ON COMPOSITION TABLEAUX BASIS FOR THE PLÜCKER ALGEBRA

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Abstract. Let $V$ be a complex vector space with dim$_C(V) = n$. The irreducible polynomial representations of $GL(V)$ are the Schur Functors $S^\lambda(V)$, with each Schur Functors labelled by a partition with parts at most $n$. The representations can be realized inside a certain subring of the polynomial ring $\mathbb{C}[x_{i,j}]$ in $n^2$ variables. This subring, $\Pi$, is called the Plücker Algebra and has a decomposition into irreducible $GL(V)$ representations as $\Pi \cong \bigoplus S^\lambda(V)$.

In $[3]$, Mason and Remmel introduce Row Strict Composition Tableaux (RSCT), a method of filling composition diagrams, which are diagrams associated to compositions of positive integers. This gives a method of realization of the Plücker Algebra by taking associated minors of the matrix in $n^2$ indeterminates. In this paper, we develop the representation theory of these RSCT fillings. For each Schur Functor $S^\lambda(V)$, we identify certain compositions, and show that the polynomials that one gets from the minors associated to the RSCT fillings of these compositions, for a basis for that Schur Functor inside $\Pi$.

1. Introduction

Fix a finite dimensional vector space $V$ over $\mathbb{C}$ and let $n = \dim_{\mathbb{C}}(V)$. The representation theory of $GL(V)$ is well understood. The finite dimensional irreducible polynomial (a polynomial representation ($\rho, W$) is one where, upon choosing coordinates, the matrix elements of the homomorphism $\rho : GL(V) \to GL(W)$ are polynomial), are given via a certain universal construction - as we now recall.

Let $\lambda$ be a partition of positive integers $m \in \mathbb{N}$, with $l(\lambda) \leq n$. For each such $\lambda$, look at $V^{\times |\lambda|}$, the $|\lambda|$ times product of $V$. We then look at linear maps $\phi : V^{\times |\lambda|} \to W$, where $W$ is an arbitrary vector space, such that $\phi$ satisfies certain conditions under the exchange of its coordinate entries (for details see $[\Pi]$). These conditions generalize the usual conditions imposed on alternating and symmetric maps; and correspondingly there exists a universal target module $S^\lambda(V)$, functorial in $V$, generalizing the construction of the alternating and symmetric powers of a vector space. This construction is called a Schur Functor. By universality, this construction is unique up to a canonical isomorphism.

Then $S^\lambda(V)$ has a natural action of $GL(V)$ arising by the functoriality of the construction. One then proves that this representation of $GL(V)$ is an irreducible polynomial representation, and in fact each irreducible polynomial representation is isomorphic to one arising via this construction.

One usually constructs the Schur Functors as certain quotients of $\bigwedge^{\lambda_1}(V) \otimes \ldots \otimes \bigwedge^{\lambda_l}(V)$, where $\lambda^T$ is the transpose of the partition $\lambda$. One would in general like a more explicit description of this module. This is done via the following construction.

Let $\text{End}(V)$ be the endomorphism ring of $V$. One has a natural left $GL(V) \times GL(V)$ action on $\text{End}(V)$ given by left and right multiplication $\rho_{(g,h)}(m) = g \cdot m \cdot h^{-1}$. Note that $\text{End}(V)$ has a natural structure of affine space, $A^2$. Therefore this action lifts to an action on the ring of regular functions, $\mathcal{R}$, and we have the following isomorphism of $GL(V) \times GL(V)$ representations $4$

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$1$For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, we denote by $|\lambda|$ the sum $\sum \lambda_i$ and by $l(\lambda)$, the size of the tuple ($k$ here). The fact that we are dealing with $\text{GL}_n(\mathbb{C})$ enforces that we deal with partitions with $l(\lambda) \leq n$.

$2$In fact we have a functorial isomorphism $\text{Sym}^\lambda(\text{End}(V)) \cong \bigoplus_{l(\lambda) \leq n} S^\lambda(V^*) \otimes_C S^\lambda(V)$.
(1) \[
\mathcal{R} \cong \bigoplus_{l(\lambda) \leq n} S^{\lambda}(V^\vee) \otimes_{\mathbb{C}} S^{\lambda}(V).
\]

Choosing coordinates we identify \(GL(V)\) with \(GL_n(\mathbb{C})\) and \(\text{End}(V)\) with \(\text{Mat}_n(\mathbb{C})\), the \(n \times n\) matrices with complex entries. Identifying \(\text{Mat}_n(\mathbb{C})\) with \(A_n^{2n}\), we see that \(\mathcal{R}\) is the complex polynomial ring in \(n^2\) variables, \(\mathbb{C}[z_{i,j}]\) where \(1 \leq i, j \leq n\). (Although we will still denote the vector space as \(V\) and not \(\mathbb{C}^n\).)

Therefore one has hope of realizing the irreducible representation \(S^{\lambda}(V)\) inside the polynomial ring \(\mathcal{R}\). A first step in achieving this is to look at a ‘refinement’ of the isomorphism in Eq. (1).

Let \(N_-\) is the subgroup of \(GL_n(\mathbb{C})\) consisting of lower triangular matrices with 1’s on the diagonal (the dual unipotent group). We look at the subring of \(\mathcal{R}\) fixed by \(N_- \times id_{GL_n(\mathbb{C})}\). We denote this ring \(\Pi\) (canonically identified with regular functions on the quotient space \((N_- \times id_{GL_n(\mathbb{C})}) \backslash \text{Mat}_n(\mathbb{C}))\). Then one has the following isomorphism:

(2) \[
\Pi \cong \bigoplus_{l(\lambda) \leq n} S^{\lambda}(V)
\]

The ring \(\Pi\) is called the \textbf{Plücker Algebra} and its elements are called \textbf{Plücker monomials}.

We will now establish the standard procedure for constructing the irreducible \(S^{\lambda}(V)\)’s inside \(\Pi\).

1.1. \textbf{Semi Standard Young’s Tableaux.} Let \(M\) be the \(n \times n\) matrix of indeterminates \([z_{ij}]\). The \(\mathbb{C}\) algebra \(\Pi\) is generated by the \(2^n\) possible top justified minors \(M\). These \(2^n\) generators of \(\Pi\), realized as matrix minors of \(M\) are called \textbf{Plücker coordinates}.

We let \([n]\) be the set of all positive integers from 1 to \(n\). We label each such minor as \(\Delta_I\) where \(I = (i_1, \ldots, i_k) \subset [n]\) is a multi index. Therefore if \(|I| = k\), \(\Delta_I\) is the \(k \times k\) top justified minor of the matrix \(M\) with columns labeled by the entries of \(I\).

**Example 1.** The Plücker coordinate \(\Delta_{1,5,7}\) is
\[
det \begin{pmatrix}
    z_{11} & z_{15} & z_{17} \\
    z_{21} & z_{25} & z_{27} \\
    z_{31} & z_{35} & z_{37}
\end{pmatrix}
\]

We identify every partition \(\lambda\) with a Young’s Diagram, with \(\lambda_i\) cells in the \(i^{\text{th}}\) row. We will now fill in the cells of the associated Young’s diagram with entries ranging from 1 to \(n\), where \(n\) is the dimension of \(GL_n(\mathbb{C})\).

**Example 2.** An example of a filling of a Young’s diagram is
\[
\begin{array}{ccc}
1 & 7 & 7 \\
2 & 3
\end{array}
\]

Let \(\mathcal{TAB}(\lambda, [n])\) be the set of all fillings \(F\) of a Young’s diagram associated to the partition \(\lambda\), with entries in \([n]\).

Given a filling \(F\) of a Young’s diagram with distinct entries along the columns we can get a Plücker monomial. We do this by taking the product of \(\lambda_1\) many Plücker coordinates, where the \(i^{\text{th}}\) Plücker coordinate, \(\Delta_I\), has its multi-index labeled by the entries of the \(i^{\text{th}}\) column. It is also clear that if we associate such a minor to a filling \(F’\) with non-distinct entries along a column, then the associated monomial \(\Delta_{F’}\) is zero.

**Example 3.** The Plücker monomial associated to the filling of the previous example is \(\Delta_{1,2}\Delta_{7,3}\Delta_7\).

\(^3\)We will now abuse notation by using \(\lambda\) interchangeably for both the partition and the associated diagram.
Let $\Pi_{T,AB}(\lambda, n)$ be the set of Plücker monomials $\Delta_F$, with $F \in T, AB(\lambda, [n])$. The irreducible representation $S^\lambda(V)$ is realized by identifying it with the vector space spanned by Plücker monomials $\Delta_F \in \Pi_{T,AB}(\lambda, n)$.

(3) $S^\lambda(V) = \text{Span}_C(\Pi_{T,AB}(\lambda, [n]))$

We can produce an explicit set of basis for the representation $V^\lambda$. To do so, we look at a special set of fillings, called the Semi-Standard Young’s Tableaux (SSYT) which satisfy the following conditions

1. The entries strictly increase down each column
2. The entries weakly increase along each row.

We call the set of all SSYT fillings $T$ of a Young’s Diagram $\lambda$, with entries in $[n]$, the set $SSYT(\lambda, [n])$.

Note that $SSYT(\lambda, [n])$ injects into $T, AB(\lambda, [n])$.

One has Plücker monomials $\Delta_T$ for $T \in SSYT(\lambda, [n])$ in the same way as an arbitrary filling. Let $\Pi_{SSYT(\lambda, [n])}$ be the set of all Plücker monomials $\Delta_Y$ labeled by a $Y \in SSYT(\lambda, [n])$. Then we note further that $\Pi_{SSYT(\lambda, [n])}$ injects into $\Pi_{T,AB}(\lambda, [n])$.

Example 4. An example of an SSYT filling of the same Young’s diagram as in Eq.(4)

\[
\begin{array}{ccc}
1 & 2 & 3 \\
7 & 9 & 1
\end{array}
\]

Our Plücker monomial is $\Delta_{1,7} \Delta_{1,9} \Delta_3$

A classical theorem of representation theory shows that $\Pi_{SSYT(\lambda, [n])}$ forms a basis for the irreducible representation $S^\lambda(V)$.

Theorem 1. $\Pi_{SSYT(\lambda, [n])}$ forms a basis for the embedding $S^\lambda(V) \hookrightarrow \Pi$.

1.2. Row Strict Composition Tableaux. In this section we will briefly describe what compositions are, and an overview of the properties that they satisfy.

A composition $\alpha$ of $m \in \mathbb{N}$ is a sequence of positive integers $\{\alpha_i\}_{i \in \mathbb{N}}$ with finite support such that $\sum_i \alpha_i = m$ and whenever $\alpha_i = 0$, then $\alpha_{i+1} = 0$. We also place the restriction that $\alpha_i \leq n$ for all $i$.

We associate to each composition, a composition diagram which is an array with $\alpha_i$ rows in the $i^{th}$ row.

Example 5. A composition of 8 is $(4, 2, 3)$. The diagram is

\[
\begin{array}{ccc}
& & \\
& & \\
& & \\
& & \\
& & \\
\end{array}
\]

We call the largest non-zero index the length of $\alpha$ and denote it as $l(\alpha)$. We call the largest non-zero element of the composition i.e the $\alpha_j$ such that $\alpha_j \geq \alpha_i$ for all $i \in \mathbb{N}$, the width of $\alpha$, and denote it $w(\alpha)$.

To each composition diagram of a given composition $\alpha$, there is a method of filling with entries in $[n]$, called the Row Strict Composition Tableaux (RSCT).

Definition 1. Given a composition diagram $\alpha$ with $l(\alpha) = l$ and $w(\alpha) = w$, we define a row-strict composition tableau (RSCT), $Y$ of shape $\alpha$, to be a filling of the cells of $\alpha$ with positive integers from $[n]$ such that

1. The entries of $Y$ strictly decrease in each row when read from left to right,
2. The entries in the leftmost column of $Y$ weakly increase when read from top to bottom,
3. and $Y$ satisfies the row-strict triple rule.
Here we say that $Y$ satisfies the row-strict triple rule if when we supplement $Y$ by adding enough cells with zero-valued entries to the end of each row so that the resulting supplemented tableau, $\overline{Y}$, is of rectangular shape $w \times l$, then for $1 \leq i < j \leq w$ and $2 \leq k \leq l$, we have

$$\overline{Y}(j, k) > \overline{Y}(i, k) \Rightarrow \overline{Y}(j, k) \geq \overline{Y}(i, k - 1).$$

We call the set of all such fillings is denoted $\mathcal{RSC}(\alpha, [n])$.

**Example 6.** An RSCT filling of the composition $(4, 2, 3)$ is

\[
\begin{array}{ccc}
5 & 3 & 2 \\
7 & 3 \\
8 & 7 & 4
\end{array}
\]

We note that to associate to an element $Y \in \mathcal{RSC}(\alpha, [n])$, an non-zero element of the Plücker Algebra we must fill take products of Plücker Coordinates $\Delta_I$, with $I$ being a multi-index of row entries rather than column entries. This is because a given RSCT may have non-distinct entries along the column, and thus any Plücker Coordinate with that entry would vanish.

Thus we associate to each $Y \in \mathcal{RSC}(\alpha, [n])$ an element $\Delta_Y$, which is a product of the $l(\alpha)$ top justified minors with the columns in the $i^{th}$ minor labeled by the entries of the $i^{th}$ row of $\alpha$.

**Example 7.** The Plücker monomial associated to the filling of the previous example is $\Delta_{1,2,3,5}\Delta_{3,7}\Delta_{4,7,8}$.

We observe that given a partition $\alpha$, there is a unique partition $\lambda$ which one can rearrange $\alpha$ to. We write $P(\alpha) = \lambda$ for the unique $\lambda$ that $\alpha$ can be rearranged into. The operation $P$ descends to an operation on the associated diagrams.

**Example 8.** The rearrangement on the Young’s Diagram associated to $(4, 2, 3)$ is

Now given a partition $\lambda$, we can take its transpose, by interchanging the rows and columns of the associated Young’s Diagram. It is clear that this is a Young’s Diagram, and we denote the associated partition by $\lambda^T$.

**Example 9.** Applying the transpose operation to the Young’s Diagram obtained in Eg.(8) we get

The two operations $P$, which is rearrangement of a composition to a partition, and $T$ which is taking the transpose of give partition, when composed allow us to get a function, $T \circ P$ from the set of all composition of a positive integer $m$ to the set of all partitions of the same integer. We denote the image of $\alpha$ under $T \circ P$ as $P(\alpha)^T$. The operation $T \circ P$ so defined descends to a function from $\mathcal{RSC}(\alpha, [n]) \to \mathcal{TAB}(P(\alpha)^T, [n]).$ Here we give an example
Example 10. We apply the above operation to the filling in Eq. (6) to get

\[
\begin{array}{ccc}
5 & 8 & 7 \\
3 & 7 & 3 \\
2 & 4 & \\
1 & \\
\end{array}
\]

Note that given a partition \( \lambda \), there might be many compositions \( \alpha \) such that \( \text{P}(\alpha)^T = \lambda \). To keep track of this information we define

\[
\mathcal{Z}(\lambda) := \{ \alpha : \text{P}(\alpha)^T = \lambda \}.
\]

Remark 1. Note that given any Plücker monomial \( \Delta_Y \) with \( Y \) an RSCT filling of a composition \( \alpha \in \mathcal{Z}(\lambda) \), for some \( \lambda \), we have that \( \Delta_Y \in \Pi_{T,AB}(\lambda,[n]) \), where the set \( \Pi_{T,AB}(\lambda,[n]) \) was defined immediately after Eq. (3).

1.3. Representation Theory. In this section we introduce the relevant representation theory of \( \text{GL}_n(\mathbb{C}) \) needed in order to motivate our result. A standard reference is [1].

Let \( H \) be the subgroup of all diagonal matrices inside \( \text{GL}_n(\mathbb{C}) \). A vector \( w \) in a \( \text{GL}_n(\mathbb{C}) \) representation \( W \) is called a weight vector with weight \( \beta = (\beta_1, \ldots, \beta_n) \), if for each matrix diag\((t_1, \ldots, t_n)\) \( \in H \), we have

\[
\rho(\text{diag}(t_1, \ldots, t_n)) \cdot w = t^\beta \cdot w,
\]

where \( \rho \) is the homomorphism \( \rho : \text{GL}_n(\mathbb{C}) \to \text{GL}(W) \), and \( t^\beta \) is the monomial \( t_1^{\beta_1} \cdots t_n^{\beta_n} \). Since the morphism \( \rho \) is algebraic, the image of the \( H \) is diagonalizable in \( \text{GL}(W) \).

Thus the action of \( H \) on \( W \) is by simultaneously commuting diagonal matrices, one has a decomposition of \( W \) into its weight spaces

\[
W = \bigoplus_{\beta} W_\beta.
\]

For irreducible representations \( \mathbb{S}^\lambda(V) \), under the identification of Eq. (3), the Plücker Monomials \( \Delta_T \) with \( T \) an SSYT filling are weight vectors with weight \( a = (a_1, \ldots, a_n) \), where \( a_i \) is the number of times \( i \) occurs in \( T \) i.e. the action of \( H \) on these Plucker monomials gives

\[
\rho(\text{diag}(t_1, \ldots, t_n)) \cdot \Delta_T = t^a \cdot \Delta_T,
\]

where the notation \( t^a \) was defined just after Eq. (5).

We also see that because of Theorem [1], these are all the weight vectors of \( \mathbb{S}^\lambda(V) \).

One defines a character \( \chi_W \) of a representation as a function from \( \text{GL}_n(\mathbb{C}) \to \mathbb{C} \), given by the formula

\[
\chi_W(g) = \text{Trace} \circ \rho(g).
\]

It is easy to see the character is in fact a polynomial in \( n \) variables determined by the density of the diagonalizable matrices

\[
\chi_W(t_1, \ldots, t_n) = \text{Trace} \circ \rho(\text{diag}(t_1, \ldots, t_n)).
\]

Changing basis to the weight spaces we see that the character has a specifically useful formula

\[
\chi_W(t_1, \ldots, t_n) = \sum_{\beta} \dim_{\mathbb{C}}(W_\beta)t^\beta,
\]

where the sum is over all weights \( \beta = (\beta_1 \ldots \beta_n) \), and \( W_\beta \) is the corresponding weight space.

The character \( \chi_V \) of a representation identifies it up to an isomorphism. For an irreducible representation \( \mathbb{S}^\lambda(V) \), our observation in Eq. (9) shows us that
\[ \chi_{S^\lambda(V)}(t_1, \ldots, t_n) = \sum_{T \in SSYT(\lambda, [n])} t^{a_T}, \]

where \( a^T = (a_1^T, \ldots, a_n^T) \), and \( a_i^T \) is the number of times \( i \) occurs in the Semi Standard Filling \( T \).

This polynomial is called a **Schur Polynomial**, and can be constructed more formally as we now describe.

We define a weight function \( w : SSYT(\lambda, [n]) \to \mathbb{Z}^{\oplus n} \), where we think of each \( \mathbb{Z} \) being indexed by an element of \([n]\), so that \( w \) sends each \( T \in SSYT(\lambda, [n]) \) to the element \((w_1(T), \ldots, w_n(T)) \in \mathbb{Z}^{\oplus n}\), where \( w_i(T) \) is the number of times \( i \) occurs in \( T \).

We let \( x^T = \prod_i x_i^{w_i(T)} \). We now define the Schur Polynomials as

\[ S_\lambda = \sum_{T \in SSYT(\lambda, [n])} x^T. \]

The **Row-Strict Quasi-Symmetric Schur** Polynomials introduced by Mason and Remmel in [4] are defined analogously as

\[ RS_\alpha = \sum_{Y \in RSCT(\alpha, [n])} x^Y, \]

where in this case we have weight function \( w : RSCT(\alpha, [n]) \to \mathbb{Z}^{\oplus n} \), defined in the obvious manner, with \( x^Y = \prod_i x_i^{w_i(Y)} \).

In [4], Mason and Remmel prove a result which implies that

\[ S_\lambda = \sum_{\alpha \in \mathbb{Z}(\lambda)} RS_\alpha. \]

Now we note that the Schur Polynomials as defined in Eq.(11) is built out of monic monomials, with each monic monomial labeled by a SSYT filling. Now we know that the coefficient of each monomial in the Schur Polynomial \( S_\lambda \) counts the dimension of the corresponding weight space (as evident from Eq.(9)), and thus each monic monomial in the sum in Eq.(11), can be thought of as a basis element.

Now since the RSCT polynomials are built up from monic monomials, with each monic monomial labeled by a RSCT filling, Eq.(13) shows us that there are as many of the RSCT fillings as there are the SSYT fillings, and therefore as many as in any minimal generating set of \( S^\lambda(V) \). Thus we can conclude a bijection as sets

\[ SSYT(\lambda, [n]) \cong \bigsqcup_{\alpha \in \mathbb{Z}(\lambda)} RSCT(\alpha, [n]). \]

Eq.(14), allows us to conclude another set theoretic bijection on the set of Plücker coordinates

\[ \Pi_{SSYT(\lambda, [n])} \cong \bigsqcup_{\alpha \in \mathbb{Z}(\lambda)} \Pi_{RSCT(\alpha, [n])}. \]

Therefore taking into account Eq.(15), Eq.(12) and the discussion in Remark [11] one can ask whether the set

\[ \Sigma_\lambda := \bigsqcup_{\alpha \in \mathbb{Z}(\lambda)} \Pi_{RSCT(\alpha, [n])}, \]

is isomorphic to the Schur Polynomials as defined in Eq.(11).

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\(^{4}\)Not to be confused with the weights of representation theory, although for the SSYT tableaux they turn out to be equivalent.
forms a basis for $S^\lambda(V)$. In this paper we will answer this question affirmatively. Our main result is

**Theorem 2.** $\Sigma_\lambda$ forms a basis for the embedding $S^\lambda(V) \hookrightarrow \Pi$.

The rest of this paper is dedicated to giving a proof of this theorem. In Section 2 we develop some machinery which will help us construct the proof and in Section 3 we give the proof.

**Remark 2.** One can ask whether the same sort of story works out also in the case of the Quasi-Symmetric Schur Functions and the associated composition tableaux, introduced for example Definition 4.1 in [3]. The answer in this context is no since there is no ‘canonical’ way to associate a matrix minor to a composition tableaux. This can be observed by noting that the following example is a composition tableaux.

```
10  6  6
11
25 25
```

We thank Professor Van Willigenberg for asking this question.

2. SOME CONSTRUCTIONS

In this section we will describe some constructions which will be essential in guiding our proof. We first fix some notation

(17) \[ \text{RSCT}[n] := \bigsqcup_{l(\alpha) \leq n} \text{RSCT}(\alpha, [n]) \]

(18) \[ \Pi_{\text{RSCT}} := \{ \Delta_J : J \in \text{RSCT}[n] \} \]

Note that $\text{RSCT}[n] \cong \Pi_{\text{RSCT}}$.

Our proof will proceed by identifying a monomial order on the Plücker Algebra.

**Definition 2.** A monomial order is an order on the monic monomials of a polynomial ring such that whenever $v \leq w$, then $vt \leq wt$, for all monomials $t$ is the polynomial ring.

For our purpose it will suffice to chose a family of ordering called the **Anti-diagonal term order**.

**Definition 3.** An anti-diagonal term order is a monomial order such that, in any minor, the anti-diagonal monomial is the initial term.

**Example 11.** Consider the Plücker Coordinate $\Delta_{1,5,7}$ We write

```
11 5 7
21 25 27
31 35 37
```

The leading term is then $z_{17}z_{25}z_{31}$.

Given a Plücker monomial $\Delta_Y \in \Pi_{\text{TAB}(\lambda,[n])}$ we will denote its initial term by $i(\Delta_Y)$. We now fix some more notation

(19) \[ \mathcal{I}(\Pi_{\text{RSCT}}) := \{ i(\Delta_J) | \Delta_J \in \text{RSCT}[n] \} \]

We will show that under this ordering, each element of $\Pi_{\text{RSCT}}$ has a unique leading term. We will do this by first examining a subset of $\text{Mat}_n(\mathbb{Z})$, the set of integer valued $n \times n$ matrices. We will call this set the **Anti-Diagonal patterns**, and denote it $\mathcal{AD}_n$. We will then construct maps $\Phi$ and $\Psi$ such that the following diagram commutes.
The map $\rho$ is the projection to initial monomials. The map $\Phi$ will be given algorithmically. We will then show that $\Phi(\mathcal{A}D_n) \subset \mathcal{R}SC\mathcal{T}[n]$. We will then show that $\Phi \circ \Psi \circ \rho = \text{id}_{\mathcal{R}SC\mathcal{T}[n]}$, which then allows us to conclude that $\rho$ is injective. This will show that each element of $\Pi_{\mathcal{R}SC\mathcal{T}}$ has a unique leading term.

This is sufficient to prove linear independence. To see this we restrict to the set $\Sigma_{\lambda}$ (see Eq.(16)), we can produce a full rank change of basis matrix by writing down a matrix with columns labeled by monomials in the $z_{ij}$’s, ordered under our monomial ordering, and labeling the rows by $\Delta_J$ as $J$ ranges over $\Sigma_{\lambda}$, with the $\Delta_J$ ordered by their leading term. Because of uniqueness of the leading term, the sub-matrix of columns which are labeled by those monomials which are leading terms is upper-triangular, with 1’s on the diagonal. Therefore we have a full rank matrix, with the image spanned by $\Sigma_{\lambda}$.

Once we have proven linear independence the equality in Eq.(15) guarantees us that the set $\Sigma_{\lambda}$ forms a basis for $S^\lambda(V)$.

We now describe the Anti-Diagonal patterns.

**Definition 4.** A matrix $(\Lambda_{i,j})_{i,j=1}^n \in \text{Mat}_n(\mathbb{Z})$ is called an **Anti-Diagonal Pattern** if-

1. Each non-zero entry is positive,
2. Let $k$ be the largest integer so that $\Lambda_{1,k} \neq 0$, then $\Lambda_{2,k} = 0$,
3. And for each $i \in [n-1]$ and $p \in [n]$, we have $\sum_{j=p+1}^n (\Lambda_{i+1,j}) \leq \sum_{j=p}^n (\Lambda_{i,j})$.

The set of all such matrices $\Lambda$ is denoted $\mathcal{A}D_n$.

**Remark 3.** Note that the set $\mathcal{A}D_n$ is closed under addition.

**Example 12.** We show an element of $\mathcal{A}D_4$

$$
\begin{pmatrix}
5 & 9 & 18 & 7 \\
0 & 13 & 6 & 0 \\
2 & 4 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
$$

**Remark 4.** The usual Gel’fand Tsetlin patterns defined, for example, in Chapter 14 of [2] can be recovered from the $\mathcal{A}D_n$. Given an Anti-Diagonal pattern, $(\Lambda_{i,j})_{i,j=1}^n \in \text{Mat}_n(\mathbb{Z})$, we get a Gel’fand Tsetlin pattern $(\Gamma_{i,j})_{i,j=1}^n$, with coordinates defined as $\Gamma_{i,j} = \sum_{j' \geq j} \Lambda_{i,j'}$.

Conversely a Gel’fand Tsetlin pattern $(\Gamma_{i,j})_{i,j=1}^n$ gives us an Anti-Diagonal pattern $(\Lambda_{i,j})_{i,j=1}^n$ with coordinates defined as $\Lambda_{i,j} = \Gamma_{i,j} - \Gamma_{i,j+1}$.

It is straightforward to check that the maps are mutual inverses.

### 3. Proof of the theorem

#### 3.1. Construction of $\Phi$

In this section we will give an algorithm for constructing the map $\Phi$. This algorithm will provide the essential ‘twist’ which will make our proof work. Before we begin, we will fix some notation.
We wish to show that given any \( \Lambda \in \mathcal{AD}_n \), we obtain a filling \( \Omega \) of a composition diagram \( \alpha \). We will then show that \( \Omega \) satisfy the conditions in Def. (1).

We will have a running example to work with in order to show how our algorithm works

**Example 13.** For our example we will show an element \( \Lambda \in \mathcal{AD}_3 \), where

\[
\Lambda = \begin{pmatrix}
0 & 2 & 4 \\
1 & 3 & 0 \\
2 & 0 & 0
\end{pmatrix}
\]

**Algorithm for \( \Phi \).** We denote by \( \Lambda_{i,j} \) the entries in the \( i^{th} \) row and \( j^{th} \) column of \( \Lambda \). Whenever \( \Lambda_{i,j} \neq 0 \), we will place \( \Lambda_{i,j} \) copies of \( j \) in the \( i^{th} \) column of \( \Omega \).

For \( i = 1 \), the procedure is straightforward. We start with an array of dimension \( l(\alpha) \times 1 \) where \( l(\alpha) = \sum_j \Lambda_{1,j} \) and is the length of the composition \( \alpha \) to which \( \Omega \) will be associated. This array will serve as the first column of \( \Omega \).

Now we go from left to right in the entries of the first row of \( \Lambda \). Every time we reach a non-zero entry \( \Lambda_{1,k} \) we place \( \Lambda_{1,k} \) copies of \( k \) in the next \( \Lambda_{1,k} \) empty (from top) cells of the array.

**Example 14.** *The first column of \( \Omega \) therefore*

\[
\begin{array}{c}
2 \\
2 \\
3 \\
3 \\
3 \\
3
\end{array}
\]

We will now move to the \( i^{th} \) column of \( \Omega \) for \( i > 1 \). We will assume that columns \( 1, \ldots, i - 1 \) have been completed. The idea will be to fill in the \( i^{th} \) column of \( \Omega \) by ensuring that the row entries strictly decrease, while being compatible with the Row Strict Triple Rule.

We will now start from the right when we reach a non-zero entry \( \Lambda_{i,j} \), we will still place \( \Lambda_{i,j} \) copies of \( j \) to the right of the first, from top, entry in the \( (i - 1)^{th} \) column of \( \Omega \) which is strictly greater than \( j \). The condition (2) in the definition of \( \mathcal{AD}_n \), Def. (4) guarantees us that we will always find at least \( \Lambda_{i,j} \) slots to do this.

**Example 15.** *The completed tableaux is*

\[
\begin{array}{ccc}
2 & 1 \\
2 \\
3 & 2 & 1 \\
3 & 2 & 1 \\
3 \\
3
\end{array}
\]

We now show that the map \( \Phi \) so constructed in 3.1 in fact lands in \( \mathcal{RSCT}[n] \).

**Proposition 1.** The co-domain of \( \Phi \) is \( \mathcal{RSCT}[n] \)

**Proof.** We have to check that the \( \Omega \) so obtained is an RSCT. We will pad \( \Omega \) with zeros but still denote it \( \Omega \). We will denote by \( \Omega_{i,j} \) the entry in the \( i^{th} \) row and \( j^{th} \) column of \( \Omega \).
The conditions (1) and (2) of Def.3 are clear by construction of the algorithm. Therefore we only need to check condition (3).

Suppose for some $i, j$ and $k$ satisfying $1 \leq i < j \leq s$ and $2 \leq k \leq m$, we have $\Omega_{j,k} > \Omega_{i,k}$.

Now note that the entries in both $\Omega_{j,k}$ and $\Omega_{i,k}$ are column indices corresponding two different entries in the $k^{th}$ row of $\Lambda$. By hypothesis, the column index of $\Omega_{i,k}$ is less than the column index of $\Omega_{j,k}$, or that the entry corresponding to $\Omega_{j,k}$ occurs to the right of the entry corresponding to $\Omega_{i,k}$.

By the algorithm for $\Phi$, constructed in 3.1, we investigated the entries in the $k^{th}$ row (for $k > 1$) of $\Lambda$ by starting from the right, and placing the column index of the entry next to the first entry (from top) of the $(k-1)^{th}$ column of $\Omega$ which was strictly greater than that column index. Since we placed $\Omega_{j,k}$ in the $j^{th}$ row of the $k^{th}$ column of $\Omega$ instead of the $i^{th}$ row, we must have $\Omega_{i,k} \leq \Omega_{j,k}$. \hfill \Box

### 3.2. The map $\Psi$

In this section we consider the map $\Psi$ which sends each element of $I(\Pi_{RSCT})$, the initial monomials of $\Pi_{RSCT}$, to $Mat_n(\mathbb{Z})$. We wish to show that the image of $\Psi$ lies in $AD_n$.

The map $\Psi$ is the restriction to $I(\Pi_{RSCT})$, of the map which sends the general monomial of form $\prod_{(i,j) \in [n] \times [n]} k_{i,j}$ to the matrix with entries $k_{i,j}$ in the $i^{th}$ row and $j^{th}$ column.

Given a generic $Y \in RSCT[n]$, the initial term of $\Delta_Y$ is a product of Plücker coordinates of the form $\Delta_{J_i}$, where $J_i$ is the index consisting of the entries of the $i^{th}$ row. Since the map $\rho$ respects multiplication, $\rho(\Delta_Y)$ is the product of $\rho(\Delta_{J_i})$. Thus we note that for $Y \in RSCT[n]$, with $\Delta_Y = \prod_i \Delta_{J_i}$, the following identity is satisfied

\begin{equation}
\Psi(\rho(\prod_i \Delta_{J_i})) = \Psi(\prod_i \rho(\Delta_{J_i})) = \sum_i \Psi(\rho(\Delta_{J_i})),
\end{equation}

where the sum on the right hand side is taken over matrices in $Mat_n(\mathbb{Z})$.

Therefore in keeping with this idea we only have to show that the image of the initial term, $\rho(\Delta_I)$ of a single Plücker coordinate $\Delta_I$ is in $AD_n$ since $AD_n$ is closed under addition.

**Proposition 2.** The image of $\Psi$ lies in $AD_n$.

**Proof.** For this we note that the initial term of a Plücker coordinate $\Delta_I$ under the anti-diagonal term order is $\prod_{i=1}^{k} a_i$, where the $a_i$ satisfy $n \geq a_i > a_{i+1}$, and $k \leq n$, where $n$ comes from $GL_n(\mathbb{C})$.

Now the conditions in the Def.3 are satisfied in the case of a single Plücker coordinate since the only non-zero entries in the matrix are 1’s in the $i^{th}$ row and $a_i^{th}$ column. Keeping Eq. (20) and Remark.3 in mind, we are done. \hfill \Box

### 3.3. Proof that $\rho$ is injective

In this section we prove that $\rho$ is injective. As discussed in Sec.3, this will follow immediately if one can prove that $\Phi \circ \Psi \circ \rho = id_{RSCT[n]}$.

Before proving this we will prove a certain 'rigidity' lemma for $RSCT[n]$. This essentially says that specifying the entries of an RSCT filling for each column identifies that filling uniquely.

**Lemma 1.** Let $Y$ and $W$ be two fillings from $RSCT[n]$, padded with zeros as in Def.1 (3) and with entries $Y_{i,j}$ and $W_{i,j}$. Suppose that for each column $i$, the row entries of $W$ are, up to a rearrangement, in bijection with the row entries of $Y$, then $Y = W$.

**Proof.** We give this proof by directly comparing the entries of $Y$ and $W$.

We proceed by induction on the columns. For $i = 1$ the entries have to be weakly increasing as we go down the rows, and there is clearly only one way to do that.

Now suppose we have equality for $i = k$. We will show that equality holds for all entries in column $i = k + 1$.

Suppose not, then there is one $j$ such that $Y_{j,k+1} \neq W_{j,k+1}$, then let us assume without loss of generality that $Y_{j,k+1} > W_{j,k+1}$.
Now the entry which corresponds to $Y_{j,k+1}$ in $W$ must occur somewhere on the $k+1^{th}$ column of $W$. If it occurs below $W_{j,k+1}$, that is in row $j'$ for $j' > j$, then we have $W_{j',k+1} > W_{j,k+1}$, but note that $Y_{j,k+1} = W_{j',k+1} < W_{j,k} = Y_{j,k}$. This contradicts the Row Strict Triple Rule.

So the entry corresponding to $Y_{j,k+1}$ can only occur above the $j^{th}$ row. Now suppose that if it occurs in row $j'$ for $j' < j$ then it occurs next to some $Y_{j',k} = W_{j',k}$. Now the entry corresponding to $Y_{j',k+1}$ in $W$ cannot occur below row $j'$ by recycling the contradiction in the preceding paragraph. Therefore it has to occur above row $j'$.

Proceeding this way for a finite number of steps, we see that at some point we will reach a contradiction wherein the entry in $W$ corresponding to the entry in $Y$ will have to occur below the row of the entry in $Y$.

Thus equality has to hold for column $k + 1$ as well. □

**Proposition 3.** $\Phi \circ \Psi \circ \rho = \text{id}_{\text{RSC}[n]}$

**Proof.** Note that by the definition of $\Psi$, the the entries which show up in the $i^{th}$ row of $\Psi(\rho(\Delta Y))$, which is a matrix, are exponents of indeterminates of forms $z_{i,b_{k,i}}$, here $b_{k,i}$ is the entry in the $i^{th}$ column and $k^{th}$ row. The exponent of $z_{i,b_{k,i}}$ is the number of times $b_{k,i}$ occurs in the $i^{th}$ column.

Now under the construction of $\Phi$ in Section 3.1, the $i^{th}$ row of an element $\Lambda \in AD_n$ determines the $i^{th}$ column of $\Phi(\Lambda)$. Since the entry in any given column of the $i^{th}$ row of $\Psi(\rho(\Delta Y))$ is exactly the number of times the column index occurs in the $i^{th}$ column of $Y$, the RSC we get from our maps, $\Phi(\Psi(\rho(\Delta Y)))$, and $Y$ have exactly the same entries in the $i^{th}$ column upto rearrangement.

Applying Lemma 1, we are done. □

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