Well-Posedness of Free Boundary Problem in Non-relativistic and Relativistic Ideal Compressible Magnetohydrodynamics

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December 30, 2019

Abstract

We consider the free boundary problem for non-relativistic and relativistic ideal compressible magnetohydrodynamics in two and three spatial dimensions with the total pressure vanishing on the plasma–vacuum interface. We establish the local-in-time existence and uniqueness of solutions to this nonlinear characteristic hyperbolic problem under the Rayleigh–Taylor sign condition on the total pressure. The proof is based on certain tame estimates in anisotropic Sobolev spaces for the linearized problem and a modification of the Nash–Moser iteration scheme. Our result is uniform in the light speed and appears to be the first well-posedness result for the free boundary problem in ideal compressible magnetohydrodynamics with zero total pressure on the free surface.

Keywords: Ideal compressible magnetohydrodynamics, Plasma–vacuum interface, Free characteristic boundary, Rayleigh–Taylor sign condition, Nash–Moser iteration

Mathematics Subject Classification (2010): 35L65, 76N10, 76W05, 35Q60, 35Q75

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The research of Yuri Trakhinin was partially supported by RFBR (Russian Foundation for Basic Research) grant 19-01-00261-a. The research of Tao Wang was partially supported by the National Natural Science Foundation of China grants 11971359 and 11731008.
This paper concerns the well-posedness of the free boundary problem for a plasma–vacuum interface in non-relativistic and relativistic, ideal (i.e., neglecting the effect of viscosity and electrical resistivity), compressible magnetohydrodynamics (MHD) for the space dimension \( d = 2, 3 \).

Let \( \Omega(t) \subset \mathbb{R}^d \) denote the moving domain occupied by the plasma. We first consider the following equations of the non-relativistic ideal compressible MHD (see Landau–Lifshitz [18, §65]):

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho v) &= 0 \quad \text{in } \Omega(t), \\
\partial_t (\rho v) + \nabla \cdot (\rho v \otimes v - H \otimes H) + \nabla q &= 0 \quad \text{in } \Omega(t), \\
\partial_t H - \nabla \times (v \times H) &= 0 \quad \text{in } \Omega(t), \\
\partial_t (\rho E + \frac{1}{2}|H|^2) + \nabla \cdot (v(\rho E + p) + H \times (v \times H)) &= 0 \quad \text{in } \Omega(t),
\end{align*}
\]

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\[
\nabla \cdot H = 0 \quad \text{in } \Omega(t),
\]

which describe the motion of a perfectly conducting fluid in a magnetic field. Here density \( \rho \), velocity \( v \in \mathbb{R}^d \), magnetic field \( H \in \mathbb{R}^d \), and pressure \( p \) are unknown functions of time \( t \) and spatial variables \( x = (x_1, \ldots, x_d) \). The symbols \( q = p + \frac{1}{2}|H|^2 \) and \( E = e + \frac{1}{2}|v|^2 \) denote the total pressure and the specific total energy, respectively, where \( e \) is the specific internal energy. Equations (1.1) constitute a closed system of \( 2d + 2 \) conservation laws through smooth constitutive relations \( \rho = \rho(p, S) \) and \( e = e(p, S) \), where \( S \) is the specific entropy. The thermodynamic variables \( \rho, p, e, \) and \( S \) satisfy the Gibbs relation

\[
\vartheta dS = de - \frac{p}{\rho^2} d\rho,
\]

where \( \vartheta > 0 \) is the absolute temperature. Identity (1.2) is preserved by the evolution, provided it holds at the initial time. Throughout this paper, we
denote by \( \partial_t := \partial/\partial t \) the time derivative and by \( \nabla := (\partial_1, \ldots, \partial_d)^T \) the gradient with \( \partial_i := \partial/\partial x_i \). See Appendix A for the conventional notation in the vector calculus.

System (1.1) is supplemented with the initial conditions

\[
\Omega(0) = \Omega_0, \quad (\rho, v, H, S)|_{t=0} = (\rho_0, v_0, H_0, S_0) \quad \text{on} \; \Omega_0, \tag{1.4}
\]

where the domain \( \Omega_0 \subset \mathbb{R}^d \) and the initial data \((\rho_0, v_0, H_0, S_0)\) are already given and satisfy constraint \( \nabla \cdot H = 0 \). On the free vacuum boundary \( \Sigma(t) \), we require the following boundary conditions:

\[
\begin{align*}
  v \cdot n &= \mathcal{V}(\Sigma(t)) \quad \text{on} \; \Sigma(t), \tag{1.5a} \\
  q &= 0 \quad \text{on} \; \Sigma(t), \tag{1.5b} \\
  H \cdot n &= 0 \quad \text{on} \; \Sigma(t), \tag{1.5c}
\end{align*}
\]

where \( n \) is the exterior unit normal to \( \Omega(t) \) on the boundary \( \Sigma(t) \) and \( \mathcal{V}(\Sigma(t)) \) is the normal velocity of \( \Sigma(t) \). Condition (1.5a) means that the surface \( \Sigma(t) \) moves with the fluid. Condition (1.5b) comes from the vanishing vacuum magnetic field; see Goedbloed et al. \[11, \S 4.6\] for physical models of plasma–vacuum interfaces. Condition (1.5c) corresponds to constraint (1.2) and should also be regarded as the constraint on the initial data.

We aim to prove the local-in-time existence and uniqueness of solutions to the free boundary problem (1.1), (1.4)–(1.5b), provided the hyperbolicity assumption \( \rho|_{\Omega(t)} > \rho_* > 0 \) and the following Rayleigh–Taylor sign condition on the total pressure hold initially:

\[
\nabla_n q \leq -\kappa_0 < 0 \quad \text{on} \; \Sigma(t), \tag{1.6}
\]

where \( \rho_* \) and \( \kappa_0 \) are positive constants, and \( \nabla_n := n \cdot \nabla \). This problem is a nonlinear hyperbolic problem with a free characteristic boundary due to condition (1.5a). Assumption (1.6) is also the natural physical condition for the incompressible MHD; see Hao–Luo \[14\] for a priori estimates through a geometrical point of view and Gu–Wang \[12\] for local well-posedness. We also refer to Hao–Luo \[15\] for a recent ill-posedness result of the 2D incompressible MHD when condition (1.6) is violated. For the vacuum free boundary problem of incompressible and compressible liquids, without magnetic fields, the natural physical assumption reads

\[
\nabla_n p \leq -\kappa_0 < 0 \quad \text{on} \; \Sigma(t), \tag{1.7}
\]

See, for instance, the works \[8, 21, 45\] and \[22, 40\] respectively for incompressible and compressible liquids with a free surface under assumption (1.7). It is important to point out that under the initial constraint \( H|_{\Sigma(0)} = 0 \) (implying \( H|_{\Sigma(t)} = 0 \) for \( t > 0 \)), stability condition (1.6) can be reduced to (1.7).

Our construction of solutions is motivated by that of the first author in \[39, 40\] for compressible current-vortex sheets and compressible Euler equations in vacuum. The approach involves, in particular, the reduction to a fixed domain, the application of the “good unknown” of Alinhac \[1\], suitable tame estimates in certain function spaces for the linearized problem, and an appropriate Nash–Moser iteration scheme. This approach has also been applied to the study of rarefaction waves \[1\], compressible vortex sheets \[7\], compressible current-vortex sheets \[4, 38, 39\], MHD contact discontinuities \[29\], relativistic vortex
sheets [5], among others. More precisely, we consider the liquids, for which the density $\rho$ is supposed to be uniformly bounded from below by a positive constant, so that equations (1.1) can be rewritten equivalently as a symmetric hyperbolic system for sufficiently smooth solutions. Furthermore, we suppose that the moving boundary $\Sigma(t)$ has the form of a graph, which enables us to reduce problem (1.1), (1.4)–(1.5b) to a fixed domain by the standard partial hodograph transformation. We first study the well-posedness for the effective linear problem, resulting from the variable coefficient linearized problem by use of the “good unknown” and neglect of some zero-th order terms. In the basic $L^2$ estimate, the sign condition (1.6) provides us a good term for the interface function. Regarding higher-order energy estimates, as in [32, 39, 44] for other characteristic problems of the ideal compressible MHD, we work in anisotropic Sobolev spaces $H^m_\ast$, introduced first by Chen [6]. These function spaces, taking into account the loss of normal derivatives to the characteristic boundary, turn out to be appropriate for investigating symmetric hyperbolic, characteristic problems; see, e.g., Secchi [33] for a general theory. Having the well-posedness and tame estimate for the effective linear problem in hand, we can deduce the local existence and uniqueness of solutions for our nonlinear problem (cf. Theorem 2.1) by a modification of the Nash–Moser iteration scheme.

Moreover, we can employ the approach outlined above to show a counterpart of Theorem 2.1 for the relativistic version of problem (1.1), (1.4)–(1.5b) in the Minkowski spacetime $\mathbb{R}^{1+d}$ (cf. Theorem 2.2). The main ingredient is that one can symmetrize the following equations of ideal relativistic magnetohydrodynamics (RMHD) (see Lichnerowicz [20, §§30–34]):

\begin{align*}
\nabla_\alpha (\rho u^\alpha) &= 0 \quad \text{(conservation of matter),} & (1.8a) \\
\nabla_\alpha T^{\alpha\beta} &= 0 \quad \text{(conservation of energy–momentum),} & (1.8b) \\
\nabla_\alpha (u^\alpha b^\beta - u^\beta b^\alpha) &= 0 \quad \text{(relevant Maxwell’s equations).} & (1.8c)
\end{align*}

The symmetrization has been derived in Freistühler–Trakhinin [10] by properly applying the Lorentz transformation; also see Appendix B for a direct verification. Here $\nabla_\alpha$ is the covariant derivative with respect to the Minkowski metric $(g_{\alpha\beta})$ with

\[(g_{\alpha\beta}) := \begin{cases} 
\text{diag} (-1,1,1) & \text{if } d = 2, \\
\text{diag} (-1,1,1,1) & \text{if } d = 3,
\end{cases} \]

the symbol $\rho$ is the particle number density, $u^\alpha$ and $b^\alpha$ are, respectively, the components of the $d$-velocity and the magnetic field $d$-vector with respect to the plasma velocity such that

\[g_{\alpha\beta} u^\alpha u^\beta = -1, \quad g_{\alpha\beta} u^\alpha b^\beta = 0. \quad (1.9)\]

The total energy–momentum tensor $T^{\alpha\beta}$ has the form

\[T^{\alpha\beta} = (\rho h + |b|^2)u^\alpha u^\beta + \epsilon^2 qg^{\alpha\beta} - b^\alpha b^\beta, \quad (1.10)\]

where $h := 1+\epsilon^2(e+p/\rho)$ is the index of the fluid, $\epsilon^{-1}$ is the speed of light, $e$ is the specific internal energy, $p$ is the pressure, $|b|^2 := g_{\alpha\beta} b^\alpha b^\beta$, and $q := p + \frac{1}{2}\epsilon^{-2}|b|^2$ is the total pressure. The density $\rho = \rho(p,S)$ and internal energy $e = e(p,S)$ are given smooth functions of $p$ and $S$ and satisfy the Gibbs relation (1.3).
Throughout this paper, we adopt the Einstein summation convention, for which Greek and Latin indices range from 0 to \(d\) and from 1 to \(d\) respectively.

It is worth mentioning that our result is uniform in the light speed and appears to be the first well-posedness result for the free boundary problem in ideal compressible magnetohydrodynamics with zero total pressure on the free surface. However, our approach relies heavily on the hyperbolicity condition \(\rho(\Omega(t)) > \rho_\ast > 0\), and it is an open problem to extend our result to the motion of gases, namely \(\rho|_{\Gamma(t)} = 0\). For ideal gases without magnetic fields, the local well-posedness for the 3D compressible Euler equations with a physical vacuum boundary has been established by Coutand–Shkoller [9] and Jang–Masmoudi [16] via the Lagrangian reformulation and nonlinear energy estimates; also see Luo ET AL. [24] for a general uniqueness result.

The case with nonvanishing total pressure on the boundary corresponds to that with nonzero vacuum magnetic field. Considering the pre-Maxwell equations for the magnetic field in vacuum, Secchi–Trakhinin proved in [36] the well-posedness of the nonlinear plasma–vacuum interface problem in ideal compressible MHD based on their linear well-posedness results in [35, 41]. As for the incompressible case, we refer to Morando ET AL. [28] for the well-posedness of the linearized problem, Hao [13] for nonlinear \(a\) priori estimates, and Sun ET AL. [37] for nonlinear well-posedness. For the plasma–vacuum interface problem in RMHD, where the vacuum electric and magnetic fields satisfy Maxwell’s equations, an \(a\) priori estimate for the linearized problem in the anisotropic Sobolev space \(H^1_m\) was provided by the first author in [42]. Notice that the results in [28, 35–37, 42] all require a non-collinearity condition, leading to the ellipticity of the symbol associated with the interface. Finally we point out that the well-posedness of the plasma–vacuum interface problem for nontrivial vacuum magnetic field without the non-collinearity condition is still unknown; see [43] for a thorough discussion of this issue.

The plan of this paper is as follows. In Section 2, we reformulate problem (1.1), (1.4)–(1.5b) to that in a fixed domain and state the main results of this paper, namely Theorems 2.1 and 2.2. Section 3 is devoted to the proof of Theorem 3.1, that is, the well-posedness of the effective linear problem in anisotropic Sobolev spaces \(H^1_m\). In Section 4, we employ a modification of the Nash–Moser iteration scheme to construct the solutions of our nonlinear problem and prove Theorem 2.1. In Section 5, we sketch the proof of Theorem 2.2 for ideal RMHD in vacuum. For the completeness of the paper, we collect the conventional notation of the vector calculus for two and three spatial dimensions in Appendix A and present a direct calculation for the symmetrization of ideal RMHD in Appendix B.

2 Nonlinear Problems and Main Theorems

This section is dedicated to reducing the free boundary problem (1.1), (1.4)–(1.5b) to a fixed domain and stating the main results of this paper.

Let us first introduce the symmetric hyperbolic form of the MHD equations (1.1). For this purpose, we suppose that the sound speed \(a := a(\rho, S)\) is smooth and satisfies

\[
a(\rho, S) := \sqrt{p(\rho, S)} > 0 \quad \text{for all } \rho \in (\rho_\ast, \rho^\ast),
\]  

(2.1)
where \( \rho_* \) and \( \rho^* \) are positive constants with \( \rho_* < \rho^* \). By virtue of (1.2)–(1.3), we have the following equivalent system for smooth solutions of (1.1):

\[
\begin{aligned}
(\partial_t + v \cdot \nabla)p + \rho a^2 \nabla \cdot v &= 0, \\
\rho(\partial_t + v \cdot \nabla)v - (H \cdot \nabla)H + \nabla q &= 0, \\
(\partial_t + v \cdot \nabla)H - (H \cdot \nabla)v + H \nabla \cdot v &= 0, \\
(\partial_t + v \cdot \nabla)S &= 0.
\end{aligned}
\]  

(2.2)

Thanks to (2.1), system (2.2) is symmetrizable hyperbolic when

\[ \rho_* < \rho < \rho^*. \]

(2.3)

In light of (1.5b), as in Secchi–Trakhinin [35], we take \( U := (q, v, H, S)^T \) as the primary unknowns and obtain from (2.2) the symmetric system

\[
A_0(U)\partial_t U + A_i(U)\partial_i U = 0 \quad \text{in } \Omega(t),
\]

(2.4)

where

\[
A_0(U) := \begin{pmatrix}
\frac{1}{\rho a^2} & 0 & -\frac{1}{\rho a^2}H^T & 0 \\
0 & \rho I_d & 0 & 0 \\
-\frac{1}{\rho a^2}H & O_d & I_d + \frac{1}{\rho a^2}H \otimes H & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

(2.5)

\[
A_i(U) := \begin{pmatrix}
v_i \\
0 \\
-\frac{v_i}{\rho a^2}H & -H I_d & v_i I_d + \frac{v_i}{\rho a^2}H \otimes H & 0 \\
0 & 0 & 0 & v_i
\end{pmatrix},
\]

(2.6)

for \( i = 1, \ldots, d \). Here and below, \( O_m \) and \( I_m \) denote the zero and identity matrices of order \( m \), respectively, \( e_i := (\delta_{i1}, \ldots, \delta_{id})^T \), and \( \delta_{ij} \) denotes the Kronecker delta.

For technical simplicity, we assume that the space domain \( \Omega(t) \) occupied by the plasma takes the form

\[
\Omega(t) := \{ x \in \mathbb{R}^d : x_1 > \varphi(t, x'), x' := (x_2, \ldots, x_d) \in \mathbb{T}^{d-1} \},
\]

where \( \mathbb{T}^{d-1} \) denotes the \((d-1)\)-torus and the interface function \( \varphi(t, x') \) is to be determined. Then the free interface \( \Sigma(t) \) is given by

\[
\Sigma(t) := \{ x \in \mathbb{R}^d : x_1 = \varphi(t, x'), x' \in \mathbb{T}^{d-1} \}.
\]

Denoting by \( N := (1, -\partial_2 \varphi, \ldots, -\partial_d \varphi)^T \) the normal to \( \Sigma(t) \), we can rewrite the boundary conditions (1.5) as

\[
\begin{aligned}
\partial_t \varphi &= v_N, & q &= 0 & \text{on } \Sigma(t), \\
H_N &= 0 & \text{on } \Sigma(t),
\end{aligned}
\]

(2.7)

(2.8)

where \( v_N := v \cdot N \) and \( H_N := H \cdot N \). Moreover, the Rayleigh–Taylor sign condition (1.6) becomes

\[
\nabla_N q := N \cdot \nabla q \geq \kappa_0 |N| > 0 \quad \text{on } \Sigma(t).
\]

(2.9)
In view of (2.7)–(2.8), the boundary matrix for problem (1.1), (1.4)–(1.5b) on \( \Sigma(t) \) reads

\[
(\partial_t \varphi A_0(U) - N_i A_i(U))|_{\Sigma(t)} = \begin{pmatrix}
0 & -N^T & 0 & 0 \\
-N & O_d & O_d & 0 \\
0 & O_d & O_d & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

which is singular, meaning that the free boundary \( \Sigma(t) \) is characteristic. Furthermore, the boundary matrix on \( \Sigma(t) \) has one positive, one negative, and \( 2d \) zero eigenvalues. Since one boundary condition is necessary for determining the unknown function \( \varphi \), we know that the correct number of the boundary conditions is two, according to the well-posedness theory for hyperbolic problems. Therefore, condition (2.8) will be taken as the constraint on the initial data rather than a real boundary condition.

It is a standard first step to reduce the free boundary problem (1.1), (1.4)–(1.5b) to an equivalent problem in a fixed domain. To this end, we replace the unknown \( U \) by

\[
U_\sharp(t, x) := U(t, \Phi(t, x), x'),
\]

where as in MÉTIVIER [25, p. 70], we choose the function \( \Phi \) to satisfy

\[
\Phi(t, x) := x_1 + \kappa_\sharp \chi(x_1) \varphi(t, x'),
\]

with positive constant \( \kappa_\sharp \) and \( C_0^\infty(\mathbb{R}) \)–function \( \chi \) satisfying

\[
\kappa_\sharp \| \varphi_0 \|_{L^\infty(T^{d-1})} \leq \frac{1}{4}, \quad \| \chi' \|_{L^\infty(\mathbb{R})} < 1, \quad \chi \equiv 1 \quad \text{on } [0, 1].
\]

This change of variables is admissible on time interval \([0, T]\), provided \( T > 0 \) is sufficiently small so that \( \kappa_\sharp \| \varphi \|_{L^\infty([0, T] \times T^{d-1})} \leq 1/2 \). Without loss of generality we set \( \kappa_\sharp = 1 \). The cut-off function \( \chi \) is introduced as in [25, 40] to avoid assumptions about the compact support of the initial data (shifted to a constant state).

Dropping the subscript “\( \sharp \)” for convenience, we reformulate the vacuum free boundary problem (1.1), (1.4)–(1.5b) as the following initial boundary value problem in a fixed domain \( \Omega := \{ x \in \mathbb{R}^d : x_1 > 0, x' \in T^{d-1} \} \):

\[
L(U, \Phi) := L(U, \Phi)U = 0 \quad \text{in } [0, T] \times \Omega,
\]

\[
B(U, \varphi) := \left( \frac{\partial_t \varphi - v_N}{q} \right) = 0 \quad \text{on } [0, T] \times \Sigma,
\]

\[
(U, \varphi) = (U_0, \varphi_0) \quad \text{on } \{ t = 0 \} \times \Omega,
\]

where \( \Sigma := \{ x \in \mathbb{R}^d : x_1 = 0, x' \in T^{d-1} \} \) denotes the boundary and the operator \( L(U, \Phi) \) is defined by

\[
L(U, \Phi) := A_0(U) \partial_t + \tilde{A}_1(U, \Phi) \partial_1 + \sum_{i=2}^d A_i(U) \partial_i,
\]

with \( \tilde{A}_1(U, \Phi) := \frac{1}{\partial_1 \Phi} \left( A_1(U) - \partial_1 \Phi A_0(U) - \sum_{i=2}^d \partial_i \Phi A_i(U) \right) \).
Here, the coefficient matrices $A_i(U)$, for $i = 0, \ldots, d$, are given in (2.5)–(2.6). But, in the relativistic case, $A_i(U)$, for $i = 0, \ldots, d$, are defined by (5.12). In the new variables, identities (1.2) and (2.8) are reduced to

$$H_N = 0 \quad \text{if } x_1 = 0, \quad (2.16)$$

$$\partial^\Phi_i H_i = 0 \quad \text{if } x_1 > 0, \quad (2.17)$$

where we denote for notational simplicity the partial differentials with respect to the lifting function $\Phi$ by

$$\partial^\Phi_i := \partial_t - \frac{\partial \Phi}{\partial t_1} \partial_1, \quad \partial^\Phi_1 := \frac{1}{\partial \Phi} \partial_1, \quad \partial^\Phi_i := \partial_t - \frac{\partial \Phi}{\partial t_1} \partial_1 \quad \text{for } i = 2, \ldots, d. \quad (2.18)$$

Thanks to the change of variables (2.10)–(2.11) and the second condition in (2.7), assumption (2.9) can be recovered from

$$\partial_1 q \geq \kappa_0 > 0 \quad \text{if } x_1 = 0. \quad (2.19)$$

The following proposition indicates that identities (2.16)–(2.17) can be regarded as the constraints on the initial data (see [39, Appendix A] for the proof).

**Proposition 2.1.** For sufficiently smooth solutions of problem (2.13) on time interval $[0, T]$, constraints (2.16)–(2.17) are satisfied for all $t \in [0, T]$ as long as they hold initially.

Let $\lfloor s \rfloor$ denote the floor function of $s \in \mathbb{R}$ that maps $s$ to the greatest integer less than or equal to $s$. We are ready to state the first main theorem of this paper.

**Theorem 2.1** (Non-relativistic case). Let $m \geq 19$ be an integer. Assume that the initial data (2.13c) satisfy the hyperbolicity condition (2.3), constraints (2.16)–(2.17), the sign condition (2.19), and the compatibility conditions up to order $m$ (see Definition 4.1). Assume further that $(U_0 - \bar{U}, \varphi_0)$ belongs to $H^m([0, T]) \times H^m([0, T])$ for some constant state $\bar{U}$ with $\rho_* < \rho(\bar{U}) < \rho^*$. Then there exists a sufficiently small constant $T > 0$, such that problem (2.13) with $A_i(U)$ defined by (2.5)–(2.6) has a unique solution $(U, \varphi)$ on the time interval $[0, T]$ satisfying

$$(U - \bar{U}, \varphi) \in H^{[m/2]}([0, T] \times \Omega) \times H^{m-8}([0, T] \times T^{d-1}).$$

For the relativistic case, as in Freistühler–Trakhinin [10], we impose the physical assumption that the relativistic sound speed $c_s = c_s(\rho, S)$ is positive and smaller than the light speed, i.e.,

$$0 < c_s(\rho, S) := \frac{a(\rho, S)}{\sqrt{\hbar}} < \epsilon^{-1} \quad \text{for all } \rho \in (\rho_*, \rho^*), \quad (2.20)$$

where $\rho_*$ and $\rho^*$ are positive constants with $\rho_* < \rho^*$, and $a(\rho, S)$ is defined by (2.1). The second main result of this paper stated below is the relativistic counterpart of Theorem 2.1.

**Theorem 2.2** (Relativistic case). Let $m \geq 19$ be an integer. Assume that the initial data (2.13c) satisfy the hyperbolicity condition (2.3), constraints (2.16)–(2.17), the sign condition (2.19), the physical constraint $|v_0| < \epsilon^{-1}$, and the compatibility conditions up to order $m$ (see Definition 4.1). Assume further
that \((U_0 - \overline{U}, \varphi_0)\) belongs to \(H^{m+3/2}(\Omega) \times H^{m+2}(\mathbb{T}^{d-1})\) for some constant state \(\overline{U}\) with \(\rho_s < \rho(\overline{U}) < \rho^*\) and \(|\bar{v}| < \epsilon^{-1}\). Then there exists a sufficiently small constant \(T > 0\) such that problem (2.13) with \(A(U)\) defined by (5.12) has a unique solution \((U, \varphi)\) on the time interval \([0, T]\) satisfying

\[
(U - \overline{U}, \varphi) \in H^{[m/2] - 4}([0, T] \times \Omega) \times H^{m-8}([0, T] \times \mathbb{T}^{d-1}).
\]

**Remark 2.1.** As a matter of fact, the constructed solution \(U\) satisfies \(U - \overline{U} \in H^{m-8}([0, T] \times \Omega)\), and hence \(U - \overline{U}\) belongs to \(H^{[m/2] - 4}([0, T] \times \Omega)\) in virtue of the embedding \(H^m \hookrightarrow H^{[m/2]}\) (see §3.1 for the definition of the anisotropic Sobolev spaces \(H^m\)).

**Remark 2.2.** We emphasize that Theorems 2.1 and 2.2 imply corresponding results for the free boundary problem (1.1), (1.4)–(1.5b) and its relativistic counterpart respectively because constraint (2.17) and \(\partial_t \Phi \geq 1/2 > 0\) hold in \([0, T] \times \Omega\) for a sufficiently small \(T > 0\).

### 3 Well-posedness of the Linearized Problem

In this section, we perform the linearization of problem (2.13) and establish the well-posedness of the linearized problem in anisotropic Sobolev spaces \(H^m\), that is, Theorem 3.1.

#### 3.1 Main Result for the Linearized Problem

Let us denote \(\Omega_T := (-\infty, T) \times \Omega\) and \(\Sigma_T := (-\infty, T) \times \Sigma\) for \(T > 0\). Let the basic state \((\bar{U}, \bar{\varphi})\) with \(\bar{U} := (\bar{q}, \bar{v}, \bar{H}, \bar{S})^T\) be sufficiently smooth and satisfy

\[
\rho_s < \rho(\bar{U}) < \rho^* \quad \text{in } \Omega_T, \tag{3.1}
\]
\[
\partial_t \bar{\varphi} = \bar{v}_N, \quad \bar{H}_N = 0 \quad \text{on } \Sigma_T, \tag{3.2}
\]
\[
\partial_t \bar{q} \geq \frac{\kappa_0}{2} > 0 \quad \text{on } \Sigma_T, \tag{3.3}
\]

where

\[
\bar{v}_N := \bar{v} \cdot \bar{N}, \quad \bar{H}_N := \bar{H} \cdot \bar{N}, \quad \bar{N} := (1, -\partial_2 \bar{\varphi}, \ldots, -\partial_d \bar{\varphi})^T.
\]

We also denote \(\tilde{\Phi} := \chi(x_1)\tilde{\varphi}(t, x')\) and \(\hat{\Phi} := x_1 + \tilde{\Phi}\), where \(\chi \in C_0^\infty(\mathbb{R})\) satisfies (2.12). Then \(\partial_t \hat{\Phi} \geq 1/2\) on \(\Omega_T\), provided we without loss of generality assume that \(\|\tilde{\varphi}\|_{L^\infty(\Sigma_T)} \leq 1/2\). Moreover, we assume that

\[
\|\bar{U}\|_{W^{3, \infty}(\Omega_T)} + \|\bar{\varphi}\|_{W^{4, \infty}(\Sigma_T)} \leq K, \tag{3.4}
\]

for some constant \(K > 0\). Note that the physical constraint \(|\bar{v}| < \epsilon^{-1}\) should be imposed for the relativistic case.

The linearized operators for (2.13a)–(2.13b) are defined by

\[
\mathbb{L}'(\bar{U}, \bar{\varphi})(V, \psi) := L(\bar{U}, \bar{\varphi})V + C(\bar{U}, \bar{\varphi})V - \frac{1}{\partial_t \bar{\varphi}}(L(\bar{U}, \bar{\varphi})\psi)\partial_t \bar{U}, \tag{3.5}
\]
\[
\mathbb{B}'(\bar{U}, \bar{\varphi})(V, \psi) := \left(\partial_t \psi + \sum_{i=2}^d \bar{v}_i \partial_i \bar{\varphi} - \bar{v} \cdot \bar{N}\right), \tag{3.6}
\]
where \( \Psi := \chi(x_1)\psi(t, x') \) and
\[
C(\hat{U}, \hat{\phi})V := \sum_{k=1}^{2d+2} V_k \left( \frac{\partial A_0}{\partial \hat{U}_k}(\hat{U})\partial_1 \hat{U} + \frac{\partial A_1}{\partial \hat{U}_k}(\hat{U}, \hat{\phi})\partial_1 \hat{U} + \sum_{i=2}^{d} \frac{\partial A_i}{\partial \hat{U}_k}(\hat{U})\partial_1 \hat{U} \right).
\]

Following ALINHAC [1] and introducing the good unknown
\[
\hat{V} := V - \frac{\partial_1 \hat{U}}{\partial_1 \hat{\phi}} \Psi,
\]
we have (cf. [25, Proposition 1.3.1])
\[
\mathbb{L}'(\hat{U}, \hat{\phi})(V, \Psi) = L(\hat{U}, \hat{\phi})\hat{V} + C(\hat{U}, \hat{\phi})\hat{V} + \frac{\Psi}{\partial_1 \hat{\phi}}\partial_1 (L(\hat{U}, \hat{\phi})\hat{U}).
\]

As in [1, 5, 7, 39], we drop the last term in (3.8) and consider the effective linear problem
\[
\mathbb{L}'(\hat{U}, \hat{\phi})\hat{V} := L(\hat{U}, \hat{\phi})\hat{V} + C(\hat{U}, \hat{\phi})\hat{V} = f \quad \text{if } x_1 > 0, \quad (3.9a)
\]
\[
\mathbb{E}_e'((\hat{U}, \hat{\phi}))(\hat{V}, \psi) := \left( (\partial_t + \sum_{i=2}^{d} \hat{v}_i \partial_i - \partial_1 \hat{v}_N)\psi - \hat{v} \cdot \hat{N} \right) = g \quad \text{if } x_1 = 0, \quad (3.9b)
\]
\[
(\hat{V}, \psi) = 0 \quad \text{if } t < 0. \quad (3.9c)
\]

In light of the results in [32, 39, 44] for other characteristic problems in ideal compressible MHD, we shall work in the anisotropic Sobolev spaces \( H^m_\alpha(\Omega) \). Throughout this paper, symbol \( D^\alpha_\ast \) means that \( \alpha := (\alpha_0, \ldots, \alpha_{d+1}) \in \mathbb{N}^{d+1} \), and
\[
D^\alpha_\ast := \partial_1^{\alpha_0}(\sigma \partial_1)^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d} \partial_{d+1}^{\alpha_{d+1}}, \quad \langle \alpha \rangle := |\alpha| + \alpha_{d+1}, \quad |\alpha| := \sum_{i=0}^{d+1} \alpha_i, \quad (3.10)
\]
where \( \sigma = \sigma(x_1) \) is an increasing smooth function on \([0, +\infty)\) and satisfies \( \sigma(x_1) = x_1 \) for \( 0 \leq x_1 \leq 1/2 \) and \( \sigma(x_1) = 1 \) for \( x_1 \geq 1 \). For any integer \( m \in \mathbb{N} \) and interval \( I \subset \mathbb{R} \), function spaces \( H^m_\alpha(\Omega) \) and \( H^m_\alpha(I \times \Omega) \) are defined by
\[
H^m_\alpha(\Omega) := \{ u \in L^2(\Omega) : D^\alpha_\ast u \in L^2(\Omega) \text{ for } \langle \alpha \rangle \leq m \text{ with } \alpha_0 = 0 \},
\]
\[
H^m_\alpha(I \times \Omega) := \{ u \in L^2(I \times \Omega) : D^\alpha_\ast u \in L^2(I \times \Omega) \text{ for } \langle \alpha \rangle \leq m \},
\]
and equipped with the norms \( \| \cdot \|_{H^m_\alpha(\Omega)} \) and \( \| \cdot \|_{H^m_\alpha(I \times \Omega)} \), respectively, where
\[
\| u \|^2_{H^m_\alpha(\Omega)} := \sum_{\langle \alpha \rangle \leq m, \alpha_0 = 0} \| D^\alpha_\ast u \|^2_{L^2(\Omega)}, \quad (3.11)
\]
\[
\| u \|^2_{H^m_\alpha(I \times \Omega)} := \sum_{\langle \alpha \rangle \leq m} \| D^\alpha_\ast u \|^2_{L^2(I \times \Omega)}. \quad (3.12)
\]

We will write \( \| \cdot \|_{m, \ast, t} := \| u \|_{H^m_\alpha(\Omega)} \) for short. By definition, we have
\[
H^m(I \times \Omega) \hookrightarrow H^m_\ast(I \times \Omega) \hookrightarrow H^{|m|/2}(I \times \Omega) \quad \text{for all } m \in \mathbb{N}, \ I \subset \mathbb{R}.
\]

We are going to prove the following result in this section.
Theorem 3.1. Let $K_0 > 0$ and $m \in \mathbb{N}$ with $m \geq 6$. Then there exist constants $T_0 > 0$ and $C(K_0) > 0$ such that if the basic state $(\hat{U}, \hat{\varphi})$ satisfies (3.1)-(3.4), $(\hat{V}, \hat{\varphi}) \in H^{m+3}(\Omega_T) \times H^{m+3}(\Sigma_T)$, and
\[
\|\hat{V}\|_{9,T} + \|\hat{\varphi}\|_{H^{0}(\Sigma_T)} \leq K_0 \quad \text{for } \hat{V} := \hat{U} - \overline{U},
\]
and the source terms $f \in H^m_*(\Omega_T)$, $g \in H^{m+1}(\Sigma_T)$ vanish in the past, for some $0 < T \leq T_0$, then problem (3.9) has a unique solution $(\hat{V}, \hat{\psi}) \in H^{m}_*(\Omega_T) \times H^{m}(\Sigma_T)$ satisfying the tame estimate
\[
\|\hat{V}, \hat{\psi}\|_{m,T} + \|\hat{\psi}\|_{H^{0}(\Sigma_T)} \leq C(K_0) \left( \|f\|_{m,T} + \|g\|_{H^{m+1}(\Sigma_T)} + \|\hat{V}, \hat{\psi}\|_{m+3,T} \right) \|f\|_{6,T} + \|g\|_{H^{7}(\Sigma_T)} \right). \quad (3.14)
\]

In the theorem above, the condition that the source terms $f$ and $g$ vanish in the past corresponds to the case of zero initial data. The case of general initial data is postponed to the subsequent nonlinear analysis. In the rest of this section, we will focus on the three-dimensional case because the 2D case ($d = 2$) can be analyzed in the same way.

### 3.2 Well-posedness in $L^2$

It is more convenient to reduce problem (3.9) to the case with homogeneous boundary conditions, namely $g = 0$. For this purpose, we employ the trace theorem for the spaces $H^m_T$ (see Lemma 3.4 below) to find a function $\hat{V}_i \in H^{m+2}_T(\Omega_T)$ vanishing in the past and satisfying
\[
\mathbb{E}^i_0(\hat{U}, \hat{\varphi})(\hat{V}_i, 0)|_{\Sigma_T} = g, \quad \|\hat{V}_i\|_s \lesssim \|g\|_{H^{s+1}(\Sigma_T)} \quad \text{for } s = 0, \ldots, m. \quad (3.15)
\]

Then the new unknown $\hat{V}_i := \hat{V} - \hat{V}_i$ solves problem (3.9) with zero boundary source term and the new internal source term $\hat{f}$, that is,
\[
\begin{align*}
\mathbb{L}_0^i(\hat{U}, \hat{\varphi})V &= \hat{f} := f - \mathbb{L}_0^i(\hat{U}, \hat{\varphi})V_i \quad \text{if } x_1 > 0, \\
\mathbb{E}^i_0(\hat{U}, \hat{\varphi})(V, \hat{\psi}) &= 0 \quad \text{if } x_1 = 0, \\
(V, \hat{\psi}) &= 0 \quad \text{if } t < 0,
\end{align*}
\]

(3.16a, 3.16b, 3.16c)

where we have dropped subscript “$i$” for notational simplicity. Furthermore, we shall introduce a new unknown $W$ in order to separate the noncharacteristic variables from others for problem (3.16). To be more precise, we set
\[
W := (q, v_1 - \partial_2 \Phi v_2 - \partial_3 \Phi v_3, v_2, v_3, H_1, H_2, H_3, S)^T,
\]
or equivalently,
\[
W := \hat{J}^{-1}V \quad \text{with } \hat{J} :=
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & \partial_2 \Phi & \partial_3 \Phi & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\quad (3.17)
\]
Then problem (3.16) can be reduced to

\[ \begin{align*}
LW & := A_0 \partial_t W + A_i \partial_i W + A_4 W = \tilde{J}^T \tilde{f} & \text{in } \Omega_T, \\
W_1 & = -\partial_t q \varphi & \text{on } \Sigma_T, \\
W_2 & = (\partial_t + \tilde{v}_2 \partial_2 + \tilde{v}_3 \partial_3) \psi - \partial_1 \tilde{v}_N \psi & \text{on } \Sigma_T, \\
(W; \psi) & = 0 & \text{if } t < 0,
\end{align*} \]

where \( A_1 := \tilde{J}^T \tilde{A}_1(\tilde{U}; \tilde{\varphi}) \tilde{J}, \ A_i := \tilde{J}^T L_i(\tilde{U}; \tilde{\varphi}) \tilde{J}, \) and \( A_i := \tilde{J}^T A_i(\tilde{U}; \tilde{\varphi}) \tilde{J} \) for \( i = 0, 2, 3. \) Notice that system (3.18a) is still symmetric hyperbolic. By virtue of (3.2), we have

\[ \tilde{A}_1(\tilde{U}; \tilde{\varphi}) = \begin{pmatrix} 0 & \tilde{N}^T & 0 & 0 \\ \tilde{N} & O_3 & O_3 & 0 \\ 0 & O_3 & O_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

on \( \Sigma_T, \)

which implies the following decomposition:

\[ A_1 = A_1^{(1)} + A_1^{(0)} \quad \text{with } A_1^{(0)} |_{x_1=0} = 0, \quad A_1^{(1)} := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & O_6 \end{pmatrix}. \]

According to the kernel of the matrix \( A_1 \) on the boundary \( \Sigma_T, \) we denote by \( W_{nc} := (W_1, W_2)^T \) the noncharacteristic variables. The boundary matrix for problem (3.18), namely \(-A_1,\) has one negative, one positive, and six zero eigenvalues on the boundary \( \Sigma_T. \) As discussed in Section 2, the correct number of boundary conditions is two, which is just the case in (3.18b)–(3.18c). Therefore, for the hyperbolic problem (3.18), the boundary is characteristic of constant multiplicity and the maximality condition is fulfilled (cf. [31, Definition 2 and (11)].

Let us turn to derive the \( L^2 \) a priori estimate for solutions of (3.18). Take the scalar product of (3.18a) with \( W \) to get

\[ \mathcal{E}_0(t) - 2 \int_{\Sigma_t} W_1 W_2 \, dx' \, ds \leq C(K) \left( \| \tilde{f} \|_{L^2(\Omega_T)}^2 + \int_0^t \mathcal{E}_0(s) \, ds \right), \]

where \( \mathcal{E}_0(t) := \int_{\Omega} A_0 W \cdot W \, dx. \) By virtue of (3.18b)–(3.18c), we obtain

\[ -2W_1 W_2 = \partial_t (\partial_t \tilde{q} \varphi^2) + \sum_{i=2}^3 \partial_i (\tilde{v}_i \partial_i \tilde{q} \psi^2) - \left( \partial_1 \partial_t \tilde{q} + \sum_{i=2}^3 \partial_i (\tilde{v}_i \partial_t \tilde{q}) + \partial_t \tilde{q} \partial_1 \tilde{v}_N \right) \psi^2 \]

on \( \Sigma_T, \)

which together with (3.21) yields

\[ \mathcal{E}_0(t) + \int_{\Sigma} \partial_t \tilde{q} \psi^2 \, dx' \leq C(K) \left\{ \| \tilde{f} \|_{L^2(\Omega_T)}^2 + \int_0^t \left( \mathcal{E}_0(s) + \| \psi(s) \|_{L^2(\Sigma_T)}^2 \right) \, ds \right\}. \]

Since the sign condition (3.3) and \( A_0 \geq \kappa \lambda I_3 \) are satisfied for some positive constant \( \kappa \) (independent of the light speed in the relativistic case), we apply Grönwall’s inequality to deduce the \( L^2 \) estimate

\[ \| W \|_{L^2(\Omega_T)} + \| \psi \|_{L^2(\Sigma_T)} \leq C(K) \| \tilde{f} \|_{L^2(\Omega_T)}. \]
The last estimate exhibits no loss of derivatives from the source term \( \tilde{f} \) to the solution \( W \), so one can apply the classical argument in [19, 31] and [3, Chapter 7] to construct the solutions to problem \((3.18)\). We only need to show that the dual problem of \((3.18)\) satisfies an \textit{a priori} estimate without loss of derivatives similar to \((3.22)\). Let us define the following dual problem for \((3.18)\):

\[
\begin{align*}
L^*U^* &= f^* & \text{in } \Omega_T, \\
\partial_t U^*_1 + \partial_2(\tilde{v}_2 U^*_1) + \partial_3(\tilde{v}_3 U^*_1) + \partial_4\tilde{q} U^*_2 + \partial_4\tilde{v}_N U^*_1 &= 0 & \text{on } \Sigma_T, \\
U^* &= 0 & \text{if } t < 0,
\end{align*}
\]

where \( L^* := -L + A_4 + A_4^\top - \partial_t A_0 - \partial_i A_i \) denotes the formal adjoint of operator \( L \). Taking the scalar product of \((3.23a)\) with \( U^* \) implies

\[
\mathcal{E}_0^*(t) - 2 \int_{\Sigma_t} U_1^* U_2^* \, dx' \, ds \leq C(K) \left( \|f^*\|_{L^2(\Omega_T)}^2 + \int_0^t \mathcal{E}_0^*(s) \, ds \right),
\]

where \( \mathcal{E}_0^*(t) := \int_{\Omega} A_0 U^* \cdot U^* \, dx \). Then we utilize \((3.23b)\) to obtain

\[
\mathcal{E}_0^*(t) + \int_{\Sigma} \frac{(U_1^*)^2}{\partial_t \tilde{q}} \, dx' \leq C(K) \left\{ \|f^*\|_{L^2(\Omega_T)}^2 + \int_0^t \left( \mathcal{E}_0^*(s) + \|U_1^*(s)\|_{L^2(\Sigma)}^2 \right) \, ds \right\},
\]

which combined with condition \((3.3)\) and Grönwall’s inequality leads to

\[
\|U^*\|_{L^2(\Omega_T)} \leq C(K) \|f^*\|_{L^2(\Omega_T)}.
\]

With the aid of the last estimate and \((3.22)\), one can deduce the following well-posedness result in \( L^2 \) for the reformulated problem \((3.18)\). We omit further details that are standard and can be found in [19, 31] and [3, Chapter 7].

\begin{theorem}
Assume that the basic state \((\bar{U}, \bar{\varphi})\) satisfies \((3.1)\)–\((3.4)\) and the source terms \( f \in L^2(\Omega_T) \), \( g \in H^1(\Sigma_T) \) vanish in the past. Then problem \((3.18)\) has a unique solution \((W, \psi) \in L^2(\Omega_T) \times L^2(\Sigma_T)\) satisfying estimate \((3.22)\).
\end{theorem}

### 3.3 Preliminaries

We shall deduce the tame estimate in \( H^m_*(\Omega_T) \) for problem \((3.18)\) with \( m \) large enough. For this purpose, in this subsection we collect the Moser-type calculus inequalities for the spaces \( H^m \) and \( H^m_* \) and the embedding and trace theorems for \( H^m_* \).

\begin{lemma}[Moser-type calculus inequalities for \( H^m \)]
Let \( m \in \mathbb{N}_+ \). Let \( \mathcal{O} \) be an open subset of \( \mathbb{R}^n \) with Lipschitz boundary. Assume that \( F \) is \( C^\infty \) in a neighborhood of the origin with \( F(0) = 0 \), and that \( u, w \in H^m(\mathcal{O}) \) with \( \|u\|_{L^\infty} \leq M \) for some constant \( M > 0 \). Then

\[
\|\partial^\alpha u \partial^\beta w\|_{L^2} + \|uw\|_{H^m} \lesssim \|u\|_{H^m} \|w\|_{L^\infty} + \|u\|_{L^\infty} \|w\|_{H^m},
\]

\[
\|F(u)\|_{H^m} \leq C(M) \|u\|_{H^m},
\]

for any multi-indices \( \alpha, \beta \in \mathbb{N}^n \) with \( |\alpha| + |\beta| \leq m \).
\end{lemma}

For the proof of the last lemma, we refer to, for instance, [2, pp. 84–89]. Here and below, we employ the symbol \( A \lesssim B \) (or \( B \gtrsim A \)) meaning that \( A \leq CB \) holds uniformly for some universal positive constant \( C \).
Lemma 3.2 (Moser-type calculus inequalities for $H^m_\ast$). Let $m \in \mathbb{N}_+$. Assume that $F$ is $C^\infty$ in a neighborhood of the origin, with $F(0) = 0$, and the functions $u, w$ belong to $H^m_\ast(\Omega_T)$ and satisfy
\[
\|u\|_{W^{1,\infty}_\ast(\Omega_T)} := \sum_{\langle \alpha \rangle \leq 1} \|D_\alpha^u\|_{L^\infty(\Omega_T)} \leq M_\ast,
\] (3.24)
for some constant $M_\ast > 0$, where $D_\alpha^u$ and $\langle \alpha \rangle$ are defined in (3.10). Then
\[
\|D_\alpha^u D_\beta^w\|_{L^2(\Omega_T)} \lesssim \|u\|_{m,s,T} \|w\|_{W^{1,\infty}_\ast(\Omega_T)} + \|w\|_{m,s,T} \|u\|_{W^{1,\infty}_\ast(\Omega_T)},
\] (3.25)
\[
\|u w\|_{m,s,T} \lesssim \|u\|_{m,s,T} \|w\|_{W^{1,\infty}_\ast(\Omega_T)} + \|w\|_{m,s,T} \|u\|_{W^{1,\infty}_\ast(\Omega_T)},
\] (3.26)
\[
\|F(u)\|_{m,s,T} \leq C(M_\ast) \|u\|_{m,s,T},
\] (3.27)
for any multi-indices $\alpha, \beta \in \mathbb{N}^5$ with $\langle \alpha \rangle + \langle \beta \rangle \leq m$.

One can find the proof of inequalities (3.25)–(3.26) in [27, Theorem B.3] and the proof of (3.27) in [39, Appendix B].

Lemma 3.3 (Embedding theorem for $H^m_\ast$). The following inequalities hold:
\[
\|u\|_{L^\infty(\Omega_T)} \lesssim \|u\|_{3,s,T}, \quad \|u\|_{W^{1,\infty}_\ast(\Omega_T)} \lesssim \|u\|_{4,s,T},
\] (3.28)
\[
\|u\|_{W^{1,\infty}_\ast(\Omega_T)} \lesssim \|u\|_{5,s,T}, \quad \|u\|_{W^{2,\infty}_\ast(\Omega_T)} \lesssim \|u\|_{6,s,T},
\] (3.29)
where $\|u\|_{W^{1,\infty}_\ast(\Omega_T)}$ is defined by (3.24), and
\[
\|u\|_{W^{2,\infty}_\ast(\Omega_T)} := \sum_{\langle \alpha \rangle \leq 1} \|D_\alpha^u\|_{W^{1,\infty}_\ast(\Omega_T)}.
\]

Thanks to [27, Theorem B.4] and $\Omega_T \subset \mathbb{R}^4$, we obtain the first inequality in (3.28), which implies the second one by definition. Observing that
\[
\|u\|_{W^{1,\infty}_\ast(\Omega_T)} \leq \sum_{\langle \alpha \rangle \leq 2} \|D_\alpha^u\|_{L^\infty(\Omega_T)},
\]
we derive (3.29) from the first inequality in (3.28).

For deriving higher-order energy estimates, we also need to use the following trace theorem for the anisotropic Sobolev spaces $H^m_\ast$.

Lemma 3.4 ([30, Theorem 1]). Let $m \geq 1$ be an integer.

a) If $u \in H^{m+1}_\ast(\Omega_T)$, then its trace $u|_{x_1 = 0}$ belongs to $H^m(\Sigma_T)$ and satisfies
\[
\|u|_{x_1 = 0}\|_{H^m(\Sigma_T)} \lesssim \|u\|_{m+1,s,T}.
\]

b) There exists a continuous operator $\mathcal{R}_T : H^m(\Sigma_T) \to H^{m+1}_\ast(\Omega_T)$ such that
\[
(\mathcal{R}_T w)|_{x_1 = 0} = w \quad \text{and} \quad \|\mathcal{R}_T w\|_{m+1,s,T} \lesssim \|w\|_{H^m(\Sigma_T)}.
\]
The same properties still hold true for the function spaces $H^{m+1}_\ast(\Omega)$ and $H^m(\Sigma)$ with the norms replaced by $\|\cdot\|_{H^{m+1}_\ast(\Omega)}$ and $\|\cdot\|_{H^m(\Sigma)}$ accordingly.
3.4 Higher-order Energy Estimates

Having Lemmas 3.1–3.4 in hand, we now derive the tame estimate in $H^m_\Omega$ for problem (3.18) and integer $m \in \mathbb{N}_+$. Let $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{N}^5$ with $\langle \alpha \rangle := \sum_{i=0}^4 \alpha_i + 2\alpha_4 \leq m$. Applying the operator $D^\alpha := \partial_t^{\alpha_0}(\sigma \partial_1)^{\alpha_1}(\partial_2)^{\alpha_2}(\partial_3)^{\alpha_3}(\partial_4)^{\alpha_4}$ to (3.18a) yields

$$A_0 \partial_t D^\alpha W + A_i \partial_i D^\alpha W = R_\alpha,$$

where

$$R_\alpha := D^\alpha (J^T \hat{f}) - D^\alpha (A_4 W) - [D^\alpha W, A_i \partial_i]W - [D^\alpha W, A_0 \partial_t]W.$$

Take the scalar product of (3.30) with $D^\alpha W$ to get

$$\mathcal{E}_\alpha(t) = \mathcal{R}_\alpha(t) + \mathcal{Q}_\alpha(t),$$

where

$$\mathcal{E}_\alpha(t) := \int_\Omega A_0 D^\alpha W \cdot D^\alpha W \, dx \gtrsim \|D^\alpha W(t)\|^2_{L^2(\Omega)},$$

$$\mathcal{Q}_\alpha(t) := 2 \int_{\Sigma_t} D^\alpha W \mathcal{K} W \, dx \, ds,$$

$$\mathcal{R}_\alpha(t) := \int_{\Omega_T} D^\alpha W \cdot \{2R_\alpha + (\partial_t A_0 + \partial_t A_i)D^\alpha W\} \, dx \, ds.$$

The following lemma provides the estimate of non-weighted tangential derivatives $D^\alpha W$ for $\alpha_1 = \alpha_4 = 0$ and $\langle \alpha \rangle \leq m$.

**Lemma 3.5.** If the assumptions in Theorem 3.1 are fulfilled, then

$$\sum_{\langle \alpha \rangle \leq m, \alpha_1 = \alpha_4 = 0} (\|D^\alpha W(t)\|^2_{L^2(\Omega)} + \|D^\alpha \psi(t)\|^2_{L^2(\Sigma)}) \leq C(K)\mathcal{M}(t)$$

for all $t \in [0, T]$ and $m \in \mathbb{N}_+$, where

$$\mathcal{M}(t) := \|\hat{f}, W\|^2_{m,s,t} + \|\psi\|^2_{H^m(\Sigma_t)} + \hat{C}_{m+2} \|\hat{f}\|^2_{W^{2,\infty}(\Omega_t)} + \hat{C}_{m+3} \|\psi\|^2_{L^\infty(\Sigma_t)},$$

with $\hat{C}_s := 1 + \|\hat{V}^s\|^2_{s,s,T}$.

**Proof.** Let $\alpha_1 = \alpha_4 = 0$ and $\langle \alpha \rangle \leq m$. It follows from definition that $\hat{C}_s \geq \|\hat{V}\|^2_{H^s(\Sigma_T)}$. We first estimate the boundary term $\mathcal{Q}_\alpha(t)$ defined by (3.33). In view of (3.18b)–(3.18c), we find

$$\mathcal{Q}_\alpha(t) = \mathcal{Q}^{(1)}_\alpha(t) + \mathcal{Q}^{(2)}_\alpha(t),$$

where

$$\mathcal{Q}^{(1)}_\alpha(t) := -2 \int_{\Sigma_t} D^\alpha (\partial_t \hat{\psi}) (\partial_t + \hat{v}_2 \partial_2 + \hat{v}_3 \partial_3) D^\alpha \psi,$$

$$\mathcal{Q}^{(2)}_\alpha(t) := -2 \int_{\Sigma_t} D^\alpha (\partial_t \hat{\psi}) \{D^\alpha \psi, \hat{v}_2 \partial_2 + \hat{v}_3 \partial_3 \psi - D^\alpha (\partial_1 \hat{\psi} N \psi)\}.$$

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Since \( D^\alpha = \partial_t^{\alpha_0} \partial_2^{\alpha_2} \partial_3^{\alpha_3} \) with \( \alpha_0 + \alpha_2 + \alpha_3 \leq m \) and \( \| \partial_t \hat{q} \|_{H^{m+1}(\Sigma)} \lesssim \| \hat{V} \|_{m+3,\ast,T} \), we use the sign condition (3.3), integration by parts, Cauchy’s inequality, and Lemma 3.1 to get

\[
Q_\alpha^{(1)}(t) \leq - \int_\Sigma \partial_t \hat{q}(D^\alpha \psi)^2 - 2 \int_\Sigma [D^\alpha, \partial_t \hat{q}] \partial_t \psi D^\alpha \psi + C(K) \| D^\alpha \psi \|_{L^2(\Sigma)}^2 + \\sum \| \partial_t (\hat{v}_i [D^\alpha, \partial_t \hat{q}] \psi) \|_{L^2(\Sigma)}^2 \\
\leq - \frac{\kappa_0}{4} \| D^\alpha \psi \|_{L^2(\Sigma)}^2 + C(K) \left( \| D^\alpha \psi \|_{L^2(\Sigma)}^2 + \| [D^\alpha, \partial_t \hat{q}] \psi \|_{H^1(\Sigma)}^2 \right) \\
\leq - \frac{\kappa_0}{4} \| D^\alpha \psi \|_{L^2(\Sigma)}^2 + C(K) \left( \| \psi \|_{H^m(\Sigma)}^2 + \tilde{C}_{m+3} \| \psi \|_{L^\infty(\Sigma)}^2 \right).
\]

Apply Lemma 3.1 to derive

\[
Q_\alpha^{(2)}(t) \leq C(K) \left( \| \psi \|_{H^m(\Sigma)}^2 + \tilde{C}_{m+3} \| \psi \|_{L^\infty(\Sigma)}^2 \right),
\]

which together with the estimate for \( Q_\alpha^{(1)}(t) \) above implies

\[
Q_\alpha(t) \leq - \frac{\kappa_0}{4} \| D^\alpha \psi(t) \|_{L^2(\Sigma)}^2 + C(K) \left( \| \psi \|_{H^m(\Sigma)}^2 + \tilde{C}_{m+3} \| \psi \|_{L^\infty(\Sigma)}^2 \right).
\]  (3.37)

Next we make the estimate of \( R_\alpha(t) \) defined by (3.34). It follows from Cauchy’s inequality that

\[
R_\alpha(t) \leq \| (J^\dagger \hat{f}, A_4 W) \|_{m,\ast,t}^2 + \sum_{i=0}^3 C(K) \| (D^\alpha W, [D^\alpha, A_3 \partial_i W]) \|_{L^2(\Omega_1)}^2,
\]  (3.38)

where we denote \( \partial_0 := \partial_t \) for simplifying the presentation. Since \( A_4 \) is a \( C^\infty \)-function of \( \hat{U}, \hat{v}_i \hat{U}, \nabla \hat{U}, \nabla \hat{\Phi}, \partial_\ast \nabla \hat{\Phi}, \nabla^2 \hat{\Phi} \) and \( J \) is a \( C^\infty \)-function of \( \nabla \hat{\Phi} \) (cf. (3.17)), we apply the Moser-type calculus inequalities (3.26)–(3.27) to obtain

\[
\| (J^\dagger \hat{f}, A_4 W) \|_{m,\ast,t}^2 \leq C(K) \left( \| (\hat{f}, W) \|_{m,\ast,t}^2 + \tilde{C}_{m+2} \| (\hat{f}, W) \|_{L^\infty(\Omega_1)}^2 \right).
\]  (3.39)

Regarding the commutator terms in (3.38), for \( i = 0, \ldots, 3 \), we have

\[
\| [D^\alpha, A_4] \partial_i W \|_{L^2(\Omega_1)}^2 \lesssim \sum_{0 < \alpha' \leq \alpha} \| D^\alpha A_4 D^{\alpha-\alpha'} \partial_i W \|_{L^2(\Omega_1)}^2.
\]  (3.40)

It follows from (3.20) that \( \partial_i A_4 |_{x_1=0} = 0 \) for \( i = 0, 2, 3 \). Since \( \langle \alpha' \rangle = 1 \) implies \( \alpha'_4 = 0 \), we get

\[
\| D^{\alpha'}_4 A_4(x_1) \|_{L^\infty((-\infty,T) \times \mathbb{T}^2)} \leq C(K) \sigma(x_1) \quad \text{for all } x_1 \geq 0, \langle \alpha' \rangle = 1.
\]  (3.41)

By virtue of the last inequality, we infer

\[
\sum_{i=0}^3 \mathcal{J}_\alpha^{(i)} \leq C(K) \sum_{i=0,2,3} \| D^{\alpha-\alpha'} \partial_i W \|_{L^2(\Omega_1)}^2 + C(K) \| \sigma D^{\alpha-\alpha'} \partial_i W \|_{L^2(\Omega_1)}^2 \\
\leq C(K) \| \bar{W} \|_{m,\ast,t}^2 \quad \text{for } \langle \alpha' \rangle = 1 \text{ and } \alpha' \leq \alpha.
\]  (3.42)
Since \( \|\partial_i W\|_{m-2,*,t} \lesssim \|W\|_{m,*,t} \) and \( A_i \) are \( C^\infty \)-functions of \((\bar{U}, \nabla \Phi)\) for \( i = 0, \ldots, 3 \), we deduce from (3.25) and (3.27) that

\[
\sum_{i=0}^3 J^{(i)}_{\alpha'} \lesssim \sum_{i=0}^3 \sum_{\beta = 2, \beta \leq \alpha'} \|D_i^{\alpha' - \beta}(D_i^\beta A_i)D_i^{\alpha - \alpha'}(\partial_i W)^2\|_{L^2(\Omega_t)}
\]

\[
\leq C(K) \left( \|W\|_{m,*,t}^2 + \hat{C}_{m+2}\|W\|_{W_{2,\infty}^2(\Omega_t)}^2 \right) \quad \text{for} \quad \langle \alpha' \rangle \geq 2. \tag{3.43}
\]

Noting that \( [D_i^\alpha, A_i, \partial_i]W = [D_i^\alpha, A_i]\partial_i W \) for \( i = 0, \ldots, 3 \) and \( \alpha_1 = 0 \), we plug (3.42)-(3.43) into (3.40) and then substitute the resulting estimate and (3.39) into (3.38) to obtain

\[
R_{\alpha}(t) \leq C(K)M(t) \quad \text{for} \quad \alpha_1 = 0. \tag{3.44}
\]

Plugging (3.32), (3.37), and (3.44) into (3.31) yields the desired estimate (3.35) and completes the proof of the lemma. \( \square \)

In the next lemma, we have an estimate of the non-weighted normal derivatives \( D_i^\alpha W \) for \( \alpha_1 = 0, \alpha_4 > 0, \) and \( \langle \alpha \rangle \leq m. \)

**Lemma 3.6.** If the assumptions in Theorem 3.1 are fulfilled, then

\[
\sum_{\langle \alpha \rangle \leq m, \alpha_1 = 0} \|D_i^\alpha W(t)\|_{L^2(\Omega)}^2 \lesssim \varepsilon \sum_{\langle \beta \rangle \leq m} \|D_i^\beta W(t)\|_{L^2(\Omega)}^2 + C(\varepsilon, K)M(t) \tag{3.45}
\]

holds for all \( t \in [0, T], \ m \in \mathbb{N}_+, \) and \( \varepsilon > 0, \) where \( M(t) \) is defined by (3.36).

**Proof.** Let \( \alpha_1 = 0, \alpha_4 > 0, \) and \( \langle \alpha \rangle \leq m. \) Then \( |\alpha| = \langle \alpha \rangle - \alpha_4 < m. \) Since estimates (3.32) and (3.44) are still valid, it remains to control the term \( Q_{\alpha}(t) \) (cf. definition (3.33)). For this purpose, we use equations (3.18a) and decomposition (3.20) to get

\[
(\partial_i W_2, \partial_i W_1, 0)^T = \bar{J}^T \bar{f} - A_i W - \sum_{i=0,2,3} A_i \partial_i W - A_i^{(0)} \partial_i W, \tag{3.46}
\]

which immediately implies

\[
Q_{\alpha}(t) \lesssim \sum_{i=0,2,3} \|D_i^{\alpha - e}(\bar{J}^T \bar{f}, A_i W, A_i \partial_i W, A_i^{(0)} \partial_i W)\|_{L^2(\Sigma_t)}^2, \tag{3.47}
\]

where \( e := (0, 0, 0, 1) \) and \( \partial_0 := \partial_t. \) Use Lemma 3.4 and (3.39) to infer

\[
\|D_i^{\alpha - e}(\bar{J}^T \bar{f}, A_i W)\|_{L^2(\Sigma_t)}^2 \leq C(K)M(t). \tag{3.48}
\]

For \( i = 0, 2, 3, \) we have

\[
\|D_i^{\alpha - e}(A_i \partial_i W)\|_{L^2(\Sigma_t)}^2 \lesssim C(K)\|D_i^{\alpha - e} \partial_i W\|_{L^2(\Sigma_t)}^2 + \sum_{0 < \alpha' \leq \alpha - e} \mathcal{K}_{\alpha'}^{(i)}, \tag{3.49}
\]

where

\[
\mathcal{K}_{\alpha'}^{(i)} := \|D_i^{\alpha'} A_i D_i^{\alpha - e - \alpha'} \partial_i W\|_{L^2(\Sigma_t)}^2 \quad \text{for} \quad \alpha' \leq \alpha - e.
\]

To estimate the first term on the right-hand side of (3.49), one can employ the argument of passing to the volume integral (cf. [39, pp. 273–274]). But
we propose here another way to deal with this term. More precisely, we use integration by parts to obtain
\[ \sum_{i=0,2,3} \|D_\epsilon^{\alpha-e} \partial_i W\|_{L^2(\Sigma_t)}^2 = \mathcal{K}_1 + \mathcal{K}_2, \]
\[ (3.50) \]
where
\[ \mathcal{K}_1 := \int_\Sigma D_\epsilon^{\alpha-e} \partial_i W \cdot D_\epsilon^{\alpha-e} W, \quad \mathcal{K}_2 := - \sum_{i=0,2,3} \int_\Sigma D_\epsilon^{\alpha-e} \partial_i^2 W \cdot D_\epsilon^{\alpha-e} W. \]

Utilize integration by parts and Lemma 3.4 to derive
\[ \mathcal{K}_1 \lesssim \epsilon \|D_\epsilon^{\alpha-e} \partial_i W\|_{L^2(\Sigma)}^2 + C(\epsilon) \|D_\epsilon^{\alpha-e} W\|_{L^2(\Sigma)}^2 \]
\[ \lesssim \epsilon \|D_\epsilon^{\alpha-e} W\|_{H^1(\Sigma)}^2 + C(\epsilon) \|D_\epsilon^{\alpha-e} W\|_{L^2(\Sigma)}^2 \]
\[ \lesssim \epsilon \sum_{\langle \beta \rangle \leq m} \|D_\epsilon^{\alpha} W(t)\|_{L^2(\Omega)}^2 + C(\epsilon) \|W\|_{L^2(\Omega)}^2 \]
\[ (3.51) \]
where the norm \( \| \cdot \|_{H^2(\Omega)} \) is defined by (3.11). Using Lemma 3.4 yields
\[ \mathcal{K}_2 \lesssim \sum_{i=0,2,3} \|D_\epsilon^{\alpha-e} \partial_i^2 W\|_{H^{-1}(\Sigma_t)} \|D_\epsilon^{\alpha-e} W\|_{H^1(\Sigma_t)} \lesssim \|W\|_{m,T}^2. \]
\[ (3.52) \]
For \( i = 0, 2, 3 \), we apply Lemma 3.4 and inequalities (3.25)–(3.27) to obtain
\[ \sum_{0 \leq \alpha' \leq \alpha-e} \mathcal{K}_{\alpha'}^{(i)} \lesssim \sum_{\langle \beta \rangle = 1, \beta \leq \alpha' \leq \alpha-e} \|D_\epsilon^{\alpha'-\beta} (D_\epsilon^\beta A_i) D_\epsilon^{\alpha-e-\alpha'} \partial_i W\|_{L^2(\Sigma_t)}^2 \]
\[ \leq C(K) \left( \|W\|_{m,T}^2 + \hat{C}_{m+2} \|W\|_{L^2(\Omega)}^2 \right) \]
\[ (3.53) \]
due to \( \langle \alpha' - \beta \rangle + \langle \alpha - e - \alpha' \rangle + 1 \leq m - 2 \). Since \( (D_\epsilon^\beta A_i^{(0)}) |_{\Sigma_T} = 0 \) for all \( \beta = (\beta_0, \beta_1, \beta_2, \beta_3, \beta_4) \) with \( \beta_4 = 0 \), we get
\[ \|D_\epsilon^{\alpha-e} (A_i^{(0)} \partial_1 W)\|_{L^2(\Sigma_t)}^2 \lesssim \sum_{e \leq \beta \leq \alpha-e} \|D_\epsilon^\beta A_i^{(0)} D_\epsilon^{\alpha-e-\beta} \partial_1 W\|_{L^2(\Sigma_t)}^2 \]
\[ \lesssim \sum_{e \leq \beta \leq \alpha-e} \|D_\epsilon^{\alpha-e} (\partial_1 A_i^{(0)}) D_\epsilon^{\alpha-e-\beta} W\|_{L^2(\Sigma_t)}^2 \leq C(K) \mathcal{M}(t). \]
\[ (3.54) \]
Plug (3.48)–(3.54) into (3.47) and combine the resulting estimate with (3.31)–(3.32), (3.44), and (3.35) to derive (3.45) and finish the proof of the lemma. \( \square \)

The estimate of the weighted derivatives \( D_\epsilon^{\alpha} W \) for \( \alpha_1 > 0 \) and \( \langle \alpha \rangle \leq m \) is provided in the following lemma.

**Lemma 3.7.** If the assumptions in Theorem 3.1 are fulfilled, then
\[ \sum_{\langle \alpha \rangle \leq m, \alpha_1 > 0} \|D_\epsilon^{\alpha} W(t)\|_{L^2(\Omega)}^2 \leq C(K) \mathcal{M}(t) \]
\[ (3.55) \]
holds for all \( t \in [0,T] \) and \( m \in \mathbb{N}_+ \), where \( \mathcal{M}(t) \) is defined by (3.36).
Proof. Let $\alpha_1 > 0$ and $(\alpha) \leq m$. Then it is easy to check that $Q_\alpha(t) = 0$. Since estimates (3.38)–(3.43) are still valid, it suffices to make the estimate of differences $[D^m_\alpha, A_i] \partial_t W - [D^n_\alpha, A_i] \partial_t W$ for $i = 0, \ldots, 3$. In the following computations, we can replace the operator $D^m_\alpha$ by $\sigma_1 \partial_1^{\alpha_1} + \alpha_0 \partial_t^{\alpha_0} \partial_2^{\alpha_2} \partial_3^{\alpha_3}$ because the corresponding norms are equivalent; see, for instance, [30, p. 394]. In view of decomposition (3.20), we compute

\[
\int\|[D^m_\alpha, A_i] \partial_t W - [D^n_\alpha, A_i] \partial_t W\|^2_{L^2(\Omega_t)} \lesssim \|A_i \sigma_1^{-1} \partial_1^{\alpha_1} + \partial_t^{\alpha_0} \partial_2^{\alpha_2} \partial_3^{\alpha_3} W\|^2_{L^2(\Omega_t)}
\]

where $W_{nc} = (W_1, W_2)^T$. Use identity (3.46) to infer

\[
\|\partial_t W_{nc}\|_{m-1, t} \lesssim \sum_{i=0,2,3} \langle J^T f, A_1 W, A_i \partial_t W, A_1^{(0)} \partial_t W \rangle_{m-1, t}.
\]

By definition, we deduce

\[
\|A_1^{(0)} \partial_t W\|^2_{m-1, t} \lesssim \sum_{\beta' \leq \beta, (\beta) \leq m-1} \|D^\beta_{n, \alpha} A_1^{(0)} D^{\beta-\beta'}_{m, \alpha} \partial_t W\|^2_{L^2(\Omega_t)}.
\]

We use (3.20) to get

\[
\|A_1^{(0)} D^\beta_{n, \alpha} \partial_t W\|^2_{L^2(\Omega_t)} \lesssim \|\sigma D^\beta_{n, \alpha} \partial_t W\|^2_{L^2(\Omega_t)} \lesssim \|W\|^2_{m, t}.
\]

Applying the Moser-type calculus inequalities (3.25)–(3.27) leads to

\[
\|D^\beta_{n, \alpha} A_1^{(0)} D^{\beta-\beta'}_{m, \alpha} \partial_t W\|^2_{L^2(\Omega_t)} \lesssim \sum_{(\gamma) = 1, (\gamma) \leq \beta'} \|D^{\beta-\gamma}_{n, \alpha} (D^\gamma_{n, \alpha} A_1^{(0)}) D^{\beta-\beta'}_{m, \alpha} \partial_t W\|^2_{L^2(\Omega_t)}
\]

owing to $(\beta' - \gamma) + (\beta - \beta') + 2 \leq m$. The terms $\|A_i \partial_t W\|_{m-1, t}$ in (3.57), for $i = 0, 2, 3$, can be estimated by means of the Moser-type calculus inequalities (3.25)–(3.27). Then we combine (3.56)–(3.60), (3.38)–(3.43), and (3.31)–(3.32) to obtain estimate (3.55) and conclude the proof of the lemma.

\[3.5 \text{ Proof of Theorem 3.1}\]

We first show the tame $a$ priori estimate for problem (3.18) in $H^m_\alpha(\Omega_T)$. By definition, for $s \in \mathbb{N}$, we get

\[
\|\hat{\psi}\|_{H^s(\Sigma)} \leq \|\hat{\psi}\|_{s, t} \lesssim \|\hat{\psi}\|_{H^s(\Sigma)} , \|\psi\|_{H^s(\Sigma)} \leq \|\psi\|_{s, t} \lesssim \|\psi\|_{H^s(\Sigma)}.
\]

Combine estimates (3.35), (3.45), and (3.55), and take $\varepsilon > 0$ small enough to derive

\[
I(t) := \sum_{(\alpha) \leq m} \|D^\alpha_\alpha W(t)\|^2_{L^2(\Omega)} + \sum_{(\alpha) \leq m, \alpha_0 = \alpha_4 = 0} \|D^\alpha_\alpha \psi(t)\|^2_{L^2(\Sigma)} \leq C(K)M(t).
\]

Noting from (3.10), (3.12), and (3.18d) that

\[
\int_0^T I(s) \, ds = \|W\|^2_{m, t} + \|\psi\|^2_{H^m(\Sigma)},
\]
we apply Grönwall’s inequality to obtain
\[ I(t) \leq C(K)e^{C(K)t}N(T) \quad \text{for } 0 \leq t \leq T, \tag{3.63} \]
where
\[ N(t) := \|\hat{f}\|_{m,s,T}^2 + \hat{C}_{m+3}\left( \|\hat{f}\|_{W^1,\infty(\Omega_t)}^2 + \|W\|_{W^2,\infty(\Omega_t)}^2 + \|\psi\|_{L^\infty(\Sigma_t)}^2 \right). \]

By virtue of the embedding inequalities (3.28)–(3.29), we have
\[ N(T) \lesssim \|\hat{f}\|_{m,T}^2 + \hat{C}_{m+3}\left( \|\hat{f}\|_{4,s,T}^2 + \|W\|_{6,s,T}^2 + \|\psi\|_{H^2(\Sigma_T)}^2 \right). \tag{3.64} \]

Integrate (3.63) over \([0,T]\), use (3.62) and (3.64), and take \(T > 0\) sufficiently small to discover
\[ \|W\|_{m,T}^2 + \|\psi\|_{H^m(\Sigma_T)}^2 \leq C(K)T e^{C(K)T} \left\{ \|\hat{f}\|_{m,s,T}^2 + \|(\hat{V},\hat{\Psi})\|_{m+3,s,T}^2 \times \left( \|\hat{f}\|_{4,s,T}^2 + \|W\|_{6,s,T}^2 + \|\psi\|_{H^2(\Sigma_T)}^2 \right) \right\}, \tag{3.65} \]
for \(m \geq 6\). In view of (3.65) for \(m = 6\), assumption (3.13), estimate (3.61), and the embedding \(H^2(\Omega_T) \hookrightarrow W^{3,\infty}(\Omega_T)\), we can find a sufficiently small constant \(T_0 > 0\), depending on \(K_0\), such that if \(0 < T \leq T_0\), then
\[ \|W\|_{6,s,T}^2 + \|\psi\|_{H^6(\Sigma_T)}^2 \leq C(K_0)\|\hat{f}\|_{6,s,T}^2. \tag{3.66} \]

Plugging (3.66) into (3.65) yields
\[ \|W\|_{m,T}^2 + \|\psi\|_{H^m(\Sigma_T)}^2 \leq C(K_0) \left( \|\hat{f}\|_{m,s,T}^2 + \|(\hat{V},\hat{\Psi})\|_{m+3,s,T}^2 \|\hat{f}\|_{6,s,T}^2 \right). \tag{3.67} \]

According to Theorem 3.2, for \((f,g) \in L^2(\Omega_T) \times H^1(\Sigma_T)\) vanishing in the past, problem (3.18) has a unique solution \((W,\psi) \in L^2(\Omega_T) \times L^2(\Sigma_T)\). Applying the arguments in [3, Chapter 7] and using the tame estimate (3.67), one can establish the existence and uniqueness of solutions of problem (3.18) in \(H^m(\Omega_T) \times H^m(\Sigma_T)\) for \(m \geq 6\).

It suffices to show the tame estimate (3.14) for problem (3.9). For this purpose, we utilize (3.15) and Lemma 3.2 to get the estimates
\[ \|\hat{V}\|_{m,T}^2 \leq C(K_0) \left\{ \|W\|_{m,T}^2 + \|W\|_{W^1,\infty(\Omega_T)}^2 \hat{C}_{m+1} + \|g\|_{H^m(\Sigma_T)}^2 \right\}, \]
\[ \|\hat{f}\|_{m,T}^2 \leq C(K_0) \left\{ \|f\|_{m,T}^2 + \|V\|_{W^2,\infty(\Omega_T)}^2 \hat{C}_{m+1} + \|g\|_{H^{m+1}(\Sigma_T)}^2 \right\}, \]
which together with (3.28)–(3.29), (3.67), and (3.61) imply the desired tame estimate (3.14). This completes the proof of Theorem 3.1.

4 Nash–Moser Iteration

This section is devoted to solving the nonlinear problem (2.13) by an appropriate Nash–Moser iteration scheme. See Alinhac–Gérard [2, Chapter 3] and Secchi [34] for a general description of the method.
4.1 Approximate Solution

To apply Theorem 3.1, that is a well-posedness result in the space of functions vanishing in the past, we should reduce the nonlinear problem \((2.13)\) to the case with zero initial data. For this purpose, we introduce the so-called approximate solutions to absorb the initial data into the interior equations.

Let \(m \geq 3\) be an integer. Assume that initial data \(U_0\) and \(\varphi_0\) satisfy \(\tilde{U}_0 := U_0 - \underline{U} \in H^{m+3/2}(\Omega)\) and \(\varphi_0 \in H^{m+2}(\mathbb{T}^{d-1})\). Define

\[
\tilde{\Psi}_0 := \chi(x_1)\varphi_0, \quad \tilde{\Phi}_0 := x_1 + \tilde{\Psi}_0,
\]

where \(\chi \in C_0^\infty(\mathbb{R})\) satisfies \((2.12)\). Without loss of generality, we assume that \(\|\varphi_0\|_{L^\infty(\mathbb{T}^{d-1})} \leq 1/4\). Then we have

\[
\partial_t \tilde{\Phi}_0 \geq \frac{3}{4} \quad \text{in } \Omega.
\]

Let us denote \(\tilde{U} := U - \underline{U}\) and define

\[
\tilde{U}(j) := \partial_t^j \tilde{U}|_{t=0}, \quad \varphi(j) := \partial_t^j \varphi|_{t=0}, \quad \tilde{\Psi}(j) := \chi(x_1)\varphi(j) \quad \text{for } j \in \mathbb{N}.
\]

Then we get \(\tilde{U}(0) = \tilde{U}_0, \varphi(0) = \varphi_0,\) and \(\tilde{\Psi}(0) = \tilde{\Psi}_0\). To introduce the compatibility conditions, we shall determine the traces \(\tilde{U}(j)\) and \(\varphi(j)\) in terms of initial perturbations \(\tilde{U}_0\) and \(\varphi_0\) through equations \((2.13a)\) and the first boundary condition in \((2.13b)\). More precisely, applying the operator \(\partial_t^j\) to the first equation in \((2.13b)\), taking the traces at the initial time and employing the Leibniz’s rule yield

\[
\varphi(j+1) = \varphi(0) - \sum_{k=0}^j \sum_{i=2}^d \binom{j}{k} \partial_t^i \varphi(0) v_i(0)|_{x_1=0},
\]

where \(\binom{j}{k}\) is the binomial coefficient. Setting \(\mathcal{W} := (\tilde{U}, \nabla \tilde{U}, \nabla \tilde{\Psi}) \in \mathbb{R}^{2d^2+5d+2}\) and assuming that the hyperbolicity condition \((2.3)\) is satisfied, we can rewrite equations \((2.13a)\) as

\[
\partial_t \tilde{U} = \mathcal{G} (\mathcal{W}),
\]

where \(\mathcal{G}\) is a \(C^\infty\)-function that vanishes at the origin. We employ the generalized Faà di Bruno’s formula (see [26, Theorem 2.1]) to obtain

\[
\tilde{U}(j+1) = \sum_{\alpha_k \in \mathbb{N}^{2d^2+5d+2}} D^{\alpha_1+\ldots+\alpha_j} \mathcal{G}(\mathcal{W}(0)) \prod_{k=1}^j \frac{j!}{\alpha_k!} \left(\frac{\mathcal{W}(k)}{k!}\right)^{\alpha_k},
\]

where \(\mathcal{W}(k) := (\tilde{U}(k), \nabla \tilde{U}(k), \nabla \tilde{\Psi}(k))\). From \((4.3)\) and \((4.5)\), we can determine \(\tilde{U}(j)\) and \(\varphi(j)\) for integers \(j\) inductively as functions of \(\tilde{U}_0, \varphi_0,\) and their space derivatives up to order \(j\). Moreover, we have the following lemma (see [25, Lemma 4.2.1] for the proof).
Lemma 4.1. Let $m \in \mathbb{N}$ with $m \geq 3$ and $(\tilde{U}_0, \varphi_0) \in H^{m+3/2}(\Omega) \times H^{m+2}(\mathbb{T}^{d-1})$. Then equations (4.3) and (4.5) determine functions $(\tilde{U}(j), \varphi(j)) \in H^{m+3/2-j}(\Omega) \times H^{m+2-j}(\mathbb{T}^{d-1})$, for $j = 1, \ldots, m$, satisfying

$$\sum_{j=0}^{m} \left( \|\tilde{U}(j)\|_{H^{m+3/2-j}(\Omega)} + \|\varphi(j)\|_{H^{m+2-j}(\mathbb{T}^{d-1})} \right) \leq CM_0,$$

(4.6)

with $M_0 := \|\tilde{U}_0\|_{H^{m+3/2}(\Omega)} + \|\varphi_0\|_{H^{m+2}(\mathbb{T}^{d-1})}$, (4.7)

where the constant $C > 0$ depends only on $m$, $\|\tilde{U}_0\|_{W^{1,\infty}(\Omega)}$, and $\|\varphi_0\|_{W^{1,\infty}(\mathbb{T}^{d-1})}$.

To construct a smooth approximate solution, it is necessary to impose certain compatibility conditions on the initial data.

Definition 4.1. Let $m \in \mathbb{N}$ with $m \geq 3$. Suppose that $\tilde{U}_0 := U_0 - \tilde{U} \in H^{m+3/2}(\Omega)$ and $\varphi_0 \in H^{m+2}(\mathbb{T}^{d-1})$ satisfy (4.1)–(4.2). The initial data $(U_0, \varphi_0)$ are said to be compatible up to order $m$ if the functions $\tilde{U}(j)$ and $\varphi(j)$ determined by (4.3) and (4.5) satisfy

$$q(j)|_{x_1=0} = 0 \quad \text{for } j = 0, \ldots, m.$$ (4.8)

It follows from Lemma 4.1 that $\tilde{q}(j) \in H^1(\Omega)$ for $j = 0, \ldots, m$. Hence, it is permissible to consider the traces of these functions on the boundary $\{x_1 = 0\}$. The zero-th order compatibility condition, namely (4.8) with $j = 0$, comes from the second component of (2.13b).

Now we are in a position to construct the approximate solution.

Lemma 4.2. Let $m \in \mathbb{N}$ with $m \geq 3$. Assume that the initial data (2.13c) satisfy (2.3), (2.16), (2.19), the compatibility conditions up to order $m$, and $(\tilde{U}_0, \varphi_0) \in H^{m+3/2}(\Omega) \times H^{m+2}(\mathbb{T}^{d-1})$ for $\tilde{U}_0 := U_0 - \tilde{U}$. Then there exist positive constants $T_1(M_0)$ and $C(M_0)$ depending on $M_0$ (cf. (4.7)) such that if $0 < T \leq T_1(M_0)$, then there are functions $U^a$ and $\varphi^a$ satisfying

$$\|\tilde{U}^a\|_{H^{m+1}(\Omega_T)} + \|\varphi^a\|_{H^{m+5/2}(\Sigma_T)} \leq C(M_0) \quad \text{for } \tilde{U}^a := U^a - \tilde{U},$$

(4.9)

$$\rho(U^a) \in (\rho_\ast, \rho^\ast), \quad \partial_t \varphi^a \geq \frac{5}{8} \quad \text{in } \Omega_T,$$ (4.10)

where $\Phi^a := x_1 + \Psi^a$ with $\Psi^a := \chi(x_1)\varphi^a$. Moreover,

$$\partial_t^k \mathbb{B}(U^a, \varphi^a)|_{t=0} = 0 \quad \text{for } k = 0, \ldots, m-1, \quad \text{in } \Omega,$$ (4.11)

$$\mathbb{B}(U^a, \varphi^a) = 0, \quad H_N^a = 0, \quad \partial_t q^a \geq \frac{3\kappa_0}{4} > 0 \quad \text{on } \Sigma_T,$$ (4.12)

$$U^a|_{t=0} = U_0 \quad \text{in } \Omega, \quad \varphi^a|_{t=0} = \varphi_0 \quad \text{on } \Sigma,$$ (4.13)

$$\mathbb{L}_H(v^a, H^a, \varphi^a) = 0 \quad \text{in } \Omega_T,$$ (4.14)

where $H_N^a := H_1^a - \sum_{i=2}^d \partial_i \Psi^a H_i^a$, and $\mathbb{L}_H$ denotes the component for the magnetic field of $\mathbb{L}$, that is,

$$\mathbb{L}_H(v, H, \Phi) = (\partial_t^\Phi + v_i \partial_i^\Phi)H - H_i \partial_t^\Phi v + H \partial_t^\Phi v_i,$$ (4.15)

with $\partial_t^\Phi$ and $\partial_i^\Phi$ defined by (2.18).
Proof. We divide the proof into four steps.

Step 1. We take \( \varphi^a \in H^{m+5/2}(\mathbb{R} \times \mathbb{T}^{d-1}) \) and \((\tilde{v}_2^a, ..., \tilde{v}_d^a, \tilde{S}^a) \in H^{m+2}(\mathbb{R} \times \Omega) \) to satisfy
\[
\partial^k \varphi^a|_{t=0} = \varphi(k), \quad \partial^k (\tilde{v}_2^a, ..., \tilde{v}_d^a, \tilde{S}^a)|_{t=0} = (\tilde{v}_2(k), ..., \tilde{v}_d(k), \tilde{S}(k)),
\]
for \( k = 0, ..., m \). Set \( \Psi^a := \chi(x_1)\varphi^a \in H^{m+5/2}(\mathbb{R} \times \Omega) \) and \( \Phi^a := x_1 + \Psi^a \).

Step 2. In view of the compatibility conditions (4.8), we can apply the lifting result in [23, Theorem 2.3] to find \( \tilde{q}^a \in H^{m+2}(\mathbb{R} \times \Omega) \) such that
\[
q^a = 0 \quad \text{on } \Sigma, \quad \partial^k \tilde{q}^a|_{t=0} = \tilde{q}(k) \quad \text{in } \Omega \quad \text{for } k = 0, ..., m.
\]
It follows from the trace theorem that
\[
w^a := \partial_t \varphi^a + \sum_{i=2}^d \partial_i \varphi^a v_i^a|_{x_1=0} \in H^{m+3/2}(\mathbb{R} \times \mathbb{T}^{d-1}).
\]
Thanks to (4.3), we can choose \( \tilde{v}_1^a \in H^{m+2}(\mathbb{R} \times \Omega) \) such that
\[
v_1^a = w^a \quad \text{on } \Sigma, \quad \partial_t \tilde{v}_1^a|_{t=0} = v_{1(k)} \quad \text{in } \Omega \quad \text{for } k = 0, ..., m.
\]

As a direct consequence, we obtain the first identity in (4.12).

Step 3. Noting that \( \tilde{v}^a \in H^{m+2}(\mathbb{R} \times \Omega) \) and \( \Psi^a = \chi(x_1)\varphi^a \in H^{m+5/2}(\mathbb{R} \times \Omega) \) have been already specified, we take \( \tilde{H}^a \in H^{m+1}(\mathbb{R} \times \Omega) \) as the unique solution of equations (4.14) supplemented with the initial data \( \tilde{H}^a|_{t=0} = \tilde{H} \). Since the second identity in (4.12) is fulfilled at \( t = 0 \), similar to the proof of Proposition 2.1, we can deduce that the second identity in (4.12) holds for all \( t \in \mathbb{R} \) by considering the restriction of equations (4.14) to the boundary \( \Sigma \).

Step 4. We now have obtained (4.13)–(4.14) and the first two identities in (4.12). Estimate (4.9) follows from (4.6) and the continuity of the lifting operators. We deduce (4.10) and the third relation in (4.12) by taking \( T > 0 \) small enough. Equations (4.5) imply (4.11). The proof of the lemma is complete. \( \square \)

The vector function \((U^a, \varphi^a)\) in Lemma 4.2 is called the approximate solution to problem (2.13). Let us define
\[
f^a := \begin{cases} 
- \mathbb{L}(U^a, \Phi^a) & \text{if } t > 0, \\
0 & \text{if } t < 0.
\end{cases}
\]
Since \((\tilde{U}^a, \varphi^a) \in H^{m+1}(\Omega_T) \times H^{m+5/2}(\Sigma_T)\), we obtain from (4.9) and (4.11) that \( f^a \in H^m(\Omega_T) \) and
\[
||f^a||_{H^m(\Omega_T)} \leq \delta_0(T),
\]
where \( \delta_0(T) \to 0 \) as \( T \to 0 \). Estimate (4.17) results from the Moser-type calculus and embedding inequalities.

Let \((U^a, \Phi^a)\) be the approximate solution defined in Lemma 4.2. By virtue of (4.11)–(4.13) and (4.16), we find that \((U, \varphi) = (U^a, \varphi^a) + (V, \psi)\) is a solution of the nonlinear problem (2.13) on \([0, T] \times \Omega\), provided \( V, \psi, \) and \( \Psi := \chi(x_1)\psi \) solve
\[
\begin{cases}
\mathcal{L}(V, \Psi) := \mathbb{L}(U^a + V, \Phi^a + \Psi) - \mathbb{L}(U^a, \Phi^a) = f^a & \text{in } \Omega_T,\\
\mathcal{B}(V, \psi) := \mathbb{B}(U^a + V, \varphi^a + \psi) = 0 & \text{on } \Sigma_T,\\
(V, \psi) = 0, & \text{if } t < 0.
\end{cases}
\]

Thanks to (4.12), we find that $(V, \psi) = 0$ satisfies (4.18) for $t < 0$. Therefore, the original problem on $[0, T] \times \Omega$ is reformulated as a problem in $\Omega_T$ whose solutions vanish in the past.

### 4.2 Iteration Scheme

We first quote the following result on the smoothing operators from [39, Proposition 10].

**Proposition 4.3.** Let $T > 0$ and $m \in \mathbb{N}$ with $m \geq 3$. Let $\mathcal{F}_s^*(\Omega_T) := \{ u \in H^s(\Omega_T) : u = 0 \text{ for } t < 0 \}$. Then there exists a family $\{ S_\theta \}_{\theta \geq 1}$ of smoothing operators from $\mathcal{F}_s^*(\Omega_T)$ to $\bigcap_{s \geq 3} \mathcal{F}_s^*(\Omega_T)$, such that

\[
\begin{align*}
\| S_\theta u \|_{k,*,T} &\leq C\theta^{(k-j)+}\| u \|_{j,*,T} \quad \text{for } k, j = 1, \ldots, m, \\
\| S_\theta u - u \|_{k,*,T} &\leq C\theta^{k-j}\| u \|_{j,*,T} \quad \text{for } 1 \leq k \leq j \leq m, \\
\frac{d}{d\theta} S_\theta u &\leq C\theta^{k-j-1}\| u \|_{j,*,T} \quad \text{for } k, j = 1, \ldots, m,
\end{align*}
\]

where $k$ and $j$ are integers, $(k-j)+ := \max\{0, k-j\}$, and constant $C$ depends only on $m$. Furthermore, there is another family of smoothing operators (still denoted by $S_\theta$) acting on the functions defined on boundary $\Sigma_T$ and satisfying the properties in (4.19) with norms $\| \cdot \|_{H^s(\Sigma_T)}$.

Now we follow [7, 39] to describe the iteration scheme for problem (4.18).

**Assumption (A-1):** Set $(V_0, \psi_0) = 0$. Let $(V_k, \psi_k)$ be given and vanish in the past, and set $\Psi_k := \chi(x_1)\psi_k$, for $k = 0, \ldots, n$.

We consider

\[ V_{n+1} = V_n + \delta V_n, \quad \psi_{n+1} = \psi_n + \delta \psi_n, \quad \delta \Psi_n := \chi(x_1)\delta \psi_n, \]

where the differences $\delta V_n$ and $\delta \psi_n$ will be specified via the effective linear problem

\[
\begin{align*}
\mathcal{L}_e(U^n + V_{n+1/2}, \Phi^a + \Psi_{n+1/2})\delta V_n &= f_n \quad \text{in } \Omega_T, \\
\mathcal{B}_e(U^n + V_{n+1/2}, \varphi^a + \psi_{n+1/2}) (\delta V_n, \delta \psi_n) &= g_n \quad \text{on } \Sigma_T, \\
(\delta \hat{V}_n, \delta \hat{\psi}_n) &= 0 \quad \text{for } t < 0,
\end{align*}
\]

with $\Psi_{n+1/2} := \chi(x_1)\psi_{n+1/2}$. Here $\delta \hat{V}_n$ is the good unknown (cf. (3.7)), i.e.,

\[ \delta \hat{V}_n := \delta V_n - \frac{\partial_t (U^n + V_{n+1/2})}{\partial_t (\Phi^a + \Psi_{n+1/2})} \delta \Psi_n, \]

and $(V_{n+1/2}, \psi_{n+1/2})$ is a smooth modified state such that $(U^n + V_{n+1/2}, \varphi^a + \psi_{n+1/2})$ satisfies constraints (3.1)–(3.4); see Proposition 4.8 for the detailed construction and estimate. The source terms $f_n$ and $g_n$ will be defined through the accumulated error terms at Step $n$ later on.

**Assumption (A-2):** Set $f_0 := S_{\theta_0} f^0$ and $(e_0, \tilde{e}_0, g_0) := 0$ for $\theta_0 \geq 1$ sufficiently large, and let $(f_k, g_k, e_k, \tilde{e}_k)$ be given and vanish in the past for $k = 1, \ldots, n-1$.

Under Assumptions (A-1)–(A-2), we can describe our iteration scheme as follows. First we compute the accumulated error terms at Step $n$ for $n \geq 1$ by

\[ E_n := \sum_{k=0}^{n-1} e_k, \quad \tilde{E}_n := \sum_{k=0}^{n-1} \tilde{e}_k. \]
Then we calculate terms \( f_n \) and \( g_n \) from
\[
\sum_{k=0}^{n} f_k + S_{\theta_n} E_n = S_{\theta_n} f^n, \quad \sum_{k=0}^{n} g_k + S_{\theta_n} \tilde{E}_n = 0,
\]
where \( S_{\theta_n} \) are the smoothing operators given in Proposition 4.3 with the sequence \( \{\theta_n\} \) defined by
\[
\theta_0 \geq 1, \quad \theta_n = \sqrt{\theta_0^2 + n}.
\]

Once \( f_n \) and \( g_n \) are specified, we can apply Theorem 3.1 to solve \((\delta \hat{V}_n, \delta \psi_n)\) from problem (4.21). Then, we obtain the function \( \delta V_n \) and \((V_{n+1}, \psi_{n+1})\) from (4.22) and (4.20).

The error terms at Step \( n \) are defined through the following decompositions:
\[
\mathcal{L}(V_{n+1}, \Psi_{n+1}) = \mathcal{L}(V_n, \Psi_n) + \mathcal{B}(V_{n+1}, \psi_{n+1}) - \mathcal{B}(V_n, \psi_n)
\]
\[
= \mathcal{B}(U^n + V_n, \Psi_n, \delta V_n, \delta \psi_n) + e_n
\]
\[
= \mathcal{B}(U^n + V_n, \Psi_n, \delta V_n, \delta \psi_n) + \tilde{e}'_n + e''_n + e'''_n,
\]
with
\[
D_{n+1/2} := \frac{1}{\partial_1 \mathcal{B}(U^n + V_{n+1/2}, \Psi_n, \delta V_{n+1/2})} \partial_1 \mathcal{B}(U^n + V_{n+1/2}, \Psi_n, \delta V_{n+1/2}).
\]
where we utilize (3.8) to obtain the last identity in (4.26). Setting
\[
e_n := e'_n + e''_n + e'''_n + D_{n+1/2} \delta \Psi_n, \quad \tilde{e}_n := \tilde{e}'_n + \tilde{e}''_n + \tilde{e}'''_n,
\]
we complete the description of the iteration scheme.

Summing (4.26) and (4.27) from \( n = 0 \) to \( N \), respectively, we use (4.21) and (4.23)–(4.24) to find
\[
\mathcal{L}(V_{N+1}, \Psi_{N+1}) = \sum_{n=0}^{N} f_n + E_{N+1} = S_{\theta_N} f^n + (I - S_{\theta_N}) E_N + e_N,
\]
\[
\mathcal{B}(V_{N+1}, \psi_{N+1}) = \sum_{n=0}^{N} g_n + \tilde{E}_{N+1} = (I - S_{\theta_N}) \tilde{E}_N + \tilde{e}_N.
\]
Since \( S_{\theta_N} \to \text{Id} \) as \( N \to \infty \), we can formally obtain the solution to problem (4.18) from \( \mathcal{L}(V_{N+1}, \Psi_{N+1}) \to f^n \) and \( \mathcal{B}(V_{N+1}, \psi_{N+1}) \to 0 \), provided \( (e_N, \tilde{e}_N) \to 0 \) as \( N \to \infty \).
4.3 Inductive Hypothesis

Let \( m \in \mathbb{N} \) with \( m \geq 12 \) and \( \bar{\alpha} := m - 4 \). Suppose that the initial data (2.13c) satisfy \( \tilde{U}_0, \varphi_0 \in H^{m+3/2}(\Omega) \times H^{m+2}(\mathbb{T}^d-1) \) for \( \tilde{U}_0 := U_0 - \overline{U} \). Thanks to Lemma 4.2, the following estimates hold:

\[
\| \tilde{U}^a \|_{H^{\bar{\alpha}+\varepsilon}(\Omega_T)} + \| \varphi^a \|_{H^{\bar{\alpha}+13/2}(\Sigma_T)} \leq C(M_0), \quad \| f^a \|_{H^{\bar{\alpha}+4}(\Omega_T)} \leq \delta_0(T),
\]

where \( M_0 \) is defined by (4.7) and \( \delta_0(T) \to 0 \) as \( T \to 0 \). Suppose further that Assumptions (A-1)–(A-2) are fulfilled. For an integer \( \alpha \geq 7 \) and a real number \( \varepsilon > 0 \), our inductive hypothesis reads

\[
(H_{n-1}) \begin{cases}
(a) \| (\delta V_k, \delta \Psi_k) \|_{s,s,T} + \| \delta \psi_k \|_{H^s(\Sigma_T)} \leq \varepsilon \theta_k^{(s-\alpha)+} & \text{if } s \neq \alpha, \\
& \varepsilon \log \theta_k & \text{if } s = \alpha,
(b) \| L(V_k, \Psi_k) - f^a \|_{s,s,T} \leq 2 \varepsilon \theta_k^{s-\alpha-1} & \text{for all } k \in \{0, \ldots, n-1\} \text{ and } s \in \{6, \ldots, \bar{\alpha}\},
(c) \| B(V_k, \psi_k) \|_{H^s(\Sigma_T)} \leq \varepsilon \theta_k^{s-\alpha-1} & \text{for all } k \in \{0, \ldots, n-1\} \text{ and } s \in \{7, \ldots, \alpha\},
\end{cases}
\]

where \( \theta_k \) is defined by (4.25) and \( \Delta_k := \theta_{k+1} - \theta_k \). Note that \( 1/3 \leq \theta_k \Delta_k \leq 1/2 \) for all \( k \in \mathbb{N} \). We will choose \( \alpha, \bar{\alpha} > \alpha, \) and \( \varepsilon \) later on.

We aim to prove that hypothesis \((H_{n-1}) \) implies \((H_n) \) and that \((H_0) \) holds, provided \( T > 0 \) and \( \varepsilon > 0 \) are sufficiently small and \( \theta_0 \geq 1 \) is large enough. Supposing that hypothesis \((H_{n-1}) \) holds, we have the following result.

**Lemma 4.4** ([39, Lemma 7]). If \( \theta_0 \) is large enough, then

\[
\| (V_k, \Psi_k) \|_{s,s,T} + \| \psi_k \|_{H^s(\Sigma_T)} \leq \begin{cases} \varepsilon \theta_k^{(s-\alpha)+} & \text{if } s \neq \alpha, \\
\varepsilon \log \theta_k & \text{if } s = \alpha, \end{cases}
\]

for all \( k \in \{0, \ldots, n-1\} \) and \( s \in \{6, \ldots, \bar{\alpha}\} \). Furthermore,

\[
\|(I - S_{\theta_k})(V_k, \Psi_k)\|_{s,s,T} + \|(I - S_{\theta_k})\psi_k\|_{H^s(\Sigma_T)} \leq C \varepsilon \theta_k^{(s-\alpha)+} & \text{if } s \neq \alpha, \\
C \varepsilon \log \theta_k & \text{if } s = \alpha,
\]

for all \( k \in \{0, \ldots, n-1\} \) and \( s \in \{6, \ldots, \bar{\alpha} + 6\} \).

4.4 Estimates of the Error Terms

For deriving \((H_n) \) from \((H_{n-1}) \), in this subsection, we estimate the quadratic error terms \( \epsilon'_k \) and \( \tilde{\epsilon}'_k \), the first substitution error terms \( \epsilon''_k \) and \( \tilde{\epsilon}''_k \), the second substitution error terms \( \epsilon'''_k \) and \( \tilde{\epsilon}'''_k \), and the last error term \( D_{k+1/2} \delta \Psi_k \) (cf. (4.26)–(4.28)). First we find

\[
\epsilon'_k = \int_0^1 \mathbb{L}'(U^a + V_k + \tau \delta V_k, \Phi^a + \Psi_k \\
+ \tau \delta \Psi_k) ((\delta V_k, \delta \Psi_k), (\delta V_k, \delta \Psi_k)) (1 - \tau) \, d\tau,
\]

\[
\tilde{\epsilon}'_k = \frac{1}{2} \mathbb{B}'((\delta V_k, \delta \psi_k), (\delta V_k, \delta \psi_k)),
\]
where $L''$ and $B''$ are the second derivatives of operators $L$ and $B$ respectively. More precisely, we define

\[
L''(\bar{U}, \bar{\phi})(V, \Psi) := \frac{d}{d\theta} L'((\bar{U} + \theta \bar{V}, \bar{\phi} + \theta \bar{\psi}))(V, \Psi) \bigg|_{\theta=0},
\]

\[
B''((V, \psi), (\bar{V}, \bar{\psi})) := \frac{d}{d\theta} B'((\bar{U} + \theta \bar{V}, \phi + \theta \bar{\psi}))(V, \psi) \bigg|_{\theta=0},
\]

where the operators $L'$ and $B'$ are given in (3.5)-(3.6). For our problem, we compute

\[
B''((V, \psi), (\bar{V}, \bar{\psi})) = \left( \sum_{i=2}^{d} \left( \bar{v}_i \partial_i \psi + \partial_i \bar{\psi}_i \right) \right),
\]

(4.34)

To control the error terms, we need estimates for operators $L''$ and $B''$. These estimates can be obtained from the explicit forms of $L''$ and $B''$ by applying the Moser-type calculus inequalities in Lemmas 3.1 and 3.2. Omitting detailed calculations, we have the following proposition.

**Proposition 4.5.** Let $T > 0$ and $s \in \mathbb{N}$ with $s \geq 6$. Assume that $(\bar{V}, \bar{\psi}) \in H^{s+2}(\Omega_T)$ satisfies

\[
\| (\bar{V}, \bar{\psi}) \|_{W^{2,\infty}(\Omega_T)} \leq \tilde{K}
\]

for some constant $\tilde{K} > 0$. Then there exists a positive constant $C$, depending on $\tilde{K}$ but not on $T$, such that, if $(V_i, \psi_i) \in H^{s+2}(\Omega_T)$ and $(W_i, \psi_i) \in H^{s}(\Sigma_T) \times H^{s+1}(\Sigma_T)$ for $i = 1, 2$, then

\[
\| L''(\bar{U} + \bar{V}, x_1 + \bar{\psi})(V_1, \Psi_1), (V_2, \Psi_2) \|_{s,s,T} \leq C \| (\bar{U}, \bar{\psi}) \|_{s+2,s,T} \| (V_1, \Psi_1) \|_{W^{2,\infty}(\Omega_T)} \| (V_2, \Psi_2) \|_{W^{2,\infty}(\Omega_T)}
\]

\[
+ C \sum_{i \neq j} \| (V_i, \Psi_i) \|_{s+2,s,T} \| (V_j, \Psi_j) \|_{W^{2,\infty}(\Omega_T)},
\]

and

\[
\| B''(W_1, \psi_1), (W_2, \psi_2) \|_{H^s(\Sigma_T)} \leq C \sum_{i \neq j} \left\{ \| W_i \|_{H^s(\Sigma_T)} \| \psi_j \|_{W^{1,\infty}(\Sigma_T)} + \| W_i \|_{L^\infty(\Sigma_T)} \| \psi_j \|_{H^{s+1}(\Sigma_T)} \right\}.
\]

We first apply Proposition 4.5 to deduce the following estimate for the quadratic error terms $e_k'$ and $e_k''$.

**Lemma 4.6.** Let $\alpha \geq 7$. If $\varepsilon > 0$ is sufficiently small and $\theta_0 \geq 1$ is suitably large, then

\[
\| e_k' \|_{s,s,T} + \| e_k'' \|_{H^s(\Sigma_T)} \lesssim \varepsilon^2 \theta_k^{\alpha+1} \Delta_k,
\]

(4.35)

for all $k \in \{0, \ldots, n-1\}$ and $s \in \{6, \ldots, \tilde{\alpha} - 2\}$, where $\zeta(s) := \max\{(s + 2 - \alpha)_+ + 10 - 2\alpha, s + 6 - 2\alpha\}$. 


Proof. In view of assumption (4.30), hypothesis \((H_{n-1})\), and estimate (4.31), we utilize (3.29) to obtain
\[
\|\tilde{U}^\alpha, V_k, \delta V_k, \Psi^\alpha, \Psi_k, \delta \Psi_k\|_{W^{2,\infty}_s(\Omega_T)} \lesssim 1.
\]
Then we can apply Proposition 4.5 and use embedding inequalities (3.29), assumption (4.30), and hypothesis \((H_{n-1})\) to get
\[
\|\varepsilon'\|_{s,*,T} \lesssim \varepsilon^2 (\theta_k^{10-2\alpha} + \theta_k^{s+1-\alpha}) \Delta_k^2 \left(1 + \|\alpha, \Psi_k\|_{s+2,*,T} + \varepsilon \theta_k^{s+1-\alpha} \Delta_k \right)
\]
\[
+ \|\delta V_k, \delta \Psi_k\|_{s+2,*,T}(\|\delta V_k, \delta \Psi_k\|_{6,*,T}) \lesssim \varepsilon^2 (\theta_k^{10-2\alpha} + \theta_k^{s+1-\alpha}) \Delta_k^2 \left(1 + \|\alpha, \Psi_k\|_{s+2,*,T} + \varepsilon \theta_k^{s+7-2\alpha} \Delta_k \right)
\]
for all \(s \in \{6, \ldots, \bar{\alpha} - 2\}\).

If \(s + 2 \neq \alpha\), then it follows from (4.31) and \(2\theta_k \Delta_k \leq 1\) that
\[
\|\varepsilon'\|_{s,*,T} \lesssim \varepsilon^2 (\theta_k^{10-2\alpha} + \theta_k^{s+6-2\alpha}) \lesssim \varepsilon^2 \theta_k^{(s+1-\alpha)} \Delta_k.
\]
If \(s + 2 = \alpha\), then we use (4.31) and \(\alpha \geq 7\) to find
\[
\|\varepsilon'\|_{\alpha-2,*,T} \lesssim \varepsilon^2 (\theta_k^{11-2\alpha} + \theta_k^{4-\alpha}) \lesssim \varepsilon^2 \theta_k^{(s+2-\alpha)} \Delta_k.
\]
Employing Proposition 4.5 and Lemma 3.4 yields the estimate for \(\tilde{e}_k'\) and completes the proof of the lemma.

For the first substitution error terms \(\varepsilon''\) and \(\tilde{e}_k''\) defined in (4.26)–(4.27), we have the following result.

Lemma 4.7. Let \(\alpha \geq 7\). If \(\varepsilon > 0\) is sufficiently small and \(\theta_0 \geq 1\) is suitably large, then
\[
\|\varepsilon''\|_{s,*,T} + \|\tilde{e}_k''\|_{H^s(\Sigma_T)} \lesssim \varepsilon^2 \theta_k^s \Delta_k,
\]
for all \(k \in \{0, \ldots, n - 1\}\) and \(s \in \{6, \ldots, \bar{\alpha} - 2\}\), where
\[
\Theta_1(s) := \max\{(s + 2 - \alpha)_+, 12 - 2\alpha, s + 8 - 2\alpha\}.
\]

Proof. First we have
\[
\varepsilon''_k = \int_0^1 L''_\alpha(U^\alpha + S_{\theta_k} V_k + \tau(I - S_{\theta_k}) V_k, \Phi^\alpha + S_{\theta_k} \Psi_k + \tau(I - S_{\theta_k}) \Psi_k) d\tau,
\]
\[
\tilde{e}_k'' = B''(\delta V_k, \delta \Psi_k, \delta V_k, \delta \Psi_k, \delta V_k, \delta \Psi_k).
\]
Thanks to (4.32)–(4.33), we have
\[
\|(S_{\theta_k} V_k, S_{\theta_k} \Psi_k, \Psi_k)\|_{W^{2,\infty}_s(\Omega_T)} \lesssim \|(S_{\theta_k} V_k, S_{\theta_k} \Psi_k, \Psi_k)\|_{6,*,T} \lesssim 1.
\]
Then we apply Proposition 4.5 and use (3.29), (4.30), hypothesis \((H_{n-1})\), and (4.32) to get
\[
\|\varepsilon''\|_{s,*,T} \lesssim \varepsilon^2 \theta_k^{11-2\alpha} \Delta_k (1 + \|(S_{\theta_k} V_k, S_{\theta_k} \Psi_k)\|_{s+2,*,T}) + \varepsilon^2 \theta_k^{s+7-2\alpha} \Delta_k
\]
for all \(s \in \{6, \ldots, \bar{\alpha} - 2\}\). Similar to the proof of Lemma 4.6, analyzing cases \(s + 2 \neq \alpha\) and \(s + 2 = \alpha\) separately, we use (4.33) to deduce (4.36) and finish the proof. 

Let us construct the smooth modified state \((V_{n+1/2}, \psi_{n+1/2})\) so that \((U^a + V_{n+1/2}, \varphi^a + \psi_{n+1/2})\) satisfies constraints (3.1)–(3.4). Since the smooth modified state will be chosen to vanish in the past and the approximate solution satisfies (4.9)–(4.10) and (4.12), state \((U^a + V_{n+1/2}, \varphi^a + \psi_{n+1/2})\) will satisfy (3.1), (3.3), and (3.4) for \(T > 0\) small enough. Consequently, we only need to focus on the constraints (3.2).

**Proposition 4.8.** Let \(\alpha \geq 8\). Then there exist functions \(V_{n+1/2}\) and \(\psi_{n+1/2}\) vanishing in the past, such that \((U^a + V_{n+1/2}, \varphi^a + \psi_{n+1/2})\) satisfies (3.2), where \((U^a, \Phi^a)\) is the approximate solution constructed in Lemma 4.2. Furthermore,

\[
\begin{align*}
\psi_{n+1/2} &= S_{\theta_n} \psi_n, & v_{i,n+1/2} &= S_{\theta_n} v_{i,n} & \text{for } i = 2, \ldots, d, \\
\|S_{\theta_n} \psi_n - \psi_{n+1/2}\|_{s,s,T} &\lesssim \epsilon \theta_n^{s-h} & \text{for } s = 6, \ldots, \tilde{\alpha} + 6, \\
\|S_{\theta_n} V_n - V_{n+1/2}\|_{s,s,T} &\lesssim \epsilon \theta_n^{s+2-h} & \text{for } s = 6, \ldots, \tilde{\alpha} + 3.
\end{align*}
\]

**Proof.** The proof is divided into three steps.

**Step 1.** We define \(\psi_{n+1/2}\) and \(v_{i,n+1/2}\) for \(i = 2, \ldots, d\) by (4.38). Let \(\Psi_{n+1/2} := \chi(x_1)\psi_{n+1/2}\). Let us define \(q_{n+1/2} := S_{\theta_n} q_n\) and \(S_{n+1/2} := S_{\theta_n} S_n\).

If \(6 \leq s \leq \tilde{\alpha}\), then we use (4.19b) and (4.31) to get

\[
\|S_{\theta_n} \Psi_n - \Psi_{n+1/2}\|_{s,s,T} \lesssim \|S_{\theta_n} - I\|_{s,s,T} \|\chi(x_1)\psi_n\|_{s,s,T} \lesssim \epsilon \theta_n^{s-h} \left(\|\chi(x_1)\psi_n\|_{s,s,T} + \|\psi_n\|_{H^T(\Sigma_T)}\right) \lesssim \epsilon \theta_n^{s-h}.
\]

If \(\tilde{\alpha} < s \leq \tilde{\alpha} + 6\), then estimate (4.33) gives

\[
\|S_{\theta_n} \Psi_n - \Psi_{n+1/2}\|_{s,s,T} \lesssim \|S_{\theta_n} \Psi_n\|_{s,s,T} \lesssim \epsilon \theta_n^{s-h}.
\]

This finishes the proof of (4.39).

**Step 2.** Next we define and estimate \(v_{1,n+1/2}\) such that the first identity in (3.2) holds for \((U^a + V_{n+1/2}, \varphi^a + \psi_{n+1/2})\). Let us define

\[
v_{1,n+1/2} := S_{\theta_n} v_{1,n} + \mathfrak{R}_T \left(\hat{w}_n - (S_{\theta_n} v_{1,n})\right)_{x_1=0},
\]

with \(\hat{w}_n := \partial_1 \psi_{n+1/2} + \sum_{i=2}^d \left((v_{i}^a + v_{i,n+1/2})\partial_i \psi_{n+1/2} + v_{i,n+1/2}\partial_i \varphi^a\right)\big|_{x_1=0},\)

where \(\mathfrak{R}_T\) is the lifting operator given in Lemma 3.4. By virtue of (4.12), we infer that \((v^a + v_{1,n+1/2}, \varphi^a + \psi_{n+1/2})\) satisfies the first constraint in (3.2).

Use (4.18) and (4.38) to get

\[
\mathcal{B} \left(S_{\theta_n} V_n, S_{\theta_n} \psi_n\right)_1 = \hat{w}_n - (S_{\theta_n} v_{1,n})\big|_{x_1=0},
\]

which together with Lemma 3.4 implies

\[
\|v_{1,n+1/2} - S_{\theta_n} v_{1,n}\|_{s,s,T} \lesssim \|\mathcal{B} \left(S_{\theta_n} V_n, S_{\theta_n} \psi_n\right)_1\|_{H^{-1}(\Sigma_T)} \quad \text{for } s \geq 6.
\]

To estimate the right-hand side, we use the decomposition

\[
\mathcal{B} \left(S_{\theta_n} V_n, S_{\theta_n} \psi_n\right)_1 = \mathcal{H}_1 + \mathcal{H}_2 + S_{\theta_n} \mathcal{B} \left(V_{n-1}, \psi_{n-1}\right)_1.
\]
We decompose \( \mathcal{H}_1 := \mathcal{B}(\mathcal{S}_\theta V_n, \mathcal{S}_\theta \psi_n) - \mathcal{S}_\theta \mathcal{B}(V_n, \psi_n) \) as
\[
\mathcal{H}_1 = \left\{ \partial_t (\mathcal{S}_\theta \psi_n) - \mathcal{S}_\theta \partial_t \psi_n \right\} - \left\{ (\mathcal{S}_\theta v_{1, n}) |_{x_1 = 0} - \mathcal{S}_\theta (v_{1, n} |_{x_1 = 0}) \right\} \\
+ \sum_{i=2}^{d} \left\{ (\mathcal{S}_\theta v_{i, n} + v_i^a) |_{x_1 = 0} \partial_t \mathcal{S}_\theta \psi_n - \mathcal{S}_\theta ((v_{i, n} + v_i^a) |_{x_1 = 0} \partial_t \psi_n) \right\} \\
+ \sum_{i=2}^{d} \left\{ (\mathcal{S}_\theta v_{i, n} |_{x_1 = 0} \partial_t \varphi^a - \mathcal{S}_\theta (v_{i, n} |_{x_1 = 0} \partial_t \varphi^a)) \right\}.
\]

Noting that \( (\tilde{U}^a, \varphi^a) \in H^{s+5}_{a+5} (\Omega_T) \times H^{s+13/2} (\Sigma_T) \), we use Lemma 3.1, the Sobolev and trace embedding theorems, and (4.33) to infer
\[
\| (\mathcal{S}_\theta v_{i, n} + v_i^a) \partial_t \mathcal{S}_\theta \psi_n \|_{H^{s-1}(\Sigma_T)} \lesssim (1 + \| \mathcal{S}_\theta v_{i, n} + v_i^a \|_{6, s, T}) \| \mathcal{S}_\theta \psi_n \|_{H^s(\Sigma_T)} \\
+ \| \mathcal{S}_\theta (v_{i, n} + v_i^a) \|_{s, s, T} \| \mathcal{S}_\theta \psi_n \|_{H^s(\Sigma_T)} \\
\lesssim \varepsilon \theta_n^{s-a} \quad \text{for}\ s = \alpha + 1, \ldots, \tilde{\alpha} + 5.
\]

Since the other terms in \( \mathcal{H}_1 \) can be estimated as in the proof of [39, Proposition 12], we omit the details and obtain
\[
\| \mathcal{H}_1 \|_{H^{s-1}(\Sigma_T)} \lesssim \varepsilon \theta_n^{s-a} \quad \text{for}\ s = 6, \ldots, \tilde{\alpha} + 5.
\]

For the term \( \mathcal{H}_2 := \mathcal{S}_\theta (\mathcal{B}(V_n, \psi_n) - \mathcal{B}(V_{n-1}, \psi_{n-1})) \), we have
\[
\mathcal{H}_2 = \mathcal{S}_\theta (\partial_t \delta \psi_{n-1}) - \mathcal{S}_\theta (\delta v_{1, n-1} |_{x_1 = 0}) \\
+ \sum_{i=2}^{d} \mathcal{S}_\theta (v_{i, n} + v_i^a) |_{x_1 = 0} \partial_t \delta \psi_{n-1} + \delta v_{i, n-1} |_{x_1 = 0} \partial_t (\psi_{n-1} + \varphi^a)).
\]

Use (4.19a), hypothesis \( (H_{n-1}) \), and Lemma 3.1 to get
\[
\| \mathcal{H}_2 \|_{H^{s-1}(\Sigma_T)} \lesssim \varepsilon \theta_n^{s-a} \quad \text{for}\ s = 6, \ldots, \tilde{\alpha} + 5.
\]

Thanks to (4.19) and hypothesis \( (H_{n-1}) \), we have
\[
\| \mathcal{S}_\theta \mathcal{B}(V_{n-1}, \psi_{n-1}) \|_{H^{s-1}(\Sigma_T)} \lesssim \theta_n^{s-6} \| \mathcal{B}(V_{n-1}, \psi_{n-1}) \|_{H^s(\Sigma_T)} \lesssim \varepsilon \theta_n^{s-a}
\]
for \( s = 6, \ldots, \tilde{\alpha} + 5 \). In conclusion, we infer
\[
\| v_{n+1/2} - \mathcal{S}_\theta v_n \|_{s, s, T} \lesssim \varepsilon \theta_n^{s-a} \quad \text{for}\ s = 6, \ldots, \tilde{\alpha} + 5. \tag{4.41}
\]

Step 2. Finally we define and estimate \( H_{n+1/2} \) such that the second identity in (3.2) holds for \( (U^a + V_{n+1/2}, \varphi^a + \psi_{n+1/2}) \). Noting that \( v_{n+1/2} \) and \( \Psi_{n+1/2} \) have been specified, we take \( H_{n+1/2} \) as the unique solution of the linear transport equations
\[
\mathbb{L}_H (v^a + v_{n+1/2}, H^a + H_{n+1/2}, \varphi^a + \psi_{n+1/2}) = 0, \tag{4.42}
\]
supplemented with zero initial data \( H_{n+1/2} |_{t=0} = 0 \), where operator \( \mathbb{L}_H \) is defined by (2.13a). Since \( (v^a + v_{n+1/2}, \varphi^a + \psi_{n+1/2}) \) satisfies the first boundary condition in (3.2), equations (4.42) do not need to be supplemented with any boundary condition.
Noting that \( H_{n+1/2} \) and \( \psi_{n+1/2} \) vanish at the initial time, by virtue of the second identity in (4.12), one can show as the proof of Proposition 2.1 that 
\( (H^a + H_{n+1/2}, \psi^a + \psi_{n+1/2}) \) satisfies the second constraint (3.2).

We now estimate \( H_{n+1/2} - S_\theta H_n \). We first utilize (4.42) to find
\[
\mathbb{L}_H(v^a + v_{n+1/2}, H_{n+1/2} - S_\theta H_n, \Phi^a + \Psi_{n+1/2}) = \mathcal{H}_3 + \mathcal{H}_4 + \mathcal{H}_5, \tag{4.43}
\]
where \( \mathcal{H}_3 := -S_\theta \mathbb{L}_H(v^a + v_n, H^a + H_n, \Phi^a + \Psi_n) \), and
\[
\begin{align*}
\mathcal{H}_4 & := -\mathbb{L}_H(v^a + v_{n+1/2}, H^a + S_\theta H_n, \Phi^a + \Psi_{n+1/2}) \\
& \quad + \mathbb{L}_H(v^a + S_\theta v_n, H^a + S_\theta H_n, \Phi^a + S_\theta \Psi_n), \\
\mathcal{H}_5 & := -\mathbb{L}_H(v^a + S_\theta v_n, H^a + S_\theta H_n, \Phi^a + S_\theta \Psi_n) \\
& \quad + S_\theta \mathbb{L}_H(v^a + v_n, H^a + H_n, \Phi^a + \Psi_n).
\end{align*}
\]

Noting \((\tilde{U}^a, \psi^a) \in H^{\bar{\alpha}+5}(\Omega_T) \times H^{\bar{\alpha}+13/2}(\Sigma_T)\) and using the Moser-type calculus inequalities (3.25)–(3.27), we get from (4.19), (4.30)–(4.33), (4.39), and (4.41) that
\[
\|\mathcal{H}_4\|_{s,T} + \|\mathcal{H}_5\|_{s,T} \lesssim \varepsilon \theta_n^{s-\alpha+2} \quad \text{for} \ s = 6, \ldots, \bar{\alpha} + 3. \tag{4.44}
\]

Thanks to (4.20), we utilize Lemma 3.2, (4.19), (4.30)–(4.33), and hypothesis \( (H_{n-1}) \) to get
\[
\|\mathcal{H}_3 - S_\theta \mathbb{L}_H(v^a + v_{n-1}, H^a + H_{n-1}, \Phi^a + \Psi_{n-1})\|_{s,T} \lesssim \varepsilon \theta_n^{s-\alpha+2} \tag{4.45}
\]
for \( s = 6, \ldots, \bar{\alpha} + 3 \). By virtue of (4.19), (4.30)–(4.33), and hypothesis \( (H_{n-1}) \), we infer
\[
\|S_\theta \mathbb{L}_H(v^a + v_{n-1}, H^a + H_{n-1}, \Phi^a + \Psi_{n-1})\|_{s,T} \lesssim \theta_n^{-6}\|\mathcal{L}(v_{n-1}, \Psi_{n-1})\|_{6,T} \lesssim \varepsilon \theta_n^{s-\alpha-1} \quad \text{for} \ s = 6, \ldots, \bar{\alpha} + 3. \tag{4.46}
\]

Plugging estimates (4.44)–(4.46) into (4.43) yields
\[
\|\mathbb{L}_H(v^a + v_{n+1/2}, H_{n+1/2} - S_\theta H_n, \Phi^a + \Psi_{n+1/2})\|_{s,T} \lesssim \varepsilon \theta_n^{s-\alpha+2}
\]
for \( s = 6, \ldots, \bar{\alpha} + 3 \). Employing the standard argument of the energy method and the Moser-type calculus inequalities (3.25)–(3.27), we deduce
\[
\|H_{n+1/2} - S_\theta H_n\|_{s,T} \lesssim \varepsilon \theta_n^{s-\alpha+1}
\]
for \( s = 6, \ldots, \bar{\alpha} + 3 \). This completes the proof of the lemma.

The next lemma concerns the estimate for the second substitution error terms \( \varepsilon''''_k \) and \( \varepsilon''''_n \) given in (4.26)–(4.27).

**Lemma 4.9.** Let \( \alpha \geq 8 \). If \( \varepsilon > 0 \) is sufficiently small and \( \theta_0 \geq 1 \) is suitably large, then
\[
\varepsilon''''_n = 0, \quad \|\varepsilon''''_n\|_{s,T} \lesssim \varepsilon^2 \theta_k^{\zeta_3(s)-1} \Delta_k, \tag{4.47}
\]
for all \( k \in \{0, \ldots, n - 1\} \) and \( s \in \{6, \ldots, \bar{\alpha} - 2\} \), where \( \zeta_3(s) := \max\{(s + 2 - \alpha)_+ + 14 - 2\alpha, s + 10 - 2\alpha\} \).
Proof. It follows from (4.38) and (3.6) that \( \varepsilon''_k = 0 \). For term \( \varepsilon''_k \), we have
\[
epsilon''_k = \int_0^1 \mathbb{I}(U^a + \tau(S_{\theta_k}V_k - V_{k+1/2}) + V_{k+1/2}, \Phi^a + \tau(S_{\theta_k}\Psi_k - \Psi_{k+1/2}) + \Psi_{k+1/2}) \left( (\delta V_k, \delta \Psi_k), (S_{\theta_k}V_k - V_{k+1/2}, S_{\theta_k}\Psi_k - \Psi_{k+1/2}) \right) d\tau.
\]
In view of (4.39)–(4.40), similar to the proof of Lemmas 4.6–4.7, we can apply Proposition 4.5 to obtain the estimate for \( \varepsilon''_k \) in (4.47).

The following lemma gives the estimate of \( D_{k+1/2}\delta \Psi_k \) defined by (4.28).

Lemma 4.10. Let \( \alpha \geq 8 \) and \( \tilde{\alpha} \geq \alpha + 2 \). If \( \varepsilon > 0 \) is sufficiently small and \( \theta_0 \geq 1 \) is suitably large, then
\[
\| D_{k+1/2}\delta \Psi_k \|_{s,*,T} \lesssim \varepsilon^2 \theta_k^{\alpha(s)-1} \Delta_k,
\]
for all \( k \in \{0, \ldots, n-1\} \) and \( s \in \{6, \ldots, \tilde{\alpha} - 2\} \), where
\[
\varsigma_4(s) := \max \{ (s-\alpha)_+ + 18 - 2\alpha, s + 12 - 2\alpha \}.
\]
Proof. We proceed as in [1, 7]. Denote \( \Omega_T^k := (0, T) \times \Omega \) and
\[
R_k := \partial_1(U^a + V_k + \Psi_k)\partial_1(U^a + V_k + \Psi_k).
\]
Since \( \| \Psi^a + \Psi_{k+1/2}\|_{H_f^2(\Omega_T)} \lesssim \| \varphi^a + \psi_{k+1/2}\|_{H_f^2(\Omega_T)} \) and \( \delta \Psi_k \) vanishes in the past, we apply (3.26) and (3.28) to get
\[
\| D_{k+1/2}\delta \Psi_k \|_{s,*,T} = \| D_{k+1/2}\delta \Psi_k \|_{H_f^2(\Omega_T^k)} \lesssim \| \delta \Psi_k \|_{4,*,T} \| R_k \|_{H_f^2(\Omega_T^k)} + \| \delta \Psi_k \|_{s,*,T} \| R_k \|_{s,*,T}
+ \| \delta \Psi_k \|_{4,*,T} \| R_k \|_{4,*,T} \| \varphi^a + \psi_{k+1/2}\|_{H_f^2(\Omega_T)}.
\]
For estimating \( R_k \), we decompose
\[
R_k = \partial_1(L(V_k, \Psi_k) - f^a + \mathcal{H}_6) \quad \text{if} \quad t > 0,
\]
with \( \mathcal{H}_6 := \int_0^1 \mathbb{I}(U^a + V_k + \tau(V_{k+1/2} - V_k), \Phi^a + \Psi_k
+ \tau(\Psi_{k+1/2} - \Psi_k) (V_{k+1/2} - V_k, \Psi_{k+1/2} - \Psi_k) d\tau.
\]
If \( s \in \{4, \ldots, \tilde{\alpha} - 4\} \), then hypothesis (\( H_{n-1} \)) leads to
\[
\| \mathcal{L}(V_k, \Psi_k) - f^a \|_{s+2,*,T} \leq 2\varepsilon \theta_k^{s+1-\alpha}.
\]
Estimating the term \( \mathcal{H}_6 \) through
\[
\| \mathbb{I}(U + V_1, \Phi + \Psi_1)(V_2, \Psi_2) \|_{s,*,T} \lesssim \sum_{i \neq j} \| (V_i, \Psi_i) \|_{6,*,T} \| (V_j, \Psi_j) \|_{s+2,*,T},
\]
we use (4.31)–(4.32), (4.38)–(4.40), and (4.51)–(4.52) to derive
\[
\| R_k \|_{H_f^2(\Omega_T^k)} \lesssim \varepsilon \left( \theta_k^{s+6-\alpha} + \theta_k^{(s+4-\alpha)_+ + \alpha} \right) \quad \text{for} \quad s = 4, \ldots, \tilde{\alpha} - 4.
\]
If \( s \in \{ \tilde{\alpha} - 3, \tilde{\alpha} - 2\} \), then we use (4.33) and (4.38)–(4.40) to obtain
\[
\| R_k \|_{s,*,T} \lesssim \| (\tilde{U}^a + V_{k+1/2}, \tilde{\Phi}^a + \Psi_{k+1/2}) \|_{s+4,*,T} \lesssim \varepsilon \theta_k^{s+6-\alpha}.
\]
Consequently, estimate (4.53) holds for \( s \in \{4, \ldots, \tilde{\alpha} - 2\} \). Substituting hypothesis (\( H_{n-1} \), (4.53), (4.33), and (4.38)–(4.40) into (4.50) implies (4.48) and completes the proof. \( \square \)
As a direct corollary to Lemmas 4.6–4.10, we have the following estimate for $e_k$ and $\tilde{e}_k$ defined by (4.29).

**Corollary 4.11.** Let $\alpha \geq 8$ and $\bar{\alpha} \geq \alpha + 2$. If $\varepsilon > 0$ is sufficiently small and $\theta_0 \geq 1$ is suitably large, then

\[
\|e_k\|_{s,s,T} \lesssim \varepsilon^2 \theta_k^{\alpha(s)-1} \Delta_k, \\
\|\tilde{e}_k\|_{H^s(\Sigma_T)} \lesssim \varepsilon^2 \theta_k^{\alpha(s)-1} \Delta_k,
\]

for all $k \in \{0, \ldots, n-1\}$ and $s \in \{6, \ldots, \bar{\alpha}-2\}$, where $\varsigma_4(s)$ and $\varsigma_2(s)$ are defined by (4.49) and (4.37) respectively.

Corollary 4.11 implies the following estimate for the accumulated error terms $E_n$ and $\tilde{E}_n$ defined by (4.23).

**Lemma 4.12.** Let $\alpha \geq 12$ and $\bar{\alpha} = \alpha + 3$. If $\varepsilon > 0$ is sufficiently small and $\theta_0 \geq 1$ is suitably large, then

\[
\|E_n\|_{\alpha+1,s,T} \lesssim \varepsilon^2 \theta_n, \\
\|\tilde{E}_n\|_{H^{\alpha+1}(\Sigma_T)} \lesssim \varepsilon^2.
\]

**Proof.** Notice that $\varsigma_4(\alpha + 1) \leq 1$ if $\alpha \geq 12$. It follows from (4.54) that

\[
\|E_n\|_{\alpha+1,s,T} \lesssim \sum_{k=0}^{n-1} \|e_k\|_{\alpha+1,s,T} \lesssim \sum_{k=0}^{n-1} \varepsilon^2 \Delta_k \lesssim \varepsilon^2 \theta_n,
\]

provided $\alpha \geq 12$ and $\alpha + 1 \leq \bar{\alpha} - 2$. Since $\varsigma_2(\alpha + 1) = 9 - \alpha \leq -3$ for $\alpha \geq 12$ and $\alpha + 1 \leq \bar{\alpha} - 2$, estimate (4.55) implies

\[
\|\tilde{E}_n\|_{H^{\alpha+1}(\Sigma_T)} \lesssim \sum_{k=0}^{n-1} \|\tilde{e}_k\|_{H^{\alpha+1}(\Sigma_T)} \lesssim \sum_{k=0}^{n-1} \varepsilon^2 \theta_k^{-4} \Delta_k \lesssim \varepsilon^2.
\]

The minimal possible $\bar{\alpha}$ is $\alpha + 3$. This completes the proof of the lemma.

### 4.5 Proof of Theorem 2.1

We first derive hypothesis $(H_n)$ from $(H_{n-1})$. For this purpose, we derive the estimates of source terms $f_n$ and $g_n$ given in (4.24).

**Lemma 4.13.** Let $\alpha \geq 12$ and $\bar{\alpha} = \alpha + 3$. If $\varepsilon > 0$ is sufficiently small and $\theta_0 \geq 1$ is suitably large, then

\[
\|f_n\|_{s,s,T} \lesssim \Delta_n \left( \theta_n^{s-a-1} \|f_a\|_{s,s,T} + \varepsilon^2 \theta_n^{s-a-1} + \varepsilon^2 \theta_n^{\varsigma_4(s)-1} \right), \\
\|g_n\|_{H^{s+1}(\Sigma_T)} \lesssim \varepsilon^2 \Delta_n \left( \theta_n^{s-a-1} + \theta_n^{\varsigma_2(s+1)-1} \right),
\]

for $s = 6, \ldots, \bar{\alpha}$, where $\varsigma_4(s)$ and $\varsigma_2(s)$ are defined by (4.49) and (4.37) respectively.

**Proof.** From (4.19), (4.24), (4.54), and (4.56), we obtain

\[
\|f_n\|_{s,s,T} \leq \|(S_{\theta_n} - S_{\theta_n-1}) f_a - (S_{\theta_n} - S_{\theta_n-1}) E_{n-1} - S_{\theta_n} e_{n-1}\|_{s,s,T} \\
\lesssim \Delta_n \theta_n^{s-a-1} \left( \|f_a\|_{s,s,T} + \theta_n^{-1} \|E_{n-1}\|_{\alpha+1,s,T} \right) + \|S_{\theta_n} e_{n-1}\|_{s,s,T} \\
\lesssim \Delta_n \left( \theta_n^{s-a-1} \|f_a\|_{s,s,T} + \varepsilon^2 \right) + \varepsilon^2 \theta_n^{\varsigma_4(s)-1}.\]
Thanks to (4.55) and (4.56), we obtain
\[
\|g_n\|_{H^{s+1}(\Sigma_T)} \leq \|(S_{\theta_n} - S_{\theta_{n-1}})\tilde{E}_{n-1} - S_{\theta_n}\tilde{e}_{n-1}\|_{H^{s+1}(\Sigma_T)} \\
\lesssim \Delta_n\theta_n^{s+\alpha-1}\|\tilde{E}_{n-1}\|_{H^{s+1}(\Sigma_T)} + \|S_{\theta_n}\tilde{e}_{n-1}\|_{H^{s+1}(\Sigma_T)} \\
\lesssim \varepsilon^2\Delta_n(\theta_n^{s+\alpha-1} + \theta_n^{\alpha(s+1)-1}),
\]
which completes the proof of the lemma.

The next lemma gives the estimate of solutions to problem (4.21) by means of tame estimate (3.14). We omit the proof for brevity, since it is similar to the proof of [39, Lemma 15].

**Lemma 4.14.** Let $\alpha \geq 12$ and $\tilde{\alpha} = \alpha + 3$. If $\varepsilon > 0$ and $\|f^a\|_{\alpha,s,T}/\varepsilon$ are sufficiently small, and if $\theta_0 \geq 1$ is suitably large, then
\[
\|\langle \delta V_n, \delta \Psi_n \rangle\|_{s,s,T} + \|\delta\psi_n\|_{H^s(\Sigma_T)} \leq \varepsilon\theta_n^{s-\alpha-1}\Delta_n
\]
for $s = 6, \ldots, \tilde{\alpha}$.

Estimate (4.59) is inequality (a) in hypothesis $(H_n)$. The other inequalities in $(H_n)$ are provided in the following lemma, whose proof can be found in [39, Lemma 16].

**Lemma 4.15.** Let $\alpha \geq 12$ and $\tilde{\alpha} = \alpha + 3$. If $\varepsilon > 0$ and $\|f^a\|_{\alpha,s,T}/\varepsilon$ are sufficiently small, and if $\theta_0 \geq 1$ is suitably large, then
\[
\|\mathcal{L}(V_n, \Psi_n) - f^a\|_{s,s,T} \leq 2\varepsilon\theta_n^{s-\alpha-1} \quad \text{for} \ s = 6, \ldots, \tilde{\alpha} - 1,
\]
\[
\|\mathcal{B}(V_n, \psi_n)\|_{H^s(\Sigma_T)} \leq \varepsilon\theta_n^{s-\alpha-1} \quad \text{for} \ s = 7, \ldots, \alpha.
\]

According to Lemmas 4.14–4.15, we have derived $(H_n)$ from $(H_{n-1})$, provided that $\alpha \geq 12$, $\tilde{\alpha} = \alpha + 3$, $\varepsilon > 0$ and $\|f^a\|_{\alpha,s,T}/\varepsilon$ are sufficiently small, and $\theta_0 \geq 1$ is large enough. Fixing the constants $\alpha$, $\tilde{\alpha}$, $\varepsilon > 0$, and $\theta_0 \geq 1$, as in [39, Lemma 17], we can show that $(H_0)$ is true for a sufficiently short time.

**Lemma 4.16.** If time $T > 0$ is small enough, then $(H_0)$ holds.

We are now in a position to conclude the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Assume that the initial data $(U_0, \varphi_0)$ satisfy all the conditions of Theorem 2.1. Let $\tilde{\alpha} = m - 4$ and $\alpha = \tilde{\alpha} - 3 \geq 12$. Then initial data $U_0$ and $\varphi_0$ are compatible up to order $m = \tilde{\alpha} + 4$. Thanks to (4.9) and (4.17), we obtain (4.30) and all the requirements of Lemmas 4.14–4.16, provided $\varepsilon > 0$ and $T > 0$ is sufficiently small and $\theta_0 \geq 1$ is large enough. Hence, for sufficiently short time $T > 0$, hypothesis $(H_n)$ holds for all $n \in \mathbb{N}$. In particular, recalling (4.25), we have
\[
\sum_{k=0}^{\infty} (\|\delta V_k, \delta \Phi_k\|_{s,s,T} + \|\delta\psi_k\|_{H^s(\Sigma_T)}) \lesssim \sum_{k=0}^{\infty} \theta_k^{s-\alpha-2} < \infty \quad \text{for} \ 6 \leq s \leq \alpha - 1.
\]
As a result, the sequence $(V_k, \psi_k)$ converges to some limit $(V, \psi)$ in $H^{\alpha-1}(\Omega_T) \times H^{\alpha-1}(\omega_T)$. Passing to the limit in (4.60)–(4.61) for $s = \alpha - 1 = m - 8$, we obtain (4.18). Therefore, $(U, \Phi) = (U^a + V, \Phi^a + \Psi)$ is a solution of the original problem (2.13) on time interval $[0, T]$. The uniqueness of solutions to problem (2.13) can be obtained by a standard argument; see, for instance, [36, §13]. This completes the proof. \qed
5 Relativistic Case

Let us first reduce the RMHD equations (1.8) to an equivalent symmetric hyperbolic system. To this end, we introduce the coordinate velocity $v := (v_1, \ldots, v_d)^T$, where

$$v_i := \epsilon^{-1} \Gamma^{-1} u^i \quad \text{for } i = 1, \ldots, d,$$

(5.1)

with the Lorentz factor $\Gamma := u^0 > 0$ thanks to (1.9). Moreover, from the first identity in (1.9), we infer

$$|v| < \epsilon^{-1}, \quad \Gamma = \Gamma(v) := (1 - \epsilon^2 |v|^2)^{-1/2}.$$  

(5.2)

Let us define $H := (H_1, \ldots, H_d)^T$ with

$$H_i := \epsilon^{-1} (u^0 b^i - u^i b^0) \quad \text{for } i = 1, \ldots, d.$$  

(5.3)

Then relations (1.9) imply

$$b^i = \epsilon \Gamma^{-1} H_i + \epsilon^3 \Gamma (v \cdot H) v_i \quad \text{for } i = 1, \ldots, d,$$  

(5.4)

$$b^0 = \epsilon^2 \Gamma (v \cdot H), \quad |b|^2 := g_{\alpha\beta} b^\alpha b^\beta = \epsilon^2 \Gamma^2 |H|^2 + \epsilon^4 (v \cdot H)^2.$$  

(5.5)

In the inertial coordinates $(x^0, \ldots, x^d)$, the covariant derivative $\nabla_\alpha$ coincides with the partial derivative $\partial/\partial x^\alpha$. We introduce the spacetime coordinates $(t, x)$, where $t = \epsilon x^0$ and $x = (x_1, \ldots, x_d)$ with $x_i := x^i$ for $i = 1, \ldots, d$. Then a straightforward computation leads to the following equivalent system of (1.8) (cf. [10, (2)–(6))):

$$\partial_t (\rho \Gamma) + \nabla \cdot (\rho \Gamma v) = 0,$$  

(5.6a)

$$\partial_t (\rho \Gamma^2 + \epsilon^2 |H|^2 - \epsilon^2 q) + \nabla \cdot (\rho \Gamma^2 v + \epsilon^2 |H|^2 v - \epsilon^2 (v \cdot H) H) = 0,$$  

(5.6b)

$$\partial_t (\rho \Gamma^2 v + \epsilon^2 |H|^2 v - \epsilon^2 (v \cdot H) H) - \nabla \cdot (\Gamma^{-2} H \otimes H) + \nabla q$$

$$+ \nabla \cdot \left( (\rho \Gamma^2 + \epsilon^2 |H|^2) v \otimes v - \epsilon^2 (v \cdot H) (H \otimes v + v \otimes H) \right) = 0,$$  

(5.6c)

$$\partial_t H - \nabla \times (v \times H) = 0,$$  

(5.6d)

and

$$\nabla \cdot H = 0,$$  

(5.7)

where $q = p + \frac{1}{2} \epsilon^{-2} |b|^2$ is the total pressure. As in [20, §34], it follows from (1.3) that $u^\alpha \nabla_\alpha S = 0$ for smooth solutions to (1.8), and hence

$$(\partial_t + v \cdot \nabla) S = 0.$$  

(5.8)

Let us impose the physical assumption (2.20) and define $V := (p, w, H, S)^T$ with

$$w := \Gamma(v) v = (1 - \epsilon^2 |v|^2)^{-1/2} v.$$  

(5.9)

By properly applying the Lorentz transformation as in [10], we can obtain the following symmetric hyperbolic system in the non-vacuum region $\{\rho_* < \rho < \rho^*\}$, which is equivalent to (5.6) (and also to (1.8)):

$$B_0(V) \partial_t V + B_i(V) \partial_i V = 0,$$  

(5.10)
where

\[
B_0(V) := \begin{pmatrix}
\frac{\Gamma}{\rho a^2} \epsilon^2 v^T & 0 & 0 \\
\frac{\epsilon^2 v}{\rho a^2} A_0 & O_d & 0 \\
0 & O_d & M_0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad B_i(V) = \begin{pmatrix}
\frac{\Gamma v_i}{\rho a^2} \epsilon_i^T & 0 & 0 \\
\epsilon_i & A_i & N_i^T \\
0 & N_i & v_i M_0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad (5.11)
\]

with \(M_0 := \Gamma^{-1}(I_d + \epsilon^2 \Gamma^2 v \otimes v)\),

\[
A_0 := \left(\rho h \Gamma + \epsilon^2 \Gamma^{-1}|H|^2\right)(I_d - \epsilon^2 v \otimes v) - \epsilon^2 \Gamma^{-1}|\dot{b}|^2 v \otimes v
- \epsilon^2 \Gamma^{-1}H \otimes H + \epsilon^4 \Gamma^{-1}(v \cdot H)(H \otimes v + v \otimes H),
\]

\[
A_i := v_i \left\{\left(\rho h \Gamma + \epsilon^2 \Gamma^{-1}|H|^2\right)(I_d - \epsilon^2 v \otimes v) + \epsilon^2 \Gamma^{-1}(|\dot{b}|^2 v \otimes v - H \otimes H)\right\}
+ \epsilon^2 \Gamma^{-1}H_i \left\{\Gamma^{-2}(H \otimes v + v \otimes H) - 2(v \cdot H)(I_d - \epsilon^2 v \otimes v)\right\}
+ \Gamma^{-1}(\epsilon^2(v \cdot H) H - |\dot{b}|^2 v) \otimes \epsilon_i + \Gamma^{-1} \epsilon_i \otimes (\epsilon^2(v \cdot H) H - |\dot{b}|^2 v),
\]

\[
N_i := \left(\Gamma^{-2} H + \epsilon^2(v \cdot H) v\right) \otimes \left(\epsilon_i - \epsilon^2 v \otimes v\right) - \Gamma^{-2} H_i I_d,
\]

for \(i = 1, \ldots, d\). Here, \(a\) and \(|b|^2\) are given in (2.1) and (5.5) respectively. Also see Appendix B for a direct verification of (5.10) and the positive-definiteness of matrix \(B_0(V)\).

We consider the free boundary problem in ideal RMHD, that is, to solve equations (5.6) (or equivalently (1.8)) in \(\Omega(t)\) supplemented with the initial and boundary conditions (1.4)–(1.5). As in the non-relativistic case, we take \(U := (q, v, H, S)^T\) as the primary unknowns and deduce from (5.10) the hyperbolic symmetric system (2.4), with coefficient matrices \(A_i(U)\) being defined by

\[
A_i(U) := J^T B_i(V) J \quad \text{with} \quad J := \frac{\partial V}{\partial U} = \begin{pmatrix}
1 & 0 & a^T & -b^T & 0 \\
0 & \Gamma^2 M_0 & O_d & 0 \\
0 & O_d & I_d & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad (5.12)
\]

for \(i = 0, \ldots, d\), where \(B_i(V)\) are given in (5.11), \(a := \epsilon^2(|H|^2 (v - (v \cdot H) H))\), and \(b := \Gamma^{-2} H + \epsilon^2(v \cdot H) v\). We can compute \(\det J = \Gamma^5 > 0\), meaning that \(J\) is invertible. In the new variables, from (2.13b) and (2.15)–(2.16), we have

\[
\tilde{A}_1(U, \Phi) := J^T \left(\begin{array}{cccc}
0 & c^T & 0 & 0 \\
c & \tilde{A}_1 & c \otimes b & 0 \\
0 & b \otimes c & O_d & 0 \\
0 & 0 & 0 & 0
\end{array}\right) J \quad \text{on} \quad [0, T] \times \Sigma,
\]

where \(c := N - \epsilon^2 v N v\) and

\[
\tilde{A}_1 := \epsilon^2 \Gamma^{-1} N \left\{2|\dot{b}|^2 v \otimes v - \epsilon^2(v \cdot H)(v \otimes H + H \otimes v)\right\}
+ \Gamma^{-1}(\epsilon^2(v \cdot H) H - |\dot{b}|^2 v) \otimes N + \Gamma^{-1} N \otimes (\epsilon^2(v \cdot H) H - |\dot{b}|^2 v).
\]

From (5.2), we get \(\Gamma M_0 c = N\) and \(M_0 \tilde{A}_1 M_0 = -\Gamma^{-3}(N \otimes a + a \otimes N)\), so that

\[
\tilde{A}_1(U, \Phi) = \begin{pmatrix}
0 & \Gamma N^T & 0 & 0 \\
\Gamma N & O_d & O_d & 0 \\
0 & O_d & O_d & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \quad \text{on} \quad [0, T] \times \Sigma. \quad (5.13)
\]

Having identity (5.13) and the symmetric form (2.4) with the matrices \(A_i(U)\) defined by (5.12) in hand, we can prove Theorem 2.2 in an entirely similar way as for the non-relativistic case in Sections 3 and 4.
Appendix A  Conventional Notation in the Vector Calculus

For readers’ convenience, we collect the conventional notation of the vector calculus. The spatial dimension is denoted by $d = 2, 3$. We abbreviate the partial differentials as

$$\partial_i := \frac{\partial}{\partial t}, \quad \partial_i := \frac{\partial}{\partial x_i} \text{ for } i = 1, \ldots, d.$$  

We denote the gradient by $\nabla := (\partial_1, \ldots, \partial_d)^T$. For any $d \times d$ matrix $F = (F_{ij})$, vectors $u = (u_1, \ldots, u_d)^T$ and $v = (v_1, \ldots, v_d)^T$, and a scalar $a$, the symbol $u \otimes v$ denotes the $d \times d$ matrix with $(i,j)$-entry $u_i v_j$, and

$$\nabla \cdot F := (\partial_j F_{1j}, \ldots, \partial_j F_{dj})^T, \quad \nabla \cdot u := \partial_i u_i,$$

$$u \times v := \begin{cases} u_1 v_2 - u_2 v_1 & \text{if } d = 2, \\ (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1)^T & \text{if } d = 3, \end{cases}$$

$$\nabla \times v := \begin{cases} \partial_1 v_2 - \partial_2 v_1 & \text{if } d = 2, \\ (\partial_2 v_3 - \partial_3 v_2, \partial_3 v_1 - \partial_1 v_3, \partial_1 v_2 - \partial_2 v_1)^T & \text{if } d = 3, \end{cases}$$

$$u \times a = -a \times u := (u_2 a_3 - u_3 a_2, u_3 a_1 - u_1 a_3, u_1 a_2 - u_2 a_1)^T,$$

$$\nabla \times a := (\partial_2 a_3 - \partial_3 a_2, \partial_3 a_1 - \partial_1 a_3, \partial_1 a_2 - \partial_2 a_1)^T \text{ if } d = 2.$$

The notation above was employed by Kawashima [17, p. 144] to write down the electromagnetic fluid system in two spatial dimensions. The compressible MHD equations (1.1) with $d = 2$ follow from the assumption that all the quantities in (1.1) are independent of $x_3$ and the components $v_3$ and $H_3$ are identically zero.

Appendix B  Symmetrization for RMHD

Let us deduce the symmetric system (5.10) from (5.6)–(5.8). First, the last equation for $S$ in (5.10) is exactly (5.8). In view of (5.2) and (5.9), we have $\Gamma = (1 + \epsilon^2 |v|^2)^{1/2}$ and $v = \Gamma^{-1} w$, so that

$$\partial_\alpha \Gamma = \epsilon^2 v_\alpha \partial_\alpha w_1, \quad \partial_\alpha v = \Gamma^{-1} \partial_\alpha w - \epsilon^2 \Gamma^{-1} v_\alpha \partial_\alpha w_1 \quad \text{for } \alpha = 0, \ldots, d. \quad (B.1)$$

It follows from identities (5.8), (5.6a), and (B.1) that

$$\Gamma (\partial_t + v_\alpha \partial_\alpha) p = -\rho a^2 (\epsilon^2 v_\alpha \partial_\alpha w_1 + \partial_\alpha w_1),$$

which immediately gives the first equation for $p$ in (5.10). Using (5.6d) and (5.7), we have $(\partial_t + v \cdot \nabla) H - (H \cdot \nabla) v + H \nabla \cdot v = 0$, which together with (B.1) yields

$$(\partial_t + v \cdot \nabla) H + M_t \partial_t w = 0, \quad (B.2)$$

with $M_t := \Gamma^{-1} \{H \otimes e_1 - H_1 I_d - \epsilon^2 (v_1 H - H_1 v) \otimes v\}$. Thanks to (5.6b) and (B.1), we infer from (5.6c) that

$$(\rho \partial_t + \epsilon^2 \Gamma^{-1} |H|^2) (I_d - \epsilon^2 v \otimes v) (\partial_t + v \cdot \nabla) w + \epsilon^2 v \partial_t p + \nabla p + \sum_{i=1}^4 T_i = 0,$$

where

$$T_1 := \frac{1}{2} v \partial_t |b|^2 - \epsilon^2 \partial_t ((v \cdot H) H), \quad T_2 := -\nabla \cdot (\Gamma^{-2} H \otimes H), \quad T_3 := \frac{1}{2} \epsilon^{-2} \nabla |b|^2,$$

$$T_4 := -\epsilon^2 \nabla \cdot ((v \cdot H)(H \otimes v + v \otimes H)) + \epsilon^2 v \nabla \cdot ((v \cdot H) H) = -\epsilon^2 (v \cdot H)((v \cdot \nabla) H + (H \cdot \nabla) v + H \nabla \cdot v) - \epsilon^2 H(v \cdot \nabla)(v \cdot H).$$

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Therefore, we obtain that

\[ T_1 = \varepsilon^2 \Gamma^{-1} \left\{ \varepsilon^2 (v \cdot H)(H \otimes v + v \otimes H) - \|b\|^2 v \otimes v - H \otimes H \right\} \partial_tw + T_{1a}, \]

\[ T_2 = -\Gamma^{-2} (H \cdot \nabla)H_i + 2\varepsilon^2 \Gamma^{-3} H_i H \otimes v \partial_tw, \]

\[ T_3 = (\Gamma^{-2} H_i + \varepsilon^2 (v \cdot H)v_i) \nabla H_i + \Gamma^{-1} \varepsilon_i \otimes \varepsilon_i (v \cdot H)H - \|b\|^2 v \partial_tw, \]

with \( T_{1a} := \varepsilon^2 (v \Gamma^{-2} H_i + \varepsilon^2 (v \cdot H)v_i - Hv_i) \partial_tw + \varepsilon^2 (v \cdot H) \partial_t H \). Then we utilize (B.1) and (B.2) for calculating the terms \( T_4 \) and \( T_{1a} \) respectively to derive the equations for \( w \) in (5.10). Noticing that \( M_0 \mathcal{M}_1 = \mathcal{N}_1 \), we obtain the equations for \( H \) in (5.10) from the left-multiplication of (B.2) by \( \mathcal{M}_0 \).

Next we show that the matrix \( B_0(V) \) is positive definite in the non-vacuum region \( \{ \rho_s < \rho < \rho^* \} \). For any \( u \in \mathbb{R}^d \setminus \{0\} \), we get \( u^T \mathcal{M}_0 u \geq \Gamma^{-1} |u|^2 \), that is, \( \mathcal{M}_0 \geq \Gamma^{-1} I_d \). Since

\[
\begin{pmatrix}
\frac{\Gamma}{\rho a^2} \varepsilon^2 v^T \\
\varepsilon^2 v \quad \mathcal{A}_0
\end{pmatrix} = P^T \begin{pmatrix}
\frac{\Gamma}{\rho a^2} & 0 \\
0 & \mathcal{A}_0 - \varepsilon^4 \rho a^2 \frac{T}{\Gamma} vv^T
\end{pmatrix} P
\]

with \( P \) defined as

\[
\begin{pmatrix}
1 & \frac{\rho a^2}{\Gamma} \varepsilon^2 v^T \\
\frac{1}{\rho a^2} \varepsilon^2 v & 0
\end{pmatrix},
\]

it suffices to show that the matrix \( \mathcal{A}_0 - \varepsilon^4 \rho a^2 \Gamma^{-1} vv^T \) is positive definite. For any \( u \in \mathbb{R}^d \setminus \{0\} \), we have

\[
u^T (\mathcal{A}_0 - \varepsilon^4 \rho a^2 \Gamma^{-1} vv^T) u = \mathcal{T}_5 + \mathcal{T}_6,
\]

where \( \mathcal{T}_5 := \rho \varepsilon \Gamma |u|^2 - (\varepsilon^2 \rho \varepsilon^2 + \varepsilon^4 \rho a^2 \Gamma^{-1})(v \cdot u)^2 \) and

\[
\mathcal{T}_6 := \varepsilon^2 \Gamma^{-1} \left\{ |u|^2 |H|^2 - \varepsilon^2 (1 + \Gamma^{-2})|H|^2 (v \cdot u)^2
\right.

\left. - \varepsilon^4 (v \cdot u)^2 (v \cdot H)^2 - (v \cdot u)^2 (v \cdot H)^2 + 2\varepsilon^2 (v \cdot H)(v \cdot u)(H \cdot u)\right\}
\]

\[
= \varepsilon^2 \Gamma^{-1} \left\{ |u|^2 |H|^2 - \varepsilon^2 (v \cdot u)^2 (1 + \Gamma^{-2})|H|^2 - (H \cdot (\varepsilon^2 (v \cdot u)v - u))^2\right\}
\]

\[
\geq \varepsilon^2 \Gamma^{-1} |H|^2 \left\{ |u|^2 - \varepsilon^2 (2 - \varepsilon^2 (v \cdot u)^2)(v \cdot u)^2 - |(v \cdot u)v - u|^2\right\} = 0,
\]

owing to (5.2). By virtue of \( (2.20) \) and (5.2), we infer

\[
\mathcal{T}_5 \geq \rho \varepsilon \Gamma |u|^2 - (\varepsilon^2 \rho \varepsilon^2 + \varepsilon^4 \rho a^2 \Gamma^{-1})|v|^2 |u|^2
\]

\[
= \rho \varepsilon \Gamma^{-1} |u|^2 (1 - c_s^2 \varepsilon^4 |v|^2) \geq \rho \varepsilon \Gamma^{-3} |u|^2 (1 - \varepsilon^2 |v|^2) = \rho \varepsilon \Gamma^{-3} |u|^2.
\]

Therefore, we obtain that \( \mathcal{A}_0 - \varepsilon^4 \rho a^2 \Gamma^{-1} vv^T \) and \( B_0(V) \) are positive definite.

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