Analytic torsion on spherical factors and tessellations

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The analytic torsion is computed on fixed–point free and non fixed–point free factors (tessellations) of the three–sphere. We repeat the standard computation on spherical space forms (Clifford–Klein spaces) by an improved technique. The transformation to a simpler form of the spectral expression of the torsion on spherical factors effected by Ray is shown to be more general than his derivation implies. It effectively allows the eigenvalues to be considered as squares of integers, and applies also to trivial twistings. The analytic torsions compute to algebraic numbers, as expected. In the case of icosahedral space, the quaternion twisting gives a torsion proportional to the fundamental unit of \( \mathbb{Q}(\sqrt{5}) \). As well as a direct calculation, the torsions are obtained from the lens space values by a character inducing procedure. On tessellations, terms occur due to edge conical singularities.

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1. Introduction.

The Reidemeister torsion was introduced originally as a secondary combinatorial topological invariant and used by Franz to distinguish between non–homeomorphic lens spaces that were cohomologically and homotopically identical. An analytical analogue was later defined by Ray and Singer in the Riemannian de Rham setting and shown by Cheeger and Müller to be equal to the Reidemeister torsion. There is a useful summary in the nice little book by Rosenberg, [1]. The evaluation for lens spaces is classic, [2].

The torsion has become of some physical significance following the work of Schwarz on topological field theory, and a physicist’s calculation of the torsion on lens spaces can be found in Nash and O’Connor, [3].

It was initially defined on manifolds without boundary, although Cheeger extended it to those with a boundary. Later, Lott and Rothenberg, Lück and Vishik showed that, in this case, when the metric is product at the boundary, there is a difference between the Reidemeister and Ray–Singer quantities which involved the boundary Euler number.

At around the same time, the second named author calculated the torsion on the tessellated three–sphere using the information gathered during a computation of the Casimir energy on orbifold factors of spheres and the results were given in his thesis of 1993, [4]. The present paper details these results and adds a few up–to–date comments in the light of continuing interest in analytic torsion, e.g. Vertman, [5], and De Melo et al, [6]. We particularly wish to draw attention to the cancellation (section 4) which improves a result of Ray, [2].

2. The tessellation.

The tessellation is classic. In related calculations it has been described in our earlier works, [7], [8], where many original references are given. In the orbifold factoring, $S^3/\Gamma$, non–trivial elements of the full reflective, polytope symmetry group, $\Gamma$, leave hyperplanes, planes, lines and points invariant (in $\mathbb{R}^4$), and the fundamental domain (a spherical tetrahedron), $M$, on $S^3$, has a boundary, as well as edges and vertices. Restricting to the rotational subgroup, $\Gamma^+$, doubles the fundamental domain and removes the odd co–dimension singular domains, in this case this means the two–dimensional boundary, $\partial M$, and the vertices. The remaining singular edges form the axes of periodic dihedral wedges (cones).
3. Analytic torsion.

We need a few, very standard formulae and preliminary results. The definition of analytic torsion, \( T \), by Ray and Singer reads, for a manifold, \( \mathcal{M} \), of dimension \( d \),

\[
\log T(\mathcal{M}, \rho) = \frac{1}{2} \sum_{p=0}^{d} (-1)^p \zeta'_{\Delta_p, \rho}(0)
\]

where \( \zeta_{\Delta_p, \rho}(s) \) is the \( \zeta \)-function for the de Rham Laplacian, twisted by a representation, \( \rho \), of \( \Gamma \), i.e.

\[
(\gamma \phi)(x) = \phi(\gamma x) = \rho(\gamma)\phi(x), \quad \forall \gamma \in \Gamma, \quad x \in \mathcal{M}.
\]

The form \( \phi \) takes values in the flat vector bundle associated with the representation \( \rho \) which is taken to be either orthogonal, \( \Gamma \to O(N) \), or unitary, \( \Gamma \to U(N) \). In the former case the form is assumed real and in the latter, complex. For complex forms, one must be careful to include a factor of two when counting dimensions.

If \( x \) is a fixed point, \( \gamma x = x \), and, if the rep is non–trivial, this implies that the form, \( \phi \), has to vanish at the fixed points i.e. , in the present situation, on the axis of the cone. In this case, normal Hodge duality applies. There is no need for absolute or relative conditions in the purely rotational case which concerns us here.

If the rep, \( \rho \), is trivial, then one has to decide on the conditions that hold at the edge singularity. We select, by default, those that yield the Friedrichs extension.

An alternative, perhaps more fundamental, form for the torsion is given in terms of the coexact \( \zeta \)-functions, \( \zeta_{p, \rho} \),

\[
\log T(\mathcal{M}, \rho) = \frac{1}{2} \sum_{p=0}^{d-1} (-1)^p \zeta'_{p, \rho}(0)
\]

which transcribes to (1) using télescopage via the well–known relation,

\[
\zeta_{\Delta_p, \rho}(s) = \zeta_{p, \rho}(s) + \zeta_{p-1, \rho}(s),
\]

between the total \( \zeta \)-function and the coexact \( \zeta \)-functions.

At this point we restrict to odd dimensional manifolds, and set \( d = 2M + 1 \). Then (1), for example, is easily rewritten as (see also [9]),

\[
\log T(\mathcal{M}, \rho) = -\sum_{p=0}^{M-1} (-1)^p \zeta'_{p, \rho}(0) + \frac{1}{2}(-1)^{M+1} \zeta'_{M, \rho}(0)
\]
using Hodge duality and télescopage.

In particular, in the three–dimensional case,

\[ \log T(M, \rho) = \frac{1}{2} \zeta_{1, \rho}(0) - \zeta_{0, \rho}(0) \]  

(4)

This is the case computed in [4] and now exposed.

To begin with, we define an intermediate quantity,

\[ \tau_{\rho}(s) = \frac{1}{2} \zeta_{1, \rho}(s) - \zeta_{0, \rho}(s). \]  

(5)

In three dimensions, one does not need the full apparatus of \( p \)-forms. A coexact one–form is a conformally coupled divergenceless (transverse) vector and the zero–form is a minimal scalar. The eigenvalues on the (unit) three–sphere are well–known and the \( \zeta \)–functions on the orbifold factors are, (cf [10]),

\[ \zeta_{0, \rho}(s) = \sum_{n=1}^{\infty} \frac{d_0(n, \rho)}{(n(n+2))^s}, \]

\[ \zeta_{1, \rho}(s) = \sum_{n=1}^{\infty} \frac{d_1(n, \rho)}{(n+1)^{2s}}. \]  

(6)

We remark that any zero mode \((n = 0)\) that might exist for the scalar, has been omitted from the sum. This would be the case on the complete sphere, \( S^d \), \((d > 1)\). For non–trivial twistings, there are no zero modes, corresponding to a trivial cohomology for (2).

The problem is to find the degeneracies, \( d_p \), and then continue the expression (5). This can be done for the \( \zeta \)–functions, (6), separately, as in [3], or for the combination, as in [2]. We choose the latter.

In the case of the full sphere, the degeneracies, \( d_p(n, \mathbf{1}) \) are \((2)N(n+1)^2\) and \((2)N2n(n+2)\), for \( p = 0 \) and 1, respectively. (The factor of 2 comes in for a \( U(N) \) bundle.)

On the factored three–sphere, the necessary formulae have been published previously in similar contexts, [11–13], so we will not give the full derivations here. The approach is the very standard one of group averaging (or projection, or symmetry adaptation) in order to achieve the twisting, (2). It is convenient (but not necessary) to use the ‘left–right’ \( \text{SU}(2) \) actions to represent the action of \( \Gamma^+ \) on \( S^3 \), which is isomorphic to \( \text{SU}(2) \).

The situation in [10] is very close to the one here. The \( \zeta \)–functions are diagonal in the (flat) vector bundle indices and, combined with the explicit discussion in [13],
this leads to the expression for the degeneracies,

\[ d_0(n - 1, \rho) = d(L, L), \quad d_1(n, \rho) = d_{\rho}(L, L + 1) + d_{\rho}(L + 1, L) \] (7)

where \( n = 2L + 1 \) with the expected group theory degeneracy,

\[ d_{\rho}(L, J) = \frac{1}{|\Gamma^+|} \sum_{\gamma \in \Gamma^+} \chi_{\rho}^*(\gamma) \chi^{(L)}(\gamma_L) \chi^{(J)}(\gamma_R) \] (8)

in terms of the character, \( \chi_{\rho} \), of the \( \rho \) rep and the SU(2) character,

\[ \chi^{(L)}(\gamma) \equiv \chi_{2L+1}(\gamma) = \frac{\sin(2L + 1)\theta}{\sin \theta} . \] (9)

We note that \( \chi_{\rho}^*(1) = N \) and, in addition, there is the factor of two for complex forms.

The angle \( \theta_L \) is the radial coordinate labelling the SU(2) element \( \gamma_L \) as a point on \( S^3 \).

A tactical decision is whether to leave the group average until last or to try to effect it earlier. In some cases, \( e.g. \) for cyclic groups, the latter is possible, and preferable, but here we leave it until last and so, for convenience, define the summands, which could be termed ‘partial’ or ‘off–diagonal’ quantities, by the formulae

\[ \tau_{\rho}(s) = \frac{1}{|\Gamma^+|} \sum_{\gamma} \chi_{\rho}^*(\gamma) \tau(\gamma; s) \]
\[ d_{\rho}(n, \rho) = \frac{1}{|\Gamma^+|} \sum_{\gamma} \chi_{\rho}^*(\gamma) d_{\rho}(\gamma; n) . \] (10)

Another decision is whether to treat the two terms in (5) separately, as in [3], or manipulate them together, as in [2]. The former is less economic and does produce interesting incidental information, however we prefer the second way. To this end, the vector (spin–one) partial degeneracy, is rewritten, from (7), (8), (9) and (10), as

\[ d_1(\gamma; n) = \chi_{n+2}(\theta_L)\chi_{n+2}(\theta_R) + \chi_n(\theta_L)\chi_n(\theta_R) - 4 \cos(n + 1)\theta_L \cos(n + 1)\theta_R \]

\[ d_0(\gamma; n) = \chi_n(\theta_L)\chi_n(\theta_R) . \]
4. The cancellation.

In Ray’s calculation, [2], through the mess of summations and integrations, a miracle occurs on p.125 in which, for the total quantity, the effective eigenvalues become those appropriate for a conformally invariant propagation equation, i.e. perfect squares. In this section we give our physicist’s version of the implied cancellation, [4], which is independent of the free/non–free behaviour of the deck group. Actually, our result goes further than Ray’s as he makes a preliminary group average for a non–trivial twisting thereby removing a particular set of terms, which then do not have to be continued. The group average has still to be performed on the remaining terms.

In contrast, our result is before any group average and refers to the complete expression. It can therefore be applied to the case of a trivial twisting, which provides a useful check, in view of Cheeger’s result, [14], Theorem 8.35 (see later).

Combining the various $\zeta$–functions one gets

$$
\tau(\gamma; s) = \sum_{n=1}^{\infty} \frac{d_1(\gamma; n)}{2(n+1)^{2s}} - \frac{d_0(\gamma; n)}{(n(n+2))^s}
$$

$$
= \sum_{n=1}^{\infty} \left[ \frac{\chi_{n+2}(\theta_L)\chi_{n+2}(\theta_R)}{2(n+1)^{2s}} - \frac{\chi_{n+1}(\theta_L)\chi_{n+1}(\theta_R)}{(n(n+2))^s} \right.

+ \frac{\chi_n(\theta_L)\chi_n(\theta_R)}{2(n+1)^{2s}} - 2\frac{\cos(n+1)\theta_L\cos(n+1)\theta_R}{(n+1)^{2s}}

\left. \right] \chi_n(\theta_L)\chi_n(\theta_R)
$$

$$
= \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{(n+1)^s} + \frac{1}{(n-1)^{2s}} - \frac{2}{(n^2-1)^s} \right] \chi_n(\theta_L)\chi_n(\theta_R)

- 2\sum_{n=1}^{\infty} \frac{\cos n\theta_L\cos n\theta_R}{n^{2s}} + \frac{1}{2^{2s+1}}
$$

If we now define two auxiliary functions, $F$ and $\tilde{\tau}$, as the quantities on the two lines in the last equality, so that,

$$
\tau(\gamma; s) = \frac{1}{2} F(\gamma; s) + \tilde{\tau}(\gamma; s),
$$

we proceed to show, quite non–rigorously, that there is no contribution to the torsion from $F(\gamma; s)$.

Now our function, $F$, can be written as,

$$
F(\gamma; s) = \sum_{n=1}^{\infty} \left[ \frac{(n-1)^{2s} + (n+1)^{2s} - 2(n^2-1)^s}{(n^2-1)^{2s}} \right] \chi_n(\theta_L)\chi_n(\theta_R),
$$
and it is obvious, by inspection, that it vanishes at \( s = 0 \). Also, taking the derivative with respect to \( s \) gives

\[
\frac{dF(\gamma; s)}{ds} = \sum_{n=2}^{\infty} \left[ \frac{2(n^2 - 1)^s - (n-1)^{2s} - (n+1)^{2s}}{(n^2 - 1)^{2s}} \right] 2 \ln(n^2 - 1) \chi_n(\theta_L) \chi_n(\theta_R)
\]

\[
-2 \sum_{n=2}^{\infty} \left[ \frac{\ln(n^2-1)(n^2-1)^s - \ln(n-1)(n-1)^{2s} - \ln(n+1)(n+1)^{2s}}{(n^2 - 1)^{2s}} \right] \chi_n(\theta_L) \chi_n(\theta_R)
\]

which at \( s = 0 \) yields

\[
\left. \frac{dF(\gamma; s)}{ds} \right|_0 = 2 \sum_{n=2}^{\infty} \left[ \ln(n^2 - 1) - \ln(n - 1) - \ln(n + 1) \right] \chi_n(\theta_L) \chi_n(\theta_R)
\]

\[
= 0.
\]

Hence

\[
F(\gamma; 0) = 0, \quad \text{and} \quad F'(\gamma; 0) = 0.
\]

(14)

and, therefore, to all intents and purposes, the contribution from \( F(\gamma; s) \) can be ignored at the point of interest, \( s = 0 \). Hence, \( \tilde{\tau} \) is related to \( \tau \) in (12) by

\[
\tilde{\tau}(\gamma; 0) = \tau(\gamma; 0), \quad \tilde{\tau}'(\gamma; 0) = \tau'(\gamma; 0)
\]

and we can work with the simpler, effective tau function, \( \tilde{\tau}(s) \), when calculating the analytic torsion which is, according to (4) and (5),

\[
\ln T(\mathcal{M}, \rho) = \tilde{\tau}'_\rho(0).
\]

(15)

5. The effective \( \tau \) function.

Putting the group average back, we have the effective function,

\[
\tilde{\tau}_{\Gamma^+}(s, \rho) = \frac{1}{|\Gamma^+|} \sum_{\gamma \in \Gamma^+} \chi^*_\rho(\gamma) \tilde{\tau}(\gamma; s)
\]

(16)

and it is now necessary to address this sum. First some old facts are needed.

2 Mathematicians would demand a higher quality of proof. Some further remarks are given in Appendix 1.
The left and right angles $\theta_L$ and $\theta_R$ are half the ‘rotation’ angles associated with the left and right SU(2) actions via the isomorphism $\text{SO}(3) \sim \text{SU}(2)/\mathbb{Z}_2$. (These rotation angles run from 0 to $4\pi$.) The $\theta_L$ and $\theta_R$ are related to angles $\alpha$ and $\beta$, which are the rotation angles in the two planes of $\mathbb{R}^4$ under an SO(4) action, by

$$\theta_L = \frac{1}{2}(\alpha - \beta), \quad \text{and} \quad \theta_R = \frac{1}{2}(\alpha + \beta),$$

both mod $\pi$.

In the usual embedding, any element of SO(4) is conjugate to a block diagonal $4 \times 4$ matrix representation of the form (see e.g. [15]),

$$R(\gamma) = \begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 & 0 \\
\sin \alpha & \cos \alpha & 0 & 0 \\
0 & 0 & \cos \beta & -\sin \beta \\
0 & 0 & \sin \beta & \cos \beta
\end{pmatrix}. \quad (17)$$

From (11) the summand in the effective $\tau$ function, (16), is a class function and so the sum can be written as one over conjugacy classes. The group $\Gamma^+$ decomposes into classes, $C_p$, as

$$\Gamma^+ = \bigoplus_p c_p C_p$$

where $c_p$ is the size of the class labeled by $p$. The total effective function (16) then reads

$$\tilde{\tau}_{\Gamma^+}(s, \rho) = \frac{1}{|\Gamma^+|} \sum_p \chi^*_\rho(p) c_p \tilde{\tau}(C_p; s)$$

$$= \sum_p \chi^*_\rho(p) \tau_p \tilde{\tau}(C_p; s) \quad (18)$$

with obvious notation and where we have introduced the class ‘density’, $\tau_p = c_p/|\Gamma^+|$.

In this approach, where the group average is performed by hand, this is the best that can be done.

From (11), the partial auxiliary function, $\tilde{\tau}$, is given by

$$\tilde{\tau}(\alpha, \beta; s) = -\sum_{n=1}^{\infty} \frac{b(\alpha, \beta, n)}{2n^{2s}} + \frac{1}{2^{2s+1}}$$

$$= \tilde{\tau}(C_p; s) \quad (19)$$

where

$$b(\alpha, \beta, n) = 4 \cos n\theta_L \cos n\theta_R = 2(\cos n\alpha + \cos n\beta)$$
is recognised as the trace of the SO(4) rep, \( R(\gamma^n) \), for a rotation \( \gamma \sim (\alpha, \beta) \). The same quantity occurs in Ray’s treatment, [2] p.125. The extension to any odd dimensional sphere is clear at this point simply by extending the block form, (17), granted the cancellation.

The summation term in (19) is essentially what Ray, [2], gets for his \( f(s; g) \) on p.125.

One now recognises that the effective \( \tau \) function, (19), involves just a sum of two one–dimensional Epstein \( \zeta \)–functions defined by, [16],

\[
Z \left| \frac{h}{\alpha/2\pi} \right|(s) = 2 \sum_{n=1}^{\infty} \frac{\cos n\alpha}{n^s},
\]

(20)

If \( h = 0 \), there is the relation with the Hurwitz–Lerch \( \zeta \)–function,

\[
Z \left| \frac{g}{0} \right|(s) = \zeta_R(s, g) + \zeta_R(s, 1-g)
\]

(21)

and, if \( g = 0 \), the \( n = 0 \) term is omitted so that

\[
Z \left| \frac{0}{\alpha/2\pi} \right|(s) = 2 \sum_{n=1}^{\infty} \frac{\cos n\alpha}{n^s},
\]

(22)

related to polylogarithms and the Lerch–Lipshitz \( \zeta \)–function. This has regularly occurred in these, and other, situations from the earliest times, e.g. [11]. When \( \alpha = 0 \) it is just twice the Riemann \( \zeta \)–function.

We thus have

\[
\bar{\tau}(\alpha, \beta; s) = -\frac{1}{2} Z \left| \frac{0}{\alpha/2\pi} \right|(2s) - \frac{1}{2} Z \left| \frac{0}{\beta/2\pi} \right|(2s) + \frac{1}{2^{2s+1}}.
\]

(23)

Our primary objective is the torsion and so we now move directly to evaluate \( \bar{\tau}'(\alpha, \beta; 0) \),

\[
\bar{\tau}'(\alpha, \beta; 0) = -Z' \left| \frac{0}{\alpha/2\pi} \right|(0) - Z' \left| \frac{0}{\beta/2\pi} \right|(0) - \ln 2.
\]

(24)

It is possible to proceed as in Ray, [2], but we prefer to streamline the analysis using some earlier results, as employed in [12] in a computation of lens space determinants.

There are a number of ways of proceeding. Here we first note that as \( s \) tends to one,

\[
Z \left| \frac{h}{0} \right|(s) \to \frac{2}{s-1} - \psi(h) - \psi(1-h), \quad h \neq 0,
\]

(23)
Use of the functional relation, for $g = 0$,

$$
Z \left| \begin{array}{c} \frac{g}{h} \\ 2s \end{array} \right| = \pi^{2s-1/2} \frac{\Gamma(s - 1/2)}{\Gamma(s)} e^{-2\pi i gh} Z \left| \begin{array}{c} \frac{h}{-g} \\ (1 - 2s) \end{array} \right|,
$$

(25)
gives, equivalently, a classic formula,

$$
Z \left| \begin{array}{c} 0 \\ h \end{array} \right| (0) = -\gamma - \ln 2\pi - \frac{1}{2} (\psi(h) + \psi(1 - h)), \quad h \neq 0,
$$

(26)

which is ready to be substituted directly into (23).

6. The partial analytic torsion.

It is now important to remark that, in the groups we consider, the angles, $(\theta_L, \theta_R)$, or $(\alpha, \beta)$, are submultiples of $2\pi$. Therefore, let, generally,

$$
\frac{\alpha}{2\pi} = \frac{a}{q} \quad \text{and} \quad \frac{\beta}{2\pi} = \frac{b}{q}
$$

(27)

where $a$ and $b$ are integers coprime to $q$. This allows one to use Gauss' famous formula for $\psi(p/q)$, $(p/q \in \mathbb{Q})$, or better, a formula that appears during a proof of this relation, [17] p.146, given below, (see also [18] p.13), $0 < p < q$,

$$
\psi\left(\frac{p}{q}\right) + \psi\left(1 - \frac{p}{q}\right) = -2\gamma - 2 \log q + 2 \sum_{k=1}^{q-1} \cos\left(\frac{2\pi pk}{q}\right) \log 2 \sin\frac{\pi k}{q}.
$$

(28)

Substituting into (23), there results the equivalent formulae,

$$
\tau'(\alpha, \beta; 0) = \sum_{k=1}^{q-1} \left( \cos k\alpha + \cos k\beta \right) \log 2 \sin\frac{\pi k}{q} + \ln(2\pi^2/q^2)
$$

$$
= 2 \sum_{k=1}^{q-1} \cos k\theta_L \cos k\theta_R \log 2 \sin\frac{\pi k}{q} + \ln(2\pi^2/q^2)
$$

$$
= \frac{1}{2} \sum_{k=1}^{q-1} b(\alpha, \beta, k) \log 2 \sin\frac{\pi k}{q} + \ln(2\pi^2/q^2)
$$

$$
\equiv \sum_{k=1}^{q-1} b(\alpha, \beta, k) T_q(k) + \ln(2\pi^2/q^2)
$$

(29)

with the angles (27).
Strictly speaking, the special cases when either, or both, of $\alpha$ and $\beta$ are zero should be treated separately. For example, equation (22) gives the classic value,

$$Z' \bigg|_{0}^{0} (0) = 2\zeta_R'(0) = -\ln 2\pi,$$

but it is easy to show using the ancient product,

$$\prod_{k=1}^{q-1} 2\sin\left(\frac{\pi k}{q}\right) = q,$$  \hspace{1cm} (30)

that (29) covers these cases.

Incidentally, in this connection, equation (24) can be extended formally to $h = 0$, and (28) to $p = 0$ and $q$, by regularising $\psi(0)$ to $\psi(1), = -\gamma$.

For example, for the full sphere, where the group action, and average, is trivial (all points are fixed),

$$\ln T(S^3, 1) = \tilde{\tau}'(0, 0; 0) = \ln 2\pi^2 = \ln |S^3|,$$  \hspace{1cm} (31)

for real forms, i.e. $\chi_\rho(1) = 1$. This falls into the result (29) on setting $q = 1$, when the sum is non–existent.

Weng and You, [19], have performed a rather involved calculation of the analytic torsion on an odd sphere.

The old result for the circle is,

$$\ln T(S^1, 1) = \ln |S^1|,$$

obtained easily from the Riemann $\zeta$–function. A simple scaling gives for the factored circle,

$$\ln T(S^1/\mathbb{Z}_q, 1) = \ln |S^1/\mathbb{Z}_q|$$  \hspace{1cm} (32)

7. The group average. Clifford–Klein spaces.

In summary, the (logarithm of the) torsion is given by the twisted group average (15) with the effective tau–function (16) or (18) and the expressions (29) for $\tilde{\tau}'(C_p, 0)$, the class $C_p$ being associated with a pair of angles, $(\alpha, \beta)$.

As an application, we look at the classic case of fixed–point free actions i.e. Clifford–Klein spaces the basic example of which are lens spaces. The evaluations
are taken a little further than we have seen in the literature and proceed at a simple level of analysis.

We first rederive Ray’s formula for the torsion of the lens space, $L(q; l_1, l_2) = S^3/(\mathbb{Z}_q \times \mathbb{Z}_q)$, which is defined by the angles

$$\alpha/2\pi = p\nu_1/q, \quad \beta/2\pi = p\nu_2/q,$$

where $p, = 0, \ldots, q−1, labels \gamma (or, equivalently, the class $C_p$). $\nu_1$ and $\nu_2$ are (fixed) integers coprime to $q$, with $l_1$ and $l_2$ their mod $q$ inverses. With these choices, there are no fixed points.

By an appropriate selection of a $q$-th root of unity, it would be possible to set $\nu_1 = 1$, i.e. $l_1 = 1$, without loss of generality. Any pair, $(\nu_1, \nu_2)$, could be reduced to $(1, \nu)$ by multiplying through by the mod $q$ inverse of $\nu_1$. The simple, one–sided lens space, $L(q; 1, 1)$, corresponds to setting $\nu = 1$ so that $\theta_L = 0, \theta_R = 2\pi p/q$.

Inserting a U(1) twisting (or, equivalently an SO(2) one), $\chi_r(p)$, the torsion reads

$$\ln T(L(q; l_1, l_2), r) = 2 \sum_{j=1}^{q-1} \sum_{p=0}^{q-1} \sum_{k=1}^{q} e^{2\pi irp/q} e^{-2\pi ik\nu_j p/q} \ln 2 \sin \frac{\pi k}{q}$$

where the integer $r$ determines the bundle twisting, the non–triviality of which ensures that the constant terms in (29) go out.

The sum over $p$ implies that $k\nu_j - r$ is a multiple of $q$ or that $k\nu_j = r \mod q$ or that $k = r/\nu_j \mod q \equiv rl_j$.\(^4\)

Therefore we get

$$\ln T(L(q; l_1, l_2), r) = 2 \sum_{j} \ln 2 \sin \frac{\pi rl_j}{q}$$

\(^3\) The $\cos k\alpha$ and $\cos k\beta$ terms in (29) can each be replaced by an exponential. Set $k \rightarrow q - k$. We have included an overall factor of two to account for the complex nature of the forms. Ray does not appear to do this overtly, although he uses a complex line bundle, as Ray and Singer, [20], mainly do. Ray and Singer in [21] use orthogonal bundles. For the single real twisting (when $q$ is even), one could remove the factor of two.

\(^4\) These manipulations can be found in Epstein, [16].
which is Ray’s value, allowing for the different definition of torsion. The extension to any odd sphere dimension is obvious.

If the U(1) twisting is trivial (set $r = 0$), (34) yields zero because $k$ never attains 0 or $q$. The torsion then arises from the average of the constant term in (29) or

$$\ln T(S^3/\mathbb{Z}_q, 1) = \ln(2\pi^2/q) - \ln q - \ln(|S^3/\mathbb{Z}_q|) - \ln q,$$

where, since the representation is real, the complex doubling has not been invoked.

The case of a trivial real flat bundle, has been considered by Cheeger, [14], Theorem 8.35 which says that the combinatorial Reidemeister torsion is given by,

$$\ln T_R(\mathcal{M}, 1) = \sum_{p=0}^{d} (-1)^{p+1} \left( \ln V_p(\mathcal{M}) + \ln O_p \right)$$

where $O_p = |\text{Tor} H^p(\mathcal{M}; \mathbb{Z})| = |\text{Tor} H_{p-1}(\mathcal{M}; \mathbb{Z})|$ is the order of the torsion subgroup of $H^p$ and the $V_p$'s are essentially volumes associated with the real/integer cohomology and reflect the existence of zero modes.

In the case that only the top and bottom real cohomology is non–trivial, one has

$$\ln T_R(\mathcal{M}, 1) = \ln |\mathcal{M}| - \sum_{p=0}^{d} (-1)^p \ln O_p$$

after noting that $V_0 = 1/\sqrt{|\mathcal{M}|}$ and $V_d = \sqrt{|\mathcal{M}|}$. (For example, the normalised 0–form zero mode is $1/\sqrt{|\mathcal{M}|}$.)

Applied to a homology lens space, $\mathcal{M} \sim L_q$, this expression gives, [23],

$$\ln T_R(\mathcal{M}) = \ln |\mathcal{M}| - \frac{d-1}{2} \ln q, \quad d = \dim \mathcal{M}$$

The explicit results, (36) and (32), of course confirm this formula.

Rosenberg mentions that the torsion suitably defined for non–exact complexes, is (see [1], p.154),

$$\sum_{p=0}^{d} (-1)^p \ln |\text{Tor} H^p(\mathcal{M}; \mathbb{Z})| = \sum_{p=0}^{d} (-1)^p \ln |\text{Tor} H_{p-1}(\mathcal{M}; \mathbb{Z})| .$$

---

5 There appears to be a square root misprint in Cheeger’s text. We refer to the elegant paper of Schwarz and Tyupkin, [22], for a physicist’s approach to these questions.
This corresponds to just the first term in Cheeger’s Theorem 8.35 and is a topological invariant, being a special case of Ray and Singer’s extension of the torsion to non-trivial cohomology. See [20] section 3.

The other Clifford–Klein spaces can be computed. The work of Tsuchiya, [24], is concerned with these but does not seem to calculate any specific values, and uses Ray’s formulae (see later). The interesting and more extensive work of Bauer and Furutani, [25], must also be mentioned as it deals with higher dimensions and contains explicit results.

For simplicity, we consider only one-sided (homogeneous) factors, \( S^3/\Gamma' \), where \( \Gamma' \) is a binary polyhedral group. To be precise, we set \( \theta_L = 0 \) i.e. \( \alpha = \beta = \theta_R \equiv \theta_\gamma \).

Prism spaces are another infinite set of spaces, for which \( \Gamma' \) is the binary dihedral group, \( D'_q \), of order \( 4q \). The generator–relation structure can be written,

\[
A^q = B^2 = (AB)^2 = Q, \quad Q^2 = E, 
\]

and thus \( D'_q \) can be formally written as the direct sum

\[
D'_q = Z_{2q} \oplus Z_{2q}B, 
\]

where \( Z_{2q} \) is generated by \( A \). To express the binary doubling, one has \( Z_{2q} = Z_q \oplus QZ_q \) where \( Z_q \) is the SO(3) cyclic rotation group and \( Q \) is a rotation in \( \mathbb{R}^3 \) through \( 2\pi \) i.e. \( \theta_Q = \pi \).

For \( D'_q \), the angles \( \theta_\gamma \) are,

\[
\theta_\gamma = \pi p/q, \quad p = 0, \ldots, 2q - 1. \tag{40}
\]

The SU(2) angle, \( \theta_\gamma \), has been left to run from 0 to \( 2\pi \), corresponding to an SO(3) rotation from 0 to \( 4\pi \). If one wishes \( \theta_\gamma \) to be restricted to the range 0 to \( \pi \), as a colatitude on \( S^3 \) should be, then it can be arranged that,

\[
\theta_\gamma = \pi p/q, \quad p = 0, \ldots, q - 1
= 2\pi - \pi p/q, \quad p = q, \ldots, 2q - 1. 
\]

For \( \gamma = A^pB \), i.e. those \( 2q \) elements containing a (binary) dihedral rotation, \( \theta_\gamma = \pi/2 \) for all \( \gamma \).

It is straightforward to compute the analytic torsion from (29), using a machine if necessary. The simplest case is the trivial bundle, and we find

\[
\ln T_{D'_q}(1) = \ln(|S^3|/4q) - 2 \ln 2 \tag{41}
\]
which agrees with (38) in view of the homology, [15],
\[ H_1(S^3/\cal{D}'_q) = \mathbb{Z}_4, \quad q \text{ odd} \]
\[ = \mathbb{Z}_2 + \mathbb{Z}_2, \quad q \text{ even}. \]

To compute the twisted torsion one needs the irreps of $\cal{D}'_q$, which are either one–, or two–dimensional. The former are generated by,
\[ \chi(A) = (-1)^a, \quad \chi(B) = i^b, \quad a = 0, 1, \quad b = 0, 1, 2, 3, \quad (42) \]
and the latter by,
\[ a(A) = \begin{pmatrix} e^{i\pi(2a+b)/q} & 0 \\ 0 & e^{-i\pi(2a+b)/q} \end{pmatrix}, \quad a(B) = \begin{pmatrix} 0 & 1 \\ (-1)^b & 0 \end{pmatrix}, \quad b = 0, 1, \quad (43) \]
the conditions being that in (42) for real representations, $b = 0, 2$, and then, if $q$ is odd, $a$ cannot equal 1, while for imaginary reps, $b = 1, 3$, then $a = 0$ is not possible and, for $a = 1$, $q$ must be odd.

In (43), we have
\[
\begin{aligned}
&\text{even } q \begin{cases} 
 b = 0, & a = 1, \ldots, q/2 - 1 \\
 b = 1, & a = 0, \ldots, q/2 - 1 
\end{cases} \\
&\text{odd } q \begin{cases} 
 b = 0, & a = 1, \ldots, (q - 1)/2 \\
 b = 1, & a = 0, \ldots, (q - 3)/2 
\end{cases}
\end{aligned}
\]
and for the traces one finds,
\[ \text{Tr} \ (A^pB) = 0, \quad \text{Tr} \ A^p = 2 \cos \left( \frac{2\pi ap + \pi bp}{q} \right). \]

Labelling the one–dimensional irreps by $(a, b)$, calculation gives the analytic torsion,
\[ T_{\cal{D}'_q}(0, 2) = 4/q, \quad \forall q \]
\[ T_{\cal{D}'_q}(1, 1) = T_{\cal{D}'_q}(1, 3) = 2^2, \quad q \text{ odd} \]
\[ T_{\cal{D}'_q}(1, 0) = T_{\cal{D}'_q}(1, 2) = 2, \quad q \text{ even}. \]
The power of 2 is a complex (or SO(2)) dimension effect.

Labelling the two–dimensional irreps again by $(a, b)$, we find, e.g.,
\[ T_{\cal{D}'_2}(0, 1) = 2 \]
\[ T_{\cal{D}'_3}(0, 1) = 1, \quad T_{\cal{D}'_3}(1, 0) = 3 \]
\[ T_{\cal{D}'_4}(1, 0) = 2, \quad T_{\cal{D}'_4}(0, 1) = 2 - \sqrt{2}, \quad T_{\cal{D}'_4}(1, 1) = 2 + \sqrt{2}. \]
The irreps are usually labelled by letters, for example for $D_4'$ the two–dimensional reps are $(1, 0) = E$, $(0, 1) = E'_1$ and $(1, 1) = E'_2$, in the notation of Landau and Lifshitz, [26].

No attempt will be made here to give an exhaustive list of all the numerical possibilities.

The remaining groups are $T'$ (octahedral space), $O'$ (truncated cube space) and $Y'$ (dodecahedral, or Poincaré, space). For the untwisted cases we find

\[
\begin{align*}
T_{T'} &= \frac{1}{3} |S^3|/24 \\
T_{O'} &= \frac{1}{2} |S^3|/48 \\
T_{Y'} &= |S^3|/120,
\end{align*}
\]

in agreement with Cheeger’s expression, (37), and the known (co)homology of the Clifford–Klein manifolds, [15]. Another way of obtaining these values is given below and a further one in Appendix 2.

We compute the twisted values to be,

\[
\begin{align*}
T_{T'}(\hat{1'}) &= (3/2)^2, & T_{T'}(\mathbf{3}) &= 2 \\
T_{T'}(\mathbf{2}_s) &= 1/2, & T_{T'}(\hat{2}'_s) &= 2^2 \\
T_{O'}(\hat{1'}) &= 4/3, & T_{O'}(\mathbf{2}) &= 3/2 \\
T_{O'}(\mathbf{3}) &= 2, & T_{O'}(\mathbf{3'}) &= 1, & T_{O'}(\mathbf{4}_s) &= 2 \\
T_{O'}(\mathbf{2}_s) &= (2 - \sqrt{2})/2, & T_{O'}(\mathbf{2}'_s) &= (2 + \sqrt{2})/2 \\
T_{Y'}(\mathbf{4}) &= 5/3, & T_{Y'}(\mathbf{2}_s) &= \frac{3 - \sqrt{5}}{4}, & T_{Y'}(\mathbf{2}'_s) &= \frac{3 + \sqrt{5}}{4} \\
T_{Y'}(\mathbf{6}_s) &= 2, & T_{Y'}(\mathbf{3'}) &= 1 - \frac{1}{\sqrt{5}}, & T_{Y'}(\mathbf{3}) &= 1 + \frac{1}{\sqrt{5}} \\
T_{Y'}(\mathbf{5}) &= 3/2, & T_{Y'}(\mathbf{4}_s) &= 1.
\end{align*}
\]

The new notation is that the rep is labelled by its dimension and, if a spinor rep, has a suffix $s$. Distinct reps with the same dimension are distinguished by dashes and a dot means that the rep is one of a conjugate pair, with the same torsion. \(^6\)

It is interesting to note that the analytic torsions of Poincaré space, for the spinor (quaternion), two–dimensional, double valued representations, $\mathbf{2}_s$ and $\mathbf{2}'_s$, \(^6\)

---

\(^6\) We have adopted a convention that the bundle is real if the twisting character is real, and that a complex character is indicated by a power of two. An alternative is to regard real twistings as particular complex ones, and then to square every torsion. One could then elect not to square everything and to work with an implied complex dimension. This we do later.
are half the conjugate fundamental units of $\mathbb{Q}(\sqrt{5})$. These values agree with the relations obtained by Tsuchiya, [24].

In order to amplify this, some explanatory remarks on signs and factors of two are needed and the actual definition of torsion is relevant. We have chosen the usual, but by no means universal, definition, (1), employed by Ray and Singer, [20]. In terms of this $T$, Ray’s, [2], definition is,

$$T\big|_{Ray} = -2 \ln T = \ln(1/T^2),$$  \hspace{1cm} (46)

assuming that the $\zeta$–functions are the same. Since both references [2] and [20] deal with complex line bundles, this should be the case. (See also [1].) However, on lens spaces, Ray calculates, [2], (11),

$$T_{Ray} = -2 \sum_j \ln 2 \sin(\pi rl_j/p)$$  \hspace{1cm} (47)

constructing the $\zeta$–functions from the degeneracies of a real line bundle, so far as we can see.

When Ray’s result is referred to, it is usually the $T$ derived from (47) using (46) that is quoted. We note that this is one half of our result, (35), which incorporates a complex dimension of two in the degeneracies; which is consistent.

In particular, Tsuchiya, [24], states that the torsion on a (one–sided) lens space, $S^3/Z_{10}$, is, from [2],

$$T_{Z_{10}}(\omega^r) = |\omega^r - 1|^2, \quad \omega^{10} = 1,$$  \hspace{1cm} (48)

in conformity with (46) and (47), rather than with (35)

8. Clifford–Klein from lens by induction.

The values of the torsion on Clifford–Klein spaces can be obtained from those on lens spaces, although not quite for free, as claimed by Ray, [2]. The general result is guaranteed by Artin’s theorem on the rational sufficiency of the representations induced from all the cyclic subgroups of the deck group and from the covering theorem of Ray and Singer, [20], theorem 2.6 (see below).

Tsuchiya, [24], on this basis, derives some relations between torsions for the icosahedral case, which we summarise, and extend here.

---

7 The machine algebra we used was not able to reduce to this form. We had to use Pell’s equation.
The basic connecting relation is the Ray–Singer covering theorem,

\[ T(\tilde{\mathcal{M}}/\Gamma_1; \text{Ind } \rho) = T(\tilde{\mathcal{M}}/\Gamma_2; \rho), \quad \Gamma_2 \subset \Gamma_1, \]

applied, in particular to the cyclic subgroups of \( \Gamma \). For example, in the present situation,

\[ T_{Y'}(\text{Ind } \rho) = T_{Z_{10}}(\rho) \tag{49} \]

where \( Z_{10} = \mathbb{Z}_5 \times \mathbb{Z}_2 \), is generated by \( A \) in the Hamilton–Coxeter presentation of \( Y' \), \( (C^2 = B^3 = A^5, ABC) \).

Choosing an irrep for \( \rho \), the dimension of the induced representation is \( 120/10 = 12 \), which is the size of the relevant classes in \( Y' \). An irrep, \( \rho \), is generated by a power, \( \omega^r \), of a primitive tenth root of unity, say \( \omega = \omega_{10} = e^{\pi i/5} \), and a simple induced character calculation yields,

\[ \text{Ind } \omega_{10} = 2_s \oplus 4_s \oplus 6_s \]
\[ \text{Ind } \omega_{10}^3 = 2_s' \oplus 4_s \oplus 6_s \]
\[ \text{Ind } \omega_{10}^5 = 6_s \oplus 6_s. \]

These decompositions translate into the torsion relations,

\[ T_{Y'}(2_s) T_{Y'}(4_s) T_{Y'}(6_s) = T_{Z_{10}}(\omega_{10}) = (3 - \sqrt{5})/2 \]
\[ T_{Y'}(2_s') T_{Y'}(4_s) T_{Y'}(6_s) = T_{Z_{10}}(\omega_{10}^3) = (3 + \sqrt{5})/2 \tag{50} \]
\[ T_{Y'}(6_s)^2 = T_{Z_{10}}(\omega_{10}^5) = 4, \]

the first two of which were given by Tsuchiya, except he does not identify the ten–dimensional rep with \( 4_s \oplus 6_s \).

The same calculation for \( Z_{6} = \mathbb{Z}_3 \times \mathbb{Z}_2 \), generated by \( B \), produces the induction \( \text{Ind } \omega_6 = 2_s \oplus 2_s' \oplus 4_s \oplus 6_s \oplus 6_s \) and thence the relation,

\[ T_{Y'}(2_s) T_{Y'}(2_s') T_{Y'}(4_s) T_{Y'}(6_s) = T_{Z_{6}}(\omega_6) = 1, \quad \omega_6 = e^{i\pi/3}. \tag{51} \]

All the spinor torsions in (45) follow immediately from (50) and (51). We note that, for numerical consistency, the implied complex dimension convention is used.

An even \( Z_{10} \) induction involves the non–spinor representations,

\[ \text{Ind } \omega_{10}^6 = 3 \oplus 4 \oplus 5, \quad \omega_{10}^6 = -\omega_{10}, \]

and it is left as an exercise to evaluate the non–spinor torsion values.
Nothing is gained by looking at say, \( Z_5 \), as this is a subgroup of \( Z_{10} \) and induction is transitive. Thus, for \( Z_5 \), the induced reps are sums of \( Z_{10} \) induced ones. For example \( (\omega_5 = e^{2\pi i/5}) \),

\[
\text{Ind} \omega_5 = \text{Ind} \omega_{10} \oplus \text{Ind}(-\omega_{10}) = (2_s \oplus 4_s \oplus 6_s) \oplus (3 \oplus 4 \oplus 5)
\]

\[
\text{Ind} \omega_3^3 = \text{Ind} \omega_{10}^3 \oplus \text{Ind}(-\omega_{10}^3) = (2_s \oplus 4_s \oplus 6_s) \oplus (3' \oplus 4 \oplus 5),
\]

and the equality of the analytic torsions is equivalent to the trivial relation, \(|\omega^2 - 1| = |\omega - 1| - \omega - 1|\).

This induction method of deriving the torsion values is another, perhaps more elegant, way of organising the information used in the brute force evaluation via (29). Either way, one needs the character tables.

A further check on the numbers is afforded by inducing from the trivial representation. Any cyclic group would do; we select \( Z_{10} \). Then characters produce the decomposition,

\[
\text{Ind}(1) = 1 \oplus 3 \oplus 3' \oplus 5. \tag{52}
\]

The lens space trivial torsion is given by (36), (not computed by Ray),

\[
T_{Z_{10}}(1) = \frac{1}{100} |S^3|
\]

whence, using (49) and the values (45), one finds

\[
T_{Y'}(1) = \frac{1}{120} |S^3|
\]

agreeing with our earlier direct evaluation, (44).

In the light of (50), the question now arises of showing that \( T_{Z_{10}}(\omega_{10}^3) \) is the fundamental unit of \( \mathbb{Q}(\sqrt{5}) \), without numerical evaluation. We do not pursue this point except to say that the appearance of \((3 \pm \sqrt{5})/2\) is perhaps not unexpected in the light of the general similarity of the torsion formula (29) to that for the class number of quadratic forms with positive discriminant \( e.g. \) Lerch, [27], p.366, Zagier, [28], p.81).
9. Polytope group averages.

In general, the group average cannot be given in abstract closed form and one has to resort to numerical addition class by class, at least in this approach. This will be so for the finite (rotational) polytope groups, \{3, 3, 3\}, \{3, 3, 4\}, \{3, 4, 3\} and \{3, 3, 5\}.

The class decompositions for these have been given in [4]. The character tables of the full, reflective polytope groups, \([p, q, r]\), were computed using the CAYLEY computer algebra system (this was superseded by MAGMA in 1993) from their Coxeter presentations and the required subgroup of index two selected. Much of the information can be found in Hurley, [29], except for \{3, 3, 5\}, which is not crystallographic. These character tables can also be found in Warner, [30] and some in Littlewood, [31]. The relevant data have been published in [13] and, being reasonably extensive, will not all be repeated here and we can be brief. We will, however, demonstrate how to calculate the torsion for \{3, 3, 3\}, the class structure of which is

\[
\{3, 3, 3\} = I \oplus 15E \oplus 20K \oplus (2 \times 12)L'.
\]

The \(b\) coefficients\(^8\) in (29), calculated using the angles \(\alpha\) and \(\beta\) for the non–trivial classes, \(E, K\) and \(L'\) are \((0, 4), (1, 1, 4)\) and \((-1, -1, -1, -1, 4)\) respectively, where we have used the order, \(q\), for each class, of 2, 3 and 5. Putting the ingredients together we find,

\[
\ln T_{\{3, 3, 3\}}(\rho) = \frac{1}{60} \left[ \ln 2 \pi^2 \sum_{\gamma} \chi^*(\gamma) - 30 \chi^*_E \ln 2 - 20 \chi^*_K \left( T_3(1) + T_3(2) + 2 \ln 3 \right) \\
12 \left( \chi^*_{L_1'} + \chi^*_{L_2'} \right) \left( T_5(1) + T_5(2) + T_5(3) + T_5(4) - 2 \ln 5 \right) \right] \\
= \frac{1}{60} \ln 2 \pi^2 \sum_{\gamma} \chi^*(\gamma) - \frac{1}{2} \left( \chi^*_E \ln 2 + \chi^*_K \ln 3 + \left( \chi^*_{L_1'} + \chi^*_{L_2'} \right) \ln 5 \right),
\]

where we have used the product, (30), to simplify things. The twisting, \(\chi\), is chosen from the character tables.

The first term on the right of (53) vanishes for a non–trivial twisting, of which there are four irreducible ones. The corresponding values of the torsion \(T\) are \(\sqrt{2/5}\) for both three dimensional irreps, \(5/\sqrt{3}\) and \(\sqrt{3/2}\) for the four–dimensional and five–dimensional ones, respectively.

\(^8\) The \(b\)s are not degeneracies.
We list the analytic torsions for all the twelve irreps of the \(\{3, 3, 3\}\) group. 
\(\sqrt{8/3}, \sqrt{3/2}, 1, 1/\sqrt{2}, 1, 1/\sqrt{2}, \sqrt{8}, 1/\sqrt{4}, \sqrt{4}, 1/\sqrt{2}, 1/\sqrt{2}\).

For the trivial representation, one finds that, for the doubled orbifold fundamental domains,

\[
\ln T_{\{3,3,3\}}(1) = \ln(2\pi^2/60) + \frac{3}{2}\ln 2 + \frac{1}{2}\ln 3
\]

\[
\ln T_{\{3,3,4\}}(1) = \ln(2\pi^2/192) + 2\ln 2 + \frac{1}{2}\ln 3
\]

the torsion parts of which are not of Cheeger’s combinatorial form, (38). That there should be some difference is not unreasonable in view of the calculations of Lück, [32], and Lott and Rothenberg, [33] on the effect of boundaries and non-free actions. We conjecture that any extra terms are consequences of the codimension-two conical singularities. The absence of a \(\ln 5\) term is noted for \(\{3, 3, 3\}\).

10. Conclusion.

The explicit computations performed here show that the twisted torsion is an algebraic number, as expected on general grounds.

On the doubled tessellations terms occur which we attribute to codimension-two singularities.

An alternative technique, which avoids the need to sum over classes and angles, is to describe the polytope groups in terms of their integer degrees, as in [13,34] and use a twisted generating function. This is left for another time.

It has to be said that our discussion is geared to the three-sphere while Ray’s applies to any dimension. The extension of our cancellation to higher dimensions via a generating function is also left for another time.

Appendix 1. The cancellation.

We wish to make some technical remarks concerning the important cancellation (14). The derivation given earlier is somewhat cavalier and needs tightening. For example, the trigonometric factor in the definition (13) which, for \(\theta_L \neq 0\) or \(\theta_R \neq 0\), provides a regulation necessary for the absence of a determinant multiplicative anomaly associated with the product of ‘eigenvalues’, \(n^2 - 1 = (n + 1)(n - 1)\). We flesh out this comment with some algebra.
There are several cases that should be considered separately. First, assume that \( \theta_L \neq 0 \) and \( \theta_R \neq 0 \) i.e. \( \alpha \neq \pm \beta \). Then write,

\[
\chi_n(\theta_L) \chi_n(\theta_R) = \frac{\cos n\alpha - \cos n\beta}{\cos \alpha - \cos \beta},
\]

so that we can concentrate on just the factor \( \cos n\alpha \).

To facilitate the analysis, we make use of the Lerch–Lipshitz \( \zeta \)-function, \( \Phi \), or rather a function simply related to it. Thus we define,

\[
\Xi(\alpha, s, w) = \frac{e^{i\alpha}}{2} \sum_{n=2}^{\infty} \frac{e^{i\alpha}}{(n+w)^s} = e^{i\alpha} \sum_{n=1}^{\infty} \frac{e^{i\alpha}}{(n+w+1)^s}
\]

\[
= e^{i\alpha} \left[ \Phi(e^{i\alpha}, s, w + 1) - \frac{1}{(w+1)^s} \right].
\]

(55)

The sums are defined only for \( \Re s > 0 \) but \( \Phi \) can be continued in a known fashion (e.g. [35], [36]).

For the third sum in \( F \), we employ a very old technique, explained in [37], and more generally in [38], which is sufficient to continue to \( s = 0 \), at least.

To this end, we write it as,

\[
-2 \lim_{u \to 1} \Re \sum_{n=2}^{\infty} \frac{e^{i\alpha}}{(n^2 - u^2)^s} \equiv -2 \lim_{u \to 1} \Re \Upsilon(\alpha, 2s, u), \quad 0 \leq |u| < 2.
\]

In terms of \( \Xi \) and \( \Upsilon \), the essential part of \( F \), (13), is

\[
F(\alpha, s) = \Re \lim_{u \to 1} \left( \Xi(\alpha, 2s, u) + \Xi(\alpha, 2s, -u) - 2\Upsilon(\alpha, 2s, u) \right).
\]

(56)

The \( \Re \) is only a convenient bookkeeping symbol, not applying to the complex nature of \( s \). It will sometimes be removed and replaced at the end of manipulations.

Expanding \( \Upsilon \) in powers of \( u^2 \) one gets,

\[
\Upsilon(\alpha, 2s, u) = \sum_{r=0}^{\infty} u^{2r} \frac{s(s+1) \ldots (s+r-1)}{r!} \Upsilon(\alpha, 2s + 2r, 0)
\]

(57)

where \( \Upsilon(\ast, \ast, 0) \) is just \( \Xi(\ast, \ast, 0) \),

\[
\Upsilon(\alpha, 2s, 0) = \Xi(\alpha, 2s, 0).
\]

(58)

This yields values at \( s = 0 \) because the continuation of \( \Xi \) is known. For \( \alpha \neq 0 \) it is an entire function of \( s \) and can be given as a contour integral, [35].
For example, at $s = 0$, the absence of poles in $\Xi$ means that,

$$-2\Upsilon(\alpha, 0, u) = -2\Xi(\alpha, 0, 0), \quad 0 \leq |u| < 2,$$

and actual calculation shows, correctly, that this cancels the first two sums in (13), i.e. (56). We soon show that such an explicit demonstration is unnecessary.

Next the derivative at $s = 0$ is required, and, again, the absence of poles in $\Xi$, implies,

$$2\Upsilon'(\alpha, 0, u) = 2\Xi'(\alpha, 0, 0) + \sum_{r=1}^{\infty} \frac{u^{2r}}{r} \Xi(\alpha, 2r, 0). \quad (59)$$

using (58).

To evaluate the final summation in (59), we proceed as in [37] and write $\Xi$ as an integral of the corresponding cylinder kernel, $T$,

$$\sum_{r=1}^{\infty} \frac{u^{2r}}{r} \Xi(\alpha, 2r, 0) = \sum_{r=1}^{\infty} \frac{u^{2r}}{r\Gamma(2r)} \int_{0}^{\infty} dt t^{2r-1} e^{-t} T(t, \alpha)$$

$$= \int_{0}^{\infty} (\cosh ut - 1) e^{-t} T(t, \alpha) \frac{dt}{t} \quad (60)$$

$$= \frac{1}{2} \lim_{s \to 0} \int_{0}^{\infty} (e^{st} + e^{-st} - 2) t^{s-1} e^{-t} T(t, \alpha) dt,$$

where $T$ is contained in Lipshitz’ formula, [36] 1.11 (4)

$$\Xi(\alpha, s, w) = \frac{1}{2\Gamma(s)} \int_{0}^{\infty} t^{s-1} e^{-wt} e^{-t} \frac{e^{i\alpha} (e^{i\alpha} - e^{-t})}{\cosh t - \cos \alpha},$$

and an auxiliary regularisation has been put in to assist the evaluation of the (finite) integral.

A further, reverse, application of Lipshitz’ formula yields, simply

$$\sum_{r=1}^{\infty} \frac{u^{2r}}{r} \Xi(\alpha, 2r, 0) = \lim_{s \to 0} \Gamma(s) (\Xi(\alpha, s, u) + \Xi(\alpha, s, -u) - 2\Xi(\alpha, s, 0)). \quad (61)$$

This quantity must be finite, and so, in particular,

$$\Xi(\alpha, 0, 1) + \Xi(\alpha, 0, -1) - 2\Xi(\alpha, 0, 0) = 0$$

which expresses the vanishing of $F(s)$ at $s = 0$ without explicit evaluation, as advertised above.

22
The value of (61) is, therefore,
\[ \sum_{r=1}^{\infty} \frac{u^{2r}}{r} \Xi(\alpha, 2r, 0) = \Xi'(\alpha, 0, u) + \Xi'(\alpha, 0, -u) - 2\Xi'(\alpha, 0, 0) \]
and simple algebra shows that this is equivalent to
\[ F'(\alpha, 0) = 0 \]
from (56), as required.

If \( \alpha = 0 \), only a slight modification is needed. Then \( \Xi(0, s, w) \) is essentially the Hurwitz \( \zeta \)-function and has a single pole, at \( s = 1 \). However this does not change anything since only poles at positive even integers can contribute to the evaluation of (57) at \( s = 0 \). The multiplicative anomaly still vanishes.

This derivation of the cancellation is not much different from the rather crude one in the main body of this paper.

The other cases should be examined separately. If \( \alpha = \pm \beta \neq 0 \), \( i.e. \) the group action is either all left or all right, then
\[ \chi_n(\theta_L) \chi_n(\theta_R) = n \frac{\sin n\alpha}{\sin \alpha} = -\frac{1}{\sin \alpha} \frac{d\cos n\alpha}{d\alpha} \]
and the previous analysis can be applied since it is valid for all \( \alpha \).

The same argument holds when \( \theta_L = \theta_R = 0 \), which gives just the unit element contribution. The degeneracies are the usual polynomials but all one needs is the statement,
\[ \chi_n(0)\chi_n(0) = n^2 = -\frac{d^2}{d\alpha^2} \cos n\alpha \bigg|_{\alpha=0} \]
To convince ourselves that this is true, it is easy to perform an explicit calculation. The quantity involved is,
\[ \sum_{n=2}^{\infty} \frac{n^2}{(n-1)^{2s}} + \frac{n^2}{(n+1)^{2s}} - 2 \frac{n^2}{(n^2 - 1)^s} \]
\[ = 2 \sum_{n=1}^{\infty} \left( \frac{n^2 + 1}{n^{2s}} - \frac{n^2}{(n^2 - 1)^s} \right) - \frac{1}{2^{2s}} \]
\[ = 2 \zeta_R(2s - 2) + 2\zeta_R(2s) - \frac{1}{2^{2s}} - 2\zeta_3(s) , \tag{62} \]
where \( \zeta_3(s) \) is the minimal scalar \( \zeta \)-function on the full three sphere.

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The derivative of this at 0 is

$$4\zeta_R'(-2) + 4\zeta_R'(0) + 2\ln 2 - 2\zeta'_3(0),$$

which vanishes using the known value of $\zeta'_3(0)$ given in [39], for example.\(^9\)

**Appendix 2. Cyclic decompositions.**

Due to the cancellation, the effective eigenvalues are squares of integers, as in (19). This means that the calculations in [7,40] on spectral problems with conformal operators in spherical factors can be applied directly to the present situation.

It was shown in [40] that the $\zeta$–functions on fixed point free, one sided factors, $S^3/\Gamma'$, i.e. homogeneous Clifford–Klein spaces, are related to the zeta functions on cyclic factors (one sided lens spaces) by going through an $S^2$ orbifold intermediary. Applying this result to the (total) effective tau function gives,

$$\tilde{\tau}_{\Gamma'}(s) = \frac{1}{2} \left( \sum_q \tilde{\tau}_{Z_2q}(s) - \tilde{\tau}_{Z_2}(s) \right) \quad (63)$$

where $\Gamma'$ is a binary polyhedral group, one of $T'$, $O'$ or $Y'$. The sum is over the set of conjugate $q$–fold rotation axes appropriate to the SO(3) rotation groups, $T$, $O$ or $Y$. (See [7].) The half and the fact that only even dimensional lens spaces occur are consequences of the binary doubling. The orbit–stabiliser relation has been used in reaching (63).

For technical reasons, only untwisted fields were considered in [40] and so the result (63) applies immediately only to trivial bundles, for which we have the lens space values (36). The values of $q$ are, in each case, $T : 2, 3, 3$, $O : 2, 3, 4$ and $Y : 2, 3, 5$. The analytic torsion is combined in the same way as (63). For example,

$$\ln T_{T'} = \frac{1}{2} \left( 2 \ln |S^3| - 2 \ln(4.6.6) + 2 \ln 2 \right)$$

$$= \ln |S^3| - \ln(4.6.6) + \ln 2$$

$$= \ln \left( |S^3|/24 \right) + \ln 24 - \ln(4.6.6) + \ln 2$$

$$= \ln \left( |S^3|/24 \right) - \ln 3,$$

\(^9\) This is really no more than an algebraic check as the method in [39] is just that used here. However there are other techniques, such as Plana summation, which yield the same value.
and similarly,
\[ \ln T_O = \ln \left( |S^3|/48 \right) - \ln 2, \quad \ln T_Y = \ln \left( |S^3|/120 \right), \]
in agreement with our previous results, (44).

The orbit–stabiliser relation does not hold for the cyclic case itself, nor for the dihedral group. For the latter, one has the relation for the conformal \( \zeta \)-functions,\(^{10}\)
\[ \zeta_{D_{2q}'} = \frac{1}{2} \left( \zeta_{Z_{2q}} + 2 \zeta_{Z_4} - \zeta_{Z_2} \right) , \]
and likewise for the effective \( \tau \) function. Calculation rapidly reproduces (41).

**Appendix 3. Some homology.**

The relation obtained in [40] between the conformal \( \zeta \)-functions on one–sided factors \( S^3/\Gamma' \) and on the orbifolds, \( S^2/\Gamma \), is paralleled by a relation between the homologies of fundamental domains. Here we just discuss the first homology group, which is sufficient for three dimensions.

The natural action of the rotation group, \( \Gamma \), on \( \mathbb{R}^3 \) has a (double) Möbius corner as its (infinite) fundamental domain, \( \mathcal{M} = \mathbb{R}^3/\Gamma \), (cf [41]). Denoting the generators of \( \Gamma \) by \( A_i \) for rotations through the angles \( 2\pi/\nu_i \), about the vertices of a given spherical triangle on \( S^2 \), we consider the presentation \( \Gamma = (A_i : A_{\nu_i}, A_1 A_2 A_3) \).

Thus \( H_1(\mathcal{M},\mathbb{Z}) \cong \Gamma/F \) where \( F \) is the commutator subgroup of \( \Gamma \), that is
\[ H_1(\mathcal{M},\mathbb{Z}) = (A_i : A_{\nu_i}^{\nu}, A_1 A_2 A_3, A_1 A_2 A_1^{-1} A_2^{-1}, (1 \rightarrow 2, 2 \rightarrow 3), (1 \rightarrow 3, 2 \rightarrow 1)) \]
from which, one finds
\[ H_1(\mathbb{R}^3/\Gamma,\mathbb{Z}) = \mathbb{Z}_3, \mathbb{Z}_2, \{\text{id}\} \quad \text{for} \ T, O, Y \quad \text{respectively}. \quad (64) \]

The cyclic group, \( \mathbb{Z}_q \) is already abelian and so, elementarily,
\[ H_1(\mathbb{R}^3/\mathbb{Z}_q,\mathbb{Z}) = \mathbb{Z}_q. \]

These results coincide with those for the corresponding Clifford–Klein \( S^3 \) factors, that is, generalising slightly,
\[ H_*(\mathbb{R}^3/\Gamma,\mathbb{Z}) = H_*(S^3/\Gamma',\mathbb{Z}). \]

\(^{10}\) One could just set \( q = 2, 2, q \) in (63).
A U(1) bundle (a complex field) can be twisted by a representation of $H_1$ through phase factors around the triangle vertices,

$$a(\gamma) = e^{2\pi i n \mu}$$

where $n = (n_1, n_2, n_3)$ is the set of occurrences of the generators, $A_1, A_2, A_3$, in the word presentation of $\gamma$ and $\mu_i = s_i/\nu_i$.

The relation condition $a(A_1 A_2 A_3) = a(E) = 1$ translates into $\sum_i \mu_i \in \mathbb{Z}$ with $0 \leq \mu_i < 1$ which determines the possible sets $s = (s_1, s_2, s_3)$ for given $\nu_i$. One straightforwardly finds, in addition to the trivial solution $(0, 0, 0)$, the two (complex) representations $s = (0, 2, 1)$ and $(0, 1, 2)$ for $T$ and the real one, $(1, 0, 2)$, for $O$. There are no such reps for $Y$. This result is equivalent to the homology (64).

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11 This just confirms the known character tables.
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