Simple positivity-preserving nonlinear finite volume scheme for subdiffusion equations on general non-conforming distorted meshes

Xuehua Yang · Haixiang Zhang · Qi Zhang · Guangwei Yuan

Received: 16 April 2021 / Accepted: 22 January 2022 / Published online: 4 April 2022
© The Author(s), under exclusive licence to Springer Nature B.V. 2022

Abstract We propose a positivity-preserving finite volume scheme on non-conforming quadrilateral distorted meshes with hanging nodes for subdiffusion equations, where the differential equations have a sum of time-fractional derivatives of different orders, and the typical solutions of the problem have a weak singularity at the initial time \( t = 0 \) for given smooth data. In this paper, a positivity-preserving nonlinear method with centered unknowns is obtained by the two-point flux technique, where a new method to handling vertex unknown including hanging nodes is the highlight of our paper. For each time derivative, we apply the L1 scheme on a temporal graded mesh. Especially, the existence of a solution is strictly proved for the nonlinear system by applying the Brouwer’s fixed point theorem. Numerical results show that the proposed positivity-preserving method is effective for strongly anisotropic and heterogeneous full tensor subdiffusion coefficient problems.

Keywords Time-fractional subdiffusion equation · L1 scheme · Positivity preserving · Non-conforming

Mathematics Subject Classification 65N12 · 65N30 · 35K61

1 Introduction

In recent years, due to the accuracy of fractional partial differential and integral equations in simulating various natural phenomena [1], fractional calculus has become the hot topic and the focus of many researchers. Due to the nonlocal and hereditary properties of fractional derivatives, fractional differential equations are more effective than inter-order differential equations in describing systems with memory and hereditary. Take the fractional subdiffusion equation as an example, which is a basic tool for modeling multiscale inhomogeneous phenomena and plays an important role in describing the process of diffusion [2], so it is increasingly used to simulate some phenomena or processes in the fields of anomalous diffusion, electrochemistry, finance, physics science, material science, biology, control, wave propagation, mechanics, etc. see [3–5] and related literature. Therefore, many researchers are interested in studying the properties of fractional calculus and in providing robust and accurate analytical and numerical methods for solution of fractional subdiffusion problems [6–11]. However, almost all numerical discrete schemes focus on the stability and

The work was supported by National Natural Science Foundation of China (12126321, 11701168) and Scientific Research Fund of Hunan Provincial Education Department (21B0550, YB2016B033).

X. Yang · H. Zhang (*) School of Science, Hunan University of Technology, Zhuzhou 412007, China e-mail: hassenzhang@163.com
Q. Zhang · G. Yuan Institute of Applied Physics and Computational Mathematics, Beijing 100088, China
high-order convergence of numerical solutions. The study of some nice properties, such as positivity preservation, local conservation, discrete extremum principle, is much less considered and developed. The positivity preservation, also called monotonicity, is an important and significant requirement for discrete schemes of the subdiffusion equations. A discrete scheme without such a property can lead to nonphysical oscillations or nonphysical numerical solutions, especially on distorted meshes. For example, the temperatures in Kelvin, temperature in thermal conduction problem, and other physical variables governed by some subdiffusion equations are generally positive quantities. For Lagrangian computation of hydrodynamic problems and reservoir simulations, meshes are usually large deformation and the subdiffusion coefficients are heterogeneous and anisotropic, which may cause the discrete schemes more readily to produce nonphysical negative solutions and oscillations. Therefore, effective numerical methods with positivity-preserving property are regarded as an indispensable requirement in constructing discrete schemes in order to avoid these problems. In the present paper, we consider a more complex model, where several time-fractional derivatives present in the differential equation.

Let $\alpha$ be a constant in $(0, 1)$, and $D_t^\alpha$ be the Caputo temporal fractional-derivative operator of order $\alpha$-th, which is defined by

$$D_t^\alpha u(t) = \int_0^t \frac{\partial u(s) (t-s)^{-\alpha}}{\partial s} \Gamma(1-\alpha) ds, \quad 0 < \alpha < 1. \quad (1.1)$$

Set $\ell$ be a positive integer, and $\{\alpha_i : i = 1, 2, \ldots, \ell\}$ be given constants that satisfy

$$0 < \alpha_1 < \alpha_{\ell-1} < \cdots < \alpha_2 < \alpha_1 < 1.$$  

We shall consider the multi-term subdiffusion problem

$$\sum_{k=1}^\ell l_k D_t^{\alpha_k} u(x, t) - \nabla \cdot (\kappa(x, t) \nabla u)$$

$$= f(x, t), \quad (x, t) \in \Omega \times (0, T], \quad (1.2)$$

$$u(x, 0) = \psi(x), \quad x \in \bar{\Omega}, \quad (1.3)$$

$$u(x, t) = \psi(x, t), \quad (x, t) \in \partial \Omega \times (0, T], \quad (1.4)$$

where

(a) $l_k$ is the given positive constants, $u = u(x, t)$ is the solution of the multi-term subdiffusion equation. Without loss of generality we can consider $l_k = 1$. 

(b) $\Omega$ is an open bounded polygonal domain of $R^2$ with boundary $\partial \Omega$, and $T$ is positive constant.

(c) $\kappa(x, t)$ is $2 \times 2$ diffusion tensor (possibly anisotropic and discontinuous on some interfaces) such that it is piecewise $C^1$ on $\Omega$ and is uniformly bounded above and below on $\Omega$. That is to say, there exist two positive constants $\lambda_{\min} > 0$ and $\lambda_{\max} > 0$ such that the set of eigenvalues of $\kappa(x, t)$ is included in $[\lambda_{\min}, \lambda_{\max}]$ with $\lambda_{\min} > 0$.

(d) The source term $f$, the initial data $\psi$, and the boundary data $\psi$ are piecewise smooth on $\Omega \times (0, T]$, respectively.

It is well known that large relative displacement tends to occur near multi-material interface or sliding interface for high temperature and high pressure Lagrangian radiation hydrodynamic problems. Because the computational grid moves with fluid flow, which causes general non-conforming meshes (see Fig. 1) to occur naturally, especially result in the appearance of hanging nodes. Therefore, efficient numerical methods with positivity-preserving property on general non-conforming meshes are necessary for solving these problems.

Positivity is an important feature of the subdiffusion equation. Recently, Luchko [12–14] established a series of interesting works, proved that the solution $u(t)$ of problem (1.2) satisfied the property of positivity preserving. Hence, it is natural to expect that numerical solutions could maintain the discrete analogy of this property of exact solution. So far, few numerical methods have considered the positivity preservation for fractional differential equation, especially on various distorted meshes. Ye et al. [15] used a fractional predictor–corrector method combining the L1 and L2 discrete schemes for the 1D multi-term time-

![Fig. 1 Non-conforming mesh](image-url)
space Riesz–Caputo fractional differential equations and proved a maximum principle on a uniform grid. Brunner et al. [16] provided a finite difference maximum principle preserving scheme for the fractional diffusion equation by introducing an equivalent definition of the Caputo fractional derivative, but there is no numerical implementation. Jin et al. [17] studied three positivity preservation numerical methods with some special meshes and geometric restriction for the problem (1.2)–(1.4), including the standard Galerkin (SG) scheme, the lumped mass (LM) scheme, and the finite volume element (FVE) scheme, however, positivity is not preserved in small time or time step for all three schemes, and positivity may reappear after a positivity threshold. Especially, for lumped mass method positivity is preserved if and only if the triangulation is of Delaunay type. Based on energy stability, Liao et al. [18,19] considered the second-order and nonuniform adaptive time-stepping maximum principle preserving scheme for time-fractional Allen–Cahn equations, where the convolution structure of consistency error is used and sharp maximum-norm error estimates with the temporal regularity is proved. Ji et al. [20] provided the fast L1 formula preserving the discrete maximum principle for the time-fractional Allen–Cahn equation with Caputo’s derivative, then extended to volume constraint problem [21]. Based on the piecewise linear interpolation, Ji et al. [22] considered an adaptive second-order Crank–Nicolson time-stepping schemes for time-fractional molecular beam epitaxial growth models, and proved the provided schemes preserve the positive semidefinite property of the integral kernel. However, all of these existing preserving positivity schemes are designed on conforming meshes, which cannot be applied directly to non-conforming grids.

It is very necessary to construct monotone diffusion scheme on general non-conforming meshes. For some applications such as inertial confinement fusion and nuclear reactor, it is important to make sure that the heat propagation travels in the right direction, that is heat should flow from higher to lower temperature on distorted, especially on non-conforming meshes. However, to the best of our knowledge, there is no numerical method to consider the positivity preservation for subdiffusion equation on non-conforming distorted meshes in the appearance of hanging nodes. In this paper, the nonlinear preserving positivity finite volume scheme will be used to non-conforming meshes, moreover, a new method will be introduced to eliminate auxiliary vertex unknowns, in particular those defined vertex unknowns at hanging nodes. We find that our new method is adapted to almost arbitrary mesh geometry including non-conforming cells, and is a cell-centered conservative scheme.

The rest of this paper is organized as follows. In Sect. 2, we first present some definitions and notations for the meshes, and then we consider the construction of the nonlinear finite volume scheme, and introduce a formula of calculating hanging-node unknowns. In Sect. 3, positivity and nonlinear iteration are studied. Existence of a solution for the positivity-preserving scheme is analyzed in Sect. 4. Numerical examples are presented to demonstrate the performance of our new methods in Sect. 5.

2 Construction of a fully positivity-preserving scheme

Firstly, we present some definitions and notations for the meshes. Set \( (\mathcal{M}, \mathcal{E}, \mathcal{P}) \) be a mesh partition of \( \Omega \), where

- \( \mathcal{M} \) is a set of polygonal cells \( K \), \( \tilde{\Omega} = \bigcup_{K \in \mathcal{M}} \tilde{K} \), and \( \partial K \) is the boundary of \( K \).
- \( \mathcal{E} \) is the set of edges of all the cells, denoted by \( \sigma \).
- \( \mathcal{P} = (x_k)_{K \in \mathcal{M}} \) is a set of points, \( \forall K \in \mathcal{M}, x_k \in K \).
- \( \mathcal{E} \) is the set of edges of all the cells, denoted by \( \sigma \).

We denote by \( \mathcal{E}_K \subset \mathcal{E} \) the set of edges of \( K \in \mathcal{M} \). Set \( \mathcal{E} = \mathcal{E}_{ext} \cup \mathcal{E}_{int} \), \( \mathcal{E}_{ext} \) and \( \mathcal{E}_{int} \) the set of boundary and interior edges, respectively.

Each cell \( K \) is a star-shaped polygon with respect to its collocation point \( x_K \); if \( \sigma \in \mathcal{E}_{ext} \), the perpendicular line of \( \sigma \), which starts from \( x_K \), intersects on the boundary of \( \Omega \), not on any interior edge.

Next, we present more notations (see Fig. 2). Let \( m(K) \) be the area of cell \( K \), \( h = (\sup_{K \in \mathcal{M}} m(K))^1/2 \). Denote the cell–vertexes by \( P_1, P_2, P_3, \ldots \) and \( A, B \). Set \( \mathbf{n}_{K,\sigma} \) (resp. \( \mathbf{n}_{L,\sigma} \)) be the unit outer normal on the edge \( \sigma \) of cell \( K \) (resp. \( L \)), and \( \mathbf{n}_{K,\sigma} = -\mathbf{n}_{L,\sigma} \) for \( \sigma = K|L| = AB \). Let \( \mathbf{t}_{K,P_j} \) and \( \mathbf{t}_{L,P_j} \) be the unit tangential vectors on the line \( KP_j \) and \( LP_j \), \( j = 1, 2, \ldots \). The distance between \( A \) and \( B \) is denoted by \( |AB| \). The unknown \( u \) defined at \( K \) is denoted by \( u_K \).
Now we describe the construction of a fully positivity-preserving scheme. Let $N$ be a positive integer. Firstly, by integrating (1.2) on the cell $K$, we have

$$
\int_K \sum_{k=1}^\ell l_k D_t^{\alpha_k} u(x, t_n) dx
- \int_K \nabla \cdot (\kappa(x, t_n) \nabla u(x, t_n)) dx
= \int_K f(x, t_n) dx, \ 1 \leq n \leq N,
$$

by using the divergence formula to equation above, we have

$$
\int_K \sum_{k=1}^\ell l_k D_t^{\alpha_k} u(x, t_n) dx
- \int_{\partial K} (\kappa(x, t_n) \nabla u(x, t_n)) \cdot n_{K, \sigma} dl
= \int_K f(x, t_n) dx, \ 1 \leq n \leq N,
$$

that is

$$
\int_K \sum_{k=1}^\ell l_k D_t^{\alpha_k} u(x, t_n) dx
- \sum_{\sigma \in \partial K} \int_{\sigma} (\kappa(x, t_n) \nabla u(x, t_n)) \cdot n_{K, \sigma} dl
= \int_K f(x, t_n) dx, \ 1 \leq n \leq N,
$$

Denote

$$
\mathcal{P}^n_{t, K} = \int_K \sum_{k=1}^\ell l_k D_t^{\alpha_k} u(x, t_n) dx, \ 1 \leq n \leq N, \quad (2.5)
$$

and let

$$
\mathcal{F}^n_{K, \sigma} = - \int_{\sigma} (\kappa(x, t_n) \nabla u(x, t_n)) \cdot n_{K, \sigma} dl, \ 1 \leq n \leq N, \quad (2.6)
$$

be the continuous subdiffusion flux going out of $K$ through on the edge $\sigma$.

To get a fully discrete scheme, we first discretize (2.5) in time. Taking into account this initial singularity of the solution of problem (1.2)–(1.4), it is well known that graded meshes produce more accurate numerical solutions than uniform meshes (see [23, Chapter 6]), and this is our motivation for choosing temporal graded mesh. Set $t_n = (n/N)^\tau T$ for $n = 0, 1, \ldots, N$, where $\tau \geq 1$ is the mesh grading constant. Let $\tau_n = t_n - t_{n-1}$, $\tau_n = (t_n - t_{n-1})$ for $n = 1, \ldots, N$, and $\tau = \max_{1 \leq k \leq N} \tau_k$.

For each $\alpha_k$ and $n \geq 1$, we approximate Caputo time-fractional derivative $D_t^{\alpha_k} u(x, t_n)$ of (2.5) by the classical L1 formula on graded meshes

$$
D_t^{\alpha_k} u(x, t_n) = \frac{1}{\Gamma(1 - \alpha_k)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \frac{\partial u(x, s)}{\partial s} (t_n - s)^{\alpha_k - 1} ds
$$

$$
= \frac{1}{\Gamma(1 - \alpha_k)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \frac{\Delta_t u(x, t_j)}{\tau_j} ds + R^n_{\tau, \alpha_k}
$$

$$
= \frac{1}{\Gamma(2 - \alpha_k)} \sum_{j=0}^{n-1} \int_{t_{n-j}}^{t_{n-j-1}} \frac{\Delta_t u(x, t_{n-j})}{\tau_{n-j}} ds + R^n_{\tau, \alpha_k},
$$

where $\Delta_t u(x, t_j) = u(x, t_{j+1}) - u(x, t_j)$. Let

$$
b^n_{j+1} = [(t_n - t_{n-j-1})^{1 - \alpha_k} - (t_n - t_{n-j})^{1 - \alpha_k}] / \tau_{n-j}, \quad j = 0, \ldots, n - 1,
$$

and

$$
b^n_j = \begin{cases} 
- \frac{(t_n - t_{n-j})^{1 - \alpha_k} - (t_n - t_{n-j})^{1 - \alpha_k}}{\tau_j}, & j = 0; \\
\frac{(t_n - t_{n-j-1})^{1 - \alpha_k} - (t_n - t_{n-j})^{1 - \alpha_k}}{\tau_{j+1}} - \frac{(t_n - t_{n-j})^{1 - \alpha_k} - (t_n - t_{n-j+1})^{1 - \alpha_k}}{\tau_n}, & 1 \leq j \leq n - 1; \\
\frac{(t_n - t_{n-1})^{1 - \alpha_k}}{\tau_n}, & j = n.
\end{cases}
$$
Note that $b_{n,\alpha k}^n = \tau_{n}^{-\alpha_k}$, since $(t-s)^{-\alpha}$ is the monotonic increasing function, it is easy to get

$$b_j^{n,\alpha_k} < 0 \quad \text{for} \quad j = 0, 1, \ldots, n - 1. \quad (2.7)$$

Also, note that $\tilde{b}_j^{n,\alpha_k} = \tau_{n}^{-\alpha_k}$, by using the mean value theorem, we easily obtain

$$\tilde{b}_{j+1}^{n,\alpha_k} < b_{j}^{n,\alpha_k}, \quad 0 \leq j \leq n - 1 \leq N - 1. \quad (2.8)$$

Denote

$$D^n_{N} u(x, t_n) = \frac{1}{\Gamma(2 - \alpha_k)} \sum_{j=0}^{n-1} b_{j+n}^{n,\alpha_k} \Delta_1 u(x, t_{n-j-1})$$

$$= \frac{1}{\Gamma(2 - \alpha_k)} \sum_{j=0}^{n} b_{j}^{n,\alpha_k} u(x, t_j),$$

where $D^n_{N} u(x, t_n)$ is an approximation to $D_t^{\alpha_k} u(x, t_n)$. Then, $\mathcal{F}_{N, k}^n$ in (2.5) can be approximated by

$$\mathcal{F}_{N, k}^n = \int_{\mathcal{K}} \sum_{k=1}^{l} l_k D^n_{N} u(x, t_n) dx$$

$$= \int_{\mathcal{K}} \sum_{k=1}^{l} l_k \frac{1}{\Gamma(2 - \alpha_k)} \sum_{j=0}^{n-1} b_{j+n}^{n,\alpha_k} \Delta_1 u(x, t_{n-j-1}) dx$$

$$= \int_{\mathcal{K}} \sum_{k=1}^{l} l_k \frac{1}{\Gamma(2 - \alpha_k)} \sum_{j=0}^{n} b_{j}^{n,\alpha_k} u(x, t_j) dx, \quad 1 \leq n \leq N. \quad (2.9)$$

Based on the proposed graded meshes above, for each $k = 1, 2, \ldots, m$, there exists a constant $C$ so that

$$R_{T, \alpha_k}^n = |D^n_{N} u(x, t_n) - D_t^{\alpha_k} u(x, t_n)| \leq C n^{-\min(2 - \alpha_k)}.$$  

(2.10)

The proof of the above results can be found in Lemma 3.1 of Huang and Martin [24].

In order to attain a fully positivity-preserving finite volume method, we also need to approximate the continuous subdiffusion flux (2.6) of the edge $\sigma$.

By using Fig. 2, the Gauss theorem, and the following well-known formula,

$$kappa(x, t_n) \nabla u(x, t_n) \cdot n_{K, \sigma}$$

$$u(x, t_n) \cdot (k^T(x, t_n) n_{K, \sigma}),$$

we have

$$\mathcal{F}_{K, \sigma}^n = -\int_{\sigma} \nabla u(x, t_n) \cdot k^T(x, t_n) n_{K, \sigma} dl, \quad 1 \leq n \leq N. \quad (2.11)$$

where $k^T(x, t_n)$ is the transpose of subdiffusion matrix $\kappa(x, t_n)$. By the mesh stencil in Fig. 2, we can obtain

$$\frac{1}{|k^T(x, t_n) n_{K, \sigma}|} k^T(x, t_n) n_{K, \sigma}$$

$$= \frac{1}{\sin \theta_K} [\sin \theta_{K_2} t_{K_1} + \sin \theta_{K_1} t_{K_2}], \quad 1 \leq n \leq N. \quad (2.12)$$

and

$$\frac{1}{|k^T(x, t_n) n_{L, \sigma}|} k^T(x, t_n) n_{L, \sigma}$$

$$= \frac{1}{\sin \theta_L} [\sin \theta_L t_{L_3} + \sin \theta_L t_{L_2}], \quad 1 \leq n \leq N. \quad (2.13)$$

For $1 \leq n \leq N$, substituting (2.12) into (2.11), we have

$$\mathcal{F}_{K, \sigma}^n$$

$$= -\int_{\sigma} |k^T(x, t_n) n_{K, \sigma}| (\sin \theta_{K_2} \nabla u(x, t_n) \cdot t_{K_1})$$

$$+ \sin \theta_{K_1} \nabla u(x, t_n) \cdot t_{K_2}) \, dl$$

$$= \frac{|k^T(L, t_n) n_{L, \sigma}|}{\sin \theta_L} \left( \sin \theta_{L_2} u(P_4, t_n) - u(L, t_n) \right)$$

$$+ \sin \theta_{L_1} \left( u(L_3, t_n) - u(L, t_n) \right) + R_{L, \sigma}^n,$$

and

$$\mathcal{F}_{L, \sigma}^n$$

$$= -\int_{\sigma} |k^T(x, t_n) n_{L, \sigma}| (\sin \theta_{L_2} \nabla u(x, t_n) \cdot t_{L_4})$$

$$+ \sin \theta_{L_1} \nabla u(x, t_n) \cdot t_{L_3}) \, dl$$

$$= \frac{|k^T(L, t_n) n_{L, \sigma}|}{\sin \theta_L} \left( \sin \theta_{L_2} u(P_4, t_n) - u(L, t_n) \right)$$

$$+ \sin \theta_{L_1} \left( u(L_3, t_n) - u(L, t_n) \right) + R_{L, \sigma}^n,$$

and

$$R_{K, \sigma}^n = O(h^2), \quad R_{L, \sigma}^n = O(h^2), \quad \text{for} \quad 1 \leq n \leq N. \quad (2.14)$$

Let

$$\tilde{F}_1^n = -\frac{\sin \theta_{K_2}}{|K P_1|} \left( \sin \theta_{K_2} (u_{P_1}^n - u_{K_1}^n) + \sin \theta_{K_1} (u_{P_2}^n - u_{K_2}^n) \right),$$

and
\[
\tilde{F}_2^n = -\frac{[\kappa^T(L, t_n)\mathbf{n}_{L, \sigma}]}{\sin \theta_L} \left( \frac{\sin \theta_{L_2}}{|L P_4|} (u_{p_4}^n - u_{n}^n) + \frac{\sin \theta_{L_1}}{|L P_3|} (u_{p_3}^n - u_{n}^n) \right),
\]
which are discrete normal flux on \(\sigma\) of cell \(K\) (resp. \(L\)). To obtain a finite volume scheme satisfying the positivity-preserving property and conservation, we use the continuity of the normal flux component on edge \(\sigma\) and introduce the weighted combination of \(\tilde{F}_1^n\) and \(\tilde{F}_2^n\) to get:
\[
F_{K, \sigma} = -F_{L, \sigma} = \mu_1^n \tilde{F}_1^n - \mu_2^n \tilde{F}_2^n,
\]
where \(\mu_1 \geq 0, \mu_2 \geq 0\) and \(\mu_1^n + \mu_2^n = 1\).

By combining the expression \(\tilde{F}_1^n\) and \(\tilde{F}_2^n\), we have
\[
F_{K, \sigma} = -F_{L, \sigma} = \mu_1^n \frac{[\kappa^T(K, t_n)\mathbf{n}_{K, \sigma}]}{|\sigma|-1 \sin \theta_K} \left( \frac{\sin \theta_{K_2}}{|K P_1|} + \frac{\sin \theta_{K_1}}{|K P_2|} u_{K}^n - \mu_2^n \frac{[\kappa^T(L, t_n)\mathbf{n}_{L, \sigma}]}{|\sigma|-1 \sin \theta_L} \left( \frac{\sin \theta_{L_2}}{|L P_4|} + \frac{\sin \theta_{L_1}}{|L P_3|} u_{L}^n \right) \right).
\]

To get the two-point flux approximation, we take
\[
\begin{align*}
-\mu_1^n \frac{[\kappa^T(K, t_n)\mathbf{n}_{K, \sigma}]}{|\sigma|-1 \sin \theta_K} \left( \frac{\sin \theta_{K_2}}{|K P_1|} u_{p_1}^n + \frac{\sin \theta_{K_1}}{|K P_2|} u_{p_2}^n \right) + & \mu_2^n \frac{[\kappa^T(L, t_n)\mathbf{n}_{L, \sigma}]}{|\sigma|} \left( \frac{\sin \theta_{L_2}}{|L P_4|} u_{p_4}^n + \frac{\sin \theta_{L_1}}{|L P_3|} u_{p_3}^n \right) = 0.
\end{align*}
\]

For the sake of writing, let
\[
a_1^n = \frac{[\kappa^T(K, t_n)\mathbf{n}_{K, \sigma}]}{\sin \theta_K} \left( \frac{\sin \theta_{K_2}}{|K P_1|} u_{p_1}^n + \frac{\sin \theta_{K_1}}{|K P_2|} u_{p_2}^n \right), \quad 1 \leq n \leq N,
\]
\[
a_2^n = \frac{[\kappa^T(L, t_n)\mathbf{n}_{L, \sigma}]}{\sin \theta_L} \left( \frac{\sin \theta_{L_2}}{|L P_4|} u_{p_4}^n + \frac{\sin \theta_{L_1}}{|L P_3|} u_{p_3}^n \right), \quad 1 \leq n \leq N.
\]

Therefore, \(\mu_1\) and \(\mu_2\) satisfy
\[
\begin{align*}
\mu_1^n + \mu_2^n &= 1, \\
-a_1^n \mu_1^n + a_2^n \mu_2^n &= 0.
\end{align*}
\]

Now we consider the solution of system (2.15). If \(a_1^n + a_2^n \neq 0\), then (2.15) has a unique solution, and reads
\[
\begin{align*}
\mu_1^n &= a_2^n/(a_1^n + a_2^n), \\
\mu_2^n &= a_1^n/(a_1^n + a_2^n),
\end{align*}
\]
for \(1 \leq n \leq N\).

If \(a_1^n + a_2^n = 0\), generally, we take \(\mu_1^n = \mu_2^n = \frac{1}{2}, 1 \leq n \leq N\). However, this may cause \(\mu_1^n\) and \(\mu_2^n\) to be discontinuous with respect to the solution \(u\), so to avoid this case we take
\[
\begin{align*}
\hat{a}_1^n &= a_1^n + h^2, \\
\hat{a}_2^n &= a_2^n + h^2, \quad 1 \leq n \leq N.
\end{align*}
\]

Also
\[
\begin{align*}
\hat{\mu}_1^n + \hat{\mu}_2^n &= 1, \\
-\hat{a}_1^n \hat{\mu}_1^n + \hat{a}_2^n \hat{\mu}_2^n &= 0,
\end{align*}
\]
then we can get a unique solution of system (2.15) as
\[
\begin{align*}
\hat{\mu}_1^n &= (a_2^n + h^2)/(a_1^n + a_2^n + h^2), \\
\hat{\mu}_2^n &= (a_1^n + h^2)/(a_1^n + a_2^n + h^2), \quad 1 \leq n \leq N.
\end{align*}
\]

Next, we prove that \(\hat{\mu}_1^n\) and \(\hat{\mu}_2^n\) are continuous with respect to the solution \(u\).

We introduce a binary auxiliary function
\[
Q(x, y) = \frac{y + h^2}{x + y + 2h^2},
\]
which is differentiable for \(x \geq 0, y \geq 0\), and
\[
\begin{align*}
\frac{\partial Q(x, y)}{\partial x} &= -\frac{y + h^2}{(x + y + 2h^2)^2}, \\
\frac{\partial Q(x, y)}{\partial y} &= \frac{x + h^2}{(x + y + 2h^2)^2}.
\end{align*}
\]

Thus, we have
\[
|\nabla Q| \leq 1/(2h^2),
\]
\[
\hat{\mu}_1^n = \hat{\mu}_1^n(u) = Q(a_1^n(u), a_2^n(u)), \quad 1 \leq n \leq N.
\]

and
\[
|\hat{\mu}_1^n(u) - \hat{\mu}_1^n(w)| = |Q(a_1^n(u), a_2^n(u)) - Q(a_1^n(w), a_2^n(w))| \leq \frac{|a_1^n(u) - a_1^n(w)| + |a_2^n(u) - a_2^n(w)|}{2h^2}, \quad 1 \leq n \leq N.
\]

By the definition of \(a_1^n\) and \(a_2^n\), there is a constant \(C\) such that
\[
\begin{align*}
|a_1^n(u) - a_1^n(w)| &\leq C \|u - w\|, \\
|a_2^n(u) - a_2^n(w)| &\leq C \|u - w\|.
\end{align*}
\]

Therefore,
\[
|\hat{\mu}_1^n(u) - \hat{\mu}_1^n(w)| \leq \frac{C}{h^2} \|u - w\|, \quad 1 \leq n \leq N,
\]
that is to say \( \tilde{\mu}_1^2(u) \) is a continuous function of \( u \). Similarly, we can prove that \( \tilde{\mu}_2^2(u) \) is also a continuous function of \( u \).

By the definitions of \( \theta_{K_1}, \theta_{K_2}, \theta_{L_1}, \theta_{L_2}, \theta_\sigma \) and \( \theta_L \) in mesh stencil (Fig. 2), we induce

\[
\sin \theta_K > 0, \quad \sin \theta_{K_1} \geq 0, \quad \sin \theta_{K_2} \geq 0,
\]

and

\[
\sin \theta_L > 0, \quad \sin \theta_{L_1} \geq 0, \quad \sin \theta_{L_2} \geq 0.
\]

Therefore, if the conditions \( a^n_{\sigma} \geq 0 \) (\( i = 1, 2, 3, 4, 1 \leq n \leq N \)) hold, then we have

\[
a^n_1 \geq 0, \quad a^n_2 \geq 0, \quad 1 \leq n \leq N,
\]

and

\[
\hat{\mu}_1^n \geq 0, \quad \hat{\mu}_2^n \geq 0, \quad 1 \leq n \leq N.
\]

We now can redefine the two-point subdiffusion flux approximation as

\[
F^n_{K,\sigma} = -F^n_{L,\sigma} = \tilde{\mu}_1^n \frac{|\kappa T(K, t_\sigma)| |\kappa L(T, t_\sigma)|}{|\sigma|^{-1} \sin \theta_K} \left( \frac{\sin \theta_{K_2}}{|K P_1|} + \frac{\sin \theta_{K_1}}{|K P_2|} \right) u^n_K
\]

\[
- \tilde{\mu}_2^n \frac{|\kappa T(K, t_\sigma)| |\kappa L(T, t_\sigma)|}{|\sigma|^{-1} \sin \theta_L} \left( \frac{\sin \theta_{L_2}}{|L P_4|} + \frac{\sin \theta_{L_1}}{|L P_3|} \right) u^n_L.
\]

Let

\[
A^n_{K,\sigma} = \tilde{\mu}_1^n \frac{|\kappa T(K, t_\sigma)|}{|\sigma|^{-1} \sin \theta_K} \left( \frac{\sin \theta_{K_2}}{|K P_1|} + \frac{\sin \theta_{K_1}}{|K P_2|} \right), \quad 1 \leq n \leq N,
\]

\[
A^n_{L,\sigma} = \tilde{\mu}_2^n \frac{|\kappa T(L, t_\sigma)|}{|\sigma|^{-1} \sin \theta_L} \left( \frac{\sin \theta_{L_2}}{|L P_4|} + \frac{\sin \theta_{L_1}}{|L P_3|} \right), \quad 1 \leq n \leq N.
\]

It is easy to see

\[
A^n_{K,\sigma} \geq 0, \quad A^n_{L,\sigma} \geq 0, \quad 1 \leq n \leq N.
\]

Then, if \( \sigma = K/L \in \mathcal{E}_{int} \), we attain

\[
F^n_{K,\sigma} = -F^n_{L,\sigma} = A^n_{K,\sigma} u^n_K - A^n_{L,\sigma} u^n_L, \quad 1 \leq n \leq N.
\] (2.17)

If \( \sigma \in \mathcal{E}_{ext} \) (see Fig. 2), let

\[
A^n_{K,\sigma} = \frac{|\kappa T(K, t_\sigma)|}{|\sigma|^{-1} \sin \theta_K} \left( \frac{\sin \theta_{K_2}}{|K P_1|} + \frac{\sin \theta_{K_1}}{|K P_2|} \right), \quad 1 \leq n \leq N,
\]

\[
a^n_{K,\sigma} = \frac{|\kappa T(K, t_\sigma)|}{|\sigma|^{-1} \sin \theta_K} \left( \frac{\sin \theta_{K_2}}{|K P_1|} + \frac{\sin \theta_{K_1}}{|K P_2|} \right), \quad 1 \leq n \leq N.
\]

Then, we obtain

\[
F^n_{K,\sigma} = \frac{|\kappa T(K, t_\sigma)|}{|\sigma|^{-1} \sin \theta_K} \left( \frac{\sin \theta_{K_2}}{|K P_1|} u^n_{P_1} + \frac{\sin \theta_{K_1}}{|K P_2|} u^n_{P_2} \right) + \frac{\sin \theta_{K_2}}{|K P_1|} u^n_{P_1} + \frac{\sin \theta_{K_1}}{|K P_2|} u^n_{P_2} = A^n_{K,\sigma} u^n_K - A^n_{K,\sigma}, \quad 1 \leq n \leq N.
\] (2.18)

Therefore, by using (2.9), (2.17) and (2.18), we can obtain the positivity-preserving nonlinear finite volume scheme for problem (1.2)–(1.4) as follows

\[
\sum_{k=1}^\ell \frac{m(K)}{\Gamma(2-\alpha_k)} \sum_{j=0}^n b_{j} \alpha_k u^n_K
\]

\[
+ \sum_{\sigma \in \mathcal{E}_K} F^n_{K,\sigma} = m(K) f^n_K, \quad \forall K \in \mathcal{M}, \quad 1 \leq n \leq N,
\] (2.19)

\[
u^n_{P_1} = \psi(P_1, t_\sigma), \quad \forall P_1 \in \partial \Omega, \quad 1 \leq n \leq N,
\] (2.20)

\[
u^n_K = \psi(K), \quad \forall K \in \Omega,
\] (2.21)

where \( f^n_K = f(K, t_\sigma). \)

**Remark** The reason the scheme is nonlinear is that these coefficients \( A^n_{K,\sigma} \) and \( A^n_{L,\sigma} \) depend on cell–vertex unknowns.

To obtain a central type scheme, next, we will discuss how to eliminate vertex unknowns in the flux expression (2.17). For cell–vertex unknowns at non-hanging nodes, we use the methods in [26] to approximate. However, for hanging nodes the method in reference [26] fails, especially for “T” style cell–vertex, that is, the hanging node lies in a straight segment of a cell edge. Now, we focus on how to eliminate the auxiliary unknowns at hanging nodes. For convenience, we first introduce a local discrete stencil, which cell–vertex \( A \) is a hanging node (see Fig. 3).

Based on the discrete stencil in Fig. 3, using the continuity condition of normal flux components, we can obtain
where $\nabla u(p_2, t_n)|_{\Delta K_1 P_1 P_2}$ is the gradient of function $u(x, t_n)$ evaluated at cell–vertex $P_2$ in the triangle $\Delta K_1 P_1 P_2$, and the other notations have similar meanings.

According to the Gauss theorem and trapezoid quadrature rule, we have

$$\nabla u(x, t_n)|_{\Delta K_1 P_1 P_2} = \frac{1}{S_{\Delta K_1 P_1 P_2}} \int_{\Delta K_1 P_1 P_2} \nabla u(x, t_n) dx$$

$$\approx \frac{1}{2 S_{\Delta K_1 P_1 P_2}} \left[ (u^n_{K_1} - u^n_{P_1}) |K_1 P_1| \mathbf{n}_{11} + (u^n_{K_1} - u^n_{P_1}) |K_1 P_1| \mathbf{n}_{12} \right],$$

(2.25)

where $u^n_{K_1} = u(K_1, t_n)$, $u^n_{P_1} = u(P_1, t_n)$ and $u^n_{P_2} = u(P_2, t_n)$, and $S_{\Delta K_1 P_1 P_2}$ is the area of the triangle $\Delta K_1 P_1 P_2$.

Similarly,

$$\nabla u(x, t_n)|_{\Delta K_2 P_2 P_3} = \frac{1}{2 S_{\Delta K_1 P_1 P_2}}$$

Thus, by substitute (2.25)–(2.27) into (2.22), since $|K_1 P_1| \mathbf{n}_{11} + |K_1 P_2| \mathbf{n}_{12} = -|P_1 P_2| \mathbf{n}_{13}$, $|K_3 P_1| \mathbf{n}_{33} + |K_3 P_2| \mathbf{n}_{34} = -|P_1 P_2| \mathbf{n}_{31}$,

then, we obtain

$$\frac{|K_3 P_2|}{2 S_{\Delta K_3 P_1 P_2}} (\kappa(K_3, t_n) \mathbf{n}_{34}) \cdot \mathbf{n}_{31}$$

$$+ \frac{|K_1 P_1|}{2 S_{\Delta K_1 P_1 P_2}} (\kappa(K_1, t_n) \mathbf{n}_{12}) \cdot \mathbf{n}_{13} u^n_{P_1}$$

$$+ \frac{|K_3 P_1|}{2 S_{\Delta K_3 P_1 P_2}} (\kappa(K_3, t_n) \mathbf{n}_{33}) \cdot \mathbf{n}_{31}$$

$$+ \frac{|K_1 P_1|}{2 S_{\Delta K_1 P_1 P_2}} (\kappa(K_1, t_n) \mathbf{n}_{11}) \cdot \mathbf{n}_{13} u^n_{K_1}$$

$$+ \frac{|P_1 P_2|}{2 S_{\Delta K_1 P_1 P_2}} (\kappa(K_3, t_n) \mathbf{n}_{31}) \cdot \mathbf{n}_{31}$$

(2.29)

Let

$$\omega_{11} = \frac{|K_3 P_2|}{2 S_{\Delta K_3 P_1 P_2}} (\kappa(K_3, t_n) \mathbf{n}_{34}) \cdot \mathbf{n}_{31}$$

$$+ \frac{|K_1 P_1|}{2 S_{\Delta K_1 P_1 P_2}} (\kappa(K_1, t_n) \mathbf{n}_{12}) \cdot \mathbf{n}_{13},$$

$$\omega_{12} = \frac{|K_3 P_1|}{2 S_{\Delta K_3 P_1 P_2}} (\kappa(K_3, t_n) \mathbf{n}_{33}) \cdot \mathbf{n}_{31}$$

$$+ \frac{|K_1 P_1|}{2 S_{\Delta K_1 P_1 P_2}} (\kappa(K_1, t_n) \mathbf{n}_{11}) \cdot \mathbf{n}_{13},$$

$$\eta_{11} = -\frac{|P_1 P_2|}{2 S_{\Delta K_1 P_1 P_2}} (\kappa(K_1, t_n) \mathbf{n}_{13}) \cdot \mathbf{n}_{13},$$

$$\eta_{13} = -\frac{|P_1 P_2|}{2 S_{\Delta K_1 P_1 P_2}} (\kappa(K_3, t_n) \mathbf{n}_{31}) \cdot \mathbf{n}_{31}.$$

Then, (2.29) can be rewritten as

$$\omega_{11} u^n_{P_1} + \omega_{12} u^n_{P_2} = \eta_{11} u^n_{K_1} + \eta_{13} u^n_{K_3}.$$
Similarly, from (2.23) and (2.24), we have
\[
\omega_{22}u_p^0 + \omega_{23}u_p^3 = \eta_{22}u_k^2 + \eta_{23}u_k^3, \tag{2.31}
\]
\[
\omega_{31}u_p^1 + \omega_{32}u_p^2 + \omega_{33}u_p^3 = \eta_{33}u_k^3. \tag{2.32}
\]

Denote \(U_p^0 = (u_{p1}^0, u_{p2}^0, u_{p3}^0)\), \(U_p^3 = (u_{k1}^1, u_{k2}^1, u_{k3}^1)\), and
\[
B_\omega = \begin{pmatrix}
\omega_{11} & \omega_{12} & 0 \\
0 & \omega_{22} & \omega_{23} \\
\omega_{31} & \omega_{32} & \omega_{33}
\end{pmatrix}, \quad B_\eta = \begin{pmatrix}
\eta_{11} & 0 & \eta_{13} \\
0 & \eta_{22} & \eta_{23} \\
0 & 0 & \eta_{33}
\end{pmatrix}
\]

By combining (2.30)–(2.32), we can get the following system
\[
B_\omega U_p^0 = B_\eta U_k^3.
\]

Then, by solving the system the vertex unknowns \(u_{p3}^n\) can be written as the convex combination of cell-centered unknowns. That is
\[
u_{p3}^n = v_1u_{k1}^n + v_2u_{k2}^n + v_3u_{k3}^n. \tag{2.33}
\]

### 3 Positivity and nonlinear iteration

Denote \(U^n = (U^n_k)_{k \in \mathcal{M}}\) be the vector discrete cell-centered unknowns at the \(n\)-th time level. \(A(U^n)\) be the matrix corresponding to the spatial discretization. \(f^n = (f^n_K)_{k \in \mathcal{M}}, U^0 = (v(K))_{k \in \mathcal{M}}\) and \(\psi^n = (\psi(K, t_n))_{K \in \mathcal{M}}\) are three vectors corresponding to the source term, initial condition and boundary condition. Then, the positivity-preserving nonlinear finite volume scheme (2.19) can be written in the following matrix form,
\[
\sum_{k=1}^{\ell} \frac{m(K)l_k}{\Gamma(2 - \alpha_k)} b^{n,\alpha_k}_n U^n + A(U^n) U^n
\]
\[
= m(K) f^n - \sum_{k=1}^{\ell} \frac{m(K)l_k}{\Gamma(2 - \alpha_k)} \sum_{j=0}^{n-1} b^{n,\alpha_k}_j U^n + \psi^n. \tag{3.34}
\]

Let \(S\) be the diagonal matrix of the areas of control volumes, \(L\) be the diagonal matrix of with a diagonal element of \(\sum_{k=1}^{\ell} \frac{m(K)l_k}{\Gamma(2 - \alpha_k)} b^{n,\alpha_k}_n\), \(L_j\) be the diagonal matrix of with a diagonal element of \(-\sum_{k=1}^{\ell} \frac{m(K)l_k}{\Gamma(2 - \alpha_k)} b^{n,\alpha_k}_j\), \(F_n\) is a vectors corresponding to the right-hand-side vector of (3.34). Finally, (3.34) can be further rewritten as
\[
(L + A(U^n)) U^n = \bar{S}^n + \psi^n + \sum_{j=0}^{n-1} L_j U^n. \tag{3.35}
\]

For any vector \(U \in \mathbb{R}^n\) and \(U \geq 0\), the matrix \(A(U)\) has the following properties:
1. All diagonal entries \((A(U))_{ii}\) of matrix \(A(U)\) are positive;
2. All off-diagonal entries \((A(U))_{ij}\) \((i \neq j)\) of \(A(U)\) are non-positive;
3. Column sum corresponding to the boundary nodes is positive, and column sum corresponding to the interior nodes is 0.

For the nonlinear algebraic system (3.35), we will use the relaxed version of Picard nonlinear iterations to solve. By choosing a small value \(\delta_{non} > 0\) and initial vector \(U^0 \geq 0\) and repeat for \(k = 1, 2, \ldots\),
1. Solve \(L + A(U^{k-1}) \bar{U}^k = \bar{S}^n + \psi^n + \sum_{j=0}^{n-1} L_j U^j\),
2. \(U^k = U^{k-1} + \vartheta(\bar{U}^k - U^{k-1}), \vartheta \in (0, 1)\) is the damping factor,
3. Stop if \(||(L + A(U^n))U^n - F|| \leq \delta_{non}||L + A(U^n)|| F||\),
4. The value of \(U^n\) is then given by the last \(U^k\).

The linear system with non-symmetric matrix \((L + A(U^n))\) is solved by the Bi-Conjugate Gradient Stabilized (BiCGStab) method, and the linear iterations are stopped when relative norm of the initial residual becomes smaller than \(\delta_{lin}\).

Based on the definition of M-matrix [27], for any vector \(U \geq 0\), \(A(U)^T\) is an M-matrix. Note that
\[
b^{n,\alpha_k}_n = \tau_{-\alpha_k} > 0,
\]
then,
\[
(L + A(U^n))^T
\]
is also an M-matrix. By using the properties of M-matrix [27], it holds that the matrix
\[
(L + A(U^n))^{-T}
\]
has nonnegative elements. Moreover, from (2.7),
\[
b^{n,\alpha_k}_j < 0 \quad \text{for} \quad j = 0, 1, \ldots, n - 1.
\]

Therefore, under the assumptions \(f^n \geq 0, \psi^n \geq 0,\) and \(U^0 \geq 0\), the linear systems in the above Picard iterations are exactly solved. Moreover, it is clear that all iterative solutions are nonnegative vectors.
4 Existence of a solution for the positivity-preserving scheme

In this section, we prove the existence of a solution for the positivity-preserving nonlinear finite volume scheme (2.19).

Theorem 1 If \( f^n \geq 0, \psi^n \geq 0 \) and \( U^0 \geq 0 \), then the nonlinear scheme (3.35) has at least one solution \( U^n \).

Proof Define a compact set in \( \mathbb{R}^N \)
\[ \mathcal{C} = \{ w = (w_K)_{K \in \mathcal{M}} \in \mathbb{R}^N : w \geq 0, w_K \leq C_1 \}, \]
where \( C_1 \) be a nonnegative constant.

Define a map \( \phi : \mathcal{C} \mapsto \mathbb{R}^N \) such that
\[ \phi(w) = (L + A(w))^{-1}(Sf^n + \psi^n + \sum_{j=0}^{n-1} L_j U^j). \]
(4.36)

Now we need to prove that \( \phi \) has a fixed point in order to prove that system (4.36) has a solution. In view of the discussion in the above section, we prove that the matrix \( (L + A(w))^{-1} \) has nonnegative elements. Since
\[ F^n = Sf^n + \psi^n + \sum_{j=0}^{n-1} L_j U^j \geq 0, \]
we have
\[ \phi(w) \geq 0, \text{ for } \forall w \in \mathcal{C}. \]

Rewritten (4.36) as
\[ (L + A(w))\phi(w) = Sf^n + \psi^n + \sum_{j=0}^{n-1} L_j U^j. \]
(4.37)

Multiplying (4.37) by the constant vector \((1, \cdots , 1)\), we have
\[ (1, \cdots , 1) \cdot (L + A(w))\phi(w) = (1, \cdots , 1) \cdot (Sf^n + \psi^n + \sum_{j=0}^{n-1} L_j U^j), \]
(4.38)

by using the facts that \( \phi(w) \geq 0 \) and all column sums in \( A(w) \) are nonnegative, the term on the left of (4.38) can be estimate as
\[ (1, \cdots , 1) \cdot L\phi(w) \leq (1, \cdots , 1) \cdot (L + A(w))\phi(w) \]
\[ = (1, \cdots , 1) \cdot (L + \tilde{A}(w))\phi(w), \]
(4.39)

where \( \tilde{A}(w) \) is the diagonal matrix of the unoffset parts of the vector \((1, \cdots , 1)\) that depend on \( u \), and all diagonal entries are non-positive.

The right-hand side of (4.39) is a nonnegative constant. We take
\[ C_0 = (1, \cdots , 1) \cdot \left( Sf^n + \psi^n + \sum_{j=0}^{n-1} L_j U^j \right). \]

Then, \( \phi \) maps \( \mathcal{C} \) into itself. The set \( \mathcal{C} \) is a convex compact subset of \( \mathbb{R}^N \). Since each coefficient of \( A(w) \) is a continuous function of \( w \) and \( A(w) \mapsto (L + A(w))^{-1} \) is continuous from the set \( \mathcal{M} \)-matrices to the set of matrices, \( \phi \) is continuous. Hence, we may apply Brouwer’s theorem, which implies that \( \phi \) has a fixed point in \( \mathcal{C} \), i.e., the scheme (3.35) has a solution. The proof of Theorem 1 is finished. \( \square \)

5 Numerical experiments

In this section, we examine numerical performance of the positivity-preserving finite volume method proposed in this paper. The positivity preservation and convergence study of the discrete solution are presented for four subdiffusion problems on both uniform and random distorted quadrilateral meshes. We use the graded meshes in time with \( N \) intervals. The optimal value of the mesh grading parameter \( r = \frac{2 - \delta_{lin}}{\delta_{lin}} \) is used in all five examples. By choosing the time step \( \tau \) and the number of cell so that the error stemming from the temporal and spatial is consistent. In this paper we take \( \delta_{non} = 1.0 \times 15 \) and \( \delta_{lin} = 1.0 \times 20 \). Moreover, to illustrate the efficiency of our scheme, we apply it to solve an equilibrium radiation subdiffusion equation.

Here for the Picard nonlinear iterations we take \( \delta_{non} = 1.0 \times 15 \), and \( \delta_{lin} = 1.0 \times 20 \) for BiCGStab iterations. We use following discrete norms to evaluate approximation errors.

For the solution \( u \), we use the following \( L_2 \)-norm and \( L_{\infty} \)-norm
\[ \epsilon^u_2 = \left\| \sum_{K \in \mathcal{F}} (u_K - u(K))^2 m(K) \right\|^{1/2}, \]
\[ \epsilon^u_{\infty} = \max_{K \in \mathcal{F}} \| u_K - u(K) \|. \]

For the flux \( F \), we use the following \( L_2 \)-norm and \( L_{\infty} \)-norm
\[ \epsilon^F_2 = \left[ \sum_{\sigma \in \mathcal{E}} (F_{K, \sigma} - F_{K, \sigma})^2 \right]^{1/2}, \]
\[ E_{\infty}^F = \max_{\sigma \in \mathcal{E}} |F_{K,\sigma} - \mathcal{F}_{K,\sigma}|. \]

In Example 1-Example 4, we cover mainly single-term ($\ell = 1$). In Example 5, we consider the multi-term subdiffusion problem.

5.1 A problem with anisotropic diffusion coefficient

**Example 1** Let us consider the problem (1.2) on $\Omega = [0, 1] \times [0, 1]$, $T = 1$ and $\ell = 1$. Take $\kappa(x, y) = RDR^T$, and

\[ R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad D = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \]

where $\theta = \frac{5\pi}{12}$, $\kappa_1(x, y) = 1 + 2x^2 + y^2$, $\kappa_2(x, y) = 1 + x^2 + 2y^2$. The solution is chosen to be $u(x, y) = t^{\alpha_1} \sin(\pi x) \sin(\pi y)$.

We take $\alpha_1 = 0.5$ and test the proposed monotone finite volume scheme on non-conforming rectangular mesh and random meshes shown in Figs. 4 and 7 for four levels of grid refinement. This random mesh is generated from a perturbation of mesh in Fig. 4. For a vertex $P$, we denote $h_P = \frac{1}{2} \min_{|P| \in \sigma} |\sigma|$, and the random mesh is defined by $x_{PR} = x_P + \beta(R - 0.5)h_P$, where $\beta \in [0, 1]$ is a parameter, and $R$ is a normalized random variable. In this test, we take $\beta = 0.5$. In Table 1, the accuracy of solution and flux are obtained for four non-conforming rectangular meshes shown in Fig. 4, starting with the initial grid and refining the grids uniformly on each successive level. In Table 2, we test the accuracy of our schemes on four non-conforming random meshes shown in Fig. 7. We can see that our scheme attain second-order accuracy for the solution.

**Table 1** Errors on non-conforming rectangular meshes for Example 1

| Cells | Accuracy of solution $\|u\|_\infty$ Order | $\|u\|_2^2$ Order | Accuracy of flux $\|F\|_\infty$ Order | $\|F\|_2^2$ Order |
|-------|---------------------------------|----------------|-------------------------------|----------------|
| 88    | 5.0854e-2                       | -              | 7.6029e-2                     | -              |
| 352   | 1.3066e-2                       | 1.9605         | 1.3260e-2                     | 2.5195         |
| 1408  | 3.4938e-3                       | 1.9029         | 3.4540e-3                     | 1.9407         |
| 5632  | 9.2512e-4                       | 1.9171         | 8.6449e-4                     | 1.9983         |

**Table 2** Errors on non-conforming random meshes for Example 1

| Cells | Accuracy of solution $\|u\|_\infty$ Order | $\|u\|_2^2$ Order | Accuracy of flux $\|F\|_\infty$ Order | $\|F\|_2^2$ Order |
|-------|---------------------------------|----------------|-------------------------------|----------------|
| 88    | 8.0850e-2                       | -              | 1.8053e-1                     | -              |
| 352   | 3.6256e-2                       | 1.1570         | 4.2362e-2                     | 2.0914         |
| 1408  | 7.9732e-3                       | 2.1850         | 8.9325e-3                     | 2.2456         |
| 5632  | 2.1873e-3                       | 1.8660         | 2.7523e-3                     | 1.6984         |

We take $\alpha_1 = 0.5$ and test the proposed monotone finite volume scheme on non-conforming rectangular mesh and random meshes shown in Figs. 4 and 7 for four levels of grid refinement. This random mesh is generated from a perturbation of mesh in Fig. 4. For a vertex $P$, we denote $h_P = \frac{1}{2} \min_{|P| \in \sigma} |\sigma|$, and the random mesh is defined by $x_{PR} = x_P + \beta(R - 0.5)h_P$, where $\beta \in [0, 1]$ is a parameter, and $R$ is a normalized random variable. In this test, we take $\beta = 0.5$. In Table 1, the accuracy of solution and flux are obtained for four non-conforming rectangular meshes shown in Fig. 4, starting with the initial grid and refining the grids uniformly on each successive level. In Table 2, we test the accuracy of our schemes on four non-conforming random meshes shown in Fig. 7. We can see that our scheme attain second-order accuracy for the solution.

**Table 1** Errors on non-conforming rectangular meshes for Example 1

| Cells | Accuracy of solution $\|u\|_\infty$ Order | $\|u\|_2^2$ Order | Accuracy of flux $\|F\|_\infty$ Order | $\|F\|_2^2$ Order |
|-------|---------------------------------|----------------|-------------------------------|----------------|
| 88    | 5.0854e-2                       | -              | 7.6029e-2                     | -              |
| 352   | 1.3066e-2                       | 1.9605         | 1.3260e-2                     | 2.5195         |
| 1408  | 3.4938e-3                       | 1.9029         | 3.4540e-3                     | 1.9407         |
| 5632  | 9.2512e-4                       | 1.9171         | 8.6449e-4                     | 1.9983         |

**Table 2** Errors on non-conforming random meshes for Example 1

| Cells | Accuracy of solution $\|u\|_\infty$ Order | $\|u\|_2^2$ Order | Accuracy of flux $\|F\|_\infty$ Order | $\|F\|_2^2$ Order |
|-------|---------------------------------|----------------|-------------------------------|----------------|
| 88    | 8.0850e-2                       | -              | 1.8053e-1                     | -              |
| 352   | 3.6256e-2                       | 1.1570         | 4.2362e-2                     | 2.0914         |
| 1408  | 7.9732e-3                       | 2.1850         | 8.9325e-3                     | 2.2456         |
| 5632  | 2.1873e-3                       | 1.8660         | 2.7523e-3                     | 1.6984         |
Fig. 5 The projection of the solution in the $x$ direction on non-conforming rectangular meshes for Example 1.
**Fig. 6** The projection of the solution at the grid point in the x direction on non-conforming rectangular meshes for Example 1.
In Fig. 5, we show the projection of the solution at the grid point in the x direction obtained by our proposed monotone finite volume scheme on four non-conforming rectangular meshes shown in Fig. 4, and the counterparts at the grid point are shown in Fig. 6. In Fig. 8, we give the projection of the solution at the grid point in the x direction on four non-conforming random meshes shown in Fig. 7, and the counterparts at the grid point are shown in Fig. 9. In these figures, we observe that the monotone finite volume scheme preserves positivity of the solution.

**Example 2** We consider the single-term model (1.2)–(1.4) on $\Omega = [0, 1] \times [0, 1]$ and $T = 0.1$, and set

$$\kappa = \begin{pmatrix} \varepsilon x^2 + y^2 & -xy + \varepsilon xy \\ -xy + \varepsilon xy & x^2 + \varepsilon y^2 \end{pmatrix}, \quad \varepsilon = 5 \times 10^{-3},$$

$$f(x, y) = \begin{cases} 1 & \text{if } x \in [3/8, 5/8], y \in [3/8, 5/8], \\ 0 & \text{otherwise,} \end{cases}$$

$v(x, y) = 0$ and $\psi(x, y, t) = 0$ on $\partial \Omega$.

In Example 2, the exact solution $u(x, y, t)$ is unknown. We test our scheme on non-conforming rectangular mesh and non-conforming random quadrilateral meshes (see Figs. 10, 11, 12, 13). We take $\alpha_1 = 0.6$ and the perturbation parameter $\beta = 0.2$. The numerical solutions are shown in Figs. 11 and 12 for non-conforming rectangular meshes and Figs. 14 and 15 for non-conforming random quadrilateral meshes, which demonstrates that our scheme preserves positivity of the solution in this case. Although our method is positive on the non-conforming rectangular meshes and non-conforming random meshes, however, we find that our method does not preserve the maximum at several distorted grid points, such as the grid point $(0.374502, 0.554261)$ in the first figure of Figs. 14 and 15, and the grid points $(0.596276, 0.171952), (0.689080, 0.090537), (0.751508, 0.069647)$ and $(0.844898, 0.020044)$ at the third figure of Figs. 14 and 15. We will discuss a finite volume scheme preserving maximum principle in our subsequent work.

In Example 3, we present numerical results for multi-term subdiffusion problem (1.2) with two fractional-derivative terms.

**Example 3** We consider the multi-term subdiffusion equation (1.2)–(1.4) on $\Omega = [0, 1] \times [0, 1]$ and $T = 0.01$ with two fractional-derivative terms

$$D_t^{\alpha_1} u + 0.1 D_t^{\alpha_2} u - \nabla \cdot (\kappa \nabla u) = f(x, y, t),$$

where $\kappa = \begin{pmatrix} 10^{-1} & 0 \\ 0 & 10^{-1} \end{pmatrix}$, the initial conditions $v(x, y) = 0$, with $(x, y) \in (0, 1) \times (0, 1)$, $\psi(x, y, t) = 0$ on $\partial \Omega$ and $f(x, y) = \begin{cases} 0.5 + t^{\alpha_1} & \text{if } x \in [0, 1/4], y \in [0, 1/4] \\ +t^{1+\alpha_2} & \text{and } x \in [3/4, 1], y \in [3/4, 1] \\ 1 & \text{if } x \in [0, 1/4], y \in [3/4, 1] \\ 0 & \text{and } x \in [3/4, 1], y \in [0, 1/4], \text{ otherwise.} \end{cases}$

The exact solution $u(x, y, t)$ is unknown. We test mainly our scheme on non-conforming random mesh can preserve the positivity. In Fig. 16, we show the numerical solutions by using our proposed scheme on non-conforming random meshes with 10240 cells in Fig. 10 for different $\alpha_1 = 0.3, 0.5, 0.7$ and $\alpha_2 = 0.2, 0.4, 0.6$. In Fig. 17, we provide the corresponding 2D contour figures. From Figs. 16 and 17, we find that our scheme preserves the positivity of the continuous solution in this case.
Fig. 8 The projection of the solution in the $x$ direction on non-conforming random meshes for Example 1.
Fig. 9 The projection of the solution at the grid point in the x direction on non-conforming random meshes for Example 1.
5.2 A problem with scalar diffusion coefficient

In the following Example 4, we compare preserving positivity obtained by our proposed nonlinear finite volume method with the methods proposed by Jin et al. [17], in which time was discretized by convolution quadrature generated by the backward Euler method and space was approximated by the three methods on different meshes, including the standard FVE method, SG method, and LM method.

**Example 4** Let us consider the model (1.2)–(1.4) on \( \Omega = [0,1] \times [0,1] \). To compare we take \( \kappa = E, \ T = 10^{-5} \), and the initial conditions \( v(x,y) = xy(1-x)(1-y), \ f = 1, \psi(x,y,t) = 0 \) on \( \partial\Omega \), the exact solution \( u(x,y,t) \) is unknown.

By taking the perturbation parameter \( \beta = 0.5 \), we show the numerical solutions by using our proposed scheme on non-conforming random meshes (see Fig. 18) with \( \alpha_1 = 0.8 \) in Figs. 19 and 20. The numerical solution attained by our FV scheme on non-conforming rectangular meshes with 10240 cells is shown in Fig. 21 for different \( \alpha_1 = 0.15, 0.45, 0.75, 0.95 \). Figure 22 provides the corresponding 2D contour figure. From Figs. 19, 20, 21, and 22, we find that our scheme preserves the positivity of the continuous solution on non-conforming random meshes. However, the numerical solutions obtained by the Jin et al. [17] can produce negative values when the time step is less than some positivity thresholds.

In Example 5, we present numerical results for multi-term subdiffusion problem (1.2) with three fractional-derivative terms.

**Example 5** We consider the multi-term subdiffusion equation (1.2)–(1.4) on \( \Omega = [0,1] \times [0,1] \) with three fractional-derivative terms

\[
D_t^{\alpha_1} u + 0.5 D_t^{0.1} u + 0.05 D_t^{0.2} u - \nabla \cdot (\kappa(x,y,t) \nabla u) = f(x,y,t).
\]

Taking \( \kappa = E, \ T = 1 \), and the initial conditions \( v(x,y) = 0, \) with \( (x,y) \in (0,1) \times (0,1) \). The function \( f(x,y,t) \) are chosen such that the solution of the problem above is \( u(x,y,t) = (t^{\alpha_1} + t^{1+\alpha_1} + t^{2+\alpha_1}) \sin(\pi x) \sin(\pi y) \).

We test the accuracy of our scheme on non-conforming rectangular mesh and random quadrilateral meshes for four levels of grid refinement with different \( \alpha_1 = 0.3, 0.5, 0.8 \). Columns 1 through 6 of Table 3 give the errors about solution by running case for four levels of grid refinement on non-conforming rectangular mesh. This table shows that our schemes almost obtain second-order accuracy. Columns 1 through 6 of Table 4 show the errors of solution on non-conforming random quadrilateral mesh. Similarly, this table shows that our schemes almost obtain second-order accuracy. From these tables, we can know that the accuracy of our scheme is consistent with the abstract and introduction for multi-term subdiffusion problem. The last three columns of Tables 3 and 4 show the corresponding CPU runtime. The third to last column represents the iteration time. The penultimate column shows the time of solving equation. The last column shows the total time. We can see that the total time is increased rapidly when the number of grids is increased. In future work, we will consider introducing fast algorithms to reduce CPU time.
Fig. 11 The projection of solution in the $x$ direction on non-conforming rectangular meshes for Example 2
**Fig. 12** The projection of solution at the grid point in the $x$ direction on non-conforming rectangular meshes for Example 2.
Fig. 13 Non-conforming random meshes for Example 2

| $\alpha_1$ | Cells | Accuracy of solution | CPU time (s) | Total time |
|------------|-------|-----------------------|--------------|------------|
|            |       | $\epsilon_{\infty}^u$ | Iteration | Solving Eq. |             |
|            |       | Order | $\epsilon_2^u$ | Order |             |             |
| 0.3        | 64    | 5.1228e-3 | – | 2.6628e-3 | – | 0 | 0 | 1.5625e-2 |
|            | 256   | 1.5011e-3 | 1.7709 | 7.5783e-4 | 1.8130 | 0 | 1.5625e-2 | 0.125 |
|            | 1024  | 3.9230e-4 | 1.9360 | 1.9963e-4 | 1.9464 | 0.2031 | 0.4375 | 28.2812 |
|            | 4096  | 9.9407e-5 | 1.9805 | 4.9733e-5 | 1.9832 | 4.5938 | 7.1250 | 1465.7969 |
| 0.5        | 64    | 2.5268e-3 | – | 1.3134e-3 | – | 0 | 1.5625e-2 | 3.12e-2 |
|            | 256   | 8.4588e-4 | 1.5788 | 4.2704e-4 | 1.6208 | 0 | 3.125e-2 | 0.45 |
|            | 1024  | 2.2708e-4 | 1.8972 | 1.1381e-4 | 1.9077 | 0.1875 | 0.4375 | 27.89 |
|            | 4096  | 7.5865e-5 | 1.9724 | 2.8950e-5 | 1.9751 | 2.9844 | 6.6406 | 1414.95 |
| 0.8        | 64    | 5.5094e-4 | – | 2.8637e-4 | – | 0 | 0 | 1.5625e-2 |
|            | 256   | 2.8016e-4 | 0.9756 | 1.4144e-4 | 1.0177 | 0 | 1.5625e-2 | 0.4844 |
|            | 1024  | 7.8901e-5 | 1.8281 | 3.9546e-5 | 1.8386 | 0.1094 | 0.4375 | 27.125 |
|            | 4096  | 1.9921e-5 | 1.9858 | 9.9663e-6 | 1.9884 | 1.8906 | 6.6406 | 1451.6562 |

Table 3 Numerical results on non-conforming rectangular meshes for Example 5

| $\alpha_1$ | Cells | Accuracy of solution | CPU time (s) | Total time |
|------------|-------|-----------------------|--------------|------------|
|            |       | $\epsilon_{\infty}^u$ | Iteration | Solving Eq. |             |
|            |       | Order | $\epsilon_2^u$ | Order |             |             |
| 0.3        | 64    | 1.3234e-2 | – | 5.1636e-3 | – | 0 | 1.5625e-2 | 6.25e-2 |
|            | 256   | 4.2126e-3 | 1.6514 | 1.1617e-3 | 2.1521 | 0.3281 | 4.6875e-2 | 1.1563 |
|            | 1024  | 1.1061e-3 | 1.9293 | 3.0409e-4 | 1.9337 | 6.5 | 0.4688 | 34 |
|            | 4096  | 2.9293e-4 | 1.9168 | 7.2378e-5 | 2.0709 | 103.7031 | 6.25 | 1443 |
| 0.5        | 64    | 7.3236e-3 | – | 2.9084e-3 | – | 0 | 1.5625e-2 | 6.25e-2 |
|            | 256   | 2.5349e-3 | 1.5306 | 6.9471e-4 | 2.0657 | 0.2031 | 4.6875e-2 | 1.1406 |
|            | 1024  | 6.7808e-4 | 1.9024 | 1.8538e-4 | 1.9059 | 4.5 | 0.4531 | 32 |
|            | 4096  | 1.8107e-4 | 1.9049 | 4.4540e-5 | 2.0573 | 61.5156 | 7.3594 | 1488.1562 |
| 0.8        | 64    | 2.8984e-3 | – | 1.1688e-3 | – | 1.5625e-2 | 0 | 4.6875e-2 |
|            | 256   | 1.1104e-3 | 1.3842 | 3.0340e-4 | 1.9458 | 9.375e-2 | 4.6875e-2 | 1.0141 |
|            | 1024  | 3.0760e-4 | 1.8520 | 8.3497e-5 | 1.8614 | 2 | 0.4375 | 29.9844 |
|            | 4096  | 8.3681e-5 | 1.8781 | 2.0529e-5 | 2.0240 | 22.6719 | 6.2656 | 1380 |
**Fig. 14** The projection of the solution in the $y$ direction on non-conforming random meshes for Example 2
Fig. 15 The projection of the solution at the grid point in the y direction on non-conforming random meshes for Example 2
Fig. 16 Numerical solution on non-conforming rectangular meshes for Example 3: a \( \alpha_1 = 0.3, \alpha_2 = 0.2 \); b \( \alpha_1 = 0.5, \alpha_2 = 0.4 \); c \( \alpha_1 = 0.7, \alpha_2 = 0.6 \)

Fig. 17 The 2D contour figure of numerical solution on non-conforming rectangular meshes for Example 3: a \( \alpha_1 = 0.3, \alpha_2 = 0.2 \); b \( \alpha_1 = 0.5, \alpha_2 = 0.4 \); c \( \alpha_1 = 0.7, \alpha_2 = 0.6 \)

Fig. 18 Non-conforming random meshes for Example 4

6 Conclusion

In this paper, we construct a nonlinear positivity-preserving finite volume scheme for subdiffusion equations on distorted non-conforming quadrilateral meshes, where a new method of eliminating vertex unknowns is constructed. Our method is adapted to subdiffusion equations with anisotropic coefficients on general non-conforming meshes with hanging nodes. Besides, the proofs of the positivity and the existence of a solution for this nonlinear finite volume scheme are also presented.

In the future, we will further explore the nonlinear finite volume scheme preserving positivity and preserv-
Fig. 19 The projection of the solution in the y direction on non-conforming random meshes for Example 4
**Fig. 20** The projection of the solution at the grid point in the $y$ direction on non-conforming random meshes for Example 4.
Fig. 21 Numerical solution on non-conforming rectangular meshes for Example 4: a $\alpha_1 = 0.15$, b $\alpha_1 = 0.45$, c $\alpha_1 = 0.75$, d $\alpha_1 = 0.95$

ing maximum principle for space fractional-derivative model or time and space fractional-derivative model on distort meshes, such as nonlocal Allen–Cahn equation [28]. Especially, we will devote to investigate the fractional Laplacian equation [29]

\[ (-\Delta)_{\alpha}^u(x) = f(x, u, \nabla u), \text{ in } \Omega, \text{ for } \alpha \in (0, 2), \]

where $(-\Delta)_{\alpha}^u$ is the nonlocal fractional Laplacian operator [30].
Fig. 22 The 2D contour figure of numerical solution on non-conforming rectangular meshes for Example 4: a $\alpha_1 = 0.15$, b $\alpha_1 = 0.45$, c $\alpha_1 = 0.75$, d $\alpha_1 = 0.95$

Acknowledgements We would like to express our sincere thanks to the referees and the editors for their valuable comments and suggestions, which helped us to improve the manuscript a lot.

Data availability The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Magin, R.L.: Fractional Calculus in Bioengineering. Begell House Publishers (2006)
2. Liu, F., Zhuang, P., Liu, Q.: Numerical Methods of Fractional Partial Differential Equations and Applications. Science Press, Beijing (2015)
3. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
4. Scalas, E., Gorenflo, R., Mainardi, F.: Fractional calculus and continuous-time finance. Phys. A 284, 376–384 (2000)
5. Uchaikin, V.V.: Fractional Derivatives for Physicists and Engineers. Higher Education Press, Beijing (2012)
6. Gao, G., Alikhanov, A., Sun, Z.: The temporal second order difference schemes based on the interpolation approximation for solving the time multi-term and distributed-order fractional sub-diffusion equations. J. Sci. Comput. 73, 93–121 (2017)

7. Zheng, R., Liu, F., Jiang, X.: A Legendre spectral method on graded meshes for the two-dimensional multi-term time-fractional diffusion equation with non-smooth solutions. Appl. Math. Lett. 104, 106247 (2020)

8. Bu, W., Shu, S., Yue, X., Xiao, A., Zeng, W.: Space-time finite element method for the multi-term time-space fractional diffusion equation on a two-dimensional domain. Comput. Math. Appl. 78, 1367–1379 (2019)

9. Zhou, J., Xu, D.: Alternating direction implicit difference scheme for the multi-term time-fractional integro-differential equation with a weakly singular kernel. Comput. Math. Appl. 79, 244–255 (2020)

10. Jin, B., Lazarov, R., Liu, Y., Zhou, Z.: The Galerkin finite element method for a multi-term time-fractional diffusion equation. J. Comput. Phys. 281, 825–843 (2015)

11. Ren, J., Sun, Z.: Efficient and stable numerical methods for multi-term time-fractional sub-diffusion equations. East Asian J. Appl. Math. 4, 242–266 (2014)

12. Luchko, Y.: Boundary value problems for the generalized time-fractional diffusion equation of distributed order. Fract. Calc. Appl. Anal. 12, 409–422 (2009)

13. Luchko, Y.: Maximum principle for the generalized time-fractional diffusion equation. J. Math. Anal. Appl. 351, 218–223 (2009)

14. Luchko, Y.: Initial-boundary problems for the generalized multi-term time-fractional diffusion equation. J. Math. Anal. Appl. 374, 538–548 (2011)

15. Ye, H., Liu, F., Anh, V., Turner, I.: Maximum principle and numerical method for the multi-term time-space Riesz-Caputo fractional differential equations. Appl. Math. Comput. 227, 531–540 (2014)

16. Brunner, H., Han, H., Yin, D.: The maximum principle for time-fractional diffusion equations and its application. Numer. Funct. Anal. Optim. 36, 1307–1321 (2015)

17. Jin, B., Lazarov, R., Thomée, V., Zhou, Z.: On nonnegativity preservation in finite element methods for subdiffusion equations. Math. Comp. 86, 2239–2260 (2017)

18. Liao, H.-L., Tang, T., Zhou, T.: Second-order and nonuniform time-stepping maximum-principle preserving scheme for time-fractional Allen-Cahn equations. J. Comput. Phys. 414, 109473 (2020)

19. Liao, H.-L., Tang, T., Zhou, T.: On energy stable, maximum-principle preserving, second order BDF scheme with variable steps for the Allen-Cahn equation. arXiv:2003.00421, (2020)

20. Ji, B., Liao, H.-L., Zhang, L.: Simple maximum principle preserving time-stepping methods for time-fractional Allen-Cahn equation. Adv. Comput. Math. 46, 37 (2020)

21. Ji, B., Liao, H.-L., Gong, Y., Zhang, L.: Adaptive linear second-order energy stable schemes for time-fractional Allen-Cahn equation with volume constraint. Commun. Nonlinear Sci. Numer. Simul. 90, 10536 (2020)

22. Ji, B., Liao, H.-L., Gong, Y., Zhang, L.: Adaptive second-order Crank-Nicolson time-stepping schemes for time-fractional molecular beam epitaxial growth models. SIAM J. Sci. Comput. 42, B738–B760 (2020)

23. Brunner, H.: Collocation methods for Volterra Integral and Related Functional Differential Equations, Cambridge Monogr. Appl. Comput. Maths. 15, Cambridge University Press, Cambridge, (2004)

24. Huang, C., Stynes, M.: Superconvergence of a finite element method for the multi-term time-fractional diffusion problem. J. Sci. Comput. 82, 1 (2020)

25. Yuan, G., Sheng, Q.: Analysis of accuracy of a finite volume scheme for diffusion equations on distorted meshes. J. Comput. Phys. 224, 1170–1189 (2007)

26. Sheng, Z., Yuan, G.: A nine point scheme for the approximation of diffusion operators on distorted quadrilateral meshes. SIAM J. Sci. Comput. 30, 1341–1361 (2008)

27. Xavier, B., Emmanuel, L.: A positive scheme for diffusion problems on deformed meshes. ZAMM-J. Appl. Math. Mech. 96, 660–680 (2016)

28. Du, Q., Ju, L., Li, X., Qiao, Z.: Maximum principle preserving exponential time differencing schemes for the nonlocal Allen-Cahn equation. SIAM J. Numer. Anal. 57(2), 875–898 (2019)

29. Yan, Y., Deng, W., Nie, D.: A finite-difference approximation for the one- and two-dimensional tempered fractional Laplacian. Commun. Appl. Math. Comput. 2, 129–145 (2020)

30. Liu, H., Sheng, C., Wang, L.L., Yuan, H.: On diagonal dominance of FEM stiffness matrix of fractional Laplacian and maximum principle preserving schemes for the fractional Allen-Cahn equation. J. Sci. Comput. 86, 19 (2021)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.