CANONICAL CARTAN CONNECTION FOR 4-DIMENSIONAL CR-MANIFOLDS BELONGING TO GENERAL CLASS II

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ABSTRACT

We study the equivalence problem for 4-dimensional CR-manifolds of CR-dimension 1 and codimension 2 which have been referred to as belonging to general class II in [9], and which are also known as Engel CR-manifolds. We construct a canonical Cartan connection on such CR-manifolds through Cartan equivalence’s method, thus providing an alternative approach to the results contained in [1]. In particular, we give the explicit expression of 4 biholomorphic invariants, the annulation of which is a necessary and sufficient condition for an Engel manifold to be locally biholomorphic to Beloshapka’s cubic in $\mathbb{C}^3$.

1. INTRODUCTION

As highlighted by Henri Poincaré [14] in 1907, the (local) biholomorphic equivalence problem between two submanifolds $M$ and $M'$ of $\mathbb{C}^N$ is to determine whether or not there exists a (local) biholomorphism $\phi$ of $\mathbb{C}^N$ such that $\phi(M) = M'$. Elie Cartan [2, 3] solved this problem for hypersurfaces $M^3 \subset \mathbb{C}^2$ in 1932, as he constructed a “hyperspherical connection” on such hypersurfaces by using the powerful technique which is now referred to as Cartan’s equivalence method.

Given a manifold $M$ and some geometric data specified on $M$, which usually appears as a $G$-structure on $M$ (i.e. a reduction of the bundle of coframes of $M$), Cartan’s equivalence method seeks to provide a principal bundle $P$ on $M$ together with a coframe $\omega$ of 1-forms on $P$ which is adapted to the geometric structure of $M$ in the following sense: an isomorphism between two such geometric structures $M$ and $M'$ lifts to a unique isomorphism between $P$ and $P'$ which sends $\omega$ on $\omega'$. The equivalence problem between $M$ and $M'$ is thus reduced to an equivalence problem between $\{e\}$-structures, which is well understood [10, 15].

We recall that a CR-manifold $M$ is a real manifold endowed with a subbundle $L$ of $\mathbb{C} \otimes TM$ of even rank $2n$ such that

(1) $L \cap \overline{L} = \{0\}$

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(2) \( L \) is formally integrable, i.e. \([L, L] \subset L\).

The integer \( n \) is the CR-dimension of \( M \) and \( k = \dim M - 2n \) is the codimension of \( M \). In a recent attempt \([9]\) to solve the equivalence problem for CR-manifolds up to dimension 5, it has been shown that one can restrict the study to six different general classes of CR-manifolds of dimension \( \leq 5 \), which have been referred to as general classes I, II, III, IV, and IV'2. The aim of this paper is to provide a solution to the equivalence problem for CR-manifolds which belong to general class II, that is the CR-manifolds of dimension 4 and of CR-dimension 1 whose CR-bundle \( L \) satisfy the additional non-degeneracy condition:

\[
\mathbb{C} \otimes TM = L + \overline{L} + [L, \overline{L}] + [L, [L, \overline{L}]],
\]

meaning that \( \mathbb{C} \otimes TM \) is spanned by \( L, \overline{L} \) and their Lie brackets up to order 3.

This problem has already been solved by Beloshapka, Ezhov and Schmalz in \([1]\), where the CR-manifolds we study are called Engel manifolds. The present paper provides thus an alternative solution to the results contained in \([1]\). The main result is the following:

**Theorem 1.** Let \( M \) be a CR-manifold belonging to general class II. There exists a 5-dimensional subbundle \( P \) of the bundle of coframes \( \mathbb{C} \otimes F(M) \) of \( M \) and a coframe \( \omega := (\Lambda, \sigma, \rho, \zeta, \overline{\zeta}) \) on \( P \) such that any CR-diffeomorphism \( h \) of \( M \) lifts to a bundle isomorphism \( h^* \) of \( P \) which satisfy \( h^*(\omega) = \omega \). Moreover the structure equations of \( \omega \) on \( P \) are of the form:

\[
\begin{align*}
\text{d}\sigma &= 3 \Lambda \wedge \sigma + \rho \wedge \zeta + \rho \wedge \overline{\zeta}, \\
\text{d}\rho &= 2 \Lambda \wedge \rho + i \zeta \wedge \overline{\zeta} \\
\text{d}\zeta &= \Lambda \wedge \zeta + \mathcal{I}_1 \sigma \wedge \rho + \mathcal{I}_2 \sigma \wedge \zeta + \mathcal{I}_3 \sigma \wedge \overline{\zeta} + \mathcal{I}_4 \rho \wedge \zeta + \mathcal{I}_5 \rho \wedge \overline{\zeta}, \\
\text{d}\overline{\zeta} &= \Lambda \wedge \overline{\zeta} + \overline{\mathcal{I}_1} \sigma \wedge \rho + \overline{\mathcal{I}_3} \sigma \wedge \zeta + \overline{\mathcal{I}_2} \sigma \wedge \overline{\zeta} + \overline{\mathcal{I}_4} \rho \wedge \zeta + \overline{\mathcal{I}_5} \rho \wedge \overline{\zeta}, \\
\text{d}\Lambda &= \frac{i}{2} \mathcal{I}_1 \sigma \wedge \zeta - \frac{i}{2} \overline{\mathcal{I}_1} \sigma \wedge \zeta - \frac{1}{3} (\mathcal{I}_2 + \overline{\mathcal{I}_3}) \rho \wedge \zeta - \frac{1}{3} (\overline{\mathcal{I}_2} + \mathcal{I}_3) \rho \wedge \zeta \\
&\quad + \mathcal{I}_0 \sigma \wedge \zeta,
\end{align*}
\]

where \( \mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5 \), are functions on \( P \).

An example of CR-manifold belonging to general class II is provided by Beloshapka’s cubic \( B \subset \mathbb{C}^3 \), which is defined by the equations:

\[
\begin{align*}
B : \quad w_1 &= \overline{w}_1 + 2i \overline{z}\overline{z}, \\
w_2 &= \overline{w}_2 + 2i \overline{z}(z + \overline{z}).
\end{align*}
\]

Cartan’s equivalence method has been applied to Beloshapka’s cubic in \([12]\) where it has been shown that the coframe \( (\Lambda, \sigma, \rho, \zeta, \overline{\zeta}) \) of theorem \([1]\) satisfy
the simplified structure equations:

\[ dσ = 3 \Lambda ∧ σ + ρ ∧ ζ + ρ ∧ \bar{ζ}, \]
\[ dp = 2 \Lambda ∧ ρ + i \zeta ∧ \bar{ζ}, \]
\[ dζ = \Lambda ∧ ζ, \]
\[ d\bar{ζ} = \Lambda ∧ \bar{ζ}, \]
\[ d\Lambda = 0, \]

corresponding to the case where the biholomorphic invariants \( I_i \) vanish identically. From this result together with theorem II we deduce the existence of a Cartan connection on CR-manifolds belonging to general class II in section 4.

We start in section 2 with the construction of a canonical \( G \)-structure \( P^1 \) on \( M \), (e.g. a subbundle of the bundle of coframes of \( M \)), which encodes the equivalence problem for \( M \) under CR-automorphisms in the following sense: a diffeomorphism

\[ h : M \longrightarrow M \]

is a CR-automorphism of \( M \) if and only if

\[ h^* : P^1 \longrightarrow P^1 \]

is a \( G \)-structure isomorphism of \( P^1 \). We refer to \([9, 6, 7]\) for details on the results summarized in this section and to \([15]\) for an introduction to \( G \)-structures. Section 3 is devoted to reduce successively \( P^1 \) to three subbundles:

\[ P^4 \subset P^3 \subset P^2 \subset P^1, \]

which are still adapted to the biholomorphic equivalence problem for \( M \). We use Cartan equivalence method, for which we refer to \([10]\). Eventually a Cartan connection is constructed on \( P^4 \) in section 4.

2. INITIAL \( G \)-STRUCTURE

Let \( M \) be a 4-dimensional CR-manifold belonging to general class II and \( \mathcal{L} \) be a local generator of the CR-bundle \( L \) of \( M \). As \( M \) belongs to general class II, the two vector fields \( \mathcal{T}, \mathcal{P} \), defined by:

\[ \mathcal{T} := i [\mathcal{L}, \mathcal{D}], \]
\[ \mathcal{P} := [\mathcal{L}, \mathcal{T}], \]

are such that:

\[ 4 = \text{rank}_\mathbb{C} \left( \mathcal{L}, \mathcal{D}, \mathcal{T}, \mathcal{P} \right), \]

namely

\( (\mathcal{L}, \mathcal{D}, \mathcal{T}, \mathcal{P}) \) is a frame on \( M \).
As a result there exist two functions $A$ and $B$ such that:

$$\mathcal{F} = A \cdot \mathcal{T} + B \cdot \mathcal{T}.$$  

From the fact that $\overline{\mathcal{F}} = \mathcal{F}$, the functions $A$ and $B$ satisfy the relations:

$$B \overline{B} = 1,$$
$$A + B A = 0.$$  

There also exist two functions $P, Q$ such that:

$$[\mathcal{L}, \mathcal{I}] = P \cdot \mathcal{T} + Q \cdot \mathcal{T}.$$  

The conjugate of $P$ and $Q$, $\overline{P}$ and $\overline{Q}$, are given by the relations:

$$\overline{Q} = \mathcal{L}(B) + B Q + 2A + \frac{\mathcal{F}(B)}{B},$$
$$\overline{P} = B \mathcal{L}(A) - A \mathcal{L}(B) - B A Q - A^2 - A \frac{\mathcal{F}(B)}{B} + \mathcal{F}(A) + B^2 P.$$  

The four functions $A, B, P, Q$ appear to be fundamental as all other Lie brackets between the vector fields $\mathcal{L}, \mathcal{F}, \mathcal{T}$ and $\mathcal{I}$ are expressed in terms of these five functions and their $\{\mathcal{L}, \mathcal{F}\}$-derivatives (17).

In the case of an embedded CR-manifold $M \subset \mathbb{C}^3$, we can give an explicit formula for the fundamental vector field $\mathcal{L}$, and hence for the functions $A, B, P, Q$, in terms of a graphing function of $M$. We refer to [8] for details on this question. Let us just mention that the submanifold $M \subset \mathbb{C}^3$ is represented in local coordinates:

$$(z, w_1, w_2) := (x + i y, u_1 + i v_1, u_2 + i v_2)$$

as a graph:

$$v_1 = \phi_1(x, y, u_1, u_2)$$
$$v_2 = \phi_2(x, y, u_1, u_2).$$

There exists then a unique local generator $\mathcal{L}$ of $T^{1,0}M$ of the form:

$$\mathcal{L} = \frac{\partial}{\partial z} + A^1 \frac{\partial}{\partial u_1} + A^2 \frac{\partial}{\partial u_2}$$

having conjugate:

$$\overline{\mathcal{L}} = \frac{\partial}{\partial \overline{z}} + \overline{A^1} \frac{\partial}{\partial \overline{u_1}} + \overline{A^2} \frac{\partial}{\partial \overline{u_2}}.$$
which is a generator of $T^{0,1}M$, where the functions $A^1$ and $A^2$ are given by the determinants:

\[
A^1 := \begin{vmatrix}
-\phi_{1,z} & \phi_{1,u_2} \\
-\phi_{2,z} & i + \phi_{2,u_2} \\
i + \phi_{1,u_1} & \phi_{1,u_2} \\
\phi_{2,u_1} & i + \phi_{2,u_2}
\end{vmatrix}, \quad A^2 := \begin{vmatrix}
i + \phi_{1,u_1} & -\phi_{1,z} \\
\phi_{2,u_1} & -\phi_{2,z} \\
i + \phi_{1,u_1} & \phi_{1,u_2} \\
\phi_{2,u_1} & i + \phi_{2,u_2}
\end{vmatrix}.
\]

Returning to the general case of abstract CR-manifolds, let us introduce the coframe \( \omega^0 := (\sigma^0, \rho^0, \zeta^0, \overline{\zeta}^0) \), as the dual coframe of \( (S, T, L, \overline{L}) \). We have [7]:

**Lemma 1.** The structure equations enjoyed by \( \omega^0 \) are of the form:

\[
d\sigma^0 = H \sigma^0 \wedge \rho^0 + F \sigma^0 \wedge \overline{\zeta}^0 + Q \sigma^0 \wedge \zeta^0 + B \rho^0 \wedge \overline{\zeta}^0 + \rho^0 \wedge \zeta^0,
\]

\[
d\rho^0 = G \sigma^0 \wedge \rho^0 + E \sigma^0 \wedge \zeta^0 + P \sigma^0 \wedge \zeta^0 + A \rho^0 \wedge \zeta^0 + i \zeta^0 \wedge \overline{\zeta}^0,
\]

\[
d\zeta^0 = 0,
\]

\[
d\overline{\zeta}^0 = 0,
\]

where the four functions:

\[ E, F, G, H, \]

*can be expressed in terms of the four fundamental functions:

\[ A, B, P, Q, \]

*and their \( \{L, \overline{L}\} \)-derivatives as:

\[ E := L(A) + B P, \]

\[ F := L(B) + B Q + A, \]

\[ G := i L(L(A)) + i P L(B) - i L(P) - i Q L(A) + i P L(B) + i B L(P), \]

\[ H := i L(L(B)) + i Q L(B) + i B L(Q) + 2i L(A) - i L(Q). \]

Let \( h : M \rightarrow M \) be a CR-automorphism of \( M \). As we have

\[
h_\ast(L) = L,
\]

there exists a non-vanishing complex-valued function \( a \) on \( M \) such that:

\[
h_\ast(L) = a L.
\]

From the definition of \( T, \overline{T} \), and the invariance

\[
h_\ast([X, Y]) = [h_\ast(X), h_\ast(Y)]
\]

for any vector fields \( X, Y \) on \( M \), we easily get the existence of four functions

\[ b, c, d, e : M \rightarrow \mathbb{C}, \]
such that:

\[
\begin{pmatrix}
L \\
T \\
S
\end{pmatrix} =
\begin{pmatrix}
a & 0 & 0 & 0 \\
0 & \bar{a} & 0 & 0 \\
b & b & a\bar{a} & 0 \\
e & d & c & a^2\bar{a}
\end{pmatrix}
\begin{pmatrix}
L \\
T \\
S
\end{pmatrix}.
\]

This is summarized in the following lemma [6]:

**Lemma 2.** Let \( h : M \to M \) a CR-automorphism of \( M \) and let \( G_1 \) be the subgroup of \( \text{GL}_4(\mathbb{C}) \)

\[
G_1 := \left\{ \begin{pmatrix}
a^2\bar{a} & 0 & 0 & 0 \\
c & a\bar{a} & 0 & 0 \\
d & b & a & 0 \\
e & d & c & a^2\bar{a}
\end{pmatrix} , \ a \in \mathbb{C} \setminus \{0\}, \ b, c, d, e \in \mathbb{C} \right\}.
\]

Then the pullback \( \omega \) of \( \omega_0 \) by \( h \), \( \omega := h^*\omega_0 \), satisfies:

\[
\omega = g \cdot \omega_0,
\]

where \( g \) is smooth (locally defined) function \( M \overset{g}{\longrightarrow} G_1 \).

This motivates the introduction of the subbundle \( P^1 \) of the bundle of coframes on \( M \) constituted by the coframes \( \omega \) of the form

\[
\omega := g \cdot \omega_0, \quad g \in G_1.
\]

The next section is devoted to reduce successively \( P^1 \) to three subbundles:

\[
P^4 \subset P^3 \subset P^2 \subset P^1,
\]

which are adapted to the biholomorphic equivalence problem for \( M \).

### 3. Reducions of \( P^1 \)

The coframe \( \omega_0 \) gives a natural (local) trivialisation \( P^1 \overset{tr}{\longrightarrow} M \times G_1 \) from which we may consider any differential form on \( M \) (resp. \( G_1 \)) as a differential form on \( P^1 \) through the pullback by the first (resp. the second) component of \( tr \). With this identification, the structure equations of \( P^1 \) are naturally obtained by the formula:

\[
d\omega = dg \cdot g^{-1} \land \omega + g \cdot d\omega_0.
\]

The term \( g \cdot d\omega_0 \) contains the so-called torsion coefficients of \( P^1 \). A 1-form \( \tilde{\alpha} \) on \( P^1 \) is called a modified Maurer-Cartan form if its restriction to any fiber of \( P^1 \) is a Maurer-Cartan form of \( G_1 \), or equivalently, if it is of the form:

\[
\tilde{\alpha} := \alpha - x_\sigma \sigma - x_\rho \rho - x_\zeta \zeta - x_{\overline{\zeta}} \overline{\zeta},
\]

where \( x_\sigma, x_\rho, x_\zeta, x_{\overline{\zeta}} \) are arbitrary complex-valued functions on \( M \) and where \( \alpha \) is a Maurer-Cartan form of \( G_1 \).
A basis for the Maurer-Cartan forms of $G_1$ is given by the following 1-forms:

\[ \alpha^1 := \frac{da}{a}, \]
\[ \alpha^2 := -\frac{bda}{a^2a} + \frac{db}{a^2}, \]
\[ \alpha^3 := \frac{cd}{a^3} - \frac{c d a}{a^2a^2} + \frac{dc}{a^2}, \]
\[ \alpha^4 = -\frac{(d a b - b c) d a}{a^3a^2} - \frac{c d b}{a^3a^2} + \frac{dd}{a^2}, \]
\[ \alpha^5 = -\frac{(e a b - B c) d a}{a^3a^2} - \frac{c d B}{a^3a^2} + \frac{de}{a^2}, \]

together with their conjugate.

We derive the structure equations of $P^1$ from the relations (3), from which we extract the expression of $d\sigma$:

\[
d\sigma = 2 \alpha^1 \wedge \sigma + \overline{\alpha^1} \wedge \sigma
+ T_{\sigma\rho}^\sigma \sigma \wedge \rho - T_{\sigma\zeta}^\sigma \sigma \wedge \zeta
- T_{\sigma\bar{\zeta}}^\sigma \sigma \wedge \bar{\zeta}
+ \rho \wedge \zeta + \frac{a}{a} B \rho \wedge \bar{\zeta},
\]
or equivalently:

\[
d\sigma = 2 \bar{\alpha}^1 \wedge \sigma + \overline{\bar{\alpha}^1} \wedge \sigma
+ \rho \wedge \zeta + \frac{a}{a} B \rho \wedge \bar{\zeta},
\]

for a modified Maurer-Cartan form $\bar{\alpha}^1$. The coefficient

\[
\frac{a}{a} B,
\]

which can not be absorbed for any choice of the modified Maurer-Cartan form $\bar{\alpha}^1$, is referred to as an essential torsion coefficient. From standard results on Cartan theory (see [10, 15]), a diffeomorphism of $M$ is an isomorphism of the $G_1$-structure $P^1$ if and only if it is an isomorphism of the reduced bundle $P^2 \subset P^1$ consisting of those coframes $\omega$ on $M$ such that

\[
\frac{a}{a} B = 1.
\]

This is equivalent to the normalization:

\[
\overline{a} = a B.
\]

A coframe $\omega \in P^2$ is related to the coframe $\omega_0$ by the relations:

\[
\begin{align*}
\sigma &= a^3 B \sigma_0 \\
rho &= c \sigma_0 + a^2 B \rho_0 \\
\zeta &= d \sigma_0 + b \rho_0 + a \zeta_0 \\
\overline{\zeta} &= e \sigma_0 + B \rho_0 + a B \overline{\zeta_0},
\end{align*}
\]
which are equivalent to:

\[
\begin{align*}
\sigma &= a'^3 \sigma_1 \\
\rho &= c' \sigma_1 + a'^2 \rho_1 \\
\zeta &= d' \sigma_1 + b \rho_1 + a' \zeta_1 \\
\overline{\zeta} &= e' \sigma_1 + \overline{b} \rho_1 + a' \overline{\zeta}_1,
\end{align*}
\]

where:

\[
\begin{align*}
\sigma_1 &:= \frac{\sigma_0}{B^2}, \quad \rho_1 := \rho_0, \quad \zeta_1 := \frac{\zeta_0}{B^2},
\end{align*}
\]

and

\[
x' := x \cdot B^1, \quad \text{for} \quad x = a, c, d, e.
\]

We notice that \(a'\) is a real parameter, and that \(\sigma_1\) is a real 1-form. Let \(\omega_1\) be the coframe \(\omega_1 := (\sigma_1, \rho_1, \zeta_1, \overline{\zeta}_1)\), and \(G_2\) be the subgroup of \(G_1\):\n
\[
G_2 := \left\{ \begin{pmatrix}
a^3 & 0 & 0 & 0 \\
c & a^2 & 0 & 0 \\
d & b & a & 0 \\
e & \overline{b} & 0 & a
\end{pmatrix}, \quad a \in \mathbb{R} \setminus \{0\}, \quad b, c, d, e \in \mathbb{C} \right\}.
\]

A coframe \(\omega\) on \(M\) belongs to \(P^2\) if and only if there is a local function \(g : M \rightarrow G_2\) such that \(\omega = g \cdot \omega_1\), namely \(P^2\) is a \(G_2\) structure on \(M\).

The Maurer-Cartan forms of \(G_2\) are given by:

\[
\begin{align*}
\beta^1 &= \frac{da}{a}, \\
\beta^2 &= -\frac{bda}{a^3} + \frac{db}{a^2}, \\
\beta^3 &= -2 \frac{cda}{a^4} + \frac{dc}{a^3}, \\
\beta^4 &= -\frac{(da^2 - bc) da}{a^6} - \frac{cd b}{a^5} + \frac{dd}{a^3}, \\
\beta^5 &= -\frac{(ea^2 - \overline{b} c) da}{a^6} - \frac{cd \overline{b}}{a^5} + \frac{de}{a^3},
\end{align*}
\]

together with \(\overline{\beta}^2, \overline{\beta}^3, \overline{\beta}^4, \overline{\beta}^5\). Using formula (3), we get the structure equations of \(P^2\):

\[
d\sigma = 3 \beta^1 \wedge \sigma \\
+ U_{\sigma \rho}^\sigma \sigma \wedge \rho + U_{\sigma \zeta}^\sigma \sigma \wedge \zeta + U_{\sigma \overline{\zeta}}^\sigma \sigma \wedge \overline{\zeta} + \rho \wedge \zeta + \rho \wedge \overline{\zeta}
\]
\[ d\rho = 2\beta^1 \wedge \rho + \beta^3 \wedge \sigma + U^\rho_{\sigma \rho} \sigma \wedge \rho + U^\rho_{\sigma \zeta} \sigma \wedge \zeta + U^\rho_{\sigma \bar{\zeta}} \sigma \wedge \bar{\zeta} + U^\rho_{\rho \zeta} \rho \wedge \zeta + U^\rho_{\rho \bar{\zeta}} \rho \wedge \bar{\zeta} + i \zeta \wedge \bar{\zeta}, \]

\[ d\zeta = \beta^1 \wedge \zeta + \beta^2 \wedge \rho + \beta^4 \wedge \sigma + U^\zeta_{\sigma \rho} \sigma \wedge \rho + U^\zeta_{\sigma \zeta} \sigma \wedge \zeta + U^\zeta_{\sigma \bar{\zeta}} \sigma \wedge \bar{\zeta} + U^\zeta_{\rho \zeta} \rho \wedge \zeta + U^\zeta_{\rho \bar{\zeta}} \rho \wedge \bar{\zeta} + i \zeta \wedge \bar{\zeta}. \]

Introducing the modified Maurer-Cartan forms:

\[ \tilde{\beta}^i = \beta^i - y^i_{\rho} \sigma - y^i_{\zeta} \zeta - y^i_{\bar{\zeta}} \bar{\zeta}, \]

the structure equations rewrite:

\[ d\sigma = 3 \tilde{\beta}^1 \wedge \sigma + (U^\sigma_{\sigma \rho} - 3 y^1_{\rho}) \sigma \wedge \rho + (U^\sigma_{\sigma \zeta} - 3 y^1_{\zeta}) \sigma \wedge \zeta + (U^\sigma_{\sigma \bar{\zeta}} - 3 y^1_{\bar{\zeta}}) \sigma \wedge \bar{\zeta} + \rho \wedge \zeta + \rho \wedge \bar{\zeta}, \]

\[ d\rho = 2\tilde{\beta}^1 \wedge \rho + \tilde{\beta}^3 \wedge \sigma + (U^\rho_{\sigma \rho} + 2 y^1_{\rho} - y^3_{\rho}) \sigma \wedge \rho + (U^\rho_{\sigma \zeta} - y^3_{\zeta}) \sigma \wedge \zeta + (U^\rho_{\sigma \bar{\zeta}} - y^3_{\bar{\zeta}}) \sigma \wedge \bar{\zeta} + (U^\rho_{\rho \zeta} - 2 y^1_{\zeta}) \rho \wedge \zeta + (U^\rho_{\rho \bar{\zeta}} - 2 y^1_{\bar{\zeta}}) \rho \wedge \bar{\zeta} + i \zeta \wedge \bar{\zeta}, \]

\[ d\zeta = \tilde{\beta}^1 \wedge \zeta + \tilde{\beta}^2 \wedge \rho + \tilde{\beta}^4 \wedge \sigma + (U^\zeta_{\sigma \rho} + y^2_{\rho} - y^4_{\rho}) \sigma \wedge \rho + (U^\zeta_{\sigma \zeta} + y^4_{\sigma} - y^4_{\zeta}) \sigma \wedge \zeta + (U^\zeta_{\sigma \bar{\zeta}} + y^4_{\sigma} - y^4_{\bar{\zeta}}) \sigma \wedge \bar{\zeta} + (U^\zeta_{\rho \zeta} + y^4_{\rho} - y^4_{\bar{\zeta}}) \rho \wedge \zeta + (U^\zeta_{\rho \bar{\zeta}} - y^4_{\bar{\zeta}}) \rho \wedge \bar{\zeta} + \zeta \wedge \bar{\zeta}. \]
which leads to the following absorption equations:

\[
\begin{align*}
3 y_1^\rho &= U_{\sigma \rho}^\sigma, & 3 y_1^\zeta &= U_{\sigma \zeta}^\sigma, & 3 y_1^\zeta &= U_{\sigma \zeta}^\sigma, \\
-2 y_1^\rho + y_3^\rho &= U_{\rho \rho}^\rho, & y_3^\zeta &= U_{\rho \zeta}^\rho, & y_3^\zeta &= U_{\rho \zeta}^\rho, \\
2 y_1^\zeta &= U_{\rho \zeta}^\zeta, & 2 y_1^\zeta &= U_{\rho \zeta}^\zeta, & -y_2^\sigma + y_4^\rho &= U_{\sigma \rho}^\zeta, \\
-y_1^\sigma + y_4^\zeta &= U_{\sigma \zeta}^\rho, & -y_1^\sigma + y_4^\zeta &= U_{\sigma \zeta}^\rho, & y_4^\zeta &= U_{\zeta \zeta}^\zeta.
\end{align*}
\]

Eliminating \( y_1^\zeta \) among these equations leads to:

\[
U_{\zeta \zeta}^\zeta = \frac{1}{2} U_{\rho \rho}^\rho = \frac{1}{3} U_{\sigma \sigma}^\sigma,
\]

from which we deduce the following normalizations:

\[
\begin{align*}
c &= a^2 C_0, \\
and \\
b &= a B_0,
\end{align*}
\]

where:

\[
C_0 := \left( \frac{1}{2} \mathcal{L}(B) - \frac{1}{2} Q B^2 \right),
\]

and

\[
B_0 := \left( \frac{i}{3} \mathcal{L}(B) \right) + \frac{i}{3} \frac{A}{B^2} - \frac{i}{6} \frac{Q}{B^2} - \frac{i}{6} \mathcal{L}(B).
\]

We introduce the coframe \( \omega_2 := (\sigma_2, \rho_2, \zeta_2, \bar{\zeta}_2) \) on \( M \), defined by:

\[
\begin{align*}
\sigma_2 &:= \sigma_1, \\
\rho_2 &:= \rho_1 + C_0 \sigma_1, \\
\zeta_2 &:= \zeta_1 + B_0 \rho_1,
\end{align*}
\]

and the 3-dimensional subgroup \( G_3 \subset G_2 \):

\[
G_3 := \left\{ \begin{pmatrix} a^3 & 0 & 0 \\ 0 & a^2 & 0 \\ d & 0 & a \end{pmatrix} : a \in \mathbb{R} \setminus \{0\}, d \in \mathbb{C} \right\}.
\]

The normalizations:

\[
b := a B_0, \quad c := a^2 C_0,
\]

amount to consider the subbundle \( P^3 \subset P^2 \) consisting of those coframes \( \omega \) of the form

\[
\omega := g \cdot \omega_2, \quad \text{where} \ g \ \text{is a function} \ \ g : M \to G_3.
\]
A basis of the Maurer Cartan forms of $G_3$ is given by:
\[ \gamma_1 := \frac{da}{a}, \quad \gamma_2 := - \frac{dda}{a} + \frac{dd}{a^3}, \quad \gamma_2. \]

The structure equations of $P^3$ are:
\[ d\sigma = 3 \gamma_1 \wedge \sigma + V^\rho_{\sigma \rho} \sigma \wedge \rho + V^\rho_{\sigma \zeta} \sigma \wedge \zeta + V^\rho_{\sigma \zeta} \sigma \wedge \zeta + \rho \wedge \zeta + \rho \wedge \zeta, \]
\[ d\rho = 2 \gamma_1 \wedge \rho + V^\rho_{\sigma \rho} \sigma \wedge \rho + V^\rho_{\sigma \zeta} \sigma \wedge \zeta + V^\rho_{\sigma \zeta} \sigma \wedge \zeta + V^\rho_{\sigma \zeta} \rho \wedge \zeta + V^\rho_{\sigma \zeta} \rho \wedge \zeta + i \zeta \wedge \zeta, \]
\[ d\zeta = \gamma_1 \wedge \zeta + \gamma_2 \wedge \sigma + V^\zeta_{\sigma \rho} \sigma \wedge \rho + V^\zeta_{\sigma \zeta} \sigma \wedge \zeta + V^\zeta_{\sigma \zeta} \sigma \wedge \zeta + V^\zeta_{\sigma \zeta} \rho \wedge \zeta + V^\zeta_{\sigma \zeta} \rho \wedge \zeta + V^\zeta_{\sigma \zeta} \zeta \wedge \zeta \]
\[ d\zeta = \gamma_1 \wedge \zeta + \gamma_3 \wedge \sigma + V^\zeta_{\sigma \rho} \sigma \wedge \rho + V^\zeta_{\sigma \zeta} \sigma \wedge \zeta + V^\zeta_{\sigma \zeta} \sigma \wedge \zeta + V^\zeta_{\sigma \zeta} \rho \wedge \zeta + V^\zeta_{\sigma \zeta} \rho \wedge \zeta + V^\zeta_{\sigma \zeta} \zeta \wedge \zeta. \]

It is straightforward to notice that $V^\rho_{\sigma \zeta}$ and $V^\rho_{\sigma \zeta}$ are two essential torsion coefficients. The first one leads to the normalization:
\[ d = a \mathcal{D}_0, \]
with
\[ \mathcal{D}_0 := \frac{i}{2} \mathcal{L}(B)^2 - \frac{1}{6} Q \mathcal{L}(B) - \frac{i}{2} \mathcal{L}(\mathcal{L}(B)) - \frac{i}{2} B \mathcal{L}(Q) + \frac{i}{2} A \mathcal{L}(B) - \frac{i}{6} A Q - i B P, \]
while the second essential torsion coefficient gives the normalization:
\[ d = a \mathcal{D}_0, \]
with:
\[ \mathcal{D}_0 := - \frac{2i}{3} \mathcal{L}(B) Q - \frac{i}{6} \mathcal{L}(B) A - \frac{i}{6} A Q - \frac{i}{6} \mathcal{L}(B) Q + \frac{i}{2} \mathcal{L}(B) \mathcal{L}(B) - \frac{i}{3} B Q^2 - \frac{i}{3} \mathcal{L}(A) - \frac{i}{3} \mathcal{L}(B) \mathcal{L}(B) + \frac{i}{2} \mathcal{L}(Q) - i B P. \]

The coherency of the above formulae can be checked using the relations (1) and (2).
Let \( G_4 \) be the 1-dimensional Lie subgroup of \( G_3 \) whose elements \( g \) are of the form:
\[
g := \begin{pmatrix}
a^3 & 0 & 0 & 0 \\
0 & a^2 & 0 & 0 \\
0 & 0 & a & 0 \\
0 & 0 & 0 & a
\end{pmatrix}, \quad a \in \mathbb{R} \setminus \{0\},
\]
and let \( \omega_3 := (\sigma_3, \rho_3, \zeta_3, \bar{\zeta}_3) \) be the coframe defined on \( M \) by:
\[
\sigma_3 := \sigma_2, \quad \rho_3 := \rho_2, \quad \zeta_3 := \zeta_2 + D_0 \sigma_2.
\]
The normalization of \( d \) is equivalent to the reduction of \( P^3 \) to a subbundle \( P^4 \) consisting of those coframes \( \omega \) on \( M \) such that:
\[
\omega := g \cdot \omega_3, \quad \text{where } g \text{ is a function } g : M \to G_4.
\]
The Maurer-Cartan forms of \( G_4 \) are spanned by:
\[
\alpha := \frac{da}{a}.
\]
Proceeding as in the previous steps, we compute the structure equations of \( P^4 \):
\[
d\sigma = 3 \frac{da}{a} \wedge \sigma \\
\quad + W^\sigma_{\sigma\rho} \sigma \wedge \rho + W^\sigma_{\sigma\zeta} \sigma \wedge \zeta + W^\sigma_{\sigma\bar{\zeta}} \sigma \wedge \bar{\zeta} + \rho \wedge \zeta + \rho \wedge \bar{\zeta},
\]
\[
d\rho = 2 \frac{da}{a} \wedge \rho + W^\rho_{\sigma\rho} \sigma \wedge \rho + W^\rho_{\rho\zeta} \rho \wedge \zeta + W^\rho_{\rho\bar{\zeta}} \rho \wedge \bar{\zeta} + i \zeta \wedge \bar{\zeta},
\]
\[
d\zeta = 2 \frac{da}{a} \wedge \zeta \\
\quad + W^\zeta_{\sigma\rho} \sigma \wedge \rho + W^\zeta_{\sigma\zeta} \sigma \wedge \zeta + W^\zeta_{\sigma\bar{\zeta}} \sigma \wedge \bar{\zeta} + W^\zeta_{\rho\zeta} \rho \wedge \zeta \\
\quad + W^\zeta_{\rho\bar{\zeta}} \rho \wedge \bar{\zeta} + W^\zeta_{\zeta\zeta} \zeta \wedge \bar{\zeta},
\]
\[
d\bar{\zeta} = \frac{da}{a} \wedge \bar{\zeta} \\
\quad + W^{\bar{\zeta}}_{\sigma\rho} \sigma \wedge \rho + W^{\bar{\zeta}}_{\sigma\zeta} \sigma \wedge \zeta + W^{\bar{\zeta}}_{\sigma\bar{\zeta}} \sigma \wedge \bar{\zeta} + W^{\bar{\zeta}}_{\rho\zeta} \rho \wedge \zeta \\
\quad + W^{\bar{\zeta}}_{\rho\bar{\zeta}} \rho \wedge \bar{\zeta} + W^{\bar{\zeta}}_{\zeta\zeta} \zeta \wedge \bar{\zeta}.
\]
Introducing the modified Maurer-Cartan form \( \Lambda \):
\[
\Lambda := \frac{da}{a} + \frac{W^\rho_{\sigma\rho}}{2} \rho - \frac{W^\rho_{\sigma\zeta}}{3} \sigma - \frac{W^\rho_{\sigma\bar{\zeta}}}{3} \zeta - \frac{W^\rho_{\rho\bar{\zeta}}}{3} \bar{\zeta},
\]
these equations rewrite in the absorbed form as:

\( d\sigma = 3 \Lambda \wedge \sigma + \rho \wedge \zeta + \rho \wedge \overline{\zeta}, \)

\( d\rho = 2 \Lambda \wedge \rho + i \zeta \wedge \overline{\zeta}, \)

\( d\zeta = \Lambda \wedge \zeta + \frac{I_1}{a^3} \sigma \wedge \rho + \frac{I_2}{a^3} \sigma \wedge \zeta + \frac{I_3}{a^3} \sigma \wedge \overline{\zeta} + \frac{I_4}{a^2} \rho \wedge \zeta + \frac{I_5}{a^2} \rho \wedge \overline{\zeta}, \)

\( d\overline{\zeta} = \Lambda \wedge \overline{\zeta} + \frac{T_1}{a^3} \sigma \wedge \rho + \frac{T_2}{a^3} \sigma \wedge \zeta + \frac{T_3}{a^3} \sigma \wedge \overline{\zeta} + \frac{T_4}{a^2} \rho \wedge \zeta + \frac{T_5}{a^2} \rho \wedge \overline{\zeta}, \)

where the invariants \( I_i, \ i = 2 \ldots 5, \) are given by:

\[
\begin{align*}
T_2 &= \frac{i}{8} \frac{Q \mathcal{L}(B)^2}{B^\frac{3}{2}} - \frac{i}{8} \frac{B^\frac{3}{2} \mathcal{L}(B) Q^2}{B^\frac{3}{2}} - \frac{3i}{4} \frac{\mathcal{L}(\mathcal{L}(B)) \mathcal{L}(B)}{B^\frac{3}{2}} + \frac{i}{4} B^\frac{3}{2} \mathcal{L}(B) \mathcal{L}(Q) \\
&\quad - \frac{i}{2} B^\frac{3}{2} P \mathcal{L}(B) - \frac{i}{4} B^\frac{3}{2} Q \mathcal{L}(\mathcal{L}(B)) - \frac{i}{4} B^\frac{3}{2} Q \mathcal{L}(\mathcal{L}(B)) \\
&\quad - \frac{3i}{4} \frac{B^\frac{3}{2} Q \mathcal{L}(Q)}{B^\frac{3}{2}} + \frac{i}{2} \frac{B^\frac{3}{2} P Q}{B^\frac{3}{2}} + \frac{3i}{8} \frac{\mathcal{L}(B)^3}{B^\frac{3}{2}} + \frac{i}{8} \frac{B^\frac{3}{2} Q^3}{B^\frac{3}{2}} \\
&\quad + \frac{i}{2} \frac{B^\frac{3}{2} \mathcal{L}(\mathcal{L}(Q))}{B^\frac{3}{2}} + \frac{i}{2} B^\frac{3}{2} \mathcal{L}(\mathcal{L}(\mathcal{L}(B))) - iB^\frac{3}{2} \mathcal{L}(P) \\
I_3 &= -D_0 C_0 + \frac{\mathcal{L}(B)}{B^\frac{3}{2}} - D_0 B^\frac{3}{2} Q D_0 + \frac{A}{B^\frac{3}{2}} D_0 - \frac{\overline{\mathcal{F}}(D_0)}{B^\frac{3}{2}} \frac{\mathcal{L}(\mathcal{L}(B))}{B^\frac{3}{2}} - iB_0 D_0 + i B_0^2 C_0 \\
&\quad - \frac{A}{B^\frac{3}{2}} B_0 C_0 + B_0 \mathcal{L}(A) + B P B_0 + \frac{\overline{\mathcal{F}}(B_0)}{B^\frac{3}{2}} C_0 + \frac{1}{2} \frac{\overline{\mathcal{F}}(B)}{B^\frac{3}{2}} B_0 C_0 \\
T_4 &= \frac{3}{4} \frac{\mathcal{L}(B)^2}{B} + \frac{1}{6} \frac{\mathcal{L}(B) Q}{B} + \frac{11}{36} i B Q^2 - i \mathcal{L}(\mathcal{L}(B)) - \frac{2}{3} i B \mathcal{L}(Q) + i B P, \\
T_5 &= \frac{i}{3} \mathcal{L}(A) + \frac{i}{3} \overline{\mathcal{F}}(Q) - i \frac{\overline{\mathcal{F}}(\mathcal{L}(B))}{B} + \frac{5}{12} i \frac{\mathcal{L}(B)^2}{B} - i \frac{3}{3} \mathcal{L}(\mathcal{L}(Q)) \\
&\quad + \frac{11}{36} i B Q^2 + i B P + \frac{2}{3} i \frac{\mathcal{L}(\overline{\mathcal{F}}(B))}{B} - i \frac{3}{3} \mathcal{L}(\mathcal{L}(B)) \\
&\quad + \frac{i}{3} \frac{A \mathcal{L}(B)}{B} + \frac{7}{18} i \mathcal{L}(B) Q - \frac{i}{9} \frac{\overline{\mathcal{F}}(B) Q}{B} + \frac{i}{9} A Q,
\end{align*}
\]

and \( I_1 \) is given by:

\( I_1 = \frac{2i}{3} (I_3) \zeta - \frac{2i}{3} (I_2) \overline{\zeta}. \)

The exterior derivative of \( \Lambda \) can be determined by taking the exterior derivative of the four equations \((4)\), which leads to the so-called Bianchi-Cartan’s identities. For example, taking the exterior derivative of the first
equation of (4), one gets:

$$0 = \left[ 3d\Lambda + \left( \frac{I_2}{a^3} + \frac{I_3}{a^3} \right) \rho \wedge \zeta + \left( \frac{\bar{I}_2}{\bar{a}^3} + \frac{\bar{I}_3}{\bar{a}^3} \right) \rho \wedge \bar{\zeta} \right] \wedge \sigma,$$

while taking the exterior derivative of the second equation gives:

$$0 = \left[ 2d\Lambda - i\frac{I_1}{a^4} \sigma \wedge \bar{\zeta} + i\frac{\bar{I}_1}{\bar{a}^4} \sigma \wedge \bar{\zeta} \right] \wedge \rho.$$

Eventually we get:

$$d\Lambda = \frac{i}{2a^4} \sigma \wedge \bar{\zeta} - \frac{I_1}{2a^4} \sigma \wedge \zeta - \frac{1}{3} \left( \frac{I_2}{a^3} + \frac{\bar{I}_3}{\bar{a}^3} \right) \rho \wedge \zeta - \frac{1}{3} \left( \frac{\bar{I}_2}{\bar{a}^3} + \frac{I_3}{a^3} \right) \rho \wedge \bar{\zeta} + \frac{I_0}{a^4} \sigma \wedge \zeta,$$

where $I_0$ is given by:

$$I_0 := -\frac{1}{2a^4} (I_1) \zeta - \frac{1}{2a^4} (\bar{I}_1) \bar{\zeta}.$$

4. Cartan Connection

We recall that the model for CR-manifolds belonging to general class II is Beloshapka’s cubic $B \subset \mathbb{C}^3$, which is defined by the equations:

$$B : \begin{align*}
    w_1 &= \overline{w_1} + 2iz\bar{\zeta}, \\
    w_2 &= \overline{w_2} + 2iz\bar{\zeta}(z + \bar{z}).
\end{align*}$$

Its Lie algebra of infinitesimal CR-automorphisms is given by the following theorem:

**Theorem 2. [12].** Beloshapka’s cubic,

$$B : \begin{align*}
    w_1 &= \overline{w_1} + 2iz\bar{\zeta}, \\
    w_2 &= \overline{w_2} + 2iz\bar{\zeta}(z + \bar{z}),
\end{align*}$$

has a 5-dimensional Lie algebra of CR-automorphisms $\text{aut}_{\text{CR}}(B)$. A basis for the Maurer-Cartan forms of $\text{aut}_{\text{CR}}(B)$ is provided by the 5 differential 1-forms $\sigma$, $\rho$, $\zeta$, $\bar{\zeta}$, $\alpha$, which satisfy the structure equations:

$$\begin{align*}
    d\sigma &= 3\alpha \wedge \sigma + \rho \wedge \zeta + \rho \wedge \bar{\zeta}, \\
    d\rho &= 2\alpha \wedge \rho + i\zeta \wedge \bar{\zeta}, \\
    d\zeta &= \alpha \wedge \zeta, \\
    d\bar{\zeta} &= \alpha \wedge \bar{\zeta}, \\
    d\alpha &= 0.
\end{align*}$$
Let us write \( \mathfrak{g} \) instead of \( \text{aut}_{\text{CR}}(B) \) for the Lie algebra of infinitesimal automorphisms of Beloshapka’s cubic, and let \( (e_\alpha, e_\sigma, e_\rho, e_\zeta, e_\overline{\zeta}) \) be the dual basis of the basis of Maurer-Cartan 1-forms: \( (\alpha, \sigma, \rho, \zeta, \overline{\zeta}) \) of \( \mathfrak{g} \). From the above structure equations, the Lie brackets structure of \( \mathfrak{g} \) is given by:

\[
\begin{align*}
[e_\alpha, e_\sigma] &= -3e_\sigma, & [e_\alpha, e_\rho] &= -2e_\rho, & [e_\alpha, e_\zeta] &= -e_\zeta, \\
[e_\alpha, e_\overline{\zeta}] &= -e_\overline{\zeta}, & [e_\rho, e_\zeta] &= -e_\rho, & [e_\rho, e_\overline{\zeta}] &= -e_\rho, \\
[e_\zeta, e_\overline{\zeta}] &= -i e_\rho,
\end{align*}
\]

the remaining brackets being equal to zero.

We refer to [5], p. 127-128, for the definition of a Cartan connection. Let \( \mathfrak{g}_0 \subset \mathfrak{g} \) be the subalgebra spanned by \( e_\alpha, G \) the connected, simply connected Lie group whose Lie algebra is \( \mathfrak{g} \) and \( G_0 \) the connected closed 1-dimensional subgroup of \( G \) generated by \( \mathfrak{g}_0 \). We notice that \( G_0 \cong G_4 \), so that \( P^4 \) is a principal bundle over \( M \) with structure group \( G_0 \), and that \( \dim G / G_0 = \dim M = 4 \).

Let \( (\Lambda, \sigma, \rho, \zeta, \overline{\zeta}) \) be the coframe of 1-forms on \( P^4 \) whose structure equation are given by (4) – (5) and \( \omega \) the 1-form on \( P \) with values in \( \mathfrak{g} \) defined by:

\[
\omega(X) := \Lambda(X) e_\alpha + \sigma(X) e_\sigma + \rho(X) e_\rho + \zeta(X) e_\zeta + \overline{\zeta}(X) e_\overline{\zeta},
\]

for \( X \in T_p P^4 \). We have:

**Theorem 3.** \( \omega \) is a Cartan connection on \( P^4 \).

**Proof.** We shall check that the following three conditions hold:

1. \( \omega(e_\alpha^*) = e_\alpha \), where \( e_\alpha^* \) is the vertical vector field on \( P^4 \) generated by the action of \( e_\alpha \).
2. \( R^*_a \omega = \text{Ad}(a^{-1}) \omega \) for every \( a \in G_0 \),
3. for each \( p \in P^4 \), \( \omega_p \) is an isomorphism \( T^*_p P^4 \rightarrow \mathfrak{g} \).

Condition (3) is trivially satisfied as \( (\Lambda, \sigma, \rho, \zeta, \overline{\zeta}) \) is a coframe on \( P^4 \) and thus defines a basis of \( T^*_p P^4 \) at each point \( p \).

Condition (1) follows simply from the fact that \( \Lambda \) is a modified-Maurer Cartan form on \( P^4 \):

\[
\Lambda = \frac{da}{a} + \frac{W^\rho}{2} \rho - \frac{W^\sigma}{3} \sigma - \frac{W^{\sigma \rho}}{3} \zeta - \frac{W^{\sigma \zeta}}{3} \overline{\zeta},
\]

so that

\[
\omega(e_\alpha^*) = \Lambda(e_\alpha^*) = e_\alpha,
\]

as

\[
\sigma(e_\alpha^*) = \rho(e_\alpha^*) = \zeta(e_\alpha^*) = \overline{\zeta}(e_\alpha^*) = 0, \quad \frac{da}{a}(e_\alpha^*) = 1,
\]
since \( e^*_\alpha \) is a vertical vector field on \( P^4 \).

Condition (2) is equivalent to its infinitesimal counterpart:

\[
\mathcal{L}_{e^*_\alpha} \omega = -\text{ad}_{e^*_\alpha} \omega,
\]

where \( \mathcal{L}_{e^*_\alpha} \omega \) is the Lie derivative of \( \omega \) by the vector field \( e^*_\alpha \) and where \( \text{ad}_{e^*_\alpha} \) is the linear map \( g \to g \) defined by: \( \text{ad}_{e^*_\alpha}(X) = [e_\alpha, X] \). We determine \( \mathcal{L}_{e^*_\alpha} \omega \) with the help of Cartan’s formula:

\[
\mathcal{L}_{e^*_\alpha} \omega = e^*\alpha \cdot d\omega + d(e^*\alpha \cdot \omega),
\]

with

\[
d(e^*\alpha \cdot \omega) = 0
\]

from condition (1). The structure equations (4)–(5) give:

\[
e^*\alpha \cdot d\omega = \begin{pmatrix}
0 \\
3\sigma \\
2\rho \\
\zeta \\
\zeta
\end{pmatrix},
\]

which is easily seen being equal to \( -\text{ad}_{e^*_\alpha} \omega \) from the Lie bracket structure of \( g \). \( \square \)

From theorem the structure equations (4) and (5), and the fact that the invariants \( I_0 \) and \( I_1 \) are expressed in terms of \( I_2, I_3, I_4, I_5 \), we have:

**Theorem 4.** A CR-manifold \( M \) belonging to general class II is locally biholomorphic to Beloshapka’s cubic \( \mathcal{B} \subset \mathbb{C}^3 \) if and only if the condition

\[
I_2 \equiv I_3 \equiv I_4 \equiv I_5 \equiv 0
\]

holds locally on \( M \).
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