On a reduction of the generalized Darboux-Halphen system

Sumanto Chanda¹, Sarbarish Chakravarty², Partha Guha¹,³

January 16, 2018

¹ S.N. Bose National Centre for Basic Sciences
JD Block, Sector-3, Salt Lake,
Calcutta-700098, INDIA.
sumanto12@boson.bose.res.in,
partha@bose.res.in

² Department of Mathematics
University of Colorado,
Colorado Springs, CO 80918,
USA.
schakrav@uccs.edu

³ IHES, Le Bois-Marie 35
route de Chartres 91440,
Bures-sur-Yvette,
France.
guha@ihes.fr

Keywords: Darboux-Halphen, Self-Dual Yang-Mills, Bianchi-IX, Lax representation, Hypergeometric function

Abstract

The equations for the general Darboux-Halphen system obtained as a reduction of the self-dual Yang-Mills can be transformed to a third-order system which resembles the classical Darboux-Halphen system with a common additive terms. It is shown that the transformed system can be further reduced to a constrained non-autonomous, non-homogeneous dynamical system. This dynamical system becomes homogeneous for the classical Darboux-Halphen case, and was studied in the context of self-dual Einstein’s equations for Bianchi IX metrics. A Lax pair and Hamiltonian for this reduced system is derived and the solutions for the system are prescribed in terms of hypergeometric functions.

1 Introduction

The Darboux-Halphen differential equations often referred to as the classical Darboux-Halphen (DH) system

\[ \dot{\omega}_i = \omega_j \omega_k - \omega_i (\omega_j + \omega_k), \quad \text{if } i \neq j \neq k = 1, 2, 3, \text{ cyclic,} \quad \dot{:=} \frac{d}{dt}, \quad (1.1) \]

was originally formulated by Darboux [1] and subsequently solved by Halphen [2]. The general solution to equation (1.1) may be expressed in terms of the elliptic modular function. In fact Halphen related the DH equation in terms of the null theta functions.
The system (1.1) has found applications in mathematical physics in relation to magnetic monopole dynamics [3], self-dual Einstein equations [4, 5], topological field theory [6] and reduction of self-dual Yang-Mills (SDYM) equations [7]. Recently in [8], the DH system was reviewed from the perspective of the self-dual Bianchi-IX metric and the SDYM field equations, describing a gravitational instanton in the former case, and a Yang-Mills instanton in the latter. All systems related to the DH system such as Ramanujan and Ramamani system were covered, as well as aspects of integrability of the DH system.

Ablowitz et al [9, 10] studied the reduction of the SDYM equation with an infinite-dimensional Lie algebra to a \( 3 \times 3 \) matrix differential equation. This work led to a generalized Darboux-Halphen (gDH) system which differs from the DH system by a common additive term. The gDH system was also solved originally by Halphen [11] in terms of general hypergeometric functions and whose general solution admits movable natural barriers which can be densely branched.

In this article, we discuss certain aspects related to the integrability of the gDH system. Some of these features were implicit in the original formulation of the system but were never made concrete. Specifically, we show that it is possible to derive naturally from the gDH system yet another reduced system of equations which satisfy a constraint. This constrained system resembles a non-autonomous Euler equation similar to that derived by Dubrovin [12] but with non-homogeneous terms. Furthermore, we derive a simple Lax pair for the constrained system. The paper is organized as follows. In Section 2, the gDH system is introduced and a constrained system is derived from it. Then the solutions of both the gDH and the constrained systems are discussed. In Section 3, we derive following [10], the gDH system from a ninth-order dynamical system that is obtained as a reduction of the SDYM field equations equation. We provide some details in our derivation that were not included in earlier papers. Then we discuss the constrained system in the framework of a fifth-order system that arise as a special case of the SDYM reduction. In Section 4, we formulate a Lax pair and a Hamiltonian for the reduced system introduced in Section 2.

## 2 The gDH system

In this section, we introduce the gDH system for the complex functions \( \omega_i(t) \)

\[
\dot{\omega}_i = \omega_j \omega_k - \omega_i (\omega_j + \omega_k) + \tau^2, \quad i \neq j \neq k = 1, 2, 3, \text{cyclic}. \quad (2.1)
\]

The common additive term \( \tau^2 \) is elaborated as

\[
\tau^2 = \alpha_1^2 x_2 x_3 + \alpha_2^2 x_3 x_1 + \alpha_3^2 x_1 x_2 \quad \text{with} \quad x_i = \omega_j - \omega_k, \quad i \neq j \neq k, \text{cyclic}, \quad x_1 + x_2 + x_3 = 0, \quad (2.2)
\]

where \( \alpha_i, i = 1, 2, 3 \) are complex constants. As mentioned in Section 1, the gDH system arises from a particular reduction of the SDYM equations [9, 10]. They also appear in the study of \( SU(2) \)-invariant, hypercomplex four-manifolds [13]. In Section 3, we will provide a derivation of the gDH system from the SDYM reductions following [10].

In the following, we derive from (2.1) a reduced system of differential equations which satisfy a constraint.
2.1 Constrained gDH system

Note that the variables $x_i$ defined in (2.2) satisfy the equations

$$\dot{x}_i = -2\omega_ix_i, \quad i = 1, 2, 3,$$

which are obtained from (2.1) by taking the difference of the equations for $\omega_j$ and $\omega_k$. Using (2.3), the gDH equations (2.1) can be re-expressed as follows:

$$\dot{\omega}_i - \frac{\omega_i}{2} \left( \frac{\dot{x}_j}{x_j} + \frac{\dot{x}_k}{x_k} \right) = \omega_j\omega_k + \tau^2.$$

Then by defining new variables $W_i$, $i = 1, 2, 3$ via

$$W_i := \frac{\omega_i}{\sqrt{x_jx_k}}, \quad i \neq j \neq k, \text{ cyclic},$$

one obtains the system

$$\dot{W}_i = x_iW_jW_k + \frac{\tau^2}{\sqrt{x_jx_k}}.$$  (2.5)

It follows from (2.5) that

$$\sum_{i=1}^{3} W_i \dot{W}_i = W_1W_2W_3\sum_{i=1}^{3} x_i - \frac{\tau^2}{2x_1x_2x_3}\sum_{i=1}^{3} \dot{x}_i = 0$$

after using (2.4), (2.3) and the fact that $x_1 + x_2 + x_3 = 0$. Thus, one finds that the quantity

$$Q := \sum_{i=1}^{3} W_i^2 = \frac{\omega_1^2}{x_2x_3} + \frac{\omega_2^2}{x_1x_3} + \frac{\omega_3^2}{x_1x_2}$$

is a constant. However, the quantity $Q$ is not a conserved quantity of (2.5), rather $Q = -1$ is an identity which follows from the definition of the variables $W_i$ in (2.4). Indeed, a direct calculation using $x_1 + x_2 + x_3 = 0$, shows that

$$Q = \frac{\omega_1^2x_1 + \omega_2^2x_2 + \omega_3^2x_3}{x_1x_2x_3} = \frac{\omega_1^2x_1 + \omega_2^2x_2 - \omega_3^2(x_1 + x_2)}{x_1x_2x_3}$$

$$= \frac{x_1(\omega_1 - \omega_3)(\omega_1 + \omega_3) + x_2(\omega_2 - \omega_3)(\omega_2 + \omega_3)}{x_1x_2x_3} = \frac{x_1x_2(\omega_2 - \omega_1)}{x_1x_2x_3} = \frac{x_1x_2x_3}{x_1x_2x_3} = -1$$

Therefore, the system in (2.5) is a reduction of the original gDH system; the reduced system can be regarded as a third order system for the $W_i$ satisfying the constraint $Q = -1$. Note that the DH equations (1.1) being a special case ($\alpha_i = 0$) of (2.1), also admits the same reduced system (2.5) as above but with $\tau = 0$.

Remark: A third order system similar to (2.5) but without the non-homogeneous term, was introduced in [14, 15] where the authors derived a family of self-dual, SU(2)-invariant, Bianchi-IX metrics obtained from solutions of a special Painlevé-VI equation. In that case, the vanishing of the anti-self-dual Weyl tensor and scalar curvature led to a sixth order system described by the classical DH system (1.1) coupled to another third order system. The $W_i$ variables represented different quantities in [14, 15] although they
were defined in the same way as in (2.4). The quantity \( Q \) was a first integral (instead of a number) in that case, depending on the initial conditions for the sixth order system. This sixth order system considered in [14, 15] also admits a special reduction to the third order DH system when the metric is self-dual Einstein. It is this latter case which corresponds to the homogeneous version of (2.5) above with \( Q = -1 \).

Next, we discuss the solution of the reduced system via the solutions of the original gDH system (2.1).

2.2 Solutions

As mentioned in Section 1, Halphen [11] solved the gDH system and expressed its solution in terms of the general hypergeometric equation. Below we discuss a method of solution first given by Brioschi [16].

Let us first introduce a function \( s(t) \) via the following ratio:

\[
s = \frac{\omega_3 - \omega_2}{\omega_1 - \omega_2} = -\frac{x_1}{x_3}, \tag{2.6}
\]

Taking the derivative of \( \ln s \) in (2.6) and then using (2.4), the \( x_i \) can be written as

\[
x_1 = -\frac{1}{2} \frac{\dot{s}}{s-1}, \quad x_2 = \frac{1}{2} \frac{\dot{s}}{s}, \quad x_3 = \frac{1}{2} \frac{\dot{s}}{s(s-1)}. \tag{2.7}
\]

Using (2.3) once more, the gDH variables \( \omega_i \) can be expressed in terms of \( s, \dot{s} \) and \( \ddot{s} \) as

\[
\omega_1 = -\frac{1}{2} \frac{d}{dt} \left[ \ln \left( \frac{\dot{s}}{s-1} \right) \right], \quad \omega_2 = \frac{1}{2} \frac{d}{dt} \left[ \ln \left( \frac{\dot{s}}{s} \right) \right], \quad \omega_3 = -\frac{1}{2} \frac{d}{dt} \left[ \ln \left( \frac{\dot{s}}{s(s-1)} \right) \right]. \tag{2.8}
\]

Substituting the above expressions for \( \omega_i \) into the gDH system (2.1) yields the following third order equation for \( s(t) \)

\[
\frac{\ddot{s}}{s} - \frac{3}{2} \left( \frac{\dot{s}}{s} \right)^2 + \frac{\dot{s}^2}{2} \left[ \frac{1 - \alpha_1^2}{s^2} + \frac{1 - \alpha_2^2}{(s-1)^2} + \frac{\alpha_1^2 + \alpha_2^2 - \alpha_3^2 - 1}{s(s-1)} \right], \tag{2.9}
\]

also known as the Schwarzian equation. Equation (2.9) can be linearized in terms of the hypergeometric equation as follows. Let \( \chi_1(s) \) and \( \chi_2(s) \) be any two linearly independent solution of the hypergeometric equation

\[
\chi'' + \left( \frac{1 - \alpha_1}{s} + \frac{1 - \alpha_2}{s-1} \right) \chi' + \frac{(\alpha_1 + \alpha_2 - 1)^2 - \alpha_3^2}{4s(s-1)} \chi = 0. \tag{2.10}
\]

If the independent variable \( t \) in the gDH system is defined by

\[
t(s) = \frac{\chi_2(s)}{\chi_1(s)}, \tag{2.11}
\]

then the inverse function \( s(t) \) satisfies the Schwarzian equation above. Thus, it is possible to express the gDH variables \( \omega_i \) in terms of the hypergeometric solution \( \chi_1 \) and its derivative. One should note that \( s(t) \) is single-valued if and only if the parameters \( \alpha_i \) in (2.9) and (2.10) are either zero or reciprocals of a positive integer.
The reduced system (2.5) takes a simple but interesting form if we consider a variable change from $t$ to $s$ and re-express the corresponding equations. First, let us define new variables

\[ \hat{W}_1 = \frac{W_1}{2}, \quad \hat{W}_2 = \frac{W_2}{2s}, \quad \hat{W}_3 = \frac{W_3}{2s}, \]

where $i := \sqrt{-1}$. Then by using the parametrization of the $x_i$ from (2.7) in (2.5), one obtains a non-autonomous, non-homogeneous version of the Euler “top” equations, namely,

\[ \hat{W}_1 = \frac{\hat{W}_2 \hat{W}_3}{s-1} + \frac{f(s)}{\sqrt{s-1}}, \quad \hat{W}_2' = \hat{W}_1 \hat{W}_3 - f(s) \hat{W}_s, \quad \hat{W}_3' = \hat{W}_1 \hat{W}_2 - \frac{f(s)}{\sqrt{s(s-1)}}, \]

where

\[ f(s) = \frac{\alpha_1^2(s-1) - \alpha_2^2 s - \alpha_3^2 s (s-1)}{4s(s-1)}, \]

and “prime” indicates derivative with respect to $s$. It follows from (2.13) that

\[ \hat{W}_1 \hat{W}_1' - \hat{W}_2 \hat{W}_2' - \hat{W}_3 \hat{W}_3' = \hat{W}_1 \hat{W}_2 \hat{W}_3 \left( \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s(s-1)} \right) + f(s) \left( \frac{\hat{W}_1}{\sqrt{s-1}} + \frac{\hat{W}_2}{\sqrt{s}} + \frac{\hat{W}_3}{\sqrt{s(s-1)}} \right) = 0. \]

The interested reader can easily verify using (2.12), (2.6) and (2.4) that the coefficient of $f(s)$ vanishes identically in above, thereby showing that $\hat{W}_1^2 - \hat{W}_2^2 - \hat{W}_3^2$ is a constant. Moreover, from (2.12) one can easily compute

\[ \gamma = \hat{W}_1^2 - \hat{W}_2^2 - \hat{W}_3^2 = \frac{1}{4} \sum_{i=1}^{3} W_i^2 = \frac{1}{4} Q = -\frac{1}{4}. \]

Thus, the reduced system (2.13) for the $\hat{W}_i$ satisfy the constraint $\gamma = -\frac{1}{4}$.

For the DH case, $f(s) = 0$ (because $\alpha_i = 0$), then (2.13) reduces to a set of homogeneous, non-autonomous equations arising in similarity reductions of certain hydrodynamic type systems [12], as well as in self-dual Einstein equations for $SU(2)$-invariant Bianchi IX metrics [14, 15, 5] (see Remark in Section 2.1). It is known that this homogeneous system can be solved in terms of a special Painlevé VI equation via a transformation discussed in [17], or from the Schlesinger equations associated with the Painlevé VI equation [5]. In general, the solution for the reduced system (2.13) can be expressed in terms of hypergeometric functions utilizing the transformation given by (2.11) and the parametrization of $x_i$ and $\omega_i$ given in (2.7) and (2.8). One also uses the relation $\dot{s} = 1/t'(s) = \chi^2/s/W$ where $\chi^2(s)$ is a solution of (2.10) and $W(s) := W(\chi_1, \chi_2)$ is the Wronskian of two independent solutions. Finally, taking into account the definitions from (2.11) and (2.12) the explicit form of the solutions are

\[ \hat{W}_1(s) = -\frac{s \sqrt{s-1}}{2} \left( \frac{2 \chi'_1}{\chi_1} \frac{W'}{W} - \frac{1}{s-1} \right), \quad \hat{W}_2(s) = \sqrt{s(s-1)} \left( \frac{2 \chi'_1}{\chi_1} \frac{W'}{W} - \frac{1}{s} \right), \quad \hat{W}_3(s) = \frac{s \sqrt{s-1}}{2} \left( \frac{2 \chi'_1}{\chi_1} \frac{W'}{W} - \frac{1}{s-1} \right). \]
Moreover, applying Abel’s formula to (2.10), \( W'/W \) is expressed as
\[
\frac{W'}{W} = - \left( \frac{1 - \alpha_1}{s} + \frac{1 - \alpha_2}{s - 1} \right).
\]

A more direct way to solve the \( \hat{W}_i \) is to reduce the system (2.13) into a single, scalar ordinary differential equation for one of the variables. Recall that the \( \hat{W}_i \) satisfy the following constraints, namely,
\[
\hat{W}_1^2 - \hat{W}_2^2 - \hat{W}_3^2 = -\frac{1}{4}, \quad \frac{\hat{W}_1}{\sqrt{s - 1}} + \frac{\hat{W}_2}{\sqrt{s}} + \frac{\hat{W}_3}{\sqrt{s(s - 1)}} = 0.
\]
(2.14)

By regarding these constraints as two equations for the \( \hat{W}_i \), it is possible to solve for any two of them, say, \( \hat{W}_1 \) and \( \hat{W}_3 \) in terms of \( \hat{W}_2 \). Thus, one obtains
\[
\hat{W}_1 = \frac{c - \sqrt{s} \hat{W}_2}{\sqrt{s - 1}}, \quad \hat{W}_3 = \frac{\hat{W}_2 - c \sqrt{s}}{\sqrt{s - 1}}, \quad c = \pm \frac{1}{2}.
\]
(2.15)

Next, substituting the expressions for \( \hat{W}_1 \) and \( \hat{W}_3 \) from (2.15) into the equation for \( \hat{W}_2 \) in (2.13), yields a Riccati equation
\[
\sqrt{s(s - 1)} \hat{W}_2' + \hat{W}_2^2 - c \frac{s + 1}{\sqrt{s}} \hat{W}_2 + (s - 1)f(s) + \frac{1}{4} = 0,
\]

where the rational function \( f(s) \) is given in (2.13). If we take \( c = \frac{1}{2} \), then the Riccati equation can be linearized by the following transformation
\[
\frac{\hat{W}_2}{\sqrt{s(s - 1)}} = \frac{1}{2} \left( \frac{1 - \alpha_2}{s - 1} - \frac{\alpha_1}{s} \right) + \frac{\chi'}{\chi}
\]
where the function \( \chi(s) \) satisfies the hypergeometric equation (2.10). If \( c = -\frac{1}{2} \), then one can still linearize the resulting Riccati equation but the parameters in the underlying hypergeometric equation are related to but are not the same as the \( \alpha_i \).

3 The DH-IX matrix system

So far we have dealt with the gDH system which consists of the DH equations together with a common additive term \( \tau^2 \) appearing in all three equations in (2.1). In this section, we will show how the gDH system can be derived from a \( 3 \times 3 \) matrix system which arise as a reduction of the SDYM field equations. We start by reviewing the reduction process on the SDYM equations following [10].

Consider a gauge group \( G \) which may be a finite or infinite-dimensional Lie group. The gauge field \( F \) is a 2-form taking values in the associated Lie algebra \( \mathfrak{g} \), and is given in terms of the \( \mathfrak{g} \)-valued connection 1-form (gauge potential) \( A \) as \( F = dA - A \wedge A \). In a local co-ordinate system \( \{ x^a \} \ a = 0, 1, 2, 3 \) the gauge field components are given by \( F_{ab} = \partial_a A_b - \partial_b A_a - [A_a, A_b] \) where \( \partial_a \) denotes partial derivative with respect to \( x^a \) and \( [\ , \] \) denotes the Lie bracket in \( \mathfrak{g} \). The self-duality condition implies that \( F = *F \) where \( *F \)
is the dual 2-form. In terms of components of $F$, the self-duality condition is equivalent to
\[ F_{0i} = F_{jk}, \quad i \neq j \neq k, \quad \text{cyclic.} \quad (3.1) \]

If the connection 1-form is restricted to depend only on the co-ordinate $x^0 := t$, then without loss of generality, one can choose a gauge where $A_0 = 0$. Consequently, $A_i = A_i(t)$ for $i = 1, 2, 3$, and (3.1) reduces to the Nahm equations \[ \dot{A}_i = [A_j, A_k], \quad i \neq j \neq k, \quad \text{cyclic.} \quad (3.2) \]

Suppose the Lie algebra $\mathfrak{g}$ is chosen to be $\mathfrak{sdiff}(S^3)$ - the infinite-dimensional Lie algebra of diffeomorphisms on $S^3$ generated by the left-invariant vector fields $X_i$ satisfying the relation $[X_i, X_j] = X_k, \ i \neq j \neq k, \ cyclic$. Furthermore, let the $A_i$ be of the form
\[ A_i = -\sum_{j,k=1}^{3} M_{ij}(t)O_{jk}X_k \quad (3.3) \]

where $M_{ij}(t)$ are the entries of a $3 \times 3$ matrix $M(t)$ and $O_{ij} \in SO(3)$ represents a point on $S^3$. Then the Nahm equations (3.2) lead to the following matrix ordinary differential equation for $M(t)$ \[ \dot{M} = C(M) + M^T M - (\text{Tr} M) M, \quad (3.4) \]

where $C(M)$ denotes the matrix of cofactors of $M$. Equation (3.4) is a ninth-order coupled system of equations for the matrix elements of $M(t)$ and was referred to as the DH-IX system in \[ gDH \] where the equations have a common additive term.

### 3.1 Reduction of DH-IX to the gDH system

Note that the equations for the off-diagonal entries in (3.5) involve symmetric and skew-symmetric combinations of the off-diagonal elements. This fact can be exploited further to simplify the matrix equation (3.4) as follows: First, the cofactor matrix $C(M) = (\text{adj} M)^T$, where the adjugate matrix can be expressed as
\[ \text{adj}M = M^2 - (\text{Tr} M) M + \frac{1}{2} \left( (\text{Tr} M)^2 - \text{Tr} M^2 \right) I \]
using the Caley-Hamilton theorem for $3 \times 3$ matrices. In above, $\text{Tr}$ denotes the matrix trace and $I$ is the identity matrix. Next, substituting the transpose of the above expression for $C(M)$ into (3.4) yields

$$\dot{M} = (M^T - (\text{Tr}M)I)(M + M^T) + \frac{1}{2} \left((\text{Tr}M)^2 - \text{Tr}M^2\right) I. \quad (3.6)$$

Equation (3.6) motivates decomposing the matrix $M$ into its symmetric and skew-symmetric parts and re-expressing the DH-IX system in terms of these components as illustrated below. Let us consider the following decomposition of $M$

$$M = M_s + M_a = P(d + a)P^{-1}, \quad (3.7)$$

where the symmetric part $M_s$ is further diagonalized by an orthogonal matrix $P$ ($P^T = P^{-1}$) and the skew-symmetric part is expressed as $M_a = PaP^{-1}$. Let $d = \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix}$, $a = \begin{pmatrix} 0 & \tau_3 & -\tau_2 \\ -\tau_3 & 0 & \tau_1 \\ \tau_2 & -\tau_1 & 0 \end{pmatrix}$. Substituting (3.7) into (3.6) yields the following set of matrix equations for $P, a$ and $d$,

$$\dot{P} + Pa = 0, \quad \dot{a} + ad + da = 0, \quad \dot{d} = 2d^2 - 2(\text{Tr}d)d + \frac{1}{2} \left(\text{Tr}d^2 - (\text{Tr}d)^2 - 2\text{Tr}a^2\right) I. \quad (3.9)$$

The last equation of (3.9) gives the gDH system (2.1) with $\tau^2 = \tau_1^2 + \tau_2^2 + \tau_3^2$. Then, using (2.3) one can integrate the second equation in (3.9), i.e.,

$$\tau_i = -\tau_i(\omega_j + \omega_k) \Rightarrow \tau_i^2 = \alpha_i^2 x_j x_k = \alpha_i^2 (\omega_j - \omega_i) (\omega_i - \omega_k), \quad i \neq j \neq k, \text{ cyclic}, \quad (3.10)$$

and where $\alpha_i$ are integration constants. Combining the last equation of (3.9) with (3.10), yields the gDH system (2.1). The first equation in (3.9) is linear and can be solved for $P$ given the $\tau_i$ although it is not possible to obtain closed form solutions for $P$ except for special cases. We illustrate one such special case in the example below.

### 3.2 The DH-V system

We now discuss a fifth order reduction of the DH-IX system where the matrix $P$ introduced in (3.7) can be expressed in closed form. Let us consider the case in which the DH-IX matrix has the special form

$$M = \begin{pmatrix} \Omega_1 & \theta & 0 \\ \phi & \Omega_2 & 0 \\ 0 & 0 & \Omega_3 \end{pmatrix}. \quad (3.11)$$

Then (3.5) becomes a fifth-order system given by

$$\begin{align*}
\dot{\Omega}_1 &= \Omega_2 \Omega_3 - \Omega_1(\Omega_2 + \Omega_3) + \phi^2 \\
\dot{\Omega}_2 &= \Omega_3 \Omega_1 - \Omega_2(\Omega_3 + \Omega_1) + \theta^2 \\
\dot{\Omega}_3 &= \Omega_1 \Omega_2 - \Omega_2(\Omega_1 + \Omega_2) - \theta \phi \\
\dot{\theta} &= - (\theta + \phi) \Omega_3 - (\theta - \phi) \Omega_2 \\
\dot{\phi} &= - (\theta + \phi) \Omega_3 + (\theta - \phi) \Omega_1
\end{align*}$$
which was introduced in [19]. We refer to system (3.11) as the DH-V system and will construct its solution based on the method discussed in Section 3.1. Due to the special block structure of $M$, its symmetric part $M_s$ can be diagonalized by an orthogonal matrix of the form

$$
P = \begin{pmatrix}
\cos \gamma & \sin \gamma & 0 \\
-\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

(3.12)

where $\gamma = \gamma(t)$ is a complex function to be determined. That is, $M_s = P d P^{-1}$ with $d$ as in (3.8). Furthermore, the skew-symmetric part $M_a$ commutes with the $P$ above so that $a = P^{-1} M_a P = M_a = \begin{pmatrix} 0 & \tau_3 & 0 \\ -\tau_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Then the first equation in (3.9),

$$
\dot{P} = -P a \Rightarrow \dot{\gamma} = \tau_3,
$$

which can be solved in terms of $s(t)$ as

$$
\gamma(s(t)) = \pm i \alpha_3 \log \left( \sqrt{s} - \sqrt{s - 1} \right) + \gamma_0,
$$

(3.15)

where $\gamma_0$ is a (complex) constant. Hence the DH-V matrix $M$ can be reconstructed in terms of the matrices $P,d$ and $a$ as follows:

$$
\begin{aligned}
\Omega_1 + \Omega_2 &= \omega_1 + \omega_2, \\
\Omega_1 - \Omega_2 &= (\omega_1 - \omega_2) \cos 2\gamma, \\
\Omega_3 &= \omega_3, \\
\theta + \phi &= (\omega_2 - \omega_1) \sin 2\gamma, \\
\theta - \phi &= 2\tau_3,
\end{aligned}
$$

(3.16)

where $\omega_i$ are given by (2.8), and $\tau_3, \gamma$ are given by equations (3.14) and (3.15), respectively. Equation (3.16) gives the complete solution of the DH-V system in terms of the solution $s(t)$ of Schwarzian equation (2.9) with $\alpha_1 = \alpha_2 = 0$.

It is also possible to express the constraint $Q$ introduced in Section 2.1 in terms of the DH-V matrix elements. Indeed, one can calculate from (3.16)

$$
\begin{aligned}
\Omega_1 + \Omega_2 &= \omega_1 + \omega_2, \\
\Omega_1 - \Omega_2 &= (\omega_1 - \omega_2) \cos 2\gamma, \\
\Omega_3 &= \omega_3, \\
\theta + \phi &= (\omega_2 - \omega_1) \sin 2\gamma, \\
\theta - \phi &= 2\tau_3,
\end{aligned}
$$

(3.17)

where $\omega_i$ are given by (2.8), and $\tau_3, \gamma$ are given by equations (3.14) and (3.15), respectively. Equation (3.16) gives the complete solution of the DH-V system in terms of the solution $s(t)$ of Schwarzian equation (2.9) with $\alpha_1 = \alpha_2 = 0$.

It is also possible to express the constraint $Q$ introduced in Section 2.1 in terms of the DH-V matrix elements. Indeed, one can calculate from (3.16)

$$
\begin{aligned}
\omega_1 &= \frac{1}{2} (\Sigma + \Delta), \\
\omega_2 &= \frac{1}{2} (\Sigma - \Delta), \\
\omega_3 &= \Omega_3, \\
\Sigma := \Omega_1 + \Omega_2, \\
\Delta := \pm \sqrt{(\Omega_1 - \Omega_2)^2 + (\theta + \phi)^2}.
\end{aligned}
$$

(3.17)

Substituting these expressions into the definition of $Q$, yields

$$
Q := \frac{\omega_1^2}{x_2 x_3} + \frac{\omega_2^2}{x_3 x_1} + \frac{\omega_3^2}{x_1 x_2}
= \frac{1}{4} (\Sigma + \Delta)^2 \left( \frac{1}{4} (\Sigma - \Delta) - \Omega_3 \right) - \frac{1}{4} (\Sigma - \Delta)^2 \left( \frac{1}{4} (\Sigma + \Delta) - \Omega_3 \right) + \Omega_3 \Delta
\Delta \left( \Omega_3 - \frac{1}{2} (\Sigma + \Delta) \right) \left( \frac{1}{2} (\Sigma - \Delta) - \Omega_3 \right) = -1
$$

after some simplification.
4 Lax pair and Hamiltonian for the constrained gDH system

In this section we derive a Lax pair for the reduced non-homogeneous system (2.13) for the \( \hat{W}_i \) introduced in Section 2.2. Furthermore, we show that (2.13) can also be regarded as a constrained Hamiltonian system in the phase space of the variables \( \hat{W}_i \).

4.1 Lax equation

Specifically, we find 3 \( \times \) 3 matrices \( U \) and \( V \) such that (2.13) is equivalent to the following matrix Lax equation

\[
U' + [U, V] = 0,
\]

where recall that “prime” denotes \( \frac{d}{ds} \). Let us choose \( U \) and \( V \) in the Lie algebra \( so(1,2) \) as follows:

\[
U = \begin{pmatrix}
0 & \hat{W}_3 & \hat{W}_2 \\
\hat{W}_3 & 0 & \hat{W}_1 \\
\hat{W}_2 & -\hat{W}_1 & 0
\end{pmatrix}, \quad V = \begin{pmatrix}
0 & v_3 & v_2 \\
v_3 & 0 & v_1 \\
v_2 & -v_1 & 0
\end{pmatrix},
\]

where the \( v_i \) are to be determined. The commutator [\( U, V \)] is also in \( so(2,1) \) and its entries should be equal to the right hand side of (2.13), which we denote by \( r_i, i = 1, 2, 3 \). This results in the following linear system

\[
Bv = r, \quad B = \begin{pmatrix}
0 & -\hat{W}_3 & \hat{W}_2 \\
-\hat{W}_3 & 0 & \hat{W}_1 \\
\hat{W}_2 & -\hat{W}_1 & 0
\end{pmatrix}, \quad v = [v_1, v_2, v_3]^T, \quad r = [r_1, r_2, r_3]^T.
\]

for the vector \( v \). Note that the matrix \( B \) is singular. In order for the linear system (4.2) to have a consistent solution, the vector \( r \) must be orthogonal to the null space of \( B^T \) by Fredholm’s alternative. The null space of \( B^T \) is spanned by the vector \( \hat{N} = [\hat{W}_1, -\hat{W}_2, -\hat{W}_3]^T \). Therefore, one must have \( \hat{N} \cdot r = \hat{W}_1 r_1 - \hat{W}_2 r_2 - \hat{W}_3 r_3 = 0 \), which is readily verified from the calculations immediately following (2.13). Thus, the linear system (4.2) admits infinitely many solutions (defined modulo the homogeneous solution spanned by the null vector \( [\hat{W}_1, \hat{W}_2, \hat{W}_3]^T \) of \( B \)). A particular choice for the vector \( v \) is given by

\[
v_1 = 0, \quad v_2 = \frac{r_3}{\hat{W}_1} = -\left( \frac{\hat{W}_2}{s(s-1)} + \frac{f(s)}{\hat{W}_1 \sqrt{s(s-1)}} \right), \quad v_3 = \frac{r_2}{\hat{W}_1} = \left( \frac{\hat{W}_3}{s} + \frac{f(s)}{\hat{W}_1 \sqrt{s}} \right),
\]

which then yields the matrix \( V \) in the Lax pair.

In a general setting, the Lax equation \( U' + [U, V] = 0 \) is useful to generate a sequence of conserved quantities \( \text{Tr} U^n, n = 1, 2, \ldots \). Indeed, by differentiating with respect to \( s \) one obtains

\[
(\text{Tr} U^n)' = n \text{Tr} \left( U^{n-1} [V, U] \right) = n \text{Tr} \left( V [U, U^{n-1}] \right) = 0.
\]

These conserved quantities are related to the symmetric functions of the eigenvalues of the matrix \( U \). In the present case, the eigenvalues of \( U \) are simply given by \( \lambda = \)
0, \sqrt{-q} = 0, \pm \frac{1}{2} \sqrt{Q} = -\frac{1}{2}. In fact, one obtains \( \text{Tr}U = 0, \text{Tr}U^2 = -2q \), and the remaining traces are polynomials in \( q \).

It is worth pointing out that (2.5) for the \( W_i \) also admits a Lax pair. Here, one chooses so(3)-valued 3 \times 3 matrices

\[
L = \begin{pmatrix}
0 & W_3 & -W_2 \\
-W_3 & 0 & W_1 \\
W_2 & -W_1 & 0
\end{pmatrix}, \quad A = \begin{pmatrix}
0 & A_3 & -A_2 \\
-A_3 & 0 & A_1 \\
A_2 & -A_1 & 0
\end{pmatrix},
\]

where the \( W_i(t) \) are defined in (2.4) and the \( A(t) \) are to be determined such that the Lax equation \( \dot{L} + [L, A] = 0 \) is equivalent to the system (2.5). The matrix \( A \) can be found by proceeding in a similar fashion as outlined above. One finds that a particular choice for the matrix elements of \( A \) is given by

\[
A_1 = 0, \quad A_2 = x_3W_2 + \frac{\tau^2}{W_1 \sqrt{x_1x_2}}, \quad A_3 = -\left( x_2W_3 + \frac{\tau^2}{W_1 \sqrt{x_1x_3}} \right).
\]

The eigenvalues of \( L \) is given by \( \lambda = 0, \pm \sqrt{-Q} = 0, \pm 1 \). Consequently, \( \text{Tr}L^n, n = 1, 2, \ldots \) are polynomials in \( Q \).

### 4.2 Hamiltonian formulation

Equations (2.13) can be also regarded as a constrained Hamiltonian system in the phase space of the variables \( \widehat{W}_i \) satisfying the constraints in (2.14). The phase space is endowed with a natural Poisson structure inherited from the Lie-Poisson structure defined on the dual space of the Lie algebra so(1,2) used to construct the Lax pair. Explicitly, the Poisson structure is given by the fundamental Poisson bracket relations

\[
\{\widehat{W}_1, \widehat{W}_2\} = \widehat{W}_3, \quad \{\widehat{W}_2, \widehat{W}_3\} = -\widehat{W}_1, \quad \{\widehat{W}_3, \widehat{W}_1\} = \widehat{W}_2.
\]

In general, the Poisson bracket of any two continuously differentiable functions \( f \) and \( g \) on the phase space, is given by

\[
\{f, g\} = J(df, dg) = \sum_{i,j,k=1}^{3} C_{ij}^k \frac{\partial f}{\partial \widehat{W}_i} \frac{\partial g}{\partial \widehat{W}_j},
\]

where \( C_{ij}^k \) are the structure constants for the Lie algebra so(1,2). The Poisson tensor \( J_{ij} := \sum_{k=1}^{3} C_{ij}^k \widehat{W}_k \) is degenerate on the three-dimensional phase space, and admits a Casimir function constructed from the Lax matrix \( U \) as follows

\[
C = -\frac{1}{2} \text{Tr}U^2 = \widehat{W}_1^2 - \widehat{W}_2^2 - \widehat{W}_3^2
\]

such that \( J(\cdot, dC) = 0 \). In other words, \( \{f, C\} = 0 \) for any smooth function \( f \) on the phase space. Note from (2.14) that \( C + \frac{1}{4} = 0 \) is one of the constraints, while the other constraint is given by \( l = 0 \), where

\[
l = -\frac{1}{2} \text{Tr}US = \frac{\widehat{W}_1}{\sqrt{s-1}} + \frac{\widehat{W}_2}{\sqrt{s}} + \frac{\widehat{W}_3}{\sqrt{s(s-1)}}
\]
\[
S = \begin{pmatrix}
0 & (s(s - 1))^{-1/2} & s^{-1/2} \\
(s(s - 1))^{-1/2} & 0 & (s - 1)^{-1/2} \\
0 & -(s - 1)^{-1/2} & 0
\end{pmatrix}.
\]

Next, we introduce a Hamiltonian function on the phase space by

\[
H = -\frac{1}{2} \text{Tr}(UIU - 4cf(s)US) = \frac{1}{2} \left( \frac{\widehat{W}_1^2}{s-1} - \frac{\widehat{W}_2^2}{s} - \frac{(2s-1)\widehat{W}_3^2}{s(s-1)} \right) - 4cf(s)l,
\]

where \( I = \text{diag}(s^{-1}, (s-1)^{-1}, 0) \), \( l \) is defined above, \( c = \pm \frac{1}{2} \), and \( f(s) \) is defined in (2.13). With the fundamental Poisson brackets given by (4.4), the reduced gDH equations (2.13) can be expressed by the following equation of motions together with the constraints

\[
\dot{\widehat{W}}_i = \{\widehat{W}_i, H\}, \quad C + \frac{1}{4} = 0, \quad l = 0,
\]

where the Hamiltonian \( H \) is given by (4.5). The equations of motions obtained from (4.6) determines the equations in (2.13) after applying the constraints. For example, one can compute using (4.4) that

\[
\{\widehat{W}_1, H\} = \frac{\widehat{W}_2\widehat{W}_3}{s-1} - 4cf(s) \left( \frac{\widehat{W}_2}{\sqrt{s(s-1)}} - \frac{\widehat{W}_3}{\sqrt{s}} \right).
\]

Upon applying the constraints, one can replace \( \widehat{W}_3 \) in the second term above by its expression from (2.15) to obtain the first equation in (2.13). The remaining equations in (4.6) lead to the corresponding equations in (2.13) in a similar fashion.

For consistency, it also needs to be checked that the constraints are satisfied by the Hamiltonian dynamics. In other words, one should have modulo the constraints

\[
dC/ds = \{C, H\} = 0, \quad dl/ds = \partial l/\partial s + \{l, H\} = 0.
\]

The first consistency condition is obviously satisfied since \( C \) is a Casimir function, the second one can also be verified by using (2.15) and after some straightforward computations.

5 Conclusion

In this note, we have discussed certain features pertaining to the integrability of the gDH system introduced in [9, 10] which contains the well-known DH system as a special case. We have also provided a detailed derivation of the gDH system from a ninth-order system which arises as a reduction of the SDYM equations associated with Lie algebra \( \text{soff}(S^3) \). Starting from the gDH system, we have derived a reduced system (2.13) which is similar to a non-autonomous, non-homogeneous Euler system with a constraint. We then give the complete solution for this system in terms of the general solution of the classical hypergeometric equations. Finally, we derive a Lax pair and a Hamiltonian for this reduced system.
Acknowledgement

The work of SC was partially supported by NSF grant No. DMS-1410862 and the work of PG was partially supported by FAPESP through IFSC, Sao Carlos with grant number 2016/06560-6. SC also thanks SNBNCBS, Kolkata, for its hospitality where this work was initiated. PG would like to express his gratitude to Professor Maxim Kontsevich and other members of the IHES for their warm hospitality where the final part of the work has been done.

References

[1] G. Darboux, Leons sur les systémes othogonaux, (2nd ed.). Gauthiers-Villars, Paris (1910).
[2] G.H. Halphen, Sur une système déquations différentielles, C. R. Acad. Sci. Paris, 92 (1881) 1101-1103.
[3] M. Atiyah and N. Hitchin, The Geometry and dynamics of magnetic monopoles, (Princeton University Press, Princeton, NJ, 1988).
[4] G.W. Gibbons and C.N. Pope, The Positive Action Conjecture and Asymptotically Euclidean Metrics in Quantum Gravity, Commun. Math. Phys. 66 (1979) 267-290.
[5] N. Hitchin, Twistor Spaces, Einstein metrics and isomonodromic deformations, J. Diff. Geom. 42 (1995) 30-112.
[6] B. A. Dubrovin, Geometry of 2D topological field theories, Lecture Notes in Math. 1620, Springer-Verlag, Berlin, Heidelberg, New York (1996), arXiv: hep-th/9407018.
[7] S. Chakravarty, M.J. Ablowitz and P.A. Clarkson, Reductions of Self-Dual Yang-Mills Fields and Classical Systems, Phys. Rev. Lett. 65 (1991) 1085-87.
[8] S. Chanda, P. Guha, R. Roychowdhury, Bianchi-IX, Darboux-Halphen and Chazy-Ramanujan, Int. J. Geom. Methods Mod. Phys. 13 (2016), no. 4, 1650042, 25 pp, arXiv:1512.01662 [hep-th].
[9] M.J. Ablowitz, S. Chakravarty and R.G. Halburd, On Painlevé and Darboux-Halphen type equation, Proc. 4th Int. Conf. on Mathematical and Numerical Aspects of Wave Propagation. 1998.
[10] M.J. Ablowitz, S. Chakravarty and R.G. Halburd, The generalized Chazy equation from the self-duality equations, Stud. Appl. Math. 103 (1999)75-88.
[11] G.H. Halphen, Sur certains systéme déquations différentielles, C. R. Acad. Sci. Paris, 92 (1881) 1404-1406.
[12] B. A. Dubrovin, Differential geometry of strongly integrable systems of hydrodynamic type, Func. Anal. Appl. 24 (1991), 280–285.
[13] N. Hitchin, Hypercomplex manifolds and the space of framings, in The geometric universe, (Oxford 1996), Oxford Univ. Press, Oxford, 1998, 9–30.
[14] S. Chakravarty, A class of integrable conformally self-dual metrics, Class. Quantum Grav. 11 (1994) L1-L6.
[15] K.P. Tod, Self-dual Einstein metrics from the Painlevé VI equation, Phys. Lett. A 190 (1994) 221-224.
[16] F. Brioschi, Sur un système d'équations différentielles, C.R. Acad. Sci. XCII (1881) 1389–93.
[17] A. S. Fokas and M. J. Ablowitz, On a unified approach to transformations and elementary solutions of Painlevé equations, J. Math. Phys. 23 (1982) 2033-42.
[18] W. Nahm, The construction of all self-dual monopoles by the ADHM method, in “Monopoles in Quantum Field Theory” Word Scientific (Singapore) (1982).
[19] M.J. Ablowitz, S. Chakravarty, and R.G. Halburd. On Painlevé and Darboux-Halphen type equations, in the Painleve property, one century later, CRM series in Mathematical Physics, Springer (Berlin) 1998.