Mismatched Models to Lower Bound the Capacity of Optical Fiber Channels

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Abstract—Lower bounds on the capacity of optical fiber channels are derived by using a correlated phase-and-additive-noise (CPAN) mismatched model that is motivated by perturbing the nonlinear Schrödinger equation with small nonlinearity. Both the phase and additive noise processes are Gaussian-Markov. The CPAN model characterizes nonlinearities better than existing models in the sense that it achieves larger information rates.

Index Terms—Achievable rate, regular perturbation, logarithmic perturbation, phase noise, particle filtering

I. INTRODUCTION

Computing the capacity of nonlinear optical channels is an open problem. The spectral efficiency of the additive white Gaussian noise (AWGN) channel, \( \log_2(1 + \text{SNR}) \) where SNR is the signal-to-noise ratio, is known to be an upper bound on the spectral efficiency of the nonlinear optical channel [1], [2]. However, all existing lower bounds reach an information rate peak at some launch power and then decrease as the launch power increases. Over the years, many simplified models have been developed to obtain better lower bounds.

Numerical lower bounds were developed in [3] by using geometric shaping and by treating nonlinear distortions as additive Gaussian noise (AGN) whose mean, variance, and pseudo-variance depend on the transmit symbol amplitude \( x \). The expectation of \( X \) conditioned on \( Y = y \) is written as \( \langle X|Y = y \rangle \). The inner product of the signals \( a(t) \) and \( b(t) \) with time parameter \( t \) is written as \( \langle a(t), b(t) \rangle \). The Euclidean norm of \( x \) is denoted by \( ||x|| \). The Fourier transform of \( u(t) \) is \( \mathcal{F}(u(t))(\Omega) \), where \( \Omega \) denotes an angular frequency. The inverse Fourier transform of \( U(\Omega) \) is \( \mathcal{F}^{-1}(U(\Omega))(t) \). The dispersion operator \( \mathcal{D}_z \) is described by

\[
\mathcal{D}_z u(t) = \mathcal{F}^{-1} \left( e^{j \frac{2\pi}{L} \Omega^2 z \mathcal{F}(u(t))} \right). \tag{1}
\]

We write \( \text{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \). We write \( \delta[t] \) for the function that maps integers to zero except for \( \delta[0] = 1 \).

We consider the following parameters. The variables \( z, t \), and \( \Omega \) are the respective position, time, and angular frequency. The symbol period is \( T \) and the bandwidth is \( \mathcal{B} \). The launch position is \( z = 0 \) and the receiver position is \( z = L \). The fiber span length is \( L_s \). The attenuation coefficient is \( \alpha \), the dispersion coefficient is \( \beta_2 \), and the nonlinearity coefficient is \( \gamma \).

II. NONLINEAR SCHRÖDINGER EQUATION AND RP

The Nonlinear Schrödinger Equation (NLSE) [13] describes propagation along an optical fiber:

\[
\frac{\partial}{\partial z} u = -j \frac{\beta_2}{2} \frac{\partial^2}{\partial t^2} u + j \gamma f(z)|u|^2 u + \frac{n(z, t)}{\sqrt{f(z)}}. \tag{2}
\]

where \( u(z, t) \) is the propagating signal and \( f(z) \) models loss and amplification along the fiber [6]. We have \( f(z) = 1 \) for ideal distributed amplification (IDA) and \( f(z) = \exp(-\alpha z + \alpha L_s |z|/L_s) \) for lumped amplification. The accumulated noise at \( z = L \) is usually dominated by amplified spontaneous emission (ASE) noise with autocorrelation function \( N_{\text{ASE}}(\mathcal{B}) \text{sinc}(\mathcal{B}(t - t')) \).

A. Continuous-Time RP model

RP [6] Sec. III] expands \( u(z, t) \) in powers of \( \gamma \)

\[
u(z, t) = u_0(z, t) + \gamma \Delta u(z, t) + \mathcal{O}(\gamma^2). \tag{3}
\]

Assuming \( \gamma \) is small, the right-hand side of (3) is placed in (2), and the equations are solved for the zeroth and first powers of \( \gamma \). The result is

\[
u(z, t) = u_0(z, t) + u_{\text{NL}}(z, t) + \mathcal{O}(\gamma^2) \tag{4}
\]

where the linear terms are

\[
u_0(z, t) = \mathcal{D}_z [u(0, t) + u_{\text{ASE}}(z, t)] \tag{5}
\]

\[
u_{\text{ASE}}(z, t) = \int_0^z \mathcal{D}_z \left( \frac{n(z', t)}{\sqrt{f(z')}} \right) \, dz' \tag{6}
\]

A. Notation

We use the following notation. The real and imaginary parts of the complex-valued \( x \) are denoted by \( \Re(x) \) and \( \Im(x) \), respectively. Random variables are written with uppercase letters such as \( X \) and their realizations with the corresponding lowercase letters \( x \). The expectation of \( X \) is denoted by \( \langle X \rangle \). The expectation of \( X \) conditioned on \( Y = y \) is written as \( \langle X|Y = y \rangle \).
and the non-linear perturbation term is
\[ u_{NL}(z, t) = j \gamma D_z \int_0^\tau f(z') D_{-z'} \left[ |u_0(z', t)|^2 u_0(z', t) \right] dz'. \]  

(7)

The non-linear term is responsible for signal-signal mixing and signal-noise mixing because \( u_0(z, t) \) includes noise. We focus on wavelength division multiplexing (WDM) systems where the limiting factor is cross-phase modulation (XPM), so we ignore signal-noise mixing and replace \( u_0(z', t) \) by \( D_{z'} u(0, t) \) in (7).

B. Discrete-Time RP Model

Consider WDM and pulse amplitude modulation (PAM) with \( C \) channels with indexes \( c \) between \( c_{\min} \leq 0 \) and \( c_{\max} \geq 0 \):
\[ c \in C = \{ c_{\min}, \ldots, -1, 0, 1, \ldots, c_{\max} \} \]  

where \( C = c_{\max} - c_{\min} + 1 \). Let \( n \) be the channel of interest have index \( c = 0 \). The launch signal including all channels is
\[ u(0, t) = \sum_m x_m s(t - mT) + \sum_{c \neq 0} c^{2\Omega_c} \sum_k b_{c,k} s(t - kT) \]  

(9)

where \( \Omega_c \) is the center frequency of channel \( c \in C \), and \( \Omega_0 = 0 \). The pulse shaping filter \( s(t) \) is taken to be a normalized Nyquist pulse, i.e., we have \( \|s(t)\| = 1 \) and \( \langle s(t), s(t+nT) \rangle = 0 \) for \( n \neq 0 \).

For the modulation, we model the sequences \( \{ X_m \} \) and \( \{ B_{c,m} \} \), \( c \in C \setminus \{ 0 \} \), as being independent, and as each having independent and identically distributed (i.i.d.) and proper complex symbols with energies \( \|X_m\|^2 = E \) and \( \|B_{c,m}\|^2 = E_c \) for all \( m \) and \( c \). Note that the optical power of channel \( c \) is \( P_c = E_c/T \). The properness ensures that the pseudo-covariances are zero. Since we will need fourth moments, we define \( \|B_{c,m}\|^2 = Q_c \) for all \( m \) and \( c \).

After digital back-propagation of the center channel, and matched filtering and sampling, the discrete-time mismatched model based on RP is (see [6] Sec. VI-VIII)
\[ y_m = x_m + w_m + \sum_{c \neq 0} \Delta x_{c,m} \]  

(10)

where the noise realization is
\[ w_m = \langle u_{ASE}(t), s(t - mT) \rangle \]  

and the noise process \( \{ W_m \} \) is i.i.d., circularly-symmetric, complex Gaussian with variance \( \sigma_w^2 = N_0 \) and \( \{ W_m \} \) is independent of \( \{ X_m \} \) and \( \{ B_{c,m} \} \) for all \( c \). The non-linear interference (NLI) terms are (see [6] Eq. (60))
\[ \Delta x_{c,m} = j \sum_{n,k,k'} C_{c,n,k,k'}^{(c)} \cdot x_{n+m} \cdot b_{c,k,k'} \cdot b_{c,k,k'}^* + m \]  

(12)

where the NLI coefficients are
\[ C_{c,n,k,k'}^{(c)} = 2 \gamma \int_0^\tau \int_{-\infty}^{\infty} f(z) \int_{-\infty}^{\infty} s(z, t)^* s(z, t - nT) \]  

\[ s(z, t - kT + \beta_2 \Omega_c z) s(z, t - k'T + \beta_2 \Omega_c z)^* dt \]  

(13)

\[ \int_{-\infty}^{\infty} f(z) dz \]

and \( s(z, t) \) is in general complex-valued even if the pulse \( s(t) \) is not. Observe also that
\[ C_{c,n,k,k'}^{(c)} = (C_{c,n',k',k}^{(c)})^* \]  

(14)

In particular, we have \( C_{0,k,k}^{(c)} = (C_{0,k',k'}^{(c)})^* \), and \( C_{0,k,k}^{(c)} \) is real-valued.

Several NLI coefficient magnitudes \( |C_{n,k,k'}^{(c)}| \) are plotted in the top of Fig. 1 with \( \Omega_1 = 2\pi 50 \) GHz. Observe that \( C_{0,k,k}^{(c)} \) has the largest magnitude. The top of Fig. 1 also shows \( |C_{n,k,k'}^{(c)}| \) for \( 0 < k \leq 680 \), but these NLI coefficients are very small. Due to different group velocities, the symbols from the channel \( \Omega_1 = 2\pi 50 \) GHz that interfere with the channel of interest are mostly past symbols (\( k \leq 0 \)). The bottom of Fig. 1 shows \( |C_{n,k,k'}^{(c)}| \) for the channel \( \Omega_{-1} = -2\pi 50 \) GHz. Note that if \( \Omega_c = -\Omega_{-c} \), then we have
\[ C_{n,k,k'}^{(-c)} = C_{-n,-k,-k'}^{(-c)} = (C_{n,n-k',k-n}^{(c)})^* \]  

(15)

where the last step is the same as (14). For example, the curve \( |C_{0,k,k}^{(-1)}| \) is the same as \( |C_{0,k,k}^{(1)}| \) but flipped at position \( k = 0 \). Similarly, the curve \( |C_{1,k,k}^{(-1)}| \) is the same as the curve \( |C_{1,k,k}^{(1)}| \) but flipped at position \( k = 1 \).

III. CPAN MISMATCHED MODEL

We separately consider the NLI involving \( x_m \) and \( \{ x_n \}_{n \neq m} \). Consider (12) and collect the terms as
\[ \Delta x_{c,m} = j x_m \theta_{c,m} + v_{c,m} \]  

(16)
We treat the conditional variance of $N_{c,m}$. The complete ISI at time $m$ is therefore

$$\sum_{n \neq 0} \left( \sum_{c \neq 0} \sum_{k} C_{n,k,k}^{(c)} E_c \right) \cdot x_{n+m}$$

We treat the conditional variance of $N_{c,m}$ below.

2) Second-Order Statistics for $\Theta_{c,m}$: Consider the (real-valued) phase noise $\{\Theta_{c,m}\}$. The covariances are

$$r_{\Theta}^{(c)}[\ell] := \langle \Theta_{c,m} \Theta_{c,m+\ell} \rangle - \langle \Theta_{c,m} \rangle \langle \Theta_{c,m+\ell} \rangle$$

$$= \sum_{k} C_{0,k,k}^{(c)} C_{0,k-k-\ell}^\ast (Q_c - E_c^2)$$

$$+ \sum_{k \neq k'} C_{0,k,k'}^{(c)} (C_{0,k-k-k'}^\ast)^2 E_c^2. \quad (28)$$

Setting $\ell = 0$, we obtain the variance of $\Theta_{c,m}$:

$$r_{\Theta}^{(c)}[0] = \sum_{k} (C_{0,k,k}^{(c)})^2 (Q_c - E_c^2) + \sum_{k \neq k'} |C_{0,k,k'}^{(c)}|^2 E_c^2. \quad (29)$$

3) Second-Order Statistics for $N_{c,m}$: Consider next the additive noise $\{N_{c,m}\}$. The pseudo crosscorrelations $\langle N_{c,m} N_{c,m+\ell} \rangle$ are zero for all $m$ and $\ell$, so that $N_{c,m}$ is proper complex (recall that $\langle N_{c,m} \rangle = 0$). The correlations and covariances are therefore

$$r_N^{(c)}[\ell] := \langle N_{c,m} N_{c,m+\ell} \rangle$$

$$= \sum_{n \neq 0} \sum_{k \neq 0} C_{n,k,k}^{(c)} (C_{c-n-k-k-\ell}^\ast E_{Q_c}$$

$$+ \sum_{k \neq k'} C_{n,k,k'}^{(c)} (C_{c-n-k-k-\ell}^\ast E_{E_c^2}$$

$$+ \sum_{k \neq k'} C_{n,k,k'}^{(c)} (C_{c-n-k-k-\ell}^\ast E_{E_c^2}^2. \quad (30)$$

On the other hand, if we condition on $\{X_\ell\} = \{x_\ell\}$ then we have the time-varying covariances

$$r_N^{(c)}[m,\ell] := \langle N_{c,m} N_{c+m+\ell} \rangle \{X_\ell\} = \{x_\ell\}$$

$$- \langle N_{c,m} \rangle \{X_\ell\} \langle N_{c+m+\ell} \rangle \{X_\ell\} = \{x_\ell\}$$

$$= \sum_{n \neq 0} \sum_{k \neq 0} C_{n,k,k}^{(c)} (C_{n-k-k-k-\ell}^\ast (Q_c - E_c^2$$

$$+ \sum_{k \neq k'} C_{n,k,k'}^{(c)} (C_{n-k-k-k-\ell}^\ast E_{E_c^2}^2$$

$$x_{n+m} x_{n+m+\ell}. \quad (31)$$

The most important terms are those with $\hat{n} = n - \ell$.

For example, for $\ell = 0$ we have

$$r_N^{(c)}[m,0] = \sum_{n \neq 0} \sum_{k \neq 0} C_{n,k,k}^{(c)} (C_{n,k,k}^\ast (Q_c - E_c^2$$

$$+ \sum_{k \neq k'} C_{n,k,k'}^{(c)} (C_{n,k,k'}^\ast E_{E_c^2}^2$$

$$x_{n+m} x_{n+m}. \quad (32)$$

In other words, the variance of the noise $N_m$ depends on the previous and past symbols.
Thus, the inter-channel phase noise processes have zero co-

crosscorrelations and pseudo-crosscorrelations are

indexes in $C_{n,k,k}$ necessarily proper complex when conditioned on the symbol
conjugation, the negations, and the swapping of the

For example, for $\ell$ we also have

The most important terms are again those with

The changes as compared to (31) are the lack of complex

Moreover, the pseudo-covariance is not necessarily zero, and therefore the additive noise is not necessarily proper complex when conditioned on the symbol sequence $\{X_\ell\} = \{x_\ell\}$ even after subtracting off the means.

The most important terms are again those with $\tilde{n} = n - \ell$. For example, for $\ell = 0$ we have

The most important terms are again those with $\tilde{n} = n - \ell$. For example, for $\ell = 0$ we have

4) Intra-Channel Crosscorrelations: The intra-channel crosscorrelations and pseudo-crosscorrelations are

but for $\ell \neq m$ we also have

Thus, the additive noise is correlated with the $X_m$.

5) Inter-Channel Crosscorrelations: Consider two channels $c$ and $c'$ where $c \neq c'$. The phase noise crosscorrelations are

Thus, the inter-channel phase noise processes have zero co-

covariance and we have

TABLE I

| Parameter                  | Symbol | Value       |
|----------------------------|--------|-------------|
| Attenuation coefficient    | $\alpha$ | 0.2 dB/km  |
| Dispersion coefficient     | $\beta_2$ | $-21.7$ ps$^2$/km |
| Nonlinear coefficient      | $\gamma$ | 1.27 W$^{-1}$km$^{-1}$ |
| Phonon occupancy factor    | $\eta$ | 1           |
| Transmit pulse shape       | $s(t)$ | Sinc        |
| Number of WDM channels     | $C$ | 5           |
| Channel bandwidth          | $B_{ch}$ | 50 GHz     |
| Channel spacing            | $B_{sp}$ | 50 GHz     |
| Channel of interest        | $c=0$ Center channel |

We similarly compute

and $\langle N_{c,m}N_{c',m+\ell}^* \rangle = 0$. In other words, the inter-channel additive noise processes are correlated and we have

Finally, we compute

so the phase and additive noise processes are uncorrelated.

B. Large Accumulated Dispersion

The paper [7] considers links with large accumulated dispersion, i.e., $\mu_c = |\beta_2\Omega_c/L| >> 1$ for all $c \in C$, and uses the seemingly coarse approximation

and all other $C_{0,k,k'}^{(c)}$ are approximated as zero. For $C - 1$ interfering channels, we thus have (cf. [7])

Fig. [2] shows that the covariance (41) is very close to the approximation (37) for a 1000-km link with IDA and the parameters in Table I.
IV. SIMPLIFIED MODELS FOR COMPUTATION

Given an input distribution \( p(x) \), an achievable rate (lower bound on capacity) of a channel with conditional distribution \( p(y|x) \) can be obtained via an auxiliary model \( q(y|x) \) as

\[
I_q(X;Y) = \text{E} \left[ \log_2 \frac{q(Y|X)}{q(Y)} \right] \leq I(X;Y)
\]

where \( q(y) = \int p(x)q(y|x) \, dx \) is the output distribution.

A. Wiener Phase Noise (WPN) Model

The covariance function of the phase noise \( \{\Theta_m\} \) is very long in time (see Fig. 2) which makes it difficult to compute achievable rates. A popular simplified model is the Wiener phase noise (WPN) model \([11]\) where \( \tilde{\Theta}_m := \Theta_m - \langle \Theta_m \rangle \) is a discrete-time Wiener process with realizations

\[
\tilde{\theta}_m = \tilde{\theta}_{m-1} + \delta_m
\]

and where the \( \{\Delta_m\} \) are i.i.d. real Gaussian variables with zero mean and variance \( \sigma^2_\Delta \). This model has memory \( \mu = 1 \).

We remark that WPN seems simple and general, but it has several issues. First, WPN has only one parameter \( \sigma^2_\Delta \) which does not permit to control the phase noise variance and correlation length simultaneously. Next, WPN is a non-stationary process in Euclidean (non-modulo) space, e.g., its variance grows with time. As a result, in phase (modulo) space the WPN steady-state distribution has uniform phase irrespective of the starting phase, which does not agree with the variance \([29]\) predicted by the LP or CPAN models.

B. Markov Phase Noise (MPN) Model

The above motivates modeling the phase noise \( \{\Theta_m\} \) as a Markov chain with memory \( \mu \):

\[
p_{\tilde{\Theta}_m|\tilde{\Theta}_{m-\mu}}(\tilde{\Theta}_m|\tilde{\Theta}_{m-\mu}) = p_{\tilde{\Theta}_m|\tilde{\Theta}_{m-\mu}}(\tilde{\Theta}_m|\tilde{\Theta}_{m-\mu})
\]

where \( \tilde{\Theta}_{m}^{n} \) is the vector \( \tilde{\Theta}_m, \ldots, \tilde{\Theta}_m \). For each \( m \), we model \( \{\Theta_{m-\mu}, \ldots, \tilde{\Theta}_m\} \) as jointly Gaussian with zero mean and with a symmetric Toeplitz covariance matrix \( C_\mu \) whose first column is \( (\rho_\Theta[0], \ldots, \rho_\Theta[\mu]) \) from \([11]\).

The conditional distribution \([50]\) can be computed from the mean vector and covariance matrix \([14]\ Ch. 2, Sec. 3.4\). The result is that \( \tilde{\Theta}_m | \tilde{\Theta}_m^{n-1} \) is Gaussian with mean \( g_\mu \tilde{\Theta}_m^{n-1} \) and variance \( \sigma^2_\nu \), where

\[
g_\mu = (\rho_\Theta[\mu], \ldots, \rho_\Theta[1]) (C_\mu)^{-1}
\]

\[
\sigma^2_\nu = r_\Theta[0] - g_\mu (r_\Theta[\mu], \ldots, r_\Theta[1])^T.
\]

This yields the recursive Markov phase noise (MPN) model

\[
\hat{\theta}_m = g_\mu \tilde{\Theta}_m^{n-1} + \sigma_\nu v_m
\]

where the \( \{v_m\} \) are i.i.d. real Gaussian variables with zero mean and unit variance. Note that \( \sigma^2_\nu \) does not depend on \( \{\theta_m\} \). Note also that we perform the computations \([51]-[52]\) in Euclidean (non-modulo) space.

For example, for memory \( \mu = 1 \), the MPN model has

\[
\hat{\theta}_m = \frac{r_\Theta[1]}{r_\Theta[0]} \hat{\theta}_{m-1} + \sqrt{r_\Theta[0] - r_\Theta[1]^2} v_m
\]

which is different than \([49]\), e.g., \( \langle \hat{\Theta}_m^2 \rangle \) does not increase with \( m \). Fig. 2 shows that the MPN model approximates the covariance function \( r_\Theta[\ell] \) well, especially for moderate \( \ell \).

C. CPAN Model with Simplified Memory

We combine the ASE noise \( w_m \) and the NLI noise \( \nu_m \) in one correlated noise term \( z_m \). The simplified CPAN model is

\[
y_m = x_m e^{j\theta_m} = z_m
\]

where \( \theta_m = \hat{\theta}_m + \langle \Theta_m \rangle \) and \( \hat{\theta}_m \) is a real zero-mean Gaussian Markov process generated according to \([52]\), and \( z_m \) is circularly-symmetric Gaussian with autocorrelation function

\[
r_Z[\ell] \triangleq \langle Z_m Z_{m+\ell} \rangle = \sigma^2_w \delta[\ell] + r_N[\ell]
\]

where \( \sigma^2_w = N_{\text{ASE}} \) and \( r_N[\ell] \) is given by \([43]\). Fig. 3 shows the simulated autocorrelation function of \( Z_m \) in the center channel of a 5-channel WDM system with \( L = 1000 \) km and the parameters in Table \([4]\). With these parameters, the imaginary part of the autocorrelation function turns out to be negligible.
To compute an achievable rate, we first compensate for \( \{ \Theta_m \} \) and whiten the noise \( \{ Z_m \} \). We use a real and normalized whitening filter \( h \) with \( \| h \|^2 = 1 \) and \( L \) taps and compute
\[
  u_m = e^{-j(\Theta_m)} \sum_{\ell=0}^{L-1} h_{\ell} y_{m-\ell} = \sum_{\ell=0}^{L-1} h_{\ell} x_{m-\ell} e^{j\theta_{m-\ell}} + z_m.
\]
(56)

Similar to \([11], [12]\), we apply particle filtering \([15]\) to obtain \( h(U|X) \), where \( U \) and \( X \) represent time-averages of the \( U_m \) and \( X_m \), respectively.

Consider two sequences \( \{ x_m \} \) and \( \{ u_m \} \) of \( M \) symbols. A particle filter tracks the phase noise \( \theta_m \) by keeping a list of \( K \) particles that are updated iteratively for every new received symbol \( u_m \). After the \( (m-1) \)-th iteration, i.e., after processing \( u_{m-1} \), the \( k \)-th particle is an ordered pair of a realization \( \{ \hat{\theta}_{m-\mu}^{(k)}, \ldots, \hat{\theta}_{m-1}^{(k)} \} \) of the tracked variable and a weight \( W_{m-1}^{(k)} \) such that the probabilities
\[
  \Pr \left[ \theta_{m-1}^{(k)} = \left( \hat{\theta}_{m-\mu}^{(k)}, \ldots, \hat{\theta}_{m-1}^{(k)} \right) \right] = W_{m-1}^{(k)} \tag{57}
\]
defined by the \( K \) particles give a good approximation of the posterior distribution \( p(\theta_{m-1}^{(k)}|X_{m-\mu}^{(k)}=x_{m-\mu}^{(k)}, \ldots, x_{m-1}^{(k)}) \).

At the \( m \)-th iteration, the particle list is updated using \( u_m \) by performing the following steps:

1) Update the \( K \) realizations with \( \hat{\theta}_m^{(k)} \), where, for each \( k \), the new \( \hat{\theta}_m^{(k)} \) is generated using \((52)\) (or \((49)\) for the WPN model) and distributed according to \((50)\):
\[
  \hat{\theta}_m^{(k)} \sim p_{\theta_m|\theta_{m-1}^{(k)}} \left( \left( \hat{\theta}_{m-\mu}^{(k)}, \ldots, \hat{\theta}_{m-1}^{(k)} \right) \right) \tag{58}
\]

2) Estimate the posterior conditional probability:
\[
  D_m = \sum_{k=1}^{K} W_{m-1}^{(k)} \pi_u \left( u_m - \sum_{\ell=0}^{L-1} h_{\ell} x_{m-\ell} e^{j\hat{\theta}_{m-\ell}} \right). \tag{59}
\]

We model \( Z_m \) as circularly-symmetric Gaussian with variance \( \sigma_z^2 \). By an argument similar to \([13]\) Eqs. (14)-(16), the value \( D_m \) is equal to
\[
  D_m = E \left[ p \left( u_m | x_{m-L+1}^m, \theta_{m-\mu}^m \right) x_{m-1}^{-m}, u_{1}^{-m-1} \right]. \tag{60}
\]

3) Update the \( K \) weights:
\[
  W_{m}^{(k)} = \frac{1}{D_m} W_{m-1}^{(k)} \pi_u \left( u_m - \sum_{\ell=0}^{L-1} h_{\ell} x_{m-\ell} e^{j\hat{\theta}_{m-\ell}} \right). \tag{61}
\]

4) If too many particles have negligible weight, i.e., if for a specified small positive \( \epsilon \) the effective number of particles \([15]\) is small:
\[
  \frac{1}{\sum_k W_{m}^{(k)}} < \epsilon K \tag{62}
\]
then resample the particles by drawing \( K \) new realizations from \( \left\{ \left( \hat{\theta}_{m-\mu+1}^{(k)}, \ldots, \hat{\theta}_{m}^{(k)} \right) \right\}_{k=1}^K \) with probabilities \( W_{m}^{(k)} \). Now set all \( W_{m}^{(k)} \) to \( 1/K \). As suggested in \([15]\), we use \( \epsilon = 0.3 \).

After the last iteration, the estimated conditional entropy is
\[
  h_q(U|X) \approx -\frac{1}{M} \sum_{m=1}^{M} \log_2(D_m) \tag{63}
\]
where \( M \) is the number of transmitted symbols.

We use circularly symmetric i.i.d. Gaussian input symbols with variance \( E = \sigma_v^2 \). In this case, the process defined by \((56)\) is also i.i.d. and circularly symmetric Gaussian with variance \( E + \sigma_w^2 \). Therefore we have
\[
  h_q(U) = \log_2 \left( \pi e (E + \sigma_w^2) \right). \tag{64}
\]

Finally, our lower bound on the capacity of the NLSE is
\[
  I_q(X;U) = h_q(U) - h_q(U|X) \tag{65}
\]
where step (a) follows by \((48)\) and step (b) follows by the data processing inequality.

VI. ESTIMATING MODEL PARAMETERS

We estimate the parameters of the simplified CPAN model \((54)\) from simulated data in a training phase. Similar to \([12]\), we use a maximum-likelihood estimator for the additive noise variance based on \( \{ |y_m| \} \) and \( \{ x_m \} \):
\[
  \hat{\sigma}_z^2 = \arg \max_{\sigma^2} \sum_{m=1}^{M} \log |y_m| |x_m| |y_m| |x_m|; \sigma^2 \tag{66}
\]
where the likelihood function is a Rice density
\[
  L |y_m| |x_m|; \sigma^2 = \frac{2|y_m| \sigma^2}{\pi \sigma^2} e^{-\frac{y_m^2+|x_m|^2}{\sigma^2}} I_0 \left( \frac{2|y_m| |x_m|}{\sigma^2} \right). \tag{67}
\]

From \((54)\), we have
\[
  \{ Y_m X_m^* \} = \langle |X_m|^2 \rangle, \langle e^{j \theta_m} \rangle, \langle e^{j \bar{\theta}_m} \rangle \tag{68}
\]
and \( \langle e^{j \Theta_m} \rangle \) is real because \( \hat{\theta}_m \) and \( -\bar{\Theta}_m \) have the same statistics. We can thus estimate \( \langle \Theta_m \rangle \) via
\[
  \langle \bar{\Theta}_m \rangle = \frac{1}{M} \sum_{m=1}^{M} y_m x_m^*. \tag{69}
\]

For the MPN model, we choose \( r_\Theta [0] \ldots r_\Theta [\mu] \) by assuming that \( r_\Theta [\ell] \) is a scaled version of \((28)\) and by minimizing \( h(Y|X) \) over the scaling factor, where \( h(Y|X) \) is computed using the particle method. We use these estimated \( r_\Theta [0] \ldots r_\Theta [\mu] \) in \((52)\) to compute the parameters \( g_\mu \) and \( \sigma_v \) of the MPN model.

We use a symmetric whitening filter with \( L = 3 \) taps:
\[
  h = \begin{bmatrix} h_1, \sqrt{1-2h_1^2} \end{bmatrix}. \tag{70}
\]

We estimate \( h_1 \) by minimizing the cost function \( h(U|X) \), which is computed using \((55)\) and particle filtering.

In a subsequent testing phase, the achievable rate is computed using \((65)\) with the estimated model parameters on a new set of simulated data to avoid overfitting.
have found in the literature by at least 0.2 bits/s/Hz. We expect that our lower bound can be further improved by using several carriers as in [11], as well as by using more sophisticated receiver filters and by shaping the symbol input distribution.

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