Singular value decomposition in parametrized tests of post-Newtonian theory

Archana Pai and K G Arun

1 Indian Institute of Science Education and Research Thiruvananthapuram, College of Engineering Campus, Trivandrum 695016, India
2 Chennai Mathematical Institute, Siruseri 603103, India

E-mail: archana@iisertvm.ac.in and kgarun@cmi.ac.in

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Abstract

Various coefficients of the 3.5 post-Newtonian (PN) phasing formula of non-spinning compact binaries moving in circular orbits is fully characterized by the two component masses. If two of these coefficients are independently measured, the masses can be estimated. Future gravitational wave observations may measure most of the eight independent PN coefficients calculated to date. These additional measurements can be used to test the PN predictions of the underlying theory of gravity with a similar PN structure. Since all of these parameters are functions of the two component masses, there is a strong correlation between the parameters when treated independently. Using singular value decomposition of the Fisher information matrix, we remove these correlations and obtain a new set of parameters which are a linear combination of the original phasing coefficients. We show that the new set of parameters can be estimated with significantly improved accuracies which have implications for the ongoing efforts to implement parametrized tests of PN theory in the data analysis pipelines.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Testing general relativity (GR) and alternative theories of gravity is one of the most important goals of directly observing gravitational wave (GW) signals (see [1, 2] for reviews). The ground-based GW detectors such as LIGO and Virgo are undergoing upgrades to second generation advanced LIGO (aLIGO) and advanced Virgo, making them capable of observing 1000 times the volume of the universe they were sensitive to in the first generation. The science case studies of a third generation GW detector, namely Einstein telescope (ET), are
being undertaken [3]. In this context, it is important to devise strategies which enhance the capabilities of these detectors to test GR and alternative theories of gravity. Inspiralling compact binaries are one of the most promising sources of GWs. In the adiabatic inspiral regime, the gravitational waveforms from these sources are accurately modelled using the post-Newtonian (PN) approximation to GR [4]. For modelled signals, matched filtering is known to be an optimal detection technique in Gaussian noise. Existing detection and parameter estimation techniques are based on matched filtering. Thus, prior predictability of the detailed nature of the waveforms becomes crucial in testing GR using inspiralling compact binaries.

Since gravitational waveforms within alternative theories of gravity are not easy to compute, one tries to parametrize the possible deviations of alternative theories of gravity from GR. There have been many proposals in the literature to test GR and its alternatives. These proposals may be classified into two types. First is to test a particular alternative theory of gravity by bounding the value(s) of some parameters which quantifies the deviation from GR. The proposals to bound scalar–tensor theories [5–11] and massive graviton theories [12, 8, 13, 14, 9, 10, 15, 16] are examples of the first type. The second type is to bound the values of a set of parameters which generically captures the deviation from GR without referring to any particular alternative theory. Such a test was first proposed in [17] where the authors discussed the possibility of measuring the 1.5PN term in the phasing formula. This was extended to include all the PN-order terms in the phasing formula in [18, 19] where the authors discussed the possibility of accurate determination of various PN coefficients of a binary inspiral waveform from GW observations and further proposed a consistency test of these phasing coefficients in the mass plane of the binary. In [20], a similar exercise was carried out by folding in the amplitude corrections to the PN waveforms. Recently, there have been other proposals in the literature that discuss generic parametrized deviations from GR and how they affect GW detection and parameter estimation. The authors in [21–23] discussed parametrized post-Einsteinian (PPE) bounds on alternative theories of gravity, while the authors in [24] discussed implementing a variant of the test proposed in [19] in the LIGO data analysis context using Bayesian methods. The PPE can be viewed as a generalization of PTPN to allow for the fact that many modified theories of gravity do not possess the same PN structure as GR. In other words, the PPE tests reduce exactly to the PTPN in the limit where the former’s exponent parameters (of the frequency) are taken to be same as those of PN series in GR. The effect of spin precession on determining the various PN parameters was studied in [25]. The authors in [26] discussed possible bounds on generic dipolar gravitational radiation from the GW observations.

This work can be seen as an extension of the idea proposed in [18] where the gravitational waveform was parametrized in terms of the phasing coefficients. In an ideal scenario, one would like to treat all the eight phasing parameters as independent as suggested in [18]. However, since all of them are functions of binary masses, there are strong correlations amongst them. This results in a near-singular Fisher information matrix whose inversion cannot be trusted. Hence, it was possible to obtain reliably the errors on all the parameters only for a limited range of masses, within this approach1.

In this work, we use the singular value decomposition (SVD) approach to circumvent this ill conditioning of the Fisher matrix. Using SVD, we construct new parameters (corresponding to the dominant singular values of the Fisher matrix) which are combinations of the original phasing parameters, and obtain the errors associated with these new parameters. We find these errors to be very small which implies that one will be able to extract these new parameters with

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1 Parameter estimation techniques such as Bayesian inference can help to deal with such situations in the case of real data [27].
very good accuracy with ground-based GW detectors such as aLIGO and ET. For example,
we focus on the first three dominant new phase parameters. We show that these new phase
parameters of GW can be estimated with relative accuracy of a few times $10^{-5}$ as emitted by
a stellar mass BHs at 100 Mpc. Furthermore, for the ET, the accuracy for the stellar mass and
intermediate BH binary inspirals at 100 Mpc would be $10^{-6} - 10^{-5}$ which would enable one to
very precisely estimate these parameters.

This paper is organized in the following way. Section 2 describes the parametrized tests
of PN theory and discusses the waveform model we use. A summary of the noise model and
the parameter estimation scheme using the Fisher matrix is presented in section 3. S V D of
the Fisher matrix is explained in detail in section 4. Our results are discussed in section 5.
Conclusions and future directions are elaborated in section 6.

2. Parametrized tests of PN theory

Fourier domain gravitational waveforms from non-spinning inspiralling compact binaries in a
circular orbit, obtained using the stationary phase approximation, may be written as

$$\tilde{h}(f) = A f^{-7/6} e^{2\Psi(f)},$$

with the amplitude $A$ and phase $\Psi(f)$ given by

$$A = \frac{C}{Dl \pi^{2/3}} \sqrt{\frac{5}{24}} M^{5/6},$$

$$\Psi(f) = 2\pi f t_c - \Phi_c + \frac{\pi}{4} + \sum_{k=0}^{7} \left[ \psi_k + \psi_{kl} \ln f \right] (k-5/3).$$

Here, $t_c$ and $\Phi_c$ are the fiducial epoch of merger and the phase of the signal at that epoch.
Chirp mass $M$ is related to the total mass $M$ by $M = M \eta^{3/5}$, where $\eta = \sqrt{m_1 m_2} / (m_1 + m_2)$ ($m_1$ and $m_2$
are individual masses of the binary) is called the symmetric mass ratio which is 0.25 for equal
mass binaries. $C$ which appears in the equation for amplitude of the waveform is a constant
with $0 \leq C \leq 1$ which depends on the sky position of the source and the geometry of the
detector and has a root-mean-square value of 2/5 when averaged over all sky locations and
source orientations. The coefficients in the PN expansion of the Fourier phase are given by

$$\psi_k = \frac{3}{256} \eta (\pi M)^{(k-5)/3} \alpha_k, \quad \psi_{kl} = \frac{3}{256} \eta (\pi M)^{(k-5)/3} \alpha_{kl},$$

where $\alpha_k$ and $\alpha_{kl}$ up to 3.5 PN read as 

$$\alpha_0 = 1, \quad \alpha_1 = 0, \quad \alpha_2 = \frac{3715}{756} + \frac{55}{9} \eta, \quad \alpha_3 = -16\pi, \quad \alpha_4 = \frac{15 293 365}{508 032} + \frac{27 145}{504} \eta + \frac{3085}{72} \eta^2;$$

2 Throughout this paper, we work in geometrical units with $G = c = 1$.}

3
\[ \alpha_5 = \pi \left( \frac{38,645}{756} - \frac{65}{9} \eta \right) \left[ 1 + \ln(6^{3/2} \pi M) \right], \]  
\( \alpha_{5l} = \pi \left( \frac{38,645}{756} - \frac{65}{9} \eta \right), \]  
\[ \alpha_6 = \frac{11,583,231,236,531}{4694,215,680} - \frac{640}{3} \pi^2 - \frac{6848}{21} C + \left( -\frac{15,737,765,635}{3048,192} + \frac{2255}{12} \pi^2 \right) \eta \]  
\[ + \frac{76,055}{1728} \eta^2 - \frac{127,825}{1296} \eta^3 - \frac{6848}{63} \ln(64 \pi M), \]  
\[ \alpha_{6l} = -\frac{6848}{63}, \]  
\[ \alpha_7 = \pi \left( \frac{77,096,675}{254,016} + \frac{378,515}{1512} \eta - \frac{74,045}{756} \eta^2 \right), \]  
\[ \alpha_{kl} = 0, \text{ for } k = 0, 1, 2, 3, 4, 7, \]  
where \( C = 0.577 \ldots \) is the Euler constant. It is evident from equation (2) that different PN-order terms occur with different frequency dependences. For non-spinning binaries, each of the PN coefficients \( \psi_k \) and \( \psi_{kl} \) depends only on the component masses of the binary as given in equation (3). Interestingly, from equation (2), the non-logarithmic term corresponding to \( k = 5 \) can be absorbed into a redefinition of the fiducial phase \( \Phi_c \) as it has no frequency dependence. Hence, there are eight parameters that characterize the phasing formula which are \( \{ \psi_0, \psi_2, \psi_3, \psi_4, \psi_{5l}, \psi_{6l}, \psi_6, \psi_7 \} \).

The usual parameter estimation problem addresses the estimation of individual masses of the binary (or particular combinations of the component masses) using the 3.5 PN accurate phasing formula. Two of the phasing coefficients are sufficient to estimate the two component masses. The additional phasing coefficients improve this estimation \([31, 32, 30, 33]\). Alternatively, one can treat each of the phasing coefficients as independent parameters in the parameter estimation problem. If this is possible, it will enable us to test the PN structure of GR by independently measuring each PN coefficient and comparing with the predictions of PN approximation to GR \([18]\). Although this idea works in principle, there will be large correlations amongst the different PN phasing coefficients because all of them are essential functions of the two component masses of the binary.

To circumvent this problem, the proposal in \([19]\) uses only three out of eight parameters as independent in the parameter estimation problem. This is achieved by re-expressing all the phasing coefficients, except the one to be used as a test parameter, in terms of two basic parameters (usually, the two lowest order phasing coefficients). The third parameter used as the test can be any of the higher order terms. Hence, the authors proposed many such tests corresponding to different higher order phasing coefficients which are used as the test parameter. Needless to say, this test is not as robust as the original test where all of the phasing coefficients were independent, but this is one of the plausible ways to overcome the problem arising from correlations between various parameters.

This paper paves foundation for a new approach of working with the original proposal where all of the phase parameters are treated independently. Using SVD, one can get rid of the correlations among the PN phasing coefficients and obtain the most dominant new phase parameters as a linear combination of the original phase parameters. The PN theory has unique predictions about the values of these new parameters and this can be compared with observations by directly measuring them from the data. Hence, these parameters can be used to do tests of PN theory.
We model the noise to be a Gaussian and stationary, random process. We are interested in the estimation of parameters of a signal buried in the noise. We use the Fisher information matrix formalism [34, 35] to estimate the errors associated with the measurement of the signal parameters. (See [36, 37] for some of the initial applications of Fisher matrix approach in the GW data analysis context. See a recent article about the caveats of using the Fisher matrix as an error estimator [38]. A method to go beyond the Fisher matrix is discussed in [39, 40].)

Let \( \hat{\theta} \) denote the ‘actual values’ of the parameters and \( \hat{\theta}_{ab} \equiv \hat{\theta}^a + \Delta \theta^a \) denote the best-fit parameters (or parameter estimate) in the presence of noise. Then, in the limit of large signal-to-noise ratios (SNRs), errors in the estimation of parameters \( \Delta \theta^a \equiv \{ \Delta \theta^a \} \) obey a multivariate Gaussian probability distribution of the form

\[
p(\Delta \theta) = N \exp[-\Gamma_{ab} \Delta \theta^a \Delta \theta^b / 2], \tag{14}
\]

where \( N \) is the normalization constant. The repeated indices are summed over. The quantity \( \Gamma_{ab} \) appearing in equation (14) is the Fisher information matrix which is given by

\[
\Gamma_{ab} = (h_a | h_b), \tag{15}
\]

where \( h_a \equiv \partial h / \partial \theta^a \). Here, \((\_ | \_)\) denotes the noise weighted inner product as defined below.

Given any two functions \( a \) and \( b \), their inner product is defined as

\[
(a | b) \equiv 4\Re \int_{f_{\text{lower}}}^{f_{\text{upper}}} \hat{a}^*(f) \hat{b}(f) S_h(f) \, df, \tag{16}
\]

where \( S_h(f) \) is the one-sided noise power spectral density of the detector. Hence, the explicit expression for the Fisher matrix is given by

\[
\Gamma_{ab} = 4\Re \int_{f_c}^{f_{\text{lower}}} \hat{h}_a^*(f) \hat{h}_b(f) S_h(f) \, df, \tag{17}
\]

where \( f_c \) is the lower cut-off frequency as given by the seismic noise limit and \( f_{\text{upper}} \) is the upper cut-off frequency which is the frequency at the last stable orbit beyond which the PN approximation breaks down. The frequency at the last stable orbit can be expressed as a function of the total mass \( M \) of the binary as \( f_{\text{upper}} = (63/2 \pi M)^{-1} \). \( \Re(.) \) denotes the real part of the quantity in brackets.

The covariance matrix is the inverse of the Fisher matrix as given below:

\[
C \equiv \{ \langle \Delta \theta^a \Delta \theta^b \rangle \} = \Gamma^{-1}, \tag{18}
\]

where \( \langle \_ \rangle \) denotes an average over the probability distribution function in equation (14). The root-mean-square error \( \sigma_a \) in the estimation of the parameters \( \theta^a \) is

\[
\sigma_a = \langle (\Delta \theta^a)^2 \rangle^{1/2} = \sqrt{C_{aa}}. \tag{19}
\]

Note that in the above expression, index \( a \) is not summed over.

For the noise power spectral densities of aLIGO and ET, we have used the analytic fit given in [20, equation (2.1)] and [20, equations (2.2) and (2.3)], respectively. The lower limit of the frequency is assumed to be given by the seismic cut-off which is taken to be 20 Hz for aLIGO and 1 Hz for ET.

4. Singular value decomposition and Fisher information matrix

SVD is a well-known technique in signal processing to transform a set of correlated variables into a set of uncorrelated variables in terms of which certain features of the data are more
evident. This technique is used both for detection purposes (see, for example, [41]) and in studies of parameter estimation [42] in the case of GW data analysis. There are techniques similar to SVD which have also been used in the case of detection [43, 44].

In this section, we recast the parameter estimation problem with the help of SVD of the Fisher matrix. We show that the truncated SVD eliminates the near-singular feature of the Fisher information discussed in section 2. To demonstrate, we opt to work in the discrete frequency domain. However, the final result does not depend on this choice.

Let \( \theta \equiv \{ \theta^i \} \) denotes the \( m \)-dimensional vector corresponding to the PN phase parameters as defined in equation (3). We divide the frequency band \((f_s, f_{\text{fin}})\) into \( N \) discrete frequency points \( f_k \) with the bin size \( \Delta f \) such that \( f_0 = f_s \) and \( f_{N-1} = f_{\text{fin}} \). In the discrete domain, equation (17) is recast as

\[
\Gamma_{ij} = 4\Re \left[ \sum_{k=0}^{N-1} \frac{\tilde{h}_j(f_k) \tilde{h}^*_i(f_k)}{\sqrt{B_k(f_k)}} \right] \Delta f,
\]

where at any frequency bin, \( f_k, \tilde{h}_j(f_k) = \partial \tilde{h}(f_k)/\partial \theta^i \) denotes the \( i \)th component of the tangent vector of the GW signal with respect to the phase parameters.

We define \( H = [H_k] = \frac{h_k(f_k) \gamma_k}{\sqrt{B_k}} \) to be a matrix of size \( m \times N \), where the row index \( i \) corresponds to the phase parameter space and the column index \( k \) is for the frequency index. The matrix \( H \) physically signifies the variation of the signal in the \( m \)-dimensional parameter space \( \theta \) when evaluated at different frequency locations. It further acts as a building block for the Fisher matrix as shown below.

The Fisher matrix re-expressed in terms of \( H \) takes the form

\[
\Gamma_{ij} = \sum_{k=0}^{N-1} [H_{ik}H^*_j + H^*_ikH_j].
\]

The main contribution to the variation of the signal in the parameter space comes from the variation of the phase. Hence, owing to the form of the signal expressed in equation (1), we note that \( H_{ik}H^*_j + H^*_ikH_j \) is a real quantity in the phase parameter space \( \theta \). In terms of \( H \), the \( m \times m \) Fisher matrix compactifies to

\[
\Gamma = 2HH^\dagger.
\]

Henceforth, we focus on \( H \). The SVD is a powerful tool to identify the subspace of \( m \times N \) space where \( H \) resides or in other words, the directions along which the signal varies dominantly in the parameter space at a given frequency location.

The method of SVD provides a factorization of any rectangular matrix into a square matrix, a rectangular diagonal matrix and another square matrix as given below:

\[
H = U_H \Sigma_H V_H^\dagger.
\]

where \( U_H \) and \( V_H \) are the left and right unitary square matrices of \( m \) and \( N \) dimensions, respectively, and the rectangular \( m \times N \) diagonal matrix \( \Sigma_H \) arranges the singular values of \( H \) in the descending order. The columns of \( U_H \) and \( V_H \) form an orthonormal basis set in \( m \)- and \( N \)-dimensional spaces, respectively, i.e. \( U^\dagger V = V^\dagger U = I \). Furthermore, they act as (orthogonal) eigenvectors of \((HH^\dagger)_{m \times m}\) and \((H^\dagger H)_{N \times N}\), respectively. Substituting equation (23) into equation (22), equation (22) emerges as an SVD of \( \Gamma \) as

\[
\Gamma = U_H \Sigma U_H^\dagger \Rightarrow \Gamma U_H = \Sigma U_H,
\]

where \( \Sigma = 2 \Sigma_H \Sigma_H^\dagger \) with the diagonal entries as \( \Sigma_{kk} = 2(\Sigma_{kk})^2 \).
We note that \( U_H \) provides the SVD for the Fisher matrix with its columns acting as eigenvectors of \( \Gamma \). In other words, they provide the principal components of \( \Gamma_1 \). Hence, we have shown an alternative way to obtain the principal components of Fisher matrix using the SVD of \( H \). We can compare and contrast our method with the principal component analysis (PCA) used, for example, in [42].

In [42], the authors focus on the PCA of the covariance matrix in the context of parameter estimation of spin-modulated chirp signals in the LISA band. Since the covariance matrix is simply the inverse of the Fisher matrix, both the methods would be equivalent for a non-singular \( \Gamma \).

4.1. Treating the ill-conditioning of the Fisher matrix

To recall, in order to test the PN theory, the ideal scenario would be to measure all the PN phase coefficients independently which was not possible due to large correlations among the parameters. The main hurdle to address the full problem was that the Fisher matrix when expressed in the eight-dimensional parameter space \( \theta \equiv \{ \psi_0, \psi_2, \psi_3, \psi_4, \psi_5l, \psi_6l, \psi_6, \psi_7 \} \) is ill-conditioned. An ill-conditioned Fisher matrix implies that \( \Gamma_1 \) cannot be inverted reliably.

For cases where \( \Gamma_1 \) is near-singular, the SVD approach provides an elegant solution to deal with the singularity and conditions the Fisher matrix. In the rest of the section, we shall focus on this.

Consider the parameter space to be \( \theta \equiv \{ \psi_0, \psi_2, \psi_3, \psi_4, \psi_5l, \psi_6l, \psi_6, \psi_7 \}, i.e. m = 8 \). We compute the SVD of \( H \) in this parameter space. In the case of the near-singular matrix, some of the diagonal entries of \( \Sigma_1 \) are very close to zero. As a result, the errors in those parameters are very large or unreliable. Furthermore, the effect of singular nature of \( \Gamma_1 \) introduces large errors in other parameters as well. Thus, to obtain the reliable errors, the first step is to condition the Fisher matrix or treat its singular nature. There are various approaches to address this problem.

We follow the truncated SVD approach in this paper [45].

The truncated SVD approach is elaborated in our context as below. We first perform the SVD of \( H \). We then examine the singular values given by the diagonal entries of \( \Omega_1 \), i.e. \((\Omega_1)_{kk}\). The total number of non-zero diagonal entries gives the rank of \( H \). We define the ratio of the singular values \((\Omega_1)_{kk}/(\Omega_1)_{11}\) and consider only those which are greater than or equal to a small number \( \epsilon \). Let the total number which satisfies this criterion be \( r \). Then, we truncate \( \Omega_1 \) to the dimensions equal to \( r \). The new truncated singular matrix \( \Omega_1^t \) is \( r \times r \) with diagonal entries. Consequently, the truncated version of left and right singular matrices \( U_H^t \) and \( V_H^t \) is a subset of \( U_H \) and \( V_H \), respectively, which corresponds to the first \( r \) dominant singular values.

The truncated SVD takes the form

\[
H = U_H^t \Omega_1^t V_H^{t\dagger},
\]

where \( U_H^t \) is \( 8 \times r \), \( \Omega_1^t \) is \( r \times r \) and \( V_H^{t\dagger} \) is \( r \times N \). Thus, in the truncated SVD approach, \( 8 \times N \)-dimensional \( H \) is reconstructed by eliminating all non-dominant/near-zero singular values which otherwise do not contribute to \( H \). This explicit elimination of zeros brings us into a subspace in which \( H \) resides. Hence, by construction, the truncated SVD given in equation (24) provides a conditioned invertible Fisher matrix

\[
\Gamma = U_H^t \Sigma^t U_H \quad \Rightarrow \quad U_H^t \Gamma U_H = \Sigma^t,
\]

3 Laser interferometer space antenna (LISA) is a proposed space-based GW detector sensitive to the low-frequency band between \(-10^{-4}\) and 1 Hz.
where \( \Sigma^T = 2 \Omega^T \Omega \) and \( U^T \) give eigenvectors corresponding to the dominant eigenvalues. Thus, the truncated SVD conditions \( \Gamma \). In fact, \( U^T \) provides the unitary transformation equation (26) which diagonalizes \( \Gamma \) with the diagonal entries \( \Sigma_{kk}^T = 2(\Omega_k^T)^2 \) for \( k = 1, \ldots, r \).

Comparing this approach with that of [42], one observes that the latter is based on the PCA of the covariance matrix which is the inverse of Fisher matrix. This method cannot be applied if the inversion of the Fisher matrix itself is unreliable, as in our case. In other words, PCA is not aimed at removing the near-singular nature of the Fisher matrix, whereas our procedure is specifically devised for that.

### 4.2. New phase parameters and the relative errors

Let \( \Delta \theta \) be the error in the phase parameters \( \theta \) which obey the multi-dimensional Gaussian distribution following equation (14) as given below:

\[
p(\Delta \theta) = N \exp[-\Gamma_{\alpha \beta} \Delta \theta_{\alpha} \Delta \theta_{\beta}/2].
\] (27)

The repeated indices are summed over and \( \alpha, \beta = 1, \ldots, 8 \).

Consider the scalar \( \Gamma_{\alpha \beta} \Delta \theta_{\alpha} \Delta \theta_{\beta} \) in equation (27). The truncated SVD approach should keep this scalar invariant as we are not removing any information but removing the zeros in the problem. We use this fact to obtain the new phase coordinates:

\[
\Delta \theta^T \Gamma \Delta \theta = \Delta \theta^T U \Sigma U^T \Delta \theta
\equiv \Delta \theta'^T \Sigma' \Delta \theta',
\] (28)

where superscript \( T \) shows the transpose of the matrix. The new phase coordinates \( \theta' \) are obtained by linearly combining the PN phase parameters \( \theta \) as

\[
\theta' = U^T \theta,
\] (29)

where \( \theta' = \{\psi'_1, \psi'_2, \psi'_3, \ldots, \psi'_r\} \) is an \( r \times 8 \)-dimensional phase vector. The Fisher matrix and hence the covariance matrix are diagonal in this new phase coordinates. Thus, \( \theta' \) phase coordinates are statistically independent.

Let the errors in parameter estimation be \( \Delta \theta' \). Owing to the statistical independence, the multi-dimensional Gaussian distribution of \( r \)-independent random variables \( \Delta \theta'_{\alpha} \) is the product of the \( k \) Gaussian probability distributions with zero mean and variance \( \sigma_{\alpha} = (\Sigma_{\alpha \alpha}^{'})^{-1/2} \) as shown below:

\[
p(\Delta \theta') = N' \Pi \alpha \exp[-(\Delta \theta'^2_{\alpha})/(2\sigma_{\alpha}^2)],
\] (30)

where \( \alpha = 1, \ldots, r \) and \( N' \) is a normalization constant.

### 4.3. Phasing formula in terms of the new phase parameters

The PN phase given in equation (3) can be expressed in terms of \( \theta \) as

\[
\Psi(f) = 2\pi f c \Phi_f + \pi/4 + \sum_{j=1}^{8} P_j \theta_j,
\] (31)

where \( \sum_{j=1}^{8} P_j \theta_j \equiv \sum_{k=0}^{7} [\psi_k + \psi_{k+} \ln f] f^{(k-5)/3} \) such that

\[
P \equiv \{f^{5/3}, f^{-1}, f^{-2/3}, f^{-1/3}, \ln f, f^{1/3}, \ln f, f^{1/3}, f^{2/3}\}
\] (32)

is a row vector as a function of frequency. Using the unitary transformation, \( \theta' = U^T \theta \) and using \( UU^T = I \), one can write the sum \( \sum_{j=1}^{8} P_j \theta_j \) as

\[
P \theta = PUU^T \theta = P \theta'.
\] (33)
Thus, in terms of the new phase parameters, the PN phase becomes

$$\Psi(f) = 2\pi f_\text{c} - \Phi_\text{c} + \frac{\pi}{4} + \sum_{k=1}^{8} P_k' \theta_k'.$$

(34)

In the next section, we give a comparison of how various $\psi_k$'s behave as a function of the total mass (for equal mass systems) in the original and new representations; see e.g. figure 3.

5. Results and discussions

In this section, we demonstrate the outcome of using the truncated SVD approach to the parameter estimation problem we were interested in and present the errors on the new phase parameters. Before we start discussing our results, it is worth recalling the typical SNRs expected from the aLIGO and ET detections as a function of the total mass. This is presented in figure 1. Roughly speaking, ET events will have an SNR which is an order of magnitude higher than that of aLIGO, thanks to the smaller seismic cut-off and improved sensitivity. In the figure, the distance to the source is assumed to be 100 Mpc and we have considered binaries with equal masses.

We now proceed and compute the Fisher matrix for a particular binary system with a fixed mass following the prescription outlined in section 3. This amounts to computing the derivatives w.r.t. the various parameters (the phasing coefficients) and performing the integration in equation (17). We then obtain the matrix $H$ and carry out the SVD using the in-built Matlab function. We observe that if we set $\epsilon \sim 10^{-3}$, the first three singular values are non-zero with the rest of the diagonal entries decreasing rapidly for all total masses considered. We retain only the first three $(k = 1, 2, 3)$ leading singular values of $\Sigma$ and truncate $\Sigma$ to a $3 \times 3$ matrix. As shown earlier, the above procedure is equivalent to first diagonalize the Fisher matrix and then rank the diagonal entries in the descending order and retain only
Figure 2. Left panel: ratio of singular values of the Fisher matrix w.r.t. the dominant one ($\frac{\sigma_k}{\sigma_1}$) as a function of $k$, where $k$ denotes the different singular values and runs from 1 to 8. Right panel: for the case of aLIGO detector, variation of relative weights as a function of total mass for various $k$ values. Both the panels are for equal mass systems ($\eta = 0.25$).

Figure 3. $|\psi_k|$ (left panel) and $|\psi'_k|$ (right panel) as functions of the total mass of the binary for the equal mass case.

the first three eigenvalues, ignoring the rest. Both the procedures are numerically tested to be equivalent. Values of the diagonal entries (normalized to 1) are plotted as a function of the index of the singular values in figure 2 for two different total masses and for aLIGO and ET configurations. From the figure, it is obvious that eigenvalues corresponding to $k \geq 4$ are much smaller than the eigenvalues of the first parameter. This means that the contribution of the parameters with $k \geq 4$ is negligible in comparison with the first three parameters. Thus, we omit all eigenvalues higher than the first three. By diagonalizing the Fisher matrix, we obtain a set of new parameters as linear combinations of the old parameters which are the new phasing coefficients.

In figure 3, we plot the old and the new $|\psi_k|$ with the total mass of the binary. For the new phasing coefficients, we have shown the curves only for the dominant three given by the truncated SVD. The original phasing coefficients have essentially power law dependences on the total mass and hence are linear in the log–log scale. The only exception to this is $\psi_6$ which has a term proportional to $M^{1/3} \ln M$ and hence not linear. The new parameters which are linear combinations of the old ones are not linear in the log–log scale. Another interesting thing to note is that the values of the old $|\psi_k|$ decrease with increasing $k$, whereas the new $|\psi'_k|$ values increase with $k$.

The inverse of the diagonalized $3 \times 3$ Fisher matrix $\Sigma'$ directly gives the covariance matrix corresponding to the new set of phase parameters $\mathbf{P}'$. The square root of the diagonal
In figure 4, we show the relative errors associated with the measurement of the three new parameters, as a function of mass for aLIGO and ET configurations. For masses $< 50M_\odot$, the relative errors in the new parameters (termed as $\psi'_1, \psi'_2, \psi'_3$) are $\lesssim 10^{-3}$ for aLIGO and $\lesssim 10^{-5}$ for ET. We note that the removal of singularity improves the relative errors immensely. This is the primary result of this paper. This new set of parameters, which can be measured more accurately, should be more useful for devising parametrized tests of PN theory.

It should be borne in mind, while looking at figure 4, that what we have shown are only the statistical errors arising from the noise. There can be systematic errors due to the fact that PN waveforms are only approximate. There have been proposals in the literature to quantify the systematic biases due to inaccurate waveform modelling [46]. We have not considered this aspect in this work.

In figure 5, we plot the contours of relative errors for a general case of unequal mass binaries in the $M - \eta$ plane. The colours of the contours show the values of the relative errors. From the contours, it is clear that the approach works well for the entire range of $\eta$ values. Throughout the parameter space, the SVD approach can provide the three dominant parameters (for $\epsilon \sim 10^{-3}$) which can be very well estimated using matched filtering. To understand the
relative error trends, we focus the reader’s attention that there are two competing factors that affect the relative errors, namely the SNR and the number of cycles spent in the detector band. For the inspiral phase, $\text{SNR} \propto M^{5/6} \eta^{1/2}$. The number of cycles $\mathcal{N}_{\text{cycles}} \propto M^{-5/3} \eta^{-1}$.

For a fixed $\eta$ case, parameters of the binaries with low masses spend many more cycles in the detector band compared to the massive system. Although the SNR increases with the total mass, the effect of these two factors together makes the relative error increase with total mass as $M^{5/6}$. In contrast, for a fixed total mass and the varying $\eta$ case, systems with smaller $\eta$ (more asymmetric) spend more time in the band (see the left panel of figure 6). Here, the number of cycles is plotted as a function of $\eta$ which varies as $\eta^{-1}$ [30]. The SNR varies as $\eta^{1/2}$. The net product gives the relative error variation as $\eta^{-1/2}$ which is inverse of the square root of the number of cycles. In the right panel of figure 6, we plot the relative error of $\psi'_1$ as a function of symmetric mass ratio $\eta$ for fixed total mass $M$ for aLIGO noise. Analytical fit of the relative error as a function of $\eta$ shows that the relative error indeed varies as $\sim \eta^{0.5}$.

Although we have considered both aLIGO and ET on equal footing for our calculations, we wish to emphasize that aLIGO is the upgraded version of initial LIGO detector and is fully funded and instrumentation for which have already started. ET, on the other hand, is a proposed third generation interferometer, still in the phase of design studies.
Figure 6. Fixed total mass (2 $M_\odot$, 10 $M_\odot$, 18 $M_\odot$) and varying $\eta$ for aLIGO. (a) Number of cycles as a function of $\eta$. (b) Relative error of $\psi'_1$ as a function of $\eta$. The analytical fit for $M = 2\, M_\odot$, error $= 7.5 \times 10^{-5} \eta^{0.4925}$, for $M = 10\, M_\odot$, error $= 17.25 \times 10^{-5} \eta^{0.4947}$ and $M = 18\, M_\odot$, error $= 22.8 \times 10^{-5} \eta^{0.5066}$.

6. Conclusions and future directions

We revisited the parameter estimation problem where each of the PN phasing coefficients is treated as independent parameters and used to devise a self-consistency test of PN approximation to GR. We showed that using singular value decomposition of the Fisher matrix, it is possible to deduce the three new combinations of the phasing coefficients which are best estimated using matched filtering. The Fisher matrix and hence the covariance matrix are diagonal in this new phase parameters which in turn are linear combinations of the original PN phasing coefficients. Pertaining to the removal of the singular nature, the new phase parameters can be estimated very accurately with very small relative errors $\sim 10^{-3}$–$10^{-6}$ for aLIGO and ET configurations. This covers the BNS and NS-stellar mass BH cases for aLIGO and BNS, NS-BH and IMBH binaries for ET. We studied how the errors in the new parameters vary as a function of the total mass of the binary as well as the symmetric mass ratio $\eta$.

However, here we have not discussed in detail how to rephrase the parametrized tests of PN theory in terms of these new parameters. This requires devising a consistency test using this new set of phasing parameters. One approach could be the traditional approach to follow the procedure outlined in [18–20] where the consistency test(s) were carried out in the mass plane of the binary. One may carry out similar tests in the $m_1 - m_2$ plane in terms of the three dominant parameters obtained using SVD. Another approach could be to fold in the new parameters in the Bayesian framework following the work of [24] in which case the formulation and the implementation of the test are entirely different from the first one. Hence, our work should be seen as a first step towards the final goal of devising tests of PN theory where we have parameters which can be estimated much more accurately than those in the original proposal. These issues will be addressed in detail in future work. We foresee that since these new phase parameters pertain to the principle components of the Fisher matrix which in turn is related to the maximal variation of the signal, they could also be useful in the detection scenario. At present, we are investigating these issues.

Lastly, we would like to emphasize that the use of SVD in the context of tests of GR is not limited to parametrized tests of PN theory. It may be even more useful, for example, in the case of PPE framework where the number of free parameters is much more due to the
generality of the waveform. It will be interesting in future to investigate the implications of SVD for PPE formalism.

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Note added. After this paper was finalized and circulated as a preprint, a related work by Brown et al [47] appeared where they addressed the detection problem of inspiralling spinning binaries. They discuss the construction of a template bank for spinning searches using a waveform where all of the PN terms in the phasing are treated as independent. They diagonalize the metric in the eight-dimensional parameter space to obtain a reduced two-dimensional parameter space which they use for the template placement and the signal detection problem. They write down the PN phasing formula in terms of a dimensionless variable $x = f/f_0$, where $f_0$ is some arbitrary frequency scale. The new parameters we defined should correspond to $f_0 = 1$ Hz in their approach.

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