), different from 2.6(2) found for the maximal fluctuation distribution in this work. We remark that the value of \( n_m \) is larger than \( n_{w_2} \), that is, \( \gamma_{w_2} > \gamma_m \), implying that the width-square distribution is more non-Gaussian than the maximal fluctuation distribution.

In summary, we investigated the statistics of the maximal fluctuation in the 2D Gaussian interface. Because of the correlations among individual fluctuations, the average maximal fluctuation as well as the whole distribution exhibit distinct behaviors from those of uncorrelated fluctuations. We also identified the appropriateness of Gumbel’s first asymptote distribution with a non-integer parameter for the maximal fluctuation statistics, which suggests its applicability to the extreme value statistics in a class of correlated systems with the parameter depending on the correlations.

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