Lie systems and Schrödinger equations

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Abstract
We prove that $t$-dependent Schrödinger equations on finite-dimensional Hilbert spaces determined by $t$-dependent Hermitian Hamiltonian operators can be described through Lie systems admitting a Vessiot–Guldberg Lie algebra of Kähler vector fields. This result is extended to other related Schrödinger equations, e.g. projective ones, and their properties are studied through Poisson, presymplectic and Kähler structures. This leads to derive nonlinear superposition rules for them depending in a lower (or equal) number of solutions than standard linear ones. Special attention is paid to applications in $n$-qubit systems.

Keywords: Hamiltonian vector field, Kähler structure, Lie system, Poisson structure, projective Schrödinger equation, superposition rule, symplectic structure, $t$-dependent Schrödinger equation, Vessiot–Guldberg Lie algebra.

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1. Introduction
It is undoubtable that geometric techniques, e.g. Lie symmetries or jet bundles, have become a standard tool in the study of differential equations and related problems [10, 40, 51]. In particular, this work focuses on the geometric analysis of Lie systems appearing in quantum mechanics [19, 21, 41, 47, 57]. A Lie system is a non-autonomous system of first-order ordinary differential equations whose general solution can be written in terms of a generic finite family of particular solutions and a set of constants via a (generally) nonlinear function, a so-called superposition rule [19, 21, 47, 57].

Lie systems occur in the research on the integrability of quantum systems [2], $t$-dependent Schrödinger equations [24], $t$-dependent frequency Smrodnisky–Winternitz oscillators [7],
several types of Ermakov systems and Milne–Pinney equations \[15, 22, 25\], string theory \[30\], deformation of mechanical systems \[33\], control systems \[50\], etcetera (see \[23\]).

Lie systems of physical or mathematical relevance can be studied via symplectic \[2, 19\], Poisson \[26\], \(k\)-symplectic \[48\], Jacobi \[39\], Dirac \[18\] and Nambu structures \[45\]. This allows one to use geometric techniques to analyse their properties, e.g. their solutions \[2\], constants of motion \[7\], Lie symmetries \[26\], and other features \[19, 21, 36, 41, 57\]. This has also led to develop new mathematical tools so as to investigate Lie systems \[48\].

Although Lie systems have already been applied in quantum mechanical systems \[11, 24, 27, 28\], there still exist many open problems. In particular, this article addresses the application of Lie systems in \(t\)-dependent Schrödinger equations on finite-dimensional Hilbert spaces and their projections/restrictions to relevant spaces, e.g. projective Schrödinger equations \[12\]. A special role is played by the use of geometric structures, e.g. Kähler structures, which enables us to calculate superposition rules through the distributional approach devised in \[21\] and its refinement for Lie systems with compatible geometric structures \[7\]. It is worth noting that Kähler structures naturally appear as a consequence of the quantum nature of the problems under study.

In geometric terms, the Lie–Scheffers Theorem \[21, 23, 47\] states that a Lie system amounts to a \(t\)-dependent vector field taking values in a finite-dimensional Lie algebra of vector fields: a Vessiot–Guldberg Lie algebra (VG–Lie algebra) \[23, 42, 43\].

A particular branch of the research on Lie systems is devoted to the study of Lie systems admitting a VG–Lie algebra of Hamiltonian vector fields with respect to a geometric structure. In the pioneering work \[19\], the authors briefly analysed Lie systems with a VG–Lie algebra of Hamiltonian vector fields with respect to a symplectic structure. The study of Lie–Hamilton systems, i.e. Lie systems with a VG–Lie algebra of Hamiltonian vector fields relative to a Poisson structure, was initiated in \[26\]. This gave rise to new methods to investigate such Lie systems \[4, 34, 35\]. Lie systems admitting a VG–Lie algebra of Hamiltonian vector fields relative to Dirac and \(k\)-symplectic structures were studied in \[18, 48\]. Recently, Lie systems with a VG–Lie algebra of Hamiltonian vector fields relative to a Nambu structure have been investigated in \[45\].

The first aim of this work is to show that \(t\)-dependent Schrödinger equations on a finite-dimensional Hilbert space \(\mathcal{H} := \mathbb{C}^n\) related to a \(t\)-dependent Hermitian Hamiltonian operator can be studied through Lie systems admitting a VG–Lie algebra \(V_{\mathcal{M}_{2n}} \simeq \mathfrak{u}(n)\) of Kähler vector fields with respect to the Kähler structure induced by the natural Hermitian product on \(\mathbb{C}^n\) \[13, 16\]. We prove that \(t\)-dependent Schrödinger equations on \(\mathbb{C}^n\) related to \(t\)-dependent traceless Hermitian Hamiltonian operators admit nonlinear superposition rules depending on \(n − 1\) particular solutions. Thus, such quantum systems can be endowed with a simple, generally nonlinear, superposition rule allowing us to recover their general solutions by means of a lower number of particular solutions than by standard linear superposition rules.

The Lie groups \(U(1)\) and \(\mathbb{R}_+\) act freely on \(\mathbb{C}^n_0 := \mathbb{C}^n \setminus \{(0, \ldots, 0)\}\), by multiplication. The corresponding spaces of orbits are denoted by \(\mathbb{C}^n_0 / U(1)\) and \(\mathbb{C}^n_0 / \mathbb{R}_+\). To highlight that previous spaces can be considered as real manifolds, they will be denoted \(\mathcal{R}_n\) and \(\mathcal{S}_n\), respectively. Likewise, the spaces \(\mathbb{C}^n, \mathbb{C}^n_0\) and \(\mathbb{C}^n_0 / \mathbb{C}_0\) will be represented by \(\mathcal{M}_{2n}, \mathcal{M}_{2n}^\times\) and \(\mathcal{P}_n\), respectively. We prove that the restriction to \(\mathcal{M}_{2n}^\times\) of the \(t\)-dependent Schrödinger
equations referred in the preceding paragraph can be projected onto $\mathcal{R}_n$ and restricted to the unit sphere $S_n$ giving rise to Lie systems admitting VG–Lie algebras of Hamiltonian vector fields relative to different geometric structures, e.g. Dirac and Poisson structures.

Subsequently, the solutions of the referred to as projective Schrödinger equations [13] are recovered through the projection onto the projective space $\mathcal{P}_n$ of the $t$-dependent Schrödinger equations on $\mathcal{M}_{2n}^\times$. This allows us to understand geometrically standard projective Schrödinger equations as Lie systems admitting a VG–Lie algebra of Kähler vector fields relative to the Kähler structure induced by the Study–Fubini metric on $\mathcal{P}_n$ [13].

Above findings suggest us to define a new type of Lie systems possessing a Lie algebra of Kähler vector fields with respect to a Kähler structure, the Kähler–Lie systems, and to use techniques from Riemannian and symplectic geometry to study them.

Using our results we derive geometrically superposition rules for $t$-dependent Schrödinger equations on $\mathcal{M}_{2n}$ related to $t$-dependent traceless Hermitian Hamiltonian operators. This allows us to obtain superposition rules without the integration of vector fields or PDEs as in standard methods [21, 57]. Similarly, we study and calculate superposition rules for the projections of the previous Schrödinger equations on certain spaces $\mathcal{S}_n$, $\mathcal{R}_n$ and $\mathcal{P}_n$. Schrödinger equations on $\mathcal{M}_4$, $\mathcal{R}_2$, $\mathcal{P}_2$ are analysed in detail. Most relevant results concerning the preceeding equations and superposition rules are summarised in Table 1. Their interest is due to its occurrence in the research on qubits.

Table 1: The following diagram illustrates the geometric structures and natural inclusions employed to study the Lie systems induced by the projection on each space of $t$-dependent Schrödinger equations on $\mathcal{M}_{2n}$ related to traceless Hermitian Hamiltonian operators. The number $m$ stands for the number of particular solutions of their superposition rules. The right column shows some known diffeomorphisms used in our work.

$$\mathcal{M}_{2n}^\times \simeq S_n \times \mathbb{R}_+,$$
$$S_n \simeq S^{2n-1} \simeq U(n)/U(n-1),$$
$$\mathcal{R}_n \simeq \mathcal{P}_n \times \mathbb{R}_+.$$
tion rules for $t$-dependent Schrödinger equations and their projections to previous spaces. Section 8 is devoted to provide superposition rules for one-qubit systems and their projections onto the above mentioned spaces. The cases of $n$-qubit systems and other $t$-dependent Schrödinger equations and their projections are analysed in Section 9. Our results and future work are summarised in Section 10.

2. Fundamentals

If not otherwise stated, we assume mathematical objects to be real, smooth, and globally defined to omit minor technical problems and to highlight main results. Systems of differential equations are assumed to be non-autonomous systems of ordinary differential equations.

Let $(V,[\cdot,\cdot])$ be a Lie algebra with Lie bracket $[\cdot,\cdot]: V \times V \to V$. For the sake of simplicity, we will denote the Lie algebra by $V$ if $[\cdot,\cdot]$ is known from context. Given subsets $\mathcal{A}, \mathcal{B} \subset V$, we write $[\mathcal{A}, \mathcal{B}]$ for the linear subspace of $V$ spanned by the Lie brackets between elements of $\mathcal{A}$ and $\mathcal{B}$, and we define $\text{Lie}(\mathcal{B},[\cdot,\cdot])$ to be the smallest Lie subalgebra of $V$ containing $\mathcal{B}$. We will simply write $\text{Lie}(\mathcal{B})$ if it is clear what we mean.

A generalised distribution $\mathcal{D}$ on a manifold $N$ is a function mapping each $x \in N$ to a linear subspace $\mathcal{D}_x \subset T_x N$. We say that $\mathcal{D}$ is regular at $x' \in N$ if $r: x \in N \mapsto \dim \mathcal{D}_x \in \mathbb{N} \cup \{0\}$ is locally constant around $x'$. Similarly, $\mathcal{D}$ is said to be regular on an open $U \subset N$ when $r$ is constant on $U$. Finally, a vector field $Y$ on $N$ takes values in $\mathcal{D}$, in short $Y \in \mathcal{D}$, if $Y_x \in \mathcal{D}_x$ for all $x \in N$.

A $t$-dependent vector field $X$ on $N$ is a map $X: (t,x) \in \mathbb{R} \times N \mapsto X(t,x) \in TN$ such that $\tau_N \circ X = \pi_2$, where $\pi_2: (t,x) \in \mathbb{R} \times N \mapsto x \in N$ and $\tau_N$ is the canonical projection of the tangent bundle on $N$. A $t$-dependent vector field $X$ on $N$ amounts to a family of vector fields $\{X_t\}_{t \in \mathbb{R}}$ on $N$, where $X_t: x \in N \mapsto X(t,x) \in TN$ for all $t \in \mathbb{R}$ \cite{23}. A $t$-dependent vector field $X$ is projectable relative to a map $\pi: N \to M$ when $X_t$ is projectable with respect to $\pi$ for each $t \in \mathbb{R}$.

The smallest Lie algebra of $X$ is the smallest real Lie subalgebra, $V^X$, containing $\{X_t\}_{t \in \mathbb{R}}$, namely $V^X = \text{Lie}(\{X_t\}_{t \in \mathbb{R}})$. Every Lie algebra $V$ of vector fields on $N$ induces an integrable generalised distribution $\mathcal{D}^V := \{X(x)|X \in V, x \in N\} \subset TN$ on $N$.

An integral curve of $X$ is an integral curve $\gamma: \mathbb{R} \mapsto \mathbb{R} \times N$ of the suspension of $X$, i.e. the vector field $X(t,x) + \partial/\partial t$ on $\mathbb{R} \times N$ \cite{11}. The curve $\gamma$ always admits a reparametrisation $\bar{t} = \bar{t}(t)$ such that

$$\frac{d(\pi_2 \circ \gamma)}{d\bar{t}}(\bar{t}) = (X \circ \gamma)(\bar{t}).$$

This system is referred to as the associated system of $X$. Conversely, a system of first-order differential equations in normal form is always the associated system of a unique $t$-dependent vector field. This induces a bijection between $t$-dependent vector fields and systems of first-order differential equations in normal form. This justifies to denote by $X$ both a $t$-dependent vector field and its associated system.

**Definition 2.1.** A superposition rule depending on $m$ particular solutions for a non-autonomous system $X$ on $N$ is a map $\Phi: (u(1), \ldots, u(m); k) \in N^m \times N \mapsto \Phi(u(1), \ldots, u(m); k) \in
N such that the general solution, \( x(t) \), of \( X \) can be written as 
\[
    x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)}(t); k),
\]
where \( x_{(1)}(t), \ldots, x_{(m)}(t) \) is a generic set of particular solutions to \( X \), and \( k \in N \).

**Example 2.1.** It is known that a Riccati equation, namely
\[
    \frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)x^2, \quad x \in \mathbb{R},
\]
where \( a_0(t), a_1(t), a_2(t) \) are \( t \)-dependent real functions satisfying \( a_0(t)a_2(t) \neq 0 \), is such that its general solution can be brought into the form
\[
    x(t) = \Phi(x_{(1)}(t), x_{(2)}(t), x_{(3)}(t); k),
\]
with \( \Phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R} \) defined by
\[
    \Phi(u_{(1)}, u_{(2)}, u_{(3)}; k) := \frac{u_{(1)}(u_{(3)} - u_{(2)}) + ku_{(2)}(u_{(1)} - u_{(3)})}{u_{(3)} - u_{(2)} + k(u_{(1)} - u_{(3)})},
\]
where \( x_{(1)}(t), x_{(2)}(t), x_{(3)}(t) \) are different particular solutions to (2.1).

**Theorem 2.2.** (The Lie–Scheffers Theorem [21, 47]) A system \( X \) on \( N \) admits a superposition rule if and only if \( X = \sum_{\alpha=1}^{r} b_{\alpha}(t)X_{\alpha} \) for a family \( b_{1}(t), \ldots, b_{r}(t) \) of \( t \)-dependent functions and a basis \( X_{1}, \ldots, X_{r} \) of a real Lie algebra of vector fields on \( N \).

If \( X \) possesses a superposition rule, then \( X \) is called a *Lie system*. The associated real Lie algebra of vector fields \( \langle X_{1}, \ldots, X_{r} \rangle \) is called a *VG–Lie algebra* of \( X \). The Lie–Scheffers theorem amounts to saying that \( X \) is a Lie system if and only if \( V^{X} \) is finite-dimensional. This fact is the keystone of the theory of Lie systems. When \( V^{X} \) consists of Hamiltonian vector fields relative to some geometric structure, much more powerful methods can be used to study Lie systems [1, 13, 34, 35, 45, 48].

**Definition 2.3.** A system \( X \) on \( N \) is a *Lie–Hamilton system* if \( V^{X} \) is a VG–Lie algebra of Hamiltonian vector fields relative to some Poisson bivector field on \( N \).

**Note 2.4.** A vector field \( X \) is *Hamiltonian* relative to a Poisson bivector \( \Lambda \) with Hamiltonian function \( h \) if \( X = -\hat{\Lambda}(dh) \) for \( \hat{\Lambda} : T^*N \to TN \), given by \( \hat{\Lambda} : \theta \in T^*N \mapsto \Lambda(\theta, \cdot) \in TN \). This is the standard convention in geometric mechanics, while the definition \( X = \Lambda(dh) \) is usually chosen in Poisson geometry [55].

**Definition 2.5.** A *Lie–Hamilton structure* is a triple \( (N, \Lambda, h) \), where \( \Lambda \) is a Poisson bivector on \( N \) and \( h : (t, x) \in \mathbb{R} \times N \mapsto h_{t}(x) := h(t, x) \in \mathbb{R} \) is such that \( \text{Lie} \{ h_{t} \}_{t \in \mathbb{R}}, \{ \cdot, \cdot \}_{\Lambda} \), where \( \{ \cdot, \cdot \}_{\Lambda} \) is the Lie bracket induced by \( \Lambda \) [55], is finite-dimensional.

**Theorem 2.6.** (Characterisation of Lie–Hamilton systems [26]) A system \( X \) on \( N \) is a Lie–Hamilton system if and only if there exists a Lie–Hamilton structure \( (N, \Lambda, h) \) such that \( X_{t} \) is a Hamiltonian vector field for the function \( h_{t} \) for each \( t \in \mathbb{R} \). We say that \( \text{Lie} \{ h_{t} \}_{t \in \mathbb{R}}, \{ \cdot, \cdot \}_{\Lambda} \) is a *Lie–Hamilton algebra* of \( X \).

Lie–Hamilton algebras can be employed to find superposition rules and constants of motion for Lie–Hamilton systems in a more easy way than by standard methods [7].
Example 2.2. A complex Riccati equation with \(t\)-dependent real coefficients \([14, 24]\) can be brought into
\[
\begin{align*}
\frac{dx}{dt} &= a_1(t) + a_2(t)x + a_3(t)(x^2 - y^2), \\
\frac{dy}{dt} &= a_2(t)y + a_3(t)2xy,
\end{align*}
\]
where \(a_1(t), a_2(t)\) and \(a_3(t)\) are arbitrary \(t\)-dependent real functions. Let us prove that (2.2) is a Lie–Hamilton system on \(\mathbb{R}^2_{y \neq 0} := \mathbb{R}^2 \setminus \{(x,0) \mid x \in \mathbb{R}\}\). The system (2.2) is associated with the \(t\)-dependent vector field \(X = \sum_{\alpha=1}^{3} a_\alpha(t)X_\alpha\), where
\[
X_1 := \frac{\partial}{\partial x}, \quad X_2 := x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_3 := (x^2 - y^2) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y}.
\]
Hence, \(X\) takes values in the Lie algebra \(V := \langle X_1, X_2, X_3 \rangle\). The vector fields \(X_1, X_2, X_3\) are Hamiltonian relative to the Poisson bivector \(\Lambda := \Lambda \) induced by \(H\) takes values in the Lie algebra \(\mathfrak{sl}(\mathbb{R}^2_{y \neq 0})\), then
\[
\{h_1, h_2\}_\Lambda = -h_1, \quad \{h_1, h_3\}_\Lambda = -2h_2, \quad \{h_2, h_3\}_\Lambda = -h_3.
\]
Thus, \((\mathbb{R}^2_{y \neq 0}, \Lambda, h := a_0(t)h_1 + a_1(t)h_2 + a_2(t)h_3)\) is a Lie–Hamiltonian structure for \(X\). If \(V_X \cong \mathfrak{sl}(2)\), then \((\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\Lambda) := ((h_1, h_2, h_3), \{\cdot, \cdot\}_\Lambda)\) is a Lie–Hamiltonian algebra for \(X\) isomorphic to \(\mathfrak{sl}(2)\).

One of the main properties of Lie systems is the existence of superposition rules. There are several methods to obtain them \([21, 23, 57]\). We here choose a procedure that can be improved by using Lie–Hamilton structures. This method is based upon the so-called diagonal prolongations of the vector fields \([21]\). Given a vector field \(X\) on \(N\) with local coordinate expression
\[
X = \sum_{j=1}^{n} X_j(x) \frac{\partial}{\partial x_j}, \quad x \in N, \quad n := \dim N,
\]
its diagonal prolongation to \(N^m := N \times \ldots \times N\) \((m\text{-times})\) is the vector field on \(N^m\):
\[
X^{(m)}(x^{(0)}, \ldots, x^{(m-1)}) := \sum_{a=0}^{m-1} \sum_{j=1}^{n} X^{(a)}_j(x^{(a)}) \frac{\partial}{\partial x^{(a)}_j} = \sum_{a=0}^{m-1} X^{(a)}(x^{(a)}), \quad (x^{(0)}, \ldots, x^{(m-1)}) \in N^m,
\]
where \(x^{(a)}_j(x^{(0)}, \ldots, x^{(m-1)}) := x_j(x^{(a)})\) for \(j \in \overline{1,n}\) and \(X^{(a)}(x^{(a)}) = X(x^{(a)})\) stands for \(X\) on the \(a\)-th copy of \(N\) within \(N^m\).
To calculate a superposition rule for a Lie system on $N$ with a VG–Lie algebra $V$, we find the smallest $m \in \mathbb{N}$ so that the diagonal prolongations of elements of $V$ to $N^m$ span an integral distribution of rank $\dim V$ at a generic point. Then, $m$ becomes the number of particular solutions involved in the superposition rule. The superposition rule can be obtained by deriving $n$ common first-integrals $I_1, \ldots, I_n$ for the diagonal prolongations to $N^{m+1}$ of the vector fields of $V$ satisfying that $\det(\partial(I_1, \ldots, I_n)/\partial(x_1^{(0)}, \ldots, x_n^{(0)})) \neq 0$. This gives the superposition rule by assuming $I_1 = k_1, \ldots, I_n = k_n$ and writing $x_1^{(0)}, \ldots, x_n^{(0)}$ in terms of the remaining variables $x_i^{(a)}$, with $1 < a \leq m$ and $k_1, \ldots, k_n$ (see [21, 23] for details and examples).

When a Lie system admits a VG–Lie algebra of Hamiltonian vector fields relative to some geometric structure, e.g. a symplectic or Kähler structure, there exist geometric and algebraic methods to obtain $I_1, \ldots, I_n$ and to simplify the description of the superposition rule [2]. This requires to prolongate geometric structures according to the following construction. Let $(E, N, \tau : E \to N)$ be a vector bundle. Its diagonal prolongation $N^m$ is a vector bundle $(E^m[N], N^m, \tau^{[m]} : E^m \to N^m)$, where $E^m[N] := E \times \cdots \times E$ ($m$-times) and $\tau^{[m]}$ is the only map satisfying that $\pi_{N,j} \circ \tau^{[m]} = \tau \circ \pi_E$ for $j = 1, m$, where $\pi_{E,j} : E^m \to E$ and $\pi_{N,j} : N^m \to N$ are the natural projections of $E^m[N]$ and $N^m$ onto the $j$-th copy of $E$ and $N$ within $E^m[N]$ and $N^m$, respectively. Every section $e : N \to E$ of $(E, N, \tau)$ has a natural diagonal prolongation to a section $e^m[N]$ of $(E^m[N], N^m, \tau^{[m]})$:

$$e^m[N](x^{(0)}, \ldots, x^{(m-1)}) := e(x^{(0)}) + \cdots + e(x^{(m-1)}).$$

This is the only section of $(E^m[N], N^m, \tau^{[m]})$ satisfying that $\pi_{E,j} \circ e^m[N] = e \circ \pi_{N,j}$ for $j = 1, m$.

Also of interest is the tensor field that transports vector fields from one copy of $N$ to another one within $N^m$. More specifically, let $T^{(1,1)}N$ denote the $(1,1)$-tensor bundle of the manifold $N$. For $r, s \in [0, m-1]$, we define the $S_{rs}$ to be the sections of $T^{(1,1)}N^m$ over $N^m$ of the following form:

$$X^{[s]} := \sum_{j=1}^{n} X_j(x^{(a)}) \frac{\partial}{\partial x_j^{(s)}} \text{ then } S_{rs}(X^{[s]}) := \sum_{j=1}^{n} X_j(x^{(a)}) \frac{\partial}{\partial x_j^{(r)}}, \quad \forall X \in \mathfrak{X}(N). \quad (2.5)$$

In coordinates, these tensor fields read

$$S_{rs} = \sum_{j=1}^{n} dx_j^{(s)} \otimes \frac{\partial}{\partial x_j^{(r)}}, \quad r, s \in [0, m-1]. \quad (2.6)$$

These objects will play a key role in the computation of constants of motion and superposition rules.

Finally, the diagonal prolongation of $f : N \to \mathbb{R}$ to $N^m$ is the function $f^m[N] : N^m \to \mathbb{R}$ given by $f^m[N](x^{(0)}, \ldots, x^{(m-1)}) := f(x^{(0)}) + \cdots + f(x^{(m-1)})$.

3. The geometrical description of quantum mechanics

We briefly present the geometrical formulation of quantum mechanics which has been developed during the last forty years (see [3, 16, 32] for details).
3.1. The linear, complex and Hermitian structure

To investigate quantum mechanics in a differential geometric way and to identify its similarities with the geometric formalism of classical mechanics, the Hilbert space $\mathcal{H}$ must be understood as a real Banach manifold and its algebraic structures as real differential geometric objects in such manifolds. In particular, if $\mathcal{H}$ is of a complex dimension $n$, $\mathcal{H}$ should be identified with a real $2n$-dimensional differentiable manifold $\mathcal{M}_{2n}$. Each point $\psi \in \mathcal{M}_{2n}$ represents an element of $\mathcal{H}$. Any Hilbert basis in the Hilbert space $\mathcal{H}$ defines a real global chart on $\mathcal{H}$ which determines its differentiable structure. Let $\{e_j\}_{j \in \mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$; the functions $q_j, p_j : \mathcal{M}_{2n} \to \mathbb{R}$ given by

$$\langle e_j, \psi \rangle = q_j(\psi) + ip_j(\psi), \quad j = 1, n, \quad \forall \psi \in \mathcal{M}_{2n},$$

(3.1)
define a real global chart of $\mathcal{M}_{2n}$ and $T_\psi \mathcal{M}_{2n} = \{\partial/\partial q_1, \partial/\partial p_1, \ldots, \partial/\partial q_n, \partial/\partial p_n\}$ at every $\psi \in \mathcal{M}_{2n}$. The complex structure on the $n$-dimensional Hilbert space $\mathcal{H}$, represented by the multiplication by the imaginary unit $i$, can be encoded in a $(1,1)$-tensor field $J$ on $\mathcal{M}_{2n}$ satisfying $J^2 = -\mathbb{I}$ with $\mathbb{I}$ being the $(1,1)$-tensor field given by the identity $\mathbb{I} : T_\psi \mathcal{M}_{2n} \to T_\psi \mathcal{M}_{2n}$ at every $\psi \in \mathcal{M}_{2n}$. This leads to a distribution $\text{Im}(T\mathcal{M}_{2n})$ on $\mathcal{M}_{2n}$, which is integrable. In the coordinate system (3.1), the complex structure $J$ reads

$$J = \sum_{j=1}^n \left( dq_j \otimes \frac{\partial}{\partial p_j} - dp_j \otimes \frac{\partial}{\partial q_j} \right).$$

(3.2)

Another important element of $\mathcal{H}$ is its Hermitian product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$. As $\mathcal{H} \simeq \mathbb{R}^{2n}$ as $\mathbb{R}$-linear spaces, there exists at each $\hat{\psi} \in \mathcal{M}_{2n}$ an $\mathbb{R}$-linear isomorphism $\psi \in \mathcal{M}_{2n} \mapsto \psi_{\hat{\psi}} \in T_{\hat{\psi}} \mathcal{M}_{2n}$, where the tangent vector $\psi_{\hat{\psi}}$ acts as a derivation

$$\psi_{\hat{\psi}} f := \frac{d}{dt} \bigg|_{t=0} f \left( \hat{\psi} + tf \right), \quad \forall f \in C^\infty(\mathcal{M}_{2n}).$$

(3.3)

This identification and the Hermitian product on $\mathcal{H}$ allow us to define a pair of tensor fields $g, \omega$ on $\mathcal{M}_{2n}$ satisfying:

$$g_\psi(\psi_1, \psi_2) := \Re \langle \psi_1, \psi_2 \rangle \quad \omega_\psi(\psi_1, \psi_2) := \Im \langle \psi_1, \psi_2 \rangle, \quad \forall \psi, \psi_1, \psi_2 \in \mathcal{M}_{2n}. \quad (3.4)$$

These tensor fields encode in geometrical terms the Hermitian product existing in the Hilbert space. In coordinates, these tensor fields read

$$g = \sum_{j=1}^n \left( dq_j \otimes dq_j + dp_j \otimes dp_j \right), \quad \omega = \sum_{j=1}^n \left( dq_j \otimes dp_j - dp_j \otimes dq_j \right) = \sum_{j=1}^n dq_j \wedge dp_j. \quad (3.5)$$

The tensor field $g$ becomes a Euclidean metric on $\mathbb{R}^{2n}$ while $\omega$ becomes a symplectic structure on $\mathbb{R}^{2n}$ with Darboux coordinates $\{q_j, p_j\}_{j \in \mathbb{N}}$. The tensor fields $g$ and $\omega$ satisfy some relations with the complex structure $J$:

$$g(JX, JY) = g(X, Y), \quad \omega(JX, JY) = \omega(X, Y), \quad \omega(X, Y) = g(JX, Y), \quad \forall X, Y \in \mathfrak{x}(\mathcal{M}_{2n}).$$
Thus, the Hermitian product on the complex Hilbert space \( \mathcal{H} \) leads to a Kähler structure on \( \mathcal{M}_{2n} \), which is typical of quantum models and richer than the standard symplectic one typical appearing in classical mechanics.

The metric tensor field \( g \) induces a bundle isomorphism \( G : \psi_{\psi} \in T \mathcal{M}_{2n} \mapsto g(\psi_{\psi}, \cdot) \in T^* \mathcal{M}_{2n} \). This can be used to transform \( g \) and \( \omega \) into two 2-contravariant tensor fields, i.e. \( G(\alpha, \beta) := g(G^{-1} \alpha, G^{-1} \beta) \) and \( \Lambda(\alpha, \beta) := \omega(G^{-1} \alpha, G^{-1} \beta) \). Their expressions in local coordinates are

\[
G = \sum_{j=1}^{n} \left( \frac{\partial}{\partial q_j} \otimes \frac{\partial}{\partial q_j} + \frac{\partial}{\partial p_j} \otimes \frac{\partial}{\partial p_j} \right), \quad \Lambda = \sum_{j=1}^{n} \frac{\partial}{\partial q_j} \wedge \frac{\partial}{\partial p_j}.
\]  

These tensor fields define a Poisson bracket and a commutative bracket on \( C^\infty(\mathcal{M}_{2n}) \), respectively:

\[
\{ f, g \} := \Lambda(df, dg), \quad \{ f, g \}_+ := G(df, dg), \quad \forall f, g \in C^\infty(\mathcal{M}_{2n}).
\]  

A third element in the description of the Hilbert space structure of \( \mathcal{H} \) is its \( \mathbb{R} \)-linear structure. Geometrically, it is induced by the so-called dilation vector field defined by \( \Delta : \psi \in \mathcal{M}_{2n} \mapsto \psi \psi \in T \mathcal{M}_{2n} \). Meanwhile, the phase-change vector field takes the form \( \Gamma : \psi \in \mathcal{M}_{2n} \mapsto J \psi \psi \in T \psi \mathcal{M}_{2n} \). In local coordinates

\[
\Delta = \sum_{j=1}^{n} \left( q_j \frac{\partial}{\partial q_j} + p_j \frac{\partial}{\partial p_j} \right), \quad \Gamma = \sum_{j=1}^{n} \left( q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j} \right).
\]  

Both vector fields satisfy the relation \( \Gamma = J(\Delta) \).

Finally, let us define two \( n \)-forms, \( \Omega_R \) and \( \Omega_I \), on \( \mathcal{M}_{2n} \) satisfying

\[
\Omega_R|_{\psi}(\psi_1, \ldots, \psi_n) := \text{Re}(\det(\psi_1, \ldots, \psi_n)), \quad \Omega_I|_{\psi}(\psi_1, \ldots, \psi_n) := \text{Im}(\det(\psi_1, \ldots, \psi_n)),
\]

for all \( \psi, \psi_1, \ldots, \psi_n \in \mathcal{M}_{2n} \). They satisfy the relation

\[
\Omega_R(JX_1, \ldots, X_n) = -\Omega_I(X_1, \ldots, X_n), \quad \forall X_1, \ldots, X_n \in \mathfrak{x}(\mathcal{M}_{2n}).
\]

It is simple to prove that they are non-degenerate and closed.

### 3.2. Observables: Hamiltonian dynamics and Killing vector fields

The real vector space \( \text{Herm}(\mathcal{H}) \) of physical observables on \( \mathcal{H} \), i.e. Hermitian operators on \( \mathcal{H} \), can also be given a tensor description. Every observable \( A \in \text{Herm}(\mathcal{H}) \) gives rise to a function on \( \mathcal{M}_{2n} \) of the form

\[
f_A(\psi) := \frac{1}{2} \langle \psi, A \psi \rangle, \quad \psi \in \mathcal{M}_{2n}.
\]  

It is worth noting that

\[
\{ f_A, f_B \} = \Lambda(df_A, df_B) = f_{[A,B]}, \quad \{ f_A, f_B \}_+ = G(df_A, df_B) = f_{[A,B]_+},
\]
for
\[ [A, B] := -i(AB - BA), \quad [A, B]_+ := AB + BA. \] (3.10)

Observe that the skew-symmetric operation has an extra factor with respect to the commutator of operators. This factor is needed to obtain an inner composition law in the space of Hermitian operators. Given a linear operator \( A \in \text{Herm}(\mathcal{H}) \), we can define the Hamiltonian vector field:
\[ X_A := -\Lambda(df_A, \cdot) = \{\cdot, f_A\}. \] (3.11)

One remarkable property of these Hamiltonian vector fields is:
\[ [X_A, X_B] = -X_{[A, B]}. \]

The vector field \( \Gamma \) defined in (3.8) is the Hamiltonian vector field associated, up to a sign, with the identity operator on \( \mathcal{H} \), i.e. \( \Gamma = -X_I \).

The integral curves of the Hamiltonian vector field, \( X_H \), associated with the quadratic form \( f_H(\psi) := \frac{1}{2}\langle \psi, H\psi \rangle \) correspond to the solutions of the Schrödinger equation
\[
\frac{d\psi}{dt} = H\psi,
\]
where we assumed, as hereafter, \( \hbar = 1 \). The evolution operator \( t \mapsto U_t \) of this equation is such that each \( U_t : \mathcal{H} \to \mathcal{H} \) is an isometry of the Hermitian product on \( \mathcal{H} \). Hence, each \( U_t \) leaves invariant its real and imaginary parts. Since each \( U_t \) is \( \mathbb{C} \)-linear, it also leaves invariant \( \omega \), \( J \), and \( g \). Therefore, \( X_H \) is also a Killing vector field relative to \( g \) giving rise to a Kähler vector field.

### 3.3. Projective Hilbert spaces as Kähler manifolds

From a physical point of view, the probabilistic interpretation requires the set of states of a quantum system to be a complex projective space. Our aim in this section is to introduce the geometrical structures arising in this case. To simplify the notation, \( B^\times \) will stand for the restriction of a structure \( B \) on \( \mathcal{M}_{2n} \) to \( \mathcal{M}_{2n}^\times \), e.g. \( G^\times \) is the tensor field on \( \mathcal{M}_{2n}^\times \) obtained by restricting the tensor \( G \) on \( \mathcal{M}_{2n} \) given in (3.6) to \( \mathcal{M}_{2n}^\times \).

The equivalence relation on \( \mathcal{H}_0 := \mathcal{H}\setminus\{0\} \) defining its projective space \( \mathbb{CP}^{n-1} \), namely
\[ \psi_1, \psi_2 \in \mathcal{H}_0 := \mathcal{H}\setminus\{0\}, \quad \psi_1 \sim \psi_2 \Leftrightarrow \psi_2 = \lambda \psi_1, \quad \lambda \in \mathbb{C}\setminus\{0\}, \] (3.12)
can be encoded at the level of our geometrical description by means of the vector fields \( \Delta^\times \) and \( \Gamma^\times \). Indeed, it follows from (3.8) that both vector fields commute and define a regular integrable distribution on \( \mathcal{M}_{2n}^\times \). Its space of leaves can be given a differentiable manifold structure becoming a differentiable manifold \( \mathcal{P}_n \) and inducing a differentiable projection \( \pi_{\mathcal{M}P} : \mathcal{M}_{2n}^\times \to \mathcal{P}_n \). Each leaf of the foliation induced by \( \Delta^\times \) and \( \Gamma^\times \) contains the set of equivalent points in \( \mathcal{H}_0 \) relative to the equivalence relation (3.12). Therefore, it is natural to consider \( \mathcal{P}_n \) as the geometrical representation of the complex projective space \( \mathbb{CP}^{n-1} \).

In what regards the tensor structures, it is well known [38] that the complex projective space is a Hermitian symmetric space [38] and therefore admits a canonical Kähler structure encoded in the Fubiny-Study metric. Hence, there will exist a Riemannian tensor, a
Nevertheless, we would like to define projectable tensor fields $G$ under dilations. They are also invariant under $\Gamma$ expectation value functions, i.e. the functions related to observables that are invariant under $\Gamma^\times$, they are not invariant under $\Delta^\times$. Since $\mathcal{L}_{\Delta^\times}G^\times = -2G^\times$ and $\mathcal{L}_{\Delta^\times}\Lambda^\times = -2\Lambda^\times$, we can define two new tensor fields by multiplication:

$$G := 2f_I(\psi)G, \quad \Lambda := 2f_I(\psi)\Lambda.$$

The tensor fields $\tilde{G}^\times$ and $\tilde{\Lambda}^\times$ on $\mathcal{M}^\times_{2n}$ are homogeneous of degree 0 and therefore invariant under dilations. They are also invariant under $\Gamma^\times$ and hence projectable onto $\mathcal{P}_n$. Nevertheless, we would like to define projectable tensor fields $G_P$ and $\Lambda_P$ on $\mathcal{M}^\times_{2n}$ satisfying that

$$G_P(d\pi^*f_1, d\pi^*f_2) := \{\pi_{\mathcal{M}P}(f_1), \pi_{\mathcal{M}P}(f_2)\}, \quad \Lambda_P(d\pi^*f_1, d\pi^*f_2) := \{\pi_{\mathcal{M}P}(f_1), \pi_{\mathcal{M}P}(f_2)\},$$

for every $f_1, f_2 \in C^\infty(\mathcal{P}_n)$. To ensure this, we define

$$G_P := 2f_I^\times(\psi)G^\times - (\Delta^\times \otimes \Delta^\times + \Gamma^\times \otimes \Gamma^\times), \quad \Lambda_P := 2f_I^\times(\psi)\Lambda^\times - (\Delta^\times \otimes \Gamma^\times - \Gamma^\times \otimes \Delta^\times).$$

(3.13)

Observables must also be represented on $\mathcal{P}_n$ by tensorial objects which are projections of tensor objects on $\mathcal{M}^\times_{2n}$ that are invariant by the vector fields $\Delta^\times$ and $\Gamma^\times$, and this is clearly not the case for the quadratic functions $f^\times_A$ defined in (3.9). Instead, we consider the set of expectation value functions, i.e. the functions related to observables $A$ of the form

$$e_A(\psi) := \frac{1}{2} \langle \psi, A\psi \rangle / \langle \psi, \psi \rangle, \quad \forall \psi \in \mathcal{M}^\times_{2n}.$$

These functions are first-integrals on $\mathcal{M}^\times_{2n}$ of both vector fields, $\Delta^\times$ and $\Gamma^\times$, and as they are projectable, they correspond to pullbacks of functions on $\mathcal{P}_n$. Furthermore they represent, up to a proportional constant, the physical magnitude known as expectation value of the observable $A$.

Finally, we can combine the expectation value functions and the tensor $\Lambda_P$ to define Hamiltonian vector fields on $\mathcal{P}_n$. Indeed, given a Hermitian operator $A$, we can define the vector field on $\mathcal{M}^\times_{2n}$:

$$X_A := -\Lambda_P(d e_A, \cdot).$$

The projections of these vector fields under $\pi_{\mathcal{M}P}$ give rise to Hamiltonian vector fields associated with the canonical Kähler structure on $\mathcal{P}_n$.

Additionally, there exists a natural action of the unitary group $U(n)$ on $\mathcal{P}_n$ of the form

$$\varphi_{\mathcal{P}_n} : U(n) \times \mathcal{P}_n \rightarrow \mathcal{P}_n, \quad (U, [\psi]_{\mathcal{P}_n}) \mapsto [U\psi]_{\mathcal{P}_n},$$

(3.14)

where $[\psi]_{\mathcal{P}} := \pi_{\mathcal{M}P}(\psi)$ denotes the equivalence class in $\mathcal{P}_n$ of the element $\psi \in \mathcal{M}^\times_{2n}$.
4. Quantum Lie systems and Kähler–Lie systems

In this section we apply the theory of Lie and Lie–Hamilton systems to a $t$-dependent Hamiltonian operator $H(t)$ that can be written as a linear combination, with some $t$-dependent real coefficients $b_1(t), \ldots, b_r(t)$, of some Hermitian operators,

$$H(t) = \sum_{k=1}^r b_k(t)H_k,$$

where the $H_k$ form a basis of a real finite-dimensional Lie algebra $V$ relative to the Lie bracket of observables, i.e. $[H_j, H_k] = \sum_{l=1}^{r} c_{jkl}H_l$, with $c_{jkl} \in \mathbb{R}$ and $j, k, l = \overline{1,r}$. The $t$-dependent operator $H(t)$, a so-called quantum Lie system [24], becomes a curve in a Lie algebra of operators: quantum VG–Lie algebra of $H(t)$. In particular, we prove that a very general class of these systems leads to define Lie systems admitting a VG–Lie algebra of Kähler vector fields with respect to a Kähler structure. In turn, this suggests us to define a new type of Lie systems: the Kähler–Lie systems.

In particular, a quantum Lie system $H(t)$ determines a $t$-dependent Schrödinger equation

$$\frac{d\psi}{dt} = -iH(t)\psi = -i\sum_{k=1}^r b_k(t)H_k\psi. \quad (4.2)$$

The isomorphism $(\psi, \phi) \in \mathcal{M}_{2n} \oplus \mathcal{M}_{2n} \mapsto \phi \circ \psi \in T\psi \mathcal{M}_{2n}$ allows us to identify the operators $-iH_k$ with the vector fields $X_k : \psi \in \mathcal{M}_{2n} \mapsto (\psi, -iH_k\psi) \in \mathcal{M}_{2n} \oplus \mathcal{M}_{2n} \simeq T\mathcal{M}_{2n}$. In turn, the $t$-dependent Schrödinger equation (4.2) becomes the associated system of the $t$-dependent vector field $X = \sum_{k=1}^r b_k(t)X_k$ on $\mathcal{M}_{2n}$.

It was stated in Section 3.2 that $\mathcal{M}_{2n}$ admits a natural symplectic structure turning the vector fields $X_k$ into Hamiltonian admitting real Hamiltonian functions $h_k(\psi) = \frac{1}{2}\langle \psi, H_k\psi \rangle$. From (3.2), the commutators of these Hamiltonian vector fields are

$$[X_j, X_k] = -X_{[H_j, H_k]} = -\sum_{l=1}^r c_{jkl}X_l, \quad j, k = \overline{1,r}. \quad (4.3)$$

Hence, $t$-dependent Schrödinger equations (4.2) are Lie–Hamilton systems. Summing up, we have this first theorem.

**Theorem 4.1.** Every $t$-dependent Schrödinger equation on $\mathcal{M}_{2n}$ determined by a quantum Lie system $H(t)$ is a Lie–Hamilton system.

**Note 4.2.** From now on we will only consider Schrödinger equations related to a quantum–Lie system.

We exemplify the above result by studying a two-level quantum system. Its possible states are described by elements $\psi \in \mathbb{C}^2$. Unitary evolution is described by the canonical action of the unitary Lie group $U(2)$ on $\mathbb{C}^2$. In consequence, the evolution of every particular solution $\psi(t)$ in $\mathbb{C}^2$ of the corresponding $t$-dependent Schrödinger equation is determined by
a curve $t \mapsto U_t$ within $U(2)$ which in turn gives rise to a curve $-iH(t) := \dot{U}_t U^{-1}_t$ in the Lie algebra $\mathfrak{u}(2)$ of $U(2)$. More specifically,

$$\frac{d\psi}{dt} = -iH(t)\psi, \quad -iH(t) \in \mathfrak{u}(2), \quad \forall t \in \mathbb{R}. \quad (4.4)$$

As $\mathfrak{u}(2)$ is the space of skew-Hermitian operators on $\mathbb{C}^2$, each $H(t)$ is Hermitian.

Consider a basis for $\text{Herm}(2)$ given by the $2 \times 2$ identity matrix $I_0$ and the traceless matrices $\{S_j := \frac{1}{2}\sigma_j\}_{j=1,2,3}$, where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices, namely

$$I_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If we consider the commutator defined in (3.10), then we find that $[I_0, \cdot] = 0$ and

$$[S_j, S_k] = \sum_{l=1}^{3} \epsilon_{jkl} S_l, \quad j, k = 1, 2, 3. \quad (4.5)$$

Every Hamiltonian in $\text{Herm}(2)$ can be brought into the form:

$$H = B_0 I_0 + \sum_{j=1}^{3} B_j S_j = B_0 I_0 + \mathbf{B} \cdot \mathbf{S}, \quad \mathbf{S} = (\sigma_1, \sigma_2, \sigma_3)/2, \quad \mathbf{B} := (B_1, B_2, B_3) \in \mathbb{R}^3, \quad B_0 \in \mathbb{R}. \quad (4.6)$$

In a physical system, $\mathbf{B}$ is identified with the magnetic field applied to a $1/2$-spin particle. To obtain a $t$-dependent Hamiltonian, the magnetic field must be $t$-dependent:

$$H(t) := B_0(t) I_0 + \mathbf{B}(t) \cdot \mathbf{S}. \quad (4.7)$$

The $t$-dependent Hamiltonian $H(t)$ is therefore a quantum Lie system. It determines a $t$-dependent Schrödinger equation of the form (4.4) in $\mathbb{C}^2$ [20].

Consider now the geometric formalism presented in the previous section. The Hilbert space $\mathbb{C}^2$ is replaced by a real manifold $\mathcal{M}_4$ with coordinates $(q_1, p_1, q_2, p_2)$. The coordinate expression of the $t$-dependent Schrödinger equation (4.4) with the quantum Lie system defined by (4.7) is

$$\frac{d}{dt} \begin{bmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 2B_0(t) + B_3(t) & -B_2(t) & B_1(t) \\ -2B_0(t) - B_3(t) & 0 & -B_1(t) & -B_2(t) \\ B_2(t) & B_1(t) & 0 & 2B_0(t) - B_3(t) \\ -B_1(t) & B_2(t) & B_3(t) - 2B_0(t) & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{bmatrix}. \quad (4.8)$$

This is the associated system of the $t$-dependent vector field $X = \sum_{\alpha=0}^{3} B_\alpha(t) X_\alpha$, with

$$X_0 = -\Gamma = p_1 \frac{\partial}{\partial q_1} - q_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial q_2} - q_2 \frac{\partial}{\partial p_2}, \quad X_1 = \frac{1}{2} \left( p_2 \frac{\partial}{\partial q_1} - q_2 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial q_2} - q_1 \frac{\partial}{\partial p_2} \right),$$

$$X_2 = \frac{1}{2} \left( -q_2 \frac{\partial}{\partial q_1} - p_2 \frac{\partial}{\partial p_1} + q_1 \frac{\partial}{\partial q_2} + p_1 \frac{\partial}{\partial p_2} \right), \quad X_3 = \frac{1}{2} \left( p_1 \frac{\partial}{\partial q_1} - q_1 \frac{\partial}{\partial p_1} - p_2 \frac{\partial}{\partial q_2} + q_2 \frac{\partial}{\partial p_2} \right). \quad (4.9)$$
spanning a Lie algebra of vector fields isomorphic to $\mathfrak{u}(2)$:

$$[X_0, \cdot] = 0, \quad [X_1, X_2] = -X_3, \quad [X_2, X_3] = -X_1, \quad [X_3, X_1] = -X_2.$$ 

Recall that $\mathcal{M}_4$ admits a Kähler structure with symplectic and Riemannian tensor fields given by

$$\omega = \sum_{j=1}^{2} dq_j \wedge dp_j, \quad g = \sum_{j=1}^{2} (dq_j \otimes dq_j + dp_j \otimes dp_j).$$

The vector fields $X_0, X_1, X_2, X_3$ are Hamiltonian with respect to $\omega$. Their Hamiltonian functions are

$$h_0(\psi) = \frac{1}{2} \langle \psi, \psi \rangle = \frac{1}{2} (q_1^2 + p_1^2 + q_2^2 + p_2^2), \quad h_1(\psi) = \frac{1}{2} \langle \psi, S_1 \psi \rangle = \frac{1}{2} (q_1 q_2 + p_1 p_2),$$

$$h_2(\psi) = \frac{1}{2} \langle \psi, S_2 \psi \rangle = \frac{1}{2} (q_3 p_2 - p_1 q_2), \quad h_3(\psi) = \frac{1}{2} \langle \psi, S_3 \psi \rangle = \frac{1}{4} (q_1^2 + p_1^2 - q_2^2 - p_2^2),$$

with $\iota_{X_\alpha} \omega = dh_\alpha$ for $\alpha = 0, 1, 2, 3$.

The Hamiltonian functions span a Lie algebra isomorphic to $\mathfrak{u}(2)$:

$$\{h_0, \cdot\} = 0, \quad \{h_1, h_2\} = h_3, \quad \{h_2, h_3\} = h_1, \quad \{h_3, h_1\} = h_2.$$ 

It will be useful to note that $h_1, h_2, h_3$ are functionally independent, but $h_0^2 = 4(h_1^2 + h_2^2 + h_3^2)$.

The $t$-dependent Schrödinger equation (4.4) enjoys an additional property: $X_0, X_1, X_2$ and $X_3$ are Killing vector fields with respect to $g$, namely $L_X g = 0$ for $\alpha = 0, 1, 2, 3$. Using this, we can easily prove in an intrinsic geometric way that

$$I_1 = g(X_0, X_0), \quad I_2 = g(X_1, X_1) + g(X_2, X_2) + g(X_3, X_3), \quad I_3 = h_1^2 + h_2^2 + h_3^2, \quad I_4 = h_0$$

are constants of the motion for $X$. This example is relevant because it illustrates how to define the above constants of the motion geometrically in terms of $g$ and the Hamiltonian functions due to $\omega$.

Note also that our real system comes from a linear complex differential equation. This gives rise to a symmetry $(q_1, p_1, q_2, p_2) \in \mathcal{M}_4 \rightarrow (-p_1, q_1, -p_2, q_2) \in \mathcal{M}_4$ of system (4.8), which is the counterpart of the multiplication by the imaginary unit in $\mathbb{C}^2$. Therefore, the Lie system preserves the complex structure $J$ in $\mathcal{M}_4$.

Finally, $\mathcal{M}_4$ admits the following symplectic forms

$$\Omega_R := dq_1 \wedge dq_2 - dp_1 \wedge dp_2, \quad \Omega_I := dq_1 \wedge dp_2 + dp_1 \wedge dq_2$$

turning $X_1, X_2, X_3$ into Hamiltonian vector fields. However, these symplectic forms are not invariant under $X_0$:

$$L_{X_0} \Omega_R = \Omega_I, \quad L_{X_0} \Omega_I = -\Omega_R.$$ (4.11)

Recall that $X_0 = -\Gamma$. This result proves that $\Omega_R$ and $\Omega_I$ are invariant under the canonical $SU(2)$-action on $\mathcal{M}_4$, but not under the $U(2)$-action.

The results here presented can be used to obtain a superposition rule for the initial system. This topic will be studied in following sections. Let us generalise the above example.
Theorem 4.3. Every Schrödinger equation on $M_{2n}$ admits a VG–Lie algebra $V_{M_{2n}} \simeq \mathfrak{u}(n)$ of Kähler vector fields relative to the Kähler structure $(g, \omega, J)$ on $M_{2n}$.

Proof. Let $\varphi_{M_{2n}} : U(n) \times M_{2n} \to M_{2n}$ be the natural action of the unitary group by unitary matrices on $\mathbb{C}^n$ understood in the natural way as a real manifold $M_{2n}$.

Every element $h \in U(n)$ induces a diffeomorphism on $M_{2n}$ leaving invariant the Hermitian product on $\mathcal{H}$. Hence, it leaves invariant its real and imaginary parts. In view of expressions (3.4), it also follows that $h_t^* \omega = \omega$ and $h_t^* g = g$ for every $t \in \mathbb{R}$ and every curve $h_t$ in $U(n)$. As a consequence, every fundamental vector field $Y$ of the Lie group action $\varphi_{M_{2n}}$ satisfies that $L_Y g = 0$ and $L_Y \omega = 0$. Hence, the fundamental vector fields of $\varphi_{M_{2n}}$ are Kähler vector fields relative to $(g, \omega, J)$.

The evolution of the Schrödinger equation

$$\frac{d\psi}{dt} = -iH(t)\psi, \quad -iH(t) \in \mathfrak{u}(n), \quad \psi \in M_{2n}, \quad (4.12)$$

takes the form $\psi(t) = \varphi_{M_{2n}}(h_t, \psi(0))$, with $h_0 = \text{Id}_{M_{2n}}$, and a certain curve $h : t \in \mathbb{R} \mapsto h_t \in U(n)$. Therefore, (4.12) is determined by a $t$-dependent vector field $X$ taking values in the Lie algebra of fundamental vector fields of $\varphi_{M_{2n}}$, namely $X = \sum_{\alpha=1}^{r} b_\alpha(t)X_\alpha$, where $X_1, \ldots, X_r$ is a basis of the Lie algebra of fundamental vector fields of $\varphi_{M_{2n}}$. Then $(X_1, \ldots, X_r)$ is a Lie algebra isomorphic to $\mathfrak{u}(n)$ and becomes a VG–Lie algebra of Kähler vector fields for $X_{M_{2n}}$.

We hereafter write $V_{M_{2n}}$ and $V_{M_{2n}^x}$ for VG–Lie algebras of Kähler vector fields relative to the standard Kähler structure on $M_{2n}$ and its natural restriction to $M_{2n}^x$, respectively.

The above examples motivate to introduce the following definition.

Definition 4.4. We call Kähler–Lie system a Lie system admitting a VG–Lie algebra of Kähler vector fields with respect to a Kähler structure.

It is therefore simple to prove the proposition below.

Proposition 4.5. The space $I_X$ of $t$-independent constants of motion for a Kähler–Lie system $X$ is a Poisson algebra with respect to the Poisson bracket of the Kähler structure and the commutative algebra relative to the bracket induced by its Riemannian structure.

In following sections, it will be shown that Kähler structures allow us to devise techniques to obtain constants of the motion and superposition rules for Kähler–Lie systems, e.g. if vector fields $Y_1, Y_2$ commute with all the elements of the VG–Lie algebra of Kähler vector fields for a Kähler–Lie system $X$, then $g(Y_1, Y_1), g(Y_2, Y_2), g(Y_1, Y_2)$ and $\omega(Y_1, Y_2)$ are constants of motion for $X$.

5. Lie systems and Schrödinger equations on $\mathcal{R}_n$ and $\mathcal{S}_n$

The unit sphere $S_n$ in $\mathbb{C}^n$ admits a natural structure as a real $(2n - 1)$-dimensional manifold. Since we study $t$-dependent Schrödinger equations with a unitary evolution and
their evolution leave $S_n$ invariant, it seems natural at first to restrict them to the unity sphere $S_n$. Nevertheless, as shown next, their restriction is generally no longer neither a Kähler–Lie system nor a Lie–Hamilton one. That is why we now introduce an alternative Schrödinger equation which possesses more useful properties to describe its superposition rules and the superposition rules for other related Schrödinger equations.

**Proposition 5.1.** A Schrödinger equation on $M_{2n}$ can be restricted to the unity sphere $S_n$ giving rise to a Lie system $X_{S_n}$ possessing a VG–Lie algebra $V_{S_n}$ of Hamiltonian vector fields with respect to the presymplectic form $i_\ast S\omega$ with $i_S : S_n \to M_{2n}$. If $V^{X_{S_n}} = V_{S_n}$, then $X_{S_n}$ is not a Lie–Hamilton system.

**Proof.** The VG–Lie algebra $V_{M_{2n}}$ of (4.12) is the Lie algebra of fundamental vector fields of the unitary action of $\varphi_{M_{2n}} : U(n) \times M_{2n} \to M_{2n}$. Therefore, $\langle \varphi_{M_{2n}}(g, \psi),\varphi_{M_{2n}}(g, \psi') \rangle = \langle \psi, \psi' \rangle$ for every $\psi \in M_{2n}$ and $g \in U(n)$. Hence, $f(\psi) := \langle \psi, \psi \rangle$ is invariant under $\varphi_{M_{2n}}$ and, in consequence, a first-integral of its fundamental vector fields, namely $X_{S_n}$ states that previous conditions are enough to ensure that $X_{S_n}$ become tangent to $S_n$ and they therefore span a finite-dimensional Lie algebra of vector fields $V_{S_n}$ on $S_n$. In the light of Theorem 4.3, the system (4.12) is related to a $t$-dependent vector field $X_{M_{2n}}$, taking values in $V_{M_{2n}}$. Therefore, system (4.12) can be restricted to a system $X_{S_n}$ on $S_n$ admitting a VG–Lie algebra $V_{S_n}$.

The embedding $i_S : S_n \to M_{2n}$ gives rise to a presymplectic structure $i_\ast S\omega$ on $S_n$, where $\omega$ is the natural symplectic structure (5.3) on $M_{2n}$. Since the elements of $V_{M_{2n}}$ are Hamiltonian vector fields on $M_{2n}$ with Hamiltonian functions $h_H(\psi) = \frac{1}{2} \langle \psi, H\psi \rangle$ with $-iH \in U(n)$, their restrictions to $S_n$ are tangent to $S_n$ and Hamiltonian relative to the presymplectic form $i_\ast S\omega$ with Hamiltonian functions $i_\ast S h_H$. Therefore, they span a VG–Lie algebra $V_{S_n}$ on $S_n$ of Hamiltonian vector fields relative to the presymplectic structure $\omega_S$.

As $S_n$ is an orbit of $\varphi_{M_{2n}}$, then $T S_n = D^V_{M_{2n}}|_{S_n}$, which is an odd $(2n - 1)$-dimensional distribution on $S_n$. From assumption $V^{X_{S_n}} = V_{S_n}$ and, hence, $D^{X_{S_n}} = D^V_{S_n} = D^V_{M_{2n}}|_{S_n} = T S_n$. The so-called no-go Theorem for Lie–Hamilton systems (see [18, Proposition 5.1]) states that previous conditions are enough to ensure that $X_{S_n}$ is not a Lie–Hamilton system. 

A Dirac structure is a generalisation of presymplectic and Poisson manifolds. In fact, presymplectic and Poisson manifolds can be naturally attached to Dirac structures whose Hamiltonian vector fields are the Hamilton vector fields of the structures originating them (see [18] for details). This fact enables us to prove the following.

**Corollary 5.2.** The Schrödinger equation (4.12) on $S_n$ is a Dirac–Lie system with respect to the Dirac structure induced by $i_\ast S\omega$.

Let us now prove that the projection of the restriction of (4.12) to $M_{2n}$ onto $\mathcal{R}_n$ exists and it is a Lie–Hamilton system that can be endowed with a natural coordinate system coming from this fact.

**Lemma 5.3.** The manifold $\mathcal{R}_n$, for $n > 1$, admits a local coordinate system on a neighborhood of each point given by $2n - 1$ functions $f_\alpha(\psi) = \frac{1}{2} \langle \psi, H_\alpha \psi \rangle$, for $\alpha = \frac{1}{2}, \ldots, 2n - 1$ for certain operators $H_\alpha \in \mathfrak{su}(n)$. 

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Proof. For \( n > 1 \) any two elements of \( \mathcal{M}_n^2 \) with the same norm can be connected by the action of an element of \( SU(n) \). Hence, the special unitary action \( \varphi : SU(n) \times \mathcal{M}_n^2 \rightarrow \mathcal{M}_n^2 \), with \( n > 1 \), has \((2n - 1)\)-dimensional orbits, which are embedded submanifolds of \( \mathcal{M}_n^2 \) because \( SU(n) \) is compact. Since \( \dim SU(n) = n^2 - 1 \geq 2n - 1 \) for \( n > 1 \), we can choose around any point of \( \mathcal{M}_n^2 \) an open neighbourhood \( A_0 \) where \( 2n - 1 \) fundamental vector fields of \( \varphi \) are linearly independent at each point. As they are also Hamiltonian vector fields, their Hamiltonian functions, which can be taken of the form \( f_n(\psi) := \frac{1}{2}\langle \psi, H_n\psi \rangle \) with \( H_n \in \mathfrak{su}(n) \), \( \phi \) in \( \mathcal{M}_n^2 \), and \( \alpha \in \mathbb{T} ; 2n - 1 \), are functionally independent on \( A_0 \). These functions are invariant under the action \( \varphi_{U(1)} : (e^{i\varphi}, \psi) \in U(1) \times \mathcal{M}_n^2 \rightarrow e^{i\varphi}\psi \in \mathcal{M}_n^2 \) and give rise to well-defined functions \( f_{1|\mathcal{R}_n} , \ldots , f_{2n-1|\mathcal{R}_n} \), on an open subset of \( \mathcal{R}_n \). As \( f_1 , \ldots , f_{2n-1} \) are constant on the leaves of \( \varphi_{U(1)} \) and functionally independent on \( A_0 \subset \mathcal{M}_n^2 \), then \( f_{1|\mathcal{R}_n} , \ldots , f_{2n-1|\mathcal{R}_n} \) are functionally independent and provide a local coordinate system on \( \mathcal{R}_n \). \( \square \)

Example 5.1. A two-level system is described by a four-dimensional manifold \( \mathcal{M}_4 \) with coordinates \( (q_1, p_1, q_2, p_2) \) given in \((4.1)\). Its dynamics is described by a \( t \)-dependent vector field \( X = \sum_{n=1}^n B_n(t)X_n \), where the vector fields \( X_1, X_2, X_3 \) are given in \((4.9)\) and admit Hamiltonian functions

\[
\begin{align*}
    h_1(\psi) &= \frac{1}{2}(q_1q_2 + p_1p_2), \\
    h_2(\psi) &= \frac{1}{2}(q_1p_2 - q_2p_1), \\
    h_3(\psi) &= \frac{1}{4}(q_1^2 + p_1^2 - q_2^2 - p_2^2), \\
    \forall \psi &\in \mathcal{M}_4
\end{align*}
\]

relative to the natural symplectic structure on \( \mathcal{M}_4 \) appearing in \((4.10)\). As stated in Lemma 5.3, these functions define a coordinate system \( \{h_1, h_2, h_3\} \) on \( \mathcal{R}_2 \). To verify it, let us consider the projection \( \pi_{MR} : \mathcal{M}_4 \rightarrow \mathbb{R}_0^3 \) by:

\[
\pi_{MR}(\psi) := (x := h_1(\psi), y := h_2(\psi), z := h_3(\psi)),
\]

and show that \( \mathcal{R}_2 \simeq \mathbb{R}_0^3 \). Since \( x^2 + y^2 + z^2 = \frac{1}{4\pi} \langle \psi, \psi \rangle^2 \) and \( (0,0) \notin \mathcal{M}_4 \), the image of \( \pi_{MR} \) does not contain the origin \((0,0,0)\) and \( \pi_{MR} \) takes values in \( \mathbb{R}_0^3 \) as assumed.

Coming back to the complex notation of \( \mathcal{H} = \mathbb{C}^2 \), we write \( \psi = (z_1, z_2) = (q_1 + ip_1, q_2 + ip_2) \). Therefore,

\[
\begin{align*}
    x(\psi) &= \frac{1}{2}\Re \langle z_1, z_2 \rangle, \\
    y(\psi) &= \frac{1}{2}\Im \langle z_1, z_2 \rangle, \\
    z(\psi) &= \frac{1}{4}(|z_1|^2 - |z_2|^2).
\end{align*}
\]

for every \( \psi = (z_1, z_2) \in \mathcal{M}_4 \). Hence, \( x, y, z \) are constant along the equivalence classes of \( \mathcal{R}_2 \) and if \( \psi, \tilde{\psi} \in \mathcal{M}_4 \) belong to the same equivalence class of \( \mathcal{R}_2 \), then \( \pi_{MR}(\psi) = \pi_{MR}(\tilde{\psi}) \). Let us additionally show that if \( \pi_{MR}(\psi) = \pi_{MR}(\tilde{\psi}) \), then \( \psi = (z_1, z_2) \) and \( \tilde{\psi} = (\tilde{z}_1, \tilde{z}_2) \) belong to the same equivalence class. Indeed, if \( \pi_{MR}(z_1, z_2) = \pi_{MR}(\tilde{z}_1, \tilde{z}_2) \), then \( \sqrt{x^2 + y^2 + z^2} = (|z_1|^2 + |z_2|^2)/4 = (|\tilde{z}_1|^2 + |\tilde{z}_2|^2)/4 \) and, since \( z(\psi) = z(\tilde{\psi}) \), we obtain \( |z_1| = |\tilde{z}_1| \) for \( i = 1, 2 \). Therefore, \( z_j = e^{iv_j} \tilde{z}_j \) for certain \( v_j \in \mathbb{R} \) with \( j = 1, 2 \). In view of this and \( (z_1, z_2) = (\tilde{z}_1, \tilde{z}_2) \), we obtain \( \varphi_1 - \varphi_2 = 2\pi k \), for \( k \in \mathbb{Z} \) and hence \( (z_1, z_2) = e^{i\varphi_1}(\tilde{z}_1, \tilde{z}_2) \). Thus, if \( \pi_{MR}^{-1}(x, y, z) \) is not empty, it gives rise to an equivalence class of \( \mathcal{R}_2 \).
Let us prove that $\pi_{MR}$ is a surjection. For every $(x, y, z) \in \mathbb{R}^3_0$, we can prove that

$$\pi_{MR}\left([2(\sqrt{x^2 + y^2 + z^2})^{1/2}, [2(\sqrt{x^2 + y^2 + z^2} - z)]^{1/2} e^{i\Theta}]\right) = (x, y, z),$$

where, the angle $\Theta \in [0, 2\pi)$ satisfies for $x^2 + y^2 \neq 0$ the relation

$$\frac{x}{\sqrt{x^2 + y^2}} = \cos \Theta, \quad \frac{y}{\sqrt{x^2 + y^2}} = \sin \Theta$$

and it is arbitrary for $x^2 + y^2 = 0$. The above expressions show that $\pi_{MR}$ is surjective. Therefore, $\pi_{MR}^{-1}(x, y, z)$ is the equivalence class of an element of $\mathcal{R}_2$ for every $(x, y, z) \in \mathcal{R}_2$ and $\mathcal{R}_2 \simeq \mathbb{R}^3_0$.

**Proposition 5.4.** The $t$-dependent Schrödinger equation (4.12), when restricted to $\mathcal{M}^x_{2n}$, can be projected onto $\mathcal{R}_n$ originating a Lie–system $X_{\mathcal{R}_n}$ possessing a VG–Lie algebra $V_{\mathcal{R}_n} \simeq \mathfrak{su}(n)$ of Hamiltonian vector fields with respect to the projection of $\Lambda^x$ on $\mathcal{M}^x_{2n}$ onto $\mathcal{R}_n$.

**Proof.** The $\mathbb{C}$-linear Lie group action $\varphi_{\mathcal{M}^x_{2n}} : U(n) \times \mathcal{M}^x_{2n} \to \mathcal{M}^x_{2n}$ induces, due to its $\mathbb{C}$-linearity, another action on $\mathcal{R}_n$ such that the map $\pi_{MR}$ is equivariant, as follows:

$$\varphi_{\mathcal{R}_n} : (g, [\psi]_\mathcal{R}) \mapsto [\varphi_{\mathcal{M}^x_{2n}}(g, \psi)]_\mathcal{R}.$$

As a consequence, the fundamental vector fields of $V_{\mathcal{M}^x_{2n}}$ project onto $\mathcal{R}_n$ giving rise to a new finite-dimensional Lie algebra of vector fields $V_{\mathcal{R}_n}$ and the projection map $\pi_{MR} : \mathcal{M}^x_{2n} \to \mathcal{R}_n$ induces a Lie algebra morphism $\pi_{MR}|_{V_{\mathcal{M}^x_{2n}}} : V_{\mathcal{M}^x_{2n}} \to V_{\mathcal{R}_n}$. Then, the restriction to $\mathcal{M}^x_{2n}$ of the Schrödinger equation on (4.12) also projects onto $\mathcal{R}_n$ giving rise to a system $X_{\mathcal{R}_n}$.

Let us prove that $X_{\mathcal{R}_n}$ admits a VG–Lie algebra isomorphic to $\mathfrak{su}(n)$. As $V_{\mathcal{M}^x_{2n}} \simeq \mathfrak{u}(n) \simeq \mathbb{R} \oplus \mathfrak{su}(n)$, the kernel of $\pi_{MR}|_{V_{\mathcal{M}^x_{2n}}}$, which is an ideal of $V_{\mathcal{M}^x_{2n}}$, may be zero, isomorphic to $\mathbb{R}$, to $\mathfrak{su}(n)$ or to $\mathfrak{u}(n)$. The one-parameter group of diffeomorphisms induced by the vector field $\Gamma^x$ is given by $F_t : \psi \in \mathcal{M}^x_{2n} \mapsto e^{it}\psi \in \mathcal{M}^x_{2n}$. Hence, $\pi_{MR}|_{\mathcal{M}^x_{2n}} \Gamma^x = 0$ and $\Gamma^x$ belongs to the center of $V_{\mathcal{M}^x_{2n}}$, i.e. $\Gamma^x \in \mathfrak{z}(V_{\mathcal{M}^x_{2n}}) \simeq \mathbb{R}$. If $n = 1$, then this shows that $\text{Im} \pi_{MR}|_{V_{\mathcal{M}^x_{2n}}} = \{0\} \simeq \mathfrak{su}(1)$ and the result follows. Meanwhile, $V_{\mathcal{R}_n} \neq 0$ for $n > 1$ and in view of the decomposition $V_{\mathcal{M}^x_{2n}} \simeq \mathbb{R} \oplus \mathfrak{su}(n)$, we get that $\ker \pi_{MR} \simeq \langle \Gamma^x \rangle$ and $\text{Im} \pi_{MR}|_{V_{\mathcal{M}^x_{2n}}} \simeq \mathfrak{su}(n)$. Thus, the projection of (4.12) onto $\mathcal{R}_n$ admits a VG–Lie algebra $V_{\mathcal{R}_n} \simeq \mathfrak{su}(n)$.

**Example 5.2.** A simple computation shows that there exist vector fields $Y_\alpha$ on $\mathcal{R}_2$ such that $\pi_{MR}(X_\alpha) = Y_\alpha$ for $\alpha = 1, 2, 3$. Indeed,

$$Y_1 = -z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}, \quad Y_2 = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}, \quad Y_3 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}. \quad (5.2)$$

The Lie brackets between these vector fields read

$$[Y_1, Y_2] = -Y_3, \quad [Y_2, Y_3] = -Y_1, \quad [Y_3, Y_1] = -Y_2.$$
that is \([Y_j, Y_k] = -\sum_{l=1}^{3} \epsilon_{jkl} Y_l\) for \(j, k, l = 1, 2, 3\). The projection of \(X = \sum_{\alpha=1}^{3} B_\alpha(t) X_\alpha\) to \(\mathcal{R}_2\), i.e. the \(t\)-dependent vector field \(X_{\mathcal{R}_2}\) satisfying \((X_{\mathcal{R}_2})_t = \pi_{\mathcal{M}^*_R} X_\alpha\), becomes

\[X_{\mathcal{R}_2} = \sum_{\alpha=1}^{3} B_\alpha(t) Y_\alpha. \tag{5.3}\]

This is exactly the same relation given in (4.5), which shows that \(\mathfrak{f} := \langle Y_1, Y_2, Y_3 \rangle \simeq \mathfrak{su}^*(2)\). Therefore, \(X_{\mathcal{R}_2}\) is a Lie system. Observe that \(Y_1, Y_2, Y_3\) span a two-dimensional distribution \(\mathcal{D}\).

The following proposition shows that \(\mathcal{R}_n\) can be endowed with a Poisson structure turning \(\mathcal{V}_{\mathcal{R}_n}\) into a Lie algebra of Hamiltonian vector fields.

**Proposition 5.5.** The system \(X_{\mathcal{R}_n}\) is a Lie–Hamilton system with respect to the Poisson bivector \(\pi_{\mathcal{M}^*_R} \Lambda^*\).

**Proof.** Since \(L_{\Gamma^*} \Lambda^* = 0\), the Poisson bivector \(\Lambda^*\) on \(\mathcal{M}^*_2\) can be projected onto \(\mathcal{R}_n\). Additionally, \(\pi_{\mathcal{M}^*_R} [\Lambda^*, \Lambda^*]_{SN} = [\pi_{\mathcal{M}^*_R} \Lambda^*, \pi_{\mathcal{M}^*_R} \Lambda^*]_{SN}\), where \([\cdot, \cdot]_{SN}\) is the Schouten-Nijenhuis bracket [55]. So, \(\pi_{\mathcal{M}^*_R} \Lambda^*\) is a Poisson bivector on \(\mathcal{R}_n\). The vector fields \(X_\alpha\) spanning the VG–Lie algebra \(V_{\mathcal{M}^*_2}\) for \(X_{\mathcal{M}^*_2}\) are Hamiltonian relative to the restrictions to \(\mathcal{M}^*_2\) of the functions \(h_\alpha\) in (4.10). Such Hamiltonian functions are invariant relative to the action of \(U(1)\) on \(\mathcal{M}^*_2\) and hence projectable onto \(\mathcal{R}_n\). The projections \(\pi_{\mathcal{M}^*_R} X_\alpha\) are also Hamiltonian vector fields with Hamiltonian functions \(x_\alpha = \pi_{\mathcal{M}^*_R} (h_\alpha^\times)\). Therefore, the VG–Lie algebra \(V_{\mathcal{R}_n}\) on \(\mathcal{R}_n\) consists of Hamiltonian vector fields relative to \(\pi_{\mathcal{M}^*_R} \Lambda^*\). \(\Box\)

**Example 5.3.** The Poisson bivector \(\Lambda^*\) on \(\mathcal{M}^*_4\) projects onto \(\mathcal{R}_2\) giving rise to the Poisson bivector

\[\hat{\Lambda} = z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} \tag{5.4}\]

in the coordinate system given in Example 5.1.

**Proposition 5.6.** The system \(X_{\mathcal{R}_n}\) consists of Killing vector fields with respect to the metric induced by the projection of the tensor field \(G^\times\) to \(\mathcal{R}_n\).

**Proof.** The Lie derivative of \(G^\times\) with respect to \(\Gamma^\times\) is zero. Hence, \(G^\times\) projects onto \(\mathcal{R}_n\). Since \(G^\times\) is Riemannian, it is non-degenerate. So is its projection onto \(\mathcal{R}_n\) giving rise to a Riemannian metric on \(\mathcal{R}_n\). The vector fields of \(V_{\mathcal{M}^*_2}\) are Killing relative to \(G^\times\) and projectable under \(\pi_{\mathcal{M}^*_R}\). Therefore, their projections, namely the elements of \(V_{\mathcal{R}_n}\), are also Killing vector fields relative to the projection of \(G^\times\) onto \(\mathcal{R}_n\) and span a VG–Lie algebra \(V_{\mathcal{R}_n}\) of Killing vector fields. \(\Box\)

**Example 5.4.** The tensor field \(G^\times\) on \(\mathcal{M}^*_4\) projects onto \(\mathcal{R}_2\) giving rise to the tensor field

\[\hat{G} = (x^2 + y^2 + z^2)^{1/2} \left[ \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \right]. \tag{5.5}\]
It naturally allows us to define a Riemannian metric $\hat{g}$ on $\mathbb{R}^2$ given by
\[
\hat{g} = \frac{dx \otimes dx + dy \otimes dy + dz \otimes dz}{(x^2 + y^2 + z^2)^{1/2}}.
\] (5.6)

It is immediate to check that the Lie derivatives of $\hat{g}$ and $\hat{G}$ with respect to the vector fields (5.2) are zero as stated in the previous proposition.

6. Lie systems and projective Schrödinger equations

Let $\hat{H}(t)$ be a $t$-dependent Hermitian operator on the Hilbert space $L^2(\mathbb{R}^3)$ of equivalence classes of square integrable measurable functions on $\mathbb{R}^3$ with respect to Lebesgue measure. Solutions to the $t$-dependent Schrödinger equation determined by $\hat{H}(t)$ differing in a proportional non-zero $t$-dependent complex factor describe the same physical state. Hence, it is natural to consider whether $t$-dependent Schrödinger equations admit such a symmetry. Nevertheless, (4.12) is not invariant under the change of phase $\psi \mapsto f(t)\psi(t)$ for a non-zero complex valued function $f$. There is however another differential equation, the so-called projective Schrödinger equation, that is invariant under such a change of phase while admitting the particular solutions to the original $t$-dependent Schrödinger equation [13]. It is given by
\[
i \left[ \psi_y \frac{d\psi_x}{dt} - \psi_x \frac{d\psi_y}{dt} \right] = \psi_y \hat{H}_x \psi_x - \psi_x \hat{H}_y \psi_y, \quad \psi : \mathbb{R}^3 \to \mathbb{C},
\] (6.1)
with $\psi_x := \psi(x)$, $\psi_y := \psi(y)$ and $x, y \in \mathbb{R}^3$. These differential equations can be further generalised while admitting the previous symmetry and covering particular solutions to $t$-dependent Schrödinger equations on (probably infinite-dimensional Hilbert spaces) by writing
\[
i \left[ \text{Id} \otimes \frac{d}{dt} - \frac{d}{dt} \otimes \text{Id} \right] \psi \otimes \psi = \left[ \text{Id} \otimes \hat{H}(t) - \hat{H}(t) \otimes \text{Id} \right] \psi \otimes \psi, \quad \forall \psi \in \mathcal{H},
\] (6.2)
where $\psi \otimes \psi \in \mathcal{H} \otimes \mathcal{H}$ and $A \otimes B$ are the tensorial products of the Hermitian operators $A, B : \mathcal{H} \to \mathcal{H}$. It said that $\psi \in \mathcal{H}$ is a solution to (6.2) is $\psi \otimes \psi$ satisfies it.

Nevertheless, (6.1) and (6.2) have not a clear geometric interpretation. Equation (6.1) depends on the values of the function $\psi$ in two different points and (6.2) is defined on tensorial products of the form $\psi \otimes \psi$. Instead, we will make use of a projection of the $t$-dependent Schrödinger equation on $\mathcal{H}_0$ onto $\mathcal{P}_n$ to recover its solutions up to a global $t$-dependent change of phase and, therefore, recovering the same solutions of (6.2).

**Lemma 6.1.** The $t$-dependent vector field $X$ on $\mathcal{M}^{\infty}_{2n}$ related to a $t$-dependent Schrödinger equation
\[
\frac{d\psi}{dt} = -iH(t)\psi, \quad -iH(t) \in \mathfrak{u}(n), \quad \psi \in \mathcal{H}_0,
\] (6.3)
is projectable under the fibration $\pi_{\mathcal{M}P} : \mathcal{M}^{\infty}_{2n} \to \mathcal{P}_n$. 

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Proof. Consider the natural actions of the unitary group $U(n)$ on $\mathcal{M}_{2n}^\otimes$ and on the projective space $\mathcal{P}_n$ given by
\[
\varphi : (U, \psi) \in U(n) \times \mathcal{M}_{2n}^\otimes \mapsto U\psi \in \mathcal{M}_{2n}^\otimes, \quad \varphi_{V_p} : (U, [\psi]_p) \in U(n) \times \mathcal{P}_n \mapsto [U\psi]_p \in \mathcal{P}_n,
\]
respectively. Then, the map $\pi_{\mathcal{M}} : \mathcal{M}_{2n}^\otimes \to \mathcal{P}_n$ is equivariant. By the definition of (6.3), the $t$-dependent vector field $X$ belongs, at each $t \in \mathbb{R}$, to the Lie algebra, $V_{\varphi_{V_p}}$ of fundamental vector fields of $\varphi$. As $\pi_{\mathcal{M}}$ is equivariant, the fundamental vector fields of the actions $\varphi$ and $\varphi_{V_p}$ are $\pi_{\mathcal{M}}$-related and each vector field of $V_{\mathcal{M}_{2n}}$ projects onto a fundamental vector field of $\varphi_{V_p}$ and vice versa. This ensures $\pi_{\mathcal{M}} X$ to exist and to admit a VG–Lie algebra $V_{\varphi_{V_p}} := V_{\varphi_{V_p}}$.

\[\Box\]

Definition 6.2. Given a $t$-dependent Schrödinger equation $X$ of the form (6.3), we call projective Schrödinger equation on $\mathcal{P}_n$ the system of differential equations
\[
\frac{d\xi}{dt} = X_{\mathcal{P}_n}(t, \xi), \quad \xi \in \mathcal{P}_n, \quad \forall t \in \mathbb{R}, \tag{6.4}
\]
where $X_{\mathcal{P}_n}$ is the projection onto $\mathcal{P}_n$ of $X$ relative to the projection $\pi_{\mathcal{M}} : \mathcal{M}_{2n}^\otimes \to \mathcal{P}_n$.

Proposition 6.3. A non-vanishing curve $\psi : \mathbb{R} \to \mathcal{M}_{2n}^\otimes$ is a particular solution to the restriction of the projective Schrödinger equation (6.2) to $\mathcal{H}_0$ if and only if $\pi_{\mathcal{M}} \circ \psi$ is a particular solution to (6.4).

Proof. The projective Schrödinger equation (6.2) can be brought into the form $[\hat{E}(t) \otimes \text{Id}](\psi \otimes \psi) = [\text{Id} \otimes \hat{E}(t)](\psi \otimes \psi)$, where $\hat{E}(t) := \partial_t - iH(t)$. Let $\psi(t)$ be a particular solution to (6.2). Then, $\hat{E}(t)\psi(t) = g(t)\psi(t)$ for a certain $t$-dependent function $g(t)$ and there always exists a $t$-dependent function $f(t)$ such that $\hat{E}(t)[f(t)\psi(t)] = 0$ and $\psi_S(t) := f(t)\psi(t)$ becomes a solution to the standard $t$-dependent Schrödinger equation. Since the Schrödinger equation is related to a $t$-dependent Hermitian Hamiltonian operator, it follows that $\psi_S(t)$ has a constant module. By assumption $\psi(t)$ does not vanishes and therefore $\psi_S(t)$ does not vanishes neither. Hence, both curves can be projected onto $\mathcal{P}_n$ and $\pi_{\mathcal{M}}(\psi(t)) = \pi_{\mathcal{M}}(\psi_S(t))$. Since $X_{\mathcal{M}_{2n}}$ projects onto $X_{\mathcal{P}_n}$, then $\pi_{\mathcal{M}}(\psi_S(t))$ is a particular solution to (6.4) and $\psi(t)$ projects onto a particular solution to (6.4).

Conversely, if $\psi(t)$ projects onto a solution $\psi_{\mathcal{P}}(t)$ to (6.4), there exists a solution $\psi_S(t)$ to the Schrödinger equation projecting onto $\psi_{\mathcal{P}}(t)$. Since $\psi(t)$ and $\psi_S(t)$ project onto $\psi_{\mathcal{P}}(t)$, they differ at each $t$ on a phase and $\psi(t) = \lambda(t)\psi_S(t)$ for a certain complex function $\lambda(t)$. Note that the fact that $\psi(t)$ projects onto $\psi_{\mathcal{P}}(t)$ implies that it does not vanish. Hence, $\psi(t)$ is a non-vanishing particular solution to (6.1).

\[\Box\]

Theorem 6.4. The system (6.4) is a Lie system related to a VG–Lie algebra $V_{\mathcal{P}_n} \simeq \text{su}(n)$ consisting of Kähler vector fields with respect to the natural Kähler structure on $\mathcal{P}_n$. 

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Proof. In view of Theorem 4.3, the vector fields of $V_{M^\times_{2n}}$ for (6.3) leave invariant $G^\times$ and $\Lambda^\times$, i.e. $L_X \Lambda^\times = L_X G^\times = 0$ for every $X \in V_{M^\times_{2n}}$. Since $[\Gamma^\times, X] = [\Delta^\times, X] = 0$, $\Gamma^\times f^\times_I(\psi) = \Delta^\times f^\times_I(\psi) = 0$ and in view of the expressions (3.13), it follows that

$$L_X \Lambda^\times_P = L_X G^\times_P = 0,$$

where $G_P$ and $\Lambda_P$ are given by (3.13). The tensor fields $\Lambda_P$ and $G_P$ on $P_n$ generate the canonical Kähler structure on $P_n$ (see [32] for details). Since the vector fields of $V_{M^\times_{2n}}$ project onto $P_n$ and in view of (6.5), it turns out that the projection onto $P_n$ of the vector fields of $V_{M^\times_{2n}}$ span a VG–Lie algebra $V_{P_n}$ for (6.4) of Kähler vector fields relative to the natural Kähler structure on $P_n$.

Let us prove that $V_{P_n} \cong \mathfrak{su}(n)$. The natural projection map $\pi_{MP} : M^\times_{2n} \to P_n$ induces a Lie algebra morphism $\pi_{MP*} |_{M^\times_{2n}} : V_{M^\times_{2n}} \to V_{P_n}$. As $V_{M^\times_{2n}} \cong \mathfrak{su}(n) \cong \mathbb{R} \oplus \mathfrak{su}(n)$, the kernel of $\pi_{MP*} |_{M^\times_{2n}}$, which is an ideal of $V_{M^\times_{2n}}$, may be zero, either isomorphic to $\mathbb{R}$, to $\mathfrak{su}(n)$ or to $\mathfrak{u}(n)$. The one-parameter group of diffeomorphism induced by the vector field $\Gamma$ on $M^\times_{2n}$ is given by $F_t : \psi \in M^\times_{2n} \mapsto e^{t\psi} \in M^\times_{2n}$. Hence, $\pi_{MP*} \Gamma^\times = 0$ and $\Gamma^\times \in \mathfrak{h}(M^\times_{2n}) \cong \mathbb{R}$. Since $V_{P_n} \neq \{0\}$ and in view of the decomposition of $V_{M^\times_{2n}}$, we get that $\ker \pi_{MP*} \cong \langle \Gamma^\times \rangle$ and $\text{Im} \pi_{MP*} |_{M^\times_{2n}} \cong \mathfrak{su}(n)$. Thus, $V_{P_n} \cong \mathfrak{su}(n)$.

The following proposition will be helpful to study projective Schrödinger equations as a restriction of a system on $\mathcal{R}_n$.

**Proposition 6.5.** Let $\pi_{SR} : S_n \to \mathcal{R}_n$ be of the form $\pi_{SR} := \pi_{MR} \circ i_S$, with $i_S : S_n \to M^\times_{2n}$ being the natural embedding of $S_n$ in $M^\times_{2n}$, and let $\pi_{RP} : [\psi]|_R \in \mathcal{R}_n \mapsto [\psi]|_P \in P_n$. There exists a differentiable embedding $\iota_P : P_n \to \mathcal{R}_n$ such that $\pi_{RP} \circ \iota_P = \iota_{P*} X_{P_n} = \pi_{SR*} X_{S_n}$.

Proof. It is immediate that the diagram aside is commutative. For every element of $P_n$ there exists an element of $S_n \subset M^\times_{2n}$ projecting to it under $\pi_{SP}$. Hence, $\pi_{RP} \circ \pi_{SR}(S_n) = P_n$ and $\pi_{RP}|_{\pi_{SR}(S_n)} : \pi_{SR}(S_n) \to P_n$ is surjective. Let us show that it is also injective. Let $[\psi]|_R$, with $\psi \in M^\times_{2n}$, be the equivalence class of functions on $M^\times_{2n}$ differing in a complex-phase of module one. If $\pi_{RP}([\psi_1]|_R) = \pi_{RP}([\psi_2]|_R)$ and $\|\psi_1\| = \|\psi_2\|$, then $\psi_1 = e^{i\phi} \psi_2$ with $\phi \in \mathbb{R}$. Hence, $[\psi_1]|_R = [\psi_2]|_R$ and $\pi_{RP}$ is a bijection when restricted to $\pi_{SR}(S_n)$ and the inverse map is defined to be $\iota_P$. From this it is immediate that $\iota_{P*} X_{P_n} = \pi_{SR*} X_{S_n}$. 

□
7. Superposition rules for special unitary Schrödinger equations

We now prove that, apart from the very well-known standard linear superposition rules, \( t \)-dependent Schrödinger equations associated with \( t \)-dependent traceless Hermitian Hamiltonian operators have other nonlinear ones which depend, generally, on fewer solutions. Superposition rules for different types of projections of Schrödinger equations are investigated.

**Theorem 7.1.** Every Schrödinger equation on \( \mathcal{M}_{2n} \), with \( n > 1 \), related to a VG–Lie algebra \( V \subset V_{\mathcal{M}_{2n}} \) isomorphic to \( \mathfrak{su}(n) \) admits a superposition rule depending on \( n - 1 \) particular solutions.

**Proof.** In light of Theorem 4.3, our Schrödinger equation admits a VG–Lie algebra \( V \subset V_{\mathcal{M}_{2n}} \) of Kähler vector fields isomorphic to \( \mathfrak{su}(n) \). To derive a superposition rule, we determine the smallest \( m \in \mathbb{N} \) so that the diagonal prolongations to \( \mathcal{M}_{2n} \) of the vector fields of \( V \) span a distribution of rank \( \dim V \) at a generic point. Since \( V \subset V_{\mathcal{M}_{2n}} \) and \( V \cong \mathfrak{su}(n) \), the elements of \( V \) are fundamental vector fields of the standard linear action of \( \mathbb{SU}(n) \) on \( \mathcal{M}_{2n} \) (thought of as a \( \mathbb{C} \)-linear space). The diagonal prolongations of \( V \) to \( \mathcal{M}_{2n} \) span the tangent space to the orbits of the Lie group action

\[
\varphi^m : \mathbb{SU}(n) \times \mathcal{M}_{2n} \rightarrow \mathcal{M}_{2n} \quad \text{(7.1)}
\]

The fundamental vector fields of this action span a distribution of rank \( \dim V \) at \( \xi \in \mathcal{M}_{2n} \) if and only if its isotropy group \( O_\xi \) at \( \xi \) is discrete. Let us set \( m := n - 1 \). The elements \( U \in O_\xi \), with \( \xi := (\psi_1, \ldots, \psi_{n-1}) \in \mathcal{M}_{2n}^{n-1} \), satisfy

\[
U\psi_j = \psi_j, \quad j = 1, n - 1.
\]

At a generic point of \( \mathcal{M}_{2n}^{n-1} \), we can assume that \( \psi_1, \ldots, \psi_{n-1} \) are linearly independent elements of \( \mathbb{C}^n \) (over \( \mathbb{C} \)). Then, the knowledge of the action of \( U \) on these elements fixes \( U \) on \( \langle \psi_1, \ldots, \psi_{n-1} \rangle \subset \mathcal{M}_{2n} \), where it acts as the identity map. If \( \psi \) is orthogonal to \( \langle \psi_1, \ldots, \psi_{n-1} \rangle \) with respect to the natural Hermitian product on \( \mathbb{C}^n \), then \( U\psi \) must also be orthogonal to \( \langle \psi_1, \ldots, \psi_{n-1} \rangle \) because of (7.1) and the unitarity of \( U \). Therefore, \( U\psi \) is proportional to \( \psi \). Since \( U \in \mathbb{SU}(n) \), then \( U\psi = \psi \) and \( U = \text{Id} \). Therefore, the isotropy group of \( \varphi^m \) is trivial at a generic point of \( \mathcal{M}_{2n}^{n-1} \), the fundamental vector fields of \( \varphi^m \) are linearly independent over \( \mathbb{R} \) and there exists a superposition rule depending on \( n - 1 \) particular solutions. \( \square \)

**Note 7.2.** It is worth noting that the isotropy group for \( \varphi^m \) is not trivial at any point of \( \mathcal{M}_{2n}^m \) for \( m < n - 1 \). Given \( m \) linearly independent elements \( \psi_1, \ldots, \psi_m \in \mathcal{M}_{2n} \) over \( \mathbb{C} \), we can construct several special unitary transformations on \( \mathcal{M}_{2n} \) acting as the identity on \( \langle \psi_1, \ldots, \psi_m \rangle \) and leaving stable its orthogonal complement. Hence, the isotropy group on any point of \( \mathcal{M}_{2n}^m \) is not discrete.
Since the elements of $U(n)$ act on $M_{2n}$ preserving the norm relative to the standard Hermitian product on $\mathbb{C}^n$, the Lie group action $\varphi^m$ given in the proof of the previous theorem can be restricted to $S_n^{m}$. In view of this, the previous proof can be slightly modified to prove that the restriction of $\varphi^m$ to $S_n^{m}$ have a trivial isotropy group at a generic point for $m = n - 1$ and $n > 1$. As a consequence, we obtain the following corollary.

**Corollary 7.3.** Every Schrödinger equation $X_{\mathbb{C}^n}$, with $n > 1$, related to a VG–Lie algebra $V \subset V_{\mathbb{C}^n}$ isomorphic to $\mathfrak{su}(n)$ admits a superposition rule depending on $n - 1$ particular solutions.

**Theorem 7.4.** Every Schrödinger equation $X_{\mathbb{R}^n}$, with $n > 1$, admits a superposition rule depending on $n$ particular solutions.

**Proof.** In view of Proposition 5.4, the Schrödinger equation under study admits a VG–Lie algebra $V_{\mathbb{R}^n}$ of fundamental vector fields isomorphic to $\mathfrak{su}(n)$. Also the proof of Proposition 5.4 shows that the diagonal prolongation of the elements of $V_{\mathbb{R}^n}$ to $R_n^m$ are the fundamental vector fields of the Lie group action

$$\varphi^m_R : \quad SU(n) \times R_n^m \quad \rightarrow \quad R_n^m$$

$$(U; [\psi_1]_R, ..., [\psi_m]_R) \quad \mapsto \quad ([U\psi_1]_R, ..., [U\psi_m]_R).$$

To derive a superposition rule for $X_{\mathbb{R}^n}$, we determine the smallest $m \in \mathbb{N}$ so that the diagonal prolongations of a basis $V_{\mathbb{R}^n}$ become linearly independent at a generic point. This occurs at $p \in R_n^m$ if and only if the isotropy group of this action at $p$ is discrete. Let us set $m = n$. The elements of the isotropy group of $\varphi^m_R$ at a generic point $p := ([\psi_1]_R, ..., [\psi_n]_R) \in R_n^m$ satisfy

$$U[\psi_j]_R = [\psi_j]_R, \quad j \in \{1, ..., n\}. \quad (7.2)$$

At a generic point of $R_n^m$, we can assume that $\psi_1, ..., \psi_n$ are linearly independent elements of $\mathbb{C}^n$ (over $\mathbb{C}$). In view of (7.2), the operator $U$ diagonalises on the basis $\psi_1, ..., \psi_n$. Since $U \in U(n)$, we have that $\langle U\psi_i, U\psi_j \rangle = \langle \psi_i, \psi_j \rangle$ for $i, j = 1, ..., n$ and all factors in the diagonal of the matrix representation of $U$ must be equal. As $U \in SU(n)$, the multiplication of such diagonal elements must be equal to 1. This fixes $U = e^{i2\pi k/n}$ for $k \in \mathbb{Z}$. Therefore, the stability group of $\varphi^m_R$ is discrete at a generic point of $R_n^m$, the fundamental vector fields of $\varphi^m_R$ are linearly independent over $\mathbb{R}$ at a generic point and $X_{\mathbb{R}^n}$ admits a superposition rule depending on $n$ particular solutions.

Recall that Proposition 6.3 states that $\mathcal{P}_n$ can be embedded naturally within $R_n$. Additionally, the projection $\pi_{\mathcal{P}^n : R_n \rightarrow \mathcal{P}_n}$ is equivariant relative to the the Lie group action of $SU(n)$ on $R_n$ and the action $\varphi_P$ of $SU(n)$ on $\mathcal{P}_n$. Using these facts, we can easily prove the following corollary by using the same line of reasoning as in Corollary 7.3.

**Corollary 7.5.** Every Schrödinger equation on $\mathcal{P}_n$, with $n > 1$, related to a VG–Lie algebra $V_{\mathcal{P}_n}$ admits a superposition rule depending on $n$ particular solutions.

The second interesting point is that the constants of motion needed to obtain a superposition rule for special unitary Schrödinger equations can be obtained from the associated Kähler structure.
8. Superposition rules for one-qubit systems

In this section we illustrate our theory by describing superposition rules for one-qubit systems and their projections onto $S_2, R_2$ and $P_2$. Observe that we can define the commutative diagram below. For the sake of completeness, we have added under each space the smallest number of particular solutions for its corresponding superposition rule.

On each space we can define a Lie system admitting Vessiot–Guldberg Lie algebras of Hamiltonian vector fields relative to different compatible geometric structures, which in turn allows us to obtain their superposition rules geometrically. The following subsections provide these superposition rules, their relevant geometric properties and their potential applications in quantum mechanics. This will be carried out by applying our previous results. Our procedures will give rise to generalisations of our methods to systems with an arbitrary number of qubits.

8.1. Superposition rule for a two-level system on $M_4^x$

Let us obtain a superposition rule for the system $X = \sum_{\alpha=1}^{3} B_\alpha(t) X_\alpha$ on $M_4^x$ given by (4.8) with $B_0(t) = 0$. Our aim is to illustrate our previous theory while showing that there exists a superposition rule for $X$ depending just on one particular solution.

It is an immediate consequence of Theorem 4.3 that the restriction of system (4.8) to $M_4^x$ is a Kähler–Lie system whose VG–Lie algebra $V = \langle X_1, X_2, X_3 \rangle$, with $X_1, X_2, X_3$ given by (4.9), consists of Kähler vector fields relative to the standard Kähler structure $(g, \omega, J)$ on $M_4^x$. Also, $X_t$ commutes with the phase change vector field $\Gamma$ and with the dilation vector field $\Delta$ for every $t \in \mathbb{R}$, namely $\Gamma$ and $\Delta$ are Lie symmetries of $X$. All these facts will be afterwards used to obtain superposition rules for (4.8) with $B_0 = 0$.

The number of particular solutions needed to obtain a superposition rule for the $t$-dependent vector field $X$ related to (4.8) with $B_0 = 0$ can be given by the smallest integer $m$ such that the diagonal prolongations to $(M_4^x)^m \simeq (\mathbb{R}_+)^m$ of $X_1, X_2, X_3$ are linearly independent at a generic point. The coordinate expressions for $X_1, X_2, X_3$, given in (4.9), show that they are already linearly independent at a generic point of $M_4^x$. Hence, the superposition rule does depend on a mere particular solution, which is better than the standard quantum linear superposition rule for the linear system (4.8), which depends on two particular solutions.

As mentioned in Section 2 the superposition rule for $X$ can be obtained from certain first-integrals for the diagonal prolongations $X_1^{[2]}, X_2^{[2]}, X_3^{[2]}$ of $X_1, X_2, X_3$ to $(M_4^x)^2 \simeq (\mathbb{R}_+)^2$. Using the definition of diagonal prolongations of vector fields and sections of vector bundles given in Section 2 we can prove that, as the Lie derivative of $g, \omega, J$ with respect to any $X \in V$ is zero, the same happens for the diagonal prolongations $g^{[2]}, \omega^{[2]}, J^{[2]}$ relative to any
$X^{[2]} \in V^{[2]} := \langle X_1^{[2]}, X_2^{[2]}, X_3^{[2]} \rangle$, where

$$\omega^{[2]} = \sum_{r=0}^{1} \sum_{j=1}^{2} dq_j^{(r)} \wedge dp_j^{(r)}, \quad g^{[2]} = \sum_{r=0}^{1} \sum_{j=1}^{2} (dq_j^{(r)} \otimes dq_j^{(r)} + dp_j^{(r)} \otimes dp_j^{(r)}),$$

$$j^{[2]} = \sum_{r=0}^{1} \sum_{j=1}^{2} \left( \frac{\partial}{\partial p_j^{(r)}} \otimes dq_j^{(r)} - \frac{\partial}{\partial q_j^{(r)}} \otimes dp_j^{(r)} \right).$$

Additionally, the vector fields $\Delta^{(0)}$, $\Delta^{(1)}$, $\Gamma^{(0)}$, and $\Gamma^{(1)}$ defined on each copy of $\mathcal{M}_4^X$ within $(\mathcal{M}_4^X)^2$, commute with $X_1^{[2]}$, $X_2^{[2]}$, $X_3^{[2]}$. The tensor field $S_{01}$, defined in (4.4) remains invariant under the evolution, namely $\mathcal{L}_{X^{[2]}} S_{01} = 0$ for any $X^{[2]} \in V^{[2]}$.

To obtain the superposition rule for $X$, four common first-integrals $I_1^c, I_1^s, I_2^c$, and $I_2^s$ for $X_1^{[2]}, X_2^{[2]}, X_3^{[2]}$ are needed. Additionally, we must demand

$$\det \left( \frac{\partial (I_1^c, I_1^s, I_2^c, I_2^s)}{\partial (q_1^{(0)}, p_1^{(0)}, q_2^{(0)}, p_2^{(0)})} \right) \neq 0. \quad (8.1)$$

Some common first-integrals for $X_1^{[2]}, X_2^{[2]}$, and $X_3^{[2]}$ can be obtained geometrically from the invariance with respect to such vector fields of several geometric structures previously described:

$$g^{[2]}(\Delta^{(0)}, \Delta^{(0)}) = g^{[2]}(\Gamma^{(0)}, \Gamma^{(0)}), \quad g^{[2]}(\Delta^{(1)}, \Delta^{(1)}) = g^{[2]}(\Gamma^{(1)}, \Gamma^{(1)}), \quad g^{[2]}(\Delta^{(0)}, S_{01} \Delta^{(1)}) = g^{[2]}(S_{01} \Delta^{(0)}, \Delta^{(1)}), \quad \omega^{[2]}(\Delta^{(0)}, S_{01} \Delta^{(1)}) = g^{[2]}(J^{[2]} \Delta^{(0)}, S_{01} \Delta^{(1)}), \quad \text{etc.} \quad (8.2)$$

A simple but long calculation shows that we cannot construct among (8.2) four functions satisfying (8.1).

Another first-integral for $X_1^{[2]}$, $X_2^{[2]}$, and $X_3^{[2]}$ can be obtained from the fact that the complex volume element on $\mathcal{M}_4^X$, understood as a complex manifold $\mathbb{C}_0^2$, reads

$$\Omega = dz_1 \wedge dz_2 = (dq_1 \wedge dq_2 - dp_1 \wedge dp_2) + i(dq_1 \wedge dp_2 + dp_1 \wedge dq_2).$$

This volume element defines two real closed non-degenerate 2-forms $\Omega_R, \Omega_I$ on $\mathcal{M}_4^X$:

$$\Omega_R := dq_1 \wedge dq_2 - dp_1 \wedge dp_2, \quad \Omega_I := dq_1 \wedge dp_2 + dp_1 \wedge dq_2,$$

which satisfy that $\mathcal{L}_Y \Omega_R = \mathcal{L}_Y \Omega_I = 0$ for any $Y \in V$. The diagonal prolongations of $\Omega_R$ and $\Omega_I$ to $(\mathcal{M}_4^X)^2$ allow us to obtain new first-integrals for $X_1^{[2]}, X_2^{[2]}, X_3^{[2]}$:

$$\Omega_R^{[2]}(\Delta^{(0)}, S_{01} \Delta^{(1)}) = -\Omega_I^{[2]}(\Gamma^{(0)}, S_{01} \Gamma^{(1)}), \quad \Omega_I^{[2]}(\Delta^{(0)}, S_{01} \Delta^{(1)}) = -\Omega_I^{[2]}(\Gamma^{(0)}, S_{01} \Gamma^{(1)}).$$
From the set of first-integrals on \( (\mathcal{M}_4^\times)^2 \) so obtained, let us choose four of them as follows:

\[
I_1^\ast(\psi(0), \psi(1)) := \frac{g[2](\Omega(0), S_0 \Omega(1))}{g[2](\Delta(0), \Delta(1))} = \frac{g[2](\Gamma(0), S_0 \Gamma(1))}{g[2](\Gamma(1), \Gamma(1))} = \frac{\sum_{j=1}^{2} (q_j(0) q_j(1) + p_j(0) p_j(1))}{\sum_{j=1}^{2} [(q_j(0))^2 + (p_j(0))^2]},
\]

\[
I_2^\ast(\psi(0), \psi(1)) := \frac{\Omega_R[2](\Omega(0), S_0 \Omega(1))}{g[2](\Delta(0), \Delta(1))} = \frac{\Omega_R[2](\Gamma(0), S_0 \Gamma(1))}{g[2](\Gamma(1), \Gamma(1))} = \frac{\sum_{j=1}^{2} (q_j(0) q_j(1) - p_j(0) p_j(1) - q_j(1) q_j(1) + p_j(0) p_j(1))}{\sum_{j=1}^{2} [(q_j(0))^2 + (p_j(0))^2]},
\]

\[
I_3^\ast(\psi(0), \psi(1)) := \frac{\Omega_R[2](\Delta(0), S_0 \Delta(1))}{g[2](\Delta(0), \Delta(1))} = \frac{\Omega_R[2](\Gamma(0), S_0 \Gamma(1))}{g[2](\Gamma(1), \Gamma(1))} = \frac{\sum_{j=1}^{2} (q_j(0) p_j(1) + p_j(0) q_j(1) - q_j(1) p_j(1) - q_j(0) p_j(1))}{\sum_{j=1}^{2} [(q_j(0))^2 + (p_j(0))^2]},
\]

\[
I_4^\ast(\psi(0), \psi(1)) := \frac{\Omega[2](\Delta(0), S_0 \Delta(1))}{g[2](\Delta(0), \Delta(1))} = \frac{\Omega[2](\Gamma(0), S_0 \Gamma(1))}{g[2](\Gamma(1), \Gamma(1))} = \frac{\sum_{j=1}^{2} (q_j(0) q_j(1) + p_j(0) p_j(1))}{\sum_{j=1}^{2} [(q_j(0))^2 + (p_j(0))^2]}.
\]

The normalization factors allow us to obtain a simple superposition rule. These functions satisfy that

\[
\det \left( \frac{\partial (I_1^\ast, I_2^\ast, I_3^\ast, I_4^\ast)}{\partial (q_1(0), p_1(0), q_2(0), p_2(0))} \right) = \left((q_1(0))^2 + (p_1(0))^2 + (q_2(0))^2 + (p_2(0))^2\right)^{-2} \neq 0.
\]

The matrix of partial derivatives is non-singular for any point in \( \mathcal{M}_4^\times \). Therefore, the system of equations

\[
I_1^\ast(\psi(0), \psi(1)) = k_1, \quad I_2^\ast(\psi(0), \psi(1)) = k_2, \quad I_3^\ast(\psi(0), \psi(1)) = k_3, \quad I_4^\ast(\psi(0), \psi(1)) = k_4,
\]

can be solved for \( \psi(0) := (q_1(0), p_1(0), q_2(0), p_2(0)) \), giving rise to the superposition rule

\[
\Phi : (\psi(1), k) \in \mathcal{M}_4^\times \times \mathcal{M}_4^\times \mapsto \psi(0) := A(k) \psi(1) \in \mathcal{M}_4^\times, \quad A(k) := \begin{bmatrix} k_1 & -k_2 & k_3 & k_4 \\ k_2 & k_1 & -k_3 & k_4 \\ -k_3 & -k_4 & k_1 & -k_2 \\ -k_4 & k_3 & k_2 & k_1 \end{bmatrix},
\]

with \( k = (k_1, k_2, k_3, k_4) \).

8.2. The superposition rule for the Lie system on \( S_2 \)

The projection \( \pi_{\mathcal{M}} : \mathcal{M}_4^\times \to S_2 \) imposes an equivalence relation between points in \( \mathcal{M}_4^\times \) that differ only on a positive real multiplicative constant. Therefore, each equivalence class of a point \( \psi \in \mathcal{M}_4^\times \) is of the form \( [\psi]_\mathcal{S} := \{\lambda \psi \mid \lambda > 0\} \). Hence, each equivalence class can be represented by its unique intersection with the unit sphere and \( \mathcal{M}_4^\times / \mathbb{R}_+ \simeq S_2 \). We will consider the natural embedding \( \iota_S : S_2 \hookrightarrow \mathcal{M}_4 \). The pullback of this embedding defines a presymplectic structure \( \iota_S^* \omega \) and a Riemannian metric \( \iota_S^* g \) on \( S_2 \).

As \( \mathcal{L}_S \) is an \( \mathbb{R} \)-linear system over \( \mathbb{R} \), then the Lie system \( X_{\mathcal{M}_4^\times} \) can be projected through \( \pi_{\mathcal{M}} \) onto a system \( X_{S_2} \) on \( S_2 \); which, indeed, is the restriction of \( \mathcal{L}_S \) to \( S_2 \), as in Proposition 5.1. This proposition also ensures that the vector fields \( X_1|_{S_2}, X_2|_{S_2}, \) and \( X_3|_{S_2} \)
are Hamiltonian with respect to the presymplectic structure $\iota^* g$, admitting Hamiltonian functions $\bar{h}_i := \iota^*_S h_i$. Also, since the vector fields $X_1, X_2,$ and $X_3$ are Killing vector fields for $g$ on $\mathcal{M}^\times_4$, then the vector fields $X_1|_{S_2}, X_2|_{S_2}$, and $X_3|_{S_2}$ are also Killing vector fields with respect to $\iota^*_S g$.

Corollary 7.3 ensures that the restrictions $X_1|_{S_2}, X_2|_{S_2}, X_3|_{S_2}$ are linearly independent at a generic point of $S_2$, then $X|_{S_2}$ admits a superposition rule depending on a unique particular solution. This superposition rule can be obtained by using a similar approach as in the above section, i.e. obtaining three common first-integrals for the diagonal prolongations $X^{[2]}_1|_{S_2}, X^{[2]}_2|_{S_2}, X^{[2]}_3|_{S_2}$, which are Killing vector fields with respect to $(\iota^*_S g)^{[2]}$ and Hamiltonian vector fields relative to the presymplectic structure $(\iota^*_S \omega)^{[2]}$. The latter can be employed to obtain the common first-integrals through invariant functions constructed through $(\iota^*_S g)^{[2]}$, $(\iota^*_S \omega)^{[2]}$. Importantly, the vector $\Delta$ is not tangent to $S_2$ and it cannot be used to construct invariants. More easily, we can obtain the pullback via $\iota_S$ of the first integrals on $(\mathcal{M}^\times_4)^2$ computed in the above section, which allows to determine the superposition rule.

Instead of the above, we will use the following approach, which allows us to obtain the superposition rule for $X|_{S_2}$ from of the superposition for $X|_{\mathcal{M}^\times_4}$. Observe that $X|_{\mathcal{M}^\times_4}$ is a Lie system on $\mathcal{M}^\times_4$ with a superposition rule $\Phi : \mathcal{M}^\times_4 \times \mathcal{M}^\times_4 \to \mathcal{M}^\times_4$ and that $(X|_{\mathcal{M}^\times_4})_t$ is tangent to a submanifold $S_2 \subset \mathcal{M}^\times_4$ for each $t \in \mathbb{R}$. Assume also that there exists $\bar{S} \subset \mathcal{M}^\times_4$ such that $\Phi(\bar{S}_2 \times \bar{S}) = S_2$. Then, the initial superposition rule can be restricted to elements on $S_2$ giving rise to a new superposition principle.

Indeed, let us consider the superposition rule $\Phi$ defined above and evaluated on points $\psi_S^{(1)}, k_S \in S_2$, i.e. $\|\psi^{(1)}_S\| = \|k_S\| = 1$. The resulting point $\Phi(\psi^{(1)}_S, k_S)$ satisfies that

$$
\|\Phi(\psi^{(1)}_S, k_S)\| = |\det A(k_S)| \|\psi^{(1)}_S\| = \|k_S\|^4 \|\psi^{(1)}_S\| = 1 \Rightarrow \Phi(\psi^{(1)}_S, k_S) \in S_2.
$$

Conversely, there always exists, for points $\psi^{(0)}_S \in S_2$ and $k_S \in S_2$, a point $\psi^{(1)}_S \in S_2$ such that $\Phi(\psi^{(1)}_S, k_S) = \psi^{(0)}_S$. Hence $X|_{S_2}$ admits a superposition rule

$$
\Phi_S : (\psi^{(1)}_S, k_S) \in S_2 \times S_2 \mapsto A(k_S)\psi^{(1)}_S \in S_2, \quad (8.4)
$$

with $A(k)$ given by (8.3).

8.3. Superposition rules on $\mathcal{R}_2$ and $\mathcal{P}_2$

Let us obtain a superposition rule for the system (5.3) on $\mathcal{R}_2$ and prove that it depends on two particular solutions.

We employ the global coordinate system $\{x, y, z\}$ on $\mathcal{R}_2$ suggested in Lemma 5.3 and presented in Example 5.1. To simplify the notation, $x := (x, y, z) \neq 0$ will represent an arbitrary point of $\mathcal{R}_2 \simeq \mathbb{R}_0^3$.

Recall that the Poisson structure $\Lambda$ on $\mathcal{M}^\times_4$ can be projected onto $\mathcal{R}_2$ giving rise to a contravariant tensor field $\hat{\Lambda} = \pi_{\mathcal{MR}}\Lambda$ on $\mathcal{R}_2$, presented in (5.4), which is a Poisson tensor field on $\mathcal{R}_2$.

To obtain a superposition rule for (5.3), we have to find the smallest $m \in \mathbb{N}$ so that $Y_1^{[m]}, Y_2^{[m]}, Y_3^{[m]}$ are linearly independent at a generic point of $\mathcal{R}_2^m$. Since $Y_1, Y_2,$ and $Y_3$ span
a two-dimensional distribution and their are linearly independent over \( \mathbb{R} \) (see Example 5.2), then their diagonal prolongations to \( \mathcal{R}_2 \) span, at least, a distribution of rank three and \( Y_1^{[2]}, Y_2^{[2]}, \) and \( Y_3^{[2]} \) are linearly independent at a generic point [21, 23]. Hence, a superposition rule for (5.3) results from giving three functionally independent common first-integrals \( I_1, I_2, I_3 : \mathcal{R}_2^3 \rightarrow \mathbb{R} \) for the diagonal prolongations

\[
Y_1^{[3]} = z(0) \frac{\partial}{\partial y(0)} - y(0) \frac{\partial}{\partial z(0)} + z(1) \frac{\partial}{\partial y(1)} - y(1) \frac{\partial}{\partial z(1)} + z(2) \frac{\partial}{\partial y(2)} - y(2) \frac{\partial}{\partial z(2)},
\]

\[
Y_2^{[3]} = x(0) \frac{\partial}{\partial x(0)} - z(0) \frac{\partial}{\partial x(1)} + x(1) \frac{\partial}{\partial x(1)} - z(1) \frac{\partial}{\partial x(2)} + x(2) \frac{\partial}{\partial x(2)} - z(2) \frac{\partial}{\partial y(2)},
\]

\[
Y_3^{[3]} = y(0) \frac{\partial}{\partial y(0)} - x(0) \frac{\partial}{\partial y(0)} + y(1) \frac{\partial}{\partial y(1)} - x(1) \frac{\partial}{\partial y(2)} + y(2) \frac{\partial}{\partial x(2)} - x(2) \frac{\partial}{\partial y(2)}.
\]

satisfying \( \det(\partial (I_1, I_2, I_3)/\partial (x(0), y(0), z(0))) \neq 0 \).

The needed first-integrals can be obtained from the diagonal prolongations of the Lie symmetry

\[
\Delta = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}.
\]

Firstly, let us define the normalization function

\[
N(x) := \hat{g}(\Delta, \Delta) = (x^2 + y^2 + z^2)^{1/2}, \quad x \in \mathcal{R}_2,
\]

with \( \hat{g} \) given by (5.6). This function is a first integral of \( X_{\mathcal{R}_2} \) and hence it can be understood as a constant of motion of any of its prolongations. Consider now the invariant functions on \( \mathcal{R}_2^3 \) of the form

\[
I_i(x^{(0)}, x^{(1)}, x^{(2)}) := N(x^{(0)})g^{[3]}(\Delta^{(0)}, S_0, \Delta^{(1)}) = x^{(0)} \cdot x^{(j)} + y^{(0)} \cdot y^{(j)} + z^{(0)} \cdot z^{(j)}, \quad j = 1, 2,
\]

\[
I_3(x^{(0)}, x^{(1)}, x^{(2)}) := N(x^{(0)})g^{[3]}(\Delta^{(0)}, \Delta^{(1)}) = (x^{(0)})^2 + (y^{(0)})^2 + (z^{(0)})^2,
\]

Since

\[
\det \left( \frac{\partial (I_1, I_2, I_3)}{\partial (x^{(0)}, y^{(0)}, z^{(0)})} \right) = \det \left[ \begin{array}{ccc}
  x^{(1)} & y^{(1)} & z^{(1)} \\
  x^{(2)} & y^{(2)} & z^{(2)} \\
  2x^{(0)} & 2y^{(0)} & 2z^{(0)}
\end{array} \right] \neq 0
\] (8.6)

at a generic point \( (x^{(0)}, x^{(1)}, x^{(2)}) \in \mathcal{R}_2^3 \), obtaining the superposition rule is equivalent to solving for \( x \) the following system of equations in \( \mathcal{R}_2^3 = \mathbb{R}_0^3 \):

\[
x \cdot x_1 = k_1, \quad x \cdot x_2 = k_2, \quad x \cdot x = k_3,
\]

(8.7)

for some \( k_1, k_2, k_3 \in \mathbb{R} \), with \( k_3 > 0 \). Since \( x_1 \) and \( x_2 \) are not collinear when (8.6) holds, the above system is easily solved in \( \mathbb{R} \) by defining an orthonormal system relative to the standard scalar product on \( \mathcal{R}_2 \simeq \mathbb{R}_2^3 \):

\[
x'_1 := \frac{x_1}{\|x_1\|}, \quad x'_2 := \frac{\|x_1\|^2 x_2 - (x_1 \cdot x_2)x_1}{\|x_1\| \sqrt{\|x_1\|^2 \|x_2\|^2 - (x_1 \cdot x_2)^2}}.
\]
These two new vectors together with their cross product, \( x'_1 \times x'_2 \), conform an orthonormal basis for \( \mathbb{R}^3 \). From (8.7), the general expression for \( x \) is
\[
x = k'_1 x'_1 + k'_2 x'_2 \pm \sqrt{k_3 - (k'_1)^2 - (k'_2)^2}(x'_1 \times x'_2),
\]
where the coefficients \( k'_1 \) and \( k'_2 \) are
\[
k'_1 = x \cdot x'_1 = \frac{k_1}{\|x_1\|}, \quad k'_2 = x \cdot x'_2 = \frac{k_2\|x_1\|^2 - k_1(x_1 \cdot x_2)}{\|x_1\|\sqrt{\|x_1\|^2\|x_2\|^2 - (x_1 \cdot x_2)^2}}.
\]
Replacing \( k'_1 \) and \( k'_2 \) in (8.8), the solution to the system of equations (8.7) is
\[
x = \frac{\delta_{12}x_1 + \delta_{21}x_2 \pm \sqrt{k_3\|x_1\|^2\|x_2\|^2 - (x_1 \cdot x_2)^2} - (k_1x_1 - k_2x_2)^2x_1 \times x_2}{\|x_1\|^2\|x_2\|^2 - (x_1 \cdot x_2)^2},
\]
where \( \delta_{ij} := k_i\|x_j\|^2 - k_j(x_i \cdot x_j) \). As the Lie system \( X_{\mathcal{R}_2} \) is linear in the chosen coordinate system and the Riemannian metric related to the standard scalar product on \( \mathcal{R}_2 \simeq \mathbb{R}^3_0 \) is invariant under the elements of \( V_{\mathcal{R}_2} \), it follows that \( \|x_1\|^2, \|x_2\|^2 \) and \( x_1 \cdot x_2 \) are constant along particular solutions of \( X_{\mathcal{R}_2} \). Then, the above expression gives rise to a superposition rule \( \Phi : (x_1, x_2, (k_1, k_2, k_3)) \in \mathcal{R}_2^2 \times A \mapsto x \in \mathcal{R}_2 \), with \( A = \{(k_1, k_2, k_3 : k_3 \neq 0)\} \), of the form
\[
x = \delta_{12}x_1 + \delta_{21}x_2 + \text{sign}(k_3)\sqrt{k_3(k_1x_1 - k_2x_2)^2(x_1 \times x_2)},
\]
where \( k_{12} := \|x_1\|^2\|x_2\|^2 - (x_1 \cdot x_2)^2 \).

In view of Proposition 6.5, deriving a superposition rule for \( X_{\mathcal{P}_2} \) amounts to obtain a superposition rule for the solutions to \( X_{\mathcal{R}_2} \) on \( \pi_{\mathcal{S}_2}(\mathcal{S}_2) \), namely those equivalence classes of \( \mathcal{R}_2 \) coming from elements of \( \mathcal{M}^x_{2n} \) with the same module. To obtain the superposition rule for the system \( X_{\mathcal{P}_2} \) on \( \mathcal{P}_2 \), consider the natural embedding of \( \mathcal{P}_2 \) into \( \mathcal{R}_2 \) whose image is the set of elements \((x, y, z) \in \mathcal{R}_2 \) such that \( x^2 + y^2 + z^2 = \langle \psi, \psi \rangle^2 / 16 = 1 \). Therefore, \( \mathcal{P}_2 \) is diffeomorphic to a sphere \( \mathcal{S}_2 \subset \mathcal{R}_2 \simeq \mathbb{R}^3_0 \). Consider the superposition rule defined for \( X_{\mathcal{R}_2} \) when restricted to points in \( \mathcal{S}_2 \), i.e. with \( \|x_1\| = \|x_2\| = 1 \). The set of constants has to be constrained in order to obtain solutions in \( \mathcal{S}_2 \). From (8.7), the constraints are \(|k_1|, |k_2| \leq 1, k_3 = 1 \). In consequence, the superposition rule for \( \mathcal{P}_2 \simeq \mathcal{S}_2 \) is
\[
x = \frac{(k_1 - k_2x_1 \cdot x_2)x_1 + (k_2 - k_1x_1 \cdot x_2)x_2 \pm \sqrt{1 - (x_1 \cdot x_2)^2 - (k_1x_1 - k_2x_2)^2}(x_1 \times x_2)}{1 - (x_1 \cdot x_2)^2},
\]
with \( x_1, x_2 \in \mathcal{P}_2 \) and \(|k_1|, |k_2| \leq 1 \).

When \( x_1 \) and \( x_2 \) are replaced by two generic particular solutions of the system within \( \mathcal{S}_2 \subset \mathcal{R}_2 \simeq \mathbb{R}^3_0 \), the general solution is obtained.

9. Superposition rules for \( n \)-levels systems on \( \mathcal{M}^x_{2n} \) and \( \mathcal{S}_n \)

Let us obtain a superposition rule for \( X_{\mathcal{M}^x_{2n}} \), where we assume \( X_1, \ldots, X_{n^2-1} \) to belong to the Lie algebra of fundamental vector fields for the action of \( SU(n) \) on \( \mathcal{M}^x_{2n} \). Our aim is
to illustrate our previous theory while obtaining the explicit expression for a superposition rule depending just on \( n - 1 \) particular solutions.

Theorem 7.4 shows that the smallest \( m \) turning linearly independent the diagonal prolongations of the vector fields \( X_\alpha \) is given by \( m = n - 1 \). Hence, the superposition rule does depend on \( n - 1 \) particular solutions, which is better than the standard linear superposition rule for the \( t \)-dependent Schrödinger equation depending on \( n \) particular solutions. In contrast with the two-level system, the superposition rule depending on \( n - 1 \) particular solutions for \( n > 2 \) is not linear.

The superposition rule can be derived through \( 2n \) common first-integrals \( I^c_1, I^c_1, \ldots, I^c_n, I^n_1 : (\mathcal{M}^\times_{2n})^n \to \mathbb{R} \) for all the diagonal prolongations \( X_\alpha^{[n]} \) on \((\mathcal{M}^\times_{2n})^n\), with \( \alpha = 1, n^2 - 1 \). Additionally, these first-integrals give rise to a superposition rule provided that

\[
\det(\partial(I^c_1, I^c_1, \ldots, I^c_n, I^n_1)/\partial(q_1^{(0)}, p_1^{(0)}, \ldots, q_n^{(0)}, p_n^{(0)})) \neq 0.
\]

Let us obtain the first integrals geometrically. Since the \( X_\alpha \) are Kähler vector fields relative to the Kähler structure \((g, \omega, J)\) on \(\mathcal{M}^\times_{2n}\), their diagonal prolongations \( X_\alpha^{[n]} \) are Kähler relative to the diagonal prolongation \((g^{[n]}, \omega^{[n]}, J^{[n]})\) to \( (\mathcal{M}^\times_{2n})^n \) of the Kähler structure \((g, \omega, J)\), namely

\[
\omega^{[n]} = \sum_{j=1}^n \sum_{a=0}^{n-1} dq_j^{(a)} \wedge dp_j^{(a)}, \quad g^{[n]} = \sum_{j=1}^n \sum_{a=0}^{n-1} (dq_j^{(a)} \otimes dq_j^{(a)} + dp_j^{(a)} \otimes dp_j^{(a)}),
\]

\[
J^{[n]} = \sum_{j=1}^n \sum_{a=0}^{n-1} \frac{\partial}{\partial q_j^{(a)}} \otimes dq_j^{(a)} - \frac{\partial}{\partial p_j^{(a)}} \otimes dp_j^{(a)}.
\]

Similarly, if \( X \) is a Hamiltonian vector field relative to \( \omega \) with Hamiltonian function \( h_X \), then \( X^{[n]} \) is a Hamiltonian vector field with Hamiltonian function \( h_X^{[n]} \).

As the vector fields \( X_\alpha^{[n]} \) are Killing vector fields with respect to \( g^{[n]} \) and symmetries of the tensor fields \( S_{rs} \) for \( r, s = 0, 1, n - 1 \) and \( r \neq s \), presented in \((2.5)\) and \((2.6)\), we can obtain the following common first-integrals for all such vector fields:

\[
I^c_k := g^{[n]}(\Delta^{(0)}, S_{0k}(\Delta^{(k)})) = \sum_{j=1}^n (q_j^{(0)} p_j^{(k)} - p_j^{(0)} q_j^{(k)}), \quad k = 1, \ldots, n - 1,
\]

\[
I^n_k := g^{[n]}(\Gamma^{(0)}, S_{0k}(\Delta^{(k)})) = \sum_{j=1}^n (q_j^{(0)} q_j^{(k)} + p_j^{(0)} p_j^{(k)}), \quad k = 1, \ldots, n - 1.
\]

These functions satisfy that

\[
J^{[n]}(dI^c_k) = dI^n_k, \quad k = 1, \ldots, n - 1.
\]

For a given value of \( k \), the functions \( I^c_k \) and \( I^n_k \) are functionally independent, while functions with different values of \( k \) involve different variables. Hence all of the functions are functionally independent among them.
Observe that the functions in (9.1) are first-integrals not only for \( X_n \), but also for \( \Gamma^n \). We can also obtain functions which are not first-integrals of \( \Gamma^n \) with help of the \( n \)-forms \( \Omega_R \) and \( \Omega_I \). Let \( I_n^c, I_n^s \) be the function defined as

\[
I_n^c := \Omega_R(\Delta(0), S_{01}(\Delta(1)), \ldots, S_{0(n-1)}(\Delta(n-1))) = \Re(\det(\psi(0), \ldots, \psi(n-1))),
\]

\[
I_n^s := \Omega_I(\Delta(0), S_{01}(\Delta(1)), \ldots, S_{0(n-1)}(\Delta(n-1))) = \Im(\det(\psi(0), \ldots, \psi(n-1))).
\]

These functions satisfy that \( J^{[n]}(dI_n^c) = dI_n^s \), so they are functionally independent among themselves. As they are not first-integrals of \( \Gamma^n \), they are functionally independent of functions in (9.1).

To sum up, there exist \( 2n \) first-integrals of the action of \( \text{su}(n) \) on \( \mathcal{M}_{2n} \), given by \( I_1^c, I_1^s, \ldots, I_n^c, I_n^s \). They satisfy that the matrix of partial derivatives of these functions with respect to the coordinates of \( \psi(0) \) is non-singular. Therefore, the solution \( \psi(0) \) to the equations

\[
I_j^c(\psi(0), \psi(1), \ldots, \psi(n-1)) = k_{2j-1}, \quad I_j^s(\psi(0), \psi(1), \ldots, \psi(n-1)) = k_{2j}, \quad j = 1, n,
\]

can be obtained, at least locally, in terms of the coordinates of \( \psi(1), \ldots, \psi(n-1) \) and \( 2n \) real constants \( k_1, \ldots, k_{2n} \).

Since all functions are linear in the coordinates of \( \psi(0) \), then \( \psi(0) \), and consequently the superposition rule, can be obtained by solving the system

\[
\begin{pmatrix}
\frac{\partial I_1^c}{\partial q_1^c} & \frac{\partial I_1^c}{\partial q_1^s} & \cdots & \frac{\partial I_n^c}{\partial q_1^c} & \frac{\partial I_n^c}{\partial q_1^s} \\
\frac{\partial I_1^s}{\partial p_1^c} & \frac{\partial I_1^s}{\partial p_1^s} & \cdots & \frac{\partial I_n^s}{\partial p_1^c} & \frac{\partial I_n^s}{\partial p_1^s} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{\partial I_1^c}{\partial q_1^c} & \frac{\partial I_1^c}{\partial q_1^s} & \cdots & \frac{\partial I_n^c}{\partial q_1^c} & \frac{\partial I_n^c}{\partial q_1^s} \\
\frac{\partial I_1^s}{\partial p_1^c} & \frac{\partial I_1^s}{\partial p_1^s} & \cdots & \frac{\partial I_n^s}{\partial p_1^c} & \frac{\partial I_n^s}{\partial p_1^s}
\end{pmatrix}
\begin{pmatrix}
q_0^c \\
p_0^c \\
q_0^s \\
p_0^s
\end{pmatrix}
= \begin{pmatrix}
q_1^c \\
p_1^c \\
q_1^s \\
p_1^s \\
q_{n-1}^c \\
p_{n-1}^c \\
q_{n-1}^s \\
p_{n-1}^s \\
q_n^c \\
p_n^c \\
q_n^s \\
p_n^s
\end{pmatrix}
\begin{pmatrix}
k_1 \\
k_2 \\
k_3 \\
k_4 \\
k_5 \\
k_6 \\
k_7 \\
k_8 \\
k_9 \\
k_{10} \\
k_{11} \\
k_{12}
\end{pmatrix}.
\]

Observe that

\[
\sum_{j=1}^n (q_j^{(r)} q_j^{(s)} + p_j^{(r)} p_j^{(s)}) = g[n](\Delta^{(r)}, S_{rs}(\Delta^{(s)})),
\]

\[
r \neq s \in \{1, \ldots, n-1\}.
\]

\[
\sum_{j=1}^n (q_j^{(r)} p_j^{(s)} - p_j^{(r)} q_j^{(s)}) = g[n](\Gamma^{(r)}, S_{rs}(\Delta^{(s)})),
\]

These values are constants of motion. Given a set of linearly independent vectors \( \psi^{(1)}, \ldots, \psi^{(n-1)} \in \mathcal{M}_{2n} \), one can always find linear combinations of them such that these functions are zero for
r \neq s$. Also, we have that
\begin{align*}
\sum_{\alpha=1}^{n} \left[ q_{\alpha}^{(i)} \frac{\partial I_n^c}{\partial q_{\alpha}^{(0)}} + p_{\alpha}^{(i)} \frac{\partial I_n^c}{\partial p_{\alpha}^{(0)}} \right] = - \sum_{\alpha=1}^{n} \left[ p_{\alpha}^{(i)} \frac{\partial I_n^a}{\partial q_{\alpha}^{(0)}} - q_{\alpha}^{(i)} \frac{\partial I_n^a}{\partial p_{\alpha}^{(0)}} \right] = \mathfrak{Re}(\det(\psi^{(i)}, \psi^{(1)}, \ldots, \psi^{(n-1)})) = 0,
\sum_{\alpha=1}^{n} \left[ p_{\alpha}^{(i)} \frac{\partial I_n^s}{\partial q_{\alpha}^{(0)}} - q_{\alpha}^{(i)} \frac{\partial I_n^s}{\partial p_{\alpha}^{(0)}} \right] = \sum_{\alpha=1}^{n} \left[ q_{\alpha}^{(i)} \frac{\partial I_n^s}{\partial q_{\alpha}^{(0)}} + p_{\alpha}^{(i)} \frac{\partial I_n^s}{\partial p_{\alpha}^{(0)}} \right] = \mathfrak{Im}(\det(\psi^{(i)}, \psi^{(1)}, \ldots, \psi^{(n-1)})) = 0,
\sum_{\alpha=1}^{n} \left( \frac{\partial I_n^c}{\partial q_{\alpha}^{(0)}} \right)^2 + \left( \frac{\partial I_n^c}{\partial p_{\alpha}^{(0)}} \right)^2 = \sum_{\alpha=1}^{n} \left[ \left( \frac{\partial I_n^s}{\partial q_{\alpha}^{(0)}} \right)^2 + \left( \frac{\partial I_n^s}{\partial p_{\alpha}^{(0)}} \right)^2 \right] = \prod_{\alpha=1}^{n-1} ||\psi^{(\alpha)}||^2,
\end{align*}

for $i \in 1, n-1$. Defining $\Theta := \prod_{\alpha=1}^{n-1} ||\psi^{(\alpha)}||^2$ and choosing $\psi^{(1)}, \ldots, \psi^{(n-1)}$, so that the quantities in (9.4) are zero, we obtain:
\begin{equation}
\left( \begin{array}{c}
q_1^{(0)} \\
p_1^{(0)} \\
\vdots \\
p_{n}^{(0)}
\end{array} \right) = \left( \begin{array}{cccc}
\frac{q_1^{(1)}}{||\psi^{(1)}||^2} & \frac{p_1^{(1)}}{||\psi^{(1)}||^2} & \cdots & \frac{q_n^{(1)}}{||\psi^{(1)}||^2} \\
\frac{p_1^{(1)}}{||\psi^{(1)}||^2} & \frac{\cdots}{\cdots} & \cdots & \frac{p_n^{(1)}}{||\psi^{(1)}||^2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{q_1^{(n-1)}}{||\psi^{(n-1)}||^2} & \frac{p_1^{(n-1)}}{||\psi^{(n-1)}||^2} & \cdots & \frac{q_{n-1}^{(n-1)}}{||\psi^{(n-1)}||^2} \\
\frac{1}{\Theta q_1^{(0)}} & \frac{1}{\Theta p_1^{(0)}} & \cdots & \frac{1}{\Theta q_n^{(0)}} \\
\frac{1}{\Theta q_1^{(0)}} & \frac{1}{\Theta p_1^{(0)}} & \cdots & \frac{1}{\Theta q_n^{(0)}} \\
\frac{1}{\Theta q_1^{(0)}} & \frac{1}{\Theta p_1^{(0)}} & \cdots & \frac{1}{\Theta q_n^{(0)}} \\
\frac{1}{\Theta q_1^{(0)}} & \frac{1}{\Theta p_1^{(0)}} & \cdots & \frac{1}{\Theta q_n^{(0)}}
\end{array} \right)^T \left( \begin{array}{c}
k_1 \\
k_2
\end{array} \right). \quad (9.5)
\end{equation}

This expression gives rise to a superposition rule $\Phi : [\mathcal{M}_{2n}]^{n-1} \times \mathcal{M}_{2n} \to \mathcal{M}_{2n}$ for $X_{\mathcal{M}_{2n}}$. Observe that when we choose particular solutions $\psi_1, \ldots, \psi_{n-1}$ with norm one, then the matrix of the above system becomes unimodular and hence, when $\sum_{\alpha=1}^{2n-1} k_\alpha^2 = 1$, we obtain that $\sum_{\alpha=1}^{n} (q_{\alpha}^{(0)})^2 + (p_{\alpha}^{(0)})^2 = 1$. This allows us to restrict the above superposition rule to a new one of the form $\Phi_S : \mathcal{S}_{n}^{n-1} \times \mathcal{S}_{n} \to \mathcal{S}_{n}$ for $X_{\mathcal{S}_{n}}$.

10. Conclusions and outlook

The present work has laid down the basis for the study of quantum systems on finite-dimensional Hilbert spaces and some of its projective spaces through the theory of Lie systems. We have proved that all such systems and their projections onto projective spaces are Lie systems and we have found some of their superposition rules.

An analogous development can be carried out for Heisenberg equations. We aim to find a formalism based on Lie systems to study those equations in the future. We also recently
found that Lie systems appear in the description of Kossakowski–Lindbland equations, which can be of interest for their analysis. Other topics concerning the geometry of Lie systems in quantum mechanics and their study from differential geometry are also in progress.

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