Solving the Helmholtz equation for membranes of arbitrary shape: numerical results

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Abstract
I calculate the modes of vibration of membranes of arbitrary shape using a collocation approach based on little sinc functions. The matrix representation of the PDE obtained using this method is explicit and does not require the calculation of integrals. To illustrate the virtues of this approach, I have considered a large number of examples, part of them are taken from the literature, and part of them new. When possible, I have tested the accuracy of these results by comparing them with the exact results (when available) or with results from the literature. In particular, in the case of the L-shaped membrane, the first example discussed in the paper, I show that it is possible to extrapolate the results obtained with different grid sizes to obtain highly precise results. Finally, I also show that the present collocation technique can be easily combined with conformal mapping to provide numerical approximations to the energies which quite rapidly converge to the exact results.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

This paper considers the problem of solving the Helmholtz equation

$$-\Delta \psi(x, y) = E\psi(x, y)$$

over a two-dimensional domain, $B$, of arbitrary shape, assuming Dirichlet boundary conditions over the border, $\partial B$. Physically, this equation describes the classical vibration of a homogeneous membrane or the behaviour of a particle confined in a region with infinite walls in quantum mechanics. Unfortunately exact solutions to this equation are available only...
in few cases, such as for a rectangular or a circular membrane, where they can be expressed in terms of trigonometric and Bessel functions, respectively [1]. In the majority of cases, in fact, only numerical approaches can be used: some of these approaches are discussed, for example, in a beautiful paper by Kuttler and Sigillito [2]. The purpose of the present paper is to introduce a different approach to the numerical solution of the Helmholtz equation (both homogeneous and inhomogeneous) and illustrate its strength and flexibility by applying it to a large number of examples.

The paper is organized as follows: in section 2, I describe the method and discuss its application to the classical problem of a L-shaped membrane; in section 3, I consider an homogeneous membrane, with the shape of Africa and calculate few states; in section 4, I consider two inequivalent membranes, which are known to be isospectral, obtaining a numerical indication of isospectrality; in section 5, I study an example of irregular drum; in section 6, the method is applied to study the emergence of bound states in a configuration of wires of negligible transverse dimension, in the presence of crossings; in section 7, I show that even more precise results can be achieved by combining the collocation method with a conformal mapping of the boundary. Finally, in section 8, I draw my conclusions.

2. The method

The method that I propose in this paper uses a particular set of functions, the little sinc functions (LSF) of [13, 14], to obtain a discretization of a finite region of the two-dimensional plane. These functions have been used with success in the numerical solution of the Schrödinger equation in one dimension, both for problems restricted to finite intervals and for problems on the real line. In particular, it has been proved that exponential convergence to the exact solution can be reached when variational considerations are made (see [13, 14]).

Although [13] contains a detailed discussion of the LSF, I will briefly review here the main properties, which will be useful in the paper. Throughout the paper, I will follow the notation of [13].

A little sinc function is obtained as an approximate representation of the Dirac delta function in terms of the wavefunctions of a particle in a box (being $2L$ the size of the box). Straightforward algebra leads to the expression

$$s_k(h, N, x) = \frac{1}{2N} \left\{ \frac{\sin((2N + 1)\chi_-(x))}{\sin \chi_-(x)} - \frac{\cos((2N + 1)\chi_+(x))}{\cos \chi_+(x)} \right\},$$

where $\chi_{\pm}(x) = \frac{\pi}{Nh}(x \pm kh)$. An alternative expression for these functions in terms of Chebyshev polynomials reads [14]

$$s_k(h, N, x) = \frac{1}{2N} [U_{2N}(\cos \chi_-(x)) - U_{2N}(\sin \chi_+(x))].$$

The index $k$ takes the integer values between $-N/2 + 1$ and $N/2 - 1$ ($N$ being an even integer). The LSF corresponding to a specific value of $k$ is peaked at $x_k = 2Lk/N = kh$, $h$ being the grid spacing and $2L$ the total extension of the interval where the function is defined. By direct inspection of equation (2) it is found that $s_k(h, N, x_j) = \delta_{kj}$, showing that the LSF takes its maximum value at the $k$th grid point and vanishes on the remaining points of the grid.

It can be easily proved that the different LSF corresponding to the same set are orthogonal [13]:

$$\int_{-L}^{L} s_k(h, N, x)s_j(h, N, x) \, dx = h\delta_{kj}$$

2
and that a function defined on $x \in (-L, L)$ may be approximated as
\[ f(x) \approx \sum_{k=-N/2+1}^{N/2-1} f(x_k) s_k(h, N, x). \]

This formula can be applied to obtain a representation of the derivative of a LSF in terms of the set of LSF as
\[ \frac{ds_k(h, N, x)}{dx} \approx \sum_j \frac{ds_j(h, N, x)}{dx} \bigg|_{x=x_j} s_j(h, N, x) \equiv \sum_j c_{kj}^{(1)} s_j(h, N, x), \]
\[ \frac{d^2s_k(h, N, x)}{dx^2} \approx \sum_j \frac{d^2s_j(h, N, x)}{dx^2} \bigg|_{x=x_j} s_j(h, N, x) \equiv \sum_j c_{kj}^{(2)} s_j(h, N, x), \]
where the expressions for the coefficients $c_{kj}^{(r)}$ can be found in [13]. Although equation (5) is approximate and the LSF strictly speaking do not form a basis, the error made with this approximation decreases with $N$ and tends to zero as $N$ tends to infinity, as shown in [13]. This is the reason the effect of this approximation is essentially to replace the continuum of an interval of size $2L$ on the real line with a discrete set of $N-1$ points, $x_k$, uniformly spaced on this interval.

Clearly these relations are easily generalized to functions of two or more variables. Since the focus of this paper is on two-dimensional membranes, I will briefly discuss how the LSF are used to discretize a region of the plane; the extension to higher dimensional spaces is straightforward. A function of two variables can be approximated in terms of $(N_x - 1) \times (N_y - 1)$ functions, corresponding to the direct product of the $N_x - 1$ and $N_y - 1$ LSF in the $x$- and $y$-axis: each term in this set corresponds to a specific point on a rectangular grid with spacings $h_x$ and $h_y$ (in this paper, I use a square grid with $N_x = N_y = N$ and $L_x = L_y = L$).

Since $(k, k')$ identifies a unique point on the grid, I can select this point using a single index
\[ K \equiv k' + \frac{N}{2} + (N - 1) \left( k + \frac{N}{2} - 1 \right) \]
which can take the values $1 \leq K \leq (N-1)^2$. I can also invert this relation and write
\[ k = 1 - N/2 + \left[ \frac{K}{N - 1 + \varepsilon} \right], \]
\[ k' = K - N/2 - (N - 1) \left[ \frac{K}{N - 1 + \varepsilon} \right], \]
where $[a]$ is the integer part of a real number $a$ and $\varepsilon \to 0$.

As a natural extension of the results presented in [13, 14], I can consider the Schrödinger equation in two dimensions
\[ \hat{H} \psi_m(x, y) = [-\Delta + V(x, y)] \psi_m(x, y) = E_m \psi_m(x, y) \]
using the convention of assuming a particle of mass $m = 1/2$ and setting $\hbar = 1$. The Helmholtz equation, which describes the vibration of a membrane, is a special case of (10), corresponding to having $V(x, y) = 0$ inside the region $B$ where the membrane lies and $V(x, y) = \infty$ on the border $\partial B$ and outside the membrane.

The discretization of equation (10) proceeds in a simple way using the properties discussed in equations (5) and (6):
\[ H_{kk', jj'} = -\left[ \left[ c_{kj}^{(2)} \delta_{k'j'} + \delta_{kj} c_{j'k}^{(2)} \right] + \delta_{kj} \delta_{k'j'} V(x_k, y_k) \right], \]
where \( k, j, k', j' \) take the values \(-N/2 + 1, \ldots, N/2 - 1\). Note that the potential part of the Hamiltonian is obtained by simply ‘collocating’ the potential \( V(x, y) \) on the grid, an operation with a limited computational price. The result shown in (11) corresponds to the matrix element of the Hamiltonian operator \( \hat{H} \) between two grid points, \((k, k')\) and \((j, j')\), which can be selected using two integer values \( K \) and \( J \), as shown in (7).

Following this procedure, the solution of the Schrödinger (Helmholtz) equation on the uniform grid generated by the LSF corresponds to the diagonalization of a \((N-1)^2 \times (N-1)^2\) square matrix, whose elements are given by equation (11).

I will now use a specific problem, the vibration of a L-shaped membrane, represented in figure 1, to illustrate the method, and discuss different implementations of the method itself. This problem has been widely used in the past to test the performance of the different numerical methods (see, for example [2–5, 7–11]) and is therefore a useful tool to assess the strength of the present approach. Because of the reentrant corner, corresponding to the angle \( \theta = 3\pi/2 \) located at \((0, 0)\), the derivatives of \( \psi(x, y) \) in the radial direction are unbounded (see [3]).

Reid and Walsh [3] obtained a numerical approximation for the two lowest modes of this membrane using finite differences and a conformal map which eliminates the reentrant corner (see figure 5 of [3]); a more precise result was later obtained by Fox, Henrici and Moler who used the method of particular solutions (MPS) in [4] exploiting the symmetries of the problem (the reader may find a detailed discussion of the symmetries for this problem in [2]): the first eight digits of the lowest eigenvalue reported by the authors are correct. Mason has obtained numerical estimates for the first few modes of the L-shaped membrane in terms of a two-dimensional Chebyshev series [5]. Milsted and Hutchinson [6] have obtained finite element
solutions to this problem. Sideridis [7] used a conformal mapping of the L-shaped region onto a square and then solved the resulting equation on a uniform rectangular mesh, obtaining the first four digits of the lowest mode. Schiff [8] has calculated the first 15 lowest modes of this membrane using finite elements, with a refined grid covering the region surrounding the reentrant corner.

More recently, Platte and Driscoll have solved the boundary value problem on the L-shaped membrane using radial basis functions [9]. Finally, Betcke and Trefethen have revisited the MPS in [10]; in that paper, they have observed that the MPS reaches a minimal error for a certain value of $N$ (the number of collocation points on each of the sides non-adjacent to the corner where the expansion is performed) but then it starts to grow as $N$ increases. The modified version of the method discussed in [10], which samples the Fourier–Bessel functions also in the interior points, corrects this problem and provides a convergent behaviour for the error. In this way, Betcke and Trefethen were able to obtain the first 14 digits of the lowest eigenvalue of the L-shaped membrane, $E_1 \approx 9.639723 844 0219$. I will use this precise result to test the accuracy of our method. Reference [11] contains precise estimates for some higher excited states of the L-shaped membrane.

I will now apply the LSF to the numerical solution of this problem: looking at figure 1, I consider the grid points which are internal to the membrane and which do not fall on the border. For a fixed $N$ there is a total of $3/4N^2 - 2N + 1$ points; the grid represented in the figure corresponds to $N = 10$ and therefore to a total of 56 internal points. In this case the collocation of the Hamiltonian on the uniform grid generated by the LSF leads to a $56 \times 56$ matrix, which can then be diagonalized. The eigenvalues of this matrix provide the lowest 56 modes of the membrane, while the eigenvectors provide the lowest 56 wavefunctions. Alternatively I can pick all the points of the grid internal to the membrane, including those falling on the border: in such a case a total of $3/4N^2 - N$ points are found, corresponding to a total of 65 points in the case of the figure.

Table 1 contains the first 108 eigenvalues of the L-shaped membrane calculated using a grid with $N = 60$ and selecting the grid points according to the prescriptions just explained. I have used the notation $E_n^{(\pm)}$ for the energy of the $n$th state when the collocation points on the border are either rejected ($E_n^{(+)}$) or kept ($E_n^{(-)}$). The notation ($\pm$) is used since the two sets approach the exact results either from above ($+$) or from below ($-$), as one can see comparing these numbers with the precise results contained in [10, 11]. The reader will certainly note that the results of table 1 contain rather large errors: in the case of the fundamental state, for example, one has an error of about 1% from $E_n^{(+)}$ and a much larger error of almost 5% for $E_n^{(-)}$.

The left panel of figure 2 shows the eigenvalues $E_n^{(+)}$ (solid line) and $E_n^{(-)}$ (dashed line) for the L-shaped membrane corresponding to a grid with $N = 60$. The reader may note that the higher end of the spectrum displays a curvature, contrary to the behaviour predicted by Weyl’s law, i.e. $\langle N \rangle \propto E$ for large energies. It is easy to show that such effect is artificial: consider, for example, the case of a particle confined in a unit square, whose energies are given by $E_{n_x,n_y} = (n_x^2 + n_y^2)\pi^2$. The diagonalization of the Hamiltonian (11) for this problem would provide the energies corresponding to the $(N - 1)^2$ states obtained taking the first $N - 1$ values of $n_x$ and $n_y$. This means that for energies higher than $E_N = [N^2 - 2N + 2]\pi^2$, the method will provide only the eigenvalues contained inside a square of side $N - 1$ (in the $(n_x,n_y)$ plane), up to a maximal energy $E_{\text{MAX}} = 2[N^2 - 2N + 2]\pi^2$. For this reason, the states above $E_N$ are incomplete and should not be taken into account for inferring the asymptotic behaviour of $\langle N \rangle$. The right panel of figure 2 displays the asymmetry defined as $A_n = 2(E_n^{(+)} - E_n^{(-)})/(E_n^{(+)} + E_n^{(-)})$ for the same grid: this quantity provides an upper estimate for the error.
Table 1. First 108 eigenvalues of the L-shaped membrane calculated with a grid with $N = 60$.

| $n$ | $E^{(+)}_n$ | $E^{(-)}_n$ | $E^{(+)}_n$ | $E^{(-)}_n$ |
|-----|-------------|-------------|-------------|-------------|
| 1   | 9.177164983| 9.725740015| 178.4849465| 184.8421046|
| 2   | 14.78073926| 15.25792488| 193.8672234| 197.9267047|
| 3   | 19.37069304| 19.79218759| 193.8814147| 197.9267047|
| 4   | 29.22316338| 29.5638567  | 201.4624222| 201.9430968|
| 5   | 30.96354699| 32.09126661| 208.0809373| 208.8812769|
| 6   | 40.02464252| 41.71342235| 208.809373  | 208.8812769|
| 7   | 43.35567534| 45.16667725| 209.9202486| 209.9202486|
| 8   | 48.49170563| 49.48205954| 218.9529607| 223.7951430|
| 9   | 48.50129154| 49.48210584| 219.7977882| 224.2020086|
| 10  | 55.00253452| 56.99285853| 230.4881589| 237.0380886|
| 11  | 64.39116656| 65.51743185| 234.522406  | 238.9225242|
| 12  | 70.17580289| 71.2539692  | 240.8305192| 247.3994490|
| 13  | 70.75367576| 71.694315   | 242.091938  | 251.9722965|
| 14  | 77.43821507| 79.16827278| 242.4936924| 253.9585512|
| 15  | 85.62385216| 89.92807671| 252.68965   | 257.8558125|
| 16  | 89.20335691| 92.6479784  | 254.3713602| 257.4891189|
| 17  | 95.0265677 | 97.66188999| 258.45965   | 267.2165568|
| 18  | 99.4402117 | 98.95181845| 262.252381  | 270.3623781|
| 19  | 101.15707293| 102.1148968| 262.9019839| 271.2576246|
| 20  | 109.8094028| 112.7440906| 276.4727322| 282.0671402|
| 21  | 112.6706295| 115.940658 | 279.3270569| 286.9614486|
| 22  | 125.8673839| 128.647868 | 281.7073779| 287.0144046|
| 23  | 126.0084139| 128.6517418| 284.1271564| 290.2236089|
| 24  | 138.4732345| 143.0937626| 287.835501  | 294.4411423|
| 25  | 148.8908462| 151.4047394| 290.802136  | 297.2785674|
| 26  | 149.3132131| 155.4149521| 301.6031656| 306.9689597|
| 27  | 157.1294641| 162.8917066| 308.637254  | 314.7477374|
| 28  | 159.2280728| 165.3935921| 310.2598585| 316.6847591|
| 29  | 166.1006112| 168.297836 | 320.973129  | 336.5004545|
| 30  | 166.9707708| 170.355339| 330.6951225| 336.5831701|
| 31  | 173.3430437| 178.122097| 340.1437751| 346.9498855|
| 32  | 175.411857 | 180.5827508| 353.4120482| 369.245389|

Figure 3 displays the ground-state energy of the L-shaped membrane as a function of the number of grid points and compares it with the precise result of [10]: as already pointed out the two sets approach the exact value from above and below.

Much more precise results can be obtained by performing an extrapolation of the results corresponding to finite grids: this is a common procedure used in the literature (see, for example [2]). I have considered four different extrapolation sets using the numerical results obtained working with grids with $N$ ranging from $N = 10$ to $N = 60$ (only even values).

Calling $h = 2L/N$ the grid spacing the sets are

$$ f_1(h) = \sum_{n=0}^{S_1} c_n h^n, $$

(12)
Figure 2. Left panel: Energy of the ground state of the L-shaped membrane as a function of the number of grid points \( N \). The horizontal line is the precise result of [10]. The set approaching the exact result from above (below) corresponds to \( E_{1}^{(+)} \) (\( E_{1}^{(-)} \)). Right panel: The asymmetry \( A_{n} = 2(E_{n}^{(+)} - E_{n}^{(-)})/(E_{n}^{(+)} + E_{n}^{(-)}) \) calculated with a grid with \( N = 60 \).

Figure 3. Energy of the ground state of the L-shaped membrane as a function of the number of grid points \( N \). The horizontal line is the precise result of [10]. The set approaching the exact result from above (below) corresponds to \( E_{1}^{(+)} \) (\( E_{1}^{(-)} \)).

\[
f_{2}(h) = \frac{\sum_{n=0}^{\bar{N}/2} c_{n} h^{n}}{1 + \sum_{n=1}^{\bar{N}/2} c_{n} h^{n}}, \tag{13}
\]

\[
f_{3}(h) = c_{0} + \sum_{n=1}^{\bar{N}} c_{n} h^{n/3 + 2/3}, \tag{14}
\]

\[
f_{4}(h) = c_{0} + \frac{\sum_{n=1}^{\bar{N}/2} c_{n} h^{n/3 + 2/3}}{1 + \sum_{n=1}^{\bar{N}/2} c_{n} h^{n/3 + 2/3}}, \tag{15}
\]

where \( \bar{N} \) is an even integer which determines the number of coefficients used in the fits.

The continuum limit is reached taking \( h \to \infty \), where only the coefficient \( c_{0} \) survives. The unknown coefficients in expressions (12)–(15) are obtained using a least square approach:
Table 2. Extrapolation of selected eigenvalues of the L-shaped membrane using the four different sets. The first six states correspond to extrapolating the results for grids going from \( N = 10 \) to \( N = 60 \), with 25 unknown coefficients; the last two states correspond to extrapolating the results for grids going from \( N = 18 \) to \( N = 60 \), with 21 unknown coefficients. For a given state, the set with the asterisk corresponds to the minimal value taken by the least squares. The results which do not converge to the exact value have been omitted.

| \( n \) | Set 1 | Set 2 | Set 3 | Set 4 |
|------|------|------|------|------|
| 1 (--) | 9.63959383529194 | 9.63970774930113 | 9.63972385784876 | 9.63972384404696* |
| 1 (+) | 9.63959513453456 | 9.63971258279395 | 9.63972384034031 | 9.63972384401891* |
| 2 (--) | 15.1972518419212 | 15.1974702475024 | 15.1972519362081 | 15.1972519266011* |
| 2 (+) | 15.1972518428845 | 15.1972519235114 | 15.1972519387503 | 15.1972519264561* |
| 3 (--) | 19.739208761784 | 19.7392088017282 | 19.7392073765870 | 19.7392088020095* |
| 3 (+) | 19.7392088019879 | 19.7392088021704 | 19.7392087962239 | 19.7392088021785* |
| 4 (--) | 29.5178267971821 | 29.5214811197206 | – | 29.5214811103487* |
| 4 (+) | 29.52148701813503 | 29.5214811126514 | 29.5214794563921 | 29.5214811141506* |
| 5 (--) | 31.915976759531 | 31.912574596685 | – | 31.91263593035* |
| 5 (+) | 31.9123209946513 | 31.912605580344 | 31.912638707453 | 31.912635971263* |
| 6 (--) | 41.474273066813 | 41.474470492213 | 41.4761914432832 | 41.4745098944789* |
| 6 (+) | 41.4742739974452 | 41.474478007070 | 41.4741677038785 | 41.4745098904487* |
| 20 (--) | 101.776561675314* | 101.60533389975 | – | 99.7713224851033 |
| 20 (+) | 101.60485353178 | 101.605223692426 | 101.673183488214 | 101.605294080845* |
| 50 (--) | – | 246.704564791939 | – | 246.602432808866* |
| 50 (+) | 250.78479377301 | 250.785244396338 | – | 250.785494606618* |
| 104 (--) | – | 410.08260648211 | – | – |
| 104 (+) | 493.480067984180* | 493.480206216096 | – | 493.488405725447 |

I show the results of this procedure in table 2. In general, the last set provides the best results and indeed reproduces the first 11 digits of \( E_3 \). In the case of \( E_3 \), for which the exact value is known \( (E_3 = 2\pi^2) \), I obtain the first 14 digits correct using \( E_3^{(+)} \) and the first 11 digits correct using \( E_3^{(-)} \).

In [15] Berry has devised an algorithm for obtaining successive approximations to the geometric properties \( K_j \) of a closed boundary \( B \) given the lowest \( N \) eigenvalues \( E_n \). The partition function \( \Phi(t) = \sum_{n=1}^{\infty} e^{-E_n t} \) obeys an asymptotic expansion for small values of \( t \)

\[
\Phi(t) \approx \frac{1}{t} \sum_{j=0}^{\infty} K_j t^{j/2},
\]

where the coefficients \( K_j \) are related to the geometric properties of \( B \). For example, \( K_0 = \lambda/4\pi \) and \( K_1 = -\gamma L/8\sqrt{\pi} \). Using this asymptotic expansion, Berry has obtained accelerated expressions for the geometrical constants of \( B \). In particular, for the area of \( B \) he has found the approximant (equation (20) of [15])

\[
A_m(t) = \frac{2\pi t}{m!} \sum_{n=1}^{\infty} e^{-\xi_n^2} \xi_n^{m-1} H_{m+1}(\xi_n),
\]

where \( \xi_n = \sqrt{E_{n1} t} \).

In the left panel of figure 4 I show the area approximant \( A_2(t) \), obtained using the expression of Berry. The thin lines correspond to using the sets \( E_{n1}^{(+)} \) and \( E_{n1}^{(-)} \) (solid and dashed lines, respectively); the thick lines correspond to using the eigenvalues obtained from the extrapolation of the sets \( E_{n1}^{(+)} \) and \( E_{n1}^{(-)} \) (solid and dashed lines, respectively). I call \( E_{n1}^{(ab)} \) the eigenvalues obtained extrapolating the eigenvalues \( E_{n1}^{(ab)} \); the extrapolation is carried out using
the results obtained with grids with \( N \) going from 48 to 60 and assuming \( E_n(N) \approx \bar{E}_n + \epsilon_n N \). The approximants obtained with the extrapolated eigenvalues provide excellent approximations to the area and perimeter of the membrane, as seen in figure 4.

Figure 5 shows the first two eigenfunctions of the L-shaped membrane obtained with a grid corresponding to \( N = 30 \). The solid lines appearing in the ‘forbidden region’ correspond to the level \( \psi(x, y) = 0 \): the effect observed in the figure is due to the approximation of working with a finite number of grid points. In fact, although a particular LSF vanishes on the points defining the grid, except on a particular point, where it reaches its maximum, it is nonzero elsewhere. This means that the numerical solution can take small values even in the region where the exact solution must vanish; however, the size of this effect decreases as the number of grid points is increased (taking into account that the computational load roughly increases as \( N^4 \)). In the appendix we propose an alternative procedure which does not involve the diagonalization of larger matrices and which can be used to improve the results obtained with a given grid.
3. The Africa drum

I will now examine the case of a membrane with an irregular shape. The application of the method proceeds exactly as in the case of the L-shaped membrane: once a grid is chosen, the points of the grid which are internal to the membrane are used to build a matrix representation of the Hamiltonian which, once diagonalized, provides the energies and wavefunctions of the problem.

As a paradigm of this class of membranes, I have studied the vibrations of a drum with the shape of Africa. Unlike in the previous example the border does not cross the grid points, a feature which affects the precision of the results. The plots in figure 6 display the energies of the first two states of the Africa drum for grids with different $N$ (the dots in the plots) and compare them with the best fit obtained assuming that $E(N) = a + b/N$, where $a$ and $b$ are constants independent of $N$. The irregularity of the border is reflected in the behaviour of the eigenvalues which decay with $N$ but at the same time oscillate.

In figure 7, I show the density plot of four different states of the Africa drum, obtained using a grid with $N = 60$. In figure 8, I show the wavefunction of the ground state of the Africa drum, obtained using a grid with $N = 60$.

4. Isospectral membranes

In a classic paper dated 1966 [16], Kac formulated an interesting question: whether it is possible to hear the shape of a drum, meaning if the spectrum of frequencies of a given drum is unique to that drum or drums with different shapes can have the same spectrum. The question was finally answered in 1992, when Gordon, Webb and Wolpert found a first example of inequivalent drums having the same spectrum [17]. An experiment made by Sridhar and Kudrolli reported in [18] used microwave cavities with the shape of the drums of [17] to verify the equality of the spectrum for the lowest 54 states. More recently, the same experiments have been carried out on isospectral cavities where the classical dynamics changes from pseudo-integrable to chaotic [19]. Numerical calculations of the first few modes

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1 As a technical remark, the shape of Africa—or of arbitrary membranes—is obtained in the Mathematica code by reading a digital image and then by generating a function inside a unit square, whose values at a given point are 0 and 1 depending if the point falls inside or outside the membrane.
of the isospectral drums found in [17] have been performed with different techniques: Wu, Sprung and Martorell [20] have used a mode matching method to calculate the first 25 states of these drums and compared the results with those obtained with finite difference; using a different approach, Driscoll [21] has also calculated the first 25 states obtaining results which are accurate to 12 digits; Betcke and Trefethen [10] have used their modified version of the method of particular solutions to obtain the first three eigenvalues of these drums, reporting results which are slightly more precise than those of Driscoll.

I will now discuss the application of the present method to the calculation of the spectrum of these isospectral membranes: whereas in the case of the L-shaped membrane the border of the membrane was sampled by the grid, regardless of the grid size (keeping $N$ even), in the case of the isospectral membranes this happens only for grids where $N = 6k$, with $k$ an integer. It is important to restrict the calculation to this class of grids to avoid the oscillations observed.

Figure 7. Density plot for the fundamental state (upper left), first excited state (upper right), 200th excited state (lower left) and 300th excited state (lower right) of the Africa-shaped membrane. In all plots the absolute value of the wavefunction is shown and a grid with $N = 60$ is used.
in the case of the Africa membrane. I have thus applied the method with grids ranging from $N = 6$ to $N = 120$.

The plot in figure 9 displays the ground-state energy of the first isospectral membrane calculated at different grid sizes. The horizontal line is the precise value given in [10]. The set approaching this value from above (below) corresponds to the application of the method rejecting (accepting) the grid points falling on the border. The corresponding plot for the second isospectral membrane is almost identical and therefore it is not presented here.

In table 3, I report the energies of the first 30 states obtained using Richardson extrapolation [22] on the results for grids going from $N = 66$ to $N = 120$. The second and third columns are the energies of the first isospectral membranes obtained with the sets which reject $(E^{(+)}_n)$ or accept $(E^{(-)}_n)$ the grid points falling on the border, which as seen in the case of the L-shaped membrane provide a sequence of numerical values approaching the exact eigenvalue from above and from below, respectively. The last two columns report the analogous results for the second isospectral membrane. Note that some of the energies in the third column are clearly incorrect.

A further empirical verification of the isospectrality of the two membranes is presented in figure 10, where I have plotted the asymmetry $A_n \equiv \left( E^{(1+)}_n - E^{(2+)}_n \right) / \left( E^{(1+)}_n + E^{(2+)}_n \right)$ for the first 2000 states of the isospectral membranes. In this case $E^{(1+)}_n$ is the energy of the $n$th state of the first (second) membrane obtained using Richardson extrapolation of the grids with $N = 114$ and $N = 120$. Figure 11 displays the wavefunctions of the ground state and 100th state of the two isospectral membranes.

---

2 The numerical results presented in the case of the L-shaped membrane were obtained with a 40-digit precision in the eigenvalues, using the command N[ , 40] of Mathematica: in this case, since I need to resort to larger grids I have worked with less digits precision using the command N[] in Mathematica.
Figure 9. Energy of the ground state of the first isospectral membrane as a function of the number of grid points $N$. The horizontal line is the precise result of \[10\]. The set approaching the exact result from above (below) corresponds to $E_{1}^{(1+)}$ ($E_{1}^{(1-)}$).

Figure 10. Left panel: log$_{10}$ of the asymmetry $A_{n} \equiv (E_{n}^{(1+)} - E_{n}^{(2+)})/(E_{n}^{(1+)} + E_{n}^{(2+)})$ for the first 2000 states of the isospectral membranes. $E_{n}^{(1+)}$ ($E_{n}^{(2+)}$) is the energy of the $n$th state of the first (second) membrane obtained using Richardson extrapolation of the grids with $N = 114$ and $N = 120$. Right panel: Blow-up of the previous plot for the first 100 states.

5. An unusual drum

I will now consider a further example by looking at a particular membrane originally studied by Trott \[23\]: this drum is shown in figure 12 and consists of a total of 308 units squares which are joined into a rather irregular form. Theoretical and experimental studies carried out on drums with fractal or irregular boundaries have shown that the wave excitations for these drums are drastically altered \[24–26\]: in particular, the Weyl law for these membranes is modified in a way which depends on the fractal dimension of the perimeter (see, for example \[27\]), the so-called Weyl–Berry–Lapidus conjecture. Recently, the vibrations of a uniform membrane contained in a Koch snowflake have been studied in two papers \[28, 29\].

The paper by Trott is both interesting in its physical and mathematical content and as an example of the excellent capabilities of Mathematica to handle heavy numerical calculations:
Table 3. First 30 eigenvalues of the isospectral membranes obtained with Richardson extrapolation of the results obtained with grids from $N = 66$ to $N = 120$.

| n  | $E_n^{(1+)}$ | $E_n^{(1-)}$ | $E_n^{(2+)}$ | $E_n^{(2-)}$ |
|----|-------------|-------------|-------------|-------------|
| 1  | 2.537938184 | 2.537859157 | 2.53790157  | 2.537924672 |
| 2  | 3.655477379 | 3.655454782 | 3.65539267  | 3.65543693  |
| 3  | 5.175456364 | 5.175515223 | 5.175489106 | 5.175528448 |
| 4  | 6.537580464 | 6.537493542 | 6.537567744 | 6.537528448 |
| 5  | 7.247973684 | 7.248012453 | 7.247966062 | 7.248007219 |
| 6  | 9.209282216 | 9.209252596 | 9.20928929  | 9.209222008 |
| 7  | 10.59698943 | 10.59697476 | 10.59692509 | 10.59694683 |
| 8  | 11.54137149 | 11.54137651 | 11.54137016 | 11.54142735 |
| 9  | 12.33702671 | 12.33696554 | 12.33700655 | 12.33698898 |
| 10 | 13.0535072  | 13.0535318  | 13.05351736 | 13.05354013 |
| 11 | 14.31383084 | 14.31384362 | 14.31380888 | 14.31380888 |
| 12 | 15.87110323 | 15.87106608 | 15.8711794  | 15.8711794  |
| 13 | 16.94182893 | -2544.66845 | 17.66503368 | 17.6650544  |
| 14 | 17.66507424 | 2544.66845  | 18.98078964 | 18.98081294 |
| 15 | 18.98079211 | 18.98083269 | 18.98081294 | 18.98081294 |
| 16 | 20.88240176 | 16.71191189 | 20.88233985 | 20.88246668 |
| 17 | 21.24773575 | 2544.66845  | 21.24772537 | 21.24764682 |
| 18 | 22.32657552 | 22.23620394 | 22.32658971 | 22.32628952 |
| 19 | 23.71129295 | 23.71127276 | 23.71130372 | 23.71130372 |
| 20 | 24.47925064 | 24.48080219 | 24.47920658 | 24.47934876 |
| 21 | 24.64106118 | 24.67245947 | 24.67401531 | 24.67403958 |
| 22 | 26.0801208 | 26.08090828 | 26.08008881 | 26.08012901 |
| 23 | 27.30391033 | 27.30298845 | 27.30390863 | 27.3039225 |
| 24 | 28.17508031 | 28.17506497 | 28.17506143 | 28.17505957 |
| 25 | 29.56976983 | 29.56970152 | 29.56970401 | 29.56905778 |
| 26 | 31.48300874 | 31.51241562 | 31.48304984 | 31.48303448 |
| 27 | 32.07624358 | 32.16454642 | 32.07622156 | 32.08006665 |
| 28 | 32.21611001 | 37.0118719 | 32.21605287 | 32.21393591 |
| 29 | 32.90353338 | 27.9888228 | 32.90357696 | 32.90354978 |
| 30 | 34.13633502 | 34.13929552 | 34.13632946 | 34.13632752 |

as a matter of fact Trott uses a finite difference approximation of the Laplacian on a uniform grid and samples the membrane in 28 521 internal points. Explicit numerical values for the first 24 modes are reported.

I have therefore considered the same problem using the LSF with grids of different size (up to $N = 250$ which leads to the same grid of [23]). Figure 13 displays the energy of the fundamental mode of this membrane as a function of the size of $N$. The dashed horizontal line in the plot represents the result of [23], $E_1 = 6.64705$: the points on the upper part of the plot correspond to $N$ going from 50 to 250, with intervals of 50. For these particular values of $N$, the border of the membrane is sampled by the grid and therefore more accurate results are expected. The grid points on the border are rejected, which leads to eigenvalues which approach the exact results from above, as seen in the previous examples. The points in the lower part of the plot correspond to grid sizes varying from $N = 52$ to $N = 148$, excluding $N = 100$: in this case, the values approach the exact result from below, although in doing so they also oscillate reflecting the treatment of the border (a behaviour already observed in the case of the Africa membrane). As mentioned above, the finest grid corresponds to sampling the membrane on 28521 internal points and therefore to working with a
28521 × 28521 square matrix. Given that the matrix obtained with the LSF is a sparse symmetrix matrix, it is possible to deal efficiently with it in Mathematica, applying the Arnoldi method to extract a limited sequence of eigenvalues/eigenvectors. The reader will note that in this example I have not considered the set corresponding to accepting the grid points falling on the border, as was done in the case of the L-shaped and of the isospectral membranes: although this set provides a sequence of values which uniformly approach the value at the continuum, the number of grid points sampled is quite large because of the large perimeter of the membrane. For example, for $N = 100$, this set samples the membrane on 7029 points, compared with the $N = 3801$ points used in the other set.

The figure also displays the improved ground-state energies obtained using the ‘mesh refinement’ procedure described in the appendix (the three green points): the eigenvector for a given grid is extrapolated to a finer grid rejecting contributions in the ‘forbidden region’ (i.e. falling outside the border of the membrane). The improved energy estimate corresponds to the expectation value of the Hamiltonian in this state and thus requires no diagonalization. The results displayed in the figure correspond to extrapolation to a grid which is twice finer.
Figure 12. The unusual drum considered by Trott [23]. The black area is the surface of the drum; the red points are the collocation points corresponding to $N = 50$.

Figure 13. Energy of the ground state of the unusual drum as a function of $N$. The horizontal line is the result of [23]; the points approaching the horizontal line from above correspond to configurations where the border is sampled by the collocation points (and as discussed in the case of the L-shaped membrane are rejected). The green points correspond to the results obtained with the ‘mesh refinement’ procedure described in the appendix.

6. Bound states in the continuum

It is well known that the spectrum of the Laplacian with Dirichlet boundary conditions may contain bound states even for open geometries, in correspondence of crossings or bendings of
the domain. For example, Schult et al [30] have studied the problem of two crossed wires, of infinite length, showing that such a geometry supports exactly one bound state, localized at the crossing. Avishai and collaborators have also proved the existence of a bound state in the broken strip configuration for arbitrarily small angles, see [31] (more recently, Levin has proved the existence of one bound state in the broken strip for any angle of the strip [32]). Goldstone and Jaffe [33] have given a variational proof of the existence of a bound state for an infinite tube in two and three dimensions, provided that the tube is not straight. Other interesting configurations which support bound states in the continuum have been studied by Trefethen and Betcke [11].

The example which I will consider here is somehow related to the crossed wires configuration studied by Schult et al. I have considered a set of horizontal and vertical wires, of negligible transverse dimension, which are contained in a square box of size 2. Calling $\bar{n}$ the number of wires in each dimension, $\bar{n}^2$ is the number of crossings between these wires (for simplicity the wires are assumed to be equally spaced). This configuration can be easily studied in the present collocation approach, by sampling the wires on a grid and by then diagonalizing the Hamiltonian obtained following this procedure. The resulting energies calculated in this way will clearly depend on the spacing of the collocation grid, $h$, and diverge as $h$ is sent to zero. To obtain finite results one need to multiply these eigenvalues by $h^2$, which eliminates the divergence caused by the shrinking of the transverse dimension. Following this procedure I have studied different configurations, corresponding to choosing different value of $\bar{n}$ (going from $\bar{n} = 1$ to $\bar{n} = 4$) and I have found that a given configuration has precisely the same number of bound states as the number of crossings. These bound states happen to be almost exactly degenerate and correspond to wavefunctions which are localized on the vertices.

In table 4, I report the energy (multiplied by $h^2$) of the bound states and of the first unbound state ($E_{gap}$) for the different configurations. These results have been obtained using a fine grid corresponding to $h = 1/300$ and show that the bound states are precisely $\bar{n}^2$ as anticipated and they are essentially degenerate; the energy of the bound states and of the gap are also found to be almost insensitive to $\bar{n}$, which can be interpreted as a sign of confinement of a state to the crossings. I have also checked the dependence of these results upon $N$ (or equivalently upon $h$) observing that the energies can be fitted excellently as $E = a + b/N^2$; for example, in the case of the ground state of the configuration with $\bar{n} = 4$, I have obtained $E = 2.59874 - 44.6364/N^2$.

In figure 14, I have plotted the wavefunction of the ground state of the configuration corresponding to $\bar{n} = 4$ using a grid with $N = 500$. The wavefunction is clearly localized at the crossings between the wires. Similar behaviour is observed for the remaining 15 bound states.

7. Collocation with conformal mapping

The examples considered in the previous sections show that it is possible to obtain the spectrum of the negative Laplacian over regions of arbitrary shape by using a collocation scheme, where the boundary conditions need not to be explicitly enforced on the border. Clearly, the precision of this approach should improve if the boundary conditions would be enforced exactly on the border of the membrane. One way of achieving this result is by mapping conformally the shape of the membrane into a square (or a rectangle), on whose border the LSF obey Dirichlet boundary conditions. I will discuss explicitly two examples of how this is done.
Figure 14. Wavefunction of the ground state of the configuration for \( \bar{n} = 4 \) using \( N = 500 \).

Table 4. Energies of the bound states for configurations with different number of crossings, using \( N = 600 \), corresponding to a spacing \( h = 1/300 \).

| \( \bar{n} \) | 1     | 2     | 3     | 4     |
|------------|-------|-------|-------|-------|
| \( h^2 E_1 \) | 2.59873 | 2.59871 | 2.59867 | 2.59862 |
| \( h^2 E_2 \) | –     | 2.59873 | 2.59869 | 2.59864 |
| \( h^2 E_3 \) | –     | 2.59873 | 2.59869 | 2.59864 |
| \( h^2 E_4 \) | –     | 2.59876 | 2.59872 | 2.59867 |
| \( h^2 E_5 \) | –     | –     | 2.59873 | 2.59868 |
| \( h^2 E_6 \) | –     | –     | 2.59873 | 2.59868 |
| \( h^2 E_7 \) | –     | –     | 2.59876 | 2.59871 |
| \( h^2 E_8 \) | –     | –     | 2.59876 | 2.59871 |
| \( h^2 E_9 \) | –     | –     | 2.59880 | 2.59874 |
| \( h^2 E_{10} \) | –     | –     | –     | 2.59874 |
| \( h^2 E_{11} \) | –     | –     | –     | 2.59875 |
| \( h^2 E_{12} \) | –     | –     | –     | 2.59877 |
| \( h^2 E_{13} \) | –     | –     | –     | 2.59877 |
| \( h^2 E_{14} \) | –     | –     | –     | 2.59881 |
| \( h^2 E_{15} \) | –     | –     | –     | 2.59881 |
| \( h^2 E_{16} \) | –     | –     | –     | 2.59887 |
| \( h^2 E_{\text{gap}} \) | 3.28997 | 3.29006 | 3.29019 | 3.29035 |

7.1. Circular membrane

As a first example I consider a circular homogeneous membrane, which is exactly solvable (see, for example [1]) and therefore it can be a useful tool to test the precision of the present method.

The function

\[
f(z) = e^{-\frac{2z}{\pi}} \text{sn}(zF\left(\sin^{-1}\left(\frac{z}{e^{-\frac{2z}{\pi}}}\right)\right) - 1) - 1)
\]

maps the unit square in the \( w \) complex plane into the unit circle in the complex \( z \) plane, as seen in figure 15. Here, \( \text{sn}(a|b) \) is the Jacobi elliptic function \( \text{sn} \) and \( F(a|b) \) is the incomplete elliptic function of first kind.
Figure 15. Unit square in the $z$ plane and the corresponding unit circle in the $w$ plane reached through transformation (18).

Under this mapping the original equation,

$$-\Delta \psi(w) = \lambda \psi(w)$$

(19)

with Dirichlet boundary conditions on the unit circle, is mapped to

$$-\Delta \chi(z) = \lambda \sigma(z) \chi(z)$$

(20)

with Dirichlet boundary conditions on the unit square. Here, $\sigma(z) \equiv \frac{d\psi}{dw}^2$ and equation (20) describes the vibrations of a non-uniform square membrane. Although in the previous sections I have restricted the application of the method to the case of uniform membranes of arbitrary shapes, the method can be applied also to inhomogeneous membranes straightforwardly. Let me briefly mention how this is done. As a first step, equation (20) may be written in the equivalent form

$$-\frac{1}{\sigma(z)} \Delta \chi(z) = \lambda \chi(z).$$

(21)

The operator $\hat{O} \equiv \frac{1}{\sigma(z)} \Delta$ is evaluated on a uniform grid in the $z$-plane using the little sinc functions (LSF). The action of the operator over a product of sinc functions can be calculated very easily, as explained in the previous sections. To make the discussion simpler, I restrict to the equivalent one-dimensional operator and make it act over a single LSF:

$$-\frac{1}{\sigma(x)} \frac{d^2}{dx^2} s_1(h, N, x) = -\sum_{jl} \frac{1}{\sigma(x_j)} c^{(2)}_{ij} s_j(h, N, N) s_i(h, N, x)$$

$$\approx -\sum_{j} \frac{1}{\sigma(x_j)} c^{(2)}_{ij} s_j(h, N, x).$$

(22)

The matrix representation of the operator over the grid may now be read explicitly from the expression above. The reader should note that the matrix will not be symmetric unless the
density is constant\(^3\). Figure 16 displays the density of the inhomogeneous square membrane which is isospectral to the homogeneous circular membrane.

Using this approach I have considered grids with \(N = 10, 20, \ldots, 80\) and I have calculated the first four even–even eigenvalues, which are shown in table 5. Taking into account the symmetry of problem I have used symmetrized LSF, which obey mixed boundary conditions (Dirichlet at one end and Neumann at the other end): in this way, for a given value of \(N\) a grid of \((N/2)^2\) points is used. As mentioned before the exact eigenvalues for this problem are known (the zeros of the Bessel functions): these are reported in the last row.

In figure 17, I have plotted the lowest eigenvalue of the circular membrane corresponding to different \(N\) and I have fitted these points using functions like \(c_0 + c_1/N^r\), with \(r = 3, 4, 5\) (the dashed, solid and dotted lines, respectively, in the plot). This plot shows that the leading (non-constant) behaviour of the numerical energy for \(N \gg 1\) is \(1/N^4\).

Taking into account this behaviour, I have considered the quantity

\[
\Xi_Q = \sum_{k=1}^{8} \left[ \alpha_1 - \sum_{n=2}^{Q} \frac{\alpha_n}{(10k)^{n+2}} \right]^2, \tag{23}
\]

where \(Q = 8\) and I have obtained the coefficients \(\alpha_n\) by minimizing \(\Xi_Q\) (note that this expression takes into account the leading \(1/N^4\) behaviour just discussed). The row marked as \(\text{LSQ}_8\) displays the quite precise results obtained following this procedure.

\(^3\) In general, the calculation of the eigenvalues and eigenvectors of non-symmetric matrices is computationally more demanding than for symmetric matrices of equal dimension.
I would like to discuss briefly a different issue. In [34], Gottlieb has used the Moebius transformation 
\[ f_g(z) = \frac{z - a}{1 - az} \] 
(24) to map the unit circle onto itself. This mapping transforms the homogeneous Helmholtz equation for a circular membrane into the inhomogeneous Helmholtz equation for a circular membrane with density 
\[ \rho(x, y) = |f'_g(z)|^2 = \rho_0 \frac{(1 - a)^2}{[(1 - ax)^2 + a^2 y^2]^2}. \] (25)

Gottlieb uses this result to conclude that membranes corresponding to different densities, i.e. different values of \( a \), are isospectral, thus providing a negative answer to the famous question ‘Can one hear the shape of a drum?’ posed by Kac [16]. I wish to move our discussion on computational grounds: for a given \( a \) the mapping of equation (24) deforms the grid inside the unit circle; as \( a \) is changed, the grid points move, as shown in figure 18. The case \( a = 0 \) is plotted in the right panel of figure 15. Clearly, if the density of the membrane is constant, or symmetric with respect to the centre, one expects that \( a = 0 \) provide the best grid. In figure 19, I have plotted the logarithm of the difference between the approximate
Figure 18. Grid obtained with the Moebius map corresponding to $a = 0.5$ (left) and $a = -0.8$ (right).

Figure 19. $\Delta \equiv \log_{10}(E_N - E_{\text{exact}})$ using three values of $a$ ($a = 0, 0.4$ and $0.8$ from bottom to top).

and exact energy for the ground state of a circular membrane, $\Delta \equiv \log_{10}(E_N - E_{\text{exact}})$, using three values of $a$ ($a = 0, 0.4$ and $0.8$). These numerical results confirm the prediction made: stated in different terms one can conclude that for a given problem one can improve the numerical accuracy of a calculation by selecting an optimal grid among those obtained through a conformal map of the region onto itself. The optimization of the parameter $a$ depending on the specific problem considered is in the same spirit of the variational approach used in [13, 14, 35] and could provide a useful computational tool to boost the precision of the results.

7.2. Circular waveguide

The second example of the application of conformal mapping to the solution of the Helmholtz equation is taken from the paper of Kuttler and Sigillito [2] (this problem was also studied earlier by Moler, in [101] of [2]).
In figure 20, two regions of the plane are displayed: the left plot corresponds to a square of side $\pi$ centred on the origin in the $z = x + iy$ plane; the right plot corresponds to a circular waveguide with circular ridges in the $w = u + iv$ plane. The function $w = \tan \frac{z}{2}$ maps the first region into the second one.

As I have shown for the case of the circular membrane, the homogeneous Helmholtz equation over the second region may be transformed into an inhomogeneous Helmholtz equation over the square:

$$-\Delta U(z) = \lambda \sigma(z) U(z).$$  \hfill (26)
Figure 22. Upper panel: Even–even wavefunctions (absolute value): ground state and 100th excited state of the circular waveguide. Lower panel: Even–even wavefunctions (absolute value): 200th and 300th excited states of the circular waveguide. A grid with $N = 80$ is used.

In the present case $\sigma(z) \equiv \left| \frac{dw}{dz} \right|^2 = (\cos x + \cosh y)^2$, and Dirichlet boundary conditions are assumed on the borders of the two regions.

In tables 1–3 of their paper, Kuttler and Sigillito report different estimates for the first 12 even–even eigenvalues, obtained using different approaches. In table 2 they also apply Richardson extrapolation to the eigenvalues obtained with finite difference. In the case of the ground state of this membrane they also mention the precise value obtained by Moler using the method of point matching

$$\lambda_1 = 7.5695769,$$

(27)

In table 6, I report the even–even eigenvalues of equation (26) obtained using collocation with different values of $N$. The results corresponding to the ground state are plotted in figure 21 and fitted using functions like $c_0 + c_1/N^r$, with $r = 3, 4, 5$ (the dashed, solid
and dotted lines, respectively, in the plot). This plot proves that the leading (non-constant) behaviour of the numerical energy for $N \gg 1$ is $1/N^4$, as for the circular membrane.

The results in the table have also been extrapolated using a least square approach

$$
\Xi_Q \equiv \sum_{k=1}^{8} \left[ \alpha_k - \sum_{n=2}^{Q} \frac{\alpha_n}{(10k)^{n/2}} \right]^2,
$$

where $Q = 7, 8$ and $\alpha_n$ are coefficients which are obtained by minimizing $\Xi_Q$. Note that this expression takes into account the leading $1/N^4$ behaviour just discussed. The rows marked as LSQ$_7,8$ display the results obtained following this procedure (the comparison between the results for $Q = 7$ and $Q = 8$ gives an indication over the precision reached): in particular, the energy of the ground state reproduces all the digits of the result obtained by Moler. It is also remarkable that the energies obtained with the conformal-collocation method decrease

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### Table 6. Even–even eigenvalues of the problem of equation (26) using collocation with the little sinc functions (LSF).

| $N$  | $E_1$          | $E_2$          | $E_3$          | $E_4$          |
|------|----------------|----------------|----------------|----------------|
| 10   | 7.575738906    | 29.3566905     | 44.9366765     | 68.9953251     |
| 20   | 7.569970385    | 29.1282337     | 44.8404770     | 67.8592485     |
| 30   | 7.569586991    | 29.1157595     | 44.8402696     | 67.8590710     |
| 40   | 7.569581767    | 29.1155901     | 44.8401955     | 67.8531950     |
| 50   | 7.569579528    | 29.1155178     | 44.8401638     | 67.8528928     |
| 60   | 7.569576902    | 29.1154828     | 44.8401485     | 67.8527471     |
| 70   | 7.569576902    | 29.1154334     | 44.8401269     | 67.8525423     |

| $N$  | $E_5$          | $E_6$          | $E_7$          | $E_8$          |
|------|----------------|----------------|----------------|----------------|
| 10   | 76.36327173    | 105.8649443    | 127.5818229    | 147.6128111    |
| 20   | 74.57343676    | 104.7105731    | 123.4501146    | 137.5136748    |
| 30   | 74.51254455    | 104.6448241    | 123.2916952    | 137.1508752    |
| 40   | 74.50340797    | 104.6345417    | 123.2690972    | 137.1033030    |
| 50   | 74.50101871    | 104.6318226    | 123.2633192    | 137.0914797    |
| 60   | 74.50017885    | 104.6308625    | 123.2613110    | 137.0874237    |
| 70   | 74.49982321    | 104.6304550    | 123.2604661    | 137.0857295    |
| 80   | 74.49965192    | 104.6302584    | 123.2600608    | 137.0849203    |
| LSQ  | 74.49941161    | 104.6299823    | 123.2594952    | 137.0837970    |

| $N$  | $E_9$          | $E_{10}$       | $E_{11}$       | $E_{12}$       |
|------|----------------|----------------|----------------|----------------|
| 10   | 152.6308731    | 175.0505071    | 202.7827432    | 229.6150278    |
| 20   | 147.1852075    | 177.5293898    | 193.4167694    | 213.4632048    |
| 30   | 147.1167888    | 177.232164     | 193.0075863    | 212.8440230    |
| 40   | 147.1064916    | 177.1901085    | 192.9541314    | 212.7718374    |
| 50   | 147.1038082    | 177.1796045    | 192.9401918    | 212.7546507    |
| 60   | 147.1028673    | 177.1752263    | 192.9364009    | 212.7488774    |
| 70   | 147.1024696    | 177.1736121    | 192.9345164    | 212.7464938    |
| 80   | 147.1022783    | 177.1728379    | 192.9336173    | 212.7453635    |
| LSQ  | 147.1020103    | 177.1715822    | 192.9323707    | 212.7438068    |
| LSQ  | 147.1020110    | 177.1715737    | 192.9323707    | 212.7438134    |
monotonically when the number of collocation points is increased (the only exception is represented by the $E_{10}$ for $N = 10$, probably due to the limited number of collocation points).

As a technical remark, one should note that the results corresponding to a given value of $N$ are obtained using a set of $N/2$ symmetric (even) functions for each direction, thus reducing the computation load by a factor of 4. The results displayed in this table should be compared with the analogous results of table 2 of [2], which were obtained using finite difference. Four different wavefunctions of the circular waveguide, corresponding to the ground, 100th, 200th and 300th even–even excited states are shown in figure 22.

8. Conclusions

In this paper, I have used a collocation method based on LSF to obtain the numerical solutions of the Helmholtz equation over two-dimensional regions of arbitrary shape. A large number of examples have been studied, illustrating the great potentialities of the present method. Among the principal virtues of this method I would like to mention its generality (it can be applied to membranes of arbitrary shapes, including inhomogeneous membranes, and to the Schrödinger equation—although I have not done this in the present paper), its simplicity (the matrix representation of the Helmholtz operator is obtained directly by collocation, and therefore it does not require the calculation of integrals) and the possibility of combining it with a conformal mapping, as done in the last section. In this last case, a rapid convergence to the exact eigenvalues is observed as the number of grid points is increased. In the case where the border is not treated exactly, it has also been observed that the method provides monotonous sequences of approximations to the exact eigenvalue either from above or from below. Readers interested to looking at more examples of application of this method may find useful to check the gallery of images which can be found at http://fejer.ucol.mx/paolo/drum.

Appendix. Mesh refinement

Although the collocation method described in this paper allows one to obtain precise solutions to the Helmholtz equation over domains of arbitrary shape, in general the Dirichlet boundary conditions are not enforced exactly over all the boundary. As discussed in section 7, the best approach consists of introducing a conformal map, which allows one to go from the original problem to an inhomogeneous Helmholtz problem over a square: in such a case, the Dirichlet boundary conditions are imposed exactly and rapid convergence to the exact solutions is observed. In general, however, finding such a conformal map can be a difficult task and therefore the first approach may be more appealing. I will discuss here a simple procedure to ‘refine’ the results obtained by direct collocation of the Helmholtz equation over the grid. The fundamental observation is that the LSF that we have used do vanish on the grid points on the border and external to the membrane, but they are nonzero in all the other points external to the membrane. Therefore, the cumulative effects of the LSF internal to the membrane can be seen also outside the membrane, although it will tend to disappear as the number of grid points is increased. This solution, to increase the number of grid points, may be the most obvious but is certainly not appealing computationally, since increasing the number of grid points strongly increases the computational cost (remember that the number of matrix elements grows as $N^4$). However, we can use much simpler procedure, which does not require any additional diagonalization. Call $N$ the parameter defining the size of the grid:
a point in this grid is described by the direct product of the LSF in the \( x \) and \( y \) directions. In the Dirac notation, we write

\[
\langle x, y | k, k' \rangle \approx s_k(h, x)s_{k'}(h, y),
\]

assuming for simplicity that the grid has the same spacing in both directions. Let us now concentrate on one of the LSF, say the one in the \( x \) direction: we take a finer grid, with a spacing \( h' = h/l \), where \( l \) is an integer. The new grid contains now \((IN - 1)\) points, including obviously the original grid points. However, it is clear that the original LSF can be decomposed in the new grid as

\[
s_k(h, x) = \sum_{j=-(IN/2)}^{lN/2-1} s_k(h, \bar{x}_j)s_j(h/l, x),
\]

where \( \bar{x}_j = 2L_j/(lN) \) are the new grid points. Note that this relation is exact.

The wavefunction of the \( n \)th state obtained from the diagonalization of the \((N-1) \times (N-1)\) Hamiltonian reads

\[
\psi_n(x, y) = \frac{1}{h} \sum_K v^{(n)}_K s_K(h, x)s_{K'}(h, y)
\]

\[
= \frac{1}{h} \sum_K v^{(n)}_K \sum_{j=-(IN/2)}^{IN/2-1} s_K(h, \bar{x}_j)s_j(h/l, x) \sum_{j'=-(IN/2)}^{IN/2-1} s_{K'}(h, \bar{y}_{j'})s_{j'}(h/l, y),
\]

where \( v^{(n)} \) is the \( n \)th eigenvector. Clearly, \( \psi_n(x, y) \) differs from 0 even in points of the refined grid which fall outside the membrane profile. We introduce a new matrix whose elements are given by

\[
\eta_{jj'} = \begin{cases} 0 & \text{if } (\bar{x}_j, \bar{y}_{j'}) \notin B \\ 1 & \text{if } (\bar{x}_j, \bar{y}_{j'}) \in B \end{cases}
\]

and rewrite the wavefunction ‘purged’ on the refined grid as

\[
\tilde{\psi}_n(x, y) = \frac{\mathcal{N}}{h} \sum_{j=-(IN/2)}^{IN/2-1} \sum_{j'=-(IN/2)}^{IN/2-1} \tilde{V}_{jj'} s_j(h/l, x)s_{j'}(h/l, y),
\]

where

\[
\tilde{V}_{jj'} \equiv \eta_{jj'} \sum_K v^{(n)}_K s_K(h, \bar{x}_j)s_{K'}(h, \bar{y}_{j'})
\]

and \( \mathcal{N} \) is a normalization constant that ensures that

\[
\int_B \psi^2_n(x, y) \, dx \, dy = 1.
\]

It is easy to show that

\[
\mathcal{N} = \frac{l}{\sqrt{\sum_{jj'} \tilde{V}_{jj'}^2}}.
\]

To simplify the notation, I define

\[
\mathcal{V}_{jj'} = \frac{\mathcal{N}}{l} \tilde{V}_{jj'}
\]

and thus write

\[
\tilde{\psi}_n(x, y) = \frac{1}{h} \sum_{j=-(IN/2)}^{IN/2-1} \sum_{j'=-(IN/2)}^{IN/2-1} \mathcal{V}_{jj'} s_j(h/l, x)s_{j'}(h/l, y).
\]
On the other hand, we may also calculate the expectation value of the Hamiltonian in this state

\[
\langle \hat{H} \rangle_n = -\int_B \bar{\psi}_n(x, y) \Delta \psi_n(x, y) \, dx \, dy \\
= -\sum_{jj'rr's'} \frac{V_{jj'}V_{rr'}}{\hbar^2} \left[ \bar{c}^{(2)}_{jj'} \delta_{rr'} + c^{(2)}_{rj} \delta_{rj} \right] \int_B s_j(h/l, x)s_j(h/l, y)s_j(h/l, x)s_j(h/l, y) \\
= -\sum_{jj'rr'} \bar{c}^{(2)}_{jj'} [V_{jj'}V_{rr'} + V_{rj}V_{rj}] \\
= -\sum_{jj'r} \bar{c}^{(2)}_{rj} [V_{jj'}V_{rj} + V_{rj}V_{rj}],
\]

(A.8)

where \( \bar{c}^{(2)} \) is the matrix for the second derivative on the refined grid. An example of application of this procedure is shown in figure 13.

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