A REMARK ON GLOBAL SOLUTIONS TO RANDOM 3D VORTICITY EQUATIONS FOR SMALL INITIAL DATA

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Abstract. In this paper, we prove that the solution constructed in [2] satisfies the stochastic vorticity equations with the stochastic integration being understood in the sense of the integration of controlled rough path introduced in [8]. As a result, we obtain the existence and uniqueness of the global solutions to the stochastic vorticity equations in 3D case for the small initial data independent of time, which can be viewed as a stochastic version of the Kato-Fujita result (see [10]).

1. Introduction. Consider the stochastic 3D Navier-Stokes equation on \((0, \infty) \times \mathbb{R}^3\):

\[
\begin{aligned}
dX - \Delta X dt + (X \cdot \nabla)X dt &= \sum_{i=1}^{N} (B_i(X) + \lambda_i X) d\beta_i^t + \nabla \pi dt, \\
\nabla \cdot X &= 0,
\end{aligned}
\]

where \(\{\beta^i\}_{i=1}^{N}\) is a system of independent Brownian motions on a probability space \((\Omega, \mathcal{F}, P)\) with normal filtration \((\mathcal{F}_t)_{t \geq 0}\), and \(\lambda_i \in \mathbb{R}, x : \Omega \to L^2(\mathbb{R}^3; \mathbb{R}^3)\) is a random variable. Here \(\pi\) denotes the pressure, \(\Delta\) is the Laplacian on \(L^2(\mathbb{R}^3; \mathbb{R}^3)\) and \(B_i\) are convolution operators given by

\[
B_i(X)(\xi) = \int_{\mathbb{R}^3} h_i(\xi - \xi') X(\xi') d\xi' = (h_i \ast X)(\xi), \quad \xi \in \mathbb{R}^3,
\]

where \(h_i \in L^1(\mathbb{R}^3), i = 1, \ldots, N\).

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Consider the vorticity field
\[ U = \nabla \times X = \text{curl}X \]
and apply the curl operator to equation (1.1). We obtain the transport vorticity equation on \((0, \infty) \times \mathbb{R}^3\):
\[
dU - \Delta U dt + ((X \cdot \nabla)U - (U \cdot \nabla)X) dt = \sum_{i=1}^{N} (h_i * U + \lambda_i U) d\beta_t^i, \tag{1.2}
\]
\[ U_0(\xi) = (\text{curl}x)(\xi), \quad \xi \in \mathbb{R}^3. \tag{1.3} \]

The vorticity \(U\) is related to the velocity \(X\) by the Biot-Savart integral operator (see [4])
\[
X_t(\xi) = K(U_t)(\xi) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\xi - \tilde{\xi}}{|\xi - \tilde{\xi}|^3} \times U_t(\tilde{\xi}) d\tilde{\xi}, \quad t \in (0, \infty), \xi \in \mathbb{R}^3. \tag{1.4}
\]
Then one can rewrite the vorticity equation (1.2) as
\[
dU - \Delta U dt + ((K(U) \cdot \nabla)U - (U \cdot \nabla)K(U)) dt = \sum_{i=1}^{N} (h_i * U + \lambda_i U) d\beta_t^i, \tag{1.4}
\]
\[ U_0(\xi) = (\text{curl}x)(\xi), \quad \xi \in \mathbb{R}^3. \tag{1.5} \]

In [2] using the transformation
\[ U_t = \Gamma_t y_t \]
with
\[ \Gamma_t = \prod_{i=1}^{N} \exp \left( \beta_t^i \tilde{B}_t - \frac{t}{2} \tilde{B}_t^2 \right), \quad \tilde{B}_t = B_t + \lambda_t I, \]
the authors transformed (1.4) into the following equation
\[
\frac{dy}{dt} - \Gamma_t^{-1}\Delta(\Gamma_t y_t) dt + \Gamma_t^{-1}((K(\Gamma_t y_t) \cdot \nabla)(\Gamma_t y_t) - (\Gamma_t y_t \cdot \nabla)K(\Gamma_t y_t)) = 0, \tag{1.5}
\]
\[ y_0 = U_0. \]

In [2] the authors proved that if the initial value is small enough (compared to a function depending on the paths of Brownian motions \(\beta_i\)), then there exists a unique solution \(y_t\) (in the mild sense) to (1.5). However, since the initial value satisfying the following condition (1.7) is not \(F_0\)-measurable, the process \(y_t\) is not \((\mathcal{F}_t)_{t \geq 0}\)-adapted. Therefore, the solution to (1.5) cannot be transformed back into (1.4).

The main aim of this paper is to obtain the stochastic version of the result of Kato-Fujita to (1.4). Let \(y\) be the solution to (1.5) obtained in [2] and define \(U_t := \Gamma_t y_t\). Since \(y_t\) is not \((\mathcal{F}_t)_{t \geq 0}\)-adapted, the corresponding \(U_t\) is also not \((\mathcal{F}_t)_{t \geq 0}\)-adapted. Therefore, the stochastic integral should be understood in the sense of a rough path integral or the Skorohod integral. To use the Skorohod integral and find a solution to (1.4) we have to use the shift operator (see [3], [12]), which breaks the result that there exists some \(C(\omega)\) independent of time such that if \(|U_0|_{3/2} \leq C(\omega)\), there exists a global solution to (1.5). Thus in this paper we understand the stochastic integral of (1.4) in the sense of a rough path integral.

**Framework and main result**

First we recall the main result in [2]. In the following we denote by \(L^p, 1 \leq p \leq \infty\) the space \(L^p(\mathbb{R}^3; \mathbb{R}^3)\) with norm \(|\cdot|_p\) and by \(C_b([0, \infty); L^p)\) the space of all
bounded and continuous functions $u : [0, \infty) \to L^p$ with the sup-norm. We also set $D_i = \frac{\partial}{\partial x_i}, \ i = 1, 2, 3$. We set for $p \in (\frac{3}{2}, 3), q \in (1, \infty)$

$$\eta_t = \|\Gamma_t\|_{L(L^p, L^p)}\|\Gamma^p_t\|_{L(L^p, L^p)}\|\Gamma^{-1}_t\|_{L(L^q, L^q)}, \ t \geq 0,$$

where $\|\cdot\|_{L(L^p, L^p)}$ is the norm of the space $L(L^p, L^p)$ of linear continuous operators on $L^p$.

For $p \in [1, \infty)$ we denote by $Z_p$ the space of all functions $y : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$t^{\frac{3}{2}} \eta_t y_t \in C_b([0, \infty); L^p),$$

$$t^{\frac{3}{2}}(1 - t) D_i y_t \in C_b([0, \infty); L^p), \ i = 1, 2, 3.$$

The space $Z_p$ is endowed with the norm

$$\|y\| = \sup\{t^{\frac{3}{2}} \eta_t |y_t|_p + t^{\frac{3}{2}}(1 - t) |D_i y_t|_p; t \in (0, \infty), i = 1, 2, 3\}.$$

In the following we take $\lambda_i \in \mathbb{R}$ such that

$$|\lambda_i| > (\sqrt{12} + 3)|h_i|, \ i = 1, 2, ..., N.$$

Consider the equation (1.5) in the following mild sense:

$$y_t = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} M\Gamma(y_s)ds, \ t \in (0, \infty), \ (1.6)$$

where

$$M(u) = -(K(u) \cdot \nabla)(u) + (u \cdot \nabla)K(u).$$

The following is the main result in [2].

**Theorem 1.1.** Let $p, q \in (1, \infty)$ such that

$$\frac{3}{2} < p < 2, \frac{1}{q} = \frac{2}{p} - \frac{1}{3}.$$

Let $\Omega_0 = \{\sup_{t \geq 0} \eta_t < \infty\}$ and consider (1.6) for fixed $\omega \in \Omega_0$. Then $P(\Omega_0) = 1$ and there exists a positive constant $C^*$ independent of $\omega \in \Omega_0$ such that, if $U_0 \in L^{3/2}$ satisfying

$$\sup_{t \geq 0} \eta_t |U_0|_{3/2} \leq C^*, \ (1.7)$$

then there exists a unique solution $y \in Z_p$ to (1.6). Moreover, for each $\varphi \in L^3 \cap L^{\frac{12}{3}},$ the function $t \to \int_{\mathbb{R}^3} y_t(\xi)\varphi(\xi)d\xi$ is continuous on $[0, \infty)$.

To formulate our first main result we introduce the following notations and definitions from rough paths theory: Fix $\frac{1}{3} < \alpha < \frac{1}{2}, \ 0 \leq s < t,$ for $X \in C([s, t], \mathbb{R}^N)$ we define

$$\delta X_{uv} := X_v - X_u, \ ||X||_{\alpha, [s, t]} := \sup_{u,v \in [s, t], u \neq v} \frac{|\delta X_{uv}|}{|u - v|^\alpha}.$$  

Moreover, for a tensor process $X \in C([s, t]^2, \mathbb{R}^{N \times N})$ we define

$$||X||_{2\alpha, [s, t]} := \sup_{u,v \in [s, t], u \neq v} \frac{|X_{uv}|}{|u - v|^{2\alpha}}.$$  

In fact, $(X, \mathbb{X})$ is an $\alpha$-Hölder rough path in the sense of [7], Def.2.1 if $||X||_{\alpha, [s, t]} < \infty,$ $||X||_{2\alpha, [s, t]} < \infty$ and the following holds for every triple of times $(u, v, w)$

$$X_{uv} - X_{uw} - X_{wu} = \delta X_{uw} \otimes \delta X_{uv}.$$
For an $N$-dimensional Brownian motion $\beta$ on the probability space $(\Omega, \mathcal{F}, P)$ and $B_{uv} := \int_u^v \delta_\beta w_r \otimes d\beta_r \in \mathbb{R}^N \times \mathbb{R}^N$, it is well known that there exists a set $\Omega_1$ with $P(\Omega_1) = 1$ such that for $\omega \in \Omega_1 (\beta(\omega), \mathcal{B}(\omega))$ is an $\alpha$-Hölder rough path (see [7], Prop. 3.4), where the stochastic integration is understood in the sense of Itô. In the following we consider the problem on $\Omega_1$ $\omega$-wise. We also introduce the following smaller space for later use: for $\varepsilon > 0$ we set

$$Z_{p}^{\varepsilon} := \{ y \in Z_{p} | \sup_{s \leq u < v \leq t} u^{2\varepsilon + 1} \frac{| \delta y_{uv} |_{p}}{| u - v |^{\varepsilon}} + (2\varepsilon + 1)^{2} \frac{\sum_{j=1}^{3} | \delta (D_j y)_{uv} |_{p}}{| u - v |^{\varepsilon}} < \infty, \quad \forall 0 < s < t \},$$

Now we recall the notion of a controlled path $Y$ relative to some reference path $X$ due to Gubinelli [8].

**Definition 1.1.** Given a path $X \in C^\alpha([s, t], \mathbb{R}^N)$, we say that $Y \in C^\alpha([s, t], \mathbb{R}^N)$ is controlled by $X$ if there exists $Y' \in C^{\alpha}(\Omega_1)$ so that the remainder term $R$, for $s \leq u < v \leq t$ given by the formula

$$\delta Y_{uv}^{\mu} = \sum_{\nu=1}^{N} Y^{\prime \mu \nu} X_{uv}^{\nu} + R_{uv}^{\mu},$$

satisfies $\| R \|_{2\alpha, [s, t]} < \infty$. Here the superscript $\mu$ and $\nu$ relates to the coordinate.

By [8], if we are given a path $Y$ controlled by $X$, then we can define the integration of $Y'$ against $(X, X)$, which is an extension of Young’s integral (see Theorem 1 and Corollary 2 in [8]): for $0 \leq s < t \leq T$

$$\int_{s}^{t} Y^{\mu} dX^{\nu} := \lim_{| P | \to 0} \sum_{i=0}^{n-1} \left( Y_{t_i}^{\mu \nu} X_{t_i, t_{i+1}}^{\nu} \right. + \sum_{\mu' = 1}^{N} Y_{t_i}^{\mu' \nu} X_{t_i, t_{i+1}}^{\nu} \Big),$$

where $P = \{ t_0, t_1, ..., t_n \}$ is a partition of the interval $[s, t]$ such that $t_0 = s, t_n = t, t_{i+1} > t_i, | P | = \sup_i | t_{i+1} - t_i |$.

Now we give the definition of solutions to equation (1.4). In the following we define the analytic weak solution to equation (1.4) and we use $\langle \cdot, \cdot \rangle$ to denote the $L^2$ inner product.

**Definition 1.2.** We say that $U$ is a solution to equation (1.4) if $\Gamma^{-1} U \in Z_{p}^{\varepsilon}$ for some $\varepsilon > 0$ and for any $\varphi \in C^\infty_\varepsilon (\mathbb{R}^3; \mathbb{R}^3)$, the function $t \to \langle \Gamma^{-1} U_t, \varphi \rangle$ is continuous on $[0, \infty)$ and for $0 \leq s < t$,

$$\langle U_t - U_s, \varphi \rangle - \int_{s}^{t} \left[ (U_r, \Delta \varphi) - \langle M(U_r), \varphi \rangle \right] dr = \sum_{i=1}^{N} \int_{s}^{t} \langle \tilde{B}_{i} U_r, \varphi \rangle d\beta^{i}_{r},$$

$$\| U|_{t=0} = U_0,$$

where the integral $\int_{s}^{t} \langle \tilde{B}_{i} U_r, \varphi \rangle d\beta^{i}_{r}$ is understood in the sense of (1.8) with respect to the rough paths $(\beta, B)$. Here for $0 < s < t$ $(\tilde{B}_{i} U_r, \varphi) \in C^\alpha([s, t])$ is controlled by $\beta$ in the sense of Definition 1.1 and

$$\delta((\tilde{B}_{i} U_r, \varphi))_{st} = \sum_{k=1}^{N} \langle \tilde{B}_{k} \tilde{B}_{i} U_s, \varphi \rangle \delta \beta^{k}_{st} + R^{i}_{st},$$

with $R$ being the remainder term satisfying

$$\| (\tilde{B}_{k} \tilde{B}_{i} U_r, \varphi) \|_{2\alpha, [s, t]} < \infty, \quad \| R^{i} \|_{2\alpha, [s, t]} < \infty.$$

**References:**

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Remark 1.2. (i) Here due to the singularity of solution $U$ at $t = 0$, the stochastic integral defined in (1.8) has some problem at $t = 0$. So, in (1.9) we only assume $0 < s < t$. Since $\Gamma^{-1}U \in \mathbb{Z}_p$, $\int_0^s (M(U_r), \varphi)dr$ is well-defined due to (2.35) in [2].

(ii) In general rough paths theory, often approximations are used to give a meaning to the solution of stochastic equations (see [7], Chapter 12). However, in this case if we need the approximation equations to be well-posed for small initial data, then the conditions on the initial value might be artificial. Therefore, since our aim is to prove a stochastic version of the Kato-Fujita result (see [10]), the above definition is more suitable. We also want to mention that such kind of definition has also been used for the linear equation in [6].

The main result of this paper is the following theorem:

Theorem 1.3. Under the condition of Theorem 1.1 and for $y$ as obtained in Theorem 1.1, for $\omega \in \Omega_0 \cap \Omega_1$, $U_t(\omega) := \Gamma_t(\omega)y_t(\omega)$ is the unique solution to (1.4) in the sense of Definition 1.2.

2. Proof of Theorem 1.3. First, we prove the following lemma.

Lemma 2.1. (mild solution $\Rightarrow$ weak solution) If $y \in \mathbb{Z}_p$ is the unique solution to (1.6), then for any $\varphi \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$

$$
\langle y_t, \varphi \rangle = \langle U_0, \varphi \rangle + \int_0^t \left[ \langle y_s, \Delta \varphi \rangle + \langle \Gamma_s^{-1}M(\Gamma_s y_s), \varphi \rangle \right] ds, \quad t \in [0, \infty). \quad (2.1)
$$

Conversely, if there exists $y \in \mathbb{Z}_p$ satisfying equation (2.1) for any $\varphi \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$, then $y$ is a solution to (1.6).

Proof. mild solution $\Rightarrow$ weak solution: By (1.6) we know that for $\varphi \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$, $T > 0$

$$
\int_0^T \langle y_t, \Delta \varphi \rangle dt = \int_0^T \langle e^{t\Delta} U_0, \Delta \varphi \rangle dt + \int_0^T \int_0^t e^{(t-s)\Delta} \Gamma_s^{-1}M(\Gamma_s y_s) ds, \varphi \rangle dt.
$$

Following similar arguments as in the proof of [5], Proposition 6.4, we have

$$
\int_0^T \langle e^{t\Delta} U_0, \Delta \varphi \rangle dt = \int_0^T \langle U_0, \frac{d}{dt} e^{t \Delta} \varphi \rangle dt = \langle e^{T \Delta} U_0, \varphi \rangle - \langle U_0, \varphi \rangle.
$$

$$
\int_0^T \int_0^t e^{(t-s)\Delta} \Gamma_s^{-1}M(\Gamma_s y_s) ds, \Delta \varphi \rangle dt = \int_0^T \langle \Gamma_s^{-1}M(\Gamma_s y_s), (e^{(T-s)\Delta} - I) \varphi \rangle ds.
$$

Combining the above arguments we have

$$
\int_0^t \langle y_s, \Delta \varphi \rangle ds = \langle e^{t\Delta} U_0, \varphi \rangle - \langle U_0, \varphi \rangle + \int_0^t \langle e^{(t-s)\Delta} \Gamma_s^{-1}M(\Gamma_s y_s), \varphi \rangle ds
$$

$$
- \int_0^t \langle \Gamma_s^{-1}M(\Gamma_s y_s), \varphi \rangle ds,
$$

which implies (2.1).

weak solution $\Rightarrow$ mild solution: By (2.1) and similar arguments as in the proof of [5], Proposition 6.3, we have for $\zeta \in C^1([0, T]; C_c^\infty(\mathbb{R}^3; \mathbb{R}^3))$, $T > 0$ and $0 < t \leq T$

$$
\langle y_t, \zeta_t \rangle = \langle U_0, \zeta_0 \rangle + \int_0^t \left[ \langle y_s, \Delta \zeta_s + \zeta'_s \rangle + \langle \Gamma_s^{-1}M(\Gamma_s y_s), \zeta_s \rangle \right] ds, \quad t \in [0, \infty).
$$

(2.2)
Choosing $\zeta_s := e^{(t-s)\Delta} \varphi$, $\varphi \in C_c(\mathbb{R}^3; \mathbb{R})$, we have

$$\langle y_t, \varphi \rangle = (U_0, e^{t\Delta} \varphi) + \int_0^t \langle e^{(t-s)\Delta} \Gamma_s^{-1} M(\Gamma_s y_s), \varphi \rangle ds.$$ 

Thus (1.6) follows.

Now we prove the following estimate for the solutions:

**Lemma 2.2.** For $T > 0$, $\varphi \in L^{q/(q-1)} \cap L^3$ and $\Omega_0$ being in Theorem 1.1, on $\Omega_0$

$$\sup_{t \in [0, T]} \|\Gamma_t y_t\| < \infty \quad \text{and} \quad y \in Z^p_\varepsilon \quad \text{for} \quad 0 < \varepsilon < \frac{1}{2} - \frac{3}{4p}, \quad \text{with} \quad p, q \text{ as in Theorem 1.1.}$$

**Proof.** We have

$$y_t = e^{t\Delta} U_0 + \int_0^t e^{(t-s)\Delta} \Gamma_s^{-1} M(\Gamma_s y_s) ds.$$ 

Then on $\Omega_0$

$$\|\Gamma_t y_t|\varphi\| \leq C\|\Gamma_t\|_{L(L^{3/2}, L^{3/2})} \|e^{t\Delta} U_0\|_{3/2} + C\|\Gamma_t\|_{L(L^q, L^q)} \int_0^t \|\Gamma_s^{-1} M(\Gamma_s y_s)\|_q ds$$

$$\leq C\|\Gamma_t\|_{L(L^{3/2}, L^{3/2})} \|U_0\|_{3/2} + C\|\Gamma_t\|_{L(L^q, L^q)} \int_0^t \|\Gamma_s^{-1} M(\Gamma_s y_s)\|_q ds$$

$$\leq C\|\Gamma_t\|_{L(L^{3/2}, L^{3/2})} \|U_0\|_{3/2} + C\|\Gamma_t\|_{L(L^q, L^q)} \|y\|^2 \sup_{s \in [0, t]} \eta_s \int_0^t s^{-5/2+3/p} ds$$

$$< \infty,$$

where in the second inequality we used (2.15) in [2] and in the third inequality we used (2.35) in [2] and in the last inequality we used that $\|y\| \leq C\|U_0\|_{3/2}$ by the proof of Theorem 1.1 in [2]. Now we prove $y \in Z^p_\varepsilon$. We have

$$\|\delta y_{uv}\|_p \leq \|(e^{v\Delta} - e^{u\Delta}) U_0\|_p + \|(e^{(v-u)\Delta} - 1) \int_0^u e^{(u-s)\Delta} \Gamma_s^{-1} M(\Gamma_s y_s) ds\|_p$$

$$+ \| \int_u^v e^{(v-s)\Delta} \Gamma_s^{-1} M(\Gamma_s y_s) ds\|_p.$$ 

For the first term we have

$$\|(e^{v\Delta} - e^{u\Delta}) U_0\|_p \leq \|(e^{(v-u)\Delta} - I) e^{u\Delta} U_0\|_p \leq C\|(e^{(v-u)\Delta} - I) e^{u\Delta} U_0\|_{B^p_{3/2}}$$

$$\leq C(v-u)^{3/2} \|e^{u\Delta} U_0\|_{B^p_{3/2}} \leq C(v-u)^{3/2} u^{-2} \|e^{u\Delta} U_0\|_p$$

$$\leq C(v-u)^{3/2} u^{-2} \eta^{1-2\varepsilon+\frac{3}{p}} \|U_0\|_{3/2},$$

where $B^s_{m,n}$ is the usual Besov space and we used Propositions 3.11 and 3.12 in [11].

For the second term similarly we have

$$\|(e^{(v-u)\Delta} - 1) \int_0^u e^{(u-s)\Delta} \Gamma_s^{-1} M(\Gamma_s y_s) ds\|_p$$

$$\leq C(v-u)^{3/2} \int_0^u \|e^{(u-s)\Delta} \Gamma_s^{-1} M(\Gamma_s y_s)\|_{B^p_{3/2}} ds$$

$$\leq C(v-u)^{3/2} \int_0^u (u-s)^{-2\varepsilon+\frac{3}{p}} \|e^{(u-s)\Delta} \Gamma_s^{-1} M(\Gamma_s y_s)\|_p ds$$

$$\leq C(v-u)^{3/2} \sup \eta_s \|y\|^2 \int_0^u (u-s)^{-2\varepsilon+\frac{3}{p}} \left( \frac{1}{s^{\frac{3}{2}}} - \frac{1}{s^{\frac{3}{2}}} \right) ds$$

$$\leq C(v-u)^{3/2} \sup \eta_s \|y\|^2.$$
where in the third inequality we used a similar calculation as (2.17) in [2]. For the third term we have

\[
\left| \int_u^v e^{(v-s)\Delta} \Gamma^{-1}_s M(\Gamma_s y_s) ds \right|_p \\
\leq C \sup_{s \geq 0} \eta_s \|y\|_2 \int_u^v (v-s)^{-\frac{1}{2}(\frac{3}{p}-1) - \frac{5}{2} + \frac{2}{p}} ds \\
= C \sup_{s \geq 0} \eta_s \|y\|_2^2 (v-u)^{\frac{3}{2} - \frac{3}{p}} \int_0^1 (1-l)^{-\frac{1}{2}(\frac{3}{p}-1)} [u+l(v-u)]^{-\frac{5}{2} + \frac{2}{p}} dl \\
\leq C \sup_{s \geq 0} \eta_s \|y\|_2^2 (v-u)^{-1-2\varepsilon + \frac{2}{p}} \int_0^1 (1-l)^{-\frac{1}{2}(\frac{3}{p}-1) l^{-\frac{2}{2} + \frac{2}{p}} + 2\varepsilon} dl,
\]

where we used interpolation and \(-1 - 2\varepsilon + \frac{2}{p} < 0, -\frac{3}{2} + \frac{3}{2p} + 2\varepsilon < 0\) in the last inequality. Combining the argument above we obtain that

\[
|\delta y_{uv}|_p \leq C (v-u)^{\varepsilon} u^{-2\varepsilon-1 + \frac{2}{p} (\|U_0\|_{3/2} + \sup_{s \geq 0} \eta_s \|y\|_2^2)}.
\]

Similarly we have

\[
|\delta(D_j y)_{uv}|_p \leq |(e^{\varepsilon \Delta} - e^{\varepsilon \Delta}) D_j U_0|_p + |(e^{(v-u)\Delta} - 1) \int_0^u e^{(v-s)\Delta} D_j \Gamma^{-1}_s M(\Gamma_s y_s) ds|_p \\
+ | \int_u^v e^{(v-s)\Delta} D_j \Gamma^{-1}_s M(\Gamma_s y_s) ds|_p \\
\leq C (v-u)^{\varepsilon} u^{-2\varepsilon - \frac{2}{2} + \frac{2}{p}} (|U_0|_{3/2} + \sup_{s \geq 0} \eta_s \|y\|_2^2),
\]

where we used a similar calculation as (2.18) in [2]. Thus the second result follows.

**Proof of Theorem 1.3** [Existence] Now we check that \(U = \Gamma y\) satisfies equation (1.9). We first calculate \(\langle \delta \Gamma y, \varphi \rangle\): for \(0 < u < v\)

\[
\langle \delta \Gamma y, \varphi \rangle = \langle \delta \Gamma y_{uv}, \varphi \rangle + \langle \Gamma_u \delta y_{uv}, \varphi \rangle + \langle \delta \Gamma_{uv} \delta y_{uv}, \varphi \rangle \\
:= I_1 + I_2 + I_3.
\]

Since \(\Gamma_u \varphi = \Pi_{i=1}^N \exp (\beta_i^u \partial_i^u - \frac{\gamma}{2} \partial^2_i) \varphi\) for \(\varphi \in C^\infty_c(\mathbb{R}^3; \mathbb{R}^3)\), by Taylor expansion we have

\[
\delta \Gamma_{uv} \varphi = \Gamma_u \left( \sum_{i=1}^N (\delta \beta_{uv}^i \partial_i^u \varphi) - \frac{(v-u)}{2} \hat{B}_i^2 \varphi + \sum_{k=1}^N \frac{1}{2} \hat{B}_i \hat{B}_k \varphi \delta \beta_{uv}^k \delta \beta_{uv}^i \right) + o(|v-u|).
\]

Here and in the following \(o(|u-v|)\) means a higher order term of \(|u-v|\). Now we recall the following result from Section 3.3 in [7]:

\[
\mathbb{E}^{ik}_{uv} + \frac{1}{2} \delta^{ik} (v-u) = \mathbb{E}_{str, uv}^{ik}, \tag{2.3}
\]

\[
\frac{1}{2} \left( \mathbb{E}_{str, uv}^{ik} + \mathbb{E}_{str, uv}^{ki} \right) = \frac{1}{2} \delta^{ik} \delta_{str, uv}, \tag{2.4}
\]

where \(\delta^{ik} = 1\) if \(i = k\), zero else, and \(\mathbb{E}_{str, uv} := \int_u^v \delta \beta_{uv} \otimes \hat{\beta} \in \mathbb{R}^{N \times N}\) with the integral in the Stratonovich sense. Then by symmetry of \(\hat{B}_i \hat{B}_k \varphi\) with respect to \(i, k\) we have

\[
\delta \Gamma_{uv} \varphi = \Gamma_u \left( \sum_{i=1}^N (\delta \beta_{uv}^i \partial_i^u \varphi) - \frac{(v-u)}{2} \hat{B}_i^2 \varphi + \sum_{k=1}^N \hat{B}_i \hat{B}_k \varphi \mathbb{E}_{str, uv}^{ik} \right) + o(|v-u|),
\]
which by (2.3) implies that

$$I_k = \sum_{i=1}^{N} (\Gamma u \hat{B}_i y_u, \varphi) \delta \beta_{uv} + \sum_{i,k=1}^{N} (\Gamma u \hat{B}_k \hat{B}_i y_u, \varphi) \mathbb{B}^i_{uv} + o(|u - v|).$$

Also since \(y\) satisfies equation (2.1) and \(y \in \mathbb{Z}^p\), we have

$$I_2 = \langle y_u, \Delta \Gamma u \varphi \rangle (v - u) + \langle \Gamma u^{-1} M(\Gamma u y_u), \Gamma u \varphi \rangle (v - u) + o(|v - u|)$$

$$= \langle y_u, \Delta \varphi \rangle (v - u) + \langle M(\Gamma u y_u), \varphi \rangle (v - u) + o(|v - u|),$$

where \(\Gamma u\) means the dual operator of \(\Gamma u\). Here in the first equality we used the following for \(u < s\)

$$|\Gamma u^{-1} M(\Gamma s y_s) - \Gamma u^{-1} M(\Gamma u y_u)|_q$$

$$\leq ||\Gamma u^{-1} - \Gamma u^{-1}||_{L(L^q, L^q)} |M(\Gamma s y_s)|_q + ||\Gamma u^{-1}||_{L(L^q, L^q)} |M(\Gamma s y_s) - M(\Gamma u y_u)|_q$$

$$\leq C_u |s - u|^\varepsilon,$$

where in the last inequality we used a similar calculation as Lemma 2.2 in [2]. By the above calculations we know that

$$I_3 = \langle \delta y_{uv}, \delta \Gamma u^* \varphi \rangle = o(|v - u|),$$

where \(\delta \Gamma u^*\) means the dual operator of \(\delta \Gamma u\). The above calculations and Lemma 2.2 and (2.35) in [2] imply that \(\delta B U, \varphi\) is controlled by \(\beta\) in the sense of Definition 1.1 and satisfies (1.10) and (1.11). By the above calculations we also obtain that for \(0 < s < t\)

$$\langle U_t, \varphi \rangle - \langle U_s, \varphi \rangle$$

$$= \sum_{[u,v] \in \mathcal{P}} \langle (\delta \Gamma y)_{uv}, \varphi \rangle$$

$$= \sum_{[u,v] \in \mathcal{P}} \left[ \sum_{i=1}^{N} (\Gamma u \hat{B}_i y_u, \varphi) \delta \beta_{uv} + \sum_{i,k=1}^{N} (\Gamma u \hat{B}_k \hat{B}_i y_u, \varphi) \mathbb{B}^i_{uv} + \langle y_u, \Delta \varphi \rangle (v - u) + \langle M(\Gamma u y_u), \varphi \rangle (v - u) + o(|v - u|) \right],$$

where \(\mathcal{P}\) is a partition of the interval \([s, t]\) similar as above. Taking the limit \(|\mathcal{P}| \to 0\), by (1.8) we obtain that \(U = \Gamma y\) satisfies the equation (1.9).

[Uniqueness] Now we prove the uniqueness of the solution. In fact by Theorem 1.1 we already know that the solution to (1.6) is unique, so we only need to prove that \(y = \Gamma^{-1} U\) satisfies (2.1), which is equivalent to (1.6) by Lemma 2.1. We have for \(0 < u < v\)

$$\langle \delta (\Gamma^{-1} U)_{uv}, \varphi \rangle = \langle \delta \Gamma^{-1} U_u, \varphi \rangle + \langle \Gamma u^{-1} \delta U_{uv}, \varphi \rangle + \langle \delta \Gamma u^{-1} \delta U_{uv}, \varphi \rangle$$

$$:= J_1 + J_2 + J_3.$$

Since \(\Gamma^{-1} U \in \mathbb{Z}^p\), we obtain the Hölder continuity of \(U_u\) when \(u > 0\). Since \(M(U_u) = M(\Gamma u y_u)\), then (2.5) implies the Hölder continuity of \(M(U_u)\) when \(u > 0\). Then by Corollary 3 in [8] we have

$$J_2 = \langle \delta U_{uv}, \Gamma^{-1} \varphi \rangle = \langle y_u, \Delta \varphi \rangle (v - u) + \langle \Gamma u^{-1} M(\Gamma u y_u), \varphi \rangle (v - u)$$

$$+ \sum_{k=1}^{N} (\hat{B}_k y_u, \varphi) \delta \beta_{uv} + \sum_{i,k=1}^{N} (\hat{B}_i \hat{B}_k y_u, \varphi) \mathbb{B}^i_{uv} + o(|u - v|).$$
where $(\Gamma_u^{-1})^*$ means the dual operator of $\Gamma_u^{-1}$. Moreover, since
\[
\Gamma_u^{-1} \varphi = \Pi_{i=1}^N \exp(-\beta_t^i B_t + \frac{u}{2} \tilde{B}_t^2) \varphi,
\]
by Taylor expansion we have
\[
\delta \Gamma_u^{-1} \varphi = \Gamma_u^{-1} \sum_{i=1}^N (-\delta \beta_{uv}^i \tilde{B}_i y_u + \frac{(v-u)}{2} \tilde{B}_t^2 y_u + \sum_{k=1}^N \frac{1}{2} \tilde{B}_k \varphi \delta \beta_{uv}^k \delta \beta_{uv}^i) + o(|v-u|).
\]
Thus, we have
\[
J_1 = \sum_{i=1}^N (-\delta \beta_{uv}^i \tilde{B}_i y_u + \frac{(v-u)}{2} \tilde{B}_t^2 y_u + \sum_{k=1}^N \frac{1}{2} \tilde{B}_k \varphi \delta \beta_{uv}^k \delta \beta_{uv}^i, \varphi) + o(|v-u|),
\]
and
\[
J_3 = \langle \delta U_{uv}, (\delta \Gamma_u^{-1})^* \varphi \rangle = -\sum_{k,i=1}^N \langle \tilde{B}_i \tilde{B}_k y_u, \varphi \rangle \delta \beta_{uv}^k \delta \beta_{uv}^i + o(|u-v|),
\]
where $(\delta \Gamma_u^{-1})^*$ means the dual operator of $\delta \Gamma_u^{-1}$. Using (2.3) and (2.4) we obtain that
\[
\sum_{i,k=1}^N \langle \tilde{B}_i \tilde{B}_k y_u, \varphi \rangle B_{uv}^{ik} = \sum_{i,k=1}^N \langle \tilde{B}_i \tilde{B}_k y_u, \varphi \rangle B_{uv}^{ik} - \frac{1}{2} \sum_{i=1}^N \langle \tilde{B}_t^2 y_u, \varphi \rangle (v-u)
\]
\[
= \sum_{i,k=1}^N \langle \tilde{B}_i \tilde{B}_k y_u, \varphi \rangle \left[ \frac{B_{uv}^{ik} + B_{uv}^{ki}}{2} \right] - \frac{1}{2} \sum_{i=1}^N \langle \tilde{B}_t^2 y_u, \varphi \rangle (v-u)
\]
\[
= \sum_{i,k=1}^N \langle \tilde{B}_i \tilde{B}_k y_u, \varphi \rangle \frac{1}{2} \delta \beta_{uv}^i \delta \beta_{uv}^k - \frac{1}{2} \sum_{i=1}^N \langle \tilde{B}_t^2 y_u, \varphi \rangle (v-u).
\]
Thus, we have that for $0 < s < t$
\[
\langle y_t, \varphi \rangle - \langle y_s, \varphi \rangle = \sum_{|u,v| \in \mathcal{P}} \langle (\delta \Gamma_u^{-1} U)_{uv}, \varphi \rangle
\]
\[
= \sum_{|u,v| \in \mathcal{P}} \left[ \langle y_u, \Delta \varphi \rangle (v-u) + \langle \Gamma_u^{-1} M(1, y_u), \varphi \rangle (v-u) + o(|u-v|) \right],
\]
where $\mathcal{P}$ is a partition of the interval $[s, t]$. Taking the limit $|\mathcal{P}| \to 0$ we obtain that for $0 < s < t$
\[
\langle y_t, \varphi \rangle = \langle y_s, \varphi \rangle + \int_s^t \left[ \langle y_r, \Delta \varphi \rangle + \langle \Gamma_r^{-1} M(1, y_r), \varphi \rangle \right] dr.
\]
Now letting $s \to 0$, by the continuity of $\langle y_s, \varphi \rangle$ and $y \in \mathcal{Z}_u$ we obtain that $y = \Gamma^{-1} U$ satisfies (2.1). Thus uniqueness follows. \qed
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