The optimized perturbation theory (OPT) at finite temperature \( T \) recently developed by the present authors is reviewed by using \( O(N) \phi^4 \) theory with spontaneous symmetry breaking. The method resums automatically higher loops (including the hard thermal loops) at high \( T \) and simultaneously cures the problem of tachyonic poles at relatively low \( T \). We prove that (i) the renormalization of the ultra-violet divergences can be carried out systematically in any given order of OPT, and (ii) the Nambu-Goldstone theorem is satisfied for arbitrary \( N \) and for any given order of OPT.

II. OPTIMIZED PERTURBATION AT \( T \neq 0 \)

A. Hard thermal loops and tachyonic poles

Let us illustrate, by using \( \phi^4 \) theory, the reason why the naive perturbation theory at finite \( T \) breaks down:

\[
\mathcal{L} = \frac{1}{2}[(\partial \phi)^2 - \mu^2 \phi^2] - \frac{\lambda}{4!}\phi^4.
\]

We first consider the case \( \mu^2 > 0 \). The lowest order self-energy diagram Fig.1 (A) is \( O(\lambda T^2) \) at high \( T \). However, Fig.1 (B) is \( O(\lambda T^2 \times \frac{T^2}{\mu^2}) \). Furthermore, higher powers of \( T/\mu \) arise in higher loops; e.g. the n-loop diagram in Fig.1 (C) is \( O(\lambda^n T^{2n-1}/\mu^{2n-3}) \). Thus, the validity of the perturbation theory breaks down when \( T > \mu/\lambda \) because the higher order diagrams are larger than lower ones. Therefore, one should at least resum cactus diagrams to get sensible results at high \( T \). Physics behind this resummation is the well-known Debye screening mass in the hot plasma.

When \( \mu^2 < 0 \) and the system has spontaneous symmetry breaking (SSB), the naive perturbation shows another problem. The tree-level mass \( m_0 \) in this case is defined as

\[
m_0^2 = \mu^2 + \frac{\lambda}{2}\xi^2(T),
\]

where \( \xi(T) \) is the thermal expectation value of \( \phi \). As \( T \) increases, \( \xi \) decreases. Then \( m_0^2 \) becomes negative (tachyonic) even before the critical temperature \( T_c \) is reached. If this happens, the naive perturbation using the tree-level propagator does not make sense and certain resummation should be carried out \([10]\). Note that, for \( T < T_c \), there is no reason to believe that only the cactus diagrams shown in Fig.1 are dominant; there exists a three-point vertex \( \lambda \xi \phi^3 \) which is not negligible for \( T \sim \xi(T) \).
B. Problems in self-consistent resummation methods

Self-consistent resummation method is a procedure to improve perturbation theory at finite $T$ and to avoid the problems in Sec. II A. However, the method has other difficulties. In the naive perturbation theory, there arises no new UV divergences at $T \neq 0$ because of the natural cutoff from the Boltzmann distribution function. Therefore, all the UV divergences at finite $T$ are canceled by the counter terms prepared at $T = 0$.

On the other hand, in self-consistent methods at $T \neq 0$, the situation is not that simple: In fact, the tree-level propagators have $T$-dependent mass (such as $m(T)$ in the above) which contains higher loop contributions through the self-consistent gap-equation. This leads to a necessity of $T$-dependent counter terms which are sometimes introduced in ad hoc ways.

Another problem is the violation of the Nambu-Goldstone (NG) theorem: In many of the self-consistent methods, resummation with keeping symmetry is a non-trivial issue, and the NG theorem is often violated.

C. New resummation method

For theories with SSB, loop-expansion rather than the weak-coupling expansion is relevant, since one needs to treat the thermal effective potential. Therefore, we developed an improved loop-expansion at finite $T$ for the purpose of resummation. The method keeps the renormalizability and guarantees the Nambu-Goldstone theorem order by order at finite $T$.

In the following, we divide our resummation procedure into three steps and apply it to $\phi^4$ theory. The case for $O(N)$ $\phi^4$ theory will be discussed in Sec. III B.

We start with the thermal effective action with an expansion parameter $\delta$:

$$\Gamma[\varphi^2] = \ln \int [d\phi] \exp \left[ \frac{1}{\delta} \int_0^{1/T} d^nx \left[ \mathcal{L}(\phi + \varphi; \mu^2) + J\phi \right] \right] ,$$

where $J = -\partial \Gamma[\varphi]/\partial \varphi$ and $\int_0^{1/T} d^nx \equiv \int_0^{1/T} dt \int d^{n-1}x$. If we explicitly write $\hbar$ in eq. (3), it appears as $\frac{1}{\hbar} \int_0^{1/T} d^nx \mathcal{L}$. Therefore, the loop-expansion by $\delta$ at finite $T$ does not coincide with the $\hbar$ expansion. The expansion by $\delta$ should be regarded as a steepest descent evaluation of the functional integral.

**Step 1**

Start with a renormalized Lagrangian with counter terms

$$\mathcal{L}(\phi; \mu^2) = \frac{1}{2}[(\partial \phi)^2 - \mu^2 \phi^2] - \frac{\lambda}{4!} \phi^4$$

$$+ \frac{1}{2} A(\partial \phi)^2 - \frac{1}{2} B\mu^2 \phi^2 - \frac{\lambda}{4!} C \phi^4 + D \mu^4 . \tag{4}$$

Here we have explicitly written the argument $\mu^2$ in $\mathcal{L}$ for later use. The $\bar{MS}$ scheme with the dimensional regularization is assumed in (4). Just for notational simplicity, the factor $\kappa^{(4-n)}$ to be multiplied to $\lambda$ is omitted ($\kappa$ is the renormalization point).

The $c$-number counter term $D \mu^4$, which was not considered in (4), is necessary to make the thermal effective potential finite. Also, it plays a crucial role for renormalization in OPT as will be shown in Sec. III D.

The renormalization constants are completely fixed at $T = 0$ and $\mu^2 > 0$, in which $A, B, C$ and $D$ are expanded as

$$\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \sum_{l=1}^{\infty} \left( \begin{array}{c} a_l \\ b_l \\ c_l \\ d_l \end{array} \right) \delta^l . \tag{5}$$

Note that (i) the coefficients $(a_l, b_l, c_l, d_l)$ are independent of $\mu^2$, since we use the mass independent renormalization scheme, (ii) the UV divergences in the symmetry broken phase ($\mu^2 < 0$) can be removed by the same counter terms determined for $\mu^2 > 0$ and (iii) $A, B, C$ and $D$ are independent of $T$ by definition.

The relations of $A, B, C$ and $D$ with the standard renormalization constants are $A = Z - 1$, $B = Z_\mu Z - 1$ and $C = Z_\chi Z^2 - 1$, where $Z's$ are defined by $\phi_0 = \sqrt{Z} \phi$, $\lambda_0 = Z_\lambda \lambda$ and $\mu_0^2 = Z_{\mu} \mu^2$ with suffix 0 indicating unrenormalized quantities.

**Step 2**

Rewrite the Lagrangian (4) by introducing a new mass parameter $m^2$ following the idea of OPT (4):

$$\mu^2 = m^2 - (m^2 - \mu^2) \equiv m^2 - \chi . \tag{6}$$

This identity should be used not only in the standard mass term but also in the counter terms (4), which is crucial to show the order by order renormalization in OPT:

$$\mathcal{L}(\phi; \mu^2) = \mathcal{L}_r + \mathcal{L}_c$$

$$\mathcal{L}_r = \frac{1}{2}[(\partial \phi)^2 - m^2 \phi^2] - \frac{\lambda}{4!} \phi^4 + \frac{1}{2} \chi \phi^2$$

$$\mathcal{L}_c = \frac{1}{2} A(\partial \phi)^2 - \frac{1}{2} B(m^2 - \chi) \phi^2 - \frac{\lambda}{4!} C \phi^4$$

$$+ D(m^2 - \chi)^2 . \tag{9}$$

$A, B, C$ and $D$ in $\mathcal{L}_c$ were already determined in Step 1.

On the basis of eq. (5), we define a “modified” loop-expansion in which the tree-level propagator has a mass $m^2 + \lambda \varphi^2/2$ instead of $\mu^2 + \lambda \varphi^2/2$. Major difference between this expansion and the naive one is the following assignment,

$$m^2 = O(\delta^0) , \quad \chi = O(\delta) . \tag{10}$$

The physical reason behind this assignment is the fact that $\chi$ reflects the effect of interactions. If one makes an
assignment, $m^2 = O(\delta^0)$, $\chi = O(\delta^0)$, the modified loop-expansion immediately reduces to the naive one.

Since eq. (11) is simply a reorganization of the Lagrangian, any Green’s functions (or its generating functional) calculated in the modified loop-expansion should not depend on the arbitrary mass $m$ if they are calculated in all orders. However, one needs to truncate perturbation series at certain order in practice. This inevitably introduces explicit $m$ dependence in Green’s functions. Procedures to determine $m$ are given in Step 3 below.

To find the ground state of the system, one should look for the stationary point of the thermal effective potential defined by $V(\varphi^2) = -\Gamma[\varphi^2 = \text{const.}] / \int_{0}^{L/T} d^4x$. As mentioned above, $V$ calculated up to $L$-th loops $V_L(\varphi^2; m)$ has explicit $m$-dependence. Thus the stationary condition reads

$$\frac{\partial V_L(\varphi^2; m)}{\partial \varphi} = 0,$$

(11)

where derivative with respect to $\varphi$ does not act on $m$ by definition. Eq. (11) gives a stationary point of $V_L$ for given $m$.

**Step 3**

The final step is to find an optimal value of $m$ by imposing physical conditions à la Stevenson (13) such as the following.

(a) The principle of minimal sensitivity (PMS): this condition requires that a chosen quantity calculated up to $L$-th loops ($O_L$) should be stationary by the variation of $m$:

$$\frac{\partial O_L}{\partial m} = 0.$$  

(12)

(b) The criterion of the fastest apparent convergence (FAC): this condition requires that the perturbative corrections in $O_L$ should be as small as possible for a suitable value of $m$.

$$O_L - O_{L-l} = 0,$$

(13)

where $l$ is chosen in the range, $1 \leq l \leq L$.

The above conditions reduce to self-consistent gap equations whose solution determine the optimal parameter $m$ for given $L$. Thus $m$ becomes a non-trivial function of $\varphi$, $\lambda$ and $T$. This together with the solution of (11) completely determine the thermal expectation value $\xi(T) \equiv \langle \phi \rangle$ as well as the optimal parameter $m(T)$. Through this self-consistent process, higher order terms in the naive loop expansion are resumed.

The choice of $O_L$ in Step 3 depends on the quantity one needs to improve most. To study the static nature of the phase transition, the thermal effective potential $V_L(\varphi^2; m)$ is most relevant. Applying the PMS condition for $V_L$ reads

$$\frac{\partial V_L(\varphi^2; m)}{\partial m} = 0,$$

(14)

which gives a solution $m = m(\varphi)$. This can be used to improve the effective potential at finite $T$ as $V_L(\varphi^2; m) \rightarrow V_L(\varphi^2; m(\varphi))$. Also, $\xi(T)$ and $m(T)$ are obtained by solving (11) together with (14). In this case, the following relation holds: $dV(\varphi^2; m(\varphi))/d\varphi|_{\varphi=\xi} = \partial V(\varphi^2; m(\varphi))/\partial \varphi|_{\varphi=\xi}$.

To improve particle properties at finite $T$, it is more efficient to apply PMS or FAC conditions directly to the two-point functions. We will use FAC for the one-loop pion self-energy in Section III to show its usefulness.

**D. Renormalization in OPT**

We now prove the order by order renormalization in OPT. Let us first rewrite eq. (3) as

$$\mathcal{L}(\phi; m^2) = \mathcal{L}(\phi; m^2) + \frac{1}{2} \chi \phi^2 + \left[ \frac{1}{2} B \chi \phi^2 + D \chi^2 - 2 D m^2 \right].$$

(15)

The UV divergences arising in the perturbation theory are classified into two classes: The divergences in the Green’s function generated by $\mathcal{L}(\phi; m^2)$, and the divergences obtained by the multiple insertion of $(1/2) \chi \phi^2$ to the Green’s function generated by $\mathcal{L}(\phi; m^2)$.

Since we use the symmetric and mass independent renormalization scheme (such as the MS scheme), any divergences in the first class are renormalized solely by the coefficients $A$, $B$, $C$ and $D$ in $\mathcal{L}(\phi; m^2)$. Although $T$-dependent divergences appear because of the $T$-dependent “resumed” mass $m^2(T)$, they are properly renormalized away since the counter terms (such as $B m^2 \phi^2$, $D m^4$ and $D m^4$) also acquires $T$-dependence through $m^2(T)$. In other words, the divergences arising from the resumed propagator is removed by the resumed counter terms. (See also, footnote 1.)

The divergences in the second class can be shown to be removed by the last three counter terms in (13). (Note that $B$ and $D$ are already fixed in Step 1, and we do not have any freedom to change them.) This is obviously related to the renormalization of composite operators. In fact, the standard method (16) tells us that necessary counter terms are written as

---

1 One may generalize Step 2 by adding and subtracting $\alpha_0 (\partial \phi)^{2}, \alpha_1 (\partial \phi)^{2}$ and $\gamma \phi^{4}$ with $\alpha_0$, $\alpha_1$ and $\gamma$ being finite parameters to be determined by the PMS or FAC conditions (see Step 3). The renormalizability can be also shown to be maintained in this case. However, we will concentrate on the simplest version ($\alpha_{0,1} = \gamma = 0$) in the following discussions.
\[
\frac{1}{2} (ZZ^{-1}_\phi^2 - 1) \chi \phi^2 + \Delta_2 \chi^2 + \Delta_1 \chi. \tag{16}
\]

Here \(Z_{\phi^2}\) is the renormalization constant for the composite operator \(\phi^2\), and removes the divergence in Fig.2(A). \(\Delta_2\) and \(\Delta_1\) are necessary to remove the overall divergences in Fig.2(B) and in Fig.2(C), respectively.

Now, one can prove that (16) coincides with the last three terms in (15):
\[
Z Z^{-1}_\phi^2 - 1 = B, \quad \Delta_2 = D, \quad \text{and} \quad \Delta_1 = -2Dm^2. \tag{17}
\]

The first equation is obtained by the definition \(B = Z_\mu Z - 1\) and an identity
\[
Z_{\phi^2} = Z^{-1}_\mu. \tag{18}
\]

The overall divergence of the vacuum diagram with no external-legs is removed by the \(c\)-number counter term \(Dm^2\) in \(\mathcal{L}(\phi; m^2)\). Therefore, the last two equations in (17) are obtained as
\[
\Delta_1 = -\left( \frac{\partial}{\partial m^2} \right) [Dm^4] = -2Dm^2, \tag{19}
\]
\[
2\Delta_2 = \left( \frac{\partial}{\partial m^2} \right)^2 [Dm^4] = 2D. \tag{20}
\]

Eq. (17) shows clearly that all the necessary counter terms in OPT are supplied solely by the original Lagrangian \(\mathcal{L}(\phi; m^2)\). Thus, we can carry out renormalization order by order even within the self-consistent method. For more detailed proof of the relations (17), see Appendix A of ref. 8.

Three comments are in order here:

(i) Because the renormalization is already carried out in Step 2, one obtains finite gap-equations from the beginning in Step 3. Our procedure “resummation after renormalization” has many advantages over the conventional procedure “resummation before renormalization” where UV divergences are hoped to be canceled after imposing the gap-equation. The difference between the two is prominent especially in higher order calculations.

(ii) The decomposition (6) should be done both in the mass term and the counter terms. This guarantees order by order renormalization in our modified loop-expansion in any higher orders. (In ref. 3, the renormalizability was checked up to the two-loop level in the \(\phi^4\) theory at high \(T\).) On the other hand, if one keeps the original counter term \((1/2)B\mu^2\phi^2 + D\mu^4\) without the decomposition (6), L-loop diagrams with \(L > M\) must be taken into account to remove the UV divergences in the \(M\)-loop order. This is an unnecessary complication due to the inappropriate treatment of the counter terms. (See e.g. ref. 17 which encounters this problem).

(iii) As far as we stay in the low energy region far below the Landau pole, we need not address the issue of the triviality of the \(\phi^4\) theory [3]. Perturbative renormalization in OPT works in the same sense as that in the naive perturbation.

E. Nambu-Goldstone theorem in OPT

The procedure and the renormalization in OPT discussed above do not receive modifications even if the Lagrangian has global symmetry. For \(O(N) \phi^4\) theory, one needs to replace \(\phi \) and \(\phi^2\) by \(\phi = (\phi_0, \phi_1, \cdots, \phi_{N-1})\) and \(\phi^2\) respectively in all the previous formulas.

In the symmetry broken phase of such theory, the Nambu-Goldstone (NG) theorem and massless NG bosons are guaranteed in each order of the modified loop-expansion in OPT for arbitrary \(N\). To show this, it is most convenient to start with the thermal effective potential \(V(\varphi^2)\). By the definition of the effective potential, \(V(\varphi^2)\) has manifest \(O(N)\) invariance if it is calculated in all orders.

In OPT, \(V\) calculated up to \(L\)-th loops \(V_L(\varphi^2; m)\) has also manifest \(O(N)\) invariance, because our decomposition (6) used in (6) does not break \(O(N)\) invariance. Once \(V_L\) has invariance under the \(O(N)\) rotation \((\varphi_i \rightarrow \varphi_i + i\theta^a T_{ij}^a \varphi_j)\), the immediate consequence is the standard identity:
\[
\frac{\partial V_L(\varphi^2; m)}{\partial \varphi_j} T_{ji}^a = -\frac{\partial^2 V_L(\varphi^2; m)}{\partial \varphi_i \partial \varphi_j} T_{jk}^a \varphi_k, \tag{21}
\]
with \(T^a\) being the generator of the \(O(N)\) symmetry. Eq. (21) is valid for arbitrary \(L\), \(m\) and \(N\).

At the stationary point where the l.h.s. of (21) vanishes, there arises massless NG bosons for \(T_{jk}^a \varphi_k \neq 0\), since the r.h.s. of (21) is equal to \(-D_{ij}^{-1}(0) T_{jk}^a \varphi_k\) where \(D_{ij}(0)\) is the Matsubara propagator at zero frequency and momentum calculated up to \(L\)-th loops. Thus the existence of the NG bosons is proved independent of the structure of the gap-equation in Step 3.

Now, let us show an example of the unjustified approximations leading to the breakdown of the NG theorem. Suppose that we make a general decomposition such as
\[
\mu^2 \delta_{ij} = m^2_{ij} - (m^2_j - \mu^2 \delta_{ij}), \tag{22}
\]
with \(m^2_{ij} \neq m^2 \delta_{ij}\). This leads to an \(O(N)\) non-invariant effective potential, and the relation (21) is not guaranteed in any finite orders of the loop-expansion. For example, when the \(O(N)\) symmetry is spontaneously broken down to \(O(N - 1)\), one may be tempted to make a decomposition \(m^2_{ij} = m^2_0 (i = j), m^2_i = m^2_j (i = j), m^2_{ij} = 0 (i \neq j)\) to impose self-consistent conditions for the radial mode and the NG mode. However, the effective potential does not have \(O(N)\) symmetry in this case and eq. (21) does not hold.
III. APPLICATION TO O(4) σ MODEL

Let us apply OPT to the O(4) σ model. The model shares common symmetry and dynamics with QCD and has been used to study the real-time dynamics and critical phenomena associated with the QCD chiral transition [19,20].

A. Parameters at \( T = 0 \)

The O(4) σ model reads

\[
\mathcal{L} = \frac{1}{2} \left[ \left( \partial \phi \right)^2 - \mu^2 \phi^2 \right] - \frac{\lambda}{4!} (\phi^2)^2 + h\sigma \\
+ \frac{1}{2} A (\partial \phi)^2 - \frac{1}{2} B \mu^2 \phi^2 - \frac{\lambda}{4!} C (\phi^2)^2 + D\mu^4,
\]

with \( \phi = (\sigma, \vec{\pi}) \). \( h\sigma \) is an explicit symmetry breaking term.

\( A, B, C \) and \( D \) in the one-loop order are

\[
A = 0, \quad B = \frac{\lambda}{16\pi^2}, \quad C = \frac{\lambda}{8\pi^2}, \quad D = -\frac{1}{16\pi^2},
\]

where \( \frac{1}{\bar{\tau}} = \frac{2}{\pi^2} \gamma + \log(4\pi) \), with \( \gamma \) being the Euler constant.

When SSB takes place (\( \mu^2 < 0 \)), the replacement \( \sigma \rightarrow \sigma + \xi \) in eq. (23) leads to the tree-level masses of \( \sigma \) and \( \pi \):

\[
m^2_{\sigma} = \mu^2 + \frac{\lambda}{2} \xi^2, \quad m^2_{\pi} = \mu^2 + \frac{\lambda}{6} \xi^2.
\]

The expectation value \( \xi \) at \( T = 0 \) is determined by the stationary condition for the standard effective potential \( \partial \mathcal{V}(\vec{\phi})/\partial \vec{\phi} = 0 \).

Later we will take a special FAC condition in which \( m^2 \) deviates from \( \mu^2 \) only at \( T \neq 0 \), so that the naive loop-expansion at \( T = 0 \) is valid. The renormalized couplings \( \mu^2, \lambda \) and \( h \) can thus be determined by the renormalization conditions in the naive loop-expansion at zero \( T \) such as (i) \( m_{\pi} = 140 \) MeV, (ii) \( f_{\pi} = 93 \) MeV, and (iii) \( \pi-\pi \) scattering phase shift [19].

Instead of (iii), one may adopt \( m^2_{\sigma} \) (the peak position of the spectral function in the \( \sigma \) channel). We take this simplified condition with three possible cases: \( m^2_{\sigma} = 550 \) MeV, 750 MeV and 1000 MeV. \( m^2_{\sigma} = 550 \) MeV and 750 MeV are consistent with recent re-analyses of the \( \pi-\pi \) scattering phase shift [21].

We still have a freedom to choose the renormalization point \( \kappa \). Instead of trying to determine optimal \( \kappa \) by the renormalization group equation for the effective potential [22], we take a simple and physical condition \( m_{0\pi} = m_{\pi} = 140 \) MeV. This choice has two advantages: (a) One-loop pion self-energy \( \Sigma_{\pi}^{\text{pole}} \) vanishes at the tree-mass; \( \Sigma_{\pi}(k^2 = m^2_{0\pi}) = \Sigma_{\pi}(k^2 = m^2_{\pi}) = 0 \), and (b) the spectral function in the \( \sigma \) channel starts from a correct continuum threshold in the one-loop level. Resultant parameters are summarized in TABLE I.

In Fig. 3, the spectral functions \( \rho_{\sigma} \) and \( \rho_{\pi} \) at \( T = 0 \), namely the \( T = 0 \) limit of eq. (26) defined below, are shown as a function of \( \sqrt{s} \equiv \sqrt{\omega^2 - k^2} \). In the \( \pi \) channel, there are one particle pole and a continuum starting from the threshold \( \sqrt{s_{\pi}} = m_{0\pi} + m_{0\sigma} \). \( \sqrt{s_{\pi}} \) is the point where the channel \( \pi + \sigma \) opens. In the \( \sigma \) channel, the spectral function starts from the threshold \( 2m_{0\sigma} = 280 \) MeV and shows a broad peak centered around \( \sqrt{s} = m^2_{\sigma} \). The large width of \( \sigma \) is due to a strong \( \sigma-2\pi \) coupling in the linear \( \sigma \) model. The corresponding \( \sigma \)-pole is located far from the real axis on the complex \( s \) plane.

Here we show the definition of the spectral function at finite \( T \):

\[
\rho_{\phi}(\omega, \mathbf{k}; T) = \frac{-1}{\pi} \text{Im} D^R_{\phi}(\omega, \mathbf{k}; T),
\]

where \( D^R_{\phi} \) is the retarded correlation function

\[
D^R_{\phi}(\omega, \mathbf{k}; T) = -i \int d^4xe^{i\mathbf{k}\cdot\mathbf{x}} \langle \phi(t, \mathbf{x})\phi(0, \mathbf{0}) \rangle,
\]

with \( \langle \cdot \rangle \) being the thermal expectation value.

B. Application of OPT

Now let us proceed to Step 2 in OPT and rewrite eq. (23) as

\[
\mathcal{L} = \frac{1}{2} \left[ \left( \partial \phi \right)^2 - m^2 \phi^2 \right] - \frac{\lambda}{4!} (\phi^2)^2 + \frac{1}{2} \chi \phi^2 + h\sigma \\
- \frac{1}{2} B m^2 \phi^2 - \frac{\lambda}{4!} C (\phi^2)^2 + D m^4.
\]

Since \( \chi \) ( = \( m^2 - \mu^2 \)) is already a one-loop order, we have neglected the terms proportional to \( B\chi \), \( D\chi^2 \) and \( D\chi^4 \) which are two-loop or higher orders.

When SSB takes place (\( \phi \rightarrow \phi + \frac{\chi}{2} \)), the tree-level masses to be used in the modified loop-expansion read

\[
m^2_{\sigma} = m^2 + \frac{\lambda}{2} \frac{\chi}{2}, \quad m^2_{\pi} = m^2 + \frac{\lambda}{6} \frac{\chi}{2}.
\]

Since \( m^2 \) will eventually be a function of \( T \), the tree-masses running in the loops are not necessary tachyonic at finite \( T \) contrary to the naive loop-expansion (see the discussion in Sec. II A).

| \( m^2_{\sigma} \) (MeV) | \( m^2 \) (MeV) | \( \lambda \) | \( h \) (MeV) | \( \kappa \) (MeV) |
|----------------|-----------|---------|---------|---------|
| 550 | -284 | 730 | 123 | 255 |
| 750 | -375 | 122 | 124 | 325 |
| 1000 | -469 | 194 | 125 | 401 |

TABLE I. Vacuum parameters corresponding to \( m^2_{\sigma} = 550, 750, 1000 \) MeV
The thermal effective potential $V(\vec{\varphi}; m)$ is calculated in the standard manner except for the extra terms proportional to $\chi$. The effective potential in the one-loop level reads

\[ V(\vec{\varphi}; m) = \frac{1}{2} \mu^2 \varphi^2 + \frac{\lambda}{4!} \varphi^4 - h\varphi \]

\[ + \frac{1}{64\pi^2} \left[ m_{0\sigma}^2 \ln \left| \frac{m_{0\sigma}^2}{\kappa^2 e^{k/\tau}} \right| + 3m_{0\pi}^2 \ln \left| \frac{m_{0\pi}^2}{\kappa^2 e^{k/\tau}} \right| \right] \]

\[ + T \int \frac{d^3k}{(2\pi)^3} \ln(1 - e^{-E/T}) + 3 \ln(1 - e^{-E/T}) \]

where $E_\phi \equiv \sqrt{k^2 + m_{0\phi}^2}$. Although this has the similar structure with the standard free energy in the naive loop-expansion, the coefficient of the first term in the r.h.s. of (30) is $\mu^2$ instead of $m^2$. This is because we have extra mass-term proportional to $\chi$ in the one-loop level. The stationary point $\xi$ is obtained by

\[ \frac{\partial V(\vec{\varphi}; m)}{\partial \varphi_i} \bigg|_{\varphi=(\xi,0)} = 0. \quad (31) \]

Since the derivative with respect to $\xi$ does not act on $m$, this gives a solution $\xi$ as a function of $T$ and $m$. By imposing another condition on $m$ (Step 3), one eventually determines both $\xi$ and $m$ for given $T$.

**C. FAC condition for $m^2$**

To resum the hard thermal loops, the PMS condition for the effective potential requires 2-loop calculation, while the FAC condition for the self-energy requires only 1-loop calculation. Therefore, we adopt the latter condition here to determine $m^2$.

The retarded self-energy $\Sigma^R_\phi$ (defined by $D^R_\phi^{-1} = s - m_{0\phi}^2 - \Sigma^R_\phi$) is related to the 11-component of the $2 \times 2$ self-energy in the real-time formalism [23].

\[ \text{Re}\Sigma^R_\phi(\omega, k; T) = \text{Re}\{\Sigma^{11}_\phi(\omega, k) + \Sigma^{11}_\phi(\omega, k; T)\} \]

\[ \text{Im}\Sigma^R_\phi(\omega, k; T) = \text{tanh}(\frac{\omega}{2T}) \text{Im}\{\Sigma^{11}_\phi(\omega, k) + \Sigma^{11}_\phi(\omega, k; T)\}. \]

Here $\Sigma^{11}_\phi(\omega, k; T)$ is a part having explicit $T$-dependence through the Bose-Einstein distribution, while $\Sigma^{11}_\phi(\omega, k)$ is a part having only implicit $T$-dependence through $m(T)$ and $\xi(T)$. In the one-loop level, $\Sigma^{11}_\phi$ can be calculated only by the 11-component of the free propagator,

\[ iD^{11}_{0\phi}(k^2; T) = \frac{i}{k^2 - m_{0\phi}^2 + i\epsilon} + 2\pi n_B \delta(k^2 - m_{0\phi}^2), \quad (33) \]

with $n_B = [e^{\omega/T} - 1]^{-1}$.

One-loop diagrams in OPT for $\Sigma^{11}$ are shown in Fig. 4. Their explicit forms are given in 5. The NG theorem discussed in Sec. II E can be explicitly checked by comparing eq.(31) and the inverse pion-propagator at zero momentum $[D^R_\pi(0, 0; T)]^{-1}$.

Let us impose the FAC condition on $\Sigma^R_\pi$. Since we chose a renormalization condition $m_{0\pi} = m_\pi = 140$ MeV at zero $T$, one may be tempted to adopt the following condition at finite $T$:

\[ \Sigma^R_\pi(\omega = m_{0\pi}, 0; T) = 0. \quad (34) \]

However, eq.(34) does not guarantees that $m^2$ is real, since the l.h.s. of eq.(34) receives an imaginary part due to the Landau damping. To avoid this problem, we take a hybrid condition:

\[ \Sigma^{11}_\pi(\omega = m_{0\pi}, 0) + \Sigma^{11}_\pi(\omega = 0, 0; T) = 0. \quad (35) \]

Note that the external energy is set to be zero in the $T$-dependent part. Because the second term in the l.h.s. vanishes at $T = 0$, the solution of eq.(35) at $T = 0$ becomes

\[ m^2(T = 0) = \mu^2. \quad (36) \]

Therefore, the FAC condition (35) does not spoil the naive-loop expansion at $T = 0$.

At high $T$ ($\xi(T) \simeq 0$), the following analytic solution is obtained as far as $T^2 \gg m^2$;

\[ m^2(T) = \mu^2 + \frac{\lambda}{12} T^2, \quad (37) \]

which implies that the Debye screening mass at high $T$ is properly taken into account. For realistic values of $\lambda$ in TABLE I, the condition $T^2 \gg m^2$ is not well satisfied and eq.(35) should be solved numerically.

For intermediate values of $T$, eq.(35) can effectively sum not only the contributions from the diagrams in Fig.4(a, b, h, i), but also from those in Fig.4(c, d, j). Thus, OPT can go beyond the cactus approximation which sums only the diagrams in Fig.4(a, b, h, i).

Two remarks are in order here.

(i) By eq.(23), only the $k$-independent part of the self-energy is resumed. If one needs to resum $k$-dependent part too, one must introduce $m^2$ which depends both on $T$ and $k$ and impose $k$-dependent FAC or PMS conditions 23.
(ii) In ref. [17], it has been studied the convergence properties of the free energy at high $T$ with a variational condition equivalent to the PMS condition here. Although the approach has a problem of renormalization as we have already mentioned, the result is suggestive in the sense that the optimized loop expansion has much better convergence properties than the loop expansion based on the hard thermal loops [25]. Better understanding of the convergence properties both in PMS and FAC conditions is an important future problem.

D. Behavior of $m(T)$, $m_{0\phi}(T)$ and $\xi(T)$

In Fig. 3(A) the tree-level masses in eq. (23), and $m^2(T)$ are shown for $m_{\pi}^{peak}(T=0) = 550$ MeV. $m_{tach}(T)$ is not tachyonic and approaches to $m^2(T)$ in the symmetric phase. This confirms that our resummation procedure cures the problem of tachyons in Sec. II A.

The solid line in Fig. 3(B) shows the chiral condensate $\xi(T)$ obtained by minimizing the effective potential with $m_{\pi}^{peak}(T=0) = 550$ MeV. $\xi(T)$ decreases uniformly as $T$ increases, which is a behavior of the chiral order parameter at finite $T$ away from the chiral limit.

As we approach the chiral limit ($\hbar \to 0$ or equivalently $m_\pi \to 0$), $\xi(T)$ develops multiple solutions for given $T$, which could be an indication of the first order transition. The critical value of the quark mass $m_{q}^{crit}$ below which the multiple solutions arise is

$$m_{q}^{crit}/m_{q}^{phys} = (m_{q}^{phys}/m_{\pi}^{phys})^2 = 0.08,$$

where we have used Gell-Mann-Oakes-Renner relation [26] to related the pion mass with the quark mass. $m_{q}^{phys}$ is the physical light-quark mass corresponding to $m_{\pi}^{phys} = 140$ MeV. The critical temperature for $m_{q}^{crit}/m_{q}^{phys} = 0.08$ is $T_c \approx 170$ MeV. The behavior of $\xi(T)$ for $m_{\pi}(T=0) = 30$ MeV (just below the critical value $m_{\pi}^{crit}$) is also shown by the dashed line in Fig. 3(B) for comparison.

E. Chiral limit

In Fig. 3 the chiral condensates are shown for the chiral limit ($m_{\pi} = 0$ MeV) and for $m_{\pi} = 10$ MeV. The phase transition looks like a first order in these cases. The existence of the multiple solutions of the gap equation for the $O(4)$ $\sigma$ model in the mean-field approach has been known for a long time [27]. Our analyses confirm this feature within the framework of OPT.

However, this first order nature is likely to be an artifact of the mean-field approach as discussed in the second reference in [27]: the higher loops of massless $\pi$ and almost massless $\sigma$ are not negligible near the critical point, and they could easily change the order of the transition. In fact, the renormalization group analyses as well as the direct numerical simulation on the lattice indicate that the $O(4)$ $\sigma$ model has a second order phase transition [28].

We have also studied the free energy as a function of $T$ near the chiral limit and found that it has a discontinuity near the critical point. This is another sign that the first-order nature is an artifact of the approximation. (Remember that, the free energy must be a continuous function of $T$ irrespective of the order of the phase transition.)

F. Spectral function at finite $T$

As one of the non-trivial applications of OPT, we show, in Fig. 4, the spectral functions of $\pi$ and $\sigma$ at finite $T$ defined in [29].

The figure shows that the spectral function of $\sigma$, which does not show a clear resonance at $T = 0$, develops a sharp enhancement near $2\pi$ threshold as $T$ approaches $T_c$. This is due to a combined effect of the partial restoration of chiral symmetry and the strong $\sigma-2\pi$ coupling. This is an typical example of the softening (or the precursor of the critical fluctuation) associated with the partial restoration of chiral symmetry [29].

The experimental relevance of this softening has been examined in the context of the low-mass diphoton production [30].

In the $\pi$-channel, a continuum develops by the scattering with thermal pions in the heat-bath; $\pi + \pi^{thermal} \to \sigma$. Because of this process, the pion acquires a width $\sim 50$ MeV at $T = 145$ MeV, while the peak position does not show appreciable modification. They are in accordance with the Nambu-Goldstone nature of the pion.

IV. SUMMARY

We have shown that the optimized perturbation theory (OPT) developed in [7] naturally cures the problems of the naive loop-expansion at finite $T$, namely, the breakdown of the naive perturbation at $T \gg T_c$ (due to the hard thermal loops) as well as at $T < T_c$ (due to the tachyonic poles).

Furthermore, OPT has several advantages over other resummation methods proposed so far:

First of all, the renormalization of the UV divergences, which is not a trivial issue in other methods, can be carried out systematically in the loop-expansion in OPT. This is because one can separate the self-consistent procedure (Step 3 in Sec. II C) from the renormalization procedure (Step 2 in Sec. II C).

Secondly, the Nambu-Goldstone (NG) theorem is fulfilled in any give order of the loop-expansion in OPT for arbitrary $N$ in $O(N)$ $\phi^4$ theory. This is because OPT
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FIG. 1. Bubble and cactus diagrams.

FIG. 2. Diagrams which contain UV divergences as a result of the multiple insertion of $\frac{1}{2} \phi^2$. (A) corresponds to a single insertion with two external lines. (B) and (C) have no external lines with a single insertion and a double insertion, respectively.

FIG. 3. Spectral functions at $T = 0$ in the $\pi$ channel (A) and in the $\sigma$ channel (B) for $m_\sigma^{peak} = 550$ MeV, 750 MeV and 1000 MeV.

$$-i \Sigma_\sigma^{11}(\omega, k) = \sigma + \sigma + \sigma + \sigma + \sigma + i \sigma + \sigma + \sigma + \sigma + \sigma + \sigma + \sigma + \sigma + \sigma + \sigma$$

$$-i \Sigma_\pi^{11}(\omega, k) = \pi + \pi + \pi + \pi + \pi + \pi + \pi + \pi + \pi + \pi + \pi + \pi + \pi + \pi + \pi + \pi$$

FIG. 4. One-loop self-energy $\Sigma^{11}$ for $\sigma$ and $\pi$ in the modified loop-expansion at finite $T$. 

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FIG. 5. (A) Masses in the tree-level $m\sigma_0(T)$ and $m\sigma_0(T)$ shown with left vertical scale, and the mass parameter $m^2(T)$ with the right vertical scale. (B) $\xi(T)$ for $m_\pi(T = 0) = 140\text{MeV}$ and $30\text{MeV}$ with $m_\sigma^{peak}(T = 0) = 550\text{MeV}$.

FIG. 6. $\xi(T)$ for $m_\sigma^{peak}(T = 0) = 550\text{MeV}$ with $m_\sigma = 0\text{MeV}$ and $m_\sigma = 10\text{MeV}$.

FIG. 7. Spectral function in the $\pi$ channel (A) and in the $\sigma$ channel (B) for $T = 50, 120, 145\text{MeV}$ with $m_\sigma^{peak}(T = 0) = 550\text{MeV}$.