EXISTENCE OF A GLOBAL ATTRACTOR FOR FRACTIONAL DIFFERENTIAL HEMIVARIATIONAL INEQUALITIES

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Abstract. In this paper, we introduce and study a new class of fractional differential hemivariational inequalities ((FDHVIs), for short) formulated by an initial-value fractional evolution inclusion and a hemivariational inequality in infinite Banach spaces. First, by applying measure of noncompactness, a fixed point theorem of a condensing multivalued map, we obtain the nonemptiness and compactness of the mild solution set for (FDHVIs). Further, we apply the obtained results to establish an existence theorem of the mild solution of a global attractor for the semiflow governed by a fractional differential hemivariational inequality ((FDHVI), for short). Finally, we provide an example to demonstrate the main results.

1. Introduction. Let \((X, \| \cdot \|)\) be a Banach space and \((Y, \| \cdot \|)\) be a separable Hilbert space, let \(K\) be a closed convex subset of \(Y\). In this paper, we consider the fractional differential hemivariational inequality ((FDHVI), for short) of the form:

\[
\begin{cases}
\frac{D^\alpha_0}{c} x(t) \in Ax(t) + F(x(t), u(t)), & t \in I = [0, b], \ 0 < \alpha < 1, \\
u(t) \in \text{SOL}(K, g(x(t), \cdot), B, J) & \text{a.e. } t \in [0, b], \\
x(0) = \xi,
\end{cases}
\]

where \(\frac{D^\alpha_0}{c}\) stands for Caputo fractional time derivatives of order \(\alpha\), \(A : D(A) \subseteq X \to X\) is the infinitesimal generator of a norm-continuous and uniformly bounded

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\(C_0\)-semigroup \(\{T(t)\}_{t \geq 0}\) on \(X\) (this implies that there exists \(M > 0\) such that \(\sup_{t \in [0, \infty)} \|T(t)\| \leq M\)), \(F : X \times Y \to \mathcal{P}(X)\) is a multi-valued function. The notation \(\text{SOL}(K, g(x(t), \cdot), B, J)\) stands for the solution set of the following hemivariational inequality ((HVI), for short): find \(u : [0, b] \to K\) such that
\[
(Bu(t) - g(x(t), u(t)), v - u(t)) + J^0(u(t); v - u(t)) \geq 0, \quad \forall v \in K,
\]
where \(B : Y \to Y^*\) and \(g : X \times Y \to Y^*\) are functions to be specified later, and \(J^0(\cdot, \cdot)\) denotes the generalized directional derivative (in the sense of Clarke) for a locally Lipschitz functional \(J : Y \to \mathbb{R}\).

The solution to problem (FDHVI) is understood in the following mild sense.

**Definition 1.1.** A pair \((x, u)\) with \(x \in C([0, b]; X)\) and \(u : [0, b] \to K\) integrable, is called a mild solution of problem (FDHVI) if
\[
x(t) = S_\alpha(t)\xi + \int_0^t (t - s)^{\alpha - 1}Q_\alpha(t - s)f(s)ds, \quad t \in [0, b],
\]
where \(u(t) \in \text{SOL}(K, g(x(t), \cdot), B, J)\) and \(f(t) \in F(x(t), u(t))\) a.e. \(t \in [0, b]\), and \(S_\alpha(t)\) and \(Q_\alpha(t)\) are operators (see Lemma 2.4).

Problem (FDHVI) originates in the concept of differential variational inequality ((DVI), for short), which appears as a system containing an evolution equation subject to a constraint formed by a variational inequality. The differential variational inequalities ((DVIs), for short) were first systematically studied by Pang and Stewart [23] as a general mathematical model for differential algebraic equations, differential complementarity systems, and evolutionary variational inequalities, etc. For this reason, various theoretical results, numerical algorithms and applications in connection with (DVIs) have been studied extensively by many authors (see, for example, [3, 6, 13, 25–27, 29] and the references therein). It is worth noting that recently Ke et al. [11] proved the existence of decay solutions to a class of fractional differential variational inequalities ((FDVIs), for short) by using the measure of noncompactness, a fixed point theorem of a condensing multivalued map. Loi et al. [17] applied topological methods to prove the solvability of (FDVIs). We mention that the models of (FDVIs) in [11,17] are formulated in finite dimensional spaces. If the evolution equation in (DVIs) represents a partial differential equation, we obtain infinite dimensional (DVIs), which are the subject of recent investigations; more precisely, Liu et al. [16] applied the measure of noncompactness and a fixed point theorem of a condensing multivalued map to show the nonemptiness and compactness for the set of solutions to partial differential hemivariational inequalities, Anh and Ke [1] applied measure of noncompactness to investigate the existence of global solution and global attractor for the semiflow governed by differential variational inequalities of parabolic-elliptic type, Liu et al. [15] obtain some existence results for partial differential variational inequalities with nonlocal boundary conditions by employing theory of measure of noncompactness and a fixed point theorem for condensing multivalued maps.

Regarding problem (FDHVI), we noted that (FDHVI) is expressed by an initial-value fractional partial differential inclusion (1) parameterized by an algebraic variable that is required to be the solution of an infinite-dimensional (HVI) in (2) containing the state variable of the system. In (1), we may assume that the operator \(A\) is an elliptic linear partial differential operator. The essential feature of problem (FDHVI) is that of mixing in an infinite dimensional setting an initial-value fractional evolution inclusion and a hemivariational inequality. From the literature
above, we also note that problem (FDHVI) is a novelty and is addressed here for the first time. This is the first motivation of the present work.

On the other hand, as Melnik and Valero [19, 20] pointed out, the question of uniqueness of solutions for evolution inclusions is no longer treated, the Lyapunov theory of stability is not a suitable choice, instead, the theories of attractors for multivalued semiflows. However, as far as we know, the existence of a global attractor for the semiflow governed by a (FDHVI) is still untreated topics in the literature and this fact is the second motivation of the present work.

The structure of this paper is as follows. Section 2 gives some preliminaries. Section 3 is dedicated to the nonemptiness and compactness of the mild solution set for problem (FDHVI). In Section 4, we present a result on the existence of a global attractor for the semiflow governed by a (FDHVI). In Section 5, we present an example to illustrate the obtained results.

2. Preliminaries. Let \((X, \| \cdot \|)\) be a Banach space and \(X^*\) be the dual space of \(X\). For \(I \subset \mathbb{R}\), let \(L^p(I, X)\) denote the Banach space of all Bochner integrable functions from \(I\) into \(X\) satisfying \((\int_I \|x(t)\|^p dt)^{\frac{1}{p}} < \infty\) and endowed with the norm \(\|x\|_{L^p(I, X)} = (\int_I \|x(t)\|^p dt)^{\frac{1}{p}}\) and \(C(I, X)\) be the space of all continuous functions from \(I\) into \(X\) equipped with the norm \(\|x\|_{C(I, X)} = \sup_{t \in I} \|x(t)\|\). Let \(\overline{D}\) be the closure of the set \(D\). We put

\[ P_{(f)}(X) = \{D \subseteq X : D\text{ is nonempty (closed)(bounded)( convex)}\}, \]

\[ K_{(v)}(X) = \{D \subseteq X : D\text{ is nonempty, compact (convex)}\}. \]

In what follows, we recall some known definitions of the fractional calculus theory.

**Definition 2.1.** ([12, 24]) The fractional integral of order \(\alpha\) with lower limit zero for a function \(x(t)\) is defined as

\[ I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)ds, \quad \alpha > 0, \quad t > 0, \]

provided the right side is point-wise defined on \([0, \infty)\), where \(\Gamma\) is the gamma function defined by

\[ \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt. \]

**Definition 2.2.** ([12, 24]) The Riemann-Liouville fractional derivative of order \(\alpha\) with the lower limit zero for a function \(x : [0, \infty) \rightarrow \mathbb{R}\) is defined as follows

\[ ^L D_0^\alpha x = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} x(s)ds, \quad t > 0, \quad 0 \leq n-1 < \alpha < n. \]

**Definition 2.3.** ([12, 24]) The Caputo fractional derivative of order \(\alpha\) with the lower limit zero for a function \(x : [0, \infty) \rightarrow \mathbb{R}\) is defined by

\[ ^c D_0^\alpha x = ^L D_0^\alpha \left( x(t) - \sum_{k=1}^{n-1} \frac{t^k}{k!} x^{(k)}(0) \right), \quad t > 0, \quad 0 \leq n-1 < \alpha < n. \]

**Remark 1.**

(i) If \(x \in C^n[0, \infty)\), then

\[ ^L D_0^\alpha x = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s)ds = I^\alpha x^{(n)}(t), \quad t > 0, \quad 0 \leq n-1 < \alpha < n; \]

(ii) The Caputo derivative of a constant is equal to zero.
(iii) If \(x\) is an abstract function which has values in \(H\), then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner’s sense.

**Lemma 2.4.** Let \(T(t) = e^{tA}, t \geq 0\), be the semigroup generated by \(A\). Put

\[ S_{\alpha}(t) = E_{\alpha,1}(t^\alpha A), \quad Q_{\alpha}(t) = E_{\alpha,\alpha}(t^\alpha A), \]

where \(E_{\alpha,\beta}\) a two-parameter function of the Mittag-Leffler type is defined by

\[ E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}. \]

Then by (30), one has the representation for \(S_{\alpha}(t)\) and \(Q_{\alpha}(t)\) as follows:

\[ S_{\alpha}(t) = \int_0^\infty \psi_{\alpha}(\theta)T(t^\alpha \theta)d\theta, \quad Q_{\alpha}(t) = \alpha \int_0^\infty \theta \psi_{\alpha}(\theta)T(t^\alpha \theta)d\theta, \]

\[ \psi_{\alpha}(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \varpi_{\alpha}(\theta^{-\frac{1}{\alpha}}) \geq 0, \]

\[ \varpi_{\alpha}(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^n \theta^{-n-\frac{1}{\alpha}} \frac{\Gamma(n\alpha + 1)}{n!} \sin(n\pi\alpha), \theta \in (0, \infty), \]

and \(\psi_{\alpha}\) is a probability density function defined on \((0, \infty)\), that is, \(\psi_{\alpha}(\theta) \geq 0\) for all \(\theta \in (0, \infty)\) such that

\[ \int_0^\infty \psi_{\alpha}(\theta)d\theta = 1. \]

According to [18], direct calculation yields that

\[ \int_0^\infty \theta \psi_{\alpha}(\theta)d\theta = \frac{1}{\Gamma(1 + \alpha)}. \]

**Lemma 2.5.** ([30], Lemmas 3.2-3.4) The operators \(S_{\alpha}\) and \(Q_{\alpha}\) have the following properties:

(i) For any fixed \(t \geq 0\), \(S_{\alpha}(t)\) and \(Q_{\alpha}(t)\) are bounded linear operators such that, for any \(x \in H\),

\[ \|S_{\alpha}x\| \leq M\|x\|, \quad \|Q_{\alpha}x\| \leq \frac{\alpha M}{\Gamma(\alpha + 1)}\|x\|; \]

(ii) \(\{S_{\alpha}(t), t \geq 0\}\) and \(\{Q_{\alpha}(t), t \geq 0\}\) are strongly continuous;

(iii) For any \(t > 0\), \(S_{\alpha}(t)\) and \(Q_{\alpha}(t)\) are compact operators provided that \(T(t)\) is compact.

In the sequel, we recall some notation of multi-valued analysis and nonsmooth analysis.

**Definition 2.6.** ([4, 22]) Let \(X\) be a Banach space with the dual space \(X^*\) and \(J : X \to \mathbb{R}\) be a locally Lipschitz functional on \(X\). The Clarke’s generalized directional derivative of \(J\) at the point \(x \in X\) in the direction \(v \in X\), denoted by \(J^0(x; v)\), is defined by

\[ J^0(x; v) = \lim_{\lambda \to 0^+} \sup_{y \to x} \frac{J(y + \lambda v) - J(y)}{\lambda}. \]

The Clarke’s generalized gradient of \(J\) at \(x \in X\), denoted by \(\partial J(x)\), is a subset of \(X^*\) given by

\[ \partial J(x) = \{x^* \in X^*: J^0(x; v) \geq \langle x^*, v \rangle, \forall v \in X\}. \]
Definition 2.7. ([8]) Let $X$ and $Z$ be two Banach spaces. A multi-valued map $G : X \to \mathcal{P}(Z)$ is said to be

(i) upper semicontinuous (for short u.s.c.), if for every open subset $D \subset Z$, the set

$$G^{-1}(D) = \{x \in X : G(x) \subset D\}$$

is open in $X$;

(ii) closed if its graph

$$\{(x, z) : x \in X, z \in G(x)\}$$

is a closed subset of the space $X \times Z$;

(iii) compact if its range $G(X)$ is relatively compact subset of $Z$, i.e. $\overline{G(X)}$ is compact subset of $Z$;

(iv) quasicompact, if its restriction to any compact subset $D \subset X$ is compact.

We now give a few facts related to the definition of measure of noncompactness.

Definition 2.8. ([8]) Let $X$ be a Banach space and $(\mathcal{A}, \leq)$ be a partially ordered set. A function $\beta : \mathcal{P}(X) \to (\mathcal{A}, \leq)$ is said to be a MNC in $X$ if

$$\beta(\text{co} \Omega) = \beta(\Omega),$$

for every $\Omega \in \mathcal{P}(X)$, where $\text{co} \Omega$ is the closure of convex hull of $\Omega$. A MNC $\beta$ is said to be

(i) monotone, if for each $\Omega_0, \Omega_1 \in \mathcal{P}(X)$ such that $\Omega_0 \subseteq \Omega_1$, one has $\beta(\Omega_0) \leq \beta(\Omega_1)$;

(ii) nonsingular, if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for any $a \in X, \Omega \in \mathcal{P}(X)$;

(iii) invariant with respect to the union with a compact set, if $\beta(D \cup \Omega) = \beta(\Omega)$ for every relatively compact subset $D$ of $X$ and $\Omega \in \mathcal{P}(X)$;

(iv) real, if $\mathcal{A} = [0, +\infty]$ with the natural ordering and $\beta(\Omega) < +\infty$ for every bounded $\Omega$;

(v) algebraically semi-additive, if $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for any $\Omega_0, \Omega_1 \in \mathcal{P}(X)$;

(vi) regular, if $\beta(\Omega) = 0$ is equivalent to the relative compactness of $\Omega$.

An example of MNC is the Hausdorff MNC:

$$\chi(\Omega) = \inf\{\epsilon : \Omega \text{ has a finite } \epsilon\text{-net}\},$$

which satisfies all above properties. If $T \in L(X)$ ($L(X)$ denotes the space of all bounded linear operators on $X$), we define the $\chi$-norm of $T$ as

$$\|T\|_\chi = \inf\{\lambda > 0 : \chi(T(\Omega)) \leq \lambda \cdot \chi(\Omega), \text{ for all bounded set } \Omega \subset X\}. \quad (4)$$

It is clear that

$$\chi(T(\Omega)) \leq \|T\|_\chi \cdot \chi(\Omega), \forall \Omega \subset X.$$

In addition, $\|T\|_\chi \leq \|T\|$ and $T$ is a compact operator if and only if $\|T\|_\chi = 0$.

Definition 2.9. ([8], Definition 2.2.6) A multi-valued map $G : D \subset X \to X$ is referred to as condensing with respect to a MNC $\beta$ ($\beta$-condensing) if, for any bounded subset $\Omega \subset D$ which is not relatively compact, we have

$$\beta(G(\Omega)) \neq \beta(\Omega).$$

Definition 2.10. ([10], Definition 2.6) Let $p \geq 1$. A sequence $\{f_n\}_{n=1}^\infty \subset L^p(I, X)$ is said to be semi-compact if
(i) it is integrable bounded, i.e. there exists \( \varphi \in L^p(I, \mathbb{R}^+) \) satisfying \( \| f_n(t) \|_X \leq \varphi(t) \) for a.e. \( t \in I \) and \( n \geq 1 \); 
(ii) the image sequence \( \{ f_n(t) \}_{n=1}^\infty \) is relatively compact in \( X \) for a.e. \( t \in I \).

Let \( p > \frac{1}{\alpha} \), we define a linear operator 
\[
W_\alpha : L^p(I, X) \to C(I, X)
\]
\[
W_\alpha(f)(t) = \int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)f(s)ds.
\]

(5)

The following lemma is a special case of Proposition 2.5 in [10].

**Lemma 2.11.** ([10], Proposition 2.5) Assume that the semigroup \( T(\cdot) \) generated by \( A \) is norm-continuous, i.e., the map \( (0, \infty) \ni t \to T(t) \in L(X) \) is continuous. Then

(i) For each bounded set \( \Omega \subset L^p(I, X) \), \( W_\alpha(\Omega) \) is an equicontinuous subset in \( C(I, X) \). Moreover, one has the following estimate
\[
\chi(W_\alpha(\Omega)) \leq 4\sup_{t \in [0, \infty]} \int_0^t (t-s)^{\alpha-1}\|Q_\alpha(t-s)\|_X \cdot \chi(\Omega(s))ds,
\]
where \( \| \cdot \|_X \) is the \( \chi \)-norm given by (4); 
(ii) If \( \{ f_n \} \subset L^p(I, X) \), \( p > 1 \), is a semicompact sequence, then \( \{ W_\alpha(f_n) \} \) is relatively compact in \( C(I, X) \). Moreover, if \( f_n \xrightarrow{weakly} f^* \) in \( L^p(I, X) \), then \( W_\alpha(f_n) \xrightarrow{weakly} W_\alpha(f^*) \) in \( C(I, X) \).

**Lemma 2.12.** ([8], Theorem 1.1.12) Let \( X \) and \( Z \) be two metric spaces and \( G : X \to K(Z) \) a closed quasicompact multi-valued map. Then \( G \) is u.s.c.

**Theorem 2.13.** ([8], Corollary 3.3.1) Let \( X \) be a Banach space, \( D \subset X \) a convex closed subset and \( G : D \to K(D) \) a u.s.c. \( \beta \)-condensing multi-valued map, where \( \beta \) is a nonsingular MNC defined on subsets of \( D \). Then the fixed point set \( \text{Fix}G\{ x : x \in Gx \} \) is nonempty.

**Theorem 2.14.** ([8], Proposition 3.5.1) Let \( X \) be a Banach space, \( D \subset X \) a closed subset and \( G : D \to K(X) \) a closed \( \beta \)-condensing multi-valued map, where \( \beta \) is a monotone MNC on \( X \). If the fixed point set \( \text{Fix}G \) is bounded, then it is compact.

We summarize some basic facts related to the theory of global attractor of multivalued semiflows ((m-semiflows), for short) [19]. Let \( \Upsilon \) be a nontrivial subgroup of the additive group of real numbers \( \mathbb{R} \) and \( \Upsilon_+ = \Upsilon \cap [0, \infty) \).

**Definition 2.15.** ([19], Remark 1) The map \( G : \Upsilon_+ \times X \to P(X) \) is said to be an m-semiflow if the following conditions are satisfied:

(i) \( G(0, w) = \{ w \} \), \( \forall w \in X \); 
(ii) \( G(t_1 + t_2, w) \subset G(t_1, G(t_2, w)) \), \( \forall t_1, t_2 \in \Upsilon_+ \), \( \forall w \in X \); 
where \( G(t, D) = \bigcup_{x \in D} G(t, x) \), \( D \subset X \). An m-semiflow is said to be strict if \( G(t_1 + t_2, w) \subset G(t_1, G(t_2, w)) \), \( \forall t_1, t_2 \in \Upsilon_+ \), \( \forall w \in X \); \( G \) is called eventually bounded if for each bounded set \( D \subset X \), there is a number \( T(D) > 0 \) such that \( \gamma^+(T(D)) \) is bounded. Here \( \gamma^+_T(D) \) is the set of orbits after time \( T(D) : \gamma^+_T(D) = \bigcup_{t \geq T(D)} G(t, D) \).

**Definition 2.16.** The set \( \mathcal{A} \) is referred to as a global attractor of an m-semiflow \( G \) if it satisfies the following conditions:
(i) \( A \) is negatively semi-invariant, i.e. \( A \subset G(t, A) \), \( \forall t \in \mathbb{Y}_+ \);  
(ii) \( A \) attracts any \( D \in \mathcal{P}_b(X) \), i.e. \( \text{dist}(G(t, D), A) \rightarrow 0 \) as \( t \rightarrow \infty \), for all bounded set \( D \subset X \), where \( \text{dist}(\cdot, \cdot) \) is the Hausdorff semi-distance of two subsets in \( X \): \( \text{dist}(\cdot, \cdot) = \sup_{x \in D_1} \inf_{y \in D_2} \|x - y\| \).

**Definition 2.17.** An m-semiflow \( G \) is said to be asymptotically upper semicompact if \( D \in \mathcal{P}_b(X) \) such that for some \( T(D) \in \mathbb{Y}_+ \), \( \gamma^{\tau}_{T(D)} \in \mathcal{P}_b(X) \), any sequence \( x_n \in G(t_n, D), t_n \rightarrow \infty \), is precompact in \( X \).

**Definition 2.18.** A bounded set \( D_1 \subset X \) which has the property that, for any bounded set \( D \subset X \) there exists \( \tau = \tau(D) \geq 0 \) such that \( \gamma^{\tau}_{T(D)} \subset D_1 \), is said to be an absorbing set of m-semiflow \( G \).

**Theorem 2.19.** Let the m-semiflow \( G \) satisfy the following properties: 
(i) \( G(t, \cdot) \) is u.s.c and has closed values for each \( t \in \mathbb{Y}_+ \);  
(ii) \( G \) admits an absorbing set;  
(iii) \( G \) is asymptotically upper semicompact.

Then it possesses a compact global attractor \( A \) in \( E \). Moreover, if \( G \) is a strict m-semiflow, then \( A \) is invariant, that is \( A = G(t, A) \), \( \forall t \in \mathbb{Y}_+ \).

3. Existence of mild solution. In this section, We study problem (FDHVI) under the following assumptions:

**HA** The operator \( A \) is a generator of a norm-continuous and uniformly bounded \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \);  

**HB** The operator \( B : Y \rightarrow Y^* \) is defined by  
\[ \langle u, Bv \rangle = a(u, v), \forall u, v \in Y, \]
where \( a : Y \times Y \rightarrow \mathbb{R} \) is a bilinear continuous function on \( Y \times Y \) satisfying  
\[ a(u, u) \geq k_B \|u\|^2_Y, \forall u \in Y \]
for some \( k_B > 0 \);  

**HF** The multi-valued map \( F : X \times Y \rightarrow \mathcal{P}(X) \) is u.s.c. with compact and convex values. Moreover,  
(F1) there exists a function \( \mu, \kappa \in L^p(I, \mathbb{R}_+) \) with \( p > \frac{1}{\alpha} \) satisfying  
\[ \chi(F(\Omega_1, \Omega_2)) \leq \mu(t)\chi(\Omega_1) + c(t)\kappa(\Omega_2) \]
for all bounded set \( \Omega_1 \subset X \) and \( \Omega_2 \subset Y \), where \( \chi \) and \( \kappa \) denote the Hausdorff MNC in the spaces \( X \) and \( Y \), respectively;  
(F2) there exist a function \( m \in L^p(I, \mathbb{R}_+) \) with \( p > \frac{1}{\alpha} \) and two nonnegative constants \( b_1, c_1 \) such that  
\[ \|F(x, u)\| = \sup\{\|f\|_X : f \in F(x, u)\} \leq m(t)\|x\|_X + b_1\|u\|_Y + c_1, \forall x \in X, \forall u \in Y; \]

**HJ** The functional \( J : Y \rightarrow \mathbb{R} \) satisfies the following conditions:  
(J1) \( J(\cdot) : Y \rightarrow \mathbb{R} \) is locally Lipschitz continuous for a.e. \( t \in I \);  
(J2) there exist a function \( \zeta \in L^p(I, \mathbb{R}_+) \) with \( p > \frac{1}{\alpha} \) and a constant \( c_2 > 0 \) such that  
\[ \|\partial J(x)\| = \sup\{\|y\| : y \in \partial J(x)\} \leq \zeta(t) + c_2\|x\|, \forall x \in Y, \text{ a.e. } t \in I; \]
(J3) there exists a function \( \delta \in L^p(I, \mathbb{R}_+) \) with \( p > \frac{1}{\alpha} \) satisfying  
\[ \chi(\partial J(D)) \leq \delta(t)\chi(D) \] for a.e. \( t \in I \),
for every bounded \( D \subset Y \), where \( \chi \) is the Hausdorff MNC on \( X \),
Throughout this paper, we denote \( \partial J(\cdot) \) satisfies the relaxed monotonicity, i.e., there exists a constant \( k_J > 0 \) such that
\[
\langle \eta_1 - \eta_2, u_1 - u_2 \rangle \geq -k_J \| u_1 - u_2 \|^2_Y, \forall u_1, u_2 \in Y, \eta_1 \in \partial J(u_1), \eta_2 \in \partial J(u_2);
\]
(Hg) The function \( g : X \times Y \to Y^* \) is Lipschitzian, i.e., there exist two nonnegative constants \( k_1 \) and \( k_2 \) such that \( k_2 < k_B - k_J \) and
\[
\| g(x, u) - g(\bar{\bar{x}}, \bar{\bar{u}}) \| \leq k_1 \| x - \bar{\bar{x}} \|_X + k_2 \| u - \bar{\bar{u}} \|_Y, \forall x, \bar{\bar{x}} \in X, \forall u, \bar{\bar{u}} \in Y.
\]
(H0) \( k_J < k_B \)

Let a multimap \( \mathcal{N}_F \) be defined as
\[
\mathcal{N}_F(x, u) = \{ f \in L^p(I, X) : f(t) \in F(x(t), u(t)) \text{ for a.e. } t \in I \},
\]
i.e., \( \mathcal{N}_F(x, u) \) is the set of all integrable selections of \( F(x(t), u(t)) \) for each \((x, u) \in C(I, X) \times C(I, Y)\).

We first consider (HVI) (2). It is easy to verify that any solutions of the following semilinear inclusion
\[
g(x(t), u(t)) \in Bu(t) + \partial J(u(t)), \quad t \in I,
\]
are the solutions of (HVI) (2). Indeed, if \( u \) is a solution of problem (HVI), then there exists \( \eta \in L^p(I, Y^*) \) such that \( \eta(t) \in \partial J(u(t)) \) and
\[
g(x(t), u(t)) = Bu(t) + \eta(t), \quad t \in I,
\]
which implies
\[
(Bu(t) - g(x(t), u(t)), v - u(t)) + \langle \eta(t), v - u(t) \rangle = 0, \forall v \in K,
\]
Since \( \eta(t) \in \partial J(u(t)) \) and \( \langle \eta(t), v - u(t) \rangle \leq J^0(u(t); v - u(t)) \), one has
\[
(Bu(t) - g(x(t), u(t)), v - u(t)) + J^0(u(t); v - u(t)) \geq 0, \forall v \in K.
\]

Throughout this paper, we denote
\[
\mathbb{S}(z) = \{ u \in Y : z \in Bu + \partial J(u) \}.
\]

The following result is a special case of Lemma 3.3 of [14], but the authors didn’t give the proof of [14, Lemma 3.3], in order to make this paper readable, we present the proof.

**Lemma 3.1.** Let conditions (HB), (HJ) and (H0) hold. Then for each \( z \in Y^* \), the solution set \( \mathbb{S}(z) \) is a singleton. Moreover, the map \( z \to \mathbb{S}(z) \) is Lipschitz continuous from \( Y^* \) to \( Y \).

**Proof.** Assume that \( u_1 \) and \( u_2 \) are solutions of \( \mathbb{S}(z) \) with \( u_1 \neq u_2 \). We have
\[
z \in Bu_i + \partial J(u_i), \quad i = 1, 2.
\]
We take \( \eta_1 \in \partial J(u_1) \) and \( \eta_2 \in \partial J(u_2) \). It follows that
\[
\langle z - Bu_1, u_1 - u_2 \rangle = \langle \eta_1, u_1 - u_2 \rangle,
\]
and
\[
\langle z - Bu_2, u_1 - u_2 \rangle = \langle \eta_2, u_1 - u_2 \rangle.
\]
Adding the two equalities above, one has
\[
\langle (z - Bu_1) - (z - Bu_2), u_1 - u_2 \rangle = \langle \eta_1 - \eta_2, u_1 - u_2 \rangle.
\]
In view of conditions (HB), (HJ) and (H0), the equality above gives
\[ k_B \|u_1 - u_2\|_Y^2 \leq a(u_1 - u_2, u_1 - u_2) = \langle Bu_1 - Bu_2, u_1 - u_2 \rangle = -\langle \eta_1 - \eta_2, u_1 - u_2 \rangle \leq k_f \|u_1 - u_2\|_Y^2, \]
which contradicts condition (H0).

Subsequently, we show the Lipschitz continuity of \( S \), that is,
\[ \|S(z_1) - S(z_2)\|_Y \leq \frac{1}{k_B - k_J} \|z_1 - z_2\|_{Y^*}, \quad \forall z_1, z_2 \in Y^*. \]  
(8)

We put \( u_1 = S(z_1), u_2 = S(z_2) \). Then, there exist \( \eta_1 \in \partial J(u_1) \) and \( \eta_2 \in \partial J(u_2) \) satisfying
\[ \langle z_1 - Bu_1, u_1 - u_2 \rangle = \langle \eta_1, u_1 - u_2 \rangle, \]
and
\[ \langle z_2 - Bu_2, u_1 - u_2 \rangle = \langle \eta_2, u_1 - u_2 \rangle. \]
Adding the two equalities above, we obtain
\[ \langle Bu_1 - Bu_2, u_1 - u_2 \rangle + \langle \eta_1 - \eta_2, u_1 - u_2 \rangle = \langle z_1 - z_2, u_1 - u_2 \rangle. \]
(9)

It follows from conditions (HB), (HJ) and (9) that
\[ \langle Bu_1 - Bu_2, u_1 - u_2 \rangle + \langle \eta_1 - \eta_2, u_1 - u_2 \rangle = a(u_1 - u_2, u_1 - u_2) + \langle \eta_1 - \eta_2, u_1 - u_2 \rangle \]
\[ \geq (k_B - k_f) \|u_1 - u_2\|_Y^2. \]
(10)

Combining (9) and (10), we get
\[ (k_B - k_J) \|u_1 - u_2\|_Y^2 \leq \langle z_1 - z_2, u_1 - u_2 \rangle, \]
which yields that (8).

Now, for a fixed \( y \in X \), we consider the original form of (7)
\[ g(y, u) \in Bu + \partial J(u). \]
(11)
The following result is a special case of Lemma 3.4 of [14], but the conditions (J2)–(J4) of this paper are different from condition H(J) in the authors [14, Lemma 3.4], in order to make this paper readable, we present the proof.

**Lemma 3.2.** Let conditions (HB), (HJ), (Hg) and (H0) hold. Then for each \( y \in X \), (11) has a unique solution \( u \in Y \). Moreover, the solution map
\[ \mathbb{V} : X \to Y, \]
\[ y \mapsto u, \]
is Lipschitz continuous, more precisely
\[ \|\mathbb{V}(y_1) - \mathbb{V}(y_2)\|_Y \leq \frac{k_1}{k_B - k_J - k_2} \|y_1 - y_2\|_X, \quad \forall y_1, y_2 \in X. \]  
(12)

**Proof.** Now, let \( y \in X \) be fixed. We consider the map \( S \circ g(y, \cdot) : Y \to Y \). It follows from (8) that
\[ \|S(g(y, u_1)) - S(g(y, u_2))\|_Y \leq \frac{1}{(k_B - k_J)} \|g(y, u_1) - g(y, u_2)\|_{Y^*}. \]
\[ \leq \frac{k_2}{(k_B - k_J)} \|u_1 - u_2\|_Y. \]
(13)
Since \( k_2 < k_B - k_J \) (see condition \((Hg)\)), \((13)\) shows that \( u \mapsto \mathcal{S} \circ g(y,u) \) is a contraction map, thus, it has a unique fixed point, which implies that \((11)\) admits a unique solution \( u \in Y \). □

In the sequel, we show that the map \( y \mapsto u \) is Lipschitz continuous. Let \( \mathcal{V}(y_1) = u_1, \mathcal{V}(y_2) = u_2 \). Then, we obtain
\[
\|u_1 - u_2\|_Y = \|\mathcal{S}(g(y,u_1)) - \mathcal{S}(g(y,u_2))\|_Y \\
\leq \frac{1}{(k_B - k_J)} \|g(y,u_1) - g(y,u_2)\|_Y \\
\leq \frac{k_1}{(k_B - k_J)} \|y_1 - y_2\|_X + \frac{k_2}{(k_B - k_J - k_2)} \|u_1 - u_2\|_Y.
\]
Therefore,
\[
\|u_1 - u_2\|_Y \leq \frac{k_1}{(k_B - k_J - k_2)} \|y_1 - y_2\|_Y.
\]
This shows the Lipschitz continuity of \( \mathcal{V} \).

In order to solve problem (FDHVI), we convert it to a differential inclusion. We consider the next multi-valued map:
\[
G(y) = F(y, \mathcal{V}(y)), \quad y \in X.
\]

From condition \((HF)\), we know that \( G : X \to \mathcal{P}(X) \) has compact and convex values. Furthermore, condition \((HF)\) together with the continuity of \( \mathcal{V} \) implies that the multimap \( G \) is u.s.c. Moreover, by (12) and the Hausdorff MNC property, we obtain
\[
\kappa(\mathcal{V}(\Omega)) \leq \frac{k_1}{k_B - k_J - k_2} \chi(\Omega), \quad \Omega \in \mathcal{P}_b(X),
\]
where \( \kappa \) is the Hausdorff MNC in \( Y \).

If the semigroup \( T(\cdot) \) is non-compact, then condition \((F1)\) follows
\[
\chi(G(\Omega)) = \chi(F(\Omega, \mathcal{V}(\Omega))) \leq \mu(t) \chi(\Omega) + c(t) \nu(\mathcal{V}(\Omega)) \\
\leq \left( \mu(t) + \frac{c(t)k_1}{k_B - k_J - k_2} \right) \chi(\Omega). \tag{14}
\]
Regarding the growth of \( G \), by condition \((F2)\) and (12), we get
\[
\|G(y)\| = \sup\{\|z\| : z \in G(y)\} \\
\leq m(t) \|y\|_X + b_1 \|\mathcal{V}(y)\|_Y + c_1 \\
\leq m(t) \|y\|_X + \frac{b_1k_1}{k_B - k_J - k_2} \|y\|_X + b_1 \|\mathcal{V}(0)\|_Y + c_1 \\
= \left( m(t) + \frac{b_1k_1}{k_B - k_J - k_2} \right) \|y\|_X + d_1, \tag{15}
\]
where \( d_1 = b_1 \|\mathcal{V}(0)\|_Y + c_1 \).

According to the aforementioned setting, problem (FDHVI) is converted to
\[
\begin{cases}
cD_0^\alpha x(t) \in Ax(t) + G(x(t)), \quad t \in I = [0, b], \quad 0 < \alpha < 1,
x(0) = \xi,
\end{cases}
\]
We define
\[
\mathcal{N}^p_G(x) = C(I, X) \to \mathcal{P}(L^p(I, X)),
\]
\[
\mathcal{N}^p_G(x) = \{ f \in L^p(I, X) : f(t) \in G(x(t)) \quad \text{for a.e.} \ t \in I \}.
\]
**Proposition 1.** If conditions (HB), (HF), (HJ), (Hg) and (H0) hold, then \( N^p_G \) is well-defined and weakly u.s.c. with weakly compact and convex values.

**Proof.** Similar to the proof of Theorem 1 in [2], we can obtain conclusion. Now we see that, a mild solution \( x \) of problem (FDHVI) is given by

\[
x(t) = S_\alpha(t)\xi + \int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)f(s)ds, \quad f \in N^p_G, \quad t \in [0,b].
\]

**Theorem 3.3.** Let conditions (HA), (HB), (HF), (HJ), (Hg) and (H0) hold. If \( \vartheta = 4 \sup\int_0^t \left( \mu(s) + \frac{c(s)k_1}{k_B - k_J - k_2} \right)(t-s)^{\alpha-1}\|Q_\alpha(t-s)\|_X ds < 1, \quad (16) \)

then the set of solutions to problem \( (FDHVI) \) is nonempty and compact for each initial datum \( \xi \in X \).

**Proof.** For any \( x \in C(I,X) \), define a multi-valued map \( \Phi : C(I,X) \to \mathcal{P}(C(I,X)) \) as follows

\[
\Phi(x) = S_\alpha(\cdot)\xi + W_\alpha \circ N^p_G(x), \quad x \in C(I,X).
\]

Now, if we show that \( \Phi \) admits a fixed point, then problem (FDHVI) has a mild solution. We divide our proof into six steps.

**Step 1.** We claim that: for each \( n \in \mathbb{N} = \{1,2,\cdots\} \), if \( \{y_n\} \subset C(I,X) \) with \( y_n \to y_* \) and \( f_n \in N^p_G(y_n) \), then \( f_n \xrightarrow{\text{weakly}} f_* \) in \( L^p(I,X) \) with \( f_* \in N^p_G(y_*) \).

Let \( \{y_n\} \subset C(I,X) \) be such that \( y_n \to y_* \), \( f_n \in N^p_G(y_n) \). By condition (HF), we see that \( \{f_n(t)\} \subset K(t) := G(\{y_n(t)\}) \) is a compact set for a.e. \( t \in I \). Condition (F2) implies that \( \{f_n\} \) is bounded by an \( L^p \)-integrable function. Thus, the sequence \( \{f_n\} \) is semi-compact (see Definition 2.10) and by Corollary 3.3 in [5], it is weakly compact in \( L^p(I,X) \). So we can assume that \( f_n \xrightarrow{\text{weakly}} f_* \) in \( L^p(I,X) \). According to Mazur’s lemma, there exists a sequence \( \tilde{f}_n \in \text{co}\{f_i : i \geq n\} \) such that \( \tilde{f}_n \to f_* \) in \( L^p(I,X) \) and so \( f_n(t) \to f_*(t) \) for a.e. \( t \in [0,b] \). Assumption (HF) together with the continuity of \( \forall \) infer that \( G \) has compact values and is u.s.c., this means that for \( \epsilon > 0 \)

\[
G(y_n(t)) \subset G(y_*(t)) + B_\epsilon,
\]

for all sufficiently large \( n \), here \( B_\epsilon \) is the ball in \( X \) centered at origin with radius \( \epsilon \).

Thus, we have

\[
f_n(t) \subset G(y_*(t)) + B_\epsilon, \quad \text{for a.e. } t \in [0,b].
\]

Due to the convexity of \( G(y_*(t)) + B_\epsilon \), we replace \( f_n(t) \) by \( \tilde{f}_n(t) \), the last inclusion still holds. Hence, \( f_* \in G(y_*(t)) + B_\epsilon \) for a.e. \( t \in [0,b] \). Since \( \epsilon \) is arbitrary, we get \( f_* \in G(y_*(t)) \) for a.e. \( t \in [0,b] \) and so \( f_* \in N^p_G(y_*) \).

**Step 2.** \( \Phi \) is closed.

Let \( y_n \to y_*, \ y_n \in \Phi(x_n) \) and \( x_n \to x_* \). We shall show that \( y_* \in \Phi(x_*) \). For each \( n \in \mathbb{N} = \{1,2,\cdots\} \), it follows from \( y_n \in \Phi(x_n) \) that there exists \( f_n \in N^p_G(x_n) \) such that, for \( t \in I \),

\[
y_n(t) = S_\alpha(t)\xi + \int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)f_n(s)ds.
\]

It follows from step 1 that \( f_n \xrightarrow{\text{weakly}} f_* \) in \( L^p(I,X) \) with \( f_* \in N^p_G(y_*) \). As argued in step 1, we know that the sequence \( \{f_n\} \) is semi-compact. This together with
Lemma 2.11 infers $W_\alpha(f_n) \to W_\alpha(f_\ast)$ in $C(I, X)$ and so $W_\alpha(f_n)(t) \to W_\alpha(f_\ast)(t)$ in $X$. Taking the limits of (17) as $n \to \infty$, we obtain

$$y_\ast(t) = S_\alpha(t)\xi + \int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)f_\ast(s)ds.$$  

As a consequence, $y_\ast \in \Phi(y_\ast)$ as asserted.

**Step 3.** $\Phi$ is $\chi$-condensing.
Let $\Omega \subset C(I, X)$ be a bounded set. For $y \in \Omega$, we can write $\Phi(y) = \Phi_1(y) + \Phi_2(y)$, where

$$\Phi_1(y)(t) = S_\alpha(t)\xi,$$
$$\Phi_2(y)(t) = W_\alpha \circ \mathcal{N}_G^p(y)(t).$$

It is easy to see that

$$\chi(\Phi_1(\Omega)) = 0. \quad (18)$$

Regarding $\Phi_2(\Omega)$, we observe that $D = \mathcal{N}_G^p(\Omega)$ is bounded in $L^p(I, X)$. Therefore, by Lemma 2.11, we have the following estimate

$$\chi(\Phi_2(\Omega)) = \chi(W_\alpha(D)) \leq 4 \sup_{t \in [0, b]} \int_0^t (t-s)^{\alpha-1}\|Q_\alpha(t-s)\chi_s\|\chi(D(s))ds. \quad (19)$$

It yields from (5) and (19) that $\chi(\Phi_2(\Omega)) = \chi(W_\alpha(D)) = \sup_{t \in [0, b]} \chi(W_\alpha(D)(t))$. Now deploying (14), we obtain

$$\chi(D(s)) \leq \chi(G(\Omega(s))) \leq \left(\mu(t) + \frac{c(t)k_1}{k_B - k_J - k_2}\right)\chi(\Omega(s)) \leq \left(\mu(t) + \frac{c(t)k_1}{k_B - k_J - k_2}\right)\chi(\Omega). \quad (20)$$

We put (20) in (19), it yields that

$$\chi(\Phi_2(\Omega)) = \chi(W_\alpha(D)) \leq \left(4 \sup_{t \in [0, b]} \int_0^t \left(\mu(s) + \frac{c(s)k_1}{k_B - k_J - k_2}\right)(t-s)^{\alpha-1}\|Q_\alpha(t-s)\|\chi(s)ds\right)\chi(\Omega). \quad (21)$$

Combining algebraically semiadditivity of measure of noncompactness $\chi$, (16), (18) and (21), we arrive at

$$\chi(\Phi(\Omega)) \leq \vartheta \chi(\Omega). \quad (22)$$

If $\chi(\Phi(\Omega)) \geq \chi(\Omega)$, then $\chi(\Omega) \leq \vartheta \chi(\Omega)$, which ensures that $\chi(\Omega) = 0$. Therefore, $\Omega$ is relatively compact. This together with assumption shows that $\Phi$ is $\chi$-condensing.

**Step 4.** $\Phi$ has compact convex values.
First, we shall show that $\Phi(x)$ is convex for each $x \in C(I, X)$. The convexity of $\mathcal{N}_G^p(x)$ (see Proposition 1) deduces that $\Phi(x)$ is convex.

In the sequel, we prove that $\Phi(x)$ has compact values. Indeed, for $y \in C(I, X)$, similar to the proof (22), one has

$$\chi(\Phi(y)) \leq \vartheta \chi(\Omega) = 0.$$ 

It yields that $\chi(\Phi(y)) = 0$ and so $\chi(\Phi(y))$ is a relatively compact subset. Therefore, the closedness of $\Phi$ (see step 2) shows that $\chi(\Phi(y))$ is compact.
Step 5. $G$ is u.s.c.

According to steps 2 and 4, Definition 2.7 (iv) and Lemma 2.12, it suffices to prove that $\Phi$ is quasicompact. Let $K \subset C(I, X)$ be compact and $y_n \in G(x_n)$ such that $x_n \in K$. Then, there exists $f_n \in \mathcal{N}_K^\alpha(x_n)$ satisfying (17). Since $K$ is compact, we may assume, passing to a subsequence, if necessary, that $x_n \to x_*$. From the proof of step 1, we can conclude that the sequence $\{f_n\}$ is semi-compact. This together with Lemma 2.11 implies that $\{W_\alpha(f_n)\}_{n=1}^\infty$ is relatively compact and $W_\alpha(f_n) \to W_\alpha(f_*)$. Moreover, it is easy to see that

$$S_\alpha(t)\xi \to S_\alpha(t)\xi.$$  \hspace{1cm} (23)

Thus, it follows from (17) and (23) that $y_n \to y_*$, where

$$y_*(t) = S_\alpha(t)\xi + \int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)f_*(s)ds.$$  

In addition, since $\Phi$ is closed (see step 2), we know that $y_* \in G(x_*)$. Now Definition 2.7 (iv) shows that $G$ is quasicompact. Consequently, it follows from Lemma 2.12 that $G$ is u.s.c.

Step 6. $\Phi$ has a fixed point.

From steps 1-5, it is easy to see that the multi-valued map $G : C(I, X) \to K_v(C(I, X))$ is u.s.c., closed and $\nu$-condensing. Now, we introduce the equivalent norm in the space $C(I, X)$ given by

$$\|x\|_* = \max_{t \in I} e^{-Lt}\|x(t)\|,$$  \hspace{1cm} (24)

where $L > 0$ is chosen such that

$$\max_{t \in I} \frac{\alpha M r_*}{\Gamma(\alpha + 1)} \int_0^t (t-s)^{\alpha-1} \left( m(s) + \frac{b_1k_1}{k_B - k_J - k_2} \right) e^{-L(t-s)}ds \leq l < 1. \hspace{1cm} (25)$$

In the space $C(I, X)$ with norm (24), let us consider the ball

$$B_r(0) = \{ x \in C(I, X) : \|x\|_* \leq r_* \},$$

where $r_* > 0$ is chosen such that

$$M \left[ \|\xi\| + \frac{d_1}{\Gamma(\alpha + 1)} b^\alpha \right] (1-l)^{-1} \leq r_*.$$

We note that the last inequality implies

$$M \left[ \|\xi\| + \frac{d_1}{\Gamma(\alpha + 1)} b^\alpha \right] + lr_* \leq r_* \hspace{1cm} (26)$$

In the sequel, we show that the multi-valued map $\Phi$ maps the ball $B_{r_*}(0)$ into itself. Indeed, for $x \in B_{r_*}(0)$ and $y \in \Phi(x)$, by (15), Hölder’s inequality and conclusion (i) of Lemma 2.5, (25) and (26), one has

$$e^{-Lt}\|y(t)\| \leq M\|\xi\| + e^{-Lt} \frac{\alpha M}{\Gamma(\alpha + 1)} \int_0^t (t-s)^{\alpha-1} \left[ \left( m(s) + \frac{b_1k_1}{k_B - k_J - k_2} \right) \|x(s)\| + d_1 \right] ds.$$
4. \textbf{Existence of global attractor.} In this section, we replace (HA) by the following assumption:
(HA*) The semigroup $T(t)$ generated by $A$ is norm-continuous, exponentially stable with exponent $\alpha$ and $\chi$-decreasing with exponent $\varsigma$, that is

$$
\|T(t)\|_{L(X)} \leq e^{-\sigma t}, \quad \|T(t)\|_{\chi} \leq c_* e^{-\varsigma t}, \quad t \geq 0.
$$

where $\sigma, \varsigma > 0$, $c_*$ is a constant, $\| \cdot \|_{\chi}$ is the $\chi$-norm defined in (4).

**Remark 2.**

(i) It is noticed that if the semigroup $T(t)$ is compact, then $\|T(t)\|_{\chi} = 0$, $\forall t > 0$. In this case, we can take $\varsigma = +\infty$.

(ii) The exponentially stability of $T(t)$ together with Proposition 4.1 in [9] implies that operators $S_\alpha(\cdot), Q_\alpha(\cdot)$ are asymptotically stable, i.e.,

$$
\lim_{t \to +\infty} \|S_\alpha(t)\|_{L(X)} = 0, \quad \lim_{t \to +\infty} \|Q_\alpha(t)\|_{L(X)} = 0.
$$

In the sequel, we define the m-semiflow concerned with problem (FDHVI) as follows:

$$
G : \mathbb{R}^+ \times X \to \mathcal{P}(X),
$$

$$
G(t, \xi) = \{x(t) : x \text{ is a mild solution of problem (FDHVI)}\}. \quad (28)
$$

It is easy to see that $G(t, \xi) = \Sigma(\xi)(t), \quad t \geq 0$. Moreover, applying the same arguments as in [19], we also see that

$$
G(t_1 + t_2, \xi) = G(t_1, G(t_2, \xi)), \quad \text{for all } t_1, t_2 \in R^+, \xi \in X,
$$

i.e., $G$ is a strict m-semiflow. In what follows, we prove $G$ is a u.s.c. map.

**Lemma 4.1.** If assumptions of Theorem 3.3 hold, then $G(t, \cdot)$ is u.s.c. with compact values for each $t > 0$.

**Proof.** By Theorem 3.4, for any $t > 0$, we know that $\Pi_t \circ \Sigma(\xi)$ is compact subset of $C([0, t]; X)$. This implies that $G(t, \xi)$ is a compact subset for each $\xi \in X$. That is, $G(t, \cdot)$ has compact values. According to Lemma 2.12, it suffices to prove that $G(t, \cdot)$ is quasi-compact and admits a closed graph.

We first show that $G(t, \cdot)$ is quasi-compact. Let $K \subset X$ is a compact subset. Let $\{z_n\} \subset G(t, \cdot)$, then we can find a sequence $\{\xi_n\} \subset K$ such that $\{z_n\} \subset G(t, \xi_n)$. Let $\xi_n \to \xi^* \in X$ and $x_n \in \Sigma(\xi_n)$ be such that

$$
x_n(0) = \xi_n, \quad x_n(t) = \xi_n. \quad (28)
$$

It follows from Theorem 3.4 that $\Pi_t \circ \Sigma(\xi)$ is compact subset of $C([0, t]; X)$. Then, there exists a subsequence of $\{x_n\}$ (denoted again by $\{x_n\}$) satisfying

$$
\Pi_t(x_n) \to x^* \in C([0, t], X).
$$

Therefore, by (28), we have $z_n \to x^*(t)$ in $X$ and $x^*(0) = \xi^*$.

Subsequently, we prove that $G(t, \cdot)$ has a closed graph. Let $\eta_n \to \eta_*, \eta_n \in G(t, \xi_n)$ and $\xi_n \to \xi_*$. We shall show that $\eta_* \in G(t, \xi_*)$. Choose $x_n \subset \Sigma(\xi_n)$ such that $\eta_n = x_n(t)$. Applying Theorem 3.4 again, $\{x_n\}$ has a convergent subsequence (also denoted by $\{x_n\}$). Assume that $x_* = \lim_{n \to \infty} x_n \in C([0, t]; X)$, then $\eta_* = x_*(t)$. It remains to prove that $x_* \in \Pi_t \circ \Sigma(\xi_*)$. Let $f_n \in \mathcal{N}_G^p(x_n)$ such that

$$
x_n = S_\alpha(\cdot)\xi_n + \mathcal{W}_\alpha(f_n).
$$

By (15) and the fact that $\{x_n\}$ is bounded, we see that $\{f_n\} \subset L^p([0, t]; X)$ is integrable bounded. Furthermore, $K(s) = G(\{x_n(s)\}), \ s \in [0, t]$, is compact and $\{f_n(s)\} \subset K(s)$. So the sequence $\{f_n\}$ is semicompact. Applying Lemma 2.11, we
obtain \( f_n \xrightarrow{\text{weakly}} f_* \) and \( W_\alpha(f_n) \to W_\alpha(f_*) \). Thus, one can pass to the limits in equality (29) to obtain
\[
x_* = S_\alpha(\cdot)\xi_* + W_\alpha(f_*).
\]
Since \( N^p_\alpha \) is weakly u.s.c., we get \( f_* \in N^p_\alpha(x_*) \). As a consequence, the last equality means \( x_* \in \Pi_t \circ \Sigma(\xi_*) \) as asserted.

For \( b > 0 \), let us define a translation operator \( G_b = G(b, \cdot) \). We apply the translation operator \( G_b = G(b, \cdot) \) to prove a condensing property of \( G_b \).

**Lemma 4.2.** Let conditions (HA*), (HB), (HF), (HJ), (Hg) and (H0) hold. If
\[
\ell = c_* E_{\alpha,1} (\cdot; t^\alpha) + 4 \sup_{t \in [0,\infty)} \int_0^t \left( \mu(s) + \frac{c(s)k_1}{k_B - k_J - k_2} \right) (t-s)^{\alpha-1} ||Q_\alpha(t-s)||_\chi ds < 1.
\]
then there exist \( b_0 > 0 \) and a number \( \ell \in [0,1) \) such that for all \( b \geq b_0 \) and for all bounded set \( \Omega \subset X \) satisfying
\[
\chi(G_b(\Omega)) \leq \ell \cdot \chi(\Omega).
\]

**Proof.** Let \( \Omega \) be a bounded subset of \( X \). We put \( D = \Sigma(\Omega) \) and have
\[
D(t) = \begin{align*}
G_t(\Omega) & \subset S_\alpha(t)\Omega + W_\alpha \circ N^p_\alpha(D) \\
& = S_\alpha(t)\Omega + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)N^p_\alpha(D)(s), \ t > 0,
\end{align*}
\]
It is easy to see that \( \Pi_t(D) \) is bounded subset of \( C([0,t]; X) \) for each \( t > 0 \). Thus, if the semigroup \( T(\cdot) \) is compact, then \( \chi(D(t)) = 0 \) for any \( t > 0 \). Consider the opposite case, i.e., the semigroup \( T(\cdot) \) is non-compact. Similar to the proof of (22), it yields from (30) and (31) that
\[
\chi(D(t)) = c_* E_{\alpha,1} (\cdot; t^\alpha) \chi(\Omega) + \chi \left( \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)N^p_\alpha(D)(s)ds \right)
\leq c_* E_{\alpha,1} (\cdot; t^\alpha) \chi(\Omega) + 4 \sup_{t \in [0,\infty)} \int_0^t \left( \mu(s) + \frac{c(s)k_1}{k_B - k_J - k_2} \right) (t-s)^{\alpha-1} ||Q_\alpha(t-s)||_\chi ds \chi(\Omega)
\leq \ell \cdot \chi(\Omega).
\]
Therefore, we get the conclusion of this lemma.

**Lemma 4.3.** If conditions of Lemma 4.2 hold, then multimap \( G(t, \cdot) \) is compact for each \( t > 0 \) and \( G \) is asymptotically upper semicompact.

**Proof.** Let \( \Omega \) be a bounded subset of \( X \) and \( \Xi_\Omega \) be the collection of all sequences \( \{\xi_j : \xi_j \in G(t_j, \xi_j), t_j \to +\infty\} \). Denote
\[
\omega = \sup \{ \chi(D) : D \in \Xi_\Omega \}.
\]
In the sequel, we prove that \( \omega = 0 \). Assume the opposite, then for each \( \vartheta^* \in (0, (1 - \ell)\omega) \), there exists \( D_{\Omega_1} = \{\xi_j\} \in \Xi_\Omega \) satisfying
\[
\chi(D_{\Omega_1}) \geq \omega - \vartheta^*.
\]
Here, \( \ell \) is given in Lemma 4.2. Taking \( b > 0 \) in the statement of Lemma 4.2, for each \( t_j \in (b, \infty) \), there is a number \( n_j \in \mathbb{N} \) satisfying \( t_j = n_j b + r_j, r_j \in [0,b) \). Let
an absorbing set, provided that

\[ \text{Theorem 4.5.} \]

\[ \text{Lemma 4.4.} \]

Let conditions of Lemma 4.2 hold. Then the m-semiflow \( \tau \)

\[ \text{Proof.} \]

The conclusion yields from theorem 3.3, Lemmas 4.1, 4.3 and 4.4.

\[ \text{Proof.} \]

Let \( \xi_j \in G(t_j, \Omega) = G(b + \tau_j, \Omega) = G(b(G(\tau_j, \Omega)). \)
Taking \( \eta_j \in G(\tau_j, \Omega) \) such that \( \xi_j \in G(b(\eta_j), \) we obtain

\[ \chi(D_\Omega) = \chi(\{\xi_j\}) \leq \chi(G(b(\{\eta_j\})) \leq E \cdot \chi(\{\eta_j\}) \leq \ell \omega < \omega - \vartheta^*. \]

This is a contradiction. Thus, we deduce that multimap \( G(t, \cdot) \) is compact for each \( t > 0. \) Therefore, this together with Proposition 1 in [19] infers that \( G \) is asymptotically upper semicompact.

**Lemma 4.4.** Let conditions of Lemma 4.2 hold. Then the m-semiflow \( G \) admits an absorbing set, provided that

\[ \iota = \sup_{t>0} \int_0^t (t-s)^{\alpha-1} \|Q_\alpha(t-s)\|_{L(X)} \left[ \left( m(s) + \frac{b_1k_1}{k_B-k_J-k_2} \right) + d_1 \right] \ ds < 1. \quad (32) \]

**Proof.** Let \( t > 0 \) and \( \Omega \subset X \) be a bounded set. Thus, there exists \( R^* > 0 \) such that, for any \( \xi \in \Omega \) and \( x \in \Sigma(\xi), \) we obtain \( \|\xi\| \leq R^* \) and

\[ \Phi(x)(t) = x(t) = S_{\alpha}(t)\xi + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s) ds. \]

for some \( f \in \mathcal{N}_G^p(x) \). By condition \((HA^*)\) and lemma 2.4, it is easy to obtain \( \|S_{\alpha}(t)\|_{L(X)} \leq E_{\alpha,1}(-\sigma^{\alpha}) \) Therefore, for \( t > 0, \) it follows from (15) and (32) that

\[ \|x(t)\| \]
\[ \leq \ R^* \|E_{\alpha,1}(-\sigma^{\alpha}) \]
\[ + \ \int_0^t (t-s)^{\alpha-1} \|Q_\alpha(t-s)\|_{L(X)} \left[ \left( m(s) + \frac{b_1k_1}{k_B-k_J-k_2} \right) \|x(s)\| + d_1 \right] ds \]
\[ \leq \ R^* \|E_{\alpha,1}(-\sigma^{\alpha}) \]
\[ + \ \|x\| \int_0^t (t-s)^{\alpha-1} \|Q_\alpha(t-s)\|_{L(X)} \left( m(s) + \frac{b_1k_1}{k_B-k_J-k_2} \right) ds \]
\[ + \ \int_0^t (t-s)^{\alpha-1} \|Q_\alpha(t-s)\|_{L(X)} d_1 ds \]
\[ \leq \ R^* \|E_{\alpha,1}(-\sigma^{\alpha}) + \iota \|x\| + 1. \]

This implies

\[ \|x(t)\| \leq \frac{R^* E_{\alpha,1}(-\sigma^{\alpha}) + 1}{1 - \iota}. \]

The last inequality ensures that the ball centered at origin with radius

\[ \frac{R^* E_{\alpha,1}(-\sigma^{\alpha}) + 1}{1 - \iota}, \]

becomes an absorbing set of an m-semiflow \( G. \) \( \square \)

**Theorem 4.5.** If conditions of Lemma 4.4 hold, then an m-semiflow \( G \) generated by problem (FDHV1) admits a compact global attractor.

**Proof.** The conclusion yields from theorem 3.3, Lemmas 4.1, 4.3 and 4.4. \( \square \)
5. An example. Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) with smooth boundary. We consider the following system: find \( x, u \in C([0, b]; L^2(\Omega)) \) satisfying

\[
\begin{cases}
  cD_0^\alpha x(t, y) \in \triangle_y f(t, y) + f(t, y), & t > 0, \ y \in \Omega, \\
  f(t, y) = \lambda f_1(y, x(t, y), u(t, y)) + (1 - \lambda)f_2(y, x(t, y), u(t, y)), & t > 0, \ y \in \Omega, \\
  -\Delta_y u(t, y) + \partial J(x(t, y)) \ni g(y, x(t, y), u(t, y)), & t \in (0, b), \ y \in \Omega \\
  x(t, y) = u(t, y) = 0, & (t, y) \in [0, b] \times \partial \Omega \\
  x(0, y) = \xi(y), \ y \in \Omega,
\end{cases}
\]

where \( \lambda \in [0, 1], f_1, f_2 : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is continuous and \( g : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is nonsmooth and nonconvex; \( \mathcal{J}(\cdot) \) sense of Clarke for a locally Lipschitz functional \( J \) is exponentially stable, i.e.,

\[
\lim_{\lambda \to 0} \sup_{\Omega} \{ \mathcal{J}(\lambda x) \} = 0.
\]

It is known that \( [28] \) the semigroup \( \sigma \) generated by \( A \) is compact and exponentially stable, i.e.,

\[
\|A(t)\|_{L(X)} \leq e^{-\sigma t},
\]

where \( \sigma = \inf\{\|\nabla \|^2_X : \|u\|_X = 1\} \). The condition \( (HA^*) \) is satisfied.

In addition, we assume that there exist two functions \( m_1, m_2 \in C(\mathbb{R}^+, \mathbb{R}^+) \) with \( p > \frac{1}{\alpha} \) and two nonnegative functions \( b_1, b_2, c_2 \in L(\Omega) \) such that

\[
\begin{align*}
  |f_1(y, z, w)| &\leq m_1(t)|y| + b_1(y)|z| + c_1(y), \\
  |f_2(y, z, w)| &\leq m_2(t)|y| + b_2(y)|z| + c_2(y).
\end{align*}
\]

It is easy to see that \( F \) is a multimap with closed convex and compact values. Moreover

\[
\|F(x, u)\| \leq \max\{\|m_1\|_{L(\mathbb{R}^+, \mathbb{R}^+)} , \|m_2\|_{L(\mathbb{R}^+, \mathbb{R}^+)}\} \|x\|_X + \max\{\|b_1\|_{L(\mathbb{R}^+, \mathbb{R}^+)} , \|b_2\|_{L(\mathbb{R}^+, \mathbb{R}^+)}\} \|u\|_X + \max\{\|c_1\|_{L(\mathbb{R}^+, \mathbb{R}^+)} , \|c_2\|_{L(\mathbb{R}^+, \mathbb{R}^+)}\} \|x\|_X + \max\{\|b_1\|_{L(\mathbb{R}^+, \mathbb{R}^+)} , \|b_2\|_{L(\mathbb{R}^+, \mathbb{R}^+)}\} \|u\|_X.
\]

Since \( f_1 \) and \( f_2 \) are continuous functions, the fact implies that \( F \) has a closed graph. Furthermore, if \( \{x_n\} \subset X, \{u_n\} \subset Y \) are convergent sequences, then we can find a sequence \( f_n \in F(x_n, u_n) \) that is convergent in \( X \) by applying the Lebesgue dominated convergence theorem. So \( F \) is quasi-compact. In view of Lemma 2.12, \( F \) is a u.s.c. multimap. So (HF) is testified.
We consider the elliptic HVI. We set \( B = -\triangle \), where \(-\triangle\) stands for the distributional Laplace operator, namely,
\[
\langle u, -\triangle v \rangle = \int_{\Omega} \nabla u(y) \nabla v(y) \, dy, \text{ for each } u, v \in H_0^1(\Omega).
\]
It is easily seen that \( \langle u, Bu \rangle \geq \sigma \| u \|_Y^2 \). So the condition (HB) is fulfilled with \( k_B = \sigma \).

Regarding nonlinear function \( g \), we assume that there exist two nonnegative functions \( k_1, k_2 \in L(\Omega) \) satisfying \( k_2 < k_B - k_J \) and
\[
|g(y, z, w) - g(y, z^*, w^*)| \leq k_1(x)|z - z^*| + k_2(x)|w - w^*|, \forall x \in \Omega, z, w, z^*, w^* \in \mathbb{R}.
\]
We consider the following abstract form of \( g \):
\[
g : X \times Y \to L(\Omega) \quad g(x, u)(y) = g(y, x(y), u(y)).
\]
It follows that
\[
\|g(x, u) - g(x^*, u^*)\|_X \leq \|k_1\|_X |x - x^*| + \|k_2\|_X |u - u^*|, \forall x, u, x^*, u^* \in X.
\]
In addition, from compactness of \( T(t) \) and Remark 2, it is easy to verify that conditions (30) and (32) hold. Therefore, problem (33) can be written as the abstract form of problem (FDHVI) (1).

Finally, we choose \( \rho_0 = 2 \times \frac{1}{2} > 1 \). Thus, summarizing the above, all assumptions of Theorems 3.3, 3.4 and 4.5 are fulfilled. Consequently, we deduce that the set of mild solutions to problem (33) is nonempty and compact for each initial datum \( \xi \in X \) and admits a compact global attractor.

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REFERENCES

[1] N. T. V. Anh and T. D. Ke, On the differential variational inequalities of parabolic-elliptic type, Math. Methods Appl. Sci., 40 (2017), 4683–4695.
[2] D. Bothe, Multivalued perturbations of m-accretive differential inclusions, Isr. J. Math., 108 (1998), 109–138.
[3] X. Chen and Z. Wang, Differential variational inequality approach to dynamic games with shared constraints, Math. Program. Ser. A, 146 (2014), 379–408.
[4] F. H. Clarke, Optimization and Nonsmooth Analysis, Wiley, New York, 1983.
[5] J. Diestel, W. M. Ruess and W. Schachermayer, Weak compactness in \( L^1(\mu, X) \), Proc. Am. Math. Soc., 118 (1993), 447–453.
[6] J. Gwinner, On a new class of differential variational inequalities and a stability result, Math. Program. Ser. B, 139 (2013), 205–221.
[7] A. Halanay, Differential Equations, Stability, Oscillations, Time Lags, Academic Press, New York and London, 1966.
[8] M. Kamenskii, V. Obukhovskii and P. Zecca, Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces, Walter de Gruyter, Berlin, New York, 2001.
[9] T. D. Ke and D. Lan, Decay integral solutions for a class of impulsive fractional differential equations in Banach spaces, Fract. Calc. Appl. Anal., 17 (2014), 96–121.
[10] T. D. Ke and D. Lan, Fixed point approach for weakly asymptotic stability of fractional differential inclusions involving impulsive effects, J. Fixed Point Theory Appl., 19 (2017), 2185–2208.
[11] T. D. Ke, N. V. Loi and V. Obukhovskii, Decay solutions for a class of fractional differential variational inequalities, Fract. Calc. Appl. Anal., 18 (2015), 531–553.
[12] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.
X. S. Li, N. J. Huang and D. O’Regan, Differential mixed variational inequalities in finite dimensional spaces, *Nonlinear Anal.*, 72 (2010), 3875–3886.

X. W. Li and Z. H. Liu, Sensitivity analysis of optimal control problems described by differential hemivariational inequalities, *SIAM J. Control Optim.*, 56 (2018), 3569–3597.

Z. H. Liu, S. Migorski and S. Zeng, Partial differential variational inequalities involving nonlocal boundary conditions in Banach spaces, *J. Differential Equations*, 263 (2017), 3989–4006.

Z. H. Liu, S. Zeng and D. Motreanu, Partial differential hemivariational inequalities, *Adv. Nonlinear Analysis*, 7 (2018), 571–586.

N. V. Loi, T. D. Ke, V. Obukhovskii and P. Zecca, Topological methods for some classes of differential variational inequalities, *J. Nonlinear Convex Anal.*, 17 (2016), 403–419.

F. Mainardi, P. Paradisi and R. Gorenflo, *Probability Distributions Generated by Fractional Diffusion Equations*, in: J. Kertesz, I. Kondor (Eds.), Econophysics: An Emerging Science, Kluwer Academic Publisher, Dordrecht Boston, London, 2000.

V. S. Melnik and J. Valero, On attractors of multivalued semi-flows and differential inclusions, *Set-Valued Anal.*, 6 (1998), 83–111.

V. S. Melnik and J. Valero, On global attractors of multivalued semiprocesses and nonautonomous evolution inclusions, *Set-Valued Anal.*, 8 (2000), 375–403.

S. Migórski, On existence of solutions for parabolic hemivariational inequalities, *J. Comput. Appl. Math.*, 129 (2001), 77–87.

S. Migórski, A. Ochal and M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities*, Models and Analysis of Contact Problems, Springer-Verlag, New York, 2013.

J. S. Pang and D. E. Stewart, Differential variational inequalities, *Math. Program. Ser. A*, 113 (2008), 345–424.

I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering, Academic Press, New York, 1999.

A. U. Raghunathan, J. R. Pérez-Correa, E. Agosin and L. T. Biegler, Parameter estimation in metabolic flux balance models for batch fermentation-formulation and solution using differential variational inequalities, *Ann. Oper. Res.*, 148 (2006), 251–270.

D. E. Stewart, Uniqueness for index-one differential variational inequalities, *Nonlinear Anal., Hybrid Syst.*, 2 (2008), 812–818.

A. Tasora, M. Anitescu, S. Negrini and D. Negrut, A compliant visco-plastic particle contact model based on differential variational inequalities, *International Journal of Nonlinear Mechanics*, 53 (2013), 2–12.

I. Vrabie, *C₀-Semigroups and Applications*, Elsevier, Amsterdam, 2003.

X. Wang and N. J. Huang, Differential vector variational inequalities in finite dimensional spaces, *J. Optim. Theory Appl.*, 158 (2013), 109–129.

Y. Zhou and F. Jiao, Existence of mild solutions for fractional neutral evolution equations, *Comput. Math. Appl.*, 59 (2010), 1063–1077.

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