In cosmology the equation of state (EoS) parameter \( \omega \), the ratio of pressure \( (p) \) to energy density \( (\rho) \), for a species of energy carrier, plays a crucial role in determining the properties of the expanding universe [1]. It determines the energy density at a given temperature since \( \rho \) evolves dramatically differently from that for photons. For \( \omega = 1 \), even if \( \rho_u(T_D) \) at a high decoupling temperature \( T_D \) is very small, it is possible to have a large relic density \( \rho_u(T_\gamma^0) \) at present photon temperature \( T_\gamma^0 \), large enough to play the role of dark matter. We calculate \( T_D \) and \( \rho_u(T_\gamma^0) \) using photon-unparticle interactions for illustration.

The concept of unparticle [4] stems from the observation that a high energy theory with a nontrivial infrared fixed-point at some scale \( A_T \) may develop a scale-invariant degree of freedom below the scale, named unparticle. The notion of mass does not apply to such an identity; instead, its kinematics is mainly determined by the naive arguments of conformal invariance are invoked or as occurring in ordinary particle physics processes, or in collider physics effects [5, 6] to theoretical issues [7, 8] and cosmological and astrophysical implications [9, 10], to mention a few [11]. In a glut of unparticle phenomenological studies, either unparticles are treated at zero temperature as occurring in ordinary particle physics processes, or the naive arguments of conformal invariance are invoked for the unparticle EoS with the massless photon as an analogue in mind. In this work we work out thermodynamics of unparticles directly from their basic properties, which turns out to be generally different from that of photons.
The thermodynamics of a gas of bosonic particles with mass \( \mu \) is determined by the partition function:

\[
\ln Z(\mu^2) = -g_s V \int \frac{d^4 p}{(2\pi)^4} 2\pi^3 p^0 \theta(p^0) \\
\times \delta(p^2 - \mu^2) \ln(1 - e^{-p^\beta}),
\]

where \( V, \beta = T^{-1} \) are the volume and inverse temperature in natural units respectively, and \( g_s \) accounts for degrees of freedom like spin. The density of states in four-momentum space is proportional to the \( \delta \) function due to the dispersion relation for particles. There is no such a constraint in the case of unparticles, whose density of states is dictated by the scaling dimension \( d_\mathcal{U} \) of the corresponding field to be proportional to \( \beta \):

\[
\frac{d^4 p}{(2\pi)^4} \theta(p^0) \theta(p^2)(p^2)^{d_\mathcal{U} - 2}.
\]

Nevertheless, we can interpret it in terms of a continuous collection of particles with the help of a spectral function \( \theta(\mu^2) \propto \theta(\mu^2)(\mu^2)^{d_\mathcal{U} - 2} \):

\[
2\pi \theta(\mu^0) \delta(p^2 - \mu^2) \frac{d^4 p}{(2\pi)^4} \phi(\mu^2) d\mu^2
\]

In this construction, compared to the case of particles of a definite mass, \( \mu^2 \) serves as a new quantum number to be summed over with the weight \( \phi(\mu^2) \).

To write down the partition function for unparticles, we have to normalize \( \phi \) correctly. Since unparticles exist only below the scale \( \Lambda_\mathcal{U} \), the spectrum must terminate there. Beyond the scale, unparticles can be resolved and are no more the suitable degrees of freedom to cope with. This also implies that we should require \( \beta/\Lambda_\mathcal{U} > 1 \) for self-consistency. We thus find the normalized spectrum,

\[
\phi(\mu^2) = (d_\mathcal{U} - 1) \Lambda_\mathcal{U}^{(2 - d_\mathcal{U})} \theta(\mu^2)(\mu^2)^{d_\mathcal{U} - 2},
\]

which has the correct limit \( \delta(\mu^2) \) as \( d_\mathcal{U} \to 1^+ \). Note that integrability at the lower end of \( \mu^2 \) requires \( d_\mathcal{U} \geq 1 \). The partition function for unparticles is

\[
\ln Z = \int_0^{\Lambda_\mathcal{U}^2} d\mu^2 \phi(\mu^2) \ln Z(\mu^2)
\]

\[
= \frac{-g_s V (d_\mathcal{U} - 1)}{4\pi^2 \beta^3 (\beta \Lambda_\mathcal{U})^{2(d_\mathcal{U} - 1)}} \int_0^{\beta \Lambda_\mathcal{U}^2} dy \ y^{d_\mathcal{U} - 2}
\]

\[
\times \int_y^\infty dx \ \sqrt{x - y} \ln(1 - e^{-\sqrt{x}})
\]

For \( \beta/\Lambda_\mathcal{U} > 1 \), the above integrals factorize to good precision due to the exponential:

\[
\ln Z = \frac{-g_s V (d_\mathcal{U} - 1)}{4\pi^2 \beta^3 (\beta \Lambda_\mathcal{U})^{2(d_\mathcal{U} - 1)}} \Gamma(2d_\mathcal{U} + 2) \zeta(2d_\mathcal{U} + 2),
\]

where \( \Gamma, B, \zeta \) are standard functions and integration by parts has been used for \( 2d_\mathcal{U} + 1 > 0 \). Using the definition of \( B \) function, the apparent singularity at \( d_\mathcal{U} = 1 \) can be removed explicitly:

\[
\ln Z = \frac{g_s V}{\beta^3 (\beta \Lambda_\mathcal{U})^{2(d_\mathcal{U} - 1)}} \frac{C(d_\mathcal{U})}{4\pi^2},
\]

with \( C(d_\mathcal{U}) = B(3/2, d_\mathcal{U}) \Gamma(2d_\mathcal{U} + 2) \zeta(2d_\mathcal{U} + 2) \). It is now straightforward to work out the quantities:

\[
p_\mathcal{U} = g_s T^4 \left( \frac{T}{\Lambda_\mathcal{U}} \right)^{2(d_\mathcal{U} - 1)} \frac{C(d_\mathcal{U})}{4\pi^2},
\]

\[
p_\mathcal{U} = (2d_\mathcal{U} + 1) g_s T^4 \left( \frac{T}{\Lambda_\mathcal{U}} \right)^{2(d_\mathcal{U} - 1)} \frac{C(d_\mathcal{U})}{4\pi^2}.
\]

Again the case of massless particles is recovered correctly by setting \( d_\mathcal{U} = 1 \) and \( C(1) = 2\pi^4/45 \). The above results imply the following EoS parameter for unparticles:

\[
\omega_\mathcal{U} = (2d_\mathcal{U} + 1)^{-1}.
\]

The results for fermionic unparticles can be obtained by replacing \( C(d_\mathcal{U}) \) by \( (1 - 2^{-2(d_\mathcal{U} + 1)})C(d_\mathcal{U}) \).

It is clear that \( \omega_\mathcal{U} \) is very different from that for photons or CDM, and generically lies in between for \( d_\mathcal{U} > 1 \). This is in contrast to the naive expectation based on conformal theory arguments and the massless photon analogue. This arises essentially from the fact that unparticles exist only below a finite energy scale \( \Lambda_\mathcal{U} \) as reflected in the spectral function \( \phi(\mu^2) \) while a conventional conformal theory is not characterized by such a scale. If the limit \( \Lambda_\mathcal{U} \to \infty \) were naively taken, which means there would be no unparticles in the infrared, \( \rho_\mathcal{U} \) would vanish trivially. This is indeed not the case interested in here. The factor \( \Lambda_\mathcal{U}^{(2 - d_\mathcal{U})} \) in \( p_\mathcal{U} \), \( \rho_\mathcal{U} \) acts as an effective parameter in the low temperature theory, and the presence of \( \Lambda_\mathcal{U} \) reflects its connection to the underlying theory that produces the unparticle. This connection between low and high energy theories is completely expected, as for instance, in thermodynamics of solids viewed from atomic physics.

The ensemble of unparticles thus provides a new form of energy density in our universe, which will have important repercussions for cosmology. We now study their implications in our expanding universe by concentrating on their contribution to the energy density in the universe. The unparticle energy density at present is determined by its initial value at the decoupling temperature \( T_D \) where unparticles drop out of the thermal equilibrium with standard model (SM) particles, and its evolution thereafter which is closely related to the EoS parameter.

In an FRW expanding universe, the energy density \( \rho(T) \) (or \( \rho(R) \)) of a species after decoupling from equilibrium is given by

\[
\rho(R) = \rho(R_D) \left( \frac{R_D}{R} \right)^{3(1 + \omega)},
\]

where \( R_D \) is the scale factor of the expanding universe at decoupling. From now on, we will interchange the notations \( \rho(T) \) and \( \rho(R) \) freely. Since photon expansion follows
if unparticles are singlets under the SM gauge group \[14\].

The interactions between unparticles and SM particles even

\[\text{Fig. 1. The double ratio } r_r(T^1_r)/r_r(T_{BBN}) \text{ as a function of } d_U.\]

A practical study with a global fitting should make a survey of all such interactions and those induced by thermal effects. For the purpose of illustration here, we consider below the unparticle-photon interactions:

\[\mathcal{L} = \lambda A^{d_u}_U F^{\mu\nu} F_{\mu\nu} + \tilde{\lambda} A^{d_s}_U \tilde{F}^{\mu\nu} F_{\mu\nu}, \]

where \( F, \tilde{F} \) are respectively the electromagnetic field tensor and its dual, and the coefficients \( \lambda, \tilde{\lambda} \) can be expressed in terms of the standard ones in Ref. \[14\]. We will treat the two interactions one by one.

The above interactions can bring photons and unparticles into equilibrium. Taking the \( \lambda \) term as an example, the cross section for \( \gamma \gamma \rightarrow U \) is

\[\sigma(s) = \frac{1}{4} \lambda^2 \left( \frac{s}{A_U} \right)^{d_U} \frac{1}{s} A_{d_a}, \]

where

\[A_{d_a} = \frac{16\pi^{5/2} \Gamma(d_U + 1/2)}{(2\pi)^{2d_a} \Gamma(d_U - 1) \Gamma(2d_U)}, \]

is a normalization factor for the unparticle density of states suggested in Ref. \[14\], and the interaction rate is

\[\Gamma \approx n_\gamma \sigma(s)c = \frac{\zeta(3) A_{d_a}}{8\pi^2} \lambda^2 T \left( \frac{2T}{\Lambda_U} \right)^{2d_u}, \]

where \( n_\gamma \) is the photon number density, and we have used \( s = (2T)^2 \).

This rate is compared with the Hubble parameter \( H = 1.66g_{_s}^{1/2}T^2/m_{Pl} \) in the radiation dominated era to determine at what temperature unparticles decouple from photons \[14\]. Here \( g_{_s} \) is the total number of degrees of freedom at the decoupling temperature and \( m_{Pl} = 1.22 \times 10^{19} \text{ GeV} \) is the Planck mass. When \( \Gamma < H \), the unparticles will decouple from photons. Taking the equal sign, one obtains the decoupling temperature,

\[T_D = \frac{1}{2} \left( \frac{1.66g_{_s}^{1/2}}{m_{pl}} \Lambda_U^{2d_u} \frac{4\pi^2}{A_{d_a}} \left( \frac{2}{\zeta(3)} \right)^{1/(2d_u-1)} \right). \]

Replacing \( \lambda \) by \( \lambda \), one obtains the decoupling temperature due to the \( \lambda \) term. In the following numerical discussions, we will take \( \lambda \) to be non-zero for illustration. The results will be the same for taking \( \lambda \) non-zero.

There are experimental constraints on the coupling \( \lambda/A^{d_u}_U \) from astrophysics \[14\], radiative positronium decay \( \circ-P \rightarrow \gamma U \) \[14\] and CERN LEP \( e^-e^+ \rightarrow \gamma U \) \[14\]. Among them, the astrophysical one by energy loss arguments in stars is most stringent. Using the numbers obtained in Ref. \[10\] we can calculate the allowed maximal coupling (\( \lambda/A^{d_u}_U \))max and the corresponding minimal decoupling temperature \( T_{\text{min}} \). The actual decoupling temperature can of course be higher than this minimal value. The results are listed in Table \[1\]. It is seen that \( T_{\text{min}} \) can vary in a big range from as large as \( 10^7 \text{ GeV} \) to as low as a few \( 10 \text{ GeV} \) depending on the value of \( d_u \).
Table 1. Upper bound \((\lambda / A_U^{\min} )_{\max}\) (in units of GeV\(^{-d_U}\)) and the corresponding \(T_{D}^{\text{min}}\) (in units of GeV) for various values for \(d_U\). Appropriate \(g_5\) has been used for the given energy with SM particles and a scalar unparticle.

| \(d_U\) | 4/3 | 5/3 | 2 |
|--------|-----|-----|---|
| \((\lambda / A_U^{\min} )_{\max}\) | 1.04 \times 10^{-14} | 7.17 \times 10^{-13} | 5.11 \times 10^{-11} |
| \(T_{D}^{\text{min}}\) | 7.37 \times 10^6 | 2.70 \times 10^3 | 3.68 \times 10^2 |

Table 2. \(\Omega_U\) and \(r_\gamma(T) = \rho_\gamma(T)/\rho_c(T)\) as functions of \(\Omega_U(T_{0})\). We have used \(\rho_c(T_{0}) = 8.0992 h^2 \times 10^{-47}\) GeV\(^4\) and taken the central value for \(h = 0.73^{+0.04}_{-0.03}\).

| \(d_U\) | \(\Omega_U(T_{0})\) | \(A_U\) (GeV) | \(r_\gamma(T_{D}^{\text{min}})\) | \(r_\gamma(T_{\text{BBN}})\) |
|--------|-----------------|-------------|----------------|----------------|
| 4/3    | 1.0             | 7.37 \times 10^6 | 4.78 \times 10^8 | 6.13 \times 10^{-2} |
| 5/3    | 0.2             | 2.48 \times 10^{-1} | 2.48 \times 10^{-3} | 3.85             |
| 2      | 1.0             | 4.31 \times 10^{-2} | 4.32 \times 10^3 | 3.05 \times 10^{-2} |

In order that the relic density of unparticles is not too large, say as large as the critical energy density \((\rho_c)\) which would over close the universe, for a given \(T_{D}\) one has to choose a big enough \(A_U\) besides the requirement \(A_U > T_{D}\). This provides a way to constrain the scale \(A_U\) directly. We illustrate our results in Table 2 for several representative values of the ratio of energy densities, \(\Omega_U(T_{0}) = \rho_U(T_{0}) / \rho_c(T_{0})\), at \(T_{0}\). In our analysis, we assume \(T_{0} = T_{D}^{\text{min}}\), as shown in Table 1. By equating \(\rho_U(T_{0})\) that is obtained via eq. (3) on the other hand with the one from backward evolution via eq. (11) on the other, we can determine \(A_U\) for each given \(d_U\). Also shown are the values of the ratio \(r_\gamma = \rho_\gamma(T_{0}) / \rho_c(T_{0})\) at \(T_{D}^{\text{min}}\) and \(T_{\text{BBN}}\). Note that we do not assume a value for the dimensionless coupling \(\lambda\); instead, it is fixed by \(A_U\) and \(T_{D} = T_{D}^{\text{min}}\) via eq. (17).

For \(d_U = 4/3\), we find that it is not possible to saturate the critical density, nor the dark matter density \(\Omega_{DM} = 0.2\) [13]. With the constraint that \(T_{D} < A_U\), the largest \(\Omega_{U}(T_{0})\) is 0.16 which occurs at \(T_{D} = A_U\). This, of course, still leaves enough room for unparticle to play a significant role as dark matter. For \(d_U = 5/3\) and 2, we see that the present unparticle relic density can easily saturate the critical density and dark matter density. In all cases, \(\rho_U(T_{0})\) is smaller (in most cases much smaller) than \(\rho_c(T_{0})\). Requiring the present relic of unmatter to be less than these densities one obtains conservative lower bounds on \(A_U\) for given \(T_{D} = T_{D}^{\text{min}}\). For small \(d_U\), the scale \(A_U\) is constrained to be very large, making low energy search for unparticle effects difficult. But for large \(d_U\) (close to 2), the scale can still be as low as a few hundred GeV which may be reached at LHC and ILC colliders.

The standard BBN theory explains data well. It is therefore important to make sure that at \(T_{BBN}\) unparticles do not cause problems. A simple criterion is to require that at this temperature the unparticle energy density be less than the photon’s. With this restriction, it is interesting to see whether one can still have large relic unmatter at present. We find this is indeed possible. Although there are many cases shown in Table 2 where \(r_\gamma(T_{BBN})\) is larger than one, circumstances with sizable \(\Omega_{U}(T_{0})\) but small \(r_\gamma(T_{BBN})\) also appear at large \(d_U\). This can easily be understood from eq. (13) and from Fig. 4. For \(d_U > 1\), a small \(r_\gamma(T_{BBN})\) can result in a sizable \(\Omega_{U}(T_{0})\). A universe dominated by unparticle between the BBN era and the matter or dark energy dominated universe is possible.

In the above discussions, the interactions of unparticles with photons lead to an interaction rate \(\Gamma \sim T^{2d_U+1}\) which brings unparticles and SM particles into equilibrium at a high temperature, and they decouple at a lower temperature if the unparticle dimension \(d_U\) is larger than 1. There are many possible ways unparticles can interact with SM particles, but not all interactions will have the same properties as far as thermal equilibrium is concerned. For example, we find that all of the operators involving SM fermions listed in Ref. [14] will result in an interaction rate proportional to \(T^{2d_U+1}\). If unparticles and SM fermions are required to be in thermal equilibrium at a high temperature and then decouple at a lower one, \(d_U\) must be larger than \(3/2\). On the contrary, if \(d_U\) is less than \(3/2\), then unparticles and SM fermions will not be in thermal equilibrium at a high temperature in the first place, but will be at a lower temperature till the epoch of matter dominated universe. Since when in thermal equilibrium, the unparticle density dilutes faster than SM particles, its relic density today will be negligibly small if the equilibrium sets in before or just after BBN with the unparticle relic density not larger than photon density. This is an interesting scenario to study, which may lead to sensitive information about the unparticle scaling dimension.

There is much to be explored for the roles that thermal unparticles can play in our universe. It is important to analyze available cosmological and astrophysical data for a global fit with unparticle energy density integrated. We leave this for a detailed future study.

To summarize, we have studied for the first time the thermal properties of unparticles. Due to its peculiar phase space structure we found that the EoS parameter \(\omega_U\) is given by \(1/(2d_U + 1)\), providing a new form of energy in our universe. In an expanding universe, the behavior of unparticle energy density \(\rho_U(T)\) is dramatically different than that for photons. For \(d_U > 1\), even if its value at a high decoupling temperature \(T_{D}\) is very small, it could evolve into a sizable relic density \(\rho_U(T_{0})\) at present, large enough to play the role of dark matter. We have exemplified this with photon-unparticle interactions, and found that it is indeed feasible to obtain a large relic energy density of unparticles with the most stringent constraints saturated.
Acknowledgments This was supported in part by the grants NSC95-2112-M-002-038-MY3, NCTS (NSC96-2119-M-002-001), NCET-06-0211, and NSFC-10775074.

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