On the numerical solution of second order two-dimensional Laplace equations using the alternating-direction implicit method

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Abstract

In this manuscript, we consider in detail numerical approach of solving Laplace’s equation in 2-dimensional region with given boundary values which is based on the Alternating Direction Implicit Method (ADI). This method was constructed using Taylor’s series expansion on the second order Laplace equation leading to a linear algebraic system. Solving the algebraic system, leads to the unknown coefficients of the basis function. The techniques of handling practical problems are considered in detail. The results obtained compared favorably with the results obtained from the Finite difference method constructed by Dhumal and Kiwne and the exact solution. Thus the ADI method can as well be used for the numerical solution of steady-state Laplace equations.

Introduction

Many applications in Science and Engineering have found Laplace’s equation very useful. A numerical solution of the equation can be useful in finding the distribution of temperature in a solid body, the potential distribution in a region of interest and so on. The Laplace’s equation is an important equation that has been associated with so many applications in the field of Science and Engineering. These applications are not only limited to electrostatics, but fluid dynamics as well as steady state heat conduction. It is a partial differential equation (PDE) of second order which is elliptic in nature. A general theory of the solution of Laplace’s equation is called potential theory. The solutions of Laplace’s equation are usually known as harmonic functions and are very useful in science and engineering as earlier mentioned. Laplace’s equation has both analytic and numerical solutions. Numerical solution of Laplace’s equation is obtained by different methods as applied to many linear PDEs. These include finite difference method (FDM), finite element method (FEM) as well as the Method of Moments (MoM). Laplace’s equation itself does not determine the potential of a system, thus, suitable boundary conditions must be applied to obtain the potential of any system of interest Griffiths (1999). A boundary condition can be Dirichlet if it is specified at the surface of the boundary of a system, it can be Neumann if the normal derivative of the function is specified at the boundary or it can be mixed if part of the boundary is Dirichlet and the rest Neumann Dhumal and Kiwne (2014). Laplace’s equation has received attention from many Researchers for different physical applications. A numerical solution of Laplace’s equation was obtained by comparing numerical result with the analytic solution and the results were similar Guthrie (2010). Firoozjaee et al (2018) and Anup et al (2019) dealt with the numerical solution of Fokker Planck equation with Caputo-Fabrizio Fractional Derivative and Nonlinear Reaction Advection Diffusion equation respectively. Patil and Prasad (2013) solved Laplace’s equation numerically using FDM, FEM and MCM. The results obtained were compared with the analytic solution and observed that the solution of each method used were close with the exact solution. The Laplace equation was solved numerically by method known as finite differences for electrical potentials in a certain region of space. Atsue et al (2018) used finite Difference Method (FDM) to discretize Laplace’s equation and the resulting equation was solved numerically using iterative Gauss Seidel and the Successive over Relaxation (SoR) methods. The results obtained revealed that SoR method was more effective in terms of accuracy and speed of convergence.
Papanikos and Gousidou-Koutita (2015) applied Finite Element Method and Finite Difference Method for 2-Dimension Elliptic Partial Differential Equations and the method was more effective and convenient. The equilibrium temperature of the Laplace’s equation for steady state heat flow in a two-dimensional region was solved with fixed temperature on the boundaries computed on a square grid using successive over relaxation with parity ordering of the grid element by Young (1971) and Lapidus and Pinder (1982). Per Brinch (1992) solved the Laplace’s equation by a Pascal algorithm and a parallel version of this algorithm has been tested on a computing surface. Young (1954) discusses the Laplace’s equation applicable to Physical phenomena that vary continuously in space and time. Mitra (2010) obtained the solution of Laplace’s equation using two numerical methods, Finite Difference Method (FDM) and Boundary Element Method (BEM), both numerical methods were compared the results obtained were both observed to have agreed with the analytic solution.

In our case, we shall use the Alternating Direction Implicit method (ADI) to discretize the Laplace’s equation in 2-Dimensions for the determination of the approximate solution.

**Derivation of the improved adi method for the laplace equation**

First we consider the second order Laplace’s equation of the form

\[ u_{xx} + u_{yy} = 0 \]  \hspace{1cm} (1.0)

To obtain its numerical solution, we replace the partial derivatives with the corresponding difference equations using the Taylor’s series expansion as

\[ u(x + h, y) = u(x, y) + hu_x(x, y) + \frac{1}{2!}h^2 u_{xx}(x, y) + \frac{1}{3!}h^3 u_{xxx}(x, y) + \ldots \]  \hspace{1cm} (1.2)

\[ u(x - h, y) = u(x, y) - hu_x(x, y) + \frac{1}{2!}h^2 u_{xx}(x, y) - \frac{1}{3!}h^3 u_{xxx}(x, y) + \ldots \]  \hspace{1cm} (1.3)

Subtracting (1.2) from (1.3) while neglecting the terms in \( h^3, h^4, \ldots \) and solving for \( u_x \) to obtain

\[ u_x(x, y) \approx \frac{1}{2h}[u(x + h, y) - u(x - h, y)] \]  \hspace{1cm} (1.4)

Similarly, we obtain

\[ u(x, y + k) = u(x, y) + ku_y(x, y) + \frac{1}{2}k^2 u_{yy}(x, y) + \ldots \]  \hspace{1cm} (1.5)

\[ u(x, y - k) = u(x, y) - ku_y(x, y) + \frac{1}{2}k^2 u_{yy}(x, y) + \ldots \]  \hspace{1cm} (1.6)

Subtracting (1.5) from (1.6) and neglecting terms in \( k^3, k^4, \ldots \) and solving for \( u_y \) to obtain

\[ u_y(x, y) \approx \frac{1}{2k}[u(x, y + k) - u(x, y - k)] \]  \hspace{1cm} (1.7)

The second derivatives is obtained by adding (1.2) and (1.3) while neglecting the terms in \( h^3, h^4, \ldots \) and solving for \( u_{xx} \) to arrive at

\[ u_{xx}(x, y) \approx \frac{1}{h^2}[u(x + h, y) - 2u(x, y) + u(x - h, y)] \]  \hspace{1cm} (1.8)

Similarly, adding (1.5) and (1.6) while neglecting terms in \( h^3, h^4, \ldots \) and solving for \( u_{yy} \) to get

\[ u_{yy}(x, y) = \frac{1}{k^2}[u(x, y + k) - 2u(x, y) + u(x, y - k)] \]  \hspace{1cm} (1.9)

Substitute (1.8) and (1.9) into the Laplace equation (1.0) and letting \( k = h \) to obtain a simple formula given by

\[ u(x - h, y) + u(x, y + h) + u(x - h, y) + u(x, y - h) - 4u(x, y) = 0 \]  \hspace{1cm} (1.10)

This is a difference equation corresponding to (1.0), \( h \) is called the step size. Equation (1.10) relates \( u(x, y) \) to \( u(x - h, y) \), \( u(x, y + h) \), \( u(x - h, y) \) and \( u(x, y - h) \) at the four neighboring points as shown in figure 1 below, \( u(x, y) \) represents the mean at the four neighboring points.

From equation (1.10), letting \( x = i \), \( y = j \) and \( h = 1 \) we have

\[ u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} + 4u_{ij} = 0 \]  \hspace{1cm} (1.11)
Rearranging (1.11) to obtain
\[ u_{i-1,j} - 4u_{i,j} + u_{i+1,j} = -u_{i,j-1} - u_{i,j+1} \]  
so that the left side belongs to \( y \rightarrow \text{row } j \) and the right side belongs to \( x \rightarrow \text{column } i \), of course, we can write (1.12) as
\[ u_{i,j-1} - 4u_{i,j} + u_{i,j+1} = -u_{i-1,j} - u_{i+1,j} \]
so that the left side belongs to column \( i \), and the right side belongs to row \( j \).

In this method, we iterate at every step point by choosing an arbitrary starting value \( u_{i,j}^0 \) where \( m = 0 \). In each step, we compute new values at all the mesh points, in the first step we use an iteration formula resulting from (1.12) and in the next step an iteration formula resulting from (1.13) and so on in alternating order.

Suppose approximations \( u_{i,j}^m \) have been computed, then to obtain the next approximation \( u_{i,j}^{(m+1)} \), we substitute \( u_{i,j}^m \) on the right side of (1.12) and solve for \( u_{i,j}^{(m+1)} \) on the left side, that is, we use...
\[ u^{(m+\frac{1}{2})}_{i-1,j} - 4u^{(m+\frac{1}{2})}_{i,j} + u^{(m+\frac{1}{2})}_{i+1,j} = -u^{m}_{i,j-1} - u^{m}_{i,j+1} \]  

(1.14)

Using (1.14) for a fix row \((j)\) and for all internal mesh points in this row \((j)\) gives a system of \(N\) equations \((N\) gives the number of internal mesh points per row) in \(N\) unknowns; with the new approximations of \(u\) at these mesh points. Equation (1.14) involves not only approximations computed in the previous step but also given boundary values.

In the next step, we alternate direction, that is, we compute the next approximate \(u^{(m+1)}_{i,j}\) column by column from \(u^{(m+\frac{1}{2})}_{i,j}\) and the given boundary values, using a formula obtained from (1.13) by substituting \(u^{(m+1)}_{i,j}\) on the right.

\[ u^{(m+1)}_{i,j-1} - 4u^{(m+1)}_{i,j} + u^{(m+1)}_{i,j+1} = -u^{m}_{i-1,j} - u^{m}_{i+1,j} \]  

(1.15)

For each fixed \(i\) gives a system of \(N\) equations \((N\) gives the number of internal mesh points per column) in \(N\) unknowns, which we solve by Gauss elimination, then proceed to the next column, and so on until all columns are computed.

**Numerical implementation of the method**

In this section, the techniques of handling practical problems are considered in detail and some examples contained in the literature are solved. Absolute errors of the approximate solutions is computed and compared with results obtained from existing methods particularly those contained in the literature.

**Problem 1.** Consider the Laplace’s initial value problem

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x, y < 4 \]

\[ u(x, 0) = u(0, y) = u(4, y) = 0, \quad u(x, 4) = \frac{1}{2} \sin \left(\frac{1}{4} \pi x\right) \]

The exact solution is given as,

\[ u(x, y) = \frac{1}{2} \sinh \pi \sin \left(\frac{1}{4} \pi x\right) \sinh \left(\frac{1}{4} \pi y\right) \]

Using the ADI method as explained with a grid step size \(h = 1\) and starting values 0.

\[ u^{(m+\frac{1}{2})}_{i-1,j} - 4u^{(m+\frac{1}{2})}_{i,j} + u^{(m+\frac{1}{2})}_{i+1,j} = -u^{m}_{i,j-1} - u^{m}_{i,j+1} \]

Now, for \(t = 1, m = 0\), and starting with \(j = 1\) and for \(i = 1, 2, 3\) in matrix, we have

\[
\begin{pmatrix}
-4 & 1 & 0 \\
1 & -4 & 1 \\
0 & 1 & -4
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} u^1_{i,1} \\
\frac{1}{2} u^2_{i,1} \\
\frac{1}{2} u^3_{i,1}
\end{pmatrix}
= 0
\Rightarrow
\begin{pmatrix}
\frac{1}{2} u^1_{i,1} \\
\frac{1}{2} u^2_{i,1} \\
\frac{1}{2} u^3_{i,1}
\end{pmatrix} = 0
\]

for \(j = 2, i = 1, 2, 3\) in matrix, we have

\[
\begin{pmatrix}
-4 & 1 & 0 \\
1 & -4 & 1 \\
0 & 1 & -4
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} u^1_{i,2} \\
\frac{1}{2} u^2_{i,2} \\
\frac{1}{2} u^3_{i,2}
\end{pmatrix}
= 0
\Rightarrow
\begin{pmatrix}
\frac{1}{2} u^1_{i,2} \\
\frac{1}{2} u^2_{i,2} \\
\frac{1}{2} u^3_{i,2}
\end{pmatrix} = 0
\]

also for \(j = 3, i = 1, 2, 3\) in matrix, we have

\[
\begin{pmatrix}
-4 & 1 & 0 \\
1 & -4 & 1 \\
0 & 1 & -4
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} u^1_{i,3} \\
\frac{1}{2} u^2_{i,3} \\
\frac{1}{2} u^3_{i,3}
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{2} \sin \left(\frac{1}{4} \pi \right) \\
\frac{1}{2} \sin \left(\frac{1}{4} \pi \right) \\
\frac{1}{2} \sin \left(\frac{3}{4} \pi \right)
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
\frac{1}{2} u^1_{i,3} \\
\frac{1}{2} u^2_{i,3} \\
\frac{1}{2} u^3_{i,3}
\end{pmatrix} = \begin{pmatrix}
0.136\ 729\ 540\ 1695 \\
0.193\ 364\ 770\ 0848 \\
0.136\ 725\ 401\ 695
\end{pmatrix}
\]
For the next iterations, putting \( m = 5 \), \( 10 \) and \( 15 \) alternating direction using equation (1.15), by choosing \( i = 1 \) and for \( j = 1, 2, 3 \) we compare the results with the Finite difference method using Scientific workplace 3.0 program as shown in tables 1–3 respectively.

**Problem 2.** Consider the boundary value problem

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 2, \quad 0 < y < 1
\]

\[
u(x, 0) = u(0, y) = u(2, y) = 0, \quad u(x, 1) = x(2 - x)
\]
The exact solution is given as

\[ u(x, y) = \frac{32}{\pi^3 \sinh \left( \frac{x}{2} \right)} \left( \sin \left( \frac{\pi x}{2} \right) \sinh \left( \frac{\pi y}{2} \right) \right) \]

Similarly using the ADI algorithm as discussed in problem 1 and setting the initial values at 0, choosing \( h = \frac{1}{3} \) we generated the results for \( m = 5 \text{ s}, 10 \text{ s} \) and \( 15 \text{ s} \) as shown in tables 4–6 respectively.

### Table 4. Results and errors obtained for problem 2 at \( t = 2, \ m = 5 \text{ s} \).

| \( u_{ij} \) | Exact | FDM | ADI | Absolute error |
|-----------|-------|-----|-----|----------------|
| \( u_{1,1} \) | 0.122 846 21 | 0.127 048 26 | 0.123 411 37 | 4.20E-3 | 5.65E-4 |
| \( u_{1,2} \) | 0.280 147 91 | 0.292 704 83 | 0.289 630 36 | 1.26E-2 | 9.48E-3 |
| \( u_{2,1} \) | 0.212 775 87 | 0.215 488 22 | 0.210 187 79 | 2.71E-3 | 3.66E-3 |
| \( u_{2,2} \) | 0.485 230 41 | 0.488 215 49 | 0.481 889 101 | 2.99E-3 | 3.34E-3 |
| \( u_{3,1} \) | 0.245 692 42 | 0.239 415 33 | 0.246 869 11 | 6.28E-3 | 9.97E-4 |
| \( u_{3,2} \) | 0.560 295 81 | 0.549 631 08 | 0.555 780 02 | 1.07E-2 | 4.52E-3 |
| \( u_{4,1} \) | 0.212 775 87 | 0.215 488 22 | 0.209 187 79 | 2.71E-3 | 3.59E-3 |
| \( u_{4,2} \) | 0.485 230 41 | 0.488 215 49 | 0.482 889 51 | 2.99E-3 | 2.34E-3 |
| \( u_{5,1} \) | 0.122 846 21 | 0.127 048 26 | 0.126 955 26 | 4.20E-3 | 5.65E-4 |
| \( u_{5,2} \) | 0.280 147 91 | 0.292 704 83 | 0.289 630 36 | 1.26E-2 | 9.48E-3 |

### Table 5. Results and errors obtained for problem 2 at \( t = 3, \ m = 10 \text{ s} \).

| \( u_{ij} \) | Exact | FDM | ADI | Absolute error |
|-----------|-------|-----|-----|----------------|
| \( u_{1,1} \) | 0.122 846 21 | 0.127 048 26 | 0.127 044 65 | 4.20E-3 | 4.17E-3 |
| \( u_{1,2} \) | 0.280 147 91 | 0.292 704 83 | 0.289 630 36 | 1.26E-2 | 1.21E-2 |
| \( u_{2,1} \) | 0.212 775 87 | 0.215 488 22 | 0.215 326 97 | 2.71E-3 | 2.49E-3 |
| \( u_{2,2} \) | 0.485 230 41 | 0.488 215 49 | 0.488 043 17 | 2.99E-3 | 2.81E-3 |
| \( u_{3,1} \) | 0.245 692 42 | 0.246 689 11 | 0.246 503 11 | 9.97E-4 | 8.11E-4 |
| \( u_{3,2} \) | 0.560 295 81 | 0.555 780 02 | 0.555 582 45 | 4.52E-3 | 4.71E-3 |
| \( u_{4,1} \) | 0.212 775 87 | 0.215 488 22 | 0.215 327 13 | 2.71E-3 | 2.55E-3 |
| \( u_{4,2} \) | 0.485 230 41 | 0.488 215 49 | 0.488 043 17 | 2.99E-3 | 2.81E-3 |
| \( u_{5,1} \) | 0.122 846 21 | 0.127 048 26 | 0.126 955 26 | 4.20E-3 | 4.11E-3 |
| \( u_{5,2} \) | 0.280 147 91 | 0.292 704 83 | 0.289 630 36 | 1.26E-2 | 1.25E-2 |

### Table 6. Results and errors obtained for problem 2 at \( t = 4, \ m = 15 \text{ s} \).

| \( u_{ij} \) | Exact | FDM | ADI | Absolute error |
|-----------|-------|-----|-----|----------------|
| \( u_{1,1} \) | 0.122 846 21 | 0.127 048 26 | 0.127 044 65 | 4.20E-3 | 4.17E-3 |
| \( u_{1,2} \) | 0.280 147 91 | 0.292 704 83 | 0.292 701 21 | 1.26E-2 | 1.26E-2 |
| \( u_{2,1} \) | 0.212 775 87 | 0.215 488 22 | 0.209 187 79 | 2.71E-3 | 2.63E-3 |
| \( u_{2,2} \) | 0.485 230 41 | 0.488 215 49 | 0.488 043 17 | 2.99E-3 | 2.65E-3 |
| \( u_{3,1} \) | 0.245 692 42 | 0.246 689 11 | 0.246 503 11 | 9.97E-4 | 9.09E-4 |
| \( u_{3,2} \) | 0.560 295 81 | 0.555 780 02 | 0.555 772 74 | 4.52E-3 | 4.46E-3 |
| \( u_{4,1} \) | 0.212 775 87 | 0.215 488 22 | 0.215 327 13 | 2.71E-3 | 2.63E-3 |
| \( u_{4,2} \) | 0.485 230 41 | 0.488 215 49 | 0.488 043 17 | 2.99E-3 | 2.63E-3 |
| \( u_{5,1} \) | 0.122 846 21 | 0.127 048 26 | 0.126 955 26 | 4.20E-3 | 4.17E-3 |
| \( u_{5,2} \) | 0.280 147 91 | 0.292 704 83 | 0.292 701 21 | 1.26E-2 | 1.26E-2 |
Conclusion

This work considered in detail numerical approach of solving Laplace’s equation in 2-dimension which was based on the Alternating Direction Implicit Method (ADI) approach. The method was constructed using Taylor’s series expansion on the second order Laplace equation leading to a linear algebraic system. Solving the algebraic system, leads to the unknown coefficients of the basis function. The techniques of handling practical problems were considered in detail. The results obtained compared favorably with the results obtained from the Finite difference method constructed by Dhumal and Kiwne and the exact solution. Thus the ADI method is considered as a numerical method for the solution of steady-state Laplace equations. The absolute errors revealed that the ADI method converges to the FDM method.

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References

Anup S, Das S, Ong S H and Jafari H 2019 Numerical solution of nonlinear reaction advection diffusion equation J. Comput. Nonlinear Dyn. 14 041003
Atsue T, Tikyaa E V and Nwokike S C 2018 A numerical solution of the 2D Laplace’s equation for the estimation of electric potential distribution Journal of Scientific and Engineering Research 5 268–76
Dhumal M L and Kiwne S B 2014 Finite difference method for Laplace equation International Journal of Statistics and Mathematics 9 11–3
Firoojaee M A, Jafari H, Lia A and Baleanu D 2018 Numerical approach of Fokker-Planck equation with caputo-fabrizio fractional derivative using ritz approximation J. Comput. Appl. Math. 339 367–73
Griffiths D J 1999 Introduction to Electrodynamics 3rd Edn (N.J: Prentice Hall Inc.) p 110
Guthrie M 2010 Solving Laplace’s equation with MATLAB Using the Method of Relaxation
Lapidus L and Pinder G F 1982 Numerical Solutions of Partial Differential Equations in Science and Engineering (Wiley-Interscience)
Mitra A K 2010 Finite Difference Method for the Solution of Laplace Equation (Department of Aerospace Engineering Iowa State University)
Papanikos G and Gousidou-Koutita M C 2015 A computational study with finite element method and finite difference method for 2D elliptic partial differential equations Applied Mathematics 6 2104
Patil P V and Prasad J S 2013 Numerical solution for two dimensional laplace equation with dirichlet boundary conditions IOSR Journal of Mathematics (IOSR-JM) 6 66–75
Peaceman—Rachford 2012 ADI scheme for the two-dimensional flow of a second-grade fluid Int. J. Numer. Methods Heat Fluid Flow 22 228
Per Brinch H 1992 ‘Numerical Solution of Laplaces Equation’ Electrical Engineering and Computer Science
Young D M 1971 Iterative Solution of Large Linear Systems (New York: Academic)
Young D M 1954 Iterative Methods for Solving Partial Differential Equations of Elliptic Type Transactions of the American Mathematics Society 76 92–111