Exact solutions of a two-dimensional Kemmer oscillator in the gravitational field of cosmic string

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The two dimensional Kemmer oscillator under the influence of the gravitational field produced by a topology such as the cosmic string spacetime and in the presence of a uniform magnetic field are investigated. The eigensolutions of our problem have been found by using the generalized parametric Nikiforov–Uvarov (NU) method, and the influence of the cosmic string space-time on the energy spectrum has been analyzed. We show that the dependence of the energy levels of the quantum system with the angular deficit \( \alpha \), which characterizes the global structure of the metric in the cosmic string spacetime, breaks the degeneracy of these levels.

I. INTRODUCTION

In relativistic quantum mechanics, the exact solutions of the wave equation are very important for the understanding of the physics that can be brought by such solutions. They are valuable tools in determining the radiative contributions to the energy. The quantum mechanics of charged, massive, and spin-1 particles in an external field has been studied in many different situations using different techniques \(^1\)\(^2\). These works have especially investigated the solutions of the equation in a magnetic field. These techniques are very complex in their determination of the eigensolutions of such particles. The relativistic wave equation for a massive spin-1 particle was initially derived by Kemmer in 1939. The Kemmer equation is a Dirac-type equation, which involves matrices obeying a different scheme of commutation rules \(^3\)\(^4\).

The analysis of gravitational interactions with a quantum mechanical system has recently attracted attention in particle physics and has been an active field of research. The general way to understand the interaction between relativistic quantum mechanical particles and gravity is to solve the general relativistic form of their wave equations. These equations may be considered insignificant at the atomic scale, where gravitational effects are weak, but the physics governing these particles plays an important role in astrophysics and cosmology, in which gravitational effects play a dominant role. In addition, studying single-particle states is important to constructing a unified theory of gravitation and quantum mechanics (see \(^5\) and references therein).

The Dirac oscillator was for the first time studied by Itô and Carriere. On the other side, Moszkowski and Szczepaniak were the first who introduced an interesting term in the Dirac equation. More specifically, they suggested to substitute in the free Dirac equation the momentum operator \( \vec{p} \) like \( \vec{p} = i \hbar \vec{\sigma} \). They could obtain a system in which the positive energy states have a spectrum similar to the one of the non-relativistic harmonic oscillator. Recently, this interaction has particularly got more interest \(^6\). It is reviewed, because of the interest in the many different domain in physics. Furthermore, the interaction of this oscillator with a gravitational field produced by topological defects, has become a well-investigated topic \(^10\)\(^19\).

The aim of the present study is to solve the Kemmer oscillator in a background produced by topological defects, such as cosmic strings and magnetic cosmic strings. Also, our contribution redresses the lack of existing literature concerning the physical properties of charged, massive scalar particles of spin-1 interacting with gravitational fields due to topological defects.

This paper is organized as follows: in Sec. II, we present a review of the the solutions of the DO in the cosmic string background. Sec. III is devoted to solve the case of the Kemmer oscillator in cosmic string space-time using NU method. Finally, Sec. IV, will be a conclusion.

II. EIGEN SOLUTIONS OF THE TWO-DIMENSIONAL DIRAC OSCILLATOR

A. The solutions without a magnetic field

In this section we review the solutions of a two-dimensional Dirac oscillator in the cosmic string background. The metric describe the cosmic string is given by:

\[
ds^2 = -dt^2 + d\rho^2 + \alpha^2 \rho^2 d\phi^2 + dz^2,\tag{1}\]

where \(-\infty < (t, z) < +\infty, 0 < \rho \leq \infty\) and \(0 \leq \phi \leq 2\pi\). The parameter \(\alpha\) is the deficit angle associated with conical geometry obeying \(\alpha = 1 - 4\eta\), and \(\eta\) is the linear mass density of the string in natural unite \(h = c = 1\).

The Dirac equation in the arbitrary curved spacetime is written by\(^11\)

\[
[i\gamma^\mu (x) (\partial_\mu - \Gamma_\mu) - m] \psi_D = 0,\tag{2}\]

where \(m\) is the mass of particles, \(\Gamma_\mu (x)\) are the spinor...
affine connections and $\gamma^\mu$ are the generalized Dirac matrices satisfying the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu},$$

and defined in terms of a set of tetrad fields by

$$\gamma^\mu(x) = e^a_\mu (x) \gamma^a,$$

where $e^a_\mu$ satisfies the relation $e^a_\mu (x) e^b_\nu (x) \eta^{ab} = g^{\mu\nu}$, and $(\mu, \nu) = (0,1,2,3)$ are tensor indices, $(a, b) = (0,1,2,3)$ are tetrad indices and $\gamma^a$ are the standard flat spacetime Dirac matrices $[12–14]$. The tensor $\eta^{ab} = \text{diag}(1, 1, 1, 1)$ is the Minkowski tensor.

For the two-dimensional case, we choose the Dirac matrices $\gamma^a$ in terms of Pauli matrices as $[20–22]$

$$\gamma^a = (\sigma^3, i\sigma^1, i\sigma^2),$$

with parameter $s$ taking the values $\pm 1$ (+1 for spin up and −1 for spin down) $[23]$.

The spinorial connection is given by $[22]$

$$\Gamma_\mu (x) = \frac{1}{8} \omega_{\mu ab} [\gamma^a, \gamma^b].$$

These components can be obtained by solving the Maurer-Cartan structure equations in the absence of torsions: $de^a + \omega^a_{\mu b} e^b = 0$, where $\omega^a_{\mu b} = \omega^a_{\mu b}(x) dx^\mu$, and $\omega^a_{\mu b}(x)$ is called connection 1-form $[24]$.

According to Eq. (1), we choose the tetrads for the line elements being

$$\hat{e}^0 = dt; \hat{e} = dp, \hat{e}^2 = p d\varphi, \hat{e}^3 = dz.$$

By solving the Maurer-Cartan structure equations, we obtain that

$$\gamma^\mu (x) \Gamma_\mu (x) = -\frac{1}{2\rho} \gamma^1.$$  

(8)

Now, to include the Dirac oscillator term $im\omega\beta \rho$ into Eq. (2), we proceed with the following substitution in the radial momentum component $\hat{\partial}_1 \rightarrow \hat{\partial}_1 + m\omega \rho$. Hence Eq. (2) is transformed into

$$\left\{ i\gamma^0 \partial_0 + i\gamma^1 \left( \hat{\partial}_1 + m\omega \rho \beta + \frac{1}{2\rho} \right) + i\gamma^2 \frac{\gamma^\mu}{\alpha \rho} \partial_2 - m \right\} \psi_D = 0.$$  

(9)

In order to solve the Eq. (9), we adopt the following Ansatz

$$\psi_D = e^{-iE t} e^{i(\frac{s}{2}) \varphi} \left( \phi(\rho) \chi(\rho) \right).$$

(10)

By substituting (10) into (9), we get the following system of equations

$$-\left\{ \partial_\rho + \frac{1}{2\rho} - m\omega \rho \right\} \phi - is \left( \frac{1 + \frac{s}{2}}{\alpha \rho} \right) \chi = 0,$$

(11)

$$+ \left\{ \partial_\rho + \frac{1}{2 \rho} + m\omega \rho \right\} + is \left( \frac{1 + \frac{s}{2}}{\alpha \rho} \right) \chi = 0.$$  

(12)

After a simple algebraic calculation, we have

$$\left[ \frac{\partial^2}{\partial \phi^2} + \frac{1}{\rho} \partial_\rho - \left( \frac{\eta^2}{4\rho^2} + \frac{1}{4} - \frac{\gamma^+}{4m\omega} \right) \right] \left( \phi(\rho) \chi(\rho) \right) = 0.$$  

(13)

and where

$$\eta_{\pm} = \lambda \pm \frac{1}{2} \left( \lambda = s + \frac{1}{2} \right),$$

(14)

By making a change of variables $\varphi = m\omega \rho^2$, Eq. (13) transforms into

$$\left[ \frac{\partial^2}{\partial \varphi^2} + \frac{1}{\varphi} \partial_\varphi - \left( \frac{\eta^2}{4\varphi^2} + \frac{1}{4} - \frac{\gamma^+}{4m\omega} \right) \right] \left( \phi(\rho) \chi(\rho) \right) = 0.$$  

(16)

Now, in order to solve the last equation, we use the well-known method based on the Nikiforov-Uvarov method (NU)[25, 26]. Thus, by comparing with the equations from (A1) to (A11), the following expressions are obtained

$$c_1 = 1, c_2 = 0, c_3 = 0,$$

(17)

$$\xi_1 = \frac{1}{4}, \xi_2 = \frac{\gamma^+}{4m\omega}, \xi_3 = \frac{\eta^2}{4},$$

(18)

$$c_4 = c_5 = 0, c_6 = \frac{1}{4}, c_7 = \frac{\gamma^-}{4m\omega},$$

(19)

$$c_8 = \frac{\eta^2}{4}, c_9 = \frac{1}{4}, c_{10} = 1 + |\eta^+|,$$

(20)

$$c_{11} = 1, c_{12} = \frac{|\eta^+|}{2}, c_{13} = \frac{1}{2},$$

(21)

$$2n + 1 - \frac{\gamma^-}{4m\omega} + |\eta^+| = 0,$$

(22)

From these equations, the form of energy levels is

$$E^2 = m^2 + 4m\omega \left( n + \frac{|\eta^+| - \eta^+}{2} + \frac{1}{2} \right),$$

(23)

with $n = (0, 1, \cdots)$, $l = (0, \pm 1, \pm 2, \cdots)$, and $s = \pm 1$. The eigenfunction is given in terms of hypergeometric confluent function as
\[
\begin{align*}
\left[ \chi (\rho) \right] & = (m\omega) \frac{|\eta|}{\rho} |\eta^\perp| e^{-\frac{\omega^2}{4}} \\
& \times F \left( -n, |\eta^\perp| + 1, m\omega \rho^2 \right). 
\end{align*}
\]

We note here that the eigensolutions obtained (Eqs. (23,24)) are similar to the one found by [11].

B. The solutions in a magnetic cosmic string background

The two-dimensional Dirac equation in the background of a cosmic string with magnetic field defined by its magnetic vector potential

\[
\vec{A}_\phi = \frac{\phi B}{2\pi \alpha \rho} \vec{e}_\phi, \tag{25}
\]
is

\[
\left\{ i\gamma^0 \partial_0 + i\gamma^1 \left( \partial_1 + m\omega \rho + \frac{1}{2\rho} \right) \right\} \psi_D + \left\{ i\gamma^2 \left( \partial_2 + i\frac{\phi B}{2\pi} - m \right) \right\} \psi_D = 0. \tag{26}
\]

Using the same Ansatz as in Eq. (11), and project it in Eq. (26), we obtain

\[
\left\{ \partial^2_\rho + \frac{1}{\rho} \partial_\rho - \left( m^2 \omega^2 \rho^2 + \frac{\eta^2}{\rho^2} - \Omega^2 \right) \right\} \chi (\rho) = 0,
\]
with the following substitutions

\[
\eta' = \frac{s}{2} \left( l + \frac{1}{2} \right) + \frac{\phi B}{2\pi} \pm \frac{1}{2} = \lambda' \pm \frac{1}{2}, \tag{27}
\]
and

\[
\Omega^2 = E^2 - m^2 + 2m\omega \left( \lambda' + \frac{1}{2} \right), \tag{28}
\]
Defining the following variable \( \varrho = m\omega \rho^2 \), Eq. (27) transforms into

\[
\left\{ \frac{\partial^2}{\partial \varrho^2} + \frac{1}{\varrho} \frac{\partial}{\varrho} - \left( \frac{\eta'}{4\varrho} + \frac{1}{4} - \frac{\Omega^2}{4m\omega \varrho} \right) \right\} \phi (\varrho) = 0. \tag{29}
\]
By using the NU method, we have that the following eigensolutions

\[
E^2 = m^2 + 4m\omega \left( n + \frac{|\eta^\perp|}{2} + \frac{\eta^\prime}{2} \right), \tag{30}
\]
with \( n = 0, 1, 2, \ldots \) and \( l = 0, \pm 1, \pm 2, \ldots \), and

\[
\eta^\prime = \lambda' \pm \frac{1}{2} = \frac{s}{2} \left( l + \frac{1}{2} \right) + \frac{\phi B}{2\pi} \right\} = \eta^\perp = \frac{s}{2} \left( lB + \frac{1}{2} \right), \tag{31}
\]
and where

\[
\left[ \chi (\rho) \right] = (m\omega) \frac{|\eta|}{\rho} |\eta^\perp| e^{-\frac{\omega^2}{4}} \\
& \times F \left( -n, |\eta^\perp| + 1, m\omega \rho^2 \right). \tag{32}
\]

Also, these eigensolutions are similar to those found in [11].

III. EIGEN SOLUTIONS OF THE TWO-DIMENSIONAL KEMMER OSCILLATOR

A. The solutions without a magnetic field

The free relativistic Kemmer equation in curved space-time is

\[
\left( i\tilde{\beta}^\mu \nabla_\mu - M \right) \psi_K = 0, \tag{33}
\]
where \( M = \text{total mass of identical spin } \frac{1}{2} \) particles and \( \tilde{\beta} \) are Kemmer matrices for the cosmic string background given by Eq. (11). They satisfy the following commutation relation

\[
\tilde{\beta}^\mu \tilde{\beta}^\nu \tilde{\beta}^\lambda + \tilde{\beta}^\lambda \tilde{\beta}^\nu \tilde{\beta}^\mu = g^{\mu\nu} \tilde{\beta}^\lambda + g^{\lambda\nu} \tilde{\beta}^\mu, \tag{34}
\]
with

\[
\tilde{\beta}^\mu = \gamma^\mu (x) \otimes \hat{I} + \hat{I} \otimes \gamma^\mu (x). \tag{35}
\]
The \( \gamma^\mu (x) \) are the Dirac matrices defined in the previous section Eq. (36), \( \hat{I} \) is a 4 \times 4 identity matrix, and \( \otimes \) indicates a direct product. The covariant derivative in equation (37) is

\[
\nabla_\mu = \partial_\mu - \Sigma_\mu, \tag{36}
\]
where the spinorial connections can be written as

\[
\Sigma_\mu = \lim_{\gamma \to \sigma} \Sigma_\mu = \left( \Gamma_\mu \otimes \hat{I} + \hat{I} \otimes \Gamma_\mu \right), \tag{37}
\]
where the spinorial connection \( \Gamma_\mu (x) \) is given by the equation (38). The stationary state \( \psi_K \) of the equation (34) is four-component wave function of the Kemmer equation, which can be written in the form

\[
\psi_K = \psi_D \otimes \psi_D = \left( \psi_1 \psi_2 \psi_3 \psi_4 \right)^T, \tag{39}
\]
with \( \psi_D \) is the solution of the Dirac equation.

Thus, the Kemmer equation in the cosmic string background is

\[
\left\{ i\tilde{\beta}^0 \partial_0 + i\tilde{\beta}^1 \partial_1 + i\tilde{\beta}^2 \left( \partial_2 - \Sigma_2 \right) - M \right\} \psi_K = 0. \tag{40}
\]

In the presence of Dirac oscillator potential, we could do the following change: \( \partial_1 \to \partial_1 + M\omega \rho B \). The operator
is chosen as $\hat{B} = \gamma^0 \otimes \gamma^0$ with $\hat{B}^2 = \hat{I}$. Hence the Kemmer equation with Dirac oscillator interaction

$$\{ i \left( \gamma^0 \otimes \hat{I} + \hat{I} \otimes \gamma^0 \right) \partial_0 + \[ + \] \} \psi_K = 0, \quad (41)$$

with

$$\[ = i \left( \gamma^1 \otimes \hat{I} + \hat{I} \otimes \gamma^1 \right) \left( \partial_1 + M \rho \partial_{\rho} \right), \quad (42)$$

$$\[ = i \left( \gamma^2 \otimes \hat{I} + \hat{I} \otimes \gamma^2 \right) \left( \partial_2 - \Sigma_2 - M \right). \quad (43)$$

Substituting Eq. (38) into Eq. (40), we obtain the following system of equations

$$(2E - M) \psi_1 - \left( \partial_1 - M \rho \frac{i \partial_{\rho}}{\alpha \rho} \right) \psi_2$$

$$- \left( \partial_2 - M \rho \frac{i \partial_{\rho}}{\alpha \rho} \right) \psi_3 = 0, \quad (44)$$

$$\left( \partial_1 + M \rho \partial_\rho + \frac{1}{\rho} + i \frac{\partial_{\rho}}{\alpha \rho} \right) \psi_1 + M \psi_2$$

$$+ \left( \partial_1 + M \rho \partial_\rho + \frac{1}{\rho} - i \frac{\partial_{\rho}}{\alpha \rho} \right) \psi_4 = 0, \quad (45)$$

$$\left( \partial_1 + M \rho \partial_\rho + \frac{1}{\rho} + i \frac{\partial_{\rho}}{\alpha \rho} \right) \psi_1 + M \psi_3$$

$$+ \left( \partial_1 + M \rho \partial_\rho + \frac{1}{\rho} - i \frac{\partial_{\rho}}{\alpha \rho} \right) \psi_4 = 0, \quad (46)$$

$$(2E + M) \psi_4 + \left( \partial_1 - M \rho \frac{i \partial_{\rho}}{\alpha \rho} \right) \psi_2$$

$$+ \left( \partial_1 - M \rho \frac{i \partial_{\rho}}{\alpha \rho} \right) \psi_3 = 0. \quad (47)$$

From these equations, we get the following results

$$\psi_2 = \psi_3, \quad (48)$$

$$\psi_1 = \frac{2 \left( \partial_1 - M \rho \frac{i \partial_{\rho}}{\alpha \rho} \right) \psi_2}{2E - M}, \quad (49)$$

$$\psi_4 = \frac{-2 \left( \partial_1 - M \rho \frac{i \partial_{\rho}}{\alpha \rho} \right) \psi_2}{2E + M}. \quad (50)$$

Putting Eqs. (48), (49) and (50) into Eq. (41), and with the following choice

$$\psi_2 = e^{iJ_\phi} \chi (\rho) \quad (51)$$

we get

$$\left( \frac{\partial_1^2 + \partial_1}{\rho} - M^2 \omega^2 \rho^2 - \frac{J^2}{\alpha^2 \rho^2} \right) \chi (\rho)$$

$$+ \left( -2M \omega + 4s E \omega J_\alpha + E^2 - \frac{M^2}{4} \right) \chi (\rho) = 0. \quad (52)$$

Now, when we use the following transformations:

$$\lambda = \frac{s J}{\alpha}, \quad (53)$$

$$\zeta = E^2 + 4E \omega \lambda - 2M \omega - \frac{M^2}{4}, \quad (54)$$

we have

$$\left\{ \frac{\partial_1^2 + \partial_1}{\rho} - \left( \frac{\lambda^2}{\rho^2} + M^2 \omega^2 \rho^2 - \zeta \right) \right\} \chi (\rho) = 0. \quad (55)$$

We remark that the last equation is similar to the Eq. (13). So, by using the same method as in the case of two-dimensional Dirac oscillator, we obtain

$$c_1 = 1, c_2 = 0, c_3 = 0, \quad (56)$$

$$\xi_1 = \frac{1}{4}, \xi_2 = \frac{s}{4M \omega}, \xi_3 = \frac{\lambda^2}{4}, \quad (57)$$

$$c_4 = c_5 = 0, c_6 = \frac{1}{4}, c_7 = -\frac{s}{4M \omega}, \quad (58)$$

$$c_8 = \frac{\lambda^2}{4}, c_9 = \frac{1}{4}, c_{10} = 1 + |\lambda|, \quad (59)$$

$$c_{11} = 1, c_{12} = \frac{|\lambda|}{2}, c_{13} = \frac{1}{2}, \quad (60)$$

$$\frac{2n + 1}{2} - \frac{s}{4M \omega} + \frac{|\lambda|}{2} = 0, \quad (61)$$

Thus, the eigensolutions are

$$\psi_2 (\rho) = e^{-iE_1 \rho} e^{iJ_\phi} (M \omega) \left( \frac{s J}{\alpha} \right) \left( \frac{s J_\alpha}{\alpha} \right) e^{-iM \omega \rho^2}$$

$$\times F \left( -n, \left| \frac{s J}{\alpha} \right| + 1, M \omega \rho^2 \right), \quad (62)$$
with \( r = \frac{M}{\omega e^{2}} \). We note here that the presence of the parameter \( \alpha \) in the spectrum of energy, breaks the degeneracy of the energy levels. Furthermore, by taking the limit \( \alpha \to 1 \) into Eq. (64), we reach the exact result of two-dimensional Kemmer oscillator in Minkowskian spacetime (27).

### B. The solutions in a magnetic cosmic string background

By adopting the same vector potential as in the Dirac oscillator, the Kemmer oscillator in a magnetic cosmic string spacetime obeys

\[
\begin{align*}
& \left( i \dot{b}^{0} \partial_{0} + i \dot{b}^{i} \partial_{i} \right) \psi_{K} \\
& + \left( i \dot{b}^{2} \left( \partial_{2} - \Sigma_{2} + i \frac{\phi_{B}}{2\pi} - M \right) \right) \psi_{K} = 0. \quad (65)
\end{align*}
\]

By using Eq. (65), the above equation transforms into

\[
\begin{align*}
(2E - M) \psi_{1} - & \left( \partial_{1} - M \omega + \frac{e \phi_{B}}{2\pi \alpha \rho} - i \frac{\delta_{2}}{\alpha \rho} \right) \psi_{2} \\
& - \left( \partial_{1} - M \omega + \frac{e \phi_{B}}{2\pi \alpha \rho} - i \frac{\delta_{2}}{\alpha \rho} \right) \psi_{3} = 0, \quad (66)
\end{align*}
\]

\[
\begin{align*}
& \left( \partial_{1} + M \omega + \frac{1}{\rho} - \frac{e \phi_{B}}{2\pi \alpha \rho} + i \frac{\delta_{2}}{\alpha \rho} \right) \psi_{1} + M \psi_{2} \\
& + \left( \partial_{1} + M \omega + \frac{1}{\rho} + \frac{e \phi_{B}}{2\pi \alpha \rho} - i \frac{\delta_{2}}{\alpha \rho} \right) \psi_{3} = 0, \quad (67)
\end{align*}
\]

\[
\begin{align*}
& \left( \partial_{1} + M \omega + \frac{1}{\rho} - \frac{e \phi_{B}}{2\pi \alpha \rho} + i \frac{\delta_{2}}{\alpha \rho} \right) \psi_{1} + M \psi_{3} \\
& + \left( \partial_{1} + M \omega + \frac{1}{\rho} + \frac{e \phi_{B}}{2\pi \alpha \rho} - i \frac{\delta_{2}}{\alpha \rho} \right) \psi_{4} = 0, \quad (68)
\end{align*}
\]

\[
\begin{align*}
(2E + M) \psi_{1} + & \left( \partial_{1} - M \omega - \frac{e \phi_{B}}{2\pi \alpha \rho} + i \frac{\delta_{2}}{\alpha \rho} \right) \psi_{2} \\
& + \left( \partial_{1} - M \omega - \frac{e \phi_{B}}{2\pi \alpha \rho} + i \frac{\delta_{2}}{\alpha \rho} \right) \psi_{3} = 0, \quad (69)
\end{align*}
\]

From these equations, we get the following results

\[
\psi_{2} = \psi_{3}, \quad (70)
\]

\[
\psi_{1} = 2 \left( \partial_{1} - M \omega + \frac{e \phi_{B}}{2\pi \alpha \rho} - i \frac{\delta_{2}}{\alpha \rho} \right) \psi_{2}, \quad (71)
\]

\[
\psi_{4} = -2 \left( \partial_{1} - M \omega - \frac{e \phi_{B}}{2\pi \alpha \rho} + i \frac{\delta_{2}}{\alpha \rho} \right) \psi_{2}. \quad (72)
\]

By putting Eqs. (71) and (72) into Eq. (65), and choosing that

\[
\psi_{2} = e^{iJ_{0} \chi(\rho)}, \quad (73)
\]

we have

\[
\left( \partial_{\rho}^{2} + \frac{\partial_{\rho}}{\rho} - M^{2} \omega^{2} \rho^{2} - 2M \omega - \frac{\psi}{\rho^{2}} + \xi \right) \chi(\rho), \quad (74)
\]

with

\[
|\xi| = \frac{J^{2}}{\alpha^{2}} + \frac{\left( \frac{s \phi_{B}}{2\pi} \right)^{2}}{\alpha^{2}} - \frac{\left( \frac{s \phi_{B}}{2\pi} \right) s J}{\alpha^{2}}, \quad (75)
\]

\[
\chi(\rho) = 4E_{\omega} \left\{ \frac{J}{\alpha} - \frac{s \phi_{B}}{2\pi} \right\} + E^{2} - \frac{M^{2}}{4}. \quad (76)
\]

Also, Eq. (74) can be rewritten by

\[
\left\{ \partial_{\rho}^{2} + \frac{1}{\rho} \partial_{\rho} - \left( \frac{\psi^{2}}{\rho^{2}} + M^{2} \omega^{2} \rho^{2} - \Lambda \right) \right\} \chi(\rho) = 0, \quad (77)
\]

with

\[
\mu = \frac{s \left( J - \frac{\phi_{B}}{2\pi} \right)}{\alpha}, \quad \Lambda = \left( E^{2} + 4E_{\omega} \mu - 2M \omega - \frac{M^{2}}{4} \right). \quad (78)
\]

According to the above case, and by using the (NU) method, the eigensolutions are

\[
E = 2\omega \mu \pm 2 \sqrt{\omega^{2} \mu^{2} + \left( \frac{M^{2}}{16} + M \omega \left( \frac{\psi}{2} + 1 + n \right) \right)}, \quad (79)
\]

\[
\chi(\rho) = (m_{\omega}) \left( \frac{s \left( J - \frac{\phi_{B}}{2\pi} \right)}{\alpha} \right) \left( \frac{s \left( J - \frac{\phi_{B}}{2\pi} \right)}{\alpha} \right) e^{-\frac{\psi^{2}}{4}} \times \]

\[
F \left( -n, \frac{s \left( J - \frac{\phi_{B}}{2\pi} \right)}{\alpha} + 1, M \omega \mu^{2} \right). \quad (80)
\]

We remark that if we put \( \phi_{B} = 0 \) (\( B = 0 \)) in Eq. (79), we recover the same results obtained in the previous section (Eq. (63)).
IV. CONCLUSION

In this work, we have considered the case of a Kemmer oscillator for vector bosons in a magnetic cosmic string space-time. The eigensolutions are obtained by using the generalized parametric NU method. We show that the quantum dynamics of a physical system depend on the topological defects features of the cosmic string, and the eigensolutions possesses an explicit dependence on the parameter $\alpha$. Furthermore, by comparing the spectrum of energy obtained in our case with those of the same problem in the flat spacetime \cite{27}, we can see that the presence of the angular deficit $\alpha$, which characterizes the global structure of the metric in the cosmic string space-time, breaks the degeneracies of these energies.

Appendix A: Review of the Nikiforov-Uvarov (NU) method

The Nikiforov-Uvarov method is based on solving the second-order differential equation of Schrödinger, Dirac and DKP equations by reduction to a generalized equation of hyper-geometric type. The following equation is a general form of the second order differential equation written for any potential as

$$\left[\frac{d^2}{ds^2} + \frac{c_1 - c_3 s}{s (1 - c_3 s)} \frac{d}{ds} + \frac{-\zeta_1 s^2 + \zeta_2 s - \zeta_3}{s (1 - c_3 s)^2}\right] \psi = 0.$$ \hspace{1cm} (A1)

According to the Nikiforov-Uvarov (NU) method, the eigenfunctions and eigenvalues are given by

$$\psi(s) = s^{c_{12}} (1 - c_3 s)^{-c_{12} - c_{11} \zeta_2 \zeta_3} \times \rho^{c_{10} - 1, c_{10} - 1} (1 - 2c_3 s),$$ \hspace{1cm} (A2)

and

$$c_{2n} - (2n + 1) c_5 + (2n + 1) (\sqrt{c_9} + c_3 \sqrt{c_8}) + n (n - 1) c_3 + c_7 + 2c_3 c_8 + 2 \sqrt{c_8 c_9} = 0.$$ \hspace{1cm} (A3)

The corresponding parameters are

$$c_4 = \frac{1}{2} (1 - c_1), c_5 = \frac{1}{2} (c_2 - 2c_3),$$ \hspace{1cm} (A4)

$$c_6 = c_5^2 + \zeta_1, c_7 = 2c_4 c_5 - \zeta_2$$ \hspace{1cm} (A5)

$$c_8 = c_4^2 + \zeta_3, c_9 = c_3 c_7 + \frac{c_2^2}{c_4} c_8 + c_6,$$ \hspace{1cm} (A6)

$$c_{10} = c_1 + 2c_4 + 2\sqrt{c_8},$$ \hspace{1cm} (A7)

$$c_{11} = c_2 - 2c_5 + 2 (\sqrt{c_9} + c_3 \sqrt{c_8}),$$ \hspace{1cm} (A8)

$$c_{12} = c_4 + \sqrt{c_8}, c_{13} = c_5 - (\sqrt{c_9} + c_3 \sqrt{c_8}).$$ \hspace{1cm} (A9)

In special case of $c_3 = 0$, when

$$\lim_{c_3 \to 0} (1 - c_3 s)^{-c_{12} - \frac{c_{13}}{c_3}} = e^{c_{12} s},$$ \hspace{1cm} (A10)

$$\lim_{c_3 \to 0} (1 - c_3 s)^{-c_{12} - \frac{c_{13}}{c_3}} P^{c_{10} - 1, c_{10} - 1} (c_{11} s) = L_n^{c_{10} - 1} (c_{11} s).$$ \hspace{1cm} (A11)

the wave function becomes

$$\psi(s) = s^{c_{12}} e^{c_{11} s} L_n^{c_{10} - 1} (c_{11} s),$$ \hspace{1cm} (A12)

where $L_n^{c_{10} - 1} (c_{11} s)$ is the generalized Laguerre polynomial.

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