Kähler Geometry and the Navier-Stokes Equations

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Abstract

We study the Navier-Stokes and Euler equations of incompressible hydrodynamics in two and three spatial dimensions and show how the constraint of incompressibility leads to equations of Monge–Ampère type for the stream function, when the Laplacian of the pressure is known. In two dimensions a Kähler geometry is described, which is associated with the Monge–Ampère problem. This Kähler structure is then generalised to ‘two-and-a-half dimensional’ flows, of which Burgers’ vortex is one example. In three dimensions, we show how a generalized Calabi–Yau structure emerges in a special case.

1 Equations for an Incompressible Fluid

Flow visualization methods, allied to large-scale computations of the three-dimensional incompressible Navier-Stokes equations, vividly illustrate the fact that vorticity has a tendency to accumulate on ‘thin sets’ whose morphology is characterized by quasi one-dimensional tubes or filaments and quasi two-dimensional sheets. This description is in itself approximate as these thin structures undergo dramatic morphological changes in time and space. The topology is highly complicated; sheets tend to roll-up into tube-like structures, while tubes tangle and knot like spaghetti boiling in a pan (Vincent & Meneguzzi 1994). Moreover, vortex tubes usually have short lifetimes, vanishing at one place and reforming at another. The behaviour of Navier-Stokes flows diverge in behaviour from Euler flows once viscosity has taken effect in reconnection processes. Nevertheless, the creation and early/intermediate evolution of their vortical sets appear to be similar.

No adequate mathematical theory has been forthcoming explaining why thin sets tend to be favoured. The purpose of this paper is to investigate this enduring question in the light of the recent advances made in in the geometry of Kähler and other complex manifolds. While many difficult questions remain to be solved and explored, we believe that sufficient evidence exists that suggests that three-dimensional turbulent vortical dynamics may be governed by geometric principles.

The incompressible Navier-Stokes equations, in two or three dimensions, are

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla P = \nu \nabla^2 \mathbf{u}, \quad (1)
\]

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \quad (2)
\]

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Here, \( u(x,t) \) is the fluid velocity, the pressure and density of the fluid are denoted by \( P(x,t) \) and \( \rho(x,t) \) respectively, \( \nabla \) is the gradient operator and \( \nu \) is the viscosity; in the inviscid case when \( \nu = 0 \) we have the Euler equations. The constraint imposed by the incompressibility condition
\[
\nabla \cdot \mathbf{u} = 0, \tag{3}
\]
is very severe. It means that the convective derivative of the density vanishes. In turn this means that an initially homogeneous (constant density) fluid remains constant for all time; \( \rho(x,0) = \rho(x,t) = \text{constant} \). Hereafter this density is taken as unity. Moreover, when (3) is applied across (1) it demands that velocity derivatives and the pressure are related by a Poisson equation
\[
-\nabla^2 P = u_{i,j} u_{j,i}, \tag{4}
\]
where \( \nabla^2 \) is the Laplace operator. Solving (4) in tandem with (1) is computationally demanding and is one of the major limiting factors in how far numerical calculations can be driven.

A considerable literature exists on the dynamics of vortex tubes, particularly on the topic of the Burgers’ vortex (Burgers 1948). In an influential paper that contains substantial references, Moffatt, et al. (1994) coined the simile Burgers’ vortices are the sinews of turbulence and thus identified the heart of the problem; that is, these filament-like vortices stitch together the large-scale anatomy of vortical dynamics. Despite the twisting, bending and tangling they undergo, they appear to be the preferred states of Navier-Stokes turbulent flows. In fact Burgers’ vortices are examples of a two-and-a-half-dimensional flow, which can be defined by the class of velocity fields written as (Gibbon et al. (1999))
\[
\mathbf{u}(x,y,z,t) = \{ u_1(x,y,t), u_2(x,y,t), z\gamma(x,y,t) \}. \tag{5}
\]
This flow is linear in \( z \) in the \( \hat{k} \)-direction; thus it is stretching (or compressing) in that direction but is linked dynamically to its cross-sectional part. The nomenclature refers to the fact that it is neither fully two- nor three-dimensional but lies somewhere in-between\(^1\)

Its components must also satisfy the divergence-free condition
\[
u_{1,x}(x,y,t) + u_{2,y}(x,y,t) + \gamma(x,y,t) = 0. \tag{6}\]
The class of velocity fields in equation (5), first used in Ohkitani & Gibbon (2000), is a more general classification of Burgers-type solutions and contains the specific form of the Burgers vortex solutions used in Moffat et al. (1994). Included in (5) are the Euler solutions of Stuart (1987, 1991), in which \( u_1 \) and \( u_2 \) are also linear in \( x \) (say) leaving the dependent variables to be functions of \( y \) and \( t \). Then stretching can occur in two directions thereby producing sheet-like vortical solutions.

The differences between the three-dimensional and two-dimensional Navier-Stokes equations are fundamental because the vortex stretching term \( \omega \cdot \nabla \mathbf{u} \) in the equation for vorticity is present in the former but absent in the latter. Nevertheless, Lundgren (1982) has shown that for two-and-a-half-dimensional flows of the type
\[
\begin{align*}
u_{1} &= -\frac{x}{2}\gamma(t) + \psi_y, & \nu_{2} &= -\frac{y}{2}\gamma(t) - \psi_x, & \nu_{3} &= z\gamma(t) \tag{7}
\end{align*}
\]
\(^1\)In the case of the three-dimensional Euler equations data can become rough very quickly; our manipulations in this paper are therefore purely formal. In fact it has been shown numerically in Ohkitani & Gibbon (2000) and analytically in Constantin (2000) that solutions of the type in (5) can become singular in a finite time, which is consistent with observations that vortex tubes have finite life-times; the singularity is not real in the full three-dimensional Euler sense as it has infinite energy but indicates that the flow will not sustain the structure (5) for more than a finite time. For the possibility of a real Euler singularity see Kerr (1983) and Kerr (2005).
can be mapped into solutions of the two-dimensional Navier-Stokes equations with $\psi(x, y, t)$ as a stream function.

To investigate the geometric structure behind these solutions requires certain technical tools; these are outlined in §2 of this paper. The constraint in equation (4) is the basis of our geometric arguments, and because it is true for both the Navier-Stokes and Euler equations, the conclusions reached in this paper are valid for both cases. It is, of course, to be expected that any geometric structure should be independent of viscosity. From now on when we refer to the Navier-Stokes equations it should be implicitly understood that the Euler equations are also included. The Kähler structure for the two-dimensional Navier-Stokes equations is described in §3 and then formulated for two-and-a-half-dimensional Navier-Stokes flows in §4. Our results show that the necessary condition on the pressure for a Kähler structure to exist in two spatial dimensions (with time entering only as a parameter) for the two-dimensional Navier-Stokes equations is $\nabla^2 P > 0$. This constraint is highly restrictive: by no means all two-dimensional Navier-Stokes flows would conform to it. More promising is the equivalent condition for two-and-a-half dimensional solutions of type (7). Theorem 1 in §4 shows that these two-and-a-half-dimensional solutions have an underlying Kähler structure if $\nabla^2 P$ has a very large negative lower bound, thus associating a wide set of ‘thin’ solutions with the Kähler property. While the existence of a negative finite lower bound suggests some work still needs to be done, this result implies that preferred vortical thin sets have a connection with a Kähler geometric structure that deserves further study.

The solutions considered in this paper represent the ideal cases of straight tubes or flat sheets; in reality, as indicated in the first paragraph of this section, these vortical objects constantly undergo processes of bending and tangling. Speculatively, it is possible that once this process is underway, solutions move from living on a Kähler manifold to other complex manifolds of a higher dimension, although this is a much more difficult mathematical problem to address. In §5 and Appendix A following Banos (2002), we describe an explicit connection between Monge–Ampère operators and complex manifolds of Calabi–Yau type which are associated with a restricted class of three-dimensional Navier-Stokes flows. This explicit connection exists only for a very special class of flows that require the pressure to satisfy $\nabla^2 P < 0$. What this limited class of flows physically represent is unclear at the present time.

The work of Roubtsov & Roulstone (1997, 2001) showed how quaternionic and hyper-Kähler structures emerge in models of nearly geostrophic flows in atmosphere and ocean dynamics. These results were based on earlier work by McIntyre and Roulstone and was reviewed by them in McIntyre & Roulstone (2002). It has also been shown that the three-dimensional Euler equations has a quaternionic structure in the dependent variables (Gibbon 2002). The use of different sets of dependent and independent variables in geophysical models of cyclones and fronts, has facilitated some remarkable simplifications of otherwise hopelessly difficult nonlinear problems (see, for example, Hoskin & Bretherton 1972). Roulstone & Sewell (1997) and McIntyre & Roulstone (2002), describe how contact and Kähler geometries provide a framework for understanding the basis of the various coordinate transformations that have proven so useful in this context. This present work has evolved from revisiting Gibbon (2002) in the light of Roubtsov & Roulstone (2001).

## 2 Differential Forms and Monge–Ampère Equations

In this section we prepare some tools that enable us to study certain partial differential equations arising in incompressible Navier-Stokes flows from the point-of-view of differential geometry. An introduction to the application of some basic elements of exterior...
calculus to the study of partial differential equations, with application to fluid dynamics, can be found in McIntyre & Roulstone (2002). Here, we shall draw largely on Lychagin et al. (1993) and Banos (2002).

A Monge–Ampère equation (MAE) is a second order partial differential equation, which, for instance in two variables, can be written as follows:

\[ A\phi_{xx} + 2B\phi_{xy} + C\phi_{yy} + D(\phi_{xx}\phi_{yy} - \phi_{xy}^2) + E = 0, \]

where \( A, B, C \) and \( D \) are smooth functions of \((x, y, \phi, \phi_x, \phi_y)\). This equation is elliptic if

\[ AC - 4B^2 - DE > 0. \]

In dimension \( n \), a Monge–Ampère equation is a linear combination of the minors of the hessian matrix \( \phi \). We shall refer to such equations as symplectic MAEs when the coefficients \( A, B, C \) and \( D \) are smooth functions of \((x, y, \phi_x, \phi_y) \in J^1\mathbb{R}^2/J^0\mathbb{R}^2\), where \( J^1\mathbb{R}^2 \) denotes the manifold of 1-jets on \( \mathbb{R}^2 \).

### 2.1 Monge–Ampère operators

Lychagin (1979) has proposed a geometric approach to these equations, using differential forms on the cotangent space (i.e. the phase space). The idea is to associate with a form \( \omega \in \Lambda^n(T^*\mathbb{R}^n) \), where \( \Lambda^n \) denotes the space of differential \( n \)-forms on \( T^*\mathbb{R}^n \), the Monge–Ampère equation \( \Delta \omega = 0 \), where \( \Delta : C^\infty(\mathbb{R}^n) \to \Omega^n(\mathbb{R}^n) \cong C^\infty(\mathbb{R}^n) \) is the differential operator, \( \Delta \), defined by

\[ \Delta(\phi) = (d\phi)^*\omega. \]

We denote by \( (d\phi)^*\omega \) the restriction of \( \omega \) to the graph of \( \phi \) (\( d\phi : \mathbb{R}^n \to T^*\mathbb{R}^n \) is the differential of \( \phi \)). A form \( \omega \in \Lambda^n(T^*\mathbb{R}^n) \) is said to be effective if \( \omega \wedge \Omega = 0 \), where \( \Omega \) is the canonical symplectic form on \( T^*\mathbb{R}^n \). Then the so called Hodge-Lepage-Lychagin theorem tells us that this correspondence between MAEs and effective forms is one to one. For instance, the Monge–Ampère equation \( (8) \) is associated with the effective form

\[ \omega = Adp \wedge dy + B(dx \wedge dp - dy \wedge dq) + Cdx \wedge dq + Ddp \wedge dq + Edx \wedge dy, \]

where \((x, y, p, q)\) is the symplectic system of coordinates of \( T^*\mathbb{R}^2 \), and on the graph of \( d\phi \), \( p = \phi_x \) and \( q = \phi_y \). So, for example, if we pull-back the one-form \( dp \) to the base space, we have \( dp = \phi_x dx + \phi_y dy \), and then \( dp \wedge dq = \text{hess}(\phi)dx \wedge dy \), where we have also used the skew symmetry of the wedge product.

### 2.2 Monge–Ampère structures

The geometry of MAEs in \( n \) variables can be described by a pair \((\Omega, \omega) \in \Lambda^2(T^*\mathbb{R}^n) \times \Lambda^n(T^*\mathbb{R}^n) \) such that

1. \( \Omega \) is symplectic; that is, nondegenerate \((\Omega \wedge \Omega \neq 0) \) and closed \((d\Omega = 0) \)

2. \( \omega \) is effective; that is, \( \omega \wedge \Omega = 0 \).

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2 We denote by \( \text{hess}(\phi) \) the determinant of the hessian matrix of \( \phi \). For example, in two variables, \( \text{hess}(\phi) = \phi_{xx}\phi_{yy} - \phi_{xy}^2 \).

3 The use of the Greek letters \( \omega \) and \( \Omega \) is common in differential geometry; these symbols should not be confused with the fluid vorticity vector \( \omega \).
Such a pair is called a Monge–Ampère structure. In four dimensions (that is \( n = 2 \)), this geometry can be either complex or real and this distinction coincides with the usual distinction between elliptic and hyperbolic, respectively, for differential equations in two variables. Indeed, when \( \omega \in \Lambda^2(T^*\mathbb{R}^2) \) is a non-degenerate 2-form \( (\omega \wedge \omega \neq 0) \), one can associate with the Monge–Ampère structure \((\Omega, \omega) \in \Lambda^2(T^*\mathbb{R}^2) \times \Lambda^2(T^*\mathbb{R}^2)\) the tensor \( I_\omega \) defined by

\[
\frac{1}{\sqrt{|\text{pf}(\omega)|}} I_\omega = \Omega(I_\cdot, \cdot)
\]

where \( \text{pf}(\omega) \) is the pfaffian of \( \omega \): \( \omega \wedge \omega = \text{pf}(\omega)(\Omega \wedge \Omega) \). Thus, for the effective form \( \omega \) associated with the MAE, the \( \text{pf}(\omega) \) coincides with \( \text{det}(\omega) \). This tensor is either an almost complex structure or an almost product structure:

1. \( \triangle_\omega \) is elliptic \( \iff \text{pf}(\omega) > 0 \iff I_\omega^2 = -Id \)
2. \( \triangle_\omega \) is hyperbolic \( \iff \text{pf}(\omega) < 0 \iff I_\omega^2 = Id \)

and it is integrable if and only if

\[
d \left( \frac{1}{\sqrt{|\text{pf}(\omega)|}} \omega \right) = 0.
\]

Given a pair of two-forms \((\Omega, \omega)\) on \( T^*\mathbb{R}^2 \), such that \( \omega \wedge \Omega = 0 \), by fixing the volume form in terms of \( \Omega \), we can define a pseudo-riemannian metric \( g_\omega \) in terms of the quadratic form

\[
g_\omega(X, Y) = \frac{\iota_X \Omega \wedge \iota_Y \omega + \iota_Y \Omega \wedge \iota_X \omega}{\Omega \wedge \Omega} \wedge \pi^*(\text{vol}), \quad X, Y \in T\mathbb{R}^2,
\]

where \( \text{vol} \) is the volume form on \( \mathbb{R}^2 \) and \( \pi : T^*\mathbb{R}^2 \mapsto \mathbb{R}^2 \). We can now identify our MAE given by \( \omega \) with an almost Kähler structure given by the triple \((\mathbb{R}^2, g_\omega, I_\omega)\) via

\[
\omega(X, Y) \equiv g_\omega(I_\omega X, Y).
\]

One can go further and in \( \mathbb{R}^4 \), one can show how a natural hyper-Kähler structure emerges by identifying points in \( \mathbb{R}^4 \) with quaternions \( \ell \in \mathbb{H} \). This structure was utilised by Roubtsov & Roulstone (1997, 2001) in their description of nearly geostrophic models of meteorological flows.

3 A Kähler Structure for two-dimensional Navier-Stokes flows

We shall now show Monge–Ampère equations, in two independent variables \( (\text{sat } (x, y) \in \mathbb{R}^2) \), arise in two-dimensional and two-and-a-half-dimensional incompressible Navier-Stokes flows. This in turn, via the relationships described in the previous section, leads us to a Kähler structure for these flows.

If the flow described by (13) is two-dimensional, and the fluid is incompressible, we can represent the velocity by

\[
u = k \times \nabla \psi,
\]

where \( \psi(x, y, t) \) is a stream function and \( k \) is the local unit vector in the vertical. If we substitute this for the velocity in (11), we get

\[
\nabla^2 P = -2(\psi_{xy}^2 - \psi_{xx} \psi_{yy})
\]
This is an equation of Monge–Ampère type (cf. 8) for $\psi$, given $\nabla^2 P (= P_{xx} + P_{yy})$, and it is an elliptic Monge–Ampère equation (cf. 8 and 9 with $E = \nabla^2 P, D = -2, A = B = C = 0$) if
\[ \nabla^2 P > 0. \] (15)

Following, for example, Lychagin & Roubtsov (1983), we introduce the usual notation for coordinates on $T^*\mathbb{R}^2, p = \psi_x, q = \psi_y$, and then we can write (14) on the graph of $d\psi$ via
\[ \omega_{2d} \equiv \nabla^2 P \, dx \wedge dy - 2dp \wedge dq; \quad \Delta_{\omega_{2d}} = 0. \] (16)

We have also on the graph of $d\psi$
\[ \Omega \equiv dx \wedge dp + dy \wedge dq; \quad \triangle_{\Omega} = 0, \] (17)

which says simply that $\psi_{xy} = \psi_{yx}$. Equations (16) and (17) define an almost complex structure, $I_{\omega_{2d}}$, on $T^*\mathbb{R}^2$, given in coordinates by
\[ I_{ik} = \frac{1}{\sqrt{2\nabla^2 P}} \Omega_{ij} \omega_{jk}. \]

We have
\[ I_{ij} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\alpha} \\ 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{pmatrix} \]

with $\nabla^2 P = 2\alpha^2$. This almost complex structure is integrable (cf. 10) in the special case
\[ \nabla^2 P = \text{constant}. \] (18)

Recall that time is merely a parameter here. When $P$ satisfies (18), we can introduce the coordinates $\mathcal{X}, \mathcal{Y}$, and a two-form $\omega_{\mathcal{X}\mathcal{Y}}$
\[ \mathcal{X} = x - i\alpha^{-1}q, \quad \mathcal{Y} = y + i\alpha^{-1}p, \quad \omega_{\mathcal{X}\mathcal{Y}} = d\mathcal{X} \wedge d\mathcal{Y}, \] (19)

then (14) together with (17) are equivalent to
\[ \triangle_{\omega_{\mathcal{X}\mathcal{Y}}} = 0. \] (20)

To summarize, the graph of $\psi$ is a complex curve in $(T^*\mathbb{R}^2, I_{\omega_{2d}})$. This is a starting point for a Kähler description of the incompressible two-dimensional Navier-Stokes equations.

4 A result for two-and-a-half dimensional flows

At this point it is appropriate to work with the two-and-a-half-dimensional Burgers solutions introduced in 11 in equations (5), (6) and (7). Based on the results of the last section, we shall prove a more realistic result for two-and-a-half-dimensional flows in Theorem 11.

Lundgren (1982) made a significant advance when he showed that the class of three-dimensional Navier-Stokes solutions (designated in §1 as two-and-a-half-dimensional)
\[ u_1(x, y, t) = -\frac{1}{2}\gamma(t)x + \psi_y; \quad u_2(x, y, t) = -\frac{1}{2}\gamma(t)y - \psi_x \] (21)
\[ u_3(x, y, t) = z\gamma(t) + \phi(x, y, t) \] (22)

\[ u_1(x, y, t) = -\frac{1}{2}\gamma(t)x + \psi_y; \quad u_2(x, y, t) = -\frac{1}{2}\gamma(t)y - \psi_x \] (21)
\[ u_3(x, y, t) = z\gamma(t) + \phi(x, y, t) \] (22)
under the limited conditions of a constant strain $\gamma(t) = \gamma_0$, can be mapped back to the two-dimensional Navier-Stokes equations under a stretched co-ordinate transformation; see also Majda (1986), Majda & Bertozzi (2002), Saffman (1993), and Pullin & Saffman (1998). In (21), $\psi = \psi(x,y,t)$ is a two-dimensional stream function. This idea was extended by Gibbon et al. (1999) to a time-dependent strain field $\gamma = \gamma(t)$ with the inclusion of a scalar $\phi(x,y,t)$ in (22). The class of solutions in (21), which are said to be of Burgers-type, is generally thought to represent the observed tube-sheet class of solutions in Navier-Stokes turbulent flows (Moffat et al. (1994) and Vincent & Meneguzzi (1994)).

Depending upon the sign of $\gamma(t)$ the vortex represented by (21) either stretches in the $z$-direction and contracts in the horizontal plane, which is the classic Burgers vortex tube, or vice-versa, which produces a Burgers' vortex shear layer or sheet. Thus $\gamma$, which can be interpreted as the aggregate effect of other vortices in the flow, acts as an externally imposed strain function or ‘puppet master’, and can switch a vortex between the two extremes of these two topologies as we discussed in §1.

This class of solutions is connected to the results of §2 through the following theorem, which is the main result of this section, and of the paper:

**Theorem 1** If a two-and-a-half-dimensional Burgers-type class of solutions has a Laplacian of the pressure that is bounded by $\nabla_3^2 P > -\frac{3}{4} \gamma^2$ then any associated underlying two-dimensional Navier-Stokes flow is of Kähler type.

**Proof:** To prove this theorem we first need two Lemmas. Firstly let $u = (u_1, u_2, u_3)$ be a candidate velocity field solution of the three-dimensional Navier-Stokes equations taken in the form

$$u_1 = u_1(x,y,t) \quad u_2 = u_2(x,y,t) \quad u_3 = z\gamma(x,y,t) + \phi(x,y,t).$$

(23)

with $z$ appearing only in $u_3$. With this velocity field the total derivative is now

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial x} + u_2 \frac{\partial}{\partial y} + (z\gamma + \phi) \frac{\partial}{\partial z}$$

(24)

and the vorticity vector $\omega$ must satisfy

$$\frac{D\omega}{Dt} = S\omega + \nu \nabla^2 \omega,$$

(25)

where $S$ is the strain matrix whose elements are $S_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$. In the following Lemma $v(x,y,t) = (u_1, u_2)$, and $\mathcal{P}(x,y,t)$ is a two-dimensional pressure variable which is related to the full pressure $P$ in (31). The material derivative is now

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \nabla$$

(26)

**Lemma 1** (see Gibbon et al. 1999) Consider the velocity field $u = (v, z\gamma + \phi)$; then $v$, $\omega_3$, $\phi$ and $\gamma$ satisfy

$$\frac{Dv}{Dt} + \nabla \mathcal{P} = \nu \nabla^2 v \quad \frac{D\omega_3}{Dt} = \gamma \omega_3 + \nu \nabla^2 \omega_3,$$

(27)

The notation used in this section is: $\nabla$ is the two-dimensional gradient and $\nabla_3$ is the three-dimensional gradient. $\nabla^2$ and $\nabla_3^2$ are the two- and three-dimensional Laplacians respectively (to avoid confusion with the symbol $\triangle$ in §2).
\[ \frac{D\phi}{Dt} = -\gamma \phi + \nu \nabla^2 \phi, \quad (28) \]
\[ \frac{D\gamma}{Dt} + \gamma^2 + P_{zz}(t) = \nu \nabla^2 \gamma. \quad (29) \]

The velocity field \( \mathbf{v} \) satisfies the continuity condition \( \text{div} \mathbf{v} = -\gamma \) and the second partial \( z \)-derivative of the pressure \( P_{zz} \) is constrained to be spatially uniform.

**Remark:** While (27) looks like a two-dimensional Navier-Stokes flow, the continuity condition implies that the two-dimensional divergence \( \text{div} \mathbf{v} \neq 0 \); thus an element of three-dimensionality remains.

**Proof:** The evolution of the third velocity component \( u_3 = \gamma z + \phi \) is given by
\[ -P_z = \frac{Du_3}{Dt} - \nu \nabla^2 u_3 = z \left( \frac{D\gamma}{Dt} + \gamma^2 - \nu \nabla^2 \gamma \right) + \left( \frac{D\phi}{Dt} + \gamma \phi - \nu \nabla^2 \phi \right) \quad (30) \]

which, on integration with respect to \( z \), gives
\[ -P(x, y, z, t) = \frac{1}{2} z^2 \left( \frac{D\gamma}{Dt} + \gamma^2 - \nu \nabla^2 \gamma \right) + z \left( \frac{D\phi}{Dt} + \gamma \phi - \nu \nabla^2 \phi \right) - P(x, y, t). \quad (31) \]

It is in this way that \( P(x, y, t) \) is related to \( P(x, y, z, t) \). However, from the first two components of the Navier-Stokes equations, we know that \( \nabla P \) must be independent of \( z \). For this to be true the coefficients of \( z \) and \( z^2 \) in (31) must necessarily satisfy
\[ \frac{D\phi}{Dt} + \gamma \phi - \nu \nabla^2 \phi = c_1(t) \quad \frac{D\gamma}{Dt} + \gamma^2 - \nu \nabla^2 \gamma = c_2(t). \quad (32) \]

\( c_1(t) \) is an acceleration of the co-ordinate frame which can be taken as zero without loss of generality. Equation (31) shows that \( c_2(t) = -P_{zz}(t) \) which restricts \( P_{zz} \) to being spatially uniform. To find the evolution of \( \omega_3 \) we consider the strain matrix \( S = \{S_{ij}\} \)
\[ S = \begin{pmatrix} u_{1,x} & \frac{1}{2} (u_{1,y} + u_{2,x}) & \frac{1}{2} (z \gamma_x + \phi_x) \\ \frac{1}{2} (u_{1,y} + u_{2,x}) & u_{2,y} & \frac{1}{2} (z \gamma_y + \phi_y) \\ \frac{1}{2} (z \gamma_x + \phi_x) & \frac{1}{2} (z \gamma_y + \phi_y) & \gamma \end{pmatrix}. \quad (33) \]

Working out the vorticity field \( \omega \) from (23) it is easily seen that \( (S \omega)_3 = \gamma \omega_3 \). Thus (25) shows that \( \omega_3 \) decouples from \( \phi \) to give the equation for \( \omega_3 \) in (27). \( \square \)

Now let us consider the class of Burgers’ velocity fields given in (21) with a stream function \( \psi(x, y, t) \). The strain rate variable \( \gamma \) is taken as a function of time only. The continuity condition is now automatically satisfied. The material derivative is given by
\[ \frac{D}{Dt} = \frac{\partial}{\partial t} - \frac{1}{2} \gamma(t) \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) + J_{x,y}(\psi, \cdot). \quad (34) \]

New co-ordinates can be taken (Lundgren’s transformation (Lundgren (1982)))
\[ s(t) = \exp \left( \int_0^t \gamma(t') \, dt' \right) \quad (35) \]
\[ \tilde{x} = s^{1/2} x \quad \tilde{y} = s^{1/2} y \quad \tilde{t} = \int_0^t s(t') \, dt', \quad (36) \]
which re-scale \( \omega_3 \) and \( \phi \) into new variables
\[
\tilde{\omega}_3(x, y, t) = s^{-1} \omega_3(x, y, t) \quad \tilde{\phi}(x, y, t) = s \phi(x, y, t).
\] (37)

The material derivative is
\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \tilde{v} \cdot \tilde{\nabla}
\] (38)

where \( \psi(x, y, t) = \tilde{\psi}(x, y, t) \); \( \tilde{v} = \left( \tilde{\psi}_y, -\tilde{\psi}_x \right) \) and \( \tilde{\nabla} = \hat{i} \partial_x + \hat{j} \partial_y \). The relation between \( v = (u_1, u_2) \) and \( \tilde{v} \) is given by
\[
u \frac{D}{Dt} \] (41) of (29) and write
\[ v \frac{D}{Dt} \] (42)

Next we appeal to the definition of the pressure \( \tilde{P} \) as
\[ \tilde{P} = s^{-1} \left[ P - \frac{1}{4}(x^2 + y^2) \left( \dot{\gamma} - \frac{1}{2} \gamma^2 \right) \right] \] (41)

our results can be summarized in our second Lemma:

**Lemma 2** The re-scaled velocity field \( \tilde{v} \) satisfies the two-dimensional re-scaled Navier-Stokes equations (\( \text{div} \tilde{v} = 0 \))
\[ \frac{D}{Dt} \tilde{v} + \tilde{\nabla} \tilde{P} = \nu \tilde{\nabla}^2 \tilde{v} . \] (42)

The vorticity \( \tilde{\omega}_3(x, y, t) = -\tilde{\nabla}^2 \tilde{\psi} \) and the passive scalar \( \tilde{\phi}(x, y, t) \) satisfy
\[ \frac{D}{Dt} \tilde{\omega}_3 = \nu \tilde{\nabla}^2 \omega_3, \quad \frac{D}{Dt} \tilde{\phi} = \nu \tilde{\nabla}^2 \phi. \] (43)

**Proof:** From (35) we note the useful result that \( Ds/Dt = \gamma s \). Using (35) we write
\[ \frac{Du_1}{Dt} - \nu \nabla^2 u_1 = -\frac{1}{4}x \left( \dot{\gamma} - \frac{1}{2} \gamma^2 \right) + s^{3/2} \left( \frac{D\tilde{v}_1}{Dt} - \nu \nabla^2 \tilde{v}_1 \right) , \] (44)
\[ \frac{Du_2}{Dt} - \nu \nabla^2 u_2 = -\frac{1}{4}y \left( \dot{\gamma} - \frac{1}{2} \gamma^2 \right) - s^{3/2} \left( \frac{D\tilde{v}_2}{Dt} - \nu \nabla^2 \tilde{v}_2 \right) . \] (45)

Next we appeal to the definition of the pressure \( \tilde{P} \) in (41) to give the velocity pressure relation in (42). The results for \( \tilde{\phi} \) and \( \tilde{\omega}_3 \) follow immediately.

The proof of Theorem 1 is now ready to be completed. To obtain the full three-dimensional Laplacian of the pressure \( \nabla^2_3 P \) we use (41) and (32) and write
\[ -\nabla^2_3 P = \frac{1}{4} \gamma^2 + s^2 \left[ \frac{D}{Dt} \left( \frac{D\tilde{v}_1}{Dt} - \nu \nabla^2 \tilde{v}_1 \right) \right] + \frac{D}{Dt} \left( \frac{D\tilde{v}_2}{Dt} - \nu \nabla^2 \tilde{v}_2 \right) \]
\[ = \frac{1}{4} \gamma^2 - 2s^2 \tilde{\nabla}^2 \tilde{P} . \] (46)

Thus if \( \nabla^2_3 P \) satisfies the condition in Theorem 1 then the corresponding Kähler positivity condition (13) on the Laplacian for two-dimensional flow is satisfied.

Lundgren’s mapping breaks down under one condition: while the strain \( \gamma(t) \) can take either sign, if it is forever negative or for long intervals, the domain \( t \in [0, \infty) \) maps on to a finite section of the \( t \)-axis. For example, if \( \gamma = -\gamma_0 = \text{const} < 0 \) then \( s = \exp(-\gamma_0 t) \) and \( \tilde{t} = \gamma_0^{-1} [1 - \exp(-\gamma_0 t)] \). Hence \( t \in [0, \infty) \) maps onto \( \tilde{t} \in [0, \gamma_0^{-1}] \).
5 Remarks on more general 3D Navier-Stokes flows

One of the features of this paper has been the discussion between the correspondence between Burgers-like flows and Kähler geometry: given the ubiquity of such flows it is reasonable to think of them as a ‘preferred’ or ‘attracting’ class of solutions. In §1 we speculated on the possibility that once the flow moves away from this class of preferred solutions then they may move onto complex manifolds of a higher dimension. One of the simplest of these is the Calabi–Yau manifold. These are smooth, compact, complex manifolds with a Ricci-flat Kähler metric with a holomorphically trivial canonical bundle.

To investigate the question of other manifolds exist in more general three-dimensional Navier-Stokes flows we exploit the fact that the incompressibility condition allows us to write $u$ in terms of a vector potential $A$

\[ u = \text{curl}A. \]  

(47)

We might suppose that the vector potential is some function of a scalar potential $\psi(x,t)$

\[ A = (F(x,t,\psi(x,t)), G(x,t,\psi(x,t)), H(x,t,\psi(x,t)) ), \]  

(48)

where $F$, $G$ and $H$ are any (appropriately differentiable) functions and our notation indicates that they may have an explicit dependence on $x$ and $t$. If we insert such a flow, using (47), into (1), then we obtain, in general, a variable-coefficient Monge–Ampère equation for $\psi$. As we explain in the Appendix (following Banos 2002), there is a correspondence between Monge–Ampère equations and Calabi–Yau geometries in six dimensions. Calabi–Yau geometries have been much studied in connection with string theory and in the Appendix we furnish an example of how such a geometry arises in the three-dimensional Navier-Stokes equations, in the case of a very special choice of vector potential.

A further variation on this theme revolves around the addition of rotation to the system, which has important meteorological applications. The Euler equations of motion now become

\[ \frac{\partial v}{\partial t} + v \cdot \nabla v + f(k \times v) + \nabla P = 0, \]  

(49)

where $\frac{1}{2} f$ is the angular frequency. If we examine these equations in two dimensions, with constant rotation, then taking the divergence of (49) gives

\[ \nabla^2 P = -2(\psi_{xy}^2 - \psi_{xx}\psi_{yy}) + f\nabla^2 \psi. \]  

(50)

This is an elliptic Monge–Ampère equation if $\nabla^2 P + f^2/2 > 0$. The associated complex structure is integrable when $\nabla^2 P$ is a constant (cf. (10)), and in this case we can introduce new complex coordinates

\[ \tilde{X} = ax + i(fy + 2q), \quad \tilde{Y} = ay - i(fx + 2p) \]  

(51)

with $a = (2\nabla^2 P + f^2)^{1/2}$. Once again, (50) together with (17) are equivalent to

\[ \omega_{\tilde{X}\tilde{Y}} \equiv d\tilde{X} \wedge d\tilde{Y}, \quad \Delta_{\tilde{X}\tilde{Y}} = 0. \]

If the pressure is zero, or harmonic, then (50) is suggestive of a special Lagrangian structure. A special Lagrangian structure has also been noted in the work of Roubtsov & Roulstone (2001), but its role in that context is obscure (see McIntyre & Roulstone (2002) equation (13.27) et seq.).
6 Summary

We have shown how Kähler geometry arises in the Navier-Stokes equations of incompressible hydrodynamics, via a Monge–Ampère equation associated with (4). Although it is certainly not the case that all two-dimensional flows will satisfy the condition for the Kähler structure to exist, the situation looks much more promising for two-and-a-half-dimensional flows, of which Burgers vortex is one example. In more general three-dimensional flows, there is a link with geometries of Calabi–Yau type, but this connection requires further investigation.

Issues relating to the existence and interpretation of Kähler structures, the integrability conditions, and related matters involving contact and symplectic structures, were discussed by McIntyre & Roulstone (2002) in connection with various Monge–Ampère equations arising in geophysical fluid dynamics. The semi-geostrophic equations of meteorology, which are a particularly useful model for studying the formation of fronts, were the starting point in McIntyre & Roulstone op. cit. for an investigation into the role of novel coordinate systems, similar to those we have found here in (19) and (51). In semi-geostrophic theories, such coordinates facilitate significant simplifications of difficult nonlinear problems, and they are associated with canonical Hamiltonian formulations of these systems. Issues relating to contact and symplectic geometry may also be relevant to the results presented in this paper, and this suggests one direction for further study.

Finally, it is perhaps appropriate to make a comment here about compressible flows. In two dimensions, a velocity potential \( \chi(x, t) \) may be introduced, so that in (1) and (2) we have \( u(x, t) = \nabla \chi + k \times \nabla \psi \). We may take the divergence of (1) and this yields an equation for \( \psi \) of the form

\[
2(\psi_{xx} \psi_{yy} - \psi_{xy}^2) + 2 \chi_{xy}(\psi_{xx} - \psi_{yy}) = R,
\]

where \( R \) is a function of \( \rho \) and \( P \), and their spatial derivatives; it is also a function of the first spatial derivatives of \( \psi \), and it will also depend on the derivatives of \( \chi \) with respect to space and time (e.g. \( \chi_x, \chi_{xy}, \chi_{xx} \) and \( \chi_{yy} \) etc.). In other words, we have a variable-coefficient Monge–Ampère-type equation for \( \psi \), when \( \rho, P \) and \( \chi \) (and their derivatives) are known. It is therefore possible, in principle, that a Kähler structure may be associated with this type of equation; however, in practice, such a structure will be difficult to identify.

A Calabi–Yau structures and 3D-Navier-Stokes flows

In six dimensions \( (T^*\mathbb{R}^3) \), there is again a correspondence between real/complex geometry and nondegenerate Monge–Ampère structures (Banos 2002). A Monge–Ampère structure \( (\Omega, \omega) \in \Lambda^2(N^6) \times \Lambda^3(N^6) \) on a 6-dimensional manifold \( N \) \( (N = T^*\mathbb{R}^3) \) is said to be nondegenerate if \( \omega \) is nondegenerate in the sense of Hitchin (2000): the Hitchin pfaffian \( \lambda(\omega) \in C^\infty(N) \) never vanishes (cf. (55)-(57) below). We note that:

1. \( \lambda(\omega) > 0 \) if and only if \( \omega \) is the sum of two decomposable forms, that is

\[
\omega = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \beta_1 \wedge \beta_2 \wedge \beta_3
\]

2. \( \lambda(\omega) < 0 \) if and only if \( \omega \) is the sum of two decomposable complex forms, that is

\[
\omega = (\alpha_1 + i\beta_1) \wedge (\alpha_2 + i\beta_2) \wedge (\alpha_3 + i\beta_3) + (\alpha_1 - i\beta_1) \wedge (\alpha_2 - i\beta_2) \wedge (\alpha_3 - i\beta_3).
\]

Hitchin has associated with such a nondegenerate form a tensor \( K_\omega : N \rightarrow TN \otimes T^*N \) which is either an almost complex structure if \( \lambda(\omega) < 0 \) or an almost product structure if
\( \lambda(\omega) > 0 \). Moreover, Lychagin and Roubtsov (1983) have defined a metric \( g_\omega \) on \( N \) which is compatible with \( K_\omega \) and which has signature \((3, 3)\) if \( \lambda(\omega) > 0 \) and \((6, 0)\) or \((4, 2)\) if \( \lambda(\omega) < 0 \). So, one can associate with a nondegenerate Monge–Ampère structure \((\Omega, \omega)\) on a 6-dimensional manifold an almost Kähler structure (which is "real" if \( \lambda(\omega) > 0 \)) whose Kähler form is \( \Omega \). Moreover this almost Kähler structure is "normalized" by the two decomposable three forms associated with \( \omega \): we use then the terminology of generalized Calabi–Yau structure (Banos 2002).

For example, the Monge–Ampère structure associated with the "real" MAE in three variables \((x, y, z)\)

\[
\text{hess}(\phi) = 1
\]

is, in the coordinates \((x, y, z, p, q, r)\), the pair

\[
\begin{cases}
\Omega = dx \wedge dp + dy \wedge dq + dz \wedge dr \\
\omega = dp \wedge dq \wedge dr - dx \wedge dy \wedge dz
\end{cases}
\]

and the real Kähler structure on \( T^*\mathbb{R}^3 \) is

\[
\begin{align*}
g_\omega & = \begin{pmatrix} 0 & Id \\ Id & 0 \end{pmatrix}, \\
K_\omega & = \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}.
\end{align*}
\]

The Monge–Ampère structure associated with the special Lagrangian equation

\[
\nabla^2 \phi - \text{hess}(\phi) = 0
\]

is the pair

\[
\begin{cases}
\Omega = dx \wedge dp + dy \wedge dq + dz \wedge dr \\
\omega = \Im((dx + idp) \wedge (dy + idq) \wedge (dz + idr))
\end{cases}
\]

and the underlying Kähler structure is the canonical Kähler structure on \( T^*\mathbb{R}^3 = \mathbb{C}^3 \):

\[
\begin{align*}
g_\omega & = \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix}, \\
K_\omega & = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}.
\end{align*}
\]

It is important to note that the geometry associated with a MAE \( \Delta_\omega = 0 \) of real type \( \lambda(\omega) > 0 \) is essentially real but it is very similar to the classic Kähler geometry. In particular, when this geometry is integrable, there exists a potential \( \Phi \) and a coordinate systems \((x_i, p_i)_{i=1,2,3}^6\) on \( T^*\mathbb{R}^6 \) such that

\[
g_\omega = \sum_{i,j} \frac{\partial^2 \Phi}{\partial x_i \partial p_j} dx_i \cdot dp_j
\]

and

\[
\det \left( \frac{\partial^2 \Phi}{\partial x_j \partial p_j} \right) = f(x)g(p).
\]

We have observed that the (elliptic) Monge–Ampère equation plays an intriguing role in incompressible two-dimensional and the two-and-a-half-dimensional Navier-Stokes equations, and that this is made explicit via the introduction of the stream function. As we
remark in §5 for more general three-dimensional Navier-Stokes flows, a stream function can play a role as suggested in (48); here we examine perhaps the simplest choice

\[ A = (-\psi, -\psi, -\psi), \quad (53) \]

where now \( \psi = \psi(x, y, z, t) \) (the minus signs are used simply to show the relationship with the two-dimensional case viz. \( \mathbf{v} = k \times \nabla \psi = \nabla \times (0, 0, -\psi) \)). Note that (53) is simply one choice that leads to some interesting features as described below. *This choice of vector potential is, of course, very special, and merely serves to illustrate the possible connection with generalized Calabi–Yau structures.*

If we substitute (47) with (53) for \( \mathbf{u} \) into (4) (and set \( \rho = 1 \)), and use the notation \( \psi_x = p, \psi_y = q, \psi_z = r \), then we find that we have another Monge–Ampère equation that can be written as

\[
\omega_{3d} \equiv \nabla^2 P \, dx \wedge dy \wedge dz + 2 \, dx \wedge dp \wedge dr + 2 \, dr \wedge dq \wedge dz + 2 \, dq \wedge dy \wedge dp
- 2 \, dy \wedge dq \wedge dr - 2 \, dp \wedge dq \wedge dx - 2 \, dr \wedge dz \wedge dp
- 2 \, dx \wedge dq \wedge dr - 2 \, dp \wedge dy \wedge dr - 2 \, dp \wedge dq \wedge dz;
\]

\[ \Delta \omega_{3d} = 0. \quad (54) \]

It is worth mentioning that since the canonical symplectic form \( \Omega = dx \wedge dp + dy \wedge dq + dz \wedge dr \) vanishes on the graph of \( \psi \), one can replace \( \omega \) by \( \omega + \theta \wedge \Omega \), with \( \theta \) any 1-form on \( T^*\mathbb{R}^3 \). Following Lychagin & Roubtsov (1983), we choose among all these forms, the unique one which is effective (its product with \( \Omega \) is zero). This form is

\[ \omega_0 = \omega - \frac{1}{2}(\perp \omega) \wedge \Omega, \]

with

\[ \perp \omega = \left( \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial p} + \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial q} + \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial r} \right) \wedge \omega. \]

Three invariants are associated with the effective form (54): the Lychagin-Roubtsov metric \( Q \), the Hitchin tensor \( K \) and the Hitchin pfaffian \( \lambda \) (see Hitchin (2000) and Banos (2002)). They are defined by

\[ Q(U, V) = -\frac{1}{4} \perp^2 ((U \omega_0) \wedge (V \omega_0)), \quad (55) \]

\[ K_{ij} = \Omega_{ik} Q_{kj}, \quad (56) \]

\[ \lambda = \frac{1}{6} \text{Tr}(K^2). \quad (57) \]

When \( \lambda < 0 \), the tensor \( J = \frac{1}{\sqrt{|\lambda|}} K \) is an almost complex structure and the pair \( (Q, J) \) describes a Calabi–Yau geometry. We obtain here \( \lambda = 128 \nabla^2 P \): our equation is associated with a complex geometry if and only if

\[ \nabla^2 P < 0. \quad (59) \]

The Lychagin-Roubtsov metric is

\[
Q = 16 \begin{pmatrix}
\alpha^2 & \alpha^2 & \alpha^2 & 0 & 0 & 0 \\
\alpha^2 & \alpha^2 & -\alpha^2 & 0 & 0 & 0 \\
\alpha^2 & -\alpha^2 & \alpha^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & -1 & 0
\end{pmatrix} \quad (60)
\]
and the Hitchin complex structure is

\[
J = \frac{1}{\sqrt{2}\alpha} \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & -1 & 0 \\
-\alpha^2 & -\alpha^2 & -\alpha^2 & 0 & 0 & 0 \\
-\alpha^2 & -\alpha^2 & \alpha^2 & 0 & 0 & 0 \\
-\alpha^2 & \alpha^2 & -\alpha^2 & 0 & 0 & 0
\end{pmatrix},
\]

(61)

with \(\nabla^2 P = -4\alpha^2\). Once again, this complex structure is integrable in the special case when \(\nabla^2 P\) is constant.

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