Superstrings in \( \text{AdS}_2 \times S^2 \times T^6 \)

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Abstract

We consider the type IIB Green–Schwarz superstring theory on \( \text{AdS}_2 \times S^2 \times T^6 \) supported by homogeneous Ramond–Ramond 5-form flux and its type IIA T-duals. One motivation is to understand the solution of this theory based on integrability. This background is a limit of a 1/4 supersymmetric supergravity solution describing four intersecting D3-branes and represents a consistent embedding of \( \text{AdS}_2 \times S^2 \) into critical superstring theory. Its \( \text{AdS}_2 \times S^2 \) part with corresponding fermions can be described by a classically integrable \( PSU(1, 1|2)/SO(1, 1) \times U(1) \) supercoset sigma model. We point out that since the RR 5-form field has non-zero components along the 6-torus directions one cannot, in general, factorize the 10D superstring theory into the supercoset part plus six bosons and six additional massless fermions. Still, we demonstrate that the full superstring model (i) is classically integrable, at least to quadratic order in fermions, and (ii) admits a consistent classical truncation to the supercoset part. Following the analogy with other integrable backgrounds and starting with the finite-gap equations of the \( PSU(1, 1|2)/SO(1, 1) \times U(1) \) supercoset, we propose a set of asymptotic Bethe ansatz equations for a subset of the quantum string states.

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1. Introduction

Recent remarkable progress in the exact solution of the maximally supersymmetric case of AdS/CFT duality (see, e.g., [1]) suggests applying similar integrability-based methods to other AdS/CFT systems. We are going to concentrate on the \( \text{AdS}_2 \times S^2 \) background, whose string sigma model, we will argue, is completely integrable. The role of \( \text{AdS}_2 \times S^2 \) as the near-horizon geometry of extremal 4D Reissner–Nordström black holes emphasizes

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the importance of understanding the corresponding AdS$_2$/CFT$_1$ duality [2]. There is a long and still unresolved controversy about the meaning of the corresponding 'CFT$_1$' (a large $N$ superconformal quantum-mechanical system or a chiral 'half' of a 2D CFT) [2–6]. One may hope to shed light on this issue by starting from the AdS$_2$ side, solving the corresponding string theory for any value of the radius or effective tension and re-interpreting the solution in terms of some dual CFT.

The first step is to embed the AdS$_2 \times S^2$ background into critical 10D superstring theory. Requiring that the bosonic part of the string sigma model should be exactly AdS$_2 \times S^2 \times T^6$ excludes embeddings with NS–NS flux$^7$. The relevant RR-flux embedding into type IIB string theory is based on the $1/4$ supersymmetric background describing four intersecting D3-branes [11]. Its ‘near-horizon’ limit is AdS$_2 \times S^2 \times T^6$ supported, like in the AdS$_5 \times S^5$ case, by a homogeneous self-dual 5-form flux. One may also consider a T-dual type IIA background, e.g. the one based on a superposition of three D4-branes and one D0-brane; in that case, the AdS$_2 \times S^2 \times T^6$ space is supported by a combination of 4-form and 2-form fluxes but should lead to an equivalent string theory$^8$. The spectrum of the corresponding BPS supergravity fluctuation modes was discussed in [14, 17–19].

A natural framework for a superstring theory on a RR background such as AdS$_2 \times S^2 \times T^6$ is the Green–Schwarz (GS) formalism. The GS action is in principle defined on any supergravity background [20], although constructing it explicitly, in general, is a technically complicated problem. One needs to know the exact form of all background superfields, which can be reconstructed from the bosonic fields by solving the supergravity constraints order by order in fermions. The expressions quickly become complicated making this direct approach impractical in the absence of extra symmetries. In the AdS$_5 \times S^5$ case, these difficulties were effectively bypassed [21] by observing that the GS action is equivalent to a supercoset sigma model on $PSU(2, 2|4)/SO(1, 4) \times SO(5)$.

Following the analogy with the construction in [21] and taking into account that the superisometries of the AdS$_2 \times S^2$ background form the $PSU(1, 1|2)$ supergroup, reference [22] found a formal 4D GS superstring action for the supercoset $PSU(1, 1|2)/SO(1, 1) \times U(1)$. A corresponding worldsheet $N = 2$ superconformal analog based on the ‘standard’ supercoset form of the action (with a quadratic kinetic term for the fermionic current which is absent from the GS action) was constructed in [23] where the $\mathbb{Z}_4$-structure of the supercoset and a local form of the Wess–Zumino term were pointed out. In addition to the supercoset part, the action of [23] included an $N = 2$ superconformal theory on CY$_3$ (or $T^6$) with six worldsheet fermions, which is completely decoupled from the supercoset sigma model. As we will comment later, the relation of this ‘hybrid’ model to the AdS$_2 \times S^2 \times T^6$ GS superstring remains an open issue.

The $PSU(1, 1|2)/SO(1, 1) \times U(1)$ supercoset sigma model was interpreted in [22] as a 4D kappa-symmetry invariant GS superstring action in the AdS$_2 \times S^2$ background supported by a RR 2-form flux. This supercoset action can be viewed as a direct (classical-level) truncation of the AdS$_5 \times S^5$ superstring theory with only four bosons and eight fermions kept non-zero and has several remarkable features. The $\mathbb{Z}_4$-structure [23] of the superalgebra $psu(1, 1|2)$ implies [24] that this theory is also classically integrable [25] and, in addition,

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$^7$ One may formally embed the 4D extremal RN black hole into 10D string theory using a 6D bosonic NS–NS background which reduces in the near-horizon limit to (an orbitfold of) an $SL(2, R) \times SU(2)$ WZW model [7] but in this case the AdS$_2 \times S^2$ coordinates will be coupled to two extra compact bosonic coordinates. The same applies to various $T$-dual backgrounds [8] and to similar heterotic string embeddings of AdS$_2 \times S^2$ [9, 10].

$^8$ There are also other $T$-dual cases, e.g. a D4D4D2D2 configuration with M5M5M2M2 as its 11D supergravity lift [8, 11] (see also [12–14]). One may also consider 10D embeddings of AdS$_2 \times S^2$ with $T^8$ replaced by a Calabi–Yau space [15, 16].
self-dual under the fermionic T-duality [26, 27]. In fact, its classical integrable structure is essentially equivalent to that of the \( N = 2 \) supersymmetric sine-Gordon theory [28] as the latter is its Pohlmeyer reduction [29].

As for the quantum level, the rigid symmetry structure of this supercoset GS sigma model implies that it should be UV finite, like its hybrid cousin in [23], and thus define a 2D conformal theory. Its one-loop beta-function was indeed shown to vanish [30] due to the vanishing Killing form of the \( \text{psu}(1,1|2) \) superalgebra. While for the 10D superstring one has \( n_b = n_f = 8 \) implying \( c_{\text{tot}} = 0 \) for the 4D GS string, one gets \( c_{\text{tot}} = (2+2) - 26 + \frac{1}{2} \times 4 \times 2 = -18 \). One may try to cancel the central charge deficit by adding extra decoupled bosons and fermions but while this may be straightforward in the NSR framework it is not clear \textit{a priori} how this can be consistently implemented in the GS case. An alternative is to use the ’hybrid’ model of [23] but as mentioned its equivalence to the critical 10D superstring theory remains an open question.

In this paper, we propose to start directly with a critical 10D superstring theory defined in the \( \text{AdS}_2 \times S^2 \times T^6 \) background supported by RR flux and explore its relation to the above supercoset theory. Since, e.g., in the type IIB embedding [11], the \( F_5 \) flux has non-zero components along the \( T^6 \) directions\(^9\), the toroidal string coordinates do not \textit{a priori} decouple from the GS fermions. This non-decoupling of the ‘flat directions’ sets the \( \text{AdS}_2 \times S^2 \times T^6 \) case apart from the previously studied coset-type critical-string backgrounds, where additional degrees of freedom could be either completely eliminated (\( \text{AdS}_4 \times \mathbb{C}P^3 \)) or decoupled (\( \text{AdS}_3 \times S^3 \times T^4 \)) from the coset by an appropriate choice of kappa-symmetry gauge. Contrary to what one might expect, in the present case it will not be possible to represent the world-sheet sigma model as a direct sum of the \( \text{PSU}(1,1|2)/SO(1,1) \times U(1) \) supercoset and additional free bosonic and fermionic modes. However, as we will show later, the GS action admits a reduction to the supercoset theory in a weaker sense as a classically consistent truncation.

The mixing between the coset and flat directions of \( T^6 \) may cast doubts on the potential integrability of the model, beyond the coset truncation. We shall find that quite remarkably the full superstring theory in the supersymmetric \( \text{AdS}_2 \times S^2 \times T^6 \) backgrounds is also classically integrable. Following [32], we will construct the Lax connection (to quadratic order in fermions) from the components of the conserved currents of the full GS action. We will show that the flatness condition for this Lax connection is equivalent to the full set of equations of motion of the GS superstring, thus proving classical integrability of the string sigma model. If integrability is not spoiled at the quantum level, the theory may eventually be solvable by Bethe ansatz techniques.

As a first step toward the exact solution, we derive the classical counterpart of the Bethe equations that describe finite-gap classical solutions of the supercoset sigma model. Under the assumption that the supercoset truncation goes through also at the level of massive quantum states with large quantum numbers, we propose a set of asymptotic Bethe equations for part of the quantum spectrum of the theory. We also discuss some preliminary consistency checks of integrability based on a one-loop semiclassical expansion.

This paper is organized as follows. In section 2, we will start with the action of a \( D = 10 \) GS superstring in a general symmetric-space background supported by RR fluxes written to quadratic order in fermions and find the conditions for its classical integrability

\(^9\) The \( T^6 \) components of the corresponding energy–momentum tensor of \( F_5 \) are of course equal to zero.
to this order by constructing a candidate Lax connection in terms of the Noether isometry currents.

In section 3, we will consider explicit examples of type IIA and type IIB backgrounds with AdS$_2 \times S^2 \times T^6$ supported by RR fluxes and verify that the conditions for classical integrability of the corresponding GS superstring action are satisfied, at least to quadratic order in fermions.

In section 4, we show that the type II GS string action for the AdS$_2 \times S^2 \times T^6$ backgrounds preserving eight supersymmetries can be consistently truncated, at the classical level, to the $PSU(1,1|2)/SO(1,1) \times U(1)$ supercoset GS sigma model with AdS$_2 \times S^2$ as the bosonic part and eight fermionic modes. We first show this to quadratic order in fermions and then extend the argument to all orders.

In section 5, we will consider two inequivalent BMN limits of the superstring action: when the center of mass (c.o.m.) of the string is moving along a big circle of $S^2$ or when it moves both in $S^2$ and in $S^1 \subset T^6$. The first case corresponds to the BPS vacuum state of the theory and preserves 1/2 of the original supersymmetry. The unbroken supersymmetry can be recast in the 2D form and leads to the Bose–Fermi degeneracy of the BMN modes. The non-coset degrees of freedom remain massless in this limit. In the second case, the gauge-fixed sigma model does not have effective 2D supersymmetry, and the degeneracy is lifted. Moreover, some of the ‘non-supercoset’ fermions acquire mass in this case.

In section 6, we will first review the classical Bethe equations describing finite gap solutions of the $PSU(1,1|2)/SO(1,1) \times U(1)$ supercoset model [33] and then propose a set of quantum asymptotic Bethe equations that potentially describe the spectrum of the string in the light-cone gauge associated with the supersymmetric BMN geodesic. These equations, however, only describe a subset of the massive states and do not capture the massless, non-coset degrees of freedom.

In section 7, we discuss various semiclassical string solutions in AdS$_2 \times S^2 \times T^6$ and clarify the validity of the assumption of decoupling of the AdS$_2 \times S^2$ sector.

There are several appendices describing our notation, the structure of the (enlarged) $\mathfrak{psu}(1,1|2)$ superalgebra and properties of Killing vectors on symmetric spaces. In appendix D, we also provide details of the derivation of the classical string equations of motion from the flatness of the Lax connection constructed in section 2.

2. GS superstrings in RR backgrounds and their classical integrability

A particular feature of GS superstrings propagating in supersymmetric backgrounds whose geometry is a direct product of an AdS space with compact manifolds is that (at least part of) their dynamics can often be described by certain supercoset sigma models whose isometry superalgebras have a $\mathbb{Z}_4$-grading. For example, the $PSU(2,2|4)/SO(1,4) \times SO(5)$ sigma model defines the maximally supersymmetric type IIB AdS$_3 \times S^5$ superstring theory [21]. Also an $OSp(6|4)/SO(1,3) \times U(3)$ sigma model with 24 supersymmetries [34, 35] represents the type IIA AdS$_4 \times CP^3$ superstring in those sectors where a partial kappa-symmetry gauge fixing is allowed to reduce the complete GS action to the $OSp(6|4)/SO(1,3) \times U(3)$ sigma model one containing 24 worldsheet fermions [34, 36–38].

In the case of an AdS$_3 \times S^3 \times T^4$ background supported by RR 3-form flux which preserves 16 out of the 32 possible supersymmetries$^{10}$, one can reduce the GS action to a sigma model

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$^{10}$This background can be regarded as a particular limit of an AdS$_3 \times S^3 \times S^3 \times S^1$ background in which one $S^3$ is ‘re-decompactified’ into $T^1$. The AdS$_3 \times S^3 \times S^3 \times S^1$ backgrounds are associated with the supercosets $D(1,2;\omega) \times D(1,2;\omega)/SO(1,2) \times SO(3) \times SO(3)$, where $D(1,2;\omega)$ is an exceptional supergroup (see [39] for more details and references).
We will demonstrate that (at least up to the second order in all the 32 fermions) the type II superstring action without requiring that the string sigma model has a coset structure.

The 2-form contributes to the WZ term of the string action.

It is complicated and requires a separate consideration, in particular because in these cases the purely bosonic NS–NS 2–form is the string action.

AdS2 is the full superstring action without requiring that the string sigma model has a coset structure. We then show that the type II superstrings in the superisometries in order to be able to construct from them a zero-curvature Lax connection, the relations that should be satisfied by components of the Noether currents of the background.

The analysis of the integrability of GS superstrings in backgrounds with NS–NS fluxes turns out to be a technically complicated problem. So far it has been solved only for two cases which can be obtained by double-dimensional reduction of the supermembrane action in corresponding AdS5 backgrounds.

The GS sigma model on PSU(1, 1|2) ∗ PSU(1, 1|2)/SU(1, 1) ∗ SU(2) (which has AdS3 × S3 as its bosonic subspace) plus the decoupled free bosonic sector on T6 [39]. To this end, one should completely gauge fix the kappa-symmetry by eliminating 16 of the 32 fermionic modes of the 10D GS superstring in an appropriate way. For some singular classical string configurations (e.g. when the string does not wind or move in T6) such kappa-symmetry gauge fixing is inadmissible. In such special cases, there are physical fermionic degrees of freedom which are not part of the supercoset sigma model and to take them into account one should start with the complete 10D GS superstring action.

The situation turns out to be more complicated in the case of AdS2 × S2 × T6 backgrounds. The supercoset sigma model on PSU(1, 1|2)/SO(1, 1) ∗ U(1) = Super(AdS2 × S2) [22] indeed possesses a Z4-grading [23] and is thus classically integrable [24], much like the GS sigma models on AdS5 × S5, AdS4 × CP3 and AdS3 × S3 × T4. The crucial difference is in the number of preserved supersymmetries. The psu(1, 1|2) superalgebra has only 8 supercharges, compared to 32 for AdS5 × S5, 24 for AdS4 × CP3 and 16 for AdS3 × S3 × T4. As a result, the 16-parametric kappa-symmetry of the GS action is not sufficient to gauge away the additional 32 – 8 = 24 worldsheet fermions associated with broken 10D supersymmetries. At least eight of the broken-symmetry fermions will remain in the physical spectrum of the string and will link the decoupled, at first sight, T6 sector to the coset.

Since extra, non-coset degrees of freedom cannot be gauged away and do not decouple, the integrability of the supercoset is not sufficient to prove the integrability of the full theory which includes the fermions associated with broken supersymmetries. To analyze integrability, we follow an alternative approach, proposed for the type IIA superstring action on AdS4 × CP3 [32]. In this approach, the Lax connection is constructed directly from the Noether currents of the background.

We will demonstrate that (at least up to the second order in all the 32 fermions) the type II D = 10 GS superstring propagating in AdS2 × S2 × T6 background (supported by type IIA or type IIB RR fluxes) is classically integrable. Thus, the integrability still applies despite the fact that the integrable PSU(1, 1|2)/SO(1, 1) ∗ U(1) sigma model cannot give a complete description of the GS superstring in the AdS2 × S2 × T6 case.

We shall also show that in the non-supersymmetric AdS2 × S2 × T6 backgrounds which are obtained from the supersymmetric ones by changing signs of some components of the RR fluxes, the same prescription for the construction of the Lax connection fails already at the quadratic order in fermions.

In this section, we will consider a D = 10 GS superstring in a generic superbackground with non-zero RR fluxes11 whose bosonic subspace is a symmetric space. We will derive relations that should be satisfied by components of the Noether currents of the background superisometries in order to be able to construct from them a zero-curvature Lax connection, up to the second order in fermions. We then show that the type II superstrings in the AdS2 × S2 × T6 backgrounds satisfy these requirements and hence are classically integrable, at least to quadratic order in fermions.

To construct a complete Lax connection to all orders in fermions one should know the explicit form of the full GS superstring action. A derivation of such an explicit form of the GS action in a D = 10 superbackground which is not maximally supersymmetric turns out to be a technically complicated problem. So far it has been solved only for two cases which can be obtained by double-dimensional reduction of the supermembrane action in corresponding D = 11 maximally supersymmetric backgrounds: a locally flat one and AdS4 × S7. They

11The analysis of the integrability of GS superstrings in backgrounds with NS–NS fluxes turns out to be more complicated and requires a separate consideration, in particular because in these cases the purely bosonic NS–NS 2–form contributes to the WZ term of the string action.
correspond to, respectively, the type IIA superstring in a 7-brane background with a magnetic RR flux \[40\] and to the type IIA superstring in the AdS4 \(\times\) CP3 superbackground \[36\] (the latter is the extension to superspace of the Hopf fibration realization of the \(D = 11\) supergravity solution \[41–43\]).

2.1. GS superstrings in RR backgrounds, equations of motion and conserved currents

The action for the GS superstring in a bosonic supergravity background (with zero NS–NS flux and constant dilaton \(\phi\)) has the following form up to quadratic order in fermions \[44, 45\]:

\[
S = - T \int \left( \frac{1}{2} * e^A e_A + i * e^A \Theta \Gamma_A D \Theta - i e^A \Theta \Gamma_A \hat{\Gamma} D \Theta \right),
\]

(2.1)

where \(e^A(X) (A = 0, 1, \ldots, 9)\) are worldsheet pull-backs of the background vielbein one-forms and

\[
D \Theta = (\nabla - \frac{1}{8} e^A F/\Theta_1 \Gamma_A) \Theta
\]

(2.2)

is the fermionic vielbein to the lowest order in fermions \(\Theta\). Here, \(\nabla = d + \omega\) is the covariant derivative containing the spin connection of the background space-time

\[
\hat{\Gamma} = \begin{cases} 
\Gamma_{11} & (\text{IIA}) \\
\sigma^3 & (\text{IIB}) 
\end{cases}
\]

(2.3)

and the coupling to the RR fields is given in terms of the matrix

\[
F = e^\phi \begin{cases} 
- \frac{1}{2} \Gamma^{AB} \Gamma_{11} F_{AB} + \frac{1}{4!} \Gamma^{ABCD} F_{ABCD} & (\text{IIA}) \\
- 3! \sigma^1 \Gamma^{ABC} F_{ABC} + \frac{i}{2} \sigma^2 \Gamma^{ABCDE} F_{ABCDE} & (\text{IIB}) 
\end{cases}
\]

(2.4)

in the type IIA and type IIB case, respectively. We describe the two Majorana–Weyl spinors in the IIA case as one 32-component Majorana spinor \(\Theta\) and in the IIB case as two 32-component Majorana spinors projected onto one chirality \(\Theta_i (i = 1, 2)\) by \(\frac{1}{2} (1 + \Gamma_{11})\). The Pauli matrices \(\sigma^1, \sigma^2\) and \(\sigma^3\) act on the IIB SO(2)-indices \(i, j = 1, 2\) which will be suppressed.

The bosonic equations of motion to the second order in \(\Theta\) are then

\[
\nabla \left( * (e^\Lambda + i \Theta \Gamma^\Lambda D \Theta) + i \Theta \Gamma^\Lambda \hat{\Gamma} D \Theta - \frac{i}{8} e^B \Theta \Gamma^A F \Gamma_B \Theta - \frac{i}{8} e^B \Theta \Gamma^A \hat{\Gamma} F \Gamma_B \Theta \right)
\]

\[
- \frac{i}{4} * e^B \Theta \Gamma^A \Gamma_{CD} R_{CDEA} + \frac{i}{4} e^B \Theta \Gamma^A \Gamma_{CD} \hat{\Gamma} R_{CDEA} = 0,
\]

(2.5)

where \(R_{CDEA}(X)\) is the curvature of the \(D = 10\) space, and the fermionic equations (linear in \(\Theta\)) are

\[
(* e^\Lambda \Gamma_A - e^\Lambda \Gamma_{11} \Gamma_A) D \Theta = 0.
\]

(2.6)

If the background has bosonic isometries, generated by Killing vectors \(K_A(X)\), the worldsheet model has the corresponding conserved Noether current one-form of the following generic form (see \[32\] for more details):

\[
J_B = J^A K_A + J^{AB} \nabla_A K_B = e^A K_A + \text{fermions},
\]

(2.7)

12 In what follows, we will mainly use the language of 2D differential forms with wedge products understood.

13 Our definition of \(\Gamma_{11}\) and type IIA RR-fluxes differs by a sign from those of \[45\].
where the second term comes from compensating Lorentz transformations of the fermionic fields $\Theta$. The $J^A$ and $J^{AB}$ terms in the current have the following form:

\[
J^A = e^A + i\Theta \Gamma^A D\Theta - \frac{1}{8} e^B \Theta \Gamma^A f \Gamma_B \Theta + i \Theta \Gamma^A \hat{\Gamma} \ast D\Theta - \frac{1}{8} e^B \Theta \Gamma^A \hat{\Gamma} f \Gamma_B \Theta, \tag{2.8}
\]

\[
J^{AB} = \frac{1}{4} (\Gamma^{AB} \Theta)^\ast \ast i_\alpha \mathcal{L} = -\frac{i}{4} e^C \Theta \Gamma^{AB} c \Theta + \frac{i}{4} e^C \Theta \Gamma^{AB} \hat{c} \hat{\Theta}, \tag{2.9}
\]

where $\mathcal{L}$ is the superstring Lagrangian in (2.1).

If the background preserves some supersymmetries generated by Killing spinors $\Xi(X)$, there is also a conserved supersymmetry current on the worldsheet which to linear order in $\Theta$ has the following form:

\[
J_{\text{susy}} = \frac{i}{2} R (e^A \Theta \Gamma_A \Xi - \ast e^A \Theta \Gamma_A \hat{\Xi}), \tag{2.10}
\]

where the dimension-of-length constant $R$ (which in our case will be the AdS radius) has been introduced to make the current dimensionless, and $\Xi$ satisfies the Killing spinor equation

\[
\nabla \Xi - \frac{1}{3} e^A \hat{\Gamma}_A \Xi = 0. \tag{2.11}
\]

As usual, the currents (2.7) and (2.10) are conserved

\[
d \ast J_B = 0, \quad d \ast J_{\text{susy}} = 0, \tag{2.12}
\]

provided that the equations of motion (2.5) and (2.6) are satisfied (and vice versa). The conservation of (2.7) and orthogonality of $\mathcal{K}_A$ and $\nabla_A K_B = [K_A, K_B]$ imply that the following relations must hold separately (see appendix C for basic relations satisfied by the Killing vectors of a symmetric space):

\[
(\nabla \ast J^{AB} - \ast J^{[A} e^{B]})) K_A K_B = 0, \tag{2.13}
\]

\[
\nabla \ast J^A - 2 R^{BCD} A \ast J^{CD} e^B = 0. \tag{2.14}
\]

### 2.2. Lax connection built from conserved currents

In many examples of interest, such as $\text{AdS}_4 \times \mathbb{C} \mathbb{P}^3$, $\text{AdS}_2 \times S^2 \times T^6$ or $\text{AdS}_3 \times S^3 \times T^4$, the complete target superspace describing the superstring background is not a supercoset manifold, so the $\mathbb{Z}_4$-graded supercoset prescription of [24] for the construction of the Lax connection does not directly apply. In [32] an alternative prescription was proposed for these cases and applied to $\text{AdS}_4 \times \mathbb{C} \mathbb{P}^3$. It uses the conserved currents as building blocks. The Lax connection has two parts

\[
L = L_B + L_F. \tag{2.15}
\]

The part $L_B$ of the Lax connection that corresponds to the bosonic isometries has the following form (similar to (2.7)):

\[
L_B = L^A K_A + L^{AB} \nabla_A K_B, \tag{2.16}
\]

where

\[
L^A = \alpha_1 e^A + \alpha_2 \ast J^A, \tag{2.17}
\]

\[
L^{AB} = \alpha_2 J^{AB} + \alpha_3 (1 + \alpha_1) \ast J^{AB}. \tag{2.18}
\]

\[14\] When reduced to the supercoset sigma model, this alternative Lax connection is related to the conventional one by a superisometry gauge transformation [32].
The part corresponding to the fermionic isometries is
\[ L_\mathcal{F} = -\alpha_3 \beta_1 J_{\text{asy}} + \alpha_2 \beta_2 * J_{\text{asy}}. \] (2.19)

The numerical parameters \( \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \) are expressed in terms of a single spectral parameter by requiring that the Lax connection (2.15) has zero curvature
\[ dL - LL = 0. \] (2.20)

In fact, the values of \( \alpha_1 \) and \( \alpha_2 \) are determined in terms of the spectral parameter already by the requirement of integrability of the purely bosonic part of the superstring sigma model. One can check that if in (2.16) we put \( \Theta \) to zero, the Lax connection reduces to its conventional form for an integrable sigma model on a symmetric space [46]:
\[ L_B|_{\Theta=0} = (\alpha_1 e^A + \alpha_2 e^B) K_A. \]

The curvature of this connection vanishes if
\[ \alpha_2^2 = 2\alpha_1 + \alpha_2 \Rightarrow \alpha_1 = \frac{2x^2}{1 - x^2}, \quad \alpha_2 = \pm \frac{2x}{1 - x^2}, \] (2.21)
where \( x \) is the spectral parameter. When the fermionic fields are non-zero, using the equations of motion, we find that up to the quadratic order in fermions the \( L_B \)-terms in the curvature (2.20) proportional to the Killing vectors \( K_A \) are
\[ (dL_B - L_BL_B)^A = 2\alpha_2 (\alpha_2^2 - 2\alpha_1 - \alpha_2^2) R_{BCD} e^A e^B. \] (2.22)

We observe that the right-hand side of (2.22) vanishes when the parameters satisfy equation (2.21). The \( L_B \)-terms of the Lax curvature proportional to the derivative of the Killing vectors \( \nabla_A K_B \) are
\[ (dL_B - L_BL_B)^{AB} = \alpha_2^2 \nabla_J^{[AB} + (J - e^A e^B]) |_{J}. \] (2.23)

If the right-hand side of (2.23) vanishes, then \( L_B \) has zero curvature and may itself play the role of the Lax connection (i.e. in (2.15) the term \( L_\mathcal{F} \) can be set to zero). An example, when this happens, is an integrable non-supersymmetric GS superstring on AdS_4 considered in [32]. In this case, there is no supersymmetry Noether current (2.10).

However, in general, the right-hand side of (2.23) is non-zero. Then we need to show that these terms in the curvature can be canceled by adding other terms to the Lax connection. When the background possesses some supersymmetry, these additional terms are built from the supersymmetry current (2.10) and have the form given in (2.20). This is what happened in AdS_4 \( \times \text{CP}^3 \) [32]. It will be possible to cancel the curvature of the Lax connection by this mechanism for the AdS_2 \( \times S^2 \times T^b \) backgrounds as well.

To see if the curvature term (2.23) can be canceled by contributions from (2.19), let us compute the curvature of the total Lax connection (2.15) for the parameters \( \alpha_1 \) and \( \alpha_2 \) related by (2.21):
\[ dL - LL = \alpha_2^2 (\nabla_J^{[AB} + (J - e^A e^B]) \nabla_A K_B + \alpha_2^2 (\beta_2^2 - \beta_1^2) J_{\text{asy}}^2 - \alpha_2 \beta_1 dJ_{\text{asy}} \]
\[ + \alpha_2 (\alpha_1 \beta_1 + \alpha_2 \beta_2) (J_B J_{\text{asy}} + J_{\text{asy}} J_B) \]
\[ - \alpha_2 (\alpha_1 \beta_2 + \alpha_2 \beta_1) (J_B * J_{\text{asy}} + *J_{\text{asy}} J_B). \] (2.24)

Now, if
\[ \beta_1 = \mp \sqrt{\frac{\alpha_1}{2}} = \pm \frac{i}{\sqrt{2^2 - 1}}, \quad \beta_2 = \pm \frac{\alpha_2}{\sqrt{2\alpha_1}} = \mp \frac{i}{\sqrt{2^2 - 1}}, \] (2.25)
the last term in equation (2.24) vanishes and the other terms cancel each other provided that
\[ dJ_{\text{asy}} = -2(J_B J_{\text{asy}} + *J_{\text{asy}} J_B). \] (2.26)
and

\[ \nabla J^{AB} + (J^A - e^A) e^B \nabla A K_B = - J_{\text{susy}}^2. \] (2.27)

These equations can be viewed as a modified version of the familiar Maurer–Cartan equation \( dJ = -2J^2 \), with \( J = J_B + \frac{1}{\sqrt{2}} J_{\text{susy}} \), and with some terms proportional to \( K_A \) removed.

Verifying that equations (2.26) and (2.27) hold is therefore enough to demonstrate the integrability at this order. We will see below that just as in the AdS \( 4 \times \mathbb{CP}^3 \) case, these equations hold for strings in AdS\( 2 \times S^2 \times T^6 \). Let us first compute the left-hand sides of these equations in a general RR background. Using the form of the supersymmetry current (2.10) and the equations of motion, we get

\[ dJ_{\text{susy}} = i 8 R (e^A e^B / \Theta^1 / \Gamma^1 A \hat{F} / \Gamma^1 B / \Xi^1) \] (2.28)

Using the form (2.9) of \( J^{AB} \), we find that

\[ \nabla J^{AB} = - i 2 e^C / \Theta^1 / \Gamma^1 B C \nabla / \Theta^1 + i 2 * e^C / \Theta^1 / \Gamma^1 B C \hat{F} / \Theta^1 \]

\[ = i e^B / \Theta^1 / \Gamma^1 A \nabla / \Theta^1 - i * e^B / \Theta^1 / \Gamma^1 A \hat{F} / \Theta^1 - i 16 e^D / \Theta^1 / \Gamma^1 B C \hat{F} / \Theta^1 / \Gamma^1 D \Theta^1 \]

\[ + i 16 * e^D / \Theta^1 / \Gamma^1 B C \hat{F} / \Theta^1 / \Gamma^1 D \Theta^1, \] (2.29)

where we have made use of the equations of motion. We can further rewrite this as

\[ \nabla J^{AB} + e^A (J^B - e^B) = - i 16 e^D / \Theta^1 / \Gamma^1 C \hat{F} / \Theta^1 / \Gamma^1 D \Theta^1 + i 16 * e^D / \Theta^1 / \Gamma^1 C \hat{F} / \Theta^1 / \Gamma^1 D \Theta^1. \] (2.30)

As we have seen, for the Lax-connection curvature (2.24) to vanish, the right-hand side of (2.30) should be the square of \( J_{\text{susy}} \) and the right-hand side of (2.28) should be the commutator of \( J_B \) with \( J_{\text{susy}} \). In the next section, we will show that this holds for various AdS\( 2 \times S^2 \times T^6 \) solutions with RR fluxes which preserve eight supersymmetries.

3. Superstrings in AdS\( 2 \times S^2 \times T^6 \) backgrounds

As we have discussed in the introduction, there are several solutions of type IIA and type IIB supergravity with the geometry of AdS\( 2 \times S^2 \times T^6 \) supported by RR fluxes which are related by T-duality. Type IIA solutions are also related by dimensional reduction to certain AdS\( 4 \times S^4 \times T^m \) solutions of \( D = 11 \) supergravity which correspond to limits of configurations of intersecting M2- and M5-branes (that upon compactification describe \( D = 4 \) black holes) [11].

We start from the most symmetric configuration which consists of four intersecting \( D3 \)-branes in type IIB string theory. The setup is as follows (with \( \times \) indicating the directions along the branes)

| \( \text{AdS}_2 \) | \( S^2 \) | \( T^6 \) |
|---|---|---|
| 0 | 1 | 2 3 4 5 6 7 8 9 |
| \( D3 \) \( \times \) | \( \times \) \( \times \) \( \times \) |
| \( D3 \) \( \times \) | \( \times \) \( \times \) \( \times \) |
| \( D3 \) \( \times \) | \( \times \) \( \times \) \( \times \) |
| \( D3 \) \( \times \) | \( \times \) \( \times \) \( \times \) |

(3.1)

and the \( F_5 \)-flux is given in (3.18). By performing T-dualities on some of the \( T^6 \)-directions we can obtain different solutions. In particular, T-dualizing along one of the \( T^6 \) directions, say
This solution can be obtained by the reduction from the 1/4 supersymmetric case. In a similar way, the type IIA solution \((3.2)\) is the dimensional reduction of the 1/4 supersymmetric \(D = 11\) supergravity solution \(AdS_3 \times S^2 \times T^6\) with the same \(F_4\)-flux. This is done by realizing \(S^3\) as a \(S^1\) Hopf fibration over \(S^2\). Dimensionally reducing on this, \(S^3\) creates an \(F_4\)-flux proportional to the \(K\)ähler form on \(S^2 \sim \text{CP}^1\) similar to what happens in the \(AdS_4 \times \text{CP}^3\) case. In a similar way, the type IIA solution \((3.2)\) is the dimensional reduction of the 1/4 supersymmetric \(D = 11\) supergravity solution \(AdS_3 \times S^2 \times T^6\) with \(F_4 = -\frac{1}{2} \epsilon^{ab} \epsilon^{\hat{a}\hat{b}} J_2\).

Note that if we change the sign of one of the fluxes in \((3.2)\) or \((3.3)\), the \(AdS_2 \times S^2 \times T^6\) space will still be a solution of the supergravity equations of motion, but the supersymmetry will be completely broken. This supersymmetry breaking is reflected in the form of \(\mathcal{F}\) in \((2.4)\) which will no longer be proportional to a projector. This will also spoil the structure of the Lax connection discussed in the previous section; in the non-supersymmetric cases, we have not been able to make it have zero curvature since it does not appear to be possible to cancel the terms on the right-hand side of equation \((2.23)\) without a supersymmetry current. This may indicate that though the purely bosonic sector of these non-supersymmetric models is classically integrable, the integrability is spoiled by the fermionic sector.

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15 We keep the explicit dependence of fluxes on the constant dilaton to indicate the dependence on the string coupling constant.

16 Explicitly, if \(z^\gamma, \bar{z}_\gamma\) are holomorphic coordinates on the torus, e.g. \(z^1 = y^4 + iy^5\), etc, we have \(J_2 = \frac{1}{4} dz \wedge d\bar{z}\).
Let us now consider the structure of the superisometry algebra $\mathfrak{psu}(1, 1|2)$ of the background (3.2) and demonstrate that it ensures the vanishing of the curvature of the Lax connection (2.15) (the analysis of the background (3.3) can be carried out in exactly the same fashion). For the supersymmetric solution (3.2), the definition of $\mathcal{F}$ in equation (2.4) gives

$$\mathcal{F} = -\frac{1}{R} \gamma \Gamma_{11} + \frac{i}{2R} \gamma^{5} \Gamma^{a'b'}J_{a'b'} = -\frac{4}{R} \mathcal{P}_{8} \gamma \Gamma_{11},$$

(3.4)

$$\mathcal{P}_{8} = \frac{1}{8} (2 - i J_{a'b'} \Gamma^{a'b'} \gamma^{7}),$$

(3.5)

where $\gamma = \Gamma^{0} \Gamma^{1}$ is the product of AdS$_{2}$ gamma matrices (so that $\gamma^{2} = 1$) and $\gamma^{5} = i \Gamma^{0} \Gamma^{1} \Gamma^{2} \Gamma^{3}$. Here $\mathcal{P}_{8}$ is a projector which for all the AdS$_{2} \times S^{2} \times T^{6}$ solutions has the following properties: it commutes with the $D = 10$ matrices $\Gamma^{a}$ ($a = 0, 1$) and $\Gamma^{\hat{a}}$ ($\hat{a} = 2, 3$) along the AdS$_{2} \times S^{2}$ directions, as well as with $\hat{\Gamma}$ ($= \Gamma_{11}$ in the IIA case):

$$[\mathcal{P}_{8}, \Gamma^{a}] = [\mathcal{P}_{8}, \Gamma^{\hat{a}}] = [\mathcal{P}_{8}, \hat{\Gamma}] = 0,$$

(3.6)

while for $\Gamma^{a'}$ with $a' = 4, \ldots, 9$ along the $T^{6}$ directions, we have

$$\mathcal{P}_{8} \Gamma^{a'} \mathcal{P}_{8} = 0.$$

(3.7)

The projector $\mathcal{P}_{8}$ singles out an eight-dimensional supersymmetric subspace of the 32-dimensional fermionic space and is defined in a similar way to the CP$_{3}$ case [36, 41] with the Kähler form of CP$_{3}$ replaced by that of T$^{6}$.\textsuperscript{17}

As we shall see, the same projector appears in the $\mathfrak{psu}(1, 1|2)$ superalgebra of the superisometries and so the extra terms in $\nabla J^{AB}$, equations (2.27) and (2.30), are of the required form to be canceled by the terms coming from $I_{\text{aux}}^{\alpha}$. A conventional form of the $\mathfrak{psu}(1, 1|2)$ algebra given in equation (B.9) of appendix B differs from what we would like to have in this case by factors of $\gamma^{5}$. A way to cure this is to take a slightly different realization of the gamma matrices in the algebra (B.9) which, of course, is an automorphism of the algebra. Taking

$$\Gamma^{a} \rightarrow i \Gamma^{a} \gamma^{5},$$

$$\Gamma^{\hat{a}} \rightarrow i \Gamma^{\hat{a}} \gamma^{5},$$

$$\Gamma^{a'} \rightarrow \Gamma^{a'}$$

(3.8)

together with the redefinition of the charge conjugation matrix $C \rightarrow i C \gamma^{5}$ preserves the Clifford algebra, symmetry properties of the gamma matrices and the form of the projector $\mathcal{P}_{8}$. The commutators involving $Q = \mathcal{P}_{8} \gamma$ in the $\mathfrak{psu}(1, 1|2)$ algebra (B.9) now become

$$[P_{A}, Q] = \frac{1}{2R} Q \gamma \Gamma_{11} \Gamma_{A} \mathcal{P}_{8}, \quad [M_{AB}, Q] = -\frac{1}{2} Q \Gamma_{AB} \mathcal{P}_{8}$$

(3.9)

$$\{Q, Q\} = 2 i (\mathcal{P}_{8} \Gamma^{A} \mathcal{P}_{8}) P_{A} + \frac{i R}{2} (\mathcal{P}_{8} \Gamma^{A} \Gamma^{B} \Gamma_{11} \mathcal{P}_{8}) R_{AB} C^{D} M_{CD},$$

and have the required form since they contain $\mathcal{F} \propto \mathcal{P}_{8} \gamma \Gamma_{11}$.

The relations between the Killing vectors $K_{A}$ and the Killing spinors $\Xi$ and the generators of the algebra are\textsuperscript{18}

$$K_{A} = k P_{A} k^{-1}, \quad i C \gamma \Gamma_{11} \Xi = k Q k^{-1}$$

(3.10)

$$\nabla_{A} K_{B} = -\frac{1}{2} R_{AB} C^{D} k M_{CD} k^{-1},$$

(3.11)

\textsuperscript{17} For the case of AdS$_{4} \times$ CP$_{3}$ we had $\mathcal{F} = -\frac{8}{3} (1 - \mathcal{P}_{8}) \gamma^{5}$. Note that in AdS$_{4} \times$ CP$_{3}$ the projector $\mathcal{P}_{8}$ singled out eight broken supersymmetry fermions.

\textsuperscript{18} In the following equations, we explicitly display the charge-conjugation matrices to avoid possible confusion.
where \( k(x) \) is a coset element of \( SO(1, 2) \times SU(2)/SO(1, 1) \times U(1) \) and the Killing vectors \( K_{a'} = P_{a'} \) generate the translation isometries of \( T^6 \).

Using these relations, we have

\[
\{\Xi, \Xi\} = 2i(\gamma^5 \gamma^A P_\beta \gamma^C) K_A + i R (P_\beta \Gamma^{AB} \gamma^5 \gamma^D P_\delta C) \nabla_A K_B
\]

\[
[K_A, \Xi] = -\frac{1}{2R} \Xi \gamma^A P_\beta \gamma^B \Gamma_{11} C.
\] (3.12)

Note that along the \( T^6 \) directions \( \nabla_{a'} K_{b'} = [K_{a'}, K_{b'}] = 0 \), as should be the case since the \( T^6 \) translational isometries are abelian.

Using this algebra together with (2.30) and (2.10), we get (up to quadratic order in fermions)

\[
J_{\text{susy}}^2 = -[\nabla J^{AB} + e^A (J^B - e^B)] \nabla_A K_B
\] (3.13)

and, using (2.28) and (2.7),

\[
J_B J_{\text{susy}} + J_{\text{susy}} J_B = -\frac{1}{2} \partial J_{\text{susy}}.
\] (3.14)

These equations agree with (2.27) and (2.26). Thus, the GS superstring action in this background admits the Lax connection (2.15)–(2.19) whose curvature is zero when the superstring equations of motion are satisfied, at least to quadratic order in fermions.

For the integrability of the system, also the inverse statement should be true, namely, that the zero-curvature condition should lead to the superstring equations of motion. In appendix D, we will show that this is indeed the case. Let us also note that when restricted to the \( \text{AdS}_2 \times S^2 \) supercoset sector, the Lax connection (2.15) differs from the one that is usually constructed in the case of the supercoset sigma models \([24, 34, 35, 39]\) by a \( \text{PSU}(1,1|2) \) gauge transformation whose parameters depend on the supercoset coordinates \( x \) and \( \vartheta \), and on the spectral parameter \( \theta \).

Let us conclude this subsection by mentioning another similar type IIA background which is also T-dual to the type IIB \( \text{AdS}_2 \times S^2 \times T^6 \) background with the RR 5-form flux discussed below. Here, \( \text{AdS}_2 \times S^2 \times T^6 \) is supported by the following 4-form flux:

\[
F_4 = -\frac{e^{-\phi}}{2R} e^b e^a \epsilon_{a b} \text{Im}(\Omega_2) - \frac{e^{\phi}}{2R} e^b e^a \epsilon_{a b} \text{Re}(\Omega_2),
\] (3.15)

where \( \Omega_2 \) is the holomorphic 2-form on \( T^4 \subset T^6 \). This solution describes the near-horizon geometry of the \( D2D2D4D4 \) brane intersection. The RR couplings in the GS action for this background are given by

\[
\mathcal{F} = -\frac{4}{R} \gamma^5 \Gamma(2) \mathcal{P}_8
\] (3.16)

with

\[
\mathcal{P}_8 = \frac{1}{8} (2 + J), \quad J = -2 \gamma^5 - i J_{a' b'} \gamma^{a' b'} \gamma^5, \quad J^2 = 12 + 4 J,
\] (3.17)

where now \( a', b' \) are \( T^4 \) indices, \( \gamma^5 \) is the product of the \( T^4 \) gamma matrices and \( \Gamma(2) = \frac{1}{2} \text{Re}(\Gamma^I) \text{Re}(\Gamma^J) \epsilon_{IJ} \) (\( I, J = 1, 2 \) are holomorphic indices), \( \Gamma_2 = 1 \). This type IIA solution can also be obtained from the \( D = 11 \) supergravity solution \( \text{AdS}_2 \times S^2 \times T^7 \) supported by the same \( F_4 \)-flux by dimensionally reducing it on an \( S^1 \subset T^7 \). The analysis analogous to the one carried out above shows that the GS superstring is integrable also in this background, at least up to the second order in fermions.

\[\text{In the AdS}_4 \times \text{CP}^3 \] case, the explicit form of a similar gauge transformation relating the two Lax connections was given in [32].
3.2. Type IIB AdS$_2 \times S^2 \times T^6$ with $F_5$ flux

Let us now consider the AdS$_2 \times S^2 \times T^6$ solution of type IIB supergravity with RR 5-form flux

$$F_5 = \frac{e^{-\phi}}{2R} e^a e^b \varepsilon_{ab} \text{Re}(\Omega_3) + \text{Hodge dual},$$

(3.18)

where $\Omega_3 = dz^1 dz^2 dz^3$ is the holomorphic 3-form on $T^6$, $R$ is the radius of AdS$_2$ and $S^2$ and (as above) $a, b = 0, 1, \hat{a}, \hat{b} = 2, 3$ and $a', b' = 4, \ldots, 9$ are the AdS$_2$, $S^2$ and $T^6$ indices, respectively. This solution also preserves eight supersymmetries$^{20}$.

Using equation (3.18) and definition (2.4) of $F$ in the IIB case, we find

$$F = \frac{4}{R} \mathcal{P}_8 \sigma^2 \gamma \Gamma_3 \left(1 - \Gamma_{11} \right),$$

(3.19)

where $\mathcal{P}_8$ is as in (3.5) and

$$\Gamma_{(3)} = \frac{i}{3!} \text{Re}(\Gamma^{ij} \text{Re}(\Gamma^{jk}) \text{Re}(\Gamma^{ik}) \varepsilon_{ijk})$$

(3.20)

and $I, J, \ldots = 1, 2, 3$ refer to holomorphic coordinates on $T^6$ so that $\Gamma^2_{(3)} = 1$ and $[\Gamma_{(3)}, \mathcal{P}_8] = 0$.

Let us now write down the (enlarged) $\text{psu}(1, 1/2)$ algebra (B.9) in a form suitable for this case. To this end, we pass to a different realization of the gamma-matrices by taking

$$\Gamma^a \rightarrow i \Gamma^a \Gamma_{(3)},$$

$$\Gamma^a \rightarrow i \Gamma^a \Gamma_{(3)},$$

$$\text{Re}(\Gamma^{ij}) \rightarrow \text{Re}(\Gamma^{ij}),$$

$$\text{Im}(\Gamma^{ij}) \rightarrow i \text{Im}(\Gamma^{ij}) \Gamma_{(3)},$$

$$\gamma^7 \rightarrow i \gamma^7 \Gamma_{(3)},$$

(3.21)

together with $C \rightarrow i C \Gamma_{(3)}$. This preserves the Clifford algebra, symmetry properties of the gamma matrices as well as the form of the projector $\mathcal{P}_8$, though this time in a less trivial way.

The fact that the original gamma-matrices give $\mathcal{P}_8 \gamma^7 \mathcal{P}_8 = i \varepsilon$ means that after redefinition (3.21), we have $\mathcal{P}_8 \gamma^7 \mathcal{P}_8 = -i \Gamma_{(3)} = -i \sigma^2 \Gamma_{(3)}$. After replacing in this way $\mathcal{P}_8 \gamma^7 \mathcal{P}_8$ in the (redefined) superalgebra (B.9), we identify the $SO(2)$ index on $Q$ on which $\varepsilon = i \sigma^7$ acts with the $SO(2)$ index labeling the Majorana–Weyl spinors in type IIB theory. The (enlarged) $\text{psu}(1, 1/2)$ algebra commutators involving $Q$ (see appendix B) then become

$$[P_A, Q] = -\frac{1}{2R} Q \gamma \sigma^2 \Gamma_{(3)} \Gamma_A \mathcal{P}_8, \quad [M_{ab}, Q] = -\frac{1}{2} Q \Gamma_{ab} \mathcal{P}_8, \quad [T, Q] = i \frac{1}{2} \sigma^2 Q,$$

(3.22)

and

$$[Q, Q] = 2i (\mathcal{P}_8 \Gamma^A \mathcal{P}_8) P_A - \frac{iR}{2} (\mathcal{P}_8 \Gamma^{AB} \gamma \Gamma_{(3)} \sigma^2 \mathcal{P}_8) R_{AB} C^D M_{CD}. $$

(3.23)

This is the form of the algebra we need in this case since it involves $F \propto \mathcal{P}_8 \sigma^2 \gamma \Gamma_{(3)}$.

Now the relations between Killing vectors and Killing spinors and the generators of the algebra are

$$K_A = k P_A k^{-1}, \quad i C \gamma \Gamma_{(3)} \sigma^2 \Xi = k Q k^{-1},$$

(3.24)

$^{20}$Note that as in the type IIA case discussed above, if we spoil the holomorphic structure of $F_5$ by changing signs of some of its components, we will get a non-supersymmetric solution. The GS string on such a background will, probably, not be integrable.
\[ \nabla_A K_B = -\frac{1}{2} R_{AB}^{\ C \ D} k M_{CD} k^{-1}. \] 

We get

\[ [K_A, \Xi] = \frac{1}{2R} \Xi \Gamma_A \sigma^2 \gamma^\Gamma_{(3)} \mathcal{P}_B C \] 

and

\[ \{\Xi, \Xi\} = 2i(\mathcal{P}_B \gamma^\Gamma_{(3)} \mathcal{P}_C) K_A - iR(\mathcal{P}_B \Gamma^{AB} \sigma^2 \gamma^\Gamma_{(3)} \mathcal{P}_C) \nabla_A K_B. \] 

Using this algebra together with (2.30) and (2.10), we find

\[ J_{\text{asy}}^2 = -[\nabla J^{AB} + e^A (J^B - e^B)] \nabla K_B \] 

and, using (2.28) and (2.7),

\[ J_{\text{asy}} J^B J_{\text{asy}} = -\frac{1}{2} dJ_{\text{asy}}, \] 

which is precisely what we need for the Lax connection (2.15) to have zero curvature. This proves the integrability of the GS superstring also in this background, at least up to the quadratic order in fermions.

4. Consistent truncation of the \( D = 10 \) superstring to the \( PSU(1, 1|2)/SO(1, 1) \times U(1) \) supercoset sigma model

Let us now show that the GS actions describing the type IIA and type IIB superstrings in the above \( AdS_2 \times S^2 \times T^6 \) backgrounds preserving eight supersymmetries can be consistently truncated, at the classical level, to the \( PSU(1, 1|2)/SO(1, 1) \times U(1) \) supercoset GS sigma model with the \( AdS_2 \times S^2 \) bosonic part and eight fermionic modes.

4.1. Quadratic fermionic part of the superstring Lagrangian

The string bosonic modes \( x^a \) along \( AdS_2 \times S^2 \) and the eight fermionic modes \( \vartheta = \mathcal{P}_g \Theta \) parameterize the \( PSU(1, 1|2)/SO(1, 1) \times U(1) \) supercoset. We would like to separate their contribution in the GS action from the coordinates \( y^a' \) along \( T^6 \) and the remaining 24 fermionic modes \( \upsilon \). Denoting

\[ \nu = (1 - \mathcal{P}_g) \Theta, \quad \vartheta = \mathcal{P}_g \Theta, \] 

and using the properties (3.6) and (3.7) of the projector \( F \sim \mathcal{P}_g \), we find for the quadratic part of the fermion Lagrangian in the type IIA background from section 3.1 (\( h_{ij} \) is the auxiliary worldsheet metric):

\[ \mathcal{L}_{\text{term}} = i\theta (\sqrt{-h} h^{ij} - \epsilon^{ij} \Gamma_{11}) e^i \Gamma_{12} \left( \nabla_j + \frac{1}{2R} \mathcal{P}_B \gamma^\Gamma_{11} \mathcal{P}_A e_j \right) \theta \]

\[ + i\theta (\sqrt{-h} h^{ij} - \epsilon^{ij} \Gamma_{11}) \Gamma_a \nabla_j \vartheta \partial_i y^{a'} + i\nu (\sqrt{-h} h^{ij} - \epsilon^{ij} \Gamma_{11}) \Gamma_a \nabla_j \vartheta \partial_i y^{a'} \]

\[ + i \frac{1}{R} \vartheta (\sqrt{-h} h^{ij} - \epsilon^{ij} \Gamma_{11}) \mathcal{P}_B \gamma^\Gamma_{11} \mathcal{P}_A \nabla_j \vartheta \partial_i y^{a'} \]

\[ + i \frac{1}{2R} \vartheta (\sqrt{-h} h^{ij} - \epsilon^{ij} \Gamma_{11}) \Gamma_a \mathcal{P}_B \gamma^\Gamma_{11} \Gamma_a \nabla_j \vartheta \partial_i y^{a'} \partial_j y^{b'}, \]

where

\[ \phi_j = \Gamma_{12} e^j (x) + \Gamma_a \partial_j y^{a'}, \quad \nabla_j = \partial_j + \frac{1}{2} \Gamma_{ab} \omega_j \partial_b (x), \]
and $\omega_{ij}(x)$ and $e_i^a(x)$ are the worldsheet pull-backs of the spin connection and the local frame in $\text{AdS}_2 \times S^2$.

We immediately see that the coset and non-coset directions do not decouple from each other. The couplings between the respective worldsheet fields (or their derivatives) have the following schematic form: $\nu \ddt \theta$, $\ddt \gamma$, $\ddt \ddt \theta$, $\ddt \gamma$, etc. The non-coset fields, however, never appear linearly in the Lagrangian and can thus be set to zero consistently with the equations of motion, which always admit the trivial solution $\gamma^\theta = 0$, $\nu = 0$. This crucially depends on the absence of linear couplings where only a single $y$ or $\nu$ field would appear together with $x$ and $\theta$.\(^{21}\)

Thus, in the case under consideration, one can consistently reduce the classical theory to the $\text{AdS}_2 \times S^2$ sector coupled to eight fermions $\theta$. As we shall see below, this reduced theory is nothing but the $\text{PSU}(1, 1|2)/SO(1, 1) \times U(1)$ supercoset sigma model. Due to the presence of the 4-component kappa-symmetry, the number of physical fermionic degrees of freedom in this model is equal to 2, i.e. is the same as the number of physical bosonic degrees of freedom.\(^{22}\)

Let us now keep only the $x$ and $\theta$ fields in the GS Lagrangian:

$$\mathcal{L}^{(2)}_{\text{GS}} = i\theta (\gamma - \frac{1}{2} \ddt \theta) \epsilon^{i j} \Gamma_i \Gamma_j \left( \nabla^j + \frac{1}{2 R} P_8 \gamma^i \Gamma_j \Gamma_a \gamma^2 \right) \theta.$$  \(^{(4.4)}\)

We are going to identify the Dirac matrices that appear in this Lagrangian with the structure constants of the $\text{psu}(1, 1|2)$ superalgebra in a suitable realization (see section 3.1 and appendix B).

Since $\text{AdS}_2 \times S^2$ is a coset $SO(1, 2) \times SO(3)/SO(1, 1) \times SO(2)$, its spin connection and the vielbein appear in the expansion of the left-invariant current in the algebra generators:

$$g^{-1} \partial g = \frac{1}{2} \omega_{ij} M_{ab} + e_i^a P_a = A_i + P_i,$$  \(^{(4.5)}\)

where $g \in SO(1, 2) \times SO(3)$ is a coset representative which defines an embedding of $\text{AdS}_2 \times S^2$ into $SO(1, 2) \times SO(3)$, and $\{P_a, M_{ab}\}$ are the generators of $\mathfrak{so}(1, 2) \oplus \mathfrak{so}(3)$, defined in appendix B. The gamma-matrices, which contract with the spin connection and the vielbein in (4.3), form a representation of $\mathfrak{so}(1, 2) \oplus \mathfrak{so}(3)$:

$$(P_a, M_{ab}) \rightarrow \left( \frac{1}{2 R} \gamma^i \Gamma_j \Gamma_a, -\frac{1}{2} \Gamma_{ab} \right).$$  \(^{(4.6)}\)

The same representation appears in the $D = 10$ form (3.9) of the $\text{psu}(1, 1|2)$ algebra. If we define the $\text{psu}(1, 1|2)$-valued fields

$$\Psi = \theta Q,$$  \(^{(4.7)}\)

\(^{21}\)The absence of such couplings and the structure of all other terms in the action is related to its invariance under the external $U(1)$-automorphism of the $\text{psu}(1, 1|2)$ algebra generated by the Kähler form $\mathcal{L}_{\mathcal{K}}$ on $T^6$ (see appendix B). With respect to this $U(1)$ the $x$ coordinates are neutral, the three complex coordinates $y$ of $T^6$ have charge 1, the complex counterparts of $\theta$ have charge 3/2 and the $\nu$'s have charge $-1/2$. The ratio $-3$ of the charges of $\theta$ and $\nu$ is explained by the fact that they are eigenfunctions of the matrix $J = -i J_{\psi} \gamma^{ab} \gamma^7$, appearing in the projector $\mathcal{P}_8$, whose eigenvalues are 6 with degeneracy 8 (associated with $\theta$) and $-2$ with degeneracy 24 (associated with $\nu$). One can see that these values of the charges forbid any term in the action with a single $y$ or $\nu$, provided that only neutral terms appear.

\(^{22}\)Let us also note that when the string moves in $T^6$ (i.e. the coordinates $y$ are not constant), it is possible to gauge-fix the eight fermions $\theta$ to zero using half of the kappa-symmetries. In such a gauge, one is left with 24 broken supersymmetry fermions $\nu$ and eight transverse worldsheet bosonic modes $x$ and $y$. It is not possible, however, to further consistently truncate the theory by freezing some of the transverse excitations $x$ and/or $y$ because of the presence of cubic terms of the forms $x \nu x$ and $y \nu y$ in action. Note that in this case we still have eight remaining kappa-symmetries so that the number of physical (on-shell) bosonic and fermionic degrees of freedom is of course 8+8.
(where $Q$ are the supersymmetry generators) the spin connection and the local frame will act on $\Psi$ as the Lie-algebra commutators:

\[
\frac{1}{2R} \gamma^{i_1} \Gamma_{i_1} \epsilon_{i_1}^{\alpha} \rightarrow [P_i, \cdot], \quad -\frac{1}{4} \Gamma_{ab} \omega_{ab} \rightarrow [A_i, \cdot]. \tag{4.8}
\]

We also introduce a parity transformation $\Omega$ of the fermion field:

\[
\Omega \Psi = -i \theta \Gamma_{11} Q.
\]  \tag{4.9}

Later, we will identify it with the $\mathbb{Z}_4$-automorphism of the $\text{psu}(1,1|2)$ superalgebra.

The Lagrangian (4.4) then takes the form

\[
\mathcal{L}_{\text{GS}}^{(2)} = \frac{1}{2} \text{Str} \left[ \Psi \left( \sqrt{-h} \gamma^i \gamma^j \Omega \right) \partial_i \partial_j \Psi \right],
\]  \tag{4.10}

where

\[
\partial_i \theta = \nabla_i \theta + [P_i, \theta], \quad \partial_j \theta = \partial_j \theta + [A_j, \theta]. \tag{4.11}
\]

and $\text{Str}(\cdot, \cdot)$ is the invariant bilinear form on $\text{psu}(1,1|2)$, which coincides with $C \gamma \Gamma_{11}$ on the eight-dimensional spinor space $Q = \mathcal{P} \mathcal{Q}$. More precisely,

\[
\text{Str} (Q_{\alpha} Q_{\beta}) = -4i R (C \gamma \Gamma_{11} \mathcal{P} \mathcal{Q})_{\alpha \beta}
\]  \tag{4.12}

in the normalization in which $\text{Str}(P_{\alpha} P_{\beta}) = \eta_{\alpha \beta}$.

We can now compare (4.10) with the Lagrangian of the $PSU(1,1|2)/SO(1,1) \times U(1)$ coset sigma model expanded to the second order in fermions.

4.2. Lagrangian of the supercoset model

The $PSU(1,1|2)/SO(1,1) \times U(1)$ coset is a semi-symmetric superspace [47], which means that its supergeometry is invariant under the action of a $\mathbb{Z}_4$-symmetry. This symmetry is inherited from the $\mathbb{Z}_4$-automorphism of $\text{psu}(1,1|2)$ that acts on the generators in the $D = 10$ notation of (3.9) and appendix B, as follows:

\[
\Omega(M_{\alpha \beta}) = M_{\alpha \beta}, \quad \Omega(P_{\alpha}) = -P_{\alpha}, \quad \Omega(Q) = -i Q \Gamma_{11}. \tag{4.13}
\]

The action of $\Omega$ preserves the commutation relations of the superalgebra and thus defines a $\mathbb{Z}_4$-decomposition of $\text{psu}(1,1|2)$:

\[
\text{psu}(1,1|2) = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3, \quad \Omega(\mathfrak{h}_p) = i^p \mathfrak{h}_p, \quad [\mathfrak{h}_p, \mathfrak{h}_q] \subset \mathfrak{h}_{(p+q) \text{mod } 4}. \tag{4.14}
\]

The invariant subspace of $\Omega$, $\mathfrak{h}_0 = \mathfrak{o}(1,1) \oplus \mathfrak{u}(1)$, is the denominator subalgebra of the supercoset. The action of the $\mathbb{Z}_4$ automorphism on the supercharges coincides with the 10D parity (cf equation (4.9)).

The string embedding in $PSU(1,1|2)/SO(1,1) \times U(1)$ is parameterized by a coset representative $G(\xi) \in PSU(1,1|2)$, where $\xi$ are the worldsheet coordinates. The coset representative is defined up to local $SO(1,1) \times U(1)$ transformations: $G(\xi) \rightarrow G(\xi) h(\xi)$. The global $PSU(1,1|2)$ symmetry acts from the left: $G(\xi) \rightarrow g G(\xi)$. The Lagrangian is constructed from the left-invariant current

\[
J_j = G^{-1} \partial_j G, \tag{4.15}
\]

with the help of the $\mathbb{Z}_4$-decomposition:

\[
J_{j,p} = \frac{1}{4} \sum_{\ell=0}^3 i^{-\ell} \Omega^\ell (J_j) \in \mathfrak{h}_p. \tag{4.16}
\]

The $\mathfrak{h}_0$-projection is the $SO(1,1) \times U(1)$ gauge field (or, equivalently, the AdS$_2 \times S^2$ spin connection), since it transforms as $J_{j,0} \rightarrow h^{-1} J_{j,0} h + h^{-1} \partial_j h$. The other components of the
current transform in the adjoint of $SO(1, 1) \times U(1)$: $J_{j_{1,2,3}} \rightarrow h^{-1} J_{j_{1,2,3}} h$. The Lagrangian of the supercoset sigma model is
\begin{equation}
\mathcal{L}_{\text{coset}} = \frac{1}{2} \text{Str}(\sqrt{-hh^{ij}}) J_{j_{1,2,3}} + \epsilon^{ij} J_{j_{1}} J_{j_{3}}. \tag{4.17}
\end{equation}

In order to compare the sigma model with the GS string Lagrangian (4.4), we should expand the sigma model Lagrangian to the second order in fermions. This can be understood as a particular case of the background field expansion, the general form of which is given in appendix B of [39]. The starting point is the following form of the coset representative:
\begin{equation}
G = g e^{X}, \tag{4.18}
\end{equation}
where $g(x) \in SO(1, 2) \times SO(3)$ is a bosonic element of the supergroup that describes a classical string solution in $\text{AdS}_2 \times S^2$. The field $X \in \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3$ describes string fluctuations around this solution. The coset gauge is fixed by requiring that $X$ has zero $h_0$ projection.

The sigma model Lagrangian, expanded to the second order in fluctuations, takes the following form (upon dropping a total derivative coming from the WZ term of (4.17)):
\begin{equation}
\mathcal{L}_{\text{fluct}} = \frac{1}{2} \text{Str}\left(\sqrt{-hh^{ij}} \left(\nabla_j X_2 \nabla_j X_2 - [P_j, X_2][P_j, X_3]\right)
+ (\sqrt{-hh^{ij}} + \epsilon^{ij})[P_j, \nabla_j X_1 + [P_j, X_3]]
+ (\sqrt{-hh^{ij}} - \epsilon^{ij})[P_j, \nabla_j X_3 + [P_j, X_1]]\right). \tag{4.19}
\end{equation}
where $A_j$ and $P_j$ are grade 0 and grade 2 components of the $\mathbb{Z}_4$-decomposition of the background Cartan form as in (4.5), and the covariant derivative contains the background gauge connection $\nabla_j = \partial_j + [A_j, \cdot]$. The fermion part of the fluctuation Lagrangian can be more compactly written in terms of
\begin{equation}
\Psi = X_1 + X_3, \quad X_{1,3} = \frac{1 \pm i\Omega}{2} \Psi. \tag{4.20}
\end{equation}
Namely,
\begin{equation}
\mathcal{L}^{(2)}_{\text{coset}} = \frac{1}{2} \text{Str} \Psi [P_j, (\sqrt{-h}h^{ij} + i\epsilon^{ij}/\Omega)(\nabla_j \Psi + [P_j, \Psi])] \tag{4.21}
\end{equation}
which coincides with (4.10).

For the type IIB background (3.18), the part of the GS action that contains the $PSU(1, 1|2)$ fields becomes
\begin{equation}
\mathcal{L}^{(2)}_{\text{coset}} = i\theta (\sqrt{-h}h^{ij} - \epsilon^{ij}\sigma^3)e^{2\Gamma_2} \left(\nabla_j - \frac{1}{2R} \mathcal{P}_b \sigma^2 \gamma_{(3)} \Gamma_{(3)} \mathcal{P}_a \right) \theta \tag{4.22}
\end{equation}
with the $\mathfrak{so}(1, 2) \times \mathfrak{so}(3)$ generators being
\begin{equation}
(P_a, M_{ab}) \rightarrow \left(\frac{1}{2R} \sigma^2 \gamma_{(3)} \Gamma_{(3)} \mathcal{P}_a, -\frac{1}{2} \Gamma_{ab}\right). \tag{4.23}
\end{equation}
This coincides with the representation of the bosonic generators of $\mathfrak{psu}(1, 1|2)$ in the type IIB realization of the superalgebra (3.22)–(3.23). Finally, the $\mathbb{Z}_4$-automorphism acts on the supercharges as
\begin{equation}
\Omega(Q) = iQ\sigma^3, \tag{4.24}
\end{equation}
and thus (4.22) is equivalent to the coset action (4.21) for the fields that satisfy $\mathcal{P}_b \Theta = \Theta$. 

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4.3. Supercoset truncation beyond the quadratic approximation

Let us now consider how one may, in principle, reconstruct the full AdS $2 \times S^2 \times T^6$ supergeometry by solving the supergravity constraints order by order in non-supersymmetry fermions $\nu$, with the $PSU(1, 1/2)/SO(1, 1) \times U(1)$ geometry taken as the initial condition. This will also demonstrate that the supercoset sigma model action is a consistent truncation of the full nonlinear type II superstring action.

Let us now demonstrate that the $PSU(1, 1/2)/SO(1, 1) \times U(1)$ supercoset action is a consistent truncation of the full nonlinear type II superstring action.

The GS superstring action in a generic type II $D=10$ superbackground is

$$S = -\frac{1}{4\pi\alpha'} \int \ast E^A E^B \eta_{A B} - \frac{1}{2\pi\alpha'} \int B_2,$$

where $E^A(X, /Theta1) = dX^M E_M^A + d\Theta /Gamma1 E^A_{\mu}$ are the worldsheet pullbacks of the vector supervielbein and the 2-form field $B_2(X, /Theta1)$ can be computed from its field strength $H_3 = dB_2$ as

$$B_2 = b_2(X) + \int_{0}^{1} dt i_\Theta H_3(X, t\Theta).$$

The vielbeins satisfy the type II supergravity torsion constraint

$$T^A = DE^A = dE^A - E^B /Omega1B^A = -iE^A E,$$

where $/Omega1BA(X, /Theta1)$ is the spin connection, and $E^\alpha$ are the 32-component spinor supervielbeins of the opposite (type IIA) or the same (type IIB) chirality. In the linear approximation $E^\alpha = D/Theta1\alpha$, where $D$ is a covariant derivative similar to (2.2). $H_3$ satisfies the following constraint:

$$H_3 = dB_2 = iE^A /Gamma1 A^A E + \frac{1}{3!} E^C E^B E^A H_{ABC}.$$

Let us now concentrate on the AdS $2 \times S^2 \times T^6$ case. To prove that the supercoset model is a consistent truncation, we should know the dependence of the supervielbeins on the 8 supercoset fermions $\varphi$ (which in general extends up to the eighth order) and their dependence on the extra 24 fermions $\nu$ and the derivatives of the $T^6$ coordinates $y$ to linear order. This dependence can be determined by a recursion procedure similar to the one used, e.g., in [48, 49] (and references there) assuming the $PSU(1, 1/2)/SO(1, 1) \times U(1)$ supergeometry as the initial condition.

One can always impose a gauge condition for the target superspace differentiations such that

$$\nu E^A_{\alpha} \equiv i_\nu E^A = 0, \quad \nu \Omega^A_{\alpha} \equiv i_\nu \Omega^A = 0,$$

where $\nu$ are again the 24 non-supercoset fermions $\nu = (1 - \bar{P})/Theta_3$ associated with the broken supersymmetries.

The torsion constraint (4.27), the $PSU(1, 1/2)$ superisometry of the background and the $/Gamma1$-matrix relations (3.6) and (3.7) then imply that the vector supervielbeins have the following structure. Along the coset direction AdS $2 \times S^2$ (which is labeled by $\varphi$) the supervielbein is

$$E^\varphi = \varphi E^A /Gamma1 A + i\nu \gamma \partial \nu + O(\nu^3),$$

where $E^\varphi(x, \varphi)$ is the complete expression for the supervielbein of the $PSU(1, 1/2)/SO(1, 1) \times U(1)$ supercoset (which can be computed from the Cartan

23 The superspace constraints used here differ from those used in [36, 37] by a shift in the fermionic supervielbein, $E = E + \frac{1}{2} E^A /Gamma1 A /Gamma1 A$, where $/Gamma1 A$ is the dilatino superfield.
form). Note that, by construction, the components of this supervielbein satisfy a condition similar to (4.29):

$$\partial^{\mu} e_{\nu}(x, \vartheta) = i_{\vartheta} \epsilon_{\Sigma} = 0.$$  

(4.31)

In the supervielbein along the coset directions, there are no terms linear in the extra 24 fermions $\upsilon$. The absence of such terms is guaranteed by the torsion constraint (4.27) and the gamma-matrix identities (3.6)–(3.7).

Along the $T^6$ directions, the supervielbein has the following form:

$$E_{\nu} = dy_{\nu} + 2i \upsilon \Gamma_{\nu} e(x, \vartheta) + O(\upsilon^2),$$  

(4.32)

where $e_{\Sigma}(x, \vartheta) = (P_8 e)^{\Sigma}$ is the spinorial supervielbein of the supercoset (contained in the Cartan form). Because of the $P_8$ projector, the spinorial index $\alpha$ here takes eight values.

The eight fermionic supervielbeins of the complete target superspace (associated with the supercoset) start from the corresponding supercoset fermionic Cartan form $e_{\Sigma}(x, \vartheta)$ and may contain terms quadratic in the fields $\upsilon$ and $dy$:

$$(P_8 E)^{\Sigma} = e_{\Sigma}(x, \vartheta) + (P_8 dy_{\nu} \Gamma_{\nu} \upsilon) e_{\Sigma} + O(\upsilon^2).$$  

(4.33)

The extra 24 fermionic supervielbeins are

$$(1 - P_8) E = (1 - P_8) D \upsilon + O(\upsilon^3).$$  

(4.34)

To get the explicit form of the spinorial supervielbeins to all orders in $\upsilon$, one should use the explicit form of the type II spinorial torsion $D E_{\nu} = T_{\nu}^\alpha$ in the AdS$_2 \times S^2 \times T^6$ backgrounds and Bianchi identities for the relevant superforms. The reconstruction of this dependence is technically rather involved and so far has not been carried out.

Finally, the term which might spoil the consistent truncation to the supercoset model could be the contribution to $B_2$ (4.26) which comes from the second term in (4.28) with all the indices along the coset directions, i.e. $\partial^{\mu} e_{\nu} e_{\Sigma} e_{\Xi} H_{\alpha\beta\gamma}$. Note that in the supercoset model, the three-form field strength $H_{\alpha\beta\gamma}$ is zero, while in the complete $D = 10$ superbackground $H_{\alpha\beta\gamma}$ may, in principle, depend linearly on $\upsilon$. However, this potentially dangerous term in the string action is zero because of equation (4.31). This is in accordance with the $U(1)$-automorphism invariance discussed in footnote 15. Thus, with the above form of the supervielbeins, one can use the same reasoning as in the quadratic approximation to show that the GS action can be consistently truncated to the complete nonlinear supercoset model.

5. BMN limits of the AdS$_2 \times S^2 \times T^6$ superstring

By analogy with the BMN expansion of the AdS$_2 \times S^5$ superstring (in which, in particular, the unbroken $PSU(2|2) \times PSU(2|2)$ symmetry plays an important role, see, e.g., [50]) it is interesting to study the BMN limits of the AdS$_2 \times S^2 \times T^6$ superstring, having in mind their possible role in its exact solution based on integrability.

In general, to describe the spectrum of GS strings one may, as in flat space, fix a kind of light-cone gauge concentrating on a subsector of string states carrying momentum in a particular compact direction. In the point-like limit, the string is then moving along a particular $D = 10$ null geodesic which runs along global time direction in AdS and along a geodesic in a compact direction. The BMN limit corresponds to expanding near this geodesic. In the AdS$_2 \times S^2 \times T^6$ case, there are several inequivalent choices for the geodesic, e.g. it may run (i) along the big circle of $S^2$, or (ii) along $S^1$ of $S^2$ and $S^1$ of $T^6$ at the same time (with the case of only $S^1$ of $T^6$ being a special case of the latter). These led to inequivalent BMN limits that we will consider in detail below.
Before getting into a detailed discussion of BMN limits, let us mention that the quadratic supercoset Lagrangian (4.19) is very convenient for analyzing the symmetries and the spectrum of the string in the first BMN limit when the geodesic runs along $S^2$ in $\text{AdS}_2 \times S^2$. To this end, it is convenient to represent $\mathfrak{psu}(1, 1|2)$ elements by $(2|2) \times (2|2)$ supermatrices. The BMN background is represented by

$$g = \exp(i \tau P_\tau), \quad P_\tau = \text{diag}(1, -1|1, -1) \in \mathfrak{h}_2.$$  

(5.1)

Then, $A_0 = 0$ and $P_\tau \sim P_\tau$. From (4.10) it follows that the components of $\Psi$ (or $\vartheta$) that commute with $P_\tau$ completely drop out of the action which is a manifestation of the kappa-symmetry. Only those components of the fermion fields that have non-trivial commutators of $P_\tau$ with $P_\tau$ correspond to the physical on-shell excitations of the string. Since, the commutant of $P_\tau$ contains exactly half of the supercharges of $\mathfrak{psu}(1, 1|2)$, the kappa-symmetry reduces the number of the $\vartheta$-fermion fields from 8 to 4 that describe two on-shell fermion states. The mass-squared matrix for the remaining coset excitations (both bosonic and fermionic) is then (see (4.19))

$$\mathcal{M}^2 = (\text{ad} P_\tau)^2.$$  

(5.2)

We thus get two bosons plus two fermions of equal mass. The commutant of $P_\tau$ corresponds to the symmetries which are left unbroken by the background. Here this commutant is $\mathfrak{su}(1|1) \times \mathfrak{su}(1|1)$, where $\mathfrak{su}(1|1)$ is an algebra of two real supercharges whose anti-commutator is a bosonic central element.

As we shall see below, in this BMN limit, the complete $D = 10$ superstring has an additional six massless bosons and six massless fermions from the bosonic fluctuations on $T^6$ and non-supersymmetric $\nu$-fermions. In the second BMN limit involving $T^6$, we find that the masses of worldsheet bosons and fermions will be different, i.e. there will be no 2D supersymmetry at the quadratic level of the expanded string action. Moreover, the coset fermions $\vartheta$ will mix with the broken-supersymmetry fermions $\nu$.

5.1. BMN limit along geodesic in $S^2$

BMN (or pp-wave) limits for $\text{AdS}_2 \times S^2 \times M^6$ backgrounds and their supersymmetries were considered, e.g., in [52]. To take the limit, let us choose the $\text{AdS}_2 \times S^2$ coordinates and the metric of $\text{AdS}_2 \times S^2 \times T^6$ as

$$dx^2 = R^2(-\cosh^2 \rho \, dr^2 + d\rho^2 + d\theta^2 + \sin^2 \theta \, d\phi^2) + dy_\nu^2 \, dy^{\nu'}.$$  

(5.3)

Consider a particle moving with the speed of light in the $\phi$-direction, set $\tilde{x}^\pm = \frac{t R}{\mu}$ and zoom-in on the geometry seen by the particle by letting

$$x^\pm = \frac{\tilde{x}^\pm}{\mu}, \quad x^\pm = \mu R \tilde{x}^\pm, \quad \rho = \frac{r}{R}, \quad \theta = \frac{\pi}{2} + \frac{z}{R}, \quad \mu R, \frac{R}{r}, \frac{R}{z} \to \infty,$$  

(5.4)

where $\mu$ is a mass scale parameter. Taking this limit, the metric becomes

$$dx^2 = -4dx^+ \, dx^- - \mu^2 (r^2 + z^2)(dx^+)^2 + dr^2 + dz^2 + dy_\nu^2 \, dy^{\nu'}.$$  

(5.5)

For the type IIA solution, the fluxes (3.2) become

$$F_2 = e^{-\vartheta} \mu \, dr \, dx^+$$  

(5.6)

$$F_4 = e^{-\vartheta} \mu \, dz \, dx^+ J_2,$$  

(5.7)

while for the type IIB solution, we get from (3.18)

$$F_5 = -e^{-\vartheta} \mu \, dr \, dx^+ \text{Re}(\Omega_3) + \text{Hodge dual}.$$  

(5.8)

24 This BMN limit in the $\text{AdS}_2 \times S^2$ supercoset subsector of the theory was discussed from a superalgebra point of view in [51].
5.1.1. Supersymmetries of the pp-wave background. To see how many supersymmetries these backgrounds preserve, we should analyze the dilatino and gravitino variations. In our case, these take the form

\[ \Gamma^A F \Gamma_A \epsilon = 0 \]  
\[ (\nabla_A - \frac{1}{8} F \Gamma_A) \epsilon = 0, \]  
where the second is simply the Killing spinor equation.

For the type IIA pp-wave, we get

\[ F = -4i \mu \gamma^7 \Gamma^2 \mathcal{P}_8 \Gamma^+. \]  
This background thus allows 16 Killing spinors which are annihilated by \( \mathcal{P}_8 \). It can be shown that these supersymmetries will always be present in the pp-wave (Penrose) limit of any background.

Since \( \Gamma^A F \Gamma_A = \cdots \Gamma^+(1 - \mathcal{P}_3) \), the dilatino equation also allows for four additional spinors satisfying \( \epsilon = \mathcal{P}_8 \epsilon \) and \( \Gamma^+ \epsilon \neq 0 \). To see whether these correspond to supersymmetries, we also need to solve the gravitino equation (5.10) for which the integrability condition reads

\[ (R_{AB} \Gamma^C \Gamma^{CD} - \frac{1}{8} F \Gamma^A \Gamma^B) \epsilon = 0. \]  
To compute the curvature, we chose the vielbeins in the form

\[ e^+ = 2 dx^+, \quad e^- = 2 \left( dx^- + \mu \frac{\gamma^2 + \gamma^3}{4} dx^+ \right), \quad e^r = dr, \quad e^z = dz, \quad e^a = dy^a. \]  
For this choice of the vielbeins, the spin connection has the following non-zero components, defined from the zero-torsion condition \( de^A + e^B \omega_B^A = 0 \):

\[ \omega^{-r} = -\frac{\mu^2}{2} r dx^+, \quad \omega^{-z} = -\frac{\mu^2}{2} z dx^+. \]  
Thus, the curvature has the following non-zero components:

\[ R^{-r} = \frac{\mu^2}{2} r dx^+, \quad R^{-z} = \frac{\mu^2}{2} z dx^+. \]  
The integrability condition (5.12) is thus trivially satisfied except for the \((+r)\)- and \((+z)\)-components. For the \((+r)\)-component, (5.12) gives

\[ (2 R_{\nu r}^{-r} \Gamma^r_{\nu -r} + 2 \mu^2 \Gamma^r \Gamma^+) \mathcal{P}_8 \epsilon = 0, \]  
and the two terms here cancel each other due to the form of the curvature. The \((+z)\)-component vanishes in the same way. This means that the eight supersymmetries of the AdS2 × S2 × T6 background are preserved by the Penrose limit and in addition the system acquires an extra 12 giving a total of 20 supersymmetries.

In the type IIB case, we get

\[ F = 4i \mu \Gamma^r \Gamma^z_3 \sigma^2 \mathcal{P}_8 \Gamma^+. \]  
and again we have at least the 16 Killing spinors which are annihilated by \( \Gamma^+ \). In the type IIB case, \( \Gamma^A F \Gamma_A = 0 \), and thus there are no terms in the supersymmetry variation of the dilatino (5.9) that depend on \( F \). To see whether there are any additional supersymmetries, we need to study further the gravitino equation. Let us first look for the supersymmetries with \( \epsilon = \mathcal{P}_8 \epsilon \)

Note that in all the following equations, the indices \((+,-)\) refer to the coordinate basis \((dx^+, dx^-)\) and not the \((e^+, e^-)\)-basis.
The only non-zero components of the spin-connection are \( 2R_{\alpha\beta} = 2\mu^2\Gamma^\alpha\Gamma^\beta \) and similarly for the \((+z)\)-component. The two terms cancel each other and we again conclude that the Penrose limit preserves the supersymmetries present in the original space, i.e. those with \( \epsilon = \mathcal{P}_0 \epsilon \) (only half of which are annihilated by \( \Gamma^\alpha \)).

We have thus seen that both the type IIA and type IIB pp-wave backgrounds preserve 20 supersymmetries: 8 original \( \text{AdS}_5 \times S^5 \times T^6 \) supersymmetries with \( \epsilon = \mathcal{P}_0 \epsilon \) and 12 additional ones with \( \epsilon = -\Gamma^\alpha \Gamma^\beta \epsilon \). Since the pp-wave backgrounds obtained in this way always preserve at least 16 supersymmetries [53] we conclude that there is an additional enhancement of supersymmetry by an extra four generators.

5.1.2. BMN limit of the superstring action. The quadratic part of the superstring action expanded near the classical solution representing the relevant geodesic is equivalent to the superstring action for the pp-wave background (evaluated in the light-cone gauge). In the pp-wave background discussed above, the bosonic part of the superstring action (2.1) becomes

\[
S_B = -\frac{1}{2} T \int [-4 \ast d\epsilon^+ \ast d\epsilon^- - \mu^2 (r^2 + z^2) \ast d\epsilon^+ \ast d\epsilon^- + 2d\tau \ast d\epsilon^- + 2d\gamma \ast d\epsilon^- \ast d\gamma^\prime \ast d\gamma^\prime_0].
\]

(5.19)

The fermionic part of the action is

\[
S_F = -T \int (i \ast dX^M \ast \Omega^M \ast D\Theta - i \ast dX^M \ast \Omega^M \ast \hat{D}\Theta).
\]

(5.20)

Fixing the light-cone kappa-symmetry gauge \( \Gamma^+ \Theta = 0 \) simplifies the action considerably. For the type IIA case, we find using (5.11)

\[
S_{F\text{I}(\text{IA})} = -T \int [-2i \ast d\epsilon^+ \ast \Theta \Gamma^- d\Theta + 2i \ast d\epsilon^+ \ast \Theta \Gamma^- \Gamma_1 d\Theta + 2\mu \ast d\epsilon^+ \ast \theta \Gamma^- \gamma^3 \gamma^\prime_1 \Theta].
\]

(5.21)

For the type IIB case (5.17) leads to

\[
S_{F\text{I}(\text{IB})} = -T \int [-2i \ast d\epsilon^+ \ast \Theta \Gamma^- d\Theta + 2i \ast d\epsilon^+ \ast \Theta \Gamma^- \sigma^3 \sigma^\prime d\Theta - 2i \ast d\epsilon^+ \ast \sigma \Gamma^- \Gamma_{1(3)} \sigma^\prime d\Theta],
\]

(5.22)

where \( \theta = \mathcal{P}_6 \Theta \) and \( \nu = (1 - \mathcal{P}_6) \Theta \). Upon fixing the light-cone gauge

\[
x^+ = \frac{b^+ \tau}{T}
\]

(5.23)

26 For completeness we should also check whether other supersymmetries of the form \( \epsilon = (1 - \mathcal{P}_6) \epsilon \) with \( \Gamma^+ \epsilon \neq 0 \) could be present. Starting with the \((\alpha')\)-component of the integrability condition \( \mathcal{P}_6 \Gamma_{\alpha'} \Gamma_\beta (1 - \mathcal{P}_6) \epsilon = 0 \) and multiplying it by \( \Gamma^\beta \) and using that \( \Gamma^\beta \mathcal{P}_6 \Gamma_{\alpha'} = 2(1 - \mathcal{P}_6) \) (following from (A.9)) we get \( \Gamma^\gamma (1 - \mathcal{P}_6) \epsilon = 0 \) which contradicts our assumption, i.e. there are no additional supersymmetries.

27 Note that the projectors \( \mathcal{P}_6 \) (having 8 non-zero eigenvalues) and \( \Gamma^+ \Gamma^- \) (having 16 non-zero eigenvalues) have 4 non-zero eigenvalues in common since they commute.

28 Note that this is different compared to the \( \text{AdS}_5 \times S^5 \) or \( \text{AdS}_4 \times \text{CP}^3 \) cases where the number of supersymmetries remained the same.

29 The only non-zero components of the spin-connection are \( \omega^- \Gamma^\alpha \Gamma^\beta \) and \( \omega^- \Gamma^\alpha \Gamma^+ \) which do not contribute to the action in the gauge \( \Gamma^+ \Theta = 0 \).
the action for the eight transverse bosonic modes and the fermionic fluctuations takes the form

\[
S = -\frac{1}{2} T \int dr \left[ \partial_\tau r \partial^\tau r + \partial_z z \partial^z z + m^2 (r^2 + z^2) + \partial_\phi \partial^\phi \phi \right] \\
+ iT \int dr \, dr \left[ \partial^\phi \phi - \partial^\phi \phi \right]
\]

(5.24)

Here, \( \partial_\pm = \partial_r \pm \partial_z \), \( m = \mu \rho / T \), and \( \partial_1 \) and \( \partial_2 \) are two-component fermions while \( \upsilon^1 \) and \( \upsilon^2 \) are six-component fermions. The indices 1 and 2 indicate that these fermions originate from the two \( D = 10 \) Majorana–Weyl spinors of the same (type IIB) or opposite (type IIA) chirality. The matrix \( \Pi \) satisfies \( \Pi^2 = 1 \) and is equal to \(-\Gamma^\nu \Gamma^{(3)}\) in the type IIA and IIB cases, respectively.

As was anticipated above, the resulting physical fluctuation spectrum contains (i) the supercoset sector represented by two bosons \((r, z)\) and two fermions \( \partial_1 \) and \( \partial_2 \), (ii) the non-supercoset sector containing six massless bosons \( y^\phi \) and six pairs of massless fermions \( \upsilon^1 \) and \( \upsilon^2 \). These two (2D supersymmetric) sets of states are decoupled in the quadratic (pp-wave) approximation but will start interacting at higher orders of the near-BMN expansion of the superstring action.

5.2. BMN limit along the geodesic in \( S^1 \times S^1 \subset S^2 \times T^6 \)

Let us now consider a more general BMN limit when the geodesic representing the c.o.m. of the string runs along a ‘diagonal’ direction in the \( S^1 \times S^1 \) torus formed by the equator \( S^1 \subset S^2 \) (with coordinate \( \phi \)) and one of the \( S^1 \subset T^6 \) directions, e.g. \( y^4 \). It can be parameterized by the ‘rotated’ coordinate \( \phi' \) as

\[
\phi' = \cos \alpha \, \phi + \sin \alpha \, \frac{y^4}{R}, \\
y^4' = -\sin \alpha \, R \phi + \cos \alpha \, y^4.
\]

(5.25)

Here, \( \alpha \) is related to the ratio of string angular momenta along the two circles\(^{30} \). Setting \( \tilde{x}^\pm = \frac{t \pm y'}{2} \) and zooming-in on the geometry seen by the particle by letting

\[
x^+ = \frac{\tilde{x}^+}{\mu}, \quad x^- = \mu R^2 \tilde{x}^-, \quad \rho = \frac{r}{\mu R},
\]

(5.26)

we get from (5.3) the following pp-wave metric

\[
ds^2 = -4dx^+ dx^- - \mu^2 (r^2 + \cos^2 \alpha \, z^2) (dx^+)^2 + dr^2 + dz^2 + dy_{x'} dy_{y'},
\]

(5.27)

which generalizes our previous example (5.5) to the case of \( \alpha \neq 0 \). Another special case is \( \alpha = \pi \) when \( \cos \alpha = 0 \) and the geodesic is running solely along \( S^1 \subset T^6 \).

The fluxes of the type IIA solution (3.2) become

\[
F_2 = e^{-\phi} \mu \, dr \, dx^+ \\
F_4 = -\mu e^{-\phi} \cos \alpha \, dx^+ \, dz \, J_2 - \mu e^{-\phi} (1 - \cos \alpha) \, dx^+ \, dy^5 \, dy^{y'},
\]

(5.28, 5.29)

\(^{30} \)The corresponding classical string solution is \( t = \pi \, \tau \), \( \phi = \mu \, \tau \), \( y_4 = \mu \, \tau \) with the Virasoro (massless geodesic) condition implying that \( \kappa^2 = \mu^2 + R^{-2} \pi^2 \) (see also section 7).
where \( J_2 = dy^5 dy^4 + dy^7 dy^6 + dy^9 dy^8 \). This gives
\[
F = -4i\mu \cos \alpha y^7 \Gamma^\varepsilon \mathcal{P}_8 \Gamma^\varepsilon - 2i\mu (1 - \cos \alpha) y^7 \Gamma^\varepsilon \mathcal{P}_{16} \Gamma^\varepsilon
\]  
(5.30)
where
\[
\mathcal{P}_{16} = \frac{i}{2} (1 - \Gamma^{6789}).
\]  
(5.31)

Note that in (5.30) the projector \( \mathcal{P}_8 \) is constructed with the gamma-matrices in the new basis corresponding to the change of variables performed in equation (5.25), namely
\[
\mathcal{P}_8 = \frac{1}{2} (1 + (\Gamma^{45} + \Gamma^{67} + \Gamma^{89}) \Gamma^{456789}), \quad \text{where} \quad \Gamma^\varepsilon = -\sin \alpha \Gamma^\varepsilon + \cos \alpha \Gamma^4.
\]  
(5.32)

This \( \mathcal{P}_8 \) differs from the original \( \mathcal{P}_8 \) projector singling out that the eight supersymmetries of \( \text{AdS}_2 \times S^2 \times T^6 \); \( \mathcal{P}_8 \) commutes with \( \Gamma^\varepsilon = \frac{1}{2} (\Gamma^0 + \Gamma^\varepsilon) \) for all \( \alpha \neq 0 \) while \( \mathcal{P}_8 \) does not. Using \( \mathcal{P}_8 \) we may again split the fermions \( \Theta \) into \( \theta' \) and \( \nu' \) which will now be linear combinations of the original supercoset \( \vartheta \) and non-supercoset \( \nu \) fermions.

As usual in the Penrose limit, the 8 supersymmetries of \( \text{AdS}_2 \times S^2 \times T^6 \) get again enhanced to 16 supersymmetries with the supersymmetry parameters satisfying \( \Gamma^\varepsilon \epsilon = 0 \). However, there is no further enhancement to 20 supersymmetries, unless \( \alpha = 0 \).

Now note that the projectors \( \mathcal{P}_{16} \) and \( \mathcal{P}_8' \) have the following property:
\[
\mathcal{P}_{16} \mathcal{P}_8' = \mathcal{P}_8'.
\]  
Hence, \( \mathcal{P}_{16} \) acts as a unit matrix on \( \theta' = \mathcal{P}_8' \Theta \), i.e.
\[
\mathcal{P}_{16} \theta' = \theta',
\]  
(5.33)
and annihilates 16 of the 24 components of \( \nu' = (1 - \mathcal{P}_8') \Theta \), i.e.
\[
\mathcal{P}_{16} \nu' = \nu',
\]  
(5.34)
where \( \nu' \) has only 8 non-zero components and therefore \( \nu' = (1 - \mathcal{P}_{16}) \nu' \) has 16 non-zero components.

We thus find that in the light-cone kappa-symmetry gauge \( \Gamma^\varepsilon \Theta = 0 \), the type IIA superstring action takes the form
\[
S_{IIA} = -T \int \left[ -2 * dx^+ dx^- - \frac{\mu^2}{2} (r^2 + \cos^2 \alpha z^2) * dx^+ dx^- 
+ \frac{1}{2} (dr d\phi + *d\Sigma d\phi + *dy^\alpha d\Sigma^\alpha) - 2i * dx^+ \Theta \Gamma^{\varepsilon} d\Theta + 2i dx^+ \Theta \Gamma^{\varepsilon} \Gamma_0 d\Theta
+ \mu (1 + \cos \alpha) * dx^+ dx^- \theta' \Gamma^{\varepsilon} \Gamma^{-\varepsilon} y^7 \theta' + \mu (1 - \cos \alpha) * dx^+ \nu' \Gamma^{\varepsilon} \Gamma^{-\varepsilon} y^7 \nu'ight]
\]  
(5.35)

In view of the properties (5.33) and (5.34) of the \( \mathcal{P}_{16} \)-projector, it follows from the action (5.35) that in the light-cone gauge \( \Gamma^{\varepsilon} \Theta = 0 \), \( \theta' \) and \( \nu' \) each describe two massive physical fermionic degrees of freedom while the remaining fermions \( \nu' \), satisfying \( \mathcal{P}_{16} \nu' = 0 \), represent four massless degrees of freedom.

To compute the masses we again impose the light-cone gauge (5.23) and reduce the string action to the following form describing only physical degrees of freedom
\[
S = -\frac{1}{2} T \int dr d\phi (\partial_\tau \theta' + \partial_\phi \theta' + m^2 r^2 + m^2 \cos^2 \alpha z^2 + \partial_\phi \nu' \partial_\phi \nu')
+ iT \int dr d\phi [\theta' \Gamma^{-\varepsilon} \partial_\tau \theta' + \partial^2 \Gamma^{-\varepsilon} \partial_\phi \nu' + m (1 + \cos \alpha) \theta' \Gamma^{-\varepsilon} \Pi \theta'^2
+ \nu' \Gamma^{-\varepsilon} \partial_\phi \nu' + \partial^2 \Gamma^{-\varepsilon} \partial_\phi \nu' + m (1 - \cos \alpha) \nu' \Gamma^{-\varepsilon} \Pi \nu'^2 + \nu' \Gamma^{-\varepsilon} \partial_\phi \nu' + \partial^2 \Gamma^{-\varepsilon} \partial_\phi \nu' \nu'^2],
\]  
(5.36)
where (as in (5.24)) the fermions are appropriately re-scaled, \( m = \mu p^*/T \) and the matrix \( \Pi \) stands for \(-I^\nu\) or \(i^\nu \Gamma_{13}^\nu\), respectively, in the type IIA or IIB case. This action thus describes two bosonic modes \( r \) and \( z \) with masses

\[
m_r = m, \quad m_z = m \cos \alpha, \tag{5.37}
\]

6 massless bosonic modes \( \psi^\nu \), 2 two-component fermions \( \dot{\psi}^{1,2} \) with mass \( m_{\dot{\psi}} \) and 2 two-component fermions \( \tilde{\psi}^{1,2} \) with mass \( m_{\tilde{\psi}} \)

\[
m_{\dot{\psi}} = \frac{m}{2} (1 + \cos \alpha), \quad m_{\tilde{\psi}} = \frac{m}{2} (1 - \cos \alpha), \tag{5.38}
\]
as well as 4 massless fermionic degrees of freedom described by the four-component \( \dot{\psi}^{1,2} \).

As the masses of bosonic and fermionic degrees of freedom are different for \( \alpha \neq 0 \), there is no effective 2D supersymmetry. The masses still satisfy the 2D mass sum rule

\[
\sum m_r^2 - \sum m_z^2 = m_r^2 + m_z^2 - 2m_{\dot{\psi}}^2 - 2m_{\tilde{\psi}}^2 = 0. \tag{5.39}
\]

This is a manifestation of the UV finiteness of the corresponding GS action: the bosonic masses originate from the curvature and the fermionic ones come from the RR background and the two are related for a supergravity solution.

Let us note that the non-coset fermions \( \nu' \) get non-trivial masses due to the non-decoupling of the broken-supersymmetry fermions from the coset part of the theory (recall that \( \nu' \) and \( \dot{\psi}' \) are a linear combination of the original coset fermions \( \psi \) and non-coset \( \nu \)). This raises an issue of possible equivalence of the present GS superstring theory and the hybrid \( \text{AdS}_2 \times S^2 \times T^6 \) model of [23] in which the \( T^6 \) part is decoupled from the supercoset part\(^{31}\).

6. Toward exact solution of the quantum superstring sigma model

The string spectrum in the integrable AdS backgrounds, such as \( \text{AdS}_2 \times S^2 \) or \( \text{AdS}_3 \times \text{CP}^1 \) can be found using Bethe-ansatz techniques. Above we provided evidence that the superstring sigma model on \( \text{AdS}_2 \times S^2 \times T^6 \) is integrable, and we may thus expect that its spectrum is also described by a set of Bethe equations. Here, we propose such a system for part of the spectrum that roughly speaking corresponds to the massive modes in the near-BMN expansion of section 5.1, and is associated with the coset part of the supergeometry. As in the case of \( \text{AdS}_3 \times S^1 \times T^3 \) and similar backgrounds studied in [39], the presence of the flat directions along \( T^6 \) and their massless superpartners constitutes an obstacle for direct application of integrability methods, which work best for massive theories where one can define scattering states and the \( S \)-matrix. Nevertheless, we will propose a set of quantum Bethe equations, from which we will easily reconstruct the massive part of the BMN spectrum in section 5.1. The same equations cannot describe the BMN spectrum in section 5.2 because of the mixing between coset and non-coset degrees of freedom in this latter case.

\(^{31}\)The hybrid model constructed in [23] consists of two completely decoupled sectors, a free worldsheet \( N = 2 \) superconformal theory on \( T^6 \) with six worldsheet spinor fermions and a \( \mathcal{FSU}(1,1;1/2) \) supercoset sigma model (in addition to the ghosts fields). In the hybrid model, the action for the supercoset sector differs from the GS supercoset action (4.17) by an additional \( J_1J_3 \) term leading to a second-order kinetic term for the coset fermions \( \psi' \). This additional term breaks kappa-symmetry but makes the worldsheet theory \( N = 2 \) superconformal invariant. The definition of the BMN limit involving a \( T^6 \) direction is not obvious in this case. It is not clear which condition on the coset fermions of the hybrid model may play the role of the light-cone kappa-symmetry gauge which reduced the GS superstring in the BMN limit to a free theory. In addition, in the BMN limit, the six non-supercoset worldsheet fermions of the hybrid model always remain massless, while two of their GS counterparts acquire mass in the generic case (see equation (5.36)). Thus, the relation between the hybrid model of [23] and the GS superstring on \( \text{AdS}_2 \times S^2 \times T^6 \), in which the \( \text{AdS}_2 \times S^2 \) and \( T^6 \) sectors do not in general decouple, remains an open issue even in the BMN limit.
Our starting point here is a restricted set of the equations of motion, with the $T^0$ directions and broken-supersymmetry fermions set to zero. As shown in section 4, this is a consistent truncation of the full theory. We can apply integrability methods to this restricted set of string configurations and derive the classical version of the Bethe equations, which describe finite-gap solutions of the sigma model [54] (see [55] for a review). A quantum counterpart of the finite-gap equations can then be reconstructed using a structural analogy with the higher-dimensional AdS models [33, 39, 56, 57]. By construction, the resulting quantum Bethe equations do not capture the massless $T^0$ modes of the string, but we conjecture that they correctly describe the massive sector of the spectrum, which may still form a closed subset of states due to integrability32.

The classical Bethe equations for the $PSU(1, 1|2)/SO(1, 1) \times U(1)$ coset were derived in [33]. We will first review this construction, and then propose a set of quantum Bethe equations that have the right classical limit and are structurally similar to the Bethe equations for AdS$_5$/CFT$_4$ [57] and AdS$_6$/CFT$_3$ [58].

6.1. Classical finite gap equations for the $PSU(1, 1|2)/SO(1, 1) \times U(1)$ supercoset model

The starting point of the finite-gap method is the Lax-pair representation for the equations of motion. We do not understand at the moment how to include the Abelian directions along $T^0$ and the non-supercoset fermions (entering the Lax connection (2.15)) in the finite-gap integration scheme, and will thus concentrate on the AdS$_2$ dimension. By construction, the resulting quantum Bethe equations can then be reconstructed using a structural analogy with the higher-dimensional AdS models [33, 39, 56, 57]. The string action then reduces to the $PSU(1, 1|2)/SO(1, 1) \times U(1)$ coset sigma model (4.17) to which we can apply the general procedure outlined in [33, 39].

The equations of motion of the $PSU(1, 1|2)/SO(1, 1) \times U(1)$ sigma model follow from the flatness condition for the Lax connection

$$L_j = J_{j0} + \frac{x^2 + 1}{x^2 - 1} J_{j2} - \frac{2x}{x^2 - 1} \frac{1}{\sqrt{-h}} h_{jk} \epsilon^{kl} J_{l2} + \sqrt{\frac{x + 1}{x - 1}} J_{j1} + \sqrt{\frac{x - 1}{x + 1}} J_{j3}. \quad (6.1)$$

The monodromy of the Lax connection defines the generating function for an infinite set of conserved charges:

$$M(x) = \exp \left( \oint \frac{dz}{i} L_j(\xi; z) \right). \quad (6.2)$$

32 Let us draw an analogy with the AdS$_5$/CFT$_4$ case. One can reconstruct the quantum Bethe ansatz for the $su(2)$ sector by studying classical strings restricted to the $R^1_{t|\mu} \times S^1$ subspace of AdS$_5 \times S^5$ [54] and then discretizing the resulting finite-gap equations [56]. A restriction to a particular set of classical configurations does not make sense in quantum theory, but at the level of Bethe equations, such a restriction is possible due to the underlying separation of variables.

33 The Virasoro conditions for the bosonic string in AdS$_n \times S^n \times T^{10-2n}$ can be written as the vanishing of the sum of the stress tensor components $T_{\pm\pm}(AdS_n) + T_{\pm\pm}(S^n) + T_{\pm\pm}(T^{10-2n}) = 0$. Each of the three classical stress tensors is separately traceless and conserved and thus $T_{\pm\pm}$ are functions of $\sigma \pm \tau$. In the case of $n = 5$ when there is no toroidal part we can make, say, $T_{\pm\pm}(AdS_5) = \mu \pm \text{const}$ by a conformal transformation $\sigma \pm \tau \rightarrow f_\pm(\sigma \pm \tau)$, and then we will have $T_{\pm\pm}(S^5) = -\mu \pm \text{const}$ as a consequence of the Virasoro condition. The classical solutions of the AdS$_5$ and $S^5$ sigma models satisfying $T_{\pm\pm} = \text{const}$ can then be described using the finite gap construction [54, 59]. For $n < 5$ for generic string motions, the stress tensor of the toroidal part is non-zero. The remaining freedom of conformal transformations then no longer allows us to make both $T_{\pm\pm}(AdS_5)$ and $T_{\pm\pm}(S^n)$ constant. We may make one of them constant or make their sum constant (by setting $T_{\pm\pm}(T^{10-2n}) = \mu \pm \text{const}$ but then the standard finite-gap construction for the AdS$_5$ and $S^5$ sigma models will not directly apply, as the finite-gap solutions require $T_{\pm\pm} = \text{const}$ for each of the two factors separately. The classical integrability of strings in AdS$_n \times S^n \times T^{10-2n}$ should allow one to construct solutions that are more general than the finite gap ones and are parameterized by additional holomorphic functions representing generic initial data (the values of $T_{\pm\pm}$ may be interpreted as the specifying part of the initial data). This should be closely related to an integrability-based solution of the Cauchy problem for classical coset-space sigma models, which is an interesting open problem.
More precisely, the generating functions of the conserved charges are quasimomenta, the eigenvalues of the monodromy matrix:
\[ M(x) = U^{-1} \exp(p_l(x) H_l) U, \quad (6.3) \]
where \( H_l \) are the Cartan elements of \( psu(1, 1|2) \).

Although the monodromy matrix itself is an analytic function of the spectral parameter (except for an essential singularity at \( x = 0 \)), its eigenvalues are not. In the defining supermatrix representation, \( M(x) \) is a \( 4 \times 4 \) supermatrix. The Cartan generators are diagonal matrices, and thus to find quasimomenta we need to diagonalize \( M(x) \). The eigenvalues of a \( 4 \times 4 \) matrix are solutions of an algebraic equation of degree 4 and are thus defined on an algebraic curve, a fourfold cover of the complex \( x \) plane. Indeed, if two eigenvalues coincide at some point \( x_\ast \), encircling this point in the complex plane will result in their permutation. Therefore, the eigenvalues of \( M(x) \) may have branch points with the monodromy in the permutation group. It is convenient to give a more abstract, group-theoretic characterization of possible monodromies, in terms of quasimomenta. The quasimomenta, by definition, parameterize the conjugacy class of the monodromy matrix viewed as a group element of \( PSU(1, 1|2) \). The set of conjugacy classes of a (super)Lie group is its maximal torus divided by the Weyl group. However, the quasimomenta defined in (6.3) map the conjugacy class of \( M(x) \) to the Cartan subalgebra of \( psu(1, 1|2) \). The map from the maximal torus/Weyl group to the Cartan algebra is a multivalued map, and consequently \( p_l(x) \) are multivalued functions of \( x \). We may make \( p_l(x) \) single-valued on the complex plane with cuts \( C_{l,i} \). As soon as we cross the cut \( C_{l,i} \) (or, better, once we encircle one of its endpoints), the quasimomentum \( p_l(x) \) undergoes a shift by \( 2\pi n_{l,i} \) (because the set of conjugacy classes is a torus) and a transformation from the Weyl group \( p_l(x) \rightarrow p_l(x) - A_{l,m} p_m(x) \) (because we need to mod out by Weyl transformations). Here, \( A_{l,m} \) is the Cartan matrix of \( psu(1, 1|2) \).

Additional constraints on the quasimomenta comes from the \( Z_4 \)-symmetry of the coset, which acts on the Cartan generators of \( psu(1, 1|2) \) as
\[ \Omega(H_l) = H_m S_{ml}. \quad (6.4) \]
The constraints of analyticity and the \( Z_4 \)-symmetry result in the following integral representation for the quasimomenta \[39]:
\[ p_l(x) = -\frac{\kappa_l x}{x^2 - 1} + \int dy \frac{\rho_l(y)}{x - y} - S_{lm} \int \frac{dy}{y^2} \frac{\rho_m(y)}{x - \frac{1}{y}}, \quad (6.5) \]
where the densities \( \rho_l(x) \) are defined on the cuts \( C_{l,i} \) and are determined by a set of singular integral equations:
\[ A_{l,m} - \int dy \frac{\rho_m(y)}{x - y} - A_{ik} S_{km} \int \frac{dy}{y^2} \frac{\rho_m(y)}{x - \frac{1}{y}} = A_{ik} \frac{\kappa_l x}{x^2 - 1} + 2\pi n_{l,i}, \quad x \in C_{l,i}. \quad (6.6) \]
The constants \( \kappa_l \) could in principle be extracted from the semiclassical analysis of the auxiliary linear problem for the Lax operator, but they are also constrained by group theory, which will be sufficient to determine them uniquely in our case. The constraints are
\[ S_{ik} \kappa_k = -\kappa_l, \quad \kappa_l A_{ik} \kappa_k = 0. \quad (6.7) \]
For the \( PSU(1, 1|2) \) coset \[33],
\[ A = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \kappa = \text{const} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \quad (6.8) \]
and the finite-gap integral equations take the form

\[ 2\pi n_{1,i} = \int dy \, \rho_2(y) K(x, y), \]

\[ \frac{2kx}{x^2 - 1} + 2\pi n_{2,i} = 2 \int dy \, \rho_2(y) K(x, y) - \int dy \, \rho_1(y) K(x, y) - \int dy \, \rho_3(y) K(x, y), \]

\[ 2\pi n_{3,i} = \int dy \, \rho_2(y) K(x, y), \]  \hspace{1cm} (6.9)

with

\[ K(x, y) = \frac{1}{x - y} + \frac{1}{y^2} \frac{1}{x - \frac{1}{y}}. \]  \hspace{1cm} (6.10)

These equations are schematically depicted in figure 1.

The equation for the bosonic \( \rho_2(x) \) density alone (with \( \rho_1, \rho_3 \) set to zero) describes classical strings on \( S^2 \times R_\text{time} \) [60], as it should.

The finite-gap equations characterize classical solutions of the sigma model in terms of the spectral data for the Lax operator. The inverse-scattering transformation allows, in principle, to reconstruct any given solution from its spectral data, but for the most part we are interested in the conserved charges that the solution carries, and those can be computed from the quasimomenta.

### 6.2. Proposal for asymptotic Bethe ansatz equations for the supercoset sector of states

We can now quantize the above classical Bethe equations using the rules of [56, 57], formulated for general \( \mathbb{Z}_4 \)-cosets in [33]. In quantum theory, the cuts are composed of discrete Bethe roots that satisfy a set of functional equations, the Bethe equations. The finite-gap equations constitute their classical limit, in the situation where the number of roots is large, and the distance between nearby roots is small such that their distribution can be characterized by a continuous density.

For strings in \( AdS_5 \times S^5 \), it was possible to reconstruct the quantum Bethe equations from their classical counterpart with the help of extra information from the underlying spin chain [57]. The same approach worked also for the \( AdS_4 \times CP^3 \) background [58]. In the present case, the spin chain description of the dual CFT is not known, but one may observe that the set of rules to convert classical finite-gap equations to quantum Bethe equations follows a uniform model-independent pattern. The same pattern arises in the case at hand, since the structural elements that appear in the finite-gap equations are similar. We may thus apply the same set of rules to convert the classical finite-gap equations into the quantum Bethe equations.

The basic variable of quantum Bethe ansatz is the rapidity \( \eta \), which is related to the spectral parameter by the Zhukowsky transformation

\[ x + \frac{1}{x} = \eta. \]  \hspace{1cm} (6.11)
The dependence on the sigma model coupling enters through the shifted variables
\[ x^\pm + \frac{1}{x^\pm} = u \pm i h, \] (6.12)
where 'the Planck constant' \( h \) is a function of the sigma model coupling, \( (T R^2)^{-1} = \frac{2\pi \alpha'}{\hbar} \) (with \( R \) being the AdS radius) which is not determined by the integrability of the model; we can only say that \( h \) coincides with the coupling up to possible higher-order \( \alpha' \) corrections, i.e. \( h = \frac{2\pi \alpha'}{\hbar} + O(\alpha'^2) \).

The dependence on the sigma model coupling enters through the shifted variables
\[ x^\pm + \frac{1}{x^\pm} = u \pm i h, \] (6.12)
where 'the Planck constant' \( h \) is a function of the sigma model coupling, \( (T R^2)^{-1} = \frac{2\pi \alpha'}{\hbar} \) (with \( R \) being the AdS radius) which is not determined by the integrability of the model; we can only say that \( h \) coincides with the coupling up to possible higher-order \( \alpha' \) corrections, i.e. \( h = \frac{2\pi \alpha'}{\hbar} + O(\alpha'^2) \).

The Zhukowsky variables determine the momentum and energy of a single worldsheet excitation:
\[ e^{ihp} = \frac{x^+}{x^-}, \quad \hbar E = \frac{i}{x^+} - \frac{i}{x^-} + 1. \] (6.13)

The rapidities of individual excitations in a multiparticle state satisfy a set of Bethe equations:
\[ 1 = \prod_{k} \frac{x_{1,j} - x_{2,k}^+}{x_{1,j} - x_{2,k}^-} \frac{1 - \frac{1}{x_{1,j}x_{2,k}}}{1 - \frac{1}{x_{2,k}x_{1,j}}}, \] (6.14)
\[ \left( \frac{x_{1,j} - x_{2,k}^+}{x_{1,j} - x_{2,k}^-} \right)^L = \prod_{k \neq j} \frac{x_{1,j} - x_{2,k}^+}{x_{1,j} - x_{2,k}^-} \frac{1 - \frac{1}{x_{2,k}x_{1,j}}}{1 - \frac{1}{x_{1,j}x_{2,k}}} \sigma_{BES}(u_{j,2}, u_{k,2}) \times \prod_{k} \frac{x_{2,j}^- - x_{1,k}}{x_{2,j}^- - x_{1,k}} \frac{1 - \frac{1}{x_{2,j}x_{1,k}}}{1 - \frac{1}{x_{1,k}x_{2,j}}} \prod_{k} \frac{x_{2,j}^- - x_{3,k}}{x_{2,j}^- - x_{3,k}} \frac{1 - \frac{1}{x_{2,j}x_{3,k}}}{1 - \frac{1}{x_{3,k}x_{2,j}}}, \]

The BES/BHL scattering phase \( \sigma_{BES}(u, v) \) [61, 62] (see [63] for a review) is a fairly complicated function of the rapidities. It admits the following integral representation [64]:
\[ \sigma_{BES}(u, v) = \exp \left( -i \sum_{r,s=\pm} \chi(x^r, y^s) \right), \] (6.15)
\[ \chi(x, y) = \frac{i}{4\pi^2} \oint_{|z|=1, |w|=1} \frac{dz \, dw}{(x - z)(y - w)} \ln \frac{\Gamma \left( 1 + \frac{i}{2\pi} \left( z + \frac{1}{z} - w - \frac{1}{w} \right) \right)}{\Gamma \left( 1 - \frac{i}{2\pi} \left( z + \frac{1}{z} - w - \frac{1}{w} \right) \right)}. \]

The above Bethe equations are supposed to describe the spectrum of the string in the light-cone gauge defined by the geodesic of a massless particle spinning around \( S^5 \). The parameter \( L \) that enters the left-hand side of the equations is the angular momentum of the ground state \( (L = J) \). Once \( x^\pm_{2,k} \) are determined, the light-cone energy of the string states for a given collection of Bethe roots is computed by
\[ E = \frac{1}{\hbar} \sum_{k} \left( \frac{i}{x^+_k} - \frac{i}{x^-_k} + 1 \right), \] (6.16)
\[ \prod_{k} \frac{x^+_k}{x^-_k} = e^{ihE} \equiv 1. \] (6.17)

The physical states should satisfy the level-matching (zero-momentum) condition:
\[ \prod_{k} \frac{x^+_k}{x^-_k} = e^{ihE} \equiv 1. \] (6.17)

Only the roots on the middle node of the Dynkin diagram \( (u_2 \text{ roots}) \) carry energy and momentum. The other two types of roots are auxiliary. They just change the flavor composition

34 The full AdS energy also contains the ‘vacuum’ \( L = J \) term.
of the state. These roots are fermionic in the sense that adding an odd number of $u_1$ and $u_3$ roots produces a fermion.

We should stress that the above Bethe ansatz equations have not been derived from first principles but rather conjectured, building upon structural analogy with other AdS backgrounds. It is a straightforward, albeit lengthy, exercise to show that in the classical limit $\hbar \to 0$, the quantum Bethe equations (6.14) reduce to the classical finite-gap equations (6.9), with the densities defined by

$$\rho_l(x) = 2\hbar \sum_k \frac{x_{l,k}^2}{x_{l,k}^2 - 1} \delta(x - x_{l,k}).$$

(6.18)

Hence, the conjectured equations correctly capture the classical spectrum of the string (this however by construction). It would be important to test them beyond the classical approximation.

Let us check that the Bethe equations (6.14) reproduce the BMN spectrum from section 5.1. Their simplest solution is a solitary $u_2$ root. According to (6.16), (6.17) and (6.12), at $\hbar \to 0$, it describes a worldsheet excitation with the momentum and energy

$$p = \frac{2x_2}{x_2^2 - 1}, \quad \varepsilon = \frac{x_2^2 + 1}{x_2^2 - 1},$$

(6.19)

which in fact parameterize the dispersion relation of a relativistic particle with mass $m^2 = 1$. The Bethe equations reduce to the quantization condition for the momentum: $p = 2\pi n / \hbar L$. This is consistent with the light-cone quantization, in which the length of the string is given by its c.o.m. momentum in the target space: length $= 2\pi \alpha'^2 L / R^2 = L \hbar$.

The Bethe equations admit solutions with the $u_1$ and/or $u_3$ roots added to the solitary $u_2$ root. The Bethe equations determine the positions of the auxiliary roots:

$$x_{1,3} = \frac{x_1^2 + x_3^2}{x_1^2 x_3^2 + 1} \approx \frac{2x_2}{x_2^2 + 1}.$$  

(6.20)

These roots change the quantum numbers of the string state without changing its energy. The 1–2 and 2–3 complexes are fermions and the 1–2–3 complex is a boson. The single type-2 root describes the transverse mode of the string on $S^2$, the 1–2–3 stack describes the AdS$_2$ mode, and the two fermionic solutions correspond to the two coset fermions that survive the kappa-symmetry gauge fixing. We thus reconstruct the $2b + 2f$ massive BMN modes but are obviously missing the $6b + 6f$ massless modes, which correspond to the string fluctuations in the $T^6$ directions and their superpartners. At the moment, we do not understand how to include these modes in the Bethe ansatz framework.

7. Remarks on semiclassical strings in AdS$_2 \times S^2 \times T^6$

As in the AdS$_5 \times S^5$ case, one may try to check the above Bethe ansatz-based solution for the string spectrum against direct superstring predictions in the limit of large (semiclassical) values of charges.

Already at the string tree level, one may compare the corresponding (Landau–Lifshitz-type) effective action following, for coherent states, from the ‘one-loop’ spin chain Hamiltonian formally associated with the Bethe ansatz in (6.14) to the corresponding ‘fast-string’ ($J \gg 1$) limit of the string action. The two are known to match [65] in the AdS$_5 \times S^5$ case due to a special non-renormalization property of the leading-order correction in the large $J$ limit.

For example, let us consider classical strings moving in the $S^2$ subspace. Following the discussion in section 5.2 of [66] and eliminating ‘fast’ angular coordinates from the phase-space string action on $R_6 \times S^2$, one finds that it reduces to the same Landau–Lifshitz model
that is found [65] for ‘slow’ coordinates of a fast-moving string in $R \times S^3$. In the $R_i \times S^3$ case, however, the two degrees of freedom are the coordinate and momentum of the ‘slow’ degree of freedom of $S^3$. The same action is known also to emerge from the ferromagnetic $XXX_{1/2}$ spin chain Hamiltonian (the one-loop Hamiltonian in the $su(2)$ sector of $\mathcal{N} = 4$ SYM theory).

This coincidence is, in fact, in agreement with the structure of the Bethe ansatz proposed in the previous section: in the formal weak-coupling limit ($\hbar = \frac{2\pi \alpha'}{\ell} + \cdots \to \infty$) the central-node part of the Bethe equations (6.14) takes the same form as for the $su(2)$ Heisenberg spin chain (or rather the $SO(3)$ subsector of the one-loop $SO(6)$ spin chain)\(^{35}\): $u_j$ and $x_j$ are then of order $\hbar$, $x_j^\pm = u_j \pm i \hbar$, the BES phase disappears, and only the first term on the r.h.s. of the second equation in (6.14) contributes. This implies that the same Landau–Lifshitz model should indeed emerge as a description of the coherent long-wavelength spin-wave states\(^{36}\).

At the one-loop string level, one may carry out a comparison between the string theory predictions for the one-loop corrections to energies of semiclassical strings moving in the $\text{AdS}_2 \times S^2$ part and the predictions of the Bethe ansatz equations (6.14). One may attempt to do this in general following the approach used in the $\text{AdS}_5 \times S^5$ and $\text{AdS}_4 \times \text{CP}^3$ cases [67–69]. This should provide, in particular, a check of the BES phase factor in (6.14).

It is useful also to look at specific examples of semiclassical quantization of the simplest string solutions. Let us consider strings moving only in the $\text{AdS}_2 \times S^2$ part with the metric given in (5.3). The corresponding classical solutions can be reconstructed, via Pohlmeyer reduction [29, 70], from solutions of the $\mathcal{N} = 2$ supersymmetric sine–Gordon theory (whose bosonic part is the direct sum of the sine-Gordon and sinh-Gordon models). The standard ‘vacuum’ (BMN) solution, i.e. the massless geodesic wrapping a big circle of $S^2$ is $t = \varphi = \nu \tau$, $\psi = \frac{J}{FR}$, $E = J$. It leads (as discussed in section 5.1) to the small-fluctuation spectrum containing two massive ($m = \nu$) bosonic modes, two massive fermionic modes and 6+6 massless modes.

In $\text{AdS}_2 \times S^2$ there is no place for rigid circular string solutions described by rational functions so the next class of solutions in terms of simplicity are rigid spinning or pulsating strings described by elliptic functions. One example is the giant magnon [71] (an open string spinning in $S^2$ with its ends on a big circle) which of course has the same classical dispersion relation as in the $\text{AdS}_2 \times S^3$ case. The corresponding small fluctuation spectrum can be analyzed as in [72, 73], now starting with the $\text{AdS}_2 \times S^2 \times T^6$ superstring action. Instead of four massive bosonic fluctuations from $\text{AdS}_3$ one finds one from $\text{AdS}_2$ with the same stability angle $\nu_1$; instead of four bosonic fluctuations from $S^5$ one gets one from $S^2$ with the same stability angle $\nu_2 = \nu_1 + 2\text{arccot} \ k$ (where $k$ is the spatial 2D momentum number); instead of eight massive fermionic modes, one gets two massive fermions from the supercoset part of the action with $\nu_1 = \nu_1 + \text{arccot} \ k$. In addition, there are six decoupled massless $T^6$ modes and six massless fermions (after $\kappa$-gauge fixing). The ‘non-coset’ fermions do not get mass from the RR coupling term in the quadratic fermionic action (due to the structure of the projector in (3.4) or (3.19)) while the induced connection term in the covariant derivative (2.2) can be rotated away. As a result, the one-loop correction to the string energy (determined by $\nu_1 + \nu_2 = 2\nu_1$) vanishes as in the $\text{AdS}_3 \times S^3$ case [73].

\(^{35}\)Here the role of the second spin component of the $su(2)$ sector or number of impurities is played by the string oscillation number.

\(^{36}\)Here, we compare the (i) Landau–Lifshitz model that comes out of the string phase-space classical action after isolating a fast degree of freedom and (ii) the Landau–Lifshitz model that comes out of a spin chain Hamiltonian that is suggested to exist by the form of the postulated BA, assuming we formally take the weak coupling limit there. The matching is then exactly as in the $\text{AdS}_3 \times S^3$ case where we may consider the $O(3)$ part of the one-loop $O(6)$ subsector of spin chain Hamiltonian, and the matching is effectively due to supersymmetry protection. It is unclear how such Hamiltonian may come out of the dual CFT in the present $\text{AdS}_2 \times S^2$ case.
Another simple elliptic solution is a folded string rotating around its c.o.m. in $S^3$ [74]:

$$t = \kappa \tau, \quad \theta = \theta(\sigma), \quad \varphi = w \tau, \quad \theta' = \kappa^2 - w^2 \sin^2 \theta, \quad (7.1)$$

$$\sin \theta = \sqrt{q} \sin(w \sigma |q), \quad q = \sin^2 \theta_0 = \frac{\kappa^2}{w^2}, \quad w = \frac{2}{\pi} K(q), \quad (7.2)$$

where $K$ is the elliptic integral. In the case of $\text{AdS}_n \times S^q \times T^{10-2n}$ with $n = 5, 3, 2$ the corresponding quadratic fluctuation spectrum can be found by imposing the static gauge on the fluctuations and turns out to be a simple truncation of the $\text{AdS}_n \times S^5$ spectrum found in [75]. It is described by a combination of massive bosonic and fermionic 2D modes with the following degeneracy $\times (\text{mass})^2$:

\begin{align*}
\text{Bosons} : \quad \text{AdS}_n : \quad & (n - 1) \times \kappa^2; \\
& S^n : \quad 1 \times \kappa^2 \left[ 1 - \frac{2(k^2 - w^2)}{\kappa^2 - w^2 \sin^2 \theta} \right]; \quad (n - 2) \times (2w^2 \sin^2 \theta - \kappa^2) \\
& T^{10-2n} : \quad (10 - 2n) \times 0.
\end{align*}

(7.3)

The resulting mass sum rule that checks the one-loop UV finiteness of the GS string in the static gauge is then universal in $n$ (see [75]):

$$\sum (m_B^2 - m_F^2) = 2 \left( \kappa^2 - w^2 \sin^2 \theta - \kappa^2 \frac{\kappa^2 - w^2}{\kappa^2 - w^2 \sin^2 \theta} \right) = \sqrt{-g} R^{(2)}. \quad (7.4)$$

Here the coset part decouples from the torus part of the model. This is due, in particular, to the possibility of rotating away the connection in the covariant derivative in the fermionic part, i.e. the masses come solely from the RR flux term coupling in the quadratic fermionic action. The resulting one-loop correction to the folded string energy in $\text{AdS}_2 \times S^2 \times T^6$ can be computed by a direct combination of the expressions given in [75].

A similar analysis can be carried out for the circular pulsating solution on $S^5$ [76] which is described in conformal gauge by (cf. (7.2))

$$t = \kappa \tau, \quad \theta = \theta(\tau), \quad \varphi = m \sigma, \quad \theta' = \kappa^2 - w^2 \sin^2 \theta, \quad (7.5)$$

$$\sin \theta = \sqrt{q} \sin(m \tau |q), \quad q = \sin^2 \theta_0 = \frac{\kappa^2}{m^2}, \quad w = \frac{2}{\pi} K \left( \frac{\kappa}{m} \right). \quad (7.6)$$

For a superstring in $\text{AdS}_n \times S^q \times T^{10-2n}$ ($n = 5, 3, 2$) the corresponding quadratic fluctuation spectrum in the static gauge is again a truncation of the $\text{AdS}_5 \times S^5$ spectrum in [75]:

\begin{align*}
\text{Bosons} : \quad \text{AdS}_n : \quad & (n - 1) \times \kappa^2; \\
& S^n : \quad 1 \times \kappa^2 \left( 1 - \frac{2}{\sin^2 \theta} \right); \quad (n - 2) \times (\kappa^2 - 2m^2 \sin^2 \theta) \\
& T^{10-2n} : \quad (10 - 2n) \times 0.
\end{align*}

Fermions : \quad $2(n - 1) \times (\kappa^2 - m^2 \sin^2 \theta); \quad (10 - 2n) \times 0. \quad (7.7)$

Again, we get a universal mass sum rule that checks the UV finiteness (see [75])

$$\sum (m_B^2 - m_F^2) = 2 \left( m^2 - \frac{\kappa^2}{\sin^2 \theta} \right) = \sqrt{-g} R^{(2)}. \quad (7.8)$$

As in the previous examples, the masses come only from the RR coupling term and the toroidal part decouples\footnote{From the algebraic curve point of view, the one-loop fluctuation frequencies will come only from the supercoset part.} and does not contribute to the one-loop correction to the string energy.
Given that the Bethe ansatz equations (6.14) were constructed on the basis of the supercoset part of the string model (i.e. under the assumption of the decoupling of the non-supercoset parts), this guarantees the agreement between their predictions and the direct superstring predictions for the one-loop energies of the above solutions. This agreement should also extend to the leading strong coupling finite size (TBA) generalization of the asymptotic Bethe ansatz equations as there should be a direct analog of the analysis in [77]. One may also hope that higher-order (two-loop, etc) string corrections to energies of semiclassical string solutions in AdS$_2$ × S$^2$ will not be sensitive to contributions of massless toroidal modes. A priori, this may not apply to the full quantum string spectrum which may be sensitive to finite-size corrections due to massless modes.

Let us now comment on the case of semiclassical strings moving also in the toroidal part. In general, the radii $r_i$ of $T^6$ (or, more generally, its constant metric) are free parameters of the model, in addition to the radius $R$ of AdS$_2$ and S$^2$. Thus, by varying $r_i / R$ we may suppress or enhance the contributions of configurations in which the string is moving in $T^6$. For example, we may generalize the above solutions to the case when the c.o.m. of the string moves in $S^2$ of $T^6$, $y \equiv \gamma \psi$, $\psi = p \tau$. That will generalize the energy relation $E = J$ we had for the BMN state in S$^2$ as follows:

$$\kappa^2 = v^2 + \gamma^2 p^2, \quad \text{i.e.} \quad E^2 = J^2 + \gamma^2 p^2, \quad \gamma \equiv \frac{r}{R}. \quad (7.9)$$

The corresponding fluctuation spectrum was already discussed in section 5.2. While the bosonic modes from $T^6$ remain massless, two of the non-supercoset-like fermions $\nu'$ get non-zero mass, i.e. the coset and non-coset sectors no longer decouple.

This non-decoupling will also occur for extended string solutions, e.g., a simple circular spinning string solution constructed by allowing the string to wind around a big circle of S$^2$ as well as around a circle in $T^6$ ($n$ and $k$ are integer winding numbers):

$$t = \kappa \tau, \quad \theta = \frac{\pi}{2}, \quad \varphi = w \tau + n \sigma, \quad \psi = p \tau - k \sigma, \quad (7.10)$$

Here again the non-coset fermions will get non-trivial masses and give a non-trivial contribution to the one-loop correction to the energy.

Imposing the conformal gauge, the general $T^6$ solution can be written as $y_{\mu'} = u_{\mu'}(\sigma_+) + \bar{u}_{\mu'}(\sigma_-)$, $\sigma_+ = \sigma + \tau$, so that $T_{\mu'}(T^6) = u^{2}(\sigma_+)$, $T_{-}(T^6) = \bar{u}^{2}(\sigma_-)$. We may fix the residual conformal diffeomorphisms by assuming that $T_{++}(\text{AdS}_2) = T_{--}(\text{AdS}_2) = -\mu^2 = \text{const}$. Then, the Virasoro conditions $T_{\pm\pm}^{(\text{tor})} = 0$ imply that $T_{++}(S^2) = (\mu^2 - u^2) \equiv h^2(\sigma_+)$, $T_{--}(S^2) = (\mu^2 - \bar{u}^2) \equiv h^2(\sigma_-) \quad (7.11)$

Since the $T_{\pm\pm}(S^2)$ components are now non-constant, as was already mentioned in section 6, in the case when the string moves in $T^6$ one cannot describe the corresponding AdS$_2$ × S$^2$ × $T^6$ solutions using the finite gap construction.

Let us mention that a related problem also appears when one tries to apply the Pohlmeyer reduction approach to describe such string solutions. Instead of the sinh-Gordon theory plus the standard sine-Gordon theory as found in the AdS$_2$ × S$^2$ case, one now end up with

$$L = \partial_\tau \chi \partial_\varphi \chi - \frac{\mu^2}{2} \cosh 2 \chi + \partial_\tau \varphi \partial_\varphi \varphi + \frac{1}{2} h(\sigma_+) \bar{h}(\sigma_-) \cos 2 \varphi. \quad (7.12)$$

In the equation for $\varphi$, one can formally replace $h(\sigma_+) \bar{h}(\sigma_-)$ by a constant performing a conformal redefinition of $\sigma_\pm$. The result will be the sine-Gordon equation on a complicated 2D domain, determined by functions that parameterize the $T^6$ solution.
8. Conclusion

In this paper, we considered GS superstrings in AdS$_2 \times S^2 \times T^6$ backgrounds that represent consistent embeddings of AdS$_2 \times S^2$ spacetime into critical string theory. We have shown that the full 10D structure of the superstring theory cannot be captured just by its effective AdS$_2 \times S^2$ ‘supercoset’ part described by an integrable sigma model on $PSU(1,1|2)/SO(1,1) \times U(1)$, though the latter is, indeed, a consistent classical truncation of the full theory. Nevertheless, we have provided direct evidence for the classical integrability of the complete theory by constructing the Lax representation of its equations of motion to the second order in fermions.

In general, the $T^6$ sector does not decouple from the AdS$_2 \times S^2$ one due to non-zero RR flux components along the $T^6$ directions which couple derivatives of the $T^6$ coordinates to the GS fermions. We illustrated this in the example of the second BMN limit in section 5.2. Still, for strings moving only in the AdS$_2 \times S^2$ subspace the decoupling does take place at the one-loop level (section 7) and that may extend to all orders in the semiclassical (large-charge) expansion.

For the part of the string spectrum which is associated with the supercoset sector we have proposed Bethe equations that are of the asymptotic Bethe ansatz type. They are supposed to describe quantum states with sufficiently large quantum numbers up to exponential corrections. In other words, they describe the string sigma model on an infinite plane rather than on a cylinder (the length of the string in the light-cone gauge is proportional to its light-cone momentum, and if the light-cone momentum is large, the internal length of the string goes to infinity). Application of a more general TBA or Y-system framework (see [78] for a review) to this theory will likely require understanding the massless $T^6$ modes and the non-coset fermionic excitations of the string that we have ignored in our analysis of the Bethe equations.

The super-AdS$_2 \times S^2$ background is one of the few GS-type cosets, which are simultaneously integrable and conformal. Other possible candidates that contain an AdS$_2$ factor are AdS$_2 \times S^2 \times S^2$ and AdS$_2 \times S^3$ [30]. It would be interesting to find critical string backgrounds which contain these cosets as consistent truncations. It would also be of interest to verify if the fermionic T-duality of the supercoset model [26] persists in the presence of the non-supercoset fermions.

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Appendix A. Main notation and conventions

We assume the metric to have the ‘almost plus’ signature $(-, +, \cdots, +)$. Generically, the tangent space vector indices are labeled by letters from the beginning of the Latin alphabet, while letters from the middle of the Latin alphabet stand for curved (world) indices. The spinor
indices are labeled by Greek letters. The direct product $\text{AdS}_2 \times S^2 \times T^6$ is parameterized, respectively, by the coordinates $x^m (m = 0, 1)$, $x^\hat{m} (\hat{m} = 2, 3)$ and $y^{m'} (m' = 4, 5, 6, 7, 8, 9)$. Its vielbeins are, respectively, $e^a = dx^m e_m^a(x)$ ($a = 0, 1$), $e^\delta = dx^{\hat{m}} e_{\hat{m}}^{\delta}(x)$ ($\hat{a} = 2, 3$) and $e^\rho (y) = dy^\rho$. The $\text{AdS}_2$ curvature is

$$R_{abc}^\bar{d} = \frac{2}{R^2} \eta_{[a} \delta_{b]}^\bar{d}, \quad R^{ab} = -\frac{1}{R^2} e^a e^b,$$  \hspace{1cm} (A.1)

where $R$ is the $\text{AdS}_2$ radius, and the $S^2$ curvature is

$$R_{abc}^\bar{d} = \frac{2}{R^2} \eta_{[a} \delta_{b]}^\bar{d}, \quad R^{ab} = \frac{1}{R^2} e^a e^b.$$  \hspace{1cm} (A.2)

The $D = 4$ gamma-matrices in $\text{AdS}_2 \times S^2$ are

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}, \quad \eta^{ab} = \text{diag}(-, +, +, +), \quad a = (a, \hat{a})$$  \hspace{1cm} (A.3)

The charge conjugation matrix $C$ is antisymmetric, the matrices $(\gamma^a)_{a\bar{a}} \equiv (C \gamma^a)_{a\bar{a}}$ and $(\gamma^{a\bar{a}})_{a\bar{b}} \equiv (C \gamma^{a\bar{a}})_{a\bar{b}}$ are symmetric and $\gamma^5_{a\bar{b}} \equiv (C \gamma^5)_{a\bar{b}}$ is antisymmetric, with $\alpha, \beta = 1, 2, 3, 4$ being the indices of a four-dimensional spinor representation of $SO(1, 3)$. These $4 \times 4$ matrices can be represented in terms of $2 \times 2 \text{AdS}_2$ gamma-matrices $\rho^\alpha (a = 0, 1)$ and the matrices $\rho^\hat{\alpha} (\rho^\delta = \sigma^1, \rho^\beta = \sigma^\beta)$ associated with $S^2$ as

$$\gamma^a = \rho^a \otimes \mathbb{1}, \quad \gamma^\hat{\alpha} = \gamma \otimes \rho^\hat{\alpha}, \quad \gamma = \rho^0 \rho^1.$$  \hspace{1cm} (A.5)

The $8 \times 8$ gamma-matrices associated with $T^6$ are

$$\{\gamma^{a'}, \gamma^{b'}\} = 2\delta^{a'b' }, \quad \delta^{a'b'} = \text{diag}(+, +, +, +, +, +, +).$$  \hspace{1cm} (A.6)

$$\gamma^7 = \frac{1}{6!} \varepsilon^{a'i_1i_2i_3i_4} \gamma^{a'_i} \ldots \gamma^{a'_4} \gamma^7 = 1.$$  \hspace{1cm} (A.7)

The charge conjugation matrix $C'$ is symmetric and the matrices $(\gamma^{a'})_{a'\bar{a}'} \equiv (C' \gamma^{a'})_{a'\bar{a}'}$ and $(\gamma^{a'\bar{a}'})_{a'\bar{b}'} \equiv (C' \gamma^{a'\bar{a}'})_{a'\bar{b}'}$ are antisymmetric, with $\alpha', \beta' = 1, \ldots, 8$ being the indices of an eight-dimensional spinor representation of $SO(6)$.

Using the matrices (A.3) and (A.6), one can construct the $D = 10$ gamma-matrices $\Gamma^A$ as

$$[\Gamma^A, \Gamma^B] = 2\eta^{AB}, \quad \Gamma^A = (\Gamma^a, \Gamma^{a'}) \equiv (\Gamma^\alpha, \Gamma^{\alpha'}),$$

$$\Gamma^a = \gamma^a \otimes \mathbb{1}, \quad \Gamma^{a'} = \gamma^5 \otimes \gamma^{a'}, \quad \Gamma^{11} = \gamma^5 \otimes \gamma^7,$$  \hspace{1cm} (A.8)

$$a = 0, 1, 2, 3; \quad a' = 4, \ldots, 9.$$

The charge conjugation matrix is $C = C \otimes C'$. In this realization, the 32-component spinor $\Theta^{a\bar{a}}$ is labeled by the 4-component $\text{AdS}_2 \times S^2$ spinor index $a$ and the 8-component $T^6$ spinor index $a'$.

Finally we can introduce a spinor projection operator which projects onto an 8-dimensional subspace of the 32-dimensional space of spinors as follows:

$$P_8 = \frac{1}{8} (2 - i J_{a\bar{b}'}) \Gamma^{a'\bar{b}'} \gamma^7,$$  \hspace{1cm} (A.9)

where $J_{a\bar{b}'}$ is the Kähler form on $T^6$. 

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Appendix B. Enlarged \( psu(1, 1|2) \) superalgebra

The \( psu(1, 1|2) \) superalgebra has the following conventional form. Its bosonic \( SO(2, 1) \times SO(3) \) subalgebra is generated by translations \( P_a = (P_a^+, P_a^-) \) and \( SO(1, 1) \times SO(2) \) rotations \( M_{ab} = (M_{ab}^+, M_{ab}^-) \) in \( \text{AdS}_2 \times S^2 \):

\[
[P_a^+, P_b^+] = -i \frac{1}{2} R_{\alpha\beta}^{\gamma\delta} M_{\gamma\delta}, \quad [M_{ab}^+, P_a^+] = \eta_{ac} P_b^+ - \eta_{bc} P_a^+, \quad [M_{ab}^+, M_{cd}^+] = \eta_{ac} M_{bd} - \eta_{bc} M_{ad} - \eta_{ad} M_{bc},
\]

where

\[
R_{\alpha\beta}^{\gamma\delta} = \left( R_{\alpha\beta}^{\gamma\delta}, R_{\alpha\beta}^{\gamma\delta}\right) = \left( \frac{2}{R^2} \delta_{\alpha\beta}^{\delta\gamma} - \frac{2}{R^2} \delta_{\alpha\beta}^{\delta \gamma} \right)
\]

is the \( \text{AdS}_2 \times S^2 \) curvature. The fermionic part of \( PSU(1, 1|2) \) is generated by eight Grassmann-odd operators \( Q_{\alpha I} \) carrying the index \( \alpha = 1, 2, 3, 4 \) of a spinorial representation of \( SO(1, 1) \times SO(2) \) and the vector index \( I = 1, 2 \) of the group \( SO(2) \) of external automorphisms of \( PSU(1, 1|2) \).\(^{38} \)

The generators \( Q \) satisfy the following (anti)commutation relations:

\[
[P_a^+, Q_I] = \frac{1}{2R} \varepsilon_{IJI} \gamma^{J} \gamma^{\alpha}, \quad [M_{ab}^+, Q_I] = -\frac{1}{2} \varepsilon_{IJI} \gamma^{\alpha}, \quad [T, Q_I] = \frac{1}{2} \varepsilon_{IJI} Q_J,
\]

\[
\{Q_I, Q_J\} = 2i \delta_{IJ} \gamma^{\alpha} P_{\alpha} + \frac{i}{2} \varepsilon_{IJI} \varepsilon_{JLJ} \gamma R_{\alpha\beta}^{\gamma\delta} M_{\gamma\delta}
\]

where the spinorial indices are suppressed, \( \gamma^\alpha \) and \( \gamma = \gamma^0 \gamma^1 \) are the \( D = 4 \) gamma matrices with \( \text{AdS}_2 \times S^2 \) indices and \( T \) is the generator of the external \( SO(2) \) automorphism of \( PSU(1, 1|2) \). As we will see below, from a ten-dimensional point of view, it is natural to include \( T \), even though it is not an element of \( psu(1, 1|2) \).

Since the GS formulation of the string in \( \text{AdS}_2 \times S^2 \times T^6 \) uses, \( a \) priori, \( D = 10 \) notation and 32-component spinors, we have found it convenient to formally lift the \( psu(1, 1|2) \) superalgebra to ten dimensions and enlarge it with additional generators \( P_{\alpha} \) and \( M_{\alpha\beta} \) of the \( U(1) \) isometries and \( SO(6) \) rotations in \( T^6 \). The \( psu(1, 1|2) \) superalgebra enlarged in such a way looks very similar to the \( OSp(6|4) \) superalgebra in the form used in \( [32] \). This similarity allows us to use the same gamma-matrix identities as those found in \( [36] \) for the \( \text{AdS}_4 \times CP^3 \) case.

To rewrite the \( psu(1, 1|2) \) superalgebra in a formally \( D = 10 \) covariant form, we replace the \( D = 4 \) gamma-matrices with their \( D = 10 \) counterparts \( \Gamma^\alpha \) and represent the 8 fermionic generators \( Q_{\alpha I} \) as 32-component spinors subject to the eight-dimensional projection

\[
Q = \mathcal{P}_8 Q,
\]

with \( \mathcal{P}_8 \) having eight non-zero eigenvalues and satisfying the commutation relations \( (3.6) \) and \( (3.7) \).

The following reasoning demonstrates the relation between the \( \mathcal{P}_8 \)-projected 32-component spinor \( Q_{\alpha'\alpha''} \) with the two 4-component \( \text{AdS}_2 \times S^2 \) spinors \( Q_{\alpha I} \). The projector \( \mathcal{P}_8 \) and \( \Gamma^{(3)} \), introduced in equation \( (3.20) \), commute and can be simultaneously diagonalized. Because the projector \( \mathcal{P}_8 \) also commutes with the 6D chirality \( \Gamma^z \), while \( \Gamma^{(3)} \) anti-commutes with it, the eigenvalues of \( \mathcal{P}_8 \) (in the 6D spinor space) are doubly degenerate, each pair having opposite values of \( \Gamma^{(3)} \). On the other hand, \( \mathcal{P}_8 \) and \( \Gamma^{(3)} \) commute with \( \Gamma^z \) and thus leave the 4D spinor index \( \alpha \) intact. Thus, under the action of \( \mathcal{P}_8 \), the 6D spinor index \( \alpha' \) then reduces to the binary index \( I = 1, 2 \) that labels the eigenvalues of \( \Gamma^{(3)} \) and is associated with the \( SO(2) \)-automorphism.

\(^{38}\) This group should not be confused with the \( SO(2) \) rotations of \( S^2 \) which is part of \( PSU(1, 1|2) \).
From the $D = 10$ perspective, the external $SO(2)$ automorphism of $\text{psu}(1, 1|2)$ becomes an Abelian subgroup of the group $SO(6)$ of rotations of the $T^6$-torus associated with the Kähler form $J_{a'b'}$ on $T^6$. Namely, the $SO(2)$ generator $T = \frac{1}{i} J_{a'b'} M_{a'b'}$ can be regarded as the part of $M_{a'b'}$ which acts non-trivially on $Q$. Rewriting the commutator $\{T, Q\}$ of equation (B.3) in the ten-dimensional notation we have

$$ [T, Q] = -\frac{i}{2} Q\gamma^7 P_8 \quad \Rightarrow \quad [M_{a'b'}, Q] := -\frac{i}{2} J_{a'b'} Q\gamma^7 P_8 = -\frac{1}{2} Q\hat{\Gamma}_{a'b'} P_8, \quad (B.6) $$

where

$$ \gamma^7 = i\Gamma^4 \cdots \Gamma^9 \quad (B.7) $$

is the product of the six gamma-matrices with $T^6$ indices. In (B.6) we have used the gamma-matrix identities

$$ P_8 \Gamma_{a'b'} P_8 = iJ^{a'b'} \gamma^7 P_8, \quad \gamma^7 P_8 = P_8 \gamma^7 \quad \text{and} \quad P_8 \gamma^7 P_8 := i\varepsilon \otimes 1. $$

The last expression relates the $2 \times 2$ antisymmetric matrix $\varepsilon$ of equations (B.3) and (B.4) times the unit $4 \times 4$ spinor matrix in $\text{AdS}_2 \times S^2$ with the $P_8$-projected matrix $\gamma^7$, equation (B.7).

Now, let us note that $Q$ commutes with the generators $P_{a'}$ of the $U(1)$ isometries of $T^6$. Then, since $P_8 \Gamma_a P_8 \equiv 0$, the zero commutator $\{P_{a'}, Q\}$ can be written as

$$ [P_{a'}, Q] = \frac{i}{2R} Q\gamma^7 \Gamma_{a'} P_8 = 0, $$

where now $\gamma = \Gamma^0 \Gamma^1 = \gamma^{01} \otimes 1$ is the product of $D = 10$ gamma-matrices carrying $\text{AdS}_2$ indices. Taking this into account we can write the enlarged $\text{psu}(1, 1|2)$ superalgebra in the following form:

$$ [P_A, P_B] = -\frac{1}{2} R_{AB}^{CD} M_{CD}, \quad [M_{AB}, P_C] = \eta_{AC} P_B - \eta_{BC} P_A, \quad (B.8) $$

$$ [M_{AB}, M_{CD}] = \eta_{AC} M_{BD} + \eta_{BD} M_{AC} - \eta_{BC} M_{AD} - \eta_{AD} M_{BC}, $$

$$ [P_A, Q] = \frac{i}{2R} Q\gamma^7 \Gamma_A P_8, \quad [M_{AB}, Q] = -\frac{1}{2} Q\hat{\Gamma}_{AB} P_8, \quad (B.9) $$

$$ \{Q, Q\} = 2i(\hat{P}_A \Gamma^A P_8) P_A + \frac{R}{2} (P_8 \Gamma^{AB} \gamma^7 P_8) R_{AB}^{CD} M_{CD}, $$

where $M_{AB} = (M_{ab}, M_{ab}, M_{a'b'})$ and $P_A = (P_a, P_a, P_a')$ are the generators of Lorentz-transformations and translations in $\text{AdS}_2$, $S^2$ and $T^6$, respectively, and

$$ R_{AB}^{CD} = (R_{ab}^{cd}, R_{ab}^{cd}, 0) = \left( \frac{2}{R^2} \delta^{cd}_{[a} \delta^{[b]}_b, -\frac{2}{R^3} \delta^{[c]}_{[a} \delta^{d]}_b, 0 \right) \quad (B.10) $$

is the curvature tensor of $\text{AdS}_2 \times S^2 \times T^6$.

To recapitulate, the above form of the $\text{psu}(1, 1|2)$ superalgebra is convenient for our purposes since it is formulated in a $D = 10$ covariant way and takes a form very similar to that of $\text{OSp}(6|4)$ [32, 36].

Appendix C. Basic relations for the Killing vectors on symmetric spaces $G/H$

Let $K_M(X)$ or $K_A(X) = e^A M(X) K_M(X)$ be the Killing vectors of a $D$-dimensional symmetric space $G/H$, where $M$ are world indices and $A$ are tangent space indices. The Killing vectors $K_M(X)$ take values in the algebra of the isometry group $G$ and the one-forms $K = dX^M K_M$ satisfy the Maurer–Cartan equations

$$ dK = -2K \wedge K, \quad dK \wedge K = K \wedge dK = -2K \wedge K \wedge K. \quad (C.1) $$
The following relations also hold:
\[
\begin{align*}
\{\nabla_A, \nabla_B\} C &= -R_{ABC} D K_D, \\
\nabla_A K_B &= [K_A, K_B], \\
\nabla_A \nabla_B C &= [\nabla_A K_B, C] + [K_B, \nabla_A C] = [\nabla_A K_C, K_B] - [\nabla_A K_C, K_B] = -2R_{[AB]C} D K_D, \\
\{\nabla_A K_B, C\} &= [[K_A, K_B], C] = -R_{ABC} D K_D, \\
[[K_A, K_B], [K_C, K_D]] &= R_{AB}[C^F [K_D], K_F] - R_{CD}[A^F [K_B], K_F].
\end{align*}
\]
where \( R_{ABC} D \) is the curvature of the symmetric space \( G/H \).

**Appendix D. Equations of motion from flatness of the Lax connection**

We have seen that the Lax connection defined in (2.15)–(2.19) is indeed flat for the type IIA and IIB superstring on AdS\(2 \times S^2 \times T^6 \) with RR-flux, provided that the equations of motion are satisfied. To have integrability, the opposite should also hold, i.e. requiring flatness of the Lax connection should imply the equations of motion for the string. Here, we will show that this is indeed the case.

Computing the curvature of the Lax-connection defined in (2.15)–(2.19), (2.21) and (2.25) without using the equations of motion, we get
\[
dl - LL = \alpha_2 (\nabla \ast J^A - e^D \ast J_{BCD}^A ) K_A - \frac{i}{2} \alpha_2 (1 + \alpha_1)^\theta \Gamma^A \hat{\not{D}} \nabla_A K_B
\]
\[
+ (-1)^\nu \frac{1}{2} \alpha_2 \Gamma^A \hat{\not{D}} \nabla_A K_B - \frac{i}{2R} \alpha_2 \beta_2 \Xi \hat{\not{D}} + (-1)^\nu \frac{1}{2} \alpha_2 \beta_1 \Xi \hat{\not{D}} \hat{\not{D}} \nabla_A K_B,
\]
\]
where \( \nu = 1(0) \) for type IIA(B) and
\[
\hat{\not{D}} \nabla A = * e^A \Gamma_A \hat{D} \nabla A - e^A \Gamma_A \hat{\not{D}} \nabla A
\]
is the quantity that is set to zero by the fermionic equations of motion.

Since the coefficients are all independent (they are different functions of the spectral parameter) each term has to vanish separately. The vanishing of the first term implies that
\[
\nabla (* e^A + i \theta \Gamma^A \ast E + i \theta \Gamma^A \hat{\not{D}} E - \frac{i}{8} \ast e^B \theta \Gamma^A \hat{\not{D}} F \Gamma_B \theta
\]
\[
- \frac{i}{8} e^B \theta \Gamma^A \hat{\not{D}} F \Gamma_B \theta - e^D \ast J^{BCD}^A = 0,
\]
which, using (2.9), is equal to the bosonic equations of motion (2.5).

The vanishing of the last two terms in the curvature implies the \( \hat{\not{D}} \) projection of the fermionic equations of motion
\[
\hat{\not{D}}\theta \hat{\not{D}} \nabla A = 0.
\]
We thus get eight fermionic equations of motion associated with the supercoset fermions \( \theta \).

To single out the remaining 24 fermionic equations we note that there are two other terms in (D.1) which have to vanish:
\[
\theta \Gamma^a \hat{\not{D}} \nabla A = 0
\]
and
\[
\theta \Gamma^b \hat{\not{D}} \nabla A = 0,
\]
where \( \theta^{(ab)} = (01), (23) \) (note that the covariant derivative of \( T^b \)-translation Killing vectors \( \nabla_A K_B \) vanishes so the corresponding terms are absent). Using the \( \hat{\not{D}} \) projection of the fermionic equation of motion, which we already found, we get the conditions
\[
\nu T^{01} (1 - \hat{\not{D}}) \nabla A = \nu T^{23} (1 - \hat{\not{D}}) \nabla A = 0.
\]

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This does not directly imply the missing 24 fermionic equations of motion. Instead we get

\[ (1 - \mathcal{P}_8) \mathcal{D} \Theta = M \nu, \]

where the matrix \( M \) has to be such that the four terms in (D.7) vanish. It is easy to see that there are non-zero \( M \) which satisfy this condition. For example, we could take, in the type IIA case, \( M \sim \gamma \Gamma_{a} \) or \( M \sim \Gamma_{ab} \Gamma_{a'b'c'}. \) It would seem from this analysis that the flatness of the Lax connection does not imply the equations of motion for the non-coset fermions but only the weaker condition (D.8). There is, however, one more condition that we have yet to impose. We can write equation (D.8) together with (D.4) as

\[ (\mathcal{D} \cdot 8) = (1 - \mathcal{P}_8) M \nu. \]

The left-hand side of this equation is annihilated by the projection matrix \( \frac{1}{2} (1 + \Gamma) \) with

\[ \Gamma = \frac{1}{2 \sqrt{-\hbar}} \epsilon^{ij} e_i^A e_j^B \Gamma_{AB} \Gamma_{11}, \]

where \( h_{ij} \) is the induced worldsheet metric (when \( M = 0 \) this is simply the statement of kappa-symmetry). Therefore, we find the condition

\[ (1 + \Gamma)(1 - \mathcal{P}_8) M \nu = 0. \]

From (D.8) we also have \( \mathcal{P}_8 M = 0. \) These two conditions imply that

\[ 0 = [1 + \Gamma, \mathcal{P}_8] M \nu = [\Gamma, \mathcal{P}_8] M \nu. \]

From the expressions for \( \Gamma \) and \( \mathcal{P}_8 \) in (D.10) and (A.9), it follows that for generic motions of the string, the commutator \([\Gamma, \mathcal{P}_8]\) is non-degenerate and this equation therefore implies that \( M = 0. \) When this is the case, equation (D.9) reduces to the fermionic equations of motion of the string, (2.6).

There are certain ‘singular’ configurations (e.g. when the string moves only in the AdS_2 \times S^2 subspace with no motion along the T^6-directions), for which \([\Gamma, \mathcal{P}_8] = 0 \) as a consequence of the form of \( \Gamma \) in (D.10) and the fact that \( \mathcal{P}_8 \) involves only T^6 gamma-matrices. For these ‘singular’ motions of the string we cannot use the above argument to conclude that \( M = 0. \) However, we do not expect this kind of a singularity in the Lax connection for certain motions of the string, i.e. the limit taken to get a ‘singular’ solution from the generic one should be smooth. So the natural conclusion is that \( M \) should be zero regardless of the solution of the bosonic equations of motion. This is also supported by the fact that if we take into consideration the Noether currents associated with rotational isometries in T^6, the conservation of these currents will require the matrix \( M \) in (D.9) to vanish. Unfortunately, it seems that it is not possible to include these currents directly into the Lax connection, since their presence breaks the flatness of the latter. This subtlety did not appear in the AdS_4 \times CP^3 case considered in [32].

We have therefore shown that the string equations of motion indeed follow from the flatness of the proposed Lax connection (to quadratic order in fermions) although there is a subtlety with certain ‘singular’ classical string solutions.

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