The Cayley graphs of a centralizer of the Burnside group $B_0(2, 5)$

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Abstract. Let $B_0(2, 5) = \langle a_1, a_2 \rangle$ be the largest two-generator Burnside group of exponent five. It has the order $5^{34}$. We define an automorphism $\varphi$ under which every generator is mapped into another generator. Let $C_{B_0(2, 5)}(\varphi)$ be the centralizer of $\varphi$ in $B_0(2, 5)$. It is known that $|C_{B_0(2, 5)}(\varphi)| = 5^{17}$. We calculated the growth function of this group relative to the minimal generating set and also the symmetric generating set. As the results, the diameters and the average diameters of $C_{B_0(2, 5)}(\varphi)$ were computed.

1. Introduction

One of the important tools for defining the structure of a group is the study of its growth with respect to a fixed generating set. Let $G = \langle X \rangle$. We call the ball $K_s$ of radius $s$ of a group $G$ the set of all its elements, which can be presented as a group of words with length $s$ in the alphabet $X$. For each non-negative integer $s$, one can define the (spherical) growth function of the group $F(s)$, which is equal to the number of elements of the group $G$ with respect to $X$, that can be represented as an irreducible group words with the length $s$. Thus,

$$F(0) = |K_0| = 1, \quad F(s) = |K_s| - |K_{s-1}| \text{ when } s \in \mathbb{N}.$$ 

As a rule, the growth function of a finite group is represented in the form of a table which contains non-zero values of $F(s)$.

Also we note that along with computing the growth function of a group, we define some characteristics of the corresponding Cayley graph, for instance, the diameter and the average diameter [1]. Let $F(s_0) > 0$, but $F(s_0 + 1) = 0$, then $s_0$ will be the diameter of the Cayley graph of the group $G$ in the generating alphabet $X$, which we will denote $D_X(G)$. Accordingly, the average diameter $\overline{D}_X(G)$ is equal to

$$\frac{1}{|G|} \sum_{s=0}^{s_0} s \cdot F(s).$$

Unfortunately, although the computation of the growth function of a large finite group is solvable, it is a rather complicated problem. This is due to the fact that, in general, the task of determining of the minimal word of a group element, as shown by S. Iven and O. Goldreich [2] is NP–hard. Thus, in the worst case, the number of elementary operations that must be performed to solve this problem is an exponential function of $|X|$.

It should be noted that during the process of computing the growth function of a group, the characteristics of the corresponding Cayley graph are defined in a parallel way – for instance,
the diameter and the average diameter. Let $F(s_0) > 0$, but $F(s_0 + 1) = 0$, then $s_0$ will be the diameter of the Cayley graph of the group $G$ in the alphabet generating $X$, which we will denote $D_X(G)$. Accordingly, the average diameter $\overline{D}_X(G)$ is equal to $\frac{1}{|G|} \sum_{s=0}^{s_0} s \cdot F(s)$.

As it is known, the Cayley graphs, due to their remarkable properties, have found wide application in the simulating the topologies of multiprocessor computing systems (MCS) – supercomputers [3] and data-centers [4]. It is likely that in the near future, knowledge on very large graphs will be needed for designing distributed systems, the peak capacity of which will approach 1 exaflops and more.

One of the more widely used topologies of MCS is the $k$–dimensional hypercube. This graph is determined by the group $B(k, 2)$, which is the $k$–generated Burnside group of period 2. $B(k, 2)$ has a simple structure and is equal to the direct product of $k$ copies of cyclic groups of order 2. In the work [5] the Cayley graphs of the groups $B(k, 3)$, i. e. groups of period 3, were studied and a comparative analysis of these graphs with respect to a hypercube was carried out. This analysis showed that the characteristics of $B(k, 3)$ are more preferable than the characteristics of $B(k, 2)$. It means that while paired comparison graphs $B(k_1, 3)$ and $B(k_2, 2)$ have approximately the same number of vertices, the first ones have a smaller diameters, the average diameters and degrees. We obtained a similar result in [6] during the study of groups of period 4. In this respect the task of the study of the Cayley graphs of finite Burnside groups of other periods is interesting.

Let $B(2, 5) = \langle a_1, a_2 \rangle$ be a free two–generating Burnside group of period 5. Until now, it is not known whether this group is finite or infinite. Further, let $B_0(2, 5) = \langle a_1, a_2 \rangle$ be the maximum finite group, whose order equals $5^{34}$ [7]. If $B(2, 5)$ is finite, then $B_0(2, 5) = B(2, 5)$.

To calculate the growth function $B_0(2, 5)$ with respect to the minimal generating set is hardly possible because the number of its elements is very large:

$$5^{34} = 58207660913467407265625 \approx 5 \cdot 10^{23}.$$ 

Earlier, using computer calculations, we have obtained the growth functions of the factor–groups of the group $B_0(2, 5)$ whose order does not exceed $5^{19}$ [8].

Let us consider the map $\varphi$ of the following form:

$$\varphi: \begin{cases} a_1 \rightarrow a_2, \\ a_2 \rightarrow a_1. \end{cases}$$

It is easy to see [10] that $\varphi$ is an involutive automorphism of the groups $B(2, 5)$ and $B_0(2, 5)$.

Let $C_{B(2,5)}(\varphi)$ and $C_{B_0(2,5)}(\varphi)$ be the centralizers of the automorphism $\varphi$ in $B(2, 5)$ $B_0(2, 5)$ respectively. According to V. P. Shunkov’s theorem [11], if $C_{B(2,5)}(\varphi)$ is a finite group, then the group $B(2, 5)$ is also finite. In other words, if $C_{B(2,5)}(\varphi) = C_{B_0(2,5)}(\varphi)$, then $B(2, 5) = B_0(2, 5)$. Considering this, the study of the growth function of $C_{B_0(2,5)}(\varphi)$ is of great interest. Further, for brevity, we will write $C$ instead of $C_{B_0(2,5)}(\varphi)$.

In [10] the structure of group $C$ has been studied with the following results:

(i) $|C| = 5^{16}$,

(ii) $C = C_0 \times \langle z \rangle$, where $|C_0| = 5^{15}$ and $|\langle z \rangle| = 5$,

(iii) $z$ is the central element of the group $B_0(2, 5)$,

(iv) $C_0 = \langle X_0 \rangle$, where $|X_0| = 4$ is the minimal number of the generators of $C_0$,

(v) power commutator presentation of $C_0$ is computed.

The objective of this paper is to study the growth function of the group $C$ with respect to the minimal generating set $X = X_0 \cup \{ z \}$ and the symmetric – $Y = X \cup X^{-1}$.
Note that in [12] a solution is proposed to a similar problem for the centralizer $C_{B_0(2,5)}(\psi)$ where $\psi$ is the automorphism, which translates generating elements into their inverses.

Since for evaluating of growth function it is required to multiply elements of the group, then for the practical implementation an efficient algorithm for multiplication is needed. There are two ways to calculate the product of elements in the group given by power commutator presentation: the collective process [3, 13] and Hall’s polynomials method [14]. Numerous computational experiments showed that the second method allows to multiply the elements in these groups much faster than in a collecting process (at least in exponent) [5, 9, 15, 16].

The documentation of the computer algebra system GAP refers to the possibility of automation of Hall’s polynomials in the simplest cases. However, in the general case, this task is not trivial, because is not reduced to routine computation and requires the involvement of programming languages that support complex regular expressions, and also systems of computer mathematics with a wide range of procedures for symbolic computation. In fact, working with a group that has a large order, usually it is required the unique revision of the code that takes into account the feature of the group structure and characteristics of the computer.

In the following section of the article the Hall’s polynomials of the groups $C_0$ are presented. They were calculated using algorithm from [15]. This algorithm in general was implemented in C++, with the exception of symbolic procedures, which were written in the MATLAB language.

In the last section the results of computer calculations of the growth function of the group of $C$ with respect to the generating set $X$ and $Y$ are given.

2. Hall’s polynomials of the group $C_0$

Let $C_0 = \langle X_0 \rangle$, where $X_0 = \{a_1, a_2, a_3, a_4\}$ be the minimal generating set of $C_0$. The following theorem is proved.

**Theorem.** Let $a_1^{x_1} \ldots a_{15}^{x_{15}}$ and $a_1^{y_1} \ldots a_{15}^{y_{15}}$ be two arbitrary elements in the group $C_0$ written in the commutator form. Then their product is equal to $a_1^{x_1} \ldots a_{15}^{x_{15}} \cdot a_1^{y_1} \ldots a_{15}^{y_{15}} = a_1^{z_1} \ldots a_{15}^{z_{15}}$, where $z_i \in \mathbb{Z}_5$ are Hall’s polynomials, given by the formulas (1–15).

\[
\begin{align*}
    z_1 &= x_1 + y_1, \\
    z_2 &= x_2 + y_2, \\
    z_3 &= x_3 + y_3, \\
    z_4 &= x_4 + y_4, \\
    z_5 &= x_5 + y_5 + x_2y_1 + 4x_3y_2, \\
    z_6 &= x_6 + y_6 + x_3y_1 + 4x_3y_2, \\
    z_7 &= x_7 + y_7 + x_4y_1, \\
    z_8 &= x_8 + y_8 + x_4y_2, \\
    z_9 &= x_9 + y_9 + x_4y_3, \\
    z_{10} &= x_{10} + y_{10} + 2x_3y_2 + 2x_4y_1 + x_7y_4 + 3x_4^2y_1 + x_4y_1y_4, \\
    z_{11} &= x_{11} + y_{11} + 4x_3y_2 + 2x_4y_2 + x_8y_4 + 3x_4^2y_2 + x_4y_2y_4, \\
    z_{12} &= x_{12} + y_{12} + 2x_3y_2 + 2x_4y_3 + x_9y_4 + 3x_4^2y_3 + x_4y_3y_4, \\
    z_{13} &= x_{13} + y_{13} + 3x_3y_2 + 2x_4y_1 + 2x_5y_4 + x_7y_2 + 4x_8y_1 + 2x_7y_4 + x_10y_4 + \\
     &+ 2x_4^2y_1 + x_4^2y_2 + 3x_7y_1^2 + 3x_4y_1y_2^2 + 3x_4^2y_1y_4 + 2x_2x_4y_1 + 2x_2y_1y_4 + x_4y_1y_2 + 4x_4y_1y_4, \\
    z_{14} &= x_{14} + y_{14} + 3x_3y_2 + 2x_4y_2 + 4x_6y_4 + 2x_7y_3 + 3x_9y_1 + 2x_8y_4 + x_{11}y_4 + \\
     &+ 2x_4^2y_2 + x_4^2y_4 + 3x_8y_2^2 + 3x_4y_2y_4^2 + 3x_4y_2y_4 + 4x_3x_4y_1 + 4x_3y_1y_4 + 2x_4y_1y_3 + 4x_4y_2y_4. 
\end{align*}
\]
\[ z_{15} = x_{15} + y_{15} + x_3 y_2 + 2 x_4 y_3 + x_8 y_3 + 4 x_9 y_2 + 2 x_9 y_4 + x_{12} y_4 + 2 x_4^2 y_3 + 
+ x_4^3 y_3 + 3 x_9 y_4^2 + 3 x_4 y_3 y_4^2 + 3 x_4^2 y_3 y_4 + 2 x_3 x_4 y_2 + 2 x_3 y_2 y_4 + x_4 y_2 y_3 + 4 x_4 y_3 y_4. \] (15)

3. Computer calculations of the growth functions of the group \( C \)

The calculation of the growth function of the group \( C \) related to \( X \) was carried out according to the algorithm [8]. For efficient multiplication of elements, the Hall’s polynomials were used obtained in section 2. The algorithm was implemented in C++. As a tool for parallelization, it was used the library OpenMP. For the calculations, it was used a computer with an 8-core processor and 64 Gb of RAM, running the Linux operating system. The program was compiled by the embedded compiler GCC. Calculating growth functions for the generating set \( X \) takes about 1.5 hours, and for \( Y \) – 3 hours. Their graphs are shown in Fig. 1–2. For clarity, an approximating Gaussian curve obtained by the method of the least squares is added on each graph.

![Figure 1. The growth function of \( C \) generated by \( X \)](image1)

![Figure 2. The growth function of \( C \) generated by \( Y \)](image2)

As already mentioned, the growth function of the group that contains information about the characteristics of the corresponding Cayley graph is:

**Corollary 1.** \( D_X(C) = 33, \ D_X(C) \approx 26. \)

**Corollary 2.** \( D_Y(C) = 21, \ D_Y(C) \approx 18. \)

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