NON-LOCAL HEAT EQUATIONS WITH MOVING BOUNDARY

GIACOME ASCIONE, PIERRE PATIE, AND BRUNO TOALDO

Abstract. In this paper we consider non-local (in time) heat equations on time-increasing parabolic sets whose boundary is determined by a suitable curve. We provide a notion of solution for these equations and we study well-posedness under Dirichlet conditions outside the domain. A maximum principle is proved and used to derive uniqueness and continuity with respect to the initial datum of the solutions of the Dirichlet problem. Existence is proved by showing a stochastic representation based on the delayed Brownian motion killed on the boundary. Several related distributional properties of the delayed Brownian motion and its crossing probabilities are also obtained. The asymptotic behaviour of the mean square displacement of the process is determined, showing that the diffusive behaviour is anomalous.

Contents

1. Introduction 1
2. Preliminaries on subordinators and non-local operators 5
   2.1. A weak maximum principle 8
3. The Brownian motion delayed by an inverse subordinator 9
   3.1. Definition and regularity of the density 9
   3.2. The semi-Markov property 11
4. The killed delayed Brownian motion and a Dynkin-Hunt formula 12
   4.1. The constant boundary case 13
   4.2. The non-decreasing and continuous boundary case 15
5. Main result 23
6. Anomalous diffusive behaviour 31
Appendix A. Proofs of some technical results 35
   A.1. Proof of Proposition 2.3 35
   A.2. Proof of Theorem 2.8 38
   A.3. Proof of Proposition 3.1 40
   A.4. Proof of Proposition 3.4 45
   A.5. Uniform convergence of monotone functions 46
   A.6. Proof of Proposition 6.1 47
Acknowledgements 47
References 47

1. Introduction

Fractional kinetic equations (FKEs) are typical of several physical systems. Indeed they naturally arise, for instance, in the context of Hamiltonian chaos as a balance in a mesoscopic scale between macroscopic deterministic dynamics leading to chaos and microscopic purely random behaviour. In [75] the author shows that, under some assumptions on the chaotic dynamics (called moderate non-linearity), the classical Fokker-Plank-Kolmogorov equations constitute a good candidate to be the kinetic equations for the system. However, if this assumption is violated, the system could exhibit an anomalous diffusive behaviour, which can be described, for instance, by the theory of Lévy walks and flights (see [1]). In [76] the author finally established the link

Date: February 6, 2025.
2020 Mathematics Subject Classification. 35R37, 60K50.
Key words and phrases. Semi-Markov process; anomalous diffusion; non-local heat equation; subordinator; inverse subordinator.
between FKEs and models of non-moderate Hamiltonian chaos (see also [77; 78] for further developments). It is interesting to observe that in this setting FKEs appear from a scaling limit procedure involving the Lévy walk model, see for instance [53]. Among the FKEs, we are interested in particular into the following one

\[
\frac{\partial^\alpha}{\partial t^\alpha} q(t, x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} q(t, x),
\]

(1.1)

that will be referred as (time-)fractional heat equation. Here, \( \alpha \in (0, 1) \) and \( \frac{\partial^\alpha}{\partial t^\alpha} \) is the well-known Caputo fractional derivative, defined as

\[
\frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} (f(\tau) - f(0)) d\tau
\]

for \( \alpha \in (0, 1) \) and a suitable function \( f \). To use a shorter notation, we will denote \( \frac{\partial^\alpha}{\partial t^\alpha} \) and \( \frac{\partial^\alpha}{\partial x^\alpha} \) as \( \partial_t^\alpha \) and we will use an analogous notation for classical derivatives. In [63] the authors provided the fundamental solution of a general FKE by means of a mixing formulation, i.e.

\[
q(t, x) = \frac{1}{t^{\alpha/2}} \int_0^t f_{\alpha} \left( \frac{s}{t^{\alpha/2}} \right) p(s, x) ds,
\]

(1.2)

where \( p(s, x) \) is the usual Gaussian heat kernel and \( f_{\alpha} \) is the density of a suitable random variable \( X \) with characteristic function \( \varphi_X(z) = E_{\alpha}(iz) = \sum_{k=0}^{\infty} \frac{(iz)^k}{\Gamma(\alpha k+1)} \Gamma(1-\alpha) \). It is clear that (1.2) can be rewritten in terms of a superposition formulation as

\[
q(t, x) = E[p(t^\alpha X, x)]
\]

By a simple conditional probability argument, one can show that \( q(t, x) \) is indeed the density of the random variable \( B(t^\alpha X) = B(L_{\alpha}(t)) \), where \( B \) is a Brownian motion and \( X \) is independent of \( B \), while \( L_{\alpha}(t) \) is an \( \alpha \)-self-similar process independent of \( B \) such that \( L_{\alpha}(1) = X \). It turns out that the process \( L_{\alpha}(t) \) can be described as the first time in which an \( \alpha \)-stable subordinator (i.e., a positively skew \( \alpha \)-stable, hence increasing, process) crosses the threshold \( t \geq 0 \), also called inverse \( \alpha \)-stable subordinator, see [50]. This relation has been deeper explored in [9], where the authors extended the superposition formula to general Feller semigroups. However, it has been shown that the process \( X_{\alpha}(t) := B(L_{\alpha}(t)) \) is the weak limit of a Continuous-Time Random Walk (CTRW) in the sense of one-dimensional distributions (see [53] for details), and also in the \( J_1 \) Skorokhod topology (see [45] and also [11] for generalizations). It is worth noticing that, due to the time-nonlocal nature of the equation, the FKE (1.1) cannot describe properly the transition law of a Markov process and furthermore does not characterize the whole process \( X_{\alpha} \), since it is not Markovian. As a consequence, the identification of \( X_{\alpha} \) as the limit of the CTRW model which leads to the FKE gives a deeper knowledge on the mechanisms behind non-moderate Hamiltonian chaos. For such a reason, the process \( X_{\alpha} \) is usually called a Fractional Kinetic Process (FKP).

Once we have this identification, it is clear that one can study kinetic equations of chaotic systems through several techniques involving scaling limits. This is the case, for instance, of the Hamiltonian systems describing cellular flows. In [30] the authors provided a stochastic representation result for the limit of an averaging-homogenization problem, which is expressed in terms of the process \( X_{1/2} \). The link between such a problem and the FKE (1.1) has been better underlined in [29].

While being one of the most representative, Hamiltonian chaos is not the only physical occurrence of FKEs as weak scaling limits of discrete models. In [54], trapping models have been used to study Debye-type ageing/relaxation properties of some glass formers. However, in [60] the authors observed that in a randomized energy environment, the same trap models exhibit a form of subaging. This has been better captured by [14], where the authors proved that these trap models with random environment converge, under some assumptions, to a FKP \( X_{\alpha} \) (see also [13] for further details). Similarly, in [20] the FKP emerges as limit of an exclusion process.

The FKE (1.1) also appears in the description of heat transfer and mass infiltration in heterogeneous media. It is the case, for instance, of [69], in which (1.1) appears in a Green-Ampt infiltration model for moisture in the soil. This is also the case of heat transfer in materials with memory. Indeed, several nonlocal generalizations of the Fourier conduction law have been considered. In particular, a widely used generalization of such a conduction law is given by

\[
q(t, x) \propto -\int_0^t (t-\tau)^{\alpha-1} \nabla_x T(\tau, x) d\tau
\]

(1.3)
where \( q \) is the heat flux and \( T \) is the temperature. In this case, (1.1) plays the role of the heat equation. This leads to a fractional theory of heat conduction, as described, for instance, in [25; 59]. It makes then sense to consider Cauchy-Dirichlet problems associated with (1.1), as they model the evolution of temperature in a non-Fickian material satisfying (1.3) subject to an external heat source: such kind of problems have been addressed, for instance, in [49]. If, furthermore, the source is sufficiently hot or cold to cause a change of state, then also latent heat has to be taken in consideration, see for instance [70]. These ideas lead to fractional Stefan problems, as discussed in [26; 71; 72] and references therein, see also [68]. Let us consider

\[
\begin{align*}
\frac{\partial_t}{\partial_t} u(t, x) &= \frac{1}{2} \frac{\partial^2}{\partial x^2} u(t, x), \quad x < \varphi(t), \ t > 0, \\
u(t, x) &= 0, \quad x \geq \varphi(t), \ t \geq 0, \\
u(0, x) &= f(x), \quad x < b, \\
\frac{\partial_t}{\partial_t} \varphi(t) &= -\frac{\partial_x}{\partial x} u(t, \varphi(t)), \ t > 0, \\
\varphi(0) &= b,
\end{align*}
\]

where \( f : (-\infty, b) \to \mathbb{R} \) and \( b \in \mathbb{R} \) are given initial data, with \( f(x) > 0 \) for all \( x < b \), and \( u \) and \( \varphi \) are the unknowns. To solve such a problem, one could use a fixed point argument, by first solving the Cauchy-Dirichlet problem given by the first three equations for a fixed interface \( \varphi \) and then finding a new interface \( \varphi' \) by solving the related Cauchy problem given by the fourth and fifth equalities: it is clear that a fixed point of such a procedure, together with the related solution of the Cauchy-Dirichlet fractional heat equation with moving boundary, is the desired solution of (1.4). To do this, we first need to find the solution of a generic Cauchy-Dirichlet fractional heat equation with moving boundary:

\[
\begin{align*}
\frac{\partial_t}{\partial_t} u(t, x) &= \frac{1}{2} \frac{\partial^2}{\partial x^2} u(t, x), \quad x < \varphi(t), \ t > 0, \\
u(t, x) &= 0, \quad x \geq \varphi(t), \ t \geq 0, \\
u(0, x) &= f(x), \quad x < \varphi(0), \\
\lim_{x \to -\infty} u(t, x) &= 0, \quad \text{locally uniformly with respect to } t \geq 0,
\end{align*}
\]

where the latter condition is required to guarantee uniqueness of the solution, as will be explained later, and \( \varphi : [0, +\infty) \to \mathbb{R} \) is a fixed interface.

To better understand this problem, we need to go back to the FKP \( X_\alpha \), or even further back, directly to the Brownian motion. Indeed, for \( \alpha = 1 \), (1.5) is the usual Cauchy-Dirichlet problem with moving boundary for the heat equation. To shorten the notation, we denote \( X_1 \equiv B \). We can kill the process \( X_\alpha \) upon crossing the moving threshold \( \varphi \) as follows: we define \( T_\alpha \) as the first time in which \( X_\alpha \) touches \( \varphi \) and then we set \( X^\dagger_\alpha(t) = X_\alpha(t) \) for \( t < T_\alpha \) and \( X^\dagger_\alpha(t) = \infty \) for \( t \geq T_\alpha \), where \( \infty \not\in \mathbb{R} \) is the cemetery state. Since \( T_\alpha \) is a stopping time with respect to the natural filtration of the process \( X_\alpha \) and for \( \alpha = 1 \) we have that \( X_1 \) is a Feller process, \( X_1 \) satisfies the Markov property in \( T_1 \) and thus the Dynkin-Hunt formula holds (see [21, Theorem 2.4] for the fixed threshold case and [42, Formula (1.1)] for the general case). Once this has been established, in [42, Chapter 1] it has been proved that, for \( \alpha = 1 \), the sub-probability density of \( X^\dagger_1 \) is the fundamental solution of (1.5). This eventually relates the heat equation (1.5) with the behaviour of the trajectories of \( X_1 \) (in particular, to its first crossing time with \( \varphi \)): this is indeed expected since \( X_1 \) is a Feller process.

On the other hand, \( X_\alpha \) for \( \alpha < 1 \) is not even Markov, and, at the same time, the time-fractional heat equation does not characterize the transition distribution of the process. Hence, at a first glance, the connection between (1.5) and \( X_\alpha \) could be unclear. It is however worth noticing that despite \( X_\alpha \) is not Markov, it still exhibits the Markov property on a suitable selection of stopping times. Indeed, \( X_\alpha \) inherits the semi-Markov property from its CTRW prelimin (see [23; 37] and references therein for a discussion on the topic). In general, \( T_\alpha \) is not a Markov time for \( X_\alpha \), unless we ask for some conditions on \( \varphi \): in such a case we are able to prove a suitable Dynkin-Hunt formula and then to relate \( T_\alpha \) with (1.5), as stated in the following theorem, that we will prove throughout the paper.

**Theorem 1.1.** Assume \( \varphi \) is continuous and either constant or strictly increasing with \( \varphi'(0+) > 0 \) and \( f \in C_c(-\infty, \varphi(0)) \). Then there exists a function \( q_\alpha : (0, +\infty) \times \mathbb{R} \times \mathbb{R} \) such that for all Borel subsets \( A \in \mathcal{B}(\mathbb{R}) \) it holds

\[
\mathbb{P}_y(X^\dagger_\alpha(t) \in A) = \int_A q_\alpha(s, x; y) \, dx.
\]
In particular, the unique solution of (1.5) is given by

\[ u(t, x) = \int_{-\infty}^{\phi(0)} f(y)q_\alpha(s, x; y) \, dy. \]

Let us observe here some crucial features of this result. First, notice that in any case (1.1) does not characterize the distribution of the trajectories of \( X_\alpha \); nevertheless (1.5) still provides a quite interesting piece of information concerning the behaviour of the trajectories of \( X_\alpha \), which is an unexpected result. This feature is shared with Markov processes even if the process \( X_\alpha \), and its generalizations considered later, are not Markovian: the semi-Markov property of \( X_\alpha \) is crucial to preserve the role of the analytical approach. The explicit use of semi-Markov property to obtain analytical results on non-local PDEs is, indeed, one of the main technical novelty concerning the proofs of our results. Furthermore, the assumption that \( \phi \) is continuous and strictly increasing is not really restrictive: heuristic arguments based on the second law of thermodynamics, which are then confirmed by a combination of a weak maximum principle and Hopf’s lemma (the latter has been shown in [61], see also [62]), tell us that \( \partial_t \phi > 0 \) in (1.4).

We also underline that the FKEs and the FKPs constitute a particular case of a much wider class of processes and equations, arising from limits of CTRWs. Indeed, in [46] more general scaling limits of CTRWs have been considered, leading to processes of the form \( X(t) := B(L(t)) \), where \( L(t) \) is the inverse of a (non necessarily stable) subordinator. Usually, the process \( X \) is called delayed Brownian motion, as in [44]. The relation between the process \( X \) and a suitable time-nonlocal heat equation has been explored in [19] (see also [4, Appendix] for further regularity results and [10; 12; 38; 39; 41; 32; 48; 58; 64] for different approaches or generalizations), while some first properties of its first exit times have been explored in [7; 57]. It is important to recall that such processes are also addressed in the context of semi-Markov processes, as observed in [51].

In this paper, we will prove a general result (by substituting the power kernel \( t^{-\alpha} \) with a more general one) that includes Theorem 1.1 as a particular case. This is done as follows. As already described for the FKP case, we use the semi-Markov property of \( X \) to show that the first crossing time with \( \phi \) is a Markov time for \( X \). Then, we use this to provide a Dynkin-Hunt-type formula for the killed process \( X^\dagger \). Finally, we use the latter formula together with the fact that the density of \( X \) solves a time-nonlocal heat equation, on the entire real line, to obtain the desired result. This latter step is actually quite technical and needs some precise estimates on the derivatives of the density of \( X \). Concerning uniqueness of the solution, this is obtained by means of a weak maximum principle: our results in this context constitute a crucial generalizations of the ones (for nonlocal operators) in [43] to more general parabolic domains. The reader can consult [18] for a maximum principle when there is also a non-local spatial component.

We plan, in future, to use our results to prove existence and uniqueness of the Stefan problem with a general memory kernel. Notice that while (1.3) is the most used non-local generalization of Fourier law, it is not the unique one. Indeed, general kernels were originally considered in [28], see also [55]. Hence, it makes sense to consider Stefan problems with sharp interface with a more general convolution kernel. Moreover, a further investigation on the relation between Hamiltonian chaos theory and this special processes \( X \) emerging from triangular array limits is needed, to capture some more particular anomalous diffusive behaviour (and not necessarily the power-type one), such as the ultraslow diffusive behaviour [47; 67] or some more particular relaxation phenomena [52].

The structure of the paper is as follows. In the forthcoming section 2 we introduce main facts on subordinators, their inverse processes and the related non-local equations. Furthermore, in Section 2.1 a maximum principle for a non-local parabolic problem on a time-dependent domain will be stated: this will be used later to prove uniqueness of our heat equation. In Section 3 we introduce the delayed Brownian motion, i.e., a Brownian motion time-changed with an independent inverse subordinator. Firstly we review some main facts on it and then, in particular, we obtain several regularity properties for the density and we formalize the semi-Markov property that will be crucial for our main results. In Section 4 we discuss the effect of killing a delayed Brownian motion, obtaining a (Dynkin-Hunt) representation for its density that will be crucial for regularity, needed in our main results. We divide the exposition between killing on a constant or time-dependent boundary. Section 5 is devoted to state and prove the main results. Finally, in Section 6 we obtain the anomalous diffusive behavior of the process, i.e., we study the asymptotycs of the mean square displacement for our killed process. In order to improve the readability, all throughout the paper several technical proofs are postponed to the Appendix A.
Throughout the paper we consider the family of filtered probability spaces \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}_x)_{x \in \mathbb{R}}\), and the canonical process \((B, \sigma)\) so that \(B\) is a Brownian motion and \(\sigma\) is an independent conservative subordinator such that \(\mathbb{P}_x(B(0) = x, \sigma(0) = 0) = 1\), i.e., \(\mathbb{P}_x = \mathbb{P}_{x,0}\) for simplicity. We recall, in particular, that a subordinator is a non-decreasing (hence positive) Lévy process. We denote, for \(\lambda \geq 0\),

\[
\Phi(\lambda) = -\log \left( \mathbb{E} \left[ e^{-\lambda \sigma(1)} \right] \right) = b\lambda + \int_{\mathbb{R}^+} (1 - e^{-\lambda \tau}) \nu(d\tau),
\]

for a constant \(b \geq 0\) and a measure \(\nu\) on \(\mathbb{R}^+ := (0, +\infty)\) such that \(\int_{\mathbb{R}^+} (\tau \wedge 1) \nu(d\tau) < \infty\). A function of the form (2.1) is usually called a Bernstein function, see [34; 66] for details. In the following, we assume the standing assumption:

\[
\Phi(0) = 0 \quad \text{and} \quad \Phi'(0) = b = 0 \quad \text{and} \quad \Phi(0, \infty) = +\infty.
\]

(A1)

i.e., the generalized inverse of \(\sigma\), has a.s. continuous paths. Furthermore, for all \(t > 0\), we know, by [46, Theorem 3.1], that \(L(t)\) is an absolutely continuous random variable whose density \(f_L(\cdot, t)\) is given by the formula

\[
f_L(s, t) = \int_0^t \tau \nu(d\tau, s),
\]

whered \(g_\sigma(A, t) = \mathbb{P}(\sigma(t) \in A)\) for all \(A \in \mathcal{B}(\mathbb{R})\) and \(t \geq 0\). We point out that \(\nu \in L^{1}_{\text{loc}}(\mathbb{R}_0^+)\), where \(\mathbb{R}_0^+ := [0, +\infty)\) and \(\lim_{\tau \to 0} \tau \nu(\tau) = 0\), see [66, Page 17]. We can then define the mapping \(I_\Phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+\) as

\[
I_\Phi(t) := \int_0^t \nu(d\tau)
\]

which we play an important role in the sequel.

We shall need several estimates of the density of \(\sigma(t), L(t)\) and their derivatives which we rely on the so-called Orey’s condition, see [56]:

There exist \(\gamma \in (1, 2), C_\gamma > 0\) and \(t_\gamma > 0\) such that \(\int_0^{t_\gamma} \tau^2 \nu(d\tau) > 4C_\gamma t_\gamma^2\), for any \(t < t_\gamma\). (A2)

**Remark 2.1.** In the original paper [56], the condition requires \(\gamma \in (0, 2)\). However, since \(\nu\) is the Lévy measure of a subordinator, it can be easily seen that it must be \(\gamma \in (1, 2)\).

We proceed with the following useful bounds.

**Lemma 2.2.** Let \(\Psi(\xi) = \Phi(-i\xi)\) for \(\xi \in \mathbb{R}\). Under (A1), there exist \(\delta_0 > 0\) and \(C_0 > 0\) such that

\[
\Re(\Psi(\xi)) \geq C_0|\xi|^2, \quad 0 < |\xi| < 1.
\]

Furthermore, if also (A2) holds, we have

\[
\Re(\Psi(\xi)) \geq C_\gamma |\xi|^{2-\gamma}, \quad |\xi| > M_\gamma.
\]

**Proof.** Let us recall, by [66, Proposition 3.5], that \(\Phi\) admits a unique continuous extension \(\Phi : \{z \in \mathbb{C} : \Re(z) \geq 0\} \to \{z \in \mathbb{C} : \Re(z) \geq 0\}\). Thus, in particular, we have that \(\Re(\Psi(\xi)) \geq 0\) for any \(\xi \in \mathbb{R}\). Moreover, if \(\xi \neq 0\), let us observe that

\[
\Re(\Psi(\xi)) = \int_0^{+\infty} (1 - \cos(\xi \tau)) \nu(d\tau) > 0.
\]

Recalling that for \(|x| < 1\) it holds \(1 - \cos x \geq |x|^2 / 4\), we get

\[
\Re(\Psi(\xi)) \geq \int_0^{+\infty} \frac{|\xi|^2 \nu(d\tau)}{4} \geq \frac{|\xi|^2}{4} \int_0^{+\infty} \tau^2 \nu(d\tau) =: C_0|\xi|^2, \quad 0 < \xi < \delta_0.
\]
If \( (A2) \) holds, then
\[
\Re(\Psi(\xi)) \geq \frac{|\xi|^2}{4} \int_0^{|\xi|^2} \tau^2 \nu(d\tau) \geq C_\gamma |\xi|^{2-\gamma}, \quad |\xi| > M_\gamma, \tag{2.8}
\]

As a consequence, as shown in [56], \( (A2) \) implies
\[
|E[e^{i\xi \sigma(t)}]| \leq e^{-C_\gamma t|\xi|^{2-\gamma}}, \quad |\xi| > M_\gamma,
\]
and then for \( t > 0 \) the r.v. \( \sigma(t) \) admits an infinitely differentiable density on \((0, \infty)\), that we denote \( g_\sigma(\cdot, t) \). With this in mind, we can provide the following estimates.

**Proposition 2.3.** Assume \( (A1) \) and \( (A2) \). Then \( f_L \) admits first-order partial derivatives and, for \( s, t > 0 \),
\[
\begin{align*}
\partial_t f_L(s, t) &= \int_0^t \nu(\tau) \partial_t g_\sigma(t-\tau, s) d\tau, \tag{2.10} \\
\partial_s f_L(s, t) &= \int_0^t \nu(\tau) \partial_s g_\sigma(t-\tau, s) d\tau. \tag{2.11}
\end{align*}
\]

In particular,
\[
\begin{align*}
|\partial_t f_L(s, t)| &\leq \frac{I_\phi(t)}{\pi} \int_0^{+\infty} \xi e^{-s\Re(\Psi(\xi))} d\xi \tag{2.12} \\
|\partial_s f_L(s, t)| &\leq \frac{I_\phi(t)}{\pi} \left( \sqrt{2} \int_0^{M_\gamma} |\Psi(\xi)| e^{-s\Re(\Psi(\xi))} d\xi \\
&\quad + \int_0^{+\infty} \left( 3|\Psi(1)| + \xi \int_0^1 \tau \nu(d\tau) + \frac{\xi^2}{2} \int_0^1 \tau^2 \nu(d\tau) \right) e^{-s\xi^{2-\gamma}} d\xi \right). \tag{2.13}
\end{align*}
\]

where \( I_\phi \) is defined in \((2.3)\). Furthermore we have that, locally uniformly for \( t \in (0, +\infty) \),
\[
\lim_{s \to \infty} sf_L(s, t) = \lim_{s \to \infty} s^2 |\partial_s f_L(s, t)| = 0.
\]
and also \( \partial_t f_L(s, \cdot), \partial_s f_L(s, \cdot) \in C([0, \infty))^2 \),
\[
\lim_{t \downarrow 0} \partial_t f_L(s, t) = \lim_{t \downarrow 0} \partial_s f_L(s, t) = 0 \text{ locally uniformly with respect to } s > 0. \tag{2.14}
\]

The proof is given in Appendix A.1.

We can now define a family of linear operators associated to a subordinator, or equivalently to a Bernstein functions. For a fixed Bernstein function \( \Phi \) satisfying \((A1)\), we define the convolution integral operator acting on \( L^1_{loc}(\mathbb{R}_0^+) \)
\[
\mathcal{I}_\phi f(t) := \int_0^t \nu(t-\tau)f(\tau) d\tau
\]
and the generalized fractional derivative
\[
\partial_\phi^t f(t) := \frac{d}{dt} \int_0^t \nu(t-\tau)(f(\tau) - f(0)) d\tau = \frac{d}{dt} \mathcal{I}_\phi (f(\cdot) - f(0))(t),
\]
provided the quantity is well-defined. It is, in any case, important to observe that if \( f \in AC[0, T] \) for some fixed \( T > 0 \), then \( \partial_\phi^t f \in L^1[0, T] \).

**Lemma 2.4.** Fix \( T > 0 \). For \( f \in AC[0, T] \) it holds \( \partial_\phi^t f \in L^1[0, T] \) with
\[
\partial_\phi^t f(t) = \int_0^t \nu(t-s) \partial_t f(s) ds, \quad \text{for a.e. } t \in [0, T]. \tag{2.15}
\]

If furthermore \( f \) is Lipschitz, then \( \partial_\phi^t f \in C[0, T] \) and \( (2.15) \) holds for all \( t \in [0, T] \).
Proof. Since \( f \in \text{AC}[0, T] \), then \( \partial_t f \in L^1[0, T] \) and \( \mathcal{I}^\Phi(\partial_t f) \in L^1[0, T] \). First, notice that

\[
\int_0^t \int_0^s \nu(\tau) |\partial_t f(s - \tau)|d\tau ds = \int_0^t \nu(\tau) \int_\tau^t |\partial_t f(s - \tau)|dsd\tau \\
= \int_0^t \nu(\tau) \int_0^{t-\tau} |\partial_t f(z)|dzd\tau \\
\leq \left( \int_0^t \nu(\tau)d\tau \right) \left( \int_0^t |\partial_t f(z)|dz \right) < +\infty.
\]

(2.16)

Hence, by Fubini’s theorem and [73, Theorem 7.29]

\[
\int_0^t \mathcal{I}^\Phi(\partial_t f)(s)ds = \int_0^t \int_0^s \nu(\tau)\partial_t f(s - \tau)d\tau ds \\
= \int_0^t \nu(\tau) \int_\tau^t \partial_t f(s - \tau)d\tau ds = \int_0^t \nu(\tau)(f(t - \tau) - f(0))d\tau.
\]

Hence we can differentiate both sides to conclude the first part of the proof. The second part of the statement is a direct consequence of [2, Proposition 1.3.2]. \( \square \)

Remark 2.5. We point out that there exist functions \( f \not\in \text{AC}[0, T] \) such that \( \partial_t^\Phi f \) is well-defined. For instance, if \( \Phi(\lambda) = \lambda^\alpha \), i.e. \( \nu(\tau) = \frac{\tau^{-\alpha}}{\Gamma(1 - \alpha)} \) for \( \tau > 0 \), it is sufficient that \( f \in C^\beta[0, T] \) for \( \beta > \alpha \), see [6, Theorem 1.18].

The relation between the operator \( \partial_t^\Phi \) and the density \( f_L \) is underlined by the next result.

Proposition 2.6. Under Assumptions (A1) and (A2) it holds

\[
\partial_t^\Phi f_L(s; t) = -\partial_s f_L(s; t) \quad s, t > 0,
\]

(2.17)

where we set \( f_L(s; 0) = f_L(0; t) = 0 \) for all \( s, t > 0 \).

Proof. First, note that if one proves the equality

\[
(t^\Phi f_L(s; \cdot))(t) = -\int_0^t \partial_s f_L(s; w)dw, \quad t, s > 0
\]

(2.18)

then (2.17) follows by differentiating both sides of the previous relation. To show (2.18), let us first observe, as a direct consequence of (2.2) that, for \( \lambda > 0 \),

\[
\int_0^{+\infty} e^{-\lambda t} f_L(s; t) dt = \frac{\Phi(\lambda)}{\lambda} e^{-s\Phi(\lambda)}
\]

hence

\[
\int_0^{+\infty} e^{-\lambda t} \left( t^\Phi f_L(s; \cdot) \right)(t) dt = \frac{\Phi^2(\lambda)}{\lambda^2} e^{-s\Phi(\lambda)}.
\]

On the other hand, from (2.13), we know that, for \( \lambda > 0 \),

\[
\int_0^{+\infty} e^{-\lambda t} |\partial_s f_L(s; t)| dt < \infty
\]

hence

\[
\int_0^t e^{-\lambda t} \int_0^s \partial_s f_L(s; w)dw dt = \frac{1}{\lambda} \int_0^{+\infty} e^{-\lambda t} \partial_s f_L(s; w)dw.
\]

Next, we observe that (2.13) also holds on incremental ratios, see the proof in Appendix A.1, so that, by a simple dominated convergence argument, we have

\[
\int_0^{+\infty} e^{-\lambda t} \int_0^t \partial_s f_L(s; w)dw dt = \frac{1}{\lambda} \partial_s \left( \int_0^{+\infty} e^{-\lambda t} f_L(\cdot; w) dt \right) (s) = -\frac{\Phi^2(\lambda)}{\lambda^2} e^{-s\Phi(\lambda)}.
\]

Hence, for fixed \( s > 0 \), we know that (2.18) holds for \( t \in \mathbb{R}^+ \setminus \mathcal{N} \), where \( |\mathcal{N}| = 0 \), by [2, Theorem 1.7.3]. However, since \( g_\sigma(\cdot; s) \) is continuous on \( \mathbb{R}^+_0 \) for \( s > 0 \), setting \( g_\sigma(0; s) = 0 \), and \( \nu \in L^1_{\text{loc}}(\mathbb{R}^+_0) \), then \( f_L(s; \cdot) \) is continuous on \( \mathbb{R}^+_0 \) by (2.2) and, as consequence, \( \mathcal{I}^\Phi f_L(s; \cdot) \) is continuous on \( \mathbb{R}^+_0 \). Hence (2.18) holds for all \( t, s > 0 \). \( \square \)
We will also make use of the following function
\[ U_p(t) = \mathbb{E}_x[L(t)^p], \quad t > 0, \] (2.19)
that is finite for all \( t > 0 \) and \( p > -1 \) by [5, Lemma 2.3] and [3, Lemma 4.1]. We need, however, the following integrability property.

**Proposition 2.7.** Under Assumption (A2) we have \( U_p \in L^1_{\text{loc}}(\mathbb{R}_0^+) \) for all \( p > -1 \).

**Proof.** Consider for \( \lambda > 0 \)
\[
\int_0^{+\infty} e^{-\lambda t} U_p(t) \, dt,
\]
that is well-defined since \( U_p(t) > 0 \) for all \( t > 0 \). By Tonelli's theorem, we have
\[
\int_0^{+\infty} e^{-\lambda t} U_p(t) \, dt = \int_0^{+\infty} s^p \int_0^{+\infty} e^{-\lambda t f_L(s; t)} \, dt \, ds = \frac{\Phi(\lambda)}{\lambda} \int_0^{+\infty} s^p e^{-s \Phi(\lambda)} \, ds = \frac{\Gamma(p + 1)}{\lambda(p + 1)} e^{\frac{-p}{\lambda}} < \infty.
\] (2.20)

In particular, for any \( T > 0 \) it holds
\[
\int_0^T U_p(t) \, dt \leq e^T \int_0^T e^{-t} U_p(t) \, dt \leq \Gamma(p + 1)(\Phi(1))^{-p} e^{T} < \infty.
\] (2.21)

\[ \square \]

### 2.1. A weak maximum principle.

In the following we will be interested in a time-nonlocal heat equation, on a specific parabolic domain, involving the operator \( \partial_t^p \). To prove uniqueness of the solution, we will make use of a weak maximum principle, which we now state for a more general advection-diffusion operator. Before doing this, we need to introduce some notation concerning parabolic domains. For \( T \in [0, +\infty] \) and \( N \in \mathbb{N} \), let \( E \subseteq [0, T] \times \mathbb{R}^N \) (if \( T = +\infty \) set \( E := \mathbb{R}_0^+ \)). We denote by \( \overline{E} \) the usual topological closure of \( E \) while \( \tilde{E} \) will be the usual topological interior of \( E \). Given \((t_0, x_0) \in [0, T] \times \mathbb{R}^N \) and \( r > 0 \), \( B_r(t_0, x_0) \) denotes the Euclidean ball of radius \( r > 0 \) and center \((t_0, x_0) \) in \( \mathbb{R}^{N+1} \). The parabolic interior \( E^* \) of \( E \) is defined as follows
\[
(t_0, x_0) \in E^* \iff \exists r > 0 : B_r(t_0, x_0) \cap \{(t, x) \in [0, T] \times \mathbb{R}^N : t \leq t_0\} \subseteq E.
\]

It is clear that \( \tilde{E} \subseteq E^* \). However, the two sets do not necessarily coincide. For instance, if \( G \subseteq \mathbb{R}^N \) is an open set and \( E = [0, T] \times G \), then \( \tilde{E} = (0, T) \times G \) and \( E^* = (0, T] \times G \). We say that \( E \) is a parabolic open set if \( E^* = E \). The parabolic boundary of \( E \) is given by \( \partial_s E := E \setminus E^* \). It is clear that \( \partial_s E \subseteq \partial E \), where \( \partial E := \overline{E} \setminus \tilde{E} \) is the usual boundary of the set \( E \). We also consider the slices
\[
E_1(x) = \{ t \in [0, T] : (t, x) \in E \} \quad x \in \mathbb{R}^N
\]
and the projections \( E_1 = \bigcup_{x \in \mathbb{R}^N} E_1(x) \) and \( E_2 = \bigcup_{t \in [0, T]} E_2(t) \). We say that \( E \) is non-decreasing with respect to the variable \( t \) if \( E_2(t_1) \subseteq E_2(t_2) \) whenever \( 0 \leq t_1 \leq t_2 \leq T \). If \( T < \infty \) and \( E \subseteq [0, T] \times \mathbb{R}^N \), given two functions \( f : (t, x) \in E \to \mathbb{R} \) and \( u : E_1 \to \mathbb{R} \), we say that \( \lim_{x \to \infty} f(t, x) = u(t) \) uniformly with respect to \( t \in [0, T] \) if for all \( \varepsilon > 0 \) there exists a compact set \( K \subseteq \bigcup_{t \in [0, T]} E_2(t) \) such that
\[
\sup_{x \notin K} |f(t, x) - u(t)| < \varepsilon.
\]

If instead \( E \subseteq \mathbb{R}_0^+ \times \mathbb{R}^N \), we say that \( \lim_{x \to \infty} f(t, x) = u(t) \) locally uniformly with respect to \( t \geq 0 \) if \( \lim_{x \to \infty} f(t, x) = u(t) \) uniformly with respect to \( t \in [0, T] \) for all \( T > 0 \).

We can now state the following result, which is a weak maximum principle (the operators \( \nabla \) and \( \Delta \) are referred to the \( x \) variable in \( \mathbb{R}^N \)).

**Theorem 2.8.** Let \( T \in [0, +\infty) \) and \( E \subseteq [0, T] \times \mathbb{R}^N \) be connected and non-decreasing with respect to the variable \( t \) and define, for \( x \in \overline{E}_2 \), \( t(x) = \min\{t \geq 0 : x \in E_2\} \). Let \( u : \mathbb{R}_0^+ \times \mathbb{R}^N \to \mathbb{R} \) be such that \( u \in C(\overline{E}) \), \( u(t, \cdot) \in C^2(E_2(t)) \) for all \( t \in E_1 \), \( u(t(x)) = u(t(x), x) \) for all \( x \in E_2 \) and \( t \in [0, t(x)] \) and \( \partial_t^p u(t, x) \) is well-defined for \((t, x) \in E \) with \( \partial_t^p u(\cdot, x) \in C(E_1(x)) \cap L^1_{\text{loc}}(\overline{E}_2(x)) \) for all \( x \in E_2 \). Furthermore, if \( E \) is not
bounded, assume that there exists a function $u_\infty \in C[0,T]$ such that $\lim_{x \to \infty} u(t,x) = u_\infty(t)$ uniformly with respect to $t \in [0,T]$. Finally, suppose that

$$\partial_t^\Phi u(t,x) - \langle p_1(x), \nabla u(t,x) \rangle - p_2(x) \Delta u(t,x) \leq 0 \forall (t,x) \in E,$$

where $p_1 : E_2 \to \mathbb{R}^N$ and $p_2 : E_2 \to \mathbb{R}_0^+$. Then, if $E$ is bounded,

$$\max_{(t,x) \in E} u(t,x) = \max_{(t,x) \in \partial_E E} u(t,x)$$

while if $E$ is unbounded

$$\sup_{(t,x) \in E} u(t,x) = \max \left\{ \max_{(t,x) \in \partial_E E} u(t,x), \max_{t \in [0,T]} u_\infty(t) \right\}.$$

We leave the proof of this technical result, which is of independent interest, in Appendix A.2.

3. The Brownian motion delayed by an inverse subordinator

3.1. Definition and regularity of the density. Let $B$ be the Brownian motion introduced in the previous section. Then we define the Brownian motion delayed by an inverse subordinator (see [44]) as

$$X_\Phi(t) := B(L(t)), \ t \geq 0.$$  

Notice that, under (A2) the process $X_\Phi$ is $\mathbb{P}_y$-a.s. continuous for any $y \in \mathbb{R}$ since it is the composition of two continuous functions. By independence of $B$ and $L$, it is immediate to notice that for any $A \in \mathcal{B}([0,T])$ and $y \in \mathbb{R}$ it holds

$$\mathbb{P}_y(X_\Phi(t) \in A) = \int_A p_\Phi(t,x;y) \, dy$$

where

$$p_\Phi(t,x;y) = \int_0^{+\infty} p(s,x;y)f_L(s;t) \, ds$$  \hspace{1cm} (3.1)

and

$$p(s,x;y) = \frac{1}{\sqrt{2\pi s}} e^{-\frac{(x-y)^2}{2s}}, \ (s,x,y) \in (0,\infty) \times \mathbb{R}^2$$

is the density of the Brownian motion $B$ under $\mathbb{P}_y$. It is clear that $p_\Phi(t,x;y)$ depends actually on $x-y$. Now we give some results concerning the regularity of the function $p_\Phi$ with respect to both the $x$ and the $t$ variable.

**Proposition 3.1.** Under (A2), $p_\Phi$, $\partial_x p_\Phi$ and $\partial_x^2 p_\Phi$ exist in $C(\mathbb{R}^+ \times (\mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2)))$, where $\text{diag}(\mathbb{R}^2) = \{(x,x), x \in \mathbb{R}\}$ and

$$\partial_x p_\Phi(t,x;y) = \int_0^{+\infty} \partial_x p(s,x;y)f_L(s;t) \, ds$$

$$\partial_x^2 p_\Phi(t,x;y) = \int_0^{+\infty} \partial_x^2 p(s,x;y)f_L(s;t) \, ds$$  \hspace{1cm} (3.2)

In particular, $p_\Phi \in C(\mathbb{R}^+ \times \mathbb{R}^2)$,

$$\lim_{t \to 0^+} p_\Phi(t,x;y) = 0 \text{ locally uniformly in } \{(x,y) \in \mathbb{R}^2 : x \neq y\}$$

and, in weak convergence,

$$\lim_{t \to 0^+} p_\Phi(t,x;y) \, dy = \delta_x(dy)$$

If also (A2) holds, then, for $x \neq y$, $p_\Phi(\cdot,x;y)$ is differentiable on $(0,\infty)$, $\partial_x p_\Phi \in C(\mathbb{R}^+ \times (\mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2)))$,

$$\partial_x p_\Phi(t,x;y) = \int_0^{+\infty} p(s,x;y)\partial_t f_L(s,t) \, ds$$  \hspace{1cm} (3.3)
and
\[ \lim_{t \to 0^+} \partial_t p_\Phi(t, x, y) = 0 \text{ locally uniformly with respect to } (x, y) \in \mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2). \] (3.5)

Furthermore, for \( x \neq y \), \( \partial^2_t p_\Phi(t, x; y) \) is well-defined on \((0, \infty)\), \( \partial^2_t p_\Phi \in C(\mathbb{R}^+ \times (\mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2))) \),
\[ \partial^2_t p_\Phi(t, x; y) = \int_0^{+\infty} p(s, x; y) \partial^2_L f_L(s, t) ds = -\int_0^{+\infty} p(s, x; y) \partial_s f_L(s, t) ds. \] (3.6)

and
\[ \lim_{t \to 0^+} \partial^2_t p_\Phi(t, x; y) = 0 \text{ locally uniformly with respect to } (x, y) \in \mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2). \] (3.7)

The technical proof is given in Appendix A.3.

In next theorem, we prove that \( p_\Phi \) satisfies pointwise a time-nonlocal heat equation for \( t > 0 \) and \( x \neq y \). To do this, however, we preliminarily set \( p_\Phi(0, x; y) = 0 \) whenever \( x \neq y \). In such a way, \( p_\Phi \in C(\mathbb{R}^+_0 \times (\mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2))) \).

**Theorem 3.2.** Under Assumptions (A2),
\[ \partial^2_t p_\Phi(t, x; y) = \frac{1}{2} \partial^2_x p_\Phi(t, x; y), \quad t > 0, \quad (x, y) \in \mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2). \] (3.8)

**Proof.** Arguing as in Proposition 2.6, it is sufficient to show that for \((x, y) \in \mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2)\) and \( t > 0 \),
\[ X^p p_\Phi(t, x; y) = \frac{1}{2} \int_0^t \partial^2_x p_\Phi(s, x; y) ds \] (3.9)

To do this, first observe that for \( \lambda > 0 \)
\[ \int_0^{+\infty} e^{-\lambda t} p_\Phi(t, x; y) dt = \frac{\Phi(\lambda)}{\lambda} \int_0^{+\infty} e^{-s\Phi(\lambda)} p(s, x; y) ds \] (3.10)

and then
\[ \int_0^{+\infty} e^{-\lambda t} X^p p_\Phi(t, x; y) dt = \frac{\Phi(\lambda)}{\lambda} \int_0^{+\infty} e^{-s\Phi(\lambda)} p(s, x; y) ds. \] (3.11)

Let us also recall that, clearly,
\[ \int_0^{+\infty} e^{-\lambda t} (|\partial_x p(t, x; y)| + |\partial^2_x p(t, x; y)|) dt < \infty \] (3.12)

and then, by (3.3),
\[ \int_0^{+\infty} e^{-\lambda t} \left( \int_0^t \partial^2_x p_\Phi(s, x; y) ds \right) dt = \frac{\Phi(\lambda)}{\lambda^2} \int_0^{+\infty} e^{-s\Phi(\lambda)} \partial^2_x p(s, x; y) ds. \] (3.13)

However, we recall that \( p(t, x; y) \) satisfies
\[ \partial_t p(t, x; y) = \frac{1}{2} \partial^2_x p(t, x; y) \quad t > 0, \quad (x, y) \in \mathbb{R}^2 \]

and (3.12) guarantees that \( \partial_t p(\cdot, x; y) \) admits Laplace transform with non-positive abscissa of convergence. Hence, by (3.13) and the fact that \( \Phi(\lambda) > 0 \), we get
\[ \int_0^{+\infty} e^{-\lambda t} \left( \int_0^t \partial^2_x p_\Phi(s, x; y) ds \right) dt = \frac{2\Phi(\lambda)}{\lambda^2} \int_0^{+\infty} e^{-s\Phi(\lambda)} \partial_s p(s, x; y) ds = \frac{2\Phi(\lambda)}{\lambda^2} \int_0^{+\infty} e^{-s\Phi(\lambda)} p(s, x; y) ds. \] (3.14)

Comparing (3.11) and (3.14), by the injectivity of the Laplace transform we get that for all \((x, y) \in \mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2)\), (3.9) holds for \( t \in \mathbb{R}^+ \setminus \mathbb{N} \), where \( |\mathbb{N}| = 0 \). However, by continuity of both sides of (3.9), that easily follows by (3.1) and the fact that \( \nu \in L^1_{\text{loc}} \); we have that \( \mathbb{N} = \emptyset \) and (3.9) follows.

**Remark 3.3.** The previous theorem also guarantees that \( \partial^2_t p_\Phi \in C(\mathbb{R}^+_0 \times (\mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2))). \)
Notice that $\partial_x p_\Phi, \partial_y^2 p_\Phi$ and $\partial_{y}p_\Phi$ can be infinite for $t > 0$ as $x - y \to 0$. To guarantee a uniform limit in this regime, we need a further condition on $\Phi$, which is expressed in terms of $f_L$.

It holds $\lim_{s \to 0^+} f_L(s, t) = \nu(t)$ locally uniformly with respect to $t \in \mathbb{R}^+$ and for any compact set $K \subset (0, +\infty)$ there exists a function $h(s)$ and a $\delta > 0$ such that $|\partial_s f_L(s, t)| \leq h(s)$ for any $(s, t) \in (0, \delta) \times K$ and $s \to s^{-\frac{1}{2}}h(s)$ is integrable in $(0, \delta)$.

With the previous assumption we can prove the following limit behaviour.

**Proposition 3.4.** Suppose Assumptions (A2) and (A3) hold. Then

$$
\lim_{x-y \to 0^\pm} \frac{\partial_x p_\Phi(t, x; y)}{2} = \frac{\nu(t)}{2} \text{ locally uniformly w.r.t. } t > 0. \tag{3.15}
$$

Furthermore, if also Assumption (A2) holds, then

$$
\lim_{x-y \to 0^\pm} \frac{\partial_y^2 p_\Phi(t, x; y)}{2} = \lim_{x-y \to 0^\pm} \partial_y p_\Phi(t, x; y) = -\frac{1}{\sqrt{2\pi}} \int_0^{+\infty} s^{-\frac{1}{2}} \partial_s f_L(s; t) \, ds. \tag{3.16}
$$

locally uniformly with respect to $t > 0$.

The technical proof of this Proposition is in Appendix A.4.

Since Assumption (A3) could be not so simple to verify, as it involves $\partial_s f_L$ which is a priori unknown, with next proposition we resume the results and the discussions in [8, Theorem 3.18 and Sections 3.2.1 and 6.2], which provide a sufficient condition, in terms of $\Phi$, for Assumption (A3) to hold, in practice proving the next result.

**Proposition 3.5.** Assume that there exists $\theta \in (0, \pi)$ such that $\Phi$ with representation (2.1) admits a holomorphic extension on the complex sector

$$
\mathbb{C}(\pi - \frac{\theta}{2}) := \{z \in \mathbb{C} \setminus \{0\} : |\text{Arg}(z)| < \pi - \frac{\theta}{2}\}
$$

which is continuous on the closure $\overline{\mathbb{C}(\pi - \frac{\theta}{2})}$. Furthermore, assume that $\lim_{z \to \infty} \frac{\Phi(z)}{z} = 0$ uniformly in $\overline{\mathbb{C}(\pi - \frac{\theta}{2})}$. Then Assumption (A3) holds. In particular Assumption (A3) holds whenever $\Phi$ is a complete Bernstein function or it has the form $\Phi(\lambda) = \Phi_1(\lambda^\alpha)$ for some $\alpha \in (0, 1)$ and any Bernstein function $\Phi_1$.

### 3.2. The semi-Markov property.

Note that $X_\Phi$ is not a Markov process. However, it can be embedded in a strong Markov process as follows. Let

$$
H(t) := \sigma(L(t)), \tag{3.17}
$$

which is called the overshooting of $\sigma$. Then the process $\{(X_\Phi(t), H(t) - t), t \geq 0\}$ is a time homogeneous strong Markov processes with respect to filtration $(\mathcal{F}_{L(t)})_{t \geq 0}$ (see [51, Section 4]) and furthermore it is a Hunt process (see [51, Theorem 3.1]). Actually, the process $X_\Phi$ still exhibits the Markov property at the random time $H(t)$. Indeed, since $(B, \sigma)$ is Markov additive in the sense of [22, Definition 1.4], $(\Omega, \mathcal{F}, (\mathcal{F}_{L(t)})_{t \geq 0}, \mathcal{M}, X_\Phi, \mathbb{P}_{x})$ is regenerative in the sense of [37, Definition 2.2 and Example 2.13], where the regenerative set $\mathcal{M}$ is given by the closure of

$$
\mathcal{M}(\omega) = \{ t \in \mathbb{R}_0^+ : t = \sigma(u, \omega) \text{ for some } u \geq 0 \}, \omega \in \Omega. \tag{3.18}
$$

In particular, $X_\Phi$ exhibits the strong regeneration property in the sense of [37, eq. (2.14)], so that the strong Markov property holds with respect to any $(\mathcal{F}_{L(t)})_{t \geq 0}$-Markov time taking values in $\mathcal{M}$. The reader should also consult [35] for instructive discussions of regenerative systems and semi-Markov property.

Let us stress that we can also express $H(t)$ in terms of the random set $\mathcal{M}$ as follows:

$$
H(t, \omega) = \inf \{ s > t : s \in \mathcal{M}(\omega) \}, \omega \in \Omega. \tag{3.19}
$$

In the following we will be interested in the process obtained by killing $X_\Phi$ upon reaching a certain moving boundary $\varphi : \mathbb{R}_0^+ \to \mathbb{R}$. In particular, we need the crossing time to be a Markov time for $X_\Phi$. Hence, let us first introduce the notation

$$
T := \inf \{ t \in \mathbb{R}_0^+ : X_\Phi(t) \geq \varphi(t) \}, \tag{3.20}
$$
where we set \( \inf \emptyset = +\infty \).

**Lemma 3.6.** Under Assumption (A2), if \( \varphi \) is non-decreasing and continuous then, \( \mathbb{P}_y \)-almost surely for all \( y \in \mathbb{R} \), either \( T \in \mathcal{M} \) or \( T = \infty \).

**Proof.** Let us first observe that if \( y = X_\varphi(0) \geq \varphi(0) \), then \( T = 0 \in \mathcal{M} \) \( \mathbb{P}_y \)-almost surely. Hence, let us assume that \( y = X_\varphi(0) < \varphi(0) \). Then either \( T = \infty \) or \( X_\varphi(T) \geq \varphi(T) \). We only need to study the latter case. For the sake of the reader, we omit the dependence on \( \omega \in \Omega \). Being both \( X_\varphi \) and \( \varphi \) continuous (a.s.), the set \( \{ t \in [0, +\infty) : X_\varphi(t) - \varphi(t) = 0 \} \) is closed and

\[
0 < T = \min\{ t \in [0, +\infty) : X_\varphi(t) - \varphi(t) = 0 \}. \tag{3.21}
\]

Now suppose that \( T \notin \overline{\mathcal{M}} \). Then we know that \( T \in (\sigma(w), \sigma(w)) \) for some \( w > 0 \) and

\[
X_\varphi(\sigma(w)) = X_\varphi(T) = \varphi(T) \geq \varphi(\sigma(w)) - \varphi(w).
\]

Combining this information with the fact that \( X_\varphi(0) < \varphi(0) \), we get that there exist \( t_0 \in (0, \sigma(w)) \), and then in particular \( t_0 < T \), such that \( X_\varphi(t_0) - \varphi(t_0) = 0 \), which is a contradiction. Hence \( T \in \overline{\mathcal{M}} \).

Now let \( \delta = \frac{2(0) - X_\varphi(0)}{2} > 0 \) and consider the sequence of stopping time

\[
T_n = \inf \left\{ t \geq 0 : X_\varphi(t) \geq \varphi(t) - \frac{\delta}{n} \right\}.
\]

It is clear that \( T_n \leq T \). Furthermore, since both \( \varphi \) and \( X_\varphi \) are (a.s.) continuous, we have

\[
X_\varphi(T_n) = \varphi(T_n) - \frac{\delta}{n} < \varphi(T_n)
\]

and then \( T_n < T \) a.s.. Moreover, \( T_n \) is an non-decreasing sequence of stopping times with \( T_n < T \) and then

\[
T_* = \sup_{n \in \mathbb{N}} T_n \leq T.
\]

However, taking the limit as \( n \) tends to infinity in (3.22) we achieve

\[
X_\varphi(T_*) = \varphi(T_*), \tag{3.23}
\]

which, by (3.21), implies \( T_* = T \). Thus \( T_n \uparrow T \) with \( T_n < T \) and then, since \( (X_\varphi(t), H(t) - t) \) is a Hunt process, \( T \) must be a continuity point for such a process. This implies that \( T \) is not isolated on the right, otherwise, if \( T \in \overline{\mathcal{M}} \setminus \mathcal{M} \), by (3.19) it would be clear that it is a discontinuity point for \( H \), which is a contradiction. \( \square \)

**Remark 3.7.** In case \( T < \infty \), we have \( H(T) = T \).

**Remark 3.8.** One can actually apply the previous proof to any fixed open set, even if \( B \) is an N-dimensional Brownian motion, to get that any exit time from an open set is a Markov time for \( X_\varphi \). The latter is called semi-Markov property of \( X_\varphi \), see [31, Definition 1, Chapter 3].

4. THE KILLED DELAYED BROWNIAN MOTION AND A DYKIN-HUNT FORMULA

As we stated before, we are interested in the process obtained by killing \( X_\varphi \) upon reaching \( \varphi \), that is to say

\[
X^\dagger_\varphi(t) := \begin{cases} X_\varphi(t), & t < T, \\ \infty, & t \geq T, \end{cases}
\]

where \( \infty \notin \mathbb{R} \) is a cemetery point. Such a process defines the family of sub-probability measures on \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) as follows

\[
\mathcal{B}(\mathbb{R}) \ni A \mapsto \mathbb{P}_y(X^\dagger_\varphi(t) \in A), \ y \in \mathbb{R}, \ t \geq 0.
\]

Now we want to prove that for each \( y \in \mathbb{R} \) and \( t > 0 \), these sub-probability measures admit a density and we want to relate such a density with the non-killed one \( p_\varphi \) through a Dynkin-Hunt-type formula. To do this, we first need to ensure that \( T \) is a continuous random variable, at least for \( y < \varphi(0) \).

**Lemma 4.1.** Under Assumption (A2), for all \( y < \varphi(0) \), \( T \) is a continuous random variable, i.e. \( \mathbb{P}_y(T = w) = 0 \) for any \( w \geq 0 \).

**Proof.** Just notice that for any \( w_0 \geq 0 \) and \( y < \varphi(0) \),

\[
\mathbb{P}_y(T \in \{w_0\}) \leq \mathbb{P}_y(X_\varphi(w_0) = \varphi(w_0)) = 0. \tag{4.1}
\]

\( \square \)
From now on, we denote by \( \mu_\varphi(\cdot; y) \) the law of \( T \) under \( P_y \), i.e. for all \( A \in \mathcal{B}(\mathbb{R}) \) we set \( \mu_\varphi(A; y) = P_y(T \in A) \). Now we are ready to prove the following Dynkin-Hunt formula.

**Theorem 4.2.** Suppose Assumption (A2) holds and \( \varphi \) is non-decreasing and continuous. Then, setting

\[
q_\varphi(t, x; y) = p_\varphi(t, x; y) - \int_0^t p_\varphi(t - w, x; \varphi(w))\mu_\varphi(dw; y) \quad t > 0, \quad (x, y) \in \mathbb{R} \times (0, \varphi(0)),
\]

and \( q_\varphi(t, x; y) = 0 \) for \( t > 0 \) and \( (x, y) \in \mathbb{R} \times [\varphi(0), +\infty) \), we have, for \( t > 0 \), \( y \in \mathbb{R} \) and \( A \in \mathcal{B}(\mathbb{R}) \),

\[
P_y\left(X^t_\varphi(t) \in A \right) = \int_A q_\varphi(t, x; y) dx.
\]

**Proof.** The statement is clear if \( x, y \) and \( \varphi \) is applied to the conditional expectation \( E \) where the function \( T \) property at \( \varphi \). Since \( \varphi \) is non-decreasing and continuous we have by Lemma 3.6 that

\[
\text{for some } t, x \quad \exists \Phi(t, x) = 0
\]

Before investigating the regularity properties of \( q_\varphi \), let us discuss separately the easier case in which \( \varphi(t) \equiv c \) for some \( c \in \mathbb{R} \). Some properties related to the constant case will be useful when dealing with the general moving boundary case.

4.1. **The constant boundary case.** Let \( c \in \mathbb{R} \) and set

\[
T_c := \inf\{t \in \mathbb{R}^+_0 : X_\varphi(t) \geq c\},
\]

which is a particular case of (3.20) when \( \varphi(t) = c \) for all \( t \geq 0 \). We are first interested in the distribution of \( T_c \). Notice that, for \( y < c \),

\[
P_y(T_c > t) = \Phi_y\left(\max_{0 \leq s \leq t} X_\varphi(s) \leq c\right) = \Phi_0\left(\max_{0 \leq s \leq t} X_\varphi(s) \leq c - y\right),
\]

hence, it could be useful to prove the analogous of the reflection principle for the Brownian motion on \( X_\varphi \).

**Proposition 4.3.** We have that, for any \( c > 0 \),

\[
P_0\left(\max_{0 \leq s \leq t} X_\varphi(s) > c\right) = 2P_0\left(X_\varphi(t) > c\right).
\]

**Proof.** First of all, notice that \( \max_{0 \leq s \leq t} X_\varphi(s) > c \) if an only if \( \max_{0 \leq s \leq L(t)} B(s) > c \), since the maximum actually coincide, as \( X_\varphi \) is the composition of \( B \), that is continuous, with \( L \), that is continuous and non-decreasing. By conditioning on \( L(t) \) and using the independence of \( B \) and \( L \) we have that

\[
P_0\left(\max_{0 \leq s \leq t} X_\varphi(s) > c\right) = P_0\left(\max_{0 \leq w \leq L(t)} B(w) > c\right)
\]
where we used (4.11) to guarantee that $E$.

Taking the modulus we have

$$
= 2 \int_0^\infty \mathbb{P}_0(B(s) > c) f_L(s, t) \, ds
$$

$$
= 2 \mathbb{P}_0(X_\Phi(t) > c).
$$

Hence, as a consequence, we have, for $y < c$,

$$
\mathbb{P}_y(T_c \leq t) = 2 \mathbb{P}_0(X_\Phi(t) > c - y).
$$

(4.8)

Let us now investigate the regularity of $T_c$. To do this we recall that, as done in [7], if we set

$$
T_c := \inf\{t \in \mathbb{R}_0^+ : B(t) \geq c\}
$$

we have that $T_c = \sigma(T_c)$. Using this relation, we can prove the following result.

**Proposition 4.4.** Under Assumptions (A2) and (A2), the r.v. $T_c$ is absolutely continuous with density $p_{T_c}(:; y)$. Furthermore, for all $T > 0$ and any compact sets $I, J \subset \mathbb{R}$ such that $z_1 < z_2$ for all $z_1 \in I$ and $z_2 \in J$,

$$
S(I, J) := \sup \sup \sup_{c \in J} p_{T_c}(t; y) < \infty
$$

(4.10)

Proof. Let us observe that since Assumptions (A2) and (A2) hold, we know that $\sigma$ admits density and then the proof of [7, Theorem 2.8] can be applied analogously. As a consequence $T_c$ is absolutely continuous. Furthermore, since $T_c = \sigma(T_c)$, then

$$
E_y \left[ e^{i \xi T_c} \right] = E_y \left[ e^{i \xi \sigma(T_c) \mid T_c} \right] = E_y \left[ e^{i T_c \Psi(\xi)} \right] = \int_0^{+\infty} e^{-\Psi(\xi)s} p_{T_c}(s; y) \, ds,
$$

where

$$
p_{T_c}(s; y) = \frac{c - y}{\sqrt{2\pi s}} e^{-\frac{(c - y)^2}{2s}}.
$$

(4.11)

Taking the modulus we have

$$
|E_y \left[ e^{i \xi T_c} \right]| \leq \int_0^{+\infty} e^{-\Psi(\xi)s} p_{T_c}(s; y) \, ds
$$

and then

$$
\int_{\mathbb{R}} |E_y \left[ e^{i \xi T_c} \right]| \, d\xi \leq 2 \int_0^{+\infty} \int_0^{+\infty} e^{-\Psi(\xi)s} p_{T_c}(s; y) \, dsd\xi
$$

$$
\leq 2 \int_0^{+\infty} \int_0^{M_\gamma} e^{-\Psi(\xi)s} p_{T_c}(s; y) \, dsd\xi + 2 \int_0^{+\infty} \int_0^{+\infty} e^{-C_\gamma \xi^{2-\gamma}} p_{T_c}(s; y) \, d\xi ds
$$

$$
\leq 2M_\gamma + \frac{2\Gamma \left( \frac{1}{2-\gamma} \right)}{(2-\gamma)C_\gamma^{\frac{1}{2-\gamma}}} \mathbb{E}_y \left[ T_c^{\frac{1}{2-\gamma}} \right] < \infty,
$$

where we used (4.11) to guarantee that $E_y \left[ T_c^{\frac{1}{2-\gamma}} \right] < \infty$. As a consequence, we can write

$$
p_{T_c}(t; y) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^{+\infty} e^{-\Psi(\xi)s-i\xi t} p_{T_c}(s; y) \, ds \, d\xi.
$$

Thus

$$
|p_{T_c}(t; y)| \leq \frac{1}{\pi} \int_0^{+\infty} \int_0^{+\infty} e^{-\Psi(\xi)s} p_{T_c}(s; y) \, ds \, d\xi \leq \frac{M_\gamma}{\pi} + \frac{\Gamma \left( \frac{1}{2-\gamma} \right)}{\pi(2-\gamma)C_\gamma^{\frac{1}{2-\gamma}}} \mathbb{E}_y \left[ T_c^{\frac{1}{2-\gamma}} \right].
$$
Now notice that if \( c \in J \) and \( y \in I \), then \( c - y \geq \min_{z_1 \in I, \ z_2 \in J} |z_1 - z_2| = \varepsilon(I, J) \). Hence

\[
E_y \left[ T_c - \frac{1}{\varepsilon(I, J)} \right] = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} (c - y)s^{-\frac{3}{2}}e^{-\frac{(c-y)^2}{2s}} ds
\]

\[
\leq \sqrt{2} e^{-\frac{1}{2\gamma}} \left( \frac{4}{2 - \gamma} \right)^{\frac{1}{2\gamma}} (c - y)^{-\frac{2}{2\gamma}} \int_0^{+\infty} \frac{(c - y)}{\sqrt{4\pi s^3}} e^{\frac{(c-y)^2}{4s}} ds
\]

\[
\leq \sqrt{2} e^{-\frac{1}{2\gamma}} \left( \frac{4}{2 - \gamma} \right)^{\frac{1}{2\gamma}} (\varepsilon(I, J))^{-\frac{2}{2\gamma}},
\]

that ends the proof.

It is relevant that, for \( \varphi(t) \equiv c \), we can explicitly determine \( q_\varphi \).

**Proposition 4.5.** Suppose Assumption (A2) holds and \( \varphi(t) = c \) for all \( t \geq 0 \). Then, for \( t > 0 \) and \( x, y < c \) it holds

\[
q_\varphi(t, x; y) = p_\varphi(t, x; y) - p_\varphi(t, 2c - x; y)
\]

**Proof.** Let

\[
B^\dagger(t) = \begin{cases} B(t) & t < T_c \\ \infty & t \geq T_c \end{cases}
\]

where \( T_c \) is defined in (4.9). Notice that

\[
B^\dagger(L(t)) = \begin{cases} X_\varphi(t) & L(t) < T_c \\ \infty & L(t) \geq T_c \end{cases}
\]

Notice that \( L(t) < T_c \) if and only if \( \sigma(T_c) > t \). Furthermore, since \( \sigma \) is stochastically continuous and \( T_c \) is independent of it, \( \sigma(T_c) = \sigma(T_c) \) a.s. Hence, a.s.

\[
B^\dagger(L(t)) = X_\varphi(t) \text{ a.s.}
\]

Now let \( q : \mathbb{R}^+ \times \mathbb{R}^2 \rightarrow \mathbb{R} \) be such that for all \( A \in \mathcal{B}(\mathbb{R}) \) it holds

\[
P_y(B^\dagger(t) \in A) = \int_A q(t, x; y) dx.
\]

Recall also that for \( x, y < c \) (see [36, Problem 8.6])

\[
q(t, x; y) = p(t, x; y) - p(t, 2c - x; y).
\]

Hence, by a simple conditioning argument, recalling that \( L(t) \) and \( B^\dagger \) are independent, we have, if \( A \subset (-\infty, c) \) and \( y < c \),

\[
P_y(X^\dagger_\varphi(t) \in A) = \int_A \int_0^{+\infty} q(s, x; y) f_L(s; t) ds dx = \int_A (p_\varphi(t, x; y) - p_\varphi(t, 2c - x; y)) dx
\]

that ends the proof.

**Remark 4.6.** The same result can be obtained by employing the reflection principle in Proposition 4.3 together with the fact that \( X_\varphi \) is Markov in \( T_c \), as a consequence of Lemma 3.6. In this case, the proof proceeds exactly as in the classical case of the Brownian motion.

### 4.2. The non-decreasing and continuous boundary case.

In this section we discuss some regularity properties of \( \mu_\varphi(\cdot; y) \) and \( q_\varphi \) for a more general moving boundary \( \varphi \). First of all, we prove the following regularity result concerning \( \mu_\varphi(\cdot; y) \).

**Theorem 4.7.** Suppose that Assumptions (A2) and (A2) hold. Then the law \( \mu_\varphi(\cdot; y) \) of \( T \) under \( P_y \) is absolutely continuous with respect to the Lebesque-Stieltjes measure \( d(\varphi + t) \), where \( \varphi(t) = t \) for any \( t \geq 0 \). Furthermore, if \( \varphi \) is absolutely continuous, then \( T \) is an absolutely continuous random variable and, if \( \varphi \) is locally Lipschitz, the density \( p_T(\cdot; y) \) of \( T \) satisfies

\[
S_T(I, T) = \sup_{t \in [0, T], y \in I} p_T(t, y) < \infty.
\]
Proof. Throughout the proof, we denote \( J = [\varphi(0), \varphi(T)] \). First of all, notice that, since \( T \) is a continuous random variable, then \( \mu_\varphi([s, t]; y) = \mu_\varphi((s, t]; y) = \mu_\varphi((s, t]; y) = \mu_\varphi((s, t]; y) \) for all \( 0 < a < b \). Hence, let us focus on the measure \( \mu_\varphi((s, t]; y) \). Now let \( \omega \in \Omega \) such that \( T(\omega) \in (s, t] \). Then \( \mathcal{T}_\varphi(\omega) \leq T(\omega) \leq \mathcal{T}_\varphi(t)(\omega) \).

Hence \( \mathcal{T}_\varphi(\omega) \leq t \) and \( \mathcal{T}_\varphi(t)(\omega) > s \). Hence

\[
\mu_\varphi((s, t]; y) \leq \mathbb{P}_y(\mathcal{T}_\varphi(s) \leq t, \mathcal{T}_\varphi(t) > s) = \mathbb{P}_y(s \leq \mathcal{T}_\varphi(s) \leq t, \mathcal{T}_\varphi(t) > s) + \mathbb{P}_y(\mathcal{T}_\varphi(s) < s, \mathcal{T}_\varphi(t) > s)
\]

Concerning \( P_1(t, s) \) we have

\[
P_1(t, s) \leq \mathbb{P}_y(s \leq \mathcal{T}_\varphi(s) \leq t) \leq S(I, J)(t - s).
\]

Now we handle \( P_2(t, s) \). We have

\[
P_2(t, s) = E_y \left[ 1_{[0, s]}(\mathcal{T}_\varphi(s)) E_y \left[ 1_{[s - \mathcal{T}_\varphi(s), +\infty)}(\mathcal{T}_\varphi(t) - \mathcal{T}_\varphi(s)) | \mathcal{G}_{\mathcal{T}_\varphi(s)} \right] \right].
\]

By Lemma 3.6, we know that \( X_\varphi \) satisfies the Markov property in \( \mathcal{T}_\varphi(s) \) and \( (X_\varphi(t), H(t) - t) \) is time-homogeneous, then, recalling that \( H(\mathcal{T}_\varphi(s)) = \mathcal{T}_\varphi(s) \), we get

\[
E_y \left[ 1_{[s - \mathcal{T}_\varphi(s), +\infty)}(\mathcal{T}_\varphi(t) - \mathcal{T}_\varphi(s)) | \mathcal{G}_{\mathcal{T}_\varphi(s)} \right] = G(\mathcal{T}_\varphi(s); s),
\]

where, since \( \mathcal{T}_\varphi(t) \) is absolutely continuous, for \( \tau \in [0, s) \)

\[
G(\tau; s) := E_y[1_{[\tau, s)}(\mathcal{T}_\varphi(t))] = \mathbb{P}_\varphi(s)(\mathcal{T}_\varphi(t) \geq s - \tau)
\]

\[
= 1 - \mathbb{P}_\varphi(s)(\mathcal{T}_\varphi(t) \leq s - \tau) = 1 - 2 \mathbb{P}_0(X_\varphi(s - \tau) > \varphi(t) - \varphi(s))
\]

where we used the reflection principle. However, notice also that

\[
2 \mathbb{P}_0(X_\varphi(s - \tau) > \varphi(t) - \varphi(s)) = \mathbb{P}_0(X_\varphi(s - \tau) > \varphi(t) - \varphi(s)) + \mathbb{P}_0(X_\varphi(s - \tau) < \varphi(s) - \varphi(t))
\]

\[
= \mathbb{P}_0([X_\varphi(s - \tau) > \varphi(t) - \varphi(s)])
\]

and then

\[
G(\tau; s) = \mathbb{P}_0([X_\varphi(s - \tau) \leq \varphi(t) - \varphi(s)]) = \int_{\varphi(s) - \varphi(t)}^{\varphi(t) - \varphi(s)} p_\varphi(s - \tau, z; 0) dz.
\]

As a consequence, we finally achieve

\[
P_2(t, s) = \int_0^s \int_{\varphi(s) - \varphi(t)}^{\varphi(t) - \varphi(s)} p_\varphi(s - \tau, z; 0) \mathcal{P}_{\mathcal{T}_\varphi(s)}(\tau; y) dz dr.
\]

By (4.10), we get

\[
P_2(t, s) \leq \frac{\sqrt{2}S(I, J)}{\sqrt{\pi}} (\varphi(t) - \varphi(s)) \int_0^s U^{-\frac{1}{2}}(s - \tau) d\tau \leq P_2(t, s)
\]

\[
\leq \frac{\sqrt{2}S(I, J)}{\sqrt{\pi}} (\varphi(t) - \varphi(s)) \int_0^T U^{-\frac{1}{2}}(\tau) d\tau < \infty,
\]

where we used that

\[
p_\varphi(s - \tau, z; 0) = E_0 \left[ \frac{1}{\sqrt{2\pi L(s - \tau)}} e^{-\frac{x^2}{2L(s - \tau)}} \right] \leq \frac{U^{-\frac{1}{2}}(s - \tau)}{\sqrt{2\pi}}
\]

and \( U^{-\frac{1}{2}} \in L^1_{\text{loc}}(\mathbb{R}_0^+) \) by Proposition 2.7.

Resuming, we have shown that for all \( T > 0 \) and any compact \( I \subset (-\infty, \varphi(0)) \), it holds

\[
\mu_\varphi((s, t]; y) \leq C(I, T)(t - s + \varphi(t) - \varphi(s)), \forall 0 \leq s < t \leq T, \ y \in I.
\]

This also shows that, setting \( i(t) = t \) and \( d(\varphi + i) \) the distributional derivative of \( \varphi + i \), which is a Radon measure, it holds \( \mu_\varphi(; y) \) is absolutely continuous with respect to \( d(\varphi + i) \) and the Radon-Nikodym derivative of \( \mu_\varphi(; y) \) with respect to \( d(\varphi + i) \) is a locally bounded function. In particular, if \( \varphi \) is an absolutely continuous
function, then $T$ is an absolutely continuous random variable and if $\varphi$ is locally Lipschitz then its density $p_T(t; y)$ satisfies for any $T > 0$ and any compact set $I \subset (-\infty, \varphi(0))$

$$\sup_{t \in [0, T], y \in I} p_T(t; y) \leq C(I, T)(1 + \text{Lip}(\varphi; T)) < \infty,$$

where

$$\text{Lip}(\varphi; T) = \sup_{t, s \in [0, T] \atop t \neq s} \frac{|\varphi(t) - \varphi(s)|}{|t - s|}.$$

With the previous result in mind, we can prove the continuity of $q_{\varphi}(.; \cdot; y)$.

**Theorem 4.8.** Suppose that Assumptions (A2) and (A2) hold and $\varphi$ is non-decreasing, continuous and locally Lipschitz. Then $q_{\varphi}(.; \cdot; y)$ is continuous on $\mathbb{R}^+ \times \mathbb{R}$ and $q_{\varphi}(t, x; y) = 0$ for all $x \geq \varphi(t)$ and $t > 0$.

**Proof.** Let us denote

$$r_{\varphi}(t, x; y) = \int_0^t p_{\varphi}(t - w, x; \varphi(w)) p_T(w; y) \, dw, \ (t, x) \in \mathbb{R}^+ \times \mathbb{R},$$

where the density $p_T(\cdot; y)$ exists by Theorem 4.7. It is clear that if $r_{\varphi}(\cdot, \cdot; y)$ is continuous in $(t, x)$, then also $q_{\varphi}(\cdot, \cdot; y)$ is continuous in $(t, x)$ and vice versa. First, consider the case of a point $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ with $x \neq \varphi(t)$. Then, if we consider a sequence $(t_n, x_n) \to (t, x)$, it is not difficult to check that there exists a compact set $K$ such that $(0, 0) \not\in K$ and $(t_n - w, x_n - \varphi(w)) \in K$ for all $n \in \mathbb{N}$ and $w \in [0, t_n]$. Hence we have

$$p_{\varphi}(t_n - w, x_n; \varphi(w)) \leq \sup_{(\tau, \xi) \in K} p_{\varphi}(\tau, \xi; 0)$$

and then $\lim_{n \to \infty} r_{\varphi}(t_n, x_n; y) = r_{\varphi}(t, x; y)$ by the dominated convergence theorem.

Now we need to handle the case $(t, x) = (t, \varphi(t))$. This requires several steps. First of all, let $x' \in \mathbb{R}$ and consider

$$|r_{\varphi}(t, x; y) - r_{\varphi}(t, \varphi(t); y)|$$

$$\leq \int_0^t \int_0^{\infty} |p(s, x', \varphi(w)) - p(s, \varphi(t); \varphi(w))| f_L(s; t - w) p_T(w; y) \, ds \, dw$$

$$\leq S_T(\{y\}, T) \int_0^t \int_0^{\infty} \left( \int_{m(w)}^{M(w)} \frac{|\xi|}{\sqrt{2\pi s^2}} e^{-\frac{\xi^2}{2s}} \, d\xi \right) f_L(s; t - w) \, ds \, dw,$$

where $m(w) = (x' - \varphi(w)) \wedge (\varphi(t) - \varphi(w))$, $M(w) = (x' - \varphi(w)) \vee (\varphi(t) - \varphi(w))$ and we used (4.12). Now fix $\varepsilon \in (0, \frac{1}{2})$ and recall that, for any $\lambda > 0$ and $\tau \geq 0$, it holds

$$\tau^{1-\varepsilon} e^{-\lambda \tau} \leq \left( \frac{1 - \varepsilon}{\lambda} \right)^{1-\varepsilon} e^{-(1-\varepsilon)}.$$

Using $\tau = s^{-1}$ and $\lambda = \xi^2$, we get

$$\frac{|\xi|}{\sqrt{2\pi s^2}} e^{-\frac{\xi^2}{2s}} \leq \frac{2^{1-\varepsilon}(1 - \varepsilon)^{1-\varepsilon} e^{-(1-\varepsilon)}}{\sqrt{2\pi s^2}}.$$ 

Notice that $M(w) > 0$ for $w \in (0, t)$ and let us distinguish among two cases. If $m(w) \geq 0$, that is to say $x' - \varphi(w) \geq 0$ then

$$\int_{m(w)}^{M(w)} \frac{|\xi|}{\sqrt{2\pi s^2}} e^{-\frac{\xi^2}{2s}} \, d\xi \leq \frac{2^{1-\varepsilon}(1 - \varepsilon)^{1-\varepsilon} e^{-(1-\varepsilon)}}{2\varepsilon \sqrt{2\pi s^2}} (M(w)^{2\varepsilon} - m(w)^{2\varepsilon})$$

$$\leq \frac{2^{1-\varepsilon}(1 - \varepsilon)^{1-\varepsilon} e^{-(1-\varepsilon)}}{2\varepsilon \sqrt{2\pi s^2}} |x' - \varphi(t)|^{2\varepsilon}.$$
If instead \( m(w) < 0 \), then \( m(w) = x' - \varphi(w) \), \( M(w) = \varphi(t) - \varphi(w) \) and we have
\[
\int_{m(w)}^{M(w)} \frac{|\xi|}{\sqrt{2\pi \sigma^2}} e^{-\frac{x^2}{2\sigma^2}} d\xi \leq \frac{2^{1-\varepsilon}(1-\varepsilon)^{1-\varepsilon}}{\varepsilon \sqrt{2\pi \sigma^2}} 
\left( \int_0^{\varphi(t)-\varphi(w)} \xi^{-1+2\varepsilon} + \int_0^{\varphi(w)-x'} \xi^{-1+2\varepsilon} \right)
= \frac{2^{1-\varepsilon}(1-\varepsilon)^{1-\varepsilon}}{\varepsilon \sqrt{2\pi \sigma^2}} \epsilon \left( (\varphi(t) - \varphi(w))^{2\varepsilon} + (\varphi(w) - x')^{2\varepsilon} \right)
\leq \frac{2^{1-\varepsilon}(1-\varepsilon)^{1-\varepsilon}}{\varepsilon \sqrt{2\pi \sigma^2}} \epsilon 2^{1-2\varepsilon} |x' - \varphi(t)|^{2\varepsilon}.
\]
Hence, in general
\[
\int_{m(w)}^{M(w)} \frac{|\xi|}{\sqrt{2\pi \sigma^2}} e^{-\frac{x^2}{2\sigma^2}} d\xi \leq \frac{2^{1-3\varepsilon}(1-\varepsilon)^{1-\varepsilon}}{\varepsilon \sqrt{2\pi \sigma^2}} |x' - \varphi(t)|^{2\varepsilon}.
\]
Going back to (4.13), we get
\[
|\varphi(t, x; y) - \varphi(t, \varphi(t); y)|
\leq S_T(|y|, T) \frac{2^{1-3\varepsilon}(1-\varepsilon)^{1-\varepsilon}}{\varepsilon \sqrt{2\pi}} |x' - \varphi(t)|^{2\varepsilon} \int_0^t U_{\frac{1}{2}-\varepsilon} (t - w) \, dw
\leq S_T(|y|, T) \frac{2^{1-3\varepsilon}(1-\varepsilon)^{1-\varepsilon}}{\varepsilon \sqrt{2\pi}} |x' - \varphi(t)|^{2\varepsilon} \int_0^T U_{\frac{1}{2}-\varepsilon} (t - w) \, dw < \infty.
\]
by Proposition 2.7. This proves that for all \( \varepsilon \in (0, \frac{1}{2}) \), \( y < \varphi(0) \) and \( T > 0 \) there exists a constant \( C_{\varepsilon}(y; T) \) such that
\[
|\varphi(t, x; y) - \varphi(t, \varphi(t); y)| \leq C_{\varepsilon}(y; T) |x' - \varphi(t)|
\]  
for \( t \in (0, T) \) and \( x' \in \mathbb{R} \). Furthermore, this guarantees that for fixed \( t > 0 \) the function \( \varphi(t, :: y) \) is continuous in \( \varphi(t) \) and thus also \( q_{\varphi}(t, :: y) \).

Now we prove that \( q_{\varphi}(t, x; y) = 0 \) for all \( x > \varphi(t) \). Indeed, fix \( t > 0 \) and let \( x > \varphi(t) \). Consider \( \delta > 0 \) such that \( x - \delta > \varphi(t) \) and let \( A = [x - \delta, x + \delta] \). Then
\[
\frac{1}{2\delta} \int_{x-\delta}^{x+\delta} q_{\varphi}(t, z; y) \, dz = \frac{1}{2\delta} \Phi_y(X_{\varphi}(t) \in A) = 0
\]
taking the limit as \( \delta \to 0 \), by continuity of \( q_{\varphi}(t, :: y) \) in \( x \), we have \( q_{\varphi}(t, x; y) = 0 \) for all \( x > \varphi(t) \). Then \( q_{\varphi}(t, \varphi(t); y) = 0 \) follows again by continuity.

Now we are ready to prove that \( \varphi(t, :: y) \) is continuous in \( (t, \varphi(t)) \). For this, let \((t_n, x_n) \to (t, \varphi(t))\) and observe that
\[
\varphi(t_n, \varphi(t_n); y) = p_{\varphi}(t_n, \varphi(t_n); y) - q_{\varphi}(t_n, \varphi(t_n); y) = p_{\varphi}(t_n, \varphi(t_n); y).
\]
Taking the limit, we have
\[
\lim_{n \to \infty} \varphi(t_n, \varphi(t_n); y) = \lim_{n \to \infty} p_{\varphi}(t_n, \varphi(t_n); y) = p_{\varphi}(t, \varphi(t); y) = \varphi(t, \varphi(t); y).
\]
Hence, it is sufficient to prove that
\[
\lim_{n \to \infty} |\varphi(t_n, x_n; y) - \varphi(t_n, \varphi(t_n); y)| = 0.
\]
However, since \( t_n \to t \), there exists \( T > 0 \) such that \( t_n, t < T \) for all \( n \in \mathbb{N} \) and then, by (4.14), for some \( \varepsilon < \frac{1}{2} \),
\[
|\varphi(t_n, x_n; y) - \varphi(t_n, \varphi(t_n); y)| \leq C_{\varepsilon}(y; T) |x_n - \varphi(t_n)| \to 0,
\]
ending the proof.

Concerning the continuity in the \( y \) variable, we need some further assumption on the moving boundary and on \( \Phi \).

**Theorem 4.9.** Suppose Assumptions (A2) and (A2) hold. Let \( \varphi : \mathbb{R}^+_0 \to \mathbb{R} \) be non-decreasing, continuous and locally Lipschitz. Assume further that
\[
\lim_{t \to \infty} \sup(X_{\varphi}(t) - \varphi(t)) = +\infty, \quad \int_1^{+\infty} (\varphi(z + t) - \varphi(z)) \nu(dt) < \infty, \forall z > 0,
\]  
(4.15)
and either \( \varphi \) is concave or \( \lim_{\lambda \to \infty} \frac{\Phi(\lambda)}{\Phi'(\lambda)} < 2 \). Then \( q_\varphi : \mathbb{R}^+ \times \mathbb{R} \times (-\infty, \varphi(0)) \to \mathbb{R}^+_0 \) is continuous.

Proof. Let us first show that for fixed \( (t, x) \in \mathbb{R}^+ \times \mathbb{R} \) the function \( y \in (-\infty, \varphi(0)) \mapsto q_\varphi(t, x; y) \) is continuous. If \( x \geq \varphi(t) \), this is obvious, hence let us assume that \( x < \varphi(t) \). Arguing as in Theorem 4.8, we only have to show that \( r_\varphi(t, x; \cdot) \) is continuous in \( y \), by Helly’s Theorem (e.g., [40, Theorem 4., page 370]), since \( \mathcal{T} \) is a continuous random variable by Lemma 4.1, it will be sufficient to prove that \( \mathbb{P}_{y_n} (\mathcal{T} \leq t) \to \mathbb{P}_y (\mathcal{T} \leq t) \) for every \( t \geq 0 \) and \( \{y_n\}_{n \in \mathbb{N}} \subset (-\infty, \varphi(0)) \) such that \( y_n \to y \in (-\infty, \varphi(0)) \).

Note that

\[
\mathbb{P}_y (\mathcal{T} \leq t) = \mathbb{P}_0 (\mathcal{T}^y \leq t) \tag{4.16}
\]

where

\[
\mathcal{T}^y := \inf \{ t \geq 0 : y + X_\varphi(t) - \varphi(t) > 0 \}. \tag{4.17}
\]

Therefore we have to prove that

\[
\mathbb{P} (\mathcal{T}^{y_n} \leq t) \to \mathbb{P} (\mathcal{T}^y \leq t) \tag{4.18}
\]

whenever \( y_n \to y \). To do this, we proceed as follows. Let \( D(\mathbb{R}^+_0) \) be the space of real valued càdlàg functions and let \( D_\infty(\mathbb{R}^+_0) \subset D(\mathbb{R}^+_0) \) the set containing elements \( d \in D(\mathbb{R}^+_0) \) such that \( \lim_{t \to +\infty} \sup_{0 \leq s \leq t} d(s) = \infty \).

Since \( \limsup_{t \to \infty} X_\varphi(t) - \varphi(t) = +\infty \) a.s., then for any \( n \in \mathbb{N} \) it holds \( y_n + X_\varphi(\cdot) - \varphi(\cdot) \in D_\infty(\mathbb{R}^+_0) \) a.s. Furthermore, it is clear that \( y + X_\varphi(\cdot) - \varphi(\cdot) \in C(\mathbb{R}^+_0) \) almost surely and that \( y_n + X_\varphi(\cdot) - \varphi(\cdot) \to y + X_\varphi(\cdot) - \varphi(\cdot) \) almost surely in the uniform topology of \( C(\mathbb{R}^+_0) \), thus implying that the convergence also holds in \( D_\infty(\mathbb{R}^+_0) \) with Skorokhod’s \( M \) topology (see [74, Chapter 12]).

Now we show that the distribution of \( \max_{0 \leq s \leq t} y + X_\varphi(s) - \varphi(s) \) cannot admit an atom at 0. Indeed

\[
\mathbb{P} \left( \max_{0 \leq w \leq t} (y + X_\varphi(w) - \varphi(w)) = 0 \right)
\leq \mathbb{P} (y + X_\varphi(w) - \varphi(w) \leq 0, \forall w \in [\mathcal{T}^y, t], \mathcal{T}^y \leq t)
\]

\[
= \int_0^t \mathbb{P} (y + X_\varphi(w) - \varphi(w) \leq 0, \forall w \in [z, t] \mid \mathcal{T}^y = z) \mathbb{P} (\mathcal{T}^y \in dz). \tag{4.19}
\]

We recall that, by Lemma 3.6, \( X_\varphi \) satisfies the strong Markov property at \( \mathcal{T}^y \), hence

\[
\mathbb{P} (y + X_\varphi(w) - \varphi(w) \leq 0, \forall w \in [z, t] \mid \mathcal{T}^y = z)
= \mathbb{P}_{\varphi(z)-y} (y + X_\varphi(w) - \varphi(w+z) \leq 0, \forall w \in [0, t-z])
\leq \mathbb{P}_{\varphi(z)-y} (\exists \tau > 0 : \forall t \in [0, \tau], y + X_\varphi(t) - \varphi(z+t) \leq 0). \tag{4.20}
\]

Now, since (4.15) holds, if \( \lim_{\lambda \to \infty} \frac{\Phi(\lambda)}{\Phi'(\lambda)} < 2 \) we use a restatement of [15, Proposition III.10] in terms of the inverse of the subordinator (as for instance done in [16, Proposition 4.4] for [15, Theorem III.9]), otherwise if \( \varphi \) is concave, then \( t \in \mathbb{R}^+ \mapsto \frac{\varphi(z+t) - \varphi(z)}{t} \in \mathbb{R} \) is non-increasing and we can use [16, Proposition 4.4]. In both cases, we get

\[
\lim_{t \to 0} \frac{L(t)}{\varphi(z+t) - \varphi(z)} = +\infty, \text{ a.s.,} \tag{4.21}
\]

from which it follows that, for almost all \( \omega \in \Omega \), there exists \( \tau_0(\omega) \) such that \( \varphi(t+z) - \varphi(z) \leq L(t, \omega) \), for all \( t \in [0, \tau_0(\omega)] \), that implies that, a.s., \( Y(t, \omega) - \varphi(t+z) \geq y + X_\varphi(t, \omega) - \varphi(z) - L(t, \omega) \), for all \( t \in [0, \tau_0(\omega)] \).

Therefore, using (4.20), we have that

\[
\mathbb{P} (y + X_\varphi(w) - \varphi(w) \leq 0, \forall w \in [z, t] \mid \mathcal{T}^y = z)
\leq \mathbb{P}_{\varphi(z)-y} (\exists \tau : \forall t \in [0, \tau], y + X_\varphi(t) - \varphi(z+t) \leq 0)
\leq \mathbb{P}_{\varphi(z)-y} (\exists \tau : \forall t \in [0, \tau], y + X_\varphi(t) - L(t) - \varphi(z) \leq 0)
= \mathbb{P}_{\varphi(z)} (\exists \tau : \forall t \in [0, \tau], B(L(t)) - L(t) \leq \varphi(z)). \tag{4.22}
\]

Now notice that

\[
\{B(L(t, \omega), \omega) - L(t, \omega) : t \in [0, \tau_0(\omega)]\} = \{B(t, \omega) - t : t \in [0, L(\tau_0(\omega), \omega)]\} \tag{4.23}
\]
hence

\[ P_{\varphi(z)} (\exists \varepsilon < \varepsilon_0 : \forall t \in [0, \varepsilon], B(L(t)) - L(t) \leq \varphi(z)) \]

\[ = P_{\varphi(z)} (\exists \varepsilon < L(\varepsilon_0) : \forall t \in [0, \varepsilon], B(t) - t \leq \varphi(z)) \]

\[ = E_0 \left[ P_{\varphi(z)} (\exists \varepsilon < L(\varepsilon_0) : \forall t \in [0, \varepsilon], B(t) - t \leq \varphi(z)) \mid \{ L(t), t \geq 0 \} \right] = 0, \]

where we notice that \( \varepsilon_0 \) is a functional of \( \{ L(t), t \geq 0 \} \) and \( L(\varepsilon_0) > 0 \). This proves that (4.22) is zero.

Now we show that \( y + X_\Phi(t) - \varphi(t) \) does not assume the value 0 in the interval \( (T^y - \varepsilon, T^y) \) a.s. Indeed, if for \( \omega \in \Omega \) there exists \( t < T^y \) such that \( y + X_\Phi(t, \omega) - \varphi(t) = 0 \), then \( \max_{0 \leq s \leq t} y + X_\Phi(s, \omega) - \varphi(s) = 0 \) for any \( t \leq z < T^y \) and, in particular, we can suppose \( z \in \mathbb{Q} \). Thus, we have

\[ P(\exists t < T^y : y + X_\Phi(t) - \varphi(t) = 0) \leq P(\exists z < T^y, z \in \mathbb{Q} : \max_{0 \leq s \leq z} y + X_\Phi(s) - \varphi(s) = 0) \]

\[ \leq \sum_{z \in \mathbb{Q}} P(\max_{0 \leq s \leq z} y + X_\Phi(s) - \varphi(s) = 0) = 0, \]

where the latter equality holds since the distribution of \( \max_{0 \leq s \leq t} y + X_\Phi(s) - \varphi(s) \) does not admit any atom at 0 for any \( t > 0 \).

Finally, once this has been verified, we have that \( P_0(T^{\varphi} \leq t) \rightarrow P_0(T^y \leq t) \) for all \( t > 0 \) by [74, Theorem 13.6.4].

It is worth noticing that actually \( P_0(T^{\varphi} \leq t) \rightarrow P_0(T^y \leq t) \) uniformly, see Appendix A.5.

Now we are ready to show that \( g_\Phi \) is continuous on \( \mathbb{R}^+ \times \mathbb{R} \times (-\infty, \varphi(0)) \). To do this, fix \( (t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times (-\infty, \varphi(0)) \) and let \( \{ (t_n, x_n, y_n) \}_{n \in \mathbb{N}} \subset \mathbb{R}^+ \times \mathbb{R} \times (-\infty, \varphi(0)) \) be such that \( (t_n, x_n, y_n) \rightarrow (t, x, y) \). Let us first observe that if \( x > \varphi(t) \), then we can consider \( \varepsilon > 0 \) such that \( x - \varphi(t) > \varepsilon \). Without loss of generality, we can assume that \( x_n - \varphi(t) > \frac{\varepsilon}{2} \) and, by continuity of \( \varphi \), \( \varphi(t_n) - \varphi(t) < \frac{\varepsilon}{4} \), so that

\[ x_n - \varphi(t_n) = x_n - \varphi(t) - (\varphi(t_n) - \varphi(t)) > \frac{\varepsilon}{4}. \]

As a consequence, \( g_\Phi(t_n, x_n; y_n) = 0 = g_\Phi(t, x; y) \) for all \( n \in \mathbb{N} \). Next, assume that \( x < \varphi(t) \). Then, it is sufficient to show that \( g_\Phi \) is continuous in \((t, x, y)\). To do this, notice that

\[ |r_\Phi(t_n, x_n; y_n) - r_\Phi(t, x; y)| \]

\[ \leq |r_\Phi(t_n, x_n; y_n) - r_\Phi(t, x; y_n)| + |r_\Phi(t, x; y_n) - r_\Phi(t, x; y)| = S_{n}^1 + S_{n}^2. \]

We have already shown that \( \lim_{n \rightarrow \infty} S_{n}^2 = 0 \). To prove that \( \lim_{n \rightarrow \infty} S_{n}^1 = 0 \), let us first consider the case in which \( t_n \uparrow t \). Then

\[ S_{n}^1 \leq \int_0^{t_n} |p_\Phi(t-w, x_n; \varphi(w)) - p_\Phi(t-w, x; \varphi(w))| P_{y_n}(\mathcal{T} \in dw) \]

\[ + \int_{t_n}^t p_\Phi(t-w, x; \varphi(w)) P_{y_n}(\mathcal{T} \in dw) = I_{n}^1 + I_{n}^2. \]

Since \( x < \varphi(t) \), let \( \varphi(t) - x > \varepsilon \) we can assume, without loss of generality, that \( \varphi(t) - x_n > \frac{\varepsilon}{2} \) and, by continuity of \( \varphi \), \( \varphi(t) - \varphi(t_n) < \frac{\varepsilon}{4} \). Then

\[ \varphi(t_n) - x_n = \varphi(t) - x_n - (\varphi(t) - \varphi(t_n)) > \frac{\varepsilon}{4} > 0. \]

As a consequence, there exists a compact \( K \subset \mathbb{R}^+_0 \times \mathbb{R}^2 \) such that all the curves \( w \in [0, t_n] \rightarrow (t_n - w, x_n, \varphi(w)) \in \mathbb{R}^+_0 \times \mathbb{R}^2 \) are contained in \( K \) and \((0, 0, 0) \notin K \). Since \( p_\Phi \) is continuous in \( K \), it is also uniformly continuous, hence there exists a non-decreasing modulus of continuity \( \varpi : \mathbb{R}^+_0 \rightarrow \mathbb{R}^+_0 \) such that \( \varpi(0) = \lim_{r \downarrow 0} \varpi(r) = 0 \) and

\[ |p_\Phi(t_n-w, x_n; \varphi(w)) - p_\Phi(t-w, x; \varphi(w))| \leq \varpi(|t_n - t| + |x_n - x|). \]

Hence

\[ I_{n}^1 \leq \varpi(|t_n-t| + |x_n-x|) P_{y_n}(\mathcal{T} \leq t_n) \leq \varpi(|t_n-t| + |x_n-x|) \rightarrow 0. \]
For $I^2_n$, notice that the curve $w \in [0, t] \mapsto (t - w, \varphi(w)) \in \mathbb{R} \times \mathbb{R}$ lies in the interior of a compact set $K \subset \mathbb{R} \times \mathbb{R}$ such that $(0, 0) \not\in K$. Hence

$$I^2_n \leq \max_{(s, \xi, \eta) \in K} p_{\varphi}(s, \xi; \eta)(P_y(t) - P_y(t)).$$

Since we have that $P_{\varphi}(T \leq \cdot)$ converge uniformly to $P_{\varphi}(T \leq \cdot)$, it is clear that $\lim_{n \to \infty} I^2_n = 0$. The same argument holds if $t_n \downarrow t$. Notice that this is enough to prove that $\lim_{n \to \infty} S^1_n = 0$. Indeed, let $\{(t_n, x_n; y_n)\}_{n \in \mathbb{N}}$ be the subsequence such that $\lim_{n \to \infty} S^1_n = \limsup_{n \to \infty} S^1_n$. Then we can consider a further, non-relabelled, subsequence $\{(t_n, x_n; y_n)\}_{n \in \mathbb{N}}$ such that $t_n$ converges monotonically towards $t$. Then, by the previous argument, $0 = \lim_{n \to \infty} S^1_n = \limsup_{n \to \infty} S^1_n$.

Now let $x = \varphi(t)$. If $x_n = \varphi(t_n)$, then we already know that $q_{\varphi}(t_n, \varphi(t_n); y_n) = 0 = q_{\varphi}(t, \varphi(t); y)$ and then $\lim_{n \to \infty} r_{\varphi}(t_n, \varphi(t_n); y_n) = r_{\varphi}(t, \varphi(t); y)$. In general, we split the absolute value as follows:

$$|r_{\varphi}(t_n, x_n; y_n) - r_{\varphi}(t, x; y)| \leq |r_{\varphi}(t_n, x_n; y_n) - r_{\varphi}(t_n, \varphi(t_n); y_n)| + |r_{\varphi}(t_n, \varphi(t_n); y_n) - r_{\varphi}(t, \varphi(t); y)| = S^3_n + S^4_n.$$

Clearly, $\lim_{n \to \infty} S^3_n = 0$. To handle $S^4_n$, notice that, arguing as in Theorem 4.8,

$$S^4_n \leq S_T(1, T)^{2 - 1 - 2\varepsilon} \int_0^T U_{-\frac{1}{2} - \varepsilon}(s) ds$$

where $\varepsilon \in (0, \frac{1}{2})$, $T > \max\{t_n, t\}$ for all $n \in \mathbb{N}$ and $\{y_n\}_{n \in \mathbb{N}} \subset I \subset (-\infty, \varphi(0))$ for a compact set $I$ and $U_p$ is defined in (2.19). Taking the limits as $n \to \infty$, we finally get $\lim_{n \to \infty} S^4_n = 0$.

**Remark 4.10.** Notice that (4.15) is for instance implied by asking that $\varphi$ is bounded. Other conditions that imply the first limit in (4.15) can be found by means of the law of the iterated logarithm for $X_{\varphi}(t)$, as given in [44, Theorem 4]. Let us also observe that the condition $\limsup_{\lambda \to \infty} \frac{\Phi(2\lambda)}{\lambda^{1/2}} < 2$ is, for instance, implied by asking that $\Phi$ is regularly varying at infinity of order $\alpha \in [0, 1)$. Furthermore, it is also equivalent to $\limsup_{\lambda \to \infty} \frac{\Phi(\lambda)}{\lambda^{1/2}} < 1$ (see [15, Exercise III.6]).

Concerning the behaviour of $q_{\varphi}$ as $t \downarrow 0$, we have the following result.

**Proposition 4.11.** Suppose Assumptions (A2) and (A2). Assume further that $\varphi$ is non-decreasing and locally Lipschitz. For any fixed $y \in (-\infty, \varphi(0))$ it holds $\lim_{t \to 0} q_{\varphi}(t, x; y) = 0$ locally uniformly with respect to $x \in \mathbb{R} \setminus \{y\}$.

**Proof.** Recall that, for $t \leq 1$,

$$r_{\varphi}(t, x; y) = \int_0^t p_{\varphi}(t - w, x; \varphi(w)) P_y(T \in dw) \leq S_T((y, 1)) \int_0^T U_{-\frac{1}{2}}(w) dw$$

where $U_p$ is defined in (2.19). Hence $\lim_{t \to 0} r_{\varphi}(t, x; y) = 0$ uniformly in $x$. Recall, however, that we have $\lim_{t \to 0} p_{\varphi}(t, x; y) = 0$ locally uniformly with respect to $x \in \mathbb{R} \setminus \{y\}$, as shown in Proposition 3.1.

**Remark 4.12.** Actually, with the same proof, we have $\lim_{t \to 0} q_{\varphi}(t, x; y) = 0$ locally uniformly with respect to $(x, y) \in (\mathbb{R} \times (-\infty, \varphi(0))) \setminus \text{diag}(\mathbb{R}^2)$.

The latter allows us to define, for $x \neq y$, $q_{\varphi}(0, x; y) = 0$. In next proposition we study the regularity in the $x$ variable of the sub-probability density $q_{\varphi}(\cdot, \cdot; y)$.

**Proposition 4.13.** Suppose Assumption (A2) holds. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous, non-decreasing and such that if there exists $0 \leq t_1 < t_2 < \infty$ such that $\varphi(t_1) = \varphi(t_2)$ then $\varphi(t_1) = \varphi(t)$ for all $t \geq t_1$. Then $\partial_{x} q_{\varphi}(t, x; y)$ is well-defined and continuous in any $(t, x)$ such that $x \neq \varphi(t)$ and $t > 0$. Then, for $x \neq \varphi(t)$ with $x \neq y$,

$$\partial_{x} q_{\varphi}(t, x; y) = \partial_{x} p_{\varphi}(t, x; y) - \int_0^t \partial_{x} p_{\varphi}(t - w, x; \varphi(w)) P_y(T \in dw).$$

If furthermore Assumptions (A2) and (A3) hold, then $\partial_{x}^2 q_{\varphi}(t, x; y)$ is well-defined and continuous in any $(t, x)$ such that $x \neq \varphi(t)$ and $t > 0$, and, for $x \neq \varphi(t)$ with $x \neq y$,

$$\partial_{x}^2 q_{\varphi}(t, x; y) = \partial_{x}^2 p_{\varphi}(t, x; y) - \int_0^t \partial_{x}^2 p_{\varphi}(t - w, x; \varphi(w)) P_y(T \in dw).$$
Proof. The case \( x > \varphi(t) \) is trivial. We only need to show the statement for \( r_\Phi \). To do this, notice that we can construct a compact set \( K \subset \mathbb{R}_+^n \times \mathbb{R} \) such that \( w \in [0, t] \mapsto (t - w, x - \varphi(w)) \in \mathbb{R}_+^n \times \mathbb{R} \) lies in \( K \) and \((0, 0) \notin K \). According to Proposition 3.1, \( \partial_x p_\Phi(t, \cdot, \cdot) \) is defined in \( K \setminus \{(t, 0), \ t \geq 0\} \), where \( |K \cap \{(t, 0), \ t \geq 0\}| = 0 \), and belongs to \( L^\infty(K) \). Hence \( \partial_x p_\Phi(t - w, x; \varphi(w)) \) is well-defined for all \( w \in [0, t] \) such that \( x \neq \varphi(w) \). However, notice that if \( x \in [\varphi(0), \varphi(t)] \) then the preimage \( \varphi^{-1}(x) = \{w_x\} \) for some \( w_x \in [0, t) \), while if \( x < \varphi(0) \), then \( \varphi^{-1}(x) = \emptyset \). In any case, \( \mathbb{P}_y(T \in \varphi^{-1}(x)) = 0 \) since \( T \) is a continuous random variable and then we can state that \( \partial_x p_\Phi(t - w, x; \varphi(w)) \) is well-defined for \( \mathbb{P}_y(T \in dw) \)-a.a. \( w \in [0, t] \). Furthermore, for fixed \( w \in [0, t] \) such that \( x \neq \varphi(w) \), the function \( \partial_x p_\Phi(t - w, \cdot; \varphi(w)) \) is continuous in a neighbourhood of \( x \) and, according to the previous argument, it is bounded by \( \sup_{(t, x) \in K} |\partial_x p_\Phi(t, x; 0)| \), which is independent of \( w \). Hence, by a simple application of the dominated convergence theorem, we have

\[
\partial_x r_\Phi(t, x; y) = \int_0^t \partial_x p_\Phi(t - w, x; \varphi(w)) \mathbb{P}_y(T \in dw).
\]

Continuity follows by exactly the same argument. Under Assumptions (A2) and (A3) and using also Proposition 3.4, the same argument holds for the second derivative. \( \square \)

Next, we investigate the action of the nonlocal operator \( \partial_t^\Phi \) on \( q_\Phi \).

Proposition 4.14. Suppose Assumption (A2), (A2) and (A3) hold. Let \( \varphi \) be non-decreasing and locally Lipschitz. Then, for \( t > 0 \) and \( x \neq y \) it holds

\[
\partial_t^\Phi q_\Phi(t, x; y) = \partial_t^\Phi p_\Phi(t, x; y) - \int_0^t \partial_t^\Phi p_\Phi(t - w, x; \varphi(w)) p_T(w; y) \, dw.
\]

Proof. Define

\[
k_\Phi(t, w, x) := 1_{\mathbb{R}_+}(t - w) \int_0^{t-w} p_\Phi(t - s - w, x; \varphi(w)) \mathbb{P}(s) \, ds. \tag{4.24}
\]

Notice that

\[
\int_0^t \mathbb{P}(s) r_\Phi(t - s, x; y) \, ds = \int_0^t \int_0^{t-s} \mathbb{P}(s) p_\Phi(t - s - w, x; \varphi(w)) p_T(w; y) \, dw \, ds = \int_0^t \int_0^{t-w} \mathbb{P}(s) p_\Phi(t - s - w, x; \varphi(w)) p_T(w; y) \, dw \, ds = \int_0^{+\infty} k_\Phi(t, w, x) p_T(w; y) \, dw.
\]

Since \( \lim_{t \to 0} r_\Phi(t, x; y) = 0 \), we have

\[
\partial_t^\Phi r_\Phi(t, x; y) = \partial_t \left( \int_0^{+\infty} k_\Phi(t, w, x) p_T(w; y) \, dw \right).
\]

Since \( x < \varphi(t) \), then either \( x < \varphi(0) \) and then \( \varphi^{-1}(x) = \emptyset \) or \( x \in [\varphi(0), \varphi(t)) \) and then \( \varphi^{-1}(x) = \{w_x\} \). For \( w \neq w_x \) and \( w \neq t \), since \( \lim_{r \to 0} p_\Phi(t, x; \varphi(w)) = 0 \), we have

\[
\partial_t k_\Phi(t, w, x) = 1_{\mathbb{R}_+}(t - w) \partial_t^\Phi p_\Phi(t - w, x; \varphi(w)). \tag{4.25}
\]

Before proceeding, let us provide a further bound on \( \partial_2^\Phi q_\Phi \). Consider two compact sets \( K_1, K_2 \subset \mathbb{R} \) with \( K_1 \cap K_2 = \emptyset \) and let \( \varepsilon = \min_{(\xi, \eta) \in K_1 \times K_2} |\xi - \eta| \). Then we have, for \( (\xi, \eta) \in K_1 \times K_2 \),

\[
|\partial_2^\Phi p_\Phi(t, \xi; \eta)| \leq \sqrt{\frac{2}{\pi}} E \left[ (L(t))^{-\frac{3}{2}} |(\xi - \eta)^2 (L(t))^{-1} - 1| e^{-\frac{(|\xi - \eta|^2)}{3\sigma^2 \tau^2}} \right] \\
\leq \sqrt{\frac{2}{\pi}} E \left[ (\xi - \eta)^2 (L(t))^{-\frac{3}{2}} e^{-\frac{(|\xi - \eta|^2)}{3\sigma^2 \tau^2}} \right] + \sqrt{\frac{2}{\pi}} E \left[ (L(t))^{-\frac{3}{2}} e^{-\frac{(|\xi - \eta|^2)}{3\sigma^2 \tau^2}} \right]. \tag{4.26}
\]

Now observe that for all \( r \geq 0 \) and \( \lambda > 0 \) it holds

\[
r^{\frac{3}{4}} e^{-\lambda r} \leq \left( \frac{5}{2e\lambda} \right)^\frac{3}{2} \quad \text{and} \quad r^{\frac{3}{2}} e^{-\lambda r} \leq \left( \frac{3}{2e\lambda} \right)^\frac{3}{2}.
\]
hence we get, setting $r = (L(t))^{-1}$ and $\lambda = \frac{(\xi - \eta)^2}{4}$, by (4.26),
\[
|\partial^2_t p_{\Phi}(t, \xi; \eta)| \leq \sqrt{\frac{2}{\pi}} \left( \left( \frac{10}{e} \right)^\frac{3}{2} + \left( \frac{6}{e} \right)^\frac{3}{2} \right) (\xi - \eta)^{-3} E \left[ e^{-\frac{(\xi - \eta)^2}{4 \pi \eta}} \right].
\]

Furthermore, by Theorem 3.2 we know that
\[
|\partial^4_t p_{\Phi}(t, \xi; \eta)| \leq \frac{1}{\sqrt{2\pi}} \left( \left( \frac{10}{e} \right)^\frac{3}{2} + \left( \frac{6}{e} \right)^\frac{3}{2} \right) \varepsilon^{-3} E \left[ e^{-\frac{\varepsilon^2}{4 \pi \eta}} \right].
\]

Going back to (4.25), notice that for $w \to t$ we have that $\varphi(w) \to \varphi(t) > x$. Hence, there exists $\delta > 0$ and two compact sets $K_1, K_2 \subset \mathbb{R}$ such that $K_1 \cap K_2 = \emptyset$, $\varphi(w) \in K_1$ for all $w \in [t - \delta, t)$ and $x \in K_2$. Hence, if $\varepsilon = \min(\xi - \eta, \eta - \delta)$, we have
\[
|\partial_t k_{\Phi}(t, w, x)| \leq \sqrt{\frac{2}{\pi}} \left( \left( \frac{10}{e} \right)^\frac{3}{2} + \left( \frac{6}{e} \right)^\frac{3}{2} \right) \varepsilon^{-3} E \left[ e^{-\frac{\varepsilon^2}{4 \pi \eta}} \right]
\]
and then
\[
\lim_{w \to t} |\partial_t k_{\Phi}(t, w, x)| = 0.
\]

If $x < \varphi(0)$, exactly the same argument guarantees that $\partial_t k_{\Phi}(t, \cdot, x)$ is bounded and continuous and that for any compact set $K \subset (0, +\infty)$ it holds $\sup_{(t, w) \in K \times \mathbb{R}^+} |\partial_t k_{\Phi}(t, w, x)| < \infty$. Hence, a simple dominated convergence argument shows that
\[
\partial^4_t r_{\Phi}(t, x; y) \, ds = \int_0^{+\infty} \partial_t k_{\Phi}(t, w, x) p_{\tau}(t; w, y) \, dw = \int_0^t \partial^4_t p_{\Phi}(t - w, x; \varphi(w)) p_{\tau}(t; w, y) \, dw.
\]

Now let us handle the case $x \in [\varphi(0), \varphi(t))$. In this case, the function $\partial_t k_{\Phi}(t, \cdot, x)$ is not defined in $w_x$, but it is still continuous in $\mathbb{R}^+ \backslash \{w_x\}$. Nevertheless, by (3.16), we know that
\[
\lim_{w \to w_x} \partial_t k_{\Phi}(t, w, x) = -\frac{1}{\sqrt{2\pi}} \int_0^{+\infty} s^{-\frac{3}{2}} \partial_s f_{\xi}(s, t) \, ds
\]
where the limit holds locally uniformly with respect to $t > 0$. Furthermore, by Assumption (A3), we know that the right-hand side of (4.29) is locally bounded with respect to $t$. As a consequence, we can extend $\partial_t k_{\Phi}(t, \cdot, x)$ by continuity in $w_x$ by setting $\partial_t k_{\Phi}(t, w, x) := -\frac{1}{\sqrt{2\pi}} \int_0^{+\infty} s^{-\frac{3}{2}} \partial_s f_{\xi}(s, t) \, ds$ and then notice that for any compact set $K \subset (0, +\infty)$ it holds $\sup_{(t, w) \in K \times \mathbb{R}^+} |\partial_t k_{\Phi}(t, w, x)| < \infty$ to obtain again (4.28). Thus, applying the operator $\partial^4_t$ on $q_{\Phi}(\cdot, x; y)$ for $x \neq y$ and using (4.2) and (4.28) we get the desired result.

\section{Main result}

We are now ready to state and prove our main result which will imply, as a particular case, the existence result in Theorem 1.1. Uniqueness will be dealt with separately.

\begin{theorem}
Suppose that (A2), (A2) and (A3) hold. Let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}$ be continuous and non-decreasing. Assume further that if there exist $t_1 < t_2$ such that $\varphi(t_1) = \varphi(t_2)$, then $\varphi(t) = \varphi(t_1)$ for all $t \geq t_1$. Let also $f \in C_c(-\infty, \varphi(0))$ and consider the time-nonlocal Cauchy-Dirichlet problem
\begin{equation}
\begin{cases}
\partial^4_t u(t, x) = \frac{1}{2} \partial^2_x u(t, x) & t > 0, \ x < \varphi(t) \\
u(t, x) = 0 & t \geq 0, \ x \geq \varphi(0) \\
u(0, x) = f(x) & x < \varphi(0) \\
\lim_{x \to -\infty} u(t, x) = 0 & \text{locally uniformly with respect to } t > 0.
\end{cases}
\end{equation}

Then there exists a function $u : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ such that, setting $E = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R} : x < \varphi(t)\}$:
\begin{enumerate}
\item $u \in C^0(E)$;
\item For all $t \in E_1$ it holds $u(t, \cdot) \in C^2(E_2(t))$;
\end{enumerate}
\end{theorem}
(3) For all $x \in E_2$, it holds $\partial_t^p u(\cdot, x) \in C(E_1(x))$; 
(4) For all $x \in E_2 \setminus \{\varphi(0)\}$, it holds $\partial_t^p u(\cdot, x) \in L^1_{\text{loc}}(E_1(x))$; 
(5) $u$ satisfies (5.1).

In particular, such a function $u$ is given by

$$u(t, x) = \int_{-\infty}^{\varphi(0)} q_\Phi(t, x; y) f(y) \, dy,$$

where $q_\Phi$ is defined in Theorem 4.2 and we set $u(0, x) = f(x)$.

In order to prove this theorem, we state and prove the following further result (which is, however of independent interest).

**Theorem 5.2.** Suppose that (A2), (A2) and (A3) hold. Let $\varphi : \mathbb{R}^+_0 \to \mathbb{R}$ be locally Lipschitz and non-decreasing. Assume further that if there exist $t_1 < t_2$ such that $\varphi(t_1) = \varphi(t_2)$, then $\varphi(t) = \varphi(t_1)$ for all $t \geq t_1$. Then the function $q_\Phi$ defined in Theorem 4.2 satisfies

$$\begin{cases}
\partial_t^p q_\Phi(t, x; y) = \frac{1}{2} \partial_x^2 q_\Phi(t, x; y) & t > 0, \; x < \varphi(t), \; y < \varphi(0), \; x \neq y \\
n_\Phi(t, x; y) = 0 & t \geq 0, \; x \geq \varphi(t) \text{ or } y \geq \varphi(0) \\
\lim_{x \to -\infty} q_\Phi(t, x; y) = 0 & \text{locally uniformly with respect to } t > 0 \text{ and } y \in \mathbb{R} \\
\lim_{t \to 0^+} q_\Phi(t, x; y) = 0 & x \neq y \\
\lim_{t \to 0^+} q_\Phi(t, x; y) \, dy = \delta_x(dy) & \text{vaguely as a measure on } (-\infty, \varphi(0)).
\end{cases}$$

**Proof.** First notice that the second equality in (5.3) is verified for $x \geq \varphi(t)$ by Theorem 4.8, while for $y \geq \varphi(0)$ this is implied by the definition of $q_\Phi$. Furthermore, the fourth equality in (5.3) is proved in Proposition 4.11. Concerning the last equality in (5.3), recall that, by Proposition 3.1, $\lim_{t \to 0^+} p_\Phi(t, x; y) \, dy = \delta_x(dy)$ weakly. Now let $f \in C_c((-\infty, \varphi(0)))$ and observe that, since $\text{supp}(f) \subset (-\infty, \varphi(0))$,

$$\int_{-\infty}^{\varphi(0)} q_\Phi(t, x; y) f(y) \, dy = \int_{-\infty}^{+\infty} p_\Phi(t, x; y) f(y) \, dy - \int_{\text{supp}(f)} r_\Phi(t, x; y) f(y) \, dy.$$

Furthermore, observe that, for $t \leq 1$, since $\varphi$ is locally Lipschitz, we can use Theorem 4.7 to obtain,

$$\int_{\text{supp}(f)} r_\Phi(t, x; y) f(y) \, dy \leq \frac{|\text{supp}(f)|}{\sqrt{2\pi}} \frac{S_T(\text{supp}(f), T) \max_{y \in \text{supp}(f)} |f(y)|}{\sqrt{2\pi}} \int_{0}^{1} U_\frac{1}{2}(w) \, dw$$

that implies

$$\lim_{t \downarrow 0^+} \int_{\text{supp}(f)} r_\Phi(t, x; y) f(y) \, dy = 0.$$

Hence

$$\lim_{t \downarrow 0^+} \int_{-\infty}^{\varphi(0)} q_\Phi(t, x; y) f(y) \, dy = \lim_{t \downarrow 0^+} \int_{-\infty}^{+\infty} p_\Phi(t, x; y) f(y) \, dy - \lim_{t \downarrow 0^+} \int_{\text{supp}(f)} r_\Phi(t, x; y) f(y) \, dy = f(x).$$

Since $f \in C_c((-\infty, \varphi(0)))$ is arbitrary, this proves the fifth equality in (5.3).

To prove the third equality in (5.1), notice that for fixed $T > 0$ and $t \in [0, T]$ we have

$$r_\Phi(t, x; y) \leq \frac{S_T(\{y\}, T)}{\sqrt{2\pi}} \int_{0}^{t} E \left[ (L(t - w))^{-\frac{1}{2}} e^{-\frac{1}{2\sigma^2} \left( \frac{x - \varphi(w)}{2\sigma \varphi(w)} \right)^2} \right] \, dw.$$

Since we want to study the limit as $x \to -\infty$, we can assume, without loss of generality, that $(x - \varphi(0))^2 \leq (x - \varphi(w))^2$. Thus it holds

$$r_\Phi(t, x; y) \leq \frac{S_T(\{y\}, T)}{\sqrt{2\pi}} \int_{0}^{t} E \left[ (L(w))^{-\frac{1}{2}} e^{-\frac{1}{2\sigma^2} \left( \frac{x - \varphi(w)}{2\sigma \varphi(w)} \right)^2} \right] \, dw \leq \frac{S_T(\{y\}, T)}{\sqrt{2\pi}} \int_{0}^{T} E \left[ (L(w))^{-\frac{1}{2}} e^{-\frac{1}{2\sigma^2} \left( \frac{x - \varphi(w)}{2\sigma \varphi(w)} \right)^2} \right] \, dw.$$
Taking the supremum over \( t \in [0, T] \) we have
\[
\sup_{t \in [0, T]} r_\Phi(t, x; y) \leq \frac{S_T(\{y\}, T)}{\sqrt{2\pi}} \int_0^T \mathbb{E} \left[ (L(w))^{-\frac{1}{2}} e^{-\frac{(x-\varphi(0))^2}{2L(w)}} \right] dw.
\]
Now, notice that the integrand can be controlled as
\[
E \left[ (L(w))^{-\frac{1}{2}} e^{-\frac{(x-\varphi(0))^2}{2L(w)}} \right] \leq U_{-\frac{1}{2}}(w)
\]
where \( U_p \) is defined in (2.19), and thus the right-hand side belongs to \( L^1[0, T] \) so that we can use the dominated convergence theorem to conclude that
\[
\lim_{x \to -\infty} \sup_{y \in \mathbb{R}} \sup_{t \in [0, T]} r_\Phi(t, x; y) = 0. \tag{5.4}
\]
On the other hand, observing that for \( r \geq 0 \) and \( \lambda > 0 \)
\[
\sqrt{r} e^{-r^2} \leq \sqrt{2 \lambda e},
\]
we know that, for \( x \neq y \),
\[
\sup_{t \in [0, T]} p_\Phi(t, x; y) \leq \frac{1}{|x - y| \sqrt{2 \pi e}}
\]
and then, for any compact set \( K \subset \mathbb{R} \),
\[
\lim_{x \to -\infty} \sup_{y \in K} \sup_{t \in [0, T]} p_\Phi(t, x; y) = 0. \tag{5.5}
\]
Combining (5.4) and (5.5) we get
\[
\lim_{x \to -\infty} \sup_{y \in K} \sup_{t \in [0, T]} q_\Phi(t, x; y) = 0,
\]
that, since \( T > 0 \) and \( K \subset \mathbb{R} \) are arbitrary, implies the third equality in (5.3).

It only remains to show the first equality in (5.3). This is done as follows: by Proposition 4.14 we know that for all \( x < \varphi(t) \) with \( x \neq y \) it holds
\[
\partial_t^\Phi q_\Phi(t, x; y) = \partial_t^\Phi p_\Phi(t, x; y) - \int_0^t \partial_y^\Phi p_\Phi(t - w, x; \varphi(w)) p_T(w; y) dw. \tag{5.6}
\]
By assumption, we know that either \( \varphi^{-1}(x) = \emptyset \) or \( \varphi^{-1}(x) = \{w_x\} \) for some \( 0 \leq w_x < t \). Hence, by Theorem 3.2 we know that for fixed \( (t, x) \in \mathbb{R}^+ \times \mathbb{R} \) with \( x < \varphi(t) \) and \( x \neq y \),
\[
\partial_y^\Phi p_\Phi(t - w, x; \varphi(w)) = \frac{1}{2} \partial_y^2 p_\Phi(t - w, x; \varphi(w)) \tag{5.7}
\]
for all \( w \in (0, t) \) except at most for \( w = w_x \), where clearly \( |\{w_x\}| = 0 \). Hence we can use (5.7), together with Theorem 3.2 again, in (5.6) to get
\[
\partial_t^\Phi q_\Phi(t, x; y) = \frac{1}{2} \left( \partial_y^2 p_\Phi(t, x; y) - \int_0^t \partial_y^2 p_\Phi(t - w, x; \varphi(w)) p_T(w; y) dw \right) = \frac{1}{2} \partial_y^2 q_\Phi(t, x; y),
\]
where we also used Proposition 4.13. Since \( t > 0 \) and \( x \in (-\infty, \varphi(t)) \setminus \{y\} \) is arbitrary, this ends the proof.

\( \square \)

Proof of Theorem 5.1. Let \( u \) be as in (5.2). Let us first show that \( u \in C(\mathbb{R}_+^+ \times \mathbb{R}) \). For \( (t, x) \in \mathbb{R}^+ \times \mathbb{R} \), notice that \( q_\Phi(\cdot, \cdot; y) \in C(\mathbb{R}^+ \times \mathbb{R}) \). Furthermore, consider any \( t_0 \in (0, t) \) and notice that
\[
\sup_{(\tau, \xi) \in [t_0, +\infty) \times \mathbb{R}} |f(y)q_\Phi(\tau, \xi; y)| \leq \frac{\|f\|_{L^\infty(\mathbb{R})}}{\sqrt{2\pi t_0}} 1_{\text{supp}(f)}(y),
\]
hence the continuity of \( u \) in \( (t, x) \) follows by a simple application of the dominated convergence theorem.

Concerning points of the form \((0, x)\) for \( x \in \mathbb{R} \), let us split the solution as follows:
\[
u(t, x) = \tilde{u}(t, x) + u_r(t, x), \tag{5.8}
\]
where
\[
\tilde{u}(t, x) = E[f(X_\Phi(t) + x)] \quad \text{and} \quad u_r(t, x) = - \int_{\mathbb{R}} r_\Phi(t, x; y)f(y) dy.
\]
Concerning \( \tilde{u}(t, x) \), by dominated convergence it is clear that
\[
\lim_{(s, \xi) \to (0, x)} \tilde{u}(t, x) = E[f(x)] = u(0, x).
\]
On the other hand, arguing as in Proposition 4.11, we have, for \( s \in (0, 1] \) and \( \xi \in \mathbb{R} \),
\[
|u_r(s, \xi)| \leq \frac{|\text{supp}(f)|S_r(\text{supp}(f), 1)}{\sqrt{2\pi}} \int_0^s U_{\frac{r}{2}}(w) \, dw
\]
where \( U_r \) is defined in (2.19), and then
\[
\lim_{(s, \xi) \to (0, x)} |u_r(s, \xi)| = 0.
\]
This shows that \( \lim_{(s, \xi) \to (0, x)} u(s, \xi) = u(0, x) \) and then \( u \in C(\mathbb{R}_0^+ \times \mathbb{R}) \) (and then, in particular, belongs to \( C(E) \)).

Next, fix \( t > 0 \) and \( x \in \mathbb{R} \setminus \{\varphi(t)\} \) and observe that, arguing as in Proposition 4.13, there exists a compact set \( K \subset \mathbb{R}_0^+ \times \mathbb{R} \) such that the curve \( w \in [0, t] \mapsto (t-w,x-\varphi(w)) \in \mathbb{R}_0^+ \times \mathbb{R} \) lies in \( K \) and \((0,0) \notin K \). Furthermore, there exists a \( \delta > 0 \) such that for \( \xi \in [x-\delta,x+\delta] \): \( I_\delta \) the curves \( w \in [0, t] \mapsto (t-w,\xi-\varphi(w)) \in \mathbb{R}_0^+ \times \mathbb{R} \) lie into \( K \). Hence, we have that, for \( \xi \in I_\delta \),
\[
\|\partial_x q_\Phi(t,:)y\|_{L^\infty([x-\delta,x+\delta])} \leq \|\partial_x p_\Phi(\cdot,:0)\|_{L^\infty(K)}
\]
where the right-hand side is independent of \( y \). Furthermore, for \( y \in \text{supp}(f) \) and \( \xi \in I_\delta \) it is clear that \( \xi - y \in I_\delta - \text{supp}(f) \), where the latter is the Minkowski sum of two compact subsets of \( \mathbb{R} \) and thus it is compact. As a consequence
\[
\|\partial_x q_\Phi(t,:)y\|_{L^\infty(I_\delta)} \leq \|\partial_x p_\Phi(t,:,0)\|_{L^\infty(I_\delta - \text{supp}(f))} + \|\partial_x p_\Phi(\cdot,:,0)\|_{L^\infty(K)}.
\]
Hence, a simple application of the dominated convergence theorem guarantees that for \( x \neq \varphi(t) \)
\[
\partial_x u(t,x) = \int_{\text{supp}(f)} \partial_x q_\Phi(t,x;y)f(y) \, dy
\]
and \( \partial_x u(t,:) \in C(\mathbb{R} \setminus \{\varphi(t)\}) \). With the same exact argument, we have, for \( x \neq \varphi(t) \),
\[
\partial_2 u(t,x) = \int_{\text{supp}(f)} \partial_2 q_\Phi(t,x;y)f(y) \, dy \tag{5.9}
\]
and \( \partial_2 u(t,:) \in C(\mathbb{R} \setminus \{\varphi(t)\}) \). This shows that \( u(t,:) \in C^2(E_2(t)) \) for all \( t \in E_1 \). Next notice that for \( x \in \mathbb{R} \) and \( t > 0 \) it holds, if the involved quantities exist,
\[
\partial_2 q_\Phi(t,x) = \partial_t \int_{-\infty}^{\varphi(0)} f(y) \left( \int_0^t q_\Phi(s,x;y)p_\Phi(s) \, ds \right) \, dy.
\]
We want to prove that we can exchange the order of the derivative and integral sign: this will also guarantee that \( \partial_2 q_\Phi(t,x) \) is well-defined. To do this, for any fixed \( t > 0 \) and \( x < \varphi(t) \), we will provide a bound on \( |\partial_2 q_\Phi(t,x;y)| \) in a suitable neighbourhood of \( t \) that is independent of \( y \in \text{supp}(f) \). Fix \( y \in \text{supp}(f) \) and \( x < \varphi(t) \) with \( x \neq y \). We notice that, by the assumptions on \( \varphi \), there exists \( \delta > 0 \) such that \( x < \varphi(\tau) \) for all \( \tau \in [t-\delta,t+\delta] \) where \( t-\delta > 0 \). Let \( k_\Phi \text{ as in (4.24)} \) and recall that
\[
\partial_2^2 r(t,x) = \partial_t \int_0^{\varphi(0)} f(y) \left( \int_0^t q_\Phi(s,x;y)p_\Phi(s) \, ds \right) \, dy.
\]
Arguing as in Proposition 4.14, we know that \( \partial_t k_\Phi(\cdot,\cdot,x) \) is continuous and
\[
\sup_{(\tau,w) \in [t-\delta,t+\delta] \times \mathbb{R}^+} |\partial_t k_\Phi(\tau,w,x)| < \infty. \label{ineq15}\]
Hence
\[
\sup_{(\tau,w) \in [t-\delta,t+\delta] \times \text{supp}(y)} |\partial_t^2 r(t,x;y)| \leq \sup_{(t,w) \in [t-\delta,t+\delta] \times \mathbb{R}^+} |\partial_t k_\Phi(\tau,w,x)|.
\]
Furthermore,
\[
\sup_{(\tau,\xi) \in [t-\delta,t+\delta] \times (x - \text{supp}(f))} |\partial_\xi^2 p_\Phi(\tau,\xi;0)| = \frac{1}{2} \sup_{(\tau,\xi) \in [t-\delta,t+\delta] \times (x - \text{supp}(f))} |\partial_\xi^2 p_\Phi(\tau,\xi;0)| < \infty.
\]
Concerning \(\partial^\Phi u(\cdot, x)\), we distinguish among two cases. If \(x < \phi\), then, by dominated convergence, we have
\[
\partial^\Phi u(t, x) = \int_{-\infty}^{\varphi(t)} f(y) \partial^\Phi q_\Phi(t, x; y) \, dy
\]
and \(\partial^\Phi u(\cdot, x) \in C(\mathbb{R}^+),\) in particular \(\partial^\Phi u(\cdot, x) \in C(E_1(x))\) for \(x \in E_2\).

Next, we verify that \(u(t, x) = f(x)\) is guaranteed by assumption. Next, we have, for any \(T > 0\),
\[
\sup_{t \in [0, T]} |u(t, x)| \leq \sup\{f|\}_{L^\infty(\mathbb{R})} \sup_{y \in \supp(f)} \sup_{t \in [0, T]} q_\Phi(t, x; y)
\]
and then, taking the limit as \(x \to -\infty\), by the third equality in (5.3), we get
\[
\lim_{x \to -\infty} \sup_{t \in [0, T]} |u(t, x)| = 0.
\]

This proves that \(u\) satisfies (5.1).

Finally, we need to show that \(\partial^\Phi u(\cdot, x) \in L^1_{\text{loc}}(\overline{E}_1(x))\) for all \(x \in E_2 \setminus \{\phi(0)\}\). To do this, let us split again \(u\) as in (5.8) and notice that
\[
\partial^\Phi u(t, x) = \partial^\Phi \tilde{u}(t, x) + \partial^\Phi u_r(t, x).
\]

Concerning \(\partial^\Phi \tilde{u}(t, x)\), by [19, Theorem 2.1] we know that
\[
\mathcal{T}^\Phi \tilde{u}(t, x) = \frac{1}{2} \int_0^T \partial^2_s \tilde{u}(s, x) \, ds
\]
for all \(x \in \mathbb{R},\) which in turn guarantees that \(\partial^\Phi \tilde{u}(\cdot, x) \in L^1_{\text{loc}}(\overline{E}_1(x))\) for \(x \in E_2\). Concerning \(u_r\), let us distinguish among two cases. If \(x < \phi(0)\), then consider any \(T > 0\). Notice that
\[
\int_0^T \int_0^t |\partial^\Phi p_\Phi(t - w, x; \varphi(w))| p_T(w; y) \, dw \, dt = \int_0^T \left( \int_0^T |\partial^\Phi p_\Phi(t - w, x; \varphi(w))| \, dt \right) p_T(w; y) \, dw
\]
\[
= \frac{1}{2} \int_0^T \left( \int_0^T |\partial^2_w p_\Phi(t - w, x; \varphi(w))| \, dt \right) p_T(w; y) \, dw
\]
\[
\leq \frac{1}{2} \sqrt{\frac{2}{\pi}} \left( \frac{10}{e} \right)^{\frac{5}{2}} + \left( \frac{6}{e} \right)^{\frac{3}{2}} \int_0^T (x - \varphi(w))^{-3} \left( \int_0^T e^{-\frac{(x-\varphi(w))^2}{4\pi}} \, dt \right) \, dt \, p_T(w; y) \, dw
\]
\[
\leq \frac{(x - \varphi(0))^{-3} T}{2} \sqrt{\frac{2}{\pi}} \left( \frac{10}{e} \right)^{\frac{5}{2}} + \left( \frac{6}{e} \right)^{\frac{3}{2}}.
\]

Integrating this against \(|f(y)|\) we get that \(u_r(\cdot, x) \in L^1_{\text{loc}}(\overline{E}_1(x))\). If \(\varphi\) is not constant and \(x \in (\varphi(0), \varphi(t))\), then let \(\varphi^{-1}(x) = \{w_x\}\) and observe that \(w_x > 0\). Then we can apply the same argument as before for \(T = \frac{w_x}{2}\): since we already know that \(\partial^\Phi u_r(\cdot, x) \in C(E_1(x))\), then this is enough to guarantee that \(\partial^\Phi u_r(\cdot, x) \in L^1_{\text{loc}}(\overline{E}_1(x))\).

Concerning uniqueness, let us first show the following general result.

**Theorem 5.3.** Under the assumptions of Theorem 5.1, there exists at most one function \(u : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}\) satisfying items (1), (2), (3) and (5) of Theorem 5.1 and
\[
(4') \text{ For all } x \in E_2 \text{ it holds } \partial^\Phi u(\cdot, x) \in L^1_{\text{loc}}(\overline{E}_1(x)).
\]
Proof. Since the equation is linear, it is sufficient to prove the statement for \( f \equiv 0 \). Precisely, we only need to prove that if \( u \) satisfies (1), (2), (3), (4'), (5) then \( u \equiv 0 \). This is clear since in our case the set \( E = \{ (t, x) \in \mathbb{R}_+^2 : x < \varphi(t) \} \) is unbounded and \( \lim_{x \to -\infty} u(t, x) = 0 \) locally uniformly with respect to \( t \geq 0 \), hence we can apply Theorem 2.8 to both \( u \) and \(-u\) to guarantee that
\[
\inf_{(t,x) \in \overline{E}} u(t,x) = \sup_{(t,x) \in \overline{E}} u(t,x) = 0.
\]
\[\square\]

A plain corollary, that implies uniqueness in Theorem 1.1 for constant \( \varphi \), is the following one.

**Corollary 5.4.** Under the assumptions of Theorem 5.1, if \( \varphi \) is constant, then (5.2) is the unique solution of (5.1) satisfying items (1), (2), (3) and (5) of Theorem 5.1 and (4') of Theorem 5.3.

**Proof.** Just notice that in such a case \( \varphi(0) \notin E_2 \) and use Theorems 5.1 and 5.3. \[\square\]

**Remark 5.5.** If \( \varphi \) is constant, notice that \( u(t, \varphi(0)) = 0 \), that implies
\[
\partial_t^\Phi u_r(\cdot, \varphi(0)) = -\partial_t^\Phi \Phi(\cdot, \varphi(0)) \in L^1_{\text{loc}}(\mathbb{R}_0^+).
\]
Hence, in such a case, \( \partial_t^\Phi u_r(\cdot, x) \in L^1_{\text{loc}}(\mathbb{R}_0^+) \) for all \( x \in \mathbb{R} \).

Now, let us consider the case in which \( \varphi \) is non-constant (and thus non-decreasing). In such a case, we need to require the additional regularity given by conditions (5.11) and (5.12) below; however, these conditions are not too restrictive and can be checked quite easily, as we show in Propositions 5.8 and 5.9 below.

**Corollary 5.6.** Suppose the assumptions of Theorem 5.1 hold true. Assume further that
\[
\int_0^1 \frac{\left| \Psi(\xi) \right|}{\sqrt{R(\Psi(\xi))}} d\xi < \infty
\]
and that for any \( y < \varphi(0) \) and any fixed threshold \( c > 0 \) it holds
\[
\mathbb{E}_y \left[ (\varphi(T_c) - \varphi(0))^{-\frac{4+\gamma}{2+\gamma}} \right] < \infty,
\]
where
\[
T_c := \inf \{ t > 0 : X(t) \geq c \}.
\]
Then (5.2) is the unique solution of (5.1) satisfying items (1), (2), (3) and (5) of Theorem 5.1 and (4') of Theorem 5.3.

**Proof.** We only need to prove that \( u \) as in (5.2) is such that \( \partial_t^\Phi u_r(\cdot, \varphi(0)) \in L^1_{\text{loc}}(E_1(\varphi(0))) \), so that the statement follows by Theorems 5.1 and 5.3. To do this, we split \( u \) as in (5.8) and we notice that the condition \( \partial_t^\Phi u_r(\cdot, 0) \in L^1_{\text{loc}}(\mathbb{R}_0^+) \) has been already proved in Theorem 5.1. Hence, we only need to show that \( \partial_t^\Phi u_r(\cdot, \varphi(0)) \in C(\mathbb{R}_+) \), it is sufficient to show that \( \partial_t^\Phi u_r(\cdot, \varphi(0)) \in L^1(0,1] \). We set
\[
\mathcal{P}_1(s) = \sqrt{2} \int_0^M \left| \Psi(\xi) \right| e^{-s R(\Psi(\xi))} d\xi
\]
and we set
\[
\mathcal{P}_2(s) = 3 \pi(1 + \xi) \int_0^1 \tau \nu(\tau) \left( \frac{\xi^2}{2} \int_0^1 \tau^2 \nu(\tau) d\tau \right) e^{-s \xi^2 - \gamma} d\xi
\]
and we notice that
\[
\int_0^1 \left| \partial_t^\Phi u_r(t, \varphi(0)) \right| dt
\]
\[
\leq \int_0^1 \int_{-\infty}^{\varphi(0)} \int_{t}^{t+\varphi(0)} p(s, \varphi(0) - \varphi(w); 0) \left| \partial_s f_L(s; t - w) \right| p_T(w; y) |f(y)| ds dw dy dt
\]
\[
= \int_{-\infty}^{\varphi(0)} \int_0^1 \int_{t}^{t+\varphi(0)} p(s, \varphi(0) - \varphi(w); 0) \left( \int_{t}^{t+\varphi(0)} \left| \partial_s f_L(s; t - w) \right| dt \right) p_T(w; y) |f(y)| ds dw dy
\]
\[
\leq \frac{1}{\pi} \int_{-\infty}^{\varphi(0)} \int_0^1 \int_{t}^{t+\varphi(0)} p(s, \varphi(0) - \varphi(w); 0) \left( \mathcal{P}_1(s) + \mathcal{P}_2(s) \right) \left( \int_t^{t+\varphi(0)} I_\Phi(t - w) dt \right) p_T(w; y) |f(y)| ds dw dy
\]
Furthermore, setting

\[ \text{by assumption.} \]

Next, define

\[ \text{and observe that} \]

\[ \text{hence} \]

\[ \text{where we also used (2.13). To prove that} I_1 < \infty, \text{notice that} \]

\[ I_1 \leq \left| I_\phi \right| \|f\|_{L^\infty} \left( \sup \{f\}, S_T(\sup \{f\}); 1 \right) \int_{M_y} \int_0^{+\infty} s^{-\frac{1}{2}} |\Psi(\xi)| e^{-sR(\Psi(\xi))} ds \, d\xi \]

Concerning \( I_2 \), notice that there exists a constant \( C > 0 \) such that

\[ \mathcal{P}_2(s) \leq C \int_{M_y} \xi^2 e^{-s\xi^2 - \gamma} d\xi \]

and then

\[ I_2 \leq C \left| I_\phi \right| \|f\|_{L^\infty} \left( \sup \{f\}, S_T(\sup \{f\}); 1 \right) \int_{M_y} \int_0^{+\infty} \xi^2 \left( \int_0^{+\infty} e^{-s\xi^2 - \gamma} p(s, \varphi(0) - \varphi(w); 0) ds \right) p_T(w; y) d\xi \, dw \, dy \]

Now recall that, by (A.58),

\[ \int_0^{+\infty} e^{-s\xi^2 - \gamma} p(s, \varphi(0) - \varphi(w); 0) ds = \frac{1}{\sqrt{2\pi\xi^2 - \gamma}} e^{-(\varphi(w) - \varphi(0))\sqrt{2\xi^2 - \gamma}} \]

hence

\[ I_2 \leq C \left| I_\phi \right| \|f\|_{L^\infty} \left( \sup \{f\}, S_T(\sup \{f\}); 1 \right) \int_{M_y} \int_0^{+\infty} \xi^2 \left( \int_0^{+\infty} \left( \frac{4 + \gamma - \varphi(w) - \varphi(0)\sqrt{2\xi^2 - \gamma}}{2 - \gamma} \right) p_T(w; y) d\xi \right) \]

Now let

\[ \mathcal{T}_{\varphi}(0) = \inf \{ t > 0 : X(t) \geq \varphi(0) \} \]

and observe that \( \mathcal{T}_{\varphi}(0) \leq \mathcal{T} \text{ a.s.} \) Thus

\[ I_2 \leq C \left| I_\phi \right| \|f\|_{L^\infty} \left( \sup \{f\}, S_T(\sup \{f\}); 1 \right) \Gamma \left( \frac{4 + \gamma}{2 - \gamma} \right) \int \left( \frac{4 + \gamma}{2 - \gamma} \right) \int_{M_y} \left[ (\varphi(T_{\varphi}(0)) - \varphi(0))^{\frac{-\gamma + 4}{2 - \gamma}} \right] dy. \]

Next, define

\[ \mathcal{T}_{\varphi}(0)(y) = \inf \{ t > 0 : X(t) + y \geq \varphi(0) \} \]

so that

\[ \mathbb{E}_y \left[ (\varphi(T_{\varphi}(0)(y)) - \varphi(0))^{\frac{-\gamma + 4}{2 - \gamma}} \right] = \mathbb{E} \left[ (\varphi(T_{\varphi}(0)(y)) - \varphi(0))^{\frac{-\gamma + 4}{2 - \gamma}} \right]. \]

Furthermore, setting \( \mathcal{Y} = \max \sup \{f\} \), we have \( T_{\varphi}(0)(\mathcal{Y}) \leq T_{\varphi}(0)(y) \) a.s. As a consequence

\[ I_2 \leq C \left| I_\phi \right| \|f\|_{L^\infty} \left( \sup \{f\}, S_T(\sup \{f\}); 1 \right) \Gamma \left( \frac{4 + \gamma}{2 - \gamma} \right) \mathbb{E} \left[ (\varphi(T_{\varphi}(0)(\mathcal{Y})) - \varphi(0))^{\frac{-\gamma + 4}{2 - \gamma}} \right] < \infty \]

by assumption.
Remark 5.7. With the same exact proof, under the assumptions of Corollary 5.6, we also have continuity with respect to initial conditions. Indeed, consider \( u_1 \) and \( u_2 \) to be the unique solutions, satisfying items (1), (2), (3) and (5) of Theorem 5.1 and (4') of Theorem 5.3, of

\[
\begin{align*}
\partial_t^\Phi u_j(t, x) &= \frac{1}{2} \partial_x^2 u_j(t, x) & t > 0, \ x < \varphi(t) \\
u_j(t, x) &= 0 & t \geq 0, \ x \geq \varphi(0) \\
u_j(0, x) &= f_j(x) & x < \varphi(0) \\
\lim_{x \to -\infty} u_j(t, x) &= 0 & \text{locally uniformly with respect to } t > 0,
\end{align*}
\]

where \( f_j \in C_c(-\infty, \varphi(0)) \) for \( j = 1, 2 \). Then, by linearity of the involved operators, \( w = u_1 - u_2 \) is the unique solution of

\[
\begin{align*}
\partial_t^\Phi w(t, x) &= \frac{1}{2} \partial_x^2 w(t, x) & t > 0, \ x < \varphi(t) \\
w(t, x) &= 0 & t \geq 0, \ x \geq \varphi(0) \\
w(0, x) &= f_1(x) - f_2(x) & x < \varphi(0) \\
\lim_{x \to -\infty} w(t, x) &= 0 & \text{locally uniformly with respect to } t > 0,
\end{align*}
\]

satisfying items (1), (2), (3) and (5) of Theorem 5.1 and (4') of Theorem 5.3. Hence, by the positive maximum principle given in Theorem 2.8, we get that for all \( T > 0 \) it holds

\[
\sup_{(t, x) \in [0, T] \times \mathbb{R}} |u_1(t, x) - u_2(t, x)| = \sup_{(t, x) \in [0, T] \times \mathbb{R}} |w(t, x)| = \max_{x \in \mathbb{R}} |f_1(x) - f_2(x)|.
\]

We state here some sufficient conditions that guarantee (5.11) and (5.12).

Proposition 5.8. Suppose the assumptions of Theorem 5.1 hold true. Assume further that

\[\int_1^{+\infty} \log(t) \nu(dt) < \infty;\]

Then (5.11) is satisfied.

Proof. Let us observe that, by Lemma 2.2, for \( \xi \in (0, 1) \),

\[\frac{|\Psi(\xi)|}{\sqrt{\Re(\Psi(\xi))}} \leq \sqrt{\Re(\Psi(\xi))} + \frac{1}{C_0} \frac{|\Im(\Psi(\xi))|}{\xi}.\]

Since \( \Re(\Psi(\cdot)) \) is continuous, it is clear that

\[\int_0^1 \sqrt{\Re(\Psi(\xi))} \, d\xi < \infty.
\]

On the other hand,

\[
\int_0^1 \frac{|\Im(\Psi(\xi))|}{\xi} \, d\xi \leq \int_0^1 \int_0^{+\infty} \frac{|\sin(t\xi)|}{\xi} \nu(dt) \, d\xi = \int_0^{+\infty} t \left( \int_0^1 \frac{|\sin(\xi t)|}{\xi t} \, d\xi \right) \nu(dt)
\]

\[= \int_0^{+\infty} \left( \int_0^t \frac{|\sin(z)|}{z} \, dz \right) \nu(dt) = I_1 + I_2 + I_3\]

where

\[I_1 = \int_0^1 \left( \int_0^t \frac{|\sin(z)|}{z} \, dz \right) \nu(dt) \quad I_2 = \int_1^{+\infty} \left( \int_0^1 \frac{|\sin(z)|}{z} \, dz \right) \nu(dt) \]

\[I_3 = \int_1^{+\infty} \left( \int_1^t \frac{|\sin(z)|}{z} \, dz \right) \nu(dt).
\]

Now recall that there exists a constant \( C > 0 \) such that for \( z \in (0, 1) \)

\[\frac{|\sin(z)|}{z} \leq C\]

and then

\[I_1 \leq C \int_0^1 t \nu(dt) < \infty, \quad I_2 \leq C \nu(1) < \infty.\]
Concerning $I_3$, notice that
\[
I_3 \leq \int_1^{+\infty} \int_1^t \frac{1}{z} \, dz \, \nu(dt) = \int_1^{+\infty} \log(t) \nu(dt) < \infty.
\]

\[\square\]

**Proposition 5.9.** Suppose the assumptions of Theorem 5.1 hold true. Assume further that there exist two constants $\beta_1, \beta_2 > 0$ such that
\[
\lim_{\lambda \to +\infty} \frac{\phi^{\beta_1}(\lambda)}{\lambda} = +\infty \quad \text{and} \quad \liminf_{w \to 0^+} \frac{\varphi(w) - \varphi(0)}{w^{\beta_2}} > 0.
\]
Then (5.12) holds true.

**Proof.** Notice that there exist two constants $C, \delta > 0$ such that for $w \in (0, \delta)$ it holds
\[
\varphi(w) - \varphi(0) \geq Cw^{\beta_2}
\]
and then (5.12) is implied by
\[
E \left[ T_{c}^{-\beta_2 \frac{2+\gamma}{\gamma}} 1_{(0,\delta)}(T_{c}) \right] < \infty.
\]
The latter turns out to be true by employing [7, Theorem 2.16]; notice that despite the assumptions considered here are lighter, the very same proof still holds. \[\square\]

In case $\phi(\lambda) = \lambda^{\alpha}$, notice that if $\varphi'(0+) > 0$, then the conditions of Proposition 5.9 hold with $\beta_1 = \frac{2}{\alpha}$ and $\beta_2 = 1$. At the same time, while it is clear that the condition of Proposition 5.8 holds true, one can also verify (5.11) by hands. As a consequence, combining Theorems 5.1 and 5.3 we get Theorem 1.1.

**Remark 5.10.** Notice that under the assumptions of Theorems 4.9 and 5.3, one can show that $q_\phi(t, x; y)$ is the unique fundamental solution of (5.1), in the sense that if $\tilde{q}_\phi : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that for any $f \in C_c^\infty(-\infty, \varphi(0))$ the function $u(t, x) = \int_{-\infty}^{\varphi(0)} \tilde{q}_\phi(t, x; y)f(y) \, dy$ satisfies (5.1), then $\tilde{q}_\phi(t, x; y) = q_\phi(t, x; y)$ for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ and $y \in (-\infty, \varphi(0))$. Indeed, by Theorem 5.3, we get that for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ and $f \in C_c^\infty(-\infty, \varphi(0))$ it holds
\[
\int_{-\infty}^{\varphi(0)} (\tilde{q}_\phi(t, x; y) - q_\phi(t, x; y))f(y) \, dy = 0,
\]
that implies, by the Fundamental Lemma of the Calculus of Variations (see [24, Theorem 1.24]), Theorem 4.9 and the required continuity of $\tilde{q}_\phi$ in the $y$ variable, the desired equality.

6. **Anomalous diffusive behaviour**

In this section, we focus on the anomalous diffusive behaviour of the processes $X_\Phi$ and $X_\Phi^\dagger$. Let us stress that the non-killed process $X$ could be an anomalous diffusion. Indeed, if we fix, without loss of generality, the initial value $y = 0$, we have, by the independence of the Brownian motion $B$ and the inverse subordinator $L$ and by [15, Proposition III.1],
\[
E_0[|X_\Phi(t)|^2] = E_0[E_0[|B(L(t))|^2 | L(t)|] = E_0[L(t)] = U_1(t) \asymp \frac{1}{\Phi(1/t)},
\]
where $U_r$ is defined in (2.19) and $f(t) \asymp g(t)$ means that there exists a constant $C > 1$ such that
\[
C^{-1} g(t) \leq f(t) \leq C g(t).
\]
Thus, since we are assuming that the subordinator $\sigma$ has 0 drift, $X_\Phi$ is a subdiffusion (in the sense that $\lim_{t \to \infty} t^{-1} E_0[|X_\Phi(t)|^2] = 0$, see [60, Remark 3.3.iv]). It remains to show that the same holds also for the killed process $X_\Phi^\dagger$. To do this, we need to handle separately the constant boundary case. We first recall the formula for the mean square displacement of the killed Brownian motion.
Proposition 6.1. Let $c > 0$ and

$$T_c := \inf \{ t > 0 : B(t) > c \}.$$

Then

$$E_0[|B(t)|^2 1_{T_c > t}] = 2t \text{erf} \left( \frac{c}{\sqrt{2t}} \right) - 2c^2 \left( 1 - \text{erf} \left( \frac{c}{\sqrt{2t}} \right) \right),$$

where

$$\text{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-s^2} ds.$$

The proof easily follows by the reflection principle and is given in Appendix A.6. It is also worth noticing that

$$\text{erf}(u) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \frac{(-1)^n u^{2n+1}}{n!(2n+1)} = \frac{2u}{\sqrt{\pi}} \left( 1 + \sum_{n=1}^{+\infty} \frac{(-1)^n u^{2n}}{n!(2n+1)} \right)$$

and then

$$\sqrt{t} \text{erf} \left( \frac{c}{\sqrt{2t}} \right) = \frac{2c}{\sqrt{2\pi}} \left( 1 + \sum_{n=1}^{+\infty} \frac{(-1)^n c^{2n}}{n!(2n+1)(2t)^n} \right).$$

In particular,

$$\lim_{t \to +\infty} \sqrt{t} \text{erf} \left( \frac{c}{\sqrt{2t}} \right) = \frac{2c}{\sqrt{2\pi}}.$$

At the same time

$$\lim_{t \to 0} \sqrt{t} \text{erf} \left( \frac{c}{\sqrt{2t}} \right) = 0$$

hence there exists a constant $E^*$, depending at most on $c$, such that

$$\sqrt{t} \text{erf} \left( \frac{c}{\sqrt{2t}} \right) \leq E^*.$$

Furthermore, for $t \geq 1$, there exists a constant $E_0 > 0$, depending at most on $c$, such that

$$\sqrt{t} \text{erf} \left( \frac{c}{\sqrt{2t}} \right) \geq E_0.$$

With this in mind, we can prove the following result on the constant boundary case.

Theorem 6.2. Let $\varphi(t) \equiv c > 0$ and suppose Assumption (A2) holds. Then

$$E_0[|X_\varphi(t)|^2 1_{T_{\varphi} > 1}] \asymp U_{\frac{1}{2}}(t),$$

where $U_{p}$ is defined in (2.19) and, for two non-negative functions $f, g : \mathbb{R}^+ \to \mathbb{R}^+$, the symbol $f \asymp g$ means

$$0 < \liminf_{t \to \infty} \frac{f(t)}{g(t)} \leq \limsup_{t \to \infty} \frac{f(t)}{g(t)} < \infty.$$

Proof. By Propositions 4.5 and 6.1, we clearly have

$$E_0[|X_\varphi(t)|^2 1_{T_{\varphi} > 1}] = E_0 \left[ 2L(t) \text{erf} \left( \frac{c}{\sqrt{2L(t)}} \right) - 2c^2 \left( 1 - \text{erf} \left( \frac{c}{\sqrt{2L(t)}} \right) \right) \right].$$

On the one hand, we get

$$E_0[|X_\varphi(t)|^2 1_{T_{\varphi} > 1}] \leq E_0 \left[ 2L(t) \text{erf} \left( \frac{c}{\sqrt{2L(t)}} \right) \right] \leq 2E^* U_{\frac{1}{2}}(t),$$

that clearly proves

$$\limsup_{t \to \infty} \frac{E_0[|X_\varphi(t)|^2 1_{T_{\varphi} > 1}]}{U_{\frac{1}{2}}(t)} \leq 2E^* < \infty.$$

On the other hand, notice that

$$E_0[|X_\varphi(t)|^2 1_{T_{\varphi} > 1}] \geq 2E_0 \left[ \sqrt{L(t)} 1_{\{L(t) \geq 1\}} \right] - 2c^2$$
Remark 6.3. Notice that differently from the non-killed $X_\varphi(t)$, which exhibits $U_1(t)$ as mean-square displacement, in case $\varphi$ is constant $X^1_\varphi(t)$ admits mean-square displacement whose asymptotic behaviour is given by $U_2(t)$. This was expected, as it also hold in the case of the Brownian motion. Indeed, while $E_0[[B(t)]^2] = t$, we have

$$\lim_{t \to \infty} E_0[[X^1_\varphi(t)]^2 1_{\{T > t\}}]/U_2(t) \geq 2E_* > 0.$$ 

$$\Box$$

**Theorem 6.4.** Let $\varphi$ be non-decreasing, continuous and such that $\varphi(0) > 0$. Suppose that Assumption (A2) holds. Then

$$E_0[[X^1_\varphi(t)]^2 1_{\{T > t\}}] \leq U_1(t) \leq \frac{C}{\Phi(1/t)}$$

(6.2)

where $C > 1$ is a suitable constant. Furthermore,

$$\liminf_{t \to \infty} E_0[[X^1_\varphi(t)]^2 1_{\{T > t\}}]/U_2(t) > 0.$$ 

(6.3)

Finally, if $\lim_{t \to \infty} \varphi(t) = \ell \in \mathbb{R}^+$, then

$$\limsup_{t \to \infty} E_0[[X^1_\varphi(t)]^2 1_{\{T > t\}}]/U_2(t) < \infty.$$ 

(6.4)

**Proof.** To prove (6.2), by [15, Proposition III.1], it is sufficient to show the first inequality. By Theorem 4.2 we know that

$$E[[X^1_\varphi(t)]^2 1_{\{T > t\}}] = \int \int x^2 q_\varphi(t, x; 0) \, dx$$

$$= \int x^2 p_\varphi(t, x; 0) \, dx - \int_0^t \int x^2 p_\varphi(t - w, x; \varphi(w)) \mu_\varphi(\, dw; 0)$$

$$= E_0[[X_\varphi(t)]^2] - \int_0^t E_{\varphi(w)}[[X_\varphi(t - w)]^2] \mu_\varphi(\, dw; 0)$$

$$= U_1(t) - \int_0^t U_1(t - w) \mu_\varphi(\, dw; 0) - E[\varphi^2(T) 1_{\{T \leq t\}}].$$

(6.5)

It is clear that the latter equality implies (6.2). To show (6.3), notice that, since $X^1_\varphi$ and $X_\varphi$ coincide up to $T$, we have

$$E[[X^1_\varphi(t)]^2 1_{\{T > t\}}] = E[[X_\varphi(t)]^2 1_{\{T > t\}}].$$

Now let

$$T_{\varphi(0)} := \inf \{ t > 0, \, X_\varphi(t) \geq \varphi(0) \}$$

and notice that $T_{\varphi(0)} \leq T$ a.s., that implies

$$1_{\{T > t\}} = 1_{\{t, +\infty\}}(T) \geq 1_{\{t, +\infty\}}(T_{\varphi(0)}) = 1_{T_{\varphi(0)}}$$

a.s.

since $1_{\{t, +\infty\}}(\cdot)$ is non-decreasing. Hence

$$\frac{E[[X^1_\varphi(t)]^2 1_{\{T > t\}}]}{U_2(t)} \geq \frac{E[[X_\varphi(t)]^2 1_{\{T_{\varphi(0)} > t\}}]}{U_2(t)},$$
that proves (6.3) by Theorem 6.2.

Finally, if \( \lim_{t \to +\infty} \varphi(t) = \ell \), we let
\[
T_\ell := \inf\{t > 0, \ X_\Phi(t) \geq \ell\}
\]
and notice that \( T_\ell \geq T \) a.s., that implies, arguing as before,
\[
1_{(T > t)} \leq 1_{T_\ell} \ a.s.
\]
As a consequence,
\[
\frac{\mathbb{E}[|X_\Phi(t)|^2 1_{(T > t)}]}{U_{\frac{1}{2}}(t)} \leq \frac{\mathbb{E}[|X_\Phi(t)|^2 1_{(T_\ell > t)}]}{U_{\frac{1}{2}}(t)},
\]
that implies (6.4) by Theorem 6.2.

\[\square\]

**Remark 6.5.** Notice that the asymptotic behaviour provided by (6.3) and (6.4) guarantees that \( X_\Phi \) exhibits a subdiffusive behaviour, since
\[
U_{\frac{1}{2}}(t) \leq \sqrt{U_1(t)} \leq C \frac{1}{\sqrt[\Phi(\frac{1}{\ell})]}$
\]
for some constant \( C > 1 \).

While this is enough to guarantee that \( X_\Phi \) is a subdiffusion, we are not aware of any explicit relation between the behaviour of \( U_{\frac{1}{2}}(t) \) and \( \Phi(z) \), except that in some specific cases.

Following [17, Chapter 2, Section 2.2], for a positive function \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) let us define \( \alpha^*(f) \) as the infimum of the \( \alpha > 0 \) such that there exists a constant \( D^*(\alpha) > 0 \) for which as \( x \to \infty \)
\[
\frac{f(\alpha x)}{f(x)} \leq D^*(\alpha)(1 + o(1)) \lambda^\alpha,
\]
locally uniformly with respect to \( \lambda \geq 1 \). On the other hand, we define \( \alpha_*(f) \) as the supremum of the \( \alpha > 0 \) such that there exists a constant \( D_*(\alpha) > 0 \) for which as \( x \to \infty \)
\[
\frac{f(\alpha x)}{f(x)} \geq D_*(\alpha)(1 + o(1)) \lambda^\alpha,
\]
locally uniformly with respect to \( \lambda \geq 1 \). The quantities \( \alpha^*(f) \) and \( \alpha_*(f) \) are called respectively the upper and lower Matuszewska indices. Notice that if \( f(\lambda) = \lambda^\alpha \), then clearly \( \alpha^*(f) = \alpha_*(f) = \alpha \). In general we say that \( f \) is \( O \)-regularly varying if both Matuszewska indices are finite. One can easily check that if \( f \) is of (extended) regular variation, then it is also \( O \)-regularly varying. For further details we refer to [17, Chapter 2]. Here, we can prove the following statement.

**Proposition 6.6.** Recall the definition of \( U_p \) in (2.19). If \( \Phi(1/\cdot) \) is \( O \)-regularly varying then
\[
U_p(t) \asymp \frac{1}{(\Phi(\frac{1}{t}))^p}.
\]

**Proof.** Notice that, for \( p > 0 \), \( U_p \) is non-decreasing and non-negative. Its Laplace-Stieltjes transform is
\[
\bar{U}_p(z) := \int_0^{+\infty} e^{-zt} U_p(dt) = z \int_0^{+\infty} e^{-zt} U_p(t)dt = \frac{\Gamma(p + 1)}{(\Phi(z))^p},
\]
where we used (2.20). Now let \( -\infty < \alpha < \alpha_*(\Phi(1/\cdot)) \). Then there exists a constant \( D_*(\alpha) \) such that, as \( z \to \infty \),
\[
\frac{\bar{U}_p(z)}{\bar{U}_p(\frac{1}{z})} = \frac{(\Phi(\frac{1}{z}))^p}{(\Phi(\frac{1}{z}))^p} \leq \frac{1}{D^*(\alpha)(1 + o(1)) \lambda^{-p\alpha}}
\]
uniformly with respect to \( \lambda \geq 0 \). Hence \( \alpha^*(\bar{U}_p(1/\cdot)) = -p\alpha_*(\Phi(1/\cdot)) \). Analogously, one can prove that \( \alpha_*(\bar{U}_p(1/\cdot)) = -p\alpha^*(\Phi(1/\cdot)) \), hence \( \bar{U}_p(1/\cdot) \) is \( O \)-regularly varying. Hence, by the de Haan-Stadtmüller Tauberian theorem (see [17, Theorem 2.10.2]), it holds
\[
U_p(t) \asymp \bar{U}_p(1/t) = \frac{\Gamma(p + 1)}{(\Phi(\frac{1}{t}))^p}.
\]

\[\square\]
As a consequence, we have the following result.

**Corollary 6.7.** Let \( \varphi \) be non-decreasing, continuous and such that \( \varphi(0) > 0 \). Suppose that Assumption (A2) holds and that \( \Phi(1/\cdot) \) is \( O \)-regularly varying. Then there exist two constants \( t_0, C \) such that
\[
\mathbb{E}_0[|X_{\Phi}(t)|^2 \mathbf{1}_{\{\tau > t\}}] \geq \frac{C}{\sqrt{\Phi(\frac{t}{h})}}, \quad t \geq t_0.
\]
If furthermore \( \varphi \) is bounded, then
\[
\mathbb{E}_0[|X_{\Phi}(t)|^2 \mathbf{1}_{\{\tau > t\}}] \geq \frac{1}{\infty \sqrt{\Phi(\frac{t}{h})}}.
\]

**Appendix A. Proofs of some technical results**

A.1. **Proof of Proposition 2.3.** Let us first show (2.10) and (2.12). Consider \( h > 0 \) and observe that, by (2.2),
\[
\frac{f_L(s; t + h) - f_L(s; t)}{h} = \int_0^t \frac{\nu(\tau) g_s(t + h - \tau; s) - g_s(t - \tau; s)}{h} d\tau + \int_t^{t+h} \frac{\nu(\tau) g_s(t + h - \tau; s)}{h} d\tau \quad (A.1)
\]
\[
= I_1(h) + I_2(h).
\]

We want to take the limit as \( h \to 0^+ \). Concerning \( I_2(h) \), we have
\[
I_2(h) = \int_0^h \nu(t + h - \tau) g_s(\tau; s) \frac{1}{h} d\tau \quad (A.4)
\]
\[
= \int_0^h (\nu(t + h - \tau) - \nu(t - \tau)) \frac{g_s(\tau; s)}{h} d\tau + \int_0^h \nu(t - \tau) \frac{g_s(\tau; s)}{h} d\tau.
\]
\[
= I_3(h) + I_4(h) \quad (A.5)
\]

It is clear that
\[
\lim_{h \to 0^+} I_4(h) = \nu(t) g_s(0^+; s) = 0.
\]

To work with \( I_3(h) \), let us observe that, by continuity of \( \nu \), for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[
|\nu(t + h - \tau) - \nu(t - \tau)| \leq 2\pi \varepsilon, \quad \forall h \in (0, \delta).
\]

Moreover, since
\[
g_s(t; s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\xi t - s\Psi(\xi)} d\xi \quad (A.10)
\]
and, in particular,
\[
|g_s(t; s)| \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-s\Re(\Psi(\xi))} d\xi, \quad (A.11)
\]
we get
\[
|I_3(h)| \leq 2\varepsilon \int_0^{+\infty} e^{-s\Re(\Psi(\xi))} d\xi, \quad h \in (0, \delta),
\]

that implies
\[
\limsup_{h \to 0^+} |I_3(h)| \leq 2\varepsilon \int_0^{+\infty} e^{-s\Re(\Psi(\xi))} d\xi.
\]

Sending \( \varepsilon \to 0^+ \) we obtain \( \lim_{h \to 0^+} I_3(h) = 0 \).

Finally let us consider \( I_1(h) \), whose integrand is dominated via the inequality
\[
\nu(\tau) \left| \frac{g_s(t + h - \tau; s) - g_s(t - \tau; s)}{h} \right| \leq \frac{1}{\pi} \nu(\tau) \int_0^{+\infty} \xi e^{-s\Re(\Psi(\xi))} d\xi
\]
\[
(A.14)
\]
that is integrable on \((0, t)\). Hence we can take the limit inside the integral sign to get

\[
\lim_{h \to 0^+} \frac{f_L(s; t + h) - f_L(s; t)}{h} = \int_0^t \pi(\tau) \partial_t g_\sigma(t - \tau; s) d\tau.
\] (A.15)

The argument for \(h < 0\) is exactly the same. Furthermore, (2.12) follows from (A.14). Concerning the continuity of \(\partial_t f_L(s; \cdot)\), notice that we have show in general that

\[
|\partial_t g_\sigma(t; s)| \leq \frac{1}{\pi} \int_0^{+\infty} \xi e^{-s\Re(\xi)} \, d\xi < \infty
\]

hence \(\partial_t f_L(s; \cdot)\) is the convolution product of a \(L^1(\mathbb{R}^+)\) function with a bounded continuous function and thus it is continuous.

Now we show (2.11) and (2.13). Let us consider \(M_\gamma > 0\) big enough such that, by Orey’s condition (2.9), it holds

\[
e^{-s\Re(\Psi(\xi))} \leq \exp\left\{-\frac{C_\gamma s^\alpha}{4}\right\}, \quad \xi > M_\gamma,
\] (A.16)

for \(\alpha = 2 - \gamma\). This implies in particular

\[
\Re(\Psi(\xi)) \geq \frac{C_\gamma}{4} \xi^\alpha, \quad \xi > M_\gamma.
\] (A.17)

Then, start again from (A.10) to get

\[
\left|g_\sigma(\tau; s) - g_\sigma(\tau; s')\right| \leq \frac{1}{2\pi} \int_\infty^{+\infty} \left|\frac{e^{-s\Re(\Psi(\xi))} - e^{-s'\Re(\Psi(\xi))}}{s - s'}\right| d\xi.
\] (A.18)

Suppose \(s > s'\). By (A.18) and the fact that \(x^{-1}(1 - e^{-x}) \leq 1\), we find

\[
\left|g_\sigma(\tau; s) - g_\sigma(\tau; s')\right| \leq \frac{1}{2\pi} \int_\infty^{+\infty} \left[\frac{e^{-s\Re(\Psi(\xi))} - e^{-s'\Re(\Psi(\xi))}}{s - s'}\right] d\xi
\]

\[
+ e^{-s\Re(\Psi(\xi))} \left|\frac{e^{-(s-s')\Re(\Psi(\xi))} - 1}{\Re(\Psi(\xi)) (s - s')}\right| \Re(\Psi(\xi)) d\xi
\]

\[
\leq \frac{1}{2\pi} \int_\infty^{+\infty} e^{-s\Re(\Psi(\xi))} \left|\Im(\Psi(\xi))\right| d\xi + \frac{1}{2\pi} \int_\infty^{+\infty} e^{-s\Re(\Psi(\xi))} \Re(\Psi(\xi)) d\xi
\]

\[
= \frac{1}{2\pi} \int_{-M_\gamma}^{M_\gamma} e^{-s\Re(\Psi(\xi))} \left|\Im(\Psi(\xi))\right| d\xi + \frac{1}{2\pi} \int_{-M_\gamma}^{M_\gamma} e^{-s\Re(\Psi(\xi))} \Re(\Psi(\xi)) d\xi
\]

\[
+ \frac{1}{2\pi} \int_{|\xi| > M_\gamma} e^{-s\Re(\Psi(\xi))} \left|\Im(\Psi(\xi))\right| d\xi
\]

\[
+ \frac{1}{2\pi} \int_{|\xi| > M_\gamma} e^{-s\Re(\Psi(\xi))} \Re(\Psi(\xi)) d\xi
\]

\[
=: I_1 + I_2 + I_3
\] (A.19)

where \(M_\gamma\) is defined in (A.17). Note now that

\[
\Im(\Psi(\xi)) = -\int_0^{+\infty} \sin(\xi t) \nu(dt)
\] (A.20)

is odd and therefore \(|\Im(\Psi(\xi))|\) is even. For \(\xi > M_\gamma\) and \(t < 1\), it is clear that \(|\sin(\xi t)| \leq \xi t\) and then

\[
|\Im(\Psi(\xi))| \leq \xi \int_0^1 t \nu(dt) + \nu(1).
\] (A.21)

It follows by (A.21) and (A.17) that

\[
I_2 \leq \frac{1}{\pi} \int_{\xi > M_\gamma} \left(\xi \int_0^1 t \nu(dt) + \nu(1)\right) e^{-s\xi^\alpha} d\xi < \infty,
\] (A.22)
where \( \alpha = 2 - \gamma \). Observe now that \( \Re(\Psi(\xi)) \) is even and that \( 1 - \cos(\xi t) \leq \frac{\xi^2 t^2}{2} \), thus

\[
\Re(\Psi(\xi)) = \int_0^{+\infty} (1 - \cos(\xi t)) \nu(\xi t) \leq \frac{\xi^2}{2} \int_0^{+\infty} t^2 \nu(t) + 2\nu(1). \tag{A.23}
\]

It follows by (A.23) and (A.17) that

\[
I_3 \leq \frac{1}{\pi} \int_{\nu > M, t} \left( \frac{\xi^2}{2} \int_0^{+\infty} t^2 \nu(t) + 2\nu(1) \right) e^{-\xi^2 \alpha} d\xi < \infty. \tag{A.24}
\]

It also clear that \( I_1 \) is finite and bounded as follows:

\[
I_1 \leq \frac{\sqrt{2}}{\pi} \int_{\nu > M} |\Psi(\xi)| e^{-s\Re(\Psi(\xi))} d\xi.
\]

Repeating the same argument for \( s' > s \) and combining the results we obtain, for fixed \( s > 0 \) and \( s' \in (s/2, 3s/2) \),

\[
\left| \frac{g(\tau, s) - g(\tau, s')}{s - s'} \right| \leq \sqrt{2} \int_0^{M, \nu} |\Psi(\xi)| e^{-s\Re(\Psi(\xi))} d\xi
\]

\[
+ \int_{M, \nu}^{+\infty} (3\nu(1) + \xi \int_0^{+\infty} \tau \nu(\tau) + \frac{\xi^2}{2} \int_0^{+\infty} \tau^2 \nu(\tau)) e^{-s\xi^2 - \gamma} d\xi. \tag{A.25}
\]

Since \( \nu \in L^1_{\text{loc}}(\mathbb{R}^+) \), then (2.11) follows by dominated convergence and (2.13) follows by (A.25). The latter also shows that

\[
\partial_s g(t, s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \xi |\Psi(\xi)| e^{-it\xi - s\Re(\Psi(\xi))} d\xi
\]

that, for fixed \( s > 0 \), is a continuous function of the variable \( t \in \mathbb{R}^+ \) by a simple application of the dominated convergence theorem. Furthermore,

\[
|\partial_s g(t, s)| \leq \sqrt{2} \int_0^{M, \nu} |\Psi(\xi)| e^{-s\Re(\Psi(\xi))} d\xi
\]

\[
+ \int_{M, \nu}^{+\infty} (3\nu(1) + \xi \int_0^{+\infty} \tau \nu(\tau) + \frac{\xi^2}{2} \int_0^{+\infty} \tau^2 \nu(\tau)) e^{-s\xi^2 - \gamma} d\xi.
\]

hence \( \partial_s |f_L(s; \cdot) \) is the convolution product of a \( L^1(\mathbb{R}^+) \) function with a bounded continuous function and thus it is continuous.

Now we prove the claimed behaviour at infinity. By (2.2) and (A.10), it holds

\[
\sup_{t \in [a, b]} s f_L(s, t) \leq \left( \int_0^b \nu(\tau) d\tau \right) \left| \int_0^{+\infty} e^{-s\Re(\Psi(\xi))} d\xi \right|
\]

\[
\leq 2 \left( \int_0^b \nu(\tau) d\tau \right) \left[ \int_0^{M, \nu} e^{-s\Re(\Psi(\xi))} d\xi + s \int_{M, \nu}^{+\infty} e^{-s\xi^2 - \gamma} d\xi \right]
\]

\[
\leq 2 \left( \int_0^b \nu(\tau) d\tau \right) \left[ \int_0^{M, \nu} e^{-s\Re(\Psi(\xi))} d\xi + \frac{1}{s^{2\gamma}} \Gamma \left( \frac{1}{2 - \gamma} \right) \right]. \tag{A.26}
\]

It is clear that (A.26) tends to zero as \( s \to \infty \) by using dominated convergence on the first integral and the fact that \( \gamma > 1 \). The last assertion follows by similar computation. Indeed, in the same spirit of (A.26), we obtain

\[
\sup_{t \in [a, b]} s^2 |\partial_s f_L(s, t)| \leq 2 \left( \int_0^b \nu(\tau) d\tau \right) \left[ \int_0^{M, \nu} |\Psi(\xi)| s^2 e^{-s\Re(\Psi(\xi))} d\xi + s^2 \int_{M, \nu}^{+\infty} |\Psi(\xi)| e^{-s\xi^2 - \gamma} d\xi \right]. \tag{A.27}
\]

Using the estimates on \( \Re(\Psi(\xi)) \) and \( \Re(\Psi(\xi)) \) determined above we obtain

\[
|\Psi(\xi)| \leq \left( 3\nu(1) + \frac{1}{M, \nu} \int_0^{+\infty} \tau \nu(\tau) + \frac{1}{2M, \nu} \int_0^{+\infty} \tau^2 \nu(\tau) \right) \xi^2. \tag{A.28}
\]
for $\xi > M_\gamma$. It follows that

\begin{equation}
\sup_{t \in [a,b]} s^2 |\partial_s f_L(s, t)| \leq 2 \left( \int_0^b \nu(\tau) d\tau \right) \left[ \int_0^{M_\gamma} |\Psi(\xi)| s^2 e^{-sR(\Psi(\xi))} d\xi \right]
+ \frac{1}{2 - \gamma} \left( 3\nu(1) + \frac{1}{M_\gamma} \int_0^1 \tau \nu(d\tau) + \frac{1}{2M_\gamma^2} \int_0^1 \tau^2 \nu(d\tau) \right) s^2 \frac{1}{\gamma^{3/2}} \Gamma \left( \frac{3}{2 - \gamma} \right).
\end{equation}

(A.29)

It is clear that (A.30) tends to zero as $s \to \infty$ by using dominated convergence on the first integral and, again, the fact that $\gamma > 1$.

Finally, notice that if $s \in [s_0, s_1]$ for some $0 < s_0 < s_1$, then by (2.12) and (2.13) we have

\begin{equation}
\sup_{s \in [s_0, s_1]} |\partial_s f_L(s, t)| \leq \frac{I_g(t)}{\pi} \int_0^{\infty} \xi e^{-\xi R(\Psi(\xi))} d\xi \tag{A.32}
+ \sup_{s \in [s_0, s_1]} |\partial_s f_L(s, t)| \leq \frac{I_g(t)}{\pi} \left( \sqrt{2} \int_0^{M_\gamma} |\Psi(\xi)| e^{-\xi R(\Psi(\xi))} d\xi \right)
\end{equation}

where the right-hand side converges to 0 as $t \downarrow 0$. This shows (2.14)
\end{proof}

A.2. Proof of Theorem 2.8. To prove Theorem 2.8, we first need to show the following auxiliary result,

Lemma A.1. Let $f : [0, T] \to \mathbb{R}$ be a continuous function and $t_0 \in (0, T]$ such that $f(t_0) = \min_{t \in [0, t_0]} f(t)$. Assume that there exists $\delta > 0$ such that $\partial^\Phi f(t)$ is well-defined, continuous on $(0, t_0]$ and belongs to $L^1_{\text{loc}} [0, t_0]$. Then $\partial^\Phi f(t_0) \geq 0$.

Proof. Let us first prove the statement under the additional assumption that $f \in \text{AC}[0, t_0] \cap C^1(0, t_0)$. Consider the function $g(\tau) = f(t_0) - f(\tau)$ for $\tau \in [0, t_0]$, so that, being $t_0$ the maximum point of $f$, it holds $g(\tau) \geq 0$ for any $\tau \in [0, t_0]$ and $\partial^\Phi g(\tau) = -\partial^\Phi f(\tau)$ for any $\tau \in (0, t_0]$. Since $t_0 > 0$, $g$ cannot be constant. Furthermore, since $f \in C^1(0, t_0)$, $g \in C^1[\epsilon, t_0]$ for any $\epsilon \in (0, t_0)$, and then, in particular, $g$ is Lipschitz on $[\epsilon, t_0]$, i.e.,

$$\forall \epsilon \in (0, T) \exists C_\epsilon > 0 : g(\tau) \leq C_\epsilon |t_0 - \tau| \ \forall \tau \in [\epsilon, T].$$

(A.31)

Note that by hypotheses $g$ is absolutely continuous on $[0, t_0]$, thus it follows that we can use (2.15) to say that

$$\partial^\Phi g(t_0) = \int_0^{t_0} \nu(t_0 - \tau) \partial_\tau g(\tau) d\tau,$$

(A.32)

Let us split the integral

$$\partial^\Phi g(t_0) = \int_0^\epsilon \nu(t_0 - \tau) \partial_\tau g(\tau) d\tau + \int_\epsilon^{t_0} \nu(t_0 - \tau) \partial_\tau g(\tau) d\tau$$

(A.33)

$$= I_1(\epsilon) + I_2(\epsilon),$$

(A.34)

where $\epsilon \in (0, t_0)$. We can clearly use dominated convergence theorem to obtain

$$I_2(\epsilon) = \lim_{a \to 0^+} \int_a^{t_0 - \epsilon} \nu(t) \partial_\tau g(t - \tau) d\tau.$$

(A.35)

Now let us recall that the function $\nu$ is non-increasing and finite in $[a, t_0 - \epsilon]$, hence it is of bounded variation with distributional derivative given by $-\nu$. Thus we can apply integration by parts (see [33, Theorem 3.3.1]) to obtain

$$I_2(\epsilon) = \lim_{a \to 0^+} \left[ \nu(a) g(t_0 - a) - \nu(t_0 - \epsilon) g(\epsilon) - \int_a^{t_0 - \epsilon} g(t_0 - z) \nu(dz) \right].$$

(A.36)

Combining inequality (A.31) with the fact that $\nu(t) \to 0$ as $t \to 0$ (see, e.g., [66, page 17]) and also using monotone convergence theorem, we get

$$I_2(\epsilon) = -\nu(t_0 - \epsilon) g(\epsilon) - \int_0^{t_0 - \epsilon} g(t_0 - \tau) \nu(d\tau).$$

(A.37)
Since $g$ is not constant on $(0, t_0)$, we know there exists $\varepsilon_0 \in (0, t_0)$ such that $g(\varepsilon_0) > 0$. Furthermore, we have for $\varepsilon \in (0, \varepsilon_0)$, by continuity of $g$ and the fact that $-\varphi(t_0 - \varepsilon)g(\varepsilon) \leq 0$,

$$I_2(\varepsilon) \leq -\int_0^{t_0 - \varepsilon_0} g(t_0 - \tau) \nu(d\tau) =: -C < 0,$$

(A.38)

where $C$ is finite since, by (A.31),

$$\int_0^{t_0 - \varepsilon_0} g(t_0 - \tau) \nu(d\tau) \leq C_{\varepsilon_0} \int_0^{t_0 - \varepsilon_0} \tau \nu(d\tau) < +\infty.$$

Now let us consider $I_1(\varepsilon)$. By absolute continuity of $g$ and monotonicity of $\varphi$ we clearly have

$$I_1(\varepsilon) \leq \varphi(t_0 - \varepsilon) \int_0^{\varepsilon} \partial_t g(\tau)d\tau \leq \varphi(t_0 - \varepsilon_0) \int_0^{\varepsilon} |\partial_x g(\tau)|d\tau.$$

(A.39)

By absolute continuity of the Lebesgue integral we have that there exists $\varepsilon \in (0, \varepsilon_0)$ such that $I_1(\varepsilon) \leq C/2$. Let us fix this $\varepsilon$ to achieve

$$-\partial_t^\Phi f(t_0) = \partial_t^\Phi g(t_0) = I_1(\varepsilon) + I_2(\varepsilon) \leq -\frac{C}{2} < 0$$

(A.40)

concluding the proof in the case $f \in AC[0, t_0] \cap C^1(0, t_0)$.

Now let us consider a generic $f : [0, T] \to \mathbb{R}$ satisfying the assumptions in the statement. However, we assume further that $t_0 < T$ and that $\partial_t^\Phi f(t)$ is well-defined and continuous in $(t_0 - 2\delta, t_0 + 2\delta)$ for some $\delta > 0$. We denote by $\ast$ the convolution product. Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a family of Friedrich’s mollifiers and consider the sequence $f_n = \varphi_n \ast f$. Clearly, $f_n \to f$ uniformly in $[0, T]$ (see [2, Lemma 1.3.3]). In particular, consider any sequence $t_0^n \in \argmax_{t \in [0, t_0]} f_n(t)$. Since $t_0^n \in [0, t_0]$ for all $n \in \mathbb{N}$, we can extract a (non-relabelled) subsequence such that $t_0^n \to t_0 \in [0, t_0]$. Furthermore, $f_n(t_0^n) \geq f_n(t_0)$ and then, taking the limit as $n \to \infty$, we get $f(t_0) \geq f(t_0^n)$, that guarantees that $t_0 = t_0$. Next, notice that

$$\varphi_n \ast \partial_t^\Phi f = \partial_t^\Phi f_n,$$

by associativity of the convolution product. By [2, Proposition 1.3.6] we get

$$\varphi_n \ast \partial_t^\Phi f = \partial_t^\Phi f_n,$$

and then, again, $\partial_t^\Phi f_n \to \partial_t^\Phi f$ uniformly in $[t_0 - \delta, t_0 + \delta]$. In particular, since $f_n \in C^\infty_c(\mathbb{R})$, we already know that $\partial_t^\Phi f_n(t_0^n) \geq 0$. Taking the limit and using the uniform convergence, we get $\partial_t^\Phi f(t_0) \geq 0$.

Now we are ready to prove the full statement. Let $f$ as requested and consider $\delta_n = \frac{f(t_0) - f(t_0^n)}{n}$ for $n \geq 2$. Clearly, $f(0) < f(t_0) - \delta_n < f(t_0)$, hence we can set $t_n = \min\{t \geq 0 : f(t) = f(t_0) - \delta_n\}$ and it holds $t_n \in (0, t_0)$. This guarantees that $\partial_t^\Phi f$ is continuous in an open neighbourhood of $t_n$. Furthermore $t_n = \min\{t \geq 0 : f(t) = f(t_0) - \delta_n\}$ and it holds $t_n \in (0, t_0)$.

(A.38)

Now we are ready to prove the weak maximum principle. Proof of Theorem 2.8. We argue by contradiction. Set $M_1 = \max_{(t, x) \in \partial_{\nu} E} u(t, x) \in E$ if $E$ is bounded and $M_1 = \max\{\max_{t \in [0, T]} u_\infty(t), \max_{(t, x) \in \partial_{\nu} E} u(t, x)\}$ if $E$ is unbounded. Suppose there exists $(t_0, x_0) \in E$ such that $u(t_0, x_0) > M_1$, define $\varepsilon = u(t_0, x_0) - M_1 > 0$ and consider the auxiliary function

$$w(t, x) = u(t, x) + \frac{\varepsilon}{2} \frac{T - t}{T} \quad \forall (t, x) \in \mathbb{E}$$

(A.41)

Notice that for $x \in E_2$ it holds

$$w(\mathbb{F}(x), x) = u(\mathbb{F}(x), x) + \frac{\varepsilon}{2} \frac{T - \mathbb{F}(x)}{T}.$$  

(A.42)

Now we extend $w$ to the whole cylinder $[0, T] \times E_2$ by setting, for $(t, x) \notin E$, $w(t, x) = w(t(x), x)$, so that $w(\cdot, x)$ is continuous on $[0, T]$ for any $x \in E_2$. Observe also that if $E$ is unbounded $\lim_{x \to \infty} w(t, x) = u_\infty(t) + \frac{\varepsilon}{2} \frac{T - t}{T} =: w_\infty(t)$ uniformly with respect to $t \in [0, T]$.  


Since \( \frac{T-t}{T}, \frac{T-t(x)}{T} < 1 \) for all \((t, x) \in [0, T] \times E_2\), we have that
\[
w(t, x) \leq u(t, x) + \frac{\varepsilon}{2} \quad \forall (t, x) \in [0, T] \times E_2.
\] (A.43)

Specifically, in \((t_0, x_0)\) it holds
\[
w(t_0, x_0) = u(t_0, x_0) + \frac{\varepsilon}{2} \frac{T-t_0}{T} \geq u(t_0, x_0) = \varepsilon + M_1
\] (A.44)
For any \((t, x) \in \partial_p E\) we have \(M_1 \geq u(t, x)\) and then, by (A.43),
\[
w(t_0, x_0) \geq \varepsilon + M_1 \geq \varepsilon + u(t, x) \geq \frac{\varepsilon}{2} + w(t, x), \quad \forall (t, x) \in \partial_p E.
\] (A.45)
Furthermore, if \(E\) is unbounded,
\[
w(t_0, x_0) \geq \varepsilon + M_1 \geq \varepsilon + u_\infty(t) \geq \frac{\varepsilon}{2} + w_\infty(t), \quad \forall t \in [0, T].
\] (A.46)

Now let us distinguish among two cases. If \(E\) is bounded, since \(w \in C^0(\overline{E})\), we know there exists a point \((t_1, x_1) \in \overline{E}\) such that \(w(t_1, x_1) = \max_{(t, x) \in \overline{E}} w(t, x)\) and \(t_1 = \min \arg \max_{t \in [0, T]} w(t, x_1)\). Furthermore, \(w(t_1, x_1) \geq w(t_0, x_0) > w(t, x)\) for all \((t, x) \in \partial_p E\), hence \((t_1, x_1) \in E^*\). If \(E\) is unbounded then there exists a subsequence \((t^n, x^n)\) such that \(\lim_{n \to \infty} w(t^n, x^n) = \sup_{(t, x) \in E} w(t, x)\). We can always extract a (non-relabelled) subsequence \((t^n, x^n)\) such that \(|x^n|\) is monotone. If \(|x^n| \uparrow +\infty\), then we can extract a (non-relabelled) subsequence \((t^n, x^n)\) such that \(|x^n| \uparrow +\infty\) and \(t^n \to t^* \in [0, T]\) so that
\[
\sup_{(t, x) \in E} w(t, x) = \limsup_{n \to \infty} w(t^n, x^n) = w_\infty(t^*) < w(t_0, x_0),
\]
that is absurd. Hence either \(|x^n|\) is non-increasing or non-decreasing and bounded. Again, we can extract a (non-relabelled) subsequence \((t^n, x^n)\) converging towards a point \((t^*, x_1) \in \overline{E}\) such that \(w(t^*, x_1) = \max_{(t, x) \in \overline{E}} w(t, x)\). If we set \(t_1 = \min \arg \max_{t \in [0, T]} w(t, x_1)\), we still have \(w(t_1, x_1) = \max_{(t, x) \in \overline{E}} w(t, x)\). Furthermore, arguing as before, \((t_1, x_1) \in E^*\).

Since \(w(t_1, \cdot) \in C^2(E_2(t_1))\), we have
\[
\begin{aligned}
\nabla w(t_1, x_1) &= 0, \\
\Delta w(t_1, x_1) &\leq 0.
\end{aligned}
\] (A.47)

Moreover, observe that
\[
\partial_1 \Phi u(t, x) = \partial_1 \Phi w(t, x) + \frac{\varepsilon}{2T} I_\Phi(t), \quad \forall (t, x) \in E.
\] (A.48)

Hence, if \(t(x_1) = 0\), then \(\partial_1 \Phi w(\cdot, x_1) \in C(0, t_1) \cap L^1[0, t_1]\) and then by Lemma A.1 we have
\[
\partial_1 \Phi w(t_1, x_1) \geq 0.
\] (A.49)

If \(t(x_1) \neq 0\), notice that setting \(\tilde{w}(t) = w(t + t(x_1), x_1)\), it is clear that \(\partial_1 \Phi w(t, x_1) = \partial_1 \Phi \tilde{w}(t - t(x_1))\). Again (A.48) guarantees that \(\partial_1 \Phi \tilde{w}(\cdot) \in C(0, t_1 - t(x_1)) \cap L^1[0, t_1 - t(x_1)]\) and \((t_1 - t(x_1)) = \min \arg \max_{t \in [0, t_1 - t(x_1)]} \tilde{w}(t)\), hence by Lemma A.1 we get again (A.49).

Finally, we get
\[
\partial_1 \Phi u(t_1, x_1) - Lu(t_1, x_1) = \partial_1 \Phi w(t_1, x_1) + \frac{\varepsilon}{2T} I(t_1) - p_2(x_1) \Delta w(t_1, x_1) > 0
\] (A.50)
which is absurd. \(\square\)

A.3. Proof of Proposition 3.1. Let \((t_n, x_n, y_n) \to (t, x, y)\) in \(\mathbb{R}^+ \times \mathbb{R}^2\). Notice that
\[
p_\Phi(t_n, x_n; y_n) = E_0[p(L(t_n), x_n; y_n)].
\]
Furthermore, there exists \(t_0 > 0\) such that \(t_n, t \geq t_0\) for all \(n \in \mathbb{N}\) and then
\[
p(L(t_n), x_n; y_n) \leq \frac{1}{\sqrt{2\pi L(t_0)}}
\]
where the right-hand side has finite expectation. Hence, by dominated convergence theorem, we have that \(p_\Phi \in C(\mathbb{R}^+ \times \mathbb{R}^2)\). Now consider a compact \(K \subset \mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2)\) and let \(\varepsilon = \min_{(x, y) \in K} |x - y|\). Then it holds
\[
\sup_{(x, y) \in K} p_\Phi(t, x; y) \leq E[p(L(t), \varepsilon; 0)].
\] (A.51)
Recalling that, for all \( r, \lambda > 0 \),
\[
\sqrt{r} e^{-\lambda r} \leq \frac{1}{\sqrt{2\lambda e}},
\]
we can use the dominated convergence theorem to ensure that the right-hand side of (A.51) converges to 0, hence getting that
\[
\lim_{t \downarrow 0} p_\phi(t, x; y) = 0 \ \text{locally uniformly in } \mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2).
\]
Next, consider any function \( f \in C_b(\mathbb{R}) \) and observe that, by a simple application of Fubini’s theorem,
\[
\int_\mathbb{R} p_\phi(t, x; y) f(y) \, dy = \mathbb{E} \left[ \int_\mathbb{R} p(L(t), x; y) f(y) \, dy \right].
\]
Notice that
\[
\int_\mathbb{R} p(L(t), x; y) |f(y)| \, dy \leq \|f\|_{\infty}
\]
hence by dominated convergence theorem we have
\[
\lim_{t \downarrow 0} \int_\mathbb{R} p_\phi(t, x; y) f(y) \, dy = \mathbb{E} \left[ \lim_{t \downarrow 0} \int_\mathbb{R} p(L(t), x; y) f(y) \, dy \right] = f(x),
\]
where we also used that \( p(t, x; y) \, dy \to \delta_x(dy) \) weakly as \( t \downarrow 0 \).

Now recall that
\[
\partial_x p(t, x; y) = -\sqrt{2\pi t^3} (x - y) e^{-\frac{(x-y)^2}{2t^2}}.
\]
Fix \( (t, x, y) \in \mathbb{R}^+ \times (\mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2)) \) and let \( \delta > 0 \) be such that \( y \notin [x - \delta, x + \delta] \). Let also \( \varepsilon = \max\{x - y - \delta, y - x - \delta\} \). Since, for all \( r, \lambda > 0 \),
\[
r e^{-\lambda r} \leq \frac{1}{\lambda e},
\]
we have, for \( z \in [x - \delta, x + \delta] \),
\[
|\partial_x p(L(t), z; y)| \leq \frac{8}{e^2 \pi L(t)} |z - y|^{-1} \leq \varepsilon^{-1} \left[ \frac{8}{e^2 \pi L(t)} \right]^\frac{1}{2}
\]
where we used the fact that \( |z - y| > \varepsilon \). Notice that the right-hand side has finite expectation. Hence, by a simple application of the dominated convergence theorem, we can derive under the expectation sign in
\[
p_\phi(t, x; y) = \mathbb{E}_0 [p(L(t), x; y)]
\]
to achieve
\[
\partial_x p_\phi(t, x; y) = \mathbb{E}_0 [\partial_x p(L(t), x; y)],
\]
that is (3.2).

Now let \( (t_n, x_n, y_n) \to (t, x, y) \) in \( \mathbb{R}^+ \times (\mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2)) \). Then there exist \( t_0 > 0 \) and a compact set \( K \subset \mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2) \) such that \( (t_n, x_n, y_n), (t, x, y) \in [t_0, +\infty) \times K \) for all \( n \in \mathbb{N} \). Let \( \varepsilon = \min_{(x, y) \in K} |x - y| > 0 \) and notice that, as before
\[
|\partial_x p(L(t_n), x_n; y_n)| \leq \varepsilon^{-1} \sqrt{\frac{8}{e^2 \pi L(t_0)}},
\]
hence continuity in \( (t, x, y) \) follows by dominated convergence. Next, consider a compact set \( K \subset \mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2) \) and let \( \varepsilon = \min_{(x, y) \in K} |x - y| \) and \( M = \max_{(x, y) \in K} |x - y| \). Then it holds
\[
\sup_{(x, y) \in K} |\partial_x p_\phi(t, x; y)| \leq \sqrt{\frac{2}{\pi}} M \mathbb{E}_0 \left[ (L(t))^{-\frac{3}{2}} e^{-\frac{\varepsilon^2}{2M}} \right]. \tag{A.52}
\]
Recalling that, for all \( r, \lambda > 0 \),
\[
r \frac{3}{2} e^{-\lambda r} \leq \left( \frac{3}{2\lambda e} \right)^{\frac{3}{2}},
\]
and then, a.s.,
\[
(L(t))^{-\frac{3}{2}} e^{-\frac{\varepsilon^2}{2M}} \leq \left( \frac{3}{\varepsilon^2 e} \right)^{\frac{3}{2}} \]
we can use the dominated convergence theorem to ensure that the right-hand side of (A.53) converges to 0, hence getting that
\[
\lim_{t \downarrow 0} \partial_x \phi(t, x; y) = 0 \text{ locally uniformly in } \mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2).
\]

Concerning the second derivative, notice that
\[
\partial_x^2 \phi(t, x; y) = \frac{2}{\pi t^3} \left( \frac{(x - y)^2}{t} - 1 \right) e^{-\frac{(x-y)^2}{2t}}
\]
and then
\[
|\partial_x^2 \phi(t, x; y)| = \frac{2}{\pi t^3} \left( \frac{(x - y)^2}{t} + 1 \right) e^{-\frac{(x-y)^2}{2t}}
\]
Fix \((t, x, y) \in \mathbb{R}^+ \times (\mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2))\) and let \(\delta > 0\) be such that \(y \notin [x - \delta, x + \delta]\). Set also \(\varepsilon = \max \{y - x - \delta, y - x + \delta\}\). Since, for all \(r, \lambda > 0\),
\[
r e^{-\lambda r} \leq \frac{1}{\lambda e} \quad \text{and} \quad r^2 e^{-\lambda r} \leq \frac{4}{\lambda^2 e^2},
\]
we have, for \(z \in [x - \delta, x + \delta]\),
\[
|\partial_x^2 \phi(L(t), z; y)| \leq \varepsilon^{-2} \sqrt{\frac{8}{\pi t^2 L(t)}} \left( 1 + \frac{8}{e} \right)
\]
where the right-hand side has finite expectation. Hence, by a simple application of the dominated convergence theorem, we can derive under the expectation sign in
\[
\partial_x \phi(t, x; y) = E_0 \left[ \partial_x \phi(L(t), x; y) \right]
\]
to achieve
\[
\partial_x^2 \phi(t, x; y) = E_0 \left[ \partial_x^2 \phi(L(t), x; y) \right],
\]
that is (3.3).

Now let \((t_n, x_n, y_n) \to (t, x, y)\) in \(\mathbb{R}^+ \times (\mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2))\). Then there exist \(t_0 > 0\) and a compact set \(K \subset \mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2)\) such that \((t_n, x_n, y_n), (t, x, y) \in [t_0, +\infty) \times K\) for all \(n \in \mathbb{N}\). Let \(\varepsilon = \min_{(x,y) \in K} |x - y| > 0\) and notice that, as before
\[
|\partial_x^2 \phi(L(t_n), x_n; y_n)| \leq \varepsilon^{-2} \sqrt{\frac{8}{\pi t^2 L(t_0)}} \left( 1 + \frac{8}{e} \right),
\]
hence continuity in \((t, x, y)\) follows by dominated convergence. Next, consider a compact set \(K \subset \mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2)\) and let \(\varepsilon = \min_{(x,y) \in K} |x - y|\) and \(M = \max_{(x,y) \in K} |x - y|\). Then it holds
\[
\sup_{(x,y) \in K} \left| \partial_x^2 \phi(t, x; y) \right| \leq \sqrt{\frac{2}{\pi}} E_0 \left[ \left( M^2(L(t))^{-\frac{1}{2}} + L(t)^{-\frac{1}{2}} \right) e^{-\frac{5}{2\varepsilon^2 L(t/2)}} \right]. \tag{A.53}
\]
Recalling that, for all \(r, \lambda > 0\),
\[
r^{\frac{3}{2}} e^{-\lambda r} \leq \left( \frac{3}{2 \lambda e} \right)^{\frac{3}{2}} \quad \text{and} \quad r^{\frac{5}{2}} e^{-\lambda r} \leq \left( \frac{5}{2 \lambda e} \right)^{\frac{5}{2}},
\]
and then, a.s.,
\[
\left( M^2(L(t))^{-\frac{1}{2}} + L(t)^{-\frac{1}{2}} \right) e^{-\frac{5}{2\varepsilon^2 L(t/2)}} \leq M^2 \left( \frac{5}{\varepsilon^2 e} \right)^{\frac{5}{2}} + \left( \frac{3}{\varepsilon^2 e} \right)^{\frac{3}{2}}
\]
we can use the dominated convergence theorem to ensure that the right-hand side of (A.53) converges to 0, hence getting that
\[
\lim_{t \downarrow 0} \partial_x^2 \phi(t, x; y) = 0 \text{ locally uniformly in } \mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2).
\]
Now we prove (3.4) under the additional assumption (A2). Assume that \(x \neq y\) and consider a compact set \([0, T] \subset (0, +\infty)\) and let \(t \in K\) and \(s \in (0, +\infty)\). By (2.12) we have that
\[
p(s, x; y) |\partial_t f_L(s, t)| \leq \frac{I_\Phi(T)}{\pi} p(s, x; y) \int_0^{+\infty} \xi e^{-s R(\Psi(\xi))} d\xi =: I_4(s, x; y), \tag{A.54}
\]
where the right-hand side is independent of \( t \in K \). Next, notice that
\[
\int_0^{+\infty} I_1(s, x; y) \, ds = \frac{I_\phi(T)}{\sqrt{2\pi s^3}} \int_0^{+\infty} s^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{2s}} - s \Re(\Psi(\xi)) \, \xi \, ds \, d\xi. \tag{A.55}
\]
Let us first evaluate the inner integral. To do this, rewrite
\[
\int_0^{+\infty} s^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{2s}} - s \Re(\Psi(\xi)) \, ds
= 2 \left( \frac{|x-y|}{\sqrt{2\Re(\Psi(\xi))}} \right)^{\frac{1}{2}} \frac{1}{2} \left( \frac{|x-y|}{\sqrt{2\Re(\Psi(\xi))}} \right)^{-\frac{1}{2}} \int_0^{+\infty} s^{-\frac{1}{2} - 1} e^{-\frac{2\Re(\Psi(\xi)) - s}{s}} \, ds.
\]
that, by [27, Formula 3.471.9], becomes
\[
\int_0^{+\infty} s^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{2s} - \Re(\Psi(\xi))} \, ds = 2 \left( \frac{|x-y|}{\sqrt{2\Re(\Psi(\xi))}} \right)^{\frac{1}{2}} K_{-\frac{1}{2}}(\sqrt{2\Re(\Psi(\xi))}|x-y|), \tag{A.56}
\]
where \( K_{\nu}(\cdot) \) is the modified Bessel function of the third kind (see [27]). In particular, it holds, by [27, Formula 8.469.3]
\[
K_{-\frac{1}{2}}(\sqrt{2\Re(\Psi(\xi))}|x-y|) = \sqrt{\frac{\pi}{\Re(\Psi(\xi))}} e^{-|x-y|/\sqrt{2\Re(\Psi(\xi))}}, \tag{A.57}
\]
\[
\int_0^{+\infty} s^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{2s} - \Re(\Psi(\xi))} \, ds = \sqrt{\frac{\pi}{\Re(\Psi(\xi))}} e^{-|x-y|/\sqrt{2\Re(\Psi(\xi))}}. \tag{A.58}
\]
Going back to (A.55) we have
\[
\int_0^{+\infty} I_1(s, x; y) \, ds = \frac{I_\phi(T)}{\pi \sqrt{2}} \int_0^{+\infty} \frac{\xi}{\sqrt{\Re(\Psi(\xi))}} e^{-|x-y|/\sqrt{2\Re(\Psi(\xi))}} \, d\xi. \tag{A.59}
\]
Next, we split the integral as
\[
\int_0^{+\infty} I_1(s, x; y) \, ds = \frac{I_\phi(T)}{\pi \sqrt{2}} \left( \int_0^{1} \frac{\xi}{\sqrt{\Re(\Psi(\xi))}} e^{-|x-y|/\sqrt{2\Re(\Psi(\xi))}} \, d\xi \right.
\]
\[
+ \int_1^{M_{\gamma}} \frac{\xi}{\sqrt{\Re(\Psi(\xi))}} e^{-|x-y|/\sqrt{2\Re(\Psi(\xi))}} \, d\xi + \int_{M_{\gamma}}^{+\infty} \frac{\xi}{\sqrt{\Re(\Psi(\xi))}} e^{-|x-y|/\sqrt{2\Re(\Psi(\xi))}} \, d\xi \right)
\]
\[
\leq \frac{I_\phi(t_1)}{\pi \sqrt{2}} \left( \frac{1}{\sqrt{C_0}} + \frac{M_{\gamma}(M_{\gamma} - 1)}{\min_{\xi \in [1, M_{\gamma}]} \Re(\Psi(\xi))} + \frac{1}{\sqrt{C_\gamma}} \int_0^{+\infty} \frac{\xi^2}{\sqrt{\min_{\xi \in [1, M_{\gamma}]} \Re(\Psi(\xi))}} \, d\xi \right) < \infty, \tag{A.60}
\]
where we used Lemma 2.2 and the fact that \( \Re(\Psi(\xi)) \) is continuous in \( \xi \) and \( \Re(\Psi(\xi)) > 0 \) for \( \xi > 0 \). Hence, observing that, for \( t \in [t_0, t_1] \) and fixed \( x, y \) such that \( x \neq y \), we proposed an upper bound in (A.54) that belongs to \( L^1(\mathbb{R}^+) \), by a simple application of the dominated convergence theorem, we achieve (3.4).

Concerning the continuity of the function \( \partial_{x}p_{\Phi}(t, x; y) \) we proceed as follows. Consider a vector \( (t, x, y) \in \mathbb{R}^+ \times (\mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2)) \) and \( \{(t_n, x_n, y_n)\}_{n \in \mathbb{N}} \subset \mathbb{R}^+ \times (\mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2)) \) such that \( (t_n, x_n, y_n) \to (t, x, y) \). Clearly there exists a compact \( K \subset \mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2) \) and \( T > 0 \) such that \( (x_n, y_n) \in K \) and \( t_n \in [0, T] \) for all \( n \in \mathbb{N} \). Let \( \varepsilon = \min_{(z_1, z_2) \in K} |z_1 - z_2| > 0 \). Then we have, by (2.12)
\[
p(s, x_n; y_n) \partial_{x}f_{L}(s; t_n) \leq \frac{I_\phi(T)}{\pi} p(s, \varepsilon; 0) \int_0^{+\infty} \xi e^{-s \Re(\Psi(\xi))} \, d\xi =: I_1(s, \varepsilon; 0).
\]
as in (A.54). Since we already know that \( I_1(\cdot, \varepsilon; 0) \in L^1(\mathbb{R}^+) \), we can use the dominated convergence theorem in (3.4) to guarantee that \( \lim_{n \to \infty} \partial_{x}p_{\Phi}(t_n, x_n; y_n) = p_{\Phi}(t, x; y) \). Since \( (t, x, y) \in \mathbb{R}^+ \times (\mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2)) \) is arbitrary, this proves that \( \partial_{x}p_{\Phi} \in C(\mathbb{R}^+ \times (\mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2))) \).

Finally, in order to show (3.5) consider a compact \( K \subset \mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2) \) and let \( \varepsilon = \min_{(x, y) \in K} |x - y| \). Then
\[
\sup_{(x, y) \in K} |\partial_{x}p_{\Phi}(t, x; y)| \leq \int_0^{+\infty} p(s, \varepsilon; 0) |\partial_{x}f_{L}(s; t)| \, ds. \tag{A.61}
\]
Let $T > 0$ and notice that for $t \leq T$ it holds
\[ p(s, \varepsilon; 0) |\partial_t f_L(s; t)| \leq I_1(s, \varepsilon; 0) \]
where the right-hand side belongs to $L^1(\mathbb{R}^+)$. Hence, by dominated convergence theorem and (A.62), we get
\[ \lim_{t \to 0} \sup_{(x,y) \in K} |\partial_t p_\phi(t, x; y)| = 0. \] (A.62)

Now observe that for $x \neq y$ we have that $p_\phi(\cdot, x; y) \in C^1(\mathbb{R}^+_0)$, once we set $\partial_t p_\phi(0, x; y) = 0$. Hence, by Lemma 2.4 and (3.4), it holds
\[ \partial_t^t p_\phi(t, x; y) = \int_0^t \nabla(t - s) \partial_t p_\phi(s, x; y) ds \]
\[ = \int_0^t \nabla(t - s) \left( \int_0^{+\infty} p(\tau, x; y) \partial_t f_L(\tau; s) d\tau \right) ds. \] (A.63)

Notice that, by (2.12) and (A.58),
\[ \int_0^t \int_0^{+\infty} \nabla(t - s)p(\tau, x; y)|\partial_t f_L(\tau; s)| d\tau ds \leq \frac{1}{\pi} \int_0^t \int_0^{+\infty} \int_0^{+\infty} \nabla(t - s)p(\tau, x; y)I_\phi(s)\xi e^{-\tau R(\Psi(\xi))} d\xi d\tau ds \]
\[ = \frac{1}{\sqrt{2\pi^2}} \left( \int_0^{+\infty} \frac{\xi}{\sqrt{2R(\Psi(\xi))}} \xi e^{-|x-y|\sqrt{2R(\Psi(\xi))}} d\xi \right) \left( \int_0^t \nabla(t - s)I_\phi(s) ds \right) < \infty, \]
where the finiteness of the first integral follows as in (A.60). Hence we can use Fubini’s theorem in (A.63) to achieve
\[ \partial_t^t p_\phi(t, x; y) = \int_0^{+\infty} p(\tau, x; y) \left( \int_0^t \nabla(t - s) \partial_t f_L(\tau; s) ds \right) d\tau. \]

Again, setting $\partial_t f_L(\tau; 0) = 0$ for $\tau > 0$, we know that $f_L(\tau, \cdot) \in C^1(\mathbb{R}^+_0)$ and we can use again Lemma 2.4 and Proposition 2.6 to get
\[ \partial_t^t p_\phi(t, x; y) = \int_0^{+\infty} p(\tau, x; y)\partial_t^t f_L(\tau; t) d\tau = -\int_0^{+\infty} p(s, x; y) \partial_s f_L(s; t) ds. \]

To show that $\partial_t^t p_\phi \in C(\mathbb{R}^+ \times \mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2))$ fix $(t, x, y) \in \mathbb{R}^+ \times (\mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2))$ and consider a sequence \{(tn, xn, yn)\}_n \subset \mathbb{R}^+ \times (\mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2)) such that $(tn, xn, yn) \to (t, x, y)$. Then there exist a compact $K \subset \mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2)$ and $T > 0$ such that $tn \in [0, T]$ and $(xn, yn) \in K$ for all $n \in \mathbb{N}$. Let also $\varepsilon = \min_{(z_1, z_2) \in K} |z_1 - z_2|$. By (2.13) we have
\[ |p(s, x; y) \partial_s f_L(s; t)| \leq I_2(s, \varepsilon; 0), \]
where
\[ I_2(s, x; y) = p(s, x; y)I_\phi(T)(\sqrt{2}J_1(s) + J_2(s)), \] (A.64)
\[ J_1(s) = \int_0^{M_\gamma} |\Psi(\xi)| e^{-s R(\Psi(\xi))} d\xi, \] (A.65)
\[ J_2(s) = \int_{M_\gamma}^{+\infty} \left( 3\sqrt{\pi} + \xi \int_0^1 \tau \nu(d\tau) + \frac{\xi^2}{2} \int_0^1 \tau^2 \nu(d\tau) \right) e^{-s\xi^2} d\xi. \] (A.66)

Now we need to show that $I_2(\cdot, \varepsilon; 0) \in L^1(\mathbb{R}^+)$. To do this, notice that
\[ \int_0^{+\infty} p(s, \varepsilon; 0)J_1(s) ds \leq M_\gamma \sup_{\xi \in [0, M_\gamma]} |\Psi(\xi)| \int_0^{+\infty} \frac{1}{\sqrt{2\pi s}} e^{-\frac{\xi^2}{2s}} ds < \infty. \]

Concencing $J_2(s)$, first notice that there exists a constant $C > 0$ such that, for $\xi \geq M_\gamma$,
\[ 3\sqrt{\pi} + \xi \int_0^1 \tau \nu(d\tau) + \frac{\xi^2}{2} \int_0^1 \tau^2 \nu(d\tau) \leq C\xi^2. \]
Furthermore, we have, arguing as in (A.58),
\[
\int_0^{+\infty} p(s, \varepsilon; 0) J_2(s) \, ds \leq C \int_{M_n}^{+\infty} \frac{\xi^2}{\sqrt{2\pi}} \left( \int_0^{+\infty} s^{-\frac{1}{2}} e^{-s^{2^\gamma} - \frac{\xi^2}{s^2}} \, ds \right) \, d\xi \\
\leq C \frac{1}{\sqrt{2}} \int_0^{+\infty} \xi^{2^\gamma+\frac{1}{2}} e^{-\xi s^{2^\gamma}} \, d\xi < \infty.
\]

Hence \( I_2(\cdot, \varepsilon; 0) \in L^1(\mathbb{R}^+) \) and then \( \lim_{n \to \infty} \partial_t^np_n(t, x_n; y_n) = \partial_t^n p_n(t, x; y) \) by dominated convergence theorem.

Finally, consider a compact set \( K \subset \mathbb{R}^2 \setminus \text{diag}(\mathbb{R}^2) \) and let \( \varepsilon = \min_{(x,y) \in K} |x - y| \). Then it holds
\[
\sup_{(x,y) \in K} |\partial_t^np_n(t, x; y)| \leq \frac{f_n(t)}{\pi} \int_0^{+\infty} p(s, \varepsilon; 0)(\sqrt{2}J_1(s) + J_2(s)) \, ds
\]
where the right-hand side clearly goes to 0 as \( t \downarrow 0 \).

\[\square\]

### A.4. Proof of Proposition 3.4.

Since \( p_n(t, x; y) = p_n(t, x - y; 0) \), it is sufficient to prove the statement for \( y = 0 \). Recall that, by (3.2), we have
\[
\partial_x p_n(t, x; 0) = -\sqrt{\frac{2}{\pi}} \int_0^{+\infty} xe^{-\frac{x^2}{s^2}} f_L(s; t) \, ds.
\]  
(A.67)

First consider \( x > 0 \). By using the change of variables \( z = \frac{x}{\sqrt{s}} \) for \( x > 0 \), the latter becomes
\[
\partial_x p_n(t, x; 0) = -\frac{4}{\sqrt{\pi}} \int_0^{+\infty} e^{-z^2} f_L \left( \frac{x}{2z^2}; t \right) \, dz.
\]  
(A.68)

Notice further that
\[
2\varphi(t) - \partial_x p_n(t, x; 0) = \frac{4}{\sqrt{\pi}} \int_0^{+\infty} e^{-z^2} \left( \varphi(t) - f_L \left( \frac{x}{2z^2}; t \right) \right) \, dz.
\]  
(A.69)

Now consider a compact set \( K \subset \mathbb{R}^+ \) and notice that
\[
\sup_{t \in K} |2\varphi(t) - \partial_x p_n(t, x; 0)| \leq \frac{4}{\sqrt{\pi}} \int_0^{+\infty} e^{-z^2} \sup_{t \in K} |\varphi(t) - f_L \left( \frac{x}{2z^2}; t \right)| \, dz.
\]  
(A.70)

By Assumption (A3) we know that
\[
\lim_{s \downarrow 0} \sup_{t \in K} |\varphi(t) - f_L(s; t)| = 0.
\]

Furthermore, by Proposition 2.3, we know that \( \lim_{s \to +\infty} \sup_{t \in K} |f_L(s; t)| = 0 \). Hence, there exists a constant \( C > 0 \) such that
\[
\sup_{t \in K} |\varphi(t) - f_L(s; t)| \leq C, \quad \forall s \geq 0.
\]

Hence, by dominated convergence theorem and (A.70) we achieve
\[
\lim_{x \downarrow 0} \sup_{t \in K} |2\varphi(t) - \partial_x p_n(t, x; 0)| = 0.
\]

The argument for \( x < 0 \) is analogous.

Now recall that, by (3.6), for \( x \neq 0 \) it holds
\[
\partial_t^np_n(t, x; 0) = -\int_0^{+\infty} p(s, x; 0)\partial_sf_L(s, t) \, ds.
\]

First, notice that by Proposition 2.3 we know that \( \lim_{s \to +\infty} s^2|\partial_sf_L(s, t)| = 0 \) locally uniformly with respect to \( t \in (0, +\infty) \). Now fix a compact set \( K \subset (0, +\infty) \) and let \( \delta > 0 \) and \( h : (0, \delta) \to \mathbb{R}^+ \) as in Assumption (A3). Then we know that there exists a constant \( C > 0 \) such that
\[
\sup_{t \in K} |f_L(s, t)| \leq Cs^{-2}, \quad s \geq \delta
\]
and then
\[
\int_0^{+\infty} s^{-\frac{1}{2}} \sup_{t \in K} |f_L(s, t)| \, ds \leq \int_0^{\delta} s^{-\frac{1}{2}} h(s) \, ds + \int_{\delta}^{+\infty} s^{-\frac{1}{2}} \, ds < \infty.
\]
Now notice that
\[
\sup_{t \in K} \left| \frac{\partial_x^p p(t, x; 0)}{\sqrt{2\pi}} \int_0^{+\infty} s^{-\frac{1}{2}} f_L(s, t) \, ds \right| \leq \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \left(1 - e^{-\frac{s^2}{2}}\right) s^{-\frac{1}{2}} \sup_{t \in K} |\partial_x f_L(s, t)| \, ds.
\]

The integrand can be controlled as follows
\[
\left(1 - e^{-\frac{s^2}{2}}\right) s^{-\frac{1}{2}} \sup_{t \in K} |\partial_x f_L(s, t)| \leq s^{-\frac{1}{2}} \sup_{t \in K} |\partial_x f_L(s, t)|,
\]
where the right-hand side is integrable, hence by dominated convergence we have
\[
\lim_{x \to 0} \sup_{t \in K} \left| \frac{\partial_x^p p(t, x; 0)}{\sqrt{2\pi}} \int_0^{+\infty} s^{-\frac{1}{2}} f_L(s, t) \, ds \right| = 0.
\]

Finally,
\[
\lim_{x \to 0} \sup_{t \in K} \left| \frac{\partial_x^2 p(t, x; 0)}{\sqrt{2\pi}} \int_0^{+\infty} s^{-\frac{1}{2}} f_L(s, t) \, ds \right| = 0
\]
follows by (A.71) and Theorem 3.2.

\section*{A.5. Uniform convergence of monotone functions.}

\begin{proposition}
Let $F_n, F : \mathbb{R}_0^+ \to \mathbb{R}$ for any $n \in \mathbb{N}$ such that $F_n(t) \to F(t)$ as $n \to \infty$, for any $t \in \mathbb{R}_0^+$. Suppose further that $F_n, F$ are continuous, non-decreasing and $\lim_{t \to \infty} F_n(t) = \lim_{t \to \infty} F(t) = l \in \mathbb{R}$. Then $F_n \to F$ uniformly.
\end{proposition}

\begin{proof}
Without loss of generality we can suppose that $l = 1$. Fix $\epsilon > 0$. Then there exists $T > 0$ such that for any $t \geq T$ it is true that $1 - F(t) < \epsilon/2$. Note that $F(t)$ is continuous in $[0, T]$ and thus uniformly continuous. It follows that there exists $\delta > 0$ such that for any choice of $t_1$ and $t_2$ in $[0, T]$ it holds
\[
|t_2 - t_1| < \delta \implies |F(t_2) - F(t_1)| < \epsilon/2.
\]
Take now a finite partition $\Pi$ of $[0, T]$ with $\text{diam}(\Pi) < \delta$. Since $F_n(t) \to F(t)$ for any $t$ we have by pointwise convergence that there exists $\nu > 0$ such that, for any $n > \nu$ one has
\[
|F_n(t_i) - F(t_i)| < \epsilon/2 \text{ for all } t_i \in \Pi.
\]
Now consider $t > T$. We have that
\[
F_n(t) \leq 1 \leq F(t) + \epsilon/2 < F(t) + \epsilon
\]
where we used the fact that since $t > T$ then $1 < F(t) + \epsilon/2$. On the other hand, for $n > \nu$
\[
F_n(t) \geq F_n(T) \\
\geq F(T) - \epsilon/2 \quad \text{by (A.73) since } T \in \Pi \\
\geq F(t) - (F(t) - F(T)) - \epsilon/2 \\
\geq F(t) - \epsilon \quad \text{since } F(t) - F(T) \leq 1 - F(T) < \epsilon/2.
\]
For $t \in [0, T]$ instead, we have that there exists $t_i, t_{i+1} \in \Pi$ such that $t \in [t_i, t_{i+1}]$. It follows that, for $n > \nu$,
\[
F_n(t) \leq F_n(t_{i+1}) \\
\leq F(t_{i+1}) + \epsilon/2 \quad \text{by (A.73) since } t_{i+1} \in \Pi \\
\leq F(t) + (F(t_{i+1}) - F(t)) + \epsilon/2 \\
\leq F(t) + \epsilon \quad \text{by (A.72) since } \text{diam}(\Pi) < \delta.
\]

With the same argument it is possible to see that, for any $n > \nu$
\[
F_n(t) \geq F(t) - \epsilon.
\]
\end{proof}
A.6. Proof of Proposition 6.1. By [36, Problem 8.6], we have that

\[
E_0[|B^1(t)|^2 1_{T_c > t}] = \int_{-\infty}^{c} x^2 p(t, x; 0) \, dx + \int_{-\infty}^{c} x^2 p(t, x; 2c) \, dx
\]

\[
= \int_{-\infty}^{c} x^2 p(t, x; 0) \, dx - \int_{-\infty}^{c} x^2 p(t, x; 0) \, dx
- 2c \int_{-\infty}^{c} x p(t, x; 0) \, dx - 4c^2 \int_{-\infty}^{c} p(t, x; 0) \, dx
\]

\[
= 2 \int_{0}^{c} x^2 p(t, x; 0) \, dx - 2c \int_{-\infty}^{c} x p(t, x; 0) \, dx - 4c^2 \int_{-\infty}^{c} p(t, x; 0) \, dx
\]

\[
= 2 \int_{0}^{c} x^2 p(t, x; 0) \, dx + 2c \left( \int_{0}^{c} x p(t, x; 0) \, dx - \int_{0}^{c} p(t, x; 0) \, dx \right)
- 4c^2 \left( \frac{1}{2} - \int_{0}^{c} p(t, x; 0) \, dx \right)
\]

\[
(A.78)
\]

Now, by [27, Formula 3.321.5],

\[
\int_{0}^{c} x^2 p(t, x; 0) \, dx = t \text{erf} \left( \frac{c}{\sqrt{2t}} \right) - c \sqrt{\frac{t}{2\pi} e^{-c^2}}.
\]

\[
(A.79)
\]

Next, notice that

\[
\int_{0}^{+\infty} x p(t, x; 0) \, dx = \sqrt{\frac{t}{2\pi}} \quad \int_{0}^{c} x p(t, x; 0) \, dx = \sqrt{\frac{t}{2\pi} \left( 1 - e^{-c^2} \right)}
\]

\[
(A.80)
\]

and furthermore

\[
\int_{0}^{c} p(t, x; 0) \, dx = \frac{1}{2} \text{erf} \left( \frac{c}{\sqrt{2t}} \right).
\]

\[
(A.81)
\]

Plugging (A.79), (A.80) and (A.81) into (A.78) we get (6.1). □

ACKNOWLEDGEMENTS

G. Ascione and B. Toaldo have been partially supported by the MIUR PRIN 2017 project “Stochastic Models for Complex Systems”, no. 2017JFFHSH.

The author B. Toaldo acknowledges financial support under the National Recovery and Resilience Plan (NRRP), Mission 4, Component 2, Investment 1.1, Call for tender No. 104 published on 2.2.2022 by the Italian Ministry of University and Research (MUR), funded by the European Union – NextGenerationEU– Project Title “Non–Markovian Dynamics and Non-local Equations” – 202277N5H9 - CUP: D53D23005670006 - Grant Assignment Decree No. 973 adopted on June 30, 2023, by the Italian Ministry of Ministry of University and Research (MUR).

B. Toaldo would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme Stochastic systems for anomalous diffusion, where work on this paper was undertaken. This work was supported by EPSRC grant EP/Z000580/1.

REFERENCES

[1] V. V. Afanasiev, R. Z. Sagdeev and G. M. Zaslavsky. Chaotic jets with multifractal space-time random walk. Chaos: An Interdisciplinary Journal of Nonlinear Science 1(2): 143–159, 1991.
[2] W. Arendt, C.J.K. Batty, M. Hieber and F. Neubrander. Vector valued Laplace transform and Cauchy problem. Second Edition. Birkhäuser, Berlin, 2010.
[3] G. Ascione. Abstract Cauchy problems for the generalized fractional calculus. Nonlinear Analysis, Theory, Methods and Applications, 209: 112339, 2021.
[4] G. Ascione, N. Leonenko and E. Pirozzi. Time-Non-Local Pearson Diffusions. Journal of Statistical Physics 183(3): 1 – 42, 2021.
[5] G. Ascione, Y. Mishura and E. Pirozzi. Convergence results for the time-changed fractional Ornstein–Uhlenbeck processes. *Theory of Probability and Mathematical Statistics* 104: 23 – 47, 2021.

[6] G. Ascione, Y. Mishura and E. Pirozzi *Fractional deterministic and stochastic calculus* Walter de Gruyter GmbH & Co KG, 2023.

[7] G. Ascione, E. Pirozzi and B. Toaldo. On the exit time from open sets of some semi-Markov processes. *Annals of Applied Probability*, 30(3): 1130 – 1163, 2020.

[8] G. Ascione, M. Savov and B. Toaldo. Regularity and asymptotics of densities of inverse subordinators *Transactions of the London Mathematical Society*, to appear, 2024. Available on arXiv:2303.13890

[9] B. Baeumer and M.M. Meerschaert. Stochastic solutions for fractional Cauchy problems. *Fractional Calculus and Applied Analysis* 4: 481–500, 2001.

[10] B. Baeumer and P. Straka. Fokker-Planck and Kolmogorov backward equations for continuous time random walk limits. *Proceedings of the American Mathematical Society*, 145: 399 – 412, 2017.

[11] P. Becker Kern, M.M. Meerschaert and H. P. Scheffler. Limit theorems for coupled continuous time random walks. *The Annals of Probability*, 32: 730 – 756, 2004.

[12] L. Beghin, C. Macci and C. Ricciuti. Random time-change with inverses of multivariate subordinators: Governing equations and fractional dynamics. *Stochastic Processes and their Applications*, 130(10): 6364 – 6387, 2020.

[13] G. Ben Arous and J. Černý Dynamics of trap models. *Les Houches* Vol. 83. Elsevier, 2006. 331-394.

[14] G. Ben Arous, M. Cabezas, J. Černý and R. Royfman Randomly trapped random walks *The Annals of Probability*, 43(5): 2405–2457, 2015.

[15] J. Bertoin. Lévy processes. *Cambridge University Press*, Cambridge, 1996.

[16] J. Bertoin. Subordinators: examples and applications. *Lectures on Probability Theory and Statistics* (Saint-Flour, 1997), 1 – 91. *Lectures Notes in Math.*, 1717, Springer, Berlin, 1999.

[17] N.H. Bingham, C.M. Goldie and J.F. Teugels. Regular variation. *Cambridge University Press*, Cambridge, 1987.

[18] A. Biswas and J. Lörinczi. Maximum principles for time-fractional Cauchy problems with spatially non-local components. *Fractional Calculus and Applied Analysis*, 21(5): 1335 – 1359, 2018.

[19] Z.-Q. Chen. Time fractional equations and probabilistic representation. *Chaos, Solitons and Fractals*, 102: 168 – 174, 2017.

[20] A. Chiarini, S. Floreani, F. Redig and F. Sau. Fractional kinetics equation from a Markovian system of interacting Bouchaud trap models. *arXiv preprint arXiv:2302.10156* (2023).

[21] K. L. Chung and Z. Zhao *From Brownian motion to Schrödinger’s equation*. Springer Science & Business Media, 2012.

[22] E. Cinlar. Markov additive processes. I. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 24: 85 – 93, 1972.

[23] E. Cinlar. Markov additive processes and semi-regeneration. *Discussion Paper No. 118*, Northwestern University, 1974.

[24] B. Dacorogna. Introduction to Calculus of Variations. *World Scientific Publishing*, 2014.

[25] F. Falcini, R. Garra and V. Voller Modeling anomalous heat diffusion: Comparing fractional derivative and non-linear diffusivity treatments. *International Journal of Thermal Sciences* 137: 584–588, 2019.

[26] R. Garra, F. Falcini, V. R. Voller, F. Pagnini A generalized Stefan model accounting for system memory and non-locality, *International Communications in Heat and Mass Transfer* 114 (2020): 104584.

[27] I.S. Gradshteyn and I.M. Ryzhik. *Table of integrals, series, and products*. Academic press, 2014

[28] M. E. Gurtin and A. C. Pipkin. A general theory of heat conduction with finite wave speeds. *Archive for Rational Mechanics and Analysis* 31:113–126, 1968.

[29] M. Hairer, G. Iyer, L. Koralov, A. Novikov, and Z. Pajor-Gyulai. A fractional kinetic process describing the intermediate time behaviour of cellular flows. *The Annals of Probability*, 46(2): 897 – 955, 2018.

[30] M. Hairer, L. Koralov and Z. Pajor-Gyulai From averaging to homogenization in cellular flows–An exact description of the transition *Annales de l’Institut Henri Poincaré-Probabilités et Statistiques*, 52(4), 2016.

[31] B. Harlamov *Continuous semi-Markov processes* John Wiley & Sons, 2013.

[32] M. E. Hernández-Hernández, V.N. Kolokoltsov and L. Tonialzi. Generalised Fractional Evolution Equations of Caputo Type. *Chaos, Solitons & Fractals*, 102: 184 – 196, 2017.
[33] E. Hille, R. S. Phillips. Functional analysis and semi-groups. *American Mathematical Society*, Vol. 31, 1996.

[34] N. Jacob. Pseudo-differential operators and Markov processes. Vol I. *Imperial College Press*, 2002.

[35] J. Jacod. Systèmes régénératifs et processus semi-Markoviens. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 31: 1 – 23, 1974.

[36] I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus*. Springer, 2014.

[37] H. Kaspi and B. Maisonneuve. Regenerative systems on the real line. *Ann. Probab.*, 16: 1306 – 1332, 1988.

[38] A.N. Kochubei. General fractional calculus, evolution equations and renewal processes. *Integral Equations and Operator Theory*, 71: 583 – 600, 2011.

[39] A.N. Kochubei, Y.G. Kondratiev and J.L. da Silva. Random time change and related evolution equations. Time asymptotic behavior. *Stochastics and Dynamics*, 20(5): 2050034, 2020.

[40] A.N. Kolmogorov and S.V. Fomin. Introductory real analysis. *Dover Publications, Inc.*, New York, 1975.

[41] V.N. Kolokoltsov. Generalized Continuous-Time Random Walks, subordination by hitting times, and fractional dynamics. *Theory of Probability and its Applications* 53:, 594–609, 2009.

[42] H.R. Lerche. Boundary Crossing of Brownian Motion. *Springer-Verlag*, 1980.

[43] Y. Luchko. Maximum principle for the generalized time-fractional diffusion equation. *Journal of Mathematical Analysis and Applications* 351.1: 218–223, 2009.

[44] M. Magdziarz and R.L. Schilling. Asymptotic properties of Brownian motion delayed by inverse subordinators. *Proceedings of the American Mathematical Society*, 143: 4485 – 4501, 2015.

[45] M. M. Meerschaert and H. P. Scheffler. Limit theorems for continuous-time random walks with infinite mean waiting times *Journal of applied probability* 41(3): 632–638, 2004.

[46] M.M. Meerschaert and H.P. Scheffler. Triangular array limits for continuous time random walks. *Stochastic Processes and their Applications*, 118(9): 1606 – 1633, 2008.

[47] M.M. Meerschaert and H.P. Scheffler. Stochastic model for ultraslow diffusion *Stochastic processes and their applications* 116(9):1215–1235, 2006.

[48] M.M. Meerschaert, E. Nane and P. Vellaisamy. The fractional Poisson process and the inverse stable subordinator. *Electronic Journal of Probability*, 16(59): 1600–1620, 2011.

[49] M.M. Meerschaert and P. Straka. Inverse stable subordinators. *Mathematical modelling of natural phenomena*, 8:2: 1 – 16, 2013.

[50] M.M. Meerschaert and P. Straka. Semi-Markov approach to continuous time random walk limit processes. *The Annals of Probability*, 42(4) : 1699 – 1723, 2014.

[51] M.M. Meerschaert and B. Toaldo. Relaxation patterns and semi-Markov dynamics. *Stochastic Processes and their Applications*,29(8): 2850 – 2879, 2019.

[52] R. Metzler and J. Klafter. The random walk’s guide to anomalous diffusion: a fractional dynamics approach. *Physics Reports*, 339: 1 – 77, 2000.

[53] C. Monthus and J. P. Bouchaud Models of traps and glass phenomenology *Journal of Physics A: Mathematical and General* 29(14): 3847, 1996.

[54] J. W. Nunziato, On heat conduction in materials with memory *Quarterly of Applied Mathematics* 29(2):187–204, 1971.

[55] S. Orey. On continuity properties of infinitely divisible distribution functions. *The Annals of Mathematical Statistics*, 39(3): 936–937, 1968.

[56] P. Patie, M. Savov and R.L. Loeffen. Extinction time of non-Markovian self-similar processes, persistence, annihilation of jumps and the Fréchet distribution. *Journal of Statistical Physics*, 175(5): 1022 – 1041, 2019.

[57] P. Patie and A. Szapiolyan. Self-similar Cauchy problems and generalized Mittag-Leffler functions. *Fractional Calculus and Applied Analysis*, 24(2): 447 - 482, 2021.

[58] Y. Povstenko, *Fractional thermoelectricity*. Springer International Publishing, 2015.

[59] B. Rinn, P. Maass and J. P. Bouchaud, Hopping in the glass configuration space: subaging and generalized scaling laws *Physical Review B* 64(10): 104417, 2001.

[60] S.D. Roscani. Hopf lemma for the fractional diffusion operator and its application to a fractional free-boundary problem. *Journal of Mathematical Analysis and Applications*, 434(1): 125 – 135, 2016.
[62] S.D. Roscani. Moving-boundary problems for the time-fractional diffusion equation. *Electronic Journal of Differential Equations*, 2017(44): 1 – 12, 2017.

[63] A. I. Saichev and G. M. Zaslavsky. Fractional kinetic equations: solutions and applications. *Chaos: an interdisciplinary journal of nonlinear science* 7(4):753–764, 1997.

[64] M. Savov and B. Toaldo. Semi-Markov processes, integro-differential equations and anomalous diffusion-aggregation. *Annales de l’Institut Henri Poincaré (B) Probability and Statistics*, 56(4): 2640 – 2671, 2020.

[65] R. L. Schilling, and L. Partzsch. *Brownian motion: an introduction to stochastic processes* Walter de Gruyter GmbH & Company KG, 2014.

[66] R.L. Schilling, R. Song and Z. Vondráček. *Bernstein functions: theory and applications*. Walter de Gruyter GmbH & Company KG, 2010.

[67] B. Toaldo. Lévy mixing related to distributed order calculus, subordinators and slow diffusions. *Journal of Mathematical Analysis and Applications*, 430(2): 1009 – 1036, 2015.

[68] C.J. Vogl, M.J. Miksis and S.H. Davis. Moving boundary problems governed by anomalous diffusion. *Proceedings of the Royal Society A*, 468: 3348 – 3369, 2012.

[69] V. R. Voller, On a fractional derivative form of the Green–Ampt infiltration model *Advances in water resources* 34(2): 257–262, 2011

[70] V. R. Voller, Introducing non-locality into solidification models *Transactions of the Indian Institute of Metals* 65: 515–529, 2012

[71] V.R. Voller Fractional Stefan problems. *International Journal of Heat and Mass Transfer*, 74: 269 – 277, 2014

[72] V.R. Voller, F. Falcini and R. Garra. Fractional Stefan problems exhibiting lumped and distributed latent-heat memory effects. *Physical Review E*, 87: 042401, 2013.

[73] R. L. Wheeden, A. Zygmund. Measure and integral. *Dekker*, New York, 1977.

[74] W. Whitt. Stochastic-Process Limits. *Springer-Verlag*, Berlin, 2002.

[75] G.M. Zaslavskii and B.V. Chirikov. Stochastic Instability of Non-linear Oscillations, *Soviet Physics Uspekhi*, 14: 549, 1972.

[76] G. M. Zaslavsky Fractional kinetic equation for Hamiltonian chaos, *Physica D: Nonlinear Phenomena* 76(1-3):110–122, 1994.

[77] G. M. Zaslavsky Chaos, fractional kinetics, and anomalous transport *Physics reports* 371(6):461–580, 2002.

[78] G. M. Zaslavsky *Hamiltonian Chaos and Fractional Dynamics* Oxford University Press, USA, 2005.