Novel Symmetries of Topological Conformal Field theories

J. Sonnenschein and S. Yankielowicz

School of Physics and Astronomy
Beverly and Raymond Sacler
Department of Exact Sciences
Ramat Aviv Tel-Aviv, 69987, Israel

ABSTRACT

We show that various actions of topological conformal theories that were suggested recently are particular cases of a general action. We prove the invariance of these models under transformations generated by nilpotent fermionic generators of arbitrary conformal dimension, $Q^{(n)}$ and $G^{(n)}$. The later are shown to be the $n^{th}$ covariant derivative with respect to “flat abelian gauge field” of the fermionic fields of those models. We derive the bosonic counterparts $W^{(n)}$ and $R^{(n)}$ which together with $Q^{(n)}$ and $G^{(n)}$ form a special $N = 2$ super $W_\infty$ algebra. The algebraic structure is discussed and it is shown that it generalizes the so called “topological algebra”.

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1. **Introduction**

Two dimensional gravity and non-critical string theories provide an interesting and useful arena for the study of topological quantum field theories (TQFT’s)\(^1\). In the opposite direction, the general covariant formulation provides an effective tool to calculate “correlation functions” in the form of powerful recursion relations\(^2\text{–}^6\). Several different starting points for topological conformal field theories (TCFT’s) were proposed. It was realized that “pure gravity”,\(^7\text{,}^8\) flat two dimensional gauge connection\(^9\text{,}^10\), twisted \(N = 2\) superconformal theories\(^11\text{,}^12\), the \(\mathcal{G}\) construction\(^13\) and topological sigma models\(^15\text{,}^7\text{,}^3\) were all examples of TCFT’s.

In this work we elaborate on the equivalence between the various models and suggest a general framework to analyze all of them. We show that all these models are invariant under transformations generated by infinitely many bosonic and fermionic generators of arbitrary integer conformal dimension. The fermionic generators are nilpotent. They have the structure of higher order covariant derivative with respect to a flat gauge connection of the fermionic fields of the models. This symmetry may be referred to as “\(N = 2\) super \(W_\infty\)” symmetry. We discuss the algebraic structure of those symmetries, which generalize the minimal “topological algebra” of ref. [5] and present several useful OPEs’. Infinite towers of bosonic symmetries as well as supersymmetries were discussed in the past.\(^16\) Both the fermionic and bosonic symmetry generators discussed here are different though possibly not unrelated to those in the literature.

The paper is organized as follows. In section 2 we describe the various formulations of conformal topological quantum field theories. We start with Einstein’s action of pure gravity, continue with the twisted \(N = 2\) models with zero and non-zero background charge, theories of flat gauge connections, the \(\mathcal{G}\) construction and finely the general BRST invariant( \(\hbar, 1 - \hbar\)) systems. The equivalence of several of these formulations is discussed in section 3. Section 4 is devoted to the symmetries of the TCFT’s generated by a set of infinitely many bosonic \(W_\infty\) as well as fermionic \(Q_\infty\) and \(G_\infty\) generators. The transformations of the various fields are
written down. It is shown that the $n^{th}$ generator in each sector can be expressed in terms of an $n^{th}$ covariant derivative with respect to a “flat gauge connection”. The structure of this infinite topological algebra is investigated in section 5. In section 6 we summarize the results and comment briefly on the possible implications. Some technical details are presented in the appendices. In Appendix A we prove the invariance under $Q^{(n)}$, construct the bosonic operator $R^{(n)}$ and exhibit the corresponding transformation properties of the various fields. In Appendix B the calculation of the anomaly term in the OPE $Q^{(n)}G^{(m)}$ is explained. Appendix C presents the OPE of $W^{(1)}$ with the rest of the operators.

2. Actions for Topological Conformal Field Theories

The most obvious TQFT in two dimensions is pure Einstein’s gravity (with no cosmological constant). In refs. [7, 8] it was realized that this action has in addition to the usual scale and reparametrization invariance a local symmetry referred to as the “topological symmetry”. The BRST quantized action of this theory was found out to be [7, 8]:

$$S_{TC} = \int d^2z [(b \bar{\partial} c + \beta \bar{\partial} \gamma + ce) + \pi (\bar{\partial} \bar{\partial} \varphi - \hat{R}^{(2)}) + \tilde{\psi} \bar{\partial} \bar{\partial} \psi]$$ (1)

where $(b, c)$ are the spin $(2, -1)$ reparametrization ghosts, $(\beta, \gamma)$ are commuting ghosts with the same spins, and $(\pi, \varphi)$, $(\tilde{\psi}, \psi)$ are Grassmann even and odd scalars. This action was derived via two stages of gauge fixing. In the first stage the gauge condition $R^{(2)} = \hat{R}^{(2)}$ was imposed to fix the “topological symmetry”. In the second stage the ordinary reparametrization was fixed $ds^2 = e^{2\varphi} dz \bar{d}z$ as well as a ghost symmetry. A different derivation of this action was given in ref. [14] where it was shown to correspond to a $C = -2$ matter theory coupled to a Liouville mode.

An alternative formulation of pure gravity was written down in terms of flat $SL(2, R)$ gauge connections [9]. This construction was a special case of topological flat gauge connection (TFC) [10] associated with the group $G$. The action of this
system is described in the “semi-classical limit” which is exact by

\[ S_{FC} = \int d^2z Tr[(A\bar{\partial}B + \Psi \bar{\partial}\Psi + c.c) + \partial \phi \bar{\partial}\phi + \partial c \bar{\partial}c] \]  

(2)

where all the fields are in the adjoint representation of the group. \( A \) is the gauge connection, \( \Psi \) is a spin 1 ghost field related to the gauge fixing of the topological symmetry, \((\bar{c}, c)\) are the usual ghost and anti-ghost of the non-Abelian symmetry and \( \tilde{\phi}, \phi \) are Grassmann even “ghost for ghosts”. A detailed discussion of this model can be found in refs. \([9, 10]\). This scenario was invoked also to describe world-sheet supergravity using the graded Lie group \( OSP(2|1) \) \([17]\). In that case the gauge connection was decomposed as follows: \( A = e^a J_a + \omega J_3 + \chi^\pm J_\pm \) were \( J_a \) \((a = 1, 2)\), \( J_3 \) and \( J_\pm \) are the generators of \( OSP(2|1) \) and \( e^a, \omega \) and \( \chi^\pm \) where interpreted as the zweibein, spin connection and the gravitino respectively. Similar construction for other interesting groups like \( SL(3, R) \) were worked out in ref. \([12]\).

Yet another formulation of topological theories was found by twisting the \( N = 2 \) superconformal field theory\([11,12]\). In the case of the minimal \( N = 2 \) models the action takes the form of

\[ S_{N=2} = \int d^2z \partial \phi \bar{\partial}\phi + i\alpha \sqrt{g} R^{(2)} \phi + (\lambda \bar{\partial} \lambda + c.c) \]  

(3)

where \( \phi, \bar{\phi} \) are commuting scalars, \( \lambda, \bar{\lambda} \) are \((1, 0)\) anti-commuting ghosts, \( R^{(2)} \) is the world sheet curvature of the background metric and \( \alpha \) is a parameter of the theory.

A different description of a TQFT which links it to some group \( G \) was found by extrapolating the \( \frac{G}{H} \) construction to the case of \( H = G \)\([13]\). It is well known that \( \frac{G}{H} \) coset models can be described in terms of a WZW model based on a group \( G \) where (an anomaly free) subgroup \( H \) is gauged\([19]\). The gauging amounts essentially to setting the \( H \)-currents to zero. Hence for the case \( H = G \) only the \( G \)-zero modes survive. In this case the system is equivalent to three decoupled systems i.e. \( G \)-WZW model at level \( k \), \( G \)-WZW at negative level \(-(k + c_G)\) and a free \((1,0)-(b,c)\)
system in the adjoint representation. Upon bosonization\textsuperscript{[20]} (assuming $G = SU(N)$) one can recast the action into a sum of the following terms.\textsuperscript{[21]} One term has the form of eqn.(3) where $\phi$ and $\tilde{\phi}$ are associated with the “hypercharge” currents of the two $SU(N)$-WZW models. (The scalars combine nicely into one complex boson). The ghost system in this term is the $(1,0)(b,c)$ system associated with the hypercharge direction. The rest of the terms are $N^2 - 2$ free$(1,0)-(b,c) + (\beta, \gamma)$ systems each per one of the extra generators of $SU(N)$. Clearly one can use the bosonization formulas to recast this form into other equivalent forms. This structure can be modified by introducing appropriate background charges which do not change the value of the total central charge $c_{tot} = 0$.

Older members of the family are the topological sigma models (TSM)\textsuperscript{[15,7,3,5]}. These models describe a special sector of the maps from two dimensional world sheet into some target manifolds. If the target manifold is taken to be flat then the expression of the corresponding action is given by:

$$S_{TSM} = \int d^2z \eta_{\mu\nu} [\partial X^\mu \bar{\partial} X^\nu + (\psi^\mu \bar{\partial} \tilde{\psi}^\nu + cc)],$$

(4)

$X^\mu$ is the target space coordinate, $\psi^\mu, \tilde{\psi}^\nu$ are world-sheet $(1,0)$ system. Just as in the case of TFC, one can deduce this quantum action via a BRST gauge fixing of a topological symmetry.

The later models as well as the previous ones exhibit an important relation to non-critical string theories and their matrix models counterparts once they are coupled to the topological two dimensional gravity\textsuperscript{[3,5]} of eqn. (1). For instance when a TSM is coupled to TG to produce a “topological string” model\textsuperscript{[3,5]} the corresponding action is the sum of the actions given in eqn. (1) and (4).

An interesting question is to what extent are ordinary critical and non-critical string theories, both bosonic and supersymmetric, a special case of TCFTs?\textsuperscript{[5]} Apart from a comment in the last section, we do not consider here string theories as TCFTs’.
3. A “unified” picture

Two questions are now in order: (i) can one “unify” the actions of the models presented in the last section and (ii) are there other topological models. A straightforward observation is that for $R^{(2)} = 0$ all the actions described in the previous section are special cases of the following general action:

$$S = \int d^2 z \sum_i \left[ \Phi^{(h_i)} \bar{\partial} \Phi^{(1-h_i)} + \Psi^{(h_i)} \bar{\partial} \Psi^{(1-h_i)} + cc \right]$$  \hspace{1cm} (5)

where $\Phi^{(h_i)}$, $\Psi^{(h_i)}$ are commuting and anti-commuting fields of dimension $(h_i)$ and similarly for the dimension $(1-h_i)$ fields $\bar{\Phi}^{(1-h_i)}$ and $\bar{\Psi}^{(1-h_i)}$. For the terms involving a pair of scalars (commuting or anti-commuting) the passage to the form of the above equation involves a simple redefinition which amounts to rewrite them is a first order form. For instance for eqn. (3) we rewrite $\partial \phi \bar{\partial} \bar{\phi} = W \bar{\partial} \bar{\phi}$ with $W = \partial \phi$. Since systems which are the same apart from their Grassmannian nature, have conformal anomaly which differ by a sign, it is obvious that the total conformal anomaly vanishes $c = \sum_i (c_i - c_i) = 0$. One can reformulate (5) as an exact form under fermionic operators $Q$ and $G$ of dimensions zero and one respectively.

$$S = \int d^2 z \sum_i \left[ Q(\Psi^{(h_i)} \bar{\partial} \Phi^{(1-h_i)}) + cc \right] = \int d^2 z \sum_i \left[ G(\bar{\Psi}^{(1-h_i)} \bar{\partial} \bar{\Phi}^{(h_i-1)}) + cc \right]$$  \hspace{1cm} (6)

where $\Phi^{(h_i)} = \partial \bar{\Phi}^{(h_i-1)}$. The $Q$ and $G$ transformations of the various fields are given by

$$\begin{align*}
\delta^Q \Psi^{(h_i)} &= \epsilon \Phi^{(h_i)} \\
\delta^Q \bar{\Phi}^{(1-h_i)} &= -\epsilon \bar{\Psi}^{(1-h_i)} \\
\delta^G \bar{\Psi}^{(1-h_i)} &= \epsilon \partial \bar{\Phi}^{(1-h_i)} \\
\delta^G \Phi^{(h_i)} &= -\epsilon \partial \bar{\Psi}^{(h_i)}
\end{align*}$$  \hspace{1cm} (7)

The fact that the action is exact under a zero dimension fermionic symmetry hints of the possibility to interpret the action as a BRST gauge fixed action. This interpretation follows the original TQFTs’ namely, that the “classical” action is zero.
and the “quantum action” is derived by gauge fixing of a “topological symmetry”. One can take $\mathcal{L}_{\text{classical}}(\Psi^{(h_i)}) = 0$ which is invariant under the “topological symmetry” $\delta \Psi^{(h_i)} = \epsilon^{(h_i)}(z, \bar{z})$ or $\mathcal{L}_{\text{classical}}(\tilde{\Phi}^{(1-h_i)}) = 0$ which is invariant under the “topological symmetry” $\delta \tilde{\Phi}^{(1-h_i)} = \epsilon^{(1-h_i)}(z, \bar{z})$. Replacing the parameters of transformations with ghost fields $\Phi^{(h_i)}$ for the first formulation and $\tilde{\Psi}^{(1-h_i)}$ for the second, imposing holomorphicity of the original fields as the gauge condition $\bar{\partial} \Psi^{(h_i)} = 0$ or $\bar{\partial} \tilde{\Phi}^{(h_i)} = 0$, and using the “BRST” transformations of eqn. (7) one gets the action (5). Notice that this prescription is different from the BRST gauge fixing that was applied for the cases of pure gravity\cite{TFT9}, TFC\cite{TFT9} and TSM\cite{TFT9}. It is thus apparent that various different starting points for TCFT lead to the same theory. This point will be further discussed in the last section.

So far we considered only the case of $R^{(2)} = 0$. For the twisted $N = 2$ action it is equivalent to taking $\alpha = 0$ which is the semiclassical limit of this action since $k = (\frac{1}{\alpha^2} - 2) \to \infty$. As we show in the next section, for pure imaginary $\alpha$, namely, negative $k$, one can generalize the construction by redefining $\tilde{\phi} \to \hat{\phi} = \tilde{\phi} - i\alpha \phi$.

4. The symmetries

By definition, all the “physical observables” of a TQFT are invariant under an arbitrary variation of the metric of the underlying manifold. (The notion of physical observables refers to correlation functions of products of operators which are scalars and gauge invariant with respect to any local symmetry in the system.) This implies that the energy momentum tensor can be expressed as an exact operator under a nilpotent fermionic symmetry

$$T_{\alpha\beta} = \{Q, G_{\alpha\beta}\}. \quad (8)$$

It is straightforward to check that eqn. (8) guarantees the metric independence\cite{TFT9}. Moreover, it is easy to see that in fact the TQFT actions given in the previous section are all exact under the fermionic symmetry. This is obviously the situation for the TG, TFC and TSM models since the quantum action by construction is BRST exact as well as for any other model following eqn. (6).
A topological conformal field theory (TCFT) is characterized by the fact that the trace of the classical energy momentum vanishes. All the TQFT models presented in the previous section share this property. In these cases $T_{\alpha\beta}$ as well as $G_{\alpha\beta}$, the BRST current $Q_{\alpha}$ and the ghost number current $J_{\alpha}$ can be split into their holomorphic and antiholomorphic parts. Hence one gets the following relations which reflect the BRST multiplet structure

$$T(z) = \{Q,G(z)\} \quad Q(z) = -[Q,J(z)]$$

By Laurent expansion of these operators one finds using Jacobi identities the TCFT algebra. This algebra together with its generalization will be presented in the next section.

Next we analyze the symmetries of the TCFT. Let us first discuss the symmetries generated by $J$, $T$, $Q$, and $G$. To simplify the notation we choose to demonstrate all the features in the twisted $N = 2$ model eqn. (3) with $\alpha = 0$ or $R^{(2)} = 0$. Later we explain how to generalize it to the case of non-flat world sheet and to the other models included in the general form of the action given in eqn. (5).

The following transformations of the fields leave the action invariant.

$$
\begin{align*}
\delta^J \lambda &= -\epsilon \lambda & \delta^J \tilde{\lambda} &= \epsilon \tilde{\lambda} & \delta^J \phi &= -\epsilon & \delta^J \tilde{\phi} &= \epsilon \\
\delta^Q \lambda &= \epsilon \partial \phi & \delta^Q \tilde{\phi} &= -\epsilon \tilde{\lambda} \\
\delta^G \tilde{\lambda} &= -\epsilon \partial \tilde{\phi} & \delta^G \phi &= \epsilon \lambda \\
\delta^T \lambda &= (\partial \epsilon \lambda + \epsilon \partial \lambda) & \delta^T \tilde{\lambda} &= \epsilon \partial \tilde{\lambda} & \delta^T \phi &= \epsilon \partial \phi & \delta^T \tilde{\phi} &= \epsilon \partial \tilde{\phi}
\end{align*}
$$

The parameters of transformation $\epsilon$ are holomorphic function $\epsilon = \epsilon(z)$. For the $J$ and $Q$ transformations $\epsilon$ has dimension zero, for $T$ and $G$ dimension one and for $Q$ and $G$ they are Grassmanian variables. Obviously the action is invariant also under similar transformations generated by the anti-holomorphic counterparts, $\tilde{J}, \tilde{T}, \tilde{Q}, \tilde{G}$. From here on we discuss only the holomorphic transformations. Notice
that unlike usual BRST transformations where the parameter of transformation $\epsilon$ is a global parameter, here $\epsilon = \epsilon(z)$ also for the fermionic symmetries generated by $Q$ and $G$. Hence they generate an infinite dimensional algebra. Using the OPE of the basic fields

$$\phi(z)\tilde{\phi}(\omega) = -\log(z - \omega) \quad \lambda(z)\tilde{\lambda}(\omega) = \frac{1}{z - \omega}$$  \hfill (11)

it is straightforward to extract the currents that generate the above transformations:

$$J = -\left(\lambda\tilde{\lambda} + a\partial\phi - \tilde{a}\partial\tilde{\phi}\right) \quad T = -\left(\partial\tilde{\phi}\partial\phi + \lambda\partial\tilde{\lambda} + a\partial^2\phi\right)$$

$$Q = \tilde{\lambda}\partial\phi + \tilde{a}\partial\tilde{\lambda} \quad G = -\left(\lambda\partial\tilde{\phi} + a\partial\lambda\right)$$  \hfill (12)

Note that the terms proportional to $a$ and $\tilde{a}$ are total derivatives so they do not contribute to the corresponding charges and therefore cannot be determined from the classical transformations alone. Even for parameters of transformations which are not global but rather are holomorphic functions, in which case the total derivative terms do contribute to the transformations, they cannot be determined. Hence one can generally multiply each of them with an arbitrary parameter. However, imposing the relations of eqn. (9) reduces the number of parameters from five to two $a$ and $\tilde{a}$ as stated in eqn. (12). These parameters will play a role in the corresponding algebra as will be discussed in the next section. In fact there are some additional relations among the symmetry generators

$$\tilde{T}(z) = \{G, Q(z)\} \quad G(z) = -[G, \tilde{J}(z)]$$  \hfill (13)

which are all summarized in the following diagram:

$$\tilde{T}, T \quad Q \quad \tilde{J}, G$$  \hfill (14)

where $A - B \rightarrow C$ denotes acting with a charge $B$ on a current $A(z)$ to generate a current $C(z)$. The currents $\tilde{T}$, $T$ and $\tilde{J}$, $J$ correspond to the same symmetry transformation and are related to one another by $\phi \leftrightarrow -\tilde{\phi}$ and $\lambda \leftrightarrow \tilde{\lambda}$. 
We wish now to address the question of whether the transformations of eqn. (10) exhaust the symmetries of the TCFT models. In what follows we consider only compact Riemann surfaces so the invariance of the action will be checked always up to total derivatives. The answer to this question is definitely no. The arsenal of symmetries is much richer. There are in fact three types of symmetry transformations: (i) bosonic or fermionic transformations which involve only the commuting or the anticommuting parts of the action like \( \delta J \) for \( a = \tilde{a} = 0 \) (ii) bosonic symmetries acting on both sectors like \( \delta T \) and (iii) fermionic symmetries mixing the two sectors like \( \delta Q \) and \( \delta G \). Before dwelling into the second and third types let us write the most general invariance of each of the sectors separately. Let us look for instance on the bosonic sector. This part of the action is invariant under the separate transformation of \( \phi \) and \( \tilde{\phi} \) as follows:

\[
\delta \phi = \epsilon \partial W \tilde{F}(\tilde{W}) \quad \delta \tilde{\phi} = \epsilon \partial \tilde{W} F(W)
\]

(15)

where \( W = \partial \phi, F(W) \) is a general function of \( W \), \( \partial W F(W) \) is its derivative with respect to \( W \) and similarly for the fields with \( \tilde{\phi} \). In particular any polynomials of \( W \) and \( \tilde{W} \) for \( F \) and \( \tilde{F} \) will do the job. Symmetries which leave the fermionic sector invariant are for example those which are generated by \( \lambda \) (or \( \tilde{\lambda} \)) \( \delta \lambda = \epsilon \) (\( \delta \tilde{\lambda} = \epsilon \)).

Next we want to check whether there are generalizations of the fermionic symmetries generated by \( Q \) and \( G \). One finds that the following currents generate such symmetries.

\[
Q^{(n)} = D^n \tilde{\lambda} \quad G^{(n)} = \tilde{D}^n \lambda
\]

(16)

where the covariant derivatives are \( D = \partial + W = \partial + \partial \phi \) and \( \tilde{D} = \partial - \tilde{W} = \partial - \partial \tilde{\phi} \) and \( D^n \) is the \( n^{th} \) power of \( D \). The fermionic generators \( Q \) and \( G \) are the special of \( n = 1 \), \( Q = Q^{(1)} \) and \( G = G^{(1)} \) with \( \tilde{a} = -a = 1 \). From the corresponding OPEs’ one gets that \( \tilde{a} = -a \). One can generalize the covariant derivative given above to incorporate \( a \neq 1 \) as in eqn. (12) and still maintain the structure of eqn. (16) by
the following redefinitions $\lambda \to \lambda' = \tilde{\alpha} \lambda$ $W \to W' = \frac{1}{\tilde{\alpha}} W$ and the same for $\tilde{\lambda}$ and $\tilde{W}$. Alternatively one can only redefine $\lambda$ and $\tilde{\lambda}$ and take $\frac{1}{\tilde{\alpha}}$ as the charge in the covariant derivative. One can view this covariant derivative as if its source is an abelian gauge field which is taken to be a pure gauge, namely, zero field strength or flat gauge connection. We denote the set of infinitely many symmetry generators $Q^{(n)}$ as $Q_{\infty}$ and $G^{(n)}$ as $G_{\infty}$. It is obvious from eqn. (16) that the $G_{\infty}$ and the corresponding transformation laws are related to those of $Q_{\infty}$ by the replacement $\phi \to -\tilde{\phi}$ and $\tilde{\lambda} \to \lambda$. We thus describe here only the $Q_{\infty}$ symmetries. Under the later only $\lambda$ and $\tilde{\phi}$ transform as follows:

$$
\delta Q^{(n)}(\lambda) = D_{-}^{(n)} \epsilon \quad \delta Q^{(n)}(\tilde{\phi}) = -\sum_{i=0}^{n-1} D_{-}^{i} \epsilon D^{(n-1-i)} \tilde{\lambda}
$$

where $D_{-} = -\partial + W$ such that $(DA)B - A(DB) = \partial(AB)$. The parameter of transformation $\epsilon$ has conformal dimensions $-(n-1)$. To be specific here are the global transformations generated by the first three lowest generators (omitting the parameter of transformations)

$$
\delta Q^{(1)}(\lambda) = W \quad \delta Q^{(1)}(\tilde{\phi}) = -\tilde{\lambda}
$$

$$
\delta Q^{(2)}(\lambda) = W^2 - \partial W \quad \delta Q^{(2)}(\tilde{\phi}) = -(\partial \tilde{\lambda} + 2W \tilde{\lambda})
$$

$$
\delta Q^{(3)}(\lambda) = W^3 - 3W \partial W + \partial^2 W \quad \delta Q^{(3)}(\tilde{\phi}) = -(\partial^2 \tilde{\lambda} + 3W \partial \tilde{\lambda} + 3W^2 \tilde{\lambda}).
$$

It is straightforward to check that these transformations leave the action invariant. In appendix A we show that the general transformations eqn. (16) are indeed symmetry transformations.

So far have we discussed the fermionic symmetries, now to complete the generalization of the relation given by eqn. (14) we define a double set of infinite bosonic operators as follows

$$
W^{(n)}(\omega) = \frac{1}{2\pi i} \oint dz Q(z) G^{(n-1)}(\omega) \quad \tilde{W}^{(n)}(\omega) = \frac{1}{2\pi i} \oint dz G(z) Q^{(n-1)}(\omega)
$$

(19)
Using the OPE’s of eqn. (11) we find for the $W_\infty$

$$W^{(n)} = \tilde{D}^{(n-1)} \tilde{W}^{(1)} + G^{(n-1)} \tilde{\lambda}$$

(20)

where $\tilde{W}^{(1)} = W - \lambda \tilde{\lambda}$ and $\tilde{D}W = (\partial - \tilde{W})W$, $\tilde{D}\tilde{\lambda} = \partial \tilde{\lambda}$. For some applications it is convenient to express $W^{(n)}$ as follows:

$$W^{(n)} = \tilde{D}^{(n-2)} W^{(2)} - \sum_{k=0}^{n-2} (n-1)^k G^{(k)} \partial^{n-1-k} \tilde{\lambda}$$

(21)

where $G^{(0)} = \lambda$. For $\tilde{W}^{(n)}$ we interchange $W$ with $-\tilde{W}$ and $\lambda$ with $\tilde{\lambda}$. The expressions for the lowest $W^{(n)}$ are

$$W^{(1)} = W + \tilde{W} - \lambda \tilde{\lambda}$$

$$W^{(2)} = \partial W - \tilde{W} \tilde{W} - \lambda \partial \tilde{\lambda}$$

(22)

$$W^{(3)} = \partial^2 W - \partial (\tilde{W} \tilde{W}) - \tilde{W} \partial W + \tilde{W}^2 W - \lambda \partial^2 \tilde{\lambda} - 2 \partial \lambda \partial \tilde{\lambda} + 2 \tilde{W} \lambda \partial \tilde{\lambda}$$

$W^{(1)}$ is not determined by eqn. (19). Its form is dictated by the algebra of the generators. Again it is easy to check that $W^{(1)} = -J$ and $W^{(2)} = T$ with $\tilde{a} = -a = 1$. The invariance under the $W^{(n)}$ transformations follows from that of $G^{(n)}$:

$$\delta^{W^{(n)}} S = [\epsilon W^{(n)}, S] = [\epsilon \{Q, G^{(n)}\}, S] = \{Q, [G^{(n)}, S]\} + \{\epsilon G^{(n)}[Q, S]\} = 0$$

(23)

since $[Q, S] = [G^{(n)} S] = 0$. In the same manner $Q^{(n)}$ invariance implies that of $\tilde{W}^{(n)}$. For completeness we write now the transformations under the $W_\infty$ symmetry of the various fields

$$\delta^{W^{(n)}} \lambda = (\tilde{D}^{(n-1)} \epsilon) \lambda - \epsilon \tilde{D}^{(n-1)} \lambda$$

$$\delta^{W^{(n)}} \tilde{\lambda} = \tilde{D}^{(n-1)} (\epsilon \tilde{\lambda}) - (\tilde{D}^{(n-1)} \epsilon) \lambda$$

$$\delta^{W^{(n)}} \tilde{\phi} = -\tilde{D}^{(n-1)} \epsilon$$

$$\delta^{W^{(n)}} \phi = -\sum_{i=0}^{n-2} \tilde{D}^{i} (\epsilon D^{(n-2-i)} \tilde{W}^{(1)} - \tilde{D}^{i} (\epsilon \tilde{\lambda}) \tilde{D}^{(n-2-i)} \lambda$$

(24)

where $\tilde{D} - \epsilon = -(\partial + \tilde{W}) \epsilon$. How can we generalize the diagram of eqn. (14)? One is tempted to think that there is the same structure also for the $n^{th}$ level.
By definition $W^{(n)}$ and $\tilde{W}^{(n)}$ are created by applying $Q$ and $G$ on $G^{(n)}$ and $Q^{(n)}$ respectively, however $Q^{(n)}$ and $G^{(n)}$ are not derivable from $W^{(n-1)}$ and $\tilde{W}^{(n-1)}$ by acting with $Q$ and $G$. One has to modify $W^{(n)}$ and $\tilde{W}^{(n)}$ in the following way to generate from them $Q^{(n)}$ and $G^{(n)}$. First note that

$$\frac{1}{2\pi i} \oint_{\omega} dz Q(z) \tilde{W}^{(n+1)}(\omega) = \left[ \{Q, G\}, Q^{(n)} \right] + \left[ \{Q, Q^{(n)}\}, G \right] = [W^{(2)}, Q^{(n)}] = \partial Q^{(n)}. \quad (25)$$

Now it is easy to see that if one adds $\lambda Q^{(n)}$ to $\tilde{W}^{(n+1)}$ one gets (see Appendix A):

$$\frac{1}{2\pi i} \oint_{\omega} dz Q(z) [\tilde{W}^{(n+1)} + \lambda Q^{(n)}](\omega) = \frac{1}{2\pi i} \oint_{\omega} dz Q(z) \tilde{R}^{(n+1)}(\omega) = \partial Q^{(n)} + WQ^{(n)} = Q^{(n+1)}. \quad (26)$$

and similarly for $G^{(n+1)}$. The explicit expressions for $R^{(n)}$ and $\tilde{R}^{(n)}$ are

$$R^{(n)} = D^{(n-1)} \hat{W}^{(1)} \quad \tilde{R}^{(n)} = D^{(n-1)} \hat{\tilde{W}}^{(1)}. \quad (27)$$

The diagram of eqn.(14) takes now the split form

$$\tilde{W}^{(n)}, W^{(n)}$$

$$Q^{(n-1)} \quad \quad G^{(n-1)} \quad \quad Q^{(n-1)} \quad \quad G^{(n-1)}$$

$$\tilde{R}^{(n-1)}, R^{(n-1)}$$

Are the $R^{(n)}$ and $\tilde{R}^{(n)}$ generators of symmetries? It is easy to see that $\delta^{R^{(n)}} S$ is closed under $Q$.

$$[Q^{(n)}, S] = [[Q, R^{(n)}], S] = [Q, [R^{(n)}, S]] - [R^{(n)}, [Q, S]] = 0. \quad (29)$$

It turns out, as we show in Appendix A, that $\delta^{R^{(n)}} S = \int d^2 \bar{\epsilon} (\epsilon D^n \bar{\lambda}) = 0$. The transformations of the various fields under $R^{(n)}$ are also written down in the appendix.
The next task is to show that all these fermionic and bosonic symmetries are in fact shared by all the models of the previous section. To prove this we first treat the general case of eqn. (9) and then we consider the case of eqn.(3) for $R^{(2)} \neq 0$ and $\alpha \neq 0$. The action (9) is clearly a sum of decoupled actions, (as long as it is not coupled to TG) so we can separate the symmetry generators for each separate part $Q_i^{(n)}$, $G_i^{(n)}$ and $W_i^{(n)}$. To construct the generators in a form similar to eqns. (16) and (21) we need dimension one fields as connections in the covariant derivatives. For this purpose one can “bosonize” the bosonic system in eqn. (9) in the following way

$$\int d^2z \sum_i [\Phi^{(h_i)}(\bar{\Phi}^{(1-h_i)})] =$$

$$\frac{1}{2} \int d^2z \sum_i [\partial \rho_i \bar{\partial} \rho_i - \frac{1}{4} Q_i \sqrt{g} R^{(2)} \rho_i + 2 \eta_i \bar{\partial} \xi_i]$$

where $\Phi^{(h_i)}(z) = e^{\rho_i(z)} \partial \xi_i$ and $\bar{\Phi}^{(1-h_i)} = e^{-\rho_i(z)} \eta_i$ and $Q_i = -(1-2h_i)$. Setting now $R^{(2)} = 0$ we take $\partial \rho_i$ as the connection of the following covariant derivatives $D_i = \partial + \partial \rho_i$ and $\bar{D}_i = \partial - \partial \rho_i$. The expression for the symmetry generators are thus

$$Q_i^{(n)} = D_i^{n} \bar{\Psi}^{(1-h_i)}$$
$$G_i^{(n)} = \bar{D}_i^{n} \Psi^{(h_i)}$$
$$W_i^{(n)} = \bar{D}_i^{(n-1)} \bar{W}_i^{(1)} + G_i^{(n-1)} \bar{\Psi}^{(1-h_i)}$$

$$= \bar{D}_i^{(n-2)} W_i^{(2)} - \sum_{k=1}^{n-1} \binom{n-1}{k} G_i^{(k)} \partial^{n-1-k} \bar{\Psi}^{(1-h_i)}$$

where now $\bar{W}_i^{(1)} = \partial \rho_i + \Psi^{(h_i)} \bar{\Psi}^{(h_i-1)}$ and $W_i^{(2)} = -[(\partial \rho_i)^2 - \partial^2 \rho_i + \Psi^{(h_i)} \partial \bar{\Psi}^{(h_i-1)}]$ Notice that unlike the discussion above here there is only one scalar for each system ($\rho_i$) rather than two ($\phi$, $\bar{\phi}$). Nonetheless, the transformations of the fields $\Psi^{(h_i)}$, $\bar{\Psi}^{(1-h_i)}$ and $\rho_i$, which are given by eqn. (17) and (24) with some obvious renaming, leave the action of eqn. (30) invariant due to the factor half in front of $\partial \rho_i \bar{\partial} \rho_i$.

We want to consider now the case of a non-flat world-sheet. Once we turn on the curvature, the parameters $\alpha$ in the model of (9) as well as $Q$ of the above
discussion play an important role. In the case of the twisted \( N = 2 \) theory the level \( k = \left( \frac{1}{\alpha^2} - 2 \right) \) determines the dimension of the (moduli) space on which all “physical” correlators are cohomologies.\(^{[12,5]}\) Do we loose the symmetry structure generated by the infinitely many generators of eqn. (31)? It turns out that those invariances persist also in the \( R^{(2)} \neq 0 \) case. To realize this phenomena we use again the conformal metric \( ds^2 = e^{\phi} dzd\bar{z} \). In this picture the action of eqn. (3) takes the form

\[
S_{(N=2)} = \int d^2z \partial \phi \bar{\partial} \phi + i\alpha \partial \bar{\partial} \phi + (\lambda \partial \bar{\lambda} + c.c) \tag{32}
\]

This redefinition make sense if \( \alpha \) is pure imaginary namely for negative \( k \). The later are natural if the starting point of the twisted \( N = 2 \) is \( SL(2,R) \) WZW model rather than an \( SU(2) \) model. Following this redefinition the form of all the generators remains the same apart form the fact that \( \hat{\phi} \) is replacing \( \tilde{\phi} \).

Two remarks are in order: (i) Since a fixed world sheet metric was introduced it is clear that the ghost sectors of the pure gravity theory have to be invoked and hence the whole action of eqn. (1) has to be added. The other remark refers to the form of \( T \). Building it from \( G \) we get now

\[
\oint \omega dz Q(z)G(\omega) = T(\omega) = \partial \phi (\partial \phi + i\alpha \partial \phi) + \lambda \partial \bar{\lambda} - \partial^2 \phi \tag{33}
\]

This may look an unfamiliar expression but in fact this is exactly what has to be achieved for this metric.\(^{[12]}\)
5. The algebraic structure

An algebra for the TCFT’s was written down in ref. [5]. This algebra can be deduced from the OPEs’ of the various pairs of operators made out of \( J, T, Q \) and \( G \), using Jacobi identities. The OPEs’ follow from the “topological condition” given in eqn. (8). The algebra is characterized by the three anomalous terms in the Kac-Moody algebra of \( J \), in \([L,J]\) and in \(\{G,Q\} \), which are all determined by one parameter denoted in ref. [5] as \( d = d_{JJ} = d_{QG} = -d_{TJ} \).

Next we want to analyze the algebraic structure of the set \( W^{(n)}(n), Q^{(n)}(n) \) and \( G^{(n)}(n) \). We first wish to confirm that the OPEs’ which led to the algebra of ref. [5] are those of the symmetry generators for \( n = 1 \) and \( W^{(2)} \). Using the definitions of eqns. (16) and (21) and the basic OPEs’ (11) it is straightforward to check that the resulting OPE’s:

\[
\begin{align*}
W^{(1)}(z)Q^{(1)}(\omega) &= \frac{Q^{(1)}(\omega)}{z-\omega} & W^{(1)}(z)G^{(1)}(\omega) &= -\frac{G^{(1)}(\omega)}{z-\omega} \\
W^{(1)}(z)W^{(1)}(\omega) &= -\frac{1}{(z-\omega)^2} & W^{(1)}(z)W^{(2)}(\omega) &= \frac{-1}{(z-\omega)^3} - \frac{W^{(1)}(\omega)}{(z-\omega)^2} \\
W^{(2)}(z)Q^{(1)}(\omega) &= \frac{Q^{(1)}(\omega)}{z-\omega} + \frac{\partial Q^{(1)}(\omega)}{z-\omega} & W^{(2)}(z)G^{(1)}(\omega) &= \frac{2G^{(1)}(\omega)}{(z-\omega)^2} + \frac{\partial G^{(1)}(\omega)}{z-\omega} \\
W^{(2)}(z)W^{(2)}(\omega) &= \frac{2W^{(2)}(\omega)}{(z-\omega)^2} + \frac{\partial W^{(2)}(\omega)}{z-\omega} \\
Q^{(1)}(z)G^{(1)}(\omega) &= \frac{-1}{(z-\omega)^3} + \frac{W^{(1)}(\omega)}{(z-\omega)^2} + \frac{W^{(2)}(\omega)}{(z-\omega)}
\end{align*}
\]

is identical to those of ref. [5]. Since when acting on \( Q^{(n)}(n) \) with \( W^{(1)} \) it is in fact only \( \hat{W}^{(1)} \) which operates, one can use the later as the “ghost number current” when acting on \( Q^{(n)} \). Similarly one can use \( \hat{W}^{(1)} \) when applied on \( G^{(n)} \). We now return to the more general form of the symmetry generators namely those with \( \hat{a} = -a \neq 1 \) given in eqn. (12) and in the discussion following eqn. (16). It is straightforward to check that for this case one derives the same OPEs’ apart from the fact that now \( d = d_{JJ} = d_{QG} = -d_{TJ} = 2a\hat{a} + 1 \) For the parametrization of
ref. [12] one thus gets $d = \frac{k}{k+2}$. Switching on $R^{(2)}$ introduces, as was explained in section 3, the redefinition of $\tilde{W} \to \tilde{W} + i\alpha \partial \phi$. It is easy to check that the OPEs’ of eqn. (34) stay in tack under the this modification.

We proceed now to the operators beyond the “minimal topological algebra” [5].

First we examine the OPE of $W^{(1)}$ and $W^{(2)}$ with the rest of the operators. In Appendix C it is shown in the context of model (3) that

$$W^{(1)}(z)Q^{(n)}(\omega) = \frac{Q^{(n)}(\omega)}{z-\omega} \quad W^{(1)}(z)G^{(n)}(\omega) = \frac{-G^{(n)}(\omega)}{z-\omega} \quad (35)$$

$$\frac{1}{2\pi i} \oint_{\omega} dz W^{(1)}(z)W^{(n)}(\omega) = 0 \quad (36)$$

It is thus clear that $W^{(1)}$ plays the role of the ghost number current and that the $Q^{(n)}$ and $G^{(n)}$ have ghost number 1, -1 respectively. It is shown in Appendix C that the term proportional to $\frac{1}{(z-\omega)}$ in $W^{(1)}(z)W^{(n)}(\omega)$ vanishes which leads to eqn. (36). Hence, as expected from the its definition, $W^{(n)}$ has a zero ghost number. Similarly it is not surprising to notice that $W^{(2)}$ is the energy momentum tensor and $Q^{(n)}$, $G^{(n)}$ and $W^{(n)}$ all carry dimension equal to $n$ and $n+1$ respectively.

$$W^{(2)}(z)Q^{(n)}(\omega) = \ldots \frac{n(n-1)Q^{(n-1)}(\omega)}{(z-\omega)^3} + \frac{nQ^{(n)}(\omega)}{(z-\omega)^2} + \frac{\partial Q^{(n)}(\omega)}{(z-\omega)} \quad (37)$$

and similarly for $W^{(2)}G^{(n)}$.

The next question of interest is whether the OPEs’ and the corresponding commutation relations are linear or whether products of generators and their derivatives show up in them. It turns out that the algebra is not linear. We demonstrate it now in the following two examples:

$$W^{(1)}(z)W^{(3)}(\omega) = \frac{-2}{(z-\omega)^4} + \frac{2W^{(1)}(\omega)}{(z-\omega)^3} - \frac{[2W^{(2)} + :\tilde{R}^{(1)}\tilde{R}^{(1)} + \partial \tilde{R}^{(1)}](\omega)}{(z-\omega)^2} \quad (38)$$

where the $:\ldots:\$ denotes normal ordering as explained in Appendix C. A similar structure show up in
\[ Q^{(2)}(z)G^{(1)}(\omega) = \frac{-4}{(z-\omega)^4} - \frac{2W^{(1)}(\omega)}{(z-\omega)^3} - \frac{[2W^{(2)} + :\bar{\tilde{R}}^{(1)}\tilde{R}^{(1)} : + \partial \bar{\tilde{R}}^{(1)}(\omega)]}{(z-\omega)^2} + \frac{\Delta^{(3)}(\omega)}{(z-\omega)} \]

Where \( \Delta^{(3)} = -W^2\bar{W} + \partial(W^2) + \partial W\bar{W} + \partial^2\bar{\lambda}\lambda + 2(\partial W)\lambda\lambda + 2W\partial\lambda\lambda \).

Another obvious property of \( Q^{(n)} \) and \( G^{(n)} \) is nilpotency. This is a special case of the anticommuting relations

\[ \{Q^{(n)}Q^{(m)}\} = 0 \quad \{G^{(n)}, G^{(m)}\} = 0. \] (40)

The derivation of \( Q^{(n)}(z)G^{(m)}(\omega) \) is straightforward though tedious. In Appendix B we present the calculation of the anomalous term.

6. Summary and Discussion

Since the original path-integral approach to TQFT’s it was known that a basic property of all the TQFTs’ is the fact that all the non-zero modes are canceled out from the “physical observables”. This characteristic feature should manifest itself in terms of a large set of symmetry constraints on physical states. In this note we have investigated the symmetry structure of topological theories. We showed that the TCFTs’ are in fact invariant under transformations generated by nilpotent pairs of fermionic operators of arbitrary conformal dimension. An interesting feature of these generators is that they are in fact the \( n^{th} \) covariant derivative on the basic fermions of the theory. The covariant derivative is with respect to a “flat abelian gauge connection”. We showed that the generic model can be derived as a BRST gauge fixed action of a theory with a “topological symmetry” in which holomorphicity condition was imposed. It is thus plausible that the later construction and the infinite tower of symmetries are related. In this case, it is not hard to envision, that all the TCFTs’ models that we considered are described by cohomologies on moduli spaces of flat connections and their generalization to
higher spin fields. The bosonic counterparts of the fermionic symmetries $W^{(n)}$ and $R^{(n)}$ where also expressed as covariant derivatives. The complete algebraic structure was not extracted in the present work. Therefore it is not clear to what extent the algebra of the bosonic generators is related to various $W_\infty$ which were discussed in the literature.\cite{16}. The implications of this very rich algebraic structure on the Hilbert space of physical states is under investigation. We believe that it is this algebraic structure which is responsible for the decoupling of all the non-zero modes from the physical observables.

We did not discuss in this work the application of the “minimal topological algebra” to string theories. It was realized\cite{5,22} that the set of $J, Q, G$ and $T$ do not close the algebra and one has to introduce additional symmetry generators. It was also found out that the non-critical string theory of $c = 1$ share a “higher symmetry”\cite{23}. The role of the new symmetries presented in this work in the realm of string theories is under current investigation.

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APPENDIX A

We want to show now that the $Q_\infty, G_\infty$ and $W_\infty$ transformations leave the TCFT models invariant. Again we present the explicit proof for the $\alpha = 0$ case of eqn. (12) and later we explain how to extend the proof to the rest of the cases. Obviously only $\lambda$ and $\tilde{\phi}$ transform by $Q^{(n)}$. Recall eqn. (17)

$$
\delta^{Q^{(n)}} \lambda = D^n \epsilon \quad \delta^{Q^{(n)}} \tilde{\phi} = - \sum_{i=0}^{n-1} D^i \epsilon D^{(n-1-i)} \tilde{\lambda}.
$$

(A.1)

The action thus transforms into

$$
\delta S_{N=2} = \int d^2 z [\bar{\partial} W \sum_{i=0}^{n-1} D^i \bar{\lambda} D^{(n-1-i)} \epsilon + D^n \epsilon \bar{\partial} \bar{\lambda}].
$$

(A.2)

Now this last expression is in fact a total derivative of the form $\bar{\partial}(D^n \epsilon \bar{\lambda})$. In order to prove that we have to show that

$$
\bar{\partial} W \sum_{i=0}^{n-1} D^i \lambda D^{(n-1-i)} \epsilon = \bar{\partial}(D^n \epsilon \bar{\lambda})
$$

(A.3)

Let us expand the term on the right of the last equation:

$$
\bar{\partial}(D^n \epsilon \bar{\lambda}) = \bar{\partial}(-\partial + W)(D^{n-1} \epsilon \bar{\lambda} = \bar{\partial} W(D^{n-1} \epsilon \bar{\lambda} + \bar{\partial}(D^{n-1} \epsilon) D \bar{\lambda})
$$

(A.4)

The term on the right is the first term in the sum of eqn. (A.1). Further iteration of expanding the term to the right generates exactly all the terms in the sum of eqn. (A.4).

The invariance of the action under $G^{(n)}$ follows from an identical proof with the obvious replacements $\lambda \rightarrow \bar{\lambda}, W \leftrightarrow -\bar{W}$.

Now the generalization to the rest of the TCFT models is very straightforward. For the $R^{(2)} \neq 0$ case one can again pass to the modified field $\hat{\phi}$. As for the general case of eqn. (13). The same reasoning of above leads to the conclusion that the variation of the action under for example $Q_i^{(n)}$ is $\delta S = \int d^2 z \bar{\partial}(D_i^n \epsilon \bar{\Psi}^{1-h_i})$. 

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Next we want to explain the relations of diagram (14). We use again the example of (12) for $R^{(2)} = 0$. Let us show first that

$$Q^{(n)} = \oint_{\omega} Q(z) \tilde{W}^{(n)}(\omega) = \oint W \tilde{\lambda}(z)[-D^{n-1} \tilde{W} - D^{n-1}(\tilde{\lambda} \lambda) + D^{n-1} \tilde{\lambda}] \lambda] \quad (A.5)$$

The first term in the integral gives $D^{n-1}(\partial \tilde{\lambda})$. Plugging the OPE of $\lambda \tilde{\lambda}$ into the other two terms, recalling that $D \lambda = \partial \lambda$ one gets for the second term $D^{n-1}(W \tilde{\lambda})$ so that altogether we get for the first two terms $D^{n-1}(\partial \tilde{\lambda} + W \tilde{\lambda}) = D^n \tilde{\lambda} = Q^{(n)}$. It is thus clear that omitting the last term $Q^{(n-1)} \lambda$ produces $R^{(n)}$. Under the interchange of $\lambda$ with $\tilde{\lambda}$ and $W$ with $-\tilde{W}$ we find in a complete analogy the same results for $G^{(n)}$.

Our next task to examine whether $R^{(n)}$ generate symmetry transformations, namely, we want to check if $\delta R^{(n)} S = \frac{1}{2\pi i} \oint_{\omega} dz [\epsilon R^{(n)}](z) S = 0$. Since we know that $\tilde{R}^{(n+1)} = \tilde{W}^{(n+1)} + \lambda Q^{(n)}$ and since we know that $\tilde{W}^{(n)}$ are symmetry generators it is enough to show that $\lambda Q^{(n)}$ leaves the action invariant. It is straightforward to realize that the later holds. The transformation of the various fields are found to be

$$\delta R^{(n)} \lambda = -(\tilde{D}^{(n-1)} \epsilon) \lambda \quad \delta R^{(n)} \tilde{\lambda} = \tilde{D}^{(n-1)}(\epsilon) \tilde{\lambda}$$

$$\delta R^{(n)} \tilde{\phi} = -\tilde{D}^{(n-1)} \epsilon \quad \delta R^{(n)} \phi = -\sum_{i=0}^{n-2} \tilde{D}^i \epsilon D^{(n-2-i)} \tilde{W}^{(1)} - \tilde{D}^i(\epsilon) \tilde{D}^{(n-2-i)}(\lambda \tilde{\lambda}) \quad (A.6)$$

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We compute the anomaly term in the OPE of \(Q^m G^n\). The notion of anomaly refers here to the term proportional to \(\frac{1}{(z-\omega)^{m+n+1}}\) which obviously is a number. One gets this term by performing a complete contraction of all the fields. For \(n > m\) the general form of a term in the expansion which can contribute to the anomalous term has the form

\[
[W^i \partial^j W \partial^{m-(i+j)} \tilde{\lambda}] (z)[\tilde{W}^i \partial^j \tilde{W} \partial^{n-(i+l)} \lambda](\omega)
\]  

(B.1)

where \(i = 0, \ldots, m - 1, j = 1, \ldots, m - (i + 1)\) and \(l = 1, \ldots, n - (i + 1)\). In addition one can have the case with no derivatives on \(W\) and \(\tilde{W}\). The contribution of a term of the form of (B.1) is found by performing all possible contractions between the fields. One gets

\[
(-1)^m [m + n - (2i + j + l)]!i!(j + 1)!(l + 1)! + (l + j + 1)!
\]  

(B.2)

The contribution of the terms with no \(W\) and \(\tilde{W}\) derivatives are \((-1)^m [m+n-2i]i!\).

What is left over to do is to figure out the multiplicity factors \(B_i\) and \(D_{ijl}\) of each of the terms and then perform the summation, namely:

\[
Anom = (-1)^m \sum_{i=0}^{m} B_i [(m+n) - 2i]i! \\
(-1)^m \sum_{i=0}^{m} \sum_{j=1}^{m-i} D_{ijl} [(m+n) - 2(i+1) - (j+l)]!i!(j + 1)!(l + 1)! + (l + j + 1)!
\]  

(B.3)

It is easy to check that \(B_i = \binom{m}{i}\binom{n}{i}\) and similarly one can get an expression for \(D_{ijl}\)
APPENDIX C

As in the previous sections we work here in the context of the flat world sheet of eqn. (3). Thus following eqn. (21) \( W^{(1)} = (W + \bar{W} - \lambda \bar{\lambda}) \). When acting on \( Q^{(n)} = D^n \bar{\lambda} \) obviously only the second and the third terms in \( W^{(1)} \) can contribute. Let us first look on the residue, namely, the \( \frac{1}{(z-\omega)} \) terms. Since following eqn. (11) the OPE \( \tilde{W}(z)W(\omega) = \frac{1}{(z-\omega)} \) and when \( \tilde{W} \) is applied on \( Q^{(n)}(\omega) \) there are no terms at \( z \) to expand, the only contributions can come from \( \lambda \bar{\lambda}(z)Q^{(n)}(\omega) \). Denoting a generic term in \( Q^{(n)}(\omega) \) as \( C_k F_k(W, \partial W) \partial^n - k \bar{\lambda} \) where \( C_k \) is some numerical coefficient and \( F_k(W, \partial W) \) is a dim \( k \) polynomial of \( W \) and derivatives of \( W \), than

\[
\lambda \bar{\lambda}(z)C_k F_k(W, \partial W) \partial^n - k \bar{\lambda}(z) = -C_k F_k(W, \partial W) \partial^n - k \bar{\lambda}(z) + \frac{[n - k]!}{(z-\omega)^{n+1-k}} \frac{1}{(z-\omega)} C_k F_k(W, \partial W) \partial^n - k \bar{\lambda}(z)
\]

(C.1)

It is thus clear that the residue is really \(-Q^{(n)}(\omega)\).

We want to show now that all the terms multiplying \( \frac{1}{(z-\omega)^l} \) for \( l > 1 \) vanish. Terms proportional to \( \frac{1}{(z-\omega)^2} \) are generated by contraction between the \( \lambda \bar{\lambda} \) and \( Q^{(n)}(\omega) \) and between \( \tilde{W} \) and powers of \( W \) in \( f_k \). Rewriting the later as \( C_k f_k = C_{i,k} W^i g_{k-i} \) we get a contribution of \(-\sum_i C_{i,k} W^i(n-k) \partial^n - k \bar{\lambda} \) where as the \( \tilde{W} \) contractions lead \( \sum_i C_{i,k} i W^{i-1}(n-k) \partial^n - k \bar{\lambda} \). Now since \((i+1)C_{i+1,k} = (n-k)C_{i,k} \) for \( k \neq i + 1 \) and \((i + 1)C_{i+1,k} = (n-i)C_{i,k} \) for \( k = i + 1 \) the two contributions cancel each other.

Next we compute the terms multiplying \( \frac{1}{(z-\omega)\!j} \) for \( j = 1, 2, 3 \). Following the same steps as for \( W^{(1)} \) one can realize that from contraction the \( \lambda \partial \bar{\lambda}(z) \) term one get \(-C_k F_k \frac{[n-k]!}{(z-\omega)^{n+1-k}} \partial^{l+1} \bar{\lambda} \). For \( l = n - k \) one can exactly the action of the derivative on \( \bar{\lambda} \) in \( \partial Q^{(n)} \). When \( \tilde{W}W \) is contracted with powers of \( W \) in \( f_k \) one gets the action of the derivative of this part and same applies for \( \partial^j W \) factors in \( f_k \). So altogether applying the chain rule one gets the term \( \frac{\partial Q^{(n)}}{(z-\omega)^j} \). Repeating the analysis now for \( j = 2, 3 \) one derives the eqn. (37).
We present here the explicit calculation of $W^{(1)}W^{(3)}$. The terms multiplying the various powers of $\frac{1}{(z-\omega)}$ are

$$\frac{1}{(z-\omega)^3} : 4\bar{W} + 2W - 2\lambda\bar{\lambda} - 2\bar{W} = 2W^{(1)}$$

$$\frac{1}{(z-\omega)^2} : 2\partial\bar{W} - \lambda\partial\bar{\lambda} + \partial\bar{W} - 2W\bar{W} - \bar{W}^2 + 2\bar{\lambda}\partial\lambda - 2\lambda\bar{W}\bar{\lambda} \quad (C.2)$$

$$- [W^{(2)} + : R^{(1)} \bar{R}^{(1)} : + \partial \bar{R}^{(1)} ]$$

where the normal order product $: R^{(1)} \bar{R}^{(1)} :$ is given by $W^2 + 2W\lambda\bar{\lambda} + \partial\lambda\bar{\lambda} + \partial\bar{\lambda}\lambda$. 

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